We study the low-energy behavior of $N = 1$ supersymmetric gauge theories with product gauge groups $SU(N)^M$ and $M$ chiral superfields transforming in the fundamental representation of two of the $SU(N)$ factors. These theories are in the Coulomb phase with an unbroken $U(1)^{N-1}$ gauge group. For $N \geq 3$, $M \geq 3$ the theories are chiral. The low-energy gauge kinetic functions can be obtained from hyperelliptic curves which we derive by considering various limits of the theories. We present several consistency checks of the curves including confinement through the addition of mass perturbations and other limits.
1 Introduction

There has been a dramatic progress in our understanding of the dynamics of supersymmetric gauge theories during the past three years. Seiberg and Witten gave a complete solution to the low-energy dynamics of $N = 2$ supersymmetric $SU(2)$ theory with or without fundamental matter fields \[1\]. This work has been generalized to pure $N = 2$ $SU(N)$ theories with and without fundamental matter fields as well as to other gauge groups \[2, 3, 4, 5\].

Following Seiberg’s work on $N = 1$ supersymmetric QCD \[6\], there is a growing number of exact results in $N = 1$ theories as well \[7, 8, 9, 10, 11\]. However in these theories one does not have a complete solution of the low-energy dynamics, but only the exact form of the superpotential. The major difference between $N = 2$ and $N = 1$ theories is that in $N = 2$ the full Lagrangian is determined in terms of a holomorphic prepotential, while in $N = 1$ the superpotential and the gauge-kinetic term are holomorphic, but the Kähler potential is not.

Intriligator and Seiberg noted that the methods which are used to solve certain $N = 2$ theories can also be applied to Coulomb branches of $N = 1$ theories \[7\]. In the Coulomb phase there are massless photons in the low-energy theory, whose couplings to the matter fields are described by the following Lagrangian:

$$ L = \frac{1}{4\pi} \text{Im} \int d^2 \theta \tau_{ij} W^i W^j, $$

where $W^i_\alpha$ is the field strength chiral superfield, corresponding to the $i$th $U(1)$ factor and $\tau_{ij}$ is the effective gauge coupling, which is a holomorphic function of the matter fields. Often this $\tau_{ij}$ can be identified with the period matrix of a hyperelliptic curve. Thus for theories in the Coulomb phase, an important part of the solution of the low-energy dynamics can be found by determining the hyperelliptic curve as a function of the moduli and the scales of the theory. The singular points of the curve usually signal the existence of massless monopole or dyon superfields, whose properties can be read off from the curve.

Except for the $N = 2$ theories based on $SU$, $Sp$ and $SO$ groups with matter fields in the fundamental representation \[1, 2, 4, 5\], there are very few theories for which the description of the Coulomb branch is known. The other examples include $N = 2$ $G_2$ theory with no matter fields \[12\]; $N = 1$
$SU$, $Sp$ and $SO$ theories with adjoint and fundamental matter and a Landau-Ginsburg type superpotential \cite{13}; and also $N = 1$ $SO(M)$ theories with $M - 2$ vectors \cite{8}.

In this paper we examine $N = 1$ theories with product gauge groups $SU(N)^M$ and $M$ matter fields, each transforming as a fundamental under exactly two $SU(N)$ factors. All of these theories are in the Coulomb phase. The $SU(N)^M$ theory contains an unbroken $U(1)^{N-1}$ gauge group. For each of these theories we identify the independent gauge invariant operators, which parameterize the moduli space. We determine the hyperelliptic curves describing the gauge coupling function by considering different limits in which the theory has to reproduce known results for other theories. We give several consistency checks for these curves. The theories where $N \geq 3$, $M \geq 3$ are the first examples of chiral theories in the Coulomb phase; thus, one might hope that they will be useful for building models of dynamical supersymmetry breaking.

The paper is organized as follows. In the next section we first review the $SU(2) \times SU(2)$ theory of Intriligator and Seiberg \cite{7} and then generalize this theory to $SU(2)^N$. We explain the $SU(2)^3$ case in detail and show that the singularities produce the expected behavior when the theory is perturbed by adding mass terms. Section 3 describes the $SU(N) \times SU(N)$ theories, while curves for the general $SU(N)^M$ theories are given in Section 4. We conclude in Section 5. An appendix contains an analysis of the D-flat conditions in the general $SU(N)^M$ theories.

2 $SU(2)^N$

In this section we first review the pure $N = 2$ $SU(N)$ theories and the $SU(2) \times SU(2)$ theory of Intriligator and Seiberg \cite{4}. Then we generalize this $SU(2) \times SU(2)$ theory to $SU(2)^N$.

The hyperelliptic curves for pure $N = 2$ $SU(N)$ theories were given in Ref. \cite{2}. This solution can be summarized as follows. The moduli space of the Coulomb branch can be parameterized by the expectation values of the independent gauge invariant operators formed from the adjoint field $\Phi$:

$$u_k = \frac{1}{k} \text{Tr} \, \Phi^k, \; k > 1.$$
The expectation value of the adjoint can always be rotated to a diagonal form

$$\Phi = \left( \begin{array}{c} a_1 \\ a_2 \\ \vdots \\ a_N \end{array} \right), \quad \sum a_i = 0,$$

where classically $u_k = \frac{1}{k} \sum_{i=1}^{N} a_i^k$. It was argued in Ref. [2] that the $N = 2$ pure $SU(N)$ Yang-Mills theory can be described in terms of a genus $N - 1$ Riemann surface. The hyperelliptic curve describing this surface is given by

$$y^2 = \prod_{i=1}^{N} (x - a_i)^2 - 4\Lambda^{2N}, \quad (1)$$

where $\Lambda$ is the dynamical scale of the $SU(N)$ theory and products of $a_i$’s are to be written in terms of the $u_k$. In terms of the variables $s_k$, which are defined in the classical regime by

$$s_k = (-1)^k \sum_{i_1 < \ldots < i_k} a_{i_1} \ldots a_{i_k}, \quad k = 2, \ldots, N, \quad (2)$$

this curve can also be conveniently expressed as

$$y^2 = (x^N + \sum_{i=2}^{N} s_i x^{N-i})^2 - 4\Lambda^{2N}. \quad (3)$$

The variables $s_k$ are related to the $u_k$’s by Newton’s formula, with $s_0 = 1$ and $s_1 = u_1 = 0,$

$$ks_k + \sum_{j=1}^{k} js_{k-j}u_j = 0, \quad (4)$$

thus defining them quantum mechanically.

Intriligator and Seiberg pointed out [7] that the techniques used for solving $N = 2$ theories can be applied to the Coulomb branches of $N = 1$ theories as well. However, in this case the determination of $\tau$ does not imply a complete solution of the theory. Intriligator and Seiberg showed several examples where the gauge coupling $\tau$ can be exactly determined. Their result for the $SU(2) \times SU(2)$ theory with $2(\square \square)$ can be summarized as follows.

The field content of the $SU(2) \times SU(2)$ theory is $(Q_i)_{\alpha \beta}$, where $i$ is the flavor index and $\alpha, \beta$ are the $SU(2)$ indices. The three independent gauge
invariant operators are
\[ M_{ij} = \frac{1}{2}(Q_i)_{\alpha\beta}(Q_j)_{\alpha'\beta'}\epsilon^{\alpha\alpha'}\epsilon^{\beta\beta'}. \]  (5)

On a generic point of the moduli space the \( SU(2) \times SU(2) \) gauge symmetry is broken to \( U(1) \); thus the theory is in an Abelian Coulomb phase. It is natural to assume that the Coulomb phase can be described by a genus one Riemann surface determined by an elliptic curve, where the coefficients of \( x \) are functions of the scales \( \Lambda_{1,2} \) and the moduli \( M_{ij} \). This curve can be determined by considering two different limits of the theory. One limit involves breaking the \( SU(2) \times SU(2) \) to the diagonal \( SU(2) \) group by giving a diagonal VEV to \( Q_1 \), while the other limit is \( \Lambda_2 \gg \Lambda_1 \), where \( SU(2)_2 \) is confining with a quantum modified constraint \( [6] \). In both limits the theory reduces to an \( SU(2) \) theory with an adjoint chiral superfield, whose elliptic curve is given in Eq. [3]. These two limits completely fix the genus one elliptic curve, whose fourth order form is given by
\[ y^2 = (x^2 - (U - \Lambda_1^4 - \Lambda_2^4))^2 - 4\Lambda_1^4\Lambda_2^4, \]  (6)
where \( U = \det M \). Note that the form of the curve is just what we would get for an \( N = 2 \) \( SU(2) \) theory, except that the modulus \( U \) (which is to be thought of as a function of the \( M \)'s) is shifted by a constant, and that the scale is the product of the scales of each \( SU(2) \) factor. This scale is determined by matching to the diagonal theory. A similar situation will hold for the more general \( SU(N) \times SU(N) \) theories, and with the help of these curves we will be able to describe a general class of \( SU(N)^M \) theories as well.

Now we generalize the \( SU(2) \times SU(2) \) theory of Intriligator and Seiberg \( [7] \) presented above to theories based on the \( SU(2)_1 \times SU(2)_2 \times \ldots \times SU(2)_N \) product group. The field content of the theory is described in the table below:

|        | \( SU(2)_1 \) | \( SU(2)_2 \) | \( SU(2)_3 \) | \ldots | \( SU(2)_N \) |
|--------|----------------|----------------|----------------|--------|----------------|
| \( Q_1 \) | \( \Box \)      | \( \Box \)      | 1              | \ldots | 1              |
| \( Q_2 \) | 1              | \( \Box \)      | \( \Box \)      | \ldots | 1              |
| \( \vdots \) | \( \vdots \) | \( \vdots \) | \( \vdots \) | \( \vdots \) | \( \vdots \) |
| \( Q_N \) | \( \Box \)      | 1              | 1              | \ldots | \( \Box \)      |

(7)

The classical moduli space of this theory can be parameterized by the following gauge invariants
\[ M_i = \det Q_i = \frac{1}{2}(Q_i)_{\alpha_i\alpha_{i+1}}(Q_i)_{\beta_i\beta_{i+1}}\epsilon^{\alpha_i\beta_i}\epsilon^{\alpha_{i+1}\beta_{i+1}}, \quad i = 1, \ldots, N \]
\[ T = \frac{1}{2}(Q_1)_{\beta_1\alpha_2}(Q_2)_{\beta_2\alpha_3}(Q_3)_{\beta_3\alpha_4} \cdots (Q_1)_{\beta_N\alpha_1} \epsilon^{\alpha_1\beta_1} \epsilon^{\alpha_2\beta_2} \cdots \epsilon^{\alpha_N\beta_N}. \]  

(8)

As shown in the appendix, generic vacuum expectation values of these operators preserve a \( U(1) \) gauge symmetry. We will describe the behavior of the holomorphic gauge coupling for the \( U(1) \) gauge group by constructing an elliptic curve.

We first analyze the \( SU(2)_1 \times SU(2)_2 \times SU(2)_3 \) theory, which can be reduced in various limits to the \( SU(2)_1 \times SU(2)_2 \) theory of Ref. [7]. By exploring the limit of large VEV for the field \( Q_3 \) and the limit \( \Lambda_3 \gg \Lambda_1, \Lambda_2 \) we will be able to determine the coefficients of the curve. We will work with the third order form of the elliptic curve, since that form is more convenient in this case. First consider the limit of large diagonal VEV, \( v \), for \( Q_3 \). In this limit \( SU(2)_1 \times SU(2)_3 \) is broken to its diagonal subgroup \( SU(2)_D \). Three components of \( Q_3 \) are eaten by the Higgs mechanism, while the remaining component is a singlet of \( SU(2)_D \). Both \( Q_1 \) and \( Q_2 \) transform as \((0,0)\) under the unbroken \( SU(2)_D \times SU(2)_2 \).

This \( SU(2) \times SU(2) \) theory is precisely the theory of Ref. [7] described above. The invariants of this theory are \( \tilde{M}_{11} = Q_1Q_1, \tilde{M}_{22} = Q_2Q_2 \) and \( \tilde{M}_{12} = Q_1Q_2 \), which can be expressed in terms of the invariants of the original \( SU(2)_1 \times SU(2)_2 \times SU(2)_3 \) theory:

\[ M_1 = \tilde{M}_{11}, \quad M_2 = \tilde{M}_{22}, \quad T = \tilde{M}_{12}v, \quad \text{and} \quad M_3 = v^2. \]

The third order curve for the \( SU(2)_D \times SU(2)_2 \) theory is

\[ y^2 = x^3 + x^2(\Lambda_D^4 + \Lambda_2^4 - \tilde{M}_{11}\tilde{M}_{22} + \tilde{M}_{12}^2) + x\Lambda_D^4\Lambda_2^4, \]

(9)

where \( \Lambda_D^4 = \Lambda_1^4\Lambda_3^4/M_3^2 \).

Let us express this limit of the elliptic curve in terms of the original gauge invariants:

\[ y^2 = x^3 + x^2(\Lambda_1^4\Lambda_3^4/M_3^2 + \Lambda_2^4 - M_1M_2 + T^2/M_3) + x\Lambda_1^4\Lambda_3^4\Lambda_2^4/M_3^2. \]

(10)

After rescaling the above curve by \( x \rightarrow x/M_3, \ y \rightarrow y/M_3^{3/2} \) we obtain

\[ y^2 = x^3 + x^2(\Lambda_1^4\Lambda_3^4/M_3^2 + \Lambda_2^4M_3 - M_1M_2M_3 + T^2) + x\Lambda_1^4\Lambda_3^4\Lambda_2^4. \]

(11)
Since this curve is only valid in the limit of large \( v \), the term \( x^2 \Lambda_1^4 \Lambda_3^4 / M_3^2 \) is of lower order than other terms proportional to \( x^2 \), and should be neglected.

The final form of the curve has to be invariant under all symmetries of the theory. For instance, simultaneous interchange of \( \Lambda_1 \) with \( \Lambda_2 \) and interchange of \( Q_2 \) with \( Q_3 \) does not change the theory, and there are other similar permutations. The only term that is not invariant under such permutations is \( \Lambda_4^4 M_3 \). The properly symmetrized combination is of the form \( \Lambda_1^4 M_2 + \Lambda_2^4 M_3 + \Lambda_3^4 M_1 \). The final expression for the curve is therefore

\[
y^2 = x^3 + x^2(\Lambda_1^4 M_2 + \Lambda_2^4 M_3 + \Lambda_3^4 M_1 - M_1 M_2 M_3 + T^2) + x \Lambda_1^4 \Lambda_2^4 \Lambda_3^4, \tag{12}
\]

while the equivalent quartic form is

\[
y^2 = (x^2 - (\Lambda_1^4 M_2 + \Lambda_2^4 M_3 + \Lambda_3^4 M_1 - M_1 M_2 M_3 + T^2))^2 - 4 \Lambda_1^4 \Lambda_2^4 \Lambda_3^4. \tag{13}
\]

It turns out that this is the complete form of the elliptic curve for the \( SU(2)^3 \) theory. All other terms consistent with the symmetries, such as \( x T^6 \), are excluded by the requirement of agreement with Eq. 11 in the limit of large VEV for \( Q_3 \).

We will present consistency checks which support our claim that the curve derived in the large VEV limit is indeed correct. First, let us consider the theory in the limit \( \Lambda_3 \gg \Lambda_1, \ \Lambda_2 \). The \( SU(2)_3 \) theory has the same number of flavors as the number of colors. Below \( \Lambda_3 \), the \( SU(2)_3 \) group is confining and we need to express the degrees of freedom in terms of confined fields subject to the quantum modified constraint [6]. The confined fields are \( (Q_2^2) \), \( (Q_3^2) \) and the \( 2 \times 2 \) matrix \( (Q_2 Q_3) \). The quantum modified constraint in an \( SU(2) \) theory with four doublets, \( \text{Pf} (q_i q_j) = \Lambda^4 \), when written in terms of the fields confined by the \( SU(2)_3 \) dynamics is

\[
\Lambda_3^4 = (Q_2^2)(Q_3^2) - (Q_2 Q_3)^2.
\]

Again, we can express invariants of the effective \( SU(2)_1 \times SU(2)_2 \) theory in terms of \( SU(2)^3 \) invariants: \( M_{11} = Q_1^2 = M_1 \), \( \mu M_{12} = Q_1 (Q_2 Q_3) = T \) and \( \mu^2 M_{22} = (Q_2 Q_3)^2 = M_2 M_3 - \Lambda_3^4 \), where the last equality makes use of the quantum modified constraint. The factors of dimensional constant \( \mu \) are included in order to make the \( M \)'s dimension two.

The elliptic curve for the \( SU(2)_1 \times SU(2)_2 \) theory is the same as in Eq. 11, except for the obvious substitution \( \Lambda_D \rightarrow \Lambda_1 \). In terms of \( SU(2)^3 \) invariants
the curve is
\[ y^2 = x^3 + x^2 \left( \Lambda_1^4 + \Lambda_2^4 - M_1 \frac{M_2 M_3 - \Lambda_3^4}{\mu^2} + \frac{T^2}{\mu^2} \right) + x \Lambda_1^4 \Lambda_2^4. \]

After rescaling \( x \rightarrow x/\mu^2 \) and \( y \rightarrow y/\mu^3 \) we obtain
\[ y^2 = x^3 + x^2 \left( \Lambda_1^4 \mu^2 + \Lambda_2^4 \mu^2 + M_1 \Lambda_3^4 - M_1 M_2 M_3 + T^2 \right) + x \mu^4 \Lambda_1^4 \Lambda_2^4. \] (14)

The symmetrized form of Eq. 14 with \( \mu = \Lambda_3 \) is identical to Eq. 12 up to the irrelevant subdominant term \( x^2 (\Lambda_1^4 + \Lambda_2^4)\Lambda_3^2 \).

As another consistency check we consider integrating out all matter fields from the SU(2)³ theory. This way we obtain three decoupled pure SU(2) Yang-Mills theories whose low-energy behavior is known and should be reproduced by the above description of the theory.

In order to integrate out the matter fields we add a tree-level superpotential
\[ W_{\text{tree}} = m_1 M_1 + m_2 M_2 + m_3 M_3 \] (15)
to the theory, which corresponds to adding mass terms for all \( Q_i \) fields. On the singular manifold of the curve there are massless monopoles or dyons which have to be included into the low-energy effective superpotential. The curve described by Eq. 12 is singular when
\[ -T^2 + M_1 M_2 M_3 - \Lambda_1^4 M_2 - \Lambda_2^4 M_3 - \Lambda_3^4 M_1 = \pm 2 \Lambda_1^2 \Lambda_2^2 \Lambda_3^2. \] (16)

Thus the low-energy effective superpotential is given by
\[
W = (-T^2 + M_1 M_2 M_3 - \Lambda_1^4 M_2 - \Lambda_2^4 M_3 - \Lambda_3^4 M_1 + 2 \Lambda_1^2 \Lambda_2^2 \Lambda_3^2) \tilde{E}_+ E_+ \\
+ (-T^2 + M_1 M_2 M_3 - \Lambda_1^4 M_2 - \Lambda_2^4 M_3 - \Lambda_3^4 M_1 - 2 \Lambda_1^2 \Lambda_2^2 \Lambda_3^2) \tilde{E}_- E_- \\
+ m_1 M_1 + m_2 M_2 + m_3 M_3,
\]
where \( \tilde{E}_+ \) and \( E_+ \) are the superfields corresponding to the massless monopoles at the first singular manifold, while \( \tilde{E}_- \) and \( E_- \) are the dyons which are massless at the second singular manifold. The equations of motion with respect to the fields \( T, M_i, \tilde{E}_\pm, E_\pm \) will determine the possible vacua of the theory.

The \( M_i \) equations require that either \( \tilde{E}_+ E_+ \) or \( \tilde{E}_- E_- \) is non-vanishing, which together with the \( \tilde{E}_\pm \) equations will fix the solutions to be on one of
the singular submanifolds. The $T$ equation sets $T$ to zero and thus we are left with the following set of equations:

\[
\begin{align*}
M_1 M_2 M_3 - \Lambda_1^4 M_2 - \Lambda_2^4 M_3 - \Lambda_3^4 M_1 &\pm 2 \Lambda_1^2 \Lambda_2^2 \Lambda_3^2 = 0 \\
(M_1 M_2 - \Lambda_1^4) e + m_3 &= 0 \\
(M_1 M_3 - \Lambda_1^4) e + m_2 &= 0 \\
(M_2 M_3 - \Lambda_3^4) e + m_1 &= 0,
\end{align*}
\]  

(17)

where $e$ is the value of the monopole condensate $\tilde{E}E$. One can show that there are eight solutions to these equations which reproduce the vacua obtained from gaugino condensation which we now derive.

For large $m_i$ the $Q_i$ fields can be integrated out, and the resulting theory consists of three decoupled pure $SU(2)$ Yang-Mills theories with scales determined by matching:

\[
\begin{align*}
\tilde{\Lambda}_1^6 &= m_1 m_3 \Lambda_1^4, \quad \tilde{\Lambda}_2^6 = m_1 m_2 \Lambda_2^4, \quad \text{and} \quad \tilde{\Lambda}_3^6 = m_2 m_3 \Lambda_3^4.
\end{align*}
\]

Gaugino condensation is then expected to produce a low-energy superpotential

\[
\begin{align*}
W &= 2 \epsilon_1 \tilde{\Lambda}_1^3 + 2 \epsilon_2 \tilde{\Lambda}_2^3 + 2 \epsilon_3 \tilde{\Lambda}_3^3 \\
&= 2 \epsilon_1 \sqrt{m_1 m_3} \Lambda_1^2 + 2 \epsilon_2 \sqrt{m_1 m_2} \Lambda_2^2 + 2 \epsilon_3 \sqrt{m_2 m_3} \Lambda_3^2, 
\end{align*}
\]

(18)

where $\epsilon_i = \pm 1$. Since the masses $m_i$ can be viewed as source terms for the gauge invariant operators $M_i$, the VEV’s of the gauge invariants are determined by $\left[\frac{\partial W}{\partial m_i}\right] = \langle M_i \rangle, \langle T \rangle = 0$. The resulting vacua

\[
\begin{align*}
\langle M_1 \rangle &= \epsilon_2 \sqrt{\frac{m_3}{m_1}} \Lambda_1^2 + \epsilon_3 \sqrt{\frac{m_2}{m_1}} \Lambda_2^2 \\
\langle M_2 \rangle &= \epsilon_1 \sqrt{\frac{m_1}{m_2}} \Lambda_2^2 + \epsilon_3 \sqrt{\frac{m_3}{m_2}} \Lambda_3^2 \\
\langle M_3 \rangle &= \epsilon_1 \sqrt{\frac{m_1}{m_3}} \Lambda_1^2 + \epsilon_2 \sqrt{\frac{m_2}{m_3}} \Lambda_3^2 
\end{align*}
\]

(19)

can be shown to exactly coincide with the solutions of Eqs. (17), providing us with a non-trivial check on the consistency of the curve for the $SU(2)^3$ theory.
It is quite straightforward to generalize the $SU(2)^3$ curve to $SU(2)^N$ theories with the matter content given in Table 7. We proceed as before and determine the curves from the limit of large diagonal VEV for one of the $Q_i$'s and the limit in which one of the $SU(2)$'s becomes strong. The resulting curve is:

$$y^2 = x^3 + x^2 \left( T^2 - \prod_i M_i + (M_i M_{i+1} \to -\Lambda_{i+1}^4) \right) + x \prod_i \Lambda_i^4,$$

where the last term in parentheses proportional to $x^2$ follows by substituting any set of nearest neighbor bilinears $M_i M_{i+1}$ in $\prod_i M_i$ by the dynamical scale $-\Lambda_{i+1}^4$ of the common gauge group that both $Q_i$ and $Q_{i+1}$ transform under. This is a consequence of the quantum modified constraint in the strong coupling limit in the $i+1$ gauge group. For example, for the case with four $SU(2)$ factors the term proportional to $x^2$ is $(T^2 - M_1 M_2 M_3 M_4 + \Lambda_1^4 M_2 M_3 + \Lambda_2^4 M_3 M_4 + \Lambda_3^4 M_4 M_1 + \Lambda_4^4 M_1 M_2 - \Lambda_1^4 \Lambda_3^4 - \Lambda_2^4 \Lambda_3^4)$.

3. $SU(N) \times SU(N)$

Next we generalize the $SU(2) \times SU(2)$ theory presented in Section 2 to $SU(N) \times SU(N)$ with fields $Q_1$ and $Q_2$ transforming as $(\square, \square)$ and $(\square, \square)$. Along generic flat directions $SU(N) \times SU(N)$ is broken to $U(1)^{N-1}$, as shown in the appendix. Therefore this theory is in the Coulomb phase. Since there is a non-anomalous $U(1)_R$ symmetry under which the fields $Q_1$ and $Q_2$ have R-charge zero, there can be no dynamical superpotential generated; thus, the Coulomb phase is not lifted.

The independent gauge invariant operators are $B_1 = \det Q_1$, $B_2 = \det Q_2$, and $T_n = \text{Tr}(Q_1 Q_2)^n$, $n = 1, \cdots, N-1$. This agrees with the counting of degrees of freedom: the fields $Q_1$ and $Q_2$ contain $2N^2$ complex degrees of freedom and there are $2(N^2-1)$ D-flat conditions. Since there is an unbroken $U(1)^{N-1}$ gauge symmetry only $2(N^2-1) - (N-1)$ of these conditions are independent and thus one expects to find $2N^2 - [2(N^2-1) - (N-1)] = (N+1)$ independent gauge invariant objects, which exactly matches the number of

[3]The original form of Eq. 20 which appeared in the first version of this paper was not general enough to correctly reproduce the curves for $N \geq 4$. This was noted by G. Hailu in [4].
operators listed above. We again assume that there is a hyperelliptic curve describing this theory involving these degrees of freedom and the scales $\Lambda_1$ and $\Lambda_2$.

We will present the $SU(3) \times SU(3)$ case in detail and then generalize to $SU(N) \times SU(N)$. The matter field content of the $SU(3) \times SU(3)$ theory is

|        | $SU(3)$ | $SU(3)$ |
|--------|---------|---------|
| $Q_1$  | $\blacksquare$ | $\blacksquare$ |
| $Q_2$  | $\blacksquare$ | $\blacksquare$ |

The independent gauge invariants are

\[
B_1 = \det Q_1, \quad B_2 = \det Q_2, \\
T_1 = \text{Tr} Q_1 Q_2, \quad T_2 = \text{Tr} (Q_1 Q_2)^2.
\]

All other gauge invariants can be expressed in terms of these four. For example, the operator $T_3 = \text{Tr} (Q_1 Q_2)^3$ is constrained classically via the identity

\[
\det M = B_1 B_2,
\]

where $M_{\alpha}^{\beta} = Q_{\alpha A}^\alpha Q_{\beta}^A$. To see this we first express $\det M$ in terms of the invariants $T_i$, $i = 1, 2, 3$ as

\[
\det M = \frac{1}{6} \left( T_1^3 - 3T_1 T_2 + 2T_3 \right).
\]

The classical constraint of Eq. 21 then yields

\[
T_3^{\text{cl}} = \frac{1}{2} \left( 6B_1 B_2 + 3T_2 T_1 - T_1^3 \right).
\]

It is natural to consider the composite field $\Phi_{\alpha}^\beta = Q_{1\alpha A}^\alpha Q_{2\beta}^A - \frac{1}{3} \text{Tr} Q_1 Q_2 \delta_{\alpha}^{\beta}$, which is a singlet under one of the $SU(3)$'s and an adjoint under the other. We define

\[
\begin{align*}
  u &= \frac{1}{2} \text{Tr} \Phi^2 = \frac{1}{2} \left( T_2 - \frac{1}{3} T_1^2 \right), \\
  v &= \frac{1}{3} \text{Tr} \Phi^3 = \frac{1}{3} \left( T_3^{\text{cl}} - T_2 T_1 + \frac{2}{9} T_1^3 \right) \\
  &= \frac{1}{3} \left( 3B_1 B_2 + \frac{1}{2} T_2 T_1 - \frac{5}{18} T_1^3 \right),
\end{align*}
\]

10
which correspond to the moduli of an $SU(3)$ theory with adjoint field $\Phi$. It turns out that the $SU(3) \times SU(3)$ curve depends only on these combinations of $T_i$ and $B_i$.

As there are generically two $U(1)$’s unbroken, we expect there to be a genus two hyperelliptic curve describing the theory, given by a sixth order polynomial in $x$. Having identified the moduli space we consider various limits to determine the coefficients of this hyperelliptic curve. Consider the limit where $Q_1$ gets a large diagonal VEV, $w$, $w \gg \Lambda_1, \Lambda_2$. Then $SU(3) \times SU(3)$ is broken to the diagonal $SU(3)_D$. Under $SU(3)_D$, $Q_1$ and $Q_2$ decompose into two singlets and two adjoints. The adjoint from $Q_1$ is eaten, leaving two singlets, which are assumed not to enter the gauge dynamics, and an adjoint, $\Phi_D = Q_2 - \frac{1}{3} \text{Tr} Q_2$. The scale of the resulting $SU(3)_D$ theory is determined by matching at the scale $w$ which gives $\Lambda_6^6 = \Lambda_1^6 \Lambda_2^6 / w^6 = \Lambda_1^6 \Lambda_2^6 / B_1^6$. The dynamics of this effective $N = 2$, $SU(3)$ gauge theory is described by the curve

$$y^2 = (x^3 - u_D x - v_D)^2 - 4\Lambda_D^6,$$  \hspace{1cm} (25)

and the invariant traces $u_D, v_D$ can easily be expressed in terms of $u, v$ of Eq. 24.

$$u_D = \frac{1}{2} \text{Tr} \Phi_D^2 = \frac{1}{2B_1^{2/3}} \left( T_2 - \frac{1}{3} T_1^2 \right) = \frac{u}{B_1^{2/3}}$$

$$v_D = \frac{1}{3} \text{Tr} \Phi_D^3 = \frac{1}{3B_1} \left( T_3^3 - T_2 T_1 + \frac{2}{9} T_1^3 \right) = \frac{v}{B_1} \hspace{1cm} (26)$$

The curve in Eq. 25 can then be written in terms of the original $SU(3) \times SU(3)$ gauge invariants and the original scales:

$$y^2 = \left( x^3 - \frac{u}{B_1^{2/3}} x - \frac{v}{B_1} \right)^2 - 4\frac{\Lambda_1^6 \Lambda_2^6}{B_1^6}.$$  \hspace{1cm} (27)

Rescaling $x \rightarrow x/B_1^{1/3}$, $y \rightarrow y/B_1$, the curve takes the form

$$y^2 = \left( x^3 - u x - v \right)^2 - 4\Lambda_1^6 \Lambda_2^6.$$  \hspace{1cm} (28)

The hyperelliptic curve of the $SU(3) \times SU(3)$ theory must reproduce Eq. 28 in the limit of large diagonal VEV $w$ for $Q_1$, but the sixth order polynomial which describes it may well contain new terms which are not
yet fixed because they are subdominant in this limit. We now write a more
general polynomial, containing all terms consistent with the $R$-symmetry
of the theory, and the assumption that the scales appear only as integer
powers of $\Lambda_1^6$ and $\Lambda_2^6$, corresponding to instanton effects. The theory has an
anomalous $U(1)_R$ symmetry in which $Q_1$, $Q_2$, $\Lambda_1$, and $\Lambda_2$ have $R$-charge one.
Covariance of the curve in the large VEV limit (Eq. 28) requires that $x$ and $y$
be assigned $R$-charges two and six, respectively. These $R$-charge assignments
are summarized in the table below.

| $U(1)_R$ | $y$ | $x$ | $T_1$ | $T_2$ | $B_1$, $B_2$ | $\Lambda_1^6$, $\Lambda_2^6$ |
|----------|-----|-----|-------|-------|--------------|----------------|
| 6        | 2   | 2   | 4     | 3     | 6            | 6              |

The most general sixth order polynomial including all terms consistent
with these requirements and the discrete $\Lambda_1 \leftrightarrow \Lambda_2$ symmetry is
\[
y^2 = x^6 - 2ux^4 - (2v + \alpha (\Lambda_1^6 + \Lambda_2^6)) x^3 + u^2 x^2 + (2uv + \beta (\Lambda_1^6 + \Lambda_2^6)) u x
\]
\[
+ v^2 - \gamma \Lambda_1^6 \Lambda_2^6 + \delta (\Lambda_1^6 + \Lambda_2^6)^2 + \epsilon (\Lambda_1^6 + \Lambda_2^6) v,
\]
with as yet undetermined coefficients $\alpha, \beta, \gamma, \delta, \epsilon$. Additional terms involving
other combinations of products of the fields and the scales are not consistent
with the large VEV limit. Other combinations of gauge invariants and scales
are excluded by the strong coupling limit, which we now describe.

Next we consider the limit where $SU(3)_2$ is strong, $\Lambda_2 \gg \Lambda_1$. $Q_1$ and $Q_2$
confine to form three singlets under the remaining $SU(3)_1$, $Q_1^3$, $Q_2^3$, $Tr Q_1 Q_2$,
and an adjoint $\Phi_1 = \frac{1}{\mu}(Q_1 Q_2 - Tr Q_1 Q_2)$, where the scale $\mu$ is introduced
to give the adjoint canonical dimension one. Below the scale $\Lambda_2$ we have an
$SU(3)$ theory with an adjoint and scale $\Lambda_1$.

The confining $SU(3)_2$ theory has a quantum modified constraint [6]
\[
\det M - B_1 B_2 = \Lambda_2^6.
\]
This quantum modified constraint will result in the expression (23) for $T_3$
being modified by the addition of $3\Lambda_2^6$.

We identify the moduli in this limit,
\[
u_1 = \frac{1}{2\mu^2} \left( T_2 - \frac{1}{3} T_1^2 \right) = \frac{u}{\mu^2}
\]
\[
v_1 = \frac{1}{3\mu^3} \left( T_3 - T_2 T_1 + \frac{2}{9} T_1^3 \right) = \frac{1}{3\mu^3} \left( 3B_1 B_2 + \frac{1}{2} T_2 T_1 - \frac{5}{18} T_1^3 + 3\Lambda_2^6 \right)
\]
\[
= \frac{1}{\mu^3} \left( u + \Lambda_2^6 \right).
\]
The curve in this limit is then, after rescaling $x \rightarrow x/\mu, y \rightarrow y/\mu^3$,

$$y^2 = \left[ x^3 - ux - v - \Lambda_1^6 - \Lambda_2^6 \right]^2 - 4\mu^6 \Lambda_1^6. \quad (32)$$

This fixes the previously undetermined parameters in Eq. 29 except for $\gamma$. At this stage, using the $\Lambda_1 \leftrightarrow \Lambda_2$ flavor symmetry and the above limits, the $SU(3) \times SU(3)$ curve takes the form

$$y^2 = \left[ x^3 - ux - v - \Lambda_1^6 - \Lambda_2^6 \right]^2 - \gamma \Lambda_1^6 \Lambda_2^6. \quad (33)$$

In order to determine the coefficient $\gamma$ we higgs the theory to $SU(2) \times SU(2)$. Consider the limit where $Q_1$ and $Q_2$ each get large VEVs of the form

$$Q_1 = Q_2 = \begin{pmatrix} w & 0 \\ 0 & 0 \end{pmatrix}. \quad (34)$$

Then $SU(3) \times SU(3)$ is broken to $SU(2) \times SU(2) \times U(1)$ with the uneaten degrees of freedom lying in the two by two lower right block of the fields $Q_1$ and $Q_2$, which we denote by $q^A_1$ and $\tilde{q}^A_2$. These remaining degrees of freedom are neutral under the $U(1)$.

The non-perturbative description of this higgs limit of the curve Eq. 33 is the following: Given values of the invariants $u$ and $v$, Newton’s formula of Eq. 4 can be used to find $s_2$ and $s_3$ and then a set of values for $a_1$, $a_2$, $a_3$ with $\sum a_i = 0$ via Eq. 2. Note that in the strong coupling regime the $a_i$’s are not the VEVs of any fundamental field, although classically they are the diagonal VEVs of the composite adjoint field $\Phi = Q_2Q_3 - \frac{1}{3} \text{Tr} Q_2Q_3$. The curve in Eq. 33 can be rewritten as

$$y^2 = \left( \prod_{i=1}^{3} (x - a_i) - \Lambda_1^6 - \Lambda_2^6 \right)^2 - \gamma \Lambda_1^6 \Lambda_2^6. \quad (35)$$

Without loss of generality we can take $a_3 = \frac{2}{3} \rho$, $a_1 = -\frac{1}{3} \rho + a$, $a_2 = -\frac{1}{3} \rho - a$. We shift $x \rightarrow x - \frac{1}{2} \rho$, and reexpress the curve as

$$y^2 = \left( (x - \rho)(x - a)(x + a) - \Lambda_1^6 - \Lambda_2^6 \right)^2 - \gamma \Lambda_1^6 \Lambda_2^6. \quad (36)$$

The higgs limit is $\rho \gg a$ in which one pair of branch points recedes toward infinity, as in [3], and monodromies and periods calculated in the finite region.
are those of the $SU(2) \times SU(2)$ theory. To see this concretely we rescale $y \to y(x - \rho)$, and assume $x \ll \rho$. In this region the curve Eq. 36 may be expressed as the approximate genus one curve

$$y^2 = \left( x^2 - a^2 + \frac{\Lambda_1^6}{\rho} + \frac{\Lambda_2^6}{\rho} \right)^2 - \gamma \frac{\Lambda_1^6 \Lambda_2^6}{\rho^2}.$$  \hspace{1cm} (37)

This agrees with Eq. 6 if we identify $a^2 = U$, $\gamma = 4$ and scales $\rho \tilde{\Lambda}_i^4 = \Lambda_i^6$. In the classical region, it is certainly the case that $a^2 = U$ and $\rho = w^2$ of Eq. 34. The scales are then related by the standard matching condition. We have now completely determined the $SU(3) \times SU(3)$ curve,

$$y^2 = \left( x^3 - ux - v - \Lambda_1^6 - \Lambda_2^6 \right)^2 - 4\Lambda_1^6 \Lambda_2^6. \hspace{1cm} (38)$$

The generalization of this analysis to $SU(N) \times SU(N)$ is straightforward. The effect of the quantum modified constraint is to shift the classical expression for $s_N$ by $(-1)^N(\Lambda_1^{2N} + \Lambda_2^{2N})$. Recall the curve of Eq. 3 for the $SU(N)$ theory with an adjoint,

$$y^2 = \left( \sum_{i=0}^{N} s_i x^{N-i} \right)^2 - 4\Lambda^{2N}. \hspace{1cm} (39)$$

The previous arguments carry through in direct analogy, resulting in the curve for the $SU(N) \times SU(N)$ theory:

$$y^2 = \left( \sum_{i=0}^{N} s_i x^{N-i} + (-1)^N \left( \Lambda_1^{2N} + \Lambda_2^{2N} \right) \right)^2 - 4\Lambda_1^{2N} \Lambda_2^{2N}, \hspace{1cm} (40)$$

where the $s_i$'s are the symmetric invariants of the composite adjoint $\Phi = Q_1 Q_2 - \frac{1}{N} \text{Tr} Q_1 Q_2$ and are to be expressed in terms of the gauge invariants $T_i$ and $B_i$ via classical expressions.

As a consistency check on the $SU(3) \times SU(3)$ curves we consider integrating out the fields $Q_1$ and $Q_2$ by adding a mass term $W_{\text{tree}} = mT_1$ to the superpotential. Then the low-energy theory will be a pure $SU(3) \times SU(3)$ Yang-Mills theory and we expect to find nine vacuum states.
The effective low-energy superpotential has to account for the monopoles and dyons which become massless along the singular surfaces of the hyperelliptic curve of Eq. [38]. These singular surfaces can be determined by finding the zeros of the discriminant $\Delta$ of the curve. For the $N = 2$ $SU(3)$ curve described by $y^2 = (x^3 - ux - v)^2 - 4\Lambda^6$, the discriminant factorizes, $\Delta \propto \Delta_+ \Delta_-$, where $\Delta_\pm = 4u^3 - 27(v \pm 2\Lambda^3)^2$. In our case $u$ and $v$ are expressed in terms of $T_1$, $T_2$, $B_1$ and $B_2$ by Eq. [24]. Thus the effective superpotential can be written as

$$W = \Delta_+ \tilde{E}_+ E_+ + \Delta_- \tilde{E}_- E_- + mT_1,$$

where the $E_+$, $\tilde{E}_+$ fields correspond to the monopoles which become massless at $\Delta_+ = 0$ and the $E_-$, $\tilde{E}_-$ fields correspond to the dyons which become massless at $\Delta_- = 0$. The $T_1$ equation of motion will force at least one of the monopole condensates to be non-vanishing. But then the $T_2$ equation will force both $\tilde{E}_+ E_+$ and $\tilde{E}_- E_-$ to be non-zero, which by the $\tilde{E}_+, \tilde{E}_-$ equations lock the fields to one of the $Z_3$ symmetric singularities $\Delta_+ = \Delta_- = 0$. The $B_1$ and $B_2$ equations just set $B_1$ and $B_2$ to zero, while $\tilde{E}_+ E_+$ and $\tilde{E}_- E_-$ can be uniquely determined once $T_1$ and $T_2$ are fixed. Thus in order to count the number of vacua one needs to solve the equations $\Delta_+ = \Delta_- = B_1 = B_2 = 0$ for the variables $T_1$ and $T_2$. Using Eq. [24] with $B_1 = B_2 = 0$ these can be written as

$$\frac{1}{2} T_2 T_1 - \frac{5}{18} T_1^3 - 3\Lambda_1^6 - 3\Lambda_2^6 = 0$$

$$T_2 - \frac{1}{3} T_1^2 = 3\omega \Lambda_1^2 \Lambda_2^2,$$

where $\omega$ is a third root of unity. One can see that for each value of $\omega$ we get a cubic equation for $T_1$, therefore we conclude that there are nine distinct vacua in agreement with the Witten index. We do not find detailed agreement with the vacua determined by the original integrating in procedure of [10], which we would expect to be at $T_2 = B_i = 0$, $T_1 = \omega_1 \Lambda_1^2 + \omega_2 \Lambda_2^2$. However, there are other examples, such as the $N = 2$, $SU(N_c)$ gauge theory with $N_c > 4$, where the operator $u_{2k}$ can mix with $u_k$, for example, in which the naive integrating in procedure does not reproduce the VEV’s of the gauge invariant operators found from the effective superpotential including massless monopoles and dyons. Similarly, in the $SU(3) \times SU(3)$ theory, $T_2$ and $T_1^2$ can mix, and we should not expect the naive integrating in procedure to work.

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[4] We thank Ken Intriligator for pointing this out to us. See also Ref. [15].
Finally, we generalize the previous analysis to $SU(N)^M$ theories with matter content given below:

$$
\begin{array}{c|cccc}
Q_1 & \Box & \Box & \mathbf{1} & \cdots & \mathbf{1} \\
Q_2 & 1 & \Box & \Box & \cdots & \mathbf{1} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
Q_M & \Box & \mathbf{1} & \mathbf{1} & \cdots & \Box \\
\end{array}
$$

The independent gauge invariants are $B_i = \det Q_i$, $i = 1, \ldots, M$; and $T_i = \text{Tr}(Q_1 \cdots Q_M)^i$, $i = 1, \ldots, N - 1$. We define the composite field

$$
\Phi = Q_1 Q_2 \cdots Q_M - \frac{1}{N} \text{Tr} Q_1 Q_2 \cdots Q_M,
$$

which is an adjoint under one of the $SU(N)$’s and invariant under the others. From $\Phi$ we form the invariants $s_i$ as in Section 2. These invariants $s_i$, when expressed in terms of the invariants $T_i$ and $B_i$, classically have the same functional form as in the $SU(N) \times SU(N)$ case, except that $B_1 B_2$ has to be replaced by the product over all the $B_i$’s.

In terms of these variables, the $SU(N)^M$ curve is given by

$$
y^2 = \left[ \sum_{i=1}^N s_i(T_j, B_j) x^{N-i} + (B_i B_{i+1} \to \Lambda_i^{2N}) \right]^2 - 4 \prod_{j=1}^M \Lambda_i^{2N}.
$$

The modulus $s_N$ is written in terms of $T_i$ and $B_i$ via classical relations, and contains the term $(-1)^N \prod_i B_i$, for example. In Eq. 42 the second term in brackets is shorthand for replacing sets of nearest neighbor bilinears $B_i B_{i+1}$ by $\Lambda_i^{2N}$. For example, we would substitute $B_1 B_2 B_3 B_4$ by $B_1 B_2 B_3 B_4 + \Lambda_1^{2N} B_2 B_3 + \Lambda_2^{2N} B_3 B_4 + \Lambda_3^{2N} B_4 B_1 + \Lambda_4^{2N} B_1 B_2 + \Lambda_1^{2N} \Lambda_2^{2N} + \Lambda_2^{2N} \Lambda_3^{2N} + \Lambda_3^{2N} \Lambda_4^{2N}$. We can check that this curve produces the correct $SU(N)^{M-1}$ curve upon higgsing the theory and in the strong $SU(N)_M$ limit. First consider breaking to $SU(N)^{M-1}$ by giving the field $Q_M$ a large diagonal VEV $w$. In this limit the degrees of freedom are $B_i/w$, $i = 1, \ldots, M - 1$; and $T_j/w^j$, $j = 1, \ldots, N - 1$.

---

As for the $SU(2)^N$ theory, some terms were missing from Eq. 42 in a previous version.
Upon rescaling the gauge invariants by appropriate powers of the VEV \( w \) and using the matching relation \( w^{2N}\Lambda_D^{2N} = B_M\Lambda_D^{2N} = \Lambda_M^{2N-1}\Lambda_M^{2N} \), the curve reproduces the correct \( SU(N)^{M-1} \) limit.

We can also check the curve in the strong \( SU(N)_M \) limit. In this limit \( SU(N)_M \) confines with a quantum modified constraint and we obtain another \( SU(N)^{M-1} \) theory in this limit. The degrees of freedom are

\[
B_i, \ i = 1, \ldots, M - 2; \ T_n, n = 1, \ldots, N - 1; \text{ and } \tilde{B}_{M-1} = \left( \frac{1}{\mu} Q_{M-1} Q_M \right)^N = \frac{1}{\mu^N}(B_{M-1} B_M + \Lambda_M^{2N}).
\]

The scale \( \mu \) is introduced as usual to give the field \( Q_{M-1} Q_M \) canonical dimension one. Comparing the curves for \( SU(N)^M \) and \( SU(N)^{M-1} \) with these degrees of freedom fixes the scale \( \mu = \Lambda_M \), and then Eq. [12] agrees with the curve for the \( SU(N)^M \) theory.

Similarly, we can consider higgsing the theory to \( SU(N - 1)^M \) by giving all the \( Q_i \)'s a VEV in one component, in which case we again find agreement amongst the curves.

5 Conclusions

We have extended the results of Ref. [4] to \( N = 1 \) supersymmetric gauge theories with product gauge groups \( SU(N)^M \) and \( M \) chiral superfields in the fundamental representation of exactly two of the \( SU(N) \) factors. These theories have an unbroken \( U(1)^{N - 1} \) gauge group along generic flat directions and are therefore in the Coulomb phase. For \( M = 2 \) there are two limits in which the low-energy degrees of freedom are those of an effective \( N = 2 \) \( SU(N) \) gauge theory, so it is natural to assume that the gauge kinetic functions are given in general by the period matrix of genus \( N - 1 \) hyperelliptic curves. We then derive those curves by studying these two limits. For \( M > 2 \) the \( SU(N)^{M-1} \) theory can be obtained by higgsing the \( SU(N)^M \) theory, so we assume again that hyperelliptic curves determine the Coulomb phase dynamics and then find the curves by studying limits.

There is a systematic pattern of curves for the \( SU(N) \times SU(N) \) models. When written in terms of trace of powers of the composite adjoint field \( Q_1 Q_2 \), the new curves are related to the known [5] curves for \( N = 2 \) \( SU(N) \) theories by a simple shift due to the quantum modified constraint in the product.
group models. For $M > 2$ the curves are entirely new, and they depend on the $N + M - 1$ invariants in a complex but systematic way.

One of the most striking aspects of the work of [1, 7] is that one can add a mass term to the original theory and demonstrate that magnetic confinement occurs. We have studied this mechanism in our $SU(2)^3$ and $SU(3)^2$ models. In the first case we find 8 confining vacua with detailed agreement between the two approaches based on the low-energy superpotential with monopole fields and the dynamical superpotential describing gaugino condensation after integrating out massive chiral fields. In the second case we find, as expected, 9 vacua from the low-energy monopole superpotential as well. For $N > 2$, $M > 2$ our models are chiral, and they are the first examples of chiral theories in the Coulomb phase.

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**Appendix: D-flat Conditions in $SU(N)^M$ theories**

In the early approach to the dynamics of SUSY gauge theories [10] the moduli space of SUSY vacua was obtained by finding the most general field configuration, up to symmetries, which satisfies the D-flat conditions. In the more modern approach [8], which emphasizes holomorphy, the moduli space is parameterized by a set of algebraically independent gauge-invariant polynomial functions of the chiral superfields [17]. Such a set of $N + 1$ polynomial invariants were specified in Section 3 for the $SU(N)^2$ models, and a set of $N + M - 1$ invariants for the $SU(N)^M$ models was given in Section 4. It is a simple but useful consistency check to study the D-flat conditions. This will confirm the dimension of the moduli space and make manifest the residual $U(1)^{N-1}$ symmetry of the Coulomb phase.
We begin with the \(SU(N) \times SU(N)\) case. As a simple variant of a standard matrix representation we know that \(Q_1\) and \(Q_2\) can be expressed in the form

\[
Q_1 = U_1 A V_2^{-1} e^{i\alpha}, \quad Q_2 = U_2 B V_1^{-1} e^{i\beta},
\]

where \(U_i\) and \(V_i\) are \(SU(N)\) matrices and \(A\) and \(B\) are real diagonal matrices whose diagonal elements \(a_i\) and \(b_i\) are ordered, i.e. \(a_1 > a_2 > \ldots > a_N > 0\).

We assume a generic configuration in which none of the \(a_i\) coincide, and the same for the \(b_i\). The \(U_i\) and \(V_i\) are still not uniquely determined; there remains the freedom \(U_1 \rightarrow U_1 C, V_2 \rightarrow V_2 C\) and \(U_2 \rightarrow U_2 C', V_1 \rightarrow V_1 C'\) where \(C\) and \(C'\) are diagonal \(SU(N)\) matrices, but this redundancy plays little role in the subsequent analysis.

The original form of the D-flat conditions is

\[
D_1^a = \text{Tr} \left( Q_1^\dagger T^a Q_1 - Q_2^\dagger T^a Q_2^\dagger \right) = 0, \quad D_2^a = \text{Tr} \left( Q_2^\dagger T^a Q_2 - Q_1^\dagger T^a Q_1^\dagger \right) = 0.
\]

Since the generators \(T^a\) of the fundamental representation of \(SU(N)\) are a complete set of \(N \times N\) Hermitian matrices, this means that \(Q_1^\dagger Q_1 - Q_2^\dagger Q_2\) and \(Q_2^\dagger Q_2' - Q_1^\dagger Q_1\) must be multiples of the identity. Using Eq. (43), these conditions read

\[
U_1 A^2 U_1^{-1} - V_1 B^2 V_1^{-1} = c_1 1, \quad V_2 A^2 V_2^{-1} - U_2 B^2 U_2^{-1} = c_2 1,
\]

where \(c_1\) and \(c_2\) are real constants. Taking traces gives \(c_1 = c_2 = c\). Suppose one brings the \(B^2\) term to the right side of the first equation. One can then see that the characteristic equations for the matrices \(U_1 A^2 U_1^{-1}\) and \(V_1 (B^2 - c_1 1) V_1^{-1}\) must have the same roots. This means that \(a_i^2 = b_i^2 + c\) for each diagonal element of \(A\) and \(B\).

Conjugate the first equation in (43) by \(U_1^{-1} \ldots U_1\) and the second by \(V_2^{-1} \ldots V_2\). Rearrange the resulting expressions to read

\[
\left[ U_1^{-1} V_1, B \right] = 0 \quad \text{and} \quad \left[ V_2^{-1} U_2, B \right] = 0.
\]

These imply that \(U_1^{-1} V_1 = C_1\) and \(V_2^{-1} U_2 = C_2\) where \(C_1\) and \(C_2\) are diagonal \(SU(N)\) matrices. We can now return to (43) and rewrite it as

\[
Q_1 = U_1 A C_2 U_2^{-1} e^{i\alpha} \quad \text{and} \quad Q_2 = U_2 B C_1^{-1} U_1^{-1} e^{i\beta}.
\]
Now make an $SU(N)_1$ gauge transformation by $U_1^{-1}$, and an $SU(N)_2$ transformation by $C_2 U_2^{-1}$ to obtain the canonical diagonal representation

$$Q_1 = \begin{pmatrix} a_1 & a_2 & \cdots & a_N \end{pmatrix} e^{i\alpha}, \quad Q_2 = \begin{pmatrix} b_1 & b_2 & \cdots & b_N \end{pmatrix} C e^{i\beta}. \quad (48)$$

with $C = C_2 C_1^{-1}$ and $a_i = (b_i^2 + c)^{1/2}$. There are $N$ real variables and $N$ phases in $Q_2$, and the additional real $c$ and phase $\alpha$. This is the correct count of independent variables of the Coulomb phase. Further one sees that the gauge is not completely fixed because the canonical representation is invariant under $Q_1 \rightarrow C' Q_1 C'^{-1}$ and $Q_2 \rightarrow C' Q_2 C'^{-1}$ where $C'$ is a diagonal matrix of the diagonal $SU(N)$ subgroup. This is just the expected residual $U(1)^{N-1}$ gauge invariance of an $N - 1$ dimensional Coulomb branch.

This approach can be extended to the $SU(N)^m$ case with chiral superfields in the $\left[ \begin{array}{c} \bullet \\ \bullet \end{array} \right]$ representation of $SU(N)_i \times SU(N)_{i+1}$, and otherwise inert, denoted by $Q_{ij}$. (We always take $j = i + 1$ for $i = 1 \ldots m - 1$, and $j = 1$ for $i = m$.) We start with representations

$$Q_{ij} = U_i A_{ij} V_j^{-1} e^{i\alpha_{ij}}, \quad (49)$$

where $U_i$ and $V_j \in SU(N)$ and $A_{ij}$ is a real diagonal matrix. As before there is a non-uniqueness $U_i \rightarrow U_i C_{ij}$, $V_j \rightarrow V_j C_{ij}$ with $C_{ij}$ diagonal and $\det C_{ij} = 1$.

There are $m$ independent D-flat conditions, and one learns, as in (47), that the $SU(N)_j$ condition implies

$$U_j A_{jk}^2 U_j^{-1} - V_j A_{ij}^2 V_j^{-1} = c_j 1. \quad (50)$$

(Again we take $j = i + 1$, and $k = j + 1$ with wraparound where required.) The sum of traces of these $j$ equations just gives $\sum_{j=1}^{m} c_j = 0$, and one finds from considering characteristic equations that diagonal elements $a_{ij,\rho}$, satisfy $a_{j,k,\rho}^2 = a_{ij,\rho}^2 + c_j$, for $\rho = 1, 2 \ldots N$.

By appropriate conjugation and rearrangement, one again finds that $U_i^{-1} V_i = C_i^{-1}$, so that Eq. (49) becomes

$$Q_{ij} = U_i A_{ij} C_j U_j^{-1} e^{i\alpha_{ij}}. \quad (51)$$
We now make a gauge transformation by $U_i$ in each $SU(N)_i$ factor group which brings us to the diagonal representation

$$Q_{ij} = A_{ij} C_j e^{i\alpha_{ij}}.$$  \hfill (52)

The final $SU(N)_1 \times SU(N)_2 \times SU(N)_m$ gauge transform by $(1, C_2, C_2C_3, \ldots, C_2C_3 \ldots C_m)$ then gives the canonical representation

$$Q_{ij} = A_{ij} e^{i\alpha_{ij}}, \quad i = 1 \ldots m - 1,$$

$$Q_{m1} = A_{m1} C e^{i\alpha_{m1}},$$  \hfill (53)

with $C = C_1C_2 \ldots C_m$. This contains $N + m - 1$ independent real variables and $m + N - 1$ phases in agreement with the number of independent holomorphic polynomials found in Section 4. There is again a residual unfixed $U(1)^{N-1}$ Coulomb phase gauge symmetry.

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