ON HELLY’S THEOREM IN GEODESIC SPACES

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(Communicated by Dmitri Burago)

ABSTRACT. In this note we show that Helly’s Intersection Theorem holds for convex sets in uniquely geodesic spaces (in particular, in CAT(0) spaces) without the assumption that the convex sets are open or closed.

1. INTRODUCTION

The classic Helly’s Intersection Theorem asserts the following: if \{A_i\} is a finite collection of convex sets in \(\mathbb{R}^n\) such that every subcollection consisting of at most \(n+1\) sets has a nonempty intersection, then \(\bigcap A_i \neq \emptyset\). This theorem has a topological generalization (found by Helly himself [7]) where convexity is replaced by the assumption that the sets \(A_i\) and their nonempty intersections are open homology cells. See [5] for a modern proof and further references.

The proof of the topological Helly’s theorem extends to CAT(0) spaces of geometric dimension \(n\), see, e.g., [9, Proposition 5.3] and [6, §3]. Thus Helly’s theorem holds for open convex sets in such spaces. Once the theorem is established for open sets, the variant with closed convex sets follows. In \(\mathbb{R}^n\), one can deduce the theorem for arbitrary convex sets by picking one point in every nonempty intersection and replacing every set by the convex hull of the marked points it contains. However, this argument does not work in CAT(0) spaces since convex hulls of finite sets are not necessarily closed.

In this note we show that Helly’s theorem holds for arbitrary (not necessarily open or closed) convex sets in CAT(0) and some other spaces. Namely, we prove the following:

**Theorem 1.1.** Let \(X\) be a uniquely geodesic space of compact topological dimension \(n < \infty\). Let \(\{A_i\}\) be a finite collection of convex sets in \(X\) such that every subcollection of cardinality at most \(n+1\) has a nonempty intersection. Then \(\bigcap A_i \neq \emptyset\).
Definitions. Here are the definitions of terms used in Theorem 1.1.

A geodesic space is a metric space $X$ such that every two points in $X$ belong to a segment, where a segment is a subset isometric to a compact interval of the real line. We say that $X$ is uniquely geodesic if for every $x, y \in X$ there is a unique segment $[xy] \subset X$ with endpoints at $x$ and $y$, and $[xy]$ depends continuously on $x$ and $y$. Note that the continuous dependence is automatic if $X$ is proper (i.e., if all closed balls are compact).

Examples of uniquely geodesic spaces are simply connected Riemannian and Finsler manifolds without conjugate points, CAT(0) spaces, balls of radius $\pi/2\sqrt{\kappa}$ in CAT($\kappa$) spaces (see [2, §III.1] for definitions), Busemann convex spaces [4], simply connected polyhedral Finsler spaces with locally unique geodesics [3].

The compact topological dimension $\dim_c X$ of $X$ is defined by

$$\dim_c X = \sup \{ \dim K : K \subset X \text{ is compact} \},$$

where $\dim$ is the Lebesgue covering dimension. For (locally) CAT($\kappa$) spaces, the compact topological dimension equals the geometric dimension and a number of other dimension-like quantities [9].

A set $A \subset X$ is convex if it contains all segments with endpoints in $A$.

The proof of Theorem 1.1 is topological – the only feature of convex sets used in the proof is that they are contractible. See Proposition 2.2 for a purely topological formulation. Proposition 2.2 is in some ways similar to the Topological Helly Theorem [5] (see also more general results in [1, 12]). However, known proofs of the Topological Helly Theorem involve computation of homology groups with techniques such as Mayer–Vietoris sequences. This approach requires the sets in question to be open or otherwise “nice.” It fails to work for arbitrary convex sets such as, for example, a Euclidean ball with a wild subset of the boundary removed.

In contrast to this, the proof of Proposition 2.2 is a combinatorial argument which does not use algebraic topology and does not require openness. Note that for open sets results stronger than Proposition 2.2 are known (see [12]).

2. Proof of the theorem

Fix $n \geq 1$ and denote by $\Delta$ the standard $(n + 1)$-dimensional simplex. By definition, $\Delta$ is the convex hull of the standard basis $\{e_i\}_{i=1}^{n+2}$ of $\mathbb{R}^{n+2}$. Let $F_i$ denote the $i^{\text{th}} n$-dimensional face of $\Delta$ (i.e., the one not containing $e_i$). For a positive integer $m$, we denote by $[m]$ the set $\{1, 2, \ldots, m\}$.

The following lemma is the only place in the proof where the dimension of the ambient space is used.

Lemma 2.1. Let $X$ be a Hausdorff space with $\dim_c X \leq n$ and $f : \Delta \to X$ a continuous map. Then $\bigcap_{i=1}^{n+2} f(F_i) \neq \emptyset$.

This lemma is apparently folklore. It can be seen as a special case ($r = 2$) of [8, Theorem 1.1]. Here we give a short proof based on Sperner’s lemma.

Proof of Lemma 2.1. We need the following fact: if $\{G_i\}_{i=1}^{n+2}$ is an open covering of $\Delta$ such that $G_i \cap F_i = \emptyset$ for each $i$, then $\bigcap G_i \neq \emptyset$. This fact is a topological variant of Sperner’s lemma and follows easily from the discrete counterpart. Alternatively, it follows from the Knaster–Kuratowski–Mazurkiewicz lemma [10], which is a slightly more general statement about open or closed coverings of the simplex.
Proof of Theorem 1.1. In a uniquely geodesic space all nonempty convex sets are contractible. Therefore Theorem 1.1 follows from Proposition 2.2.

Proposition 2.2. Let $X$ be a contractible Hausdorff space with $\dim X < n$. Let $\{A_i\}_{i=1}^m$ be a finite collection of contractible sets in $X$ such that the intersection of every subcollection is either contractible or empty. Suppose that $m > n + 2$ and for every set $I \subset [m]$ with $|I| = n + 1$ one has $\bigcap_{i \in I} A_i \neq \emptyset$. Then $\bigcap_{i=1}^m A_i \neq \emptyset$.

Proof. First consider the case $m = n + 2$. For a nonempty set $I \subset [m] = [n + 2]$, denote by $\Delta_I$ the convex hull of $\{e_i\}_{i \in I}$ and let $P_I = \bigcap_{i \in [m]\setminus I} A_i$ if $I \neq [m]$. In addition, define $P_{[m]} = X$. By the assumptions of the proposition, $P_I$ is contractible for every nonempty set $I \subset [m]$.

We construct a continuous map $f : \Delta \to X$ such that $f(\Delta_I) \subset P_I$ for every $I \subset [m]$. First for each $i \in [m]$ pick a point $f(e_i) = f(\Delta_{\{i\}})$ from the set $P_{\{i\}}$, which is nonempty by the assumptions of the proposition. Then extend the map by induction as follows. Assuming that $f$ is already defined on the $(k-1)$-skeleton of $\Delta$, where $1 \leq k \leq n + 1$, consider a $k$-simplex $\Delta_k$ where $I \subset [m]$, $|I| = k + 1$. Observe that $f(\partial \Delta_I) \subset P_I$ because $\partial \Delta_I = \bigcup_{i \in I} \Delta_{\{i\}}$ and $f(\Delta_{\{i\}}) \subset P_{\{i\}} \cap A_i$ for every $i \in I$. Since $P_I$ is contractible, $f|_{\partial \Delta_I}$ can be extended to a continuous map from $\Delta_k$ to $P_I$. Applying this extension procedure to all $k$-dimensional faces for $k = 1, 2, \ldots, n + 1$, one gets the desired map $f : \Delta \to X$.

By Lemma 2.1, we have $\bigcap_{i=1}^m f(F_i) \neq \emptyset$ where $F_i = \Delta_{[m]\setminus\{i\}}$. By construction, $f(F_i) \subset P_{[m]\setminus\{i\}} = A_i$ for each $i$, therefore $\bigcap_{i=1}^m A_i \neq \emptyset$. This completes the proof in the case $m = n + 2$.

The general case follows by induction in $m$. Suppose that $m > n + 2$ and a collection $\{A_i\}_{i=1}^m$ satisfies the assumptions of the proposition. Then, since the case $m = n + 2$ is already done, every subcollection of cardinality $n + 2$ has a nonempty intersection. Therefore the collection $\{A'_i\}_{i=1}^{m-1}$ where $A'_i = A_i \cap A_{m}$ satisfies the assumptions as well. Applying the induction hypothesis to $\{A'_i\}$ yields that the intersection $\bigcap_{i=1}^{m-1} A'_i = \bigcap_{i=1}^m A_i$ is nonempty.

Proof of Theorem 1.1. In a uniquely geodesic space all nonempty convex sets are contractible. This is ensured by the requirement that segments depend continuously on their endpoints. Intersections of convex sets are obviously convex and hence contractible. Therefore Theorem 1.1 follows from Proposition 2.2.

References

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