On Determinants of Laplacians on Compact Riemann Surfaces Equipped with Pullbacks of Conical Metrics by Meromorphic Functions

Victor Kalvin *

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Department of Mathematics and Statistics, Concordia University, 1455 de Maisonneuve Blvd. West, Montreal, Quebec, H3G 1M8 Canada

Abstract. Let \( m \) be any conical (or smooth) metric of finite volume on the Riemann sphere \( \mathbb{C}P^1 \). On a compact Riemann surface \( X \) of genus \( g \) consider a meromorphic function \( f : X \to \mathbb{C}P^1 \) such that all poles and critical points of \( f \) are simple and no critical value of \( f \) coincides with a conical singularity of \( m \) or \( \{\infty\} \). The pullback \( f^*m \) of \( m \) under \( f \) has conical singularities of angles \( 4\pi \) at the critical points of \( f \) and other conical singularities that are the preimages of those of \( m \). We study the \( \zeta \)-regularized determinant \( \text{Det}' \Delta_F \) of the (Friedrichs extension of) Laplace-Beltrami operator on \( (X, f^*m) \) as a functional on the moduli space of pairs \((X, f)\) and obtain an explicit formula for \( \text{Det}' \Delta_F \).

1 Introduction

The problem of explicit evaluation of the determinants of Laplacians on Riemann surfaces has received considerable attention. For smooth metrics the determinants were thoroughly studied, see e.g. [4, 5, 23, 22]. Over the past decade significant progress was also achieved for flat conical metrics, see e.g. [1, 6, 8, 9, 13, 15, 18, 7]. The problem of explicit evaluation of the determinants for conical metrics of constant positive curvature has also attracted some attention [26, 14, 10]. In this paper we derive an explicit formula for the determinant of Laplacian corresponding to the pullback of a finite volume conical metric on the Riemann sphere by a meromorphic function. In particular, the corresponding flat, constant positive curvature, and hyperbolic conical metrics are included into consideration.

*E-mail: vkalvin@gmail.com, victor.kalvin@concordia.ca
Let $X$ be a compact Riemann surface $X$ of genus $g$. Following [27], we say that $\mu$ is a conical metric on $X$ if for any point $P \in X$ there exist a neighbourhood $U$ of $P$, a local parameter $x \in \mathbb{C}$, and a real-valued function $\varphi \in L^1(U)$ such that $x(P) = 0$, $\mu = e^{2\varphi}|x|^{2\beta}|dx|^2$ in $U$ with some $\beta > -1$, and $\partial_x \partial_x \varphi \in L^1(U)$. If $\beta = 0$, then the point $P$ is regular. If $\beta \neq 0$, then $P$ is a conical singularity of total angle $2\pi(\beta + 1)$. A function $K : X \to \mathbb{R}$ defined by

$$K = e^{-2\varphi}|x|^{-2\beta}(-4\partial_x \partial_x \varphi)$$

is the curvature of $\mu$ in the neighbourhood $U$ ($K$ does not depend on the choice of $x$).

Consider a conical metric $m = \rho(z, \bar{z})|dz|^2$ on the Riemann sphere $\mathbb{C}P^1$ with conformal factor

$$\rho(z, \bar{z}) = e^{2u(z, \bar{z})} \prod_{j=1}^{n} |z - p_j|^{2\beta_j},$$

where $n \geq 0$, $\beta_j \in (-1, 0) \cup (0, \infty)$ and $u \in C(\mathbb{C}P^1)$ is a real-valued function. At $z = p_j$ the metric $m$ has a conical singularity of total angle $2\pi(\beta_j + 1)$. We define the curvature $K(z, \bar{z})$ of $m$ by the formula

$$K = e^{-2u} \prod_{j=1}^{n} |z - p_j|^{-2\beta_j}(-4\partial_z \partial_{\bar{z}} u)$$

and assume that $K \in C^\infty(\mathbb{C}P^1)$ and $\text{Vol}_m(\mathbb{C}P^1) < \infty$.

Consider a meromorphic function $f : X \to \mathbb{C}P^1$ of degree $N$ with simple poles and simple critical points. Assume that the critical values of $f$ are finite and do not coincide with conical singularities of $m$. Then the pullback $f^*m$ of $m$ by $f$ is a conical metric with $M = 2N + 2g - 2$ conical singularities of angles $4\pi$ at the critical points of $f$ and $N \times n$ conical singularities of angles $2\pi(\beta_j + 1)$ at the preimages $f^{-1}(p_j)$ of the conical points $z = p_j$ of $m$. It is easy to see that the curvature of $f^*m$ is smooth and at $P \in X$ it coincides with the curvature of $m$ at $z = f(P)$. In particular, if $m$ is a conical metric of constant curvature on $\mathbb{C}P^1$, then $f^*m$ is a conical metric of the same constant curvature on $X$.

We study the $\zeta$-regularized determinant $\text{Det}' \Delta_F$ of the (Friedrichs extension of) Laplace-Beltrami operator on $(X, f^*m)$ as a functional on the Hurwitz moduli space $H_{g,N}(1, \ldots, 1)$ of pairs $(X, f)$. The space $H_{g,N}(1, \ldots, 1)$ (here 1 is repeated $N$ times, i.e. there are exactly $N$ distinct preimages $f^{-1}(\infty)$ of the point $\{\infty\} \in \mathbb{C}P^1$) is a connected complex manifold locally coordinatized by the critical values $z_1, \ldots, z_M$ of the function $f$. The main result of this paper is the following explicit formula

$$\text{Det}' \Delta_F = C \det \mathfrak{B} |\tau|^2 \prod_{k=1}^{M} \zeta(\rho(z_k, \bar{z}_k)). \tag{1.1}$$

Here $C$ is a constant independent of the point $(X, f)$ of $H_{g,N}(1, \ldots, 1)$, $\mathfrak{B}$ is the matrix of $b$–periods of the Riemann surface $X$ (for genus zero surface $X$ the factor $\det \mathfrak{B}$ should be omitted), $\tau$ stands for the Bergman tau-function on $H_{g,N}(1, \ldots, 1)$, and $\rho(z, \bar{z})$ is the conformal factor of the metric $m$ on $\mathbb{C}P^1$. For the explicit expressions of $\tau$ in terms of basic objects on the Riemann surface (prime form, theta-functions, etc.) and the divisor of the meromorphic differential $df$, we refer the interested readers to [16, 17] for genus zero and one surfaces, and to [19, 20] if $g > 1$. 

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As a simple illustrating example consider the map $f : \mathbb{C}P^1_w \to \mathbb{C}P^1_z$, where

$$z = f(w) := \frac{z_2 w^2 + z_1 (z_2 - z_1)}{w^2 + z_2 - z_1}.$$ 

The function $f$ has simple critical points at $w = 0$ and $w = \infty$, the corresponding critical values are $z_1$ and $z_2$. We have $\tau(z_1, z_2) = \sqrt{z_1 - z_2}$ (see [16, f-la (3.38)]) and (1.1) takes the form

$$\text{Det}' \Delta_F = C \sqrt{|z_1 - z_2|} \sqrt[8]{\rho(z_1, \bar{z}_1) \cdot \rho(z_2, \bar{z}_2)}.$$  \hspace{1cm} (1.2)

Let, for instance, $m$ be the hyperbolic metric on $\mathbb{C}P^1_z$ with three conical points at $z = p_j$ of angles $2\pi(\beta_j + 1)$ satisfying $\beta_1 + \beta_2 + \beta_3 < -2$ and $\beta_j \in (-1, 0)$. Then the hyperbolic metric $f^*m$ on $\mathbb{C}P^1_w$ has conical singularities of angles $2\pi(\beta_j + 1)$ at

$$w = \pm \sqrt{(z_1 - z_2) \frac{z_1 - p_j}{z_2 - p_j}}, \quad j = 1, 2, 3,$$

and two conical points of angles $4\pi$ at $w = 0$ and $w = \infty$. Substituting the corresponding explicit formula [21] for $\rho(z, \bar{z})$ into (1.2) we obtain an explicit formula for the determinant of $\Delta_F$ on $(\mathbb{C}P^1_w, f^*m)$ with a constant $C$ that does not depend on the parameters $z_1$ and $z_2$ in $\mathbb{C}P_2 \setminus \{p_1, p_2, p_3\}$, $z_1 \neq z_2$.

Let us also notice that by setting $\rho(z, \bar{z}) = 4(1 + |z|^2)^{-2}$ in (1.1) (i.e. by taking the standard curvature 1 metric as the metric $m$ on the Riemann sphere $\mathbb{C}P^1$) we immediately obtain the main result of the recent paper [10].

The plan of this paper is as follows. In Section 2 entitled “Preliminaries” we list some known results on asymptotic expansions of solutions near conical singularities at the critical points $P_k$ of $f$. In Section 3 which is the core of the present paper, we find an explicit formula for a coefficient that is responsible for the metric dependent factor $\prod_{k=1}^M \sqrt[8]{\rho(z_k, \bar{z}_k)}$ in the explicit formula (1.1) for $\text{Det}' \Delta_F$. In Section 4 we study dependence of the eigenvalues of $\Delta_F$ on the moduli parameters $z_1, \ldots, z_M$. Finally, in Section 5 we first define the modified zeta regularized determinant $\text{Det}'\Delta_F$ of the Friedrichs Laplacian on $(X, f^*m)$ and then prove the formula (1.1).

## 2 Preliminaries

Let $\Delta_F$ stand for the Friedrichs selfadjoint extension of the Laplace-Beltrami operator $\Delta$ on $(X, f^*m)$. In this subsection we list some results on asymptotic expansions at the critical points of $f$ for functions in the domain of $\Delta_F$. These results are similar to those that previously appeared in the context of flat conical metrics [6, 9] and standard curvature 1 metric on $\mathbb{C}P^1$ [10]. We only state the results, for the proofs we refer to [10 Sec. 2] as they can be repeated here almost verbatim.

In a vicinity of a critical point $P_k$, $k = 1, \ldots, M$, of $f : X \to \mathbb{C}P^1$ we introduce the distinguished local parameter

$$x = \sqrt{f(P) - f(P_k)} = \sqrt{z - z_k}.$$ \hspace{1cm} (2.1)

Since the critical value $z_k = f(P_k)$ of $f$ does not coincide with any conical singularity $p_j$ of $m$, we have $f^*m(x, \bar{x}) = 4\rho(x, \bar{x}) |x|^2 |dx|^2$ with smooth near $x = 0$ function $\hat{\rho}(x, \bar{x}) = \rho(x^2 + z_k, \bar{x}^2 + \bar{z}_k)$, where $\rho(z, \bar{z})$ is the conformal factor of $m$. Thus $f^*m$ has a conical singularity of angle $4\pi$ at $P_k$ (i.e. at $x = 0$).
Lemma 1. Denote by \( \mathcal{D}^* \) the domain of the operator in \( L^2(X) \) adjoint to the operator \( \Delta \) defined on \( C_0^\infty(X \setminus \{P_1, \ldots, P_k\}) \). For \( u \in \mathcal{D}^* \) in a small vicinity of \( x = 0 \) we have

\[
u(x, \bar{x}) = a_{-1} \bar{x}^{-1} + b_{-1} x^{-1} + a_0 \ln |x| + b_0 + a_1 \bar{x} + b_1 x + R(x, \bar{x}), \tag{2.2}\]

where \( a_k \) and \( b_k \) are some coefficients and the remainder \( R \) satisfies \( R(x, \bar{x}) = O(|x|^{2-\epsilon}) \) with any \( \epsilon > 0 \) as \( x \to 0 \). Moreover, the equality (2.2) can be differentiated and the remainder satisfies \( \partial_x R(x, \bar{x}) = O(|x|^{1-\epsilon}) \) and \( \partial_{\bar{x}} R(x, \bar{x}) = O(|x|^{1-\epsilon}) \) with any \( \epsilon > 0 \) if \( \Delta^* u = \lambda u \).

For \( u, v \in \mathcal{D}^* \) consider the form \( q[u, v] := (\Delta u, v) - (u, \Delta v) \); here and elsewhere \((\cdot, \cdot)\) stands for the inner product in \( L^2(X) \). By Lemma 1 we have (2.2) and

\[
v(x, \bar{x}) = c_{-1} \bar{x}^{-1} + d_{-1} x^{-1} + c_0 \ln |x| + d_0 + c_1 \bar{x} + d_1 x + \tilde{R}(x, \bar{x}). \tag{2.3}\]

The Stokes theorem implies

\[
q[u, v] = 2i \lim_{\epsilon \to 0^+} \int_{|x| = \epsilon} (\partial_x u)v \, dx + u(\partial_{\bar{x}} v) \, d\bar{x},
\]

where we express the right hand side in terms of the coefficients in (2.2), (2.3) and obtain

\[
q[u, v] = 4\pi (-a_{-1} \bar{d}_1 - b_{-1} \bar{c}_1 - b_0 \bar{c}_0/2 + a_0 \bar{d}_0/2 + b_1 \bar{c}_{-1} + a_1 \bar{d}_{-1}). \tag{2.4}\]

For the domain \( \mathcal{D} \) of the Friedrichs extension \( \Delta_F \) we have \( \mathcal{D} = \mathcal{D}^* \cap H^1(X) \), where \( H^1(X) \) is the domain of the closed densely defined quadratic form of \( \Delta \). Hence for any \( u \in \mathcal{D} \) we have (2.2) with \( a_{-1} = b_{-1} = a_0 = 0 \).

For a sufficiently small \( \delta > 0 \) we take a cut-off function \( \chi \in C_0^\infty(X) \) supported in the neighbourhood \( |x| < 2\delta \) of \( P_k \) and such that \( \chi(|x|) = 1 \) for \( |x| < \delta \). Denote the spectrum of \( \Delta_F \) by \( \sigma(\Delta_F) \) and introduce

\[
Y(\lambda) = \chi x^{-1} - (\Delta_F - \lambda)^{-1}(\Delta - \lambda)\chi x^{-1}, \quad \lambda \notin \sigma(\Delta_F), \tag{2.5}\]

where the function \( \chi x^{-1} \) is extended from the support of \( \chi \) to \( X \) by zero. It is clear that \( Y(\lambda) \in \mathcal{D}^* \) and \( Y(\lambda) \neq 0 \) as \( \chi x^{-1} \notin \mathcal{D} \). By Lemma 1 we have

\[
Y(x, \bar{x}; \lambda) = x^{-1} + c(\lambda) + a(\lambda) \bar{x} + b(\lambda) x + O(|x|^{2-\epsilon}), \quad x \to 0, \quad \epsilon > 0. \tag{2.6}\]

Lemma 2. The function \( Y(\lambda) \) in (2.5) and the coefficient \( b(\lambda) \) in (2.6) are analytic functions of \( \lambda \) in \( \mathbb{C} \setminus \sigma(\Delta_F) \) and in a neighbourhood of zero. Besides, we have

\[
4\pi \frac{d}{d\lambda} b(\lambda) = (Y(\lambda), \overline{Y(\lambda)}). \tag{2.7}\]

Lemma 3. Let \( \{\Phi_j\}_{j=0}^\infty \) be a complete set of real normalized eigenfunctions of \( \Delta_F \) and let \( \{\lambda_j\}_{j=0}^\infty \) be the corresponding eigenvalues, i.e. \( \Delta_F \Phi_j = \lambda_j \Phi_j, \Phi_j = \overline{\Phi_j} \), and \( \|\Phi_j; L^2(X)\| = 1 \). Then for the coefficients \( a_j \) and \( b_j = \bar{a}_j \) in the asymptotic

\[
\Phi_j(x, \bar{x}) = c_j + a_j x + \bar{a}_j \bar{x} + O(|x|^{2-\epsilon}), \quad x \to 0, \quad \epsilon > 0, \tag{2.8}\]

we have

\[
16\pi^2 \sum_{j=0}^\infty \frac{b_j^2}{(\lambda_j - \lambda)^2} = \left(Y(\lambda), \overline{Y(\lambda)}\right), \tag{2.9}\]

where the series is absolutely convergent.
3 Explicit formula for $b(-\infty)$

In this section we first study the behaviour of the coefficient $b(\lambda)$ from (2.6) as $\lambda \to -\infty$ and obtain an explicit formula for the limit $b(-\infty) = \lim_{\lambda \to -\infty} b(\lambda)$. This is the core of the problem. For singular flat and round metrics the corresponding explicit formulas were derived with the help of separation of variables [6, 9, 10]. In our setting this approach does not work and we use a different idea: In a neighbourhood of $x=0$ that shrinks to zero as $\lambda \to -\infty$ we construct an approximation for $Y(\lambda)$ that is sufficiently good to find $b(-\infty)$.

It is also important to notice that an expression for $b(-\infty)$ needs to be integrated (i.e. written as the derivative of some function with respect to $z_k$) before it can be used in explicit formulas for the determinant $\det' \Delta_F$. This can be not an easy task even if an explicit expression for $b(-\infty)$ was found by separation of variables [11] (e.g. separation of variables can be used to find $b(-\infty)$ if $m$ is a constant curvature singular metric) and hence one may want to use the general integrated explicit expression given in Lemma 4 below.

**Lemma 4.** As $\lambda \to -\infty$ for the coefficient $b(\lambda)$ in (2.6) we have

$$b(-\infty) = \partial_z \ln(\rho(z_k, z_k))^{-1/4}, \quad b(\lambda) = b(-\infty) + O(|\lambda|^{-1/2}),$$

where $\rho(z_k, z_k)$ is the value of the conformal factor of the metric $m(z, \tilde{z}) = \rho(z, \tilde{z})|dz|^2$ at the critical value $z_k$ of $f : X \to \mathbb{C}P^1$.

**Proof.** Consider a metric plane $(\mathbb{C}, m_0)$ isometric to an euclidean cone of total angle $4\pi$. Let $m_0(x, \tilde{x}) = 4\rho_0 |x|^2|\tilde{x}|^2$ with $\rho_0 := \rho(z_k, \tilde{z}_k)$, i.e. $m_0(x, \tilde{x})$ is the metric $f^*m(x, \tilde{x}) = 4\tilde{\rho}(x, \tilde{x})|x|^2|\tilde{x}|^2$ with conformal factor $\tilde{\rho}(x, \tilde{x})$ frozen at $x = 0$. Let

$$Y_0(k, x, \tilde{x}) = \frac{1}{x} e^{-kx\tilde{x}}, \quad k = \sqrt{-\rho_0 \lambda}, \quad 0 > \lambda \to -\infty.$$  

Clearly $Y_0(x, \tilde{x}; \lambda) = x^{-1} + O(1)$ as $x \to 0$ (cf. (2.6)) and $\Delta_0 Y_0 = \lambda Y_0$, where

$$\Delta_0 = -\frac{1}{\rho_0} \frac{1}{|x|^2} \partial_x \partial_{\tilde{x}}$$

is the Laplacian on $(\mathbb{C}, m_0)$. In the remaining part of the proof we first show that $Y_0$ is a sufficiently good approximation of $Y$ and then use $Y_0$ to calculate $b(-\infty)$.

Let $\chi_j \in C^\infty(\mathbb{R}^+), \ j = 1, 2$, be cut-off functions such that $\chi_j(t) = 1$ for $t \leq j$ and $\chi_j(t) = 0$ on $t \geq 3j/2$. For $x \in \mathbb{C}$ set $\chi_j(k, x, \tilde{x}) := \tilde{\chi}_j(k^{1/4}|x|)$, where $k > 0$ is a parameter. Clearly $\chi_1 \chi_2 = \chi_1$ and the support of $\chi_j$ shrinks to $x = 0$ as $k \to +\infty$ (i.e. as $\lambda \to -\infty$).

For all sufficiently large $k > 0$ we extend the functions $\chi_j Y_0$, $(\Delta_0 - \lambda)\chi_j Y_0 = [\Delta_0, \chi_j] Y_0$ (here $[a, b] = ab - ba$), and $(\Delta - \Delta_0)\chi_j Y_0$ (considered as functions of the distinguished local parameter (2.1)) from their supports in a small vicinity of $x = 0$ to $X$ by zero. From the explicit formula for $Y_0$ we immediately see that

$$||[\Delta_0, \chi_j] Y_0; L_2(X)|| = O(|\lambda|^{-\infty}) \quad \text{as} \ \lambda \to -\infty. \quad (3.1)$$

Hence

$$(\Delta - \Delta_0)\chi_j Y_0 = (\rho_0/\tilde{\rho} - 1) \Delta_0 \chi_j Y_0$$

$$= (\rho_0/\tilde{\rho} - 1)(\chi_j \Delta_0 Y_0 + [\Delta_0, \chi_j] Y_0)$$

$$= (\rho_0/\tilde{\rho} - 1) \chi_j Y_0 + O(|\lambda|^{-\infty}) \in L^2(X).$$
On the next step we estimate \( \| (\hat{\rho}_0/\hat{\rho} - 1) \chi_j \lambda Y_0; L^2(X) \| \).

On the support of \( \chi_j \) we have \( k^{1/4} |x| \leq 3j/2 \). This together with

\[
\hat{\rho}_0/\hat{\rho}(x, \bar{x}) = 1 - \frac{p_{2k}(z_k, \bar{z}_k)}{\hat{\rho}_0} x^2 - \frac{p_{2k}(z_k, \bar{z}_k)}{\hat{\rho}_0} \bar{x}^2 + O(|x|^4)
\]

implies

\[
\hat{\rho}_0/\hat{\rho}(x, \bar{x}) - 1 = \frac{p_{2k}(z_k, \bar{z}_k)}{\hat{\rho}_0} x^2 - \frac{p_{2k}(z_k, \bar{z}_k)}{\hat{\rho}_0} \bar{x}^2 + O(k^{-1}), \quad x \in \text{supp} \chi_j.
\]

After the change of variables \((x, \bar{x}) \mapsto (k^{-1/2}x, k^{-1/2}\bar{x})\) in

\[
\| (\hat{\rho}_0/\hat{\rho} - 1) \chi_j \lambda Y_0; L^2(X) \|^2
= \int_{\mathbb{C}} \left( \frac{p_{2k}(z_k, \bar{z}_k)}{\hat{\rho}_0} x^2 + \frac{p_{2k}(z_k, \bar{z}_k)}{\hat{\rho}_0} \bar{x}^2 + O(k^{-1}) \right) dx \wedge d\bar{x}
\]

we obtain

\[
\| (\hat{\rho}_0/\hat{\rho} - 1) \chi_j \lambda Y_0; L^2(X) \|^2 \leq C \int_{\mathbb{C}} k^{-2} |\lambda|^2 e^{-2|\lambda|^2} k^{-1} dx \wedge d\bar{x} = O(k) = O(|\lambda|^{1/2}).
\]

We demonstrated that

\[
\| (\Delta - \Delta_0) \chi_j Y_0; L^2(X) \| = O(|\lambda|^{1/4}) \quad \text{as} \quad \lambda \to -\infty.
\]

We represent the function \( Y \) from (3.3) in the form

\[
Y = \chi_j Y_0 + (\Delta_F - \lambda)^{-1} (\Delta - \lambda) \chi_j Y_0.
\]

Since

\[
(\Delta - \lambda) \chi_j Y_0 = (\Delta - \Delta_0) \chi_j Y_0 + [\Delta_0, \chi_j] Y_0,
\]

the estimates (3.1) and (3.2) together with \( \| (\Delta_F - \lambda)^{-1} \|; \mathcal{B}(L^2(X)) = O(|\lambda|^{-1}) \) and (3.3) imply

\[
\| Y - \chi_j Y_0; L^2(X) \| = O(|\lambda|^{-3/4});
\]

in this sense \( Y_0 \) is a good approximation of \( Y \).

Let \( b(\lambda) \) be the coefficient from the asymptotic (2.9) of \( Y \). We have

\[
4\pi b(\lambda) = q \left[ Y - \chi_1 Y_0, Y \right] = ((\Delta - \lambda)(Y - \chi_1 Y_0), Y) = (-(\Delta - \lambda) \chi_1 Y_0, Y)
\]

\[
= -
\left([-\Delta_0, \chi_1] Y_0 + (\Delta - \Delta_0) \chi_1 Y_0,
\right.
\]

\[
\left.\chi_2 Y_0 + (\Delta_F - \lambda)^{-1} ([\Delta_0, \chi_2] Y_0 + (\Delta - \Delta_0) \chi_2 Y_0) \right).
\]

Thanks to (3.1), (3.2), and (3.4), the right hand side of (3.5) takes the form

\[-(\Delta - \Delta_0) \chi_1 Y_0, Y \] + O(|\lambda|^{-1/2}) = -(\hat{\rho}_0/\hat{\rho} - 1) \chi_1 \lambda Y_0, Y\] + O(|\lambda|^{-1/2}).

It remains to calculate the value of the inner product in the right hand side. We have

\[-(\hat{\rho}_0/\hat{\rho} - 1) \chi_1 \lambda Y_0, Y\] = - \int_{\mathbb{C}} (\hat{\rho}_0 - \hat{\rho}(x, \bar{x})) \chi_1 (k^{1/4} |x|) \lambda \frac{1}{x^2} e^{-2|\lambda|^2} dx \wedge d\bar{x}.
\]
Here we substitute \( x = k^{-1/2}r e^{i\phi} \) (recall that \( \lambda = -k^2/\hat{\rho}_0 \)) and obtain
\[
4 \int_0^\infty \int_0^{2\pi} \left( 1 - \hat{\rho}(k^{-1/2}r, \phi)/\hat{\rho}_0 \right) \tilde{\chi}_1(k^{-1/4}r) k e^{-2i\phi} e^{-2r^2} r d\phi dr \\
= -\frac{4}{\hat{\rho}_0} \int_0^\infty \int_0^{2\pi} \left( \rho_{zk}(z_k, \bar{z}_k) \frac{r^2}{k} e^{2i\phi} \right. \\
\left. + \rho_{zk}(z_k, \bar{z}_k) \frac{r^2}{k} e^{-2i\phi} + O\left( \frac{|r|^4}{k^2} \right) \right) \\
\times \tilde{\chi}_1(k^{-1/4}r) k e^{-2|\phi|} e^{-2r^2} r d\phi dr \\
= -\frac{4}{\hat{\rho}_0} \int_0^\infty \left( 2\pi \rho_{zk}(z_k, \bar{z}_k) r^2 + O\left( \frac{|r|^4}{k} \right) \right) \tilde{\chi}_1(k^{-1/4}r) e^{-2r^2} r dr \\
= -\frac{8\pi \rho_{zk}(z_k, \bar{z}_k)}{\hat{\rho}_0} \int_0^{\infty} e^{-2r^2} r^3 dr + O(k^{-1}) = -\frac{\rho_{zk}(z_k, \bar{z}_k)}{\rho(z_k, \bar{z}_k)} + O(|\lambda|^{-1/2}).
\]

\[\square\]

4 Variation of eigenvalues under perturbation of conical singularities

Pick a noncritical value \( z_0 \in \mathbb{C} \) of \( f \) such that the critical values \( z_1, \ldots, z_M \) of \( f \) are the end points but not internal points of the line segments \([z_0, z_k], k = 1, \ldots, M\). The complement \( X \setminus f^{-1}(U) \) of the preimage \( f^{-1}(U) \) of the unit \( U = \bigcup_{k=1}^M [z_0, z_k] \) in \( X \) has \( N \) connected components (\( N \) sheets of the covering) and \( f \) is a biholomorphic isometry from each of these components equipped with metric \( f^*m \) to \( \mathbb{C}P^1 \setminus U \) equipped with metric \( m \). Thus the Riemann manifold \((X, f^*m)\) is isometric to the one obtained by gluing \( N \) copies of the Riemann sphere \((\mathbb{C}P^1, m)\) along the cuts \( U \) in accordance with a certain gluing scheme. Perturbation of the conical singularity at \( P_k \) is a small shift by \( w \in \mathbb{C}, |w| << 1 \), of the end \( z_k \) of the cut \([z_0, z_k]\) to \( z_k + w \) on those two copies of the Riemann sphere \((\mathbb{C}P^1, m)\) that produce \( 4\pi \)-conical angle at \( P_k \) after gluing along \([z_0, z_k]\).

Let \( g \in C_0^\infty(\mathbb{R}) \) be a cut-off function such that \( g(r) = 1 \) for \( x < \epsilon \) and \( g(r) = 0 \) for \( r > 2\epsilon \), where \( \epsilon \) is small. Consider the selfdiffeomorphism
\[\phi_w(z, \bar{z}) = z + g(|z - z_k|)w\]
of the Riemann sphere \( \mathbb{C}P^1 \), where \( w \in \mathbb{C} \) and \( |w| \) is small. On those two copies of the Riemann sphere that produce the conical singularity at \( P_k \) after gluing along \([z_0, z_k]\), we shift \( z_k \) to \( z_k + w \) by applying \( \phi_w \). We assume that the support of \( g \) and the value \( |w| \) are so small that only \([z_0, z_k]\) and no other cuts are affected by \( \phi_w \). Let \((X, f_w^*m)\) stand for the perturbed manifold, where \( f_w : X \to \mathbb{C}P^1 \) is the meromorphic function with critical values \( z_1, \ldots, z_k-1, z_k + w, z_k+1, \ldots, z_M \). Consider \((X, f_w^*m)\) as \( N \) copies of the Riemann sphere \( \mathbb{C}P^1 \) glued along the (unperturbed) cuts \( U \), however \( N - 2 \) copies are endowed with metric \( m \) and 2 certain copies (mutually glued along \([z_0, z_k]\)) are endowed with pullback \( \phi_w^*m \) of \( m \) by \( \phi_w \). By \( \Delta_w \) we denote the Friedrichs extension of Laplace-Beltrami operator on \((X, f_w^*m)\) and consider \( \Delta_w \) as a perturbation of \( \Delta_0 \) on \((X, f^*m)\).

In a vicinity of \( P_k \) for the matrix of \( f_w^*m \) we have
\[ [f_w^*m](x, \bar{x}) = \rho \circ \phi_w(x^2 - z_k, \bar{x}^2 - \bar{z}_k) \left( \phi_w'(x, \bar{x}) \right)^* \phi_w'(x, \bar{x}), \]
where \( x \) is the local distinguished parameter \((21)\) and
\[ \phi_w'(x, \bar{x}) = \text{Id} + \frac{g'(|x|^2)}{2|x|^2} \begin{bmatrix} wx^2 & w\bar{x}^2 \\ \bar{w}x^2 & \bar{w}\bar{x}^2 \end{bmatrix} \]

\[\text{7}\]

\]
is the Jacobian matrix; i.e. \( f_w^*m = 2[x\bar{d}x \ xdx][f_w^*m][xdx \ x\bar{d}x]^T \).

Notice that \( f^*m \neq f_w^*m \) only in a small neighborhood of \( P_k \), where \( |x| \leq \sqrt{2\epsilon} \), and \( f_w^*m \) is not in the conformal class of \( f^*m \) only for \( \sqrt{\epsilon} \leq |x| \leq \sqrt{2\epsilon} \), where \( g'(\epsilon^2) \neq 0 \). The norms in the spaces \( L^2(X, f^*m) \) and \( L^2(X, f_w^*m) \) are equivalent and the spaces can be identified.

A direct computation shows that

\[
\Delta_w - \Delta_0 = \left\{ -\frac{\partial (|x|^2)}{\partial x, \bar{x}} \left( \frac{\hat{\rho}_x(x, \bar{x})}{2x} w + \frac{\hat{\rho}_{\bar{x}}(x, \bar{x})}{2\bar{x}} \bar{w} \right) + \frac{\partial'(|x|^2)}{2|x|^2} (\hat{x}^2 w + \bar{x}^2 \bar{w}) \right\} \Delta_0
\]

\[
+ \frac{1}{2|x|^2 \hat{\rho}(x, \bar{x})} \left( \partial_x \partial_x' (|x|^2) w \partial_x + \partial_{\bar{x}} \partial_{\bar{x}}' (|x|^2) \bar{w} \partial_{\bar{x}} \right) + O(|w|^2),
\]

where \( \hat{\rho}(x, \bar{x}) = \rho(x^2 + z_k, \bar{x}^2 + \bar{z}_k) \) as before and \( O(|w|^2) \) stands for a second order operator with smooth coefficients supported in \( |x| \leq \sqrt{2\epsilon} \) and uniformly bounded by \( C|w|^2 \). Now it is easy to see that not the domain \( D_w^* \) of the operator adjoint to \( \Delta_w \mid_{C^0(X \setminus \{P_1, ..., P_k\})} \) nor the Sobolev space \( H^1(X, f_w^*m) \) (the domain of the closure of \( \Delta_w \mid_{C^0(X \setminus \{P_1, ..., P_k\})} \)) depend on \( w \). Hence the domain \( \mathcal{D} = D_w^* \cap H^1(X, f_w^*m) \) of the Friedrichs extension \( \Delta_w \) does not depend on \( w \) either. From now on we consider \( \Delta_w \) as an operator in \( L^2(X) := L^2(X, f^*m) \) with domain \( \mathcal{D} \).

Let \( \Gamma \) be a closed curve enclosing an eigenvalue \( \lambda \) of \( \Delta_0 \) of multiplicity \( m \) and not any other eigenvalues of \( \Delta_0 \). The resolvent \( (\Delta_w - \xi)^{-1} \) exists for all \( \xi \in \Gamma \) provided \( |w| \) is sufficiently small. Moreover, as \( |w| \to 0 \) the difference \( (\Delta_w - \xi)^{-1} - (\Delta_0 - \xi)^{-1} \) tends to zero in the norm of \( \mathcal{B}(L^2(X), \mathcal{D}) \) uniformly in \( \xi \in \Gamma \); here \( \mathcal{D} \) is equipped with the graph norm of \( \Delta_0 \). The continuity of the total projection

\[
P_w = -\frac{1}{2\pi i} \int_{\Gamma} (\Delta_w - \xi)^{-1} d\xi
\]

implies that \( \dim P_w L^2(X) = \dim P_0 L^2(X) = m \); i.e. the sum of multiplicities of the eigenvalues of \( \Delta_w \) lying inside \( \Gamma \) is equal to \( m \) (provided \( |w| \) is small). These eigenvalues are said to form a \( \lambda \)-group, see e.g. [12].

**Lemma 5.** Consider a \( \lambda \)-group \( \lambda_1(w), ..., \lambda_m(w) \), here \( \lambda_j(w) \to \lambda_j = \lambda \) as \( w \to 0 \). Let \( \Phi_1, ..., \Phi_m \) be (real) normalized eigenfunctions of \( \Delta_0 \) corresponding to \( \lambda \), i.e. \( \Phi_j = \Phi_j, \|\Phi_j; L^2(X)\| = 1, \) and \( \text{span}\{\Phi_1, ..., \Phi_m\} = P_0 L^2(X) \). Then as \( w \to 0 \) we have

\[
\sum_{j=1}^{m} \frac{1}{(\xi - \lambda_j(w))^2} = \frac{m}{(\xi - \lambda)^2} + \frac{2(Aw + B\bar{w})}{(\xi - \lambda)^3} + O(|w|^2),
\]

where \( A = \sum_{j=1}^{m} A_j \) and \( B = \sum_{j=1}^{m} B_j \) with coefficients \( A_j \) and \( B_j \) from the expansion

\[
((\Delta_w - \Delta_0)\Phi_j, \Phi_j)_{L^2(X)} = A_j w + B_j \bar{w} + O(|w|^2).
\]

**Proof.** For the proof we refer to [10] Lemmas 5.1—5.3. In the proof of [10] Lemma 5.1 the formula (4.3) was obtained (see [10] f-la (28)) and then the coefficients \( A \) and \( B \) were calculated for curvature 1 conical metrics. That calculation does not work for metrics we study here and therefore should be omitted. With formulas for coefficients replaced by (4.3) Lemmas 5.1—5.3 in [10] and their proofs can be repeated here verbatim. The assertion of this Lemma 5 is then the one of [10] Lemma 5.3. \( \square \)
As a consequence of (4.1) for the coefficients $A_j$ and $B_j$ in (4.3) we obtain

$$A_j = -\int_{|x|<\sqrt{2}r} \left[ \left( 2x \partial_x (|x|^2) \rho_x + 2x^2 \partial_x' (|x|^2) \rho \right) \lambda \Phi_j^2 + 2 \partial_x' (|x|^2) \left( \partial_x \Phi_j \right)^2 \right] \frac{dx \wedge d\bar{x}}{-2i},$$

$$B_j = -\int_{|x|<\sqrt{2}r} \left[ \left( 2x \partial_x (|x|^2) \rho_x + 2x^2 \partial_x' (|x|^2) \rho \right) \lambda \Phi_j^2 + 2 \partial_x' (|x|^2) \left( \partial_x \Phi_j \right)^2 \right] \frac{dx \wedge d\bar{x}}{-2i}.$$

This together with Stokes theorem gives

$$A_j = i \lim_{\epsilon \to 0^+} \oint_{|x|=\epsilon} \frac{1}{x} (\partial_x \Phi_j)^2 \, dx - \lambda \Phi_j^2 \rho \, dx,$$

$$B_j = -i \lim_{\epsilon \to 0^+} \oint_{|x|=\epsilon} \frac{1}{x} (\partial_x \Phi_j)^2 \, d\bar{x} - \lambda \Phi_j^2 \rho \, dx.$$ 

Now we use the asymptotic (2.8) of $\Phi_j(x, \bar{x}, \rho(x, \bar{x}))$ to conclude that $\Phi_j^2(x, \bar{x}, \rho(x, \bar{x}))$ is bounded near $x = 0$. By Lemma 1 the asymptotic can be differentiated. Thus for any $\epsilon > 0$ we have $\partial_x \Phi_j = b_j + O(|x|^{1-\epsilon})$ as $x \to 0$. We arrive at

$$A_j = i \lim_{\epsilon \to 0^+} \oint_{|x|=\epsilon} \frac{1}{x} (\partial_x \Phi_j)^2 \, dx = 2\pi b_j^2,$$

$$B_j = -i \lim_{\epsilon \to 0^+} \oint_{|x|=\epsilon} \frac{1}{x} (\partial_x \Phi_j)^2 \, d\bar{x} = 2\pi \bar{b}_j^2.$$ 

5 Explicit formula for $\text{Det}' \Delta_F$

In this subsection we first define the modified (i.e. with zero eigenvalue excluded) zeta regularized determinant $\text{Det}' \Delta_F$ and then prove the explicit formula (1.1).

By [27] Theorem 4.1, near any conical singularity of $f^*m$ there exist smooth local polar geodesic coordinates $(r, \theta)$ such that

$$f^*m = dr^2 + h^2(r, \theta) d\theta^2,$$

where

$$\lim_{r \to 0} \frac{h(r, \theta)}{r} = 1, \quad h_r(0, \theta) = 1, \quad h_{rr}(0, \theta) = 0$$

for $\theta \in [0, \gamma)$; here $h_r = \partial_r h$, $h_{rr} = \partial_r^2 h$, and $\gamma = 2\pi (\beta_j + 1)$ (resp. $\gamma = 4\pi$) if the point $r = 0$ is a preimage $f^{-1}(p_j)$ of conical singularity of $m$ (resp. a critical point of $f$).

Let $\chi \in C^\infty(X)$ be a cutoff function supported in a small neighbourhood $r < \epsilon$ of the conical singularity and such that $\chi = 1$ for $r < \epsilon/2$. Then $\chi \Delta_F$ can be considered as the operator

$$\chi h^{-1/2} \left( -\partial_r^2 + r^{-2} A(r) \right) h^{1/2}$$

acting in $L^2(h(r, \theta) \, dr \, d\theta)$, where

$$[0, \epsilon] \ni r \mapsto A(r) = -r^2 \left( \frac{h_r^2}{4h^2} + \frac{h_{rr}}{2h} + h^{1/2} \left( \frac{1}{h} \partial_r \right)^2 h^{-1/2} \right).$$

9
is a smooth family of operators on the circle $\mathbb{R}/\gamma \mathbb{Z}$. Therefore the results of [3] are applicable. By [3, Thm 5.2 and Thm 7.1] we have

$$
\text{Tr } \chi e^{-\Delta_F t} \sim \sum_{j=0}^{\infty} A_j t^{i \frac{3}{2}} + \sum_{j=0}^{\infty} B_j t^{-\frac{\alpha_j+4}{2}} + \sum_{j: \alpha_j \in \mathbb{Z}} C_j t^{\alpha_j+4} \log t \quad \text{as } t \to 0^+ \quad \text{(5.1)}
$$

with some coefficients $A_j, B_j,$ and $C_j$, and a sequence of complex numbers $\{\alpha_j\}, \Re \alpha_j \to -\infty$. The coefficient $C_j$ before $t^0 \log t$ in (5.1) is given by $\frac{1}{4} \text{Res } \zeta(-1)$, where $\zeta$ is the $\zeta$-function of $(A(0) + 1/4)^{1/2}$; see [3, f-la (7.24)]. Since $A(0) = -\partial_{\theta}^2 - 1/4$, we have

$$
\zeta(s) = 2 \sum_{j \geq 1} (\pi j/\gamma)^{-s} = 2(\gamma/\pi)^s \zeta_R(s),
$$

where $\zeta_R$ stands for the Riemann zeta function. Thus $\text{Res } \zeta(-1) = 0$ and the coefficient $C_j$ before $t^0 \log t$ is zero.

For any cut-off function $\chi \in C^\infty(X)$ supported outside of conical singularities of $f^*m$ the short time asymptotic $\text{Tr } \chi e^{-\Delta_F t} \sim \sum_{j \geq -2} a_j t^{j/2}$ can be obtained in the standard well known way, see e.g. [24, Problem 5.1] or [25].

In summary, there is no term $C_j t^0 \log t$ in the short time asymptotic of $\text{Tr } e^{-\Delta_F t}$ and hence the $\zeta$-function

$$
\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1}(\text{Tr } e^{-t\Delta_F} - 1) \, dt
$$

has no pole at zero. Now we are in position to define the modified determinant

$$
\text{Det}' \Delta_F := \exp\{-\zeta'(0)\}.
$$

Chose oriented paths $a_1, \ldots, a_g$ and $b_1, \ldots, b_g$ marking the Riemann surface $X$ of genus $g$. Let $\{v_1, \ldots, v_g\}$ be the normalized basis of holomorphic differentials on $X$, i.e. $\int_{a_t} v_m = \delta_{tm}$, where $\delta_{tm}$ is the Kronecker delta. Introduce the $g \times g$-matrix $B = (B_{tm})$ of $b$-periods with entries $B_{tm} = \int_{b_t} v_m$. The following theorem is the main result of this paper.

**Theorem 1.** Let $m$ stand for a conical metric on the Riemann sphere $\mathbb{CP}^1$ such that the curvature of $m$ is smooth and the volume is finite. Consider a meromorphic function $f : X \to \mathbb{CP}^1$ of degree $N$ with simple poles and $M = 2N + 2g - 2$ simple critical points $P_k$. Assume that the critical values $z_k = f(P_k)$ of $f$ are finite and do not coincide with conical singularities of $m$. Consider the determinant $\text{Det}' \Delta_F$ of the Friedrichs extension $\Delta_F$ of the Laplacian on $(X, f^*m)$ as a functional on the Hurwitz moduli space $H_{g,N}(1, \ldots, 1)$ of pairs $(X, f)$ with local coordinates $z_1, \ldots, z_M$. Then the explicit formula

$$
\text{Det}' \Delta_F = C \text{det } \Im B |\tau|^2 \prod_{k=1}^M \sqrt[8]{\rho(\bar{z}_k, z_k)}
$$

holds true, where $C$ is independent of the point $(X, f)$ of $H_{g,N}(1, \ldots, 1)$, $B$ is the matrix of $b$-periods on $X$, and $\tau$ is the Bergman $\tau$-function on $H_{g,N}(1, \ldots, 1)$. (The factor $\text{det } \Im B$ should be omitted if $X$ is a genus zero surface.)
Proof. The proof runs along the lines of [10] Proposition 6.1 and Theorems 6.2 and 6.3 with some minor changes due to replacement of the formula \( b(-\infty) = \frac{1}{2\pi i} \int_{|z_k|}^{\infty} \) (valid in the case of standard round metric \( m \) on \( \mathbb{CP}^1 \)) by the general formula obtained in Lemma 4. Notice also that the final results of Section 4 i.e. f-las (4.2) and (4.3), turn out to be the same as in the case of standard round metric, thus requiring no changes in the proof. For these reasons we only outline the main steps of the proof below and refer to [10] Proposition 6.1 and Theorems 6.2, 6.3 for details.

First we compute the partial derivative of zeta function with respect to \( z_k \). Let \( \Gamma_\lambda \) be a contour running clockwise at a sufficiently small distance \( \epsilon > 0 \) around the cut \((-\infty, \lambda] \). Thanks to Lemma 5 and (4.4) we obtain

\[
\partial_{z_k} \zeta(s; \Delta_F - \lambda) = \frac{-i}{16\pi^2} \int_{\Gamma_\lambda} (\xi - \lambda)^{-s} \partial_{z_k} \text{Tr}(\Delta_F - \xi)^{-2} d\xi
\]

We integrate in the right hand side by parts and then use the equality (2.9) from Lemma 3 together with Lemma 2 and the estimate from Lemma 4. We get

\[
\partial_{z_k} \zeta(s; \Delta_F - \lambda) = \frac{-i}{16\pi^2} \int_{\Gamma_\lambda} (\xi - \lambda)^{-s} \left( \frac{d}{d\xi} \left( b(\xi) - b(-\infty) \right) \right) d\xi
\]

Since \( \xi \mapsto b(\xi) \) is holomorphic in \( \mathbb{C} \setminus \sigma(\Delta_F) \) and in a neighbourhood of zero (Lemma 2), the Cauchy Theorem implies

\[
\partial_{z_k} \zeta'(0; \Delta_F) = \frac{1}{4\pi i} \int_{\Gamma_\lambda} (\xi - \lambda)^{-1} \left( b(\xi) - b(-\infty) \right) d\xi = \frac{b(-\infty) - b(0)}{2}.
\]

The coefficient \( b(0) \) is conformally invariant \( (Y(0) \) is a harmonic function bounded everywhere on \( X \) except for the point \( P_k \); see (2.5), (2.6), and Lemma 2) and thus the equality

\[
b(0) = \partial_{z_k} \ln(\det \Im B |^2 \tau |^2)
\]

can be obtained in exactly the same way as in [10] Lemma 4.2 or [9] Prop. 6]. Since \( \text{Det}' \Delta = \exp\{-\zeta'(0)\} \), this together with formula for \( b(-\infty) \) (see Lemma 4) gives

\[
\partial_{z_k} \ln \text{Det}' \Delta_F = \partial_{z_k} \ln(\det \Im B |^2 \tau |^2) - \partial_{z_k} \ln(\rho(z_k, \bar{z_k}))^{-1/8}, \quad k = 1, \ldots, M.
\]

Similarly, \( \partial_{z_k} \ln \text{Det}' \Delta_F \) is given by the conjugate of the right hand side in (5.2). This completes the proof.

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