Quantum Communication and Decoherence

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1.1 Introduction

The problem of decoherence is an integral part of the theory of quantum computation and communication. The potential of a quantum computer lies in its ability to process information in the form of a coherent superposition of quantum mechanical states. Quantum algorithms such as Shor’s algorithm make use of the interference of different “computational paths”, which can strongly enhance their efficiency compared to classical algorithms. Because quantum coherence and interference play a central role in a quantum computer, decoherence is a major threat to its proper functioning.

A similar situation prevails in quantum communication. The central problem of quantum communication is how to faithfully transmit unknown quantum states through a noisy quantum channel. While quantum information is sent through a channel such as an optical fiber, the carriers of the information (e.g. photons) interact with the channel and get entangled with its many degrees of freedom, which gives rise to the phenomenon of decoherence on the state space of the information carriers. An initially pure state becomes a mixed state when it leaves the channel. For quantum communication purposes, it is however essential that the transmitted qubits retain their genuine quantum properties, for example in form of an entanglement with qubits on the other side of the channel.

To deal with the problem of decoherence, two methods have been developed, known as quantum error correction and entanglement purification, respectively. In quantum error correction, which will be discussed in the next section, quantum information is encoded in the joint state of several two-state particles, forming a so-called quantum error correcting code, before it is sent through the channel. By measuring certain joint observables of the particles (the so-called stabilizer of the code), it is thereby possible to “reset” the state of the information carriers after a given time, by projecting their joint state onto certain subspaces of their Hilbert space without destroying the coherence of the encoded information. Even though quantum error correction can be used, in principle, to send quantum information through a noisy channel, it has been primarily developed to stabilize a quantum computer against the effect of decoherence. Entanglement purification, on the other hand, together with the method of teleportation, is a powerful tool that is particularly suitable for quantum communication.
The idea of entanglement purification is to “distill” from an ensemble of low-fidelity Einstein-Podolsky-Rosen (EPR) pairs, which have been distributed through some noisy channel, a smaller ensemble of high-fidelity EPR pairs which may then be used for faithful teleportation or for quantum cryptography. This distillation process requires certain unitary operations and measurements to be performed on the qubits at each side of the channel, and a process of postselection, which also requires classical communication between the parties.

Both methods, quantum error correction and entanglement purification, fight decoherence by a process of controlled disentanglement of the information carriers from the quantum channel. This process involves the action of some apparatus that is used to transform and measure the state of the particles, for example via tunable interactions of the particles with each other and with external fields. Real apparatuses are themselves sources of noise, which complicates the situation considerably. From a general perspective, the apparatuses used by Alice and Bob must themselves be considered as part of the noisy communication channel. Under realistic circumstances, the information carriers will thus always become entangled with other degrees of freedom and therefore suffer from a certain amount of decoherence. The question is therefore not whether decoherence can be avoided at all, but whether its influence can be kept on a tolerable level.

What “tolerable” means depends on the context. In quantum computation, for example, the effect of decoherence may be tolerable as long as the fidelity of the output of a quantum algorithm is above a certain value, allowing one to extract the desired result with the corresponding probability. In quantum cryptography the effect of the channel cannot, in principle, be distinguished from an intelligent third party who manipulates the transmitted quantum systems to gain information about their state. All noise of a channel is therefore attributed — this is the pessimistic attitude of the cryptologist — to an adversary. Decoherence is thereby considered due to entanglement of the information carriers with degrees of freedom controlled by an adversary. As we will show in the later part of this review, the security of quantum cryptography is in fact closely connected to the disentanglement of the degrees of freedom of the information carriers, on one side, and the channel, on the other side. Even though we cannot avoid all residual entanglement with the channel, we can distinguish between residual entanglement with the apparatus, which is harmless, and residual entanglement with the part of the channel accessible to an eavesdropper, which is potentially harmful.

In the following, we will give a brief introduction to the methods of quantum error correction and entanglement purification, and to the basic protocols of quantum cryptography. We will then discuss a recent security proof for entanglement-based quantum communication through noisy channel.

1 For a more comprehensive introduction into these fields of quantum information theory, see, for example, Ref. [12].
nels, which explicitly takes into account the role of noisy apparatus. We try to pay particular attention to conceptual issues but skip some of the technical details, which can be found in the literature.

1.2 Quantum Error Correction

Quantum mechanical entanglement is exploited in quantum algorithms and in many protocols for quantum communication such as teleportation or entanglement-based quantum key distribution. It also plays a fundamental role in quantum error correction, where the coding operations are themselves simple quantum algorithms. Let us illustrate the basic principles at the example of the first quantum error correcting code found by Peter Shor in 1995 \cite{2}. To protect quantum information that is represented by the state of a particle (central qubit in Fig. 1.1) against decoherence, the information is first distributed or delocalized over several particles. In Fig. 1.1 this is done with the help of the network $\text{ENC}$, which realizes the following mapping:

$$\text{ENC}: (\alpha|0\rangle + \beta|1\rangle)|0\rangle|0\rangle \cdots |0\rangle \mapsto \alpha|0\rangle_S + \beta|1\rangle_S$$

in which the states

$$|0\rangle_S = 2^{-3/2}(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)$$

$$|1\rangle_S = 2^{-3/2}(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)(|000\rangle - |111\rangle).$$

denote the so-called code words of the (9-bit) Shor code. The encoding transformation thus corresponds to an embedding $\mathcal{H} \ni |\phi\rangle = \alpha|0\rangle + \beta|1\rangle \mapsto \alpha|0\rangle_S + \beta|1\rangle_S = |\phi\rangle_S \in \mathcal{H}_S \subset \mathcal{H}^{\otimes 9}$ of the two-dimensional Hilbert space $\mathcal{H} \simeq \mathbb{C}^2$ of the central qubit into the higher-dimensional Hilbert space of all 9 qubits. After the transformation the quantum information lies in a two-dimensional subspace $\mathcal{H}_S$ of a $2^9$ dimensional Hilbert space. The code words $|0\rangle_S$ and $|1\rangle_S$ are tensor products of entangled three-qubit states of the form $|000\rangle \pm |111\rangle$, the so-called Greenberger-Horne-Zeilinger (GHZ) states \cite{14}, which play a prominent role for the interpretation of quantum mechanics. \cite{14,15}. One can easily check that after the encoding (see dotted line in Fig. 1.1) the reduced density operator of each of the qubits is totally mixed; that is, the individual state of the particles carries no information about $|\phi\rangle$.\footnote{Quantum error correcting codes are indeed constructed in such a way that the state of individual qubits in a codeword becomes completely undetermined. As was shown by DiVincenzo and Peres \cite{16}, the codewords satisfy generalized Mermin relations \cite{15} that exclude the possibility of consistently assigning a predetermined value to complementary observables of each qubit. From the measurement of an individual qubit one can thus not gain any information about $|\phi\rangle$. In the positive sense this means that an uncontrolled interaction of the environment with one of the qubits does not (necessarily) lead to an irreversible loss of information.}
Fig. 1.1. Quantum logic network of the Shor code and quantum error correction. A “random rotation” $\sigma_{\mu,j}$ on qubit $j$ in the encoded state translates into a certain “error syndrome” $\epsilon_1, \ldots, \epsilon_8$ and a corresponding unitary operation $U = U(\epsilon_1, \ldots, \epsilon_8)$ on the central qubit (see text). The network uses the Hadamard-Rotation $R_j = 1/\sqrt{2}(\sigma_x,j + \sigma_z,j)$ and the CNOT gate ($\sigma = \text{CNOT}_{i,j} = 1 + \sigma_z,i \sigma_x,j$).

For simplicity let us consider an error model where random rotations are applied to the individual qubits with a certain “error rate”. This model is more general than it appears to be at first sight but it needs a justification to which we shall return below. Suppose that, after the encoding circuit of Fig. 1.1, one of the four Pauli-Rotations $\sigma_{\mu,j}$ ($\mu = 0, 1, 2, 3$) is applied to one of the nine qubits, where both $j$ and $\mu$ are random and unknown to us. The question is this: Can we still extract $|\phi\rangle$ from the joint state of the particles? If such a random rotation is applied to the particle before the encoding, it is clear that the information is lost, for $\frac{1}{4} \sum_{\mu} \sigma_{\mu,j} |\phi\rangle \langle \phi| \sigma_{\mu} = \frac{1}{4} I$. If it is applied to one of the particles of the encoded state, however, the information can still be rescued from the joint state of all 9 particles. A possibility to do this is shown in Fig. 1.1 There the inverse network of ENC is applied to the code which transforms the corrupt state $\sigma_{\mu,j} |\phi\rangle_S$, for arbitrary $\sigma_{\mu,j}$, back to a product state:

$$ENC^{-1} : \sigma_{\mu,j} |\phi\rangle_S \longrightarrow |\phi'\rangle |\epsilon_1\rangle |\epsilon_2\rangle \ldots |\epsilon_8\rangle.$$  \hspace{1cm} (1.3)

After this decoding transformation, the state of the neighboring qubits carries the so-called error syndrome $\epsilon_1 \ldots \epsilon_8$. The central qubit is in state $|\phi'\rangle = U(\epsilon_1 \ldots \epsilon_8) |\phi\rangle$, where the unitary transformation $U(\epsilon_1 \ldots \epsilon_8) \in \{I, \sigma_x, \sigma_y, \sigma_z\}$, is uniquely determined by the error syndrome.

By reading off the error syndrome, i.e. measuring the state of all neighboring qubits, and subsequently applying the correction operation $U^{-1}(\epsilon_1 \ldots \epsilon_8)$, the central qubit is transformed back to its initial state. Please note that the central qubit remains unmeasured, and no information about the state $|\phi\rangle$ is
obtained at any step of the protocol. By iteration of the sequence decoding $\rightarrow$ syndrome measurement $\&$ correction $\rightarrow$ encoding [1] an unknown quantum state can thus be protected against decoherence over a time significantly longer than the decoherence time.

The effect of the random rotations $\sigma_{\mu,j}$ is to map the code space $H_S$ to a set of orthogonal error spaces $\sigma_{\mu,j}H_S \perp H_S$. The images of the code words thereby satisfy the following orthogonality relations $S\langle 0|\sigma_{\mu,j}\sigma_{\nu,k}|1\rangle_S = 0$ and $S\langle 0|\sigma_{\mu,j}\sigma_{\nu,k}|0\rangle_S = \langle 1|\sigma_{\mu,j}\sigma_{\nu,k}|1\rangle_S$ for all $j, k, \mu, \nu$. These relations ensure [6,18], that all errors $\sigma_{\mu,j}$ can, in fact, be corrected. The Shor code was the first quantum error correcting code found that can correct all of the four errors (spin flip, phase flip, spin&phase flip, identity) on any one of the qubits. Independent of Shor, Steane [4] found a code that achieves the same task using only 7 qubits. Later, the theory of quantum error correcting codes was further developed [3,19], establishing in particular the connection with classical coding theory. A number of other codes were found, among them a so-called ‘perfect’ code using a minimum number of only 5 qubits [17,6]. One can also construct codes that are able to correct more than a single qubit error. These satisfy a similar set of orthogonality relations of the form given above, and the code words are entangled states of an increasing number of qubits. An introduction to the theory of quantum error correction can be found in the articles by Steane [20,21] and by Gottesman [22], for example.

In quantum error correction we exploit, as in quantum algorithms, the possibility to manipulate superpositions of states of a quantum register and to measure joint observables which describe joint properties of several qubits. The operation $ENC^{-1}$ for example, followed by one-qubit measurements, corresponds to the measurement of the “parity” of different qubits, as was shown by Gottesman [23]. The joint observables are here:

$$\begin{align*}
M_1 &= \sigma_{z,1}\sigma_{z,2}, \\
M_2 &= \sigma_{z,2}\sigma_{z,3}, \\
M_3 &= \sigma_{z,4}\sigma_{z,5}, \\
M_4 &= \sigma_{z,6}, \\
M_5 &= \sigma_{z,7}\sigma_{z,8}, \\
M_6 &= \sigma_{z,9}, \\
M_7 &= \sigma_{x,1}\sigma_{x,2}\sigma_{x,3}\sigma_{x,4}\sigma_{x,5}\sigma_{x,6}, \\
M_8 &= \sigma_{x,6}\sigma_{x,7}\sigma_{x,8}\sigma_{x,9}. 
\end{align*}$$

(1.4)

The error spaces $\sigma_{\mu,j}H_S$ are eigenspaces of these observables with eigenvalues $\pm 1$. The observable $M_1 = \sigma_{z,1}\sigma_{z,2}$, for example, tells us whether on qubit 1 or 2 a spin flip has occurred without revealing any information on which of the qubits: $M_1(|000\rangle + |111\rangle) = +(|000\rangle + |111\rangle)$, $M_1(|100\rangle + |011\rangle) = -(|000\rangle + |111\rangle)$. These observables generate an Abelian group, the so-called stabilizer of the Shor code [23].
\(-(|100⟩ + |011⟩), \ M_1(|010⟩ + |101⟩) = -(|010⟩ + |101⟩).\) By measuring both observables \(M_1 = \sigma_{z,1}\sigma_{z,2}\) and \(M_2 = \sigma_{z,2}\sigma_{z,3}\) one can find out whether, and on which of the qubits 1, 2, 3 a spin flip has taken place.

The measurement of the eigenvalues of these joint observables can be realized, as described in Fig. [1.1], by the method “decode and subsequently measure the individual state of the surrounding qubits”. This strategy has the disadvantage that the decoding leaves the logical qubit in an unprotected state, exposing it directly to the influence of decoherence. There are different methods (or networks, respectively) which use so-called ancillas to perform the error detection and correction on the encoded state directly. This is the subject of fault-tolerant quantum error correction and, more generally, fault-tolerant quantum computation. It takes into account the fact that the elementary operations which are part of the encoding and decoding network may themselves be imperfect and subject to errors. Thus one needs to make sure that the correction operations do not introduce more errors into the system than they extract. A description of the theory of fault-tolerant quantum computation is beyond the scope of this introduction. A central result states that it is possible, by using concatenated encoding strategies, to maintain the coherence of the logical state of a quantum computer over an arbitrarily long time, given that the error probability (noise level) of the elementary operations (quantum gate, measurement) is below a certain threshold. The price one has to pay for this is a certain overhead in the number of auxiliary (physical) qubits that scales polynomially [or under certain circumstances even polylogarithmically] with the time of computation. The threshold is very small and of the order of \(10^{-4} - 10^{-5}\). The theory of fault-tolerant computation is considered as the general solution of the problem of decoherence and imperfect apparatus for quantum computation. An introduction into the subject is given by Preskill [24], for example.

**What is an Error?** Let us return to the question whether the model of an error as a random unitary rotation is reasonable. The interaction of the qubits with the environment can be described as a unitary evolution in the Hilbert space of the total system consisting of both the qubits and the environment. Where and in what sense do “errors” happen in this picture? This question is certainly justified. One can show, however, that any interaction of the qubits and the environment can be written in the (integrated) form

\[
|\phi⟩_S |u⟩_\text{env} \rightarrow \sum_k (F_k|\phi⟩_S)|u_k⟩_\text{env}
\] (1.5)

where the operators \(F_k\) are tensor products of Pauli operators and \(|u_k⟩_\text{env}\) states of the environment which in general are neither orthogonal nor normalized [21]. This result remains true if \(|\phi⟩_S\) is replaced by an arbitrary multi-qubit state [21]. In case of a quantum error correcting code, such as the Shor code, one has the additional property that \(\langle \phi|F_k^\text{1bit}|\phi⟩_S = 0\), for all
“1-bit operations” $F_k^{1\text{bit}}$ which contain a non-trivial Pauli operator only at a single position (that is, for $F_k^{1\text{bit}} \sim I \otimes \cdots \otimes I \otimes \sigma_\mu \otimes I \otimes \cdots \otimes I$). For weak interactions and for uncorrelated noise, these are the terms of first order in an expansion with respect to the interaction strength. Since the overall evolution on the space of the qubits plus the environment is unitary, the picture of randomly occurring errors, which are subsequently corrected, is only a helpful way of thinking about the problem. The “digitalization of noise” [21] is in fact only introduced via the measurement of certain observables such as the $M_k$. By measuring the $M_k$ the state of the system is projected “back” into the code space $H_S$ or any of the error spaces $F_k^{1\text{bit}}H_S \bot H_S$ that are orthogonal to it. It is the disentanglement of the qubits from the environment which is the crucial process. The “error” is introduced by the fact that one does not always project back to the code space but sometimes also into an orthogonal error space, so that subsequently a unitary correction operation has to be applied to rotate the state back into the code space. On the Hilbert space of the qubits alone, the entire process can effectively be described as if the environment would apply random rotations $\sigma_{\mu,j}$ on the code, which we then check and possibly correct. Similar remarks apply to codes that correct several errors at the same time, which take into account terms of higher order in the expansion [13].

1.3 Entanglement Purification

In quantum communication, entanglement between distant parties plays a predominant role. In the following, we will concentrate on communication scenarios which involve two parties, Alice and Bob. What does it mean when we say that Alice and Bob have entanglement at their disposal? Usually, this means that they own quantum systems whose state is entangled, or, in technical terms, that the density operator which describes the state of the two quantum systems cannot be written as a convex combination of product states [22]. The two entangled quantum systems are usually called EPR pairs, due to the famous paper by Einstein, Podolsky and Rosen [9]. In the context of quantum information theory, the EPR pairs often consist of two entangled two-level systems (qubits), one owned by Alice, and the other by Bob. Maximally entangled two-qubit states are called Bell states; one can find four orthogonal Bell states, which form a basis of the two-qubit Hilbert space, the Bell basis.

The importance of entanglement is due to the fact that it is a resource which is equivalent to a quantum channel: If Alice and Bob are connected with a quantum channel, Alice can create an EPR pair locally and send one half through the quantum channel to Bob. On the other hand, if Alice and Bob own EPR pairs, they can use them to teleport qubits [8], even when they are not connected via some “real” quantum channel like an optical fiber.
The question remains however, how can Alice and Bob obtain perfect EPR pairs if they can only communicate via a noisy channel? Any real quantum channel interacts with the quantum systems which are sent through it: it becomes entangled with them. This fact is important if Alice uses the channel in order to distribute EPR pairs. If the EPR pairs are subsequently used for teleportation, then the teleported qubits become entangled with the quantum channel.

Entanglement purification protocols [5,6,7] can be used to overcome this problem. Simply speaking, these protocols create an ensemble of highly entangled pairs out of a larger ensemble of pairs with low fidelity. The fidelity of a quantum state $\rho$ is defined as its overlap with a given Bell state $|\Phi^+\rangle$, say, i.e. $F = \langle \Phi^+ | \rho | \Phi^+ \rangle$.

The purified pairs provide Alice and Bob with a purified quantum teleportation channel. If this channel is used for quantum communication, the qubits are protected against an unwanted interaction with the channel. In the next sections, we will see that this fact can be exploited for quantum cryptography protocols.

In order to perform an entanglement purification protocol, classical communication between Alice and Bob is necessary. This means, that both Alice and Bob perform measurements on their respective qubits, and tell each other the measurement outcomes. For some protocols only one-way communication is required, i.e. only Alice will send classical messages to Bob. It has been shown by Bennett et al. [6], that these one-way entanglement purification protocols are equivalent to quantum error correcting codes (see Section 1.2).

A tutorial introduction to the basic idea of entanglement purification is given in Ref. [26].

### 1.3.1 2-Way Entanglement Purification Protocols

The two-way entanglement purification protocols (2-EPP) which we present here have been developed by Bennett et al. [4] and, later, by Deutsch et al. [7]. Since these protocols work in recursive way, they are often referred to as recurrence protocols. In order to distinguish between both protocols, we will call them IBM and Oxford protocol, respectively. The IBM protocol introduces a twirling operation after each purification step, which transforms the state of the EPR pairs into the Werner form. Since Werner states [25] are described by only one real parameter, all calculations can be done analytically. A disadvantage of the IBM protocol is that it is less efficient in producing pure states from noisy ones than the Oxford protocol. Qualitatively, there is no difference between both protocols.

To be precise, we want to distinguish between the purification protocol and the distillation process (see Fig. 1.2).

In each step, the purification protocol acts on two pairs of qubits. For the sake of simplicity, we shall assume that these two pairs are described by the
Fig. 1.2. (a) The entanglement purification protocol is a (probabilistic) protocol, which creates a higher entangled pair of qubits out of two pairs with lower entanglement. Usually these pairs are called source and target pair, respectively. Through an interaction between the qubits of the source and the target pair, realizing a so-called CNOT operation on each side, the states of all four qubits become correlated. By measuring the qubits of the target pair, the source pair is probabilistically projected into a new state $\rho'_{AB}$, which is more entangled than the original state $\rho_{AB}$.

(b) The distillation process consists of several rounds. In each round, the pairs are combined into groups of two at a time, and the purification protocol is applied to them. From round to round, the entanglement of the remaining pairs is increased.

density operator $\rho_{AB} \otimes \rho_{AB}$, which is thus a four-qubit density operator. The Oxford protocol (see Fig. 1.2) consist of the following steps:

1. Alice and Bob perform one-qubit $\pi/4$ rotations about the $x$-axis on each of their qubits (in opposite directions). If the qubits were stored in atomic/ionic degrees of freedom inside a trap, this could be implemented by (simple) laser pulses.

2. Both Alice and Bob perform a CNOT-operation (controlled NOT) [2], where they use their respective particle of pair one (two) as the source (target). This is the part of the protocol which is most difficult to perform experimentally.

3. Finally, both Alice and Bob measure the qubits which belong to pair two in the $\sigma_z$-basis, and tell each other the results (two-way communication). Whenever the results coincide, the keep pair one, otherwise they discard it. In either case, they have to discard the second pair, because it is projected onto a product state by the measurement.

In order to see how this protocol works, it is useful to write the density matrices in the Bell basis, i.e. in the basis of the two qubit Hilbert space, which consists of the four Bell states $|\Phi^{\pm}\rangle = 1/\sqrt{2}(|00\rangle \pm |11\rangle)$ and $|\Psi^{\pm}\rangle =$
\[ \rho_{AB} = A |\Phi^+\rangle\langle \Phi^+ | + B |\Psi^-\rangle\langle \Psi^- | + C |\Phi^-\rangle\langle \Phi^- | + D |\Phi^-\rangle\langle \Phi^- | + \text{off-diag. elements} \]

The coefficients \(A, B, C, \) and \(D\) are called the Bell diagonal elements of the density matrix \(\rho_{AB}\). For any physical state, these coefficients have to fulfill the normalization condition \(\text{tr} \rho_{AB} = A + B + C + D = 1\).

As it turns out, the Bell diagonal elements \(A', B', C', \) and \(D'\) of the remaining pair do not depend on the off-diagonal elements of \(\rho_{AB}\). For this reason, we can find a recurrence relation for the Bell diagonal elements, which describes their evolution during the distillation process (the index \(n\) belongs to the state of the pairs at the beginning of round number \(n\) in the distillation process):

\[
\begin{align*}
A_{n+1} &= \frac{A_n^2 + B_n^2}{N_n}, \\
B_{n+1} &= \frac{2C_nD_n}{N_n}, \\
C_{n+1} &= \frac{C_n^2 + D_n^2}{N_n}, \\
D_{n+1} &= \frac{2A_nB_n}{N_n}
\end{align*}
\]

The normalization \(N_n = (A_n + B_n)^2 + (C_n + D_n)^2\) is equal to the probability \(p_{\text{success}}\), that Alice and Bob obtain the same measurement results in step 3 of the protocol. Even though no analytical solution has been found for this recurrence relation, it has been shown (numerically in [8] and later analytically [27]) that it converges to the fixpoint \(A^\infty = 1, B^\infty = C^\infty = D^\infty = 0\), whenever the initial fidelity is greater than 1/2. In this case, also the off-diagonal elements will vanish, since the density matrix has to be positive. In other words, whenever Alice and Bob are supplied with EPR pairs with a fidelity of more than 50%, they can distill (asymptotically) perfect EPR pairs.

For the IBM protocol, one only needs one recurrence relation, since (one-parametric) Werner states, described by \(A = F, B = C = D = (1 - F)/3\) and vanishing off-diagonal elements in (1.6), are mapped onto Werner states. This map is shown in Fig. (1.3a). The map has tree fixpoints. Two of these fixpoints are attractive (at \(F = 1/4\) and \(F = 1\)), and the remaining one (at \(F = 1/2\)) is repulsive. Thus, if one starts the distillation process with a fidelity greater than 1/2, one will finally reach EPR pairs in a pure state. If the initial fidelity is smaller than 1/2, one will finally be left with completely depolarized pairs, which correspond to a Werner state with a fidelity of 1/4.

1.3.2 Purification with Imperfect Apparatus

Up to now, we have assumed that the only source of decoherence is the quantum channel which connects Alice and Bob. For practical implementations, however, this is an over-simplification. Indeed, there are many operations involved in the distillation process: Qubits have to be stored for a certain time,
Fig. 1.3. The purification curve for the IBM protocol [5,6] for perfect (i.e. noiseless) apparatus (a). The staircase denotes how the fidelity increases from round to round in the distillation process of Fig. 1.3b). If the apparatus is imperfect, the purification curve is “pulled down” (b) and the fixpoints move towards each other. The upper fixpoint of the curves indicates the maximum achievable fidelity $F_{\text{max}}$, which can be reached asymptotically by the respective purification protocols; $F_{\text{max}}$ decreases with an increasing noise level. Attractive fixpoints are denoted by black circles, repulsive fixpoints by white circles.
one- and two-qubit unitary operations will act on them, and there are measurements. Each of these operations is a source of noise by itself. It would be inconsistent to ignore this source of noise. So the following question arises: What are the conditions which we have to impose on the apparatus so that entanglement distillation works at all?

As we have mentioned in the context of fault tolerant quantum computation, there exists a certain noise threshold for the elementary operations, below which fault tolerant quantum computation is possible. In the case of 2-EPP we will find a threshold which is much more favorable than the threshold for fault tolerant quantum computation.

In order to get a qualitative understanding of the influence of noisy operation on the entanglement distillation process, we look again at the purification curve (Fig. (1.3)). The curve shows how the fidelity after a purification step depends on the previous fidelity. If noise is introduced in the purification process itself, it is intuitively clear that only a smaller increase in fidelity can be achieved: the purification curve is “pulled down”. In Fig. (1.3) this is shown schematically. We thus expect that in the case of noisy operations, one has to start with a greater initial fidelity in order to purify at all, and that the maximum fidelity which can be reached will be smaller than unity.

If the noise level is increased, one reaches the situation that two of the fixpoints will merge. At even higher noise levels, the purification curve has only the trivial fixpoint which corresponds to completely depolarized pairs: the distillation process breaks down and does not work any longer.

The quantitative investigation of entanglement purification with noisy apparatus [28,29] shows that the above considerations are qualitatively correct. For the calculation, the following noise model has been assumed [29]:

- The unitary evolution of the qubits is accompanied by a depolarizing channel. It is well-known that this can be written in a time-integrated form

\[
\rho_{AB} \rightarrow p \, U_A \rho_{AB} U_A^{-1} + \frac{1-p}{d} \mathbb{1}_A \otimes \text{tr}_A \rho_{AB}.
\]  

(1.8)

Here, \( \rho_{AB} \) is the density operator which describes the state of a bipartite quantum system, \( U_A \) is the desired unitary operation (which is assumed to act only on the quantum system at party A), \( d \) is the dimension of the Hilbert space of A’s system, and \( p \) is the reliability of the quantum operation. For \( p = 1 \), there is no noise at all, and for \( p = 0 \), the quantum system at A becomes completely depolarized.

- Measurements give the correct results only with a certain probability \( \eta \). This can be conveniently described in terms of a POVM (positive operator valued measure [30]),

\[
M_0 = \eta |0\rangle \langle 0| + (1-\eta) |1\rangle \langle 1| \\
M_1 = \eta |1\rangle \langle 1| + (1-\eta) |0\rangle \langle 0|,
\]  

(1.9)
for one-qubit measurements in the $\sigma_z$ basis. Here, $\text{tr}(M_j \rho)$ describes the probability with which the detector indicates the result “$j$” for the measured qubit.

As one can see from Eq. (1.8), we have to distinguish between one- and two-qubit operations, if they are accompanied by noise: a two-qubit depolarizing channel is different from two one-qubit depolarizing channels. The first is an example of a correlated noise channel, the latter of an uncorrelated noise channel. The reliability of one- and two-qubit operations is referred to as $p_1$ and $p_2$, respectively. Whether or not entanglement purification is possible with a certain protocol, depends on the three parameters $p_1$, $p_2$, and $\eta$. For all these parameters, one gets a noise threshold in the percent regime, which is about two orders of magnitude better than the noise threshold for fault tolerant quantum computation.

1.4 Quantum Cryptography

One of the practically most advanced fields in quantum communication is quantum cryptography. In this section, we will describe the two basic protocols of quantum cryptography. We show that decoherence in the (untrusted) quantum channel as well as in the (trusted) apparatus plays an important role in the security analysis of quantum cryptography protocols.

The communication scenario in the cryptographic context looks as follows: Alice wants to send a confidential message (clear-text) to Bob, while a third communication party, Eve, wants to listen in and learn as much as possible about the message. In order to achieve her goal, Alice encrypts the message using some cryptographic method. The encrypted message is called ciphertext. A cryptographic protocol is considered good, if it is possible to restrict the information which Eve can obtain to any desired level.

There exist several categories of classical cryptographic protocols; these include symmetric key ciphers, asymmetric key ciphers and one-time pads. All these protocols have advantages and disadvantages, but the most eminent advantage of the one-time pad is that it has been proved to be secure in the information theoretical sense: one can show that an eavesdropper can gain no information (zero bits of information) about the message, even if he or she knows every single bit of the encrypted message. To this end, it is however necessary that Alice and Bob share a secret and random key, which must at least be as long as the message which Alice wants to transmit, and that this key will only be used once (thus the name one-time pad).

The one-time pad works as follows: As a key, Alice and Bob share a secret string of zeros and ones $s = (s_1, s_2, \ldots, s_N)$. Similarly, Alice can write the clear-text (like any piece of information) as a string of zeros and ones, using some encoding which Alice and Bob agree on publicly. The clear-text is thus given in a binary representation $t = (t_1, t_2, \ldots, t_N)$. For the ciphertext,
Alice adds the key and the clear-text bitwise modulo 2: $c = (s_1 \oplus t_1, s_2 \oplus t_2, \ldots, s_N \oplus t_N)$. In order to decrypt the message, Bob simply adds the key bitwise (modulo 2) to the ciphertext, and gets back the binary representation of the clear-text.

The key used in the one-time pad protocol is a valuable resource, to both the legitimate communication parties and to an eavesdropper: Alice and Bob use up the key during the communication. In order to supply themselves with a new key, they have to meet each other physically. On the other hand, if Eve knows the key, the communication between Alice and Bob is no longer a secret for her; for this reason, the cryptographic key might be a valuable target for theft or bribery. The aim of quantum cryptography is to solve this shortcoming of classical cryptography. In most quantum cryptography protocols, the quantum part of the protocol is related to the distribution of a key (quantum key distribution, QKD), which can afterwards, as soon as it is established, be used for a classical one-time pad protocol.

### 1.4.1 The BB84 Protocol

The first protocol for quantum key distribution was given by Bennett and Brassard in 1984 [10]. This so-called BB84 protocol is widely used in quantum cryptography, since all security considerations are well analyzed, and it is easy to understand.

The protocol works as follows: Alice prepares two random binary strings, the key string $(k_1, k_2, \ldots, k_N)$ and the basis string $(b_1, b_2, \ldots, b_N)$. The randomness of the bits is crucial for the security of the protocol; they may thus not be chosen by a pseudo random number generator.

There are 4 different quantum states which Alice can prepare:

- $|s_{00}\rangle = |0\rangle$
- $|s_{01}\rangle = |1\rangle$
- $|s_{10}\rangle = |+\rangle \equiv 1/\sqrt{2}(|0\rangle + |1\rangle)$
- $|s_{11}\rangle = |-\rangle \equiv 1/\sqrt{2}(|0\rangle - |1\rangle)$.

For simplicity, we will now consider the case of qubits which are represented in the polarization degree of freedom of a photon. In this case, the four states which Alice can prepare are horizontally, vertically, or $\pm 45^\circ$ polarized photons.

Alice sends $N$ photons through the quantum channel to Bob. The state in which the qubits are prepared depends on the key- and and the basis string: the $i$th qubit is prepared in the state $|s_{b_i,k_i}\rangle$.

Bob can measure each photon that arrives in his laboratory either in the $|0\rangle / |1\rangle$-basis (i.e. in the horizontal/vertical basis), or in the $|+\rangle / |-\rangle$-basis (i.e. in the $\pm 45^\circ$ polarized basis). For each individual photon, he selects the measurement basis randomly, and he writes down the chosen basis and the measurement result. When Bob has received and measured the $N$ photons, he is left with two strings of $N$ bits: the “basis” string and the “result” string.

Alice and Bob exchange their respective basis strings through a classical channel, which may be public; for example, they might announce the basis strings in a newspaper. It is no security breach if Eve knows both basis strings. However, Alice and Bob must make sure that Eve cannot alter these
messages. One possibility to achieve this goal is that Alice and Bob possess an initial shared secret, which can be used to check the authenticity and integrity of the basis strings. During the key distribution task, this initial shared secret can be recreated, so that it is not used up; rather, it plays the role of a catalyst. By comparing their basis strings, Alice and Bob can see which photons have been measured in the same basis in which they have been prepared. Whenever the preparation basis and the measurement basis are different, Bob’s measurement result is completely random and cannot be used. On the other hand, if the two bases are the same, Bob’s measurement result will be strictly correlated with Alice’s key bit for the respective photon: Alice’s key bits and Bob’s measurement results for these photons can be used as a secret key.

Before the key can be used, Alice and Bob have to make sure that the quantum channel has not been eavesdropped. One way to do this is the following: Alice chooses a certain number of the key bits randomly and sends them to Bob through the classical public channel. Bob compares Alice’s key bits with his result bits, and if they are equal, they can be sure that there was no eavesdropper who tapped the quantum channel. This is due to the fact that the only quantum operation which does not disturb non-orthogonal quantum states is the identity. In other words: if Eve does not want to disturb the non-orthogonal quantum states which Alice sends, she has to leave them alone.

1.4.2 The Ekert Protocol

The main difference between the BB84 protocol and the so-called E91 protocol found by Ekert in 1991 is that it does not use single photons which one communication party sends to the other, but pairs of entangled photons. While its experimental realization is more difficult than the BB84 protocol, it has a theoretical advantage: the security of the E91 protocol is related to the fact that there exists no local realistic theory which explains the outcomes of Bell-type experiments. While in the BB84 protocol one has to believe that the quantum mechanical description of photons is complete (i.e. that there exist no (local) variables — “hidden” or not — which could be used to predict Bob’s measurement outcomes), the E91 protocol performs a Bell experiment at the same time, which assures that there cannot exist (local) hidden variables.

4 For a recent review of experiments testing Bell’s inequalities, see e.g. [31].

5 In experiments, classical information about the state which has been prepared might leak out of Alice’s laboratory through different degrees of freedom, like the frequency of the photon, or the polarization of a second photon in a multi-photon pulse. This information could in principle be exploited by Eve without introducing noise. For the E91 protocol, this leakage problem does not exist, since such information does not exist until Alice and Bob perform their measurement.
In the E91 protocol, pairs of entangled photons are prepared, for example in the state $|\Psi^-\rangle = (|01\rangle - |10\rangle)/\sqrt{2}$. It does not matter whether these pairs are produced in Alice’s or Bob’s laboratory, or by a (potentially untrusted) source in between. One photon of each pair is sent to Alice, the other to Bob. For each photon, Alice and Bob choose one out of a set of three measurement directions at random, and measure the polarization of the photon in this direction (see Fig. 1.4). As in the BB84 protocol, Eve must not be able to predict the choice of the measurement directions. As soon as all pairs are sent to Alice and Bob and they acknowledge that they have performed the measurements, the information about the measurement directions is exchanged (through a public classical channel). Alice and Bob check for which pairs their respective measurement directions were the same; for all pairs where they have chosen different measurement directions, also the measurement outcomes are exchanged through the public classical channel. With these results, Alice and Bob check that the Bell inequalities are violated. The measurement results for the pairs where they have chosen the same measurement direction are strictly anti-correlated, and can be used as a key.

### 1.4.3 Security Proofs

As we have seen above, the quantum key distribution protocols allow for secure communication, as long as Alice and Bob are connected by a noiseless quantum channel. This is a remarkable result – however, it would be useless for all practical purposes, since all quantum channels are a source of noise. Since Alice and Bob trust only the equipment in their laboratories, they cannot be sure that the noise which they measure can be attributed to the channel. It is in principle impossible to distinguish between noise introduced by the quantum channel or by an eavesdropper. For this reason, the communication parties have to deal with the worst case scenario of an eavesdropper, who is present all the time and everywhere, except for the laboratories, which are secure by assumption. The eavesdropper might be
hidden behind the noise of the quantum channel, and she might gain partial knowledge of the cryptographic key and, later, of the secret message.

The simplest way to deal with this situation would be to use a better quantum channel. In a practical setting, however, when Alice and Bob are connected by a given quantum channel (e.g., an optical fiber), this possibility is ruled out. In this situation, Alice and Bob can use privacy amplification methods, where a shorter and perfectly secure key is distilled out of a longer key, about which Eve might have had considerable knowledge. So-called “ultimate” or “unconditional” security proofs of quantum cryptography show that such protocols do exist.

The first of these proofs has been given by Mayers in 1996 [34] for the BB84 protocol. Shor and Preskill gave a physical interpretation of this proof, as they showed that it could a posteriori be understood as a restricted, albeit sufficient, form of quantum error correction and one-way entanglement purification.

A different approach has been taken by Deutsch et al. in 1996 [7]. They employ a two-way entanglement purification protocol (2-EPP, see Sec. 1.3) in order to distill almost pure EPR pairs out of many imperfect pairs. If the purified pairs are used for teleportation, the resulting quantum channel is perfectly secure: Since the EPR pairs are in a pure state, they cannot be entangled with any other quantum system. The eavesdropper is thus “factored out” in the total Hilbert space, which we write symbolically as

$$\rho_{\text{Alice}, \text{Bob}, \text{Eve}} \xrightarrow{2\text{-EPP}} |\Psi^+\rangle_{AB} \langle \Psi^+ | \otimes \rho_{\text{Eve}}.$$  

As we have already seen in Sec. 1.3.2, in a realistic setting the purification protocol does not converge to perfect EPR pairs, but to some more or less mixed state in the Hilbert space of Alice’s and Bob’s qubits. But that means that the argument given above does no longer guarantee that Eve is factored out: a priori, there could exist residual entanglement with Eve.

### 1.5 Private Entanglement

In the last section we have seen that entanglement purification (using noisy apparatus) does not per se guarantee a provably private communication channel. Nevertheless, in this section we will show that it suffices for the creation of “private entanglement”, i.e., imperfect EPR pairs which are not entangled with an eavesdropper. Private entanglement can thereby serve as noisy but secure quantum channel.

The general idea is the following. Since Alice and Bob use noisy apparatuses for the entanglement distillation process, it is clear that the pairs become entangled with some degree of freedom of the laboratory. However, we will see that the total state of the laboratory and of the (distilled) pairs converges to a pure state, and then the same argument holds as in the case of
noiseless entanglement purification: a quantum system in a pure state cannot be entangled with any other quantum system. In particular, Eve cannot be entangled with the distilled pairs. These pairs can then be used for secure, albeit noisy quantum teleportation.

In our analysis it is necessary to keep track of the state of the laboratory. This seems to be a difficult task, since the details of the structure of the laboratory are unknown and complicated. For this reason one does usually not take care of these details, and describes the system of qubits on which the noisy apparatus acts as an open quantum system, with a master equation that describes their time evolution \[35\]. As an alternative, in the framework of quantum information theory, we use the concept of completely positive maps \[36\].

1.5.1 The Lab Demon

In this section, we give a simple model of a noisy laboratory, which allows us to keep track of its state in terms of classical variables.

As long as one cannot “look into” the device that introduced the noise, there is no way to distinguish it from a different device whose action is described by the same positive map. For this reason, our simple noise model is sufficient for the proof, and we need not delve into the complicated details of noisy quantum devices.

In order to keep the argument as transparent as possible, we will restrict our attention to the case that only Alice’s laboratory is a source of noise; it would be easy to extend the argument to two noisy laboratories.

Let us assume that in Alice’s lab there is a little demon. The lab demon kicks and shakes the qubits from time to time, and is thus a source of noise. However, there are no other sources of noise, and even the lab demon acts on the qubits in a very controlled way: let us assume that the demon has a random number generator that generates in each time step pairs of numbers \((\mu, \nu) \in \{0, 1, 2, 3\}^2\), according to a given probability distribution \(f_{\mu\nu}\) which obeys the normalization condition \(\sum_{\mu, \nu} f_{\mu\nu} = 1\). The lab demon then applies the (unitary) error operation \(\sigma^{(a_1)}_{\mu} \sigma^{(a_2)}_{\nu}\) to the two qubits \(a_1\) and \(a_2\), on which Alice acts in the entanglement purification protocol (see Section 1.3.1). For \(\mu \in \{1, 2, 3\}\), the operators \(\sigma^{(a_i)}_{\mu}\) denote the Pauli matrices acting on qubit \(a_i\), and \(\sigma^{(a_i)}_0 = 1^{(a_i)}\). In addition, the lab demon writes down which error operations he had applied to which qubits, since he will need this information later.

Alice does not know which of the error operations have been applied to the qubits, and she describes the action of the demon by the average map

\[
\rho_{a_1a_2...} \rightarrow \sum_{\mu, \nu=0}^3 f_{\mu\nu}\sigma^{(a_1)}_{\mu} \sigma^{(a_2)}_{\nu}\rho_{a_1a_2...}\sigma^{(a_1)}_{\mu} \sigma^{(a_2)}_{\nu}. \tag{1.10}
\]
The ellipsis (\ldots) denotes other degrees of freedom, on which Alice’s lab
demon does not act (like Bob’s qubits, or some quantum system in Eve’s
hands). We call the noise channel given by this equation the \textit{correlated two
qubit Pauli channel}. It includes, for special choices of the probability distri-
bution $f_{\mu\nu}$, the one- and two-qubit depolarizing channel, and combinations
thereof, which have been studied in the context of entanglement purifica-
tion using imperfect apparatus in \cite{29}.

As mentioned above, we introduced the lab demon as a simplified noise
model in order to keep track of the internal state of the lab. For that reason,
we assume that the lab demon attaches an \textit{error flag} $\lambda$ to each qubit. The
error flag will represent four different values, and it is convenient to divide it
into two classical bits.

\subsection*{1.5.2 The State of the Qubits Distributed Through the Channel}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig1_5.png}
\caption{(a) The state of $N$ pairs which are distributed through the quantum
channel is in the worst case scenario a general $2^N$-qubit state, which might moreover
be entangled with degrees of freedom under Eve’s control. (b) After step 1, the state
of the $N$ pairs is a classically correlated ensemble of pure Bell states.}
\end{figure}

In the worst case scenario, all pairs which are distributed between Alice
and Bob have been prepared by Eve (see Fig. 1.5a). For that reason, the total
state of all pairs is given by a general $2N$-qubit density operator, which can
be written in the form

$$
\rho_{AB} = \sum_{\mu_1, \ldots, \mu_N} \alpha_{\mu_1, \ldots, \mu_N} |B_{\mu_1}^{(a_1 b_1)} \cdots B_{\mu_N}^{(a_N b_N)} \rangle \langle B_{\mu_1}^{(a_1 b_1)} \cdots B_{\mu_N}^{(a_N b_N)}|.
$$

\begin{equation}
(1.11)
\end{equation}
Here, $|B_{\mu_j}^{(a_j b_j)}\rangle$, $\mu_j = 00, 01, 10, 11$ denote the 4 Bell states associated with the two qubits $a_j$ and $b_j$ and $j = 1, \ldots, N$. Specifically, $|B_{00}\rangle \equiv |\Phi^+\rangle$, $|B_{01}\rangle \equiv |\Psi^+\rangle$, $|B_{10}\rangle \equiv |\Phi^-\rangle$, $|B_{11}\rangle \equiv |\Psi^-\rangle$.

In general, (1.11) will be an entangled $2^N$-qubit state, which might moreover be entangled with additional quantum systems in Eve’s hands. For the security analysis of the entanglement purification protocol, this state is too complicated and cannot be handled. It would be helpful if there was no entanglement between the different pairs. Fortunately, Alice and Bob can apply the following protocol to the pairs, in order to handle this situation:

**Step 1:** On each pair of particles $(a_j, b_j)$, they apply randomly one of the four bi-lateral Pauli rotations $\sigma_k^{(a_j)} \otimes \sigma_k^{(b_j)}$, where $k = 0, 1, 2, 3$.

**Step 2:** Alice and Bob randomly renumber the pairs, $(a_j, b_j) \rightarrow (a_{\pi(j)}, b_{\pi(j)})$ where $\pi \in S(N)$ is a permutation which has been chosen at random.

It is important to note that Alice and Bob deliberately discard the knowledge about which permutation and which of the Pauli rotations have been applied to the pairs. Obviously, they cannot force Eve to do the same thing. So Eve might have a better description of the state of the pairs than Alice and Bob. Thus the question remains whether this additional knowledge might help Eve. It is easy to see that this is not the case: Eve’s description of the qubits has to be *statistically consistent* with the state which Alice and Bob or the lab demon assign to the to qubits. As we are going to show, at the end of the distillation process, the lab demon knows that the pairs are pure EPR pairs. Eve can thus not have more information about the pairs than the lab demon.

After step 1, Alice’s and Bob’s knowledge about the state is summarized by the density operator

$$\tilde{\rho}_{AB} = \sum_{\mu_1 \cdots \mu_N} p_{\mu_1 \cdots \mu_N} |B_{\mu_1}^{(a_1 b_1)} \cdots B_{\mu_N}^{(a_N b_N)}\rangle \langle B_{\mu_1}^{(a_1 b_1)} \cdots B_{\mu_N}^{(a_N b_N)}| \quad (1.12)$$

which corresponds to a *classically correlated ensemble* of pure Bell states (see Fig. 1.5b). The fact that the pairs are classically correlated means that the order in which they appear in the numbered ensemble may have some pattern, which may have been imposed by Eve or by the channel itself. By applying step 2, the order of the pairs is “randomized”; this will prevent Eve from making use of any possibly pre-arranged order of the pairs, which Alice and Bob are meant to follow in the course of the distillation process: they simply ignore this order.

The only correlation which remains is in the number of pairs which are in a specific Bell state. In the limit of large $N$, it is consistent for all relevant statistical predictions to describe the ensemble with the density operator

---

6 In fact, Eve has *less* information than the lab demon, because she does not know the results of his random number generator.
\[ \tilde{\rho}_{AB} = \left( \sum_{\mu} p_\mu |B_\mu\rangle\langle B_\mu| \right) \otimes^N \equiv (\rho_{ab}) \otimes^N. \quad (1.13) \]

For finite \( N \), the form of the state after step 2 is more complicated; however, the subsequent arguments are also valid in that case.

At this stage, Alice and Bob have to check whether the pairs are “good enough” for the distillation process, i.e. they have to make sure that the fidelity \( F_0 \) of the pairs is above the purification/security threshold (which coincide for all practical purposes \[37\]). They can do this by local measurements on a fraction of the pairs and classical communication.

In order to separate conceptual from technical considerations and to obtain analytical results, we will first concentrate on a toy model where all the pairs are either in the state \(|\Phi^+\rangle\) or \(|\Psi^+\rangle\). In this case, we talk about binary pairs.

### 1.5.3 Binary Pairs

Let us assume that Alice and Bob initially share pairs in the state

\[ \rho_{AB} = A |\Phi^+\rangle_{AB}\langle \Phi^+| + B |\Psi^+\rangle_{AB}\langle \Psi^+| \quad (1.14) \]

(binary pairs) with \( A = 1 - B > 1/2 \), and that the noise is of the form \[\text{(1.10)}\] with the restriction that the error operators consist only of the identity and spin flip operators:

\[ \rho_{a_1a_2...} \rightarrow \sum_{\mu,\nu=0}^{1} f_{\mu\nu} \sigma^{(a_1)}_{\mu} \sigma^{(a_2)}_{\nu} \rho_{a_1a_2...} \sigma^{(a_1)}_{\mu} \sigma^{(a_2)}_{\nu}. \quad (1.15) \]

Eq. \[\text{(1.15)}\] describes a two-bit correlated spin-flip channel. The indices 1 and 2 indicate the source and target bit of the bilateral CNOT operation, respectively. It is straightforward to show that, using this error model in the 2–EPP, binary pairs will be mapped onto binary pairs.

At the beginning of the distillation process, Alice and Bob share an ensemble of pairs described by \[\text{(1.14)}\]. In this special case, one bit suffices for the error flag. At this stage, all of these bits are set to zero. This reflects the fact that the lab demon has the same \textit{a priori} knowledge about the state of the ensemble as Alice and Bob.

In each purification step, two of the pairs are combined. The lab demon first simulates the noise channel \[\text{(1.14)}\] on each pair of pairs by using the random number generator as described. Whenever he applies a \( \sigma_x \) operation to a qubit, he inverts the error flag of the corresponding pair. Alice and Bob then apply the 2–EPP to each pair of pairs; if the measurement results in
the last step of the protocol coincide, the source pair will be kept. Obviously,
the error flag of that remaining pair will also depend on the error flag of the
the target pair, i.e. the error flag of the remaining pair is a function of the
error flags of both “parent” pairs, which we call the flag update function. In
the case of binary pairs, the flag update function maps two bits (the error
flags of both parents) onto one bit. In total, there exist 16 different functions
\( g : \{0,1\}^2 \rightarrow \{0,1\} \). From these, the lab demon chooses the logical AND
function as the flag update function, i.e. the error flag of the remaining pair
is set to “1” if and only if both parent’s error flags had the value “1”.

After each purification step, the lab demon divides all pairs into two
subensembles, according to the value of their error flags. By a straightfor-
ward calculation, we obtain for the coefficients \( A_i \) and \( B_i \), which completely
describe the state of the pairs in the subensemble with error flag \( i \), the fol-
lowing recurrence relations:

\[
A'_0 = \frac{1}{N} \left( f_{00}(A_0^2 + 2A_0A_1) + f_{11}(B_0^2 + 2B_0B_1) \right.
\]
\[
+ f_s(A_0B_1 + A_1B_1 + A_0B_0) \bigg) \]
\[
A'_1 = \frac{1}{N} \left( f_{00}A_1^2 + 2f_{11}B_0^2 + f_sA_1B_0 \bigg) \right)
\]
\[
B'_0 = \frac{1}{N} \left( f_{00}(B_0^2 + 2B_0B_1) + f_{11}(A_1^2 + 2A_0A_1) \right.
\]
\[
+ f_s(B_0A_1 + B_1A_1 + B_0A_0) \bigg) \]
\[
B'_1 = \frac{1}{N} \left( f_{00}B_1^2 + 2f_{11}A_0^2 + f_sB_1A_0 \bigg) \right)
\]

with
\[
N = (f_{00} + f_{11})(A_0 + A_1)^2 + (B_0 + B_1)^2 + 2f_s(A_0 + A_1)(B_0 + B_1) \quad \text{and} \quad f_s = f_{01} + f_{10}.
\]

Since Alice and Bob do not know the values of the error flags, they describe
the pairs in terms of \( A = A_0 + A_1 \) and \( B = B_0 + B_1 = 1 - A \) as in Eq. (1.14).
The fidelity \( F \) is thus given by \( F = A \).

For the case of uncorrelated noise, the error operations are applied inde-
pendently and with probability \( f_{\mu}(\mu = 0,1) \) to both qubits. This means that
the probability distribution \( f_{\mu\nu} \) factorizes into \( f_{\mu\nu} = f_{\mu}f_{\nu} \). In this special
case we obtain the following expression for fixpoints of this map:

\[
A_0^\infty = \frac{1}{2} \pm \frac{\sqrt{f_0 - 3/4}}{f_0 - 1} \quad \text{or} \quad A_0^\infty = \frac{1}{2},
\]
\[
A_1^\infty = 0, \quad B_0^\infty = 0, \quad B_1^\infty = 1 - A_0^\infty.
\]

The fixpoint of this map that is “relevant” for our discussion is defined by
the plus sign in the expression for \( A_0^\infty \) above. It is not \textit{per se} clear that a
fixpoint is also an attractor. In fact, we find that Eq. (1.17) gives a non-
trivial fixpoint of (1.16) for \( f_0 \geq 3/4 \), but this fixpoint is an attractor only
for \( f_0 > f_0^{\text{crit}} = 0.77184451 \) [27].
Fig. 1.6. The values of $A_0, A_1, B_0, B_1, F = A_0 + A_1, F_{\text{cond}} = A_0 + B_1$ at the fixpoint as a function of the noise parameter $f_0$. For $f_0 < 0.75$, the values of $A_1$ and $B_1$, as well as the values of $A_0$ and $B_0$ are equal, and the respective lines lie on top of each other. One can clearly see that for $f_0 < 0.75$, the fidelity becomes $1/2$, and the pairs converge to the completely mixed state $\frac{1}{2}(|\psi^+\rangle\langle\psi^+| + |\phi^+\rangle\langle\phi^+|)$: the protocol is not in the purification regime. For $f_0 > 0.75$, the maximum achievable fidelity increases, and approaches unity for $f_0 \to 1$. This corresponds to the fact that the protocol is in the purification regime, and that it works better if the apparatus is more reliable. However, the fidelity is strictly smaller than unity for $f_0 < 1$. For the conditional fidelity $F_{\text{cond}} = A_0 + B_1$, however, the situation is different: above the critical value $f_0^{\text{crit}}$, it becomes strictly equal to unity. Since $F_{\text{cond}}$ is the fidelity or the pairs from the lab demon’s point of view, any eavesdropper is factored out, and we call this regime the security regime. The regime, where the protocol purifies but does not provide secure EPR pairs is called intermediate regime (highlighted in grey). The inset shows the same graphs on a logarithmic scale. In this graph, one can see that the parameters $A_1$ and $B_0$ do not vanish only asymptotically, but become zero at $f_0 = f_0^{\text{crit}}$. 
To summarize, we can identify three regimes for values of the noise parameter (see Fig. 1.6): for a high noise level, when \( f_0 < \frac{3}{4} \), the protocol is not in the purification regime. From Alice’s and Bob’s point of view, the protocol converges to the completely mixed binary state \( \frac{1}{2} (|\Psi^+\rangle \langle \Psi^+| + |\Phi^+\rangle \langle \Phi^+|) \).

For a low noise level, when \( f_0 > f_0^{\text{crit}} \), the protocol converges to a state where \( A_0 + B_1 = 1 \) and \( A_1 = B_0 = 0 \). This means that all pairs in the subensemble 0 are in the state \( |\Phi^+\rangle \), and all pairs in the subensemble 1 are in the state \( |\Psi^+\rangle \): From the lab demon’s point of view, all pairs are in a pure state! For that reason, we will call this regime the security regime of the entanglement purification protocol. For \( \frac{3}{4} \leq f_0 \leq f_0^{\text{crit}} \), the protocol is in the intermediate regime. This regime is of no practical interest, since in this regime, the protocol converges very slowly. However, the mere existence of the intermediate regime is interesting, as it shows that purification and security are not trivially related to each other.

As we have already seen, the sum \( A_0 + B_1 \) is a measure for the purity of these pairs from the lab demon’s point of view. We call this sum the conditional fidelity \( F^{\text{cond}} \), since this is the fidelity which Alice and Bob would assign to the pairs if they knew the values of the error flags.

We have also evaluated (1.16) numerically in order to investigate correlated noise (see Fig. 1.7). Like in the case of uncorrelated noise, we found that the coefficients \( A_0 \) and \( B_1 \) reach, during the distillation process, some finite value, while the coefficients \( A_1 \) and \( B_0 \) decrease exponentially fast, whenever the noise level is moderate.

![Fig. 1.7.](image)

**Fig. 1.7.** The evolution of the four parameters \( A_0, A_1, B_0, \) and \( B_1 \) in the security regime. Note that both \( A_1 \) and \( B_0 \) decrease exponentially fast in the number of rounds. The initial fidelity was 80%, and the values of the noise parameters were \( f_{00} = 0.8575 \), \( f_{01} = f_{10} = f_{11} = 0.0475 \).
The distillation process which was described in Fig. 1.2b now looks as in
Fig. 1.8, where the ensemble of pairs is now supplemented with an error flag
for each pair. One can see that in the course of the distillation process, strict
correlations are built up between the state of the pairs and the error flags \( \lambda_i \). In the asymptotic limit, each flag identifies the state of the corresponding
pair unambiguously.

In other words, whenever the noise level is moderate, the conditional
fidelity converges to unity: entanglement purification can be used to create
private entanglement.

### 1.5.4 Bell-Diagonal Initial States

In the previous section we have considered the special case of binary pairs.
For arbitrary Bell diagonal states (Eq. (1.13)) and for noise of the form (1.10),
the results are quite similar. However, the most important difference is that
in the general case the intermediate regime is much smaller than in the case
of binary pairs.

As already mentioned, in general the error flag consists of two classical
bits. This means that the lab demon has to use a more complicated flag
update function than in the case of binary pairs. In this case, the flag update
Fig. 1.9. Typical evolution of the 16 parameters $A^{ij}, B^{ij}, C^{ij}, D^{ij}$ with $i, j \in \{0, 1\}$ under the purification protocol. As in Eq. (1.6), the coefficients $A, B, C, D$ correspond to the four Bell states which are indicated by “Phi+”, “Psi-”, “Psi+”, and “Phi-” on one axis. The other axis shows the error flag $\lambda \in \{00, 01, 10, 11\}$. As one can see, only the diagonal elements survive, which means that the error flag identifies the states of all pair unambiguously. The noise parameters in this plot are $f_{00} = 0.83981, f_{01} = f_{10} = 0.021131$ and $f_{11} = 0.003712$ for $i, j \in \{1, 2, 3\}$.

function has been found by looking at how errors are propagated during the course of the distillation process. The details of this calculation can be found in [37].

Since the error flag represents four different values, the lab demon divides all pairs into four subensembles, according to the value of their error flag $\lambda$. In each of the subensembles the pairs are described by a Bell diagonal density operator, like in Eq. (1.6), which now depends on the subensemble. That means, in order to completely specify the state of all four subensembles, we need 16 real numbers $A^{ij}, B^{ij}, C^{ij}, D^{ij}$ with $i, j \in \{0, 1\}$.

Fig. 1.9 shows how these 16 parameters evolve under the action of the distillation process: If the protocol is in the security regime, only the “diagonal” elements survive and are identified by unambiguously by the corresponding error flag. Again, this means that from the lab demons point of view, all pairs are in a pure state.

To summarize, we have found that in the entanglement distillation process, entanglement is redistributed in the following sense (see Fig. 1.10): in the beginning, there exists (unwanted) entanglement between the EPR pairs and the quantum channel (Eve). The entanglement distillation process is not capable of creating perfect EPR pairs, since the pairs become entangled with the laboratories, due to uncontrolled interactions. Despite this fact, Eve is factored out, and all entanglement between her and the EPR pairs is lost: Alice and Bob succeeded in creating private entanglement and have thus a private, albeit noisy, quantum channel.

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Fig. 1.10. The entanglement distribution process *redistributes* entanglement: at the beginning, the EPR pairs are entangled with the communication channel, or maybe even with an eavesdropper. In the end, however, the remaining EPR pairs are only entangled with the laboratories, and the eavesdropper is *factored out.*
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