Hashing with Linear Probing and Referential Integrity

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Abstract
We describe a variant of linear probing hash tables that never moves elements and thus supports referential integrity, i.e., pointers to elements remain valid while this element is in the hash table. This is achieved by the folklore method of marking some table entries as formerly occupied (tombstones). The innovation is that the number of tombstones is minimized. Experiments indicate that this allows an unbounded number of operations with bounded overhead compared to linear probing without tombstones (and without referential integrity).

1 Introduction
Hash tables are among most fundamental and widely used data structures. Refer to [3] for examples and more detailed discussion of the basic techniques. While there is a plethora of hash table data structures, hashing with linear probing is the most efficient one in many practical situations. This is due to its simplicity, cache efficiency, absence of overhead for internally used pointers, and because only a single hash function evaluation is needed for a search or insert operation.

However, deletion is problematic for linear probing. There are two main known approaches. Usually the preferred one is to rearrange elements so that the main data structure invariants are maintained [1, 3]. However this destroys referential integrity – pointers to moved elements are no longer valid. A folklore solution is to simply mark table entries previously occupied by deleted elements using a special tombstone value. This has the drawback that elements are never ever freed and thus search costs grow until, eventually, the entire table has to be searched during unsuccessful searches. Addressing this problem by reorganizing the table from time to time also destroys referential integrity.

The idea behind the present paper is to use tombstones but to remove those that are not needed to maintain the data structure invariants. Experiments (Section 4) indicate that this is sufficient to keep search costs bounded as long as the table is not too full.

1As an illustrative example where referential integrity is relevant, consider a LRU (least recently used) cache. An efficient folklore implementation consists of a hash table with one entry for each cached object and a doubly linked list that remembers how recently elements have been accessed (e.g., [5]). Directly storing list items in the hash table requires referential integrity.
Procedure delete(e)
    for i := h(e) while t[i] ≠ e do
        if t[i] = ⊥ then return
        t[i] := ⊤
        -- search for e
    h := m
    -- initialize smallest hash function value encountered
    for j := i + 1 while t[j] ≠ ⊥
        if t[j] ≠ ⊤ then if h(t[j]) < h then h := h(t[j])
        -- scan to the right
        for k := i downto h(e)
        if t[j] = ⊤ then if h > k then t[j] := ⊥
        else if h(t[j]) < h then h := h(t[j])
        -- scan to the left
        -- remove tombstone
        -- update smallest hash function value
    -- update smallest hash function value

Figure 1: Pseudocode for deletion in linear probing with referential integrity. To keep the code simple, we describe a variant without wrap-around, i.e., t is allocated sufficiently large such that overflowing elements always find free entries there; see also [3].

2 Preliminaries

Suppose we want to store n elements in a table t[0..m − 1] of elements where m > n is the table size. Let h : Element → 0..m − 1 denote the hash function. We also consider special elements ⊥ for empty table entries and ⊤ for tombstones. The invariant governing the implementation of linear probing is that if element e is stored at table entry t[i] then the entries cyclically between t[h(e)] and t[i] are nonempty. Searching an element e then amounts to scan t cyclically starting at t[h(e)]. “Cyclically” here means the the search wraps around when the end of the table is reached. This process stops if either e is found or an empty table entry is found. In the latter case, the invariant guarantees that e is not in the table. Inserting an element e works similarly to searching. If an elements with the same key exists, depending on the desired semantics of insertion, nothing is done or the element is updated. Inserting a new element overwrites the first empty table entry or tombstone at t[h(e)] or cyclically to the right.

Several implementations for deleting an element e are possible. Two extremes are simply replacing it by a tombstone or rearranging the elements so as to avoid tombstones altogether [1, 3].

3 Linear Probing with Referential Integrity

Our variant of linear probing is based on two principles:

1. Never move elements. This entails referential integrity.

2. Use tombstones only when necessary. This means, we maintain the invariant that when t[i] = ⊤ then there is an element e in the table such that h(e) is i or cyclically to the left yet e is stored cyclically to the right of i.
The main issue is now how to maintain this invariant efficiently. Searches, updates and insertions do not affect the invariant and can be implemented as before.

Deletion has two aspects relevant for the invariant. On the one hand, a deleted element \( e \) previously stored at position \( i \) has to become a tombstone if there are elements hashed to \( i \) or cyclically to the left but stored cyclically to the right of \( i \). On the other hand, tombstones cyclically between \( h(e) \) and \( i \) may become unnecessary because they were only needed to be able to find \( e \). Both aspects can be handled in a uniform way. Let \( j \) denote the position of the first empty cell cyclically to the right of \( i \). Initially, \( t[i] \) is replaced by a tombstone. We then scan \( t[i + 1..j - 1] \) and compute the hash function value \( \hat{h} \) occurring there that is cyclically farthest to the left. Then we scan \( t[h(e)..i] \) from right to left. During this scan we update \( \hat{h} \). Moreover, when encountering a tombstone at position \( k \), it is replaced by \( \bot \) if \( \hat{h} \) is cyclically to the right of \( k \). Figure 1 gives pseudocode for the deletion operation.

For correctness, first note that only in the range \( h(e)..i \) can tombstones become unnecessary because only \( e \) is deleted and other positions are irrelevant for keeping \( e \) searchable. When a tombstone at position \( k \) is removed, this is save since we know that no element stored cyclically to the right of \( k \) is hashed to position \( k \) or cyclically to the left. On the other hand, when a tombstone is kept, this is necessary since we encountered an element cyclically to the right of \( k \) that is hashed to \( k \) or cyclically to the left.

4 Experiments

We have performed a number of simple experiments in order to test the hypothesis that search times remain bounded regardless of the number of operations performed on the hash table (in contrast to naively always putting tombstones). We also wanted to get an idea about the overheads compared to avoiding tombstones altogether. In the experiments below, we begin by inserting \( n \) elements. Then we alternate between removing the least recently inserted element and inserting a new element.

Figure 2-top shows the average number of table entries accessed for successful search. The cost increase is moderate up to \( n/m \approx 60\% \). Figure 2-bottom gives the corresponding numbers for unsuccessful searches. Consistently with linear probing in general, these costs are considerably higher. Indeed, they get very high for load factor 80\%. Figure 3 indicates that the average cost bounds are independent of the overall table size.

We have also performed similar experiments where the deleted element is chosen randomly. The results are qualitatively similar – bounded access times independent of \( n \) for fixed \( n/m \). Quantitatively, successful search times are significantly smaller whereas unsuccessful search times are slightly larger.

\(^2\)The source code is available at [http://algo2.iti.kit.edu/sanders/programs/hash/](http://algo2.iti.kit.edu/sanders/programs/hash/) To avoid conceivable issues due to weak hash functions, we use tabulated values of a high quality pseudo random number generator.\(^2\).
Figure 2: Average number of table entries accessed during successful search (top) and unsuccessful search (bottom) as a function of the number of deletions performed ($m = 10^6$). For $n/m = 80\%$, the unsuccessful access cost converges to around 210.

5 Conclusions

We have presented a variant of linear probing hash tables that preserve referential integrity under deletions. First experiments indicate that this yields good performance as long as the table is not too full. Further experiments could give additional evidence. Even better would be a theoretical analysis. Note that analyzing deletions in open addressing hashing is not easy in general. Perhaps one can adapt the approach of Mitzenmacher [4] used for Robin Hood hashing and assuming that deletions pick random elements and that insertions are new elements.
Figure 3: Average number of table entries accessed during (un)successful search as a function of the number of deletions performed. We have $n/m = 50\%$ in all these experiments and vary $m$.

References

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