1. INTRODUCTION

The evaluation of the resistance of plates against in-plane load is crucial in structural design. For the assessment of plate stability, linear or nonlinear buckling analysis techniques are generally required so that the critical buckling load of the plates can be determined. The linear buckling analysis is enough to be used in the design process but the nonlinear buckling analysis may be required to check the ultimate safety of structure. The closed form solution has been often used to assess the plate stability. However, its accuracy is heavily relying on the displacement function selected in the solution process. On the other hand, the finite element (FE) technology has been used in more general situations and also it showed a great possibility to be extended.

So far, the stability analysis of plate has been carried out by many researchers (Bulson, 1970) and mostly Kirchhoff and Reissner-Mindlin assumptions have been adopted (Leissa, 1982). The governing equations of an initially stressed Mindlin plate were provided by Brunelle and Robertson (1974). The stability of moderately thick rectangular plates was investigated by the triangular FE (Rao et al., 1975). The modified complementary energy principle was also introduced in the FE buckling analysis of the thin and moderately thick plates (Luo, 1982). The Rayleigh–Ritz method and the finite strip method were adopted to carry out the elastic buckling analysis of rectangular Mindlin plates (Dawe & Roupaeil, 1982). The buckling and free vibrations of Mindlin plates were also performed by using the Rayleigh–Ritz method (Xiang, 1993). For the calculation of exact buckling load and natural frequency, the higher-order shear deformation theory (Reddy and Phan, 1985) was introduced for simply supported rectangular plates. The elastic buckling analysis of rectangular Mindlin plates with mixed boundary conditions was carried out by using integral equations combined with a numerical integration technique for the analysis (Sakiyama and Matsuda, 1987). Thin plate FE was then provided and used to predict the buckling capacity of arbitrarily shaped thin-walled structural members under general load and boundary conditions (Chin et al., 1993). In their study, the linear and geometric stiffness matrices for the thin-plate element were derived explicitly based on the principle of minimum total potential energy. General buckling solutions for in-plane loaded, isotropic Mindlin plates was derived with different shape of plates such as rectangular, polygonal, elliptical, semicircular and annular shapes (Wang et al., 1994). The buckling loads of rectangular Mindlin plates for different boundary conditions were investigated (Liew et al., 1996). The classical power series method was used to produce the exact solutions for free vibration and buckling of plates with several boundary conditions (Leissa and Kang, 2002; Kang and Leissa, 2001). The stability and vibration of the plates based on the first-order and higher-order plate theory was investigated by using the extended Kantorovich method (Shufrin and Eisenberger, 2005).

From previous studies, the utilization of RM plate element technologies in the linear buckling analysis has been mostly achieved with the reduced integration technique for the alleviation of locking phenomenon. Therefore, we here aim to introduce an enhanced plate element for the accurate prediction of buckling load for plates without locking phenomenon. All terms and descriptions...
for the linear buckling analysis with the enhanced plate element are thoroughly described here. Numerical examples are carried out to test the performance of the Q9-ANS FE in linear buckling analysis and the results are summarized.

2. REISSNER-MINDLIN PLATE THEORY

2.1 Assumptions
Since the transverse shear deformation of plate is considered, the following Reissner-Mindlin assumptions (Reissner, 1945; Mindlin, 1951) are adopted:

- The stress normal to the plate mid-surface is negligible
- Normal to the mid-surface of plate remains straight but not necessarily normal to the mid-surface after deformation
- Displacement are small compared to the plate thickness

The total domain ($\Omega$) of the plate can be written as

$$\Omega = \{(x_1, x_2, x_3) | (x_1, x_2) \in \bar{\Omega}, x_3 \in \left[ -\frac{h}{2}, \frac{h}{2} \right] \} \quad (1)$$

where $\bar{\Omega}$ is the xy-plane and $h$ is denoted as thickness of plate. Note that the domain ($\Omega$) consists of the mid-surface ($\bar{\Omega}$) and the thickness ($h$) as shown in Figure 1.

![Geometry of the plate and sign convention](image)

2.2 Displacement
The displacement field can be written as

$$u_1(x_1, x_2, x_3) = x_3 \theta_1 \left( x_1, x_2 \right)$$
$$u_2(x_1, x_2, x_3) = -x_3 \theta_2 \left( x_1, x_2 \right)$$
$$u_3(x_1, x_2, x_3) = \bar{u}_3 \left( x_1, x_2 \right) \quad (2)$$

where $\bar{u}_3$ is the transverse displacement of a point on the mid-surface, $\theta_1$ is the normal rotation in $x_1 - x_3$ plane and $\theta_2$ is the normal rotation in $x_2 - x_3$ plane.

2.3 Strain
The strain definition can be written as

$$\varepsilon_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) = \frac{1}{2} \left( u_{i,j} + u_{j,i} \right). \quad (3)$$

Substituting (2) into (3) yields

$$\varepsilon_p = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{bmatrix} = x_3 \begin{bmatrix} \theta_{1,1} \\ -\theta_{2,2} \\ \theta_{3,1} - \theta_{2,1} \end{bmatrix} = x_3 \kappa \quad (4)$$
$$\varepsilon_s = \begin{bmatrix} \gamma_{13} \\ \gamma_{23} \end{bmatrix} = \begin{bmatrix} \theta_{1} + \bar{u}_{3,1} \\ -\theta_{2} + \bar{u}_{3,2} \end{bmatrix} = \gamma$$

in which $\varepsilon_p$ is the in-plane strain term and $\varepsilon_s$ denotes the transverse shear strain term.

2.4 Constitutive equation
In this study, the plate is assumed as isotropic material and the normal transverse stress ($\sigma_3$) is assumed to be negligible. Therefore, the constitutive equation can be written as

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \\ \tau_{13} \\ \tau_{23} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & 0 & 0 & 0 \\ C_{21} & C_{22} & 0 & 0 & 0 \\ 0 & 0 & C_{33} & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 \\ 0 & 0 & 0 & 0 & C_{55} \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \\ \gamma_{13} \\ \gamma_{23} \end{bmatrix} \quad (5)$$

in which $C_{ij}$ are the components of rigidity matrix as follows

$$C_{11} = \frac{E}{1 - \nu^2}, \quad C_{22} = \frac{E}{1 - \nu^2},$$
$$C_{12} = \frac{\nu E}{1 - \nu^2} = C_{21},$$
$$C_{33} = C_{44} = C_{55} = G = \frac{E}{2(1 + \nu)} \quad (6)$$

where $E$ is the Young modulus, $G$ is the shear modulus and $\nu$ is the Poisson ratio.

2.5 Stress resultant
The stress resultants are calculated by integration of the stresses through thickness direction of the plate and stress resultant terms such as $M_i$, $Q_i$ can be therefore written as follows

$$M = \begin{bmatrix} M_1 \\ M_2 \\ M_{12} \end{bmatrix} = \int_{-h/2}^{h/2} x_3 \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{bmatrix} dx_3, \quad (7)$$
$$Q = \begin{bmatrix} Q_{13} \\ Q_{23} \end{bmatrix} = \int_{-h/2}^{h/2} \begin{bmatrix} \tau_{13} \\ \tau_{23} \end{bmatrix} dx_3.$$

The above stress resultant terms can be rewritten in the matrix form:
Buckling Analysis of Rectangular Plates using an Enhanced 9-node Element

where the components of the rigidity matrices \( \bar{D}, \bar{G} \) can be written as follows

\[
\bar{D} = \frac{Eh^3}{12(1-\nu^2)} \begin{bmatrix}
1 & \nu & 0 \\
\nu & 1 & 0 \\
0 & 0 & (1-\nu)
\end{bmatrix}
\]

(9)

\[
\bar{G} = \frac{Eh}{2(1+\nu)} \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\]

3. ENHANCED 9-NODE PLATE ELEMENT

3.1 Element geometry

The geometry of the 9-node element is illustrated in Figure 2. The present element has nine nodes which are numbered from 1 to 9 in counter-clockwise from the corner \( (\xi_1 = -1, \xi_2 = -1) \) to the center of the element.

![Figure 2. Geometry of the plate element](image)

All the quantities of the element are interpolated by using the following interpolation function (Lee, 2014)

\[
N^a = \frac{1}{2} \xi_1^a \xi_2^a (1 + \xi_1^a \xi_1) + (1 - \xi_2^a)(1 - (\xi_2^a)^2)
\]

(10)

where \( \xi_1^a, \xi_2^a \) are the natural coordinate values at the node \( a \).

3.2 Kinematics and displacement field

The position vector \( x \) of the present element can be defined in the following form:

\[
x = \sum_{a=1}^{9} N^a x^a
\]

(11)

where \( N^a \) is the shape function associate with node \( a \) and the position vector \( x^a \) associated with node \( a \) has two components such as

\[
x^a = (x_1^a, x_2^a)
\]

(12)

The displacement field of the present element can be defined in the following form:

\[
u = \sum_{a=1}^{9} N^a u^a
\]

(13)

where the displacement vector \( u^a \) associated with node \( a \) has three components such as

\[
u^a = (u_1^a, \theta_1^a, \theta_2^a)^T
\]

(14)

3.3 Strain-displacement relationship matrix

Using (13), the strains of (4) can be rewritten in the form of the strain-displacement relation matrix \( B \) as follows

\[
\varepsilon_p = \sum_{a=1}^{9} B_p^a u^a, \quad \varepsilon_s = \sum_{a=1}^{9} B_s^a u^a
\]

(15)

where the sub-matrices of \( B_p^a \) and \( B_s^a \) are

\[
B_p^a = \begin{bmatrix}
0 & 0 & N_1^a \\
0 & -N_2^a & 0 \\
0 & N_2^a & -N_1^a
\end{bmatrix}, \quad B_s^a = \begin{bmatrix}
N_1^a & 0 & N_a \\
N_2^a & 0 & 0
\end{bmatrix}
\]

(16)

in which \( N_a = \frac{\partial u^a}{\partial x_1} \).

3.4 Assumed strains

Because of the element deficiencies inherited in RM plate element, the standard linear strain-displacement matrix is substituted with assumed strains in this study. The sampling points used in the formation of assumed strains are presented in Figure 3.

![Figure 3. The sampling point for assumed strains: (left) \( \xi_{11}^a \) (right) \( \xi_{22}^a \)](image)
The interpolation functions used in formulating the assumed strains are based on Lagrange interpolation polynomials as used by Lee and Kanok-Nuchulchai (1998). Consequently, the substitute assumed strains \( \varepsilon^A \) can be defined in the following form:

\[
\varepsilon^A_{13} = \sum_{i=1}^{2} \sum_{j=1}^{3} P_i(\xi_1) Q_j(\xi_2) \varepsilon^0_{13} \\
\varepsilon^A_{23} = \sum_{i=1}^{2} \sum_{j=1}^{3} P_i(\xi_2) Q_j(\xi_1) \varepsilon^0_{23} \tag{17}
\]

in which \( \delta = 2(j - 1) + \iota \) denotes the position of the sampling point as shown in Figure 3 and the shape functions \( P_i(\xi) \) and \( Q_i(\xi) \) are

\[
P_i(\xi) = \frac{1}{2} (1 + \sqrt{3}\xi); \quad P_2(\xi) = \frac{1}{2} (1 - \sqrt{3}\xi); \\
Q_1(\xi) = \frac{1}{2} (\xi + 1); \quad Q_2(\xi) = \frac{1}{2} (1 - \xi^2); \\
Q_3(\xi) = \frac{1}{2} (\xi - 1). \tag{18}
\]

The assumed strains \( \varepsilon^A \) of (17) are used in the present plate element instead of the strains \( \varepsilon_3 \) of (15) obtained from the displacement field.

### 3.5 Strain energy I

The strain energy of the plate induced by the deformation can be written as

\[
U_E = \frac{1}{2} \int_{\Omega} \varepsilon^T D \varepsilon \, d\Omega + \frac{1}{2} \int_{\Omega} \varepsilon_3^T D_3 \varepsilon_3 \, d\Omega \\
= \frac{1}{2} \int_{\Omega} \kappa^T D \kappa \, d\Omega + \frac{1}{2} \int_{\Omega} \gamma^T D_3 \gamma \, d\Omega . \tag{19}
\]

The structural stiffness matrix can be written as

\[
K = K^{ab} = K^{ab}_p + K^{ab}_s \\
= \int_{\Omega} (B^a_p)^T D B^a_p \, d\Omega + \int_{\Omega} (B^a_s)^T D_3 B^a_s \, d\Omega . \tag{20}
\]

The above equation can be rewritten as follows

\[
K = \bigcup_{e=1}^{ne} [K^{ab}(e)] \bigcup_{e=1}^{ne} \left[ (B^a_p)^T D B^a_p \bigcup (B^a_s)^T D_3 B^a_s \right] \det(\jmath) \, d\xi_1 \, d\xi_2 \\
+ \bigcup_{e=1}^{ne} \left[ (B^a_p)^T D B^a_p \bigcup (B^a_s)^T D_3 B^a_s \right] \det(\jmath) \, d\xi_1 \, d\xi_2 \tag{21}
\]

where \( ne \) is the number of element, \( J \) is Jacobian matrix between \( x_1-x_2 \) and \( \xi_1-\xi_2 \) and \( U \) is the FE assembly operator.

### 3.6 Strain energy II

The strain energy induced by the initial stress can be written as

\[
U_0 = \int_{\Omega} (\sigma^0)^T \varepsilon \, d\Omega \tag{22}
\]

where the initial stress matrix are

\[
\sigma^0 = \{\sigma_1^0, \sigma_2^0, \tau_{12}^0\}^T \tag{23}
\]

and the higher order terms of strain definition are

\[
\varepsilon^L = \{\varepsilon_1^L, \varepsilon_2^L, \gamma_{12}^L\}^T \tag{24}
\]

in which the component of higher order strain term is

\[
\varepsilon_1^L = \frac{1}{2} \left( \frac{\partial u_1}{\partial x_1} \right)^2 + \left( \frac{\partial u_2}{\partial x_1} \right)^2 + \left( \frac{\partial u_3}{\partial x_1} \right)^2 \\
\varepsilon_2^L = \frac{1}{2} \left( \frac{\partial u_1}{\partial x_2} \right)^2 + \left( \frac{\partial u_2}{\partial x_2} \right)^2 + \left( \frac{\partial u_3}{\partial x_2} \right)^2 \\
\gamma_{12}^L = \frac{\partial u_1}{\partial x_1} \frac{\partial u_2}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \frac{\partial u_3}{\partial x_2} + \frac{\partial u_3}{\partial x_1} \frac{\partial u_1}{\partial x_2} . \tag{25}
\]

Substituting (2) into (25) yields

\[
\varepsilon_1^L = \frac{x_1^2}{2} \left( \frac{\partial \theta_1}{\partial x_1} \right)^2 + \left( \frac{\partial \theta_2}{\partial x_1} \right)^2 + \frac{1}{2} \left( \frac{\partial u_3}{\partial x_1} \right)^2 \\
\varepsilon_2^L = \frac{x_2^2}{2} \left( \frac{\partial \theta_1}{\partial x_2} \right)^2 + \left( \frac{\partial \theta_2}{\partial x_2} \right)^2 + \frac{1}{2} \left( \frac{\partial u_3}{\partial x_2} \right)^2 \\
\gamma_{12}^L = \frac{\partial \theta_1}{\partial x_1} \frac{\partial \theta_2}{\partial x_2} + \frac{\partial \theta_2}{\partial x_1} \frac{\partial \theta_1}{\partial x_2} + \frac{\partial u_3}{\partial x_1} \frac{\partial u_3}{\partial x_2} . \tag{26}
\]

We can obtain the following equation using (22), (23), (24) and (26) through thickness integration

\[
U_0 = \frac{1}{2} \int_A \left( \frac{\partial u_3}{\partial x_1} \frac{\partial u_3}{\partial x_2} \left[ \sigma_1^0 \frac{\partial \theta_1}{\partial x_1} + \sigma_2^0 \frac{\partial \theta_2}{\partial x_2} \right] + \frac{\partial \theta_1}{\partial x_1} \frac{\partial \theta_2}{\partial x_2} \right) \left[ \sigma_1^0 \frac{\partial \theta_1}{\partial x_1} + \sigma_2^0 \frac{\partial \theta_2}{\partial x_2} \right] h^3 \, dA \tag{27}
\]
Therefore, the geometric element stiffness matrix of the plate can be written as

\[ K_G^{ab(e)} = K_G^{ab(e)} + K_G^{ab(e)} + K_G^{ab(e)} \]  

where the terms of (28) are

\[ K_G^{ab(e)} = h \int_{A(e)} \begin{bmatrix} N_1^a & N_2^a \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \sigma_1^0 & \tau_{12}^0 \\ \tau_{12}^0 & \sigma_2^0 \end{bmatrix} \begin{bmatrix} N_1^b \\ N_2^b \end{bmatrix} dA(e) \]

\[ K_G^{ab(e)} = h^3 \int_{A(e)} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \sigma_{11}^0 & \tau_{12}^0 \\ \tau_{12}^0 & \sigma_{22}^0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} dA(e) \]

\[ K_G^{ab(e)} = h^3 \int_{A(e)} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} dA(e). \]

The above equation can be assembled as follows

\[ K_G = \sum_{e=1}^{n_e} \left[ \int_{A(e)} [(G_1^e)^T \sigma G_1^e] \det(J) d\xi d\eta \right. \]

\[ + \sum_{e=1}^{n_e} \left[ \int_{A(e)} [(G_2^e)^T \sigma G_2^e] \det(J) d\xi d\eta \right. \]

\[ + \sum_{e=1}^{n_e} \left[ \int_{A(e)} [(G_3^e)^T \sigma G_3^e] \det(J) d\xi d\eta \right. \]

where \( \phi \) is a buckling mode, \( \lambda \) is the critical buckling load and \( K \) and \( K_G \) are global stiffness and geometric stiffness matrices which contain contributions from element stiffness and geometric stiffness matrices.

### 4. NUMERICAL EXAMPLES

The buckling loads of rectangular plates are calculated by using the enhanced 9-node plate element. Several important parameters such as width-thickness ratio, different boundary condition and additional lateral load are mainly considered in the tests.

#### 4.1 Rectangular plate with the uniaxial stress

In this example, we consider rectangular plates under the uniaxial stress. The geometry of the plate and FE mesh are illustrated in Figure 4. A half of the plate is used for the FE analysis and 45 nodes and 8 elements are used to discretize a half of the plate. All edges of plate are simply supported. Uniaxial compressive load are applied to two edges of the plate. Ten aspect ratios such as \( \frac{a}{b} = 0.3, 0.4, 0.5, 0.6, 0.8, 1.0, 1.4, 2.0, 3.0, 4.0 \) are used in the analysis. In particular, the width-thickness ratio \( \frac{b}{h} = 1000 \) is used for this example.

The present solutions with ten aspect ratios are illustrated in Figure 5. Note that all the present solutions are the first eigenvalues obtained from ten linear buckling analyses. The dimensionless buckling parameters are then calculated by using

\[ k_\sigma = \frac{\lambda^2 \sigma_{cr}}{\pi^2 D} \]

where \( \lambda = \frac{b}{h} \) is width-thickness ratio and \( D = \frac{E}{12(1-\nu)} \) is the flexural rigidity.

The four closed form solutions with different half waves \( (m = 1, 2, 3, 4) \) are also plotted in Figure 5. Note that \( m \) is the value which is correlated to the number of half wave lengths along \( x \).

The present FE solutions are located on the curves produced by the closed form solutions. More specifically, the present solutions with seven aspect ratios \( (\frac{a}{b} = 0.3, 0.4, 0.5, 0.6, 0.8, 1.0) \) are on the curve with \( m = 1 \). Other three solutions are respectively on the curves with \( m = 2, 3, 4 \). In thin plate situation, the present solutions have therefore a good agreement with the closed form solutions with a few FE elements.

The buckling modes for four cases such as \( m = 1, 2, 3, 4 \) are illustrated in Figure 6.
We can see that the larger aspect ratio clearly triggers more number of half wave in x-direction.

4.2 Rectangular plate with different thicknesses

In this example, the effect of plate thickness on the critical buckling load is investigated. Four values of plate thicknesses are used with four aspect ratios. The same plate geometry of the previous example is used here. Numerical results are illustrated and summarized with other reference solutions in Figure 7 and Table 1 respectively.

Table 1. The buckling parameter of plate using different thicknesses-span ratios \((h/b)\) with four aspect ratios.

| a/b | h/b | Ref0 | Ref1 | Ref2 | Ref3 | Present |
|-----|-----|------|------|------|------|---------|
| 1   | 0.2 | 3.150| 3.1471| 3.2637| 3.1255| 3.128   |
|     | 0.1 | 3.741| 3.7270| 3.7865| 3.7314| 3.734   |
|     | 0.05| 3.911| 3.9293| 3.9444| 3.9287| 3.932   |
|     | 0.001| 4.000| -     | -     | -     | 4.003   |
| 2   | 0.2 | -   | 3.0783| 3.2637| 3.1255| 3.131   |
|     | 0.1 | -   | 3.6797| 3.7865| 3.7314| 3.741   |
|     | 0.05| -   | 3.8657| 3.9444| 3.9287| 3.941   |
|     | 0.001| -   | -     | -     | -     | 4.013   |
| 3   | 0.2 | -   | 3.0783| 3.2637| 3.1255| 3.142   |
|     | 0.1 | -   | 3.6797| 3.7865| 3.7314| 3.769   |
|     | 0.05| -   | 3.8657| 3.9444| 3.9287| 3.975   |
|     | 0.001| -   | -     | -     | -     | 4.049   |
| 4   | 0.2 | -   | 3.0306| 3.2421| 3.0877| 3.437   |
|     | 0.1 | -   | 3.7311| 3.8683| 3.8014| 4.091   |
|     | 0.05| -   | 3.9600| 4.0645| 4.0449| 4.125   |
|     | 0.001| -   | -     | -     | -     | 4.206   |

Note: Ref0: 3D solution (Srinivas and Rao, 1969); Ref1: the collocation with radial basis function (Kitipornchai et al., 1993); Ref2: Rayleigh-Ritz method (Ferreira et al. 2011); Ref3: isogeometric solution (Lee & Kim, 2012).

From numerical results, the thin plate closed form solutions appear to overestimate the buckling loads since it did not consider the transverse shear deformation. Therefore, the RM finite element is necessarily required to assess the buckling load of moderately thick plates accurately.

4.3 Rectangular plate: effect of boundary condition

The effect of boundary conditions on the buckling load is investigated. For the purpose of concise identification, each set of the boundary condition is denoted by four letters which stand for the four sides in the rectangular plate. The sequence begins with the edge A and proceeds counter-clockwise around the plate in Figure 8.
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The load edges are simply supported (S) or clamped (C) boundary conditions. The unloaded edges of plated can be either simply supported, clamped or free (F). Five combinations of boundary condition such as C/C/C/C, C/S/C/S, S/S/S/S, C/F/C/F and S/F/S/F are used in the analysis. Eight aspect ratios such as \( \frac{a}{b} = 0.5, 0.6, 0.8, 1.0, 1.4, 2.0, 3.0, 4.0 \) are used in the analysis. The buckling coefficient is plotted against the plate aspect ratios for the cases using different boundary conditions in Figure 8 and summarized numerically in Table 2.

| \( \frac{a}{b} \) | C/C/C/C | C/S/C/S | S/S/S/S | C/F/C/F | S/F/S/F |
|-----------------|-----------|-----------|-----------|-----------|-----------|
| 0.5             | 6.256     | 19.609    | 17.973    | 5.079     | 16.081    |
| 0.6             | 5.142     | 15.093    | 13.302    | 3.704     | 11.146    |
| 0.8             | 4.206     | 11.245    | 8.738     | 2.266     | 6.246     |
| 1.0             | 4.003     | 10.246    | 6.778     | 1.555     | 3.984     |
| 1.4             | 4.474     | 8.935     | 5.509     | 0.890     | 2.021     |
| 2.0             | 4.013     | 8.468     | 4.995     | 0.501     | 0.983     |
| 3.0             | 4.049     | 8.318     | 4.693     | 0.264     | 0.433     |
| 4.0             | 4.206     | 9.041     | 4.508     | 0.167     | 0.242     |

Table 2. The buckling parameter of plate with different boundary condition with eight aspect ratios.

From numerical results, the harder boundary condition produces the higher level of critical buckling load and the higher aspect ratio produces the lower level of critical buckling load. The buckling modes for the plates ( \( \frac{a}{b} = 1.0, 2.0, 3.0, 4.0 \) ) are illustrated in Figure 9. From numerical results, the boundary conditions significantly effect on not only the critical buckling loads but also the buckling modes. We can see that the plate buckling propagates into the flexible boundary edge and the integrity of the plate with softer boundary will be collapsed earlier than the plate with harder boundary condition.

4.4 Rectangular plate under the biaxial stress

The buckling behaviour of the plate under biaxial load is investigated. Twenty cases with the combination of ten different aspect ratios and two stress ratios are used in the test. The fundamental buckling loads are calculated and the dimensionless buckling parameters are calculated by using (32) and the results are illustrated in Figure 10. The buckling mode shapes are illustrated in Figure 11.

From numerical result, the biaxial stress produces the lower level of buckling load compare to the uniaxial stress. It means that the
stress distribution can significantly affect on the critical buckling load. In addition, the buckling mode shapes are greatly affected by the existence of lateral axial load so that the plates have only one half waves regardless of aspect ratios from 1 to 4. This means that the introduction of pre-stressing in plates can completely change the characteristics of plate buckling mode shapes.

5. CONCLUSIONS

The enhanced 9-node plate element is introduced to predict the buckling loads and mode shapes of rectangular plates. The formulation of the enhanced 9-node plate element for linear buckling analysis is described in detail. A series of the linear buckling analysis of rectangular plates is then carried out with various important parameters such as width-thickness ratio, different boundary condition, different aspect ratio and biaxial stress ratios. From numerical tests, the following conclusions are readily drawn:

- In very thin plate situation (\(h/\bar{h} = 1000\)), the present element shows a very good agreement with the closed form solution based on thin plate theory regardless of aspect ratios.
- Assumed strain adopted in the present element can rectify the shear locking phenomenon inherited in the thick plate element as summarized in Section 4.2 and the FE solutions with assumed strains, which is close to the 3D solution, are newly provided.
- The rectangular plate with the combination of four aspect ratios (\(a/b = 1.0, 2.0, 3.0, 4.0\)) and four thickness values (\(h = 0.001, 0.05, 0.1, 0.2\)) are used to produce new FE solutions with assumed strains and summarized in Section 4.2.
- The effect of boundary condition on the buckling load of rectangular plate is investigated and the results are summarized as new reference solutions. From numerical results, the harder boundary condition produces the higher level of buckling load. It is also found to be that the buckling mode shape is greatly influenced by boundary conditions.
- Additional application of lateral load can trigger the lower level of buckling load and it can completely change the fundamental buckling mode shapes of rectangular plates with large aspect ratios.

From numerical tests, the present enhanced 9-node plate element can successfully produce the accurate values of buckling load for the rectangular plates. Any locking phenomenon does not appear and all numerical results possess enough accuracy regardless of any thickness values, aspect ratios and boundary conditions. As further study, the present element can be possibly extended to the problem dealing with branched plates and the plate with cutout.

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