Density Fluctuations in Inflationary Models with Multiple Scalar Fields

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Abstract

Making use of the primordially isocurvature fluctuations, which are generated in inflationary models with multiple scalar fields, we make a phenomenological model that predicts formation of primordial black holes which can account for the massive compact halo objects recently observed.
In inflationary universe models [1] with multiple scalar fields, not only adiabatic [2] but also primordially isocurvature fluctuations [3] are generated in general. For example, if we consider ordinary slow roll-over inflation models in Brans-Dicke gravity [4], the Brans-Dicke dilaton acquires isocurvature fluctuation out of quantum fluctuation in the inflationary regime. In this particular model, however, the isocurvature fluctuation does not play any important role in structure formation because the energy density of the Brans-Dicke field remains as small as \(\sim \omega^{-1}\) times the total energy density, where \(\omega \gtrsim 500\) is the Brans-Dicke parameter [5].

On the other hand, primordially isocurvature fluctuations can be cosmologically important if energy density of its carrier becomes significant in a later epoch [3]. Furthermore it is relatively easier to imprint a nontrivial feature on the spectral shape of the isocurvature fluctuations. Making use of this property here we construct a model which possesses a peak in the spectrum of total density fluctuation at the horizon crossing. Then we can produce a significant amount of primordial black holes (PBHs) around the horizon mass scale when the fluctuation at the horizon crossing becomes maximal. To be specific, we choose the model parameters so that one can account for the observed massive compact halo objects (MACHOs) [7] in terms of PBHs thus produced. That is, our goal here is to produce PBHs of mass \(\sim 0.1 M_\odot\) with the abundance of \(\sim 20\%\) of the halo mass, corresponding to about \(10^{-3}\) times the critical density [8].

PBHs are formed if initial density fluctuations grow sufficiently and a high density region collapses within its gravitational radius. First let us review its formation process. The background spacetime of the early universe dominated by radiation is satisfactorily described by the spatially-flat Friedmann universe,

\[
ds^2 = -dt^2 + a^2(t) \left[ dr^2 + r^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right) \right],
\]

whose expansion rate is given by

\[
H^2(t) \equiv \left( \frac{\dot{a}}{a} \right)^2 = \frac{8\pi G}{3} \rho(t),
\]

with \(\rho(t)\) being the background energy density and a dot denotes time derivation. Following Carr [9], let us consider a spherically symmetric high density region with its initial radius, \(R(t_0)\), larger than the horizon scale \(\sim t_0\). The assumption of spherical symmetry will be justified below. Then the perturbed region locally constitutes a spatially closed Friedmann universe with a metric,

\[
ds^2 = -dt'^2 + R^2(t') \left[ \frac{dr'}{1 - \kappa r'^2} + r'^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right], \quad \kappa > 0,
\]
Then the Einstein equation reads

\[ H^2(t') \equiv \left( \frac{1}{R} \frac{dR}{dt'} \right)^2 = \frac{8\pi G}{3} \rho_+(t') - \frac{\kappa}{R^2(t')}, \]

there, where \( \rho_+ \) is the local energy density. One can choose the coordinate so that both the background and the perturbed region have the same expansion rate initially at \( t = t' = t_0 \). Then the initial density contrast, \( \delta_0 \), satisfies

\[ \delta_0 \equiv \frac{\rho_{+0} - \rho_0}{\rho_0} = \frac{\kappa}{H_0^2 R_0^2}, \]

where a subscript 0 implies values at \( t_0 \). It can be shown that the two time variables are related by \(^{(10)}\)

\[ (1 + \delta_0)^2 \frac{dt'}{R(t')} = \frac{dt}{a(t)}. \]

The perturbed region will eventually stop expanding at \( R \equiv R_c \), which is obtained from

\[ 0 = \frac{8\pi G}{3} \rho_0 (1 + \delta_0) \left( \frac{R_0}{R_c} \right)^4 - \frac{\kappa}{R_c^2} = H_0^2 (1 + \delta_0) \left( \frac{R_0}{R_c} \right)^4 - \frac{R_0^2 H_0^2}{R_c^2} \delta_0, \]

as

\[ R_c = \sqrt{\frac{1 + \delta_0}{\delta_0}} R_0 \simeq \delta_0^{-\frac{1}{2}} R_0, \]

corresponding to the epoch

\[ t_c \simeq \frac{t_0}{\delta_0}. \]

The perturbed region must be larger than the Jeans scale, \( R_J \), in order to contract further against the pressure gradient, while it should be smaller than the horizon scale to avoid formation of a separate universe. We thus require

\[ R_J \simeq c_s t_c \lesssim R_c \lesssim t_c, \]

or

\[ c_s \lesssim \frac{R_c}{t_c} \simeq \frac{R_0}{t_0} \delta_0^{\frac{1}{2}} \lesssim 1, \]

where \( c_s \) is the sound velocity equal to \( 1/\sqrt{3} \) in the radiation dominated era. Since \( R_0 \delta_0^{1/2}/t_0 \) is time independent, it suffices to calculate the constraint on \( \delta \) at a specific epoch, say, when the region enters the Hubble radius, \( 2R = 2t \). We find that the amplitude should lie in the range

\[ \frac{1}{3} \lesssim \delta(R = t) \lesssim 1. \]
The gravitational radius, $R_g$, of the perturbed region in the beginning of contraction with $R_c \simeq c_s t_c$ is given by

$$R_g = 2GM \simeq H^2 R_c^3 \simeq \frac{R_c^3}{t_c^2} \simeq c_s^2 R_c \lesssim R_c.$$ \hspace{1cm} (13)

This is somewhat smaller than $R_c$ but it also implies that a black hole will be formed soon after the high density region starts contraction. Thus we expect that a black hole with a mass around a horizon mass at $t = t_c$ will result and it has in fact been shown by numerical calculations \[11\] that the final mass of the black hole is about $\mathcal{O}(10^{10.5})$ times the horizon mass at that time. It has been discussed that these black holes do not accrete surrounding matter very much and that their mass do not increase even one order of magnitude \[9\]. Note also that evaporation due to the Hawking radiation is unimportant for $M \gg 10^{15}$ g.

Since the horizon mass at the time $t$ is given by

$$M_{\text{hor}} = 10^5 \left( \frac{t}{1 \text{sec}} \right) M_\odot ,$$ \hspace{1cm} (14)

what is required in order to produce a significant number of PBHs with mass $M \sim 0.1 M_\odot$ is the sufficient amplitude of density fluctuations on the horizon scale at $t \sim 10^{-6}$ sec. Because the initial mass fraction of PBHs, $\beta$, is related with the present fraction $\Omega_{\text{BH}}$ as

$$\beta = \frac{a(10^{-6} \text{sec})}{a(t_{\text{eq}})} \Omega_{\text{BH}} \simeq 10^{-8} \Omega_{\text{BH}},$$ \hspace{1cm} (15)

where $t_{\text{eq}}$ is the equality time, only an extremely tiny fraction of the universe, $\beta \simeq 10^{-11}$ should collapse into black holes.

That is, the probability of having a density contrast $1/3 \lesssim \delta \lesssim 1$ on the horizon scale at $t \sim 10^{-6}$ sec should be equal to $\beta$. Let us assume density fluctuations on the relevant scale obey the Gaussian statistics with the dispersion $\delta_{\text{BH}} \ll 1$, which is the correct property for the model introduced here. Then the probability of PBH formation is estimated as

$$\beta = \int_{1/3}^1 \frac{1}{\sqrt{2\pi} \delta_{\text{BH}}} \exp \left( -\frac{\delta^2}{2\delta_{\text{BH}}^2} \right) d\delta$$

$$\simeq \int_{1/3+\mathcal{O}(\delta_{\text{BH}}^2)}^1 \frac{1}{\sqrt{2\pi} \delta_{\text{BH}}} \exp \left( -\frac{\delta^2}{2\delta_{\text{BH}}^2} \right) d\delta$$

$$\simeq \delta_{\text{BH}} \exp \left( -\frac{1}{18\delta_{\text{BH}}^2} \right),$$ \hspace{1cm} (16)
which implies that we should have
\[ \delta_{\text{BH}} \simeq 0.05, \]  
(17)
to produce appropriate amount of PBHs. Although it is true that an exponential accuracy is required on the amplitude of fluctuations in order to produce a desired amount of PBHs we have not been able to obtain the correspondence between \( \delta_{\text{BH}} \) and \( \Omega_{\text{BH}} \) with such an accuracy because numerical coefficients appearing in the above expressions, such as 18 in (16), have been calculated based on a rather qualitative discussion. We therefore will not attempt exceedingly quantitative analysis in what follows.

Note also that for \( \delta_{\text{BH}} = 0.05 \) the threshold of PBH formation, \( \delta = 1/3 \), corresponds to 6.4 standard deviation. It has been argued by Doroshkevich [12] that such a high peak has very likely a spherically symmetric shape. Thus the assumption of spherical symmetry in the above discussion is justified and it is also expected that gravitational wave produced during PBH formation is negligibly small.

Since the primordial amplitude of density perturbations on large scales probed by the anisotropy of the background radiation [13] is known to be \( \bar{\delta} \simeq 10^{-5} \), the primordial fluctuations must have such a spectral shape that it has an amplitude of \( 10^{-5} \) on large scales, sharply increases by a factor of \( 10^4 \) on the mass scale of PBHs, and decreases again on smaller scales at the time of horizon crossing. It is difficult to produce such a spectrum of fluctuations in inflationary cosmology with a single component.

In generic inflationary models with a single scalar field \( \phi \), which drives inflation with a potential \( V[\phi] \), the root-mean-square amplitude of adiabatic fluctuations generated is given by
\[ \frac{\delta \rho (r)}{\rho (r)} \equiv \bar{\delta} (r(\phi)) \simeq \frac{8 \sqrt{6 \pi V[\phi]^{3/2}}}{V'[\phi] M_{Pl}^3}, \]  
(18)
on the comoving scale \( r(\phi) \) when that scale reenters the Hubble radius [2]. The right-hand-side is evaluated when the same scale leaves the horizon during inflation. Because of the slow variation of \( \phi \) and rapid cosmic expansion during inflation, (18) implies an almost scale-invariant spectrum in general. Nonetheless one could in principle obtain various shapes of fluctuation spectra making use of the nontrivial dependence of \( \bar{\delta} (r(\phi)) \) on \( V[\phi] \) [14]. In order to obtain a desirable spectrum for PBH formation with a mountain on a particular scale, we must employ a scalar potential with two breaks and a plateau in between [13]. Such a solution is not aesthetically appealing.
Here we instead consider an inflation model with multiple scalar fields in which not only adiabatic but also primordially isocurvature fluctuations are produced. In fact it is much easier to imprint nontrivial structure on the isocurvature spectrum as mentioned in the beginning.

We introduce three scalar fields $\phi_1$, $\phi_2$, and $\phi_3$ in order to generate the desired spectrum of density fluctuation. $\phi_1$ is the inflaton field which induces the new inflation \[16\] with a double-well potential, starting its evolution near the origin where its potential is approximated as

$$U[\phi_1] \approx V_0 - \frac{\lambda_1}{4} \phi_1^4,$$

so that the Hubble parameter during inflation, $H_I$, is given by

$$H_I^2 \approx \frac{8\pi V_0}{3M_{Pl}^2}.$$ 

On the other hand, $\phi_2$ is a long-lived scalar field which induces primordially isocurvature fluctuations that contribute to black hole formation later. Finally $\phi_3$ is an auxiliary field coupled to both $\phi_1$ and $\phi_2$ and it changes the effective mass of the latter to imprint a specific feature on the spectrum of its initial fluctuations.

We adopt the following model Lagrangian.

$$L = \frac{1}{2}(\partial \phi_1)^2 + \frac{1}{2}(\partial \phi_2)^2 + \frac{1}{2}(\partial \phi_3)^2 - V[\phi_1, \phi_2, \phi_3] + L_{\text{int}},$$

where $L_{\text{int}}$ represents interaction of $\phi_j$’s with other fields. Here $V[\phi_1, \phi_2, \phi_3]$ is the effective scalar potential which is a polynomial of $\phi_j$’s only up to the fourth order except for the inflaton sector $U[\phi_1]$:

$$V[\phi_1, \phi_2, \phi_3] = U[\phi_1] + \frac{\epsilon}{2}(\phi_1^2 - \phi_{1c}^2)\phi_3^2 + \frac{\lambda_3}{4} \phi_3^4 - \nu \frac{\phi_3^2 \phi_2^2}{2} + \frac{\lambda_2}{4} \phi_2^4 + \frac{1}{2} m_2^2 \phi_2^2,$$

where $\lambda_j$, $\epsilon$, $\nu$, $\phi_{1c}$, and $m_2$ are positive constants. $m_2$ is assumed to be much smaller than the scale of inflation, $H_I$, and it does not affect the dynamics of $\phi_2$ during inflation. Hence we ignore it for the moment.

Let us briefly outline how the system evolves before presenting its detailed description. In the early inflationary stage $\phi_1$ is smaller than $\phi_{1c}$, and $\phi_3$ has its potential minimum off the origin. Then $\phi_2$ also settles down to a nontrivial minimum, where it can have an effective mass larger than $H_I$ so that its quantum fluctuation is suppressed. As $\phi_1$ becomes larger than $\phi_{1c}$, $\phi_3$ rolls down to its origin. Then the potential of $\phi_2$ also becomes convex and its amplitude gradually decreases due to its quartic term. However, since its potential is now nearly flat,
its motion is extremely slow with its effective mass smaller than $H_I$ well until the end of inflation. In this stage quantum fluctuations are generated to $\phi_2$ with a nearly scale-invariant spectrum. Thus the initial spectrum of the isocurvature fluctuations has a scale-invariant spectrum with a cut-off on a large scale.

After inflation, the Hubble parameter starts to decrease in the reheating processes. As it becomes smaller than the effective mass of $\phi_2$, the latter starts rapid coherent oscillation. $\phi_2$ dissipates its energy in the same way as radiation in the beginning when its oscillation is governed by the quartic term. But later on when $\lambda_2 \phi_2^2$ becomes smaller than $m_2^2$, its energy density decreases more slowly in the same manner as nonrelativistic matter. Thus $\phi_2$ contributes to the total energy density more and more later, which implies that the total density fluctuation due to the primordially isocurvature fluctuations or $\phi_2$ grows with time. Since what is relevant for PBH formation is the magnitude of fluctuations at the horizon crossing, we thus obtain a spectrum with a larger amplitude on larger scale until the cut-off scale in the initial spectrum is reached, that is, it has a single peak on the mass scale of PBH formation.

In the above scenario we have assumed that $\phi_2$ survives until after the PBH formation. On the other hand, were $\phi_2$ stable, it would soon dominate the total energy density of the universe in conflict with the successful nucleosynthesis. As a natural possibility we assume that $\phi_2$ decays through gravitational interaction, so that it does not leave any unwanted relics with its only trace being the tiny amount of PBHs produced.

Having stated the outline of the scenario, we now proceed to its detailed description and obtain constraints on the model parameters to produce the right amount of PBHs on the right scale. First we consider the evolution of the homogeneous part of the fields. During inflation, the evolution of the inflaton is governed by the $U[\phi_1]$ part of the potential. Solving the equation of motion with the slow-roll approximation,

$$3H_I \dot{\phi}_1 \simeq -U'[\phi_1] \simeq \lambda_1 \phi_1^3,$$

we find

$$\lambda_1 \phi_1^2(t) = \frac{\lambda_1 \phi_{1i}^2}{1 + \frac{2\lambda_1 \phi_{1i}^2}{3H_I^2} H(t - t_i)},$$

where $\phi_{1i}$ is the field amplitude at some initial epoch $t_i$. The above approximate solution remains valid until $|U''[\phi_1]|$ becomes as large as $9H_I^2$ at $t \equiv t_f$, when inflationary expansion is terminated and we find $\phi_1(t_f) = 3H_I^2/\lambda_1$. Then (23) can
also be written as

\[ \lambda_1 \varphi_1^2(t) = \frac{3H_i^2}{2H_i(t_f - t) + 1} \equiv \frac{3H_i^2}{2\tau(t) + 1} \approx \frac{3H_i^2}{2\tau(t)}, \quad (24) \]

where \( \tau(t) \) is the \( e \)-folding number of exponential expansion after \( t < t_f \) and the last approximation is valid when \( \tau \gg 1 \). From now on, we often use \( \tau(t) \) as a new time variable or to refer to the comoving scale leaving the Hubble radius at \( t \). Note that it is a decreasing function of \( t \).

As stated above, we are taking a view that \( \varphi_1 \) determines fate of \( \varphi_3 \) and that \( \varphi_3 \) controls evolution of \( \varphi_2 \) but not vice versa. In order that \( \varphi_3 \) does not affect evolution of \( \varphi_1 \) the inequality

\[ \lambda_1 \lambda_3 \gg \epsilon^2, \quad (25) \]

must be satisfied, while we must have

\[ \lambda_2 \lambda_3 \gg \nu^2, \quad (26) \]

so that \( \varphi_2 \) does not affect the motion of \( \varphi_3 \). We assume these inequalities hold below.

When \( \varphi_1 < \varphi_{1c} \), both \( \varphi_2 \) and \( \varphi_3 \) have nontrivial minima which we denote by \( \varphi_{2m} \) and \( \varphi_{3m} \), respectively. From

\[ V_2 = -\nu \varphi_2^2 \varphi_2 + \lambda_2 \varphi_2^3 = 0, \quad (27) \]

and

\[ V_3 = \epsilon(\varphi_1^2 - \varphi_{1c}^2) \varphi_3 + \lambda_3 \varphi_3^3 - \nu \varphi_2^2 \varphi_3 = 0, \quad (28) \]

with \( V_j \equiv \partial V/\partial \varphi_j \), we find

\[ \lambda_2 \varphi_{2m}^2(t) = \nu \varphi_{3m}^2(t) \quad (29) \]

\[ \lambda_3 \varphi_{3m}^2(t) = \epsilon(\varphi_{1c}^2 - \varphi_1^2(t)) + \nu \varphi_{2m}^2(t) \approx \epsilon(\varphi_{1c}^2 - \varphi_1^2(t)), \quad (30) \]

where (26) was used in the last expression. In the early inflationary stage when \( \varphi_1 \ll \varphi_{1c} \), the effective mass-squared of \( \varphi_j \), \( V_{jj} \), at the potential minimum is given by

\[ V_{22}[\varphi_1, \varphi_{2m}, \varphi_{3m}] = 2\lambda_2 \varphi_{2m}^2 = \frac{2\nu \epsilon}{\lambda_3} (\varphi_{1c}^2 - \varphi_1^2) \approx \frac{2\nu \epsilon}{\lambda_3} \varphi_{1c}^2 = \frac{3\nu \epsilon}{\lambda_1 \lambda_3 \tau_c} H_i^2, \quad (31) \]

\[ V_{33}[\varphi_1, \varphi_{2m}, \varphi_{3m}] = \frac{3\epsilon}{\lambda_1 \tau_c} H_i^2, \quad (32) \]
where $\tau_c$ is the epoch when $\phi_1 = \phi_{1c}$. We choose parameters such that

$$\nu \epsilon > \lambda_1 \lambda_3 \tau_c, \quad \text{and} \quad \epsilon > \lambda_1 \tau_c.$$  \hfill (33)

Then $V_{22}$ and $V_{33}$ are larger than $H_I^2$ initially at the potential minimum, so that both $\phi_2$ and $\phi_3$ settle down to $\phi_{2m}(t)$ and $\phi_{3m}(t)$, respectively.

$\phi_{3m}(t)$ decreases down to zero at $\tau = \tau_c$ when $V_{33}[\phi_{3m}]$ also vanishes. Then $V_{33}[\phi_{3m} = 0]$ starts to increase according to $\phi_1$ and soon acquires a large positive value, which implies that $\phi_3$ practically traces the evolution of $\phi_{3m}(t)$ down to zero without delay. On the other hand, $\phi_2$ evolves somewhat differently because it does not acquire a positive effective mass from $\phi_3$ at the origin. Although $\phi_2(t)$ traces $\phi_{2m}(t)$ initially, as $V_{22}[\phi_{2m}]$ becomes smaller it can no longer catch up with $\phi_{2m}(t)$. From a generic property of a scalar field with a small mass in the De Sitter background, one can show that this happens when the inequality

$$\left| \frac{1}{\phi_{2m}(t)} \frac{d\phi_{2m}(t)}{dt} \right| > \frac{V_{22}[\phi_{2m}]}{3H},$$  \hfill (34)

gets satisfied, or at

$$\tau = \left( 1 + \sqrt{\frac{\lambda_1 \lambda_2}{2 \nu \epsilon}} \right) \tau_c \equiv \tau_l \approx \tau_c,$$  \hfill (35)

with

$$\phi_2^2(\tau_l) = \phi_{2l}^2 = \frac{\nu \epsilon}{2 \lambda_1 \lambda_3} \frac{3H_I^2}{2 \lambda_2 \tau_l}.$$  \hfill (36)

Thus $\phi_2$ slows down its evolution. In the meantime $\phi_3$ vanishes. Then $\phi_2$ is governed by the quartic potential. We can therefore summarize its evolution during inflation as

$$\phi_2^2(\tau) \equiv \begin{cases} 
\phi_{2m}^2(\tau) = \frac{3\nu \epsilon H_I^2}{2 \lambda_1 \lambda_3} \left( \frac{1}{\tau_c} - \frac{1}{\tau} \right), & \tau \gtrsim \tau_l, \\
\phi_{2l}^2 \left[ 1 + \frac{2 \lambda_2 \phi_{2l}^2}{3H_I^2}(\tau_l - \tau) \right]^{-1}, & 0 < \tau \lesssim \tau_l.
\end{cases}$$  \hfill (37)

Since $\lambda_2 \phi_{2l}^2$ is adequately smaller than $H_I^2$, $\phi_2$ remains practically constant in the latter regime. Let us also write down time dependence of its effective mass-squared for later use.

$$V_{22}[\phi_2] \equiv \begin{cases} 
\frac{\nu \epsilon H_I^2}{\lambda_1 \lambda_3} \left( \frac{1}{\tau_c} - \frac{1}{\tau} \right), & \tau \gtrsim \tau_l, \\
3 \lambda_2 \phi_{2l}^2 \left[ 1 + \frac{2 \lambda_2 \phi_{2l}^2}{3H_I^2}(\tau_l - \tau) \right]^{-1}, & 0 < \tau \lesssim \tau_l.
\end{cases}$$  \hfill (38)
Next we consider fluctuations in both the scalar fields and metric variables in a consistent manner. We adopt Bardeen’s gauge-invariant variables $\Phi_A$ and $\Phi_H$ [17], with which the perturbed metric can be written as

$$ds^2 = -(1 + 2\Phi_A)dt^2 + a(t)^2(1 + 2\Phi_H)dx^2,$$

in the longitudinal gauge. In this gauge scalar field fluctuation $\delta\phi_j$ coincides with the corresponding gauge-invariant variable by itself.

Assuming an $\exp(ikx)$ spatial dependence and working in the Fourier space, the perturbed Einstein and scaler field equations are given by

$$\Phi_A + \Phi_H = 0 \quad (40)$$

$$\dot{\Phi}_H + H\Phi_H = -4\pi G(\dot{\phi}_1\delta\phi_1 + \dot{\phi}_2\delta\phi_2 + \dot{\phi}_3\delta\phi_3) \quad (41)$$

$$\ddot{\delta}\phi_j + 3H\dot{\delta}\phi_j + \left(\frac{k^2}{a^2(t)} + V_{jj}\right)\delta\phi_j = 2V_j\Phi_A + \Phi_A\dot{\phi}_j - 3\dot{\Phi}_H\dot{\phi}_j - \sum_{i\neq j} V_{ji}\delta\phi_i. \quad (42)$$

Note that all the fluctuation variables are functions of $k$ and $t$.

The above system is quite complicated at a glance. However, using constraints on various model parameters we have obtained so far, i.e. (25), (26), and (33), it can somewhat be simplified. First, since $\phi_3$ has an effective mass larger than $H_I^2$ during inflation except in the vicinity of $\tau = \tau_c$, quantum fluctuation on $\delta\phi_3$ is suppressed and moreover its energy density practically vanishes by the end of inflation. We can therefore neglect fluctuations in $\phi_3$. On the other hand, we can show that

$$|\dot{\phi}_2| \sim \sqrt{\frac{\nu}{\lambda_2\lambda_3}}|\dot{\phi}_1| \ll |\dot{\phi}_1|, \quad (43)$$

with the help of (26) and (33). Hence (11) and (42) with $j = 1$ reduce to

$$\dot{\Phi}_H + H\Phi_H = -4\pi G\dot{\phi}_1\delta\phi_1 \quad (44)$$

$$\ddot{\delta}\phi_1 + 3H\dot{\delta}\phi_1 + \left(\frac{k^2}{a^2(t)} + V_{11}\right)\delta\phi_1 = 2V_1\Phi_A - 4\dot{\Phi}_H\dot{\phi}_1. \quad (45)$$

Thus only $\phi_1$ contributes to adiabatic fluctuations and it can be calculated in the same manner as in the new inflation model with a single scalar field. This is as expected because $\phi_1$ dominates the energy density during inflation. In fact since we are only interested in the growing mode on the super-horizon regime which turns out to be weakly time-dependent as can be seen from the final result, we can
consistently neglect time derivatives of metric perturbations and terms with two time derivatives in (44) and (45) during inflation. We thus find
\[
\Phi_H = -\Phi_A \approx -\frac{4\pi G \cdot \dot{\phi}_1 \delta \phi_1}{H_I}.
\] (46)

The resultant amplitude of scale-invariant adiabatic fluctuations depends on \(\lambda_1\) and one can normalize its value using the COBE observation \([13]\) as
\[
\lambda_1 = 1.3 \times 10^{-13}.
\] (47)

On the other hand, \(\delta \phi_2\) satisfies (42) with \(j = 2\). From quantum field theory in De Sitter spacetime, it has an RMS amplitude \(\delta \phi_2 \approx (H^2/2k^3)^{1/2}\) when the \(k\)-mode leaves the Hubble radius if \(V_{22}\) is not too large. Since \(V_2\) vanishes when \(\phi_2 = \phi_{2a}\) and \(V_2 \Phi_A\) remains small even for \(\tau \leq \tau_l\), we can neglect all the terms in the right-hand side, to yield
\[
\ddot{\delta \phi}_2 + 3H_I \dot{\delta \phi}_2 + V_{22} \delta \phi_2 \approx 0,
\] (48)

when \(k \ll a(t)H_I\). We can find a WKB solution with the appropriate initial condition,
\[
\delta \phi_2(k, t) \approx \sqrt{\frac{H_I^2}{2k^3}} \left(\frac{S(t_k)}{S(t)}\right)^{1/2} \exp \left\{\int_{t_k}^{t} \left[S(t')H_I - \frac{3}{2}H_I\right] dt'\right\},
\] (49)
\[
S(t) = \frac{3}{2}\sqrt{1 - \frac{4V_{22}}{9H^2}},
\]

where \(t_k\) is the time when \(k\)-mode leaves the Hubble radius: \(k = a(t_k)H_I\). The above expression is valid when \(|\dot{S}| \ll S^2\). In terms of \(\tau\), (49) can be expressed as
\[
\delta \phi_2(k, \tau) \approx \sqrt{\frac{H_I^2}{2k^3}} \left(\frac{S(\tau_k)}{S(\tau)}\right)^{1/2} \exp \left\{\int_{\tau}^{\tau_k} S(\tau') d\tau' - \frac{3}{2}(\tau_k - \tau)\right\},
\] (50)
\[
S(\tau) = \begin{cases} 
\frac{3}{2} \left(1 - \frac{\tau}{\tau_c} + \frac{\tau}{\tau}\right)^{1/2}, & \tau \gtrsim \tau_l, \\
\frac{3}{2} \left(1 - \frac{3\lambda \phi^2}{3H_I^2} \left[1 + \frac{3\lambda \phi^2}{3H_I^2} (\tau_l - \tau)\right]^{-1}\right)^{1/2}, & 0 < \tau \lesssim \tau_l,
\end{cases}
\]

where \(\tau_k \equiv \tau(t_k)\) and \(\tau_s \equiv \frac{4\nu_c}{3\lambda_1 \lambda_3}\). The above equality is valid until the end of inflation at \(t = t_f\) or \(\tau = 0\).
Let us assume the universe is rapidly and efficiently reheated at \( t = t_f \) for simplicity to avoid further complexity. (see, e.g. [18] for recent discussion on efficient reheating.) Then the reheat temperature is given by

\[
T_R \approx 0.1 \sqrt{H_1 M_{Pl}}.
\] (51)

If there is no further significant entropy production later, one can calculate the epoch, \( \tau(L) \), when the comoving length scale corresponding to \( L \) pc today left the Hubble horizon during inflation as

\[
\tau(L) = 37 + \ln \left( \frac{L}{1 \text{pc}} \right) + \frac{1}{2} \ln \left( \frac{H_I}{10^{10} \text{ GeV}} \right).
\] (52)

Then the comoving horizon scale at \( t = 10^{-6} \text{ sec} \), or \( L = 0.03 \text{ pc} \) corresponds to \( \tau \approx 34 \equiv \tau_m \) and the present horizon scale \( \simeq 3000 \text{ Mpc} \) to \( \tau \approx 59 \).

On the other hand, \( \phi_2(t) \) and \( \delta \phi_2(k, t) \) evolve according to

\[
\ddot{\phi}_2 + 3H \dot{\phi}_2 + \lambda_2 \phi_2^3 + m_2^2 \phi_2 = 0,
\] (53)

\[
\ddot{\delta \phi}_2 + 3H \dot{\delta \phi}_2 + (3\lambda_2 \phi_2^2 + m_2^2) \delta \phi_2 \approx 0,
\] (54)

where the Hubble parameter is now time-dependent: \( H = 1/2t \), and the latter equation is valid for \( k \ll aH \).

When \( H^2 \) becomes smaller than \( \lambda_2 \phi_2^2 (\gg m_2^2) \), both \( \phi_2 \) and \( \delta \phi_2 \) start rapid oscillation around the origin. Using (51) one can express the amplitude of the gauge-invariant comoving fractional density perturbation of \( \phi_2 \), \( \Delta_2 \), as

\[
\Delta_2 = \frac{1}{\rho_2} \left( \dot{\phi}_2 \delta \phi_2 - \dot{\delta \phi}_2 \phi_2 - \phi_2 \Phi_A \right) \approx 4 \left. \frac{\delta \phi_2}{\phi_2} \right|_{\tau=0} \equiv 4 \frac{\delta \phi_2}{\phi_2},
\] (55)

in the beginning of oscillation. Here

\[
\rho_2 \equiv \frac{1}{2} \phi_2^2 + \frac{\lambda_2}{4} \phi_2^4 + \frac{1}{2} m_2^2 \phi_2^2
\] (56)

is the energy density of \( \phi_2 \).

Using the virial theorem one can easily show that it decreases in proportion to \( a^{-4}(t) \) as long as \( \lambda_2 \phi_2^2 \gg m_2^2 \). Thus the amplitude of \( \phi_2 \) decreases with \( a^{-1}(t) \). On the other hand, \( \delta \phi_2 \) has a rapidly oscillating mass term when \( \lambda_2 \phi_2^2 \gg m_2^2 \), which causes parametric amplification. We have numerically solved equations (53) and (54) with various initial conditions with \( |\phi_2| \gg |\delta \phi_2| \) initially. We have found
that in all cases the amplitude of $\delta\phi_2$ remains constant as long as $m_2$ is negligible. Thus $\Delta_2$ increases in proportion to $t^{1/2}$ in this regime and $\Delta_2$ becomes as large as

$$\Delta_2 \cong 4\frac{\delta\phi_{2f}}{\phi_{2f}} \left( \frac{\lambda_2 \phi^2_{2f}}{m_2^2} \right)^{1/2},$$

(57)

while the ratio of $\rho_2$ to the total energy density, $\rho_{\text{tot}}$, which is now dominated by radiation, remains constant:

$$\frac{\rho_2}{\rho_{\text{tot}}} = 2\pi \left( \frac{\phi_{2f}}{M_{Pl}} \right)^2.$$  

(58)

As $\lambda_2 \phi_2^2$ becomes smaller than $m_2^2$, $\phi_2$ and $\delta\phi_2$ come to satisfy the same equation of motion, see (53) and (54), and $\Delta_2$ saturates to the constant value (57). At the same time $\rho_2$ starts to decrease less rapidly than radiation, in proportion to $a^{-3}(t)$. Since $\Delta_2$ contributes to the total comoving density fluctuation by the amplitude

$$\Delta \cong \frac{\rho_2}{\rho_{\text{tot}}} \Delta_2,$$

(59)

it increases in proportion to $a(t) \propto t^{1/2}$. In the beginning of this stage, we find $H^2 \cong m_2^4/\lambda_2 \phi_{2f}^2$, to yield

$$\Delta \cong 8\pi \frac{\delta\phi_{2f}}{\phi_{2f}} \left( \frac{\lambda_2 \phi^2_{2f}}{m_2^2} \right)^{1/2} \left( \frac{\phi_{2f}}{M_{Pl}} \right)^2 \left( \frac{2m_2^2 t}{\sqrt{\lambda_2 \phi_{2f}}} \right)^{1/2},$$

(60)

at a later time $t$.

In order to relate it with the initial condition required for PBH formation, we must estimate it at the time $k$-mode reenters the Hubble radius, $t_k^*$, defined by

$$k = 2\pi a(t_k^*) H(t_k^*) = \frac{\pi a_f}{(t_k^* t_f)^{1/2}}.$$  

(61)

Since $k$ can also be expressed as $k = 2\pi a_f e^{-\tau_k} H_I$, the amplitude of comoving density fluctuation at $t = t_k^*$ is given by

$$\Delta(k, t_k^*) \cong 8\pi \frac{\delta\phi_{2f}}{\phi_{2f}} \left( \frac{\sqrt{\lambda_2 \phi_{2f}}}{H_I} \right)^{1/2} \left( \frac{\phi_{2f}}{M_{Pl}} \right)^2 e^{\tau_k}.$$ 

(62)

In the beginning of this article we discussed the necessary condition on the amplitude of fluctuations for PBH formation using the uniform Hubble constant gauge. Hence we should calculate the predicted amplitude in this gauge, which is a linear combination of $\Delta$ and the gauge-invariant velocity perturbation. However, in
the present case in which \( \Delta \) grows in proportion to \( a(t) \) in the radiation-dominant universe, one finds that the latter quantity vanishes and that density fluctuation in the uniform Hubble constant gauge coincides with \( \Delta \). Thus we finally obtain the quantity to be compared with \( \delta_{\text{BH}} \) in (10), namely, the root-mean-square amplitude of density fluctuation on scale \( r = 2\pi/k \) at the horizon crossing, \( \overline{\delta}(r) \), as

\[
\overline{\delta}(r) = \left[ \frac{4\pi k^3}{(2\pi)^3} |\Delta(k, t^*)|^2 \right]^{\frac{1}{2}} \cong 4 \left( \frac{\sqrt{\lambda_2} H_I \phi_{2f}^3}{M_{\text{Pl}}^4} \right)^{\frac{1}{2}} e^{\tau_k} C_f(\tau_k, \tau_c, \tau_*) ,
\]

with

\[
C_f(\tau_k, \tau_c, \tau_*) \equiv \left( \frac{S(\tau_k)}{S(0)} \right)^{\frac{1}{2}} \exp \left\{ \int_{0}^{\tau_k} S(\tau') d\tau' - \frac{3}{2} \tau_k \right\} ,
\]

where we have used (50). We also find

\[
\phi_{2f}^2 = \sqrt{\frac{3}{8}} \left( 1 + \sqrt{\frac{2}{3\tau_*}} \right)^{-1} \left( 1 + \sqrt{\frac{3\tau_*}{8}} \right)^{-1} \frac{3H_I^2}{2\lambda_2 \tau_c} ,
\]

from (30) and (67).

The remaining task is to choose values of parameters so that \( \overline{\delta}(r) \) has a peak on the comoving horizon scale at \( t = 10^{-6}\) sec, which we denote by \( r_m \), corresponding to \( \tau_k = \tau_m \cong 34 \), with its amplitude \( \overline{\delta}(r_m) \cong 0.05 \). We thus require

\[
\frac{d\ln \overline{\delta}(r)}{d\tau_k} = \frac{S'(\tau_k)}{2S(\tau_k)} + S(\tau_k) - \frac{1}{2}
\]

\[= -\frac{1}{4\tau_k} \left[ \frac{\tau_* \tau_c}{\tau_c - \tau_k} + \frac{\tau_* \tau_c}{\tau_c + \tau_* - \tau_k} \right] + \frac{3}{2} \left( 1 - \frac{\tau_*}{\tau_c} + \frac{\tau_*}{\tau_k} \right)^{\frac{1}{2}} - \frac{1}{2}
\]

vanishes at \( \tau_k = \tau_m \), which gives us a relation between \( \tau_c \) and \( \tau_* \). Since \( \tau_c \) roughly corresponds to the comoving scale where scale-invariance of primordial fluctuation \( \Delta_2 \) is broken, the peak at \( \overline{\delta}(r_m) \) becomes the sharper, the closer \( \tau_c \) approaches \( \tau_m \).

For example, if we take \( \tau_c = 30 \) we find

\[
\tau_* = \frac{4\nu^c}{3\lambda_1 \lambda_2} = 200,
\]

so that

\[
C_f = 0.13 \quad \text{and} \quad \phi_{2f}^2 = 0.045 \frac{H_I^2}{\lambda_2}.
\]

In order to have \( \overline{\delta}(r_m) = 0.05 \), we require

\[
\frac{1}{\sqrt{\lambda_2}} \left( \frac{H_I}{M_{\text{Pl}}} \right)^2 = 1.7 \times 10^{-15} ,
\]
which can easily be satisfied with some reasonable choices of $\lambda_2$ and $H_I$. However, it is not the final constraint. Since we are assuming that the universe is dominated by radiation at this time, we require

$$\frac{\rho_2}{\rho_{tot}} = 2\pi \left( \frac{\phi_{2f}}{M_{Pl}} \right)^2 \left( \frac{m_2^2}{\sqrt{\lambda_2 \phi_{2f} H_I}} \right)^{\frac{1}{2}} e^{\tau_m} \ll 1. \quad (70)$$

Furthermore $\phi_2$ should decay some time after $t = 10^{-6}$ sec so as not to dominate the energy density of the universe which would hamper the primordial nucleosynthesis. Assuming that it decays only through gravitational interaction, its life time is given by

$$\tau_{\phi_2} \cong \frac{M_{Pl}^2}{m_2^2} = 10^{-5.5} \left( \frac{m_2}{10^{6.5} \text{GeV}} \right)^{-3} \text{sec}. \quad (71)$$

Now we have displayed all the necessary equalities and inequalities the model parameters should satisfy. Since there is a wide range of allowed region in the multi-dimensional space of parameters, we do not work out the details of the constraints but simply give one example of their values with which all the requirements are satisfied.

$$
\begin{align*}
H_I &= 1.7 \times 10^{10} \text{GeV}, \\
m_2 &= 3.2 \times 10^6 \text{GeV}, \\
\lambda_1 &= 1.3 \times 10^{-13}, \\
\lambda_2 &= 1.4 \times 10^{-6}, \\
\lambda_3 &= \nu = 6.7 \times 10^{-8}, \\
\epsilon &= 2.0 \times 10^{-11},
\end{align*}
$$

(72)

for which $\rho_2/\rho_{tot} = 0.1$ at $t = 10^{-6}$ sec and inequalities (25) and (26) are maximally satisfied.

Thus we have reached a model with the desired feature making use of a simple polynomial potential (21). In order to set the order of magnitude of the mass scale of the black holes and that of their abundance correctly, we must tune some combinations of model parameters such as (67) and (69) with two digits’ accuracy. However, there exists a wide range of allowed region in the parameter space to realize it.

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