Low-frequency phase diagram of irradiated graphene and periodically driven spin-1/2 XY chain

Bhaskar Mukherjee\(^1\), Priyanka Mohan\(^2\), Diptiman Sen\(^3\), and K. Sengupta\(^1\)

\(^1\)Theoretical Physics Department, Indian Association for the Cultivation of Science, Jadavpur, Kolkata 700032, India
\(^2\)Harish-Chandra Research Institute, HBNI, Chhatnag Road, Jhunsi, Allahabad 211019, India
\(^3\)Center for High Energy Physics, Indian Institute of Science, Bengaluru, 560012, India

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We study the Floquet phase diagram of two-dimensional Dirac materials such as graphene and the one-dimensional (1D) spin-1/2 XY model in a transverse field in the presence of periodic time-varying terms in their Hamiltonians in the low drive frequency (\(\omega\)) regime where standard 1/\(\omega\) perturbative expansions fail. For graphene, such periodic time dependent terms are generated via the application of external radiation of amplitude \(A_0\) and time period \(T = 2\pi/\omega\), while for the 1D XY model, they result from a two-rate drive protocol with time-dependent magnetic field and nearest-neighbor couplings between the spins. Using the adiabatic-impulse method, whose predictions agree almost exactly with the corresponding numerical results in the low-frequency regime, we provide several semi-analytic criteria for the occurrence of changes in the topology of the phase bands (eigenstates of the evolution operator \(U\)) of such systems. For irradiated graphene, we point out the role of the symmetries of the instantaneous Hamiltonian \(H(t)\) and the evolution operator \(U\) behind such topology changes. Our analysis reveals that at low frequencies, topology changes of irradiated graphene phase bands may also happen at \(t = T/3, 2T/3\) (apart from \(t = T\)) showing the necessity of analyzing the phase bands of the system for obtaining its phase diagrams. We chart out the phase diagrams at \(t = T/3, 2T/3,\) and \(T,\) where such topology changes occur, as a function of \(A_0,\) and \(T\) using exact numerics, and compare them with the prediction of the adiabatic-impulse method. We show that several characteristics of these phase diagrams can be analytically understood from results obtained using the adiabatic-impulse method and point out the crucial contribution of the high-symmetry points in the graphene Brillouin zone to these diagrams. We study the modes which can appear at the edges of a finite-width strip of graphene and show that the change in the number of such modes agrees with the change in the Chern number of bulk graphene as we go across a phase band crossing. Finally we study the 1D XY model with a two-rate driving protocol. After studying the symmetries of the system, we use the adiabatic-impulse method and exact numerics to study its phase band crossing which occurs at \(t = T/2\) and \(k = \pi/2.\) We also study the end modes generated by such a drive and show that there can be anomalous modes whose Floquet eigenvalues are not equal to \(\pm 1.\) We suggest experiments to test our theory.

I. INTRODUCTION

The physics of closed quantum systems driven out of equilibrium has attracted a lot of theoretical interest in recent times\(^{14}\) Such dynamics becomes particularly interesting during the passage of the system through a quantum critical point where it becomes non-adiabatic. The excess energy \(\delta E\) and density of excitations \(n\) through such a critical point obey, for slow, linear or non-linear, power-law quenches, universal scaling laws\(^{5–11}\) More recently, analogous scaling laws have also been derived for fast quenches\(^{12,13}\) The properties of such driven systems have also been studied following a sudden quench where a parameter of the system is changed instantly. Such sudden quenches lead to transient oscillations whose amplitude peaks when the final Hamiltonian is near a critical point\(^{14–16}\) in addition they lead to interesting steady states\(^{16}\) Moreover, the work statistics of such systems is tied to the Loschmidt echo and may display edge singularities\(^{17}\) Such out-of-equilibrium dynamics also leads to dynamical transitions which have no counterparts in equilibrium systems\(^{18–20}\) More recently, such studies has been extended to cases where two parameters of the system Hamiltonian are varied as functions of time with different rates; such dynamics leads to a generalization of the well-known Kibble-Zurek scaling law\(^{21}\) and provide a route to realization of quantum dynamics with controlled fidelity\(^{22}\) Furthermore, there have been several studies on the applicability of renormalization group methods for such out-of-equilibrium systems; such studies are expected to shed light on the possibility extending the concept of universality to such driven systems\(^{24–25}\) The motivation for such theoretical studies has received experimental support from ultracold atom system\(^{26}\) The isolated nature of such systems makes them perfect test beds for studying coherent quantum dynamics of closed non-equilibrium systems\(^{27–28}\)

More recently, a significant emphasis in theoretical studies of driven closed quantum systems has shifted to systems driven out of equilibrium using a periodic protocol. The chief reason for this stems from the recognition that such drives lead to a host of interesting phenomena which have no counterparts in aperiodically driven systems. For example, periodically driven systems exhibit Stuckelberg interference phenomenon\(^{29}\) the signature of such interference phenomenon leads to experimentally discernible features in their excitation densities and the statistics of work distribution\(^{30–31}\) Moreover, such driven integrable systems exhibit a separate class of dynamical transition which leaves its signature on local correlation functions\(^{32}\) In addition, they also exhibit dynamic freezing at specific frequencies; at these frequencies the wave function of the system after a single or multiple drive period(s) exhibits a near unity overlap with the initial wave function\(^{33–35}\) Furthermore, such drives may also lead to novel steady states which do not have any aperiodic counterparts\(^{36}\)

An aspect of periodically driven closed quantum systems
that has gained recent attention involves a change in topology and the concomitant generation of edge modes of these systems as a function of the drive frequency. This phenomenon has been mostly studied either in the context of graphene or topological insulators (whose low-energy quasiparticles obey a Dirac-like equation) in the presence of circularly polarized light or for model Hamiltonians with engineered drive protocols which allow for a simple analysis of the phenomenon. A central role in such studies is played by the time-evolution operator of the driven system which can be expressed, in terms of its Hamiltonian $H(t)$, as

$$U(t, 0) = T e^{-i \int_0^t dt' H(t')/\hbar},$$  \hspace{1cm} (1)$$

where $T$ denotes time ordering. The Floquet Hamiltonian $H_F$, which describes the properties of the driven system at the end of an integer number of drive periods can be read off from $U$ using the definition

$$U(nT, 0) = e^{-inH_FT/\hbar}, \hspace{1cm} n \in \mathbb{Z}. \hspace{1cm} (2)$$

Initial works on the subject analyzed such changes of topology via a study of the properties of $H_F$. However, it was later found that the study of the Floquet Hamiltonian, which amounts to a stroboscopic tracking of the time evolution after an integer number of time periods, is not always adequate for this purpose. Instead, it is sometimes necessary to track the time evolution of the phase bands, i.e., the eigenvalues $U(t, 0)$, which control the dynamics of the system within a single time period $T$. The crossing of such phase bands has been shown to be intimately tied to the change in the topology of the driven system. The precise conditions for the occurrence of such phase band crossings leading to a change in the topology of the system and the generation of edge modes has been charted out for cases where a single parameter of the system Hamiltonian is driven periodically. However such an analysis has not been extended to two-rate protocols, i.e., to situations where two parameters of the system Hamiltonian are driven periodically.

Examples of the latter class of driven systems include graphene in the presence of circularly polarized external radiation as studied in Refs. and. We note here that the grapheme in the presence of circularly polarized external radiation has also not been studied so far in the low drive frequency regime. The plan of this paper is as follows. In Sec. we chart the symmetries of the instantaneous Hamiltonian and the evolution operator describing graphene under external radiation. This is followed by Sec. where we present details of the adiabatic-impulse approximation and chart out the generic conditions for topology change of driven systems based on this approximation. Next, in Sec. we compare the results obtained from the adiabatic-impulse approximation with exact numerics, chart out semi-analytic conditions for phase band crossings, and present the phase diagram of irradiated graphene obtained using exact numerics and adiabatic-impulse approximation method. This is followed by Sec. where we study the driven 1D XY model and discuss its phase band crossings and end modes. Finally we summarize our results, discuss their experimental implications, and conclude in Sec.
We will choose the Brillouin zone to be a hexagon as shown in Fig. 1. To this end, we take the nearest-neighbor spacing in graphene to be $a_0$ and the hopping amplitude to be $-\gamma$. (We shall henceforth set $\gamma$ and $a_0$ equal to 1 unless mentioned otherwise). Given a site on the $A$ sublattice, we choose the vectors to its three nearest-neighbor sites on the $B$ sublattice to be

$$\vec{a}_1 [2] = (1/2, +[-\sqrt{3}/2]), \quad \vec{a}_3 = (-1, 0).$$

The reciprocal lattice vectors are

$$\vec{G}_1 = (2\pi/3, 2\pi/\sqrt{3}), \quad \vec{G}_2 = (2\pi/3, -2\pi/\sqrt{3}).$$

We will choose the Brillouin zone to be a hexagon as shown in Fig. 1.

We now apply circularly polarized electromagnetic radiation to graphene. We take the vector potential $\vec{A} = (A_x, A_y)$ to be of the form $A_x = A_0 \cos(\omega t)$ and $A_y = A_0 \sin(\omega t)$, where $A_0$ and $\omega = 2\pi/T$ are respectively the strength and frequency of the radiation and $T$ is its period time. Following the Peierls prescription, the vector potential is incorporated into the hopping amplitude as phases given by $-(e/c)\vec{a}_j \cdot \vec{A}$ on a bond labeled as $j$, where $-e$ is the charge of the electron and $c$ is the speed of light. Let us define the parameter $\alpha = eA_0/c$. In momentum space, the hopping amplitude on bond $j$ takes the form $-\gamma \exp[i\vec{a}_j \cdot (\vec{k} + \alpha \vec{A})]$. The Hamiltonian of graphene then takes the form

$$H_G(t) = \sum_{\vec{k}} \psi^\dagger_{\vec{k}}(t) H_{\vec{k}}(t) \psi_{\vec{k}},$$

$$H_{\vec{k}}(t) = Z_{\vec{k}}(t) \tau_+ + Z_{\vec{k}}^*(t) \tau_-, $$

$$Z_{\vec{k}}(t) = -e^{i(\alpha \cos(\omega t - \pi/3) + (k_x + \sqrt{3}k_y)/2)} - e^{i(\alpha \cos(\omega t + \pi/3) + (k_x - \sqrt{3}k_y)/2)} - e^{-i(\alpha \cos(\omega t) + k_x)},$$

where $\tau_{\pm} = (1/2)(\tau_x \pm i\tau_y)$. Defining $B_{\vec{k}}(t) = \text{Re}[Z_{\vec{k}}(t)]$ and $C_{\vec{k}}(t) = \text{Im}[Z_{\vec{k}}(t)]$, we obtain

$$H_{G}(t) = B_{\vec{k}}(t) \tau_x - C_{\vec{k}}(t) \tau_y.$$  

We can perform a unitary transformation to convert this to

$$H_{G}(t) = -C_{\vec{k}}(t) \tau_z + B_{\vec{k}}(t) \tau_x.$$  

We will use the form in Eq. (7) in the rest of this section; the fact that $\tau_x$ and $\tau_z$ are real and symmetric will prove to be very useful. Note that the instantaneous eigenvalues of the Hamiltonian in Eq. (7) are given by $E_{\vec{k}}^\pm = \pm |Z_{\vec{k}}(t)|$. We note here that both the diagonal and off-diagonal terms in Eq. (7) depends explicitly on time; thus irradiated graphene constitutes an example of the periodic version of the two-rate protocol studied in context of other models in Ref. [22].

In the rest of this section we will discuss the symmetries of the momentum space Hamiltonian $H_{G}(t)$ and the corresponding time evolution operator

$$U_{\vec{k}}(t, 0) \equiv U_{\vec{k}}(t) = T_\alpha \exp[-i \int_0^t dt' H_{\vec{k}}(t')]$$

at various points $\vec{k} = (k_x, k_y)$ in the Brillouin zone. Note that under $\vec{k} \rightarrow \vec{k} + \vec{G}_1$ or $\vec{k} + \vec{G}_2$, $Z_{\vec{k}}(t) \rightarrow e^{-i2\pi/3}Z_{\vec{k}}(t)$. Hence the symmetries of $H_{G}(t)$ are similar for points in $\vec{k}$ space which are related to each other by reciprocal lattice vectors. When looking for symmetries, we will consider arbitrary values of $t$ in Eq. (8), not just $t = T$. Further, we will look for symmetries which hold for all values of $\alpha$ and $T$.

**Γ point:** The Hamiltonian $H_{G}$ at the Γ point $[(k_x, k_y) = (0, 0)]$ has several underlying symmetries which can be deduced by an inspection of $H_{G}$ and $Z_{\vec{k}}$ (Eq. (5)). These are
given by

\[ H_\mathbf{c}(T-t) = H_\mathbf{c}(t) \quad \text{[since } Z_\mathbf{c}(T-t) = Z_\mathbf{c}(t) \text{]} \]  
\[ H_\mathbf{c}(T/6 - t) = H_\mathbf{c}(T/6 + t) = \tau_x H_\mathbf{c}(t) \tau_x \]  
\[ = [\text{since } Z_\mathbf{c}(T/6 - t) = Z_\mathbf{c}(t)]. \]  
\[ H_\mathbf{c}(T/3 - t) = H_\mathbf{c}(T/3 + t) = H_\mathbf{c}(t). \]  
\[ H_\mathbf{c}(T/2 - t) = H_\mathbf{c}(T/2 + t) = \tau_x H_\mathbf{c}(t) \tau_x. \]  
\[ H_\mathbf{c}(2T/3 - t) = H_\mathbf{c}(2T/3 + t) = H_\mathbf{c}(t). \]  
\[ H_\mathbf{c}(5T/6 - t) = H_\mathbf{c}(5T/6 + t) = \tau_x H_\mathbf{c}(t) \tau_x. \]  

The above relations imply that the instantaneous eigenvalues of \( H_\mathbf{c}(t) \) are identical at times \( nT/6 \pm t \) for \( n = 1, 2, \ldots, 5 \), and at \( T-t \). These relations also lead to the following symmetry properties of \( U(t) \). On the right hand side of Eq. (8), we can divide the integral in the exponential into \( N_t \) steps, each of size \( \Delta t = t/N_t \). Defining \( t_j = (j-1/2)\Delta t \), we can write Eq. (8) as

\[ U(t) = e^{-i\Delta t H(t_{N_t})} e^{-i\Delta t H(t_{N_t-1})} \ldots \]  
\[ \ldots e^{-i\Delta t H(t_1)} e^{-i\Delta t H(t_0)} \]  

in the limit \( N_t \rightarrow \infty \). Eq. (9) and the symmetry of the Pauli matrices in Eq. (7) implies that

\[ H(t_{N_t+1-j}) = H(t_j) = [H(t_j)]^T, \]  

where the superscript \( T \) means transpose. This, in turn, indicates that

\[ [U(T)]^T = U(T), \]  

which means that \( U(T) \) is of the form

\[ U(T) = \pm \exp[i(d_1 \tau_x + d_3 \tau_y)], \]  

namely, \( \tau_y \) does not appear in the exponential. In Eq. (18), the parameters \( d_1, d_3 \) are real and satisfy \( 0 \leq \sqrt{d_1^2 + d_3^2} < \pi \). We have allowed for a \( \pm \) sign in Eq. (18) to ensure that \( \sqrt{d_1^2 + d_3^2} \) is strictly less than \( \pi \).

Since only two parameters, \( d_1 \) and \( d_3 \), appear in Eq. (18), there is a possibility that we can make both parameters equal to zero by suitably choosing the two parameters \( \alpha \) and \( T \) appearing in the definition of \( U(T) \) in Eq. (8). This means that there is a possibility of varying \( \alpha \) and \( T \) to make \( U(T) = \pm I \) (\( I \) denotes the \( 2 \times 2 \) identity matrix), so that there is a phase band crossing at \( t = T \).

Similar arguments based on Eqs. (11) and (12) imply that \( U(T/3) \) and \( U(2T/3) \) also have the two-parameter form in Eq. (18) and can therefore be made equal to \( \pm I \) by choosing \( \alpha \) and \( T \) appropriately. From Eqs. (11) and (13), we also find that

\[ U(2T/3) = [U(T/3)]^2 \quad U(T) = [U(T/3)]^3. \]  

Hence, \( U(T/3) = \pm I \) implies that \( U(T) = \pm I \); hence a phase band crossing at \( t = T/3 \) implies a crossing at \( t = T \). The converse is not necessarily true; we can have \( U(T) = \pm I \) without having \( U(T/3) = \pm I \).

Next, we can use Eq. (12) and the expression in Eq. (15) to show that

\[ U(T/2) = \tau_x [U(T/2)]^T \tau_x. \]  

This implies that \( U(T/2) \) has the form

\[ U(T/2) = \pm \exp[i(d_1 \tau_x + d_3 \tau_y)], \]  

namely, \( \tau_z \) does not appear in the exponential. Thus \( U(T/2) \) also has a two-parameter form; hence it may be possible to find \( \alpha \) and \( T \) so that \( U(T/2) = \pm I \) and thereby have a phase band crossing at \( t = T/2 \). Eq. (12) also implies that

\[ U(T) = \tau_x U(T/2) \tau_x U(T/2). \]  

Eqs. (21) and (22) imply that if we can make \( d_1 = 0 \), we will have \( U(T) = I \). Thus we only need to make one parameter, \( d_1 \), equal to zero by varying \( \alpha \) and \( T \) in order to make \( U(T) = I \) and so have a phase band crossing at \( t = T \). Hence we expect that there may be a line in the \( \alpha - T \) plane where there is a phase band crossing at \( t = T \). Similar arguments based Eqs. (10) and (14) imply that \( U(T/6) \) and \( U(5T/6) \) also have the two-parameter form in Eq. (21) and can therefore be made equal to \( \pm I \) by choosing \( \alpha \) and \( T \) appropriately.

A key point that emerges from the discussion above is that whenever we find a fraction \( f \) (lying in the range \( 0 < f \leq 1 \)) such that either \( Z_\mathbf{c}(fT-t) = Z_\mathbf{c}(t) \) (implying \( H_\mathbf{c}(fT-t) = H_\mathbf{c}(t) \)) or \( Z_\mathbf{c}(fT-2T) = Z_\mathbf{c}(t) \) (implying \( H_\mathbf{c}(fT-2T) = \tau_x H_\mathbf{c}(t) \tau_x \)), \( U(fT) \) will have a two-parameter form given by either Eq. (18) or (21). One then expects that there would be points in the \( \alpha - T \) plane where \( U(fT) = \pm I \) so that there are phase band crossings at \( t = fT \). However, we note that such a two-parameter form of \( U(fT) \) is not sufficient to have a phase band crossing; we shall discuss this point in detail in Sec. III.

Dirac (K and K′) points: These are the two inequivalent points in the Brillouin zone where the conduction and valence bands of graphene touch in the absence of any external radiation; their positions in the Brillouin zone are given by \( \mathbf{k} = (0, \pm 4\pi/(3\sqrt{3})) \). Considering the point \( K = (0, 4\pi/(3\sqrt{3})) \), we find the following symmetries.
Hence the eigenvalues of \( H(t) \) are identical at the times \( T/6 - t, T/3 + t, T/2 - t, \) and \( 5T/6 - t \). However, from the point of view of phase band crossings, the only useful relation is Eq. (23). This implies that \( U(T/2) \) is of the form given in Eq. (21), and we can have a phase band crossing at \( t = T/2 \) (apart from those at \( t = T \)) at certain points in the \( \alpha - T \) plane. The symmetries of the \( K' \) points are identical.

\( M \) points: There are 3 sets of inequivalent points, namely \( M_{1,2,3} \). Out of these, the points \( M_3 \) lie at \( \vec{k} = (\pm 2\pi/3, 0) \). To be specific, we consider the point at \( \vec{k} = (2\pi/3, 0) \). We find the following symmetries.

\[
H_{\vec{k}}(T - t) = H_{\vec{k}}(t) \tag{28}
\]

\[
H_{\vec{k}}(T/2 - t) = H_{\vec{k}}(T/2 + t) = [(1/2)\tau_x - (\sqrt{3}/2)\tau_z]
\times H_{\vec{k}}(t)(1/2)\tau_x - (\sqrt{3}/2)\tau_z. \tag{29}
\]

Hence the eigenvalues of \( H(t) \) are identical at \( T - t \) and \( T/2 \pm t \). Eq. (28) implies that \( U(T) \) is of the form given in Eq. (18), and we can have a phase band crossing at \( t = T \) at certain points in the \( \alpha - T \) plane.

By rotating the \( M_3 \) points by \( \pm 2\pi/3 \), we obtain four other points; these are related pairwise by reciprocal lattice vectors. Hence we only have to consider two points, say, \( \vec{k} = (\pi/3, \pm \sqrt{3}/3) \). These are the \( M_{1,2} \) points alluded to in the last paragraph. We find the following symmetries at \( (\pi/3, \pi/\sqrt{3}). \)

\[
H_{\vec{k}}(T/3 - t) = H_{\vec{k}}(t) \tag{30}
\]

\[
H_{\vec{k}}(T/2 + t) = H_{\vec{k}}(5T/6 - t) = [(1/2)\tau_x + (\sqrt{3}/2)\tau_z]
\times H_{\vec{k}}(t)(1/2)\tau_x - (\sqrt{3}/2)\tau_z. \tag{31}
\]

Hence the eigenvalues of \( H(t) \) are identical at the times \( T/3 - t, T/2 + t, \) and \( 5T/6 - t \). Further, \( U(T/3) \) has the form given in Eqs. (18), implying that there can be phase band crossings at \( t = T/3 \) at certain points in the \( \alpha - T \) plane.

**X point:** As shown in Fig. 1, this point corresponds to the midpoint of the line joining the \( T \) point and the Dirac point lying at \((k_x, k_y) = (2\pi/3, 2\pi/3(\sqrt{3}))\); it has coordinates given by \((k_x, k_y) = (\pi/3, \pi/(3\sqrt{3}))\). At this point we find that \( Z_{\vec{k}}(T/6 - t) = Z_{\vec{k}}(t) \) which leads to

\[
H_{\vec{k}}(T/6 - t) = H_{\vec{k}}(7T/6 - t) = \sigma_x H_{\vec{k}}(t) \sigma_x. \tag{32}
\]

We can therefore expect phase band crossings at \( t = T/6 \) and \( t = 7T/6 \) at some suitably chosen points in the \( \alpha - T \) plane.

**Line given by \( \vec{k} = (k_x, 0) \):** We find only the symmetry \( Z_{\vec{k}}(T-t) = Z_{\vec{k}}(t) \) and \( H_{\vec{k}}(T-t) = H_{\vec{k}}(t) \) for an arbitrary point on the line \( k_y = 0 \) (but not at the Dirac points which have a larger symmetry as we have seen above). Hence the eigenvalues of \( H(t) \) and \( H(T-t) \) are identical, and \( U(T) \) has the form given in Eq. (18).

**Line given by \( \vec{k} = (0, k_y) \):** We find only the symmetry \( Z_{\vec{k}}(T/2 - t) = Z_{\vec{k}}(t) \) and \( H_{\vec{k}}(T/2 - t) = \tau_x H_{\vec{k}}(t) \tau_x \) for an arbitrary point on the line \( k_x = 0 \) (but not at the Dirac points which have a larger symmetry as discussed above). Hence the eigenvalues of \( H(t) \) and \( H(T/2-t) \) are therefore identical, and \( U(T/2) \) has the form given in Eq. (21).

Finally, in all the cases where at some value of \( \vec{k}, U_T(\vec{k}) \) has the form given in either Eq. (18) or Eq. (21) and we have a phase band crossing at \( t = T \) at a certain point in the \( \alpha - T \) plane, we can find two more points in the \( \vec{k} \) space where there will be a phase band crossing at the same values of \( \alpha \) and \( T \). This is because \( Z_{\vec{k}}(t) \) in Eq. (5) is invariant under a simultaneous rotation by \( 2\pi/3 \) in \( \vec{k} \) space and a shift in \( t \) by \( T/3 \). More precisely, we find that

\[
Z(k_x, k_y, t) = Z(-k_x/2 + \sqrt{3}k_y/2, -k_y/2 - \sqrt{3}k_x/2, t-T/3). \tag{33}
\]

Next, we note that if we hold \( \vec{k} \) fixed, set \( t = T \) and shift \( t' \rightarrow t' + s \) in Eq. (8), thus defining

\[
U_{\vec{k}}(T; s) = \exp[-i \int_0^{T+s} dt' H_{\vec{k}}(t')], \tag{34}
\]

then \( U_T(\vec{k}) \) in Eq. (8) and \( U_{\vec{k}}(T; s) \) in Eq. (34) have the same eigenvalues. (Their eigenvectors are related by a unitary transformation by the operator \( T \exp[-i \int_0^T dt H_{\vec{k}}(t')] \)). This means that if \( U(k_x, k_y, T) = \pm I \) with eigenvalues \( \pm 1 \) for some value of \( \alpha \) and \( T \), then \( U(-k_x/2 + \sqrt{3}k_y/2, -k_y/2 - \sqrt{3}k_x/2, t-T/3) \).
\[ \sqrt{3 k_x / 2}, T \] will also have the eigenvalues \pm 1. Thus a phase band crossing at \( t = T \) at some value of \( k \) means that there is also a phase band crossing at \( t = T \) at values of \( \vec{k} \) obtained by \( \pm 2\pi / 3 \) rotations. (This is ultimately related to the fact that the graphene lattice is invariant under rotations by \( \pm 2\pi / 3 \)). Thus, for example, the discussion of the line given by \( \vec{k} = (k_x, 0) \) show that we can also have phase band crossings at \( t = T \) on the two lines obtained by rotating the line \( \vec{k} = (k_x, 0) \) by \( \pm 2\pi / 3 \).

In the next section we use these symmetry properties to understand the condition of crossing of the phase bands of \( U_{\vec{k}}(t) \) at different symmetry points.

III. ADIABATIC-IMPULSE METHOD

In this section, we develop the adiabatic-impulse method and use it to compute the phase bands for a Dirac Hamiltonian in the presence of radiation in the low-frequency regime. The advantage of this method lies in the fact that it becomes accurate in the low-frequency regime where standard \( 1 / \omega \) expansions fail; thus this method serves as a complimentary analytic tool for understanding the low-frequency response of periodically driven systems. As shown in Refs. 42 and 45 we can relate \( \tilde{c}_{\vec{k}}(t) = (c_{1,\vec{k}}(t), c_{2,\vec{k}}(t))^T \) to \( \tilde{c}_{\vec{k}}(0) \) through a evolution matrix \( \tilde{U}_{\vec{k}}(t) \) given by

\[ \tilde{c}_{\vec{k}}(t) = \tilde{U}_{\vec{k}}(t, 0) \tilde{c}_{\vec{k}}(0). \]  

Note that \( \tilde{U}_{\vec{k}}(t, 0) \) is related to the evolution operator \( U_{\vec{k}}(t, 0) \) defined as

\[ \Psi_{\vec{k}}(t) = U_{\vec{k}}(t, 0) \Psi_{\vec{k}}(0) \]  

through the overlap \( \eta_{\vec{k}}(t) \) of the ground states in the adiabatic and diabatic bases as

\[ U_{\vec{k}}(t, 0) = \left[ \eta_{\vec{k}}(t) I - i \tau_y \sqrt{1 - \eta_{\vec{k}}^2(t)} \right] \tilde{U}_{\vec{k}}(t, 0), \]

\[ \eta_{\vec{k}}(t) = (\psi_{\vec{k}}^* (t))^T \psi_{\vec{k}}^0(t) = u_{\vec{k}}^*-t(t)u_{\vec{k}}^0(t) + v_{\vec{k}}^*-t(t)v_{\vec{k}}^0(t), \]

where \( \tau_{x,y,z} \) denote Pauli matrices and \( I \) is the \( 2 \times 2 \) identity matrix.

We now envisage a situation where the system goes through \( n \) avoided level crossings during a drive cycle. These avoided level crossings allow us to divide the time evolution of the system within a drive period into \( n + 1 \) regions which separate these crossings. The key assumption of the adiabatic-impulse approximation is that in these \( n + 1 \) regions, the systems undergoes adiabatic evolution and no excitations are produced. The evolution operator in the \( m \)th region can thus be written in the adiabatic basis as

\[ \tilde{U}_{\vec{k}}^{(m)}(t, t_0) = \exp[-i \tau_z \xi_{\vec{k}}^{(m)}(t, t_0)], \]

\[ \xi_{\vec{k}}^{(m)}(t, t_0) = \int_{t_0}^{t} dt' E_{\vec{k}}(t'). \]  

Note that \( \xi_{\vec{k}}^{(m)}(t, t_0) \) denotes the kinematic phase picked up by the wave function during the evolution, and the superscript \( m \) indicates that both \( t \) and \( t_0 \) lie in the \( m \)th adiabatic regime. For later use, we also define \( \xi_{\vec{k}}^{(i)}(t_{i\vec{k}}, t_{i-1\vec{k}}) \equiv \xi_{i\vec{k}} \) and \( \xi_{\vec{k}}^{(i)}(t, t_{i-1\vec{k}}) \equiv \xi_{i\vec{k}}(t). \)

At the avoided level crossing points separating two adiabatic regions, the evolution becomes non-adiabatic. The approximation involved in the present method is that it treats the impulse regions as isolated points where the avoided level crossings occur. Clearly, such an approximation becomes better at lower frequencies, and hence this method works well at small \( j \). The excitation probability at these points is usually estimated by linearizing the drive term around these regions.
which allows for an analytic calculation of the excitation probability (when a single (diagonal) term of the system Hamiltonian varies with time) via the Kibble-Zurek approach\(^\text{[12-15]}\). However, the present class of systems, where both diagonal and off-diagonal elements of the Hamiltonian vary with time, calls for a modification of this procedure. To demonstrate this modification, we consider the Hamiltonian given by Eq. (7) near the \(j^\text{th}\) avoided level crossing which separates the \((j-1)^\text{th}\) and \(j^\text{th}\) adiabatic regions. The time \(t_{j\tilde{k}}\) at which such a crossing occurs can be obtained by setting \(dE_k(t)/dt = 0\); this leads to (Eq. (45))

\[
C_k(t_{j\tilde{k}}) + \frac{dC_k(t_{j\tilde{k}})}{dt} + B_k(t_{j\tilde{k}}) - \frac{dB_k(t_{j\tilde{k}})}{dt} = 0. \quad (41)
\]

We now follow Ref. \(^\text{[45]}\) to linearize \(H_{\tilde{k}}(t)\) around \(t = t_{j\tilde{k}}\). This leads to an effective Hamiltonian \(H_{\text{eff}}^k(t)\) given by

\[
H_{\text{eff}}^k(t) = \left[C_k(t_{j\tilde{k}}) + (t - t_{j\tilde{k}})\frac{dC_k(t_{j\tilde{k}})}{dt}\right] \tau_x + \left[B_k(t_{j\tilde{k}}) + (t - t_{j\tilde{k}})\frac{dB_k(t_{j\tilde{k}})}{dt}\right] \tau_z. \quad (42)
\]

Note that \(H_{\text{eff}}^k(t)\), which determines the excitation probability near \(t_{j\tilde{k}}\), is not of the Kibble-Zurek form in the sense that both its diagonal and off-diagonal terms depend explicitly on time. To cast it into this form, we define a new set of Pauli matrices \(\vec{\sigma}\); using these matrices we can write \(H_{\text{eff}}^k(t)\) as

\[
H_{\text{eff}}^k(t) = \nu_{1\tilde{k}}(t - t_{j\tilde{k}}) \sigma_z + \nu_{2\tilde{k}} \sigma_x, \quad (43)
\]

where \(\nu_{1(2)\tilde{k}}\) are independent of time\(^\text{[22]}\). A comparison between Eqs. (42) and (43) shows that

\[
\begin{align*}
\sigma_z \nu_{1\tilde{k}} &= \tau_x \frac{dC_k(t_{j\tilde{k}})}{dt} + \tau_x \frac{dC_k(t_{j\tilde{k}})}{dt}, \\
\sigma_x \nu_{2\tilde{k}} &= \tau_x C_k(t_{j\tilde{k}}) + \tau_x B_k(t_{j\tilde{k}}).
\end{align*} \quad (44)
\]

Using the identities \(\text{Det} \sigma_3 = \text{Det} \sigma_x = -1\), we can then determine

\[
\begin{align*}
\nu_{1\tilde{k}} &= \left[\frac{dC_k(t_{j\tilde{k}})}{dt}\right]^2 + \left[\frac{dB_k(t_{j\tilde{k}})}{dt}\right]^2, \\
\nu_{2\tilde{k}} &= \left[B_k(t_{j\tilde{k}}) + C_k(t_{j\tilde{k}})\right]^{1/2}.
\end{align*} \quad (45)
\]

Note that the anticommutation relation \(\{\sigma_z, \sigma_x\} = 0\) is satisfied due to Eq. (41). We can then read off the Landau-Zener excitation probability from Eq. (45),

\[
p_{j\tilde{k}} = e^{-2\pi r_{j\tilde{k}}}, \quad r_{j\tilde{k}}^2 = (2\nu_{1\tilde{k}}). \quad (46)
\]

Having obtained these probabilities, we follow Refs. \(^\text{[22]}\) and \(^\text{[45]}\) to construct a transfer matrix \(S_{j\tilde{k}}\) which relates \(\vec{c}_k(t_{j\tilde{k}} - \epsilon) = \vec{c}_{j\tilde{k}} \) to \(\vec{c}_k(t_{j\tilde{k}} + \epsilon) = \vec{c}_{j+1\tilde{k}}\)

\[
\vec{c}_j^{j+1} = S_{j\tilde{k}} \vec{c}_j, \quad (47)
\]

\[
S_{j\tilde{k}} = \left[1 - p_{j\tilde{k}} e^{i\tau_x \Phi_{j\tilde{k}}} - i\tau_y \sqrt{p_{j\tilde{k}}} \Phi_{j\tilde{k}}\right],
\]

\[
\Phi_{j\tilde{k}} = r_{j\tilde{k}}(1 - \ln r_{j\tilde{k}}) + \text{Arg}(1 - i|\vec{r}_{j\tilde{k}}|) - 3\pi/4.
\]

Here \(\Phi_{j\tilde{k}}\) is the Stuckelberg phase\(^\text{[29,45]}\) generated at the \(j^\text{th}\) avoided crossing, and \(S_{j\tilde{k}}\) can be viewed as the transfer matrix which takes the wave function across such a crossing.

Combining Eqs. (47) and (48), the coefficients \(\vec{c}_k\) after \(n\) crossings at \(t = t_f\) are found to be

\[
\vec{c}_k(t_f) = U_{\tilde{k}}^{(n+1)\text{rad}}(t_f, t_n, \vec{c}_k) (S_{n\tilde{k}})^T \cdots S_{1\tilde{k}} U_{\tilde{k}}^{(1)\text{rad}}(t_1, \vec{c}_k) (S_{0\tilde{k}})^T \cdot \frac{1}{0} = U_{\text{rad}}(t_f, 0) \left( \begin{array}{c} 1 \\ 0 \end{array} \right), \quad (48)
\]

where \(U_{\text{rad}}(t_f, 0)\) is the evolution operator in the adiabatic basis, and \(S^T\) denotes the transpose of \(S\). Using Eqs. (39) and (48), we may obtain the evolution operator for the system at \(t = t_f\) in terms of \(r_{j\tilde{k}}\) (in Eqs. (46) and (47)) and \(\xi_{\tilde{k}}(t_1, t_2)\) (in Eq. (40)). The eigenvalues of \(U_{\tilde{k}}(t_f, 0)\) can be obtained by diagonalizing the \(2 \times 2\) matrix obtained. The unitarity of \(U_{\tilde{k}}(t_f, 0)\) ensures that these eigenvalues or phase bands are given by

\[
\lambda_{\tilde{k}}(\vec{c}_k, t) = \exp(\pm i\phi(\vec{c}_k, t)), \quad \cos(\phi(\vec{c}_k, t)) = \text{Re}[U_{\tilde{k}}(t_0, 0)],
\]

\[
\eta_{\tilde{k}}(t_f) c_{1\tilde{k}}(t_f) + \sqrt{1 - \eta_{\tilde{k}}^2(t_f)} c_{2\tilde{k}}(t_f).
\]

The details of the computation of the phase bands using Eqs. (39), (47), and (48) are charted out in the Appendix. Here we present the analytic expression for the case \(n = 2\). For \(t_{2\tilde{k}} \leq t_f \leq T = 2\pi/\omega\), the expression for the phase bands are given by

\[
\cos(\phi^{(2)}(\vec{c}_k, t_f)) = \eta_{\tilde{k}}(t_f) \left( \sqrt{1 - p_{1\tilde{k}}(1 - p_{2\tilde{k}})} \cos[\Phi_{1\tilde{k}} + \xi_{\tilde{k}}(t_f)] + \sqrt{p_{1\tilde{k}} p_{2\tilde{k}}} \cos[\xi_{\tilde{k}}(t_f) - 2\xi_{\tilde{k}}] \right),
\]

\[
+ \sqrt{1 - \eta_{\tilde{k}}^2(t_f)} \left( \sqrt{p_{1\tilde{k}}(1 - p_{2\tilde{k}})} \cos[\xi_{2\tilde{k}} + \xi_{3\tilde{k}}(t_f) - \xi_{1\tilde{k}} - \Phi_{1\tilde{k}}] - \sqrt{p_{2\tilde{k}}(1 - p_{1\tilde{k}})} \cos[\Phi_{1\tilde{k}} + \xi_{1\tilde{k}} + \xi_{2\tilde{k}} - \xi_{3\tilde{k}}] \right),
\]
where $\Phi_k^a = \sum_{i=1,2} \Phi_{ik}^a$, and $\xi_k^a(t_f) = \sum_{i=1,2} \xi_i^a + \xi_{2i+1}^a(t_f)$. We shall analyze Eq. (50) (and its counterparts for different $n$) to obtain general phase band crossing conditions for irradiated graphene in Sec. [IV].

To find the conditions for phase band crossings, we note that Eq. (49) indicates that the condition for $\cos[\phi^{(n)}(\vec{k}, t)] = \pm 1$ can be understood by finding its maximum/minimum values; these occur when

$$\eta_m = \pm \text{Re}[c_{1\vec{k}}(t_f)] / \sqrt{(\text{Re}[c_{1\vec{k}}(t_f)])^2 + (\text{Re}[c_{2\vec{k}}(t_f)])^2}. \quad (51)$$

The corresponding extremum value of $\cos[\phi^{(n)}(\vec{k}, t)]$ is given by

$$\cos[\phi^{(n)}(\vec{k}, t)] = \pm \sqrt{(\text{Re}[c_{1\vec{k}}(t_f)])^2 + (\text{Re}[c_{2\vec{k}}(t_f)])^2}. \quad (52)$$

Note that since $|c_{1\vec{k}}(t)|^2 + |c_{2\vec{k}}(t)|^2 = 1$, the condition

$$\cos[\phi^{(n)}(\vec{k}, t_f)] = \pm 1$$

thus requires $c_{1\vec{k}}(t)$ and $c_{2\vec{k}}(t)$ to be real (apart from a possible irrelevant global phase). For a generic time evolving wave function, this is most likely when one of its components vanish. This leaves us with two possibilities. The first is $\eta_{\vec{k}E}(t_f) = 0 = \text{Re}[c_{1\vec{k}}(t_f)]$ and the second is $\eta_{\vec{k}E}(t_f) = 1 = \pm \text{Re}[c_{1\vec{k}}(t_f)]$. The former possibility requires that the excited state of $H_{\vec{k}E}(t_f)$ be the same as the ground state of $H_{\vec{k}E}(0)$, and this is not guaranteed by any symmetry of the Hamiltonian unless $p_{\vec{k}} = 1$ at some crossing point. The latter condition, in contrast, is generic at $t_f = T$ for any $\vec{k}$ and at $t = nT/3$ for any integer $n$ at the $\Gamma$ point. As we shall see numerically, this latter condition is always satisfied at all the phase band crossings that we find. In what follows, we thus find the generic expression for the phase bands after $n$ avoided level crossings when $\eta_{\vec{k}E}(t_f) = 1$. The detailed method of doing so is sketched in the Appendix. The final result that we obtain from this procedure is as follows. For an even number $(2n)$ of avoided level crossings, denoting $\phi^{(2n)}(\vec{k}, t_f) \equiv \phi^o$, we get

$$\cos(\phi^o) = \sum_{j_{max}=0,2,\ldots,2n} \sum_{a=1}^{j_{max}} \prod_{j_a=1}^{2n-j_{max}} (1 - p_{j_a\vec{k}}) \left( \prod_{j'_a \neq j_a=1}^{2n-j_{max}} p_{j'_a\vec{k}} \right)^{1/2} \cos \left[ \Phi_{\vec{k}}^a + \xi_{\vec{k}}^a(t_f) - \sum_{a}(\gamma_a \Phi_{a\vec{k}} + \delta_a \xi_{a\vec{k}}(t_f)) \right], \quad (53)$$

where the sum over the index $a$ represents a sum over all possible permutations of $j_a$ and $j'_a$ for a fixed $j_{max}$, and

$$\Phi_{\vec{k}}^a = \sum_{i=1,2n} \Phi_{i\vec{k}}^a, \quad \xi_{\vec{k}}^a(t_f) = \sum_{i=1,2n} \xi_i^a + \xi_{2i+1}^a(t_f), \quad \phi_{\vec{k}}^o = \sum_{i=1,2n} \Phi_{i\vec{k}}, \quad \xi_{\vec{k}}^o = \sum_{i=1,2n+1} \xi_i^o. \quad (54)$$

In Eq. (53), $n_1 = \text{Max}[j'_a] - \text{Min}[j'_a] + 1$ provided $\text{Min}[j'_a] \neq 0$ and is 0 otherwise, and the coefficients $\gamma_a$ and $\delta_a$ for any given permutation $a$ are given by

$$\gamma_a = \begin{cases} 1 & \text{for } a \in j'_a, \\ 2 & \text{for } a \in j_a \text{ with } j'_a < j_a < j''_a, \\ 0 & \text{otherwise}, \end{cases} \quad (55)$$

$$\delta_a = \begin{cases} 2 & \text{for } a \in j_a, j'_a \text{ with } \text{Min}[j'_a] < j_a, j'_a \leq \text{Max}[j'_a], \\ 0 & \text{if } a \in j_a, j'_a \text{ with } a - 1 \in j''_a \text{ and } \delta_{a-1} = 2, \end{cases}$$

where $j''_a$ denotes any odd occurrence of $j'$ during a permutation, and $j''_a$ denotes its next occurrence in that permutation. Note that for $n = 1$, Eq. (55) reproduces Eq. (50) for $\eta_{\vec{k}E}(t_f) = 1$.

A similar analysis charted out in the Appendix shows that for an odd number $(2n+1)$ of level crossings, the phase bands $\phi^{(2n+1)}(k, t) \equiv \phi^o$ are given by
FIG. 2: Plot of instantaneous ground (red solid line) and excited (black solid line) state energies and the wave function overlap $\eta_\vec{k}(t)$ (green dashed line) as a function of $t/T$ for (a) $\Gamma$ point with $(k_x, k_y) = (0, 0)$ for the dimensionless radiation amplitude $\alpha = 2.35$, (b) Dirac ($K$) point with $(k_x, k_y) = (0, 4\pi/3\sqrt{3})$ for $\alpha = 2$, (c) $M$ point with $(k_x, k_y) = (2\pi/3, 0)$ for $\alpha = 2.5$, and (d) $X$ point with $(k_x, k_y) = (\pi/3, \pi/3\sqrt{3})$ for $\alpha = 2.5$. Note that the number of avoided crossings of the instantaneous eigenvalues can be clearly read off from these figures; this number varies with $\vec{k}$ as can be seen by comparing the plots from different panels. All energies are in units of $\gamma$.

\[
\cos(\phi^n) = \sum_{j_{\max}=0}^{2n-1} \sum_{\alpha} \prod_{j_\alpha=1}^{j_{\max}} (1 - p_{j_\alpha k_\alpha})^{1/2} \prod_{j'_\alpha \neq j_\alpha=1}^{2n-1-j_{\max}} p_{j'_\alpha k_{j'_\alpha}}^{1/2} (-1)^{n_2(1 - \delta_{n_{j_{\max}}, 1})} \cos \left[ \Phi_\vec{k}^\alpha + \xi_\vec{k}^\alpha(t) - \sum_a (\gamma_a \Phi_{a\vec{k}} + \delta_a \xi_{a\vec{k}}(t)) \right],
\]

where $n_{j'_\alpha}$ denotes the number of occurrence of $j'$, (i.e., the number of $\sqrt{p_{j_{\alpha} k_\alpha}}$ factors) in a permutation $\alpha$, and $n_2 = n_{j_{\max}}$ with $\text{Max}[j'_{\alpha}] < j_{\alpha} < \text{Min}[j'_{\alpha}]$.

Eqs. (53) and (56) represent the main results of this section. They allow us to chart out semi-analytic conditions for phase band crossings and hence the topology change of a class of driven integrable quantum models when the drive frequency is low compared to the natural energy scale of the system Hamiltonian. These conditions can be summarized as follows. For any $\vec{k}$, these driven models, at low frequency, will exhibit a topology change at a time $t_f$ during a drive which has been preceded by $n$ avoided level crossings of its instantaneous eigenvalues if

\[ \eta_\vec{k} = 1 \text{ and } \cos[\phi(n)(\vec{k}, t_f)] = \pm 1, \]

where the expressions for $\cos[\phi(n)(\vec{k}, t_f)]$ are given in Eq. (53) for even $n$ and Eq. (56) for odd $n$. In the next section, we shall compare the predictions of these equations for specific values of $n$ in the context of graphene.

IV. PHASE DIAGRAM FOR GRAPHENE WITH EXTERNAL RADIATION

In this section, we study the phase diagram of graphene in the presence of external radiation in the low-frequency regime. In doing so, we shall compare and contrast between the results obtained in Sec. III and their counterparts from exact numerical solution of the time-dependent Schrödinger equation which reads (Eq. (5))

\[
\frac{i\hbar}{\partial t} \frac{du_{\vec{k}}(t)}{dt} = Z_\vec{k}(t)v_{\vec{k}}(t), \quad \frac{i\hbar}{\partial t} \frac{dv_{\vec{k}}(t)}{dt} = Z_\vec{k}^*(t)u_{\vec{k}}(t).
\]

\[ (58) \]
Fig. 3: Comparison of the phase bands $\cos(\phi_{\vec{k}}(t))$ obtained from exact numerics (solid lines) and adiabatic-impulse approximation (dots) as a function of $t/T$ for (a) $\Gamma$ point with $\alpha = 2.35$, (b) Dirac ($K$) point with $\alpha = 2$, (c) $M$ point with $\alpha = 2.5$, and (d) $X$ point with $\alpha = 2$. The different colors in each plot correspond to different adiabatic regions separated by avoided level crossings; see text for details.

Such a comparative analysis between the adiabatic-impulse method and exact numerics will serve to check the accuracy of the former in the low-frequency regime. In what follows, we shall compare the two methods in Sec. IV A. This will be followed by an analysis of the phase band crossing conditions for several high-symmetry points in the Brillouin zone of graphene in Sec. IV B. Finally, based on the results obtained in these two sections, we shall present the phase diagram of graphene under external radiation in Sec. IV C.

### A. Comparison between adiabatic-impulse and exact numerical results

In order to compare the results obtained by the adiabatic-impulse and exact numerics, we focus on four representative points in the graphene Brillouin zone and choose four representative amplitudes of radiation. These are (a) $\Gamma$ point with $(k_x, k_y) = (0, 0)$ and $\alpha = 2.35$, (b) Dirac ($K$) point $(k_x, k_y) = (0, 4\pi/3\sqrt{3})$ and $\alpha = 2$, (c) $M$ point $(k_x, k_y) = (2\pi/3, 0)$ with $\alpha = 2.5$, and (d) $X$ point with $(k_x, k_y) = (\pi/3, \pi/3\sqrt{3})$ with $\alpha = 2$; these points are shown in Fig. 1.

The expressions for the phase bands obtained from the adiabatic-impulse approximation involve finding the number of avoided crossings of the instantaneous energy eigenvalues within a given period. Thus we first chart these out in Fig. 2 for the above-mentioned points in the Brillouin zone. In addition, we also show the wave function overlap $\eta_{\vec{k}}(t)$ for each of these points. The symmetry of the instantaneous eigenvalues becomes clear from these plots; these will be discussed in Sec. IV B.

Here we note that after obtaining the number of such avoided crossings for the specified $\vec{k}$ values, we can use the results of Sec. III to obtain the evolution matrix $U_{\vec{k}}(t, 0)$. We then diagonalize $U_{\vec{k}}(t, 0)$ to obtain the expressions for the phase bands as given by the adiabatic-impulse method. This can be done either analytically for a small number of crossings ($n < 3$) using Eqs. (A2), (A4) and (A6) or numerically using Eqs. (39), (45), and (49).

These results for the phase bands obtained using the adiabatic-impulse method is then compared against their numerical counterparts. The latter procedure involves a numerical solution of the Schrödinger equation (Eq. (58)), followed by a construction of $U_{\vec{k}}(t)$ from the final wave function using (Ref. 32).

Finally, we numerically diagonalize the matrix $U_{\vec{k}}(t, 0)$ for various values of $\vec{k}$. This leads to numerically exact expressions for the phase bands.

A comparison between $\cos(\phi_{\vec{k}}(t))$ obtained using these two methods, shown in Fig. 3, shows a near-exact match; this indicates that the adiabatic-impulse method reproduces the phase bands accurately for all $t \leq T$. We have numerically checked this result for several other amplitudes and Brillouin zone points; the phase bands obtained via the adiabatic-impulse approximation always shows a very good agreement with its exact numerical counterpart. A similar comparative plot of $\cos(\phi_{\vec{k}}(T))$, shown in Fig. 4, indicates a near-exact match of these bands for a wide range of drive frequencies. We have numerically verified that the expressions for the phase band obtained using the adiabatic-impulse method matches its numerical counterparts for all $\omega \leq 1$ and for $\alpha \leq 5$. Thus the adiabatic-impulse approximation seems to be accurate at low drive frequencies; we shall use this fact in Sec. IV B to analyze the phase band crossings of graphene.

### B. Phase band crossing conditions

In this section, we provide explicit phase band crossing conditions for several high-symmetry points in the graphene Brillouin zone and check their validity by comparison with exact numerical results. This will be followed by general comments about generic $\vec{k}$ points in the Brillouin zone.

**Gamma point**: As shown in Sec. III, the graphene Hamiltonian exhibits $T/3$ periodicity at the $\Gamma$ point since $H_{\vec{k}}(t) = H_{\vec{k}}(t + T/3)$ for $\vec{k} = (0, 0)$ (Eq. (11)). Thus we only need to track the evolution for $t \leq T/3$ for identifying the phase band crossings. Furthermore, as shown in Fig. 2 (a) and derived in Sec. III (Eq. (10)), the instantaneous energy eigenvalues $E_{\vec{k}}(0, t) = E_{\vec{k}}(t)$ satisfy $E_{\vec{k}}(nT/6 \pm t) = E_{\vec{k}}(t)$ for any integer $n$. Thus the kinematic phase, $\chi_{\Gamma}(t, 0)$, picked up by the system at the $\Gamma$ point during the adiabatic evolution satis-
fies

\[
\xi_1(t, 0) = \xi_1(t \pm nT/6, 0),
\]

\[
\xi_{1\Gamma} = \xi_{2\Gamma}/2 = \xi_{1\Gamma},
\]

(60)

where in the last line we have used the notation \(\xi_{1\Gamma}(t_{1\Gamma}, t_{-1\Gamma}) = \xi_{1\Gamma}^*\), and \(t_{1\Gamma}\) denotes the time of the \(i^{\text{th}}\) avoided level crossing at the \(\Gamma\) point. From Fig. 2(a), we find that \(t_{1\Gamma} = T/12\) and \(t_{2\Gamma} = T/4 = t_{1\Gamma} + T/6\) which leads to Eq. (60). Further, it is easy to check that \(|dE_\Gamma(t)/dt|\) satisfies \(|dE_\Gamma(t + nT/6)/dt| = |dE_\Gamma(t)/dt|\); using this it is possible to check that the Landau-Zener excitation probabilities \(p_{1\Gamma}(k = 0, 0) \equiv p_{1\Gamma}\) for \(i = 1, 2\) (corresponding to times \(t_{1\Gamma} = T/12\) and \(t_{2\Gamma} = T/4\) and the corresponding Stuckelberg phases \(\Phi_{1\Gamma}(k = 0, 0) \equiv \Phi_{1\Gamma}\) at the \(\Gamma\) point satisfy (Eqs. (46) and (47))

\[
p_{1\Gamma} = p_{2\Gamma} = p_\Gamma, \quad \Phi_{1\Gamma} = \Phi_{2\Gamma} = \Phi_\Gamma.
\]

(61)

Using Eqs. (40) and (41), we then obtain an expression for the phase bands (Eq. (A7)) at \(T = T/3\) where \(\eta = 1\) as

\[
\cos(\phi_{\Gamma}(T/3)) = p_\Gamma + (1 - p_\Gamma)\cos(2\Lambda_\Gamma),
\]

(62)

where \(\Lambda_\Gamma = \Phi_\Gamma + 2\xi_\Gamma\).

Eq. (62) can be used to relate the phase band crossing condition to the Landau-Zener probability \(p_\Gamma\) and the Stuckelberg phase \(\Phi_\Gamma\). To see this, we first note that a class of such crossings occurs at the Floquet zone center (namely, \(\cos(\phi_{\Gamma}(T/3)) = 1\), so that \(\phi_{\Gamma}(T/3) = 2\pi n\) at the crossing). From Eq. (62), we find that the condition for such crossings is given by

\[
\Lambda_\Gamma = m\pi, \quad \text{where} \quad m \in Z.
\]

(63)

We would like to point out here that the condition \(\phi_{\Gamma}(T/3) = (2n + 1)\pi\) is untenable since this would require \(\cos(2(\Phi_\Gamma + \xi_\Gamma)) = -1\) for any \(\Phi_\Gamma > 0\). Thus our analysis predicts that all phase band crossings at the \(\Gamma\) point which occur at \(t = T/3\) must be through the Floquet zone center. Note that all such crossings through zone center at \(t = T/3\) also imply corresponding crossings at \(t = 2T/3\) and \(t = T\) as can be seen from Eq. (19).

The second class of crossings that occurs at the \(\Gamma\) point involves a phase band crossing at \(t = T\) or \(t = 2T/3\) without an analogous crossing at \(t = T/3\). Such crossings, at \(t = 2T/3\), always occur through the Floquet zone edges (\(\cos(\phi_{\Gamma}) = -1\), so that \(\phi_{\Gamma} = (2n + 1)\pi\) at the crossing), and they can be understood as follows. First, we note that from Eq. (19), we have \(\phi_{\Gamma}(T/3) = \phi_{\Gamma}(2T/3)/2 = \phi_{\Gamma}(T)/3\). Thus it is possible to have a phase band crossing at \(t = 2T/3\) without any crossing.

FIG. 4: Comparison of the phase bands \(\cos(\phi_{\Gamma}(T))\) obtained from exact numerics (black solid lines) and adiabatic-impulse approximation (dotted lines) as a function of \(T\) (in units of \(\hbar/\gamma\)). All parameters are the same as in Fig. 2.

FIG. 5: Comparison of phase band crossing conditions from adiabatic-impulse approximation (Eqs. (63), (64), and (65)) with exact numerics. In all plots \(t\) and \(T\) are in units of \(\hbar/\gamma\). (a) Phase band crossings at \(t = T/3\) from Floquet zone center (\(\phi_{\Gamma}(T/3) = 2n\pi\)). Black solid line indicates numerical plot of the phase bands at \(t = T/3\) as a function of \(T\). The red dotted lines show a plot of \(\cos(2(\Phi_{\Gamma} + \xi_{\Gamma}))\) whose value is predicted to be unity when Eq. (63) is satisfied. (b) Plot of phase band crossings which occur at \(t = 2T/3\) through the Floquet zone edge (\(\phi_{\Gamma}(2T/3) = (2n + 1)\pi\)). The black solid line indicates \(\cos(\phi_{\Gamma}(2T/3))\) while the red dotted lines show the left side of Eq. (65) which touches zero at all phase band crossings in accordance with the adiabatic-impulse prediction. (c) Phase band crossings at \(t = T\) through the Floquet zone edge for which \(\phi_{\Gamma}(T/3) = \pi/3\). The black solid line indicates \(\cos(\phi_{\Gamma}(2T/3))\) while the red dotted lines show the left side of Eq. (65) which touches zero at each phase band crossing. (d) Same as (c) but for \(\phi_{\Gamma}(T/3) = 2\pi/3\) for which the crossing condition is given by Eq. (66).
at $t = T/3$ if $\phi_T(T/3) = \pi/2$. This requires

$$\cos(2\Lambda_T) + p_T/(1 - p_T) = 0.$$  \hfill (64)

Note that such crossings can occur for $p_T \leq 1/2$; further for any $p_T < 1/2$ we expect two solutions to Eq. (64) leading to a pair of possible crossings for a given $p_T$. Second, for $t = T$, we can have similar crossings for which $\phi_T(T/3) = \pi/3$ and $\phi_T(T/3) = 2\pi/3$ which lead to the conditions

$$\cos(2\Lambda_T) - \frac{1 - 2p_T}{2(1 - p_T)} = 0, \text{ for } \phi_T(T/3) = \frac{\pi}{3},$$  \hfill (65)

$$\cos(2\Lambda_T) + \frac{1 + 2p_T}{2(1 - p_T)} = 0, \text{ for } \phi_T(T/3) = \frac{2\pi}{3}. \hfill (66)$$

We note Eq. (65) predicts that such crossings through Floquet zone edge can only occur for $p_T \leq 3/4$, and for $p_T < 3/4$, there are a pair of crossings for a given value of $p_T$. In contrast, Eq. (66) shows that for $\phi_T = 2\pi/3$, crossings occur through the Floquet zone center in pairs if $p_T \leq 1/4$. The crossing conditions charted out in Eqs. (64) and (65) constitute an example of the response of the system at fractional frequencies $2\omega/3$ (Eq. (64)) and $\omega/3$ (Eqs. (65)); these occur since the evolution operator $U$ does not have the same periodicity as $H$ at these drive frequencies.

A comparison between the crossing conditions obtained above and the exact numerical result is shown in Fig. 6. The top panel of this figure shows that all the phase band crossings for $t = T/3$ at the $\Gamma$ point is consistent with Eq. (63) for a wide range of $T \geq 2\pi$ as shown in Fig. 5. We find that the analytic condition presented in Eqs. (63) (66) is exactly satisfied for all the phase band crossings that we find using exact numerics. Furthermore, we do not find any phase band crossings at $t = T/3$ which occur through the Floquet zone edge which is consistent with our expectation from the adiabatic-impulse theory.

**Dirac points:** For the Dirac points, we find from Fig. 2(b) that there are three avoided level crossings ($j = 1, 2, 3$) which divide the evolution into four regions denoted as $i = 1, 2, 3, 4$. We have confirmed numerically that the number of such crossings does not change for $\alpha \leq 5$ and our subsequent discussions will hold in this regime. Also, in what follows, we shall explicitly study the $K$ point for which $(k_x, k_y) = (0, 4\pi/(3\sqrt{3})$); all our results will also hold for the $K'$ point.

The symmetries of the instantaneous Hamiltonian $H_{K}(t)$ at the $K$ point are listed in Eqs. (23-27). Using these conditions, it is easy to see that $E_{K}(t) = E_{K}(t + T/3)$ for all $t$. Thus the kinematic phase $\xi_{iK} \equiv \xi(k_x, k_y) = (0, 4\pi/(3\sqrt{3}))$ (4K, t - 1K) picked up in the $i$th region satisfies

$$\xi_{2K} = \xi_{3K} = \xi_{K}, \xi_{1K} = \xi_{K}, \xi_{4K} = \xi_{K} - \xi_{K}. \hfill (67)$$

Furthermore since the avoided level crossings occur at time differences $\Delta t = T/3$, we find that the Landau-Zener probabilities $\rho_{jK}$ for $j = 1, 2, 3$ and the corresponding Stuckelberg phases $\Phi_{jK}$ at each of the avoided crossings are identical. We therefore denote

$$\rho_{jK} = \rho_{K} \text{ and } \Phi_{jK} = \Phi_{K}. \hfill (68)$$

Using these symmetries and following the method charted out in the Appendix, we find that the expression for the phase band at $t = T$ where $\eta_{K} = 1$ is given by

$$\cos \phi_{K}(T) = \sqrt{1 - p_{K}}[1 - p_{K}] \cos(3\Lambda_{K}) - 3p_{K} \cos \Lambda_{K}], \hfill (69)$$

where $\Lambda_{K} = \xi_{K} + \Phi_{K}$. The conditions for phase band crossings at the Dirac point, through the Floquet zone center ($\cos[\phi_{K}(T)] = 1$) or the Floquet zone edge ($\cos[\phi_{K}(T)] = -1$), are therefore given by

$$\pm \frac{1}{\sqrt{1 - p_{K}}} + \cos(3\Lambda_{K}) - 4p_{K} \cos^{3} \Lambda_{K} = 0 \hfill (70)$$

where the upper (lower) sign corresponds to crossings through the Floquet zone center (edge). We note that Eq. (70) predicts that no phase band crossings occur for $p_{K} > 3/4$. A plot of the left side of Eq. (70) as a function of $T$ is shown in Fig. 6. We find that these curves touch zero and hence satisfy Eq. (70) precisely at the locations of the phase band crossings as predicted by exact numerics.

**M point:** At the $M$ point for which $(k_x, k_y) = (2\pi/3, 0), \hfill (105)
we find, from Fig. 2(c), four avoided level crossings corresponding to \( j = 1, 2, 3, 4 \), which divide the evolution into five adiabatic regions denoted by \( i = 1, 2, \cdots, 5 \). The symmetries of the instantaneous Hamiltonian have been listed in Eqs. (28) and (29). These symmetries ensure that at the \( M \) point \( E_M(T - t) = E_M(t) \) and \( E_M(T/2 + t) = E_M(T/2 - t) = E(t) \). The kinematic phases picked up in the adiabatic regions \( \xi_{iM} = \xi_{M}(t_i, t_{i-1}) \) for \( i = 1, 2, \cdots, 5 \) thus satisfy

\[
\xi_{2M} = \xi_{4M} = \xi_M', \quad \xi_{1M} = \xi_{5M} = \xi_{3M}/2 = \xi_M. \tag{71}
\]

Further, since the four avoided crossings occur at times \( t_0, T/2 - t_0, t_0 + T/2, \) and \( T - t_0 \) where \( t_0 \approx 0.15T \), they have the same Landau-Zener probabilities and Stuckelberg phases: we therefore denote \( p_{JM} = p_M \) and \( \Phi_{JM} = \Phi_M \). Using these symmetries, we can compute the phase bands, following the method outlined in Sec. III and the Appendix, to obtain

\[
\cos \phi_M(T) = (1 - p_M^2) \cos(2\Lambda_{JM} + \xi_M) + p_M^2 \cos(2\Lambda_{JM} - \xi_M) - 2p_M(1 - p_M)\xi_M, \tag{72}
\]

where \( \Lambda_{JM} = \Phi_{JM} + 2\xi_M, \Lambda_{JM} = \Phi_{JM} + \xi_M \) and \( \Lambda_{JM} = \Lambda_{JM} \pm \xi_{JM} \). This leads to the phase band crossing conditions

\[
\pm \frac{1}{(1 - p_M^2) + \cos(2\Lambda_{JM}) + p_M^2} \cos(2\Lambda_{JM} - \xi_M) - \frac{2p_M(1 - p_M)}{1 - p_M}\xi_M = 0, \tag{73}
\]

where the upper (lower) sign holds for phase band crossing through the Floquet zone center (edge) \([\cos(\phi_M(T))] = 1(-1)\). Numerically, we find that Eq. (73) can be satisfied for all \( p_M \leq 0.9492116 \). A plot of the left side of Eq. (73) is shown in Fig. 7, these curves are found to touch zero (and hence satisfy Eq. (73)) at the positions of the phase band crossings obtained from exact numerics.

**X point:** As shown in Fig. 2(d), for the X point where \((k_x, k_y) = (\pi/3, \pi/(3\sqrt{3}))\), we have, for \( \alpha \leq 2.5 \), three avoided level crossings corresponding to \( j = 1, 2, 3, 4 \), which divide the evolution into four adiabatic regions which we label as \( i = 1, 2, 3, 4 \). In this case, using Eq. (32), we find \( E_X(t) = E_X(T/6 - t) = E_X(7T/6 - t) \). Using this we can establish that the kinematic phases in regions 1, 2, 3, and 4 satisfy

\[
\xi_{2X} = \xi_{1X} + \xi_{3X} = \xi_X, \quad \xi_{3X} = \xi_X. \tag{74}
\]

Further, since the avoided crossings corresponding to \( j = 2 \) and \( j = 3 \) occur at \( t = t_0 \approx 0.3T \) and \( t \approx 0.85T = 7T/6 - t_0 \), we have

\[
\begin{align*}
p_{2X} &= p_{3X} = p_X, \quad \text{and} \quad p_{1X} = p_X', \\
\Phi_{2X} &= \Phi_{3X} = \Phi_X, \quad \text{and} \quad \Phi_{1X} = \Phi_X'. \tag{75}
\end{align*}
\]

Using these symmetries and following the method charted out in the Appendix we obtain, at \( t = T \) where \( \eta_X = 1 \),

\[
\begin{align*}
\cos \phi_X(T) &= \sqrt{1 - p_X^2} \cos(2\Lambda_{1X} + \Lambda_{2X}) \\
-2\sqrt{p_X p_X'}(1 - p_X) &\cos \Lambda_{1X} - p_X \sqrt{1 - p_X^2} \cos \Lambda_{2X}, \tag{76}
\end{align*}
\]

where \( \Lambda_{1X} = \Phi_X + \xi_X \) and \( \Lambda_{2X} = \Phi_X + 2\xi_X - \xi_X' \). Thus the phase band crossing condition is given by

\[
\begin{align*}
\pm \frac{1}{\sqrt{1 - p_X^2}(1 - p_X)} \cos(2\Lambda_{1X} + \Lambda_{2X}) + 2\sqrt{p_X p_X'}(1 - p_X)(1 - p_X') \cos \Lambda_{1X} + \frac{p_X}{1 - p_X} \cos \Lambda_{2X} &= 0, \tag{77}
\end{align*}
\]

where the \(+(-)\) sign corresponds to a phase band crossing through the zone center (edge). A plot of the left side of Eq. (77) is shown in Fig. 8 for \( \alpha = 2 \); we find that they never touch zero within the range of \( T \) shown in the figure. This is consistent with the fact that there are no phase band crossings within this range as can be seen from a plot of \( \cos(\phi_X(T)) \); in fact, we have numerically checked that for \( \alpha = 2 \), the phase bands do not cross for any \( T \geq 2\pi \). Numerically, we find very few such crossings which will be discussed in Sec. IV C.

The decrease in the number of phase band crossings at the
X point is a consequence of the lack of a symmetry necessary to make all \( p_{iX} \) and \( \Phi_{iX} \) equal; in fact such a reduction resonates with the fact that phase band crossings are much more difficult to find at arbitrary points in the Brillouin zone which are not equal to any of the high symmetry points. The crucial role of symmetry for phase band crossings can be understood as follows. Consider a case of two avoided level crossings each having a probability \( p_i \) and a Stueckelberg phase \( \Phi_i \) for \( i = 1, 2 \). Let the phases picked up in the corresponding adiabatic regions I, II and III to be \( \xi_j \) where \( j = 1, 2, 3 \). Then for a situation with no symmetry, the phase bands are given by (Eq. (A6))

\[
\cos(\phi) = \sqrt{(1 - p_1)(1 - p_2)} \cos(\sum_{i=1,2} \phi_i + \sum_{i=1,3} \xi_i) + \sqrt{p_1 p_2} \cos(\xi_1 + \xi_3 - \xi_2)
\]

Now we look for possibility of tuning \( \cos(\phi) = \pm 1 \) by varying \( \alpha \) and \( T \). Since \( \Phi_i \) is essentially a function of \( p_i \), we have five independent quantities, \( p_{1,2} \) and \( \xi_{1,2,3} \) to tune by varying \( \alpha \) and \( T \). This requires fine tuning. In contrast, in the presence of symmetries as in the case of the \( \Gamma \) point, where \( p_1 = p_2 = p \) and \( \xi_1 = \xi_2/2 = \xi_3 = \xi \), one has to just tune two parameters by varying \( \alpha \) and \( \Gamma \). This does not require fine tuning. The argument easily extends to a larger number of crossings since the number of quantities to tune increases rapidly with the number of crossings in the absence of any symmetry. Thus, we generally expect phase band crossings to happen only at \( \vec{k} \) points which have the requisite symmetries to make most \( p_i \) and \( \xi \) equal. We note that our numerical search in the graphene Brillouin zone confirms this expectation. We shall not discuss the non-generic phase band crossings at arbitrary low-symmetry Brillouin zone points further in this work.

### C. Phase diagram

In this section, we chart out the phase diagram obtained using exact numerics and the adiabatic-impulse method. At the outset, we note that all results obtained using exact numerics concur almost exactly for all \( \omega/\gamma \leq 1 \) with those obtained from the adiabatic-impulse approximation discussed in Sec. 3. One of the central results of our work is the change in topology of the driven system at \( t = T/3 \) and \( 2T/3 \) as shown in Figs. 9, 10, 11, 12 and 13. Moreover, we also provide an analysis of the phase diagram of the system at \( t = T \), and we show in Fig. 14 the contributions to this phase diagram from different high-symmetry points in the graphene Brillouin zone. In what follows we list the salient features of the low-frequency phase diagram of irradiated graphene.

For the computation of such phase diagrams, the standard method followed in the literature involves two widely followed procedures. The first involves putting the time-dependent graphene Hamiltonian on a lattice which is periodic (spatially) along one direction (taken to be \( y \) here) and has an edge (either zigzag or armchair) along the other direction, \( x \). One then calculates the evolution operator numerically for such a system and numerically diagonalizes it to obtain the phase bands. It is well-known that a change in the bulk Chern band would involve a change in the number of edge states of the phase bands (or equivalently the Floquet Hamiltonian if we focus on \( t = T \)). The second method involves a direct numerical computation of the bulk Chern number in each of the phases separated by a Floquet topological transition; the phase diagram can then be charted out by computing the change in the Chern number across the transition induced by phase band crossings. The computation of the bulk Chern number involves an integration of the Berry curvature \( \vec{B}(k_x, k_y) = \nabla_{\vec{k}} \times \vec{A}(k_x, k_y) \) over the Brillouin zone, where the Berry potential \( \vec{A} = (A_x, A_y) \) can be expressed in terms of the system wave function \( |\psi\rangle \) as \( A_i = \partial_{k_i} |\psi(k_x, k_y; t_0)\rangle \), where the Chern number is computed at \( t = t_0 \). This is typically done by dividing the Brillouin zone into a mesh and summing up the integrand over all the points on the mesh. The choice of the mesh size is of key importance in this procedure and the optimal choice depends, among other things, on the drive frequency.

It turns out that both these methods lead to significant computational difficulties at low drive frequencies due to the following reasons. First, we note that as the frequency is lowered, the Berry curvature becomes an increasingly rugged function of \( \vec{k} \) and develops several sharp and near-singular features at different points in the Brillouin zone whose locations depend on both the drive frequency and parameters of
The number requires a mesh size which decreases rapidly with integral of $B$ the system Hamiltonian. Thus a numerical computation of the change in Chern number and number of edge states decreases by 1 across the $T = 0$ band crossing. The number of edge states decreases by 1 across the phase band crossing at $T < T_c$ before $(T < T_c)$ and after $(T > T_c)$ the phase band crossing. The number of edge states decreases by 1 across the transition as can be seen by comparing the number of such states at $E = 0$ in (c) and (d).

The phase diagram at $t = T/3$ showing such a topology change of the phase bands is shown in Fig. 9. We numerically find that the entire contribution to this phase diagram occurs from the $Γ$ point which is in accordance with the prediction from the adiabatic-impulse method. We have checked that no crossings occur for $t = T/3$ at the $K$, $M$, and $X$ points. The change in the Chern number occurs along the dotted line and the corresponding phase band crossing always occur through the zone center ($\cos φ = 1$); these are also in accordance with the predictions of the adiabatic-impulse method. The computation of the change in Chern number at a representative point $α = 2.4$ and $T = 11.2$ is shown in Fig. 10. Figure 10 (a) shows the point where the phase bands cross through the zone edge. The corresponding Chern number changes from 0 to $-1$ at this point as shown in Fig. 10 (b). Figures 10 (c) and (d) show that number of edge state changes by $-1$ by tabulating their numbers before and after the crossing. These computations were carried out by standard numerical methods following Refs. [37][38][39] and [46] as discussed earlier in this section. We have checked that for a few representative points with $T/(2π) < 3$ that such a change in the Chern number and the number of edge states is consistent with the position of the phase band crossings in Fig. 9. However, for $T/(2π) ≥ 3$, these computations become numerically difficult for the reasons mentioned above. Finally, we note that no phase bands crossings occur at $ω/γ ≤ 1$ (or $T ≤ 2π$) within the range of $α$ considered here indicating that no topology change occurs at high frequencies at $t = T/3$ within this range. However, at higher $α$, it is possible that such a topology change may occur at $t = T/3$ at higher $ω$.

We note that near $α = 4π/(3\sqrt{3}) ≈ 2.418$, all the lines in the phase diagram show sharp bends; this feature originates from the fact at this value of $α$ and all values of $T$, there are unavoidable crossings of the instantaneous energy levels (i.e., the ground and excited energy levels become exactly degenerate) at $t = T/12$ and $t = T/4$ which lead to $p_f = 1$. To see this more clearly, we compare the numerical phase-diagram with that obtained using the adiabatic-impulse method in Fig. 11. We find that there is a near exact match for all values of $α$ and $T$ except near $α ≈ 2.4$; near this line, $p_f = 1$ and the adiabatic-impulse method predicts phase band crossings for...
all values of $T$ as can be seen from Eq. (62). This leads to a vertical line in the $\alpha - T$ plane as shown in Fig. 11. However, exact numerics shows crossings at isolated points on this line; the density of such points on the line rapidly increases with $T$ showing that the two results are expected to match at high $T$. This discrepancy between the results of the adiabatic-impulse method and exact numerics can be understood to be the result of unavoided crossings of energy levels. In such a situation, the two instantaneous energy bands come close to each other for a wide range of values of $t/T$ around the crossing point. Thus a transition from the ground state to the excited state can occur over a wide range of $t/T$ during the evolution. This invalidates the assumption of a narrow impulse region, unless $T$ is small.

Next, the phase diagram for $t = 2T/3$ is shown in Fig. 12. Here, in contrast to the phase diagram for $t = T/3$, phase band crossings can occur both through the zone center (black dots) and zone edges (red dots). The crossings through the zone centers are same as those seen in Fig. 9 since by symmetry, the phase picked between $t = 0$ and $T/3$ is same as that between $t = T/3$ and $2T/3$ at the $\Gamma$ point. Thus all phase band crossings for which $\cos(\phi(T/3)) = 1$ repeat themselves at $t = 2T/3$. In addition, for $t = 2T/3$, phase band crossings may also occur at the zone edge ($\cos(\phi) = -1$). We find that these crossings enclose some closed regions in the $\alpha - T$ plane. Each of these regions terminates at some specific maximal values of $\alpha = \alpha_{\text{max}}(T)$ in the $\alpha - T$ plane. The adiabatic-impulse method predicts that $p_\Gamma(\alpha_{\text{max}}(T), T) = 1/2$ which matches quite well with the numerical values of $p_\Gamma$ at these points as can be seen by comparing the position of the $p_\Gamma$ line to these points (Fig. 12). The changes in the Chern number and the number of edge states across the transition is shown across a representative crossing point in Fig. 13. In contrast to the case of the phase diagram at $t = T/3$, topology change occurs here at high frequencies ($\omega/\gamma \gtrsim 1$) for the range of $\alpha$ considered through the zone edges, as can be seen from Fig. 12.

Next, we consider the phase diagram at $t = T$. Here phase band crossings occur at several points on the graphene Brillouin zone. We find, in accordance with the results of the adiabatic-impulse analysis, that the number of such crossings is large at high symmetry points. To see this clearly, we plot the phase band crossings at $t = T$ at the $\Gamma$, $K$, $M$ and $X$ points in Fig. 14. The plot in Fig. 14 (a) shows the phase band crossings that occur at the $\Gamma$ point. The closed black and red dotted regions in the plot correspond to $\phi(T/3) = \pi/3$ (Eq. (65)) and $\phi(T/3) = 2\pi/3$ (Eq. (66)) respectively. As discussed in Sec. IV B, the adiabatic-impulse method predicts that such regions should terminate, for a given $T$, at $\alpha = \alpha_{\text{max}}(T)$ whose positions should coincide with $p_\Gamma = 1/4$ $(3/4)$ for regions corresponding to $\phi(T/3) = 2\pi/3$ $(\pi/3)$. This prediction of the adiabatic-impulse method matches exact numerics quite well as can be seen by the positions of the $p_\Gamma = 1/4$ $(3/4)$ curves (green (blue) lines) in Fig. 14 (a). In Fig. 14 (b), we show similar crossings at the $K$ point. The closed regions in the $\alpha - T$ plane for crossing through both the zone center (black dots) and the zone edge (red dots) terminate at some maximal values of $\alpha = \alpha_{\text{max}}(T)$ for a given $T$. The positions of these points are predicted to coincide with the $p_K = 3/4$ line in the $\alpha - T$ plane (blue line in Fig. 14 (b)) by the adiabatic-impulse theory. We find that this prediction matches exact numerics quite well at high $T$; moreover we find there are no crossings in the region in $\alpha - T$ plane for which $p_K > 3/4$ which is also in ac-
while the former occurs at small $T$ center and zone edge crossings; the latter happens only at high numerically that there is a clear demarcation between the zone
predictions of the adiabatic-impulse method. We also find that the crossings do not constitute closed regions or checked that their positions for $T/2\pi > 1$ coincides with the prediction of the adiabatic-impulse method. However, a detailed calculation of these Chern at high values of $T/2\pi$ in the phase diagram turns out to be numerically

cordance with the prediction of the adiabatic-impulse method. Next, in Fig. 14 (c), we plot the crossings at the $M$ point. Here we find that the crossings do not constitute closed regions or do not terminate along some specific lines; however, we have checked that their positions for $T/(2\pi) > 1$ coincides with the predictions of the adiabatic-impulse method. We also find numerically that there is a clear demarcation between the zone center and zone edge crossings; the latter happens only at high $T$ while the former occurs at small $T$. Finally, in Fig. 14 (d), we plot the phase band crossings at the $X$ point which is a lower-symmetry point compared to $\Gamma$, $K$, and $M$. Here we find that phase band crossings occur at isolated points in the $\alpha - T$ plane; moreover, the number of such crossings is much smaller than those at the high symmetry points. This is in accordance with the general expectation that phase band crossings are more likely to happen at high symmetry points in the graphene Brillouin zone.

Finally, we present the phase diagram for $t = T$ in a region in the $\alpha - T$ plane shown in Fig. 15. We note that for this region of the $\alpha - T$ plane, phase band crossings occur at $\Gamma$, $K$, and $M$ points; there are no crossings from the $X$ point. We find that the phase diagram constitutes closed regions each of which is expected to have a definite Chern number. Each of the lines in the diagram corresponds to a change in Chern number by $\Delta C = \pm 1$; when two or more such lines coalesce, it indicates a possible Chern number change by $\pm 2, \pm 3, \ldots$. However, a detailed calculation of these Chern at high values of $T/2\pi$ in the phase diagram turns out to be numerically
difficult for reasons discussed earlier in the text and we have not attempted it here. We also note that the phase diagram

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**FIG. 13:** Computation of the Chern number and number of edge states for $t = 2T/3$ at $\alpha = 1.8$ and $T = 4.76$ (in units of $\hbar/\gamma$). (a) Position of the phase band crossing for which the Chern number is computed. (b) Change in Chern number of the system across the phase band crossing at $T = T_c$ showing $\Delta C = 1$. (c) and (d) The edge state structure around $E = 0$ and $\pi/T$ before ($T < T_c$) and after ($T > T_c$) the phase band crossing. The number of edge states increases by 1 across the transition as can be seen by comparing the number of such states at $E = \pi/T$ in (c) and (d).

**FIG. 14:** Contributions to the phase diagram at $t = T$ in the $\alpha - T$ plane from (a) $\Gamma$, (b) $K$, (c) $M$ and (d) $X$ points. All crossings through the zone center (edge) are indicated by black (red) dots, and $T$ is in units of $\hbar/\gamma$. The blue (green) solid lines in panel (a) represents the $p_\Gamma = 1/4$ (3/4) curves in the $\alpha - T$ plane, while the violet line corresponds to $p_\Gamma = 1$. The black solid line in panel (b) represents the $p_K = 3/4$ curve in the $\alpha - T$ plane. The region with no phase band crossing corresponds to $p_K > 3/4$.

**FIG. 15:** Phase diagram at $t = T$ in the $\alpha - T$ plane. The contributions from the $\Gamma$ point for crossings through the zone center (edge) are given by black (red) dots. The corresponding crossings from $K$ points are given by green (zone center) and blue (zone edge) dots and those from $M$ points correspond to violet (zone center) and orange (zone edge) dots. See text for details.
may change due to possible non-generic phase band crossings at other low-symmetry points of the graphene Brillouin zone. Such crossings are expected to be very few in number for reasons discussed earlier in Sec. IV B and our numerics have not found any for the graphene Brillouin zone points that we have checked. However, we note that it is not numerically possible to completely ascertain their non-existence for all points in the graphene Brillouin zone and for the entire \( \alpha - T \) plane.

V. 1D XY MODEL IN A TRANSVERSE FIELD

We now turn to the one-dimensional spin-1/2 model with XY nearest-neighbor interactions and a magnetic field \( \mu \) applied along the \( \hat{z} \) direction. The Hamiltonian for a system with \( N \) sites and open boundary conditions is given by

\[
H = -\sum_{n=1}^{N-1} \left[ J_x \sigma^x_n \sigma^x_{n+1} + J_y \sigma^y_n \sigma^y_{n+1} \right] - \mu \sum_{n=1}^{N} \sigma^z_n, \tag{79}
\]

where the \( \sigma^a_n \)'s denote Pauli spin matrices at site \( n \). This can be mapped to a system of spinless fermions by the Jordan-Wigner transformation; the fermionic Hamiltonian is given by\[87\]

\[
H = \sum_{n=1}^{N-1} \left[ \gamma (f_n^\dagger f_{n+1} + H.c.) + \Delta (f_n f_{n+1} + H.c.) \right] - \mu \sum_{n=1}^{N} (2f_n^\dagger f_n - 1), \tag{80}
\]

where \( \gamma = J_x + J_y, \Delta = J_y - J_x \), and the fermionic operators satisfy the usual anticommutation relations \( \{ f_n, f_m \} = \{ f_n^\dagger, f_m^\dagger \} = 0 \) and \( \{ f_n, f_m^\dagger \} = \delta_{mn} \). In this fermionic language, \( \gamma \) is a nearest-neighbor hopping amplitude, \( \Delta \) is a \( p \)-wave superconducting pairing, and \( 2\mu \) is as a chemical potential. It is convenient to define two Majorana fermion operators at each site, \( a_n \) and \( b_n \), as

\[
f_n = \frac{1}{2} (a_n + ib_n) \quad \text{and} \quad f_n^\dagger = \frac{1}{2} (a_n - ib_n). \tag{81}
\]

These operators are Hermitian and satisfy the relations \( \{ a_m, a_n \} = \{ b_m, b_n \} = 2 \delta_{mn} \) and \( \{ a_m, b_n \} = 0 \). In terms of these operators, the Hamiltonian takes the form

\[
H = -\frac{i}{2} \sum_{n=1}^{N-1} \left[ (\gamma + \Delta) a_n b_{n+1} + (\gamma - \Delta) a_{n+1} b_n \right] - i\mu \sum_{n=1}^{N} a_n b_n. \tag{82}
\]

Note that the Hamiltonian is invariant under a parity transformation which reflects the system about its midpoint, namely, \( a_n \rightarrow b_{N+1-n} \) and \( b_n \rightarrow -a_{N+1-n} \).

For a system with \( N \) sites and periodic boundary conditions, we can write the Hamiltonian in momentum space as follows. Defining the Fourier transform as \( f_k = \frac{1}{N} \sum_{n=1}^{N} f_n e^{-i k n} \), we find that Eq. \( [80] \) can be rewritten as

\[
H = \sum_{0 \leq k \leq \pi} \left( f_k^\dagger f_{-k} + f_{-k}^\dagger f_k \right),
\]

\[
H_k = 2(\gamma \cos k - \mu) \tau_z + 2\Delta \sin k \tau_y, \tag{83}
\]

where the \( \tau_n \)'s denote Pauli pseudospin matrices in the particle-hole space. It is convenient to do a unitary transformation by rotating around \( \hat{z} \) to convert \( \tau_y \) \( \rightarrow \) \( \tau_x \) so that the Hamiltonian in \( k \) space is given by

\[
H_k = 2(\gamma \cos k - \mu) \tau_z + 2\Delta \sin k \tau_x. \tag{84}
\]

Since this \( H_k \) is a real and symmetric matrix, it will be easier to derive the symmetry properties of the corresponding Floquet operator \( U_k(t) \).

A. Two-rate protocol

In what follows, we shall consider the driving of the chemical potential and superconducting pairing with two different frequencies \( \omega \) and \( r \omega \), where \( r \) is an integer, so that

\[
\mu = A \cos(\omega t) \quad \text{and} \quad \Delta = B \cos(r \omega t). \tag{85}
\]

This allows us to write

\[
H_k(t) = f_3(k, t) \tau_z + f_1(k, t) \tau_x, \tag{86}
\]

\[
f_3(k, t) = 2[\gamma \cos k - A \cos(\omega t)],
\]

\[
f_1(k, t) = 2B \cos(r \omega t) \sin k. \tag{87}
\]

We now consider the time evolution operator

\[
U_k(t) = T_t \exp[-i \int_0^t dt' H_k(t')]. \tag{88}
\]

We will be particularly interested in the conditions under which \( U_k(t) \) will be equal to \( \pm I \), giving a phase band crossing.

A simple way of getting \( U_k(t) = \pm I \) is to set \( k = k_0 \) where \( k_0 = 0 \) or \( \pi \); then \( \sin k_0 = 0 \) and \( f_1(k_0, t) = 0 \) for all \( t \). This gives

\[
U_{k_0}(t) = \exp[-i \tau_z \int_0^t dt' f_3(k_0, t')]. \tag{89}
\]

The phase band crossing condition then takes the form

\[
- \int_0^t dt' f_3(k_0, t') dt' = n\pi, \tag{90}
\]

where \( n \) is an integer. For Eq. \( [86] \), we can analytically find if Eq. \( [89] \) has solutions for any value of \( t \). Such phase band crossings which occur at the edge or center of the 1D Bril-
louin zone has been studied in details in Ref. 42; we shall not discuss them further here.

In addition to phase band crossings through the zone edge or center discussed above, we find that two-rate protocols may lead to additional phase band crossings at \( k \neq 0, \pi \). For Eq. (86), there exists such a phase band crossing at \( k = \pi / 2 \). For odd integer \( r \), we see that

\[
H_{\pi/2}(T/2 - t) = - H_{\pi/2}(t).
\]  

Using Eq. (90), and the form of the time evolution operator given in Eq. (15), we then see that

\[
[U_{\pi/2}(T/2)]^{-1} = U_{\pi/2}(T/2).
\]  

This implies that \( U_{\pi/2}(T/2) = \pm I \). Thus Eq. (86) gives a phase band crossing at \( k = \pi / 2 \) and \( t = T/2 \) for any value of \( \gamma, A, B \) and \( T \), if \( r \) is an odd integer. Further, we have

\[
H_{\pi/2}(T/2 + t) = - H_{\pi/2}(t)
\]  

if \( r \) is an odd integer. This implies that

\[
U_{\pi/2}(T) = [U_{\pi/2}(T/2)]^T U_{\pi/2}(T/2).
\]  

This, combined with \( U_{\pi/2}(T/2) = \pm I \), implies that \( U_{\pi/2}(T) = I \) for any value of \( \gamma, A, B \) and \( T \).

Assuming \( r \) to be an integer, we see from Eq. (86)

\[
H_k(T - t) = H_k(t)
\]  

for any value of \( k \). Following the arguments presented in Eqs. (15) and (18) we see that \( U(T) \) must again be of the form given in Eq. (18). Hence we expect that we should be able to find a phase band crossing at \( t = T \) by varying two parameters, such as \( \omega \) and \( k \).

Next, we analyze such phase band crossings for Hamiltonian given by Eq. (86). To this end, we first study the eigenvalues of the instantaneous Hamiltonian \( H_k(t) \) as a function of \( t/T \) for \( k = \pi/2 \). This is shown in Fig. 16. There are seven regions, denoted by \( R \), \( i = I, II, III, \cdots, VII \), where there is a substantial gap between the ground and excited state energies and the evolution of the system is adiabatic. Any two such regions \( R-i \) and \( R-(i+1) \) are separated by an avoided crossing point \( t_i \) where the gap of the instantaneous Hamiltonian has a minimum. There are six such times, \( \{t_i, i = 1, \cdots, 6\} \), and there is a small region around each of these times where we can use the impulse approximation to calculate the Landau-Zener transition probability. The analysis follows the same route as charted out in Sec. III and the Appendix. In what follows, we shall just present the final results obtained by calculating the phase band \( \cos(\phi(t)) \) for \( k = \pi/2 \) and \( t = T/2 \).

We first show a comparison in Fig. 17 between the numerically calculated plot of \( \cos(\phi(t)) \) versus \( t/T \) and the result obtained by the adiabatic-impulse method for \( k = \pi/2, A = 0.4, r = 3, \) and \( \omega = 0.1 \). We see that the match is excellent except in the small impulse regions. In particular, the agreement between the numerical and adiabatic-impulse results is found to be extremely good in region-IV around the phase band crossing at \( t = T/2 \). In fact, using the different symmetries which are clearly visible in Fig. 16 (such as \( \xi_k(t_2, t_1) = \xi_k(t_3, t_2) \), \( p_{1k} = p_{3k} \) and \( \phi_{1k} = \phi_{3k} \) for \( k = \pi/2 \)), one can obtain a simple expression for \( \cos(\phi(t = T/2)) \),

\[
\cos(\phi(T/2)) = \cos[\xi(T/2, t_3) - \xi(t_1, 0)],
\]  

where

![Graph showing instantaneous energies](image-url)
A times has end modes depending on the different parameters. As we go away from the phase band crossing point there is an increasing deviation between the two results increased as shown in Fig. 18. We see that at higher frequencies and adiabatic-impulse results changes as the frequency is increased at this point and is valid at all frequencies. This is corroborated by checking how the match between the numerical results for any \( k \neq 0, \pi/2, \pi \) for odd integer \( r \); this is consistent with the symmetry analysis carried out earlier in this section.

Here we have used the fact that at \( t = T/2 \) we have \( \eta = 0 \) and \( \xi(T/2, t_3) = \xi(t_1, 0) \). Hence \( \cos(\phi(T/2)) = 1 \) at all frequencies. Thus the adiabatic-impulse method becomes exact at this point and is valid at all frequencies. This is corroborated by checking how the match between the numerical and adiabatic-impulse results changes as the frequency is increased as shown in Fig. 18. We see that at higher frequencies there is an increasing deviation between the two results as we go away from the phase band crossing point \( t = T/2 \) on both sides, as is expected; however the match at the crossing point remains exact. Thus we find that both adiabatic-impulse and general symmetry arguments predict a line of phase band crossings at \( t = T/2 \) and \( k = \pi/2 \) for any \( T \) in this model; we have checked that this prediction agrees with exact numerics.

\[ \begin{align*}
\xi_k(t_f, t_i) &= \int_{t_i}^{t_f} dt [(A \cos(\omega t) - \cos k)^2 + \cos^2(\omega t \sin^2 k)], \\
p_{ik} &= \exp[-2\pi \delta_{ik}], \quad \delta_{ik} = |\beta_{ik}|^2/(2|\alpha_{ik}|), \\
\alpha_{ik} &= \sqrt{A^2 \omega^2 \sin^2(\omega t) + r^2 \sigma^2 \cos^2(2\beta_{ik} \sin k)}, \\
\beta_{ik} &= \sqrt{\left[ A \cos(\omega t_1) - A \omega \sin(\omega t_1) t_1' \right]^2 + \left[ \cos(r \omega t_1) \sin k - r \omega \sin(r \omega t_1) t_1' \sin k \right]^2}, \\
t'_1 &= \left[ A^2 \omega \sin(\omega t_1) \cos(\omega t_1) + r \omega \sin(r \omega t_1) \cos(r \omega t_1) \sin k \right]/(\alpha_{ik}), \\
\phi_{ik} &= -3\pi/4 + \delta_{ik}[\ln(\delta_{ik}) - 1] + \text{Arg}\Gamma(1 - i\delta_{ik}).
\end{align*} \]

An end mode is an eigenvector of the time evolution operator \( U(t) \) whose wave function is localized near one of the two

\[ \text{FIG. 18: Comparison of numerical and adiabatic-impulse results at higher frequencies, (a) } \omega = 1 \text{ (top) and (b) } \omega = 10 \text{ (bottom), for } k = \pi/2, A = 0.4 \text{ and } r = 3. \]
Eq. (97) then implies that band crossing occurring at, say, \( t_{\text{end}} \) mode can appear or disappear when we go across a phase larger than the decay length of each mode. We note that the two modes do not hybridize if the system size is much larger than the decay length of each mode. Due to the parity symmetry of Eq. (82), the isolated eigenvalues (shown as red dots) correspond to end modes, while the continuous arcs of eigenvalues correspond to bulk modes.

To understand the nature of the end modes, it is useful to write the Hamiltonian \( H \) and \( U \) in terms of Majorana operators \(^{[11]}\). We define a \( 2N \)-dimensional column \( c \) whose entries \( c_n \) are given by \( a_1, b_1, a_2, b_2, \ldots, a_N, b_N \). The Hamiltonian in Eq. (82) can then be written in the form

\[
H = \frac{i}{2} \sum_{m,n=1}^{2N} c_m M_{mn} c_n, \tag{97}
\]

where \( M \) is a real antisymmetric matrix.

Next, we allow \( H \) and \( M \) to vary with time. The Heisenberg operators \( c_n(t) \) satisfy the equations

\[
\frac{dc_n(t)}{dt} = i[H(t), c_n(t)]. \tag{98}
\]

Eq. (97) then implies that

\[
\frac{dc_m(t)}{dt} = 2 \sum_{n=1}^{2N} M_{mn}(t)c_n(t). \tag{99}
\]

In terms of the column \( c(t) \) and the matrix \( M(t) \), the solution of Eq. (99) can be written as

\[
c(t) = U(t) \, c(0),
\]

where

\[
U(t) = T e^{\int_0^t dt' M(t')} \tag{100}
\]

We thus see that \( U(t,0) \) is a real and unitary matrix; hence it is also orthogonal.

We now look at the eigenvectors and eigenvalues of \( U(t) \). If \( x \) is an eigenvector of \( U(t) \) with eigenvalue \( e^{i\theta} \), the fact that \( U(t) \) is real implies that \( x^* \) is an eigenvector of \( U(t) \) with eigenvalue \( e^{-i\theta} \). This implies that for eigenvalues equal to \( \pm 1 \), \( x \) and \( x^* \) are degenerate; hence the eigenvectors can be chosen to be real by taking the combinations \( x + x^* \) and \( i(x - x^*) \). In particular, end modes with eigenvalues equal to \( \pm 1 \) will have real eigenvectors; such modes with real wave functions are called Majorana end modes \(^{[11]}\), in analogy with the Majorana end modes of time-independent Hamiltonians with time-reversal symmetry \(^{[17]}\).

Sometimes we find end modes for which the eigenvalues of \( U(t) \) are not equal to \( \pm 1 \); these are called anomalous end modes \(^{[5]}\). Such modes always occur in pairs at each end of the system, with the eigenvalues of the pair being complex conjugates of each other. Further, the wave functions of such modes are necessarily complex.

Fig. 19 shows the eigenvalues \( e^{i\theta} \) of the Floquet operator \( U(T) \) for a 200-site system driven as described in Eqs. (82) and (85). The parameter values are \( \gamma = 1 \), \( A = 0.4 \), \( B = 1 \), \( r = 3 \), \( \omega = 0.3 \), and (a) \( t/T = 0.45 \), (b) \( t/T = 0.5 \), (c) \( t/T = 0.55 \). The isolated eigenvalues (shown as red dots) correspond to end modes, while the continuous arcs of eigenvalues correspond to bulk modes.

FIG. 19: Real and imaginary parts of the eigenvalues \( e^{i\theta} \) of the Floquet operator \( U(T/2) \) for a 200-site system driven as in Eqs. (82) and (85). We have taken \( \gamma = 1 \), \( A = 0.4 \), \( B = 1 \), \( r = 3 \), \( \omega = 0.3 \), and (a) \( t/T = 0.45 \), (b) \( t/T = 0.5 \), and (c) \( t/T = 0.55 \). The isolated eigenvalues (shown as red dots) correspond to end modes, while the continuous arcs of eigenvalues correspond to bulk modes.
and (c) $t/T = 0.55$. In all three cases, a large number of end modes are present; these modes are visible in the figures as isolated eigenvalues which are shown as red dots. All these eigenvalues have at least a two-fold degeneracy due to the parity symmetry, i.e., there are always a pair of modes lying at opposite two ends of the system with the same Floquet eigenvalue. In Fig. 19 (a), we see eight anomalous end modes (each of the four isolated eigenvalues has a two-fold degeneracy). In Fig. 19 (b), we see four Majorana end modes with Floquet eigenvalue equal to $-1$ (this eigenvalue has a four-fold degeneracy). In Fig. 19 (c), we see two Majorana end modes with Floquet eigenvalue equal to $+1$ (with a two-fold degeneracy) and four anomalous end modes (each of the two eigenvalues has a two-fold degeneracy). In Figs. 19 (a) and (c), we see that the continuous parts of the Floquet eigenvalue spectrum (these correspond to the bulk modes) form two disjoint arcs which are separated by gaps around $e^{i\theta}$ equal to both $+1$ and $-1$ (i.e., zone center and zone edge). In Fig. 19 (b), the continuous part forms a single arc with no gap around $e^{i\theta} = 1$ (zone center); thus there is a phase band crossing at $t = T/2$ with $U_{k=\pi/2} = +i$. For all the end modes, we have checked numerically that the Majorana modes with Floquet eigenvalues equal to $\pm 1$ have purely real wave functions, while the anomalous modes appear in pairs with complex conjugate eigenvalues ($e^{\pm i\theta}$) and their wave functions are complex. Interestingly, we observe that there are no anomalous end modes at $t/T = 0.5$ where a phase band crossing occurs (Fig. 19 (b)).

VI. DISCUSSION

In this work, we have studied a class of driven closed quantum integrable systems in the presence of either an external radiation or a two-rate drive protocol at low frequencies. Such systems have been studied before in the regime where the frequency of radiation is high or for models where a single parameter of the Hamiltonian is driven periodically. Our study therefore provides a complementary set of results to the existing literature.

For graphene in the presence of external radiation, we study the change in topology of the system by studying its time evolution operator $U$. Such studies have been carried out in the literature earlier in the high-frequency regime where perturbative $1/\omega$ expansions work. In contrast, our work addresses this phenomenon in the low-frequency regime where such perturbative treatments fail. To obtain an analytical understanding of the phase diagram of graphene in the low-frequency regime we therefore have used the adiabatic-impulse approximation by appropriately modifying it for the present drive protocol. We note that such an approximation yields results for the phase bands of the system which provides a near exact match with exact numerics at low frequencies; moreover, it allows us to provide semi-analytic criteria for the conditions of the phase band crossings in graphene. Our analysis also shows that the phase band crossings, leading to change in Chern number of the phase bands in graphene, are generically expected to occur at the high-symmetry points in the graphene Brillouin zone such as $\Gamma$, $K$ and $M$ points. We also provide, for each of these points, analytic criteria for such crossings. Our results indicate the presence of such crossings at $t = T/3$ and $2T/3$ (apart from those at $t = T$) indicating the inadequacy of a Floquet Hamiltonian based analysis. Such crossings at $t = T/3$ and $t = 2T/3$ lead to a distinct phase diagram whose contribution comes solely from the $\Gamma$ point in the graphene Brillouin zone; such crossings are shown not to occur at high frequencies for a range of radiation amplitude $\alpha$. This explains why such diagrams cannot be obtained by perturbative techniques which rely on some form of $1/\omega$ expansions. In contrast at $t = T$, we find that the phase band crossings may occur at other high symmetry points such as $K$ and $M$ in the graphene Brillouin zone; the general phase diagram at $t = T$ receives contribution from all such points. We emphasize that several aspects of such phase band crossings can be analytically understood from an analysis using the adiabatic-impulse method which we carried out in this work; they lead to semi-analytic conditions for several aspects of the phase diagram (Eqs. (63, 66), (70), and (75)) which match almost exactly with numerics. Moreover, such an analysis indicates the role of symmetry of the irradiated graphene Hamiltonian behind such phase band crossings in the low-frequency regime and predicts exact semi-analytic conditions (Eqs. (57), (53) and (56)) for phase band crossings for any $t \leq T$. Our study takes note of the difficulty in generic numerical computation of the Chern number and its change in the low-frequency regime; this also allows us to point out the benefit determining the position of the phase band crossings (which can be done reliably for any frequency) in determining the structure of the graphene phase diagram at low frequency. We also find that whereas generic phase band crossings are expected to occur only at high symmetry points, one can not rule out the presence of accidental crossings at other times and for other values of $k$; however, the number of such accidental crossings are expected to be much lower than the generic ones.

For the 1D $XY$ model we have shown that a two-rate drive protocol leads to additional phase band crossings at $t = T/2$ for $k = \pi/2$. For any drive frequency when the ratio $r$ of the two drive frequencies is an odd integer. In contrast to the earlier studied phase band crossings in this model for a single parameter drive protocol which occur at $k = 0, \pi$, for a two-rate drive, we find crossings at $k = \pi/2$ and $t = T/2$ which have no counterpart for single parameter drive protocols. Moreover such crossings at $t = T/2$ lead to additional phase band crossings at $t = T$ for $k = \pi/2$; this was shown via a symmetry analysis of the model. These phase band crossings for $k = \pi/2$ occur for any value of $T$ as long as the ratio of drive frequencies, $r$, is an odd integer. We have also studied the end modes of such a driven model for a finite chain with $N$ sites. We have found, apart from the usual Majorana end modes, the existence of anomalous end modes. No anomalous end modes are found at the phase band crossing points at $t = T/2$ where only Majorana modes exist; across this point the num-
ber of end modes change confirming a topological phase transition.

There are several experiments which may confirm our theoretical results. The simplest among them would be for the phase diagrams of irradiated graphene at $t = T/3$ and $t = 2T/3$ at low drive frequencies $\omega/\gamma \simeq 0.1$ and for drive amplitudes $2 \leq \alpha \leq 2.5$; these are shown in Figs. [9] and [12]. These phase diagrams would be much simpler to verify experimentally since the only contribution to them comes from the $\Gamma$ point in the graphene Brillouin zone. We note in this context that angle-resolved photoemission spectroscopy (ARPES) has already been performed on irradiated graphene leading to a detection of Chern number change via photoelectron intensity measurements [48]. In fact such experiments, by a variation of the intensity of the applied photon, can pick out selective contributions to the Chern number change from different points in the graphene Brillouin zone [50]. For example, the contribution to the phase diagram coming from the $K$ and $K'$ points would show up in such measurements; thus we expect such measurement to reproduce the phase diagram in Fig. [14] (b).

The presence of chiral edge modes in such driven systems has also been experimentally verified recently both in phononic crystal [43] and in transport experiments on topological insulator surfaces [55]. A change in their number across a phase band crossing should also lead to a reconstruction of the phase diagrams that we provide here provided that such experiments can be suitably modified and designed for graphene-like systems.

Our study also leads to several open questions which we intend to study in the future. For example, it would be interesting to study the properties of transport in irradiated graphene-like systems in the presence of disorder. An analogous problem has been studied for high radiation frequencies in Refs. [44] however, given the complexity of the phase diagram, its low-frequency counterpart is expected to lead to several and yet unexplored features in the transport. Further, the effect of a weak interaction in these Dirac systems which breaks its integrability would be interesting to study; such studies are clearly numerically difficult in higher dimensions, and we expect a suitably modified version of the adiabatic-impulse approximation to shed some light in this matter.

To conclude, we have studied the low-frequency phase diagram of irradiated graphene and a driven $XY$ model. Our study constitutes an application of a suitably modified adiabatic-impulse approximation to address the dynamics of these models. In the low-frequency regime, for both systems, the analytical results obtained using this approximation provides a near-exact match with numerics. This allows us to provide semi-analytic criteria for phase band crossings and hence a change in the topology of the wave function of such systems. Our analysis predicts a change in the topology for irradiated graphene at $t = T/3$ and $2T/3$ and has provided the corresponding phase diagrams; such diagrams indicate the inadequacy of a Floquet Hamiltonian based analysis for these systems which can only provide information about the phase diagram at $t = T$. Our work also shows that such a change in the topology of the 1D $XY$ model driven using a two-rate protocol may occur at $t = T/2$ and points out a change in the end mode structure across this transition for a $XY$ chain with a finite length. Finally we have suggested several experiments which may test our theory.

**Appendix A: Computation of phase bands for $n$ avoided crossings**

In this appendix, we provide expressions for the eigenvalues of the unitary evolution operators $U_{k}(t_{f}, 0)$ for $n$ avoided level crossings, where $t_{f}$ denotes the time at which the phase bands need to be computed. To this end, we first construct $U^{ad}_{k}(t_{f}, 0)$ for each of the adiabatic regimes between two avoided level crossings using Eq. (48), and then relate it to $U_{k}(t_{f}, 0)$ using Eq. (39).

In the adiabatic region prior to the first avoided crossing, the wave function merely gathers some kinematic phase. Thus we have, from Eq. (48) for $t_{f} \leq t_{1k}$:

$$
\hat{c}_{k}^{(1)}(t_{f}) = e^{-i\gamma \xi_{k}(t_{f})} \left( \begin{array}{c} 1 \\ 0 \end{array} \right),
\xi_{1k}(t_{f}) = \int_{0}^{t_{f}} E_{k}(t') dt',
$$

(A1)

where $E_{k}(t')$ is given by Eq. (35). Using Eqs. (A1) and (39), we obtain

$$
[U_{k}^{(1)}(t_{f}, 0)]_{11} = \eta_{k}(t_{f}) e^{-i\xi_{k}(t_{f})},
\cos \phi_{k}^{(0)}(t_{f}) = \eta_{k}(t_{f}) \cos[\xi_{1k}(t_{f})].
$$

(A2)

Note that for the phase bands to cross, i.e., for $\phi_{k}^{(0)}(t_{f}) = p\pi$, we necessarily require a perfect overlap between the instantaneous and initial ground state wave functions since we need $\eta_{k}(t_{f}) = 1$ at the crossing time.

Next, in region 2, between the first and the second avoided crossings, the evolution operator in the adiabatic basis can be read off from Eq. (48) as

$$
\hat{c}_{k}^{(2)}(t) = U_{k}^{ad}(t, t_{1k}) S^{ad}_{1k} U_{k}^{ad}(t_{1k}, 0) \left( \begin{array}{c} 1 \\ 0 \end{array} \right)
= \left( e^{-i(\xi_{k} + \xi_{2k}(t_{f}) + \phi_{i1})} \sqrt{1 - p_{1k}} \\
- \sqrt{p_{1k}} e^{-i(\xi_{k} - \xi_{2k}(t_{f}))} \right).
$$

(A3)

Using Eqs. (A3), (39) and (49), some straightforward algebra yields, for $t_{1k} \leq t \leq t_{2k}$,

$$
[U_{k}^{(2)}(t, 0)]_{11} = \eta_{k}(t) \sqrt{1 - p_{1k} e^{-i(\xi_{k} + \xi_{2k}(t_{f}) + \phi_{i1})}}
- \sqrt{p_{1k}} (1 - \eta_{k}(t)^{2}) e^{-i(\xi_{k} - \xi_{2k}(t_{f}))},
$$

(A4)

$$
\cos \phi_{k}^{(1)}(t) = \eta_{k}(t) \sqrt{1 - p_{1k} \cos[\xi_{1k} + \xi_{2k}(t_{f}) + \phi_{i1}]}
- \sqrt{p_{1k}} (1 - \eta_{k}^{2}(t)) \cos[\xi_{1k} - \xi_{2k}(t_{f})].
$$
Next, we discuss the situation in region 3 for $t_{3k} \leq t \leq t_{2k}$ for which there are two prior avoided level crossings. To obtain the expressions for the phase bands, we note that the wave function after two such level crossings must come back to itself if $p_k = 1$ for each crossing. Also we use a simplified notation where we denote $\xi_k(t_{1(2)k}, 0) \equiv \xi_{1(2)k}$. $\xi_k(t_{1(2)k}, t_{2k})$, $\xi_k^p(t_{1k}) = \sum_{j=1,2} \xi_k(t_f)$, and $\phi_k = \sum_{j=1,2} \phi_{j,k}$. This yields, after some algebra,

$$
\mathcal{C}_k^{(3)}(t_f) = U_1^{3d}(t_f, t_{2k}) S_2^T U_1^{3d}(t_{2k}, t_{1k}) S_1 U_1^{3d}(t_{1k}, 0) \left( \begin{array}{c} 1 \\ 0 \end{array} \right)
$$

$$
= \left( \begin{array}{c}
\sqrt{1 - p_{1k}} \left( e^{-i \xi_k(t_f)} \right) + \sqrt{p_{1k}p_{2k}} e^{-i (\xi_k(t_f) - 2 \xi_k)} \\
\sqrt{p_{2k}} (1 - p_{1k}) \left( e^{i \xi_k(t_f)} \right) - \sqrt{p_{1k}} (1 - p_{2k}) e^{i (\phi_k + \xi_k)}
\end{array} \right).
$$

Using Eqs. (A5), (39), and (49), we can obtain the expressions for the phase bands. The final result reads

$$
\cos \phi_k^{(2)}(t_f) = \eta_k(t_f) \left[ (1 - p_{1k})(1 - p_{2k}) \cos[\xi_k^p(t_f)] + \eta_k^2 \left( e^{i \xi_k(t_f)} \right) + \sqrt{p_{1k}p_{2k}} \cos[\phi_k + \xi_k + \xi_{2k} - \xi_{3k}(t_f)] \right]
$$

$$
+ \sqrt{1 - \eta_k^2(t_f)} \left[ \sqrt{p_{2k}} (1 - p_{1k}) \cos[\phi_k + \xi_k + \xi_{2k} - \xi_{3k}(t_f)] - \sqrt{p_{1k}} (1 - p_{2k}) \cos[\phi_k + \xi_k - \epsilon_{1k} + \xi_{3k}(t_f)] \right].
$$

Next, we consider the case where $p_{1k} = p_{2k} = p_k$ and $\phi_k = \phi_k$. Such equalities can be justified for high-symmetry points in the graphene Brillouin zone as noted in Secs. II and IV. In this case we obtain simpler expressions for the phase band given by

$$
\cos(\phi^{(2)}(k, t)) = \eta_k(t) \left( (1 - p_k) \cos(\mu_k^p(t)) + p_k \cos(\mu_k^d(t)) \right)
$$

$$
+ \sqrt{p_k(1 - p_k)(1 - \eta_k^2(t))} \left[ \cos(\mu_k^p(t) - \phi_k) - \cos(\mu_k^d(t) - \phi_k) \right],
$$

$$
\mu_k^p(t) = 2\phi_k^2 + \xi_k^p(t), \quad \mu_k^d(t) = \xi_{1k} - \xi_{2k} + \xi_{3k}(t_f).
$$

For larger $n$ we can, in principle, write down the expressions for $\cos(\phi^{(n)}(k, t))$ following the same procedure. However, these expressions get more complicated with increasing $n$. In what follows, we concentrate on the expressions for the phase bands at $t = t_f$ for which $\eta_k(t_f) = 1$ since, as argued in the main text, the generic phase band crossings require this condition. In this case, we first note that for $\eta_k(t_f) = 1$, after $n$ avoided crossings, the matrix element of the evolution operator $[U_k(t_f, 0)]_{11}$ operators is related to the wave function $c_k^{(n+1)}(t_f)$ in the adiabatic basis: $[U_k(t_f, 0)]_{11} = c_k^{(n+1)}(t_f)$. Next, we note that after $n$ crossings, $c_k^{(n+1)}(t_f)$ is given by

$$
c_k^{(n+1)}(t_f) = U_1^{3d}(t_f, t_{n-1k}) S_2^T \ldots \times U_1^{3d}(t_{2k}, t_{1k}) S_1 U_1^{3d}(t_{1k}, 0) \left( \begin{array}{c} 1 \\ 0 \end{array} \right).
$$

A few lines of algebra then lead to
where the sum over the index $\alpha$ represents a sum over all possible permutations of $j_\alpha$ and $j'_\alpha$ for a fixed $j_{\text{max}}$, $n_1 = \text{Max}[j'_\alpha] - \text{Min}[j'_\alpha] + 1$ provided $\text{Min}[j'_\alpha] \neq 0$ and is 0 otherwise, $n_{j'_\alpha}$ denotes the number of occurrence of $j'$ (i.e., the number of $\sqrt{p_{j'_\alpha}}$ factors) in a permutation $\alpha$, and $n_2 = n_{j_\alpha}$ with $\text{Max}[j'_\alpha] < j_\alpha < \text{Min}[j'_\alpha]$. The coefficients $\gamma_\alpha$ and $\delta_\alpha$ for any given permutation $\alpha$ are given by

$$
\gamma_\alpha = \begin{cases} 
1 & \text{for } a \in j'_\alpha, \\
2 & \text{for } a \in j_\alpha \text{ with } j''_\alpha < j_\alpha < j''_\alpha', \\
0 & \text{otherwise},
\end{cases}
$$

$$
\delta_\alpha = \begin{cases} 
2 & \text{for } a \in j_\alpha, j'_\alpha \text{ with } \text{Min}[j'_\alpha] < j_\alpha, j'_\alpha < \text{Max}[j'_\alpha] \\
0 & \text{if } a \in j_\alpha, j'_\alpha \text{ with } a - 1 \in j''_\alpha \text{ and } \delta_{a-1} = 2,
\end{cases}
$$

where $j''_\alpha$ denotes any odd occurrence of $j'$ during a permutation, and $j''_\alpha'$ denotes its next occurrence in that permutation.

The expressions for the phase bands can then be obtained from Eqs. (A9) by using $\cos[\phi^{(n)}(k,t)] = \text{Re}[[U^{(n)}(t_f,0)c_{\mathbf{k}}^{(n)}(t_f)]] = \text{Re}[c_{\mathbf{k}}^{(n+1)}(t_f)]$, and leads to Eqs. (53) and (56) in the main text.

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