Neutrino mass and mixing in the 3-3-1 model and $S_3$ flavor symmetry with minimal Higgs content

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A new $S_3$ flavor model based on SU(3)$_C \otimes$ SU(3)$_L \otimes$ U(1)$_X$ gauge symmetry responsible for fermion masses and mixings different from our previous work [12, 13] is constructed. The new feature is a two-dimensional representation of a Higgs anti-sextet under $S_3$ which responsible for neutrino masses and mixings. The neutrinos acquire small masses from only an anti-sextet of SU(3) which is in a doublet under $S_3$. If the difference of components of the anti-sextet is regarded as a small perturbation, $S_3$ is equivalently broken into identity, the corresponding neutrino mass mixing matrix acquires the most general form and the model can fit the latest data on neutrino oscillation. This way of the symmetry breaking helps us to reduce a content in the Higgs sector, only one an anti-sextet instead of two as in our previous work [13]. Our results show that the neutrino masses are naturally small and a small deviation from the tri-bimaximal neutrino mixing form can be realized. The Higgs potential of the model as well as the minimization conditions and gauge boson masses and mixings is also considered.

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I. INTRODUCTION

The experiments of neutrino oscillations have indicated that the neutrinos have small masses and mixings [1-4], and therefore the standard model of fundamental particles and interactions must be extended. Among this direction, there have been various models proposed, such as [5, 6] and others. An alternative is to extend the electroweak symmetry SU(2)$_L \otimes$ U(1)$_Y$ into SU(3)$_L \otimes$ U(1)$_X$, in which to complete the fundamental representations of SU(3)$_L$ with the standard-model doublets.
so as to obtain the neutral fermions. This proposal has nice features and has been extensively studied over the last two decades, is called 3-3-1 models, in which the number of fermion families has been proven to be three.

The parameters of neutrino oscillations such as the squared mass differences and mixing angles are now very constrained. The data in Ref. imply that

\[
\sin^2(2\theta_{12}) = 0.857 \pm 0.024 \quad (t_{12} \simeq 0.6717), \\
\sin^2(2\theta_{13}) = 0.098 \pm 0.013 \quad (s_{13} \simeq 0.1585), \\
\sin^2(2\theta_{23}) > 0.95,
\]

\[
\Delta m^2_{21} = (7.50 \pm 0.20) \times 10^{-5}\text{eV}^2, \quad \Delta m^2_{32} = (2.32^{+0.12}_{-0.08}) \times 10^{-3}\text{eV}^2.
\]

These large neutrino mixing angles are completely different from the quark mixing ones defined by the CKM matrix. Therefore, it is very important to find a natural model that leads to these mixing patterns of quarks and leptons with good accuracy. Small non-Abelian discrete symmetries are considered to be the most attractive choice for the flavor sector. The simplest explanation for these conclusions is probably due to an $S_3$ flavor symmetry which is the smallest non-Abelian discrete group. In fact, there is an approximately maximal mixing of two flavors $\mu$ and $\tau$ as given above which can be connected by a 2 irreducible representation of $S_3$. Besides the 2, the group $S_3$ can provide two inequivalent singlet representations 1 and $1'$ which play a crucial role in reproducing consistent fermion masses and mixings. The $S_3$ models have been studied extensively over the last decade. In we have proposed two 3-3-1 models, with either neutral fermions or right-handed neutrinos, based on $S_3$ flavor symmetry, in which there is a large number of Higgs triplets was required. In this paper, we propose a new $S_3$ flavor symmetry in the 3-3-1 model with neutral fermions, in which the number of Higgs triplets required is less and the Higgs potential of the model is therefore much simpler than the previous ones.

The motivation for extending the above application to the 3-3-1 models with the neutral fermions $N_R$ is mentioned in. In this paper, we investigate simpler choices for Higgs multiples of $S_3$ in which the unique anti-sextet responsible for neutrino mass and mixing lying in 2 under $S_3$ and the difference between two VEV components of anti-sextet plays the role of perturbation. It is also noted that the numbers of fermion families in the 3-3-1 models have an origin from the anomaly-free gauge symmetry naturally meet our criteria on the dimensions of flavor group representations as such $S_3$, unlike the others in the literature, mostly imposed by hand.

The rest of this work is as follows. In Sec. we present the necessary elements of the 3-3-1 model with neutral fermions $N_R$ under the $S_3$ symmetry and introduce the necessary Higgs...
fields responsible for the charged-lepton and quark masses. Section III is devoted to the neutrino mass and mixing. In Sec. IV we consider the Higgs potential and minimization conditions. We summarize our results and make conclusions in the section VI.

II. THE MODEL

The fermion content of the model is similar to that in [13]: the fermions in the model transform under respective $[SU(3)_L, U(1)_X, U(1)_L, S_3]$ symmetries as

\[
\psi_{1L} = (\nu_L, l_{1L}, N_{1R}^c)^T \sim [3, -1/3, 2/3, 1], \quad l_{1R} \sim [1, -1, 1, 1],
\]
\[
\psi_{aL} = (\nu_{aL}, l_{aL}, N_{aR}^c)^T \sim [3, -1/3, 2/3, 2], \quad l_{aR} \sim [1, -1, 1, 2],
\]
\[
Q_{1L} = (u_{1L}, d_{1L}, U_L)^T \sim [3, 1/3, -1/3, 1],
\]
\[
u_{1R} \sim [1, 2/3, 0, 1], \quad d_{1R} \sim [1, -1/3, 0, 1], \quad U_R \sim [1, 2/3, -1, 1],
\]
\[
Q_{aL} = (d_{aL}, -u_{aL}, D_{aL})^T \sim [3^*, 0, 1/3, 3],
\]
\[
u_{aR} \sim [1, 2/3, 0, 2], \quad d_{aR} \sim [1, -1/3, 0, 2], \quad D_{aR} \sim [1, -1/3, 1, 2].
\]

where $a = 2, 3$ is a family index of the last two lepton and quark families, which are defined as the components of the $2$ representations. We note that the $2$ for quarks satisfies the requirement of anomaly cancelation, where the last two left-quark families are in $3^*$ and the first one as well as the leptons are in $3$. All the $L$ charges of the model multiplets are listed in the square brackets. In what follows, we consider possibilities for generating the fermion masses. The scalar multiplets needed for this purpose are to be introduced accordingly.

To generate masses for the charged leptons, we introduce two $SU(3)_L$ scalar triplets $\phi$ and $\phi'$ respectively lying in $1$ and $1'$ under $S_3$, with the VEVs $\langle \phi \rangle = (0 \ v \ 0)^T$ and $\langle \phi' \rangle = (0 \ v' \ 0)^T$ [13]. From the invariant Yukawa interactions for the charged leptons, we obtain $m_e = h_1v$, $m_\mu = hv - h'v'$, $m_\tau = hv + h'v'$, and the left and right-handed charged leptons mixing matrices are diagonal, $U_{1L} = U_{1R} = 1$. The charged leptons $l_{1,2,3}$ therefore by themselves are the physical mass eigenstates and the lepton mixing matrix depends on only that of the neutrinos, which is studied in the next section.

In similarity to the charged lepton sector, to generate the quark masses, we additionally introduce the three scalar Higgs triplets $\chi$, $\eta$, $\eta'$ respectively lying in $1$, $1$ and $1'$ under $S_3$. Quark masses can be derived from the invariant Yukawa interactions for quarks, assuming that the VEVs of $\eta$, $\eta'$ and $\chi$ are $u$, $u'$ and $w$, where $u = \langle \eta_1^0 \rangle$, $u' = \langle \eta_1^0 \rangle$, and $w = \langle \chi_3^0 \rangle$ and the other VEVs $\langle \eta_3^0 \rangle$, $\langle \eta_3^0 \rangle$. 
\langle \eta_3^0 \rangle, \text{ and } \langle \chi_1^0 \rangle \text{ vanish due to the lepton parity conservation. The exotic quarks therefore acquire masses } m_U = f_1 w \text{ and } m_{D_{1,2}} = f w. \text{ The masses of ordinary up-quarks and down-quarks are}
\begin{align*}
m_u &= h_1^u u, \quad m_c = h_1^u v + h_1^u v', \quad m_t = h_1^v v - h_1^v v', \\
m_d &= h_1^d v, \quad m_s = h_1^d u + h_1^d u', \quad m_b = h_1^u u - h_1^u u'.
\end{align*}

The unitary matrices which couple the left-handed quarks $u_L$ and $d_L$ to those in the mass bases are unit ones. The CKM quark mixing matrix at the tree level is then $U_{\text{CKM}} = U_{dL}^\dagger U_{uL} = 1$. The lepton parity breaking due to the odd VEVs $\langle \eta_3^0 \rangle, \langle \eta_3^0 \rangle, \langle \chi_1^0 \rangle$, or a violation of $L$ and/or $S_3$ symmetry in terms of Yukawa interactions would disturb the tree-level matrix, resulting in a mixing between the SM and exotic quarks and/or possibly providing the desirable quark mixing pattern $\bar{Q}_1^L \chi u_1^R, \bar{Q}_L^L \chi^* d_1^R, \bar{Q}_1^L \chi u_1^R$, with a mixing between SM and exotic quarks. To obtain a realistic pattern of the SM quarks mixing, we should add radiative correction or use the effective six-dimensional operators (see Ref. [18] for details). However, we leave this problem for the future work. A detailed study on charged lepton and quark masses can be found in Ref. [13]. In this paper, we consider a new representation for the anti-sextet responsible for neutrino masses and mixings that are different from those in Ref. [13].

III. NEUTRINO MASSES AND MIXING

The neutrino masses arise from the couplings of $\bar{\psi}_{\alpha L}^c \psi_{\alpha L}, \bar{\psi}_{1L}^c \psi_{1L}$ and $\bar{\psi}_{iL}^c \psi_{iL}$ to scalars, where $\bar{\psi}_{\alpha L}^c \psi_{\alpha L}$ transforms as $3^* \oplus 6$ under SU(3)$_L$ and as $1 \oplus 1'$ under $S_3$; $\bar{\psi}_{1L}^c \psi_{1L}$ transforms as $3^* \oplus 6$ under SU(3)$_L$ and as $1$ under $S_3$, and $\bar{\psi}_{iL}^c \psi_{iL}$ transforms as $3^* \oplus 6$ under SU(3)$_L$ and as $2$ under $S_3$. For the known scalar triplets ($\phi, \phi', \chi, \eta, \eta'$), the available interactions are only $(\bar{\psi}_{\alpha L}^c \psi_{\alpha L}) \phi$ and $(\bar{\psi}_{iL}^c \psi_{iL}) \phi'$, but are explicitly suppressed because of the $\mathcal{L}$-symmetry. We therefore propose a new SU(3)$_L$ antisextet coupling to $\bar{\psi}_L^c \psi_L$ responsible for the neutrino masses lying in either $1$, $1'$, or $2$ under $S_3$. To obtain a realistic neutrino spectrum with minimal Higgs content, we introduce the Higgs anti-sextet

\[
s_i = \begin{pmatrix}
0 & s_{11}^0 & s_{12}^0 & s_{13}^0 \\
s_{11}^+ & s_{12}^+ & s_{13}^+ & s_{14}^+ \\
s_{12}^+ & s_{22}^+ & s_{23}^+ & s_{24}^+ \\
s_{13}^+ & s_{23}^+ & s_{33}^+ & s_{34}^+
\end{pmatrix}_i \sim [6^*, 2/3, -4/3, 2], \quad (i = 1, 2)
\]
where the numbered subscripts on the component scalars are the SU(3)$_L$ indices, whereas $i = 1, 2$ is that of $S_3$. The VEV of $s$ is set as $(⟨s_1⟩, ⟨s_2⟩)$ under $S_3$, with

$$
⟨s_i⟩ = \begin{pmatrix}
\lambda_i & 0 & v_i \\
0 & 0 & 0 \\
v_i & 0 & \Lambda_i
\end{pmatrix}, \quad (i = 1, 2).
$$

(3)

Following the potential minimization conditions, we have several VEV alignments. The first one is that $⟨s_1⟩ = ⟨s_2⟩$; then $S_3$ is broken into $Z_2$ consisting of the identity element and one transposition (out of the three) of $S_3$. The second one is that $⟨s_1⟩ \neq 0 = ⟨s_2⟩$ or $⟨s_1⟩ = 0 \neq ⟨s_2⟩$; then $S_3$ is broken into $Z_3$ as in the case of the charged lepton sector. The third one is that $⟨s_1⟩ \neq ⟨s_2⟩$; then $S_3$ is broken into the identity. In our previous work [13], we have argued that both breakings $S_3 → Z_2$ and $S_3 → Z_3$ must take place, and hence, to obtain a realistic neutrino spectrum, we additionally introduced a triplet ($\rho$) and an anti-sextet ($s$) that lie in $1'$ and $2$ under $S_3$. With these alignments, the number of Higgs multiplets required is eight. In this work, we propose that both the first and the third direction take place. The Yukawa interactions are

$$
- \mathcal{L}_\nu = \frac{x}{2}(\bar{\psi}_1 s)\psi_{1L} + \frac{y}{2}(\bar{\psi}_3 s)\psi_{3L} + h.c
$$

$$
= \frac{x}{2}\bar{\psi}_1^c(\psi_{2L}s_2 + \psi_{3L}s_1) + \frac{y}{2}(\bar{\psi}_2^c\psi_{2L}s_1 + \bar{\psi}_3^c\psi_{3L}s_2) + h.c,
$$

(4)

where the Yukawa coupling $x$ is that of lepton flavor changing interactions. The mass Lagrangian for the neutrinos is given by

$$
- \mathcal{L}_\nu^{\text{mass}} = \frac{1}{2}x(\lambda_2 \bar{\nu}_1^c \nu_2 + v_2 \bar{\nu}_1^c N_{2R}^c + v_2 \bar{N}_1^c \nu_2 + \Lambda_2 \bar{N}_1^c N_{2R}^c)
$$

$$+ \frac{1}{2}y(\lambda_1 \bar{\nu}_1^c \nu_3 + v_1 \bar{\nu}_1^c N_{3R}^c + v_1 \bar{N}_1^c \nu_3 + \Lambda_1 \bar{N}_1^c N_{3R}^c)
$$

$$+ \frac{1}{2}y(\lambda_1 \bar{\nu}_2^c \nu_2 + v_1 \bar{\nu}_2^c N_{2R}^c + v_1 \bar{N}_2^c \nu_2 + \Lambda_1 \bar{N}_2^c N_{2R}^c)
$$

$$+ \frac{1}{2}y(\lambda_2 \bar{\nu}_3^c \nu_3 + v_2 \bar{\nu}_3^c N_{3R}^c + v_2 \bar{N}_3^c \nu_3 + \Lambda_2 \bar{N}_3^c N_{3R}^c) + h.c.
$$

(5)

and also by

$$
- \mathcal{L}_\nu^{\text{mass}} = \frac{1}{2}\bar{\chi}_L^c M_\nu \chi_L + h.c., \quad \chi_L \equiv \begin{pmatrix}
\nu_L \\
N_{L}^c
\end{pmatrix}, \quad M_\nu \equiv \begin{pmatrix}
M_L & M_T^T \\
M_D & M_R
\end{pmatrix},
$$

(6)

where $\nu = (\nu_1, \nu_2, \nu_3)^T$ and $N = (N_1, N_2, N_3)^T$. The mass matrices are then obtained by

$$
M_{L,R,D} = \begin{pmatrix}
0 & a_{L,R,D} & b_{L,R,D} \\
a_{L,R,D} & c_{L,R,D} & 0 \\
b_{L,R,D} & 0 & d_{L,R,D}
\end{pmatrix},
$$

(7)
with
\[ a_L = \frac{x}{2} \lambda_s \equiv \frac{x}{2} \lambda_2, \quad a_D = \frac{x}{2} \nu_s = \frac{x}{2} \nu_2, \quad a_R = \frac{x}{2} \lambda_s \equiv \frac{x}{2} \lambda_2, \]
\[ b_L = \frac{x}{2} \lambda_1, \quad b_D = \frac{x}{2} \nu_1, \quad b_R = \frac{x}{2} \Lambda_1, \]
\[ c_L = y \lambda_1, \quad c_D = y \nu_1, \quad c_R = y \Lambda_1, \]
\[ d_L = y \nu_2, \quad d_D = y \nu_2, \quad d_R = y \Lambda_2. \quad (8) \]

In general, three active-neutrinos therefore gain masses via a combination of type I and type II seesaw mechanisms, derived from (6) and (7) as

\[ M_{\text{eff}} = M_L - M_D^T M_R^{-1} M_D = \begin{pmatrix} A & B_1 & B_2 \\ B_1 & C_1 & D \\ B_2 & D & C_2 \end{pmatrix}, \quad (9) \]

where
\[ A = \frac{(a_R b_D - a_D b_R)^2}{b_R^2 c_R + a_R^2 d_R}, \]
\[ B_1 = \frac{b_R [a_R^2 b_D c_D + a_L b_R c_R - a_D (b_R c_D + b_D c_R)] + a_R (a_L a_R - a_D^2) d_R}{b_R^2 c_R + a_R^2 d_R}, \]
\[ B_2 = \frac{a_D b_R (a_R b_D c_D + a_L b_R c_R - a_D (b_R c_D + b_D c_R)) + a_R (a_L a_R - a_D^2) d_R}{b_R^2 c_R + a_R^2 d_R}, \]
\[ C_1 = \frac{b_R^2 (c_L c_R - c_D^2) + (a_R^2 c_L + a_D^2 c_R - 2 a_D a_R c_D) d_R}{b_R^2 c_R + a_R^2 d_R}, \]
\[ C_2 = \frac{2 b_D b_R c_R d_D + b_R^2 c_R d_R + a_R^2 d_R d_R - a_D^2}{b_R^2 c_R + a_R^2 d_R}, \]
\[ D = \frac{(a_R c_D - a_D c_R) (b_R d_D - b_D d_R)}{b_R^2 c_R + a_R^2 d_R}. \quad (10) \]

The neutrino mass matrix in (9) is similar to the one in Ref. [13] but the broken symmetry directions are different. Indeed, in this model there are two broken symmetry directions as follows.

- If \( S_3 \) is broken to \( Z_2 \) (the subgroup \( Z_2 \) is unbroken), then we have \( A = D = 0, B_1 = B_2 \) and \( C_1 = C_2 \).

- If \( S_3 \to \{\text{Identity}\} \) (or, equivalently, \( Z_2 \to \{\text{Identity}\} \)), then we have \( A \neq 0, D \neq 0, B_1 \neq B_2 \) and \( C_1 \neq C_2 \), but \( A \) and \( D \) are close to zero, and \( B_1, B_2, C_1, C_2 \) kept close to each other in pairs. In this case, the disparity between \( \langle s_1 \rangle \) and \( \langle s_2 \rangle \) is very small and can be regarded as a small perturbation.

We next divide our considerations into two cases to fit the data: the first case is where only \( S_3 \) is broken to \( Z_2 \), and the second case is a combination of the both \( S_3 \to Z_2 \) and \( Z_2 \to \{\text{Identity}\} \).
A. Experimental constraints under $S_3 \to Z_2$

In the case $S_3 \to Z_2$, $\lambda_1 = \lambda_2 \equiv \lambda_s$, $v_1 = v_2 \equiv v_s$, $\Lambda_1 = \Lambda_2 \equiv \Lambda_s$, we have $A = D = 0$, $B_1 = B_2 \equiv B$, $C_1 = C_2 \equiv C$, and $M_{\text{eff}}$ in (9) reduces to

$$M_{\text{eff}} = \begin{pmatrix} 0 & B & B \\ B & C & 0 \\ B & 0 & C \end{pmatrix},$$

with

$$B = \left( \lambda_s - \frac{v^2_s}{\Lambda_s} \right) \frac{x}{2}, \quad C = \left( \lambda_s - \frac{v^2_s}{\Lambda_s} \right) y.$$  \hspace{1cm} (12)

We can diagonalize the matrix $M_{\text{eff}}$ in (11) as

$$U^T M_{\text{eff}} U = \text{diag}(m_1, m_2, m_3),$$

where

$$m_1 = \frac{1}{2} \left( C - \sqrt{C^2 + 8B^2} \right) = \left( \lambda_s - \frac{v^2_s}{\Lambda_s} \right) \frac{y + \sqrt{y^2 + 2x^2}}{2},$$

$$m_2 = \frac{1}{2} \left( C + \sqrt{C^2 + 8B^2} \right) = \left( \lambda_s - \frac{v^2_s}{\Lambda_s} \right) \frac{y - \sqrt{y^2 + 2x^2}}{2},$$

$$m_3 = C = \left( \lambda_s - \frac{v^2_s}{\Lambda_s} \right) y,$$

and the neutrino mixing matrix takes the form

$$U_0 = \begin{pmatrix} \frac{|K|}{\sqrt{|K|^2 + 2}} & 0 & -\frac{\sqrt{2}}{\sqrt{|K|^2 + 2}} \\ \frac{1}{\sqrt{|K|^2 + 2}} & \frac{1}{\sqrt{|K|^2 + 2}} & -\frac{|K|}{\sqrt{2}} \\ \frac{1}{\sqrt{|K|^2 + 2}} & \frac{1}{\sqrt{|K|^2 + 2}} & \frac{1}{\sqrt{2}} \end{pmatrix}, \quad K = -\frac{C + \sqrt{C^2 + 8B^2}}{2B} \hspace{1cm} (14)$$

We note that $m_1 m_2 = -2B^2$. This matrix can be parameterized by three Euler’s angles, which implies

$$\theta_{13} = 0, \quad \theta_{23} = \pi/4, \quad \tan \theta_{12} = \frac{\sqrt{2}}{|K|}.$$  \hspace{1cm} (15)

This case coincides with the data because $\sin^2(2\theta_{13}) < 0.15$ and $\sin^2(2\theta_{23}) > 0.92$. For the remaining constraints, taking the central values from the data,

$$\sin^2(2\theta_{12}) \approx 0.87, \quad (s^2_{12} = 0.32),$$

$$\Delta m^2_{21} = 7.59 \times 10^{-5} \text{ eV}^2, \quad \Delta m^2_{32} = 2.43 \times 10^{-3} \text{ eV}^2,$$
we have a solution

\[ m_1 = 0.0280284 \text{ eV}, \; m_2 = 0.0293347 \text{ eV}, \; m_3 = 0.0573631 \text{ eV}, \] (16)

and \( B = -0.0202757i \text{ eV}, \; C = 0.0573631 \text{ eV}, \; K = 1.44667, \; |x/y| = 0.707087. \) It follows that \( \tan \theta_{12} = 0.977565, \; (\theta_{12} \simeq 44.35^0), \) and the neutrino mixing matrix form is very close to that of the bi-maximal mixing pattern mentioned in Ref. [21]:

\[
U = \begin{pmatrix}
0.715083 & -0.69904 & 0 \\
0.494296 & 0.50564 & -\frac{1}{\sqrt{2}} \\
0.494296 & 0.50564 & \frac{1}{\sqrt{2}}
\end{pmatrix} \approx \begin{pmatrix}
\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\
\frac{1}{2} & \frac{1}{2} & -\frac{1}{\sqrt{2}} \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}}
\end{pmatrix}. \tag{17}
\]

Now, it is natural to choose \( \lambda_s, v_s^2/\Lambda_s \) in eV order, and suppose that \( \lambda_s > v_s^2/\Lambda_s \). We assume that \( \lambda_s - v_s^2/\Lambda_s = 0.1, \) we have \( x = -0.573631 \) and \( y = 0.399403i. \)

It was assumed in recent analyses that \( \theta_{13} \neq 0, \) but is small, as in Ref. [4]. If this is correct, then that case will fail. But the direction of the breakings \( S_3 \to \{\text{Identity}\} \) can improve this.

**B. Experimental constraints under \( S_3 \to \{\text{Identity}\} \)**

If both \( S_3 \to Z_2 \) and \( Z_2 \to \{\text{Identity}\} \) directions are realized, then \( \lambda_1 \neq \lambda_2 \equiv \lambda_s, \; v_1 \neq v_2 \equiv v_s \) and \( \Lambda_1 \neq \Lambda_2 \equiv \Lambda_s \) and, consequently \( A \approx 0, \; D_1 \approx 0, \; B_1 \approx B_2 \) and \( C_1 \approx C_2, \) and the general neutrino mass matrix in [9] can be rewritten in the form

\[
M_{\text{eff}} = \begin{pmatrix}
0 & B & B \\
B & C & 0 \\
B & 0 & C
\end{pmatrix} + \begin{pmatrix}
r_1 & p_1 & p_2 \\
p_1 & q_1 & r_2 \\
p_2 & r_2 & q_2
\end{pmatrix}. \tag{18}
\]

where \( B \) and \( C \) are given by (12), to match the case \( S_2 \to Z_2 \) as in (11). The last matrix in (18) is a deviation from the contribution due to the disparity of \( \langle s_1 \rangle \) and \( \langle s_2 \rangle, \) namely

\[
p_1 = B_1 - B, \; B_2 - B = p_1, \; C_1 - C = q_1, \; C_2 - C = q_2, \; r_1 = A, \; r_2 = D. \tag{19}\]

With the \( A, D \) and \( B_{1,2}, C_{1,2} \) defined in (10), it corresponds to \( S_3 \to \{\text{Identity}\}. \) Substituting (10) and (12) into (19) with the help of (8), we obtain

\[
r_1 = -\frac{(\Lambda_s v_1 - \Lambda_1 v_s)^2}{\Lambda_1^2 + \Lambda_s^2} x^2 = -\frac{\Lambda_1^2 \Lambda_s^2}{\Lambda_1^3 + \Lambda_s^3} \left( \frac{v_1}{\Lambda_1} - \frac{v_s}{\Lambda_s} \right)^2 \frac{x^2}{4y}, \tag{20}\]

\[
r_2 = -\frac{(\Lambda_s v_1 - \Lambda_1 v_s)^2 y}{\Lambda_1^2 + \Lambda_s^2} = -\frac{\Lambda_1^2 \Lambda_s^2}{\Lambda_1^3 + \Lambda_s^3} \left( \frac{v_1}{\Lambda_1} - \frac{v_s}{\Lambda_s} \right)^2 y. \tag{21}\]
Indeed, if $S_3 \to Z_2$, then the deviations $p_i$, $q_i$, and $r_i \ (i = 1, 2)$ vanish, and therefore the mass matrix $M_{\text{eff}}$ in (9) reduces to its first term coinciding with (11). The first term in (18) provides a bi-maximal mixing pattern with $\theta_{13} = 0$ as shown in Sect. IIIA. The others, proportional to $p_i$, $q_i$, $r_i$ due to contribution of the disparity of $\langle s_1 \rangle$ and $\langle s_2 \rangle$ take the role of perturbation for such a deviation of $\theta_{13}$. Hence, in this work we consider the disparity of $\langle s_1 \rangle$ and $\langle s_2 \rangle$ contribution as a small perturbation and truncate the theory at the first order.

In Ref. [18] we considered the case of $S_4 \to K_4$ breaking corresponding to $S_3 \to \{\text{Identity}\}$ with $\lambda_1 \neq \lambda_s$ but $v_1 = v_s$ and $\Lambda_1 = \Lambda_s$. Then $r_1 = r_2 = p_1 = q_2 = 0$, $p_2 = \frac{x}{2} q_1$, $q_1 = (\lambda_1 - \lambda_s) y \equiv \epsilon y$ with $\epsilon = \lambda_1 - \lambda_s$ being a small parameter that plays the role of a perturbation. In this paper, we consider the more general case, in which all elements of $\langle s_1 \rangle$ and $\langle s_2 \rangle$ are different from each other.

If $|\langle s_1 - s_2 \rangle| \ll |\langle s_1 \rangle - \langle s_2 \rangle|$ and $\frac{\alpha}{\Lambda_1} \sim \frac{\alpha}{\Lambda_s} \ll 1$, then we can evaluate $r_1$, $r_2$, $p_1$, $q_2 \ll 1$ which are of the second order in the perturbation and are therefore ignored. The remaining parameters $p_2$, $q_1$ are easily obtained as

$$p_2 = \alpha \frac{x}{2}, \quad q_1 = \alpha y,$$

where

$$\alpha = (\lambda_1 - \lambda_s) - \left( \frac{v_1^2 + 2 \frac{\Lambda_s}{\Lambda_1} v_1 v_s}{\Lambda_1 + \frac{\Lambda_s^2}{\Lambda_1} \Lambda_s} \right) + \left( \frac{\Lambda_s^2}{\Lambda_1} + \frac{\Lambda_s}{\Lambda_1} + \frac{\Lambda_s^2}{\Lambda_1} \right) \frac{v_s^2}{\Lambda_1 + \frac{\Lambda_s^2}{\Lambda_1} \Lambda_s}.$$ (27)

The matrix $M_{\text{eff}}$ in (18) thus reduces to

$$M_{\text{eff}} = \begin{pmatrix} 0 & B & B \\ B & C & 0 \\ B & 0 & C \end{pmatrix} + \alpha \begin{pmatrix} 0 & 0 & \frac{x}{2} \\ 0 & y & 0 \\ \frac{x}{2} & 0 & 0 \end{pmatrix} \equiv M_{\text{eff}}^0 + \alpha M^{(1)}.$$

(28)
Evaluating $\alpha$ shows that it is a small parameter, which can be regarded as a small perturbation. Within the perturbation theory up to the first order of $\alpha$, the physical neutrino masses are obtained as

$$m_1' = \lambda_1 = m_1 + \alpha \left( \frac{Kx + y}{K^2 + 2} \right),$$
$$m_2' = \lambda_2 = m_2 + \frac{\alpha K (Ky - 2x)}{2(K^2 + 2)},$$
$$m_3' = \lambda_3 = m_3 + \frac{\alpha y}{2},$$

where $m_{1,2,3}$ are the mass values as in the case $S_3 \to Z_2$ given by (10). For the corresponding perturbed eigenstates, we set

$$U \to U' = U + \Delta U,$$

where $U$ is defined by (14), and

$$\Delta U = \begin{pmatrix} \Delta U_{11} & \Delta U_{12} & \Delta U_{13} \\ \Delta U_{21} & \Delta U_{22} & \Delta U_{23} \\ \Delta U_{31} & \Delta U_{32} & \Delta U_{33} \end{pmatrix},$$

where

$$\Delta U_{11} = -\alpha \frac{(K^2 - 2)x + 2Ky}{2(K^2 + 2)^{3/2}(m_1 - m_2)},$$
$$\Delta U_{21} = -\alpha \frac{(Kx - 2y)}{4\sqrt{K^2 + 2}(m_1 - m_3)} + \alpha \frac{K[(K^2 - 2)x + 2Ky]}{4(K^2 + 2)^{3/2}(m_1 - m_2)},$$
$$\Delta U_{31} = \alpha \frac{(Kx - 2y)}{4\sqrt{K^2 + 2}(m_1 - m_3)} + \alpha \frac{K[(K^2 - 2)x + 2Ky]}{4(K^2 + 2)^{3/2}(m_1 - m_2)},$$
$$\Delta U_{12} = -\alpha \frac{K[[K^2 - 2)x + 2Ky]}{2\sqrt{2}(K^2 + 2)^{3/2}(m_1 - m_2)},$$
$$\Delta U_{22} = \frac{\alpha}{2\sqrt{2}} \frac{Ky + x}{\sqrt{K^2 + 2}(m_2 - m_3)} - \frac{\alpha}{2\sqrt{2}} \frac{(K^2 - 2)x + 2Ky}{(K^2 + 2)^{3/2}(m_1 - m_2)},$$
$$\Delta U_{32} = -\frac{\alpha}{2\sqrt{2}} \frac{Kx - 2y}{\sqrt{K^2 + 2}(m_1 - m_3)} - \frac{\alpha}{2\sqrt{2}} \frac{Ky + x}{(K^2 + 2)^{3/2}(m_1 - m_2)},$$
$$\Delta U_{13} = -\frac{\alpha}{2\sqrt{2}} \frac{Kx - 2y}{(K^2 + 2)(m_1 - m_3)} - \frac{\alpha}{2\sqrt{2}} \frac{Kx - 2y}{(K^2 + 2)(m_2 - m_3)},$$
$$\Delta U_{23} = \Delta U_{33} = \frac{\alpha}{2\sqrt{2}} \frac{Kx - 2y}{(K^2 + 2)(m_1 - m_3)} + \frac{\alpha}{2\sqrt{2}} \frac{Kx - 2y}{(K^2 + 2)(m_2 - m_3)}.$$ 

(31)

In this case, the lepton mixing matrix $U'$ can still be parameterized in terms of three new Euler’s angles $\theta'_{ij}$, which are also a perturbation from the $\theta_{ij}$ in the case 1, defined by

$$s_{13}' = -U_{13}' = \Delta U_{13} = -\frac{\alpha y}{2\sqrt{2}B},$$
\[ t'_{12} = -\frac{U'_{12}}{U'_{11}} = - \left[ 4\alpha B^2 Cx + \alpha C^2 (C + \sqrt{C^2 + 8B^2})x + 2BC(C + \sqrt{C^2 + 8B^2})(2C - \alpha y) \\
+ 8B^3(4C + 4\sqrt{C^2 + 8B^2} - \alpha y) \right] / \left\{ \sqrt{2} \left[ 64B^4 + 2C^3(C + \sqrt{C^2 + 8B^2}) \right] - \alpha BC(C + \sqrt{C^2 + 8B^2})x + 2B^2(12C^2 + 8C\sqrt{C^2 + 8B^2}) + \alpha Cy + \alpha y\sqrt{C^2 + 8B^2} \right\}, \]
\[ t'_{23} = -\frac{U'_{23}}{U'_{33}} = \frac{4B^2 + \alpha (Bx - Cy)}{4B^2 - \alpha (Bx - Cy)}. \]

It is easily shown that our model is consistent because the five experimental constraints on the mixing angles and squared mass differences of neutrinos can be respectively fitted with two Yukawa coupling parameters \( x, y \) of the antisextet scalar \( s \), if the VEVs are previously given. Indeed, taking the data in \(|1\) we obtain \( \alpha \simeq 0.0692, \ x \simeq 0.0728, \ y \simeq -0.1562, \) and \( B \simeq -0.0241, \ C = 0.022, \ K = 1.943, \) and \( t'_{23} = 0.9045 \ (\theta'_{23} \simeq 42.13^\circ, \ sin^2(2\theta'_{23}) = 0.98999 \) satisfying the condition \( sin^2(2\theta'_{23}) > 0.95 \) as in \(|1\)). The neutrino masses are explicitly given as \( m'_1 \simeq -0.02737 \text{eV}, \ m'_2 \simeq -0.02870 \text{eV} \) and \( m'_3 \simeq -0.05607 \text{eV} \), which are in a normal ordering. The neutrino mixing matrix then takes the form:

\[ U = \begin{pmatrix} 0.8251 & -0.5657 & -0.1585 \\
0.3302 & 0.6781 & -0.6716 \\
0.4697 & 0.4888 & 0.7426 \end{pmatrix}. \]  

(32)

IV. SCALAR POTENTIAL

To be complete, we write the scalar potentials of both the models mentioned. It is also noted that \((\text{Tr}A)(\text{Tr}B) = \text{Tr}(A\text{Tr}B)\) and \(V(X \to X', Y \to Y', \cdots) \equiv V(X,Y, \cdots)|_{X=X',Y=Y',\cdots} \). The general potential invariant under any subgroup takes the form

\[ V_{\text{total}} = V_{\text{tri}} + V_{\text{sext}} + V_{\text{tri-sext}}, \]  

(33)

where \( V_{\text{tri}} \) comes from only contributions of SU(3)\(_L\) triplets given as a sum of the following terms:

\[
V(\chi) = \mu_\chi^2 \chi^\dagger \chi + \lambda^\chi (\chi^\dagger \chi)^2, \\
V(\phi) = V(\chi \to \phi), \ V(\phi') = V(\chi \to \phi'), \ V(\eta) = V(\chi \to \eta), \ V(\eta') = V(\chi \to \eta'), \\
V(\phi, \chi) = \lambda_{\phi \chi}^\phi (\phi^\dagger \phi)(\chi^\dagger \chi) + \lambda_{\phi \chi}^\phi (\phi^\dagger \phi)(\chi^\dagger \phi), \\
V(\phi', \chi) = V(\phi \to \phi', \chi), \ V(\chi, \eta) = V(\chi, \phi \to \eta), \ V(\chi, \eta') = V(\chi, \phi \to \eta'), \\
V(\phi, \phi') = V(\phi, \chi \to \phi') + \lambda_{\phi \phi'}^\phi (\phi^\dagger \phi')(\phi^\dagger \phi') + \lambda_{\phi \phi'}^\phi (\phi^\dagger \phi')(\phi^\dagger \phi), \\
V(\phi, \eta) = V(\phi, \chi \to \eta), \ V(\phi, \eta') = V(\phi, \chi \to \eta'), \\
V(\phi, \eta) = V(\phi, \chi \to \eta), \ V(\phi, \eta') = V(\phi, \chi \to \eta'), \\
V(\phi, \eta) = V(\phi, \chi \to \eta), \ V(\phi, \eta') = V(\phi, \chi \to \eta').
\]
\[ V(\phi', \eta) = V(\phi \to \phi', \chi \to \eta), \quad V(\eta', \eta') = V(\phi \to \phi', \chi \to \eta') \]
\[ V(\eta, \eta') = V(\phi \to \eta, \chi \to \eta') + \lambda_{\eta \eta'}^0 (\eta^t \eta') (\eta^t \eta') + \lambda_{\eta \eta'}^1 (\eta^t \eta) (\eta^t \eta), \]
\[ V_{\chi \phi \eta \eta'} = \mu_1 \chi \eta + \mu_1' \chi \phi \eta' + \lambda_1^2 (\phi^t \phi) (\eta^t \eta) + \lambda_1^3 (\phi^t \eta) (\eta^t \phi') + \lambda_1^4 (\phi^t \eta') (\eta^t \phi') + h.c. \quad (36) \]

The \( V_{\text{sex}} \) is given by only \( V(s) \):
\[ V(s) = \mu^2 \chi \eta + \lambda_1^0 \chi \eta + \lambda_2^0 (\phi^t s^t) (s^t \phi). \]
\[ V(\phi^t, s) = V(\phi \to \phi^t, s), \quad V(\chi, s) = V(\phi \to \chi, s), \]
\[ V(\eta, s) = V(\phi \to \eta, s), \quad V(\eta', s) = V(\phi \to \eta', s), \]
\[ V_{s\chi \phi \eta \eta'} = (\lambda_1^0 \phi^t \phi' + \lambda_2^0 \eta^t \eta') \chi \eta^t \eta + \lambda_1^0 (\phi^t s^t) (s^t \phi), \]
\[ = h.c. \]

Next, \( V_{\text{trip}} \) is a sum of all the terms connecting both the sectors:
\[ V(\phi, s) = \lambda_1^3 \phi (\phi^t \phi) \chi \eta^t \eta + \lambda_2^3 (\phi^t s^t) (s^t \phi), \]
\[ V(\phi^t, s) = V(\phi \to \phi^t, s), \quad V(\chi, s) = V(\phi \to \chi, s), \]
\[ V(\eta, s) = V(\phi \to \eta, s), \quad V(\eta', s) = V(\phi \to \eta', s), \]
\[ V_{s\chi \phi \eta \eta'} = (\lambda_1^0 \phi^t \phi' + \lambda_2^0 \eta^t \eta') \chi \eta^t \eta + \lambda_1^0 (\phi^t s^t) (s^t \phi), \]
\[ = h.c. \]

To provide the Majorana masses for the neutrinos, the lepton number must be broken. This can be achieved via the scalar potential violating \( U(1)_L \), but the other symmetries should be conserved.

The \( \mathcal{L} \) violating potential is given by
\[ \mathcal{V} = [\lambda_1 \chi \eta^t \chi + \lambda_2 \eta^t \eta + \lambda_3 \eta^t \eta + \lambda_4 \eta^t \eta' + \lambda_5 \eta^t \eta' + \lambda_6 \eta^t \eta + \lambda_7 \phi^t \phi + \lambda_8 \phi^t \phi + \lambda_9 \phi^t \phi + \lambda_{10} \phi^t \phi + \chi + \lambda_{11} \chi \eta^t \chi + \lambda_{12} \eta^t \eta + \lambda_{13} \eta^t \eta + \lambda_{14} \eta^t \eta + \lambda_{15} \eta^t \eta + \lambda_{16} \eta^t \eta + \lambda_{17} \phi^t \phi + \lambda_{18} \phi^t \phi + \lambda_{19} \phi^t \phi + \lambda_{20} \phi^t \phi + \chi \eta^t \eta + \lambda_{21} (\eta^t \phi) (\phi^t \chi) + \lambda_{22} (\eta^t \phi) (\phi^t \chi) + \lambda_{23} (\eta^t \phi) (\phi^t \chi) + \lambda_{24} (\eta^t \phi) (\phi^t \chi) + \lambda_{25} (\eta^t \phi) (\phi^t \chi) + \lambda_{26} (\phi^t \chi) + \chi \eta^t \eta + h.c. \quad (38) \]

We have not pointed it out, but there must additionally exist the terms in \( \mathcal{V} \) explicitly violating the only \( S_3 \) symmetry or both the \( S_3 \) and \( \mathcal{L} \)-charge. In what follows, most of them will be omitted, only the terms of interest to us are provided.

We now consider the potential \( V_{\text{trip}} \). The flavons \( \chi, \phi, \phi', \eta, \eta' \) with their VEVs aligned in the same direction (all of them are singlets) are a mathematical solution of the minimization conditions for \( V_{\text{trip}} \). To explicitly see this, in the system of equations for minimization, we set \( v^* = v, v'^* = v', u^* = u, u'^* = u' \), and \( v^*_\chi = v_\chi \). Then the potential minimization conditions for triplets reduces to
\[ \frac{\partial V_{\text{trip}}}{\partial \omega} = 4 \lambda^1 \omega^3 + 2 \left( \mu^2 + \lambda_1 u^2 + \lambda_2 u'^2 + \lambda_3 v^2 + \lambda_4 v'^2 \right) \omega - \mu_1 u v - \mu_1' u' v' = 0, \quad (39) \]
\[
\frac{\partial V_{\text{tri}}}{\partial v} = 4\lambda^\phi v^3 + 2 \left[ \mu_2^2 + \lambda_1^{\phi\eta} u^2 + \lambda_1^{\phi\phi'} u'^2 + (\lambda_1^{\phi\phi'} + \lambda_2^{\phi\phi'} + \lambda_3^{\phi\phi'} + \lambda_4^{\phi\phi'}) v^2 + \omega^2 \lambda_1^{\phi\eta} \right] v \\
+ (\lambda_1^2 + \lambda_2^2) u u' v' - \mu_1 \omega_1 u = 0, \\
\frac{\partial V_{\text{tri}}}{\partial v'} = 4\lambda^\phi v'^3 + 2 \left[ \mu_2^2 + \lambda_1^{\phi\eta} u^2 + \lambda_1^{\phi\phi'} u'^2 + (\lambda_1^{\phi\phi'} + \lambda_2^{\phi\phi'} + \lambda_3^{\phi\phi'} + \lambda_4^{\phi\phi'}) v^2 + \omega^2 \lambda_1^{\phi\eta} \right] v' \\
+ (\lambda_1^2 + \lambda_2^2) u u' v - \mu_1 \omega_1 u' = 0, \\
\frac{\partial V_{\text{tri}}}{\partial u} = 4\lambda^\eta u^3 + 2 \left[ \mu_2^2 + (\lambda_1^{\eta\eta} + \lambda_2^{\eta\phi} + \lambda_3^{\eta\phi'} + \lambda_4^{\eta\phi'}) u'^2 + \lambda_1^{\phi\eta} v^2 + \lambda_1^{\phi\eta'} v'^2 + \omega^2 \lambda_1^{\phi\eta} \right] u \\
+ (\lambda_1^2 + \lambda_2^2) u u' v' - \mu_1 \omega_1 v = 0, \\
\frac{\partial V_{\text{tri}}}{\partial u'} = 4\lambda^\eta u'^3 + 2 \left[ \mu_2^2 + (\lambda_1^{\eta\eta} + \lambda_2^{\eta\phi} + \lambda_3^{\eta\phi'} + \lambda_4^{\eta\phi'}) u'^2 + \lambda_1^{\phi\eta} v^2 + \lambda_1^{\phi\eta'} v'^2 + \omega^2 \lambda_1^{\phi\eta} \right] u' \\
+ (\lambda_1^2 + \lambda_2^2) u u' v' - \mu_1 \omega_1 v' = 0.
\]

It is easily to see that the derivatives of \( V_{\text{tri}} \) with respect to the variables \( u, u', v, v' \) shown in (40), (41), (42), and (43) are symmetric with respect to one another. System of equations (39) - (43) always has the solution \((u, v, u', v')\) as expected, even though it is complicated. We also note that the above alignment is only one of the conditions to be imposed to have the desirable results. We have evaluated that Eqs. (40) - (44) have the same structure of solutions. Consequently, to have a simple solution, we can assume that \( u = u' = v = v' \). In this case, Eqs. (40) - (43) reduce to a single equation, and system of equations (39) - (43) becomes

\[
\frac{\partial V_{\text{tri}}}{\partial \omega} = 4\lambda^\chi \omega^3 + 2\omega [\mu_2^2 + (2\lambda_1^{\chi\eta} + 2\lambda_1^{\chi\phi}) v^2] - 2\mu_1 v^2 = 0, \\
\frac{\partial V_{\text{tri}}}{\partial v} = 2\omega [2\omega^2 (\lambda_1^{\chi\eta} + \lambda_1^{\chi\phi}) + 2(\mu_2^2 + \mu_3^2) + 2 \left( \lambda_1^{\chi\eta} + \lambda_1^{\chi\phi} + 4\lambda_2^{\phi\eta} + \lambda_3^{\phi\eta} + \lambda_4^{\phi\eta} \right) v^2 + \omega^2 \lambda_1^{\phi\eta} \right] v^2 - 2\mu_1 \omega = 0.
\]

This system has the solution

\[
u = u' = v = v = \pm \sqrt{\omega (\mu_2^2 + \lambda^\chi \omega^2) / \sqrt{\mu_1 - 2\omega (\lambda_1^{\chi\eta} + \lambda_1^{\chi\phi})}},
\]

\[
\omega = \frac{\alpha \mu_1}{2(\alpha^2 - \beta \lambda^\chi)} - \frac{\Omega}{3 \times 2^{2/3}(\alpha^2 - \beta \lambda^\chi) \left( \Gamma + \sqrt{\Gamma^2 + 4\Omega^2} \right)^{1/3}} + \frac{\Gamma + \sqrt{\Gamma^2 + 4\Omega^2}}{6 \times 2^{1/3}(\alpha^2 - \beta \lambda^\chi)^{1/3}},
\]

where

\[
\Gamma = 54\alpha \beta \mu_1 (\lambda \mu_1^2 + \alpha^2 \mu_2^2 - \beta \lambda \mu_3^2) - 108 \lambda \mu_1 \beta \gamma (\alpha^2 - \lambda \beta), \\
\Omega = 6(\alpha^2 - \beta \lambda^\chi)(2\alpha \gamma + \mu_2^2 - \beta \mu_3^2) - 9\alpha^2 \mu_1^2, \\
\alpha = \lambda_1^{\chi\eta} + \lambda_1^{\chi\phi}, \\
\beta = \lambda_1^{\phi\eta} + 4\lambda_1^{\phi\phi} + \lambda_2^{\phi\phi} + \lambda_3^{\phi\phi} + \lambda_4^{\phi\phi} + 2(\lambda_1^{\eta} + \lambda_1^{\phi}), \\
\lambda^{\phi\phi'} = \lambda_1^{\phi\phi'} + \lambda_2^{\phi\phi'} + \lambda_3^{\phi\phi'} + \lambda_4^{\phi\phi'}, \\
\lambda^{\eta\eta'} = \lambda_1^{\eta\eta'} + \lambda_2^{\eta\eta'} + \lambda_3^{\eta\eta'} + \lambda_4^{\eta\eta'}. 
\]
We next consider the potential $V_{s sext}$ and $V_{tri-sext}$. By imposing the conditions

$$
\lambda_1^* = \lambda_1, \lambda_2^* = \lambda_2, v_1^* = v_1, v_2^* = v_2, \Lambda_1^* = \Lambda_1, \Lambda_2^* = \Lambda_2,
$$

$$
v^* = v, v'^* = v', u^* = u, u'^* = u', v_\chi^* = v_\chi, v_\rho^* = v_\rho,
$$

we obtain a system of equations of the potential minimization for anti-sextets:

$$
\begin{align}
\frac{\partial V_1}{\partial \lambda_1} &= 2 \left\{ \lambda_2 \left[ \lambda_1^s \omega^2 + \mu_2^2 + (\lambda_1^s + \lambda_2^s + \lambda_3^s)u^2 + (\lambda_2^s + \lambda_1^s)uv' + (\lambda_1^s + \lambda_2^s + \lambda_3^s)u'^2 + \lambda_1^s v v' \\
&+ \lambda_1^{s,\prime} v^2 + \lambda_1^{s,\prime} v'^2 + 4 \lambda_2^s \lambda_1 \Lambda_2 + 2(3 \lambda_1^s + \lambda_2^s + \lambda_3^s + 4 \lambda_4^s) v_1 v_2 \right] + 2 \Lambda_2 (\lambda_1^s - \lambda_2^s + \lambda_3^s) v_1 v_2 \\
+ 2 \Lambda_4 (\lambda_1^s + \lambda_2^s) v_1^2 + 2 \lambda_1 \left[ \lambda_1^s \Lambda_2 + \lambda_2^s (2 \lambda_1^s + \lambda_3^s + 2 \lambda_4^s + \lambda_6^s) + (\lambda_1^s - \lambda_2^s + \lambda_3^s + 2 \lambda_6^s) v_1^2 \right] \right\},
\end{align}
$$

$$
\begin{align}
\frac{\partial V_1}{\partial \lambda_2} &= 2 \left\{ \lambda_1 \left[ \lambda_1^s \omega^2 + \mu_2^2 + (\lambda_1^s + \lambda_2^s + \lambda_3^s)u^2 + (\lambda_2^s + \lambda_1^s)uv' + (\lambda_1^s + \lambda_2^s + \lambda_3^s)u'^2 + \lambda_1^s v v' \\
&+ \lambda_1^{s,\prime} v^2 + \lambda_1^{s,\prime} v'^2 + 4 \lambda_2^s \lambda_1 \Lambda_2 + 2(3 \lambda_1^s + \lambda_2^s + \lambda_3^s + 4 \lambda_4^s) v_1 v_2 \right] + 2 \Lambda_1 (\lambda_1^s - \lambda_2^s + \lambda_3^s) v_1 v_2 \\
+ 2 \Lambda_2 (\lambda_1^s + \lambda_2^s) v_2^2 + 2 \lambda_2 \left[ \lambda_1^s \Lambda_2 + \lambda_2^s (2 \lambda_1^s + \lambda_3^s + 2 \lambda_4^s + \lambda_6^s) + (\lambda_1^s - \lambda_2^s + \lambda_3^s + 2 \lambda_6^s) v_1^2 \right] \right\},
\end{align}
$$

$$
\begin{align}
\frac{\partial V_1}{\partial v_1} &= 2 \left\{ v_2 \left[ (2 \lambda_1^s + \lambda_2^s + \lambda_3^s) \omega^2 + 2 \mu_2^2 + (2 \lambda_1^s + \lambda_2^s + \lambda_3^s) u^2 + (2 \lambda_2^s + \lambda_1^s) uu' \\
&+ (2 \lambda_1^s + \lambda_2^s + \lambda_3^s) u'^2 + 2 \lambda_1^s v v' + 2 \lambda_1^s v'^2 + 2(\lambda_1 \lambda_2 + \lambda_2 \Lambda_1) (\lambda_1^s - \lambda_2^s + \lambda_3^s) \\
+ 2(\lambda_1 \lambda_2 + \lambda_1 \Lambda_2) (3 \lambda_1^s + \lambda_2^s + \lambda_3^s + 4 \lambda_4^s) \right] + 2 \left[ 2 \lambda_2 A_2 (\lambda_1^s + \lambda_3^s) + (\lambda_2 + \lambda_2^s) (\lambda_1^s - \lambda_2^s) \\
+ \lambda_2^s + 2 \lambda_6^s) v_1 + 4(2 \lambda_1^s + \lambda_3^s + 4 \lambda_2^s + 2 \lambda_6^s) v_1 v_2 \right] \right\},
\end{align}
$$

$$
\begin{align}
\frac{\partial V_1}{\partial v_2} &= 2 \left\{ v_1 \left[ (2 \lambda_1^s + \lambda_2^s + \lambda_3^s) \omega^2 + 2 \mu_2^2 + (2 \lambda_1^s + \lambda_2^s + \lambda_3^s) u^2 + (2 \lambda_2^s + \lambda_1^s) uu' \\
&+ (2 \lambda_1^s + \lambda_2^s + \lambda_3^s) u'^2 + 2 \lambda_1^s v v' + 2 \lambda_1^s v'^2 + 2(\lambda_1 \lambda_2 + \lambda_2 \Lambda_1) (\lambda_1^s - \lambda_2^s + \lambda_3^s) \\
+ 2(\lambda_1 \lambda_2 + \lambda_1 \Lambda_2) (3 \lambda_1^s + \lambda_2^s + \lambda_3^s + 4 \lambda_4^s) \right] + 2 \left[ 2 \lambda_1 A_1 (\lambda_1^s + \lambda_3^s) + (\lambda_2 + \lambda_2^s) (\lambda_1^s - \lambda_2^s) \\
+ \lambda_2^s + 2 \lambda_6^s) v_2 + 4(2 \lambda_1^s + \lambda_3^s + 4 \lambda_2^s + 2 \lambda_6^s) v_1 v_2 \right] \right\},
\end{align}
$$

$$
\begin{align}
\frac{\partial V_1}{\partial \Lambda_1} &= 2 \left\{ \lambda_2 \left[ (\lambda_1^s + \lambda_2^s + \lambda_3^s) \omega^2 + 2 \mu_2^2 + \lambda_1^{s,\prime} u^2 + \lambda_2^{s,\prime} u'^2 + \lambda_1^{s,\prime} v v' + \lambda_1^{s,\prime} v'^2 + \lambda_1^{s,\prime} v'^2 \\
&+ 4 \lambda_2^s \lambda_1 \Lambda_2 + 2(3 \lambda_1^s + \lambda_2^s + \lambda_3^s + 4 \lambda_4^s) v_1 v_2 \right] + 2 \Lambda_2 (\lambda_1^s - \lambda_2^s + \lambda_3^s) v_1 v_2 + 2 \lambda_1 (\lambda_1^s + \lambda_2^s) v_2^2 \\
+ 2 \Lambda_4 \left[ \lambda_1^s \Lambda_2 + \lambda_2^s (2 \lambda_1^s + \lambda_3^s + 2 \lambda_4^s + \lambda_6^s) + (\lambda_1^s - \lambda_2^s + \lambda_3^s + 2 \lambda_6^s) v_2^2 \right] \right\},
\end{align}
$$

$$
\begin{align}
\frac{\partial V_1}{\partial \Lambda_2} &= 2 \left\{ \lambda_1 \left[ (\lambda_1^s + \lambda_2^s + \lambda_3^s) \omega^2 + 2 \mu_2^2 + \lambda_1^{s,\prime} u^2 + \lambda_2^{s,\prime} u'^2 + \lambda_1^{s,\prime} v v' + \lambda_1^{s,\prime} v'^2 + \lambda_1^{s,\prime} v'^2 \\
&+ 4 \lambda_1^s \lambda_1 \Lambda_2 + 2(3 \lambda_1^s + \lambda_2^s + \lambda_3^s + 4 \lambda_4^s) v_1 v_2 \right] + 2 \Lambda_1 (\lambda_1^s - \lambda_2^s + \lambda_3^s) v_1 v_2 + 2 \lambda_2 (\lambda_1^s + \lambda_2^s) v_1^2 \\
+ 2 \Lambda_2 \left[ \lambda_1^s \lambda_2^s + \lambda_1^s (2 \lambda_1^s + \lambda_3^s + 2 \lambda_4^s + \lambda_6^s) + (\lambda_1^s - \lambda_2^s + \lambda_3^s + 2 \lambda_6^s) v_1^2 \right] \right\},
\end{align}
$$
\[ V_1 = V_{\text{sext}} + V_{\text{tri-sext}} \]

It is easily to see that Eqs. (52) - (56) take the same form pairwise. This system of equations yields the relations

\[ \lambda_1 = \kappa \lambda_2, \quad v_1 = \kappa v_2, \quad \Lambda_1 = \kappa \Lambda_2, \quad (57) \]

with \( \kappa \) is a constant. It means that there are several alignments for VEVs. In this paper, to obtain the desired results, we impose the two directions for breaking \( S_3 \to Z_2 \) and \( Z_2 \to \{\text{Identity}\} \) as mentioned, in which \( \kappa = 1 \) and \( \kappa \neq 1 \) but approximates to the unit. In the case where \( \kappa = 1 \) or \( \lambda_1 = \lambda_2 = \lambda_s, \quad v_1 = v_2 = v_s \) and \( \Lambda_1 = \Lambda_2 = \Lambda_s \), system of equations (51) - (56) reduces to a system for the potential minimal consisting of three equations:

\[
\begin{align*}
\lambda_s \left[ A_\omega + \mu_s^2 + 2A_s \lambda_s^2 + 2(A_s + B_s) \lambda_s^2 + A_v + 4(A_s + B_s) v_s^2 \right] + 2B_s A_s v_s^2 &= 0, \\
2(A_\omega + B_\omega) + 2 \mu_s^2 + A_v + A'_v + 4B_s \lambda_s A_s + 4(A_s + B_s)(\lambda_s^2 + v_s^2 + \Lambda_s^2) &= 0, \\
\Lambda_s \left[ A_\omega + B_\omega + \mu_s^2 + 2A_s \lambda_s^2 + 2(A_s + B_s) \Lambda_s^2 + A'_v + 4(A_s + B_s) v_s^2 \right] + 2B_s \lambda_s v_s^2 &= 0,
\end{align*}
\]

where

\[
\begin{align*}
A_\omega &= \lambda_1^s \omega^2, \quad B_\omega = (\lambda_2^s + \lambda_3^s) \omega^2, \quad A_s = 2\lambda_4^s + \lambda_6^s, \quad B_s = 2\lambda_1^s + \lambda_3^s, \\
A_v &= (\lambda_1^s + \lambda_2^s + \lambda_3^s + \lambda_4^s + \lambda_5^s + \lambda_6^s + \lambda_7^s + \lambda_8^s + \lambda_9^s + \lambda_1^s + \lambda_2^s + \lambda_3^s + \lambda_4^s) v^2, \\
A'_v &= (\lambda_1^s + \lambda_2^s + \lambda_3^s + \lambda_4^s + \lambda_5^s + \lambda_6^s + \lambda_7^s + \lambda_8^s + \lambda_9^s + \lambda_1^s + \lambda_2^s + \lambda_3^s + \lambda_4^s) v^2.
\end{align*}
\]

System of equations (58) - (60) always has the solution \((\lambda_s, v_s, \Lambda_s)\) as expected, even though it is complicated. We also note that the above alignment is only one of the conditions to be imposed to have the desired results.

**V. GAUGE BOSONS**

The covariant derivative of the triplet is given by

\[ D_\mu = \partial_\mu - ig \frac{\lambda_a}{2} W_{\mu a} - ig_X X \frac{\lambda_a}{2} B_\mu = \partial_\mu - iP_\mu, \quad (61) \]

where \( \lambda_a = \sqrt{\frac{2}{3}} \text{diag}(1, 1, 1) \) and \( \lambda_a (a = 1, 2, \ldots, 8) \) are Gell-Mann matrices that satisfy the relations \( \text{Tr} \lambda_a \lambda_b = 2\delta_{ab} \) and \( \text{Tr} \lambda_a \lambda_9 = 2 \), and \( X \) is \( U(1)_X \) -charge of Higgs triplets.
We can rewrite $P_\mu$ in a convenient form as follows:

\[
\frac{g}{2} \begin{pmatrix}
W_{\mu3} + \frac{W_{\mu8}}{\sqrt{3}} + t \sqrt{\frac{2}{3}} XB_\mu & \sqrt{2} W^+_{\mu} & \sqrt{2} X^0_\mu \\
\sqrt{2} W^0_\mu & W_{\mu3} + \frac{W_{\mu8}}{\sqrt{3}} + t \sqrt{\frac{2}{3}} XB_\mu & \sqrt{2} Y^{+}_\mu \\
\sqrt{2} X^*_\mu & \sqrt{2} Y^{-}_\mu & -2^\frac{2}{\sqrt{3}} W_{\mu8} + t \sqrt{\frac{2}{3}} XB_\mu
\end{pmatrix},
\]  

(62)

where we set

\[
W^{+}_\mu = \frac{W_{\mu1} - i W_{\mu2}}{\sqrt{2}}, \quad X^0_\mu = \frac{W_{\mu4} - i W_{\mu5}}{\sqrt{2}},
\]

\[
Y^{-}_\mu = \frac{W_{\mu6} - i W_{\mu7}}{\sqrt{2}}, \quad W^+_\mu = (W^{+}_\mu)^*, \quad Y^{-}_\mu = (Y^{-}_\mu)^*.
\]  

(63)

and $t = g_X/g$. We note that $W_4$ and $W_5$ are respectively purely real and imaginary parts of $X^0$ and $X^{0*}$. The covariant derivative for the antisextet with a VEV part is \([22,23]\)

\[
D_\mu (s_i) = \frac{ig}{2} \{ W^a_\mu \lambda^a_i (s_i) + \langle s_i \rangle W^a_\mu \lambda^a_i \} - ig X T_0 X B_\mu (s_i).
\]  

(64)

The covariant derivative (64) acting on the antisextet VEVs is given by

\[
[D_\mu (s_i)]_{11} = ig \left( \lambda_i W_{\mu3} + \frac{\lambda_i}{\sqrt{3}} W_{\mu8} + \sqrt{\frac{2}{3}} t \lambda_i B_\mu + \sqrt{2} v_i X^{0*} \right),
\]

\[
[D_\mu (s_i)]_{12} = ig \left( \frac{\lambda_i}{\sqrt{2}} W^0_{\mu} + v_i Y^{+}_\mu \right),
\]

\[
[D_\mu (s_i)]_{13} = ig \left( v_i W_{\mu3} - \frac{v_i}{\sqrt{3}} W_{\mu8} + \frac{2}{3} \sqrt{\frac{2}{3}} t v_i B_\mu + \sqrt{2} \lambda_i X^0_\mu + \sqrt{2} \Lambda_i X^{0*} \right),
\]

\[
[D_\mu (s_i)]_{21} = [D_\mu (s_i)]_{12},
\]

\[
[D_\mu (s_i)]_{22} = 0,
\]

\[
[D_\mu (s_i)]_{23} = ig \left( v_i W^0_{\mu} + \Lambda_i Y^{+}_\mu \right),
\]

\[
[D_\mu (s_i)]_{31} = [D_\mu (s_i)]_{13},
\]

\[
[D_\mu (s_i)]_{32} = [D_\mu (s_i)]_{23},
\]

\[
[D_\mu (s_i)]_{33} = ig \left( -\frac{2}{\sqrt{3}} \Lambda_i W_{\mu8} + \sqrt{\frac{2}{3}} t \Lambda_i B_\mu + \sqrt{2} v_i X^0_\mu \right).
\]

The masses of gauge bosons in this model are defined as

\[
\mathcal{L}_{mass}^{GB} = (D_\mu (\phi))^+ (D^\mu (\phi)) + (D_\mu (\phi'))^+ (D^\mu (\phi')) + (D_\mu (\chi))^+ (D^\mu (\chi))
\]

\[
+ (D_\mu (\eta))^+ (D^\mu (\eta)) + (D_\mu (\eta'))^+ (D^\mu (\eta'))
\]

\[
+ \text{Tr}[(D_\mu (s_1))^+ (D^\mu (s_1))] + \text{Tr}[(D_\mu (s_2))^+ (D^\mu (s_2))].
\]  

(65)

Substituting the Higgs VEVs of the model in (63) yields

\[
\mathcal{L}_{mass}^{GB} = \frac{v^2}{324} \left[ 81g^2 (W^2_{\mu1} + W^2_{\mu2}) + 81g^2 (W^2_{\mu6} + W^2_{\mu7}) + (-9g W_{\mu3} + 3 \sqrt{3} g W_{\mu8} + 2 \sqrt{6} g_X B_{\mu})^2 \right]
\]
\[
\frac{v^2}{324} \left[ 81g^2(W_{\mu 1}^2 + W_{\mu 2}^2) + 81g^2(W_{\mu 6}^2 + W_{\mu 7}^2) + (-9gW_{\mu 3} + 3\sqrt{3}gW_{\mu 8} + 2\sqrt{6}g_B X B_{\mu})^2 \right] \\
+ \frac{\omega^2}{108} \left[ 27g^2(W_{\mu 1}^2 + W_{\mu 5}^2) + 27g^2(W_{\mu 6}^2 + W_{\mu 7}^2) + 36g^2W_{\mu 8}^2 + 12\sqrt{2}gg_B W_{\mu 8} B_{\mu} + 2g^2_B B_{\mu}^2 \right] \\
+ \frac{u^2}{324} \left[ 81g^2(W_{\mu 1}^2 + W_{\mu 2}^2) + 81g^2(W_{\mu 4}^2 + W_{\mu 5}^2) + (-9gW_{\mu 3} - 3\sqrt{3}gW_{\mu 8} + \sqrt{6}g_B B_{\mu})^2 \right] \\
+ \frac{u^2}{324} \left[ 81g^2(W_{\mu 1}^2 + W_{\mu 2}^2) + 81g^2(W_{\mu 4}^2 + W_{\mu 5}^2) + (-9gW_{\mu 3} - 3\sqrt{3}gW_{\mu 8} + \sqrt{6}g_B B_{\mu})^2 \right] \\
+ \frac{g^2}{6} \left[ 2(\Lambda_1 v_1 + \Lambda_2 v_2) \left( 3W_{\mu 3}W_{\mu 4} + 3W_{\mu 4}W_{\mu 6} - 3W_{\mu 2}W_{\mu 7} - 5\sqrt{3}W_{\mu 4}W_{\mu 8} \right) \\
+ 3(v_1^2 + v_2^2 + \lambda_1^2 + \lambda_2^2)W_{\mu 1}^2 + 3(v_1^2 + v_2^2 + \lambda_1^2 + \lambda_2^2)W_{\mu 2}^2 + 3(v_1^2 + v_2^2 + 2\lambda_1^2 + 2\lambda_2^2)W_{\mu 3}^2 \\
+ 3(4v_1^2 + 4v_2^2 + \lambda_1^2 + \lambda_2^2 + \Lambda_1^2 + \Lambda_2^2 + 2\Lambda_1\Lambda_1 + 2\Lambda_2\Lambda_2)W_{\mu 4}^2 \\
+ 3 \left( 4v_1^2 + 4v_2^2 + \lambda_1^2 + \lambda_2^2 + \Lambda_1^2 + \Lambda_2^2 - 2\Lambda_1\Lambda_1 - 2\Lambda_2\Lambda_2 \right) W_{\mu 5}^2 \\
+ 3(v_1^2 + v_2^2 + \lambda_1^2 + \lambda_2^2 + \Lambda_1^2 + \Lambda_2^2)W_{\mu 6}^2 + 3(v_1^2 + v_2^2 + \lambda_1^2 + \lambda_2^2)W_{\mu 7}^2 \\
+ 2\sqrt{3}(-v_1^2 - v_2^2 + 2\lambda_1^2 + 2\lambda_2^2)W_{\mu 8}^2 + (v_1^2 + v_2^2 + 2\lambda_1^2 + 2\lambda_2^2 + 8\Lambda_1^2 + 8\Lambda_2^2)W_{\mu 8}^2 \\
+ 18(\lambda_1 v_1 + \lambda_2 v_2)W_{\mu 3}W_{\mu 4} + 6(\lambda_1 v_1 + \lambda_2 v_2)W_{\mu 4}W_{\mu 6} - 6(\lambda_1 v_1 + \lambda_2 v_2)W_{\mu 2}W_{\mu 7} \\
+ 2\sqrt{3}(\lambda_1 v_1 + \lambda_2 v_2)W_{\mu 4}W_{\mu 8} \right] \\
+ \frac{2}{27}t^2 g^2 (\lambda_1^2 + \lambda_2^2 + \lambda_1^2 + \lambda_2^2 + 2v_1^2 + 2v_2^2) B_{\mu}^2 - \frac{2}{6} \sqrt{3}tg^2 (\lambda_1^2 + \lambda_2^2 + v_1^2 + v_2^2) W_{\mu 3} B_{\mu} \\
- \frac{4}{3} \sqrt{2}tg^2 [(\lambda_1 + \lambda_2)v_1 + (\lambda_2 + \lambda_2)v_2] W_{\mu 4} B_{\mu} \\
- \frac{2\sqrt{3}}{9}tg^2 (\lambda_1^2 + \lambda_2^2 - v_1^2 - v_2^2 - 2\lambda_1^2 - 2\lambda_2^2) W_{\mu 8} B_{\mu}. \tag{66}
\]

We can split \( \mathcal{L}_{mass}^{GB} \) in (66) as

\[
\mathcal{L}_{mass}^{GB} = \mathcal{L}_5 + \mathcal{L}_{mix}^{CGB} + \mathcal{L}_{mix}^{NGB}, \tag{67}
\]

where \( \mathcal{L}_5 \) is the Lagrangian part of the imaginary part \( W_5 \). This boson is decoupled with its mass given by

\[
\mathcal{L}_5 = \frac{g^2}{4} \left( \omega^2 + u^2 + u'^2 + 8v_1^2 + 8v_2^2 + 2\lambda_1^2 + 2\lambda_2^2 + 2\lambda_1^2 + 2\lambda_2^2 - 4\Lambda_1\lambda_1 - 4\Lambda_2\lambda_2 \right) W_{\mu 5}^2.
\]

Hence,

\[
M_{W_5}^2 = \frac{g^2}{2} \left( \omega^2 + u^2 + u'^2 + 8v_1^2 + 8v_2^2 + 2\lambda_1^2 + 2\lambda_2^2 + 2\lambda_1^2 + 2\lambda_2^2 - 4\Lambda_1\lambda_1 - 4\Lambda_2\lambda_2 \right). \tag{68}
\]

In the limit \( \lambda_1, \lambda_2, v_1, v_2 \to 0 \), we have

\[
M_{W_5}^2 = \frac{g^2}{2} \left( \omega^2 + u^2 + u'^2 + 2\lambda_1^2 + 2\lambda_2^2 \right). \tag{69}
\]
Next,
\[
\mathcal{L}_{\text{mix}}^{\text{CGB}} = \frac{g^2}{4} \left[ v^2 + v'^2 + u^2 + u'^2 + 2(v_1^2 + v_2^2 + \lambda_1^2 + \lambda_2^2) \right] (W_{\mu1}^2 + W_{\mu2}^2)
+ \frac{g^2}{4} \left[ v^2 + v'^2 + \omega^2 + 2(v_1'^2 + v_2^2 + \Lambda_1^2 + \Lambda_2^2) \right] (W_{\mu6}^2 + W_{\mu7}^2)
+ g^2(\Lambda_1 v_1 + \Lambda_1 v_1' + \Lambda_2 v_2 + \lambda_2 v_2) (W_{\mu1} W_{\mu6} - W_{\mu2} W_{\mu7})
\]
(70)
is the Lagrangian part of the charged gauge bosons \(W\) and \(Y\), which can be rewritten in matrix form as
\[
\mathcal{L}_{\text{mix}}^{\text{CGB}} = \frac{g^2}{4} (W_{\mu}^\prime - Y_{\mu}^\prime) M_{WY}^2 \left( \begin{array}{cc} W_{\mu}^\prime & Y_{\mu}^\prime \end{array} \right)^T,
\]
where
\[
M_{WY}^2 = 2 \begin{pmatrix}
v^2 + v'^2 + u^2 + u'^2 + 2(v_1^2 + v_2^2 + \lambda_1^2 + \lambda_2^2) & 2(\Lambda_1 v_1 + \Lambda_1 v_1 + \Lambda_2 v_2 + \lambda_2 v_2) \\
2(\Lambda_1 v_1 + \Lambda_1 v_1 + \Lambda_2 v_2 + \lambda_2 v_2) & v^2 + v'^2 + \omega^2 + 2(v_1'^2 + v_2^2 + \Lambda_1^2 + \Lambda_2^2) \end{pmatrix}
\]
(71)
The matrix \(M_{WY}^2\) in (71) can be diagonalised as
\[
U_2^T M_{WY}^2 U_2 = \text{diag}(M_W^2, M_Y^2),
\]
where
\[
M_W^2 = \frac{g^2}{4} \left\{ 2(\lambda_1^2 + \lambda_2^2 + 2v_1^2 + 2v_2^2 + \Lambda_1^2 + \Lambda_2^2) + \omega^2 + u^2 + u'^2 + 2(v_1'^2 + v_2'^2) - \sqrt{\Gamma} \right\},
\]
\[
M_Y^2 = \frac{g^2}{4} \left\{ 2(\lambda_1^2 + \lambda_2^2 + 2v_1^2 + 2v_2^2 + \Lambda_1^2 + \Lambda_2^2) + \omega^2 + u^2 + u'^2 + 2(v_1'^2 + v_2'^2) + \sqrt{\Gamma} \right\},
\]
(72)
with
\[
\Gamma = 4\lambda_1^4 + 4\lambda_2^4 + (2\lambda_2^2 - 2\lambda_1^2 - \omega^2 + u^2 + u'^2)^2 - 4\lambda_1^2 (2\lambda_1^2 - 2\lambda_2^2 + 2\Lambda_2^2 + \omega^2 + u^2 - 4v_1^2) - 4\lambda_2^2 (2\lambda_1^2 - 2\lambda_2^2 - \omega^2 + u^2 + u'^2 - 4v_1^2) + 32\Lambda_1 (\lambda_2 + \Lambda_2) v_1 v_2
+ 16(\lambda_2 + \Lambda_2) v_2^2 + 32\Lambda_1 v_1 (\Lambda_1 v_1 + \lambda_2 v_2 + \Lambda_2 v_2).
\]
(73)
In our model, the following limits are often used:
\[
\lambda_{1,2}^2, v_{1,2}^2 \ll u^2, u'^2, v^2, v'^2,
\]
(74)
\[
u^2, u^2, v^2, v'^2 \ll \omega^2 \sim \Lambda_{1,2}^2.
\]
(75)
With the help of (74), \(\Gamma\) in (73) becomes
\[
\Gamma \simeq (2\Lambda_1^2 + 2\Lambda_2^2 + \omega^2 - u^2 - u'^2) + \frac{16\Lambda_1 \Lambda_2 v_1 v_2 + 8\lambda_2^2 v_2^2}{2\Lambda_1^2 + 2\Lambda_2^2 + \omega^2 - u^2 - u'^2},
\]
(76)
We note that in the limit $v$ and we have

$$M_W^2 \simeq \frac{g^2}{2} \left( u^2 + u'^2 + v^2 + v'^2 \right) - \frac{g^2}{2} \Delta M^2,$$  \hspace{1cm} (77)

with

$$\Delta M^2 = \frac{4(2\Lambda_1 \Lambda_2 v_1 v_2 + \Lambda_2^2 v_2^2)}{2\Lambda_1^2 + 2\Lambda_2^2 + \omega^2 - u^2 - u'^2}. \hspace{1cm} (78)$$

The corresponding eigenstates are arranged into the charged gauge boson mixing matrix

$$U_2 = \left( \begin{array}{cc} \frac{\mathcal{R}}{\sqrt{\mathcal{R}^2 + 1}} & - \frac{1}{\sqrt{\mathcal{R}^2 + 1}} \\ \frac{1}{\sqrt{\mathcal{R}^2 + 1}} & \frac{\mathcal{R}}{\sqrt{\mathcal{R}^2 + 1}} \end{array} \right) \equiv \left( \begin{array}{cc} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{array} \right),$$

where

$$\mathcal{R} = \frac{2\lambda_1^2 - 2\lambda_2^2 + 2\lambda_3^2 - \omega^2 - u^2 + u'^2 - \sqrt{\Gamma}}{4(\lambda_1 + \Lambda_1) v_1 + 4(\lambda_2 + \Lambda_2) v_2}. \hspace{1cm}$$

The physical charged gauge bosons is defined as

$$W^-_\mu = \cos \theta W'_{\mu}^{' -} + \sin \theta Y_{\mu}^{Y -},$$

$$Y_{\mu}^- = -\sin \theta W'_{\mu}^{' -} + \cos \theta Y_{\mu}^{Y -}.$$

The mixing angle $\theta$ is given by

$$\tan \theta = \frac{1}{\mathcal{R}} = \frac{4(\lambda_1 + \lambda_1) v_1 + 4(\lambda_2 + \Lambda_2) v_2}{2\lambda_1^2 - 2\lambda_1^2 + 2\lambda_2^2 - \omega^2 - u^2 - u'^2 - \sqrt{\Gamma}} \sim \frac{v_i}{\Lambda_i}, \quad (i = 1, 2) \hspace{1cm} (79)$$

We note that in the limit $v_1, v_2 \to 0$ the mixing angle $\theta$ tends to zero, $\Gamma = 2\Lambda_1^2 + 2\Lambda_2^2 + \omega^2 - u^2 - u'^2$, and we have

$$M_W^2 = \frac{g^2}{2} \left( u^2 + u'^2 + v^2 + v'^2 \right),$$

$$M_Y^2 = \frac{g^2}{2} \left( 2\Lambda_1^2 + 2\Lambda_2^2 + \omega^2 + v^2 + v'^2 \right). \hspace{1cm} (80)$$

There is a mixing among the neutral gauge bosons $W_3, W_8, B$ and $W_4$. The mass Lagrangian in this case has the form

$$\mathcal{L}_{\text{mix}}^{\text{NGB}} = \frac{v^2}{324} \left( 81g^2 W_{\mu 3}^2 + 27g^2 W_{\mu 8}^2 + 24g_X B_{\mu} - 54\sqrt{3}g^2 W_{\mu 3} W_{\mu 8} - 36\sqrt{6}gg_X W_{\mu 3} B_{\mu} \right. \right.$$  

$$+ \left. 36\sqrt{2}gg_X W_{\mu 8} B_{\mu} \right) + \frac{v'^2}{324} \left( 81g^2 W_{\mu 3}^2 + 27g^2 W_{\mu 8}^2 + 24g_X B_{\mu} - 54\sqrt{3}g^2 W_{\mu 3} W_{\mu 8} \right. \right.$$  

$$- \left. 36\sqrt{6}gg_X W_{\mu 3} B_{\mu} + 36\sqrt{2}gg_X W_{\mu 8} B_{\mu} \right) + \frac{\omega^2}{108} \left( 27g^2 W_{\mu 4}^2 + 36g^2 W_{\mu 8}^2 \right). \right.$$

$$$$
+ 12\sqrt{2}gg_xW_{\mu 8}B_\mu + 2g_2^2B_\mu^2) + \frac{u^2}{324} \left( 81g_2^2W_{\mu 4}^2 + 81g_2^2W_{\mu 3}^2 + 27g_2^2W_{\mu 8}^2 + 6g_2^2B_\mu^2 \right) \\
+ 54\sqrt{3}g_2^2W_{\mu 3}W_{\mu 8} - 18\sqrt{6}gg_xW_{\mu 3}B_\mu - 18\sqrt{2}W_{\mu 8}B_\mu \right) + \frac{u^2}{324} \left( 81g_2^2W_{\mu 4}^2 \\
+ 81g_2^2W_{\mu 3}^2 + 27g_2^2W_{\mu 8}^2 + 6g_2^2B_\mu^2 + 54\sqrt{3}g_2^2W_{\mu 3}W_{\mu 8} - 18\sqrt{6}gg_xW_{\mu 3}B_\mu - 18\sqrt{2}W_{\mu 8}B_\mu \right) \\
+ \frac{g^2}{6} \left[ 2(\Lambda_1 v_1 + \Lambda_2 v_2) \left( 3W_{\mu 3}W_{\mu 4} - 5\sqrt{3}W_{\mu 4}W_{\mu 8} \right) + 3(v_1^2 + v_2^2 + 2\lambda_1^2 + 2\lambda_2^2)W_{\mu 3}^2 \\
+ 3(4v_1^2 + 4v_2^2 + \lambda_1^2 + \lambda_2^2 + \Lambda_1^2 + \Lambda_2^2 + 2\lambda_1\lambda_1 + 2\lambda_2\lambda_2)W_{\mu 4}^2 \\
+ 2\sqrt{3}(-v_1^2 - v_2^2 + 2\lambda_1^2 + 2\lambda_2^2)W_{\mu 3}W_{\mu 8} + (v_1^2 + v_2^2 + 2\lambda_1^2 + 2\lambda_2^2 + 8\Lambda_1^2 + 8\Lambda_2^2)W_{\mu 8}^2 \\
+ 18(\Lambda_1 v_1 + \Lambda_2 v_2)W_{\mu 3}W_{\mu 4} + 2\sqrt{3}(\Lambda_1 v_1 + \Lambda_2 v_2)W_{\mu 4}W_{\mu 8} \right] \\
+ \frac{2}{27}g^2(\lambda_1^2 + \lambda_2^2 + \Lambda_1^2 + \Lambda_2^2 + 2v_1^2 + 2v_2^2)B_\mu^2 - \frac{2}{3}\sqrt{\frac{2}{3}}g^2(\lambda_1^2 + \lambda_2^2 + v_1^2 + v_2^2)W_{\mu 3}B_\mu \\
- \frac{4}{3}\sqrt{\frac{2}{3}}g^2[(\Lambda_1 + \Lambda_1)v_1 + (\Lambda_2 + \Lambda_2)v_2]W_{\mu 4}B_\mu \\
- \frac{2\sqrt{3}}{9}g^2(\lambda_1^2 + \lambda_2^2 - v_1^2 - v_2^2 - 2\Lambda_1^2 - 2\Lambda_2^2)W_{\mu 8}B_\mu. \quad (81)

In the basis of \( (W_{\mu 3}, W_{\mu 8}, B_\mu, W_{\mu 4}) \), \( \mathcal{L}^{NGB}_{\text{mix}} \) can be rewritten in matrix form:

\[
\mathcal{L}^{NGB}_{\text{mix}} \equiv \frac{1}{2} V^T M^2 V, 
\]

where

\[
V^T = (W_{\mu 3}, W_{\mu 8}, B_\mu, W_{\mu 4}),
\]

\[
M^2 = \frac{g^2}{4} \begin{pmatrix}
M_{11}^2 & M_{12}^2 & M_{13}^2 & M_{14}^2 \\
M_{12}^2 & M_{22}^2 & M_{23}^2 & M_{24}^2 \\
M_{13}^2 & M_{23}^2 & M_{33}^2 & M_{34}^2 \\
M_{14}^2 & M_{24}^2 & M_{34}^2 & M_{44}^2
\end{pmatrix}, \quad (82)
\]

with

\[
M_{11}^2 = 2(v_1^2 + v_2^2 + u_1^2 + u_2^2 + 2v_1^2 + 2v_2^2 + 4\lambda_1^2 + 4\lambda_2^2),
\]

\[
M_{12}^2 = -\frac{2\sqrt{3}}{3} \left( v_1^2 + v_2^2 - u_1^2 - u_2^2 + 2v_1^2 + 2v_2^2 - 4\lambda_1^2 - 4\lambda_2^2 \right),
\]

\[
M_{13}^2 = -\frac{2}{3}\sqrt{\frac{2}{3}} \left( 2v_1^2 + 2v_2^2 + u_1^2 + u_2^2 + 4\lambda_1^2 + 4\lambda_2^2 + 4v_1^2 + 4v_2^2 \right),
\]

\[
M_{14}^2 = 4(\Lambda_1 v_1 + \Lambda_2 v_2) + 12(\Lambda_1 v_1 + \Lambda_2 v_2),
\]

\[
M_{22}^2 = \frac{2}{3} \left( v_1^2 + v_2^2 + 4\omega^2 + u_1^2 + u_2^2 + 2v_1^2 + 2v_2^2 + 4\lambda_1^2 + 4\lambda_2^2 + 16\Lambda_1^2 + 16\Lambda_2^2 \right),
\]

\[
M_{23}^2 = \frac{2\sqrt{2}t}{9} \left( 2v_1^2 + 2v_2^2 + 2\omega^2 - u_1^2 - u_2^2 - 4\lambda_1^2 - 4\lambda_2^2 + 4v_1^2 + 4v_2^2 + 8\Lambda_1^2 + 8\Lambda_2^2 \right),
\]

\[
M_{24}^2 = \frac{4}{\sqrt{3}} [\lambda_1 v_1 + \lambda_2 v_2 - 5(\Lambda_1 v_1 + \Lambda_2 v_2)].
\]
\[ M_{23}^2 = \frac{4t^2}{27} \left( 4v^2 + 4v'^2 + \omega^2 + u^2 + u'^2 + 4\lambda_1^2 + 4\lambda_2^2 + 4\Lambda_1^2 + 4\Lambda_2^2 + 8v_1^2 + 8v_2^2 \right), \]

\[ M_{34}^2 = -\frac{16}{3} \sqrt{\frac{2}{3}} t (\lambda_1 v_1 + \Lambda_1 v_1 + \lambda_2 v_2 + \Lambda_2 v_2), \]

\[ M_{44}^2 = 2(\omega^2 + u^2 + u'^2 + 8v_1^2 + 8v_2^2 + 2\lambda_1^2 + 2\lambda_2^2 + 2\Lambda_1^2 + 2\Lambda_2^2 + 4\lambda_1 \lambda_1 + 4\lambda_2 \lambda_2). \] (83)

The matrix \( M^2 \) in (82) with the elements in (83) has one exact eigenvalue, which is identified with the photon mass,

\[ M_\gamma^2 = 0. \] (84)

The corresponding eigenvector of \( M_\gamma^2 \) is

\[ A_\mu = \begin{pmatrix} \frac{\sqrt{3}t}{\sqrt{4t^2 + 18}} - \frac{t}{\sqrt{4t^2 + 18}} & 3\sqrt{\frac{2}{3}} \end{pmatrix}^T. \] (85)

We note that in the limit \( \lambda_{1,2}, v_{1,2} \to 0 \), \( M_{14}^2 = M_{24}^2 = M_{34}^2 = 0 \) and \( W_4 \) does not mix with \( W_{3\mu}, W_{8\mu} \) and \( B_\mu \). In the general case \( \lambda_{1,2}, v_{1,2} \neq 0 \), the mass matrix in (82) contains one exact eigenvalues as in (81) with the corresponding eigenstate defined in (85).

The diagonalization of the mass matrix \( M^2 \) in (82) is done in two steps. In the first step, the basis \( (W_{3\mu}, W_{8\mu}, B'_\mu, W_{4\mu}) \) is transformed into the basis \( (A_\mu, Z_\mu, Z'_\mu, W_{4\mu}) \) by the matrix

\[ U_{NGB} = \begin{pmatrix} s_W \sqrt{\frac{2}{3}} & -c_W & 0 & 0 \\ -c_W \sqrt{\frac{2}{3}} & s_W \sqrt{\frac{2}{3}} & \sqrt{1 - \frac{t^2_W}{3}} & 0 \\ c_W \sqrt{1 - \frac{t^2_W}{3}} & s_W \sqrt{1 - \frac{t^2_W}{3}} & \frac{t_W}{\sqrt{3}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \] (86)

The eigenstates are defined as

\[ A_\mu = s_W W_{3\mu} + c_W \left( -\frac{t_W}{\sqrt{3}} W_{8\mu} + \sqrt{1 - \frac{t^2_W}{3}} B_\mu \right), \]

\[ Z_\mu = -c_W W_{3\mu} + s_W \left( -\frac{t_W}{\sqrt{3}} W_{8\mu} + \sqrt{1 - \frac{t^2_W}{3}} B_\mu \right), \]

\[ Z'_\mu = \sqrt{1 - \frac{t^2_W}{3}} W_{8\mu} + \frac{t_W}{\sqrt{3}} B_\mu. \] (87)

To obtain (86) and (87) we used the continuation of the SU(3)_L gauge coupling constant \( g \) to the spontaneous symmetry breaking point, where

\[ t = \frac{3\sqrt{2}s_W}{\sqrt{3 - 4s_W^2}}. \] (88)
In this basis, the mass matrix $M^2$ in (82) becomes

$$M^2 = U^+_{NGB} M^2 U_{NGB} = \frac{g^2}{4} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & M^2_{22} & M^2_{23} & M^2_{24} \\ 0 & M^2_{23} & M^2_{33} & M^2_{34} \\ 0 & M^2_{24} & M^2_{34} & M^2_{44} \end{pmatrix}, \quad (89)$$

where

$$M^2_{22} = \frac{4(2t^2 + 9)}{t^2 + 18} \left( u^2 + u'^2 + v^2 + v'^2 + 4\lambda_1^2 + 4\lambda_2^2 + 2v_1^2 + 2v_2^2 \right) = \frac{2}{c_W^2} \left( u^2 + u'^2 + v^2 + v'^2 + 4\lambda_1^2 + 4\lambda_2^2 + 2v_1^2 + 2v_2^2 \right),$$

$$M^2_{23} = \frac{4}{3\sqrt{3}} \frac{\sqrt{2t^2 + 9}}{(t^2 + 18)} \left[ (t^2 - 9)(4\lambda_1^2 + 4\lambda_2^2 + u^2 + u'^2) + (2t^2 + 9)(v^2 + v'^2) \right] = \frac{2}{c_W^2} \left( (1 - 2\alpha_0^2)(u^2 + u'^2 + 4\lambda_1^2 + 4\lambda_2^2) + v^2 + v'^2 + v_1^2 + v_2^2 \right) \sqrt{\alpha_0},$$

$$M^2_{24} = -4\sqrt{2} \frac{\sqrt{2t^2 + 9}}{t^2 + 18} \left( \Lambda_1 v_1 + 3\lambda_1 v_1 + \Lambda_2 v_2 + 3\lambda_2 v_2 \right) = -\frac{4}{c_W^2} \left( \Lambda_1 v_1 + \Lambda_2 v_2 + 3\lambda_1 v_1 + 3\lambda_2 v_2 \right),$$

$$M^2_{33} = \frac{4}{27(t^2 + 18)} \left[ 4\lambda_1^2(t^2 - 9)^2 + 4\Lambda_1^2(t^2 + 18)^2 + 81(4\lambda_2^2 + 16\Lambda_2^2 + 4\omega^2 + u^2 + u'^2 + v^2 + v'^2) + 18t^2 \left( 8\Lambda_2^2 + 2\omega^2 - u^2 - u'^2 + 2v^2 + 2v'^2 + 4v_1^2 + 4v_2^2 \right) + 4\lambda_1^2 t^2 (t^2 - 18) + t^4 \left( 4\Lambda_2^2 + \omega^2 + u^2 + u'^2 + 4v^2 + 4v'^2 + 8v_1^2 + 8v_2^2 \right) \right],$$

$$= 32(\lambda_1^2 + \Lambda_2^2)c_W^2 \alpha_0 + 8\omega^2 c_W^2 \alpha_0 + \frac{2}{c_W^2} (v^2 + v'^2 + 2v_1^2 + 2v_2^2) \alpha_0 + \frac{2}{c_W^2} (2c_W^2 - 1)(u^2 + u'^2) \alpha_0 + \frac{8(2c_W^2 - 1)^3}{c_W^4} (\lambda_1^2 + \lambda_2^2) \alpha_0,$$

$$M^2_{34} = -\frac{4\sqrt{2}}{3\sqrt{3}} \frac{1}{\sqrt{t^2 + 18}} \left[ (4t^2 - 9)(\lambda_1 v_1 + \lambda_2 v_2) + (4t^2 + 45)(\Lambda_1 v_1 + \Lambda_2 v_2) \right] = -\frac{4\sqrt{2}}{3\sqrt{3}} \left[ x_0(\Lambda_1 v_1 + \Lambda_2 v_2) + \left( 2 - \frac{1}{\alpha_0} \right) (\lambda_1 v_1 + \lambda_2 v_2) \right],$$

$$M^2_{44} = 2 \left[ 2(\lambda_1 + \Lambda_1)^2 + 2(\lambda_2 + \Lambda_2)^2 + \omega^2 + u^2 + u'^2 + 8v_1^2 + 8v_2^2 \right] = 2 \left( u^2 + u'^2 + \omega^2 + 2\lambda_1^2 + 2\lambda_2^2 + 2\Lambda_1^2 + 2\Lambda_2^2 + 4\lambda_1 \Lambda_1 + 4\lambda_2 \Lambda_2 + 4\Lambda_1 \Lambda_1 + 4\Lambda_2 \Lambda_2 + 8v_1^2 + 8v_2^2 \right). \quad (90)$$

In the approximation $\lambda_{1,2}^2, v_{1,2}^2 \ll \Lambda_{1,2}^2 \sim \omega^2$, we have

$$M^2_{22} = \frac{2}{c_W^2} \left( u^2 + u'^2 + v^2 + v'^2 \right),$$

$$M^2_{23} = \frac{2}{c_W^2} \left( (1 - 2c_W^2)(u^2 + u'^2) + v^2 + v'^2 \right) \sqrt{\alpha_0},$$
with
\[ s_W = \sin \theta_W, \quad c_W = \cos \theta_W, \quad t_W = \tan \theta_W, \]
\[ x_0 = 4c_W^2 + 1, \quad \alpha_0 = (4c_W^2 - 1)^{-1}. \] (92)

From (89), there exist mixings between \( Z_\mu, Z'_\mu \) and \( W_\mu \). It is noteworthy that in the limit \( v_{1,2} = 0 \), the elements \( M^2_{21} \) and \( M^2_{33} \) vanish, and there is no mixing between \( W_4 \) and \( Z_\mu, Z'_\mu \).

In the second step, three remain neutral gauge bosons gain masses via seesaw mechanism:
\[ M^2_Z = g^2 \left( M^2_{22} - (M^\text{off})^T (M^2_{2x2})^{-1} M^\text{off} \right), \] (93)
where
\[ M^\text{off} = \begin{pmatrix} M^2_{23} \\ M^2_{24} \end{pmatrix}, \quad M^2_{2x2} = \begin{pmatrix} M^2_{33} & M^2_{34} \\ M^2_{34} & M^2_{44} \end{pmatrix}. \] (94)

Combining (93) and (94) yields
\[ M^2_Z = g^2 \left( M^2_{22} + \frac{(M^2_{24})^2 M^2_{33} - 2 M^2_{23} M^2_{24} M^2_{34} + (M^2_{23})^2 M^2_{44}}{(M^2_{34})^2 - M^2_{33} M^2_{44}} \right), \]
\[ = \frac{g^2 (u^2 + u'^2 + v^2 + v'^2)}{2c_W^2} - \frac{g^2}{2c_W^2} \Delta M^2_z, \]
where
\[ \Delta M^2_z = \frac{4 \Delta^2_z (4c_W^4 x_3 - x_0 x_1 + x_4) + x_1 [x_2 x_1 - 4 \Delta^2_x x_0]}{x_2 (x_4 + 4c_W^4 x_3) - 4 \Delta^2_x x_0}, \]
\[ = \frac{4 \Delta^2_z (4c_W^4 x_3 - 2x_0 x_1 + x_4) + x_1^2 x_2}{x_2 (x_4 + 4c_W^4 x_3) - 4 \Delta^2_x x_0}, \] (95)
with
\[ x_1 = (1 - 2c_W^2)(u^2 + u'^2) + v^2 + v'^2, \]
\[ x_2 = 2\Lambda_1 (2\lambda_1 + \Lambda_1) + 2\Lambda_2 (2\lambda_2 + \Lambda_2) + \omega^2 + u^2 + u'^2, \]
\[ x_3 = 4\Lambda_1^2 + 4\Lambda_2^2 + \omega^2 + u^2 + u'^2, \quad x_4 = (1 - 4c^2)(u^2 + u'^2) + v^2 + v'^2, \]
\[ \Delta^2_z = \Lambda_1 v_1 + \Lambda_2 v_2. \]
The $\rho$ parameter in our model is given by

$$\rho = \frac{M_W^2}{M_2^2 \cos^2 \theta_W} = 1 + \delta_{\text{tree}}, \quad (96)$$

where

$$\delta_{\text{tree}} = \frac{\delta_{wz}}{M_2^2}, \quad \delta_{wz} = \frac{g^2}{2c_W^2} \left( \Delta M_2^2 - \Delta M_3^2 \right). \quad (97)$$

Using approximations (74) and (75), we have

$$\Delta M_2^2 - \Delta M_3^2 \simeq 8(k^2 + 1)v_s c_W \left\{ \begin{array}{l} \frac{v_s}{2(k^2 + 2)\Lambda_s} + \frac{(k^2 + 1)(4k^2 + 5)c_W^2 \Lambda_s v_s}{2(8k^4 + 22k^2 + 15)c_W^4 \Lambda_s^2 - (k^2 + 1)^2(4c_W^2 + 1)^2 v_s^2} \\ \frac{(4k^2 + 4\Lambda_s^2 + \omega^2)(4c_W^2 - 1)c_W^2 (\Lambda_1 v_1 + \Lambda_2 v_2)}{2(4c_W^2 - 1)\left[(2k^2 + 2\Lambda_s^2 + \omega^2)(4k^2 + 4\Lambda_s^2 + \omega^2)c_W^4 - (4c_W^2 + 1)^2(\Lambda_1 v_1 + \Lambda_2 v_2)^2\right]} \end{array} \right\}. \quad (98)$$

We assume relations (57) and $v_2 = v_s$, $\omega = \Lambda_2 \equiv \Lambda_s$; then

$$\Delta M_2^2 - \Delta M_3^2 \simeq 8(k^2 + 1)v_s c_W \left\{ \begin{array}{l} -\frac{v_s}{2k^2 + 3} + \frac{(k^2 + 1)(4k^2 + 5)c_W^2 \Lambda_s v_s}{2(8k^4 + 22k^2 + 15)c_W^4 \Lambda_s^2 - (k^2 + 1)^2(4c_W^2 + 1)^2 v_s^2} \\ \frac{(4k^2 + 4\Lambda_s^2 + \omega^2)(4c_W^2 - 1)c_W^2 (\Lambda_1 v_1 + \Lambda_2 v_2)}{2(4c_W^2 - 1)\left[(2k^2 + 2\Lambda_s^2 + \omega^2)(4k^2 + 4\Lambda_s^2 + \omega^2)c_W^4 - (4c_W^2 + 1)^2(\Lambda_1 v_1 + \Lambda_2 v_2)^2\right]} \end{array} \right\}. \quad (99)$$

$$\Delta M_2^2 - \Delta M_3^2 \simeq 8(k^2 + 1)v_s c_W \left\{ \begin{array}{l} -\frac{v_s}{2k^2 + 3} + \frac{(k^2 + 1)(4k^2 + 5)c_W^2 \Lambda_s v_s}{2(8k^4 + 22k^2 + 15)c_W^4 \Lambda_s^2 - (k^2 + 1)^2(4c_W^2 + 1)^2 v_s^2} \\ \frac{(4k^2 + 4\Lambda_s^2 + \omega^2)(4c_W^2 - 1)c_W^2 (\Lambda_1 v_1 + \Lambda_2 v_2)}{2(4c_W^2 - 1)\left[(2k^2 + 2\Lambda_s^2 + \omega^2)(4k^2 + 4\Lambda_s^2 + \omega^2)c_W^4 - (4c_W^2 + 1)^2(\Lambda_1 v_1 + \Lambda_2 v_2)^2\right]} \end{array} \right\}. \quad (100)$$

From (97) and (100), we have

$$\delta_{\text{tree}} = \frac{g^2}{2c_W^2} \frac{1}{M_2^2} 8(k^2 + 1)v_s^2 \left\{ \begin{array}{l} -\frac{v_s}{2k^2 + 3} + \frac{(k^2 + 1)(4k^2 + 5)c_W^2 \Lambda_s v_s}{2(8k^4 + 22k^2 + 15)c_W^4 \Lambda_s^2 - (k^2 + 1)^2(4c_W^2 + 1)^2 v_s^2} \\ \frac{(4k^2 + 4\Lambda_s^2 + \omega^2)(4c_W^2 - 1)c_W^2 (\Lambda_1 v_1 + \Lambda_2 v_2)}{2(4c_W^2 - 1)\left[(2k^2 + 2\Lambda_s^2 + \omega^2)(4k^2 + 4\Lambda_s^2 + \omega^2)c_W^4 - (4c_W^2 + 1)^2(\Lambda_1 v_1 + \Lambda_2 v_2)^2\right]} \end{array} \right\}. \quad (101)$$

The experimental value of the $\rho$ parameter and $M_W$ are given in Ref. [4]:

$$\rho = 1.0004^{+0.0003}_{-0.0004}, \quad (\delta_{\text{tree}} = 0.0004^{+0.0003}_{-0.0004}),$$

$$s_W^2 = 0.23116 \pm 0.00012,$n\)  \quad (102)$$

Hence,

$$0 \leq \delta_{\text{tree}} \leq 0.0007. \quad (103)$$
From (102) and (103), we can deduce the relations between $v, g$ and $k$. Indeed,

$$v = \pm \frac{c_W^2 \sqrt{\delta_{\text{tree}}} \sqrt{2k^2 + 3M_Z}}{g\sqrt{2k^2 + 2\sqrt{k^2 + 1 - 2c_W^2}}}$$

The Fig. 1 gives the relation between $v_s$ and $g, k$ with $g = 0.5$ and $k \in (0.9, 1.1)$, for $|v_s| \in (0, 8) \text{ Gev}$. Conditions (74) and (75) are then satisfied. The Fig. 2 gives the relation between $g$ and $\delta_{\text{tree}}, v_s$

FIG. 1: The relation between $v_s$ and $g, k$ with $g = 0.5$ and $k \in (0.9, 1.1)$

with $k = 1$ and $\delta_{\text{tree}} \in (0, 0.0007)$, $v_s \in (0, 8.0) \text{ GeV}$, for $|g| \in (0, 2) \text{ GeV}$. Conditions (74) and (75) are then satisfied. The Fig. 3 gives the relation between $k$ and $g, v_s$ with $\delta_{\text{tree}} = 0.0005$ and $g \in (0.4, 0.6), v_s \in (0, 8.0) \text{ GeV}$, for $k \in (1, 3) \text{ GeV}$ ($k$ is a real number, Fig. 3a) or $k = ik_1, k_1 \in (-1.2, -1.05) \text{ GeV}$ ($k$ is a pure complex number, Fig. 3b). Conditions (74) and (75) are then
satisfied. From Fig. 3 we see that many values of \( k \) that are different from close to unity still can fit the recent experimental data [4]. This means that the difference of \( \langle s_1 \rangle \) and \( \langle s_1 \rangle \) as mentioned in this work is necessary.

![Diagram](image)

FIG. 3: The relation between \( k \) and \( g, v_s \) provided \( \delta_{\text{tree}} = 0.0005 \) and \( g \in (0.4, 0.6), v_s \in (0, 8.0) \text{ GeV} \)

Diagonalizing the mass matrix \( M_{2\times2}^2 \), we obtain two new physical gauge bosons

\[
Z''_\mu = \cos \phi Z'_\mu + \sin \phi W'_{\mu 4}, \\
W'_{\mu 4} = -\sin \phi Z'_\mu + \cos \phi W'_{\mu 4}. \tag{104}
\]

The mixing angle \( \phi \) is given by

\[
\tan \phi = \frac{4\sqrt{\alpha_0 c_W (\Lambda_1 v_1 + \Lambda_2 v_2)x_0}}{-4c_W^4 \alpha_0 x_3 + c_W^2 x_2 - \alpha_0 x_4 + \sqrt{F}}. \tag{105}
\]

where

\[
F = \left( 4c_W^4 \alpha_0 x_3 - c_W^2 x_2 + \alpha_0 x_4 \right)^2 + 16\alpha_0 c_W^2 (\Lambda_1 v_1 + \Lambda_2 v_2)^2 x_0^2.
\]

If \( \lambda_{1,2}^2, v_{1,2}^2, u^2, u'^2, v^2, v'^2 \ll \omega^2 \sim \Lambda_s^2 \sim \Lambda_\sigma^2 \) then

\[
\sqrt{F} \simeq c_W^2 [2\Lambda_1^2 + 2\Lambda_2^2 + \omega^2 - 4\alpha_0 c_W^2 (4\Lambda_1^2 + 4\Lambda_2^2 + \omega^2)],
\]

and we can evaluate

\[
\tan \phi \simeq -\frac{2\sqrt{\alpha_0 (\Lambda_1 v_1 + \Lambda_2 v_2)x_0}}{c_W [2(8\alpha_0 c_W^2 - 1)(\Lambda_1^2 + \Lambda_2^2) + (4\alpha_0 c_W^2 - 1)\omega^2]} \sim \frac{v_i}{\Lambda_i} \quad (i = 1, 2). \tag{106}
\]

The physical mass eigenvalues are defined by

\[
M_{Z''_{\mu}, W'_{\mu 4}}^2 = \frac{g^2}{4c_W^4} \left\{ 4\alpha_0 c_W^4 x_3 + c_W^2 x_2 + \alpha_0 x_4 \pm \sqrt{F} \right\}. \tag{107}
\]
In the limit $\lambda_{1,2}, v_{1,2} \to 0$, the mixing angle $\phi$ tends to zero, and $M^2_{Z''_{\mu \nu}}$, in (107) reduces to

$$M^2_{Z''_{\mu \nu}} = \frac{g^2}{2\epsilon_W} \left[ \epsilon^2 W (u^2 + u'^2) + v^2 + v'^2 + 4\epsilon^4_W (4\Lambda_1^2 + 4\Lambda_2^2 + \omega^2) \right] \alpha_0,$$

$$M^2_{W_{\mu 4}} = \frac{g^2}{2} \left( u^2 + u'^2 + \omega^2 + 2\Lambda_1^2 + 2\Lambda_2^2 \right). \quad (108)$$

From (69) and (108), the $W_{\mu 4}$ and $W_{\mu 5}$ components have the same mass, and hence, in this approximation we should identify the linear combination

$$\sqrt{2} X^0_\mu = W'_{\mu 4} - iW_{\mu 5} \quad (109)$$

as a physical neutral non-Hermitian gauge boson. The subscript "0" indicates the neutrality of the gauge boson $X_\mu$. We note that the identification in (109) can only be acceptable in the limit $\lambda_{1,2}, v_{1,2} \to 0$. In general, it is not true because of the difference in masses of $W'_{\mu 4}$ and $W_{\mu 5}$ as in (68) and (107).

Expressions (79) and (106) show that, with the limits (74) and (75), the mixings between the charged gauge bosons $W - Y$ and the neutral ones $Z' - W_4$ are of the same order because they are proportional to $v_i/\Lambda_i \quad (i=1,2)$. In addition, from (108)

$$M^2_{Z''_{\mu \nu}} \simeq g^2 (4\Lambda_1^2 + 4\Lambda_2^2 + \omega^2)$$

is somewhat bigger than

$$M^2_{W_{\mu 4}} \simeq \frac{g^2}{2} (\omega^2 + 2\Lambda_1^2 + 2\Lambda_2^2)$$

(or $M^2_{X^0_\mu}$), and

$$|M^2_Y - M^2_{X^0_\mu}| = \frac{g^2}{2} (u^2 + u'^2 - v^2 - v'^2)$$

is slightly smaller than

$$M^2_W = \frac{g^2}{2} (u^2 + u'^2 + v^2 + v'^2).$$

In that limit, the masses of $X^0_\mu$ and $Y$ degenerate.

VI. CONCLUSIONS

We have studied new features of the 3-3-1 model with a neutral fermion based on the $S_3$ flavor symmetry in which the anti-sextet responsible for neutrino mass and mixing lies in the $2$
representation under $S_3$ and the number of Higgs multiplets required is reduced. If the $S_3$ symmetry is violated as a perturbation by the difference in components of the anti-sextet, $S_3$ is equivalently broken into identity, the corresponding neutrino mass mixing matrix acquires the most general form. This way of the symmetry breaking helps us reduce the content in the Higgs sector: only one anti-sextet instead of three multiplets (two anti-sextets and one triplet) as in our previous work. By assuming that the VEVs of the anti-sextet differ from each other and regarding the difference between these VEVs as a small perturbation, we can make the model fit the latest data on neutrino oscillations. Our results show that the neutrino masses are naturally small and a deviation from the tri-bimaximal neutrino mixing form can be realized. The Higgs potential of the model and minimization conditions are also considered.

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