Geometric modeling of torse surfaces in BN-calculus

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Abstract. The paper presents a general approach to geometric modelling of torse surfaces in BN-calculus, based on the definition of the torse as the geometric location of tangents to its edge of return. In this paper, a method for determining the curves – edges of the return of the torse surface, using the geometric properties of the point definition of the curve in plane and spatial simplices, is proposed in general form. Examples of constructing geometric models of torse surfaces, for which algebraic and transcendental spatial curves were used as the return edge, are given.

1. Introduction
Torse surfaces, by their nature – linear and deployable, are widely used in engineering practice due to the possibility of manufacturing complex surfaces of technical forms from sheet material by stamping. The importance of analytical determination of torse surfaces was noted in [1-2] in relation to the use of torse surfaces in shipbuilding and aircraft construction, industrial and agricultural engineering, construction and architecture, light industry and sculpture. In the works of Professor Podgorny suggests the use of algebraic torse surfaces of the same slope for modeling solar ray surfaces in insolation problems. To determine them, we use the methods of geometric, mathematical and computer modeling [3-9]. In many cases, the analytical determination of torse surfaces precedes its computer model.

In BN-calculus (another name is "Point calculus" [10]), modeling of torse surfaces using 3rd-order curves as a return edge. In addition, the BN-calculus has a number of narrowly focused studies in the form of point descriptions of torse surfaces, which on the one hand rely on a graphical algorithm for constructing torse using the kinematic method, and on the other hand – do not use the definition of the torse directly as a basic geometric tool. In this way, it is possible to study individual properties of torse surfaces, but the possibility of implementing a General approach to modeling torse surfaces based on their geometric definition is excluded.

2. Point definition of the torse surface return edge
In general, the torse surface is the geometric place of tangents to its edge of return. According to [2], any spatial curve can be taken as a return edge, the tangents to which will generate the torse surface. Here it is important to highlight, and what exactly is meant by a spatial curve? Traditionally, spatial curves are lines of two-fold curvature (2-curvature), the points of which do not lie in the same plane. However, for algebraic curves whose order is higher than 3, a more General definition is also valid, which leads to the creation of torse hypersurfaces of a multidimensional space with a return edge in the form of a curve of the \( n \)-th curvature. For example, a 4th-order algebraic curve can generally belong to a 4-dimensional space and be a line of 3-curvature. In a particular case, it can also be a projection on a 3-dimensional space and, accordingly, will be a line of 2-curvature. In the BN-
calculus, it is quite easy to determine whether a curve is spatial and to which space it generally belongs from the condition that a geometric object belongs to a space of a certain dimension. Consider the point equation of a plane curve in a simplex $ABC$:

$$M = (A-C)p(u) + (B-C)q(u) + C = Ap(u) + Bq(u) + Cr(u),$$  \hspace{1cm} (1)

where $A, B, C$ – points forming a flat simplex; 
$p(u), q(u), r(u)$ – arbitrary continuous functions from a parameter that changes from 0 to 1 within the simplex.

The condition for a geometric object (in this case, a curve) to belong to a simplex $ABC$ is that the sum of functions from the parameter is equal to one $u$:

$$p(u) + q(u) + r(u) = 1$$  \hspace{1cm} (2)

This condition follows from the geometric meaning of the definition of the current $M$ point in the simplex $ABC$ [10], which has a direct generalization to a multidimensional space.

Thus, analyzing the point equation (1), it can be unambiguously stated that: first, the described geometric object is a line (a one-parameter set of points), and second, this object is a flat line if condition (2) is met. However, such a flat line can be in space and higher dimensions while remaining flat. As an example, consider the same line in 3-dimensional simplex space $ABCD$:

$$M = Ap(u) + Bq(u) + Cr(u) = (A-D)p(u) + (B-D)q(u) + (C-D)r(u) + D.$$  \hspace{1cm} (3)

The resulting expression, on the one hand, describes a spatial curve, but on the other hand, if condition (2) is met, the resulting curve will be flat, but located in 3-dimensional space. From here we get the necessary condition for detecting spatial lines in the simplex of 3-dimensional space $ABCD$.

**Statement.** To select only spatial lines from the set of lines in the $ABCD$ simplex that can serve as a return edge of the torse surface, it is necessary and sufficient to fulfill the following condition of the components that make up the point equation: $p(u) + q(u) + r(u) \neq 1$.

Note that usually to determine whether the current point belongs to the plane, use a determinant composed of the corresponding coordinates of points, which must be equal to zero. In other words, in order to determine whether the curve is flat or spatial, you need to take any 4 points on it and calculate the determinant. If the determinant is zero, the curve is flat. In the case of using BN-calculus any of the keys do not have to be. Just check that condition (2) is met. Thus, all continuous curves satisfying the condition $p(u) + q(u) + r(u) \neq 1$, they can serve as a return edge for modeling torse surfaces in the BN-calculus.

3. **The general approach to the construction of torse surfaces in BN-calculus**

Based on the definition of the torse surface, in addition to the return edge itself, it is also necessary to determine the tangent to it, which will form the torse surface. The tangent to the curve in the BN-calculus [10] is determined by differentiating the original curve by the current parameter, followed by parallel transfer of the resulting segment to the tangent point (Figure 1).

![Figure 1. Geometric interpretation of the tangent definition in the BN-calculus](image)
Let be given some flat curved line in the simplex \(ABC\) by the current point \(M\) using equation (1). In this case, the simplex \(ABC\) itself belongs to the global coordinate system (or global Cartesian simplex) with the beginning at a point \(O\) (Figure 1). Differentiating the point equation of the curve of the form (1) by parameter, we get another line with the current point \(\dot{M}\):

\[
\dot{M} = (A - C) \dot{p}(u) + (B - C) \dot{q}(u),
\]

(4)

where \(\dot{p}(u), \dot{q}(u)\) – derivatives of continuous functions \(p(u)\) and \(q(u)\) in the parameter \(u\).

In accordance with the rule of parallel transfer of the BN-calculus, given that all coordinates of the starting point \(O\) are equal to zero, we get:

\[
M + \dot{M} = (A - C)\left[p(u) + \dot{p}(u)\right] + (B - C)\left[q(u) + \dot{q}(u)\right] + C,
\]

(5)

where \(M + \dot{M}\) – this is also a point born by parallel transfer of the segment \(OM\) to the point of the original curve \(M\).

Expressions (4) and (5) are easily generalized to a multidimensional space due to the invariant properties of the tangent with respect to parallel projection. Then the point equation for determining the tangent to the spatial curve in 3-dimensional space will take the following form:

\[
M + \dot{M} = (A - D)\left[p(u) + \dot{p}(u)\right] + (B - D)\left[q(u) + \dot{q}(u)\right] + (C - D)\left[r(u) + \dot{r}(u)\right] + D,
\]

(6)

Using equation (6), we define the torse surface generating line using a parameter \(v\), thus forming a two-parameter set of points generated by the movement of the current point \(T\) when the parameters change \(0 \leq u \leq 1\) and \(0 \leq v \leq 1\).

\[
T = Mv + (M + \dot{M})v
\]

(7)

Substituting equations (3) and (6) into equation (7), we obtain a point equation of the torse surface in General form with free functions \(p(u), q(u)\) and \(r(u)\), defining the return edge of the torse surface:

\[
T = (A - D)\left[p(u) + \dot{p}(u)v\right] + (B - D)\left[q(u) + \dot{q}(u)v\right] + (C - D)\left[r(u) + \dot{r}(u)v\right] + D.
\]

(8)

It should be noted that with this approach, the length of the generative is variable and depends on the derivatives of free functions \(p(u), q(u)\) and \(r(u)\). If you need to fix it, then you should make appropriate adjustments to the equation of the generating line (7).

4. Modeling torse surfaces based on algebraic curves

Let’s consider an example of modeling a torse surface in BN-calculus with an algebraic curve as the return edge. The simplest of the 3rd-order spatial lines that can serve as a return edge is the 3rd-order Bezier curve. Its point equation in the simplex \(ABCD\) has the following form:

\[
M = At + 3Bu + 3Cu\bar{u}^2 + Du^3,
\]

(9)

where \(\bar{u} = 1 - u\) – addition of the parameter \(u\) to 1.

Let’s check the curve (9) for compliance with condition (2).

\[
\bar{u}^3 + 3\bar{u}^2u + 3\bar{u}u^2 + u^3 \neq 1.
\]

Therefore, the Bezier curve of the 3rd order is a spatial curve and can serve as a return edge of the torse surface.

To determine the point equation of the torse surface, we use equation (8), based on the fact that \(p(u) = \bar{u}^3, q(u) = 3\bar{u}^2u\) и \(r(u) = 3\bar{u}u^2\):
We perform a coordinate calculation of the point equation (10) for a 3-dimensional space. As a result, we get the following system of parametric equations:

\[
\begin{align*}
T &= \left(-u^3 + (-3v + 3)u^2 + (6v - 3)u - 3v + 1\right)A + 3\left(u^3 + (3v - 2)u^2 + (1 - 4v)u + v\right)B + \\
&\quad + 3u\left(-u^2 + (1 - 3v)u + 2v\right)C + u^2\left(u + 3v\right)D. \\
\end{align*}
\]  

(10)

Substituting the coordinates of the simplex points \(ABCD\) into the system of equations (11), we obtain various variations of the torse surface. For example, figure 2 shows a computer visualization of a torse surface constructed on a unit cube (the origin and unit points on the coordinate axes are used as simplex points).

5. Modeling torse surfaces based on transcendental spatial curves

The approach proposed above is universal in relation to the choice of continuous functions for determining the return edge, which can be used not only algebraic, but also transcendental. As an example, let's consider the construction of a torse surface with a return edge in the form of a transcendent curve:

\[
M = A\sin\left(3\pi u\right) + B\cos\left(3\pi u\right) + 3Cu^2 + Du^3. 
\]  

(12)

It is obvious that the curve defined by the point equation (12) is spatial, since \(\sin\left(3\pi u\right) + \cos\left(3\pi u\right) + 3Cu^2 \neq 0\).

By analogy with algebraic curves, using (8) and (12), we define the point equation of the torse surface with a transcendent edge:

![Figure 2. Computer visualization of the torse surface with a return edge in the form of a 3rd-order Bezier curve](image)
Using a computer algebra system, we perform a computer visualization of the resulting torse surface (Figure 3).

\[
T = (A - D) \left( -\sin(u^3 - 1) - 3u^2\cos(u^3 - 1) v \right) + \\
+ (B - D) \left( \cos(3(1 - u^2)u) - (6(1 - u^2 u + 3(1 - u^2)u) \sin(3(1 - u^2)u) v) + \\
+ (C - D) \left( 3(1 - u^2 u^2 + (-3u^2 + 6(1 - u^2)u) v) + D. \\
\]

Using a computer algebra system, we perform a computer visualization of the resulting torse surface (Figure 3).

![Computer visualization of the transcendent torse surface](image)

Figure 3. Computer visualization of the transcendent torse surface

### 6. Conclusion
The approach to modeling torse surfaces in BN-calculus, which includes the definition of return edges in the form of spatial algebraic and transcendental curves, allows on the one hand to expand the possibilities of modeling torse surfaces, and on the other hand to facilitate this process by more simple definition of return edges. The prospect of further research is to model torse surfaces with pre-defined geometric properties, for example, passing through pre-defined points.

### 7. References

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