Acyclic matchings in graphs of bounded maximum degree*

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Abstract

A matching $M$ in a graph $G$ is acyclic if the subgraph of $G$ induced by the set of vertices that are incident to an edge in $M$ is a forest. We prove that every graph with $n$ vertices, maximum degree at most $\Delta$, and no isolated vertex, has an acyclic matching of size at least $(1 - o(1)) \frac{2n}{\Delta^2}$, and we explain how to find such an acyclic matching in polynomial time.

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1 Introduction

We consider simple, finite, and undirected graphs, and use standard terminology. Let $M$ be a matching in a graph $G$, and let $H$ be the subgraph of $G$ induced by the set of vertices that are incident to an edge in $M$. If $H$ is a forest, then $M$ is an acyclic matching in $G$ \[7\], and, if $H$ is 1-regular, then $M$ is an induced matching in $G$ \[14\]. If $\nu(G)$, $\nu_{ac}(G)$, and $\nu_s(G)$ denote the largest size of a matching, an acyclic matching, and an induced matching in $G$, respectively, then, since every induced matching is acyclic, we have

$$\nu(G) \geq \nu_{ac}(G) \geq \nu_s(G).$$

In contrast to the matching number $\nu(G)$, which is a well known classical tractable graph parameter, both, the acyclic matching number $\nu_{ac}(G)$ as well as the induced matching number $\nu_s(G)$ are computationally hard \[3, 7, 13, 14\]. While induced matchings have been studied in great detail, see, in particular, \[8–11\] for lower bounds on $\nu_s(G)$ for graphs $G$ of bounded maximum degree as well as the references therein, only few results are known on the acyclic matching number. While the equality $\nu(G) = \nu_s(G)$ can be decided efficiently for a given graph $G$ \[2, 12\], it is NP-complete to decide whether $\nu(G) = \nu_{ac}(G)$ for a given bipartite graph $G$ of maximum degree at most 4 \[6\], and efficient algorithms computing the acyclic matching number are known only for certain graph classes \[1,4,6,13\].

It is known \[1\] that $\nu_{ac}(G) \geq \frac{m}{2\Delta}$ for a graph $G$ with $m$ edges and maximum degree $\Delta$, which was improved \[5\] to $\frac{m}{6}$ for connected subcubic graphs $G$ of order at least 7. Since, for every $\Delta$-regular graph $G$ with $m$ edges, a simple edge counting argument implies $\nu_{ac}(G) \leq \frac{m-1}{2(\Delta-1)}$, the constructive proofs of these bounds yield an efficient $\frac{\Delta^2}{2(\Delta-1)}$-factor approximation algorithm for $\Delta$-regular graphs, and an efficient $\frac{3}{2}$-factor approximation algorithm for cubic graphs for the maximum acyclic matching problem.

In the present paper we show a lower bound on the acyclic matching number of a graph $G$ with $n$ vertices, maximum degree $\Delta$, and no isolated vertex, which is inspired by a result of Joos \[9\] who proved

$$\nu_s(G) \geq \frac{n}{(\lceil \frac{\Delta}{2} \rceil + 1)(\lfloor \frac{\Delta}{2} \rfloor + 1)} \quad (1)$$

provided that $\Delta \geq 1000$. \[11\] is tight for the graph that arises by attaching $\lfloor \frac{\Delta}{2} \rfloor$ new vertices of degree 1 to every vertex of a complete graph of order $\lceil \frac{\Delta}{2} \rceil + 1$. In view of these graphs, we conjectured \[11\] that twice the right hand side of \[11\] should be the right lower bound on the acyclic matching number of the considered graphs for sufficiently large $\Delta$, that is, we believe that our following main result can be improved by a factor of roughly $\frac{4}{3}$.

**Theorem 1.** If $G$ is a graph with $n$ vertices, maximum degree at most $\Delta$, and no isolated vertex, then

$$\nu_{ac}(G) \geq \frac{6n}{\Delta^2 + 12\Delta^2}.$$ 

Note that, for graphs that are close to $\Delta$-regular, the bound $\nu_{ac}(G) \geq \frac{m}{2\Delta}$ is stronger than Theorem \[11\]. We prove Theorem \[11\] in the next section. In the conclusion we discuss algorithmic aspects of its proof and possible generalizations to so-called degenerate matchings \[11\].
2 Proof of Theorem \[\text{I}\]

We prove the theorem by contradiction. Therefore, suppose that \(G\) is a counterexample of minimum order. Clearly, \(G\) is connected. If \(\Delta = 1\), then \(G\) is \(K_2\), and, hence, \(\nu_{ac}(G) = \frac{n}{2}\). If \(\Delta = 2\), then \(G\) is a path or a cycle, which implies \(\nu_{ac}(G) \geq \frac{n-2}{2}\). These observations imply \(\Delta \geq 3\). At several points within the proof we consider an acyclic matching \(M\) in \(G\), and we consistently use

- \(V_M\) to denote the set of vertices of \(G\) that are incident to an edge in \(M\),
- \(N_M\) to denote the set of vertices in \(V(G) \setminus V_M\) that have a neighbor in \(V_M\),
- \(G_M\) to denote the graph \(G - (V_M \cup N_M)\),
- \(I_M\) to denote the set of isolated vertices of \(G_M\), and
- \(G'_M\) to denote the graph \(G_M - I_M\).

Since \(G'_M\) is no counterexample, and the union of \(M\) with any acyclic matching in \(G'_M\) is an acyclic matching in \(G\), we obtain

\[
\frac{6n}{\Delta^2 + 12 \Delta \frac{\Delta}{2}} > \nu_{ac}(G) \geq |M| + \frac{6(n - |V_M \cup N_M \cup I_M|)}{\Delta^2 + 12 \Delta \frac{\Delta}{2}},
\]

which implies

\[
|V_M| + |N_M| + |I_M| > \left(\frac{\Delta^2}{6} + 2 \Delta \frac{\Delta}{\Delta}\right)|M|. \tag{2}
\]

Claim 1. For every edge \(uv\) in \(G\), we have \(d_G(u) + d_G(v) > 2\sqrt{\Delta}\).

Proof. Suppose, for a contradiction, that \(d_G(u) + d_G(v) \leq 2\sqrt{\Delta}\) for some edge \(uv\) of \(G\). For \(M = \{uv\}\), we obtain \(|V_M| + |N_M| + |I_M| \leq 2 + \left(2\sqrt{\Delta} - 2\right) + \left(2\sqrt{\Delta} - 2\right)(\Delta - 1) \leq 2\Delta^2\), contradicting (2). \(\square\)

Let \(S\) be the set of vertices of degree at most \(\sqrt{\Delta}\). By Claim \(\text{I}\) the set \(S\) is independent.

Claim 2. \(S\) is not empty.

Proof. Suppose, for a contradiction, that the minimum degree \(\delta\) of \(G\) is larger than \(\sqrt{\Delta}\). Let \(uv\) be an edge of \(G\) such that \(u\) is of minimum degree. Let \(M = \{uv\}\). Since every vertex in \(I_M\) has degree at least \(\delta\), we have

\[
|V_M| + |N_M| + |I_M| \leq 2 + (\Delta + \delta - 2) + \frac{(\Delta + \delta - 2)(\Delta - 1)}{\delta} \leq \frac{(\Delta + \delta)^2}{\delta}.
\]

If \(\Delta = 3\), then \(\delta\) is 2 or 3, and in both cases \(2 + (\Delta + \delta - 2) + \frac{(\Delta + \delta - 2)(\Delta - 1)}{\delta}\) is less than the right hand side of (2), contradicting (2). For \(\Delta \geq 4\), we obtain that \(\frac{(\Delta + \delta)^2}{\delta} \leq \frac{(\Delta + \sqrt{\Delta})^2}{\sqrt{\Delta}}\) is less than the right hand side of (2). Hence, also in this case, we obtain a contradiction (2). \(\square\)

Let \(N\) be the set of vertices that have a neighbor in \(S\), and, for a vertex \(v\) in \(G\), let \(d_S(v)\) be the number of neighbors of \(v\) in \(S\). Since \(S\) is independent, the sets \(S\) and \(N\) are disjoint.

Claim 3. \(\max\{d_S(v) : v \in V(G)\} = \alpha \Delta\) for some \(\alpha\) with \(0.2 \leq \alpha \leq 0.8\).

In other words, we have \(d_S(v) \leq 0.8\Delta\) for every vertex \(v\) of \(G\), and \(d_S(v) \geq 0.2\Delta\) for some vertex \(v\) of \(G\).
Proof. Let the vertex \( v \) maximize \( d_S(v) \). Suppose, for a contradiction, that \( d_S(v) = \alpha \Delta \) for some \( \alpha \) with either \( \alpha < 0.2 \) or \( \alpha > 0.8 \). Let \( u \) be a neighbor of \( v \) of minimum degree. By Claim 2 we have \( d_S(v) \geq 1 \), which implies \( d_G(u) \leq \sqrt{\Delta} \). Let \( M = \{uv\} \). Clearly,
\[
|V_M| + |N_M| \leq \sqrt{\Delta} + \Delta.
\]

Let \( I_1 \) be the set of vertices in \( I_M \) that have a neighbor in \( N_G(u) \cup (N_G(v) \cap S) \), let \( I_2 = (I_M \setminus I_1) \cap S \), and let \( I_3 = I_M \setminus (I_1 \cup I_2) \).

We obtain
\[
|I_1| \leq (\Delta - 1)(d_G(u) - 1) + \left(\sqrt{\Delta} - 1\right) |N_G(v) \cap S|
\leq (\Delta - 1) \left(\sqrt{\Delta} - 1\right) + (\sqrt{\Delta} - 1) \alpha \Delta
\leq (1 + \alpha)\Delta^{\frac{3}{2}} - \left(\sqrt{\Delta} + \Delta\right).
\]

Let \( N' = N_G(v) \setminus (N_G(u) \cup S) \). Note that \( |N'| \leq (1 - \alpha)\Delta \), and that the vertices in \( I_2 \cup I_3 \) have all their neighbors in \( N' \). By the choice of \( v \), every vertex in \( N' \) has at most \( \alpha \Delta \) neighbors in \( S \), which implies
\[
|I_2| \leq \alpha \Delta |N'| \leq \alpha (1 - \alpha)\Delta^2.
\]

Since there are at most \( \Delta |N'| \) edges between \( N' \) and \( I_3 \), and every vertex in \( I_3 \) has degree more than \( \sqrt{\Delta} \), we obtain
\[
|I_3| \leq \frac{\Delta |N'|}{\sqrt{\Delta}} \leq (1 - \alpha)\Delta^\frac{3}{2}.
\]

Altogether, we obtain
\[
|V_M| + |N_M| + |I_M| \leq \sqrt{\Delta} + \Delta + (1 + \alpha)\Delta^{\frac{3}{2}} - \left(\sqrt{\Delta} + \Delta\right) + \alpha(1 - \alpha)\Delta^2 + (1 - \alpha)\Delta^\frac{3}{2}
\leq \alpha(1 - \alpha)\Delta^2 + 2\Delta^{\frac{3}{2}}
\leq 0.16\Delta^2 + 2\Delta^{\frac{3}{2}},
\]
contradicting (2).

Note that, so far in the proof of each claim, we had \( |M| = 1 \), and iteratively applying the corresponding reductions would eventually lead to an induced matching in \( G \) similarly as in [9]. In order to improve (1), we now choose \( M \) non-locally in some sense: Let \( M \) be an acyclic matching in \( G \) such that

(i) \( M \) only contains edges incident to a vertex in \( S \),

(ii) every vertex in \( V_M \cap S \) has degree one in the subgraph of \( G \) induced by \( V_M \),

(iii) every vertex \( v \) in \( V_M \cap N \) satisfies \( d_S(v) \geq 0.2\Delta \), and

\( M \) maximizes
\[
\sum_{v \in V_M \cap N} d_S(v).
\]

among all acyclic matchings satisfying (i), (ii), and (iii). By Claim 3 the matching \( M \) is non-empty.

We now define certain relevant sets, see Figure 4 for an illustration.
• Let $X$ be the set of vertices in $N_M$ that are not adjacent to a vertex in $V_M \cap S$ and that have at least one neighbor in $S$ that is not adjacent to a vertex in $V_M$.

(Note that $X \subseteq N$, and that the edges between vertices in $X$ and suitable neighbors in $S$ are possible candidates for modifying $M$.)

• Let $Y$ be the set of vertices in $N_M \setminus X$ that are not adjacent to a vertex in $V_M \cap S$.

(Note that $Y$ contains $N_M \setminus N = (N_M \cap S) \cup (N_M \setminus (S \cup N))$.)

• Let $Z = (N \cap N_M) \setminus (X \cup Y)$.

(Note that $Z$ consists of the vertices in $N_M$ that have a neighbor in $V_M \cap S$.)

• Let $I_1$ be the set of vertices in $I_M \cap S$ that have a neighbor in $N_M \setminus X$.

(Note that, by the definition of $X$, no vertex in $I_1$ can have a neighbor in $Y \cap N$, which implies that every vertex in $I_1$ has a neighbor in $Z$.)

• Let $I_2$ be the set of vertices in $I_M \setminus S$ that have a neighbor in $Z$.

• Let $I_3$ be the set of vertices in $I_M \cap S$ that only have neighbors in $X$.

(Note that $I_1 \cup I_3 = I_M \cap S$.)

• Finally, let $I_4 = I_M \setminus (I_1 \cup I_2 \cup I_3)$.

\[
I_1 \quad I_2 \quad I_3 \quad I_4
\]

\[
S \quad N
\]

\[
M
\]

\[
N_M
\]

\[
X
\]

\[
Y
\]

\[
I_1
\]

\[
I_2
\]

\[
I_3
\]

\[
I_4
\]

Figure 1: An illustration of the different relevant sets.

Clearly,

\[
|V_M| + |N_M| \leq (\sqrt{\Delta} + \Delta) |M|.
\]  

(4)

Since every vertex in $I_1 \cup I_2$ has a neighbor in $Z$, and every vertex in $Z$ has a neighbor in $V_M \cap S$, we have

\[
|I_1 \cup I_2| \leq (\Delta - 1)|Z| \leq (\Delta - 1) \left(\sqrt{\Delta} - 1\right) |M| = \left(\Delta^{\frac{3}{2}} - \Delta - \sqrt{\Delta} + 1\right) |M|.
\]  

(5)
Since every vertex in $I_4$ has degree more than $\sqrt{\Delta}$ and has all its neighbors in $X \cup Y$, and every vertex in $X \cup Y$ has a neighbor in $V_M \cap N$, we have

$$|I_3| \leq \frac{(\Delta - 1)|X \cup Y|}{\sqrt{\Delta}} \leq \frac{(\Delta - 1)^2|M|}{\sqrt{\Delta}} = \left(\Delta^2 - 2\sqrt{\Delta} + \frac{1}{\sqrt{\Delta}}\right)|M|. \quad (6)$$

Combining (4), (5), and (6), we obtain

$$|V_M| + |N_M| + |I_M| - |I_3| \leq 2\Delta^3. \quad (7)$$

In order to estimate $|I_3|$, we partition the set $X$ as follows:

- Let $X_1$ be the set of vertices $v$ in $X$ with $d_S(v) < 0.2\Delta$,
- let $X_2$ be the set of vertices in $X \setminus X_1$ with at least four neighbors in $V_M$, and
- let $X_3 = X \setminus (X_1 \cup X_2)$.

For a vertex $v$ in $V_M \cap N$, let $d_3(v)$ be the number of neighbors of $v$ in $X_3$.

**Claim 4.** $|I_3| \leq 0.2\Delta |X_1| + 0.8\Delta |X_2| + \frac{2}{3} \sum_{v \in V_M \cap N} d_S(v)d_3(v)$.

**Proof.** By Claim 3 we obtain that

$$|I_3| \leq \sum_{w \in X} d_S(w) = \sum_{w \in X_1 \cup X_2 \cup X_3} d_S(w) \leq 0.2\Delta |X_1| + 0.8\Delta |X_2| + \sum_{w \in X_3} d_S(w).$$

Let $w$ be a vertex in $X_3$. By the definition of $X$, the vertex $w$ has a neighbor $u$ in $S$ that is not adjacent to a vertex in $V_M$. If $w$ has only one neighbor in $V_M$, then $M \cup \{wu\}$ is an acyclic matching satisfying (i), (ii), and (iii) that has a larger value in $\mathbb{R}$, contradicting the choice of $M$. Hence, we may assume that $w$ has either $k = 2$ or $k = 3$ neighbors $v_1, \ldots, v_k$ in $V_M$. Let $u_1v_1, \ldots, u_kv_k$ be edges in $M$, and suppose that $d_S(v_1) \leq \ldots \leq d_S(v_k)$. Since

$$M' = (M \cup \{wu\}) \setminus \{u_1v_1, \ldots, u_{k-1}v_{k-1}\}$$

is an acyclic matching satisfying (i), (ii), and (iii), the choice of $M$ implies that the value of $M'$ in $\mathbb{R}$ is at most the one of $M$, which implies

$$d_S(w) \leq \sum_{i=1}^{k-1} d_S(v_i) \leq \frac{k-1}{k} \sum_{i=1}^{k} d_S(v_i) \leq \frac{2}{3} \sum_{i=1}^{k} d_S(v_i).$$

Now, we obtain

$$\sum_{w \in X_3} d_S(w) \leq \frac{2}{3} \sum_{w \in X_3} \sum_{v \in V_M \cap N \setminus N_G(w)} d_S(v) \leq \frac{2}{3} \sum_{v \in V_M \cap N} d_3(v)d_S(v),$$

which completes the proof. \qed

For a vertex $v$ in $V_M \cap N$, let $d_1(v)$ be the number of neighbors of $v$ in $X_1 \cup X_2$. By property (iii), we have $d_S(v) \geq 0.2\Delta$, which implies that $d_1(v) \leq 0.8\Delta$. Using Claim 4 $xy \leq \frac{(x+y)^2}{4}$ for $x, y \geq 0$, and
\[ d_S(v) + d_1(v) + d_3(v) \leq \Delta \text{ and } d_1(v)^2 \leq 0.8\Delta d_1(v) \text{ for } v \in V_M \cap N, \text{ we obtain} \]
\[
|I_3| \leq 0.2\Delta|X_1| + 0.8\Delta|X_2| + \frac{2}{3} \sum_{v \in V_M \cap N} d_S(v) d_3(v) \\
\leq 0.2\Delta(|X_1| + 4|X_2|) + \frac{1}{6} \sum_{v \in V_M \cap N} (d_S(v) + d_3(v))^2 \\
\leq 0.2\Delta \sum_{v \in V_M \cap N} d_1(v) + \frac{1}{6} \sum_{v \in V_M \cap N} (\Delta - d_1(v))^2 \\
= \frac{\Delta^2}{6} |M| + \Delta \left( \frac{1}{5} - \frac{1}{3} \right) \sum_{v \in V_M \cap N} d_1(v) + \frac{1}{6} \sum_{v \in V_M \cap N} d_1(v)^2 \\
\leq \frac{\Delta^2}{6} |M| + \Delta \left( \frac{2}{15} - \frac{2}{15} \right) \sum_{v \in V_M \cap N} d_1(v) \\
= \frac{\Delta^2}{6} |M|,
\]
and together with (7), we obtain a final contradiction to (2) completing the proof. □

3 Conclusion

While the choice of \( M \) after Claim 3 in the proof is non-constructive, the proof of Theorem 1 easily yields an efficient algorithm that returns an acyclic matching in a given input graph \( G \) as considered in Theorem 1 with size at least \( \frac{6m}{\Delta^2 + 12\Delta} \). If the statements of Claims 1, 2, or 3 fail, then their proofs contain simple reduction rules, each fixing one edge in the final acyclic matching and producing a strictly smaller instance \( G'_M \). Adding that fixed edge to the output on the instance \( G'_M \) yields the desired acyclic matching. The matching \( M \) chosen after Claim 3 can be initialized as any acyclic matching satisfying (i), (ii), and (iii). If Claim 4 fails, then its proof contains simple update procedures that increase the value in (3). Since this value is integral and polynomially bounded, after polynomially many updates the statement of Claim 4 holds, and adding \( M \) to the output on the instance \( G'_M \) yields the desired acyclic matching.

The acyclic matchings \( M \) produced by the proof of Theorem 1 actually have a special structure because the subgraph \( H \) of \( G \) induced by the set of vertices that are incident to an edge in \( M \) is not just any forest but a so-called corona of a forest, that is, every vertex \( v \) of \( H \) of degree at least 2 in \( H \) has a unique neighbor \( u \) of degree 1 in \( H \), and all the edges \( uv \) form \( M \).

As a generalization of acyclic matchings, [1] introduced the notion of a \( k \)-degenerate matching as a matching \( M \) in a graph \( G \) such that the subgraph \( H \) of \( G \) defined as above is \( k \)-degenerate. If the \( k \)-degenerate matching number \( \nu_k(G) \) of \( G \) denotes the largest size of a \( k \)-degenerate matching in \( G \), then \( \nu_1(G) \) coincides with the acyclic matching number. We conjecture that

\[
\nu_k(G) \geq \frac{(k + 1)n}{\left(\left\lfloor \frac{n}{2} \right\rfloor + 1\right) \left(\left\lceil \frac{n}{2} \right\rceil + 1\right)}
\]

for every graph \( G \) with \( n \) vertices, sufficiently large maximum degree \( \Delta \), and no isolated vertex. A
straightforward adaptation of the proof of Theorem II yields

$$\frac{\nu_k(G)}{n} \geq \begin{cases} 
(1 - o(1)) \frac{4(k + 3)}{3\Delta^2} & \text{for } k \in \{2, 3, 4, 5, 6\} \text{ and } \\
(1 - o(1)) \frac{k + 4}{\Delta^2} & \text{for } k \geq 7.
\end{cases}$$

for these graphs $G$.

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