The logistic-sigmoid function

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Abstract—The variants of sigmoid functions used in artificial neural networks are, by definition, limited by vanishing gradients. Defining the sigmoid function to become n-times repeated over a finite input-output map can significantly reduce the presence of this limitation. This function mapping as proposed in this paper is the logistic-sigmoid function.

Index Terms—Activation Functions; Vanishing Gradients; Nonlinear Functions; Neural Networks.

I. INTRODUCTION

ACITIVATION functions (AFs) follow the basis that neurons in the brain are input-output dynamic systems [1]. The choice of such computational transfer-function mappings is of critical importance in the design of any artificial neural network (ANN) architecture, with significant control over the network’s deep pattern learning ability or performance [2]–[4]. Logistic-sigmoid functions such as the simple logistic-sigmoid function (1) and the hyperbolic-tangent function (2) are common sigmoid-variant activation functions used in ANNs. It can be argued that sigmoid functions are popular, partly due to the fact that their derivatives make them compatible with gradient-descent algorithms [5], [6]. In addition, they are known to possess inherent output boundedness, continuously-differentiable \( C^\infty(\mathbb{R}) \), smooth, monotonic and nonlinear properties [4], [7], [8].

\[
\begin{align*}
y &= f(x) = 1/(1 + e^{-x}) \\
y &= f(x) = \tanh(x) = (2/(1 + e^{2x})) - 1
\end{align*}
\]

Therefore, logistic-sigmoid functions have found application in diverse areas, such as in physics, statistics, neuroscience, computer graphics, signal processing, feedback control and robotics. For example, the logistic-sigmoid function is the most common describing nonlinear regression function [9] for the biological baroreflex feedback regulation system in humans and animals [10]. It is also a recurring shape in nature, commonly observed in the motion-profile of ant swarms [11]. Further, sigmoid functions have been used to model actuating behaviours of high gain nonlinear operational amplifiers (amplifiers with saturation) [12]–[14], which function as signal limiting or relay (switching) functions. It has also been used to achieve smooth path-planning for mobile robots [15].

Likewise, the literature on sigmoid activation functions is rich. Considerable research have been dedicated to the application and improvement of sigmoid functions, see [16]–[21]. An efficient approximation scheme for the VLSI implementation of neural networks with the hyperbolic-tangent function was proposed in [22]. The function approximation capability of sigmoids have also been investigated in relatively recent works. [23] using a broad class of common AFs, explored the smooth function approximation ability of deep neural networks. They further classify AFs into two groups: piecewise linear (rectified linear-unit (ReLU) variants) and locally quadratic (sigmoid variants). Notably, for best function approximation and faster convergence, it has been suggested that sigmoid functions should be symmetric about the origin of the function’s input-output map [5]. Further, in application to multi-path signal propagation, [24] showed that the optimum AF for vector equalization using recurrent neural networks (RNNs) can be realised as a sum of shifted hyperbolic tangent functions.

Nevertheless, two notable inherent drawbacks affect sigmoid functions. As the first short-coming: formulations of sigmoid functions possess a fundamental learning (optimization) disadvantage in which their derivatives exponentially approach zero, especially as the network becomes deeper. This famous problem defined as the vanishing (exponential decreasing) gradient problem was first formally treated in [25], leading to the gradient-based Long Short-Term Memory (LSTM) recurrent neural network (RNN). More recently, this problem was also considered in [26]. Several proposals to partially overcome this issue have been explored in the literature. Some of the workaround solutions, apart from LSTM networks, reviewed in [27] are: deep-belief networks, hessian-free optimisation, random-weight guessing and leverage of GPU-based computational power. Consequently, more research led to the proposals of different alternative activation functions such as in [26], [28]–[32]. A concise survey of these proposed AFs in order to reduce the effect of vanishing gradients can be found in [33]. On the contrary, a common problem with most of these alternative AFs is that they cause more noisy outputs [26].

The second short-coming lies in the computational expensive arithmetic, in form of the natural exponential term \( e^x \). This is why among AFs, the attractive property of low computational complexity and fast computational speed makes ReLU function variants popular as default choices in many deep neural-network architectures. Notwithstanding, sigmoid variants still stand out as the choice in architectures such as RNNs [34], [35]. Fortunately, through computer arithmetic techniques of function approximation, reduced computational complexity can be realized in software within acceptable levels of accuracy [34], [35].

In this brief, contrasted with the above discussed approaches in the literature, we treat the first shortcoming: the vanishing
gradient problem as a flaw in the logistic-sigmoid function definition. Specifically, for suitable use both as a activation and actuating function, we provide a more general and formal definition for such logistic-sigmoids over a finite input-output space. This is in form of the \textbf{n}logistic sigmoid function, illustrated in Fig. 1.

In section II, we start with a preliminary discussion on the classic sigmoid function. The proposed function is then defined in section III. It will be shown that by definition, the proposed function has over its input-output map, a \(n\)-times repeating gradient surface, with a maximum gradient of 3 which is a factor of 12 and 3 greater than (1) and (2) respectively. This implies that the proposed function can, to a large extent, reduce the vanishing occurrence of its gradients. Finally, in section IV this paper is concluded. Useful for library developers, we provide an open-source implementation of this function in MATLAB and C/C++ [57].

II. CLASSIC LOGISTIC-SIGMOID

The classic logistic-sigmoid function given by (3) involve mapping a bounded output co-domain \(y \in \mathbb{R}\) in the range \([y_{\min}, y_{\max}]\), where \(y_{\min}, y_{\max}\) are the minimum and maximum bounds or asymptotes of \(y\) respectively, to an unspecified or unbounded input domain \(x \in \mathbb{R}\) in the range \((-\infty, \infty)\), centered by an offset parameter \(\delta \in \mathbb{R}\), and scaled by a logistic growth-rate parameter \(\alpha \in \mathbb{R}\).

\[
y = f(x) = y_{\min} + \frac{y_{\max} - y_{\min}}{1 + e^{\alpha (x - \delta)}}
\]

The derivative \(g\) with respect to the input is also given by (4). The maximum value of the derivative \(g\) occurs at the inflection point (mid-point), where \(x = \delta\). It can be easily shown that \(g = 0.25\alpha (y_{\max} - y_{\min})\). At this point the corresponding value of \(y\) is \((y_{\max} + y_{\min})/2\).

\[
g = \frac{dy}{dx} = \frac{\alpha}{y_{\max} - y_{\min}} (y - y_{\min}) (y_{\max} - y)
\]

For the simple logistic function (1), where \(y_{\max} = 1, y_{\min} = 0, \alpha = 1, \text{ and } \delta = 0\), we have \(g = 0.25\). On the other hand, for the hyperbolic tangent function (2), where \(y_{\max} = 1, y_{\min} = -1, \alpha = 2, \text{ and } \delta = 0\), we have \(g = 1\). These formulations are, therefore, limiting in the sense that the maximum gradient is both small and that the gradient exponentially decreases to zero as the value of the input \(x\) deviates from \(\delta\).

These two problems can be solved to a large degree with the use of the \textbf{n}logistic-sigmoid, which features: one, a tunable logistic growth-rate constant, such that the maximum gradient is neither too large nor too small; and two: a \(n\)-times occurring gradient map over the finite input-output space (universe).

III. \textbf{n}LOGISTIC-SIGMOID

Instead of the classic single sigmoid, a repeating sigmoid behaviour found in the motion-profile of ant-swarms can be emulated within a defined finite input-output universe specified by \([x_{\min}, x_{\max}]\) and \([y_{\min}, y_{\max}]\).

The specified finite input and output universe (space) is partitioned into \(n\) equal sub-spaces with \(n\) inflection points at \(x = \delta_i\) leading to \(v_i\) sub-sigmoids, where \(i = 1, \ldots, n\). This transforms the classic logistic-sigmoid function to a sum of \(n\) shifted sigmoids each with a continuous gradient map over the surface of the input-output universe.

The interval spacing for each sub-sigmoid partition in the finite input and output universe can be defined respectively as (5) and (6).

\[
\Delta_x = \frac{(x_{\max} - x_{\min})}{n} \quad (5)
\]

\[
\Delta_y = \frac{(y_{\max} - y_{\min})}{n} \quad (6)
\]

A formal definition for the periodic odd function that results from the sum of shifted sigmoids, each having the same property, can then be provided.

**Definition 1.** The \textbf{n}logistic-sigmoid function, with odd output \(y\) and even output \(g\), denoted as \(\textbf{n}\text{lsig} \pm : \mathbb{R} \rightarrow \mathbb{R}\), where \(n \in \mathbb{N}^+\) and \(\lim_{x \rightarrow x_{\max}} g = y_{\max}, \lim_{x \rightarrow x_{\min}} y = y_{\min}\) is:

\[
\text{\textbf{n}lsig} \pm : y = f(x) = \kappa_y + \sum_{i=1}^{n} v_i \quad (7)
\]

\[
g = \tau \sum_{i=1}^{n} v_i (\Delta_y - v_i) \quad (8)
\]

where,

\[
v_i = \frac{\Delta_y}{1 + e^{u}}, \quad u = \alpha (x - \delta_i) \quad (9)
\]

\[
\delta_i = \kappa_x + \Delta_x \left( i - \frac{1}{2} \right) \quad (10)
\]

\[
\alpha = \frac{2\lambda}{\Delta_x}, \quad \tau = \frac{\alpha}{\Delta_y} \quad (11)
\]

and,

\[
\kappa_x = x_{\min}, \kappa_y = y_{\min}, \lambda = 6.
\]

The boundary (limit) values of the input-output universe are defined as:

\[
y_{\max} = (1 - \xi) y_{\max}, \quad y_{\min} = (1 - \xi) y_{\min} \quad (12)
\]

\[
x_{\max} = (1 - \xi) x_{\max}, \quad x_{\min} = (1 - \xi) x_{\min} \quad (13)
\]

where
\[ \xi = \frac{\xi_c}{c}, \quad -c \leq \xi \leq c \] (14)

and

\[ c = 100 \quad \text{and} \quad \xi = 0 \text{(default)} \] (15)

with \( \xi, c \in \mathbb{N}^+ \), and \( x, y, g, v, \alpha, \delta, \tau \in \mathbb{R} \).

The midpoint of the input-output limits is \((\bar{x}, \bar{y})\).

\[ \bar{x} = \frac{x_{\max} + x_{\min}}{2} \] (16)

\[ \bar{y} = \frac{y_{\max} + y_{\min}}{2} \] (17)

There is no central inflection point when \( n \) is even. The central inflection point \( \delta \), when \( n \) is odd is equivalent to \( \bar{x} \).

\[ \delta = \frac{1}{n} \sum_{i=1}^{n} \delta_i = \bar{x} \] (18)

The parameter \( n \) defines the number of sub-sigmoid in the function, \( \tau \) is the derivative constant, \( v_i \) is partial output representing the \( i \)th sub-sigmoid, \( u \) is the natural exponential input, and \( \xi \) is the limit damper constant.

From the definitions in (16) and (17), it is also straightforward to see that, to ensure the curve is symmetric at a central midpoint of \( \delta = 0 \), then the minimum limit must always be set as \( x_{\min} = -x_{\max} \) and \( y_{\min} = -y_{\max} \).

The output shape of the \( n \)logistic-sigmoid function is illustrated for an arbitrary finite input-output universe \([-10, 10]\) in Fig. 1B.

The best input-output sigmoidal approximation occurs when \( \lambda = 6 \).

It can be shown that the value of the maximum gradient \( g \) is proportional to the value of \( \lambda \).

**Proof.** To start with, we note that the sigmoid shape repeats itself \( n \)-times in the input-output universe. Since, this is true, then without any loss of generality, we can assume \( n = 1 \). The derivative becomes:

\[ g = \tau v (D_y - v) \]

where

\[ v = y - y_{\min}, \quad D_y = y_{\max} - y_{\min} \]

At the interval boundaries or limits, when \( y = y_{\max} \) or \( y = y_{\min}, \) then \( g = 0 \).

Also, at the inflection-point (mid-point), when \( x = \delta \), the output value is \( y = (y_{\max} + y_{\min})/2 \), then \( g = (\lambda D_y) / (2D_x) \).

For a symmetric map, centered at zero, with \( x_{\max} \equiv x_{\min} \) and \( x_{\min} \equiv y_{\min} \), we have \( D_x = D_y \), and so \( g = \lambda / 2 \).

Since \( \lambda = 6 \), the value of the derivative at the midpoint becomes \( g = 3 \). Therefore, in one line

\[ \bar{g} = 0.5 \lambda \frac{D_y}{D_x} = 0.5 \lambda \frac{y_{\max}}{x_{\max}} = 0.5 \lambda = 3 \]

It follows that, if \( n > 1 \), then the maximum gradient will occur at each \( x = \delta_i \), which represent the centre of each sub-sigmoid. Also, as \( \lambda = 6 \), the maximum gradient \( \bar{g} \) is always 3 at \( x = \delta_i \). This completes the proof. \( \square \)

Therefore, with the definition of the \( n \)logistic-sigmoid function, gradients do not vanish except at the boundary of the maximum or minimum of the input-output universe. This is visually illustrated in Fig. 3A. Consequently, to a large extent, because of the way the gradient is controlled and repeated over the input-output map, the frequent occurrence of vanishing gradients in the network’s gradient flow and local gradient will be far reduced, compared to the classic sigmoid use-case.

Since this function definition is the intended contribution of this paper, we leave the empirical evaluation and comparison of its performance to the classic sigmoid variants and rectified-linear unit variants on benchmark ANN data-sets as future work.

Further, we note that as \( n \) increases in the \( n \)logistic-sigmoid function, approximately linear regions around the inflection points become more evident as shown in Fig. 2A. This behaviour is influenced by the logistic growth-rate constant \( \lambda \), and offers the advantage of better function approximation as discussed in [23]. Lyapunov stability is also guaranteed by the \( n \)logistic-sigmoid function under certain conditions for RNNs as shown for shifted sigmoids in [24], [38].

Finally, we note that compared to rectified linear-units, and the classic logistic-sigmoid, the main inherent drawback of the proposed function is the increased arithmetic and exponential operations as the \( n \) parameter is increased. Notwithstanding this short-coming, satisfactory computational-speed and accuracy can be obtained from computing the natural exponential term by using a combination of software computing techniques: ones-normalization, bit-manipulation and exponentiation-by-squaring.

**IV. Conclusions**

The \( n \)logistic-sigmoid function which extends the classic sigmoid function to a function of \( n \) sigmoids over a defined finite input-output map was contributed in this brief. It was analytically shown by definition and also by visual illustration that this function inherently overcomes to a large extent the famous vanishing-gradient problem of sigmoid functions. Apart from the intended popular use as a activation function and reward function in deep neural network architectures, the \( n \)logistic-sigmoid will also find suitable application as a fuzzy membership function, a nonlinear estimator, a smooth control law, a signal limiter, and as a nonlinear filter.

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Fig. 2. Output and Gradient Plot of the \textit{n}-logistic-sigmoid: \(n = 1\) in Fig. (2a), \(n = 2\) in Fig. (2b), \(n = 7\) in Fig. (2c), and \(n = 16\) in Fig. (2d).

Fig. 3. Full plot of the \textit{n}-logistic-sigmoid for \(n = 1\) to 16. Fig. (3b) shows a detailed zoomed view of the output.

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