Abstract

For the complex Clifford algebra $\text{Cl}(p,q)$ of dimension $n=p+q$ we define a Hermitian scalar product. This scalar product depends on the signature $(p,q)$ of Clifford algebra. So, we arrive at unitary spaces on Clifford algebras. With the aid of Hermitian idempotents we suggest a new construction of, so called, normal matrix representations of Clifford algebra elements. These representations take into account the structure of unitary space on Clifford algebra.
Unitary spaces on Clifford algebras

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Clifford algebras were invented by W. K. Clifford [1] in 1878. One of authors uses the Clifford algebra $\mathcal{A}(1,3)$ in the field theory [2]-[6] (Dirac-Yang-Mills equations). In the mentioned papers a notion of unitary space on Clifford algebra was developed for the Clifford algebra $\mathcal{A}(1,3)$. Several structure equalities were found for $\mathcal{A}(1,3)$.

In the present paper the notion of unitary space on Clifford algebra is generalized for Clifford algebras $\mathcal{A}(p,q)$ of dimensions $n = p + q > 4$ and for

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different signatures \((p, q)\). Also we prove several structure equalities for the Clifford algebra (theorems 1-8).

With the aid of Hermitian idempotents (sections 6,7) we suggest a new construction of, so called, normal matrix representations of Clifford algebra elements. These representations take into account the structure of unitary space on Clifford algebra.

Note that in papers [9, 7, 8] there are several partial cases of the considered structure equalities of the Clifford algebra. Namely, in [9] we see the following formula for the Clifford product of two Clifford algebra elements

\[
k \ell \quad k \ell \quad k-l \quad k-l+2 \quad \ldots \quad k+l
\]

\[
UV = W + W + \ldots + W,
\]

where \(W = 0\) for \(m > n\) and for \(m < 0\). This formula can be derived from Theorem 2 of the present paper.

Also, well known formulas

\[
e^a e_a = n, \quad e^a \ell e_a = (-1)^{n+1} n \ell,
\]

where \(e^a\) are generators of Clifford algebra \(\mathcal{C}(p, q)\), \(p+q = n\) and \(\ell = e^1 \ldots e^n\), can be considered as partial cases of the proposition of Theorem 5.

Theorem 7 defines an operation of Hermitian conjugation \(U \rightarrow U^\dagger\) of Clifford algebra elements. In particular, for \(\mathcal{C}(n)\) we get the operation \(U^\dagger = U^*\) and for \(\mathcal{C}(1, n-1)\) we get the operation \(U^\dagger = e^1 U^* e^1\). In both cases these operations are well known in the literature (see, for example, [8]).

## 1 Clifford algebras

Let \(\mathbb{F}\) be the field of real numbers \(\mathbb{R}\) or the field of complex numbers \(\mathbb{C}\) and let \(n\) be a natural number. Consider the \(2^n\) dimensional vector space \(\mathcal{E}\) over the field \(\mathbb{F}\) with a basis

\[
e, e^a, e^{a_1 a_2}, \ldots, e^{1 \ldots n}, \quad a_1 < a_2 < \ldots,
\]

with elements numbered by ordered multi-indices of length from 0 to \(n\). Indices \(a, a_1, \ldots\) take values form 1 to \(n\). Let \(p, q\) be nonnegative integer numbers and \(p + q = n\). Consider a diagonal matrix of dimension \(n\)

\[
\eta = \eta(p, q) = \text{diag}(1, \ldots, 1, -1, \ldots, -1)
\]
with $p$ pieces of 1 and $q$ pieces of $-1$ on the diagonal. By $\eta^{ab} = \eta_{ab}$ denote elements of $\eta$. Following rules define product $U, V \to UV$ of elements of the vector space $E$:

1. $\forall U, V, W \in E$
   
   $U(V + W) = UV + UW, \quad (U + V)W = UW + VW,$
   
   $(UV)W = U(VW).$

2. $\forall U \in E \ eU = U e = U.$

3. $e^a e^b + e^b e^a = 2\eta^{ab}e$ for $a, b = 1, \ldots, n$.

4. $e^{a_1} \ldots e^{a_k} = e^{a_1 \ldots a_k}$ for $1 \leq a_1 < \cdots < a_k \leq n$.

This algebra is called the Clifford algebra denoted by $\mathcal{C}^F(p, q)$ (if $F = \mathbb{C}$, then $\mathcal{C}(p, q) = \mathcal{C}^\mathbb{C}(p, q)$). For $p = n, q = 0$ we use notation $\mathcal{C}^F(n) = \mathcal{C}^F(n, 0)$. Elements $e^a$ is called generators of Clifford algebra $\mathcal{C}^F(p, q)$. $\mathcal{C}^F(p, q)$ is real Clifford algebra and $\mathcal{C}(p, q)$ is complex Clifford algebra.

Any element $U \in \mathcal{C}^F(p, q)$ can be written in the form

$$U = ue + u_a e^a + \sum_{a_1 < a_2} u_{a_1 a_2} e^{a_1 a_2} + \ldots + u_{1\ldots n} e^{1\ldots n}$$

with coefficients $u, u_a, u_{a_1 a_2}, \ldots, u_{1\ldots n} \in F$, which numbered by ordered multi-indices of length form 0 to $n$.

Denote by $\mathcal{C}^F_k(p, q)$, $(k = 0, \ldots, n)$ subspaces of the vector space $\mathcal{C}^F(p, q)$ that span over basis elements $e^{a_1 \ldots a_k}$. Elements of $\mathcal{C}^F_k(p, q)$ are called elements of rank $k$. Sometimes it is suitable to denote $U \in \mathcal{C}^F_k(p, q)$. We have

$$\mathcal{C}^F(p, q) = \mathcal{C}^F_0(p, q) \oplus \ldots \oplus \mathcal{C}^F_n(p, q) = \mathcal{C}^F_{\text{even}}(p, q) \oplus \mathcal{C}^F_{\text{odd}}(p, q),$$

where

$$\mathcal{C}^F_{\text{even}}(p, q) = \mathcal{C}^F_0(p, q) \oplus \mathcal{C}^F_2(p, q) \oplus \ldots, \quad \mathcal{C}^F_{\text{odd}}(p, q) = \mathcal{C}^F_1(p, q) \oplus \mathcal{C}^F_3(p, q) \oplus \ldots$$

and

$$\dim \mathcal{C}^F_k(p, q) = C^k_n, \quad \dim \mathcal{C}^F_{\text{even}}(p, q) = \dim \mathcal{C}^F_{\text{odd}}(p, q) = 2^{n-1},$$

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$C^k_a$ are binomial coefficients. Let us take antisymmetric coefficients $u_{a_1 \ldots a_k} = u_{[a_1 \ldots a_k]} \in \mathbb{F}$, where square brackets denote operation of alternation. Consider an element

$$U = \sum_{a_1 \ldots a_k}^k u_{a_1 \ldots a_k} e^{a_1 \ldots a_k} \in \mathcal{C}_k^p \mathbb{F}(p, q).$$

We have

$$U = \sum_{a_1 \ldots a_k}^k u_{a_1 \ldots a_k} e^{a_1 \ldots a_k} = \frac{1}{k!} u_{b_1 \ldots b_k} e^{b_1} \ldots e^{b_k},$$

and

$$U = u e + u_a e^a + \frac{1}{2!} u_{a_1 a_2} e^{a_1} e^{a_2} + \ldots + \frac{1}{n!} u_{a_1 \ldots a_n} e^{a_1} \ldots e^{a_n}.$$ 

for $U$ from (3).

**The exterior product of Clifford algebra elements.** Let us define

$$e^{i_1} \wedge e^{i_2} \wedge \ldots \wedge e^{i_k} = e^{[i_1 i_2 \ldots i_k]}.$$ 

(4)

In particular,

$$e^{i_1} \wedge e^{i_2}\;=\;\frac{1}{2}(e^{i_1}e^{i_2} - e^{i_2}e^{i_1}) = e^{i_1}e^{i_2} - \eta^{i_1i_2}e,$$

(5)

$$e^{i_1} \wedge e^{i_2} \wedge e^{i_3} \;=\; \frac{1}{6}(e^{i_1}e^{i_2}e^{i_3} + e^{i_3}e^{i_1}e^{i_2} + e^{i_2}e^{i_3}e^{i_1} - e^{i_2}e^{i_3}e^{i_1} - e^{i_3}e^{i_1}e^{i_2} - e^{i_1}e^{i_2}e^{i_3}) = e^{i_1}e^{i_2}e^{i_3} - \eta^{i_2i_3}e^{i_1} + \eta^{i_1i_2}e^{i_3} - \eta^{i_1i_3}e^{i_2},$$

$$e^{i_1} \wedge e^{i_2} \wedge e^{i_3} \wedge e^{i_4} \;=\; \frac{1}{24}(e^{i_1}e^{i_2}e^{i_3}e^{i_4} + \ldots) = e^{i_1}e^{i_2}e^{i_3}e^{i_4} - \eta^{i_3i_4}e^{i_1}e^{i_2} + \eta^{i_2i_4}e^{i_1}e^{i_3} - \eta^{i_2i_3}e^{i_1}e^{i_4} - \eta^{i_1i_4}e^{i_2}e^{i_3} + \eta^{i_1i_3}e^{i_2}e^{i_4} - \eta^{i_1i_2}e^{i_3}e^{i_4} + (\eta^{i_1i_4}\eta^{i_2i_3} - \eta^{i_1i_3}\eta^{i_2i_4} + \eta^{i_1i_2}\eta^{i_3i_4})e$$

From these formulas we get

$$e^{i_1} \wedge e^{i_2} = -e^{i_2} \wedge e^{i_1} \quad \text{for} \quad i_1, i_2 = 1, \ldots, n;$$

$$e^{i_1} \wedge \ldots \wedge e^{i_k} = e^{i_1} \ldots e^{i_k} = e^{i_1 \ldots i_k} \quad \text{for} \quad i_1 < \ldots < i_k.$$

So, we arrive at the $2^n$ dimensional vector space $\mathcal{E}$ with basis (1) and with two products of elements (the Clifford product and the exterior product).
2 Commutators and anti-commutators.

There are well known formulas for the exterior product

\[ U \wedge V = W \]  \hspace{1cm} (6)

and for the Clifford product \([9]\)

\[ UV = W + W + \ldots + W. \]  \hspace{1cm} (7)

Consider the commutator and the anti-commutator of Clifford algebra elements

\[ [U, V] = UV - VU, \quad \{U, V\} = UV + VU, \quad UV = \frac{1}{2}[U, V] + \frac{1}{2}\{U, V\}. \]

**Theorem 1.** If \(U \in \mathcal{O}_k^\mathbb{R}(p, q), V \in \mathcal{O}_2^\mathbb{R}(p, q),\) then

\[ [U, ^2V] = W, \quad \text{for} \quad 1 \leq k \leq n - 1, \]

where \(n = p + q.\)

**Proof.** We must prove that

\[ [e^{a_1 \ldots a_k}, e^{b_1 b_2}] \in \mathcal{O}_k^\mathbb{R}(p, q). \]  \hspace{1cm} (8)

If the multi-index \(a_1 \ldots a_k\) do not contain neither of indices \(b_1, b_2,\) or contain both indices \(b_1, b_2,\) than the commutator [8] is equal to zero. If the \(a_1 \ldots a_k\) contain one of indices \(b_1, b_2,\) then

\[ e^{a_1 \ldots a_k e^{b_1 b_2}} e^{b_1 b_2 e^{a_1 \ldots a_k}} \in \mathcal{O}_k^\mathbb{R}(p, q). \]

It follows from this theorem that the set \(\mathcal{O}_2^\mathbb{R}(p, q)\) is closed with respect to the commutator and, hence, can be considered as a Lie algebra. This Lie algebra is the real Lie algebra of spinor Lie groups \(\text{Pin}(p, q), \text{Pin}_+(p, q), \text{Spin}(p, q), \text{Spin}_+(p, q) [7].\)
Theorem 2. Let \(k, l, r\) be elements of \(\mathcal{C}^\mathbb{F}(p, q)\), \(p + q = n\) of ranks \(k, l, r\) respectively. Then for all nonnegative integer \(k \geq l\) the following formulas are valid (let us remind that \(W = 0\) for \(k > n\) and for \(k < 0\)):

\[
\begin{align*}
[k, U, l V] &= \begin{cases}
W + W + \ldots + W, & l \text{ is even}; \\
W + W + \ldots + W, & k \text{ is even and } l \text{ is odd}; \\
W + W + \ldots + W, & k, l \text{ are odd}.
\end{cases}
\end{align*}
\]

\[
\begin{align*}
U, V \{k, l\} &= \begin{cases}
2 W + W + \ldots + W, & k \text{ is even}; \\
2 W + W + \ldots + W, & k \text{ is odd}.
\end{cases}
\end{align*}
\]

At the right hand parts of these formulas there are sums of elements with different ranks and the increment between ranks is equal to 4.

In particular, for \(k = l\) we have formulas:

\[
\begin{align*}
[k, U, k V] &= \begin{cases}
2 + 6 W + \ldots + 2 k, & k \text{ is even}; \\
2 + 6 W + \ldots + 2 k, & k \text{ is odd}.
\end{cases}
\end{align*}
\]

\[
\begin{align*}
U, V \{k, k\} &= \begin{cases}
0 + 4 W + \ldots + 2 k, & k \text{ is even}; \\
0 + 4 W + \ldots + 2 k, & k \text{ is odd}.
\end{cases}
\end{align*}
\]

Let us write down ranks of commutators and anti-commutators of Clifford
algebra elements of ranks from 1 to 4.

\[
\begin{align*}
[U, V] &= W \\
[\hat{U}, V] &= W \\
[U, V] &= W \\
[U, V] &= W + W \\
[U, V] &= W + W \\
[U, V] &= W + W \\
[U, V] &= W + W \\
[U, V] &= W + W \\
[U, V] &= W + W \\
\end{align*}
\]

\[
\begin{align*}
\{U, V\} &= W \\
\{\hat{U}, V\} &= W + 4W \\
\{U, V\} &= W \\
\{U, V\} &= W + 5W \\
\{U, V\} &= W + 4W \\
\{U, V\} &= W + 4W + 8W \\
\end{align*}
\]

**Proof** of Theorem 2. The formulas \(e^a e^b + e^b e^a = \eta^{ab} e\) lead to the following formulas:

\[
\begin{align*}
[e^{a_1 \ldots a_k}, e^{b_1 \ldots b_l}] &= (1 - (-1)^{kl-s}) e^{a_1 \ldots a_k} e^{b_1 \ldots b_l}, \\
\{e^{a_1 \ldots a_k}, e^{b_1 \ldots b_l}\} &= (1 + (-1)^{kl-s}) e^{a_1 \ldots a_k} e^{b_1 \ldots b_l}, \\
e^{a_1 \ldots a_k} e^{b_1 \ldots b_l} &\in \mathcal{O}^p_{k+l-2s}(p, q),
\end{align*}
\]

where \(s\) is a number of coincide indices in the ordered multi-indices \(a_1 \ldots a_k\) and \(b_1 \ldots b_l\) and \(0 \leq s \leq \min(k, l)\). From these formulas we get

\[
\begin{align*}
[e^{a_1 \ldots a_k}, e^{b_1 \ldots b_l}] &= \begin{cases} \mathcal{O}^p_{k+l-2s}, & \text{if } kl - s \text{ is odd} \\ 0, & \text{if } kl - s \text{ is even} \end{cases}, \\
\{e^{a_1 \ldots a_k}, e^{b_1 \ldots b_l}\} &= \begin{cases} \mathcal{O}^p_{k+l-2s}, & \text{if } kl - s \text{ is even} \\ 0, & \text{if } kl - s \text{ is odd} \end{cases}.
\end{align*}
\]

Considering all possible variants of evenness of \(k, l, s\) in the last formulas, we conclude the proof of Theorem 2.

\[\square\]
Let us consider in more details commutators and anti-commutators of Clifford algebra elements for small dimensions $1 \leq n \leq 5$.

We take a Clifford algebra element of rank $k$

\[
U = \frac{1}{k!} u_{a_1...a_k} e^{a_1} \wedge \ldots \wedge e^{a_k} \in \mathcal{Cl}_k(p, q),
\]

where $u_{a_1...a_k} = u[a_1...a_k]$ and $n = p + q$. Let $\star : \mathcal{Cl}_k(p, q) \rightarrow \mathcal{Cl}_{n-k}(p, q)$ be the Hodge operation

\[
\star U = \frac{1}{k!(n-k)!} \varepsilon_{a_1...a_n} u^{a_1...a_k} e^{a_{k+1}} \wedge \ldots \wedge e^{a_n},
\]

where $\varepsilon_{a_1...a_n}$ is a completely antisymmetric value, $\varepsilon_{1...n} = 1$ and $u^{a_1...a_k} = \eta^{a_1b_1} \ldots \eta^{a_kb_k} u_{b_1...b_k}$.

Also, we need a bilinear operation $\text{Com} : \mathcal{Cl}_2^p(p, q) \times \mathcal{Cl}_2^q(p, q) \rightarrow \mathcal{Cl}_2^p(p, q)$

\[
\text{Com}(\frac{1}{2} u_{a_1a_2} e^{a_1} \wedge e^{a_2}, \frac{1}{2} v_{b_1b_2} e^{b_1} \wedge e^{b_2}) = \frac{1}{2} u_{a_1a_2} v_{b_1b_2} (-\eta^{a_1b_1} e^{a_2} \wedge e^{b_2} - \eta^{a_2b_2} e^{a_1} \wedge e^{b_1} + \eta^{a_1b_2} e^{a_2} \wedge e^{b_1} + \eta^{a_2b_1} e^{a_1} \wedge e^{b_2}),
\]

where $u_{a_1a_2} = u[a_1a_2], v_{b_1b_2} = v[b_1b_2]$. Evidently, $\text{Com}(U, V) = -\text{Com}(V, U)$.

Let $\varrho$ be the sign ($\pm 1$) of $\text{det} \eta$.

**Theorem 3.** For $1 \leq n \leq 5$ all commutators $[U, V]$ and all anti-commutators $\{U, V\}$ of Clifford algebra elements $U \in \mathcal{Cl}_k^p(p, q), V \in \mathcal{Cl}_k^q(p, q)$ can be expressed with the aid of the exterior product, the Hodge $\star$ operation, and the Com operation:

For $n = 1$

\[
[U, V] = 0, \quad \{U, V\} = 2 \star (U \wedge \star V) \varrho,
\]

For $n = 2$

\[
[U, V] = 2 [U, V], \quad \{U, V\} = 2 \star (U \wedge \star V) \varrho
\]

\[
[U, V] = 2 \star (U \wedge \star V) \varrho, \quad \{U, V\} = 0
\]

\[
[U, V] = -2 \star (U \wedge \star V) \varrho, \quad \{U, V\} = 0
\]

\[
[U, V] = 0, \quad \{U, V\} = -2 \star (U \wedge \star V) \varrho,
\]

\[
\{U, V\} = -2 \star (U \wedge \star V) \varrho.
\]

9
\(n = 3\)

\[
\begin{align*}
[1\bar{U}, 1\bar{V}] &= 2 \ 1\bar{U} \wedge 1\bar{V} \\
[1\bar{U}, 2\bar{V}] &= -2 \ast (1\bar{U} \wedge \ast 2\bar{V}) \\
[1\bar{U}, 3\bar{V}] &= 0 \\
[2\bar{U}, 1\bar{V}] &= -2 \ast (\ast 2\bar{U} \wedge 1\bar{V}) \\
[2\bar{U}, 2\bar{V}] &= -2\ast 1\bar{U} \wedge \ast 2\bar{V} \\
[2\bar{U}, 3\bar{V}] &= 0 \\
[3\bar{U}, 1\bar{V}] &= 0 \\
[3\bar{U}, 2\bar{V}] &= 0 \\
[3\bar{U}, 3\bar{V}] &= 0
\end{align*}
\]
\( n = 4 \)

\[
\begin{align*}
[1, 1] & = 2 \ 1 \land 1 \\
[1, 2] & = 2 \ast (1 \land \ast 2) \\
[1, 3] & = 2 \ 1 \land 3 \\
[1, 4] & = 2 \ast (1 \land \ast 4) \\
[2, 1] & = -2 \ast (\ast 2 \land 1) \\
[2, 2] & = \text{Com}(2, 2) \\
[2, 3] & = -2 \ast 2 \land \ast 3 \\
[2, 4] & = 0 \\
[3, 1] & = 2 \ 3 \land 1 \\
[3, 2] & = 2 \ast 3 \land \ast 2 \\
[3, 3] & = -2 \ast 3 \land \ast 3 \\
[3, 4] & = -2 \ast 3 \land \ast 4 \\
[4, 1] & = -2 \ast 4 \land \ast 1 \\
[4, 2] & = 0 \\
[4, 3] & = 2 \ast 4 \land \ast 3 \\
[4, 4] & = 0
\end{align*}
\]

\[
\begin{align*}
\{1, 1\} & = 2 \ast (1 \land \ast 1) \\
\{1, 2\} & = 2 \ 1 \land 2 \\
\{1, 3\} & = 2 \ast (1 \land \ast 3) \\
\{1, 4\} & = 0 \\
\{2, 1\} & = 2 \ 2 \land 1 \\
\{2, 2\} & = 2 \ast 2 \land \ast 2 \\
\{2, 3\} & = 2 \ast (2 \land \ast 3) \\
\{2, 4\} & = -2 \ast 2 \land \ast 4 \\
\{3, 1\} & = -2 \ast (\ast 3 \land 1) \\
\{3, 2\} & = 2 \ast (\ast 2 \land 1) \\
\{3, 3\} & = -2 \ast 3 \land \ast 3 \\
\{3, 4\} & = -2 \ast 3 \land \ast 4 \\
\{4, 1\} & = 0 \\
\{4, 2\} & = -2 \ast 4 \land \ast 2 \\
\{4, 3\} & = 0 \\
\{4, 4\} & = 2 \ast 4 \land \ast 4
\end{align*}
\]
\[ n = 5 \]

\[
[1, 1] = 2 \ U \land \ V \]
\[
[1, 2] = -2 \ast (U \land \ast \ 2) \phi \quad \{ U, V \} = 2 \ast (U \land \ast \ 2) \phi
\]
\[
[1, 3] = 2 \ U \land \ast \ 3 \quad \{ U, V \} = 2 \ast (U \land \ast \ 3) \phi
\]
\[
[1, 4] = -2 \ast (U \land \ast \ 4) \phi \quad \{ U, V \} = 2 \ast (U \land \ast \ 4) \phi
\]
\[
[1, 5] = 0 \quad \{ U, V \} = 2 \ast (U \land \ast \ 5) \phi
\]
\[
[2, 1] = -2 \ast (\ast \ 2 \land \ 1) \phi \quad \{ U, V \} = 2 \ast (\ast \ 2 \land \ 1) \phi
\]
\[
[2, 2] = \text{Com}(2, 2) \quad \{ U, V \} = 2 \ast (\ast \ 2 \land \ 2) \phi
\]
\[
[2, 3] = \ast \text{Com}(\ast \ 3, 2) \quad \{ U, V \} = 2 \ast (\ast \ 3 \land \ 2) \phi
\]
\[
[2, 4] = -2 \ast (\ast \ 4) \phi \quad \{ U, V \} = -2 \ast (\ast \ 4) \phi
\]
\[
[2, 5] = 0 \quad \{ U, V \} = -2 \ast (U \land \ast \ 5) \phi
\]
\[
[3, 1] = 2 \ U \land \ast \ 1 \quad \{ U, V \} = 2 \ast (\ast \ 1 \land \ 1) \phi
\]
\[
[3, 2] = \ast \text{Com}(\ast \ 3, 2) \quad \{ U, V \} = 2 \ast (\ast \ 3 \land \ 2) \phi
\]
\[
[3, 3] = \text{Com}(3, 3) \quad \{ U, V \} = 2 \ast (\ast \ 3 \land \ 3) \phi
\]
\[
[3, 4] = 2 \ast (U \land \ast \ 4) \phi \quad \{ U, V \} = 2 \ast (U \land \ast \ 4) \phi
\]
\[
[3, 5] = 0 \quad \{ U, V \} = -2 \ast (U \land \ast \ 5) \phi
\]
\[
[4, 1] = -2 \ast (\ast \ 4 \land \ 1) \phi \quad \{ U, V \} = 2 \ast (\ast \ 4 \land \ 1) \phi
\]
\[
[4, 2] = -2 \ast (\ast \ 4 \land \ 2) \phi \quad \{ U, V \} = -2 \ast (\ast \ 4 \land \ 2) \phi
\]
\[
[4, 3] = 2 \ast (U \land \ast \ 3) \phi \quad \{ U, V \} = -2 \ast (\ast \ 4 \land \ 3) \phi
\]
\[
[4, 4] = 2 \ast (U \land \ast \ 4) \phi \quad \{ U, V \} = 2 \ast (U \land \ast \ 4) \phi
\]
\[
[4, 5] = 0 \quad \{ U, V \} = 2 \ast (\ast \ 4 \land \ 5) \phi
\]
\[
[5, 1] = 0 \quad \{ U, V \} = 2 \ast (\ast \ 5 \land \ 1) \phi
\]
\[
[5, 2] = 0 \quad \{ U, V \} = -2 \ast (\ast \ 5 \land \ 2) \phi
\]
\[
[5, 3] = 0 \quad \{ U, V \} = -2 \ast (\ast \ 5 \land \ 3) \phi
\]
\[
[5, 4] = 0 \quad \{ U, V \} = 2 \ast (U \land \ast \ 4) \phi
\]
\[
[5, 5] = 0 \quad \{ U, V \} = 2 \ast (U \land \ast \ 5) \phi
\]
Proof. The proof is by direct calculation.

Note that using Theorem 3 and the formula

\[ \frac{k}{2} U V = \frac{1}{2} [U, V] + \frac{1}{2} \{U, V\} \]

we can express a Clifford algebra elements product \( \frac{k}{2} U V \) via the exterior product, the Hodge \( \star \) operation, and the Com operation.

**Theorem 4.** If \( k \) is an integer \( 1 \leq k \leq n - 1 \) and \( \frac{k}{2} U V = 0 \) for all \( U \in \mathcal{C}_k^p(p, q) \), then \( \frac{2}{V} = 0 \).

**Proof.** Assume that \( \frac{2}{V} \neq 0 \). Let us prove that for every \( 1 \leq k \leq n - 1 \) there exists \( U \in \mathcal{C}_k(p, q) \) such that \( \frac{k}{2} U V \neq 0 \). Let indices \( l < m \) be such that \( v_{lm} \neq 0 \) and \( \frac{2}{V} = v_{lm} e^{lm} \), where \( t = 0 \) or \( t = \sum_{r<s,(r,s)\neq(l,m)} v_{rs} e^{rs} \). Consider an element \( e^l e^{a_1 \ldots a_{k-1}} \in \mathcal{C}_k(p, q) \), \( k > 0 \), where \( \{m, l\} \cap \{a_1 \ldots a_{k-1}\} = \emptyset \). We have

\[
[e^l e^{a_1 \ldots a_{k-1}}, e^{lm}] = e^l e^{a_1 \ldots a_{k-1}} e^{lm} - e^{lm} e^l e^{a_1 \ldots a_{k-1}} = (-1)^{k-1} \eta^{ll} e^{a_1 \ldots a_{k-1}} e^{m} + \eta^{ll} e^{m} e^{a_1 \ldots a_{k-1}} = 2 \eta^{ll} e^{m} e^{a_1 \ldots a_{k-1}} \neq 0.
\]

This is true for \( k = 1, \ldots, n - 1 \). Further,

\[
\frac{2}{V} = [e^l e^{a_1 \ldots a_{k-1}}, v_{lm} e^{lm} + t] = [e^l e^{a_1 \ldots a_{k-1}}, v_{lm} e^{lm}] + [e^l e^{a_1 \ldots a_{k-1}}, t] = v_{lm} [e^l e^{a_1 \ldots a_{k-1}, e^{lm}}] + [e^l e^{a_1 \ldots a_{k-1}, t}]
\]

Let us prove that the first and the second summands at the right hand part of this identity are linear independent and, hence, \( \frac{2}{V} \neq 0 \). The second summand is a sum of commutators of the form \( v_{ij} [e^l e^{a_1 \ldots a_{k-1}, e^{ij}}] \). Suppose that \( l \neq i \) and \( l \neq j \). Then we get that terms \( [e^l e^{a_1 \ldots a_{k-1}, e^{ij}}] \) are linear independent with \( [e^l e^{a_1 \ldots a_{k-1}, e^{lm}}] = 2 \eta^{ll} e^{m} e^{a_1 \ldots a_{k-1}} \) as indices \( l, m, a_1, \ldots, a_{k-1} \) are mutually different.
Without lost of generality, suppose that $i = l$ and $j = c$, where $0 \leq c \leq n$, $c \neq l$. Then the terms $[e^{l} e^{a_1...a_{k-1}}, e^{l_m}]$ and $v_{ij}[e^{l} e^{a_1...a_{k-1}}, e^{j}]$ are linear independent if $e^{l} e^{a_1...a_{k-1}} e^{c}$ and $e^{m} e^{a_1...a_{k-1}}$ are linear independent. That means $c \neq m$ and the basis element $e^{lm}$ is already considered in the decomposition (9). Therefore we have prove that the commutator (9) is not equal to zero. This completes the proof of Theorem 4.

3 Generators contraction formulas.

A volume element. For $\mathfrak{C}^{\mathbb{F}}(p, q), \ p + q = n$ the basis element of rank $n$ is called the volume element and denoted by

\[ \ell = e^{1...n} = e^{1} \wedge ... \wedge e^{n} = \frac{1}{n!} \varepsilon_{a_1...a_n} e^{a_1} \ldots e^{a_n} = \frac{1}{n!} \varepsilon_{a_1...a_n} e^{a_1} \wedge ... \wedge e^{a_n}, \]

where $\varepsilon_{a_1...a_n}$ is completely antisymmetric and $\varepsilon_{1...n} = 1$. We have formulas

\[
\ell^2 = (-1)^{\frac{n(n-1)}{2}}\det \eta e, \\
\ell^* = (-1)^{\frac{n(n-1)}{2}} \ell, \\
\ell^k U = (-1)^{k(n+1)} U^k \ell.
\]

If $n$ is odd, then the volume element $\ell$ commutes with all elements of $\mathfrak{C}^{\mathbb{F}}(p, q)$. If $n$ is even, then $\ell$ commutes ($[\ell, U] = 0$) with all even elements from $\mathfrak{C}^{\mathbb{F}}_{\text{even}}(p, q)$ and anticommutes ($\{\ell, U\} = 0$) with all odd elements from $\mathfrak{C}^{\mathbb{F}}_{\text{odd}}(p, q)$.

For even $n$ the center of algebra $\mathfrak{C}^{\mathbb{F}}(p, q)$ coincides with $\mathfrak{C}^{\mathbb{F}}_{0}(p, q)$ and for odd $n$ the center of algebra $\mathfrak{C}^{\mathbb{F}}(p, q)$ coincides with $\mathfrak{C}^{\mathbb{F}}_{0}(p, q) \oplus \mathfrak{C}^{\mathbb{F}}_{n}(p, q)$.

Let $e^a$ be generators of $\mathfrak{C}(p, q), \ p + q = n$. Denote $e^{a} = \eta_{ab} e^{b}$.

**Theorem 5.** (Generators contraction formulas). For any $\ell \in \mathfrak{C}^{\mathbb{F}}(p, q)$

\[ e_{a} \ell^{k} e^{a} \ell^{k} e_{a} = (-1)^{k(n-2k)} U^{k}. \]  

**Proof.** Let us prove that $e_{a} e^{b_1} \ldots e^{b_k} e_{a} = (-1)^{k(n-2k)} e^{b_1} \ldots e^{b_k}$ for $b_1 < \ldots < b_k$. We use the method of mathematical induction. For $k = 0,$
using the relation $\eta_{ab} = \eta_{ba}$, we get

$$e_a e^a = e^b \eta_{ab} e^a = \frac{1}{2} \eta_{ab} e^b e^a + \frac{1}{2} \eta_{ab} e^b e^a$$

$$= \frac{1}{2} \eta_{ab} (e^b e^a + e^a e^b) = n.$$  

Hence, the formula (10) is valid for $k = 0$. Suppose that formula (10) is valid for some $k > 0$. Let us prove the validity of formula (10) for $k + 1$. We have

$$e_a e^{b_1} \ldots e^{b_k} e^{b_{k+1}} e^a = e_a e^{b_1} \ldots e^{b_k} (-e^a e^{b_{k+1}} + 2\eta_{ab_{k+1}})$$

$$= -e_a e^{b_1} \ldots e^{b_k} e^a e^{b_{k+1}} + 2\eta_{ab_{k+1}} e_a e^{b_1} \ldots e^{b_k}$$

$$= -(-1)^k (n - 2k) e^{b_1} \ldots e^{b_k} e^{b_{k+1}} + 2 e^{b_{k+1}} e^{b_1} \ldots e^{b_k}$$

$$= (-1)^k (-n + 2k + 2) e^{b_1} \ldots e^{b_{k+1}}$$

$$= (-1)^{k+1} (n - 2(k + 1)) e^{b_1} \ldots e^{b_{k+1}}$$

This completes the proof of Theorem 5.

Let us note some partial cases of formula (10).

- if $n$ is even and $k = n/2$, then $e_a U e^a e^{k} U e_a = 0$;
- $e^a e_a = n$;
- $e^a \ell e_a = (-1)^{n+1} n \ell$;
- for $n = 4$ we have

$$e^a (U + U + \bar{U} + \bar{U} + \bar{U}) e_a = 4 U - 2 \bar{U} + 2 \bar{U} - 4 \bar{U}.$$

4 Conjugation operators in Clifford algebras

Projection operators to vector subspaces $\mathcal{C}^\ell_F(p, q)$. Suppose $U \in \mathcal{C}^\ell_F(p, q)$ is written in the form (3). Then denote

$$\langle U \rangle_k = U = \sum_{a_1 < \ldots < a_k} u_{a_1 \ldots a_k} e^{a_1 \ldots a_k} \in \mathcal{C}^\ell_k(p, q).$$
From formulas (6), (7) we have
\[ \langle k U \rangle_{k+l} = U \wedge V. \]

Using the projection operator to the one dimensional vector subspace \( \mathcal{E}_0^\mathbb{F} \), we define an operation \( \text{Tr} : \mathcal{E}_0^\mathbb{F} \to \mathbb{F} \)
\[ \text{Tr}(U) = \langle U \rangle_0|_{e=1}. \]

We say that \( \text{Tr}(U) \) is the trace of an element \( U \). For example,
\[ \text{Tr}(ue + u_\alpha e^\alpha + \ldots) = u. \]

The following formulas give the main property of the Tr operation
\[ \text{Tr}(UV) = \text{Tr}(VU), \quad \text{Tr}([U, V]) = 0. \]

These formulas follow from Theorem 2.

**Theorem 6**. If an element \( B \in \mathcal{E}_0^\mathbb{F}(p, q) \) satisfies conditions
\[ [B, e^a] = C^a, \quad a = 1, \ldots, n \] (11)
for some given \( C^a \in \mathcal{E}_0^\mathbb{F}(p, q) \), \( \text{Tr} C^a = 0, a = 1, \ldots, n \), then
\[ B = \sum_{k=1}^{n} \frac{1}{n + (-1)^{k+1}(n - 2k)} \langle C^a e_a \rangle_k + \alpha e \quad n, \] (12)
\[ B = \sum_{k=1}^{n-1} \frac{1}{n + (-1)^{k+1}(n - 2k)} \langle C^a e_a \rangle_k + \alpha e + \beta \ell \quad n, \]
where \( \alpha, \beta \in \mathbb{F} \) and \( \ell = e^1 \ldots n \).

In other words, formulas (12) define \( B \) up to a term from the center of \( \mathcal{E}_0^\mathbb{F}(p, q) \).

**Proof.** Let us multiply left and right hand parts of (11) by \( e_a \) and sum with respect to \( a \). Then we get
\[ Be^a e_a - e^a Be_a = C^a e_a. \]
Now we use formulas (10)

\[ Be^ae_a = \sum_{k=0}^{n} n \langle B \rangle_k, \]

\[ e^aBe_a = \sum_{k=0}^{n} (-1)^k (n - 2k) \langle B \rangle_k. \]

\[ \]

**Operations of conjugation.** Consider the following operations of conjugation in \( \mathcal{O}(p, q) \):

\[ U^\wedge = U| e^a \rightarrow -e^a, \quad U^\sim = U| e^{a_1 \ldots a_r} \rightarrow e^{a_r \ldots a_1}, \quad \bar{U} = U| u^{a_1 \ldots a_r} \rightarrow \bar{u}^{a_1 \ldots a_r}. \]

In \( \bar{u}^{a_1 \ldots a_r} \) the bar means the complex conjugation. The operation \( U \rightarrow U^\wedge \) is called the *grade involution*. The superposition of conjugations \( U \rightarrow U^\sim \) and \( U \rightarrow \bar{U} \) gives the *Clifford conjugation* \( U \rightarrow U^* = \bar{U}^\sim \). We have

\[ U^\wedge = \sum_{k=0}^{n} (-1)^k \langle U \rangle_k = \langle U \rangle_0 - \langle U \rangle_1 + \langle U \rangle_2 - \langle U \rangle_3 + \langle U \rangle_4 - \ldots, \]

\[ U^\sim = \sum_{k=0}^{n} (-1)^{\frac{k(k-1)}{2}} \langle U \rangle_k = \langle U \rangle_0 + \langle U \rangle_1 - \langle U \rangle_2 - \langle U \rangle_3 + \langle U \rangle_4 + \ldots \]

and

\[ U^{\wedge \wedge} = U, \quad U^{\sim \sim} = U, \quad \bar{U} = U, \quad U^{* *} = U, \]

\[ (UV)^\wedge = U^\wedge V^\wedge, \quad (UV)^\sim = U^\sim V^\sim, \quad (UV)^* = V^* U^*, \]

\[ (U \wedge V)^\wedge = U^\wedge \wedge V^\wedge, \quad (U \wedge V)^\sim = U^\wedge \wedge V^\sim, \quad (U \wedge V)^* = V^* \wedge U^*. \]

We see that presented conjugation operations have the same properties with respect to the Clifford multiplication and with respect to the exterior multiplication.

Note that

\[ \text{Tr}(U^*) = \overline{\text{Tr}(U)}. \]

### 5 Unitary (Euclidean) spaces on Clifford algebras

Denote \( \mathcal{O}(n) = \mathcal{O}(n, 0) \). Consider the following operation \( \mathcal{O}(n) \times \mathcal{O}(n) \rightarrow \mathbb{F} \)

\[ (U, V) = \text{Tr}(U^* V). \quad (13) \]
Lemma. The operation $U, V \to (U, V)$ is a Hermitian (Euclidean) scalar product of elements of $\mathcal{C}^\mathbb{C}(n)$ ($\mathcal{C}^{\mathbb{R}}(n)$).

Proof. We must prove that the properties
\[
(U, V) = \overline{(V, U)}, \\
(U, \lambda V) = \lambda(U, V), \\
(U + V, W) = (U, W) + (V, W), \\
(U, U) > 0 \quad \text{for} \quad U \neq 0.
\] (14)
are valid for all $U, V, W \in \mathcal{C}^\mathbb{F}(n), \lambda \in \mathbb{F}$. The first three properties are evidently valid. To prove (14) it is sufficient to prove that basis (1) is orthonormal with respect to the operation $(\cdot, \cdot)$
\[
(e^{i_1\ldots i_k}, e^{j_1\ldots j_l}) = \begin{cases} 0, & \text{if } (i_1\ldots i_k) \neq (j_1\ldots j_l); \\ 1, & \text{if } (i_1\ldots i_k) = (j_1\ldots j_l). \end{cases}
\]
If multi-indices $i_1\ldots i_k$ and $j_1\ldots j_l$ have $r$ common indices, then
\[
e^{i_1\ldots i_k}e^{j_1\ldots j_l} \in \mathcal{C}^\mathbb{F}_{k+l-2r}(n),
\]
i.e., $(e^{i_1\ldots i_k}, e^{j_1\ldots j_l}) = 0$ for $k + l - 2r > 0$. We have $k + l - 2r = 0$ iff multi-indices $i_1\ldots i_k$ and $j_1\ldots j_l$ are identical. In this case
\[
(e^{i_1\ldots i_k}, e^{j_1\ldots j_l}) = \text{Tr}(e^{i_k}\ldots e^{i_1}e^{j_l}\ldots e^{j_1}) = \text{Tr}(e) = 1.
\] (15)
Hence, basis (1) is orthonormal and for $U \in \mathcal{C}^\mathbb{F}(n)$ we have
\[
(U, U) = \sum_{k=0}^{n} \sum_{a_1\ldots a_k} |u_{a_1\ldots a_k}|^2 > 0.
\] (16)
This completes the proof of the Lemma.

For Clifford algebras $\mathcal{C}^\mathbb{F}(p, q)$ with $q > 0$ property (14) is not valid. In this case we define an operation $\dagger : \mathcal{C}^\mathbb{F}(p, q) \to \mathcal{C}^\mathbb{F}(p, q)$ with the aid of the formulas
\[
(e^{i_1\ldots i_k})^\dagger = e_{i_k}\ldots e_{i_1}, \quad \lambda^\dagger = \overline{\lambda},
\] (17)
where $\lambda \in \mathbb{C}$ and $e_a = \eta_{ab}e^b$. We say that $\dagger$ is the operation of Hermitian conjugation of Clifford algebra elements. It is easy to see that
\[
(UV)^\dagger = V^\dagger U^\dagger, \quad U^{\dagger\dagger} = U.
\]
Now we can define the Hermitian (Euclidean) scalar product of Clifford algebra elements by the formula
\[(U, V) = \text{Tr}(U \dagger V).\]

In this case we have
\[(e^{i_1\ldots i_k}, e^{i_1\ldots i_k}) = \text{Tr}(e_{i_1} \ldots e_{i_k} e^{i_1} \ldots e^{i_k}) = \text{Tr}(e) = 1.\]
(no summation w.r.t. \(i_1, \ldots, i_k\)). Basis (1) of \(\mathcal{O}_p^q(p, q)\) is orthonormal with respect to this scalar product and property (16) is valid. For generators \(e^a\) formula (17) gives
\[(e^a)^\dagger = e^a \quad \text{for} \quad a = 1, \ldots, p; \quad (18)\]
\[(e^a)^\dagger = -e^a \quad \text{for} \quad a = p + 1, \ldots, n.\]

We present formulas (19), (20), which are equivalent to (17).

**Theorem 7.** Let \(p \geq 0, q \geq 0, n = p + q \geq 1\) be integer numbers. Let us define the operation of Hermitian conjugation \(\dagger : \mathcal{O}_p^q(p, q) \rightarrow \mathcal{O}_p^q(p, q)\), \(k = 0, \ldots, n\), with the aid of the following formulas:

\[
U^\dagger = \begin{cases} 
U^* & \text{for} \ (p, q) = (n, 0); \\
-e^n U^{*\wedge} e^n & \text{for} \ (p, q) = (n - 1, 1); \\
e^n e^{n-1} U^* e^{n-1} e^n & \text{for} \ (p, q) = (n - 2, 2); \\
-e^n e^{n-2} U^{*\wedge} e^{n-2} e^{n-1} e^n & \text{for} \ (p, q) = (n - 3, 3); \\
\ldots \\
(-1)^q e^n \ldots e^1 U^{*\sharp} e^1 \ldots e^n & \text{for} \ (p, q) = (0, n).
\end{cases}
\]  
(19)

If \(q\) is odd, then \(\sharp\) is the operation of grade involution \(\wedge\). Also, we may use the following equivalent formulas:

\[
U^\dagger = \begin{cases} 
e^n \ldots e^1 U^{*\sharp} e^1 \ldots e^n & \text{for} \ (p, q) = (n, 0); \\
\ldots \\
e^3 e^2 e^1 U^* e^1 e^2 e^3 & \text{for} \ (p, q) = (3, n - 3); \\
e^2 e^1 U^{*\wedge} e^1 e^2 & \text{for} \ (p, q) = (2, n - 2); \\
e^1 U^{*\sharp} e^1 & \text{for} \ (p, q) = (1, n - 1); \\
U^{*\wedge} & \text{for} \ (p, q) = (0, n).
\end{cases}
\]  
(20)

where \(\sharp\) is the operation \(\wedge\) for an even \(p\). In this case

\[U^{\dagger\dagger} = U, \quad (UV)^\dagger = V^\dagger U^\dagger, \quad (U + V)^\dagger = U^\dagger + V^\dagger,\]
\[(\lambda U)\dagger = \lambda U\dagger, \quad e\dagger = e \quad \text{for} \quad U, V \in \mathcal{C}_F^F(p, q), \lambda \in F\]

and the operation \((\cdot, \cdot) : \mathcal{C}_F^F(p, q) \times \mathcal{C}_F^F(p, q) \to F\)

\[(U, V) = \text{Tr}(U\dagger V) \quad (21)\]

gives a Hermitian (Euclidian for \(F = \mathbb{R}\)) scalar product in Clifford algebra \(\mathcal{C}_F^F(p, q)\).

**Proof.** It is sufficient to establish that formula (17) is equivalent to the formula

\[(e^{i_1\ldots i_k})\dagger = e^p\ldots e^1(e^{i_1\ldots i_k})^* e^1\ldots e^p,\]

where \(\#\) is \(\wedge\) for even \(p\) and to the formula

\[(e^{i_1\ldots i_k})\dagger = (-1)^q e^n\ldots e^{p+1}(e^{i_1\ldots i_k})^* e^{p+1}\ldots e^n,\]

where \(\#\) is \(\wedge\) for all odd \(q\). Let \(s\) be the number of common elements in sets \(\{i_1\ldots i_k\}\) and \(\{1\ldots p\}\). Using the identities \(\eta^{11} = \ldots = \eta^{pp} = 1, \quad \eta^{p+1p+1} = \ldots = \eta^{nn} = -1\), we transform the above formulas to the same form:

\[e^p\ldots e^1(e^{i_1\ldots i_k})^* e^1\ldots e^p = e_p\ldots e^1(e^{i_k}\ldots e^{i_1})^* e^1\ldots e^p = (-1)^{(p+1)k} e^p\ldots e^1 e^{i_k}\ldots e^{i_1} e^1\ldots e^p = (-1)^{(p+1)k} (-1)^{kp-s} e^{i_k} \ldots e^{i_1} = (-1)^{k-s} e^{i_k} \ldots e^{i_1}.

\[-1]^q e^n\ldots e^{p+1}(e^{i_1\ldots i_k})^* e^{p+1}\ldots e^n = (-1)^q e^n\ldots e^{p+1}(e^{i_k}\ldots e^{i_1})^* e^{p+1}\ldots e^n = (-1)^q (-1)^qket e^n\ldots e^{p+1} e^{i_k} \ldots e^{i_1} e^{p+1} \ldots e^n = (-1)^q (-1)^q (-1)^{kq-(k-s)} (-1)^q e^{i_k} \ldots e^{i_1} e^{p+1} \ldots e^n = (-1)^{k-s} e^{i_k} \ldots e^{i_1}.

\[e_{i_k} \ldots e_{i_1} = \eta_{i_1i_1} \ldots \eta_{i_ki_k} e^{i_k} \ldots e^{i_1} = (-1)^{k-s} 1^* e^{i_k} \ldots e^{i_1} = (-1)^{k-s} e^{i_k} \ldots e^{i_1}\]
(no summation over \(i_1 \ldots i_k\)). This completes the proof.

Note that for the Hermitian scalar product (21) we have
\[
(AU, V) = (U, A^\dagger V) \quad \forall A, U, V \in \mathcal{C}(p, q).
\]

Consider an element \(U \in \mathcal{C}(p, q)\). If \(U = U^\dagger\), then the element \(U\) is called Hermitian. If \(U = -U^\dagger\), then the element \(U\) is called antiHermitian. Any element \(\mathcal{C}(p, q)\) can be decomposed into the sum of Hermitian and antiHermitian elements
\[
U = \frac{1}{2}(U + U^\dagger) + \frac{1}{2}(U - U^\dagger).
\]

6 Hermitian idempotents and related structures

In what follows we consider only complex Clifford algebras \(\mathcal{C}(p, q) = \mathcal{C}(p, q)\). The element \(t \in \mathcal{C}(p, q)\) is said to be the Hermitian idempotent if
\[
t^2 = t, \quad t^\dagger = t.
\]

We say that two Hermitian idempotents \(t\) and \(\hat{t}\) are of the same type, if there exists a unitary element \(U \in \mathcal{C}(p, q)\), \(U^\dagger = U^{-1}\) such that
\[
\hat{t} = U^{-1}tU.
\]

It can be shown that for Clifford algebra \(\mathcal{C}(1, 3)\) there exist four types of Hermitian idempotents.

The set of clifford algebra elements
\[
I(t) = \{U \in \mathcal{C}(p, q) : U = Ut\}
\]
is called the left ideal of Clifford algebra (generated by the Hermitian idempotent \(t\)).

Let us define the set of Clifford algebra elements, which depend on a Hermitian idempotent \(t\)
\[
K(t) = \{U \in \mathcal{C}(p, q) : U = tUt\}.
\]

It is evident that \(K(t) \subseteq I(t)\). Note that \([U, t] = 0\) for \(U \in K(t)\).
A left ideal that doesn’t contain other left ideals except itself and the trivial ideal (generated by \( t = 0 \)), is called a \textit{minimal left ideal}. A Hermitian idempotent, which generates a minimal left ideal is called \textit{primitive}. The main property of a left ideal \( I(t) \): if \( U \in I(t) \) and \( V \in \mathcal{A}(p,q) \), then \( VU \in I(t) \).

The left ideal \( I(t) \) is a vector space. The Hermitian scalar product \( U, V \in I(t) \rightarrow (U,V) = \text{Tr}(U^\dagger V) \), gives us the structure of unitary space on \( I(t) \). Let us take an orthonormal basis \( \tau_1, \ldots, \tau_d \in I(t) \), where \( d = \dim I(t), \tau^l = \tau_l \)

\[ (\tau_k, \tau^l) = \delta^l_k, \quad k, l = 1, \ldots, d. \] (22)

In the sequel we consider matrix representations of Clifford algebra elements. Let \( \text{Mat}(d, \mathbb{C}) \) be the algebra of \( d \)-dimensional matrices with complex elements. A matrix \( Q \in \text{Mat}(d, \mathbb{C}) \) has elements \( q^l_k, k, l = 1, \ldots, d \), enumerated by two indices. The upper (first) index enumerates rows of the matrix and the lower (second) index enumerates columns of the matrix. The product \( P = QR \) of two matrices \( Q = \| q^l_k \|, R = \| r^l_k \| \in \text{Mat}(d, \mathbb{C}) \) is defined by the usual formula

\[ p^l_k = q^l_k r^l_k, \]

where at the right hand part we have summation over \( l \) (from 1 to \( d \)). With the aid of the basis (22) we may define three linear maps

\[ \gamma : \mathcal{A}(p,q) \rightarrow \text{Mat}(d, \mathbb{C}), \]
\[ \theta : K(t) \rightarrow \text{Mat}(d, \mathbb{C}), \]
\[ \rho : I(t) \rightarrow \mathbb{C}^d. \]

Here the \( d \)-dimensional complex vector space \( \mathbb{C}^d \) is considered as the set of complex matrices with one column and \( d \) raws. We define the map \( \gamma \) by the formula

\[ U \tau_k = \gamma(U)^l_k \tau_l, \] (23)

where \( U \in \mathcal{A}(p,q) \) and \( \gamma(U) = \| \gamma(U)^l_k \| \in \text{Mat}(d, \mathbb{C}) \). Therefore,

\[ \gamma(U)^l_k = (\tau^l_k, U \tau_l). \] (24)

We claim that

\[ \gamma(UV) = \gamma(U) \gamma(V). \]

Indeed,

\[ (UV) \tau_k = U(V \tau_k) = U \tau_l \gamma(V)^l_k = \gamma(U)^m_k \gamma(V)^l_m \tau_m. \]
Hence, we get a matrix representation of Clifford algebra elements. The dimension of this representation is equal to the dimension of left ideal $I(t)$. A minimal left ideal gives the matrix representation of Clifford algebra elements of the minimal dimension.

Denote

$$\gamma^a = \gamma(e^a), \quad 1 = \gamma(e),$$

where $1$ is the identity matrix. Relations for Clifford algebra generators $e^a e^b + e^b e^a = \eta^{ab} e$ give relations for matrices

$$\gamma^a \gamma^b + \gamma^b \gamma^a = \eta^{ab} 1.$$

Suppose that a representation $\gamma : \mathcal{O}(p, q) \to \text{Mat}(d, \mathbb{C})$ is generated by an orthonormal basis $\tau_1, \ldots, \tau_d$ of a left ideal with the aid of formula (23). Then

$$\gamma(U^\dagger) = \gamma(U)^\dagger, \quad \forall U \in \mathcal{O}(p, q), \quad (25)$$

where $U^\dagger$ is the Hermitian conjugated element of Clifford algebra $\mathcal{O}(p, q)$ (see Theorem 7) and $\gamma(U)^\dagger$ is the Hermitian conjugated matrix. To prove this fact we rewrite (24) in the form

$$\gamma(U)_{kl} = (\tau_k, U \tau_l), \quad (26)$$

numerating elements of the matrix $\gamma(U)$ by two lower indices. The operation of Hermitian scalar product

$$(A, B) = \text{Tr}(A^\dagger B), \quad A, B \in \mathcal{O}(p, q)$$

has properties

$$(A, UB) = (U^\dagger A, B), \quad (A, B) = (B, A).$$

Therefore, from (26) we get

$$\gamma(U)_{kl} = (U^\dagger \tau_k, \tau_l), \quad \overline{\gamma(U)}_{kl} = (\tau_l, U^\dagger \tau_k).$$

Comparing the last formula with formula (26), we obtain identity (25).

Let us define the map $\rho : I(t) \to \mathbb{C}^d$. If we take the decomposition of a left ideal element by the basis

$$\Omega = \omega^k \tau_k \in I(t),$$

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then $\rho(\Omega)$ is the column

$$\rho(\Omega) = (\omega^1 \ldots \omega^d)^T,$$

where $A^T$ is the transposed matrix. In particular, we have

$$\rho(\tau_k) = (0 \ldots 1 \ldots 0)^T$$

with only 1 on the $k$-th place of the column.

If $U \in \mathcal{A}(p,q)$ and $\Omega \in I(t)$, then

$$U\Omega = U\omega^n\tau_n = \omega^n\gamma(U)^k_{n}\tau_k \in I(t).$$

That means

$$\rho(U\Omega) = \gamma(U)\rho(\Omega).$$

Let $V \in K(t)$. Now we may define the map $\theta : K(t) \to \text{Mat}(d, \mathbb{C})$ by the following formula:

$$\tau_n V = \theta(V)^k_{n}\tau_k.$$  

From this formula we get

$$\theta(V)^k_{n} = (\tau^k, \tau_n V).$$

If $V \in K(t)$ and $\Omega \in I(t)$, then

$$\Omega V = \omega^n\tau_n V = \omega^n\theta(V)^k_{n}\tau_k \in I(t).$$

Therefore,

$$\rho(\Omega V) = \theta(V)\rho(\Omega).$$

If $U, V \in K(t)$, $\Omega \in I(t)$, then

$$\rho(\Omega UV) = \theta(V)\theta(U)\rho(\Omega) = \theta(UV)\rho(\Omega).$$

Thus we have

$$\theta(UV) = \theta(V)\theta(U),$$

i.e., at the right hand part we see multiplies in reverse order and so the map $\theta$ in an antirepresentation of elements of $K(t)$.

If $U \in \mathcal{A}(p,q)$, $V \in K(t)$, $\Omega \in I(t)$, then $U\Omega \in I(t)$, $\Omega V \in I(t)$ and

$$\rho(U\Omega V) = \gamma(U)\rho(\Omega V) = \gamma(U)\theta(V)\rho(\Omega),$$

$$\rho(U\Omega V) = \theta(V)\rho(U\Omega) = \theta(V)\gamma(U)\rho(\Omega).$$

It now follows that for all $U \in \mathcal{A}(p,q)$, $V \in K(t)$

$$[\gamma(U), \theta(V)] = 0.$$
7 Normal representations of Clifford algebra elements

Consider the Clifford algebra $\mathcal{C}(p, q)$, $p + q = n$. It is well known \[7\], that Clifford algebra elements can be represented by complex matrices of the minimal dimension $2\left\lceil \frac{n+1}{2} \right\rceil$. For odd $n = 2k + 1$ Clifford algebra elements can be represented by $2^{k+1}$ dimensional block-diagonal complex matrices with two blocks of dimension $2^k$ on the diagonal (other elements are zero).

Let $\gamma : \mathcal{C}(p, q) \to \text{Mat}(d, \mathbb{C})$ be a representation of Clifford algebra elements that satisfy the following conditions:

- $d = 2^{\left\lceil \frac{n+1}{2} \right\rceil}$.
- $\gamma(U)\dagger = \gamma(U\dagger)$ for all $U \in \mathcal{C}(p, q)$, where $\gamma(U)\dagger$ is the Hermitian conjugated matrix and $U\dagger$ is the Hermitian conjugated element of the Clifford algebra (see Theorem 7).

Then $\gamma$ is called a normal representation of Clifford algebra elements.

In this section for any Clifford algebra $\mathcal{C}(p, q)$, $p + q = n$ we give some (standard) Hermitian idempotent $t$ and the corresponding (standard) orthonormal basis of the left ideal $I(t)$. This basis defines the normal representation for $\mathcal{C}(p, q)$.

Consider generators $e^1, \ldots, e^p, e^{p+1}, \ldots, e^n$ of $\mathcal{C}(p, q)$, $p + q = n$ such that $e^a e^b = -e^b e^a$ for $a \neq b$ and

$$(e^a)^2 = e \quad \text{for} \quad a = 1, \ldots, p; \quad (e^a)^2 = -e \quad \text{for} \quad a = p + 1, \ldots, n.$$  

For $n = 1$ we take $t = e$ and for $n > 1$ we take

$$t = \frac{1}{2} (e + i^a e^1) \prod_{k=1}^{[\frac{n}{2}]-1} \frac{1}{2} (e + i^{b_k} e^{2k} e^{2k+1}) \in \mathcal{C}(p, q), \quad (27)$$

where

$$a = \begin{cases} 0 & \text{for } p \neq 0; \\ 1 & \text{for } p = 0, \end{cases} \quad b_k = \begin{cases} 0 & \text{for } 2k = p; \\ 1 & \text{for } 2k \neq p. \end{cases}$$

In product (27) all terms are commute. Using formula (18), we get

$$\left(\frac{1}{2}(e + i^a e^1)\right)^2 = \frac{1}{2}(e + i^a e^1) = (\frac{1}{2}(e + i^a e^1))\dagger,$$

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\[
\left(\frac{1}{2}(e + i b_k e^{2k} e^{2k+1})\right)^2 = \frac{1}{2}(e + i b_k e^{2k} e^{2k+1}) = \left(\frac{1}{2}(e + i b_k e^{2k} e^{2k+1})\right)^\dagger.
\]

Therefore, \(t\) is a Hermitian idempotent

\[t^2 = t, \quad t^\dagger = t.\]

It can be shown that for even \(n\) the idempotent \(t\) is primitive.

Let us discuss some notations. If \(e_1, \ldots, e_m\) are generators of the Clifford algebra \(\mathcal{A}(u, v), u + v = m\), then for this Clifford algebra we may use the notation

\[
\mathcal{A}(e_1, \ldots, e_m) = \mathcal{A}_{\text{even}}(e_1, \ldots, e_m) \oplus \mathcal{A}_{\text{odd}}(e_1, \ldots, e_m). \tag{28}
\]

The following elements:

\[e, e^a, e^{a_1} e^{a_2}, \ldots, e^1 \ldots e^n, \quad a_1 < a_2 < \ldots \tag{29}\]

are basis elements of \(\mathcal{A}(e_1, \ldots, e_n)\).

Now we may denote basis elements \([29]\) by \(c_k, k = 1, \ldots, 2^m\).

Consider the Clifford algebra

\[Q = \mathcal{A}(e^2, e^4, \ldots, e^n) \quad \text{for even } n,\]
\[Q = \mathcal{A}(e^2, e^4, \ldots, e^{n-1}, e^n) \quad \text{for odd } n.\]

The complex dimension of \(Q\) is equal to \(2^{\frac{n+1}{2}}\). Let \(c_k, k = 1, \ldots, 2^{\frac{n+1}{2}}\) be basis elements of the Clifford algebra \(Q\).

For even \(n\) the Clifford algebra \(Q\) is defined by the set of generators \(e^a\) with even indices \(2 \leq a \leq n\). For odd \(n\) the Clifford algebra \(Q\) is defined by the set of generators \(e^a\) that consists of all generators with even indices \(2 \leq a \leq n - 1\) and one generator \(e^n\). The Clifford algebra \(Q\) is a subalgebra of \(\mathcal{A}(p, q)\). The dimension of the algebra \(Q\) is equal to \(2^{\frac{(n+1)}{2}}\). Let us denote by \(c_k, k = 1, \ldots, 2^{\frac{(n+1)}{2}}\) the basis elements of \(Q\). Suppose that in the sequence \(c_k\) at first we take \(2^{\frac{n+1}{2}}-1\) even elements

\[e, e^2 e^4, \ldots \tag{30}\]

and at second we take \(2^{\frac{n+1}{2}}-1\) odd elements

\[e^2, e^4, \ldots \tag{31}\]
Theorem 8. The following elements of the left ideal \( I(t) \):
\[
\tau_k = (\sqrt{2})^{[n/2]} c_k t, \quad k = 1, \ldots, 2^{[n+1/2]}
\]
form an orthonormal basis of \( I(t) \).

Proof. We have
\[
(\tau_k, \tau_l) = \text{Tr}(\tau_k^\dagger \tau_l) = (\sqrt{2})^{2[n/2]} \text{Tr}(t^\dagger c_k^\dagger c_l t) = (\sqrt{2})^{2[n/2]} \text{Tr}(c_k^\dagger c_l t).
\]
Let us show that
\[
\begin{cases}
c_k^\dagger c_l = e & \text{for } k = l; \\
\text{Tr}(c_k^\dagger c_l t) = 0 & \text{for } k \neq l.
\end{cases}
\]
If \( c_k = e^{a_1} \cdots e^{a_r} \), then, according to formula (17), \( c_k^\dagger = c_k \) for even \( n \), \( c_k^\dagger = \pm e^{a_1} \cdots e^{a_s} e^{a_{r+1}} \cdots e^{a_r} \) for odd \( n \), and
\[
(\tau_k, \tau_k) = (\sqrt{2})^{2[n/2]} \text{Tr} t = 1, \quad k = 1, \ldots, 2^{[(n+1)/2]}
\]
Consider the case \( k \neq l \). We see that
\[
c_k^\dagger c_l = \pm e^{a_1} \cdots e^{a_s} \quad \text{for even } n,
\]
\[
c_k^\dagger c_l = \pm e^{a_1} \cdots e^{a_s}, \quad \text{or} \quad c_k^\dagger c_l = \pm e^{a_1} \cdots e^{a_s} e^n \quad \text{for odd } n,
\]
where \( a_1 < \ldots < a_s \) and \( a_1, \ldots, a_s \) are even indices. The right hand part of formula (32) contains, at least, one multiplier and, hence, \( \text{Tr}(c_k^\dagger c_l) = 0 \).

Let us write down the idempotent \( t \) from (27) in the form
\[
t = 2^{-[n/2]} e + \sum_{r=1}^{n} \sum_{b_1 < \ldots < b_r} \lambda_{b_1 \ldots b_r} e^{b_1} \ldots e^{b_r},
\]
where \( \lambda_{b_1 \ldots b_r} \in \mathbb{C} \) and every term \( \lambda_{b_1 \ldots b_r} e^{b_1} \ldots e^{b_r} \) contains, at least, one generator \( e^b \) with odd index (for odd \( n \) the idempotent \( t \) doesn’t contain the generator \( e^n \)). We get
\[
c_k^\dagger c_l t = 2^{-[n/2]} c_k^\dagger c_l + (\sum_{r=1}^{n} \sum_{b_1 < \ldots < b_r} \lambda_{b_1 \ldots b_r} c_k^\dagger c_l e^{b_1} \ldots e^{b_r}).
\]
If we write the expression in brackets as a sum of the basis elements of $\mathcal{A}(e^1, \ldots, e^n)$, then every addend contains as a multiplier, at least, one generator $e^b$ with odd index. That means the trace of every addend is equal to zero and
\[ \text{Tr}(c_k^l c_l) = 0, \quad \text{for} \quad k \neq l. \]
This completes the proof of the Theorem.

Thus with the aid of the Hermitian idempotent $t$ and the orthonormal basis $\tau_k$ of the left ideal $I(t)$ we give the normal representation $U \rightarrow U$ of Clifford algebra elements with matrices form $\text{Mat}(2^{\frac{n+1}{2}}, \mathbb{C})$. This representation gives us possibility to transfer all main notions of the matrix algebra to the Clifford algebra.

For $U \in \mathcal{A}(p, q)$ the complex number
\[ \det U := \det U \in \mathbb{C}. \]
is called the determinant of $U$. It can be shown that $\det U \in \mathbb{R}$ for $U \in \mathcal{A}^\mathbb{R}(p, q)$. A complex number $\lambda \in \mathbb{C}$ such that
\[ \det(U - \lambda e) = 0 \]
is called an eigen-value of an element $U \in \mathcal{A}(p, q)$. The set of all eigen-value of an element $U \in \mathcal{A}(p, q)$ is called the spectrum of $U$. The spectrum of an element $U \in \mathcal{A}(p, q)$ consists of $2^{\frac{n+1}{2}}$ complex numbers. A Hermitian element $U = U^\dagger \in \mathcal{A}(p, q)$ has a real spectrum.

If $\lambda \in \mathbb{C}$ is an eigen-value of an element $U \in \mathcal{A}(p, q)$ and an element $V \in \mathcal{A}(p, q)$, $V \neq 0$ satisfies the equality
\[ (U - \lambda e)V = 0, \]
then $V$ is called a left eigen-element of $U$. The left eigen-element $V$ belongs to some left ideal of the Clifford algebra $\mathcal{A}(p, q)$ and this left ideal is not coincides with $\mathcal{A}(p, q)$.

Consider matrix representations of Clifford algebra generators for small dimensions $n = p + q$.

For $n = 2$ we take the idempotent $t = 1/2(e + e^1)$ for the signatures $(p, q) = (2, 0)$ and $(1, 1)$. And we take $t = 1/2(e + ie^1)$ for the signature $(0, 2)$. In this case we have the following basis of the left ideal $I(t)$: $\tau_1 =$
\[ \sqrt{2et}, \tau_2 = \sqrt{2e^2t}. \] This basis gives us the following matrix representations of Clifford algebra generators.

For \((p, q) = (2, 0)\) we have
\[
\varepsilon^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \varepsilon^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
\]

For \((p, q) = (1, 1)\) we have
\[
\varepsilon^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \varepsilon^2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.
\]

For \((p, q) = (0, 2)\) we have
\[
\varepsilon^1 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}, \quad \varepsilon^2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.
\]

For \(n = 4\) we take \(t = 1/2(e + e^1)1/2(e + ie^{23})\) for the signatures \((4,0), (1,3), (3,1); t = 1/2(e + e^1)1/2(e + e^{23})\) for the signature \((2,2); t = 1/2(e + ie^1)1/2(e + ie^{23})\) for the signature \((0,4).\) We have the following basis of the left ideal \(I(t):\)
\[ \tau_1 = 2et, \quad \tau_2 = 2e^{24}t, \quad \tau_3 = 2e^2t, \quad \tau_4 = 2e^4t. \]

For \((p, q) = (4, 0)\) we have
\[
\varepsilon^1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \varepsilon^2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},
\]
\[
\varepsilon^3 = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \\ -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}, \quad \varepsilon^4 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.
\]

For \((p, q) = (3, 1)\) we have
\[
\varepsilon^1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \varepsilon^2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},
\]
\[ e^3 = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \\ -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}, \quad e^4 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \]

For \((p, q) = (2, 2)\) we have

\[ e^1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad e^2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \]

\[ e^3 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad e^4 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \]

For \((p, q) = (1, 3)\) we have

\[ e^1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad e^2 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \]

\[ e^3 = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}, \quad e^4 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \]

For \((p, q) = (0, 4)\) we have

\[ e^1 = \begin{pmatrix} -i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \end{pmatrix}, \quad e^2 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \]

\[ e^3 = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}, \quad e^4 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \]
If for the case \((p, q) = (1, 3)\) we take the following basis of the left ideal \(I(t)\): \(\tau_1 = -2et, \tau_2 = 2e^{24}t, \tau_3 = 2e^4t, \tau_4 = 2e^2t\), then we get the well known Dirac representation of generators

\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1
\end{bmatrix}, \quad \begin{bmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & -1 & 0 & 0 \\
-1 & 0 & 0 & 0
\end{bmatrix}, \quad \begin{bmatrix}
0 & 0 & 0 & -i \\
0 & 0 & i & 0 \\
0 & i & 0 & 0 \\
-i & 0 & 0 & 0
\end{bmatrix}, \quad \begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1 \\
-1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{bmatrix}
\]

Now let us consider matrix representations of Clifford algebra generators for odd \(n = p + q = 1, 3, 5\). We get block-diagonal matrices.

For \((p, q) = (1, 0)\) we have

\[t = e, \tau_1 = (1/\sqrt{2})(e + e^1)t, \tau_2 = (1/\sqrt{2})(e - e^1)t\]

and

\[e^1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \]

For \((p, q) = (0, 1)\) we have

\[t = e, \tau_1 = (1/\sqrt{2})(e - ie^3)t, \tau_2 = (1/\sqrt{2})(e + ie^1)t\]

and

\[e^1 = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \]

For \((p, q) = (3, 0)\) we have

\[t = 1/2(e + e^1), \tau_1 = (e - ie^{23})t, \tau_2 = (e^2 - ie^3)t, \tau_3 = (e^2 + ie^3)t, \tau_4 = (e + ie^{23})t\]

and

\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}, \quad \begin{bmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0
\end{bmatrix}, \quad \begin{bmatrix}
0 & -i & 0 & 0 \\
i & 0 & 0 & 0 \\
0 & 0 & 0 & -i \\
0 & 0 & i & 0
\end{bmatrix}.
\]
For \((p, q) = (2, 1)\) we have
\[
t = \frac{1}{2}(e + e^1), \quad \tau_1 = (e + e^{23})t, \quad \tau_2 = (e^2 + e^3)t, \quad \tau_3 = (e^2 - e^3)t, \quad \tau_4 = (e - e^{23})t
\]
and
\[
e^1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad e^2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad e^3 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.
\]

For \((p, q) = (1, 2)\) we have
\[
t = \frac{1}{2}(e + e^1), \quad \tau_1 = (e - ie^{23})t, \quad \tau_2 = (e^2 + ie^3)t, \quad \tau_3 = (e^2 - ie^3)t, \quad \tau_4 = (e + ie^{23})t
\]
and
\[
e^1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad e^2 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & i \end{pmatrix}, \quad e^3 = \begin{pmatrix} 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \end{pmatrix}.
\]

For \((p, q) = (0, 3)\) we have
\[
t = \frac{1}{2}(e + ie^1), \quad \tau_1 = (e - ie^{23})t, \quad \tau_2 = (e^2 + ie^3)t, \quad \tau_3 = (e^2 - ie^3)t, \quad \tau_4 = (e + ie^{23})t
\]
and
\[
e^1 = \begin{pmatrix} -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix}, \quad e^2 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad e^3 = \begin{pmatrix} 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \end{pmatrix}.
\]

For \((p, q) = (5, 0)\) we have
\[
t = \frac{1}{2}(e + e^1)\frac{1}{2}(e + ie^{23}), \quad \tau_1 = \sqrt{2}(e - ie^{45})t, \quad \tau_2 = \sqrt{2}(e^{24} - ie^{25})t, \quad \tau_3 = \sqrt{2}(e^2 - ie^{245})t,
\]
\[
\tau_4 = \sqrt{2}(e^4 - ie^5)t, \quad \tau_5 = \sqrt{2}(e^4 + ie^5)t, \quad \tau_6 = \sqrt{2}(e^2 + ie^{245})t,
\]
\[
\tau_7 = \sqrt{2}(e^{24} + ie^{25})t, \quad \tau_8 = \sqrt{2}(e + ie^{45})t
\]

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and

\[ e^1 = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix},
\]

\[ e^2 = \begin{pmatrix}
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0
\end{pmatrix},
\]

\[ e^3 = \begin{pmatrix}
0 & 0 & i & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -i & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & i & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & i & 0 & 0 \\
0 & 0 & 0 & 0 & -i & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -i & 0 & 0 \\
0 & 0 & 0 & 0 & i & 0 & 0 & 0
\end{pmatrix},
\]

\[ e^4 = \begin{pmatrix}
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{pmatrix},
\]

\[ e^5 = \begin{pmatrix}
0 & 0 & 0 & -i & 0 & 0 & 0 & 0 \\
0 & 0 & -i & 0 & 0 & 0 & 0 & 0 \\
0 & i & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -i & 0 \\
0 & 0 & 0 & 0 & 0 & -i & 0 & 0 \\
0 & 0 & 0 & 0 & i & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & i & 0 & 0 & 0
\end{pmatrix}.
\]

For \((p, q) = (4, 1)\) we have

\[ t = \frac{1}{2}(e + e^3)/2(e + ie^{23}), \quad \tau_1 = \sqrt{2}(e - e^{45})t, \quad \tau_2 = \sqrt{2}(e^{24} - e^{25})t, \quad \tau_3 = \sqrt{2}(e^2 - e^{245})t, \]

\[ \tau_4 = \sqrt{2}(e^4 - e^5)t, \quad \tau_5 = \sqrt{2}(e^4 + e^5)t, \quad \tau_6 = \sqrt{2}(e^2 + e^{245})t, \]

\[ \tau_7 = \sqrt{2}(e^{24} + e^{25})t, \quad \tau_8 = \sqrt{2}(e + e^{45})t. \]

In this case the first four generators have the same representation as in the previous case of signature \((5, 0)\) and the last generator has the following
representation

\[
e^5 = \begin{pmatrix}
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0
\end{pmatrix}.
\]

For \((p, q) = (3, 2)\) we have

\[
t = 1/2(e + e^1)1/2(e + ie^{23}), \quad \tau_1 = \sqrt{2}(e - ie^{45})t, \quad \tau_2 = \sqrt{2}(e^2 - ie^{245})t, \quad \tau_3 = \sqrt{2}(e^4 + ie^5)t,
\]

\[
\tau_4 = \sqrt{2}(e^{24} + ie^{25})t, \quad \tau_5 = \sqrt{2}(e^{24} - ie^{25})t, \quad \tau_6 = \sqrt{2}(e^4 - ie^5)t,
\]

\[
\tau_7 = \sqrt{2}(e^2 + ie^{45})t, \quad \tau_8 = \sqrt{2}(e + ie^{45})t
\]

and

\[
e^1 = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}, \quad e^2 = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix},
\]

\[
e^3 = \begin{pmatrix}
0 & i & 0 & 0 & 0 & 0 & 0 & 0 \\
-i & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & i & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -i & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -i & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & i & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -i & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & i & 0
\end{pmatrix}, \quad e^4 = \begin{pmatrix}
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0
\end{pmatrix}.
\]
$$e^5 = \begin{pmatrix}
0 & 0 & -i & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & i & 0 & 0 & 0 & 0 & 0 \\
-i & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & i & 0 & 0 & 0 & 0 & -i & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & i & 0 \\
0 & 0 & 0 & 0 & -i & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & i & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & i & 0
\end{pmatrix}.$$  

For \((p, q) = (2, 3)\) we take \(t = 1/2(e + e_1)1/2(e + e_23)\) and we have the same basis as in the case of signature \((3, 2)\). Representations of all generators, except \(e^3\), have the same form as for the signature \((3, 2)\) and

$$e^3 = \begin{pmatrix}
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0
\end{pmatrix}.$$  

For \((p, q) = (1, 4)\) we take \(t = 1/2(e + e_1)1/2(e + ie^{23})\). We have the same basis as for \((3, 2)\) and

$$e^1 = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}, \quad e^2 = \begin{pmatrix}
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0
\end{pmatrix}.$$
Consider the set of Clifford algebra elements 
\[ U_{\mathbb{C}^\ell}(p, q) = \{ U \in \mathbb{C}^\ell(p, q) : U^\dagger U = e \} \].

This set is closed with respect to the Clifford product and forms a group (Lie group), which is called the unitary group of Clifford algebra.
Let \( t \in \mathcal{C}(p, q) \) be a Hermitian idempotent, \( I(t) \) be the left ideal, and \( \tau_k \) be the orthonormal basis of \( I(t) \). This basis gives us the matrix representation of Clifford algebra elements (see Theorem 8)

\[
\gamma : \mathcal{C}(p, q) \rightarrow \text{Mat}(2^{\frac{n+1}{2}}, \mathbb{C})
\]
such that

\[
(\gamma(U))^\dagger = \gamma(U^\dagger), \quad \forall U \in \mathcal{C}(p, q).
\]

In particular, we may take the standard basis and the standard matrix representation from the previous section.

Let us take an element \( U \in \mathcal{U}\mathcal{C}(p, q) \) and the matrix \( \gamma(U) \in \text{Mat}(2^{\frac{n+1}{2}}, \mathbb{C}) \)

\[
U\tau_k = \gamma(U)_k^l \tau_l.
\]  

(33)

The properties \( U^\dagger U = e \) and \( (\gamma(U))^\dagger = \gamma(U^\dagger) \) leads to the property \( \gamma(U)^\dagger \gamma(U) = 1 \), where \( 1 \) is the identity matrix of dimension \( 2^{\frac{n+1}{2}} \). That means \( \gamma(U) \) is a unitary matrix.

For even \( n = p + q \) formula (33) establishes the isomorphism

\[
\mathcal{U}\mathcal{C}(p, q) \sim \mathcal{U}(2^n),
\]

where \( \mathcal{U}(2^n) \) is the group of unitary matrices of dimension \( 2^n \).

For odd \( n = p + q \) formula (33) establishes the isomorphism

\[
\mathcal{U}\mathcal{C}(p, q) \sim \mathcal{U}(2^{\frac{n-1}{2}}) \oplus \mathcal{U}(2^{\frac{n-1}{2}}),
\]

where \( \mathcal{U}(2^{\frac{n-1}{2}}) \oplus \mathcal{U}(2^{\frac{n-1}{2}}) \) is the set of block-diagonal matrices \( \text{diag}(W, V) \) and \( W, V \in \mathcal{U}(2^{\frac{n-1}{2}}) \).

With the aid of an element \( U \in \mathcal{U}\mathcal{C}(p, q) \) we may define a new orthonormal basis of \( I(t) \)

\[
\hat{\tau}_k = U\tau_k = \gamma(U)_k^l \tau_l,
\]

\[
(\hat{\tau}_k, \hat{\tau}_l) = \text{Tr}((U\tau_k)^\dagger U\tau_l) = \text{Tr}(\tau_l^k \tau_l^l) = \delta_k^l.
\]

The basis \( \hat{\tau}_k = \hat{\tau}^k \) defines the new matrix representation \( \hat{\gamma} : \mathcal{C}(p, q) \rightarrow \text{Mat}(2^{\frac{n+1}{2}}, \mathbb{C}) \)

\[
V\hat{\tau}_k = \hat{\gamma}(V)_k^l \hat{\tau}_l.
\]

The representations \( \gamma(V) \) and \( \hat{\gamma}(V) \) are connected with each other by the formula

\[
\hat{\gamma}(V) = \gamma(U)^{-1} \gamma(V) \gamma(U).
\]
If we replace the orthonormal basis \( \tau_k \) of left ideal \( I(t) \) by the orthonormal basis \( \tilde{\tau}_k = \tau_k U^{-1} \) of left ideal \( I(UtU^{-1}) \), then we get the matrix representation \( V \rightarrow \tilde{\gamma}(V) \)

\[
V \tilde{\tau}_k = V \tau_k U^{-1} = \tilde{\gamma}(V)_k \tilde{\eta}_l = \tilde{\gamma}(V)_k \tau_l U^{-1}.
\]

Comparing this formula with formula (33), we see that

\[
\tilde{\gamma}(V) = \gamma(V).
\]

Finally, if we replace the orthonormal basis \( \tau_k \) of left ideal \( I(t) \) by the orthonormal basis \( \hat{\tau}_k = U \tau_k U^{-1} \) of left ideal \( I(UtU^{-1}) \), then we get the matrix representation \( V \rightarrow \hat{\gamma}(V) \)

\[
\hat{\gamma}(V) = \gamma(U^{-1}) \gamma(V) \gamma(U).
\]

If the initial matrix representation \( \gamma : \mathcal{O}(p, q) \rightarrow \text{Mat}(2^{[\frac{n}{2}]}, \mathbb{C}) \) is normal, then the matrix representations \( \tilde{\gamma}, \hat{\gamma}, \hat{\gamma} \) are also normal.

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