Determination of the bottom scattering coefficient discontinuity lines in the multibeam ocean sounding

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Abstract. In this paper the problems of constructing sonar images of the seabed according to measurements of the multibeam side scan sonar are considered. The inverse problem for the non-stationary equation of radiation transfer with the diffuse reflection conditions at the boundary which consists in finding the discontinuity lines of the bottom scattering coefficient is investigated. A numerical algorithm for solving the inverse problem is developed, and an analysis of the quality of reconstructing the boundaries of inhomogeneities of the seabed is carried out, depending on the number of views and the width of a radiation pattern and the sounding range.

1. Introduction
In this paper, a mathematical model [1–7] describing a high-frequency acoustic sensing process of the sea bottom using the side scan sonar (SSS) installed on autonomous unmanned underwater vehicles (AUUV) is investigated. These types of research are relevant for the development of new algorithms that could improve quality of sonar images of the sea bottom from measurements of the SSS [7, 8].

The mathematical model consists of integro-differential transport equation, initial condition and boundary condition describing the diffuse reflection on the bottom surface. The objective of the inverse problem is to find a coefficient of bottom scattering having some additional redefinition conditions of solving of the radiation transfer equation [7]. The physical meaning of the problem is the sea bottom mapping from the data of measurements of a back reflected signal.

An explicit solution for a narrow beam pattern in the single scattering approximation to find a coefficient of the diffuse reflection is obtained [7]. Using this solution in the inverse problem leads to defocusing the sea bottom objects in the sonar images with increasing the beam width.

To solve this problem the SSS are equipped with multibeam echo sounders or the number of tacks can be increased when monitoring water areas. In the papers [9, 10] an approximate method for finding the coefficient of the sea bottom scattering was developed. This method can be effective but it has several disadvantages because it requires a lot of a priori information about the acoustic characteristics of the environment.

In this paper, the authors offer to change the formulation of the direct problem using a minimum of a priori information. The desired characteristic in the new formulation of the problem is not the reflection coefficient, but only its singular support. The information about the discontinuity lines of the coefficient is sufficient to determine the location of the target object.
The interest to inverse problems is quite high in those problems where it is needed to find the singular support of the function that is a characteristic of a medium or process. First of all, it is connected with problems of the integral geometry, in which it is necessary to find a function by its integrals [11–13] and non-classical problems of a low-angle tomography for the radiation transfer equation [14,15].

2. Direct and inverse problems for the non-stationary radiation transfer equation

The mathematical model considered in this paper is based on the unsteady equation of radiation transfer and describes the propagation of high-frequency acoustic wave fields in scattering media [7,9,10].

\[
\left( \frac{1}{c} \frac{\partial}{\partial t} + \mathbf{k} \cdot \nabla_r + \mu \right) I(r, k, t) = \frac{\sigma}{4\pi} \int_\Omega I(r, k', t)dk' + J(r, k, t),
\]

where \(r \in G \subset \mathbb{R}^3, t \in [0, T]\) and the wave vector \(k\) belongs to the unit sphere \(\Omega = \{ k \in \mathbb{R}^3 : |k| = 1 \}\). The function \(I(r, k, t)\) is interpreted as the wave energy flux density propagating in the direction \(k\) with the velocity \(c\) at the time moment \(t\) at the point \(r\). The quantities \(\mu\) and \(\sigma\) have the meaning of the attenuation and scattering coefficients, and the function \(J\) describes the source of the sound field.

The area \(G\) is the upper half-space bounded by the horizontal plane \(\gamma = \{ r = (r_1, r_2, r_3) \in \mathbb{R}^3 : r_3 = -l\}, l > 0\). Equation (1) is supplemented by the initial and boundary conditions [9,10]

\[
I^-(r, k, t)|_{t=0} = 0, \quad (r, k) \in G \times \Omega, \quad (y, k, t) \in \Gamma^-, \quad I^+(y, k', t)(y, k', t)dk',
\]

Formulas (2),(3) use the notation \(I^\pm(y, k, t) = \lim_{\varepsilon \to 0} I(y \pm \varepsilon k, k, t \pm \varepsilon/c)\),

\[
\Gamma^\pm = \{(y, k, t) \in \gamma \times \Omega_{\pm} \times (0, T), \quad \Omega_\pm = \{ k \in \Omega : \text{sgn}(n \cdot k) = \pm 1\},
\]

where \(n = (0, 0, -1)\) is the unit vector of the outward normal to the boundary of the region \(G\).

Condition (2) means that there is no signal emission in the medium at the initial moment of time, and the boundary condition (3) describes the effects of diffuse reflection on the sea bottom according to Lambert’s law with the reflection coefficient \(\sigma_d(y)\). The value \(\sigma_d(y)\) is also called the bottom scattering coefficient.

Equation (1) forms the initial boundary value problem for finding the unknown function \(I\) on the set \(G \times \Omega \times (0, T)\) with the initial and boundary conditions (2), (3) with the given parameters \(\mu, \sigma, \sigma_d, J, c\). Initial boundary value problem (1), (2), (3) is called the direct problem for the radiation transfer equation.

Assume that the function \(J\) describes a point sound source moving with the constant velocity \(V\) in the direction of the axis \(r_2\)

\[
J(r, k, t) = \delta(r - Vt) \sum_{i=1}^m \delta(t - t_i), \quad V = (0, V, 0), \quad t_i > 0,
\]

where \(\delta\) is the Dirac delta function. In addition to the system of relations (1)–(3)

\[
\int_\Omega S_j(k)I^+(Vt, k, t)dk = P_j(t), \quad j = 1, ..., q,
\]
where $S_j(k)$ is nonzero in the subdomain $\Omega_j \subset \Omega$. Further two inverse problems are formulated.

The first problem is to find the function $\sigma_d(y)$ from relations (1), (2), (3), (5) with given $\mu, \sigma, c, V, J, l$ and $S_j, P_j(t)$. The second problem is to define a singular support of the function $\sigma_d(y)$ from relations (1), (2), (3), (5) with given $S_j, P_j(t)$.

The formulated inverse problems have various physical applications. For instance, when modeling the processes of acoustic sounding of the sea bottom by a SSS, which moves in a straight line with a constant velocity $V$ and sounds the surrounding space with impulse signals. Antennas are located on the support that measure the total intensity $P_j(t)$ in the sector $\Omega_j$ at the moment of time $t$. If $q = 2$ and the set $\Omega_1 = \{k \in \Omega : k_1 < 0\}$, $\Omega_2 = \{k \in \Omega : k_1 > 0\}$, then it is a simple case of a SSS with one receiving antenna on each side of the sonar board [8].

In the second inverse problem, it is required to determine not the function $\sigma_d$, but only the location of the lines of its discontinuities. At the same time, much less information is required about the initial data of the problem. In particular, the quantities $\mu, \sigma, J, P_j, S_j$ remain unknown and are not defined in this formulation of the problem. Only some knowledge of the structure of these functions is required, for example, that the function $J$ has the form of (4).

3. Expressions for the total power $P_j$ in the approximation of single scattering and a narrow radiation pattern

As a rule, the velocity of the antenna support $V$, the environment parameters $\mu, \sigma, c$ and the sounding periods $t_{i+1} - t_i$ are such that $(\sigma/\mu)^2 \ll 1$, $V/c \ll 1$ and $\exp(-\mu c/|t-t_i|) \ll 1$, $t \notin (t_i, t_{i+1})$. This gives reason to apply the single scattering approximation and to neglect the echolocation signals from other sounding periods over the current sounding interval [7–10].

Taking these restriction in the single scattering approximation into account the functions $P_j$ satisfy the following relations [9, 10]:

$$P_j(t) = P_{j,\gamma}(t) + P_{j,G}(t),$$

where

$$P_{j,\gamma}(t) = \frac{c^2 \exp(-\mu c|t-t_i|)}{2\pi (c(t-t_i)/2)^5} \int_{0}^{2\pi} S_j(k(\varphi, \theta_i)) \sigma_d(y(\varphi, \theta_i)) d\varphi,$$

$$P_{j,G}(t) = \frac{\sigma c \exp(-\mu c|t-t_i|)}{8\pi (c(t-t_i)/2)^2} \int_{0}^{\pi} \int_{\theta_i}^{\theta} S_j(k(\theta, \varphi)) \sin \theta d\theta d\varphi,$$

where $k(\theta, \varphi) = (-\sin \varphi \sin \theta, \cos \varphi \sin \theta, \cos \theta)$, the angle $\theta$ changes from $\theta_i = \arccos \left( \frac{2l}{c(t-t_i)} \right)$ to $\pi$, and the angle $\varphi \in [0, 2\pi)$. If the receiving antenna pattern $S_j$ is narrowly directed in the plane perpendicular to the sea bottom surface $r_3 = -l$, then $S_j(k(\theta, \varphi)) = \delta(\varphi - \varphi_j)$, where $\delta(\varphi - \varphi_j)$ is the Dirac delta function and formulas (7), (8) take a simple form with $t \in (t_i + 2l/c, t_{i+1})$

$$P_{j,\gamma}(t) = \frac{c^2 \exp(-\mu c|t-t_i|)}{2\pi (c(t-t_i)/2)^5} \sigma_d(y(\varphi_j, \theta_i)),$$

$$P_{j,G}(t) = \frac{\sigma c \exp(-\mu c|t-t_i|)}{8\pi (c(t-t_i)/2)^2} \left( 1 + \frac{2l}{c(t-t_i)} \right).$$

In this case, from (6), (9), (10), the authors get an explicit formula to define the function $\sigma_d$ for
any value \( j \) \([9,10]\) 

\[
\sigma_{d,j}(y) = \left( P_j(t) - \frac{\sigma c \exp(-2\mu|y - Vt|)}{8\pi|y - Vt|^2} \left( 1 + \frac{l}{|y - Vt|} \right) \right) \times \\
\times \left( \frac{c \ell^2 \exp(-2\mu|y - Vt|)}{2\pi|y - Vt|^5} \right)^{-1}, \tag{11}
\]

where \( t = (y_2 + y_1\text{ctg}\varphi_j)/V \).

Obviously, in this case, two receiving antennas located on different sides of the support are enough for measurement, for example, when \( S_1 = \delta(\varphi - \pi/2) \) and \( S_2 = \delta(\varphi - 3\pi/2) \). It is a widespread method for constructing sonar images — sequentially tack by tack, perpendicular to the movement of the support.

With an increase in the width of the radiation pattern, the calculation of the function \( \sigma_d \) by formula (11) leads to an increase in the error. In particular, the effect of ”blur” or defocus of an image appears when restoring contrast structures on sonar images.

To eliminate image defects, various focusing methods are used, which are mathematically reduced to solving an integral equation of the first kind. When discretizing the required function, the solution of the integral equation is equivalent to the solution of the system of linear equations. As a rule, it is ill-conditioned and significantly complicates the definition of the required function.

According to the authors, another drawback of the algorithm for recovering the sea bottom scattering coefficient according to formula (11) is caused by the fact that the information on the attenuation and volume scattering coefficients is required. As a rule, coefficients of equation (1) describing the ocean radiation interaction are known only approximately, therefore the reconstruction of the bottom scattering coefficient according to formula (11) leads to large distortions of tomographic images.

Our further objective is to construct a numerical algorithm for finding the singular support of the function \( \sigma_d \), which requires much less information about the initial data and is free from most of the indicated drawbacks.

4. The numerical algorithm for finding the singular support of the function \( \sigma_d \)

In this section, we describe a numerical algorithm for reconstructing the discontinuity lines of the bottom scattering coefficient. The problem under consideration can be attributed to the problems of small-angle tomography, and the algorithm for its solution is close to the methods given in \([14,15]\).

The plane \( \gamma \) represented as a union of two-dimensional convex subareas \( \gamma_i, i = 1, ..., p, \)

\( \gamma_i \cap \gamma_j = \emptyset, i \neq j, \) and the set \( \gamma_0 = \gamma \setminus \bigcup_{i=1}^{p} \gamma_i. \) In each of the areas \( \gamma_i \) the function \( \sigma_d(y) \) is constant and the problem is to determine the boundaries of the subareas \( \gamma_i, i = 1, ..., p \) by the given values \( P_j(t), j = 1, ..., q. \) The receiving antenna pattern \( S_j(k(\varphi, \theta)) \) is equal to \( 2\beta \) at \( \varphi \in (\varphi_j - \beta, \varphi_j + \beta) \) and is equal to zero outside of the interval \( (\varphi_j - \beta, \varphi_j + \beta) \), at which the support of each of the function \( S_j \) is concentrated either on the interval \( 0 < \pi/2 - \delta/2 < \varphi < \pi/2 + \delta/2, \) or on the interval \( 0 < 3\pi/2 - \delta/2 < \varphi < 3\pi/2 + \delta/2, \) where \( \delta < \pi. \)

The last limitation is typical of the sea bottom scanning by a SSS and allows the restoration of the function \( \sigma_d \) separately at \( r_1 > 0 \) and \( r_1 < 0 [7]. \)

Taking into account the formulated restrictions and the fairness of the relation \( |Vt - y|^2 = y_1^2/\sin^2\varphi_j + t^2 \) at \( t = (y_2 + y_1\text{ctg}\varphi_j)/V, \) the expression for the functions \( P_{j,\gamma}(y_2 + y_1\text{ctg}\varphi_j)/V) \)
and $P_{j,G}((y_2 + y_1 \operatorname{ctg} \varphi_j)/V)$ can be written down as

$$
P_{j,\gamma}((y_2 + y_1 \operatorname{ctg} \varphi_j)/V) = \frac{c_2}{2\pi} \exp\left(-\frac{2\mu\sqrt{y_1^2 - \sin^2 \varphi_j + l^2}}{y_1^2 / \sin^2 \varphi_j + l^2} \right) \times$$

$$
\int_{\varphi_j - \beta}^{\varphi_j + \beta} \sigma_d(|y_1| \sin \varphi/|\sin \varphi_j|, y_2 + y_1 \operatorname{ctg} \varphi_j - |y_1| \cos \varphi/|\sin \varphi_j|) d\varphi =$$

$$
= \frac{c_2}{2\pi} \frac{\exp\left(-\frac{2\mu\sqrt{y_1^2 - \sin^2 \varphi_j + l^2}}{y_1^2 / \sin^2 \varphi_j + l^2} \right)}{y_1^2 / \sin^2 \varphi_j + l^2} \times$$

$$
\int_{\varphi_j - \beta}^{\varphi_j + \beta} \sigma_d(|y_1| \sin \varphi/|\sin \varphi_j|, y_2 + |y_1| \cos \varphi_j - \cos \varphi_j/|\sin \varphi_j|) d\varphi, \quad (12)$$

$$
P_{j,G}((y_2 + y_1 \operatorname{ctg} \varphi_j)/V) =$$

$$
= \frac{c_2}{8\pi} \frac{\exp\left(-\frac{2\mu\sqrt{y_1^2 - \sin^2 \varphi_j + l^2}}{y_1^2 / \sin^2 \varphi_j + l^2} \right)}{y_1^2 / \sin^2 \varphi_j + l^2} \left(1 + \frac{l}{\sqrt{y_1^2 / \sin^2 \varphi_j + l^2}}\right). \quad (13)$$

Let us denote by

$$
\tilde{P}_{j,\gamma}(y) \equiv P_{j,\gamma}((y_2 + y_1 \operatorname{ctg} \varphi_j)/V), \quad \tilde{P}_{j,G}(y) \equiv P_{j,G}((y_2 + y_1 \operatorname{ctg} \varphi_j)/V),
$$

where $y = (y_1, y_2, -l)$ and the functions $P_{j,\gamma}, P_{j,G}$ are defined in (12), (13). An approximate method for solving the inverse problem is based on a preliminary calculation for each $k_j = (\cos \varphi_j, \sin \varphi_j, 0)$ of a function as follows

$$
\tilde{\sigma}_{d,j}(y) = |\nabla \tilde{P}_{j,\gamma}(y) \cdot k_j|, \quad (14)
$$

and the subsequent construction of the indicator function

$$
\tilde{\sigma}_d(y) = \sum_{j=1}^{q} \tilde{\sigma}_{d,j}(y). \quad (15)
$$

It is easy to see that the functions $\nabla \tilde{P}_{j,G}(y) \cdot k_j$ are bounded on the entire set of their arguments, and functions $\nabla \tilde{P}_{j,\gamma}(y) \cdot k_j$ can indefinitely grow only if the line

$$
\begin{align*}
z_1 &= |y_1| \sin \varphi_j, \\
z_2 &= y_2 + |y_1| \cos \varphi_j - \cos \varphi_j/|\sin \varphi_j|, \quad \varphi \in [\varphi_j, -\beta, \varphi_j + \beta], \\
z_3 &= -l,
\end{align*}
$$

crosses the discontinuity line of the function $\sigma_d(y)$, $y = (y_1, y_2, -l)$. When the point $y$ tends to the discontinuity line of the function $\sigma_d(y)$ the number of the terms $\tilde{\sigma}_{d,j}(y)$ in sum (15) with a specified property will increase, which leads to an increase in the function $\tilde{\sigma}_d(y)$. The function growth rate $\tilde{\sigma}_{d,j}(y)$ depends on the angle between the curve (16) and the discontinuity line in the point of their intersection. And the growth is maximum if the angle value is equal to zero that is the curves touch each other. A rigorous description of the noted facts is complicated by the limitation associated with the discreteness of a set of sounding directions is rather cumbersome and goes beyond the scope of this problem. In the next section, on a number of computational experiments, the authors will give a numerical confirmation of the efficiency of the constructed algorithm.
5. Computational experiments

To demonstrate the efficiency of the algorithm for solving the inverse problem, several series of experiments were done. Testing the algorithm was for the function $\sigma_d(r)$, a graphical representation under $r_1 > 0$ is shown in figure 1. The function $\sigma_d$ took the value 0.9 in all three inclusions $\gamma_i, i = 1, 2, 3$. The type of inclusions is "sunken plane" and they are located at a distances of 50, 150, 250 meters from axis $r_1 = 0$, respectively, and $\sigma_d = 0.1$ in inclusion $\gamma_0 = \gamma \setminus \left( \gamma_1 \cup \gamma_2 \cup \gamma_3 \right)$. The remaining quantities take on values typical of acoustic sounding in the oceanic medium at frequencies of about 100kHz [7,9,10]: $\mu = 0.018 m^{-1}$, $\sigma = 0.1 \mu$, $c = 1500 m/s$.

Figure 1. Graphical representation of the bottom scattering coefficient $\sigma_d(y)$ (original). Linear dimensions of the surveyed area of 300m $\times$50m bottom.

When monitoring water areas by a SSS the value of support velocity, the height of its trajectory above the bottom and sounding interval depend on the target of mission, the dimensions of the investigated pool and the features of the equipment. In this research the values were chosen as follows [7,9,10]: $V = 1 m/s$, $l = 12m$, $t_{i+1} - t_i = 0.4s$.

In the experiments, the angle $\beta$ was equal to $1/\pi$ and $2.5/\pi$. Thus, the width of a receiving antenna of pattern in the horizontal plane was equal to 2 and 5 degrees, respectively.

In figure 2 and 3 graphical images are shown for the function $\tilde{\sigma}_d(y)$ for different values of the parameter $q$, where the angles $\varphi_j, j = 1, ..., q$ are located uniformly on the interval $[\pi/5, 5\pi/6]$. 

Figure 2. The function $\tilde{\sigma}_d(y)$, characterizing the singular support of the function $\sigma_d(y)$, for different numbers of the sounding angles $q$ with the beam width of 2 degrees.
Figure 3. The function $\tilde{\sigma}_d(y)$, characterizing the singular support of the function $\sigma_d(y)$, for different numbers of sounding angles $q$ with beam width 5 degrees.

As expected, the quality of focusing the objects increases with an increase in the number of sound waves angles and a decrease in the beam width. The improvement in the image quality with an increase in $q$ is especially pronounced at a sensing range exceeding 200 meters. When focusing the objects at a distance of up to 100m, it is sufficient to use a small number of probing angles.

6. Conclusion
In this paper the authors propose an algorithm for finding a discontinuity line of the sea bottom scattering coefficient based on the construction of indicator function indicating to the location of the desired lines.

The numerical methods used show that the algorithm proposed can be used for finding any objects with a SSS at a range of about 200 meters with the currently available instrument a panel. The quality of focusing increases with an increase in the number of sounding angles. Moreover, to detect nearby objects, one can limit yourself to a relatively small number of sounding angles.

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