UNIRATIONALITY OF CERTAIN SUPERSINGULAR $K3$
SURFACES IN CHARACTERISTIC 5

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Abstract. We show that every supersingular $K3$ surface in characteristic 5
with Artin invariant $\leq 3$ is unirational.

1. Introduction

We work over an algebraically closed field $k$.

A $K3$ surface $X$ is called supersingular (in the sense of Shioda [22]) if the Picard
number of $X$ is equal to the second Betti number 22. Supersingular $K3$ surfaces
exist only when the characteristic of $k$ is positive. Artin [3] showed that, if $X$
is a supersingular $K3$ surface in characteristic $p > 0$, then the discriminant of the
Néron-Severi lattice $NS(X)$ of $X$ is written as $-p^{2\sigma(X)}$, where $\sigma(X)$ is a positive
integer $\leq 10$. (See also Illusie [9, Section 7.2].) This integer $\sigma(X)$ is called the
Artin invariant of $X$.

A surface $S$ is called unirational if the function field $k(S)$ of $S$ is contained in a
purely transcendental extension field of $k$, or equivalently, if there exists a dominant
rational map from a projective plane $P^2$ to $S$. Shioda [22] proved that, if a smooth
projective surface $S$ is unirational, then the Picard number of $S$ is equal to the
second Betti number of $S$. Artin and Shioda conjectured that the converse is true
for $K3$ surfaces (see, for example, Shioda [23]):

Conjecture 1.1. Every supersingular $K3$ surface is unirational.

In this paper, we consider this conjecture for supersingular $K3$ surfaces in char-
acteristic 5.

From now on, we assume that the characteristic of $k$ is 5. Let $k[x]_6$ be the space
of polynomials in $x$ of degree 6, and let $U \subset k[x]_6$ be the space of $f(x) \in k[x]_6$
such that the quintic equation $f'(x) = 0$ has no multiple roots. It is obvious that $U$
is a Zariski open dense subset of $k[x]_6$. For $f \in U$, we denote by $C_f \subset P^2$
the projective plane curve of degree 6 whose affine part is defined by

$$y^5 - f(x) = 0.$$ 

Let $Y_f \to P^2$ be the double covering of $P^2$ whose branch locus is equal to $C_f$, and
let $X_f \to Y_f$ be the minimal resolution of $Y_f$.

Theorem 1.2. If $f$ is a polynomial in $U$, then $X_f$ is a supersingular $K3$ surface
with $\sigma(X_f) \leq 3$. Conversely, if $X$ is a supersingular $K3$ surface with $\sigma(X) \leq 3$,
then there exists $f \in U$ such that $X$ is isomorphic to $X_f$. 

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The affine part of \( Y_f \) is defined by \( w^2 = y^5 - f(x) \). Hence the function field \( k(X_f) \)
is equal to \( k(w, x, y) \), and it is contained in the purely transcendental extension field 
\( k(w^{1/5}, x^{1/5}) \) of \( k \). Therefore we obtain the following corollary:

**Corollary 1.3.** Every supersingular K3 surface in characteristic 5 with Artin invariant \( \leq 3 \) is unirational.

The unirationality of a supersingular K3 surface \( X \) in characteristic \( p > 0 \) with Artin invariant \( \sigma \) has been proved in the following cases: (i) \( p = 2 \), (ii) \( p = 3 \) and \( \sigma \leq 6 \), and (iii) \( p \) is odd and \( \sigma \leq 2 \). In the cases (i) and (ii), the unirationality was proved by Rudakov and Shafarevich [15], [16] by showing that there exists a structure of the quasi-elliptic fibration on \( X \). The case (iii) follows from the result of Ogus [13], [14] that a supersingular K3 surface in odd characteristic with Artin invariant \( \leq 2 \) is a Kummer surface associated with a supersingular abelian surface, and the result of Shioda [24] that such a Kummer surface is unirational. The unirationality of \( X \) in the case \((p, \sigma) = (5, 3)\) proved in this paper seems to be new.

In [19], we have shown that a supersingular K3 surface in characteristic 2 is birational to a normal K3 surface with 21A1-singularities, and that such a normal K3 surface is a purely inseparable double cover of \( \mathbb{P}^2 \). In [20], we have proved that a supersingular K3 surface in characteristic 3 with Artin invariant \( \leq 6 \) is birational to a normal K3 surface with 10A2-singularities, and it is also birational to a purely inseparable triple cover of \( \mathbb{P}^1 \times \mathbb{P}^1 \). These yield an alternative proof to the results of Rudakov and Shafarevich [15], [16] in the cases (i) and (ii) above.

In this paper, we show that a supersingular K3 surface in characteristic 5 with Artin invariant \( \leq 3 \) is birational to a normal K3 surface with 5A4-singularities that is a double cover of \( \mathbb{P}^2 \), and then prove that such a normal K3 surface is isomorphic to \( Y_f \) for some \( f \in U \). The first step follows from the structure theorem of the Néron-Severi lattices of supersingular K3 surfaces due to Rudakov and Shafarevich [16]. For the second step, we investigate projective plane curves of degree 6 with 5A4-singularities in Section 2.

### 2. Projective plane curves with 5A4-singularities

**Definition 2.1.** A germ of a curve singularity in characteristic \( \neq 2 \) is called an \( A_n \)-singularity if it is formally isomorphic to

\[
y^2 - x^{n+1} = 0,
\]

(see Artin [4], and Greuel and Kröning [8].)

We assume that the base field \( k \) is of characteristic 5 until the end of the paper.

**Proposition 2.2.** Let \( C \subset \mathbb{P}^2 \) be a reduced projective plane curve of degree 6. Then the following conditions are equivalent to each other.

(i) The singular locus of \( C \) consists of five \( A_4 \)-singular points.

(ii) There exists \( f \in U \) such that \( C = C_f \).

For the proof, we need the following result due to Wall [26], which holds in any characteristic. Let \( D \subset \mathbb{P}^2 \) be an integral plane curve of degree \( d > 1 \), and let \( I_D \subset \mathbb{P}^2 \times (\mathbb{P}^2)^\vee \) be the closure of the locus of all \((x, l) \in \mathbb{P}^2 \times (\mathbb{P}^2)^\vee \) such that \( x \) is
a smooth point of $D$ and $l$ is the tangent line to $D$ at $x$. Let $D^\vee \subset (\mathbb{P}^2)^\vee$ be the image of the second projection

$$\pi_D : I_D \to (\mathbb{P}^2)^\vee.$$  

We equip $D^\vee$ with the reduced structure, and call it the dual curve of $D$. Note that the first projection $I_D \to D$ is birational. Therefore, by the projection $\pi_D$, we can regard the function field $k(D)$ as an extension field of the function field $k(D^\vee)$. The corresponding rational map from $D$ to $D^\vee$ is called the Gauss map. We put

$$\deg \pi_D := [k(D) : k(D^\vee)].$$  

We choose general homogeneous coordinates $[w_0 : w_1 : w_2]$ of $\mathbb{P}^2$, and let $F(w_0, w_1, w_2) = 0$ be the defining equation of $D$. We denote by $D_Q \subset \mathbb{P}^2$ the curve defined by

$$\frac{\partial F}{\partial w_2} = 0,$$

which is called the polar curve of $D$ with respect to $Q = [0 : 0 : 1]$.

**Proposition 2.3** (Wall [26]). For a singular point $s$ of $D$, we denote by $(D.D_Q)_s$ the local intersection multiplicity of $D$ and $D_Q$ at $s$. Then we have

$$\deg \pi_D \cdot \deg D^\vee = d(d - 1) - \sum_{s \in \text{Sing}(D)} (D.D_Q)_s.$$

**Remark 2.4.** If $s \in D$ is an $A_n$-singular point, then the polar curve $D_Q$ is smooth at $s$ and the local intersection multiplicity $(D.D_Q)_s$ is $n + 1$.

**Proof of Proposition 2.3.** Suppose that $C$ has $5A_4$-singular points as its only singularities. Since an $A_4$-singular point is unibranched, $C$ is irreducible. By Proposition 2.3 and Remark 2.4 we have

$$\deg \pi_C \cdot \deg C^\vee = 5.$$  

Suppose that $(\deg \pi_C, \deg C^\vee) = (1, 5)$. Let $\nu : \widetilde{C} \to C$ be the normalization of $C$. Since $\deg \pi_C = 1$, we can consider $\widetilde{C}$ as a normalization of $C^\vee$. We denote by

$$\nu^\vee : \widetilde{C} \to C^\vee$$

the morphism of normalization. Let $s$ be a singular point of $C$, and let $\widetilde{s} \in \widetilde{C}$ be the point of $\widetilde{C}$ that is mapped to $s$ by $\nu$. We can choose affine coordinates $(x, y)$ of $\mathbb{P}^2$ with the origin $s$ and a formal parameter $t$ of $\widetilde{C}$ at $\widetilde{s}$ such that $\nu$ is given by

$$t \mapsto (x, y) = (t^2, t^5 + c_6 t^6 + c_7 t^7 + \cdots).$$

Let $(u, v)$ be the affine coordinates of $(\mathbb{P}^2)^\vee$ such that the point $(u, v) \in (\mathbb{P}^2)^\vee$ corresponds to the line of $\mathbb{P}^2$ defined by $y = ux + v$. Then $\nu^\vee$ is given at $\widetilde{s}$ by

$$t \mapsto (u, v) = (3 c_6 t^4 + \cdots, t^5 + \cdots).$$

(See, for example, Namba [10, p. 78].) Therefore $\nu^\vee(\widetilde{s})$ is a singular point of $C^\vee$ with multiplicity $\geq 4$. We choose distinct two points $s_1, s_2 \in \text{Sing}(C)$. There exists a line of $(\mathbb{P}^2)^\vee$ that passes through both of $\nu^\vee(s_1) \in C^\vee$ and $\nu^\vee(s_2) \in C^\vee$. This contradicts Bezout’s theorem, because $\deg C^\vee = 5 < 4 + 4$. Therefore we have $(\deg \pi_C, \deg C^\vee) = (5, 1)$. Then there exists a point $P \in \mathbb{P}^2$ such that we have

$$l \in C^\vee \iff P \in l.$$  

We choose homogeneous coordinates $[w_0 : w_1 : w_2]$ of $\mathbb{P}^2$ in such a way that $P = [0 : 1 : 0]$. Let $L_{\infty}$ be the line $w_2 = 0$, and let $(x, y)$ be the affine coordinates on
\( \mathbb{A}^2 := \mathbb{P}^2 \setminus L_\infty \) given by \( x := w_0/w_2 \) and \( y := w_1/w_2 \). Suppose that \( C \) is defined by \( h(x, y) = 0 \) in \( \mathbb{A}^2 \). From (2.1), we have

\[
(2.2) \quad h(a, b) = 0 \implies \frac{\partial h}{\partial y}(a, b) = 0.
\]

Let \( U_C \subset \mathbb{A}^1 \) be the image of the projection \( (C \setminus \text{Sing}(C)) \cap \mathbb{A}^2 \to \mathbb{A}^1 \) given by \( (a, b) \mapsto a \). Note that \( U_C \) is Zariski dense in \( \mathbb{A}^1 \). Let \( (a_0, b_0) \) be a smooth point of \( C \cap \mathbb{A}^2 \). By (2.2), we have

\[
\frac{\partial h}{\partial x}(a_0, b_0) \neq 0.
\]

Hence there exists a formal power series \( \gamma(\eta) \in k[[\eta]] \) such that \( C \) is defined by \( x - a_0 = \gamma(y - b_0) \) locally around \( (a_0, b_0) \). By (2.2) again, \( \gamma'(\eta) \) is constantly equal to 0, and hence there exists a formal power series \( \beta(\eta) \in k[[\eta]] \) such that \( \gamma(\eta) = \beta(\eta)^5 \). Therefore the local intersection multiplicity of the line \( x - a_0 = 0 \) and \( C \) at \( (a_0, b_0) \) is \( \geq 5 \). Thus we obtain the following:

\[
(2.3) \quad \text{If } a \in U_C, \text{ then the equation } h(a, y) = 0 \text{ in } y
\]

has a root of multiplicity \( \geq 5 \).

We put

\[
h(x, y) = cy^6 + g_1(x)y^5 + \cdots + g_5(x)y + g_6(x),
\]

where \( c \) is a constant, and \( g_i(x) \in k[x] \) is a polynomial of degree \( \leq \nu \). Suppose that \( c \neq 0 \). We can assume \( c = 1 \). By (2.3), we have \( g_2(a) = g_3(a) = g_4(a) = 0 \) and \( g_1(a)g_5(a) = g_6(a) \) for any \( a \in U_C \). Since \( U_C \) is Zariski dense in \( \mathbb{A}^1 \), we have \( g_2 = g_3 = g_4 = 0 \) and \( g_5, g_6 \). Then we have \( h(x, y) = (y^5 + g_5(x))(y + g_1(x)) \), which contradicts the irreducibility of \( C \). Thus \( c = 0 \) is proved. Then, by (2.3), we have \( g_1 \neq 0 \) and \( g_2 = g_3 = g_4 = g_5 = 0 \). We put \( g_1 = Ax + B \), and define a new homogeneous coordinate system \( \{z_0 : z_1 : z_2\} \) of \( \mathbb{P}^2 \) by

\[
(z_0, z_1, z_2) := (w_0, w_1, Awv_0 + Bw_2) \quad \text{if } B \neq 0;
\]

\[
(z_0, z_1, z_2) := (w_0, w_1, Aw_0) \quad \text{if } B = 0.
\]

Then \( C \) is defined by a homogeneous equation of the form

\[
z_2z_1^5 - F(z_0, z_2) = 0,
\]

where \( F(z_0, z_2) \) is a homogeneous polynomial of degree 6. We put \( L'_\infty := \{z_2 = 0\} \). Defining the affine coordinates \( (x, y) \) on \( \mathbb{P}^2 \setminus L'_\infty \) by \( (x, y) := (z_0/z_2, z_1/z_2) \), we see that the affine part of \( C \) is defined by \( y^5 - f(x) \) for some polynomial \( f(x) \) of degree \( \leq 6 \). If \( \deg f < 6 \), then \( L'_\infty \) would be an irreducible component of \( C \) because \( \deg C = 6 \). Therefore we have \( \deg f = 6 \). Then \( C \cap L'_\infty \) consists of a single point \( [0 : 1 : 0] \), and \( C \) is smooth at \( [0 : 1 : 0] \). Therefore we have

\[
\text{Sing}(C) = \{ (\alpha, f(\alpha)^{1/5}) \mid f'(\alpha) = 0 \}.
\]

Since \( C \) has five singular points, we have \( f \in U \).

Conversely, suppose that \( f \in U \). We show that \( \text{Sing}(C_f) \) consists of 5.A₄-singular points. Let \( L_\infty \subset \mathbb{P}^2 \) be the line at infinity. It is easy to check that \( C_f \cap L_\infty \) consists of a single point \( [0 : 1 : 0] \), and \( C_f \) is smooth at this point. Therefore we have \( \text{Sing}(C_f) = \{ (\alpha, f(\alpha)^{1/5}) \mid f'(\alpha) = 0 \} \). In particular, \( C_f \) has exactly five singular points. Let \( (\alpha, \beta) \) be a singular point of \( C_f \). Since \( \alpha \) is a simple root of the quintic equation \( f'(x) = 0 \), there exists a polynomial \( g(x) \) with \( g(\alpha) \neq 0 \) such that

\[
f(x) = f(\alpha) + (x - \alpha)^2 g(x).
\]
Because $\beta^5 = f(\alpha)$, the defining equation of $C$ is written as

$$(y - \beta)^5 - (x - \alpha)^2g(x) = 0.$$ 

Therefore $(\alpha, \beta)$ is an $A_4$-singular point of $C_f$. □

3. Proof of Theorem 1.2

First we show that, if $f \in \mathcal{U}$, then $X_f$ is a supersingular $K3$ surface with Artin invariant $\leq 3$. Since the sextic double plane $Y_f$ has only rational double points as its singularities by Proposition 2.2, its minimal resolution $X_f$ is a $K3$ surface by the results of Artin [1], [2]. Let $\Sigma_f$ be the sublattice of the Néron-Severi lattice $NS(X_f)$ of $X_f$ that is generated by the classes of the $(-2)$-curves contracted by $X_f \rightarrow Y_f$. Then $\Sigma_f$ is isomorphic to the negative-definite root lattice of type $5A_4$ by Proposition 2.2. In particular, $\Sigma_f$ is of rank 20, and its discriminant is $5^5$. Let $H_f \subset X_f$ be the pull-back of a line of $\mathbb{P}^2$, and put

$$h_f := [H_f] \in NS(X_f).$$

Since the line at infinity $L_\infty \subset \mathbb{P}^2$ intersects $C_f$ at a single point $[0 : 1 : 0]$ with multiplicity 6, and $[0 : 1 : 0]$ is a smooth point of $C_f$, the pull-back of $L_\infty$ to $X_f$ is a union of two smooth rational curves that intersect each other at a single point with multiplicity 3. Let $L_f$ be one of the two rational curves, and put

$$l_f := [L_f] \in NS(X_f).$$

Then $h_f$ and $l_f$ generate a lattice $\langle h_f, l_f \rangle$ of rank 2 in $NS(X_f)$ whose intersection matrix is equal to

$$\begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}.$$ 

In particular, the discriminant of $\langle h_f, l_f \rangle$ is $-5$. Note that $\Sigma_f$ and $\langle h_f, l_f \rangle$ are orthogonal in $NS(X_f)$. Therefore $NS(X_f)$ contains a sublattice $\Sigma_f \oplus \langle h, l \rangle$ of rank 22 and discriminant $-5^6$. Thus $X_f$ is supersingular, and $\sigma(X_f) \leq 3$.

In order to prove the second assertion of Theorem 1.2 we define an even lattice $S_0$ of rank 22 with signature $(1, 21)$ and discriminant $-5^6$ by

$$S_0 := \Sigma_{5A_4} \oplus \langle h, l \rangle,$$

where $\Sigma_{5A_4}$ is the negative-definite root lattice of type $5A_4$, and $\langle h, l \rangle$ is the lattice of rank 2 generated by the vectors $h$ and $l$ satisfying

$$h^2 = 2, \quad l^2 = -2, \quad hl = 1.$$ 

Remark 3.1. This lattice $\langle h, l \rangle$ is the unique even indefinite lattice of rank 2 with discriminant $-5$. See Edwards [7], or Conway and Sloane [5, Table 15.2a].

Claim 3.2. For $\sigma = 1, 2, 3$, there exists an even overlattice $S^{(\sigma)}$ of $S_0$ with the following properties:

(i) the discriminant of $S^{(\sigma)}$ is $-5^{2\sigma}$,

(ii) the Dynkin type of the root system $\{ r \in S^{(\sigma)} \mid rh = 0, r^2 = -2 \}$ is $5A_4$,

(iii) the set $\{ e \in S^{(\sigma)} \mid ce = 1, c^2 = 0 \}$ is empty.
Here we prove that $S(3) = S_0$ satisfies (ii) and (iii). Let $v = s + xh + yl$ be a vector of $S(3) = S_0$, where $s \in \Sigma_{\mathbb{A}_4}$ and $x, y \in \mathbb{Z}$. If $vh = 0$ and $v^2 = -2$, then we have $2x + y = 0$ and $s^2 - 10x^2 = -2$. Since $s^2 \leq 0$, we have $x = y = 0$ and hence $v$ is a root in $\Sigma_{\mathbb{A}_4}$. Therefore $S(3) = S_0$ satisfies (ii). If $vh = 1$ and $v^2 = 0$, then we have $2x + y = 1$ and $s^2 - 10x^2 + 10x - 2 = 0$. Since $s^2 \leq 0$, there is not such an integer $x$. Hence $S(3) = S_0$ satisfies (iii). Thus Claim 3.2 for $\sigma = 3$ has been proved. For the cases $\sigma = 2$ and $\sigma = 1$, see Proposition 4.1 in the next section.

Let $X$ be a supersingular $K3$ surface with $\sigma = \sigma(X) \leq 3$. By the results of Rudakov and Shafarevich [16], the isomorphism class of the lattice $NS(X)$ is characterized by the following properties:

(a) even and signature $(1, 21)$,

(b) the discriminant group is isomorphic to $\mathbb{Z}_{2}^{2\sigma}$.

Since the discriminant group of $S(\sigma)$ is a quotient group of a subgroup of the discriminant group $\mathbb{Z}_{2}^{2\sigma}$ of $S_0$, the lattice $S(\sigma)$ has also these properties. Therefore there exists an isomorphism

$$\phi : S(\sigma) \cong NS(X).$$

By [16] Proposition 3 in Section 3, we can assume that $\phi(h)$ is the class $[H]$ of a nef divisor $H$. Note that $H^2 = h^2 = 2$. If the complete linear system $|H|$ had a fixed component, then, by Nikulin [22] Proposition 0.1, there would be an elliptic pencil $[E]$ and a $(-2)$-curve $\Gamma$ such that $|H| = 2[E] + \Gamma$ and $E\Gamma = 1$, and the vector $e \in S(\sigma)$ that is mapped to $[E]$ by $\phi$ would satisfy $eh = 1$ and $e^2 = 0$. Therefore the property (iii) of $S(\sigma)$ implies that the linear system $|H|$ has no fixed components (see also Urabe [25] Proposition 1.7). Then, by Saint-Donat [17] Corollary 3.2, $|H|$ is base point free. Hence we have a morphism $\Phi_{|H|} : X \to \mathbb{P}^2$ induced by $|H|$. Let

$$X \to Y_H \to \mathbb{P}^2$$

be the Stein factorization of $\Phi_{|H|}$. Then $Y_H \to \mathbb{P}^2$ is a finite double covering branched along a curve $C_H \subset \mathbb{P}^2$ of degree 6. By the property (ii) of $S(\sigma)$, we see that Sing($Y_H$) consists of $5\mathbb{A}_4$-singular points, and hence Sing($C_H$) also consists of $5\mathbb{A}_4$-singular points. By Proposition 2.2 there exists an element $f \in U$ such that $C_H$ is isomorphic to $C_f$. Then $X$ is isomorphic to $X_f$. □

Remark 3.3. In [21], it is proved that a normal $K3$ surface with $5\mathbb{A}_4$-singular points exists only in characteristic 5.

4. Classification of overlattices

Let $F \subset S_0$ be a fundamental system of roots of $\Sigma_{\mathbb{A}_4} \subset S_0$ (see Ebeling [6] for the definition and properties of a fundamental system of roots.) Then $F$ consists of $4 \times 5$ vectors

$$e^{(j)}_i \quad (i = 1, \ldots, 4, \ j = 1, \ldots, 5)$$

such that

$$e^{(j)}_i e^{(j')}_i = \begin{cases} 0 & \text{if } j \neq j' \text{ or } |i - i'| > 1, \\ 1 & \text{if } j = j' \text{ and } |i - i'| = 1, \\ -2 & \text{if } j = j' \text{ and } i = i', \end{cases}$$

(see Figure 4.1). We put
$Aut(F,h) := \{ g \in O(S_0) \mid g(F) = F, g(h) = h \}$,

where $O(S_0)$ is the orthogonal group of the lattice $S_0$. Then $Aut(F,h)$ is isomorphic to the automorphism group of the Dynkin diagram of type $5A_4$, and hence it is isomorphic to the semi-direct product $\{\pm 1\}^5 \rtimes S_5$. Note that $Aut(F,h)$ acts on the dual lattice $(S_0)^\vee$ of $S_0$ in a natural way, and hence it acts on the set of even overlattices of $S_0$. We classify all even overlattices of $S_0$ with the properties (ii) and (iii) in Claim 3.2 up to the action of $Aut(F,h)$. The main tool is Nikulin’s theory of discriminant forms of even lattices [31].

The set $F \cup \{h,l\}$ of vectors form a basis of $S_0$. Let

$$(e_i^{(j)})^\vee \ (i = 1, \ldots, 4, \ j = 1, \ldots, 5), \ h^\vee \ and \ l^\vee$$

be the basis of $(S_0)^\vee$ dual to $F \cup \{h,l\}$. We denote by $G$ the discriminant group $(S_0)^\vee/S_0$ of $S_0$, and by

$$pr : (S_0)^\vee \to G$$

the natural projection. Then $G$ is isomorphic to $\mathbb{F}_5^{\oplus 5} \oplus \mathbb{F}_5$ with basis

$$pr((e_1^{(1)})^\vee), \ldots, pr((e_5^{(5)})^\vee), \ pr(h^\vee).$$

With respect to this basis, we denote the elements of $G$ by $[x_1, \ldots, x_5 | y]$ with $x_1, \ldots, x_5, y \in \mathbb{F}_5$. The discriminant form $q : G \to \mathbb{Q}/2\mathbb{Z}$ of $S_0$ is given by

$$q([x_1, \ldots, x_5 | y]) = -\frac{4}{5}(x_1^2 + \cdots + x_5^2) + \frac{2}{5}y^2 \mod 2\mathbb{Z}$$

The action of $Aut(F,h)$ on $G = \mathbb{F}_5^{\oplus 5} \oplus \mathbb{F}_5$ is generated by the multiplications by $-1$ on $x_i$, and the permutations of $x_1, \ldots, x_5$. We define subgroups $H_0, \ldots, H_8$ of $G$ by their generators as follows:

$$H_0 := \{0\},$$

$$H_1 := \langle [0,0,2,2,2 | 2] \rangle,$$

$$H_2 := \langle [2,2,2,2,0] \rangle,$$

$$H_3 := \langle [0,1,2,2,2 | 1] \rangle,$$

$$H_4 := \langle [1,2,2,2,2] \rangle,$$

$$H_5 := \langle [0,1,1,2,2 | 0] \rangle,$$

$$H_6 := \langle [1,0,1,2,2 | 0], [0,1,2,1,3 | 0] \rangle,$$

$$H_7 := \langle [1,0,0,1,1 | 1], [0,1,1,3,3] \rangle,$$

$$H_8 := \langle [1,0,1,1,2 | 2], [0,1,1,3,3 | 0] \rangle.$$

We then put

$$S_i := pr^{-1}(H_i) \subset (S_0)^\vee.$$
Using a computer, we make the complete list of subgroups of $G$. \begin{table}[h]
\begin{center}
| the $(a,b,y)$-type | the roots in $h^\perp$ | the set $E$ |
|-------------------|----------------------|--------|
| $(0,0,0)$         | $5A_4$              | empty  |
| $(0,2,\pm 1)$    | $A_9 + 3A_4$        | empty  |
| $(0,3,\pm 2)$    | $5A_4$              | empty  |
| $(0,5,0)$        | $5A_4$              | empty  |
| $(1,1,0)$        | $E_8 + 3A_4$        | empty  |
| $(1,3,\pm 1)$    | $5A_4$              | empty  |
| $(1,4,\pm 2)$    | $5A_4$              | empty  |
| $(2,0,\pm 2)$    | $A_9 + 3A_4$        | empty  |
| $(2,2,0)$        | $5A_4$              | empty  |
| $(3,0,\pm 1)$    | $5A_4$              | empty  |
| $(3,1,\pm 2)$    | $5A_4$              | empty  |
| $(4,1,\pm 1)$    | $5A_4$              | empty  |
| $(5,0,0)$        | $5A_4$              | empty  |
\end{center}
\caption{The isotropic vectors in $(G,q)$}
\end{table}

**Proposition 4.1.** The submodules $S_0, \ldots , S_8$ of $(S_0)^\vee$ are even overlattices of $S_0$ with the properties (ii) and (iii) in Claim 3.2. The discriminant of $S_1$ is $-5^6$ for $i = 0, -5^4$ for $i = 1, \ldots , 5$, and $-5^2$ for $i = 6, \ldots , 8$.

Conversely, if $S$ is an even overlattice of $S_0$ with the properties (ii) and (iii), then there exists a unique $S_i$ among $S_0, \ldots , S_8$ such that $S = g(S_i)$ holds for some $g \in \text{Aut}(F,h)$.

**Proof.** The mapping $S \mapsto S/S_0$ gives rise to a one-to-one correspondence between the set of even overlattices $S$ of $S_0$ and the set of totally isotropic subgroups $H$ of $(G,q)$. The inverse mapping is given by $H \mapsto \text{pr}^{-1}(H)$. If $\dim_{F_5} H = d$, then the discriminant of $\text{pr}^{-1}(H)$ is equal to $-5^{6-2d}$ (see Nikulin [11]).

For $v = [x_1, \ldots , x_5 | y] \in G$, we put \[
\delta(v) := (a,b,y) \in \Z_{\geq 0} \times \Z_{\geq 0} \times F_5,
\] where $a$ is the number of $\pm 1 \in F_5$ among $x_1, \ldots , x_5$ and $b$ is the number of $\pm 2 \in F_5$ among $x_1, \ldots , x_5$. Note that $\delta(v) = \delta(w)$ holds if and only if there exists $g \in \text{Aut}(F,h)$ such that $g(v) = w$. A vector $v \in G$ is isotropic with respect to $q$ if and only if $\delta(v)$ appears in the first column of Table 4.1. For each $(a,b,y)$-type $\alpha$ in Table 4.1, we choose a vector $v \in G$ such that $\delta(v) = \alpha$, and calculate the even overlattice $S_\alpha := \text{pr}^{-1}(\{v\})$ of $S_0$. The second column of Table 4.1 presents the Dynkin type of the root system $\{r \in S_\alpha | rh = 0, r^2 = -2\}$, and the third column presents the set $E := \{e \in S_\alpha | eh = 1, e^2 = 0\}$. Hence we see that the following two conditions on a subgroup $H$ of $G$ are equivalent:

(I) The corresponding submodule $\text{pr}^{-1}(H)$ of $(S_0)^\vee$ is an even overlattice of $S_0$ with the properties (ii) and (iii) in Claim 3.2.

(II) For any $v \in H$, $\delta(v)$ is an $(a,b,y)$-type with $* \in$ Table 4.1.

Using a computer, we make the complete list of subgroups of $G$ that satisfy the condition (II) up to the action of $\text{Aut}(F,h)$. The complete set of representatives is $\{H_0, \ldots , H_8\}$ above. \qed
Remark 4.2. Since there exist no even unimodular lattices of signature $(1, 21)$ (see Serre [18, Theorem 5 in Chapter V]), all totally isotropic subgroups of $(G, q)$ are of dimension $\leq 2$ over $\mathbb{F}_5$.

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