Isoperimetry for asymptotically flat 3-manifolds with positive ADM mass

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Abstract
Let \((M^3, g)\) be an asymptotically flat 3-manifold with positive ADM mass. In this paper, we show that each leaf of the canonical foliation consisting of stable constant mean curvature spheres is the unique isoperimetric surface for the volume it encloses. Our proof is based on “fill-in” argument and sharp isoperimetric inequality on asymptotically flat 3-manifold with non-negative scalar curvature.

Keywords Isoperimetric surface · Uniqueness · Canonical foliation · Volume comparison

1 Introduction
A three manifold \((M, g)\) is said to be asymptotically flat if there are a compact subset \(K \subseteq M\) and a chart
\[
M \setminus K \cong \mathbb{R}^3 \setminus \overline{B_{\frac{1}{2}}(0)}
\]
so that the components of the metric tensor have the form
\[
g_{ij} = \delta_{ij} + \sigma_{ij},
\]
where
\[
|x|^\alpha |\partial^\alpha \sigma_{ij}(x)| = O(|x|^{-\tau}), \quad \text{as} \quad |x| \to \infty
\]
for some \(\tau > 1/2\) and all multi-indices \(\alpha\) with \(|\alpha| = 0, 1, 2\). We also require that the scalar curvature of \((M, g)\) is integrable. The ADM-mass (after Arnowitt et al. [1]) of
such an asymptotically flat manifold \((M, g)\) is given by

\[
m_{\text{ADM}} = \lim_{\rho \to \infty} \frac{1}{16\pi \rho} \int_{|x| = \rho} \sum_{i,j=1}^{3} (\partial_i g_{ij} - \partial_j g_{ii}) x^j
\]

where integration is with respect to the Euclidean metric.

In their seminal paper [16], Huisken and Yau proved that if \((M, g)\) is \(C^4\)-asymptotic to Schwarzschild of mass \(m > 0\), then out of some compact set, \(M\) can be foliated by a family of strictly volume preserving stable constant mean curvature spheres \(\Sigma H \subset H_0\). Moreover, the leaves of this foliation are the unique volume preserving stable CMC spheres of their mean curvature within a large class of surfaces. Their uniqueness result was later strengthened by Qing and Tian [23]. Various extensions of these results that allow for weaker asymptotic conditions have been proven in [13, 18, 19]. The following optimal existence and uniqueness results for general asymptotically flat 3-manifolds was established by Nerz in [22].

**Theorem 1.1** Let \((M, g)\) be a complete Riemannian 3-manifold that is asymptotically flat at rate \(\tau > 1/2\) and which has positive mass. Suppose the scalar curvature of \((M, g)\) is nonnegative or satisfies \(R(g) = O(|x|^{-\frac{5}{2}-\tau})\). Then for some compact \(L\), \(M^3 \setminus L\) can be foliated by stable CMC spheres \(\{\Sigma_\sigma\}_{\sigma > \sigma_*}\) with \(\frac{2}{\sigma} \ll 1\) being the mean curvature of \(\Sigma_\sigma\). Moreover, any large stable CMC sphere with mean curvature \(\frac{2}{\sigma}\) and which is geometrically close to \(S_\sigma\) must coincides with \(\Sigma_\sigma\).

**Remark 1.1** In [22], Nerz showed that the decay assumptions are optimal and cannot be weakened to guarantee the existence of canonical foliation.

It’s a very natural question to ask whether the leaves of the canonical foliation are isoperimetric surfaces for the volume they enclose, i.e., given \(V \gg 1\), whether the isoperimetric profile \(A(V)\) can be achieved by some leaf \(\Sigma_V\) enclosing volume \(V\) of the canonical foliation. Here \(A : [0, \infty) \to [0, \infty)\) is defined by

\[
A(V) = \inf\{\mathcal{H}^2(\partial^* \Omega) : \Omega \subset M \text{ is a compact region and } \mathcal{L}^3(\Omega) = V\},
\]

where \(\mathcal{H}^2\) is 2-dimensional Hausdorff measure for the reduced boundary of \(\Omega\), and \(\mathcal{L}^3(\Omega)\) is the Lebesgue measure of \(\Omega\) with respect to metric \(g\).

In the asymptotically Schwarzschild setting, the study of isoperimetric structure on asymptotically flat Riemannian 3-manifold \((M^3, g)\) may date back to Bray’s work. In [2], Bray showed that the isoperimetric surfaces of spatial Schwarzschild manifold with positive mass are exactly round centered spheres. He deduced that if \((M^3, g)\) is the compact perturbations of the exact Schwarzschild metric with mass \(m > 0\) then the large isoperimetric surfaces are also round centered spheres. By isoperimetric technique, Bray gave a proof of Penrose inequality using positive mass theorem. He conjectured that the volume-preserving stable constant mean curvature spheres constructed by Huisken and Yau [16] are isoperimetric surfaces. Building on Bray’s volume comparison, Eichmair and Metzger [9, 10] obtained global uniqueness of large
solutions of the isoperimetric problem in any dimension for \((M, g)\) asymptotic to Schwarzschild with mass \(m > 0\) and they gave a confirm answer to Bray’s conjecture.

To study the isoperimetric properties of 3-manifolds with general asymptotics, Huisken [14] introduced the concepts of quasilocal isoperimetric mass and isoperimetric mass which only require very low regularity.

**Definition 1.1** (Huisken) Let \((M^3, g)\) be a \(C^0\)-asymptotically flat manifold and \(\Omega\) be a smooth bounded domain. The quasilocal isoperimetric mass of \(\Omega\) is

\[
m_{iso}(\Omega) = \frac{2}{\mathcal{H}^2(\partial \Omega)} \left( L^3(\Omega) - \frac{1}{6\sqrt{\pi}} \mathcal{H}^2(\partial \Omega)^{3/2} \right).
\]

The isoperimetric mass of \((M^3, g)\) is defined by

\[
m_{iso}(M, g) = \sup_{\{\Omega_i\}_{i=1}^{\infty}} \left( \limsup_{i \to \infty} m_{iso}(\Omega_i) \right),
\]

where \(\{\Omega_i\}_{i=1}^{\infty}\) is an exhaustion of \((M, g)\).

Subsequent to the work of Huisken, Fan et al. [11] observed that the “lim sup” in Huisken’s definition recovers the ADM-mass of the initial data set when evaluated along exhaustions by concentric coordinate balls in an asymptotic coordinate system. Hence, \(m_{ADM}(M, g) \leq m_{iso}(M, g)\). The following result was proposed by Huisken [14, 15] and proven by Lee and Jauregui [17]. An alternative proof was given by the author in joint work with Chodosh et al. [7]. (Both approaches also use an important insight by Fan et al. [11].)

**Theorem 1.2** Let \((M, g)\) be an asymptotically flat Riemannian 3-manifold at decay rate \(\tau > \frac{1}{2}\) and which has non-negative scalar curvature. Then

\[
m_{ADM}(M, g) = m_{iso}(M, g).
\]

Let \((M, g)\) be an asymptotically flat Riemannian 3-manifold with non-negative scalar curvature and \(\Omega \subset M\) be a compact region. An immediate consequence of the theorem above is

**Theorem 1.3** (Sharp isoperimetric inequality)

\[
V(\Omega) \leq \frac{A(\partial \Omega)^{3/2}}{6\sqrt{\pi}} + \frac{m_{ADM}}{2} A(\partial \Omega) + o(1) A(\partial \Omega)
\]

as \(V(\Omega) \to \infty\).

In a recently notable paper, Shi [25] established the isoperimetric inequality on asymptotically flat 3-manifolds with non-negative scalar curvature, based on which Carlotto et al. [3] showed that for any \(V > 0\), there always exists a smooth isoperimetric region \(\Omega\) of volume \(V\). In a jointed work with Chodosh et al. [7], we gave a complete characterization of isoperimetric structure in large scale for asymptotically flat Riemannian 3-manifold with non-negative scalar curvature.
Theorem 1.4 Let \((M, g)\) be a complete Riemannian 3-manifold that is asymptotically flat at rate \(\tau > 1/2\) and which has non-negative scalar curvature and positive mass. There is \(V_0 > 0\) with the following property. Let \(V \geq V_0\). There is a unique isoperimetic region \(\Omega_V\) with \(V(\Omega_V) = V\) whose boundary consists of the horizon \(\partial M\) and a leaf of the canonical foliation of the end of \(M\).

Without scalar curvature assumption, the general existence of large isoperimetric regions was established by Carlotto et al. [3]:

**Theorem 1.5** Let \((M^3, g)\) be an asymptotically flat Riemannian 3-manifold with horizon boundary, integrable scalar curvature, and positive ADM-mass. For all \(V > 0\) sufficiently large there is a smooth isoperimetric region of volume \(V\).

The main theorem of this paper can be stated as follows:

**Theorem 1.6** Let \((M^3, g)\) be an asymptotically flat Riemannian 3-manifold with positive ADM mass. Suppose the scalar curvature of \((M^3, g)\) is nonnegative or satisfies \(R(g) = O(|x|^{-\frac{5}{2}} - \tau)\). Then there exists some \(V_0 > 0\) such that for any \(V > V_0\) there is a unique isoperimetric region \(\Omega_V\) whose boundary is a leaf in the canonical foliation \(\{\Sigma_{\sigma}\}_{\sigma > \sigma_*}\).

**Remark 1.2** We emphasize that our proof of the theorem is completely different from [7] in the case that the scalar curvature of \((M^3, g)\) is nonnegative. In fact, we give a unified proof of the theorem.

As a corollary, we immediately have

**Corollary 1.1** Let \((M^3, g)\) be an asymptotically flat Riemannian 3-manifold with positive mass. Then

\[m_{ADM}(M, g) = m_{iso}(M, g).\]

One ingredient in our proof is the “fill-in” argument and we are partially inspired by the recent work [26]. To the author’s best knowledge, this argument is completely new in dealing with isoperimetric problems on asymptotically flat manifolds. Consider some leaf \(\Sigma_\sigma\) of the canonical foliation, cut the domain enclosed by the leaf and fill \(\Sigma_\sigma\) with a suitable metric in a canonical way. As the metric we construct has corners, we need to smooth the metric and then take conformal deformations to get a family of asymptotically flat metrics with nonnegative scalar curvature. The ADM mass of deformed metrics are strictly less than the mass of initial metric if the leaf \(\Sigma_\sigma\) we choose is far away enough. Building on the sharp isoperimetric inequality, we can show that the isoperimetric regions must look like Euclidean balls \(B_1(0)\) when scaled by their volume.

We remark here that recent breakthrough was made by Chodosh and Eichmair [5, 6]. They established the optimal, global result for stable constant mean curvature spheres in initial data asymptotic to Schwarzschild with nonnegative scalar curvature. Finally, we mention some recent progress in the asymptotically hyperbolic setting. Chodosh [4] has shown that large isoperimetric surfaces are centered coordinate spheres in...
the special case where the metric is isometric to Schwarzschild-anti-de Sitter outside of a compact set. Under the assumption that the manifold \((M^3, g)\) is asymptotic to Schwarzschild-anti-de Sitter with scalar curvature \(R \geq -6\), Chodosh et al. [8] showed that the leaves of the canonical foliation constructed by Rigger [24] are unique isoperimetric surfaces for the volume they enclose. In their case, the scalar curvature assumption is necessary.

The remains of papers are organized as follows: in Sect. 2, we construct a family of metrics \(\{g_{\sigma}\}\) with coincide with \(g\) outside of \(\Sigma_{\sigma}\) and have nice behaviour in the domain enclosed by \(\Sigma_{\sigma}\). In Sect. 3, we construct a family of asymptotically flat metrics with nonnegative scalar curvature by deforming \(\{g_{\sigma}\}\) as Miao did in [21] and show that the mass of deformed metrics can be strictly less than \(m\). We give the proof of the main theorem in the last section.

2 Gluing the metric

Let \(\omega_{\sigma}\) be the induced metric on \(\Sigma_{\sigma}\). Set \(\omega_{\sigma} = e^{2u_{\sigma}} \sigma^2 g_*, \) here \(g_*\) is some round metric on \(S^2\) with area \(4\pi\) and \(u_{\sigma}\) is a function defined on \(S^2\). Then Nerz [22] showed that for some fixed \(\alpha \in (0, 1)\), it holds

\[||\sigma^{-2}\omega_{\sigma} - g_*||_{C^2(S^2)} \leq C \sigma^{-\tau}, \text{ for } \sigma \gg 1.\]  

(5)

Here and in the following, we always use \(C\) to denote universal constants depending only on \((M^3, g)\) which may vary from line to line. Then we have

\[|u_{\sigma}|_{C^2(S^2)} \leq C \sigma^{-\tau}, \text{ for } \sigma \gg 1.\]  

(6)

We need the following result obtained by Mantoulidis and Schoen in [20].

**Lemma 2.1** Let \(\omega = e^{2u_{\sigma}} \sigma^2 g_*\) be as above. Then there exists a smooth path of metrics \(t \mapsto \omega(t)\) such that

\[\omega(0) = \omega, \ \omega\left(\frac{\sigma}{2}\right) \text{ round}, \ \frac{d}{dt} dA_{\omega(t)} = 0, \ \text{for all } t \in [0, \frac{\sigma}{2}],\]

where \(dA_{\omega(t)}\) denotes the area form for a metric \(\omega(t)\).

**Proof** The argument here follows from [20]. Consider

\[\tilde{\omega}(t) = e^{2u_{\sigma}(1-\frac{t}{\sigma})+2a(t)} \sigma^2 g_*\]

with \(a(t)\) chosen so that \(a(0) = 0\) and

\[a'(t) = \frac{2}{\sigma} \int_{S^2} u_{\sigma} dA_{\tilde{\omega}(t)} = O(\sigma^{-1-\tau}).\]
Consider the following equation

$$\Delta_{\tilde{\omega}(t)} \psi(t, \cdot) = \frac{4u_\sigma}{\sigma} - 2a'(t).$$  \hfill (7)

It is solvable since the integral of the righthand term vanishes. Then the standard elliptic estimates in [12] gives

$$|\psi(t, \cdot)|_{C^2(S^2, \tilde{\omega})} \leq C \sigma^{-1-\tau}, \quad \text{for} \quad t \in [0, \frac{\sigma}{2}].$$  \hfill (8)

Take $X_t = \nabla \tilde{\omega}(t) \psi(t, \cdot)$ and let $\phi_t$ be the one-parameter diffeomorphism group generated by $X_t$. Consider $\omega(t) = \phi_t^* \tilde{\omega}(t)$. Then

$$\frac{d}{dt} dA_{\omega(t)} = \frac{d}{dt} \phi_t^* dA_{\tilde{\omega}(t)} = \phi_t^* \left[ \frac{d}{dt} dA_{\tilde{\omega}(t)} + L_{\dot{\phi}_t} dA_{\tilde{\omega}(t)} \right]
= \phi_t^* \left[ \frac{1}{2} \text{tr}_{\tilde{\omega}(t)} \dot{\omega}(t) dA_{\tilde{\omega}(t)} + \text{div}_{\tilde{\omega}(t)} \dot{\phi}_t dA_{\tilde{\omega}(t)} \right]
= \phi_t^* \left[ 2a'(t) - \frac{4u_\sigma}{\sigma} + \Delta_{\tilde{\omega}(t)} \psi(t, \cdot) \right] dA_{\tilde{\omega}(t)} = 0,$$

where $L$ denotes the Lie derivative on $S^2$. Hence, we complete the proof.

Now we define a metric on $\Sigma_{\sigma} \times [0, \frac{\sigma}{2}]$ by

$$\gamma = f(t)\omega(t) + dt^2 = \left(1 - \frac{t}{\sigma} \right)^2 \omega(t) + dt^2.$$

Let $\Omega_{\sigma}$ be the domain enclosed by $\Sigma_{\sigma}$. Denote the surface $\Sigma_{\sigma} \times \{\frac{\sigma}{2}\}$ by $\Sigma'_\sigma$ and fill $\Sigma'_\sigma$ with Euclidean ball $(g_E, \Omega'_\sigma)$ such that $g_E|\Sigma'_\sigma = \gamma|\Sigma'_\sigma$. We define a family of asymptotically flat metrics $\{g_\sigma\}$ with corners as follows:

$$g_\sigma = \begin{cases} 
  g_E, & x \in \Omega'_\sigma, \\
  \gamma, & x \in \Omega_{\sigma} \setminus \Omega'_\sigma, \\
  g, & x \in M^3 \setminus \Omega_{\sigma}.
\end{cases} \hfill (9)$$

**Lemma 2.2** The scalar curvature of $\gamma$ satisfies

$$R_\gamma = O(\sigma^{-2-\tau}) \quad \text{for} \quad \sigma \gg 1.$$

**Proof** Set $h(t) = f(t)\omega(t)$. Then differentiating with respect to $t$ gives

$$\dot{h} = f'\omega + f\dot{\omega}.$$
Recall that $tr\omega\dot{\omega} = 0$. It follows

$$tr_h\dot{h} = \frac{2f'}{f} \quad \text{and} \quad |\dot{h}|_h^2 = \frac{2f'^2}{f^2} + |\dot{\omega}|_\omega^2.$$ 

Differentiating and tracing again,

$$\ddot{h} = f''\omega + 2f'\dot{\omega} + f\ddot{\omega} \quad \text{and} \quad tr_h\ddot{h} = \frac{2f''}{f} + tr\omega\ddot{\omega}. $$

Using $tr\omega\dot{\omega} = 0$, we get

$$tr\omega\ddot{\omega} = |\dot{\omega}|_\omega^2.$$ 

The scalar curvature of $\gamma$ is given by

$$R_\gamma = 2K_h - tr_h\ddot{h} \quad \text{with} \quad |\dot{h}|_h^2 = \frac{1}{4}(tr_h\dot{h})^2 + \frac{3}{4}|\dot{h}|_h^2.$$ 

$$= 2K_h - \frac{2f''}{f} - |\dot{\omega}|_\omega^2 - \frac{f'^2}{f^2} + \frac{3}{4}\left(\frac{2f'^2}{f^2} + |\dot{\omega}|_\omega^2\right)$$

$$= 2K_h + \frac{f'^2}{2f^2} - \frac{2f''}{f} - \frac{1}{4}|\dot{\omega}|_\omega^2.$$ 

$$= 2K_h - \frac{2}{f\sigma^2} - \frac{1}{4}|\dot{\omega}|_\omega^2.$$ 

(10)

By (6), the Gauss curvature of $h$ can be estimated by

$$K_h = f^{-1}\phi^*_t\left[e^{-2u_\sigma\left(\frac{1}{\sigma} - \frac{2}{\sigma}\right) - 2a(t)\sigma^2} \left(1 - 2\left(1 - \frac{2t}{\sigma}\right)\Delta S_\sigma^2 u_\sigma\right)\right]$$

$$= \frac{1}{f\sigma^2} + O(\sigma^{-2-\tau}).$$ 

(11)

Note that

$$\dot{\omega} = \phi^*_t\left[e^{2u_\sigma\left(\frac{1}{\sigma} - \frac{2}{\sigma}\right) + 2a(t)\sigma^2} \left(2a'(t) - \frac{4u_\sigma}{\sigma}\right) g^*_t + \nabla^2_{\omega}\psi\right].$$

Then it follows from (8) that

$$|\dot{\omega}|_\omega^2 = O(\sigma^{-2-2\tau}).$$ 

(12)

Substituting (11) and (12) into (10) gives the desired estimate.
Lemma 2.3  Let \(-\frac{\partial}{\partial t}\) be the outer normal vector field on \(\Sigma_\sigma \times [0, \frac{\sigma}{2}]\). Then for \(\sigma \gg 1\), we have

\[
H(\Sigma_\sigma, \gamma) = \frac{2}{\sigma} = H(\Sigma_\sigma, g) \quad \text{and} \quad H(\Sigma'_\sigma, g_E) = \frac{4}{R_\sigma} \geq \frac{4}{\sigma} = H(\Sigma'_\sigma, \gamma),
\]

where \(R_\sigma\) is the constant such that \(A(\Sigma_\sigma, g) = 4\pi R_\sigma^2\).

**Proof**  A direct computation shows that

\[
H(\Sigma_\sigma, \gamma) = -\frac{f'(0)}{f(0)} = \frac{2}{\sigma} = H(\Sigma_\sigma, g).
\]

The mean curvature of \(\Sigma'_\sigma\) with respect to metric \(\gamma\) and \(g_E\) are respectively given by

\[
H(\Sigma'_\sigma, \gamma) = \frac{-f'(\frac{\sigma}{2})}{f(\frac{\sigma}{2})} = \frac{4}{\sigma} \quad \text{and} \quad H(\Sigma'_\sigma, g_E) = \frac{2}{\sqrt{f(\frac{\sigma}{2}) R_\sigma}} = \frac{4}{R_\sigma}. \tag{13}
\]

Note the Hawking mass of \(\{\Sigma_\sigma\}\) satisfy

\[
m_H(\Sigma_\sigma) = \sqrt{\frac{A(\Sigma_\sigma, g)}{16\pi}} \left(1 - \frac{A(\Sigma_\sigma, g) H^2(\Sigma_\sigma, g)}{16\pi}\right) \rightarrow m, \quad \text{as} \quad \sigma \rightarrow \infty.
\]

Then we have

\[
1 - \frac{R_\sigma^2}{\sigma^2} \geq \frac{m}{\sigma} \quad \text{for} \quad \sigma \gg 1. \tag{14}
\]

This finishes the proof.

By our construction, we have

**Corollary 2.1**  For \(\sigma \gg 1\), the Sobolev Constant of \(\{g_\sigma\}\) is controlled by \(C\) depending only on \((M^3, g)\).

### 3 Smoothing \(g_\sigma\) and conformal deformation

In this section, we establish some estimates for certain conformal deformation equations. To begin with, we smooth the metric \(g_\sigma\) across \(\Sigma_\sigma\) and \(\Sigma'_\sigma\) as Miao did in [21]. Namely, we have the following proposition.

**Proposition 3.1**  There exists a family of \(C^2\) metrics \(\{g_{\sigma, \delta}\}_{0 < \delta \leq \delta_*}\) on \(\mathbb{R}^3\) so that \(g_{\sigma, \delta}\) is uniformly close to \(g_\sigma\) on \(\mathbb{R}^3\), \(g_{\sigma, \delta} = g_\sigma\) outside \((\Sigma_\sigma \cup \Sigma'_\sigma) \times (-\delta, \delta)\) (Gaussian
coordinates) and the scalar curvature of $g_{\sigma, \delta}$ satisfies

$$R_{\sigma, \delta}(z, t) = \begin{cases} 
O(1), & \text{for } (z, t) \in (\Sigma_{\sigma} \cup \Sigma'_{\sigma}) \times \{ \delta^2 < |t| < \delta \}, \\
O(1) + \frac{H(z, \gamma) - H(z, \gamma)}{\delta^2} \phi \left( \frac{t}{\delta^2} \right), & \text{for } (z, t) \in \Sigma_{\sigma} \times [-\delta^2, \delta^2], \\
O(1) + \frac{H(z, g_E) - H(z, \gamma)}{\delta^2} \phi \left( \frac{t}{\delta^2} \right), & \text{for } (z, t) \in \Sigma'_{\sigma} \times [-\delta^2, \delta^2], \\
0, & \text{otherwise.} 
\end{cases}$$

where $O(1)$ represents quantities bounded by constants depending only on $g$, but not on $\delta$ or $\sigma$, and $\phi \in C^\infty_c([-1, 1])$ is a standard mollifier satisfying $0 \leq \phi \leq 1$, $\phi \equiv 1$ in $[-\frac{1}{3}, \frac{1}{3}]$, and $\int_{-1}^1 \phi = 1$.

Then by Lemma 2.3, we have

$$R_{\sigma, \delta} = \begin{cases} 
O(|x|^{-2-\tau}), & \text{in } \Sigma_{\sigma} \times [\delta, \frac{\sigma}{2} - \delta] \text{ or outside } \Sigma_{\sigma} \times \{-\delta\}, \\
O(1) + \frac{4}{\delta^2} \left( \frac{1}{R_{\sigma}} - \frac{1}{\sigma} \right) \phi \left( \frac{t}{\delta^2} \right), & \text{in } \Sigma'_{\sigma} \times [-\delta^2, \delta^2], \\
O(1), & \left( \Sigma_{\sigma} \times [-\delta, \delta] \right) \cup \left( \Sigma'_{\sigma} \times \{ \delta^2 < |t| < \delta \} \right), \\
0, & \text{otherwise.} 
\end{cases} \tag{15}$$

We choose some $C^2$ function $f_{\sigma, \delta}$ satisfying $f_{\sigma, \delta} = \frac{R_{\sigma, \delta}}{8}$ outside $\Sigma'_{\sigma} \times [-\delta, \delta]$ and

$$-C_0 \leq f_{\sigma, \delta} \leq \min \left\{ \frac{R_{\sigma, \delta}}{8}, C_0 \right\} \text{ for } x \in \Sigma'_{\sigma} \times [-\delta, \delta]$$

for some uniform $C_0 > 0$. Then

$$\int |f_{\sigma, \delta}|^{\frac{3}{2}} dG_{\sigma, \delta} \leq C(\sigma^2 \delta)^{\frac{3}{2}} + \frac{C}{\sigma}.$$ 

Thus, for $\sigma \gg 1$ and $\delta < \frac{1}{C_{\sigma}}$, we have

$$\int |f_{\sigma, \delta}|^{\frac{3}{2}} \leq \frac{C}{\sigma} \to 0 \text{ as } \sigma \to \infty. \tag{16}$$

Consider the following equation

$$\begin{align*}
\Delta_{g_{\sigma, \delta}} u_{\sigma, \delta} - f_{\sigma, \delta} u_{\sigma, \delta} &= 0, \\
u_{\sigma, \delta}(\infty) &= \lim_{x \to \infty} u_{\sigma, \delta}(x) = 1. \tag{17}
\end{align*}$$

The solvability of equation (17) is guaranteed by the following lemma due to Schoen and Yau [27].
Lemma 3.1  Let \((N, g_N)\) be an asymptotically flat 3-manifold and \(h\) be a function that has the same decay rate at \(\infty\) as \(R(g_N)\). Then there exists a number \(\epsilon_N > 0\) depending only on the \(C^0\) norm of \(g_N\) and the decay rate of \(g_N, \partial g_N\) and \(\partial^2 g_N\) at \(\infty\) so that if
\[
\left( \int_N |h|^{-\frac{3}{2}} \, d\mu_{g_N} \right)^{\frac{2}{3}} < \epsilon_N,
\] (18)
then
\[
\left\{ \begin{array}{l}
\Delta_{g_N} u - hu = 0 \text{ in } N, \\
u \to 1 \text{ at } \infty.
\end{array} \right.
\]
has a \(C^2\) positive solution \(u\) that
\[
u(x) = 1 + \frac{A}{|x|} + B
\]
for some constant \(A\) and some function \(B\), where \(B = O(|x|^{-2})\) and \(\partial B = O(|x|^{-3})\).

Set \(\tilde{g}_{\sigma, \delta} = u^4_{\sigma, \delta} g_{\sigma, \delta}\). Then \(\{\tilde{g}_{\sigma, \delta}\}\) is a family of asymptotically flat metrics with non-negative scalar curvature. Let \(v_{\sigma, \delta} = u_{\sigma, \delta} - 1\).

Lemma 3.2  For \(\sigma \gg 1\) and \(\delta \ll \frac{1}{\sigma^3}\), it holds
\[
|v_{\sigma, \delta}(x)| \leq C \sigma^{-\frac{1}{2}}, \quad \text{for } |x| \geq \frac{\sigma}{3}.
\]

Proof  We divide our proof into two steps.

Step1: We aim to get the \(L^6\) estimate of \(v_{\sigma, \delta}\). By (17),
\[
\Delta_{g_{\sigma, \delta}} v_{\sigma, \delta} - f_{\sigma, \delta} v_{\sigma, \delta} = f_{\sigma, \delta}.
\] (19)

Multiplying the equation above with \(v_{\sigma, \delta}\) and integrating on \(\mathbb{R}^3\) give
\[
\int (v_{\sigma, \delta} \Delta_{g_{\sigma, \delta}} v_{\sigma, \delta} + \int f_{\sigma, \delta} v_{\sigma, \delta}^2) \, dg_{\sigma, \delta} = \int f_{\sigma, \delta} v_{\sigma, \delta} \, dg_{\sigma, \delta}.
\]

Integrating by parts and using Holder Inequality, we have that
\[
\int |\nabla_{\tilde{g}_{\sigma, \delta}} v_{\sigma, \delta}|^2 \, dg_{\sigma, \delta} \leq \left( \int |f_{\sigma, \delta}|^\frac{2}{3} \right)^\frac{3}{2} \left( \int v_{\sigma, \delta}^6 \, dg_{\sigma, \delta} \right)^\frac{1}{3}
\]
\[
+ \left( \int |f_{\sigma, \delta}|^\frac{6}{5} \right)^\frac{5}{6} \left( \int v_{\sigma, \delta}^6 \, dg_{\sigma, \delta} \right)^\frac{1}{6}.
\] (20)

The Sobolev inequality gives that
\[
\left( \int v_{\sigma, \delta}^6 \, dg_{\sigma, \delta} \right)^\frac{1}{6} \leq C_{\sigma, \delta} \int |\nabla_{\tilde{g}_{\sigma, \delta}} v_{\sigma, \delta}|^2 \, dg_{\sigma, \delta},
\]
where $C_{\sigma, \delta}$ denotes the Sobolev Constant of the metric $g_{\sigma, \delta}$. Then we have

$$\left( \int v_{\sigma, \delta}^6 d g_{\sigma, \delta} \right)^{\frac{1}{3}} \leq C_{\sigma, \delta} \left( \int |f_{\sigma, \delta}|^3 d g_{\sigma, \delta} \right)^{\frac{1}{3}} \left( \int v_{\sigma, \delta}^6 d g_{\sigma, \delta} \right)^{\frac{1}{3}}$$

$$+ \frac{1}{16} C_{\sigma, \delta}^2 \left( \int |f_{\sigma, \delta}|^6 d g_{\sigma, \delta} \right)^{\frac{5}{3}} + \frac{1}{2} \left( \int v_{\sigma, \delta}^6 d g_{\sigma, \delta} \right)^{\frac{1}{3}}. \quad (21)$$

Note $g_{\sigma, \delta}$ is uniformly close to $g_{\sigma}$. Then $C_{\sigma, \delta}$ is uniformly bounded by Corollary 2.1. By (16), for $\sigma \gg 1$ and $\delta < \frac{1}{8 C_{\sigma}^3}$, we have

$$C_{\sigma, \delta} \left( \int |f_{\sigma, \delta}|^3 \right)^{\frac{2}{3}} \leq \frac{1}{4}.$$

Then it follows that for $\sigma \gg 1$ and $\delta < \frac{1}{\sigma^3}$

$$\left( \int v_{\sigma, \delta}^6 d g_{\sigma, \delta} \right)^{\frac{1}{3}} \leq C \left( \int |f_{\sigma, \delta}|^6 \right)^{\frac{5}{3}} \leq C (\sigma^{-\frac{6}{5}(\tau - \frac{1}{2})} + \sigma^2 \delta)^{\frac{5}{3}} = o(1), \quad \text{as} \quad \sigma \to \infty.$$

**Step 2** We use Moser iteration to improve the estimate. We omit the lower index for simplicity. By (19), we have

$$\varphi^2 v^{2p-1} \Delta v = f \varphi^2 v^2 + f \varphi v^{2p-1},$$

where $\varphi$ is a $C^2$ function supported in $B_{\frac{\sigma}{4}}(x)$ and $p \geq 3$ is positive integer. Then Stokes’ formula implies that

$$- \int \varphi^2 v^{2p-1} \Delta v = (2p - 1) \int \varphi^2 v^{2p-2} |\nabla v|^2 + 2 \int \varphi v^{2p-1} \nabla \varphi \nabla v.$$

It follows that

$$(2p - 1) \int \varphi^2 v^{2p-2} |\nabla v|^2$$

$$= -2 \int \varphi v^{2p-1} \nabla \varphi \nabla v - \int f \varphi^2 v^2 - \int f \varphi v^{2p-1}$$

$$\leq \frac{2p - 1}{2} \int \varphi^2 v^{2p-2} |\nabla v|^2 + \frac{2}{2p - 1} \int |\nabla \varphi|^2 v^{2p} + \int |f| \varphi^2 (v^{2p} + |v|^{2p-1}). \quad (22)$$

On the other hand, using Sobolev inequality, we have

$$\left( \int (\varphi v^p)^6 \right)^{\frac{1}{3}} \leq C \int |\nabla (\varphi v^p)|^2$$

$$\leq C \left( \int |\nabla \varphi|^2 v^{2p} + \int p^2 v^{2p-2} |\nabla v|^2 \right). \quad (23)$$
Combining the two inequalities above gives
\[
\left( \int (\varphi v^6)^{\frac{1}{3}} \right) \leq C_1 p^2 \int |f| \varphi^2 v^{2p} + C_1 p^2 \int |f| \varphi^2 |v|^{2p-1} + C_1 \int |\nabla \varphi|^2 v^{2p}. \tag{24}
\]

By Holder inequality,
\[
\int |f| \varphi^2 |v|^{2p-1} \leq \left( \int |f| \varphi^2 v^{2p} \right)^{\frac{2p-1}{2p}} \left( \int |\varphi|^2 \right)^{\frac{1}{2p}},
\]
\[
\leq \sigma^\tau \int |f| \varphi^2 v^{2p} + \sigma^{\tau-2p\tau} \int |f| \varphi^2,
\]
\[
\leq \int \sigma^\tau |f| \varphi^2 v^{2p} + \sigma^{1-2p\tau}. \tag{25}
\]

Then
\[
\left( \int (\varphi v^6)^{\frac{1}{3}} \right) \leq 2C_1 p^2 \int \sigma^\tau |f| \varphi^2 v^{2p} + C_1 \int |\nabla \varphi|^2 v^{2p} + C_1 p^2 \sigma^{1-2p\tau}.
\]

Using Holder inequality,
\[
p^2 \int \sigma^\tau |f| \varphi^2 v^{2p} \leq p^2 \int_{\text{supp} \varphi} \left( \sigma^{2\tau} |f|^2 \right)^{\frac{1}{2}} \left( \int |\varphi v^6| \right)^{\frac{1}{3}} \left( \int \varphi^2 |v|^{2p} \right)^{\frac{1}{3}}
\]
\[
\leq \varepsilon \left( \int |\varphi v^6| \right)^{\frac{1}{3}} + \varepsilon^{-4} p^8 \left( \int_{\text{supp} \varphi} \sigma^{2\tau} |f|^2 \right)^{\frac{1}{2}} \int \varphi^2 v^{2p}. \tag{26}
\]

Take \( \varepsilon \) such that \( 2C_1 \varepsilon = \frac{1}{2} \). Then we have
\[
\left( \int (\varphi v^6)^{\frac{1}{3}} \right) \leq C_2 p^8 \left( \int \varphi^2 \sigma^{2\tau} |f|^2 \right)^{2} \int \varphi^2 v^{2p} + C_2 \int |\nabla \varphi|^2 v^{2p} + C_2 p^2 \sigma^{1-2p\tau}
\]
\[
\leq C_3 p^8 \left( \sigma^{-2} \int \varphi^2 v^{2p} + \int |\nabla \varphi|^2 v^{2p} + \sigma^{1-2p\tau} \right). \tag{27}
\]

We choose \( \varphi_i \in C^2_c (B_{\frac{\sigma}{4}} (x)) \) to be the cut-off function depending only on the distance to \( x \) such that
\[
\varphi_i (x) = \begin{cases} 
1, & x \in B_{r_i+1} (x), \\
0, & x \notin B_{r_i} (x),
\end{cases}
\]
and \( |\nabla \varphi_i| \leq \frac{C_2 i}{\sigma} \) for some uniform \( C \). Here \( r_i \) is defined by
\[
r_i = \frac{\sigma}{4} \left( 1 - \sum_{k=1}^{i} \frac{1}{2^{k+1}} \right).
Set
\[ p = 3^i \quad \text{and} \quad I_{i+1} = \sigma^{-3} \int_{B_{r_{i+1}}} |v|^{2.3^{i+1}} + \sigma^{-2} 3^{i+1}. \]

Then it follows from (27) that
\[
I_{i+1} \leq C_3 3^{24i} \sigma^{-3} \left[ (\sigma^{-2} \int \varphi^2 v^{2.3^i} + \int |\nabla \varphi_i|^2 v^{2.3^i} + \sigma^{-1} 2^{3^{i+1}})^3 + \sigma^{-2} 3^{i+1} \right] \\
\leq C_4 3^{24i} 8^i \left[ \sigma^{-9} \left( \int_{\text{supp} \varphi_i} v^{2.3^i} \right)^3 + \sigma^{-2} 3^{i+1} \right] \\
\leq C_4 3^{24i} 8^i \left[ \sigma^{-3} \int_{\text{supp} \varphi_i} v^{2.3^i} + \sigma^{-2} 3^{i+1} \right]^3 \\
\leq C_4 3^{24i} 8^i I_i^3. \quad (28)
\]

It’s easy to show that
\[
I_i^{\frac{1}{2.3^i}} \leq C_5 I_1^\frac{1}{5}.
\]

Sending \( i \) to \( \infty \) gives
\[
|v|(x) \leq C_5 \left( \sigma^{-\frac{1}{2}} \left( \int v^6 \right)^\frac{1}{6} + \sigma^{-\tau} \right) \leq C_6 \sigma^{-\frac{1}{2}}.
\]

Hence, we finish the proof.

**Theorem 3.1** We can find some \( \sigma_0 \gg 1 \) such that for any \( \sigma > \sigma_0 \) and \( \delta \leq \frac{1}{C\sigma^3} \), it holds
\[
m(\tilde{g}_{\sigma, \delta}) \leq \frac{7}{8} m(g_{\sigma, \delta}).
\]

**Proof** Using the definition of mass, a straightforward calculation yields
\[
m(\tilde{g}_{\sigma, \delta}) = m(g_{\sigma, \delta}) + 2 A_{\sigma, \delta},
\]
where \( A_{\sigma, \delta} \) is given by the expansion \( u_{\sigma, \delta}(x) = 1 + \frac{A_{\sigma, \delta}}{|x|} + O(\frac{1}{|x|^2}) \). Note that \( f_{\sigma, \delta} = \frac{1}{8} R_{\sigma, \delta} \) outside \( (\Sigma_{\sigma} \cup \Sigma_{\delta}') \times [-\delta, \delta] \). Applying integration by parts to (17) multiplied by \( u_{\sigma, \delta} \), we have that
\[
4\pi A_{\sigma, \delta} = \int \left( - f_{\sigma, \delta} u_{\sigma, \delta}^2 - |\nabla g_{\sigma, \delta} u_{\sigma, \delta}|^2 \right) dg_{\sigma, \delta} \\
\leq 2 \int_{(\Sigma_{\sigma} \cup \Sigma_{\delta}') \times [-\delta, \delta]} |f_{\sigma, \delta}| dg_{\sigma, \delta} - \frac{1}{8} \int_{\Sigma_{\sigma} \times [\delta, \frac{\sigma}{2} - \delta]} R_{\sigma, \delta} u_{\sigma, \delta}^2 dg_{\sigma, \delta}.
\]
\[ + 2 \int_{M'} |R_{\sigma, \delta}| dg_{\sigma, \delta} \]
\[ \leq C \sigma^2 \delta + 2 \int_{M'} |R_g| dg - \frac{1}{8} \int_{\Sigma_{\sigma} \times [\delta, \frac{\sigma}{2} - \delta]} R_{\sigma, \delta} u_{\sigma, \delta}^2 dg_{\sigma, \delta}, \]  
(29)

where \( M' \) consists of the points outside \( \Sigma_{\sigma} \times \{-\delta\} \). Recall that

\[ g_{\sigma, \delta} = \gamma = h(t) + dt^2 = f(t)\omega(t) + dt^2, \quad (x, t) \in \Sigma_{\sigma} \times \left[ \delta, \frac{\sigma}{2} - \delta \right]. \]

Then by (10),

\[ \int_{\Sigma_{\sigma} \times [\delta, \frac{\sigma}{2} - \delta]} R_{\sigma, \delta} u_{\sigma, \delta}^2 dg_{\sigma, \delta} = \int_{\Sigma_{\sigma} \times [0, \frac{\sigma}{2}]} R_\gamma u_{\sigma, \delta}^2 dg_\gamma + O(\sigma^2 \delta) \]
\[ = \int_{\Sigma_{\sigma} \times [0, \frac{\sigma}{2}]} R_\gamma \left( 1 + O(\sigma^{-\frac{1}{2}}) \right) dg_\gamma + O(\sigma^2 \delta) \]
\[ = \int_0^{\sigma} \int_{\Sigma_{\sigma} \times \{t\}} \left[ 2K_{h(t)} - \frac{2}{f_\omega^2} \right] dA_h(t) dt + O(\sigma^2 \delta) + O\left( \sigma^{\frac{1}{2} - \tau} \right) \]
\[ = 4\pi \sigma - \frac{2}{\sigma^2} \int_0^{\sigma} \int_{\Sigma_{\sigma} \times \{t\}} dA_{\omega(t)} dt + O(\sigma^2 \delta) + O\left( \sigma^{\frac{1}{2} - \tau} \right) \]
\[ \geq 4\pi m(g_{\sigma, \delta}) + O(\sigma^2 \delta) + O\left( \sigma^{\frac{1}{2} - \tau} \right), \]  
(30)

where we have used

\[ \text{Area}(\Sigma_{\sigma} \times \{t\}, \omega(t)) = 4\pi R_\sigma^2, \quad \text{for} \quad t \in \left[ 0, \frac{\sigma}{2} \right]. \]

As \( \int_M |R| dg \) is finite, we can choose \( \sigma_0 \gg 1 \) such that for any \( \sigma \geq \sigma_0 \) and \( \delta < \frac{1}{C\sigma^3} \),

\[ C \sigma^2 \delta + \int_{M'} |R| dg + O(\sigma^2 \delta) + O(\sigma^{\frac{1}{2} - \tau}) \leq \frac{\pi m(g_{\sigma, \delta})}{4}. \]

Combining the inequality above with (29) and (30) yields the desired estimate.

4 Proof of Theorem 1.6

Now we turn to the proof of the main theorem. Let \( \{\Omega_{V_k}\} \) be a sequence of isoperimetric regions of volumes \( V_k \) in \( (M, g) \) and \( \{\Omega_k\} \) be the unique large component of \( \{\Omega_{V_k}\} \) with \( V(\Omega_k) = \frac{4\pi R_{\Omega_k}^3}{3} \rightarrow \infty \). We know that \( \Omega_k \) is connected. Let \( \Omega_k \) be the subset of
$\{ x \in \mathbb{R}^3 : \rho_k |x| > 1/2 \}$ such that

$$\Omega_k \setminus K \cong \{ \rho_k x : x \in \tilde{\Omega}_k \}.$$ 

Then upon passing to a subsequence,

$$\tilde{\Omega}_k \rightarrow B_1(\xi) \quad \text{in} \quad C^{2,\alpha}_{loc}(\mathbb{R}^3 \setminus \{0\}) \quad \text{for some} \quad \xi \in \mathbb{R}^3.$$ 

as $k \rightarrow \infty$. Our goal will be to show that $\xi = 0$.

**Proposition 4.1** $\xi = 0$.

**Remark 4.1** An analogous result holds in all dimensions provided the sharp isoperimetric inequality holds for any $n$-dimensional AF manifold $(M, g)$ with nonnegative scalar curvature and positive mass. If one can construct some scalar nonnegative metric $\tilde{g} = u^{4/3} \tilde{g}$ with $\tilde{g}$ being some compact perturbation of $g$ and $u = 1 + A|x|^{2-n} + O(|x|^{1-n})$ being some conformal deformation satisfying $A < 0$, then large isoperimetric regions in AF manifolds with positive mass must be close to the corresponding centered coordinate balls.

**Proof** Assume that $\xi \neq 0$. By Theorem 3.1, we can choose some fixed $\sigma_0 \gg 1$ and $\delta_0 \leq \frac{1}{C\sigma_0}$ such that the conformal metric

$$\tilde{g}_{\sigma_0,\delta_0} = u^4_{\sigma_0,\delta_0} g_{\sigma_0,\delta_0} = \left(1 + \frac{A_{\sigma_0,\delta_0}}{|x|} \right)^4 g_{\sigma_0,\delta_0} + O(|x|^{-2})$$

satisfies

$$m(\tilde{g}_{\sigma_0,\delta_0}) = m(g_{\sigma_0,\delta_0}) + 2A_{\sigma_0,\delta_0} = m(g_{\sigma_0,\delta_0})(1 - \varepsilon_0), \quad (31)$$

for some $\varepsilon_0 > 0$. Set $\Omega'_k = \Omega_k \setminus B_{1/2}(0)$. Then we have

$$V(\Omega'_k, \tilde{g}_{\sigma_0,\delta_0}) = V(\Omega_k, g_{\sigma_0,\delta_0}) - 3\varepsilon_0 m(g_{\sigma_0,\delta_0}) \int_{\Omega'_k} \frac{1}{|x|} + o(\rho_k^2),$$

$$A(\partial \Omega'_k, \tilde{g}_{\sigma_0,\delta_0}) = A(\partial \Omega_k, g_{\sigma_0,\delta_0}) - 2\varepsilon_0 m(g_{\sigma_0,\delta_0}) \int_{\partial \Omega'_k} \frac{1}{|x|} + o(\rho_k).$$

On the other hand, $\{\Omega_k\}$ is a sequence of isoperimetric regions in $(M^3, g)$ and $g = g_{\sigma_0,\delta_0}$ outside of some compact set. Then

$$V(\Omega_k, g_{\sigma_0,\delta_0}) \geq \frac{1}{6\sqrt{\pi}} A^2(\partial \Omega_k, g_{\sigma_0,\delta_0}) + \frac{m(g_{\sigma_0,\delta_0})}{2} A(\partial \Omega_k, g_{\sigma_0,\delta_0}) + o(\rho_k^2).$$

Combining the above three inequalities yields

$$V(\Omega'_k, \tilde{g}_{\sigma_0,\delta_0}) - \frac{1}{6\sqrt{\pi}} A^2(\partial \Omega'_k, \tilde{g}_{\sigma_0,\delta_0}) - \frac{m(\tilde{g}_{\sigma_0,\delta_0})}{2} A(\partial \Omega'_k, \tilde{g}_{\sigma_0,\delta_0})$$
\[ V(\Omega_k, g_{\sigma_0, \delta_0}) - \frac{1}{6\sqrt{\pi}} A(\partial \Omega_k, g_{\sigma_0, \delta_0}) - 2\varepsilon_0 m(g_{\sigma_0, \delta_0}) \int_{\partial \Omega_k'} \frac{1}{|x|} + o(\rho_k^2) = \] 
\[ \leq V(\Omega_k, g_{\sigma_0, \delta_0}) - \frac{m(g_{\sigma_0, \delta_0})}{2} A(\partial \Omega_k, g_{\sigma_0, \delta_0}) + \frac{\varepsilon_0 m(g_{\sigma_0, \delta_0})}{2} A(\partial \Omega_k', g_{\sigma_0, \delta_0}) 
- 3\varepsilon_0 m(g_{\sigma_0, \delta_0}) \int_{\Omega_k'} \frac{1}{|x|} + o(\rho_k^2) \]

\[ \geq V(\Omega_k, g_{\sigma_0, \delta_0}) - \frac{1}{6\sqrt{\pi}} A(\partial \Omega_k, g_{\sigma_0, \delta_0}) - \frac{m(g_{\sigma_0, \delta_0})}{2} A(\partial \Omega_k, g_{\sigma_0, \delta_0}) 
+ \varepsilon_0 m(g_{\sigma_0, \delta_0}) \left( 2\pi \rho_k^2 + \rho_k \int_{\partial \Omega_k'} \frac{1}{|x|} - 3 \int_{\Omega_k'} \frac{1}{|x|} \right) + o(\rho_k^2) \]

\[ \geq \varepsilon_0 m(g_{\sigma_0, \delta_0}) \left( 2\pi \rho_k^2 + \rho_k \int_{\partial \Omega_k'} \frac{1}{|x|} - 3 \int_{\Omega_k'} \frac{1}{|x|} \right) + o(\rho_k^2). \quad (32) \]

Note that \( \tilde{\Omega}_k \rightarrow B_1(\xi) \) in \( C^{2,\alpha}_{\text{loc}}(\mathbb{R}^3 \setminus \{0\}) \). Then

\[ \rho_k \int_{\partial \Omega_k'} \frac{1}{|x|} = \rho_k \int_{S_{\rho_k}(\rho_k \xi)} \frac{1}{|x|} + o(\rho_k^2) = \begin{cases} 4\pi \rho_k^2 + o(\rho_k^2), & |\xi| \leq 1, \\ 4\pi \rho_k^2 + o(\rho_k^2), & |\xi| \geq 1. \end{cases} \quad (33) \]

Similarly,

\[ \int_{\Omega_k'} \frac{1}{|x|} = \int_{B_{\rho_k}(\rho_k \xi)} \frac{1}{|x|} + o(\rho_k^2) = \begin{cases} 2\pi \rho_k^2 (1 - \frac{|\xi|^2}{3}) + o(\rho_k^2), & |\xi| \leq 1, \\ 4\pi \rho_k^2 + o(\rho_k^2), & |\xi| \geq 1. \end{cases} \quad (34) \]

Hence, we always have

\[ V(\Omega_k', \tilde{g}_{\sigma_0, \delta_0}) - \frac{1}{6\sqrt{\pi}} A(\partial \Omega_k', \tilde{g}_{\sigma_0, \delta_0}) - \frac{m(\tilde{g}_{\sigma_0, \delta_0})}{2} A(\partial \Omega_k', \tilde{g}_{\sigma_0, \delta_0}) \]

\[ \geq \frac{2\varepsilon_0 \pi m(g_{\sigma_0, \delta_0}) |\xi|^2 \rho_k^2}{1 + |\xi|^2} + o(\rho_k^2), \quad (35) \]

which contradicts with the sharp isoperimetric inequality (4) on \( (\mathbb{R}^3, \tilde{g}_{\sigma_0, \delta_0}) \).

**Proof of Theorem 1.6** By Proposition 4.1, we see that every sufficiently large isoperimetric region is connected and close to the centered coordinate ball \( B_1(0) \) after suitable scaling in the chart at infinity (1). The uniqueness of large stable constant mean curvature spheres obtained by Nerz in [22] shows the boundary of such an isoperimetric region must be a leaf of the canonical foliation.

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