First-order invariants of differential 2-forms

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Abstract
Let $M$ be a smooth manifold of dimension $2n$, and let $O_M$ be the dense open subbundle in $\wedge^2 T^*M$ of 2-covectors of maximal rank. The algebra of Diff $M$-invariant smooth functions of first order on $O_M$ is proved to be isomorphic to the algebra of smooth $Sp(\Omega_x)$-invariant functions on $\wedge^3 T_x^*M$, $x$ being a fixed point in $M$, and $\Omega_x$ a fixed element in $(O_M)_x$. The maximum number of functionally independent invariants is computed.

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1 Reduction to symplectic group
Let $M$ be a $C^\infty$ manifold and let

$$G^r_x = G^r(x)(M) = \{ j^r_x \phi \in J^r(M, M) : \phi(x) = x, \det \phi_x \neq 0 \}, \quad r \geq 1,$$

be the Lie group of $r$-jets of diffeomorphisms at $x \in M$. If $r \geq s$, then $G^r_x$ denotes the kernel of the natural projection $G^r_x \rightarrow G^s_x$.

In particular, for every $r \geq 2$, $G^{r-1}_x$ is isomorphic to the vectorial group $S^r T^*_x(M) \otimes T_x(M)$, as $J^r(M, M) \rightarrow J^{r-1}(M, M)$ is an affine bundle modelled over $S^r T^*(M) \otimes J^{r-1}(M, M) T(M)$ and, therefore for every $j^r_x \phi \in G^{r-1}_x$ there exists a unique $t \in S^r T^*_x(M) \otimes T_x(M)$ such that $t + j^r_x(1_M) = j^r_x \phi$. Hence we can identify $j^r_x \phi$ to $t$.

**Theorem 1.** Let $M$ be a $C^\infty$ manifold of dimension $2n$, and let $p \colon O_M \rightarrow M$ be the dense open subbundle in $\wedge^2 T^*M$ of 2-covectors of maximal rank. Given a point $x \in M$, the map $\delta_x : J^2_x O_M \rightarrow \wedge^3 T^*_x M$, $\delta_x (j^2_x \Omega) = (d\Omega)_x$, is a $G^2_x$-equivariant $G^2_x$-invariant epimorphism.

**Proof.** The $G^2_x$-equivariance of $\delta_x$ is a consequence of the following well-known property: $(d(\phi^\ast \Omega))_x = \phi^\ast ((d\Omega)_{\phi(x)})$. In fact, for every $\phi \in \text{Diff}_x M$ one obtains

$$\delta_x \left( j^2_x \phi \cdot j^2_x \Omega \right) = \delta_x \left( j^2_x ((\phi^{-1})^\ast \Omega) \right) = (d((\phi^{-1})^\ast \Omega))_x = (\phi^{-1})^\ast ((d\Omega)_{\phi^{-1}(x)}) = j^2_x \phi \cdot \delta_x (j^2_x \Omega).$$

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Next, we show that $\delta_x$ is surjective. Let $w_3 \in \bigwedge^3 T^*_x M$. Let $(x^1)^2_{i=1}$ be a coordinate system centred at $x$. If $w_3 = \sum_{h<i<j} \lambda_{hij}(dx^h)_x \wedge (dx^i)_x \wedge (dx^j)_x$, then $w_3 = \delta_x(j^1 \Omega)$, where $\Omega = \sum_{h<i<j} \lambda_{hij} x^h dx^i \wedge dx^j$. If $(y_{hij})_{1 \leq h < i < j \leq 2n}$ is the coordinate system on $\bigwedge^3 T^*_x M$ given by

$$w_3 = \sum_{h<i<j} y_{hij}(w_3)(dx^h)_x \wedge (dx^i)_x \wedge (dx^j)_x, \quad \forall w_3 \in \bigwedge^3 T^*_x M,$$

then the equations for $\delta_x$ are $y_{abc} \circ \delta_x = y_{ac,b} - y_{ab,c}$, $a < b < c$.

This proves that $\delta_x$ is the restriction of a linear mapping. Moreover, the projection $p^{10}: J^1(\bigwedge^2 T^* M) \to J^0(\bigwedge^2 T^* M) = \bigwedge^2 T^* M$ is an affine bundle modelled over $T^* M \otimes \bigwedge^2 T^* M$, where the sum operation is defined as follows:

(1) $$(df)_x \otimes w_2 + j^1_x \Omega = j^1_x [(f - f(x)) \Omega' + \Omega],$$

$\Omega'$ being any 2-form such that $\Omega' = w_2 \in \bigwedge^2 T^* M$.

If $j^2_x \phi \in G^2_1$, then $(\phi^* \Omega)_x = \Omega_x$, i.e., $p^{10}(j^2_x \phi) = p^{10}(j^2_x (\phi^* \Omega))$; hence there exists a unique $\tau = \sum_{j<k} \tau_{jkl} (dx^j)_x \otimes (dx^k)_x \wedge (dx^l)_x \in T^*_x M \otimes \bigwedge^2 T^*_x M$ such that $j^k_x (\phi^* \Omega) = \tau + j^1_x \tau$. If $\Omega = \sum_{h<i} F_{hi} dx^h \wedge dx^i$, then

$$\phi^* \Omega = \sum_{j<k} F_{jk} dx^j \wedge dx^k, \quad F_{jk} = \sum_{h<i} (F_{hi} \circ \phi) \det \frac{\partial (\phi^h)}{\partial (\phi_i)}(x), \quad \phi^h = x^h \circ \phi,$$

and taking derivatives for $F_{jk}$ and evaluating at $x$, one obtains

$$\frac{\partial F_{jk}}{\partial x^h}(x) = \sum_{h<i} \frac{\partial F_{hk}}{\partial x^h}(\phi(x)) \frac{\partial (\phi_i)}{\partial (\phi_j)}(x) \det \frac{\partial (\phi^h)}{\partial (\phi_i)}(x) + \sum_{h<i} F_{hi}(\phi(x)) \left[ \frac{\partial^2 (\phi^h)}{\partial x^j \partial x^i} - \frac{\partial (\phi^h)}{\partial x^j} \frac{\partial (\phi^i)}{\partial x^j} \right]$$

As $j^2_x \phi \in G^2_1$, it follows: $\phi(x) = x$, $\frac{\partial (\phi^h)}{\partial x^j}(x) = \delta^h_j$, and from the previous formula one thus deduces $\tau_{jkl} = F_{hk}(x) \frac{\partial^2 (\phi^h)}{\partial x^j \partial x^k}(x) - F_{hj}(x) \frac{\partial^2 (\phi^h)}{\partial x^k \partial x^j}(x)$. As $\tau_{jkl}$ is alternate on $j, k$, one can write $\tau = \frac{1}{2} \tau_{jkl} (dx^j)_x \otimes (dx^k)_x \wedge (dx^l)_x$, and recalling that the coordinates are centred at $x$, taking the formula $[\square]$ into account, it follows: $j^1_x \tau = \tau$, where $\tau$ is the 2-form given by

(2) $$\tau = \frac{1}{2} x^j \left\{ F_{hk}(x) \frac{\partial^2 (\phi^h)}{\partial x^j \partial x^k}(x) - F_{hj}(x) \frac{\partial^2 (\phi^h)}{\partial x^k \partial x^j}(x) \right\} dx^j \wedge dx^k$$

$$= \frac{1}{2} x^j d \left( \frac{\partial (\phi^h)}{\partial x^j} \right) \wedge \left\{ F_{hk}(x) dx^k + F_{hj}(x) dx^j \right\}$$

$$= d \left\{ \frac{1}{2} x^i \frac{\partial (\phi^h)}{\partial x^i} - \phi^h \right\} \left( F_{hk}(x) dx^k + F_{hj}(x) dx^j \right).$$

Hence, $\delta_x (j^1_x (\phi^* \Omega)) = \delta_x (j^1_x (\Omega' + \Omega)) = \delta_x (j^1_x \Omega)$.

Let $M$ be an arbitrary $C^\infty$-manifold and let $\phi: \bigwedge^2 T^* M \to \bigwedge^2 T^* M$ be the natural lift of a diffeomorphism $\phi \in \text{Diff} M$; i.e., $\phi^*(w) = (\phi^{-1})^* w$ for every 2-covector $w \in \bigwedge^2 T^* M$. If $\Omega$ is a 2-form on $M$, then $\phi \circ \Omega \circ \phi^{-1} = (\phi^{-1})^* \Omega$. Let

$$J^1 \phi: J^1(\bigwedge^2 T^* M) \to J^1(\bigwedge^2 T^* M),$$

$$J^1 \phi(j^1_x \Omega) = j^1_x (\phi \circ \Omega \circ \phi^{-1}),$$

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be the 1-jet prolongation of $\tilde{\phi}$. A subset $S \subseteq J^1(\Lambda^2 T^* M)$ is said to be natural if $(J^1 \tilde{\phi})(S) \subseteq S$ for every $\phi \in \text{Diff} M$. Let $S \subseteq J^1(\Lambda^2 T^* M)$ be a natural embedded submanifold. A smooth function $I: S \to \mathbb{R}$ is said to be an invariant of first order under diffeomorphisms or even $\text{Diff} M$-invariant if $I \circ J^1 \tilde{\phi} = I$, $\forall \phi \in \text{Diff} M$. If we set $I(\Omega) = I \circ J^1 \tilde{\Omega}$, for a given 2-form $\Omega$ on $M$, then the previous invariance condition reads as $I((\phi^{-1})^* \Omega)(\phi(x)) = I(\Omega)(x)$, for all $x \in M$, $\phi \in \text{Diff} M$, thus leading us to the naive definition of an invariant, as being a function depending on the coefficients of $\Omega$ and its partial derivatives up to first order, which remains unchanged under arbitrary changes of coordinates.

**Theorem 2.** Let $M$ be a smooth connected manifold of dimension $2n$. The ring of invariants of first order on $O_M$ is isomorphic to $C^\infty (\Lambda^3 T_x^* M)^{Sp(\Omega_x)}$, where $\Omega_x$ is a fixed element in $O_M$.

**Proof.** As $M$ is connected, the group $\text{Diff} M$ acts transitively on $M$. Therefore, it suffices to fix a point $x \in M$ and to compute $\text{Diff}_x M$-invariant functions in $C^\infty(J^1_x O_M)$. From the very definitions it follows:

$$C^\infty(J^1_x O_M)^{\text{Diff}_x M} = C^\infty(J^1_x O_M)G^2_x,$$

and by virtue of Theorem 1 the map $\delta_x: J^1_x O_M \to (O_M)_x \times \Lambda^3 T_x^* M$, defined as follows: $\delta_x(j^1 \Omega) = (\Omega_x, (d\Omega)_x)$, is $G^2_x$-invariant and surjective; hence the induced homomorphism $(\delta_x)^*: C^\infty((O_M)_x \times \Lambda^3 T_x^* M) \to C^\infty(J^1_x O_M)^{G^2_x}$ is injective. Next, we shall prove that $(\delta_x)^*$ is also surjective, by showing that every $I \in C^\infty(J^1_x O_M)^{G^2_x}$ takes constant value on the fibres of $\delta_x$, as in this case $I$ induces $\bar{I} \in C^\infty((O_M)_x \times \Lambda^3 T_x^* M)$ such that $I = \bar{I} \circ \delta_x = (\delta_x)^*(\bar{I})$. Actually, this is a consequence of the fact that the fibres of $\delta_x$ coincide with the orbits of $G^2_x$ on $J^1_x O_M$. To prove this, we first observe that every $j^1 \Omega \in \delta_x^{-1}(\delta_x(j^1 \Omega))$ can be written as $j^1 \bar{\Omega} = j^1(\tilde{\Omega} + \Omega)$ with $\bar{\Omega}_x = 0$, $(d\bar{\Omega})_x = 0$, and the proof reduces to show the existence of a 2-form $\Omega'$ given by the formula 2 such that $j^1 \Omega' = j^1 \tilde{\Omega}$, since we have seen that $j^1 (\phi^* \Omega) = j^1 (\Omega' + \Omega)$, for some $j^1 \phi \in G^2_x$.

The rank of $\Omega$ being $2n$, there exists a coordinate system $(x^n)_{i=1}^{2n}$ centred at $x$ such that $\Omega_x = \sum_{i=1}^n(dx^{2i-1})_x \wedge (dx^{2i})_x$, or equivalently, $F_{2i-1, 2i}(x) = 1$, and $F_{jk}(x) = 0$, $1 \leq j < k \leq 2n$ otherwise. We have $\Omega = \sum_{1 < j} \lambda_{ijk} x^i dx^j \wedge dx^k$, or terms of order $\geq 2$, because $\Omega$ vanishes at $x$, and by imposing $\bar{\Omega}$ to be closed at $x$, we obtain

$$0 = \lambda_{ijk} - \lambda_{kij} + \lambda_{kij}, \quad 1 \leq k < i < j \leq 2n. \tag{3}$$

Then, the equation $j^1 \Omega' = j^1 \tilde{\Omega}$ is equivalent to the system

$$F_{hk}(x) \frac{\partial^2 \phi^h}{\partial x^i \partial x^j}(x) - F_{kj}(x) \frac{\partial^2 \phi^j}{\partial x^i \partial x^k}(x) = \lambda_{ijkl}, \quad 1 \leq j < k \leq 2n,$$

or equivalently, for $1 \leq j' < k' \leq n$,

$$\frac{\partial^2 \phi^{2k' - 1}}{\partial x^{2i} \partial x^{2k'}}(x) - \frac{\partial^2 \phi^{2j' - 1}}{\partial x^{2i} \partial x^{2j'}}(x) = \lambda_{2j', 2k', i},$$

$$\frac{\partial^2 \phi^{2j'}}{\partial x^{2i} \partial x^{2k'}}(x) - \frac{\partial^2 \phi^{2k' - 1}}{\partial x^{2i} \partial x^{2j'}}(x) = \lambda_{2j', 2k' - 1, i},$$

$$\frac{\partial^2 \phi^{2k' - 1}}{\partial x^{2i} \partial x^{2j'}}(x) + \frac{\partial^2 \phi^{2j'}}{\partial x^{2i} \partial x^{2k'}}(x) = \lambda_{2j' - 1, 2k', i},$$

$$\frac{\partial^2 \phi^{2j'}}{\partial x^{2i} \partial x^{2j'}}(x) + \frac{\partial^2 \phi^{2k' - 1}}{\partial x^{2i} \partial x^{2k'}}(x) = \lambda_{2j' - 1, 2k' - 1, l}. \tag{4}$$
as follows by taking derivatives with respect to $x^l$, $1 \leq l \leq 2n$, and evaluating at $x$, in the coefficient of $dx^j \wedge dx^k$ in the right-hand side of the first equation in \[.\] Furthermore, as a computation shows, the equations \[ are seen to be the compatibility conditions of the system \[, thus concluding that an element $\phi \in G_2^j$ satisfying such a system really exists.

Therefore, a $G_2^j$-equivariant isomorphism of algebras holds:

\[
(\delta_x)^*: C^\infty((O_M)_x \times \wedge^3 T_x^* M) \xrightarrow{\cong} C^\infty(J_x^1 O_M)^{G_2^j}.
\]

Moreover, as $G_2^j$ is the semidirect product $G_2^j = G_2^{j_1} \times G_2^{j_2}$, taking invariants with respect to $G_2^j$ being any transformation verifying $\Omega_x \in G_2^j$, we finally obtain an isomorphism

\[
(\delta_x)^*: C^\infty((O_M)_x \times \wedge^3 T_x^* M)^{G_2^j} \xrightarrow{\cong} \left(C^\infty(J_x^1 O_M)^{G_2^{j_1}}\right)^{G_2^{j_2}} = C^\infty(J_x^1 O_M)^{G_2^{j_2}}.
\]

Once an element $\Omega_x \in (O_M)_x$ has been fixed, the following injective map is defined:

\[
\alpha_{\Omega_x}^1: \wedge^3 T_x^* M \rightarrow (O_M)_x \times \wedge^3 T_x^* M,
\]

such that $L_A \circ \alpha_{\Omega_x}^1 = \alpha_{A, \Omega_x}^1 \circ L_A$, $\forall A \in G_2^j$; in particular, if $A \in Sp(\Omega_x)$, we have $L_A \circ \alpha_{\Omega_x}^1 = \alpha_{\Omega_x}^1 \circ L_A$. Hence the map $\alpha_{\Omega_x}^1$ is $Sp(\Omega_x)$-equivariant. If $f \in C^\infty((O_M)_x \times \wedge^3 T_x^* M)^{G_2^j}$, then $f \circ \alpha_{\Omega_x}^1 \in C^\infty(\wedge^3 T_x^* M)^{Sp(\Omega_x)}$. In fact, if $A \in Sp(\Omega_x)$, then for every $\theta_x \in \wedge^3 T_x^* M$ we have

\[
(f \circ \alpha_{\Omega_x}^1)(A \cdot \theta_x) = f(\Omega_x, A \cdot \theta_x) = f(A \cdot \Omega_x, A \cdot \theta_x) = f(\Omega_x, \theta_x) = (f \circ \alpha_{\Omega_x}^1)(\theta_x).
\]

Therefore, the $Sp(\Omega_x)$-equivariant ring-homomorphism

\[
(\alpha_{\Omega_x}^1)^*: C^\infty((O_M)_x \times \wedge^3 T_x^* M) \rightarrow C^\infty(\wedge^3 T_x^* M)
\]

induced by $\alpha_{\Omega_x}^1$ maps $C^\infty((O_M)_x \times \wedge^3 T_x^* M)^{G_2^j}$ into $C^\infty(\wedge^3 T_x^* M)^{Sp(\Omega_x)}$, and the restriction of $(\alpha_{\Omega_x}^1)^*$ to $C^\infty(Sp(\Omega_x))$ is denoted by

\[
(\alpha_{\Omega_x}^1)^{**}: C^\infty((O_M)_x \times \wedge^3 T_x^* M)^{G_2^j} \rightarrow C^\infty(\wedge^3 T_x^* M)^{Sp(\Omega_x)}.
\]

We prove that $(\alpha_{\Omega_x}^1)^{**}$ is injective. Actually, if $(\alpha_{\Omega_x}^1)^{**}(f) = 0$, then

\[
f(\Omega_x, \theta_x) = 0, \quad \forall \theta_x \in \wedge^3 T_x^* M.
\]

As $f$ is invariant under the action of $G_2^j$, we also have $f(A \cdot \Omega_x, A \cdot \theta_x) = 0$, $\forall A \in G_2^j$, $\forall \theta_x \in \wedge^3 T_x^* M$; as $G_2^j$ operates transitively on $(O_M)_x$, it follows:

\[
f = 0, \quad \text{because given an arbitrary point } (\Omega_x', \theta_x') \in (O_M)_x \times \wedge^3 T_x^* M \text{ there exists } A \in G_2^j \text{ such that } A \cdot \Omega_x = \Omega_x' \text{ and by taking } \theta_x = A^{-1} \cdot \theta_x', \text{ we conclude } f(\Omega_x', \theta_x') = 0.
\]

The map $(\alpha_{\Omega_x}^1)^{**}$ is also surjective: For every $g \in C^\infty(\wedge^3 T_x^* M)^{Sp(\Omega_x)}$, we define $f: (O_M)_x \times \wedge^3 T_x^* M \rightarrow \mathbb{R}$ as follows: $f(\Omega_x', \theta_x) = g(A^{-1} \cdot \theta_x), \quad A \in G_2^j$. Being any transformation verifying $\Omega_x' = A \cdot \Omega_x$, the definition is correct, since if $B$ also verifies the equation $\Omega_x' = B \cdot \Omega_x$, then $A^{-1} B \in Sp(\Omega_x)$ and $g$ being invariant under the action of $Sp(\Omega_x)$, we have $g(B^{-1} \cdot \theta_x) = g((A^{-1} B)^{-1} A^{-1} \cdot \theta_x) = g(A^{-1} \cdot \theta_x)$. Furthermore, $f$ is $G_2^j$-invariant, as $f(A' \cdot \Omega_x, A' \cdot \theta_x) = g((A')^{-1} A \cdot \theta_x) = f(\Omega_x, \theta_x), \forall A' \in G_2^j$, thus concluding.
2 The number of invariants

2.1 Infinitesimal invariants

From now onwards, $V$ denotes a real vector space of dimension $2n$ and $\Omega \in \wedge^2 V^*$ denotes a non-degenerate skew-symmetric bilinear form on $V$.

Let $(v_i)_{i=1}^{2n}$ be a basis for $V$ with dual basis $(v^i)_{i=1}^{2n}$. We define coordinate functions $y_{abc}$, $1 \leq a < b < c \leq 2n$, on $\wedge^3 V^*$ by setting

\[
\theta = \sum_{1 \leq a < b < c \leq 2n} y_{abc}(\theta) \ (v^a \wedge v^b \wedge v^c) \in \wedge^3 V^*.
\]

If $A \in GL(V)$, then for $1 \leq a < b < c \leq 2n$ we have

\[
A \cdot (v^a \wedge v^b \wedge v^c) = (A^{-1})^* v^a \wedge (A^{-1})^* v^b \wedge (A^{-1})^* v^c
\]
\[
= (v^a \circ A^{-1}) \wedge (v^b \circ A^{-1}) \wedge (v^c \circ A^{-1})
\]
\[
= (\lambda_a^h v^h) \wedge (\lambda_b^i v^i) \wedge (\lambda_c^j v^j)
\]
\[
= \sum_{1 \leq h < i < j \leq 2n} \begin{vmatrix} \lambda_{ah} & \lambda_{bh} & \lambda_{ch} \\ \lambda_{ai} & \lambda_{bi} & \lambda_{ci} \\ \lambda_{aj} & \lambda_{bj} & \lambda_{cj} \end{vmatrix} v^h \wedge v^i \wedge v^j,
\]

where $(\lambda_{ij})_{i,j=1}^{2n}$ is the matrix of $(A^{-1})^T$ in the basis $(v_i)_{i=1}^{2n}$ and the superscript $T$ means transpose. In what follows we assume $\Omega = \sum_{i=1}^{2n} v^i \wedge v^{n+i}$.

A function $I: \wedge^3 V^* \to \mathbb{R}$ is $Sp(\Omega)$-invariant if $I(A \cdot \theta) = I(\theta)$, $\forall \theta \in \wedge^3 V^*$, $\forall A \in Sp(\Omega)$.

Lemma 3. A smooth function $I: \wedge^3 V^* \to \mathbb{R}$ is $Sp(\Omega)$-invariant if and only if $I$ is a first integral of the distribution spanned by the following vector fields:

\[
U^* = \sum_{1 \leq h < i < j \leq 2n} \left( \sum_{1 \leq a < b < c \leq 2n} U_{hij}^{abc} y_{abc}(\theta) \right) \frac{\partial}{\partial y_{hij}},
\]

\[
U = (u_{ij})_{i,j=1}^{2n} \in \mathfrak{sp}(2n, \mathbb{R}),
\]

where the functions $U_{hij}^{abc}$ are given by the formulas

\[
U_{hij}^{abc} = \begin{vmatrix} h_{ab} & h_{bc} & h_{ca} \\ a_{ia} & a_{ib} & a_{ic} \\ u_{ja} & u_{jb} & u_{jc} \end{vmatrix} - \begin{vmatrix} h_{ai} & h_{ib} & h_{ic} \\ a_{ia} & a_{ib} & a_{ic} \\ u_{ja} & u_{jb} & u_{jc} \end{vmatrix} - \begin{vmatrix} h_{ai} & h_{ib} & h_{ic} \\ h_{ai} & h_{ib} & h_{ic} \\ u_{ja} & u_{jb} & u_{jc} \end{vmatrix}.
\]

Proof. If $I$ is invariant, then, in particular, we have $I(\exp(tU) \cdot \theta) = I(\theta)$, $\forall t \in \mathbb{R}$, $U = (u_{ij})_{i,j=1}^{2n} \in \mathfrak{sp}(\Omega)$. If $A(t) = \exp(-tU^T)$, then

\[
I(\sum_{1 \leq h < i < j \leq 2n} y_{abc} \left( \lambda_{ah}(t) & \lambda_{bh}(t) & \lambda_{ch}(t) \\ \lambda_{ai}(t) & \lambda_{bi}(t) & \lambda_{ci}(t) \\ \lambda_{aj}(t) & \lambda_{bj}(t) & \lambda_{cj}(t) \right) v^h \wedge v^i \wedge v^j) = I(\theta),
\]

and taking derivatives at $t = 0$, it follows:

\[
0 = \sum_{1 \leq a < b < c \leq 6, 1 \leq h < i < j \leq 6} U_{hij}^{abc} y_{abc}(\theta) \frac{\partial I}{\partial y_{hij}}(\theta),
\]

$U_{hij}^{abc}$ being as in the statement. The converse follows from the fact that the symplectic group is connected and hence, every symplectic transformation is a product of exponentials of matrices in the symplectic algebra. \qed
Theorem 4. The distribution $\mathcal{D} \subset T(\wedge^3 V^*)$ whose fibre $\mathcal{D}_\theta$ over $\theta \in \wedge^3 V^*$ is the subspace $(U^*)_\theta$, $U \in \mathfrak{sp}(2n, \mathbb{R})$, is involutive and of locally constant rank on a dense open subset $\mathcal{O} \subset \wedge^3 V^*$.

The number $N_{2n}$ of functionally independent $\mathfrak{sp}(\Omega)$-invariant functions defined on $\mathcal{O}$ is equal to $N_{2n} = \binom{2n}{3} - \text{rank} \mathcal{D}|_{\mathcal{O}}$.

Proof. Every pair of vector fields $U'^*,$ $U''^*$ belonging to $\mathcal{D}$ on an open subset $O \subset \wedge^3 V^*$ can be written as $U'^* = \sum_{h=1}^{n(2n+1)} j^h(U_h)^*, \quad U''^* = \sum_{i=1}^{n(2n+1)} g^i(U_i)^*$, with $f^h, g^i \in C^\infty(O)$, where $(U_1, \ldots, U_{n(2n+1)})$ is a basis of $\mathfrak{sp}(2n, \mathbb{R})$. As $[U'^*, U''^*] = -[U_h, U_i]^*$, it follows that $[U'^*, U''^*]$ can be written as a linear combination of $(U_h)^*, (U_i)^*$, and $[U_h^*, U_i^*] = -[U_h, U_i]^* = -c^j_{hi}(U_j)^*$, where $c^j_{hi}$ are the structure constants of $\mathfrak{sp}(2n, \mathbb{R})$ on this basis. This proves that $\mathcal{D}$ is involutive. Moreover, we first recall that the dimension of the vector spaces $\{D_\theta : \theta \in \wedge^3 V^*\}$ is uniformly bounded by $\dim(\wedge^3 V^*) = \binom{2n}{3}$. Let $O \subset \wedge^3 V^*$ be the subset defined as follows: A point $\theta \in \wedge^3 V^*$ belongs to $O$ if and only if $\theta$ admits an open neighbourhood $N$ such that $d = \dim D_\theta = \max_{\theta' \in N} (\dim D_{\theta'})$. We claim that $O$ is an open subset. Actually, there exists an open neighbourhood $N' \subset \theta$ such that the dimension of the fibres of $\mathcal{D}$ over the points $\theta' \in N'$ is at least $d$, as $(X_i)^*|_{\theta'}$, $1 \leq i \leq d$, is a basis for $\mathcal{D}$ at $\xi$, for certain $X_i \in \mathfrak{sp}(2n, \mathbb{R})$, $1 \leq i \leq d$, then the vector fields $(X_i)^*$, $1 \leq i \leq d$, are linearly independent at each point of a neighbourhood of $\theta$. From the definition of $O$ we thus conclude that if $\theta \in O$, then we have $\dim D_{\theta'} = d$ for every $\theta' \in N \cap N'$; hence $N \cap N' \subset O$. The same argument proves that the rank of $\mathcal{D}$ is locally constant over $O$. Next, we prove that $O$ is dense. Let $N$ be an open neighbourhood of an arbitrarily chosen point $\theta \in \wedge^3 V^*$ and let $\theta'$ be a point in $N$ such that the rank of $\mathcal{D}|_{\mathcal{N}}$ takes its greatest value at $\theta'$. By proceeding as above, we deduce that $\theta'$ belongs to $O$. Finally, the formula for the number of invariants in the statement now follows from Frobenius’ theorem. \hfill \square

Remark 5. We have $N_2 = 0$, as $\wedge^3 V^* = \{0\}$ if $\dim V = 2$, and $N_4 = 0$, as $\mathfrak{sp}(2n)$ acts transitively on $\wedge^3 V^\ast \setminus \{0\}$ if $\dim V = 2n = 4$. Furthermore, as a consequence of the results obtained in [5], it follows that the generic rank of $\mathcal{D}$ for $\dim V = 2n = 6$ is 18; hence $N_6 = 2$.

2.2 $N_{2n}$ computed

Theorem 6. We have

$$N_{2n} = \begin{cases} 0, & 1 \leq n \leq 2, \\ 2, & n = 3, \\ \frac{n(4n^2-12n-1)}{3}, & n \geq 4. \end{cases}$$

Proof. The formula in the statement for $1 \leq n \leq 3$ follows from Remark 5. Hence we can assume $n \geq 4$. For every 3-covector $\theta \in \wedge^3 V^*$, let us define $\mu_\theta : \mathfrak{sp}(\Omega) \to \wedge^3 V^*$, $\mu_\theta(U) = U \cdot \theta$, $\forall U \in \mathfrak{sp}(\Omega)$.

It suffices to prove that there exists a dense open subset $O' \subset \wedge^3 V^*$ such that $\mu_\theta$ is an immersion if $\theta \in O'$, since $\mu_\theta$ is a vector field on $\mathfrak{sp}(\Omega)$, and $\mu_\theta(\mathfrak{sp}(\Omega)) = \mathcal{D}_0$. \hfill \square
By expanding on (6) it follows:

\[-U^* = \sum_{1 \leq a < b < c \leq 2n} \sum_{1 \leq d < b} u_{da} y_{abc} Y_{dcb} + \sum_{1 \leq a < b < c \leq 2n} \sum_{c < d \leq 2n} u_{da} y_{abc} Y_{hcd} - \sum_{1 \leq a < b < c \leq 2n} \sum_{b < d < c} u_{da} y_{abc} Y_{bdc} + \sum_{1 \leq a < b < c \leq 2n} \sum_{a < d < c} u_{dc} y_{abc} Y_{adc} - \sum_{1 \leq a < b < c \leq 2n} \sum_{c < d \leq 2n} u_{db} y_{abc} Y_{dc} + \sum_{1 \leq a < b < c \leq 2n} \sum_{a < d < c} u_{db} y_{abc} Y_{dac} + \sum_{1 \leq a < b < c \leq 2n} \sum_{b < d < c} u_{dc} y_{abc} Y_{abcd} - \sum_{1 \leq a < b < c \leq 2n} \sum_{a < d < c} u_{dc} y_{abc} Y_{adbc}\]

where \( Y_{hij} = \frac{\partial}{\partial \theta_{hij}}, 1 \leq h < i < j \leq 2n \). Given indices \( 1 \leq \alpha < \beta < \gamma \leq 2n \), the coefficient of \( Y_{\alpha \beta \gamma} \) in (7) is

\[
C_{\alpha \beta \gamma} = \sum_{\alpha = 1}^{\beta - 1} u_{\alpha \alpha} y_{\alpha \beta \gamma} + \sum_{\alpha = 1}^{\beta - 1} u_{\gamma \alpha} y_{\alpha \beta \alpha} - \sum_{\alpha = 1}^{\beta - 1} u_{\beta \alpha} y_{\alpha \alpha \gamma} + \sum_{\alpha = \beta + 1}^{\gamma - 1} u_{\gamma \alpha} y_{\alpha \beta \alpha} - \sum_{\alpha = \beta + 1}^{\gamma - 1} u_{\alpha \alpha} y_{\alpha \beta \gamma} + \sum_{\alpha = \gamma + 1}^{2n} u_{\gamma \alpha} y_{\alpha \beta \alpha} + \sum_{\alpha = \gamma + 1}^{2n} u_{\alpha \alpha} y_{\alpha \beta \gamma} - \sum_{\alpha = \gamma + 1}^{2n} u_{\beta \alpha} y_{\alpha \alpha \gamma},
\]

\( 1 \leq \alpha < \beta < \gamma \leq 2n \).

As the matrix \( U = (u_{ij})_{i,j=1}^{2n} \) is symplectic, the following symmetries hold:

\[
u_{j,n+i} = u_{i,n+j}, \quad \nu_{n+j,i} = u_{n+i,j}, \quad \nu_{n+j,n+i} = -u_{i,j}, \quad 1 \leq i < j \leq n.
\]

A vector \( \theta^* \) belongs to \( \ker(\mu_\theta)_* \) if and only if \( C_{\alpha \beta \gamma} = 0 \) for every system of indices \( 1 \leq \alpha < \beta < \gamma \leq 2n \). We thus obtain a homogeneous linear system \( S_{2n} \) of \( \binom{2n}{3} \) linear equations in the \( n(2n + 1) \) unknowns \( u_{ij}, i,j = 1, \ldots, n; \) \( u_{n+i,j}, u_{n+j,i}, u_{n+j,n+i}, 1 \leq i \leq j \leq n \), and we have \( \binom{2n}{3} > n(2n + 1) \) for every \( n \geq 4 \). Evaluating \( S_{2n} \) at the 3-covector \( \theta^0 \) of coordinates \( y_{abc}(\theta^0) = a + b + c, 1 \leq a < b < c \leq 2n \), as a numerical calculation shows, the only solution to \( S_{2n}(\theta^0) \) is given by \( u_{ij} = 0, i,j = 1, \ldots, n \). We can thus conclude by simply applying the formula (1) for \( N_{2n} \) in Theorem 4.

References

[1] J. Muñoz Masqué, L. M. Pozo Coronado, A new look at the classification of the tri-covectors of a 6-dimensional symplectic space, Linear and Multilinear Algebra, DOI:doi.org/10.1080/03081087.2018.1440517 (to appear).