Quench Dynamics of Entanglement in an Opened Anisotropic
Spin-1/2 Heisenberg Chain

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Abstract

The quantum entanglement dynamics of a one-dimensional spin-1/2 anisotropic XXZ model is studied using the method of the adaptive time-dependent density-matrix renormalization-group when two cases of quenches are performed in the system. An anisotropic interaction quench and the maximum number of domain walls of staggered magnetization quench are considered. The dynamics of pairwise entanglement between the nearest two qubits in the spin chain is investigated. The entanglement of the two-spin qubits can be created and oscillates in both cases of the quench. The anisotropic interaction has a strong influence on the oscillation frequency of entanglement.

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I. INTRODUCTION

Entanglement generation and distribution is an important problem in performing quantum-information tasks, such as quantum computation and quantum teleportation [1][3]. Many results showed that entanglement existed naturally in the spin chain when the temperature is at zero. Due to its potential applications, the pairwise entanglement of anisotropic Heisenberg model was extensively studied recently [4]. A typical example is the exactly solved one-dimensional infinite-lattice anisotropic $XY$ model using the Jordan-Wigner transform to obtain the pairwise entanglement [5][8]. By using the Bethe ansatz solution, the entanglement in infinite isotropic Heisenberg rings was calculated [9]. In recent years, the study of the dynamics of entanglement has attracted much attention due to the manipulation of quantum systems. The dynamics of entanglement in different systems were investigated [10][19]. The effects of anisotropy interaction on the evolution of entanglement were investigated [19], such a global quench could actually be realized for atoms in optical lattices [20]. Moreover, the effects of a bond defect on the entanglement dynamics in a local quench were studied [21] and the evolution of one-dimensional quantum lattice systems was investigated when the ground state was perturbed by altering one site in the middle of the chain [22][27]. Recently, Bose exploited a global quench dynamics in spin chains for distant pairwise entanglement, which could be used for quantum communication [28]. It would be interesting to investigate the overall pairwise entanglement dynamics between the nearest two qubits in the spin chain when several typical quenches are performed in the system.

In this paper, quench dynamics of entanglement in an opened anisotropic Heisenberg spin chain is analyzed. In section II, the Hamiltonian of an opened anisotropic Heisenberg spin chain is presented. In section III, the dynamics of pairwise entanglement between the nearest two qubits in the spin chain is studied when two kinds of typical quench are performed in the system. In section IV, a discussion concludes the paper.

II. HAMILTONIAN OF THE SYSTEM

The Hamiltonian of an opened Heisenberg XXZ chain with $N$ sites is given by

$$H = \sum_{i=1}^{N-1} [J_{xy}(\sigma_{i}^{x}\sigma_{i+1}^{x} + \sigma_{i}^{y}\sigma_{i+1}^{y}) + J_{z}\sigma_{i}^{z}\sigma_{i+1}^{z}],$$

(1)
where $\sigma_\alpha^i (\alpha = x, y, z)$ are Pauli operators on the $i$-th site, $N$ is the length of the spin chain, $J_{xy}, J_z$ denotes the couplings in the $xy$-plane and the $z$-axis respectively. For simplicity, the couplings in the plane $J_{xy} = 1$ is considered. An opened boundary condition (OBC) is assumed because the antiferromagnetic Heisenberg spin chain with OBC can be achieved artificially in the experiment [29].

In the paper, the concurrence is chosen as a measurement of the pairwise entanglement [3]. The concurrence $C$ is defined as

$$C = \max\{\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4, 0\},$$

where the quantities $\lambda_i (i = 1, 2, 3, 4)$ are the square roots of the eigenvalues of the operator $
ue = \rho_{12}(\sigma_1^y \otimes \sigma_2^y)\rho_{12}^*(\sigma_1^y \otimes \sigma_2^y)$. They are in descending order. The case of $C = 1$ corresponds to the maximum entanglement between the two qubits, while $C = 0$ means that there is no entanglement between the two qubits.

### III. PAIRWISE ENTANGLEMENT DYNAMICS

In this section, the entanglement dynamics of a spin-$1/2$ antiferromagnetic Heisenberg chain is analyzed when two kinds of quenches are performed in the system. One kind of quench is performed by abruptly varying the anisotropic interaction from very large value down to finite values. Another kind of quench is performed by sudden release of the domain walls of staggered magnetization. It is known that it is hard to calculate the dynamics of entanglement because of the lack of knowledge of eigenvalues and eigenvectors of the Hamiltonian. For models that are not exactly solvable, most of researchers resort to exact diagonalization to obtain the ground state for small system size. While for small system size, some phenomena might be missing. If the system size is large, other kinds of phenomena can appear and can be seen much clearer.

For large system size, the adaptive time-dependent density-matrix renormalization-group (t-DMRG) can be applied with a second order Trotter expansion of Hamiltonian as described in [30, 31]. In order to check the accuracy of the results of t-DMRG, the results of exact diagonalization can be considered as a benchmark for a small size system. In the simulation, a Trotter slicing $\delta t = 2.5 \cdot 10^{-2}$ and Matlab codes of t-DMRG with double precision are performed with a truncated Hilbert space of $m = 60$. It turns out that a typically discarded
weight of $\delta \rho = 10^{-8}$ can keep the relative error $\delta C$ in $C$ below $10^{-6}$ for a chain of $L = 60$ sites with time $t \leq 40/J$.

A. Anisotropic Interaction Quench

To start t-DMRG, some initial states need to be prepared. Firstly, the initial state can be prepared to be a perfect antiferromagnetic Néel state. The perfect antiferromagnetic Néel state is the ground state of Eq. (1) with $J_z \to +\infty$. It is given by

$$|\text{ini}\rangle_1 = |\uparrow\downarrow\uparrow\downarrow\cdots\uparrow\downarrow\rangle.$$  \hspace{1cm} (3)

Such a state has been achieved with high fidelity by using decoupled double wells [32]. The procedure adopted for the initial state $|\text{ini}\rangle_1$ is to calculate it as the ground state of a suitably chosen Hamiltonian $H_{\text{ini}} = H(J_z \to +\infty)$. The quantum quench is performed when the anisotropic interaction parameter is abruptly varied from an initially very large value down to finite values [28].

The entanglement $C_{i,i+1}$ between the nearest two qubits is plotted in Fig. 1 as a function of the spin site $i$ and the time $t$ when $J_z = 1.0$. The three dimensional entanglement $C_{i,i+1}$ is plot in Fig. 1(a). The contour line is plotted in Fig. 1(b). From Fig. 1, it is clear that there is an entanglement wall for small time $t$. It seems that the sudden change of $J_z$ from extremely large value ($J_z \to +\infty$) to a finite value creates the entanglement wall [7, 8]. The entanglement in the range of $10 < i < 50$ is quite similar. When the time $t = 0$, the entanglement is zero. When the time increases, the entanglement creates and increases from zero. The entanglement reaches a peak at the time $t = 0.7$. When the time increases further, the entanglement value maintains in the extent $[0,0.1]$. In Fig. 1, it could also be seen that when $i < 10$ and $50 > i$, the entanglement oscillates acutely. The maximal value of entanglement is higher than the maximal value in the extent of $10 < i < 50$. This is due to the effects of the open boundary condition [19, 33]. From Fig. 1, it is clear that the entanglement $C_{i,i+1}$ is symmetric about $i = 30$. This is mainly due to the fact that several sources in a lattice emit oppositely moving pairs of entangled quasi-particles [8].

In order to avoid the boundary effect, the entanglement $C_{30,31}$ of the central two qubits is plotted as a function of the time $t$ for different anisotropic interaction in Fig. 2. The anisotropic interaction has little influence on the height of the first peak of the entangle-
When $J_z = 0$, the entanglement drops down to zero after reaching the peak. The entanglement cannot generate at any longer time. When $J_z = 1.0$, the entanglement can appear and disappear alternatively. When $J_z = 1.5, 2.0$, the entanglement can create, oscillate and does not disappear for longer time. Except the first peak in $C_{30,31}$, the height of oscillation of entanglement with $J_z = 2.0$ is higher than that with $J_z = 1.0, 1.5$. This is due to the strong dependence of the damping coefficient of the entanglement wave on the anisotropic parameter. When $t$ increases, the relative height of the first peak and the average value of $C_{30,31}$ becomes smaller with increasing $J_z$. The increase of $J_z$ reduces the relative height in $C_{30,31}$.\[8\]

**B. Domain Walls of Staggered Magnetization Quench**

Secondly, the initial state can be prepared to be an inhomogeneous initial state. It can be given by

$$|\text{ini}\rangle_2 = |\uparrow_1 \cdots \uparrow_{N/2}\!\downarrow_{N/2+1} \cdots \downarrow_N\rangle; \quad (4)$$

where all spins on the left half are pointing up along the z axis, while all spins on the right half pointing down. The state contains many high-energy excitations and is thus far from equilibrium. It can be considered as a state with almost the maximum number of domain walls of staggered magnetization. The quantum quench is performed when the domain walls are suddenly released in a Heisenberg XXZ chain with different anisotropic interaction [35].

The entanglement $C_{i,i+1}$ between the nearest two qubits is plotted as a function of spin site $i$ and time $t$ in Fig. 3 when $J_z = 0$. The three dimensional plot of $C_{i,i+1}$ is shown in Fig. 3(a). The counter line of $C_{i,i+1}$ is plotted in Fig. 3(b). From Fig. 3, it is seen that the entanglement $C_{i,i+1}$ can be generated. The entanglement is mirror symmetric about the line of $i = 30$. The entanglement $C_{30,31}$ generates firstly, then the entanglement $C_{29,30}$ and $C_{31,32}$ generate, and so on. The period of entanglement $C_{i,i+1}$ generation is a linear function of $|i - 30|$. The notion of diffusive dynamics of pairwise entanglement in the system can be clearly seen [35, 36]. The two central neighboring sites are in the state $|\uparrow_{N/2}\!\downarrow_{N/2+1}\rangle$. Due to the exchange interaction, they will be entangled after the time $t = 0$. The entanglement, initially localized on the two central neighboring sites of the chain will spread owing to the exchange interaction [7, 8].
In order to see the effects of the anisotropic interaction, the entanglement $C_{30,31}$ and the entanglement $C_{25,26}$ of the central two qubits is plotted in Fig. 4 as a function of the time $t$ for different anisotropic interaction. In Fig. 4(a), the entanglement $C_{30,31}$ generates immediately after the time $t = 0$. The entanglement reaches the first peak rapidly and then oscillates around a constant. The anisotropic interaction has a weak influence on the first peak of the entanglement. The first peak decreases slightly when the anisotropic interaction increases. When $J_z = 0.0$, the entanglement oscillates around 0.35. When $J_z = 0.5$, the entanglement oscillates around 0.12. When $J_z = 1.5, 2.0$, the entanglement oscillates around 0.40 and exists all the time. For the above cases, the entanglement maintains for a long time. While for $J_z = 1$, the entanglement appears and disappears alternatively. After $t = 37$, the entanglement disappears all the time. It is clear that the entanglement $C_{30,31}$ of the central two qubits decreases as the anisotropic interaction $J_z$ increases and then increases as $J_z$ increases further and finally almost saturates. In Fig. 4(b), the entanglement $C_{25,26}$ generates after a short time. This time delay is caused by the finite propagation speed of the entanglement from the central pair to side pairs. The entanglement reaches the first peak rapidly and then oscillates around a constant. The anisotropic interaction has a strong influence on the first peak of the entanglement. The height of the first peak decreases when the anisotropic interaction increases. When $J_z = 0$, the first peak is $C_{25,26}^{peak(1)} = 0.427$, while the peak is $C_{25,26}^{peak(1)} = 0.259$ when $J_z = 0.5$. The anisotropic interaction has a feeble influence on the period when the entanglement reaches the first peak. The time of the entanglement reaching the first peak decreases slightly when the anisotropic interaction increases. After the entanglement reaches the first peak, the entanglement oscillates around 0.33 when $J_z = 0.0$. When $J_z = 0.5$, the entanglement oscillates around 0.11. For the above cases, the entanglement exists all the time. While for $J_z = 1$, the entanglement appears and disappears alternatively. After $t = 25$, the entanglement disappears all the time. When $J_z = 1.5, 2.0$, the entanglement disappears and never appears after the first peak emergences. The entanglement decreases as the anisotropic interaction increases. It is noted that the propagation velocity of the entanglement is slightly influenced by the anisotropy parameter, but the damping coefficient of the entanglement wave is strongly dependent on the anisotropic parameter[7, 8].
IV. DISCUSSION

By using the method of the adaptive time-dependent density-matrix renormalization-group, the time evolution of the entanglement in a one-dimensional spin-1/2 anisotropic XXZ model is investigated when two quenches are performed. One quench is a sudden change of the anisotropic interaction, while another is the abrupt release of the maximum number of domain walls of staggered magnetization. The dynamics of pairwise entanglement between the two nearest qubits in the spin chain is studied. The entanglement of the two-spin qubits can be created and oscillates after one of the two quenches is performed. In both cases of the quench, the time evolution of the entanglement is symmetric about the central qubit. For the anisotropic interaction quench, an entanglement wall creates. Except the first peak, the entanglement increases and oscillates as the anisotropic interaction $J_z$ increases. For small $J_z$, the entanglement may disappear at longer time. In the quench of domain walls of staggered magnetization, the entanglement of central two qubits appears first, after a short period, the entanglement of the next two qubits appears, and so on. The height of the first peak in the entanglement also decreases as the pair of two qubits moves away from the center. The entanglement of the central two qubits decreases when the anisotropic interaction increases, and then increases when the anisotropic interaction increases further. For two kinds of the quenches, the oscillation frequency of the entanglement is strongly influenced by the anisotropy. This phenomenon may be used to control the dynamics of the entanglement by varying the anisotropic interaction of the Heisenberg spin chain.

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FIG. 1. The pairwise entanglement $C_{i,i+1}$ between the two nearest qubits is plotted as a function of spin site $i$ and time $t$ with $J_z = 1$ when there is anisotropic interaction quench. The size of the system is $N = 60$. (a). The three dimensional plot of $C_{i,i+1}$. (b). The contour line of $C_{i,i+1}$. 
FIG. 2. The pairwise entanglement $C_{30,31}$ of the two central qubits is plotted as a function of time $t$ for different anisotropic interaction $J_z$ when there is anisotropic interaction quench. The sizes of the system is $N = 60$. 
FIG. 3. The pairwise entanglement \( C_{i,i+1} \) between the two nearest qubits is plotted as a function of spin site \( i \) and time \( t \) with \( J_z = 1.0 \) when there is domain walls quench. The size of the system is \( N = 60 \). (a). Three dimensional plot of \( C_{i,i+1} \). (b). The contour plot of \( C_{i,i+1} \).
FIG. 4. The pairwise entanglement is plotted as a function of time $t$ for different anisotropic interaction $J_z$ when there is domain walls quench. The size of the system is $N = 60$. (a). The pairwise entanglement $C_{25,26}$. (b). The pairwise entanglement $C_{30,31}$ of the two central qubits.