Abstract

The paper consists of two parts. Part I develops efficient optimization based methods for the dissipative control and observation of linear systems with general delays of finite length via the Krasovskii functional (KL) approach. Our model imposes no limits to the numbers of pointwise and distributed delays (DDs) at the states, inputs, regulated and measured outputs, where the DDs integral kernels can contain any number of square-integrable functions. The key to handle DDs is a novel decomposition scheme allowing users to simultaneously factorize or approximate different integral kernels. Moreover, this approach allows one to construct complete KLs with more general forms, whose integral kernels can contain any number of differentiable linearly independent functions. Several theorems and algorithms are proposed for the solutions of dissipative state-feedback, observation and observer-based controller design problems, where the synthesis conditions can be computed without the use of nonlinear solvers. More importantly, the proposed methods can simultaneously compute controller/observer gains subject to dissipative constraints.

Keywords: General Linear Time-Delay Systems; Distributed Delays; Observer Based Controller Design, Dissipativity.

1. Introduction

Generally speaking, two types of delays, pointwise and distributed delays (DDs), have been utilized to model transport, propagation or aftereffects in dynamical systems. The mathematical nature of a pointwise-delay \( x(t - r) \) is explained in Richard (2003) where it is denoted by a transport equation with appropriate boundary conditions. This may intuitively explain why the dimension of a system with delays can be infinite Curtain & Zwart (1995). Meanwhile, time delays can be effected by transporting media with more complex structures. A DD is denoted by an integral \( \int_{-r}^{0} F(\tau)x(t + \tau)d\tau \) over a delay interval \([-r, 0]\) with a matrix-valued function \( F(\cdot) \), which takes into account a segment of the past dynamics’s information. Systems with DDs have many important applications such as modeling biological processes Elazzouzi et al. (2019), chemical reaction networks Lipták et al. (2019) and predictor controllers Kharitonov (2017). However, dealing with these integrals is much more challenging than pointwise delays due to the complexity carried by \( F(\cdot) \), even if only a linear system is concerned.

Most methodologies for linear time delay systems (LTDSs) are carried out in the traditional sense of the time or frequency domain, where delays are treated as linear operators. To the best of the author’s
knowledge, the newest trend of frequency-domain-based methods can be summarized by the results in Gumussoy & Michiels (2011); Michiels & Niculescu (2014) and Apkarian et al. (2018); Apkarian & Noll (2018). These works are mainly nourished by the recent development of numerical algorithms for non-smooth optimizations Noll & Apkarian (2005a,b); Lewis (2013). However, no general DDs \( \int_{-\infty}^{0} F(\tau)x(t+\tau)d\tau \) have been considered by the above publications, which could be attributed to the difficulties in dealing with the Laplace transform of \( \int_{-\infty}^{0} F(\tau)x(t+\tau)d\tau \) since the integral \( \int_{-\infty}^{0} F(\tau)e^{\tau s}d\tau \) may not have a closed-form expression.

For time-domain based approaches Gu et al. (2003); Kharitonov (2012), the use of Krasovskii functionals (KF) has been proven as an effective tool for the stability analysis and stabilization of LTDSs Seuret et al. (2015); Feng & Nguang (2016); Feng et al. (2020); Peet (2019) supported by efficient numerical algorithms for the computation of semidefinite programming (SDPs) Boyd et al. (1994); Parrilo (2003). For a comprehensive collection of the existing literature on this topic, see the monographs in Briat (2014). In contrast to the Lyapunov approach for an LTI system, only sufficient conditions are attainable for an infinite dimensional system (LTDSs) via the KL approach, where the level of conservatism is largely based on the generality of the predetermined-form of KFs Kharitonov (2012) and the integral inequalities Feng & Nguang (2016) utilized to construct them. Because KFs with more sophisticated structures Seuret et al. (2015); Feng et al. (2020) have been increasingly adopted, it is usually the case that the SDPs conditions for the stability analysis of a LTDS may not be easily modified via congruence transformations to form convex synthesis conditions for computing controller or observer gains. Finally, a very interesting method is set out in Vite et al. (2021) for the stabilization of LTDSs with DDs, where it can be considered as a combination of both time and frequency domain approaches based on the concept of smoothed spectral abscissa Vanbiervliet et al. (2009); ? and delay Lyapunov matrix Egorov et al. (2017); Gomez et al. (2019, 2021).

Nevertheless, it is safe to say that there are no effective solutions in the literature for the control and observation of LTDSs with general DDs, especially for the case of dissipative observer-based controller design with an unlimited number of delays. Even if only stability analysis is concerned, most existing KL approaches impose restrictions on the structure of state space parameters Münz et al. (2009) or DD kernels Feng & Nguang (2016) or the number of DD kernels and delays Seuret et al. (2015); Feng et al. (2020). Moreover, the novel controller synthesis method in Vite et al. (2021) requires the numerical computation of the delay Lyapunov matrix with its derivatives. It has not been elaborated in Vite et al. (2021) how the computations can be carried out for an LTDS with general DDs where all the delay values are non-commensurate. Finally, the well-established optimal control Delfour (1986) and infinite-dimensional system Pritchard (1987); Salamon (1987) approaches for LTDSs require solving operator-valued Riccati-type equations, which is extremely difficult to be realized numerically.

Motivated by the aforementioned observations, we propose our solutions for the control and observation problems of LTDSs based on the LKF approach, where the model can possess unlimited numbers of pointwise and general DDs at the states, inputs and outputs. The DDs can contain any number of \( L^2 \) function over bounded intervals where users have the liberty to choose which functions are approximated or directly factorized. The main results of our works are summarized in several theorems and algorithms concerning dissipative state-feedback control (DSFC), dissipative state observer design (DSOD) and dissipative observer-based control, (DOBC). The solutions to these three problems are obtained by numerically solving convex SDPs without using nonlinear solvers. Finally, the solutions have also been extended to
construct controllers/observers possessing delays for LTDSs without delays at the inputs and outputs.

The major contributions of this paper are summarized as follows:

- Apart from the early works in Delfour (1986); Pritchard (1987); Salamon (1987), we believe the control and observation problems in this work have not been researched in the literature. This is particularly true for the cases of dissipative observer design and observer-based control. The structure of our LTDS is sufficiently general with the consideration of dissipativity, imposing no constraints on the number of pointwise and distributed delays. Moreover, all DDs can contain any number of $L^2$ function over a bounded delay interval, which is clearly adequate for the modeling of many practical delay systems.

- The $L^2$ DD kernels are handled by a novel decomposition approach, which significantly generalizes the approximation scheme in Feng et al. (2020). Unlike Feng et al. (2020) where all $L^2$ DD kernels are approximated by $f(\cdot)$ satisfying $\exists M \in \mathbb{R}^{d \times d}, \frac{df(\tau)}{d\tau} = M f(\tau)$ and $\int_{-\tau}^{\tau} f(\tau)f^T(\tau) d\tau \succ 0$, the proposed approach allows users to determine which kernels are directly extracted and which are approximated by any number of differentiable and linearly independent functions with their derivatives. Moreover, the proposed scheme also enables one to construct KFs with substantial generality, where the integral kernels can contain any number of differentiable and linearly independent functions which can be totally independent of the functions inside of the DDs.

- For each case of DSFC, DSOD and DOBC, we propose two theorems with an iterative algorithm (offline). Specifically, the second theorem in each case is derived from convexifying the first theorem via Projection Lemma Gahinet & Apkarian (1994), without weakening the generality of the matrix parameters of the LKFs. Moreover, the first theorem in each case can be solved by the iterative algorithms initiated by a feasible solution of the second theorem. As a result, the proposed methodologies do not require the use of nonlinear SDP solvers. Finally, our method allows the simultaneous computation of both controller/observer gains subject to dissipative constraints, which is a very attractive feature if optimal stabilization is required.

- The proposed synthesis methods have also been extended to construct controllers/observers with DDs for an LTDS without delays at the input and measured output. This is important since the use of static controllers/observers may not be sufficient for a distributed delay system in terms of stability or performance.

- Due to the connection between LTDSs and other types of systems, it has been shown that advanced treatment of LTDSs with general DDs Seuret et al. (2015); Feng & Nguang (2016); Feng et al. (2020) can lead to efficient solutions of many engineering problems such as networked control system Yan et al. (2019), Neural Networks Li et al. (2021) and PDE-ODE coupled system Baudouin et al. (2019). As a result, the proposed methods can serve as a blueprint for the future development of new solutions of real-world and engineering problems.

The organization of the rest of the paper is outlined as follows. Preliminaries are first presented in Section 2 including the open-loop system and the novel decomposition approach proposed for tackling DDs. The main results concerning DSFC, DSOD and DOBC are set out in Sections 3–5, respectively. Finally, the computation results of two numerical examples are summarized in Section 6 prior to the final conclusion. Note that we place some important lemmas and proofs in the appendices.
2. Preliminaries

2.1. Open-loop LTDS

In this paper, we seek to design controllers and observers for an LTDS

\[
\begin{align*}
\dot{x}(t) &= \sum_{i=0}^{\nu} A_i x(t - r_i) + \sum_{i=1}^{\nu} \int_{-r_i}^{t} A_i(\tau) x(t + \tau) d\tau \\
&+ \sum_{i=0}^{\nu} B_i u(t - r_i) + \sum_{i=1}^{\nu} \int_{-r_i}^{t} B_i(\tau) u(t + \tau) d\tau + D_1 w(t), \quad \forall t \geq t_0 \in \mathbb{R} \\
\dot{z}(t) &= \sum_{i=0}^{\nu} C_i x(t - r_i) + \sum_{i=1}^{\nu} \int_{-r_i}^{t} C_i(\tau) x(t + \tau) d\tau \\
&+ \sum_{i=0}^{\nu} D_i u(t - r_i) + \sum_{i=1}^{\nu} \int_{-r_i}^{t} D_i(\tau) u(t + \tau) d\tau + D_2 w(t) \quad (1)
\end{align*}
\]

where \(\psi(\cdot) \in C([-r_0, 0] ; \mathbb{R}^n)\), and \(r_0 > r_{\nu-1} > \cdots > r_2 > r_1 > r_0 = 0\) are given constants with \(\nu \in \mathbb{N}\). Furthermore, \(x(\cdot) : [t_0 - r_\nu, \infty) \to \mathbb{R}^n\) satisfies (1), \(u(t) \in \mathbb{R}^p\) denotes the control input, \(w(\cdot) \in L^2(\mathbb{R}_{\geq t_0} ; \mathbb{R}^q)\).

\[\sqrt{X}^{-1} = \left(\sqrt{X}\right)^{-1} \text{ for any } X > 0 \text{ based on the application of eigendecomposition of } X > 0\]

\[\text{Note that } \sqrt{X}^{-1} \text{ for any } X > 0 \text{ based on the application of eigendecomposition of } X > 0\]
represents an exogenous disturbance, \( z(t) \in \mathbb{R}^m \) is the regulated output. The size of the matrices in (1) is determined by the values of \( n \in \mathbb{N} \) and \( m; p; q \in \mathbb{N}_0 := \mathbb{N} \cup \{0\} \). Finally, the DDs in (1) satisfy

\[
\forall i = 1, \ldots, \nu, \quad \tilde{A}_i(\cdot) \in L^2([-r_i, -r_{i-1}] ; \mathbb{R}^{n \times n}), \quad \tilde{C}_i(\cdot) \in L^2([-r_i, -r_{i-1}] ; \mathbb{R}^{m \times n})
\]

\[
\tilde{B}_i(\cdot) \in L^2([-r_i, -r_{i-1}] ; \mathbb{R}^{n \times p}), \quad \tilde{S}_i(\cdot) \in L^2([-r_i, -r_{i-1}] ; \mathbb{R}^{m \times p}).
\]  

It is obvious to see the generality of the TDS model in (1), as the number of delays \( \nu \in \mathbb{N} \) is unlimited and all DDs in (2) can contain any \( L^2 \) function. As a matter of fact, a general linear time delay system in the form of Lebesgue-Stieltjes integrals (see Hale & Lunel (1993)) can be denoted by (1) using the property of distributions (delta functions). Note that the functional differential equation in (1) holds for almost all \( t \geq t_0 \) with respect to the Lebesgue measure since \( w(\cdot) \in L^2(\mathbb{R}_{\geq t_0}; \mathbb{R}^q) \).

**Remark 1.** Many practical systems can be modeled by (1), such as the SIR model with vaccination and treatment in (Elazzouzi et al., 2019, eq.(7)) or the networked control system described in Goebel et al. (2011), or the chemical reaction networks in (Lipták et al., 2019, eq.(30)), etc.

### 2.2. A novel decomposition scheme for DDs

Since the kernels of DDs in (2) are infinite-dimensional, including them in any synthesis (stability) condition results in solving optimization problems with infinite dimension. To circumvent this obstacle, a novel decomposition scheme is proposed as follows, where we show that all the DDs in (2) can be parameterized by matrices with finite dimensions.

**Proposition 1.** The conditions in (2) hold if and only if there exist functions \( f_i(\cdot) \in C^1([-r_i, -r_{i-1}] ; \mathbb{R}^{d_i}) \), \( \varphi_i(\cdot) \in L^2([-r_i, -r_{i-1}] ; \mathbb{R}^{d_i}) \), \( \phi_i(\cdot) \in L^2([-r_i, -r_{i-1}] ; \mathbb{R}^{\nu}) \) and matrices \( M_i \in \mathbb{R}^{d_i \times \kappa} \), \( \tilde{A}_i \in \mathbb{R}^{n \times \kappa \times n} \), \( \tilde{B}_i \in \mathbb{R}^{n \times \kappa \times p} \), \( \tilde{C}_i \in \mathbb{R}^{m \times \kappa \times n} \), \( \tilde{S}_i \in \mathbb{R}^{m \times \kappa \times p} \) with \( i = 1, \ldots, \nu \) such that

\[
\tilde{A}_i(\tau) = \tilde{A}_i(g_i(\tau) \otimes I_n), \quad \tilde{B}_i(\tau) = \tilde{B}_i(g_i(\tau) \otimes I_p),
\]

\[
\tilde{C}_i(\tau) = \tilde{C}_i(g_i(\tau) \otimes I_n), \quad \tilde{S}_i(\tau) = \tilde{S}_i(g_i(\tau) \otimes I_p),
\]

\[
\frac{d f_i(\tau)}{d \tau} = M_i \tilde{f}_i(\tau), \quad \tilde{f}_i(\tau) = \begin{bmatrix} \varphi_i(\tau) \\ \phi_i(\tau) \end{bmatrix},
\]

\[
\mathcal{G}_i := \int_{-r_i}^{0} g_i(\tau) g_i^T(\tau) d \tau > 0, \quad g_i(\tau) = \begin{bmatrix} \varphi_i(\tau) \\ \phi_i(\tau) \end{bmatrix}
\]  

hold for all \( i = 1, \ldots, \nu \) and \( \forall \tau \in [-r_i, -r_{i-1}] \), where \( \kappa_i = \delta_i + \delta_i + \mu_i, \ k_i = d_i + \delta_i + \mu_i \) with \( d_i \in \mathbb{N} \) and \( \delta_i; \mu_i \in \mathbb{N}_0 := \mathbb{N} \cup \{0\} \) for all \( i = 1, \ldots, \nu \). Moreover, the derivatives in (5) at \( \tau = -r_i \) and \( \tau = -r_{i-1} \) are one-sided derivatives, and this proposition is always true for the case of \( \mu_i = 0 \).

**Proof.** See Appendix B.

**Remark 2.** Proposition 1 can be applied to any DD satisfying (2) without suffering any conservatism. From Appendix B, we have seen that any \( f_i(\cdot) \in C^1([-r_i, -r_{i-1}] ; \mathbb{R}^{d_i}) \) can be used in (5) as part of \( g_i(\cdot) \) even if no functions in \( f_i(\cdot) \) can be found among the kernels of (2). This is because one can add unlimited number of new functions to \( f_i(\cdot) \) or \( \varphi_i(\cdot) \) and (5) can still be satisfied by some \( M_i \in \mathbb{R}^{d_i \times \kappa} \). As long as all the kernels in (2) are “covered” by some of the functions in \( g_i(\cdot) \), then the matrices in Proposition 1 can be constructed accordingly. Finally, \( \mathcal{G}_i \) in (6) are the Gramian matrices of \( g_i(\cdot) \) where the inequalities indicate the functions in \( g_i(\cdot) \) in (5) are linearly independent in a Lebesgue sense over \([-r_i, -r_{i-1}] \) for each \( i = 1, \ldots, \nu \).
3. Dissipative state feedback controller (DSFC) design

3.1. Problem Formulation

Inspired by the state variables of the ODE-PDE coupled system in Safi et al. (2017), let \( \chi(t, \theta) = \text{Col}_i \varphi, x(t + \hat{r}_i \theta - r_{i-1}) \in \mathbb{R}^{m \nu} \) with \( \theta \in [-1, 0] \). Now apply a state feedback controller \( u(t) = K x(t) \) with \( K \in \mathbb{R}^{p \times n} \) to (1) and utilize Proposition 1, then the resulting closed-loop system is obtained as

\[
\dot{x}(t) = (A_0 + B_0 K) x(t) + \left[ \text{Row}_i (A_i + B_i K) \right] \chi(t, -1) + D_1 w(t)
\]

\[
+ \sum_{i=1}^{\nu} \int_{-r_i}^{r_i-1} \left( \tilde{A}_i + \tilde{B}_i (I_{K_i} \otimes K) \right) (g_i(\tau) \otimes I_n) x(t + \tau) d\tau, \quad \forall t \geq t_0
\]

\[
z(t) = (C_0 + \mathcal{B}_0 K) x(t) + \left[ \text{Row}_i (C_i + \mathcal{B}_i K) \right] \chi(t, -1) + D_2 w(t)
\]

\[
+ \sum_{i=1}^{\nu} \int_{-r_i}^{r_i-1} \left( \tilde{C}_i + \tilde{\mathcal{B}}_i (I_{K_i} \otimes K) \right) (g_i(\tau) \otimes I_n) x(t + \tau) d\tau
\]

\[
\forall \theta \in [-r_i, 0], \ x(t_0 + \theta) = \psi(\tau)
\]

where the decomposition of DDs has also utilized

\[
(g_i(\tau) \otimes I_p) K = (g_i(\tau) \otimes I_p) (1 \otimes K) = I_{K_i} g_i(\tau) \otimes K I_n = (I_{K_i} \otimes K) (g_i(\tau) \otimes I_n), \quad i = 1, \ldots, \nu.
\]

The functions \( \varphi_i(\cdot) \) and \( \phi_i(\cdot) \) are separated in \( g_i(\tau) \) because they will receive different mathematical treatment in this paper. Specifically, \( \phi_i(\tau) \) is always approximated by \( \hat{f}_i(\tau) \) via

\[
\phi_i(\tau) = \Gamma_i F_i^{-1} \hat{f}_i(\tau) + e_i(\tau), \quad i = 1, \ldots, \nu, \quad \tau \in [-r_i, r_i-1]
\]

with

\[
\mathbb{R}^{\mu_i \times \kappa_i} \ni \Gamma_i := \int_{-r_i}^{r_i-1} \phi_i(\tau) \hat{f}_i(\tau) d\tau, \quad F_i = \int_{-r_i}^{r_i-1} \hat{f}_i(\tau) \hat{f}_i(\tau)^T d\tau
\]

based on the application of least-square approximation. (See page 182 of Muscat (2014) for the expression of the approximation). Note that \( e_i(\tau) = \phi_i(\tau) - \Gamma_i F_i^{-1} \hat{f}_i(\tau) \) defines the error, and \( F_i > 0 \) always holds because of (6). Moreover, we utilize

\[
\mathbb{S}^{\mu_i} \ni E_i := \int_{-r_i}^{r_i-1} e_i(\tau) e_i(\tau)^T d\tau = \int_{-r_i}^{r_i-1} \left( \phi_i(\tau) - \Gamma_i F_i^{-1} \hat{f}_i(\tau) \right) \left( \phi_i(\tau) - \Gamma_i F_i^{-1} \hat{f}_i(\tau) \right)^T d\tau
\]

\[
= \int_{-r_i}^{r_i-1} \phi_i(\tau) \phi_i(\tau)^T d\tau - \mathbb{S} \left( \int_{-r_i}^{r_i-1} \phi_i(\tau) \hat{f}_i(\tau) d\tau F_i^{-1} \Gamma_i^T \right) + \Gamma_i F_i^{-1} \int_{-r_i}^{r_i-1} \hat{f}_i(\tau) \hat{f}_i(\tau)^T d\tau F_i^{-1} \Gamma_i^T
\]

\[
= \int_{-r_i}^{r_i-1} \phi_i(\tau) \phi_i(\tau)^T d\tau - \Gamma_i F_i^{-1} \Gamma_i^T
\]

(11) to measure the approximation error residual, where \( E_i > 0 \) always holds due to (Feng et al., 2020, eq.(18)).
Remark 3. The expressions of the least square approximation in (9)--(11) are well defined with $\mu_i = 0$, $\phi_i(\cdot) = \|0_{x1} \) corresponding to the case that no functions are approximated in $g_i(\cdot)$. Such a case is always usable with Proposition 1 since one can add unlimited number of $\mathbb{L}^2$ linearly independent functions to $\varphi_i(\cdot)$. This is a perfect example on the advantage of adopting empty matrices, as the cases of $\phi_i(\tau) = \|0_{x1} \) and $\phi_i(\tau) \neq \|0_{x1} \) can be treated with a unified framework. In conclusion, Proposition 1 allows users to decide which $\mathbb{L}^2$ functions in (2) are approximated (the ones in $\phi_i(\cdot)$) by $\hat{f}(\cdot)$ and which are factorized directly (the ones in $\varphi_i(\cdot)$).

Remark 4. If we assume $\delta_i = 0$ in Proposition 1, then (9)--(12) with $\nu = 1$ becomes identical to the approximation scheme in Feng et al. (2020). However, the absence of $\varphi_i(\cdot)$ severely limits the generality of the approximator $f_i(\cdot)$, since the condition $\frac{\partial f_i(\tau)}{\partial \tau} = M_i f_i(\tau)$ cannot be satisfied by all differentiable functions for some $M \in \mathbb{R}^{d_i \times d_i}$. This shows the significant generality of Proposition 1 over the approach in Feng et al. (2020).

Now by (5) and (9), we have
\[ g_i(\tau) = \begin{bmatrix} \phi_i(\tau) \\ \tilde{f}_i(\tau) \end{bmatrix} = \begin{bmatrix} \Gamma_i F_i^{-1} \tilde{f}_i(\tau) \\ \tilde{f}_i(\tau) \end{bmatrix} + \begin{bmatrix} \varepsilon_i(\tau) \\ 0_{\alpha_i} \end{bmatrix} = \Gamma_i \tilde{f}_i(\tau) + \tilde{I}_i \varepsilon_i(\tau), \] (12)
which further gives the identity
\[ \forall i = 1, \ldots, \nu, \quad (I_{\alpha_i} \otimes K) (g_i(\tau) \otimes I_n) = (I_{\alpha_i} \otimes K) \left[ \left( \Gamma_i \tilde{f}_i(\tau) + \tilde{I}_i \varepsilon_i(\tau) \right) \otimes I_n \right] \]
\[ = (I_{\alpha_i} \otimes K) \left( \tilde{I}_i \otimes I_n \right) \left( \tilde{f}_i(\tau) \otimes I_n \right) + \left( I_{\alpha_i} \otimes K \right) \left( I_i \otimes I_n \right) \left( \varepsilon_i(\tau) \otimes I_n \right) \]
\[ = \left( \Gamma_i \otimes I_p \right) \left( I_{\alpha_i} \otimes K \right) \left( \tilde{f}_i(\tau) \otimes I_n \right) + \left( \tilde{I}_i \otimes I_p \right) \left( I_{\alpha_i} \otimes K \right) \left( \varepsilon_i(\tau) \otimes I_n \right) \] (13)
\[ g_i(\tau) \otimes I_n = \left( \tilde{I}_i \tilde{f}_i(\tau) + \tilde{I}_i \varepsilon_i(\tau) \right) \otimes I_n + \left( \tilde{I}_i \otimes I_n \right) \left( \tilde{f}_i(\tau) \otimes I_n \right) + \left( \tilde{I}_i \otimes I_n \right) \left( \varepsilon_i(\tau) \otimes I_n \right) \] (14)
via (8) and (A.1). Using (12)--(13) and (A.1) to the DDs in (7), we can conclude that
\[ \left( \hat{A}_i + \hat{B}_i \left( I_{\alpha_i} \otimes K \right) \right) \left( g_i(\tau) \otimes I_n \right) = \left( \hat{A}_i \left( \tilde{I}_i \otimes I_n \right) + \hat{B}_i \left( \tilde{I}_i \otimes K \right) \right) \left( \tilde{f}_i(\tau) \otimes I_n \right) \]
\[ + \left[ \hat{A}_i \left( \tilde{I}_i \otimes I_n \right) + \hat{B}_i \left( \tilde{I}_i \otimes K \right) \right] \left( \varepsilon_i(\tau) \otimes I_n \right) = \left( \hat{A}_i \left( \tilde{I}_i \otimes I_n \right) + \hat{B}_i \left( \tilde{I}_i \otimes K \right) \right) \left( \sqrt{F_i^{-1}} \tilde{f}_i(\tau) \otimes I_n \right) \]
\[ + \left( \hat{A}_i \left( \tilde{I}_i \sqrt{E_i} \otimes I_n \right) + \hat{B}_i \left( \tilde{I}_i \sqrt{E_i} \otimes K \right) \right) \left( \sqrt{E_i^{-1}} \varepsilon_i(\tau) \otimes I_n \right) . \] (15)
\[ \left( \hat{C}_i + \hat{B}_i \left( I_{\alpha_i} \otimes K \right) \right) \left( g_i(\tau) \otimes I_n \right) = \left( \hat{C}_i \left( \tilde{I}_i \otimes I_n \right) + \hat{B}_i \left( \tilde{I}_i \otimes K \right) \right) \left( \sqrt{F_i^{-1}} \tilde{f}_i(\tau) \otimes I_n \right) \]
\[ + \left( \hat{C}_i \left( \tilde{I}_i \sqrt{E_i} \otimes I_n \right) + \hat{B}_i \left( \tilde{I}_i \sqrt{E_i} \otimes K \right) \right) \left( \sqrt{E_i^{-1}} \varepsilon_i(\tau) \otimes I_n \right) . \] (16)
for all $i = 1, \ldots, \nu$, where $\tilde{I}_i = \tilde{I}_i \sqrt{T_i} = \left[ \begin{bmatrix} \sqrt{F_i^{-1}} \tilde{f}_i(\tau) \\ \sqrt{F_i^{-1}} \tilde{I}_i \end{bmatrix} \right].$

Now by (15)--(16) with (A.1), the DDs in (7) can be further denoted as
\[
\sum_{i=1}^{\nu} \int_{-r_i}^{-r_i-1} \left( \tilde{A}_i + \tilde{B}_i \left( I_{\kappa_i} \otimes K \right) \right) \left( g_i(\tau) \otimes I_n \right) x(t + \tau) d\tau \\
= \sum_{i=1}^{\nu} \left[ \text{Row} \left[ \tilde{A}_i \left( I_{\kappa_i} \otimes I_n \right) + \tilde{B}_i \left( I_{\kappa_i} \otimes K \right) \right] \text{Col} \left[ \int_{-r_i}^{-r_i-1} \left( \sqrt{F_i^{-1}\tilde{f}_i(\tau)} \otimes I_n \right) x(t + \tau) d\tau \right] \\
+ \text{Row} \left[ \tilde{A}_i \left( I_{\kappa_i} \sqrt{E_i} \otimes I_n \right) + \tilde{B}_i \left( I_{\kappa_i} \sqrt{E_i} \otimes K \right) \right] \text{Col} \left[ \int_{-r_i}^{-r_i-1} \left( \sqrt{E_i^{-1} e_i(\tau)} \otimes I_n \right) x(t + \tau) d\tau \right] \right] (17)
\]

\[
\sum_{i=1}^{\nu} \int_{-r_i}^{-r_i-1} \left( \tilde{C}_i + \tilde{D}_i \left( I_{\kappa_i} \otimes K \right) \right) \left( g_i(\tau) \otimes I_n \right) x(t + \tau) d\tau \\
= \sum_{i=1}^{\nu} \left[ \text{Row} \left[ \tilde{C}_i \left( I_{\kappa_i} \otimes I_n \right) + \tilde{D}_i \left( I_{\kappa_i} \otimes K \right) \right] \text{Col} \left[ \int_{-r_i}^{-r_i-1} \left( \sqrt{F_i^{-1}\tilde{f}_i(\tau)} \otimes I_n \right) x(t + \tau) d\tau \right] \\
+ \text{Row} \left[ \tilde{C}_i \left( I_{\kappa_i} \sqrt{E_i} \otimes I_n \right) + \tilde{D}_i \left( I_{\kappa_i} \sqrt{E_i} \otimes K \right) \right] \text{Col} \left[ \int_{-r_i}^{-r_i-1} \left( \sqrt{E_i^{-1} e_i(\tau)} \otimes I_n \right) x(t + \tau) d\tau \right] \right] (18)
\]

Note that \((\bigoplus_{i=1}^{\nu} X_i) \otimes I_n = \bigoplus_{i=1}^{\nu} (X_i \otimes I_n)\) for all \(X_i \in \mathbb{C}^{n \times \nu},\) hence the above form can be written as \(\bigoplus_{i=1}^{\nu} X_i \otimes I_n\) by dropping the parenthesis.

Now using (17)–(18) to (7) produces

\[
\dot{x}(t) = (A + B_1 [(I_{1+\nu+\kappa} \otimes K) \otimes O_q]) \theta(t), \quad z(t) = (C + B_2 [(I_{1+\nu+\kappa} \otimes K) \otimes O_q]) \theta(t), \quad \forall t \geq t_0
\]

\[
\forall \theta \in [-r_{\nu}, 0], \quad x(t_0 + \theta) = \psi(\theta)
\]

with \(t_0\) and \(\psi(\cdot)\) in (1), where \(\kappa = \sum_{i=1}^{\nu} \kappa_i\) with \(\kappa_i = d_i + b_i + \mu_i\) and

\[
A = \begin{bmatrix}
\text{Row} A_i \\
\text{Row} A_i \\
\text{Row} A_i
\end{bmatrix}
\begin{bmatrix}
\text{Row} \tilde{A}_i \left( I_{\kappa_i} \otimes I_n \right) \\
\text{Row} \tilde{A}_i \left( I_{\kappa_i} \sqrt{E_i} \otimes I_n \right) \\
\text{Row} \tilde{A}_i \left( I_{\kappa_i} \sqrt{E_i} \otimes K \right)
\end{bmatrix}
\begin{bmatrix}
D_1 \\
O_{n,q} \\
D_2
\end{bmatrix} (20)
\]

\[
B_1 = \begin{bmatrix}
\text{Row} B_i \\
\text{Row} B_i \\
\text{Row} B_i
\end{bmatrix}
\begin{bmatrix}
\text{Row} \tilde{B}_i \left( I_{\kappa_i} \otimes I_p \right) \\
\text{Row} \tilde{B}_i \left( I_{\kappa_i} \sqrt{E_i} \otimes I_p \right) \\
\text{Row} \tilde{B}_i \left( I_{\kappa_i} \sqrt{E_i} \otimes K \right)
\end{bmatrix}
\begin{bmatrix}
O_{n,q} \\
D_2 \\
O_{m,q}
\end{bmatrix} (21)
\]

\[
C = \begin{bmatrix}
\text{Row} C_i \\
\text{Row} C_i \\
\text{Row} C_i
\end{bmatrix}
\begin{bmatrix}
\text{Row} \tilde{C}_i \left( I_{\kappa_i} \otimes I_n \right) \\
\text{Row} \tilde{C}_i \left( I_{\kappa_i} \sqrt{E_i} \otimes I_n \right) \\
\text{Row} \tilde{C}_i \left( I_{\kappa_i} \sqrt{E_i} \otimes K \right)
\end{bmatrix}
\begin{bmatrix}
D_2 \\
D_2 \\
O_{m,q}
\end{bmatrix} (22)
\]

\[
B_2 = \begin{bmatrix}
\text{Row} B_i \\
\text{Row} B_i \\
\text{Row} B_i
\end{bmatrix}
\begin{bmatrix}
\text{Row} \tilde{B}_i \left( I_{\kappa_i} \otimes I_p \right) \\
\text{Row} \tilde{B}_i \left( I_{\kappa_i} \sqrt{E_i} \otimes I_p \right) \\
\text{Row} \tilde{B}_i \left( I_{\kappa_i} \sqrt{E_i} \otimes K \right)
\end{bmatrix}
\begin{bmatrix}
O_{n,q} \\
O_{m,q} \\
O_{m,q}
\end{bmatrix} (23)
\]

\[
\theta(t) = \text{Col}_{i=1}^{\nu} \left[ \int_{-r_i}^{-r_i-1} \left( \sqrt{E_i^{-1} e_i(\tau)} \otimes I_n \right) x(t + \tau) d\tau \right] (24)
\]

\[
\omega(t) = \text{Col}_{i=1}^{\nu} \left[ \int_{-r_i}^{-r_i-1} \left( \sqrt{F_i^{-1}\tilde{f}_i(\tau)} \otimes I_n \right) x(t + \tau) d\tau \right] (25)
\]

3.2. Main results on the design of dissipative state feedback controllers

To verify the stability of the closed-loop system in (19), a KF based stability criterion is presented as follows.
Lemma 1. Let $w(t) \equiv 0_q$ in (19) and all delay values be given, then the trivial solution $x(t) \equiv 0_n$ of (19) is uniformly asymptotically (exponentially) stable with any $\psi(\cdot) \in C([-r_v, 0] \times \mathbb{R}^n)$ if there exist $\epsilon_1; \epsilon_2; \epsilon_3 > 0$ and a differentiable functional $v : C([-r_v, 0] \times \mathbb{R}^n) \to \mathbb{R}$ with $v(O_n(\cdot)) = 0$ such that

$$
\forall \psi(\cdot) \in C([-r_v, 0] \times \mathbb{R}^n), \quad \epsilon_1 \|\psi(\cdot)\|_2^2 \leq v(\psi(\cdot)) \leq \epsilon_2 \|\psi(\cdot)\|_\infty^2, \quad (26)
$$

$$
\overline{\forall} t \geq t_0, \quad d\|v(x(\cdot))\| \leq -\epsilon_3 \|x(t)\|_2^2 \quad (27)
$$

for any $\psi(\cdot) \in C([-r_v, 0] \times \mathbb{R}^n)$ in (19), where $\|\psi(\cdot)\|_\infty := \sup_{-r_v \leq \tau \leq 0} \|\psi(\cdot)\|_2^2$. Furthermore, $x_i(\cdot)$ in (27) is defined by $\overline{\forall} t \geq t_0, \forall \theta \in [-r_v, 0], \quad x_i(\theta) = x(t + \theta)$ in which $x : [t_0 - r_v, \infty) \to \mathbb{R}^n$ satisfies (19) with $w(t) \equiv 0_q$.

Proof. The proof is similar to the proof of Corollary 1 in Feng et al. (2020).

The following definition of dissipativity is based on the original framework outlined in Willems (1972).

Definition 1. The closed-loop system in (19) with a supply rate function $s(z(t), w(t))$ is said to be dissipative if there exists a differentiable functional $v : C([-r_v, 0] \times \mathbb{R}^n) \to \mathbb{R}$ such that

$$
\overline{\forall} t \geq t_0, \quad v(x(\cdot)) - s(z(t), w(t)) \leq 0 \quad (28)
$$

with $t_0 \in \mathbb{R}, z(t)$ and $w(t)$ in (19). Moreover, $x_i(\cdot)$ in (28) is defined by the equality $\forall t \geq t_0, \forall \theta \in [-r_v, 0], \quad x_i(\theta) = x(t + \theta)$ with $x(t)$ satisfying (19).

Note that (28) implies the original definition of dissipativity by taking Lebesgue integrations at both side. In this paper, the supply function is denoted as

$$
s(z(t), w(t)) = \begin{bmatrix}
    z(t) \\
    w(t)
\end{bmatrix}^T \begin{bmatrix}
    J^T J_1^{-1} \gamma \quad J_2 \\
    * \quad J_3
\end{bmatrix} \begin{bmatrix}
    z(t) \\
    w(t)
\end{bmatrix}, \quad (29)
$$

$$
\overline{J}^T J_1^{-1} \gamma \leq 0, \quad J_1^{-1} \prec 0, \quad \gamma \in \mathbb{R}^{m \times m}, \quad J_2 \in \mathbb{R}^{m \times q}, \quad J_3 \in \mathbb{S}^q
$$

whose structure is based on the quadratic constraints in Scherer et al. (1997). The structure in (29) features numerous performance criteria such as

- $L^2$ gain performance: $J_1 = -\gamma I_m, \quad \gamma \in \mathbb{R}^{m \times m}, \quad J_2 = O_{m \times q}, \quad J_3 = \gamma I_q$ with $\gamma > 0$

- Passivity: $J_1 \prec 0, \quad \gamma \in \mathbb{R}^{m \times m}, \quad J_2 = I_m, \quad J_3 = O_m$ with $m = q$.

Next, the main results on DSFC are presented in Theorem 1–2 and Algorithm 1, where Theorem 2 is proposed as a convexification of Theorem 1 which can be further solved by Algorithm 1.

Theorem 1. Let all the parameters in Proposition 1 be given, then the closed-loop system in (19) with the supply rate function in (29) is dissipative, and the trivial solution of (19) with $w(t) \equiv 0_q$ is uniformly asymptotically (exponentially) stable if there exist a controller gain $K \in \mathbb{R}^{p \times n}$ and matrix parameters $P_1 \in \mathbb{S}^n, \quad P_2 \in \mathbb{R}^{n \times e}, \quad P_3 \in \mathbb{S}^e$ with $q = n \sum_{i=1}^\nu d_i$ and $Q_i; R_i \in \mathbb{S}^e, \quad i = 1, \ldots, \nu$ such that

$$
\begin{bmatrix}
    P_1 & P_2 \\
    * & P_3
\end{bmatrix} + \begin{bmatrix}
    O_n \oplus \bigoplus_{i=1}^\nu I_{d_i} \oplus Q_i
\end{bmatrix} \succ 0, \quad (30)
$$

$$
Q = \bigoplus_{i=1}^\nu Q_i \succ 0, \quad R = \bigoplus_{i=1}^\nu R_i \succ 0, \quad (31)
$$
\[ \begin{bmatrix} \Psi \times \Omega \rightarrow \tilde{J} \end{bmatrix} = \Sigma \left[ \begin{bmatrix} P^\top \Pi \right] + \Phi < 0 \right. \] (32)

where \( \Sigma = C + B_2 [(I_{1+\nu+\kappa} \otimes K) \otimes O_q] \) with \( C, B_2 \) in (22)-(23), and

\[
\Psi = Sy \left[ \begin{bmatrix} I_n & O_{n,q} \\ O_{\nu n,n} & O_{\nu n,q} \\ O_{\mu n,n} & O_{\mu n,q} \\ O_{\eta n,n} & O_{\eta n,q} \end{bmatrix} + \begin{bmatrix} P_1 & P_2 \\ P_3 & \Omega \end{bmatrix} \left[ \begin{bmatrix} \tilde{I} \otimes I_n \\ O_{\gamma n,n} \end{bmatrix} + \begin{bmatrix} O_{\beta (n\mu+q),m} \\ J_2^\top \end{bmatrix} \right] \right] + \Xi (33)
\]

\[
\Xi = \left[ (Q + R \Lambda) \oplus O_n \oplus O_{\kappa n} \oplus O_q \right] - \left[ O_n \oplus Q \oplus \left( \bigoplus_{i=1}^\nu I_{\kappa_i} \otimes R_i \right) \oplus \left( \bigoplus_{i=1}^\nu I_{\mu_i} \otimes R_i \right) \oplus J_3 \right], (34)
\]

\[
\tilde{I} = \left( \bigoplus_{i=1}^\nu \sqrt{F_{i}^{-1}} \left[ O_{d_i, d_i} \right] \right) \otimes I_n, \quad \Lambda = \bigoplus_{i=1}^\nu \tilde{r}_i I_{\kappa_i}, \quad \tilde{r}_i = r_i - r_{i-1} (35)
\]

\[
\tilde{F} = \left[ \bigoplus_{i=1}^\nu \sqrt{F_{i}^{-1}} f_i(-r_{i-1}) \right] \oplus O_{d, d_n} = \left[ O_d \bigoplus_{i=1}^\nu \sqrt{F_{i}^{-1}} f_i(-r_{i-1}) \right] (36)
\]

with \( \kappa = \sum_{i=1}^\nu \kappa_i \) and \( \mu = \sum_{i=1}^\nu \mu_i \) and \( \Omega = A + B_1 [(I_{1+\nu+\kappa} \otimes K) \otimes O_q] \) with \( A, B_1 \) in (20)-(21), and \( F_i = \int_{-r_{i-1}}^{-r_i} f_i(\tau)f_i(\tau) d\tau, \forall i = 1, \ldots, \nu \). Moreover, the rest of the parameters in (32) is defined as

\[
P = \begin{bmatrix} P_1 & O_{n,\nu n} & P_2 \tilde{I} & O_{n, (n\mu+q+m)} \end{bmatrix}, \quad \Pi = [\Omega, O_{n,m}] (37)
\]

and

\[
\Phi = Sy \left[ \begin{bmatrix} P_2 \\ O_{\nu n,q} \\ \tilde{I}^\top P_3 \\ O_{(n\mu+q+m),q} \end{bmatrix} \right] \left[ \begin{bmatrix} \tilde{I} \otimes I_n \\ O_{\gamma n,n} \end{bmatrix} + \begin{bmatrix} O_{(n+\nu+n\kappa),m} \\ -J_2^\top \end{bmatrix} \right] + \Xi \oplus (-J_1). (38)
\]

Finally, the number of unknown variables in Theorem 1 is \((0.5d^2 + 0.5d + \nu + 0.5)n^2 + (0.5d + 0.5 + \nu + p)n \in O(d^2n^2)\), where \( d = \sum_{i=1}^\nu d_i \).

**Proof.** The proof of Theorem 1 is based on the construction of the KF:

\[
\nu(x_i(\cdot)) = \eta^\top(t) \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} \eta(t) + \sum_{i=1}^\nu \int_{-r_i}^{-r_{i-1}} x^\top(\tau) Q_i x(\tau) d\tau + \sum_{i=1}^\nu \int_{-r_i}^{-r_{i-1}} (\tau + r_i) x^\top(\tau + r_i) R_i x(\tau + r_i) d\tau (39)
\]

where \( x_i(\cdot) \) follows the same definition in (28), and \( P_1 \in S^n, P_2 \in \mathbb{R}^{n \times q}, P_3 \in S^\nu \) and \( Q_i \in S^n, R_i \in S^n, \ i = 1, \ldots, \nu \) and

\[
\eta(t) := \text{Col} \left( \begin{bmatrix} x(t) \\ \int_{-r_{i-1}}^{-r_i} \sqrt{F_i^{-1}} f_i(\tau) \otimes I_n \end{bmatrix} x(t + \tau) d\tau \right) (40)
\]

with \( F_i = \int_{-r_{i-1}}^{-r_i} f_i(\tau)f_i(\tau) d\tau, \forall i = 1, \ldots, \nu \). Note that \( \sqrt{F_i^{-1}} \) are well defined and unique because of (6).

From \( \chi(t, \tau) = \text{Col} \nu_{i=1,0} x(t + \tilde{r}_i \tau - r_{i-1}) \), we have

\[
\sum_{i=1}^\nu \frac{d}{dt} \int_{-r_{i-1}}^{-r_i} x^\top(\tau) Q_i x(\tau) d\tau = \sum_{i=1}^\nu x^\top(t - r_{i-1}) Q_i x(t - r_{i-1}) - \sum_{i=1}^\nu x^\top(t - r_i) Q_i x(t - r_i)
\]

\[
= \chi^\top(t, 0) Q \chi(t, 0) - \chi^\top(t, -1) Q \chi(t, -1), (41)
\]
where $Q = \bigoplus_{i=1}^\nu Q_i$, and $R = \bigoplus_{i=1}^\nu R_i$ and $\Lambda$ is defined in (35). 

Given the relations in (40)–(42), differentiating (weak derivative) $v(x_i(\cdot))$ along the trajectory of (19) and consider $s(x(t), u(t))$ in (29) produces

$$\dot{\tilde{\nu}} t \geq t_0, \quad \tilde{v}(x_i(\cdot)) = s(x(t), u(t)) =$$

$$\left( \begin{array}{c} I_n \\ O_{m \times n} \\ O_{m \times q} \\ O_{m \times q} \end{array} \right) \left( \begin{array}{c} P_1 \\ P_2 \\ P_3 \end{array} \right) \left[ \begin{array}{c} A + B_1 \left[ (I_{1+n+v+K} \otimes K) \otimes O_q \right] \\ \hat{F} \otimes I_n \\ O_{p \times (nu+q)} \end{array} \right] \left[ \begin{array}{c} O_{(nu+nu+K),m} \\ J^\nu \\ \Sigma \end{array} \right] \vartheta(t)$$

$$+ \chi^T(t,0) (Q + R \Lambda) \chi(t,0) - \chi^T(t,-1) Q \chi(t,-1)$$

$$- \sum_{i=1}^\nu \int_{t_i}^{t_{i-1}} \dot{x}^T(t,\tau) R_i x(t,\tau) d\tau - w^T(t) J \omega(t) - \vartheta(t) \Sigma^T \tilde{J}^\nu \tilde{J}^\nu \vartheta(t)$$

(43)

where $\chi(t,0) = \text{Col}_{i=1}^\nu x(t-r_{i-1})$, $\chi(t,-1) = \text{Col}_{i=1}^\nu x(t-r_i)$ and $\vartheta(t)$ is given in (24) and $\Sigma$, $\tilde{J}$ and $\hat{F}$ are defined in the statements of Theorem 1. Note that $\tilde{J}$ and $\hat{F}$ in (35)–(36) are obtained by the identities

$$\text{Col}_{i=1}^\nu \left( \int_{-r_i}^{-r_{i-1}} \sqrt{F_i^{-1} f_i(\tau) \otimes I_n} x(t+\tau,\tau) d\tau \right) = \int_{-r_i}^{0} \left( \text{Col}_{i=1}^\nu \sqrt{F_i^{-1} f_i(\tau, t-r_i) \otimes I_n} \right) \lambda(\chi(t,\tau)) d\tau$$

$$= \left( \bigoplus_{i=1}^\nu \sqrt{F_i^{-1} \otimes I_n} \right) \int_{-r_i}^{0} \left( \bigoplus_{i=1}^\nu f_i(\tau, t-r_i) \otimes I_n \right) \lambda(\chi(t,\tau)) d\tau$$

$$= \left( \bigoplus_{i=1}^\nu \sqrt{F_i^{-1} \otimes I_n} \right) \int_{-r_i}^{0} \left( \bigoplus_{i=1}^\nu f_i(\tau, t-r_i) \otimes I_n \right) \lambda(\chi(t,\tau)) d\tau$$

$$= \left( \bigoplus_{i=1}^\nu \sqrt{F_i^{-1} \otimes I_n} \right) \int_{-r_i}^{0} \left( \bigoplus_{i=1}^\nu f_i(\tau, t-r_i) \otimes I_n \right) \lambda(\chi(t,\tau)) d\tau$$

(44)

$$\int_{-r_i}^{0} \left( \text{Col}_{i=1}^\nu \sqrt{F_i^{-1} f_i(\tau, t-r_i) \otimes I_n} \right) \lambda \frac{d(\chi(t,\tau))}{d\tau} d\tau = \int_{-r_i}^{0} \left( \text{Col}_{i=1}^\nu \sqrt{F_i^{-1} f_i(\tau, t-r_i) \otimes I_n} \right) \frac{d(\chi(t,\tau))}{d\tau} d\tau$$

$$= \left( \bigoplus_{i=1}^\nu \sqrt{F_i^{-1} f_i(\tau, t-r_i) \otimes I_n} \right) \chi(t,0) - \left( \bigoplus_{i=1}^\nu \sqrt{F_i^{-1} f_i(\tau, t-r_i) \otimes I_n} \right) \chi(t,-1)$$

$$- \left( \bigoplus_{i=1}^\nu \sqrt{F_i^{-1} M_i \otimes I_n} \right) \left[ \int_{-r_i}^{-r_{i-1}} \left( \bigoplus_{i=1}^\nu \sqrt{F_i^{-1} f_i(\tau) \otimes I_n} \right) \lambda(\chi(t,\tau)) d\tau \right]$$

$$- \left[ \bigoplus_{i=1}^\nu \sqrt{F_i^{-1} f_i(\tau, t-r_i) \otimes I_n} \right] \omega(t)$$

$$- \left[ \bigoplus_{i=1}^\nu \sqrt{F_i^{-1} f_i(\tau, t-r_i) \otimes I_n} \right] \omega(t)$$

(45)
with \( \tilde{f}_i(\cdot) \) in (5) and \( F_i \) in (10), which are derived via the fact that \( f_i(\hat{r}_i \tau - r_{i-1}) = [O_{d_i} \delta_i \ I_{d_i}] \tilde{f}_i(\hat{r}_i \tau - r_{i-1}) \) and the application of (58)–(6) and (A.1)–(A.3) and

\[
\int_{-r_i}^{0} \frac{d}{d\tau} \left( \sqrt{F_i^{-1} f_i(\hat{r}_i \tau - r_{i-1}) \otimes I_n} \right) \chi(t, \tau) d\tau \\
= \left( \bigoplus_{i=1}^\nu \sqrt{F_i^{-1} M_i \sqrt{F_i} \otimes I_n} \right) \int_{-r_i}^{0} \left( \bigoplus_{i=1}^\nu \sqrt{F_i^{-1} \tilde{f}_i(\hat{r}_i \tau - r_{i-1}) \otimes I_n} \right) \chi(t, \tau) d\tau \\
= \left( \bigoplus_{i=1}^\nu \sqrt{F_i^{-1} M_i \sqrt{F_i} \otimes I_n} \right) \sum_{l=1}^\nu \left( \int_{-r_i}^{r_{i-1}} \left[ \sqrt{F_i^{-1} \tilde{f}_i(\tau) \otimes I_n} \right] x(t + \tau) d\tau \right). (46) \]

Note that also the parameters \( A, B_1, C \) and \( B_2 \) in (43) are given in (20)–(23).

Assume (31) is true, apply (A.6) with \( \varepsilon(\tau) = 1 \) and \( g_i(\tau) = \phi_i(\tau), f_i(\tau) = \tilde{f}_i(\tau), i = 1, \ldots, \nu \) to the integral terms \( \sum_{i=1}^\nu \int_{-r_i}^{r_{i-1}} x^\top(t + \tau) R_i x(t + \tau) d\tau \) in (43), then we have

\[
\sum_{i=1}^\nu \int_{-r_i}^{r_{i-1}} x^\top(t + \tau) R_i x(t + \tau) d\tau \geq \\
\sum_{i=1}^\nu \int_{-r_i}^{r_{i-1}} x^\top(t + \tau) \left( \tilde{f}_i(\tau) \sqrt{F_i^{-1} \otimes I_n} \right) d\tau \left( \int_{-r_i}^{r_{i-1}} \left( \sqrt{F_i^{-1} \tilde{f}_i(\tau) \otimes I_n} \right) x(t + \tau) d\tau \right) \\
+ \sum_{i=1}^\nu \int_{-r_i}^{r_{i-1}} x^\top(t + \tau) \left( \varepsilon_i(\tau) \sqrt{E_i^{-1} \otimes I_n} \right) d\tau \left( \int_{-r_i}^{r_{i-1}} \left( \sqrt{E_i^{-1} \varepsilon_i(\tau) \otimes I_n} \right) x(t + \tau) d\tau \right) \\
= \left[ \bigoplus_{i=1}^\nu I_{\chi_i} \otimes R_i \right] \sum_{l=1}^\nu \left( \int_{-r_i}^{r_{i-1}} \left( \sqrt{F_i^{-1} \tilde{f}_i(\tau) \otimes I_n} \right) x(t + \tau) d\tau \right) \\
+ \left[ \bigoplus_{i=1}^\nu I_{\chi_i} \otimes R_i \right] \sum_{l=1}^\nu \left( \int_{-r_i}^{r_{i-1}} \left[ \sqrt{E_i^{-1} \varepsilon_i(\tau) \otimes I_n} \right] x(t + \tau) d\tau \right). (47) \]

By using (47) to (43), one can conclude

\[
\tilde{v}(t) \geq t_0, \quad \varepsilon(\tau) - s(z(t), w(t)) \leq \vartheta(t) \left( \Psi - \Sigma^T \hat{J}^T J_i^{-1} \hat{J} \Sigma \right) \vartheta(t) (48) \]

with \( \Psi \) in (33) and \( \vartheta(t) \) in (24). Now it is obvious that if (31) and \( \Psi - \Sigma^T \hat{J}^T J_i^{-1} \hat{J} \Sigma < 0 \) are satisfied, then

\[
\exists \epsilon_3 > 0 : \tilde{v}(t) \geq t_0, \quad \varepsilon(\tau) - s(z(t), w(t)) \leq -\epsilon_3 \| x(t) \|_2. (49) \]

Moreover, considering the structure of \( \Psi - \Sigma^T \hat{J}^T J_i^{-1} \hat{J} \Sigma < 0 \) with \( \vartheta(t) \) in (24), (49) infers

\[
\exists \epsilon_3 > 0, \quad \frac{d}{dt} \varepsilon(\chi(t)) \bigg|_{t=t_0, \chi(t_0) = \phi(\cdot)} \leq -\epsilon_3 \| \phi(\cdot) \|_2. (50) \]

for any \( \phi(\cdot) \in C([-r_i, 0] ; \mathbb{R}^n) \) in (19) with \( w(t) \equiv 0_q \). Note that \( \chi(t) \) in (50) is in line with the definition in (27). As a result, if (31) and \( \Psi - \Sigma^T \hat{J}^T J_i^{-1} \hat{J} \Sigma < 0 \) are feasible, then \( \varepsilon(\chi(t)) \) in (39) satisfies (27)–(28).

Finally, applying the Schur complement to \( \Psi - \Sigma^T \hat{J}^T J_i^{-1} \hat{J} \Sigma < 0 \) with (31) and \( J_i^{-1} < 0 \) yields (32). Hence we have proved that if (31)–(32) are feasible, then there exists \( \epsilon_3 > 0 \) such that \( \varepsilon(\chi(t)) \) in (39) satisfies (27)–(28).

Now we start to show that there exist \( \epsilon_1, \epsilon_2 > 0 \) such that \( \varepsilon(\chi(t)) \) in (39) satisfies (26) if (30)–(31) are feasible. Consider the structure of \( \varepsilon(\chi(t)) \) in (39) with \( t = t_0 \), it follows that there exists \( \lambda > 0 \) such that
\[
\nu(x_0) = \nu(\phi) \leq \eta^T(t_0)\lambda \eta(t_0) + \int_{-r_i}^0 \phi^T(\tau)\lambda \phi(\tau) d\tau \leq \lambda \|\phi(0)\|_2^2 + \lambda r_\nu \|\phi\|_\infty^2 \\
+ \sum_{i=1}^\nu \int_{-r_i}^{t-1} \phi^T(\tau) \left( \sqrt{F_i^{-1} f_i(\tau) \otimes I_n} \right)^T d\tau (I_{d_i} \otimes \lambda I_n) \int_{-r_i}^{t-1} \left( \sqrt{F_i^{-1} f_i(\tau) \otimes I_n} \right) \phi(\tau) d\tau \\
\leq \lambda \|\phi(0)\|_2^2 + \lambda r_\nu \|\phi\|_\infty^2 + \sum_{i=1}^\nu \int_{-r_i}^{t-1} \phi^T(\tau) \lambda \phi(\tau) d\tau \\
\leq (\lambda + \lambda r_\nu) \|\phi\|_\infty^2 + \lambda \int_{-r_i}^0 \phi^T(\tau) \phi(\tau) d\tau \leq (\lambda + 2\lambda r_\nu) \|\phi\|_\infty^2 (51)
\]

for any \( \phi(\cdot) \in C([-r_i, 0]; \mathbb{R}^n) \) in (19), where (51) is derived via the property of \( \forall X \in S^n, \exists \lambda > 0 : \forall x \in \mathbb{R}^n \setminus \{0\}, x^T (\lambda I_n - X) x > 0 \) and (A.6) with \( \varpi(\tau) = 1 \) and \( f_i(\tau) = \sqrt{F_i^{-1} f_i(\tau)}. \) Consequently, (51) shows that there exists \( \epsilon_2 \) such that \( \nu(x_0(\cdot)) \) in (39) satisfies (26).

Now we show that if (30)–(31) are feasible, then \( \nu(x_0(\cdot)) \) in (39) satisfies (26) with some \( \epsilon_1; \epsilon_2 > 0. \)

Applying (A.6) to (39) with \( \varpi(\tau) = 1, g_i(\cdot) = [0, 1) \) and \( f_i(\tau) = \sqrt{F_i^{-1} f_i(\tau)} \) produces

\[
\sum_{i=1}^\nu \int_{-r_i}^{t-1} x^T(t + \tau) Q_i x(t + \tau) d\tau \geq \\
\sum_{i=1}^\nu \int_{-r_i}^{t-1} x^T(t + \tau) \left( f_i^T(\tau) \sqrt{F_i^{-1} \otimes I_n} \right) d\tau (I_{d_i} \otimes Q_i) \int_{-r_i}^{t-1} \left( \sqrt{F_i^{-1} f_i(\tau) \otimes I_n} \right) x(t + \tau) d\tau \\
= [\epsilon] \left( \bigoplus_{i=1}^\nu I_{d_i} \otimes Q_i \right) \text{col} \left( \int_{-r_i}^{t-1} \left( \sqrt{F_i^{-1} f_i(\tau) \otimes I_n} \right) x(t + \tau) d\tau \right) (52)
\]

provided that (31) holds. Moreover, by utilizing (52) to (39) with (31) and (51), it is clear to see that the existence of the feasible solutions of (30)–(31) infer that \( \nu(x_0(\cdot)) \) in (39) satisfies (26) with some \( \epsilon_1; \epsilon_2 > 0. \)

In conclusion, we have shown that the existence of the feasible solutions of (30)–(32) infers the existence of the Krasovskii functional \( \nu(x_0(\cdot)) \) in (39) and \( \epsilon_1; \epsilon_2 > 0 \) satisfying the dissipative condition in (28) and the stability criteria in (26)–(27). As a result, the trivial solution of (19) with \( w(t) \equiv 0 \) is uniformly asymptotically (exponentially) stable, and (19) with (29) is dissipative if (30)–(32) are true.

Note that Theorem 1 is valid for the closed-loop system in (7), not an approximated system. This is because (19) is identical to the original system in (7) and the approximation errors \( E_i \) in (11) are incorporated by the synthesis condition. As a matter of fact, (32) would be infeasible if the eigenvalues of \( E_i \) are too large.

**Remark 5.** The functional in (40) is a realization of the complete Krasovskii functional Kharitonov (2012) defined for \( \dot{x}(t) = \sum_{i=0}^\nu A_i x(t - r_i) \) via the "basis" functions \( f_i(\tau) \). Since \( f_i(\tau) \) in (3)–(4) and (40) can contain any number of differentiable and linearly independent functions even if they are not included by the DIs in (2), as we have discussed in Remark 2. Hence the generality of the functional in (39) is substantially greater than the ones\(^2\) in Seuret et al. (2015); Feng & Nguang (2016); Feng et al. (2020) even for the case of \( \nu = 1. \) Meanwhile, we can select \( g_i(\tau) \) in Proposition 1 in relation to the functions included by the DIs in (2). Since the dimensions of the matrix parameters in (39) are determined not by the dimensions of \( g_i(\cdot) \) but \( f_i(\cdot), \) hence the number of unknowns in Theorem 1 is of \( O(d^2 n^2) \), which is independent of the dimensions of \( \phi_i(\cdot) \) \( \phi_i(\cdot). \)

\(^2\)The equivalent \( f_i(\cdot) \) functions in Seuret et al. (2015); Feng & Nguang (2016); Feng et al. (2020) are the special case of the proposed \( f_i(\cdot) \) in this work.
The inequality in (32) is bilinear if we want to compute $K$, which cannot be solved numerically via standard SDP solvers. In Theorem 2, we convexify the BMI in (32) via the application of Projection Lemma Gahinet & Apkarian (1994), which produces a convex dissipative synthesis condition.

**Lemma 2 (Projection Lemma).** Gahinet & Apkarian (1994) Given $n; p; q \in N$, $\Pi \in S^n$, $P \in R^{q \times n}$, $Q \in R^{P \times n}$, there exists $\Theta \in R^{P \times q}$ such that the following two propositions are equivalent:

$$
\Pi + P^T \Theta^T Q + Q^T \Theta P \prec 0,
$$

$$
P_1^1 \Pi P_1 \prec 0 \text{ and } Q_1^1 \Pi Q_1 \prec 0,
$$

where the columns of $P_1$ and $Q_1$ contain bases of null space of matrix $P$ and $Q$, respectively, which means that $PP_1 = 0$ and $QQ_1 = 0$.

**Proof.** Refer to Gahinet & Apkarian (1994) and Briat (2014). □

**Theorem 2.** Given $\{\alpha_i\}_{i=1}^{1+\nu+\kappa} \subset R$ and the functions and parameters in Proposition 1, then the closed-loop system in (19) with the supply rate function in (29) is dissipative and the trivial solution of (19) with $w(t) \equiv 0$, $q$ is uniformly asymptotically (exponentially) stable if there exists $\hat{P}_1; X \in S^n$, $\hat{P}_2 \in R^{n \times q}$, $\hat{P}_3 \in S^q$ and $\hat{Q}_i; \hat{R}_i \in S^n$, $i = 1, \ldots, \nu$, $\nu = nd$ and $V \in R^{P \times n}$ such that

$$
\begin{bmatrix}
\dot{P}_1 & \dot{P}_2 \\
\dot{P}_1 & \hat{P}_3
\end{bmatrix} + \begin{bmatrix}
O_n \oplus \left( \bigoplus_{i=1}^{\nu} I_{d_i} \otimes \hat{Q}_i \right)
\end{bmatrix} \succ 0,
$$

$$
\dot{\hat{Q}}_i = \bigoplus_{i=1}^{\nu} \hat{Q}_i \succ 0, \quad \dot{\hat{R}}_i = \bigoplus_{i=1}^{\nu} \hat{R}_i \succ 0
$$

$$
Sy \left( \begin{bmatrix}
I_n \\
\text{Col}_{i=1}^{1+\nu+\kappa} \alpha_i I_n \\
0_{(q+m),n}
\end{bmatrix} \right) \left[ -X \ I \ I \right] + \begin{bmatrix}
O_n & \hat{P} & \Phi
\end{bmatrix} \prec 0
$$

where $\hat{P} = \begin{bmatrix}
\hat{P}_1 & 0_{n \times \nu} & \hat{P}_2 & \hat{P}_3 & 0_{n \times (n\mu+q+m)}
\end{bmatrix}$ and

$$
\hat{\Pi} = [A \ [(I_{1+\nu+\kappa} \otimes \ X) \oplus I_q] + B_1 \ [(I_{1+\nu+\kappa} \otimes \ V) \oplus O_q] \ O_{n,m}]
$$

with $\hat{\Pi}$ in (35) and

$$
\hat{\Phi} = Sy \left( \begin{bmatrix}
0_{n \times \nu} & I_{\nu \times \nu} & \hat{P}_3
\end{bmatrix} \left[ \begin{bmatrix}
\hat{P} & I_n & 0_{n \times (n\mu+q+m),q}
\end{bmatrix} + \begin{bmatrix}
O_{(n+n\nu+n\kappa),m} & -J_2 & \Sigma \ O_m
\end{bmatrix}
\right] \bigotimes \left( \begin{bmatrix}
\hat{Q}_i & \hat{R}_i
\end{bmatrix} \otimes J_3 \oplus (-J_1)
\right) \bigoplus \left( \hat{Q} + \Lambda \hat{R} \oplus O_{n \times \kappa \times n} \oplus O_{q+m} \right)
\right)
$$

with $\hat{\Pi}$ in (36) and $\hat{\Sigma} = C \ [(I_{1+\nu+\kappa} \otimes \ X) \oplus I_q] + B_2 \ [(I_{1+\nu+\kappa} \otimes \ V) \oplus O_q]$ and $A, B_1, B_2, C$ in (20)–(23). The controller gain is calculated via $K = V X^{-1}$. Finally, the number of unknown variables in Theorem 2 is $(0.5d^2 + 0.5d + \nu + 1)n^2 + (0.5d + 1 + \nu + p)n \in O(d^2 n^2)$, where $d = \sum_{i=1}^{\nu} d_i$.

**Proof.** First of all, note that the inequality $Sy \left( \begin{bmatrix}
\Pi^T \Pi & \Phi
\end{bmatrix} \right) \prec 0$ in (32) can be reformulated as

$$
Sy \left( \begin{bmatrix}
\Pi & \Phi
\end{bmatrix} \right) \bigotimes \begin{bmatrix}
O_{n \times \nu} & \Pi & I_{n \times (n\mu+q+m)}
\end{bmatrix} \prec 0.
$$

(59)
It is easy to see that the structure of (59) is similar to one of the inequalities in (54). Given that two matrix inequalities are presented in (54), thus a new matrix inequality must be constructed to utilize Lemma 2. Now by considering the structure of \( \Phi \), we have

\[
\mathbf{Y}^T \begin{bmatrix}
  0_n & \mathbf{P} \\
  * & \Phi
\end{bmatrix} \mathbf{Y} = \begin{bmatrix}
  -J_3 - \mathbf{S} \mathbf{y}(D_2^T J_2) & D_2^T J_1
\end{bmatrix} < 0
\]

(60)

where \( \mathbf{Y}^T := \begin{bmatrix} O_{(q+m),(2n+\nu+\kappa n)} & I_{q+m} \end{bmatrix} \). Since the matrix in (60) corresponds to the 2 \( \times \) 2 block matrix at the bottom-right corner of the matrices \( \mathbf{S} \mathbf{y}(\mathbf{P}^T \mathbf{II} + \Phi) \) or \( \Phi \), hence the inequality in (60) is implied by (59) or (32). On the other hand, the following identities

\[
\begin{bmatrix}
  -I_n & \mathbf{II} \\
  I_{n+\nu+\kappa n+q+m} & I_{n+\nu+\kappa n+q+m} \\
\end{bmatrix} = O_{n \times (n+\nu+\kappa n+q+m)}, \quad \begin{bmatrix}
  -I_n & \mathbf{II} \\
  I_{n+\nu+\kappa n+q+m} & I_{n+\nu+\kappa n+q+m} \\
\end{bmatrix} = \begin{bmatrix}
  O_{(2n+\nu+\kappa n),(q+m)} & I_{q+m} \\
  I_{2n+\nu+\kappa n} & O_{(2n+\nu+\kappa n),(q+m)} \\
\end{bmatrix} \mathbf{Y} = O_{(2n+\nu+\kappa n),(q+m)},
\]

\[
\begin{bmatrix}
  I_{2n+\nu+\kappa n} & O_{(2n+\nu+\kappa n),(q+m)} \\
  O_{(q+m),(2n+\nu+\kappa n)} & I_{q+m} \\
\end{bmatrix} \mathbf{Y} = O_{(2n+\nu+\kappa n),(q+m)},
\]

(61)

which satisfy \( \text{rank} \left( \begin{bmatrix}
  -I_n & \mathbf{II} \\
\end{bmatrix} \right) = n \) and \( \text{rank} \left( \begin{bmatrix}
  I_{2n+\nu+\kappa n} & O_{(2n+\nu+\kappa n),(q+m)} \\
\end{bmatrix} \right) = 2n+\nu+\kappa n \), imply that \( \begin{bmatrix}
  -I_n & \mathbf{II} \\
\end{bmatrix} \text{ and } \begin{bmatrix}
  I_{2n+\nu+\kappa n} & O_{(2n+\nu+\kappa n),(q+m)} \\
\end{bmatrix} \text{ can be utilized by Lemma 2} \) given the rank nullity theorem.

Applying Lemma 2 to (59)–(61) yields the conclusion that (59)–(60) are true if and only if

\[
\exists \mathbf{W} \in \mathbb{R}^{(2n+\nu+\kappa n) \times n} : \mathbf{S}( \mathbf{P}^T \mathbf{II} + \Phi) \begin{bmatrix}
  -I_n & \mathbf{II} \\
\end{bmatrix} + \begin{bmatrix}
  0_n & \mathbf{P} \\
  * & \Phi
\end{bmatrix} < 0.
\]

(62)

Now (62) is still bilinear due to the product between \( \mathbf{W} \) and \( \mathbf{II} \). To convexify (62), let

\[
\mathbf{W} = \left[ \mathbf{Col}_{i=1}^{1+\nu+\kappa} \alpha_i \mathbf{W} \right]_{(1+\nu+\kappa) \times n}
\]

(63)

with \( \mathbf{W} \in \mathbb{S}^n \) and \( \{\alpha_i\}_{i=1}^{1+\nu+\kappa} \subset \mathbb{R} \). With (63), the inequality in (62) becomes

\[
\Theta = \mathbf{S}( \mathbf{P}^T \mathbf{II} + \Phi) \begin{bmatrix}
  -I_n & \mathbf{II} \\
\end{bmatrix} + \begin{bmatrix}
  0_n & \mathbf{P} \\
  * & \Phi
\end{bmatrix} < 0.
\]

(64)

which implies (59). Note that (64) is only a sufficient condition implying (59) or (32) due to the structural constraints in (63). Note that also the invertibility of \( \mathbf{W} \in \mathbb{S}^n \) is guaranteed by (64) since \(-2\mathbf{W}\) is the only element at the first diagonal block of \( \Theta \).

Let \( X^T = \mathbf{W}^{-1} \), we apply appropriate congruence transformations to the matrix inequalities in (30)–(31) and (64). Then

\[
(1) \mathbf{Q} (1) > 0, \quad (1) \mathbf{R} (1) > 0,
\]

\[
\left[ (1+\nu+\kappa) (1) \right] \Theta \left[ (1+\nu+\kappa) (1) \right] \sim 0, \quad [s] \begin{bmatrix}
  \hat{P}_1 & \hat{P}_2 \\
  * & \hat{P}_3
\end{bmatrix} (1+\nu+\kappa) > 0
\]

(65)

hold if and only if (30)–(31) and (64) hold. Moreover, with (A.1) and the definitions

\[
\begin{bmatrix}
  \hat{P}_1 & \hat{P}_2 \\
  * & \hat{P}_3
\end{bmatrix} := [s] \begin{bmatrix}
  P_1 & P_2 \\
  * & P_3
\end{bmatrix} (1+\nu+\kappa), \quad \mathbf{Q} = \bigoplus_{i=1}^{\nu} \hat{Q}_i, \quad \mathbf{R} = \bigoplus_{i=1}^{\nu} \hat{R}_i, \quad \mathbf{X} = \bigoplus_{i=1}^{\nu} X_i
\]

(66)
the inequalities in (65) can be rewritten as (55)–(56) and
\[ *[\Theta] [(I_{2+\nu+\kappa} \otimes X) \oplus I_{q+m}] = \hat{\Theta} = SY \left( \begin{bmatrix} I_n & 0 \\ \text{Col}^{1+\nu+\kappa} A, I_n \\ O_{(q+m),n} \end{bmatrix} \right) [-X \ 1] \right) + \left[ O_n \ 1 \right] \begin{bmatrix} \hat{P} \\ \hat{\Phi} \end{bmatrix} \leq 0 \] (67)
where
\[ \hat{P} = XP [(I_{1+\nu+\kappa} \otimes X) \oplus I_{q+m}] = \begin{bmatrix} \hat{P}_1 & O_{\nu,n} & \hat{P}_2 \hat{\Theta} & O_{q,n} & O_{n,m} \end{bmatrix} \] (68)
and
\[ \hat{\Theta} = \Pi [I_{1+\nu+\kappa} \otimes X) \oplus I_{q+m}] = \begin{bmatrix} A [(I_{1+\nu+\kappa} \otimes X) \oplus I_q] + B_1 [(I_{1+\nu+\kappa} \otimes KX) \oplus O_q] \end{bmatrix} O_{n,m} 
= \begin{bmatrix} A [(I_{1+\nu+\kappa} \otimes X) \oplus I_q] + B_1 [(I_{1+\nu+\kappa} \otimes V) \oplus O_q] \end{bmatrix} O_{n,m} \] (69)
with \( V = KX \) and \( \hat{\Phi} \) in (58). Note that (67) is the same as (57), and the form of \( \hat{\Phi} \) in (58) is derived via the relations \( \hat{\Theta} (I_{\nu} \otimes X) = (I_{\nu} \otimes X) \hat{\Theta} \) and
\[ \hat{\Phi} \otimes I_n \right) [(I_{1+\nu+\kappa} \otimes X) \oplus I_{q+m}] = \left( I_{\nu} \otimes X \right) \begin{bmatrix} \hat{\Phi} \otimes I_n & O_{(q+m)} \end{bmatrix} \] (70)
which are derived from (A.1) and (A.3). Furthermore, since \(-2X\) is the only term at the first diagonal block of \( \hat{\Theta} \) in (57), thus \( X \) is invertible if (57) holds. This is consistent with the fact that the invertibility of \( W \) is implied by (64).

As a result, we have shown the equivalence between (30)–(31) and (55)–(56). Meanwhile, it has been shown that (57) is equivalent to (64) which infers (32). Consequently, (30)–(32) are satisfied if (55)–(57) hold for some \( W \in S^n \) and \( \{\alpha_i\}_{i=1}^{1+\nu+\kappa} \subset \mathbb{R} \). This finishes the proof of this theorem.

Note that Theorem 2 is designed to compute \( K \). If one wants to analyze the stability of the open-loop system in (2) with \( B_i = \hat{B}_i(\tau) = O_{n,p} \) and \( \mathcal{B}_i = \hat{\mathcal{B}}_i(\tau) = O_{m,p}, i = 1, \ldots, \nu \), then Theorem 1 should be applied. This is because Theorem 1 is convex in such a case and the slack variables in Theorem 2 do not provide extra feasibility compared to Theorem 1.

Remark 6. Though the simplification at the step (63) can introduce conservatism, the form of the variables in (55) remains identical to (30). As a result, the use of Lemma 2 at (62) does not degenerate the structure of the KL functional in (39), thereby causing less conservatism compared to simplifying \( P_1, P_2 \) to convexify the BMI in (32) via congruence transformations.

Remark 7. For \( \{\alpha_i\}_{i=1}^{1+\nu+\kappa} \subset \mathbb{R} \) in Theorem 2, a simple assignment can be \( \alpha_i = 0 \) for \( i = 2 \cdots 1 + \nu + \kappa \) with an adjustable \( \alpha_1 \in \mathbb{R} \setminus \{0\} \). Note that \( \alpha_1 \neq 0 \) is necessary since \( \alpha_1 = 0 \) nullifies the feasibility of the diagonal block related to \( A_0 \) in (57).

3.3. An inner convex approximation solution of Theorem 1

The simplification at (63) can introduce conservatism. In this subsection, Algorithm 1 is proposed based on the idea outlined in Dinh et al. (2012) to solve the BMI in Theorem 1, which can be initiated by a feasible solution of Theorem 2. Thus the advantage of both Theorem 1 and 2 are combined together without asking for a nonlinear SDP solver.

First of all, we want to point out that the inequality in (32) is nonconvex in general whereas (30)–(31) remain convex even when a synthesis problem is considered. Now we reformulate the inequality in (32) as
\[ U(H, K) := SY \left( P^\top \Pi \right) + \Phi = SY \left( P^\top B \left( (I_{1+\nu+\kappa} \otimes K) \oplus O_{p+m} \right) \right) + \hat{\Phi} < 0 \] (71)
with $B := \begin{bmatrix} B_1 & O_{n,m} \end{bmatrix}$ and $\tilde{\Phi} := \text{Sy} \left( P^T \begin{bmatrix} A & O_{n,m} \end{bmatrix} \right) + \Phi$, where $P$ is given in (37), and $A$ and $B_1$ are given in (20)–(21), and $H := \begin{bmatrix} P_1 & P_2 \end{bmatrix}$ with $P_1$ and $P_2$ in Theorem 1. It is important to stress that $\tilde{\Phi}$ contains no non-convexities. Utilizing the results of Example 3 in Dinh et al. (2012), one can conclude that 

$$
\Delta \left( \ast, \tilde{G}, \ast, \tilde{\Gamma} \right), \text{ which is defined as }
$$

\[ S^{\ell \times \ell} \ni \Delta \left( G, \tilde{G}, \Gamma, \tilde{\Gamma} \right) := \left[ G^T - \tilde{G}^T \quad \Gamma^T - \tilde{\Gamma}^T \right] \left[ Z \oplus (I_n - Z) \right]^{-1} \left[ G - \tilde{G} \quad \Gamma - \tilde{\Gamma} \right] + \text{Sy} \left( G^T \Gamma + \tilde{G}^T \tilde{\Gamma} - \tilde{G}^T \tilde{\Gamma} \right) + T
\]

(72)

with $Z \oplus (I_n - Z) \succ 0$ satisfying

\[ \forall G; \tilde{G} \in \mathbb{R}^{n \times \ell}, \forall \Gamma; \tilde{\Gamma} \in \mathbb{R}^{n \times \ell}, \quad T + \text{Sy} \left( G^T \Gamma \right) := T + \text{Sy} \left( G^T \Gamma \right) \preceq \Delta \left( G, \tilde{G}, \Gamma, \tilde{\Gamma} \right) \]

(73)

is a psd-convex overestimate of $\Delta(G, \Gamma) = T + \text{Sy} \left( G^T \Gamma \right)$ with respect to the parameterization

\[
\begin{bmatrix}
\text{vec}(\tilde{G}) \\
\text{vec}(\tilde{\Gamma})
\end{bmatrix} =
\begin{bmatrix}
\text{vec}(G) \\
\text{vec}(\Gamma)
\end{bmatrix}.
\]

(74)

Now let

\[ T = \tilde{\Phi}, \quad G = P, \quad \tilde{G} = \tilde{P}, \quad H = \begin{bmatrix} P_1 & P_2 \end{bmatrix}, \quad H := \begin{bmatrix} P_1 & \tilde{P}_2 \end{bmatrix}, \quad \tilde{P}_1 \in \mathbb{S}^n, \quad \tilde{P}_2 \in \mathbb{R}^{n,dn}\]

(75)

\[ \Gamma = BK, \quad K = (I_{1+n+\kappa} \otimes K) \oplus O_{p+m}, \quad \tilde{\Gamma} = BK, \quad \tilde{K} = (I_{1+n+\kappa} \otimes \tilde{K}) \oplus O_{p+m} \]

in (72) with $\ell := n + n\nu + n\kappa + q + m$ and $Z \oplus (I_n - Z) \succ 0$ and $\tilde{\Phi}, H$ and $K$ are in line with the definition in (71). Then one can obtain

\[ \mathcal{U}(H, K) = \tilde{\Phi} + \text{Sy} \left[ P^T B \left[ (I_{1+n+\kappa} \otimes K) \oplus O_{p+m} \right] \right] \preceq \mathcal{S} \left( H, \tilde{H}, K, \tilde{K} \right) := \tilde{\Phi} + \text{Sy} \left( \tilde{P}^T \Gamma + P^T \tilde{\Gamma} - \tilde{P}^T \tilde{\Gamma} \right) + \left[ P^T - \tilde{P}^T \quad \Gamma^T - \tilde{\Gamma}^T \right] \left[ Z \oplus (I_n - Z) \right]^{-1} \ast 
\]

(76)

by (73), where $\mathcal{S}(\ast, \tilde{H}, \ast, \tilde{K})$ is a psd-convex overestimate of $\mathcal{U}(H, K)$ in (71) with respect to the parameterization

\[
\begin{bmatrix}
\text{vec}(\tilde{H}) \\
\text{vec}(\tilde{K})
\end{bmatrix} =
\begin{bmatrix}
\text{vec}(H) \\
\text{vec}(K)
\end{bmatrix}.
\]

(77)

From (76), it is obvious that $\mathcal{S} \left( H, \tilde{H}, K, \tilde{K} \right) \prec 0$ infers (71). Moreover, it is also true that $\mathcal{S} \left( H, \tilde{H}, K, \tilde{K} \right) \prec 0$ holds if and only if

\[
\begin{bmatrix}
\tilde{\Phi} + \text{Sy} \left( \tilde{P}^T \Gamma + P^T \tilde{\Gamma} - \tilde{P}^T \tilde{\Gamma} \right) & P^T - \tilde{P}^T & \Gamma - \tilde{\Gamma}^T \\
\ast & -Z & O_n \\
\ast & * & Z - I_n
\end{bmatrix} \prec 0
\]

(78)

holds with $\Gamma, \tilde{\Gamma}$ in (75) based on the application of the Schur complement given $Z \oplus (I_n - Z) \succ 0$. As a result, (71) is inferred by (78) which can be handled by standard numerical SDP solvers provided that the values of $\tilde{H}$ and $\tilde{K}$ are given.
By compiling all the aforementioned procedures according to the expositions in Dinh et al. (2012), Algorithm 1 is established where $x$ consists of all the variables in $P_3$, $Q_1$, $Q_2$, $R_1$, $R_2$ in Theorem 1 and $Z$ in (78). Furthermore, $\rho_1$, $\rho_2$ and $\varepsilon$ are given constants for regularizations and indicating error tolerance, respectively.

Algorithm 1: An inner convex approximation solution for Theorem 1

begin

solve Theorem 2 with given $\alpha_i$ to obtain $K$ and then solve Theorem 1 with the previous $K$ to obtain $P_1$, $P_2$. Finally, solve Theorem 1 again with the previous $P_1$, $P_2$ to obtain $K$.

update $\tilde{H} \leftarrow H = [P_1 \quad P_2]$, $\tilde{K} \leftarrow K$,

solve $\min_{x,H,K} \text{tr} [\rho_1[s](H - \tilde{H})] + \text{tr} [\rho_2[s](K - \tilde{K})]$ subject to (30)–(31) and (78) with (75) and the parameters in Theorem 1 to obtain $H$ and $K$

while $\left\| \begin{bmatrix} \text{vec}(H) \\ \text{vec}(K) \end{bmatrix} \right\|_\infty \geq \varepsilon$ do

update $\tilde{H} \leftarrow H$, $\tilde{K} \leftarrow K$,

solve $\min_{x,H,K} \text{tr} [\rho_1[s](H - \tilde{H})] + \text{tr} [\rho_2[s](K - \tilde{K})]$ subject to (30)–(31) and (78) with (75) and the parameters in Theorem 1 to obtain $H$ and $K$

end

end

3.4. A variant scheme of dissipative state feedback controller design (DSFC)

If there are no delays at the control input of (1), we can modify Theorem 1–2 and Algorithm 1 to solve a different DSFC problem where the controller contains $\nu$ pointwise and distributed delays.

Specifically, let $B_i = \tilde{B}_i(\tau) = O_{n,p}$, $\mathcal{B}_i = \tilde{\mathcal{B}}_i(\tau) = O_{m,p}$, $i = 1, \ldots, \nu$ in (1), which corresponds to a distributed-delay system without input delays. Now we want to construct

$$u(t) = \sum_{i=0}^{\nu} K_i x(t - r_i) + \sum_{i=1}^{\nu} \int_{-r_i}^{0} \tilde{K}_i(\tau) x(t + \tau) d\tau$$

(79)

to stabilize the open-loop system in (1), where $K_i \in \mathbb{R}^{p \times n}$, $\tilde{K}_i(\cdot) \in \mathcal{L}^2([-r_i, r_i-1])$. The above case is vitally important for the research on LTDSs, as early results in Olbrot (1978) have indicated that a controller with delays could be necessary in order to stabilize certain unstable LTDS.

By the proof of Proposition 1 in Appendix B, one can conclude that (2) and $\tilde{K}_i(\cdot) \in \mathcal{L}^2([-r_i, r_i-1])$. If Proposition 1 is true and

$$\forall \tau \in [-r_i, r_i-1], \quad \tilde{K}_i(\tau) = \mathcal{K}_i(\mathcal{g}_i(\tau) \otimes I_n), \quad i = 1, \ldots, \nu$$

(80)

for some $\mathcal{K}_i \in \mathbb{R}^{n \times n \delta_i}$, $i = 1, \ldots, \nu$. 

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Now by using the above conclusion with (9)–(12) and (80), the closed-loop system with (79) is
\[
\dot{x}(t) = (A + B_0 K) \vartheta(t), \quad z(t) = (C + \mathcal{B}_0 K) \vartheta(t), \quad \forall t \geq t_0 \in \mathbb{R}
\]
where \( A, C, \vartheta(t) \) follows the same definitions in (20)–(24).

Since the structure in (81) is essentially the same as to (19), then one can modify Theorem 1 and 2 to handle (81) accordingly. This yields the following two corollaries.

**Corollary 1.** Let all the parameters in Proposition 1 and (80) be given. Then the closed-loop system (81) with the supply rate function in (29) is dissipative and the trivial solution of (81) with \( w(t) \equiv 0 \) is uniformly asymptotically (exponentially) stable if there exist \( P_1 \in \mathbb{S}^n, P_2 \in \mathbb{R}^{n \times e}, P_3 \in \mathbb{S}^e, K_0, K_1 \in \mathbb{R}^{p \times n}, K_2 \in \mathbb{R}^{p \times m}, i = 1, \ldots, \nu \) such that (30)–(32) hold with \( \Omega = A + B_0 K \) and \( \Sigma = C + \mathcal{B}_0 K \) where \( A \) and \( C \) are given in (20)–(22) and \( K \) is given in (81). Finally, the number of unknown variables in Corollary 1 is \( (0.5d^2 + 0.5d + \nu + 0.5)n^2 + (0.5d + 0.5 + \nu + p + \nu p + \nu p + \nu p) n \in \mathcal{O}(d^2 n^2) \), where \( d = \sum_{i=1}^{\nu} d_i \).

**Proof.** The corollary is derived via the direction substitutions \( \Omega = A + B_0 K \) and \( \Sigma = C + \mathcal{B}_0 K \) in (32).

**Corollary 2.** Given the conditions in Proposition 1 with (80) and known parameters \( \{\alpha_i\}_{i=1}^{1+\nu+\kappa} \). Then the closed-loop system (81) with the supply rate function in (29) is dissipative and the trivial solution of (81) with \( w(t) \equiv 0 \) is uniformly asymptotically (exponentially) stable if there exist \( P_1; X \in \mathbb{S}^n, P_2 \in \mathbb{R}^{n \times e}, P_3 \in \mathbb{S}^e, Q_i; R_i \in \mathbb{S}^n, V_0; V_i \in \mathbb{R}^{p \times n}, V_i \in \mathbb{R}^{p \times m}, i = 1, \ldots, \nu \) such that (55)–(57) hold with
\[
\dot{\Pi} = [A \{(I_{1+\nu+\kappa} \otimes X) \oplus I_q\} + B_0 V \quad \Omega_{n,m}] + \mathcal{B}_0 V \quad \Sigma_{p,q}, \quad \dot{\Omega} = C \{(I_{1+\nu+\kappa} \otimes X) \oplus I_q\} + \mathcal{B}_0 V
\]

where \( A \) and \( C \) are given in (20),(22) and \( V \) in (82) where \( V_0 = K_0 X, V_i = K_i X \) and \( V_i = K_i \{I_{\kappa_i} \otimes X\} \) for \( i = 1, \ldots, \nu \). Finally, the number of unknown variables in Corollary 2 is \( (0.5d^2 + 0.5d + \nu + 1 + \nu) n^2 + (0.5d + 1 + \nu + p + \nu p + \nu p) n \in \mathcal{O}(d^2 n^2) \), where \( d = \sum_{i=1}^{\nu} d_i \).

**Proof.** The proof of this corollary is straightforward considering the procedure of proving Theorem 2. Note that the corresponding step at (69) is
\[
\dot{\Pi} = [A \{\{(I_{1+\nu+\kappa} \otimes X) \oplus I_q\} + B_0 K \{(I_{1+\nu+\kappa} \otimes X) \oplus I_q\} \quad \Omega_{n,m}] = [A \{\{(I_{1+\nu+\kappa} \otimes X) \oplus I_q\} + B_0 V \quad \Omega_{n,m}]
\]

with \( V \) in (82) where \( V_0 = K_0 X, V_i = K_i X \) and \( V_i = K_i \{I_{\kappa_i} \otimes X\} \) for \( i = 1, \ldots, \nu \). Note that the equality \( K \{(I_{1+\nu+\kappa} \otimes X) \oplus I_q\} = V \) with \( K \) in (81) and \( V \) in (82) can be proved by the application of (A.1).

Finally, Corollary 1 can be solved by a modified version of Algorithm 1 using the substitutions \( E = B_0 \{K \quad \Omega_{p \times m}\} \), \( \tilde{E} = B_0 \{K \quad \Omega_{p \times m}\} \), \( K \leftrightarrow \tilde{K} \), \( \mathcal{K} \leftrightarrow \mathcal{K} \) for the condition in (78) with the parameters in Corollary 1, where \( K \) is given in (81) and
\[
\begin{align*}
\tilde{K} & = \left[ \begin{array}{cccc}
\mathcal{K}_{i} & \mathcal{K}_{i} & \mathcal{K}_{i} & \mathcal{K}_{i} \\
\mathcal{K}_{i} & \mathcal{K}_{i} & \mathcal{K}_{i} & \mathcal{K}_{i}
\end{array} \right] \\
\mathcal{K} & = \left[ \begin{array}{cccc}
\mathcal{K}_{i} & \mathcal{K}_{i} & \mathcal{K}_{i} & \mathcal{K}_{i} \\
\mathcal{K}_{i} & \mathcal{K}_{i} & \mathcal{K}_{i} & \mathcal{K}_{i}
\end{array} \right].
\end{align*}
\]

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4. Dissipative State Observer Design (DSOD)

As we have solved the DSFC problem, the strategy will be employed in this section to find solutions for the DSOD problem. Unlike the case of LTI delay-free systems, the DSOD problem is not strictly a dual problem of DSFC.

Assume a measured output

\[ \mathbb{R}^l \ni y(t) = \sum_{i=0}^{\nu} \mathcal{C}_i x(t-t_i) + \sum_{i=1}^{\nu} \int_{-r_i}^{-r_i-1} \tilde{\mathcal{C}}_i(\tau) x(t+\tau) d\tau + D_4 w(t) \]  

(85)

for (1), where \( \mathcal{C}_i \in \mathbb{R}^{l \times n} \), \( \tilde{\mathcal{C}}_i(\cdot) \in \mathbb{L}^2([-r_i, -r_i-1]; \mathbb{R}^{l \times n}) \), \( \forall i = 1, \ldots, \nu \) and \( D_4 w(t) \) is an exogenous disturbance affecting the measured output with \( D_4 \in \mathbb{R}^{l \times q} \). Note that we do not use the form of \( z(t) \) in (1) for the DSOD problem. Now consider the following equation

\[ \hat{x}(t) = \sum_{i=0}^{\nu} A_i \hat{x}(t-t_i) + \sum_{i=1}^{\nu} \int_{-r_i}^{-r_i-1} \tilde{A}_i(\tau) \hat{x}(t+\tau) d\tau 
+ \sum_{i=0}^{\nu} B_i u(t-t_i) + \sum_{i=1}^{\nu} \int_{-r_i}^{-r_i-1} \tilde{B}_i(\tau) u(t+\tau) d\tau - L (y(t) - \hat{y}(t)) + D_3 w(t) \]  

\[ \hat{y}(t) = \sum_{i=0}^{\nu} \mathcal{C}_i \hat{x}(t-t_i) - \sum_{i=1}^{\nu} \int_{-r_i}^{-r_i-1} \tilde{\mathcal{C}}_i(\tau) \hat{x}(t+\tau) d\tau \]  

(86)

(87)

describing the dynamics of an observer operating in a disturbance-existing environment, where \( L \in \mathbb{R}^{n \times l} \) is the observer gain and \( \tilde{A}_i(\cdot), \tilde{B}_i(\cdot) \) are given in (1). To ensure the proposed observer guarantees certain dissipative constraints or possesses disturbance attenuation properties, \( D_3 \in \mathbb{R}^{n \times q} \) is included in (86) which will be taken into account by the synthesis conditions. Nevertheless, it is very crucial to emphasize that the observer itself does not contain \( D_3 w(t) \), rather \( D_3 w(t) \) is motivated by modeling an observer operating in a realistic environment. Once the value of \( L \) is obtained, the observer is realized via the equations in (86) with \( D_3 = 0_{n \times q} \).

Now combine the open-loop system in (1) with the observer in (86) and the measured output in (85) with \( e(t) := x(t) - \hat{x}(t) \), we have

\[ \dot{e}(t) = \sum_{i=0}^{\nu} (A_i + L \mathcal{C}_i) e(t-t_i) + \sum_{i=1}^{\nu} \int_{-r_i}^{-r_i-1} (\tilde{A}_i(\tau) + L \tilde{\mathcal{C}}_i(\tau)) e(t+\tau) d\tau + (D_1 + L D_4 - D_3) w(t). \]  

(88)

where the DDs satisfy

\[ \forall i = 1, \ldots, \nu, \quad \tilde{A}_i(\cdot) \in \mathbb{L}^2([-r_i, -r_i-1]; \mathbb{R}^{n \times n}), \quad \tilde{\mathcal{C}}_i(\cdot) \in \mathbb{L}^2([-r_i, -r_i-1]; \mathbb{R}^{l \times n}). \]  

(89)

Similar to Proposition 1, the following proposition can be established for the DDs in (88).

**Proposition 2.** (89) is true if and only if there exist \( f_i(\cdot) \in \mathbb{C}^l([-r_i, -r_i-1]; \mathbb{R}^d) \), \( \varphi_i(\cdot) \in \mathbb{L}^2([-r_i, -r_i-1]; \mathbb{R}^l) \), \( \phi_i(\cdot) \in \mathbb{L}^2([-r_i, -r_i-1]; \mathbb{R}^\kappa) \), \( M_i \in \mathbb{R}^{d \times \kappa} \), \( \tilde{A}_i \in \mathbb{R}^{n \times n} \), \( \tilde{\mathcal{C}}_i \in \mathbb{R}^{l \times n} \) with \( i = 1, \ldots, \nu \) such that

\[ \tilde{A}_i(\tau) = \tilde{A}_i(g_i(\tau) \otimes I_n), \quad \tilde{\mathcal{C}}_i(\tau) = \tilde{\mathcal{C}}_i(g_i(\tau) \otimes I_n), \]  

(90)

and (5)–(6) hold for all \( i = 1, \ldots, \nu \) and for all \( \tau \in [-r_i, -r_i-1] \), where \( \kappa, \kappa \) are defined in the same way as in Proposition 1. Finally, the statement of this proposition is always true for the case of \( \mu_i = 0 \) or \( \delta_i = 0 \).
Now assume a simple regulated output\(^3\) \(z(t) = e(t)\) for the closed-loop system in (88). By using the scheme in (9)–(12) and Proposition 2, the closed-loop system with \(z(t) = e(t)\) is denoted by

\[
\begin{align*}
\dot{e}(t) &= (A_0 + L\mathcal{C}_0) e(t) + \sum_{i=1}^\nu (A_i + L\mathcal{C}_i) \chi(t, -1) \\
&\quad + \sum_{i=1}^\nu \int_{-r_i}^{t} \left[ \left( \hat{A}_i + L\mathcal{E}_i \right) \left( g_i(\tau) \otimes I_n \right) \right] x(t + \tau) d\tau + (D_1 + LD_4 - D_3) w(t),
\end{align*}
\]

(91)

\[z(t) = e(t)\]

which can be further written as

\[
\begin{align*}
\dot{e}(t) &= (A + L\mathcal{C}) \vartheta(t), \quad z(t) = \begin{bmatrix} I_n & O_{n,(\nu n + n\kappa + q)} \end{bmatrix} \vartheta(t), \quad \forall t \geq t_0 \\
\forall \theta \in [-r_\nu, 0], \quad e(t_0 + \theta) = \psi(\theta)
\end{align*}
\]

(92)

with \(t_0 \in \mathbb{R}\) and \(\psi(\cdot) \in \mathcal{C}([-r_\nu, 0] \times {\mathbb{R}^n}), \) where \(\kappa = \sum_{i=1}^\nu \kappa_i\) and

\[
\begin{align*}
\mathbf{A} &= \begin{bmatrix}
\text{Row } A_i & \text{Row } \hat{A}_i \left( \bar{I}_i \otimes I_n \right) & \text{Row } \hat{A}_i \left( \bar{I}_i \sqrt{E}_i \otimes I_n \right) & D_1 - D_3
\end{bmatrix} \\
\mathbf{C} &= \begin{bmatrix}
\text{Row } \mathcal{C}_i & \text{Row } \bar{E}_i \left( \bar{I}_i \otimes I_n \right) & \text{Row } \bar{E}_i \left( \bar{I}_i \sqrt{E}_i \otimes I_n \right) & D_4
\end{bmatrix}
\end{align*}
\]

(93)

(94)

Since the structure of (92) is compatible with the one in (19), hence the results in Section 3 can be modified to compute observer gains for (92). This leads to the following theorems and algorithm.

**Theorem 3.** Let all the parameters in Proposition 2 be given, then the closed-loop system (92) with the supply rate function \(s(e(t), w(t))\) is dissipative and the trivial solution of (92) with \(w(t) \equiv 0\) is uniformly asymptotically (exponentially) stable if there exist \(L \in \mathbb{R}^{n \times l}\) and \(P_1 \in \mathbb{S}^n, \ P_2 \in \mathbb{R}^{n \times e}, \ P_3 \in \mathbb{S}^e\) with \(q = n \sum_{i=1}^\nu d_i\) and \(Q_i; R_i \in \mathbb{S}^n, \ i = 1, \ldots, \nu\) such that (30)–(32) hold with the variable assignment \(\Omega = A + L\mathcal{C}\) and \(\Sigma = \left[ I_n \quad O_{n,(\nu n + n\kappa + q)} \right],\) where \(\mathbf{A}\) and \(\mathbf{C}\) are given in (93)–(94). Finally, the number of unknown variables in Theorem 3 is \((0.5d_l^2 + 0.5d + \nu + 0.5)n^2 + (0.5d + 0.5 + \nu + l)n \in \mathcal{O}(d^2n^2),\) where \(d = \sum_{i=1}^\nu d_i\).

Proof. Using the assignments \(\Omega = A + L\mathcal{C}\) and \(\Sigma = \left[ I_n \quad O_{n,(\nu n + n\kappa + q)} \right]\) in (32) proves this theorem since the structure of the system in (92) is covered by (19).

**Remark 8.** The inclusion of \(D_3 w(t)\) and \(D_4 w(t)\) in (85)–(86) is motivated by the objective that the observer computed by the proposed methods can satisfy dissipative constraints or possess anti-disturbance properties if the observer is operating in a disturbance existing environment. On the other hand, users always have the liberty to set \(D_1 = O_{n,q}\) or \(D_4 = O_{l,q}\) if they believe the consideration of disturbance is redundant. Finally, it is important to emphasize that the values of \(D_3, D_4\) do not affect the stability of the closed-loop system in (92) with \(w(t) \equiv 0\).

\[\text{More complex structures can be employed to } z(t) \text{ considering the form of } z(t) \text{ in (1).}\]
Similar to Theorem 1, the optimization constraints in Theorem 3 contain a BMI due to the product $P^\top \Pi$ with $\Omega = A + L\mathcal{C}$. Hence the following theorem is set out based on what Theorem 2 has been established.

**Theorem 4.** Let all the parameters in Proposition 2 and $\{\alpha_i\}_{i=1}^{1+\nu+k}$ be given, then the closed-loop system (92) with the supply rate function $s(e(t), w(t))$ is dissipative and the trivial solution of (92) with $w(t) \equiv 0$ is uniformly asymptotically (exponentially) stable if there exist $U_1 \in \mathbb{R}^{n \times 1}$ and $P_1, P_2 \in \mathbb{R}^{n \times \nu}$, $P_3 \in \mathbb{R}^\nu$ with $\nu = n \sum_{i=1}^{\nu} d_i$, and $Q_i, R_i: W \in \mathbb{S}^n$, $i = 1, \ldots, \nu$ such that (30)–(31) and

$$
\begin{align*}
\Sigma_y \left( \begin{bmatrix} I_n & \text{Col}^{1+\nu+k} \alpha_i I_n \\ \Omega_{(q+m),n} \end{bmatrix} \left[ -W & W A + U \mathcal{C} \\ O_{n,m} \right] \right) + \left[ O_{n \times \nu} & P^\top \right] < 0
\end{align*}
$$

(96)

hold with $A, \mathcal{C}$ in (93)–(94) and $\Sigma = [I_n \quad O_{n,(n\nu+n\kappa+q)^\top}]$, where the structure of $P$ and $\Phi$ in (96) is the same in (37)–(38). Finally, the observer gain is obtained via $L = W^{-1}U$, and the number of unknown variables in Theorem 4 is $(0.5d^2 + 5.5d + \nu + 1)n^2 + (0.5d + 1 + \nu + 1)n \in \mathcal{O}(d^2n^2)$, where $d = \sum_{i=1}^{\nu} d_i$.

**Proof.** The theorem is proved by letting $U = LW$ and using the procedures from (59) to (64) with the closed-loop system in (92). Note that no congruence transformations are needed to derive (96) compared to the step in (65). This is because the product $LW$ can be directly obtained at (38) with the parameters in (92). Finally, $W^{-1}$ is always well defined if (96) holds. \hfill \blacksquare

Given what has been presented in subsection 3.3, the following iterative algorithm can be easily constructed for solving Theorem 3 with the substitutions

$$
\Gamma = L \left[ \mathcal{C} \quad O_{l,m} \right], \quad \tilde{\Gamma} = \tilde{L} \left[ \mathcal{C} \quad O_{l,m} \right]
$$

(97)

**Algorithm 2:** An inner convex approximation solution for Theorem 3

```plaintext
begin
solve Theorem 4 with given $\alpha_i$ to obtain $L$ and then solve Theorem 3 with the previous $L$ to obtain $P_1, P_2$. Finally, solve Theorem 3 again with the previous $P_1, P_2$ to obtain a new $L$.
update $\tilde{H} \leftarrow H = [P_1 \quad P_2]$, $\tilde{L} \leftarrow L$,
solve $\min_{H \in \mathcal{H}, L}$ $\tr \left( \alpha_i [H - \tilde{H}] \right) + \tr \left( \rho_2 [L - \tilde{L}] \right)$ subject to (30)–(31) and (78) with (97) and the parameters in Theorem 3 to obtain $H$ and $L$
while $\left\| \begin{bmatrix} \vec(H) \\ \vec(L) \end{bmatrix} - \begin{bmatrix} \vec(\tilde{H}) \\ \vec(\tilde{L}) \end{bmatrix} \right\|_\infty \geq \varepsilon$ do
\hspace{1cm} update $\tilde{H} \leftarrow H$, $\tilde{L} \leftarrow L$,
solve $\min_{H \in \mathcal{H}, L}$ $\tr \left( \alpha_i [H - \tilde{H}] \right) + \tr \left( \rho_2 [L - \tilde{L}] \right)$ subject to (30)–(31) and (78) with (97) and the parameters in Theorem 3 to obtain $H$ and $L$;
end
```

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4.1. A Variant of Dissipative State Observer Design (DSOD)

Similar to subsection 3.4, a different DSOD problem can be resolved by a modified version of Theorem 3–4 and Algorithm 2 if no delays exist at the measured output \( y(t) \) in (85).

Specifically, let \( D_A = O_{l,q} \) and \( \xi_i = \bar{\xi}_i(\tau) = O_{l_i}, \ i = 1, \ldots, \nu \) in (85) which gives a static measured output \( y(t) = C_0x(t) \). The dynamics of the observer we want to construct, operating in a disturbance existing environment, are denoted as

\[
\dot{x}(t) = \sum_{i=0}^{n} A_i \dot{x}(t - r_i) + \sum_{i=1}^{n} \int_{-r_i}^{t} \dot{A}_i(\tau) \dot{x}(t + \tau) d\tau + \sum_{i=1}^{n} B_i u(t - r_i) + \sum_{i=1}^{n} \int_{-r_i}^{t} \dot{B}_i(\tau) u(t + \tau) d\tau
\]

\[
- \sum_{i=1}^{n} L_i (y(t - r_i) - \dot{y}(t - r_i)) - \sum_{i=1}^{n} \int_{-r_i}^{t} \dot{L}_i(\tau) (y(t + \tau) - \dot{y}(t + \tau)) d\tau + D_3 w(t)
\]

\[
\dot{y}(t) = C_0 x(t)
\]

where \( L_i \in \mathbb{R}^{n \times l_i}, \bar{L}_i(\tau) \in \mathbb{L}^2([-r_i, -r_i - 1] ; \mathbb{R}^{n \times l_i}) \) for \( i = 1, \ldots, \nu \) are the observer gains. Again, we want to stress that \( D_3 w(t) \) in (98) is not part of the observer similar to the case we have explained in (86). In parallel to the ideas outlined in (80), we know that (2) and \( \bar{L}_i(\tau) \in \mathbb{L}^{2}([-r_i, -r_i - 1] ; \mathbb{R}^{n \times l_i}) \) are true if and only if Proposition 2 and

\[
\forall \tau \in [-r_i, -r_i - 1], \bar{L}_i(\tau) = \mathcal{L}_i(g_i(\tau) \otimes I_i), \ i = 1, \ldots, \nu
\]

are true for some \( \mathcal{L}_i \in \mathbb{R}^{n \times l_i}, \ i = 1, \ldots, \nu \). Moreover, we have \( \bar{L}_i(\tau) y(t) = \mathcal{L}_i(g_i(\tau) \otimes I_i) C_0 x(t) = \mathcal{L}_i(I_i \otimes I_n)(g_i(\tau) \otimes I_n) x(t) \).

Now utilize the above conclusion with (9)–(12), then the closed-loop system consisting of (1) and (86) with a regulated output \( z(t) = e(t) = x(t) - \dot{x}(t) \) is denoted by

\[
\dot{e}(t) = (A + L) \dot{\theta}(t), \ z(t) = \left[ I_n, O_{n,(n\nu+n\kappa+q)} \right] \dot{\theta}(t), \ \forall t \geq t_0 \in \mathbb{R}
\]

\[
\forall \theta \in [-r_{\nu}, 0], \ x(t_0 + \theta) = \psi(\theta), \ \psi(\cdot) \in \mathcal{C}([-r_{\nu}, 0]; \mathbb{R}^n)
\]

\[
L = \left[ \text{Row}_{i=0}^{\nu} L_i e_0, \text{Row}_{i=1}^{\nu} \mathcal{L}_i \left( \tilde{I} \otimes e_0 \right), \text{Row}_{i=1}^{\nu} \mathcal{L}_i \left( \tilde{I} \sqrt{E_i} \otimes e_0 \right), O_{n,q} \right]
\]

where \( A, \dot{\theta}(t) \) follows the same definitions in (93)–(95).

Similar to subsection 3.4, the following two corollaries can be set out via modifying the constraints in Theorem 3–4 to compute the observer gains in (99).

**Corollary 3.** Let all the parameters in Proposition 2 and (99) and \( \{\alpha_i\}_{i=1}^{1+n+\kappa} \) be given. Then the closed-loop system (100) with the supply rate function \( s(e(t), w(t)) \) is dissipative and the trivial solution of (100) with \( w(t) \equiv 0_q \) is uniformly asymptotically (exponentially) stable if there exist \( P_1 \in \mathbb{S}^n, P_2 \in \mathbb{R}^{n \times n}, P_3 \in \mathbb{S}^n \) with \( \rho = n \sum_{i=1}^{d_i} d_i \), and \( Q_i, R_i \in \mathbb{S}^n, L_0; L_i \in \mathbb{R}^{n \times l_i}, \mathcal{L}_i \in \mathbb{R}^{n \times l_i}, i = 1, \ldots, \nu \) such that (30)–(32) hold with \( \Omega = A + L \) and \( \Sigma = [I_n, O_{n,(n\nu+n\kappa+q)}] \) where \( A, L \) are given in (93) and (100), respectively. Finally, the number of unknown variables in Corollary 3 is \((0.5d^2 + 0.5d + \nu + 0.5)n^2 + (0.5d + 0.5 + n + v + l + (v + l)\nu)n \in O(d^2n^2)\), where \( d = \sum_{i=1}^{\nu} d_i \).

**Proof.** Letting \( \Omega = A + L \) and \( \Sigma = [I_n, O_{n,(n\nu+n\kappa+q)}] \) in (32) proves this corollary, where \( A, L \) are given in (93) and (100), respectively. 

**Corollary 4.** Let all the parameters in Proposition 2 and (99) and \( \{\alpha_i\}_{i=1}^{1+n+\kappa} \) be given, then the closed-loop system (100) with the supply rate function \( s(e(t), w(t)) \) is dissipative and the trivial solution of (100) with \( w(t) \equiv 0_q \) is
uniformly asymptotically (exponentially) stable if there exist $P_1 \in \mathbb{S}^n$, $P_2 \in \mathbb{R}^{n \times q}$, $P_3 \in \mathbb{S}^q$ with $q = n \sum_{i=1}^\nu d_i$, and $Q_i; R_i; W \in \mathbb{S}^n, U_0; U_i \in \mathbb{R}^{n \times l_i}, l_i \in \mathbb{R}^{n \times l_i}, i = 1, \ldots, \nu$ such that (30)--(31) and

$$ Sy \left[ \begin{bmatrix} I_n & 0 \\ \text{Col}_{i=1}^{1+\nu+\kappa} \alpha_i I_n \end{bmatrix} \begin{bmatrix} -W & WA + U & O_{n,m} \end{bmatrix} \right] + \begin{bmatrix} O_n & P \end{bmatrix} \Phi \prec 0 \quad (101) $$

hold with $A$ in (93) and $\Sigma = [I_n \quad O_{n,(n+nx+nq)}]$ and

$$ U = \begin{bmatrix} \nu \text{Row} U_i e_0 & \nu \text{Row} \bar{U}_i \left( \bar{T}_i \otimes e_0 \right) & \nu \text{Row} \bar{U}_i \left( \bar{T}_i \sqrt{E_i} \otimes e_0 \right) & O_{n,q} \end{bmatrix}, \quad (102) $$

where the form of $P$ and $\Phi$ in (101) is identical to the one in (37)--(38). Finally, the observer gains are calculated via the relations $L_0 = W^{-1} U_0$ and $L_1 = W^{-1} U_i$ and $L_1 = W^{-1} U_i$ for $i = 1, \ldots, \nu$. Finally, the number of unknown variables in Corollary 4 is $0.5d^2 + 0.5d + \nu + 1) n^2 + (0.5d + 1 + \nu + l + \nu l + \kappa l)n \in O(d^2 n^2)$, where $d = \sum_{i=1}^\nu d_i$.

Proof. By replacing $U \psi$ with $U e_0$ in (96), then the corollary is derived. Note that the relation $L = W^{-1} U$ is utilized to calculate the observer gains.

Similar to the algorithm proposed for Corollary 1 we have mentioned in subsection 3.4, Corollary 3 can be solved by a modified version of Algorithm 2 using the substitutions $\Gamma = \begin{bmatrix} L & O_{n,m} \end{bmatrix}$, $\Gamma = \begin{bmatrix} L & O_{n,m} \end{bmatrix}$, $L \leftarrow \mathcal{L}$, $\tilde{L} \leftarrow \tilde{\mathcal{L}}$ with the parameters in Corollary 3 where

$$ \tilde{L} = \begin{bmatrix} \nu \text{Row} \tilde{L}_i e_0 & \nu \text{Row} \tilde{L}_i \left( \tilde{T}_i \otimes e_0 \right) & \nu \text{Row} \tilde{L}_i \left( \tilde{T}_i \sqrt{E_i} \otimes e_0 \right) & O_{n,q} \end{bmatrix}, \quad \mathcal{L} = \begin{bmatrix} \nu \text{Row} L_i e_0 & \nu \text{Row} \bar{L}_i \left( \bar{T}_i \otimes e_0 \right) & \nu \text{Row} \bar{L}_i \left( \bar{T}_i \sqrt{E_i} \otimes e_0 \right) & O_{n,q} \end{bmatrix}, \quad (103) $$

5. Observer-Based Dissipative Controller Design (DOBC)

Having presented the solutions for DSFC and DSOD, now the most important problem in this work, dissipative observer-based controller design (DOBC), will be solved in this section. We will first see that separation principle is preserved with the previous proposed controller/observer. Then Theorem 5 and Algorithm 3 are established which can simultaneously compute controller/observer gains subject to dissipative constraints.

Consider (1) with the measured output $y(t)$ in (85) and the equation of observation$^4$ in (86) with a controller $u(t) = K \hat{x}(t)$. This scenario is suitable to be employed if no states of $x(t)$ is measurable. After combine (1), (85) and $u(t) = K \hat{x}(t)$, the dynamics of the resulting closed-loop system operating in an

$^4$Again, the observation equations consist of the equations of observer plus the exogenous disturbance terms introduced by the operating environment

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disturbance existing environment are denoted as

\[
\begin{bmatrix}
\dot{e}(t) \\
\dot{z}(t)
\end{bmatrix} = \sum_{i=0}^\nu \begin{bmatrix}
A_i + B_i K & -B_i K \\
O_n & A_i + L \xi_i
\end{bmatrix} \begin{bmatrix}
e(t - r_i) \\
e(t - r_i)
\end{bmatrix} \\
+ \sum_{i=1}^\nu \int_{-r_i}^0 \begin{bmatrix}
\tilde{A}_i(\tau) + \tilde{B}_i(\tau) K & -\tilde{B}_i(\tau) K \\
O_n & \tilde{A}_i(\tau) + \tilde{L} \xi_i(\tau)
\end{bmatrix} \begin{bmatrix}
x(t + \tau) \\
e(t + \tau)
\end{bmatrix} d\tau + \begin{bmatrix}
D_1 \\
D_1 + L D_4 - D_3
\end{bmatrix} w(t),
\]

(104)

where the DDs satisfy (2) and (89).

By observing the mathematical structure of the spectrum of the nominal system in (104), it is clear that the spectrum of the nominal system in (104) is identical to the one in (19) and (86) combined. As a result, the separation principle holds for (104) that the controller/observer computed by Algorithm 1–2 do not destroy the stability of the closed-loop system in (104) with \( w(t) \equiv 0_q \).

By combining Proposition 1–2, the following proposition is put forward which can handle the DDs in (104) all together.

**Proposition 3.** (2) and (89) hold if and only if there exist \( f_i(\cdot) \in C^1([-r_i, -r_{i-1}] ; \mathbb{R}^d) \), \( \varphi_i(\cdot) \in L^2([-r_i, -r_{i-1}] ; \mathbb{R}^d) \), \( \phi_i(\cdot) \in L^2([-r_i, -r_{i-1}] ; \mathbb{R}^d) \), \( M_i \in \mathbb{R}^{d \times \kappa_i} \), \( \tilde{A}_i \in \mathbb{R}^{n \times \kappa_i} \), \( \tilde{B}_i \in \mathbb{R}^{n \times \kappa_i} \), \( C_i \in \mathbb{R}^{m \times \kappa_i} \), \( \tilde{B}_i \in \mathbb{R}^{m \times \kappa_i} \) and \( \tilde{\xi}_i \in \mathbb{R}^{l \times \kappa_i} \) with \( i = 1, \ldots, \nu \) such that (3)–(6) and (90) hold for all \( i = 1, \ldots, \nu \) and for all \( \tau \in [-r_i, -r_{i-1}] \), where \( \kappa_i; \kappa_i \) are defined the same way as in Proposition 1.

Again the scheme in (9)–(11) is employed to \( g_i(t) \) in Proposition 3. Using the commutation matrices \( K_{(n,d)} \) defined in Lemma 3 of Feng et al. (2020), we have

\[
\begin{bmatrix}
\hat{f}_i(\tau) \otimes I_n \\
\varepsilon_i(\tau) \otimes I_n
\end{bmatrix} = \begin{bmatrix}
K_{(\kappa_i,n)} & K_{(\kappa_i,n)} \\
K_{(\mu_i,n)} & K_{(\mu_i,n)}
\end{bmatrix} \begin{bmatrix}
I_n \otimes \hat{f}_i(\tau) \\
I_n \otimes \varepsilon_i(\tau)
\end{bmatrix}
\]

(105)

Now by applying Proposition 3 to the DDs in (104) with (8)–(11) and the relations in (105)–(106), then (104)
can be rewritten as

\[
\begin{bmatrix}
\dot{x}(t) \\
\dot{e}(t)
\end{bmatrix} = \begin{bmatrix}
A + B_1 [(I_{2+2\nu+2\kappa} \otimes K) \oplus O_q] + (I_2 \otimes L) \mathbf{C}
\end{bmatrix} Y \theta(t)
\]

\[
x(t) = (C + B_2 [(I_{2+2\nu+2\kappa} \otimes K) \oplus O_q]) Y \theta(t), \quad \forall t \geq t_0, \quad t_0 \in \mathbb{R}
\]

(107)

\[
Y = I_{2n+2\nu} \oplus \bigoplus_{i=1}^{\nu} (K_{\kappa_i,n} \oplus K_{\kappa_i,n}) K_{2n,\kappa_i} \oplus \bigoplus_{i=1}^{\nu} (K_{\mu_i,n} \oplus K_{\mu_i,n}) K_{2n,\mu_i} \oplus I_q
\]

where \( \kappa = \sum_{i=1}^{\nu} \kappa_i \) and \( \nu = \sum_{i=1}^{\nu} \nu_i \) with \( \kappa_i = d_i + \bar{d}_i + \mu_i \) and \( \nu_i = d_i + \delta_i \) and

\[
A = \begin{bmatrix}
\text{Row} I_2 \otimes \hat{A}_i \left( \hat{I}_i \otimes I_n \right) & \text{Row} I_2 \otimes \hat{A}_i \left( \hat{I}_i \otimes I_n \right) & \text{Row} I_2 \otimes \hat{A}_i \left( \hat{I}_i \otimes I_n \right) & \left[ D_1 \\ D_1 - D_3 \right]
\end{bmatrix}
\]

(108)

\[
\mathbf{C} = \begin{bmatrix}
\text{Row} \left( O_{l,n} \otimes \hat{c}_i \right) & \text{Row} \left( O_{l,n} \otimes \hat{c}_i \right) & \text{Row} \left( O_{l,n} \otimes \hat{c}_i \right) & \text{Row} \left( O_{l,q} \right) & \left[ D_1 \\ D_4 \right]
\end{bmatrix}
\]

(109)

\[
B_1 = \begin{bmatrix}
\text{Row} \left( B_{i_1} - B_{i_2} \right) & \text{Row} \left( B_{i_1} - B_{i_2} \right) & \text{Row} \left( B_{i_1} - B_{i_2} \right) & \text{Row} \left( B_{i_1} - B_{i_2} \right) & \text{Row} \left( B_{i_1} - B_{i_2} \right) & \text{Row} \left( B_{i_1} - B_{i_2} \right)
\end{bmatrix}
\]

(110)

\[
C = \begin{bmatrix}
\text{Row} \left( \hat{C}_i \right) & \text{Row} \left( \hat{C}_i \right) & \text{Row} \left( \hat{C}_i \right) & \text{Row} \left( O_{m,n} \right) & \left[ D_1 \\ D_2 \right]
\end{bmatrix}
\]

(111)

\[
B_2 = \begin{bmatrix}
\text{Row} \left( \bar{B}_i \right) & \text{Row} \left( \bar{B}_i \right) & \text{Row} \left( \bar{B}_i \right) & \text{Row} \left( \bar{B}_i \right) & \text{Row} \left( \bar{B}_i \right) & \text{Row} \left( \bar{B}_i \right)
\end{bmatrix}
\]

(112)

Given the mathematical structure in (107), the results in Sections 3–4 can be extended accordingly resulting in the following theorem solving the DOBC problem.

**Theorem 5.** Let all the parameters in Proposition 3 be given. Then the closed-loop system (107) with the supply rate function in (29) is dissipative and the trivial solution of (107) with \( \psi(t) \equiv 0 \) is uniformly asymptotically (exponentially) stable if there exist \( K \in \mathbb{R}^{p\times n} \), \( L \in \mathbb{R}^{n\times l} \) and \( P_1 \in \mathbb{S}^{2n} \), \( P_2 \in \mathbb{S}^{2n\times \nu} \), \( P_3 \in \mathbb{S}^p \) with \( \theta = 2n \sum_{i=1}^{d_i} d_i \), and \( Q_i, R_i \in \mathbb{S}^{2n} \), \( i = 1, \ldots, \nu \) such that (30)–(32) hold with the substitutions \( n \leftarrow 2n \) and \( \Omega = (A + B_1 [(I_{2+2\nu+2\kappa} \otimes K) \oplus O_q] + (I_2 \otimes L) \mathbf{C}) Y \) and \( \Sigma = (C + B_2 [(I_{2+2\nu+2\kappa} \otimes K) \oplus O_q]) Y \) with the parameters \( A, B_1, C, \Sigma, B_2, Y \) in (108)–(112). Finally, the number of unknown variables in Theorem 5 is \( (2d^2 + 2d + 4\nu + 2)n^2 + (d + 1 + 2\nu + p + l)n \in \mathcal{O}(d^2n^2) \), where \( d = \sum_{i=1}^{\nu} d_i \).

**Remark 9.** The use of the commutation matrices in (105)–(106) is critical in ensuring the consistency between the DDs in (104) and (107) so that Theorem 5 can be established from Theorem 1 using simple substitutions. This clearly shows the intricacy in dealing with distributed delays.
The constraints in Theorem 5 contain two BMIs related to \( K \) and \( L \). To overcome this issue, Algorithm 3 is set out for solving Theorem 5 based on how Algorithm 1 is assembled. Algorithm 3 concerns the parameters in (108)–(112) and \( \rho_1 > 0, \rho_2 > 0, \rho_1 > 0, \varepsilon > 0, Z \in \mathbb{S}^{2n} \) and \( \Sigma \) in Theorem 5 with the substitutions \( n \leftarrow 2n \) and \( A \leftarrow AY \). Moreover,

\[ \Gamma = \left( B_1 \left( I_{2+2n+2\kappa} \otimes K \right) \otimes O_q \right) + \left( I_2 \otimes L \right) e \right) Y \ dO_{2n,m} \]
\[ \tilde{\Gamma} = \left( B_1 \left( I_{2+2n+2\kappa} \otimes \tilde{K} \right) \otimes O_q \right) + \left( I_2 \otimes \tilde{L} \right) e \right) Y \ dO_{2n,m} \]  

(114)
in Algorithm 3 with \( K, \tilde{K} \in \mathbb{R}^{p \times n} \) and \( L, \tilde{L} \in \mathbb{R}^{n \times l} \). Note that Algorithm 3 can compute \( K \) and \( L \) simultaneously satisfying (optimizing) an objective. This is a very attractive feature, as the gains computed by Theorem 2 and 4 may not provide the best optimal controller/observer gains solution for Theorem 5.

**Algorithm 3: An inner convex approximation solution for Theorem 5**

begin
  Use Algorithm 1 and 2 to compute \( K \) and \( L \), then solve Theorem 5 with the previous \( K, L \) to obtain \( \bar{H} = \left[ P_1 \ P_2 \right] \in \mathbb{R}^{2n \times (2n+2\kappa)} \). Finally, solve Theorem 5 again with the previous \( H \) to obtain \( \bar{H}, \bar{K}, \bar{L} \).

update \( \bar{H} \leftarrow H, \bar{K} \leftarrow K, \bar{L} \leftarrow L \)

solve \( x, H, K, L \) subject to (30)–(31) and (78) with \( n \leftarrow 2n, A \leftarrow AY \) and (114) and the parameters in Theorem 5 to compute \( H, K \) and \( L \)

while \( \left\| \begin{bmatrix} \text{vec}(H) \\ \text{vec}(K) \\ \text{vec}(L) \end{bmatrix} \right\|_{\infty} \geq \varepsilon \) do
  update \( \bar{H} \leftarrow H, \bar{K} \leftarrow K, \bar{L} \leftarrow L \);
  solve \( x, H, K, L \) subject to (30)–(31) and (78) with \( n \leftarrow 2n, A \leftarrow AY \) and (114) and the parameters in Theorem 5 to compute \( H, K \) and \( L \)
end

5.1. A Variant Scheme of Observer Based Control Design (OBCD)

Assume that there are no delays at the control input in (1) and the measured output in (85). Then the results in Theorem 5 and Algorithm 3 can be modified similar to subsection 3.4 and 4.1 to construct controller/observer simultaneously.

Specifically, let \( B_i = \tilde{B}_i(\tau) = O_{n,p} \) and \( B_i = \tilde{B}_i(\tau) = O_{m,n}, i = 1, \ldots, \nu \) for the open-loop system in (1) and \( D_i = O_{l,q}, C_i = \tilde{C}_i(\tau) = O_{n}, i = 1, \ldots, \nu \) for the measured output in (85). Now let’s assume that none of the states is measurable, so a controller

\[ u(t) = \sum_{i=0}^{\nu} K_i \dot{x}(t - r_i) + \sum_{i=1}^{\nu} \int_{-r_{i-1}}^{-r_i} \tilde{K}_i(\tau) \dot{x}(t + \tau) d\tau \]  

(115)
and the observer in (98) are utilized together to stabilize the open-loop system with dissipative constraints. This gives the following closed-loop system

\[
\begin{align*}
\dot{x}(t) &= \sum_{i=0}^{\nu} \left[ A_i + B_0 K_i - B_0 \tilde{K}_i \right] x(t - r_i) \\
\dot{e}(t) &= \sum_{i=0}^{\nu} \int_{t-r_i}^{t} \left[ \tilde{A}_i(\tau) + B_0 \tilde{K}_i(\tau) - B_0 \tilde{K}_i(\tau) \right] x(t + \tau) + \left[ D_1 \right] w(t), \\
z(t) &= \sum_{i=0}^{\nu} \left[ C_i + \mathcal{B}_0 K_i - \mathcal{B}_0 \tilde{K}_i \right] \left[ x(t - r_i) - e(t - r_i) \right] \\
&+ \sum_{i=0}^{\nu} \int_{t-r_i}^{t} \left[ \tilde{C}_i(\tau) + \mathcal{B}_0 \tilde{K}_i(\tau) - \mathcal{B}_0 \tilde{K}_i(\tau) \right] \left[ x(t + \tau) - e(t + \tau) \right] d\tau + D_2 w(t), \\
\forall \theta \in [-r_\nu, 0], \quad \begin{bmatrix} x(t_0 + \theta) \\ e(t_0 + \theta) \end{bmatrix} &= \psi(\theta), \quad \psi(\cdot) \in \mathcal{C}([-r_\nu, 0]; \mathbb{R}^{2n}).
\end{align*}
\] (116)

As we have discussed in the previous subsections, it is obvious to conclude that (2) and $\tilde{L}_i(\tau) \in \mathbb{L}^2([-r_i, -r_{i-1}]; \mathbb{R}^{n \times 1})$ and $\tilde{K}_i(\tau) \in \mathbb{L}^2([-r_i, -r_{i-1}]; \mathbb{R}^{n \times 1})$ are true if and only if Proposition 3 and (80) and (99) are true. Now apply the above conclusion with (9)–(12), then (116) can be rewritten as

\[
\begin{align*}
\dot{x}(t) &= (A + B_0 \mathcal{K} + \mathcal{L}) \psi(t), \\
z(t) &= (C + B_0 \mathcal{K} + \mathcal{L}) \psi(t), \quad \forall \theta \in [-r_\nu, 0], \\
\psi(\cdot) \in \mathcal{C}([-r_\nu, 0]; \mathbb{R}^{2n})
\end{align*}
\] (117)

where $\psi$ is given in (107) and

\[
\begin{align*}
B_1 &= \begin{bmatrix} 3^\nu \\ \text{Row} \end{bmatrix} \left[ \begin{array}{cc} B_0 & -B_0 \\ \mathcal{O}_{n,p} & \mathcal{O}_{n,p} \end{array} \right] \mathcal{O}_{2n,q}, \\
B_2 &= \begin{bmatrix} 3^\nu \\ \text{Row} \end{bmatrix} \left[ \begin{array}{cc} \mathcal{B}_0 & -\mathcal{B}_0 \\ \mathcal{O}_{m,q} & \mathcal{O}_{m,q} \end{array} \right] \\
\mathcal{K} &= \bigoplus_{i=0}^{\nu} (I_2 \otimes K_i) \bigoplus \bigoplus_{i=0}^{\nu} \left[ I_2 \otimes \mathcal{K}_i \left( \tilde{I}_i \otimes \mathcal{I}_0 \right) \right] + \bigoplus_{i=0}^{\nu} \left[ I_2 \otimes \mathcal{K}_i \left( \tilde{I}_i \otimes \mathcal{E}_0 \right) \right] + \mathcal{O}_q \\
\mathcal{L} &= (\mathcal{O}_n \otimes I_n) \begin{bmatrix} \text{Row} \\ \text{Row} \end{bmatrix} \left[ \begin{array}{cc} I_2 \otimes \mathcal{L}_0 \\ \mathcal{I}_0 \otimes \mathcal{I}_0 \end{array} \right] + \begin{bmatrix} \text{Row} \\ \text{Row} \end{bmatrix} \left[ I_2 \otimes \mathcal{L}_0 \left( \tilde{I}_i \otimes \mathcal{E}_0 \right) \right] + \mathcal{O}_{2n,q}
\end{align*}
\] (118)

and the rest of the parameters are defined in (108)–(113).

Since the structure of (117) is compatible with (107), hence the following corollary can be immediately obtained via Theorem 5 with modified parameters.

**Corollary 5.** Let all the parameters in Proposition 3 be given. Then the closed-loop system in (116) with the supply rate function in (29) is dissipative and the trivial solution of (116) with $w(t) \equiv 0$ is uniformly asymptotically (exponentially) stable if there exist $P_i \in \mathbb{S}^{2n}$, $P_0 \in \mathbb{R}^{2n \times 2n}$, $P_3 \in \mathbb{S}^0$ with $\phi = 2n \sum_{i=1}^{d_i}$, and $Q_i; R_i \in \mathbb{S}^{2n}$, $K_i; \mathcal{K}_i \in \mathbb{R}^{p \times n}$, $\mathcal{L}_i \in \mathbb{R}^{n \times p}$, $\mathcal{E}_i \in \mathbb{R}^{p \times n}$, $i = 1, \ldots, \nu$ such that (30)–(32) hold with the substitutions $n \leftarrow 2n$ and $\Omega = (A + B_0 \mathcal{K} + \mathcal{L}) \psi$ and $\Sigma = (C + B_0 \mathcal{K} + \mathcal{L}) \psi$ with the parameters $A, B_1, B_2, C, \mathcal{K}, \mathcal{L}, \psi$ in (108)–(113) and (118). Finally, the number of unknown variables in Theorem 5 is $(2d^2 + 2d + 4\nu + 2)n^2 + (d + 1 + 2\nu + p + \nu p + \kappa p + l + \nu l + \kappa l)n \in \mathcal{O}(d^2n^2)$, where $d = \sum_{i=1}^{d_i}$. 

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Likewise, a modified version of Algorithm 3 can be established to solve Corollary 5 using the substitutions $K \leftarrow \mathbf{K}, \tilde{K} \leftarrow \tilde{\mathbf{K}}, L \leftarrow \mathbf{L}$, and $\tilde{L} \leftarrow \tilde{\mathbf{L}}$, with the parameters in Theorem 5 and

$$
\boldsymbol{\Sigma} = (\mathbf{C} + \mathbf{B}_2 \mathbf{K}) \mathbf{Y}, \quad \mathbf{T} = \left[ (\mathbf{B}_1 \mathbf{K} + \mathbf{L}) \mathbf{Y} \quad \mathbf{O}_{2n,m} \right], \quad \tilde{\mathbf{\Gamma}} = \left[ (\mathbf{B}_1 \mathbf{\tilde{K}} + \mathbf{\tilde{L}}) \mathbf{Y} \quad \mathbf{O}_{2n,m} \right]
$$

$$
\tilde{\mathbf{\Sigma}} = \left( \bigoplus_{i=0}^{\mathbf{\nu}} (\mathbf{I}_2 \otimes \tilde{\mathbf{K}}_i) \right) \oplus \left( \bigoplus_{i=1}^{\mathbf{\nu}} [\mathbf{I}_2 \otimes \tilde{\mathbf{X}}_i (\tilde{\mathbf{\Gamma}}_i \otimes \mathbf{I}_n)] \right) \oplus \left( \bigoplus_{i=1}^{\mathbf{\nu}} [\mathbf{I}_2 \otimes \tilde{\mathbf{K}}_i (\tilde{\mathbf{\Gamma}}_i \sqrt{\mathbf{E} \otimes \mathbf{I}_n})] \right) \oplus \mathbf{O}_q
$$

$$
\tilde{\mathbf{\mathbf{\Sigma}}} = (\mathbf{O}_n \oplus \mathbf{I}_n) \left[ \mathbf{\text{Row}} \left( \mathbf{I}_2 \otimes \tilde{\mathbf{L}}, \mathbf{c}_0 \right) \mathbf{\text{Row}} \left( \mathbf{I}_2 \otimes \tilde{\mathbf{X}}_i (\tilde{\mathbf{\Gamma}}_i \otimes \mathbf{c}_0) \right) \mathbf{\text{Row}} \left( \mathbf{I}_2 \otimes \tilde{\mathbf{K}}_i (\tilde{\mathbf{\Gamma}}_i \sqrt{\mathbf{E} \otimes \mathbf{c}_0}) \right) \mathbf{O}_{2n,q} \right]
$$

where $\mathbf{K}, \tilde{\mathbf{K}}$ and $\mathbf{L}, \tilde{\mathbf{L}}$ are given in (84) and (103).

6. Numerical examples

In this section, we present two numerical examples to show the effectiveness of our proposed methodologies. The examples concern using Algorithm 1–3 to attain dissipative observer-based controller (DOBC) designs, thereby involving all the components (DSFC, DSOD) of the proposed methods. All the computations are carried out in Matlab© using Yalmip Löberg (2004) as the optimization interface, and SDPT3, Mosk Toh et al. (2012); Mosk (2022) as the numerical solvers for SDPs.

6.1. Dissipativity analysis of an LTDSs with multiple pointwise and distributed delays

Consider a system in the form of (1) and $y(t)$ described by (85) with $r_1 = 1, r_2 = 1.7$ and the state space matrices

$$
\begin{align*}
A_0 & = \begin{bmatrix} -2 & 0 \\ 2 & 0.01 \end{bmatrix}, \quad A_1 = \begin{bmatrix} -1 & 0.1 \\ 0.2 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -0.1 & 0 \\ 0 & -0.2 \end{bmatrix}, \quad B_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0.01 \\ 0.1 \end{bmatrix} \\
\tilde{A}_1(\tau) & = \begin{bmatrix} 0.1 + 3 \sin(20\tau) & 0.8e^{\sin(20\tau)} - 0.3e^{\cos(20\tau)} \\ 0.3 + \frac{\sin^2(1.2\tau)}{2} + 1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} -1 \\ 0.1 \end{bmatrix} \\
\tilde{A}_2(\tau) & = \begin{bmatrix} 0.2 \sin(18\tau) & 0.3e^{\cos(18\tau)} - \frac{1}{\cos^2(0.7\tau) + 1} \\ 0.1e^{\sin(18\tau)} & 0.2 - 10 \cos(18\tau) \end{bmatrix}, \quad D_1 = \begin{bmatrix} 0.2 \\ 0.3 \end{bmatrix}, \quad D_2 = \begin{bmatrix} 0.12 \\ 0.1 \end{bmatrix} \\
\tilde{B}_1(\tau) & = \begin{bmatrix} 0.01 \tau - \frac{0.01}{\sin^2(1.2\tau) + 1} + 1 \\ 0.17 + \frac{0.02}{\sin^2(1.2\tau) + 1} \end{bmatrix}, \quad \tilde{B}_2(\tau) = \begin{bmatrix} 0.2e^{\cos(18\tau)} + 0.01e^{\sin(18\tau)} + \frac{0.01}{\cos^2(0.7\tau) + 1} \\ 0.1e^{\cos(18\tau)} + 0.02e^{\sin(18\tau)} \end{bmatrix} \\
C_0 & = \begin{bmatrix} -0.1 & 0.2 \\ 0 & 0.1 \end{bmatrix}, \quad C_1 = \begin{bmatrix} -0.1 & 0 \\ 0 & 0.2 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 0 & 0.1 \end{bmatrix}, \quad D_3 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \quad D_4 = 0.1, \\
\tilde{C}_1(\tau) & = \begin{bmatrix} 0.7 + \cos(20\tau) - \frac{0.2}{\sin^2(1.2\tau) + 1} \\ 0.4 - 0.5e^{\sin(20\tau)} - 0.8 - \sin(20\tau) \end{bmatrix}, \quad \tilde{C}_2(\tau) = \begin{bmatrix} 0.2 + \sin(18\tau) & 0.3 + e^{\cos(18\tau)} \\ 0 & 0.1 - \frac{1}{\cos(0.7\tau) + 1} \end{bmatrix} \\
\tilde{B}_1(\tau) & = \begin{bmatrix} 0.1e^{\sin(20\tau)} - \frac{0.1}{\sin^2(1.2\tau) + 1} \\ 0.2e^{\sin(20\tau)} \end{bmatrix}, \quad \tilde{B}_2(\tau) = \begin{bmatrix} 0.2e^{\cos(18\tau)} + 0.01e^{\sin(18\tau)} + \frac{0.1}{\cos^2(0.7\tau) + 1} \\ 0.02e^{\sin(18\tau)} + 0.2e^{\cos(18\tau)} \end{bmatrix} \\
\tilde{C}_1(\tau) & = \begin{bmatrix} 0.1e^{\sin(20\tau)} + 0.1e^{\cos(20\tau)} - \frac{0.4}{\sin^2(1.2\tau) + 1} + 1.0 \end{bmatrix}, \quad \tilde{C}_2(\tau) = \begin{bmatrix} 0.2 & e^{\cos(18\tau)} + 0.2e^{\sin(18\tau)} + 0.3 \end{bmatrix}
\end{align*}
$$
with \( n = m = 2, p = l = q = 1 \). By using the numerical toolbox of the spectral method proposed in Breda et al. (2015), it shows the nominal system is unstable. Moreover, we utilize

\[
\gamma > 0, \quad J_1 = -\gamma J_2, \quad \tilde{J} = I_2, \quad J_2 = 0_2, \quad J_3 = \gamma
\]  

(121)

for the supply function in (29) where \( \gamma \) is the \( L^2 \) gain to be minimized.

**Remark 10.** The parameters in (120) are chosen with sufficient degree of complexity in order to demonstrate the strength of our proposed method. It is important to point out that many practical examples such as the ones mentioned in Remark 1 can be deal with by the proposed methods, where the DDs therein are usually much simpler than the DDs in (120). Note that no existing methods can effectively solve the DOBC problem of an LTDS with (120) due to the complexity of the DDs with multiple non-commensurate delays and the fact that \( A_0 \) is non-Hurwitz.

Assuming all the states in \( x(t) \) cannot be measured, we want to compute a controller/observer to stabilize the open-loop system while minimizing \( L^2 \). By observing the functions inside of the DDs, let \( \varphi_1(\tau) = 1/(\sin^2 1.2\tau + 1) \) and \( \varphi_2(\tau) = 1/(\cos^2 0.7\tau + 1) \) and

\[
\phi_1(\tau) = \begin{bmatrix} e^{i\sin(20\pi \tau)} \\ e^{i\cos(20\pi \tau)} \end{bmatrix}, \quad \phi_2(\tau) = \begin{bmatrix} e^{i\sin(18\pi \tau)} \\ e^{i\cos(18\pi \tau)} \end{bmatrix}, \quad f_1(\tau) = \begin{bmatrix} \text{Col}_{i=0}^{d_1} \tau^i \\ \text{Col}_{i=1}^{\lambda_1} \sin 20\pi \tau \\ \text{Col}_{i=1}^{\lambda_1} \cos 20\pi \tau \end{bmatrix}, \quad f_2(\tau) = \begin{bmatrix} \text{Col}_{i=0}^{d_2} \tau^i \\ \text{Col}_{i=1}^{\lambda_2} \sin 18\pi i \tau \\ \text{Col}_{i=1}^{\lambda_2} \cos 18\pi i \tau \end{bmatrix}
\]

(122)

for the parameters in Proposition 1 with

\[
M_1 = \begin{bmatrix} 0_2^T & 0 \\ \Theta_{d_1}^{\lambda_1} i & 0_{d_1} \end{bmatrix}, \quad M_2 = \begin{bmatrix} 0_2^T & 0 \\ \Theta_{d_2}^{\lambda_2} i & 0_{d_2} \end{bmatrix}
\]

(123)

in (5). Now let us first solve the DOBC problem that using Algorithm 1 to find a controller gain \( K \) for Algorithm 3. By (122) and (A.1), we can construct

\[
\hat{A}_1 = \begin{bmatrix} 0.8 & 0 & 0 & -0.3 & 0 & 0 & 0.1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0.3 & 0 & 0 & 0 \end{bmatrix}, \quad \hat{A}_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}
\]

\[
\hat{B}_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0.7 & -0.2 & 0 \end{bmatrix}^T, \quad \hat{B}_2 = \begin{bmatrix} 0 & 0.01 & 0.2 & 0.01 & 0 \end{bmatrix}^T
\]

\[
\hat{C}_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0.4 & 0.8 & 0 & 0 \end{bmatrix}^T, \quad \hat{C}_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T
\]

\[
\hat{B}_1 = \begin{bmatrix} 0.1 & 0 & -0.1 & 0 & 0.2 \end{bmatrix}, \quad \hat{B}_2 = \begin{bmatrix} 0.01 & 0.2 & 0.1 & 0 & 0 \end{bmatrix}
\]

(124)

satisfying (3)-(4).
Remark 11. The choice for the basis functions in (122) is motivated by several reasons. First of all, the functions in $\psi_i(\cdot)$ can be well approximated via appropriate trigonometric functions together with polynomials, some of which are presented in the DDs in (120). On the other hand, the functions in $\varphi_i(\cdot)$ are directly factorized since it is very difficult to be approximated without exceptionally large $d_i$, $\lambda_i$. As a result, the choice in (122) involves all the components in Proposition 1, which balances feasibility and the implied computational complexity $O(d^2n^2)$ affected by the dimensions of $f_i(\tau)$. This serves as a good example demonstrating the effectiveness of the proposed decomposition approach over the existing approaches in Seuret et al. (2015); Feng & Nguang (2016); Feng et al. (2020).

First of all, we want to find a controller gain $K$ stabilizing the closed loop system in (19) with (120) minimizing $\gamma$. Now apply Theorem 2 to (19) with $d_1 = d_2 = \lambda_1 = \lambda_2 = 1$ and $\alpha_i = 0$, $i = 2 \cdots 1 + \nu + \kappa$, $\alpha_1 = 5$ and the parameters in (120)–(124), and employ the scheme in (9)–(12) where the matrices therein are computed via the vpaintegral function in Matlab©. The optimization program yields $K = \begin{bmatrix} -1.3794 & -1.8668 \end{bmatrix}$ with $\min \gamma = 0.8986$. By further using Algorithm 1 with 20 iterations, it produces

$$K = \begin{bmatrix} -1.5032 & -1.9813 \end{bmatrix}, \text{ with } \min \gamma = 0.651$$  \hspace{1cm} (125)$$

which will be used for the first step in Algorithm 3.

Now we want to compute an observer gain $L$ for the first step of Algorithm 3. Firstly, we choose the same parameters as outlined in (122)–(123) for using the decomposition approach, by which we can construct

$$\bar{E}_1 = \begin{bmatrix} 0.1 & 0 & 0.1 & 0 & 0.4 & 0 & 1 \end{bmatrix}, \quad \bar{E}_2 = \begin{bmatrix} 0 & 0.2 & 0 & 1 & 0 & 0.2 & 0.3 \end{bmatrix} \cdot 0_{2d_2 + 4\lambda_2}$$ \hspace{1cm} (126)$$

for the DDs in the measured output satisfying (90).

Apply Theorem 4 to the closed-loop system in (100) with $d_1 = d_2 = 1$, $\lambda_1 = \lambda_2 = 1$ and $\alpha_i = 0$, $i = 2 \cdots 1 + \nu + \kappa$, $\alpha_1 = 50$ and the parameters in (120)–(123), (126), it yields $L = - \begin{bmatrix} 25.7029 & 38.8817 \end{bmatrix}^\top$ with $\min \gamma = 0.0358$. By using Algorithm 2 to the same system with 20 iterations, it produces $L = - \begin{bmatrix} 39.9499 & 60.0118 \end{bmatrix}^\top$ with $\min \gamma = 0.0207$ which will be used for the first step of Algorithm 3.

As both $K$ and $L$ are now available, we employ Algorithm 3 to the closed-loop system in (107) with the parameters in (120)–(126), which produces the results in Table 1–2, where SA stands for the Spectral Abscissa of the resulting closed-loop system with $w(t) \equiv 0_q$ and Nols for the number of iteration in the while loop. It clearly shows the superior performance of the controller/observer pair computed via Algorithm 3 over the controller in (125) by Algorithm 1, as Algorithm 3 simultaneously computes optimal $K$ and $L$ minimizing $\min \gamma$. Moreover, the values of $\min \gamma$ in Table 1–2 show that adding more functions (larger $\lambda_1, \lambda_2$) to $f_i(\cdot)$ may increase the feasibility of the synthesis conditions leading to smaller $\min \gamma$.

For numerical simulation, we consider the closed-loop systems in (107) with $K = \begin{bmatrix} -1.4434 & -1.9992 \end{bmatrix}^\top$ and $L = \begin{bmatrix} -33.3577 & -50.1559 \end{bmatrix}^\top$ in Table 2, and the state space parameters in (120). Moreover, let $t_0 = 0$, $z(t) = 0_2$, $t < 0$, and $\psi(\tau) = \begin{bmatrix} 50 & 30 & 30 & 20 \end{bmatrix}^\top$, $\tau \in [-r_2, 0]$ for the initial condition, and $w(t) = 50 \sin 10t(1(t) - 1(t - 5))$ as the disturbance where $1(t)$ is the Heaviside step function. The simulation is performed in Simulink via the ODE solver ode8 with 0.002 as the fundamental sampling time. The result is summarized in Figures 1–3 concerning the trajectories of the states, regulated outputs and

---

All results of SA are calculated via the spectral method in Breda et al. (2015)
Controller gain $K$

$$\begin{bmatrix}
-1.4253 \\
-1.9829
\end{bmatrix}^{\top}$$

Observer gain $L$

$$\begin{bmatrix}
-33.2508 \\
-49.9964
\end{bmatrix}^{\top}$$

$$\begin{bmatrix}
-33.0844 \\
-49.7443
\end{bmatrix}^{\top}$$

$$\begin{bmatrix}
-32.9287 \\
-49.5104
\end{bmatrix}^{\top}$$

$$\begin{bmatrix}
-32.7784 \\
-49.2857
\end{bmatrix}^{\top}$$

$\min \gamma$

$$\begin{bmatrix}
0.6289 \\
0.6286
\end{bmatrix}^{\top}$$

$$\begin{bmatrix}
0.6285 \\
0.6284
\end{bmatrix}^{\top}$$

$\text{SA}$

$$\begin{bmatrix}
-0.7360 \\
-0.7369
\end{bmatrix}^{\top}$$

$$\begin{bmatrix}
-0.7374 \\
-0.7377
\end{bmatrix}^{\top}$$

$\text{Nols}$

$$\begin{bmatrix}
5 \\
10
\end{bmatrix}$$

$$\begin{bmatrix}
15 \\
20
\end{bmatrix}$$

Table 1: Controller and observer gains with $\min \gamma$ produced with $d_1 = d_2 = \lambda_1 = \lambda_2 = 1$

Controller gain $K$

$$\begin{bmatrix}
-1.4517 \\
-1.9962
\end{bmatrix}^{\top}$$

Observer gain $L$

$$\begin{bmatrix}
-33.7152 \\
-50.6877
\end{bmatrix}^{\top}$$

$$\begin{bmatrix}
-33.5992 \\
-50.515
\end{bmatrix}^{\top}$$

$$\begin{bmatrix}
-33.4790 \\
-50.3361
\end{bmatrix}^{\top}$$

$$\begin{bmatrix}
-33.3577 \\
-50.1559
\end{bmatrix}^{\top}$$

$\min \gamma$

$$\begin{bmatrix}
0.6093 \\
0.6092
\end{bmatrix}^{\top}$$

$$\begin{bmatrix}
0.6092 \\
0.6091
\end{bmatrix}^{\top}$$

$\text{SA}$

$$\begin{bmatrix}
-0.7297 \\
-0.7301
\end{bmatrix}^{\top}$$

$$\begin{bmatrix}
-0.7304 \\
-0.7306
\end{bmatrix}^{\top}$$

$\text{Nols}$

$$\begin{bmatrix}
5 \\
10
\end{bmatrix}$$

$$\begin{bmatrix}
15 \\
20
\end{bmatrix}$$

Table 2: Controller and observer gains with $\min \gamma$ produced with $d_1 = d_2 = 1, \lambda_1 = \lambda_2 = 2$

the observer’s states in the closed-loop system. Note that all the DDs are discretized in simulation via the trapezoidal rule with 200 sample points.

Figure 1: The close-loop system’s trajectories including the observer errors $e(t)$
6.2. Observer-based dissipative stabilization of time-delay system via controllers with delays

The purpose of this subsection is to show the advantage of adding delays to controllers/observers when the system has no delays at the input and measured output.

Consider a system with the same parameters in subsection 6.1 except for $B_i = \tilde{B}_i(\tau) = O_{n,p}$ and $\mathfrak{B}_i = \tilde{\mathfrak{B}}_i(\tau) = O_{m,n}, i = 1, \ldots, \nu$ and $D_i = \tilde{D}_i(\tau) = O_{l,q}$, $C_i = \tilde{C}_i(\tau) = O_{l,n}, i = 1, \ldots, \nu$. This allows one to construct a controller in the form of (79)–(80) with an observer in (98)–(99) minimizing the same objective function $\gamma$ in (121).

Given the results in subsection 6.1, the procedures of computing controller/observer gains here are entirely identical to the previous subsection apart from utilizing the variant of Algorithm 1-3 to the closed-loop systems in (81) and (100) and (117) with Corollary 1–5 developed in subsections 3.4, 4.1, 5.1 instead
of Theorem 1–5. Specifically, we apply Proposition 1 with (80) and (99) for the DDs in (120), (79) and (98) using the same parameters in (122) for $g_i(\cdot)$ and $M_i$. This leads to the same parameters $\tilde{A}_i$, $\tilde{C}_i$, in (124), whereas $\mathcal{K}_i$ in (80) and $\mathcal{L}_i$ in (99) are the unknowns to be computed for the controllers/observers.

Next, we assume $\alpha_i = 0$, $i = 2, \cdots, 1 + \nu + \kappa$, $\alpha_1 = 5$ and $\alpha_1 = 50$, respectively, when Corollary 2 and 4 are applied. The numerical results are summarized in Table 3–6 where the resulting controller/observer gains are omitted due to limit space. Note that the comparison between the results in Table 3–4 and Table 5–6 shows that using the delay structures in (79) and (98) can materially improve the performance of $\min \gamma$ compared to the use of a static controller/observer $K_0/L_0$. This justify the use of the delay structures in (79) and (98) even though it requires more resources for implementing the resulting controller/observer with delays.

**Remark 12.** Since the closed-loop delay systems in (81) and (100) and (117) belong to the retarded type, their nominal stability is guaranteed by Kharitonov (2015) with the numerical implementation of the DDs in (80),(99) as long as the numerical accuracy reach certain degree. This property ensures that the resulting controllers/observers in (81),(100),(117) can be materialized numerically for real-world applications.

| $\min \gamma$ | 0.4935 | 0.4928 | 0.4925 | 0.4922 |
| SA          | −0.7552 | −0.7536 | −0.7538 | −0.7544 |
| NoIs        | 5       | 10      | 15      | 20     |

Table 3: $\min \gamma$ produced by the variant of Algorithm 3 to (5.1) with $d_1 = d_2 = \lambda_1 = \lambda_2 = 1$

| $\min \gamma$ | 0.4912 | 0.4906 | 0.4903 | 0.4927 |
| SA          | −0.7371 | −0.7453 | −0.7479 | −0.7483 |
| NoIs        | 5       | 10      | 15      | 20     |

Table 4: $\min \gamma$ produced by the variant of Algorithm 3 to (5.1) with $d_1 = d_2 = 1, \lambda_1 = \lambda_2 = 2$ with $d_1 = d_2 = \lambda_1 = \lambda_2 = 1$ and $K_i = \tilde{K}_i(\tau) = O_{p,n}$, $L_i = \tilde{L}_i(\tau) = O_{n,l}$

| $\min \gamma$ | 0.5313 | 0.5313 | 0.5312 | 0.5312 |
| SA          | −0.7412 | −0.7417 | −0.7421 | −0.7424 |
| NoIs        | 5       | 10      | 15      | 20     |

Table 5: $\min \gamma$ produced by the variant of Algorithm 3 to (5.1) with $d_1 = d_2 = 1, \lambda_1 = \lambda_2 = 2$ and $K_i = \tilde{K}_i(\tau) = O_{p,n}$, $L_i = \tilde{L}_i(\tau) = O_{n,l}$

| $\min \gamma$ | 0.5231 | 0.5230 | 0.5229 | 0.5228 |
| SA          | −0.7477 | −0.7481 | −0.7486 | −0.7489 |
| NoIs        | 5       | 10      | 15      | 20     |

Table 6: $\min \gamma$ produced by the variant of Algorithm 3 to (5.1) with $d_1 = d_2 = 1, \lambda_1 = \lambda_2 = 2$ and $K_i = \tilde{K}_i(\tau) = O_{p,n}$, $L_i = \tilde{L}_i(\tau) = O_{n,l}$
Figure 4: The close-loop system’s trajectories including the observer errors $e(t)$

Figure 5: The trajectory of the observer’s states $\hat{x}(t)$
Figure 6: The regulated output of the closed-loop system $z(t)$

For numerical simulation, we consider the closed-loop systems in (116) with

$$K_0 = \begin{bmatrix} -5.9332 & -16.0079 \end{bmatrix}, \quad K_1 = \begin{bmatrix} 0.3877 & 0.2920 \end{bmatrix}, \quad K_2 = \begin{bmatrix} 0.1147 & -0.0035 \end{bmatrix}$$

$$K_1 = \begin{bmatrix} -0.3753 & -0.1711 & -0.6469 & -0.5271 & -1.5646 & -0.4321 & 3.6833 & -2.7850 & -2.3409 & -2.2794 & \cdots \\
-0.5546 & -1.1529 & -0.0477 & -0.3417 & 1.2123 & 1.2468 & 0.1759 & 0.0138 \\
\cdots \cdots \end{bmatrix}$$

$$K_2 = \begin{bmatrix} 0.3358 & 0.3354 & -0.3809 & -0.9398 & -0.7564 & -0.7966 & 7.2214 & -1.1190 & 5.2653 & -1.5049 & \cdots \\
-0.3859 & 1.0201 & -0.0827 & 0.3584 & 1.5520 & 4.1739 & -0.2891 & -0.7145 \\
\cdots \cdots \end{bmatrix}$$

$$L_0 = \begin{bmatrix} -75.2687 & -112.4781 \end{bmatrix}^\top, \quad L_1 = \begin{bmatrix} 0.7118 & 1.4812 \end{bmatrix}^\top, \quad L_2 = \begin{bmatrix} -0.0770 & 0.1311 \end{bmatrix}^\top$$

$$L_1 = \begin{bmatrix} -0.0468 & 0.7910 & 0.8041 & -27.4224 & -14.6889 & -0.2046 & -0.0350 & -1.6277 & 1.0984 \\
-0.0869 & 1.1998 & 3.3425 & -39.6207 & -16.1770 & -10.4573 & 0.6329 & -1.8172 & 4.0465 \\
\cdots \cdots \end{bmatrix}$$

$$L_2 = \begin{bmatrix} 1.2091 & -6.8747 & -95.6277 & -9.8827 & -55.3697 & -3.3184 & 4.9281 & -1.1427 & -2.8585 \\
-1.5661 & -8.1022 & -145.0497 & 3.8740 & -91.2384 & -18.7310 & -0.5970 & 15.8409 & -5.9459 \\
\cdots \cdots \end{bmatrix}$$

(127)

The rest of the setting is identical to the one in subsection 6.1, where the simulation results are summarized in Figures 4–6.

7. Conclusion

This work has set out general solutions for the DSFC, DSOD, DOBC problems of a general LTDS with delays at the states, inputs and outputs, where the number of delays is unlimited and the DDs can contain any number of $L^2$ function over bounded intervals. Apart from the novel decomposition in Proposition 1, which gives users the liberty to use different ways to handle the DDs, the proposed approaches also allow the simultaneous computation of controller/observer gains both subject to dissipative constraints, without requiring nonlinear SDP solvers. Because of the generality and scope of the proposed method, it could be considered as a milestone of the SDP-based solutions for LTDSs similar to the SDP approach for an LTI
delay-free system. Since we have seen an efficient treatment of DDs in this work, we will further develop this idea to design augmented dynamical state controllers/observers for LTDs in part II of the paper, where the effects of delays at the input and measured output can be totally nullified via the structures similar to predictor controllers.

Appendix A. Important lemmas and definition

Some lemmas are presented here which are crucial for the derivations of the results in this paper. A novel integral inequality is proposed to construct lower bounds for integrals defined over \([-r_i, -r_{i-1}].\)

**Lemma 3.** \(\forall X \in \mathbb{R}^{n \times m}, \forall Y \in \mathbb{R}^{m \times p}, \forall Z \in \mathbb{R}^{q \times r},\)

\[
(X \otimes I_q)(Y \otimes Z) = XY \otimes Z = XY \otimes ZI_r = (X \otimes Z)(Y \otimes I_r).
\] (A.1)

\[
(X \otimes I_q)(Y \otimes Z) = XY \otimes Z = ImX \otimes (ZI_r) = (Im \otimes Z)(XY \otimes I_r).
\] (A.2)

Moreover, \(\forall X \in \mathbb{R}^{n \times m},\) we have

\[
\begin{bmatrix}
A & B \\
C & D
\end{bmatrix} \otimes X = 
\begin{bmatrix}
A \otimes X & B \otimes X \\
C \otimes X & D \otimes X
\end{bmatrix}, \quad I_n \otimes X = \bigoplus_{i=1}^{n} X
\] (A.3)

for any \(A, B, C, D\) with appropriate dimensions which make the block matrix at the left hand of the equality in (A.3) to be compatible.

Firstly, we define the following weighted Lebesgue function space

\[
L^2_{w_0}(\mathcal{K} \ni \mathbb{R}^d) := \left\{ \phi(\cdot) \in M_{L(\mathbb{R})/B(\mathbb{R})}(\mathcal{K} \ni \mathbb{R}^d) : \|\phi(\cdot)\|_{2, w_0} < \infty \right\}
\] (A.4)

with \(d \in \mathbb{N}\) and \(\|\phi(\cdot)\|_{2, w_0} := \int_{\mathcal{K}} \varpi(\tau) \phi^\top(\tau) \phi(\tau) d\tau\) where \(\varpi(\cdot) \in M_{L(\mathbb{R})/B(\mathbb{R})}(\mathcal{K} \ni \mathbb{R}_{\geq 0})\) and the function \(\varpi(\cdot)\) has only countably infinite or finite number of zero values. Furthermore, \(\mathcal{K} \subseteq \mathbb{R} \cup \{\pm \infty\}\) and the Lebesgue measure of \(\mathcal{K}\) is non-zero.

**Lemma 4.** Given \(\mathcal{K}\) and \(\varpi(\cdot)\) in (A.4) and \(U \in S^{n}_{\geq 0} := \{X \in S^n : X \succeq 0\}\) with \(n \in \mathbb{N}\). Let \(f_i(\cdot) \in L^2_{w_0}(\mathcal{K} \ni \mathbb{R}^l)\) and \(g_i(\cdot) \in L^2_{w_0}(\mathcal{K} \ni \mathbb{R}^n)\) with \(l_i \in \mathbb{N}\) and \(\lambda_i \in \mathbb{N}_0\), \(i = 1, \ldots, \nu\), in which the functions \(f_i(\cdot)\) and \(g_i(\cdot)\) satisfy

\[
\int_{\mathcal{K}} \varpi(\tau) \left[ g_i(\tau) \right]^\top \left[ f_i(\tau) \right] d\tau \succ 0, \quad i = 1, \ldots, \nu
\] (A.5)

which implies \(\int_{\mathcal{K}} \varpi(\tau) f_i(\tau) f_i^\top(\tau) d\tau \succ 0, \ i = 1, \ldots, \nu\). Then the inequality

\[
\int_{\mathcal{K}} \varpi(\tau) x^\top(\tau) \left[ \sum_{i=1}^{\nu} U_i \right] x(\tau) d\tau \geq \left[ \left( \sum_{i=1}^{\nu} T_i^{-1} \right) \otimes \left( \sum_{i=1}^{\nu} U_i \right) \right] \left( \int_{\mathcal{K}} \varpi(\tau) \left[ \sum_{i=1}^{\nu} f_i(\tau) \right] \otimes I_n \right) x(\tau) d\tau \\
+ \left[ \left( \sum_{i=1}^{\nu} E_i^{-1} \right) \otimes \left( \sum_{i=1}^{\nu} U_i \right) \right] \left( \int_{\mathcal{K}} \varpi(\tau) \left[ \sum_{i=1}^{\nu} e_i(\tau) \right] \otimes I_n \right) x(\tau) d\tau \\
\geq \left[ \left( \sum_{i=1}^{\nu} T_i^{-1} \right) \otimes \left( \sum_{i=1}^{\nu} U_i \right) \right] \left( \int_{\mathcal{K}} \varpi(\tau) \left[ \sum_{i=1}^{\nu} f_i(\tau) \right] \otimes I_n \right) x(\tau) d\tau
\] (A.6)

holds for all \(x(\cdot) \in L^2_{w_0}(\mathcal{K} \ni \mathbb{R}^n)\), where \(T_i = \int_{\mathcal{K}} \varpi(\tau) f_i(\tau) f_i^\top(\tau) d\tau \succ 0.\) In addition, \(e_i(\cdot) = g_i(\cdot) - A_i f_i(\cdot) \in \mathbb{R}^{\lambda_i}\) and \(A_i = \int_{\mathcal{K}} \varpi(\tau) g_i(\tau) f_i^\top(\tau) d\tau, T_i \in \mathbb{R}^{\lambda_i \times \lambda_i}\) and \(E_i := \int_{\mathcal{K}} \varpi(\tau) e_i(\tau) e_i^\top(\tau) d\tau = \|e_i(\tau)\|_{1 \times 1}^2\).  

**Proof.** By using the inequality in (Feng et al., 2020, eq.(17)) \(\nu\) times, then (A.6) is obtained. Note that the definition of \(T_i\) here is the inverse of the one in Feng et al. (2020). Furthermore, \(E_i^{-1}\) in (A.6) is well defined with \(g_i(\tau) = \|e_i(\tau)\|_{1 \times 1}.\)
Appendix B. Proof of Proposition 1

First of all, it is obvious that (2) is implied by (3)–(6) because of the definitions of $\varphi_i(\cdot)$, $f_i(\cdot)$, $\phi_i(\cdot)$ and the fact that $C^1([-r_i, -r_{i-1}] \setminus \mathbb{R}^d_i) \subset \mathbb{L}^2([-r_i, -r_{i-1}] \setminus \mathbb{R}^d_i)$. So the necessity part of the statement is proved.

Now we start to prove the sufficiency part of the statement. Namely, the condition in (2) implies the existence of the parameters in Proposition 1 satisfying (3)–(6). Given any $f_i(\cdot) \in C^1([-r_i, -r_{i-1}] \setminus \mathbb{R}^d_i)$, $i = 1, \ldots, \nu$, $d \in \mathbb{N}$ satisfying $\int_{-r_{i-1}}^{-r_i} f_i^*(\tau)f_i(\tau)d\tau > 0$, one can always construct appropriate $\phi_i(\cdot)$ and $\varphi_i(\cdot)$ in $\mathbb{L}^2([-r_i, -r_{i-1}] \setminus \mathbb{R}^d_i)$ with $M_i \in \mathbb{R}^{d \times \kappa_i}$ such that the conditions in (5)–(6) are satisfied. Note that $\int_{-r_{i-1}}^{-r_i} f_i^*(\tau)f_i(\tau)d\tau > 0$ is implied by the matrix inequalities in (6) which indicate that the functions in $g_i(\cdot)$ in (5) are linearly independent in a Lebesgue sense over $[-r_i, -r_{i-1}]$ for each $i = 1, \ldots, \nu$. The aforementioned conclusion is true because $\frac{df_i(\tau)}{d\tau} \in C([-r_i, -r_{i-1}] \setminus \mathbb{R}^d_i) \subset \mathbb{L}^2([-r_i, -r_{i-1}] \setminus \mathbb{R}^d_i)$ for all $i = 1, \ldots, \nu$, and the restrictions of $\varphi_i(\tau)$ and $\phi_i(\tau)$, $i = 1, \ldots, \nu$ can be arbitrarily enlarged with more linearly independent functions. Note that $\varphi_i(\tau)$ or $\phi_i(\tau)$ can be any matrix $[\tau]_{0 \times 1}$. Now since $\dim (g_i(\tau))$ in (5)–(6) can be arbitrarily increased, (appropriate new functions can always be added) there always exist $\hat{A}_{i,j} \in \mathbb{R}^{n \times n}$, $\hat{C}_{i,j} \in \mathbb{R}^{m \times n}$, $\hat{B}_{i,j} \in \mathbb{R}^{n \times p}$, $\hat{B}_i \in \mathbb{R}^{m \times p}$ and $g_i(\tau) = \text{Col}_{j=1}^{\nu} g_{i,j}(\tau)$ in (6) for the distributed delay terms in (2) such that

$$\forall \tau \in [-r_i, -r_{i-1}], \quad \hat{A}_i(\tau) = \sum_{j=1}^{\kappa_i} \hat{A}_{i,j} g_{i,j}(\tau), \quad \hat{C}_i(\tau) = \sum_{j=1}^{\kappa_i} \hat{C}_{i,j} g_{i,j}(\tau),$$

$$\forall \tau \in [-r_i, -r_{i-1}], \quad \hat{B}_i(\tau) = \sum_{j=1}^{\kappa_i} \hat{B}_{i,j} g_{i,j}(\tau), \quad \hat{B}_i(\tau) = \sum_{j=1}^{\kappa_i} \hat{B}_{i,j} g_{i,j}(\tau)$$

with $\kappa_i \in \mathbb{N}_0$ for $i = 1, \ldots, \nu$, where $\varphi_i(\cdot) \in \mathbb{L}^2([-r_i, -r_{i-1}] \setminus \mathbb{R}^d_i)$, $f_i(\cdot) \in C^1([-r_i, -r_{i-1}] \setminus \mathbb{R}^d_i)$, $\phi_i(\cdot) \in \mathbb{L}^2([-r_i, -r_{i-1}] \setminus \mathbb{R}^\nu)$ satisfy (5)–(6) for some $M_i \in \mathbb{R}^{d \times \kappa_i}$, $i = 1, \ldots, \nu$. Moreover, (B.1) can be rewritten as

$$\forall \tau \in [-r_i, -r_{i-1}], \quad \hat{A}_i(\tau) = \left[ \text{Row}_{i=1}^{\kappa_i} \hat{A}_{i,j} \right] (g(\tau) \otimes I_n), \quad \hat{C}_i(\tau) = \left[ \text{Row}_{i=1}^{\kappa_i} \hat{C}_{i,j} \right] (g(\tau) \otimes I_n)$$

$$\forall \tau \in [-r_i, -r_{i-1}], \quad \hat{B}_i(\tau) = \left[ \text{Row}_{i=1}^{\kappa_i} \hat{B}_{i,j} \right] (g(\tau) \otimes I_p), \quad \hat{B}_i(\tau) = \left[ \text{Row}_{i=1}^{\kappa_i} \hat{B}_{i,j} \right] (g(\tau) \otimes I_p)$$

(B.2)

which are in line with the decompositions in (3)–(4) by letting $\hat{A}_i = \text{Row}_{i=1}^{\kappa_i} \hat{A}_{i,j}$, $\hat{C}_i = \text{Row}_{i=1}^{\kappa_i} \hat{C}_{i,j}$, $\hat{B}_i = \text{Row}_{i=1}^{\kappa_i} \hat{B}_{i,j}$, $\hat{B}_i = \text{Row}_{i=1}^{\kappa_i} \hat{B}_{i,j}$ and $\hat{B}_i = \text{Row}_{i=1}^{\kappa_i} \hat{B}_{i,j}$ for all $i = 1, \ldots, \nu$. Finally, the conclusion in (B.1) is true for the case of $\mu_i = 0$ or $\delta_i = 0$. Given all the aforementioned statements we have presented, then Proposition 1 is proved.

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\[^{6}\text{See Theorem 7.2.10 in Horn & Johnson (2012) for more information}\]
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