An efficient polynomial-time approximation scheme for parallel multi-stage open shops

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Abstract

Various new scheduling problems have been arising from practical production processes and spawning new research areas in the scheduling field. We study the parallel multi-stage open shops problem, which generalizes the classic open shop scheduling and parallel machine scheduling problems. Given \(m\) identical \(k\)-stage open shops and a set of \(n\) jobs, we aim to process all jobs on these open shops with the minimum makespan, i.e., the completion time of the last job, under the constraint that job preemption is not allowed. We present an efficient polynomial-time approximation scheme (EPTAS) for the case when both \(m\) and \(k\) are constant. The main idea for our EPTAS is the combination of several categorization, scaling, and linear programming rounding techniques. Jobs and/or operations are first scaled and then categorized carefully into multiple types so that different types of jobs and/or operations are scheduled appropriately without increasing the makespan too much.

Keywords:
Scheduling; Parallel multi-stage open shops; Makespan; Linear programming; Efficient polynomial-time approximation scheme

1. Introduction

Scheduling plays a crucial role in manufacturing and service industries. Incorporating new features that reflect modern industrial processes to classic models has been receiving considerable attentions from researchers. Open shop scheduling \(^{12}\) is one of the main branches of scheduling problems. An open shop consists of multiple machines (or processors), each of which processes a different operation (or task) of every job, and every job has to go through all machines without ordering restrictions on which operation should be processed first. Considering the situation

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where multiple identical open shops are serving in production processes, we investigate \textit{m-parallel k-stage open shops} problem.

Formally, we are given \( m \) identical \( k \)-stage open shops \( \{S_1, S_2, \ldots, S_m\} \) and a set of \( n \) jobs \( J = \{J_1, J_2, \ldots, J_n\} \). Each \( k \)-stage open shop consists of \( k \) machines. Denote the open shop \( S_i \) by its \( k \) machines, i.e., \( S_i = \{M_{i,1}, M_{i,2}, \ldots, M_{i,k}\} \). Every job \( J_i \) contains \( k \) operations \( \{O_{i,1}, O_{i,2}, \ldots, O_{i,k}\} \) and the operation \( O_{i,j} \) needs to be processed by the machine \( M_{i,j} \) if \( J_i \) is assigned to the open shop \( S_i \). Without loss of generality, assume \( O_{i,j} \) requires to occupy a \( j \)-th stage machine \( p_{i,j} \) time non-preemptively, \( \forall \ i \in [n], j \in [k] \). Here \( [x] \) denotes the number set \( \{1, 2, \ldots, x\} \) for any positive integer \( x \). Then we represent each job \( J_i \) as a \( k \)-tuple \((p_{i,1}, p_{i,2}, \ldots, p_{i,k})\). Once a job is assigned to an open shop, it has to stay on this open shop until the job is completed. The goal is to minimize the \textit{makespan}, i.e., the completion time of the last job. Following the standard 3-field notation \( \alpha | \beta | \gamma \) introduced by Graham et al. \cite{Graham1969}, where \( \alpha \) describes the machine environment, \( \beta \) refers to job characteristics, and \( \gamma \) indicates objectives or optimality criteria, the \textit{m-parallel k-stage open shops} problem is represented as \( P_m(O_k) \parallel C_{\text{max}} \) when both \( m \) and \( k \) are constant.

For a minimization optimization problem \( \Pi \), a polynomial-time algorithm \( \mathcal{A} \) has an approximation ratio \( \alpha \) where \( \alpha = \min_{I \in \mathcal{I}} \{\mathcal{A}(I)/\text{OPT}(I)\} \), where \( \mathcal{I} \) is the set of all instances. If there is a family of algorithms \( \{\mathcal{A}_\epsilon, \epsilon > 0\} \) such that each algorithm \( \mathcal{A}_\epsilon \) has an approximation ratio of \( 1 + \epsilon \) for any \( \epsilon > 0 \) and its time complexity is \( O(n^{f(1/\epsilon)}) \) for the instance size \( n \) and some polynomial-time computable function \( f \), then the minimization problem \( \Pi \) admits a \textit{polynomial-time approximation scheme} (PTAS). We say a PTAS is an \textit{efficient polynomial-time approximation scheme} (EPTAS) if the running time of \( \mathcal{A}_\epsilon \) is in the form of \( f(1/\epsilon) \cdot n^{O(1)} \). A \textit{fully polynomial-time approximation scheme} (FPTAS) is a PTAS with its running time in form of \( (1/\epsilon)^{O(1)} \cdot n^{O(1)} \), i.e., a polynomial in \( 1/\epsilon \) and \( n \).

Previously, only the case when \( k = 2 \) was investigated for \( P_m(O_k) \parallel C_{\text{max}} \). Chen et al. \cite{Chen2009} introduced the parallel open shops problem \( P_m(O_2) \parallel C_{\text{max}} \) for the first time. Inspired by the observation found by Gonzalez and Sahni \cite{Gonzalez1976} that the optimal makespan of \( O_2 \parallel C_{\text{max}} \) is either determined by a single job or determined by the total running time on one machine of the open shop, Chen et al. \cite{Chen2009} proposed a 3/2-approximation algorithm for \( P_2(O_2) \parallel C_{\text{max}} \) and a 2-approximation algorithm for \( P_m(O_2) \parallel C_{\text{max}} \) for any \( m \) even when \( m \) is part of the input. Both these two approximation algorithms run in \( O(n \log n) \) time. Later, Dong et al. \cite{Dong2015} presented a pseudo-polynomial time dynamic programming algorithm and adopted a standard scaling technique to develop an FPTAS for \( P_m(O_2) \parallel C_{\text{max}} \) when the number of open shops is any fixed constant. Dong et al.’s algorithm was inspired by Gonzalez and Sahni \cite{Gonzalez1976}’s linear-time optimal algorithm for \( O_2 \parallel C_{\text{max}} \) the two-stage open flowshop scheduling with makespan minimization. Because \( O_2 \parallel C_{\text{max}} \) can be solved optimally, Dong et al. \cite{Dong2015} defined an important concept so-called \textit{critical-job} to help determining the makespan on one two-stage open shop quantitatively. When \( k \geq 3 \), the multi-stage open flowshop scheduling with makespan minimization (or \( O_k \parallel C_{\text{max}} \)) becomes (weakly) NP-hard \cite{Gonzalez1976}, which makes the problem \( O_k \parallel C_{\text{max}} \) loss good properties related to \textit{critical-job}. Thus, Dong et al.’s idea for FPTAS \cite{Dong2015} cannot be extended to the general
$P_m(O_k) \parallel C_{\text{max}}$ even when both $m$ and $k$ are constants.

Our main contribution is an EPTAS for $P_m(O_k) \parallel C_{\text{max}}$. We combine several categorization, scaling, and linear programming rounding techniques, some of which are inspired by the PTAS for the open shop problem $O_k \parallel C_{\text{max}}$ [29] and the PTAS for the flow shop problem $F_k \parallel C_{\text{max}}$ [18]. First, the processing time of each job is scaled in a way such that we only need to consider the schedules with makespan at most 1. Then the job set is categorized into a set of big jobs and a set of small jobs. This categorization makes sure the number of big jobs is a constant, which enables to enumerate all possible schedules of big jobs in constant time. Next, the operations of small jobs are categorized carefully into different types, which will be scheduled in different phases. Roughly, the operation of a small job is either simply scheduled to the end of the current schedule or “densely” fitted into the “gaps” or idle time intervals introduced by the schedule of big jobs. The above scaling and categorization steps aim to define an abstract representative for each feasible schedule. Instead of inspecting an exponential number of all feasible schedules, our EPTAS only checks a polynomial number of abstract representatives and converts each abstract representative into a near-optimal feasible schedule within the represented group of schedules. To obtain the smallest possible makespan without increasing the time complexity too much, each step of our EPTAS is well-designed and the overall analysis is involved. More details can be found in Section 4.

The remaining context is organized as follows. In Section 2, we review most related works. Section 3 introduces necessary terminologies, concepts, and preprocessing steps. The details of our EPTAS and its analysis are provided in Section 4. Finally, we make a conclusion in Section 5.

2. Related Work

It is easy to observe that $P_m(O_k) \parallel C_{\text{max}}$ reduces to the $k$-stage open shop scheduling problem (denoted as $O_k \parallel C_{\text{max}}$) [12] when $m = 1$ and to the $m$-parallel identical machine scheduling problem (denoted as $P_m \parallel C_{\text{max}}$) [11] when $k = 1$. Both have been explored extensively in the literature.

For $O_k \parallel C_{\text{max}}$, it admits a linear-time optimal algorithm when $k = 2$ but becomes weakly NP-hard when $k \geq 3$ [12]. It is still an open question whether $O_k \parallel C_{\text{max}}$ with $k \geq 3$ is strongly NP-hard [34]. Sevastianov and Woeginger [29] presented a PTAS for the case when $k \geq 3$ is a constant. This problem is denoted as $O \parallel C_{\text{max}}$ when $k$ is part of the input. Williamson et al. [33] showed its strong NP-hardness and an inapproximability of 1.25. For other results on the open shop problem, we refer the readers to the survey by Ellur and Ramasamy [3].

For $P_m \parallel C_{\text{max}}$, Graham [13] proposed the famous LIST-SCHEDULING algorithm with an approximation ratio $2 - 1/m$, for arbitrary $m$. When $m \geq 2$ is a constant, Garey and Johnson [11] proved $P_m \parallel C_{\text{max}}$ is weakly NP-hard and designed a pseudo-polynomial time exact algorithm. Sahni [26] presented an FPTAS. Hochbaum and Shmoys [20] studied the case when $m$ is part of the input, denoted as $P \parallel C_{\text{max}}$. They proved its strong NP-hardness and proposed a PTAS.
3. Preliminary

Recall that each operation \( O_{i,j}, i \in [n], j \in [k], \) has a processing time \( p_{i,j}. \) Define \( P_i = \sum_{j=1}^{k} p_{i,j} \) to be the total processing time of job \( J_i \) on an open shop and \( P = \sum_{i=1}^{n} P_i \) to be the total processing time of all jobs. Suppose a \( P_m(O_k) || C_{\text{max}} \) instance is denoted by its job set \( J. \) Given any instance \( J, \) let \( \pi(J) \) denote a (feasible) schedule and \( C_{\text{max}}^*(J) \) be its makespan. Similarly, we define \( \pi^*(J) \) and \( C_{\text{max}}^{\pi^*}(J) \) as the optimum schedule and its makespan respectively. \( J \) will be omitted if there is no confusion in the context.

Because every job needs to go through all machines of a \( k \)-stage open shop, we have \( C_{\text{max}}^{\pi^*} \geq \max_{1 \leq i \leq n} P_i. \) As the average workload is \( \frac{P}{mk}, \) \( C_{\text{max}}^{\pi^*} \geq \frac{P}{mk}. \) On the other hand, an instance of the

Please refer to the survey \[24\] for more results.

Chen and Strusevich \[6\] introduced the multiprocessor open shop problem, which generalizes \( O_k || C_{\text{max}} \) and \( P_m || C_{\text{max}} \) in another way. Instead of allowing multiple identical open shops in \( P_m(O_k) || C_{\text{max}} \), the multiprocessor open shop problem considers only one open shop and allows each stage to have multiple identical machines. The multiprocessor open shop problem is also studied under the names flexible open shop, multi-machine open shop, or open shop with parallel machines. Under the standard 3-field notation, the multiprocessor open shop problem can be denoted as \( O_k(P) || C_{\text{max}}. \) Chen and Strusevich \[6\] designed a \((2-2/\text{m}^2)\)-approximation algorithm for \( O_2(P) || C_{\text{max}}, \) where \( \text{m} \) is the upper bound of the number of machines in each stage. For the case when \( k \) is part of input, Chen and Strusevich \[6\] presented a \((2 + \epsilon)\)-approximation algorithm. These two approximation ratios were improved to \( 3/2 + \epsilon \) and \( 2 \) respectively by Woeginger \[27\]. When \( k \) is a constant and the number of machines in each stage is part of input, Jansen and Sviridenko \[22\] designed a PTAS. For more results on the multiprocessor open shop problem and its variants, we would like to refer readers to a recent survey by Adak et al. \[1\].

Changing the open shop environment to flow shop environment in \( P_m(O_k) || C_{\text{max}} \) and \( O_k(P) || C_{\text{max}} \) results in two more variants, i.e. the \( m\)-parallel \( k\)-stage flow shops problem and the multiprocessor flow shop problem denoted as \( P_m(F_k) || C_{\text{max}} \) and \( F_k(P) || C_{\text{max}} \) respectively under the 3-field notation. It has attracted quite a few attentions \[23, 19, 31, 2, 39, 10, 9, 30, 36, 35, 37, 38\]. For the case \( k = 2 \), FPTASes were designed independently by Kovalyov \[23\], Dong et al. \[10\], and Wu et al. \[36\] when \( m \) is a fixed constant; and Dong et al. \[9\] presented a PTAS when \( m \) is part of the input. For the case \( k \geq 3 \), Tong et al. \[30\] proposed a PTAS when both \( m \) and \( k \) are constant. All the mentioned FPTASes and PTASes are best possible approximability result as the \( m\)-parallel \( k\)-stage flow shops problem is weakly NP-hard when \( m \geq 2 \) and becomes strongly NP-hard if \( m \) is part of the input or \( k \geq 3 \) \[9\]. \( F_k(P) || C_{\text{max}} \) and its special cases also attracted considerable attentions from researchers, including but not limited to \[12, 17, 6, 21, 16, 18, 28, 32, 25\]. In particular, Hall \[18\] claimed a PTAS for \( F_k(P) || C_{\text{max}} \) when \( k \) is constant and the number of machines in each stage is also constant. Ruiz and Vázquez-Rodríguez \[25\] surveyed plenty of heuristics for the general \( F_k(P) || C_{\text{max}} \) problem.
The parallel machine scheduling problem can be constructed by adding a constraint that every open shop must complete a job before starting processing another job. Then the famous List-Scheduling algorithm \[13\] produces a schedule with makespan at most \[\frac{P}{m} + (1 - \frac{1}{m}) \max_{1 \leq i \leq n} P_i \leq \frac{P}{m} + \max_{1 \leq i \leq n} P_i.\] To wrap up, \(C^\pi_{\text{max}}\) for parallel open shops can be bounded in Lemma 1.

**Lemma 1.** For the \(P_m(O_k)||C_{\text{max}}\) problem, we have the following bounds for the minimum makespan \(C^\pi_{\text{max}}\):

\[
\max \left\{ \frac{P}{mk}, \max_{1 \leq i \leq n} P_i \right\} \leq C^\pi_{\text{max}} \leq \frac{P}{m} + \max_{1 \leq i \leq n} P_i
\]  

(1)

We scale the processing time of each operation by dividing \(2 \cdot \max \left\{ \frac{P}{mk}, \max_{1 \leq i \leq n} P_i \right\}\). Then the optimal makespan \(C^*_{\text{max}}\) can be normalized as

\[
\frac{1}{2k} \leq C^*_{\text{max}} \leq 1.
\]

(2)

In the sequel, we consider only those feasible schedules having a makespan less than or equal to 1.

Using a parameter \(\gamma\), which will be determined later in Algorithm Reduce, we categorize the job set \(J\) into two sets, named as big job set \(B^\gamma\) and small job set \(S^\gamma\), respectively. If \(\gamma\) is clear from the context, we may leave out the superscript for simplicity.

\[
B^\gamma = \{ J_i \in J \mid \exists j \in [k], p_{i,j} \geq \gamma \},
\]

\[
S^\gamma = J - B^\gamma.
\]

As there are \(mk\) machines in total and each machine has a load at most 1, the total processing time of all jobs is at most \(mk\). Each big job contains at least one operation with processing time greater than \(\gamma\). Thus, the number of big jobs is at most \(mk/\gamma\), which is summarized in Lemma 2.

**Lemma 2.** Given \(\gamma\), there are at most \(mk/\gamma\) big jobs.

### 4. Algorithm description and analysis

In this section, an EPTAS for the \(P_m(O_k)||C_{\text{max}}\) problem is presented. We name it as Reduce-Assign-Dense-Greedy, because it mainly consists of four procedures: a reduction procedure to massage the given instance into a more structured one; an assignment procedure to generate a “rough” and possibly infeasible schedule (named as assignment) for most jobs; a dense scheduling procedure to turn the assignment into a feasible schedule for those jobs; a greedy procedure to greedily process the unassigned jobs and/or operations. Roughly, the main purpose of the first two procedures is to categorize the feasible schedules into groups and extract an abstract representative for each group of schedules. The last two procedures recover a feasible schedule from the abstract representative such that the makespan is near-optimal with respect
to the represented group of schedules. Detailed description and analysis for each procedure is presented in Sections 4.1, 4.2, 4.3, and 4.4 respectively.

4.1. Description and Analysis for REDUCE

Recall that any operation of a small job can be processed within $\gamma$ time by definition. We further categorize the operations of small jobs into two types. More specifically, we call an operation $\gamma^2$-small if its processing time is less than $\gamma^2$, or $\gamma^2$-big otherwise.

Algorithm REDUCE identifies a specific value of $\gamma$ to categorize the job set. Then, it converts any $P_m(O_k)||C_{\max}$ instance $J$ into another instance $J'$ such that (1) every operation of small jobs in $J'$ has a processing time less than than $\gamma^2$; (2) the processing time of every operation of big jobs in $J'$ is a multiple of $\gamma^2$; (3) the optimal makespan is increased by a little amount.

Consider a sequence of candidates for $\gamma$

$$\gamma_x = \delta^{2^x}, \quad \delta = \frac{\epsilon}{14mk^3}, \quad x \in \mathbb{Z}_{\geq 0}. \quad (3)$$

We will search for an $x$ and assign $\gamma_x$ to $\gamma$. We choose the value of $\epsilon$ such that $1/\epsilon$ is an integer, which implies both $mk/\delta$ and $mk/\gamma_x$ are integers as well. Given a $\gamma_x$, we first categorize jobs into $B^{\gamma_x}$ and $S^{\gamma_x}$ and then define $\gamma^2_x$-big and $\gamma^2_x$-small operations for jobs in $S^{\gamma_x}$.

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**REDUCE**

**Input:** a general $P_m(O_k)||C_{\max}$ instance $J$, parameter $\epsilon$;

**Output:** parameter $\gamma$, $\gamma^2$-big operation set $O$, a special $P_m(O_k)||C_{\max}$ instance $J'$ with (1) $p_{i,j} \leq \gamma^2$ for $\forall j \in [k]$, $J_i \in S^{\gamma}$; (2) $p_{i,j}$ is a multiple of $\gamma^2$ for $\forall j \in [k]$, $J_i \in B^{\gamma}$.

```latex
1: Set $\delta = \frac{\epsilon}{14mk^3}$;
2: Define a sequence of real numbers $\gamma_x = \delta^{2^x}, x \in \mathbb{Z}_{\geq 0}$;
3: for $x \in \{0, 1, \ldots, \frac{mk}{\delta}\}$ do
4: \hspace{1em} Categorize $J$ into $B^{\gamma_x}$ and $S^{\gamma_x}$;
5: \hspace{1em} Let $O$ denote the set of all $\gamma^2$-big operations;
6: \hspace{1em} Let $L(\gamma_x)$ denote the total processing time of all operations in $O_x$;
7: \hspace{1em} if $L(\gamma_x) \leq \delta$ then
8: \hspace{2em} $\gamma \leftarrow \gamma_x$
9: \hspace{2em} for $O_{i,j} \in O$ do
10: \hspace{3em} $p_{i,j} \leftarrow 0$;
11: \hspace{2em} end for
12: \hspace{2em} for $J_i \in B^{\gamma}$ and $O_{i,j} \in J_i$ do
13: \hspace{3em} $p_{i,j} \leftarrow \lceil p_{i,j}/\gamma^2 \rceil \cdot \gamma^2$; \hspace{1em} \textcircled{Round $p_{i,j}$ up to the nearest multiple of $\gamma^2$}
14: \hspace{2em} end for
15: $J' \leftarrow B^{\gamma} \cup S^{\gamma}$;
16: return $\gamma$, $O$, and the new job set $J'$.
17: end if
18: end for
```

**Lemma 3.** Each operation in any job is $\gamma^2_x$-big for at most one index $x$.  

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Proof. Consider any job $J_i$ and follow the introduced rules for job categorization and operation categorization, if there is any operation of $J_i$ is $\gamma_x^2$-big for an index $x$, $J_i$ must be categorized as a small job by $\gamma_x$. That is, $\forall j \in [k], \ p_{i,j} < \gamma_x$ and $\exists j' \in [k], p_{i,j'} \geq \gamma_x^2$. We argue that any operation of $J_i$ cannot be $\gamma_y^2$-big for any $y \neq x$. Recall that $\gamma_x = \delta_2 x$ and $\delta = \frac{\epsilon}{14mk} < 1$.

For any index $y > x$, the job $J_i$ is categorized as a big job with respect to $\gamma_y$ as $\exists j' \in [k], p_{i,j'} \geq \gamma_x^2 = \gamma_x + 1 \geq \gamma_y$. Therefore, $J_i$ has no $\gamma_y^2$-big operations.

For any index $y < x$, the job $J_i$ is categorized as a small job with respect to $\gamma_y$ but every operation of $J_i$ has a processing time less than $\gamma_x = \gamma_x^2 - 1 \leq \gamma_y^2$, which implies $J_i$ has no $\gamma_y^2$-big operations.

This completes the proof.

Lemma 4. Algorithm Reduce terminates in $O(n/\epsilon)$ time and generates a special instance $J'$ with its optimal makespan bounded by

$$C^\pi_{\max}(J') \leq C^\pi_{\max}(J) + mk^2 \gamma.$$

Proof. We first analyze the time complexity of Reduce. Let $L(\gamma_x)$ denote the total processing time of all $\gamma_x^2$-big operations. Assume $L(\gamma_x) > \delta$ holds for all $x \in \{0, 1, \ldots, \frac{mk}{\gamma}\}$. By Lemma 4, the total processing time of all operations will be greater than $\sum_{x=0}^{\frac{mk}{\gamma}} \delta > mk$, which contradicts to the fact that the total processing time of all operations is at most $mk$. Therefore, there exists one $x < \frac{mk}{\delta}$ such that $L(\gamma_x) \leq \delta$. Considering $\delta = \frac{\epsilon}{14mk}$, it takes $O(1/\epsilon)$ iterations to find such a special index $x$. Since each iteration of the first for-loop takes $O(n)$ to compute $L(\cdot)$, the first for-loop takes $O(n/\epsilon)$ time. It is easy to observe that the second and third for-loops take $O(n)$ time in total. Therefore, the time complexity of Reduce is $O(n/\epsilon)$.

Now we show the upper bound of the optimal makespan of the constructed instance $J'$. Turning the processing time of each $\gamma^2$-big operation to zero does not increase the makespan. Consider an optimal schedule $\pi^*(J)$ and assume jobs are scheduled according to $\pi^*(J)$. Starting from time 0, we scan through the operations on each open shop. For every operation of big jobs, delaying the remaining schedule by less than $\gamma^2$ will generate enough space to process the rounded operation. By Lemma 2, there are $mk/\gamma$ big jobs and we delay the (partial) schedule on one open shop at most $mk^2/\gamma$ times, one delay for each operation of big jobs. This results a feasible schedule for $J'$ and increases the makespan by at most $mk^2 \gamma$.

This completes the proof.

4.2. Description and Analysis for Dense

In this section, we introduce several essential concepts, including restricted schedule and assignment of jobs. The main idea is to enumerate all possible restricted schedules for big jobs and then use the “gaps”, formed on each machine between two consecutively scheduled operations, to fit small jobs “densely”.

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4.2.1. Restricted Schedule and Assignment

Define a right half-open interval of length $\gamma^2$ as a $\gamma^2$-interval. The time interval $[0, 2)$ can be partitioned into $2/\gamma^2$ consecutive $\gamma^2$-intervals $[(t - 1) \cdot \gamma^2, t \cdot \gamma^2), t \in [2/\gamma^2]$. We say a feasible schedule is restricted if every operation of big operations starts processing at the beginning of some $\gamma^2$-interval.

**Lemma 5.** For any special $P_m(O_k)||C_{\max}$ instance $J'$, the minimum makespan of restricted schedules is bounded above by

$$C_{\max}^{\pi^*(J')} + mk^2\gamma.$$

**Proof.** The argument is similar to the proof for the upper bound in Lemma 4. Starting from time 0, we scan through the operations following the schedule $\pi^*(J')$ on each open shop. For every operation of big jobs, we delay the remaining schedule by less than $\gamma^2$ such that the operation starts processing at the beginning of some $\gamma^2$-interval. As there are at most $mk^2/\gamma$ times of delays, the makespan is increased by at most $mk^2\gamma$. $\square$

By Eq. (2), Lemma 4, Lemma 5, and the definition of $\gamma$, we consider special $P_m(O_k)||C_{\max}$ instances and restricted schedules with makespan at most 2 by default in the following context.

Given a feasible restricted schedule $\pi$, for each big job $J_i \in B$, its assignment is defined as $X_i = (\ell, \tau_1, \ldots, \tau_k)$, where $\ell$ is the index of the open shop to which the job $J_i$ assigned in $\pi$, and $\tau_j$ is the index of the $\gamma^2$-interval in which the $j$-th operation $O_{i,j}$ of $J_i$ starts processing.

Then the multi-set of assignments for jobs in $B$, denoted by $X_B = \{X_i | J_i \in B\}$, is called an assignment for $B$. We say $X_B$ is associated with a feasible schedule $\pi$ if $X_B$ can be defined from $\pi$.

**Lemma 6.** There are at most $m^{mk/\gamma} \cdot (2/\gamma^2)^{mk^2/\gamma}$ distinct assignments for $B$.

**Proof.** For each big job, there are at most $m(2/\gamma^2)^k$ distinct assignments. From Lemma 2, the number of big jobs is at most $mk/\gamma$. Therefore, the number of all possible assignments for $B$ is no greater than $m^{mk/\gamma} \cdot (2/\gamma^2)^{mk^2/\gamma}$. $\square$

We make a guess of $C_{\max}^{\pi^*(J')}$, denoted by $C$, which is used to define the assignment for small jobs. Given an assignment for $B$, we schedule the big jobs on open shops accordingly with each operation starting processing at the beginning of some $\gamma^2$-interval. Every machine, say $M_{\ell,j}$, may contain idle intervals, each formed between a pair of consecutively scheduled operations. There may be two more intervals, one between the time 0 and the time starting processing the first operation on $M_{\ell,j}$, the other one between completing the last operation and the guessed makespan $C$ on $M_{\ell,j}$. Define such an idle interval as a gap. By Lemma 2, there are at most $mk/\gamma + 1$ gaps. Let $G_{g,\ell,j} = [s_{g,\ell,j}, e_{g,\ell,j}]$ denote the $g$-th gap on the machine $M_{\ell,j}$, where $s_{g,\ell,j}$ and $e_{g,\ell,j}$ are the starting and ending time of this gap respectively.

Given a feasible restricted schedule $\pi$, an associated assignment for $B$, and an estimate $C$ of the makespan of $\pi$, we define the assignment for each small job $J_i \in S$ as $X_i = (\ell, \tau_1, \ldots, \tau_k)$ similar to the assignment for a big job. The difference is that $\tau_j$ records the index of the gap
instead of the $\gamma^2$-interval. Let $X_S = \{X_i \mid J_i \in S\}$ denote the assignment for small jobs. Obviously, there may be an exponential number of distinct assignments for small jobs, as $|S|$ may be linear in $n$. We say $X_S$ is associated with $X_B$ and $C$ if $X_S$ can be defined from $X_B$ and $C$.

$X_B$ is said to be feasible if it is associated with a feasible schedule. $X_S$ is said to be feasible if every gap introduced by the associated $X_B$ and $C$ provides enough space for the operations of jobs $S$ that are assigned to this gap, i.e.,

$$e_{g,\ell,j} - s_{g,\ell,j} \geq \sum_{J_i \in S, X_i = (\ell, r_1, \ldots, r_\ell = g, \ldots, k)} p_{i,j}.$$  

(4)

As $C_{\max}^{\pi^*(J')} \pi^*(J') \max C^\pi(J') \pi^*(J')$ is unknown, it is possible the guess $C$ is not correct. As long as $C \geq C_{\max}^{\pi^*(J')} \pi^*(J')$, there must be a feasible assignment for small jobs. When the guess $C$ is too small, say $C < C_{\max}^{\pi^*(J')} \pi^*(J')$, the defined $X_S$ may be not feasible. We will argue later in Algorithm Assign to search for an accurate enough guess such that $C \leq C_{\max}^{\pi^*(J')} \pi^*(J') + \epsilon$ and a feasible assignment exists for most small jobs.

4.2.2. Algorithm Dense

Suppose $X_B$ is associated with a feasible schedule $\pi$. Assume $X_S$ is also feasible and associated with $X_B$ and $C$, where $C$ is an estimate of the makespan of $\pi$. Let $O_{g,\ell,j}$ denote the set of operations assigned to start processing in the gap $G_{g,\ell,j}$, $g \in [mk/\gamma + 1], \ell \in [m], j \in [k]$. Requiring the operations in $O_{g,\ell,j}$ either scheduled in the gap $G_{g,\ell,j}$ or left unscheduled, we will generate a dense schedule for almost all operations in every gap. In a dense schedule, any machine becomes idle only because there is no operation that is ready to be processed on the machine. Note that the leftover operations from each set $O_{g,\ell,j}$ will be processed later greedily in a post-processing stage.

Dense schedules operations while scanning through the gaps. We mark a gap $G_{g,\ell,j}$ as “scanned” once Dense cannot find an operation in $O_{g,\ell,j}$ to fit in the gap $G_{g,\ell,j}$. Let $G_{\text{scanned}}$ record the scanned gaps and $O_{\text{remaining}}$ collect the leftover operations from the set $O_{g,\ell,j}$ after finishing scanning each gap $G_{g,\ell,j}$. Both $G_{\text{scanned}}$ and $O_{\text{remaining}}$ are initialized by empty sets.

Dense (Lines 5-20) repeats the following steps until all gaps have been scanned. Every time some machine, say $M_{\ell,j}$, becomes idle at the earliest while ignoring the scanned gaps. We update $T$ as this moment. Note that in the case when multiple machines becomes idle at the same time, we pick on machine arbitrarily and $T$ may stay the same for multiple iterations. Suppose $T \in [s_{g,\ell,j}, e_{g,\ell,j}]$ for some $g$ without loss of generality. We check whether there exists an unprocessed operation $O_{i,j} \in O_{g,\ell,j}$ such that no other operation of the corresponding job $J_i$ is currently being processed on some other machine of $S_\ell$. If at least one such operations exists, we pick one arbitrarily, start processing it at time $T$, and remove it from $O_{g,\ell,j}$. Otherwise, $G_{g,\ell,j}$ and $O_{g,\ell,j}$ are added to $G_{\text{scanned}}$ and $O_{\text{remaining}}$, respectively. If $O_{g,\ell,j}$ becomes empty, $G_{g,\ell,j}$ is also added to $G_{\text{scanned}}$.

After all gaps has been scanned, the remaining operations in $O_{\text{remaining}}$ will be processed
after the time $C$ sequentially such that for each open shop only one machine is actively processing operations at any moment. Refer to Algorithm DENSE for the detailed description. We will argue in Lemma 8 that for each gap most assigned operations will be scheduled in this gap.

**Algorithm DENSE**

**Input:** $C$, feasible $X_B$, feasible $X_S$;

**Output:** a feasible schedule $\pi'$ with makespan at most $C + mk^2\gamma + k\gamma^2$.

1: $G_{\text{scanned}} = \emptyset$;
2: $O_{\text{remaining}} = \emptyset$;
3: Let $\pi'$ denote the initial partial feasible schedule formed by $X_B$;
4: Let $T$ be the earliest idle time over all machines;
5: while $T < C$ do
6: Suppose $M_{\ell,j}$ is the earliest idle machine at $T$ without loss of generality;
7: Suppose $T \in [s_{g,\ell,j}, e_{g,\ell,j}]$;
8: if $\exists O_{i,j} \in O_{g,\ell,j}$ such that $p_{i,j} \leq e_{g,\ell,j} - T$ and no other operations of $J_i$ is currently being processed on $S_{\ell}$ then
9: Start processing $O_{i,j}$ at time $T$;
10: $O_{g,\ell,j} \leftarrow O_{g,\ell,j} - O_{i,j}$;
11: Update $\pi'$ accordingly;
12: else $\triangleright$ Leave the remaining operations to the post-processing stage
13: $G_{\text{scanned}} \leftarrow G_{\text{scanned}} + G_{g,\ell,j}$;
14: $O_{\text{remaining}} \leftarrow O_{\text{remaining}} + O_{g,\ell,j}$ $\triangleright$ $O_{g,\ell,j}$ may be empty
15: end if
16: Update $T$ as the earliest idle time over all machines while skipping all scanned gaps;
17: end while
18: $T = C$ $\triangleright$ Start the post-processing stage to schedule $O_{\text{remaining}}$
19: while $O_{\text{remaining}} \neq \emptyset$ do
20: Pick arbitrarily $O_{g,\ell,j} \in O_{\text{remaining}}$;
21: Start processing all operations $O_{g,\ell,j}$ in any order at time $T$ on $M_{\ell,j}$;
22: $T \leftarrow T + \sum_{O_{i,j} \in O_{g,\ell,j}} p_{i,j}$;
23: $O_{\text{remaining}} \leftarrow O_{\text{remaining}} - O_{g,\ell,j}$;
24: Update $\pi'$ accordingly;
25: end while
26: return a feasible schedule $\pi'$.

**Lemma 7.** For any gap, there are less than $k$ leftover operations after the first While-loop of DENSE.

**Proof.** A gap, say $G_{g,\ell,j}$, is added to $G_{\text{scanned}}$ when DENSE cannot find available operations to fit in this gap and all the remaining operations $O_{g,\ell,j}$, which may be empty, will be delayed to be scheduled at the post-processing stage. Therefore, there is at most one idle interval inside each gap after the first While-loop of DENSE.

For any gap $G_{g,\ell,j}$, if it has at least one operation remained after DENSE scans through the time range $[0, C]$, suppose the idle interval formed inside $[s_{g,\ell,j}, e_{g,\ell,j}]$ after the execution of
DENSE is \([T_{g,\ell,j}, e_{g,\ell,j}]\). Because both \(X_B\) and \(X_S\) are feasible. By Eq. (4), every gap has enough space to process all operations assigned to this gap. Every leftover operation from \(O_{g,\ell,j}\) cannot start processing at the time point \(T_{g,\ell,j}\) because another operation \(O_{i,j'}\) belonging to the same job \(J_i\) is being processed on another machine of the same open shop.

Assume there are at least \(k\) remaining operations in \(O_{g,\ell,j}\) after the first While-loop of DENSE. Each of these operations belongs to a different small job. At the time point \(T_{g,\ell,j}\), at most \(k - 1\) other machines of the open shop \(S_{\ell}\) are busy. This implies there is at least one of the leftover operations can be scheduled at \(T_{g,\ell,j}\), which is a contradiction.

This proves the lemma.

Lemma 8. Suppose \(X_B\) is associated with a feasible restricted schedule \(\pi\) and \(X_S\) is feasible and associated with \(X_B\) and \(C\). Algorithm DENSE takes \(O(1/\gamma^2 + n^2)\) time and returns another feasible schedule \(\pi'\) (which may be the same as \(\pi\)) such that

\[C_{\max}^{\pi'(B\cup S)} \leq C + mk^3\gamma + k^2\gamma^2.\]

Proof. By Lemma 7, for any gap, there are less than \(k\) leftover operations after the first While-loop of DENSE. DENSE simply processes these leftover operations in any order after time \(C\) on \(M_{\ell,j}\), which increases the makespan by at most \((k - 1)\gamma^2 < k\gamma^2\). Considering each open shops contains \(k\) machines and each machine has at most \(mk/\gamma + 1\) gaps, the makespan of \(\pi'\) is upper bounded by \((mk/\gamma + 1) \cdot k \cdot k\gamma^2 = mk^3\gamma + k^2\gamma^2\).

Now we analyze the time complexity of DENSE. The first While-loop scans through the gaps and tries to fit every gap densely while maintaining the feasibility of the current schedule. The first While-loop starts a new iteration after either one operation is assigned to some gap or a gap completes scanning. As there are at most \(n\) small jobs and at most \((mk/\gamma + 1)\cdot mk = m^2k^2/\gamma + mk\) gaps, the first While-loop has at most \(nk + m^2k^2/\gamma + mk\) iterations. Since each iteration needs \(O(1/\gamma + n)\) time, the first While-loop takes \(O(1/\gamma^2 + n^2)\) time. The second While-loop simply processes the unprocessed operations from previous gaps sequentially after the time \(C\) such that for each open shop at most one machine is working at any moment, which obviously takes linear time \(O(n)\) and maintains the feasibility of the current schedule. Therefore, the returned restricted schedule \(\pi'\) is feasible and the overall time complexity is \(O(1/\gamma^2 + n^2)\).

4.3. Description and Analysis for Assign

DENSE is based on the given feasible assignments for big jobs and small jobs. Nevertheless, there may be an exponential number of distinct assignments for small jobs. In this section, given only \(X_B\) associated with a feasible schedule \(\pi\), we first search for an estimate \(C\) of the makespan of \(\pi\) such that \(C \leq C_{\max}^{\pi(J')} + \epsilon/2\) and then obtain a feasible assignment for most small jobs by solving a linear program. The detailed procedure is described in Algorithm Assign.
**Assign**

**Input:** parameter $\epsilon$, $J'$, $X_B$, which is associated with a feasible schedule $\pi$;

**Output:**
- an estimate $C$ of the makespan of $\pi$ such that $C \leq C_{\max}^{\pi(J')} + \epsilon/2$;
- integrally assigned small jobs, denoted by $S^{\text{one}}$;
- fractionally assigned small jobs, denoted by $S^{\text{fractional}}$;
- feasible assignment for $S^{\text{one}}$ associated with $X_B$ and $C$, denoted by $X_{S^{\text{one}}}$.

1: Binary search for a $C$ value in the range $(0, 2)$ such that the constructed LP has a basic feasible solution and $C \leq C_{\pi(J')}^{\max} + \epsilon/2$;

2: Let $y$ be a basic feasible solution to the constructed LP with the chosen $C$ value;

3: $S^{\text{one}} = \{ J_i \in S \mid \exists y_i, X = 1 \}$;

4: $X_{S^{\text{one}}} = \{ X \mid \forall J_i \in S^{\text{one}}, \exists y_i, X = 1 \}$;

5: $S^{\text{fractional}} = S \setminus S^{\text{one}}$;

6: return $C, S^{\text{one}}, S^{\text{fractional}}, X_{S^{\text{one}}}$.

Define the binary variable $y_i, X$ to indicate whether the job $J_i \in S$ has an assignment $X$ in the gaps formed by $X_B$ and $C$. We construct a linear program, denoted by LP. The first set of constraints in LP makes sure every small job is assigned and the second set of constraints in LP guarantees there is enough space to process all operations assigned to each gap. In the constructed LP, the number of non-trivial constraints is only $|S| + m^2k^2/\gamma + km$ while the number of variables $(mk/\gamma + 1)^k m|S|$ is considerably larger. Then, any basic feasible solution to LP has at most $|S| + m^2k^2/\gamma + km$ positive values.

(LP)

$$\sum_X y_i, X = 1, \quad \forall J_i \in S;$$
$$\sum_{J_i \in S, X = (\ell, \tau_1, \ldots, \tau_j = g, \ldots, s_k)} p_{ij} y_i, X \leq e_{g, \ell, j} - s_{g, \ell, j}, \quad \forall (g, \ell, j) \in [mk/\gamma + 1] \times [m] \times [k];$$
$$y \geq 0.$$

Given a basic feasible solution to LP, let $S^{\text{one}}$ and $S^{\text{fractional}}$ be the set of small jobs that are assigned integrally and fractionally, respectively. From the definition, we have $|S^{\text{one}}| + |S^{\text{fractional}}| = |S|$. For each fractionally assigned small job, it corresponds to at least two positive variables. The number of variables with positive values is at least $|S^{\text{one}}| + 2|S^{\text{fractional}}|$. Therefore, $|S| + m^2k^2/\gamma + km \geq |S^{\text{one}}| + 2|S^{\text{fractional}}| = |S| + |S^{\text{fractional}}|$, which implies

$$|S^{\text{fractional}}| \leq m^2k^2/\gamma + km.$$

**Lemma 9.** Suppose $B$ is the big job set of $J'$ and $X_B$ is associated with a feasible schedule $\pi$. Algorithm Assign takes $n^{O(1)} \cdot (1/\gamma)^{O(1)}$ time to return
- an estimate $C$ of the makespan of $\pi$ such that $C \leq C_{\max}^{\pi(J')} + \epsilon/2$;
- integrally assigned small jobs $S^{\text{one}}$ and its feasible assignment $X_{S^{\text{one}}}$ associated with $X_B$ and $C$;
- fractionally assigned small jobs $S^{\text{fractional}}$ with $|B^{\text{fractional}}| \leq m^2k^2/\gamma + km$.

**Proof.** We only need to argue for the time complexity and how to estimate $C$ accurately, as the other parts of this lemma follows immediately from the previous discussions.

Given $X_B$ and a $C$ value, the constructed LP contains $|S| + m^2k^2/\gamma + km$ non-trivial constraints and $(mk/\gamma + 1)^m|S|$ variables. It takes polynomial time in $n$ and $1/\gamma$, denoted by $n^{O(1)} \cdot (1/\gamma)^{O(1)}$, to solve this LP.

Whenever LP has a basic feasible solution, the conclusions regarding $S^{\text{one}}$ and $S^{\text{fractional}}$ hold. That is, most small jobs can be assigned integrally. Due to the second set of constraints of LP, the assignment for $S^{\text{one}}$ is feasible. Whether LP has a feasible solution can be used to tell whether the input $C$ value is able to generate a feasible assignment for $S^{\text{one}}$. On the other hand, when $C \geq C^{\pi(\mathcal{J}^{'})}_{\max}$, a feasible solution to LP can be easily constructed from the feasible schedule $\pi$ and thus a feasible assignment for $S^{\text{one}}$ can be guaranteed. Therefore, binary searching a value for $C$ in the range $(0, 2)$ gives us a desired precision of $C$ after $\log 1/\epsilon$ guesses. The overall time complexity of Assign is $n^{O(1)} \cdot (1/\gamma)^{O(1)}$.

4.4. **Description and Analysis for Our EPTAS**

**REDUCE-ASSIGN-DENSE-GREEDY**

**Input:** a $P_m(O_k)||C_{\max}$ instance $\mathcal{J}$, parameter $\epsilon$;

**Output:** a feasible schedule $\pi$ with makespan at most $(1 + \epsilon) \cdot C^{\pi^{*}(\mathcal{J})}_{\max}$.

1. $\gamma$, $\gamma^2$-big operations $\mathcal{O}$, a special instance $\mathcal{J}' \leftarrow \text{REDUCE}(\mathcal{J}, \epsilon)$;
2. $\pi$ is initialized by None;
3. Set $C_{\max} = \infty$;
4. for any assignment $X_B$ for big jobs do
5. $C$, $S^{\text{one}}$, $S^{\text{fractional}}$, $X_{S^{\text{one}}} \leftarrow \text{ASSIGN}(\epsilon, X_B, \mathcal{J}^{'})$;
6. a feasible restricted schedule $\pi'$ for $B \cup S^{\text{one}} \leftarrow \text{DENSE}(X_B, X_{S^{\text{one}}}, C)$;
7. Divide $S^{\text{fractional}}$ evenly into $m$ partitions;
8. Schedule each partition at the end of one open shop “sequentially”;
9. if $C_{\max} > C^{\pi'}_{\max}$ then
10. $\pi \leftarrow \pi'$;
11. $C_{\max} \leftarrow C^{\pi}_{\max}$;
12. end if
13. end for
14. Schedule the $\gamma^2$-big operations $\mathcal{O}$ at the end of open shops “sequentially”;
15. Update $\pi$ accordingly
16. **return** the final schedule $\pi$

Now we are ready to provide a detailed description and analysis for our EPTAS. **REDUCE-DENSE-ASSIGN-GREEDY** takes in any $P_m(O_k)||C_{\max}$ instance $\mathcal{J}$ and any small number $\epsilon \in (0, 1)$. First, **REDUCE** is invoked to find an appropriate value of $\gamma$, categorize jobs $\mathcal{J}$ into a set $B$ of big jobs and a set $S$ of small jobs, obtain a set $\mathcal{O}$ of $\gamma^2$-big operations from small jobs, and construct a special instance $\mathcal{J}^{'}$. 

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Then, enumerating all possible assignments of big jobs and we repeat the following procedure for each assignment $X_B$ to find a feasible restricted schedule $\pi$ with the minimum makespan. For each assignment $X_B$, Assign guesses an estimate $C$ of the makespan of an associated feasible schedule and generates (integral) assignments for most small jobs via solving a linear program. $S^{one}$ and $S^{fractional}$ denote the integrally and fractionally assigned small jobs. As the (integral) assignments for jobs in $X_B \cup S^{one}$ are known, Dense is called to return a feasible restricted schedule for $X_B \cup S^{one}$. The fractionally assigned small jobs $S^{fractional}$ is then partitioned evenly into $m$ subsets, each of which is greedily scheduled to the end of one open shop in a sequential manner such that the open shop is dedicated to process only one job at any moment.

Finally, using the previously obtained feasible restricted schedule $\pi$ for the special instance $J'$, we construct a feasible schedule for the original instance $J$. We simply schedule each $\gamma^2$-big operation with its original processing time to the end of the open shop, to which its associated job is assigned, such that the open shop is dedicated to process only one operation at any moment.

**Theorem 1.** Algorithm Reduce-Assign-Dense-Greedy is an EPTAS.

**Proof.** We first analyze the time complexity for Reduce-Assign-Dense-Greedy. By Lemma 4, Reduce takes $O(n/\epsilon)$ time. Lemma 6 states there are at most $m^{\frac{mk}{\gamma}} \cdot \frac{2}{\gamma^2} \frac{mk^2}{\gamma}$ distinct assignments for $B$, which bounds the number of iterations of the For-loop. Each iteration calls both Assign and Dense, taking $n^{O(1)}(1/\gamma)^{O(1)}$ and $O(1/\gamma^2 + n^2)$ time respectively. Greedily scheduling the jobs in $S^{fractional}$ or the $\gamma^2$-big operations $O$ takes constant time. Therefore, the overall time complexity of Reduce-Assign-Dense-Greedy is $n^{O(1)}(1/\gamma)^{O(1/\gamma)}$, where the term $(1/\gamma)^{O(1/\gamma)}$ is a function of $\epsilon$ by the definition of $\gamma$.

When we greedily schedule the jobs in $S^{fractional}$ or the $\gamma^2$-big operations $O$ to the end of open shops, only one machine is active on every open shop and therefore the feasibility of the constructed schedule is maintained. Then the feasibility of the returned schedule follows from Lemma 8 and Lemma 9. Next, we estimate the quality of the returned schedule or the makespan.

By Lemma 4 and Lemma 5, the optimal restricted schedule for the constructed special instance $J'$ has a makespan bounded by

$$C_{\pi^*(J')} = C_{\pi^*(J)} + 2mk^2\gamma. \quad (5)$$

Enumerating all possible assignments for big jobs, including the one associated with the optimal restricted schedule $\pi^*(J')$, Reduce-Assign-Dense-Greedy chooses the feasible schedule with the minimum makespan. Without loss of generality, we assume $X_B$ is associated with $\pi^*(J')$ in the sequel. From Lemma 8 and Lemma 9, the makespan of $\pi$ when scheduling only $B \cup S^{one}$ is at most

$$C_{\pi(B \cup S^{one})} = C + mk^3\gamma + k^2\gamma^2 \leq C_{\max} + \epsilon/2 + mk^3\gamma + k^2\gamma^2. \quad (6)$$

By Lemma 9, $|B^{fractional}| \leq m^2k^2/\gamma + km$ and therefore at most $mk^2/\gamma + k$ small jobs are assigned to each open shop. We greedily schedule these small jobs at the end of the current
schedule \(\pi(B \cup S^{one})\) in a sequential manner such that at most one machine is active on every open shop at any moment. As the total processing time of a small job is at most \(k \gamma^2\), the makespan increases by at most \((mk^2/\gamma + k) \cdot k \gamma^2\). That is,

\[
C^\pi_{\text{max}}(J) \leq C^\pi_{\text{max}}(B \cup S^{one}) + (mk^2/\gamma + k) \cdot k \gamma^2.
\] (7)

To obtain a feasible schedule for the original instance \(J\), we schedule the \(\gamma^2\)-big operations \(O\) at the end of \(\pi(J')\) sequentially. As the total processing time of operations in \(O\) is at most \(\delta\), the makespan increases by at most \(\delta\). That is,

\[
C^\pi_{\text{max}}(J) \leq C^\pi_{\text{max}}(J') + \delta.
\] (8)

Combining Eq.(5), Eq.(6), Eq.(7), and Eq.(8), the makespan of the returned schedule can be estimated as follows.

\[
C^\pi_{\text{max}}(J) \leq C^\pi_{\text{max}}(J) + 2mk^2 \gamma + \epsilon/2 + mk^3 \gamma + k^2 \gamma^2 + (mk^2/\gamma + k) \cdot k \gamma^2 + \delta
\]

\[
\leq C^\pi_{\text{max}}(J) + \epsilon/2 + 7mk^3 \delta
\]

\[
\leq C^\pi_{\text{max}}(J) + \epsilon,
\]

where the last two inequalities are because of \(\gamma = \delta^2 \leq \delta = \frac{\epsilon}{14mk^3} < 1\).

This completes the proof.

\[\square\]

5. Conclusion

We investigate the parallel multi-stage open shops (denoted by \(P_m(O_k) || C_{\text{max}}\)) problem, a meaningful hybrid of the classic open shop scheduling and parallel machine scheduling, which takes in multiple identical open shops and schedules jobs to achieve the minimum makespan. Note that shifting a job between open shops and preemptive processing an operation are not allowed. As \(P_m(O_k) || C_{\text{max}}\) can be treated as the parallelized version of the classic open shop scheduling problem, it inherits the computational complexity directly from the latter problem and thus it is NP-hard in the weak sense when \(k \geq 3\) is a constant. When both \(m\) and \(k\) are constant, we design an efficient polynomial-time approximation scheme (EPTAS) for \(P_m(O_k) || C_{\text{max}}\).

There are two interesting open questions that worth further exploration in the future.

1. Because the classic open shop scheduling problem is strongly NP-hard when \(k \geq 3\) has been open since the 1970s \[34\]. As a generalized variant, whether \(P_m(O_k) || C_{\text{max}}\) is strongly NP-hard is also unknown. On the other hand, \(P_m || C_{\text{max}}\) is only weakly NP-hard when \(m\) is fixed \[11\]. It is worth investigating whether \(P_m(O_k) || C_{\text{max}}\) is NP-hard in a strong sense for constant \(m\) and \(k \geq 3\). If it is not strongly NP-hard, can we design an FPTAS?

2. When \(m\) is part of the input, \(P(O_k) || C_{\text{max}}\) generalizes the parallel machine scheduling
\( P \parallel C_{\text{max}} \) and thus inherits the strong NP-hardness. It would be interesting to investigate whether a PTAS exists for \( P(O_k) \parallel C_{\text{max}} \). Perhaps we can approach the problem starting from the special case when \( k = 2 \).

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References

[1] Z. Adak, M. Ö. Arıoğlu Akan, and S. Bulkan. Multiprocessor open shop problem: literature review and future directions. *Journal of Combinatorial Optimization*, 40:547–569, 2020.

[2] A. Al-Salem. A heuristic to minimize makespan in proportional parallel flow shops. *International Journal of Computing & Information Sciences*, 2(2):98, 2004.

[3] E. Anand and R. Panneerselvam. Literature review of open shop scheduling problems. *Intelligent Information Management*, 7(01):33, 2015.

[4] D. Cao and M. Chen. Parallel flowshop scheduling using tabu search. *International Journal of Production Research*, 41(13):3059–3073, 2003.

[5] B. Chen. Analysis of classes of heuristics for scheduling a two-stage flow shop with parallel machines at one stage. *Journal of the Operational Research Society*, 46(2):234–244, 1995.

[6] B. Chen and V. A. Strusevich. Worst-case analysis of heuristics for open shops with parallel machines. *European Journal of Operational Research*, 70(3):379–390, 1993.

[7] Y. Chen, A. Zhang, G. Chen, and J. Dong. Approximation algorithms for parallel open shop scheduling. *Information Processing Letters*, 113(7):220–224, 2013.

[8] J. Dong, R. Jin, J. Hu, and G. Lin. A fully polynomial time approximation scheme for scheduling on parallel identical two-stage openshops. *Journal of Combinatorial Optimization*, 37(2):668–684, 2019.

[9] J. Dong, R. Jin, T. Luo, and W. Tong. A polynomial-time approximation scheme for an arbitrary number of parallel two-stage flow-shops. *European Journal of Operational Research*, 281(1):16–24, 2020.

[10] J. Dong, W. Tong, T. Luo, X. Wang, J. Hu, Y. Xu, and G. Lin. An FPTAS for the parallel two-stage flowshop problem. *Theoretical Computer Science*, 657:64–72, 2017.
[11] M. R. Garey and D. S. Johnson. *Computers and intractability*, volume 174. Freeman San Francisco, 1979.

[12] T. Gonzalez and S. Sahni. Open shop scheduling to minimize finish time. *Journal of the ACM*, 23:665–679, 1976.

[13] R. L. Graham. Bounds for certain multiprocessing anomalies. *Bell system technical journal*, 45(9):1563–1581, 1966.

[14] R. L. Graham, E. L. Lawler, J. K. Lenstra, and A. R. Kan. Optimization and approximation in deterministic sequencing and scheduling: a survey. In *Annals of discrete mathematics*, volume 5, pages 287–326. 1979.

[15] J. N. Gupta. Two-stage, hybrid flowshop scheduling problem. *Journal of the operational Research Society*, 39(4):359–364, 1988.

[16] J. N. Gupta, A. Hariri, and C. N. Potts. Scheduling a two-stage hybrid flow shop with parallel machines at the first stage. *Annals of Operations Research*, 69:171–191, 1997.

[17] J. N. Gupta and E. A. Tunc. Schedules for a two-stage hybrid flowshop with parallel machines at the second stage. *The International Journal of Production Research*, 29(7):1489–1502, 1991.

[18] L. A. Hall. Approximability of flow shop scheduling. *Mathematical Programming*, 82(1):175–190, 1998.

[19] D. W. He, A. Kusiak, and A. Artiba. A scheduling problem in glass manufacturing. *IIE Transactions*, 28:129–139, 1996.

[20] D. S. Hochbaum and D. B. Shmoys. Using dual approximation algorithms for scheduling problems theoretical and practical results. *Journal of the ACM*, 34(1):144–162, 1987.

[21] J. A. Hoogeveen, J. K. Lenstra, and B. Veltman. Preemptive scheduling in a two-stage multiprocessor flow shop is NP-hard. *European Journal of Operational Research*, 89(1):172–175, 1996.

[22] K. Jansen and M. I. Sviridenko. Polynomial time approximation schemes for the multiprocessor open and flow shop scheduling problem. In *Annual Symposium on Theoretical Aspects of Computer Science*, pages 455–465, 2000.

[23] M. Y. Kovalyov. Efficient epsilon-approximation algorithm for minimizing the makespan in a parallel two-stage system. *Vesti Akademii navuk Belaruskai SSR. Seria phizikamatematichnikh navuk*, 1985.

[24] E. Mokotoff. Parallel machine scheduling problems: A survey. *Asia-Pacific Journal of Operational Research*, 18(2):193, 2001.
[25] R. Ruiz and J. A. Vázquez-Rodríguez. The hybrid flow shop scheduling problem. *European journal of operational research*, 205(1):1–18, 2010.

[26] S. K. Sahni. Algorithms for scheduling independent tasks. *Journal of the ACM*, 23(1):116–127, 1976.

[27] P. Schuurman and G. J. Woeginger. Approximation algorithms for the multiprocessor open shop scheduling problem. *Operations Research Letters*, 24(4):157–163, 1999.

[28] P. Schuurman and G. J. Woeginger. A polynomial time approximation scheme for the two-stage multiprocessor flow shop problem. *Theoretical Computer Science*, 237(1-2):105–122, 2000.

[29] S. V. Sevastianov and G. J. Woeginger. Makespan minimization in open shops: A polynomial time approximation scheme. *Mathematical Programming*, 82(1):191–198, 1998.

[30] W. Tong, E. Miyano, R. Goebel, and G. Lin. An approximation scheme for minimizing the makespan of the parallel identical multi-stage flow-shops. *Theoretical Computer Science*, 734:24–31, 2018.

[31] G. Vairaktarakis and M. Elhafsi. The use of flowlines to simplify routing complexity in two-stage flowshops. *Iie Transactions*, 32(8):687–699, 2000.

[32] H. Wang. Flexible flow shop scheduling: optimum, heuristics and artificial intelligence solutions. *Expert Systems*, 22(2):78–85, 2005.

[33] D. P. Williamson, L. A. Hall, J. A. Hoogeveen, C. A. Hurkens, J. K. Lenstra, S. V. Sevast’janov, and D. B. Shmoys. Short shop schedules. *Operations Research*, 45(2):288–294, 1997.

[34] G. J. Woeginger. The open shop scheduling problem. In *35th Symposium on Theoretical Aspects of Computer Science (STACS 2018)*, 2018.

[35] G. Wu, J. Chen, and J. Wang. On scheduling inclined jobs on multiple two-stage flowshops. *Theoretical Computer Science*, 786:67–77, 2019.

[36] G. Wu, J. Chen, and J. Wang. Scheduling two-stage jobs on multiple flowshops. *Theoretical Computer Science*, 776:117–124, 2019.

[37] G. Wu, J. Chen, and J. Wang. Improved approximation algorithms for two-stage flowshops scheduling problem. *Theoretical Computer Science*, 806:509–515, 2020.

[38] G. Wu, J. Chen, and J. Wang. On scheduling multiple two-stage flowshops. *Theoretical Computer Science*, 818:74–82, 2020.

[39] X. Zhang and S. van de Velde. Approximation algorithms for the parallel flow shop problem. *European Journal of Operational Research*, 216(3):544–552, 2012.