Operator algebras associated with the 
Klein-Gordon position representation 
in relativistic quantum mechanics 

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We initiate a mathematically rigorous study of Klein-Gordon position operators 
in single-particle relativistic quantum mechanics. Although not self-adjoint, these 
operators have real spectrum and enjoy a limited form of spectral decomposition. 
The associated C*-algebras are identifiable as crossed products. We also introduce 
a variety of non self-adjoint operator algebras associated with the Klein-Gordon 
position representation; these algebras are commutative and continuously (but not 
homeomorphically) embeddable in corresponding function algebras. Several open 
problems are indicated.

1. Physical background

Consider a free, spinless, nonrelativistic quantum mechanical particle in $\mathbb{R}^n$. Its state 
is represented by a normalized function $\psi$ in $L^2(\mathbb{R}^n)$, such that the probability of the 
particle being found in a region $S \subset \mathbb{R}^n$ is

$$\int_S |\psi(\vec{x})|^2 d\vec{x}. $$

Similarly, the probability that a measurement of its momentum will return a value in $S$ is 
given by

$$\int_S |\phi(\vec{p})|^2 d\vec{p},$$

where $\phi = \hat{\psi}$ is the Fourier transform of $\psi$. (We adopt units in which $\hbar = c = 1$ and 
normalize the Fourier transform to be unitary from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$.) The associated 
nonrelativistic position operators are the multiplication operators $M_{x_i}$, $1 \leq i \leq n$ on 
$L^2(\mathbb{R}^n)$.

The relativistic picture is more complicated (see, e.g., [20]). The momentum representa-
tion now involves the weighted $L^2$ space $L^2(\mathbb{R}^n, d\vec{p}/E)$ where $E = E(\vec{p}) = \sqrt{|\vec{p}|^2 + m^2}$ 
is the relativistic energy, $m$ being the mass of the particle. (We consider throughout only 
the simplest case of positive mass and zero spin.) Thus, the probability of a momentum 
measurement yielding a value in $S \subset \mathbb{R}^n$ is

$$\int_S |\phi(\vec{p})|^2 \frac{d\vec{p}}{\sqrt{|\vec{p}|^2 + m^2}},$$

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where $\phi \in L^2(\mathbb{R}^n, d\vec{p}/E)$ is the normalized state of the particle in this momentum representation. The use of the measure $d\vec{p}/E$ is dictated by the requirement that Lorentz transformations must induce unitary transformations of state space; one can regard $\phi$ as being a function on the Lorentz-invariant positive mass shell

$$\{(\vec{p}, E) \in \mathbb{R}^{n+1} : E > 0, E^2 - |\vec{p}|^2 = m^2\},$$
onumber

on which $d\vec{p}/E$ is the essentially unique Lorentz-invariant measure.

The problem of determining a valid relativistic position representation has long been controversial. Those oriented in the direction of mathematical rigor generally seem to prefer the representation advanced by Newton and Wigner [15] and Wightman [22]. In this picture the position representation of the state of a particle with momentum representation $\phi \in L^2(\mathbb{R}^n, d\vec{p}/E)$ is

$$\psi_{NW} = (\phi/\sqrt{E})^\sim \in L^2(\mathbb{R}^n),$$
onumber

and the position operators are exactly the same as in the nonrelativistic case. Here $^\sim$ denotes the inverse Fourier transform. This representation has the advantage that the map $\phi \mapsto \psi_{NW}$ is unitary from $L^2(\mathbb{R}^n, d\vec{p}/E)$ to $L^2(\mathbb{R}^n)$. Moreover, Newton, Wigner, and Wightman established that their position representation was the only one which satisfied a small set of seemingly undebatable postulates.

Nonetheless, most physicists have evidently continued to prefer the so-called “Klein-Gordon” position representation [19], in which the state is represented as

$$\psi_{KG} = \check{\phi}.$$

(See [13] for a review of other proposed position representations, and [3] for more recent references.) Among the difficulties associated with this representation are the fact that $\check{\phi}$ need not belong to $L^2(\mathbb{R}^n)$, or even to be well-defined except as a distribution, and the failure of the corresponding position operators (these will be described in the next section) to be self-adjoint.

A third oddity of the Klein-Gordon position representation, and the cause of its failure to satisfy the Newton-Wigner-Wightman postulates, is the fact that states localized in disjoint spatial regions are in general not orthogonal. At first sight this seems to violate relativistic causality. However, the mere nonorthogonality of distant spatially separated states does not actually imply any mechanism for superluminal signalling. Moreover, a frustrating and seemingly unavoidable feature of essentially any relativistic notion of position for single particles — including both the Newton-Wigner and the Klein-Gordon proposals — is the instantaneous spreading throughout space of any free particle which is localized in a bounded spatial region at some time [9]. In particular, the state of a free particle localized in some bounded region at some time in the Newton-Wigner sense will, in general, not be orthogonal to a state which is localized in an arbitrarily distant region at a slightly later time. This fact calls into question the necessity of requiring orthogonality at a fixed time and makes the nonorthogonality exhibited by Klein-Gordon localized particles seem less objectionable in comparison to the Newton-Wigner picture.
Philips [16] took advantage of this point by proposing an alternative set of postulates for localization, which are satisfied by the Klein-Gordon representation but not the Newton-Wigner representation and which replace the orthogonality requirement mentioned above by a requirement of “Lorentz invariance.” Informally, this condition asserts that a state which appears as a delta function (i.e., is localized at a single spatial point) at some time according to one inertial observer must appear as a delta function at the same space-time event according to all inertial observers. Now since delta states are non-physical, it seems unlikely that this postulate has any direct physical content. In particular, one cannot reformulate the Lorentz invariance condition in terms of states localized in arbitrarily small but finite spatial regions: that is, one cannot require that states supported sufficiently near a given space-time event according to one observer must be supported near that event according to all observers. This is because the phenomenon of instantaneous propagation mentioned above implies that a state localized in a small region at some time according to one inertial observer will in fact not be localized in any bounded region at any time according to other observers, regardless of the position representation used. It therefore appears that, contrary to a claim often made in the literature, the requirement of Lorentz invariance is not actually necessary on physical grounds, nor even directly physically interpretable.

Thus, the fact that localized states instantaneously propagate throughout space calls into question, in different ways, the physical justification for both the Newton-Wigner and the Klein-Gordon position representations. Analyzing the localization problem in terms of field theory also fails to resolve the conflict: although one has mathematically rigorous models of free fields which identify the observables that can be measured in any given spatial region ([8]; see [21] for an elementary treatment), it turns out that for any spatial region $S$, the entire one-particle Hilbert space of a free field is generated by those one-particle states which can be detected with certainty by field observables localized in $S$. This phenomenon suggests that definite physical localization of one-particle states is simply not possible, so that the debate between rival one-particle localization schemes is operationally meaningless.

Of course, this negative conclusion does not deny that various position representations may be theoretically useful in different ways. However, if the issue is one of theoretical utility, and this apparently is the only real issue, we must face the uncomfortable fact that the Newton-Wigner notion of position, despite its manifest advantages from the mathematical point of view, has not been taken up by physicists with much enthusiasm. In fact the vast majority of works on relativistic quantum mechanics adopt the Klein-Gordon representation without comment, and the bulk of those which do discuss position or localization explicitly also favor this representation (see, e.g., [2], [7], [10], [11], [12], [14], [18]). Even a brief glance at any standard reference on relativistic quantum mechanics will show how essential the Klein-Gordon representation is, and how peripheral the Newton-Wigner representation, to its usual treatment.

In this light, it is rather surprising that the Klein-Gordon position representation for one-particle states has, up to now, not been explored from a serious mathematical point of view. The goal of this paper is to begin to study the Klein-Gordon representation in
a mathematically rigorous manner, and in particular to place it in an operator algebra context. As will be seen, many interesting questions emerge, several of which we have been unable to answer. Moreover, especially in later sections, the reader will find it easy to pose natural questions not addressed here.

2. Spectral subspaces

We will mainly work in the Fourier transformed (i.e., momentum) picture. As explained in Section 1, this involves the Hilbert space $L^2(\mathbb{R}^n, d\vec{p}/E)$. Now the Klein-Gordon position operators are, heuristically, coordinate multiplication operators in the untransformed picture, and they can therefore be rigorously defined as differentiation operators in the transformed picture. Similarly, translations and convolutions in the transformed picture can be viewed as different kinds of multiplication operators in the untransformed picture.

Definition 1.

(a) The relativistic momentum Hilbert space is the space $L^2(\mathbb{R}^n, d\vec{p}/E)$, where $E(\vec{p}) = \sqrt{|\vec{p}|^2 + m^2}$ and $m > 0$ is fixed. For simplicity we will write $L^2_r(\mathbb{R}^n) = L^2(\mathbb{R}^n, d\vec{p}/E)$. The norm of $\phi \in L^2_r(\mathbb{R}^n)$ will be denoted

$$\|\phi\|_r = \left( \int |\phi(\vec{p})|^2 \frac{d\vec{p}}{E(\vec{p})} \right)^{1/2}.$$

(b) The Klein-Gordon position operators are the unbounded operators $Q_i$ ($1 \leq i \leq n$) on $L^2_r(\mathbb{R}^n)$ defined by

$$Q_i \phi = i \frac{\partial \phi}{\partial p_i},$$

with domain $D(Q_i)$ the set of $\phi \in L^2_r(\mathbb{R}^n)$ whose restrictions to lines parallel to the $i$th coordinate axis are almost all locally absolutely continuous, and which satisfy $\partial \phi / \partial p_i \in L^2_r(\mathbb{R}^n)$.

(c) For $\vec{a} \in \mathbb{R}^n$ let $T_{\vec{a}}$ be the translation operator on $L^2_r(\mathbb{R}^n)$ defined by $T_{\vec{a}} \phi(\vec{p}) = \phi(\vec{p} - \vec{a})$.

(d) Let $\mathcal{S}(\mathbb{R}^n)$ denote the class of Schwartz functions on $\mathbb{R}^n$, and for $f \in \mathcal{S}(\mathbb{R}^n)$ let $K_f$ be the convolution operator (with respect to Lebesgue measure) acting on $L^2_r(\mathbb{R}^n)$ defined by

$$K_f \phi(\vec{p}) = f * \phi(\vec{p}) = \int_{\mathbb{R}^n} f(\vec{p'}) \phi(\vec{p} - \vec{p'}) d\vec{p'}.$$

In line with the previous comment, $T_{\vec{a}}$ is to be thought of heuristically as multiplication by $e^{i\vec{a} \cdot \vec{x}}$, and $K_f$ as multiplication by $\hat{f}$, in the position representation (untransformed picture).

Remarks 2.
(a) The usual Hilbert space $L^2(\mathbb{R}^n)$ is densely contained in $L^2_r(\mathbb{R}^n)$. Moreover, the set of compactly supported functions in $L^2(\mathbb{R}^n)$ equals the set of compactly supported functions in $L^2_r(\mathbb{R}^n)$.

(b) An equivalent definition of $D(Q_i)$ is: $\phi \in D(Q_i)$ if for every $N > 0$ the restriction $\phi|_{[-N,N]^n}$ belongs to the maximal domain of $i\partial/\partial p_i$ on $L^2([-N,N]^n)$, and $\partial\phi/\partial p_i \in L^2_r(\mathbb{R}^n)$. 

(c) The adjoint of $Q_i$ is the operator $M_E Q_i M_{1/E}$, i.e., the operator $\phi \mapsto E \cdot (i\partial/\partial p_i)(\phi/E)$. Thus, $Q_i$ is not self-adjoint, nor even normal.

(d) A simple calculation shows that the operators $T_{\bar{a}}$ ($\bar{a} \in \mathbb{R}^n$) are bounded, and the $K_f$ ($f \in S(\mathbb{R}^n)$) are bounded by [5, Theorem II.1.6] (cf. Lemma 36). Note that the $T_{\bar{a}}$ are not unitary.

Although the $Q_i$ are not normal, it is interesting to note that their spectra are real, just as in the nonrelativistic case.

**Proposition 3.** For each $i$, the operator $Q_i$ is closed and its spectrum is $\mathbb{R}$.

**Proof.** Both claims are fairly standard. To verify closure, suppose $\phi_k \to \phi$ and $Q_i \phi_k \to \psi$ in $L^2_r(\mathbb{R}^n)$. If $S = [-N,N]^n \subset \mathbb{R}^n$ then $L^2(S, dp/E) \cong L^2(S)$, so $\phi_k|_S \to \phi|_S$ and $(Q_i \phi_k)|_S \to \psi|_S$ in $L^2(S)$. Letting $A$ be the maximal version of $i\partial\phi/\partial p_i$ on $L^2(S)$, we have $\phi_k|_S \in D(A)$ and $A(\phi_k)|_S = (Q_i \phi_k)|_S$ for all $k$. Closure of $A$ implies that $\phi|_S \in D(A)$ and $i\partial\phi/\partial p_i = \psi$ on $S$. It follows that $\phi \in D(Q_i)$ and $Q_i \phi = \psi$, so $Q_i$ is closed.

Let $z \in \mathbb{C}$ and suppose $\text{Im} \ z > 0$. Then the operator $R_z$ defined by

$$R_z \phi(p) = \int_0^\infty e^{itz} \phi(p + te_i) \, dt,$$

where $e_i$ is the canonical basis vector, is a bounded inverse of $Q_i - zI$. A similar expression accomplishes the same result when $\text{Im} \ z < 0$. Thus the spectrum of $Q_i$ is contained in $\mathbb{R}$. Conversely, for any $a \in \mathbb{R}$ the functions

$$f_k(p) = e^{iap} e^{-|p|^2/k}$$

are approximate eigenvectors for $Q_i$ with approximate eigenvalue $a$. This implies that $(Q_i - aI)^{-1}$ is not bounded. So the spectrum of $Q_i$ equals $\mathbb{R}$. 

In fact the $Q_i$ even have simultaneous approximate eigenvectors, for instance the functions $e^{iap} e^{-|p|^2/k}$. This raises the question, to what extent do the $Q_i$ support a spectral decomposition of $L^2_r(\mathbb{R}^n)$? The following seems a reasonable notion of spectral subspaces associated to the Klein-Gordon position operators.

**Definition 4.** For any positive measure set $S \subset \mathbb{R}^n$, let $H_S \subset L^2_r(\mathbb{R}^n)$ be the closure (with respect to $\| \cdot \|_r$) of the set $\{ \hat{\psi} : \psi \in L^2(S) \}$. 

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Proposition 5. Let \( S \subset \mathbb{R}^n \) be a bounded measurable set. Then \( H_S \) is contained in the joint domain of the \( Q_i \), and the distributional equation \( Q_i \phi = (x_i \phi)^\wedge \) holds for all \( \phi \in H_S \).

Proof. Fix \( 1 \leq i \leq n \) and \( f \in S(\mathbb{R}^n) \) such that \( f(\vec{x}) = x_i \) on \( S \). Then let \( \phi \in H_S \) and find a sequence \( (\psi_k) \) in \( L^2(S) \) such that \( \hat{\psi}_k \to \phi \) in \( L^2_r(\mathbb{R}^n) \). We have \( f \psi_k = x_i \psi_k \) for all \( k \), and hence \( K_f \hat{\psi}_k = Q_i \hat{\psi}_k \) for all \( k \). This implies that the sequence \( (Q_i \hat{\psi}_k) \) converges in \( L^2_r(\mathbb{R}^n) \) (since \( K_f \) is bounded) and therefore, since \( Q_i \) is closed, that \( \phi \in D(Q_i) \). The equation \( Q_i \phi = (x_i \phi)^\wedge \) holds by continuity. \( \square \)

Now disjointness of \( S \) and \( T \) does not imply that \( H_S \) and \( H_T \) are orthogonal; this corresponds to the nonorthogonality of disjointly supported states mentioned in Section 1. However, we can still hope that \( H_S \cap H_T = \{0\} \) when \( S \) and \( T \) are disjoint. This is at least true to the following extent:

Proposition 6. Let \( S, T \subset \mathbb{R}^n \) be disjoint closed sets. Then \( H_S \cap H_T = \{0\} \).

Proof. Let \( \phi \in H_S \cap H_T \) and suppose \( \phi \neq 0 \). Fix \( \vec{x} \) in the support of the distribution \( \hat{\phi} \). Since \( S \) and \( T \) are disjoint, without loss of generality we can assume \( \vec{x} \not\in S \). Let \( f \) be a smooth, compactly supported function which is constantly zero on \( S \) and satisfies \( f(\vec{x}) \neq 0 \). Then \( f \hat{\phi} \neq 0 \), so \( f \ast \phi \neq 0 \) by the definition of the convolution of a function with a distribution. But since \( \phi \in H_S \) we can find a sequence \( (\psi_k) \) in \( L^2(S) \) such that \( \hat{\psi}_k \to \phi \) in \( L^2_r(\mathbb{R}^n) \). Then \( f \psi_k = 0 \) for all \( k \), which implies

\[
K_f \hat{\psi}_k = (f \psi_k)^\wedge = 0
\]

for all \( k \), which implies \( K_f \phi = 0 \) by continuity. This is a contradiction, and we conclude that \( \phi = 0 \). \( \square \)

The technique of the preceding proof can be adapted to the following situation as well.

Proposition 7. Let \( S, T \subset \mathbb{R}^n \) be disjoint measurable sets and suppose the boundary of \( S \) is piecewise linear. Then \( H_S \cap H_T = \{0\} \).

Proof. Let \( \phi \in H_S \cap H_T \) and suppose \( \phi \neq 0 \). The argument in the proof of Proposition 6 shows that the support of \( \hat{\phi} \) must be contained in both the closure of \( S \) and the closure of \( T \), and hence it must be contained in the boundary of \( S \). Since this boundary is piecewise linear, we can find a smooth function \( f \) with compact support such that \( f \hat{\phi} \) is nonzero and is supported in a hyperplane. But the Fourier transform of \( f \hat{\phi} \) then has constant modulus in directions perpendicular to the hyperplane, so \( f \ast \phi = K_f \phi \) cannot belong to \( L^2_r(\mathbb{R}^n) \), contradicting boundedness of \( K_f \). We conclude that \( \phi = 0 \). \( \square \)

Problem 8. Does \( H_S \cap H_T = \{0\} \) hold for any disjoint positive measure sets \( S, T \subset \mathbb{R}^n \)? Does it hold if \( S \) has smooth boundary?

We note that the answer to Problem 8 depends on the rate of decrease of the weight function \( 1/E(\rho) \). If we took \( E(\rho) \equiv 1 \), then \( L^2_r(\mathbb{R}^n) \) would equal \( L^2(\mathbb{R}^n) \) and we would
certainly have $H_S \cap H_T = \{0\}$ whenever $S$ and $T$ were disjoint. On the other hand, if we took $E(p) = 1 + p^2$ in the case $n = 1$, then the function $\phi(p) \equiv 1$ would belong to $L^2_\nu(\mathbb{R})$ and would be in $H_S \cap H_T$ for $S = (0, 1)$ and $T = (-1, 0)$. This shows that even Proposition 7 does not hold for weights of more rapid decrease.

3. C*- and von Neumann algebras

In this section we identify the structure of some self-adjoint operator algebras affiliated with the Klein-Gordon position operators. Although the non self-adjoint operator algebras to be discussed in later sections seem more interesting, we deal with the self-adjoint case first since these algebras are easier.

Specifically, the algebras we consider in this section are the C*-algebras

$$C^*(T_{\vec{a}} : \vec{a} \in \mathbb{R}^n)$$

and

$$C^*(K_f : f \in \mathcal{S}(\mathbb{R}^n))$$

generated by the operators $T_{\vec{a}}$ and $K_f$, and the von Neumann algebra

$$W^*(T_{\vec{a}} : \vec{a} \in \mathbb{R}^n) = W^*(K_f : f \in \mathcal{S}(\mathbb{R}^n))$$

generated by either set. (It is straightforward to verify that every $T_{\vec{a}}$ can be approximated in the strong operator topology by operators of the form $K_f$, and that every $K_f$ can be strong operator approximated by finite linear combinations of operators of the form $T_{\vec{a}}$ — just approximate the integral $K_f = \int f(\vec{a})T_{\vec{a}}\,d\vec{a}$ by Riemann sums — so that the two definitions of the von Neumann algebra do coincide.)

In order to analyze these algebras, it is helpful to pass to a different Hilbert space. The desired transformation is this:

**Definition 9.** Regard the multiplication operator $M_{\sqrt{E}}$ as a unitary map from the usual Hilbert space $L^2(\mathbb{R}^n)$ to the relativistic space $L^2_\nu(\mathbb{R}^n)$. Then define operators $\tilde{T}_{\vec{a}}$ and $\tilde{K}_f$ on $L^2(\mathbb{R}^n)$ by

$$\tilde{T}_{\vec{a}} = M_{\sqrt{E}}^{-1}T_{\vec{a}}M_{\sqrt{E}}$$

and

$$\tilde{K}_f = M_{\sqrt{E}}^{-1}K_fM_{\sqrt{E}}$$

for $\vec{a} \in \mathbb{R}^n$ and $f \in \mathcal{S}(\mathbb{R}^n)$.

We first show that the von Neumann algebra generated by the $T_{\vec{a}}$ (or equivalently, the $K_f$) is trivial.

**Theorem 10.** $W^*(\tilde{T}_{\vec{a}} : \vec{a} \in \mathbb{R}^n) = B(L^2(\mathbb{R}^n))$.

**Proof.** Let $\mathcal{M} = W^*(\tilde{T}_{\vec{a}} : \vec{a} \in \mathbb{R}^n) \subset B(L^2(\mathbb{R}^n))$. Simple calculations show that

$$\tilde{T}_{\vec{a}}\phi(\vec{p}) = \sqrt{\frac{E(\vec{p} - \vec{a})}{E(\vec{p})}}\phi(\vec{p} - \vec{a})$$
and

$$\tilde{T}_{\tilde{a}}^*\phi(\tilde{p}) = \sqrt{\frac{E(\tilde{p})}{E(\tilde{p} + \tilde{a})}} \phi(\tilde{p} + \tilde{a})$$

for all $\phi \in L^2(\mathbb{R}^n)$. Therefore the operator $\tilde{T}_{\tilde{a}}^*\tilde{T}_{\tilde{a}}$ is multiplication by the function $E(\tilde{p})/E(\tilde{p} + \tilde{a})$. These functions separate points in $\mathbb{R}^n$, and therefore $\mathcal{M}$ must contain all bounded multiplication operators. In particular, it contains multiplication by the function $\sqrt{E(\tilde{p})/E(\tilde{p} - \tilde{a})}$ for every $\tilde{a}$; taking the product of this operator with $\tilde{T}_{\tilde{a}}$, we deduce that every translation operator on $L^2(\mathbb{R}^n)$ belongs to $\mathcal{M}$.

Now since $\mathcal{M}$ contains every multiplication operator, any operator in the commutant of $\mathcal{M}$ must itself be a multiplication operator. But $\mathcal{M}$ also contains all translations, and the only multiplication operators which commute with all translations are the scalar multiples of the identity. Thus $\mathcal{M}' = C \cdot I$ and $\mathcal{M} = \mathcal{M}' = B(L^2(\mathbb{R}^n))$.

Next we characterize the structure of the two C*-algebras mentioned above.

**Theorem 11.** $C^*(\tilde{T}_{\tilde{a}} : \tilde{a} \in \mathbb{R}^n)$ is the C*-algebra generated by the translations on $L^2(\mathbb{R}^n)$ together with the multiplication operators by continuous functions vanishing at infinity. It is *-isomorphic to the crossed product $C_0(\mathbb{R}^n)^+ \times_\lambda \mathbb{R}_d^\gamma$, where $C_0(\mathbb{R}^n)^+$ is the unitization of $C_0(\mathbb{R}^n)$, $\mathbb{R}_d^\gamma$ is $\mathbb{R}^n$ with the discrete topology, and $\lambda$ is the action of $\mathbb{R}_d^\gamma$ on $C_0(\mathbb{R}^n)^+$ by translations.

**Proof.** Let $\mathcal{A}_1 = C^*(\tilde{T}_{\tilde{a}} : \tilde{a} \in \mathbb{R}^n)$. As in the proof of Theorem 10, $\mathcal{A}_1$ contains multiplication by the function $E(\tilde{p})/E(\tilde{p} + \tilde{a})$, for every $\tilde{a} \in \mathbb{R}^n$. These functions separate points and they all converge to 1 at infinity (and are not all 1 at any other point), so they generate the C*-algebra of multiplications by continuous functions on $\mathbb{R}^n$ which are continuously extendible to the one-point compactification of $\mathbb{R}^n$. In particular, $\mathcal{A}_1$ contains multiplication by any function in $C_0(\mathbb{R}^n)$; also, as in the proof of Theorem 10 we can deduce that $\mathcal{A}_1$ contains all translation operators. Thus one containment of the first assertion of the theorem holds. The reverse containment is easy: any C*-algebra which contains the translations must contain the identity operator, and if it also contains multiplications by functions in $C_0(\mathbb{R}^n)$ then it contains multiplications by all continuous functions which converge at infinity, in particular by the functions $\sqrt{E(\tilde{p} - \tilde{a})}/E(\tilde{p})$. From the expression for $\tilde{T}_{\tilde{a}}$ given in the proof of Theorem 10, we conclude that the C*-algebra generated by $C_0(\mathbb{R}^n)$ and the translations contains $\mathcal{A}_1$. This completes the proof of the first assertion of the theorem.

For the second assertion, recall that the crossed product $C_0(\mathbb{R}^n)^+ \times_\lambda \mathbb{R}_d^\gamma$ is the universal C*-algebra generated by a copy of $C_0(\mathbb{R}^n)^+$ and a (not continuous) one-parameter group of unitaries $U_{\tilde{a}} (\tilde{a} \in \mathbb{R}^n)$ which satisfy the commutation relations $U_{\tilde{a}} f = f_{\tilde{a}} U_{\tilde{a}}$, where $f_{\tilde{a}}$ is the translation of $f \in C_0(\mathbb{R}^n)^+$ by $\tilde{a}$. Now $\mathcal{A}_1$ is, by the preceding paragraph, generated by a copy of $C_0(\mathbb{R}^n)^+$ and the (unitary) translations on $L^2(\mathbb{R}^n)$, which satisfy the same commutation relations. Thus $\mathcal{A}_1$ is naturally a *-homomorphic image of the crossed product C*-algebra. To show isomorphism, we must verify that the kernel of this *-homomorphism is zero.
Let $\mathcal{I}$ be this kernel. Consider the action $\alpha$ of $\mathbb{R}^n$ on $C_0(\mathbb{R}^n)^+ \times \gamma \mathbb{R}^n_\alpha$ defined by $\alpha_{\tilde{a}}(fU_\tilde{b}) = e^{i\tilde{a} \cdot \tilde{b}}fU_\tilde{b}$. Every element of the crossed product is almost periodic for this action, so taking mean-value integrals yields a mean value map $m$ from the crossed product to the fixed point algebra for $\alpha$. But the mean value map is the identity on $C_0(\mathbb{R}^n)^+$ and annihilates all elements of the form $fU_\tilde{b}$ for $\tilde{b} \neq 0$. Thus $C_0(\mathbb{R}^n)^+$ is the fixed point algebra.

Now suppose $\mathcal{I} \neq \{0\}$. Then $\mathcal{I}$ contains a positive operator $A$, and so $m(A) > 0$ by a standard property of the mean value map. Let $f \in C_0(\mathbb{R}^n)$ be compactly supported and satisfy $f \geq 0$ and $f^2m(A) \neq 0$.

We claim that $\mathcal{I}$ contains $m(fAf) = f^2 \cdot m(A)$. It will suffice to verify that $\mathcal{I}$ contains $\alpha_{\tilde{a}}(fAf)$ for all $\tilde{a}$. Find $g_{\tilde{a}} \in C_0(\mathbb{R}^n)$ such that $g_{\tilde{a}}f = e^{i\tilde{a} \cdot \tilde{b}}f$. Then $\alpha_{\tilde{a}}(fAf) = g_{\tilde{a}}fAf g_{\tilde{a}} \in \mathcal{I}$ since $\mathcal{I}$ is an ideal. Thus $\mathcal{I}$ contains $f^2m(A)$, which is a nonzero element of $C_0(\mathbb{R}^n)^+$. But the quotient map from $C_0(\mathbb{R}^n)^+ \times \gamma \mathbb{R}^n_\alpha$ to $\mathcal{A}_1$ is clearly isometric on $C_0(\mathbb{R}^n)^+$, a contradiction. We conclude that $\mathcal{I}$ is zero and the quotient map is a $*$-isomorphism.

For any operator $A \in B(L^2(\mathbb{R}^n))$, let $\hat{A}$ denote its conjugation by the Fourier transform; that is, $\hat{A}\phi = (A\phi)^\ast$. Also write $\hat{\mathcal{A}} = \{\hat{A} : A \in \mathcal{A}\}$ for any subalgebra $\mathcal{A}$ of $B(L^2(\mathbb{R}^n))$.

**Theorem 12.** $C^*(\hat{K}_f : f \in \mathcal{S}(\mathbb{R}^n)) = \overline{C_0(\mathbb{R}^n)^+} + K(L^2(\mathbb{R}^n))$, regarding $C_0(\mathbb{R}^n)$ as acting on $L^2(\mathbb{R}^n)$ by multiplication. It is $*$-isomorphic to the crossed product $C_0(\mathbb{R}^n)^+ \times \gamma \mathbb{R}^n$, where $C_0(\mathbb{R}^n)^+$ is the unitization of $C_0(\mathbb{R}^n)$ and $\lambda$ is the action of $\mathbb{R}^n$ on $C_0(\mathbb{R}^n)^+$ by translations (and the group $\mathbb{R}^n$ has its usual topology).

**Proof.** Let $\mathcal{A}_2 = C^*(\hat{K}_f : f \in \mathcal{S}(\mathbb{R}^n))$. A straightforward computation shows that
\[
\hat{K}_f \phi(\tilde{p}) = \int f(\tilde{p'}) \sqrt{\frac{E(\tilde{p} - \tilde{p'})}{E(\tilde{p})}} \phi(\tilde{p} - \tilde{p'}) \, d\tilde{p'}
\]
for all $\phi \in L^2(\mathbb{R}^n)$, and its adjoint satisfies
\[
\hat{K}_f^* \phi(\tilde{p}) = \int f^*(\tilde{p'}) \sqrt{\frac{E(\tilde{p})}{E(\tilde{p} - \tilde{p'})}} \phi(\tilde{p} - \tilde{p'}) \, d\tilde{p'}
\]
where $f^*(\tilde{p}) = f(-\tilde{p})$. Thus, for any $f \in \mathcal{S}(\mathbb{R}^n)$ we have
\[
(\hat{K}_f - \hat{K}_f^*) \phi(\tilde{p}) = \int f(\tilde{p'}) \left[ \sqrt{\frac{E(\tilde{p} - \tilde{p'})}{E(\tilde{p})}} - \sqrt{\frac{E(\tilde{p})}{E(\tilde{p} - \tilde{p'})}} \right] \phi(\tilde{p} - \tilde{p'}) \, d\tilde{p'}.
\]
This is an integral operator on $L^2(\mathbb{R}^n)$ with kernel in $C_0(\mathbb{R}^{2n})$. If the kernel function is multiplied by the characteristic function of the ball of radius $R$ about the origin in $\mathbb{R}^{2n}$, then the corresponding integral operator will be compact, and these truncated operators converge to the original operator in norm by an estimate using [5, Theorem II.1.6]. Hence $\hat{K}_f - \hat{K}_f^*$ is compact (and nonzero if $f$ is nonzero). By Theorem 10, $\mathcal{A}_2$ acts irreducibly.
on \( L^2(\mathbb{R}^n) \). As we have just shown that \( \mathcal{A}_2 \) nontrivially intersects the compact operators, it follows that \( \mathcal{A}_2 \) contains \( K(L^2(\mathbb{R}^n)) \) [1, Corollary 2, p. 18].

Now for any \( f \in S(\mathbb{R}^n) \) the operator \( A_f \in B(L^2(\mathbb{R}^n)) \) defined by

\[
A_f \phi(\vec{p}) = \int f(\vec{p}') \left[ \sqrt{\frac{E(\vec{p} - \vec{p}')} {E(\vec{p})}} - 1 \right] \phi(\vec{p} - \vec{p}') \, d\vec{p}'
\]

is another integral operator which is compact by a similar argument to the one indicated above, and hence it belongs to \( \mathcal{A}_2 \). It follows that the difference \( K_f - A_f \), which is simply convolution by \( f \), also belongs to \( \mathcal{A}_2 \) and that \( \mathcal{A}_2 \) is generated by \( K(L^2(\mathbb{R}^n)) \) together with the convolution operators by \( f \in S(\mathbb{R}^n) \). The \( C^* \)-algebra generated by the latter operators is just \( \overline{C_0(\mathbb{R}^n)} \), so we conclude that \( \mathcal{A}_2 = \overline{C_0(\mathbb{R}^n)} + K(L^2(\mathbb{R}^n)) \). (Note that we do not have to take the closure; it is a basic exercise in \( C^* \)-algebra theory to show that this sum is already closed.)

The crossed product \( C^* \)-algebra \( C_0(\mathbb{R}^n) \times_{\chi} \mathbb{R}^n \) is \( * \)-isomorphic to \( K(L^2(\mathbb{R}^n)) \) [6, Theorem 8.4.3]. Thus the larger crossed product \( C_0(\mathbb{R}^n)^+ \times_{\chi} \mathbb{R}^n \) is generated by \( K(L^2(\mathbb{R}^n)) \) together with \( C^*(\mathbb{R}^n) \cong C_0(\mathbb{R}^n) \). This shows that the latter crossed product is also isomorphic to \( \overline{C_0(\mathbb{R}^n)} + K(L^2(\mathbb{R}^n)) \). (The isomorphism is natural; it is implemented by the following map. For any \( f \in S(\mathbb{R}^n) \) with compact support, define a function \( \omega_f \in C_c(\mathbb{R}^n, C_0(\mathbb{R}^n)^+) \) by

\[
\omega_f(\vec{a}) = f(\vec{a}) \sqrt{\frac{E(\vec{p} - \vec{a})} {E(\vec{p})}}.
\]

Regarding \( C_0(\mathbb{R}^n)^+ \times_{\chi} \mathbb{R}^n \) as a completion of \( C_c(\mathbb{R}^n, C_0(\mathbb{R}^n)^+) \), the map \( \tilde{K}_f \mapsto \omega_f \) extends to a \( * \)-isomorphism from \( \mathcal{A}_2 \) onto \( C_0(\mathbb{R}^n)^+ \times_{\chi} \mathbb{R}^n \).)

4. A non self-adjoint analog of \( L^\infty(\mathbb{R}^n) \)

We mentioned at the start of the last section that the self-adjoint algebras affiliated with the Klein-Gordon position operators seem less interesting than the non self-adjoint algebras. One reason for this is that the non self-adjoint algebras play a functional calculus role that the self-adjoint algebras do not. Intuitively, the operators \( T_\vec{a} \) and \( K_f \) can be regarded as arising from the \( Q_i \) by the equations \( T_\vec{a} = e^{i\vec{a} \cdot \vec{Q}} \) and \( K_f = \tilde{f}(\vec{Q}) \), where \( \vec{Q} = (Q_1, \ldots, Q_n) \). Their adjoints have no such interpretation. Thus, various non self-adjoint algebras can be seen as the set of all operators arising from the \( Q_i \) via functional calculi involving various classes of functions. We begin in this section with the largest (bounded) class of interest, and discuss other classes in the next section.

**Definition 13.**

(a) Denote by \( L^\infty_c(\mathbb{R}^n) \) the commutant of the set \( \{T_\vec{a} : \vec{a} \in \mathbb{R}^n\} \), or equivalently, of the set \( \{K_f : f \in S(\mathbb{R}^n)\} \).
(b) Let $\gamma$ be the action of $\mathbb{R}^n$ on $L^\infty_r(\mathbb{R}^n)$ defined by

$$\gamma_{\hat{a}}(A) = M_{e^{-i\sigma \hat{a}}} A M_{e^{i\sigma \hat{a}}}.$$ 

As we noted at the beginning of Section 3, the $K_f$ and linear combinations of the $T_\hat{a}$ mutually strong operator approximate each other. Thus the commutants of the two sets are the same.

The set span$\{T_\hat{a} : \hat{a} \in \mathbb{R}^n\}$ is stable under $\gamma$, and it follows that $L^\infty_r(\mathbb{R}^n)$ is also stable under $\gamma$. The action $\gamma$ is, in the Fourier transform picture, analogous to the action of $\mathbb{R}^n$ on $L^\infty(\mathbb{R}^n)$ by translations. It will play a basic role in the sequel.

Remarks 14.

(a) The action $\gamma$ on $L^\infty_r(\mathbb{R}^n)$ arises via conjugation by a unitary action of the group of translations of $\mathbb{R}^n$ on $L^2_r(\mathbb{R}^n)$. The latter extends to a natural representation of the $(n+1)$-dimensional Poincaré group on $L^2_r(\mathbb{R}^n)$. To see this, regard elements of $L^2_r(\mathbb{R}^n)$ as functions on the positive mass shell $\{(\hat{p},E) \in \mathbb{R}^{n+1} : E > 0, E^2 - |\hat{p}|^2 = m^2\} \subset \mathbb{R}^{n+1}$. Then Lorentz transformations act on $L^2_r(\mathbb{R}^n)$ by composition, and these actions are unitary by Lorentz invariance of the measure $d\hat{p}/E$. The translation of $\mathbb{R}^{n+1}$ by $(\hat{a},a_{n+1}) \in \mathbb{R}^{n+1}$ acts on $L^2_r(\mathbb{R}^n)$ via the multiplication operator $M_{e^{i(\hat{a} \cdot \hat{p} + a_{n+1} E)}}$.

(b) This action of the Poincaré group on $L^2_r(\mathbb{R}^n)$ does not induce an action on $L^\infty_r(\mathbb{R}^n)$, since the latter is, in general, not stable under conjugation by the above unitaries. However, the group of isometries of $\mathbb{R}^n$, regarded as a subgroup of the Poincaré group, does act on $L^\infty_r(\mathbb{R}^n)$ by isometries in this way.

(c) The group $GL(n,\mathbb{R})$ also acts boundedly on $L^2_r(\mathbb{R}^n)$ by composition, and conjugation by these operators yields an action of $GL(n,\mathbb{R})$ on $L^\infty_r(\mathbb{R}^n)$ by (non-isometric) isomorphisms.

We begin with a simple fact about $L^\infty_r(\mathbb{R}^n)$.

**Proposition 15.** $L^\infty_r(\mathbb{R}^n)$ is a dual space.

**Proof.** It is easy to check that $L^\infty_r(\mathbb{R}^n)$ is a weak operator closed linear subspace of $B(L^2_r(\mathbb{R}^n))$. This automatically implies that $L^\infty_r(\mathbb{R}^n)$ is weak* (ultraweakly) closed. A weak* closed subspace of a dual space is always a dual space.

We proceed immediately to our main result on $L^\infty_r(\mathbb{R}^n)$, which states that it consists of convolutions by distributions which are Fourier transforms of functions in $L^\infty(\mathbb{R}^n)$. Thus, $L^\infty_r(\mathbb{R}^n)$ can be (non-homeomorphically) embedded as a subalgebra of $L^\infty(\mathbb{R}^n)$.

Recall the notation $\hat{\mathbb{A}}$ defined just before Theorem 12.

**Theorem 16.** Regarding $L^2(\mathbb{R}^n)$ as a subset of $L^2_r(\mathbb{R}^n)$, for every $A \in L^\infty_r(\mathbb{R}^n)$ we have $A(L^2(\mathbb{R}^n)) \subset L^2(\mathbb{R}^n)$. The restriction $A|_{L^2(\mathbb{R}^n)}$ is a bounded operator on $L^2(\mathbb{R}^n)$, and the
restriction map \( \hat{\sigma} : A \mapsto A|_{L^2(\mathbb{R}^n)} \) is an injective, contractive, weak*-continuous algebra homomorphism from \( L^\infty(\mathbb{R}^n) \) into \( \widehat{L^\infty(\mathbb{R}^n)} \).

**Proof.** Let \( A \in L^\infty(\mathbb{R}^n) \) and let \( \phi \in L^2(\mathbb{R}^n) \) be nonzero and have compact support (so that \( \phi \in L^2(\mathbb{R}^n) \) also). Let \( T_t \) be the operator of translation by \( t \) units in the first coordinate. Then writing \( \| \cdot \| \) for the norm in \( L^2(\mathbb{R}^n) \), we have

\[
\lim_{t \to \infty} \sqrt{t}\|T_t\phi\|_r = \|\phi\|
\]

(using the fact that \( \phi \) has compact support), and

\[
\liminf_{t \to \infty} \sqrt{t}\|T_tA\phi\|_r \geq \|A\phi\|
\]

(even if, hypothetically, \( \|A\phi\| \) is infinite), so that

\[
\liminf_{t \to \infty} \frac{\|AT_t\phi\|_r}{\|T_t\phi\|_r} = \liminf_{t \to \infty} \frac{\|T_tA\phi\|_r}{\|T_t\phi\|_r} \geq \|A\phi\| / \|\phi\|.
\]

This shows that \( \|A\phi\| / \|\phi\| \leq \|A\| \) (the norm of \( A \) as an element of \( L^\infty(\mathbb{R}^n) \)) for all compactly supported, nonzero \( \phi \), but we cannot immediately deduce that \( A|_{L^2(\mathbb{R}^n)} \) is bounded on \( L^2(\mathbb{R}^n) \). For arbitrary nonzero \( \phi \) we must make the following argument. Let \( \phi \in L^2(\mathbb{R}^n) \) and find a sequence \( (\phi_k) \) of compactly supported functions which converge to \( \phi \) in \( L^2(\mathbb{R}^n) \) (and hence in \( L^2_r(\mathbb{R}^n) \)); by the above the sequence \( (A\phi_k) \) is Cauchy in \( L^2(\mathbb{R}^n) \), and \( A\phi_k \to A\phi \) in \( L^2(\mathbb{R}^n) \); this implies that \( A\phi \in L^2(\mathbb{R}^n) \) and \( \|A\phi\| / \|\phi\| \leq \|A\| \). We now conclude that \( \hat{\sigma}(A) = A|_{L^2(\mathbb{R}^n)} \) is a bounded operator on \( L^2(\mathbb{R}^n) \) and \( \|\hat{\sigma}(A)\| \leq \|A\| \). Injectivity of \( \hat{\sigma} \) follows from the fact that \( L^2(\mathbb{R}^n) \) is dense in \( L^2_r(\mathbb{R}^n) \), and \( \hat{\sigma} \) is trivially an algebra homomorphism. Next, it is clear that \( \hat{\sigma}(T_{\vec{a}}) = \hat{M}_{\vec{a},\vec{x}} \in L^\infty(\mathbb{R}^n) \) for any \( \vec{a} \in \mathbb{R}^n \). But any \( A \in L^\infty(\mathbb{R}^n) \) commutes with all \( T_{\vec{a}} \), and hence \( \hat{\sigma}(A) \) commutes with all \( \hat{\sigma}(T_{\vec{a}}) \), so we conclude \( A \in L^\infty(\mathbb{R}^n) \) since the latter is a maximal abelian subalgebra of \( B(L^2(\mathbb{R}^n)) \) and is generated as a von Neumann algebra by the operators \( \hat{\sigma}(T_{\vec{a}}) \). Weak*-continuity of \( \hat{\sigma} \) holds for bounded nets since the weak* topology agrees with the weak operator topology on bounded sets, and it then holds in general by the Krein-Smulian theorem. \( \blacksquare \)

**Definition 17.** Define a map \( \sigma : L^\infty_r(\mathbb{R}^n) \to L^\infty(\mathbb{R}^n) \) by composing \( \hat{\sigma} \) with the natural isomorphism of \( L^\infty(\mathbb{R}^n) \) with \( L^\infty(\mathbb{R}^n) \). We call \( \sigma \) the symbol map.

We will retain both notations \( \sigma \) and \( \hat{\sigma} \), as the latter will occasionally continue to be useful.

We give three easy corollaries of Theorem 16.

**Corollary 18.** \( L^\infty_r(\mathbb{R}^n) \) is a maximal abelian subalgebra of \( B(L^2_r(\mathbb{R}^n)) \).

(It is abelian since \( \phi \) is injective, and maximality is then an immediate consequence of Definition 13 (a).)
Corollary 19. Let $S \subset \mathbb{R}^n$ be measurable and let $H_S \subset L^2_\gamma(\mathbb{R}^n)$ be as in Definition 4. Then $H_S$ is invariant for every $A \in L_\infty(\mathbb{R}^n)$, and the distributional equation

$$A\phi = (\sigma(A)\check{\phi})^\wedge$$

holds for all $\phi \in H_S$.

This holds since the restriction of $A$ to $L^2(\mathbb{R}^n)$ is multiplication by $\sigma(A)$ in the untransformed picture. It follows that $L^2_\gamma(S)$ is invariant for $A|_{L^2(\mathbb{R}^n)}$, and hence $H_S$ is invariant for $A$ by continuity. The equation for $A\phi$ also holds by continuity.

An algebra is ergodic for a group action if the only fixed points of the action are scalar multiples of the identity. As the action of $\mathbb{R}^n$ on $L_\infty(\mathbb{R}^n)$ by translations is ergodic and pulls back under $\sigma$ (which is injective) to the action $\gamma$ on $L_\infty^\gamma(\mathbb{R}^n)$, the following corollary is immediate.

Corollary 20. The action of $\gamma$ on $L_\infty^\gamma(\mathbb{R}^n)$ is ergodic.

The preceding result is in contrast with the von Neumann algebra discussed in the last section, which contains all bounded multiplication operators on $L^2(\mathbb{R}^n)$, and hence is vastly non-ergodic for this action.

Next we give a more serious corollary.

Corollary 21. The sets $\{K_f : f \in S(\mathbb{R}^n)\}$ and $\text{span}\{T_{\bar{a}} : \bar{a} \in \mathbb{R}^n\}$ are strong operator dense in $L^\infty_\gamma(\mathbb{R}^n)$.

Proof. It will suffice to verify that the $K_f$ are dense in $L^\infty_\gamma(\mathbb{R}^n)$.

Let $f$ be a smooth, compactly supported function on $\mathbb{R}^n$ such that $\int f = 1$. Then define a sequence $(f_k)$ by $f_k(\bar{p}) = k^n f(\bar{k}\bar{p})$.

Let $A \in L^\infty_\gamma(\mathbb{R}^n)$. We will show that $A$ is strong operator approximated by convolution operators by Schwartz functions. Observe first that the sequence $(K_{f_k})$ strong operator converges to the identity operator on $L^2_\gamma(\mathbb{R}^n)$, since $\|K_{f_k}\| \to 1$ (a consequence of the elementary estimate [5, Theorem II.1.6]; see Lemma 36) and $K_{f_k}g \to g$ in $S(\mathbb{R}^n)$ for any Schwartz function $g$. So $K_{f_k}A \to A$ strong operator and it will suffice to approximate $K_{f_k}A$ for arbitrary $k$. Thus, fix $k$ and let

$$A_j = \int_{\mathbb{R}^n} f_j(\bar{p})\gamma_{\bar{p}}(K_{f_k}A) \, d\bar{p}.$$ 

This integral can be defined weakly, i.e., by the formula

$$\langle A_j \phi, \psi \rangle = \int_{\mathbb{R}^n} f_j(\bar{p})\langle \gamma_{\bar{p}}(K_{f_k}A)\phi, \psi \rangle \, d\bar{p}$$

for $\phi, \psi \in L^2_\gamma(\mathbb{R}^n)$. Since $\gamma$ is implemented by unitary operators on $L^2_\gamma(\mathbb{R}^n)$, it follows that $\|A_j\| \leq \|f\|_1 \|K_{f_k}A\|$. Now if $\sigma(A) = g \in L_\infty(\mathbb{R}^n)$, then a classical computation shows
that the restriction of \( K_{f_k} A \) to \( L^2(\mathbb{R}^n) \subset L^2_\sigma(\mathbb{R}^n) \) is \( \hat{M}_{f_k \sigma} \), whereas the restriction of \( A_j \) is \( \hat{M}_{f_j \sigma(f_k \sigma)} \). It therefore follows that \( A_j \to K_{f_k} A \) strongly on vectors in \( L^2(\mathbb{R}^n) \), and since the latter is a dense subspace of \( L^2_\sigma(\mathbb{R}^n) \) and the sequence \( (A_j) \) is bounded, we conclude that \( A_j \to K_{f_k} A \) strongly. Moreover, \( \hat{f}_j(f_k \sigma \hat{g}) \in \mathcal{S}(\mathbb{R}^n) \) and \( A_j = K_{\hat{f}_j(f_k \sigma \hat{g})} \) (on \( L^2(\mathbb{R}^n) \)), and hence by continuity on \( L^2_\sigma(\mathbb{R}^n) \)), so we conclude that \( A \) is a strong operator limit of convolution operators by Schwartz functions.

Next, it is interesting to note that although the algebra \( L^\infty_\sigma(\mathbb{R}^n) \) is not closed under adjoints, its image under \( \sigma \) is. Thus, \( L^\infty_\sigma(\mathbb{R}^n) \) is a Banach *-algebra in a natural way.

**Proposition 22.** The subalgebra \( \sigma(L^\infty_\sigma(\mathbb{R}^n)) \subset L^\infty(\mathbb{R}^n) \) is self-adjoint.

**Proof.** Let \( F : L^2_\sigma(\mathbb{R}^n) \to L^2_\sigma(\mathbb{R}^n) \) be the antiunitary flip map \( F\phi(\tilde{p}) = \overline{\phi(-\tilde{p})} \) and observe that \( F\lambda T_{\tilde{a}} F = \lambda T_{-\tilde{a}} \) for any \( \lambda \in \mathbb{C} \) and \( \tilde{a} \in \mathbb{R}^n \). It follows that \( L^\infty_\sigma(\mathbb{R}^n) \) is stable under conjugation by \( F \). Regarding \( F \) also as an operator on \( L^2(\mathbb{R}^n) \subset L^2_\sigma(\mathbb{R}^n) \), we have \( FBF = B^* \) for any \( B \in L^\infty(\mathbb{R}^n) \subset B(L^2(\mathbb{R}^n)) \). We conclude that

\[
\hat{\sigma}(A)^* = F\hat{\sigma}(A)F = \hat{\sigma}(FAF) \in \hat{\sigma}(L^\infty_\sigma(\mathbb{R}^n))
\]

for any \( A \in L^\infty_\sigma(\mathbb{R}^n) \).

The preceding proof shows that the pullback of the involution on \( L^\infty(\mathbb{R}^n) \) to \( L^\infty_\sigma(\mathbb{R}^n) \) is given by conjugation with \( F \). Thus, it is isometric on \( L^\infty_\sigma(\mathbb{R}^n) \).

We close this section with a problem on the existence of idempotents in \( L^\infty_\sigma(\mathbb{R}^n) \).

**Remarks 23.**

(a) The map \( \sigma : L^\infty(\mathbb{R}^n) \to L^\infty(\mathbb{R}^n) \) is not surjective. If it were, then by the open mapping theorem it would be an isomorphism and the inverse map would be bounded. However, \( \|\sigma(T_{\tilde{a}})\| = 1 \) for all \( \tilde{a} \in \mathbb{R}^n \), while \( \|T_{\tilde{a}}\| \to \infty \) as \( |\tilde{a}| \to \infty \), so the inverse map is unbounded.

(b) It follows that \( \sigma(L^\infty_\sigma(\mathbb{R}^n)) \) does not contain every projection in \( L^\infty(\mathbb{R}^n) \), since these span \( L^\infty(\mathbb{R}^n) \). In fact, one can show by direct computation that in the case \( n = 1 \) the characteristic function of any interval (besides the empty interval and the entire real line) fails to belong to \( \sigma(L^\infty_\sigma(\mathbb{R})) \).

We sketch the construction. First, if the characteristic function of any half-infinite interval belonged to \( \sigma(L^\infty_\sigma(\mathbb{R})) \) then so would any translation of it, and hence, taking differences, the characteristic function of some bounded interval would also belong to \( \sigma(L^\infty_\sigma(\mathbb{R})) \). Thus, it suffices to consider the bounded case. Now the Fourier transform \( f \) of any bounded interval \( J \) decays like \( 1/p \). We must find \( \phi \in L^2_\sigma(\mathbb{R}) \) such that \( \|f * \phi\|_r/\|\phi\|_r \) is arbitrarily large. For \( N \) large and \( a, b > 0 \) depending on \( f \) we let \( \phi \) be the characteristic function of a disjoint union of \( N \) intervals \( I_j \) \((1 \leq j \leq N)\), each of length \( b \) and distance \( ja \) from the origin, such that \( f \approx c/ja \) around the interval \(-I_j \) for some nonzero complex scalar \( c \), and hence \( f * \chi_{I_j} \) is approximately \( bc/ja \) on \([0, b]\). (Here \( \chi \) denotes
characteristic function.) Then $\|\phi\|_r^2$ is roughly $(b/a)\sum_1^N 1/j$ and $\|f * \phi\|_r^2$ is roughly at least $(|c|b^3/a^2)(\sum_1^N 1/j)^2$. As $N$ goes to infinity, the divergence of $\sum_1^\infty 1/j$ implies that $\|f * \phi\|_r/\|\phi\|_r$ goes to infinity. So convolution by $f$ is not bounded on $L_r^2(\mathbb{R})$.

**Problem 24.** Does $L_r^\infty(\mathbb{R}^n)$ contain any nontrivial idempotents?

5. Other non self-adjoint algebras

Using the basic ingredients introduced in previous sections (the algebra $L_r^\infty(\mathbb{R}^n)$, the action $\gamma$, the operators $T_\alpha$ and $K_f$) one can identify a large variety of operator algebras which are analogous to various classical function algebras. We do this now. In each case, it is easy to see (or vacuous) that the analogous classical construction produces the analogous function algebra.

**Definition 25.** Let $T_\alpha$ and $K_f$ be as in Definition 1 and $L_r^\infty(\mathbb{R}^n)$ and $\gamma$ as in Definition 13. In all cases except (e) and (f), we give the following spaces the operator norm they inherit from $L_r^\infty(\mathbb{R}^n) \subset B(L_r^2(\mathbb{R}^n))$.

(a) Let $C_{0,r}(\mathbb{R}^n)$ be the norm closure of the set $\{K_f : f \in S(\mathbb{R}^n)\}$.

(b) Let $AP_r(\mathbb{R}^n)$ be the norm closure of span$\{T_\alpha : \vec{a} \in \mathbb{R}^n\}$.

(c) Let $UC_r(\mathbb{R}^n)$ be the set of $A \in L_r^\infty(\mathbb{R}^n)$ such that the function $\vec{a} \mapsto \gamma_{\vec{a}}(A)$ is continuous for the norm topology.

(d) Let $C_{b,r}(\mathbb{R}^n)$ be the set of operators $A \in L_r^\infty(\mathbb{R}^n)$ such that $B \in C_{0,r}(\mathbb{R}^n)$ implies $AB \in C_{0,r}(\mathbb{R}^n)$.

(e) Define $C_{0,r}^k(\mathbb{R}^n)$ inductively by setting $C_{0,r}^0(\mathbb{R}^n) = C_{0,r}(\mathbb{R}^n)$ and letting $C_{0,r}^{k+1}(\mathbb{R}^n)$ be the set of operators $A \in C_{0,r}(\mathbb{R}^n)$ such that the norm limit

$$D_iA = \lim_{t \to 0} \frac{1}{t}(\gamma_{t\vec{e}_i}(A) - A)$$

$(1 \leq i \leq n)$ exists and belongs to $C_{0,r}^k(\mathbb{R}^n)$. We give $C_{0,r}^{k+1}(\mathbb{R}^n)$ the norm

$$\|A\|_{k+1} = \max(\|A\|, \|D_1A\|, \ldots, \|D_nA\|_k).$$

(f) Let $\text{Lip}_r(\mathbb{R}^n)$ be the set of operators $A \in L_r^\infty(\mathbb{R}^n)$ such that

$$L(A) = \sup_{\vec{a} \in \mathbb{R}^n} \frac{1}{|\vec{a}|} \|\gamma_{\vec{a}}(A) - A\|$$

is finite (using the convention $0/0 = 0$). We give it the norm $\|A\|_L = \max(\|A\|, L(A))$.

(g) Let $H_r^\infty(\mathbb{R})$ be the strong operator closure of span$\{T_\alpha : \alpha \geq 0\} \subset L_r^\infty(\mathbb{R})$.

This list could easily be extended further.
$AP$ stands for “almost periodic,” $UC$ for “uniformly continuous,” $C_b$ for “continuous and bounded,” and Lip for “Lipschitz.”

The next proposition is straightforward and we omit its proof.

**Proposition 26.** Each of the spaces in Definition 25 is a Banach algebra. The images of all but $H_r^\infty(\mathbb{R}^n)$ under $\sigma$ are self-adjoint subalgebras of $L^\infty(\mathbb{R}^n)$.

(Note that in cases (e) and (f) the Banach algebra law holds only in its weak form, i.e., $\|xy\| \leq C\|x\|\|y\|$ for some constant $C$. Of course, this occurs classically as well.)

Next we observe that the expected containments hold. This uses a lemma which is of independent interest.

**Lemma 27.** Let $f \in \mathcal{S}(\mathbb{R}^n)$ be compactly supported and satisfy $\int f = 1$. Then $A \in L^\infty_r(\mathbb{R}^n)$ belongs to $C_{0,r}(\mathbb{R}^n)$ if and only if $A \in UC_r(\mathbb{R}^n)$ and $K_{f_k}A \to A$ in norm, where $f_k(\vec{p}) = k^n f(k\vec{p})$.

**Proof.** Observe first that $K_{f_k}K_g = K_{f_k*g} \to K_g$ in norm for all $g \in \mathcal{S}(\mathbb{R}^n)$; this follows from the fact that $f_k * g \to g$ in $\mathcal{S}(\mathbb{R}^n)$. Moreover, using the fact that $\|K_{f_k}\| \to 1$, an $\varepsilon/3$ argument shows that the set of $A$ which satisfy $K_{f_k}A \to A$ is closed in norm. This proves that every $A \in C_{0,r}(\mathbb{R}^n)$ satisfies $K_{f_k}A \to A$. Also, direct computation shows that $\gamma_{\sigma}(K_f) = K_{e^{-i\sigma f}}$, and this is norm continuous in $\sigma$ for any $f \in \mathcal{S}(\mathbb{R}^n)$ by [5, Theorem II.1.6] (cf. Lemma 36). Using the fact that $UC_r(\mathbb{R}^n)$ is norm closed, it follows that $C_{0,r}(\mathbb{R}^n)$ is contained in $UC_r(\mathbb{R}^n)$. This completes the proof of the forward implication.

For the reverse implication, let $A \in UC_r(\mathbb{R}^n)$ and suppose $K_{f_k}A \to A$ in norm. It will suffice to show that $K_{f_k}A \in C_{0,r}(\mathbb{R}^n)$ for arbitrary $k$. Thus, fix $k$ and let

$$A_j = \int_{\mathbb{R}^n} f_j(\vec{p})\gamma_{\vec{p}}(K_{f_k}A) \, d\vec{p}$$

as in the proof of Corollary 21. Now $K_{f_k}A = \int f_k(\vec{p})K_{f_k}A \, d\vec{p}$ so

$$\|A_j - K_{f_k}A\| \leq \int_{\mathbb{R}^n} |f_j(\vec{p})|\|\gamma_{\vec{p}}(K_{f_k}A) - K_{f_k}A\| \, d\vec{p}.$$ 

This goes to zero as $j \to \infty$ since $K_{f_k}A \in UC_r(\mathbb{R}^n)$ implies that the function $\vec{p} \mapsto \gamma_{\vec{p}}(K_{f_k}A) - K_{f_k}A$ is continuous in norm, and it is clearly bounded and zero at $\vec{0} = 0$. However, $A_j = K_{\hat{f}_j(f_k*g)}$ where $g = \sigma(A) \in L^\infty(\mathbb{R}^n)$; and $\hat{f}_j(f_k*g) \in \mathcal{S}(\mathbb{R}^n)$, so $A_j$ is of the form $K_f$ for $f \in \mathcal{S}(\mathbb{R}^n)$. We conclude that $K_{f_k}A$, and hence $A$, belongs to $C_{0,r}(\mathbb{R}^n)$.

The condition $A \in UC_r(\mathbb{R}^n)$ in Lemma 27 can be weakened to $A \in C_{b,r}(\mathbb{R}^n)$. In fact the proof of the backward implication becomes easier, for then we know immediately that $K_{f_k}A \in C_{0,r}(\mathbb{R}^n)$. The point is that proving $UC_r(\mathbb{R}^n) \subset C_{b,r}(\mathbb{R}^n)$ is tantamount to showing that $A \in UC_r(\mathbb{R}^n)$ implies $K_{f_k}A \in C_{0,r}(\mathbb{R}^n)$.
Proposition 28. We have

\[ AP_r(\mathbb{R}^n), \text{Lip}_r(\mathbb{R}^n), C_{0,r}(\mathbb{R}^n) \subset UC_r(\mathbb{R}^n) \subset C_{b,r}(\mathbb{R}^n). \]

Proof. We showed that \( C_{0,r}(\mathbb{R}^n) \subset UC_r(\mathbb{R}^n) \) in the lemma. The proof of \( AP_r(\mathbb{R}^n) \subset UC_r(\mathbb{R}^n) \) is similar, now using the fact that \( \gamma_{\vec{a}}(T_{\vec{b}}) = e^{-i\vec{a} \cdot \vec{b}}T_{\vec{b}} \) is norm continuous in \( \vec{a} \).

The containment \( \text{Lip}_r(\mathbb{R}^n) \subset UC_r(\mathbb{R}^n) \) holds because

\[ \| \gamma_{\vec{a}}(A) - \gamma_{\vec{b}}(A) \| = \| \gamma_{\vec{b}}(\gamma_{\vec{a} - \vec{b}}(A) - A) \| \leq |\vec{a} - \vec{b}| L(A) \]

for all \( \vec{a}, \vec{b} \in \mathbb{R}^n \) and all \( A \in \text{Lip}_r(\mathbb{R}^n) \).

For the final containment, let \( A \in UC_r(\mathbb{R}^n) \) and \( B \in C_{0,r}(\mathbb{R}^n) \); we must show that \( AB \in C_{0,r}(\mathbb{R}^n) \). As above, \( B \in UC_r(\mathbb{R}^n) \), so \( AB \in UC_r(\mathbb{R}^n) \) by Proposition 26. Also \( K_{f_k} B \to B \) implies \( K_{f_k} AB = AK_{f_k} B \to AB. \) So \( AB \in C_{0,r}(\mathbb{R}^n) \) by the lemma.

The next result is also basic.

Proposition 29. \( \text{Lip}_r(\mathbb{R}^n) \) and \( H_r^\infty(\mathbb{R}) \) are dual spaces.

Proof. It is standard that any strong operator closed subspace of \( B(H) \) is weak operator closed, and hence weak* closed, as in the proof of Proposition 15. This shows that \( H_r^\infty(\mathbb{R}) \) is a dual space. For \( \text{Lip}_r(\mathbb{R}^n) \), let \( \mathcal{M} = \bigoplus \gamma_{\vec{a}} L_r^\infty(\mathbb{R}^n) \) be the \( l^\infty \) direct sum over \( \vec{a} \in \mathbb{R}^n, \vec{a} \neq 0 \), and consider the map \( d : \text{Lip}_r(\mathbb{R}^n) \to \mathcal{M} \) defined by \( (dA)_{\vec{a}} = (\gamma_{\vec{a}}(A) - A)/|\vec{a}| \). Then \( \|A\|_L = \max(\|A\|, \|dA\|) \), i.e., \( \| \cdot \|_L \) is the graph norm. As \( \gamma_{\vec{a}} \) is weak* continuous, it easily follows that the graph of \( d \) is weak* closed in \( L_r^\infty(\mathbb{R}^n) \oplus \mathcal{M} \), so this graph, which is isometric to \( \text{Lip}_r(\mathbb{R}^n) \), is a dual space.

We note that according to [4, Proposition 3.1.23], \( \text{Lip}_r(\mathbb{R}^n) \) is equivalently the set of operators \( A \) in \( L_r^\infty(\mathbb{R}^n) \) such that the partial derivatives \( D_i A \) \((1 \leq i \leq n)\) as in Definition 25 (e) exist as weak operator limits.

We now turn to the relation of the above spaces to the symbol map \( \sigma \). The first result is easy.

Proposition 30. Let \( X_r \) be any of the spaces in Definition 25 and let \( X \) be its classical analog. Then \( \sigma(X_r) \subset X \).

The unboundedness argument of Remark 23 (a) applies to every algebra in Definition 25 to show that \( \sigma(X_r) \) must be a proper subset of \( X \). A basic question is whether \( \sigma(X_r) = X \cap \sigma(L_r^\infty(\mathbb{R}^n)) \). We first show that this holds for \( H_r^\infty(\mathbb{R}) \).

Proposition 31. \( H_r^\infty(\mathbb{R}) \) consists of precisely those operators \( A \in L_r^\infty(\mathbb{R}) \) such that

\[ \inf \text{supp} A \phi \geq \inf \text{supp} \phi \]
for all $\phi \in L^2_r(\mathbb{R})$ with support bounded from below. We have

$$\sigma(H^\infty_r(\mathbb{R})) = H^\infty(\mathbb{R}) \cap \sigma(L^\infty_r(\mathbb{R})).$$

Proof. It is clear that every $T_a$, $a \geq 0$, translates supports of functions in $L^2_r(\mathbb{R})$ to the right. It follows that any finite linear combination of the $T_a$ respects lower bounds of supports, and, taking strong operator limits, that this is true of any operator in $H^\infty(\mathbb{R})$. Conversely, let $A \in L^\infty_r(\mathbb{R})$ and suppose $\inf \sup \phi \geq \inf \sup \phi$ for all $\phi \in L^2_r(\mathbb{R})$ with support bounded from below. Then this is true in particular for all $\phi \in L^2(\mathbb{R}) \subset L^2_r(\mathbb{R})$, so that $\hat{\sigma}(A) \in H^\infty(\mathbb{R}) \subset L^\infty(\mathbb{R})$. Adopting the notation of the proof of Corollary 21, we have that $A$ is strong operator approximated by operators of the form $K_{f_i} \hat{f_k}$ where $\sigma(A) = g \in H^\infty(\mathbb{R})$. Since the $f_k$ have compact support, the support of $f_k$ must converge to zero as $k \to \infty$, and it follows that $A$ is strong operator approximated by operators of the form $K_h$ with $\supp h \subset [0, \infty)$. This implies that $A \in H^\infty_r(\mathbb{R})$.

The equation $\sigma(H^\infty_r(\mathbb{R})) = H^\infty(\mathbb{R}) \cap \sigma(L^\infty_r(\mathbb{R}))$ follows easily.

Next, we have partial positive results to the same question for $C_{0,r}(\mathbb{R}^n)$ and $AP_r(\mathbb{R}^n)$.

Lemma 32. If $f \in L^\infty(\mathbb{R}^n)$ and $\hat{f}$ has compact support then $f \in \sigma(L^\infty_r(\mathbb{R}^n))$. Moreover, if $S \subset \mathbb{R}^n$ is compact then there is a constant $C > 0$ such that $\|\sigma^{-1}(f)\| \leq C \cdot \|f\|_\infty$ for all $f \in L^\infty(\mathbb{R}^n)$ with $\supp \hat{f} \subset S$.

Proof. Fix a compact set $S \subset \mathbb{R}^n$ and let $C_1 = [-N/2, N/2]^n$ be a cube which contains $S$. Then the cubes $C_1 + N \cdot \mathbf{Z}^n$ tile $\mathbb{R}^n$. Let $C_2, \ldots, C_{3^n}$ be the cubes adjacent to $C_1$ in this tiling.

Fix an arbitrary $f \in L^\infty(\mathbb{R}^n)$ such that $\supp \hat{f} \subset S$ (in the distributional sense). Regard $\hat{M}_f$ as acting either on $L^2(\mathbb{R}^n)$ or, perhaps unboundedly, on $L^2_r(\mathbb{R}^n)$. For a cube $C \subset \mathbb{R}^n$ of any size and location, let $P_C : \phi \mapsto \chi_C \cdot \phi$ be the orthogonal projection given by restriction to $C$ ($\chi$ denotes characteristic function) acting on either $L^2(\mathbb{R}^n)$ or $L^2_r(\mathbb{R}^n)$. We claim that there is a cube $C$ whose side has length $3N$ such that the norm of $P_C \hat{M}_f P_C$ acting on $L^2_r(\mathbb{R}^n)$ is at least $3^{-n}$ times the norm of $\hat{M}_f$ acting on $L^2_r(\mathbb{R}^n)$ (or infinite if the latter is infinite). To see this, for $1 \leq i \leq 3^n$ let $A_i = \sum P_C \hat{M}_f P_C$, taking the sum over all cubes $C$ belonging to the tiling $C_i + 3N \cdot \mathbf{Z}^n$ where $C_i$ is the cube with side length $3N$ consisting of $C_i$ and its neighbors. Now $\supp \hat{f} \subset C_1$ implies that for any $\phi \in L^2_r(\mathbb{R}^n)$ supported outside $C_i$ we must have $\hat{M}_f \phi = \hat{f} \ast \phi = 0$ on $C_i$. Thus,

$$P_{C_i} \hat{M}_f P_{C_i} \phi = P_{C_i} \hat{M}_f \phi,$$

that is, $A_i \phi = \hat{M}_f \phi$ on $C_i$, for any $\phi \in L^2_r(\mathbb{R}^n)$; more generally, $A_i \phi$ must agree with $\hat{M}_f \phi$ on the set $C_i + 3N \cdot \mathbf{Z}^n$. Since these $3^n$ sets ($1 \leq i \leq 3^n$) tile $\mathbb{R}^n$, it follows that

$$\|A_1\|^2 + \cdots + \|A_{3^n}\|^2 \geq \|\hat{M}_f\|^2$$
on $L^2_0(\mathbb{R}^n)$, and hence $\|A_i\| \geq 3^{-n/2} \|\widehat{M}_f\|$ (or is infinite, if $\|\widehat{M}_f\|$ is infinite) for some $i$. But $\|A_i\|$ is the supremum of the norms $\|P_C \widehat{M}_f P_C\|$ where $C$ ranges over the sum which defines $A_i$. Thus, there is some cube $C$ whose side has length $3N$ such that $\|P_C \widehat{M}_f P_C\| \geq 3^{-n} \|\widehat{M}_f\|$, as claimed.

Find a constant $C > 0$ such that if $C$ is any cube whose side has length $3N$, and $a$ and $b$ are respectively the maximum and minimum of the function $E(\vec{p})$ on $C$, then $a/b \leq C$. This is possible because the ratio goes to 1 as the cube goes to infinity. Now, continuing to let $f \in L^\infty(\mathbb{R}^n)$ be arbitrary such that supp $\hat{f} \subset C$, find a cube $C$ so that $\|P_C \widehat{M}_f P_C\| \geq 3^{-n} \|\widehat{M}_f\|$ on $L^2_0(\mathbb{R}^n)$, and let $\phi \in L^2_0(\mathbb{R}^n)$ be supported on $C$. Let $a$ and $b$ be the respective maximum and minimum of $E(\vec{p})$ on this cube. Then the norm of $\phi$ in $L^2(\mathbb{R}^n)$ is related to its norm in $L^2_0(\mathbb{R}^n)$ by $\|\phi\|_r \geq \sqrt{b} \|\phi\|$. Similarly, we have $\|P_C \widehat{M}_f \phi\|_r \leq \sqrt{a} \|P_C \widehat{M}_f \phi\|$. But $\|P_C \widehat{M}_f \phi\| \leq \|f\|_\infty \|\phi\|$, so that

$$\|P_C \widehat{M}_f \phi\|_r \leq \sqrt{a} \|P_C \widehat{M}_f \phi\| \leq \sqrt{a} \|f\|_\infty \|\phi\| \leq \sqrt{C} \|f\|_\infty \|\phi\|_r.$$ 

Together with the choice of $C$, we conclude that the norm of $\widehat{M}_f = \sigma^{-1}(f)$ on $L^2_0(\mathbb{R}^n)$ is at most $3^n \sqrt{C} \|f\|_\infty$. This completes the proof.

Note that if $\hat{f}$ has compact support then $f$ is infinitely differentiable [17, Theorem 7.23]. This aspect of the above result will be strengthened in Corollary 37.

**Corollary 33.** Let $X_r$ be any of the spaces of Definition 25 and let $X$ be its classical counterpart. Then $\sigma(X_r)$ contains every $f \in X$ such that $\hat{f}$ has compact support.

**Theorem 34.** We have

$$\sigma(C_{0,r}(\mathbb{R}^n)) = C_0(\mathbb{R}^n) \cap \sigma(UC_r(\mathbb{R}^n))$$

and

$$\sigma(AP_r(\mathbb{R}^n)) = AP(\mathbb{R}^n) \cap \sigma(UC_r(\mathbb{R}^n)).$$

**Proof.** In both cases, one containment follows from Propositions 28 and 30. For the other direction, choose $g$ in $C_0(\mathbb{R}^n) \cap \sigma(UC_r(\mathbb{R}^n))$ (respectively, $AP(\mathbb{R}^n) \cap \sigma(UC_r(\mathbb{R}^n))$) and let $A = \sigma^{-1}(g)$. Choose $f \in S(\mathbb{R}^n)$ such that $\int f = 1$ (i.e., $\hat{f}(0) = 1$) and $\hat{f}$ has compact support, and define $f_k(\vec{p}) = k^n f(k\vec{p})$. Then $A \in UC_r(\mathbb{R}^n)$ implies that the sequence

$$A_j = \int_{\mathbb{R}^n} f_j(\vec{p}) \gamma_{\vec{p}}(A) \, d\vec{p}$$

converges to $A$ in norm, as in the proof of Lemma 27. Moreover, $A_j = K_{f_j,\hat{g}}$, so that $\sigma(A_j) = \hat{f}_j \hat{g}$ has compact support, while $f_j * g \in C_0(\mathbb{R}^n)$ (respectively, $AP(\mathbb{R}^n)$) is clear. Corollary 33 then implies that $A_j$ is in $C_{0,r}(\mathbb{R}^n)$ (respectively, $AP_r(\mathbb{R}^n)$). Taking $j \to \infty$, we find that $A$ belongs to the same space, as desired.
Corollary 35. $C_{0,r}(\mathbb{R}^n)^+$ and $AP_r(\mathbb{R}^n)$ are inverse closed subalgebras of $UC_r(\mathbb{R}^n)$.

We now proceed to a negative result which states that there exists a function in $\sigma(C_{0,r}(\mathbb{R}^n))$ which belongs to $C^d_{0,r}(\mathbb{R}^n)$ but not $\sigma(C^d_{0,r}(\mathbb{R}^n))$.

Lemma 36. Let $f : \mathbb{R}^n \to \mathbb{C}$ be measurable and suppose
\[ \int_{\mathbb{R}^n} (1 + |\vec{p}|^{1/2}) |f(\vec{p})| d\vec{p} < \infty. \]
Then $K_f$, the operator of convolution by $f$ on $L^2_r(\mathbb{R}^n)$, belongs to $L^\infty_r(\mathbb{R}^n)$ and
\[ \|K_f\| \leq \int_{\mathbb{R}^n} \left( 1 + \frac{|\vec{p}|}{m} + \frac{|\vec{p}|^2}{m^2} \right)^{1/4} |f(\vec{p})| d\vec{p}. \]

Proof. It is enough to show that $K_f$ is bounded on $L^2_r(\mathbb{R}^n)$; commutation with the operators $T_\vec{a}$ is then trivial and implies $K_f \in L^\infty_r(\mathbb{R}^n)$.

For the sharpest bound on $\|K_f\|$, we pass to the operator $\tilde{K}_f = M_\sqrt{E} K_f M_\sqrt{E}$ on $L^2(\mathbb{R}^n)$ as in Section 3. Recall that
\[ \tilde{K}_f \phi(\vec{p}) = \int f(\vec{p}') \sqrt{E(\vec{p}' - \vec{p})} \phi(\vec{p} - \vec{p}') d\vec{p}' = \int f(\vec{p}' - \vec{p}') \sqrt{E(\vec{p})} \phi(\vec{p}') d\vec{p}', \]
so $\tilde{K}_f$ is an integral operator with kernel
\[ K(\vec{p}, \vec{p}') = f(\vec{p}' - \vec{p}) \sqrt{E(\vec{p}')} / E(\vec{p}). \]
We claim that the integrals $\int |K(\vec{p}, \vec{p}')| d\vec{p}$ and $\int |K(\vec{p}, \vec{p}')| d\vec{p}'$ are uniformly bounded by
\[ \int_{\mathbb{R}^n} \left( 1 + \frac{|\vec{p}|}{m} + \frac{|\vec{p}|^2}{m^2} \right)^{1/4} |f(\vec{p})| d\vec{p}. \]
By [5, Theorem II.1.6] the norm of $\tilde{K}_f$, and hence of $K_f$, is then less than or equal to this uniform bound. The claim follows by a short computation using the elementary inequality
\[ \sqrt{\frac{E(\vec{p} - \vec{p}')}{E(\vec{p}')}} \leq \left( 1 + \frac{|\vec{p}|}{m} + \frac{|\vec{p}|^2}{m^2} \right)^{1/4} \]
and the assumption on integrability of $f$ then implies that the bound on $K_f$ is finite. 

The argument in the above proof, while easier than the one in the proof of Lemma 32, does not apply there because in that situation $\hat{f}$ (which plays the role played by $f$ here) is in general not a function.
Corollary 37. Let $f$ be a compactly supported, $d$-times continuously differentiable function on $\mathbb{R}^n$, where $d > (n + 1)/2$. Then $f \in \sigma(C_{0,r}(\mathbb{R}^n))$. 

Proof. Observe first that

$$\int |\vec{p}|^{1/2} |\hat{f}(\vec{p})| d\vec{p} \leq \left( \int \frac{d\vec{p}}{1 + |\vec{p}|^{2d-1}} \right)^{1/2} \left( \int (1 + |\vec{p}|^{2d-1}) |\hat{f}(\vec{p})|^2 d\vec{p} \right)^{1/2}$$

by the Cauchy-Schwarz inequality. The first integral on the right is finite by our choice of $d$. For the second, observe that

$$\int |\vec{p}|^{2d} |\hat{f}(\vec{p})|^2 d\vec{p} \leq \int n^d \max_{1 \leq i \leq n} |p_i^d \hat{f}(\vec{p})|^2 d\vec{p} \leq n^d \sum_{1 \leq i \leq n} \left| \frac{\partial^d f}{\partial x_i^d}(\vec{x}) \right|^2 d\vec{x} < \infty.$$ 

Since $\hat{f}$ is continuous, and hence bounded in a neighborhood of the origin, this implies that $\int |\vec{p}| |\hat{f}(\vec{p})|^2 d\vec{p}$ is also finite, hence $\int |\vec{p}|^{1/2} |\hat{f}(\vec{p})| d\vec{p}$ is finite and therefore so is $\int |\hat{f}(\vec{p})| d\vec{p}$. We conclude that $\hat{f}$ satisfies the hypothesis of Lemma 36. Thus $K_{\hat{f}} \in L_\sigma^\infty(\mathbb{R}^n)$, and to $f \in \sigma(L_\sigma^\infty(\mathbb{R}^n))$.

To show that $f \in \sigma(C_{0,r}(\mathbb{R}^n))$, by Theorem 34 it is enough to show that $f \in \sigma(UC_\sigma(\mathbb{R}^n))$, i.e., that $K_f \in UC_\sigma(\mathbb{R}^n)$. Now

$$\gamma_{\vec{a}}(K_f) - K_{\hat{f}} = K_{(e^{-i\vec{a} \cdot \vec{x}} - 1)\hat{f}}.$$ 

Substituting $(e^{-i\vec{a} \cdot \vec{p}} - 1)\hat{f}$ for $f$ in the hypothesis of Lemma 36, the dominated convergence theorem implies that the bound on $\|K_{(e^{-i\vec{a} \cdot \vec{x}} - 1)\hat{f}}\|$ given there goes to zero as $\vec{a} \to 0$. So $\gamma_{\vec{a}}(K_f) \to K_f$ as $\vec{a} \to 0$. This shows that $K_f \in UC_\sigma(\mathbb{R}^n)$, as desired. 

Theorem 38. Let $d > (n + 1)/2$. Then $\sigma(C_{0,r}^d(\mathbb{R}^n))$ is properly contained in $C_0^d(\mathbb{R}^n) \cap \sigma(C_{0,r}(\mathbb{R}^n))$. 

Proof. By Corollary 37, $\sigma(C_{0,r}(\mathbb{R}^n))$ contains every compactly supported function in $C_0^d(\mathbb{R}^n)$. Thus, we need only show that there exists such a function which is not in $\sigma(C_{0,r}^d(\mathbb{R}^n))$. Let $S$ be the unit ball in $\mathbb{R}^n$ and let $X$ be the set of operators $A$ in $C_{0,r}^d(\mathbb{R}^n)$ such that $\sigma(A)$ is supported in $S$. We claim that $X$ is a closed subspace of $C_{0,r}^d(\mathbb{R}^n)$. To see this, let $(A_k)$ be a Cauchy sequence in $X$, say $A_k \to A \in C_{0,r}^d(\mathbb{R}^n)$, and let $f \in S(\mathbb{R}^n)$ be any Schwartz function which is constantly 1 on $S$. Then $K_f A_k = A_k$ for all $k$, so $K_f A = A$ by continuity. It follows that $f \cdot \sigma(A) = \sigma(A)$, and we deduce that $\sigma(A)$ is supported on $S$, i.e., $A \in X$. Thus $X$ is closed. Suppose $\sigma$ maps $X$ onto the closed subspace $Y$ of $C_0^d(\mathbb{R}^n)$ consisting of all the functions in $C_0^d(\mathbb{R}^n)$ which are supported on $S$. Then the open mapping theorem implies that $\sigma$ is an isomorphism from $X$ onto $Y$.

Let $g \in Y$ be arbitrary but nonzero. Then the norm of $g_k = k^{-d} e^{ikx_1} g$ in $Y$ is bounded as $k \to \infty$, but $\|D_1^d \sigma^{-1}(g_k)\| \sim \sqrt{k}$. This shows that $\sigma^{-1}$ is not bounded from $Y$ to $X$, ...
contradicting the conclusion of the last paragraph. Thus \( \sigma(X) \), and hence \( \sigma(C^{d}_{0,r}(\mathbb{R}^n)) \),
do not contain all functions in \( C^{d}_{0}(\mathbb{R}^n) \) which are supported on \( S \), as claimed.

This still leaves our main question largely unanswered.

**Problem 39.** For which relativistic spaces \( X_r \subset L^{\infty}_r(\mathbb{R}^n) \) with classical analog \( X \) does \( \sigma(X_r) = X \cap \sigma(L^{\infty}_r(\mathbb{R}^n)) \)?

### 6. Ideals and subalgebras

We begin by identifying the maximal ideals of \( C^{0}_r(\mathbb{R}^n) \).

**Theorem 40.** The maximal ideal space of \( C^{0}_r(\mathbb{R}^n) \) can be identified with \( \mathbb{R}^n \), and the symbol map \( \sigma : C^{0}_r(\mathbb{R}^n) \to C^{0}(\mathbb{R}^n) \) can be identified with the Gelfand transform.

**Proof.** For every \( \vec{a} \in \mathbb{R}^n \), the map \( A \mapsto \sigma(A)(\vec{a}) \) is a complex homomorphism on \( C^{0}_r(\mathbb{R}^n) \). Since \( \sigma(C^{0}_r(\mathbb{R}^n)) \) contains the Schwarz functions on \( \mathbb{R}^n \), it follows that the preceding homomorphisms are distinct. We must show that every complex homomorphism on \( C^{0}_r(\mathbb{R}^n) \) is of this form.

Let \( \omega : C^{0}_r(\mathbb{R}^n) \to \mathbb{C} \) be a homomorphism, and let \( \mathcal{A} \subset C^{0}_r(\mathbb{R}^n) \) be the dense subalgebra consisting of all operators of the form \( K_f \) such that \( f \in S(\mathbb{R}^n) \) has compact support. We claim that \( \| \omega(K_f) \| \leq \| f \|_{1} \) for every \( K_f \in \mathcal{A} \). Letting \( f^{*k} \) be the \( k \)th convolution power of \( f \), we have \( K^k_f = K_{f^{*k}} \), and if \( f \) is supported on \([-N,N]^n \) then \( f^{*k} \) is supported on \([-kN,kN]^n \), so as \( k \to \infty \)

\[
\| K^k_f \| = \| f^{*k} \|_1 \cdot O((nkN/m)^{1/2}) \leq \| f \|^k \cdot O(\sqrt{k})
\]

by Lemma 36. We are using the fact that the dominant term in the bound on \( \| K^k_f \| \) given by Lemma 36 is \( (|p|^2/m^2)^{1/4} = (|p|/m)^{1/2} \). It follows that

\[
|\omega(K_f)|^k = |\omega(K^k_f)| \leq C \sqrt{k} \| f \|^k_1
\]

and taking \( k \)th roots and letting \( k \to \infty \) yields \( |\omega(K_f)| \leq \| f \|_1 \), as claimed.

If follows that \( \omega \) extends by continuity to a complex homomorphism on the convolution algebra \( L^1(\mathbb{R}^n) \). These are well-known to be given by point evaluations on \( \mathbb{R}^n \) composed with the Fourier transform. Thus \( \omega \) applied to elements of \( \mathcal{A} \) must be a point evaluation on \( C^{0}(\mathbb{R}^n) \) composed with \( \sigma \), and hence this holds on \( C^{0}_r(\mathbb{R}^n) \). This completes the proof.

Now we turn to more general ideals. We have the following examples of ideals of \( L^{\infty}_r(\mathbb{R}^n) \) and \( C^{0}_r(\mathbb{R}^n) \).

**Definition 41.** For any measurable \( S \subset \mathbb{R}^n \), let \( \mathcal{I}(S) \) be the set of all \( A \in L^{\infty}_r(\mathbb{R}^n) \) such that \( \sigma(A)|_S \equiv 0 \).
Proposition 42. Let $S \subset \mathbb{R}^n$ be measurable. Then $\mathcal{I}(S)$ is a weak* closed ideal of $L_r^\infty(\mathbb{R}^n)$ and $\mathcal{I}(S) \cap C_{0,r}(\mathbb{R}^n)$ is a norm closed ideal of $C_{0,r}(\mathbb{R}^n)$.

Proof. Let $T$ be the restriction map from $L^\infty(\mathbb{R}^n)$ to $L^\infty(\mathbb{R}^n - S)$. It is weak* continuous, hence $T \circ \sigma$ is weak* continuous, so $\mathcal{I}(S) = \ker(T \circ \sigma)$ is a weak* closed ideal of $L_r^\infty(\mathbb{R}^n)$. The second assertion of the proposition follows easily.

Problem 43. Are there any weak* closed ideals of $L_r^\infty(\mathbb{R}^n)$ not of the form $\mathcal{I}(S)$, or any norm closed ideals of $C_{0,r}(\mathbb{R}^n)$ not of the form $\mathcal{I}(S) \cap C_{0,r}(\mathbb{R}^n)$?

Remarks 44.
(a) If $\overline{S}$ is the closure of $S$, then Proposition 30 implies that $\mathcal{I}(S) \cap C_{0,r}(\mathbb{R}^n) = \mathcal{I}(\overline{S}) \cap C_{0,r}(\mathbb{R}^n)$. Conversely, if $S$ and $S'$ are distinct closed sets then there is a Schwarz function which vanishes on one but not the other, and hence $\mathcal{I}(S) \cap C_{0,r}(\mathbb{R}^n) \neq \mathcal{I}(S') \cap C_{0,r}(\mathbb{R}^n)$.
(b) Taking advantage of Corollary 19, we can also define spaces $L_r^\infty(S)$ and $C_{0,r}(S)$ as, respectively, the strong operator and norm closures of the set $\{K_f|_{H_S} : f \in \mathcal{S}(\mathbb{R}^n)\} \subset B(H_S)$.

Among the variety of possible constructions of other related operator algebras (such as those just exhibited in Remark 44 (b)), perhaps the most interesting are a family of algebras on the $n$-torus arising from the operators $T_{\bar{a}}$ for $\bar{a} \in \mathbb{Z}^n$. Because harmonic analysis techniques are generally more powerful on $\mathbb{Z}^n$ than on $\mathbb{R}^n$, these algebras are somewhat more tractable than the algebras in Sections 4 and 5: most of the results in those sections can be proven more easily in the torus case. However, instead of giving direct proofs we find it simpler to reduce to the real case.

We adopt the convention that $T^n = \mathbb{R}^n/2\pi\mathbb{Z}^n$.

Definition 45.
(a) Let $L_r^\infty(T^n)$ be the strong operator closure of span$\{T_{\bar{a}} : \bar{a} \in \mathbb{Z}^n\}$ in $L_r^\infty(\mathbb{R}^n)$.
(b) Let $C_r(T^n)$ be the norm closure of span$\{T_{\bar{a}} : \bar{a} \in \mathbb{Z}^n\}$.
(c) Define $C_r^k(T^n)$ inductively by setting $C_r^0(T^n) = C_r(T^n)$ and letting $C_r^{k+1}(T^n)$ be the set of operators $A$ in $C_r^k(T^n)$ such that $D_iA$ exists as a norm limit, as in Definition 25 (e).
(d) Let Lip$_{r}(T^n)$ be the set of operators $A$ in $L_r^\infty(T^n)$ such that $L(A) < \infty$, as in Definition 25 (f).
(e) Let $A_r(T)$ be the norm closure of span$\{T_a : a \in \mathbb{N}\}$.
(f) Let $H_{r}^\infty(T)$ be the strong operator closure of span$\{T_a : a \in \mathbb{N}\}$.

Remarks 46.
(a) Composition with any invertible linear transformation on $\mathbb{R}^n$ defines a bounded operator on $L_r^2(\mathbb{R}^n)$ (Remark 14 (c)). Thus, if in the above definitions $\mathbb{Z}^n$ were replaced by any cocompact lattice, the resulting spaces would have equivalent norms.
(b) Similarly, $L^2_r(\mathbb{R}^n)$ is naturally isomorphic (but not isometric) to the tensor product $l^2(\mathbb{Z}^n, 1/E(\vec{k})) \otimes L^2([0, 1]^n)$. The corresponding representation of the above algebras on this product is trivial on the second factor, so it follows that they are all isomorphically represented on $l^2(\mathbb{Z}^n, 1/E(\vec{k}))$ in the obvious way.

For $A \in L_r^\infty(T^n)$, define the Nth Cesàro mean of $A$ by

$$\tau_N(A) = \frac{1}{(2\pi)^n} \int_{T^n} \gamma(t) K_N^n(d\vec{t})$$

where $K_N$ is the Fejér kernel

$$K_N(t) = \sum_{m=-N}^{N} \left(1 - \frac{|m|}{N + 1}\right) e^{imt} = \frac{1}{N + 1} \left(\frac{\sin((N + 1)t/2)}{\sin(t/2)}\right)^2$$

($t \in T$) and $K_N^n(\vec{t}) = K_N(t_1) \cdots K_N(t_n)$. Observe that if $A$ is in $\text{span}\{T_{\vec{a}} : \vec{a} \in \mathbb{Z}^n\}$ then $\tau_N(A)$ is a linear combination of the $T_{\vec{a}}$ with $\vec{a} \in \mathbb{Z}^n$ and $-N \leq a_i \leq N$ for all $i$. By dominated convergence, the map $A \mapsto \tau_N(A)$ is weak* continuous, so we conclude that $\tau_N(A)$ is such a linear combination for all $A \in L_r^\infty(T^n)$.

**Theorem 47.** Regard $L^\infty(T^n)$ as the subalgebra of $2\pi$-periodic functions in $L^\infty(\mathbb{R}^n)$. The following equalities hold:

\[
\begin{align*}
L_r^\infty(T^n) &= \sigma^{-1}(L^\infty(T^n)) \\
C_r(T^n) &= UC_r(\mathbb{R}^n) \cap L_r^\infty(T^n) \\
\text{Lip}_r(T^n) &= \text{Lip}_r(\mathbb{R}^n) \cap L_r^\infty(T^n) \\
H_r^\infty(T) &= H_r^\infty(\mathbb{R}) \cap L_r^\infty(T).
\end{align*}
\]

**Proof.** Since $\sigma(T_{\vec{a}}) \in L^\infty(T^n)$ for any $\vec{a} \in \mathbb{Z}^n$, linearity and weak* continuity of $\sigma$ imply that $\sigma(L_r^\infty(T^n)) \subset L^\infty(T^n)$. Conversely, let $A \in L_r^\infty(\mathbb{R}^n)$ and suppose $\sigma(A) \in L^\infty(T^n)$. Although we do not yet know $A \in L_r^\infty(T^n)$, define $\tau_N(A)$ by the formula preceding the theorem, and define $\tau_N(\sigma(A)) \in L^\infty(\mathbb{R}^n)$ similarly. Then $\sigma(A) \in L^\infty(T^n)$ implies that $\tau_N(\sigma(A)) = \sigma(\tau_N(A))$ is a linear combination of the functions $\sigma(T_{\vec{a}}) = e^{i\vec{a} \cdot \vec{t}}$ with $\vec{a} \in \mathbb{Z}^n$ and $-N \leq a_i \leq N$ for all $i$. Applying $\sigma^{-1}$, we infer that $\tau_N(A)$ is in the span of the corresponding $T_{\vec{a}}$. Now $\|\tau_N(A)\| \leq \|A\|$ since $K_N \geq 0$ and $\int_T K_N(t) dt = 2\pi$, and $\sigma(\tau_N(A)) = \tau_N(\sigma(A)) \to \sigma(A)$ weak* is classical, so we must also have $\hat{\sigma}(\tau_N(A)) \to \hat{\sigma}(A)$ weak operator, and hence $\tau_N(A) \to A$ weak operator since $(\tau_N(A))$ is bounded and $L^2(\mathbb{R}^n)$ is dense in $L_r^2(\mathbb{R}^n)$. It follows that $A$ is in the weak operator closure of elements of span$\{T_{\vec{a}} : \vec{a} \in \mathbb{Z}^n\}$, so $A \in L_r^\infty(T^n)$. This proves the first equality.

The forward containment of the second equality follows from the fact that $C_r(T^n) \subset AP_r(\mathbb{R}^n) \subset UC_r(\mathbb{R}^n)$ (Proposition 28). For the reverse containment, let $A \in UC_r(\mathbb{R}^n) \cap L_r^\infty(T^n)$. By the remark preceding the theorem, it will suffice to show that $\tau_N(A) \to A$ in norm. Observe that

$$A = \frac{1}{(2\pi)^n} \int_{T^n} AK_N^n(d\vec{t})$$

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since \( \int_T K_N(t) \, dt = 2\pi \). Thus

\[
A - \tau_N(A) = \int_{T^n} (A - \gamma(t)(A)) K^N_N(t) \, dt.
\]

This integral goes to zero in norm as \( N \to \infty \) because the map \( t \mapsto A - \gamma(t)(A) \) is norm continuous and is zero when \( t = 0 \), while for any \( \epsilon > 0 \) we have \( \int_{|t| \geq \epsilon} K^n_N(t) \, dt \to 0 \) as \( N \to \infty \). Thus \( \tau_N(A) \to A \) in norm.

The third equality is trivial, as is the forward containment of the fourth. For the reverse containment, let \( A \in H^\infty_r(R) \cap L^\infty_r(T) \). Then \( \sigma(A) \in H^\infty(T) \) and \( \sigma(\tau_N(A)) = \tau_N(\sigma(A)) \) is a linear combination of \( \sigma(T_0), \ldots, \sigma(T_N) \). Thus \( \tau_N(A) \) is a linear combination of \( T_0, \ldots, T_N \), and \( \tau_N(A) \to A \) weak operator as in an earlier part of the proof, so we conclude that \( A \in H^\infty_r(T) \).

We can prove \( C_r(T^n) = C_{b,r}(R^n) \cap L^\infty_r(T^n) \) by a somewhat more complicated argument involving passage to \( l^2(Z^n, 1/E(k)) \otimes L^2([0, 1]^n) \). The point is that multiplication by \( K_f \), for \( f \) sufficiently close to a delta function at the origin, is an isometry on \( L^\infty_r(T^n) \) with the norm it receives in this representation. Then we use the fact that \( A \in C_{b,r}(R^n) \) implies \( AK_f \in C_{0,r}(R^n) \), which implies norm continuity of \( \gamma_t(AK_f) \); together with \( \gamma(t)(AK_f) \approx \gamma(t)(A)K_f \) for small \( |t| \), this allows us to conclude that \( \gamma(t)(A) \to A \) in norm as \( t \to 0 \).

**Corollary 48.** Let \( X_r \) be any of the spaces in Definition 45 and let \( X \) be its classical analog. Then \( \sigma(X_r) \subset X \).

**Corollary 49.** The images under \( \sigma \) of all of the spaces in Definition 45, except \( A_r(T) \) and \( H^\infty_r(T) \), are self-adjoint subalgebras of \( L^\infty(T^n) \).

**Corollary 50.** We have

\[
C^1_r(T^n) \subset \text{Lip}_r(T^n) \subset C_r(T^n).
\]

(For the first containment, use the comment following Proposition 29.)

**Corollary 51.** Let \( S \subset R^n \) be measurable and periodic. Then \( \mathcal{I}(S) \cap L^\infty_r(T^n) \) is a weak* closed ideal of \( L^\infty_r(T^n) \) and \( \mathcal{I}(S) \cap C_r(T^n) \) is a norm closed ideal of \( C_r(T^n) \).

Our last three results are not corollaries of Theorem 47. The first two can be proven directly using the techniques of Corollary 37 and Theorem 38 in the \( l^2(Z^n, 1/E(k)) \) model of \( L^\infty_r(T^n) \). The proof of the third resembles the proof of Theorem 40 in outline, but is easier. The key step is to show that any complex homomorphism \( \omega : C_r(T^n) \to C \) takes \( T_{\vec{e}_i} \) into the unit circle \( (1 \leq i \leq n) \). This follows from the fact that \( \|T_{\vec{e}_i}^k\| = \|T_{k\vec{e}_i}\| \sim \sqrt{k} \):
then \(|\omega(T_{e_i})^k| = |\omega(T_{e_i}^k)| = O(\sqrt{k})\) implies \(|\omega(T_{e_i})| \leq 1\), and the same argument applied to \(T_{-e_i} = T_{e_i}^{-1}\) shows the reverse inequality.

**Proposition 52.** Let \(d > (n + 1)/2\). Then

\[ C^d(T^n) \subset \sigma(C_r(T^n)). \]

**Proposition 53.** Let \(d > (n + 1)/2\). Then \(\sigma(C^d_r(T^n))\) is properly contained in \(C^d(T^n) \cap \sigma(C_r(T^n))\).

**Proposition 54.** The maximal ideal space of \(C_r(T^n)\) can be identified with \(T^n\), and the symbol map \(\sigma : C_r(T^n) \to C(T^n)\) can be identified with the Gelfand transform.

As a closing remark, we consider the possibility of using other weight functions on \(\mathbb{R}^n\) besides \(1/E(\vec{p})\). For reasons explained in Section 1, this is the only weight that interests us, but our methods and results could apply to other weights \(w(\vec{p})\) as well. What properties of the function \(1/E(\vec{p})\) were used?

Remark 14 (a) specifically requires \(w(\vec{p}) = 1/E(\vec{p})\), and the asymptotic approximation \(w(\vec{p}) \sim |\vec{p}|^{-1}\) for large \(|\vec{p}|\) is used in Remark 14 (c), Remark 23 (b), Lemma 36, Corollary 37, Theorem 38, and Remark 46 (a). However, the bulk of the paper is valid more broadly. The most important conditions are that \(w(\vec{p})\) be bounded (needed so that \(L^2(\mathbb{R}^n) \subset L^2_r(\mathbb{R}^n)\)) and that \(w(\vec{p} + \vec{a})/w(\vec{p}) \to 1\) as \(\vec{p} \to \infty\), for all \(\vec{a}\) (needed for boundedness of \(T_{\vec{a}}\) and \(K_f\), and in Theorem 16 and Lemma 32).

Various other properties of \(1/E(\vec{p})\) appear sporadically. Proposition 7 requires that the weight not be integrable, Theorems 10, 11, and 12 require that it not be constant, Remark 14 (b) requires that \(w\) be a function of \(|\vec{p}|\), Proposition 22 requires that \(w(-\vec{p})/w(\vec{p})\) be bounded, and Remark 23 (a) requires that \(w(\vec{p}) \to 0\) as \(\vec{p} \to \infty\).

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