Iteration of $z \mapsto \lambda + z + \tan z$: Topologically hyperbolic maps

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Abstract

Iteration of the function $f_\lambda(z) = \lambda + z + \tan z, z \in \mathbb{C}$ is investigated in this article. It is proved that for every $\lambda$, the Fatou set of $f_\lambda$ has a completely invariant Baker domain $B$; we call it the primary Fatou component. The rest of the results deals with $f_\lambda$ when it is topologically hyperbolic. For all real $\lambda$ or $\lambda$ such that $\lambda = \pi k + i\lambda_2$ for some integer $k$ and $0 < \lambda_2 < 1$, the only other Fatou component is shown to be another completely invariant Baker domain.

It is proved that if $|2 + \lambda^2| < 1$, then the Fatou set is the union of $B$ and infinitely many invariant attracting domains. Every such domain $U$ has exactly one invariant access to infinity and is unbounded in a special way; $\{\Im(z) : z \in U\}$ is unbounded whereas $\{\Re(z) : z \in U\}$ is bounded.

If $\Im(\lambda) > \sqrt{2} + \sinh^{-1} 1$ then it is found that the primary Fatou component is the only Fatou component and the Julia set is disconnected. For every natural number $k$, the Fatou set of $f_\lambda$ for $\lambda = k\pi + i\frac{\pi}{2}$ is shown to contain $k$ wandering domains with distinct grand orbits. These wandering domains are found to be escaping. The Fatou set is the union of $B$, these wandering domains and their pre-images.

Keywords: Baker domain, wandering domain, unbounded set of singular values.

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1 Introduction

A transcendental meromorphic map \( f : \mathbb{C} \rightarrow \hat{\mathbb{C}} \) with a single essential singularity is called general meromorphic if it has at least two poles or exactly one pole that is not an omitted value. We choose the essential singularity to be at \( \infty \). The Fatou set, denoted by \( \mathcal{F}(f) \), is the set of all points in a neighborhood of which \( \{ f^n \}_{n>0} \) is normal. Its complement in \( \hat{\mathbb{C}} \) is the Julia set of \( f \) and it is denoted by \( \mathcal{J}(f) \). For general meromorphic maps, the backward orbit of \( \infty \), \( \{ z : f^n(z) = \infty \text{ for some natural number } n \} \) is an infinite set and its closure turns out to be the Julia set of \( f \). By the dynamics of a function, we mean its Fatou set and the Julia set.

A maximally connected subset of the Fatou set is called a Fatou component. For a given \( n \), \( U_n \) denotes the Fatou component containing \( f^n(U) \). A Fatou component \( U \) is said to be \( p \)–periodic if \( p \) is the smallest natural number such that \( U_p = U \). If \( p = 1 \) then \( U \) is called invariant. An invariant Fatou component \( U \) is called completely invariant if \( f^{-1}(U) \subseteq U \).

A periodic Fatou component can be an attracting domain, a parabolic domain, a rotational domain (a Herman ring or a Siegel disk) or a Baker domain. For a point \( z_0 \) if \( p \) is the smallest natural number such that \( f^p(z_0) = z_0 \) then \( z_0 \) is called a \( p \)–periodic point of \( f \). A \( 1 \)–periodic point is called a fixed point. An important number associated to \( z_0 \) is its multiplier \( \alpha_{z_0} = (f^p)'(z_0) \). The \( p \)–periodic point \( z_0 \) is called attracting, indifferent or repelling if \( |\alpha_{z_0}| < 1, = 1 \) or \( > 1 \) respectively. An indifferent \( p \)–periodic point is called parabolic if \( \alpha_{z_0} = e^{2\pi i \beta} \) for some rational number \( \beta \). A \( p \)–periodic attracting domain contains an attracting \( p \)–periodic point whereas a \( p \)–periodic parabolic domain contains a parabolic \( p \)–periodic point on its boundary. Similarly, a Siegel disc always contains a non-parabolic indifferent periodic point. A periodic Fatou component \( U \) is called a Baker domain if for some \( U_k \), \( f^{np}(z) \rightarrow \infty \) uniformly on every compact subset of \( U_k \). A Fatou component \( W \) is called wandering if \( W_m \neq W_n \) for \( m \neq n \). Further details can be found in [11].

The map \( i + z + \tan z \) is the Newton method of \( \exp( - \int^z_0 \frac{du}{i+\tan u} ) \) and it is reported in [2, 3] that this map has an invariant Baker domain but no wandering domain. It is proved in [3] that the upper half plane is an invariant Baker domain for \( z + \tan z \) and the positive imaginary axis is an invariant, but not a strongly invariant access to \( \infty \). An access from a simply connected Fatou component \( U \) to one of its boundary points \( a \) is a homotopic class of curves in \( U \) tending to \( a \). An access is strongly invariant if it contains the image.
of each curve in it, in some way (for definition see Section 2). Gillen and Sixsmith have recently shown that for \( f(z) = z + \tan z \), there are infinitely many disjoint simply connected domains \( \{U_n\}_{n \geq 1} \) such that \( f^{-1}(U_n) \) is connected for all \( n \) \[7\]. This gives a positive answer to a question raised by Eremenko: Does there exist a non-constant meromorphic function having three disjoint simply-connected regions each with connected preimage? The above mentioned functions are two particular members of the one parameter family given by

\[
f_{\lambda}(z) = \lambda + z + \tan z \quad \text{for} \quad \lambda \in \mathbb{C}.
\]

This article undertakes a systematic study of the Fatou set and the Julia set of \( f_{\lambda} \) for most of the values of \( \lambda \).

A point \( z \) is called a critical point of \( f \) if \( f'(z) = 0 \) and the image of a critical point is known as a critical value of the function. A point \( a \in \hat{\mathbb{C}} \) is called an asymptotic value of \( f \) if there exists a curve \( \eta : [0, \infty) \to \mathbb{C} \) with \( \lim_{t \to \infty} \eta(t) = \infty \) such that \( \lim_{t \to \infty} f(\eta(t)) = a \). A subtle situation arises when the point at \( \infty \) is an asymptotic value. The set of all the singular values of \( f \), denoted by \( S_f \) consists of all the critical values, asymptotic values and their limit points. It is important to note that at every point of \( S_f \), at least one branch of \( f^{-1} \) fails to be defined. The postsingular set of \( f \), denoted by \( P(f) \) is the closure of the set \( \cup_{s \in S_f} \{ f^n(s) : \ n \geq 0 \} \).

Most of the research on the dynamics of general transcendental maps have been focussed on those with a bounded set of singular values; the set of all such functions is well-known as the Eremenko-Lyubich class. A Baker domain \( U \) is special in the sense that the essential singularity \( \infty \) is always a limit function of \( \{f^n\}_{n \geq 0} \) on \( U \). Every limit function of \( \{f^n\}_{n \geq 0} \) on a wandering domain is always constant and the set of all such limits can be an infinite and unbounded set \[1\]. The Fatou set of functions having only finitely many singular values cannot contain any Baker domain or any wandering domain. In order to have a Baker domain or a wandering domain, a map in the Eremenko-Lyubich class has to have infinitely many singular values. Several results on the relation of these types of Fatou components with the postsingular set are obtained in \[2\] though a complete understanding is yet to be arrived at. Some other aspects of dynamics of functions in the Eremenko-Lyubich class have also been investigated and a number of tools are developed. However, the maps outside this class i.e., with an unbounded set of singular values mostly remain unexplored. One of the motivations for taking up \( f_{\lambda}(z) = \lambda + z + \tan z \) is that it is one such map. For suitable values of \( \lambda \), the existence of Baker domain and wandering domain for \( f_{\lambda} \) is
established in this article.

The study of the dynamics of specific functions have been immensely useful, not only for predicting results for a class of functions containing them but also often provides clues for their proofs. The first general transcendental meromorphic map subjected to a systematic investigation from a dynamical point of view is probably $z \mapsto \lambda \tan z$ for $\lambda \in \mathbb{C}$, which has only two singular values (in fact asymptotic values) \[9\]. Later on, Sajid and Kapoor undertook the study of other maps including some with infinitely many singular values, namely $\lambda \frac{\sinh^2 z}{z^4}$ and $\lambda \frac{\sinh z}{z^2}$ \[14, 15\]. However, all these maps are in the Eremenko-Lyubich class. Nayak and Prasad investigated some meromorphic maps with an unbounded set of singular values, namely $z \mapsto \lambda \frac{z^m}{\sinh^m z}$ for real $\lambda$ and the non-existence of Baker domain and wandering domain is established among other results in \[12\].

The function $f_\lambda$ considered in this article has an unbounded set of singular values. This is one of the motivation for studying the dynamics of these functions. A transcendental meromorphic map $f$ is said to be topologically hyperbolic if $P(f) \cap J(f) \cap \mathbb{C} = \emptyset$. This article deals with $f_\lambda$ that are topologically hyperbolic.

For real $\lambda$, the Fatou set of $f_\lambda$ is the union of two completely invariant Baker domains. To see it, note that $\Im(f_\lambda(z)) > 0$ (or $< 0$) if and only if $\Im(z) > 0$ (or $< 0$ respectively) for all $\lambda \in \mathbb{R}$. Therefore, the upper half plane and the lower half plane are the two completely invariant Fatou components of $f_\lambda$, by the Fundamental Normality Test (Lemma 2.1). Since all the fixed points of $f_\lambda$ are real and repelling, none of the Fatou components is either an attracting domain or a parabolic domain. A completely invariant Fatou component cannot be a rotational domain and this gives that both the Fatou components are Baker domains. Clearly, the extended real line $\mathbb{R} \cup \{\infty\}$ is the Julia set.

The functions $f_\lambda$ and $f_{-\lambda}$ are conformally conjugate via $z \mapsto -z$, i.e., $-f_{-\lambda}(-z) = -(-\lambda - z - \tan(z)) = f_\lambda(z)$. This means that $-f_{-\lambda}^n(-z) = f_\lambda^n(z)$ for all $n$ and the dynamical behaviour (the Fatou and the Julia set) of $f_\lambda$ is essentially the same as that of $f_{-\lambda}$. In view of this, now onwards, we assume $\Im(\lambda) > 0$.

The following is a straightforward observation and forms the basis of subsequent results.

**Theorem 1.1.** For $\Im(\lambda) > 0$, there is a completely invariant Baker domain $B_\lambda$ of $f_\lambda$ containing the upper half plane.

We call the completely invariant Baker domain $B_\lambda$ of $f_\lambda$, as the primary Fatou component and denote it by $B$ whenever $\lambda$ is understood. Let us call a Fatou component...
non-primary if it is different from $B$. Before looking into the non-primary Fatou components, we make few remarks.

**Remark 1.1.**

1. Since the Julia set is the boundary of every completely invariant Fatou component, $\mathcal{J}(f_{\lambda}) = \partial B$.

2. Every Fatou component of $f_{\lambda}$ different from $B$ is simply connected. In particular, there is no Herman ring in the Fatou set of $f_{\lambda}$.

3. All the critical points of $f_{\lambda}$ with positive imaginary part are in $B$.

The function $f_{\lambda}$ has infinitely many fixed points for each $\lambda \neq i$. These are the solutions of $\tan z = -\lambda$. But the multiplier of each fixed point is $2 + \lambda^2$ leading to some amount of advantage. First we consider $|2 + \lambda^2| < 1$. The set of all such values of $\lambda$ in the upper half plane is a bounded simply connected domain. The following theorem demonstrates that non-primary Fatou components do exist and it describes all of them.

**Theorem 1.2.** Let $|2 + \lambda^2| < 1$. Then,

1. there are infinitely many invariant attracting domains of $f_{\lambda}$ and each such attracting domain $U$ is unbounded in such a way that $\{\Im(z) : z \in U\}$ is unbounded but $\{\Re(z) : z \in U\}$ is bounded. Further, there is exactly one invariant access from this attracting domain to $\infty$.

2. $f_{\lambda}$ does not have any other periodic Fatou component or any wandering domain.

In other words, the Fatou set of $f_{\lambda}$ is the union of the primary Fatou component, all the invariant attracting domains and their pre-images.

The attracting domains (in blue) along with the primary Fatou component (in red) of $f_{0.1+i\frac{\pi}{2}}$, $f_{1.5i}$ and $f_{-0.1+i\frac{\pi}{2}}$ are given in Figure 1(a), Figure 1(b) and Figure 1(c) respectively.

**Remark 1.2.** The boundary of the set $A = \{\lambda : \Im(\lambda) > 0 \text{ and } |2 + \lambda^2| < 1\}$ contains $i$ and $\sqrt{3}i$ and for every $\lambda \in A, 1 < \Im(\lambda) < \sqrt{3}$. In particular, if $0 < \Im(\lambda) < 1$ or $\Im(\lambda) \geq \sqrt{2 + \sinh^{-1}1} > \sqrt{3}$ then all the fixed points of $f_{\lambda}$ are repelling.

It is important to note that for a large set of parameters $\lambda$ (i.e., $|2 + \lambda^2| > 1$), all the fixed points of $f_{\lambda}$ are repelling and that calls for further effort to determine the dynamics. However, the situation is relatively simple if the imaginary part of such a parameter is either sufficiently large or sufficiently small. The following theorem makes it precise.
Theorem 1.3. 1. For $0 < \Im(\lambda) < 1$, the Fatou set of $f_\lambda$ contains an invariant Baker domain $\tilde{B}$ different from $B$. Further, if $\Re(\lambda) = \pi k$ for some integer $k$ then $\tilde{B}$ is the only non-primary Fatou component and the Julia set is connected.

2. For $\Im(\lambda) > \sqrt{2} + \sinh^{-1} 1$, the primary Fatou component is the only Fatou component and the Julia set is not connected.

The Julia sets of $f_\lambda$ for $\lambda = \pi + i(\sqrt{2} + \sinh^{-1} 1)$ is given as the complement of the yellow region—it is disconnected and is given in Figure 2(a). The connected Julia set of $f_\lambda$ for $\lambda = \pi + 0.99i$ is shown as the boundary of the yellow and the green region in Figure 2(b).

Every limit function of $\{f^n\}_{n>0}$ on each wandering domain of $f$ is always constant [16], one of which can be $\infty$. For a wandering domain $W$, let $L_W$ denote the set of all limits of $\{f^n\}_{n>0}$ on $W$. A wandering domain $W$ is called escaping if $L_W = \{\infty\}$. It is called oscillating if $L_W$ contains $\infty$ and at least one other point. If $\infty \notin L_W$ then $W$ is called dynamically bounded. Though the escaping and the oscillating wandering domains appear in the literature [5, 10], the existence of dynamically bounded wandering domain is not known. The following theorem proves the existence of escaping wandering domains for some values of $\lambda$ with $\Im(\lambda) = \pi/2$. We say a Fatou component $U$ lands on a Fatou component $V$ if $U_n = V$ for some natural number $n$. The grand orbit of a wandering domain $W$ is the set of all wandering domains landing on $W$ or on one of its iterated forward images. Note that the grand orbit of two Fatou components are either identical or disjoint.

Theorem 1.4. For every natural number $k$, there is a $\lambda$ such that $f_\lambda$ has $k$ many wandering domains with distinct grand orbits. If $W$ is such a wandering domain then it has the following properties.

1. Each $W$ is escaping.

2. There is a two sided sequence of unbounded wandering domains $\{W_n\}_{n \in \mathbb{Z}}$ in the grand orbit of $W$ such that $f_\lambda : W_n \to W_{n+1}$ is a proper map with degree 2.

3. If $W'$ is a wandering domain in the grand orbit of $W$ and different from all $W_n$s then $f_\lambda$ is one-one on $W'$.

The Fatou set is the union of the primary Fatou component and these $k$ many grand orbits of wandering domains.
For a complex number $z$, $\Im(z)$ and $\Re(z)$ denote the imaginary and real part of $z$ respectively. Let $H^+ = \{z \in \mathbb{C} : \Im(z) > 0\}$ and $H^- = \{z \in \mathbb{C} : \Im(z) < 0\}$ be the upper and the lower half plane respectively. For any set $A \subset \hat{\mathbb{C}}$, the boundary of $A$ is denoted by $\partial A$. For a complex number $w$, let $A + w = \{z + w : z \in A\}$. Let $D(a, r)$ denote the disc centered at $a$ and with radius $r$ and $\mathbb{D}$ denotes the unit disc. The set of integers is denoted by $\mathbb{Z}$.

2 Preliminaries

2.1 Some useful results

We start with a useful result known as the Fundamental Normality Test.

Lemma 2.1. (Fundamental Normality Test) If $f : \mathbb{C} \to \hat{\mathbb{C}}$ is a meromorphic function and $D$ is a domain such that $\cup_{n>0}\{f^n(z) : z \in D\}$ does not contain at least three points of $\hat{\mathbb{C}}$ then $\{f^n\}_{n>0}$ is normal in $D$.

A point $a$ on the boundary of a simply connected domain $U$ is called accessible from $U$ if there exists a curve $\gamma : [0, 1] \to \hat{\mathbb{C}}$ such that $\gamma([0,1]) \subset U$ and $\lim_{t \to 1^-} \gamma(t) = a$. We say that $\gamma$ lands at $a$. There are simply connected domains such that a point on its boundary is not accessible. In particular, $\lim_{t_n \to 1^-} \gamma(t)$ may be different for different sequences $t_n$ converging to 1 from the left hand side. Some such examples can be found in [11]. For an accessible point, there are uncountably many curves landing on it. What is important is the set of homotopically equivalent classes of such curves.

Definition 2.1. (Access) For a simply connected domain $U$, let $z_0 \in U$ and $a \in \partial U$ be an accessible point. An access $A$ from $U$ to $a$ is the class of all curves $\gamma : [0, 1] \to \hat{\mathbb{C}}$ homotopic to each other such that $\gamma([0,1]) \subset U$, $\gamma(0) = z_0$ and $\lim_{t \to 1^-} \gamma(t) = a$.

This article is concerned with simply connected domains which are in fact Fatou components of a meromorphic function.

Definition 2.2. (Invariant and strongly invariant access) Let $U$ be a simply connected and invariant Fatou component of a meromorphic function $f$. An access $A$ from $U$ to one of its boundary points $a$ is called invariant if there exists $\gamma \in A$ such that $f(\gamma) \cup \gamma_1 \in A$, where $\gamma_1 : [0, 1] \to U$ is a curve contained in $U$ such that $\gamma_1(0) = z_0$ and $\gamma_1(1) = f(z_0)$. If $f(\gamma) \cup \gamma_1 \in A$ for every $\gamma \in A$ then $A$ is called a strongly invariant access.
For an invariant simply connected Fatou component $U$ of $f$, if $\phi : \mathbb{D} \to U$ is the Riemann map, then the inner function $g : \mathbb{D} \to \mathbb{D}$ associated with $f$ is defined as $g = \phi^{-1} \circ f \circ \phi$. We need the following result (Theorem B, [3]) relating the behaviour of $g$ on the unit circle to that of $f$ on the boundary of $U$. A fixed of $f$ is called weakly repelling if it is either repelling or is parabolic with multiplier equal to 1.

**Theorem 2.1.** Let $U$ be a simply connected and invariant Fatou component of $f$ and $g = \phi^{-1} \circ f \circ \phi$ be the inner function associated with $f|_U$. If the degree $d$ of $f$ on $U$ is finite and $d_1$ is the number of fixed points of $g$ in $\partial \mathbb{D}$ then $f$ has exactly $d_1$ invariant accesses, and $d - 1 \leq d_1 \leq d + 1$. Moreover, every invariant access of $f$ from $U$ either lands at $\infty$ or at a weakly repelling fixed point of $f$.

Recall that $S_f$ is the set of singular values of $f$. The post singular set of $f$, denoted by $P(f)$ is the closure of the set

$$\bigcup_{s \in S_f} \{f^n(s) : n \geq 0\}.$$ 

Here is a well-known result.

**Lemma 2.2.** Every attracting domain and parabolic domain of a meromorphic function intersects the set $S_f$. If $U$ is a rotational domain then $\partial U \subset P(f)$. In particular, the Fatou set of a topologically hyperbolic map can not contain any rotational domain.

The following lemma proved in [2] reveals the connection of the singular values with the Fatou components. In particular, this is more relevant for Baker and wandering domains for topologically hyperbolic meromorphic maps.

**Lemma 2.3.** Let $U$ be a Fatou component of a topologically hyperbolic meromorphic map $f$ such that $U_n \cap P(f) = \emptyset$ for all $n > 0$. Then for every compact set $K \subset U$ and every $r > 0$, there exists $n_0$ such that for every $z \in K$ and every $n \geq n_0$, $D(f^n(z), r) \subset U_n$.

We end this subsection by stating a very important result. For a continuous map $f : V \to U$ between two open connected subsets of $\mathbb{C}$ if the pre-image of each compact subset of $U$ is compact in $V$ then $f$ is called proper. Further, if $f$ is analytic then there is a $d$ such that every element of $U$ has $d$ preimages counting multiplicity. Here, the multiplicity of a point $z$ is the local degree of $f$ at $z$. This number $d$ is known as the degree of $f : V \to U$. The following lemma proved in [6] is to be applied repeatedly.
Lemma 2.4. (Riemann-Hurwitz formula)

Let $f : \mathbb{C} \to \hat{\mathbb{C}}$ be a transcendental meromorphic function. If $V$ is a component of the pre-image of an open connected set $U$ and $f : V \to U$ is a proper map of degree $d$, then $c(V) - 2 = d(c(U) - 2) + n$, where $n$ is the number of critical points of $f$ in $V$ counting multiplicity and $n \leq 2d - 2$. Here, the multiplicity of a critical point is one less than the local degree of $f$ at the critical point.

2.2 Some basic properties of $f_\lambda$

We make few preliminary observations on $f_\lambda(z) = \lambda + z + \tan z$ for $\Im(\lambda) > 0$. First note that $\tan(z + \pi) = \tan z$ for all $z$ and for $z = x + iy$,

$$
\Re(\tan z) = \frac{\sin 2x}{\cos 2x + \cosh 2y} \quad \text{and} \quad \Im(\tan z) = \frac{\sinh 2y}{\cos 2x + \cosh 2y}.
$$

Lemma 2.5. The Fatou set $\mathcal{F}(f_\lambda)$ is invariant under $z \mapsto \pi$ i.e., $z \in \mathcal{F}(f_\lambda)$ if and only if $z + \pi \in \mathcal{F}(f_\lambda)$. If a Fatou component $U$ contains a point $z$ and its $k\pi$-translate $z + k\pi$ for some non-zero $k \in \mathbb{Z}$ then $\{\Re(z) : z \in U\} = \mathbb{R}$. In particular, this is true if $U$ contains a horizontal line segment of length bigger than $\pi$.

Proof. Note that $f_\lambda(z + \pi) = f_\lambda(z) + \pi$ which gives $f_\lambda^n(z + \pi) = f_\lambda^n(z) + \pi$ for all $n$. Hence $z \in \mathcal{F}(f_\lambda)$ if and only if $z + \pi \in \mathcal{F}(f_\lambda)$. If a Fatou component $U$ contains $z$ as well as $z + k\pi$ for some non-zero integer $k$ then for a curve $\gamma \subset U$ joining and containing these two points, we have $\bigcup_{n \in \mathbb{Z}} \gamma + n\pi \subset U$. Thus $\{\Re(z) : z \in \bigcup_{n \in \mathbb{Z}} \gamma + n\pi\} = \mathbb{R}$. \hfill \Box

The following describes the behaviour of $f_\lambda$ on some vertical lines. For a vertical line $l$ and a real number $r$, let $l + r = \{z + r : z \in l\}$.

Lemma 2.6. Let $m$ be an integer and $l_{m\pi} = \{z : \Re(z) = m\pi\}$.

1. The function $f_\lambda$ maps the line $l_{m\pi}$ bijectively onto $l_{m\pi + \Re(\lambda)}$.

2. If $\lambda = k\pi + i\lambda_2$ for some $k \in \mathbb{Z}$ and $\lambda_2 > 1$ then $\lim_{n \to \infty} \Im(f_\lambda^n(z)) = +\infty$ for all $z \in l_{m\pi}$.

Proof. For $z = m\pi + iy$, $f_\lambda(z) = \lambda + m\pi + iy + i \tanh y$. Define $\phi : \mathbb{R} \to \mathbb{R}$ by $\phi(y) = \Im(\lambda) + y + \tanh y$. This is a strictly increasing function satisfying $\lim_{y \to -\infty} \phi(y) = -\infty$ and $\lim_{y \to \infty} \phi(y) = \infty$. In particular, this is a bijection.
1. Since \( \phi(y) \) is a bijection of the real line onto itself, \( f_\lambda \) maps \( l_{m\pi} \) bijectively onto \( l_{m\pi+\Re(\lambda)} \).

2. For \( \lambda = k\pi + i\lambda_2, \; k \in \mathbb{Z} \) and \( \lambda_2 > 1 \), \( \phi(y) = \lambda_2 + y + \tanh y > y \) for all \( y \). This (non-existence of any fixed point) along with the strict increasingness of \( \phi \) implies that \( \lim_{n \to \infty} \phi^n(y) = +\infty \). Since \( \Im(f_\lambda(m\pi + iy)) = \phi(y) \), \( \Im(f_\lambda^2(m\pi + iy)) = \phi^2(y) \) and in general, \( \Im(f_\lambda^n(m\pi + iy)) = \phi^n(y) \) for all \( n > 0 \), \( \lim_{n \to \infty} \Im(f_\lambda^n(z)) = +\infty \) for all \( z \in l_{m\pi} \).

\[ \square \]

To determine all the singular values of \( f_\lambda \), let \( \overline{C} \) denote the set \( \{ \overline{z} : z \in C \} \) whenever \( C \) is a set of complex numbers. Recall that we have assumed \( \Im(\lambda) > 0 \).

**Lemma 2.7.** 1. The set of all critical points of \( f_\lambda \) is \( C \cup \overline{C} \) where \( C = \left\{ \frac{\pi}{2} + n\pi + i\sinh^{-1} 1 : n \in \mathbb{Z} \right\} \). The critical values are \( \lambda + \frac{\pi}{2} + n\pi \pm i(\sinh^{-1} 1 + \sqrt{2}) \) where \( n \in \mathbb{Z} \).

2. The point at infinity is the only asymptotic value of \( f_\lambda \) and there is only one transcendental singularity lying over it.

**Proof.** 1. The solutions of \( f_\lambda(z) = 0 \) are precisely those satisfying \( \cos z = i \) or \( -i \). Since \( \cos \overline{z} = \cos z \) for all \( z \in \mathbb{C} \), we have \( \cos z = i \) if and only if \( \cos \overline{z} = -i \).

Let \( \cos z = i \). Then \( \cos x \cos y - i \sin x \sin y = i \). As \( \cosh y \) is never zero, \( \cos x = 0 \) and \( \sin x \sinh y = -1 \). The first equation gives that \( x = x_n = \frac{\pi}{2} + n\pi \) for all \( n \in \mathbb{Z} \).

If \( n \) is odd then \( \sin x_n = -1 \) and \( \sinh y = 1 \) and the solution is \( \frac{\pi}{2} + n\pi + i\sinh^{-1} 1 \).

Similarly for even \( n \), \( \sin x_n = 1 \), \( \sinh y = -1 \) and we have \( \frac{\pi}{2} + n\pi + i\sinh^{-1}(-1) \) as the solution of \( \cos z = i \). Taking the complex conjugate of these solutions, the set of all critical points of \( f_\lambda \) is now found to be \( C \cup \overline{C} \) where \( C = \{ c_n = \frac{\pi}{2} + n\pi + i\sinh^{-1} 1 : n \in \mathbb{Z} \} \).

Since \( \tan c_n = i \coth(\sinh^{-1} 1) = i\sqrt{2} \), \( f_\lambda(c_n) = \lambda + \frac{\pi}{2} + n\pi + i\sinh^{-1} 1 + \tan(c_n) = \lambda + \frac{\pi}{2} + n\pi + i(\sinh^{-1} 1 + \sqrt{2}) \).

Similarly \( f_\lambda(\overline{c_n}) = \lambda + \frac{\pi}{2} + n\pi - i(\sinh^{-1} 1 + \sqrt{2}) \).

2. For every unbounded curve \( \gamma : [0, 1] \to \mathbb{C} \) with \( \lim_{t \to 1^-} \gamma(t) = \infty \), it is not difficult to see that \( \lim_{t \to 1^-} f_\lambda(\gamma(t)) = \infty \). This gives that \( \infty \) is the only asymptotic value of \( f_\lambda \).

We are to show that there is only one singularity lying over it.

Let \( D \) be a disc centered at \( \infty \) with respect to the spherical metric. Then there exists a \( \delta > 0 \) such that the half planes \( H_\delta = \{ z : \Im(z) > \delta \} \) and \( \overline{H_\delta} = \{ \overline{z} : z \in H_\delta \} \) are
contained in $D$. Since $H_\delta$ is invariant under $f_\lambda$ (as $\Im(\lambda) > 0$), $f_\lambda^{-1}(D)$ contains $H_\delta$. Note that if $\Im(z) < -\delta - \Im(\lambda)$ then $\Im(f_\lambda) < -\delta + \Im(\tan z) < -\delta$. In other words, the half plane $H_{-\delta-\Im(\lambda)} = \{z : \Im(z) < -\delta - \Im(\lambda)\}$ is mapped into $\overline{H_\delta} \subset D$ giving that $H_{-\delta-\Im(\lambda)} \subset f_\lambda^{-1}(D)$. Therefore,

$$H_\delta \cup H_{-\delta-\Im(\lambda)} \subset f_\lambda^{-1}(D). \quad (1)$$

The disc $D$ contains the left half plane $H_\alpha = \{z : \Re(z) < \alpha\}$ and $-H_\alpha = \{-z : z \in H_\alpha\}$ for some $\alpha > 0$. There is a natural number $m_0$ (depending on $\alpha$ and $\lambda$) such that the line $l_{m\pi+\Re(\lambda)} = \{z : \Re(z) = m\pi + \Re(\lambda)\}$ is contained in $D$ for all integers $m$ with $|m| > m_0$. By Lemma 2.6(1), we have

$$l_{m\pi} = \{z : \Re(z) = m\pi\} \subset f_\lambda^{-1}(D) \quad \text{for infinitely many values of } m. \quad (2)$$

Now it follows from Equation(1) and Equation(2) that there is a unique unbounded component of $f_\lambda^{-1}(D)$. In other words, there is a only one essential singularity lying over $\infty$.

\[\square\]

**Remark 2.1.**  
1. All the critical points are simple, i.e., the local degree of $f_\lambda$ is two at every critical point.

2. Note that $C \subset H^+$ and $\overline{C} \subset H^-$. The critical values corresponding to the critical points belonging to $C$ are in $H^+$ whenever $\Im(\lambda) > 0$. The other critical values are on the same horizontal line but may not be in $H^+$.

Now we determine some properties of the fixed points of $f_\lambda$.

**Lemma 2.8.** For each $\lambda \neq i$ with $\Im(\lambda) > 0$, $f_\lambda$ has infinitely many fixed points. Moreover, the following are true.

1. The multiplier of each fixed point is $2 + \lambda^2$. In other words, all the fixed points of $f_\lambda$ are attracting, repelling or indifferent together.

2. A point $z$ is a fixed point of $f_\lambda$ if and only if $z + n\pi$ is so for all $n \in \mathbb{Z}$.

3. All the fixed points of $f_\lambda$ are in $H^-$ whenever $\Im(\lambda) > 0$.
Proof. 1. The fixed points of \( f_\lambda \) are the solutions of \( \tan z = -\lambda \). Since \( \lambda \neq i \) and \( \Im(\lambda) > 0 \), there are infinitely many fixed points. The multiplier of each fixed point is \( f'_\lambda(z) = 1 + \sec^2 z = 2 + \lambda^2 \). It depends on the value of \( \lambda \) but not on any fixed point. All the fixed points are attracting, repelling or indifferent if and only if \(|2 + \lambda^2| < 1\), \(|2 + \lambda^2| > 1\) or \(|2 + \lambda^2| = 1\) respectively.

2. This follows from the fact that \( \tan z \) is \( \pi \)-periodic.

3. The is so because all the solutions of \( \tan z = -\lambda, \Im(\lambda) > 0 \) are in \( H^- \).

Remark 2.2. The fixed points of \( f_\lambda \) are real if and only if \( \lambda \) is real.

3 The proofs

Here is the proof of Theorem 1.1.

Proof of Theorem 1.1. Note that for all \( z \in H^+ \), \( \Im(f_\lambda(z)) > \Im(\lambda) + \Im(z) > 0 \). The family \( \{f^n\}_{n\geq0} \) is normal in \( H^+ \) by the Fundamental Normality Test. Since \( \Im(f^n_\lambda(z)) > n\Im(\lambda) + \Im(z) \) for all \( n, f^n_\lambda(z) \to \infty \) as \( n \to \infty \) for all \( z \in H^+ \). Thus \( f_\lambda \) has an invariant Baker domain containing the upper half plane. This is the primary Fatou component and we denote it by \( B \).

In order to show that \( B \) is backward invariant, let \( B_{-1} \) be a component of \( f^{-1}_\lambda(B) \). It is known that if \( U \) and \( V \) are two Fatou components of a meromorphic function \( f \) such that \( f: U \to V \) then \( V \setminus f(U) \) contains at most two points (Theorem 1, [1]). Therefore, \( B \setminus f_\lambda(B_{-1}) \) contains at most two points. Consider \( \epsilon_1 < \epsilon_2 \) and the horizontal line segment \( l = \{w : \epsilon_1 \leq \Re(w) \leq \epsilon_2 \text{ and } \Im(w) = \Im(\lambda)\} \subset f_\lambda(B_{-1}) \). Note that \( l \subset H^+ \subset B \). For \( w \in l \), let \( z \) be such that \( \lambda + z + \tan z = w \). Since \( \Im(w) = \Im(\lambda), \Im(z) + \Im(\tan z) = 0 \) and it gives that \( z \) is a real number. Each real number except the poles of \( f_\lambda \) is mapped into \( H^+ \) by \( f_\lambda \) and therefore \( B \) contains the real line except the poles. Thus the full pre-image \( f^{-1}_\lambda(l) \) of \( l \) is contained in \( B \). On the other hand the set \( B_{-1} \) intersects \( f^{-1}_\lambda(l) \) which gives that \( B_{-1} \) intersects \( B \). Thus \( B \) is backward invariant. Therefore \( B \) is a completely invariant Baker domain. \( \square \)
The following lemma states that the set of all pre-images of every point in the lower half plane is spread horizontally.

**Lemma 3.1.** For $\Im(\lambda) > 0$ and $w \in H^-$, if $f_\lambda(z) = w$ then $\Im(z) > \Im(w) - \Im(\lambda)$.

**Proof.** If $f_\lambda(z) = w$ then $z \in H^-$ (because $f_\lambda(H^+) \subset H^+$) and $\Im(\tan z) < 0$. Now, $\Im(w) = \Im(\lambda) + \Im(z) + \Im(\tan z) < \Im(\lambda) + \Im(z)$. This is what is claimed. 

Every point in a non-primary Fatou component has negative imaginary part. Note that a Fatou component containing the image or any pre-image of a non-primary Fatou component is also non-primary. A non-primary Fatou component is called *horizontally spread* if there is a $\delta < 0$ such that $\{\Re(z) : z \in U \text{ and } \Im(z) > \delta\}$ is unbounded. Horizontally spread Fatou components are unbounded in a special way. The existence of a sequence of points $z_n$ in $U$ with $\Im(z_n) \to -\infty$ as $n \to \infty$ is not ruled out and $U$ is allowed to contain even a half plane of the form $\{z : \Im(z) < \delta'\}$ for some $\delta' < 0$. The following describes some useful properties of horizontally spread Fatou components that are to be used in the proof of Theorem 1.2.

**Lemma 3.2.** For $\Im(\lambda) > 0$, let $U$ be a non-primary Fatou component of $f_\lambda$.

1. If $U$ is horizontally spread and is not invariant under $z \mapsto z + \pi$ then $f_\lambda$ has an invariant Baker domain.

2. If $U$ is not horizontally spread then $f_\lambda : U \to U_1$ is a proper map with degree 1 or 2.

**Proof.** 1. If $U$ is horizontally spread then all its $k\pi$-translates $U + k\pi = \{z + \pi k : z \in U\}$ are also horizontally spread. Since $U$ is not invariant under $z \mapsto z + \pi$, $U + k\pi \cap U + k'\pi = \emptyset$ for all $k \neq k'$ (by Lemma 2.5). Now, if $\{\Im(z) : z \in U\}$ is unbounded then we can find an unbounded Jordan curve $\gamma \subset U$ which separates the primary Fatou component $B$ from $U'$ where $U' = U + \pi$ or $U - \pi$; i.e., one component of $\hat{C} \setminus \gamma$, say $B'$ contains $B$ whereas the other contains $U'$. This means that $\mathcal{J}(f_\lambda) = \partial B$ which is contained in the closure of $B'$ which contradicts the fact that the other component of $\hat{C} \setminus \gamma$ contains some points of the Julia set, namely those on the boundary of $U'$. Thus, the set $\{\Im(z) : z \in U\}$ and therefore $\{\Im(z) : z \in U + k\pi\}$ for all $k$ is bounded. Using the same argument, it can be seen that $\{\Re(z) : z \in U\} = \mathbb{R}$ is not possible and there is a $\delta$ such that $\Re(z) > \delta$ or $\Re(z) < \delta$ for all $z \in U$. Without
loss of generality we assume that $\Re(z) > \delta$ for all $z \in U$. This is clearly true for all $U + k\pi$.

Let $\partial_k$ be the boundary of $U + k\pi$ and $\alpha$ be the set of all the limit points of $\partial_k$, i.e., $\alpha = \{z : \lim_{n \to \infty} z_{k,n} = z\}$. This $\alpha$ is an unbounded connected subset of the Julia set. Further, $\{\Re(z) : z \in \alpha\} = \mathbb{R}$ and $\{\Im(z) : z \in \alpha\}$ is bounded. Now one component of $\hat{\mathbb{C}} \setminus \alpha$ contains the primary Fatou component $B$ and the other component must be a Fatou component, say $\tilde{B}$. This $\tilde{B}$ contains a lower half plane $H_{\beta} = \{z : \Im(z) < \beta\}$ for some $\beta < 0$. Since $\Im(f_{\lambda}(z)) = \Im(\lambda) + \Im(z) + \Im(\tan z)$, we can choose a $z \in \tilde{B}$ (depending on $\lambda$) with imaginary part sufficiently near to $-\infty$ such that its image is in $\tilde{B}$. For example, take $\beta_1 < \beta - \lambda$ such that $\Im(\tan z) \in (-1.1, -0.9)$ for $\Im(z) < \beta_1$. This shows that $\tilde{B}$ is invariant. If $\lim_{n \to \infty} f_{\lambda}^n(z)$ is a fixed point $z_0$ for some $z \in H_{\beta_1} \subset \tilde{B}$ then $z + \pi \in H_{\beta_1}$ and $\lim_{n \to \infty} f_{\lambda}^n(z + \pi)$ is $z_0 + \pi$, which is also a fixed point. This cannot be true if $\tilde{B}$ is either an attracting domain or a parabolic domain. Similarly, it can be seen that it is also not a Siegel disc. Therefore $\tilde{B}$ is a Baker domain.

2. If $U$ is not horizontally spread then it follows from Lemma 3.1 that every point of $U_1$, the Fatou component containing $f_{\lambda}(U)$, has finitely many pre-images in $U$. Hence $f_{\lambda} : U \to U_1$ is proper (by Theorem 1, [6]). Since $U$ and $U_1$ are simply connected (by Remark 2.1(2)), it follows from the Riemann-Hurwitz formula (Lemma 2.4) that $\deg f_{\lambda}|_U = N + 1$ where $N$ is the number of critical points of $f_{\lambda}$ in $U$ counting multiplicity. Since all the critical points of $f_{\lambda}$ are simple (Remark 2.1), the number $N$ here is in fact the number of distinct critical points.

If $U$ contains two critical points then it contains all the critical points (as the Fatou set is $\pi$-invariant and any two consecutive critical points are with the same imaginary part but with real parts differing by $\pi$ (See Lemma 2.7)) and becomes horizontally spread. Therefore $N = 0$ or 1, and the degree $d$ of $f_{\lambda} : U \to U_1$ is 1 or 2 respectively.

\[\square\]

**Remark 3.1.** If an unbounded Fatou component $U$ is not horizontally spread then $\{\Im(z) : z \in U\}$ is unbounded but $\{\Re(z) : z \in U\}$ is bounded.

For proving Theorem 1.2 we also need the following.
Lemma 3.3. Let $f_\lambda$ be a topologically hyperbolic map for some $\lambda$. Then for every wandering domain $W$ there is an $n \geq 0$ such that $W_n \cap P(f_\lambda) \neq \emptyset$.

Proof. Suppose on the contrary that $W$ is a wandering domain of $f_\lambda$ such that $W_n \cap P(f_\lambda) = \emptyset$ for all $n \geq 0$. Since $f_\lambda$ is topologically hyperbolic, it follows from Lemma 2.3 that there exists an $n_0$ such that for all $n \geq n_0$, $W_n$ contains a disc of radius $\pi$. In particular, $W_n$ contains a horizontal line segment including its end point with length $\pi$. Since the Fatou set $\mathcal{F}(f_\lambda)$ is $\pi$–invariant (Lemma 2.5), $W_n$ contains a horizontal line unbounded in both the directions for all $n \geq n_0$. The horizontal strip bounded by two such lines $l_{n_0} \subset W_{n_0}$ and $l_{n_0+1} \subset W_{n_0+1}$ contains a point of the Julia set, namely a point on the boundary of $W_{n_0}$. It follows from the fact $\mathcal{J}(f_\lambda) = \partial B$ (by Remark 1.1(1)) that this strip contains a point of $B$. Now $l_{n_0} \cup l_{n_0+1} \cup \{\infty\}$ is a closed curve in $\hat{\mathbb{C}} \setminus B$ separating $B$. However, this is not possible as $B$ is connected.

Proof of Theorem 1.2. It follows from Lemma 2.8 that for $|2 + \lambda^2| < 1$, $f_\lambda$ has infinitely many attracting fixed points. The attracting domains corresponding to these attracting fixed points are distinct.

The point at $\infty$ is the only asymptotic value of $f_\lambda$ and is in the Julia set. It follows from Lemma 2.5 that if $c$ is a critical point such that $f_\lambda^n(c)$ converges to an attracting fixed point $z_0$ then $\lim_{n \to \infty} f_\lambda^n(c + k\pi) = z_0 + k\pi$ for each $k \in \mathbb{Z}$. Recall that $z_0 + \pi k$ is an attracting fixed point if and only if $z_0$ is so. Note that every critical point of $f_\lambda$ in the lower half plane is of the form $c + k\pi$ for some $k \in \mathbb{Z}$. Since each invariant attracting domain contains a critical point, each critical point in the lower half plane is in an invariant attracting domain. Also each critical point in the upper half plane is in the primary Fatou component. Thus $f_\lambda$ is a topologically hyperbolic map for $|2 + \lambda^2| < 1$.

1. Let $U$ be an invariant attracting domain. If $U$ is horizontally spread then $f_\lambda$ has an invariant Baker domain $\tilde{B}$ containing a lower half plane by Lemma 3.2. If there is a $\delta < 0$ such that $\Im(f_\lambda^{n_k}(z)) > \delta$ for some subsequence $n_k$ and some $z \in \tilde{B}$ then the topologically hyperbolicity of $f_\lambda$ gives that $\tilde{B}$ contains the disc $D(z_{n_k}, |\delta|)$ for a sufficiently large $k$ (by Lemma 2.3). However $D(z_{n_k}, |\delta|)$ contains a real number and that is either a pole or belongs to $B$. None of this can be true. Therefore,

$$\text{for each } \delta < 0 \text{ there is an } n_\delta \text{ such that } \Im(f_\lambda^n(z)) < \delta \text{ for all } n > n_\delta.$$  (3)
Now, choose a suitable $\delta_0 < 0$ such that $\Im(\tan z) > -\Im(\lambda)$ for all $z$ with $\Im(z) < \delta_0$. This is possible because $\tan z \to -i$ as $\Im(z) \to -\infty$ and $-\sqrt{3} < -\Im(\lambda) < -1$. For such a $z$, let $z_n = f_\lambda^n(z)$ and observe that $\Im(z_1) > \Im(z)$. If $n_0$ is such that $\Im(z_n) < \delta_0$ for all $n > n_0$ then $\{\Im(z_n)\}_{n>n_0}$ is strictly increasing and bounded above by $\delta_0$. This sequence converges to some number less than or equal to $\delta_0$, which is a contradiction to Equation (3) for a $\delta < \delta_0$. Thus, the attracting domain $U$ is not horizontally spread.

By Lemma 3.2(2), $f_\lambda : U \to U$ is a proper map of degree 1 or 2. Since $U$ contains exactly one critical point of $f_\lambda$ by Lemma 2.2, it follows from the Riemann-Hurwitz formula that the degree of $f_\lambda : U \to U$ is 2.

It follows from Theorem 2.1 that the number of invariant accesses from $U$ to its boundary points is 1, 2 or 3. Further, each of these boundary points is either a weakly repelling fixed point or $\infty$. Since $f_\lambda$ has no weakly repelling fixed point, all these accesses are to $\infty$. Now, if there are more than one access to $\infty$ then for two curves $\gamma_1, \gamma_2$ in $U$ with a common starting point and landing at $\infty$, each component of $\hat{\mathbb{C}} \setminus (\gamma_1 \cup \gamma_2)$ would intersect the boundary of $U$. This is not possible as $\partial U \subset \partial B$. Thus there is exactly one invariant access from $U$ to $\infty$. In particular, $U$ is unbounded.

As $U$ is unbounded but not horizontally spread, it follows from Remark 3.1 that $\{\Im(z) : z \in U\}$ is unbounded but $\{\Re(z) : z \in U\}$ is bounded.

![Figure 1: Julia sets](image)

(a) The attracting domains of $f_{0.1+i\frac{\pi}{2}}$ seen in blue. (b) The attracting domains of $f_{1.5i}$ seen in blue. (c) The attracting domains of $f_{-0.1+i\frac{\pi}{2}}$ seen in blue.

2. The existence of any attracting domain with period more than 1 or any parabolic
domain is therefore ruled out by Lemma 2.2. Also by the same lemma, $f_\lambda$ has neither any Siegel disc nor any Herman ring. The non-existence of any Baker domain (other than $B$) or any wandering domain remains to be looked into.

Let $V$ be a $p$-periodic Baker domain of $f_\lambda$ such that $\lim_{n \to \infty} f_\lambda^{np}(z) = \infty$ uniformly on $V$. Since $f_\lambda$ is topologically hyperbolic, it follows from Lemma 2.3 that $V$ contains a disc of radius more than $\pi$. Since $\mathcal{F}(f_\lambda)$ is $\pi$-invariant (Lemma 2.5), $V$ contains a horizontal line which is unbounded in both the directions. This line separates $\mathbb{C} \cap \partial B$ from the boundary of each invariant attracting domain since $\{ \Im(z) : z \in U \}$ is unbounded. Again $\partial V \subset \partial B$ implies that $V$ contains a half plane of the form $\{ z : \Im(z) < M < 0 \}$. But this is not true as there is a sequence of points in the invariant attracting domain whose imaginary parts tends to $-\infty$. Thus $f_\lambda$ does not have any Baker domain.

There cannot be any wandering domain of $f_\lambda$ by Lemma 3.3.

Now the proof of Theorem 1.3 is presented.

**Proof of Theorem 1.3**

1. Let $0 < \Im(\lambda) < 1$. Since $\lim_{\Im(z) \to -\infty} \Im(\tan z) = -1$, choose $\delta < 0$ such that the image of $H_\delta = \{ z : \Im(z) < \delta \}$ under $\tan z$ is contained in the half plane $\{ z : \Im(z) < \Im(-\lambda) \}$. This is also true for all smaller values of $\delta$. Then the image of $H_\delta$ under $z + \tan z$ is contained in $\{ z : \Im(z) < \Im(-\lambda) + \delta \}$ and consequently, $f_\lambda(H_\delta) \subset H_\delta$. By the Fundamental Normality Test, the half plane $H_\delta$ is contained in the Fatou set of $f_\lambda$. The Fatou component containing $H_\delta$, call it $\tilde{B}$, is invariant. This Fatou component $\tilde{B}$ is simply connected by Remark 1.1(1). In particular, it is not a Herman ring. If an invariant Fatou component is a Siegel disc, an attracting domain or a parabolic domain then its closure contains a non-repelling fixed point. Since all the fixed points of $f_\lambda$ are repelling by Remark 1.2, $\tilde{B}$ can neither be a Siegel disc, an attracting domain nor a parabolic domain. Thus, $\tilde{B}$ is an invariant Baker domain.

Note that each critical point with positive imaginary part is contained in $B$.

Let $\lambda = k\pi + i\lambda_2$ for some $k \in \mathbb{Z}$ and $\lambda_2 > 0$. If $m$ is an integer then

$$f_\lambda(m\pi + \frac{\pi}{2} + iy) = k\pi + m\pi + \frac{\pi}{2} + i(\lambda_2 + y + \coth y).$$  \hspace{1cm} (4)
Here $0 < \lambda_2 < 1$. Let $L_{m\pi} = \{ m\pi + \frac{\pi}{2} + iy : y < 0 \}$ and $L_{(m+k)\pi} = k\pi + L_{m\pi}$. Then $f_\lambda(L_{m\pi}) \subset L_{(m+k)\pi}$ and $f_\lambda^n(L_{m\pi}) \subset L_{(m+kn)\pi}$ for all $n \geq 1$. For every $m$ and $z \in L_{m\pi}$, the sequence of real parts of $f_\lambda^n(z)$ tends to $\infty$ (or $-\infty$) as $n \to \infty$ when $k > 0$ (or $k < 0$ respectively). We are to show that,

$$\lim_{n \to \infty} \Im(f_\lambda^n(z)) = -\infty \text{ for every } z \in L_{m\pi}. \quad (5)$$

For this, consider $\phi : (-\infty, 0) \to (-\infty, 0)$ defined by $\phi(y) = \lambda_2 + y + \coth y$ where $0 < \lambda_2 < 1$. It is clear that for $z \in L_{m\pi}$, $\Im(f_\lambda^2(z)) = \phi^2(y)$ and in general $\Im(f_\lambda^n(z)) = \phi^n(y)$ for all $n > 0$. Our claim $\Box$ will be proved by showing that $\lim_{n \to \infty} \phi^n(y) = -\infty$ for all $y < 0$. Since $\phi'(y) = 1 - \cosech^2(y)$, $\phi$ has a unique critical point and that is $y_0 = -\sinh^{-1} 1$. Further, it increases in $(-\infty, y_0)$, attains its maximum at $y_0$ and then decreases. Note that $\lim_{y \to 0^-} \phi(y) = -\infty = \lim_{y \to -\infty} \phi(y)$. The image of $(-\infty, 0)$ under $\phi$ is strictly contained in $(-\infty, \phi(y_0))$. Since $\lambda_2 + \coth y_0 < 0$, $\phi(y_0) = \lambda_2 + y_0 + \coth y_0 < y_0$ and $\phi$ is strictly increasing in $(-\infty, y_0)$, we have $\phi^n(y_0) \to -\infty$ as $n \to \infty$. This gives that $\lim_{n \to \infty} \phi^n(y) = -\infty$ for all $y < y_0$. Thus $\lim_{n \to \infty} \phi^n(y) = -\infty$ for all $y < 0$.

Since each critical point $c$ in the lower half plane belongs to $L_{m\pi}$ for some integer $m$, it follows from $(5)$ that $\lim_{n \to \infty} \Im(f_\lambda^n(c)) = -\infty$. Note that $L_{m\pi} \subset \tilde{B}$ for all $m$ and in particular, $\tilde{B}$ contains all the critical points (with negative imaginary part) and their forward orbits. This proves that $f_\lambda$ is topologically hyperbolic.

By the similar argument as used in Theorem $1.2(2)$ and Lemma $3.3$ we conclude that $f_\lambda$ does not have any non-primary periodic Fatou component other than $\tilde{B}$ or any wandering domain.

Every pole is of the form $m\pi + \frac{\pi}{2}$ and is an end point of $L_{m\pi}$ for some $m$. This gives that the boundary of $\tilde{B}$ contains a pole. As $\tilde{B}$ is simply connected, the Julia component (i.e., a maximally connected subset of the Julia set) containing a pole is unbounded. If there is a multiply connected Fatou component $V$ of a general meromorphic function then consider a Jordan curve which is not contractible in $V$. Arguing as in Lemma 1(13), one finds that some iterated (forward) image of this curve surrounds a pole. This means that there is a bounded Julia component containing a pole, which is not possible. Thus the primary Fatou component $B$ and hence all the Fatou components are simply connected. Therefore, the Julia set of $f_\lambda$
is connected whenever \( \lambda = k\pi + i\lambda_2 \) for \( 0 < \lambda_2 < 1 \).

2. By Lemma 2.7, the critical values of \( f_\lambda \) corresponding to the critical points in the lower half plane are \( \lambda + \frac{\pi}{2} + n\pi - i(\sqrt{2} + \sinh^{-1} 1) \) where \( n \) is an integer. For \( \Im(\lambda) > \sqrt{2} + \sinh^{-1} 1 \approx 2.295 \), the imaginary part of each such critical value is non-negative. Hence all these critical values are in the primary component \( B \). Thus \( B \) contains all the critical values of the function.

Clearly, \( f_\lambda \) is topologically hyperbolic. By Lemma 3.3, \( f_\lambda \) has no wandering domain.

Let \( f_\lambda \) have a non-primary \( p \)-periodic Baker domain and \( z \) be a point in it. Without loss of generality assume that \( \lim_{n \to \infty} z_n = \infty \) where \( z_n = f_\lambda^{np}(z) \). If there is a \( \delta < 0 \) such that \( \Im(z_{n_k}) > \delta \) for some subsequence \( n_k \) then the topologically hyperbolicity of \( f_\lambda \) gives that the assumed Baker domain contains the disc \( D(z_{n_k}, |\delta|) \) for a sufficiently large \( k \) (by Lemma 2.3). However \( D(z_{n_k}, |\delta|) \) contains a real number and that is either a pole or belongs to \( B \). None of this can be true. Therefore, for each \( \delta < 0 \) there is an \( n_0 \) such that \( \Im(z_n) < \delta \) for all \( n > n_0 \). In other words,

\[
\Im(z_n) \to -\infty \text{ as } n \to \infty. \tag{6}
\]

Now, choose a sufficiently large \( n_0 \) such that \( \Im(\tan z_n) > -2 \) for all \( n > n_0 \). This is because \( \tan z \to -i \) as \( \Im(z) \to -\infty \). Since \( \Im(\lambda) > 2 \), we have \( \Im(z_{n+1}) = \Im(\lambda) + \Im(z_n) + \Im(\tan z_n) > \Im(z_n) \) for all \( n > n_0 \). This is a contradiction to (6). Thus \( f_\lambda \) does not have any Baker domain. Therefore \( B \) is the only Fatou component of \( f_\lambda \) for \( \Im(\lambda) > \sqrt{2} + \sinh^{-1} 1 \).

That the Julia set is disconnected will be established by proving the existence of a bounded component of the Julia set. This is because \( \infty \in \mathcal{J}(f_\lambda) \). This desired Julia component is going to be the one containing a pole of the function.

Since the Fatou set is connected, no Julia component separates the plane, i.e., its complement is connected. Let \( J \) be a connected subset of \( \mathcal{J}(f_\lambda) \cap \mathbb{C} \) containing a pole. If \( J \) contains another pole then by Lemma 2.5, it contains all the poles of \( f_\lambda \) and then it separates the plane. However this is not possible implying that \( J \) contains exactly one pole, say \( z_0 \). Let \( J_0 \) be a connected subset of \( J \setminus \{z_0\} \). Then \( f_\lambda(J_0) \subset H^- \) and all the critical values of \( f_\lambda \) are in \( H^+ \). Take a point \( z' \in J_0 \) and consider a branch \( g \).
of $f^{-1}_\lambda$ defined in a neighborhood of $f_\lambda(z')$ such that $g(f_\lambda(z')) = z'$. This $g$ can be analytically continued to the whole of $H^-$ by the Monodromy theorem. In particular, $g$ is analytically defined in a simply connected domain in $H^-$ containing $f_\lambda(J_0)$. In other words, the function $f_\lambda$ is one-one on $J_0$.

Now assuming that $J_0$ is unbounded, consider two connected subsets $J_{z_0}$ and $J_\infty$ of $J_0$ containing $z_0$ and $\infty$ in their closures respectively. Observe that $f_\lambda(J_{z_0})$ and $f_\lambda(J_\infty)$ are both unbounded and connected. Further $f_\lambda(J_{z_0}) \cap f_\lambda(J_\infty) = \emptyset$. Now $f_\lambda(J_0)$ is a connected subset of $\mathcal{J}(f_\lambda) \cap \mathbb{C}$ containing two disjoint and connected subsets, each of which is unbounded. Thus $f_\lambda(J_0)$ and hence the Julia component containing it, separates the plane. This is a contradiction. This proves that every connected subset of $J \setminus \{z_0\}$ is bounded. Therefore $J$ is bounded and the proof completes.

Here is a remark on $f_\lambda$ for $\lambda$ with real part different from any integral multiple of $\pi$.

**Remark 3.2.** For $0 < \Im(\lambda) < 1$, consider the critical point $c_0 = \frac{\pi}{2} - i \sinh^{-1} 1 \in \overline{\mathbb{C}}$ of $f_\lambda$. Now $f_\lambda(c) = \lambda + c + \tan\left(\frac{\pi}{2} - i \sinh^{-1} 1\right) = \lambda + c + \cot(i \sinh^{-1} 1) = \lambda + c - i \coth(i \sinh^{-1} 1)$. This gives that $\Im(f_\lambda(c_0)) = \Im(\lambda) - \Im(c_0) - \coth(i \sinh^{-1} 1)$. Since $\coth(x) > 1$ for all $x > 0$, $\Im(\lambda) - \coth(i \sinh^{-1} 1) < 0$ giving that $\Im(f_\lambda(c_0)) < \Im(c_0)$. It now follows from Lemma 2.7 and Lemma 2.5 that $\Im(f_\lambda(c)) < \Im(c)$ for all $c \in \overline{\mathbb{C}}$. However, this argument seems to fail to conclude anything about the iterated images of the critical values.
To prove Theorem 1.4 we need two lemmas.

**Lemma 3.4.** If \( \lambda = k\pi + i\frac{\pi}{2} \) for a non-zero integer \( k \) then the following are true.

1. The vertical line \( l_{m\pi} = \{ z : \Re(z) = m\pi \} \) is contained in \( B \) for all integers \( m \).

2. The vertical half line \( l_{m\pi + \frac{\pi}{2}} = \{ z : \Re(z) = m\pi + \frac{\pi}{2} \) and \(-\infty < \Im(z) \leq -\sinh^{-1} 1 \} \) is mapped into the half line \( l_{(m+k)\pi + \frac{\pi}{2}} = \{ z : \Re(z) = (m+k)\pi + \frac{\pi}{2} \) and \( \Im(z) < 0 \} \) for all integers \( m \).

3. None of the critical points in the lower half plane is contained in \( B \).

**Proof.**

1. It follows from Lemma 2.6(2) that for all \( z \in l_{m\pi} \), \( \lim_{n\to\infty} \Im(f^m_\lambda(z)) = +\infty \).

   We are done since \( l_{m\pi} \cap B \neq \emptyset \) for all integers \( m \).

2. For \( z \in l_{m\pi + \frac{\pi}{2}} \), \( f_\lambda(z) = k\pi + i\frac{\pi}{2} + m\pi + \frac{\pi}{2} + i\Im(z) + \tan(m\pi + \frac{\pi}{2} + i\Im(z)) = (k + m)\pi + \frac{\pi}{2} + i\{\Im(z) + \frac{\pi}{2} + \coth \Im(z)\} \). Note that \(-\infty < \coth \Im(z) < -1 \) for all \( z \) with \( \Im(z) < 0 \). Therefore, \( \Im(f_\lambda(z)) < -\sinh^{-1} 1 + \frac{\pi}{2} - 1 < 0 \) for all \( z \in l_{m\pi + \frac{\pi}{2}} \). Thus \( f_\lambda \) maps \( l_{m\pi + \frac{\pi}{2}} \) into \( l_{(m+k)\pi + \frac{\pi}{2}} \).
3. The critical points of $f_\lambda$ in the lower half plane are $m\pi + \frac{\pi}{2} - i\sinh^{-1}1$ for $m \in \mathbb{Z}$.

For each $z \in l_{m\pi + \frac{\pi}{2}} = \{ z : \Re(z) = m\pi + \frac{\pi}{2} \text{ and } \Im(z) \leq -\sinh^{-1}1 \}$, $\Re(f_\lambda(z)) = (m+k)\pi + \frac{\pi}{2}$ and $\Im(f_\lambda(z)) = \frac{\pi}{2} + \Im(z) + \coth(\Im(z))$.

Consider the function $g(y) = \frac{\pi}{2} + y + \coth y, y < 0$ and $h(y) = g(y) - y$. Note that $\lim_{y \to -\infty} h(y) = \frac{\pi}{2} + \lim_{y \to -\infty} \coth y = \frac{\pi}{2} - 1 > 0$ and $\lim_{y \to 0^-} h(y) = -\infty$. By the Intermediate Value Theorem, there is a negative real number $y_0$ such that $h(y_0) = 0$.

This $y_0$ is a fixed point of $g(y)$. Since $h'(y) = -\cosech^2 y < 0$ for all $y < 0$, $y_0$ is unique. Note that $y_0 = \frac{1}{2} \ln \frac{\pi}{\pi+2} \approx -0.7524$. Note that $g'(y) = 2 - \coth^2 y$. The multiplier of $y_0$, $g'(y_0) = 2 - \coth^2 y_0 = 2 - \frac{\pi^2}{4} \in (-1, 0)$ which means that $y_0$ is an attracting. Now $g''(y) = 2 \coth y \cosech^2 y < 0$ for all $y < 0$. Then $g'$ has a unique root and,

$$g'(y) \begin{cases} > 0 & \text{for all } y < -\sinh^{-1}1 \approx -0.8814, \\ = 0 & \text{for } y = -\sinh^{-1}1, \\ < 0 & \text{for all } -\sinh^{-1}1 < y < 0. \end{cases} \tag{7}$$

Note that $g([-0.8814, y_0]) = [y_0, -0.7248]$ and $g(-0.7248) > -0.8814$. Since $g$ is decreasing in $(-0.8814, 0)$, $g([-0.7248, y_0]) \subseteq [-0.8814, y_0]$ and it follows that $g^{n+1}([-0.8814, -0.7248]) \subseteq g^n([-0.8814, -0.7248])$ for all $n$. Thus $g^n(y) \to y_0$ for all $y \in [-0.8814, -0.7248]$.

Since the image of $l_{m\pi + \frac{\pi}{2}}$ is $l_{(m+k)\pi + \frac{\pi}{2}}$ under $f_\lambda$, $\Im(f_\lambda^n(z)) = g^n(\Im(z))$ for all $z \in l_{m\pi + \frac{\pi}{2}}$ and all $n$. Let $c_0 = \frac{\pi}{2} - i\sinh^{-1}1$. Note that $f_\lambda(c_0) = k\pi + i\frac{\pi}{2} + \frac{\pi}{2} - i\sinh^{-1}1 + \tan(\frac{\pi}{2} - i\sinh^{-1}1) = k\pi + \frac{\pi}{2} + i(\frac{\pi}{2} - \sinh^{-1}1 - \coth(\sinh^{-1}1)) = k\pi + \frac{\pi}{2} - 0.7248i$.

As $\Im(f_\lambda(c_0)) \in [-0.8814, -0.7248]$ then $\Im(f_\lambda^n(c_0)) \to y_0 \approx -0.7524$ and hence $c_0$ is not contained in $B$. It follows from Lemma 2.5 that none of the critical points in the lower half plane is contained in $B$.

Here are some estimates of three functions in suitable intervals.

**Lemma 3.5.** 1. If $x \leq -0.6658$ then $0 < -\frac{\sin \frac{\pi}{8}}{\cos \frac{\pi}{8} + \cosh 2x} < \frac{\pi}{8}$.

2. For all $x \leq -0.6658$, $\frac{\pi}{2} + x + \frac{\sinh 2x}{\cos \frac{\pi}{8} + \cosh 2x} \leq -0.6658$.  

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3. If \( m \) is an integer and \(|x-(m\pi+\frac{\pi}{2})| \leq \frac{\pi}{16}\) then \( \frac{\pi}{2} - 0.6658 - \frac{\sinh 1.3316}{\cos 2x + \cosh 2x} < 0.6658 \).

**Proof.**

1. For \( h(x) = \frac{\sin \frac{x}{8}}{-\cos \frac{x}{8} + \cosh 2x}, \ h'(x) = -2\sin \frac{x}{8} \frac{\sinh 2x}{(-\cos \frac{x}{8} + \cosh 2x)^2} > 0 \) for all \( x < 0 \).

   The function \( h \) is strictly increasing. Further, \( \lim_{x \to -\infty} h(x) = 0 \) and \( h(-0.6658) \approx 0.3473 \). This gives that \( 0 < h(x) \leq 0.3473 < \frac{\pi}{8} \) for \( x \leq -0.6658 \).

2. Let \( h(x) = \frac{\pi}{2} + x + \frac{\sinh 2x}{-\cos \frac{x}{8} + \cosh 2x} \). Then \( h'(x) = 1 + 2\frac{1-\cosh 2x \cos \frac{x}{8}}{(-\cos \frac{x}{8} + \cosh 2x)^2} \) and \( h''(x) = 4\sinh 2x \frac{\cos^2 \frac{x}{8} + \cos \frac{x}{8} \cosh 2x - 2}{(-\cos \frac{x}{8} + \cosh 2x)^3} \). The function \( \cos^2 \frac{x}{8} + \cos \frac{x}{8} \cosh 2x - 2 \) is a strictly decreasing function with its minimum value approximately equal to 0.7249 achieved at \( -0.6658 \). Thus \( h''(x) < 0 \) for all \( x \leq -0.6658 \) giving that \( h' \) is a strictly decreasing function. As \( \lim_{x \to -\infty} h'(x) = 1 \) and \( h'(-0.6658) \approx -0.4359 \), there exists a unique \( x_0 \leq -0.6658 \) such that \( h'(x_0) = 0 \). Computationally, it is found that \( x_0 \approx -0.804 \). This proves that \( h \) attains maximum at \( x_0 \) and the maximum value is \( \approx -0.6658 \). Thus \( h(x) \leq -0.6658 \) for all \( x \leq -0.6658 \).

3. Let \( h(x) = \frac{\pi}{2} - 0.6658 - \frac{\sinh 1.3316}{\cos 2x + \cosh 1.3316} \) for \( x \in I_m = \{ x : |x-(m\pi+\frac{\pi}{2})| \leq \frac{\pi}{16}\} \).

Then \( h'(x) = -2\sinh 1.3316 \frac{\sin 2x}{(\cosh 1.3316 + \cos 2x)^2} \) is 0 only when \( x = m\pi + \frac{\pi}{2} \). Further, \( h'(x) < 0 \) for \( x < m\pi + \frac{\pi}{2} \) and \( h'(x) > 0 \) for \( x > m\pi + \frac{\pi}{2} \) giving that \( h \) attains its minimum at \( m\pi + \frac{\pi}{2} \). As \( h(m\pi + \frac{\pi}{2} - \frac{\pi}{16}) = h(m\pi + \frac{\pi}{2} + \frac{\pi}{16}) \approx -0.6939 \), we have \( h(x) < 0.6939 < -0.6658 \) for all \( x \in I_m \).

\[ \square \]

**Proof of Theorem 1.4.** Let \( \lambda = \pi k + i\frac{\pi}{2} \) for a natural number \( k \). Firstly, we show that certain regions outside the primary Fatou component are in the Fatou set of \( f_\lambda \). Consider the region \( R_m = \{ z : |\Re(z) - (m\pi + \frac{\pi}{2})| < \frac{\pi}{16} \) and \( \Im(z) \leq -0.6658 \} \). Note that \( R_m \) does not contain any pole of \( f_\lambda \). Our intention is to show that \( f_\lambda(R_m) \subset R_{m+k} \). Let

\[ l_1 = \{ z : \Re(z) = m\pi + \frac{\pi}{2} - \frac{\pi}{16} \text{ and } \Im(z) \leq -0.6658 \}, \]

\[ l_2 = \{ z : \Re(z) = m\pi + \frac{\pi}{2} + \frac{\pi}{16} \text{ and } \Im(z) \leq -0.6658 \} \]

and

\[ l_3 = \{ z : |\Re(z) - (m\pi + \frac{\pi}{2})| \leq \frac{\pi}{16} \text{ and } \Im(z) = -0.6658 \}. \]

The boundary of \( R_m \) is \( l_1 \cup l_2 \cup l_3 \cup \{ \infty \} \).

For \( z \in l_1 \), \( \Re(f_\lambda(z)) = (k+m)\pi + \frac{\pi}{2} - \frac{\pi}{16} + \Re(\tan z) = (k+m)\pi + \frac{\pi}{2} - \frac{\pi}{16} + \frac{\sin \frac{\pi}{8}}{-\cos \frac{\pi}{8} + \cosh 2\Im(z)} \)

and \( \Im(f_\lambda(z)) = \frac{\pi}{2} + \Im(z) + \frac{\sinh 2\Im(z)}{-\cos \frac{\pi}{8} + \cosh 2\Im(z)} \). It follows from Lemma 3.5(1) that \( (m+k)\pi + \frac{\pi}{2} - \frac{\pi}{16} + \Re(\tan z) \leq \frac{\pi}{8} \) and \( \Im(z) + \frac{\sinh 2\Im(z)}{-\cos \frac{\pi}{8} + \cosh 2\Im(z)} \leq -0.6658 \).
\[
\frac{\pi}{16} \leq \Re(f_\lambda(z)) \leq (k+m)\pi + \frac{\pi}{2} + \frac{\pi}{16}.
\]
Similarly, Lemma \[3.5(2)\] gives that \( \Im(f_\lambda(z)) \leq -0.6658 \) for all \( z \in l_1 \).

Now, for \( z \in l_2 \), \( \Re(f_\lambda(z)) = (k+m)\pi + \frac{\pi}{2} + \frac{\pi}{16} - \frac{\sin \frac{\pi}{2}}{\cos \frac{\pi}{2} + \cosh 2\Im(z)} \) and \( \Im(f_\lambda(z)) = \frac{\pi}{2} + \Im(z) + \frac{\sinh 2\Im(z)}{\cosh \frac{\pi}{2} + \cosh 2\Im(z)} \). By Lemma \[3.5(1)\], \((k+m)\pi + \frac{\pi}{2} - \frac{\pi}{16} \leq \Re(f_\lambda(z)) \leq (m+k)\pi + \frac{\pi}{2} + \frac{\pi}{16} \).

Similarly, Lemma \[3.5(2)\] gives that \( \Im(f_\lambda(z)) \leq -0.6658 \) for all \( z \in l_2 \).

If \( z \in l_3 \) then \( \Im(f_\lambda(z)) = \frac{\pi}{2} + \Im(z) + \Im(\tan z) = \frac{\pi}{2} - 0.6658 - \frac{\sin 1.3316}{\cos 2x + \cosh 1.3316} \). It follows from Lemma \[3.5(3)\] that \( \Im(f_\lambda(z)) < -0.6658 \) for all \( z \in l_3 \).

Thus \( f_\lambda(R_m) \subset R_{m+k} \) and \( \cup_{n \in \mathbb{Z}} R_{m+nk} \) is invariant under \( f_\lambda \) giving that \( R_m \) is in the Fatou set of \( f_\lambda \) for every integer \( m \) by the Fundamental Normality Test.

For each integer \( m \), the line \( L_m = \{ z : \Re(z) = m\pi + \frac{\pi}{2} \text{ and } \Im(z) \leq -0.6658 \} \) is contained in \( R_m \). Between any two such consecutive lines \( L_m \) and \( L_{m+1} \), there is a vertical line \( l_{(m+1)\pi} \) which is in the primary Fatou component (by Lemma \[3.4(1)\]). In other words, for \( m \neq m' \), the Fatou components containing \( R_m \) is different from that containing \( R_{m'} \).

Let \( W \) be the Fatou component containing \( R_0 \). Then all the \( W_n \)s are distinct giving that \( W \) is a wandering domain.

1. Note that \( R_{nk} \) is in the Fatou set and is contained in \( W \) for each \( n \). Further, \( f_\lambda^n \to \infty \) on \( W \). Thus, \( W \) is escaping.

2. Since each \( R_{nk} \) contains a critical point of \( f_\lambda \), each \( W_n \) contains a critical point. It cannot contain more than one critical point as each two critical point are separated by a vertical line contained in \( B \). For the same reason, no \( W_n \) is horizontally spread. By Lemma \[3.2(2)\], \( f_\lambda : W_n \to W_{n+1} \) is proper. Its degree is 2 by the Riemann Hurwitz formula. Let, for a natural number \( n \), \( W_- \) be the wandering domain containing \( R_{-kn} \) such that \( f_\lambda^n(W_-) = W \). The above argument gives that \( f_\lambda : W_m \to W_{m+1} \) is proper map with degree 2 for all negative integer \( m \).

3. If \( W' \) is a wandering domain in the grand orbit and is different from all \( W_n \) then there is no critical point in \( W' \) and the map \( f_\lambda \) is one-one on \( W' \) by the Riemann Hurwitz formula.

It can be seen that, for \( i \in \{1, 2, \ldots, k - 1\} \), the Fatou component containing \( R_i \) is also a wandering domain \( W^i \) and their forward orbits are disjoint from each other and also from \( W \). Thus, there are \( k \) wandering domains with distinct forward orbits. Clearly, their grand orbits are also different.
Note that $f_\lambda$ is topologically hyperbolic. Using similar argument as described in Theorem 1.2(2), it can be shown that $f_\lambda$ does not have any periodic Fatou component except $B$ or any other wandering domain.

Figure 3: Wandering domains of $f_\lambda$ for $\lambda = \pi + i \frac{\pi}{2}$ in green.

Remark 3.3. For $k < 0$, there are wandering domains $W$ with the same properties except that $\Re(f^n_\lambda) \to -\infty$ on $W$ as mentioned in Theorem 1.4.

4 Concluding remarks

We first summarize the dynamics of $f_\lambda$ in terms of the parameter $\lambda$ for $\Im(\lambda) > 0$ (Figure 4). Since $f_\lambda$ has a completely invariant Baker domain, the primary Fatou component for every $\lambda$, we describe the other Fatou components only. An archetype of the parameter plane is described below. The parameters in the strip $\{\lambda : 0 < \Im(\lambda) < 1\}$ (seen in yellow) correspond to $f_\lambda$ with an invariant Baker domain as mentioned in Theorem 1.3. This is the only non-primary Fatou component if $\Re(\lambda) = k\pi$ whenever $k \in \mathbb{Z}$. The parameters in the yellow region $\{\lambda : |2 + \lambda^2| < 1\}$, we call this the attracting lobe, correspond to the existence of infinitely many invariant attracting domains as described in Theorem 1.2. For a fixed integer $k$, $f_{\lambda+k\pi}^n(z) = nk\pi + f_\lambda^n(z)$ for every natural number $n$ and $z \in \mathbb{C}$. If $|2 + \lambda^2| < 1$ and $A_\lambda$ is an attracting domain of $f_\lambda$ then $f_{k\pi+k\lambda}^n \to \infty$ uniformly on $A_\lambda$. In other words, all the attracting domains of $f_\lambda$ are contained in the Fatou set of $f_{\lambda+k\pi}$. For $k \neq 0$, with some extra effort these attracting domains of $f_\lambda$ have been shown to be wandering domains for $f_{i\frac{\pi}{2}+k\pi}$ in Theorem 1.4. Further, since all the critical points...
in $H^-$ of $f_\lambda$ are in the invariant attracting domains, the function $f_{\lambda+k\pi}$ is topologically hyperbolic. Other details of its dynamics is to be taken up later. The primary Fatou component is the only Fatou component of $f_\lambda$ and the Julia set is disconnected whenever $\lambda$ is in the yellow strip $\{\lambda : \Im(\lambda) > \sqrt{2} + \sinh^{-1} 1\}$ above the attracting lobe. This is given in Theorem 1.3. It is important to note that the attracting lobe does not touch this strip. The situation for $f_\lambda$ is the same when $\Im(\lambda) = \sqrt{2} + \sinh^{-1} 1$ but $\Re(\lambda) \neq k\pi + \frac{\pi}{2}, k \in \mathbb{Z}$. For $\lambda = k\pi + \frac{\pi}{2} + i(\sqrt{2} + \sinh^{-1} 1)$, the poles become the critical values and the function is no longer topologically hyperbolic. But the dynamics seems to be tractable!

![Figure 4: The parameter plane](image)

Some of the dynamically crucial properties of $f_\lambda$ are due to $\tan z$. In place of $\tan z$, one may consider a periodic meromorphic function $h$ such that $1 + h'(z) = g(h)$ for an entire function $g$. If $F_\lambda(z) = \lambda + z + h(z)$ is such a function then the following are true.

1. The function $F_\lambda$ has infinitely many fixed points for all except possibly two values of $\lambda$ and the multiplier of every fixed point is $g(-\lambda)$. To see it, note that every fixed point $z_0$ of $F_\lambda$ satisfies $h(z_0) = -\lambda$ and since $h$ is meromorphic, for all but atmost two values of $\lambda$, $h(z_0) = -\lambda$ has infinitely many solutions. The multiplier of $z_0$ is $F'_\lambda(z_0) = 1 + h'(z_0) = g(h(z_0)) = g(-\lambda)$.

2. The Fatou set (and therefore the Julia set) of $F_\lambda$ is $w$-invariant where $w$ is the period of $h$. This follows from the fact that $F^n_\lambda(z + w) = w + F^n_\lambda(z)$ for all $n$ and $z \in \mathbb{C}$. 

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3. The set of all the singular values of $F_\lambda$ is unbounded whenever $g$ has at least three distinct roots. To see it, first note that the critical points of $F_\lambda$ are the solutions of $g(h(z)) = 0$. Since $g$ has at least three distinct roots, there is a solution of $g(h(z)) = 0$.

If $g(h(c)) = 0$ for some $c$ then for each $n \geq 0$, $g(h(c + nw)) = g(h(c)) = 0$ and $c + nw$ is a critical point of $F_\lambda$. The critical values are $F_\lambda(c + nw) = \lambda + c + nw + h(c)$. We are done as the set $\{\lambda + c + nw + h(c) : n \geq 0\}$ of critical values of $F_\lambda$ is unbounded.

The dynamics of $F_\lambda$ can be studied possibly under some additional conditions on $h$.

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