Regular independent sets

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Abstract

The regular independence number, introduced by Albertson and Boutin in 1990, is
the size of a largest set of independent vertices with the same degree. Lower bounds
were proven for this invariant, in terms of the order, for trees and planar graphs. In
this article, we generalize and extend these results to find lower bounds for the regu-
lar $k$-independence number for trees, forests, planar graphs, $k$-trees and $k$-degenerate
graphs.

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1 Introduction and benchmark bounds

Albertson and Boutin \cite{1} introduced the parameter $\alpha_{\text{reg}}(G)$ as the maximum cardinality
of an independent set in a graph $G$ in which all vertices have equal degree in $G$. An
independent set whose vertices all have equal degree in $G$ is called a regular independent
set. A $k$-independent set is a set of vertices whose induced subgraph has maximum degree at
most $k$. Let us define the regular $k$-independence number, denoted $\alpha_{k-\text{reg}}(G)$, as the size of
a largest $k$-independent set of vertices which have the same degree in $G$. More particularly,
we denote by $\alpha_{k,j}(G)$ the size of a largest $k$-independent set in the subgraph induced by the
vertices of degree $j$ in $G$. Thus, $\alpha_{k-\text{reg}}(G) = \max\{\alpha_{k,j}|\delta \leq j \leq \Delta\}$, where $j$ is an integer,
$\delta$ is the minimum degree and $\Delta$ is the maximum degree. Now, when $k = 0$, we see that
$\alpha_{0-\text{reg}}(G) = \alpha_{\text{reg}}(G)$. In \cite{1}, the authors proved the following:

(1) If $G$ is a planar graph on $n$ vertices, then $\alpha_{\text{reg}}(G) \geq \frac{2}{5}n$.

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If $G$ is a maximal planar graph on $n$ vertices, then $\alpha_{\text{reg}}(G) \geq \frac{3}{61}n$.

(3) If $T$ is a tree on $n$ vertices, then $\alpha_{\text{reg}}(T) \geq \frac{n+2}{4}$ and this is sharp.

(4) If $G$ is a connected graph on $n$ vertices and maximum degree $\Delta$, then $\alpha_{\text{reg}}(G) \geq \frac{n}{\left(\Delta+1\right)}$ and this is sharp.

They left open the problems whether the results in (1) and (2) are best possible and constructed an example of maximal planar graphs $G$ for which $\alpha_{\text{reg}}(G) = \frac{n}{16}$. Our intention is a far reaching extension of the issue raised by Albertson and Boutin, namely we extend the outlook to a broader families of graphs, including forests, $k$-trees and $k$-degenerate graphs, while on the way improving item (1) for minimum degree $\delta = 2, 3, 4, 5$ and item (2) for minimum degree $\delta = 4, 5$. We also extend the problem from regular independent sets to regular $k$-independent sets. We mention in passing that related papers were written (that came from distinct inspiration), see e.g. [4, 5, 7], also using induced forests (see [1, 3]) and its many interesting descendants, which, in conjunction with [7] and averaging arguments, give some results which are still weaker than the current approach.

Note also that the parameter $\alpha_{\text{reg}}(G)$ is closely related to the newly introduced parameter $\chi_k(G)$, the fair domination number $\text{fd}(G)$. A fair dominating set $S \subseteq V(G)$ is a dominating set such that all vertices $v \in V(G) \setminus S$ have exactly the same number of neighbors in $S$. The fair domination number $\text{fd}(G)$ is the cardinality of a minimum fair dominating set of $G$. When $\delta(G) \geq 1$ and $R$ is a maximum regular independent set of $G$, then $V(G) \setminus R$ is a fair dominating set and hence $\text{fd}(G) \leq n - \alpha_{\text{reg}}(G)$. Hence, any lower bound on $\alpha_{\text{reg}}(G)$ yields an upper bound on $\text{fd}(G)$ (and any lower bound on $\text{fd}(G)$ yields an upper bound on $\alpha_{\text{reg}}(G)$).

The following proposition gives a basic result, part of which already appeared in the literature ([2]). This, together with the above mentioned results of Albertson and Boutin, serves as a benchmark to our work. Hereby, we define $\text{rep}(G)$ as the maximum number of vertices with equal degree, while $\chi_k(G)$ is the $k$-chromatic number of $G$, i.e. the minimum number of colors needed to color the vertices of the graph $G$ such that the graphs induced by equally colored vertices have maximum degree at most $k$.

**Proposition 1.1** Let $G$ be a graph with average degree $d$ and minimum degree $\delta$. Then

$$\alpha_{k-\text{reg}}(G) \geq \frac{\text{rep}(G)}{\chi_k(G)} \geq \frac{n}{(2d - 2\delta + 1)\chi_k(G)}.$$ 

**Proof.** Let $R$ be a set of vertices of equal degree such that $|R| = \text{rep}(G)$. Let $R = \bigcup_{i=1}^{t} R_i$ be a partition of the set $R$ into $\chi_k(G) = t$ $k$-independent sets $R_i$, $1 \leq i \leq t$. Then we obtain

$$\alpha_{k-\text{reg}}(G) \geq \max\{|R_i| : 1 \leq i \leq t\} \geq \frac{|R|}{\chi_k(G)} = \frac{\text{rep}(G)}{\chi_k(G)}.$$ 

The bound $\text{rep}(G) \geq \frac{n}{2d - 2\delta + 1}$ proved in [3] gives the second inequality. $\square$
This paper is organized as follows. After this introduction section, we deal in Section 2 with trees and forests, where we generalize and extend the results of Albertson and Boutin in [2] to $\alpha_{k-reg}(G)$. In Section 3, we present a lower bound on $\alpha_{reg}(G)$ for $k$-trees and give analogous results for $k$-degenerate graphs and planar graphs, refining hereby the method used in [2] and generalizing and improving the bounds on $\alpha_{reg}(G)$ given there, too. In Section 4 we give lower bounds on $\alpha_{2-reg}(G)$ for planar and outerplanar graphs. Finally, Section 5 collects some open problems on regular $k$-independent sets.

2 Trees and forests

In this section, we generalize and extend the result that $\alpha_{reg}(T) \geq \frac{n+2}{4}$ for any tree $T$, obtained by Albertson and Boutin in [2], to regular $k$-independence number in both trees and forests. These results appear below. Here and throughout, we will use $n_i(G)$ to denote the number of vertices of degree $i$ in $G$. When the context is clear, $n_i(G)$ will be abbreviated to $n_i$.

Theorem 2.1 For every tree $T$ on $n \geq 2$ vertices,

(i) $\alpha_{reg}(T) \geq \frac{n+2}{4}$ (Albertson and Boutin [2]),
(ii) $\alpha_{1-reg}(T) \geq \frac{2(n+2)}{7}$, and
(iii) $\alpha_{k-reg}(T) \geq \frac{n+2}{3}$ for $k \geq 2$.

Moreover, all bounds are sharp.

Proof. Let $T$ be a tree on $n \geq 2$ vertices. The proof of item (i) and the sharpness of the result are found in [2], and are only mentioned here for completeness.

To prove (ii), we first assume that $n_1 \geq \frac{2(n+2)}{7}$. Since the subgraph induced by the vertices of degree one is a 1-independent set, $\alpha_{1,1} = n_1$. However, since $\alpha_{1-reg} \geq \alpha_{1,1}$ by definition, we are done. Next, we assume that $n_1 < \frac{2(n+2)}{7}$. Denote by $N_3$ the set of vertices in $T$ of degree at least 3 (these are the branch points of $T$). Then, we make use of the observations that $n_1 \geq N_3 + 2$ and $n_2 = n - n_1 - N_3$ to deduce that $n_2 \geq n - 2n_1 + 2$. Finally, since the vertices of degree two in $T$ induce a collection of paths, and the 1-independence number of a path is at least $\frac{2}{3}$ of its order, $\alpha_{1,2}(T) \geq \frac{2}{3}n_2$. Putting all this together, we complete the proof as follows:

$$\alpha_{1-reg}(T) \geq \alpha_{1,2}(T) \geq \frac{2}{3}n_2 \geq \frac{2}{3}(n - 2n_1 + 2) > \frac{2}{3} \left( n - \frac{4(n+2)}{7} + 2 \right) = \frac{2(n+2)}{7}.$$ 

To see that this bound is sharp, let $p$ be a positive integer and $n = 7p + 5$. Start with a path $P$ with $7p+5$ vertices and two copies, $T_1$ and $T_2$, of a tree with $4\sqrt{7p}$ vertices which has a unique degree two vertex and maximum degree at most three (these trees exist since we
can start with a path on three vertices and continually add two leaves to an endpoint. Now join one endpoint of \( P \) to the unique degree two vertex of one \( T_1 \) and the other endpoint of \( P \) to the unique degree two vertex of \( T_2 \). Now, for this new tree we constructed, \( n_1 = \frac{2n_2+4}{7}, n_2 = \frac{2n_3+6}{7}, n_3 = \frac{2n_1-10}{7} \), and \( \alpha_{1-reg} = \alpha_{1,1} = \alpha_{1,2} = \frac{2n+4}{7} \) — which shows the bound is sharp (see Figure 1).

![Figure 1: Trees with \( \alpha_{1-reg}(T) = \frac{2(n(T)+2)}{7} \). The gray and the black vertices each form a different \( \alpha_{1-reg}(T) \)-set.](image-url)

To prove (iii), let \( k \geq 2 \) and assume first that \( n_1 \geq \frac{n+2}{3} \). Since the subgraph induced by the vertices of degree one is a 1-independent set, and therefore also a \( k \)-independent set, \( \alpha_{k-reg} \geq \alpha_{k,1} = n_1 \), and so we are done in this case. Next, we assume that \( n_1 < \frac{n+2}{3} \). Now, we have as before that \( n_2 \geq n - 2n_1 + 2 \), since \( n_1 \geq N_3 + 2 \) and \( n_2 = n - n_1 - N_3 \). Finally, since the subgraph induced by the vertices of degree two is a 2-independent set, we complete the proof as follows:

\[
\alpha_{k-reg}(T) \geq \alpha_{k,2}(T) = n_2 \geq n - 2n_1 + 2 \geq n - \frac{2n + 4}{3} + 2 = \frac{n + 2}{3}.
\]

To see that this bound is sharp, we observe that the following family of graphs gives us examples of equality. Let \( p \) be a non-negative integer. Starting with a path on \( r = 2p + 4 \) vertices labeled 1, 2, \ldots, \( r \), attach a leaf (or pendant vertex) to each of the first \( p \) vertices with even labels (the first graph in this family is the path on four vertices). This family of trees has the following properties; \( n_1 = n_2 = n_3 + 2 = p + 2, n = 3p + 4, \) and
\[ \alpha_{k \text{-reg}} = \alpha_{k,1} = \alpha_{k,2} = p + 2 = \frac{n+2}{3} \] – which shows the bound is sharp (see Figure 2). \( \Box \)

Figure 2: Trees with \( \alpha_{k \text{-reg}}(T) = \frac{n(T)+2}{3} \) for \( k \geq 2 \). The gray and the black vertices each form a different \( \alpha_{k \text{-reg}}(T) \)-set.

**Theorem 2.2** For every forest \( F \),

(i) \( \alpha_{\text{reg}}(F) \geq \frac{n+2}{3} \),

(ii) \( \alpha_{1 \text{-reg}}(F) \geq \frac{2(n+2)}{9} \), and

(iii) \( \alpha_{k \text{-reg}}(F) \geq \frac{n+2}{4} \) for \( k \geq 2 \).

Moreover, all bounds are sharp.

**Proof.** Let \( F \) be a forest with \( n \) vertices, \( n_0 \) isolated vertices, \( t \) isolated edges (isolated complete graphs on two vertices) and \( r \) trees with at least three vertices. If \( n \leq 2 \), all three parts are trivially true, so we may assume throughout the proof that \( n \geq 3 \). Furthermore, we again make use of the equation \( n = n_0 + n_1 + n_2 + N_3 \), where \( N_3 \) is the number of vertices of degree three or more.

To prove (i), we first suppose \( r = 0 \), which in turn implies that \( n = n_0 + 2t \). Now, if \( n_0 > \frac{n}{3} \), we are done since the degree zero vertices are independent and \( \frac{n}{3} \geq \frac{n_0}{3} \) when \( n \geq 3 \). So assume \( n_0 \leq \frac{n}{3} \). This implies \( 2t = n - n_0 \geq n - \frac{n}{3} = \frac{2n}{3} \) so that \( t \geq \frac{n}{3} \). Now, since taking one vertex of degree one from each of the \( t \) isolated edges yields an independent set, \( \alpha_{\text{reg}} \geq t \geq \frac{n}{3} \geq \frac{n+2}{4} \) when \( n \geq 3 \). This proves (i) when \( r = 0 \), so we may assume that \( r \geq 1 \).

Let \( T_1, \ldots, T_r \) be the \( r \) components of \( F \) with at least three vertices and let \( t_i \) denote the number of vertices of degree one in \( T_i \). Then, we have the following,

\[ n = n_0 + n_1 + n_2 + N_3 = n_0 + 2t + \sum_{i=1}^{r} t_i + n_2 + N_3. \] (1)

Note that in each component tree \( T_i \), the number of vertices of degree at least three is at most \( t_i - 2 \), with equality holding if and only if no vertices in that component have degree four or more – equivalently, \( n_3 = N_3 \). Hence, using also Equation (1) we get;

\[ N_3 \leq \sum_{i=1}^{r} (t_i - 2) = \sum_{i=1}^{r} t_i - 2r = n_1 - 2(t + r). \] (2)
Combining Equations 1 and 2

\[ n = n_0 + n_1 + n_2 + N_3 \leq n_0 + 2n_1 + n_2 - 2(t + r) \]

so that,

\[ n_2 \geq n - n_0 - 2n_1 + 2(t + r). \]  

(3)

Since the degree zero vertices are independent, if \( n_0 > \frac{n+2}{5} \), we are done – so assume \( n_0 \leq \frac{n+2}{5} \). Next, if \( n_1 > \frac{n+2+5t}{5} = \frac{n+2-5t}{5} + 2t \), then there are at least \( \frac{n+2+5t}{5} + t = \frac{n+2}{5} \) independent vertices of degree one, implying \( \alpha_{reg} \geq \frac{n+2}{5} \). So we may also assume that \( n_1 \leq \frac{n+2+5t}{5} \). Under these assumptions, and using Equation 3, we get:

\[ n_2 \geq n - \frac{n + 2}{5} - \frac{2(n + 2 + 5t)}{5} + 2(t + r) = \frac{2n}{5} - \frac{6}{5} + 2r. \]

Clearly, at least half of the degree two vertices are an independent set so that \( \alpha_{reg} \geq \frac{n+2}{5} \). Finally, because in this case, \( r \geq 1 \),

\[ \alpha_{reg} \geq \frac{n_2}{2} \geq n - \frac{3}{5} + r \geq n - \frac{3}{5} + 1 = \frac{n + 2}{5}, \]

which completes the proof of (i).

To see that this bound is sharp, consider the following family. Starting with a tree \( T \) on \( 2p \) vertices all having degree one or three, subdivide each edge joining a leaf to its support vertex twice and call this tree \( T^\ast \). Clearly, in both \( T \) and \( T^\ast \), \( n_1 = p + 1 \) and \( n_3 = p - 1 \). On the other hand, in \( T^\ast \), \( n_2 = 2n_1 = 2p + 2 \). Thus, \( T^\ast \) has a total of \( 4p + 2 \) vertices, where the degree two vertices induce a matching. Now add \( p + 1 \) isolated vertices to make a forest \( F \) of order \( n = 5p + 3 \) where \( n_0 = n_1 = \frac{n_2}{2} = n_3 = 2 + p = 1 + n + 2 \). Since \( n_3 < p + 1 \), the degree zero vertices are independent, the degree one vertices are independent, and exactly half of the degree two vertices are independent, \( \alpha_{reg}(F) = \frac{n+2}{5} \) – which shows the bound is sharp (see \( F_1 \) in Figure 8).

To prove (ii), notice that if either \( n_0 \geq \frac{2(n+2)}{9} \) or \( n_1 \geq \frac{2(n+2)}{9} \), we are done since both of the subgraphs induced by the vertices of degree zero and one respectively are \( 1 \)-independent sets and \( \alpha_{1-reg}(F) \geq \max\{\alpha_{1,0}, \alpha_{1,1}\} = \max\{n_0, n_1\} \). So we may assume that \( n_0 < \frac{2(n+2)}{9} \) and \( n_1 < \frac{2(n+2)}{9} \). As in the proof of the previous proposition, we use the observations that \( n_1 \geq N_3 + 2 \) and \( n_2 = n - n_0 - n_1 - N_3 \) to deduce that \( n_2 \geq n - n_0 - 2n_1 + 2 \). Finally, we use the fact that \( \alpha_{1,2}(F) \geq \frac{2}{3}n_2 \) since the degree two vertices induce a collection of paths to complete the proof as follows:

\[ \alpha_{1-reg}(F) \geq \alpha_{1,2}(F) \geq \frac{2}{3}n_2 \geq \frac{2}{3}(n - n_0 - 2n_1 + 2) \]

\[ > \frac{2}{3}(n - \frac{2(n + 2)}{9} - \frac{4(n + 2)}{9} + 2) = \frac{2(n + 2)}{9}. \]

To see that this bound is sharp, let \( p \) be a positive integer. We will construct a forest \( F \) with \( n = 9p + 7 \) vertices where equality holds. First, observe that there is a tree with exactly
\(4(n + 2) \over 9\) – 2 vertices all of which having degree one or degree three. To see this is true, note that we can start with a double star on six vertices with \(n_1 = 4\) and \(n_3 = 2\), corresponding to the case that \(n = 16\), and then attach two leaves to each of two existing leaves each time \(p\) is incremented. The trees formed this way will always have \(4p + 2 = \frac{4(n + 2)}{9} - 2\) vertices with \(2p + 2\) leaves and \(2p\) degree three vertices. Now take a path on \(\frac{3(n + 2)}{9}\) vertices and join an endpoint of this path to a leaf of the previously formed tree. Finally, add \(\frac{2(n + 2)}{9}\) isolated vertices. This graph is a forest with exactly, \(n\) vertices such that \(n_0 = \frac{2(n + 2)}{9}\), \(n_1 = \frac{2(n + 2)}{9}\), \(n_2 = \frac{3(n + 2)}{9}\), and \(N_3 = n_3 = \frac{2(n + 2)}{9} - 2\.

Since the degree two vertices induce a path whose order is a multiple of three, and the 1-independence number of such paths is easily seen to be \(\frac{2}{3}\) of its order, we find that \(\alpha \over 1, 2(F) = \frac{n(F_1) + 2}{5}\), \(\alpha \over 1 - \text{reg}(F_2) = \frac{2(n(F_2) + 2)}{9}\), and \(\alpha \over k - \text{reg}(F_3) = \frac{n(F_3) + 2}{4}\) for \(k \geq 2\). Equal color vertices, aside from the vertices colored white, form different \(\alpha \text{reg}(F_1)\)-sets, \(\alpha \text{reg}(F_2)\)-sets and \(\alpha \text{reg}(F_3)\)-sets in each case.

![Forest](image)

Figure 3: Forests satisfying equality in Theorem 2.2 \(\alpha \text{reg}(F_1) = \frac{n(F_1) + 2}{5}\), \(\alpha \text{reg}(F_2) = \frac{2(n(F_2) + 2)}{9}\), and \(\alpha \text{reg}(F_3) = \frac{n(F_3) + 2}{4}\) for \(k \geq 2\). Equal color vertices, aside from the vertices colored white, form different \(\alpha \text{reg}(F_1)\)-sets, \(\alpha \text{reg}(F_2)\)-sets and \(\alpha \text{reg}(F_3)\)-sets in each case.

To prove (iii), let \(k \geq 2\) and notice that if either \(n_0 \geq \frac{n + 2}{4}\) or \(n_1 \geq \frac{n + 2}{4}\), we are done since both of the subgraphs induced by the vertices of degree zero and one respectively are \(k\)-independent sets and \(\alpha \text{reg}(F) \geq \max\{\alpha_{k,0}, \alpha_{k,1}\} = \max\{n_0, n_1\}\). So we may assume that \(n_0 < \frac{n + 2}{4}\) and \(n_1 < \frac{n + 2}{4}\). We again make use of \(n_2 \geq n - n_0 - 2n_1 + 2\) together with \(\alpha_{k,2} = n_2\) to complete the proof as follows:

\[
\alpha_{k - \text{reg}}(F) \geq \alpha_{k,2}(F) \geq n_2 \geq n - n_0 - 2n_1 + 2 > n - \frac{n + 2}{4} - \frac{2(n + 2)}{4} + 2 = \frac{n + 2}{4}.
\]

To see that this bound is sharp, start with a path on \(q \geq 1\) vertices and attach a path of length two to each vertex. Now attach an additional leaf to one endpoint of the
initial path and add $q + 1$ isolated vertices. This family has the following properties:

\[ n_0 = n_1 = n_2 = n_3 + 2 = q + 1, \quad n = 4q + 2 \quad \text{and} \quad \alpha_{k-reg} = \alpha_{k,0} = \alpha_{k,1} = \alpha_{k,2} = q + 1 = \frac{n+2}{4} \]

which shows the bound is sharp (see $F_3$ in Figure 3).

To end this section, we would like to mention that the bounds given here for trees and forests can be extended easily to $n$-vertex graphs with $o(n)$ cycles. This can be done by deleting one by one the edges from of the $o(n)$ cycles until one is left with a tree or, in case the original graph was disconnected, with a forest. Then, a regular $k$-independent set of the obtained tree/forest can be transformed into a regular $k$-independent set of the original graph by deleting only the $o(n)$ vertices whose degree was affected in the process of deleting edges. Doing so, we obtain a regular $k$-independent set whose size is $(1 - o(1))$ times the size of the regular $k$-independent set of the tree/forest.

3 $k$-trees, $k$-degenerate graphs and planar graphs

In this section, we will refine the method used by Alberson and Boutin in [2] and we will give lower bounds on $\alpha_{reg}$ for $k$-trees, $k$-degenerate graphs and planar graphs. Since the bound on $k$-trees is the more involved one and needs some previous theory, we present it here fully, while for the other graph classes the procedure is similar and quite more simple, and so we give the obtained bounds in a table.

Throughout this and the next section, we will use $V_i(G)$ to denote the set of vertices of degree $i$ in $G$ and $n_i(G) = |V_i(G)|$. When the context is clear, $V_i(G)$ will be abbreviated to $V_i$ and $n_i(G)$ to $n_i$. Moreover, for any set $S \subseteq V(G)$, we will write $\chi_k(S)$ instead of $\chi_k(G[S])$. Recall that a $k$-tree may be formed by starting with a complete graph $K_{k+1}$ and then adding repeatedly vertices in such a way that each added vertex has exactly $k$ neighbors that form a clique. A $k$-degenerate graph is a graph all of whose induced subgraphs have minimum degree at most $k$. A maximal $k$-degenerate graph is a $k$-degenerate graph with the maximum possible number of edges, which is $kn - \frac{k(k+1)}{2}$, where $n$ is the order of the graph. Note also that a (maximal) $k$-degenerate graph may be formed by starting with a vertex (a $(k+1)$-clique) and then adding repeatedly vertices in such a way that each added vertex has at most (exactly) $k$ neighbors. In particular, a $k$-tree is a certain kind of maximal $k$-degenerate graph.

**Lemma 3.1** Let $G$ be a $k$-tree on $n = k + t + 2$ vertices, where $t \geq 1$ is an integer. Then $n_{k+t} \leq t + 1$.

**Proof.** We will prove the statement by induction on $t$. If $t = 1$ and $n = k + 3$, it is not difficult to see that there are only two non-isomorphic $k$-trees on $k + 3$ vertices and that they have at most 2 vertices of degree $k + 1$. Hence $n_{k+1} \leq 2$ and the beginning of induction is clear. Suppose that for any $k$-tree on $n - 1 = k + (t - 1) + 2$ vertices, where $t - 1 \geq 1$, there are at most $t$ vertices of degree $k + t - 1$. Let $G$ be a $k$-tree on $n = k + t + 2$ vertices and let $x$ be a vertex of degree $k$ in $G$. Let $G^* = G - x$. Then
Let $G$ be a $k$-tree of order $n \geq k + t + 2$, where $t \geq 0$ is an integer. Then

$$\chi(V_{k+t}(G)) \leq \frac{1}{2}(t^2 + t + 2).$$

**Proof.** The proof goes through induction on $t$. If $t = 0$, then, since $n \geq k + 2$, the vertices of degree $k$ form an independent set and thus we have clearly $\chi(V_k(G)) \leq 1 = \frac{1}{2}(t^2 + t + 2)$. Let $t \geq 1$. Assume that

$$\chi(V_{k+t-1}(G)) \leq \frac{1}{2}(t - 1)^2 + (t - 1) + 2 = \frac{1}{2}(t^2 - t)$$

for every $k$-tree $G$ on $n \geq k + (t - 1) + 2 = k + t + 1$ vertices. Now we will show that $\chi(V_{k+t}(G)) \leq \frac{1}{2}(t^2 + t + 2)$ for every $k$-tree $G$ on $n \geq k + t + 2$ vertices. Again, we use induction but now on $n$.

If $G$ is a $k$-tree on $n = k + t + 2$ vertices, then Lemma 3.1 yields $n_{k+t}(G) \leq t + 1$ and we obtain $\chi(V_{k+t}(G)) \leq t + 1 \leq \frac{1}{2}(t^2 + t + 2)$, which settles the beginning of induction. Assume now that $\chi(V_{k+t}(G)) \leq \frac{1}{2}(t^2 + t + 2)$ also holds. Let $x \in V(G)$ be a vertex of degree $k$ and let $G^* = G - x$. Then $n(G^*) = n, \chi(V_{k+t-1}(G^*)) \leq \frac{1}{2}(t^2 - t)$ because of (4) and, by induction, $\chi(V_{k+t}(G^*)) \leq \frac{1}{2}(t^2 + t + 2)$. Then, since $N(x)$ is a clique, $|V_{k+t-1}(G^*) \cap N(x)| \leq \frac{1}{2}(t^2 - t)$. Note that the vertices of degree $k + t$ in $G$ are either contained in $V(G) \setminus N[x]$ and have degree $k + t$ in $G^*$ or they are contained in $N(x)$ and have degree $k + t - 1$ in $G^*$. Observe also that every vertex $u \in V_{k+t}(G) \cap N(x)$ has exactly $t$ neighbors in $V(G) \setminus N[x]$ and hence it has at most $t$ neighbors in $V_{k+t}(G) \setminus N[x]$. Thus, we can transform a proper coloring of $V_{k+t}(G^*)$ with $\frac{1}{2}(t^2 + t + 2)$ colors into a proper coloring of $V_{k+t}(G)$ with $\frac{1}{2}(t^2 + t + 2)$ colors the following way. Every vertex contained in $V_{k+t}(G) \setminus N[x] = V_{k+t}(G^*) \setminus N[x]$ remains colored with the same color. Since each vertex of $V_{k+t}(G) \cap N(x)$ has at most $t$ neighbors in $V_{k+t}(G) \setminus N[x]$ and since $|V_{k+t}(G) \cap N(x)| = |V_{k+t-1}(G^*) \cap N(x)| \leq \frac{1}{2}(t^2 - t)$, it follows that the vertices of $V_{k+t}(G) \cap N(x)$ have at most $\frac{1}{2}(t^2 - t) - 1 + t < \frac{1}{2}(t^2 + t + 2)$ neighbors of degree $k + t$ in $G$. Hence, coloring the vertices of $V_{k+t}(G) \cap N(x)$ one after the other, we can assign each vertex a color that does not appear on its neighbors.

Hence, it follows by induction that $\chi(V_{k+t}(G)) \leq \frac{1}{2}(t^2 + t + 2)$ for every $k$-tree on $n \geq k + t + 2$ vertices. □
Remark 3.3 Note that in Lemma 3.2 we prove something stronger, namely that, given a $k$-tree $G$ of order $n \geq k + t + 2$, $t \geq 0$, the graph $G[V_{k+t}]$ is $\frac{1}{2}(t^2 + 2)$-degenerate. Hence, clearly $\chi(V_{k+t}) \leq \frac{1}{2}(t^2 + t + 2)$.

Lemma 3.2 yields us the following corollary.

Corollary 3.4 Let $G$ be a $k$-tree on $n \geq k + q(k) + 2$ vertices, where $q(k) = \lfloor \frac{-1 + \sqrt{8k - 7}}{2} \rfloor$. Then

$$n_{k+t}(G) \leq \frac{1}{2} (t^2 + t + 2) \alpha_{\text{reg}}(G), \quad \text{for } 0 \leq t \leq q(k),$$

and

$$n_{k+t}(G) \leq (k+1) \alpha_{\text{reg}}(G), \quad \text{for } t \geq q(k) + 1.$$

Proof. Let $q(k)$ be the maximum integer for which $\frac{1}{2} (q(k)^2 + q(k) + 2) \leq k$. Then $q(k) = \lfloor \frac{-1 + \sqrt{8k - 7}}{2} \rfloor$. Since $n_{k+t} \leq \chi(V_{k+t}(G)) \alpha_{\text{reg}}(G)$, Lemma 3.2 yields

$$n_{k+t}(G) \leq \frac{1}{2} (t^2 + t + 2) \alpha_{\text{reg}}(G)$$

for $0 \leq t \leq q(k)$. For $t \geq q(k) + 1$, we use the fact that every $k$-tree on $n \geq k + 3$ vertices is $(k+1)$-colorable and we obtain $n_{k+t} \leq \chi(V_{k+t}(G)) \alpha_{\text{reg}}(G) \leq (k+1) \alpha_{\text{reg}}(G)$. □

Corollary 3.4 enables us to compute a bound on $\alpha_{\text{reg}}(G)$ for $k$-trees, which will be given in the next theorem. The proof of this theorem, adapted for each particular case, contains the essence of the computations of all other bounds that are given further on for planar graphs, outerplanar graphs and $k$-degenerate graphs.

Theorem 3.5 Let $G$ be a $k$-tree on $n \geq k + q(k) + 2$ vertices, where $q(k) = \lfloor \frac{-1 + \sqrt{8k - 7}}{2} \rfloor$ and $k \geq 2$. Then

$$\alpha_{\text{reg}}(G) > \frac{24k}{48k^3 + 84k^2 - 72k - (16k^2 - 13)\sqrt{8k - 7} + 36} n.$$

Proof. Since $G$ is a $k$-tree on $n$ vertices, $e(G) = kn - \frac{k(k+1)}{2}$. Hence, we have

$$\sum_{i \geq k} in_i = 2 \left( kn - \frac{k(k+1)}{2} \right) = 2kn - k(k+1) = \sum_{i \geq k} 2kn_i - k(k+1)$$

and thus

$$\sum_{i \geq 2k+1} (i - 2k)n_i = \sum_{i=k}^{2k} (2k - i)n_i - k(k+1).$$
This implies, for any index \( r \geq 2k + 1 \),

\[
(r - 2k) \sum_{i \geq r} n_i \leq \sum_{i \geq r} (i - 2k) n_i = \sum_{i \geq 2k + 1} (i - 2k) n_i - \sum_{i = 2k + 1}^{r-1} (i - 2k) n_i = \sum_{i = k}^{r-1} (2k - i) n_i - k(k + 1),
\]

yielding

\[
\sum_{i \geq r} n_i \leq \frac{1}{r - 2k} \left( \sum_{i = k}^{r-1} (2k - i) n_i - k(k + 1) \right).
\]

Thus, for any \( r \geq 2k + 1 \), we have

\[
n = \sum_{i \geq r} n_i = \sum_{i = k}^{r-1} n_i + \sum_{i \geq r} n_i \leq \sum_{i = k}^{r-1} n_i + \frac{1}{r - 2k} \left( \sum_{i = k}^{r-1} (2k - i) n_i - k(k + 1) \right) = \frac{1}{r - 2k} \left( \sum_{i = k}^{r-1} (r - i) n_i - k(k + 1) \right).
\]

We will now use the inequalities given in Corollary 3.4 to bound the above inequality. Let \( q(k) = \lfloor -1 + \sqrt{8k - 7} \rfloor \).

\[
n \leq \frac{1}{r - 2k} \left( \sum_{i = k}^{r-1} (r - i) n_i - k(k + 1) \right)
\]

\[
\leq \frac{1}{r - 2k} \sum_{i = k}^{r-1} (r - i) n_i
\]

\[
= \frac{1}{r - 2k} \left( \sum_{i = k}^{q(k)} (r - i) n_i + \sum_{i = q(k) + 1}^{r-1} (r - i) n_i \right)
\]

\[
\leq \frac{\alpha_{\text{reg}}(G)}{r - 2k} \left( \sum_{i = k}^{k+q(k)} \frac{1}{2} (r - i)((i - k)^2 + (i - k) + 2) + \sum_{i = k+q(k)+1}^{r-1} (r - i)(k + 1) \right).
\]

Note that the sum \( \sum_{i = k+q(k)+1}^{r-1} (r - i)(k + 1) \) can be written as \( \frac{r^2 - (1 + 2k + 2q(k))r + (k + q(k))^2 + k + q(k)}{2(k + 1)} \).

Now consider the function \( f_k(x) = \frac{a_k(x) + b_k(x)}{x - 2k} \) for \( x \geq 2k + 1 \), where

\[
a_k(x) = \sum_{i = k}^{k+q(k)} \frac{1}{2} (x - i)((i - k)^2 + (i - k) + 2),
\]

\[
b_k(x) = \sum_{i = k+q(k)+1}^{r-1} (r - i)(k + 1).
\]
Then, since \( b_k(r) = \frac{r^2 - (1 + 2k + 2q(k))r + (k + q(k))^2 + k + q(k)}{2(k+1)} \), we have

\[
\sum_{i=k+q(k)+1}^{r-1} (r - i)(k + 1),
\]

The minimum of the function \( f_k(r) \) is attained when

\[
x = \frac{1}{(k+1)} \left( 2k^2 + 2k + \sqrt{k^3 + R(k)} \right),
\]

where \( R(k) = O(k^{7/4}) \). Hence, the minimum of the function has order \( O(3k) \). Therefore, we set \( r = 3k \) and calculate \( f_k(3k) \), which gives

\[
f_k(3k) = \frac{1}{24k} \left( 48k^3 + 24k^2 + 24k - 3q(k)^4 + (-10 + 8k)q(k)^3 + (36k - 9)q(k)^2 - (48k^2 + 2 - 28k)q(k) \right).
\]

As \( q_1(k) = \frac{-3 + \sqrt{8k - 7}}{2} \leq q(k) = \lfloor \frac{-1 + \sqrt{8k - 7}}{2} \rfloor \leq \frac{-1 + \sqrt{8k - 7}}{2} = q_2(k) \) and since the coefficients \((-10 + 8k), (36k - 9)\) and \((48k^2 + 2 - 28k)\) are all positive for \( k \geq 2 \), we obtain

\[
f_k(3k) \leq \frac{1}{24k} \left( 48k^3 + 24k^2 + 24k - 3q_1(k)^4 + (-10 + 8k)q_2(k)^3 + (36k - 9)q_2(k)^2 \right)
\]

\[
- (48k^2 + 2 - 28k)q_2(k)
\]

\[
= \frac{48k^3 + 84k^2 - 72k - (16k^2 - 13)\sqrt{8k - 7} + 36}{24k}.
\]

Setting this into the inequality (5), yields finally

\[
\alpha_{reg}(G) > \frac{24k}{48k^3 + 84k^2 - 72k - (16k^2 - 13)\sqrt{8k - 7} + 36} n.
\]

\( \square \)

In Table 1 we give more accurate lower bounds on the regular independence number of \( k \)-trees of order \( n \) with \( 1 \leq k \leq 10 \). This is done by a more detailed analysis of the function \( f_k(x) \) used in the proof of Theorem 3.5 and by calculating the bound from the inequality \( n \leq \alpha_{reg}(G)f_k(r) - \frac{k(k+1)}{r-2k} \). For instance, for \( k = 2 \), we have \( f_2(x) = \frac{3x^2 - 15x + 20}{2(r-4)} \) for which \( x = 6 \) is the integer that is closest to its minimum. This yields \( n \leq \alpha_{reg}(G)f_2(6) = \frac{3\alpha_{reg}(G)(10)}{2} - 3 \) and, hence, we obtain the lower bound \( \alpha_{reg}(G) \geq \frac{2}{19} (n + 3) \) for a 2-tree \( G \) on \( n \) vertices.

We remark that, when \( k = 1 \), the bound given in Table 1 is precisely the bound for trees given in [2] (and in Theorem 2.1 (i)).
Table 1: Lower bounds on $\alpha_{\text{reg}}(G)$ for $k$-trees, $1 \leq k \leq 10$.

| $k$ | Bound                  | $k$ | Bound                  |
|-----|------------------------|-----|------------------------|
| 1   | $\frac{1}{4}(n+2)$    | 6   | $\frac{5}{37}(n+\frac{42}{5})$ |
| 2   | $\frac{2}{13}(n+3)$   | 7   | $\frac{1}{37}(n+\frac{56}{9})$ |
| 3   | $\frac{2}{5}(n+6)$    | 8   | $\frac{1}{33}(n+12)$    |
| 4   | $\frac{3}{20}(n+\frac{20}{3})$ | 9   | $\frac{1}{120}(n+\frac{90}{7})$ |
| 5   | $\frac{4}{179}(n+\frac{15}{2})$ | 10  | $\frac{1}{172}(n+\frac{55}{4})$ |

In a similar way as for the $k$-trees, we can compute lower bounds on $\alpha_{\text{reg}}(G)$ for planar graphs, outerplanar graphs and $k$-degenerate graphs. The following lemmas and the next corollary give us the needed background theory to achieve this goal. Recall that a maximum planar graph is a planar graph with the maximum possible number of edges, namely $3n - 6$ where $n$ is the order of the graph. Further, an (maximum) outerplanar graph is a triangulation of the polygon and it has at most (exactly) $2n - 3$ edges, where, again, $n$ is the number of vertices.

**Lemma 3.6** Let $G$ be a connected graph of order $n$ which is not an odd cycle or a complete graph. Then $\chi(V_i(G)) \leq \min\{i, \chi(G)\}$.

**Proof.** If $\chi(G) \leq i$, then evidently $\chi(V_i) \leq \chi(G) \leq i$ and we are done. Hence we may assume that $\chi(G) \geq i$. Since $\Delta(G[V_i]) \leq i$, Brook’s Theorem implies that $\chi(V_i) = i + 1$ if $G[V_i]$ contains either a component which is a complete graph on $i + 1$ vertices or $i = 2$ and $G[V_2]$ contains a component which is an odd cycle. Both cases are impossible as $G$ is connected and is neither a complete graph nor an odd cycle. Hence $\chi(V_i) \leq i$. \(\square\)

**Corollary 3.7** Let $G$ be a connected graph of order $n$ and which is not an odd cycle. Then the following statements hold:

1. If $G$ is planar and $n \geq 5$, then $\chi(V_i(G)) \leq i$ for $i = 1, 2, 3$ and $\chi(V_i) \leq 4$ for $i \geq 4$.
2. If $G$ is outerplanar and $n \geq 4$, then $\chi(V_1(G)) = 1$, $\chi(V_2(G)) \leq 2$ and $\chi(V_i) \leq 3$ for $i \geq 3$.
3. If $G$ is $k$-degenerate and $n \geq k + 2$, then $\chi(V_i(G)) \leq i$ for $i \leq k$ and $\chi(V_i) \leq k + 1$ for $i \geq k + 1$.

When $G$ is a maximal planar, maximal outerplanar or a maximal $k$-degenerate graph, we also know the following facts:
Lemma 3.8 The following statements are valid:

(1) If $G$ is a maximal planar graph on $n \geq 5$ vertices, then $\chi(V_3(G)) = 1$, $\chi(V_4(G)) \leq 3$ and $\chi(V_i(G)) \leq 4$ for $i \geq 5$ (see [2]).

(2) If $G$ is a maximal outerplanar graph on $n \geq 3$ vertices, then $\chi(V_3(G)) \leq 2$ and $\chi(V_i(G)) \leq 3$ for $i \geq 4$.

(3) If $G$ is a maximal $k$-degenerate graph on $n \geq k + 2$ vertices, then $\chi(V_k(G)) = 1$ and $\chi(V_i(G)) \leq k + 1$ for $i \geq k + 1$.

Proof. (1) The proof of this item is given in [2], but for completeness we present it here again. In a maximal planar graph on $n \geq 5$ vertices, the vertices of degree 3 are independent. Also it is easy to check that each component of $G[V_4(G)]$ is either $K_3$-free or is a $K_3$. Since, by Grötzsch’s Theorem, $K_3$-free graphs are 3-colorable, $G[V_4(G)]$ is 3-colorable. Moreover, $\chi(V_i(G)) \leq \chi(G) \leq 4$ by the Four-Color-Theorem.

(2) The cases $n = 3$ and $n = 4$ are trivial. For $n \geq 5$, $\chi(V_3(G)) \leq 2$ follows from Lemma 3.2 and the fact that maximal outerplanar are 2-trees (see also [6] for an explicit proof). Moreover, since maximal outerplanar graphs are 3-colorable, we have $\chi(V_i(G)) \leq 3$ for $i \geq 4$.

(3) Since in a maximal $k$-degenerate graph $G$ the vertices of degree $k$ are independent and $\chi(G) \leq k + 1$, we have $\chi(V_k(G)) = 1$ and $\chi(V_i(G)) \leq \chi(G) \leq k + 1$ for $i \geq k + 1$. □

Using $n_i \leq \alpha_{\text{reg}}(G)\chi(V_i)$ together with Corollary 3.7 and Lemma 3.8 we can bound the number of vertices of degree $i$ for connected planar graphs, outerplanar graphs and $k$-degenerate graphs. Proceeding as we did with the $k$-trees in Theorem 3.5 we can find lower bounds on the regular independence number for each of these graph types. Our results are listed in Table 2.

We remark that, while the bound on planar graphs with $\delta = 1$ and the bound on maximal planar graphs and $\delta = 3$ are only very tiny refinements of Albertson and Boutin’s results (1) and (2) mentioned in the Introduction, the other bounds on planar graphs improve upon them considerably. Further, although for general planar graphs with $\delta = 4$ and $\delta = 5$ the improvement is modest, all bounds obtained via the procedure of Theorem 3.5 are better than the benchmark bound from Proposition 1.1. Note also that the bounds on $k$-degenerate graphs with $\delta = k$ and on maximal $k$-degenerate graphs generalize Alberson and Boutin’s bound for trees (see (3) in the Introduction and Proposition 2.1 (i)). Furthermore, we remark that the bound on maximum outerplanar graphs is the complementary result to the one about the fair domination number obtained in [6] and the same bound from Table 1 when $k = 2$ (derived from the fact that maximal outerplanar graphs are a special kind of 2-trees).

Observe that it is important to have connected graphs for the bounds given in Table 2 since in general it is not true that, for two disjoint graphs $G$ and $H$, $\alpha_{\text{reg}}(G \cup H) = \alpha_{\text{reg}}(G) + \alpha_{\text{reg}}(H)$, as $\alpha_{\text{reg}}(G)$ and $\alpha_{\text{reg}}(H)$ could be attained by sets of vertices each with a different degree. This phenomena already occurred in the case of trees versus forests in the previous section.

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In this section, we present some lower bounds on \( \alpha_{2-\text{reg}}(G) \) for planar and outerplanar graphs \( G \). Recall that the \( k \)-chromatic number \( \chi_k(G) \) is the maximum number of colors needed to color the vertices of the graph \( G \) such that the graphs induced by equally colored vertices have maximum degree at most \( k \). In [11], Lovász shows that

\[
\chi_k(G) \leq \left\lceil \frac{\Delta(G) + 1}{k + 1} \right\rceil.
\]

Dealing with the more general concept of defective colourings, also called improper colourings (see [8–10, 12]), Cowen, Cowen and Woodall [8] show that \( \chi_2(G) \leq 2 \) for any outerplanar graph \( G \). Moreover, they prove that there are outerplanar graphs \( G \) with \( \chi_1(G) = 3 \) and show that \( \chi_2(G) \leq 3 \) for a planar graph \( G \) and that there are planar graphs \( G \) with \( \chi_1(G) = 4 \).

**Lemma 4.1** The following statements hold:

1. If \( G \) is a planar graph, then \( \chi_2(V_1) = \chi_2(V_2) = 1, \chi_2(V_i) \leq 2, \) for \( 3 \leq i \leq 5, \) and \( \chi_2(V_i) \leq 3, \) for \( i \geq 6. \) If \( G \) is maximal planar, then \( \chi_2(V_3) = 1 \) also holds.
(2) If \( G \) is an outerplanar graph, then \( \chi_2(V_1) = \chi_2(V_2) = 1 \) and \( \chi_2(V_i) \leq 2 \), for \( i \geq 3 \).

**Proof.** From Lovasz’s bound above, we derive \( \chi_2(V_1) = \chi_2(V_2) = 1 \) and \( \chi_2(V_i) \leq 2 \), for \( 3 \leq i \leq 5 \). By the above cited results, we have \( \chi_2(V_i) \leq \chi_2(G) \leq 3 \) for all \( i \geq 6 \) when \( G \) is planar, and \( \chi_2(V_i) \leq \chi(G) \leq 2 \) for \( i \geq 3 \) when \( G \) is outerplanar. If \( G \) is maximal planar, then from Lemma 3.8 we obtain \( \chi_2(V_3) \leq \chi(V_3) = 1 \) and thus \( \chi_2(V_3) = 1 \). □

The previous lemma allows us to compute some lower bounds on \( \alpha_{2-reg}(G) \) for planar and outerplanar graphs \( G \), while for the case \( k = 1 \), with the current ideas and techniques alone, we cannot do better than the bounds which were already obtained for \( \alpha_{reg}(G) \). Table 3 collects the bounds on \( \alpha_{2-reg}(G) \) we have computed for planar graphs and outerplanar graphs.

| \( \delta \) | Bench Mark from Prop. 1.1 | Bound obtained using Lemma 4.1 | Bound provided from Prop. 1.1 |
|---|---|---|---|
| \( \delta = 1 \) | \( \frac{1}{17} n \) | \( \frac{4}{55}(n + 3) \) | – |
| \( \delta = 2 \) | \( \frac{1}{27} n \) | \( \frac{3}{55}(n + 4) \) | – |
| \( \delta = 3 \) | \( \frac{1}{21} n \) | \( \frac{1}{16}(n + 4) \) | \( \frac{1}{14}(n + 6) \) |
| \( \delta = 4 \) | \( \frac{1}{15} n \) | \( \frac{2}{23}(n + 6) \) | \( \frac{2}{23}(n + 6) \) |
| \( \delta = 5 \) | \( \frac{1}{9} n \) | \( \frac{1}{7}(n + 12) \) | \( \frac{1}{7}(n + 12) \) |

Table 3: Lower bounds on \( \alpha_{2-reg}(G) \)

## 5 Open problems

We close this paper with the following open problems.

Lemma 3.2 is of interest of its own as it gives some non-trivial structural results about \( k \)-trees. Thus, it would be a desire to further improve the bound given in Lemma 3.2 or to show its optimality.

**Problem 1** Let \( G \) be a \( k \)-tree of order \( n \geq k + t + 2 \), where \( t \geq 0 \) is an integer. Is the bound \( \chi(V_{k+t}(G)) \leq \frac{1}{2}(t^2 + t + 2) \) optimal or can it be improved?
Observe that, any improvement of Lemma 3.2 would result in an improvement on the corresponding bound on $\alpha_{\text{reg}}(G)$.

While the case of the best lower bound for $\alpha_{k-\text{reg}}(G)$ is solved for every $k$ in the classes of forests and trees, already $\alpha_{\text{reg}}(G)$ is far from being solved for $k$-trees and in particular for outerplanar graphs, maximal outerplanar graphs and planar graphs. The connection established to fair dominating sets makes this problem more interesting, and any improvement of our lower bounds for $\alpha_{\text{reg}}(G)$ in these classes will require some new ideas. Also, it would be interesting to extend the bounds on $\alpha_{\text{reg}}(G)$ for $k$-trees to $\alpha_{j-\text{reg}}(G)$, for $j > 0$ and $k > 1$. Thus, we close with the following problems.

**Problem 2** Improve upon the bounds on $\alpha_{\text{reg}}(G)$ given in Section 3.

**Problem 3** Improve upon the benchmark bounds on $\alpha_{j-\text{reg}}(G)$ for $k$-trees when $j > 0$ and $k > 1$.

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