Variational thermodynamics of relativistic thin disks

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We present a relativistic model describing a thin disk system composed of two fluids. The system is surrounded by a halo in the presence of a non-trivial electromagnetic field. We show that the model is compatible with the variational multi-fluid thermodynamics formalism, allowing us to determine all the thermodynamic variables associated with the matter content of the disk. The asymptotic behaviour of these quantities indicates that the single fluid interpretation should be abandoned in favour of a two-fluid model.

The problem of finding exact solutions in relativity which are consistently applicable in the context of astrophysics remains a topical problem [1]. Most systems of astrophysical relevance are studied through various assumptions of symmetry. Of special interest, are those which are approximately axially symmetric, such as accretion disks and galaxies in thermodynamic equilibrium. In addition, these systems can also be approximated by a thin disk. The conventional treatment of galaxies modelled as a thin disk has been largely studied using Newtonian dynamics. However, there are only a handful of physical solutions which are mathematically simple and fully relativistic. Moreover, most efforts in the understanding of the physical properties of such objects rely on the input provided through an equation of state. In this work we present a new exact solution for a relativistic thin disk surrounded by an electromagnetic halo. This solution has a number of interesting features. Firstly, it is notably simple in its mathematical form, making it useful for testing various matter models in a straightforward manner. Secondly, it generalises the commonly used pressure free (dust) models to a perfect fluid with non-vanishing pressure, allowing a more detailed physical description [2, 3]. Thirdly, we make a novel analysis by considering a multi-fluid system to describe the thermodynamics associated with the matter content of the solution. In this manner, we use a new thermodynamic criterion to exclude some dynamically unconstrained models in favour of those which are thermodynamically sound. Moreover, the multi-fluid formalism is suitable to extend the analysis beyond equilibrium thermodynamics, providing us with a tool to test new directions in the context of relativistic astrophysics.

To obtain the solution, we solved the distributional Einstein-Maxwell field equations assuming axial symmetry and that the derivatives of the metric and electromagnetic potential across the disk space-like hyper-surface are discontinuous. Here, the energy-momentum tensor is taken to be the sum of two distributional components, the purely electromagnetic part and a ‘material’ part. We solve the Einstein-Maxwell equations in a conformastatic space-time background through the introduction of an auxiliary harmonic function that determines the functional dependence of the metric components and the electromagnetic potential. Let us assume that the solution has the general form

$$\text{ds}^2 = -e^{2\phi}\text{dt}^2 + e^{-2\beta}\left\{r^2\text{d}\varphi^2 + \text{dr}^2 + \text{dz}^2\right\}, \quad (1)$$

where the metric function $\phi$ depends only on $r$ and $z$, $\beta$ is taken to be a constant and the electromagnetic potential, $A_\alpha = (A_0, A_0, 0, 0)$, is time-independent.

In order to analyse the physical characteristics of the system, it is convenient to work out all the relevant quantities in terms of the orthonormal tetrad of the “locally static observers” (LSO) [4], i.e. observers at rest with respect to infinity.

The system under consideration is solvable in terms of various solutions of the Laplace equation. Let us consider the particular case of the Kuzmin solution [5] such that the metric potential $\phi(r, z)$ is

$$(1 + \beta) \phi = \ln \left( \frac{\sqrt{r^2 + (|z| + a)^2}}{\sqrt{r^2 + (|z| + a)^2} - m} \right). \quad (2)$$

With respect to the LSO, the diagonal components of the three-energy-momentum tensor (evaluated on the surface of the disk), can be interpreted as the energy den-
$\rho \geq 0, \rho + \varphi > 0 \quad \beta \in (-\infty, -1] \cup [0, \infty)$

$\rho + \varphi \geq 0, \quad 1 \geq 0$ (no information)

$\rho + 2\varphi \geq 0, \rho + \varphi \geq 0 \quad \beta \in [-1, \infty)$

$\rho \geq 0, \rho \geq |\varphi| \quad \beta \in (-\infty, -1] \cup [1/3, \infty)$

**TABLE I.** The range of $\beta$ for which the energy conditions hold.

density and the pressures of the disk given by the expressions

$$\rho(r) = 4\beta maF(r), \quad \varphi(r) = \frac{(1 - \beta)}{2\beta} \rho(r),$$

where the function $F(r)$ is

$$F(r) = \frac{(a^2 + r^2)^{\frac{1+\beta}{2\beta}}}{(1 + \beta) (\sqrt{r^2 + a^2} - m)^{\frac{1+\beta}{2\beta}}}.$$

The parameter $\beta$ of the metric produces a non-zero pressure and, from the point of view of the LSO, it has the same value in the radial and angular directions. Note that the particular case of $\beta = 1$ corresponds to a preliminary study of this type of solutions found in [3]. We can use the energy conditions on the disk to obtain the physical range of possible values of the parameter $\beta$, these values are shown in Table I.

Moreover, we can read off the value of the adiabatic speed of sound for the LSO from equation (8). It follows that, in order to satisfy simultaneously the energy conditions and the causality requirement for the speed of sound, we must require that $\beta \in (1/3, \infty)$.

The main point we want to address here arises from the fact that we can use the variational techniques for relativistic thermodynamics [2, 3] to obtain the thermal properties of the material content of the disk. The central role in such a formalism is played by the so-called ‘master function’ $\Lambda$, the Lagrangian density of the matter content in the Einstein-Hilbert action. Let us assume that the disk consists of a multicomponent fluid described by its particle number and entropy density currents, denoted by $n^a$ and $s^a$, respectively, together with their corresponding conjugate momenta $\mu_a = \partial \Lambda / \partial n^a$ and $\theta_a = \partial \Lambda / \partial s^a$. Since the configuration at hand is (conformal)static, these currents cannot depend on time and must be aligned with the time-like Killing vector field of the metric. We carry out our thermodynamic analysis from the point of view of an observer moving with the fluid. Thus, the currents are completely specified by their time-like components which we assume to be functions of $r$, namely $n(r)$ and $s(r)$.

Substituting the variational definition of the energy-momentum tensor (c.f. equation (2.13) in [3]), as the source of the Einstein field equations restricted to the disk surface and using the solution (2), it follows that the master function is simply $\Lambda = -\rho(r)$, in agreement with the definition of local thermodynamic equilibrium [5]. We also obtain a single differential equation stemming from the identification of the generalised pressure $\Psi = \Lambda - \mu_a n^a - \theta_a s^a$ with $\varphi$ [c.f. equation (3)], i.e. we have

$$\left(\frac{d}{dr} \ln \rho \right) \left(\frac{d}{dr} n s \right) = \left(\frac{1 + \beta}{2\beta} \right) \frac{dn}{dr} \frac{ds}{dr}. $$

This equation admits solutions of the form

$$n = A \rho^{\kappa_n} \quad \text{and} \quad s = B \rho^{\kappa_s},$$

where $A$, $B$, $\kappa_n$ and $\kappa_s$ are constants, and $\kappa_n$ and $\kappa_s$ satisfy the relation

$$\frac{\kappa_n + \kappa_s}{\kappa_n \kappa_s} = \frac{1 + \beta}{2\beta}.$$

If we assume that the fluid is made of ‘charged dust’, the energy density must be proportional to the particle number density, setting $\kappa_n = 1$ and the entropy of electromagnetic radiation is such that $s \propto \rho^{3/4}$, thus $\kappa_s = 3/4$ and from (5) it follows that $\beta \geq 3/11$. To verify this, let us compute the adiabatic speed of sound for each component (c.f. equation (29) in [3])

$$c_i^2 = \frac{1 - \kappa_i}{\kappa_i}, \quad \text{with} \quad i = n, s.$$

Thus, for the constants we have chosen, we have $c_n^2 = 0$ and $c_s^2 = 1/3$, in agreement with our physical interpretation. Moreover, from equations (9), it is straightforward to obtain the range of $\beta$ for which the multi-fluid interpretation is causal. That is, $1/2 \leq \kappa_i \leq 1$, and therefore $1/5 \leq \beta \leq 1/3$. This result implies that the single fluid interpretation of the system is incompatible with the multi-fluid thermodynamic description. We now show that the latter has a richer physical content in the sense that the thermodynamics associated with the multifluid model is consistent with the matter content assumed for the disk.

Integrating equations (9) and substituting $\Lambda = -\rho$ we obtain the material fundamental thermodynamic relation

$$\rho(n, s) = \mu_0 n^{1/\kappa_n} + \theta_0 s^{1/\kappa_s},$$

where $\mu_0$ and $\theta_0$ are integration constants. Now, taking the projection of each conjugate momenta into the co-moving frame we get the chemical potential and temperature, respectively [5]. Thus, using the fundamental relation (10) and the solutions (7) we obtain

$$\mu = \frac{\mu_0}{\kappa_n} A^{1-\kappa_n} \rho^{1-\kappa_n},$$

$$T = \frac{\theta_0}{\kappa_s} B^{1-\kappa_s} \rho^{1-\kappa_s}.$$
In the case of $\kappa_n = 1$ and $\kappa_s = 3/4$ we observe that the chemical potential of the dust is constant across the disk and the temperature is proportional to $\rho^{1/4}$.

The solutions (7) together with their thermodynamic conjugate quantities (11) and (12), satisfy Euler’s identity $\rho + \Psi = \mu n + T s$, for any value of $\beta$ [see figure 2]. In figure 1 we show the various thermodynamic quantities [equations (7) and (11) - (12)] corresponding to different values of $\beta$.

This exercise shows that the scheme is consistent with the physical input we considered. Furthermore, this solution is suitable for a vast family of master functions (fundamental relations) for the material content of the disk provided the relation between the constants, equation (8), is satisfied. Thus for multicomponent models for which one component is proportional to the energy density of the disk ($\kappa_n = 1$), the pressure must be generated by the other component. In the multi-fluid thermodynamic interpretation, this component is typically identified with the entropy of the disk [8] and is controlled by the parameter $\kappa_s$. Thus, fixing the particular solution compatible with the matter content.

In sum, we have presented an exact solution for modelling relativistic thin disks. The relevance of the solution is essentially two-fold. On the one hand, it has a remarkably simple mathematical form. The solution (2) determines the behaviour of the material content through the function $F(r)$, equation (5). On the other hand, the multi-fluid interpretation of the solution has allowed us for the first time to give a complete thermodynamic description of the system in terms of two parameters which determine the matter content of the disk. It remains to give a complete thermodynamic treatment for the halo. This work serves as a ‘proof of principle’ that gives a solid footing for a fuller study of relativistic disks, in particular, for a later study of the more realistic stationary solution.

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FIG. 1. Thermodynamic quantities on the disk for different values of $\beta$. In the top part we show the particle number and entropy densities whilst in the bottom their corresponding conjugate quantities, chemical potential and temperature. The plots in the left are well behaved for any value of $\beta \in [1/5, 1]$, the chemical potential is constant. The plots on the right are not defined for the case $\beta = 1$. Moreover there are divergences for values of $\beta > 1/3$. In all cases, the solid line represents the electromagnetic case we study. Here, we have used the values $a = 1$ and $m = 0.75$ [c.f. equation (2)].

FIG. 2. Euler’s identity. The plots for $\rho + \Psi$ are perfectly overlapped with those of $\mu n + T s$ for the same values of $\beta$ as in figure 1.

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