GRAVITY AND INSTANTONS

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Conventional non-Abelian SO(4) gauge theory is able to describe gravity provided
the gauge field possesses a specific polarized vacuum state in which the instantons
have a preferred orientation. Their orientation plays the role of the order parameter
for the polarized phase of the gauge field. The interaction of a weak and smooth
gauge field with the polarized vacuum is described by an effective long-range action
which is identical to the Hilbert action of general relativity. In the classical limit
this action results in the Einstein equations of general relativity. Gravitons appear
as the mode describing propagation of the gauge field which strongly interacts with
the oriented instantons. The Newton gravitational constant describes the density
of the considered phase of the gauge field. The radius of the instantons under
consideration is comparable with the Planck radius.

1 Instantons in an external field

This work reviews the idea proposed in [1, 2] which suggests a new explanation
for the origin of gravitational forces. In the proposed scenario gravity arises
as a particular effect in the conventional Yang-Mills gauge theory [3] formulated in flat space-time. All geometrical objects necessary for the gravitational
phenomenon can originate from the gauge degrees of freedom if a particular nontrivial vacuum state, which we will call the vacuum with polarized instantons
develops in the SO(4) gauge theory.

Consider Euclidean formulation of the SO(4) gauge theory. The gauge
algebra for SO(4) gauge group consists of two su(2) gauge subalgebras, so(4) = su(2) + su(2). The instantons and antiinstantons can belong to any one of these
two available su(2) gauge subalgebras. It is convenient to choose the generators
for one su(2) gauge subalgebra to be (−1/2)η_{aij} and the generators for the
other one to be (−1/2)\bar{\eta}^{aij}. To distinguish between these two subalgebras
we will refer to them as su(2)η and su(2)\bar{\eta}. Symbols η^{aij}, \bar{\eta}^{aij} are the usual 't Hooft symbols, a = 1, 2, 3, i, j = 1, · · · , 4. In this notation the gauge potential
and the strength of the gauge field are

\[ A_{\mu}^{ij} = \frac{1}{2}(A_{\mu}^{a} \eta^{aij} + \bar{A}_{\mu}^{a} \bar{\eta}^{aij}), \]
\[ F_{\mu\nu}^{ij} = \frac{1}{2}(F_{\mu\nu}^{a} \eta^{aij} + \bar{F}_{\mu\nu}^{a} \bar{\eta}^{aij}), \]

where \( A_{\mu}^{a} \) and \( F_{\mu\nu}^{a} \) belong to su(2)η and \( \bar{A}_{\mu}^{a}, \bar{F}_{\mu\nu}^{a} \) belong to su(2)\bar{\eta}. The Yang-
Mills action reads $S = 1/(4g^2) \int F_{\mu \nu}^{ij}(x) F_{\mu \nu}^{ij}(x) \, d^4x$. The Latin indexes $i, j = 1, \cdots, 4$ label the isotopic indexes, while the Greek indexes $\mu, \nu = 1, \cdots, 4$ label the indexes in Euclidean coordinate space. Remember that we consider the usual gauge field theory in flat space-time. For the chosen normalization of generators the relation between the gauge potential and the field strength reads

$$F_{\mu \nu}^{ij} = \partial_\mu A_{\nu}^{ij} - \partial_\nu A_{\mu}^{ij} + A_{\nu}^{ik} A_{\mu}^{kj} - A_{\nu}^{ik} A_{\mu}^{kj}.$$  

Consider interaction of an instanton with an external gauge field which has trivial topological structure and is weak and smooth. Thus formulated problem was first considered by Callan, Dashen and Gross where it was shown that the interaction of an instanton with the external field is described by an effective action

$$\Delta S = \frac{2\pi^2 \rho^2}{g^2} \bar{\eta}^{a \mu \nu} D_{\mu \nu}^{ab} F_{\mu \nu}^b(x_0).$$  

(3)

Here $\rho$ and $x_0$ are the radius and position of the instanton. The matrix $D_{\mu \nu}^{ab} \in SO(3)$ describes the orientation of the instanton in the $su(2)$ gauge subalgebra where the instanton belongs. $F_{\mu \nu}^b(x)$ is the gauge field in the subalgebra where the instanton belongs. This field has to be taken in the singular gauge. The interaction of an antiinstanton with an external field is described similarly. The only distinction is that it produces the corresponding term with the ‘t Hooft symbol $\eta^{a \mu \nu}$ instead of $\bar{\eta}^{a \mu \nu}$ which stands in (3).

Let us now generalize the problem. Suppose that there is a number of instantons and antiinstantons which belong to either $su(2)\eta$ or $su(2)\bar{\eta}$ gauge algebras. Assuming that the dilute gas approximation is valid we find that the contribution to the action reads

$$\Delta S = -\frac{\pi^2}{g^2} \sum_k \eta^{a \mu \nu} \eta^{B ij} T_k^{AB} \rho_k^2 F_{\mu \nu}^j(x_k).$$  

(4)

Here $k$ runs over all instantons and antiinstantons which have radiuses and coordinates $\rho_k$ and $x_k$. To simplify notation the ‘t Hooft symbols are enumerated as 6-vectors $\eta^A = (\eta^a, \bar{\eta}^\beta)$, $A = 1, \cdots, 6$; $a, b = 1, 2, 3$. To describe an orientation of every (anti)instanton it is convenient to introduce a $6 \times 6$ matrix $T_k^{AB}$, $A, B = 1, \cdots, 6$

$$T_k^{AB} \equiv T_k = \begin{pmatrix} C_k & D_k \\ \bar{D}_k & \bar{C}_k \end{pmatrix}$$  

(5)

as a set of four $3 \times 3$ matrices $C_k, C_k, D_k, \bar{D}_k$. For any given $k$-th (anti)instanton only one of these four matrices is essential while the other three are equal to zero. This nonzero matrix belongs to $SO(3)$ and describes the orientation of the $k$-th topological object in the gauge algebra where it belongs. For example, if the $k$-th object is an antiinstanton in the $su(2)\eta$ gauge subalgebra,
then we assume that $C_k \in SO(3)$ describes its orientation in the $su(2)\eta$ while $\bar{C}_k, D_k, \bar{D}_k = 0$.

Let us consider the behavior of the ensemble of instantons in the vacuum state assuming that there exists a weak and smooth topologically trivial gauge field $F_{i\mu}^j(x)$. One can derive the effective action which describes interaction of the vacuum with this field averaging (6) over short-range quantum fluctuations in the vacuum. The result can be presented as the effective action

$$\Delta S = - \int \eta^{A\mu\nu} \eta^{Aij} \mathcal{M}^{AB}(x) F_{i\mu}^j(x) d^4 x.$$  

where the matrix $\mathcal{M}^{AB}(x)$ is

$$\mathcal{M}^{AB}(x) = \pi^2 \langle \frac{1}{g^2} \rho^2 T^{AB} n(\rho, T, x) \rangle.$$  

The brackets $\langle \rangle$ here describe averaging over quantum fluctuations whose wavelength is shorter than a typical distance describing variation of the external field. These fluctuations in the dilute gas approximation for instantons should include averaging over positions, radiuses and orientations of instantons. In (7) $n(\rho, T, x)$ is the concentration of (anti)instantons which have the radius $\rho$ and the orientation described by the matrix $T \equiv T^{AB}$. In the usual vacuum states the concentration of instantons does not depend on the orientation, $n(\rho, T, x) \equiv n(\rho, x)$. In that case an averaging over orientations gives the trivial result $\mathcal{M}^{AB}(x) \equiv 0$, as mentioned by Vainshtein et al.\cite{10}. The main goal of this paper is to investigate what happens if the concentration $n(\rho, T, x)$ does depend on the orientation $T = T^{AB}$ providing the nonzero value for the matrix $\mathcal{M}^{AB}(x)$.

It is shown below that interesting physical consequences arise if one assumes that the matrix $\mathcal{M}^{AB}(x)$ satisfies the following condition

$$\mathcal{M}^{AB}(x) = \frac{1}{4} f M^{AB}(x),$$  

where $f$ is a positive constant and $M^{AB}(x) \in SO(3, 3)$, which means that $M \Sigma M^T = \Sigma$, where the matrix $\Sigma^{AB}$ is defined as

$$\Sigma = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$  

Condition (8) is the main assumption about properties of the vacuum state of the $SO(4)$ gauge theory. Now we can clarify the meaning of the term “polarization of instantons” which was introduced above intuitively. We say
that there is the polarization of instantons with the $SO(3,3)$ symmetry, if (5) is valid. One can interpret (5) as the statement that there exists the new nontrivial phase of the $SO(4)$ gauge theory. The matrix $M(x)$ plays the role of the order parameter for this phase. The (anti)instantons which contribute to the nontrivial value of the matrix $T^{AB}$ in (6) can be looked at as a specific condensate. The constant $f$ characterizes the density of this condensate.

Notice that existence of a state with polarized instantons does not come into contradiction with general principle of gauge invariance which in the context considered is known as the Elitzur theorem, see also the book of Polyakov and recent lectures of Hamer, because the orientation of an instanton is a gauge invariant parameter.

It is instructive to re-write (4) in another form. Notice that there exists a relation between matrixes belonging to $SO(3,3)$ and matrixes belonging to $SL(4)$ groups which is well known, see a book of Gilmore. It can be presented as a statement that for any $M^{AB} \in SO_+ (3,3), A, B = 1, \cdots, 6$ there exists some matrix $H^{i\mu} \in SL(4), i, \mu = 1, \cdots, 4$ satisfying

\[ H^{i\mu} H^{j\nu} - H^{i\nu} H^{j\mu} = \frac{1}{2} \eta^{A\mu\nu} \eta^{Bij} M^{AB}. \] (10)

Identifying $M^{AB}(x) = M^{AB}$ one finds from (10) $H^{i\mu} = H^{i\mu}(x) \in SL(4)$ which can be considered as another representation for the order parameter. Substituting $H^{i\mu}(x)$ defined by (10) into (8) one can rewrite the action (6) in the following useful form

\[ \Delta S = -f \int H^{i\mu}(x) H^{j\nu}(x) F^{ij}_{\mu\nu}(x) d^4x. \] (11)

Up to now our consideration was restricted by orthogonal coordinates. It is instructive however to present the action (11) in arbitrary coordinates. Under the coordinate transformation the matrix $H^{i\mu}(x)$ is transformed as $H^{i\mu}(x) \rightarrow h^{i\mu}(x') = (\partial x'^{\mu}/\partial x^{\lambda}) H^{i\lambda}(x)$, where the transformed matrix is called $h^{i\mu}(x)$. Using it one can present the action (11) in arbitrary coordinates $x$ in the following form

\[ \Delta S = -f \int h^{i\mu}(x) h^{j\nu}(x) F^{ij}_{\mu\nu}(x) \det h(x) d^4x. \] (12)

We use notation in which $h^{i\mu}(x)$ is understood as the matrix inverse to $h^{i\mu}(x)$, i.e. $h^{i\mu}(x) h^{j\nu}(x) = \delta_{ij}$. The determinant in (12) is defined as a determinant of this inverse matrix, $\det h = \det[h^{ij}_{\mu}]$. Thus the factor $\det h(x)$ in (12) simply accounts for the variation of the phase volume under the coordinate transformation.
2 The Riemann geometry and the Einstein equations

Excitations above the polarized vacuum should possess interesting properties because variation of the gauge field results in the contribution to the action (12) which is linear in the field. This is in contrast to the standard quadratic behavior of the conventional Yang-Mills action. Let us examine the properties of excitations in the classical approximation.

We will assume that the fields considered vary on the macroscopic distances, say \( \sim 1 \text{cm} \), and their magnitude can be roughly estimated as \( |F_{ij}^{\mu\nu}| \sim 1/\text{cm}^2 \). We will see below that the constant \( f \) which was defined in (7),(8) is large, \( f \sim 1/r_P^2 \), where \( r_P \) is the Planck radius. This shows that for weak fields the integrand in the Yang-Mills action is suppressed compared to the Yang-Mills action by a drastic factor \( |F_{ij}^{\mu\nu}|/f \sim (r_P/\text{1cm})^2 = 10^{-64} \). Therefore our priority is to take into account the action (12) which describes interaction of the weak field with polarized instantons, neglecting the Yang-Mills action.

Let us consider the action (12) as a functional which depends on the weak field vector potential and the matrix \( h_{i\mu}^j(x) \) which describes orientations of instantons, \( \Delta S = \Delta S(\{A_{ij}^{\mu}(x)\}, \{h^{i\mu}(x)\}) \). Classical equations for this functional read:

\[
\frac{\delta(\Delta S)}{\delta A_{ij}^{\mu}(x)} = 0, \quad (13)
\]

\[
\frac{\delta(\Delta S)}{\delta h^{i\mu}(x)} = 0. \quad (14)
\]

In order to present classical equations in a convenient form let us define three quantities, \( g_{\mu\nu}(x), \Gamma^\lambda_{\mu\nu}(x), \) and \( R^\lambda_{\rho\mu\nu}(x) \):

\[
g_{\mu\nu}(x) = h_{i\mu}^i(x)h_{i\nu}^i(x), \quad (15)
\]

\[
\Gamma^\lambda_{\mu\nu}(x) = h^{i\lambda}(x)h_{i\mu}^j(x)A^{ij}_{\nu}(x) + h^{i\lambda}(x)\partial_{\nu}h_{i\mu}^j(x), \quad (16)
\]

\[
R^\lambda_{\rho\mu\nu}(x) = h^{i\lambda}(x)h_{i\rho}^j(x)F^{ij}_{\mu\nu}(x). \quad (17)
\]

After simple calculations the first classical equation (13) may be presented in the form \( \Gamma^\gamma_{\mu\nu}(x) = 1/2g^\rho\tau(x) [\partial_\gamma g_{\tau\rho}(x) + \partial_\rho g_{\gamma\tau}(x) - \partial_\tau g_{\gamma\rho}(x)] \), in which the matrix \( g^{\mu\nu}(x) \) is defined as \( g^{\mu\nu}(x) = h^{i\mu}(x)h^{i\nu}(x) \). Clearly the found relation is identical to the usual expression for the Christoffel symbol in terms of the Riemann metric for some Riemann geometry, see for definitions Landau and Lifshitz. Moreover, it is easy to verify that the quantity \( R^\lambda_{\rho\mu\nu}(x) \) can be presented in terms of \( g_{\mu\nu}(x) \) as well \( R^\lambda_{\rho\mu\nu}(x) = \partial_\rho \Gamma^\lambda_{\mu\nu} - \partial_\nu \Gamma^\lambda_{\rho\mu} + \Gamma^\sigma_{\mu\nu} \Gamma^\rho_{\sigma\lambda} - \Gamma^\sigma_{\rho\nu} \Gamma^\sigma_{\mu\lambda}. \)

One recognizes in this relation the usual connection between the Riemann ten-
sor and the Riemann metric. We see that equation (13) shows that the introduced in (15) quantity $g_{\mu\nu}(x)$ can be considered as a metric for some Riemann geometry with the Christoffel symbol $\Gamma^\lambda_{\mu\nu}(x)$ and the Riemann tensor $R^\lambda_{\nu\mu}(x)$.

Consider now the second classical equation (14). It is easy to verify that it can be presented in the following form

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 0,$$

in which $R_{\mu\nu}(x)$ and $R(x)$ are the Ricci tensor and the curvature of the Riemann geometry. The found equation (18) is identical to the Einstein equations of general relativity in the absence of matter.

Moreover, the effective action (12) may also be presented in geometrical terms. To see this consider the action when (13) is valid. It is clear from (12) that the integrand of the action (12) proves be proportional to the integrand of the usual Hilbert action of general relativity (15). One can consider the action (12) and the Hilbert action as same quantity if the Newton gravitational constant $k$ is identified with the constant $f$ which characterize the density of the polarized condensate of instantons $k = 1/(16\pi f)$. This relation shows that $f = 2/r_P^2$. Remember that the constant $f$ introduced in (7),(8) depends on the typical radiuses and separations of the polarized instantons. We see that these radiuses and separations should be comparable with the Planck radius.

3 Discussion of results

We come to the interesting result. The first classical equation of motion for the gauge field (13) shows that particular combinations of the gauge field variables (13),(14),(15) are identical to the Riemann metric, the Christoffel symbol, and the Riemann tensor for some Riemann space. The second classical equation (14) proves be identical to the Einstein equations for this Riemann metric. Validity of the Einstein equations guarantees that long-range excitations are massless spin-2 excitations. The found effective action turns out to be identical to the Hilbert action of general relativity. These facts altogether permit one to identify the found excitations with gravitons. This indicates that gravity arises in the framework of the gauge theory. It is very important that the dynamics of general relativity, its action and equations of motion, originate directly from the dynamics of the gauge field. All these results follow from assumption (8) which has been interpreted above as the $SO(3,3)$ polarization of instantons.

In order to justify the considered scenario one needs to find gauge models in which polarization of instantons takes place. The necessary model should
satisfy several demanding conditions. One of them is a necessity that the polarization of (anti)instantons remains non-trivial even in the simplest case when gravitational field is absent. In this case the polarization of instantons simplifies to be a constant $SO(3) \times SO(3)$ matrix. To meet this requirement the least one needs is a $SU(2)$ gauge theory model in which instantons are polarized, while antiinstantons remain non-polarized. A candidate for such a model has been proposed in [2].

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