ON THE ABSENCE OF UNIFORM DENOMINATORS
IN HILBERT’S 17TH PROBLEM

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(Communicated by Michael Stillman)

Abstract. Hilbert showed that for most \((n, m)\) there exist positive semidefinite forms \(p(x_1, \ldots, x_n)\) of degree \(m\) which cannot be written as a sum of squares of forms. His 17th problem asked whether, in this case, there exists a form \(h\) so that \(h^2 p\) is a sum of squares of forms; that is, \(p\) is a sum of squares of rational functions with denominator \(h\). We show that, for every such \((n, m)\) there does not exist a single form \(h\) which serves in this way as a denominator for every positive semidefinite \(p(x_1, \ldots, x_n)\) of degree \(m\).

1. Introduction

Let \(H_d(\mathbb{R}^n)\) denote the set of real homogeneous forms of degree \(d\) in \(n\) variables (“\(n\)-ary \(d\)-ics”). By identifying \(p \in H_d(\mathbb{R}^n)\) with the \(N = \binom{n+d-1}{n-1}\)-tuple of its coefficients, we see that \(H_d(\mathbb{R}^n) \approx \mathbb{R}^N\). Suppose \(m\) is an even integer. A form \(p \in H_m(\mathbb{R}^n)\) is called positive semidefinite or \(psd\) if \(p(x_1, \ldots, x_n) \geq 0\) for all \((x_1, \ldots, x_n) \in \mathbb{R}^n\). Following [1], we denote the set of psd forms in \(H_m(\mathbb{R}^n)\) by \(P_{n,m}\). Since \(P_{n,m}\) is closed under addition and closed under multiplication by positive scalars, it is a convex cone. In fact, \(P_{n,m}\) is a closed convex cone: if \(p_n \to p\) coefficient-wise, and each \(p_n\) is psd, then so is \(p\). A psd form is called positive definite or \(pd\) if \(p(x_1, \ldots, x_n) = 0\) implies \(x_j = 0\) for \(1 \leq j \leq n\). The \(pd\) \(n\)-ary \(m\)-ics are the interior of the cone \(P_{n,m}\).

A form \(p \in H_m(\mathbb{R}^n)\) is called a sum of squares or \(sos\) if it can be written as a sum of squares of polynomials; that is, \(p = \sum h_k^2\). It is easy to show in this case that each \(h_k \in H_m(\mathbb{R}^n)\). Again following [1], we denote the set of sos forms in \(H_m(\mathbb{R}^n)\) by \(\Sigma_{n,m}\). Clearly, \(\Sigma_{n,m}\) is a convex cone; less obviously, it is a closed cone, a result proved in general by R. M. Robinson [22], although shown for \(\Sigma_{3,6}\) by Hilbert [9].

In light of the inclusion \(\Sigma_{n,m} \subseteq P_{n,m}\), let \(\Delta_{n,m} = P_{n,m} \setminus \Sigma_{n,m}\). It was well known by the late 19th century that \(P_{n,m} = \Sigma_{n,m}\) when \(m = 2\) or \(n = 2\). In 1888, Hilbert proved [22] that \(\Sigma_{3,4} = P_{3,4}\); more specifically, every \(p \in P_{3,4}\) can be written as the sum of three squares of quadratic forms. (An elementary proof, with “five”...
squares is in \cite{2} pp. 16-17; for modern expositions of Hilbert’s proof, see \cite{26} and \cite{22}.) Hilbert also proved in \cite{9} that the preceding are the only cases for which $\Delta_{n,m} = \emptyset$. That is, if $n \geq 3$ and $m \geq 6$ or $n \geq 4$ and $m \geq 4$, then there exist psd $n$-ary $m$-ics that are not sos.

In 1893, Hilbert \cite{10} generalized his three-square result for $P_{3,4}$ to ternary forms of higher degree. Suppose $p \in P_{3,m}$ with $m \geq 6$. Then there exist $p_1 \in P_{3,m-4}$ and $h_{1k} \in H_{m-2}(\mathbb{R}^3)$, $1 \leq k \leq 3$, so that

$$p_1 p = h_{11}^2 + h_{12}^2 + h_{13}^2.$$

(Hilbert’s proof seems to be non-constructive and lacks a modern exposition. In the very recent paper \cite{11}, de Klerk and Pasechnik discuss the implementation of an algorithm to find $p_1$ so that $p_1 p$ is sos, though not necessarily as a sum of three squares. This paper uses Hilbert’s result without giving an independent proof.)

If $m = 6$ or 8, then $p_1$ is a sum of three squares of forms, and hence (as Landau later noted \cite{12}), the four-square identity implies that $p_1^2 p = p_1(p_1 p)$ is the sum of four squares of forms. If $m \geq 10$, then the argument can be applied to $p_1$: there exists $p_2 \in P_{3,m-8}$ with $p_2 p_1 = h_{21}^2 + h_{22}^2 + h_{23}^2$. Thus, if $m = 10$ or 12 (so that $P_{3,m-8} = \Sigma_{3,m-8}$), then $(p_1 p_2)^2 p = p_2(p_2 p_1)(p_1 p)$ is the sum of four squares of forms. An easy induction shows that there exists $q \in H_t(\mathbb{R}^3)$ with $t = \lfloor (m-2)^2\rfloor$ so that $q^2 p$ is the sum of four squares of forms.

Hilbert’s 17th Problem asked whether this generalizes to $n > 3$ variables; that is, if $p \in P_{n,m}$, must there exist some form $q$ so that $q^2 p$ is sos? Artin proved that there must be, in a way that gives no information about $q$. Much more on the history of this subject can be found in the survey paper \cite{20}.

This discussion leads to two closely related questions. Suppose $p \in P_{n,m}$. Can we find a form $h$ such that $h p$ is sos? Can we find a form $q$ so that $q^2 p$ is sos? If we’ve answered the second, we’ve answered the first. Conversely, if $p \neq 0$ is psd and $h p$ is sos, then $h$ is psd. But it needn’t be sos; indeed, a trivial answer to the first question is to take $h = p$. Stengle proved \cite{25} that if $p(x,y,z) = x^3 y^3 + (y^2 z - x^3 - z^2 x)^2$, then $p^{2s+1} \in \Delta_{3,6(2s+1)}$ for every integer $s$. That is, $p^{2s+1} \cdot p$ is sos, but $p^{2s+1}$ is not. Choi and Lam showed \cite{1} that for $S \in \Delta_{3,6}$ (see (3) below), the product $S(x,y,z) S(x,z,y)$ is actually sos.

The author gratefully acknowledges correspondence with Chip Delzell, Pablo Parrilo, Vicki Powers, Marie-Françoise Roy and Claus Scheiderer. Their suggestions have made this a better paper.

2. What is known about the denominator

The first concrete result about a denominator in Hilbert’s 17th Problem was found by Pólya \cite{17}. He showed that if $f \in H_d(\mathbb{R}^n)$ is positive on the unit simplex $\{(x_1, \ldots, x_n) \mid x_j \geq 0, \sum x_j = 1\}$, then for sufficiently large $N$, $(\sum x_j)^N f$ has positive coefficients. Replacing each $x_j$ by $x_j^2$, we see that if $p \in H_{2d}(\mathbb{R}^n)$ is an even positive definite form, then $(\sum x_j^2)^N p$ is a sum of even monomials with positive coefficients, and so, as it stands, is a sum of squares of monomials. Taking even $N$, we see that $q = (\sum x_j^2)^{N/2}$ is a denominator for $p$. Habicht \cite{7} generalized Pólya’s proof to give an alternate solution to Hilbert’s 17th Problem for pd forms; however, $h$ is not readily constructible and in general is no longer a power of $\sum x_j^2$. Except for one example, Pólya did not attempt to determine an explicit value of $N$. A good exposition of the theorems of Pólya and Habicht can be found in \cite{8}.
For positive definite \( p \in P_{n,m} \), let
\[
\epsilon(p) := \frac{\inf \{ p(u) : u \in S^{n-1} \}}{\sup \{ p(u) : u \in S^{n-1} \}}
\]
measure how “close” \( p \) is to having a zero. The author [19] showed that if
\[
N \geq \frac{nm(m-1)}{4(\log 2)\epsilon(p)} - \frac{n+m}{2},
\]
then \((\sum x_j^2)^N p\) is a sum of \((m+2N)\)-th powers of linear forms, and so is sos. A
similar lower bound has been shown to apply in Polya’s Theorem; the bound goes
to infinity as \( p \) approaches the boundary of \( P_{n,m} \). (See papers by de Loera and
Santos [13] and by Powers and the author [18].)

The restriction to positive definite forms is necessary. There exist psd forms \( p \)
in \( n \geq 4 \) variables so that, if \( h^2p \) is sos, then \( h \) must have a specified zero. The
existence of these unavoidable singularities, or so-called “bad points”, insures that
\((\sum x_j^2)^r p\) can never be a sum of squares of forms for any \( r \). Habicht’s Theorem
implies that no positive definite form can have a bad point. Bad points were first
noted by Straus and have been extensively studied by Delzell; see, e.g. [5, 6].

Little specific is known about the degree of the denominator in more than 3
variables. A. Robinson proved [21, p. 268] that there exists \( d(n,m) \) so that \( p \in P_{n,m} \)
implies that there exists \( q \in H_{d(n,m)}(\mathbb{R}^n) \) so that \( q^2 p \) is sos. Moreover, \( d(n,m) \)
is a general recursive function of \( n \) and \( m \). Various improvements have been made
in the description of \( d \), but no “practical” bounds are known. See [1] \S\S 5.4–5.6,
5.11–5.13, 9.1–9.7 for a detailed survey. The existence of \( d(n,m) \) is also a special
case of a quantitative version of the Positivstellensatz constructed by Lombardi and
Roy [14].

3. Recent results and a new theorem

Scheiderer has shown in very recent work [24] that for \( p \in P_{n,m} \), there exists
\( N = N(p) \) so that \((x^2+y^2+z^2)^N p(x,y,z) \) is sos; indeed, \( x^2+y^2+z^2 \) can be
replaced by any positive definite form. This is a strong refutation to the existence
of bad points for ternary forms.

Suppose \((n,m)\) is such that \( \Delta_{n,m} \neq \emptyset \). Theorem 1 below states that there is
no single form \( h \) so that, if \( p \in P_{n,m} \), then \( hp \) is sos. Corollary 2 says that there
is not even a finite set of forms \( H \) so that, if \( p \in P_{n,m} \), then there exists \( h \in H \)
so that \( hp \) is sos. In particular, there does not exist a finite set of denominators
which apply to all of \( P_{n,m} \). This result implies that \( N(p) \) in Scheiderer’s theorem
is not bounded as \( p \) ranges over \( P_{n,m} \). It also implies that the denominators in the
Lombardi-Roy theorem cannot be chosen from a finite, predetermined set.

The proof of the theorem is elementary and relies on a few simple observations.
If \( p \neq 0 \) is psd and \( hp \) is sos, then \( h \) is psd. As previously noted, \( \Sigma_{n,m} \) is a closed
cone for all \((n,m)\). This cone is invariant under the action of taking invertible
linear changes of variable. Thus, if \( h' \) is derived from \( h \) by such a linear change,
and if \( hp \) is sos for every \( p \in P_{n,m} \), then so is \( h'p \). Suppose \( \ell \) is a linear form,
\( p = \sum_k q_k^{g_k} \) is sos, and \( \ell \mid p \). Then \( \ell^2 \mid p \) and \( \ell \mid g_k \) for each \( k \), and by induction,
\( \ell^{2s} \mid p \implies \ell^s \mid g_k \). Thus, we can “peel off” squares of linear factors from any sos
form; this is a common practice, dating back at least to [22, p. 267]. We use this
observation in the contrapositive: if \( p \in \Delta_{n,m} \), then \( \ell^{2s} p \in \Delta_{n,m+2s} \).
Theorem 1. Suppose $\Delta_{n,m} \neq \emptyset$. Then there does not exist a non-zero form $h$ so that if $p \in P_{n,m}$, then $hp$ is sos.

Proof. Suppose to the contrary that such a form $h$ exists. Since $h \neq 0$, there exists a point $a \in \mathbb{R}^n$ so that $h(a) \neq 0$. By making an invertible linear change of variables, we can take $a = (1, 0, \ldots, 0)$. Thus, we may assume without loss of generality that $h(x_1,0,\ldots,0) = \alpha x_1^d$, where $\alpha > 0$ and $d$ is even. In the sequel, we distinguish $x_1$ from the other variables.

Choose $p \in P_{n,m} \setminus \Sigma_{n,m}$. Then

$$h(x_1, x_2, \ldots, x_n)p(x_1, rx_2, \ldots, rx_n)$$

is sos for every $r \in \mathbb{N}$. By making the change of variables $x_i \to x_i/r$ for $i \geq 2$, we see that

$$h(x_1, r^{-1}x_2, \ldots, r^{-1}x_n)p(x_1, x_2, \ldots, x_n)$$

is also sos. Since

$$\lim_{r \to \infty} h(x_1, r^{-1}x_2, \ldots, r^{-1}x_n) = h(x_1, 0, \ldots, 0) = \alpha x_1^d,$$

and since $\Sigma_{n,m+d}$ is closed, it follows that

$$\lim_{r \to \infty} h(x_1, r^{-1}x_2, \ldots, r^{-1}x_n)p(x_1, x_2, \ldots, x_n) = \alpha x_1^d p(x_1, \ldots, x_n)$$

is sos. Thus $p$ is sos, a contradiction. \hfill \Box

The following elegant proof is due to Claus Scheiderer and is included with his permission; it supersedes the proof in an earlier version of this manuscript.

Corollary 2. Suppose $\Delta_{n,m} \neq \emptyset$. Then there does not exist a finite set of non-zero forms $\mathcal{H} = \{h_1, \ldots, h_N\}$ with the property that, if $p \in P_{n,m}$, then $h_k p$ is sos for some $h_k \in \mathcal{H}$.

Proof. Suppose $\mathcal{H}$ exists. For each $k$, there exists non-zero $p \in \Delta_{n,m}$ so that $h_k p$ is sos. (Otherwise, we may delete $h_k$ harmlessly from $\mathcal{H}$.) Thus, each $h_k$ is psd, and there exists a form $q_k$ so that $q_k^2 h_k$ is sos. Define $h = \prod_k q_k^2 h_k$. We now show that for every $p \in P_{n,m}$, $hp$ is sos: this contradicts Theorem 1 and proves the corollary. By hypothesis, there exists $h_j \in \mathcal{H}$ so that $h_j p$ is sos. Thus,

$$hp = \left( \prod_{k \neq j} q_k^2 h_k \right) \cdot q_j^2 \cdot h_j p$$

is a product of sos factors, and so is sos. \hfill \Box

Finally, we know by Hilbert’s theorem that for $p \in P_{3,6}$, there exists quadratic $h$ so that $hp \in \Sigma_{3,8}$. The three simplest forms in $\Delta_{3,6}$ are

1. $M(x,y,z) = x^4y^2 + x^2y^4 + z^6 - 3x^2y^2z^2$, due to Motzkin [15];
2. R. M. Robinson’s [22] simplification of Hilbert’s construction
3. $R(x,y,z) = x^6 + y^6 + z^6 - (x^4y^2 + x^2y^4 + x^4z^2 + x^2z^4 + y^4z^2 + y^2z^4) + 3x^2y^2z^2$; and
4. $S(x,y,z) = x^4y^2 + y^4z^2 + z^4x^2 - 3x^2y^2z^2$, due to Choi and Lam [1, 2].
It is not too difficult to consider $qM, qR, qS$ for $q(x, y, z) = a^2x^2 + b^2y^2 + c^2z^2$ and determine whether these are sos using the algorithm of [3] directly or its implementation in, e.g., [10].

Interestingly enough, these conditions are the same in each case: the forms are sos if and only if

$$2(a^2b^2 + a^2c^2 + b^2c^2) \geq a^4 + b^4 + c^4.$$ 

This expression factors rather neatly into

$$(a + b + c)(a + b - c)(b + c - a)(c + a - b) \geq 0,$$

so if $a \geq b \geq c \geq 0$ without loss of generality, the only non-trivial condition is that $b + c \geq a$; that is, there is a (possibly degenerate) triangle with sides $a, b, c$. (Robinson [22, p. 273] has a superficially similar condition, but note that his multiplier is $ax^2 + by^2 + cz^2$.)

If we scale variables as in the proof of Theorem 1, it follows from this computation that the three forms

$$(x^2 + y^2 + z^2)M(x, \lambda y, \lambda z), \quad (x^2 + y^2 + z^2)R(x, \lambda y, \lambda z), \quad (x^2 + y^2 + z^2)S(x, \lambda y, \lambda z)$$

are sos if and only if $0 \leq |\lambda| \leq 2$.

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