THE HIGHER-DIMENSIONAL RUDNICK-KURLBERG CONJECTURE

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preliminary version

Abstract

In this paper we give a proof of the Hecke quantum unique ergodicity conjecture for the multidimensional Berry-Hannay model. A model of quantum mechanics on the 2n-dimensional torus. This result generalizes the proof of the Rudnick-Kurlberg Conjecture given in [GH3] for the 2-dimensional torus.

Contents

0 Introduction 2
  0.1 Berry-Hannay model 2
  0.2 Quantum chaos 2
  0.3 Hecke quantum unique ergodicity 2
  0.4 Motivation 3
  0.5 Results 3

1 Classical Torus 4
  1.1 Classical mechanical system 4

2 Quantization of the Torus 5
  2.1 The Weyl quantization model 6
  2.2 Equivariant Weyl quantization of the torus 7
  2.3 Quantum mechanical system 9
0 Introduction

0.1 Berry-Hannay model

In the paper “Quantization of linear maps on the torus - Fresnel diffraction by a periodic grating”, published in 1980 (see [BH]), the physicists M.V. Berry and J. Hannay explore a model for quantum mechanics on the 2-dimensional symplectic torus $\left( T, \omega \right)$. Berry and Hannay suggested to quantize simultaneously the functions on the torus and the linear symplectic group $G = SL(2, \mathbb{Z})$.

0.2 Quantum chaos

One of the motivations for considering the Berry-Hannay model was to study the phenomenon of quantum chaos in this model (see [R2] for a survey).

Consider a classical ergodic mechanical system. In what sense the ergodicity property is reflected in the corresponding quantum system ?. Or phrase it differently, is there a meaningful notion of Quantum Ergodicity ?. This is a fundamental meta-question in the area of quantum chaos.

0.3 Hecke quantum unique ergodicity

For the specific case of the 2-dimensional Berry-Hannay model, the quantum ergodicity question was addressed in a paper by Rudnick and Kurlberg [KR]. In that paper they formulated a rigorous definition of quantum ergodicity. We call this notion Hecke Quantum Unique Ergodicity. A focal step in their
work was to introduce a group of hidden symmetries they named the Hecke group. The statement of Hecke Quantum Unique Ergodicity concerns the semi-classical convergence of certain matrix coefficients which are defined in terms of the Hecke group. In their paper they proved a bound on the rate of convergence. In Z. Rudnick’s lectures at MSRI, Berkeley 1999 [R1] and ECM, Barcelona 2000 [R2] he conjectured that a stronger bound should hold true.

In [GH1] we have constructed the 2-dimensional Berry-Hannay model and in [GH3] we have proved the 2-dimensional Rudnick-Kurlberg rate conjecture. We established the correct bound, on the semi-classical asymptotic of the Hecke matrix coefficients, as stated in Rudnick’s lectures [R1, R2].

0.4 Motivation

Our motivation in this paper is to generalize the Rudnick-Kurlberg Hecke theory to the Multidimensional Berry-Hannay Model (see [GH2]). More specifically we study the higher-dimensional generalization of the Rudnick-Kurlberg Conjecture [GH3] about Hecke quantum unique ergodicity.

0.5 Results

In [GH2] we have constructed the 2n-dimensional Berry-Hannay model. We have quantized simultaneously the functions on the 2n-dimensional torus and the linear symplectic group \( G = Sp(2n, \mathbb{Z}) \). In this paper we develop the Hecke theory for this model and we manifest the algebro-geometric tools to deal with quantitative quantum mechanical statistical questions arising in the higher-dimensional Berry-Hannay model. As an application we prove the higher-dimensional Hecke Quantum Unique Ergodicity Conjecture for ergodic element \( A \in Sp(2n, \mathbb{Z}) \) which has an irreducible characteristic polynomial over \( \mathbb{Q} \).

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1 Classical Torus

Let \((T,\omega)\) be the 2n-dimensional symplectic torus. Together with its linear symplectomorphisms \(G \simeq Sp(2n, \mathbb{Z})\) it serves as a simple model of classical mechanics (a compact version of the phase space of the harmonic oscillator). More precisely, let \(T = V/\Lambda\), where \(V\) is a 2n-dimensional real vector space, \(V \simeq \mathbb{R}^{2n}\) and \(\Lambda\) is a full lattice in \(V\), \(\Lambda \simeq \mathbb{Z}^{2n}\). The symplectic form on \(T\) we obtain by taking a non-degenerate symplectic form on \(V\),

\[\omega: V \times V \to \mathbb{R}\]

We require \(\omega\) to be integral, namely \(\omega: \Lambda \times \Lambda \to \mathbb{Z}\) and normalized such that \(\text{Vol}(T) = 1\).

Let \(Sp(V, \omega)\) be the group of linear symplectomorphisms, \(Sp(V, \omega) \simeq Sp(2n, \mathbb{R})\). Consider the subgroup \(G \subset Sp(V, \omega)\) of elements which preserve the lattice \(\Lambda\), i.e \(G(\Lambda) \subseteq \Lambda\). Then \(G \simeq Sp(2n, \mathbb{Z})\). The subgroup \(G\) is exactly the group of linear symplectomorphisms of \(T\).

We denote by \(\Lambda^* \subseteq V^*\) the dual lattice, \(\Lambda^* = \{\xi \in V^*| \ \xi(\Lambda) \subseteq \mathbb{Z}\}\). The lattice \(\Lambda^*\) is identified with the lattice of characters of \(T\) by the following map:

\[\xi \in \Lambda^* \mapsto e^{2\pi i \langle \xi, \cdot \rangle} \in T^*, \tag{1}\]

where \(T^* = \text{Hom}(T, \mathbb{C}^\times)\).

1.1 Classical mechanical system

We consider a very simple discrete mechanical system. An element \(A \in G\), that has no roots of unity as eigenvalues, generates an ergodic discrete
dynamical system. The *ergodic theorem* asserts that:

$$\lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} f(A^k x) = \int_T f \omega, \quad (2)$$

for every $f \in \mathcal{F}(T)$ and almost every point $x \in T$. Here $\mathcal{F}(T)$ stands for a good class of functions, for example trigonometric polynomials or smooth functions.

## 2 Quantization of the Torus

Quantization is one of the big mysteries of modern mathematics, indeed it is not clear at all what is the precise structure which underlies quantization in general. Although physicists have been quantizing for almost a century, for mathematicians the concept remains all-together unclear. Yet in specific cases there are certain formal models for quantization which are well justified on the mathematical side. The case of the symplectic torus is one of these cases. Before we employ the formal model, it is worthwhile to discuss the general phenomenological principles of quantization which are surely common for all models.

Let us start from a model of classical mechanics, namely a symplectic manifold, serving as classical phase space. In our case this manifold is the symplectic torus $T$. Principally quantization is a protocol by which one associates to the classical phase space $T$ a quantum "phase" space $\mathcal{H}$, where $\mathcal{H}$ is a Hilbert space. In addition the protocol gives a rule by which one associates to every classical observable, namely a function $f \in \mathcal{F}(T)$ a quantum observable $\mathcal{O}_f : \mathcal{H} \rightarrow \mathcal{H}$, an operator on the Hilbert space. This rule should send a real function into a self adjoint operator.

To be a little bit more precise, quantization should be considered not as a single protocol, but as a one parametric family of protocols, parameterized by $\hbar$, the Planck constant. For every fixed value of the parameter $\hbar$ there is a protocol which associates to $T$ a Hilbert space $\mathcal{H}_\hbar$ and for every function $f \in \mathcal{F}(T)$ an operator $\mathcal{O}_f^\hbar : \mathcal{H}_\hbar \rightarrow \mathcal{H}_\hbar$. Again the association rule should send real functions to self adjoint operators.
Accepting the general principles of quantization, one searches for a formal model by which to quantize, that is a mathematical model which will manufacture a family of Hilbert spaces $\mathcal{H}_h$ and association rules $\mathcal{F}(T) \to \text{End}(\mathcal{H}_h)$. In this work we employ a model of quantization called the \textit{Weyl Quantization model}.

### 2.1 The Weyl quantization model

The Weyl quantization model works as follows. Let $\mathcal{A}_h$ be one parametric deformation of the algebra $\mathcal{A}$ of trigonometric polynomials on the torus. This algebra is known in the literature as the Rieffel torus [Ri]. The algebra $\mathcal{A}_h$ is constructed by taking the free algebra over $\mathbb{C}$ generated by the following symbols \{ $s(\xi) \mid \xi \in \Lambda^*$ \} and quotient out by the relation $s(\xi + \eta) = e^{\pi \hbar \omega(\xi,\eta)} s(\xi) s(\eta)$. We point out two facts about this algebra $\mathcal{A}_h$. First when substituting $\hbar = 0$ one gets the group algebra of $\Lambda^*$, which is exactly equal to the algebra of trigonometric polynomials on the torus. Second the algebra $\mathcal{A}_h$ contains as a standard basis the lattice $\Lambda^*$:

$$ s : \Lambda^* \to \mathcal{A}_h $$

(3)

So one can identify the algebras $\mathcal{A}_h \simeq \mathcal{A}$ as vector spaces. Therefore every function $f \in \mathcal{A}$ can be viewed as an element of $\mathcal{A}_h$.

For a fixed $\hbar$ a representation $\pi_\hbar : \mathcal{A}_h \to \text{End}(\mathcal{H}_h)$ serves as a quantization protocol, namely for every function $f \in \mathcal{A}$ one has:

$$ f \in \mathcal{A} \simeq \mathcal{A}_h \longrightarrow \pi_\hbar(f) \in \text{End}(\mathcal{H}_h) $$

(4)

An equivalent way of saying this, is that:

$$ f \mapsto \sum_{\xi \in \Lambda^*} a_\xi \pi_\hbar(\xi) $$

(5)

where $f = \sum_{\xi \in \Lambda^*} a_\xi \cdot \xi$ is the Fourier expansion of $f$.

To summarize, every family of representations $\pi_\hbar : \mathcal{A}_h \to \text{End}(\mathcal{H}_h)$ gives us a complete quantization protocol. But now a serious question rises, namely what representations to pick? Is there a correct choice of representations, both on the mathematical side, but also perhaps on the physical side?. A possible restriction on the choice is to pick an irreducible representation. Yet
some ambiguity still remains, because there are several irreducible classes for specific values of $\hbar$.

We present here a partial solution to this problem in the case where the parameter $\hbar$ is restricted to take only rational values $[\hbar]$. Even more particularly, we take $\hbar$ to be of the form $\hbar = \frac{1}{p}$, where $p$ is an odd prime number. Before any formal discussion, recall that our classical object is the symplectic torus $T$ together with its linear symplectomorphisms $G$. We would like to quantize not only the observables $A$, but also the symmetries $G$. Next we are going to construct an equivariant quantization of $T$.

2.2 Equivariant Weyl quantization of the torus

Fix $\hbar = \frac{1}{p}$. We give here a slightly different presentation of the algebra $A_{\hbar}$. Write $\nu = \frac{p + 1}{2}$. Let $A_{\hbar}$ be the free $\mathbb{C}$-algebra generated by the symbols \{\(s(\xi) \mid \xi \in \Lambda^*\}\} and the relations \(s(\xi + \eta) = e^{2\pi i \hbar \omega(\xi, \eta)} s(\xi)s(\eta)\). The lattice $\Lambda^*$ serves as a standard basis for $A_{\hbar}$:

\[
s : \Lambda^* \longrightarrow A_{\hbar}.
\] (6)

The group $G$ acts on the lattice $\Lambda^*$, therefore it acts on $A_{\hbar}$. It is easy to see that $G$ acts on $A_{\hbar}$ by homomorphisms of the algebra. For an element $B \in G$, we denote by $f \mapsto f^B$ the action of $B$ on an element $f \in A_{\hbar}$.

An equivariant quantization of the torus is a pair:

\[
\pi_{\hbar} : A_{\hbar} \longrightarrow \text{End}(\mathcal{H}_{\hbar});
\] (7)

\[
\rho_{\hbar} : G \longrightarrow \text{PGL}(\mathcal{H}_{\hbar}),
\] (8)

where $\pi_{\hbar}$ is a representation of $A_{\hbar}$ and $\rho_{\hbar}$ is a projective representation of $G$. These two should be compatible in the following manner:

\[
\rho_{\hbar}(B) \pi_{\hbar}(f) \rho_{\hbar}(B)^{-1} = \pi_{\hbar}(f^B),
\] (9)

for every $B \in G$, and $f \in A_{\hbar}$. Equation (9) is called the Egorov identity.

We give here a construction of an equivariant quantization of the torus.
Given a representation $\pi : \mathcal{A}_h \rightarrow \text{End}(\mathcal{H})$, and an element $B \in G$, we construct a new representation $\pi^B : \mathcal{A}_h \rightarrow \text{End}(\mathcal{H})$:

$$\pi^B(f) := \pi(f^B).$$

(10)

This gives an action of $G$ on the set $\text{Irr}(\mathcal{A}_h)$ of equivalence classes of irreducible representations. The set $\text{Irr}(\mathcal{A}_h)$ has a very regular structure, it is a principal homogeneous space over $T$. Moreover every irreducible representation of $\mathcal{A}_h$ is finite dimensional and of dimension $p^n$. The following theorem plays a central role in the construction.

**Theorem 2.1 (Canonical invariant representation [GH2])** Let $h = \frac{1}{p}$.

There exists a unique (up to isomorphism) irreducible representation $(\pi_h, \mathcal{H}_h)$ of $\mathcal{A}_h$ for which its equivalence class is fixed by $G$.

Let $(\pi_h, \mathcal{H}_h)$ be a representative of the fixed irreducible equivalence class. This means that for every $B \in G$:

$$\pi_h^B \simeq \pi_h.$$  

(11)

Hence for every element $B \in G$ there exists an operator $\rho_h(B) : \mathcal{H}_h \rightarrow \mathcal{H}_h$ which realizes the isomorphism (11). The collection $\{\rho_h(B) : B \in G\}$ constitutes a projective representation:

$$\rho_h : G \rightarrow \text{PGL}(\mathcal{H}_h)$$

(12)

Equations (10), (11) also implies the Egorov identity (9).

We want to understand if the projective representation (12) of $G$ can be lifted (linearized) into a honest representation. The next theorem claims the existence of a canonical linearization.

**Theorem 2.2 (Canonical linearization)** Let $h = \frac{1}{p}$, $p > 2$. there exists a unique linearization, which we denote also by $\rho_h$,

$$\rho_h : G \rightarrow \text{GL}(\mathcal{H}_h),$$

(13)
characterized by the property that it is factorized through the quotient group $Sp(2n, \mathbb{F}_p)$:

\[ \begin{array}{ccc}
G & \longrightarrow & Sp(2n, \mathbb{F}_p) \\
\rho_h & \downarrow & \bar{\rho}_h \\
& GL(H_h) & \end{array} \]

**Summary.** Theorem 2.1 claims the existence of a unique invariant representation of $A_\hbar$, for every $\hbar = \frac{1}{p}, p > 2$. This gives a canonical equivariant quantization $(\pi_\hbar, \rho_\hbar, H_\hbar)$. Moreover, by Theorem 2.2, the projective representation $\rho_\hbar$ can be linearized in a canonical way to give an honest representation of $G$ which factories through $Sp(2n, \mathbb{F}_p)$. Altogether this gives a pair:

\[ \pi_\hbar : A_\hbar \longrightarrow \text{End}(H_\hbar) \]  

\[ \rho_\hbar : Sp(2n, \mathbb{F}_p) \longrightarrow \text{PGL}(H_\hbar) \]

satisfying the following compatibility condition (Egorov identity):

\[ \rho_\hbar(B)\pi_\hbar(f)\rho_\hbar(B)^{-1} = \pi_\hbar(f^B). \]  

For every $B \in Sp(2n, \mathbb{F}_p), f \in A_\hbar$. Here the notation $\pi_\hbar(f^B)$ means, to take any pre-image $\bar{B} \in G$ of $B \in Sp(2n, \mathbb{F}_p)$ and act by it on $f$, $\pi_\hbar(f^B)$ does not depend on the choice of $\bar{B}$. For what follows, we denote $\bar{\rho}_\hbar$ by $\rho_\hbar$, and consider $Sp(2n, \mathbb{F}_p)$ to be the default domain.

### 2.3 Quantum mechanical system

Let $(\pi_\hbar, \rho_\hbar, H_\hbar)$ be the canonical equivariant quantization. Let $A$ be our fixed ergodic element, considered as an element of $Sp(2n, \mathbb{F}_p)$ . The element $A$ generates a quantum dynamical system. For every (pure) quantum state $v \in S(H_\hbar) = \{ v \in H_\hbar : \|v\| = 1 \}$:

\[ v \mapsto v^A = \rho_\hbar(A)v \]
3 Hecke Quantum Unique Ergodicity

The main silent question of this paper is whether the system (17) is quantum ergodic. Before discussing this question, one is obliged to define a notion of quantum ergodicity. As a first approximation just copy the classical definition, but replace each classical notion by its quantum counterpart. Namely, for every $f \in \mathcal{A}_\hbar$ and almost every quantum state $v \in S(\mathcal{H}_\hbar)$ the following holds:

$$\lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} <v|\pi_\hbar(f^{A_k})v> = \int_T f \omega. \quad (18)$$

Unfortunately (18) is literally not true. The limit is never exactly equal the integral for a fixed $\hbar$. Next we give a true statement which is a slight modification of (18), and is called Hecke Quantum Unique Ergodicity. First rewrite (18) in an equivalent form. Using the Egorov identity (9) we have:

$$<v|\pi_\hbar(f^{A_k})v> = <v|\rho_\hbar(A_k)\pi_\hbar(f)\rho_\hbar(A_k)^{-1}v> \quad (19)$$

The elements $A^k$ runs inside the finite group $Sp(2n, \mathbb{F}_p)$. Denote by $\langle A \rangle \subseteq Sp(2n, \mathbb{F}_p)$ the cyclic subgroup generated by $A$. It is easy to see, using (19), that:

$$\lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} <v|\pi_\hbar(f^{A_k})v> = \frac{1}{|\langle A \rangle|} \sum_{B \in \langle A \rangle} <v|\rho_\hbar(B)\pi_\hbar(f)\rho_\hbar(B)^{-1}v> \quad (20)$$

Altogether (18) can be written in the form:

$$\mathbf{A}v_{(A)} (<v|\pi_\hbar(f)v>) = \int_T f \omega \quad (21)$$

where $\mathbf{A}v_{(A)}$ denote the average with respect to the group $\langle A \rangle$.

3.1 Hecke theory

Denote by $C_A = \{B \in Sp(2n, \mathbb{F}_p) : BA = AB\}$ the centralizer of $A$ in $Sp(2n, \mathbb{F}_p)$. The finite group $C_A$ is an algebraic group. More particularly,
as an algebraic group, it is a torus. We call $C_A$ the Hecke torus. One has, $\langle A \rangle \subseteq C_A \subseteq Sp(2n, \mathbb{F}_p)$. Now, in (21), average with respect to the group $C_A$ instead of the group $\langle A \rangle$. We assume the characteristic polynomial $P_A$ of $A$ is irreducible over $\mathbb{Q}$. Then the precise statement of the Rudnick-Kurlberg conjecture (which naturally generalize [GH3]) is given in the following theorem:

Theorem 3.1 (Hecke Quantum Unique Ergodicity) Let $\hbar = \frac{1}{p}$, $p > 2$. For every $f \in A_\hbar$ and $v \in S(H_\hbar)$, the following holds:

$$\left\| \text{Av}_{c_A}(\langle v | \pi_\hbar(f)v >) - \int_T f \omega \right\| \leq \frac{C_f}{p^n/2},$$

(22)

where $C_f$ is an explicit constant depends only on $f$.

The rest of this paper is devoted to proving Theorem 3.1.

4 Proof of the Hecke Quantum Unique Ergodicity Conjecture

The proof is given in two stages. The first stage is a preparation stage and consists of mainly linear algebra considerations. We massage statement (22) in several steps into an equivalent statement which will be better suited to our needs. In the second stage we introduce the main part of the proof. Here we invoke tools from algebraic geometry in the framework of $\ell$-adic sheaves and $\ell$-adic cohomology (cf. [M]).

4.1 Preparation stage

Step 1. It is enough to prove (3.1) for $f$ a character, $0 \neq f = \xi \in \Lambda^*$. Because $\int_T \xi \omega = 0$, statement (22) becomes:

$$\left\| \text{Av}_{c_A}(\langle v | \pi_\hbar(\xi)v >) \right\| \leq \frac{C_\xi}{p^n/2}$$

(23)

The statement for general $f \in A_\hbar$ follows directly from the triangle inequality and the rapid decrease of the Fourier coefficients of $f$. 
Step 2. It is enough to prove (23) in case $v \in S(\mathcal{H}_h)$ is a Hecke eigenvector. To be more precise: the Hecke torus $\mathcal{C}_A$ acts semisimply on $\mathcal{H}_h$ via the representation $\rho_h$, thus $\mathcal{H}_h$ decomposes into a direct sum of character spaces:

$$\mathcal{H}_h = \bigoplus_{\chi : \mathcal{C}_A \rightarrow \mathbb{C}^*} \mathcal{H}_\chi. \quad (24)$$

The sum in (24) is over multiplicative characters of the torus $\mathcal{C}_A$. For $v \in \mathcal{H}_\chi, B \in \mathcal{C}_A$,

$$\rho_h(B)v = \chi(B)v \quad (25)$$

Taking $v \in \mathcal{H}_\chi$, statement (23) becomes:

$$\| <v | \pi_h(\xi)v > \| \leq \frac{\xi}{p^{n/2}} \quad (26)$$

Here $C_\xi = 2^n$.

The averaged operator:

$$\frac{1}{|\mathcal{C}_A|} \sum_{B \in \mathcal{C}_A} \rho_h(B)\pi_h(\xi)\rho_h(B)^{-1} \quad (27)$$

is essentially diagonal in the Hecke base. Knowing this then statement (23) follows from (26) by invoking the triangle inequality.

Step 3. Let $P_\chi : \mathcal{H}_h \rightarrow \mathcal{H}_h$ be the orthogonal projector on the eigenspace $\mathcal{H}_\chi$.

Lemma 4.1 For $\chi \neq 1$ dim $\mathcal{H}_\chi = 1$.

Using lemma 4.1 we can rewrite (26) in the form:

$$\| \text{Tr}(\pi_h(\xi)P_\chi) \| \leq \frac{\xi}{p^{n/2}}. \quad (28)$$

The projector $P_\chi$ can be defined in terms of the representation $\rho_h$:

$$P_\chi = \frac{1}{|\mathcal{C}_A|} \sum_{B \in \mathcal{C}_A} \chi(B)\rho_h(B). \quad (29)$$
Now rewrite (26):

\[ \frac{1}{|C_A|} \left| \sum_{B \in C_A} \text{Tr}(\pi_h(\xi)\rho_h(B))\chi(B) \right| \leq \frac{C_\xi}{p^{n/2}} \]  \hspace{1cm} (30)

Or noting that $|C_A| = p^n$, multiplying both sides of (30) by $|C_A|$ we get:

**Theorem 4.2 (Hecke Quantum Unique Ergodicity (Restated))** Let $h = \frac{1}{p^k}$, $p > 2$ and let $\chi$ be a character of $C_A$. For every $0 \neq \xi \in \Lambda^*$ the following holds:

\[ \left| \sum_{B \in C_A} \text{Tr}(\pi_h(\xi)\rho_h(B))\chi(B) \right| \leq Cp^{n/2}, \]  \hspace{1cm} (31)

where $C = 2^n$.

We prove the Hecke ergodicity theorem in the form of Theorem 4.2.

### 4.2 Proof of Theorem 4.2

We prove Theorem 4.2 using sheaf theoretic techniques. Before diving into geometric considerations, we investigate further the ingredients appearing in Theorem 4.2. For what follows we fix a character $\chi : C_A \rightarrow \mathbb{C}^\times$.

Denote by $F(\xi, B) = \text{Tr}(\pi_h(\xi)\rho_h(B))$, which is a function of two variables, $F : \Lambda^* \times Sp(2n, \mathbb{F}_p) \rightarrow \mathbb{C}$.

Denote by $\Lambda^*_p := \Lambda^*/p\Lambda^*$. The quotient lattice $\Lambda^* \simeq \mathbb{F}_p^{2n}$. Set $Y_0 = \Lambda^* \times Sp(2n, \mathbb{F}_p)$, and $Y = \Lambda^*_p \times Sp(2n, \mathbb{F}_p)$. One has the quotient map:

\[ Y_0 \rightarrow Y. \]  \hspace{1cm} (32)

**Lemma 4.3** The function $F : Y_0 \rightarrow \mathbb{C}$ factories through the quotient $Y$. 

\[ F \]

\[ \begin{array}{ccc}
Y_0 & \xrightarrow{F} & Y \\
\downarrow & & \downarrow \\
\mathbb{C} & \xrightarrow{F} & 
\end{array} \]
Denote \( \overline{F} \) also by \( F \), and from now on \( Y \) will be considered as the default domain.

The function \( F : Y \rightarrow \mathbb{C} \) is invariant under a certain group action of \( Sp(2n, \mathbb{F}_p) \). To be more precise, let \( S \in Sp(2n, \mathbb{F}_p) \),

\[
\text{Tr}(\pi_h(\xi)\rho_h(B)) = \text{Tr}(\rho_h(S)\pi_h(\xi)\rho_h(S)^{-1}\rho_h(S)\rho_h(B)\rho_h(S)^{-1})
\]  

(33)

Applying the Egorov identity (16) and using the fact that \( \rho_h \) is a representation we get:

\[
\text{Tr}(\rho_h(S)\pi_h(\xi)\rho_h(S)^{-1}\rho_h(S)\rho_h(B)\rho_h(S)^{-1}) = \text{Tr}(\pi_h(S\xi)\rho_h(SBS^{-1}))
\]  

(34)

Altogether we have:

\[
F(\xi, B) = F(S\xi, SBS^{-1})
\]  

(35)

Putting (35) in a more diagrammatic form: there is an action of \( Sp(2n, \mathbb{F}_p) \) on \( Y \) given by the following formula:

\[
Sp(2n, \mathbb{F}_p) \times Y \xrightarrow{\alpha} Y
\]

\[
(S, (\xi, B)) \xrightarrow{\alpha} (S\xi, SBS^{-1})
\]

(36)

Consider the following diagram:

\[
Y \xleftarrow{pr} Sp(2n, \mathbb{F}_p) \times Y \xrightarrow{\alpha} Y
\]

(37)

Where \( pr \) is the projection on the \( Y \) variable. Formula (35) can be stated equivalently as:

\[
\alpha^*(F) = pr^*(F),
\]

(38)

where \( \alpha^*(F) \) and \( pr^*(F) \) are the pullbacks of the function \( F \) on \( Y \) via the maps \( \alpha \) and \( pr \) respectively.
4.3 Geometrization (Sheafification)

What we are going to do next is to replace statement (4.2) by a geometric statement, by which it will be implied. Going into the geometric setting we replace the set \( Y \) by algebraic variety and the functions \( F, \chi \) by sheaf theoretic objects, also of a geometric flavor.

**Step 1.** Notice that the set \( Y \) is not an arbitrary finite set, but is the set of rational points of an algebraic variety \( \overline{Y} \) defined over \( \mathbb{F}_p \). To be more precise: \( \overline{Y} \cong \mathbb{F}_p^{2n} \times Sp(2n, \mathbb{F}_p) \). The variety \( \overline{Y} \) is equipped with an endomorphism:

\[
\text{Fr} : \overline{Y} \longrightarrow \overline{Y}
\]  

(39)

The endomorphism \( \text{Fr} \) is called Frobenius. The set \( Y \) is identified with the set of fixed points of Frobenius:

\[
Y = \overline{Y}^{\text{Fr}} = \{ y \in \overline{Y} : \text{Fr}(y) = y \}.
\]  

(40)

Note that the finite group \( Sp(2n, \mathbb{F}_p) \) is the set of rational points of the algebraic group \( Sp(2n, \mathbb{F}_p) \). The finite quotient lattice \( \Lambda^*_p \) is the set of rational points of the affine space \( \mathbb{F}_p^{2n} \).

Having all finite sets replaced by corresponding algebraic varieties, we want to replace functions by sheaf theoretic objects. This we do next.

**step 2.** The following theorem, tells us about the appropriate sheaf theoretic object standing in place of the function \( F : Y \longrightarrow \mathbb{C} \). Denote by \( D^{b}_{c,w}(\overline{Y}) \) the bounded derived category of constructible, \( \ell \)-adic Weil sheaves (cf. [M]).

**Theorem 4.4 (Deligne [D1])** There exists a unique, up to an isomorphism, object \( F \in D^{b}_{c,w}(\overline{Y}) \), which is of weight \( w(F) \leq 0 \) and is associated, by sheaf-to-function correspondence, to the function \( F : Y \longrightarrow \mathbb{C} \):

\[
f^F = F
\]  

(41)

We give here an intuitive explanation of Theorem 4.3 part by part, as it was stated. By means of an object \( F \in D^{b}_{c,w}(\overline{Y}) \) think of \( F \) as a vector
bundle over $\overline{Y}$:

$$
\begin{array}{c}
F \\
\downarrow \\
\overline{Y}
\end{array}
\quad \text{(42)}
$$

The letter $w$ in the notation $\mathcal{D}_{c,w}^b$, means that $\mathcal{F}$ is a Weil sheaf, namely it is a sheaf equipped with a lifting of the Frobenius:

$$
\begin{array}{c}
\mathcal{F} \\
\downarrow \\
\overline{Y}
\end{array} \xrightarrow{\text{Fr}} \begin{array}{c}
\mathcal{F} \\
\downarrow \\
\overline{Y}
\end{array}
\quad \text{(43)}
$$

To be even more precise, think of $\mathcal{F}$ not as a single vector bundle, but as a complex of vector bundles over $\overline{Y}$, $\mathcal{F} = \mathcal{F}^*$:

$$
\begin{array}{c}
d \rightarrow \mathcal{F}^{-1} \\
d \rightarrow \mathcal{F}^0 \\
d \rightarrow \mathcal{F}^1 \\
d \rightarrow \ldots
\end{array}
\quad \text{(44)}
$$

$\mathcal{F}^*$ is equipped with a lifting of Frobenius:

$$
\begin{array}{c}
d \rightarrow \mathcal{F}^{-1} \\
d \rightarrow \mathcal{F}^0 \\
d \rightarrow \mathcal{F}^1 \\
d \rightarrow \ldots
\end{array} \xrightarrow{\text{Fr}} \begin{array}{c}
d \rightarrow \mathcal{F}^{-1} \\
d \rightarrow \mathcal{F}^0 \\
d \rightarrow \mathcal{F}^1 \\
d \rightarrow \ldots
\end{array}
\quad \text{(45)}
$$

Here the Frobenius commutes with the differentials.

Next we explain what is the meaning of the statement $w(\mathcal{F}) \leq 0$. Let $y \in \overline{Y}^{\text{Fr}} = Y$ be a fixed point of Frobenius. Denote by $\mathcal{F}_y$ the fiber of $\mathcal{F}$ at the point $y$. Thinking of $\mathcal{F}$ as a complex of vector bundles, it is clear what one means by taking the fiber at a point. The fiber $\mathcal{F}_y$ is just a complex of vector spaces. Because the point $y$ is fixed by the Frobenius, it induces an
endomorphism of $\mathcal{F}_y$:

$$
\begin{array}{cccccc}
\vdots & d & \to & \mathcal{F}^{-1}_y & d & \to & \mathcal{F}^0_y & d & \to & \mathcal{F}^1_y & \vdots \\
\downarrow & & & \downarrow & & & \downarrow & & & \downarrow \\
\vdots & d & \to & \mathcal{F}^{-1}_y & d & \to & \mathcal{F}^0_y & d & \to & \mathcal{F}^1_y & \vdots
\end{array}
$$

The Frobenius acting as in (46) commutes with the differentials, so it induces an action on cohomologies. For every $i \in \mathbb{Z}$ we have an endomorphism:

$$
\text{Fr} : H^i(\mathcal{F}_y) \longrightarrow H^i(\mathcal{F}_y) \quad (47)
$$

Saying an object $\mathcal{F}$ has weight $w(\mathcal{F}) \leq \omega$ means that for every point $y \in \overline{Y}^{\text{Fr}}$, and for every $i \in \mathbb{Z}$ the absolute values of the eigenvalues of Frobenius acting on the $i$'th cohomology (47) satisfy:

$$
\left| \text{e.v.}(\text{Fr}|_{H^i(\mathcal{F}_y)}) \right| \leq \sqrt{p}^{\omega+i} \quad (48)
$$

In our case $\omega = 0$ and so:

$$
\left| \text{e.v.}(\text{Fr}|_{H^i(\mathcal{F}_y)}) \right| \leq \sqrt{p}^i \quad (49)
$$

The last part of Theorem 4.4 concerns a function $f^F : Y \longrightarrow \mathbb{C}$ associated to the sheaf $\mathcal{F}$. To define $f^F$, we have to describe its value at every point $y \in Y$. Let $y \in Y = \overline{Y}^{\text{Fr}}$. Frobenius acts on the cohomologies of the fiber $\mathcal{F}_y$ (cf. (47)). Now put:

$$
f^F(y) = \sum_{i \in \mathbb{Z}} (-1)^i \text{Tr}(\text{Fr}|_{H^i(\mathcal{F}_y)}).
$$

In words: $f^F(y)$ is the alternating sum of traces of Frobenius acting on the cohomologies of the fiber $\mathcal{F}_y$. We call this sum the Euler characteristic of Frobenius and denote it by:

$$
f^F(y) = \chi_{\text{Fr}}(\mathcal{F}_y) \quad (50)
$$

Theorem 4.4 asserts that $f^F$ is the function $F$ defined earlier. Associating the function $f^F$ on the set $\overline{Y}^{\text{Fr}}$ to the sheaf $\mathcal{F}$ on $\overline{Y}$ is a particular case of a general procedure called Grothendieck’s Sheaf-to-Function Correspondence [G]. Because we are going to use this procedure later, next we spend some
time explaining it in greater generality (see also [Ga]).

**Grothendieck’s Sheaf-to-Function Correspondence.**

Let $X$ be an algebraic variety defined over $\mathbb{F}_p$. This means that there exists a Frobenius endomorphism:

$$\text{Fr} : X \rightarrow X$$

(52)

The set $X = X^{\text{Fr}}$ is called the set of rational points of $X$. Let $\mathcal{L} \in \mathcal{D}_{c,w}^b(X)$ be a Weil sheaf. One can associate to $\mathcal{L}$ a function $f^\mathcal{L}$ on the set $X$ by the following formula:

$$f^\mathcal{L}(x) = \sum_{i \in \mathbb{Z}} (-1)^i \text{Tr}(\text{Fr}|_{H^i(\mathcal{L})})$$

(53)

This procedure is called *Sheaf-to-Function correspondence*. Next we describe some important functorial properties of the procedure:

Let $X_1$, $X_2$ be algebraic varieties defined over $\mathbb{F}_p$. Let $X_1 = X_1^{\text{Fr}}$, and $X_2 = X_2^{\text{Fr}}$ be the corresponding sets of rational points. Let $\pi : X_1 \rightarrow X_2$ be a morphism of algebraic varieties. Denote also by $\pi : X_1 \rightarrow X_2$ the induced map on the level of sets.

**First statement.** Say we have a sheaf $\mathcal{L} \in \mathcal{D}_{c,w}^b(X_2)$. The following holds:

$$f'^\pi(\mathcal{L}) = \pi^*(f^\mathcal{L})$$

(54)

where on the function level $\pi^*$ is just pull back of functions. On the sheaf theoretic level $\pi^*$ is the pull-back functor of sheaves (think of pulling back a vector bundle). Equation (54) says that *Sheaf-to-Function Correspondence* commutes with the operation of pull back.

**Second statement.** Say we have a sheaf $\mathcal{L} \in \mathcal{D}_{c,w}^b(X_1)$. The following holds:

$$f'^\pi(\mathcal{L}) = \pi_!(f^\mathcal{L})$$

(55)
where on the function level $\pi_1$ means to sum up the values of the function along the fibers of the map $\pi$. On the sheaf theoretic level $\pi_1$ is compact integration of sheaves (here we have no analogue under the vector bundle interpretation). Equation (55) says that Sheaf-to-Function Correspondence commutes with integration.

**Third statement.** Say we have two sheaves $\mathcal{L}_1, \mathcal{L}_2 \in \mathcal{D}_{c,w}(\overline{X}_1)$. The following holds:

$$f^{\mathcal{L}_1 \otimes \mathcal{L}_2} = f^{\mathcal{L}_1} \cdot f^{\mathcal{L}_2}$$  \hspace{1cm} (56)

In words: Sheaf-to-Function Correspondence takes tensor product of sheaves to multiplication of the corresponding functions.

### 4.4 Geometric statement

Fix an element $0 \neq \xi \in \Lambda^*$. Denote by $i_\xi$ the inclusion map:

$$i_\xi : \{\xi\} \times \mathcal{C} \longrightarrow Y$$  \hspace{1cm} (57)

and the canonical projection:

$$p_\xi : \{\xi\} \times \mathcal{C} \longrightarrow pt$$  \hspace{1cm} (58)

Going back to Theorem 4.2 and putting its content in a diagrammatic form, we obtain the following inequality:

$$\|p_\xi(i^*_\xi(F) \cdot \chi)\| \leq 2^n p^{n/2}$$  \hspace{1cm} (59)

In words: taking the function $F : Y \longrightarrow \mathbb{C}$.

- Restrict $F$ to $\{\xi\} \times \mathcal{C}$ and get $i^*_\xi(F)$
- Multiply $i^*_\xi(F)$ by the character $\chi$ to get $i^*_\xi(F) \cdot \chi$
- Integrate $i^*_\xi(F) \chi$ to the point, that is sum all the values, to get a scalar $a_\chi = p_\xi(i^*_\xi(F) \cdot \chi)$
Theorem 4.2 claims that the scalar $a_\chi$ is of absolute value less than $2^n p^{n/2}$.

Now do the same thing on the geometric level. We have the closed embedding:

$$i_\xi : \{\xi\} \times \mathcal{C}_A \longrightarrow \mathcal{Y}$$

And the canonical projection:

$$p_\xi : \{\xi\} \times \mathcal{C}_A \longrightarrow pt$$

Take the sheaf $\mathcal{F}$ on $\mathcal{Y}$,

- Pull-back $\mathcal{F}$ to the closed subvariety $\{\xi\} \times \mathcal{C}_A$, to get the sheaf $i_\xi^*(\mathcal{F})$.
- Take a tensor product of $i_\xi^*(\mathcal{F})$ with the Kummer sheaf $\mathcal{L}_\chi$, to get $i_\xi^*(\mathcal{F}) \otimes \mathcal{L}_\chi$.
- Integrate $i_\xi^*(\mathcal{F}) \otimes \mathcal{L}_\chi$ to the point, to get the sheaf $p_\xi! (i_\xi^*(\mathcal{F}) \otimes \mathcal{L}_\chi)$ on the point.

The Kummer sheaf $\mathcal{L}_\chi$ is associated via Sheaf-to-Function Correspondence to the character $\chi$.

Recall that Sheaf-to-Function Correspondence commutes both with pullback (54), with integration (55), and takes tensor product of sheaves to multiplication of functions (56). This means that it commutes the operations done on the level of sheaves with those done on the level of functions. The following diagram describe pictorially, what has been said so far:

$$
\begin{array}{ccc}
\mathcal{F} & \xrightarrow{\chi_{p\xi}} & \mathcal{F} \\
\uparrow i_\xi & & \uparrow i_\xi \\
i_\xi^*(\mathcal{F}) \otimes \mathcal{L}_\chi & \xrightarrow{\chi_{p\xi}} & i_\xi^*(\mathcal{F}) \cdot \chi \\
\downarrow p_\xi & & \downarrow p_\xi \\
p_\xi! (i_\xi^*(\mathcal{F}) \otimes \mathcal{L}_\chi) & \xrightarrow{\chi_{p\xi}} & p_\xi! (i_\xi^*(\mathcal{F}) \cdot \chi)
\end{array}
$$
Denote by $\mathcal{G} = p_{\xi!}(i^*(\mathcal{F}) \otimes \mathcal{L}_\chi)$. It is an object in $\mathcal{D}_{\xi,w}^b(pt)$. This means it is merely a complex of vector spaces, $\mathcal{G} = \mathcal{G}^\bullet$, together with an action of Frobenius:

$$
\begin{array}{cccccccc}
\cdots & d & \rightarrow & \mathcal{G}^{-1} & d & \rightarrow & \mathcal{G}^0 & d & \rightarrow & \mathcal{G}^1 & d & \rightarrow & \cdots \\
\downarrow \text{Fr} & & \downarrow \text{Fr} & & \downarrow \text{Fr} & & \downarrow \text{Fr} & & \downarrow \text{Fr} & & \downarrow \text{Fr} & & \downarrow \text{Fr} \\
\cdots & d & \rightarrow & \mathcal{G}^{-1} & d & \rightarrow & \mathcal{G}^0 & d & \rightarrow & \mathcal{G}^1 & d & \rightarrow & \cdots
\end{array}
$$

(63)

The complex $\mathcal{G}^\bullet$ is associated by Sheaf-to-Function correspondence to the scalar $a_\chi$:

$$
a_\chi = \sum_{i \in \mathbb{Z}} (-1)^i \text{Tr}(\text{Fr}|_{H^i(\mathcal{G})})
$$

(64)

At last we can give the geometric statement about $\mathcal{G}$, which will imply Theorem 4.2.

**Lemma 4.5 (Geometric Lemma)** Let $\mathcal{G} = p_{\xi!}(i^*(\mathcal{F}) \otimes \mathcal{L}_\chi)$. There exist a natural number $0 \leq m(\chi) \leq n$ such that:

(a) All cohomologies $H^i(\mathcal{G})$ vanish except for $i = n + m(\chi)$. Moreover $H^i(\mathcal{G})$ is a $2^n$-dimensional vector space.

(b) The sheaf $\mathcal{G}$ has weight $w(\mathcal{G}) \leq -2m(\chi)$.

Theorem 4.2 now follows easily. Having $w(\mathcal{G}) \leq -2m(\chi)$ means that:

$$
\left| \text{e.v}(\text{Fr}|_{H^i(\mathcal{G})}) \right| \leq \sqrt{p}^{-2m(\chi)}.
$$

(65)

By Lemma 4.5 only the cohomology $H^{n+m(\chi)}(\mathcal{G})$ does not vanish and it is $2^n$-dimensional. The eigenvalues of Frobenius acting on $H^{n+m(\chi)}(\mathcal{G})$ are of absolute value $\leq \sqrt{p}^{-m(\chi)}$, (65). It now follows, using formula (63), that:

$$
|a_\chi| \leq 2^n p^{n/2}
$$

(66)

for all $\chi$.

The rest of the paper is devoted to the proof of Lemma 4.5.
4.5 Proof of the Geometric Lemma

The proof will be given in several steps, reducing the Geometric Lemma to the case n=1.

**Step 1.** The sheaf $\mathcal{F}$ is $Sp(2n, \mathbb{F}_p)$-equivariant sheaf.

Recall that the function $F : Y \rightarrow \mathbb{C}$ is invariant under a group action of $Sp(2n, \mathbb{F}_p)$ on Y. One can define the analogue group action in the geometric setting. Here the algebraic group $Sp(2n, \mathbb{F}_p)$ acts on the variety $\overline{Y}$. The formulas are the same as those given in (36):

$$Sp(2n, \mathbb{F}_p) \times \overline{Y} \xrightarrow{\alpha} \overline{Y}$$

$$(S, (\xi, B)) \rightarrow (S\xi, SBS^{-1})$$

(67)

It turns out that the invariance property of the function $F$ is a manifestation of a finer geometric phenomena. Namely the sheaf $\mathcal{F}$ is equivariant with respect to the action $\alpha$. More precisely, we have the diagram:

$$\overline{Y} \xleftarrow{pr} Sp(2n, \mathbb{F}_p) \times \overline{Y} \xrightarrow{\alpha} \overline{Y}$$

(68)

There exists an isomorphism $\theta$:

$$\theta : \alpha^*(\mathcal{F}) \simeq pr^*(\mathcal{F})$$

(69)

**Step 2.** All tori in $Sp(2n, \mathbb{F}_p)$ are conjugated over $\mathbb{F}_q$, where $q = 2n$. In particular there exists an element $S \in Sp(2n, \mathbb{F}_q)$ conjugating the Hecke torus $\overline{C}_A \subset Sp(2n, \mathbb{F}_p)$ with the standard torus $\overline{T} \subset Sp(2n, \mathbb{F}_p)$,

$$S\overline{C}_A S^{-1} = \overline{T}$$

(70)

The standard torus is:

$$\overline{T} = \left\{ \begin{pmatrix} a_1 & \cdots & \cdots & \cdots & \cdots \\ a_1^{-1} & & & & \\ \vdots & & & & \\ \vdots & & & & \\ a_n & \cdots & \cdots & \cdots & \cdots \\ a_n^{-1} & & & & \end{pmatrix} : a_i \in \mathbb{F}_p^\times \right\}$$

(71)
The situation is displayed in the following diagram:

\[
\begin{array}{c}
\overline{N}_p \times Sp(2n, \mathbb{F}_p) \\
\uparrow_{i_\xi} \quad \uparrow_{i_\eta} \quad \uparrow_{p_\xi} \quad \downarrow_{p_\eta} \quad \downarrow_{pt}
\end{array}
\begin{array}{c}
{\{\xi\}} \times \mathcal{C}_A \\
\alpha_S \quad \alpha_S \\
{\{\eta\}} \times \mathcal{T}
\end{array}
\]

(72)

Where \( \eta = S \cdot \xi \), and \( \alpha_s \) is the restriction of the action \( \alpha \), (67) to the element \( S \).

**Step 3.** Denote by \( G' \) the object in \( \mathcal{D}^b_{c,w}(pt) \) defined by:

\[
G' := p_{\eta!}(i^*(\mathcal{F}) \otimes \alpha_{S_1}(\mathcal{L}_\chi))
\]

(73)

Then \( G \) and \( G' \) are isomorphic as "quasi Weil sheaves". Namely there exist an isomorphism:

\[
\mathcal{G} \simeq \mathcal{G}'
\]

(74)

which commutes with the action of \( \text{Fr}_q := \text{Fr}^{2n} \).

By base change:

\[
p_{\xi!}(i_\xi^*(\mathcal{F}) \otimes \mathcal{L}_\chi) \simeq p_{\eta!}(\alpha_{S_1}(i_\xi^*(\mathcal{F}) \otimes \mathcal{L}_\chi))
\]

(75)

By usual property:

\[
p_{\eta!}(\alpha_{S_1}(i_\xi^*(\mathcal{F}) \otimes \mathcal{L}_\chi)) \simeq p_{\eta!}(\alpha_{S_1}(i_\xi^*(\mathcal{F})) \otimes \alpha_{S_1}(\mathcal{L}_\chi))
\]

(76)

By base change:

\[
\alpha_{S_1}(i_\xi^*(\mathcal{F})) \simeq i_\eta^*(\alpha_{S_1}(\mathcal{F}))
\]

(77)

By the equivariance property of the sheaf \( \mathcal{F} \), (30), we have an isomorphism:

\[
\theta_s : \alpha_{S_1}(\mathcal{F}) \simeq \mathcal{F}
\]

(78)
Combining (75), (76), (77), (78) we get:

\[ p_\xi(i_\xi^*(\mathcal{F}) \otimes \mathcal{L}_\chi) \simeq p_\eta(i_\eta^*(\mathcal{F}) \otimes \alpha_{s_\xi}(\mathcal{L}_\chi)) \]  

(79)
as claimed.

**Step 4.** It is enough to prove the Geometric Lemma for the sheaf $G'$. Knowing the weight $w(G')$, then using the isomorphism (74) we know the weight $w(G)$ with respect to the action of Fr$_q$ and hence also for the action of Fr.

**Step 5. (Factorization).** The sheaf $G'$ factors:

\[ G' \simeq \bigotimes_{j=1}^n G'_j \]  

(80)

where the sheaves $G'_j$ are the sheaves on pt corresponds to the case $n=1$ over $\mathbb{F}_q$. This fact follows from factorization properties of the sheaves $\mathcal{F}$ and $\mathcal{L}_\chi$, which we are able to show over the torus $\overline{T}$ where we have explicit formulas for the sheaves involved.

**Step 6.** The geometric Lemma holds for any of the sheaves $G'_j$.

First we compute the sheaf $i_\eta^*(\mathcal{F})$.

Denote by $\psi : \mathbb{F}_q \to \mathbb{C}^\times$ the additive character $\psi(t) = e^{2\pi i \text{tr}(t)/p}$. Write $\eta$ in the coordinates of the standard basis of $\mathbb{F}_q^2$, say $\eta = (\lambda, \mu)$. Let $\mathcal{F}_\psi$ be the Artin-Schreier sheaf corresponding to the character $\psi$. Consider the morphism:

\[ f : \overline{T} - \{\text{Id}\} \longrightarrow \overline{\mathbb{F}_p} \]  

(81)
given by the formula:

\[ f \left( \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \right) = \frac{1}{2}(\lambda \cdot \mu) \cdot \frac{1 + a}{1 - a} \]  

(82)

We have:

\[ i_\eta^*(\mathcal{F}) = f^*(\mathcal{F}_\psi) \]  

(83)

Note that the sheaf $f^*(\mathcal{F}_\psi)$ although apriori defined over $\overline{T} - \{\text{Id}\}$, has all its extensions to $\overline{T}$ coincide. So in (83) we consider some extension to $\overline{T}$,
and there is no ambiguity here.

About the sheaf $\alpha_{s_{1}}(\mathcal{L}_{\chi})$, it is isomorphic to an explicit Kummer sheaf on $\mathcal{T}$ correspond to a character $\chi' : \mathcal{T}(\mathbb{F}_{q}) \to \mathbb{C}^{\times}$. Altogether we get that $\mathcal{G}'$ is a kind of a Gauss-sum sheaf. More precisely this is a sheaf given by explicit formula. By direct computation we prove the Geometric Lemma [1.3] for this sheaf. We found that $m(\chi) = 1$ if $\chi'$ is trivial and $m(\chi) = 0$ otherwise.

Now using the property:

$$w(\mathcal{G}_{1} \otimes \mathcal{G}_{2}) = w(\mathcal{G}_{1}) + w(\mathcal{G}_{2})$$

and Kunneth formula:

$$H^{i}(\mathcal{G}_{1} \otimes \mathcal{G}_{2}) = \sum_{k+l=i} H^{k}(\mathcal{G}_{1}) \otimes H^{l}(\mathcal{G}_{2})$$

The proof is completed. ■

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