ZARISKI EQUISINGULARITY AND LIPSCHITZ STRATIFICATION OF A FAMILY OF SURFACE SINGULARITIES.

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Abstract. We consider the generic Zariski equisingular families of surface, not necessarily isolated, singularities in $\mathbb{C}^3$. We show that a natural stratification of such a family given by the singular set and the generic family of polar curves provides a Lipschitz stratification in the sense of Mostowski. In particular, such families are bi-Lipschitz trivial by trivializations obtained by integrating Lipschitz vector fields.

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1. SET-UP AND STATEMENT OF RESULTS

Let \( f(x, y, z, t) : (\mathbb{C}^{3+l}, 0) \to (\mathbb{C}, 0) \) be analytic. We consider such \( f \) as an analytic family \( f_t(x, y, z) = f(x, y, z, t) \) of analytic function germs, where \( t \in \mathbb{C}^l \) is a parameter. In what follows we suppress for simplicity the germ notation.

We denote by \( X = f^{-1}(0) \) and by \( \Sigma \) the singular set of \( X \). We always assume that the germs \( f_t \) are reduced, and that the system of coordinates is sufficiently generic, see the Transversality Assumptions below for a precise statement. In particular we assume that the projection \( \pi(x, y, z, t) = (x, y, t) \) restricted to \( X \) is finite.

Denote by \( C \) the polar set of \( \pi|_X \), i.e. the closure of the critical locus of the projection \( \pi \) restricted to the regular part of \( X \). The set \( C \) can be understood as a family of space curves (polar curves) parameterized by \( t \). Let

\[
S = \{ f(x, y, z; t) = f'_x(x, y, z; t) = 0 \} = \Sigma \cup C.
\]

The main goal of this paper is to show the following result.

**Theorem 1.1.** Suppose that the family \( X_t = f_t^{-1}(0) \) is generically Zariski equisingular. Then it is bi-Lipschitz trivial. That is, there are neighbourhoods \( \Omega \) of \( 0 \) in \( \mathbb{C}^3 \times \mathbb{C}^l \), \( \Omega_0 \) of \( 0 \) in \( \mathbb{C}^3 \), and \( U \) of \( 0 \) in \( \mathbb{C}^l \), and a bi-Lipschitz homeomorphism

\[
\Phi : \Omega_0 \times U \to \Omega,
\]

satisfying \( \Phi(x, y, z, t) = (\Psi(x, y, z, t), t), \Phi(x, y, z, 0) = (x, y, z, 0) \), such that

\[
\Phi(X_0 \times U) = X.
\]

Moreover, \( \{ X \setminus S, S \setminus T, T \} \), where \( T = \{ 0 \} \times \mathbb{C}^l \), defines a Lipschitz stratification of \( X \) in the sense of Mostowski, c.f. [4, 8]. In particular, the homeomorphism \( \Phi \) can be obtained by the integration of Lipschitz vector fields.

The non parameterized version, i.e. for \( l = 0 \), of Theorem 1.1 was proven in [5], and the general version, as stated above, was conjectured by J.-P Henry and T. Mostowski more than ten years ago. The bi-Lipschitz triviality for families of normal surface singularities in \( \mathbb{C}^3 \) was announced in [6]. Our proof is using some ideas of [6] and [1], in particular that of polar wedges. Nevertheless, our main idea of proof is different from that of [6]. Moreover, we show a much stronger bi-Lipschitz property, the existence of a Lipschitz stratification in the sense of Mostowski. This is important because not all bi-Lipschitz trivializations are obtained by integration of Lipschitz vector fields.

In order to show that \( \{ X \setminus S, S \setminus T, T \} \) is a Lipschitz stratification we do not use Mostowski’s regularity conditions on tangent spaces but an equivalent characterization based on the extension of stratified Lipschitz vector fields, see subsection 1.2. For this we use two, in a way complementary constructions, the polar wedges of [1] and [6] and the quasi-wings of [4]. The polar wedges cover neighbourhoods of the critical loci of linear projections, the quasi-wings their complements. Both can be understood as a generalized version of the classical wings. Actually we need a strong analytic form of the latter given by [9], in order to construct for an arbitrary real analytic arc, not contained in polar wedges, first a complex analytic wing and then a quasi-wing containing it, see Proposition 6.4.

There is one important feature, essential for our proof, that we would like to stress here. Virtually in all the geometric constructions of the proof, including the description of stratified Lipschitz vector fields to polar wedges in Proposition
1.1. Zariski equisingularity. Given a family of reduced analytic functions germs \( f_t(x, y, z) : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0) \) as above, we denote by \( \Delta(x, y, t) \) the discriminant of the projection \( \pi \) restricted to \( X \). This is a family of plane curve singularities parameterized by \( t \). We say that the family \( X_t \) is Zariski equisingular \((\text{with respect to the projection } \pi)\) if \( t \to \{ \Delta(x, y, t) = 0 \} \) is an equisingular family of plane curves, that is satisfying one of the standard equivalent definitions, see \([12, 10], \text{p. 623}\). We shall use in this paper the fact that a family of equisingular plane curves admits Puiseux expansion with parameter in the sense of \([9, \text{Theorem 2.2}]\). We shall use in this paper the fact that a family of equisingular plane curves admits Puiseux expansion with parameter in the sense of \([9, \text{Theorem 2.2}]\).

We say that the family \( X_t \) is \textit{generically Zariski equisingular} if it is Zariski equisingular after a generic linear change of coordinates \( x, y, z \).

In the proof of Theorem 1.1 we use the following precise assumptions on \( f \), called Transversality Assumptions, that are implied by the generic Zariski equisingularity.

Let us denote by \( \pi_b \) the projection \( \mathbb{C}^3 \times \mathbb{C}^l \to \mathbb{C}^2 \times \mathbb{C}^l \) parallel to \((0, -b, 1, 0)\), that is \( \pi_b(x, y, z, t) = (x, y + bz, t) \). We denote by \( \Delta_b(x, y, t) \) the discriminant of the projection \( \pi_b \) restricted to \( X \).

**Transversality Assumptions.** The tangent cone \( C_0(X_0) \) to \( X_0 = f_0^{-1}(0) \) does not contain the \( z \)-axis and that, for \( b \) and \( t \) small, the family of the discriminant loci \( \Delta_b = 0 \) is an equisingular family of plane curve singularities with respect to both parameters \( b, t \). Moreover, we suppose that \( \Delta_0 = 0 \) is transverse to the \( y \)-axis and that \( x = 0 \) is not a limit of tangent spaces to \( X_{reg} \).

**Remark 1.2.** Since Zariski equisingular families are equimultiple, see \([13, \text{or }]\) [Proposition 1.13], the above assumptions imply the following. The tangent cone \( C_0(X_t) \) does not contain \((0, -b, 1)\), for \( t \) and \( b \) small. The \( y \)-axis is transverse to every \( \{(x, y); \Delta_b(x, y, t) = 0 \} \), also for \( t \) and \( b \) small.

We now show that a generically Zariski equisingular family satisfies, after a linear change of coordinates \( x, y, z \), the Transversality Assumptions. First we need the following lemma.

**Lemma 1.3.** The family \( f_t(x, y, z) = 0 \) is generically Zariski equisingular if and only if, after a linear change of coordinates \( x, y, z \), the family \( f(x + az, y + bz, z, t) = 0 \) is Zariski equisingular with respect to parameters \( a, b, t \).

**Proof.** The ”if” part is obvious. We show the ”only if”. Let \( \Delta(x, y, a, b, t) \) be the reduced discriminant of \( f(x + az, y + bz, z, t) \). By assumption there is an open subset \( U \subset \mathbb{C}^2 \) such that this family of plane curve germs \( \Delta(x, y, a, b, t) = 0 \) is equisingular with respect to \( t \) for every \((a, b) \in U \). Fix a small neighbourhood \( V \) of the origin in \( \mathbb{C}^l \) so that the subset of parameters \((a, b, t) \in U \times V \) such that \( \Delta(x, y, a, b, t) = 0 \) changes the equisingularity type is a proper analytic subset of \( Y \subset U \times V \). Then \( Y \) cannot contain \( U \times \{ 0 \} \), this would contradict the Zariski equisingularity of \( \Delta = 0 \) for \((a, b) \in U \) arbitrary and fixed. This shows the claim. \( \square \)

Suppose now that the family \( f_t = 0 \) is generically Zariski equisingular and choose a generic line \( l \) in the parameter space of \((a, b) \in U \) in the notation of the proof of the above lemma. This line corresponds to a hyperplane \( H \subset \mathbb{C}^3 \). Then we choose coordinates \( x, y, z \) so that \( H = \{ x = 0 \} \) and \( l \) corresponds to the pencil of
projections parallel to \((0, -b, 1) \in H\). Then in this system of coordinates \((x, y, z)\), \(f\) satisfies the Transversality Assumptions.

1.2. **Lipschitz stratification.** In [4] T. Mostowski introduced a sequence of conditions on the tangent spaces to the strata of a stratified subset of \(\mathbb{C}^n\) that, if satisfied, imply the Lipschitz triviality of the stratification along each stratum. T. Mostowski showed the existence of such stratifications for germs of complex analytic subsets of \(\mathbb{C}^n\), though there is no canonical Lipschitz stratification in the sense of Mostowski in general. We do not state these conditions in this paper since we are going to use an equivalent definition of Lipschitz stratification. We refer the interested reader to [4], [7], [8], [3].

In [5] Mostowski gave a criterion for the codimension one stratum of Lipschitz stratification of a complex surface germ in \(\mathbb{C}^3\), see the second example on pages 320-321 of [5]. This criterion implies that a generic polar curve can be chosen as such a stratum. It is not difficult to complete Mostowski’s argument and show Theorem 1.1 in the non-parameterized case \((l = 0)\). In subsection 5.1 we give a different proof that implies the parameterized case.

Mostowski’s conditions imply the existence of extensions of Lipschitz stratified vector fields from lower dimensional to higher dimensional strata, property which was shown in [7] equivalent to Mostowski’s conditions. Let us recall this equivalent definition. It is convenient to express Mostowski’s stratification in terms of its skelata, that is the union of strata of dimension \(\leq k\). Let \(X \subset \mathbb{C}^n\) be a complex analytic subset of dimension \(d\) and let

\[
X = X^d \supset X^{d-1} \supset \cdots \supset X^l \neq \emptyset,
\]

\(l \geq 0\), \(X^{l-1} = \emptyset\), be its filtration by complex analytic sets such that each \(X^k \setminus X^{k-1}\) is either empty or nonsingular of pure dimension \(k\). Then, by Proposition 1.5 of [7], [2] is a Lipschitz stratification if and only if one of the following equivalent conditions hold:

(i) There exists \(c > 0\) such that for every \(W \subset X\) satisfying \(X^{j-1} \subseteq W \subset X^j\), each Lipschitz stratified vector field on \(W\) with a Lipschitz constant \(L\), bounded on \(W \cap X^j\) by \(K\), can be extended to a Lipschitz stratified vector field on \(X^j\) with a Lipschitz constant \(c(L + K)\).

(ii) There exists \(c > 0\) such that for every \(W = X^{j-1} \cup \{q\}, q \in X^j\), each Lipschitz stratified vector field on \(W\) with a Lipschitz constant \(L\), bounded on \(W \cap X^j\) by \(K\), can be extended to a Lipschitz stratified vector field on \(W \cup \{q\}, q' \in X^j\), with a Lipschitz constant \(c(L + K)\).

Here by a stratified vector field we mean a vector field tangent to strata. In our particular case, stratification \(\{X \setminus S, S \setminus T, T\}\) it Lipschitz if and only if there is a constant \(c > 0\) such that:

(L1) for every couple of points \(q, q' \in S \setminus T\), every stratified Lipschitz vector field on \(T \cup \{q\}\), with Lipschitz constant \(L\) and bounded by \(M\), can be extended to a Lipschitz stratified vector field on \(T \cup \{q, q'\}\) with Lipschitz constant \(c(L + M)\).

(L2) for every couple of points \(q, q' \in X \setminus S\), every stratified Lipschitz vector field on \(S \cup \{q\}\) with Lipschitz constant \(L\) and bounded by \(M\), can be extended to a Lipschitz vector field on \(S \cup \{q, q'\}\) with Lipschitz constant \(c(L + M)\).
In order to show the conditions (L1) and (L2) we consider two geometric constructions, the quasi-wings of Mostowski [4] and the polar wedges of [1] and [6] that, as sets, together cover the whole $\mathcal{X}$. We first show the (L1) condition in general and the (L2) condition on polar wedges. This part of the proof is based on a complete description of the stratified Lipschitz vector fields on polar wedges in terms of their parameterizations, see Section 4. Note that we compare points in polar wedges, work with fractional powers, using parameterizations over the same allowable sector, see the Subsection 3.1 for more details. For the proof of (L2) on the quasi-wings we employ the following strategy. If Mostowski’s conditions fail to hold then they would fail along real analytic arcs $q(s), q'(s), s \in [0, \varepsilon)$, see [4] Lemma 6.2 or the valuative Mostowski’s conditions of [3]. For such arcs, however, if they are not in the union of polar wedges, we can construct quasi-wings containing them. Let us denote those quasiwings by $\mathcal{QW}$ and $\mathcal{QW}'$. Then we show that the stratification $\{\mathcal{QW} \cup \mathcal{QW}' \setminus S, S \setminus T, T\}$ satisfies criterion (L2) on the arcs $q(s), q'(s)$.

1.3. Notation and conventions. In what follows we often use the following notations. For two complex function germs $f, g : (\mathbb{C}^k, 0) \to (\mathbb{C}, 0)$ we write:

(1) $|f(x)| \lesssim |g(x)|$ (or $f = O(g)$) if $|f(x)| \leq c|g(x)|$, $c > 0$ a given constant, in a neighbourhood of 0 (we also use $|f(x)| \gtrsim |g(x)|$ for $|g(x)| \lesssim |f(x)|$).

(2) $|f(x)| \sim |g(x)|$ if $|f(x)| \lesssim |g(x)| \lesssim |f(x)|$ in a neighbourhood of 0.

(3) $|f(x)| \ll |g(x)|$ (or $f = o(g)$) if the ratio $|f(x)| / |g(x)| \to 0$ as $\|x\| \to 0$.

While parameterizing analytic curve singularities or families of such singularities in $\mathbb{C}^2$ and $\mathbb{C}^3$ using Puiseux Theorem, we usually write $x = u^a$. We often have to replace such an exponent $a$ by its multiple in order for such parameterizations to remain analytic, but we keep denoting it by $a$ for simplicity. This makes no harm since we always work over an admissible sector as explained in subsection 3.1.

2. Families of polar curves

In this section we discuss how the families of polar curves to $\mathcal{X}$, associated to the projections $\pi_b$, $b \in \mathbb{C}$, depend to $b$. The main result is Proposition 2.3 (non parameterized case) and Proposition 2.4 (parameterized case). The proposition in the non parameterized case appeared in the proof of the Polar wedge lemma, i.e. Proposition 3.4, of [1]. The proofs of Propositions 2.3 and 2.4 are based on the key lemma, see Lemma 2.1 below, due to [2] and [11].

2.1. Non parameterized case. For simplicity we consider first the case of $f(x, y, z)$ without parameter. We assume that the coordinate system satisfies the Transversality Assumptions and therefore the family

$$(3) \quad F(X, Y, Z, b) := f(X, Y + bZ, Z),$$

parameterized by $b \in \mathbb{C}$ is Zariski equisingular for $b$ small. By this assumption the zero set of the discriminant $\Delta_F(X, Y, b)$ of $F$ satisfies the Puiseux with parameter theorem. The set $F = F'_x = 0$, is the union $S_F = \Sigma_F \cup C_F$ of the singular set $\Sigma_F$ of $F$ and the family of the polar curves $C_F$. It consists of finitely many irreducible components parameterized by

$$(4) \quad (u, b) \to (u^n, Y_l(u, b), Z_l(u, b), b)$$
with $Y_i, Z_i$ analytic. Then $(u^n, Y = Y_i(u, b), b)$ parameterizes a component of the discriminant locus $\Delta_F = 0$ of $F$.

The below key lemma is a version of the first formula on page 278 of [2] or a formula on page 465 of [11].

**Lemma 2.1.**

\[ Z_i = -\frac{\partial Y_i}{\partial b}. \]

**Proof.** We have

\[ F(u^n, Y_i, Z_i, b) = 0 = F'_Z(u^n, Y_i, Z_i, b). \]

We differentiate the first identity with respect to $b$ and use the second one to simplify the result

\[ 0 = F'_Y \frac{\partial Y_i}{\partial b} + F'_Z \frac{\partial Z_i}{\partial b} + F_b = f'_y(u^n, Y_i + bZ_i, Z_i) \left( \frac{\partial Y_i}{\partial b} + Z_i \right) \]

If $f'_y(u^n, Y_i + bZ_i, Z_i) \neq 0$ then the formula (5) holds. Note that in this case (4) parameterizes a family of polar curves $C_f$.

If $f'_y(u^n, Y_i + bZ_i, Z_i) = 0$ then, in addition to (5), we have $F'_Y(u^n, Y_i, Z_i, b) = 0$. Thus in this case (4) parameterizes a component of $\Sigma_F$. By the formula

\[ F'_Z(X, Y, Z, b) = bf'_y(X, Y + bZ, Z) + f'_z(X, Y + bZ, Z), \]

$(X, Y, Z, b) \in \Sigma_F$ if and only if $(x, y, z) = (X, Y + bZ, Z) \in \Sigma$, the singular set of $f$. Thus in this case the map

\[ (u, b) \rightarrow (u^n, y_i(u, b), \cdot b), \quad y_i = Y_i + bZ_i, \quad z_i = Z_i, \]

parameterizes a component of $\Sigma$. Moreover, by the Transversality Assumptions, the projection of $\Sigma$ on the $x$-axis is finite. Consequently, both $y_i = Y_i + bZ_i$, and $Z_i$ are independent of $b$ and (5) trivially holds. \( \square \)

We note that, if $f'_y(u^n, Y_i + bZ_i, Z_i) \neq 0$, i.e. if (4) parameterizes a component of $C_f$, then (8) parameterizes a family of polar curves in $f^{-1}(0)$ defined by the projections $\pi_b$. In both cases, the functions $y_i(u, b), z_i(u, b) = Z_i(u, b)$, and $Y_i(u, b)$ are related by

\[ z_i = -\partial Y_i/\partial b, \quad y_i = Y_i + bZ_i, \quad \partial y_i/\partial b = b\partial z_i/\partial b. \]

In particular, the expansion of $y_i$ cannot have a term linear in $b$.

By the Zariski equisingularity assumption for any two distinct branches $Y_i(u, b), Y_j(u, b)$ there is $k_{ij} \in \mathbb{N}_{\geq 0}$ such that $Y_i(u, b) - Y_j(u, b) = u^{k_{ij}} \text{unit}(u, b)$. By (9) this implies the following result.

**Lemma 2.2.** For $i \neq j$ there is $k_{ij} \in \mathbb{N}_{\geq 0}$ such that

\[ y_i(u, b) - y_j(u, b) = u^{k_{ij}} \text{unit}(u, b) \]

\[ z_i(u, b) - z_j(u, b) = O(u^{k_{ij}}). \]

The next result, that we prove later in the more general parameterized case, is crucial.
Proposition 2.3. There are integers \( m_i \in \mathbb{N}_{\geq 0} \) such that
\[
\begin{align*}
\varphi_i(u, b, t) &= y_i(u, 0, t) + b^2 u^{m_i} \varphi_i(u, b, t) \\
z_i(u, b, t) &= z_i(u, 0, t) + bu^{m_i} \psi_i(u, b, t)
\end{align*}
\]
with either \( \varphi_i(0, 0) \neq 0, \psi_i(0, 0) \neq 0 \) or, if \( \varphi_i \) parameterizes a component of \( \Sigma \) then \( \varphi_i \equiv \psi_i \equiv 0 \).

2.2. Parameterized case. We extend the results of the previous subsection to the parameterized case family
\[
F(X, Y, Z, b, t) := f(X, Y + bZ, Z, t),
\]
with \( f \) satisfying the Transversality Assumptions. Thus \( F \) is now Zariski equisingular with respect to the parameters \( b \) and \( t \) and therefore the discriminant \( \Delta(X, Y, b, t) \) of \( F \) with respect to \( Z \) satisfies the Puiseux with parameter theorem. Similarly to the non-parameterized case, \( S_F = \{ F = F' = 0 \} \) is parameterized by
\[
(u, b, t) \mapsto (u^n, \psi_i(u, b, t), Z_i(u, b, t), b, t)
\]
and consists of the singular locus \( \Sigma_F \) and a family \( C_F \) of polar curves, now parameterized by \( b \) and \( t \).

The lemma 2.1 still holds (with the same proof) so we have \( Z_i = -\partial Y_i/\partial b \). Then
\[
(u, b) \mapsto p_i(u, b, t) = (u^n, y_i(u, b, t), z_i(u, b, t), t), \quad y_i = Y_i + bZ_i, \quad z_i = Z_i.
\]
parameterize in \( \mathbb{C}^3 \times \mathbb{C}^l \) the families of polar curves with respect to the projections \( \pi_b \) with \( t \) being a parameter, or the branches of the singular locus \( \Sigma \). The relations \( \varphi \) are still satisfied.

Also the counterpart of Proposition 2.3 holds. We give its proof below.

Proposition 2.4. There are integers \( m_i \in \mathbb{N}_{\geq 0} \) and functions \( \varphi_i(u, b, t), \psi_i(u, b, t) \) such that
\[
\begin{align*}
y_i(u, b, t) &= y_i(u, 0, t) + b^2 u^{m_i} \varphi_i(u, b, t) \\
z_i(u, b, t) &= z_i(u, 0, t) + bu^{m_i} \psi_i(u, b, t).
\end{align*}
\]
Moreover, either \( \varphi_i \equiv \psi_i \equiv 0 \) if \( \ref{14} \) parameterizes a branch of \( a \) of \( \Sigma_f \) or \( \varphi_i(0, 0) \neq 0, \psi_i(0, 0) \neq 0 \) if \( \ref{14} \) parameterizes a family of polar curves.

Proof. If \( y_i(u, b, t) \) and \( z_i(u, b, t) \) are independent of \( b \) then \( \ref{14} \) parameterizes a branch of the singular locus \( \Sigma \). Therefore we suppose that one of them, and hence by \( \ref{8} \) both of them, depend not trivially on \( b \). Expand \( \frac{\partial f}{\partial b}(u, b, t) = \sum k \geq m a_k(b, t) u^k \) with \( a_m(b, t) \neq 0 \). To show the lemma it suffices to show that \( a_m(0) \neq 0 \).

Suppose, by contradiction, that \( a_m(0) = 0 \). Then there is a non-trivial solution \( (b(u), t(u)) \), with \( (b(0), t(0)) = 0 \), of the equation \( \frac{\partial f}{\partial b}(u, b, t) = 0 \). By the last identity of \( \ref{8} \), \( (b(u), t(u)) \) also solves \( \frac{\partial f}{\partial b} = 0 \). Recall that \( f'_z + b f'_y \) vanishes identically on \( \Sigma_f \). Thus computing \( \frac{\partial}{\partial b}(f'_z + b f'_y) \) on \( \ref{14} \), and replacing \( (u, b, t) \) by \( (u, b(u), t(u)) \) we get
\[
0 = \frac{\partial}{\partial b}(f'_z + b f'_y) = (f''_y + b f'_{yy}) \frac{\partial}{\partial b} + (f''_y + b f'_{yz}) \frac{\partial}{\partial b} + f'_y = f'_y.
\]
Therefore, in this case, \( \ref{14} \) parameterizes a component of \( \Sigma_f \). \( \square \)
Corollary 2.5.

(17) \( Y_i(u, b, t) = y_i(u, b, t) - bz_i(u, b, t) = y_i(u, 0, t) - bz_i(u, 0, t) + b^2 u^{m_i} \text{unit}(u, b, t). \)

Proof. It follows from (15) and (5). \( \square \)

The following lemma follows from the Zariski equisingularity assumption.

Lemma 2.6.

(18) \( y_i(u, b, t) - y_j(u, b, t) = u^{k_{ij}} \text{unit}(u, b, t) \)
\( z_i(u, b, t) - z_j(u, b, t) = O(u^{k_{ij}}) \)
\( Y_i(u, b, t) - Y_j(u, b, t) = u^{k_{ij}} \text{unit}(u, b, t) \)

and \( y_i(u, b, t) = O(u^n), z_i(u, b, t) = O(u^n). \)

Lemma 2.7. Let \( p_i(u, 0, t) = (u^n, y_i(u, 0, t), z_i(u, 0, t)) \) parameterize a family of polar curves. Then \( \text{dist}(p_i(u, 0, t), \Sigma) \geq |u|^{m_i}. \)

Proof. Fix a component \( \Sigma_x \) of \( \Sigma \) parameterized by \( (u^n, \tilde{y}_r(u, t), \tilde{z}_r(u, t), t). \) By Proposition 2.3 and Zariski equisingularity
\( y_i(u, b, t) - \tilde{y}_r(u, t) = (y_i(u, 0, t) - \tilde{y}_r(u, t)) + u^{m_i} b^2 \text{unit} = u^{k_{ir}} \text{unit}, \)
that is possible only if \( m_i \geq k_{ir}. \) This implies \( \text{dist}(p_i(u, 0, t), \Sigma_k) \geq |u|^{m_i}, \) since, for \( x \neq 0, \) we may suppose that \( u \) is unique. \( \square \)

3. Polar wedges

In this section we consider the polar wedges in the sense of [1] and [6]. The polar wedges are neighbourhoods of the polar curves that play a crucial role in our proof of Theorem 1.1. The formal definition is the following.

Definition 3.1 (Polar wedge). We call a polar wedge and denote it by \( \mathcal{PW}_i \) the image of the map \( p_i(u, b, t) \) defined by (14), that parameterizes a family of polar curves associated to the projection \( \pi_b, \) together with this parametrization.

Thus if \( p_i(u, b, t) \) of (14) is independent of \( b, \) that is it parameterizes a branch of the singular set \( \Sigma_f, \) then it does not define a polar wedge. Two polar wedges either coincide or are disjoint for \( u \neq 0. \) Moreover, either \( k_{ij} \leq \min\{m_i, m_j\} \) or \( k_{ij} > m_i = m_j. \)

3.1. Allowable sectors. In order to compare two polar wedges we use allowable sectors. Let \( \mathcal{PW}_i \) be a polar wedge parameterized by \( p_i \) and let \( \theta \) be an \( n \)-th root of unity. Then \( p_i(\theta u, b, t) \) is also a parameterization of \( \mathcal{PW}_i \) (it can be identical to \( p_i(u, b, t). \)) We say that a sector \( \Xi = \Xi_I = \{ u \in \mathbb{C}; \text{arg } u \in I \} \) is allowable if the interval \( I \subset \mathbb{R} \) is of length strictly smaller than \( 2\pi/n. \) If we consider only \( u \in \Xi \) then \( x = u^n \neq 0 \) uniquely defines \( u. \) Then we may write the parameterization (14) in terms of \( x, b, t \) assuming implicitly that we work over a sector \( \Xi \)

(19) \( p_i(x, b, t) = (x, y(x, b, t), z(x, b, t), t) \)

with

(20) \( y_i(x, b, t) = y_i(x, 0, t) + b^2 x^{m_i/n} \varphi_i(x, b, t) \)
\( z_i(x, b, t) = z_i(x, 0, t) + bx^{m_i/n} \psi_i(x, b, t). \)
We note that any two points in polar wedges $p_i(u_1, b_1, t_1) \in \mathcal{PW}_i$ and $p_j(u_2, b_2, t_2) \in \mathcal{PW}_j$ can be compared using parameterizations over the same allowable sector. Indeed, given nonzero $u_1, u_2$ there exists always an $n$-th root of unity $\theta$ and an allowable sector $\Xi_t$ that contains $u_1$ and $\theta u_2$.

### 3.2. Distance in polar wedges.

Having an allowable sector fixed we show below formulas for the distance between points inside one polar wedge and the distance between points of different polar wedges. Note that these formulas imply, in particular, that different polar wedges do not intersect outside $T = \{x = y = z = 0\}$. In order to avoid a heavy notation we do use special symbols for the restriction of a polar wedge to an allowable sector.

**Proposition 3.2.** For every polar wedge $\mathcal{PW}_i$ and for $x_1, x_2, b_1, b_2, t_1, t_2$ sufficiently small

\[
\|p_i(x_1, b_1, t_1) - p_i(x_2, b_2, t_2)\| \approx \max\{|t_1 - t_2|, |x_1 - x_2|, |b_1 - b_2| |x_1|^{m/n}\}
\]

For every pair of polar wedges $\mathcal{PW}_i, \mathcal{PW}_j$, if $k_{ij} \leq \min\{m_i, m_j\}$ (in particular if $m_i \neq m_j$) then

\[
\|p_i(x_1, b_1, t_1) - p_j(x_2, b_2, t_2)\| \approx \max\{|t_1 - t_2|, |x_1 - x_2|, |x_1|^{k_{ij}/n}\}
\]

and if $m_i = m_j = m$ then

\[
\|p_i(x_1, b_1, t_1) - p_j(x_2, b_2, t_2)\| \approx \max\{|t_1 - t_2|, |x_1 - x_2|, |x_1|^{k_{ij}/n}, |b_1 - b_2| |x_1|^{m/n}\}
\]

**Corollary 3.3.**

\[
\|p_i(x_1, b_1, t_1) - p_j(x_2, b_2, t_2)\|
\approx \|p_i(x_1, b_1, t_1) - p_j(x_1, b_1, t_1)\| + \|p_j(x_1, b_1, t_1) - p_j(x_2, b_2, t_2)\|.
\]

**Corollary 3.4.** [Lipschitz property]

There is $c > 0$ such that for all $x_1, x_2, b_1, b_2, t$ sufficiently small

\[
\|p_i(x_1, b_1, 0) - p_j(x_2, b_2, 0)\| \leq c\|p_i(x_1, b_1, t) - p_j(x_2, b_2, t)\|
\]

\[
\leq c^2\|p_i(x_1, b_1, 0) - p_j(x_2, b_2, 0)\|
\]

**Proof of Proposition 3.2.** We divide the proof in four steps.

1. **First reduction.**

   It suffices to show the formulas \((21), (22), (23)\) for $t_1 = t_2$. Indeed, it follows from the following observations. Firstly, $p(x, b, t_1) - p(x, b, t_2) = O(t_1 - t_2)$ because $p(u^n, b, t)$ is analytic. Therefore

   \[
   |t_1 - t_2| \leq \|p_i(x_1, b_1, t_1) - p_j(x_2, b_2, t_2)\|
   \leq \|p_i(x_1, b_1, t_1) - p_i(x_1, b_1, t_2)\| + \|p_i(x_1, b_1, t_2) - p_j(x_2, b_2, t_2)\|
   \lesssim |t_1 - t_2| + \|p_i(x_1, b_1, t_2) - p_j(x_2, b_2, t_2)\|,
   \]

   that shows the claim.
2. Second reduction.
We show that it suffices to show the formulas of the above proposition for the case
\( t = t_1 = t_2, x_1 = x_2 \). The argument is similar to the one above and is based on the
observation that
\[
\| p_i(x_1, b, t) - p_i(x_2, b, t) \| \sim |x_1 - x_2| \leq \| p_i(x_1, b, t) - p_j(x_2, b_2, t) \|.
\]

3. Proof of (21) and (22).
We assume \( t = t_1 = t_2, x = x_1 = x_2 \). Then (21) follows from (15) and (22) follows from
\[
y_i(x, b_1, t) - y_j(x, b_2, t) = (y_i(x, 0, t) - y_j(x, 0, t)) + (b_1^2 x^{m_1/n} \varphi_i(x, b_1, t) - b_2^2 x^{m_2/n} \varphi_j(x, b_2, t))
\]
and a similar formula for \( z_i(x, b_1, t) - z_j(x, b_2, t) \).

4. Proof of (23).
We assume \( t = t_1 = t_2, x = x_1 = x_2 \) and \( m = m_1 = m_2 \). Then
\[
(24) \quad y_i(x, b_1, t) - y_j(x, b_2, t) = (y_i(x, b_1, t) - y_j(x, b_1, t)) + (y_j(x, b_1, t) - y_j(x, b_2, t))
\]
\[
= x^{k_{ij}/n} \text{unit} + x^{m/n}(b_1^2 \varphi_i(x, b_1, t) - b_2^2 \varphi_j(x, b_2, t))
\]
\[
= x^{k_{ij}/n} \text{unit} + x^{m/n}(b_1 - b_2) O(\| (b_1, b_2) \|).
\]

\[
(25) \quad z_i(x, b_1, t) - z_j(x, b_2, t) = O(x^{k_{ij}/n}) + x^{m/n}(b_1 - b_2)(\text{unit} + O(\| (b_1, b_2) \|))
\]
Now (23) follows from (24), (25). Indeed, we may consider separately the cases:
\( |x|^{k_{ij}/n} \sim |b_1 - b_2||x|^{m/n} \), \( |x|^{k_{ij}/n} \text{ dominant} \), and \( |b_1 - b_2||x|^{m/n} \text{ dominant} \), and suppose that \( b_1, b_2 \) are small in comparison to the units. \( \square \)

4. STRATIFIED LIPSCHITZ VECTOR FIELDS ON POLAR WEDGES
In this section we describe completely the stratified Lipschitz vector fields on polar
wedges in terms of their parameterizations. Note that these descriptions are valid
only over allowable sectors.

Let \( \mathcal{PW}_i \) be a polar wedge parameterized by (14). We call the polar set \( C_i \),
parameterized by \( p_i(u, 0, t) \), the spine of \( \mathcal{PW}_i \). A vector field on \( \mathcal{PW}_i \) is stratified
if it is tangent to the strata: \( T, C_i \setminus T \), and to \( \mathcal{PW}_i \setminus C_i \).

4.1. Stratified Lipschitz vector fields on a single polar wedge. Let \( p_{i\ast}(v) \) be
a vector field defined on a subset of \( \mathcal{PW}_i \), where
\[
v(x, b, t) = \alpha(x, b, t) \frac{\partial}{\partial t} + \beta(x, b, t) \frac{\partial}{\partial x} + \delta(x, b, t) \frac{\partial}{\partial b}.
\]
We always suppose that the vector field \( p_{i\ast}(v) \) is well defined on \( \mathcal{PW}_i \), that is
independent of \( b \) if \( x = 0 \), and that it is stratified. These requirements mean that
\( \alpha(0, b, t) \) is independent of \( b \), \( \beta(0, b, t) = 0 \), \( \delta(0, b, t) = 0 \) if \( m_i = n \) in the notation of
(14), and that \( \delta(x, 0, t) = 0 \).

Suppose that a function \( h(x, b, t) \) defines a function \( \tilde{h} = h \circ p_{i\ast}^{-1} \) on \( \mathcal{PW}_i \), that is
\( h(0, b, t) \) does not depend on \( b \). Then, after Proposition 3.2 \( \tilde{h} \) is Lipschitz iff
\[
|h(x_1, b_1, t_1) - h(x_2, b_2, t_2)| \lesssim |t_1 - t_2| + |x_1 - x_2| + |b_1 - b_2||x_2|^{m/n}.
\]

Proposition 4.1. The vector fields \( p_{i\ast}(\frac{\partial}{\partial t}), p_{i\ast}(x \frac{\partial}{\partial x}), p_{i\ast}(b \frac{\partial}{\partial b}) \) are stratified Lipschitz
on \( \mathcal{PW}_i \).
Lemma 4.3. \[ \text{Lipschitz, we need the following elementary generalization.} \]

Proof. We show that each coordinate of these vector fields is Lipschitz. For this computation it is more convenient to use the parameter \( u \) instead of \( x \) since these vector fields are analytic in \( u, b, t \). For clarity we also drop the index \( i \) coming from the parameterization \( X \).

The \( t \)-coordinate of \( p_\alpha(\partial \overline{t}) \) equals \( 1 = \frac{\partial \alpha}{\partial t} \) and is Lipschitz. The \( x \)-coordinate of \( p_\alpha(\partial \overline{t}) \) vanishes identically. Let us show that the \( y \)-coordinate of \( p_\alpha(\partial \overline{t}) \) is Lipschitz (the argument for the \( z \) coordinate is similar).

\[
\left| \frac{\partial y}{\partial t}(u_1, b_1, t_1) - \frac{\partial y}{\partial t}(u_2, b_2, t_2) \right| \\
\leq \left| \frac{\partial y}{\partial t}(u_1, b_1, t_1) - \frac{\partial y}{\partial t}(u_1, b_1, t_2) \right| + \left| \frac{\partial y}{\partial t}(u_1, b_1, t_2) - \frac{\partial y}{\partial t}(u_2, b_1, t_2) \right| \\
+ \left| \frac{\partial y}{\partial t}(u_2, b_1, t_2) - \frac{\partial y}{\partial t}(u_2, b_2, t_2) \right| \lesssim \max\{ |t_1 - t_2|, |u_1^n - u_2^n|, |b_1 - b_2||u_2|^m \}.
\]

A similar computation works for \( p_\alpha(x \partial \overline{t}) = \frac{1}{n} p_\alpha(u \partial \overline{t}) \)

\[
\left| \frac{u}{\partial x}(u_1, b_1, t_1) - \frac{\partial y}{\partial x}(u_2, b_2, t_2) \right| \\
\leq \left| \frac{u}{\partial u}(u_1, b_1, t_1) - \frac{\partial y}{\partial x}(u_1, b_1, t_2) \right| + \left| \frac{u}{\partial u}(u_1, b_1, t_2) - \frac{u}{\partial x}(u_2, b_1, t_2) \right| \\
+ \left| \frac{u}{\partial u}(u_2, b_1, t_2) - \frac{\partial y}{\partial u}(u_2, b_2, t_2) \right| \lesssim \max\{ |t_1 - t_2|, |u_1^n - u_2^n|, |b_1 - b_2||u_2|^m \}.
\]

All the other cases can be checked in a similar way. \( \square \)

Proposition 4.2. The vector field of the form \( p_\alpha(v) \), defined on a subset \( U \) of \( \mathcal{PW}_i \) containing \( C_i \), is stratified Lipschitz iff the following conditions are satisfied:

1) \( \alpha \) satisfies (26);
2) \( |\beta| \lesssim |x| \) and \( \beta \) satisfies (20);
3) \( |\delta| \lesssim |b| \) and \( \delta x^{m/n} \) satisfies (20).

Proof. If \( p_\alpha(v) \) is Lipschitz then so is its \( t \)-coordinate, that is \( \alpha \). We claim that if \( \alpha \) satisfies (26) so do \( \alpha \frac{\partial \alpha}{\partial t} \) and \( \alpha \frac{\partial \alpha}{\partial u} \). This follows from Proposition 4.1 because the product of two Lipschitz functions is Lipschitz. This shows that \( p_\alpha(\alpha \frac{\partial \alpha}{\partial t}) \) is Lipschitz. By subtracting it from \( p_\alpha(v) \) we may assume that \( \alpha \equiv 0 \).

If \( p_\alpha(v) \) is Lipschitz then so is its \( x \)-coordinate, that is \( \beta \). Let \( (x, b, t) \in p_1^{-1}(U) \). Then, by the Lipschitz property between \( p(x, b, t) \) and \( p(0, b, t) \), we have \( |\beta| \lesssim |x| \) as claimed.

To use a similar argument to the previous "the product of Lipschitz functions is Lipschitz", we need the following elementary generalization.

Lemma 4.3. Suppose \( h : X \to \mathbb{C} \) is a Lipschitz function on a metric space \( X \) and let \( L_h := \{ f : X \to \mathbb{C}; \text{Lipschitz on } X, |f| \lesssim |h| \} \). If \( f, g \in L_h \), then \( \xi := fg/h \in L_h \) (here \( \xi \) is understood to be equal to 0 on the zero set of \( h \)).
Proof. Suppose $|h(q_2)| \geq |h(q_1)|$. Then $|fg(q_2) - fg(q_1)| \lesssim |h(q_2)| \text{dist}(q_1, q_2)$ and

$$|\xi(q_2) - \xi(q_1)| \leq \frac{|fg(q_2)h(q_1) - fg(q_1)h(q_2)|}{|h(q_1)h(q_2)|} \leq \frac{|fg(q_2)h(q_1) - fg(q_1)h(q_1)| + |fg(q_1)h(q_1) - fg(q_1)h(q_2)|}{|h(q_1)h(q_2)|} \lesssim \text{dist}(q_1, q_2).$$

We apply the above lemma to $\beta$, $p_{is}(x \frac{\partial}{\partial x})$, and $x$ respectively, to complete the proof of 2). Thus, by subtracting $p_{is}(\beta \frac{\partial}{\partial x})$ from $p_{is}(v)$ we may assume that $\beta \equiv 0$.

Consider now $p_{is}(\delta \frac{\partial}{\partial \theta}) = (0, \delta \frac{\partial}{\partial \theta}, \delta \frac{\partial}{\partial \theta}, 0)$. By Proposition 4.4, $p_{is}(\beta \frac{\partial}{\partial \theta})$ is Lipschitz and by (15) it satisfies $\|p_{is}(\beta \frac{\partial}{\partial \theta})\| \lesssim \|b\|^{m/n}|$. Therefore, by Lemma 4.3, if $\delta x^{m/n}$ satisfies (26), then $p_{is}(\delta \frac{\partial}{\partial \theta})$ is Lipschitz if we apply the lemma to $\delta x^{m/n}$, $p_{is}(\beta \frac{\partial}{\partial \theta})$ and $bx^{m/n}$.

Reciprocally, if $p_{is}(\delta \frac{\partial}{\partial \theta})$ is Lipschitz so is its $z$-coordinate $\delta \frac{\partial}{\partial \theta}$. Therefore, if we apply the Lemma 4.3 to $\delta \frac{\partial}{\partial \theta}$, $bx^{m/n}$ and $b \frac{\partial}{\partial \theta}$, then $\delta x^{m/n}$ satisfies (26). ☐

4.2. Lipschitz vector fields on the union of two polar wedges. Consider two polar wedges $PW_i$ and $PW_j$ parameterized by $p_i(x, b, t)$ and $p_j(x, b, t)$.

Let $\hat{h}$ be a function defined on a subset of $PW_i \cup PW_j$ by two functions $h_k(x, b, t)$, $k = i, j$. Then, after the Proposition 3.2, $\hat{h}$ is Lipschitz iff so are its restrictions $\hat{h}_i$ and $\hat{h}_j$ to $PW_i$ and $PW_j$ respectively, and

$$(27) \quad |h_i(x_1, b_1, t_1) - h_j(x_2, b_2, t_2)| \lesssim |t_1 - t_2| + |x_1 - x_2| + |x_1|^{k_i/n} + |b_1 - b_2||x_2|^{m/n},$$

where $m = \min\{m_i, m_j\}$.

Proposition 4.4. The vector fields given by $p_{ks}(v)$, $k = i, j$, where $v$ are $\frac{\partial}{\partial x}$, $x \frac{\partial}{\partial x}$, or $b \frac{\partial}{\partial \theta}$, are Lipschitz on $PW_i \cup PW_j$.

Proof. By Corollary 4.3 and Proposition 4.1 it suffices to check only the condition (27) for $t = t_1 = t_2$, $u = u_1 = u_2$, and $b = b_1 = b_2$. In this case the result follows from the fact that $\|p_i - p_j\| \lesssim u^{k_j}$ and the analyticity of $p_{is}(v)(u, b, t)$ and $p_{js}(v)(u, b, t)$.

For $k = i, j$ let $p_{ks}(v_k)$ be a vector field on a subset of $W_{\Xi, k}$ given by

$$v_k(x, b; t) = \alpha_k \frac{\partial}{\partial t} + \beta_k \frac{\partial}{\partial x} + \gamma_k \frac{\partial}{\partial \theta}.$$

Proposition 4.5. The vector field given by $p_{ks}(v_k)$, $k = i, j$, defined on a subset $U$ of $PW_i \cup PW_j$ containing $C_i \cup C_j$, is stratified Lipschitz iff the following conditions are satisfied:

0) each $p_{is}(v_k)$ is stratified Lipschitz on $U \cap PW_k$;
1) $\alpha_i, \alpha_j$ satisfy (27);
2) $\beta_i, \beta_j$ satisfy (27);
3) $\delta x^{m/n}, \delta x^{m/n}$ satisfy (27).

Proof. The proof is similar to the proof of Proposition 4.2 and it is based on Lemma 4.3 and Proposition 4.4. ☐
Remark 4.6. If \( \tilde{h}_i, \tilde{h}_j \) are stratified Lipschitz on \( \mathcal{PW}_i \) and \( \mathcal{PW}_j \) respectively, then, by Corollary 3.3, it suffices to check (27) for \( t = t_1 = t_2, u = u_1 = u_2, \) and \( b = b_1 = b_2. \) Therefore, in this case, (27) can be replaced by
\[
|h_i(x, b, t) - h_j(x, b, t)| \lesssim |x|^{k_i/n}.
\]

5. Proof of Theorem 1.1. Part I.

5.1. Extension of stratified Lipschitz vector fields on polar wedges in the non parameterized case. Let \( X = \{ f(x, y, z) = 0 \}, S_0 = \{ f(x, y, z) = f'_i(x, y, z) = 0 \}, \) and \( f \) satisfies the Transversality Assumptions. We show that \( \{ \mathcal{PW} \setminus S, S \setminus \{0\}, \{0\} \} \) is a Lipschitz stratification of \( \mathcal{PW} \cup \Sigma_f \) in the sense of Mostowski.

Given \( q_0 \in S \setminus \{0\} \) and a vector \( v_0 = v(q_0) \) tangent to \( S. \) Suppose \( q_0 \) belongs to a component \( S_i \) (a polar curve or a branch of the singular locus) of \( S \) parameterized by
\[
p_i(x) = (x, y_i(x), z_i(x)), \quad q_0 = p_i(x_0)
\]
and \( v_0 = p_i(\beta_0 \frac{\partial}{\partial x}). \) Then the vector on \( S \) defined on each \( S_j \) by \( v_j = p_{j*}(\beta x \frac{\partial}{\partial x}), \) with \( \beta = \beta_0/x_0, \) defines a Lipschitz extension of \( v_0. \) This shows (L1).

Consider a stratified Lipschitz vector field on \( S \cup \{q_0\} \) with \( q_0 = p_i(x_0, b_0) \in \mathcal{PW}_i \) defined by \( p_{j*} v_j \) on the component \( S_j \) of \( S, \) where
\[
v_j(x, b) = \beta_j \frac{\partial}{\partial x} + \delta_j \frac{\partial}{\partial b}.
\]
Thus, for \( j \neq i, \) the functions \( \beta_j \) and \( \delta_j \) are defined only for \( b = 0 \) (and hence \( \delta_j = 0 \) since the vector field is stratified). The functions \( \beta_i \) and \( \delta_i \) are defined on \( \{(x, b); b = 0\} \cup \{(x_0, b_0)\}. \) Denote \( \beta_0 = \beta_i(x_0, b_0), \) \( \delta_0 = \delta_i(x_0, b_0). \) By Propositions 4.2 and 4.5 it suffices to extend \( \beta_j \) and \( \delta_j \) to two families of functions, still denoted by \( \beta_j, \delta_j, \) that satisfy the conditions given in those propositions. We define
\[
\beta_j(x, b) = (\beta_0 - \beta_i(x_0, 0)) \frac{b}{b_0} \frac{x^m/n}{x_0^m/n} + \beta_j(x, 0)
\]
\[
\delta_j(x, b) = (\delta_0)/b_0.
\]
Then, because \( |\beta_0 - \beta_i(x_0, 0)| \lesssim |b_0||x_0|^{m/n}, \) the first summand of the right-hand side of (29) satisfies 2) of Propositions 4.2 and 4.5. The argument for (30) is similar because \( |\delta_0| \lesssim |b_0|. \) This completes the proof of Theorem 1.1 for \( \mathcal{PW} \cup \Sigma_f \) in the non-parameterized case.

Remark 5.1. If \( X \) has isolated singularity but there is an \( m_i > n \) then \( \{X \setminus \{0\}, \{0\}\} \) is not a Lipschitz stratification of \( X \) in the sense of Mostowski.

We show the claim of Remark 5.1. Let \( q_0 = p(x_0, b_0) \in X \setminus \{0\} \) be on the polar wedge parameterized by \( p(x, b) = (x, y(x, b), z(x, b)), \) \( x = u^n, \) where \( y, z \) are as in (11). Let \( v_0 = p_*(\frac{\partial}{\partial x}) \) be the vector tangent at \( q_0 = p(x_0, b_0) \) to \( X. \) We extend it by 0 to \( \{0\} \) and get a Lipschitz vector field on \( \{0\} \cup \{q_0\} \) with Lipschitz constant \( L = Cx_0^{m/n-1}, \) where \( C > 0 \) depends only on the polar wedge. Suppose we extend this vector field to \( q_1 = p(x_1, b_1), \) \( x_0 = x_1, \) by \( v_1 = p_*(\alpha_1 \frac{\partial}{\partial x} + \delta_1 \frac{\partial}{\partial b}) \) so that the extended
vector field has Lipschitz constant $L_1 = C_1 L$. By the Lipschitz property of the $x$-coordinate of this vector field $|\alpha_1| \leq C_1 L \|q_0 - q_1\| \sim C_1 L |b_0 - b_1| |x_0|^{m/n}$. Therefore, we can subtract from $v_1$ the vector $p_+(\alpha_1 \frac{\partial}{\partial x})$ without changing significantly the Lipschitz constant (just changing $C_1$). Thus we may assume that $\alpha_1 = 0$. By the Lipschitz property of the $y$ and $z$-coordinates of this vector field

\begin{align}
\tag{31} b_0 x_0^{m/n} \tilde{\varphi}(x_0, b_0) - \delta_1 b_1 x_0^{m/n} \tilde{\varphi}(x_0, b_1) &= O(\|b_0 - b_1\| |x_0|^{m/n}) L_1 \\
x_0^{m/n} \tilde{\psi}(x_0, b_0) - \delta_1 x_0^{m/n} \tilde{\psi}(x_0, b_1) &= O(\|b_0 - b_1\| |x_0|^{m/n}) L_1,
\end{align}

where $\tilde{\varphi}, \tilde{\psi}$ are units. Considering (31) as a system of linear equations with the unknowns 1 (in front of the first summand of both equations) and $\delta_1$, by Cramer’s rule,

$$1 \lesssim |x_0^{m/n-1}|, \quad |\delta_1| \lesssim |x_0^{m/n-1}|$$

that is impossible if we allow $x_0 \to 0$.

5.2. Parameterized case. By Corollary 3.4 and Propositions 4.2, 4.5, the map given $\mathcal{X}_0 \times T \to \mathcal{X}$, restricted to $(\mathcal{PW} \cup \Sigma_f) \cap \mathcal{X}_0$, defined in terms of the parameterizations of polar wedges by

$$(p_1(0, x, b), t) \to p_1(x, b; t),$$

is not only Lipschitz but also establishes a bijection between the Lipschitz vector fields. Therefore, $\{\mathcal{PW} \cup \Sigma_f \setminus S, S \setminus T, T\}$ is a Lipschitz stratification if and only if so is its intersection with $\mathcal{X}_0$ and the latter is a Lipschitz stratification by the non-parameterized case.

6. Quasiwings.

Let $\Delta(x, y, t)$ denote the discriminant of $f(x, y, z, t)$. Then the discriminant locus $\Delta = 0$ is the union of families of finitely many analytic curves parameterized by

\begin{align}
\tag{32} & (u, t) \to (u^n, y_i(u, t), t). \nonumber
\end{align}

By the Zariski equisingularity assumption we have

$$y_i(u, t) - y_i(u, t) = u^{k_i} \text{unit}(u, t)$$

and by the Transversality Assumptions $y_i(u, t) = O(u^n)$. Note that $y_i$ of (32) is either the projection of a polar branch, the one denoted by $y_i(u, 0, t)$ in (15), or parameterizes the projection of a branch of the singular locus $\Sigma$.

Given a parameterization of a family of analytic curves or a simple wing

\begin{align}
\tag{33} q(u, t) = (u^n, y(u, t), t). \nonumber
\end{align}

We assume $y(u, t) = O(u^n)$ and that for each discriminant branch (32), $y(u, t)$ satisfies

$$y(u, t) - y_i(u, t) = u^{l_i} \text{unit}(u, t).$$

Let $l \geq \max_i l_i$. Consider a map

\begin{align}
\tag{34} q(u, v, t) = (u^n, y(u, t) + u^l v, t) = (u^n, y(u, t) + u^l v, t), \nonumber
\end{align}

defined for complex $|v| < \varepsilon$ with $\varepsilon > 0$ small. Geometrically the image of $q$ is a wedge around the wing, the image of (33), inside the complement of the discriminant locus $\Delta = 0$. 

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Lemma 6.1. Let \( g(u, v, z, t) = f(q(u, v, t), z) \). Then the discriminant of \( g \) satisfies
\[
\Delta_g = u^N \text{unit}(u, v, t).
\]

Proof. Write the discriminant of \( f \)
\[
\Delta(u^n, y, t) = \text{unit}(u, y, t) \prod_i (y - y_i(u, t))^{d_i}.
\]
Then, by assumption \( l \geq \max_i l_i \),
\[
\Delta_g(u, v, t) = \Delta(u^n, y(u, t) + vu^l, t) = u^{\sum l_i d_i} \text{unit}(u, v, t).
\]
provided \( l > \max_{i \neq i_0} k_{i_0} \).
\[\square\]
Thus, after a ramification in \( u \), we may assume that the roots of \( g \) are analytic functions of the form \( z_\tau(u, v, t) = z_\tau(u^n, y(u, t)) \) are analytic and that
\[
z_\tau(u, v, t) - z_\nu(u, v, t) \simeq u^{\tau - \nu}.
\]
Moreover, by transversality of projection \( \pi \), \( z_\tau(u, t) = O(u^n) \).

Proposition 6.2. Suppose moreover that \( l_i \leq m_i \) for every polar discriminant branch \((32)\). Then the (first order) partial derivatives of the roots \( z_\tau(x, y, t) \) of \( f \) over the image of \((34)\) are bounded.

Therefore, in this case, the roots of \( g \) are of the form
\[
z_\tau(u, v, t) = z_\tau(u, t) + vu^l \tilde{\psi}(u, v, t).
\]

Proof. Let us denote the image of \((34)\) by \( W_q \). The derivative \( \frac{\partial}{\partial t}(z_\tau(x, y; t)) \) is bounded on \( W_q \) because \( z_\tau(u, v; t) \) is analytic in \( t \). Similarly \( x \frac{\partial}{\partial x}(z_\tau(x, y; t)) \) is bounded by \( x \) because \( z_\tau(u, v; t) \) is analytic in \( u \) and
\[
x \frac{\partial z_\tau}{\partial x} \simeq u \frac{\partial z_\tau}{\partial u} \lesssim u^n.
\]

Finally, \( \frac{\partial}{\partial y}(z_\tau(x, y, t)) \) is bounded on \( W_q \) by the conditions \( l_i \leq m_i \), \( l_i \leq l \), and \((15)\). Indeed, if this derivative were big, say \( |\frac{\partial}{\partial y}(z_\tau(x, y, t))| > N \), then the graph of \( z_\tau(x, y, t) \) on \( W_q \) would intersect a polar wedge \( PW_i \) for small \( b \), say for \( |b| < \varepsilon_N \). This is only possible if \( l_i \geq \max\{l, m_i\} \). Consequently we may assume that \( l = m_i = l_i \). In this case if we suppose both \( b \) and \( v \) sufficiently small this intersection is empty.
\[\square\]

We introduce below a version of quasi-wings and nicely-situated quasi-wings of [4].

Definition 6.3 (Quasi-wings). We say that the image of \( q(u, v, t) \) of \((31)\) is a regular wedge \( W_q \) if satisfies the assumptions of Proposition 6.2. Then a quasi-wing \( QW_\tau \) over \( W_q \) is the image of the map \( p_\tau(u, v, t) = (q(u, v, t), z_\tau(u, v, t)) \), where \( z_\tau \) is a root of \( f(q_\tau(u, v), z) \).

We say that quasi-wings \( QW_\tau, QW_\nu \) are nicely-situated if they lie over the same regular wedge \( W_q \).
6.1. Construction of quasi-wings. Consider a real analytic arc $\gamma(s)$, $s \in [0, \varepsilon)$, of the form
\begin{equation}
\gamma(s) = (s^n, y(s), z(s); t(s)), \quad y(s) = O(s^n), z(s) = O(s^n).
\end{equation}
We suppose, moreover, that $y(s) = O(s^n)$, $z(s) = O(s^n)$. Complexify $\gamma$ by setting $\gamma(u) = (u^n, y(u), z(u); t(u))$.

Let $(u^n, y_i(u, t), z_i(u, t), t)$ be a parameterization of the polar branch $C_i$, and let $(u^n, \tilde{y}_k(u, t), \tilde{z}_k(u, t), t)$ be a parameterization of the branch $\Sigma_k$ of the singular set $\Sigma$. Set
\[ l_i := \text{ord}_s \text{dist}(\gamma(s), C_i), \quad \tilde{l}_k := \text{ord}_s \text{dist}(\gamma(s), \Sigma_k). \]
Recall that $\pi(x, y, z, t) = (x, y, t)$. We shall make the following assumption.

**Assumption:** $\text{ord}_s \text{dist}(\pi(\gamma(s)), \pi(C_i)) = l_i$ and $\text{ord}_s \text{dist}(\pi(\gamma(s)), \pi(\Sigma_k)) = \tilde{l}_k$ and, moreover, $l_i \leq m_i$ for all $i$.

**Proposition 6.4.** If the arc $\gamma(s)$ satisfies the above assumption then there is a quasi-wing that contains it.

**Proof.** By [9] there is an arc-wise analytic local trivialization $\Phi : \mathbb{C}^3 \times T \to \mathbb{C}^3 \times T$ of $\mathcal{X}$. More precisely, there is a local homeomorphism $\Phi$ of the form
\begin{equation}
\Phi(x, y, z, t) = (\Psi_1(x, t), \Psi_2(x, y, t), \Psi_3(x, y, z, t), t),
\end{equation}
complex analytic with respect to $t$, such that both $\Phi$ and its inverse $\Phi^{-1}$ are real analytic on real analytic arcs. The homeomorphism $\Phi$ trivializes $\mathcal{X} = f^{-1}(0)$, the singular locus $\Sigma$ and the polar set $C$. It is a lift of a local arc-wise analytic trivialization $\tilde{\Phi} = (\Psi_1(x, t), \Psi_2(x, y, t), t) : \mathbb{C}^2 \times T \to \mathbb{C}^2 \times T$ the discriminant locus $\Delta = 0$.

By the arc analyticity of $\tilde{\Phi}^{-1}$, there exists an analytic arc $(s^n, \tilde{y}(s), t(s))$ such that $\tilde{\Phi}(s^n, \tilde{y}(s), t(s)) = (s^n, y(s), t(s))$. Then, by the arc-wise analyticity of $\tilde{\Phi}$, the map $h(s, t) = \tilde{\Phi}(s^n, \tilde{y}(s), t)$ is analytic in both $s$ and $t$, and its complexification $H(u, t)$ is a complex analytic wing containing $\pi(\gamma)$.

We note that for each polar component $C_i$
\begin{equation}
s^i \sim \text{dist}(\pi(\gamma(s)), \pi(C_i)) \sim |y(s) - y_i(s, t(s))| \sim |\tilde{y}(s) - y_i(s, 0)|,
\end{equation}
because $|y(s) - y_i(s, t(s))| = |\Psi_2(s^n, \tilde{y}(s), t(s)) - \Psi_2(s^n, \tilde{y}(s), t(s))|$. A similar property holds for each component $\Sigma_k$ of the singular locus. Denote $y(s, t) := \Psi_2(s^n, \tilde{y}(s), t)$ and by $y(u, t)$ its complexification. Then
\begin{equation}
y(u, t) - y_i(u, t) = u^i \text{unit}(u, t).
\end{equation}

Let $l \geq \max\{\max_i l_i, \max_k \tilde{l}_k\}$. The map
\[ q(u, v, t) = (u^n, y(u, t) + u^i v, t) \]
for $v$ small, parameterizes a regular wedge $\mathcal{W}_q$. Its inverse image $\pi^{-1}(\mathcal{W}_q)$ is a finite union of nicely-situated quasi-wings and one of them contains $\gamma$. \hfill \Box

**Corollary 6.5.** Suppose a real analytic arc $\gamma(s) = (s^n, y(s), z(s), t(s))$, not contained in the singular locus $\Sigma$, satisfies for every polar branch $C_i$, $\text{ord}_s \text{dist}(\gamma(s), C_i) = l_i \leq m_i$.

Then, for $b$ small and generic, $\gamma$ belongs to a quasi-wing in the coordinates $x, Y_b, z; t$, where $Y_b := y - bz$. 

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(Here by generic we mean in \( \{ b \in \mathbb{C}; |b| < \varepsilon \} \setminus A \), where \( A \) is finite. Moreover, one may choose \( \varepsilon > 0 \) independent of \( \gamma \).)

**Proof.** We denote \( \pi_b(x, y, z, t) := (x, y - bz, t) \). By the assumption one of \( y(s) - y_i(st) \) or \( z(s) - z_i(s, t) \) equals \( s^{1/3} \text{unit}(t, s) \) and the other one is \( O(s^{\delta}) \). Consider the expression

\[
|y(s) - y_i(s, t) - b(z(s) - z_i(s, t)) + b^2 s^{m_i}(\psi_i(s, b; t) - \varphi_i(s, b, t))|,
\]

that is the distance from \( \pi_b(\gamma) \) to the component \( \Delta_{b,i} \) of the discriminant locus \( \Delta_b \) of \( \pi_b \) corresponding to the polar wing \( \mathcal{W}_i \). By Corollary 2.29 \( \psi_i(0) - \varphi_i(0) \neq 0 \) and therefore for \( b \) small and generic the expression of (42) is \( \sim s^{\delta} \). More precisely, this may fail for at most two values of \( b \).

A similar but simpler argument can be applied to the distance of \( \gamma \) to the branches of singular set \( \Sigma \).

Thus the statement follows from Proposition 6.3. \( \square \)

7. Lipschitz vector fields on quasi-wings.

Let the quasi-wings \( \mathcal{QW}_r \) over a fixed regular wedge \( \mathcal{W}_q \) parameterized by (33) be given by

\[
p_r(u, v, t) = (u^n, y(u, v, t), z_r(u, v, y), t).
\]

We consider such parameterizations for \( u \) in an allowable sector \( \Xi = \Xi_I = \{ u \in \mathbb{C}; \text{arg} \ u \in I \} \). Then we may write these parameterizations in terms of \( t, x, v \) assuming implicitly that we work over a sector \( \Xi \) and, moreover, that \( z_r(x, v, t) \) is a single valued functions. Again, in order to avoid heavy notation we do use special symbols for the restriction of a quasi-wing parameterization to an allowable sector.

**Proposition 7.1.** For all \( \tau \) and for all \( x_1, x_2, v_1, v_2, t_1, t_2 \) sufficiently small

\[
\|p_r(x_1, v_1, t_1) - p_r(x_2, v_2, t_2)\| \sim \max\{|t_1 - t_2|, |x_1 - x_2|, |v_1 - v_2||x_2|^{1/n}\}.
\]

For every pair of parameterizations \( p_r, p_v \)

\[
\|p_r(x_1, v_1, t_1) - p_v(x_2, v_2, t_2)\|
\sim \|p_r(x_1, v_1, t_1) - p_r(x_2, v_2, t_2)\| + \|p_r(x_2, v_2, t_2) - p_v(x_2, v_2, t_2)\|
\sim \max\{|t_1 - t_2|, |x_1 - x_2|, |x_2|^{r_{rv}/n}, |v_1 - v_2||x_2|^{1/n}\},
\]

where \( r_{rv} \) are given by (36).

Given two well-situated quasi-wings. Let \( h \) be a function defined on a subset of \( \mathcal{QW}_r \cup \mathcal{QW}_v \) whose restrictions to \( \mathcal{QW}_r, \mathcal{QW}_v \) we denote by \( h_r(x, v, t), h_v(x, v, t) \) respectively. Then, after Proposition 7.1 \( h \) is Lipschitz iff so are its restrictions \( h_r, h_v \) and

\[
|h_r(x_1, v_1, t_1) - h_v(x_2, v_2, t_2)| \lesssim |t_1 - t_2| + |x_1 - x_2| + |x_2|^{r_{sv}/n} + |v_1 - v_2||x_2|^{1/n}.
\]

**Proposition 7.2.** The vector fields given on \( \mathcal{QW}_r \cup \mathcal{QW}_v \) by \( p_k(v) \), \( k = \tau, v \), where \( v \) are \( \frac{\partial}{\partial v}, \frac{\partial}{\partial x}, \frac{\partial}{\partial t} \), are Lipschitz.
Proof. First we check that the partial derivatives \( \frac{\partial}{\partial t}, \frac{\partial}{\partial x} = nu u^{-1} \frac{\partial}{\partial u}, \frac{\partial}{\partial y} = u^{-1} \frac{\partial}{\partial u} \) of the coefficients of these vector fields are bounded. For the latter two it would be more convenient to check that \( u \frac{\partial}{\partial u} \) is bounded by \( x = u^n \), and \( \frac{\partial}{\partial u} \) is bounded by \( u' \). Then the claim follows from the facts that \( y(u, v, t), z_\tau(u, v, t) \) are analytic and divisible by \( u^n \), and \( \frac{\partial}{\partial u} y(u, v, t), \frac{\partial}{\partial v} z_\tau(u, v, t) \) are divisible by \( y' \). (Note that we need the bounds for the second order partial derivatives since the coefficients of these vector fields are the ones of the first order.) This shows that these vector fields are Lipschitz on each wing \( QW_\tau, QW_\nu \).

To obtain the Lipschitz property between the points of \( QW_\tau \) and \( QW_\nu \) we use a similar argument. Namely, we show that \( \frac{\partial}{\partial t}(z_\tau - z_\nu), \frac{\partial}{\partial u}(z_\tau - z_\nu), \frac{\partial}{\partial v}(z_\tau - z_\nu) \) are bounded (up to a constant) by \( z_\tau - z_\nu \).

Let \( p_{\tau,\nu}(w) \) be a vector field on \( QW_\tau \), where

\[
(47) \quad w(x, v, t) = \alpha \frac{\partial}{\partial t} + \beta \frac{\partial}{\partial x} + \gamma \frac{\partial}{\partial v}.
\]

We will describe in the terms of \( \alpha, \beta, \gamma \) the property for \( w \) to be Lipschitz. We shall always assume that \( p_{\tau,\nu}(w) \) is stratification compatible in the following sense.

**Definition 7.3.** We call such a vector field \( (47) \) stratification compatible if \( |\beta| \lesssim |x| \) and \( |\gamma| \) is bounded.

The property that \( |\beta| \lesssim |x| \) follows, for Lipschitz vector fields, from the requirement that \( p_\tau(w) \) is tangent to \( T \). The fact that \( |\gamma| \) is bounded expresses the requirement that \( p_\tau(w) \) is the restriction of a Lipschitz vector field defined on \( QW_\tau \cup S \). (Note the difference between being a stratified vector field, i.e. tangent to the strata, the property we impose to vector fields on polar wedges, and the property of being stratification compatible. The latter is the property of vector fields on quasi-wings that we would like to be restrictions of Lipschitz vector fields from bigger sets (that contain \( S \)).)

By Proposition \( 7.1 \) that \( h_\tau(x, v; t) \) defines a Lipschitz function on the quasiwing \( QW_\tau \) iff

\[
(48) \quad |h_\tau(x_1, v_1, t_1) - h_\tau(x_2, v_2, t_2)| \lesssim (t_1 - t_2) + |x_1 - x_2| + |v_1 - v_2| \lesssim |t_1 - t_2| + |x_1 - x_2| + |v_1 - v_2||x_2|^{1/n}.
\]

The following results easily follows from \( (48) \). Their proofs are left to the reader.

**Proposition 7.4.** The vector field on \( QW_\tau \) of the form \( p_\tau(w) \) is Lipschitz and stratification compatible iff:

1) \( \alpha \) satisfies \( (48) \);
2) \( |\beta| \lesssim |x| \) and \( \beta \) satisfies \( (48) \);
3) \( |\gamma| \) is bounded and \( \gamma x^{1/n} \) satisfies \( (48) \).

**Proposition 7.5.** The vector field on \( QW_\tau \cup QW_\nu \) given by \( p_{\tau,\nu}(w_\tau), p_{\nu,\tau}(w_\nu) \) is Lipschitz and stratification compatible iff:

0) \( p_{\tau,\nu}(w_\tau) \) and \( p_{\nu,\tau}(w_\nu) \) are Lipschitz and stratification compatible;
1) \( \alpha_\tau, \alpha_\nu \) satisfy \( (46) \);
2) \( \beta_\tau, \beta_\nu \) satisfy \( (46) \);
3) \( \gamma_\tau x^{1/n}, \gamma_\nu x^{1/n} \) satisfy \( (46) \).
Corollary 7.6 (Extension of Lipschitz vector fields on quasi-wings). Let $QW_\tau$, $QW_\nu$ be nicely-situated quasi-wings. Suppose that $p_{\tau}(w_0)$, with $w_0(x,0,t)$ of the form \( \gamma \), be a Lipschitz stratification compatible vector field defined on the wing $p_{\tau}(x,0,t)$. Then $p_{\nu}(w)$, $p_{\nu}(w)$, with $w(x,v,t) = w_0(x,0,t)$, defines a Lipschitz stratification compatible vector field on the union $QW_\tau \cup QW_\nu$. \hfill \Box

8. Proof of Theorem 1.1, Part II.

We complete the proof of Theorem 1.1. Let $\gamma(s), \gamma'(s)$ two real analytic arcs in $\mathcal{X}$ that are not included neither in the union of polar wedges nor in the singular locus. We want to show that any stratified Lipschitz vector field defined on the union of $\gamma$ and $\gamma'$ extends to $\gamma'$. We consider two cases.

**Case 1.** \( \text{dist}(\gamma(s), \gamma'(s)) \gtrsim \text{dist}(\gamma'(s), S) \).

Then we may forget about $\gamma$ and extend the vector field directly from $S$. For this we construct a quasi-wing containing $\gamma$.

**Case 2.** \( \text{dist}(\gamma(s), \gamma'(s)) \ll \text{dist}(\gamma'(s), S) \).

Then it suffices to extend the vector field from $\gamma$ to $\gamma'$. For this we use one quasi-wing or two nicely situated quasi-wings containing $\gamma$ and $\gamma'$.

Note that we may suppose for both arcs $\gamma$, $\gamma'$ that $y = O(x), z = O(x)$, that is, they are in the form \( \gamma \). Indeed, if this is not the case, then by the transversality of the projection $\pi$, $x = o(y), z = O(y)$ and then we change the system of coordinates to $(X_0, y, z, t) = (x - ay, y, z, t)$, for $a \neq 0$ and small.

Then we proceed as follows. In the next subsection we show in Proposition 8.1 that if $\gamma(s) \not\subset \mathcal{PW}_i$ then $\text{dist}(C_i, \gamma(s)) \gtrsim s^m$. This would allow to reduce the problem to the problem of extension of Lipschitz stratification compatible vector fields on nicely situated quasi-wings and use Corollary 7.6

8.1. Distance to polar wedges. Recall that we denote by $\mathcal{PW}$ the union of polar wedges, i.e. $\mathcal{PW} = \Sigma \cup \bigcup \mathcal{PW}_i$ and that for each $\mathcal{PW}_i$ the polar set $C_i$, parameterized by $p_i(u,0;t)$, is the spine of $\mathcal{PW}_i$.

Our goal is to show the following result.

**Proposition 8.1.** Let $\gamma(s) = (x(s), y(s), z(s), t(s)), s \in [0, \varepsilon)$, be a real analytic arc at the origin. If $\gamma(s) \not\subset \mathcal{PW} \cup \Sigma$ then for all $j$,

\[
\text{dist}(\gamma(s), C_j) \gtrsim \|(x(s), y(s), z(s))\|^{m_j/n}.
\]

**Lemma 8.2.** If the polar set $C_i$ minimizes the distance of $\gamma$ to $S$ and if this distance satisfies

\[
\text{dist}(\gamma(s), S) = \text{dist}(\gamma(s), C_i) \ll \|(x(s), y(s), z(s))\|^{m_i/n},
\]

then $\gamma(s)$ is contained, for small $s$, in $\mathcal{PW}$.

By (49) we mean that there is $\delta > 0$ such that

\[
\text{dist}(\gamma(s), C_i) \leq \|(x(s), y(s), z(s))\|^\delta + m_i/n.
\]

**Proof.** We write the proof in the non-parameterized case. The proof in the parameterized case is similar.

We may suppose that the arc $\gamma$ is of the form $\gamma(s) = (s^n, y(s), z(s))$ with $y(s) = O(s^n), z(s) = O(s^n)$. Indeed, otherwise $\text{dist}(\gamma(s), S) \sim \|(x(s), y(s), z(s))\|$. 

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First we complexify $\gamma$ by setting $\gamma(u) = (u^n, y(u), z(u))$. Then, as in the proof of Corollary 6.5 we construct a quasi-wing $QW$ containing $\gamma$ by changing the system of coordinates, that is replacing $y$ by $Y = y - b_0 z$, for $b_0$ sufficiently generic. In this new system of coordinates the parameterizations of $PW_i$ and $QW$ are, $x = u^n$ and, respectively,

\begin{align}
Y_i(u, b) &= y_i(u, b) - b_0 z_i(u, b) \\
&= (y_i(u) - b_0 z_i(u)) + u^{m_i}(b^2 \varphi_i(u, b) - b b_0 \psi_i(u, b)) \\
z_i(u, b) &= z_i(u) + b u^{m_i} \psi_i(u, b).
\end{align}

\begin{align}
Y(u, v) &= (y(u) - b_0 z(u)) + v u^{m_i} \\
z(u, v) &= z(u) + v u^{m_i} \tilde{\psi}_i(u, v).
\end{align}

(Since by $\{49\}$ $l = l_i = m_i$.)

Therefore, the intersection $PW_i \cap QW$, defined by $Y_i(u, b) = Y(u, v)$ and $z_i(u, b) = z(u, v)$, is given by the following system of equations.

\begin{align}
(b^2 \varphi_i(u, b) - b b_0 \psi_i(u, b)) - v &= o(u) \\
b \tilde{\psi}_i(u, b) - v \tilde{\psi}_i(u, v) &= o(u).
\end{align}

There are two cases:

1. By the Implicit Function Theorem there is a solution $(b, v) = (b(u), v(u))$ of (52), such that $b(u) \to 0$ and $v(u) \to 0$ as $u \to 0$. This happens if the jacobian determinant of the LHS of (52), with respect to variables $b, v$ is nonzero at $u = b = v = 0$. Then the intersection $PW_i \cap QW$ contains the curve $(u^n, Y_i(u, b(u)), z_i(u, b(u))) = (u^n, Y(u, v(u)), z(u, v(u)))$. Since both $PW_i$ and $QW$, for $u \neq 0$, are parameterizations of the regular part of $X$, if their intersection is non-empty they have to coincide. This shows that $\gamma \in PW_i$.

2. We suppose that the jacobian determinant of the LHS of (52) vanish at $u = b = v = 0$. Then the partial derivatives

\[
\frac{\partial}{\partial b} u^{-m_i}(Y_i(u, b), z_i(u, b)), \quad \frac{\partial}{\partial v} u^{-m_i}(Y(u, v), z(u, v)),
\]

that are both non-zero at $u = b = v = 0$, are proportional. This means that the limits of tangent spaces to $X$ along $C_i$, i.e. at $(u^n, y_i(u, 0), z_i(u, 0))$ as $u \to 0$, and at $\gamma(u)$ as $u \to 0$, coincide. This limit is transverse to $H = \{x = 0\}$ since $H$ is not a limit of tangent spaces by the Transversality Assumptions. Hence so are the tangent spaces to $X$ at $\gamma(u)$ for small $u$ that contain vectors of the form $(0, -b, 1)$ with $b \to 0$ as $u \to 0$. This shows that $\gamma \in PW$ (but not necessarily $\gamma \in PW_i$).

The proof of lemma is now complete. \qed

**Proof of Proposition 8.7.** The proof is the same in the parameterized and the non-parameterized case. We may suppose again that $\gamma(s) = (s^n, y(s), z(s))$ with $y(s) = O(s^n)$, $z(s) = O(s^n)$.

If $\text{dist}(\gamma(s), S) = \text{dist}(\gamma(s), C_i)$ then the conclusion for $j = i$ follows directly from Lemma 8.2. Then consider $j \neq i$. If the conclusion is not satisfied then

\[
s^{m_i} \lesssim \text{dist}(C_i, \gamma(s)) \leq \text{dist}(C_j, \gamma(s)) \ll s^{m_j}.
\]
In particular, \( m_i > m_j \), and therefore by Proposition 2.4, \( k_{ij} \leq m_j < m_i \). But this is impossible since then

\[
s^{m_j} \lesssim s^{k_{ij}} \simeq \text{dist}(p_i(s), p_j(s)) \lesssim \text{dist}(C_j, \gamma(s)) + \text{dist}(C_i, \gamma(s)) \ll s^{m_j},
\]

where \( p_i, p_j \) denote parameterizations of \( C_i \) and \( C_j \) respectively. This ends the proof in this case.

If \( \text{dist}(\gamma(s), S) = \text{dist}(\gamma(s), \Sigma_k) \) then the conclusion follows by the second part of Lemma 2.7.

\[\square\]

8.2. **End of proof.** First we consider **Case 1**. Let \( v \) be a stratified Lipschitz vector field on \( S \). We extend it on \( \gamma' \). By Propositions 8.1, \( \gamma' \) satisfies the assumptions of Corollary 6.5. Thus there exists a quasiwing \( QW \) containing \( \gamma' \) and, moreover, \( \text{dist}(\gamma'(s), S) = \text{dist}(\Sigma_b(\gamma'(s)), \Delta_b) \sim s^{l} \), where \( l = \max\{\max l_i, \max l_k\} \) and \( \Delta_b \) is the discriminant \( \Sigma_b \). Then any stratification compatible, see Definition 7.3, Lipschitz vector field on \( QW \) defines the needed extension.

We apply exactly the same strategy in **Case 2**, first by coconstructing a quasi-wing \( QW \) containing \( \gamma \). By the assumption \( \text{dist}(\Sigma_b(\gamma'(s)), \Sigma_b(\gamma(s))) \sim \text{dist}(\gamma(s), \gamma'(s)) \sim s^{l} \), and therefore, \( \gamma' \) is contained either in \( QW \) or in another quasi-wing \( QW' \) such that \( QW \) and \( QW' \) are nicely-situated. Then we apply Corollary 7.6.

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