Maximizing Communication Throughput in Tree Networks

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Abstract

A widely studied problem in communication networks is that of finding the maximum number of communication requests that can be concurrently scheduled, provided that there are at most \( k \) requests that pass through any given edge of the network. In this work we consider the problem of finding the largest number of given subtrees of a tree that satisfy given load constraints. This is an extension of the problem of finding a largest induced \( k \)-colorable subgraph of a chordal graph (which is the intersection graph of subtrees of a tree). We extend a greedy algorithm that solves the latter problem for interval graphs, and obtain an \( M \)-approximation for chordal graphs where \( M \) is the maximum number of leaves of the subtrees in the representation of the chordal graph. This implies a 2-approximation for \( V_{pt} \) graphs (vertex-intersection graphs of paths in a tree), and an optimal algorithm for the class of directed path graphs (vertex-intersection graphs of paths in a directed tree) which in turn extends the class of interval graphs.

In fact, we consider a more general problem that is defined on the subtrees of the representation of chordal graphs, in which we allow any set of different bounds on the vertices and edges. Thus our algorithm generalizes the known one in two directions: first, it applies to more general graph classes, and second, it does not require the same bound for all the edges (of the representation). Last, we present a polynomial-time algorithm for the general problem where instances are restricted to paths in a star.

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1 Introduction

1.1 The framework

Consider a communication graph $G$, in which the vertices represent telephone stations (trunks) or internet computers and the edges represent connection lines. Each station and connection line has a given capacity to handle transmissions. Given a set of transmission requests as paths between vertices, the problem is to find the maximum set of transmissions (paths in $G$) which can be handled at any moment. Thus, we must find a maximum set of paths that do not overload the capacity of the vertices or edges.

This scenario occurs in numerous applications in communication networks, and it can be interpreted also in the context of a production line, where vertices are associated with machines, and bounds with number of units that can be produced by a machine during one time unit. In these applications, the aim is to maximize the benefit during one time unit. In addition, such algorithms can be used in successive steps, so as to schedule all of the given requests.

Stated in graph-theoretical terms, this is a special case of the fundamental class of graph optimization problems, in which the objective is to find a maximum number of given subgraphs under a given set of constraints. Specifically, when the communication graph $G$ is a tree and the given subgraphs are subtrees of $G$, the problem is a generalization of the maximum $k$-colorable set problem in chordal graphs. In this problem, we look for the maximum number of vertices of a given graph that can be colored with $k$ colors (such that no two vertices of the same color are adjacent).

The problem has been studied since around the 1980’s, in theoretical as well as practical works. We now mention some theoretical studies that involve this problem. In [4], the problem is studied for interval graphs. In [14], the problem is used in the context of scheduling theory. Specifically, it is used as a subroutine to get an approximation algorithm for the minimum sum coloring problem in interval graphs. This is related to the problem of minimizing the average delay in scheduling communication requests on a path with bounded capacity. In [15], it was used in the context of parameterized algorithms. Examples of works in which the maximum $k$-colorable set problem is used in applications are [6], in the context of optical networks with wavelength division multiplexing (WDM), where multiple connections can share a link if they are transmitted on different wavelengths, and the aim is to satisfy a maximum number of connection requests in a tree network; [2], where it was used in the context of traffic grooming in optical networks; and in [13], in the context of register allocation, where it corresponds to the case where the input programs are in static single assignment form.

1.2 Graphs representing intersections of subtrees of a tree

In this paper, we study classes of graphs that can be represented as intersections of subtrees of a tree, and also circular arc graphs. Trees and rings are fundamental topologies under which many networking problems have been studied. In our study, paths correspond to calls between pairs of vertices, and sub-trees correspond to subsets of vertices that need to be connected in the form of multicast communication. Following are the classes of graphs used in this paper.

- **Chordal graphs**: a graph is chordal if it does not contain induced cycles of length four or more. In other words, in a chordal graph every cycle with more than three vertices contains a chord (i.e., an edge that joins two non-adjacent vertices of the cycle). It is
known that a graph is chordal if and only if it is the vertex-intersection graph of subtrees of a tree \cite{9}. We refer the reader to \cite{11} for more details on chordal graphs.

- **Vpt graphs**: A Vpt graph is the vertex-intersection graph of paths in a tree. In other words, a Vpt graph is a chordal graph that has a representation in which every subtree is a path.

- **Directed path graphs**: a directed path graph is the vertex-intersection graph of (directed) paths in a rooted tree. We emphasize that a directed path graph is an undirected graph. In fact, it is easy to see that it is a Vpt graph. Clearly, an interval graph is a directed path graph.

- **Ept graphs**: an Ept graph is an edge-intersection graph of paths of a tree; such a graph is not necessarily chordal.

### 1.3 Previous Work

In \cite{23} it is shown that determining whether there is a set, of at least a given size, of paths that can be $k$ colored, is NP-hard for chordal graphs, even for those who can be represented as intersection of subtrees of a star graph. (We term this problem as MAX $k$-COLORABLE SET in Section 2.) They also present a greedy algorithm that solves the problem for interval graphs.

Ept graphs were defined in \cite{10,12}, where it was shown that - as opposed to chordal graphs - their recognition problem is NP-hard and so is the chromatic number problem. In \cite{7} and \cite{17}, it is shown that it is NP-hard to determine the chromatic number of an Ept graph even if it has a representation on a star. The reader is referred to \cite{3} as an excellent survey on path coloring problems on trees.

### 1.4 Our Contribution

We extend the greedy algorithm of \cite{23} (for interval graphs) to the class of chordal graphs. We obtain an $M$-approximation, where $M$ is the maximum number of leaves of the trees in the representation. This implies a 2-approximation for Vpt graphs, and an optimal algorithm for directed path graphs. In addition to the rigorous analysis, we tested our greedy algorithm on random data and real-life data. We found out that its performance is very close to optimum.

In fact, we consider a more general problem that is defined on the subtrees of the representation of chordal graphs. Whereas the algorithm in \cite{23} deals with the case where there is the same bound $k$ for the number of intersections in any vertex, our extension allows an upper bound to be specified for each vertex individually. Therefore, our algorithm generalizes the greedy algorithm of \cite{23} in two directions: first it works on wider classes of graphs and second, it works for more general constraints.

For a star network, we show that the problem of finding the largest number of paths can be solved in polynomial time. On the other hand, when the subtrees are claws (i.e. sub-stars with 3 leaves) and all the loads are 1 except for the center vertex, the problem becomes APX-hard, and in general, it can not be approximated to within $n^{1-\epsilon}$ for any $\epsilon > 0$ where $n$ is the number of subtrees.

In Section 2 we present basic terms and notations, and formally present the optimization problems. In Section 3 we present our results for the generalized problem where different capacities can be defined on the vertices and edges of the tree, and in Section 4 we consider the problem of finding a $k$-colorable set of vertices of maximum cardinality. We summarize the theoretical results and present simulation results in Section 5.
2 Preliminaries

Sets: For two non-negative integers $n_1, n_2$, denote by $[n_1, n_2]$ the set of integers that are not smaller than $n_1$ and not larger than $n_2$. We also use $[n]$ as a shorthand for $[1, n]$.

Graphs: We use standard terminology and notation for graphs, see for instance [5]. Given a simple undirected graph $G$, we denote by $V(G)$ the set of vertices of $G$ and by $E(G)$ the set of the edges of $G$. Denote an edge between two vertices $u$ and $v$ as $uv$. We say that the edge $uv \in E(G)$ is incident to $u$ and $v$, $u$ and $v$ are the endpoints of $uv$, and $u$ is adjacent to $v$ (and vice versa). We use the term object to refer to a vertex or an edge.

A coloring of $G$ is a function from $V(G)$ into the set of positive natural numbers. A coloring of $G$ is proper if $u$ and $v$ are assigned different colors by the coloring whenever $uv$ is an edge of $G$. A graph is $k$-colorable if it has a proper coloring of its vertices using colors from $[k]$. The chromatic number of a graph $G$, denoted by $\chi(G)$, is the smallest integer $k$ such that $G$ is $k$-colorable. The clique number $\omega(G)$ of a graph $G$ is the size of its largest clique.

Trees, subtrees and their intersection graphs: Let $U$ be a set of subtrees of a tree $T$, and $o$ an object of $T$. Denote as $U_o$ the set of subtrees in $U$ that contain the object $o$, and by load($U,o$) their number $|U_o|$ which is termed as the load of $o$ induced by $U$. Denote by load($U$) the vector consisting of the values load($U,o$) indexed by the objects of $T$, and by $LV(U)$ (resp. $LE(U)$) the maximum of load($U,o$) over all vertices (resp. edges) $o$ of $T$.

Let $\mathcal{T} = \{T_1, \ldots, T_n\}$ be a set of subtrees of a tree $T$. Let $G = (V,E)$ be a graph over the vertices $\{v_1, \ldots, v_n\}$ such that $v_i$ and $v_j$ are adjacent if and only if $T_i$ and $T_j$ have a common vertex. Then, $G$ is termed as the vertex-intersection graph of $(T,\mathcal{T})$ and conversely $(T,\mathcal{T})$ is termed a subtree intersection representation, or simply a representation of $G$. Whenever there is no confusion, we will denote the representation $(T,\mathcal{T})$ simply as $\mathcal{T}$.

Graph classes: In Section 1.2 we introduced the following graph classes: Chordal graphs, VPT graphs, directed path graphs, and EPT graphs. Let us now mention some of their properties needed for this paper.

Let $G$ be a chordal graph with a representation $(T,\mathcal{T})$, and $v$ a vertex of $T$. Clearly, the set of $T_v$ of subtrees in $\mathcal{T}$ that contain the vertex $v$ corresponds to a clique of $G$. It is also known that there is a representation (where $T$ is termed the clique-tree of $G$) such that every maximal clique of $G$ corresponds to $T_v$ for some vertex $v$ of $T$. Therefore, the clique number $\omega(G)$ of $G$ is equal to $LV(T)$. Since chordal graphs are perfect, we have $\chi(G) = \omega(G)$. It is known that one can find a representation of a given chordal graph $G$ in polynomial time, and such a representation $(T,\mathcal{T})$ can be used to determine the chromatic number of $G$, since $\chi(G) = \omega(G) = LV(T)$.

A graph is termed a VPS graph if it is a VPT graph, with a representation $(T,\mathcal{T})$ where $T$ is a star.

Let $G$ be an EPT graph with a representation $(T,\mathcal{T})$. Clearly, for an edge $e$ of $T$ the set $T_e$ corresponds to a clique of $G$. However, this correspondence is not a bijection as the following simple example shows. Consider a star with three leaves and three paths each of which joins a distinct pair of leaves. The edge-intersection graph of these paths is a triangle, i.e. a clique. However, no edge of the star contains all paths.

Problem Statement: Our main goal is to solve the following problem.
Maximum k-Colorable Set (Max k-Colorable Set)

**Input:** A graph $G$, a positive integer $k$.

**Output:** A $k$-colorable induced subgraph of $G$ of maximum order.

Since the decision version of the chromatic number problem is equivalent to asking whether the maximum $k$-colorable set is the entire vertex set, we have the following observation.

> **Observation 1.** If the chromatic number problem is $\text{NP}$-hard when restricted to a class of graphs $\mathcal{C}$, then Max $k$-Colorable Set, when restricted to $\mathcal{C}$, is also $\text{NP}$-hard.

Recall that a representation $\langle T, \mathcal{T} \rangle$ of a chordal graph $G$ can be found in polynomial time and a set of vertices of $G$ is $k$-colorable if and only if the corresponding set $U \subseteq \mathcal{T}$ of subtrees satisfies $\text{load}(U, v) \leq k$ for every node $v$ of $T$. Therefore, we define the following problem that generalizes the Max $k$-Colorable Set problem in chordal graphs. Our results will lead to an approximation algorithm for Ept graphs.

**Maximum Subtrees with Bounded Load (MaxSubtreesBL)**

**Input:** A triple $(T, \mathcal{T}, \mathbf{k})$ where

- $T$ is a tree,
- $\mathcal{T}$ is a set of subtrees of $T$,
- $\mathbf{k}$ is a vector of non-negative integers indexed by the objects of $T$.

**Output:** A maximum cardinality subset $U$ of $\mathcal{T}$ such that $\text{load}(U) \leq \mathbf{k}$.

Given an instance $(T, \mathcal{T}, \mathbf{k})$ of MaxSubtreesBL, a set $U \subseteq \mathcal{T}$ of subtrees and an object $o$ of $T$, we say that $o$ is overloaded (resp. tight, underloaded) in $U$ if $\text{load}(U, o) > k_o$ (resp. $\text{load}(U, o) = k_o$, $\text{load}(U, o) < k_o$). A set $U \subseteq \mathcal{T}$ of subtrees of $T$ is feasible (for the instance $(T, \mathcal{T}, \mathbf{k})$) if no object of $T$ is overloaded in $U$.

Throughout this paper, $T$ is a tree with an arbitrary vertex $r$ chosen as its root and $\mathcal{T} = \{T_1, \ldots, T_n\}$ is a set of subtrees of $T$. For every subtree $T_i \in \mathcal{T}$ of $T$ we denote by $\text{root}(T_i)$ its (unique) vertex that is closest to $r$, and by $\text{leaves}(T_i)$ the number of leaves of $T_i$ that are different from $\text{root}(T_i)$. Finally, $M \overset{\text{def}}{=} \max_{T_i \in \mathcal{T}} \text{leaves}(T_i)$ is the maximum number of leaves of a subtree, except its root.

## 3 Maximum number of subtrees with bounded load

We now consider the MaxSubtreesBL problem on chordal graphs. In the next section we consider the Max k-Colorable Set problem.

In this section, we present a greedy algorithm for the MaxSubtreesBL problem and prove that it is an $M$-approximation. This implies a) a 2-approximation whenever the subtrees are paths, and b) optimality whenever the tree is directed according to some root and all the paths follow the direction. Since MaxSubtreesBL generalizes Max k-Colorable Set for chordal graphs, these results imply the same approximation ratios for Max k-Colorable Set in the families of chordal, Vpt and directed path graphs respectively. We then consider the special case of paths in a star and provide an optimal algorithm for this case.

We use the main result of this section also in the following section in which we consider non-chordal graphs.

Assume without loss of generality, that the subtrees $\mathcal{T}$ are numbered in the order they are visited by BottomUpGreedy, i.e., in some bottom-up order of their roots in $T$, and let $T_i$ be
Note that the first and last conditions above, together imply that $\text{ALG}_i = \text{ALG} \cap T_i$.

Lemma 1. There exists a sequence $S_0, S_1, \ldots, S_n$ of feasible subsets of $\mathcal{T}$ such that for every $i \in [0, n]$ we have

1. $S_i \setminus T_i \subseteq \text{OPT}$,
2. $|\text{OPT}| - |S_i| \leq (M - 1)|\text{ALG}_i|$, and
3. $S_i \cap T_i = \text{ALG}_i$.

Note that the first and last conditions above, together imply that $S_i \setminus \text{ALG}_i$ is contained in $\text{OPT}$.

Proof. We define $S_i$ inductively, and prove by induction on $i$. We start with $S_0 = \text{OPT}$. Clearly, $S_0$ is feasible, and it is easy to verify that it satisfies all three conditions. For $i \in [n]$ we proceed as by dividing into cases.

- $T_i \notin \text{ALG}$ and $T_i \notin S_{i-1}$ or $T_i \in \text{ALG}$ and $T_i \in S_{i-1}$: in this case we define $S_i = S_{i-1}$ which is feasible by the inductive hypothesis. Clearly, $S_{i-1} \cap \{T_i\} = \text{ALG} \cap \{T_i\}$ in both cases. We now verify that all the conditions are satisfied by $S_i$.

1. $S_i \setminus T_i = S_{i-1} \setminus T_i = S_{i-1} \setminus T_{i-1} - T_i \subseteq S_{i-1} \setminus T_{i-1} \subseteq \text{OPT}$.
2. $|\text{OPT}| - |S_i| = |\text{OPT}| - |S_{i-1}| \leq (M - 1)|\text{ALG}_{i-1}| \leq (M - 1)|\text{ALG}_i|$.
3. $S_i \cap T_i = S_{i-1} \cap T_i = S_{i-1} \cap (T_{i-1} + T_i) = (S_{i-1} \cap T_{i-1}) \cup (S_{i-1} \cap \{T_i\}) = \text{ALG}_{i-1} \cup (\text{ALG} \cap \{T_i\}) = \text{ALG}_i$.

- $T_i \notin \text{ALG}$ and $T_i \in S_{i-1}$: in this case we have

$$S_{i-1} = (S_{i-1} \cap T_{i-1}) \cup (S_{i-1} \setminus T_{i-1}) \supseteq (S_{i-1} \cap T_{i-1}) + T_i = \text{ALG}_{i-1} + T_i.$$ Since, by the inductive hypothesis, $S_{i-1}$ is feasible, we conclude that $\text{ALG}_{i-1} + T_i$ is also feasible. Then, by the greedy behaviour of the algorithm, we have $T_i \in \text{ALG}$, a contradiction.
Theorem 2. BottomUpGreedy is an $M$-approximation algorithm for MaxSubtreesBL.

Proof. Let $S_0, \ldots, S_n$ be the sets whose existences are guaranteed by Lemma [1]. Then, $S_n = S_n \cap T_n = ALG_n = ALG$, and we conclude

$$|\text{Opt}| - |ALG| = |\text{Opt}| - |S_n| \leq (M - 1) |ALG_n| = (M - 1) |ALG|.$$
We now show that the bound proven in Theorem 2 is tight for \( M = 2 \).

\begin{theorem}
The approximation ratio of \texttt{BottomUpGreedy} is exactly 2 whenever \( M = 2 \).
\end{theorem}

\begin{proof}
Consider the instance \((T, T, k)\) of \texttt{MaxSubtreesBL} where \( T \) is a star with center 0 and 4 leaves 1, 2, 3, 4. Let also \( k_0 = 2 \) and \( k_o = 1 \) for every other object \( o \). \texttt{BottomUpGreedy} processes the paths of \( T \) in an arbitrary order since \( \text{root}(P) = 0 \) for every \( P \in T \). The algorithm may consider first the path \( P_{2,3} \) and return \( \{P_{2,3}\} \) as a solution. On the other hand, \( \{P_{1,2}, P_{3,4}\} \) is a feasible solution. Therefore, the approximation ratio of \texttt{BottomUpGreedy} is at least 2. Combining with Theorem 2 we get the result.
\end{proof}

In the sequel, we present an optimal algorithm for \texttt{MaxSubtreesBL} where \( T \) is a star.

\begin{algorithm}[VPSByDegreeConstrainedSubgraph]
\begin{algorithmic}
\Require An instance \((T, T, k)\) of \texttt{MaxSubtreesBL}
\Ensure Return a subset \( U \) of \( T \) such that \( \text{load}(U) \leq k \).
\State \( U \leftarrow \emptyset \).
\While {\( T \) contains at least one path with less than three vertices}
\If {there is an object \( o \) of \( T \) with \( k_o = 0 \)}
\State remove \( o \) from \( T \) and from \( k \)
\State remove all the paths using \( o \) from \( T \)
\ElseIf {there is a trivial path \( P_i \in T \) consisting of vertex \( j \)}
\State \( U \leftarrow U + P_i; \ T \leftarrow T - P_i \)
\State \( k_j \leftarrow k_j - 1 \)
\Else
there is a path \( P_i \in T \) with exactly two vertices \( 0, j \)
\State \( U \leftarrow U + P_i; \ T \leftarrow T - P_i \)
\State \( k_0 \leftarrow k_0 - 1; \ k_j \leftarrow k_j - 1; \ k_{0j} \leftarrow k_{0j} - 1 \)
\EndIf
\EndWhile
\State Construct the graph \( H \) whose vertices are the leaves of \( T \)
\State and it has an edge \( ij \) for every path \( i0j \) of \( T \).
\State \( d_i \leftarrow k_i \) for every leaf \( i \) of \( T \).
\State \( E \leftarrow \) a solution of \texttt{MaxDegConstSG} on \((H, d)\)
\If {\( |E| > k_0 \)}
\State \( E \leftarrow \) an arbitrary subset of \( E \) with \( k_0 \) paths
\EndIf
\State \( U_E \leftarrow \) the paths of \( T \) corresponding to the edges \( E \)
\State \Return \( U \cup U_E \).
\end{algorithmic}
\end{algorithm}

\begin{theorem}
Algorithm \texttt{VPSByDegreeConstrainedSubgraph} returns an optimal solution of \texttt{MaxSubtreesBL} instances in which \( T \) is a star.
\end{theorem}

\begin{proof}
Let \((T, T, k)\) be an instance of \texttt{MaxSubtreesBL} where \( T \) is a star with center 0 and \( p \) leaves 1, \ldots, \( p \), and \( T \) is a set of \( n \) paths \( P_1, \ldots, P_n \) of \( T \). Clearly, if \( k_o = 0 \) for some
object \( o \) of \( T \) the object \( o \) and all the paths using it can be removed from the instance as done by \textsc{VPSByDegreeConstrainedSubgraph} at line 4. Therefore, we may assume that \( k_0 > 0 \) for every object \( o \) of \( T \). First observe that there exists an optimal solution \( \text{Opt} \) such that if \( P_i \) is a (not necessarily proper) sub-path of \( P_j \) then either \( P_i \in \text{Opt} \) or \( P_j \notin \text{Opt} \). This is true since otherwise \( \text{Opt} - P_j + P_i \) is also an optimal solution. Then, for every trivial path \( P_i \), there is an optimal solution that contains it. Therefore, one can include any trivial path in the solution and proceed to the remaining instance as done by \textsc{VPSByDegreeConstrainedSubgraph} at line 7. In the rest of the proof, we further assume that \( \mathcal{T} \) does not contain trivial paths. We now show that paths with two vertices can be processed similarly, as done by \textsc{VPSByDegreeConstrainedSubgraph} at line 10.

Let \( P_i \) be a path with two vertices 0, \( j \), and let \( \text{Opt} \) be an optimal solution that does not contain \( P_i \). Recall that \( k_0, k_j > 0 \). \( \text{Opt} \) contains at least one path that intersects \( P_i \), since otherwise \( \text{Opt} + P_i \) is a feasible solution, contradicting the optimality of \( \text{Opt} \). Since such a path is not trivial, it intersects \( P_i \) at vertex 0 and possibly at vertex \( j \). Let \( Q \) be a path of \( \text{Opt} \) with the largest intersection with \( P_i \) (in terms of number of vertices). If \( Q \) intersects \( P_i \) in both vertices, than \( P_i \) is a sub-path of \( Q \), and there is an optimal solution that contains \( P_i \). Otherwise, \( Q \) intersects \( P_i \) only at the center of the star and no path of \( \text{Opt} \) intersects \( P_i \) at the leaf \( j \). Then, \( \text{Opt} - Q + P_i \) is another optimal solution that contains \( P_i \).

After this processing, we remain with an instance where all the paths of \( \mathcal{T} \) have exactly three vertices. We now observe that this instance is almost equivalent to an instance of the following degree-constrained maximum subgraph problem: we are given a graph with an integer \( d_v \geq 1 \) on every vertex \( v \) of it, and our goal is to find a subset of edges with maximum cardinality, such that every vertex \( v \) is incident to at most \( d_v \) of these edges. This problem can be solved in polynomial time \cite{19}. We now note that if \( k_0 \geq n \) our instance is equivalent to the degree constrained subgraph problem on the graph \( H \) whose vertices correspond to the leaves of \( T \), its edges correspond to the paths of \( \mathcal{T} \) (each of which has three vertices), and finally \( d_j = k_j \) for every \( j \in \mathcal{P} \). If the maximum degree constrained subgraph contains more than \( k_0 \) edges, only an arbitrary subset of \( k_0 \) edges is a feasible solution of our problem. Lines 13 through 19 of \textsc{VPSByDegreeConstrainedSubgraph} achieve exactly this goal.

We complement the above result by showing that \textsc{MaxSubtreesBL} is \( \text{APX} \)-hard even if \( T \) is a star and every subtree in \( \mathcal{T} \) has at most three leaves.

\begin{theorem}
1. \textsc{MaxSubtreesBL} is not approximable within \( |\mathcal{T}|^{1-\varepsilon} \) for every \( \varepsilon > 0 \) even when
   \begin{itemize}
   \item \( T \) is a star, and
   \item \( k_o = 1 \) for every object \( o \) of \( T \) except its center.
   \end{itemize}
2. \textsc{MaxSubtreesBL} is \( \text{APX} \)-hard even when
   \begin{itemize}
   \item \( T \) is a star,
   \item \( k_o = 1 \) for every object \( o \) of \( T \) except its center, and
   \item \( M = 3 \).
   \end{itemize}
\end{theorem}

\begin{proof}
By approximation preserving reduction from the maximum independent set problem which is known not to be approximable within \( n^{1-\varepsilon} \) for any \( \varepsilon > 0 \) \cite{15}, where \( n \) is the number of vertices of the graph, and also in \( \text{APX} \)-hard in graphs with maximum degree three \cite{19}. Given a graph \( G \) on \( n \) vertices which is an instance of the maximum independent set problem, we construct a star \( T \), every edge of which corresponds to an edge of \( G \). Now a vertex \( v \) of \( G \) corresponds to a sub-star \( T_v \) of \( T \) that consists of the edges corresponding to the edges of \( G \) incident to \( v \). Then \( G \) is the edge-intersection graph of the sub-stars \( \mathcal{T} = \{ T_v | v \in V(G) \} \).

\small
\hrule

\end{proof}
We note that the degree of a vertex $v$ of $G$ is equal to the number of leaves of the sub-star $T_v$. A set of vertices of $G$ is an independent set if and only if they are pairwise non-adjacent if and only if the corresponding sub-stars of $T$ are pairwise non-edge-intersecting if and only if they induce a load of at most 1 on every object except the center of $T$. Therefore, $G$ has an independent set of $k$ vertices if and only if the $T$ contains $k$ sub-stars that induce load of at most 1 on the leaves of $T$. We conclude that this reduction preserves approximation ratio.

4 Maximum $k$-colorable set

In this section, we consider the $\text{Max } k\text{-Colorable Set}$ problem. Given a chordal graph $G$, one can find a representation $(T, T)$ of it in polynomial time. Let $k$ be the vector with $k_e = n$ for every edge $e$ of $T$ and $k_v = k$ for every vertex $v$ of $T$. We recall that, in this context, the following statements are equivalent:

- A subset $U$ of vertices of $G$ is $k$-colorable.
- The set $T_U$ satisfies $\text{load}(T_U, v) \leq k$ for every vertex $v$ of $T$.
- The set $T_U$ is feasible for the instance $(T, T, k)$ of $\text{MaxSubtreesBL}$.

Therefore, one can run $\text{BottomUpGreedy}$ on the instance $(T, T, k)$ to get an $M$-approximation for $\text{Max } k\text{-Colorable Set}$ in chordal graphs where $M$ is the maximum number of leaves over all subtrees of a representation. Whenever $G$ is a Vpt graph, one can find in polynomial time a representation $T$ in which every subtree is a path. Since the number of leaves of a path is at most 2, we obtain the following corollary.

$\triangleright$ **Corollary 6.** There is a 2-approximation algorithm for $\text{Max } k\text{-Colorable Set}$ in Vpt graphs.

Since in a directed tree one of the two leaves of a path $T_i$ is $\text{root}(T_i)$, we obtain

$\triangleright$ **Corollary 7.** $\text{Max } k\text{-Colorable Set}$ can be solved optimally in directed path graphs.

We now proceed with two graph classes that are not chordal, namely Ept graphs and circular arc graphs. Since the chromatic number problem is NP-hard for Ept graphs \cite{13} and also for circular arc graphs \cite{8}, by Corollary 1 the $\text{Max } k\text{-Colorable Set}$ problem is NP-hard when restricted to each of these graph classes. We provide two constant approximation algorithms for these classes in Theorem 8 and Theorem 9.

Since the recognition problem of Ept graphs is NP-hard, in this section we assume that an Ept graph is given with a representation $(T, T)$.

$\triangleright$ **Theorem 8.** There is a $\frac{2k}{k+3}$-approximation algorithm for $\text{Max } k\text{-Colorable Set}$ in Ept graphs.

**Proof.** Let $\text{Opt}$ be an optimal solution for $\text{Max } k\text{-Colorable Set}$, and let $\text{Opt}_1, \ldots, \text{Opt}_k$ be the color classes of some $k$-coloring of $\text{Opt}$. Assume without loss of generality $|\text{Opt}_1| \geq |\text{Opt}_2| \geq \ldots \geq |\text{Opt}_k|$. Let $\text{Opt}' = \bigcup_{i=1}^{2k/3} \text{Opt}_i$. Clearly,

$$|\text{Opt}'| \geq \frac{2k/3}{k} |\text{Opt}|$$

and $\text{Opt}'$ is $2k/3$-colorable. In particular, $\text{load}(\text{Opt}', v) \leq 2k/3$ for every $v \in T$. Let $k$ be such that $k_e = 2k/3$ for every edge $e$ of $T$, and $k_v = n$ for every vertex $v$ of it. By running $\text{BottomUpGreedy}$ on the instance $(T, T, k)$, one gets a solution $\text{ALG}$ with maximum load
at most $2k/3$. In [21], it is shown that every set of paths with maximum edge load at most $L_E$ can be colored using at most $3L_E/2$ colors. Therefore, $ALG$ is $k$-colorable. Furthermore, $ALG$ is a $2$-approximation for MaxSubtreesBL. We conclude,

$$|ALG| \geq \frac{|Opt'|}{2} \geq \frac{2k/3}{2k} |Opt|.$$ 

\hfill ▶

\section{Theorem 9.} There is a $2$-approximation algorithm for Max $k$-Colorable Set in circular arc graphs.

\textbf{Proof.} Let $G = (V,E)$ be a circular arc graph represented as circular arcs of a circle $C$. Choose an arc $e$ of $C$ that is contained in a smallest set of circular arcs, and let $K_e$ be the clique of $G$ that corresponds to the circular arcs containing $e$. Then $G[V \setminus K_e]$ is an interval graph. Compute a maximum $k$-colorable set $S_I$ of $G[V \setminus K_e]$. Let $S_e$ be a subset of $K_e$ with \min$(|K_e|, k)$ vertices. We claim that a biggest set among $S_I$ and $S_e$ is a $2$-approximation. Indeed, any $k$-colorable set $S \subset V$ is partitioned into two sets $S \setminus K_e$ and $S \setminus K_e$, each of which is clearly $k$-colorable. We conclude,

$$|S| = |S \setminus K_e| + |S \cap K_e| \leq |S_I| + |S_e| \leq 2 \max(|S_I|, |S_e|).$$ 

\hfill ▶

\section{Conclusion, Simulations and Open Problems}

In this paper, we presented approximation algorithms for the MaxSubtreesBL problem, and for the Max $k$-Colorable Set problem on chordal graphs, Vpt graphs, Ept graphs and circular arc graphs. On directed path graphs, our algorithm gives an optimal solution for both problems.

We also presented an optimal algorithm VPSByDegreeConstrainedSubgraph for a sub-family of Vpt graphs in which the tree is a star. It is evident that our algorithm for this case works also in the case where the capacities of all internal vertices and internal edges of the tree are unbounded, or at least sufficient capacity is provisioned in the internal vertices and edges to handle all possible communication requests between the leaves of the tree. Indeed, such an instance is equivalent to a star in which the capacity of the center of the star is unbounded. This is actually the case in data center architectures in practice (e.g., [22]).

We conducted simulations to observe the approximation ratio of the greedy algorithm BottomUpGreedy in practice for paths on a tree with high capacities. We compared the performance of the greedy algorithm to the optimum which is computed by using algorithm VPSByDegreeConstrainedSubgraph. The approximation ratio was at most 1.6 with an average of 1.25 on random instances, and at most 1.005 in data center traffic as reported by [22]. Plots of these results can be found in Figure 2 and Figure 3 in the Appendix.

Two open problems immediately related to our work are:

- Is the Max $k$-Colorable Set problem NP-hard when restricted to Vpt graphs?
- Given a set of circular arcs on a cycle $C$, and an integer $k$, can one find in polynomial time a maximum number of arcs that induce a load of at most $k$ on every edge of $C$?
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A Simulation Results

Figure 2 Approximation ratio of the greedy algorithm on random VPT graphs. The histogram is for 30,000 instances on random trees of 50 to 150 nodes, and number of paths from twice to four times the size of the tree.
Figure 3  Approximation ratio of the greedy algorithm for 180 instances on data center traffic according to the distributions in [22]. The individual histograms show the ratio for different traffic distributions resulting from three different applications.