A geometric application of Nambu mechanics: the motion of three point vortices in the plane

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Abstract
In this paper the dynamics of three point vortices in the plane is analysed in terms of Nambu mechanics and compared to the classical Hamiltonian dynamics. Two new aspects are introduced: (i) the motion of three point vortices can be classified as non-canonical Nambu mechanics and (ii) Nambu mechanics leads to a geometric representation of the trajectories in an adapted three-dimensional phase space without explicitly solving the differential equations. Thereby, the point vortex trajectory is given by the intersection of two conserved quantities as surfaces in the phase space. These constitutive quantities are the total energy and an angular momentum based Casimir function of the dynamical system. The topological structure of the last surface represents a one- or two-sheeted hyperboloid, a cone or an ellipsoid in the phase space. Examples of the periodic motion and a novel aspect of the collapse of three point vortices in the unbounded plane are discussed. Furthermore, an approach to generalize Nambu mechanics for an arbitrary number of point vortices is proposed.

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(Some figures may appear in colour only in the online journal)
1. Introduction

Vortices have applications in many disciplines and different spatio-temporal scales; DNA-strings, magnetic flows and further fields of study. A simple concept that we can use to understand vortex motion are discrete point vortices. The first investigations on the dynamics of point vortices were introduced by Helmholtz in 1858 [8]. Twenty years later Kirchhoff represented the general Hamiltonian structure of N point vortices [9]. Shortly after Kirchhoff, Gröbli analysed in detail the motion of three point vortices and introduced the equations of motions of the intervortical distances. In his pioneering work he studied the self-similar contraction of the triangle spanned by three point vortices among other things. Today, this self-similar contraction is known as collapsed motion [7]. Since then, numerous papers have been published on the motion of three point vortices: see, for example, the work of Aref, Blackmore, Newton or Novikov [2, 4, 15, 17]. In this classical analysis of three point vortex motion, the relevant conservation laws are not considered as equitable quantities. In 1949 Synge introduced trilinear coordinates based on the three relative distances to describe the motion of three point vortices in terms of Hamiltonian dynamics [3, 4, 15, 19]. Also Aref used these trilinear coordinates to describe trajectories in a phase plane [1]. Thereby, the phase space coordinates represent the distances from the three sides of the triangle. Synge and Aref show that the physical regions of the vortex motions in this trilinear coordinate plane are bounded by conic sections (ellipse, parabola or hyperbola) [3, 19]. Examples of meteorological applications of point vortex dynamics to analyse large scale atmospheric flows are given in the works of Obukhov et al., Kuhlbrodt and Névir and Newton [10, 16, 18].

In 1973 Nambu published a generalization of canonical Hamiltonian mechanics of discrete systems satisfying Liouville’s theorem [12]. This extension describes conservative systems with odd or even degrees of freedom, where the motion of a system of N degrees of freedom is described by N − 1 conserved quantities. In 1998, Névir and Makhaldiani simultaneously applied the so-called Nambu mechanics to three point vortices [11, 14]. Whereas Makhaldiani picked up Nambu’s general idea to represent the equations of motion in terms of the conserved quantities, we build on Névir’s non-canonical classification and analysis of Nambu mechanics in the context of point vortex motion. In contrast to Hamiltonian mechanics, applying Nambu mechanics for N = 3, the two conserved quantities are considered equitable. In order to give a geometric application in a natural three-dimensional phase space we represent the conserved quantities as surfaces. That way, the point vortex trajectory is given by the intersection of these two surfaces and the type of motion can be determined without solving the system of nonlinear differential equations.

Let \( \mathbf{v} \) a solenoidal vector field, i.e. \( \nabla \cdot \mathbf{v} = 0 \). Then the vorticity vector is given by \( \mathbf{\omega} = \nabla \times \mathbf{v} \). The circulation \( \Gamma \) of a fluid around a closed curve \( C \) in the plane is defined by:

\[
\Gamma = \oint_C \mathbf{v} \cdot ds = \int_A \mathbf{\omega} \cdot \mathbf{n} \ dS,
\]

where Stokes theorem was applied to achieve the right-hand side with area \( A \) and normal vector \( \mathbf{n} \). Assuming ideal incompressible fluids with conservative forces, Kelvin’s circulation theorem states that the circulation \( \Gamma \) around a closed material curve moving with the fluid is constant. Contracting local vorticity fields with same sign to singular points leads to the conceptual model of point vortices. During this limiting process the circulation (1.1) remains constant. Thereby, this asymptotic transition leads to a physical discretization of the vorticity field, where the points are characterized by their circulation \( \Gamma_i, i = 1, 2, \ldots, n \). Whereas the mass which characterize the Newtonian description of point dynamics is always positive, the characteristic quantity of point vortices is given by the circulation and can take positive or negative values. Therefore, a whole point vortex system can also have zero circulation.
In the following we will denote $\mathbf{x}_i = (x_i, y_i)^T$ the local coordinates of the $i$th point vortex in the plane and $r_{ij} = ((x_i - x_j)^2 + (y_i - y_j)^2)^{1/2}$ the relative distance of the $i$th and $j$th point vortex ($i, j = 1, \ldots, n$). The equations of motion derived in 1858 by Helmholtz [8] are given by:

$$\frac{dx_j}{dt} = -\frac{1}{2\pi} \sum_{i \neq j, i, j = 1}^{N} \frac{\Gamma_i (y_j - y_i)}{r_{ij}^2}, \quad \frac{dy_j}{dt} = +\frac{1}{2\pi} \sum_{i \neq j, i, j = 1}^{N} \frac{\Gamma_j (x_j - x_i)}{r_{ij}^2}. \quad (1.2)$$

Kirchhoff [9] established the Hamiltonian representation of these equations of motion as nonlinear coupled system of $2N$ ordinary differential equations:

$$\Gamma_j \frac{dx_j}{dt} = -\frac{\partial H}{\partial y_j}, \quad \Gamma_j \frac{dy_j}{dt} = \frac{\partial H}{\partial x_j}. \quad (1.3)$$

Here, the total energy $H$ of the $N$ point vortex system can be derived by Green’s function and is given by

$$H = -\frac{1}{4\pi} \sum_{i \neq j, i, j = 1}^{N} \Gamma_i \Gamma_j \ln(r_{ij}). \quad (1.4)$$

For a one-vortex system, the motion of the point vortex is clockwise if $\Gamma < 0$ and anticlockwise if $\Gamma > 0$. An interesting aspect is the dependency of the energy of the relative distances. This motivates to consider a phase space of intervortical distances appropriate for Nambu mechanics in section 3.

Already Kirchoff has shown the conservation of the zonal momentum $P_z$, the meridional momentum $P_\phi$ and the vertical component of the angular momentum $L_z$ [9]

$$P_z(x_i, y_i) = \sum_{i = 1}^{N} \Gamma_i y_i \quad (1.5a)$$

$$P_\phi(x_i, y_i) = -\sum_{i = 1}^{N} \Gamma_i x_i \quad (1.5b)$$

$$L_z(x_i, y_i) = -\frac{1}{2} \sum_{i = 1}^{N} \Gamma_i (x_i^2 + y_i^2). \quad (1.5c)$$

Moreover, Kelvin’s circulation theorem allows us to conclude that the following scalars are conserved:

$$\Gamma := \sum_{i = 1}^{N} \Gamma_i, \quad V := \frac{1}{2} \sum_{i, j = 1}^{N} \Gamma_i \Gamma_j. \quad (1.6)$$

The first quantity above is the total circulation of the system whereas the second is a quadratic sum of all circulations and can be related to the enstrophy of the system. Let $\mathbf{k} = (0, 0, 1)$ and $\mathbf{r}_j = (x_j, y_j, 0), \quad i = 1, \ldots, n$. Then $V$ is equivalent to [6]:

$$2\pi \sum_{j = 1}^{N} \Gamma_j \mathbf{k} \cdot \left( \mathbf{r}_j \times \frac{d\mathbf{r}_j}{dt} \right) = \frac{1}{2} \sum_{j, k = 1}^{N} \Gamma_j \Gamma_k = V. \quad (1.7)$$

Moreover, deriving the point vortex equations (1.2) by a variational principle, the corresponding Lagrange-function is given by the sum of $V$ and the total energy $H$ [5]. This shows that the conservation of $V$ is an non-trivial aspect of point vortex dynamics.
The conservation of the centre of circulation $C$ can be derived by $P_x$, $P_y$ and $\Gamma$ and is given by:

$$C = \sum_{i=1}^{N} \frac{\Gamma_i x_i}{\sum_{i=1}^{N} \Gamma_i}. \quad (1.8)$$

If the total circulation $\Gamma$ is not equal to zero, the point vortices move around the centre of circulation. The orientation of an $N$-vortex system depends on the sign of the circulation of highest absolute value. For $\Gamma \to 0$, $C$ approaches infinity. Therefore, if $\Gamma = 0$, each vortex may rotate, but the geometric central point of the $N$-vortex system translates. Because of the conservation of the centre of circulation, one single point vortex always remains in its initial condition [6].

The canonical Poisson bracket of two functions $f = f(x_i, y_i)$ and $g = g(x_i, y_i)$, $i = 1, \ldots, N$ of position variable $(x_i, y_i)$ with respect to the $i\text{th}$ vortex with circulation $\Gamma_i$ is defined by:

$$\{f, g\} = \sum_{i=1}^{N} \frac{1}{\Gamma_i} \left( \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial y_i} - \frac{\partial f}{\partial y_i} \frac{\partial g}{\partial x_i} \right). \quad (1.9)$$

The properties symmetry, linearity, Jacobi’s identity and Leibniz identity all hold. The conserved quantities depending on the local coordinates $P_x$, $P_y$, $L_z$ and $H$ determine the motion of the vortices. The point vortex dynamics is constrained by a further quantity $M$ which is well known (cf [3]) but will be identified here as Casimir function of the system. Elements commuting with all other elements in the bracket are called Casimir functions. Thereby, Casimir functions also commute with the energy. Thus, they are always conserved quantities. The non-trivial Poisson brackets depending on $P_x$, $P_y$ and $\Gamma$ are given by:

$$\{P_x, P_y\} = \Gamma, \quad \{L_z, P_x\} = P_y, \quad \{L_z, P_y\} = -P_x.$$

The total energy $H$ commutes with these three quantities. The Casimir function of point vortex dynamics is the quantity $M$ and given by:

$$M = -\Gamma L_z - \frac{1}{2} (P_x^2 + P_y^2) = \frac{1}{4} \sum_{i \neq j}^{N} \Gamma_i \Gamma_j r_{ij}^2. \quad (1.10)$$

This quantity commutes with the linear momentums $P_x$ and $P_y$, the angular momentum $L_z$ and the Hamilton function $H$, i.e.:

$$\{M, P_x\} = 0, \quad \{M, P_y\} = 0, \quad \{M, L_z\} = 0, \quad \{M, H\} = 0.$$ 

Therefore, $M$ is a conserved quantity and in addition to the energy, the second conserved quantities depending on the relative distances $r_{ij}$ of the vortices. In a phase space determined by the intervortical distances of the vortices, we use both quantities for the Nambu representation of the three point vortex system. An interesting question is the name of the quantity $M$. For example, consider the rigid body rotation. Here, the Casimir function of the three-dimensional rotation group is the squared angular momentum. Since the quantity $M$ is identified as Casimir function, we would like to suggest calling $M$ the squared relative angular momentum with respect to the centre of circulation.

2. Nambu mechanics

In 1973 Nambu generalized the Hamilton dynamic by replacing the bilinear, antisymmetric Poisson bracket by a trilinear, twice antisymmetric bracket [12]. This bracket is called the
Nambu bracket. Twenty years later, Névir and Blender introduced a generalization of Nambu mechanics to describe continuous fluid mechanical models [13]. We will give a brief summary of Nambu mechanics and the reader is directed towards [12, 14, 20] for more detailed description.

Begin with a classical mechanical system with \( n \) degrees of freedom:

\[
x(t) = (x_1, \ldots, x_n)
\]

and let \( S \) denote the phase space of the system with \( n - 1 \) conserved quantities \( H_1(x), H_2(x), \ldots, H_{n-1}(x) \), \( H_j : S \rightarrow \mathbb{R}, j = 1, \ldots, n - 1 \). The equation of motion can be written in terms of Nambu formalism

\[
\frac{dx_i}{dt} = \frac{\partial (x_i, H_1, H_2, \ldots, H_{n-1})}{\partial (x_1, \ldots, x_n)}, \quad i = 1, \ldots, n.
\]

We can calculate the derivative of an arbitrary dynamic function \( F(x_1, x_2, \ldots, x_n), F : S \rightarrow \mathbb{R} \) by:

\[
\frac{dF}{dt} = \frac{\partial F}{\partial x_1} \frac{dx_1}{dt} + \frac{\partial F}{\partial x_2} \frac{dx_2}{dt} + \cdots + \frac{\partial F}{\partial x_n} \frac{dx_n}{dt}
\]

and apply (2.2). Then the Nambu bracket describes the time evolution the function \( F \):

\[
\frac{dF}{dt} = \{ F, H_1, H_2, \ldots, H_{n-1} \} = \frac{\partial (F, H_1, H_2, \ldots, H_{n-1})}{\partial (x_1, \ldots, x_n)}.
\]

The functions \( H_j, j = 1, \ldots, n - 1 \) are automatically conserved quantities since the Jacobi-determinant of two similar arguments becomes zero. The Nambu bracket is linear and antisymmetric in all arguments. Furthermore, the Leibniz identity and the so-called Takhtajan identity holds [20].

2.1. Transition to canonical Hamiltonian dynamics

Take a system with \( n = 2 \) degrees of freedom and one conserved quantity \( H = H(x) \). If we apply (2.2)

\[
\frac{dx_i}{dt} = \frac{\partial (x_i, H)}{\partial (x_1, x_2)}, \quad i = 1, 2
\]

and identify \( x_1 = q \) and \( x_2 = p \), we obtain the well known formula of the canonical conjugated Hamiltonian differential equations. These Hamiltonian equations can also be expressed by an antisymmetric, second-order Poisson tensor \( P \):

\[
\frac{dx_i}{dt} = P_{ij} \cdot \frac{\partial H}{\partial x_j}, \quad P_{ij} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.
\]

The characteristic properties of this canonical Poisson tensor are its independence from the phase space coordinates and its non degeneracy (det(\( P_{ij} \)) \( \neq 0 \)).

2.2. Nambu mechanics for \( n = 3 \) degrees of freedom

In the first physical application for \( n = 3 \) degrees of freedom, Nambu discussed the Euler equation for a rigid rotator [12]. Another example for \( n = 3 \) is given by Névir and Blender who used the Nambu representation to analyse the non-dissipative Lorenz equations (cf [13]). In the following we will shortly summarize the main equations for the case \( n = 3 \) to apply the theory to the motion of three point vortices in the next section.
Let a system with $n = 3$ degrees of freedom. Let $\mathbf{x} = (x_1, x_2, x_3)$. Let $H_1(x)$ and $H_2(x)$ denote two conserved quantities, $H_i : \mathcal{S} \rightarrow \mathbb{R}$, $i = 1, 2$. Applying formula (2.2), we obtain for $i, j, k = 1, 2, 3$, $i \neq j \neq k$:

$$\frac{dx_i}{dt} = \partial(H_1, H_2) = \frac{\partial H_1}{\partial x_j} \frac{\partial H_2}{\partial x_k} - \frac{\partial H_1}{\partial x_k} \frac{\partial H_2}{\partial x_j}.$$  \hfill (2.6)

The three-dimensional gradient in the state space $\mathcal{S}$ is given by:

$$\nabla = e_1 \frac{\partial}{\partial x_1} + e_2 \frac{\partial}{\partial x_2} + e_3 \frac{\partial}{\partial x_3}.$$  \hfill (2.7)

Therefore, for $n = 3$ the canonical Nambu representation of the time evolution can be written as:

$$\frac{d\mathbf{x}}{dt} = \nabla H_1 \times \nabla H_2.$$  \hfill (2.8)

Let $F(x)$ an arbitrary function, $F : \mathcal{S} \rightarrow \mathbb{R}$.

$$\frac{dF}{dt} = [F, H_1, H_2] := \nabla F \cdot (\nabla H_1 \times \nabla H_2) = 0.$$ \hfill (2.9)

Because of the cross product, the Nambu bracket is antisymmetric and also trilinear. The divergence of (2.8) generates Liouville’s theorem of Nambu mechanics:

$$\nabla \cdot \dot{x} = \nabla \cdot (\nabla H_1 \times \nabla H_2) = 0.$$ \hfill (2.10)

Therefore, the state space can be regarded as an incompressible fluid.

The central property of Nambu mechanics is the equal status of two conserved quantities, whereas in Hamiltonian mechanics only one conserved quantity appears in the equations and a second does not appear explicitly. The singularity of the Poisson tensor with vanishing determinant leads to the second conserved quantity, which is called distinguished or Casimir function in terms of Hamiltonian dynamics. Thus, the main advantage of the equality of two conserved quantities in Nambu formulation, is the representation of the phase space trajectory as intersection line of two surfaces based on the conserved quantities. Therefore, this geometric application illustrates the kind of motion without explicitly solving the equations of motion.

### 2.3. Non-canonical Nambu mechanics for $n = 3$

In contrast to Hamiltonian dynamics that is characterized by the general antisymmetric second-order Poisson tensor (2.5), Nambu mechanics is determined by an antisymmetric, third-order tensor [14]. This tensor is called Nambu tensor and denoted by $N_{ijk}$, $i, j, k = 1, 2, 3$. Additional conserved quantities determining the dynamics in Nambu mechanics increase the order of the tensor by one dimension.

Let $F(x), G(x)$ and $H(x)$ now denote three arbitrary functions, mapping from the phase space to $\mathbb{R}$. Then the canonical Nambu bracket is defined by:

$$[F, G, H] = \varepsilon_{ijk} \frac{\partial F}{\partial x_i} \frac{\partial G}{\partial x_j} \frac{\partial H}{\partial x_k}.$$  \hfill (2.11)

For general Nambu systems, the total antisymmetric tensor of third order can depend on the phase space coordinates. Therefore, we will introduce the non-canonical tensor $N_{ijk}$.

$$N_{ijk} = c \varepsilon_{ijk} = \Lambda^a_i \Lambda^b_j \Lambda^c_k \varepsilon_{abc} = \det \left( \Lambda^a_i \right) \varepsilon_{ijk} = c = \det \left( \Lambda^a_i \right).$$  \hfill (2.12)
Substituting $\epsilon_{ijk}$ by $N_{ijk}$ in (2.11) we get the canonical Nambu bracket for $\det(\Lambda^\alpha_i) = 1$ and $\det(\Lambda^\alpha_i) \neq 1$ leads to the non-canonical Nambu bracket:

$$\{ F, G, H \} = N_{ijk}(x_i) \frac{\partial F}{\partial x_j} \frac{\partial G}{\partial x_k} \frac{\partial H}{\partial x_l}. \quad (2.13)$$

The dependency on the phase space variables of the Nambu tensor is a generalization of the mechanics introduced by Nambu in 1973 \[12\]. This generalization is relevant for the motion of three point vortices.

If $H(x_i)$ and $C(x_i)$ denote two conserved quantities, then the above generalized Nambu bracket describes the motion of the function $F$, i.e. $dF/dt = \{ F, G, H \}$. Therefore, the equation of motion for the phase space coordinates is given by:

$$\frac{dx_i(t)}{dt} = \{ x_i, C, H \} = N_{ijk}(x_i) \frac{\partial H}{\partial x_j} \frac{\partial C}{\partial x_l}. \quad (2.14)$$

Inserting the phase space coordinates $x_i$ into the Nambu bracket (2.14) shows the equivalence of the Nambu tensor and the fundamental Nambu bracket:

$$\{ x_i, x_j, x_k \} = N_{ijk}(x_i). \quad (2.15)$$

The decomposition of the phase space coordinates $x_i$ in the canonical triple $q_i(x_i)$, $p_i(x_i)$ and $r_i(x_i)$ leads to the constant canonical Nambu tensor which is identical to the total antisymmetric third-order tensor $\{ q_i, p_j, r_k \} = N^C_{ijk} = \epsilon_{ijk}$.

3. Nambu mechanics applied to point vortices

In the following, we will represent the motion of three point vortices as a geometrical application of Nambu mechanics. Thereby, the phase space is spanned by the relative distances of three point vortices. Similar to the Nambu representation of the time evolution of a force-free rigid body, where the phase space dimension is reduced from six to three dimensions (because of symmetry properties), choosing the relative distances as phase space coordinates of three point vortices leads to a reduction from six to three dimensions (see (1.2)). We show a geometrical Nambu representation of the motion of three vortices depending on two conserved quantities.

3.1. Equations of motion

Consider the well known equations motion for a three point vortex system in the plane depending on the relative distances \[2\]. Denote $A_{123} = A(r_{12}, r_{23}, r_{31})$ as the area of the triangle with the intervortical distances as side lengths. Let $\sigma$ the orientation of the triangle, where $\sigma = \sigma_{ijk} = 1$ for $\Gamma_i, \Gamma_j, \Gamma_k$ arranged counter-clockwise and $\sigma = \sigma_{ijk} = -1$ for $\Gamma_i, \Gamma_j, \Gamma_k$ appearing clockwise. Further, let $i, j, k = 1, \ldots, n$ in cyclic order, $i \neq j \neq k$. The time evolution of the squared relative distances $r_{ij}^2 = ((x_i - x_j)^2 + (y_i - y_j)^2)^{1/2}$ of three point vortices is given by

$$\frac{dr_{ij}^2}{dt} = 2\pi \sigma \Gamma_{A123} \left( \frac{r_{jk}^2}{r_{ik}^2} - \frac{1}{r_{ik}^2} \right). \quad (3.1)$$

Apply the chain rule and denote $\rho := \rho(r_{ij}, r_{jk}, r_{ki}) = \frac{r_{ij}r_{jk}r_{ki}}{4\pi A}$ (which represents the inscribed circle radius of the triangle with side length $r_{ij}, r_{jk}, r_{ki}$). Then we obtain:

$$\frac{dr_{ij}}{dt} = \frac{\sigma \Gamma_k}{4\pi \rho} \left( \frac{r_{ij}}{r_{jk}} - \frac{r_{jk}}{r_{ki}} \right). \quad (3.2)$$
In the following we will analyse the relative motion of three point vortices spanned by the phase space coordinates \( \mathbf{r} = (r_{12}, r_{23}, r_{31}) \). Applying Nambu mechanics to a three-dimensional phase space, we need two conserved quantities to characterize the motion of three point vortices. The only conserved quantities depending on the relative distances are the energy \( H \) and the total momentum \( M \):

\[
M = M(r_{12}, r_{23}, r_{31}) = \frac{1}{2} \left( \Gamma_1 \Gamma_2 r_{12}^2 + \Gamma_2 \Gamma_3 r_{23}^2 + \Gamma_3 \Gamma_1 r_{31}^2 \right) \quad (3.3)
\]

\[
H = H(r_{12}, r_{23}, r_{31}) = -\frac{1}{2\pi} \left( \Gamma_1 \Gamma_2 \ln(r_{12}) + \Gamma_2 \Gamma_3 \ln(r_{23}) + \Gamma_3 \Gamma_1 \ln(r_{31}) \right). \quad (3.4)
\]

The time evolution of the relative distances of three point vortices in Nambu representation can be written as:

\[
\frac{d \mathbf{r}_{ij}}{d t} = \frac{\sigma}{2\Gamma_1 \Gamma_2 \Gamma_3 \rho} \left( \frac{\partial M}{\partial r_{jk}} \frac{\partial H}{\partial r_{ki}} - \frac{\partial M}{\partial r_{ki}} \frac{\partial H}{\partial r_{jk}} \right) \quad (3.5)
\]

with \( i, j, k = 1, 2, 3 \) in cyclic order and \( i \neq j \neq k \). In contrast to Makhaldiani, we scale the time \( t \) by a constant factor \( t' = \alpha t \) with \( \alpha = \sigma / 2\Gamma_1 \Gamma_2 \Gamma_3 \). Because the inscribed circle radius \( \rho = \rho(r_{12}, r_{12}, r_{31}) \) depends on the relative distances, we get a non-canonical Nambu representation of three point vortices:

\[
\rho \frac{d \mathbf{r}}{d t'} = \nabla M \times \nabla H. \quad (3.6)
\]

This Nambu-representation motivates the representation of the trajectory of three point vortices as intersection of two surfaces given by \( M \) and \( H \) in the phase space of intervortical distances. Applying the generalized non-canonical Nambu-bracket (2.13) respectively (2.14), it follows that the Nambu-tensor for point vortices is given by:

\[
N_{ijk}(r_{12}, r_{12}, r_{31}) = \frac{1}{\rho} \epsilon_{ijk}, \; i, j, k = 1, 2, 3. \quad (3.7)
\]

Therefore we classify the relative motion of three point vortices as non-canonical Nambu dynamics.

### 3.2. Geometric representation of three point vortex motion

First we will give a general geometric description of the motion of three point vortices with respect to Nambu mechanics. Applying formula (3.6), the time evolution of three point vortices in the plane will be determined by the quantity \( M \) and the energy \( H \) in the phase space \( \mathbf{r} = (r_{12}, r_{23}, r_{31}) \).

Set \( b := -2\pi H / (\Gamma_1 \Gamma_2 \Gamma_3) \) and \( c := 2M / (\Gamma_1 \Gamma_2 \Gamma_3) \). Now we consider the conserved quantities \( H \) and \( M \):

\[
H : \frac{\ln(r_{12})}{\Gamma_3} + \frac{\ln(r_{23})}{\Gamma_1} + \frac{\ln(r_{31})}{\Gamma_2} = b, \quad M : \frac{r_{12}^2}{\Gamma_3} + \frac{r_{23}^2}{\Gamma_1} + \frac{r_{31}^2}{\Gamma_2} = c. \quad (3.8)
\]

The surface of \( M \) represents a quadric and the sign of the circulations characterize the topological structure of the surface representation of \( M \) (an ellipsoid, a one-sheeted hyperboloid, a cone, a two-sheeted hyperboloid, see figure 1). The different circulations and occurring motions are summarized in table 1.

Since both conserved quantities depend on the relative distances, the intersection \( M \cap H \) yields the relative motion of the point vortices in the phase space. Because the relative distances \( r_{ij} \) (\( i, j = 1, 2, 3, \; i \neq j \)) are positive, in each case the \( M \)-surface is a subsurface of the surfaces mentioned before. Table 2 summarizes the conditions of the quantities \( M \) and \( H \) for the different kind of motions and will be discussed in the following.
Figure 1. The $M$-surface represents one of the four types of quadrics: (from lhs to rhs) an ellipsoid, a one-sheeted hyperboloid, a cone, a two-sheeted hyperboloid.

Table 1. Geometry of the quantity $M$ depending on the sign of the circulations of the point vortices.

| Sign of the circulations | Surface                    | Possible motion            |
|--------------------------|----------------------------|----------------------------|
| $\Gamma_1, \Gamma_2, \Gamma_3 > 0, c > 0$ | Ellipsoid                  | Periodic motion/Equilibrium |
| $\Gamma_1, \Gamma_3 > 0, \Gamma_2 < 0, c > 0$ | One-sheeted hyperboloid    | Periodic motion/Equilibrium |
| $\Gamma_1, \Gamma_3 > 0, \Gamma_2 < 0, c = 0$ | Cone                       | Collapse/Expanding          |
| $\Gamma_1, \Gamma_3 > 0, \Gamma_2 < 0, c < 0$ | Two-sheeted hyperboloid    | Periodic motion/Equilibrium |
| $\Gamma_1, \Gamma_2, \Gamma_3 > 0, c < 0$ |                           | No real solution            |

Table 2. Necessary conditions for the conserved quantities.

| Motion                  | Properties                        |
|-------------------------|-----------------------------------|
| Periodic motion         | $M \neq 0, H \neq 0$              |
| Relative equilibrium    | $M = \lambda H, \lambda \in \mathbb{R} \setminus \{0\}$ |
| Collapsed motion        | $M = 0, H \neq 0, \sigma > 0$    |
| Expanding motion        | $M = 0, H \neq 0, \sigma < 0$    |

3.3. Periodic motion

Closed intersection lines in the phase space represent periodic point vortex motions. If the signs of the circulations are all positive and if the value of $c$ is greater than zero, the surface of $M$ becomes an ellipsoid (see (3.8)). Therefore, the intersection with $H$ is given by a closed line leading to a periodic motion. Moreover, if the circulations have different signs, $M$ represents a one-sheeted hyperboloid in case of $c > 0$ and a two-sheeted hyperboloid if $c < 0$. In this case, periodic motions also can occur. In figure 2 an example of a periodic motion with initial circulations $\Gamma_1 = 1, \Gamma_2 = 2, \Gamma_3 = 1$ and initial distances $r_{12} = 4, r_{23} = 3, r_{31} = 5$ is shown. On the figure on the left the point vortex trajectory in the phase space is given by the intersection line of the two surfaces of $M$ (red surface) and $H$ (blue surface). Because the signs of all circulations are positive, the $M$-surface represents a part of an ellipsoid. Therefore, the intersection line is a closed curve, i.e. the motion is periodic. The figure on the right represents the classical plot of time evolution of the relative displacements. To illustrate the point vortex motion in the $x$-$y$-plane or the temporal change of the intervortical distance (figures 2–4) it is necessary to solve nonlinear differential equations as (1.2) or (3.2). This of course costs more effort than simply illustrating the surfaces $H$ and $M$ to obtain the kind of the motion by the trajectories in phase space.
3.4. Relative equilibrium

In case the two surfaces intersect in one point in the phase space, three point vortices generate a relative equilibrium configuration.

It is well known that the relative equilibrium of three point vortices occur either for collinear initial states or for three point vortices forming an equilateral triangle. In case of an equilateral triangle and total circulation unequal zero, the three vortex system rotates about their centre of circulation with a rotation- frequency \( \omega = \Gamma / 2\pi r^2 \), where \( r \) is the triangle side [15]. In case of zero total circulation, the centre of circulation lies in infinity. In that case, instead of rotating around the centre of circulation, the vortices translate with velocity

\[
v = 1/2 (\Gamma_1^2 + \Gamma_2^2 + \Gamma_3^2)^{1/2} / 2\pi r.
\]

This can be shown by applying \( r_{12} = r_{23} = r_{31} = 1 \) to (1.2).

Let now the x-axis be on the line \( \Gamma_1 \Gamma_3 \), i.e. the straight line through the vortices of the same sign of circulation passing the origin (see figure 3). The point \( P_0 = (x_0, y_0) \) with circulation \( \Gamma_1 \) lies in the origin. Set \( r = r_{12} = r_{23} = r_{31} \). Applying (1.2), the velocity vector \( v \) in \( P_0 \) and the slope \( m \) of the line lying on the velocity vector are given by:

\[
v = 1/4\pi r \left( \sqrt{3} \Gamma_2 \right), \quad m = \frac{dy}{dx} = \frac{\Gamma_1 - \Gamma_3}{\sqrt{3} \Gamma_2}.
\]

Therefore, the angle of translation \( \alpha \) between the velocity vector and the x-axis is given by:

\[
\alpha = \arctan(m) = \arctan \left( \frac{\Gamma_1 - \Gamma_3}{\sqrt{3} \Gamma_2} \right).
\]
To apply the Nambu formulation in an equilibrium state we take a look at the relevant conserved quantities $H$ and $M$:

$$H = -\frac{\ln(r)}{2\pi} V, \quad M = \frac{r^2}{2} V \quad \Rightarrow \quad H = \lambda \cdot M, \quad \lambda \in \mathbb{R}. \quad (3.10)$$

In terms of Nambu mechanics for every energy level and an arbitrary but fixed $V$ it exists a solution of a relative equilibrium if and only if the $M$-surface is tangent to the $H$-surface. Because the intersection is given by a fixed point, no time evolution is possible.

### 3.5. Collapse and expanding state

Special cases are the collapse and the expanding motion that are characterized by self-similarity. An interesting question is how the self-similar motion is expressed in terms of the geometrical view of Nambu mechanics. First we will discuss the behaviour of the conserved quantities. Necessary and sufficient condition for the self-similar collapse of three vortices are shown in [1, 15, 19]. The first condition is given by the quantity $M$ being zero, i.e.

$$M = \frac{1}{2} \sum_{i,j=1, i \neq j}^{3} \Gamma_i \Gamma_j r_{ij}^2 = 0. \quad (3.11)$$

Second, the harmonic mean needs to be zero:

$$h = \frac{1}{3} \left( \frac{1}{\Gamma_1} + \frac{1}{\Gamma_2} + \frac{1}{\Gamma_3} \right) = 0 \quad \iff \quad V = \sum_{i,j=1, i \neq j}^{N} \Gamma_i \Gamma_j = 0. \quad (3.12)$$

And, of course, the initial configuration must not be an equilibrium. Aref established the self-similarity of the collapse [1] with respect to the intervortical distances:

$$r_{ij}(t) = f(t) r_{ij}(0), \quad i, j = 1, 2, 3, \quad (3.13)$$

where $f(t) = \sqrt{1 - t/\tau}$ and $\tau$ is the collapse time. The conservation of the energy during the whole collapse process is an interesting fact and results from the self-similarity and the collapse condition $V = 0$ [1]:

$$H = -\frac{1}{4\pi} \sum_{i,j=1}^{N} \Gamma_i \Gamma_j \ln(f(t) r_{ij}(0)) = -\frac{1}{4\pi} \left[ \ln(f(t)) \sum_{i,j=1}^{N} \Gamma_i \Gamma_j + \sum_{i,j=1}^{N} \Gamma_i \Gamma_j \ln(r_{ij}(0)) \right]$$

$$= -\frac{1}{4\pi} \sum_{i,j=1}^{N} \Gamma_i \Gamma_j \ln(r_{ij}(0)) = \text{const.} \quad (3.14)$$

It is well known that if these constraints are fulfilled and the vortices $\Gamma_1$, $\Gamma_2$ and $\Gamma_3$ appear counter-clockwise, the vortices expand $(\sigma = -1)$. Otherwise they collapse $(\sigma = 1)$. The orientation of the rotation of the whole vortex system is determined by the sign of the circulation of highest absolute value of the three vortices.

The Nambu equation of the time evolution of three point vortices is given by (3.6):

$$\frac{d\rho}{dt} = \nabla M \times \nabla H. \quad (3.16)$$

Before we interpret the right-hand side of this term for the geometrical representation, apply (3.13) to $\rho$ that was defined for the classical equations of motion (3.2):

$$\rho = \frac{r_{12}(t') r_{23}(t') r_{31}(t')}{4A(t')} = f(t') \rho_0 \quad (3.17)$$
where \( \rho_0 = \rho(t' = 0) \). Therefore, the left-hand side of (3.16) is given by:

\[
\frac{dr}{dt'} = \rho_0 f(t') \frac{df(t')}{dt'} = \rho_0 f(t') r_0 \frac{df(t')}{dt'} = -\frac{\rho_0}{2r} r_0.
\]

(3.18)

Thus, the time evolution of the relative distances multiplied by \( \rho \) is constant and the phase space trajectory can not be a closed curve, i.e. periodic motion.

Now we will analyse the right-hand side of equation (3.16) in terms of the geometrical interpretation. Because the quantity \( V \) is equal zero, the circulations have different signs. Therefore, in case of collapse/expanding the \( M \)-surface is given by

\[
M : \frac{r_{12}^2}{\Gamma_1} + \frac{r_{23}^2}{\Gamma_2} + \frac{r_{31}^2}{\Gamma_3} = 0
\]

(3.19)

and represents a cone; more precisely, since the relative distances are all positive, a part of a cone. The intersection of a cone with the \( H \)-surface leads to two lines passing the origin as trajectories in the phase space. Figure 4 shows the intersection lines of the two surfaces of the energy (blue surface) and the quantity \( H \) (red surface), representing a subset of a light cone; more precisely, since the relative distances are all positive, a part of the \( M \)-surface is given by

\[
M : \frac{r_{12}^2}{\Gamma_1} + r_{23}^2 + \frac{r_{31}^2}{\Gamma_3} = 0
\]

(3.19)

and represents a cone; more precisely, since the relative distances are all positive, a part of a cone. The intersection of a cone with the \( H \)-surface leads to two lines passing the origin as trajectories in the phase space. Figure 4 shows the intersection lines of the two surfaces of the energy (blue surface) and the quantity \( H \) (red surface). Here, the circulations \( \Gamma_1 = 12, \Gamma_2 = -3, \Gamma_3 = 4 \) satisfy \( V = 0 \) and together with the initial values of the intervortical distances \( r_{12} = \sqrt{2.5}, r_{23} = \sqrt{4.5}, r_{31} = \sqrt{3} \) also \( M \) is equal zero. It can be seen that the \( M \)-surface is part of the upper cone and the trajectory of three point vortices in the phase space consists of two lines. Since we know that the initial conditions lead to a collapse, the phase space coordinate \( r_{0.1} = (\sqrt{2.5}, \sqrt{4.5}, \sqrt{3})^T \) moves towards the origin (see figure 4). But why does the geometrical representation shows two lines? We have two constraints given by the surfaces of \( H \) and \( M \); set \( \Gamma_1, \Gamma_2, \Gamma_3 \) and \( r_{31} \), as above and let \( r_{12}, r_{23} \) two variables. Solving the \( H \)-equation (3.8) for \( r_{12} \), we get \( r_{12} = \exp(4b) r_{13}^{1/3} \). Now we insert \( r_{12} \) into the \( M \)-equation (3.19). On condition of \( M = 0 \) this leads to two solutions of \( r_0 = (r_{12}(0), r_{23}(0), r_{31}(0)) \), namely, as expected the above solution \( r_{0.1} = (\sqrt{2.5}, \sqrt{4.5}, \sqrt{3})^T \) and a second solution, numerically given by the approximated values \( r_{0.2} = (1.84, 1.33, \sqrt{3})^T \). As we can see, the magnitude relative to the fixed value \( r_{31} \) interchanges, i.e. it is \( r_{12} < r_{31} < r_{12} \), but \( r_{23} < r_{23} < r_{12} \). Let the orientation of the vortices of these two solution be fixed. Then, putting these values in the equations of motion (3.1), we get:

\[
\frac{dr_1^2}{dt} \bigg|_{t=0} < 0, \quad \frac{dr_2^2}{dt} \bigg|_{t=0} > 0.
\]

(3.20)
Therefore, for fixed $\Gamma_1, \Gamma_2, \Gamma_3$ and $r_{31}$ we get one solution leading to a collapse and another solution leading to an expanding of three point vortices (see figure 4 on the right-hand side). This explains that there are two intersection lines. In contrast to solving a system of differential equations by time-stepping using a single initial condition, applying Nambu mechanics we obtain a whole set of all possible initial conditions and trajectories for the collapse and expanding motions for fixed values of $M$ and $H$. Moreover, beyond the well known effect of $\sigma$, we show, as a new aspect, that the order of magnitude of the intervortical distances also differentiates between the collapse and the expanding motion. In contrast to the change of $\sigma$ that can be interpreted simply as time reversal, interchanging this order of magnitude leads to two structural different collapse and expanding motion. In figure 4 the intersection line on the left-hand side represents the collapse and the other intersection line represents the expanding state. We conclude that for every fixed $r_{31}$ there are two initial configurations leading to a collapse and an expanding at every energy level.

### 3.6. Nambu representation for an arbitrary number of point vortices

Nambu representation for an arbitrary number of point vortices It would be interesting to apply Nambu mechanics to an arbitrary number of point vortices. Therefore, for fixed $\Gamma_1$ and $\Gamma_2$, the Nambu representation for an arbitrary number of point vortices is given by the circulation $A$ defined by a surface integral over the vortex vector and it is conserved on any material surface. Contracting this area to a point leads to the definition of the point vortex. Therefore, the dynamics of point vortices can also be classified by the evolution of material surfaces. Let $A_{ijk} = A_{ijk}(t)$ for $i,j,k = 1,2,3$. The area spanned by three point vortices is determined by the time rate of change of all possible solutions leading to an expanding of three point vortices.

Equation (3.1) for $N$ vortices can be written as

$$\frac{dA_{ijk}}{dt} = \frac{64}{\pi} \sum_{k=1}^{N} \sigma_{ijk} A_{ijk} \frac{\partial (M, H)}{\partial (r_{jk}, r_{ki})}.$$  \hspace{1cm} (3.21)

with the Jacobi-determinant $(\partial (M, H) / \partial (r_{jk}, r_{ki}))$.

We first derive the time evolution of the area for three point vortices. We denote $A = A_{123}$ and apply (3.21) to

$$\frac{dA}{dt} = \frac{\partial A}{\partial r_{12}} \frac{dr_{12}}{dt} + \frac{\partial A}{\partial r_{23}} \frac{dr_{23}}{dt} + \frac{\partial A}{\partial r_{31}} \frac{dr_{31}}{dt}.$$  \hspace{1cm} (3.22)

We get

$$\frac{dA}{dt} = \frac{192}{\Gamma_1 \Gamma_2 \Gamma_3} \sigma_{123} \frac{\partial (A, M, H)}{\partial (r_{12}, r_{23}, r_{31})}.$$  \hspace{1cm} (3.23)

Now we can generalize (3.23) to describe the interaction of $N \geq 4$ point vortices in terms of the two conserved quantities $M$ and $H$. In this approach, applying (3.21) the nonlinear dynamics of the area of one triangle is determined by the time rate of change of all possible triangles spanned by the fixed $i$th and $j$th vortices:

$$\frac{dA_{ijk}}{dt} = \sum_{k=1}^{N} \frac{192}{\Gamma_i \Gamma_j \Gamma_k} \sigma_{ijk} A_{ijk} \frac{\partial (A_{ijk}, M, H)}{\partial (r_{ij}, r_{jk}, r_{ki})}.$$  \hspace{1cm} (3.24)

Analogue to (3.24), we can also derive the time evolution in dependency of the non-quadratical distances:

$$\frac{dA_{ijk}}{dt} = \sum_{k=1}^{N} \frac{6}{\rho \Gamma_i \Gamma_j \Gamma_k} \sigma_{ijk} A_{ijk} \frac{\partial (A_{ijk}, M, H)}{\partial (r_{ij}, r_{jk}, r_{ki})}.$$  \hspace{1cm} (3.25)
Even though we lose the information of the local position and the intervortical distances, we can still classify the special kinds of motion given by the relative equilibrium ($\dot{\lambda} = 0$), collapse and expanding motion ($\dot{\lambda} < 0, \dot{\lambda} > 0$).

The Nambu-representation of $N$ point vortices based on the area spanned by three point vortices is quite natural for two reasons. On the one hand, the dynamics can be understood as interactions of integrable subsystems with, in general, maximum number of degrees of freedom. On the other hand contrary to mass point dynamics, the evolution of point vortices is based on the circulation and therefore it can be seen as evolution of material surfaces according to Lagrangian’s view of fluid mechanics.

4. Conclusion

In this work we have described a three point vortex system based on Nambu mechanics. In terms of Nambu mechanics only two conserved quantities ($H, M$) in a three-dimensional phase space spanned by intervortical distances suffice for the integrability of a three point vortex system. Thereby, the energy $H$ and the quantity $M$ that we identify as Casimir of the Hamiltonian system have equal status. In contrast, using Hamiltonian mechanics, the integrability of the three point vortex system is assured by three conserved quantities ($H, L, p_x^2 + p_y^2$) in a six-dimensional phase space spanned by the $(x, y)$-coordinates in the plane. Generally, discrete Nambu mechanics leads to a geometrical representation of singular Hamiltonian dynamics in a reduced phase space.

Our first result is the classification of the evolution of three point vortices as non-canonical Nambu mechanics. Second, we have shown that the phase space trajectory is generated by the intersection of two surfaces represented by two conserved quantities $H$ and $M$. Thus, in order to specify the motion, it is not necessary to solve the nonlinear differential equations of motion. Topologically, the surface represented by $M$ is a quadric giving rise to the different classes of motion. The trajectory of a periodic point vortex motion is always a closed line in the phase space. If the two surfaces intersect in one point, an equilibrium is given. Moreover, there are a collapse and an expanding motion, each of them represented by one line passing the origin. To the best of our knowledge, we have shown for the first time that besides the orientation $\sigma$, the order of magnitude of the intervortical distances distinguishes between collapse and expanding motion. In contrast to solving a system of differential equations by time-stepping using a single initial condition, applying Nambu mechanics we obtain a whole set of all possible initial conditions and trajectories for fixed values of $M$ and $H$. Especially, this is interesting for analysing collapsed and expanding motion. Finally, an approach to generalize Nambu mechanics for an arbitrary number of point vortices is proposed. Thereby, the motion of $N$ point vortices is given by the interaction of surfaces spanned by three point vortex subsystems.

Independent of the complexity of the surface, Nambu mechanics offers a change from local numerical time stepping methods to global geometric solutions which is a focus of modern topological fluid mechanics.

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