Frequency and Wavelength of Light
in Relativistically Rotating Frames

Robert D. Klauber
1100 University Manor Dr., 38B, Fairfield, IA 52556, USA
email: rklauber@netscape.net

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Abstract

Non-time-orthogonal frame analysis is applied to determine the frequency and wavelength of light as observed i) in a relativistically rotating frame when emission is from a source fixed in the non-rotating frame, ii) in a non-rotating frame when emission is from a source fixed in the rotating frame, and iii) when both source and observer are fixed in the rotating frame and the source emission direction varies with respect to the rotating frame. Appropriate Doppler effects are demonstrated, and second order differences from translating (time-orthogonal) frame analysis are noted.

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1 INTRODUCTION

1.1 Background

An analysis[1] [2] [3] has been carried out of the non-time-orthogonal metric (i.e., \(g_{0i} \neq 0\), time is not orthogonal to space) obtained when one makes a straightforward transformation from the lab to a relativistically rotating frame. Rather than assuming, as have other researchers, that it is then necessary to transform to locally time orthogonal (i.e., time is orthogonal to space) frames, one can proceed by considering the non-time-orthogonal (NTO) metric to be a physically valid representation of the rotating frame.

When this is done, one finds the usual time dilation and mass-energy dependence[4] on tangential speed \(\omega r\), in full accord with the test data from numerous cyclotron experiments. One also finds resolutions of paradoxes inherent in the traditional analytical treatment of rotating frames. Further, the analysis predicts at least one experimental result[5] [6] that, in the context of the traditional analysis, has heretofore been considered inexplicable.

NTO frame analysis is in full accord with fundamental principles of relativity theory, and makes many of the same predictions as the traditional analysis for rotating frames. It does not conflict with recognized analyses of time-orthogonal (TO) frames, including those described by Lorentz, Schwarzchild, and Friedman metrics. Just as for TO frames, the NTO line element remains invariant, and differential geometry reigns as the appropriate descriptor of non-inertial systems.

However, NTO analysis does predict some behavior that may seem strange from a traditional relativistic standpoint, though it appears corroborated by both gedanken and physical experiments[7] [8] [9]. In particular, NTO analysis finds the specific result for the speed of light in the circumferential direction for rotating (NTO) frames to be non-invariant, non-isotropic, and equal to:

\[u_{\text{light,circum}} = \frac{c \pm \omega r}{\sqrt{1 - (\omega r)^2/c^2}} = \frac{c \pm v}{\sqrt{1 - v^2/c^2}},\]  

(1)

where the sign before \(v = \omega r\) depends on the circumferential direction of the light ray at \(r\) relative to the tangential speed \(v\). Note the circumferential light speed in the rotating frame varies to first order with \(\omega r\).

This result agrees with Ashby’s research with the global positioning system. Ashby notes “.. the principle of the constancy of \(c\) cannot be applied in a rotating reference frame ..” [8]. He also states “Now consider a process
in which observers in the rotating frame attempt to use Einstein synchronization [constancy of the speed of light] ..... Simple minded use of Einstein synchronization in the rotating frame ... thus leads to a significant error”[7].

1.2 Overview

In the present article NTO analysis is used to determine frequency and wavelength of light i) emitted from a source in the lab and observed from the rotating frame, ii) emitted from a source in the rotating frame and observed in the lab, and iii) emitted from a source in the rotating frame that turns relative to the rotating frame. Consistency is evidenced in that multiplication of frequency and wavelength thus obtained for the rotating frame yields (1).

As background, and as an aide for comparison, Section 2 provides a summary of speed, frequency, wavelength, and Doppler shift of waves propagating through elastic media for various observers in a Newtonian universe. Section 3 provides a similar summary for waves propagating in vacuum (i.e., non-elastic waves) for both Newtonian and Lorentzian observers. The transformation between the lab and the rotating frames, as well as the resulting NTO metric for the rotating frame, as derived in cited references, are listed in Section 4. That section also includes a summary of the differences between the generalized coordinate components of vectors used in mathematical analysis and the physical components that equal the values actually measured in physical experiments. Section 4 then prescribes the method for converting coordinate components to physical components and vice versa, as well as the procedure for using such conversions to solve problems of a most general nature. The mathematical relations and methodology of Section 4 are subsequently applied in Section 5 to determine appropriate wave frequencies and wavelengths of light as seen by lab and rotating frame observers. Section 6 addresses the case where both source and observer are in the rotating frame and the light source is turned relative to that frame.

2 NEWTONIAN WAVES IN ELASTIC MEDIA

With minimal comment we present Table 1, a rather elementary summary[11] of elastic wave propagation in Galilean frames, which will prove of value for comparison with the results of Section 5. Prior to Section 6 we treat only cases in which motions of the observer, the source, and the medium, if any, are along the line of sight between the observer and source. In Table 1 the light source is to the left of the observer, the wave travels toward the right, \( v_m \) is the speed of the wave within the elastic medium (i.e., relative to the medium), and \( v \) is the speed of the source toward (approaching) the observer. Positive displacement, and hence velocity, is to the right. Quantities in the source frame \( K \) are unprimed, in the observer frame \( K' \) are primed, and in the medium have a subscript \( m \). In all tables presented herein source and observer receding from one another implies \( v \) becomes \( -v \) in all blocks within a given table.

Note that in all cases multiplication of frequency by wavelength as seen either in the source frame, or in the observer frame, results in the correct wave speed for the given frame. Note further in Table 1 the second order difference in Doppler effect seen by the observer when the observer is fixed in the medium (Case 2) as opposed to when the source is fixed in the medium (Case 1).

\[
\begin{array}{|c|c|c|}
\hline
\text{Source frame } K & \text{Case 1} & \text{Case 2} & \text{Case 3} \\
\hline
& \text{Fixed to medium} & \text{Not fixed to medium} & \text{Not fixed to medium} \\
\hline
\text{Observer } K' & \text{Not fixed to medium} & \text{Fixed to medium} & \text{Not fixed to medium} \\
\hline
\text{Motion} & \text{Source (medium) toward observer at } v & \text{Source toward observer (medium) at } v & \text{Source, observer both at rest.} \\
& \text{Medium at } v. \\
\hline
\text{Wave speed} & V = v_m & V = v_m - v & V = V' = v_m + v \\
& V' = v_m + v & V' = v_m & \\
\hline
\text{Wave length} & \lambda' = \lambda & \lambda' = \lambda & \lambda' = \lambda = \lambda_m \\
\hline
\text{Frequency} & f' = f (1 + v/v_m) & f' = \frac{f}{1 - v/v_m} & f' = f = f_m (1 + v/v_m) \\
\hline
\text{Doppler shift} & f' = f (1 + v/v_m) & f' = f (1 + v/v_m + v^2/v_m^2 + ... ) & \text{None} \\
\text{observer sees} & & & \\
\hline
\end{array}
\]
3 WAVES WITHOUT ELASTIC MEDIA

Table 2 summarizes the behavior of light waves that propagate through vacuum without an underlying supporting medium for both a Newtonian and a relativistic universe. Note that for Lorentz frames the wavelength appears to the observer to have different length than it does in the source frame. As for elastic waves, in each case multiplication of frequency times wavelength equals wave speed for a given frame. The second order dependence for the relativistic Doppler shift and how it differs from either Case 1 or Case 2 in Table 1 is well known, and thereby provides a means to test special relativity.

Table 2. Summary: Waves without Media, Galilean and Lorentzian

| Source K | Galilean Frames | Lorentz Frames |
|----------|----------------|----------------|
| Observer K' | No medium | No medium |
| Motion | Source toward observer at v | Source toward observer at v |
| Wave speed | \( V = c \) | \( V = c \) |
| Wave length | \( \lambda' = \lambda \) | \( \lambda' = \lambda \sqrt{1-v/c} = \lambda \sqrt{1-v^2/c^2} \) |
| Frequency | \( f' = f (1 + v/c) \) | \( f' = f \sqrt{1-v/c} \sqrt{1+v/c} = f \sqrt{1-v^2/c^2} \) |
| Doppler shift observer sees | \( f' = f (1 + v/c) \) | \( f' = f (1 + v/c + \frac{1}{2}v^2/c^2 + ...) \) |

4 ROTATING FRAMES

4.1 Transformation and metric

We adopt notation in which the Minkowski metric for a Lorentz frame has form \( \eta_{\alpha\beta} = \text{diag} (-1,1,1,1) \). For the rotating frame analysis we employ cylindrical coordinates with \((cT,R,\Phi,Z)\) for the lab frame \(K\) and \((ct,r,\phi,z)\) for the rotating frame \(k\). The transformation between the lab and rotating frame having angular velocity \(\omega\) in the \(Z\) direction is

\[
\begin{align*}
cT &= ct \\
R &= r \\
\Phi &= \phi + \omega t \\
Z &= z .
\end{align*}
\]

In matrix form this may be expressed as

\[
\Lambda^\alpha_B = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
-\frac{\omega}{c} & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}, \quad \Lambda^A_\beta = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix},
\]

where \(A\) and \(B\) here are upper case Greek, \(\Lambda^A_\beta\) transforms a contravariant lab vector \([eT,eR,e\Phi,eZ]^T\) to the corresponding contravariant rotating frame vector \([dx^\alpha = (c dt, dr, d\phi, dz)^T]\), and \(\Lambda^\alpha_B\) transforms the latter back from the rotating frame to the lab.

The following relations, which we will use in subsequent sections, were first championed by Langevin, can be found in many sources, and are shown in Klauber to be derivable from \(\Lambda^\alpha_B\). The rotating frame coordinate metric \(g_{\alpha\beta}\) and its inverse \(g^{\alpha\beta}\) are
\[ g_{\alpha\beta} = \begin{bmatrix} -\left(1 - \frac{r^2 \omega^2}{c^2}\right) & 0 & \frac{r^2 \omega}{c} & 0 \\ 0 & 1 & 0 & 0 \\ \frac{r^2 \omega}{c} & 0 & r^2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad g^{\alpha\beta} = \begin{bmatrix} -1 & 0 & \frac{\omega}{c} & 0 \\ 0 & 1 & 0 & 0 \\ \frac{\omega}{c} & 0 & \left(1 - \frac{r^2 \omega^2}{c^2}\right) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{4} \]

The off-diagonal terms imply that time and space are not orthogonal in the rotating frame. Although a little unusual, the rotating frame metric is not alone in this regard, and shares this NTO characteristic with the spacetime metric around a massive body such as a star or black hole that possesses angular momentum.[17]

For completeness, the lab metric \( G_{AB} \) and its inverse \( G^{AB} \) are

\[ G_{AB} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & R^2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad G^{AB} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{R^2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{5} \]

### 4.2 Mathematical vs. measured components

When working with rotating frames, we need to keep two things in mind that are usually irrelevant for Minkowski metrics in Lorentz frames, but are quite relevant for NTO frames. Both of these concern the relationship between generalized components of four-vectors (i.e., the mathematical components one works with in analyses, which are called coordinate components) and physical components (i.e., the components one would actually measure with standard instruments in an experiment.) Contravariant/covariant coordinate components of a vector do not equal physical components except for the special case where the coordinate system basis is orthonormal, such as in Minkowski coordinates.

### 4.3 Contravariant vs. covariant four-vectors

The first of the aforementioned concerns lies with the covariant or contravariant nature of the vector components.[18] Generalized coordinates (e.g., \( x^\alpha \)) are expressed as contravariant quantities, and generalized four-velocity \( u^\alpha \) is simply the derivative of these coordinates with respect to the invariant scalar quantity \( \tau_p \) (proper time of the particle.) In the strictest and most general sense, four-velocities only represent (proper) time derivatives of the coordinates if they are expressed in contravariant form. For example, in an NTO frame lowering the index of \( u^\alpha \) via the metric \( g_{\alpha\beta} \) gives components \( u_\alpha \) which are not the proper time derivatives of their respective coordinate values. This is true because \( g_{\alpha\beta} \) is not the identity matrix. Note that in Minkowski coordinates \( g_{\alpha\beta} = \eta_{\alpha\beta} \), which is, apart from the sign of the \( g_{00} \) component, an identity matrix. In a coordinate frame with such a Minkowski metric the covariant form of the four-velocity is identical to the contravariant form except for the sign of the timelike component. In NTO frames, however, the difference is much more significant, care must be taken, and one must recognize that four-velocity is contravariant, not covariant, in form.

Four-momentum, on the other hand, must be treated as a covariant vector. This is because the four-momentum is the canonical conjugate of the four-velocity. In brief, if the Lagrangian of a given system is

\[ L = L(x^\alpha, u^\alpha, \tau_p), \tag{6} \]

then the conjugate momentum is

\[ p_\alpha = \frac{\partial L}{\partial u^\alpha}. \tag{7} \]

This is covariant, not contravariant in form. Hence, it is imperative in an NTO system such as a rotating frame that one use covariant components for the four-momentum. Contravariant components in such a system, unlike that of a system with Minkowski metric, will not represent the physical quantities of energy and linear momentum. This is demonstrated explicitly in reference [1], section 4.3.4, where it is shown that \( p_0 \), and not \( p^0 \), represents the energy of a particle fixed to a rotating disk.
4.4 Relation between physical and coordinate components

Getting the correct contravariant or covariant components is not quite enough, however, in order to compare theoretical results with measured quantities. If a given basis vector does not have unit length, the magnitude of the corresponding component will not equal the physical quantity measured. For example, a vector with a single non-zero component value of 1 in a coordinate system where the corresponding basis vector for that component has length 3 does not have an absolute (physical) length equal to 1, but to three.

In general, physical components (those measured with physical instruments in the real world) are the components associated with unit basis vectors, and generalized coordinate basis vectors are generally not of unit length. As shown in the Appendix A (see also, texts cited in footnote [19], Malvern[20], Misner, Thorne and Wheeler[21], and Klauber[22]) physical components are found from generalized coordinate components (those used in generalized coordinate mathematical analysis) via the relations

\[
\begin{align*}
\hat{v}_i &= \sqrt{g_{ii}} v_i \\
\hat{v}_0 &= \sqrt{-g_{00}} v_0,
\end{align*}
\]

where carets over indices designate physical quantities, underlining implies no summation, Roman indices have values 1,2,3 and the negative signs arise on the RHS because \(g_{00}\) and \(g^{00}\) are negative.

4.5 Steps in general analysis

Hence, in order to compare theoretical component values with experiment, it is necessary to use contravariant components for coordinate differences and four-velocity, covariant components for four-momentum, and physical components of all component quantities whether covariant or contravariant.

It is important to note, however, that while coordinate components transform as true vectors, physical components do not[23]. So, while physical components are needed to compare theory with experiment, coordinate components are needed to carry out vector/tensor analysis.

Steps in NTO analysis therefore comprise
i) conversion of known (measured) physical components to coordinate components via (8),
ii) appropriate vector/tensor analysis using coordinate components, and
iii) conversion of the coordinate component answer back to physical component form via (8) in order to compare with experiment.

We note that the speed of light in (1) can be derived[24] using the above steps and that said speed is a physical, not coordinate, value.

5 WAVES IN ROTATING FRAMES

5.1 Overview of procedure

In order to transform frequencies and wavelengths of light from one frame to another we first express those frequencies and wavelengths, via the Planck energy and DeBroglie wave relations, as energy and momentum, respectively. We use those energy and momentum values to determine appropriate components of the generalized four-momentum \(p_\mu\). We can then simply apply the transformations (3) to transform the four-momentum from the lab to rotating frame, and vice versa. Converting the resulting four-momentum components back to frequency and wavelength form then reveals Doppler and other wave effects from rotation.

5.2 Lab emission, rotating observer

Consider a photon emitted in the lab in the negative \(\Phi\) direction while the rotating frame observer is moving in the positive \(\Phi\) direction such that the observer is approaching the light source. The light with wavelength \(\lambda\) and frequency \(f\) as measured in the lab frame \(K\) has four-momentum physical components of

\[
P_\hat{A} = \begin{bmatrix}
-\frac{hf}{c} \\
0 \\
-\frac{hf}{c} \\
0
\end{bmatrix}, \tag{9}
\]
where $h$ is Planck’s constant. From (8) and (10), the coordinate components are

$$P_A = \begin{bmatrix} -\frac{hf}{c} & \frac{1}{\sqrt{1-v^2/c^2}} \\ 0 & 0 \\ \frac{h}{c} & \frac{1}{\sqrt{1-v^2/c^2}} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -\frac{hf}{c} \\ 0 \\ 0 \\ -\frac{hR}{\lambda} \end{bmatrix}. \quad (10)$$

We need to raise the index in order to transform to the rotating frame $k$, as our transformations (3) are specifically for contravariant vectors. With (8), we have

$$P^A = G^{AB}P_B = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -\frac{hf}{c} \\ 0 \\ 0 \\ -\frac{hR}{\lambda} \end{bmatrix} = \begin{bmatrix} \frac{hf}{c} \\ 0 \\ 0 \\ -\frac{hR}{\lambda} \end{bmatrix}. \quad (11)$$

Using (3) and $R = r$ from (4) to transform to the rotating frame, we get

$$p^\alpha = \Lambda^\alpha_\beta P^\beta = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{\omega}{c} & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{hf}{c} \\ 0 \\ 0 \\ -\frac{hR}{\lambda} \end{bmatrix} = \begin{bmatrix} \frac{hf}{c} \\ 0 \\ -\frac{h}{\lambda} \left(\frac{\omega}{c} + \frac{1}{r}\right) \end{bmatrix}. \quad (12)$$

Lowering this to get the necessary covariant form for the four-momentum yields

$$p_\alpha = g_{\alpha\beta}p^\beta = \begin{bmatrix} (1 - \frac{r^2\omega^2}{c^2}) & 0 & \frac{r^2\omega}{c} & 0 \\ 0 & 1 & 0 & 0 \\ \frac{r^2\omega}{c} & 0 & r^2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{hf}{c} \\ 0 \\ -\frac{h}{\lambda} \left(\frac{\omega}{c} + \frac{1}{r}\right) \end{bmatrix} = \begin{bmatrix} \frac{hf}{c} \left(1 + \frac{\omega}{c}\right) \\ 0 \\ -\frac{h}{\lambda} \left(\frac{\omega}{c} + \frac{1}{r}\right) \end{bmatrix}. \quad (13)$$

We take physical components of (13) to obtain what an observer in the rotating frame would measure with physical instruments, i.e.,

$$p_\hat{\alpha} = \begin{bmatrix} \sqrt{-g_{00}} \frac{hf}{c} \left(1 + \frac{\omega}{c}\right) \\ 0 \\ -\sqrt{g_{22}} \frac{h}{\lambda} \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{hf}{c} \left(1 + \frac{\omega}{c}\right) \\ 0 \\ -\frac{h}{\lambda} \sqrt{1 - \frac{\omega^2}{c^2}} \end{bmatrix}. \quad (14)$$

We then need to express (14) in terms of the frequency $f_k$ and wavelength $\lambda_k$ measured by an observer in the rotating frame $k$. We can not simply assume the Planck and DeBroglie relations hold in a rotating frame, as there is no guarantee that they have the same form in an NTO frame. That is, we can not presume that $p_0 c = E_k$ equals $hf_k$, nor that $p_2$ equals $h/\lambda_k$.

Instead we employ a thought experiment and physical world logic. We know time in the rotating frame runs more slowly than time in the lab by the inverse of the Lorentz factor $1/\gamma = \sqrt{1 - v^2/c^2}$, and this should increase the frequency of light as seen on the rotating frame $(f_k)$ over that of the lab $(f)$. Independent of that, a rotating frame observer moving toward a lab oscillator should see that oscillator beating faster than would an observer in the lab. At least at low speeds, this increase should be by the factor $(1 + v/c) = (1 + \omega r/c)$, as shown in Tables 1 and 2. Based on these physical considerations, we propose a relationship between lab and rotating frame frequencies for a given photon of light as

$$f_k = \frac{f \left(1 + \frac{\omega}{c}\right)}{\sqrt{1 - v^2/c^2}}. \quad (15)$$

Given (15) and the fact that $f_k \lambda_k = u_{\text{light,circum}}$ of (1), it follows that

$$\lambda_k = \lambda. \quad (16)$$

Thus (14) becomes, in terms of rotating frame observable quantities (RHS below),
\[ p_\alpha = \begin{bmatrix} \frac{-h \sqrt{1-v^2/c^2}}{c} & 0 \\ 0 & \frac{-h \sqrt{1-v^2/c^2}}{\lambda} \\ \frac{-hf}{c} & 0 \\ 0 & \frac{-h \sqrt{1-v^2/c^2}}{\lambda} \end{bmatrix} \]

(17)

From this the Planck and DeBroglie relations for the NTO rotating disk frame are seen to be

\[ E_k = hf_k \sqrt{1 - v^2/c^2} \]

\[ p_{\text{phys.circum}} = \frac{h \sqrt{1 - v^2/c^2}}{\lambda_k} \]

(18)

Similar results for the radially directed light are derived in Appendix B.

The frequency and wavelength results above are summarized in Case 1 of Table 3. In Table 3, non-subscripted quantities and velocity \( V \) refer to the lab frame \( K \), and the subscript \( k \) designates rotating frame quantities. Note if \( v \) were in the negative \( \Phi \) direction, or if the rotating frame observer were moving away from the light emission source in the lab, then \( v \rightarrow -v \) in (15) and throughout Table 3.

### 5.3 Rotating frame emission, lab observer

Consider now a photon emitted from the rotating frame in the positive \( \Phi \) direction from a source on the rotating frame that is approaching a lab observer. We can simply reverse the steps (14) to (9) of Section 5.2 to relate wavelengths and frequencies in the rotating and lab frames, taking care that the sign for linear momentum changes from the earlier case. The reader can either carry out these steps to justify the results summarized in Case 2 of Table 3, or consider the following logic.

We know from (1) that the speed of light in the rotating frame in this case is

\[ u_{\text{light,circum}} = v_k = \frac{c - v}{\sqrt{1 - v^2/c^2}}. \]

(19)

We also know that frequency should increase by first order in \( v/c \) as seen by the lab observer because the source is approaching, yet it should decrease by the inverse Lorentz factor since time runs more slowly in the rotating frame. Further, the wavelengths in the rotating and lab frames should be related mathematically in the very same way as (16). Thus,

\[ f_k = \frac{f(1 - v/c)}{\sqrt{1 - v^2/c^2}} \]

\[ \lambda_k = \lambda \]

(20)

such that (16) holds, i.e.,

\[ f_k \lambda_k = \frac{f\lambda(1 - v/c)}{\sqrt{1 - v^2/c^2}} = \frac{c - v}{\sqrt{1 - v^2/c^2}} = v_k. \]

(21)

Hence, frequency seen in the lab from emission in the rotating frame is

\[ f = \frac{f_k \sqrt{1 - v^2/c^2}}{1 - v/c} = f_k (1 + v/c - \frac{1}{2}v^2/c^2 + ...). \]

(22)

These results are summarized in the right hand column of Table 3.

The relation (22) could also have been deduced directly from the LHS of (15) with the realization that changing the direction of the photon has the same effect mathematically as changing the direction of rotation. That is \( v \rightarrow -v \) in (15) yields (22). Note that higher order Doppler shift effects for rotating frames differ from those of both Sections 2 and 3.

| Table 3. Circumferentially Directed Waves in Relativistically Rotating Frames |
|-------------------------------------------------|-----------------|
| **Case 1** | **Case 2** |
| Source | Lab frame K | Rotating frame k |
| Observer | Rotating frame k | Lab frame K |
6 Turning the Light Source in the Rotating Frame

For completeness, we consider the effect on light speed, frequency and wavelength measurements in the rotating frame when a source in the rotating frame is turned. This can be valuable for evaluation of Brillet and Hall\[25\] type experiments in which test apparatus is turned within the rotating frame of the earth. One part of the apparatus is the light source; another part the sensing equipment.

We consider two cases: the circumferential direction and the radial direction. The vertical (z) direction parallels that of the radial direction. We take it as an axiom that an observer who is fixed relative to a source detects no Doppler shift in frequency from that source. This axiom appears to have been tested to extremely high accuracy by Chen and Liu[26] although there are certain caveats\[27\] regarding the relevance of that test to the present article.

6.1 Circumferential Direction

In the circumferential direction, light speed is given by (1). For observers on the rotating frame the frequency of light emitted from a source on the rotating frame is \( f_k \). Hence, the wavelength in the circumferential direction, determined solely by measurements on the rotating frame must be

\[
\lambda_{k,\text{circum}} = \frac{u_{\text{light, circum}}}{f_k} = \frac{c \pm v}{f_k \sqrt{1 - v^2/c^2}}.
\]

(23)

6.2 Radial Direction

6.2.1 Radial Direction Light Speed

From the rotating (NTO) frame metric of (4), the line element is

\[
ds^2 = -c^2(1 - r^2 \omega^2/c^2)dt^2 + dr^2 + r^2 d\phi^2 + 2r \omega d\phi dt + dz^2 = g_{\alpha\beta}dx^\alpha dx^\beta
\]

(24)

where \( t \) is coordinate time and equals that on standard clocks in the lab. Note that the time on a standard clock at a fixed 3D location on the rotating disk, found by taking \( ds^2 = -c^2 dt \) and \( dr = d\phi = dz = 0 \), is

\[
d\tilde{t} = \frac{d\tilde{t}}{d\tilde{t}} = \sqrt{1 - r^2 \omega^2/c^2} dt,
\]

(25)

where the caret over \( dt \) indicates physical time (i.e., time measured with standard clocks fixed in the rotating frame.)

For a radially directed ray of light, \( d\phi = dz = 0 \), and \( ds = 0 \). Solving for \( dr/dt \) one obtains

\[
\frac{dr}{dt} = c \sqrt{1 - r^2 \omega^2/c^2}.
\]

(26)

Since \( g_{rr} = 1 \), the physical component, (measured with standard meter sticks) for radial displacement \( \tilde{d}r \) equals the coordinate radial displacement \( dr \). The physical (measured) speed of light in the radial direction is therefore

| | Observer toward source at \( v = \omega r \) | Source toward observer at \( v = \omega r \) |
|---|---|---|
| Wave speed | \( V = c \) \( \nu_k = \frac{c + v}{\sqrt{1 - v^2/c^2}} \) | \( V = c \) \( \nu_k = \frac{c - v}{\sqrt{1 - v^2/c^2}} \) |
| Wave length | \( \lambda_k = \lambda \) \( \lambda_k,\text{circum} = \frac{u_{\text{light, circum}}}{f_k} = \frac{c \pm v}{f_k \sqrt{1 - v^2/c^2}} \) | \( \lambda_k = \lambda \) |
| Frequency | \( f_k = \frac{f(1 + v/c)}{\sqrt{1 - v^2/c^2}} \) | \( f = \frac{f_k \sqrt{1 - v^2/c^2}}{1 - v/c} \) |
| Doppler shift observer sees | \( f_k = f \left( 1 + v/c - \frac{v^2}{2c^2} + \ldots \right) \) | \( f = f_k \left( 1 + v/c + \frac{v^2}{2c^2} + \ldots \right) \) |
\[ \nu_{\text{light, radial, phys}} = \frac{d\hat{r}}{dt} = \frac{dr}{d\tau} = \frac{dr}{\sqrt{1 - r^2 \omega^2/c^2} dt} = c. \] (27)

This can be further justified by physical considerations of the path followed by such a light ray as seen in the lab frame, along with (25).

### 6.2.2 Radial Direction Wavelength

As the apparatus containing the light source and detector is turned from the circumferential to radial direction, the speed of light in the rotating frame changes from that of (1) to that of (27). The frequency remains unchanged. Hence, the wavelength changes to

\[ \lambda_{k, \text{radial}} = \frac{c}{f_k}. \] (28)

From (23) we see that

\[ \lambda_{k, \text{circum}} = \frac{1 \pm v/c}{\sqrt{1 - v^2/c^2}} \lambda_{k, \text{radial}}. \] (29)

The results of this and the prior section are summarized in Table 4.

#### Table 4. Turning Light Source in Relativistically Rotating Frame

|                    | Case 1 Circumferential Direction | Case 2 Radial Direction |
|--------------------|----------------------------------|------------------------|
| Source             | Rotating frame k                 | Rotating frame k       |
| Observer           | Rotating frame k                 | Rotating frame k       |
| Motion             | Source and observer both fixed in k | Source and observer both fixed in k |
| Wave speed         | \( \nu_{k, \text{circum}} = \frac{c \pm v}{\sqrt{1 - v^2/c^2}} \) | \( \nu_{k, \text{radial}} = c \) |
| Wave length        | \( \lambda_{\text{observe, circum}} = \lambda_{k, \text{circum}} = \lambda_{k, \text{source, circum}} \) | \( \lambda_{\text{observe, radial}} = \lambda_{k, \text{radial}} = \frac{c \pm v/c}{\sqrt{1 - v^2/c^2}} \lambda_{k, \text{radial}} \) |
| Frequency          | \( f_{\text{observe, circum}} = f_{k, \text{circum}} = f_{\text{source, circum}} \) | \( f_{\text{observe, radial}} = f_{k, \text{radial}} = f_{\text{source, radial}} = f_k \) |
| Doppler shift      | None                             | None                   |
| observer sees      |                                  |                        |

#### 6.3 Possible First Order Test

The difference in wavelength between the radial and circumferential directions shown above is first order in \( v/c \). Although tests of light speed \( \text{per se} \) are limited to round trip measurements, and hence must be of second order, there are a number of tests, such as the Young double slit experiment, that detect differences in wavelength. As readily seen in (23) and (28) above, wavelength is proportional to one way light speed. Thus, a test that measures wavelength could be a first order test of light speed. Such an experiment could be performed using the earth as the rotating frame and turning the test apparatus relative to the earth.

### 7 SUMMARY AND CONCLUSIONS

Comparison of Tables 1, 2 and 3 shows that to first order rotating frame Doppler effects are equivalent to those of classical and relativistic analyses. Higher order effects, however, vary among the three. As seen in Table 4, turning the light source within the rotating frame changes light speed and wavelength, but not frequency,
for rotating frame based observers. Multiplication of frequency by wavelength in all cases equals wave speed, providing corroboration for the methodology employed.

APPENDIX A. PHYSICAL COMPONENTS

Consider an arbitrary vector \( \mathbf{v} \) in a 2D space

\[
\mathbf{v} = v^1 \mathbf{e}_1 + v^2 \mathbf{e}_2 = v^\hat{1} \hat{\mathbf{e}}_1 + v^\hat{2} \hat{\mathbf{e}}_2
\]

where \( \mathbf{e}_i \) are coordinate basis vectors and \( \hat{\mathbf{e}}_i \) are unit length (non-coordinate) basis vectors pointing in the same respective directions. That is,

\[
\hat{\mathbf{e}}_i = \frac{\mathbf{e}_i}{|\mathbf{e}_i|} = \frac{\mathbf{e}_i}{\sqrt{\mathbf{e}_i \cdot \mathbf{e}_i}} = \mathbf{e}_i \sqrt{g_{ii}}
\]

where underlining implies no summation. Note that \( \mathbf{e}_1 \) and \( \mathbf{e}_2 \) here do not, in general, have to be orthogonal. Note also, that physical components are those associated with unit length basis vectors and hence are represented by indices with carets in (30).

Substituting (31) into (30), one readily obtains

\[
v^\hat{i} = \sqrt{g_{ii}} v^i
\]

Relation (32) between physical and coordinate components is valid locally in curved, as well as flat, spaces and can be extrapolated to 4D general relativistic applications, to higher order tensors, and to covariant components.

APPENDIX B. RADIAL DIRECTION: LAB VS. ROTATING FRAME

We would be remiss if we did not include an analysis similar to that of sections 5.2 and 5.3 for the radial direction. We outline this below.

Photon in Lab Frame Radial Direction

Parallel to (1) through (18), we consider a photon emitted in the lab in the \( R \) direction. We determine the physical components of the 4 momentum vector in the lab, convert to coordinate components, raise the index, transform to the rotating frame, lower the index, and find physical components.

\[
p_{\hat{A}} = \begin{bmatrix} -\frac{hf}{c} \\ \frac{h}{\lambda} \\ 0 \\ 0 \end{bmatrix} \rightarrow p_A = \begin{bmatrix} -\frac{hf}{c} \\ \frac{h}{\lambda} \\ 0 \\ 0 \end{bmatrix} \rightarrow p^A = \begin{bmatrix} -\frac{hf}{c} \\ \frac{h}{\lambda} \\ 0 \\ 0 \end{bmatrix}
\]

\[
\rightarrow p^\alpha = \begin{bmatrix} -\frac{hf}{c} \\ \frac{h}{\lambda} \\ \frac{\omega hf}{c^2} \\ 0 \end{bmatrix} \rightarrow p_\alpha = \begin{bmatrix} -\frac{hf}{c} \\ \frac{h}{\lambda} \\ 0 \\ 0 \end{bmatrix} \rightarrow p_\alpha = \begin{bmatrix} -\frac{hf}{c} \\ \frac{h}{\lambda} \\ 0 \\ 0 \end{bmatrix}
\]

We conclude that the speed, energy and three-momentum of a light ray directed radially in the lab is identical to that measured on the rotating disk. This is interesting since this photon as seen in the rotating frame is not moving in a purely radial direction, but has a circumferential component. To prove this we can not simply transform the four velocity for the photon since four velocity \( U^A = dX^A/d\tau_p \), where \( \tau_p \) is the proper length of the path in spacetime, and for a photon \( d\tau_p = 0 \). Consider instead the displacement of the photon having three velocity in the radial direction \( dR/dT = c \). The 4D displacement vector in time \( dT \) shown on the left side of (34) below can then be transformed to the rotating frame and the physical components determined in that frame as follows
\[dX^\hat{A} = dX^A = \begin{bmatrix} \frac{cdT}{\omega dt} & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow dx^\alpha = \Lambda_\alpha^\beta dX^\beta = \begin{bmatrix} \frac{cdt}{\omega dt} \\ 0 \end{bmatrix} \Rightarrow dx^\hat{\alpha} = \begin{bmatrix} \sqrt{1 - \frac{c^2}{v^2}} c dt \\ \frac{cdt}{\omega rdT} \\ 0 \end{bmatrix} \]. \quad (34)

The physical time of a standard clock fixed in the rotating frame \(28\) is \(d\tau = \sqrt{1 - \frac{v^2}{c^2}} dt\) where \(v = \omega r\). Hence the physical speeds of the photon in the rotating frame in the radial and circumferential directions are, respectively,

\[v_{\text{light,phys,radial}} = \frac{dx^i}{d\tau} = \frac{cdt}{\sqrt{1 - \frac{v^2}{c^2}}}, \quad v_{\text{light,phys,circum}} = \frac{dx^\hat{\alpha}}{d\tau} = -\frac{v dt}{\sqrt{1 - \frac{v^2}{c^2}}}. \quad (35)\]

**Photon in Rotating Frame Radial Direction**

Consider, on the other hand, a photon in the lab with a circumferential speed component of \(v = \omega r\). One might expect this to travel along a purely radial direction in the rotating frame, and it does, as we prove at the end of this appendix. Note that the radial velocity component in the lab must be \(c\sqrt{1 - \frac{v^2}{c^2}}\) in order for the photon to have speed \(c\) along its direction of travel in the lab.

The energy and momentum in the rotating frame for this photon may be found in parallel fashion to that for the lab purely radial direction photon above. Starting with the physical 4 momentum in the lab we find

\[p^\hat{A} = \begin{bmatrix} -\frac{hf}{\sqrt{1 - \frac{v^2}{c^2}} R^2} \\ \frac{v h}{c} \\ 0 \end{bmatrix} \Rightarrow p_\alpha = \begin{bmatrix} -\frac{hf}{\sqrt{1 - \frac{v^2}{c^2}} R^2} \\ \frac{v h}{c} \\ 0 \end{bmatrix} \Rightarrow p^\alpha = \begin{bmatrix} \frac{hf}{\sqrt{1 - \frac{v^2}{c^2}} R^2} \\ \frac{v h}{c} \\ 0 \end{bmatrix}. \quad (36)\]

As a check, one can see that \(p^2 = 0\) in both frames, as it must. The differences in physical values of energy and momentum between the lab and rotating frames are second order. And though the photon has purely radial direction of travel in the rotating frame, it has a circumferential momentum component in that frame \(29\).

We can then, as before, determine the physical 4 displacement in time \(dT\) in the lab, convert to coordinate displacement, transform to the rotating frame, and find the physical components in that frame.

\[dX^\hat{A} = \begin{bmatrix} \frac{cdT}{\omega dt} & 0 \\ \frac{c}{\omega T} dT & 0 \end{bmatrix} \Rightarrow dX^A = \begin{bmatrix} \frac{cdT}{\omega dt} & 0 \\ \frac{c}{\omega T} dT & 0 \end{bmatrix} \]

\[\rightarrow dx^\alpha = \begin{bmatrix} \frac{c}{\omega T} dT \\ 0 \end{bmatrix} \rightarrow dx^\hat{\alpha} = \begin{bmatrix} \frac{cdT}{\omega dt} \\ \frac{c}{\omega T} dT \end{bmatrix}. \quad (37)\]

Thus, we see that the 3 velocity in the rotating frame has only a component in the radial direction

\[v_{\text{light,phys,circum}} = \frac{c}{\sqrt{1 - \frac{v^2}{c^2}} dT} = c, \quad (38)\]

as asserted, and this agrees with the earlier result \(27\).
References

[1] Robert D. Klauber, “New perspectives on the relatively rotating disk and non-time-orthogonal reference frames”, Found. Phys. Lett. 11(3), 405-443 (Oct 1998). gr-qc/0103076

[2] Robert D. Klauber, “Comments regarding recent articles on relativistically rotating frames”, Am. J. Phys. 67(2), 158-159 (Feb 1999). gr-qc/9812025.

[3] Robert D. Klauber, “Non-time-orthogonality and tests of special relativity”, gr-qc/0006023.

[4] Ref. [1], Section 4.3, pp. 425-429.

[5] Ref. [1], Section 6, pp. 434-436.

[6] Ref. [3], Section 5.

[7] Neil Ashby, “Relativistic Effects in the Global Positioning System”, 15th Intl. Conf. Gen. Rel. and Gravitation, Pune, India (Dec 15-21, 1997), available at www.colorado.edu/engineering/GPS/Papers/RelativityinGPS.ps. See pp. 5-7.

[8] Neil Ashby, “Relativity and the Global Positioning System”, Phys. Today, May 2002, 41-47. See pg 44.

[9] Robert D. Klauber, “Non-time-orthogonal reference frames in the theory of relativity”, gr-qc/0005121, section II.

[10] Ref. [1], pg. 425, eq. (19) modified by the time dilation factor discussed in the subsequent paragraphs therein to yield physical velocity, and pg. 430, eq. (33).

[11] David Halliday and Robert Resnick, Fundamentals of Physics (John Wiley & Sons, New York, 1974), p.334-337. The authors employ different symbols from those of Table 1. For example, they use $v$ as the wave velocity within the medium, whereas we use $v_m$. They also show that the wavelength in Case 2 of Table 1 is shorter than that of Case 1, whereas we emphasize that in Galilean theory a wave has the same length for the observer and the source frames in each case.

[12] Ibid., pp.660-663.

[13] Herbert E. Ives and G. R. Stilwell, “An Experimental Study of the Rate of a Moving Atomic Clock”, J. Opt. Soc. Am. 28(8), 215-226 (July 1938). ”An Experimental Study of the Rate of Moving Atomic Clock II”, J. Opt. Soc. Am. B 31, 369-374 (May 1941).

[14] Paul Langevin, “Relativité – Sur l’expérience de Sagnac”, Academie des sciences comptes rendus des seances., Vol. 205, 304-306 (2 Aug 1937.)

[15] Ref. [6], for one.

[16] Ref. [1], Section 4.1, pp. 420-422.

[17] Charles W. Misner, Kip S. Thorne, and John Archibald Wheeler, Gravitation (W. H. Freeman and Co., New York, 1973), Chap. 19, p.448-459.

[18] Robert D. Klauber, "Generalized Tensor Analysis Method and the Wilson and Wilson Experiment", gr-qc/0107035 (2001). See Section 2.1 and Appendix A therein for more detailed presentation of the physical relevance of contravariant and covariant components in non-orthonormal coordinates.

[19] The texts listed below are among those that discuss physical vector and tensor components (the values one measures in experiment) and the relationship between them and coordinate components (the mathematical values that depend on the generalized coordinate system being used.) For a vector this relationship is is $u^\hat{\mu} = \sqrt{g_{\mu\nu}}u^\nu$ where the caret over the index indicates a physical vector component and underlining implies no summation. See I.S. Sokolnikoff, Tensor Analysis, (Wiley & Sons, 1951) pp. 8, 122-127, 205; G.E. Hay, Vector and Tensor Analysis, (Dover, 1953) pp 184-186; A. J. McConnell, Application of Tensor Analysis, (Dover, 1947) pp. 303-311; Carl E. Pearson, Handbook of Applied Mathematics, (Van Nostrand Reinhold, 1983 2nd ed.), pp. 214-216; Murry R. Spiegel, Schaum’s Outline of Vector Analysis, (Schaum) pg. 172; Robert C. Wrede, Introduction to Vector and Tensor Analysis, (Dover 1972), pp. 234-235.
L. E. Malvern, *Introduction to the Mechanics of a Continuous Medium* (Prentice-Hall, Englewood Cliffs, New Jersey, 1969), Appendix I, Sec. 5, p.606-613.

Reference [21]. Physical components are introduced on pg. 37, and used in many places throughout the text, e.g. pp. 821-822. Note, however, that the authors confine their explanation and use of physical components to the special case of time orthogonal systems. As noted on page 606 in Ref. [20], the definitions presented in the present article are applicable to any coordinate system, orthogonal or non-orthogonal, and reduce to those of Ref. [17] for orthogonal coordinates.

Reference [22]. Robert D. Klauber, “Physical Components, Coordinate Components, and the Speed of Light”, gr-qc/0105071 (2001).

Reference [23]. Y. C. Fung, *Foundations of Solid Mechanics* (Prentice-Hall, Inc., Englewood Cliffs, NJ, 1965), p.53.

Reference [24]. Ref. [1], Sections 4.2.5 and 5.1, and Ref. [1], Section 5.

Reference [25]. A. Brillet and J. L. Hall, “Improved laser test of the isotropy of space,” *Phys. Rev. Lett.*, 42(9), 549-552 (1979).

Reference [26]. Shaoguang Chen and Baocheng Liu, “A New Method for Inspect Spatial Isotropy”, Peking University, http://www.pku.edu.cn/academic/xb/96/e96510.html. “Experimental Test of the Isotropy of Two-way Speed of Light”, Peking University, http://www.pku.edu.cn/academic/xb/97/97e509.html. The web sites cited contain only abstracts. The full reports may only be published in the Chinese language. The authors note experimental verification of isotropy of light frequency for $\Delta f/f$ of $1\times10^{-18}$.

Reference [27]. The Chen and Liu tests were for two way light transmission, whereas the present article addresses one way transmission from source to observer. It is also not apparent whether the test apparatus rotated relative to the earth fixed reference frame or the earth frame rotation was used to rotate the apparatus relative to the heavens. This is relevant as NTO analysis predicts the effects described herein only for true rotating frames (in which angular velocity and centrifugal acceleration may be sensed) such as the earth fixed frame. NTO analysis predicts no difference from traditional time-orthogonal relativity theory for gravitational orbits (in which neither angular orbital velocity nor centrifugal acceleration may be sensed.) See ref. [1], section 6.

Reference [28]. See ref. [1] Sections 3.3.3, 4.3.1, and 5.1.

Reference [29]. The two cases evaluated in Appendix B suggest that momentum in the inertial lab frame has, to first order, a certain absolute quality to it with respect to rotating frames. This has a parallel in non-relativistic mechanics where the angular momentum of a rotating body (integral of $(\omega r \times r)dm$) calculated in the lab is also the angular momentum seen in the rotating frame of the body itself. This implies linear momentum $\omega r dm$ is the same non-relativistically in both frames as well. This agrees with findings of ref. [1], section 4.3.4 for a massive particle fixed to the disk. The analysis herein therefore appears consistent in the low velocity limit with classical dynamics.