On the minimal positive standardizer of a parabolic subgroup of an Artin-Tits group

María Cumplido
Université Rennes 1, Universidad de Sevilla

December 22, 2017

Abstract

The minimal standardizer of a curve system on a punctured disk is the minimal positive braid that transforms it into a system formed only by round curves. We give an algorithm to compute it in a geometrical way. Then, we generalize this problem algebraically to parabolic subgroups of Artin-Tits groups of spherical type and we show that, to compute the minimal standardizer of a parabolic subgroup, it suffices to compute the \( p_n \)-normal form of a particular central element.

1 Introduction

Let \( D \) be the disk in \( \mathbb{C} \) with diameter the real segment \([0, n + 1]\) and let \( D_n = D \setminus \{1, \ldots, n\} \) be the \( n \)-punctured disk. The \( n \)-strand braid group, \( B_n \), can be identified with the mapping class group of \( D_n \) relative to \( \partial D_n \). \( B_n \) acts on the right on the set of isotopy classes of simple closed curves in the interior of \( D_n \). The result of the action of a braid \( \alpha \) on the isotopy class \( C \) of a curve \( C \) will be denoted by \( C^\alpha \) and it is represented by the image of the curve \( C \) under any automorphism of \( D_n \) representing \( \alpha \). We say that a curve is non-degenerate if it is not homotopic to a puncture, to a point or to the boundary of \( D_n \), in other words, if it encloses more than one and less than \( n \) punctures. A curve system is a collection of isotopy classes of disjoint non-degenerate simple closed curves, pairwise non-isotopic.

Curve systems are very important as they allow to use geometric tools to study braids. From Nielsen-Thurston theory (Thurston, 1988), every braid can be decomposed along a curve system, so that each component becomes either periodic or pseudo-Anosov. The simplest possible scenario appears when the curve is standard:

Definition 1. A simple closed curve in \( D_n \) is called standard if it is isotopic to a circle centered at the real axis. A curve system containing only isotopy classes of standard curves is called standard.

Every curve system can be transformed into a standard one by the action of a braid, as we shall see. Let \( B_n^+ \) be the submonoid of \( B_n \) of positive braids, generated by \( \sigma_1, \ldots, \sigma_{n-1} \) (Artin, 1947). We can define a partial order \( \preceq \) on \( B_n \), called prefix order, as follows: for \( \alpha, \beta \in B_n \), \( \alpha \preceq \beta \) if there is \( \gamma \in B_n^+ \) such that \( \alpha \gamma = \beta \). This partial order endows \( B_n \) with a lattice structure, i.e., for each pair \( \alpha, \beta \in B_n \), their gcd \( \alpha \wedge \beta \) and their lcm \( \alpha \lor \beta \) with respect to \( \preceq \) exist and are unique. Symmetrically, we can define the suffix order \( \succeq \) as follows: for \( \alpha, \beta \in B_n \), \( \beta \succeq \alpha \) if there is \( \gamma \in B_n^+ \) such that \( \gamma \alpha = \beta \). We will focus on \( B_n \) as a lattice with respect to \( \preceq \), and we remark that \( B_n^+ \) is a sublattice of \( B_n \). In 2008, Lee and Lee proved the following:
The Coxeter group \( (A, \Sigma) \) is fixed. If \( \Sigma \) is irreducible, then \( \alpha A \Sigma \beta^{-1} \) is also irreducible. In particular, Artin-Tits groups of spherical type are completely classified (Lee & Lee, 2008).

The first aim of this paper is to give a direct algorithm to compute the \( \preceq \)-minimal element of \( \text{St}(S) \), for a curve system \( S \). The algorithm, explained in Section 4, is inspired by Dynnikov and Wiest algorithm to compute a braid given its curve diagram (Dynnikov & Wiest, 2007) and the modifications made in (Caruso, 2013).

The second aim of the paper is to solve the analogous problem for Artin-Tits groups of spherical type.

**Definition 3.** Let \( S \) be a finite set and \( M = (m_{i,j})_{i,j \in S} \) a symmetric matrix with \( m_{i,i} = 1 \) and \( m_{i,j} \in \{2, \ldots, \infty\} \) for \( i \neq j \). Let \( \Sigma = \{\sigma_i \mid i \in S\} \). The Artin-Tits system associated to \( M \) is \( (A, \Sigma) \), where \( A \) is a group (called Artin-Tits group) with the following presentation

\[
A = \langle \sigma_i | \sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j \forall i, j \in S, i \neq j, m_{i,j} \neq \infty \rangle.
\]

For instance, \( B_n \) has the following presentation (Artin, 1947)

\[
B_n = \left\langle \sigma_1, \ldots, \sigma_{n-1} \right| \sigma_i \sigma_j = \sigma_j \sigma_i, \sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j, |i - j| > 1 \right\rangle.
\]

The Coxeter group \( W \) associated to \((A, \Sigma)\) can be obtained by adding the relations \( \sigma_i^2 = 1 \):

\[
W = \langle \sigma_i^2 = 1 \forall i \in S; \sigma_i \sigma_j \sigma_i \sigma_j = \sigma_j \sigma_i \sigma_j \sigma_i \forall i, j \in S, i \neq j, m_{i,j} \neq \infty \rangle.
\]

If \( W \) is finite, the corresponding Artin-Tits group is said to have spherical type. We will just consider Artin-Tits groups of spherical type, assuming that a spherical type Artin-Tits system is fixed. If \( A \) cannot be decomposed as direct product of non-trivial Artin-Tits groups, we say that \( A \) is irreducible. Irreducible Artin-Tits groups of spherical type are completely classified (Coxeter, 1935).

Let \( A \) be an Artin-Tits group of spherical type. A **standard parabolic subgroup**, \( A_X \), is the subgroup generated by some \( X \subseteq \Sigma \). A subgroup \( P \) is called **parabolic** if it is conjugated to a standard parabolic subgroup, that is, \( P = \alpha A_Y \alpha^{-1} \) for some standard parabolic subgroup \( A_Y \) and some \( \alpha \in A \). Notice that we may have \( P = \alpha A_Y \alpha^{-1} = \beta A_Z \beta^{-1} \) for distinct \( Y, Z \subseteq \Sigma \) and distinct \( \alpha, \beta \in A \). We will write \( P = (Y, \alpha) \) to express that \( A_Y \) and \( \alpha \) are known data defining the parabolic subgroup \( P \). The conjugation of \( P \) by \( \alpha \) will also be noted by \( P^\alpha \).

There is a natural way to associate a parabolic subgroup of \( B_n \) to a curve system. Suppose that \( A = B_n \) and let \( A_X \) be the standard parabolic subgroup generated by \( \{\sigma_i, \sigma_{i+1}, \ldots, \sigma_j\} \subseteq \{\sigma_1, \ldots, \sigma_{n-1}\} \). Let \( C \) be the isotopy class of the circle enclosing the punctures \( i, \ldots, j + 1 \) in \( D_n \). Then \( A_X \) fixes \( C \) and we will say that \( A_X \) is the parabolic subgroup associated to \( C \). If \( C' = C^\alpha \) for some \( \alpha \in B_n \), then \( (A_X)^\alpha := \alpha^{-1} A_Y \alpha \) is the parabolic subgroup associated to \( C' \). The parabolic subgroup associated to a system of non-nested curves is the direct sum of the subgroups associated to each curve. Notice that this is a well defined subgroup of \( B_n \), as the involved subgroups commute. In this way, parabolic subgroups play a similar role, in Artin-Tits groups, to the one played by systems of curves in \( B_n \).
Our second purpose in this paper is to give a fast and simple algorithm to compute the minimal positive element that conjugates a given parabolic subgroup to a standard parabolic subgroup. The central Garside element of a standard parabolic subgroup \( A_X \) will be denoted by \( c_X \) and is to be defined in the next section. Having a generic parabolic subgroup, \( P = (X, \alpha) \), the central Garside element will be denoted by \( c_P \). We also define the minimal standardizer of the parabolic subgroup \( P = (X, \alpha) \) to be the minimal positive element that conjugates \( P \) to a standard parabolic subgroup. The existence and uniqueness of this element will be shown in this paper. Keep in mind that the \( pn \)-normal form of an element is a particular decomposition of the form \( ab^{-1} \), where \( a \) and \( b \) are positive and have no common suffix. The main result of this paper is the following:

**Theorem 4.** Let \( P = (X, \alpha) \) be a parabolic subgroup. If \( c_P = ab^{-1} \) is in \( pn \)-normal form, then \( b \) is the minimal standardizer of \( P \).

Thus, the algorithm will take a parabolic subgroup \( P = (X, \alpha) \) and will just compute the normal form of its central Garside element \( c_P \), obtaining immediately the minimal standardizer of \( P \).

The paper will be structured in the following way: In section 2 some results and concepts about Garside theory will be recalled. In sections 3 and 4 the algorithm for braids will be explained. In section 5 the algorithm for Artin-Tits groups will be described and, finally, in section 6 we will bound the complexity of both procedures.

## 2 Preliminaries about Garside theory

Let us briefly recall some concepts from Garside theory. A group \( G \) is called a **Garside group** with Garside structure \((G, \mathcal{P}, \Delta)\) if it admits a submonoid \( \mathcal{P} \) of positive elements such that \( \mathcal{P} \cap \mathcal{P}^{-1} = \{1\} \) and a special element \( \Delta \in \mathcal{P} \), called Garside element, with the following properties:

- There is a partial order in \( G \), \( \preceq \), defined by \( a \preceq b \iff a^{-1}b \in \mathcal{P} \) such that for all \( a, b \in G \) it exists a unique gcd \( \land a \land b \) and a unique lcm \( \lor a \lor b \) with respect to \( \preceq \). This order is called prefix order and it is invariant by left-multiplication.
- The set of simple elements \([1, \Delta] = \{a \in G \mid 1 \preceq a \preceq \Delta\}\) generates \( G \).
- \( \Delta^{-1}\mathcal{P}\Delta = \mathcal{P} \).
- \( \mathcal{P} \) is atomic: If we define the set of atoms as the set of elements \( a \in \mathcal{P} \) such that there is no non-trivial elements \( b, c \in \mathcal{P} \) such that \( a = bc \), then for every \( x \in \mathcal{P} \) there is an upper bound on the number of atoms in the decomposition \( x = a_1a_2\cdots a_n \), where each \( a_i \) is an atom.

The conjugate by \( \Delta \) of an element \( x \) will be denoted \( \tau(x) = x\Delta = \Delta^{-1}x\Delta \).

In a Garside group, the monoid \( \mathcal{P} \) also induces a partial order invariant under right-multiplication, the suffix order \( \succeq \). This order is defined by \( a \succeq b \iff ab^{-1} \in \mathcal{P} \), and for all \( a, b \in G \) there exists an unique gcd \( (a \land \, b) \) and an unique lcm \( (a \lor \, b) \) with respect to \( \succeq \). We say that a Garside group has **finite type** if \([1, \Delta]\) is finite. It is well known that Artin-Tits groups of spherical type admit a Garside structure of finite type (Brieskorn & Saito, 1972, Dehornoy & Paris, 1999). Moreover:

**Proposition 5** (Van der Lek, 1983). A parabolic subgroup \( A_X \) of an Artin-Tits group of spherical type is an Artin-Tits group of spherical type whose Artin-Tits system is \((A_X, X)\).
Proposition 6 (Brieskorn & Saito, 1972). Let \((A_\Sigma, \Sigma)\) be an Artin-Tits system where \(A_\Sigma\) is of spherical type. Then, a Garside element for \(A_\Sigma\) is:

\[
\Delta_\Sigma = \bigvee_{\sigma_i \in \Sigma} \sigma_i = \bigvee_{\sigma_i \in \Sigma} \lnot \sigma_i,
\]

and the submonoid of positive elements is the monoid generated by \(\Sigma\). Moreover, if \(A_\Sigma\) is irreducible, then \((\Delta_\Sigma)^e\) generates the center of \(A_\Sigma\), for some \(e \in \{1, 2\}\).

**Definition 7.** We define the right complement of a simple element \(a\) as \(\partial(a) = a^{-1}\Delta\) and the left complement as \(\partial^{-1}(a) = \Delta a^{-1}\).

**Remark 8.** Observe that \(\partial^2 = \tau\) and that, if \(a\) is simple, then \(\partial(a)\) is also simple, i.e., \(1 \preceq \partial(a) \preceq \Delta\). Both claims follow from \(\partial(a)\tau(a) = \partial(a)\Delta^{-1}a\Delta = \Delta\) since \(\partial(a)\) and \(\tau(a)\) are positive.

**Definition 9.** Given two simple elements \(a, b\), the product \(a \cdot b\) is said to be in left (resp. right) normal form if \(ab \wedge \Delta = a\) (resp. \(ab \wedge \Delta = b\)). This is equivalent to say that \(\partial(a) \wedge b = 1\) (resp. \(a \wedge \partial^{-1}(b) = 1\)).

We say that \(x = \Delta^k x_1 \cdots x_r\) is in left normal form if \(k \in \mathbb{Z}\), \(x_i \notin \{1, \Delta\}\) is a simple element for \(i = 1, \ldots, r\), and \(x_i x_{i+1}\) is in left normal form for \(0 < i < r\).

Analogously, \(x = x_1 \cdots x_r \Delta^k\) is in right normal form if \(k \in \mathbb{Z}\), \(x_i \notin \{1, \Delta\}\) is a simple element for \(i = 1, \ldots, r\), and \(x_i x_{i+1}\) is in right normal form for \(0 < i < r\).

It is well known that the normal form of an element is unique (Dehornoy & Paris, 1999, Corollary 7.5). Moreover, the numbers \(r\) and \(k\) do not depend on the normal form (left or right). We define the infimum, the canonical length and the supremum of \(x\) respectively as \(\inf(x) = k\), \(\ell(x) = r\) and \(\sup(x) = k + r\).

Let \(a\) and \(b\) be two simple elements such that \(a \cdot b\) is in left normal form. One can write its inverse as \(b^{-1}a^{-1} = \Delta^{-2}\partial^{-3}(b)\partial^{-1}(a)\). This is in left normal form because \(\partial^{-1}(b)\partial(a)\) is in normal form by definition and \(\tau = \partial^2\) preserves \(\preceq\). More generally (see Elrifai & Morton, 1994), if \(x = \Delta^k x_1 \cdots x_r\) is in left normal form, then the left normal form of \(x^{-1}\) is

\[
x^{-1} = \Delta^{-(k+r)}(x_r)\partial^{-2(k+r-1)}(x_r)\partial^{-2(k+r-2)}(x_{r-1})\cdots\partial^{-2k-1}(x_1)
\]

For a right normal form, \(x = x_1 \cdots x_r \Delta^k\), the right normal form of \(x^{-1}\) is:

\[
x^{-1} = \partial^{2k+1}(x_1)\partial^{2(k+1)+1}(x_{r-1})\cdots\partial^{2(k+r-1)+1}(x_1)\Delta^{-(k+r)}
\]

**Definition 10** (Charney, 1995, Theorem 2.6). Let \(a, b \in P\), then \(x = a^{-1}b\) is said to be in np-normal form if \(a \wedge b = 1\). Similarly, we say that \(x = ab^{-1}\) is in pm-normal form if \(a \wedge \lnot b = 1\).

**Definition 11.** Let \(\Delta^k x_1 \cdots x_r\) with \(r > 0\) be the left normal form of \(x\). We define the initial and the final factor respectively as \(\iota(x) = \tau^{-k}(x_1)\) and \(\varphi(x) = x_r\). We will say that \(x\) is rigid if \(\varphi(x) \cdot \iota(x)\) is in left normal form or if \(r = 0\).

**Definition 12** (Elrifai & Morton, 1994, Gebhardt & González-Meneses, 2010b). Let \(\Delta^k x_1 \cdots x_r\) with \(r > 0\) be the left normal form of \(x\). The **cycling** of \(x\) is defined as

\[
c(x) = x^{\iota(x)} = \Delta^k x_2 \cdots x_r \iota(x).
\]

The **decycling** of \(x\) is \(d(x) = x^{(\varphi(x)^{-1})} = \varphi(x)\Delta^k x_1 \cdots x_{r-1}\). We also define the preferred prefix of \(x\) as

\[
p(x) = \iota(x) \land \iota(x^{-1}).
\]

The **cyclic sliding** of \(x\) is defined as the conjugate of \(x\) by its preferred prefix:

\[
s(x) = x^p(x) = p(x)^{-1}xp(x).
\]
Let $G$ be a Garside group and $x \in G$. Keep in mind that $\inf_s(x)$ and $\sup_s(x)$ denote respectively the maximal infimum and the minimal supremum in the conjugacy class $x^G$.

- The super summit set of $x$ is
  \[
  SSS(x) = \{ y \in x^G \mid \ell \text{ is minimal in } x^G \} 
  = \{ y \in x^G \mid \inf(y) = \inf_s(x) \text{ and } \sup(y) = \sup_s(y) \} 
  \]

- The ultra summit set of $x$ is
  \[
  USS(x) = \{ y \in SSS(x) \mid c^m(y) = y \text{ for some } m \geq 1 \} 
  \]

- The set of sliding circuits of $x$ is
  \[
  SC(x) = \{ y \in x^G \mid s^m(y) = y \text{ for some } m \geq 1 \} 
  \]

These sets are finite if the set of simple elements is finite and their computation is very useful to solve the conjugacy problem in Garside groups. They satisfy the following inclusions:

\[
SSS(x) \supseteq USS(x) \supseteq SC(x). 
\]

**The braid group, $B_n$**

A braid with $n$ strands can be seen as a collection of $n$ disjoint paths in a cylinder, defined up to isotopy, joining $n$ points at the top with $n$ points at the bottom, running monotonically in the vertical direction.

Each generator $\sigma_i$ represents a crossing between the strands in positions $i$ and $i + 1$ with a fixed orientation. The generator $\sigma_i^{-1}$ represents the same crossing with the opposite orientation. When considering a braid as a mapping class of $D_n$, these crossings are identified with the swap of two punctures in $D_n$ (See Figure 1).

![Diagram of the braid $\sigma_1 \sigma_2^{-1}$ and how it acts on a curve in $D_3$.](image)

**Remark 13.** After the above results, we see that the standard Garside structure of the braid group $B_n$ is $(B_n, B_n^+, \Delta_n)$ where

\[
\Delta_n = \sigma_1 \lor \cdots \lor \sigma_{n-1} = (\sigma_1 \sigma_2 \cdots \sigma_{n-1})(\sigma_1 \sigma_2 \cdots \sigma_{n-2}) \cdots (\sigma_1 \sigma_2) \sigma_1
\]

The simple elements in this case are also called **permutation braids** (Elrifai & Morton, 1994), because the set of simple braids is in one-to-one correspondence with the set of permutations of $n$ elements. Later we will use the following result:

**Lemma 14** (Elrifai & Morton, 1994, Lemma 2.4). If $s$ is a simple braid and its strands $j$ and $j + 1$ cross, then $\sigma_j \preceq s$. 
3 Detecting bending points

In order to describe a non-degenerate closed curve \( C \) in \( D_n \), we will use a notation introduced in (Fenn et al., 1999). Recall that \( D_n \) has diameter \([0, n+1]\) and that the punctures of \( D_n \) are placed at \( 1, 2, \ldots, n \in \mathbb{R} \). Choose a point on \( C \) intersecting the real axis and choose an orientation for \( C \). We will obtain a word \( W(C) \) representing \( C \), on the alphabet \( \{⌣,⌢,0,1,\ldots,n\} \), by running along the curve, starting and finishing at the chosen point. We write down a symbol \( ⌣ \) for each arc on the lower half plane, a symbol \( ⌢ \) for each arc on the upper half plane, and a number \( m \) for each intersection of \( C \) with the real segment \((m, m+1)\). An example is provided in Figure 2.

Figure 2: \( W(C) = 0 \⌣ 6 \⌣ 4 \⌣ 2 \⌣ 1 \⌣ 4 \⌣ 5 \⌣ 1 \⌣ \).

For an isotopy class of curves \( C \), \( W(C) \) is the word associated to a reduced representative \( C^{red} \), i.e., a curve in \( C \) which has minimal intersection with the real axis. \( C^{red} \) is unique up to isotopy of \( D_n \) fixing the real diameter setwise (Fenn et al., 1999), and \( W(C) \) is unique up to cyclic permutation and reversing.

Remark 15. Notice that if a curve \( C \) does not have minimal intersection with the real axis, then \( W(C) \) contains a subword of the form \( p \⌣ p \⌣ \) or \( p \⌣ p \⌣ \). Hence, the curve can be isotoped by “pushing” this arc in order not to intersect the real axis. This is equivalent to remove the subword mentioned before from \( W(C) \). In fact, we will obtain \( W(C) \) by removing all subwords of this kind from \( W(C) \). The process of removing \( p \⌣ p \⌣ \) (resp. \( p \⌣ p \⌣ \)) from \( W(C) \) is called relaxation of the arc \( p \⌣ p \) (resp. \( p \⌣ p \)).

Definition 16. Let \( C \) be a non-degenerate simple closed curve. We say that there is a bending point (resp. reversed bending point) of \( C \) at \( j \) if we can find in \( W(C) \), up to cyclic permutation and reversing, a subword of the form \( i \⌣ j \⌣ k \) (resp. \( i \⌣ j \⌣ k \)) for some \( 0 \leq i < j < k \leq n \) (Figure 3).

We say that a curve system has a bending point at \( j \) if one of its curves has a bending point at \( j \).

The algorithm we give in Section 4 takes a curve system \( S \) and ‘untangles’ it in the shortest (positive) way. That is, it gives the shortest positive braid \( \alpha \) such that \( S^\alpha \) is standard, i.e., the minimal element in \( \text{St}(S) \). Bending points are the key ingredient of the algorithm. We
will show that if a curve system $S$ has a bending point at $j$, then $\sigma_j$ is a prefix of the minimal element in $\text{St}(S)$. This will allow to untangle $S$ by looking for bending points and applying the corresponding $\sigma_j$ to the curve until no bending point is found. The aim of this section is to describe a suitable input for this algorithm and to show the following result.

**Proposition 17.** A curve system is standard if and only if its reduced representative has no bending points.

### 3.1 Dynnikov coordinates

We have just described a non-degenerate simple closed curve in $D_n$ by means of the word $W(C)$. There is a different and usually much shorter way to determine a curve system $S$ in $D_n$: its Dynnikov coordinates (Dehornoy et al., 2008, Chapter 7). The method to establish the coordinates of $C$ is as follows. Take a triangulation of $D_n$ as in Figure 4 and let $x_i$ be the number of times the curve system $S$ intersects the edge $e_i$. The Dynnikov coordinates of the curve system are given by the $t$-uple $(x_0, x_1, \ldots, x_{3n-4})$. There exists a reduced version of these coordinates, namely $(a_0, b_0, \ldots, a_{n-1}, b_{n-1})$, where

$$
a_i = \frac{x_{3i-1} - x_{3i}}{2}, \quad b_i = \frac{x_{3i-2} - x_{3i+1}}{2}, \quad \forall i = 1, \ldots, n - 2
$$

and $a_0 = a_{n-1} = 0$, $b_0 = -x_0$ and $b_{n-1} = x_{3n-4}$. See an example in Figure 5.

![Figure 4: Triangulation used to define Dynnikov coordinates.](image)

Furthermore, there are formulae determining how these coordinates change when applying $\sigma_j^\pm$, to the corresponding curve, for $0 < j < n$.

**Proposition 18 (Dehornoy et al., 2002, Proposition 8.5.4).** For $c = (a_0, b_0, \ldots, a_{n-1}, b_{n-1})$, we have

$$
c^{\sigma_j^\pm} = (a'_0, b'_0, \ldots, a'_{n-1}, b'_{n-1}),
$$

with $a'_j = a_j$, $b'_j = b_j$ for $j \notin \{k - 1, k\}$, and

$$
\begin{align*}
a'_{k-1} &= a_{k-1} + (\delta + b_{k-1})^+, \\
a'_k &= a_k - (\delta - b_k)^+, \\
b'_{k-1} &= b_{k-1} - (\delta')^+ + \delta^+, \\
b'_k &= b_k + (\delta')^+ - \delta^+,
\end{align*}
$$

where $\delta = a_k - a_{k-1}$, $\delta' = a'_k - a'_{k-1}$ and $x^+ = \max(0, x)$.

We also have

$$
c^{\sigma_k} = c^{\lambda \sigma_k^{-1} \lambda}
$$

with $(a_1, b_1, \ldots, a_{n-1}, b_{n-1})^{\lambda} = (-a_1, b_1, \ldots, -a_{n-1}, b_{n-1})$. 


Remark 19. Notice that the use of $\sigma_k^{-1}$ in the first equation above is due to the orientation of the strands crossings that we are taking for our braids (see Figure 1), which is the opposite of the orientation used in (Dehornoy et al., 2002).

![Figure 5: The Dynnikov coordinates and reduced Dynnikov coordinates of C are, respectively, $(x_0, \ldots, x_8) = (1, 2, 4, 2, 6, 9, 3, 12, 6)$; $(a_0, b_0, a_1, a_2, b_2, a_3, b_3) = (0, -1, -2, 3, -3, 0, 6)$.

Let us see how to detect a bending point of a curve system $S$ with these coordinates. Notice that there cannot be a bending point at 0 or at $n$. It is easy to check that there is a bending point at 1 if and only if $x_2 < x_3$ (Figure 6a). Actually, if $R$ is the number of subwords of type $0 \rightarrow 1 \rightarrow k$ for some $1 < k \leq n$, then $x_3 = x_2 + 2R$. Symmetrically, there is a bending point at $n - 1$ if and only if $x_{3n-6} < x_{3n-7}$.

![Figure 6: Detecting bending points with Dynnikov coordinates.](image)

A bending point at $i$, for $1 < i < n - 1$, is detected by comparing the coordinates $a_{i-1}$ and $a_i$ (Figure 6b). Notice that arcs not intersecting $e_{3i-2}$ affect neither $a_{i-1}$ nor $a_i$, and arcs not intersecting the real line do not affect the difference $a_{i-1} - a_i$. Hence, there is a bending point of $S$ at $i$ if and only if $a_{i-1} - a_i > 0$. Using a similar argument we can prove that there is a reversed bending point of $S$ at $i$ if and only if $a_{i-1} - a_i < 0$. Moreover, each bending point (resp. reversed bending point) at $i$ increases (resp. decreases) by 1 the difference $a_{i-1} - a_i$. We have just shown the following result:

Lemma 20 (Bending point with Dynnikov coordinates). Let $S$ be a curve system on $D_n$ with reduced Dynnikov coordinates $(a_0, b_0, \ldots, a_{n-1}, b_{n-1})$. For $j = 1, \ldots, n - 1$ there are exactly $R$ bending points (resp. reversed bending points) of $S$ at $j$ if and only if $a_{j-1} - a_j = R$ (resp. $a_{j-1} - a_j = -R$).

Lemma 21. Let $S$ be a curve system as above. Then $S$ is symmetric with respect to the real axis if and only if $a_i = 0$, for $0 < i < n$. 

Proof. Just notice that a symmetry with respect to the real axis does not affect $b$-coordinates and changes the sign of every $a_i$, for $0 < i < n$.

**Lemma 22.** A curve system is standard if and only if it is symmetric with respect to the real axis.

Proof. For every $m = 0, \ldots, n$, we can order the finite number of elements in $\mathcal{S} \cap (m, m+1)$ from left to right, as real numbers. Given an arc $a \leadsto b$ in $W(\mathcal{S})$, suppose that it joins the $i$-th element in $\mathcal{S} \cap (a, a+1)$ with the $j$-th element in $\mathcal{S} \cap (b, b+1)$. The symmetry with respect to the real axis preserves the order of the intersections with the real line, hence the image $a \leadsto b$ of the above upper arc will also join the $i$-th element in $\mathcal{S} \cap (a, a+1)$ with the $j$-th element in $\mathcal{S} \cap (b, b+1)$. This implies that both arcs $a \sim b$ and $a \sim b$ form a single standard curve $a \leadsto b \leadsto$. As this can be done for every upper arc in $\mathcal{S}$, it follows that $\mathcal{S}$ is standard.

Proof of Proposition 17. If the curve system is standard, then it clearly has no bending points. Conversely, if it has no bending points, by Lemma 20 the sequence $a_0, \ldots, a_{n-1}$ is non-decreasing, starting and ending at 0, so it is constant. By Lemmas 21 and 22, the curve system is standard.

4 Standardizing a curve system

We will now describe an algorithm which takes a curve system $\mathcal{S}$, given in reduced Dynnikov coordinates, and finds the minimal element in $\text{St}(\mathcal{S})$. The algorithm will do the following: Start with $\beta = 1$. Check whether the curve has a bending point at $j$. If so, multiply $\beta$ by $\sigma_j$ and restart the process with $\mathcal{S} \sigma_j$. A simple example is provided in Figure 7. The formal way is described in Algorithm 1.

![Figure 7: A simple example of how to find the minimal standardizer of a curve.](image)

The minimality of the output is guaranteed by the following theorem, which shows that $\sigma_j$ is a prefix of the minimal standardizer in $\text{St}(\mathcal{S})$, provided $\mathcal{S}$ has a bending point at $j$.

**Theorem 23.** Let $\mathcal{S}$ be a curve system with a bending point at $j$. Then $\sigma_j$ is a prefix of $\alpha$, for every positive braid $\alpha$ such that $\mathcal{S} \alpha$ is standard.

To prove the theorem we will need a result from (Calvez, 2012).

**Definition 24.** We will say that a simple braid $s$ is compatible with a bending point at $j$ if the strands $j$ and $j+1$ of $s$ do not cross in $s$. That is, if $\sigma_j \nsubseteq s$. 
Algorithm 1: Rounding algorithm

Input : The reduced coordinates \((a_0, b_0, \ldots, a_{n-1}, b_{n-1})\) of a curve system \(S\) on \(D_n\).
Output: The \(\prec\)-minimal element of \(St(S)\).

\[ c = (a_0, b_0, \ldots, a_{n-1}, b_{n-1}); \]
\[ \beta = 1; \]
\[ j = 1; \]
while \(j < n\) do
  if \(a_j < a_{j-1}\) then
    \[ c = c \sigma_j; \] (use Proposition 18)
    \[ \beta = \beta \cdot \sigma_j; \]
    \[ j = 1; \]
  else
    \[ j = j + 1; \]
return \(\beta\);
1. $g^{-1}A_Xg \subseteq A_Y$;
2. $g^{-1}c_Xg \in A_Y$;
3. $g = xy$ where $y \in A_Y$ and $x$ conjugates $X$ to a subset of $Y$.

The above proposition is a generalization of (Paris, 1997, Theorem 5.2) and implies, as we will see, that conjugating standard parabolic subgroups is equivalent to conjugating their central Garside elements. This will lead us to the definition of the central Garside element for a non-standard parabolic subgroup as given in Proposition 35. In order to prove the following results, we need to define an object that generalizes to Artin-Tits groups of spherical type some operations used in braid theory:

**Definition 29.** Let $X \subset \Sigma$, $t \in \Sigma$. We define

$$r_{X,t} = \Delta_{X \cup \{t\}} \Delta_X^{-1}.$$ 

**Remark 30.** In the case $t \notin X$, this definition is equivalent to the definition of positive elementary ribbon (Godelle, 2003, Definition 0.4). Notice that if $t \in X$, $r_{X,t} = 1$.

**Proposition 31.** There is a unique $Y \subset X \cup \{t\}$ such that $r_{X,t}X = Y r_{X,t}$.

**Proof.** Given $Z \subset \Sigma$, conjugation by $\Delta_Z$ permutes the elements of $Z$. Let us denote by $Y$ the image of $X$ under the permutation of $X \cup \{t\}$ induced by the conjugation by $\Delta_{X \cup \{t\}}$. Then

$$r_{X,t}X r_{X,t}^{-1} = \Delta_{X \cup \{t\}} \Delta_X^{-1} X \Delta_X \Delta_{X \cup \{t\}}^{-1} = \Delta_{X \cup \{t\}} X \Delta_{X \cup \{t\}}^{-1} = Y.$$ 

Artin-Tits groups of spherical type can be represented by Coxeter graphs. Recall that such a group, $A$, is defined by a symmetric matrix $M = (m_{ij})_{i,j \in S}$ and the set of generators $\Sigma = \{\sigma_i \mid i \in S\}$, where $S$ is a finite set. The Coxeter graph associated to $A$ is denoted $\Gamma_A$. The set of vertices of $\Gamma_A$ is $\Sigma$, and there is an edge joining two vertices $s, t \in \Sigma$ if $m_{st} \geq 3$. The edge will be labelled with $m_{st}$ if $m_{st} \geq 4$. We say that the group $A$ is indecomposable if $\Gamma_A$ is connected and decomposable otherwise. If $A$ is decomposable, then there exists a non trivial partition $\Sigma = X_1 \cup \cdots \cup X_k$ such that $A$ is isomorphic to $A_{X_1} \times \cdots \times A_{X_k}$, where each $A_{X_j}$ is indecomposable (each $X_j$ is just the set of vertices of a connected component of $\Gamma_{X_j}$). Each $A_{X_j}$ is called an indecomposable component of $A$.

**Lemma 32.** Let $X, Y \subset \Sigma$ and let $X = X_1 \cup \cdots \cup X_n$ and $Y = Y_1 \cup \cdots \cup Y_m$ be the partitions of $X$ and $Y$, respectively, inducing the indecomposable components of $A_X$ and $A_Y$. Then, for every $g \in A$, the following conditions are equivalent:

1. $g^{-1}A_Xg = A_Y$.
2. $m = n$ and $g = xy$, where $y \in A_Y$ and the parts of $Y$ can be reordered so that we have $x^{-1}X_i x = Y_i$ for $i = 1, \ldots, n$.
3. $m = n$ and $g = xy$, where $y \in A_Y$ and the parts of $Y$ can be reordered so that we have $x^{-1}A_X x = A_Y$, for $i = 1, \ldots, n$.

**Proof.** Suppose that $g^{-1}A_Xg = A_Y$. By Proposition 28, we can decompose $g = xy$ where $y \in A_Y$ and $x$ conjugates the set $X$ to a subset of the set $Y$. Since conjugation by $y$ induces an automorphism of $A_Y$, it follows that $x$ conjugates $A_X$ isomorphically onto $A_Y$, so it conjugates $X$ to the whole set $Y$. Then conjugation by $x$ sends indecomposable components of $X$ onto
indecomposable components of $Y$. Hence $m = n$ and $x^{-1}X_i x = Y_i$ for $i = 1, \ldots, n$ (reordering the indecomposable components of $Y$ in a suitable way), as we wanted to show. Thus, statement 1 implies statement 2.

Now the relations satisfied by the elements of $X$ are also satisfied (through conjugation by $x$) by their images in $Y$, and viceversa. Since the connected components of $\Gamma_X$ (resp. $\Gamma_Y$) are determined by the commutation relations among the letters of $X$ (resp. $Y$), it follows that conjugation by $x$ sends indecomposable components of $X$ onto indecomposable components of $Y$. Therefore statement 2 implies 3.

Finally, statement 3 implies 1 because $A_X = A_{X_1} \times \cdots \times A_{X_n}$ and $A_Y = A_{Y_1} \times \cdots \times A_{Y_n}$.

\begin{lemma}
Let $X, Y \subseteq \Sigma$, $g \in A$. Then,
\[ g^{-1} A_X g = A_Y \iff g^{-1} c_X g = c_Y.\]
\end{lemma}

\textbf{Proof.} Suppose that $g^{-1} c_X g = c_Y$. Then, by Proposition 28, we have $g^{-1} A_X g \subseteq A_Y$ and also $g A_Y g^{-1} \subseteq A_X$. As conjugation by $g$ is an isomorphism of $A$, the last inclusion is equivalent to $A_Y \subseteq g^{-1} A_X g$. Thus, $g^{-1} A_X g = A_Y$, as desired.

Conversely, suppose that $g^{-1} A_X g = A_Y$. By using Lemma 32, we can decompose $g = xy$ where $y \in A_Y$ and $x$ is such that $x^{-1} A_X x = A_{Y_1}$, where $A_{X_i}$ and $A_{Y_1}$ are indecomposable components of $A_X$ and $A_Y$ for $i = 1, \ldots, n$. As the conjugation by $x$ defines an isomorphism between $A_{X_i}$ and $A_{Y_1}$, we have that $x^{-1} Z(A_{X_i}) x = Z(A_{Y_1})$. Hence, we have $x^{-1} c_{X_i} x = \Delta_{Y_1}^{e_i}$ for some $k \in \mathbb{Z}$. Let $c_X = \Delta_X^{e_1}$ and $c_Y = \Delta_Y^{e_2}$. As $A_{X_i}$ and $A_{Y_1}$ are isomorphic, then $e_1 = e_2$. Also notice that in an Artin-Tits group of spherical type the relations are homogeneous and so $k = e_1 = e_2$, having $x^{-1} c_{X_i} x = c_{Y_1}$. Let
\[ e = \max \{ e_i \mid c_{X_i} = \Delta_X^{e_i} \} = \max \{ e_i \mid c_{Y_1} = \Delta_Y^{e_i} \}, \]
and denote $d_{X_i} = \Delta_X^{e_i}$ and $d_{Y_1} = \Delta_Y^{e_i}$ for $i = 1, \ldots, n$. Notice that $d_{X_i}$ is equal to either $c_{X_i}$ or $(c_{X_i})^2$, and the same happens for each $d_{Y_1}$, hence $x^{-1} d_{X_i} x = d_{Y_1}$ for $i = 1, \ldots, n$. Then, as $c_X = \prod_{i=1}^n d_{X_i}$ and $c_Y = \prod_{i=1}^n d_{Y_1}$, it follows that $x^{-1} c_{X} x = c_{Y}$, hence $g^{-1} c_X g = y^{-1}(x^{-1} c_X x)y = y^{-1} c_{Y} y = c_Y$. Therefore, $g^{-1} A_X g = A_Y$, as desired.

\begin{lemma}
Let $P = (X, \alpha)$ be a parabolic subgroup and $A_Y$ be a standard parabolic subgroup of an Artin-Tits group $A$ of spherical type. Then we have
\[ g^{-1} P g = A_Y \iff g^{-1} c_{X, \alpha} g = c_Y.\]
\end{lemma}

\textbf{Proof.} If $P = (X, \alpha)$, it follows that $g^{-1} P g = A_Y$ if and only if $g^{-1} \alpha A_X \alpha^{-1} g = A_Y$. By Lemma 33, this is equivalent to $g^{-1} c_{X, \alpha} g = c_Y$, i.e., $g^{-1} c_{X, \alpha} g = c_Y$.

\begin{proposition}
Let $P = (X, \alpha) = (Y, \beta)$ be a parabolic subgroup of an Artin-Tits group of spherical type. Then $c_{X, \alpha} = c_{Y, \beta}$ and we can define $c_P := c_{X, \alpha}$ to be the central Garside element of $P$.
\end{proposition}

\textbf{Proof.} Suppose that $g$ is a standardizer of $P$ such that $g^{-1} P g = A_Z$. By using Lemma 34, we have that $c_Z = g^{-1} c_{X, \alpha} g = g^{-1} c_{Y, \beta} g$. Thus, $c_{X, \alpha} = c_{Y, \beta}$.

By Lemma 34, to find the minimal standardizer of a parabolic subgroup $P = (X, \alpha)$, we just need to find the minimal positive element conjugating $c_P$ to some $c_Y$. Let
\[ C_{X, \alpha}^+(c_P) = \{ s \in P \mid s = u^{-1} c_P u, u \in A_X \} \]
be the set of positive elements conjugate to $c_P$ (which coincides with the positive elements conjugate to $c_X$). Firstly, let us compute the minimal conjugator from $c_P$ to this set. That is, the shortest positive element $u$ such that $u^{-1} c_{X, \alpha} u \in P$. 

\[ C_{X, \alpha}^+(c_P) = \{ s \in P \mid s = u^{-1} c_P u, u \in A_X \} \]
**Proposition 36** (Franco & González-Meneses, 2003, Proposition 4.8). For any \( \alpha \in A_\Sigma \) conjugated to a positive element, it exists a unique \( \sim \)-minimal positive element conjugating \( \alpha \) to \( C^-\alpha \).

**Proposition 37.** If \( x = ab^{-1} \) is in \( \Sigma \)-normal form and \( x \) is conjugated to a positive element, then \( b \) is a prefix of every positive element conjugating \( x \) to \( C^-\alpha \).

**Proof.** Suppose that \( \rho \) is a positive element such that \( \rho^{-1}x\rho \) is positive. Then \( 1 \sim \rho^{-1}x\rho \). Multiplying from the left by \( \rho \), we obtain \( a^+ \rho \sim \rho \), and, since \( \rho \) is positive, \( \rho^{-1} \) is positive, \( \rho^{-1}x\rho \). Hence \( x^{-1} \not\sim \rho \) or, in other words \( ba^{-1} \not\sim \rho \). On the other hand, by definition of \( \Sigma \)-normal form, we have \( a \wedge b = 1 \), which is equivalent to \( a^{-1} \vee b^{-1} = 1 \) (Gebhardt & González-Meneses, 2010a, Lemma 1.3). Multiplying from the left by \( b \), we obtain \( ba^{-1} \not\sim 1 \).

Finally, notice that \( ba^{-1} \not\sim \rho \) and also \( 1 \not\sim \rho \). Hence \( b = ba^{-1} \not\sim 1 \). Since \( b \) is a prefix of \( \rho \) for every positive \( \rho \) conjugating \( x \) to a positive element, the result follows.

**Lemma 38.** Let \( A_X \) be a standard parabolic subgroup and \( t \in \Sigma \). If \( \alpha \Delta_X \not\sim t \), then \( \alpha \not\sim r_X,t \).

**Proof.** Since the result is obvious for \( t \in X \) (\( r_X,t = 1 \)), suppose \( t \notin X \). Trivially, \( \alpha \Delta_X \not\sim \Delta_X \). As \( \alpha \Delta_X \not\sim t \), we have that \( \alpha \Delta_X \not\sim \Delta_X \vee t \). By definition, \( \Delta_X \vee t = \Delta_X \cup \{t\} = r_X,t \Delta_X \). Thus, \( \alpha \Delta_X \not\sim r_X,t \Delta_X \) and so \( \alpha \not\sim r_X,t \), because \( \not\sim \) is invariant under right-multiplication.

**Theorem 4.** Let \( P = (X, \alpha) \) be a parabolic subgroup. If \( c_P = ab^{-1} \) is in \( \Sigma \)-normal form, then \( b \) is the \( \sim \)-minimal standardizer of \( P \).

**Proof.** We know from Proposition 37 that \( b \) is a prefix of any positive element conjugating \( c_P \) to a positive element, which guarantees its \( \sim \)-minimality. We also know from Lemma 34 that any standardizer of \( P \) must conjugate \( c_P \) to a positive element, namely to the central Garside element of some standard parabolic subgroup. So we only have to prove that \( b \) itself conjugates \( c_P \) to the central Garside element of some standard parabolic subgroup. We assume \( \alpha \) to be positive, because there is always some \( k \in \mathbb{N} \) such that \( \Delta^{2k} \alpha \) is positive and, as \( \Delta^2 \) lies in the center of \( A \), \( P = (X, \alpha) = (X, \Delta^{2k} \alpha) \).

The \( \Sigma \)-normal form of \( c_P = \alpha \Delta_X \alpha^{-1} \) is obtained by cancelling the greatest common suffix of \( \alpha \Delta_X \) and \( \alpha \). Suppose that \( t \in \Sigma \) is such that \( \alpha \not\sim t \) and \( \alpha \Delta_X \not\sim t \).

If \( t \notin X \), then \( r_X,t \neq 1 \) and by Lemma 38 we have that \( \alpha \not\sim r_X,t \), i.e., \( \alpha = \alpha_1 r_X,t \) for some \( \alpha_1 \in A_\Sigma \). Hence,

\[ \alpha \Delta_X \alpha^{-1} = \alpha_1 r_X,t \Delta_X \alpha^{-1} = \alpha_1 \Delta_X \alpha^{-1} \]

for some \( X_1 \subset \Sigma \). In this case, we reduce the length of the conjugator (by the length of \( r_X,t \)).

If \( t \in X \), then \( t \) commutes with \( \Delta_X \), which means that

\[ \alpha \Delta_X \alpha^{-1} = \alpha_1 \Delta_X \alpha^{-1} = \alpha_1 \Delta_x \alpha^{-1} \]

where \( \alpha_1 \) is one letter shorter than \( \alpha \) and \( X_1 = X \).

We can repeat the same procedure for \( \alpha_i \Delta_X \alpha_{i}^{-1} \), where \( X_i \subset \Sigma \), \( t_i \in \Sigma \) such that \( \alpha_i \not\sim t_i \) and \( \alpha_i \Delta_X \not\sim t_i \). As the length of the conjugator decreases at each step, the procedure must stop, having as a result the \( \Sigma \)-normal form of \( c_P \), which will have the form

\[ c_P = (\alpha_k \Delta_X) \alpha_{k}^{-1} \]

for \( k \in \mathbb{N} \), \( X_k \subset \Sigma \).

Then, \( \alpha_k = b \) clearly conjugates \( c_P \) to \( \Delta_X \), which is the central Garside element of a standard parabolic subgroup, so \( b \) is the \( \sim \)-minimal standardizer of \( P \).

We end this section with a result concerning the conjugacy classes of elements of the form \( c_P \). As all the elements of the form \( c_Z \), \( Z \subset X \), are rigid (Definition 11), using the next theorem we can prove that the set of sliding circuits of \( c_P \) is equal to its set of positive conjugates.
Theorem 39 (Gebhardt & González-Meneses, 2010b, Theorem 1). Let \( G \) be a Garside group of finite type. If \( x \in G \) is conjugate to a rigid element, then \( SC(x) \) is the set of rigid conjugates of \( x \).

Corollary 40. Let \( P = (X, \alpha) \) be a parabolic subgroup of an Artin-Tits group of spherical type. Then

\[
C^+_{A^+_\Sigma}(c_P) = SSS(c_P) = USS(c_P) = SC(c_P) = \{c_Y \mid Y \in \Sigma, c_Y \text{ conjugate to } c_X\}.
\]

Proof. By Theorem 39, it suffices to prove that \( C^+_{A^+_\Sigma}(c_P) \) is composed only by rigid elements of the form \( c_Z \). Let \( P' = (X, \beta) \) and suppose that \( c_{P'} \in C^+_{A^+_\Sigma}(c_P) \). Let \( b \) be the minimal standardizer of \( c_{P'} \). By Proposition 37 and Theorem 4, \( b \) is the minimal positive element conjugating \( c_{P'} \) to \( C^+_{A^+_\Sigma}(c_P) \), which implies that \( b = 1 \), so \( P' \) is standard. Hence, all positive conjugates of \( c_{P'} \) are equal to \( c_Y \) for some \( Y \), therefore they are rigid.

Corollary 41. Let \( P = (X, \alpha) \) be a parabolic subgroup of an Artin-Tits group of spherical type. Then the set of positive standardizers of \( P \),

\[\text{St}(P) = \{\alpha \in A^+_\Sigma, \mid c^\alpha_P \equiv c_Y, \text{ for some } Y \subseteq \Sigma\},\]

is a sublattice of \( A^+_\Sigma \).

Proof. Let \( s_1 \) and \( s_2 \) be two positive standardizers of \( P \) and let \( \alpha := s_1 \wedge s_2 \) and \( \beta := s_1 \vee s_2 \). By Corollary 40 and, for example, (Gebhardt & González-Meneses, 2010b, Proposition 7, Corollary 8), we have that \( c_P^\alpha = c_Y \) and \( c_P^\beta = c_Z \) for some \( Y, Z \subseteq \Sigma \). Hence \( \alpha, \beta \in \text{St}(P) \), as we wanted to show.

6 Complexity

In this section we will describe the computational complexity of the algorithms which compute minimal standardizers of curves and parabolic subgroups. Let us start with Algorithm 1, which computes the minimal standardizer of a curve system.

Notice that Algorithm 1 takes at each step the leftmost bending point of the curve system. However, Theorem 23 allows us to take any bending point. The complexity of Algorithm 1 will depend on the length of the output, which is the number of steps of the algorithm. To bound this length, we will compute a positive braid which belongs to \( \text{St}(S) \). This will bound the length of the minimal standardizer of \( S \).

The usual way to describe the length (or the complexity) of a curve system consists in counting the number of intersections with the real axis, i.e., \( \ell(S) = \#(S \cap \mathbb{R}) \). For integers \( 0 \leq i < j < k \leq n \), we define the following braid (see Figure 8):

\[s(i,j,k) = (\sigma_j\sigma_{j-1}\cdots\sigma_{i+1})(\sigma_{j+1}\sigma_{j}\cdots\sigma_{i+2})\cdots(\sigma_{k-1}\sigma_{k-2}\cdots\sigma_{i+k-j})\]

Lemma 42. Applying \( s = s(i,j,k) \) to a curve system \( S \), when \( i \prec j \prec k \) is a bending point, decreases the length of the curve system at least by two.

Proof. We will describe the arcs of the curves of \( S \) in a new way, by associating a real number \( c_p \in (0, n+1) \) to each of the intersections of \( S \) with the real axis, where \( p \) is the position of the intersection with respect to the other intersections: \( c_1 \) is the leftmost intersection and \( c_{\ell(S)} \) is the rightmost one. We will obtain a set of words representing the curves of \( S \), on the alphabet \( \{\searrow, \nearrow, c_1, \ldots, c_{\ell(S)}\} \), by running along the curve, starting and finishing at the same point. As
before, we write down a symbol $\sim$ for each arc on the lower half plane, and a symbol $\sim$ for each arc on the upper half plane. We also define the following function that sends this alphabet to the former one:

$$L: \{\sim, \sim, c_1, \ldots, c_{\ell(S)}\} \rightarrow \{\sim, \sim, 0, \ldots, n\}$$

$$L(\sim) = \sim, \quad L(\sim) = \sim, \quad L(c_p) = \lfloor c_p \rfloor.$$ 

Take a disk $D$ such that the $\partial(D)$ intersects the real axis at two points, $x_2$ and $x_3$. Consider another point $x_1$ on the real axis such that $L(x_1) < L(x_2)$. Suppose that there are no arcs of $S$ on the upper-half plane intersecting the arc $x_1 \sim x_2$ and there are no arcs of $S$ on the lower-half plane intersecting the arc $x_2 \sim x_3$. We denote $I_1 = (0, x_1)$, $I_2 = (x_1, x_2)$, $I_3 = (x_2, x_3)$ and $I_4 = (x_3, n + 1)$ and define $|I_i|$ as the number of punctures that lie in the interval $I_i$.

We consider an automorphism of $D_n$, called $d = d(x_1, x_2, x_3)$, which is the final position of an isotopy that takes $D$ and moves it trough the upper half-plane to a disk of radius $\epsilon$ centered at $x_1$, which contains no point $c_p$ and no puncture, followed by an automorphism which fixes the real line as a set and takes the punctures back to the positions $1, \ldots, n$. This corresponds to “placing the interval $I_3$ between the intervals $I_1$ and $I_2$”. Firstly, we can see in Figure 9 that the only modifications that the arcs of $S$ can suffer is the shifting of their endpoints. By hypothesis, there are no arcs in the upper half-plane joining $I_2$ with $I_j$ for $j \neq 2$, and there are no arcs in the lower half-plane joining $I_3$ with $I_j$ for $j \neq 3$. Any other possible arc is transformed by $d$ into a single arc, so every arc is transformed in this way. Algebraically, take an arc of $S$, $c_a \sim c_b$ (resp. $c_a \sim c_b$), such that $L(c_a) = \tilde{a}$ and $L(c_b) = \tilde{b}$. Then, its image under $d$ is $c'_a \sim c'_b$ (resp. $c'_a \sim c'_b$) where

![Figure 8: Applying $s(0, 3, 6)$.](image)

![Figure 9: How the automorphism $d(x_1, x_2, x_3)$ acts on the arcs of $C$.](image)
Observe in Figure 11 that the only types of arcs that can appear in the region between the lines if and only if $|a|_{nikov coordinates. In fact, there are exactly $R$ intersecting both

If $i,k > j$ of $W$ We say that there is a

Definition 43. Let $S$ be a curve system on $D_n$ represented by the reduced Dynnikov coordinates $(a_0, b_0, \ldots, a_{n-1}, b_{n-1})$. Then $\ell(S) \leq \sum_{i=0}^{n-1} (2|a_i| + |b_i|)$.}

Proposition 44. Let $S$ be a curve system on $D_n$ represented by the reduced Dynnikov coordinates $(a_0, b_0, \ldots, a_{n-1}, b_{n-1})$. Then $\ell(S) \leq \sum_{i=0}^{n-1} (2|a_i| + |b_i|)$.}

Proof. Notice that each intersection of a curve $C$ with the real axis corresponds to a subword of $W(C)$ of the form $i \sim j \sim k$ or $i \sim j \sim k$. If $i < j < k$ the subword corresponds to a bending point or a reversed bending point respectively. If $i,k > j$, there is a left hairpin at $j+1$. Similarly, if $i,k < j$, there is a right hairpin at $j$.

Recall that Lemma 20 already establishes how to detect bending points with reduced Dynnikov coordinates. In fact, there are exactly $R$ bending points (including reversed ones) at $i$ if and only if $|a_{i-1} - a_i| = R$. We want to detect also hairpins in order to determine $\ell(S)$. Observe in Figure 11 that the only types of arcs that can appear in the region between the lines $e_{3j-5}$ and $e_{3j-2}$ are left or right hairpins and arcs intersecting both $e_{3j-5}$ and $e_{3j-2}$. The arcs intersecting both $e_{3j-5}$ and $e_{3j-2}$ do not affect the difference $x_{3j-5} - x_{3j-2}$ whereas each left

![Figure 10: Applying $s(i,j,k)$ to a curve is equivalent to permute their intersections with the real axis and then make the curve tight.](image-url)
Corollary 45. Let \( S \) be a curve system on \( D_n \) represented by the reduced Dynnikov coordinates \((a_0, b_0, \ldots, a_{n-1}, b_{n-1})\). Then, the length of the minimal standardizer of \( S \) is at most
\[
\frac{1}{2} \sum_{i=0}^{n-1} (2|a_i| + |b_i|)(n - 1)^2.
\]

Proof. By Lemma 42, the length of the minimal standardizer of \( S \) is at most \( \frac{1}{2} \ell(S)(n - 1)^2 \). Consider the bound for \( \ell(S) \) given in Proposition 44 and the result will follow. ■

Remark 46. To check that this bound is computationally optimal we need to find a case where at each step we can only remove a single bending point, i.e., we want to find a family of curve systems \( \{S_k\}_{k>0} \) such that the length of the minimal standardizer of \( S_k \) is quadratic on \( n \) and linear on \( \ell(S) \). Let \( n = 2t + 1 \), \( t \in \mathbb{N} \). Consider the following curve system on \( D_n \),
\[ S_0 = \{ t \sim n \sim \} \]
and the braid \( \alpha = s(0, t, n - 1) \). Now define \( S_k = (S_0)^{\alpha^k} \). The curve \( S_k \) is called a spiral with \( k \) twists (see Figure 12) and is such that \( \ell(S_k) = 2(k + 1) \). Using Algorithm 1, we obtain that the minimal standardizer of this curve is \( \alpha^k \), which has \( k \cdot t^2 \) factors. Therefore, the number of factors of the minimal standardizer of \( S_k \) is of order \( O(\ell(S_k) \cdot n^2) \).

Corollary 47. Let \( S \) be a curve system on \( D_n \) represented by the reduced Dynnikov coordinates \((a_0, b_0, \ldots, a_{n-1}, b_{n-1})\). Let \( m = \sum_{i=0}^{n-1} (|a_i| + |b_i|) \). Then, the complexity of computing the minimal standardizer of \( S \) is \( O(n^2 m \log(m)) \).

Proof. First notice that
\[
\ell(S) \leq \sum_{i=0}^{n-1} (2|a_i| + |b_i|) \leq 2 \sum_{i=0}^{n-1} (|a_i| + |b_i|) = 2m,
\]
and that the transformation described in Proposition 18 involves a finite number of basic operations (addition and max). Applying \( \sigma_j \) to the Dynnikov coordinates modifies only four such coordinates, and each maximum or addition between two numbers is linear on the number of digits of its arguments. This means that applying \( \sigma_j \) to the curve has a cost of \( O(\log(M)) \), where \( M = \max\{|a_i|, |b_i| \mid i = 0, \ldots, n-1\} \). By Corollary 45, the number of iterations performed by the algorithm is \( O(n^2m) \). Hence, as \( M \leq m \), computing the minimal standardizer of \( \mathcal{S} \) has complexity \( O(n^2m \log(m)) \).

To find the complexity of the algorithm which computes the minimal standardizer of a parabolic subgroup \( P = (X, \alpha) \) of an Artin-Tits group \( A \), we only need to know the cost of computing the pm-normal form of \( c_P \). If \( x_r \cdots x_1 \Delta^{-p} \) with \( p > 0 \) is the right normal form of \( c_P \), then its pm-normal form is \( (x_r \cdots x_{p+1})(x_p \cdots x_1\Delta^{-p}) \). Hence, we just have to compute the right normal form of \( c_P \) in order to compute the minimal standardizer. It is well known that this computation has quadratic complexity (for a proof, see (Dehornoy, 2008)). Thus, we have the following:

**Proposition 48.** Let \( P = (X, \alpha) \) be a parabolic subgroup of an Artin-Tits group of spherical type, and let \( \ell = \ell(\alpha) \) be the canonical length \( \alpha \). Computing the minimal standardizer of \( P \) has a cost of \( O(\ell^2) \).

**Acknowledgements.** This research was supported by a PhD contract founded by Université Rennes 1, Spanish Projects MTM2013-44233-P, MTM2016-76453-C2-1-P, FEDER and French-Spanish Mobility programme “Mérimée 2015”. I also thank my PhD advisors, Juan González-Meneses and Bert Wiest, for helping and guiding me during this research work.

**References**

Artin, Emil. 1947. Theory of Braids. *Ann. of Math.(2)*, 48, 101–126.

Brieskorn, Egbert, & Saito, Kyoji. 1972. Artin-gruppen und Coxeter-gruppen. *Invent. Math.*, 17(4), 245–271.

Calvez, Matthieu. 2012. Dual Garside structure and reducibility of braids. *J. Algebra*, 356(1), 355–373.

Caruso, Sandrine. 2013. *Algorithmes et généricité dans les groupes de tresses*. Ph.D. thesis, Université Rennes 1.
REFERENCES

Charney, Ruth. 1995. Geodesic automation and growth functions for Artin groups of finite type. *Math. Ann.*, **301**, 307–324.

Coxeter, Harold S. M. 1935. The Complete Enumeration of Finite Groups of the Form $r_i^2 = (r_i r_j)^{k_{ij}} = 1$. *J. Lond. Math. Soc.*, **1-10**(1), 21–25.

Dehornoy, Patrick. 2008. Efficient solutions to the braid isotopy problem. *Discrete Appl. Math.*, **156**(16), 3091–3112.

Dehornoy, Patrick, & Paris, Luis. 1999. Gaussian Groups and Garside Groups, two Generalisations of Artin Groups. *Proc. Lond. Math. Soc. (3)*, **79**, 569–604.

Dehornoy, Patrick, Dynnikov, Ivan, Rolfsen, Dale, & Wiest, Bert. 2002. *Why are braids orderable?* Panoramas et Synthèses, vol. 14. Paris: Société Mathématique de France.

Dehornoy, Patrick, Dynnikov, Ivan, Rolfsen, Dale, & Wiest, Bert. 2008. *Ordering braids.* Mathematical surveys and monographs, vol. 148. American Mathematical Society.

Dynnikov, Ivan, & Wiest, Bert. 2007. On the complexity of braids. *J. Eur. Math. Soc. (JEMS)*, **9**(4), 801–840.

Elrifai, Elsayed A., & Morton, Hugh R. 1994. Algorithms for positive braids. *Q. J. Math.*, **45**(4), 479–497.

Fenn, Roger, Greene, Michael T, Rolfsen, Dale, Rourke, Colin, & Wiest, Bert. 1999. Ordering the braid groups. *Pacific J. Math.*, **191**(1), 24.

Franco, Nuno, & González-Meneses, Juan. 2003. Conjugacy problem for braid groups and Garside groups. *J. Algebra*, **266**, 112–132.

Gebhardt, Volker, & González-Meneses, Juan. 2010a. Solving the conjugacy problem in Garside groups by cyclic sliding. *J. Symbolic Comput.*, **45**(6), 629–656.

Gebhardt, Volker, & González-Meneses, Juan. 2010b. The cyclic sliding operation in Garside groups. *Math. Z.*, **265**, 85–114.

Godelle, Eddy. 2003. Normalisateur et groupe d’Artin de type sphérique. *J. Algebra*, **269**(1), 263–274.

Lee, Eon-Kyung, & Lee, Sang-Jin. 2008. A Garside-theoretic approach to the reducibility problem in braid groups. *J. Algebra*, **320**(2), 783–820.

Paris, Luis. 1997. Parabolic Subgroups of Artin Groups. *J. Algebra*, **196**(2), 369–399.

Thurston, William P. 1988. On the geometry and dynamics of diffeomorphisms of surfaces. *Bull. Amer. Math. Soc. (N.S.)*, **19**(2), 417–432.

Van der Lek, Harm. 1983. *The Homotopy Type of Complex Hyperplane Complements.* Ph.D. thesis, Nijmegen.

María Cumplido, UFR Mathématiques, Université de Rennes 1, France, and Departamento de Álgebra, Universidad de Sevilla, Spain
E-mail address: maria.cumplidocabello@univ-rennes1.fr