Anisotropic energy transfers in quasi-static magnetohydrodynamic turbulence

K. Sandeep Reddy, 1, a) Raghwendra Kumar, 2, b) and Mahendra K. Verma 3, c)
1) Department of Mechanical Engineering, Indian Institute of Technology, Kanpur 208016, India
2) Theoretical Physics Division, Bhabha Atomic Research Centre, Mumbai 400 085, India
3) Department of Physics, Indian Institute of Technology, Kanpur 208016, India

(Dated: 3 December 2014)

We perform direct numerical simulations of quasi-static magnetohydrodynamic turbulence, and compute various energy transfers including the ring-to-ring and conical energy transfers, and the energy fluxes of the perpendicular and parallel components of the velocity field. We show that the rings with higher polar angles transfer energy to ones with lower polar angles. For large interaction parameters, the dominant energy transfer takes place near the equator (polar angle $\theta \approx \frac{\pi}{2}$). The energy transfers are local both in wavenumbers and angles. The energy flux of the perpendicular component is predominantly from higher to lower wavenumbers (inverse cascade of energy), while that of the parallel component is from lower to higher wavenumbers (forward cascade of energy). Our results are consistent with earlier results, which indicate quasi two-dimensionalization of quasi-static magnetohydrodynamic (MHD) flows at high interaction parameters.

I. INTRODUCTION

Liquid-metal flows under strong magnetic field occur in geophysics, metallurgical applications like metal-plate rolling, heat exchangers of the proposed fusion reactor ITER, etc. These flows are described by magnetohydrodynamics (MHD), which involves equations for the velocity and magnetic fields. Liquid metals have small magnetic Prandtl numbers $P_m$, which is the ratio of the kinematic viscosity $\nu$ to the magnetic diffusivity $\eta$. 

The flow velocity in a typical industrial application is rather small. Hence the magnetic Reynolds number $R_m \equiv UL/\eta$, where $U$ and $L$ are the large-scale velocity and length scales respectively) for such flows is quite small. A limiting case of such flows, called the quasi-static limit 1,2 $(R_m \to 0)$, provides further simplification; here the time derivative of the magnetic field is negligible compared to the magnetic diffusion term. Experiments 3,4 and numerical simulations 5-7 show that the flow becomes quasi-two-dimensional when subjected to a strong mean magnetic field. In the present paper, we discuss the energy transfers in the quasi-static MHD. We highlight the energy transfers responsible for making the flow quasi-two-dimensional.

The external magnetic field makes the flow anisotropic. For a strong magnetic field, Moffatt 8 predicted a rapid decay of isotropic three-dimensional turbulence to a two-dimensional state. Kit and Tsinober 9 analyzed several experimental results and argued that MHD flow under strong magnetic field is two-dimensional. Alemany et al. 3 performed experiment on mercury and obtained a $k^{-3}$ energy spectrum. Alemany et al. 3 and Moreau 10 however, explained this spectrum by arguing that the nonlinear transfer time is independent of the wavenumber $k$, not due to the two-dimensionality of the flow; they proposed that the quasi-static MHD is quasi two-dimensional. Sommeria and Moreau 11 studied conditions when the MHD turbulence at low-$R_m$ becomes two-dimensional. Klein and Pothérat 12 and Pothérat and Klein 13 studied the three dimensionalization of wall-bounded MHD flows in a quasi two-dimensional flow of liquid metals; these works as well as Pothérat 14 emphasize the role of boundary walls in the dynamics of quasi-static MHD.

The aforementioned quasi two-dimensionalization has been studied using direct numerical simulations. Burattini et al. 15,16 computed the kinetic energy spectrum and showed how the anisotropy varies with respect to the direction of the external magnetic field. Favier et al. 7,17 studied this phenomena using direct numerical simulations (DNS) and eddy-damped quasi-normal Markovian (EDQNM) model. Zikanov and These 6 showed that for moderate interaction parameters, the turbulence remains quasi two-dimensional for several eddy turnover times before it is interrupted by strong bursts of three dimensional turbulence. Reddy and Verma 18 quantified the energy distribution using ring spectrum, and show that the energy is concentrated near the equator. They also showed that they energy spectrum is exponential $(\exp(-bk))$ for a very large magnetic field.

The above simulations, performed using pseudospectral method in a periodic box, capture the properties of the bulk flow quite well. For example, steepening of the energy spectrum with the increase of interaction parameter is observed in all the simulations 15,18 as well as in experiments. 19,20 However, the Hartmann layers cannot be studied using periodic box simulations. Dymkov and Potherat 21 and Kornet and Potherat 22 have developed numerical techniques to simulate wall bounded MHD flows using least dissipative modes. Boeck et al. 23 performed DNS of quasi-static MHD flow in a channel with no-slip walls and observed recurring transitions be-

---

a)Electronic mail: ksreddy@iitk.ac.in
b)Electronic mail: raghav@barc.gov.in
c)Electronic mail: mkv@iitk.ac.in
between two-dimensional and three-dimensional states in the flow.

However, a word of caution is in order. Most of the aforementioned simulations have been performed on a periodic box. The flow structures with realistic boundary conditions (e.g., no-slip walls) differ significantly from those with periodic domains, since boundary effects are completely ignored in periodic box simulations. In a wall-bounded low-Rm liquid-metal MHD flow, the Hartmann layers at walls restrict the elongation of two-dimensional structures; these features are not captured in periodic box simulations. The structures longer than the length of the domain are cut at the periodic boundaries and appear as 2D structures.

Yet, periodic box computations provide interesting insights into energy transfers in the bulk flow. The energy spectrum computed using the periodic box simulations are in general agreement with those computed in experiments, for example, quasi two-dimensionalization of the flow is captured successfully in periodic box simulations.

The energy spectrum of liquid-metal flows has been studied by a large number of scientists and engineers (see above). However, diagnostics like energy flux, shell-to-shell energy transfer, etc. are much less studied in this field. In fluid turbulence, the turbulence is homogeneous and isotropic in the inertial range. Also, in the inertial range, Kolmogorov’s flux is constant, and the shell-to-shell energy transfer is forward and local (maximum transfers between the neighboring shells). However, in liquid-metal flows, the mean magnetic field induces anisotropic energy transfers, which are quantified using the angular-dependent energy flux and ring-to-ring transfers. We use the formalism proposed by Dar et al., Verma, and Teaca et al. to compute these quantities.

For magnetohydrodynamic flows with unit magnetic Prandtl number, Teaca et al. computed the energy transfers among the spectral rings (see Fig. 1). These rings are specified by their radii and sector indices (see Fig. 2). For convenience, we refer to the rings near the pole as “polar rings” (θ ≈ 0), and those near the equator as “equatorial rings” (θ ≈ π/2). In this paper, we compute the energy transfers among the rings, and show that the energy transfers are dominant near the plane perpendicular to the external magnetic field when the external field is large. We also compute other quantities, like, the energy flux, conical energy flux, and ring dissipation rates. These results provide newer insights into the quasi-two-dimensional nature of quasi-static MHD turbulence at high interaction parameters. Note, however, that our work differs from that of Favier et al. We explicitly compute the energy transfers (in contrast to Favier et al. who focus on the energy spectra of the poloidal and toroidal components), anisotropy of the flow, as well as nonlinear transfer spectrum.

The paper is organized as follows: In Sec. II, we present the formalism of ring-to-ring energy transfers, conical energy flux, and parallel and perpendicular energy fluxes. Section III contains the details of our numerical simulations. We present the results of our numerical computations in Sec. IV, and summarize the results in Sec. V.

II. THEORETICAL FRAMEWORK

A. Governing equations

The governing equations of low-Rm liquid-metal flows under quasi-static approximation are:

\[
\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\nabla(p/\rho) - \frac{\sigma B_0^2}{\rho} \Delta^{-1} \frac{\partial^2 \mathbf{u}}{\partial z^2} + \nu \nabla^2 \mathbf{u} + \mathbf{f},
\]

\[
\nabla \cdot \mathbf{u} = 0,
\]

where \( \mathbf{u} \) is the velocity field, \( B_0 = B_0 \hat{z} \) is the constant external magnetic field, \( p \) is the pressure, \( \rho, \nu, \sigma \) are the density, kinematic viscosity, conductivity of the fluid, respectively, \( \Delta^{-1} \) is the inverse of the Laplacian operator, and \( \mathbf{f} \) is the forcing. We also assume that the flow is incompressible, i.e., the density of the fluid is constant.
The above equations are nondimensionalized using the characteristic velocity $U_0$ as the velocity scale, the box dimension $L_0$ as the length scale, and $L_0/U_0$ as the time scale. As a result, the non-dimensional equations are

$$\frac{\partial \hat{U}_i(k)}{\partial T} = -ik_j \sum \hat{U}_j(q) \hat{U}_i(k-q) - ik_i \hat{P}(k) - B_0^2 \cos^2(\theta) \hat{U}_i(k) - \nu' k^2 \hat{U}_i(k) + \hat{f}_i^r(k),$$

$$k_i \hat{U}_i(k) = 0,$$

where non-dimensional variables are $U = u/U_0$, $\nabla' = L_0 \nabla$, $\Delta' = \Delta/L_0^2$, $T = t(U_0/L_0)$, $B_0^2 = \sigma B_0^2 L_0/(\rho U_0)$, and $\nu' = \nu/(U_0 L_0)$.

In quasi-static MHD turbulence, there is an interplay between the Joule dissipation, viscous dissipation, and the non-linear energy transfers at various scales. It is convenient to analyze the aforementioned processes in the wavenumber or the Fourier space. The non-dimensional equations in the spectral space\cite{3, 6, 28} are

$$\frac{\partial \hat{U}_i(k)}{\partial T} = -ik_j \sum \hat{U}_j(q) \hat{U}_i(k-q) - ik_i \hat{P}(k) - B_0^2 \cos^2(\theta) \hat{U}_i(k) - \nu' k^2 \hat{U}_i(k) + \hat{f}_i^r(k),$$

$$k_i \hat{U}_i(k) = 0,$$

where $\hat{U}_i(k)$, $\hat{P}(k)$, and $\hat{f}_i^r(k)$ are the Fourier transforms of the velocity, pressure, and force fields, respectively, and $\theta$ is the angle between wavenumber vector $\mathbf{k}$ and the external magnetic field $\mathbf{B}_0$.

The Reynolds number, which is the ratio of the nonlinear term to the viscous term, is a measure of nonlinearity in the flow. The interaction parameter, which is the ratio of the Lorentz force to the nonlinear term, quantifies the strength of the Lorentz force. The interaction parameter $N$ is defined as

$$N = \frac{B_0^2 L}{U_0^2},$$

where $U_0$ is the root mean square (rms) of the velocity defined\cite{15, 29} as

$$\frac{3}{2} U_0^2 = E = \int_0^\infty E(k) dk,$$

and $L$ is the the non-dimensional integral length scale defined as

$$L = \frac{\pi}{2U_0^2} \int_0^{k_{max}} \frac{E(k)}{k} dk,$$

where $E(k)$ is the one-dimensional energy spectrum. The energy equation corresponding to Eq. (5) is

$$\frac{\partial E(k)}{\partial T} = T(k) - 2B_0^2 \cos^2(\theta) E(k)$$

$$-2\nu' k^2 E(k) + F(k),$$

where $E(k) = |\hat{U}(k)|^2/2$, $F(k)$ is energy supply rate due to external forcing $\mathbf{f}'$, and $T(k)$ is the net nonlinear energy transfer rate to a mode $\mathbf{k}$. The energy equation contains two dissipative terms: the Joule dissipation rate

$$\epsilon_J(k) = 2B_0^2 \cos^2(\theta) E(k),$$

and viscous dissipation rate

$$\epsilon_v(k) = 2\nu' k^2 E(k).$$

The nonlinear interactions among the Fourier modes yield energy transfers among the modes. We quantify these transfers using energy flux, shell-to-shell and ring-to-ring energy transfers, etc. which will be described below.

B. Shell-to-shell and ring-to-ring energy transfers, and conical energy flux

We can study the energy transfers in the Fourier space in detail using the “mode-to-mode” energy transfer proposed by Dar et al.\cite{25} and Verma.\cite{26} For a triad $(\mathbf{k}, \mathbf{p}, \mathbf{q})$, \n
$$S(k|p|q) = 3\{[k \cdot \hat{U}(q)][\hat{U}^*(k) \cdot \hat{U}(p)]\},$$

is the mode-to-mode energy transfer rate from the mode $\mathbf{p}$ to the mode $\mathbf{k}$ with the mode $\mathbf{q}$ acting as a mediator.\cite{25, 26} Here, $\exists$ and * represent the imaginary part and the complex conjugate of a complex number, respectively. Note that $\mathbf{k} = \mathbf{p} + \mathbf{q}$.

The shell-to-shell energy transfer rate from all the modes in the $m^{th}$ shell to the modes in the $n^{th}$ shell is defined as

$$T_{mn} = \sum_{\mathbf{k} \in m} \sum_{\mathbf{p} \in m} S(k|p|q).$$

The shell-to-shell energy transfer provides an average energy transfer over all angles. To diagnose the angular dependence of the energy transfer, we divide the wavenumber shells into rings, as shown in Fig. 1. A ring is an intersection of a shell and a sector (see Fig. 2), hence it is characterized by $(m, \alpha)$, where $m$ denotes the shell index, and $\alpha$ represents the sector index. The ring-to-ring energy transfer rate from the ring $(m, \alpha)$ to the ring $(n, \beta)$ is\cite{27}

$$T_{mn}^{(m, \alpha)} = \sum_{\mathbf{k} \in (n, \beta)} \sum_{\mathbf{p} \in (m, \alpha)} S(k|p|q).$$

The ring-to-ring energy transfers are normalized using $A_i = [\cos(\theta_i) - \cos(\theta_{i+1})]$ to compensate for the uneven distribution of modes in the rings.\cite{27} The rings closer to the equator have more Fourier modes than those near the poles. Hence, we define a normalized ring energy transfer function as

$$\tilde{T}_{mn}^{(m, \alpha)} = \frac{1}{A_\alpha A_\beta} T_{mn}^{(m, \alpha)}.$$
transfers using the energy fluxes of the parallel and perpendicular components of the velocity field (see Appendix A). In brief, the energy equations for the perpendicular and parallel components of the velocity field are

\[
\frac{\partial E_{\perp}(k)}{\partial t} = \sum_{k=\text{p}+q} S_{\perp}(k|p|q) - 2B^2_0 \cos^2(\theta) E_{\perp}(k) + P_{\perp}(k)
- 2\nu' k^2 E_{\perp}(k) + \Re\{\hat{f}'_{\perp}(k) \cdot \hat{U}_{\perp}^*(k)\},
\]

\[
\frac{\partial E_{\parallel}(k)}{\partial t} = \sum_{k=\text{p}+q} S_{\parallel}(k|p|q) - 2B^2_0 \cos^2(\theta) E_{\parallel}(k) + P_{\parallel}(k)
- 2\nu' k^2 E_{\parallel}(k) + \Re\{\hat{f}'_{\parallel}(k) \hat{U}_{\parallel}^*(k)\},
\]

respectively, where \(E_{\perp}(k) = \frac{1}{2} |\hat{U}_{\perp}(k)|^2\) and \(E_{\parallel}(k) = \frac{1}{2} |\hat{U}_{\parallel}(k)|^2\) are the energies of the perpendicular and parallel components of the velocity field, respectively, and

\[
S_{\perp}(k|p|q) = \Im\{[k \cdot \hat{U}(q)][\hat{U}_{\perp}^*(k) \cdot \hat{U}_{\perp}(p)]\},
\]

\[
S_{\parallel}(k|p|q) = \Im\{[k \cdot \hat{U}(q)][\hat{U}_{\parallel}^*(k) \hat{U}_{\parallel}(p)]\},
\]

\[
P_{\perp}(k) = \Im\{[\hat{k} \cdot \hat{U}_{\perp}^*(k)] \hat{P}(k)\},
\]

\[
P_{\parallel}(k) = \Im\{[\hat{k} \cdot \hat{U}_{\parallel}^*(k)] \hat{P}(k)\},
\]

and \(\Re, \Im, \ast\) represent the real and imaginary parts, and the complex conjugate a complex number, respectively. In the above equations we have replaced \(k'\) and \(\hat{U}(k')\) in the equations of Appendix A with \(-k\) and \(\hat{U}^*(k)\) respectively. Also note that Eqs. (25,26) and the condition \(k \cdot \hat{U}(k) = 0\) imply that

\[
P_{\perp}(k) = -P_{\parallel}(k).
\]

We interpret the above result as following. The energy gained by the perpendicular component \(\hat{U}_{\perp}^*(k)\) via pressure is equal and opposite to the energy lost by the parallel component. The magnitude of the transfer to the parallel component via pressure is given by Eq. (25). Thus pressure facilitates energy transfers between the parallel and perpendicular components of the velocity field. Note that there is no direct energy transfer between \(\hat{U}_{\perp}\) and \(\hat{U}_{\parallel}\).

The energy flux \(\Pi_{\perp}(k_0)\) for the perpendicular component of the velocity field for a wavenumber sphere of radius \(k_0\) is defined as the net energy transfer from the modes \(U_{\perp}(p)\) residing inside the sphere to the modes \(U_{\perp}(k)\) outside the sphere, i.e.,

\[
\Pi_{\perp}(k_0) = \sum_{|k| \geq k_0} \sum_{|p| < k_0} S_{\perp}(k|p|q).
\]

A similar formula for the flux of the parallel velocity component, \(\Pi_{\parallel}(k_0)\), is

\[
\Pi_{\parallel}(k_0) = \sum_{|k| \geq k_0} \sum_{|p| < k_0} S_{\parallel}(k|p|q).
\]

We will compute these quantities using our simulation data.

In the following section, we describe the details of simulation method employed for the present study.
III. DETAILS OF NUMERICAL SIMULATIONS

We use pseudo-spectral code Tarang\textsuperscript{30} to solve the non-dimensional quasi-static MHD equations (Eqs. (3) and (4)) in a cubical box on a 256\textsuperscript{3} grid. Periodic boundary conditions are applied in all the three directions. We use the fourth-order Runge-Kutta method for time-stepping, Courant-Friedrichs-Lewy (CFL) condition for calculating time-step ($\Delta t$), and the $3/2$ rule for dealiasing.\textsuperscript{31,32} We start our simulation for $N = 0$ using a model energy spectrum\textsuperscript{33} as the initial condition:

$$E(k) = C \epsilon^{2/3} k^{-5/3} f_L(kL) f_\eta(k\eta),$$

(30)

with the Kolmogorov constant $C = 1.5$, and the energy supply rate $\epsilon = 1.0$. $f_L, f_\eta$ are defined as

$$f_L(kL) = \left(\frac{kL}{(kL)^2 + c_L}\right)^{5/3+p_0},$$

(31)

$$f_\eta(k\eta) = \exp(-\beta k\eta),$$

(32)

where $c_L = 1.5$, $p_0 = 2$ and $\beta = 5.2$. The initial phases of the velocity Fourier modes are randomly generated.

In order to achieve a steady-state, the velocity field is randomly forced using a scheme similar to that followed by Burattini et al.,\textsuperscript{15} Vorobev et al.,\textsuperscript{29} and Carati et al.,\textsuperscript{34} which is,

$$\hat{\mathbf{F}}(\mathbf{k}) = \gamma(\mathbf{k}) \hat{\mathbf{U}}(\mathbf{k}),$$

\begin{equation}
\gamma(\mathbf{k}) = \frac{\epsilon_{in}}{n_f \hat{U}(\mathbf{k}) \cdot \hat{\mathbf{U}}(\mathbf{k})},
\end{equation}

(33)

where $n_f$ is total number of modes inside the forcing wavenumber band. We choose the energy input rate $\epsilon_{in} = 0.016$, and the forcing band as $1 \leq |k| \leq 3$ for the shell-to-shell, ring-to-ring, and conical flux studies. However, we choose the forcing band as $8 \leq |k| \leq 9$ with $\epsilon_{in} = 0.072$, for the computation of the energy fluxes of the parallel and perpendicular components of the velocity field.

| $B'_0$ | $k_f$ | $N$ | $N_0$ | $U'$ | $\tau$ | $k_{max}$ | $\eta$ |
|-------|-------|-----|-------|------|-------|---------|--------|
| 2.29  | [1,3] | 1.7 | 1.0   | 0.39 | 0.32  | 2.4     |
| 3.60  | [1,3] | 5.5 | 2.5   | 0.35 | 0.43  | 2.8     |
| 5.15  | [1,3] | 11  | 5.0   | 0.39 | 0.39  | 2.9     |
| 6.26  | [1,3] | 14  | 7.5   | 0.45 | 0.37  | 2.9     |
| 7.28  | [1,3] | 18  | 10.0  | 0.51 | 0.33  | 2.8     |
| 10.23 | [1,3] | 27  | 20.0  | 0.65 | 0.26  | 2.6     |
| 25.1  | [1,3] | 130 | –     | 0.86 | 0.21  | 2.4     |
| 32.6  | [1,3] | 220 | –     | 0.87 | 0.21  | 2.4     |
| 19.6  | [8,9] | 100 | 30    | 0.64 | 0.26  | 2.1     |

IV. NUMERICAL RESULTS

We compute various energy transfer rates for $N = 1.7, 5.5, 11, 14, 18, 27, 130$, and 220. A detailed description of each transfer is described in the following subsections.

A. Anisotropic energy spectrum

The external magnetic field induces a strong anisotropy in the flow. A systematic study of anisotropic energy spectrum for various $N$’s have been presented in Reddy and Verma.\textsuperscript{18} In Fig. 4, we exhibit the density and contour plots of the energy spectra for $N = 18$ and 130. These figures illustrate the energy concentrated near the equator,\textsuperscript{15,36,37} but there is a significant energy
away from the equator. This is the essential nature of quasi two-dimensional quasi-static MHD at high interaction parameters.

In the next subsection, we will investigate how energy exchange takes place among the Fourier modes.

![Image](image1)

**FIG. 4.** Density (left) and contour (right) plots of the energy spectrum for: (a) \(N = 18\) and (b) \(N = 130\).

**B. Shell-to-shell energy transfers**

In Fig. 5, we present the shell-to-shell energy transfer rates for \(N = 1.7, 11, 18,\) and \(130\). We observe that the \(n^{th}\) shell gives energy to the \((n+l)^{th}\) shells \((l > 0)\), and it receives energy from the \((n-l)^{th}\) shells. Thus, the shell-to-shell energy transfer for quasi-static MHD is forward. We also observe that the maximum energy transfer is to the nearest neighbor, i.e., the \(n^{th}\) shell gives maximum positive energy transfer to the \((n+1)^{th}\) shell, and maximum negative energy to the \((n-1)^{th}\) shell. Hence, the shell-to-shell energy transfer is also local. Our results are consistent with those of Burattini et al.\(^{15}\)

![Image](image2)

**FIG. 5.** Forward and local shell-to-shell energy transfer rates \(T^{i,j}_{m,n}\) for: (a) \(N = 1.7\), (b) \(N = 11\), (c) \(N = 18\), and (d) \(N = 130\). Here, \(m\) and \(n\) are the giver and receiver shells, respectively, and \(k\) is the wavenumber of the outer radius of the corresponding shell.

**C. Ring-to-ring energy transfers**

The angular dependence of the energy transfers can be computed using the ring-to-ring transfers. In Figs. 6, 7 and 8, we illustrate the normalized ring-to-ring energy transfers \(T^{(m,n)}_{(\alpha,\beta)}\) from the rings of the \(9^{th}\) shell \((m = 9)\) to the rings of the shells \(n = 9, 10,\) and \(8\), respectively. This analysis has been performed for \(N = 1.7, 11, 18,\) and \(130\). In these figures, the vertical axis represents the sector index of the giver ring \((\alpha)\), while the horizontal axis represents the sector index for the receiver ring \((\beta)\).

First, we discuss \(T^{(9,\alpha)}_{(9,\beta)}\), i.e., the energy transfers among the rings with shell index 9. Figure 6 shows that the energy transfer from the ring \(\alpha\) to the ring \((\alpha - 1)\) is positive \((T^{(9,\alpha)}_{(9,\alpha-1)} > 0)\), while that from the ring \(\alpha\) to the ring \((\alpha + 1)\) is negative \((T^{(9,\alpha)}_{(9,\alpha+1)} < 0)\). Hence, the ring-to-ring energy transfer within a shell is from the equatorial region to the polar region. Among the rings,
the most significant energy transfers occur between the neighboring rings, i.e., from a ring with index $\alpha$ to the rings with index $\alpha \pm 1$. Hence, the energy transfer is local in the angular direction as well. Another important conclusion that can be drawn from the above computation is that for large $N$ ($N = 11, 18, 130$), the dominant energy transfers take place from the rings closer to the equator to their neighbors (lower $\theta$).

Figure 7 illustrates $T_{(10,\beta)}^{(9,\alpha)}$, i.e., the energy transfers from the rings in the 9th shell to those in the 10th shell. The figure shows that $T_{(10,\beta)}^{(9,\alpha)} > 0$, and that they are most dominant for the equatorial rings ($\alpha, \beta \approx 15$). Since $T_{(10,\beta)}^{(9,\alpha)}$ dominates for $\alpha = \beta$, we conclude that the energy is transferred dominantly along a sector near the equator. Hence, the energy transfers are forward along the sectors as well. This feature is reinforced by $T_{(8,\beta)}^{(9,\alpha)}$, illustrated in Fig. 8, where we observe a negative energy being transferred diagonally from the rings of shell 9 to the rings of shell 8. Thus, the ring-to-ring transfers are local and forward. For large $N$, these transfers tend to be dominant near the equator.

In the next subsection, we will describe conical energy flux.

D. Conical Energy Flux

We can integrate the ring energy transfers over sectors and compute the conical energy flux [see Eq. (17)]. This quantity describes the energy flux leaving a cone in the Fourier space (see Fig. 3). In Fig. 9, we plot the normalized flux $\Pi(\theta)/\max(\Pi(\theta))$. The figure shows that for $N = 1.7$ to 130, the above flux is negative, indicating that the energy is transferred from the modes outside the cone to the modes inside the cone. Note that $\Pi(\theta)/\max(\Pi(\theta))$ is monotonic, except for $N = 1.7$ (due to the relatively weak magnetic field). We also observe that the maximal energy transfer takes place for the cone with a semi-vertical angle $\theta \approx \pi/2$. Hence, the modes near the equatorial region transfer maximal energy towards the regions of smaller $\theta$. This energy gets dissipated by Joule heating, as well as it trickles down to the polar region.

In Fig. 10, we plot the net energy transferred from the cone with the largest semi-vertical angle

$$\Pi_{eq} = \sum_{\theta_0 < \pi/2} \sum_{s^2 \geq \pi/2} S(\mathbf{k} | p | q).$$  \hspace{1cm} (35)$$

The quantity $-\Pi_{eq}$ quantifies the energy transfer from the equatorial region to the modes inside the largest cone. The figure indicates that $|\Pi_{eq}|$ decreases very sharply with $N$ and follows $|\Pi_{eq}(N)| \propto N^{-1.2}$.

The decrease in $|\Pi_{eq}|$ can be understood qualitatively using the energy distribution in the Fourier space. In Fig. 11, we plot the total energy and energy contained in the equatorial region. The remaining energy, $E_{non-eq} = E - E_{eq}$, is also plotted in the figure. We find that $E_{non-eq}$ decreases sharply with $N$ ($E_{non-eq} \propto N^{-1.8}$). Since the energy flux is a sum of $E(p) E(q)$, $E(k) E(p)$ and $E(k) E(q)$, apart from some other factors (here $k = p + q$), and the receiver energy spectrum $E_{eq} \propto N^{-1.8}$, it is reasonable that the conical energy flux $|\Pi_{eq}|$ decreases very sharply. Thus, we provide a qualitative explanation for the sharp decline of $|\Pi_{eq}|$ with the interaction parameter. This observation also

FIG. 7. Local ring-to-ring energy transfers $T_{(10,\beta)}^{(9,\alpha)}$ from the rings of the 9th shell to the rings of the 10th shell for: (a) $N = 1.7$, (b) $N = 11$, (c) $N = 18$ and (d) $N = 130$. Note that $T_{(10,\beta)}^{(9,\alpha)} > 0$.

FIG. 8. Local ring-to-ring energy transfers $T_{(8,\beta)}^{(9,\alpha)}$ from the rings of the 9th shell to the rings of the 8th shell for: (a) $N = 1.7$, (b) $N = 11$, (c) $N = 18$ and (d) $N = 130$. Note that $T_{(8,\beta)}^{(9,\alpha)} < 0$. 
explains why quasi-static MHD is quasi-two-dimensional for large \( N \).

E. Energy fluxes of the parallel and perpendicular components

Many experiments\(^3,4\) and numerical simulations\(^6,7,18\) indicate that quasi-static MHD exhibits quasi-two-dimensional behavior for large \( N \). To probe the physics of energy transfers for large \( N \) in detail, we perform a numerical simulation for \( N = 100 \) with forcing applied at intermediate length scales (8.0 \( \leq |k_f| \leq 9.0 \)) to resolve the inverse and forward cascade regimes. We take the final state of hydrodynamic simulation as an initial condition (see Sec. III) and apply an external magnetic field. The simulation is carried out till a final (quasi-steady) state is reached, which occurs at \( t_\text{final} \approx 400 \). The Joule dissipation, which is active at all scales, balances the energy growth due to the inverse cascade.

In Fig. 12, we plot the energy spectrum of the parallel and perpendicular components of the velocity field for \( N = 100 \). The figure indicates that \( E_\perp \gg 2E_\parallel \) for \( k < k_f \), but \( E_\perp \ll 2E_\parallel \) for \( k > k_f \). We also observe that \( E_\perp(k) \) follows \( k^{-5/3} \) for \( k < k_f \). This feature demonstrates the quasi-two-dimensionalization of quasi-static MHD turbulence at high interaction parameters in periodic domains. Our results are consistent with those of Favier et al.\(^7\) To probe the physics of the flow further, we compute the energy fluxes of the parallel and perpendicular components of the velocity field.

Figure 13 exhibits the energy fluxes for the parallel and perpendicular components of the velocity field (\( \Pi_\parallel \) and \( \Pi_\perp \), respectively). We observe that the \( k < k_f \) and \( k > k_f \) regions are dominated by the \( \Pi_\perp \) and \( \Pi_\parallel \) fluxes, respectively. The dominance of the negative energy flux for \( \Pi_\perp \) in \( k < k_f \) is consistent with the dominance of the inverse cascade of \( U_\perp \), while \( \Pi_\parallel > \Pi_\perp \) in the \( k > k_f \) region indicates the dominance of the forward cascade for \( U_\parallel \). The aforementioned energy flux computations are consistent with the simulation results that \( E_\perp(k) \gg E_\parallel(k) \) for lower wavenumbers, and \( E_\perp(k) \ll E_\parallel(k) \) for higher wavenumbers (see Fig. 12), which is consistent with the quasi-two-dimensional nature of quasi-static MHD turbulence at high interaction parameters.

In Fig. 13, we also plot \( P_\parallel(k) \), which is the energy transferred to \( U_\parallel(k) \) from \( U_\perp(k) \) via pressure. We observe that \( P_\parallel(k) \) is positive for \( k \geq k_f \). Hence, \( U_\parallel(k) \) receives energy from \( U_\perp(k) \), which is consistent with the nature of the energy fluxes \( \Pi_\parallel \) and \( \Pi_\perp \) described above.
FIG. 12. Plots of $E_{\perp}(k)$ and $2E_{\perp}(k)$ for $N = 100$. $E_{\perp}(k) > E_{\parallel}(k)$ for $k < k_f$, with $E_{\perp}(k) \sim k^{-5/3}$, but $E_{\perp}(k) < E_{\parallel}(k)$ for $k > k_f$. The shaded region exhibits the forcing band $k_f \in [8, 9]$.

FIG. 13. Plots of the energy fluxes $\Pi(k), \Pi_{\perp}(k), \Pi_{\parallel}(k)$, and $P_{\perp}(k)$ for $N = 100$. $\Pi_{\perp}(k) < 0$ for $k < k_f$, indicating an inverse cascade for $U_{\perp}$, while $\Pi_{\parallel}(k) > 0$ for $k > k_f$, indicating a forward cascade for $U_{\parallel}$. $P_{\perp}(k) > 0$ for $k > k_f$, indicating an energy transfer from $U_{\perp}$ to $U_{\parallel}$ via pressure.

F. Dissipation rates

The aforementioned preferential energy transfer from the equatorial region to the polar region can be understood using the distribution of the Joule dissipation $\epsilon_J$, which is proportional to $(\cos^2 \theta)E(k)$ [see Eq. (11)]. Clearly, $\epsilon_J$ vanishes at the equatorial plane, where $\theta = \pi/2$. However, $E(k)$ increases monotonically with $\theta$. As a result, the Joule dissipation $\epsilon_J$ reaches a maximum near $\theta \approx \pi/2$, but not at $\theta = \pi/2$ itself. To maintain a steady state, $\epsilon_J$ is balanced by a nonlinear energy transfer from the equatorial region. This is the reason why the energy flows maximally from the equator towards the polar region (see Figs. 6 and 9).

For large $N$, $E(k)$ is concentrated near the equator. Therefore, $\epsilon_J$ peaks near $\theta = \pi/2$. As a result, the ring-to-ring energy transfers are localized near the equator, as exhibited in Figs. 6(c,d), 7(c,d), and 8(c,d). These results are consistent with the quasi-two-dimensional behavior of the quasi-static MHD flow for large $N$.

Lastly we study the viscous and Joule dissipation rates for a large interaction parameter, here $N = 27$. Since

$$\frac{\epsilon_v(k, \theta)}{\epsilon_J(k, \theta)} = \frac{2\nu'k^2E(k)}{2B_0^2\cos^2 \theta E(k)} = \frac{2\nu'k^2}{2B_0^2\cos^2 \theta}$$

(36)

$\epsilon_J(k, \theta)$ dominates $\epsilon_v(k, \theta)$ for

$$k < k_* = \frac{B_0 \cos \theta}{\sqrt{\nu'}}$$

(37)

and vice versa. This is expected, since the Joule dissipation is active at all wavenumbers, but the viscous dissipation acts strongly only at large wavenumbers. In Fig. 14, we plot $\epsilon_v(k, \theta)/\epsilon_J(k, \theta)$ as a function of the wavenumber $k$ for various sectors. The mean angles of the chosen sectors are $\theta = 0.05, 0.48, 0.99, \text{ and } 1.41$.

FIG. 14. For $N = 27$, $\epsilon_v(k, \theta)/\epsilon_J(k, \theta)$ vs. $k$ for various sectors. $\epsilon_v(k, \theta)/\epsilon_J(k, \theta) \sim k^2$.

For a given sectorial angle $\theta$, the ratio $\epsilon_v(k, \theta)/\epsilon_J(k, \theta) \propto k^2$ because the viscous dissipation is proportional to $k^2$. Consequently, the Joule dissipation dominates at small wavenumbers, but the viscous dissipation takes over at large wavenumbers. For a given wavenumber $k$, the ratio $\epsilon_v(k, \theta)/\epsilon_J(k, \theta) \propto 1/\cos^2 \theta$; or, $\epsilon_v(k, \theta) \gg \epsilon_J(k, \theta)$ for the equatorial region ($\theta \approx \pi/2$) and vice versa for the polar region ($\theta \approx 0$). The figure also indicates that the transition wavenumber $k_*$ decreases with increasing $\theta$, which is consistent with Eq. (37).

Our results are schematically illustrated in Fig. 15. The energy of the perpendicular component of the velocity cascades to smaller wavenumbers, while the energy of the parallel component cascades to larger wavenumbers.
where it gets depleted by the Joule dissipation via energy cascades to the polar region.

V. CONCLUSIONS

Earlier experiments and numerical simulations revealed that quasi-static MHD exhibits quasi-two-dimensional behavior at high interaction parameters.\textsuperscript{3,6,7} In this paper, we have studied the energy transfer mechanisms operating in quasi-static MHD and show them to be consistent with the aforementioned anisotropic energy distribution. Here, we have studied the shell-to-shell and ring-to-ring energy transfers, as well as the conical flux. We have also studied the energy fluxes of the parallel and perpendicular components of the velocity field. For most of our runs, our forcing wavenumber band lies in the small-wavenumber regime.

The main results of our paper are:

1. We have developed a formalism to compute the conical energy transfer. We also provided a scheme to compute the energy fluxes for the parallel and perpendicular components of the velocity field.

2. Earlier, Burattini et al.\textsuperscript{15} showed that the shell-to-shell energy transfer is local. In this paper, we show that the ring-to-ring energy transfers are forward and local, both in wavenumber shells and angles. Within a shell, the ring-to-ring transfers are from higher polar angles to lower polar angles (i.e., from the equatorial region to the polar region). For the rings across shells, it is dominantly along the same sector or neighboring sectors.

3. When the flow is forced at an intermediate wavenumber band, for large $N$, we observe that the inverse cascade at low wavenumbers is dominated by the negative energy flux of the perpendicular component of velocity, while the forward cascade at large wavenumbers is dominated by the positive energy flux of the parallel component.

In conclusion, the energy transfers in quasi-static MHD provide valuable insights into the physics of the flow. The energy transfers in quasi-static MHD have similarities with full MHD, rotating, and stratified turbulence. Hence the tools developed in the present paper may be useful for such studies.

ACKNOWLEDGMENTS

We thank D. Carati, B. Knaepen, B. Teaca, P. Perlekar, P. Satyamurthy, and D. Biswas for useful discussions and the anonymous referee for helpful comments. RK thanks S. V. G. Menon, former Head, Theoretical Physics Division BARC for the encouragement and support. This work was supported by Board of Research in Nuclear Science, Department of Atomic Energy, Govt. of India through research grant 2009/36/81-BRNS. All the simulations were performed on the HPC system and Chaos cluster of IIT Kanpur.

APPENDIX A: MODE-TO-MODE ENERGY TRANSFERS FOR THE PERPENDICULAR AND PARALLEL COMPONENTS OF THE VELOCITY FIELD

In this appendix, we derive formulas for the energy transfers for the perpendicular and parallel components of the velocity field. We focus on a triad $(k, p, q)$ under the limit $\nu = 0$ and $B_0 = 0$. Note that $k' + p + q = 0$, and $k' = -k$.

Following Dar et al.\textsuperscript{25} and Verma,\textsuperscript{26} we derive the following equations from Eqs. (3,4):

$$\frac{\partial E_\perp(k')}{\partial t} = S_\perp(k'|p,q) + S_\perp(q'|p) + P_\perp(k'), \quad (38)$$

$$\frac{\partial E_\parallel(k')}{\partial t} = S_\parallel(k'|p,q) + S_\parallel(q'|p) + P_\parallel(k'), \quad (39)$$

where $E_\perp(k) = E_\perp(k') = \frac{1}{2} |\hat{\bf U}_\perp(k)|^2$ and $E_\parallel(k) = E_\parallel(k') = \frac{1}{2} |\hat{\bf U}_\parallel(k)|^2$ are the energies of the perpendicular and parallel components of the velocity field, respectively, and

$$S_\perp(k'|p,q) = -3 \{ [k' \cdot \hat{\bf U}_\perp(q)][\hat{\bf U}_\perp(k') \cdot \hat{\bf U}_\perp(p)] \}, \quad (40)$$

$$S_\parallel(k'|p,q) = -3 \{ [k' \cdot \hat{\bf U}_\parallel(q)][\hat{\bf U}_\parallel(k') \cdot \hat{\bf U}_\parallel(p)] \}, \quad (41)$$

$$P_\perp(k') = -3 \{ |k'| [\hat{\bf U}_\perp(k') \cdot \hat{\bf U}_\perp(p)] \}, \quad (42)$$

$$P_\parallel(k') = -3 \{ |k'| [\hat{\bf U}_\parallel(k') \cdot \hat{\bf U}_\parallel(p)] \}, \quad (43)$$

where $\Re$, $\Im$, * represent the real part and imaginary part, and the complex conjugate of a complex number, respectively. Equations (38,39) indicate that the mode $k'$ receives energy from modes $p$ and $q$. Similarly, we can also...
Using the above, we can conclude that:

\[
\frac{\partial E_\perp(p)}{\partial t} = S_\perp(p|q'k') + S_\perp(p'k'|q) + P_\perp(p),
\]

\[
\frac{\partial E_\parallel(p)}{\partial t} = S_\parallel(p|q'k') + S_\parallel(p'k'|q) + P_\parallel(p),
\]

\[
\frac{\partial E_\perp(q)}{\partial t} = S_\perp(q'k'|q) + S_\perp(q|p'k') + P_\perp(q),
\]

\[
\frac{\partial E_\parallel(q)}{\partial t} = S_\parallel(q|k'|q) + S_\parallel(q|p'k') + P_\parallel(q).
\]

Using \( \mathbf{k} \cdot \hat{U}(\mathbf{k}) = 0 \), we can show that:

\[
P_\perp(k') + P_\parallel(k') = 0,
\]

\[
S_\perp(k'|p|q) = -S_\perp(p|k'|q),
\]

\[
S_\parallel(k'|p|q) = -S_\parallel(p|k'|q).
\]

Using the above, we can conclude that:

\[
\frac{\partial}{\partial t} [E_\perp(k') + E_\perp(p) + E_\perp(q)] = P_\perp(k') + P_\parallel(p) + P_\perp(q),
\]

\[
\frac{\partial}{\partial t} [E_\parallel(k') + E_\parallel(p) + E_\parallel(q)] = -[P_\perp(k') + P_\parallel(p) + P_\perp(q)].
\]

Therefore, we can make the following conclusions regarding the energy transfers for the parallel and perpendicular components of the velocity field:

1. The sum of Eqs. (51, 52) shows that the total energy (sum of the perpendicular and parallel components) for a triad is conserved. However, there is an energy transfer between the perpendicular and parallel components via pressure.

2. The perpendicular component \( \hat{\mathbf{U}}_\perp(k') \) receives energy by an amount \( S_\perp(k'|p|q) \) from \( \hat{\mathbf{U}}_\perp(p) \) with \( \hat{\mathbf{U}}(q) \) as a mediator. Symmetrically, it also receives energy by an amount \( S_\perp(k'|q|p) \) from \( \hat{\mathbf{U}}_\perp(q) \) via \( \hat{\mathbf{U}}(p) \).

3. Equation (38) implies that the perpendicular component \( \hat{\mathbf{U}}_\perp(k') \) gains energy from the \( P_\perp(k') \) term, which arises due to the pressure. Since \( P_\perp(k') = -P_\parallel(k') \), the energy gained by \( \hat{\mathbf{U}}_\perp(k') \) via pressure is the same as the energy lost by \( \hat{\mathbf{U}}_\parallel(k) \) (see Eq. (39)). Hence, the energy transfer between the parallel and perpendicular components occurs via pressure.

We use these formulas to compute the energy fluxes of the perpendicular and parallel components of the velocity field.

1. P. H. Roberts, An Introduction to Magnetohydrodynamics (Elsevier, New York, 1967).
2. B. Knaepen and R. Moreau, Ann. Rev. Fluid Mech. 40, 25 (2008).
3. A. Alemany, R. Moreau, P. L. Sulem, and U. Frisch, J. Méc. 18, 277 (1979).
4. V. Kolesnikov and A. Tsinob, Fluid Dynamics 9, 621 (1974).
5. U. Schumann, J. Fluid Mech. 74, 31 (1976).
6. O. Zikanov and A. Thess, J. Fluid Mech. 358, 299 (1998).
7. B. Favier, F. S. Godeferd, C. Cambon, and A. Delache, Phys. Fluids 22, 075104 (2010).
8. H. K. Moffatt, J. Fluid Mech. 28, 571 (1967).
9. M. Kits and A. A. Tsinob, Magnitnaya Gidrodinamika 3, 27 (1971).
10. R. Moreau, Magnetohydrodynamics (Kluwer Academic Publishers, Dordrecht, 1990).
11. J. Sommeria and R. Moreau, J. Fluid Mech. 118, 507 (1982).
12. R. Klein and A. Pothérat, Phys. Rev. Lett. 104, 034502 (2010).
13. A. Pothérat and R. Klein, arXiv:1305.7105 (2013).
14. A. Pothérat, Magnetohydrodynamics 48, 13 (2012).
15. P. Burattini, M. Kinet, D. Carati, and B. Knaepen, Physica D 237, 2062 (2008).
16. P. Burattini, M. Kinet, D. Carati, and B. Knaepen, Phys. Fluids 20, 065110 (2008).
17. B. Favier, F. S. Godeferd, C. Cambon, A. Delache, and W. J. T. Bos, J. Fluid Mech. 681, 434 (2011).
18. K. S. Reddy and M. K. Verma, Phys. Fluids 26, 025109 (2014).
19. H. Branover, A. Eidelmann, M. Nagorny, and M. Kireev, Progress in Turbulence Research 162, 64 (1994).
20. Eckert, G. Gerbenth, W. Witke, and H. Langenbrunner, International Journal of Heat and Fluid Flow 22, 358 (2001).
21. Y. Dymkou and A. Pothérat, Theoretical and Computational Fluid Dynamics 23, 535 (2009).
22. K. Kornet and A. Pothérat, arXiv:1403.4129 (2014).
23. T. Boeck, D. Krasnov, A. Thess, and O. Zikanov, Phys. Rev. Lett. 101, 244501 (2008).
24. M. Lesieur, Turbulence in Fluids (Kluwer Academic, Dordrecht, 1990).
25. G. Dar, M. Verma, and V. Eswaran, Physica D 157, 207 (2001).
26. M. K. Verma, Phys. Rep. 401, 229 (2004).
27. B. Teaca, M. K. Verma, B. Knaepen, and D. Carati, Phys. Rev. E 79, 046312 (2009).
28. B. Knaepen, S. Kassinos, and D. Carati, J. Fluid Mech. 513, 199 (2004).
29. A. Vorobee, O. Zikanov, P. A. Davidson, and B. Knaepen, Phys. Fluids 17, 125105 (2005).
30. M. K. Verma, A. Chatterjee, K. S. Reddy, K. Y. Yadav, S. Paul, M. Chandra, and R. Santaney, Pramana 81, 617 (2013).
31. C. Canuto, M. Y. Hussaini, A. Quarteroni, and T. A. Zhang, Spectral Methods in Fluid Turbulence (Springer-Verlag, Berlin, 1998).
32. I. P. Boyd, Chebyshev and Fourier Spectral Methods (Dover Publishers, New York, 2001).
33. B. Pope, Turbulent Flows (Cambridge University Press, Cambridge, UK, 2000).
34. D. Carati, S. Ghosal, and P. Moin, Phys. Fluids 7, 606 (1995).
35. J. Jiménez, A. A. Wray, P. G. Saffman, and R. S. Rogallo, J. Fluid Mech. 255, 65 (1993).
36. C. Casper and A. Alemany, J. Mech. Theor. Appl. 4, 175 (1985).
37. A. Pothérat and V. Dymkou, J. Fluid Mech. 655, 174 (2010).