BOUND ED SETS OF SHEAVES ON COMPACT KÄHLER
MANIFOLDS

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ABSTRACT. We show that any set of quotients with fixed Chern classes
of a given coherent sheaf on a compact Kähler manifold is bounded in a
sense which we define. The result is proved by adapting Grothendieck’s
boundedness criterium expressed via the Hilbert polynomial to the Kähler
set-up. As a consequence we obtain the compactness of the connected
components of the Douady space of a compact Kähler manifold.

1. INTRODUCTION

Let \((X, \mathcal{O}_X(1))\) be a projective scheme endowed with a very ample line
bundle. In [Gro61] Grothendieck constructed the Hilbert scheme of \(X\) which
parametrizes the closed subschemes of \(X\). He also showed that by fixing the
Hilbert polynomial of the closed subschemes which are to be parametrized, one
gets a projective subscheme of the Hilbert scheme. In particular the connected
components of the Hilbert scheme are projective.

Let now \(X\) be a compact analytic space. The set of subspaces of \(X\) was en-
dowed with a natural analytic structure by Douady in [Dou66]. Its irreducible
components were shown to be compact by Fujiki when \(X\) is a Kähler space
and more generally when \(X\) belongs to the class \(\mathcal{C}\), [Fuj84]. Here we show
that if \(X\) is a compact Kähler manifold, then the connected components
of the Douady space of \(X\) are compact as well. See the last section for more
precise statements.

Our proof relies on a boundedness criterium similar to [Gro61] Thm. 2.1.
We basically follow Grothendieck’s approach but several changes and new
arguments are needed; one important technical tool in the projective case for
instance, for which we could find no good substitute in the Kähler case, is the
use of linear projections. The main idea is to reduce the problem to bounding
the volume of some appropriate analytic cycles and use the compactness of
the associated cycle space provided by Bishop’s theorem, cf. [Lê77], [Fuj84].
We start by reviewing some properties of the homology Todd class which will serve as a replacement of the Hilbert polynomial. In the next section boundedness for sets of isomorphism classes of coherent sheaves is introduced and some basic properties are proven. A boundedness result for reflexive sheaves of rank 1 on compact Kähler normal spaces is also included. We prove the boundedness criterion in section 4 and we end with the compactness result for the connected components of the Douady space of a compact Kähler manifold.

Acknowledgements: I wish to thank Daniel Barlet and Julien Grivaux for several discussions and particularly Jon Magnusson for arousing my interest in these topics.

2. Homology classes and degrees

The Grothendieck-Riemann-Roch theorem for singular varieties was proved by Baum, Fulton and MacPherson [BFM75], [BFM79] in the projective case and by Levy [Lev87] in the complex analytic case. One way to formulate it is that there exists a natural transformation of functors \( \tau : K_0 \to H^2_\ast ( ; \mathbb{Q}) \) such that for any compact complex space \( X \) the diagram

\[
\begin{array}{ccc}
K_0^X \otimes K_0^X & \xrightarrow{\otimes} & K_0^X \\
\downarrow \mathrm{ch}_\otimes & & \downarrow \tau \\
H^{2\ast}(X; \mathbb{Q}) \otimes H_2^\ast(X; \mathbb{Q}) & \xrightarrow{\sim} & H_2^\ast(X; \mathbb{Q})
\end{array}
\]

commutes and if \( X \) is nonsingular then \( \tau (\mathcal{O}_X) = \mathrm{Td}(X) \sim [X] \), where \( K_0^X \), \( K_0^X \) are the Grothendieck groups generated by holomorphic vector bundles and coherent sheaves respectively and \( \mathrm{Td}(X) \) is the (cohomology) Todd class of the tangent bundle to \( X \). Naturality means that for each proper morphism \( f : X \to Y \) of complex spaces the diagram

\[
\begin{array}{ccc}
K_0^X & \xrightarrow{\tau} & H_2^\ast(X; \mathbb{Q}) \\
\downarrow f_! & & \downarrow f_* \\
K_0^Y & \xrightarrow{\tau} & H_2^\ast(Y; \mathbb{Q})
\end{array}
\]

commutes, where \( f_! \) is defined by \( f_!(\mathcal{F}) = \sum_i (-1)^i [R^if_*(\mathcal{F})] \) for any coherent sheaf \( \mathcal{F} \) on \( X \). (In the non-compact case \( \tau \) takes values in the Borel-Moore homology.) We refer to the original papers and to the books [Ful], [FL], [DV] for a thorough treatment of these facts.
For a coherent sheaf $\mathcal{F}$ on a compact complex space $X$ we shall call $\tau(\mathcal{F}) := \tau([\mathcal{F}])$ the homology Todd class of $\mathcal{F}$. We list some of its properties:

1. When $\mathcal{F}$ is locally free and $X$ smooth and connected, $\tau(\mathcal{F})$ is the Poincaré dual of $\text{ch}(\mathcal{F}) \cdot \text{Td}(X) \in H^{2*}(X; \mathbb{Q})$.
2. If $f : X \to Y$ is an embedding then $\tau(f_*\mathcal{F}) = f_*(\tau(\mathcal{F}))$.
3. $\tau(\mathcal{F})_r = 0$ for $r > \dim \text{Supp} \mathcal{F}$.
4. $\tau$ is additive on exact sequences.
5. If $X$ is irreducible then $\tau(\mathcal{F})|_{\dim X} = \text{rank}(\mathcal{F})[X] \in H_{2\dim X}(X; \mathbb{Q})$.
   (This may be deduced from the previous properties via some modification of $X$ which desingularizes $X$ and flattens $\mathcal{F}$.)
6. The component of $\tau(\mathcal{F})$ in degree $\dim \text{Supp} \mathcal{F}$ is the homology class of an effective analytic cycle.

(In order to see this one makes the following reduction steps. For any integer $r$ we set $N_r(\mathcal{F})$ to be the sheaf of sections of $\mathcal{F}$ whose support have dimension less than $r$ and $\mathcal{F}_r := \mathcal{F}/N_r(\mathcal{F})$. If $r = \dim \text{Supp} \mathcal{F}$ we may assume that $\mathcal{F} = \mathcal{F}_r$. Let now $\mathcal{I}$ be the reduced ideal of $\text{Supp} \mathcal{F}$. We may also work with the graduation of $0 = \mathcal{I}^{k} \mathcal{F} \subset \mathcal{I}^{k-1} \mathcal{F} \subset \ldots \subset \mathcal{I} \mathcal{F} \subset \mathcal{F}$ instead of $\mathcal{F}$ and in particular we may suppose that $\mathcal{I} \mathcal{F} = 0$. If $Y_1, \ldots, Y_n$ are the irreducible components of $\text{Supp} \mathcal{F}$ and $\iota_j : Y_j \to X$ their respective embeddings, we consider $\mathcal{F}_j := \text{Im}(\mathcal{F} \to \iota_j^* \mathcal{F})$. The assumption $\mathcal{F} = \mathcal{F}_r$ now implies that $\mathcal{F}$ embeds into $\oplus_j \mathcal{F}_j$ and $\dim \text{Supp}((\oplus_j \mathcal{F}_j)/\mathcal{F}) < r$ thus reducing the situation to the case of an irreducible support.)

In the complex analytic setting a definition of the homology Todd class may be given using real-analytic locally free resolutions; see also [AH61] for the analogous construction of the cohomological Grothendieck element $\gamma_X(\mathcal{F})$. Using this approach one sees easily that property (1) above holds also for arbitrary coherent sheaves when $X$ is smooth and connected. This allows one to prove the following result on the variation of the homology Todd classes in a flat family.

**Proposition 2.1.** Let $X$ and $S$ be complex manifolds with $X$ compact and $S$ connected and $\mathcal{F}$ be a coherent sheaf on $X \times S$ flat over $S$. Then the class $\tau(\mathcal{F}_s) \in H_{2*}(X; \mathbb{Q})$ is independent of $s \in S$.

Let $(X, \omega)$ be a Kähler compact analytic space. The class $[\omega] \in H^2(X, \mathbb{R})$ allows us to define degrees of a coherent sheaf $\mathcal{F}$ on $X$ in the following way.
For each $r \in \mathbb{N}$ we define the $r$-degree of $\mathcal{F}$ by
\[
\deg_r(\mathcal{F}) := [\omega^r] \smile \tau_r(\mathcal{F}) \in \mathbb{R}.
\]
Notice that in case $(X, \mathcal{O}_X(1))$ is polarized, smooth, projective and $[\omega] = c_1(\mathcal{O}_X(1))$ one recovers the coefficient of the Hilbert polynomial of $\mathcal{F}$ in degree $r$ as
\[
\frac{\deg_r(\mathcal{F})}{r!}.
\]

3. Bounded families

According to Grothendieck’s definition a family of coherent sheaves on a scheme $X$ (over a field $k$ for instance) is bounded if it can be parametrized by a scheme of finite type over $k$. One feature of this context is that the topology of the parameter space is noetherian, i.e. every descending chain of closed subspaces is stationary. In the complex analytic setting a possible substitute would be to consider compact parameter spaces and work with their Zariski topology. We chose to work with semi-analytic Stein compacta, which are better adapted to our purposes; see [BS] 5.1.f for the definition. Their Zariski topology is known to be noetherian, cf. [BS] 5.3, p.220.

**Definition 3.1.** Let $X$ be an analytic space proper over an analytic space $S$ and $E$ a set of isomorphy classes of coherent sheaves on the fibers $X_s$ of $X \to S$. We say that the set $E$ is bounded if a semi-analytic Stein compactum $K$ over $S$ exists together with a coherent sheaf $\mathcal{G}$ in a neighbourhood of $X \times_S K$ such that $E$ is contained in the set of isomorphy classes defined by the sheaf $\mathcal{G}$ on the fibers of $X \times_S K \to K$. A set of complex subspaces of the fibers of $X \to S$ is called bounded if the set of isomorphy classes of their structure sheaves is bounded.

(We will be loose on the above terminology and often say “sheaves” instead of “isomorphy classes of sheaves”.)

In the above definition it is clear that $E$ can be viewed as set of sheaves defined on (some of) the fibers of $X \times_S K \to K$.

Let $X$ be an analytic space over $S$ and $\mathcal{F}$ a coherent sheaf on $X$. If $T \to S$ is a morphism, we will write as usual $X_T := X \times_S T$ and $\mathcal{F}_T$ for the base change. The projections $X_T \to T$ will be denoted by $p_T$.

When speaking of morphisms defined on Stein compacta $K$ or sheaves over $K$ we mean of course that such objects are defined on some analytic space containing $K$.
Remark 3.2. Suppose $X$ is an analytic space proper over a reduced analytic space $S$ and $\mathcal{F}$ a coherent sheaf on $X$. Then there exists a nowhere dense closed analytic subspace $T$ of $S$ such that over $S \setminus T$ the sheaf $\mathcal{F}_{S \setminus T}$ is flat and base change holds for the sheaves $R^q p_{S \setminus T *} (\mathcal{F})$ for all $q$.

Indeed, by Frisch’s theorem on generic flatness, there exists a nowhere dense analytic subset $T' \subset S$ such that the sheaf $\mathcal{F}$ is flat on $S \setminus T'$. We apply Grauert’s semicontinuity theorem and get a nowhere dense analytic subset $T''$ of Zariski open subset of $S \setminus T'$ such that the sheaves $R^q p_{S \setminus T' \setminus T''} (\mathcal{F})$ are locally free and base change holds over $S \setminus T' \setminus T''$. Now by Hironaka’s flattening theorem, [Hir75], after some proper modification $S' \to S$ not affecting $S \setminus T'$ the sheaf $\mathcal{F}_{S'}$ modulo $S'$-torsion becomes flat over $S'$. This implies that $T := T' \cup T''$ is a closed analytic subspace of $S$.

In fact a proof of this remark may be given using an older result of Kiehl-Verdier and Schneider, see [BS] 3.4.1, and avoiding Hironaka’s flattening theorem.

The following proposition gathers some basic properties of bounded sets of coherent sheaves, which will be needed in the sequel.

Proposition 3.3. Let $X$ be an analytic space proper over an analytic space $S$ and $E, E'$ two bounded sets of isomorphy classes of sheaves on the fibers of $X \to S$. Then the following sets are bounded as well:

1. The sets of kernels, cokernels and images of sheaf homomorphisms $\mathcal{F} \to \mathcal{F'}$, when the isomorphy classes of $\mathcal{F}$ and $\mathcal{F'}$ belong to $E$ and $E'$ respectively.
2. The set of isomorphy classes of extensions of $\mathcal{F}$ by $\mathcal{F'}$, for $\mathcal{F}$ and $\mathcal{F'}$ as above.
3. The set of isomorphy classes of tensor products $\mathcal{F} \otimes \mathcal{F'}$.

Proof. It is clear that one can find a base extension $K \to S$ from a semi-analytic Stein compactum together with coherent sheaves $\mathcal{G}$ and $\mathcal{G'}$ on $X_K := X \times_S K$ such that $E$ and $E'$ are contained in the set of isomorphy classes defined by the sheaves $\mathcal{G}$ and $\mathcal{G'}$ on the fibers of $X \to S$. We may assume $K$ smooth and connected.

1. By Remark 3.2 there exists an analytic subset $K_1 \subset K$ such that the sheaves $\mathcal{O}_{X_K}$, $\mathcal{G}$, $\mathcal{G'}$, and $\mathcal{Hom}(\mathcal{G}, \mathcal{G'})$ are flat over $K \setminus K_1$ and base change holds for these sheaves over $K \setminus K_1$. Working by descending induction on $\dim K$ we only need to check boundedness for the family of possible kernels and cokernels over $K \setminus K_1$. 

Let $T := \mathbb{P}((p_K)_* \mathcal{H}om(\mathcal{G}, \mathcal{G}'))^\vee$ and $((p_K)_* \mathcal{H}om(\mathcal{G}, \mathcal{G}'))^\vee_T \rightarrow \mathcal{L}$ be the universal quotient. The semi-analytic compactum $T$ is not Stein in general but we may cover it by a finite number of semi-analytic Stein compacta.

We get an universal section $O_T \rightarrow (((p_S)_* \mathcal{H}om(\mathcal{G}, \mathcal{G}'))^\vee_T \otimes \mathcal{L}$ which restricted over the part of $T$ lying over $K \setminus K_1$ induces an universal family of morphisms $\mathcal{G}_{X_s} \rightarrow \mathcal{G}'_{X_s}$. We look for a coherent sheaf $\mathcal{K}$ on $X_T$ which extends to $T$ the family of kernels of these morphisms. We may therefore reduce ourselves to the situation of a proper morphism $T \rightarrow K$ together with a section $\sigma$ of $((p_T)_* \mathcal{H}om(\mathcal{G}_T, \mathcal{G}'_T))^\vee$. Let $T$ and $C$ be the kernel and the cokernel of the natural morphism $((p_T)_* \mathcal{H}om(\mathcal{G}_T, \mathcal{G}'_T)) \rightarrow ((p_T)_* \mathcal{H}om(\mathcal{G}_T, \mathcal{G}'_T))^\vee$ of coherent sheaves on $T$. Let further $U$ be a Stein open neighbourhood of some point $t \in T$ and $f$ any function in $\text{Ann}(\mathcal{C})(U)$. Then $f\sigma$ has a lift to $((p_T)_* \mathcal{H}om(\mathcal{G}_T, \mathcal{G}'_T))(U) = \mathcal{H}om(\mathcal{G}_T, \mathcal{G}'_T)(X_U)$ which we denote by $f\sigma$ for simplicity. Let now $f_i$ and $g_j$ run through two finite sets of generators of $\text{Ann}(\mathcal{C})_t$ and $\text{Ann}(T)_t$ respectively and suppose that they still generate these sheaves at any point of $U$. On $X_U$ we define

$$\mathcal{K} := \cap_{i,j} \text{Ker}(g_j f_i \sigma : \mathcal{G}_U \rightarrow \mathcal{G}'_U).$$

It is easy to check that the definition does not depend on the chosen generators, hence the existence of the desired sheaf $\mathcal{K}$ on the compact space $X_T$. For a global substitute of a cokernel one may consider

$$\mathcal{G}'_U / \sum_{i,j} \text{Im}(g_j f_i \sigma : \mathcal{G}_U \rightarrow \mathcal{G}'_U).$$

Then the boundedness holds over the open set of $T_{K \setminus K_1}$ where the cokernel is flat and we continue by noetherian induction on $K$.

(2) When $\mathcal{O}_{X_K}$, $\mathcal{G}$, $\mathcal{G}'$ are flat over $K$ and $\mathcal{E}xt^1(p_K; \mathcal{G}, \mathcal{G}')$ commutes with base change, families of extensions of $\mathcal{G}$ by $\mathcal{G}'$ are parametrized by $H^0(K; \mathcal{E}xt^1(p_K; \mathcal{G}, \mathcal{G}'))$, cf. [Lan83], see also [BPS80] and [Fle81]. Such a family of extensions comes from a global extension on $X_K$ when $H^2(K; (p_K)_* \mathcal{H}om(\mathcal{G}, \mathcal{G}')) = 0$ which is the case in our situation. Considering $\mathbb{P}((\mathcal{E}xt^1(p_K; \mathcal{G}, \mathcal{G}'))^\vee)$ we obtain as before a semi-analytic compactum $T$ containing some nontrivial Zariski open subset which
parametrizes a universal family of classes of extensions of $\mathcal{G}$ by $\mathcal{G}'$. Let $T'$ be the complement of this open subset, $t$ be some point of $T$ and $U$ some Stein neighbourhood of $t$. The universal section $\xi$ of $\mathcal{E}xt^1(p_{T'T}; \mathcal{G}, \mathcal{G}')$ multiplied again by some function $g\xi$ on $U$ extends as a section $gf\xi$ of $\mathcal{E}xt^1(p_{U}; \mathcal{G}, \mathcal{G}')$ over $U \setminus T'$ and thus as an element of $\mathcal{E}xt^1(X_U; \mathcal{G}, \mathcal{G}')$, since $U$ is Stein and we have an exact sequence

$$\mathcal{E}xt^1(X_U; \mathcal{G}, \mathcal{G}') \to H^0(\mathcal{E}xt^1(p_{U}; \mathcal{G}, \mathcal{G}')) \to H^2(U, p_{U*}\mathcal{H}om(\mathcal{G}, \mathcal{G}')) \Rightarrow \mathcal{E}xt^{p+q}(X_U; \mathcal{G}, \mathcal{G}').$$

We want to use the elements $gf\xi$ in order to get some coherent sheaf on $X_U$ whose restriction to $X_U \setminus T'$ would be precisely the extension defined by $\xi$. For this we recall the construction of this extension. Let $0 \to \mathcal{I}_0 \to \mathcal{I}_1 \to ...$ an injective resolution of $\mathcal{G}'$ and $\delta : \mathcal{I}_0 \to \mathcal{I}_1$ the first map. Then one may view $\xi$ as a homomorphism $\mathcal{G} \to \text{Im}(\delta)$ and the induced extension is

$$0 \to \mathcal{G}' \to \text{Ker}(\delta + \xi) : \mathcal{I}_0 \oplus \mathcal{G} \to \text{Im}\delta \to \mathcal{G} \to 0.$$

Over $U \setminus T'$ we have a commutative diagram with exact rows

$$\begin{array}{cccccc}
0 & \longrightarrow & \text{Ker}(\delta + g\xi) & \longrightarrow & \mathcal{I}_0 \oplus \mathcal{G} & \longrightarrow & \text{Im}(\delta) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \text{Ker}(\delta + \xi) & \longrightarrow & \mathcal{I}_0 \oplus \mathcal{G} & \longrightarrow & \text{Im}(\delta)
\end{array}$$

and we consider the image $\mathcal{E}_{gf}$ of $\text{Ker}(\delta + g\xi)$ in $\mathcal{I}_0 \oplus \mathcal{G}$ by the vertical arrow $id \oplus g\xi id$. It is clear that $\mathcal{E}_{gf}$ coincides with $\text{Ker}(\delta + \xi)$ away from the vanishing locus of $g\xi$. Then $\mathcal{E} := \sum_{i,j} \mathcal{E}_{gf, i}$ provides the desired substitute. Here we have considered the functions $f_i, g_j$ as above. It is again easy to check that $\mathcal{E}$ does not depend on the chosen systems of generators. Although $\mathcal{E}$ might not sit as middle term of a sheaf extension of $\mathcal{G}$ by $\mathcal{G}'$, it exists globally on $X_U$ and gives sheaf extensions in each fiber of $p_{U \setminus T'}$.

(3) This is clear since $(\mathcal{G} \otimes \mathcal{O}_X \mathcal{G}') \otimes \mathcal{O}_X \mathcal{O}_{X_s} \cong \mathcal{G}_s \otimes \mathcal{O}_{X_s} \mathcal{G}'_s$.

□

We mention one more boundedness result in the Kähler case which is of independent interest. For this we will need the following lemma.
Lemma 3.4. Let \( f : X' \to X \) be a desingularization of the normal space \( X \) and \( \mathcal{F} \) a reflexive sheaf of rank 1 on \( X \). Let \( \mathcal{F}' := (f^*\mathcal{F})^\vee\vee \). Then an invertible sheaf \( \mathcal{L} \) on \( X' \) has the property \( f_*\mathcal{L} = \mathcal{F} \) if and only if there exists an effective divisor \( D \) supported on the exceptional locus of \( f \) such that \( \mathcal{L} \cong \mathcal{F}'(D) \).

Proof. Suppose that the invertible sheaf \( \mathcal{L} \) on \( X' \) has the property \( f_*\mathcal{L} = \mathcal{F} \). Then the map \( f^*\mathcal{F} \cong f^*f_*\mathcal{L} \to \mathcal{L} \) vanishes on torsion and thus factorizes through \( \mathcal{F}' \). It is moreover clear that this map is an isomorphism away from the exceptional locus of \( f \).

Conversely, if \( D \) is an effective divisor supported on the exceptional locus of \( f \), we get natural morphisms
\[
\mathcal{F} \hookrightarrow f_*f^*\mathcal{F} \to f_*\mathcal{F}' \hookrightarrow (f_*\mathcal{F}'(D))^\vee\vee \cong \mathcal{F}
\]
the last isomorphism being a consequence of the unicity of the extension of a reflexive sheaf over a codimension 2 locus; cf. [BS] Prop. 8.3.5. Since the global endomorphisms of \( \mathcal{F} \) are homotheties, we get our conclusion. \( \square \)

Proposition 3.5. Let \( X \) be a compact Kähler normal space of dimension \( d \). Then the set of reflexive sheaves of rank one of fixed homology Todd class \( \tau_{d-1} \) on \( X \) is bounded.

Proof. Let \( f : X' \to X \) be a desingularization \( X \) and suppose that the exceptional divisor \( E \) of \( f \) has \( n \) components. Denote by \( \text{Pic}(X' \setminus E, X') \) the group of invertible sheaves on \( X' \setminus E \) which may be extended as invertible sheaves to \( X' \). It is clear that \( \text{Pic}(X' \setminus E, X') \) is isomorphic to the group of reflexive sheaves of rank one on \( X \). The restriction \( r : \text{Pic}(X') \to \text{Pic}(X' \setminus E, X') \) induces an exact sequence of groups
\[
\mathbb{Z}^n \to \text{Pic}(X') \to \text{Pic}(X' \setminus E, X') \to 0.
\]
Note however that the natural map \( \text{Pic}(X' \setminus E, X') \to \text{Pic}(X') \), \( \mathcal{F} \mapsto \mathcal{F}' \), defined in Lemma 3.4 is not a group morphism in general.

Let \( \mathcal{L} \) be any invertible sheaf on \( X' \). The class \( \tau_{d-1}(f_*\mathcal{L}) = f_*\tau_{d-1}(\mathcal{L}) \) only depends on \( r(\mathcal{L}) \). In order to see this one may use the following natural diagram in singular homology:
\[
\begin{array}{ccc}
H_{2d-2}(E; \mathbb{Q}) & \longrightarrow & H_{2d-2}(X'; \mathbb{Q}) \\
\downarrow f_* & & \downarrow f_* \\
H_{2d-2}(X; \mathbb{Q}) & \cong & H_{2d-2}(X, \text{Sing}(X); \mathbb{Q})
\end{array}
\]
or the corresponding diagram in Borel-Moore homology, cf. [Ful] 19.1. Consider the set of reflexive sheaves of rank one of fixed homology Todd class $\tau_{d-1} = a$ on $X$, let $\mathcal{L}$ be an invertible sheaf on $X'$ such that $\tau_{d-1}(f_*\mathcal{L}) = a$ and consider the subset $S$ of elements of $\text{Pic}(X')$ having the same homology Todd class as $\mathcal{L}$. Then $S$ is a finite union of components of $\text{Pic}(X')$ and any reflexive sheaf of rank one of homology Todd class $\tau_{d-1} = a$ on $X$ is of the form $(f_*\mathcal{L}')^\vee$ for some $\mathcal{L}'$ whose isomorphy class lies in $S$. Denote the Poincaré invertible sheaf on $X \times S$ by $P$. For each $s \in S$ the natural morphism $(f_s)_*P_s \to (f_{S,s})_*P_s$ is an isomorphism over the regular part of $X$. Moreover for $s \in S$ general we have $(\text{Hom}_{O_{X \times S}}(P, O_{X \times S}))_s \cong \text{Hom}_{O_{X_s}}(P_s, O_{X_s})$ and the desired boundedness follows in the usual way. 

4. A boundedness criterium

Our main result is the following boundedness criterion.

**Theorem 4.1.** Let $(X, \omega)$ be a Kähler compact complex manifold and $F$ a set of isomorphy classes of coherent sheaves on $X$. Then the set $F$ is bounded if and only if the following two conditions are fulfilled:

1. There exists a bounded set $G$ of classes of coherent sheaves on $X$ such that each element of $F$ is a quotient of an element of $G$.

2. The degrees of the sheaves of $F$ are bounded from above.

In general we shall call a set $F$ of isomorphy classes of coherent sheaves on the fibers of $X \to S$ dominated if it satisfies the first condition of the criterium. By our discussion on Chern classes and since a semi-analytic Stein compactum has only finitely many connected components, it is clear that the two conditions are necessary.

For the converse we will need some preparations.

Recall that $F$ is called pure of dimension $d$ if $F$ as well as all its non-trivial coherent subsheaves are of dimension $d$. We introduce the following notation. As in Section 2 we denote by $N_r(F)$ the sheaf of sections of a coherent sheaf $F$ whose support have dimension less than $r$ and $F_{(r)} := F/N_r$. If $F$ is of dimension $d$ then $F_{(d)}$ is pure of dimension $d$.

**Lemma 4.2.** Let $d$ be a non-negative integer, $(X, \omega)$ as above and $S$ a reduced analytic space. Let further $W \subset X \times S$ be a reduced analytic subset of $X$ such that the second projection $W \to S$ is equidimensional of relative dimension $d$ with irreducible general fibers and $F$ a dominated set of classes of pure $d$-dimensional sheaves $F$ on the irreducible fibers of $W$ over $S$. Then:
The set of classes $\tau_d(\mathcal{F})$ is finite.

(2) The degrees $\deg_{d-1}(\mathcal{F})$ are bounded from below.

(3) If the degrees $\deg_{d-1}(\mathcal{F})$ are bounded from above, then the set $\mathcal{F}$ is bounded.

Proof. By Hironaka’s desingularization and flattening theorems we may assume that $S$ is smooth and irreducible containing a semi-analytic Stein compactum $A$, $W$ is flat over $S$ and that there exists a coherent sheaf $\mathcal{G}$ on $W$ flat over $S$ such that each sheaf $\mathcal{F}$ in $\mathcal{F}$ is a quotient of some $\mathcal{G}_s$, $s \in A$. For instance in order to obtain $W$ flat over $S$ we consider the locus $S_1 \subset S$ of non-flatness of $W \to S$, flatify $W \to S$ to obtain $W' \to S'$, deal with $W_s \to S_1$ separately by noetherian induction and consider the resulting space over the disjoint union of $S'$ and $S'$. In the same way we may replace $\mathcal{G}$ by $\mathcal{G}_{(\dim W)}$ since the sheaves $\mathcal{F}$ are pure. Moreover the arguments of [HL] 1.1.6-14 adapt to our case to show that for general $s \in S$ the fibers $\mathcal{G}_s$ are pure as well. Hence as before we may suppose that each $\mathcal{G}_s$ we are concerned with is pure.

For the first assertion consider the exact sequence

$$0 \to \mathcal{E} \to \mathcal{G}_s \to \mathcal{F} \to 0$$

exhibiting $\mathcal{F}$ as a quotient of $\mathcal{G}_s$. Then $\tau_i(\mathcal{G}_s) = \tau_i(\mathcal{F}) + \tau_i(\mathcal{E})$ and for $i = d$ both $\tau_i(\mathcal{F})$ and $\tau_i(\mathcal{E})$ are positive. In fact these classes correspond to integer combinations of $r$-dimensional irreducible components of $W_s$. Now the classes $\tau_i(\mathcal{G}_s)$ range through a finite set by Proposition 2.1. Since the $d$-degrees of the sheaves $\mathcal{F}$ are bounded from above by the $d$-degrees of the $\mathcal{G}_s$, Bishop’s theorem implies that the set of classes $\tau_d(\mathcal{F})$ must be finite.

For the next assertions we will need some further reductions: take an embedded desingularization $\hat{W} \to W$ of $W \subset X \times S$ and a flattening $\hat{W} \to \hat{W}$ of $\mathcal{G}_W$ over $\hat{W}$. Thus $\hat{\mathcal{G}} := \mathcal{G}_W / \text{Tors} \mathcal{G}_W$ is locally free on $\hat{W}$. Denote by $p : \hat{W} \to W$ the projection. Notice that the topological invariants of the general fiber of $\hat{W} \to S$ are independent of the sheaves $\mathcal{F}$. The general fibers of $\hat{W} \to S$ are smooth and connected. Take now a sheaf $\mathcal{F}$ on some $W_s$ such that $\mathcal{G}$ is flat over a neighbourhood of $s$ in $S$ and $\mathcal{G}_s$ is pure and suppose that the corresponding $\hat{W}_s$ is smooth and connected. The projection $p_s : \hat{W}_s \to W_s$ factors through the normalization $g : W_s^{\text{norm}} \to W_s$ as

$$\hat{W}_s \overset{f}{\longrightarrow} W_s^{\text{norm}} \overset{g}{\longrightarrow} W_s.$$
Let $\mathcal{E}$ be the kernel of $\mathcal{G}_{s} \to \mathcal{F}$, $\hat{\mathcal{E}}$ the image of the composition $p_{s}^{*}\mathcal{E} \to p_{s}^{*}\mathcal{G}_{s} \to \hat{\mathcal{G}}_{s}$ and $\mathcal{C}$ the cokernel of $\mathcal{E} \to (p_{s})_{s}\hat{\mathcal{E}}$. We have exact sequences of $\mathcal{O}_{W_{s}}$-modules:

$$0 \to \mathcal{E} \to \mathcal{G}_{s} \to \mathcal{F} \to 0,$$

$$0 \to \mathcal{E} \to (p_{s})_{s}\hat{\mathcal{E}} \to \mathcal{C} \to 0$$

hence $(p_{s})_{s}(\tau_{d-1}(\hat{\mathcal{E}})) = \tau_{d-1}((p_{s})_{s}\hat{\mathcal{E}}) = \tau_{d-1}(\mathcal{E}) + \tau_{d-1}(\mathcal{C})$. Notice that $g^{*}\mathcal{F}$ is pure but $p_{s}^{*}\mathcal{F}$ might have torsion and its support is contained in the exceptional divisor $E$ of $f$. Let then $\hat{\mathcal{E}}^{\text{sat}}$ be the saturation of $\hat{\mathcal{E}}$ so that the sequence

$$0 \to \hat{\mathcal{E}}^{\text{sat}} \to \hat{\mathcal{G}}_{s} \to p_{s}^{*}\mathcal{F}/\text{Tors}(p_{s}^{*}\mathcal{F}) \to 0$$

is exact. Set $\hat{\mathcal{F}} := p_{s}^{*}\mathcal{F}/\text{Tors}(p_{s}^{*}\mathcal{F})$ and $\mathcal{L} := (\Lambda^{\text{rank}_{\mathcal{E}}}\hat{\mathcal{G}}_{s})^{\vee}$. It is an invertible subsheaf of $\Lambda^{\text{rank}_{\mathcal{E}}}\hat{\mathcal{G}}_{s}$ whose first Chern class $[c_{1}(\mathcal{L})]$ equals that of $\hat{\mathcal{E}}^{\text{sat}}$.

Let now $\hat{\mathcal{P}} := \mathbb{P}(\Lambda^{\text{rank}_{\mathcal{E}}}\hat{\mathcal{G}}_{s})$ be the projectivized bundle of $\Lambda^{\text{rank}_{\mathcal{E}}}\hat{\mathcal{G}}_{s}$ and $\mathbb{P}$ the irreducible component of $\mathbb{P}(\Lambda^{\text{rank}_{\mathcal{E}}}\hat{\mathcal{G}}_{s})$ which covers $W_{s}$. We have a commutative diagram of natural morphisms

$$\begin{array}{ccc}
\hat{\mathbb{P}} & \xrightarrow{\hat{p}_{s}} & \mathbb{P} \\
\downarrow{\hat{q}} & & \downarrow{q} \\
\hat{W}_{s} & \xrightarrow{p_{s}} & W_{s}.
\end{array}$$

The inclusion $\mathcal{L} \hookrightarrow \Lambda^{\text{rank}_{\mathcal{E}}}\hat{\mathcal{G}}_{s}$ gives a section $\phi \in H^{0}(X; \Lambda^{\text{rank}_{\mathcal{E}}}\hat{\mathcal{G}}_{s} \otimes \mathcal{L}^{\vee}) \cong H^{0}(\hat{\mathbb{P}}; \mathcal{O}_{\hat{\mathbb{P}}}(1) \otimes q^{*}\mathcal{L}^{\vee})$ which vanishes on some irreducible divisor $\hat{D}$ on $\hat{\mathbb{P}}$. In fact $\hat{D}$ corresponds to the canonical embedding of $\mathbb{P}(\Lambda^{\text{rank}_{\mathcal{E}}}\hat{\mathcal{G}}_{s}/\mathcal{L})$ in $\hat{\mathbb{P}}$. Denote by $D$ the image of $\hat{D}$ in $\mathbb{P}$.

Set $n := \text{rank}(\Lambda^{\text{rank}_{\mathcal{E}}}\hat{\mathcal{G}}_{s})$ and let $\omega$ be the restriction of the Kähler form to $W_{s}$. One may choose Chern forms $\eta_{\mathcal{P}}, \eta_{\hat{\mathbb{P}}}$ of the tautological line bundles $\mathcal{O}_{\mathbb{P}}(1)$ and $\mathcal{O}_{\hat{\mathbb{P}}}(1)$ respectively which are positive on the fibers of $q$ and $\hat{q}$ respectively, [B83] Lemma 4.19. Notice that $(p_{s}^{*}\omega)^{d-1}$ vanishes on the exceptional divisor $E$. Replacing $\eta_{\mathcal{P}}, \eta_{\hat{\mathbb{P}}}$ by some small multiples of theirs if necessary we obtain a positive $(d + n - 2, d + n - 2)$-form $\Omega := q^{*}\omega^{d-1} \wedge \eta_{\mathcal{P}}^{n-1} + q^{*}\omega^{d} \wedge \eta_{\mathcal{P}}^{n-2}$ on $\mathbb{P}$.

We compute the volume of $D$ with respect to $\Omega$:

$$\text{vol}(D) = [\Omega] \sim (\hat{p}_{s})_{s}\hat{D} = (\hat{p}_{s})_{s}(\hat{p}_{s}^{*}([\Omega]) \sim \hat{D}) = \hat{p}_{s}^{*}([\Omega])([\eta_{\mathcal{P}}] - \hat{q}^{*}[c_{1}(\mathcal{L})]) =$$

$$= (\hat{p}_{s}^{*}([\Omega])([\eta_{\mathcal{P}}] - p_{s}^{*}([\omega^{d-1}])[c_{1}(\mathcal{L})] = (\hat{p}_{s}^{*}([\Omega])([\eta_{\mathcal{P}}] - (p_{s}^{*}([\omega^{d-1}])[c_{1}(\hat{\mathcal{E}}^{\text{sat}})]) =$$

$$= (\hat{p}_{s}^{*}([\Omega])([\eta_{\mathcal{P}}] + \frac{1}{2}(p_{s}^{*}([\omega^{d-1}]c_{1}(\hat{W}_{s}) - (p_{s}^{*}([\omega^{d-1}]) \sim \tau_{d-1}(\hat{\mathcal{E}}^{\text{sat}}) =$$
where $C$ is some constant not depending on $\mathcal{F}$, cf. [Tel08] 2.1 for a similar computation in the smooth case. This proves the second assertion.

The same formula shows that an upper bound on $\deg_{d-1}(\mathcal{F})$ leads to an upper bound on $\mathrm{vol}(D)$. Then the divisors $D$ may be seen as points in a part $T$ proper over $S$ of the relative cycle space of $\mathbb{P}(\mathcal{G})$ over $S$. The universal family $U$ of the corresponding $(d-1)$-cycles sits in a commutative diagram

$$
\begin{array}{ccc}
U & \longrightarrow & \mathbb{P}(\mathcal{G}) \\
\downarrow & & \downarrow \\
T & \longrightarrow & S
\end{array}
$$

as a universal divisor. It lifts to a universal divisor $D$ in $\mathbb{P}(\hat{\mathcal{G}})_T$ over $T$ and we consider the sheaf of ideals $\mathcal{I}$ of the image of $D$ in $\hat{W}_T$. Then for any $\mathcal{E}$ as above on a fiber $W_s$ over $s$ in some Zariski open set of $S$, there is some $t \in T$ lying over $s$ such that

$$\mathcal{L} := \left( \bigwedge_{\mathcal{E}_{\text{sat}}} \hat{\mathcal{E}} \right)^{\vee \vee} = \hat{q}_t^*(\mathcal{O}_{\mathbb{P}(\hat{\mathcal{G}})}(1) \otimes \mathcal{I}_t)$$

which shows that the set of sheaves $\det(\hat{\mathcal{F}})$ is bounded. We have denoted by $\hat{q}_t$ the induced projection over $\hat{W}_t$.

Let $r$ be the rank of $\mathcal{F}$. One checks directly then that $\hat{\mathcal{F}}$ is the image of the composition of the natural maps

$$
\hat{G}_s \rightarrow \mathcal{H}om\left( \bigwedge^{r-1} \hat{G}_s, \bigwedge^r \hat{G}_s \right) \rightarrow \mathcal{H}om\left( \bigwedge^{r-1} \hat{G}_s, \det(\hat{\mathcal{F}}) \right).
$$

Since all terms above sit in bounded sets we get the boundedness of the sheaves $\hat{\mathcal{F}}$ by Proposition 3.3. Let $\tilde{\mathcal{F}}$ be a family containing these sheaves. We suppose that the family $\tilde{\mathcal{F}}$ is over $S$ for simplicity of notation. The composition

$$
\mathcal{G}_s \rightarrow (p_*p^*\mathcal{G})_s \rightarrow (p_s)_*(p^*\mathcal{G})_s \cong (p_s)_*(p^*\mathcal{G}_s) \rightarrow (p_s)_*(\hat{\mathcal{F}}_s) \cong (p_s)_*(p^*_s\mathcal{F}/\mathrm{Tors}(p^*_s\mathcal{F}))
$$

factorizes as

$$
\mathcal{G}_s \rightarrow (p_*p^*\mathcal{G})_s \rightarrow (p_*\tilde{\mathcal{F}})_s \rightarrow (p_s)_*(\tilde{\mathcal{F}}_s)
$$

and factorizes also through the surjection $\mathcal{G}_s \rightarrow \mathcal{F}$. Since $\mathcal{F}$ and for $s$ general also $(p_*\tilde{\mathcal{F}})_s$ are pure the image of this composition is isomorphic to both $\mathcal{F}$ and $(p_*\tilde{\mathcal{F}})_s$.

The boundedness of the set $\mathcal{F}$ now follows. \qed
Lemma 4.3. Let $X$ be a compact Kähler manifold, $r$ an integer and $F$ a set of reduced subspaces of $X$ of bounded degree and all of whose components are of dimension $r$. Then $F$ is bounded.

Proof. The part $S$ of the Barlet space parameterizing the analytic cycles of dimension $r$ on $X$ and of degree bounded by some constant $M$ is compact by [Fuj78]. Moreover by [Bar75] Thm. 1, p. 38 the set $W := \{(x, s) \in X \times S \mid x \in |Z_s|\}$ is analytic and closed in $X \times S$, see also [Fuj78]. Here $|Z_s|$ denotes the support of the cycle corresponding to $s \in S$. Furthermore, there exist positive integers associated to the irreducible components $W_j$ of $W$ and a dense Zariski open subset $V \subset S$ such that for each $s \in V$ the multiplicities of the irreducible components of $Z_s$ contained in $W_j$ are precisely $n_j$.

Consider the (reduced) subspace $W' \subset W$ consisting of those components $W_j$ for which $n_j = 1$. The general fibers of $W' \to S$ are reduced and cover part of our set $F$. We deal with the rest by noetherian induction on $S$. □

We can now prove Theorem 4.1.

Proof. Let $F$ be a set of isomorphy classes of sheaves $\mathcal{F}$ dominated by a bounded set $G$ on $X$. We suppose that for all $n \in \mathbb{Z}$ the degrees $\deg_n(\mathcal{F})$ are bounded from above.

We will show that the set $F$ is bounded working by induction on $r := \max_{\mathcal{F} \in E} \dim \text{Supp} \mathcal{F}$.

For $r < 0$ there is nothing to prove.

Suppose that $r \geq 0$ and that the statement holds for all sets $F'$ as above and with $\max_{\mathcal{F} \in E'} \dim \text{Supp} \mathcal{F} < r$.

By Lemma 4.3 the set of the $r$-dimensional parts of the supports $\text{Supp} \mathcal{F}$ hence also the set of their corresponding ideal sheaves $\mathcal{I}$ are bounded. The sets of ideals of type $\mathcal{I}^j$ are equally bounded by Proposition 3.3. Using the boundedness assumption on the degrees of $\mathcal{F}$ and the devissage $\mathcal{I}^j \mathcal{F} \subset \mathcal{I}^{j-1} \mathcal{F} \subset \ldots \subset \mathcal{I}^k \mathcal{F} \subset \mathcal{F}$ we find the existence of some integer $k$ such that $\dim \text{Supp} \mathcal{I}^k \mathcal{F} < r$ for all $\mathcal{F} \in E$. Let $\mathcal{F}_j := (\mathcal{I}^j \mathcal{F})/(\mathcal{I}^j \mathcal{F})$. The sets of sheaves $\mathcal{F}_j$ as well as $\mathcal{I}^k \mathcal{F}$ are dominated and

$$\deg_d \mathcal{F} = \sum_{j=1}^{k} \deg_d \mathcal{F}_j + \deg_d \mathcal{I}^k \mathcal{F}$$

for all $d \in \mathbb{Z}$. 
Let $Y_1, ..., Y_n$ denote the irreducible components of $\text{Supp}(\mathcal{O}_X/I)$ and for $1 \leq j \leq k$ and $1 \leq i \leq n$ set

$$\mathcal{F}_{j,i} := \sum_{1 \leq l \leq n, l \neq i} I_{Y_l} \mathcal{F}_j$$

and $\mathcal{F}'_j := \text{Im}(\oplus_i \mathcal{F}_{j,i} \to \mathcal{F}_j)$. Then the sets of sheaves $\mathcal{F}'_j$ and $\mathcal{F}_j / \mathcal{F}'_j$ are dominated, $\dim \text{Supp}(\mathcal{F}_j / \mathcal{F}'_j) < r$ and

$$\deg_d \mathcal{F} = \sum_{j=1}^k \deg_d \mathcal{F}'_j + \sum_{j=1}^k \deg_d (\mathcal{F}_j / \mathcal{F}'_j) + \deg_d (I^k \mathcal{F})$$

for all $d \in \mathbb{Z}$.

By applying the functor $-(r)$ to the sequence of maps $\oplus_i \mathcal{F}_{j,i} \to \mathcal{F}'_j \to \mathcal{F}_j$ one sees that $(\mathcal{F}'_j)(r) \cong \oplus_i (\mathcal{F}_{j,i})(r)$, hence the sheaves $(\mathcal{F}'_j)(r)$ are direct sums of pure sheaves of dimension $r$ each supported on some component $Y_i$ of $\text{Supp}(\mathcal{O}_X/I)$. Moreover the relation

(1)

$$\deg_d \mathcal{F} = \sum_{j,i} \deg_d (\mathcal{F}_{j,i})(r) + \sum_{j=1}^k \deg_d N_i (\mathcal{F}_j) + \sum_{j=1}^k \deg_d (\mathcal{F}_j / \mathcal{F}'_j) + \deg_d (I^k \mathcal{F})$$

holds for all $d \in \mathbb{Z}$.

We will next use this formula for $d = r - 1$ and apply Lemma 4.2 to show that the sheaves $(\mathcal{F}_{j,i})(r)$ form a bounded set. Lemma 4.3 gives us an analytic family of supports $W \subset X \times S$ parametrized by some compact analytic space $S$ and we may view the sheaves $(\mathcal{F}_{j,i})(r)$ as sheaves on the fibers of $W \to S$. We may also suppose $S$ and $W$ reduced and irreducible. The first two assertions of Lemma 4.2 show that the $(r - 1)$-degrees of the sheaves $(\mathcal{F}_{j,i})(r)$ are bounded from below. Using this and the upper bound on $\deg_{r-1} \mathcal{F}$ in formula (1) imply that they are also bounded from above. Then the third assertion of Lemma 4.2 says that after some base change $K \to S$ to a semi-analytic Stein compactum $K$ the sheaves $(\mathcal{F}_{j,i})(r)$ occur as fibers of a family $\mathcal{H}$ over $W_K$. Using generic flatness we see that our sheaves are also fibers of some family over $X \times K'$ for some possibly new semi-analytic Stein compactum $K'$ and our claim is proven.

Let $\mathcal{G}_j$ be a bounded set of sheaves dominating the set of sheaves $\mathcal{F}'_j$. Then the kernels $\mathcal{K}_j$ of the surjections $\mathcal{G} \to (\mathcal{F}'_j)(r)$ also form a bounded family.
From the diagram

\[
\begin{array}{ccccccccc}
0 & \rightarrow & K_j & \rightarrow & G_j & \rightarrow & (F_j')(r) & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \text{id} & & \\
0 & \rightarrow & N_r(F_j') & \rightarrow & F_j' & \rightarrow & (F_j')(r) & \rightarrow & 0
\end{array}
\]

we see that the set of sheaves \(N_r(F_j')\) is dominated as well.

The \(2k + 1\) sheaves of dimension \(< r\) appearing in formula 1 may be each considered on a different copy of \(X\) thus giving a sheaf on the disjoint union \(X^{(2k+1)}\) of these copies. These sheaves on \(X^{(2k+1)}\) have bounded degrees and are dominated, hence they form a bounded set by the induction hypothesis. But then each of the sets of sheaves \(N_r(F_j'), F_j/F_j', I^kF\) is bounded on \(X\). Now the exact sequences

\[
\begin{align*}
0 & \to N_r(F_j') \to F_j' \to (F_j')(r) \to 0, \\
0 & \to F_j' \to F_j \to F_j/F_j' \to 0, \\
0 & \to I^jF \to I^{j-1}F \to F_j \to 0
\end{align*}
\]

allow us to reconstruct \(F\) and deduce the desired boundedness. \(\Box\)

5. Corollaries

We start with a direct consequence of Theorem 4.1.

**Corollary 5.1.** Let \((X, \omega)\) be a Kähler compact complex manifold and \(F\) a dominated set of isomorphy classes of coherent sheaves on \(X\). Then the following assertions are equivalent:

1. The set \(F\) is bounded.
2. The set of Chern classes of the sheaves of \(F\) is finite.
3. The set of homology Todd classes of the sheaves of \(F\) is finite.
4. The degrees of the sheaves of \(F\) are bounded.
5. The degrees of the sheaves of \(F\) are bounded from above.

We finally come to our main application: the compactness of the connected components of the Douady space of a compact Kähler manifold.

**Corollary 5.2.** Let \(X\) be a Kähler, compact, complex manifold, \(G\) a coherent sheaf on \(X\) and \(b\) some real number. Then the Douady space \(D(G)_{\leq b}\) of quotients of \(G\) with degrees bounded from above by \(b\) is compact. In particular the Douady space \(D(G)_\alpha\) of quotients of \(G\) with fixed homology Todd class equal
to $\alpha \in H_*(X, \mathbb{Q})$ is compact, especially the connected components of the whole Douady space $D(G)$ are compact.

Proof. By our boundedness criterion, the sheaves which are quotients of $G$ and whose degrees are bounded from above by $b$ are fibers over a semi-analytic Stein compactum $K$ of some family $\mathcal{F}$ over $X \times S$ with $S$ smooth. Here we may suppose that $K$ is contained in $S$. By noetherian induction we may suppose that $\mathcal{F}$ is flat over $S$. To any complex space $T$ over $S$ we associate the set $\text{Hom}_{X \times T}(G_T, \mathcal{F}_T)$. This defines a contravariant functor which is represented by a linear space $V$ over $S$, cf. [Ple81] 3.2. It is clear that we may find a finite number of semi-analytic Stein compacta in $V$ covering $K$ and such that up to some multiplicative constant each non-zero morphism in $\text{Hom}_X(G, \mathcal{F}_s)$, $s \in K$, is represented by some element in this union; see for instance the construction of the projective variety over $S$ associated to $V$ in [Fi] 1.9. Let $K'$ be the union of these semi-analytic Stein compacta and $\mathcal{F}'$ the image of the universal morphism restricted to $K'$. By flatenning and noetherian induction again we may assume that $\mathcal{F}'$ is flat over a neighbourhood $U$ of $K'$ and that each quotient of $G$ whose class is in $D(G)_{\leq b}$ is represented by some morphism $G \to \mathcal{F}'_s$ for some $s \in K'$. The universal property of the whole Douady space $D(G)$ of quotients of $G$ now gives us a a morphism $U \to D(G)$ whose restriction to $K'$ covers $D(G)_{\leq b}$. Thus $D(G)_{\leq b}$ is compact. \[\square\]

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