Multi-item fuzzy inventory model by Kuhn-Tucker method

K. Kalaiarasi,¹* M. Sumathi² and M. Sabina Begum³

Abstract
Form on the generalized defuzzifying approach developed in this paper, we substantiate the fuzzy economic order quantity (EOQ) model with backorder level. Holding cost, ordering cost, shortage back order cost and demand are taken in terms of fuzzy numbers. Our main object is to find fuzzy total cost and fuzzy order quantity. When the order quantity is a crisp number, we use the direct derivation method to obtain the optimal solution. When the order quantity is a fuzzy number, we use the extension of the Kuhn-tucker method and signed distance method to solve the inequality constraints. A numerical example is given to illustrate the significance of the proposed model.

Keywords
Multi-item inventory model, trapezoidal fuzzy numbers, signed distance, Kuhn-tucker condition.

AMS Subject Classification
03E72.

1. Introduction
Inventory management is one of the widely learned topics in operations research. Many researchers tried to explore the different aspects of this wide topic. In earlier days of the operations research, researchers considered all parameters constant and tried to optimize the system. Then, some researchers started using variables and considered the changing nature of parameters.

Recently, researchers in production and operations research are mainly focusing on cost minimization and profit maximization for production/inventory modeling. Fuzzy set theory originally introduced by Zadeh in 1965, the application of fuzzy set theory to inventory problems has been proposed by Kacpryzk and Park. Park [1987] Examine the EOQ formula in the fuzzy set theoretic perspective associating the fuzziness with the cost data. The Kuhn tucker conditions are necessary conditions for identifying stationary points of a non-linear constrained problem subject to inequality constraints.

This paper is organized as follows. In section 2, we give a brief introduction to fuzzy number, membership function and fuzzy arithmetical operations and obtain the generalized defuzzifying expression whether the fuzzy number is the trapezoid.

In section 3, we establish the classical EOQ models and respectively use the direct derivation method and the extension of the Kuhn-tucker method to obtain the optimal solutions when the order quantity is a crisp number or a fuzzy number.

We observe that the results are same. In section 4, one calculation example is given. Finally, some conclusions are drawn in section 5.

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1. Introduction

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In section 3, we establish the classical EOQ models and respectively use the direct derivation method and the extension of the Kuhn-tucker method to obtain the optimal solutions when the order quantity is a crisp number or a fuzzy number.

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2. Definitions and Preliminaries

Definition 2.1. **Fuzzy Set**: A fuzzy set $\tilde{A}$ on the given universal set $X$ is a set of ordered pairs $\tilde{A} = \{ (x, \mu_\tilde{A}(x)) : x \in X \}$ where $\mu_\tilde{A} : x \to [0, 1]$ is called membership function or grade membership. The membership function is also a degree of compatibility or a degree of truth of $x$ in $\tilde{A}$.

Definition 2.2. **Fuzzy Number [FN]**: A fuzzy number is a generalization of a regular real number and which does not refer to a single value but rather to a connected set of possible values, where each possible value has its weight between 0 and 1. This weight is called the membership function.

A fuzzy number is a convex normalized fuzzy set on the real line $R$ such that:

- There exist at least one with $x \in R$ with $\mu_\tilde{A}(x) = 1$.
- $\mu_\tilde{A}(x)$ is piecewise continuous.

Definition 2.3. **Trapezoidal Fuzzy number**: A trapezoidal fuzzy number is a fuzzy set $\tilde{A} = (a_1, a_2, a_3, a_4)$ where $a_1, a_2, a_3, a_4$ are real numbers and its membership function is given by:

$$
\mu_\tilde{A}(x) = \begin{cases} 
0, & x < a_1 \text{ or } x > a_4 \\
\frac{x-a_1}{a_2-a_1}, & a_1 \leq x \leq a_2 \\
1, & a_2 \leq x \leq a_3 \\
\frac{x-a_3}{a_4-a_3}, & a_3 \leq x \leq a_4 
\end{cases}
$$

Definition 2.4. **Signed distance method**: Let $\tilde{A} = (a_1, a_2, a_3, a_4)$ be a trapezoidal fuzzy numbers. Then the signed distance representation

$$
d(\tilde{A}, 0) = \frac{1}{2} \int_0^1 (A_L(\alpha) + A_R(\alpha))d\alpha
$$

Definition 2.5. **The Kuhn-Tucker conditions**: Table 1: The objective function and the solution space

| Sense of optimization | Required conditions | Objective function | Solution space |
|-----------------------|---------------------|--------------------|---------------|
| Maximization          | Concave or convex set| $f(x)$             | $\tilde{A}$    |
| Minimization          | Convex or convex set | $f(x)$             | $\tilde{A}$    |

(i). Problem

$$
\text{max } z = f(x), \text{ subject to } h'_i(x) \leq 0 \\
x \geq 0, i = 1, 2, ..., m
$$

Kuhn-Tucker conditions

$$
\frac{\partial}{\partial x_j} f(x) - \sum_{i=1}^{m} \lambda_i \frac{\partial}{\partial x_j} h'_i(x) = 0 \\
\lambda_i \geq 0, i = 1, 2, ..., m
$$

(ii). Problem

$$
\text{min } z = f(x), \text{ subject to } h'_i(x) \geq 0 \\
x \geq 0, i = 1, 2, ..., m
$$

Kuhn-Tucker conditions

$$
\frac{\partial}{\partial x_j} f(x) - \sum_{i=1}^{m} \lambda_i \frac{\partial}{\partial x_j} h'_i(x) = 0 \\
\lambda_i \geq 0, i = 1, 2, ..., m
$$

Definition 2.6. **Arithmetic operations on trapezoidal fuzzy numbers**: Suppose $\tilde{A} = (a_1, a_2, a_3, a_4)$ and $\tilde{B} = (b_1, b_2, b_3, b_4)$ are two trapezoidal fuzzy numbers. Then

1. $\tilde{A} \oplus \tilde{B} = (a_1 + b_1, a_2 + b_2, a_3 + b_3, a_4 + b_4)$
2. $\tilde{A} \otimes \tilde{B} = (a_1 b_1, a_2 b_2, a_3 b_3, a_4 b_4)$
3. $\tilde{A} \ominus \tilde{B} = (a_1 - b_1, a_2 - b_2, a_3 - b_3, a_4 - b_4)$
4. $\frac{1}{\tilde{B}} = \tilde{B}^{-1} = \left( \begin{array}{cccc}
\frac{1}{b_1} & 1 & 1 & 1 \\
\frac{1}{b_1} & \frac{1}{b_2} & \frac{1}{b_3} & \frac{1}{b_4} \\
\frac{1}{b_1} & \frac{1}{b_2} & \frac{1}{b_3} & \frac{1}{b_4} \\
\frac{1}{b_1} & \frac{1}{b_2} & \frac{1}{b_3} & \frac{1}{b_4}
\end{array} \right) \\
\tilde{A} \ominus \tilde{B} = (\frac{\alpha_1}{b_4}, \frac{\alpha_2}{b_3}, \frac{\alpha_3}{b_2}, \frac{\alpha_4}{b_1})$

3. Mathematical fuzzy model

3.1 Parameters

$K$: ordering cost per order

$D$: demand rate per unit time

$R$: order quantity

$h$: holding cost per unit time

$p$: the probability that a supply batch

$J$: backorder level

$b$: shortage backorder cost per unit time

$T(R, J)$: total average cost
The average total cost is

\[ T(R,J) = \frac{K D}{R} + \frac{h(1-p)(R-J)^2}{2R} + bJp + \frac{bp(1+p)R}{2(1-p)} \]

For crisp order quantity

\[ R = \sqrt{\frac{(1-p)(2KD + (1-p)J^2)(h+b)}{h(1-p)^2 + bp(1+p)}} \]

### 3.2 Fuzzy EOQ model using singed distance method

Here

\[ K = (k_1, k_2, k_3, k_4), \]
\[ h = (h_1, h_2, h_3, h_4), \]
\[ p = (p_1, p_2, p_3, p_4), \]
\[ D = (d_1, d_2, d_3, d_4), \]
\[ b = (b_1, b_2, b_3, b_4) \]

are fuzzy parameters.

\[ \hat{T}(R,J) = \left[ \begin{array}{c} \frac{D}{R} + \frac{h_1(1-p)(R-J)^2}{2R} + b_1Jp_1 + \frac{b_1p(1+p)R}{2(1-p)} \\ \frac{D}{R} + \frac{h_2(1-p)(R-J)^2}{2R} + b_2Jp_2 + \frac{b_2p(1+p)R}{2(1-p)} \\ \frac{D}{R} + \frac{h_3(1-p)(R-J)^2}{2R} + b_3Jp_3 + \frac{b_3p(1+p)R}{2(1-p)} \\ \frac{D}{R} + \frac{h_4(1-p)(R-J)^2}{2R} + b_4Jp_4 + \frac{b_4p(1+p)R}{2(1-p)} \end{array} \right] \]

By singed distance method, total cost is given by

\[ \bar{T}_s(R,J) = \frac{1}{4} \left[ \begin{array}{c} \frac{D}{R} + \frac{h_1(1-p)(R-J)^2}{2R} + b_1Jp_1 + \frac{b_1p(1+p)R}{2(1-p)} \\ \frac{D}{R} + \frac{h_2(1-p)(R-J)^2}{2R} + b_2Jp_2 + \frac{b_2p(1+p)R}{2(1-p)} \\ \frac{D}{R} + \frac{h_3(1-p)(R-J)^2}{2R} + b_3Jp_3 + \frac{b_3p(1+p)R}{2(1-p)} \\ \frac{D}{R} + \frac{h_4(1-p)(R-J)^2}{2R} + b_4Jp_4 + \frac{b_4p(1+p)R}{2(1-p)} \end{array} \right] \]

To minimize total cost per unit time, the order quantity is obtained by solving the following equation:

\[ \frac{d}{dR} \bar{T}_s(R,J) = 0 \]

Then the result is,

\[ R^* = \sqrt{\frac{(2(K_1D_1 + K_2D_2 + K_3D_3 + K_4D_4) + J^2 + J^2 + \frac{h_1(1-p)(R-J)^2}{2R} + \frac{h_2(1-p)(R-J)^2}{2R} + \frac{h_3(1-p)(R-J)^2}{2R} + \frac{h_4(1-p)(R-J)^2}{2R} + J^2 + J^2 + \frac{b_1Jp_1}{2(1-p)} + \frac{b_2Jp_2}{2(1-p)} + \frac{b_3Jp_3}{2(1-p)} + \frac{b_4Jp_4}{2(1-p)}}{p(1-p)(b_1 + b_2 + b_3 + b_4)}} \]

Next the fuzzy EOQ \( R^* \) be a trapezoidal fuzzy numbers \( R^* = (r_1, r_2, r_3, r_4) \) with \( 0 \leq r_1 \leq r_2 \leq r_3 \leq r_4 \).

The fuzzy total average cost is

\[ T_1(R,J) = \left[ \begin{array}{c} K_1 \frac{D_1}{r_4} + \frac{h_1(1-p)(r_1-J)^2}{2r_4} + b_1Jp_1 + \frac{b_1p(1+p)r_1}{2(1-p)} \\ K_2 \frac{D_2}{r_3} + \frac{h_2(1-p)(r_2-J)^2}{2r_3} + b_2Jp_2 + \frac{b_2p(1+p)r_2}{2(1-p)} \\ K_3 \frac{D_3}{r_2} + \frac{h_3(1-p)(r_3-J)^2}{2r_2} + b_3Jp_3 + \frac{b_3p(1+p)r_3}{2(1-p)} \\ K_4 \frac{D_4}{r_1} + \frac{h_4(1-p)(r_4-J)^2}{2r_1} + b_4Jp_4 + \frac{b_4p(1+p)r_4}{2(1-p)} \end{array} \right] \]

We defuzzify the above total cost by using singed distance method, we get

\[ \bar{T}_s(Q,J) = \frac{1}{4} \left[ \begin{array}{c} K_1 \frac{D_1}{r_4} + \frac{h_1(1-p)(r_1-J)^2}{2r_4} + b_1Jp_1 + \frac{b_1p(1+p)r_1}{2(1-p)} \\ K_2 \frac{D_2}{r_3} + \frac{h_2(1-p)(r_2-J)^2}{2r_3} + b_2Jp_2 + \frac{b_2p(1+p)r_2}{2(1-p)} \\ K_3 \frac{D_3}{r_2} + \frac{h_3(1-p)(r_3-J)^2}{2r_2} + b_3Jp_3 + \frac{b_3p(1+p)r_3}{2(1-p)} \\ K_4 \frac{D_4}{r_1} + \frac{h_4(1-p)(r_4-J)^2}{2r_1} + b_4Jp_4 + \frac{b_4p(1+p)r_4}{2(1-p)} \end{array} \right] \]

With \( 0 \leq r_1 \leq r_2 \leq r_3 \leq r_4 \). It will not change the meaning of above, if we replace inequality constrains \( 0 \leq r_1 \leq r_2 \leq r_3 \leq r_4 \) into the following inequality constrains \( r_3 - r_1 \geq 0, r_4 - r_2 \geq 0, r_3 - r_2 \geq 0 \) and \( r_1 \geq 0 \). The Kuhn-tucker conditions are, \( \lambda \leq 0 \).

\[ \nabla f(\bar{T}_s(R,J)) - \lambda_i \nabla g_i(R) = 0 \]
\[ \lambda_i g_i(R) = 0 \]

These conditions simplify to the following, \( \lambda_1, \lambda_2, \lambda_3, \lambda_4 \leq 0 \)

\[ \frac{1}{4} \left( K_1 \frac{D_1}{r_4} + \frac{h_1(1-p)(r_1-J)^2}{2r_4} + \frac{b_1Jp_1}{2(1-p)} + \frac{b_1p(1+p)r_1}{2(1-p)} \right) + \frac{b_1Jp_1}{2(1-p)} + \frac{b_1p(1+p)r_1}{2(1-p)} = 0 \]
\[ \frac{1}{4} \left( K_2 \frac{D_2}{r_3} + \frac{h_2(1-p)(r_2-J)^2}{2r_3} + \frac{b_2Jp_2}{2(1-p)} + \frac{b_2p(1+p)r_2}{2(1-p)} \right) + \frac{b_2Jp_2}{2(1-p)} + \frac{b_2p(1+p)r_2}{2(1-p)} = 0 \]
\[ \frac{1}{4} \left( K_3 \frac{D_3}{r_2} + \frac{h_3(1-p)(r_3-J)^2}{2r_2} + \frac{b_3Jp_3}{2(1-p)} + \frac{b_3p(1+p)r_3}{2(1-p)} \right) + \frac{b_3Jp_3}{2(1-p)} + \frac{b_3p(1+p)r_3}{2(1-p)} = 0 \]
\[ \frac{1}{4} \left( K_4 \frac{D_4}{r_1} + \frac{h_4(1-p)(r_4-J)^2}{2r_1} + \frac{b_4Jp_4}{2(1-p)} + \frac{b_4p(1+p)r_4}{2(1-p)} \right) + \frac{b_4Jp_4}{2(1-p)} + \frac{b_4p(1+p)r_4}{2(1-p)} = 0 \]

If \( r_1 > 0 \) and \( \lambda_4 r_1 = 0 \) then \( \lambda_4 = 0 \).

If \( \lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0 \). Then \( r_4 < r_3 < r_2 < r_1 \). It does not satisfy the constrains \( 0 \leq r_1 \leq r_2 \leq r_3 \leq r_4 \).

Therefore \( r_2 = r_1, r_3 = r_2, r_4 = r_3 \).
Then
\[ R^* = \left( \frac{1}{2} \left[ (K_1D_1 + K_2D_2 + K_3D_3 + K_4D_4) \right] \right. \]
\[ + J^2 \left( (h_1 + b_1) + (h_2 + b_2) \right) \]
\[ + \left. b_3 + (h_4 + b_4)) \right) (1 - p) \]
\[ (1 - p)^2 (h_1 + h_2 + h_3 + h_4) \]
\[ + p (1 - p) (b_1 + b_2 + b_3 + b_4) \]

Hence the order quantity of the above equation is,
\[ R = (8, 10, 14, 16), \]
\[ h = 0.6, \]
\[ b = (3, 5, 7, 9), \]
\[ D = (75, 150, 175, 200), \]
\[ K = Fuzzy model: \]
\[ b = 0.6, \]
\[ J = (14 units) \]
\[ Rs.1512.3/- \]

Algorithm for finding fuzzy total cost and fuzzy optimal order quantity

**Step 1:** Calculate the total cost for the given crisp values of \( K, D, h, \) and \( b. \)

**Step 2:** Find fuzzy total cost using fuzzy arithmetic operations on fuzzy demand, fuzzy shortage backorder cost and fuzzy ordering cost and fuzzy holding cost taken as trapezoidal fuzzy numbers.

**Step 3:** Use signed distance method, then we find the fuzzy order quantity which can be obtained by putting the first derivative is equal to 0.

**Step 4:** Use Kuhn-Tucker condition, to check whether the order quantity and total cost in fuzzy sense is as same as the crisp sense.

### 4. Numerical example

#### Crisp model:

Let
\[ K = Rs.150/- \text{ per unit,} \]
\[ D = 400 \text{ unit,} \]
\[ h = Rs.12/- \text{ per unit,} \]
\[ p = 0.6, \]
\[ J = 6, \]
\[ b = Rs.6/- \text{ per unit.} \]

Then \( R = 79.14 \text{ units} \)
\[ \text{And } T(R, J) = Rs.1512.3/- \]

#### Fuzzy model:

Let
\[ K = (75, 150, 175, 200), \]
\[ D = (333, 367, 400, 450), \]
\[ h = (8, 10, 14, 16), \]
\[ b = (3, 5, 7, 9), \]
\[ p = 0.6, \]
\[ J = 6. \]

Then \( \tilde{R}^* = 79.14 \text{ units} \)
\[ \tilde{T}_{11}(R, J) = Rs.1512.3/- \]

### 5. Conclusion

In this paper, we have thought-out fuzzy total cost and fuzzy order quantity. Cost components are fuzzified with the help of the trapezoidal fuzzy numbers and defuzzified by using signed distance method. This method worked-out by Kuhn-tucker conditions. We conclude that the fuzzy estimates all closer to the crisp estimates.

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