On the Minimization of Convex Functionals of Probability Distributions Under Band Constraints

Michael Fauß, Member, IEEE and Abdelhak M. Zoubir, Fellow, IEEE

Abstract—The problem of minimizing convex functionals of probability distributions is solved under the assumption that the density of every distribution is bounded from above and below. First, a system of sufficient and necessary first order optimality conditions, which characterize global minima as solutions of a fixed-point equation, is derived. Based on these conditions, two algorithms are proposed that iteratively solve the fixed-point equation via a block coordinate descent strategy. While the first algorithm is conceptually simpler and more efficient, it is not guaranteed to converge for objective functions that are not strictly convex. This shortcoming is overcome in the second algorithm, which uses an additional outer proximal iteration, and, which is proven to converge under very mild assumptions. Two examples are given to demonstrate the theoretical usefulness of the optimality conditions as well as the high efficiency and accuracy of the proposed numerical algorithms.

Index Terms—Robust statistics, model uncertainties, band model, convex optimization, block coordinate descent

I. INTRODUCTION

FUNCTIONALS of probability distributions play a central role in probability theory and statistics. To clarify, a functional is a mapping from a vector space to the real line, i.e., a function which maps an element of a possibly high or infinite dimensional space to a scalar value. Omnipresent examples are the moments of a real-valued random variable, which map a distribution to a real number.

Convex functionals of probability distributions occur naturally in problems of statistical inference and decision making. In general, the expected cost of any inference procedure with a convex loss function can be shown to be a convex functional of probability distributions [1]. Consequently, examples of convex functionals can be found in detection [2], estimation [3], and joint detection and estimation theory as well as in Bayesian inference [4].

In practice, it is often the case that the distributions of random variables that describe a random phenomenon are not known exactly, but are subject to uncertainty. This uncertainty can, for example, be caused by a lack of information about the random phenomenon or the absence of an appropriate model for its mathematical description. Uncertainty can also be introduced intentionally in order to safeguard an inference procedure against deviations from the underlying assumptions. The type and degree of uncertainty is usually specified by means of an uncertainty set that contains all feasible distributions. Given such a set, a question that naturally arises is which distributions are most favorable and which are least favorable. In order to answer this question, a given cost function has to be minimized or maximized over the uncertainty set. This is the problem addressed in this paper.

The uncertainty model that is assumed to hold throughout the paper restricts the densities of feasible distributions to lie within a band that is defined by two non-intersecting functions, which bound the density from above an below. It can be thought of as a confidence interval for the true density function. This model is known as the density band model and is commonly used in robust statistics [5]. In the context of robust hypothesis testing it was first studied in [6] and was recently revisited in [7]. It is discussed in more detail in Section II.

The problem that motivated the work in this paper relates to the design of minimax optimal sequential tests for multiple hypotheses. The least favorable distributions for this type of test can be shown to be minimizers of functionals of the form

\[
\int_{\Omega} f(\omega, \frac{dP_1}{dP}(\omega), \ldots, \frac{dP_N}{dP}(\omega)) \, dP(\omega),
\]

where \(P_1, P_2, \ldots, P_N\) denote distributions that are subject to uncertainty and the function \(f\) is jointly convex in the likelihood ratios \(dP_n/dP\). A particular difficulty that arises in the design of minimax sequential tests is that \(f\) is itself the solution of an optimization problem so that its value and its derivatives can only be evaluated numerically. Minimizing \(f\) over \(P_1, P_2, \ldots, P_N\), and under band constraints, is a challenging task because it involves two nested optimization problems. Deriving analytic solutions, or determining approximate solutions is usually not possible. Therefore, a suitable numerical algorithm is required that is

- accurate enough to closely approximate continuous density functions,
- efficient enough to handle multiple distributions and reasonably fine grids for their discretization,
- parallelizable, in order to leverage modern hardware,
- robust against mild numerical noise in the evaluation of \(f\) and its derivatives.

When looking into existing convex optimization frameworks and algorithms, it was found that most off-the-shelf methods did not satisfy these requirements. On the one hand, high performance solvers such as Gurobi, MOSEK, or CPLEX were found to be too restrictive in terms of feasible objective functions. On the other hand, commonly used generic convex optimization algorithms, such as interior point, steepest descent or conjugate gradient methods, turned out to be too

\footnote{For binary tests, this result can be found in \[8\]. The corresponding results on tests for multiple hypotheses are to be presented in a forthcoming publication.}
inefficient to be useful in practice. In addition, all algorithms considered in the survey suffered from severe accuracy issues, especially in the tails of the optimal distributions.

In this work, an approach for the minimization of convex functionals of probability distributions, under density band uncertainty, is detailed, which is efficient, reliable, and applicable beyond the particular use case of robust sequential hypothesis testing. The proposed algorithms offer a good trade-off between being generic and specific, in the sense that they heavily exploit assumptions about the structure of the objective function and the constraints, while at the same time providing enough flexibility to be applicable to a large class of problems in statistical signal processing and robust statistics in particular.

The existing literature on convex functionals is large and dispersed. While in the functional analysis literature the term convex functionals prevails [9], in statistics, signal processing and information theory, similar classes of functions often go by the names divergence, distance, dissimilarity, or disparity. Early results on the subject are due to, among others, Pearson [10], Mahalanobis [11], Shannon [12], and Kullback [13]. See [14], [15] and references therein for a detailed treatment. The minimization of convex functionals, and their relation to robust decision making, has been addressed by Huber [16]. Poor [17], Kassam [6] and more recently Gunetbouyina [18] to name just a few. More recently, a closely related convex optimization problem was investigated in [19] under slightly different assumptions and constraints.

The paper is organized as follows: The density band uncertainty model is briefly reviewed in Section II. In Section III the functional minimization problem is stated and the corresponding optimality conditions are derived. These conditions are a useful result in itself and constitute a main contribution of the paper. In Section IV an algorithm is proposed that iteratively solves the optimality conditions using a block coordinate descent (BCD) strategy. Since the latter is not guaranteed to converge to a global minimum for general convex objective functions, a second algorithm is introduced that augments the objective function with an additional proximal term. Guaranteed convergence of the proximal algorithm is shown subsequently. In Section V two examples are provided to illustrate how the optimality conditions as well as the proposed numerical algorithms can be used in practice. Section VI concludes the paper.

Notation: Probability distributions are denoted by upper case letters, their densities by the corresponding lower case letters. Superscripts are used as a shorthand notation for Cartesian products, i.e., $A^2 = A \times A$. Boldface is used to indicate vectors, i.e., $\mathbf{x} = (x_1, \ldots, x_N)$, $N \geq 1$. No distinction is made between row and column vectors. The notation $x_{[n]}$ is used to denote a vector whose $n$th element has been removed, i.e., $x_{[n]} = (x_1, \ldots, x_{n-1}, x_{n+1}, \ldots, x_N)$. For functions that are defined directly on the sample space, the explicit argument $\omega \in \Omega$ is often omitted for the sake of a more compact notation, in particular if the function is integrated or is itself an argument of a higher order function. Inequalities of vectors and functions are defined element-wise. Finally, $\partial_{x_n} f(\mathbf{x})$ denotes the subdifferential of a convex function $f: \mathcal{X} \subset \mathbb{R}^N \rightarrow \mathbb{R}$ with respect to $x_n$ at $\mathbf{x}$, i.e.,

$$\partial_{x_n} f(\mathbf{x}) := \{ \gamma \in \mathbb{R} : \frac{f(\mathbf{y}) - f(\mathbf{x})}{y_n - x_n} \leq \gamma \ \forall \mathbf{y} \in \mathcal{X} \setminus \{ \mathbf{x} \} \}.$$ 

II. THE DENSITY-BAND UNCERTAINTY MODEL

Let $(\Omega, \mathcal{F})$ be a measurable space and let $\mu$ be an absolutely continuous $\sigma$-finite measure on this space. Further, let $\mathcal{L}^1$ denote the set of all functions $\Omega \rightarrow \mathbb{R}$ that are $\mu$-integrable. The density-band uncertainty model specifies sets of the form

$$\mathcal{P}^\gamma = \{ p \in \mathcal{L}^1 : p' \leq p \leq p'' \},$$

where $0 \leq p' \leq p'' \leq \infty$, $\int p' \, d\mu \leq 1$, $\int p'' \, d\mu \geq 1$. (3)

Thus, all feasible densities are upper bounded by $p''$ and lower bounded by $p'$. The case where the upper bound is infinity and the lower bound is zero is the unconstrained case.

The band model is useful for several reasons. First, it provides a great amount of flexibility to the designer of an inference procedure, as it allows for varying local degrees of uncertainty on different regions of the sample space. Depending on the application, it can be constructed by hand, based on expert knowledge, or statistically, via confidence interval estimators. Second, in contrast to many parametric uncertainty models, it provides clear visualization and easy interpretation. Third, from a theoretical point of view, the density band model is of interest because it generalizes several popular uncertainty models such as the $\varepsilon$-contamination model [20] and the bounded distribution function model [21]. A more detailed discussion of the band model and its properties can be found in [6] and [7].

III. PROBLEM FORMULATION AND OPTIMALITY CONDITIONS

In this section, the functional minimization problem is stated in a formal manner and a system of necessary and sufficient optimality conditions is derived.

Let

$$I_f(p_1, \ldots, p_N) := \int_\Omega f(\omega, p_1(\omega), \ldots, p_N(\omega)) \, d\mu(\omega),$$

where

$$f: \Omega \times [0, \infty)^N \rightarrow (-\infty, \infty]$$

$$(\omega, x_1, \ldots, x_N) \mapsto f(\omega, x_1, \ldots, x_N)$$

(5)

is a function that is convex with respect to $(x_1, \ldots, x_N)$. In order to guarantee that (4) is well defined, it is further assumed that $f(\omega, p_1(\omega), \ldots, p_N(\omega))$ is $\mu$-measurable for all feasible densities $p_1, \ldots, p_N$. A proof, and a more in-depth analysis, of the existence and well-definedness of (4) is detailed in [2].

Also note that $f$ being convex with respect to $(x_1, \ldots, x_N)$ does not imply that $f(\omega, p_1(\omega), \ldots, p_N(\omega))$ is convex with respect to $\omega$. To facilitate compact notation, the arguments of $f$ are occasionally written in vector notation and the direct dependence on $\omega$ is omitted, i.e.,

$$f(\mathbf{x}) := f(x_1, \ldots, x_N) := f(\omega, x_1, \ldots, x_N).$$
It is important to note that $I_f(p_1, \ldots, p_N)$ is used instead of $I_f(P_1, \ldots, P_N)$. The latter notation is commonly used in the context of distance measures between distributions, such as $f$-divergences \((22)\) and $f$-dissimilarities \((23)\), in order to emphasize that the distance does not depend on how the reference measure $\mu$ is chosen. However, since $f$ in \((5)\) is not assumed to be homogeneous and is allowed to directly depend on $\omega$, this independence does not hold in general.

The optimization problem considered in this work is

$$
\min_{p_n \in \mathcal{P}^n} I_f(p_1, \ldots, p_N),
$$

where $n \in \{1, \ldots, N\}$ and all $\mathcal{P}^n$ are of the form \((2)\). Expressing the objective function and constraints explicitly, \((6)\) becomes

$$
\min_{p_n \in \mathcal{L}^1} \int_{\Omega} f(\omega, p_1(\omega), \ldots, p_N(\omega)) \, d\mu(\omega)
$$

s.t. $p_n \geq p''_n$, $p_n \leq p''_n$, $\int p_n \, d\mu = 1$.

The optimality conditions for \((7)\) are given in the following theorem.

**Theorem 1:** A sufficient and necessary condition for the densities $q = (q_1, \ldots, q_N)$ to be a solution of \((7)\) is that they satisfy

$$
q_n = \begin{cases}
p''_n, & f_n(q) < c_n \\
f_n^{-1}(q_n), c_n), & f_n(q) = c_n \\
p'_n, & f_n(q) > c_n
\end{cases}
$$

for some $c_1, \ldots, c_N \in \mathbb{R}$ and all $n \in \{1, \ldots, N\}$. Here $f_n$ denotes a partial subderivative of $f$ with respect to $x_n$, i.e.,

$$
f_n(x) \in \partial_{x_n} f(x),
$$

and $f_n^{-1}$ denotes the inverse of $f_n$ with respect to $x_n$, i.e.,

$$
f_n^{-1}(x_n), c_n) \in \{x_n \in [0, \infty) : f_n(x_1, \ldots, x_N) = c_n\}.
$$

A proof for Theorem 1 is detailed in Appendix A. Making use of the fact that $f_n$ is a subderivative of a convex function and, hence, is nondecreasing, Theorem 1 can be further written in an alternative, more expressive form that eliminates the need for the explicit case-by-case definition.

**Corollary 1:** A sufficient and necessary condition for the densities $q = (q_1, \ldots, q_N)$ to be a solution of \((7)\) is that they satisfy

$$
q_n = \min\{p''_n, \max\{f_n^{-1}(q_n), c_n\}, p'_n\}
$$

for some $c_1, \ldots, c_N \in \mathbb{R}$ and all $n \in \{1, \ldots, N\}$.

Corollary 1 is proven in Appendix B. It gives an expression for $q_n$, solely in terms of the remaining optimal densities $q_{[n]}$ and the scalar $c_n$. That is, knowing $q_{[n]}$, the missing density $q_n$ can be found via a search over $c_n$. The iterative algorithms presented in the next section are based on this idea. It can further be seen from \((9)\) that $q_n$ is a projection of $f_n^{-1}(q_{[n]}, c_n)$ onto the band of feasible densities $\mathcal{P}^n$. In the limit, i.e., $p''_n \to \infty$ and $p'_n \to 0$, it follows that $q_n = f_n^{-1}(q_{[n]}, c_n)$. The example in Section V-A demonstrates this.

### IV. Calculation of the Optimal Densities

In this section, it is shown how the optimality conditions derived in the previous section can be used to calculate the optimal densities. For functions $f$ with analytically invertible subderivatives, and simple bounds $p''_n$, $p'_n$, it can be possible to solve the optimality conditions in Theorem 1 or Corollary 1 by evaluating them on the grid \(\{\omega_1, \ldots, \omega_K\}\).

In order to make \((7)\) tractable for numerical optimization techniques, it first needs to be reduced to a finite dimensional problem. To achieve this, the densities $p_n$ are written in terms of a basis expansion of the form

$$
p_n(\omega) = p_{a_n}(\omega) := \sum_{k=1}^{K} a_{n,k} \psi_k(\omega - \omega_k), \quad K > 1,
$$

where $\psi_1, \ldots, \psi_K$ denote $K$ nonnegative $\mu$-integrable basis functions centered at grid points $\omega_1, \ldots, \omega_K$ and $a_n = (a_{n,1}, \ldots, a_{n,K})$ are nonnegative weight coefficients. It is further assumed that

$$
p_{a_n}(\omega_k) = a_{n,k}
$$

for all $k \in \{1, \ldots, K\}$ and all $n \in \{1, \ldots, N\}$. That is, the basis functions are chosen such that

$$
\psi_k(\omega_l - \omega_k) = \begin{cases} 1, & l = k \\
0, & l \neq k
\end{cases}
$$

This assumption is relatively mild and is satisfied by many common sets of basis functions, in particular by appropriately chosen M-splines and squared sinc functions. Note that \((11)\) does not imply orthogonality.

Let $\omega = (\omega_1, \ldots, \omega_K)$ denote the vector of grid points and let $\mu = (\mu_1, \ldots, \mu_K)$, with

$$
\mu_k := \int_{\Omega} \psi_k(\omega) \, d\mu(\omega),
$$

denote the vector of basis function masses. The minimization in \((6)\) can then be written as the finite dimensional convex optimization problem

$$
\min_{a_n \geq 0} \int_{\Omega} f(\omega, p_{a_1}(\omega), \ldots, p_{a_N}(\omega)) \, d\mu
$$

s.t. $a_n \geq p''_n(\omega)$, $a_n \leq p''_n(\omega)$, $\langle a_n, \mu \rangle = 1$, for all $n \in \{1, \ldots, N\}$, where $\langle \cdot, \cdot \rangle$ denotes the dot product.

Since the degrees of freedom in \((12)\) are reduced to $NK$, the constraints on each density $p_{a_n}$ can only be satisfied at the $K$ grid points. Accordingly, the optimality conditions for problem \((12)\) are obtained from the general conditions in Theorem 1 or Corollary 1 by evaluating them on the grid \(\{\omega_1, \ldots, \omega_K\}\).
A. Block Coordinate Descent

The first proposed algorithm for approximating the optimal densities is detailed below. It is based on the observation that if \( N - 1 \) optimal densities are given, then the remaining one can be determined by choosing the corresponding \( c_n \) such that the right hand side of (9) is a valid density. In each iteration of Algorithm 1, the weight vector \( a_n^i \) is chosen such that \( p_{a_n^i}^i \) satisfies the \( n \)-th optimality condition in Corollary 1 on all the \( K \) grid points and, therefore, is a coordinate-wise minimum of (12). The joint minimum is approached by cyclically minimizing over individual densities while keeping the remaining ones fixed. Consequently, Algorithm 1 implements a block coordinate descent (BCD) in the vectors \( a_1, \ldots, a_N \). The termination criterion in line 10 is intentionally left vague. Any strictly convex vector norm can be used to determine whether all \( a_n^i \) are sufficiently close to convergence.

Algorithm 1 Block coordinate descent

1: input partial subderivatives \((f_1, \ldots, f_N)\) lower density bounds \((p_1^0, \ldots, p_N^0)\) upper density bounds \((p_1^0', \ldots, p_N^0')\)
2: initialize
   Choose feasible initial weight vectors \( a_n^0 \) and set \( i \leftarrow 0 \).
3: repeat
4:   for \( n = 1 \) to \( N \) do
5:     Solve \( \{ \min \{ p_n', \max \{ f_n^{-1}(a_{n|n}, c_n), p_n' \} \} , \mu \} = 1 \)
6:     for \( c_n \) and set \( a_n^i \leftarrow \min \{ p_n', \max \{ f_n^{-1}(a_{n|n}, c_n), p_n' \} \} \).
7:   end for
8:   set \( a_n^{i+1} \leftarrow a_n^i \) for all \( n \in \{1, \ldots, N\} \).
9:   set \( i \leftarrow i + 1 \).
10: until \( a_n^i \approx a_n^{i-1} \) for all \( n \in \{1, \ldots, N\} \)
11: return \((a_1^i, \ldots, a_N^i)\)

The central task in Algorithm 1 is the evaluation of \( f_n^{-1}(a_{n|n}, c_n) \). In analogy to (8), \( f_n^{-1}(a_{n|n}, c_n) \) represents a vector \( a_n \) whose elements \( a_n,k \) satisfy

\[
 f_n(a_{1,k}^i, \ldots, a_{n-1,k}^i, a_{n,k}^i, a_{n+1,k}^i, \ldots, a_{N,k}^i) = c_n. \tag{13}
\]

This condition is obtained by restricting \( f_n^{-1}(q_{n|n}, c_n) \) to the finite grid \( \{\omega_1, \ldots, \omega_K\} \) and making use of property (11). In the case of the inverse of \( f_n \) has an analytical from, the corresponding explicit function can simply be substituted for \( f_n^{-1}(a_{n|n}, c_n) \). An example for this case is given in Section V-A. Otherwise, \( f_n^{-1}(a_{n|n}, c_n) \) needs to be evaluated numerically by solving (13) for \( a_n \). Since there is no coupling between the elements of \( a_n \), the \( K \) equations in (13) can be solved individually for \( c_{n,1}, \ldots, c_{n,K} \) via standard one-dimensional root-finding. This decoupling makes it possible to numerically evaluate \( f_n^{-1}(a_{n|n}, c_n) \) in a highly parallel manner, using up to \( K \) compute cores simultaneously. In addition, the memory requirements per core are minimal since apart from \( f_n \) and \( c_n \), only the vector \( (a_{1,k}^i, \ldots, a_{N,k}^i) \) needs to be stored, which is of dimension \( N \), irrespective of how \( K \) is chosen.

In order to obtain \( a_n^i \), the vector \( f_n^{-1}(a_{n|n}, c_n) \) is projected onto the density band \( P_n^- \), and it then follows that no root-finding needs to be performed if the root is guaranteed to lie outside the feasible interval \([p_n', \omega_k], p_n' \omega_k] \). Whether or not this is the case can be determined by simply evaluating \( f_n \) at the endpoints of the interval. Since \( f_n \) is nondecreasing in \( x_n \), it follows from

\[
 f_n(a_{1,k}^i, \ldots, a_{n-1,k}^i, p_n' \omega_k(a_{n,k}^i), a_{n+1,k}^i, \ldots, a_{N,k}^i) \geq c_n \nRightarrow a_{n,k} < p_n' \omega_k \leftarrow p_n'(\omega_k) \quad \text{so that, after the projection onto the feasible band, } a_{n,k}^i = p_n'(\omega_k).
\]

Analogously, it follows from

\[
 f_n(a_{1,k}^i, \ldots, a_{n-1,k}^i, p_n'(\omega_k), a_{n+1,k}^i, \ldots, a_{N,k}^i) \leq c_n. \tag{14}
\]

Whether or not \( a_{n,k}^i = p_n'(\omega_k) \). These additional checks simplify the evaluation of \( f_n^{-1}(a_{n|n}, c_n) \) significantly.

Algorithm 1 is a straightforward attempt at solving the system of optimality conditions in Theorem 1. By construction, its limit points satisfy (9) and, hence, are global minimizers of (12). For general band models and convex functions \( f \), however, its convergence to a limit point cannot be guaranteed. First, the block coordinate descent strategy implemented in Algorithm 1 is not guaranteed to converge for objective functions that are not strictly convex (24). Second, the density bands can be chosen such that no \( c_n \) exists for which the projection of \( f_n^{-1}(a_{n|n}, c_n) \) onto the band \( P_n^- \) yields a feasible density. If this is the case, the equation in line 5 cannot be solved for \( c_n \) and Algorithm 1 cannot proceed. The latter problem is likely to arise if some density bands have lost or overlap; see (9) and (10) for more details. In the next section, a proximal algorithm that addresses these shortcomings at the cost of a slightly reduced efficiency, is detailed.

B. Block Coordinate Descent With Guaranteed Convergence

The algorithm presented in this section solves the system of optimality conditions by means of a proximal iteration instead of a regular fixed-point iteration. Proximal algorithms are well studied and offer a reliable tool for iteratively solving problems that are not strictly convex. The underlying idea is to augment the objective function with a strictly convex term such that the solution of the augmented problem is guaranteed to be unique. More precisely, in each iteration, a proximal algorithm seeks to minimize a weighted sum of the objective function and a term that penalizes some distance between the current and the previous iterate. The additional distance term ensures that the problem is strictly convex and automatically vanishes when a minimum of the objective function is approached. A comprehensive introduction to proximal algorithms can be found in [25].

For the function \( f_f \) in (4), a proximal operator based on the \( L^2 \)-norm can be defined as

\[
 \text{prox}_{f_f}(h_1, \ldots, h_n) := \arg \min_{p_n \in P_n^-} \left( f_f(p_1, \ldots, p_N) + \frac{1}{2} \sum_{n=1}^{N} \|p_n - h_n\|^2 \right)
\]

\[= \arg \min_{p_n \in P_n^-} \left( \int f(p) + \frac{1}{2} \sum_{n=1}^{N} (p_n - h_n)^2 \, d\mu \right). \tag{14}
\]
The proximal algorithm iteratively solves this operator for its
fixed-point solution, i.e.,
\[(h_1^{i+1}, \ldots, h_N^{i+1}) = \text{prox}_f(h_1^i, \ldots, h_N^i), \quad i \geq 0.\]

In order to implement an iteration of the proximal algorithm,
the inner problem in (14) needs to be solved. By inspection,
this problem is equivalent to the original problem (7) with \(f_1\) replaced by
\[\tilde{f}(\omega, x) := f(\omega, x) + \frac{1}{2} \sum_{n=1}^{N} (x_n - h_n(\omega))^2,\]
which is strictly convex in \(x\) for every convex function \(f\) and
admits the partial subderivatives
\[\tilde{f}_n(\omega, x) = f_n(\omega, x) + x_n - h_n(\omega).\]
(15)
The proximal block coordinate descent algorithm is denoted
Algorithm 2 and is specified below.

**Algorithm 2 Proximal BCD with guaranteed convergence**

1: \textbf{input} partial subderivatives \((f_1, \ldots, f_N)\)
2: \text{lower density bounds} \((p_1', \ldots, p_N')\)
3: \text{upper density bounds} \((p_1'', \ldots, p_N'')\)

4: \textbf{initialize} Choose feasible initial weight vectors \(a_0^n\) and set \(i \leftarrow 0\).

5: \textbf{repeat}

6: \textbf{for} all \(n \in \{1, \ldots, N\}\) \textbf{set}

7: \(\tilde{f}_n(\omega, x_1, \ldots, x_N) \leftarrow f_n(\omega, x_1, \ldots, x_N) + x_n - p_n(\omega)\),

8: \text{then set}

9: \textbf{input} \(\left\{ (\tilde{f}_1, \ldots, \tilde{f}_N), (p_1', \ldots, p_N'), (p_1'', \ldots, p_N'') \right\} \).

10: \textbf{Use Algorithm 1 to obtain}

11: \((a_1^{i+1}, \ldots, a_N^{i+1}) \leftarrow \text{Algorithm 1 input,})

12: \textbf{Set} \(i \leftarrow i + 1\).

13: \textbf{until} \(a_i^n \approx a_{i-1}^n\) for all \(n \in \{1, \ldots, N\}\)

14: \textbf{return} \((a_1^0, \ldots, a_N^0)\)

Convergence of Algorithm 2 is established in the following theorem.

**Theorem 2**: Let the basis functions in (10) be chosen such
that \(\mu > 0\). For all convex functions \(f\) and all bands \(\mathcal{P}=\) that satisfy
\[\langle \mu''(\omega), \mu \rangle \geq 1 \quad \text{and} \quad \langle \mu'(\omega), \mu \rangle \leq 1,\]
Algorithm 2 is guaranteed to converge to a global minimizer of (12).

A proof for Theorem 2 can be found in Appendix C. The two conditions on the density bands are vectorized versions of the integral inequalities in (5) and ensure that in each iteration a feasible density of the form (10) can be constructed.

The price for the guaranteed convergence of the proximal version of the block coordinate descent algorithm is reduced efficiency. In our experiments, Algorithm 2 consistently required more iterations than Algorithm 1 to achieve the same level of accuracy. Some comparisons are shown in Section VI. Moreover, in some cases an analytic expression for the inverse of \(f_n\) exists, but not for the inverse of \(\tilde{f}_n\). In general, the use of Algorithm 2 is recommended only when Algorithm 1 fails to converge.

**C. Remarks**

There are several ways to further improve the performance of the presented algorithms. First, the speed of the coordinate-wise minimization can be improved by not performing it cyclically, but using a suitable selection rule [26], [27]. Second, in Algorithm 2 the weight of the penalty term can be reduced in order to increase the performance. In practice, this means trading off speed of convergence for numerical stability. Whether or not these improvements warrant the additional complexity needs to be decided on a case-by-case basis. Finally, the initial vectors \(a_0^n\) have a non-negligible impact on the speed of convergence.

Apart from the two algorithms presented in this section, the finite-dimensional problem (12) can, in principle, be solved by many off-the-shelf convex optimization programs. The proposed algorithms are preferable over generic solvers for the following reasons:

First, the approach presented in this paper often allows an analytical general form for the optimal densities to be obtained. The numerical part of the solution process then reduces to determining the scalars \(c_1, \ldots, c_N\), which usually is a much simpler problem. Moreover, having a parametric expression for the least favorable densities can facilitate the derivation of bounds or approximations to the exact solution. An example where the system of optimality conditions leads to a useful lower bound is given in the next section.

Second, the accuracy of generic convex optimization algorithms can be hard to control for certain density bands. If, for example, in some region of the sample space the width of the band is of the same order of magnitude as the tolerance of the solver, all points within the band become equivalent. While this effect is negligible in terms of the shape of the optimal densities themselves, it can be critical when calculating functions that involve products or ratios of optimal densities. Although the amount of numerical noise in the solution can be reduced by applying suitable variable transformations and carefully tuning the absolute and relative tolerances of the solvers, the general issue of having to deal with, potentially, very badly scaled problems is an important consideration.

The presented algorithms avoid scaling problems by reducing the entire optimization process to repeated searches for the root of a nondecreasing real-valued function. In each iteration, both \(c_n\) and, if necessary, \(a_{n,k}\) are determined via one-dimensional root-finding. The latter is a standard problem in numerical mathematics and can be solved fast and reliably—see, for example, [28], Chapter 8.3 for an overview of suitable algorithms. In cases where \(f_n\) can only be evaluated with (numerical) noise, stochastic root-finding methods, which handle the additional uncertainty in a systematic manner [29], can be applied. Finally, owing to the explicit projection on
The optimal densities are exact in regions of $\Omega$ where the band constraints are active.

In addition to the increased accuracy, the proposed algorithms can also be significantly faster than generic solvers. The most substantial performance gains can be achieved using Algorithm 1 if an analytic expression for the inverse of $f_n$ exists; see the example in the next section. However, even if $f_n$ has to be inverted numerically, this can be done in a highly parallel manner that facilitates the use of high performance graphics processing units and distributed computing systems.

In summary, the proposed approach to the minimization of convex functionals offers useful theoretical insights and at the same time provides a fast and reliable way to obtain accurate numerical results.

V. EXAMPLES

In this section, the usefulness of the results presented in the previous sections is demonstrated by means of two examples. First, the convex functional is chosen as a weighted sum of Kullback–Leibler divergences. This example shows how analytical results can be obtained by means of Theorem 1 and it highlights the increased efficiency and accuracy of the proposed algorithms in comparison to generic solvers. In the second example, the least favorable distributions for a binary decision making problem with an observation dependent cost function are derived. This example illustrates how the concept of minimizing convex functionals is applicable beyond its traditional context of statistical distance measures.

In order to simplify the presentation, we restrict ourselves to the sample space $\Omega = \mathbb{R}$. For the finite-dimensional representation of the densities, a regular grid with step size $\omega_k - \omega_{k-1} = \Delta \omega$ is used in combination with a linear interpolation scheme, i.e.,

$$
\psi_k(\omega - \omega_k) = \begin{cases} 
1 - \frac{\omega - \omega_k}{\Delta \omega}, & \omega \in [\omega_{k-1}, \omega_{k+1}] \\
0, & \text{otherwise}
\end{cases}
$$

Here, $\mu_k = \Delta \omega$ for all $k$. As a convergence criterion for Algorithm 1 and Algorithm 2, the supremum norm of the difference of the left and right hand side of (9) is used with Algorithm 1 and Algorithm 2, the supremum norm of the reference distribution $P$ of fields, including minimax robust statistics [18], geoscience [30] and biology [31]. In this example, $I_f$ is a weighted sum of Kullback-Leibler divergences with respect to a common reference distribution $P_N$, i.e.,

$$
I_f(p_1, \ldots, p_N) = \sum_{n=1}^{N-1} \alpha_n D_{KL}(p_n || P_N),
$$

where $D_{KL}(\cdot || \cdot)$ denotes the Kullback-Leibler divergence and $\alpha_1, \ldots, \alpha_{N-1}$ are convex combination weights that satisfy $\alpha_1, \ldots, \alpha_{N-1} \geq 0$, $\sum_{n=1}^{N-1} \alpha_n = 1$.

The corresponding function $f$ is given by

$$
f(\omega, x_1, \ldots, x_N) = \sum_{n=1}^{N-1} \alpha_n \log \left( \frac{x_N}{x_n} \right) x_N.
$$

Its partial derivatives are

$$
f_n(\omega, x_1, \ldots, x_N) = -\alpha_n \frac{x_N}{x_n}
$$

for $n \in \{1, \ldots, N-1\}$ and

$$
f_N(\omega, x_1, \ldots, x_N) = 1 + \sum_{n=1}^{N-1} \alpha_n \log \left( \frac{x_N}{x_n} \right).
$$

The inverse functions are obtained by solving $f_n = c_n$ for $x_n$ and are

$$
f_n^{-1}(x_n, c_n) = \frac{\alpha_n}{c_n} x_N =: b_n x_N
$$

for $n \in \{1, \ldots, N-1\}$ and

$$
f_N^{-1}(x_N, c_n) = e^{c_n-1} \prod_{n=1}^{N-1} x_n^\alpha_n =: b_N \prod_{n=1}^{N-1} x_n^\alpha_n.
$$

Here $b_1, \ldots, b_{N-1} \in \mathbb{R}$ and $b_N > 0$ are introduced for the sake of a more compact notation. From Corollary 1 it follows that the optimal densities are of the form

$$
q_n = \min \{ p_n'', \max \{ b_n q_N, p_n' \} \}
$$

for $n \in \{1, \ldots, N-1\}$ and

$$
q_N = \min \{ p_N'', \max \{ b_N q_1^{\alpha_1} \cdots q_{N-1}^{\alpha_{N-1}}, p_N' \} \}. \quad (17)
$$

From this, it follows that $q_1, \ldots, q_{N-1}$ are the projections of $q_N$ onto the bands $P_1^{-}, \ldots, P_{N-1}^{-}$, respectively, and $q_N$ is the projection of the weighted geometric mean of $q_1, \ldots, q_{N-1}$ onto the band $P_N$.

Before presenting numerical results, it is shown that the above expressions for the optimal densities can be used to derive a tight lower bound on (16) for the special case where $p_1, \ldots, p_{N-1}$ are given and the optimization is performed only over $p_N$. Problems of this kind are, for example, considered in [18] in order to derive lower bounds on the minimax risk of a decision making procedure. Making use of (17), it can be shown that

$$
\min_{p_N \in P_N} I_f(p_1, \ldots, p_N) \geq \min_{p_N \in P_N} I_f(p_1, \ldots, p_N)
$$

$$
= \sum_{n=1}^{N-1} \alpha_n D_{KL}(p_n || b_N p_1^{\alpha_1} \cdots p_{N-1}^{\alpha_{N-1}})
$$

$$
= -\log b_N,
$$

where

$$
b_N = \int p_1^{\alpha_1} \cdots p_{N-1}^{\alpha_{N-1}} d\mu
$$

is a generalized version of the Bhattacharyya coefficient [32]. To the best of our knowledge, this bound has not been stated in the literature so far.
For the numerical solution of (16) consider \( N = 3 \) densities and uncertainty bands defined by scaling and shifting Gaussian densities according to

\[
\begin{align*}
p_1' &= 0.8 p_N(-0.5, 1), \quad p_1'' = 1.2 p_N(-0.5, 1), \\
p_2' &= 0.8 p_N(0.5, 1), \quad p_2'' = 1.2 p_N(0.5, 1), \\
p_3' &= 0.8 p_N(0, 1), \quad p_3'' = 1.2 p_N(0, 1),
\end{align*}
\]

where \( p_N(m, \sigma^2) \) denotes the density function of a Gaussian distribution with mean \( m \) and variance \( \sigma^2 \).

Three triplets of optimal densities, for different weights \( \alpha_1 \) and \( \alpha_2 \), are depicted in Fig. 1. As can be seen, \( q_1 \) and \( q_2 \) are independent of the weights in this particular example, but \( q_3 \) changes significantly and different combinations push it either towards \( q_1 \) or \( q_2 \).

The densities in Fig. 1 were calculated using Algorithm 1 on the interval \([-5, 5]\) with step size \( \Delta \omega = 0.01 \) and initial densities \( q_0' = p_N(-0.5, 1), \quad q_0'' = p_N(0.5, 1), \quad q_0 = p_N(0, 1) \). Convergence was reached after 20, 35 and 103 iterations, respectively, for the three choices of \( \alpha_1 \) and \( \alpha_2 \) given in Fig. 1.

In this example, the inverse functions \( f^{-1}_n(\mathbf{x}[n], c_n) \) do not need to be evaluated numerically and Algorithm 1 is highly efficient. Our MATLAB\textsuperscript{®} implementation terminated in well under a second on a regular desktop computer.\textsuperscript{2} For comparison, the minimization problem was also solved using version 2.0.4 of the ECOS solver \textsuperscript{33}, which is a state-of-the-art software package for solving conic optimization problems and is written in C. The reason for choosing ECOS over other options is that it is one of the few high performance solvers that support the exponential cone and, hence, logarithmic objective functions. The average run-times of Algorithm 1 and the ECOS solver are given in Table I. The results were obtained on the same machine with absolute tolerances set to 10\(^{-7}\) for all algorithms. The ECOS solver was called via its CVX interface \textsuperscript{34}, but only the time spent in the C routine was used for the benchmarking. As can be seen, Algorithm 1 is consistently faster than the ECOS solver, despite the additional handicap of being written in an interpreted language. Especially for large numbers of grid points \( K \), the advantage of the proposed approach becomes obvious.

Table I also includes the run-times of Algorithm 2 for which it can be shown that

\[
\tilde{f}^{-1}_n(\mathbf{x}[n], c_n) = \frac{c_n + h(\omega)}{2} + \sqrt{\left( \frac{c_n + h(\omega)}{2} \right)^2 + \alpha_n x^N}.
\]

for \( n \in \{1, \ldots, N\} \) and

\[
\tilde{f}^{-1}_N(\mathbf{x}[N], c_N) = W \left( e^{c_N + h(\omega) - 1} \prod_{n=1}^{N-1} x_n^N \right).
\]

Here, \( W \) denotes the Lambert W function. For \( \alpha_1 = \alpha_2 = 0.5 \), Algorithm 2 required 37 outer and, in total, 205 inner iterations for convergence. For the two cases with asymmetric weights, these numbers become 46 (321) and 102 (880). Owing to this increase, and the computationally costly evaluation of the Lambert W function, Algorithm 2 is significantly slower than Algorithm 1. However, for medium to large problem sizes its performance is comparable to, or even better than, that of the ECOS solver.

These results arise from the chosen example and it is noted that raw execution time is not a reliable performance metric. Nevertheless, the example is non-trivial and the fact that the proposed algorithms are able to outperform an optimized software package is indicative of a good performance in general.

Another important aspect of the proposed algorithms is that they lead to results with high accuracy. This can be seen by

\textsuperscript{2} All simulations were performed on an Intel\textsuperscript{®} Core\textsuperscript{TM} i5-760@2.80GHz using MATLAB\textsuperscript{®} 2016a.
inspection of the ratios of the optimal densities, which are of particular interest in detection problems, where they determine the optimal test statistic. In Fig. 2, the log-likelihood ratios

\[ \log \frac{q_1(\omega)}{q_3(\omega)} \quad \text{and} \quad \log \frac{q_2(\omega)}{q_3(\omega)} \]

(19)

are depicted as calculated by Algorithm 1 and the ECOS solver. Note that the interval of the sample space is increased to \([-10, 10]\). It can clearly be seen how the ECOS solver produces artifacts at the tails of the densities, where their values fall below the tolerance of \(10^{-7}\). Algorithm 1, in contrast, correctly identifies the tails as regions where the band constraints are active and is, therefore, able to calculate the likelihood ratios exactly. If at all possible, obtaining results of comparable quality with generic solvers requires careful parameter tuning and tolerances close to the machine precision.

B. Minimax Detection with Observation Dependent Cost

The purpose of the second example is to show that the proposed algorithms handle cases where \(f\) directly depends on \(\omega\) and that this dependence widens the scope of problems which can be solved within the presented framework.

Consider a binary decision making problem with a cost function

\[ r(\delta) = E_{P_1}[\delta(\omega)r_1(\omega)] + E_{P_2}[(1-\delta(\omega))r_2(\omega)], \]

(20)

where \(E_P\) denotes expectation with respect to a distribution \(P, \delta: \Omega \to [0, 1]\) denotes a decision rule, and \(r_0, r_1: \Omega \to [0, \infty)\) denote observation dependent costs for each decision. Cost functions of this form occur, for example, in detection problems, where the cost for an incorrect decision depends on the true state of the system. In a collision avoidance system for cars, for instance, the cost for not detecting an obstacle becomes higher when the obstacle is close to the car and the car is moving faster.

The optimal decision rule for the cost function (20) can be shown to be given by

\[ \delta^*(\omega) = \begin{cases} 1, & r_1(\omega)p_1(\omega) < r_2(\omega)p_2(\omega) \\ \kappa \in [0, 1], & r_1(\omega)p_1(\omega) = r_2(\omega)p_2(\omega) \\ 0, & r_1(\omega)p_1(\omega) > r_2(\omega)p_2(\omega) \end{cases} \]

Without loss of generality, it is assumed that \(\kappa = 1\). Using this decision rule, the expected cost is given by

\[ \int \min\{r_1p_1, r_2p_2\} \, d\mu = \int r^*(p_1, p_2) \, d\mu. \]

(21)

It is assumed that \(p_1\) and \(p_2\) are subject to uncertainties of the density band type. In order to design a minimax detector, i.e., a detector that minimizes the worst-case cost, (21) needs to be maximized with respect to the densities \(p_1\) and \(p_2\). This problem is of the form (4) with

\[ -f(\omega, x_1, x_2) = \min\{r_1(\omega)x_1, r_2(\omega)x_2\} \]

(22)

and

\[ -f_1(\omega, x_1, x_2) = \begin{cases} r_1(\omega)x_1, & r_1(\omega)x_1 \leq r_2(\omega)x_2 \\ 0, & \text{otherwise} \end{cases}, \]

\[ -f_2(\omega, x_1, x_2) = \begin{cases} r_2(\omega)x_2, & r_1(\omega)x_1 > r_2(\omega)x_2 \\ 0, & \text{otherwise} \end{cases}. \]

For illustration purposes, the cost functions are chosen as

\[ r_1(\omega) = 1 + \cos(\pi \omega) \quad \text{and} \quad r_2(\omega) = 2 \exp(-|\omega|). \]

(23)

and their graphs are shown in Fig. 3. The same density bands as in (18) are used to constrain \(p_1\) and \(p_2\). The grid for the discrete representation is again constructed on \([-5, 5]\) with step size \(\Delta \omega = 0.01\). Although \(f\) in (22) is not strictly convex, Algorithm 1 with initial densities \(q_1^0 = p_N(-0.5, 1), q_2^0 = p_N(0.5, 1)\) converges after a mere three iterations. The resulting least favorable densities are depicted in Fig. 4. Their effect on the cost function can be seen in Fig. 5, where \(r^*(p_0, p_1)\) is plotted for the least favorable and two Gaussian densities. Interestingly, the shape of the cost function \(r^*\) is preserved, despite the “ragged” shape of the least favorable densities. Again, the proposed algorithm is able to accurately identify abrupt changes as well as smooth variations in the optimal densities.

VI. CONCLUSION

The problem of minimizing convex functionals of probability distributions under density band constraints was analyzed and a system of sufficient and necessary first order optimality conditions was derived. These conditions were used to characterize global minimizers of the constrained optimization problem as solutions of a nonlinear fixed-point equation. Two
algorithms were proposed that iteratively solve this equation using a block coordinate descent strategy. While the first algorithm proved to be efficient in practice, it is not guaranteed to converge for objective functions that are not strictly convex. This problem was overcome by introducing an additional outer proximal iteration. The modified algorithm was then shown to admit guaranteed convergence properties for all band constraints and all convex objective functions, at the cost of a reduced efficiency. Two examples were given to illustrate how the optimality conditions can be used in practice to derive analytical results and to demonstrate the high efficiency and accuracy of the proposed numerical algorithms.

APPENDIX A
PROOF OF THEOREM 1

Theorem 1 is proven by showing that if the densities \((q_1, \ldots, q_N)\) satisfy the conditions in Theorem 1, they also satisfy the Karush–Kuhn–Tucker (KKT) conditions of (7). Since \(I_f\) is convex in \(\{p_1, \ldots, p_N\}\) by assumption and the constraints in (7) are linear, the KKT conditions are necessary and sufficient for \((q_1, \ldots, q_N)\) to be a global minimizer [35].

The proof makes use of some basic results of infinite-dimensional optimization theory, more precisely, the theory of Lagrange multipliers on Banach spaces. An introduction to the topic is beyond the scope of this work. A comprehensive treatment can be found, for example, in [36]. An elegant standalone proof for the method of Lagrange multipliers and the sufficiency of the KKT conditions is given in [37] and [38], respectively. In brief, the method of Lagrange multipliers can be applied to convex functions on Banach spaces by using Fréchet subderivatives instead of subgradients and elements of the dual space instead of scalar or vector valued Lagrange multipliers.

Let

\[
\mathcal{L}^\infty = \left\{ u : \Omega \to \mathbb{R} : \sup_{B \in F} \sup_{\omega \in B} |u(\omega)| < \infty \right\}
\]

denote the space of all essentially bounded functions on \(\Omega\), which is the dual space of \(\mathcal{L}^1\). The Lagrange function \(L : (\mathcal{L}^1 \times \mathcal{L}^\infty \times \mathcal{L}^\infty) \to \mathbb{R}\) of (7) is

\[
L(p, u, v, c) = \int \left( f(p) + \sum_{n=1}^N w_n(p_n) \right) \, d\mu + \sum_{n=1}^N c_n,
\]

where

\[
w_n(p_n) := (p_n - p''_n)u_n - (p_n - p'_n)v_n - p_n c_n
\]

and \(u_n, v_n \in \mathcal{L}^\infty\), \(c_n \in \mathbb{R}\) denote the Lagrange multipliers corresponding to the constraints \(p_n - p''_n \leq 0\), \(p_n - p'_n \geq 0\) and \(\int p_n \, d\mu = 1\), respectively. The dual problem is given by

\[
\max_{u_n, v_n \in \mathcal{L}^\infty} \left\{ \min_{p_n \in \mathcal{L}^1} L(p, u, v, c) \right\} \quad \text{s.t.} \quad u_n, v_n \geq 0
\]

for all \(n \in \{1, \ldots, N\}\). The partial Fréchet-subdifferential of \(L\) with respect to \(p_n\) can be shown to be

\[
\partial_{p_n} L(p, u, v, c) = \partial_x f(p) + u_n - v_n - c_n,
\]

The KKT conditions for the optimal densities require that for all \(n \in \{1, \ldots, N\}\)

\[
f_n(q) + u_n - v_n - c_n = 0 \quad \text{(stationarity)}
\]

\[
p'_n \leq q_n \leq p''_n, \quad \int q_n \, d\mu = 1 \quad \text{(primal feasibility)}
\]

\[
\begin{align*}
&u_n, v_n \geq 0 \quad \text{(dual feasibility)} \\
&(q_n - p''_n)u_n = (q_n - p'_n)v_n = 0 \quad \text{(compl. slackness)}
\end{align*}
\]

Let all \(q_n\) and \(c_n\) be chosen such that they comply with the conditions in Theorem 1. By construction, this implies that \(q_n\) satisfies the primal feasibility constraints. Since \(f\) is convex and \(\mu\) continuous, it follows from Rademacher’s theorem [39, Chapter 9.J.] that the partial subderivatives of \(f\) are finite \(\mu\)-almost everywhere, i.e., \(f_n \in \mathcal{L}^\infty\) for all \(n\). Without violating dual feasibility, the functions \(v_n\) and \(u_n\) can be chosen as

\[
v_n = \max\{f_n(q) - c_n, 0\}, \quad -u_n = \min\{f_n(q) - c_n, 0\}
\]

so that

\[
u_n - u_n = f_n(q) - c_n.
\]

Inserting (25) back into the stationarity condition yields

\[
f_n(q) + u_n - v_n - c_n = 0
\]

for all \(n \in \{1, \ldots, N\}\). The last step in the proof is to show that these choices for \(q_n\), \(u_n\) and \(v_n\) also satisfy the complementary slackness constraints, i.e.,

\[
u_n(\omega) > 0 \Rightarrow q_n(\omega) = p''_n(\omega), \quad v_n(\omega) > 0 \Rightarrow q_n(\omega) = p'_n(\omega),
\]
for all $\omega \in \Omega$. By construction of $u_n$ and $v_n$, $u_n(\omega) > 0$ implies $f_n(q(\omega)) < c_n$, which in turn implies $q_n = p_n'$. Analogously, $v_n(\omega) > 0$ implies $f_n(q(\omega)) > c_n$ and in turn $q_n(\omega) = p_n''(\omega)$.

**APPENDIX B**

**PROOF OF COROLLARY**

Corollary [1] is a consequence of the fact that $f_n$, being a subderivative of a convex function, is nondecreasing in $x_n$. For the three cases in Theorem [1] it then follows that

\begin{align*}
&f_n(q(\omega)) < c_n \Rightarrow q_n(\omega) = p_n''(\omega) \leq f_n^{-1}(q_n(\omega), c_n), \\
&f_n(q(\omega)) = c_n \Rightarrow q_n(\omega) = f_n^{-1}(q_n(\omega), c_n), \\
&f_n(q(\omega)) > c_n \Rightarrow q_n(\omega) = p_n''(\omega) \geq f_n^{-1}(q_n(\omega), c_n).
\end{align*}

The expression for $q_n$ given in Corollary [1] is merely a more compact way of writing the equation system on the right hand side of (26).

**APPENDIX C**

**PROOF OF CONVERGENCE OF ALGORITHM**

The convergence of proximal iterations is a well established result in the convex optimization literature. It follows directly from the contractive property of the proximal operator and can be applied to Algorithm [2] in a straightforward manner. For a selection of convergence proofs, see, for example, [25] Chapter 2.3, [40]–[42], and the references therein.

In order to prove that Algorithm 2 converges, it hence suffices to show that Algorithm 1 indeed solves the inner minimization in [14]. This is guaranteed if, first, the function $f$ is strictly convex and, second, if, in every iteration, the equation in line 5 of Algorithm [1] has a solution. The first condition ensures that the algorithm is able to find a block-coordinate-wise minimum in each iteration; the second condition ensures that the block-coordinate-wise descent indeed converges to a global minimum.

The first condition is fulfilled by construction. Since $\tilde{f}$ is chosen such that it is strictly convex in $(x_1, \ldots, x_n)$, $I_f(p_{a_1}, \ldots, p_{a_N})$ is strictly convex in $(a_1, \ldots, a_N)$. The second condition can be shown as follows: For $\tilde{f}_n$ as defined in [15], the elements of $\tilde{f}_n^{-1}(a_{n}, c_n)$ are obtained by solving

\begin{align*}
&f_n(a_{1,k}, \ldots, a_{n,k}, \ldots, a_{N,k}) + a_{n,k} - h_n(\omega_k) = c_n
\end{align*}

for $a_{n,k}$. First, note that in the $i$th iteration it holds that

\begin{align*}
h_n(\omega_k) = q_{a_{n,k}^{-1}}(\omega_k) = a_{n,k}^{-1}
\end{align*}

Since $q_{a_{n,k}^{-1}}$ is a probability density, $a_{n,k}^{-1}$ is bounded by

\begin{align*}
0 \leq a_{n,k}^{-1} \leq \frac{1}{\mu_k}.
\end{align*}

Hence, $h_n(\omega_k)$ is guaranteed to be finite as long as $\mu_k > 0$, which is the case by assumption. Since $f_n$ is continuous and nondecreasing in $x_n$, the left hand side of (27) is continuous and increasing in $a_{n,k}$. Since $h_n(\omega_k)$ is finite, it is further unbounded from below and above. Consequently, for every $c_n \in \mathbb{R}$, there exists some $a_{n,k} \in \mathbb{R}$ that solves (27).

The final step is to show, in each iteration, that $c_n$ can be chosen such that the projection of $\tilde{f}_n^{-1}(a_{n,k}, c_n)$ on $\mathcal{P}_n$ is a valid density. This follows again from the monotonicity of (27). The function on the left hand side is continuous, unbounded and increasing in $a_{n,k}$. By the inverse function theorem [43] Chapter 14, this implies that every element of $\tilde{f}_n^{-1}(a_{n,k}, c_n)$ is continuous, unbounded and increasing in $c_n$. Therefore, it holds that

\begin{align*}
&\lim_{c_n \to \infty} \{\min \{p_n''(\omega), \max \{f_n^{-1}(a_{n,k}, c_n), p_n''(\omega)\}\}, \mu\} \\
= &\{p_n''(\omega), \mu\} \geq 1
\end{align*}

and

\begin{align*}
&\lim_{c_n \to \infty} \{\min \{p_n''(\omega), \max \{f_n^{-1}(a_{n,k}, c_n), p_n''(\omega)\}\}, \mu\} \\
= &\{p_n''(\omega), \mu\} \leq 1,
\end{align*}

where the inequalities hold by assumption. Consequently, some $c_n \in \mathbb{R}$ is guaranteed to exists that solves the equation in line 5 of Algorithm [1].

REFERENCES

[1] X. Nguyen, M. J. Wainwright, and M. I. Jordan, “On surrogate loss functions and $f$-divergences,” The Annals of Statistics, vol. 37, no. 2, pp. 876–904, 2009.

[2] B. C. Levy, Principles of Signal Detection and Parameter Estimation, 1st ed. New York, New York, USA: Springer, 2008.

[3] L. B. Klebanov, “Unbiased estimates and convex loss functions,” Journal of Soviet Mathematics, vol. 9, no. 6, pp. 870–880, 1978.

[4] M. D. Reid and R. C. Williamson, “Information, divergence and risk for binary experiments,” Journal of Machine Learning Research, vol. 12, pp. 731–817, 2011.

[5] S. Kassam and H. Poor, “Robust techniques for signal processing: A survey,” Proceedings of the IEEE, vol. 73, no. 3, pp. 433–481, 1985.

[6] S. Kassam, “Robust hypothesis testing for bounded classes of probability densities,” IEEE Transactions on Information Theory, vol. 27, no. 2, pp. 424–247, 1981.

[7] M. Fauß and A. M. Zoubir, “Old bands, new tracks—revisiting the band model for robust hypothesis testing,” IEEE Transactions on Signal Processing, vol. 64, no. 22, pp. 5875–5886, 11 2016.

[8] M. Fauß, “Design and analysis of optimal and minimax robust sequential hypothesis tests,” Ph.D. dissertation, Technische Universität Darmstadt, Darmstadt, Germany, 2016. [Online]. Available: http://huprints.ulb.tu-darmstadt.de/5494.

[9] R. T. Rockafellar, “Integrals which are convex functionals,” Pacific Journal of Mathematics, vol. 24, no. 3, pp. 525–539, 1968.

[10] K. Pearson, “On the criterion that a given system of deviations from the probable in the case of a correlated system of variables is such that it can be reasonably supposed to have arisen from random sampling,” Philosopical Magazine Series 5, vol. 50, no. 302, pp. 157–175, 1900.

[11] P. C. Mahalanobis, “On tests and measures of groups divergence,” Journal of the Asiatic Society of Bengal, vol. 26, pp. 49–55, 1930.

[12] C. Shannon, “A mathematical theory of communication,” The Bell System Technical Journal, vol. 27, no. 3, pp. 379–423, 1948.

[13] S. Kullback and R. A. Leibler, “On information and sufficiency,” The Annals of Mathematical Statistics, vol. 22, no. 1, pp. 79–86, 1951.

[14] F. Liese and I. Vajda, Convex Statistical Distances. Leipzig, Germany: Teubner, 1987.

[15] L. Pardo, Statistical Inference Based on Divergence Measures. Boca Raton, Florida, USA: CRC Press, 2005.

[16] P. J. Huber and V. Strassen, “Minimax tests and the Neyman–Pearson lemma for capacities,” The Annals of Statistics, vol. 1, no. 2, pp. 251–263, 1973.

[17] H. Poor, “Robust decision design using a distance criterion,” IEEE Transactions on Information Theory, vol. 26, no. 5, pp. 575–587, 1980.

[18] A. Guntuboyina, “Lower bounds for the minimax risk using $f$-divergences, and applications,” IEEE Transactions on Information Theory, vol. 57, no. 4, pp. 2386–2399, 2011.
[19] A. A. D’Amico, L. Sanguinetti, and D. P. Palomar, “Convex separable problems with linear and box constraints,” in *IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP)*, May 2014, pp. 5641–5645.

[20] P. J. Huber, “A robust version of the probability ratio test,” *The Annals of Mathematical Statistics*, vol. 36, no. 6, pp. 1753–1758, 1965.

[21] F. Österreicher, “On the construction of least favourable pairs of distributions,” *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete*, vol. 43, no. 1, pp. 49–55, 1978.

[22] S. M. Ali and S. D. Silvey, “A general class of coefficients of divergence of one distribution from another,” *Journal of the Royal Statistical Society. Series B*, vol. 28, no. 1, pp. 131–142, 1966.

[23] L. Györfi and T. Nemetz, “f-dissimilarity: a general class of separation measures of several probability distributions,” *Colloquia of the János Bolyai Mathematical Society: Topics in Information Theory*, vol. 16, pp. 309–321, 1977.

[24] P. Tseng, “Convergence of a block coordinate descent method for nondifferentiable minimization,” *Journal of Optimization Theory and Applications*, vol. 109, no. 3, pp. 475–494, 2001.

[25] N. Parikh and S. Boyd, “Proximal algorithms,” *Foundations and Trends in Optimization*, vol. 1, no. 3, pp. 127–239, 2014.

[26] A. Saha and A. Tewari, “On the nonasymptotic convergence of cyclic coordinate descent methods,” *SIAM Journal on Optimization*, vol. 23, no. 1, pp. 576–601, 2013.

[27] J. Nutini, M. Schmidt, I. Laradji, M. Friedlander, and H. Koepke, “Coordinate descent converges faster with the Gauss–Southwell rule than random selection,” in *Proc. of the International Conference on Machine Learning (ICML)*, D. Blei and F. Bach, Eds. JMLR Workshop and Conference Proceedings, 2013, pp. 1632–1641.

[28] J. Monahan, *Numerical Methods of Statistics*, ser. Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press, 2001, no. Bd. 1.

[29] R. Waeder, “Probabilistic bisection search for stochastic root-finding,” Ph.D. dissertation, Cornell University, 2013. [Online]. Available: https://people.orie.cornell.edu/shane/theses/ThesisRolfWaeder.pdf

[30] I. Karoui, R. Fablet, J.-M. Boucher, and J.-M. Augustin, “Seabed segmentation using optimized statistics of sonar textures,” *IEEE Transactions on Geoscience and Remote Sensing*, vol. 47, no. 6, pp. 1621–1631, 2009.

[31] H. Suzuki, R. Saito, and M. Tomita, “The weighted sum of relative entropy: a new index for synonymous codon usage bias,” *Gene*, vol. 335, pp. 19–23, 2004.

[32] A. Bhattacharyya, “On a measure of divergence between two statistical populations defined by their probability distributions,” *Bulletin of Calcutta Mathematical Society*, vol. 35, pp. 99–109, 1943.

[33] A. Domahidi, E. Chu, and S. Boyd, “ECOS: An SOCP solver for embedded systems,” in *European Control Conference (ECC)*, 2013, pp. 3071–3076.

[34] CVX Research, Inc., “CVX: Matlab software for disciplined convex programming, version 3.0 beta,” [http://cvxr.com/cvx](http://cvxr.com/cvx), 2012.

[35] M. Guignard, “Generalized Kuhn–Tucker conditions for mathematical programming problems in a Banach space,” *SIAM Journal on Control*, vol. 7, no. 2, pp. 232–241, 1969.

[36] V. Barbu and T. Precupanu, *Convexity and Optimization in Banach Spaces*, 4th ed., ser. Springer Monographs in Mathematics. Houten, Netherlands: Springer Netherlands, 2012.

[37] O. Brezhneva and A. A. Tret’yakov, “An elementary proof of the Lagrange multiplier theorem in normed linear spaces,” *Optimization*, vol. 61, no. 12, pp. 1511–1517, 2012.

[38] ——, “An elementary proof of the Karush–Kuhn–Tucker theorem in normed linear spaces for problems with a finite number of inequality constraints,” *Optimization*, vol. 60, no. 5, pp. 613–618, 2011.

[39] R. T. Rockafellar and R. J.-B. Wets, *Variational Analysis*. New York City, New York, USA: Springer, 1998. [Online]. Available: [http://math.ucdavis.edu/~rjbw](http://math.ucdavis.edu/~rjbw)

[40] S. Reich and S. Sabach, “Two strong convergence theorems for a proximal method in reflexive Banach spaces,” *Numerical Functional Analysis and Optimization*, vol. 31, no. 1, pp. 22–44, 2010.

[41] A. N. Iusem and E. Resmerita, “A proximal point method in nonreflexive Banach spaces,” *Set-Valued and Variational Analysis*, vol. 18, no. 1, pp. 109–120, 2010.