REDDUCTION TECHNIQUES FOR THE FINITISTIC DIMENSION

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Abstract. In this paper we develop new reduction techniques for testing the finiteness of the finitistic dimension of a finite dimensional algebra over a field. Viewing the latter algebra as a quotient of a path algebra, we propose two operations on the quiver of the algebra, namely arrow removal and vertex removal. The former gives rise to cleft extensions and the latter to recollements. These two operations provide us new practical methods to detect algebras of finite finitistic dimension. We illustrate our methods with many examples.

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1. Introduction

One of the longstanding open problems in representation theory of finite dimensional algebras is the Finitistic Dimension Conjecture. Let Λ be a finite dimensional algebra over a field. The finitistic dimension \( \text{fin. dim} \Lambda \) of Λ is defined as the supremum of the projective dimension of all finitely generated right modules of finite projective dimension. The finitistic dimension conjecture asserts that the latter supremum is finite, i.e. \( \text{fin. dim} \Lambda < \infty \). Our aim in this paper is to present some new reduction techniques for detecting the finiteness of the finitistic dimension.

The finitistic dimension conjecture has a long and interesting history. Already in the beginning of the sixties, it became apparent that the finitistic dimension provides a measure of the complexity of the module category. In the commutative noetherian case, it has been proved basically by Auslander and Buchbaum [11] that the finitistic dimension equals the depth of the ring. It was Bass that emphasized the role of this homological dimension in the non-commutative setup. For more on the history of the finitistic dimension conjecture we refer to Zimmermann-Huisgen’s paper [28].

The finitistic dimension conjecture is known to be related with other important problems concerning the homological behaviour and the structure theory of the module category of a finite dimensional algebra. In the hierarchy of the homological
conjectures in representation theory, the finitistic dimension conjecture plays a central role. More precisely, we have the following diagram which shows that almost all other homological conjectures for finite dimensional algebras are implied by the finitistic dimension conjecture (FDC):

$$
(FDC) \xrightarrow{\text{WTC}} (GSC) \\
\downarrow \\
(NuC) \xrightarrow{\text{SNC}} (ARC) \xrightarrow{\text{NC}}
$$

We write (SNC) for the strong Nakayama conjecture, (NC) for the Nakayama conjecture, (ARC) for the Auslander-Reiten conjecture, (WTC) for the Wakamatsu tilting conjecture, (NuC) for the Nunke condition and (GSC) for the Gorenstein symmetry conjecture. The above diagram is not complete, we refer to [2, 11, 12, 25] and references therein for more information on the hierarchy of homological conjectures.

In the middle of the seventies, Fossum–Griffith–Reiten [6] proved for a triangular matrix algebra $\Lambda = (R \stackrel{M}{\leftarrow} S)$ that the finitistic dimension of $\Lambda$ is less or equal of the finitistic dimensions of $R$ and $S$ plus one. Thus, for this particular class of algebras we can test finiteness of the finitistic dimension by computing the finitistic dimension of the corner algebras. This result should be considered as the first reduction technique for the finitistic dimension. Subsequently, but almost twenty years after, Happel [12] showed that if a finite dimensional algebra $\Lambda$ admits a recollement of bounded derived categories ($D^b(\text{mod-}\Lambda'''), D^b(\text{mod-}\Lambda), D^b(\text{mod-}\Lambda')$), where $\Lambda'$ and $\Lambda''$ are finite dimensional algebras, then the finitistic dimension of $\Lambda$ is finite if and only if the same holds for $\Lambda'$ and $\Lambda'''$. Clearly this is again a reduction technique for the finitistic dimension. However, it is in general a difficult problem to decompose the bounded derived category of an algebra in such a recollement situation. On the other hand, Happel’s technique can be considered as a natural extension of Fossum-Griffith-Reiten’s result, since triangular matrix algebras (under mild conditions on the bimodule) induces a recollement at the level of bounded derived categories.

In the beginning of the nineties, Fuller and Saorín [7] introduced the idea of illuminating simples of projective dimension less or equal to one. In particular, picking the idempotent $f$ corresponding to such a simple and considering the idempotent $e = 1 - f$, they showed that $\dim \Lambda \leq e\Lambda\leq < \infty$ implies $\dim \Lambda < \infty$. This is clearly a reduction technique for the finitistic dimension. We extend this result by putting it in the general context of recollements of abelian categories. Clearly, this result is the predecessor of the vertex removal operation. More recently, Xi in a series of papers [23–26] introduced various methods for detecting finiteness of the finitistic dimension. It should be noted that Xi has connected the finitistic dimension of an algebra of the form $e\Lambda$, where $e$ is an idempotent, with other homological dimensions, for instance, finiteness of the global dimension of $\Lambda$ (less or equal to four), finiteness of the representation dimension of $\Lambda/e\Lambda$ (less or equal to three) and several other interesting relations. On the other hand, he considers pairs of algebras $(B, A)$ where $A$ is an extension of $B$ and the Jacobson radical $\text{rad}(B)$ is a left ideal in $A$. Then, under some further conditions he shows that $\dim A < \infty$ implies $\dim B < \infty$. Roughly speaking, Xi’s philosophy is to control the finitistic dimension by certain extension of algebras. Clearly, this machinery is again another reduction technique for testing the finiteness of the finitistic dimension.

At this point we would like to mention Xi’s comment [23] on the available techniques that we have for the finitistic dimension. Very briefly, he writes that “not many practical methods are available so far to detect algebras of finite finitistic dimension and it is necessary to develop some methods even for some concrete
examples”. The central point for us in this work is exactly the lack of practical methods to estimate the finitistic dimension of an algebra.

Let $\Lambda$ be a finite dimensional algebra over an algebraically closed field $k$. Thus $\Lambda$ is Morita equivalent to an admissible quotient $kQ/I$ of a path algebra $kQ$ over $k$. Moreover, if $e$ is a trivial path in $kQ$, then $v_e$ denotes the corresponding vertex in $Q$. Our approach on reducing the finitistic dimension is based on two operations on the quiver of the algebra. It is natural to consider how the vertices and the arrows contribute to the finitistic dimension. The idea is to remove those that don’t contribute and thus the finiteness of the finitistic dimension is reduced to a simpler algebra at least in terms of size. This is clearly a practical method that can be applied easily to any algebra.

Let $a$ be an arrow in $Q$, which does not occur in a minimal generating set of $I$, and consider the arrow removal $\Gamma = \Lambda/\langle a \rangle$. The abstract categorical framework of this operation is the concept of cleft extension of abelian categories. Our first main result provides a ring theoretical characterization of the arrow removal operation. In addition, it reduces the finiteness of the finitistic dimension of $\Lambda$ to the one of the arrow removal. The first part of Theorem A is proved in Proposition 4.5. The second part is stated in Theorem 4.7 and follows from Theorem 4.1 (a general result on the finitistic dimension for cleft extensions) and Proposition 4.6 (a result on the precise properties of the arrow removal as a cleft extension).

Theorem A. (Arrow Removal) Let $\Lambda = kQ/I$ be an admissible quotient of a path algebra $kQ$ over a field $k$. Let $a: v_e \rightarrow v_f$ be an arrow in $Q$ and define $\Gamma = \Lambda/\langle a \rangle$ the arrow removal. The following hold.

(i) The arrow $a: v_e \rightarrow v_f$ in $Q$ does not occur in a set of minimal generators of $I$ in $kQ$ if and only if $\Lambda$ is isomorphic to the trivial extension $\Gamma \bowtie P$, where $P = \Gamma e \otimes_k f \Gamma$ with $\text{Hom}_\Gamma(e\Gamma, f\Gamma) = (0)$.

(ii) If the arrow $a$ does not occur in a set of minimal generators of $I$ in $kQ$, then $\text{fin.dim.} \Lambda < \infty$ if and only if $\text{fin.dim.} \Gamma < \infty$.

We would like to mention that the arrow removal operation has been considered in the work of Diracca–Koenig [5]. Their focus was removing arrows in a monomial relation and homological reductions towards the strong no loop conjecture.

Our second operation is the vertex removal. Let $\Lambda$ be a quotient of a path algebra as above and take $e$ a sum of vertices. Then the vertices in the quiver of $e\Lambda$ correspond to the ones occurring in $e$ and therefore the vertices occurring in $1 - e$ are removed. The transition from $\Lambda$ to $e\Lambda$ is what we call vertex removal. The abstract categorical framework of this operation is the concept of recollements of abelian categories. In our second main result we show that removing the vertices which correspond to simples of finite injective dimension provides a reduction for the finitistic dimension. This is our second new practical method for testing the finiteness of the finitistic dimension. The result is basically proved in Theorem 5.8 in the setting of recollements of abelian categories.

Theorem B. (Vertex Removal) Let $\Lambda$ be a finite dimensional algebra over an algebraically closed field and $e$ an idempotent element. Then

$$\text{fin.dim.} \Lambda \leq \text{fin.dim.} e\Lambda + \sup \{ \text{id}_\Lambda S \mid S \text{ simple } \Lambda/e\Lambda\text{-module} \}$$

The above result shows an interesting interplay between the finitistic dimension and the injective dimension of some simples. Recently, a similar connection was observed by Rickard [21]. In particular, he showed that if the injectives over a finite dimensional algebra generate its unbounded derived category, then the finitistic dimension conjecture holds.
We have already mentioned several times the practical issue of our main results. To see this, we advise the reader to look at the last section where we present several examples where our methods can be applied without much effort. We have also tested our techniques on some known examples from the literature.

Our reduction techniques can be applied to any finite dimensional algebra over an algebraically closed field. Given such an algebra, we can iterate the reductions (vertex and arrow removal) to obtain a reduced algebra, see Definition 6.1. As a consequence of our work, to prove or disprove the finitistic dimension conjecture it suffices to consider the class of reduced algebras.

The contents of the paper section by section are as follows. Sections 2 and 3 are devoted for the abstract categorical framework of the arrow and vertex removal operations on a quiver. In Section 2 we study cleft extensions of abelian categories. More precisely, we recall and prove several properties of cleft extensions that are used later in Section 4. We analyze carefully the associated endofunctors that this data carries and we settle the necessary conditions on a cleft extension that the arrow removal operation requires.

In Section 3 we study recollements of abelian categories. We introduce a relative (injective) homological dimension in a recollement situation and show that it provides interesting homological properties in the abelian categories involved in a recollement, see Proposition 3.5. We also recall the notion of a functor between abelian categories being an eventually homological isomorphism and we characterize when the quotient functor in a recollement is an eventually homological isomorphism, see Proposition 3.11. The latter result is used in Section 5.

Section 4 is devoted to arrow removal and the finitistic dimension. This section is divided into two subsections. In the first one, we investigate the behaviour of the finitistic dimension of the abelian categories in a cleft extension (under certain conditions), see Theorem 4.1. This is the first key result for showing Theorem A (ii). In the second subsection, we study arrow removals of quotients of path algebras. We first characterize arrow removals as trivial extensions with projective bimodules admitting special properties, see Corollary 4.3 and Propositions 4.4 and 4.5. Then, we show that arrow removals gives rise to cleft extensions satisfying the needed properties for applying Theorem 4.1. This is done in Proposition 4.6. Finally, we summarise our results on reducing the finitistic dimension by removing arrows in Theorem 4.7, where Theorem A (ii) is a special case.

In Section 5 we investigate the vertex removal operation with respect to the finitistic dimension. This section is divided into four subsections. In the first subsection, we reprove the main result of [10] on reducing the finitistic dimension via the homological heart using the reduction techniques of Fossum–Griffith–Reiten and Happel. In the second subsection, we generalize the result of Fuller–Saorín [7, Proposition 2.1] (vertex removal, projective dimension at most one) from the case of artinian rings to the general context of recollements of abelian categories, see Theorem 5.5. We remark that we also provide a lower bound. The third subsection is about removing vertices of finite injective dimension. In particular, in Theorem 5.3 we show Theorem B stated above in the general context of recollements of abelian categories. Note that Theorems 5.3 and 5.5 (as well as Theorem 4.1 for the arrow removal) can be applied to the big finitistic dimension as well. The last subsection is about eventually homological isomorphisms in recollements of abelian categories and invariance of finiteness of the finitistic dimension between the middle category and the quotient category. In particular, we show that fin. dim. $\Lambda < \infty$ if and only if $\text{fin. dim. } e\Lambda e < \infty$, where $\Lambda$ is an Artin algebra and $e$ an idempotent element, provided that the quotient functor $e(-) : \text{mod } \Lambda \to \text{mod } e\Lambda e$ is an eventually homological isomorphism, see Theorem 5.11.
The last section, Section 5, is devoted to examples. We provide a variety of new and known examples where we reduce the finiteness of the finitistic dimension using our techniques of arrow and vertex removal. In particular, we show in Example 6.4 that the finitistic dimension of a reduced algebra can be arbitrarily large. We also present an example showing that a reduced algebra is not unique.

In an appendix we provide a short introduction to non-commutative Gröbner basis for path algebras. We recall some notions and results from the theory of Gröbner basis that are used in Section 4.

Conventions and Notation. For a ring $R$ we work usually with right $R$-modules and the corresponding category is denoted by $\text{Mod}-R$. The full subcategory of finitely presented $R$-modules is denoted by $\text{mod}-R$. By a module over an Artin algebra $\Lambda$, we mean a finitely presented (generated) right $\Lambda$-module. Our abelian categories are assumed to have enough projectives and enough injectives. Given an abelian category $\mathcal{A}$, we denote by $\text{Proj}\mathcal{A}$ (resp. $\text{Inj}\mathcal{A}$) the full subcategory consisting of projective (resp. injective) objects. For an additive functor $F: \mathcal{A} \to \mathcal{B}$ between additive categories, we denote by $\text{Im} F = \{ B \in \mathcal{B} \mid B \cong F(A) \text{ for some } A \in \mathcal{A} \}$ the essential image of $F$ and by $\text{Ker} F = \{ A \in \mathcal{A} \mid F(A) = 0 \}$ the kernel of $F$. For a path algebra $kQ$, we denote by $v_e$ the vertex corresponding to the primitive idempotent $e$.

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2. CLEFT EXTENSIONS OF ABELIAN CATEGORIES

In this section we study cleft extensions of abelian categories. This concept was introduced by Beligiannis in [3], and it generalizes trivial extension of abelian categories due to Fossum-Griffith-Reiten [4]. We start by recalling and reviewing some of the known results about cleft extensions of abelian categories from [3, 4]. Then endofunctors of the two categories occurring in a cleft extension are constructed and some properties are derived, which are used in Section 4 to investigate the finitistic dimensions in a cleft extension.

2.1. Basis properties. We first recall the definition of cleft extensions of abelian categories.

Definition 2.1. ([3 Definition 2.1]) A cleft extension of an abelian category $\mathcal{B}$ is an abelian category $\mathcal{A}$ together with functors:

![Diagram of cleft extension]

henceforth denoted by $(\mathcal{B}, \mathcal{A}, e, l, i)$, such that the following conditions hold:

(a) The functor $e$ is faithful exact.
(b) The pair $(l, e)$ is an adjoint pair of functors, where we denote the adjunction by $\theta_{B,A}: \text{Hom}_\mathcal{A}(l(B), A) \cong \text{Hom}_\mathcal{B}(B, e(A))$.
(c) There is a natural isomorphism $\varphi: e i \to \text{Id}_\mathcal{A}$ of functors.
Denote the unit \( \theta_{B,B_1}(1_{B_1}) \) and the counit \( \theta_{\epsilon(A),A}^{-1}(1_{\epsilon(A)}) \) of the adjoint pair \((l, \epsilon)\) by \( \nu: 1_{\mathcal{B}} \rightarrow l e \) and \( \mu: l e \rightarrow 1_{\mathcal{A}} \), respectively. The unit and the counit satisfy the relations
\[
1_{i(B)} = \mu_B(\nu_B)
\]
and
\[
1_{\epsilon(A)} = e(\mu_A)\epsilon(\nu_A)
\]
for all \( B \) in \( \mathcal{B} \) and \( A \) in \( \mathcal{A} \). From (2.2) the morphism \( e(\mu_A) \) is an (split) epimorphism. Using that \( e \) is faithful exact, we infer that \( \mu_A \) is an epimorphism for all \( A \) in \( \mathcal{A} \). Hence we have for every \( A \) in \( \mathcal{A} \) the following short exact sequence
\[
0 \rightarrow \text{Ker} \mu_A \rightarrow l e(A) \overset{\mu_A}{\rightarrow} A \rightarrow 0
\]
(2.3)

In the next result we collect some basic properties of a cleft extension. Most of these properties follow from Definition 2.1 and are discussed in [3, 4] but for completeness and the reader’s convenience we provide a proof.

**Lemma 2.2.** Let \( \mathcal{A} \) be a cleft extension of \( \mathcal{B} \). Then the following hold.

(i) The functor \( e: \mathcal{A} \rightarrow \mathcal{B} \) is essentially surjective.

(ii) The functor \( i: \mathcal{B} \rightarrow \mathcal{A} \) is fully faithful and exact.

(iii) The functor \( l: \mathcal{B} \rightarrow \mathcal{A} \) is faithful and preserves projective objects.

(iv) There is a functor \( q: \mathcal{A} \rightarrow \mathcal{B} \) such that \( (q, i) \) is an adjoint pair.

(v) There is a natural isomorphism \( q l \cong \text{Id}_{\mathcal{B}} \) of functors.

**Proof.**

(i) Since \( ei \cong \text{Id}_{\mathcal{A}} \), it follows immediately that \( e \) is essentially surjective.

(ii) **faithful:** Let \( f: B \rightarrow B' \) be in \( \mathcal{B} \), and assume that \( i(f) = 0 \). Then we have that \( e(i(f)) = 0 \), and we have a commutative diagram
\[
\begin{array}{ccc}
B & \xrightarrow{f} & B' \\
\downarrow{\varphi_B} & & \downarrow{\varphi_{B'}} \\
\varphi_B & & \varphi_{B'} \\
\end{array}
\]
It follows that \( f = 0 \) and \( i \) is faithful.

(i) **full:** Let \( g: i(B) \rightarrow i(B') \). Then straightforward arguments show that
\[
g = i(\varphi_{B'}e(g)\varphi_B^{-1}).
\]
This shows that \( i \) is full.

(i) **exact:** First note the following. Since \( e: \mathcal{A} \rightarrow \mathcal{B} \) is faithful, the kernel of \( e \) is consisting only of the zero object. Let
\[
\eta: 0 \rightarrow B_1 \xrightarrow{f} B_2 \xrightarrow{g} B_3 \rightarrow 0
\]
be an exact sequence in \( \mathcal{B} \). Since \( ei \cong \text{Id}_{\mathcal{A}} \), we infer that \( e(i(\eta)) \) is a short exact sequence. The complex
\[
0 \rightarrow i(B_1) \xrightarrow{i(f)} i(B_2) \xrightarrow{i(g)} i(B_3) \rightarrow 0
\]
gives rise to the exact sequences
\[
0 \rightarrow \text{Ker} i(f) \rightarrow \text{Ker} i(g) \rightarrow 0;
\]
\[
0 \rightarrow \text{Im} i(f) \rightarrow \text{Im} i(g) \rightarrow 0;
\]
\[
0 \rightarrow \text{Coker} i(g) \rightarrow 0.
\]
Applying \( e \) to these sequences and using that \( e \) is faithful exact and that \( e(i(\eta)) \) is exact, we conclude that \( \text{Ker} i(f) = 0 \), \( \text{Im} i(f) \cong \text{Ker} i(g) \), \( \text{Im} i(g) \cong i(B_3) \) and \( \text{Coker} i(g) = 0 \). Consequently, the functor \( i: \mathcal{B} \rightarrow \mathcal{A} \) is exact.
(iii) faithful: Let \( f : B \to B' \) be in \( \mathcal{B} \). For \( X \) in \( \mathcal{B} \) let \( \mu' : \text{el}(X) \to X \) be defined by the following diagram:

\[
\begin{array}{ccc}
\text{el}(\text{el}(X)) & \xrightarrow{\text{el}(\varphi_X)} & \text{el}(X) \\
\downarrow & & \downarrow \\
\text{el}(X) & \xrightarrow{\mu'_X} & X
\end{array}
\]

Then we have the following commutative diagram

\[
\begin{array}{ccc}
\mathcal{B}(B, B') & \xrightarrow{1} & \mathcal{A}(l(B), l(B')) \\
\downarrow \mathcal{B}(B, f) & & \downarrow \mathcal{A}(l(B), l(f)) \\
\mathcal{B}(B, B) & \xrightarrow{\mathcal{A}(B, f')} & \mathcal{B}(B, B')
\end{array}
\]

Starting with \( 1_B \) in the upper left corner and tracing this element to the upper right corner we obtain \( 1_B \) using (2.2) for \( A = i(B) \), hence we get \( f \) in the lower right corner. Starting with \( 1_B \) in the upper left corner and tracing it around the first square we obtain \( l(f) \). Using that \( l(f) \) is mapped to \( f \) on the lower row, it follows that the functor \( 1 \) is faithful.

 preserves projectives: Since we have the adjoint pair \((l, e)\) and the functor \( e \) is exact, it follows that the functor \( 1 : \mathcal{B} \to \mathcal{A} \) preserves projective objects.

(iv) We show that there is a functor \( q : \mathcal{A} \to \mathcal{B} \) such \( (q, i) \) is an adjoint pair. Consider first the maps \( e\mu_A \) and \( e\mu_{ie(A)} \) in the following not necessarily commutative diagram:

\[
\begin{array}{ccc}
\text{ele}(A) & \xrightarrow{e\mu_A} & e(A) \\
\downarrow \simeq & & \downarrow \simeq \\
e\varphi_{ie(A)} & \sim & \varphi_{ie(A)} \\
\text{ele}(A) & \xrightarrow{\mu_{ie(A)}} & e(A)
\end{array}
\]

For simplicity, we identify the vertical isomorphisms and we consider the following exact sequence:

\[
\text{ele}(A) \xrightarrow{e\mu_A - e\mu_{ie(A)}} e(A) \xrightarrow{\kappa_A} \text{Coker} (e\mu_A - e\mu_{ie(A)}) \to 0 \tag{2.4}
\]

Then there is a functor \( q : \mathcal{A} \to \mathcal{B} \) defined on objects by the assignment \( A \mapsto q(A) := \text{Coker} (e\mu_A - e\mu_{ie(A)}) \). Given a morphism \( f : A \to A' \) in \( \mathcal{A} \), then \( q(f) \) is the induced morphism in \( \mathcal{B} \) between the cokernels \( \text{q}(f) : q(A) \to q(A') \). For \( A = i(B) \) in (2.4), it follows that the map \( e\mu_{ie(B)} - e\mu_{ie(i(B))} = 0 \), since \( \varphi_{ie(B)} = ei(\varphi_B) \). Therefore we get that the map \( \kappa_{i(B)} : B \to q(i(B)) \) is an isomorphism. We define a natural morphism:

\[
F_{A,B} : \text{Hom}_{\mathcal{A}}(i,B) \to \text{Hom}_{\mathcal{B}}(q(A), B), \ f \mapsto \kappa_{i(B)}^{-1} q(f) : q(A) \to B
\]

Since \( \kappa_{i(B)} e(f) = q(f) \kappa_A \) and \( e \) is faithful, the map \( F_{A,B} \) is injective. To show that \( F_{A,B} \) is also an epimorphism we need some more work. For every \( A \) in \( \mathcal{A} \) we have the short exact sequence (2.3). In particular, using that \( \mu_A : \text{le}(A) \to A \) is surjective for all objects \( A \) in \( \mathcal{A} \) we get the following exact sequence:

\[
\text{le}(A) \xrightarrow{i\mu_A - \mu_{ie(A)}} \text{le}(A) \xrightarrow{i\kappa_A} i\text{q}(A) \to 0
\]

It is shown in [3 Proposition 2.3] that the next composition is zero:

\[
\text{le}(A) \xrightarrow{i\mu_A - \mu_{ie(A)}} \text{le}(A) \xrightarrow{i\kappa_{ie(A)}^{-1}} \text{le}(A) \xrightarrow{\mu_{ie(A)}} \text{le}(A) \xrightarrow{i\kappa_A} i\text{q}(A)
\]
This implies that there is map \( \lambda_A : A \rightarrow \text{iq}(A) \) such that
\[
\lambda_A \mu_A = i \kappa_A \mu_{\text{iq}(A)} \nu_{\text{e}(A)}\tag{*}
\]
Consider now a morphism \( g : q(A) \rightarrow B \). Then the composition map \( i(g) \lambda_A : A \rightarrow i(B) \) is such that \( F_{A,B}(i(g) \lambda_A) = g \). To see this there is a series of computations that the reader needs to verify. First, we compute that \( F_{A,B}(i(g) \lambda_A) = \kappa_{i(B)}^{-1} q(i(g) q(\lambda_A)) \) and we have to show that the latter morphism is \( g \). The first observation is that \( \kappa_{i(B)}^{-1} q(i(g) q(\lambda_A)) \). Using the natural isomorphism \( \varphi_B \) and \( \varphi_{\text{e}(A)} \) we obtain that the desired composition \( \kappa_{i(B)}^{-1} q(i(g) q(\lambda_A)) = g \kappa_{\text{e}(A)}^{-1} q(\lambda_A) \). Using the relation (3), it follows that \( e \lambda = \kappa \). Since \( \kappa_{\text{e}(A)} e \lambda A = q(\lambda_A) \kappa_A \) using the identification \( e \lambda = \kappa \) we get that \( q(\lambda_A) \kappa_A = \kappa_{\text{e}(A)}^{-1} \kappa_A \). Since the map \( \kappa_A \) is an epimorphism, it follows that \( q(\lambda_A) = \kappa_{\text{e}(A)} \). Thus the desired composition \( g \kappa_{\text{e}(A)}^{-1} q(\lambda_A) \) gives the map \( g \). This shows that \( F_{A,B} \) is surjective. The details are left to the reader, see also the proof of [3, Proposition 2.3]. We infer that \( (q, i) \) is an adjoint pair.

(v) From the adjoint pairs \((l, e), (q, i)\) and since we have the natural isomorphism \( e i \simeq \text{Id}_{\mathscr{A}} \), it follows that there is a natural isomorphism \( q i \simeq \text{Id}_{\mathscr{B}} \) of functors.

\[\square\]

2.2. Endofunctors. We saw in (2.3) that there is a short exact sequence
\[
0 \rightarrow \ker \mu_A \rightarrow l e(A) \rightarrow A \rightarrow 0
\]
for all \( A \) in \( \mathscr{A} \). The assignment \( A \mapsto \ker \mu_A \) defines an endofunctor \( G : \mathscr{A} \rightarrow \mathscr{A} \).
Consider now an object \( B \) in \( \mathscr{B} \). Then we have the short exact sequence in \( \mathscr{A} \):
\[
0 \rightarrow G(i(B)) \rightarrow l e(i(B)) \rightarrow i(B) \rightarrow 0
\]
and applying the exact functor \( e : \mathscr{A} \rightarrow \mathscr{B} \) we obtain the exact sequence
\[
0 \rightarrow e(G(i(B))) \rightarrow e(l e(i(B))) \rightarrow e(i(B)) \rightarrow 0
\]
We denote by \( F(B) \) the object \( e(G(i(B))) \). Then the assignment \( B \mapsto F(B) \) defines an endofunctor \( F : \mathscr{B} \rightarrow \mathscr{B} \). Viewing the natural isomorphism \( e i(B) \simeq B \) as an identification, we obtain the exact sequence:
\[
0 \rightarrow F(B) \rightarrow e(l i(B)) \rightarrow e(i(B)) \rightarrow B \rightarrow 0 \tag{2.5}
\]
The following lemma is an immediate consequence of (2.2).

Lemma 2.3. Let \((\mathscr{B}, \mathscr{A}, e, l, i)\) be a cleft extension of abelian categories. Then the exact sequence (2.5) splits.

We end this section by discussing cleft extensions with special properties. In the application discussed in Section 4 the square of \( F \) is zero, and the functor \( l \) is exact and the functor \( e \) preserves projectives. The remaining results concern cleft extensions having some of these properties or generalizations thereof.

Lemma 2.4. Let \((\mathscr{B}, \mathscr{A}, e, l, i)\) be a cleft extension of abelian categories. The following statements hold.

(i) For any \( n \geq 1 \), there is a natural isomorphism \( e G^n \simeq F^n e \).

(ii) Let \( n \geq 1 \). Then \( F^n = 0 \) if and only if \( G^n = 0 \).

Proof. (i) Let \( A \) be an object in \( \mathscr{A} \). We first show that \( e(G(A)) \simeq F(e(A)) \). From (2.3) and (2.7) we obtain the following two short exact sequences:
\[
0 \rightarrow e(G(A)) \rightarrow e l e(A) \rightarrow e i(A) \rightarrow 0
\]
and

\[ 0 \to F(e(A)) \to \text{ele}(e(A)) \to \text{e}(\nu_{\text{e}(A)}) \to \text{e}(A) \to 0 \]

where by definition \( F(e(A)) = eG(e(A)) \). From Lemma 2.3 the above two exact sequences split. Then using the natural isomorphism \( \varphi : \text{e}I \to \text{Id}_{\mathcal{B}} \) we obtain the following exact commutative diagram:

\[
\begin{array}{ccc}
0 & \to & \text{e}(A) \\
\downarrow & & \downarrow \\
0 & \to & \text{e}(A)
\end{array}
\]

From the Snake Lemma it follows that \( e(G(A)) \simeq F(e(A)) \). By induction it follows that \( eG^n(A) \simeq F^n(e(A)) \) for all \( n \geq 1 \). Hence \( eG^n \simeq F^n(e) \) for all \( n \geq 1 \).

(ii) Let \( n \geq 1 \). If \( F^n = 0 \), then \( eG^n = 0 \) by (i). Since \( e \) is faithful, \( G^n = 0 \). If \( G^n = 0 \), then \( F^n = 0 \) by (i). Since \( e \) is essentially surjective by Lemma 2.2 (i), we infer that \( F^n = 0 \). This completes the proof.

\[ \square \]

**Remark 2.5.** There is a dual notion of a cleft extension for abelian categories. More precisely, a cleft coextension of an abelian category \( \mathcal{B} \) is an abelian category \( \mathcal{A} \) together with functors:

\[
\begin{array}{ccc}
\mathcal{B} & \to & \mathcal{A} \\
\iota & & \eta \\
\mathcal{A} & \to & \mathcal{B}
\end{array}
\]

such that the functor \( e \) is faithful exact, \((e, r)\) is an adjoint pair of functors, and there is a natural isomorphism \( eI \simeq \text{Id}_{\mathcal{B}} \) of functors. In this case, we can derive as in Lemma 2.2 similar properties for a cleft coextension. It is very interesting when a cleft extension of abelian categories \((\mathcal{B}, \mathcal{A}, e, l, i)\) is also a cleft coextension. Indeed, this holds if and only if the endofunctor \( F \) appearing in the split exact sequence (2.5) has a right adjoint. In this case, we have the following diagram of functors:

\[
\begin{array}{ccc}
F & \to & \mathcal{B} \\
\iota & & \eta \\
\mathcal{A} & \to & \mathcal{B}
\end{array}
\]

All these concepts are due to Beligiannis [3][4], in particular, see [4] subsection 2.4] for a thorough discussion of cleft coextensions of abelian categories. We remark that diagram (2.6) will be studied further in Section 4.2.

From now on we make the following assumption on a cleft extension \((\mathcal{B}, \mathcal{A}, e, l, i)\) of abelian categories:

The functor \( l \) is exact and the functor \( e \) preserves projectives. \hspace{1cm} (2.7)

Note that if a cleft extension \((\mathcal{B}, \mathcal{A}, e, l, i)\) is also a cleft coextension, i.e. we have diagram (2.6), then \( r \) being exact implies that \( e \) preserves projectives.

We continue with the following useful results.

**Lemma 2.6.** Let \((\mathcal{B}, \mathcal{A}, e, l, i)\) be a cleft extension of abelian categories such that condition (2.7) holds. Then the functor \( F \) is exact and preserves projective objects.

**Proof.** From Lemma 2.3 it follows that there is an isomorphism of functors \( eI \simeq F \oplus 1_{\mathcal{B}} \). Since the functors \( e \) and \( l \) are exact, we infer that \( F \) is exact.

By Lemma 2.2 (iii) the functor \( l \) preserves projective objects, and by assumption the functor \( e \) has the same property. Then the isomorphism \( eI \simeq F \oplus 1_{\mathcal{B}} \) show that \( F \) preserves projective objects. \( \square \)
Lemma 2.7. Let \((\mathcal{A}, \mathcal{B}, \mathcal{C})\) be a cleft extension of abelian categories such that condition (2.7) holds. Moreover, assume that \(F^n = 0\) for some \(n \geq 2\). Let \(A\) be an object in \(\mathcal{A}\). Then the following statements are equivalent:

(i) \(G^{n-1}(A)\) lies in \(\text{Proj} \mathcal{A}\).
(ii) \(F^{n-1}e(A)\) lies in \(\text{Proj} \mathcal{B}\).

Proof. (i)\(\Rightarrow\)(ii) Assume that \(G^{n-1}(A)\) is projective in \(\mathcal{A}\). By Lemma 2.2 (vii) the functor \(e\) preserves projective objects, hence \(eG^{n-1}(A)\) is projective in \(\mathcal{B}\). By Lemma 2.3 (i) the objects \(eG^{n-1}(A)\) and \(F^{n-1}e(A)\) are isomorphic, and therefore \(F^{n-1}e(A)\) is projective in \(\mathcal{B}\).

(ii)\(\Rightarrow\)(i) Assume that \(F^{n-1}e(A)\) is projective in \(\mathcal{B}\). Then \(F^{n-1}e(A)\) lies in \(\text{Proj} \mathcal{A}\) by Lemma 2.2 (iii), and therefore from Lemma 2.4 the object \(eG^{n-1}(A)\) is projective. Moreover, Lemma 2.3 (ii) yields that \(G^n = 0\) by our assumption. Then, if we consider the exact sequence (2.3) for the object \(G^{n-1}(A)\), it follows that the map \(\mu_{G^{n-1}(A)} \colon leG^{n-1}(A) \rightarrow G^{n-1}(A)\) is an isomorphism. Hence, the object \(G^{n-1}(A)\) is projective. \(\Box\)

3. Recollements of abelian categories

We start this section by recalling the definition of a recollement situation in the context of abelian categories, see for instance [17] and references therein, we fix notation and recall some well known properties of recollements which are used later in the paper. We also introduce a relative homological dimension in a recollement \((\mathcal{A}, \mathcal{B}, \mathcal{C})\) which is relevant for the inclusion functor \(i : \mathcal{A} \rightarrow \mathcal{B}\) to be a homological embedding and provides other properties of the recollement. Finally, we recall when a functor between abelian categories is an eventually homological isomorphism and characterize this property for the quotient functor in a recollement.

We begin by recalling the definition of a recollement of abelian categories. 

Definition 3.1. A recollement situation between abelian categories \(\mathcal{A}, \mathcal{B}\) and \(\mathcal{C}\) is a diagram

\[\begin{array}{ccc}
\mathcal{A} & \xrightarrow{q} & \mathcal{B} \\
\downarrow{i} & & \downarrow{e} \\
\mathcal{C} & \xleftarrow{l} & \mathcal{B}
\end{array}\]

henceforth denoted by \((\mathcal{A}, \mathcal{B}, \mathcal{C})\), satisfying the following conditions:

(a) \((l, e)\) and \((e, r)\) are adjoint pairs.
(b) \((q, i)\) and \((i, p)\) are adjoint pairs.
(c) The functors \(i, l, \) and \(r\) are fully faithful.
(d) \(\text{Im} i = \text{Ker} e\).

We collect some basic properties of a recollement of abelian categories. They can be derived easily from Definition 3.1 for more details see [17].

Proposition 3.2. Let \((\mathcal{A}, \mathcal{B}, \mathcal{C})\) be a recollement of abelian categories. Then the following hold.

(i) The functors \(i : \mathcal{A} \rightarrow \mathcal{B}\) and \(e : \mathcal{B} \rightarrow \mathcal{C}\) are exact.
(ii) The compositions \(e_i, q_l\) and \(p_r\) are zero.
(iii) The functor \(e : \mathcal{B} \rightarrow \mathcal{C}\) is essentially surjective.
(iv) The units of the adjoint pairs \((i, p)\) and \((l, e)\) and the counits of the adjoint pairs \((q, i)\) and \((e, r)\) are isomorphisms:

\[\begin{array}{ccc}
\text{Id}_{\mathcal{A}} & \cong & \text{Id}_{\mathcal{B}} \\
\cong & \text{Id}_{\mathcal{C}} & \cong \\
\text{qi} & \cong & \text{Id}_{\mathcal{A}} \\
\cong & \text{Id}_{\mathcal{C}} & \cong \\
\text{er} & \cong & \text{Id}_{\mathcal{C}}
\end{array}\]
(v) The functors \( l : \mathcal{C} \to \mathcal{B} \) and \( q : \mathcal{B} \to \mathcal{A} \) preserve projective objects and the functors \( r : \mathcal{C} \to \mathcal{B} \) and \( p : \mathcal{B} \to \mathcal{A} \) preserve injective objects.

(vi) The functor \( i : \mathcal{A} \to \mathcal{B} \) induces an equivalence between \( \mathcal{A} \) and the Serre subcategory \( \text{Ker } e = \text{Im } i \) of \( \mathcal{B} \). Moreover, \( \mathcal{A} \) is a localizing and colocalizing subcategory of \( \mathcal{B} \) and there is an equivalence of categories \( \mathcal{B}/\mathcal{A} \simeq \mathcal{C} \).

(vii) For every \( B \in \mathcal{B} \) there are objects \( A \) and \( A' \) in \( \mathcal{A} \) such that the units and counits of the adjunctions induce the following exact sequences:

\[
0 \to i(A) \to \text{le}(B) \to B \to \text{iq}(B) \to 0 \quad (3.1)
\]

and

\[
0 \to \text{ip}(B) \to B \to \text{re}(B) \to i(A') \to 0
\]

A well-known example of a recollement of abelian categories is induced from a ring and an idempotent as we explain next.

**Example 3.3.** Let \( R \) be a ring with an idempotent element \( e \in R \). Then the diagram

\[
\begin{array}{ccc}
\text{Mod-}R/ReR & \xrightarrow{\text{inc}} & \text{Mod}-R \\
\xleftarrow{\text{Hom}_R(R/ReR,-)} & & \xleftarrow{\text{Hom}_{eRe}(Re,-)}\
\end{array}
\]

is a recollement of abelian categories. It should be noted that this recollement is the universal example of a recollement situation with terms categories of modules. Indeed, from [18] we know that any recollement of module categories is equivalent, in an appropriate sense, to one induced by an idempotent element as above.

There are even further functors that are naturally associated to a recollement of abelian categories \((\mathcal{A}, \mathcal{B}, \mathcal{C})\). One such functor is obtained from the exact sequence \((3.1)\): We let \( H : \mathcal{B} \to \mathcal{B} \) be the endofunctor defined by the short exact sequence

\[
0 \to H(B) \to B \to \text{iq}(B) \to 0 \quad (3.2)
\]

on all objects \( B \) of \( \mathcal{B} \). The endofunctor \( H \) is an idempotent radical subfunctor of the identity functor \( \text{Id}_\mathcal{B} \), see [16, Proposition 3.5]. This endofunctor is useful in connection with the next relative dimensions, see Proposition 3.5 below.

Let \((\mathcal{A}, \mathcal{B}, \mathcal{C})\) be a recollement of abelian categories. In [16] the notion of \( \mathcal{A} \)-relative global dimension \( \text{pgl.dim}_{\mathcal{A}} \) of \( \mathcal{B} \) was defined as follows:

\[
\text{pgl.dim}_{\mathcal{A}} := \sup \{ \text{pd}_{\mathcal{A}}(A) \mid A \in \mathcal{A} \}
\]

We call this the \( \mathcal{A} \)-relative projective global dimension of \( \mathcal{B} \). Dually, we define the \( \mathcal{A} \)-relative injective global dimension of \( \mathcal{B} \) by

\[
\text{igl.dim}_{\mathcal{A}} := \sup \{ \text{id}_{\mathcal{A}}(A) \mid A \in \mathcal{A} \}.
\]

Our aim is to explore the homological behaviour of a recollement \((\mathcal{A}, \mathcal{B}, \mathcal{C})\) under the finiteness of \( \text{pgl.dim}_{\mathcal{A}} \) or \( \text{igl.dim}_{\mathcal{A}} \).

From Proposition 3.2 we know that the functor \( r : \mathcal{C} \to \mathcal{B} \) preserves injective objects. Under the finiteness of \( \text{igl.dim}_{\mathcal{A}} \), we show that the functor \( r \) preserves objects of finite injective dimension even though it is not an exact functor in general. Similar considerations hold for the functor \( l \).

**Lemma 3.4.** Let \((\mathcal{A}, \mathcal{B}, \mathcal{C})\) be a recollement of abelian categories.

\[\text{Note that in [16] the notation was pgl.dim}_{\mathcal{A}} \mathcal{B}].}
(i) If $\text{igl. dim}_{\mathcal{A}} \mathcal{A} < \infty$, then the functor $r: \mathcal{C} \to \mathcal{B}$ preserves objects of finite injective dimension.
(ii) If $\text{pgl. dim}_{\mathcal{A}} \mathcal{A} < \infty$, then the functor $l: \mathcal{C} \to \mathcal{B}$ preserves objects of finite projective dimension.

Proof. We only prove (i) as (ii) is shown by similar arguments.

For any object $C$ in $\mathcal{C}$ we show that the following formula holds:

$$\text{id}_{\mathcal{A}}r(C) \leq \text{id}_{\mathcal{A}}C + \text{igl. dim}_{\mathcal{A}} \mathcal{A} + 1 \quad (3.3)$$

Then it follows immediately that the functor $r$ preserves objects of finite injective dimension. Note that the dual formula is proved in [16, Lemma 4.3], so (ii) follows as well. For readers convenience we prove formula (3.3).

Assume that $\text{igl. dim}_{\mathcal{A}} \mathcal{A} = n < \infty$. If the object $C$ is injective, then $r(C)$ is injective and therefore the relation (3.3) holds. Suppose that the object $C$ has injective dimension one. This means that there is an exact sequence

$$0 \longrightarrow C \xrightarrow{a^0} I^0 \xrightarrow{a^1} I^1 \longrightarrow 0$$

with $I^0$ and $I^1$ in $\text{Inj} \mathcal{C}$. Applying the left exact functor $r$ we obtain the following exact sequence

$$0 \longrightarrow r(C) \xrightarrow{r(a^0)} r(I^0) \xrightarrow{r(a^1)} r(I^1) \longrightarrow R^1r(C) \longrightarrow 0$$

where $R^1r(C)$ is the first right derived functor of $r$. If we apply the exact functor $e: \mathcal{B} \to \mathcal{C}$ and by Proposition 5.2 (iv), we derive that the object $R^1r(C)$ is annihilated by $e$. This implies that $R^1r(C)$ lies in $(\mathcal{A})$ which by our assumption satisfies $\text{id}_{\mathcal{A}}R^1r(C) \leq n$. Thus, we have $\text{id}_{\mathcal{A}} \text{Coker} r(a^0) \leq n + 1$ and therefore we conclude that $\text{id}_{\mathcal{A}}r(C) \leq 1 + n + 1$. Continuing inductively on the length of the injective resolution of $C$ we get formula (3.3) and this completes the proof. $\square$

For an injective coresolution $0 \longrightarrow B \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \cdots$ of $B$ in $\mathcal{B}$, the image of the morphism $I^{m-1} \longrightarrow I^m$ is an $m$-th cosyzygy of $B$ and is denoted by $\Sigma^m(B)$. The full subcategory of $\mathcal{B}$ consisting of the $m$-th cosyzygy objects is denoted by $\Sigma^m(\mathcal{B})$.

The finiteness of the $\mathcal{A}$-relative projective global dimension of $\mathcal{B}$ implies also that $\text{sup}(\text{pd}_{\mathcal{A}} i(P)) | P \in \text{Proj} \mathcal{A}$ is finite. In the following result we characterize the finiteness of the above number, compare this with [16, Remark 4.6, Proposition 4.15]. Recall from [16] that an exact functor $i: \mathcal{A} \to \mathcal{B}$ between abelian categories is a homological embedding if the map $i_X^Y: \text{Ext}^n_{\mathcal{A}}(X,Y) \to \text{Ext}^n_{\mathcal{B}}(i(X),i(Y))$ is an isomorphism for every pair of objects $X,Y \in \mathcal{A}$ and for all $n \geq 0$. To this end we need to introduce the following notation.

Let $X \subseteq \mathcal{A}$ be a full subcategory. For integers $0 \leq i \leq k$ we denote by $X^{i+k}$ the full subcategory of $\mathcal{A}$ which is defined by

$$X^{i+k} = \{ A \in \mathcal{A} | \text{Ext}^n_{\mathcal{A}}(X,A) = 0, \forall i \leq n \leq k \}.$$

We also denote by $X^{i+\infty}$ the full subcategory of $\mathcal{A}$ defined by

$$X^{i+\infty} = \{ A \in \mathcal{A} | \text{Ext}^n_{\mathcal{A}}(X,A) = 0, \forall n \geq 1 \}.$$

Similarly we define the full subcategories $^{i+k}X$ and $^{i+\infty}X$.

**Proposition 3.5.** Let $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ be a recollement of abelian categories. Assume that $\mathcal{B}$ has enough projectives and injectives. The following are equivalent:

(i) $\text{sup}(\text{pd}_{\mathcal{A}} i(P)) | P \in \text{Proj} \mathcal{A}$ $\leq m$.
(ii) $\Sigma^m(\mathcal{B}) \subseteq i(\text{Proj} \mathcal{A})^{i+\infty}$.
(iii) For every projective object $P$ in $\mathcal{B}$, we have $\text{pd}_{\mathcal{A}} H(P) \leq m - 1$ where the functor $H: \mathcal{B} \to \mathcal{B}$ is given in [32].
If $m = 1$, then the above conditions are equivalent to the following one.

(iv) The functor $i: \mathcal{A} \rightarrow \mathcal{B}$ is a homological embedding and the quotient functor $e: \mathcal{B} \rightarrow \mathcal{C}$ preserves projective objects.

Proof. The equivalence of statements (i) and (ii) follows from the isomorphism

\[ \text{Ext}^n_{\mathcal{A}}(i(P), \Sigma^m(B)) \simeq \text{Ext}^{n+m}_{\mathcal{A}}(i(P), B) \]

for all $n \geq 1$, $P$ in Proj $\mathcal{B}$ and $B$ in $\mathcal{B}$.

(ii)$\Rightarrow$(iii) Let $B$ be an object in $\mathcal{B}$ and consider an injective coresolution in $\mathcal{B}$:

\[ 0 \rightarrow B \rightarrow I^0 \rightarrow \ldots \rightarrow I^{m-1} \rightarrow \Sigma^m(B) \rightarrow 0 \]

Let $P$ be a projective object in $\mathcal{B}$. From our assumption and by dimension shift

we have $\text{Ext}^n_{\mathcal{A}}(iq(P), \Sigma^m(B)) \simeq \text{Ext}^{n+m}_{\mathcal{A}}(iq(P), B) \simeq \text{Ext}^{n+1}_{\mathcal{A}}(iq(P), B) = 0$. From [8] there is an exact sequence

\[ 0 \rightarrow H(P) \rightarrow P \rightarrow iq(P) \rightarrow 0 \]  \hspace{1cm} (3.4)

Applying the functor $\text{Hom}_{\mathcal{A}}(-, B)$ we get the long exact sequence:

\[ \cdots \rightarrow \text{Ext}^n_{\mathcal{A}}(P, B) \rightarrow \text{Ext}^n_{\mathcal{A}}(H(P), B) \rightarrow \text{Ext}^{n+1}_{\mathcal{A}}(iq(P), B) \rightarrow \text{Ext}^{n+1}_{\mathcal{A}}(P, B) \rightarrow \cdots \]

and therefore $\text{Ext}^m_{\mathcal{A}}(H(P), B) = 0$. We infer that $\text{pd}_{\mathcal{A}} H(P) \leq m - 1$.

(iii)$\Rightarrow$(ii) Let $B$ be an object in $\mathcal{B}$ and consider an injective coresolution as above. From [16] Remark 2.5 we have $\text{Proj} \mathcal{A} = \text{add} q(\text{Proj} \mathcal{B})$, so it suffices to show that $\text{Ext}^n_{\mathcal{A}}(iq(P), B) = 0$ for all $n \geq 1$ and any object $P$ in $\text{Proj} \mathcal{B}$. From the exact sequence (3.4) and our assumption it follows that $\text{pd}_{\mathcal{A}} iq(P) \leq m$. Then the result follows easily as above by dimension shift.

Assume that $m = 1$. We show (iii)$\Rightarrow$(iv). Let $P$ be a projective object in $\mathcal{B}$ and let $A$ be an object in $\mathcal{A}$. Consider the exact sequence (3.4) and apply the functor $\text{Hom}_{\mathcal{A}}(-, i(A))$. Since $H(P)$ is projective, the long exact sequence of homology yields that $\text{Ext}^n_{\mathcal{A}}(iq(P), i(A)) = 0$ for all $n \geq 1$. Furthermore, it shows that $\text{Ext}^1_{\mathcal{A}}(iq(P), i(A))$ is a quotient of $\text{Hom}_{\mathcal{A}}(H(P), i(A))$, which is isomorphic to $\text{Hom}_{\mathcal{A}}(iq(H(P)), A)$ by the adjunction $(q, i)$. Using that $H(P)$ is a quotient of $\text{Ext}^1_{\mathcal{A}}$, the functor $q$ is right exact and $ql = 0$, it follows that $q(H(P)) = 0$. Hence $\text{Ext}^n_{\mathcal{A}}(iq(P), i(A)) = 0$ for all $n \geq 1$. Then [10] Theorem 3.9 implies that the functor $i$ is a homological embedding.

It remains to show that $e(P)$ lies in Proj $\mathcal{C}$. From Proposition 3.2 (iii) we have that the functor $e: \mathcal{B} \rightarrow \mathcal{C}$ is essentially surjective. Moreover, since the composition $ei = 0$, see Proposition 3.2 (ii), if we apply the exact functor $e$ to the sequence (3.4) we get that $\text{Ext}^n_{\mathcal{A}}(H(P), Y) \simeq \text{Ext}^n_{\mathcal{A}}(e(H(P)), e(Y)) = 0$ for any $n \geq 1$ and $Y$ in $\mathcal{B}$. Since the functor $i$ is a homological embedding, it follows from [10] Corollary 3.11 that there is an isomorphism

\[ \text{Ext}^n_{\mathcal{A}}(H(P), Y) \xrightarrow{\sim} \text{Ext}^n_{\mathcal{A}}(e(H(P)), e(Y)) \]  \hspace{1cm} (3.5)

for any $n \geq 1$. Since by (iii), the object $H(P)$ is projective it follows from the isomorphism (3.5) that the object $e(P)$ is projective.

Finally, the implication (iv)$\Rightarrow$(iii) follows immediately from the isomorphism (3.5) using again that $e(H(P)) \simeq e(P)$.

Let $\mathcal{A}$ be an abelian category with enough projective objects. Let $A$ be an object in $\mathcal{A}$ and $\cdots \rightarrow P_1 \rightarrow P_0 \rightarrow A \rightarrow 0$ a projective resolution of $A$ in $\mathcal{A}$. The kernel of the morphism $P_{n-1} \rightarrow P_{n-2}$ is an $n$-th syzygy of $A$ and is denoted by $\Omega^n(A)$. Also, if $X$ is a class of objects in $\mathcal{A}$, then we denote by $\perp X = \{ A \in \mathcal{A} \mid \text{Hom}_{\mathcal{A}}(A, X) = 0\}$ the left orthogonal subcategory of $X$.

We now recall some basic facts about projective covers in the setting of abelian categories. We refer to [15] for more details. Let $X$ be an object in an abelian category $\mathcal{A}$. The radical of $X$, denoted by $\text{rad} X$, is the intersection of all its
maximal subobjects. Clearly, if $S$ is a simple object then $\text{rad} S = 0$. Recall also that an epimorphism $f: P \rightarrow X$ is a projective cover, if $P$ is projective and the map $f$ is an essential epimorphism. In this case, the kernel $\text{Ker} f$ is contained in the radical of $P$, i.e. $\text{Ker} f \subseteq \text{rad} P$. Moreover, given a morphism $g: X \rightarrow Y$ in $\mathcal{A}$ we always have $g(\text{rad}X) \subseteq \text{rad}Y$.

**Lemma 3.6.** Let $\mathcal{A}$ be an abelian category with projective covers and let $S$ be a simple object. Then for every $n \geq 1$ there is an isomorphism

$$\text{Ext}^n_{\mathcal{A}}(X, S) \simeq \text{Hom}_{\mathcal{A}}(\Omega^n(X), S)$$

*Proof.* Let $X$ be an object in $\mathcal{A}$. Consider a projective resolution of $X$ by taking projective covers

$$\cdots \rightarrow P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} X \rightarrow 0 \quad (3.6)$$

This means that the map $d_0$ is a projective cover, the map $P_1 \rightarrow \Omega(X)$ is a projective cover and so on. Applying the functor $\text{Hom}_{\mathcal{A}}(-, S)$ we claim that the following complex

$$0 \rightarrow \text{Hom}_{\mathcal{A}}(P_0, S) \xrightarrow{(d_1)_*} \text{Hom}_{\mathcal{A}}(P_1, S) \xrightarrow{(d_2)_*} \text{Hom}_{\mathcal{A}}(P_2, S) \rightarrow \cdots \quad (3.7)$$

has zero differentials. By the construction of the resolution $(3.6)$ we have that $\text{Im} d_n = \Omega^n(X) \subseteq \text{rad}P_{n-1}$ for all $n \geq 1$. Let $f$ be a map in $\text{Hom}_{\mathcal{A}}(P_n, S)$. Then $(d_{n+1})_*(f) = fd_{n+1}$ and we compute that

$$\text{Im}(d_{n+1})_*(f) = f(\text{Im} d_{n+1}) \subseteq f(\text{rad} P_n) \subseteq \text{rad} f(P_n) \subseteq \text{rad}S = 0$$

This implies that the complex $(3.7)$ has zero differential and therefore we get that

$$\text{Ext}^n_{\mathcal{A}}(X, S) \simeq \text{Hom}_{\mathcal{A}}(P_n, S)$$

Consider now the exact sequence $P_{n+1} \rightarrow P_n \rightarrow \Omega^n(X) \rightarrow 0$. Applying the functor $\text{Hom}_{\mathcal{A}}(-, S)$ we obtain the exact sequence

$$0 \rightarrow \text{Hom}_{\mathcal{A}}(\Omega^n(X), S) \rightarrow \text{Hom}_{\mathcal{A}}(P_n, S) \rightarrow \cdots$$

and thus an isomorphism $\text{Hom}_{\mathcal{A}}(\Omega^n(X), S) \simeq \text{Hom}_{\mathcal{A}}(P_n, S)$. This completes the proof of the desired isomorphism. □

In the rest of this section we are interested in the eventual homological behaviour of the category $\mathcal{B}$ is a recollement $(\mathcal{A}, \mathcal{B}, \mathcal{C})$. To this end the following result from [19] is useful.

**Lemma 3.7.** ([19] Theorem 3.4]) Let $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ be a recollement of abelian categories with enough projectives. The following statements are equivalent:

(i) The object $B$ has a projective resolution of the form

$$\cdots \rightarrow I(Q_1) \xrightarrow{i(Q_1)} I(Q_0) \rightarrow P_{n-1} \rightarrow \cdots \rightarrow P_0 \rightarrow B \rightarrow 0$$

where each $Q_j$ is a projective object in $\mathcal{C}$.

(ii) $\text{Ext}^j_{\mathcal{B}}(B, i(A)) = 0$ for every $A \in \mathcal{A}$ and $j > n$, and there exists an $n$-th syzygy of $B$ lying in $i(\mathcal{A})$.

Our aim is to provide a bound for the finitistic projective dimension of an abelian category $\mathcal{B}$ in a recollement situation $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ when the $\mathcal{A}$-relative injective global dimension $\text{igl. dim}_{\mathcal{A}} \mathcal{A}$ of $\mathcal{B}$ is finite.

Let $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ be a recollement of abelian categories. Let $B$ be an object in $\mathcal{B}$ and $I$ an object in $\text{Inj} \mathcal{A}$. Then by dimension shift we have the isomorphism $\text{Ext}^n_{\mathcal{A}}(\Omega^n(B), i(I)) \simeq \text{Ext}^{n+m}_{\mathcal{A}}(B, i(I))$ for every $n \geq 1$. This implies that $\text{igl. dim}_{\mathcal{A}} \mathcal{A} = t$ if and only if $\Omega^t(\mathcal{B}) \subseteq i^{1 \sim i(\text{Inj} \mathcal{A})}$. The latter is equivalent to
Consider the following statements for an object of a category and assume that a characterization is provided for when the functor \( t \) induces isomorphisms of extension groups in almost all degrees.

**Theorem 3.10.** Let \((\mathcal{A}, \mathcal{B}, \mathcal{C})\) be a recollement of abelian categories such that \( \mathcal{A} \) is a finite length category. Assume that \( \text{igl.dim}_{\mathcal{A}} t = t < \infty \) and that \( \mathcal{B} \) has projective covers. Then any object of \( \mathcal{B} \) has a projective resolution as follows:

\[
\cdots \longrightarrow t(Q_1) \longrightarrow t(Q_0) \longrightarrow P_1 \longrightarrow \cdots \longrightarrow P_0 \longrightarrow B \longrightarrow 0
\]

where each \( Q_j \) belongs to Prot \( \mathcal{C} \).

**Proof.** From Lemma 3.8 it suffices to show that \( \text{Ext}^j_{\mathcal{B}}(B, i(A)) = 0 \) for all \( A \) in \( \mathcal{A} \) and \( j > t + 1 \), and there is an \( (t + 1) \)-syzygy of \( B \) lying in \( \perp i(\mathcal{A}) \). Since \( \text{id}_{\mathcal{A}}(A) \leq t \), it follows that \( \text{Ext}^j_{\mathcal{B}}(B, i(A)) = 0 \) for all \( j > t \). Since every object in \( \mathcal{A} \) is filtered in finitely many steps by simple objects, it follows from Lemma 3.8 that \( \text{Hom}_{\mathcal{A}}(\Omega^{t+1}(B), i(A)) = 0 \) for all \( A \) in \( \mathcal{A} \). We infer that \( \Omega^{t+1}(B) \) lies in \( \perp i(\mathcal{A}) \) and this completes the proof. \( \square \)

We start by recalling from [19] when a functor is an eventually homological isomorphism. Note that we define the latter in the context of abelian categories.

**Definition 3.9.** Let \( E : \mathcal{B} \to \mathcal{C} \) be a functor between abelian categories. The functor \( E \) is called an eventually homological isomorphism if there is a positive integer \( t \) and a group isomorphism

\[ \text{Ext}^n_{\mathcal{B}}(B, B') \simeq \text{Ext}^n_{\mathcal{C}}(E(B), E(B')) \]

for every \( n > t \) and for all objects \( B, B' \) in \( \mathcal{B} \). For the minimal such \( t \) it is called a \( t \)-eventually homological isomorphism.

Note that we do not require these isomorphisms to be induced by the functor \( E \), see Remark 3.12 below. We continue with the next result from [19] where a characterization is provided for when the functor \( e \) in a recollement \((\mathcal{A}, \mathcal{B}, \mathcal{C})\) induces isomorphisms of extension groups in almost all degrees.

**Theorem 3.10.** ([19] Theorem 3.4]) Let \((\mathcal{A}, \mathcal{B}, \mathcal{C})\) be a recollement of abelian categories and assume that \( \mathcal{B} \) and \( \mathcal{C} \) have enough projective and injective objects. Consider the following statements for an object \( B \) of \( \mathcal{B} \) and two integers \( n \) and \( m \):

(i) The map \( e_{B,B'} : \text{Ext}^j_{\mathcal{B}}(B, B') \to \text{Ext}^j_{\mathcal{C}}(E(B), E(B')) \) is an isomorphism for every object \( B' \) in \( \mathcal{B} \) and every integer \( j > m + n \).

(ii) The object \( B \) has a projective resolution of the form

\[
\cdots \longrightarrow t(Q_1) \longrightarrow t(Q_0) \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow B \longrightarrow 0
\]

where each \( Q_j \) lies in Prot \( \mathcal{C} \).

(iii) \( \text{Ext}^j_{\mathcal{B}}(B, i(A)) = 0 \) for every \( A \) in \( \mathcal{A} \) and \( j > n \), and there exists an \( n \)-th syzygy of \( B \) lying in the left orthogonal subcategory \( \perp i(\mathcal{A}) \).

(iv) \( \text{Ext}^j_{\mathcal{B}}(B, i(I)) = 0 \) for every \( I \) in \( \text{inj}\mathcal{A} \) and \( j > n \), and there exists an \( n \)-th syzygy of \( B \) lying in the left orthogonal subcategory \( \perp i(\text{inj}\mathcal{A}) \).

Then the following statements are equivalent: (ii) \( \iff \) (iii) \( \iff \) (iv). If one of these holds and in addition we have \( \text{pd}_{\mathcal{C}} e(P) \leq m \) for every projective object \( P \) in \( \mathcal{B} \), then statement (i) holds.

In [19] Corollary 3.12] the authors characterized when the multiplication functor \( e(-) : \text{mod-}\Lambda \to \text{mod-}\Lambda e \) (see Example 3.3), where \( \Lambda \) is an Artin algebra, is an eventually homological isomorphism. The following result generalizes [19] Corollary 3.12 from Artin algebras to abelian categories with certain conditions.
Proposition 3.11. Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be a recollement of abelian categories. Assume that $\mathcal{A}$ is a finite length category and that $\mathcal{B}$ has projective covers. The following statements are equivalent:

(i) There is an integer $t$ such that for every pair of objects $X$ and $Y$ in $\mathcal{B}$, and every $j > t$, the map

$$e^j_{X,Y} : \text{Ext}^j_\mathcal{B}(X,Y) \rightarrow \text{Ext}^j_\mathcal{B}(e(X), e(Y))$$

is an isomorphism.

(ii) The functor $e : \mathcal{B} \rightarrow \mathcal{C}$ is an eventually homological isomorphism.

(iii) $(\alpha)$ $\text{igl.dim}_{\mathcal{A}} < \infty$ and $(\beta)$ $\sup\{\text{pd}_e e(P) \mid P \in \text{Proj} \mathcal{B}\} < \infty$.

(iv) $(\gamma)$ $\text{gpl.dim}_{\mathcal{A}} < \infty$ and $(\delta)$ $\sup\{\text{id}_e e(I) \mid I \in \text{Inj} \mathcal{B}\} < \infty$.

In particular, if the functor $e$ is a $s$-homological isomorphism, then each of the dimensions in (iii) and (iv) are at most $s$. The bound $t$ in (i) is bounded by the sum of the dimensions occurring in (iii), and also bounded by the sum of the dimensions occurring in (iv).

Proof. The implication $(i) \Rightarrow (ii)$ is clear.

$(ii) \Rightarrow (iii)$: Let $A$ be an object of $\mathcal{A}$. Then, there is a positive integer $t$ such that for every object $B$ in $\mathcal{B}$ and for all $j > t$, we have the isomorphism

$$\text{Ext}^j_\mathcal{B}(B, i(A)) \simeq \text{Ext}^j_{\mathcal{C}}(e(B), e(A))$$

The latter extension group vanishes since $e = 0$ by Proposition 3.2 (ii). This implies that $\text{id}_{\mathcal{B}} i(A) \leq t$ and therefore $(\alpha)$ holds. For $(\beta)$, let $P$ be a projective object of $\mathcal{B}$. For every $C$ in $\mathcal{C}$ and $j > t$, we have the following isomorphism

$$\text{Ext}^j_{\mathcal{C}}(e(P), C) \simeq \text{Ext}^j_{\mathcal{C}}(e(P), e(C)) \simeq \text{Ext}^j_\mathcal{B}(P, i(C)) = 0$$

since $e \simeq \text{Id}_\mathcal{C}$ by Proposition 3.2 (iv). We infer that $\text{pd}_e e(P) \leq t$ and therefore statement $(\beta)$ holds. Similarly we show that $(ii) \Rightarrow (iv)$.

$(iii) \Rightarrow (i)$: Assume that $\text{igl.dim}_{\mathcal{A}} = m - 1 < \infty$ and let $S$ be a simple object in $\mathcal{A}$. Since the object $i(S)$ is simple in $\mathcal{B}$ by Lemma 3.2 (vi), Lemma 3.6 provides us the isomorphism:

$$\text{Ext}^j_\mathcal{B}(B, i(S)) \simeq \text{Hom}_{\mathcal{A}}(\Omega^j(B), i(S))$$

Since $\text{id}_{\mathcal{B}} i(S) \leq m - 1$, it follows that $\text{Hom}_{\mathcal{A}}(\Omega^j(B), i(S)) = 0$ for all $j > m - 1$. Since every object $A$ in $\mathcal{A}$ has a finite composition series, it follows that $\text{Hom}_{\mathcal{A}}(\Omega^j(B), i(A)) = 0$ for all $j > m - 1$. So far we have shown that $\text{Ext}^j_\mathcal{B}(B, i(A)) = 0$ for every $A$ in $\mathcal{A}$ and $j > m$, and there exists an $m$-th syzygy of $B$ lying in $\text{Im}(\mathcal{A})$.

Using now assumption $(\beta)$, Theorem 3.10 implies (i).

Similarly we show that $(iv) \Rightarrow (i)$ and then the four statements are equivalent. 

Remark 3.12. (i) The implication $(ii) \Rightarrow (i)$ in Proposition 3.11 shows that whenever we know that the quotient functor $e$ is an eventually homological isomorphism, we can obtain the desired isomorphisms from the functor $e$. This explains why in Definition 3.9 we don’t require these isomorphisms to be induced by the involved functor.

(ii) It is natural to consider if any other pair of the four conditions $(\alpha) - (\delta)$ in Proposition 3.11 implies that the functor $e$ is an eventually homological isomorphism. We refer to [19, Subsection 8.1] for examples showing that this is not the case.
4. Arrow removal and finitistic dimension

In this and the following section the finitistic dimension of an abelian category with enough projectives is discussed. Recall that given an abelian category $\mathcal{A}$ with enough projectives, the finitistic projective dimension of $\mathcal{A}$ is defined by

$$\text{Fin. dim } \mathcal{A} = \sup \{ \text{pd}_\mathcal{A} A \mid \text{pd}_\mathcal{A} A < \infty \}. $$

In the first subsection we begin by investigating cleft extensions of abelian categories where the images of the endofunctors $F$ and $G$ have bounded projective dimension. Such cleft extensions occur when factoring out ideals generated by arrows not occurring in a set of minimal relations (with some conditions) of finite dimensional quotients of path algebras. These special cleft extensions we study and characterize in the second and the final subsection.

4.1. Cleft extensions with special endofunctors $F$ and $G$. The exact sequences (2.5) and (2.3) give rise to the following exact sequences of functors

\[
\begin{align*}
0 \rightarrow & \quad F \rightarrow \text{el} \rightarrow \text{Id}_\mathcal{A} \rightarrow 0 \quad (4.1) \\
0 \rightarrow & \quad G \rightarrow \text{le} \rightarrow \text{Id}_\mathcal{A} \rightarrow 0 \quad (4.2)
\end{align*}
\]

Consider the following

\[
\text{Im } F \subseteq \mathcal{P}^{n.(\mathcal{B})} \quad \text{and} \quad \text{Im } G \subseteq \mathcal{P}^{n.(\mathcal{A})}. \quad (4.3)
\]

This means that all objects in $\text{Im } F$ and $\text{Im } G$ have finite projective dimension and there is a uniform bound for the length of the shortest projective resolutions, which is $n_{\mathcal{A}}$ and $n_{\mathcal{B}}$, respectively. This is a general version of Theorem A (ii) presented in the Introduction.

**Theorem 4.1.** Let $(\mathcal{B}, \mathcal{A}, e, l, i)$ be a cleft extension $\mathcal{A}$ of $\mathcal{B}$, where $\mathcal{A}$ and $\mathcal{B}$ are abelian categories with enough projectives, such that condition (4.3) holds. Assume in addition the conditions (4.1). Then:

(i) $\text{Fin. dim } \mathcal{A} \leq \max \{ \text{Fin. dim } \mathcal{B}, n_{\mathcal{A}} + 1 \}$.

(ii) $\text{Fin. dim } \mathcal{B} \leq \max \{ \text{Fin. dim } \mathcal{A}, n_{\mathcal{B}} + 1 \}$.

In particular,

$$\text{Fin. dim } \mathcal{A} < \infty \text{ if and only if } \text{Fin. dim } \mathcal{B} < \infty.$$  

**Proof.** For any object $B$ in $\mathcal{B}$, using that the functors $l$ and $e$ are exact and preserve projective objects (by Lemma 2.2 (iii) and (vii)), we have

$$\text{pd}_{\mathcal{B}} e(B) \leq \text{pd}_{\mathcal{A}} l(B) \leq \text{pd}_{\mathcal{B}} B.$$ 

Similarly, for any object $A$ in $\mathcal{A}$ we have

$$\text{pd}_{\mathcal{A}} e(A) \leq \text{pd}_{\mathcal{B}} l(A) \leq \text{pd}_{\mathcal{A}} A.$$ 

Let $B$ be an object in $\mathcal{B}$ of finite projective dimension. Then we obtain that

$$\text{pd}_{\mathcal{A}} l(B) \leq \text{pd}_{\mathcal{B}} B \leq \text{Fin. dim } \mathcal{B}$$

since the projective dimension of $B$ is finite. This implies that

$$\text{pd}_{\mathcal{A}} e(B) \leq \text{Fin. dim } \mathcal{A}.$$ 

Similarly, if $A$ is an object in $\mathcal{A}$ of finite projective dimension, then we get that

$$\text{pd}_{\mathcal{B}} e(A) \leq \text{Fin. dim } \mathcal{A} \quad \text{and therefore} \quad \text{pd}_{\mathcal{A}} e(A) \leq \text{Fin. dim } \mathcal{B}.$$ 

Then, if $\text{pd}_{\mathcal{B}} B < \infty$, from the exact sequence (4.1) it follows that

$$\text{pd}_{\mathcal{A}} B \leq \max \{ \text{pd}_{\mathcal{A}} e(B), \text{pd}_{\mathcal{A}} F(B) + 1 \} \leq \max \{ \text{Fin. dim } \mathcal{A}, n_{\mathcal{A}} + 1 \}.$$ 

This implies that

$$\text{Fin. dim } \mathcal{B} \leq \max \{ \text{Fin. dim } \mathcal{A}, n_{\mathcal{A}} + 1 \}. \quad (4.4)$$

Suppose now that $\text{pd}_{\mathcal{A}} A < \infty$. Then from the exact sequence (4.2) it follows that

$$\text{pd}_{\mathcal{A}} A \leq \max \{ \text{pd}_{\mathcal{A}} e(A), \text{pd}_{\mathcal{A}} G(A) + 1 \} \leq \max \{ \text{Fin. dim } \mathcal{B}, n_{\mathcal{B}} + 1 \},$$

\[
\text{pd}_{\mathcal{B}} B \leq \max \{ \text{pd}_{\mathcal{B}} e(B), \text{pd}_{\mathcal{B}} F(B) + 1 \} \leq \max \{ \text{Fin. dim } \mathcal{A}, n_{\mathcal{A}} + 1 \}.
\]
and therefore we have
\[ \text{Fin. dim } \mathcal{A} \leq \max \{ \text{Fin. dim } \mathcal{B}, n_{\mathcal{A}} + 1 \}. \]  
(4.5)
From the relations (4.4) and (4.5) it follows that \( \text{Fin. dim } \mathcal{A} < \infty \) if and only if \( \text{Fin. dim } \mathcal{B} < \infty \). \( \square \)

4.2. Arrow removal. Now we apply the result from the previous subsection to quotients of path algebras by admissible ideals, where we factor out an ideal generated by arrows which do not occur in a minimal set of relations with additional properties.

Let \( \Lambda = kQ/I \) be an admissible quotient of a path algebra \( kQ \). If \( a \) is an arrow in \( Q \) which is not occurring in a minimal generating set for \( I \), then we refer to the quotient \( \Gamma = \Lambda/(a) \) as arrow removal. In this subsection we have two aims:

1) To show that arrow removal is a cleft extension satisfying the conditions in Theorem 4.1.

2) Characterize arrow removal as trivial extensions with projective bimodules with special properties.

We start with the second goal. We remark that Corollary 4.3 and Proposition 4.4 provide a general version of Theorem A (i), as presented in the Introduction, for removing multiple arrows that do not occur in a minimal set of relations with additional properties.

Let \( \Gamma = kQ^*/I^* \) be an admissible quotient of a path algebra \( kQ^* \). First we analyse trivial extensions \( \Lambda = \Gamma \ltimes P \) with a projective \( \Gamma \Gamma \)-bimodule \( P \) and show it is the same as adding one arrow for each indecomposable direct summand of \( P \) to \( Q^* \) and adding the relations \( akQ^*b \) for all the different arrows \( a \) and \( b \) that we are adding.

Proposition 4.2. Let \( \Gamma = kQ^*/I^* \) for a field \( k \), a finite quiver \( Q^* \) and an admissible ideal \( I^* \) in \( kQ^* \). Let \( P \) be the projective \( \Gamma^\text{env} \)-module given by \( P = \bigoplus_{i=1}^t \Gamma e_i \otimes_k f_i \Gamma \) for some trivial paths \( e_i \) and \( f_i \) in \( Q^* \) for \( i = 1, 2, \ldots, t \), where \( \Gamma^\text{env} = \Gamma^0 \otimes_k \Gamma \). Then the trivial extension \( \Lambda = \Gamma \ltimes P \) is isomorphic to a quotient \( kQ/I \), where \( Q \) has the same vertices and the same arrows as \( Q^* \) with one arrow \( a_i : v_e \rightarrow v_f \), added for each indecomposable summand \( \Gamma e_i \otimes_k f_i \Gamma \) of \( P \) and \( I = \langle I^*, \sum_{i,j=1}^t a_i kQ^* a_j \rangle \) in \( kQ \).

Proof. The radical \( \text{rad}(\Lambda) \) of \( \Lambda \) is given by \( \text{rad}(\Gamma) \ltimes P \), and therefore
\[ \text{rad}(\Lambda)^2 = \text{rad}(\Gamma)^2 \ltimes (\text{rad}(\Gamma)P + P\text{rad}(\Gamma)). \]
This gives that \( \Lambda/\text{rad}(\Lambda) \simeq kQ^*_0 \) and
\[ \text{rad}(\Lambda)/\text{rad}(\Lambda)^2 \simeq \text{rad}(\Gamma)/\text{rad}(\Gamma)^2 \ltimes P/(P\text{rad}(\Gamma)P + P\text{rad}(\Gamma)). \]
\[ \simeq kQ^*_1 \ltimes \bigoplus_{i=1}^t (kQ^*_0 e_i \otimes_k f_i kQ^*_0). \]
It follows from this that the vertices in \( \Lambda \) are the same as the vertices in \( Q^* \) and that \( Q \) has the same arrows as \( Q^* \) and in addition has one additional arrow \( a_i \) for each indecomposable direct summand \( \Gamma e_i \otimes_k f_i \Gamma \) of \( P \) corresponding to the vectorspace \( 0 \ltimes kQ^*_0 e_i \otimes_k f_i kQ^*_0 \). The new arrow \( a_i \) correspond to the element \( e_i x_i f_i = (0, e_i \cdot e_i \otimes f_i \cdot f_i) = (0, e_i \otimes f_i) = x_i. \)
Hence \( a_i : v_{e_i} \rightarrow v_{f_i} \). This shows that there is a surjective homomorphism of algebras \( \varphi : kQ \rightarrow \Lambda \) defined by
\[ \varphi(x) = \begin{cases} (x, 0), & \text{if } x \in Q^*_0 \cup Q^*_1, \\ (0, e_i \otimes f_i), & \text{if } x = a_i \text{ for } i = 1, 2, \ldots, t. \end{cases} \]
It is clear that $\langle I^* \rangle$ and $\sum_{i,j=1}^{t, t} a_i kQ^* a_j$ are in the kernel of $\varphi$, since $(0 \otimes P)^2 = 0$. We want to show that

$$\text{Ker } \varphi = \langle I^*, \sum_{i,j=1}^{t, t} a_i kQ^* a_j \rangle.$$ 

Let $z$ be an arbitrary element in the ideal generated by the arrows in $kQ$ and in the kernel of $\varphi$. Given that we know $\langle I^*, \sum_{i,j=1}^{t, t} a_i kQ^* a_j \rangle$ is in the kernel of $\varphi$, we only need to consider elements $z$ of the form

$$z = r + \sum_{i,j=1}^{t, s_i} r_{1ij} a_i r_{2ij}.$$ 

for $r, r_{1ij}$ and $r_{2ij}$ in $kQ^*$, where $r_{1ij} = r_{1ij} e_i$ and $r_{2ij} = f_i r_{2ij}$ for $1 \leq j \leq s_i$. By the algebra homomorphism $\varphi$ this element is mapped to

$$0 = (r, \sum_{i,j=1}^{t, s_i} r_{1ij} x_i r_{2ij}).$$ 

Then we must have that $r = 0$ in $\Gamma$ and $\sum_{i,j=1}^{t, s_i} r_{1ij} x_i r_{2ij} = 0$ in $P$. We have that

$$r_{1ij} x_i r_{2ij} = r_{1ij} (0, e_i \otimes f_i) r_{2ij} = (0, r_{1ij} e_i \otimes f_i r_{2ij}) = (0, r_{1ij} \otimes r_{2ij}),$$

and this gives that

$$\sum_{i,j=1}^{t, s_i} r_{1ij} x_i r_{2ij} = \sum_{i,j=1}^{t, s_i} (0, r_{1ij} \otimes r_{2ij}) = (0, \sum_{i=1}^{t} \sum_{j=1}^{s_j} r_{1ij} \otimes r_{2ij})$$

and in particular that $\sum_{j=1}^{s_j} r_{1ij} \otimes r_{2ij} = 0$ in $\Gamma e_i \otimes_k f_i \Gamma$ for each $i$. Since

$$\Gamma e_i \otimes_k f_i \Gamma \simeq kQ^* e_i \otimes_k f_i kQ^*/(I^* e_i \otimes_k f_i kQ^* + kQ^* e_i \otimes_k f_i I^*),$$

it follows from the above that kernel of $\varphi$ is $\langle I^*, \sum_{i,j=1}^{t, t} a_i kQ^* a_j \rangle$ in $kQ$. This completes the proof of the proposition.

Now we prove that a trivial extension of an algebra $\Gamma$ with a finitely generated projective bimodule $P$ with certain properties gives rise to an algebra where arrows can be removed. This is the first part of the characterization of arrow removal.

Corollary 4.3. Let $\Gamma = kQ^*/I^*$ for a field $k$, a finite quiver $Q^*$ and an admissible ideal $I^*$ in $kQ^*$. Let $P$ be the projective $\Gamma_{\text{env}}$-module given by $P = \oplus_{i=1}^{t} \Gamma e_i \otimes_k f_i \Gamma$ for some trivial paths $e_i$ and $f_i$ in $Q^*$ for $i = 1, 2, \ldots, t$, where $\Gamma_{\text{env}} = \Gamma^{\text{op}} \otimes_k \Gamma$. Suppose that $\text{Hom}_\Gamma(e_i \Gamma, f_j \Gamma) = 0$ for all $i$ and $j$ in $\{1, 2, \ldots, t\}$.

(i) The trivial extension $\Lambda = \Gamma \otimes P$ is isomorphic to a quotient $kQ/I$, where $Q$ has the same vertices and the same arrows as $Q^*$ with one arrow $a_i : v_{e_i} \rightarrow v_{f_i}$, added for each indecomposable summand $\Gamma e_i \otimes_k f_i \Gamma$ of $P$ and $I = \langle I^* \rangle$ in $kQ$. Here $v_{e_i}$ and $v_{f_i}$ are the vertices corresponding to the primitive idempotents $e_i$ and $f_i$, respectively.

(ii) The arrows $a_i$ do not occur in a minimal set of generators for the relations $I$, and $\text{Hom}_\Lambda(e_i \Lambda, f_j \Lambda) = 0$ for all $i$ and $j$ in $\{1, 2, \ldots, t\}$.

Proof. (i) We have that $0 = \text{Hom}_\Gamma(e_i \Gamma, f_j \Gamma) \simeq f_j \Gamma e_i$ for all $i$ and $j$, which is equivalent to saying that $f_j kQ^* e_i$ is contained in $I^*$ for all $i$ and $j$. We infer from this that $a_i kQ^* a_j = a_j (f_j kQ^* e_i) a_i$ is in the ideal $\langle I^* \rangle$ in $kQ$. The claim then follows immediately from Proposition 4.2.
(ii) The first claim follows directly from (i). We have that
\[
\text{Hom}_\Lambda(e_i \Lambda, f_j \Lambda) \simeq f_j \Lambda e_i
\]
\[
= f_j (\Gamma \otimes P) e_i
\]
\[
= (f_j \Gamma e_i) \otimes (f_j P e_i)
\]
\[
= (f_j \Gamma e_i) \otimes (f_j \Gamma e_i) \oplus (f_j \Gamma e_i) f_i \Gamma e_i)
\]
By assumption \(f_r \Gamma e_s = 0\) for all \(r\) and \(s\) in \(\{1, 2, \ldots, t\}\), so we obtain that
\[
\text{Hom}_\Lambda(e_i \Lambda, f_j \Lambda) = 0
\]
for all \(i\) and \(j\) in \(\{1, 2, \ldots, t\}\). This completes the proof of the proposition. \(\square\)

Next we prove the converse of the above result. This needs some preparation. Let \(\Lambda = kQ/I\) be an admissible quotient of the path algebra \(kQ\) over a field \(k\). Suppose that there are arrows \(a_i : v_{e_i} \rightarrow v_{f_j}\) in \(Q\) for \(i = 1, 2, \ldots, t\) which do not occur in a set of minimal generators of \(I\) in \(kQ\) and \(\text{Hom}_\Lambda(e_i \Lambda, f_j \Lambda) = 0\) for all \(i\) and \(j\) in \(\{1, 2, \ldots, t\}\). Let \(\Gamma = \Lambda / \Lambda \{\pi_i\}_{i=1}^t \Lambda\) for \(\pi_i = a_i + I\) in \(\Lambda\). We have the natural surjective algebra homomorphism \(\pi : \Lambda \rightarrow \Gamma\), and we claim that there is a natural algebra inclusion \(\nu : \Gamma \hookrightarrow \Lambda\) such that \(\pi \nu = \text{id}_\Gamma\). Let \(Q^*\) be the subquiver of \(Q\), where the arrows \(\{a_i\}_{i=1}^t\) have been removed. The quiver inclusion morphism \(Q^* \rightarrow Q\) induces an inclusion \(kQ^* \rightarrow kQ\) of path algebras. This further induces an inclusion
\[
\nu' : kQ^*/(kQ^* \cap I) \rightarrow kQ/I = \Lambda.
\]
We want to show that \(\Gamma \simeq kQ^*/(kQ^* \cap I)\). We have that
\[
\Gamma = \Lambda / \Lambda \{\pi_i\}_{i=1}^t \Lambda = (kQ/I) / ((kQ/I) \{\pi_i\}_{i=1}^t (kQ/I))
\]
\[
= (kQ/I) / (kQ \{a_i\}_{i=1}^t kQ + I) / I
\]
\[
\simeq kQ/(kQ \{a_i\}_{i=1}^t kQ + I)
\]
Furthermore,
\[
kQ = kQ \{a_i\}_{i=1}^t kQ + kQ^*,
\]
where the sum is direct as vectorspaces, hence \(kQ = kQ \{a_i\}_{i=1}^t kQ \oplus kQ^*\). In addition, since \(I\) is generated by \(I \cap kQ^*\), it follows that
\[
kQ \{a_i\}_{i=1}^t kQ + I = kQ \{a_i\}_{i=1}^t kQ + (I \cap kQ^*)
\]
in \(kQ\). As above,
\[
kQ \{a_i\}_{i=1}^t kQ + I = kQ \{a_i\}_{i=1}^t kQ \oplus (I \cap kQ^*),
\]
and this implies that
\[
\Gamma \simeq kQ/(kQ \{a_i\}_{i=1}^t kQ + I) = (kQ \{a_i\}_{i=1}^t kQ \oplus kQ^*)/(kQ \{a_i\}_{i=1}^t kQ \oplus (I \cap kQ^*))
\]
\[
\simeq kQ^*/(I \cap kQ^*).\]
We infer from this that the epimorphism \(\varphi : kQ^*/(I \cap kQ^*) \rightarrow \Lambda / \Lambda \{a_i\}_{i=1}^t \Lambda = \Gamma\) given by \(\varphi(\overline{p}) = \overline{p} + I\) for \(p\) in \(kQ^*\) is an isomorphism. The inclusion \(kQ^* \rightarrow kQ\) induces an inclusion \(\nu : kQ^*/(I \cap kQ^*) \rightarrow \Lambda\) in such a way that the composition \(\varphi^{-1} \pi \nu = \text{id}\). If we now identify \(\Gamma\) with \(kQ^*/(I \cap kQ^*)\), we have our desired result.

The exact sequence
\[
\eta : 0 \rightarrow \Lambda \{\pi_i\}_{i=1}^t \Lambda \rightarrow \Lambda \rightarrow \Gamma \rightarrow 0 \quad (4.6)
\]
can be considered as a sequence of \(\Gamma\)-\(\Gamma\)-bimodules. This sequence splits as an exact sequence of one-sided \(\Gamma\)-modules, so that \(\Lambda \simeq \Lambda \{\pi_i\}_{i=1}^t \Lambda \oplus \Gamma\) as a left and as a right \(\Gamma\)-module. Next we prove that \(\Lambda \{\pi_i\}_{i=1}^t \Lambda\) is a projective \(\Gamma\)-\(\Gamma\)-bimodule and that we have the converse of Corollary 4.3.
Proposition 4.4. Let $\Lambda = kQ/I$ be an admissible quotient of the path algebra $kQ$ over a field $k$. Suppose that there are arrows $a_i : v_{e_i} \rightarrow v_{f_i}$ in $Q$ for $i = 1, 2, \ldots, t$ which do not occur in a set of minimal generators of $I$ in $kQ$ and $\text{Hom}_\Lambda(e_i, f_j\Lambda) = 0$ for all $i$ and $j$ in $\{1, 2, \ldots, t\}$. Let $\Gamma = \Lambda/\Lambda\{v_i\}_{i=1}^t$. Then the following assertions are true.

(i) $f_j\Gamma e_i \simeq \text{Hom}_\Gamma(e_i, f_j\Gamma) = 0$ for all $i$ and $j$ in $\{1, 2, \ldots, t\}$.

(ii) \[
\sum_{i=1}^t \Lambda\pi_i\Lambda \simeq \otimes_{i=1}^t \Lambda\pi_i\Lambda
\]

and

\[
\Lambda\pi_i\Lambda \simeq \Gamma e_i \otimes_k f_i\Gamma
\]
as $\Gamma$-$\Gamma$-bimodules. In particular, $\sum_{i=1}^t \Lambda\pi_i\Lambda$ is a projective $\Gamma$-$\Gamma$-bimodule.

(iii) $\Lambda$ is isomorphic to the trivial extension $\Gamma \ltimes P$, where $P = \oplus_{i=1}^t \Gamma e_i \otimes_k f_i\Gamma$ with $\text{Hom}_\Gamma(e_i, f_j\Gamma) = 0$ for all $i$ and $j$ in $\{1, 2, \ldots, t\}$.

Proof. (i) Since $f_j\Lambda e_i = 0$ for all $i$ and $j$ in $\{1, 2, \ldots, t\}$ and $\nu : \Gamma \rightarrow \Lambda$ is an inclusion, it follows that $0 = f_j\Gamma e_i \simeq \text{Hom}_\Gamma(e_i, f_j\Gamma)$ for all $i$ and $j$ in $\{1, 2, \ldots, t\}$.

(ii) First we argue that $\sum_{i=1}^t \Lambda\pi_i\Lambda$ is a direct sum. Since $f_j\Lambda e_i = 0$ for all $i$ and $j$ in $\{1, 2, \ldots, t\}$, it follows that $a_j\Lambda a_i = 0$ for all $i$ and $j$ in $\{1, 2, \ldots, t\}$. This implies that

\[
\Lambda\pi_i\Lambda = \Gamma e_i \Gamma
\]
as a $\Gamma$-$\Gamma$-bimodule. Let

\[
x = \sum_{i=1}^t \sum_{j=1}^{t_i} c_{j_i} \lambda_{j_i} x_{j_i}^i
\]
be in $\sum_{i=1}^t \Lambda\pi_i\Lambda$, where $c_{j_i}$ is in $k \setminus \{0\}$, and the elements $\lambda_{j_i}$ and $x_{j_i}^i$ are in $\text{Nontip}(I)$ by Lemma A.1 (ii). Assume that $x = 0$, or equivalently, when $x$ is viewed as an element in $kQ$, then $x$ is in $I$. Then the tip of $x$ is divisible by a tip of an element of a Gröbner basis $\mathfrak{g}$ for $I$ in $kQ$ by the definition of a Gröbner basis (see Definition A.2). The tip of $x$ is of the form $\lambda_{j_0} a \lambda'_{j_0} = p \mathfrak{g}(g)p'$ for some integers $i_0$ and $j_0$, some paths $p$ and $p'$ and some $g$ in $\mathfrak{g}$ by Lemma A.3 (ii). Since the arrows $a_i$ do not occur in a set of minimal generators for $I$, the element $\mathfrak{g}(g)$ must divide $\lambda_{j_0} a \lambda'_{j_0}$ by Lemma A.2. Since $\lambda_{j_0} a \lambda'_{j_0}$ are in $\text{Nontip}(I)$, this is a contradiction. Hence, we must have $c_{j_0} = 0$, which is another contradiction. It follows that the sum $\sum_{i=1}^t \Lambda\pi_i\Lambda$ is direct.

Now we show that $\Lambda\pi_i\Lambda \simeq \Gamma e_i \otimes_k f_i\Gamma$ for all $i$ in $\{1, 2, \ldots, t\}$. This shows that $\sum_{i=1}^t \Lambda\pi_i\Lambda$ is a projective $\Gamma$-$\Gamma$-bimodule.

Consider the map

\[
\psi : \Gamma e_i \otimes_k f_i\Gamma \rightarrow \Lambda\pi_i\Lambda
\]
given by $\psi(\gamma e_i \otimes f_i\gamma') = \gamma e_i \alpha_i f_i\gamma'$ for $\gamma e_i \otimes f_i\gamma'$ in $\Gamma e_i \otimes_k f_i\Gamma$. Any element $x$ in $\Gamma e_i \otimes_k f_i\Gamma$ can be written as

\[
x = \sum_{r=1}^n \alpha_r (\gamma_r e_i \otimes f_i\gamma'_r)
\]
with $\alpha_r$ in $k \setminus \{0\}$ and $\gamma_r$ and $\gamma'_r$ in $\text{Nontip}(I) \cap kQ^*$ by Lemma A.3. If $x$ is in $\text{Ker} \psi$, then

\[
\psi(x) = \sum_{r=1}^n \alpha_r \gamma_r e_i \alpha_i f_i\gamma'_r = 0
\]
in $\Lambda\pi_i\Lambda$, or equivalently that $\psi(x)$ is in $I$. The tip of $\psi(x)$ is $\text{Tip}(\psi(x)) = \gamma_r a \gamma'_r$ for some $r_0$. As $\psi(x)$ is in $I$, this tip must be divisible by some tip of an element $g$ in $\mathfrak{g}$, a Gröbner basis for $I$. The arrow $a_i$ does not occur in any element in $\mathfrak{g}$.
by Lemma [A.4] so we infer that \( \text{Tip}(g) \) divides \( \gamma_{ro} \) or \( \gamma'_{ro} \). But this is impossible, since \( \gamma_{ro} \) and \( \gamma'_{ro} \) are elements in \( \text{Nontip}(I) \). It follows that \( \alpha_{ro} = 0 \), which is a contradiction. Hence we have that \( \text{Ker} \psi = 0 \) and \( \Gamma e \otimes_k f \Gamma \simeq \Lambda \pi_\Lambda \), which is a projective \( \Gamma - \Gamma \)-bimodule. This completes the proof of (ii).

(iii) This follows from the comments before this proposition and parts (i) and (ii). □

Now we look at the special case removing only one arrow \( a : v_c \rightarrow v_f \) not occurring in a set of minimal generators of \( I \) and \( \text{Hom}_\Lambda(e \Lambda, f \Lambda) = 0 \). We show that the second condition is superfluous. This is Theorem A (i) as stated in the Introduction.

Proposition 4.5. Let \( \Lambda = k \Lambda/\Lambda I \) be an admissible quotient of a path algebra \( k \Lambda \) over a field \( k \). Then an arrow \( a : v_c \rightarrow v_f \) in \( Q \) does not occur in a set of minimal generators of \( I \) in \( k \Lambda \) if and only if \( \Lambda \) is isomorphic to the trivial extension \( \Gamma \ltimes P \), where \( \Gamma \simeq \Lambda/\Lambda \pi_\Lambda \) and \( P = \Gamma e \otimes_k f \Gamma \) with \( \text{Hom}_\Gamma(e \Gamma, f \Gamma) = (0) \).

Proof. Assume that the arrow \( a : v_c \rightarrow v_f \) in \( Q \) does not occur in a set of minimal generators of \( I \) in \( k \Lambda \). Given the assumption on \( a \), a minimal set of generators for a Gröbner basis \( \mathcal{G} \) for the ideal \( I \) (using length left-lexicographic ordering on the paths in \( Q \) ) does not contain any elements in which the arrow \( a \) appears by Lemma [A.3].

First we show that \( \text{Hom}_\Lambda(e \Lambda, f \Lambda) = (0) \). We have that \( \text{Hom}_\Lambda(e \Lambda, f \Lambda) \simeq f \Lambda e \). Since \( a \) is not occurring in set of minimal generators of \( I \) in \( k \Lambda \), the multiplication map \( f \Lambda e \rightarrow \pi \Lambda e \) given by left multiplication by \( a \), is an isomorphism. Hence, \( f \Lambda e = 0 \) if and only if \( \pi \Lambda e = 0 \).

Assume that \( \pi \Lambda e \neq 0 \), that is, there is some path \( p \) in \( Q \) such that \( \overline{p} \neq 0 \) and \( p \) ends in \( v_c \). Since \( I \) is an admissible ideal, we have that \( \overline{(ap)^t} = 0 \) in \( \Lambda \) or equivalently \( (ap)^t \) is in \( I \) for some \( t \geq 1 \). In particular, reducing the element \( (ap)^t \) modulo a minimal set of generators for a Gröbner basis for \( I \) gives zero. Choose \( p \) minimal with the property that \( ap \notin I \). Reducing \( (ap)^t \) modulo \( I \) means to subtract elements of the form \( p'gp'' \) for some paths \( p' \) and \( p'' \) in \( Q \) and \( g \) in \( \mathcal{G} \), where

\[
\text{Tip}((ap)^t) = (ap)^t \cdot \text{Tip}(p'gp'') = p'Tip(g)p''.
\]

Since \( \text{Tip}(g) \) does not contain \( a \) by Lemma [A.4] we must have that \( \text{Tip}(g) \mid p \), where this means \( \text{Tip}(g) \) divides \( p \), that is, \( q \text{Tip}(g)q' = p \) for some path \( q \) and \( q' \) in \( Q \). Hence we can write

\[
p = q \text{Tip}(g)q' \]
\[
p' = (ap)^t aq \]
\[
p'' = q'(ap)^t-r-1.
\]

Then

\[
(ap)^t - p'gp'' = (ap)^t a(p - qgg')(ap)^t-r-1
\]
\[
= (ap)^t a(p - \text{Tip}(gg')) - \left\{ \sum \text{smaller paths than } p \right\}(ap)^t-r-1
\]
\[
= -(ap)^t a\left( \sum \text{smaller paths than } p \right)(ap)^t-r-1.
\]

Since for all the paths \( s \) that are smaller than \( p \) the elements \( as \) are in \( I \), it follows from the above that \( ap \) is in \( I \). This is a contradiction, so \( \overline{p} \Lambda e = 0 \). Hence \( f \Lambda e = 0 \) and \( \text{Hom}_\Lambda(e \Lambda, f \Lambda) = 0 \).

From the above and Proposition [4.4] it follows that \( \Lambda \) is isomorphic to the trivial extension \( \Gamma \ltimes P \), where \( \Gamma \simeq \Lambda/\Lambda \pi_\Lambda \) and \( P = \Gamma e \otimes_k f \Gamma \) with \( \text{Hom}_\Gamma(e \Gamma, f \Gamma) = (0) \).
Conversely, assume that $\Lambda$ is isomorphic to the trivial extension $\Gamma \ltimes P$, where $\Gamma \simeq \Lambda/\Lambda\Lambda\Lambda$ and $P = \Gamma e \otimes_k f\Gamma$ with $\text{Hom}_\Gamma(e\Gamma, f\Gamma) = \{0\}$. Using Corollary 4.3, the claim follows. $\square$

Let the setting be as above, $\Lambda = kQ/I$ be an admissible quotient of the path algebra $kQ$ over a field $k$. Suppose that there are arrows $a_i : v_i \rightarrow v_f$ in $Q$ for $i = 1, 2, \ldots, t$ which do not occur in a set of minimal generators of $I$ in $kQ$ and $\text{Hom}_\Lambda(e_i\Lambda, f_j\Lambda) = 0$ for all $i$ and $j$ in $\{1, 2, \ldots, t\}$. Let $\Gamma = \Lambda/\Lambda(\pi_i)_{i=1}^t$. Now we want to address the first aim of this section, namely that arrow removal $\Lambda \rightarrow \Gamma$ is a cleft extension satisfying the conditions in Theorem 1.1.

In the above situation we have the functors

\[
\begin{array}{c}
\xymatrix{F \ar@/_1pc/[rr]_e \ar@/^1pc/[rr]_l & & \text{mod-}\Gamma & \ar@/_1pc/[ll]_l & \ar@/^1pc/[ll]_e \text{mod-}\Lambda & \ar@/^1pc/[ll]_r & \text{mod-}\Gamma & \ar@/_1pc/[ll]_r & \ar@/^1pc/[ll]_l \text{mod-}\Lambda} \\
\end{array}
\]

where $F$ is given by the exact sequence

\[
0 \longrightarrow F \longrightarrow e l \longrightarrow \text{Id}_{\text{mod-}\Gamma}.
\]

**Proposition 4.6.** Let $\Lambda = kQ/I$ be an admissible quotient of the path algebra $kQ$ over a field $k$. Suppose that there are arrows $a_i : v_i \rightarrow v_f$ in $Q$ for $i = 1, 2, \ldots, t$ which do not occur in a set of minimal generators of $I$ in $kQ$ and $\text{Hom}_\Lambda(e_i\Lambda, f_j\Lambda) = 0$ for all $i$ and $j$ in $\{1, 2, \ldots, t\}$. Let $\Gamma = \Lambda/\Lambda(\pi_i)_{i=1}^t$. Then the following assertions hold.

(i) $e$ is faithful exact,

(ii) $(l, e)$ is an adjoint pair of functors,

(iii) $e i \simeq \text{Id}_{\text{mod-}\Gamma}$,

(iv) $l$ and $r$ are exact functors,

(v) $e$ preserves projectives,

(vi) $\text{Im} F \subseteq \text{proj}(\Gamma)$.

(vii) $F^2 = 0$.

In particular,

\[
\mathcal{C} = (\text{mod-}\Gamma, \text{mod-}\Lambda, e = \text{Hom}_\Lambda(\Gamma\Lambda, \Lambda_\Lambda), l = - \otimes_{\Gamma} \Gamma\Lambda\Lambda, i = \text{Hom}_\Gamma(\Lambda\Gamma, \Lambda_\Lambda))
\]

is a cleft extension $\text{mod-}\Lambda$ of $\text{mod-}\Gamma$ satisfying (2.7) and (4.3) with both bounds being zero, that is, $\text{Im} F \subseteq \text{proj}(\Gamma)$ and $\text{Im} G \subseteq \text{proj}(\Lambda)$.

**Proof.** (i) The functor $e : \text{mod-}\Lambda \rightarrow \text{mod-}\Gamma$ is faithful exact, since it is given by the restriction along the algebra inclusion $\Gamma \rightarrow \Lambda$.

(ii) This is immediate from the definitions of the functors $l$ and $e$.

(iii) Since the composition of the algebra homomorphisms $\nu: \Gamma \rightarrow \Lambda$ and $\pi: \Lambda \rightarrow \Gamma$ is the identity on $\Gamma$, it follows that $e i \simeq \text{Id}_{\text{mod-}\Gamma}$.

(iv) By the split exact sequence (4.1) as left $\Gamma$-modules, we have the isomorphism $\Gamma\Lambda \simeq \Gamma \oplus \Gamma(\pi_i)_{i=1}^t$. By Proposition 4.3(ii) we have that

\[
\Gamma\Lambda(\pi_i)_{i=1}^t \simeq \oplus_{i=1}^t \Gamma \otimes_k f_i \Gamma.
\]

so that $\Gamma\Lambda$ is a projective left $\Gamma$-module. Since

\[
l = - \otimes_{\Gamma} \Gamma\Lambda\Lambda : \text{mod-}\Gamma \rightarrow \text{mod-}\Lambda,
\]

the functor $l$ is exact. Using similar arguments as above we show that $\Lambda\Gamma$ is a projective $\Gamma$-module, hence $r = \text{Hom}_\Gamma(\Lambda\Gamma, \Lambda_\Lambda)$ is an exact functor.

(v) Since $(e, r)$ is an adjoint pair and $r$ is exact by (iv), it follows that $e$ preserves projectives.
(vi) Since $e$ and $l$ commute with finite direct sums and they are exact, we infer that $F$ also commutes with finite direct sums and is exact. By Watt’s theorem $F \simeq - \otimes \Gamma F(\Gamma) : \operatorname{mod} \Gamma \to \operatorname{mod} \Gamma$, where $F(\Gamma) \simeq \Lambda \{ \pi_i \}_{i=1}^t \Lambda$ as $\Gamma$-$\Gamma$-bimodules. We have that

$$F(B) = B \otimes \Gamma \Lambda \{ \pi_i \}_{i=1}^t \Lambda \simeq \oplus_{i=1}^t B \otimes \gamma_i \Gamma e_i \otimes_k f_i \Gamma$$

by Proposition 4.4 (ii). The claim follows from this.

(vii) We have that

$$F^2(B) = B \otimes \Gamma \Lambda \{ \pi_i \}_{i=1}^t \Lambda \otimes \Gamma \Lambda \{ \pi_i \}_{i=1}^t \Lambda.$$

Since $\pi_j \Lambda = \pi_j \Gamma = \pi_j f_j \Gamma$ and $\Lambda \pi_i = \Gamma e_i \pi_i$, we have that

$$B \otimes \Gamma \Lambda \pi_j \Lambda \otimes \Gamma \Lambda \pi_i \Lambda = B \otimes \Gamma \Lambda \pi_j \Gamma \otimes \Gamma \Gamma e_i \pi_i \Lambda = B \otimes \Gamma \Lambda \pi_j f_j \Gamma \otimes \Gamma \pi_i \Lambda = B \otimes \Gamma \Lambda \pi_j f_j e_i \otimes \Gamma \pi_i \Lambda.$$ 

From this we infer that $F^2(B) = 0$ as $f_j \Gamma e_i = 0$ by Proposition 4.3 (i) for all $i$ and $j$ in $\{1, 2, \ldots, t\}$.

For the final claim, (i)–(iii) show that $G$ is a cleft extension. Conditions (iv) and (v) show that (2.7) holds.

By (vi) we have that $\operatorname{Im} F \subseteq \operatorname{Proj}(\Gamma)$. Therefore applying (vii) and Lemma 2.7 we have $\operatorname{Im} G \subseteq \operatorname{Proj}(\Lambda)$. This shows that (4.3) holds with $n_{\text{mod} \Gamma} = n_{\text{mod} \Lambda} = 0$. 

We end this section with the following consequence of the previous result and Theorem 4.1. This is Theorem A (ii) presented in the Introduction.

**Theorem 4.7.** Let $\Lambda = kQ/I$ be an admissible quotient of the path algebra $kQ$ over a field $k$. Suppose that there are arrows $a_i : v_{e_i} \to v_{f_i}$ in $Q$ for $i = 1, 2, \ldots, t$ which do not occur in a set of minimal generators of $I$ in $kQ$ and $\operatorname{Hom}_\Lambda(e_i \Lambda, f_j \Lambda) = 0$ for all $i$ and $j$ in $\{1, 2, \ldots, t\}$. Let $\Gamma = \Lambda/\Lambda \{ \pi_i \}_{i=1}^t \Lambda$.

(i) $\operatorname{fin} \dim \Lambda \leq \max \{ \operatorname{fin} \dim \Gamma, 1 \}$.

(ii) $\operatorname{fin} \dim \Gamma \leq \max \{ \operatorname{fin} \dim \Lambda, 1 \}$.

In particular,

$$\operatorname{fin} \dim \Lambda < \infty \text{ if and only if } \operatorname{fin} \dim \Gamma < \infty.$$

**Remark 4.8.** The arrow removal operation has also been considered by Diracca and Koenig in [5]. They considered the notion of an exact split pair $(i, e)$, i.e. a pair of exact functors $i : \mathcal{B} \to \mathcal{A}$ and $e : \mathcal{A} \to \mathcal{B}$ between abelian categories such that the composition $ie$ is an auto-equivalence of $\mathcal{B}$. Up to this auto-equivalence, $e$ being faithful and the existence of a left adjoint of $e$, this is a cleft extension as defined in Definition 2.1. Arrow removal of an arrow $a$ which only occurs in monomial relations gives rise to an exact split pair, see [5] Proposition 5.4 (b)]. We now show that the induced cleft extension of such an arrow removal does not in general satisfy the conditions of Theorem 4.1. For example, consider the algebra $\Lambda = k \left\{ a \bigcup a \right\}/(a^2)$. The abelian category $\operatorname{mod} \Lambda$ is a cleft extension of $\operatorname{mod} k$, in particular, we have the following diagram:

$$
\begin{array}{ccc}
\text{mod-}k & \xrightarrow{q=-\otimes \Lambda \Lambda k} & \text{mod-} \Lambda \\
\text{mod-}k & \xrightarrow{p=\operatorname{Hom}_\Lambda(\Lambda k, -)} & \text{mod-} \Lambda \\
\text{mod-}k & \xrightarrow{r=\operatorname{Hom}_\Lambda(\Lambda k, -)} & \text{mod-} \Lambda \\
\end{array}
$$

The right hand side is induced from the natural inclusion $k \to \Lambda$. Then, the functor $l$ is exact, the functor $r$ is exact and therefore the functor $e$ preserves projectives. Moreover, since $\operatorname{mod} k$ is a semisimple category the image of the endofunctor $F$ is
projective. Let us now compute the endofunctor $G$. Let $X$ be a $\Lambda$-module. From the exact sequence (4.2) we have the map
\[
\begin{align*}
\text{le}(X) = X|_k \otimes_k \Lambda &\longrightarrow X \longrightarrow 0, \quad x \otimes \lambda \mapsto x\lambda \\
\end{align*}
\]
and the endofunctor $G$ is the kernel. Clearly, $X|_k \otimes_k \Lambda$ is a finitely generated projective $\Lambda$-module. Thus, $G(X)$ is the first syzygy of $X$ plus some projective. This shows that for all $X$ in mod-$\Lambda$ the first syzygy of $X$ is a direct summand of $G(X)$ and therefore $G(X)$ is not projective. Hence, the second part of condition (4.3) is not satisfied.

5. Vertex removal and finitistic dimension

This section is devoted to discussing reduction techniques for finitistic dimension in abelian categories with enough projectives occurring in recollement situations. This is done in four situations, two of which have occurred in the literature already (see [3, Corollary 4.21]), [7, Proposition 2.1]) and two are new. These are applied to finite dimensional algebras.

5.1. Triangular reduction. A form of triangular reduction was considered by Happel in [12] via recollements:

**Theorem 5.1.** If a finite dimensional algebra $\Lambda$ occur in a recollement of bounded derived categories like
\[
\begin{align*}
\text{D}^b(\text{mod}-\Lambda') &\longrightarrow \text{D}^b(\text{mod}-\Lambda) &\longrightarrow \text{D}^b(\text{mod}-\Lambda') \\
\text{r} &\quad &\text{i} \\
\text{p} &\quad &\text{q} \\
\end{align*}
\]
then fin. dim $\Lambda < \infty$ if and only if fin. dim $\Lambda' < \infty$ and fin. dim $\Lambda' < \infty$.

When such a recollement exists is characterized in [14], but it seems hard to apply. A reduction formula for the finitistic dimension of triangular matrix rings is given by the following classical result due to Fossum, Griffith and Reiten.

**Theorem 5.2 ([3, Corollary 4.21]).** Let $\Lambda = (\begin{smallmatrix} R & 0 \\ M & S \end{smallmatrix})$ for rings $R$ and $S$ and a non-zero $S$-$R$-bimodule $M$. Then
\[
\begin{align*}
\text{fin. dim } R &\leq \text{fin. dim } \Lambda \leq 1 + \text{fin. dim } R + \text{fin. dim } S. \\
\end{align*}
\]

Let $\Lambda$ be as in Theorem 5.2 and let $e = (\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix})$ an idempotent element in $\Lambda$. The triangular ring gives rise to a recollement situation as in Example 5.3 with $(1 - e)\Lambda(1 - e) \simeq \Lambda/\Lambda e \Lambda$. We apply this reduction technique to finite dimensional quotients $kQ/I$ of path algebras to show that $Q$ is path connected if no triangular reduction is possible (first proved in [10]).

Given an idempotent $e$ in a finite dimensional algebra $\Lambda$ we can view $\Lambda$ as the matrix ring
\[
\begin{pmatrix}
 e\Lambda e & e\Lambda(1 - e) \\
 (1 - e)\Lambda e & (1 - e)\Lambda(1 - e)
\end{pmatrix}.
\]
If $e\Lambda(1 - e)$ (or $(1 - e)\Lambda e$) equals zero for any idempotent $e \neq 0, 1$, then we say that $\Lambda$ has a triangular structure. Furthermore, an algebra $\Lambda$ is said to be triangular reduced if $\Lambda$ has no non-trivial triangular structures. We use Theorem 5.2 to reprove the following result from [10]: If a finite dimensional algebra $\Lambda = kQ/I$ has no non-trivial triangular structure, then the quiver $Q$ is path connected.
Let $\Lambda = kQ/I$ be an admissible quotient of the path algebra $kQ$ over a field $k$, and let $e$ be a sum of vertices in $Q$ with $e \neq 0,1$. Assume that $e\Lambda(1-e) = (0)$. If $(1-e)\Lambda e = (0)$ instead, interchange the role of $e$ and $1-e$. Then

$$\Lambda \simeq \left(\frac{e\Lambda e}{(1-e)\Lambda e(1-e)}\right).$$

Then $\text{fin.dim}\, \Lambda$ is finite if $\text{fin.dim}\, e\Lambda e$ and $\text{fin.dim}(1-e)\Lambda(1-e)$ are finite. Next, a triangular reduced admissible quotient $\Lambda = kQ/I$ of a path algebra is shown to have a path connected quiver $Q$.

**Proposition 5.3** ([10]). Let $\Lambda = kQ/I$ be an admissible quotient of the path algebra $kQ$ over a field $k$. Assume that $\Lambda$ is triangular reduced. Then $Q$ is path connected.

**Proof.** Let $Q_0 = \{v_1, v_2, \ldots, v_n\}$. Let $e = v_1$. Then $v_1\Lambda(\sum_{i=2}^n v_i) \neq (0)$. Hence there is an arrow $v_1 \to v_{i_2}$ for $v_{i_2} \neq v_1$. Let $i_1 = 1$. Now let $e = v_{i_1} + v_{i_2}$. Again using that $e\Lambda(1-e) \neq (0)$, we infer that there is an arrow $v_{i_1} \to v_{i_3}$ or $v_{i_2} \to v_{i_3}$ for some $v_{i_3} \not\in \{v_{i_1}, v_{i_2}\}$. We can continue this process and get subsets $V_1 = \{v_{i_1}, v_{i_2}, \ldots, v_{i_t}\}$ in $Q_0$ and a vertex $v_{i_{t+1}} \not\in V_t$ with an arrow from some $v_{i_j} \to v_{i_{j+1}}$ for $j \in \{1, 2, \ldots, t\}$. We can only continue this construction until we reach $V_n$, since then $e = \sum_{j=1}^n v_{i_j} = 1$ and $1-e = 0$. In other words, there is a path from the vertex $v_1$ to any other vertex different from $v_1$ in $Q$. Since $v_1$ can be chosen to be any vertex in $Q$, the quiver $Q$ is path connected. \qed

A concept of a homological heart of a quotient of a path algebra is introduced in [10], which we recall and discuss next. As above let $\Lambda = kQ/I$ be an admissible quotient of the path algebra $kQ$ over a field $k$. The homological heart of $Q$ is given as follows: Let

$$X = \{v \in Q_0 \mid v \text{ is a vertex on a non-trivial oriented cycle in } Q\}$$

and let

$$Y = \{v \in Q_0 \mid y \text{ is a vertex on a path starting and ending in } X\}.$$

Then the homological heart $H(Q)$ of $Q$ is the subquiver of $Q$ with vertex set $Y$.

In order to discuss the properties of the homological heart of a quiver, we need to introduce the following. Let $\Gamma$ be a full subquiver of $Q$. Define the following three full subquivers of $Q$ by their corresponding vertex sets:

$$\Gamma_0^+ = \{v \in Q_0 \mid v \not\in \Gamma_0, \exists \text{ path } \Gamma_0 \leadsto v\}$$

$$\Gamma_0^- = \{v \in Q_0 \mid v \not\in \Gamma_0, \exists \text{ path } v \leadsto \Gamma_0\}$$

$$\Gamma_0^0 = \{v \in Q_0 \mid v \not\in \Gamma_0, \not\exists \text{ path } \Gamma_0 \leadsto v \text{ and } \not\exists \text{ path } v \leadsto \Gamma_0\}$$

Let $e^+(\Gamma)$, $e^-\Gamma)$ and $e^0(\Gamma)$ be the sum of all the vertices in $\Gamma^+$, $\Gamma^-$ and $\Gamma^0$, respectively.

The homological heart $H = H(Q)$ is a full subquiver. Let $e^+$, $e^-$ and $e^0$ the corresponding idempotents defined above for $\Gamma = H$ and $e$ the sum of the vertices in $H$. Then the following is proved in [10, Theorem 5.9 (2)].

**Theorem 5.4** ([10, Theorem 5.9 (2)]). Let $\Lambda = kQ/I$ be an admissible quotient of the path algebra $kQ$ over a field $k$ with homological heart $H = H(Q)$ and $e$ the sum of the vertices in $H$. Then $\text{fin.dim}\, \Lambda < \infty$ if and only if $\text{fin.dim}\, e\Lambda e < \infty$.

We want to show that this result can be obtained from the reduction techniques presented in this paper. Let

$$\sum_{\lambda} 1_{\lambda} = e^+ + e^0 + e + e^-,$$
then
\[ \Lambda \simeq \begin{pmatrix} e^+\Lambda e^+ & e^+\Lambda e^o & e^+\Lambda e & e^+\Lambda e^- \\ e^o\Lambda e^+ & e^o\Lambda e^o & e^o\Lambda e & e^o\Lambda e^- \\ e\Lambda e^+ & e\Lambda e^o & e\Lambda e & e\Lambda e^- \\ e^-\Lambda e^+ & e^-\Lambda e^o & e^-\Lambda e & e^-\Lambda e^- \end{pmatrix} \]

Here
\[ e^+\Lambda e = e^+\Lambda e^- = e^o\Lambda e = e^o\Lambda e^- = e^+\Lambda e^o = e\Lambda e^- = e\Lambda e^o = (0). \]

Hence
\[ \Lambda \simeq \begin{pmatrix} e^+\Lambda e^+ & 0 & 0 & 0 \\ e^o\Lambda e^+ & e^o\Lambda e^o & 0 & 0 \\ e\Lambda e^+ & 0 & e\Lambda e & 0 \\ e^-\Lambda e^+ & e^-\Lambda e^o & e^-\Lambda e & e^-\Lambda e^- \end{pmatrix} \]

By [10, Proposition 5.1 (4)] the full subquiver with vertex set \( H_0^+ \cup H_0^- \cup H_0^0 \) has no oriented cycles, therefore the algebras \( \begin{pmatrix} e^+\Lambda e^+ & e^o\Lambda e^o \\ e^-\Lambda e^+ & e^-\Lambda e^- \end{pmatrix} \) and \( e\Lambda e^- \) have finite global dimension. Iterated use of Theorem 5.2 show that if \( \text{fin. dim } e\Lambda e < \infty \), then \( \text{fin. dim } \Lambda < \infty \).

Assume now conversely that \( \text{fin. dim } \Lambda < \infty \) and consider the idempotent element of \( \Lambda \):
\[ f = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \]

Then by Example 3.3 we have the following recollement of module categories:
\[ \text{mod-}A \xrightarrow{i} \text{mod-}e\Lambda e^- \xrightarrow{f=f(-)} \text{mod-}e^-\Lambda e^- \]

where
\[ A = \begin{pmatrix} e^+\Lambda e^+ & 0 & 0 \\ e^o\Lambda e^+ & e^o\Lambda e^o & 0 \\ e\Lambda e^+ & 0 & e\Lambda e \end{pmatrix}. \]

Since \( \Lambda \) is a triangular matrix algebra, it follows by [16, Theorem 3.9] that the functor \( i \) is a homological embedding. Since \( \text{gl. dim } e^-\Lambda e^- < \infty \), by [16, Theorem 7.2 (ii)] we get a lifting of the above recollement to a recollement situation at the level of bounded derived categories as follows:
\[ \text{D}^b(\text{mod-}A) \xrightarrow{i} \text{D}^b(\text{mod-}\Lambda) \xrightarrow{f} \text{D}^b(\text{mod-}e^-\Lambda e^-) \]

By Theorem 5.1 it follows that \( \text{fin. dim } \Lambda < \infty \). Then, as above, the module category of \( A \) admits the following recollement situation:
\[ \text{mod-}A/\text{Af}^\prime A \xrightarrow{f(-)} \text{mod-}A/\text{f}^\prime A\text{f} \]

where \( f^\prime = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \), \( A/\text{Af}^\prime A \simeq \begin{pmatrix} e^+\Lambda e^+ & e^o\Lambda e^o & 0 \\ e^-\Lambda e^+ & e^-\Lambda e^- & e^-\Lambda e^- \end{pmatrix} \) and \( f^\prime A f^\prime \simeq e\Lambda e \). The left adjoint \( l \) of \( f^\prime(-) \) is an exact functor. Then from [16, Theorem 5.1], or by just using that \( (l, f^\prime(-)) \) is an adjoint pair and \( l \) is exact which preserves projective modules, we get that \( \text{fin. dim } e\Lambda e \leq \text{fin. dim } A \). We infer that \( \text{fin. dim } e\Lambda e < \infty \).
5.2. Vertex removal, projective dimension at most 1. Let $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ be a recollement of abelian categories. Recall that
\[
\text{pgl. dim}_{\mathcal{A}} \mathcal{A} = \{ \text{pd}_{\mathcal{A}} i(A) \mid A \in \mathcal{A} \}
\]
denotes the $\mathcal{A}$-relative projective global dimension of $\mathcal{B}$. If $\text{pgl. dim}_{\mathcal{A}} \mathcal{A} \leq 1$ we show that $\text{Fin. dim } \mathcal{B}$ is finite if and only if $\text{Fin. dim } \mathcal{C}$ is finite. Since $\mathcal{C} \simeq \mathcal{B}/\mathcal{A}$, we can interpret the result as follows: We can remove $\mathcal{A}$ from $\mathcal{B}$ and not lose any information about the finiteness of the finitistic dimension. For an algebra $\Lambda$ it means that $\Lambda$ and $e\Lambda e$ has mutually finite finitistic dimensions for an idempotent $e$ whenever $\text{pd}_{\Lambda}(1-e)\Lambda/(1-e)r \leq 1$. If $\Lambda$ is a quotient of a path algebra and $e$ is a sum of vertices, then the vertices in the quiver of $e\Lambda e$ correspond to the ones occurring in $e$, that is, the vertices occurring in $1-e$ are removed. This is why we call the transition from $\Lambda$ to $e\Lambda e$ vertex removal.

We start with the general situation of a recollement of abelian categories.

**Theorem 5.5.** Let $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ be a recollement of abelian categories. Assume that $\text{pgl. dim}_{\mathcal{A}} \mathcal{A} \leq 1$. Then the following hold:

\[\max\{\text{Fin. dim } \mathcal{A}, \text{Fin. dim } \mathcal{C}\} \leq \text{Fin. dim } \mathcal{B} \leq \text{Fin. dim } \mathcal{C} + 2.\]

In particular, we have that $\text{Fin. dim } \mathcal{B} < \infty$ if and only if $\text{Fin. dim } \mathcal{C} < \infty$.

**Proof.** Since $\text{pgl. dim}_{\mathcal{A}} \mathcal{A} \leq 1$, it follows from Proposition 3.3 (iv) that the quotient functor $e: \mathcal{B} \to \mathcal{C}$ preserves projective objects. Then from [16, Theorem 5.5] (ii) it follows that $\text{Fin. dim } \mathcal{B} \leq \text{Fin. dim } \mathcal{C} + 2$. Also from [16, Theorem 5.5] (i) we have $\text{Fin. dim } \mathcal{A} \leq \text{Fin. dim } \mathcal{C}$. It remains to show that $\text{Fin. dim } \mathcal{C} \leq \text{Fin. dim } \mathcal{B}$.

Let $C$ be an object in $\mathcal{C}$ of finite projective dimension. Suppose that $\text{Fin. dim } \mathcal{B} = m < \infty$. By the formula $\text{pd}_{\mathcal{B}} l(C) \leq \text{pd}_{\mathcal{C}} C + \text{pgl. dim}_{\mathcal{A}} \mathcal{A} + 1$, which is the dual of formula 5.3 in the proof of Lemma 3.3, we have that $\text{pd}_{\mathcal{C}} l(C) \leq \text{pd}_{\mathcal{C}} C + 2$ and therefore $\text{pd}_{\mathcal{C}} l(C) \leq m$. Thus we have an exact sequence

\[0 \to P_m \to \cdots \to P_0 \to l(C) \to 0\]

with $P_i$ in $\text{Proj } \mathcal{B}$. Applying the functor $e$ and using Proposition 3.2 (iv) we get the exact sequence

\[0 \to e(P_m) \to \cdots \to e(P_0) \to C \to 0\]

where each $e(P_i)$ lies in $\text{Proj } \mathcal{C}$. We infer that $\text{pd}_{\mathcal{C}} C \leq m = \text{Fin. dim } \mathcal{B}$ and this completes the proof.

**Remark 5.6.** The inequality of Theorem 5.5 was proved in [16, Proposition 4.15] to hold for the global dimension. Note also that the upper bound of Theorem 5.5 generalizes [21, Proposition 2.1] from basic left Artin rings to the setting of abstract abelian categories, thus to any recollement of module categories with $R/ReR$-relative projective global dimension $\text{pgl. dim}_{R/ReR} R \leq 1$.

Let $\Lambda$ be an artin algebra with an idempotent $e$. As in Example 3.3 it gives rise to the recollement $(\text{mod-}\Lambda/\Lambda e\Lambda, \text{mod-}\Lambda, \text{mod-}e\Lambda e)$, where $\Lambda/e\Lambda$ is also an artin algebra and therefore all modules are filtered in semisimple modules. Then $(1-e)\Lambda/(1-e)r$ is right $\Lambda/e\Lambda$-module. In fact it is semisimple, where all simple $\Lambda/e\Lambda$-modules occur as a direct summand. Then $\text{pgl. dim}_{\text{mod-}e\Lambda e} \mod-\Lambda \leq 1$ if and only if $\text{pd}_{\Lambda}(1-e)\Lambda/(1-e)r \leq 1$. Using this we have the following immediate consequence of the above.

**Corollary 5.7.** Let $\Lambda$ be an artin algebra with an idempotent $e$. Assume that $\text{pd}_{\Lambda}(1-e)\Lambda/(1-e)\text{rad}(\Lambda) \leq 1$. Then $\text{fin. dim } \Lambda < \infty$ if and only if $\text{fin. dim } e\Lambda e < \infty$. 

5.3. Vertex removal, finite injective dimension. Let \((\mathcal{A}, \mathcal{B}, \mathcal{C})\) be a recollement of abelian categories with enough injectives and projectives. Recall that

\[ \text{igl.} \dim_{\mathcal{A}} \mathcal{A} = \{ \text{id}_{\mathcal{A}}(A) \mid A \in \mathcal{A} \} \]

denotes the \(\mathcal{A}\)-relative injective global dimension of \(\mathcal{B}\). If \(\text{igl.} \dim_{\mathcal{A}} \mathcal{A} \leq 1\), the same statement as in Theorem 5.2 is true, namely \(\text{Fin. dim } \mathcal{B}\) is finite if and only if \(\text{Fin. dim } \mathcal{C}\) is finite, see Remark 5.9 below. This has the same translation for an artin algebra \(\Lambda\) as in Corollary 5.7, namely through the condition \(\text{id}_{\Lambda}(1 - e)\Lambda/(1 - e)e < \infty\).

Next we prove the main result of this subsection which gives an upper bound for the finitistic dimension of \(\mathcal{B}\) using the finiteness of the \(\mathcal{A}\)-relative injective global dimension of \(\mathcal{B}\). This is a general version of Theorem B presented in the Introduction.

**Theorem 5.8.** Let \((\mathcal{A}, \mathcal{B}, \mathcal{C})\) be a recollement of abelian categories such that \(\mathcal{A}\) is a finite length category and that \(\mathcal{B}\) has projective covers. Then we have

\[ \text{Fin. dim } \mathcal{B} \leq \text{Fin. dim } \mathcal{C} + \text{igl. dim}_{\mathcal{A}} \mathcal{A} \]

**Proof.** Assume that \(\text{igl.} \dim_{\mathcal{A}} \mathcal{A} = \sup\{ \text{id}_{\mathcal{A}}(A) \mid A \in \mathcal{A} \} = t < \infty\). Let \(X\) be an object in \(\mathcal{B}\) of finite projective dimension. Then from Lemma 5.8 there is a finite projective resolution as follows:

\[ \cdots \to \text{l}(Q_1) \to \text{l}(Q_0) \to P_t \to \cdots \to P_0 \to X \to 0 \quad (5.1) \]

with \(Q_1\) and \(Q_0\) in \(\text{Proj } \mathcal{C}\). Applying the functor \(e: \mathcal{B} \to \mathcal{C}\) to (5.1) and using Proposition 3.2 (iv), we obtain the exact sequence

\[ \cdots \to Q_1 \to Q_0 \to e(\Omega^t(X)) \to 0 \]

which is a finite projective resolution of \(e(\Omega^t(X))\). This implies that \(\text{pd}_{\mathcal{C}} e(\Omega^t(X)) \leq \text{Fin. dim } \mathcal{C}\). Hence, the length of the resolution (5.1) is bounded by \(\text{Fin. dim } \mathcal{C} + t\).

We conclude that \(\text{pd}_{\mathcal{C}} X \leq \text{Fin. dim } \mathcal{C} + t\), and the claim follows. \(\square\)

**Remark 5.9.** It is natural to ask if in Theorem 5.8 we can get a lower bound for \(\text{Fin. dim } \mathcal{B}\). We show that this is indeed the case if we assume that \(\text{igl.} \dim_{\mathcal{A}} \mathcal{A} \leq 1\). Assume that \(\text{Fin. dim } \mathcal{B} = m < \infty\) and let \(C\) be an object of \(\mathcal{C}\) of finite projective dimension. Since the injective relative dimension \(\text{igl.} \dim_{\mathcal{A}} \mathcal{A} \leq 1\), it follows from the dual of Proposition 5.3 that the functor \(i: \mathcal{A} \to \mathcal{B}\) is a homological embedding and the functor \(e: \mathcal{B} \to \mathcal{C}\) preserves injective objects. The latter condition is equivalent to the functor \(i: \mathcal{C} \to \mathcal{B}\) being exact. Since \(i\) is exact and preserves projectives, we get that \(\text{pd}_{\mathcal{A}} C = \text{pd}_{\mathcal{A}} i(C) \leq \text{Fin. dim } \mathcal{B} = m < \infty\). This implies that \(\text{Fin. dim } \mathcal{C} \leq \text{Fin. dim } \mathcal{B}\) and so we are done.

We close this subsection with the following consequence of Theorem 5.8 for Artin algebras.

**Corollary 5.10.** (Solberg [22]) Let \(\Lambda\) be an Artin algebra and \(e\) an idempotent element. Then

\[ \text{fin. dim } \Lambda \leq \text{fin. dim } e\Lambda e + \sup \{ \text{id}_{\Lambda} S \mid S \text{ simple } \Lambda/e\Lambda-\text{module} \} \]

5.4. Vertex removal, eventually homological isomorphism. Let \((\mathcal{A}, \mathcal{B}, \mathcal{C})\) be a recollement of abelian categories with enough injectives and projectives. Recall that the functor \(e: \mathcal{B} \to \mathcal{C}\) is a \(t\)-eventually homological isomorphism for some \(t\) if (a) \(\text{igl.} \dim_{\mathcal{A}} \mathcal{A} < \infty\) and (b) \(\sup \{ \text{pd}_{\mathcal{C}} e(P) \mid P \in \text{Proj } \mathcal{B} \} < \infty\), see Proposition 5.11. We see that this includes the condition \(\text{igl.} \dim_{\mathcal{A}} \mathcal{A} < \infty\) from the previous subsection. Hence if \(e: \mathcal{B} \to \mathcal{C}\) is a \(t\)-eventually homological isomorphism, there should be an even closer relationship between finitistic dimension of \(\mathcal{B}\) and \(\mathcal{C}\). This is shown in the next result.
Theorem 5.11. Let \( \mathcal{A}, \mathcal{B}, \mathcal{C} \) be a recollement of abelian categories such that \( \mathcal{A} \) is a finite length category and \( \mathcal{B} \) has projective covers. Assume that the functor \( e: \mathcal{B} \rightarrow \mathcal{C} \) is a \( t \)-eventually homological isomorphism. Then the following statements hold.

(i) \( \text{Fin.dim} \, \mathcal{C} \leq \max\{\text{Fin.dim} \, \mathcal{B}, t\} \).

(ii) \( \text{Fin.dim} \, \mathcal{B} \leq \max\{\text{Fin.dim} \, \mathcal{C}, t\} \).

In particular, \( \text{Fin.dim} \, \mathcal{B} \) is finite if and only if \( \text{Fin.dim} \, \mathcal{C} \) is finite.

Proof. By assumption there is an isomorphism \( \text{Ext}^n_{\mathcal{A}}(B, B') \simeq \text{Ext}^n_{\mathcal{C}}(e(B), e(B')) \)
for every \( n > t \) and for all objects \( B, B' \) in \( \mathcal{B} \).

(i) Assume that the finitistic dimension of \( \mathcal{B} \) is finite. Let \( Y \) an object in \( \mathcal{C} \) of finite projective dimension. By Proposition 5.1 and Proposition 5.11 we have

\[
\text{Ext}^n_{\mathcal{A}}(l(Y), B') \simeq \text{Ext}^n_{\mathcal{C}}(Y, e(B'))
\]

for all integers \( n > t \) and for all objects \( B' \) in \( \mathcal{B} \). Since the functor \( e \) is essentially surjective by Proposition 5.12 (iii), it follows from (5.2) that \( \text{pd}_{\mathcal{A}} l(Y) < \infty \) and in particular \( \text{pd}_{\mathcal{A}} l(Y) \leq \text{Fin.dim} \, \mathcal{B} \) From the isomorphism (5.2) it follows that

\[
\text{pd}_{\mathcal{A}} Y \leq \max\{\text{Fin.dim} \, \mathcal{B}, t\}.
\]

We infer that \( \text{Fin.dim} \, \mathcal{B} \leq \max\{\text{Fin.dim} \, \mathcal{C}, t\} \).

(ii) Assume that the finitistic dimension of \( \mathcal{C} \) is finite. Let \( B \) be an object in \( \mathcal{B} \) of finite projective dimension. Then for any object \( B' \) in \( \mathcal{B} \) we have the isomorphism \( \text{Ext}^n_{\mathcal{B}}(B, B') \simeq \text{Ext}^n_{\mathcal{C}}(e(B), e(B')) \) for every \( n > t \). As above, we obtain that \( \text{pd}_{\mathcal{B}} e(B) \leq \text{Fin.dim} \, \mathcal{C} \) and therefore

\[
\text{pd}_{\mathcal{A}} B \leq \max\{\text{Fin.dim} \, \mathcal{C}, t\}
\]

Hence, we conclude that \( \text{Fin.dim} \, \mathcal{B} \leq \max\{\text{Fin.dim} \, \mathcal{C}, t\} \).

The last claim follows directly from (i) and (ii).

As a consequence of Theorem 5.11 we have the following result for Artin algebras.

Corollary 5.12. Let \( \Lambda \) be an artin algebra with an idempotent \( e \). Assume that the functor \( e: \text{mod-\Lambda} \rightarrow \text{mod-\Lambda}e \) is a \( t \)-eventually homological isomorphism. Then the following statements hold.

(i) \( \text{fin.dim} \, e\Lambda e \leq \max\{\text{fin.dim} \, \Lambda, t\} \).

(ii) \( \text{fin.dim} \, \Lambda \leq \max\{\text{fin.dim} \, e\Lambda e, t\} \).

In particular, \( \text{fin.dim} \, \Lambda \) is finite if and only if \( \text{fin.dim} \, e\Lambda e \) is finite.

Using Corollary 5.12 we can reprove Corollary 5.7 in the following way. Assume that \( \text{pd}_{\Lambda}(1-e)\Lambda/(1-e)\text{rad}(\Lambda) \leq 1 \). By Proposition 5.5 the inclusion functor inc: \( \text{mod-\Lambda}/\text{e\Lambda} \rightarrow \text{mod-\Lambda} \) is a homological embedding and the quotient functor \( e: \text{mod-\Lambda} \rightarrow \text{mod-e\Lambda} \) preserves projectives. Hence, from Proposition 5.11 we have the properties \( \beta \) and \( \gamma \) but in general this is not enough to obtain an eventually homological isomorphism. However, it follows from Lemma 8.9 and Corollary 8.8 in [19] that the functor \( e \) is an eventually homological isomorphism and therefore we can apply Corollary 5.12.

Remark 5.13. All the reductions that we have discussed give at least one thing, depending on your point of view. If you believe the finitistic dimension conjecture is true, then you “only” need to prove it for \( \Lambda = kQ/I \) where \( Q \) is path connected, all simple modules have infinite injective dimension and projective dimension at least 2, and all arrows occur in a given minimal generating set for the relations. If you believe the finitistic dimension conjecture is false, then you “only” need to search for/find a counter example with all the properties just given.
6. Examples

This section is devoted to giving examples illustrating the reduction techniques
we have discussed in the previous sections. In this context we introduce the follow-
ing notion.

**Definition 6.1.** An algebra in called **reduced** if

(a) all arrows occur in some relation in a minimal set of relations,
(b) all simple modules have infinite injective dimension and projective dimension
   at least 2,
(c) no triangular reductions are possible.

The first two examples are not possible to reduce using the earlier known tech-
niques with vertex reduction for vertices corresponding to simple modules of pro-
jective dimension at most 1 or triangular reduction. In these examples one must
use the new techniques with vertex reduction for vertices corresponding to simple
modules of finite injective dimension or arrow removal, respectively. Reduced al-
gebras $\Lambda_n$ for all positive integers $n$ are constructed with the finitistic dimension
equal to $n$. This shows that the finitistic dimension can be arbitrary for a reduced
algebra. Then, all the rest of the examples are from the existing literature, and
they are shown to be reducible using our techniques. In most cases a bound for the
finitistic dimension of the algebra is given.

In all the examples the algebras $\Lambda$ are given by quivers and relations. Then
denote by $S_i$ the simple module associated to vertex number $i$ and by $e_i$ the cor-
responding primitive idempotent. Furthermore, $P_i$ denotes the indecomposable
projective modules $e_i\Lambda$.

First we give two examples which can only be reduced by applying the new
reduction techniques introduced in this paper, namely vertex removal corresponding
to a simple module with finite injective dimension and arrow removal.

**Example 6.2.** Let $\Lambda$ be given as $k\Gamma/I$ for a field $k$, where $\Gamma$ is the quiver

\[
\begin{array}{ccccccc}
 & & 1 & & 2 & & 3 \\
 & k & & a & & b & \\
 & & & & c & & d \\
 & j & i & 4 & & e & \\
 & & & & & f & \\
 & 5 & & & & h & \\
 & & & & & & g \\
7 & & & & & & 6
\end{array}
\]

and $I = \langle ac - bd, bc, cf, df - eg, fh, egb, ghi, hik, ij, ikb, j^2, jkb, ka, kbd \rangle$. One can show that (i) all simple $\Lambda$-modules have infinite injective dimension, except $S_7$, which has injective dimension 3, (ii) all simple $\Lambda$-modules have (here in fact infinite) projective dimension at least 2, (iii) no triangular reductions are possible and (iv) no arrow removal is possible. Therefore the only available reduction is vertex removal corresponding to vertex 7, where the simple module has injective dimension 3.

The finitistic dimension is $\Lambda$ is 2, while the finitistic dimension of $\Lambda^{op}$ is 4.

In the next example only arrow removal is possible.
Example 6.3. Let $\Lambda$ be given as $k\Gamma/I$ for a field $k$, where $\Gamma$ is

$\begin{array}{c}
6 \quad g \\
5 \quad f_1 \\
1 \\
3 \\
2 \\
4 \\
\end{array}$

and $I = \langle ac - bd, ce f_1, dc, ef_1 g, f_1 gb, ga \rangle$. Then all the simple modules are either $\Omega$-periodic ($\Omega^{-1}$-periodic) or eventually $\Omega$-periodic ($\Omega^{-1}$-periodic) of periode 11. Hence all the simple modules have infinite injective and infinite projective dimension and the algebra is reduced with respect to vertex removal. Furthermore, the algebra is triangular reduced. Therefore the only possible reduction we can perform is arrow removal as $f_2$ is not occurring in any relations.

By Theorem 4.7 the inequality $\text{fin. dim } \Lambda \leq \max\{\text{fin. dim } \Lambda/(f_2)\}$ is true. Since $\text{fin. dim } \Lambda/(f_2) = 1$, it follows that $\text{fin. dim } \Lambda \leq 1$.

Next we construct algebras $\Lambda_n$ for all positive integers $n$, which are reduced with $\text{fin. dim } \Lambda_n = n$. Hence, a reduced algebra can have arbitrarily large finitistic dimension.

Example 6.4. Let $\Lambda_n$ be given as $k\Gamma_n/I_n$, where $\Gamma_n$ is given by

and the ideal $I_n$ in $k\Gamma_n$ is generated by $\{a_i c_i - b_i d_i\}^n_{i=1}$, $\{b_i b_{i+1}\}^n_{i=1}$, $\{c_i c_{i+1}\}^n_{i=1}$ and $\{c_ne, d_ne, ea_1, eb_1\}$. All the algebras $\Lambda_n$ are reduced, Koszul and of finite representation type. Since $\text{fin. dim } \Lambda_n = n$, a reduced algebra can have arbitrarily large finitistic dimension.

All the remaining examples are taken from the existing literature. We shall see that all of them can be reduced using one of our reduction techniques.
Example 6.5. This example is Example 1 from [24]. Define $\Lambda$ by the following quiver and relations over a field $k$.

\[
\Lambda = k \left( \begin{array}{c}
\alpha \\
2 \\
\gamma \\
\beta \\
\eta \\
\delta \\
\xi \\
3 \\
4 \\
5 \\
\end{array} \right) / \langle \alpha^3, \alpha \delta, \beta \delta, \eta \xi - \gamma \delta \rangle
\]

The injective and the projective dimensions of the simple $\Lambda$-modules are given as follows.

| $S_1$ | $S_2$ | $S_3$ | $S_4$ | $S_5$ |
|-------|-------|-------|-------|-------|
| pd    | 2     | $\infty$ | 1     | 0     | 1     |
| id    | 0     | $\infty$ | 1     | $\infty$ | 0     |

Using that the projective dimension of the simple modules $\{S_3, S_4, S_5\}$ are at most 1, we can remove the vertices $\{3, 4, 5\}$ and obtain the algebra

\[
\Lambda_1 = k \left( \begin{array}{c}
\alpha \\
2 \\
\gamma \\
\beta \\
\eta \\
\delta \\
\psi \\
3 \\
4 \\
\xi \\
\end{array} \right) / \langle \alpha^3 \rangle.
\]

Which can be further reduced to the local algebra $e_2 \Lambda_1 e_2$, since $S_1$ is injective.

Alternatively, using that the injective dimension of the simple modules $\{S_3, S_4, S_5\}$ are at most 1 in the original algebra, the vertices $\{1, 3, 5\}$ can be removed and we obtain the algebra

\[
\Lambda_2 = k \left( \begin{array}{c}
\alpha \\
2 \\
\delta \\
4 \\
\xi \\
\end{array} \right) / \langle \alpha^3, \alpha \delta \rangle.
\]

Which can be further reduced to the local algebra $e_2 \Lambda_2 e_2$, since $S_4$ is projective.

Independent of which reduction we carry out, the original algebra has indeed finite finitistic dimension. In addition, whatever order we carry out the reductions, we end up with the same algebra up to isomorphism.

Using Theorem 5.5 and Theorem 5.8 we obtain that $\text{fin. dim } \Lambda \leq 3$.

In addition, applying triangular reduction with $e = e_3 + e_4 + e_5$ we obtain

\[
\text{fin. dim } \Lambda \leq 1 + \text{fin. dim } e \Lambda e + \text{fin. dim } (1 - e) \Lambda (1 - e)
\]

where $(1 - e) \Lambda (1 - e) = \Lambda_1$ and $e \Lambda e$ is hereditary. We can again reduce $\Lambda_1$ by triangular reduction and get $\text{fin. dim } \Lambda_1 \leq 1 + \text{fin. dim } k + \text{fin. dim } k[x]/\langle x^3 \rangle = 1$.

Collecting these observations we obtain, as above, $\text{fin. dim } \Lambda \leq 3$.

Example 2 from [25] is very similar, and it can be reduced in a similar fashion. Example 2 from [26] is different, but it admits similar reductions.

Example 6.6. This is Example 1 from [25]. Define $\Lambda$ by the following quiver and relations over a field $k$.

\[
\Lambda = k \left( \begin{array}{c}
\tau \\
2 \\
\alpha \\
\beta \\
\gamma \\
\delta \\
\phi \\
\xi \\
\eta \\
\psi \\
\xi \\
\phi \\
\gamma \\
\psi \\
\end{array} \right) / \langle \delta \beta - \eta \gamma, \varphi \delta, \varphi \eta, \varphi \psi, \epsilon^2, \psi \epsilon, \varphi \alpha, \varphi \xi, \beta \eta, \beta \psi, \beta \delta, \gamma \delta, \gamma \psi, \gamma \eta, \tau^2, \alpha \tau, \xi \tau \rangle
\]

Here all simple modules have infinite injective dimension and projective dimension at least 2, and all arrows occur in some generator of a minimal set of generators for
the relations. But $e_2\Lambda(1-e_2) = (0)$, so that we can perform a triangular reduction and obtain

$\text{fin. dim } \Lambda \leq 1 + \text{fin. dim } e_2\Lambda e_2 + \text{fin. dim } (1-e_2)\Lambda(1-e_2) = 1 + \text{fin. dim } (1-e_2)\Lambda(1-e_2)$,

since $e_2\Lambda e_2$ is a local algebra. The algebra $(1-e_2)\Lambda(1-e_2)$ is reduced.

**Example 6.7.** This example is Example 4.4 from [7]. Define $\Lambda$ by the following quiver and relations over a field $k$.

$$\Lambda = k\begin{pmatrix} 1 \alpha \rightarrow \gamma \beta \rightarrow \delta \gamma \rightarrow \beta \gamma \rightarrow \epsilon \gamma \\ \beta \rightarrow \gamma \rightarrow \epsilon \rightarrow \gamma \rightarrow \epsilon \end{pmatrix}/\langle \alpha \delta, \alpha \gamma, \beta \epsilon \delta \epsilon, \gamma \alpha - \delta \epsilon, \gamma \beta, \delta \epsilon \delta, \epsilon \gamma \beta \rangle$$

This algebra is triangular reduced, so the only possible reductions are vertex removal or arrow removal. All arrows occur in a minimal set of relations, so this only leaves us with vertex removal reduction.

The injective and the projective dimensions of the simple $\Lambda$-modules are given as follows.

|       | $S_1$ | $S_2$ | $S_3$ |
|-------|-------|-------|-------|
| pd    | $\infty$ | $\infty$ | 1     |
| id    | $\infty$ | 3     | $\infty$ |

Using that the projective dimension of the simple module $S_3$ is 1, the vertex 3 can be removed to obtain the algebra

$$\Lambda_1 = k\begin{pmatrix} 1 \alpha \rightarrow \beta \rightarrow \gamma \rightarrow \delta \gamma \rightarrow \beta \gamma \rightarrow \epsilon \gamma \\ \beta \rightarrow \gamma \rightarrow \epsilon \rightarrow \gamma \rightarrow \epsilon \end{pmatrix}/\langle a_\alpha a_\gamma, a_\gamma a_\beta \delta \epsilon, a_\beta \delta a_\gamma \rangle.$$

The indices $\sigma$ on $a_\sigma$ for the arrows in the quiver of $\Lambda_1$ and later $\Lambda_2$ refer to which basis elements in $\Lambda$ they correspond to. This is the opposite of the algebra in Example 3 in [25], which originally appeared in [13]. The algebra $\Lambda_1$ cannot be reduced further.

Alternatively, using that the injective dimension of the simple $\Lambda$-module $S_2$ is 3, the vertex 2 can be removed to obtain the algebra

$$\Lambda_2 = k\begin{pmatrix} 1 \alpha \rightarrow \beta \rightarrow \gamma \rightarrow \delta \gamma \rightarrow \beta \gamma \rightarrow \epsilon \gamma \\ \beta \rightarrow \gamma \rightarrow \epsilon \rightarrow \gamma \rightarrow \epsilon \end{pmatrix}/\langle a_\alpha a_\gamma, a_\gamma a_\beta \delta \epsilon, a_\beta \delta a_\gamma \rangle.$$

The algebra $\Lambda_2$ cannot be reduced further. Hence, different paths of reduction to a reduced algebra, do not give a unique algebra up to isomorphism.

As $\Lambda_1$ and also $\Lambda_2$ are monomial algebras, it follows easily from each of the different reductions that $\Lambda$ has finite finitistic dimension. One can show that no indecomposable projective module can be a submodule of a finitely generated projective module without being a direct summand. Hence, the finitistic dimension of $\Lambda_2$ is 0, so that $\text{fin. dim } \Lambda \leq 3$ by Corollary 5.10.

Not all examples in the existing literature are reducible, for instance, the examples [23, example at the end of Section 3] and [27, Example 3.1 and 3.2] seem to be reduced.

**Appendix A. Gr"obner basis**

In this appendix we recall some basic facts about Gr"obner basis that we use in Section 4. For further details and a proper introduction to the Gr"obner basis we refer the reader to [9].
Let $Q$ be a quiver and $kQ$ the corresponding path algebra over a field $k$. Then the set $B$ of all the paths in $Q$ is a $k$-basis of the path algebra $kQ$. Gröbner basis theory is based on having a totally ordered basis with special properties, namely an admissible ordering of the basis. For a path algebra $kQ$ we obtain such a basis by giving $B$ the length-left-lexicographic ordering $\succ$ by describing a total order on all vertices and a total order on all arrows with all arrows bigger than any vertex. Then, given any non-zero element

$$r = \sum_{p \in B} a_p p$$

in $kQ$ with almost all $a_p$ in $k$ being zero and $p$ in $B$, we define the tip of $r$ to be the element $\text{Tip}(r) = p$ if $a_p \neq 0$ and $p \succ q$ for all $q$ with $a_q \neq 0$. For any subset $X$ in $kQ$, then the set of tips of $X$ is given as

$$\text{Tip}(X) = \{ \text{Tip}(r) \mid r \in X \setminus \{0\} \}$$

and the set of non-tips of $X$ is given as

$$\text{Nontip}(X) = B \setminus \text{Tip}(X).$$

Having the notions of tips and non-tips of a set give rise to the following fundamental result (see [9, Lemma 5.1] for (i) and [8, Theorem 2.1] for (ii) and (iii)).

**Lemma A.1.** Let $I$ be an ideal in $kQ$ with $\succ$ an admissible ordering of the $k$-basis $B$ consisting of all paths in $Q$.

(i) For two paths $p$ and $p'$ in $Q$ and an element $x$ in $kQ$ such that $pxp'$ is non-zero, the tip of $pxp'$ is

$$\text{Tip}(pxp') = p \text{Tip}(x)p'.$$

(ii) As a $k$-vector space $kQ$ can be decomposed as

$$kQ = I \oplus \text{Span}_k(\text{Nontip}(I)),$$

where $\text{Span}_k(\text{Nontip}(I))$ denotes the $k$-linear span of $\text{Nontip}(I)$.

(iii) As a $k$-vector space $kQ/I$ can be identified with $\text{Span}_k(\text{Nontip}(I))$. In particular, any element $r + I$ in $kQ/I$ can be represent uniquely by $N(r)$ as

$$r + I = N(r) + I,$$

where $N(r)$ is called the normal form of $r$ and $N(r) \in \text{Span}_k(\text{Nontip}(I))$.

A Gröbner basis of the ideal $I$ facilitates a way of computing the normal form of any element in $kQ$. A Gröbner basis of an ideal $I$ in $kQ$ is defined as follows.

**Definition A.2.** Let $I$ be an ideal in $kQ$ with $\succ$ an admissible ordering of the $k$-basis $B$ consisting of all paths in $Q$. If $\mathcal{G}$ is a subset of $I$, then $\mathcal{G}$ is a Gröbner basis for $I$ with respect to $\succ$ if the ideal generated by $\text{Tip}(\mathcal{G})$ equals the ideal generated by $\text{Tip}(I)$.

Having a finite Gröbner basis $\mathcal{G}$ for an ideal $I$ in $kQ$ with an admissible ordering $\succ$, gives a finite algorithm for computing the normal form of any element in $kQ$ by iteratively applying of the reduction described in statement (iii) of the result below.

**Lemma A.3.** Let $I$ be an admissible ideal in $kQ$ with $\succ$ an admissible ordering of the $k$-basis $B$ consisting of all paths in $Q$.

(i) Then there exists a finite Gröbner basis $\mathcal{G}$ of $I$ in $kQ$.

For a Gröbner basis $\mathcal{G}$ of $I$ in $kQ$ and any non-zero element $x$ in $I$, the following hold.

(ii) The element $\text{Tip}(x)$ is in $\text{Tip}(\mathcal{G})$. 

(iii) There exist c in k, paths p and p' in Q and g in $\mathcal{G}$ such that
\[ \text{Tip}(x) = p \text{Tip}(g)p' \]
and
\[ \text{Tip}(x - cpgp') \prec \text{Tip}(x), \]
whenever $x - cpgp' \neq 0$.

For a proof of (i) see [8, Corollary 2.2]. The statement in (ii) is a consequence of the definition of a Gröbner basis, and the statement in (iii) is a consequence of (ii) and the division algorithm in [8, Division algorithm 2.3.2].

From a generating set $\mathcal{F}$ of an ideal in $kQ$ using the Buchberger-algorithm a Gröbner basis $\mathcal{G}$ for $I$ can be constructed (see [8, 2.4.1]). Using this algorithm it is easy to see the following result, which we need in Section 4. Recall that an element $u$ in $kQ$ is called uniform if $u = euf$ for some trivial paths $e$ and $f$ in $kQ$.

Lemma A.4. Let $\mathcal{F}$ be a generating set of uniform elements of an ideal $I$ in $kQ$. If an arrow $a$ does not occur in any path of any element of $\mathcal{F}$, then there is a Gröbner basis $\mathcal{G}$ of $I$ such that $a$ does not occur in any path of any element in $\mathcal{G}$.

The second result we need in Section 4 is an easy consequence of the above.

Lemma A.5. Let $\Lambda = kQ/I$ for an admissible ideal $I$, and let $a$ be an arrow in $Q$ not occurring in any path of any element of a minimal set of generators for $I$. Denote by $Q^*$ the quiver $Q$ with the arrow $a$ removed, and let $I^* = kQ^* \cap I$. Then
\[ \text{Nontip}(I^*) = \text{Nontip}(I) \cap kQ^*. \]

Proof. Let $x$ be in $\text{Nontip}(I^*)$ and assume that $x$ is in $\text{Tip}(I)$. Then $x = p \text{Tip}(g)p'$ for some paths $p$ and $p'$ and $g$ in $\mathcal{G}$. Since $x$ is in $kQ^*$, the paths $p$ and $p'$ are also in $kQ^*$. By Lemma A.4 the Gröbner basis $\mathcal{G}$ for $I$ is also a Gröbner basis for $I^*$. We infer that $x$ is a tip of an element in $I^*$, which is a contradiction, and $x$ is in $\text{Nontip}(I) \cap kQ^*$.

Conversely, assume that $x$ is in $\text{Nontip}(I) \cap kQ^*$. Clearly $x$ is the set $\mathcal{B}^*$ of all paths in $Q^*$ and not in $\text{Tip}(I)$. Since $I^* \subseteq I$, the inclusion $\text{Tip}(I^*) \subseteq \text{Tip}(I)$ holds. If $x$ is in $\text{Tip}(I^*)$, then $x$ is in $\text{Tip}(I)$. This is a contradiction, so that $x$ is in $\text{Nontip}(I^*)$.

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