MINIMAL GRAVITO-MAGNETISM

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We show that Feynman’s proof applies to Newtonian gravitation, implying thus the existence of gravitational analogous of the electric and magnetic fields and the corresponding Lorentz-like force. Consistency of the formalism require particular properties of the electric and magnetic-like fields under Galilei transformations, which coincide with those obtained in previous analysis of Galilean electromagnetism.

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I. INTRODUCTION

Historically the analogies between gravity and electromagnetism have played an important role in the development of Gravito-magnetism, even though is known that such analogies are necessarily incomplete and therefore the natural framework where this topic has developed is general relativity [1]. However it has been recognized that any theory including Newtonian gravitation and Lorentz invariance in a consistent framework must contain gravito-magnetism in some form [2] and even, a close relation to Coriolis force has been remarked [3, 4]. Thus, one wonders what is the minimal framework where the phenomenon of gravitomagnetic forces occur. Our interest in this short note is to point the relevance to gravito-magnetism of what is known as Feynman’s proof [5]. In fact, once one goes through the demonstration the conclusion is evident, a particular form of gravito-magnetism follows just from Galilei invariance and Newton’s second law.

In 1990, F. Dyson [5] published an original proof given by Feynman in 1948 of the homogeneous Maxwell equations (divergenceless magnetic field and Faraday’s law) and the Lorentz Force Law. The motivation of Feynman was to discover a brand new theory starting form simple assumptions, but the result was nothing but the same old theory, and therefore, from his point of view, the proof was more a failure than a success. Even though the proof is mathematically correct, it requires some clarification [6]. The proof is based on two essential parts: 1) second Newton’s law, 2) the commutator between components of the position operator and between position and velocity. There is an apparent inconsistency in these assumptions since the first is purely classical while the second comes from a quantum theory, however Bracken[7] remarked that it is possible to substitute the quantum commutators by classical Poisson brackets.

According to our view, the logic of the proof is the following.

- Galilean relativity is the basis of the formulation.
- Galilean invariance is enough to derive minimal coupling, *i.e.* to introduce electromagnetic interactions.
- Newton’s second law and minimal coupling are consistent with the Lorentz force and two homogeneous Maxwell equations, provided the electric and magnetic field transforms appropriately under boosts.

It is important to remark that, in the second and third points above, we can change the electromagnetic interactions – Maxwell equations and Lorentz force – by more generic terms since no where in the proof the electromagnetic nature of the vector potential is invoked. That has motivated us to pursue the implications of Feynman’s proof to the gravitomagnetic interaction, a point that seems to be overlooked so far [8].

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In order to introduce the notation and for easy of reading, in this short note we first formulate Feynman’s proof and summarize the ingredients of the Galilei group required for the presentation. With these tools at hand we show that the assumptions required in Feynman’s proof can be derived from the Galilei algebra; in particular, minimal coupling is discussed following the approach by Levy-Leblond [9]. Requiring consistency of the whole approach we derive the properties of the electric and magnetic fields under boost transformations, a point that deserves attention since the non-relativistic limit of the transformation of the fields is ambiguous, a point discussed long time ago [10]. Finally the applicability of Feynman’s proof to gravito-magnetism is discussed, in particular the conditions for its validity.

A. Feynman’s proof

Assume a particle exists with position $x_j$ ($j=1,2,3$) and velocity $\dot{x}_j$ satisfying Newton’s Second Law

$$m\ddot{x}_j = F_j(x, \dot{x}, t), \quad (1)$$

with Poisson brackets

$$\{x_i, x_j\} = 0 \quad (2)$$

$$m\{x_i, \dot{x}_j\} = \delta_{ij}. \quad (3)$$

Then there exists fields $E(x,t)$ and $B(x,t)$ satisfying the Lorentz force and Maxwell equations

$$F_j = E_j + \epsilon_{jkl}\dot{x}_kB_l, \quad (4)$$

$$\nabla \cdot B = 0, \quad (5)$$

$$\frac{\partial B}{\partial t} + \nabla \times E = 0. \quad (6)$$

Proof. From Eqs. (1,3) it follows:

$$\{x_j, F_k\} + m\{\dot{x}_j, \dot{x}_k\} = 0. \quad (7)$$

The Jacobi identity

$$\{x_l, \{\dot{x}_j, \dot{x}_k\}\} + \{\dot{x}_j, \{\dot{x}_k, x_l\}\} + \{\dot{x}_k, \{x_l, \dot{x}_j\}\} = 0, \quad (8)$$

together with Eq. (3) and Eq. (7) imply

$$\{x_l, \{x_j, F_k\}\} = 0, \quad (9)$$

while Eq. (7) allow us to conclude that

$$\{x_j, F_k\} = -\{x_k, F_j\}, \quad (10)$$

and therefore, we can write

$$\{x_j, F_k\} = -\frac{1}{m}\epsilon_{jkl}\dot{x}_kB_l. \quad (11)$$

Eq. (11) is the definition of $B$, which by virtue of Eq. (8) can be written as

$$B_l = \frac{m^2}{2}\epsilon_{jkl}\{\dot{x}_j, \dot{x}_k\} \quad (12)$$
On the other hand Eq. (9) can also be expressed as follows
\[ \{x_j, B_l\} = 0, \tag{13} \]
which means that \( B \) is only a function of \( x \) and \( t \).

Defining \( E \) by Eq. (4), which guarantee the Lorentz Force is correctly incorporated, and using Eqs. (3, 11) and Eq. (13), it follows that \( E \) is only a function of \( x \) and \( t \).
\[ \{x_j, E_l\} = 0, \tag{14} \]
Moreover, using the expression for \( B \), Eq. (12) the Jacobi identity :
\[ \epsilon_{jkl} \{\dot{x}_l, \{\dot{x}_j, \dot{x}_k\}\} = 0. \tag{15} \]
can be cast in the form:
\[ \{\dot{x}_l, B_l\} = 0, \tag{16} \]
which is equivalent to the Maxwell equation (5).

The time evolution of \( B \) is obtained from the time derivative of Eq. (12). This gives :
\[ \frac{\partial B_l}{\partial t} + \frac{\partial B_l}{\partial x_m} \dot{x}_m = m^2 \epsilon_{jkl} \{\dot{x}_j, \dot{x}_k\}. \tag{17} \]
Now by Eq. (1) and Eq. (4), Eq. (17) becomes
\[ \frac{\partial B_l}{\partial t} + \dot{x}_m \frac{\partial B_l}{\partial x_m} = m \epsilon_{jkl} \{E_j + \epsilon_{jmn} \dot{x}_m B_n, \dot{x}_k\} \]
\[ = m \left( \epsilon_{jkl} \{E_j, \dot{x}_k\} + \{\dot{x}_kB_l, \dot{x}_k\} - \{\dot{x}_l B_k, \dot{x}_k\}\right) \]
\[ = \epsilon_{jkl} \frac{\partial E_j}{\partial x_k} + \dot{x}_k \frac{\partial B_l}{\partial x_k} - \dot{x}_l \frac{\partial B_k}{\partial x_k} - m B_k \{\dot{x}_l, \dot{x}_k\}. \tag{18} \]
Using Eq. (12) one shows the last term is zero by symmetry while the third term vanishes because of Eq. (16). Thus:
\[ \frac{\partial B_l}{\partial t} = \epsilon_{jkl} \frac{\partial E_j}{\partial x_k}, \tag{19} \]
which is equivalent to Eq(6). End of proof.

II. GALILEI GROUP IN 3+1 D

The three dimensional Galilei group is defined as the ten parameter Lie group of the space time transformations of the form:
\[ x' = R(\theta, \varphi, \psi)x + vt + u \]
\[ t' = t + \tau \tag{20} \]
where \( R \) is a SO(3) rotation matrix. The Lie algebra of the three dimensional Galilei group is ordinarily referred to a conventional basis consisting of ten generators: time and space translations \( H, P_i \), rotations \( J_i \), and boosts \( K_i \). It is well known that Galilei group possess a family of nontrivial projective representations [9] characterized by a real number \( m \), which in physical systems is interpreted as the particle mass. The corresponding Poisson brackets are:
\[ \{J_i, J_j\} = \epsilon_{ijk} J_k, \quad \{J_i, K_j\} = \epsilon_{ijk} K_k, \quad \{H, P_j\} = 0, \quad \{J_i, P_j\} = \epsilon_{ijk} P_k, \]
\[ \{H, J_i\} = 0, \quad \{H, K_j\} = -P_j, \quad \{K_i, K_j\} = 0, \quad \{P_i, P_j\} = 0, \quad \{K_i, P_j\} = m \delta_{ij}. \tag{21} \]

The localization properties of the system can be investigated looking for a position function \( x_i \) \( (i = 1, 2, 3) \) in the enveloping Galilei Lie algebra. The natural requirements \( x_i \) must obey, in order to be identified with the spatial position, are [9]:

\[ \{x_j, B_l\} = 0, \tag{13} \]
\[ \{x_j, E_l\} = 0, \tag{14} \]
\[ \epsilon_{jkl} \{\dot{x}_l, \{\dot{x}_j, \dot{x}_k\}\} = 0. \tag{15} \]
\[ \{\dot{x}_l, B_l\} = 0, \tag{16} \]
\[ \frac{\partial B_l}{\partial t} = \epsilon_{jkl} \frac{\partial E_j}{\partial x_k}, \tag{19} \]
1. A state localized at \( x_i \) transforms under a translation by \( u_i \) in a state localized at \( x_i + u_i \). In other words, it requires the validity of the Poisson bracket rule

\[
\{ x_i, P_j \} = \delta_{ij}, \tag{22}
\]

2. It should transform like a vector under spatial rotations, or equivalently,

\[
\{ J_i, x_j \} = \epsilon_{ijk} x_k. \tag{23}
\]

3. An instantaneous \((t = 0)\) boost, must leave invariant the position, i.e.

\[
\{ K_i, x_j \}_{t=0} = 0. \tag{24}
\]

These conditions are fulfilled by the following function:

\[
x_i = \frac{K_i}{m} \bigg|_{t=0} . \tag{25}
\]

Once the relation between the boost generator and the position is established, it is clear that the mass as a central extension — last relation in Eq. (21)— plays an important role in the classical relation that is used in [7] to replace the quantum commutator in Feynman’s assumptions [5].

### III. FEYNMAN’S PROOF AND GALILEAN INVARIANCE

We are now ready to analyze the hypotheses of Feynman’s proof with to the light of galilean relativity. The second assumption Eq. (2) is an immediate consequence of Eqs. (24,25), which define the action of instantaneous boosts (at \( t = 0 \)) on the particle position:

\[
\left\{ x_i, \frac{k_j}{m} \right\} \bigg|_{t=0} = \{ x_i, x_j \} = 0. \tag{26}
\]

The first assumption, Eq. (1), involves in fact two relations:

- Newton’s second law

\[
F_i = \frac{d\pi_i}{dt} \quad (i = 1, 2, 3), \tag{27}
\]

where \( \pi_i \) \((i = 1, 2, 3)\) is the kinematical momentum of the particle, not necessarily equal to the canonical momentum \( p_i \),

- and the statement that \( \pi_i \) is related to the velocity of the particle \( \dot{x}_i \) according to

\[
F_i = \frac{d\pi_i}{dt} = m \frac{d\dot{x}_i}{dt} \quad (i = 1, 2, 3), \tag{28}
\]

or equivalently

\[
\pi_i = m \dot{x}_i + c_i, \tag{29}
\]

where \( c_i \) is a constant vector that can be absorbed into the definition of \( \pi_i \):

\[
\pi_i = m \dot{x}_i. \tag{30}
\]
Notice that the relation between velocity and kinematical momentum Eq. (30) severely restricts the form of the Hamiltonian since we have
\[ \pi_i = m \{ x_i, H \}. \quad (31) \]

The connection of the third assumption Eq. (3) with Galilei algebra goes through the relation among velocity and momentum; therefore in order to proceed, we need to know in first place the relation between canonical \( p_i \) and kinematical momentum \( \pi_i \). The following argument due to Lévy-Leblond [9], provides the desired link. Indeed, it is enough to demand the existence of instantaneous boost transformations of momentum and position
\[ p_i \rightarrow p_i + mv_i \]
\[ x_i \rightarrow x_i, \quad (32) \]
and to postulate that the kinematical momentum transforms in the same way, not only for the free particle, but also when interactions are introduced
\[ \pi_i \rightarrow \pi_i + mv_i. \quad (33) \]

In this case, the transformation of \( \pi_i \) is noting but the familiar velocity composition under a boost. As required, Eqs. (22,23,24) remain valid, and therefore, comparing Eq. (32) and Eq. (33), we conclude that under a boost:
\[ p_i - \pi_i \rightarrow p_i - \pi_i, \quad (34) \]

thus, the functions \( A_i = p_i - \pi_i \ (i = 1, 2, 3) \) satisfy
\[ \{ k_i, A_j \}_{t=0} = m \{ x_i, A_j \} = m \frac{\partial A_j}{\partial p_i} = 0, \quad (35) \]
that is, \( A \) is a function of \( x \) alone (and possibly of time). Then, the relation between the canonical momentum and the kinematical momentum is, using Eq. (30),
\[ p_i = \pi_i + A_i(x, t) = m \dot{x}_i + A_i(x, t). \quad (36) \]

This is nothing but minimal coupling, which has been obtained from Galilei relativity plus plausible assumptions (a formal proof of the derivation of minimal coupling based on Galilei relativity can be found in [11]). Now the third assumption, Eq. (3), may be seen as a consequence of Eqs. (22,36).
\[ \{ x_i, p_j \} = \{ x_i, m \dot{x}_j \} + \{ x_i, A_j(x, t) \} \]
\[ = m \{ x_i, \dot{x}_j \} = \delta_{ij}. \quad (37) \]

So far we have shown that besides Newton’s second Law, the hypothesis used by Feynman follow from the 3+1 dimensional Galilei algebra. In order to check the consistency of the output we now investigate the transformation laws for the Electric and Magnetic fields. Since the properties of the differential operators under boosts follow from Eqs. (20,32)
\[ \frac{\partial}{\partial x'} = \frac{\partial}{\partial x'} \quad \frac{\partial}{\partial y'} = \frac{\partial}{\partial y'} - v_i \frac{\partial}{\partial x'}, \quad \frac{\partial}{\partial p_i'} = \frac{\partial}{\partial p_i}. \quad (38) \]
then, the Lorentz force Eq. (4) transforms as:
\[ F'_i = \frac{d\pi'_i}{dt'} = \frac{d\pi_i}{dt} = F_i. \quad (39) \]

On the other hand, according to the definition of \( B \) and \( \pi \), Eqs. (12,30):
\[ B_l = \frac{1}{2} \epsilon_{klj} \{ \pi_j, \pi_k \}, \quad (40) \]
then, the transformation law for the magnetic field is

\[ B'_l = \frac{1}{2} \varepsilon_{jkl} \{ \pi'_j, \pi'_k \}' \]

\[ = \frac{1}{2} \varepsilon_{jkl} \{ \pi_j, \pi_k \} = B_l. \]

Finally, the transformation of the Electric field, under boosts is

\[ E'_i = F'_i - \frac{\varepsilon_{ikl}}{m} \pi'_k B'_l = F_i - \frac{\varepsilon_{ikl}}{m} (\pi_k + mv_k) B_l \]

\[ = E_i - \varepsilon_{ikl} v_k B_l. \]

Thus, Lorentz force and the two homogeneous Maxwell equations, are consistent with Galilean relativity if the electric and magnetic field transform according to:

\[ E' = E - v \times B, \]

\[ B' = B. \]

Can these transformation properties be identified with the non-relativistic limit of Maxwell equations? It turns out [10] that two such limits exist (these can be traced back to the relation \( \varepsilon_0 \mu_0 c^2 = 1 \), since in the \( c \rightarrow \infty \) limit \( \varepsilon_0 \) and \( \mu_0 \) can not remain finite simultaneously). In none of these non-relativistic limits Eqs. (5, 6) and the Lorentz force Eq. (4) can be obtained together with the transformation rules Eq. (38) and Eq. (43). Thus, the transformation properties of the fields Eq. (43) can not be obtained from a non-relativistic limit; therefore care must be exercised when considering the non-relativistic theory of Maxwell equations, which should be defined not only as the \( c \) large limit, but also as the limit that ensures the correct transformation properties under the Galilei group.

### IV. GRAVITO-MAGNETISM

The relation to gravito-magnetism arises from the observation that the vector potential involved in the derivation of minimal coupling, as presented in the previous section or in Ref. [11], has nothing to do with electromagnetism, therefore gravito-magnetism should also be derived solely from Newton’s equation of motion and Galilei invariance [8].

The derivation of Maxwell-type gravitational equations and Lorentz-like force has a long history [12, 13]. These relations, and others applying in the relativistic domain, are usually derived starting with the gravitational field equations. Here we restrict our attention to Electromagnetic-like effects of a stationary space-time in the low velocity and weak field approximation.

The characteristic features of a stationary space-time are the following [3]:

- its metric tensor is independent of time, that is

\[ ds^2 = g_{\mu\nu}(x) dx^\mu dx^\nu, (\mu, \nu = 0, \cdots, 3; i = 1, \cdots, 3). \]

- it is always possible to find a canonical form of the metric in which, according to time dilation, the component \( g_{00} \) can be parameterized as

\[ g_{00} = e^{\frac{2\Phi(x)}{c^2}}, \]

where \( \Phi(x) \) is the clock-rate function (in the approximation we work, the gravitational potential).

The so-called canonical form of the stationary space-time metric is, then,

\[ ds^2 = e^{2\Phi(x)/c^2} \left( c dt - \frac{1}{c^2} w_i(x) dx^i \right)^2 - k_{ij}(x) dx^i dx^j, \]
where \( w_i(x) \) and \( k_{ij}(x) \) are time independent coefficients. It can be shown that under the transformation of the time coordinate of the stationary metric:
\[
t \rightarrow \kappa [t + f(x)],
\]
Equation (46) remain invariant provided the functions \( \Phi \), \( w_i \) and \( k_{ij} \) transform as follows:
\[
\Phi \rightarrow \Phi - c^2 \ln \kappa \\
w_i \rightarrow \kappa (w_i + c^3 \frac{2f_i}{dx^i}) \\
k_{ij} \rightarrow k_{ij}.
\]
In the slow-motion and weak field approximation, the metric Eq. (46) can be replaced by
\[
ds^2 = \left(1 + 2\Phi(x) c^2\right) \left(1 - \frac{1}{c^2} \frac{dx^i}{dt} \frac{dx^j}{dt}\right) c^2 dt^2 - dx^2 \\
\simeq \left(1 + \frac{2\Phi(x)}{c^2} - \frac{1}{c^2} w \cdot \dot{x}\right) c^2 dt^2 - dx^2.
\]
Then, the action of a massive particle that moves between points \( P_1 \) and \( P_2 \) — under the action of a weak gravitational field — is
\[
S[t, x(t)] = -m \int_{P_1}^{P_2} ds \simeq -m_0 c \left\{ t_2 - t_1 - \int_{P_1}^{P_2} \left[ \frac{\dot{x}^2}{2} - \Phi(x) - \frac{w \cdot \dot{x}}{c^2}\right] dt \right\}.
\]
The corresponding variational principle yields the Lorentz-type force equation
\[
f = m \ddot{x} = -m \nabla \Phi + \frac{m}{c} \dot{x} \times (\nabla \times w).
\]
w is subject to the condition \( \partial w / \partial t = 0 \) and plays the role of a gravito-magnetic vector potential, which turns out to be related to the local rotation rate of the reference frame. From the point of view of Newtonian mechanics, if a point \( P \) of a rigid reference frame \( L \) travels with acceleration \( \mathbf{a} = \mathbf{E}/m \) through an inertial frame while \( L \) rotates about \( P \) at angular velocity \( \Omega = \mathbf{B}/2m \), then a free particle of mass \( m \) at \( P \) moving relative to \( L \) at velocity \( \dot{x} \) experiences a force
\[
f = m \ddot{x} = \mathbf{E} + \dot{x} \times \mathbf{B},
\]
where the last term corresponds to the well known Coriolis force. Comparison of Eqs. (51) and (52) makes it possible to establish the relations:
\[
\mathbf{E}(x) = -m \nabla \Phi(x)
\]
\[
\mathbf{B}(x) = \frac{m}{c} [\nabla \times w(x)],
\]
therefore, by construction, the fields \( \mathbf{E}(x) \) and \( \mathbf{B}(x) \) satisfy
\[
\nabla \cdot \mathbf{B} = 0,
\]
\[
\nabla \times \mathbf{E} = 0.
\]
Thus, if we suppose that the force does not depend explicitly on time, i.e. if we replace the assumption Eq. (1) by the more restrictive one
\[
m \ddot{x}_j = F_j(x, \dot{x}),
\]
then Feynman’s proof is able to reproduce the stationary, weak field and low velocity partial description of gravito-magnetism, given by Eqs(52, 55 and 56). In this case again, the argument has nothing to say regarding the relation between the fields \( \mathbf{B}(x) \) and \( \mathbf{E}(x) \) and the sources (the complementary inhomogeneous Maxwell-like gravito-magnetic equations).

Finally it is important to remark that in the non-stationary case Feynman’s proof is incompatible with gravito-magnetism, because in such a case [13], Eq. (56) is not replaced by Eq. (6) but instead by
\[
\frac{1}{4} \frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E} = 0.
\]
V. DISCUSSION

In this paper we have shown that:

- The assumptions used in Feynman’s proof are either consistent with Galilei invariance (Newton’s second law), or derived from it (Eqs. (2,3)).
- Minimal coupling is derived from Galilei invariance. Although a formal proof of this is given in [11], here we presented the argument following Levy-Leblond [9].
- Appropriated transformations under boosts of the electric and magnetic fields exist such that the Lorentz force and Maxwell equations are consistent with Galilean relativity.
- Feynman’s proof is able to reproduce the stationary, weak field and slow-motion approximation of gravito-magnetism, assuming the fields do not depend explicitly on time.

A question that comes to mind immediately is if the derivation of minimal coupling (gauge principle) starting from the Galilei group can be extended to the relativistic domain. As far as we know all attempts in this direction have failed [11], for they cannot incorporate the inherent reparametrization invariance of the relativistic theory [14].

We conclude that Feynman’s proof is valid in the framework of Galilean relativity (Dyson’s statement referring to Feynman’s proof [5] “The proof begins with assumptions invariant under Galilean transformations and ends with equations invariant under Lorentz transformations” turns out to be incorrect) and that Feynman’s proof applies to Newtonian gravitation, implying thus the existence of gravitational analogous of the electric and magnetic fields and the corresponding Lorentz-like force.

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