Dynamic and static response of a slightly compressible hyperelastic solid

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Abstract. The paper presents dynamic and static response of a slightly compressible hyperelastic solid. Material models under consideration are obtained by extending incompressible material models which are consistent second and third order approximation of existing stored energy function in terms of a deformation measure, namely MCMV and MCIZ. The problems of a uniaxial compression of a rod and a twisting column are solved using ABAQUS which is a powerful finite element program designed for general use in nonlinear problems. The models are defined with a help of user subroutine UHYPER. In the case of including dynamic effects, an implicit integration scheme available in the software is used. For the twisting column problem, results are compared against available in the ABAQUS library polynomial model where the volumetric stored energy function does not meet basic growth conditions. Experimental data on synthetic rubber neoprene published in the literature is utilized.

1. Introduction
The aim of this paper is to present dynamic and static response of slightly compressible hyperelastic material models [1]. Discussed models are obtained by extending incompressible material models which are consistent second and third order approximations of existing stored energy function in terms of a deformation measure. These are widely applied to model rubber-like materials, e.g. elastomers which are characterized by the volumetric compressibility modulus, couple orders of magnitude larger than the shear modulus, $K_0 \gg \mu_0$ [2]. Therefore, the stored energy function (SEF) is assumed to be sum of two functions: the function of the isochoric deformation $\bar{W}$ and the volumetric one $W_{vol}$ [3]. It means that volumetric and isochoric parts of the energy are decoupled. Such approach leads to a clear physical interpretation of basic experimental tests when determining material parameters.

We focus on material models, namely MCMV and MCIZ, that were proposed in [4,5]. Example initial-value problems, which concern a simple compression of a rod and a twisting column, are solved using ABAQUS/Standard which is a finite element program designed for general use in nonlinear problems [6,7]. In the case of including dynamic effects, we use an implicit integration scheme available in the software. Since the models are not available in ABAQUS library, we utilize a user subroutine UHYPER which is suitable for definition of isotropic material model. Moreover, in the second example we compare results against available in the library polynomial model where the volumetric stored energy function does not meet basic growth conditions [8]. We use experimental data on synthetic rubber neoprene published in the literature [9,10].
2. Basic equations of hyperelasticity

A description of finite deformations in the framework of continuum mechanics requires to distinct configurations of a deformed and undeformed body. As a consequence, different tensors appear in equilibrium equations, i.e. the first Piola-Kirchhoff stress tensor \( S \) (unsymmetric) and the Cauchy stress tensor \( \sigma \) (symmetric) in the reference configuration and in the spatial one, respectively. The equations of elastodynamics, which may be written in the Lagrangian (reference) and Eulerian (spatial) coordinates, with absence of body forces have the form

\[
\rho_0 \frac{\partial^2 \mathbf{u}}{\partial t^2} = \text{Div} S, \quad \rho \text{div} \mathbf{u} = \text{div} \sigma
\]

where \( \rho_0, \rho \) are densities in the reference and actual configurations, \( \mathbf{u} = \mathbf{u}(t, \mathbf{X}) \) is the displacement vector. ‘Div’ and ‘div’ denote the divergence operator in the Lagrangian and Eulerian coordinates, respectively [8].

A deformation of a body is described by so called deformation gradient given by

\[
F = \frac{\partial \mathcal{X}(\mathbf{x}, t)}{\partial \mathbf{X}}, \quad \dot{F} = \mathbf{L} F, \quad \mathbf{L} = \frac{\partial \mathbf{u}}{\partial \mathbf{x}} = \frac{\partial \mathbf{u}}{\partial \mathbf{x}}
\]

with the relation between the Eulerian and the Lagrangian coordinates systems \( \mathbf{u}(t, \mathbf{X}) = \mathbf{x} - \mathbf{X} \). The velocity gradient tensor is denoted by \( \mathbf{L} \). Since \( J = \det F > 0 \), the mapping \( \mathbf{x} = \mathcal{X}(\mathbf{X}, t) \) is orientation-preserving and locally invertible. The polar decomposition states \( F = \mathbf{R} \mathbf{U} = \mathbf{V} \mathbf{F} \), which means that any \( F \) can be multiplicatively decomposed into a rotation tensor \( \mathbf{R} \in SO(3) \) and \( \mathbf{U} = \sqrt{\mathbf{F}^T \mathbf{F}}, \mathbf{V} = \sqrt{\mathbf{FF}^T} \) the stretch tensors. Assumption of existence of a stored energy function \( W = W(F) \) such that

\[
S = \frac{\partial W(F)}{\partial F}
\]

leads to the hyperelastic material model. The energy is defined per unit volume in the reference configuration. In order to avoid interpenetration of matter, it is necessary to put some restrictions on the stored energy function

\[
W(F) \to +\infty \quad \text{as} \quad J \to 0^+, \quad W(F) \to +\infty \quad \text{as} \quad \|F\| \to +\infty
\]

Here, we also assume that stored-energy function is isotropic, i.e.

\[
W(F) = \tilde{W}(C) = \tilde{W}(B) = \tilde{W}(I_1, I_2, I_3),
\]

where \( I_1 = \text{tr} \mathbf{C} = \text{tr} \mathbf{B} \), \( I_2 = \text{tr} \text{cof} \mathbf{C} = \text{tr} \text{cof} \mathbf{B} \) and \( I_3 = J^2 \) are the invariants of each of the left and right Cauchy-Green deformation tensors \( \mathbf{C} = \mathbf{F}^T \mathbf{F} \) and \( \mathbf{B} = \mathbf{F} \mathbf{F}^T \). There exists a solution for a wide class of boundary values problems associated with hyperelastic model when \( W \) has the form \( W(F) = W(\mathbf{F}, \text{cof} \mathbf{F}, \text{det} \mathbf{F}) \) for some convex \( \tilde{W} \). Such functions are said to be polyconvex [11]. We emphasize that \( \mathbf{F}, \text{cof} \mathbf{F}, \text{det} \mathbf{F} \) describe deformation of fibres, surface and volume of a body, respectively. Since \( S \) is non-symmetric, it is convenient to introduce the second Piola-Kirchhoff stress tensor

\[
T = \frac{\partial \tilde{W}(\mathbf{E})}{\partial \mathbf{E}} \bigg|_{\mathbf{E} = \mathbf{F}^T} = 2 \frac{\partial \tilde{W}(\mathbf{C})}{\partial \mathbf{C}} \bigg|_{\mathbf{C} = \mathbf{F}^T} = 2 \left( \gamma_1 I + \gamma_2 C + \gamma_3 C^2 \right)
\]

\[
\gamma_1 = \frac{\partial \tilde{W}}{\partial I_1} + \frac{\partial \tilde{W}}{\partial I_2} I_1 + \frac{\partial \tilde{W}}{\partial I_3} I_2, \quad \gamma_2 = \left( \frac{\partial \tilde{W}}{\partial I_2} + \frac{\partial \tilde{W}}{\partial I_1} I_3 \right), \quad \gamma_3 = \frac{\partial \tilde{W}}{\partial I_3}
\]
where \( E = (C - I) / 2 \) denotes the Lagrangian strain tensor. The relationship between the Cauchy stress tensor \( \sigma \), the first and second Piola-Kirchhoff ones, namely \( J\sigma = S\sigma^T = FTF^T \), implies a constitutive relation in the spatial description such that

\[
J\sigma = 2B \frac{\partial \tilde{W}(B)}{\partial B} \bigg|_{B=B^T} = 2 \left( \beta_1 I + \beta_2 B + \beta_3 B^2 \right)
\]

(7)

\[
\beta_1 = \frac{\partial \tilde{W}}{\partial I_3}, \quad \beta_2 = \frac{\partial \tilde{W}}{\partial I_1} + \frac{\partial \tilde{W}}{\partial I_2} I_1, \quad \beta_3 = -\frac{\partial \tilde{W}}{\partial I_3}
\]

From the principle of conservation of mechanical energy and the principle of conservation of mass the following work conjugations arise

\[
\frac{1}{J} \tilde{W} = \sigma \cdot D = \frac{1}{J} S \cdot F = \frac{1}{2J} T \cdot \tilde{C} = \frac{1}{J} T \cdot \mathcal{E}
\]

(8)

A symmetric part of the velocity gradient tensor is denoted by \( D \). It should be noted that \( \tilde{C} = 2F^T DF = 2\mathcal{E} \).

3. Slightly compressible hyperelastic material models

3.1. Constitutive relations of slightly compressible materials

Models of isotropic slightly compressible materials are a simple generalisation of models describing incompressible materials. In the case we assume that no coupling between the stored energy function of isochoric \( \tilde{W} \) and volumetric \( W_{vol} \) deformations occurs [1]. Therefore, it is convenient to express stored energy function in the form

\[
\tilde{W}\left(I_1, I_2, J\right) = \tilde{W}\left(I_1, I_2\right) + W_{vol}\left(J\right)
\]

(9)

where the modified invariants are given by \( I_1 = \text{tr}\tilde{B} \), \( I_2 = \text{tr}\tilde{B}^{-1} \) and \( \tilde{B} = FF^T \), \( F = J^{1/3}F \). These can be explicitly written as

\[
I_1 = J^{2/3}I_1, \quad I_2 = J^{4/3}I_2
\]

(10)

Due to the form of SEF (9), the constitutive relationship of a slightly compressible material in the current configuration yields

\[
\sigma = \frac{\partial W_{vol}}{\partial J} I + \frac{2}{J} \left( \frac{\partial \tilde{W}}{\partial I_1} \tilde{B} + \frac{\partial \tilde{W}}{\partial I_2} \tilde{B}^{-1} \right)
\]

(11)

Tensor \( \tilde{B}_D \) stands for deviatoric part of the modified left Cauchy-Green tensor \( \tilde{B} \) as follows

\[
\tilde{B}_D = \tilde{B} - \frac{1}{3} I_1, \quad \tilde{B}_D^{-1} = \tilde{B}^{-1} - \frac{1}{3} I_2
\]

(12)

3.2. The MCMV material model

In the article we consider so called MCMV model [4] which isochoric part is defined by the stored energy function of the form
\[ W(\bar{\mathbf{C}}) = b + c \operatorname{tr}\bar{\mathbf{C}} + d_1 (\operatorname{tr}\bar{\mathbf{C}})^2 + d_2 \operatorname{tr}\bar{\mathbf{C}}^2 + e_1 (\operatorname{tr}\bar{\mathbf{C}})^3 + e_2 \operatorname{tr}\bar{\mathbf{C}}\operatorname{tr}\bar{\mathbf{C}}^2 \]  
\hspace{4cm} (13)

The function is a consistent third order approximation in terms of deformation tensor \( \bar{\mathbf{C}} \) norm such that

\[ |W(\bar{T}_1, \bar{T}_2) - \bar{W}(\bar{T}_1, \bar{T}_2)| = O\left(||\bar{\mathbf{C}}||^4\right) \]  
\hspace{4cm} (14)

Assuming the existence of natural state

\[ \bar{W}(I) = b + 3c + 9d_1 + 9d_2 + 27e_1 + 27e_2 = 0 \]  
\hspace{4cm} (15)

we eliminate one of the constants obtaining

\[ \bar{W}(\bar{\mathbf{C}}) = c (\operatorname{tr}\bar{\mathbf{C}} - 3) + d_1 \left[(\operatorname{tr}\bar{\mathbf{C}})^2 - 9\right] + d_2 (\operatorname{tr}\bar{\mathbf{C}}^2 - 9) + \\ + e_1 \left[(\operatorname{tr}\bar{\mathbf{C}})^3 - 27\right] + e_2 (\operatorname{tr}\bar{\mathbf{C}}\operatorname{tr}\bar{\mathbf{C}}^2 - 27) \]  
\hspace{4cm} (16)

Substituting definitions of the invariants into (16) yields five parameters model

\[ \bar{W}(\bar{\mathbf{C}}) = \bar{\mathbf{W}}(\bar{T}_1, \bar{T}_2) = \\
\frac{1}{2} \left[a_1 (\bar{T}_1 - 3) + \frac{1}{2} a_2 (\bar{T}_1^2 - 9) + \frac{1}{3} a_3 (\bar{T}_1^3 - 27) + a_4 (\bar{T}_2 - 3) + a_5 (\bar{T}_2^2 - 9)\right] \]  
\hspace{4cm} (17)

Similarly, as a special case of (17) we obtain isochoric part of SEF of the Ishihara-Zahorski (MCIZ) model [5]

\[ \bar{W}(\bar{T}_1, \bar{T}_2) = \frac{1}{2} \left[a_1 (\bar{T}_1 - 3) + \frac{1}{2} a_2 (\bar{T}_1^2 - 9) + a_4 (\bar{T}_2 - 3)\right] \]  
\hspace{4cm} (18)

which states the second order approximation.

The part of SEF describing volumetric changes should meet certain mathematical conditions. Firstly, we assume that \( W_{\text{vol}}(J) \) is differentiable with respect to \( J \) and \( W_{\text{vol}}(1) = 0 \). Based on assumption of existence of natural state and convexity of the function we have

\[ \frac{\partial W_{\text{vol}}(J)}{\partial J} \bigg|_{J=1} = 0, \quad \frac{\partial^2 W_{\text{vol}}(J)}{\partial J^2} \geq 0 \]  
\hspace{4cm} (19)

To ensure proper growth, the conditions should hold

\[ W_{\text{vol}}(J \to 0) = +\infty, \quad \frac{\partial W_{\text{vol}}(J)}{\partial J} \bigg|_{J=0} = -\infty \]
\hspace{4cm} (20)

\[ W_{\text{vol}}(J \to +\infty) = +\infty, \quad \frac{\partial W_{\text{vol}}(J)}{\partial J} \bigg|_{J=+\infty} = +\infty \]

For such models the initial bulk modulus is given by

\[ K_0 = \frac{\partial^2 W_{\text{vol}}(J)}{\partial J^2} \bigg|_{J=1} \]  
\hspace{4cm} (21)

In the case of rubber-like materials, the initial bulk modulus is several orders of magnitude greater than the initial shear modulus. Therefore, these are typically described taking into consideration the
isochoric-volumetric split. The approach significantly simplifies interpretation of basic experimental results for such materials.

Figure 1 presents nominal stress vs stretch plots for synthetic rubber neoprene [9] in the case of incompressibility. Parameter values are summarized in table 1. For more details we refer the reader to [10].

### Table 1. Parameters of MCMV and MCIZ for neoprene [9,10].

| Model | $a_1$ [MPa] | $a_2$ [MPa] | $a_3$ [MPa] | $a_4$ [MPa] | $a_5$ [MPa] |
|-------|-------------|-------------|-------------|-------------|-------------|
| MCIZ  | 2.486 $10^{-1}$ | 3.054 $10^{-4}$ | -           | 1.839 $10^{2}$ | -           |
| MCMV  | 3.402 $10^{-1}$ | -9.644 $10^{-3}$ | 1.685 $10^{4}$ | 2.902 $10^{2}$ | -1.762 $10^{4}$ |

**Figure 1.** Nominal stress vs principal stretch for uniaxial (UT) and biaxial (BT) tensile test for neoprene in the case of incompressibility.

In the literature, there are a large number of propositions of volumetric stored energy function that meet the mathematical conditions. For examples presented in the following part of the article we choose function of the form

\[
W_{vol}(J) = K_v \left[ \frac{1}{4} (J^2 - 1) - \frac{1}{2} \ln J \right]
\]  

(22)

We leave it to the reader to verify that the function meets discussed conditions.

### 4. Example initial-boundary value problems

In this section we present solutions of two initial-boundary value problems which illustrates application of the discussed models. The example problems are solved using ABAQUS/Standard which is a finite element program designed for general use in nonlinear problems [6,7]. In case of including dynamic effects, we use an implicit integration scheme available in the software. The MCMV and MCIZ models are defined with a help of user subroutine UHYPER.

#### 4.1. Uniaxial compression of a rod

The first problem illustrating the behaviour of the MCMV material model is a simple uniaxial compression of a rectangular rod with dimensions $L \times L / 20 \times L / 20$. The FEM model consists of 450 C3D8H (hybrid formulation) elements. The initial bulk modulus is step up to be
\( K_0 = 1000 \mu_0 = 340.74 \text{ MPa} \) and density is assumed to be \( \rho = 1000 \text{ kg/m}^3 \) in the case of initial-boundary value problem. Values of the parameters are given in table 1.

Over the marked face A, the prescribed displacement is applied \( u_3 = -L/2 \). On the face B \( u = 0 \) is imposed and zero traction is prescribed over the other faces. In the case of dynamic problem, zero initial velocity is given which means that dynamic effect is due to the inertial forces.

![Figure 2. Finite element mesh for uniaxial compression of a rectangular rod.](image)

**Figure 3.** Plot of averaged stress \( \bar{\sigma} \) as a function normalized displacement \( u_3 / L \).

Figure 3 shows plots of averaged stress \( \bar{\sigma} \) as a function of displacement of the face A. The averaged stress values are obtained by averaging values of the component \( \sigma_{33} \) from the Gauss points of finite elements at face B. In the case of the static problem, we obtain solution which results with a simple uniaxial deformation. This is not unique solution because of nonlinear nature of the problem. Taking into consideration dynamics effects, we observe a loss of stability at \( u_3 / L = -0.4 \) after uniaxial deformation. In order to obtain the solution in the static case an initial imperfection should be applied.
4.2. **Twisting column**

The second example concerns a twisting column 100x100x600 mm clamped on its bottom face [12]. An initial sinusoidal rotational velocity field relative to the origin is given by

\[ \omega_0 = \begin{bmatrix} 0, 0, \Omega \sin \left( \frac{\pi Z}{200} \right) \end{bmatrix}^T \]  

(23)

where \( \Omega = 300 \) rad/s represents the magnitude of initial velocity. Finite elements C3D8H with hybrid formulation are used for the problem. Material parameters and density are assumed to be the same as for the previous example.

Figure 5 presents change of a height of the column for different models. We observe that there is no significant difference between proposed volumetric stored energy function and the one available in ABAQUS. Results of MCMV and MCIZ do not align, because mainly isochoric deformation occurs. However, there a significant difference between values of total energy when comparing the models. The initial bulk modulus \( K_0 \) has high impact on the total energy, see figure 7.

![Figure 4. Twisting column – mesh and initial velocity.](image)

![Figure 5. Twisting column: change of a height of the column for different models.](image)
Figure 6. Twisting column: a sequence of deformations with Huber-Mises stress map [MPa].
Figure 7. Twisting column: plots of total energy.
5. Conclusions
The article is intended as an attempt to present a dynamic and static response of certain slightly compressible hyperelastic models. These are obtained by extending incompressible material models which are consistent second and third order approximations of existing stored energy function in terms of a deformation measure, namely MCMV and MCIZ models. The approach leads to clear physical interpretation of a response of a material, i.e. volumetric and isochoric parts of deformation are decoupled. The description is especially suitable for rubber-like material, because of their nearly incompressibility. Moreover, the approach significantly simplifies interpretation of experimental results for such materials.

Example initial-value problems, which concern a simple compression of a rod and a twisting column, are solved using ABAQUS/Standard. In the case of including dynamic effects, we use an implicit integration scheme available in the software. Since the models are not available in ABAQUS library, we utilize a user subroutine UHYPER which is suitable for definition of isotropic material model. Moreover, in the second example we compare results against available in the library polynomial model where the volumetric stored energy function does not meet basic growth conditions.

In the case of the static uniaxial compression of a rod, we obtain solution which results with a simple uniaxial deformation. This is not unique solution because of nonlinear nature of the problem. Taking into consideration dynamics effects, we observe a loss of global stability after uniaxial deformation. In order to obtain the solution in the static case an initial imperfection should be applied. In the case of dynamically twisted columns we observe that there is no significant difference between discussed models when observing change of a height of the column. Results of MCMV and MCIZ do not align, because mainly isochoric deformation occurs. However, there a significant difference between values of total energy when comparing the models. The initial bulk modulus has high impact on the total energy.

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