f-KENMOTSU MANIFOLDS WITH THE SCHOUTEN–VAN KAMPEN CONNECTION

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Abstract. We study 3-dimensional f-Kenmotsu manifolds with the Schouten–van Kampen connection. With the help of such a connection, we study projectively flat, conharmonically flat, Ricci semisymmetric and semisymmetric 3-dimensional f-Kenmotsu manifolds. Finally, we give an example of 3-dimensional f-Kenmotsu manifolds with the Schouten–van Kampen connection.

1. Introduction

The Schouten–van Kampen connection is one of the most natural connections adapted to a pair of complementary distributions on a differentiable manifold endowed with an affine connection [2, 4, 11]. Solov’ev investigated hyperdistributions in Riemannian manifolds using the Schouten–van Kampen connection [12–15]. Then Olszak studied the Schouten–van Kampen connection to an almost contact metric structure [8]. He characterized some classes of almost contact metric manifolds with the Schouten–van Kampen connection and found certain curvature properties of this connection on these manifolds.

On the other hand, let $M$ be an almost contact manifold, i.e., $M$ is a connected $(2n+1)$-dimensional differentiable manifold endowed with an almost contact metric structure $(\phi, \xi, \eta, g)$ [1]. Denote by $\Phi$ the fundamental 2-form of $M$, $\Phi(X, Y) = g(X, \phi Y)$, $X, Y \in \chi(M)$, $\chi(M)$ being the Lie algebra of differentiable vector fields on $M$.

For further use, we recall the following definitions [1, 3, 10]. The manifold $M$ and its structure $(\phi, \xi, \eta, g)$ is said to be:

i) normal, if the almost complex structure defined on the product manifold $M \times \mathbb{R}$ is integrable (equivalently $[\phi, \phi] + 2d\eta \otimes \xi = 0$),

ii) almost cosymplectic, if $d\eta = 0$ and $d\Phi = 0$.

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iii) cosymplectic, if it is normal and almost cosymplectic (equivalently, $\nabla \phi = 0$, $\nabla$ being covariant differentiation with respect to the Levi-Civita connection).

The manifold $M$ is called locally conformal, cosymplectic (respectively almost cosymplectic), if $M$ has an open covering $\{U_i\}$ endowed with differentiable functions $\sigma_i: U_i \to \mathbb{R}$ such that over each $U_i$ the almost contact metric structure $(\phi_i, \xi_i, \eta_i, g_i)$ defined by

$$\phi_i = \phi, \quad \xi_i = e^{\sigma_i} \xi, \quad \eta_i = e^{-\sigma_i} \eta, \quad g_i = e^{-2\sigma_i} g$$

is cosymplectic (respectively almost cosymplectic).

Also, Olszak and Rosca [9] studied normal locally conformal almost cosymplectic manifolds. They given a geometric interpretation of $f$-Kenmotsu manifolds and studied some curvature properties. Among others they proved that a Ricci symmetric $f$-Kenmotsu manifold is an Einstein manifold.

By an $f$-Kenmotsu manifold, we mean an almost contact metric manifold which is normal and locally conformal almost cosymplectic manifold.

In the present paper we study some curvature properties of 3-dimensional $f$-Kenmotsu manifolds with the Schouten–van Kampen connection. The paper is organized as follows: after introduction, we give the Schouten–van Kampen connection and $f$-Kenmotsu manifolds. Then we adapt the Schouten–van Kampen connection on 3-dimensional $f$-Kenmotsu manifolds. In section 5 we study projectively flat 3-dimensional $f$-Kenmotsu manifolds with the Schouten–van Kampen connection. In section 6 we consider conharmonically flat 3-dimensional $f$-Kenmotsu manifolds with the Schouten–van Kampen connection. Section 7 is devoted to study Ricci semisymmetric 3-dimensional $f$-Kenmotsu manifolds with the Schouten–van Kampen connection and we prove that if a 3-dimensional $f$-Kenmotsu manifold is Ricci semisymmetric, then it is an $\eta$-Einstein manifold. In section 8 we study semisymmetric 3-dimensional $f$-Kenmotsu manifolds with the Schouten–van Kampen connection. Finally, we give an example of a 3-dimensional $f$-Kenmotsu manifold with the Schouten–van Kampen connection which verifies Theorem 5.1 and Theorem 6.1.

2. The Schouten–van Kampen connection

Let $M$ be a connected pseudo-Riemannian manifold of an arbitrary signature $(p, n - p)$, $0 \leq p \leq n$, $n = \dim M \geq 2$. By $g$ and $\nabla$ we denote the pseudo-Riemannian metric and Levi-Civita connection induced from the metric $g$ on $M$ respectively. Assume that $H$ and $V$ are two complementary, orthogonal distributions on $M$ such that $\dim H = n - 1$, $\dim V = 1$, and the distribution $V$ is non-null. Thus $TM = H \oplus V$, $H \cap V = \{0\}$ and $H \perp V$. Assume that $\xi$ is a unit vector field and $\eta$ is a linear form such that $\eta(\xi) = 1$, $g(\xi, \xi) = \varepsilon = \pm 1$ and

$$H = \ker \eta, \quad V = \text{span}\{\xi\}.$$  

We can always choose such $\xi$ and $\eta$ at least locally (in a certain neighborhood of an arbitrarily chosen point of $M$). We also have $\eta(X) = \varepsilon g(X, \xi)$. Moreover, it holds that $\nabla_X \xi \in H$. 

For any $X \in TM$, by $X^h$ and $X^v$ we denote the projections of $X$ onto $H$ and $V$, respectively. Thus, we have $X = X^h + X^v$ with
\begin{equation}
X^h = X - \eta(X)\xi, \quad X^v = \eta(X)\xi.
\end{equation}
The Schouten–van Kampen connection $\nabla$ associated to the Levi-Civita connection $\nabla$ and adapted to the pair of the distributions $(H, V)$ is defined by
\begin{equation}
\nabla_X Y = (\nabla_X Y^h)^h + (\nabla_X Y^v)^v,
\end{equation}
and the corresponding second fundamental form $B$ is defined by $B = \nabla - \nabla$. Note that condition (2.2) implies the parallelism of the distributions $B$ and the corresponding second fundamental form $\tilde{B}$ with respect to the Schouten–van Kampen connection $\nabla$.

From (2.1), one can compute
\begin{align*}
(\nabla_X Y^h)^h &= \nabla_X Y - \eta(\nabla_X Y)\xi - \eta(Y)\nabla_X \xi, \\
(\nabla_X Y^v)^v &= (\nabla_X \eta)(Y)\xi + \eta(\nabla_X Y)\xi,
\end{align*}
which enables us to express the Schouten–van Kampen connection with help of the Levi-Civita connection in the following way
\begin{equation}
\nabla_X Y = \nabla_X Y - \eta(Y)\nabla_X \xi + (\nabla_X \eta)(Y)\xi.
\end{equation}
Thus, the second fundamental form $B$ and the torsion $\tilde{T}$ of $\nabla$ are
\begin{align*}
B(X, Y) &= \eta(Y)\nabla_X \xi - (\nabla_X \eta)(Y)\xi, \\
\tilde{T}(X, Y) &= \eta(X)\nabla_Y \xi - \eta(Y)\nabla_X \xi + 2\eta(X, Y)\eta.
\end{align*}
With the help of the Schouten–van Kampen connection (2.3), many properties of some geometric objects connected with the distributions $H, V$ can be characterized (2.14). Probably, the most spectacular is the following statement: $g, \xi$ and $\eta$ are parallel with respect to $\nabla$, that is, $\nabla g = 0$, $\nabla \xi = 0$, $\nabla \eta = 0$.

3. $J$-Kenmotsu manifolds

Let $M$ be a real $(2n + 1)$-dimensional differentiable manifold endowed with an almost contact structure $(\phi, \xi, \eta, g)$ satisfying
\begin{align*}
\phi^2 &= -I + \eta \otimes \xi, \\
\eta(\xi) &= 1, \\
\phi \xi &= 0, \\
\eta \circ \phi &= 0,
\end{align*}
for any vector fields $X, Y \in \chi(M)$, where $I$ is the identity of the tangent bundle $TM$, $\phi$ is a tensor field of $(1, 1)$-type, $\eta$ is a $1$-form, $\xi$ is a vector field and $g$ is a metric tensor field. We say that $(M, \phi, \xi, \eta, g)$ is a $J$-Kenmotsu manifold if the Levi-Civita connection of $g$ satisfy
\begin{equation}
(\nabla_X \phi)(Y) = f(g(\phi X, Y)\xi - \eta(Y)\phi X),
\end{equation}
where $f \in C^\infty(M)$ such that $df \wedge \eta = 0$. If $f = \alpha = \text{constant} \neq 0$, then the manifold is an $\alpha$-Kenmotsu manifold [5]. 1-Kenmotsu manifold is a Kenmotsu manifold [6]. If $f = 0$, then the manifold is cosymplectic [5]. An $J$-Kenmotsu manifold is said to be regular if $f' + f' \neq 0$, where $f' = \xi(f)$.
For an $f$-Kenmotsu manifold from (3.1) it follows that
\begin{equation}
\nabla_X \xi = f \{ X - \eta(X) \xi \}.
\end{equation}
Then using (3.2), we have
\begin{equation}
(\nabla_X \eta)(Y) = f \{ g(X, Y) - \eta(X) \eta(Y) \}.
\end{equation}
The condition $df \wedge \eta = 0$ holds if $\dim M \geq 5$. This does not hold in general if $\dim M = 3$.

As is well known, in a 3-dimensional Riemannian manifold, we always have
\begin{equation}
R(X, Y)Z = g(Y, Z)QX - g(X, Z)QY + S(Y, Z)X - S(X, Z)Y
- \frac{\tau}{2} \{ g(Y, Z)X - g(X, Z)Y \}.
\end{equation}
In a 3-dimensional $f$-Kenmotsu manifold $M$, we have \[9\]
\begin{align}
R(X, Y)Z &= \left( \frac{\tau}{2} + 2f^2 + 2f' \right) \{ g(Y, Z)X - g(X, Z)Y \} \\
&\quad - \left( \frac{\tau}{2} + 3f^2 + 3f' \right) \{ g(Y, Z) \eta(X) \xi - g(X, Z) \eta(Y) \xi \\
&\qquad + \eta(Y) \eta(Z)X - \eta(X) \eta(Z)Y \},
\end{align}
\begin{equation}
S(X, Y) = \left( \frac{\tau}{2} + f^2 + f' \right) g(X, Y) - \left( \frac{\tau}{2} + 3f^2 + 3f' \right) \eta(X) \eta(Y),
\end{equation}
\begin{equation}
QX = \left( \frac{\tau}{2} + f^2 + f' \right) X - \left( \frac{\tau}{2} + 3f^2 + 3f' \right) \eta(X) \xi,
\end{equation}
where $R$ denotes the curvature tensor, $S$ is the Ricci tensor, $Q$ is the Ricci operator and $\tau$ is the scalar curvature of $M$.

From (3.4) and (3.5), we obtain
\begin{align}
R(X, Y) \xi &= -(f^2 + f') \{ \eta(Y)X - \eta(X)Y \}, \\
S(X, \xi) &= -2(f^2 + f') \eta(X).
\end{align}

4. 3-dimensional $f$-Kenmotsu manifolds with the Schouten–van Kampen connection

Let $M$ be a 3-dimensional $f$-Kenmotsu manifold with the Schouten–van Kampen connection. Then using (3.2) and (3.3) in (2.3), we get
\begin{equation}
\tilde{\nabla}_X Y = \nabla_X Y + f \{ g(X, Y) \xi - \eta(Y)X \}.
\end{equation}
Let $R$ and $\tilde{R}$ be the curvature tensors of the Levi-Civita connection $\nabla$ and the Schouten–van Kampen connection $\tilde{\nabla}$,
\begin{equation}
R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}, \quad \tilde{R}(X, Y) = [\tilde{\nabla}_X, \tilde{\nabla}_Y] - \tilde{\nabla}_{[X,Y]}.
\end{equation}
Using (4.1), by direct calculations, we obtain the following formula connecting $\tilde{R}$ and $R$ on a 3-dimensional $f$-Kenmotsu manifold $M$,
\begin{align}
\tilde{R}(X, Y)Z &= R(X, Y)Z + f^2 \{ g(Y, Z)X - g(X, Z)Y \} \\
&\quad + f' \{ g(Y, Z) \eta(X) \xi - g(X, Z) \eta(Y) \xi \\
&\qquad + \eta(Y) \eta(Z)X - \eta(X) \eta(Z)Y \}.
\end{align}
We will also consider the Riemann curvature \((0, 4)\)-tensors \(\tilde{R}, R\), the Ricci tensors \(\tilde{S}, S\), the Ricci operators \(\tilde{Q}, Q\) and the scalar curvatures \(\tilde{\tau}, \tau\) of the connections \(\tilde{\nabla}\) and \(\nabla\) are given by
\[
(4.3) \quad \tilde{R}(X, Y, Z, W) = R(X, Y, Z, W) + f^2 \{g(Y, Z) g(X, W) - g(X, Z) g(Y, W)\} \\
+ f' \{g(Y, Z) \eta(X) \eta(W) - g(X, Z) \eta(Y) \eta(W) \\
+ g(X, W) \eta(Y) \eta(Z) - g(Y, W) \eta(X) \eta(Z)\},
\]
\[
(4.4) \quad \tilde{S}(Y, Z) = S(Y, Z) + (2f^2 + f') g(Y, Z) + f' \eta(Y) \eta(Z),
\]
\[
(4.5) \quad \tilde{Q}X = QX + (2f^2 + f')X + f' \eta(X) \xi;
\]
respectively, where
\[
\tilde{R}(X, Y, Z, W) = g(\tilde{R}(X, Y)Z, W) \quad \text{and} \quad R(X, Y, Z, W) = g(R(X, Y)Z, W).
\]

5. Projectively flat 3-dimensional \(f\)-Kenmotsu manifolds with the Schouten–van Kampen connection

In this section, we study projectively flat 3-dimensional \(f\)-Kenmotsu manifolds with respect to the Schouten–van Kampen connection. In a 3-dimensional \(f\)-Kenmotsu manifold, the projective curvature tensor with respect to the Schouten–van Kampen connection is given by
\[
(5.1) \quad \tilde{P}(X, Y)Z = \tilde{R}(X, Y)Z - \frac{1}{2}\{\tilde{S}(Y, Z)X - \tilde{S}(X, Z)Y\}.
\]
If \(\tilde{P} = 0\), then the manifold \(M\) is called \emph{projectively flat} manifold with respect to the Schouten–van Kampen connection.

Let \(M\) be a projectively flat manifold with respect to the Schouten–van Kampen connection. From (5.1), we have
\[
(5.2) \quad \tilde{R}(X, Y)Z = \frac{1}{2}\{\tilde{S}(Y, Z)X - \tilde{S}(X, Z)Y\}.
\]
Using (4.3) and (4.4) in (5.2), we get
\[
(5.3) \quad g(R(X, Y)Z, W) + f^2 \{g(Y, Z) g(X, W) - g(X, Z) g(Y, W)\} \\
+ f' \{g(Y, Z) \eta(X) \eta(W) - g(X, Z) \eta(Y) \eta(W) \\
+ g(X, W) \eta(Y) \eta(Z) - g(Y, W) \eta(X) \eta(Z)\} \\
= \frac{1}{2}\{S(Y, Z) g(X, W) - S(X, Z) g(Y, W) \\
+ [2f^2 + f'] \eta(Y) \eta(Z) g(X, W) - g(X, Z) g(Y, W)] \\
+ f' \eta(Y) \eta(Z) g(X, W) - \eta(X) \eta(Z) g(Y, W)\}.
\]
Now putting \(W = \xi\) in (5.3), we obtain
\[
(5.3) \quad \{g(X, Z) \eta(Y) - g(Y, Z) \eta(X)\} + (f^2 + f') \{g(Y, Z) \eta(X) - g(X, Z) \eta(Y)\} \\
= \frac{1}{2}\{S(Y, Z) \eta(X) - S(X, Z) \eta(Y) + (2f^2 + f') \eta(Y) \eta(Z) g(X, W) - g(X, Z) g(Y, W)] \\
+ f' \eta(Y) \eta(Z) g(X, W) - \eta(X) \eta(Z) g(Y, W)\}.
\]
which gives
\[ S(Y, Z) \eta(X) - S(X, Z) \eta(Y) + (2f^2 + f') [g(Y, Z) \eta(X) - g(X, Z) \eta(Y)] = 0. \]
Again putting \( X = \xi \) in (5.4), we get
\[ S(Y, Z) = -(2f^2 + f') g(Y, Z) - f' \eta(Y) \eta(Z). \]
Thus \( M \) is an \( \eta \)-Einstein manifold with respect to the Levi-Civita connection.

Also, using (5.5) in (4.4), we obtain
\[ \tilde{S}(Y, Z) = 0. \]

Thus the manifold \( M \) is a flat manifold with respect to the Schouten–van Kampen connection.

Conversely, let \( M \) be a flat manifold with respect to the Schouten–van Kampen connection. Then we say that the manifold \( M \) is a Ricci-flat manifold with respect to the Schouten–van Kampen connection. Hence from (5.1), we get
\[ \tilde{P}(X, Y)Z = 0, \]
that is, the manifold \( M \) is a projectively flat manifold with respect to the Schouten–van Kampen connection. Thus we have the following:

**Theorem 5.1.** Let \( M \) be a 3-dimensional \( f \)-Kenmotsu manifold with the Schouten–van Kampen connection. Then the following statements are equivalent:

i) \( M \) is projectively flat with respect to the Schouten–van Kampen connection,

ii) \( M \) is Ricci flat with respect to the Schouten–van Kampen connection,

iii) \( M \) is flat with respect to the Schouten–van Kampen connection.

6. Conharmonically flat 3-dimensional \( f \)-Kenmotsu manifolds with the Schouten–van Kampen connection

In this section, we study conharmonically flat 3-dimensional \( f \)-Kenmotsu manifolds with respect to the Schouten–van Kampen connection. In a 3-dimensional \( f \)-Kenmotsu manifold the conharmonic curvature tensor with respect to the Schouten–van Kampen connection is given by
\[ \tilde{K}(X, Y)Z = \tilde{R}(X, Y)Z - \{ \tilde{S}(Y, Z)X - \tilde{S}(X, Z)Y + g(Y, Z) \tilde{Q}X - g(X, Z) \tilde{Q}Y \}. \]

If \( \tilde{K} = 0 \), then the manifold \( M \) is called *conharmonically flat* manifold with respect to the Schouten–van Kampen connection.

Let \( M \) be a conharmonically flat manifold with respect to the Schouten–van Kampen connection. From (6.1), we have
\[ \tilde{R}(X, Y)Z = \tilde{S}(Y, Z)X - \tilde{S}(X, Z)Y + g(Y, Z) \tilde{Q}X - g(X, Z) \tilde{Q}Y. \]

Using (4.3), (4.4) and (4.5) in (6.2), we get
\[ R(X, Y)Z + f^2 \{ g(Y, Z)X - g(X, Z)Y \} + f' \{ g(Y, Z) \eta(X) \xi - g(X, Z) \eta(Y) \eta(X) \eta(Y) \} + \eta(Y) \eta(Z) X - \eta(X) \eta(Z) Y + (4f^2 + 2f') \{ g(Y, Z)X - g(X, Z)Y \} = S(Y, Z)X - S(X, Z)Y + \left( 4f^2 + 2f' + f^2 + f' \right) \{ g(Y, Z)X - g(X, Z)Y \} \]
Now putting $X = \xi$ in (6.3), we obtain

\[
R(\xi, Y)Z + (f^2 + f')\{g(Y, Z)\xi - \eta(Z)Y\}
\]

\[
= S(Y, Z)\xi - S(\xi, Z)Y
\]

\[
+ \left(4f^2 + 2f' + \frac{\tau}{2} + f^2 + f'\right)\{g(Y, Z)\xi - \eta(Z)Y\}
\]

\[
+ f'(\eta(Y)\eta(Z)\xi - \eta(Z)Y)
\]

\[
+ \left(f' - \frac{\tau}{2} - 3f^2 - 3f'\right)\{g(Y, Z)\xi - \eta(Z)\eta(Y)\xi\}.
\]

Using (3.4) and (3.7) in (6.4), we get

\[
S(Y, Z)\xi - S(\xi, Z)Y + \left(4f^2 + 2f' + \frac{\tau}{2} + f^2 + f'\right)\{g(Y, Z)\xi - \eta(Z)Y\}
\]

\[
+ f'(\eta(Y)\eta(Z)\xi - \eta(Z)Y)
\]

\[
+ \left(f' - \frac{\tau}{2} - 3f^2 - 3f'\right)\{g(Y, Z)\xi - \eta(Z)\eta(Y)\xi\} = 0.
\]

Taking the inner product with $\xi$ in (6.5), we have

\[
S(Y, Z) + 2(f^2 + f')\eta(Y)\eta(Z) + (2f^2 + f')\{g(Y, Z) - \eta(Y)\eta(Z)\} = 0,
\]

which gives

\[
S(Y, Z) = -(2f^2 + f')g(Y, Z) - f'\eta(Y)\eta(Z).
\]

Thus $M$ is an $\eta$-Einstein manifold with respect to the Levi-Civita connection.

Using (6.6) in (4.4), we obtain $\tilde{S}(Y, Z) = 0$. Hence the manifold $M$ is a Ricci-flat manifold with respect to the Schouten–van Kampen connection. Then from (6.2) the manifold $M$ is a flat manifold with respect to the Schouten–van Kampen connection.

Conversely, let $M$ be a flat manifold with respect to the Schouten–van Kampen connection. Then we say that the manifold $M$ is a Ricci-flat manifold with respect to the Schouten–van Kampen connection. Hence from (6.1), we get $\tilde{K}(X, Y)Z = 0$. i.e., the manifold $M$ is a conharmonically flat manifold with respect to the Schouten–van Kampen connection. Thus we have the following:

**Theorem 6.1.** Let $M$ be a 3-dimensional $f$-Kenmotsu manifold with the Schouten–van Kampen connection. Then the following statements are equivalent:

i) $M$ is conharmonically flat with respect to the Schouten–van Kampen connection,

ii) $M$ is Ricci flat with respect to the Schouten–van Kampen connection,

iii) $M$ is flat with respect to the Schouten–van Kampen connection.
7. Ricci semisymmetric 3-dimensional $f$-Kenmotsu manifolds with the Schouten–van Kampen connection

A $f$-Kenmotsu manifold with the Schouten–van Kampen connection is called Ricci semisymmetric if $\tilde{R}(X,Y) \cdot \tilde{S} = 0$, where $\tilde{R}(X,Y)$ is treated as a derivation of the tensor algebra for any tangent vectors $X, Y$. Then

$$(7.1) \quad \tilde{S}(\tilde{R}(X,Y)Z,W) + \tilde{S}(\tilde{R}(X,Y)W,Z) = 0.$$ 

Using (4.3) and (4.4) in (7.1), we get

$$S(R(X,Y)Z,W) + S(Z,R(X,Y)W) + f'\{\eta(R(X,Y)Z) \eta(W) + f'\eta(R(X,Y)W) \eta(Z)\} + f^2\{S(X,W)g(Y,Z) - S(Y,W)g(X,Z) + S(X,Z)g(Y,W) - S(Y,Z)g(X,W)\} - f'(f^2 + f')\{g(Y,Z) \eta(X) \eta(W) - g(X,Z) \eta(Y) \eta(W) + g(Y,W) \eta(X) \eta(Z) - g(X,W) \eta(Y) \eta(Z)\} = 0.$$ 

Let $M$ be Ricci semisymmetric with respect to the Levi-Civita connection. Then we have

$$(7.2) \quad f'\{\eta(R(X,Y)Z) \eta(W) + f'\eta(R(X,Y)W) \eta(Z)\} + f^2\{S(X,W)g(Y,Z) - S(Y,W)g(X,Z) + S(X,Z)g(Y,W) - S(Y,Z)g(X,W)\} - f'(f^2 + f')\{g(Y,Z) \eta(X) \eta(W) - g(X,Z) \eta(Y) \eta(W) + g(Y,W) \eta(X) \eta(Z) - g(X,W) \eta(Y) \eta(Z)\} = 0.$$ 

Putting $W = \xi$ in (7.2), we obtain

$$f'\eta(R(X,Y)Z) + f^2\{S(X,\xi)g(Y,Z) - S(Y,\xi)g(X,Z) + S(X,Z)\eta(Y) - S(Y,Z)\eta(X)\} - f'(f^2 + f')\{g(Y,Z) \eta(X) - g(X,Z) \eta(Y)\} + f'\{S(X,\xi)\eta(Y)\eta(Z) - S(Y,\xi)\eta(Z) + S(X,Z)\eta(Y) - S(Y,Z)\eta(X)\} = 0.$$ 

After some calculations, we get

$$(7.3) \quad 2(f^2 + f')^2\{g(Y,Z) \eta(X) - g(X,Z) \eta(Y)\} - (f^2 + f')\{S(Y,Z)\eta(X) - S(X,Z)\eta(Y)\} = 0.$$ 

Again putting $X = \xi$ in (7.3), we have

$$2(f^2 + f')^2\{g(Y,Z) - \eta(Y)\eta(Z)\} - (f^2 + f')\{S(Y,Z) + 2(f^2 + f')\eta(Y)\eta(Z)\} = 0,$$

which gives

$$(7.4) \quad (f^2 + f')\{S(Y,Z) + 4(f^2 + f')\eta(Y)\eta(Z) - 2(f^2 + f')g(Y,Z)\} = 0.$$
Let $f^2 + f' \neq 0$, then from (7.4), we get
\begin{equation}
S(Y, Z) = 2(f^2 + f') g(Y, Z) - 4(f^2 + f') \eta(Y) \eta(Z).
\end{equation}
Hence the manifold is an $\eta$-Einstein manifold with respect to the Levi-Civita connection.

Using (7.5) in (4.4), we obtain
\begin{equation}
\hat{S}(Y, Z) = (4f^2 + 3f') g(Y, Z) - (4f^2 + 3f') \eta(Y) \eta(Z).
\end{equation}
Thus we have the following:

**Theorem 7.1.** Let $M$ be a Ricci semisymmetric 3-dimensional regular $f$-Kenmotsu manifold with the Schouten–van Kampen connection. If $M$ is a Ricci semisymmetric 3-dimensional $f$-Kenmotsu manifold with respect to the Levi-Civita connection, then $M$ is an $\eta$-Einstein manifold with respect to the Schouten–van Kampen connection.

8. **Semisymmetric 3-dimensional $f$-Kenmotsu manifolds with the Schouten–van Kampen connection**

In this section, we study a semisymmetric regular 3-dimensional $f$-Kenmotsu manifold with the Schouten–van Kampen connection. If a 3-dimensional $f$-Kenmotsu manifold with the Schouten–van Kampen connection is *semisymmetric* then we can write
\begin{equation}
(\tilde{R}(X, Y) \cdot \tilde{R})(Z, U)W = 0,
\end{equation}
which gives
\begin{equation}
\tilde{R}(X, Y)\tilde{R}(Z, U)W - \tilde{R}(\tilde{R}(X, Y)Z, U)W - \tilde{R}(Z, U)\tilde{R}(X, Y)W = 0.
\end{equation}
Using (4.2) in (8.1), we have
\begin{equation}
\tilde{R}(X, Y)\tilde{R}(Z, U)W - R(\tilde{R}(X, Y)Z, U)W - R(Z, U)\tilde{R}(X, Y)W = 0,
\end{equation}
which gives
\begin{equation}
(\tilde{R}(X, Y) \cdot R)(Z, U)W = 0.
\end{equation}
Again using (4.2) in (8.2), we obtain
\begin{equation}
R(X, Y)R(Z, U)W - R(R(X, Y)Z, U)W - R(Z, R(X, Y)U)W
- R(Z, U)R(X, Y)W + f^2 \{g(R(Z, U)W, Y)X - g(R(Z, U)W, X)Y
- g(Y, Z)R(X, U)W + g(X, Z)R(Y, U)W - g(Y, Z)R(Z, X)W
+ g(X, U)R(Z, Y)W - g(Y, W)R(Z, U)X + g(X, W)R(Z, U)Y
+ f' \{g(R(Z, U)W, Y) \eta(X) \xi - g(R(Z, U)W, X) \eta(Y) \xi + \eta(R(Z, U)W) \eta(Y) \xi
- \eta(R(Z, U)W) \eta(X) - g(Y, Z) \eta(R(X, U)W) \eta(Y) \xi + g(X, Z) \eta(R(Y, U)W) \xi
- \eta(Y) \eta(Z) R(X, U)W + \eta(X) \eta(Z) R(Y, U)W - g(Y, U) \eta(R(Z, X)W) \xi
- \eta(Y) \eta(Z) R(X, U)W + \eta(X) \eta(Z) R(Y, U)W - g(Y, U) \eta(R(Z, X)W) \xi.
\end{equation}
\[ + g(X, U) \eta(R(Z, Y)W) \xi - \eta(Y) \eta(U) R(Z, X) W + g(X) \eta(U) R(Z, Y) W \\
- g(Y, W) \eta(R(Z, U)X) \xi + g(X, W) \eta(R(Z, U)Y) \xi \\
- \eta(Y) \eta(W) R(Z, U) X + \eta(X) \eta(W) R(Z, U) Y \} = 0. \]

Now from (8.3), we can say:

If \( 0 \neq f = \text{constant} \) (say \( f = \alpha \)), then \( f' = 0 \). Hence we get \( R \cdot R = -\alpha^2 Q(g, R) \).

Therefore the manifold \( M \) is a pseudo-symmetric \( \alpha \)-Kenmotsu manifold.

If \( f \) is not constant, then using \( X = \xi \) in (8.3), we get

\[ R(\xi, Y) R(Z, U) W - R(R(\xi, Y) Z, U) W - R(Z, R(\xi, Y) U) W \\
- R(Z, U) R(\xi, Y) W + f^2 \{ g(R(Z, U) W, Y) \xi - g(R(Z, U) W, \xi) Y \\
- g(Y, Z) R(\xi, U) W + g(\xi, Z) R(Y, U) W - g(Y, U) R(Z, \xi) W \\
+ g(\xi, U) R(Z, Y) W - g(Y, W) R(Z, U) \xi + g(\xi, W) R(Z, U) Y \} \\
+ f' \{ g(R(Z, U) W, Y) \xi - g(R(Z, U) W, \xi) Y + \eta(R(Z, U) W) \eta(Y) \xi \\
- \eta(R(Z, U) W) Y - g(Y, Z) \eta(R(\xi, U) W) + g(\xi, Z) \eta(R(Y, U) W) \xi \\
- \eta(Y) \eta(Z) \eta(R(\xi, U) W) + \eta(Z) R(Y, U) W - g(Y, U) \eta(R(Z, \xi) W) \xi \\
+ g(\xi, U) \eta(R(Z, Y) W) - \eta(Y) \eta(U) R(Z, \xi) W + \eta(U) \eta(R(Z, Y) W) \\
- g(Y, W) \eta(R(Z, U) \xi) + g(\xi, W) \eta(R(Z, U) Y) \\
- \eta(Y) \eta(W) \eta(R(Z, U) \xi) + \eta(W) \eta(R(Z, U) Y) \} = 0. \]

Taking the inner product with \( \xi \) in (8.4), we obtain

\[ \eta(R(\xi, Y) R(Z, U) W) - \eta(R(R(\xi, Y) Z, U) W) - \eta(R(Z, R(\xi, Y) U) W) \\
- \eta(R(Z, U) R(\xi, Y) W) + f^2 \{ g(R(Z, U) W, Y) - g(R(Z, U) W, \xi) Y \\
- g(Y, Z) \eta(R(\xi, U) W) + g(\xi, Z) \eta(R(Y, U) W) - g(Y, U) \eta(R(Z, \xi) W) \\
+ g(\xi, U) \eta(R(Z, Y) W) - g(Y, W) \eta(R(Z, U) \xi) + g(\xi, W) \eta(R(Z, U) Y) \} \\
+ f' \{ g(R(Z, U) W, Y) - g(R(Z, U) W, \xi) Y + \eta(R(Z, U) W) \eta(Y) \\
- \eta(R(Z, U) W) \eta(Y) - g(Y, Z) \eta(R(\xi, U) W) + g(\xi, Z) \eta(R(Y, U) W) \\
- \eta(Y) \eta(Z) \eta(R(\xi, U) W) + \eta(Z) \eta(R(Y, U) W) - g(Y, U) \eta(R(Z, \xi) W) \xi \\
+ g(\xi, U) \eta(R(Z, Y) W) - \eta(Y) \eta(U) R(Z, \xi) W + \eta(U) \eta(R(Z, Y) W) \\
- g(Y, W) \eta(R(Z, U) \xi) + g(\xi, W) \eta(R(Z, U) Y) \\
- \eta(Y) \eta(W) \eta(R(Z, U) \xi) + \eta(W) \eta(R(Z, U) Y) \} = 0. \]

Let \( \{ e_i \} \) \((1 \leq i \leq 3)\) be an orthonormal basis of the tangent space at any point of \( M \). Then the sum for \( 1 \leq i \leq 3 \) of the relation (8.5) for \( Y = Z = e_i \) gives

\[ \eta(R(\xi, e_i) R(e_i, U) W) - \eta(R(R(\xi, e_i) e_i, U) W) - \eta(R(e_i, R(\xi, e_i) U) W) \\
- \eta(R(e_i, U) R(\xi, e_i) W) + f^2 \{ g(R(e_i, U) W, e_i) - g(R(e_i, U) W, \xi) \eta(e_i) \\
- g(e_i, e_i) \eta(R(\xi, U) W) + g(\xi, e_i) \eta(R(e_i, U) W) - g(e_i, U) \eta(R(e_i, \xi) W) \\
+ g(\xi, U) \eta(R(e_i, e_i) W) - g(e_i, W) \eta(R(e_i, U) \xi) + g(\xi, W) \eta(R(e_i, U) e_i) \} \]
After some calculations, we obtain

\[ 2(f^2 + f')\{S(U, W) - 2g(R(\xi, W)U, \xi)\} \]
\[ - f^2\{S(U, W) - 2g(R(\xi, W)U, \xi) - 2(f^2 + f')\eta(U)\eta(W)\} \]
\[ - f'\{S(U, W) - 2g(R(\xi, W)U, \xi) - 2(f^2 + f')\eta(U)\eta(W)\} \]

which gives

\[ (f^2 + f')\{S(U, W) - 2g(R(\xi, W)U, \xi) + 2(f^2 + f')\eta(U)\eta(W)\} = 0. \]

Let \( f^2 + f' \neq 0 \). Then from (8.6), we get

\[ S(U, W) - 2g(R(\xi, W)U, \xi) + 2(f^2 + f')\eta(U)\eta(W) = 0. \]

Using (3.6) in (8.7), we obtain

\[ S(U, W) = -2(f^2 + f')g(U, W). \]

Thus we have the following:

**Theorem 8.1.** Let \( M \) be a 3-dimensional regular \( f \)-Kenmotsu manifold with the Schouten–van Kampen connection. If \( M \) is semisymmetric with respect to the Schouten–van Kampen connection, then:

i) If \( 0 \neq f = \alpha = \text{constant} \), then the manifold \( M \) is a pseudosymmetric \( \alpha \)-Kenmotsu manifold, or,

ii) If \( f \) is not constant, then the manifold \( M \) is an Einstein manifold.

**9. An example of a 3-dimensional \( f \)-Kenmotsu manifold with the Schouten–van Kampen connection**

We consider the 3-dimensional manifold \( M = \{ (x, y, z) \in \mathbb{R}^3, z \neq 0 \} \), where \((x, y, z)\) are the standard coordinates in \( \mathbb{R}^3 \). The vector fields

\[ e_1 = z^2 \frac{\partial}{\partial x}, \quad e_2 = z^2 \frac{\partial}{\partial y}, \quad e_3 = \frac{\partial}{\partial z} \]

are linearly independent at each point of \( M \). Let \( g \) be the Riemannian metric defined by

\[ g(e_1, e_3) = g(e_2, e_3) = g(e_1, e_2) = 0, \quad g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = 1. \]

Let \( \eta \) be the 1-form defined by \( \eta(Z) = g(Z, e_3) \) for any \( Z \in \chi(M) \). Let \( \phi \) be the \((1,1)\) tensor field defined by \( \phi(e_1) = -e_2, \phi(e_2) = e_1, \phi(e_3) = 0 \). Then using linearity of \( \phi \) and \( g \) we have

\[ \eta(e_3) = 1, \quad \phi^2 Z = -Z + \eta(Z)e_3, \quad g(\phi Z, \phi W) = g(Z, W) - \eta(Z) \eta(W), \]
for any $Z, W \in \chi(M)$. Now, by direct computations we obtain
\[
[e_1, e_2] = 0, \quad [e_2, e_3] = -\frac{2}{z} e_2, \quad [e_1, e_3] = -\frac{2}{z} e_1.
\]

The Riemannian connection $\nabla$ of the metric tensor $g$ is given by Koszul’s formula which is

\[
(9.1) \quad 2g(\nabla_X Y, Z) = X g(Y, Z) + Y g(Z, X) - Z g(X, Y) - g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]).
\]

Using $(9.1)$, we have
\[
2g(\nabla_{e_1} e_3, e_1) = 2g\left(-\frac{2}{z} e_1, e_1\right), \quad 2g(\nabla_{e_1} e_3, e_2) = 0 \quad \text{and} \quad 2g(\nabla_{e_1} e_3, e_3) = 0.
\]

Hence $\nabla_{e_1} e_3 = -\frac{2}{z} e_1$. Similarly, $\nabla_{e_2} e_3 = -\frac{2}{z} e_2$ and $\nabla_{e_3} e_3 = 0$. $(9.1)$ further yields

\[
(9.2) \quad \nabla_{e_1} e_2 = 0, \quad \nabla_{e_2} e_2 = \frac{2}{z} e_3, \quad \nabla_{e_3} e_2 = 0,
\]
\[
\nabla_{e_1} e_3 = \frac{2}{z} e_3, \quad \nabla_{e_2} e_1 = 0, \quad \nabla_{e_3} e_1 = 0.
\]

From $(9.2)$, we see that the manifold satisfies $\nabla_X \xi = f(X - \eta(X)\xi)$ for $\xi = e_3$, where $f = -\frac{2}{z}$. Hence we conclude that $M$ is an $f$-Kenmotsu manifold. Also $f^2 + f' \neq 0$. Hence $M$ is a regular $f$-Kenmotsu manifold [16].

It is known that

\[
R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z.
\]

With the help of the above formula and using $(9.3)$, it can be easily verified that

\[
R(e_1, e_2)e_3 = 0, \quad R(e_2, e_3)e_3 = -\frac{6}{z^2} e_2,
\]
\[
R(e_1, e_3)e_3 = -\frac{6}{z^2} e_1, \quad R(e_1, e_2)e_2 = -\frac{4}{z^2} e_1,
\]
\[
(9.4) \quad R(e_3, e_2)e_2 = -\frac{6}{z^2} e_3, \quad R(e_1, e_3)e_2 = 0,
\]
\[
R(e_1, e_2)e_1 = \frac{4}{z^2} e_2, \quad R(e_2, e_3)e_1 = 0,
\]
\[
R(e_1, e_3)e_1 = \frac{6}{z^2} e_3.
\]

Now the Schouten–van Kampen connection on $M$ is given by

\[
\hat{\nabla}_{e_1} e_3 = \left(-\frac{2}{z} - f\right)e_1, \quad \hat{\nabla}_{e_2} e_3 = \left(-\frac{2}{z} - f\right)e_2,
\]
\[
\hat{\nabla}_{e_3} e_3 = -f(e_3 - \xi), \quad \hat{\nabla}_{e_1} e_2 = 0,
\]
\[
(9.5) \quad \hat{\nabla}_{e_2} e_2 = \frac{2}{z}(e_3 - \xi), \quad \hat{\nabla}_{e_3} e_2 = 0,
\]
\[
\hat{\nabla}_{e_1} e_1 = \frac{2}{z}(e_3 - \xi), \quad \hat{\nabla}_{e_2} e_1 = 0,
\]
\[
\hat{\nabla}_{e_3} e_1 = 0.
\]
From (9.5), we can see that \( \tilde{\nabla}_e e_j = 0 \) for \( 1 \leq i, j \leq 3 \) and \( \xi = e_3 \) and \( f = -\frac{2}{z} \). Hence \( M \) is a 3-dimensional \( f \)-Kenmotsu manifold with respect to the Schouten–van Kampen connection. Also using (9.4), it can be seen that \( \tilde{R} = 0 \). Thus the manifold \( M \) is a flat manifold with respect to the Schouten–van Kampen connection. Since a flat manifold is a Ricci-flat manifold with respect to the Schouten–van Kampen connection, the manifold \( M \) is both a projectively flat and a conharmonically flat 3-dimensional \( f \)-Kenmotsu manifold with respect to the Schouten–van Kampen connection. So, from Theorems 5.1 and 6.1, \( M \) is an \( \eta \)-Einstein manifold with respect to the Levi-Civita connection.

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