Time-delay version of the integrable discrete Lotka-Volterra system in terms of the LR transformations

Masaki Sekiguchi, Kurumi Oka, Masashi Iwasaki and Emiko Ishiwata

1. Introduction

The Lotka-Volterra (LV) system is a well-known mathematical model of predator-prey interactions. A skillful discretization of the LV system describing the interaction of $2m - 1$ species leads to the integrable discrete LV (dLV) system with discretization parameter $\delta^{(n)}$:

$$
\begin{align*}
    u_k^{(n+1)}(1 + \delta^{(n+1)}u_{k-1}^{(n+1)}) &= u_k^{(n)}(1 + \delta^{(n)}u_{k+1}^{(n)}), \quad k = 1, 2, \ldots, 2m - 1, \\
    u_0^{(n)} &= 0, \quad u_1^{(n)} = 0, \quad n = 0, 1, \ldots
\end{align*}
$$

(1)

where the subscript $k$ and the superscript $(n)$ respectively denote species index and discrete-time and the variable $u_k^{(n)}$ corresponds to the population of the $k$th species at discrete-time $n$. In this paper, for simplicity, we focus on the case where $\delta^{(n)} = 1$ in the dLV system (1). Here, 'integrable' means that the solution can be explicitly expressed. Using the Hankel determinants:

$$
H_k^{(n)} := \begin{vmatrix}
    a_0^{(n)} & a_1^{(n)} & \cdots & a_k^{(n)} \\
    a_1^{(n)} & a_2^{(n)} & \cdots & a_{k+1}^{(n)} \\
    \vdots & \vdots & \ddots & \vdots \\
    a_{k-1}^{(n)} & a_k^{(n)} & \cdots & a_{2k-2}^{(n)}
\end{vmatrix},
\hat{H}_k^{(n)} := \begin{vmatrix}
    a_1^{(n)} & a_2^{(n)} & \cdots & a_k^{(n)} \\
    a_2^{(n)} & a_3^{(n)} & \cdots & a_{k+1}^{(n)} \\
    \vdots & \vdots & \ddots & \vdots \\
    a_k^{(n)} & a_{k+1}^{(n)} & \cdots & a_{2k-2}^{(n)}
\end{vmatrix},
$$

\[H_0^{(n)} := 1, \quad \hat{H}_0^{(n)} := 1, \quad H_{m+1}^{(n)} := 0,\]

where the sequences $\{a_k^{(n)}\}_{n=0,1,\ldots}$ satisfy the recursion formula:

$$
a_k^{(n+1)} - a_k^{(n)} = a_{k+1}^{(n)}
$$

(2)

we can write the solution to the dLV system (1) as:

$$
\begin{align*}
    u_k^{(n+1)} &= \frac{\hat{H}_k^{(n)}H_{k+1}^{(n+1)}}{H_k^{(n)}\hat{H}_{k+1}^{(n+1)}}, \quad u_{2k}^{(n)} = \frac{H_k^{(n+1)}\hat{H}_{k-1}^{(n+1)}}{\hat{H}_k^{(n+1)}H_k^{(n+1)}}.
\end{align*}
$$

(3)

The Hankel determinants enable us to express the solution to the discrete Toda equation [1–3], which is one of the most famous discrete integrable systems. Extending the Hankel determinants, we can grasp the solutions to...
the discrete relativistic Toda equation [4, 5] and the discrete hungry Toda equation [6], which are extensions of the discrete Toda equation. Similarly, we can derive the determinantal solution to the discrete hungry Lotka-Volterra system [6], which is an extension of the dLV system (1). The discrete Korteweg–de Vries equation, which often appears in the study of integrable systems, is also shown to have determinantal solution [7]. The viewpoint of determinantal structure is furthermore applied in finding the solutions to the discrete Painlevé equations [8, 9].

With respect to the continuous LV system, various types of time delay have been considered, and the preserveability and convergence of the solutions have been investigated [10–12]. However, to the best of our knowledge, the discrete analogues have been extensively clarified and time delays in discrete integrable systems have not yet been discussed. One of the authors proved [13–15] that, under the initial settings ̅ > ̅ > ̅ > ... > ̅ > ̅ > ̅ > ̅ > ... > ̅ and ̅ > ̅ > ... > ̅ , respectively, the terms ̅ and ̅ converge to positive constants ̅ and ̅ as ̅ → ∞, where ̅ > ̅ > ... > ̅ and ̅ > ̅ > ... > ̅ coincide with eigenvalues of tridiagonal matrices and squares of singular values of bidiagonal matrices. The key point in this asymptotic analysis is to relate the dLV system (1) to a sequence of the LR transformations for computing eigenvalues of tridiagonal matrices. In this paper, we introduce a time delay into the dLV system (1) in terms of the LR transformations. We also show asymptotic convergence of the delay dLV (ddLV) variables to matrix eigenvalues and singular values.

The remainder of this paper is organized as follows. In section 2, we briefly explain the LR transformations related to the dLV system (1) and their applications to computing matrix eigenvalues and singular values. In section 3, we derive a time-delay version of the dLV system (1) by considering that of the LR transformations related to the dLV system (1). We then clarify the matrix structure to which the delay LR transformation can be applied. In section 4, we show asymptotic convergence in the ddLV system, and present an example for numerical verification. Finally, we give concluding remarks in section 5.

2. LR transformations derived from the dLV system

In this section, we briefly review [13–15] an application of the dLV system (1) to computing eigenvalues and singular values of matrices.

Let us introduce new variables ̅ and ̅ given using the dLV variables ̅ as:

\[
\begin{align*}
q_k^{(n)} &:= (1 + u_{k+2k-2}^{(n)})(1 + u_{k-2}^{(n)}), \quad k = 1, 2, \ldots, m, \\
e_k^{(n)} &:= u_{k-2k+2}^{(n)}u_{k+2}^{(n)}, \quad k = 1, 2, \ldots, m - 1.
\end{align*}
\]

Since the dLV system (1) immediately leads to:

\[
\begin{align*}
(1 + u_{2k-2}^{(n+1)})(1 + u_{2k-1}^{(n+1)} + u_{2k+1}^{(n+1)})u_{2k-2}^{(n+1)} = (1 + u_{2k-2}^{(n+1)})(1 + u_{2k-1}^{(n+1)} + u_{2k+1}^{(n+1)}), \\
& \quad k = 1, 2, \ldots, m, \\
(1 + u_{2k-2}^{(n+1)})(1 + u_{2k-1}^{(n+1)})u_{2k-2}^{(n+1)}u_{2k}^{(n+1)} = (1 + u_{2k}^{(n+1)})(1 + u_{2k+1}^{(n+1)}u_{2k-1}^{(n+1)}), \\
& \quad k = 1, 2, \ldots, m - 1,
\end{align*}
\]

we can easily derive a recursion formula with respect to discrete-time evolutions from ̅ to ̅ and from ̅ to ̅:

\[
\begin{align*}
q_k^{(n+1)} + e_k^{(n+1)} &= q_k^{(n)} + e_k^{(n)}, \quad k = 1, 2, \ldots, m, \\
e_k^{(n+1)} + e_k^{(n+1)} &= q_k^{(n)} + e_k^{(n)}, \quad k = 1, 2, \ldots, m - 1, \\
e_0^{(n)} := 0, \quad e_m^{(n)} := 0, \quad n = 0, 1, \ldots,
\end{align*}
\]

equation (5) is simply the discrete Toda equation [16], which coincides with the recursion formula of the quotient-difference (qd) algorithm [17] for computing tridiagonal eigenvalues. Equation (4) is called the Miura transformation and can be regarded as the transformation from the dLV system (1) to the discrete Toda equation (5).

Now, let us prepare lower and upper bidiagonal matrices with the discrete Toda variables ̅ and ̅:

\[
\mathcal{L}^{(n)} := \begin{pmatrix}
0 & & & \\
1 & q_1^{(n)} & & \\
& \ddots & \ddots & \\
& & 1 & q_m^{(n)}
\end{pmatrix}, \quad \mathcal{R}^{(n)} := \begin{pmatrix}
1 & & & \\
& e_1^{(n)} & & \\
& & \ddots & \\
& & & e_{m-1}^{(n)}
\end{pmatrix}.
\]

Then, \(\mathcal{L}^{(n+1)}\mathcal{R}^{(n+1)}\) and \(\mathcal{R}^{(n)}\mathcal{L}^{(n)}\) are both tridiagonal matrices whose \((k + 1, k)\) entries are all 1. The \((k, k)\) entries of \(\mathcal{L}^{(n+1)}\mathcal{R}^{(n+1)}\) and \(\mathcal{R}^{(n)}\mathcal{L}^{(n)}\) are ̅ and ̅, respectively. The \((k + 1, k)\) entries of
\( \mathcal{L}^{(n+1)} \mathcal{R}^{(n+1)} \) and \( \mathcal{R}^{(n)} \mathcal{L}^{(n)} \) are \( \delta_{k+1}^{(n+1)} \) and \( \delta_{k}^{(n)} \), respectively. Thus, we can write:

\[
\mathcal{L}^{(n+1)} \mathcal{R}^{(n+1)} = \mathcal{R}^{(n)} \mathcal{L}^{(n)}. \tag{6}
\]

Noting that the inverse \( (\mathcal{R}^{(n)})^{-1} \) exists and letting \( \mathcal{Y}^{(n)} := \mathcal{L}^{(n)} \mathcal{R}^{(n)} - I \), where \( I \) is the identity matrix, we obtain:

\[
\mathcal{Y}^{(n+1)} = (\mathcal{R}^{(n)})^{-1} \mathcal{Y}^{(n)} (\mathcal{R}^{(n)})^{-1},
\]

which implies that the discrete Toda equation (5) generates a similarity transformation from \( \mathcal{Y}^{(n)} \) to \( \mathcal{Y}^{(n+1)} \) if \( \delta_{k}^{(n)} \) and \( \epsilon_{k}^{(n)} \) are given from the entries of \( \mathcal{Y}^{(n)} \). The dLV variables \( u_{0}^{(n)} \) are also uniquely determined from the entries of \( \mathcal{Y}^{(n)} \) because the latter can be expressed in terms of the dLV variables \( u_{k}^{(n)} \) as:

\[
\mathcal{Y}^{(n)} = \begin{pmatrix}
    w_{1}^{(n)} & w_{1}^{(n)} & w_{2}^{(n)} \\
    1 & w_{2}^{(n)} + w_{3}^{(n)} & w_{4}^{(n)} \\
    \vdots & \vdots & \vdots \\
    1 & w_{2m-4} + w_{2m-3} & w_{2m-2} + w_{2m-1} \\
    \end{pmatrix},
\]

\[
w_{k}^{(n)} := u_{k}^{(n)}(1 + u_{k-1}^{(n)}).
\]

The discrete-time evolutions in the dLV system (1) with suitable settings of \( u_{0}^{(0)} \), \( u_{1}^{(0)} \), \ldots, \( u_{2m-1}^{(0)} \) implicitly give a sequence of the LR transformations (6). The resulting \( \mathcal{Y}^{(1)}, \mathcal{Y}^{(2)}, \ldots \) are similar to \( \mathcal{Y}^{(0)} \).

Moreover, by limiting the cases where \( \mathcal{Y}^{(n)} \) are positive-definite, we can symmetrize \( \mathcal{Y}^{(n)} \) using the diagonal matrix \( \mathcal{D}^{(n)} := \text{diag}(\prod_{k=1}^{m-1} \sqrt{w_{2k-1}^{(n)}}, \prod_{k=2}^{m-1} \sqrt{w_{2k-1}^{(n)}}, \ldots, \sqrt{w_{2m-3}^{(n)}w_{2m-2}^{(n)}}, 1) \) as:

\[
\mathcal{Y}_{s}^{(n)} := (\mathcal{D}^{(n)})^{-1} \mathcal{Y}^{(n)} \mathcal{D}^{(n)} = \begin{pmatrix}
    \sqrt{w_{1}^{(n)}} & \sqrt{w_{2}^{(n)}} \\
    \sqrt{w_{3}^{(n)}} & \sqrt{w_{4}^{(n)}} \\
    \vdots & \vdots \\
    \sqrt{w_{2m-4}^{(n)}} & \sqrt{w_{2m-3}^{(n)}} \\
    \sqrt{w_{2m-2}^{(n)}} & \sqrt{w_{2m-1}^{(n)}} \\
\end{pmatrix},
\]

Since \( u_{2k-1}^{(n)} \) and \( u_{2k}^{(n)} \) converge to some positive constants \( c_{k} \) and 0 as \( n \to \infty \), respectively, we can observe that \( \mathcal{Y}_{s}^{(n)} = \text{diag}(c_{1}, c_{2}, \ldots, c_{m}) \) as \( n \to \infty \). We thus find that \( u_{1}^{(n)}, u_{3}^{(n)}, \ldots, u_{2m-1}^{(n)} \) converge to eigenvalues of \( \mathcal{Y}_{s}^{(n)} \) as \( n \to \infty \). From the Cholesky decomposition:

\[
\mathcal{Y}_{s}^{(n)} = (\mathcal{B}^{(n)})^{T} \mathcal{B}^{(n)},
\]

\[
\mathcal{B}^{(n)} := \begin{pmatrix}
    \sqrt{w_{1}^{(n)}} & \sqrt{w_{2}^{(n)}} \\
    \sqrt{w_{3}^{(n)}} & \sqrt{w_{4}^{(n)}} \\
    \vdots & \vdots \\
    0 & \sqrt{w_{2m-1}^{(n)}} \\
\end{pmatrix},
\]

we further find that \( \sqrt{u_{1}^{(n)}}, \sqrt{u_{3}^{(n)}}, \ldots, \sqrt{u_{2m-1}^{(n)}} \) converge to singular values of \( \mathcal{B}^{(n)} \) as \( n \to \infty \).

### 3. Discrete-time delay in the dLV system

In this section, we introduce a time delay into the dLV system (1) by considering the related LR transformations.

Let us introduce \( (m_{1} + m_{2} + \ldots + m_{r+1}) \)-by-\( (m_{1} + m_{2} + \ldots + m_{r+1}) \) block diagonal matrices:

\[
L^{(n)} := \begin{pmatrix}
    L_{1}^{(n)} & & \\
    & \ddots & \\
    & & L_{r}^{(n)} \\
\end{pmatrix}, \quad
R^{(n)} := \begin{pmatrix}
    R_{1}^{(n)} & & \\
    & \ddots & \\
    & & R_{r}^{(n)} \\
\end{pmatrix},
\]
where $L_j^{(n)}$ and $R_j^{(n)}$ are respectively the $m_j$-by-$m_j$ lower and upper bidiagonal matrices given as:

$$L_j^{(n)} := \begin{pmatrix} d_{i,j}^{(n)} & q_{i+1,j}^{(n)} & \ldots & q_{m,j}^{(n)} \\ q_{i,j}^{(n)} & d_{i+1,j}^{(n)} & \ldots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ q_{m-1,j}^{(n)} & \ldots & q_{m-1,j}^{(n)} & d_{m,j}^{(n)} \end{pmatrix}, \quad R_j^{(n)} := \begin{pmatrix} e_{i,j}^{(n)} & 1 & \ldots & \vdots \\ 1 & e_{i+1,j}^{(n)} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ \vdots & \ldots & 1 & e_{m-1,j}^{(n)} \end{pmatrix}. \tag{7}$$

Now, let us consider the matrix equality with nonnegative integer $\tau$:

$$L_j^{(n+1)}R_j^{(n+1)} = R_j^{(n-\tau)}L_j^{(n-\tau)}, \tag{8}$$

which is equivalent to (6) with $m = m_j$ if $\tau = 0$. Then, $L_j^{(n+1)}R_j^{(n+1)}$ and $R_j^{(n-\tau)}L_j^{(n-\tau)}$ are clearly both block diagonal matrices. Since the $j$th diagonal blocks of $L_j^{(n+1)}R_j^{(n+1)}$ and $R_j^{(n-\tau)}L_j^{(n-\tau)}$ are respectively $L_j^{(n+1)}R_j^{(n+1)}$ and $R_j^{(n-\tau)}L_j^{(n-\tau)}$, it holds that:

$$L_j^{(n+1)}R_j^{(n+1)} = R_j^{(n-\tau)}L_j^{(n-\tau)}. \tag{9}$$

Both $L_j^{(n+1)}R_j^{(n+1)}$ and $R_j^{(n-\tau)}L_j^{(n-\tau)}$ are tridiagonal matrices with superdiagonal entries 1. Observing the diagonal and subdiagonal entries of $L_j^{(n+1)}R_j^{(n+1)}$ and $R_j^{(n-\tau)}L_j^{(n-\tau)}$, we obtain:

$$\begin{align*}
q_{i,j}^{(n+1)} + e_{i-1,j}^{(n+1)} &= d_{i,j}^{(n-\tau)} + e_{i,j}^{(n-\tau)}, \quad i = 1, 2, \ldots, m_j, \quad j = 1, 2, \ldots, \tau + 1, \\
e_{i,j}^{(n+1)} &= q_{i+1,j}^{(n-\tau)} + e_{i,j}^{(n-\tau)}, \quad i = 1, 2, \ldots, m_j - 1, \quad j = 1, 2, \ldots, \tau + 1, \\
e_{m,j}^{(n+1)} &= 0, \quad e_{m,j}^{(n+1)} = 0, \quad n = 0, 1, \ldots. \tag{10}
\end{align*}$$

We note that (9) with $\tau = 0$ is equivalent to the discrete Toda equation (5). Equation (9) with nonzero $\tau$ differs from the discrete Toda equation (5) in that it requires the $q$ and $e$ values at discrete-time $n - \tau$ rather than discrete-time $n$. Thus, we can regard (9) as a time-delay extension of the discrete Toda equation (5). Since it is obvious from (7) that $(R_j^{(n-\tau)})^{-1}$ exists, we can rewrite (7) as $Y^{(n+1)} = R_j^{(n-\tau)}Y^{(n-\tau)}(R_j^{(n-\tau)})^{-1}$, where $Y^{(n)} := I^{(n)}R_j^{(n)} = I$. This implies that discrete-time evolutions in the delay discrete Toda equation (9) generate similarity transformations from $Y^{(n+1)}$ to $Y^{(n)}$.

Let us prepare auxiliary variables $u_{i,j}^{(n)}$ such as:

$$\begin{align*}
u_{i,j}^{(n)} &= (1 + u_{i,j}^{(n)})(1 + u_{i+1,j}^{(n)}), \quad i = 1, 2, \ldots, m_j, \quad j = 1, 2, \ldots, \tau + 1, \\
e_{i,j}^{(n)} &= u_{i+1,j}^{(n)} - u_{i,j}^{(n)}, \quad i = 1, 2, \ldots, m_j - 1, \quad j = 1, 2, \ldots, \tau + 1. \tag{11}
\end{align*}$$

Combining this with (9), we obtain:

$$\begin{align*}
u_{i,j}^{(n+1)}(1 + u_{i+1,j}^{(n+1)}) + u_{i,j}^{(n+1)}(1 + u_{i+1,j}^{(n+1)}) &= \nu_{i,j}^{(n-\tau)}(1 + u_{i,j}^{(n-\tau)}) + u_{i,j}^{(n-\tau)}(1 + u_{i,j}^{(n-\tau)}), \\
u_{i+1,j}^{(n+1)}(1 + u_{i+1,j}^{(n+1)}) &= \nu_{i,j}^{(n-\tau)}(1 + u_{i,j}^{(n-\tau)}) + u_{i,j}^{(n-\tau)}(1 + u_{i,j}^{(n-\tau)}),
\end{align*}$$

where $i = 1, 2, \ldots, m_j, \quad j = 1, 2, \ldots, \tau + 1, \quad n = 0, 1, \ldots.$

It is important to note here that (10) holds if:

$$u_{i,j}^{(n+1)}(1 + u_{i,j}^{(n+1)}) = u_{i,j}^{(n-\tau)}(1 + u_{i,j}^{(n-\tau)}), \quad i = 1, 2, \ldots, m_j - 1, \quad j = 1, 2, \ldots, \tau + 1. \tag{12}$$

Replacing $u_{i,j}^{(n+1)}$, $u_{i+1,j}^{(n+1)}$, ..., $u_{i+\tau,j}^{(n+1)}$, with $u_{i,j}^{(n)}$, $u_{i+1,j}^{(n-\tau)}$, ..., $u_{i+\tau,j}^{(n-\tau)}$, respectively, we can simplify (11) as:

$$\begin{align*}
u_{i,j}^{(n+1)} &= (1 + u_{i+1,j}^{(n+1)})(1 + u_{i,j}^{(n+1)}), \quad i = 1, 2, \ldots, m_j - 1, \quad j = 1, 2, \ldots, \tau + 1, \\
u_{0,j}^{(n+1)} &= 0, \quad u_{m,j}^{(n-\tau)} = 0, \quad n = 0, 1, \ldots. \tag{13}
\end{align*}$$

In (13), $u_{i,j}^{(n+1)}$ are uniquely given from $u_{i,j}^{(n-\tau)}$ rather than $u_{i,j}^{(n)}$. That is, (13) incorporates a time delay into the dLV system (1). We hereinafter refer to (13) as the delay dLV (ddLV) system.
Let us prepare \((m_1 + m_2 + \cdots + m_{r+1})\)-by-(\((m_1 + m_2 + \cdots + m_{r+1})\)) diagonal matrices:

\[
D^{(n)} := \begin{pmatrix}
D_1^{(n)} & 0 & \cdots & 0 \\
0 & D_2^{(n)} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & D_{r+1}^{(n)}
\end{pmatrix},
\]

where \(D_j^{(n)} := \text{diag}(\Pi_{\ell=1}^{n-1} \sqrt{w_{2\ell-2j}^{(n)}w_{2\ell-2j+2}^{(n)}}, \Pi_{\ell=2}^{n-1} \sqrt{w_{2\ell-2j}^{(n)}w_{2\ell-2j+2}^{(n)}}, \cdots, \sqrt{w_{2n-3j}^{(n)}w_{2n-3j+2}^{(n)}, 1})\) and \(w_{kj}^{(n)} := u_k^{(n+j-r-1)}(1 + u_k^{(n+j-r-1)}).\) Then, using the diagonal matrices \(D_j^{(n)}\), we can symmetrize block diagonal matrices \(Y_j^{(n)}\) as:

\[
Y_j^{(n)} := (D_j^{(n)})^{-1}Y_j^{(n)}D_j^{(n)} = \begin{pmatrix}
Y_{s,1}^{(n)} \\
Y_{s,2}^{(n)} \\
\vdots \\
Y_{s,r+1}^{(n)}
\end{pmatrix},
\]

\[
Y_{s,j}^{(n)} := \begin{pmatrix}
\sqrt{w_{1,j}^{(n)}w_{2,j}^{(n)}} & \sqrt{w_{2,j}^{(n)}w_{3,j}^{(n)}} & \cdots & \sqrt{w_{2m-3j}^{(n)}w_{2m-2j}^{(n)}} \\
\sqrt{w_{2m-5j}^{(n)}w_{2m-4j}^{(n)}} & \sqrt{w_{2m-3j}^{(n)}w_{2m-2j}^{(n)}} & \cdots & \sqrt{w_{2m-2j}^{(n)}w_{2m-1,j}^{(n)}}
\end{pmatrix}.
\]

Recall that \(Y^{(n+1)} = R^{(n-r)}Y^{(n-r)}(R^{(n-r)})^{-1}.\) Then, we can derive:

\[
Y_j^{(n+1)} = (G^{(n-r)})^{-1}Y_j^{(n)}G^{(n-r)},
\]

where \(G^{(n-r)} := (D_j^{(n)})^{-1}(R^{(n-r)})^{-1}D_j^{(n+1)}.\) This implies that the dLV system (12) also generates similarity transformations from \(Y_j^{(n-r)}\) to \(Y_j^{(n+1)}.\) The case where \(\tau = 0,\) of course, equates to that the dLV system (1), which gives similarity transformations of symmetric tridiagonal matrices. Since the blocks \(Y_{s,j}^{(n)}\) can be decomposed as:

\[
Y_{s,j}^{(n)} = (B_j^{(n)})^{-1}B_j^{(n)},
\]

\[
B_j^{(n)} := \begin{pmatrix}
\sqrt{w_{1,j}^{(n)}} & \sqrt{w_{2,j}^{(n)}} & \cdots & \sqrt{w_{2m-2j}^{(n)}} \\
\sqrt{w_{2m-3j}^{(n)}} & \sqrt{w_{2m-2j}^{(n)}} & \cdots & \sqrt{w_{2m-1,j}^{(n)}}
\end{pmatrix},
\]

the singular values of \(B_j^{(n)} = \text{diag}(B_1^{(n)}, B_2^{(n)}, \ldots, B_{r+1}^{(n)})\) are the same as those of \(B^{(0)}.\)

### 4. Asymptotic convergence

In this section, we investigate asymptotic convergence as \(n \to \infty\) in the dLV system (12), and then relate it to eigenvalues and singular values of matrices.

First, we split the dLV system (12) as:

\[
\begin{cases}
\begin{aligned}
u_i^{(n+1)}(1 + \nu_i^{(n+1)}) &= \nu_i^{(n-r)}(1 + \nu_i^{(n-r)}), \\
i &= 1, 2, \ldots, 2m_i - 1, \quad n = 0, \tau + 1, 2\tau + 2, \ldots
\end{aligned}
\end{cases}
\]

\[
\begin{cases}
\begin{aligned}
u_1^{(n+1)}(1 + \nu_1^{(n+1)}) &= \nu_i^{(n-r)}(1 + \nu_i^{(n-r)}), \\
i &= 1, 2, \ldots, 2m_1 - 1, \quad n = 1, 2\tau + 2, \ldots
\end{aligned}
\end{cases}
\]

\[
\vdots
\]

\[
\begin{cases}
\begin{aligned}
u_i^{(n+1)}(1 + \nu_i^{(n+1)}) &= \nu_i^{(n-r)}(1 + \nu_i^{(n-r)}), \\
i &= 1, 2, \ldots, 2m_r - 1, \quad n = \tau, 2\tau + 3, \ldots
\end{aligned}
\end{cases}
\]

\[
\begin{cases}
\begin{aligned}
u_1^{(n+1)}(1 + \nu_1^{(n+1)}) &= \nu_i^{(n-r)}(1 + \nu_i^{(n-r)}), \\
i &= 1, 2, \ldots, 2m_r - 1, \quad n = \tau, 2\tau + 3, \ldots
\end{aligned}
\end{cases}
\]

\[
\begin{cases}
\begin{aligned}
u_i^{(n+1)}(1 + \nu_i^{(n+1)}) &= \nu_i^{(n-r)}(1 + \nu_i^{(n-r)}), \\
i &= 1, 2, \ldots, 2m_{r+1} - 1, \quad n = \tau, 2\tau + 3, \ldots
\end{aligned}
\end{cases}
\]

\[
\begin{cases}
\begin{aligned}
u_i^{(n+1)}(1 + \nu_i^{(n+1)}) &= \nu_i^{(n-r)}(1 + \nu_i^{(n-r)}), \\
i &= 1, 2, \ldots, 2m_{r+1} - 1, \quad n = \tau, 2\tau + 3, \ldots
\end{aligned}
\end{cases}
\]

\[
u_0^{(n+1)} := 0, \quad \nu_{2m_j}^{(n-r)} := 0, \quad j = 1, 2, \ldots, r, \quad n = 0, 1, \ldots
\]
Preparing an auxiliary discrete-time $N$ such that $n = N(\tau + 1) - \tau$, we can rewrite the first equation of (13) as:

\[
\begin{align*}
\begin{cases}
    u^{(N+1)}_n (1 + u^{(N+1)}_{i+1}) = u^{(N)}_n (1 + u^{(N)}_{i+1}), & i = 1, 2, \ldots, m_1 - 1, \\
    u^{(N+1)}_0 = 0, & N = 0, 1, \ldots
\end{cases}
\end{align*}
\]  

(14)

which is equivalent to the dLV system (1). The other equations of (13) are similarly equivalent. Thus, from the asymptotic convergence in the dLV system (1), it is obvious that, for $j = 0, 1, \ldots, \tau$, the sequences 

\[
\begin{align*}
    \{u^{(n)}_1\}_{n=0}^\infty = \tau_j, j+1, j+1, \ldots, \{u^{(n)}_2\}_{n=0}^\infty = \tau_j, j+1, j+1, \ldots, \{u^{(n)}_m\}_{n=0}^\infty = \tau_j, j+1, j+1, \ldots, \{u^{(n)}_{m+1}\}_{n=0}^\infty = \tau_j, j+1, j+1, \ldots
\end{align*}
\]

respectively converge to some positive constants $c_1, c_2, \ldots, c_m$, under the initial settings $u^{(N)}_j > 0$, $u^{(N)}_{j+1} > 0$, $\ldots$, $u^{(N)}_{j+\tau} > 0$. It follows that the sequences $\{u^{(n)}_1\}_{n=0}^\infty = \tau_j, j+1, j+1, \ldots$, $\{u^{(n)}_2\}_{n=0}^\infty = \tau_j, j+1, j+1, \ldots$, $\{u^{(n)}_m\}_{n=0}^\infty = \tau_j, j+1, j+1, \ldots$, $\{u^{(n)}_{m+1}\}_{n=0}^\infty = \tau_j, j+1, j+1, \ldots$, all converge to all 0. Therefore, by combining this with section 3, we see that $c_1, c_2, \ldots, c_m$ are eigenvalues of $Y^{(0)}_t$ and squares of singular values of $B^{(0)}$. The determinant solution to the ddLV system (12) can be expressed as (3) with the sequences $\{a^{(n)}_k\}_{n=-\tau-1, \ldots}$ satisfying the recursion formula:

\[
\begin{align*}
a^{(n+1)}_k - a^{(n-\tau)}_k = a^{(n-1)}_{k+1},
\end{align*}
\]

instead of (2).

We now give a numerical example to demonstrate the asymptotic convergence as $n \to \infty$ in the ddLV system (12). As parameters in the dLV system (12), let $\tau = 2$ and $m_1 = m_2 = m_3 = 2$. For the initial settings, let $u_k^{(-2)} = 5$, $u_k^{(-1)} = 4$, $u_k^{(0)} = 11/2$, $u_k^{(-2)} = 3/10$, $u_k^{(-1)} = 1/20$, $u_k^{(0)} = 49/143$, $u_k^{(-2)} = 32/13$, $u_k^{(-1)} = 25/7$, $u_k^{(0)} = 36/11$. Since $w_{1,1}^{(0)} = 5$, $w_{2,1}^{(0)} = 9/5$, $w_{1,2}^{(0)} = 16/5$, $w_{2,2}^{(0)} = 4$, $w_{1,3}^{(0)} = 14/4$, $w_{2,3}^{(0)} = 15/4$, $w_{1,3}^{(0)} = 16/2$, $w_{2,3}^{(0)} = 36/2$, and $w_{3,3}^{(0)} = 36/11$, it follows that:

\[
Y^{(0)}_t = \begin{pmatrix} 5 & 3 & 0 & 0 & 0 & 0 \\ 3 & 5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 1 & 0 & 0 \\ 0 & 0 & 1 & 4 & 0 & 0 \\ 0 & 0 & 0 & 1 & 7 & 2 \\ 0 & 0 & 0 & 7 & 11 & 2 \end{pmatrix}.
\]

Figures 1 and 2 show the values of $u_{1}^{(n)}$, $u_{2}^{(n)}$, and $u_{3}^{(n)}$ at $n = -2, -1, \ldots, 60$. From figure 1, the sequences $\{u_{1}^{(n)}\}_{n=-2,1,4,\ldots}$, $\{u_{3}^{(n)}\}_{n=-2,1,4,\ldots}$, $\{u_{2}^{(n)}\}_{n=-2,1,4,\ldots}$, and $\{u_{3}^{(n)}\}_{n=-2,1,4,\ldots}$ respectively approach 8, 2, 5, 3, 9, and 2, which are eigenvalues the 1st, 2nd, and 3rd blocks of $Y^{(0)}_t$. From figure 2, the sequences $\{u_{1}^{(n)}\}_{n=-2,1,4,\ldots}$, $\{u_{2}^{(n)}\}_{n=-2,1,4,\ldots}$, and $\{u_{3}^{(n)}\}_{n=-2,1,4,\ldots}$ all tend to 0.

Figure 1. Discrete-time $n$ (x-axis) versus values of $u_{1}^{(n)}$, $u_{2}^{(n)}$, and $u_{3}^{(n)}$ (y-axis). Circles: $\{u_{1}^{(n)}\}_{n=-2,1,4,\ldots}$; filled circles: $\{u_{1}^{(n)}\}_{n=-2,1,4,\ldots}$; triangles: $\{u_{2}^{(n)}\}_{n=-2,1,4,\ldots}$; filled triangles: $\{u_{2}^{(n)}\}_{n=-2,1,4,\ldots}$; squares: $\{u_{3}^{(n)}\}_{n=-2,1,4,\ldots}$; and filled squares: $\{u_{3}^{(n)}\}_{n=-2,1,4,\ldots}$.
5. Concluding remarks

In this paper, we proposed a time-delay version of the integrable discrete Lotka-Volterra (dLV) system by introducing a time delay into the LR transformations derived from the dLV system. The resulting delay dLV (ddLV) system is also an integrable system. This is because the solution can be explicitly expressed using the Hankel determinants, similarly to the original dLV system. We clarified the asymptotic convergence as discrete-time goes to infinity in the ddLV system, and showed an application for computing eigenvalues of block diagonal matrices whose diagonal blocks are tridiagonal matrices. In future work, by focusing on the LR transformations of other structured matrices, we plan to design time-delay versions of more complicated integrable discrete systems, such as the discrete hungry Toda equation, discrete hungry Lotka-Volterra system, and the discrete relativistic Toda equation. These integrable studies are also expected to support the development of discrete time-delay frameworks in mathematical and theoretical biology.

Acknowledgments

The authors would like to thank the reviewers for their careful reading and beneficial comments. This research was partially supported by Grants-in-Aid for Scientific Research Number 18K03424 from the Japan Society for the Promotion of Science.

Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

ORCID iDs

Masaki Sekiguchi https://orcid.org/0000-0003-4632-3385

References

[1] Henrici P 1974 Applied and Computational Complex Analysis 1 (New York, NY: Wiley)
[2] Hirota R, Tsujimoto S and Imai T 1993 Future Directions of Nonlinear Dynamics in Physical and Biological Systems, NATO ASI Series 312 ed P L Christiansen, J C Eilbeck and R D Parmentier (Boston, MA: Springer) 7–15
[3] Sogo K 1993 Toda molecule equation and quotient-difference method J. Phys. Soc. Jpn. 52 1081–4
[4] Maruno K, Kajiwara K and Oikawa M 1998 Casorati determinant solution for the discrete-time relativistic Toda lattice equation Phys. Lett. A 241 335–43
[5] Minesaki Y and Nakamura Y 2001 The discrete relativistic Toda molecule equation and a Padé approximation algorithm Numer. Algori. 27 219–35
[6] Shinjo M, Nakamura Y, Iwasaki M and Kondo K 2018 of non-autonomous discrete hungry integrable systems of non-autonomous discrete hungry integrable systems J. Integr. Syst. 3 xxy001
[7] Ohta Y and Hirota R 1991 A discrete KdV equation and Its Casorati determinant solution J. Phys. Soc. Jpn. 60 2095
[8] Kajiwara K, Yamamoto K and Ohta Y 1997 Rational solutions for the discrete Painlevé II equation Phys. Lett. A 232 189–99
[9] Kajiwara K and Ohta Y 1998 Determinant structure of the rational solutions for the Painlevé IV equation J. Phys. A 31 2431
[10] Nakata Y and Muroya Y 2010 Permanence for nonautonomous Lotka-Volterra cooperative systems with delays Nonl. Anal. RWA 11 528–34
[11] Saito Y, Hara T and Ma W 1999 Necessary and sufficient conditions for permanence and global stability of a Lotka-Volterra system with two delays J. Math. Anal. Appl. 236 534–56
[12] Saito Y 2002 The necessary and sufficient condition for global stability of Lotka-Volterra cooperative or competition system with delays J. Math. Anal. Appl. 268 109–24
[13] Iwakashi M and Nakamura Y 2002 On the convergence of a solution of the discrete Lotka-Volterra system Inverse Prob. 18 1569–78
[14] Iwakashi M and Nakamura Y 2004 An application of the discrete Lotka-Volterra system with variable step-size to singular value computation Inverse Prob. 20 553–63
[15] Tsujimoto S, Nakamura Y and Iwasaki M 2002 The discrete Lotka-Volterra system computes singular values Inverse Prob. 17 53–8
[16] Hirota R 1981 Discrete analogue of a generalized Toda equation J. Phys. Soc. Jpn. 50 3785–91
[17] Rutishauser H 1990 Lectures on Numerical Mathematics (Boston: Birkhäuser)