Constituent monopoles without gauge fixing

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We discuss the recent construction of new exact finite temperature instanton solutions with a non-trivial value of the Polyakov loop at infinity. They can be shown, in a precise and gauge invariant way, to be formed by the superposition of $n$ BPS monopoles for an $SU(n)$ gauge group.

1. Introduction

Instantons at finite temperature (or calorons) are constructed on $\mathbb{R}^3 \times S^1$, taking a periodic array of instantons. For $SU(2)$ the five parameter Harrington-Shepard solution \cite{1} can be formulated within the ‘t Hooft ansatz. New exact solutions with a non-trivial value of the Polyakov loop at infinity \cite{2} were only constructed very recently, either using \cite{3} results due to Nahm \cite{4} or by using \cite{5} the well-known ADHM construction \cite{6}, translated by Fourier transformation to the Nahm language. Thus mapped to an Abelian problem on the circle, the quadratic ADHM constraint is solved \cite{5c}.

2. New caloron solutions

In the periodic gauge, $A_\mu(x+\beta)=A_\mu(x)$, the Polyakov loop at spatial infinity

$$P_\infty = \lim_{|x| \to \infty} P \exp(\int_0^\beta A_0(\vec{x},t)dt),$$

after a constant gauge transformation, is characterised by ($\sum_{m=1}^n \rho_m = 0$)

$$P_\infty^0 = \exp[2\pi i \text{diag}(\mu_1, \ldots, \mu_n)],$$

$$\mu_1 < \ldots < \mu_n < \mu_{n+1} \equiv \mu_1 + 1.$$ 

A non-trivial value, $P \notin \mathbb{Z}_n$, acts like a Higgs field. We found \cite{5c} a remarkably simple formula for the action density, valid for arbitrary $SU(n)$.

Using the classical scale invariance to put $\beta = 1$,

$$A_m \equiv \frac{1}{r_m} \begin{pmatrix} r_m & |\vec{y}_m - \vec{y}_{m+1}| \\ 0 & r_{m+1} \end{pmatrix} \begin{pmatrix} c_m & s_m \\ s_m & c_m \end{pmatrix},$$

with $r_m = |\vec{x} - \vec{y}_m|$ the center of mass radius of the $m$th constituent monopole, which can be assigned a mass $16\pi^2 \nu_m$, where $\nu_m = \mu_{m+1} - \mu_m$. Also $r_{n+1} \equiv r_1$, $\vec{y}_{n+1} \equiv \vec{y}_1$, $c_m \equiv \cosh(2\pi \nu_m r_m)$ and $s_m \equiv \sinh(2\pi \nu_m r_m)$. The order of matrix multiplication is crucial here, $\prod_{m=1}^n A_m \equiv A_n \ldots A_1$.

Figure 1. Action densities for the $SU(3)$ caloron on equal logarithmic scales, cut off at $1/e$, for $t = 0$ in the plane defined by $\vec{y}_1 = (-\frac{1}{2}, \frac{1}{2}, 0)$, $\vec{y}_2 = (0, \frac{1}{2}, 0)$ and $\vec{y}_3 = (\frac{1}{2}, -\frac{1}{2}, 0)$, in units of $\beta$, for $\beta = 1/4$, $1/3$ and $2/3$ from top to bottom, using $(\mu_1, \mu_2, \mu_3) = (-17, -2, 19)/60$. 

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For $P_\infty = \exp(2\pi i \omega_3)$ the $SU(2)$ gauge field reads [5a], in terms of the anti-selfdual 't Hooft tensor $\bar{\eta}$ and Pauli matrices $\tau_a$,

$$A_\mu(x) = \frac{i}{2} \bar{\eta}_{\mu\nu} \tau_a \partial_\nu \log \phi(x) + \frac{i}{2} \phi(x) \text{Re} \left( (\bar{\eta}_{\mu\nu} - i \bar{\eta}_{\mu\nu})(r_1 + i r_2) \partial_\nu \chi(x) \right),$$

where $\phi^{-1} = 1 - \frac{\pi \rho^2}{\psi} \left( \frac{s_1 c_2}{r_1} + \frac{s_2 c_1}{r_2} + \frac{\pi \rho^2 s_1 s_2}{r_1 r_2} \right)$ and $\chi = \frac{\pi \rho^2}{\psi} \left( e^{-2\pi it \frac{s_1}{r_1}} + e^{2\pi i \nu_1 t} \right)$, with $\nu_1 = 2\omega$, $\nu_2 = 1 - 2\omega$ and $\pi \rho^2 = |\vec{y}_2 - \vec{y}_1|$. The solution is presented in the “algebraic” gauge, $A_\mu(x + \beta) = P_\infty A_\mu(x) P_\infty^{-1}$.

For $\rho$, equivalent to large $\beta$, the caloron approaches the ordinary single instanton solution, with no dependence on $P_\infty$. Finite size effects set in roughly when $\rho = \frac{1}{4} \beta$. At this point, for $\nu_i \neq 0$, two lumps ($n$ for $SU(n)$) are formed, whose separation grows as $\pi \rho^2 / \beta$. When $P_\infty = (-1)$ for $SU(2)$, one of the lumps disappears, as $\nu_1 = 0$, and the spherically symmetric Harrington-Shepard solution is retrieved.

A non-trivial value of $P_\infty$ will modify the vacuum fluctuations and thereby leads to a non-zero vacuum energy density as compared to $P \in \mathbb{Z}_n$. A dilute, semi-classical instanton calculation is no longer considered a reliable starting point for QCD. Rather, it is the monopole constituent nature from which we should draw important lessons for QCD [5b].

3. Monopoles from instantons

At small $\beta$ the solution becomes static and the lumps are well separated and spherically symmetric. As they are self-dual, they are necessarily BPS monopoles [6]. Also, when sending (at fixed $\beta$) one of the constituents to infinity, $|\vec{y}_n| \rightarrow \infty$, the solution becomes static and yields a simple way to obtain $SU(n)$ monopole solutions [5c]. Explicitly we find (assuming $\nu_n \neq 0$) in the limit $|\vec{y}_n| \rightarrow \infty$, which removes the $n$-th constituent,

$$A_n \rightarrow 2c_n \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \quad A_{n-1} \rightarrow |\vec{y}_n| \begin{pmatrix} c_{n-1} & s_{n-1} \\ s_{n-1} & c_{n-1} \end{pmatrix},$$

implying $\psi(x) \rightarrow 2|\vec{y}_n| \exp(2\pi \nu_n |\vec{y}_n - \vec{x}|) \tilde{\psi}(\vec{x})$, with

$$\tilde{\psi}(\vec{x}) = \text{tr} \left\{ \frac{1}{r_{n-1}} \begin{pmatrix} c_{n-1} & s_{n-1} \\ 0 & 0 \end{pmatrix} \prod_{m=1}^{n-2} A_m \right\}.$$

As was emphasised in ref. [5c], the energy density of the $SU(n)$ monopole is easily found from eq. (3) (for a detailed description of some special cases see ref. [6])

$$E(\vec{x}) = -\frac{1}{4} \text{tr} P_{\mu\nu}^2(\vec{x}) = -\frac{1}{4} \Delta^2 \log \tilde{\psi}(\vec{x}).$$

4. Instantons from monopoles

The new caloron solutions provide examples of gauge fields with topological charge built out of monopole fields, a construction going back to
Taubes [9]. Non-trivial $SU(2)$ monopole fields are classified by the winding number of maps from $S^2$ to $SU(2)/U(1) \sim S^2$, where $U(1)$ is the unbroken gauge group. Isospin orientations for a configuration made out of monopoles with opposite charges behave as shown in fig. 4 (top), in a suitable gauge and sufficiently far from the core of both monopoles. Taubes constructed topologically non-trivial configurations by creating a monopole anti-monopole pair, bringing them far apart, rotating one of them over a full rotation and finally bringing them together to annihilate (cmp. fig. 5). We can describe this as a closed monopole line (or loop) with the orientation of the core defined by $SU(2)/U(1) \sim S^2$, “twisting” along the loop, thus describing a Hopf fibration [5b] (see fig. 4 (bottom)). The only topological invariant available to characterise the homotopy type of this Hopf fibration is the Pontryagin index. It prevents full annihilation of the “twisted” monopole loop.

For large $\rho$, eq. (4) gives up to exponential corrections, i.e. outside the cores of the constituents,

$$A_\mu = \frac{i}{2} e_3 \bar{h}_{\mu \nu} \partial_\nu \log \phi_0, \quad \phi_0 = \frac{r_1 + r_2 + \pi \rho^2}{r_1 + r_2 - \pi \rho^2}. \quad (8)$$

This describes two Abelian Dirac monopoles and one easily verifies $\log \phi_0$ is harmonic, as required by self-duality. Furthermore $\phi_0^{-1}$ vanishes on the line connecting the two monopole centers, giving rise to return flux, absent in the full theory. The relative phase $e^{-2\pi i t}$ in the expression for $\chi$ given before, describes the full rotation of the core of a constituent monopole, required so as to give rise to non-trivial topology.

A conjectured QCD application, in the form of a hybrid monopole-instanton liquid, was discussed in ref. [5b]. Abelian projection applied to our solutions was also discussed at this conference [10].

Figure 4. Topological charge constructed from oppositely charged monopoles by rotating one of them. For a closed monopole line, the embedding of the unbroken subgroup makes a full rotation.

![Figure 4](image-url)

Figure 5. Action density in the $z-t$–plane for $x = y = 0$, $\omega = \frac{1}{4}$, $\rho = \frac{3}{4}$ and $\beta = 1$ on a linear scale. One can trace the constituent monopoles in the low field regions, “annihilating” to give an instanton.

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