SPECTRAL EXPONENTIAL SUMS ON HYPERBOLIC SURFACES

IKUYA KANEKO

Abstract. We study an exponential sum over Laplacian eigenvalues \( \lambda_j = 1/4 + t_j^2 \) with \( t_j \leq T \) for Maass cusp forms on \( \Gamma \setminus \mathbb{H} \), where \( \Gamma \) is a cofinite Fuchsian group acting on the upper half-plane \( \mathbb{H} \). The aim is to establish an asymptotic formula which expresses spectral exponential sums in terms of an oscillatory component, von Mangoldt-like functions and Selberg zeta functions. The behaviour is determined by whether \( \Gamma \) is essentially cuspidal or not.

1. Introduction

1.1. Motivation. This article introduces a new idea to prove an asymptotic law for the spectral exponential sum

\[
S(T, X) = \sum_{t_j < T} X^{-t_j^2}
\]

in the spectral aspect. Here \( \lambda_j = s_j(1-s_j) = 1/4 + t_j^2 \) are non-exceptional eigenvalues of the hyperbolic Laplacian acting on \( L^2(\Gamma \setminus \mathbb{H}) \) for a congruence subgroup \( \Gamma \subseteq \text{SL}_2(\mathbb{Z}) \) and the upper half-plane \( \mathbb{H} \). We henceforth exploit the sign convention \( t_j > 0 \). There are various applications of spectral exponential sums, but we elaborate on the Prime Geodesic Theorem for ease of exposition. This concerns the asymptotic behaviour of the counting function

\[
\pi_T(X) = \#\{ P : N(P) \leq X \},
\]

where \( \{ P \} \) signifies a primitive hyperbolic conjugacy class in \( \Gamma \) and \( N(P) \) stands for its norm. We call \( \{ P \} \) primitive if a hyperbolic element \( P \in \Gamma \) cannot be expressed as \( Q^j \) with \( j \geq 2 \) for some \( Q \in \Gamma \); hence every hyperbolic conjugacy class is a power of some primitive class. By partial summation, one passes between asymptotic results for \( \pi_T(X) \) and the allied counting function

\[
\Psi_T(X) = \sum_{N(P) \leq X} \Lambda_{\Gamma}(P),
\]

where the sum runs over all hyperbolic classes and \( \Lambda(\cdot) \) is the analogue of the von Mangoldt function defined by \( \Lambda_{\Gamma}(P) = \log N(P_0) \) if \( \{ P \} \) is a power of a primitive hyperbolic class \( \{ P_0 \} \).

In his seminal work, Iwaniec [10] obtained the explicit formula for \( \Psi_T(X) \) as a corollary of the Selberg trace formula and an inequality à la Brun–Titchmarsh; see [3, 10, 13]. For any congruence subgroup \( \Gamma \), we have that

\[
\Psi_T(X) = X + \sum_{1/2 < s_j < 1} \frac{X^{s_j}}{s_j} + \sum_{|t_j| < T} \frac{X^{t_j}}{t_j} + O \left( \frac{X}{T} \log X \right)^2
\]

for an auxiliary parameter \( 1 \leq T \leq X^{1/2} (\log X)^{-2} \). The first sum on the right-hand side arises from the exceptional eigenvalues of the Laplacian and the sum \( \sum_{|t_j| < T} \) denotes that it is symmetrised by including both \( t_j \) and \( -t_j \). The explicit formula (1.2) implies that even when \( T \) is suitably controlled, one cannot reach an error term smaller than \( O(X^{3/4 + \epsilon}) \) without considering any cancellation in the spectral exponential sum. It behoves us to mention that the corresponding barrier for \( S(T, X) \) is \( O(T^2) \), which is called the trivial bound.

We recall that the Selberg zeta function is built out of prime geodesics as follows:

\[
Z_T(s) = \prod_{\{ P_0 \}} \prod_{k=0}^\infty \left( 1 - N(P_0)^{-s-k} \right),
\]

Date: December 31, 2021.
2010 Mathematics Subject Classification. Primary 11M36; Secondary 11F72.
Key words and phrases. Spectral exponential sum, Prime Geodesic Theorem, Weyl’s law.
The author is supported in part by the Masason Foundation and the Spirit of Ramanujan STEM Talent Initiative.
where the outer product runs over all primitive hyperbolic classes. Given the analogue of the Riemann hypothesis for the Selberg zeta functions of congruence subgroups (apart from a finite number of exceptional zeroes), we should conjecture that \( \Psi_\Gamma(X) = X + O(X^{1/2+\varepsilon}) \). This remains an impenetrable open problem. For convenience, we define \( \eta \) to be such that the following formula holds:

\[
\pi_\Gamma(X) = \text{li}(X) + O_\varepsilon(X^{\eta+\varepsilon}), \quad \text{li}(X) = \int_2^X \frac{dt}{\log t}, \tag{1.4}
\]

In broad strokes, the optimal value of \( \eta \) so far established marks the level of current technology.

If \( \Gamma \) is arithmetic, an improvement over the 3/4-barrier can be deduced by appealing to the Kuznetsov formula. A general rule of thumb is that using the Selberg trace formula leads to cleaner formulæ than using the Kuznetsov formula — for estimations the latter is known to be more beneficial nonetheless. In retrospect, the first landmark triumph to go beyond the 3/4-barrier was due to Ivaniec \cite{Iv}, who showed for \( \Gamma = \text{PSL}_2(\mathbb{Z}) \) that

\[
S(T, X) \ll \varepsilon TX^{11/48+\varepsilon}
\]

with \( T, X \gg 1 \). This bound was improved in the celebrated work of Luo–Sarnak \cite{LS3}:

\[
S(T, X) \ll X^{1/8}T^{5/4}(\log T)^2.
\]

This is also achievable for any arithmetic group \( \Gamma \subset \text{PSL}_2(\mathbb{R}) \) such as a congruence subgroup and a subgroup arising from a quaternion division algebra. They showed \( \eta = 7/10 \) according to the optimal choice \( T = X^{3/10} \). The crucial step in all of these works was to showcase a nontrivial bound on \( S(T, X) \). Soundararajan–Young \cite{SY} succeeded in showing the best known bound for \( \text{PSL}_2(\mathbb{Z}) \), namely \( \eta = 2/3 + \theta/6 \) with \( \theta \) a subconvex exponent for quadratic Dirichlet \( L \)-functions. The Weyl-strength exponent \( \theta = 1/6 \) of Conrey–Iwaniec \cite{CI} yields \( \eta = 25/36 \). The conjectural exponent \( \eta = 2/3 \) follows from the Lindelöf hypothesis for Dirichlet \( L \)-functions. The author \cite{K1} recently established an exact connection between pointwise and second moment bounds for the error term in (1.4).

It is important to speculate what the correct order of \( S(T, X) \) should be. Petridis–Risager \cite[Conjecture 2.2]{PR} have conjectured square root cancellation, namely

\[
S(T, X) \ll _\varepsilon T(TX)^{\varepsilon}.
\]

Moreover, the bound (1.5) yields not only the best possible error term \( O(X^{1/2+\varepsilon}) \) in the Prime Geodesic Theorem, but also the best error term on average for the hyperbolic lattice point counting. In the appendix of \cite{K1}, Laaksonen has proven via the Selberg trace formula that the conjecture (1.5) is true for a fixed \( X > 1 \) in the spectral aspect.

1.2. Results. This work considers the congruence subgroups \( \Gamma_0(q), \Gamma_1(q) \) and \( \Gamma(q) \) defined in Section 2.2. Just before stating our results, we introduce the following standard notation. Let \( \Lambda(X) \) be the von Mangoldt function extended to \( \mathbb{R} \) by defining it to be 0 when \( X \) is not a prime power. Define an analogous function \( \Lambda_\Gamma(X) \) for norms of hyperbolic conjugacy classes in \( \Gamma \) as

\[
\Lambda_\Gamma(X) := \begin{cases} 
\log N(P_0) & \text{if } X = N(P)^j, \ j \geq 1, \\
0 & \text{otherwise}.
\end{cases}
\]

One of our aims is to establish that the spectral exponential sum \( S(T, X) \) associated to a congruence subgroup \( \Gamma \) obeys a conjectural bound in the spectral aspect to which we have already alluded.

Theorem 1.1. For \( \Gamma = \Gamma_0(q), \Gamma_1(q), \Gamma(q) \), we define

\[
S(T) = \frac{1}{\pi} \arg Z_\Gamma(\frac{1}{2} + iT) \quad \text{and} \quad \mathcal{G}(T) = \int_{1/2}^{T/2} \log |Z_\Gamma(\sigma + iT)|d\sigma.
\]

For a fixed \( X > 1 \), we then have

\[
S(T, X) = \frac{\text{vol}(\Gamma \backslash \mathbb{H})}{2\pi i \log X} X^{iT} + \frac{T}{2\pi} (X^{1/2} - X^{-1/2})^{-1} \Lambda_\Gamma(X) + \frac{T}{\pi} X^{-1/2} \Lambda(X^{1/2}) \sum_{\psi} \Re(\psi(X^{1/2})) + X^{iT} S(T) + O(\mathcal{G}(T)),
\]

where the sum runs over Dirichlet characters \( \psi \) to some modulus dividing \( q \) and the meaning of \( \bullet \) on the sum is described in Lemma 2.2.
The formula (1.6) for $\Gamma = \text{PSL}_2(\mathbb{Z})$ was announced without a proof by Fujii in 1984 in his report [7]. Note that his theorem recovers Laaksonen’s result. In Fujii’s work, the additional term $X^{1/2}S(T)$ was replaced by $O(T/\log T)$, which follows from the trivial bounds $S(T) \ll T/\log T$ and $\mathcal{S}(T) \ll T/\log T$. We may revisit the issue of bounding $\mathcal{S}(T)$ in a cleverer manner, elsewhere. Theorem 1.1 generalises his result to the congruence subgroups $\Gamma_0(q)$, $\Gamma_1(q)$ and $\Gamma(q)$. It makes us believe that the extremely strong bound $\Psi(\xi) - X \ll X^{1/2+\epsilon}$ may hold.

**Remark 1.2.** For a fixed $X > 1$, the spectral exponential sum $S(T, X)$ has a peak of order $T$ whenever $X$ is equal to a power of a norm of a primitive hyperbolic class in $\Gamma$, or to an even power of a prime number which comes from the determinant of the scattering matrix associated with $\Gamma$. At the peak point $X$ corresponding to the peak of $S(T, X)$, the second or the third term in the asymptotic formula (1.6) does not vanish. Indeed, $S(T, X)$ does not always exhibit a clear peak structure; see [16, Appendix].

Remark 1.2 for $\Gamma = \text{PSL}_2(\mathbb{Z})$ is in accordance with the theorems by Chazarain [5], where for the wave kernel the singularities occur at the lengths of closed geodesics. Via Theorem 1.1, one can utilise $S(T, X)$ as a roundabout way of detecting $N(P)$. There must be a connection underneath the surface between spectral parameters $t_j$ and the length spectrum in $\mathbb{H}$. The spectral exponential sum $S(T, X)$ differs from the classical one counted with Riemann zeroes where the peaks are known to be at all prime powers. By Theorem 1.1, we conjecture the following square root cancellation for $S(T, X)$, which generalises the conjecture of Petridis–Risager for $\Gamma = \text{PSL}_2(\mathbb{Z})$.

**Conjecture 1.3.** Let $X > 2$. For any arithmetic hyperbolic surface, the spectral exponential sum $S(T, X)$ exhibits square root cancellation in $T$ with uniform dependence on $X$ up to a factor of $X^\epsilon$, namely

$$S(T, X) \ll T^{1+\epsilon} X^\epsilon.$$

**Acknowledgements**

Thanks are owed to Shin-ya Koyama for pointing out errors in an earlier version of this article.

2. Sketch of requisites

We provide background material which we shall need later to establish Theorem 1.1.

2.1. Weyl’s law. The spectrum of the Laplacian on hyperbolic surfaces has a connection with certain objects in number theory such as $L$-functions and exponential sums. There are a number of conjectures centered around the structure of the discrete spectrum. For a general cofinite group, it remains an open problem which spectrum is larger in order of magnitude; see Section 4. Weyl’s law renders asymptotic behaviour of both discrete and continuous spectrum in an expanding window. For any Fuchsian group $\Gamma$ of the first kind, it asserts that

$$N_\Gamma(T) + M_\Gamma(T) \sim \frac{\text{vol}(\Gamma \backslash \mathbb{H})}{4\pi} T^2$$

(2.1) as $T \to \infty$, where $N_\Gamma(T) := \#\{ j : t_j \leq T \}$ and $M_\Gamma(T)$ is the winding number which accounts for the contribution of the continuous spectrum:

$$M_\Gamma(T) := \frac{1}{4\pi} \int_{-T}^T - \frac{\varphi'}{\varphi} \left( \frac{1}{2} + it \right) \, dt.$$

Here $\varphi$ signifies the determinant of the scattering matrix $\Phi$ of $\Gamma$. In order to establish the asymptotic formula (2.1), the complete spectral decomposition of an automorphic kernel is needed. The Selberg trace formula is a beneficial tool for deducing (2.1). If $\Gamma \backslash \mathbb{H}$ is compact, $M_\Gamma(T)$ vanishes and the identity (2.1) can be reduced to the asymptotic behaviour of $N_\Gamma(T)$. We note that $M_\Gamma(T)$ is real and approximately equal to the number of scattering poles on the left of the critical line $\Re(s) = 1/2$ with heights at most $T$.

As we shall see later, the formula (2.1) is not satisfactory for our purpose. We recall a refined version of Weyl’s law (cf. [21, Theorem 5.2.1], [22, Theorem 7.2], [11, (11.3)]), which is available for the general cofinite scenario:

**Theorem 2.1.** Let $\Gamma$ be a cofinite subgroup of $\text{PSL}_2(\mathbb{R})$. We then have that

$$N_\Gamma(T) + M_\Gamma(T) = \frac{\text{vol}(\Gamma \backslash \mathbb{H})}{4\pi} T^2 - \frac{h}{\pi} T \log T + c_\Gamma T + S(T) + \text{const.} + w(T)$$

with $w(T) \ll T^{1/2+\epsilon}$.
as $T \to \infty$, where $h$ is the number of inequivalent cusps and $c_\Gamma$ is a certain constant dependent only on $\Gamma$, $S(T)$ is the same as in the introduction, and $w(T)$ can be chosen such that $w'(T) \ll T^{-2}$.

Theorem 2.1 can be deduced almost verbatim the argument used to establish (2.1).

2.2. **Scattering determinants.** In this subsection, we contemplate the congruence subgroups

$$\Gamma_0(q) := \left\{ \begin{array}{c} a \ b \\ c \ d \end{array} \right\} \in \text{PSL}_2(\mathbb{Z}) : c \equiv 0 \pmod{q},$$

$$\Gamma_1(q) := \left\{ \begin{array}{c} a \ b \\ c \ d \end{array} \right\} \in \text{PSL}_2(\mathbb{Z}) : a, d \equiv 1 \pmod{q}, \ c \equiv 0 \pmod{q},$$

$$\Gamma(q) := \left\{ \begin{array}{c} a \ b \\ c \ d \end{array} \right\} \in \text{PSL}_2(\mathbb{Z}) : a, d \equiv 1 \pmod{q}, \ b, c \equiv 0 \pmod{q}. $$

Recall that

$$\text{vol}(\Gamma \backslash \mathbb{H}) = \frac{\pi}{3} [\text{PSL}_2(\mathbb{Z}) : \Gamma] = \begin{cases} \frac{\pi}{3} q \prod_{p\mid q} \left( 1 + \frac{1}{p} \right) & \text{for } \Gamma = \Gamma_0(q), \\ \frac{\pi}{3} q^2 \prod_{p\mid q} \left( 1 - \frac{1}{p^2} \right) & \text{for } \Gamma = \Gamma_1(q), \\ \frac{\pi}{3} q^3 \prod_{p\mid q} \left( 1 - \frac{1}{p^2} \right) & \text{for } \Gamma = \Gamma(q). \end{cases}$$

For $\Gamma = \Gamma_0(q)$, $\Gamma_1(q)$, or $\Gamma(q)$, an accurate calculation of $\varphi$ was executed by Hejhal and Huxley for squarefree $q$ and for every $q$, respectively. For notational convenience, we only record the result of Huxley.

**Theorem 2.2** (Huxley [9]). Let $\Gamma = \Gamma_0(q)$, $\Gamma_1(q)$, or $\Gamma(q)$. Then the scattering determinant of $\Gamma$ is given by

$$\varphi(s) = (-1)^{(h-h_0)/2} \left( \frac{\Gamma(1-g)}{\Gamma(s)} \right)^h \left( \frac{A}{\pi^n} \right)^{1-2s} \prod_{\psi} L(2-2s, \psi) \frac{L(2s, \psi)}{L(2s, \psi)}. \quad (2.2)$$

Notation is as follows: $h$ equals the number of inequivalent cusps, and $h_0$ is an integer for which

$$\text{Tr}(\Phi(s)) \to -h_0 \quad \text{as} \quad s \to \frac{1}{2}.$$

The Dirichlet characters $\psi$ appearing in the product can be expressed as

$$\psi(n) = \psi_1(n)\psi_2(n)\omega_{m_1m_2}(n) \quad \text{for} \quad n \in \mathbb{N},$$

where $\psi_\ell$ ($\ell = 1, 2$) is a primitive Dirichlet character modulo $q_\ell$ and $\omega_{m_1m_2}$ is the trivial character modulo $m_1m_2$. As for the product over $\psi$, the variables $q_1, q_2, m_1,$ and $m_2$ are over all positive integers satisfying the conditions below, and $\psi_\ell$ ($\ell = 1, 2$) runs over all possible Dirichlet characters.

(a) $(m_1, m_2) = 1, m_1q_1 \mid q, m_2q_2 \mid q$ for $\Gamma = \Gamma(q),$

(b) $(a)$ and $m_1 = 1, q_1 \mid m_2$ for $\Gamma = \Gamma_1(q),$

(c) $(b)$ and $q_1 = q_2, \psi_1 = \psi_2$ for $\Gamma = \Gamma_0(q).$

The product in (2.2) has $h$ terms and $A$ is a positive integer composed of primes dividing $q$:

$$A = \begin{cases} \prod_{(a)} m_1m_2q_1q & \text{for } \Gamma = \Gamma(q), \\ \prod_{(b)} q_1q & \text{for } \Gamma = \Gamma_1(q), \\ \prod_{(c)} (m_2, q/m_2) & \text{for } \Gamma = \Gamma_0(q). \end{cases}$$
3. Proof of Theorem 1.1

This section is aimed at proving Theorem 1.1 with the tools in §2 in mind.

**Proof of Theorem 1.1.** Using Theorem 2.1, one obtains

\[
S(T, X) = \int_1^T X^it \, dN_\Gamma(t) = \frac{\vol(\Gamma \backslash \mathbb{H})}{2\pi} \int_1^T X^it \, dt - \int_1^T X^it \, dM_\Gamma(t) + \int_1^T X^it \, dS(T) + O(\log T) \quad (3.1)
\]

\[
= \mathcal{L}^1 + \mathcal{L}^2 + \mathcal{L}^3 + O(\log T).
\]

We first handle \( \mathcal{L}^1 \) for which we use integration by parts to deduce

\[
\mathcal{L}^1 = \frac{\vol(\Gamma \backslash \mathbb{H})}{2\pi i} \frac{X^iT}{\log X} T + O(1).
\]

As for the second term \( \mathcal{L}^2 \), we calculate the scattering determinant via Theorem 2.2:

\[
-\frac{\varphi}{\varphi} \left( \frac{1}{2} + i \right) = 2 \log \frac{A}{\pi h} + \frac{\Gamma'}{\Gamma} \left( \frac{1}{2} + it \right) + 2 \sum_\psi \left( \frac{L'}{L} (1 - 2it, \overline{\psi}) + \log \frac{L(1 + 2it, \psi) L(1 + 2it, \overline{\psi})}{L(1 + 2it, \psi) L(1 - 2it, \overline{\psi})} \right). 
\]

Hence we exploit the Stirling asymptotics, deriving

\[
M_\Gamma(t) = \frac{1}{4\pi i} \log \prod_\psi \left( \frac{L(1 + 2it, \psi) L(1 + 2it, \overline{\psi})}{L(1 - 2it, \psi) L(1 - 2it, \overline{\psi})} \right) + O(t \log t). \quad (3.2)
\]

We use integration by parts again and then bounding trivially the Dirichlet \( L \)-functions in (3.2) yields

\[
\mathcal{L}^2 = -\frac{1}{4\pi i} \sum_\psi \int_1^T X^it \, d \log \left( \frac{L(1 + 2it, \psi) L(1 + 2it, \overline{\psi})}{L(1 - 2it, \psi) L(1 - 2it, \overline{\psi})} \right) + O(\log T)
\]

\[
= \log X \sum_\psi \int_1^T X^it \left( \log(L(1 + 2it, \psi) L(1 + 2it, \overline{\psi})) - \log(L(1 - 2it, \psi) L(1 - 2it, \overline{\psi})) \right) dt + O(\log T).
\]

Letting \( \delta = (\log X)^{-1} \), the first integration involving \( \log(L(1 + 2it, \psi) L(1 + 2it, \overline{\psi})) \) equals

\[
\frac{1}{2i} \int_{1+2i}^{1+2i+2T} X^{(t-1)/2} \log(L(t, \psi) L(t, \overline{\psi})) \, dt
\]

\[
= \frac{1}{2i} \int_{1+2i}^{1+2i+2T} - \int_{1+2i}^{1+2i+2i} + \int_{1+2i}^{1+2i+2i} X^{(t-1)/2} \log(L(t, \psi) L(t, \overline{\psi})) \, dt
\]

\[
= \frac{1}{2i} \int_{1+2i}^{1+2i+2i} X^{(t-1)/2} \log(L(t, \psi) L(t, \overline{\psi})) \, dt + O(\log T)
\]

\[
= X^{\delta/2} \sum_{n=2}^{\infty} \frac{\varphi(n) + \overline{\varphi(n)} \Lambda(n)}{n^{1+\delta} \log n} \int_1^T \exp(it(\log X - 2 \log n)) \, dt + O(\log T)
\]

\[
= \frac{2T}{\log X} X^{-\delta/2} \left( \varphi(X^{1/2}) + \overline{\varphi(X^{1/2})} \right) \Lambda(X^{1/2}) + O(\log T),
\]

where \( \Lambda(X) \) is the classical von Mangoldt function. The second integral involving \( \log(L(1 - 2it, \psi) L(1 - 2it, \overline{\psi})) \) is obviously bounded, since the corresponding integrand becomes \( \exp(it(\log X + 2 \log n)) \).

It remains to deal with the third term \( \mathcal{L}^3 \). By partial integration, we have

\[
\mathcal{L}^3 = X^{iT} S(T) - i \log X \int_1^T X^it S(T) \, dt + O(1) = X^{iT} S(T) + \mathcal{L}^4 \log X - i \mathcal{L}^5 \log X + O(1),
\]

where

\[
\mathcal{L}^4 = \int_1^T \sin(t \log X) S(T) \, dt \quad \text{and} \quad \mathcal{L}^5 = \int_1^T \cos(t \log X) S(T) \, dt.
\]
For the Selberg zeta function in (1.3), we define
\[ F(z) = \log \Lambda_1(z) \sin \left( \left( \frac{1}{2} - z \right) i \log X \right), \]
where we choose the principal value of the logarithm and the branch of \( \log \Lambda_1(z) \) is taken such that \( \log \Lambda_1(z) \) is real for every \( z > 1 \). Mimicking the treatment of \( \mathcal{L}^2 \), we consider the rectangle with vertices \( 1 + \delta + it, 1 + \delta + iT, 1/2 + iT \) and \( 1/2 + i \) with \( \delta = (\log X)^{-1} \). Hence, it follows that
\[
\mathcal{L}^4 = \mathcal{S} \left( \frac{1}{\pi i} \int_{1/2+i}^{1+\delta+iT} F(z)dz \right) = \mathcal{S} \left( \frac{1}{\pi i} \int_{1+\delta+i}^{1+\delta+iT} \int_{1/2+i}^{1+\delta+iT} F(z)dz \right).
\]
The third integral is bounded. From the definition (1.3), we compute the first integral as
\[
\frac{1}{\pi} \int_{1}^{T} \log \Lambda_1(1 + \delta + it) \sin \left( \left( t - \left( \frac{1}{2} + \delta \right) i \right) \log X \right) dt
= -\frac{1}{2\pi i} \sum_{(P)} \frac{\Lambda_1(P)(1 - N(P)^{-1})^{-1}}{N(P)^{1+\delta} \log N(P)} \int_{1}^{T} \exp \left( \left( t - \left( \frac{1}{2} + \delta \right) i \right) \log X \right) dt
+ \frac{1}{2\pi i} \sum_{(P)} \frac{\Lambda_1(P)(1 - N(P)^{-1})^{-1}}{N(P)^{1+\delta} \log N(P)} \int_{1}^{T} \exp \left( \left( t - \left( \frac{1}{2} + \delta \right) i \right) \log X \right) dt
= -\frac{X^{1/2+\delta}}{2\pi i} \sum_{(P)} \frac{\Lambda_1(P)(1 - N(P)^{-1})^{-1}}{N(P)^{1+\delta} \log N(P)} \int_{1}^{T} \exp(it(\log X - \log N(P))) dt + O(1)
= -\frac{\Lambda_1(X)}{2\pi i X^{1/2} \log X} \left( X^{1/2} - X^{-1/2} \right)^{-1} T + O(1),
\]
whence we obtain
\[
\mathcal{S} \left( \frac{1}{\pi i} \int_{1+\delta+i}^{1+\delta+iT} F(z)dz \right) = \frac{T}{2\pi \log X} X^{-1/2} \left( X^{1/2} - X^{-1/2} \right)^{-1} \Lambda_1(X) + O(1).
\]
Finally, the second integral can be bounded as
\[
\frac{1}{\pi i} \int_{1/2+iT}^{1+\delta+iT} F(z)dz = \frac{1}{\pi} \int_{1/2}^{1+\delta+i} \log \Lambda_1(\sigma + iT) \sinh((\sigma - 1 + iT) \log X) d\sigma \ll \mathcal{S}(T),
\]
where \( \mathcal{S}(T) \) is the same as in the introduction. The integral \( \mathcal{L}^5 \) is estimated in a similar manner, namely \( \mathcal{L}^5 \ll \mathcal{S}(T) \). Collecting those estimates concludes the proof of Theorem 1.1. \( \square \)

4. Counterexamples

4.1. Phillips–Sarnak theory. We are interested in the validity of the asymptotic law \( N_\Gamma(T) \sim \text{vol}(\Gamma \backslash \mathbb{H}) T^2 / 4\pi \). In a major breakthrough, Phillips–Sarnak [17] innovated a new methodology of the real analytic deformation of discrete groups in \( \text{PSL}_2(\mathbb{R}) \). They examined the behaviour of a Maaß cusp form for \( \Gamma_0(p) \) under quasi-conformal deformations \( \Gamma_\tau \), with \( 0 \leq \tau \leq 1 \). The folklore theory of Phillips–Sarnak manifests that Maaß cusp forms are rare whose existence should be restricted to certain arithmetic groups. The work of Luo [14] yields that Weyl’s law
\[ N_{\Gamma_\tau}(T) \sim \frac{\text{vol}(\Gamma \backslash \mathbb{H})}{4\pi} T^2 \]
cannot hold for generic \( \Gamma_\tau \) if one assumes that the eigenvalue multiplicities for \( \Gamma_0(p) \backslash \mathbb{H} \) are bounded.

Observation 1. Non-arithmetic groups should not be essentially cuspidal and there should only be a finite number of Maaß cusp forms in such a case. If the latter assertion is true, the bound \( \mathcal{S}(T, X) \ll 1 \) holds.

We say that \( \Gamma \) is essentially cuspidal if \( N_\Gamma(T) \) dominates the behaviour of \( N_\Gamma(T) + M_\Gamma(T) \), namely if
\[ N_\Gamma(T) \sim \frac{\text{vol}(\Gamma \backslash \mathbb{H})}{4\pi} T^2. \]
Here we consider the eigenvalue multiplicity problem for arithmetic groups with an emphasis on the case of \( \Gamma_0(q) \).
The case of \( q = 1 \) was first considered by Cartier [4] and consequently he conjectured that the cuspidal spectrum is
simple based on limited numerical data. Nowadays, intensive numerical computations of Then [20] are available, determining the first 53000 eigenvalues for $\text{PSL}_2(\mathbb{Z}) \backslash \mathbb{H}$.

For $q \gg 2$, the situation is entirely different, since the cuspidal spectrum for $\Gamma_0(q) \backslash \mathbb{H}$ is not necessarily simple, which follows from the existence of newforms and oldforms having the same Laplace eigenvalue. Nevertheless, one can ask whether the new part of the cuspidal spectrum consisting of eigenvalues for $\Gamma_0(q) \backslash \mathbb{H}$ associated to newforms is simple. This conjecture should be true for squarefree $q$, which is manifested in the work of Bolte–Johansson [1, 2] and Strömbergsson [19]. We also refer the reader to the recent work of Humphries [8].

Since the estimate $S(T) \ll T/\log T$ holds for all cofinite surfaces, our proof of Theorem 1.1 works for more extensive context, although the integral entailing the winding number remains. One then derives from (3.1) that

$$S(T, X) = \frac{\text{vol}(\Gamma \backslash \mathbb{H})}{2\pi} \frac{X^T}{\log X} T + \frac{T}{2\pi} \left( X^{1/2} - X^{-1/2} \right)^{-1} \Lambda_f(X) - \int_1^T X^{\alpha(t)} dM_f(t) + O \left( \frac{T}{\log T} \right).$$ (4.1)

If $\Gamma$ is essentially cuspidal, the spectral exponential sum $S(T, X)$ asymptotically equals the first term in (4.1). As an important instance, if $\Gamma \backslash \mathbb{H}$ is compact, then the aforementioned formula can be reduced to

$$S(T, X) = \frac{\text{vol}(\Gamma \backslash \mathbb{H})}{2\pi} \frac{X^T}{\log X} T + \frac{T}{2\pi} \left( X^{1/2} - X^{-1/2} \right)^{-1} \Lambda_f(X) + O \left( \frac{T}{\log T} \right).$$

**References**

[1] J. Bolte and S. Johansson. A spectral correspondence for Maass waveforms. *Geom. Funct. Anal.*, 9(6):1128–1155, 1999.

[2] J. Bolte and S. Johansson. Theta-lifts of Maass waveforms. In D. A. Hejhal, J. Friedman, M. C. Gutzwiller, and A. M. Odlyzko, editors, *Emerging Applications of Number Theory* (Minneapolis, MN, 1996), volume 109 of *IMA Vol. Math. Appl.*, pages 39–72, New York, NY, 1999. Springer.

[3] V. A. Bykovskii. Density theorems and the mean value of arithmetic functions in short intervals (Russian). *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI)*, 212, 1994. Anal. Teor. Chisel i Teor. Funktsii. 12, 196:56–70; translation in J. Math. Sci. (N.Y.), 83(6):720–730, 1997.

[4] P. Cartier. Some numerical computations relating to automorphic functions. In A. O. L. Atkin and B. J. Birch, editors, *Computers in number theory*, Proceedings of the Science Research Council Atlas Symposium No. 2 held at Oxford, August 18–23, 1969, pages 37–48. Academic Press, London-New York, 1971.

[5] J. Chazarain. Formule de Poisson pour les variétés riemanniennes. *Invent. Math.*, 24(1):65–82, 1974.

[6] J. B. Conrey and J. Iwaniec. The cubic moment of central values and the Weyl law. *J. reine angew. Math.*, 154(2):477–502, 2001.

[7] A. Fujii. Zeros, eigenvalues and arithmetic. *Proc. Japan Acad. Ser. A Math. Sci.*, 60(1):22–25, 1984.

[8] P. Humphries. Spectral multiplicity for Maass newforms of non-squarefree level. *Int. Math. Res. Not. IMRN*, 2019(18):5703–5743, 2019.

[9] M. N. Huxley. Scattering matrices for congruence subgroups. In R. A. Rankin, editor, *Modular forms (Durham, 1983)*, Ellis Horwood Ser. Math. Appl., pages 141–156. Statist. Oper. Res., Horwood, Chichester, 1984.

[10] H. Iwaniec. Prime geodesic theorem. *J. reine angew. Math.*, 349:136–159, 1984.

[11] H. Iwaniec. Spectral methods of automorphic forms, volume 53 of Graduate Studies in Mathematics. Amer. Math. Soc., Providence, RI, Revista Matemática Iberoamericana, Madrid, 2 edition, 2002.

[12] I. Kaneko. The Prime Geodesic Theorem for $\text{PSL}_2(\mathbb{Z})$ and spectral exponential sums. *to appear in Algebra Number Theory*, 38 pages, 2021. [https://arxiv.org/abs/1909.05111](https://arxiv.org/abs/1909.05111).

[13] I. Kaneko and S. Koyama. Euler products of Selberg zeta functions in the critical strip. *arXiv e-prints*, 24 pages, 2018. [https://arxiv.org/abs/1809.10140](https://arxiv.org/abs/1809.10140).

[14] W. Luo. Nonvanishing of L-values and the Weyl law. *Ann. of Math. (2)*, 154(2):477–502, 2001.

[15] W. Luo and P. Sarnak. Quantum ergodicity of eigenfunctions on $\text{PSL}_2(\mathbb{R}) \backslash \mathbb{H}$, *Publ. Math. Inst. Hautes Études Sci.*, 81:207–237, 1995.

[16] Y. N. Petridis and M. S. Risager. Local average in hyperbolic lattice point counting, with an appendix by Niko Laaksonen. *Math. Z.*, 285(3–4):1319–1344, 2017.

[17] R. Phillips and P. Sarnak. The Weyl theorem and the deformation of discrete groups. *Comm. Pure Appl. Math.*, 38(6):853–866, 1985.

[18] K. Soundararajan and M. P. Young. The prime geodesic theorem. *J. reine angew. Math.*, 676:105–120, 2013.

[19] A. Strömbergsson. Some remarks on a spectral correspondence for Maass waveforms. *Int. Math. Res. Not. IMRN*, 2001(10):505–517, 2001.

[20] H. Then. Maass cusp forms for large eigenvalues. *Math. Comp.*, 74(249):363–381, 2005.

[21] A. B. Venkov. *Spectral Theory of Automorphic Functions*, volume 153 of *Trudy Math. Inst. Steklov*. Amer. Math. Soc., 1982.

[22] A. B. Venkov. *Spectral Theory of Automorphic Functions and its applications*, volume 51 of *Mathematics and its Applications (Soviet Series)*. Kluwer Academic Publishers Group, 1990. Translated from the Russian by N. B. Lebedinskaya.
DEPARTMENT OF MATHEMATICS, CALIFORNIA INSTITUTE OF TECHNOLOGY, 1200 E CALIFORNIA BLVD, PASADENA, CA 91125, USA

Email address: ikuyak@icloud.com

URL: https://sites.google.com/view/ikuyakaneko/