GENERALIZATION OF SPECIAL FUNCTIONS AND ITS APPLICATIONS TO
MULTIPLICATIVE AND ORDINARY FRACTIONAL DERIVATIVES

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Abstract

The goal of this paper is to extend the classical and multiplicative fractional derivatives. For this purpose, it is introduced the new extended modified Bessel function and also given an important relation between this new function $I_\nu(q;x)$ and the confluent hypergeometric function $1F_1(\alpha, \beta, x)$. Besides, it is used to generalize the hypergeometric, the confluent hypergeometric and the extended beta functions by using the new extended modified Bessel function. Also, the asymptotic formulae and the generating function of the extended modified Bessel function are obtained. The extensions of classical and multiplicative fractional derivatives are defined via extended modified Bessel function and, first time the fractional derivative of rational functions is explicitly given via complex partial fraction decomposition.

1 Introduction

1.1 Generalized Special Functions

Especially, in the last two decades, several generalizations of the well-known special functions have been studied by different authors. In 1997, Chaudhry [8]...
have introduced the extension of Euler’s beta function by
\[
B_p(x, y) = \int_0^1 t^{x-1}(1 - t)^{y-1} e^{-\frac{p}{t(1-t)}} dt.
\] (1)
\[(Re(p) > 0, Re(x) > 0, Re(y) > 0)\]

It is clear that the special case \(p = 0\) gives the Euler’s beta function \(B_0(x, y) = B(x, y)\).

Then, the authors in (24) extended beta functions and hypergeometric functions as
\[
B_p^{(\alpha, \beta)}(x, y) = \int_0^1 t^{x-1}(1 - t)^{y-1} F_1(\alpha; \beta; \frac{-p}{t(1-t)}) dt
\] (2)
\[(Re(p) > 0, Re(x) > 0, Re(y) > 0, Re(\alpha) > 0, Re(\beta) > 0)\]

Lee et al. in (18) introduced the more generalized Beta type function as follows:
\[
B_p^{(\alpha, \beta; m)}(x, y) = \int_0^1 t^{x-1}(1 - t)^{y-1} F_1(\alpha; \beta; \frac{-p}{t^m(1-t)^m}) dt.
\] (3)
\[(Re(p) > 0, Re(x) > 0, Re(y) > 0, Re(\alpha) > 0, Re(\beta) > 0)\]

Consequently, Luo et. al. in (19) generalized extended beta function (3) (as well as (1) and (2)) by introducing
\[
B_{b; \rho; \lambda}^{(\alpha, \beta)}(x, y) = \int_0^1 t^{x-1}(1 - t)^{y-1} F_1(\alpha; \beta; \frac{-b}{t^\rho (1-t)^\lambda}) dt.
\] (4)
\[(\rho \geq 0, \lambda \geq 0, \min\{Re(\alpha), Re(\beta)\} > 0, Re(x) > -Re(\rho \alpha), Re(y) > -Re(\lambda \alpha))\]

Recently, Parmar in (30) introduced very interesting special function consisting Bessel function of second kind as
\[
B_v(x, y; p) = \sqrt{\frac{2p}{\pi}} \int_0^1 t^{x-\frac{3}{2}}(1 - t)^{y-\frac{3}{2}} K_v^{\frac{1}{2}}(\frac{p}{t(1-t)^{\frac{1}{2}}}) dt.
\] (5)
\[(Re(p) > 0)\]

Finally, we refer the papers (21) and (23) for more properties of extended Gauss hypergeometric and extended confluent hypergeometric functions.

In this paper, we introduce extended special functions as generalizations of modified Bessel Functions, Beta functions, hypergeometric functions and confluent hypergeometric functions. We would like to mention an interesting remark from Qadir (31) who explains the importance of generalization of the special functions as ”Notice that the generalization of the other special functions has proved even more useful than the separate special functions themselves”. We refer the paper (31) for more details about generalization of the special functions.

Consequently, we define extended fractional derivative and extended multiplicative fractional derivative.
1.2 Multiplicative (geometric) Calculus and Fractional Derivatives

Multiplicative calculus has improved rapidly over the past 10 years. In this period, superiority of the multiplicative calculus over ordinary calculus was proved by many studies. The most significant among these studies are \[15\] in Biomedical Image Analysis, \[6\] in complex analysis, \[14\] in growth phenomena, \[20\], \[32\], \[28\] and \[25\] in numerical analysis, \[4\] in actuarial science, finance, demography etc., \[13\] in biology, recently \[29\] in accounting. In order for multiplicative calculus to be used efficiently in all respects more studies needs to be done in various field. Recently, multiplicative Laplace transform

\[
L_m \{ f(t) \} = \exp \left( \int_0^\infty e^{-st \ln f(t)} dt \right)
\]

\((f(t) \in [0, \infty)).\)

has been introduced and applied in optics in \[33\].

First application of the multiplicative calculus to fractional derivative is executed by Abdeljawad and Grossman in \[1\]. In this paper, the Riemann-Liouville fractional integral of order \(\alpha \in \mathbb{C}\) has been defined as

\[
(aI^\alpha_x f)(x) = e^{\frac{x}{(x-t)^{\alpha-1}(\ln \circ f)(x)}} \int_a^x (x-t)^{\alpha-1} \ln f(x) dx, x > a.
\]

In this article multiplicative fractional derivative \((7)\) is extended. The connection between multiplicative and ordinary fractional derivatives is presented with assertion of some essential characteristics of fractional derivative. As a result, the generalized ordinary fractional derivative introduced in section 3 is applied to introduce multiplicative generalized fractional derivative.

2 Extension of Special Functions

In this section, we introduce special functions which will be generalization of the functions \((1)-(5)\).

2.1 Extended Modified Bessel Function

We here introduce new extended of modified Bessel functions as follows.

**Definition 1** The function

\[
I_v(q; x) = \frac{(\frac{q}{\pi})^v}{\sqrt{\pi} \Gamma(v + \frac{1}{2})} \int_{-1}^{1} (1 - t^2)^{v-\frac{1}{2}} (1 - t)^{q-\frac{1}{2}} \exp(-x (t - 1)) dt, (8)
\]

\((\text{Re}(v + q) > 0, \text{Re}(v) > \frac{1}{2})\)

is called extended modified Bessel function whenever integral exists.
It is clear that the function (8) reduce to Bessel function when \( q = \frac{1}{2} \). Explicitly, \( I_{\frac{1}{2}; x} = \exp(x) I_{\frac{1}{2}}(x) \).

**Corollary 2** We have the following integral representation for \( I_\nu(q; x) \):
\[
I_\nu(q; x) = \left( \frac{x}{2} \right)^\nu \frac{2^{2\nu+q-\frac{1}{2}}}{\sqrt{\pi} \Gamma\left( \nu + \frac{1}{2} \right)} \int_0^1 t^{\nu+q-1} (1-t)^{-\frac{1}{2}} \exp(2xt) \, dt.
\]
(9)
\[
(\text{Re}(\nu + q) > 0, \text{Re}(\nu) > -\frac{1}{2})
\]
(10)

**Proof.** By using the transformation \( t \to 1 - 2t \), the statement can be obtained. \( \blacksquare \)

**Theorem 3** The extended modified Bessel function \( I_\nu(q; x) \) has power series representation as follows:
\[
I_\nu(q; x) = \frac{\left( \frac{x}{2} \right)^\nu}{2^{2\nu+\frac{1}{2}}} \sum_{n=0}^\infty \frac{\Gamma(2\nu + 2q + 2n)}{\Gamma(\nu + q + n + \frac{1}{2}) \Gamma(2\nu + q + n + \frac{1}{2})} \left( \frac{x}{2} \right)^n.
\]
(11)

**Proof.** From the representation (9), we can write the following relation consisting of the power series of the function \( \exp(2xt) \)
\[
\int_0^1 t^{\nu+q-1} (1-t)^{-\frac{1}{2}} \exp(2xt) \, dt = \sum_{n=0}^\infty \frac{(2x)^n}{n!} \int_0^1 t^{\nu+q+n-1} (1-t)^{-\frac{1}{2}} \, dt
\]
\[
= \sum_{n=0}^\infty B\left( \nu + q + n, \nu + \frac{1}{2} \right) \frac{(2x)^n}{n!}
\]
\[
= \sum_{n=0}^\infty \frac{\Gamma(\nu + q + n) \Gamma(\nu + \frac{1}{2})}{\Gamma(2\nu + q + n + \frac{1}{2})} \frac{(2x)^n}{n!}.
\]
(12)

Using the Legendre’s duplication formula, we get
\[
\Gamma(\nu + q + n) = \Gamma\left( \nu + q + n - \frac{1}{2} + \frac{1}{2} \right)
\]
\[
= \sqrt{\pi} \frac{\Gamma(2\nu + 2q + 2n - 1)}{\Gamma(2\nu + q + n - \frac{1}{2})}
\]
\[
= \frac{\sqrt{\pi} \Gamma(2\nu + q + 2n)}{2^{2\nu+2q+2n-1} \Gamma(\nu + q + n + \frac{1}{2})}.
\]
(13)

Substituting equations (12) and (13) into equation (11), we obtain
\[
I_\nu(q; x) = \frac{\left( \frac{x}{2} \right)^\nu}{2^{2\nu+\frac{1}{2}}} \sum_{n=0}^\infty \frac{\Gamma(2\nu + 2q + 2n)}{\Gamma(\nu + q + n + \frac{1}{2}) \Gamma(2\nu + q + n + \frac{1}{2})} \left( \frac{x}{2} \right)^n.
\]
(14)
Theorem 4  The relation between the extended modified Bessel function $I_\nu(q;x)$ and the confluent hypergeometric function $\, _1F_1(\alpha, \beta, x)\,$ is

$$I_\nu(q;x) = \left(\frac{x}{2}\right)^\nu 2^{2\nu+q-\frac{1}{2}} \frac{\Gamma(v+q)}{\Gamma(2v+q+\frac{1}{2})} _1F_1 \left( v+q, 2v+q+\frac{1}{2}; 2x \right). \quad (15)$$

Proof. Recall that

$$I_\nu(q;x) = \left(\frac{x}{2}\right)^\nu 2^{2\nu+q-\frac{1}{2}} \frac{\Gamma(v+q)}{\Gamma(v+\frac{1}{2})} \int_0^1 t^{v+q-\frac{1}{2}} (1-t)^{v-\frac{1}{2}} \exp(2xt) \, dt. \quad (16)$$

Consider the representation of the function $\, _1F_1(\alpha, \beta, x)\,$ as

$$\, _1F_1(\alpha, \beta, x) = \frac{\Gamma(\beta)}{\Gamma(\alpha) \Gamma(\beta-\alpha)} \int_0^1 t^{\alpha-1} (1-t)^{\beta-\alpha-1} \exp(2xt) \, dt. \quad (17)$$

Hence, the special cases $\alpha = v+q$ and $\beta = 2v+q+\frac{1}{2}$ give us a new relation between the extended modified Bessel function and the confluent hypergeometric function as follows:

$$I_\nu(q;x) = \left(\frac{x}{2}\right)^\nu 2^{2\nu+q-\frac{1}{2}} \frac{\Gamma(v+q)}{\Gamma(2v+q+\frac{1}{2})} _1F_1 \left( v+q, 2v+q+\frac{1}{2}; 2x \right) \quad (18)$$

which proves the theorem.

The relation (15) provides a wide range of applications of the function (9). Since $I_\nu(q;x)$ represents both modified Bessel and confluent hypergeometric functions, the special function $I_\nu(q;x)$ can effectively used to generalize many special functions.

Example 5  For some values of $\nu$ and $q$ we can easily write the following relations

$$I_{1/2}(1/2; x) = e^{x} I_{1/2}(x) = \sqrt{\frac{2}{\pi x}} e^{x} \sinh x,$$

$$I_0(1/2; x) = e^{x} J_0(x), \quad I_1(1/2; x) = e^{x} J_1(x), \quad I_0(3/2; x) = e^{x} (J_0(x) + J_1(x)),$$

$$I_\nu(1/2; x) = e^{x} J_\nu(x), \quad \text{Re}(\nu) > -\frac{1}{2},$$

$$I_0(q; x) = \frac{2^{q-\frac{1}{2}}}{\sqrt{\pi}} \, _1F_1(q, q+\frac{1}{2}, 2x), \quad \text{Re}(q) > 0.$$

$$I_{1/2}(q; x) = (-x)^{-q-\frac{1}{2}} \sqrt{\frac{x}{2\pi}} \left( \Gamma(q+\frac{1}{2}) - \Gamma(q+\frac{1}{2}, -2x) \right), \quad \text{Re}(q) > -\frac{1}{2}.$$

The next theorem deals with an asymptotic formula of extended modified Bessel function.
Theorem 6  The special function $I_v(q; x)$ as $x \to \infty$ approaches to

$$I_v(q; x) \to \frac{\Gamma(v+q)}{\sqrt{2\pi x} \Gamma\left(v + \frac{1}{2}\right)}.$$  

Proof. Consider the integral representation of the $I_v(q; x)$ as

$$I_v(q; x) = \left(\frac{x}{2}\right)^v 2^{2v+q-\frac{1}{2}} \int_0^1 t^{v+q-1} (1-t)^{v-\frac{1}{2}} \exp(2xt) \, dt.$$  

Let $I = \int_0^1 t^{v+q-1} (1-t)^{v-\frac{1}{2}} \exp(2xt) \, dt$. By using the substitution $t = \frac{u}{u-x}$, the integral $I$ will be

$$I = \int_0^\infty \frac{u^{v+q-1}}{(u-x)^{v+q-1}} \left(1 - \frac{u}{u-x}\right)^{v-\frac{1}{2}} \exp\left(\frac{2xu}{u-x}\right) \left(-\frac{x}{(u-x)^2}\right) \, du$$

$$= \int_0^\infty \frac{u^{v+q-1}x^{v+\frac{1}{2}}}{x^{2v+q+\frac{1}{2}}(1-x)^{2v+q+\frac{1}{2}}} \exp\left(\frac{2xu}{u-x}\right) \, du$$

$$= \frac{1}{x^{v+q}} \int_0^\infty u^{v+q-1} \exp(-2u) \, du$$

where $\frac{u}{x} \to 0$ and $\frac{2xu}{u-x} \to -2u$ for large number $x$. Since $\int_0^\infty u^{v+q-1} \exp(-2u) \, du = 2^{-v-q}\Gamma(v+q)$, we have

$$I_v(q; x) = \left(\frac{x}{2}\right)^v 2^{2v+q-\frac{1}{2}} \frac{2^{-v-q}\Gamma(v+q)}{x^{v+q}}$$

which proves the theorem. \[\square\]

Remark 7  If we consider the relation

$$I_v(\frac{1}{2}; x) = e^x I_v(x),$$

the corresponding asymptotic formula of modified Bessel function of first kind can easily be derived as

$$I_v(x) \to \frac{e^x}{\sqrt{2\pi x}}, \ x \to \infty.$$  

Theorem 8  For $|z| < 1$, the following generating function holds true:

$$\sum_{n=-\infty}^{\infty} I_{n+\frac{1}{2}}(-n + \frac{1}{2}; x)z^n = \sqrt{\frac{2}{\pi x}} \cdot \left(\frac{ze^{\frac{z^2}{2}}}{z - 2}\right).$$  

(19)  

Proof. By using Legendre's duplication formula, the series representation of $I_v(q; x)$ can be given as

$$I_v(q; x) = \sum_{n=0}^{\infty} \frac{2^{v+q+n-\frac{1}{2}}\Gamma(v+q+n)}{\sqrt{\pi\Gamma(2v+q+n+\frac{1}{2})} n!} p^{n+v}.$$
Consequently,
\[
\sum_{n=-\infty}^{\infty} I_{n+\frac{1}{2}}(-n+\frac{1}{2};x)z^n = \sum_{n=-\infty}^{\infty} \left( \sum_{k=0}^{\infty} \frac{\Gamma(k+1)2^{k+\frac{1}{2}}}{\sqrt{\pi k!}\Gamma(n+k+2)}x^{n+k+\frac{1}{2}} \right) z^n
\]
\[
= \sum_{n=-\infty}^{\infty} \left( \sum_{m=n+k+1}^{\infty} \frac{2^{k+\frac{1}{2}}}{\sqrt{\pi m(m+1)}}x^{m-\frac{1}{2}} \right) z^n
\]
\[
= \sqrt{\frac{2}{\pi x}} \left( \sum_{m=0}^{\infty} \frac{(xz)^m}{m!} \sum_{k=0}^{\infty} 2^k z^{-k} \right). \quad (20)
\]

Since \( |\frac{2}{z} | < 1 \), the geometric series
\[
\sum_{k=0}^{\infty} 2^k z^{-k} = \frac{1}{1-\frac{2}{z}}.
\]

Hence,
\[
\sum_{n=-\infty}^{\infty} I_{n+\frac{1}{2}}(-n+\frac{1}{2};x)z^n = \sqrt{\frac{2}{\pi x}} e^{xz} \frac{z}{z-2}.
\]

Next, we attempt to find generating functions involving the special function \( I_v(q; x) \), mainly motivated by the paper of Agarwal et al. \([3]\).

**Theorem 9.** For \( v, q \in \mathbb{C} \), the following generating function holds true:
\[
\sum_{k=0}^{\infty} \frac{1}{\Gamma(k+1)} I_{v-k}(q+2k;x)t^k = \left(1 - \frac{2t}{x}\right)^{-v-q} I_v(q; \frac{x^2}{x-2t}). \quad (21)
\]

**Proof.** By using Legendre’s duplication formula, the series representation of \( I_v(q; x) \) can be given as
\[
I_v(q; x) = \sum_{n=0}^{\infty} \frac{2^{v+q+n+\frac{1}{2}}\Gamma(v+q+n)}{\sqrt{\pi}\Gamma(2v+q+n+\frac{1}{2}) n!} x^n.
\]

Consequently, by a little simplifications,
\[
\sum_{k=0}^{\infty} \frac{1}{\Gamma(k+1)} I_{v-k}(q+2k;x)t^k = \sum_{n=0}^{\infty} \frac{2^{v+q+n+\frac{1}{2}}x^{n+v}}{\sqrt{\pi}\Gamma(2v+q+n+\frac{1}{2}) n!} \sum_{k=0}^{\infty} \frac{\Gamma(v+q+n+k)2^k t^k}{k!x^k}.
\]

Since \( \sum_{k=0}^{\infty} \frac{\Gamma(\lambda+k)t^k}{k!x^k} = (1-t)^{\lambda} \) for \( \lambda \in \mathbb{C} \),
\[
\sum_{k=0}^{\infty} \frac{\Gamma(v+q+n+k)2^k t^k}{k!x^k} = \Gamma(v+q+n) \sum_{k=0}^{\infty} \frac{\Gamma(v+q+n+k) \left(\frac{2t}{x}\right)^k}{k!}.
\]
Therefore the infinite sum (22) becomes
\[
\sum_{k=0}^{\infty} \frac{1}{\Gamma(k+1)} I_{v-k}(q + 2k; x) t^k = \sum_{n=0}^{\infty} \frac{2^{v+q+n+\frac{1}{2}} x^{n+v} \Gamma(v + q + n)}{\sqrt{\pi} (2v + q + n + \frac{1}{2}) n!} \cdot \left[ \left( 1 - \frac{2t}{x} \right)^{-(v+q+n)} \right] \\
= \sum_{n=0}^{\infty} \frac{2^{v+q+n+\frac{1}{2}} x^{v} \left( \frac{x}{1-x} \right)^{n} \Gamma(v + q + n)}{\sqrt{\pi} (2v + q + n + \frac{1}{2}) n!} \cdot \left[ \left( 1 - \frac{2t}{x} \right)^{-(v+q)} \right]
\]
which gives the generating function given in (23). \[\blacksquare\]

We aim to continue to generate a new generating function involving confluent hypergeometric function \(1F_1(\alpha, \beta, x)\) via generating function (21).

**Theorem 10** For \(\alpha, \beta \in \mathbb{C}\), the following generating function holds true:
\[
\sum_{k=0}^{\infty} \frac{\Gamma(k+\alpha)}{\Gamma(k+1)} F_1(\alpha, \beta + k, x) \cdot z^k = (1 - z)^{-\alpha} \Gamma(\alpha) \cdot F_1(\alpha, \beta, \frac{x}{1-z}). \tag{23}
\]

**Proof.** Use the generating function (21) together with relation (15) and set \(\alpha \rightarrow v + q, \beta \rightarrow 2v + q + \frac{1}{2}, z \rightarrow \frac{2t}{x}. \) \[\blacksquare\]

For one interesting reference paper about generating functions of special functions, we refer the paper Cohl. et al. (12).

### 2.2 Extended Hypergeometric, Confluent Hypergeometric and Beta Functions via extended modified Bessel Function

We use the new extension of extended beta function to generalize the hypergeometric, confluent hypergeometric and extended beta functions as follows.

**Definition 11** The extended beta-hypergeometric function
\[
B_{\mu, \sigma}^{(\nu, q)}(x, y; p) = \sqrt{\frac{2}{\pi}} \int_{0}^{1} t^{\nu}(1-t)^{y} I_{v+\frac{1}{2}}(q; \frac{-p}{t \nu (1-t)^{\sigma}}) dt, \tag{24}
\]
\[
(\text{Re}(p) > 0, \, \mu, \sigma \geq 0, \min\{\text{Re} \left( v + q + \frac{1}{2} \right), \text{Re} \left( 2v + q + \frac{3}{2} \right) \} > 0, \text{Re}(x + \mu q) > -1, \text{Re}(y + \sigma q) > -1)
\]
is defined.

**Remark 12** The necessary conditions for existence of integral given in (24) can also be derived by using the relation (13) and the paper (19, pp. 633, theorem 2.1).
It is clear that the new extension \((24)\) reduces to many defined special functions as

**Case 13** function \((1)\) in the paper when \(v = 0, q = \frac{1}{2}\) and \(\mu, \sigma = 1, 2\).

**Case 14** generalized of extended beta function in Lee et al. \((18), pp. 189, equations (1.13)\) when \(v = 0, q = \frac{1}{2}\) and \(\mu, \sigma = m, \rho, \lambda\).

**Case 15** function \((4)\) when \(q = 2\alpha - \beta + \frac{1}{2}, v = \beta - \alpha - \frac{1}{2}\) and \(\mu = \rho; \sigma = \lambda\).

**Case 16** function \((5)\) in the paper by using the relation

\[
K_v(x) = \pi \exp(-x) \frac{\sin(v\pi)}{2} \left(I_{v+\frac{1}{2}}(1, x) - I_{v+\frac{1}{2}}(1, x)\right)
\]

where \(v \notin \mathbb{Z}\).

Consequently, we use the function \((24)\) to extend the hypergeometric functions and beta functions as follows:

\[
F_{v, q; p}^{(\mu, \sigma)}(a, b; c; z) = \sum_{n=0}^{\infty} \binom{a}{n} B_{v, q}^{(\mu, \sigma)}(b + n, c - b; p) \frac{z^n}{n!}, \quad (26)
\]

\[
(\text{Re}(p) > 0, |z| < 1, \min\{\text{Re}(v + q + \frac{1}{2}), \text{Re}(2v + q + \frac{3}{2})\}), \text{Re}(\mu), \text{Re}(\sigma) \geq 0, \text{Re}(c) > \text{Re}(b) > 0, \text{Re}(a) > 0)
\]

and

\[
\Phi_{v, q; p}^{(\mu, \sigma)}(b; c; z) = \sum_{n=0}^{\infty} \binom{b}{n} B_{v, q}^{(\mu, \sigma)}(b + n, c - b; p) \frac{z^n}{n!}, \quad (27)
\]

\[
(\text{Re}(p) > 0, \min\{\text{Re}(v + q + \frac{1}{2}), \text{Re}(2v + q + \frac{3}{2})\}), \text{Re}(\mu), \text{Re}(\sigma) \geq 0, \text{Re}(c) > \text{Re}(b) > 0, \text{Re}(a) > 0).
\]

**Theorem 17** The special functions \((26)\) and \((27)\), respectively, has the following integral representation

\[
F_{v, q; p}^{(\mu, \sigma)}(a, b; c; z) = \frac{\sqrt{\pi}}{B(b, c - b)} \int_0^1 t^b(1-t)^{c-b}(1-zt)^{-a} I_{v+\frac{1}{2}}(q; -\mu, \frac{-p}{(1-t)^{\mu}}) dt,
\]

\[
(\text{Re}(p) > 0, |\arg(1-z)| < \pi, \text{Re}(\mu), \text{Re}(\sigma) \geq 0, \text{Re}(c) > \text{Re}(b) > 0, \text{Re}(v + q) > 0, \text{Re}(v) > \frac{-1}{2}, \text{Re}(a) > 0).
\]

and
$$\Phi_{v,q,p}(b;c;z) = \sqrt{\frac{2}{\pi}} \frac{1}{B(b,c-b)} \int_0^1 t^b (1-t)^{c-b} e^{zt} I_{v+\frac{1}{2}}(q; \frac{-p}{t^\mu (1-t)}) \, dt, \quad (28)$$

$$(\text{Re}(p) > 0, \text{Re}(\mu), \text{Re}(\sigma) \geq 0, \text{Re}(c) > \text{Re}(b) > 0,$$

$$\text{Re}(v+q) > 0, \text{Re}(v) > -\frac{1}{2}, \text{Re}(a) > 0). \quad (29)$$

**Proof.** Substituting the function (24) with $x \to b+n, y \to c-b$ into function (26), we have after interchanging the order of summation and integration which is guaranteed

$$F_{v,q,p}(a,b;c;z) = \sqrt{\frac{2}{\pi}} \frac{1}{B(b,c-b)} \int_0^1 t^b (1-t)^c e^{-b} e^{-z} I_{v+\frac{1}{2}}(q; \frac{-p}{t^\mu (1-t)}) \sum_{n=0}^{\infty} (a)_n \frac{(zt)^n}{n!} \, dt$$

$$= \sqrt{\frac{2}{\pi}} \frac{1}{B(b,c-b)} \int_0^1 t^b (1-t)^c e^{-b} e^{-z} \sum_{n=0}^{\infty} (a)_n \frac{(zt)^n}{n!} \, dt,$$

where $(1-zt)^{-a} = \sum_{n=0}^{\infty} (a)_n \frac{(zt)^n}{n!}, \forall |zt| < 1$. Similarly, from the definitions of the functions (24) and (27), we can derive the integral representation (28) with $\exp(zt) = \sum_{n=0}^{\infty} \frac{(zt)^n}{n!}$. Also, it can be easily seen that the new extensions (26) and (28) reduce to the following special functions as

**Case 18** extended Gauss hypergeometric function and extended confluent hypergeometric function in Lee et al. ([18], pp. 189, Equations. (1.11) and (1.12)) respectively when $v=0, q=\frac{1}{2}$ and $\mu = \sigma = 1$.

**Case 19** new generalized beta function in Özergin et al. ([24]; pp. 4607, Equations. (11)) when $q = 2a - 2 + \frac{1}{2}, v = \beta - \alpha - \frac{1}{2}$ and $\mu = \sigma = 1$.

**Remark 20** An interesting generalization of extension of gamma function and generalized gamma function given together in the paper ([24]) can be considered as

$$\Gamma_{v,q}(x) : = \int_0^1 t^{x-1} I_{v+\frac{1}{2}}(q; -(t + \frac{p}{x})) \, dt,$$

$$(\text{Re}(p) > 0, \text{Re}(x) > 0, \text{Re}(v+q) > -\frac{1}{2}, \text{Re}(2v+q) > -\frac{3}{2}).$$

### 2.3 The Mellin and Laplace Transforms

In this section, we derive the Mellin and Laplace transforms of extended modified Bessel and extended beta-hypergeometric functions. The necessary conditions for their existences can be followed through existences of the special functions appearing in their respective formulæ.
Theorem 21 The Mellin transform of

\[ M[I_v(q; x); s] = \int_0^\infty x^{s-1} I_v(q; x) dx \]

\[ = \frac{(-1)^{v+s-1} 2^{q-s-\frac{1}{2}} \Gamma(v) \Gamma(q-s)}{\sqrt{\pi} \Gamma(q+v-s+\frac{1}{2})} \]

whenever integral exists.

**Proof.** Assume that the Mellin transform of \( I_v(q; x) \) exists. Then,

\[ M[I_v(q; x); s] = \int_0^\infty x^{s-1} I_v(q; x) dx \]

\[ = \frac{2^{v+q-\frac{1}{2}}}{\sqrt{\pi} \Gamma(v+\frac{1}{2})} \int_0^1 x^s \int_0^1 x^r \exp(2xt) dt dx. \]

By using uniform convergency of the integration with substitutions \( \sigma = -2xt \) and \( \lambda = t \), we have

\[ \int_0^\infty x^{s-1} I_v(q; x) dx = \frac{2^{q-s-\frac{1}{2}} (-1)^{v+s-1}}{\sqrt{\pi} \Gamma(v+\frac{1}{2})} \int_0^1 \lambda^{q-s-1} \Gamma(q-v-s+\frac{1}{2}) \int_0^\infty \sigma^{v+s-1} \exp(-\sigma) d\sigma. \]

Hence,

\[ \int_0^\infty x^{s-1} I_v(q; x) dx = \frac{2^{q-s-\frac{1}{2}} (-1)^{v+s-1} \Gamma(v) \Gamma(q-s)}{\sqrt{\pi} \Gamma(v+\frac{1}{2})} \cdot \frac{\Gamma(q-v-s+\frac{1}{2})}{\Gamma(q+v-s+\frac{1}{2})} \]

\[ = \frac{2^{q-s-\frac{1}{2}} (-1)^{v+s-1} \Gamma(v) \Gamma(q-s)}{\sqrt{\pi} \Gamma(q+v-s+\frac{1}{2})}. \]

In the paper ([2], pp. 410, equation (3.3)), the Mellin transform of the hypergeometric function \( _1F_1(\alpha, \beta, -b) \) is used as

\[ \int_0^\infty b^{s-1} \frac{\Gamma(\alpha-s) \Gamma(\beta) \Gamma(s)}{\Gamma(\alpha) \Gamma(\beta-s)} dB = \frac{\Gamma(\alpha-s) \Gamma(\beta) \Gamma(s)}{\Gamma(\alpha) \Gamma(\beta-s)}. \quad (30) \]

By using Mellin transform of \( I_v(q; x) \), the Mellin transform of \( _1F_1 \) can easily be showed by the following corollary.

**Corollary 22** From the Mellin transform of \( I_v(q; x) \), we can easily derive the Mellin transform of \( _1F_1(\alpha, \beta, -p) \) as

\[ \int_0^\infty p^{s-1} \frac{\Gamma(\alpha-s) \Gamma(\beta) \Gamma(s)}{\Gamma(\alpha) \Gamma(\beta-s)} dp = \frac{\Gamma(\alpha-s) \Gamma(\beta) \Gamma(s)}{\Gamma(\alpha) \Gamma(\beta-s)}. \]

**Proof.** Considering the Mellin transform of \( I_v(q; x) \) with relation (15), we have

\[ \int_0^\infty p^{s-1} \frac{(p)^V 2^{v+q-\frac{1}{2}} \Gamma(v+q)}{\sqrt{\pi} \Gamma(2v+q+\frac{1}{2})} _1F_1 \left( v+q, 2v+q+\frac{1}{2}, 2p \right) dp = \frac{2^{q-s-\frac{1}{2}} (-1)^{v+s-1} \Gamma(v) \Gamma(q-s)}{\sqrt{\pi} \Gamma(q+v-s+\frac{1}{2})}. \]
If we consider the substitutions \( p \to -\frac{\pi}{2}, \alpha = v + q, \beta = 2v + q + \frac{1}{2} \) for above integration, we have

\[
\int_0^\infty \frac{p^{(s+v)-1} \Gamma(\alpha) 2^{q-s-\frac{1}{2}} (-1)^{v+s-1}}{\sqrt{\pi} \Gamma(\beta)} \ \,_{1}F_{1}(\alpha,\beta,-p) \, dp = \frac{2^{q-s-\frac{1}{2}} (-1)^{v+s-1} \Gamma(v+s) \Gamma(q-s)}{\sqrt{\pi} \Gamma(q+v-s+\frac{1}{2})}
\]

which gives Mellin transform \([27]\).

**Theorem 23** The Mellin transform of

\[
M \left[ B^{(\mu,\sigma)}_{v,q}(x,y;p) \right] = \int_0^\infty p^{s-1} B^{(\mu,\sigma)}_{v,q}(x,y;p) \, dp
\]

whenever integral exists.

**Proof.** In the light of Mellin transform of \( I_v(q;x) \), the Mellin transform of \( B^{(\mu,\sigma)}_{v,q}(x,y;p) \) can be represented as

\[
M \left[ B^{(\mu,\sigma)}_{v,q}(x,y;p) \right] = \sqrt{\frac{2}{\pi}} \int_0^1 t^\mu (1-t)^{\nu} \int_0^\infty p^{s-1} I_{v+\frac{1}{2}}(q; -\frac{p}{t^\nu (1-t)^\nu}) \, dp \, dt.
\]

Let \( p = \Theta (t^\mu (1-t)^\nu) \) and \( \lambda = t \) (\( dp = d\Theta (t^\mu (1-t)^\nu) \) and \( d\lambda = dt \)). Then,

\[
M \left[ B^{(\mu,\sigma)}_{v,q}(x,y;p) \right] = \sqrt{\frac{2}{\pi}} \int_0^1 \lambda^{x+\mu s} (1-\lambda)^{y+\sigma s} \, d\lambda \left[ \frac{2^{q-s-\frac{1}{2}} (-1)^v \Gamma(v+s) \Gamma(q-s)}{\sqrt{\pi} \Gamma(q+v-s+\frac{1}{2})} \right]
\]

\[
= \sqrt{\frac{2}{\pi}} \left[ \frac{2^{q-s-\frac{1}{2}} (-1)^v \Gamma(v+s) \Gamma(q-s)}{\sqrt{\pi} \Gamma(q+v-s+\frac{1}{2})} \right] \int_0^1 \lambda^{x+\mu s} (1-\lambda)^{y+\sigma s} \, d\lambda
\]

\[
= \frac{\sqrt{2} 2^{q-s-\frac{1}{2}} (-1)^v \Gamma(v+s) \Gamma(q-s)}{\sqrt{\pi} \Gamma(q+v-s+\frac{1}{2})} B(x + \mu s + 1, y + \sigma s + 1).
\]

**Remark 24** Since the special function \([23]\) is extension of some recently introduced special functions, the Mellin transform of these covered functions can be derived.

**Theorem 25** The Laplace transform, if exists, of extended modified Bessel function is

\[
L [ I_v(q;x) \{ s \} ] = \int_0^\infty e^{-sx} I_v(q;x) \, dx
\]

\[
= \frac{2^{q+v-\frac{1}{2}} \Gamma(v+q) \Gamma(v+1)}{\sqrt{\pi} s^{v+1} \Gamma(q+2v+\frac{1}{2})} F \left( v+1, v+q; 2v+q+\frac{1}{2}; \frac{2}{s} \right).
\]

where \( F(a,b;c;z) \) is Gauss hypergeometric function (see \([21]\), pp.11, equation \((2)\)).
Proof. Consider the Laplace transform of $I_v(q; x)$

$$L\{I_v(q; x)\} = \int_0^\infty e^{-sx} I_v(q; x) dx = \int_0^\infty e^{-s\frac{x}{2}} \frac{2^{2v+q} \sqrt{\pi}}{\Gamma \left( v + \frac{1}{2} \right)} \int_0^1 t^{v+q-1} (1-t)^{-\frac{1}{2}} \exp(2xt) \, dt \, dx.$$ 

By using uniform convergency of the integration, we have

$$\int_0^\infty e^{-sx} I_v(q; x) dx = \frac{2^{v+q-\frac{1}{2}}}{\sqrt{\pi} \Gamma \left( v + \frac{1}{2} \right)} \int_0^1 t^{v+q-1} (1-t)^{-\frac{1}{2}} \, dt \int_0^\infty (x)^v \exp(-s^2t) \, dx, (s > 2t)$$

By using the substitutions $x = \frac{s}{\sqrt{2\pi}}$ and $\lambda = t$, we have

$$\int_0^\infty e^{-sx} I_v(q; x) dx = \frac{2^{v+q-\frac{1}{2}}}{\sqrt{\pi} s^{v+1} \Gamma \left( v + \frac{1}{2} \right)} \int_0^1 \lambda^{v+q-1} (1-\lambda)^{-\frac{1}{2}} (s-2\lambda)^{-v-1} \, d\lambda \int_0^\infty (\sigma)^v e^{-\sigma} \, d\sigma.$$

Since

$$F(a, b; c; z) = \frac{1}{B(b, c-b)} \int_0^1 t^{b-1} (1-t)^{c-b-1} (1-z\lambda)^{-a} \, dt \text{ with } |\arg(1-z)| < \pi,$$

then

$$\int_0^\infty e^{-sx} I_v(q; x) dx = \frac{2^{v+q-\frac{1}{2}} \Gamma (v + 1)}{\sqrt{\pi} s^{v+1} \Gamma \left( v + \frac{1}{2} \right)} \Gamma (v + q) \Gamma \left( v + \frac{1}{2} \right) \Gamma (v + q + \frac{1}{2})$$

which gives the formula (31).

As an particular case, the Laplace transform of $I_v(q; x)$ for $v = 0$ and $q = \frac{1}{2}$ gives

$$L\{I_0(\frac{1}{2}; x)\} = \frac{1}{\sqrt{s^2 - 2s}}.$$

Consequently, Laplace transform of modified Bessel function of the first kind for $v = 0$ via Laplace transform of $I_v(q; x)$ can easily be obtained

$$L\{I_0(x); s\} = L\{e^{-x} I_0(\frac{1}{2}; x); s\} = \frac{1}{\sqrt{(s + 1)^2 - 2(s + 1)}} = \frac{1}{\sqrt{s^2 - 1}}.$$

Corollary 26 The Laplace transform of modified Bessel function is

$$L\{I_v(x); s\} = \frac{1}{2^v (s + 1)^{v+1} \Gamma (q + 2v + \frac{1}{2})} F \left( v + 1, v + \frac{1}{2}; 2v + 1; \frac{2}{s + 1} \right).$$
Proof. Assume that Laplace transform of \(I_v(q;x)\) exists and equals to \(F(s)\). Consequently,

\[ L\{I_v(x);s\} = L\{e^{-x}I_v\left(\frac{1}{2},x\right);s\} = F(s+1). \]

By using formula (31) together with Legendre’s duplication formula, we derive the corresponding formula.

3 Generalization of Fractional Derivatives

3.1 Extended Fractional Derivative via Extended Modified Bessel Function

In this section, we introduce an interesting extended fractional derivative which can be a generalization of a large set of fractional derivatives. Let \(z > 0\) then the new extension of Riemann-Liouville fractional derivative \(D_{v,q}^{\alpha,\eta,p} f(z)\) is defined as follows:

\[
\mu,\sigma D_{v,q}^{\alpha,\eta,p} f(z) := \sqrt{\frac{2}{\pi}} \frac{\Gamma(\alpha)}{\Gamma(\alpha)} \int_0^z f(t)(z-t)^{\alpha-1} t^\eta I_{v+\frac{1}{2}}(q; \frac{-p z^{\mu+\alpha}}{t^\mu(z-t)}) dt, \quad (32)
\]

(min{Re(\(\alpha\)) > 0, Re(\(p\)), Re(\(\eta\)) > 0, Re(\(v + q + \frac{1}{2}\)) > 0, Re(2v + q + \frac{3}{2}) > 0}, \(\mu, \sigma \geq 0\))

and \(n - 1 < Re(\(\alpha\)) < n \quad (n = 1, 2, 3, \ldots)\).

Now, we start with the extended fractional derivative of elementary function \(f(z) = z^\lambda\).

**Corollary 27** Let Re\((\eta + \lambda + \mu q) > -1\). Then

\[
\mu,\sigma D_{v,q}^{\alpha,\eta,p} (z^\lambda) = \frac{z^{\eta+\lambda+\alpha}}{\Gamma(\alpha)} B_{v,q}(\eta + \lambda, \alpha - 1; p)
\]

whenever the function \(B_{v,q}(\eta + \lambda, \alpha - 1; p)\) exists.

**Proof.** Consider the fractional derivative (32), we get

\[
\mu,\sigma D_{v,q}^{\alpha,\eta,p} (z^\lambda) = \sqrt{\frac{2}{\pi}} \frac{z^{\eta+\lambda+\alpha}}{\Gamma(\alpha)} \int_0^z (z-t)^{\alpha-1} t^\eta I_{v+\frac{1}{2}}(q; \frac{-p z^{\mu+\alpha}}{t^\mu(z-t)}) dt.
\]

Taking \(t = zu\), after a little simplification, gives

\[
\mu,\sigma D_{v,q}^{\alpha,\eta,p} (z^\lambda) = \frac{\sqrt{\frac{2}{\pi}} z^{\eta+\lambda+\alpha}}{\Gamma(\alpha)} \int_0^1 (1-u)^{\alpha-1} u^{\eta+\lambda} I_{v+\frac{1}{2}}(q; \frac{-p}{u^\mu(1-u)}) du
\]

Taking \(t = zu\), after a little simplification, gives

\[
\mu,\sigma D_{v,q}^{\alpha,\eta,p} (z^\lambda) = \frac{\sqrt{\frac{2}{\pi}} z^{\eta+\lambda+\alpha}}{\Gamma(\alpha)} B_{v,q}(\eta + \lambda, \alpha - 1; p).
\]
Corollary 28 Let $\xi \neq 0$ and $\xi \in \mathbb{C}$. Then

$$\mu,\sigma D_{v,q}^{\alpha,\eta,p}((z - \xi)^r) := \frac{(-\xi)^r B(\eta, \alpha - 1)z^{\eta + \alpha}}{\Gamma(\alpha)} F_{v,q,p}^{(\mu,\sigma)}(-r, \eta; \eta + \alpha - 1; \frac{z}{\xi})$$  \hspace{1cm} (33)$$

whenever the function $F_{v,q,p}^{(\mu,\sigma)}$ exists.

**Proof.** Consider the fractional derivative 32, we get

$$D_{z}^{\mu,\eta,p}((z - \xi)^r)$$

$$= \frac{\sqrt{2}}{\Gamma(\alpha)} \int_{0}^{\infty} (z - t)^{\alpha - 1}(t - \xi)^{r} I_{v+\frac{1}{2}} q; \frac{-pz^{\mu+\sigma}}{t(z - t)^{\sigma}}) dt$$

$$= \frac{\sqrt{2}}{\Gamma(\alpha)} (-\xi)^{r} z^{\eta + \alpha} \int_{0}^{1} (1 - u)^{\alpha - 1}(1 - \frac{z}{\xi} u) I_{v+\frac{1}{2}} q; \frac{-p}{u^\mu (1 - u)^{\eta}}) du (t = uz)$$

$$= (-\xi)^{r} B(\eta, \alpha - 1)z^{\eta + \alpha} \frac{\Gamma(\alpha)}{\Gamma(\alpha)} F_{v,q,p}^{(\mu,\sigma)}(-r, \eta; \eta + \alpha - 1; \frac{z}{\xi}).$$

\[\square\]

The special case of new extension 32 with $p \to 2p, \mu = \sigma = 1; v = 0, q = \frac{1}{2}; \alpha = -\mu - \frac{1}{2}, \eta = -\frac{1}{2}$ reduces the generalized Riemann-Liouville fractional derivative which is defined by Özarslan et al (22) as

$$D_{z}^{\mu,\eta,p} f(z) := \frac{1}{\Gamma(-\mu)} \int_{0}^{z} f(t)(z - t)^{-\mu - 1} \exp\left(\frac{-p z^2}{t(z - t)}\right) dt,$$ \hspace{1cm} (34)$$

(Re(\mu) < 0, Re(p) > 0).$$

Also, the particular case $\mu = \sigma = 0; v = 0, q = \frac{1}{2}; \alpha = -\mu, \eta = 0$ for extended fractional derivative 32 reduces the Riemann-Liouville fractional derivative

$$D_{z}^{\mu} f(z) := \frac{1}{\Gamma(-\mu)} \int_{0}^{z} f(t)(z - t)^{-\mu - 1} dt,$$ \hspace{1cm} (35)$$

(Re(\mu) < 0).$$

It is also important to note that the extended fractional derivative 32 reduces to extended fractional derivative

$$I_{z}^{\mu,b} \{f(z)\} := \frac{1}{\Gamma(\mu)} \int_{0}^{z} f(t)(z - t)^{\mu - 1} \binom{b}{\frac{z^{\mu+\lambda}}{t(z - t)^{\lambda}}} dt,$$ \hspace{1cm} (36)$$

(\rho > 0, \lambda > 0, \min\{Re(\gamma), Re(\beta), Re(\mu), Re(b)\} > 0)$$

defined in (171,p.647) when $p \to \frac{h}{2}, \mu = \rho, \sigma = \lambda; v = 0, q = 2\gamma - \beta + \frac{1}{2}, v = \beta - \gamma - \frac{1}{2}; \alpha = \mu + \beta\lambda - \gamma\lambda - \frac{1}{2} = \beta\rho - \gamma\rho - \frac{1}{2}.$
Finally, Katugampola in the paper ([17]) introduced a new fractional integral operator given by,

\[(\rho \mathcal{I}_a^\alpha + f)(x) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^x \frac{\tau^{\rho-1} f(\tau)}{(x^{\rho} - \tau^{\rho})^{1-\alpha}} d\tau,\]

which is generalization of the Riemann-Liouville and the Hadamard fractional integrals. The extended fractional derivative (32) reduces to the fractional derivative (37) when \(z \to x^{\rho - a^\rho}, f(z) \to f\left((z + a^\rho)\frac{1}{\rho}\right); \alpha \to \alpha, \eta = 0, v = 0, q = \frac{1}{2}\) and \(\mu = \sigma = 0\).

In the light of these reductions, we can easily understand that the extended fractional derivative (32) is generalization of many defined fractional derivatives.

### 3.2 Fractional derivative of Rational Functions

In this section, we will derive the extended fractional derivative of arbitrary rational functions. Consequently, the general representation of fractional derivatives of many defined fractional derivatives of arbitrary rational functions can firstly be derived.

Assume that \(P(z)\) and \(Q(z)\) are polynomials such that \(\deg(P) < \deg(Q)\). In this case, the real partial fraction decomposition of the rational function \(\frac{P(z)}{Q(z)}\) can be represented as

\[
\frac{P(z)}{Q(z)} = \sum_{i=1}^{p} \sum_{r=1}^{k_i} \frac{a_{ir}}{(z - x_i)^r} + \sum_{j=1}^{q} \sum_{s=1}^{l_j} \frac{\beta_{js} z + \gamma_{js}}{(z^2 - 2\Re(z_j)z + |z_j|^2)^s},
\]

where \(a_{ir}, \beta_{js}, \gamma_{js} \in \mathbb{R}\). In the representation (38), the inverse of quadratic functions can not be worked well in many calculations. Because of this quadratic functions in the denominators, for example, we can not derive the fractional derivatives of rational functions. In this paper, we will use complex partial fraction decomposition method together with formula (33) to derive extended fractional derivatives of rational functions. In the paper ([27]), the complex partial fraction decomposition of arbitrary rational fraction was derived by the following theorem:

**Theorem 29** Let \(x_1, \ldots, x_p\) be pairwise different real numbers and \(z_1, \ldots, z_q \in \mathbb{C} \setminus \mathbb{R}\) be also pairwise different. If \(P(x)\) is a polynomial with real coefficients whose degree satisfies the inequality \(\deg(P(x)) < p + 2(l_1 + \cdots + l_q)\), then there exists \(a_{ir}, \beta_{js}, \gamma_{js} \in \mathbb{R}\) and \(b_{js} \in \mathbb{C}\) such that

\[
\frac{P(x)}{Q(x)} = \sum_{i=1}^{p} \sum_{r=1}^{k_i} \frac{a_{ir}}{(x - x_i)^r} + \sum_{j=1}^{q} \sum_{s=1}^{l_j} \frac{\beta_{js} x + \gamma_{js}}{(x^2 - 2\Re(z_j) x + |z_j|^2)^s} = \sum_{i=1}^{p} \sum_{r=1}^{k_i} \frac{a_{ir}}{(x - x_i)^r} + \sum_{j=1}^{q} \sum_{s=1}^{l_j} \left( \frac{b_{js}}{x - z_j} + \frac{\bar{b}_{js}}{x - \bar{z}_j} \right).
\]
where
\[ Q(x) = (x-x_1)^{k_1} \ldots (x-x_p)^{k_p} (x^2-2\text{Re}(z_1)x+|z_1|^2)^{l_1} \ldots (x^2-2\text{Re}(z_q)x+|z_q|^2)^{l_q}. \]

The relations between the coefficients of the real partial fraction decomposition and the coefficients of the complex partial fraction decomposition are:

\[
\begin{align*}
b_{j1} &= \sum_{s=2}^{l} \beta_{js} \omega_j^2 |\omega|^2 C_{2s-3}^{s-2} + \sum_{s=1}^{l} (\beta_{js} \omega_j^2 \omega_j \gamma_{js}) |\omega_j|^{2(s-1)C_{2(s-1)}^{s-1}}, \\
b_{j2} &= \sum_{s=3}^{l} \beta_{js} \omega_j^3 |\omega_j|^{2(s-3)C_{2s-4}^{s-3}} + \sum_{s=2}^{l} (\beta_{js} \omega_j^2 z + \omega_j^2 \gamma_{js}) |\omega_j|^{2(s-2)C_{2s-3}^{s-2}}, \\
&\vdots \\
b_{jl-1} &= \beta_{jl} \omega_j^l + \beta_{jl} \omega_j^{l-1} + \omega_j^{l-1} \gamma_{jl-1} + \beta_{jl} \omega_j^{l-1} \gamma_{jl-1}, \\
b_{lj} &= \beta_{jl} \omega_j^l z + \omega_j^l \gamma_{jl} ,
\end{align*}
\]

where \( \omega_j = \frac{1}{2i \text{Im}(z_j)}. \)

**Theorem 30** Let \( \text{Re}(\eta) > 0 \) and \( \text{Re}(\alpha) > 0. \) The extended fractional derivative of arbitrary rational function satisfying previous theorem is

\[
\begin{align*}
\mu,\sigma D^{\alpha,\eta,p}_{v,q,z} \left( \frac{P(z)}{Q(z)} \right) &= \sum_{s=1}^{p} \sum_{r=1}^{k_s} a_{ir} B(\eta, \alpha-1) z^{\eta+\alpha} \left( -x_i \right)^{r \Gamma(\alpha)} B_p^{(\mu,\sigma)}(r,\eta;\eta+\alpha-1; \frac{z}{x_i}) \\
&+ \sum_{j=1}^{q} \sum_{s=1}^{l_j} b_{js} B(\eta, \alpha-1) z^{\eta+\alpha} \left( -z_j \right)^{s \Gamma(\alpha)} B_p^{(\mu,\sigma)}(s,\eta;\eta+\alpha-1; \frac{z}{z_j}) \\
&+ \bar{b}_{js} B(\eta, \alpha-1) z^{\eta+\alpha} \left( -z_j \right)^{s \Gamma(\alpha)} B_p^{(\mu,\sigma)}(s,\eta;\eta+\alpha-1; \frac{z}{z_j})
\end{align*}
\]

whenever the extended hypergeometric functions \( B_p^{(\mu,\sigma)} \) exist.

**Proof.** Considering the formula (33) and complex partial fraction decomposition, the proof of the theorem can easily be done. \( \blacksquare \)

A numerical example of extended fractional derivative of the rational function given in (27) will be derived by the following example.

**Example 31** Consider the rational function given in (27)

\[
f(z) = \frac{2z + 1}{(z^2 + 6z + 10)^3}, \quad (39)
\]
The complex partial fraction decomposition of function (26) can be given as

\[
f(z) = \sum_{s=1}^{3} \left( \frac{z_s}{(x - (-3 + i))^s} + \frac{\bar{z}_s}{(x - (-3 - i))^s} \right)
\]

(40)

where \( z_1 = \frac{15}{16}i \), \( z_2 = \frac{15}{16} - \frac{i}{8} \) and \( z_3 = -\frac{1}{4} - \frac{5}{8}i \). By using the decomposition (40), the extended fractional derivative of rational function (39) can be given as

\[
\mu,\sigma \cdot D^{\alpha,\eta,p}_{v,q;x}(f(z)) = \sum_{s=1}^{3} \left( z_s \cdot B(\eta,\alpha - 1)z^{\eta+\alpha} \cdot F^{(\mu,\sigma)}(s,\eta + 1; \frac{z}{\eta + 1}) + \bar{z}_s \cdot B(\eta,\alpha - 1)z^{\eta+\alpha} \cdot F^{(\mu,\sigma)}(s,\eta + 1; \frac{z}{\eta + 1}) \right)
\]

3.3 Multiplicative Extended Fractional Derivative (MEFD)

In this section, we analogously introduce the extended multiplicative fractional derivative in the multiplicative sense. Recall that if \( f \) is positive and Riemann integrable on \([a, b]\), then it is \(^*\) integrable (multiplicative integrable) on \([a, b]\) and

\[
\int_{a}^{b} f(x) dx = e^{\int_{a}^{b} (\ln f(x)) \cdot dx}.
\]

Recently, Abdeljawad and Grossman in [1] introduced Caputo, Riemann and Letnikov multiplicative fractional derivatives and gave their properties. The multiplicative Riemann-Liouville fractional derivative for \( f(z) \) of order \( \alpha \), \( \text{Re}(\alpha) > 0 \), \( n - 1 < \text{Re}(\alpha) < n \), starting from 0 is represented as

\[
\mu,\sigma \cdot D^{\alpha,n,p}_{v,q;x}(f(z)) = e^{\int_{0}^{t} (\ln f(t)) \cdot dt}
\]

also it can be represented as

\[
\mu,\sigma \cdot D^{\alpha,n,p}_{v,q;x}(f(z)) = e^{\int_{0}^{t} (\ln f(t)) \cdot dt}, \quad \text{Re}(\alpha) < 0.
\]

(41)

For more details we refer the recent paper [1].

In the lights of multiplicative fractional derivative [1], we introduce multiplicative extended fractional derivative (MEFD) as follows:

\[
\mu,\sigma \cdot D^{\alpha,n,p}_{v,q;x}(f(z)) = e^{\int_{0}^{t} (\ln f(t)) \cdot dt}, \quad \text{Re}(\alpha) < 0.
\]

(42)
Theorem 32 Suppose that $f$ is a positive function on $I$. The multiplicative generalized fractional derivative exists if and only if ordinary generalized fractional derivative exists and

$$
\mu, \sigma D_{v, q, z}^{\alpha, \eta, p}(f(z)) = \mu, \sigma D_{v, q, z}^{\alpha, \eta, p}(e^{f(z)}).
$$

(43)

Remark 33 The existence of the extended fractional derivative of a positive function is equivalent to existence of multiplicative fractional derivative of the same function.

Remark 34 If we set $\mu = \sigma = 1; v = 0$, $q = \frac{1}{2}; \alpha = -\mu - \frac{1}{2}; \eta = \frac{1}{2}$ in (42), then it gives alternative extended multiplicative fractional derivative

$$
\mu, \sigma D_{v, q, z}^{\alpha, \eta, p}(f(z)) = \exp \left( \frac{\sqrt{\pi}}{\Gamma(\alpha)} \int_0^z (\ln f(t))(z-t)^{\alpha-1} \exp \left( -\frac{pz^2}{t(z-t)} \right) dt \right)
$$

which is analogous of generalized fractional derivative

$$
D_{\mu, \sigma} f(z) := \frac{1}{\Gamma(\alpha)} \int_0^z f(t)(z-t)^{\alpha-1} \exp \left( -\frac{pz^2}{t(z-t)} \right) dt
$$

given in [22].

The following formulae can easily be derived for MEFD.

Corollary 35 Let $\Re(\mu) < 0$ and $\Re(\lambda) > 0$. Then

$$
\mu, \sigma D_{v, q, z}^{\alpha, \eta, p}(e^{z}) := \exp \left( \frac{z^{\mu \lambda + \alpha}}{\Gamma(\alpha)} B(\mu, \lambda)(\eta + \alpha - 1; p) \right).
$$

(44)

Corollary 36 Let $a > 0$ and $\xi \neq 0$. Then,

$$
\mu, \sigma D_{v, q, z}^{\alpha, \eta, p}(a^{z-\xi}) := e^{\frac{\ln(a-\xi)^{\mu \lambda + \alpha}}{\Gamma(\alpha)} B(\mu, \lambda)(\eta + \alpha - 1; p)} F_{\nu, q, p}^{(\mu, \lambda)(\eta + \alpha - 1; \xi)}.
$$

(45)

Now, we give some properties of MEFD given in [12]. Let $f$ and $g$ be $\ast$-integrable on $[a, b]$. Then, the following properties/rules of $\mu, \sigma D_{v, q, z}^{\alpha, \eta, p}(f(z))$ can be ordered as:

(a) $\mu, \sigma D_{v, q, z}^{\alpha, \eta, p}(f(z)^k) = (\mu, \sigma D_{v, q, z}^{\alpha, \eta, p}(f(z)))^k, k \in \mathbb{R} - \{0\}$

(b) $\mu, \sigma D_{v, q, z}^{\alpha, \eta, p}(f(z)g(z)) = (\mu, \sigma D_{v, q, z}^{\alpha, \eta, p}(f(z))) \cdot (\mu, \sigma D_{v, q, z}^{\alpha, \eta, p}(g(z)))$

(c) $\mu, \sigma D_{v, q, z}^{\alpha, \eta, p} \left( \frac{f(x)}{g(x)} \right) = \frac{\mu, \sigma D_{v, q, z}^{\alpha, \eta, p}(f(z))}{\mu, \sigma D_{v, q, z}^{\alpha, \eta, p}(g(z))}, g(z) \neq 0,$

The proofs of properties can be easily obtained from the definition of MEFD. For example, the proof of the rule (b) follows from

$$
\mu, \sigma D_{v, q, z}^{\alpha, \eta, p}(f(z)g(z)) = e^{\frac{\sqrt{\pi}}{\Gamma(\alpha)} \int_0^z (\ln f(t)+\ln g(t))(z-t)^{\alpha-1} \exp \left( -\frac{pz^2}{t(z-t)} \right) dt} \mu, \sigma D_{v, q, z}^{\alpha, \eta, p}(f(z)) \cdot \mu, \sigma D_{v, q, z}^{\alpha, \eta, p}(g(z))
$$

Next, we give the following theorem for the analytic function $f(z)$. 

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Theorem 37 Let $f(z)$ be an analytic function on an open interval $I$ for $|z| < 1$. If $f(z)$ has Maclaurin’s series as

$$f(z) = \sum_{k=0}^{\infty} a_k z^k,$$

then from the uniformly convergence of the integral

$$\mu, \sigma \ast D^{\alpha, \eta, p}_{v, q; z}(e^{f(z)}) = e^{1/(\Gamma(\alpha))} \sum_{k=0}^{\infty} a_k z^{\eta+k+\alpha} B^{(\mu, \sigma)}_{v, q}(\eta+k, \alpha-1; p).$$

Proof. Suppose that $f(z)$ is analytic function over the interval $I$ and it has series representation as $f(z) = \sum_{k=0}^{\infty} a_k z^k$. Then,

$$\mu, \sigma \ast D^{\alpha, \eta, p}_{v, q; z}(e^{f(z)}) = e^{1/(\Gamma(\alpha))} \sum_{k=0}^{\infty} a_k \int_0^z (z-t)^{\alpha-1} t^{\eta+k} I_{v+\frac{1}{2}}(q \mu^{-\frac{\eta+k}{\mu}}(z-t)) dt).$$

Choosing $t = zu$ and interchanging the summation and integral which is guarantee, then

$$\mu, \sigma \ast D^{\alpha, \eta, p}_{v, q; z}(e^{f(z)}) = e^{1/(\Gamma(\alpha))} \sum_{k=0}^{\infty} a_k z^{\eta+k+\alpha} B^{(\mu, \sigma)}_{v, q}(\eta+k, \alpha-1; p).$$

This completes the proof. ■

4 Conclusion

Recently, the investigation for extension of some special functions have become important. Thus, many extensions of special functions have been obtained by the authors in different studies. From this point of view, we present extended modified Bessel function $I_v(q; x)$ which generalizes the Bessel And modified Bessel functions, by using an additional parameter in the integral representation. An extensions of the well-known functions in the literature such as the hypergeometric, the confluent hypergeometric and the extended beta functions are also given via extended modified Bessel function. A necessary relation between extended modified Bessel function $I_v(q; x)$ and the confluent hypergeometric function $\mathbf{1}_F(\alpha, \beta, x)$ is easily given. Moreover, Mellin and Laplace transforms for some newly derived special functions are obtained as a common coverage. We determine asymptotic formulae and also the generating functions of the extended modified Bessel function. Hence, a lot of relations with respect to this new function can be proved by using its generating functions. In last section, we introduce new extensions of the classical and multiplicative
Riemann-Liouville fractional derivatives via defined extended special function $I_{q}(q;x)$. The fractional derivative of rational functions is explicitly found by using the new definition of fractional derivative and complex partial fraction decomposition. It can be easily seen that the results obtained in this paper are new and effective mathematical tools and, also extensions of many results in the literature.

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