The graph theoretic moment problem∗

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Abstract

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We study an analogue of the classical moment problem in the framework where moments are indexed by graphs instead of natural numbers. We study limit objects of graph sequences where edges are labeled by elements of a topological space. Among other things we obtain strengthening and generalizations of the main results of previous papers characterizing reflection positive graph parameters, graph homomorphism numbers, and limits of simple graph sequences. We study a new class of reflection positive partition functions which generalize the node-coloring models (homomorphisms into weighted graphs).

1 Introduction

To study very large graphs, a natural way to obtain information about them is sampling. In the case of dense simple graphs, a natural way to sample is to pick $k$ random nodes and look at the subgraph induced by them. A sequence $G_1, G_2, \ldots$ of simple graphs with $|V(G_n)| \to \infty$ is called convergent if the distribution of this random induced subgraph is convergent for every $k$. To every convergent sequence of simple graphs one can assign a limit object in the form of a 2-variable real function [14].

Instead of the induced subgraph samples, one can consider homomorphism densities of various “small” graphs. While for simple graphs they trivially carry the same information as the samples described above (connected by a simple inclusion-exclusion), their algebraic properties are quite different and often more useful. These densities are very good 2-variable analogues of moments of 1-variable functions (see Section 1.1).

It turns out that in a more general setting, moment sequences can be indexed by multigraphs rather than simple graphs. Let $X$ be a random variable. A moment of $X$ (in a slightly generalized sense) is the expected value of $p(X)$ where $p$ is a polynomial in $\mathbb{R}[x]$. The classical moment problem can be phrased as follows: which functions $\alpha : \mathbb{R}[x] \to \mathbb{R}$ can be represented by a real valued random variable $X$ so that $\alpha(p) = E(p(X))$ for all $p \in \mathbb{R}[x]$. The necessary and sufficient condition is that $\alpha$ is linear, normalized ($\alpha(1) = 1$) and positive definite ($\alpha(p^2) \geq 0$ for very polynomial $p$).

Consider a symmetric measurable 2-variable function $W : [0,1]^2 \to [0,1]$. Let $X_1, X_2, X_3, \ldots$ be random independent elements from $[0,1]$. The random variables $Z_{i,j} = W(X_i, X_j)$ ($i \neq j$) have all the same distribution but they are not all independent (for example, $Z_{1,2}$ and $Z_{2,3}$ are correlated in general). Note that by the symmetry of $W$, we have $Z_{i,j} = Z_{j,i}$ for every $i$ and $j$.

It is natural to define the moments of $W$ as expected values of multi-
variate polynomials in the variables $Z_{i,j}$. As in the one-variable case, $W$ induces a linear map from the polynomial ring $\mathbb{R}[\{z_{i,j}|1 \leq i < j\}]$ to the real numbers by

$$t(p,W) = E(p(\{Z_{i,j}|1 \leq i < j\})),$$

and this moment function is determined by its values on monomials. Every monomial in this ring corresponds to a multigraph, and if two such monomials correspond to isomorphic graphs, then the moment function has the same value on them.

So, just like in the one-variable case, $W$ has a countable number of “moments”, but instead of forming a single sequence, they are indexed by (finite) multigraphs.

1.1 Moments indexed by simple graphs

Somewhat surprisingly, if we want to define moments of a 2-variable function $W$, it is often enough to restrict ourselves to simple graphs (in other words, to multilinear polynomials $p \in Z^{(2)}$). In this section we recall various results that can be viewed as supporting this claim. (We’ll return to why moments indexed by multigraphs are needed, and how to treat them.)

Recall that a graph parameter is a map from the set of finite graphs to the real numbers, invariant under isomorphism. A simple graph parameter is only defined on simple graphs.

Let $\mathcal{W}$ be the space of bounded symmetric measurable functions $W: [0,1]^2 \to \mathbb{R}$, and let $F$ be a simple graph with $k$ nodes. We define

$$t(F,W) = \int_{[0,1]^V(F)} \prod_{ij \in E(F)} W(x_i, x_j) \, dx.$$  \hspace{1cm} (2)

We call $t(F,W)$ as the $F$-moment of the function $W$. While this definition is meaningful for every (multi)-graph $F$, we’ll restrict our attention for the time being to simple graphs.

There is an obvious relation between these moments: if $F_1$ and $F_2$ are two graphs and $F_1F_2$ denotes their disjoint union, then

$$t(F_1F_2, W) = t(F_1, W)t(F_2, W).$$

We call this relation the multiplicativity of the moments. Using this relation, we can restrict our attention to moments defined by connected graphs.

Let us compare some basic properties of these moments with the analogous properties of moments of one-variable functions.
**Property 1** Moment sequences are interesting. For example, the Fibonacci sequence is a moment sequence. Moment parameters are also interesting. The number of $q$-colorings of a graph $F$, divided by $q^{|V(F)|}$, is a moment parameter; more generally, the number of homomorphisms $\text{hom}(F, G)$ of a graph $F$ into a fixed (for simplicity, simple) graph $G$ (appropriately normalized) is a moment parameter. To be precise, if

$$t(F, G) = \frac{\text{hom}(F, G)}{|V(G)||V(F)|},$$

then $t(F, G) = t(F, W_G)$ for an appropriate function $W_G$. The number of nowhere-zero $k$-flows is an important graph parameter representable this way.

To show a moment sequence of a non-step-function with combinatorial significance, let us quote the following example from [14]: the number

$$2^{|E(F)|} t(F, \cos(2\pi(x - y)))$$

is the number of eulerian orientations of the graph $F$.

**Property 2** Any finite number of moments are independent: no finite number of moments determine any other. This is also true in the 2-variable case: For any finite set $F_1, \ldots, F_k$ of connected graphs, the set of vectors $(t(F_1, W), \ldots, t(F_k, W))$ has a nonempty interior in $\mathbb{R}^k$ (Erdős, Lovász and Spencer [6]). This shows that each of this countable, but “large” set of moments carries information that is not implied by a finite number of others. So in a sense this large set of moments is indeed needed (instead of, say, a two-parameter family).

**Property 3** The moments determine the function up to a measure preserving transformation of the variable. (For one-variable functions, this is equivalent to saying that they determine the distribution of the function values, but this would be too weak for two-variable functions.) To be more precise, it is well known that if $f, g : [0, 1] \to \mathbb{R}$ are two (for simplicity, bounded) measurable functions such that $\int_0^1 f^k = \int_0^1 g^k$ for all $k$, then there is a third bounded measurable function $h : [0, 1] \to \mathbb{R}$ and measure-preserving maps $\varphi, \psi : [0, 1] \to [0, 1]$ such that $f(x) = h(\varphi(x))$ and $g(x) = h(\psi(x))$ for almost all $x$.

This fact generalizes to two-variable functions (Borgs, Chayes and Lovász [3]): If $U, W \in \mathcal{W}$ such that for every simple graph $F$, $t(F, U) = t(F, W)$, then there exists a function $V \in \mathcal{W}$ and two measure preserving maps $\varphi, \psi : [0, 1] \to [0, 1]$ such that $U(x, y) = V(\varphi(x), \varphi(y))$ and $W(x, y) = V(\psi(x), \psi(y))$ almost everywhere.
Property 4 Moment sequences can be characterized by inclusion-exclusion. Hausdorff [9] proved that a sequence \((a_0, a_1, \ldots)\) is the moment sequence of a function \(f\) with \(0 \leq f \leq 1\) if and only if \(a_0 = 1\), and the following inequality holds for all \(0 \leq k \leq n\):

\[
\sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} a_{n+j} \geq 0.
\]

(cf. Diaconis and Freedman [5]).

The following analogue of this for graph parameters was proved by the authors in [14]: A simple graph parameter \(f\) can be represented as \(f = t(\cdot, W)\) with some \(W \in \mathcal{W}_0\) if and only if \(f(K_1) = 1\), \(f\) is multiplicative, and the following inequality holds for all simple graphs \(F\):

\[
\sum_{F' \supseteq F, V(F') = V(F)} (-1)^{|E(F') \setminus E(F)|} f(F') \geq 0.
\]

Property 5 Moment sequences can be characterized by a semidefiniteness condition. Hausdorff gave another characterization as well: a sequence \((a_0, a_1, \ldots)\) is the moment sequence of a function \(f\) with \(0 \leq f \leq 1\) if and only if \(a_0 = 1\), and the (infinite) matrix \(A\) defined by \(A_{ij} = a_{i+j-2}\) \((i, j = 1, 2, \ldots, \infty)\) is positive semidefinite.

An analogue for graph parameters was proved by the authors in [14]. We need to define what replaces adding up indices \(i\) and \(j\). To this end, we define \(k\)-labeled simple graph \((k \geq 0)\) is a finite graph in which \(k\) nodes are labeled by \(1, 2, \ldots, k\) (it can have any number of unlabeled nodes). The simple product \(F_1 F_2\) of two \(k\)-labeled graphs \(F_1\) and \(F_2\) is defined by taking their disjoint union, and then identifying nodes with the same label; if we get parallel edges, then their multiplicity is suppressed. (For 0-labeled graphs product means disjoint union.)

Let \(f\) be any simple graph parameter and \(k \geq 0\). We define the following (infinite) matrix \(M(f, k)\). The rows and columns are indexed by isomorphism types of \(k\)-labeled simple graphs. The entry in the intersection of the row corresponding to \(F_1\) and the column corresponding to \(F_2\) is \(f(F_1 F_2)\).

With this notation, we can state the following characterization of moment parameters [14]: A simple graph parameter \(f\) can be represented as \(f = t(\cdot, W)\) with some \(W \in \mathcal{W}_0\) if and only if \(f(K_1) = 1\), \(f\) is multiplicative, and the (infinite) matrix \(M(f, k)\) is positive semidefinite for each \(k\).
Property 6 A sequence is the moment sequence of a stepfunction if and only if the matrix $A$ defined above is semidefinite and has finite rank. To state an analogous assertion for two-variable functions, we call a symmetric measurable function $W : [0,1] \times [0,1] \to [0,1]$ a stepfunction if there is a finite partition $[0,1] = \bigcup_{i=1}^{r} S_i$ into measurable sets such that $W$ is constant on every $S_i \times S_j$. The following was proved for simple graph parameters by Lovász and Schrijver [12] (paralleling an earlier result by Freedman, Lovász and Schrijver [7] for multigraph parameters, see Theorem 1.2 below): A simple graph parameter is the moment parameter of a stepfunction with $q$ steps if and only if the matrix $M(f,k)$ is semidefinite and has rank at most $q^k$ for every $k \geq 0$.

Considering stepfunctions points at other interesting analogies with the one-variable case. It is not hard to see that a one-variable function is a stepfunction if and only if it is determined by a finite set of its moments. The “only if” part of the analogous statement for 2-variable functions was proved (in graph-theoretic terms) for two-variable functions by Lovász and Sós [13]: For every stepfunction $U \in W$ there is a finite set $F_1,\ldots,F_m$ of simple graphs such that if $t(F_j,U) = t(F_j,W)$ for some $W \in W$ for $j = 1,\ldots,m$, then $t(F,U) = t(F,W)$ for every simple graph $F$. However, the converse fails to hold [16].

Property 7 Convergence in moments implies convergence. More exactly, if $X_1, X_2, \ldots$ are uniformly bounded random variables such that $E(X_n^k)$ is convergent for every $k$, then $X_n$ tends to a limit in distribution. Analogously, if $(W_n)$ is a uniformly bounded sequence of functions in $W$, then $t(F,W_n)$ is convergent for every simple graph $F$ if and only if there are measure preserving maps $\varphi_n : [0,1] \to [0,1]$ such that the functions $W'_n(x,y) = W_n(\varphi_n(x),\varphi_n(y))$ are convergent in an appropriate norm (the $\|\cdot\|$).

This fact is closely related to limits of graph sequences. In fact, if $(G_n)$ is a sequence of simple graphs for which $t(F,G_n)$ is convergent for every simple graph $F$, then there is a function $W \in W$ such that $t(F,G_n) \to t(F,W)$ for every $F$ [14]. This result can be extended to the case when $(G_n)$ is a sequence of weighted graphs with uniformly bounded edgeweights [4].

1.2 Moments indexed by multigraphs

We have seen that the densities of simple graphs in symmetric measurable functions $W : [0,1]^2 \to [0,1]$ can be considered as an analogue of moments. The formula [2] defining $F$-moments makes sense for all (multi)graphs $F$, and there are many reasons why we don’t want to restrict ourselves to just
simple graph moments. For example, we may be interested in the “ordinary” moments of a function \( W \in \mathcal{W} \) (considered as a function in a single variable defined on the probability space \([0, 1]^2\), rather than a 2-variable function). These moments can be expressed as

\[
\int_{[0, 1]^2} W(x, y)^n \, dx \, dy = t(K_n^2, W),
\]

where \( K_n^2 \) consists of two nodes connected by \( n \) parallel edges. By Property 3, this is determined by the simple graph moments, but it can be seen (using a slight extension of the results mentioned in Property 2) that no finite number of them determines \( t(K_n^2, W) \).

Another reason for considering multigraphs is that we want to think of a polynomial \( p \) in variables \( z_{i,j} \) \( (1 \leq i < j \leq 1) \) as a formal linear combination of multigraphs. Then every multigraph parameter \( f \) can be extended linearly to these polynomials.

1.2.1 Limits of weighted graphs

Suppose that the sequence \( t(F, G_n) \) is convergent for every multigraph \( F \) (rather than for every simple graph \( F \)). Does this imply that there exists a limit function \( W \) that encodes the limiting values?

To illustrate the difficulty, let \( G_n \) be a random graph on \( n \) nodes, with edge probability 1/2. It is easy to see that with probability 1,

\[
t(F, G_n) \to 2^{-|E(F)|} = t(F, 1/2) \quad (n \to \infty)
\]

for every simple graph \( F \) (here 1/2 denotes the identically 1/2 function). It can be shown that this is the only limit function (e.g. by Property 3 above).

Suppose that \( F \) has multiple edges, and let \( F' \) denote the simple graph obtained from \( F \) by suppressing the edge multiplicities. Then \( t(F, G_n) = t(F', G_n) \), so \( t(F, G_n) \to 2^{-|E(F')|} \); but \( t(F, 1/2) = 2^{-|E(F)|} \), so while the sequence \( t(F, G_n) \) is convergent for every multigraph \( F \), its limit is not \( t(F, 1/2) \) if multiple edges are present. By the uniqueness of the limit function, this means that the limit cannot be described by a single function in \( \mathcal{W} \).

In [17], limit objects for moments indexed by multigraphs with bounded edge multiplicities are described. Let \( \mathcal{W}(d) \) denote the set of symmetric measurable functions \( W : [0, 1] \times [0, 1] \to [-d, d] \). A moment function sequence is a sequence \((W_0, W_1, \ldots)\) of functions such that \( W_i \in \mathcal{W}(d^i) \) and \((W_0(x, y), W_1(x, y), \ldots)\) is a moment sequence for almost all pairs
(x, y) ∈ [0, 1]^2. Then the limit object of a graph sequence with edge weights uniformly bounded by d can be described by a moment function sequence. (As in the one-variable and also in the simple-graph case, these objects are not uniquely determined by their moments since any measure preserving transformation of [0, 1] yields another object which has the same moments.)

It is also shown in [17] that moment function sequences can be represented essentially uniquely by functions \( W : [0, 1]^2 \rightarrow \mathcal{P}(d) \), where \( \mathcal{P}[-d, d] \) is the set of probability distributions on the Borel sets of \([-d, d]\) (we endow \( \mathcal{P}[-d, d] \) with the week topology, and require \( W \) to be measurable as a map into the Borel sets of \( \mathcal{P}[-d, d] \)). Such a function is called a \([-d, d]-\text{graphon}\).

There is a third representation which is unique and is analogous to the distribution of a random variable: This is a probability distribution on infinite edge-weighted graphs on the node set \( \mathbb{N} \) that is symmetric under the permutations of the node set and has the property that disjoint subsets of \( \mathbb{N} \) span independent (in the probability sense) labeled weight ed graphs. (Again, the edge weights are between \( d \) and \(-d\).) The above can be viewed as a natural characterization of these homogeneous infinite random graph models.

### 1.2.2 Characterizing moment parameters

One of the goals of this paper is to characterize moment parameters indexed by multigraphs. Here are some basic properties of graph parameters of the form \( t(\cdot, W) \), where \( W \in \mathcal{W}(d) \) (see Proposition 3.4):

1. \( t(K_1, W) = 1 \) where \( K_1 \) is the one-node graph (\( t(\cdot, W) \) is normalized).
2. \( t(F_1 \cup F_2, W) = t(F_1, W)t(F_2, W) \) for all \( F_1 \) and \( F_2 \), where \( F_1 \cup F_2 \) is the disjoint union of \( F_1 \) and \( F_2 \) (multiplicativity).
3. \( t(p^2, W) \geq 0 \) for all polynomials \( p \in \mathbb{Z}^{(2)} \) (weak reflection positivity).
4. \( |t(K^n_2, W)| \leq d^n \) (exponentially bounded growth on the \( n \)-fold edge).

Let \( \mathcal{T}_3(d) \) denote the of multigraph parameters with these four properties, and let \( f \in \mathcal{T}_3(d) \). Can \( f \) be represented as \( t(\cdot, W) \) with some function \( W : \Omega \times \Omega \rightarrow [-d, d] \)? The parameter \( 2^{-|E(F')|} \) discussed above shows that these conditions are not sufficient; however they are not very far from being sufficient. We will show (Theorem 3.2) that the set of graph parameters of the form \( t(\cdot, W) \) (\( W \in \mathcal{W}(d) \)) is dense with respect to the pointwise
convergence in $T_3(d)$. We will also show that graph parameters in $T_3(d)$ can be represented by $[-d,d]$-graphons.

**Remark 1.1** There is another generalization of the classical moment problem, the theory of positive definite functions on semigroups $[1,10]$. Although our context does not entirely fit into the framework of that theory, we will make use of a theorem about exponentially bounded positive definite functions $[2]$ (see $[12]$ for results on semigroups that are related to both that theory and our framework).

### 1.2.3 Homomorphisms and stepfunctions

One can define the homomorphism number $\text{hom}(F,H)$ from a multigraph into a weighted graph, as well as connection matrices $M(f,k)$ for multigraph parameters $f$, analogously to the simple case. The analogue of Property 5 above holds (Freedman, Lovász and Schrijver $[7]$):

**Theorem 1.2** A multigraph parameter is of the form $\text{hom}(.,H)$ for some weighted graph with $q$ nodes if and only if the multigraph connection matrix $M(f,k)$ is semidefinite and has rank at most $q^k$ for every $k \geq 0$.

In this paper we prove extensions of this theorem. To state our results, we need the notion of a *randomly weighted graph*: a graph whose nodes are weighted with nonnegative real numbers, and edges are weighted by random variables with values from a finite set of real numbers. A weighted graph is a special case when all these distributions are concentrated on a single value. We say that the randomly weighted graph is proper, if it is not an ordinary weighted graph.

Multigraph moments $t(F,H)$ of a randomly weighted graph $H$ can be defined; they will be multiplicative, normalized, reflection positive graph parameters.

Our main result (Theorem 3.10) describes multigraph parameters $f$ that are multiplicative, normalized, reflection positive, and whose second connection matrix $M(f,2)$ has finite rank (it is enough to require the finiteness of certain very simple submatrices). The theorem gives two alternatives: such a graph parameter is either

- of the form $t(.,H)$ for some weighted graph $H$, in which case $\text{rk}(M(f,k))^{1/k} \to c \geq 1$ as $k \to \infty$, or
- of the form $t(.,H)$ for some proper randomly weighted graph $H$, in which case $\text{rk}(M(f,k))^{1/k^2} \to c > 1$ as $k \to \infty$. 

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In particular, the finiteness of the rank of $M(f,2)$ implies the finiteness of the ranks of all higher connection matrices $M(f,k)$.

2 Preliminaries

2.1 Graphs and homomorphisms

We consider four types of graphs. A simple graph is a finite undirected graph without loops or multiple edges. In a multigraph multiple edges are allowed but loop edges are excluded. The edge set $E(G)$ of a multigraph $G$ is a multiset of unordered pairs $ij$ where $i,j$ are distinct elements of the node set. A weighted graph $H$ on node set $V = V(H)$ is given by an assignment of positive nodeweights ($\alpha_i : i \in V$) and an assignment of real edgeweights $\beta_{ij} : i,j \in V$). We consider $i,j \in V$ as adjacent if $\beta_{ij} \neq 0$. Note that we allow loop edges in weighted graphs, but if $\beta_{ii} = 0$ for all $1 \leq i \leq n$, then we say that $H$ is loopless. Every multigraph $F$ can be considered as a weighted graph with nodeweights 1 and nonnegative integral edgeweights (multiplicities) $F_{i,j}$.

We say that $H$ is a randomly weighted graph if its nodes are weighted by nonnegative real numbers $\alpha_i$, and its edges are weighted by independent random variables $B_{i,j}$ with finite distribution. We can also think of randomly weighted graphs as graphs whose edges are labeled by moment sequences of random variables with a finite range, showing that these are discrete versions of $[-d,d]$-graphons.

Also note that ordinary weighted graphs can be regarded as randomly weighted graphs in which the edgeweights are single-valued random variables. An important parameter of randomly weighted graph $H$ will be $p_{i,j}$, the number of values $B_{ij}$ takes with positive probability, and $p(H)$, the maximum of the $p_{i,j}$. Ordinary weighted graphs are just those random weighted graphs with $p(H) = 1$.

Throughout this paper, if we say just graph, we mean a multigraph.

For an arbitrary multigraph $F$ and weighted graph $H$, the homomorphism number from $F$ to $H$ is defined by

$$\text{hom}(F,H) = \sum_{\varphi : V(F) \rightarrow V(H)} \prod_{i \in V(F)} \alpha_{\varphi(i)} \prod_{(i,j) \in E(F)} \beta_{\varphi(i), \varphi(j)}.$$  \hspace{1cm} (3)

Sometimes it is convenient to normalize the graph parameter $\text{hom}(G,H)$ and to introduce the homomorphism density

$$t(F,H) = \frac{\text{hom}(F,H)}{(\sum_i \alpha_i)^{|V(F)|}}.$$
Note that $t(F, H) = \text{hom}(F, H')$ where $H'$ is obtained from $H$ by dividing the node weights by $\alpha$. A weighted graph is called normalized if the sum of its node weights is 1.

For an arbitrary graph $F$ with $m$ nodes we define an injective version of these numbers by the formula

$$\text{inj}(F, H) = \sum_{\varphi: V(F) \to V(H)} \prod_{(i,j) \in E(F)} \beta_{\varphi(i), \varphi(j)},$$

where $\varphi$ ranges over all injective functions from $V(F)$ to $V(H)$. Again, we can normalize to get

$$t_{\text{inj}}(F, H) = \frac{\text{inj}(F, H)}{\sigma_{|V(F)|}(\alpha)},$$

where $\sigma_k(\alpha)$ denotes the $k$-th elementary symmetric polynomial of the $\alpha_i$.

For a randomly weighted graph we define the homomorphism number $\text{hom}(F, H)$ as

$$\text{hom}(F, H) = \sum_{\varphi: V(F) \to V(H)} \prod_{i \in V(F)} \alpha_{\varphi(i)} \prod_{ij \in E(F)} E(B_{\varphi(i), \varphi(j)})^{A_{\varphi(i), \varphi(j)}}. \quad (4)$$

Setting $\beta_{i,j,k} = E(B_{i,j}^k)$, we have

$$\text{hom}(F, H) = \sum_{\varphi: V(F) \to V(H)} \prod_{i \in V(F)} \alpha_{\varphi(i)} \prod_{ij \in E(F)} \beta_{\varphi(i), \varphi(j), F_{i,j}}.$$

Similarly as before, we introduce the scaled version

$$t(F, H) = \frac{\text{hom}(F, H)}{(\sum_i \alpha_i)^{|V(F)|}}.$$

**Remark 2.1** It is not quite evident where to put the expectation in (4). Moving it further in like

$$\sum_{\varphi: V(F) \to V(H)} \prod_{i \in V(F)} \alpha_{\varphi(i)} \prod_{ij \in E(F)} E(B_{\varphi(i), \varphi(j)})^{A_{\varphi(i), \varphi(j)}}$$

would of course just reduce the issue to an ordinary weighted graph, where each random variable $B_{i,j}$ is replaced by its expectation. Moving it further out like

$$E\left( \sum_{\varphi: V(F) \to V(H)} \prod_{i \in V(F)} \alpha_{\varphi(i)} \prod_{ij \in E(F)} B_{\varphi(i), \varphi(j)}^{A_{\varphi(i), \varphi(j)}} \right)$$

would destroy multiplicativity.
With every normalized randomly weighted graph $H$ we can associate a $[-d,d]$-graphon $W_H$, by splitting the unit interval into $|V(H)|$ intervals $S_i$ of length $\alpha_i$, and assigning the random variable $B_{i,j}$ to each point in $S_i \times S_j$. This graphon $W_H$ has two finiteness properties: $W_H(x,y)$ has a finite range for all $x$ and $y$, and there are only a finite number of different distributions $W_H(x,y)$. In the special case when $H$ is a weighted graph, we get a function $W \in \mathcal{W}$. It is easy to see that this representation has the property that for every multigraph $F$, $t(F,W_H) = \text{hom}(F,H) = t(F,H)$.

### 2.2 Quantum graphs and reflection positivity

Let $\mathcal{G}_n \ (n = 0, 1, 2, \ldots)$ denote the set of multigraphs in which $n$ different nodes are labeled by the natural numbers $\{1, 2, \ldots, n\}$ (the graphs may have an arbitrary number of unlabeled nodes). Note that $\mathcal{G}_0$ is the set of (isomorphism classes of) graphs without labeled nodes. Let $\mathcal{F}_n \subset \mathcal{G}_n$ denote the set of graphs whose node set is $\{1, 2, \ldots, n\}$. For two graphs $F_1, F_2 \in \mathcal{G}_n$ we define their product $F_1 F_2$ as follows: we take their disjoint union and then we identify nodes with identical labels.

The set $\mathcal{G}_n$ endowed with this multiplication forms a commutative semigroup with a unit element in which $\mathcal{F}_n$ is a sub-semigroup. We denote by $Q_n$ the semigroup algebra $\mathbb{R}[\mathcal{G}_n]$ and by $P_n$ the semigroup algebra $\mathbb{R}[\mathcal{F}_n]$. The elements of these algebras are formal linear combinations of (partially) labeled graphs, and for this reason we call them quantum graphs.

Let us fix a number $n$ and let $z_{i,j} \ (1 \leq i < j \leq n)$ denote the graph with $V(z_{i,j}) = \{n\}$ with a single edge connecting $i$ and $j$. It is clear that $P_n$ is generated freely by $\{z_{i,j}|1 \leq i < j \leq n\}$ as a commutative algebra and thus it is isomorphic to the polynomial ring $\mathbb{R}[\{z_{i,j}|1 \leq i < j \leq n\}]$. Note that the monomials of this polynomial ring are in a one-to-one correspondence with graphs in $\mathcal{F}_n$.

A **graph parameter** is a map from the set of multigraphs to the real numbers. Any graph parameter $f$ can be extended linearly to the vector spaces $Q_n$ and $P_n$ for all $n \geq 0$. We say that $f$ is **reflection positive** (resp. **weakly reflection positive**) if $f(p^2) \geq 0$ holds for all natural numbers $n$ and quantum graphs $p \in Q_n$ (resp. $p \in P_n$).

Any graph parameter $f$, as we have seen, extends linearly to $Q_n$. In addition, $f$ induces a bilinear form $\langle \cdot, \cdot \rangle_f$ on $Q_n$ by $\langle p, q \rangle_f = f(pq)$. This form has the property that $\langle pq, r \rangle_f = \langle p, qr \rangle_f$. Note that the reflection positivity (resp. weak reflection positivity) of $f$ is equivalent to the positive semidefinitness of the bilinear forms $\langle \cdot, \cdot \rangle_f$ on the algebras $Q_n$ (resp. $P_n$).
Let
\[ I(P_n, f) = \{ x | x \in P_n, \langle x, P_n \rangle_f = 0 \}. \]
It is clear that \( I(Q_n, f) \) is an ideal of the algebra \( Q_n \), and we can consider the factor \( Q_n/f = Q_n/I(Q_n, f) \). Clearly \( \text{dim}(Q_n/f) \) is the rank of the bilinear form \( \langle \cdot, \cdot \rangle_f \) on \( Q_n \). We can carry out these constructions with \( P_n \) instead of \( Q_n \). In the case when \( f = \text{hom}(\cdot, H) \) for some randomly weighted graph \( H \), we also denote \( Q_n/f \) by \( Q_n/H \).

The algebras \( Q_n/f \) and the numbers \( \text{dim}(Q_n/f) \) were introduced in [7].

Basic properties of these algebras can also be expressed in terms of certain matrices. The \( n \)-th connection matrix of a graph parameter \( f \) is an infinite matrix \( M(n, f) \) whose rows and columns are indexed by the elements of \( G_n \) and the entry in the intersection of the row corresponding to \( F_1 \) and column corresponding to \( F_2 \) is \( f(F_1 F_2) \). The rank of this matrix is equal to \( \text{dim}(Q_n/f) \), and this matrix is positive semidefinite if and only if so is the bilinear form \( \langle \cdot, \cdot \rangle_f \) on \( Q_n \).

A graph parameter \( f \) is multiplicative if \( f(F_1 \cup F_2) = f(F_1)f(F_2) \) where \( F_1 \cup F_2 \) is the disjoint union of \( F_1 \) and \( F_2 \). We call \( f \) normalized if takes the value 1 on a single node. It is clear that \( f \) is multiplicative and normalized if and only if the induced map \( f_0 : Q_0 \to \mathbb{R} \) is an algebra homomorphism. It is also easy to see that \( f \) is multiplicative if and only if \( \text{dim}(Q_0, f) \leq 1 \).

### 2.3 Semidefinite functions on polynomial rings

Let \( n \) be a fixed natural number and let \( x_1, x_2, \ldots, x_n \) be variables. A polynomial expression of \( W \) in \( n \) variables is a polynomial of the functions \( \{ W(x_i, x_j) \mid 1 \leq i < j \leq n \} \). Note that these \( n \)-variable functions form a commutative algebra with the pointwise multiplication and addition. We define the moment of \( W \) corresponding to a polynomial expression \( p(x_1, x_2, \ldots, x_n) \) as
\[
\int_{[0,1]^n} p \, dx_1 \ldots dx_n.
\]

The ring of the polynomial expressions of \( W \) in \( n \) variables is a homomorphic image of \( P_n \) where the homomorphism is given by \( z_{i,j} \to W(x_i, x_j) \). Composing the moment map with this homomorphism we obtain a linear map \( t_W : P_n \to \mathbb{R} \). Since the moments of a polynomial expression are invariant under any permutation of the variables, we obtain that for an element \( F \in F_n \) the value \( t_W(F) = t(F, W) \) does not depend on the labeling of \( F \), only on its isomorphism class. For this reason we can also regard \( t_W \) as a
which implies $1 \geq |\alpha|$ and so positive semidefiniteness of $\beta$. For all possible sequences $r$ on the index of the last nonzero $i$ for all $n$. It is easy to see that $t_W$ is concentrated on finitely many points. Let us introduce the linear function $\beta : \mathbb{R}[x_1, x_2, \ldots, x_n] \to \mathbb{R}$ by

$$\beta(x_1^{r_1} x_2^{r_2} \ldots x_n^{r_n}) = d^{-(r_1+r_2+\cdots+r_n)} \alpha(x_1^{r_1} x_2^{r_2} \ldots x_n^{r_n}).$$

It is easy to see that $\beta$ is a positive semidefinite function and $|\beta(x_i^r)| \leq 1$ for all $i$ and $r$. We show that $|\beta(x_1^{r_1} x_2^{r_2} \ldots x_n^{r_n})| \leq 1$. We do it by induction on the index of the last nonzero $r_i$. Assume that $|\beta(x_1^{r_1} x_2^{r_2} \ldots x_i^{r_i})| \leq 1$ for all possible sequences $r_1, r_2, \ldots, r_i$. Let $p = x_1^{r_1} x_2^{r_2} \ldots x_i^{r_i}$. It follows by the positive semidefiniteness of $\beta$ that

$$\beta((p \pm x_i^{r_i+1})^2) \geq 0$$

and so

$$2 \geq \beta(p^2) + \beta(x_i^{2r_i+1}) = \pm 2px_i^{r_i+1}$$

which implies $1 \geq |px_i^{r_i+1}|$. As a consequence we get that

$$|\alpha(x_1^{r_1} x_2^{r_2} \ldots x_n^{r_n})| \leq d^{r_1+r_2+\cdots+r_n}.$$
This means that $\alpha$ is an exponentially bounded positive semidefinite function on the semigroup of the monomials which is isomorphic to $\mathbb{N}_0^n$. Now 2.5. Theorem (…) completes the proof. \hfill $\square$

Note that any probability measure $\mu$ on $[-d, d]^n$ defines a semidefinite function $\alpha$ on $\mathbb{R}[x_1, x_2, \ldots, x_n]$ by \eqref{5}.

We will need two further well-known facts.

**Lemma 2.3** Assume that the map $\alpha : \mathbb{R}[x_1, x_2, \ldots, x_n] \to \mathbb{R}$ is defined by

$$\alpha(p) = \sum_{i=1}^{k} h_i p(a_i),$$

where $a_1, \ldots, a_k \in \mathbb{R}^n$ are different real vectors, and the weights $h_i$ are positive real numbers. Then $\alpha$ is a positive semidefinite function and the corresponding bilinear form

$$\langle p_1, p_2 \rangle = \alpha(p_1 p_2) \quad (p_1, p_2 \in \mathbb{R}[x_1, x_2, \ldots, x_n])$$

has rank $k$.

**Lemma 2.4** Let $d > 0$ be a fixed real number. A sequence of measures $\mu_1, \mu_2, \ldots$ on $[-d, d]^n$ is weakly convergent if and only if $\lim_{k \to \infty} \int f d\mu_k$ exists for every monomial $f$.

### 2.4 Two-variable functions as operators

Any bounded symmetric function $W$ on $[0, 1]^2$ gives rise to a symmetric integral kernel operator $T_W$ on the Hilbert space $L_2([0, 1])$, by

$$T_W(f)(x) = \int_{0}^{1} W(y, x) f(y) \, dy.$$ 

It follows from the Hilbert-Smith condition that such an operator is always compact, and so it has a countable set of nonzero eigenvalues \{\lambda_1, \lambda_2, \lambda_3 \ldots\}, where we may assume that $|\lambda_1| \geq |\lambda_2| \geq \ldots$. It is known that $\lambda_k \to 0$, and so every nonzero eigenvalue has finite multiplicity. We will need the well-known fact that for $n \geq 2$,

$$t(C_n, W) = \sum_{k=1}^{\infty} \lambda_k^n.$$ \hfill (6)

The operator rank of $T_W$ and the matrix rank of $C(T_W)$ are either both infinite or both finite. More exactly,
Lemma 2.5 The rank of $C(t_W)$ is between the number of different nonzero eigenvalues and the number of all nonzero eigenvalues of $T_W$.

Proof. By (6),

$$C(t_W)_{i,j} = \sum_k \lambda_k^{i+j} = \sum_k m_k \lambda_k^{i+j}.$$ 

If this sum is finite, i.e., $\text{rk}(T_W) = m$ is finite, then $C(t_W)$ is the sum of $m$ matrices of rank 1, and so it has rank at most $m$.

Conversely, suppose that $C(t_W)$ has finite rank $n$. Then there is a linear dependence between its first $n+1$ columns, which means that we have a relation

$$\sum_{j=1}^{n+1} a_j \sum_k \lambda_k^{i+j} = 0$$

valid for all $i \geq 1$. We can rewrite this as

$$\sum_k p(\lambda_k) \lambda_k^{i+1} = 0,$$

where $p$ is a polynomial of degree at most $n$.

We claim that $p(\lambda_k) = 0$ for all $k$. Suppose not, and let $r$ be the first index for which $p(\lambda_r) \neq 0$, and let $a$ and $b$ be the multiplicities of the eigenvalues $\lambda_r$ and $-\lambda_r$ ($a \geq 1, b \geq 0$). Then we have

$$ap(\lambda_r) + (-1)^{i+1}bp(-\lambda_r) = -\sum_{k:|\lambda_k|<|\lambda_r|} p(\lambda_k) \left( \frac{\lambda_k}{\lambda_r} \right)^{i+1}.$$ 

Here the right hand side tends to 0 as $i \to \infty$, implying that $a+b=0$ and also $a-b=0$, which is a contradiction.

So every nonzero eigenvalue of $T_W$ is a root of $p$, which means that their number is at most $\text{deg}(p) \leq n$. □

This implies

Corollary 2.6 Let $W \in \mathcal{W}(d)$ and assume that $C(t_W)$ has finite rank. Then the kernel operator $T_W$ is of finite rank and there is a finite sequence of pairwise orthogonal functions $g_1, g_2, \ldots, g_k \in L^2([0,1])$ and numbers $\nu_i \in \{d, -d\}$ such that

$$W(x,y) = \sum_{i=1}^{k} \nu_i g_i(x) g_i(y)$$

almost everywhere on $[0,1]^2$.  

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The product of two operators $T_{W_1}$ and $T_{W_2}$ is $T_{W_1 \circ W_2}$ where $W_1 \circ W_2$ is given by

$$(W_1 \circ W_2)(x, y) = \int_0^1 w_1(x, z) w_2(z, y) \, dz.$$ 

Let $F'$ denote the graph which is obtained from $F$ by subdividing each edge in $E(G)$. It will be useful to note that

**Lemma 2.7** If $W \in \mathcal{W}$ and $F$ is any graph then $t(F', W) = t(F, W \circ W)$.

### 3 Results and proofs

#### 3.1 Moments and moment-like graph parameters

Let $\mathcal{W}(d)$ denote the set of 2-variable measurable functions $W : [0, 1]^2 \to [-d, d]$ that are symmetric in the sense that $W(x, y) = W(y, x)$ for all $x, y \in [0, 1]$. We denote by $\mathcal{W}$ the union of the sets $\mathcal{W}(d)$ over all real numbers $d$.

Let us define four sets of graph parameters:

- $\mathcal{T}_0(d) = \{t(., H) : H \text{ is a } [-d, d]-\text{weighted graph}\}$,
- $\mathcal{T}_1(d) = \{t(., H) : H \text{ is a randomly } [-d, d]-\text{weighted graph}\}$,
- $\mathcal{T}_2(d) = \{t(., W) : W \in \mathcal{W}(d)\}$,
- $\mathcal{T}_3(d) = \{t(., W) : W \text{ is a } [-d, d]-\text{graphon}\}$.

Clearly $\mathcal{T}_0(d) \subseteq \mathcal{T}_1(d), \mathcal{T}_2(d) \subseteq \mathcal{T}_3(d)$. We prove that equality almost holds here. Let us quote Theorem 2.6 from [17], applied to our case:

**Theorem 3.1** Let $W_1, W_2, \ldots$ be a sequence of $[-d, d]$-graphons such that $(t(F, W_1), t(F, W_2), \ldots)$ is a convergent sequence for every multigraph $F$. Then there is a $[-d, d]$-graphon $W$ such that $t(F, W_n) \to t(F, W)$ for every $F$.

This theorem implies that $\mathcal{T}_3(d)$ is a closed subset of the space of graph parameters under pointwise convergence. We are going to prove:

**Theorem 3.2** The set $\mathcal{T}_3(d)$ is the closure of $\mathcal{T}_0(d)$.

We are also going to prove

**Theorem 3.3** A graph parameter $f$ belongs to $\mathcal{T}_3(d)$ if and only if it is normalized, multiplicative, weakly reflection positive and satisfies $|f(K^k_2)| \leq d^k$. 

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By Tychonov’s Compactness Theorem, $\mathcal{T}_3(d)$ is compact as a closed subspace of the compact space $\prod_F [-d|E(F)|, d|E(F)|]$.

Both theorems will follow if we prove two facts:

**Proposition 3.4** Every graph parameter $f \in \mathcal{T}_3(d)$ is normalized, multiplicative, reflection positive and satisfies $|f(K_2^k)| \leq d^k$.

**Proposition 3.5** Every normalized, multiplicative, weakly reflection positive graph parameter satisfying $|f(K_2^k)| \leq d^k$ is the limit of graph parameters in $\mathcal{T}_0$.

Before proving these propositions, we state an easy lemma about homomorphism densities and injective homomorphism densities, which follows from Lemma 2.1 in [14] by scaling the edgeweights.

**Lemma 3.6** Let $H$ be a weighted graph with $n$ nodes such that all the nodeweights are 1 and the edgeweights are in $[-d,d]$. Then for an arbitrary multigraph $F$ with $m$ nodes,

$$|t(F,H) - t_{\text{inj}}(F,H)| \leq 2\left(\frac{m}{2}\right)\frac{1}{n}d|E(F)|.$$ 

**Proof of Proposition 3.4.** It is clear that for every $[-d,d]$-graphon $W$, the parameter $f = t(.,W)$ is multiplicative and normalized, and satisfies $|f(K_2^k)| \leq d^k$. We prove that $M(f,n)$ is positive semidefinite.

Let $F$ be a graph in $\mathcal{G}_n$ such that $V(F) = [m]$ for some natural number $m \geq n$. For every choice of the variables $x_1, \ldots, x_n$ we define

$$t_{x_1,\ldots,x_n}(F,W) = \int_{[0,1]^{m-n}} \prod_{1 \leq i<j \leq m} W_{F_{i,j}}(x_i,x_j) \, dx_{n+1} \ldots dx_m.$$ 

We have

$$t(FF',W) = \int_{[0,1]^n} t_{x_1,\ldots,x_n}(F,W)t_{x_1,\ldots,x_n}(F',W)$$

$$\times \prod_{1 \leq i<j \leq n} W_{F_{i,j}+F'_{i,j}}(x_i,x_j) \, dx_1 \ldots dx_n.$$ 

where $F$ and $F'$ are graphs in $\mathcal{G}_n$.

For every $x \in [0,1]^n$, let $M(x)$ denote the $\mathcal{G}_n \times \mathcal{G}_n$ matrix in which

$$M(x)_{F,F'} = t_x(F,W)t_x(F',W) \prod_{1 \leq i<j \leq n} W_{F_{i,j}+F'_{i,j}}(x_i,x_j).$$
From the above formulas one obtains that
\[ M(f, n) = \int_{[0,1]^n} M(x) \, dx, \tag{7} \]
so it suffices to prove that \( M(x) \) is positive semidefinite for every \( x \).

For \( 1 \leq i < j \leq n \), let \( M(x, i, j) \) denote the \( G_n \times G_n \) matrix in which
\[ M(x, i, j)_{F, F'} = W_{F_{ij} + F'_{ij}}(x_i, x_j). \]

Since \( M(x, i, j) \) is essentially (up to repetition of rows and columns) the moment matrix of the random variable \( W_{F_{ij} + F'_{ij}}(x_i, x_j) \), it is positive semidefinite. We get \( M(x) \) from the Schur product of the matrices \( M(x, i, j) \) over all possible pairs \( 1 \leq i < j \leq n \), and scaling the rows and columns. This shows that \( M(x) \) is indeed positive semidefinite. \( \square \)

Proof of Proposition 3.5. It suffices to consider the case \( d = 1 \), since we can scale the edgeweights by \( 1/d \). Let \( f \) be a weakly reflection positive, normalized multiplicative graph parameter with \( f(K^2_k) \leq 1 \) for all \( k \geq 1 \).

We prove that there is a sequence of stepfunctions \( U_1, U_2, \ldots \) in \( W(1) \) such that \( \lim_{n \to \infty} t(F, U_n) = f(F) \) for every graph \( F \).

The weak reflection positivity of \( f \) means that \( f \) is a semidefinite function on the polynomial ring \( P_n \) for every natural number \( n \). Using that \( f(K^2_k) \leq 1 \) and Theorem 2.2 for \( P_n \) we obtain that there is a unique probability measure \( \mu_n \) on \( [-1,1]^{(n)} \) such that
\[ f(F) = E\left( \prod_{1 \leq i < j \leq n} z_{F_{ij}}^{i,j} \right) \]
for every graph \( F \in \mathcal{F}_n \), where the \( z_{i,j} \) are regarded as random variables whose joint distribution is given by \( \mu_n \). Let \( Z_n \) be a random weighted graph (not a randomly weighted graph!) on \( [n] \) with nodeweights \( 1/n \) and edgeweights \( z_{i,j} \). Since \( f \) is invariant under relabeling the nodes of \( F \) we get that
\[ f(F) = E\left( \frac{1}{n!} \sum_{\sigma \in \mathcal{S}_n} \prod_{1 \leq i < j \leq n} z_{\sigma(i),\sigma(j)}^{F_{ij}} \right) = E(t_{\text{inj}}(F, Z_n)). \]

Fix a graph \( F \in \mathcal{F}_m \) and for every \( n \geq m \), define the graph \( F_n \in \mathcal{F}_n \) by adding \( n - m \) isolated labeled nodes to \( F \). It is clear that
\[ t_{\text{inj}}(F_n, Z_n) = t_{\text{inj}}(F, Z_n) \]
and (using the properties of $f$) that

$$f(F) = f(F_n) = \mathbb{E}(t_{\text{inj}}(F_n, Z_n)) = \mathbb{E}(t_{\text{inj}}(F, Z_n))$$ (8)

for all $n \geq m$. Let $F^2$ denote the disjoint union of $F$ with itself. Using that $f(F^2) = f(F)^2$ and (8) we get that

$$\text{Var}(t_{\text{inj}}(F, Z_n)) = \mathbb{E}(t_{\text{inj}}(F, Z_n)^2) - \mathbb{E}(t_{\text{inj}}(F, Z_n))^2$$

$$\quad = \mathbb{E}(t_{\text{inj}}(F, Z_n)^2) - f(F^2) = \mathbb{E}(t_{\text{inj}}(F, Z_n)^2 - t_{\text{inj}}(F^2, Z_n)).$$

From Lemma 3.6 it follows that

$$|t(F, Z_n)^2 - t_{\text{inj}}(F, Z_n)^2| \leq \left|2 \left(\frac{m}{2}\right)^2 \frac{1}{n} (t(F, Z_n) + t_{\text{inj}}(F, Z_n))\right| \leq \frac{4}{n} \left(\frac{m}{2}\right).$$

and similarly

$$|t(F^2, Z_n) - t_{\text{inj}}(F^2, Z_n)| \leq \frac{2}{n} \left(\frac{2m}{2}\right).$$

Using that $t(F^2, Z_n) = t(F, Z_n)^2$, we get

$$|t_{\text{inj}}(F, Z_n)^2 - t_{\text{inj}}(F^2, Z_n)| \leq \frac{6m^2}{n}.$$

Thus

$$\text{Var}(t_{\text{inj}}(F, Z_n)) = \mathbb{E}(t_{\text{inj}}(F, Z_n)^2 - t_{\text{inj}}(F^2, Z_n)) \leq \frac{6m^2}{n}.$$ 

By Chebyshev’s inequality, we have for every $\varepsilon > 0$,

$$\mathbb{P}(|t_{\text{inj}}(F, Z_n) - f(F)| > \varepsilon) \leq \frac{6m^2}{\varepsilon^2 n}.$$ 

It follows by the Borel-Cantelli lemma that

$$\lim_{n \to \infty} t_{\text{inj}}(F, Z_n^2) = f(F)$$

with probability 1.

Since there are only countably many different graphs $F$, we obtain that the above convergence holds simultaneously for all graphs with probability 1. By Lemma 3.6 we get that the graph parameter $t(., Z_n^2)$ converges to $f(F)$ with probability one in the space of graph parameters. Thus $f$ is in the closure of $T_0(d)$.

$\square$
3.2 Finiteness conditions

In a sense, the classes $T_0(d)$ and $T_1(d)$ are finite versions of the classes $T_2(d)$ and $T_3(d)$. Theorem 1.2 tells us that a graph parameter $f \in T_2(d)$ belongs to $T_0(d)$ if and only if there is a positive integer $q$ such that $\text{rk}(M(f,k)) \leq q^k$ for all $k$. We prove that a much weaker condition is sufficient.

For a graph parameter $f$, we define three infinite matrices $E(f)$, $C(f)$ and $B(f)$, in each of which the rows and columns are indexed by the natural numbers $0, 1, 2, 3, \ldots$, and the entries are defined by three one-parameter families of graphs. Let $K^n_n$ consist of 2 nodes joined by $n$ edges, let $C_n$ be the $n$-cycle, and let $K_{a,b}$ be the complete bipartite graph with color classes if sizes $a$ and $b$. We define

$$E(f)_{ij} = f(K_2^{i+j}),$$
$$C(f)_{ij} = f(C_{i+j-1}),$$
$$B(f)_{ij} = f(K_{i+j,2}).$$

Note that all three matrices are submatrices of the multigraph connection matrix $M(f,2)$.

**Theorem 3.7** For a graph parameter $f \in T_2(d)$, the following are equivalent:

(a) $f \in T_0(d)$;
(b) both $C(f)$ and $E(f)$ have finite rank;
(c) both $B(f)$ and $E(f)$ have finite rank;
(d) $M(f,2)$ has finite rank.

Even though the graphs used in condition (b) in Theorem 3.7 are smaller, condition (c) may be more useful because it doesn’t use graphs with multiple edges.

**Proof.** It is trivial that (a) implies (d), which in turn implies both (b) and (c).

(b)$\Rightarrow$(a). Let $f = t(.,W)$ and let $X_W$ denote the random variable $W(X_1,X_2)$ where $X_1$ and $X_2$ are chosen uniformly at random from $[0,1]$. The $n$-th moment of $X_W$ is

$$\int_{[0,1]^2} W(x_1,x_2)^n \, dx_1 \, dx_2 = t(K_2^n,W).$$
Let a linear map $\alpha : \mathbb{R}[x] \to \mathbb{R}$ be defined by

$$\alpha(x^n) = t(K^n_2, W).$$

Since the matrix $E(tW)$ is the matrix of the bilinear form $\langle f, g \rangle = \alpha(fg)$ in the basis $1, x, x^2, \ldots$ it follows from Theorem 2.22 that the distribution of $X_W$ is concentrated on some finite set $S$. Since $C(T_W)$ has finite rank, Corollary 2.6 implies that there is a finite system of one-variable functions $g_1, g_2, \ldots, g_k$ and signs $\nu_i \in \{1, -1\}$ such that

$$W(x, y) = \sum_{i=1}^{k} \nu_i g_i(x)g_i(y)$$

almost everywhere. By changing $W$ on a zero measure set, we can assume that the previous equality holds everywhere. Since the set $\{ (x, y) : W(x, y) \notin S \}$ has measure 0, there is a set $Z \subseteq [0, 1]$ with measure 0 such that for all $x \in [0, 1] \setminus Z$, the function $W(x, .)$ is measurable and the set $\{ y \in [0, 1] : W(x, y) \notin S \}$ has measure 0. For every fixed $x$, the function $W(x, .)$ is an element of the finite dimensional subspace generated by $g_1, g_2, \ldots, g_k$, and so there are points $x_1, x_2, \ldots, x_k \in [0, 1] \setminus Z$ such that every function $W(x, .), x \in [0, 1] \setminus Z$ is a linear combination of the functions $W(x_i, .)$.

For each $i$, there is partition $\{U_0^i, U_1^i, \ldots, U_s^i\}$ of $[0, 1] \setminus Z$ into measurable sets, where $s = \lvert S \rvert$, such that $\lambda(U_0^i) = 0$ and $W(x_i, .)$ is constant on each $U_j^i$, $1 \leq j \leq s$. Combining the sets $U_j^i$ into a single 0-measure set, and taking a common refinement of the partitions on the rest, we get a finite partition $\{U_0, U_1, \ldots, U_N\}$ such that $\lambda(U_0) = 0$ and each $W(x_i, .)$ is constant on each $U_j$, $1 \leq j \leq N$. Hence every function $W(x, .), x \in [0, 1] \setminus Z$, is constant on every set $U_j$, $1 \leq j \leq N$. From the symmetry of $W$ it follows that $W$ is constant on every set $U_i \times U_j$, and so $W$ is equal to a stepfunction almost everywhere.

(c)$\Rightarrow$(a). It follows from Lemma 2.7 that the matrix $B(t_W)$ is the same as $E(t_{W^2})$ and that $C(t_{W^2})$ is a submatrix of $C(t_W)$. So $W \circ W$ satisfies (b), and so we already know that $W \circ W$ is a stepfunction. Thus $T_{W^2}$ has finite rank and every eigenvector of $T_{W^2}$ corresponding to a nonzero eigenvalue is a one-variable stepfunction. Since $T_{W^2}$ is the square of $T_W$, it follows that the same statement holds for $T_W$. This implies that $W$ is a stepfunction. \[\square\]
3.3 Homomorphisms into randomly weighted graphs

We prove the following generalization of Theorem 1.2.

**Theorem 3.8** Let $f$ be a graph parameter. Then the following are equivalent.

1. There is a randomly weighted graph $H$ such that $f(F) = \text{hom}(F, H)$ for all graphs $F$.
2. $f$ is reflection positive, multiplicative and $\text{rk}(M(2, f)) < \infty$.
3. $f$ is weakly reflection positive, multiplicative and $\text{rk}(M(2, f)) < \infty$.

As a corollary, we obtain the following characterization of simple graph parameters representable as homomorphism functions:

**Corollary 3.9** If $f$ is weakly reflection positive, multiplicative and $\text{rk}(M(2, f)) < \infty$, then there is a weighted graph $H$ such that $\text{hom}(F, H) = f(F)$ for all simple graphs $F$.

**Proof of Theorem 3.8**  

(1)⇒(2) Let $H$ be a randomly weighted graph. We may scale the nodeweights so that they sum to 1. Then $f = t(., W_H)$. By Proposition 3.4 we know that $f$ is multiplicative, normalized and reflection positive. We need to prove that $M(f, 2)$ has finite rank.

This follows easily by looking at the proof of Proposition 3.4 carefully. In (7), the integral can be replaced by a finite sum with $|V(H)|^n$ terms, since the integrand depends only on the nodes of $H$ represented by the intervals containing each $x_i$. Furthermore, each matrix $M(x)$ is the Schur product of a finite number of matrices $M(x, i, j)$. Each $M(x, i, j)$ is a moment matrix of a random variable with finite range, and hence it has finite rank. Hence every matrix $M(x)$ has finite rank, and so $M(f, n)$ has finite rank.

(2)⇒(3) is trivial.

(3)⇒(1) If $f$ is identically zero, then we regard it as the homomorphism function into the empty graph. Assume that $f$ is not identically zero. First we prove that $f(K_1) > 0$ where $K_1$ is the one-node graph. Regarding $K_1$ as an element of the algebra $Q_1$, we get from the weak reflection positivity of $f$ that $f(K_1)^2 = f(K_1) \geq 0$. Now assume that $f(K_1) = 0$. Let $F \in \mathcal{F}_n$ be a graph with $f(F) \neq 0$ for some natural number $n$ and let $e_n \in \mathcal{F}_n$ denote graph with $n$ labeled nodes and no edge (the unit element in $\mathcal{F}_n$). By multiplicativity, $f(e_n) = 0$. By weak reflection positivity we get for every real $\lambda$ that

$$0 \leq f((F - \lambda e_n)^2) = f(F^2) - 2\lambda f(F),$$

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which is a contradiction.

Replacing $f$ by $f/f(K_1)^{|V(F)|}$, we may assume that $f$ is normalized. The matrix $E(f)$ is positive semidefinite with finite rank. This implies that the sequence $f(K^n_2), \ n = 0, 1, 2, \ldots$ is the moment sequence of some random variable $X$ whose values are from a finite set. It follows that there is a number $d > 0$ such that $|f(K^n_2)| \leq d^n$ for every $n$. By Theorem [3.3] this implies that there is a $[-d, d]$-graphon $W$ such that $f(F) = t(F, w)$ for all graphs $F$.

Let $(W_0, W_1, \ldots) \in \mathcal{M}(d)$ be the moment function sequence representing $W$. We show that each function $W_i$ is a stepfunction. By Theorem [3.7] it is enough to show that $C(t_{W_i})$ and $B(t_{W_i})$ have finite rank. This will follow if we show that both are submatrices of $M(2, f)$.

Let $P_{a;i} \in \mathcal{G}_2$ denote the path of length $a$ in which each edge is $i$-fold and the two endpoints are labeled by 1 and 2. Let $K_{a;i} \in \mathcal{G}_2$ denote the complete bipartite graph $K_{2,a}$ in which each edge is $i$-fold and the nodes from the color class with two nodes are labeled by 1 and 2. It is clear from the definitions that the $\{P_{a;i}|a \geq 2\} \times \{P_{a;i}|a \geq 2\}$ sub-matrix of $M(2, f)$ is identical with $C(t_{W_i})$ and the $\{K_{a;i}|a \geq 0\} \times \{K_{a;i}|a \geq 0\}$ sub-matrix of $M(2, f)$ is identical with $B(t_{W_i})$ for all $i$. This proves that each $W_i$ is a stepfunction.

Next we argue that the $W_i$ can be considered stepfunctions with the same steps. For every pair $x, y \in [0, 1]$, $W(x, y)$ is a random variable with values in $[-d, d]$. Let $Y$ be the random variable which is obtained by selecting two random points $x, y$ uniformly form $[0, 1]$ and then evaluating the random variable $W(x, y)$. It is clear that

$$
E(Y^i) = \int_{[0,1]^2} W_i(x, y) = f(K^n_2) = E(X^i),
$$

and thus the distribution of $Y$ is the same as the distribution of $X$, which is concentrated on the finite set $S$. It follows that for almost all pairs $x, y \in [0, 1]$ the distribution of $W(x, y)$ is concentrated on $S$ and at such places the first $|S|$ moments $W_1(x, y), \ldots, W_{|S|}(x, y)$ of $W(x, y)$ determine all other moments. This means that all of the functions $W_i$ are stepfunctions with the same steps that are intersections of the steps of $W_1, W_2, \ldots, W_{|S|}$.

Thus there is a partition $[0, 1] = P_1 \cup P_2 \cup \cdots \cup P_t$ such that the variables $W(x, y)$ are constant on $P_i \times P_j$ for all $1 \leq i, j \leq t$. This defines the structure of a randomly weighted graph $H$ on $\{1, 2, \ldots, t\}$, in which the node weights are the sizes of the sets $P_i$, and the edgeweight of $ij$ is the random variable $W(x, y)$ for any $(x, y) \in P_i \times P_j$. It is clear that $f(F) = t(F, W) = \text{hom}(F, H)$ for every graph $F$. \qed
3.4 The growth rate of connection ranks

We have seen (Theorem 1.2) that for a graph parameter \( f \in T_0(d) \), the connection ranks \( \text{rk}(M(f,k)) \) are bounded by \( q^n \) for an appropriate \( q \). What can we say about graph parameters in the larger class \( T_1(d) \)? The following theorem gives the answer.

**Theorem 3.10** If \( f \) is a weakly reflection positive and multiplicative graph parameter, then \( f \) belongs to one of the following three types.

1. \( \text{rk}(M(f,n)) = \infty \) for all \( n \geq 2 \).
2. \( \text{rk}(M(f,n))^{1/n} \rightarrow c \) (\( n \rightarrow \infty \)) with some \( c \geq 1 \), and there is a weighted graph \( H \) such that \( f = \text{hom}(.,H) \).
3. \( \text{rk}(M(f,n))^{1/n^2} \rightarrow c \) (\( n \rightarrow \infty \)) with some \( c > 1 \), and there is a proper randomly weighted graph \( H \) such that \( f = \text{hom}(.,H) \).

**Remark 3.11** Finiteness of the rank of the first connection matrix \( M(f,1) \) is not enough here. In fact, let \( W \in \mathcal{W} \) be a function such that its measure preserving automorphism group (the group of invertible measure preserving maps \( \varphi : [0,1] \rightarrow [0,1] \) such that \( W(\varphi(x),\varphi(y)) = W(x,y) \)) is transitive. (For example, \( W(x,y) = |x-y| \) is such a function.) Then \( M(t_W,1) \) has rank 1. However, such functions may be far from being stepfunctions.

Before proving Theorem 3.10, we remark that the limiting constants \( c \) in (2) and (3) can be described easily, once we know that \( f \) is given by a randomly weighted graph. In the case when \( f = t(.,H) \) for an ordinary weighted graph \( H \), the rank of \( M(f,n) \) was described in [11]. We may assume that \( H \) has no twin nodes, since we can identify twin nodes in \( H \) without changing \( f \). Let us state also a description the dimension of \( P_n/f \).

**Lemma 3.12** Let \( f = t(.,H) \), where \( H \) is a weighted graph without twin nodes. Then

(a) The dimension of \( Q_n/f \) is equal to the number of non-equivalent maps \( [n] \rightarrow V(H) \), where two maps \( \varphi,\psi \) are equivalent if there is an automorphism \( \alpha \) of \( H \) such that \( \varphi\alpha = \psi \).

(b) The dimension of \( P_n/f \) is equal to the number of non-equivalent maps \( [n] \rightarrow V(H) \), where two maps \( \varphi,\psi \) are equivalent if there is an isomorphism \( \alpha \) between the subgraphs of \( H \) induced by \( \text{Rng}(\varphi) \) and \( \text{Rng}(\psi) \) such that \( \varphi\alpha = \psi \).

Part (a) was proved in [11]. We prove part (b) in a more general form, for randomly weighted graphs:
Lemma 3.13 Let $H$ be a randomly weighted graph with edge weights $B_{i,j}$ and node weights $\alpha_i$, let $f = \text{hom}(. , H)$, and let $n$ be a natural number. Then $\dim(P_n/f)$ is equal to the number of different weighted graphs $L$ on the node set $[n]$ with node weights 1 for which there is a function $\varphi : [n] \to V(H)$ such that each edge weight $\lambda_{i,j}$ of $L$ is an element of the range of $B_{\varphi(i), \varphi(j)}$.

We will not need to extend part (a) of Lemma 3.12 to randomly weighted graphs, but we believe this is possible.

Proof. Let $f = \text{hom}(. , H)$. Let $\mathbb{R}^{(2)}_n$ denote the polynomial ring $\mathbb{R}\{z_{i,j} | 1 \leq i < j \leq n\}$, which is isomorphic to the algebra $P_n$. Let $A$ denote the set of all possible pairs $(L, \varphi)$ where $L$ is a weighted graph on $\{1, 2, \ldots, n\}$, $\varphi : \{1, 2, \ldots, n\} \to V(H)$ is a function such that every edge weight $\lambda_{i,j}$ of $L$ is in the range of $B_{\varphi(i), \varphi(j)}$. To each element $(L, \varphi) \in A$ we introduce the weight

$$h(L, \varphi) = \prod_{i=1}^{n} \alpha_{\varphi(i)} \prod_{1 \leq i < j \leq n} P(B_{\varphi(i), \varphi(j)} = \lambda_{i,j}),$$

which is always a positive number. By substituting the definition of moments into the formula 4 one obtains that if $p$ is an arbitrary element of $R$ then

$$f(p) = \sum_{(L, \varphi) \in A} h(L, \varphi)p(\lambda)$$

where $p(\lambda)$ denotes the substitution of $z_{i,j} = \lambda_{i,j}$ in the polynomial $p$. Note that these substitutions are not always different for two different elements of $A$ but after sorting the sum according to different substitutions they can’t cancel each other because the weights $h(L, \varphi)$ are all positive. Using Lemma 2.3 we get that $\dim(P_n/f)$ is equal to the number of different labeled weighted graphs $L$ occurring in the first coordinate of the elements of $A$. This is exactly the statement of the lemma. \qed

Using this lemma, we can derive bounds on the rank of $M(f,n) = \dim(Q_n/f)$, where $f = \text{hom}(. , H)$ for a randomly weighted graph $H$. Define

$$A(H) = \max \left\{ \frac{1}{2} \sum_{u,v \in V(H)} x_u x_v \log p_{u,v} : x \geq 0, \sum_{u \in V(H)} x_u = 1 \right\}.$$  \hspace{1cm} (9)

Lemma 3.14 Let $H$ be a randomly weighted graph, $f = \text{hom}(. , H)$, and $n \in \mathbb{N}$. Then

$$\frac{2^{n^2 A(H)}}{p(H)^{2n}} \leq \dim(P_n/f) \leq \dim(Q_n/f) \leq |V(H)|^n 2^{n^2 A(H)}.$$
Proof. We may assume for convenience that $H$ is normalized. The upper bound follows from an even more careful look at the proof of Theorem 3.8 part (1)⇒(2). Each point $x \in [0,1]$ defines a map $\varphi : [n] \rightarrow V(H)$, and $M(x)$ depends on this $\varphi$ only. Each matrix $M(x,i,j)$ is a moment matrix of a random variable with finite range of size $p_{\varphi(i),\varphi(j)}$, and hence it has rank $p_{\varphi(i),\varphi(j)}$. Hence the rank of $M(x)$ is at most

$$\text{rk}(M(x)) \leq \prod_{1 \leq i < j \leq n} p_{\varphi(i),\varphi(j)}.$$ 

Let $n_u = |\varphi^{-1}(u)|$ ($u \in V(H)$), then we get

$$\text{rk}(M(x)) \leq \prod_{u,v \in V(H)} \frac{1}{p_{u,v}} \prod_{u \in V(H)} \left( \frac{n_u}{n} \right) \prod_{u,v \in V(H)} p_{u,v}.$$ 

Here

$$\log \prod_{u,v \in V(H)} \frac{1}{p_{u,v}} = \sum_{u,v \in V(H)} n_u n_v \log p_{u,v} = n^2 \sum_{u,v \in V(H)} \frac{n_u n_v}{n} \log p_{u,v} \leq n^2 A(H),$$

and so $\text{rk}(M(x)) \leq 2n^2 A(H)$. Since there are at most $|V(H)|^n$ different matrices $M(x)$, the upper bound follows.

To prove the lower bound, let $x \in \mathbb{R}^{V(H)}$ be the vector that attains the maximum in (9). Let $n_u$ ($u \in V(H)$) be integers such that $|nx_u - n_u| < 1$. Fix a map $\varphi : [n] \rightarrow V(H)$ such that $|\varphi^{-1}(u)| = n_u$ for all $u$. It is clear that we can create at least

$$N = \prod_{u,v \in V(H)} \frac{1}{p_{u,v}} \prod_{u \in V(H)} \left( \frac{n_u}{2} \right)$$

different weighted graphs on $[n]$ satisfying the condition of lemma 3.13 by choosing the edge weights between $\varphi^{-1}(u)$ and $\varphi^{-1}(v)$ independently from the range of $B_{\varphi(u),\varphi(v)}$. We have

$$\log N = \frac{1}{2} \sum_{u,v \in V(H)} n_u n_v \log p_{u,v} + \sum_{u \in V(H)} \left( \frac{n_u}{2} \right) \log p_{u,u}$$

$$= \frac{1}{2} \sum_{u,v \in V(H)} n_u n_v \log p_{u,v} - \sum_{u \in V(H)} n_u \log p_{u,u}.$$
Here
\[
\frac{1}{2} \sum_{u,v \in V(H)} n_u n_v \log p_{u,v} - n^2 A(H)
\]
\[
= \frac{1}{2} \sum_{u,v \in V(H)} n_u n_v \log p_{u,v} - n^2 \frac{1}{2} \sum_{u,v \in V(H)} x_u x_v \log p_{u,v}
\]
\[
= \frac{1}{2} \sum_{u,v \in V(H)} n_u (n_v - n x_u) \log p_{u,v} + \frac{1}{2} \sum_{u,v \in V(H)} (n_u - n x_u) n x_v \log p_{u,v}
\]
\[
\leq \sum_{u,v \in V(H)} n_u \log p_{u,v} \leq n \log p(H),
\]
and
\[
\sum_{u \in V(H)} n_u \log p_{u,u} \leq n \log p(H),
\]
showing that
\[
\log N \geq n^2 A(H) - 2n \log p(H).
\]
By Lemma 3.13 this proves that
\[
\dim(P_n/f) \geq N \geq \frac{2^n A(H)}{p(H)^{2n}}.
\]

Proof of Theorem 3.10. Let \( f \) be a weakly reflection positive and multiplicative graph invariant. Assume that \( \text{rk}(M(f,n)) < \infty \) for some integer \( n \geq 2 \). Since \( M(f,2) \) is a submatrix of \( M(f,n) \), we have that \( \text{rk}(M(f,2)) < \infty \). By Theorem 3.8 we obtain that there is a randomly weighted graph \( H \) such that \( f = \text{hom}(. , H) \). If \( H \) is a weighted graph, then by Lemma 3.12 it follows that
\[
\frac{|V(H)|^n}{|V(H)!|} \leq \dim(P_n/f) \leq \dim(Q_n/f) \leq |V(H)|^n,
\]
and hence both \( \dim(P_n/f)^{1/n} \) and \( \dim(Q_n/f)^{1/n} \) tend to \( \log |V(H)| \).

On the other hand, if \( H \) is a proper randomly weighted graph, then by Lemma 3.14 we have
\[
\frac{2^A(H)}{p(H)^{2/n}} \leq \dim(P_n/f)^{1/n^2} \leq \dim(Q_n/f)^{1/n^2} \leq |V(H)|^{1/n} 2^A(H),
\]
and so both \( \dim(P_n/f)^{1/n^2} \) and \( \dim(Q_n/f)^{1/n^2} \) tend to \( A(H) \).
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