\( \mathcal{N} = 1 \) Supersymmetric Vacua in Heterotic M–Theory

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Abstract

In the first lecture, we derive the five–dimensional effective action of strongly coupled heterotic string theory for the complete \((1,1)\) sector of the theory by performing a reduction, on a Calabi–Yau three–fold, of M–theory on \(S^1/Z_2\). The correct effective theory is a gauged version of five–dimensional \(\mathcal{N} = 1\) supergravity coupled to Abelian vector multiplets, the universal hypermultiplet and four–dimensional boundary theories with gauge and gauge matter fields. The supersymmetric ground state of the theory is a multi–charged BPS three–brane domain wall, which we construct in general. In this first lecture, we assume the “standard” embedding of the spin connection into the \(E_8\) gauge connection on one orbifold fixed plane. In the second lecture, we generalize these results to “non–standard” embeddings. That is, we allow for general \(E_8 \times E_8\) gauge bundles and for the presence of five–branes. The five–branes span the four-dimensional uncompactified space and are wrapped on holomorphic curves in the Calabi–Yau manifold. Properties of these “non–perturbative” vacua, as well as of the resulting low–energy theories, are discussed. Characteristic features of the low–energy theory, such as the threshold corrections to the gauge kinetic functions, are significantly modified due to the presence of the five–branes, as compared to the case of standard or non–standard embeddings without five–branes. In the last lecture, we review the spectral cover formalism for constructing both \(U(n)\) and \(SU(n)\) holomorphic vector bundles on elliptically fibered Calabi–Yau three–folds which admit a section. We discuss the allowed bases of these three–folds and show that physical constraints eliminate Enriques surfaces from consideration. Restricting the structure group to \(SU(n)\), we derive, in detail, a set of rules for the construction of three–family particle physics theories with phenomenologically relevant gauge groups. We illustrate these ideas by constructing several explicit three–family non–perturbative vacua.

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Introduction:

Heterotic M–theory, first discussed by Hořava and Witten [1, 2, 3], holds great promise as the starting point for phenomenological investigations of low energy particle physics and cosmology. In several papers [4, 5], the five–dimensional effective action of Hořava–Witten theory has been constructed by dimensional reduction of \( D = 11, \mathcal{N} = 1 \) supergravity on Calabi–Yau three–folds. The resulting theory was shown to admit a pair of BPS three–branes located at the orbifold fixed planes as its minimal static vacuum state. In subsequent work [6], we explored “non-perturbative” vacua, which consist of an arbitrary number of BPS threebranes in addition to the orbifold fixed planes. Within the context of both the minimal and non-perturbative vacua, we discussed the need for, and analyzed, non-standard embedding; that is, we do not require that the Calabi–Yau spin connection be embedded in one of the \( E_8 \) gauge groups. Finally, in a series of papers [7, 8, 9], the generic mathematical structure of non-standard embeddings were presented within the context of holomorphic vector bundles. In these papers, we gave rules for the construction of three family particle physics models with realistic unification groups. In the present lectures we review this work, discussing the five–dimensional effective theory and its three–brane BPS solutions in Lecture 1, non-perturbative vacua and non-standard embedding in Lecture 2 and, finally, in Lecture 3 we discuss the mathematics and physics of holomorphic vector bundles.

Lecture 1: Heterotic M–Theory in Five Dimensions

One of the phenomenologically most promising corners of the M–theory moduli space, in addition to the weakly coupled heterotic string, is the point described at low-energy by eleven-dimensional supergravity on the orbifold \( S^1/Z_2 \) due to Hořava and Witten [1, 2]. This theory gives the strongly coupled limit of the heterotic string with, in addition to the bulk supergravity, two sets of \( E_8 \) gauge fields residing one on each of the two ten–dimensional fixed hyperplanes of the orbifold. It has been shown [3] that this theory has phenomenologically interesting compactifications on deformed Calabi–Yau three–folds times the orbifold to four dimensions. Matching the 11–dimensional Newton constant \( \kappa \), the Calabi–Yau volume and the orbifold radius to the known values of the Newton constant and the grand unification coupling and scale leads to an orbifold radius which is about an order of magnitude or so larger than the two other scales [3, 10]. This suggests that, near this “physical” point in moduli space, the theory appears effectively five–dimensional in some intermediate energy regime.

In previous papers [4, 5] we have derived this five–dimensional effective theory for the first time by directly reducing Hořava–Witten theory on a Calabi–Yau three–fold. This calculation included all \((1,1)\) moduli as well as the universal hypermultiplet. We showed that a non–zero mode of the antisymmetric tensor field strength has to be included for a consistent reduction from
eleven to five dimensions and that the correct five–dimensional effective theory of strongly coupled heterotic string is given by a gauged version of five–dimensional supergravity. A reduction of pure eleven–dimensional supergravity on a Calabi–Yau three–fold [11], on the other hand, leads to a non–gauged version of five–dimensional supergravity. Therefore, while this provides a consistent low–energy description of M–theory on a smooth manifold, it is not the correct effective theory for M–theory on $S^1/Z_2$. The necessary additions are chiral four–dimensional boundary theories with potential terms for the bulk moduli and, most importantly, the aforementioned non–zero mode, living solely in the Calabi-Yau three–fold, which leads to the gauging of the bulk supergravity. As pointed out in ref. [4, 5], this theory is the correct starting point for strongly coupled heterotic particle phenomenology as well as early universe cosmology. Moreover, we have shown that contact with four–dimensional physics should not be made using flat space–time but rather via a domain–wall solution as the background configuration. This domain wall arises as a BPS state of the five–dimensional theory [4, 5] and its existence is intimately tied to the gauging of the theory. A reduction to four dimensions on this domain wall has been performed in [3, 12] to lowest non–trivial order. The result agrees with ref. [13] where the complete four–dimensional effective action to that order has been derived directly from eleven dimensions.

Various other aspects of the Hoˇ rava–Witten description of strongly coupled heterotic string theory have been addressed in the literature such as the structure of the four–dimensional effective action, its relation to 10–dimensional weakly coupled heterotic string, gaugino condensation, and anomaly cancelation [13–38]. Aspects of five–dimensional physics motivated by Hoˇ rava–Witten theory and related to particle phenomenology have been discussed in ref. [10, 32, 39, 40]. In refs. [41, 42, 43, 44] five–dimensional early universe M–theory cosmology have been investigated. Recently, aspects of five–dimensional physics have also been discussed in ref. [45].

In this first lecture, we review the work presented in ref. [4, 5]. Our central result is to obtain the five–dimensional effective theory of strongly coupled heterotic string for all $(1,1)$ moduli fields and the universal hypermultiplet, and construct its fundamental BPS domain wall three–brane solutions. We show that, in the bulk, this theory is indeed a form of gauged supergravity.

Let us now summarize our conventions. We will consider eleven–dimensional spacetime compactified on a Calabi–Yau space $X$, with the subsequent reduction down to four dimensions effectively provided by a double-domain-wall background, corresponding to an $S^1/Z_2$ orbifold. We use coordinates $x^I$ with indices $I, J, K, \ldots = 0, \ldots , 9, 11$ to parameterize the full 11–dimensional space $M_{11}$. Throughout this paper, when we refer to orbifolds, we will work in the “upstairs” picture with the orbifold $S^1/Z_2$ in the $x^{11}$–direction. We choose the range $x^{11} \in [-\pi \rho, \pi \rho]$ with the endpoints being identified. The $Z_2$ orbifold symmetry acts as $x^{11} \rightarrow -x^{11}$. There then exist two ten–dimensional hyperplanes fixed under the $Z_2$ symmetry which we denote by $M^{(n)}_{10}$, $n = 1, 2$. Locally, they are specified by the conditions $x^{11} = 0, \pi \rho$. Upon reduction on a Calabi–Yau space to five dimensions
they lead to four–dimensional fixed hyperplanes $M_4^{(n)}$. Barred indices $\bar{I}, \bar{J}, \bar{K}, \ldots = 0, \ldots, 9$ are used for the ten–dimensional space orthogonal to the orbifold. Upon reduction on the Calabi-Yau space we have a five-dimensional spacetime $M_5$ labeled by indices $\alpha, \beta, \gamma, \ldots = 0, \ldots, 3, 11$. The orbifold fixed planes become four-dimensional with indices $\mu, \nu, \rho, \ldots = 0, \ldots, 3$. We use indices $A, B, C, \ldots = 4, \ldots, 9$ for the Calabi–Yau space. Holomorphic and anti–holomorphic indices on the Calabi–Yau space are denoted by $a, b, c, \ldots$ and $\bar{a}, \bar{b}, \bar{c}, \ldots$, respectively. The harmonic $(1, 1)$–forms of the Calabi–Yau space on which we will concentrate throughout this paper are indexed by $i, j, k, \ldots = 1, \ldots, h^{1,1}$.

The 11-dimensional Dirac–matrices $\Gamma^I$ with $\{\Gamma^I, \Gamma^J\} = 2g^{IJ}$ are decomposed as $\Gamma^I = \{\gamma^\alpha \otimes \lambda, 1 \otimes \lambda^A\}$ where $\gamma^\alpha$ and $\lambda^A$ are the five– and six–dimensional Dirac matrices, respectively. Here, $\lambda$ is the chiral projection matrix in six dimensions with $\lambda^2 = 1$. Spinors in eleven dimensions are Majorana with 32 real components throughout the paper. In five dimensions we use symplectic–real spinors [46]. Fields will be required to have a definite behavior under the $Z_2$ orbifold symmetry in $D = 11$. We demand a bosonic field $\Phi$ to be even or odd; that is, $\Phi(x^{11}) = \pm \Phi(-x^{11})$. For a spinor $\Psi$ the condition is $\Gamma_{11}\Psi(-x^{11}) = \pm \Psi(x^{11})$ and, depending on the sign, we also call the spinor even or odd. The projection to one of the orbifold planes leads then to a ten–dimensional Majorana–Weyl spinor with definite chirality. Similarly, in five dimensions, bosonic fields will be either even or odd, and there is a corresponding orbifold condition on spinors.

1 Eleven–Dimensional Supergravity on an Orbifold

In this section we briefly review the formulation of the low–energy effective action of strongly coupled heterotic string theory as eleven–dimensional supergravity on the orbifold $S^1/Z_2$ due to Hořava and Witten [1, 2].

The bosonic part of the action is given by

$$S = S_{SG} + S_{YM}$$

where $S_{SG}$ is the familiar 11–dimensional supergravity action

$$S_{SG} = -\frac{1}{2\kappa^2} \int_{M^{11}} \sqrt{-g} \left[ R + \frac{1}{24} G_{IJKL} G^{IJKL} + \frac{\sqrt{2}}{1728} \epsilon^{I_1 \ldots I_{11}} C_{I_1 I_2 I_3} G_{I_4 \ldots I_7} G_{I_8 \ldots I_{11}} \right]$$

(1.2)
and $S_{YM}$ describes the two $E_8$ Yang–Mills theories on the orbifold planes, explicitly given by

$$
S_{YM} = -\frac{1}{8\pi\kappa^2} \left( \frac{\kappa}{4\pi} \right)^{2/3} \int_{M_{10}^{(1)}} \sqrt{-g} \left[ \tr(F(1))^2 - \frac{1}{2} \tr R^2 \right] - \frac{1}{8\pi\kappa^2} \left( \frac{\kappa}{4\pi} \right)^{2/3} \int_{M_{10}^{(2)}} \sqrt{-g} \left[ \tr(F(2))^2 - \frac{1}{2} \tr R^2 \right].
$$

(1.3)

Here $F_{ij}^{(n)}$ are the two $E_8$ gauge field strengths and $C_{IJK}$ is the 3–form with field strength $G_{IJKL} = 24 \partial_I C_{JKL}$. The above action has to be supplemented by the Bianchi identity

$$
(dG)_{11IJKL} = -\frac{1}{2\sqrt{2\pi}} \left( \frac{\kappa}{4\pi} \right)^{2/3} \left\{ J^{(1)} \delta(x^{11}) + J^{(2)} \delta(x^{11} - \pi p) \right\}_{IJKL}
$$

(1.4)

where the sources are defined by

$$
J^{(n)} = \tr F^{(n)} \wedge F^{(n)} - \frac{1}{2} \tr R \wedge R.
$$

(1.5)

Note that, in analogy with the weakly coupled case, the boundary $\tr R^2$ terms in eq. (1.3) are required by supersymmetry as pointed out in ref. [13]. Under the $Z_2$ orbifold symmetry, the field components $g_{Ij}$, $g_{11,11}$, $C_{Ij1}$ are even, while $g_{I11}$, $C_{IJK}$ are odd. The above action is complete to order $\kappa^{2/3}$ relative to the bulk. Corrections, however, will appear as higher–dimension operators at order $\kappa^{4/3}$.

The fermionic fields of the theory are the 11–dimensional gravitino $\Psi_I$ and the two 10–dimensional Majorana–Weyl spinors $\chi^{(n)}$, located on the boundaries, one for each $E_8$ gauge group. The components $\Psi_I$ of the gravitino are even while $\Psi_{11}$ is odd. The gravitino supersymmetry variation is given by

$$
\delta \Psi_I = D_I \eta + \frac{\sqrt{2}}{288} (\Gamma_{IJKLM} - 8g_{ij1} \Gamma_{KLM}) G^{JKLM} \eta + \cdots,
$$

(1.6)

where the dots indicate terms that involve fermion fields. The spinor $\eta$ in this variation is $Z_2$ even.

The appearance of the boundary source terms in the Bianchi identity has a simple interpretation by analogy with the theory of $D$-branes. It is well known that the $U(N)$ gauge fields describing the theory of $N$ overlapping $Dp$-branes encode the charges for lower-dimensional $D$-branes embedded in the $Dp$-branes. For instance, the magnetic flux $\tr F$ couples to the $p-1$-form Ramond-Ramond potential, so describes $D(p-2)$-brane charge. Higher cohomology classes $\tr F \wedge \cdots \wedge F$ describe the embedding of lower-dimensional branes. Furthermore, if the $Dp$-brane is curved, then the

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1We note that there is a debate in the literature about the precise value of the Yang–Mills coupling constant in terms of $\kappa$. While we quote the original value $[2, 47]$ the value found in ref. [19] is smaller. In the second case, the coefficients in the Yang-Mills action (1.3) and the Bianchi identity (1.4) should both be multiplied by $2^{-1/3}$. This potential factor will not be essential in the following discussion as it will simply lead to a redefinition of the five–dimensional coupling constants. In the following, we will give the necessary modifications where appropriate.
cohomology classes of the tangent bundle also contribute. For instance \( \text{tr} R \wedge R \) induces \( D(p-4) \)-brane charge. We recall that in eleven dimensions it is M five-branes which are magnetic sources for \( G_{IJKL} \). Thus we can interpret the magnetic sources in the Bianchi identity (1.4) as five-branes embedded in the orbifold fixed planes.

2 The Five–Dimensional Effective Theory

As mentioned above, matching of scales suggests that strongly coupled heterotic string theory appears effectively five–dimensional in some intermediate energy range. In this section we derive the five–dimensional effective theory in this regime obtained by a compactification on a Calabi–Yau three–fold. We expect that this should lead to a theory with bulk \( \mathcal{N} = 1 \) five-dimensional supersymmetry and four-dimensional \( \mathcal{N} = 1 \) supersymmetry on the orbifold fixed planes. As we will see, doing this compactification consistently requires the inclusion of non–zero modes for the field strength of the anti–symmetric tensor field. These non–zero modes appear in the purely internal Calabi–Yau part of the anti–symmetric tensor field and correspond to harmonic \((1,1)\) forms on the Calabi–Yau three–fold. Consequently, to capture the complete structure of these non–zero modes, we will have to consider the full \((1,1)\) sector of the theory. We will not, however, explicitly include the \((2,1)\) sector as it is largely unaffected by the specific structure of Hořava–Witten theory. Instead, we comment on the additions necessary to incorporate this sector along the way.

To make contact with the compactifications to four-dimensions discussed by Witten [3], one embeds the spin-connection of the Calabi–Yau manifold in the gauge connection of one of the \( E_8 \) groups breaking it to \( E_6 \). This is the so-called standard embedding and, in this lecture, we will restrict our discussion to it. In general, this implies that there is a non-zero instanton number on one of the orbifold planes. From the discussion of the previous section, this can be interpreted as including five-branes living in the orbifold plane in the compactification. It is this additional element to the compactification which introduces the non–zero mode and leads to much of the interesting structure of the five-dimensional theory. We note that the presence of five-brane charge is really unavoidable. Even without exciting instanton number, the curvature of the Calabi–Yau three–fold leads to an induced magnetic charge in the Bianchi identity (1.4), forcing us to include non-zero modes.

We wish to emphasize that the standard embedding is not in any way required by heterotic M–theory. In fact, unlike the case of weakly coupled heterotic string theory where the choice of the standard embedding greatly simplifies the vacuum, in M–theory this embedding is somewhat unnatural and is not singled out in any way. We employ it in this lecture only because of its familiarity and the fact that it was used in refs. \([1,2]\). The following two lectures will be devoted to generalizing heterotic M–theory to “non-standard” embeddings.
2.1 Zero Modes

Let us now explain the structure of the zero mode fields used in the reduction to five dimensions. We begin with the bulk. The background space–time manifold is $M_{11} = X \times S_1/Z_2 \times M_4$, where $X$ is a Calabi–Yau three–fold and $M_4$ is four–dimensional Minkowski space. Reduction on such a background leads to eight preserved supercharges and, hence, to minimal $\mathcal{N} = 1$ supergravity in five dimensions. Due to the projection condition, this leads to four preserved supercharges on the orbifold planes implying four–dimensional $\mathcal{N} = 1$ supersymmetry on those planes. Including the zero modes, the metric is given by

$$ds^2 = V^{-2/3}g_{\alpha\beta}dx^\alpha dx^\beta + g_{AB}dx^A dx^B$$

(2.1)

where $g_{AB}$ is the metric of the Calabi–Yau space $X$. Its Kähler form is defined by $\omega_{\alpha\bar{\beta}} = ig_{\alpha\bar{\beta}}$ and can be expanded in terms of the harmonic $(1,1)$–forms $\omega_i AB$, $i = 1, \cdots, h^{1,1}$ as

$$\omega_{AB} = a^i \omega_i AB.$$  

(2.2)

The coefficients $a^i = a^i(x^\alpha)$ are the $(1,1)$ moduli of the Calabi–Yau space. The Calabi–Yau volume modulus $V = V(x^\alpha)$ is defined by

$$V = \frac{1}{v} \int_X \sqrt{g}$$

(2.3)

where $\sqrt{g}$ is the determinant of the Calabi–Yau metric $g_{AB}$. In order to make $V$ dimensionless we have introduced a coordinate volume $v$ in this definition which can be chosen for convenience. The modulus $V$ then measures the Calabi–Yau volume in units of $v$. The factor $V^{-2/3}$ in eq. (2.1) has been chosen such that the metric $g_{\alpha\beta}$ is the five–dimensional Einstein frame metric. Clearly $V$ is not independent of the $(1,1)$ moduli $a^i$ but it can be expressed as

$$V = \frac{1}{6} \mathcal{K}(a), \quad \mathcal{K}(a) = d_{ijk} a^i a^j a^k$$

(2.4)

where $\mathcal{K}(a)$ is the Kähler potential and $d_{ijk}$ are the Calabi–Yau intersection numbers. Their definition, along with a more detailed account of Calabi–Yau geometry, can be found in appendix A of ref. [5].

Let us now turn to the zero modes of the antisymmetric tensor field. We have the potentials and field strengths,

$$C_{\alpha\beta\gamma}, \quad G_{\alpha\beta\gamma\delta},$$

$$C_{\alpha AB} = \frac{1}{6} A_{\alpha}^i \omega_i AB, \quad G_{\alpha\beta AB} = F_{\alpha\beta}^i \omega_i AB$$

(2.5)

$$C_{abc} = \frac{1}{6} \Omega_{abc}, \quad G_{aabc} = X_a \Omega_{abc}.$$
The five–dimensional fields are therefore an antisymmetric tensor field $C_{\alpha\beta\gamma}$ with field strength $G_{\alpha\beta\gamma\delta}$, $h^{1,1}$ vector fields $A^i_\alpha$ with field strengths $F^i_{\alpha\beta}$ and a complex scalar $\xi$ with field strength $X_\alpha$ that arises from the harmonic $(3, 0)$ form denoted by $\Omega_{abc}$. In the bulk the relations between those fields and their field strengths are simply

$$G_{\alpha\beta\gamma\delta} = 24 \partial_{[\alpha} C_{\beta\gamma\delta]}$$
$$F^i_{\alpha\beta} = \partial_\alpha A^i_\beta - \partial_\beta A^i_\alpha$$
$$X_\alpha = \partial_\alpha \xi \ .$$

These relations, however, will receive corrections from the boundary controlled by the 11–dimensional Bianchi identity \([1, 4]\). We will derive the associated five–dimensional Bianchi identities later.

Next, we should set up the structure of the boundary fields. The starting point is the standard embedding of the spin connection in the first $E_8$ gauge group such that

$$\text{tr} F^{(1)} \wedge F^{(1)} = \text{tr} R \wedge R .$$

As a result, we have an $E_6$ gauge field $A^{(1)}_a$ with field strength $F^{(1)}_{\mu\nu}$ on the first hyperplane and an $E_8$ gauge field $A^{(2)}_\mu$ with field strength $F^{(2)}_{\mu
u}$ on the second hyperplane. In addition, there are $h^{1,1}$ gauge matter fields from the $(1, 1)$ sector on the first plane. They are specified by

$$A^{(1)}_b = \bar{A}_b + \omega_{ib} T_{cp} C^p$$

where $\bar{A}_b$ is the (embedded) spin connection. Furthermore, $p, q, r, \ldots = 1, \ldots, 27$ are indices in the fundamental $27$ representation of $E_6$ and $T_{ap}$ are the $(3, 27)$ generators of $E_8$ that arise in the decomposition under the subgroup $SU(3) \times E_6$. Their complex conjugate is denoted by $T^{ap}$. The $C^p$ are $h^{1,1}$ complex scalars in the $27$ representation of $E_6$. Useful traces for these generators are $\text{tr}(T_{ap} T^{bq}) = \delta^{b}_a \delta^p_q$ and $\text{tr}(T_{ap} T_{bq} T_{cr}) = \Omega_{abc} f_{pqr}$ where $f_{pqr}$ is the totally symmetric tensor that projects out the singlet in $27^3$.

### 2.2 The Nonzero Mode

So far, what we have considered is similar to a reduction of pure 11–dimensional supergravity on a Calabi–Yau space, as for example performed in ref. \([1]\), with the addition of gauge and gauge matter fields on the boundaries. An important difference arises, however, because the standard embedding \((2.7)\), unlike in the case of the weakly coupled heterotic string, no longer leads to vanishing sources in the Bianchi identity \((1.4)\). Instead, there is a net five-brane charge, with opposite sources on each fixed plane, proportional to $\pm \text{tr} R \wedge R$. The nontrivial components of the Bianchi identity \((1.4)\) are given by

$$(dG)_{11ABCD} = -\frac{1}{4\sqrt{2}\pi} \left( \frac{\kappa}{4\pi} \right)^{2/3} \left\{ \delta(x^{11}) - \delta(x^{11} - \pi \rho) \right\} (\text{tr} R \wedge R)_{ABCD} .$$

7
As a result, the components $G_{ABCD}$ and $G_{ABC11}$ of the antisymmetric tensor field are nonvanishing. More precisely, the above equation has to be solved along with the equation of motion.

$$D_I G^{IJKL} = 0.$$  \hfill (2.10)

(Note that the Chern–Simons contribution to the antisymmetric tensor field equation of motion vanishes if $G_{ABCD}$ and $G_{ABC11}$ are the only nonzero components of $G_{IJKL}$.) The general solution of these equations is quite complicated and has been given in ref. [13] as an expansion in Calabi–Yau harmonic functions. For the present purpose of deriving a five–dimensional effective action, we are only interested in the zero mode terms in this expansion because the heavy Calabi–Yau modes decouple as a result of the consistent Kaluza-Klein truncation to $D = 5$. To work out the zero mode part of the solution, we note that $tr R \wedge R$ is a $(2,2)$ form on the Calabi–Yau space (since the only nonvanishing components of a Calabi–Yau curvature tensor are $R_{a\bar{b}cd}$). Let us, therefore, introduce a basis $\nu^i$, $i = 1, \ldots, h^{2,2} = h^{1,1}$ of harmonic $(2,2)$ forms and corresponding four–cycles $C_i$ such that

$$\int_X \omega^i \wedge \nu^j = \delta^i_j, \quad \int_{C_i} \nu^j = \delta^i_j.$$  \hfill (2.11)

The zero mode part $tr R \wedge R|_0$ of the source can then be expanded as

$$tr R \wedge R|_0 = -8 \sqrt{2} \pi \left( \frac{4\pi}{\kappa} \right)^{2/3} v^i \alpha^i \nu^i$$  \hfill (2.12)

where the numerical factor has been included for convenience. The expansion coefficients $\alpha_i$ are

$$\alpha_i = -\frac{\pi}{\sqrt{2}} \left( \frac{\kappa}{4\pi} \right)^{2/3} \frac{1}{v^{2/3}} \beta_i, \quad \beta_i = -\frac{1}{8\pi^2} \int_{C_i} tr R \wedge R.$$  \hfill (2.13)

Note that $\beta_i$ are integers, characterizing the first Pontrjagin class of the Calabi-Yau. It is then straightforward to see that the zero mode part of the Bianchi identity (2.9) and the equation of motion (2.10) are solved by

$$G_{ABCD}|_0 = \alpha_i \nu^i_{ABCD} \epsilon(x^{11}) = \frac{1}{4V} \alpha^i \epsilon_{ABCD}^{EF} \omega_i^{EF} \epsilon(x^{11})$$  \hfill (2.14)

$$G_{ABC11}|_0 = 0.$$  \hfill (2.15)

Here $\epsilon(x^{11})$ is the step function which is +1 for positive $x^{11}$ and −1 otherwise. The index of the coefficient $\alpha^i$ in the second part of the first equation has been raised using the metric

$$G_{ij}(a) = \frac{1}{2V} \int_X \omega_i \wedge (\ast \omega_j)$$  \hfill (2.16)

on the $(1,1)$ moduli space. Note that, while the coefficients $\alpha_i$ with lowered index are truly constants, as is apparent from eq. (2.13), the coefficients $\alpha^i$ depend on the $(1,1)$ moduli $a^i$ since the metric (2.16) does. From the expansion (2.12) we can derive an expression for the boundary
\(\text{tr}F^2\) and \(\text{tr}R^2\) terms in the action \((1.3)\) which will be essential for the reduction of the boundary theories. We have

\[
\text{tr}R_{AB}R^{AB}\big|_0 = \text{tr}F^{(1)}_{AB}F^{(1)AB}\big|_0 = -4\sqrt{2}\pi \left(\frac{4\pi}{\kappa}\right)^{2/3} V^{-1} \alpha_i \omega^{AB} \omega_{iAB} \quad (2.17)
\]

while, of course

\[
\text{tr}F^{(2)}_{AB}F^{(2)AB} = 0 \quad (2.18)
\]

The expression \((2.14)\) for \(G_{ABCD}\) with \(\alpha_i\) as defined in \((2.13)\) is the new and somewhat unconventional ingredient in our reduction. Using the terminology of ref. \[49\] we call this configuration for the antisymmetric tensor field strength a nonzero mode. Generally, a nonzero mode is defined as a nonzero internal antisymmetric tensor field strength \(G\) that solves the equation of motion. In contrast, conventional zero modes of an antisymmetric tensor field, like those in eq. \((2.6)\), have vanishing field strength once the moduli fields are set to constants. Since the kinetic term \(G^2\) is positive for a nonzero mode it corresponds to a nonzero energy configuration. Given that nonzero modes, for a \(p\)-form field strength, satisfy

\[
dG = d^*G = 0 \quad (2.19)
\]

they correspond to harmonic forms of degree \(p\). Hence, they can be identified with the \(p\)th cohomology group \(H^p(X)\) of the internal manifold \(X\). In the present case, we are dealing with a four–form field strength on a Calabi–Yau three–fold \(X\) so that the relevant cohomology group is \(H^4(X)\). The expression \((2.14)\) is just an expansion of the nonzero mode in terms of the basis \(\{\nu^i\}\) of \(H^4(X)\). The appearance of all harmonic \((2,2)\) forms shows that it is necessary to include the complete \((1,1)\) sector into the low energy effective action in order to fully describe the nonzero mode, as argued in the beginning of this section. On the other hand, harmonic \((2,1)\) forms do not appear here and are hence less important in our context. We stress that the nonzero mode \((2.14)\), for a given Calabi–Yau space, specifies a fixed element in \(H^4(X)\) since the coefficients \(\alpha_i\) are fixed in terms of Calabi–Yau properties. In fact, they are related to the integers \(\beta_i\) characterizing the first Pontrjagin class of the tangent bundle. Thus we see that, correctly normalized, \(G\) is in the integer cohomology of the Calabi-Yau. This quantization condition has been described in \[50\]

In a dimensional reduction of pure 11–dimensional supergravity, non–zero modes can be considered as well but are usually dismissed as non–zero energy configuration. Compactifications of 11–dimensional supergravity on various manifolds including Calabi–Yau three–folds with non–zero modes have been considered in the literature \[51\]. The difference in our case is that we are not free to turn off the non–zero mode. Its presence is simply dictated by the nonvanishing boundary sources.
2.3 The Five–Dimensional Effective Action

Let us now summarize the field content which we have obtained above and discuss how it fits into the multiplets of five–dimensional $\mathcal{N} = 1$ supergravity. The form of these multiplets and in particular the conditions on the fermions is discussed in more detail in appendix B of ref. [5]. We know that the gravitational multiplet should contain one vector field, the graviphoton. Thus since the reduction leads to $h^{1,1}$ vectors, we must have $h^{1,1} - 1$ vector multiplets. This leaves us with the $h^{1,1}$ scalars $a^i$, the complex scalar $\xi$ and the three-form $C_{\alpha\beta\gamma}$. Since there is one scalar in each vector multiplet, we are left with three unaccounted for real scalars (one from the set of $a^i$, and $\xi$) and the three-form. Together, these fields form the “universal hypermultiplet;” universal because it is present independently of the particular form of the Calabi-Yau manifold. From this, it is clear that it must be the overall volume breathing mode $V = \frac{1}{6}d_{ijk}a^ia^ja^k$ that is the additional scalar from the set of the $a^i$ which enters the universal multiplet. The three-form may appear a little unusual, but one should recall that in five dimensions a three-form is dual to a scalar $\sigma$. Thus, the bosonic sector of the universal hypermultiplet consists of the four scalars ($V, \sigma, \xi, \bar{\xi}$).

The $h^{1,1} - 1$ vector multiplet scalars are the remaining $a^i$. More properly, since the breathing mode $V$ is already part of a hypermultiplet it should be first scaled out when defining the shape moduli

$$b^i = V^{-1/3} a^i .$$

Note that the $h^{1,1}$ moduli $b^i$ represent only $h^{1,1} - 1$ independent degrees of freedom as they satisfy the constraint

$$\mathcal{K}(b) \equiv d_{ijk}b^ib^jb^k = 6 .$$

Alternatively, as described in appendix B of ref. [5], we can introduce $h^{1,1} - 1$ independent fields $\phi^x$ with $b^i = b^i(\phi^x)$. The bosonic fields in the vector multiplets are then given by $(\phi^x, b^x_i A^i_\alpha)$ ($b^x_i$ represents a projection onto the $\phi^x$ subspace). Meanwhile the graviton and graviphoton of the gravity multiplet are given by $(g_{\alpha\beta}, \frac{2}{3}b_i A^i_\alpha)$.

Therefore, in total, the five dimensional bulk theory contains a gravity multiplet, the universal hypermultiplet and $h^{1,1} - 1$ vector multiplets. The inclusion of the (2,1) sector of the Calabi–Yau space would lead to an additional $h^{2,1}$ set of hypermultiplets in the theory. Since they will not play a prominent rôle in our context they will not be explicitly included in the following.

On the boundary $M_4^{(1)}$ we have an $E_6$ gauge multiplet $(A^{(1)}_\mu, \chi^{(1)})$ and $h^{1,1}$ chiral multiplets $(C^{ip}, \eta^{ip})$ in the fundamental 27 representation of $E_6$. Here $C^{ip}$ denote the complex scalars and $\eta^{ip}$ the chiral fermions. The other boundary, $M_4^{(2)}$, carries an $E_8$ gauge multiplet $(A^{(2)}_\mu, \chi^{(2)})$ only. Inclusion of the (2,1) sector would add $h^{2,1}$ chiral multiplets in the $\overline{27}$ representation of $E_6$ to the field content of the boundary $M_4^{(1)}$. Any even bulk field will also survive on the boundary. Thus,
in addition to the four–dimensional part of the metric, the scalars $b^i$ together with $A^i_{11}$, and $V$ and $\sigma$ survive on the boundaries. These pair into $h^{1,1}$ chiral multiplets.

After this survey we are ready to derive the bosonic part of the five–dimensional effective action for the $(1,1)$ sector. Inserting the expressions for the various fields from the previous subsection into the action (1.1), using the formulae given in appendix A of ref. [5] and dropping higher derivative terms we find

\[ S_5 = S_{\text{grav,vec}} + S_{\text{hyper}} + S_{\text{bound}} + S_{\text{matter}} \]  

with

\[ S_{\text{grav,vec}} = -\frac{1}{2\kappa_5^2} \int_{M_5} \sqrt{-g} \left[ R + G_{ij} \partial_\alpha b^i \partial^{\alpha} b^j + G_{ij} F^i_{\alpha\beta} F^{j\alpha\beta} + \frac{\sqrt{2}}{12} \alpha^{\beta\gamma\delta} d_{ijk} A^i_{\alpha} F^j_{\beta} F^k_{\gamma} \right] \]  

\[ S_{\text{hyper}} = -\frac{1}{2\kappa_5^2} \int_{M_5} \sqrt{-g} \left[ \frac{1}{2} V^{-2} \partial_\alpha V \partial^{\alpha} V + 2V^{-1} X_\alpha \bar{X}^\alpha + \frac{1}{24} V^2 G_{\alpha\beta\gamma\delta} G^{\alpha\beta\gamma\delta} \right. \\
+ \frac{\sqrt{2}}{24} \alpha^{\beta\gamma\delta} G_{\alpha\beta\gamma\delta} (i(\xi \bar{X} - \bar{\xi} X) + 2\epsilon(x^{11})_{\alpha} A^i_{\alpha}) \\
+ \frac{1}{2} V^{-2} G^{ij} \alpha_i \alpha_j \right] \]  

\[ S_{\text{bound}} = \frac{\sqrt{2}}{\kappa_5^2} \int_{M_4^{(1)}} \sqrt{-g} V^{-1} X^i \bar{X}^i - \frac{\sqrt{2}}{\kappa_5^2} \int_{M_4^{(2)}} \sqrt{-g} V^{-1} \alpha_i b^i \]  

\[ S_{\text{matter}} = -\frac{1}{16\pi\alpha_{\text{GUT}}} \sum_{n=1}^{2} \int_{M_4^{(n)}} \sqrt{-g} V \text{tr} \bar{F}_{\mu\nu}^2 \\
- \frac{1}{2\pi\alpha_{\text{GUT}}} \int_{M_4^{(1)}} \sqrt{-g} \left[ G_{ij}(D_{\mu} C)^i (D_{\nu} \bar{C})^j \\
+ V^{-1} G^{ij} \frac{\partial W}{\partial C^p} \frac{\partial \bar{W}}{\partial \bar{C}^p} + D^{(u)} D^{(u)} \right] . \]  

All fields in this action that originate from the 11–dimensional antisymmetric tensor field are subject to a nontrivial Bianchi identity. Specifically, from eq. (1.4) we have

\[ (dG)^{11\mu\nu\rho} = -\frac{\kappa_5^2}{4\sqrt{2}\pi\alpha_{\text{GUT}}} \left\{ J^{(1)} \delta(x^{11}) + J^{(2)} \delta(x^{11} - \pi p) \right\}_{\mu\nu\rho} \]  

\[ (dF^i)^{11\mu} = -\frac{\kappa_5^2}{4\sqrt{2}\pi\alpha_{\text{GUT}}} J^i_{\mu} \]  

\[ (dX)^{11\mu} = -\frac{\kappa_5^2}{4\sqrt{2}\pi\alpha_{\text{GUT}}} J_{\mu} \]
with the currents defined by

\[ J^{(n)}_{\mu\nu\rho\sigma} = \left( \text{tr} F^{(n)} \wedge F^{(n)} - \frac{1}{2} \text{tr} R \wedge R \right)_{\mu\nu\rho\sigma} \]  

\[ J^i_{\mu\nu} = -2iV^{-1} \Gamma^i_{jk} \left( (D_\mu C)^{jp}(D_\nu \bar{C})^k_p - (D_\mu \bar{C})^k_p(D_\nu C)^{jp} \right) \]  

\[ J_\mu = -\frac{i}{2} V^{-1} d_{ijk} f_{pq} (D_\mu C)^{jp} C^{jq} C^{kr}. \]  

The five–dimensional Newton constant \( \kappa_5 \) and the Yang–Mills coupling \( \alpha_{\text{GUT}} \) are expressed in terms of 11–dimensional quantities as

\[ \kappa_5^2 = \frac{\kappa^2}{v}, \quad \alpha_{\text{GUT}} = \frac{\kappa^2}{2v} \left( \frac{4\pi}{\kappa} \right)^{2/3}. \]  

We still need to define various quantities in the above action. The metric \( G_{ij} \) is given in terms of the Kähler potential \( K \) as

\[ G_{ij} = -\frac{1}{2} \frac{\partial}{\partial b^i} \frac{\partial}{\partial b^j} \ln K. \]  

The corresponding connection \( \Gamma^i_{jk} \) is defined as

\[ \Gamma^i_{jk} = \frac{1}{2} G^i_{il} \frac{\partial G_{jk}}{\partial b^l}. \]  

We recall that

\[ K = d_{ijk} b^i b^j b^k, \]  

where \( d_{ijk} \) are the Calabi–Yau intersection numbers. All indices \( i, j, k, \cdots \) in the five–dimensional theory are raised and lowered with the metric \( G_{ij} \). A more explicit form of this metric can be found in appendix A of ref. [5]. We also recall that the fields \( b^i \) are subject to the constraint

\[ K = 6 \]  

which should be taken into account when equations of motion are derived from the above action. Most conveniently, it can be implemented by adding a Lagrange multiplier term \( \sqrt{-g} \lambda (K(b) - 6) \) to the bulk action. Furthermore, we need to define the superpotential

\[ W = \frac{1}{6} d_{ijk} f_{pq} C^{ip} C^{jq} C^{kr}. \]  

---

3These relations are given for the normalization of the 11–dimensional action as in eq. (1.1). If instead the normalization of [19] is used the expression for \( \alpha_{\text{GUT}} \) gets rescaled to \( a_{\text{GUT}} = 2^{1/3} \left( \frac{\kappa^2}{2v} \right) \left( \frac{4\pi}{\kappa} \right)^{2/3} \). Otherwise the action and Bianchi identities are unchanged, except that in the expression [2.13] for \( \alpha_i \) the RHS is multiplied by \( 2^{1/3} \).
and the D–term

$$D^{(u)} = G_{ij} C^i T^{(u)} C^j$$  \hspace{1cm} (2.32)$$

where $T^{(u)}$, $u = 1, \ldots, 78$ are the $E_6$ generators in the fundamental representation. The consistency of the above theory has been explicitly checked by a reduction of the 11–dimensional equations of motion.

The most notable features of this action, at first sight, are the bulk and boundary potentials for the $(1, 1)$ moduli $V$ and $b^i$ that appear in $S_{\text{hyper}}$ and $S_{\text{bound}}$. Those potentials involve the five–brane charges $\alpha_i$, defined by eq. (2.13), that characterize the nonzero mode. The bulk potential in the hypermultiplet part of the action arises directly from the kinetic term $G^2$ of the antisymmetric tensor field with the expression (2.14) for the nonzero mode inserted. It can therefore be interpreted as the energy contribution of the nonzero mode. The origin of the boundary potentials, on the other hand, can be directly seen from eq. (2.17) and the boundary actions (1.3). Essentially, they arise because the standard embedding leads to nonvanishing internal boundary actions due to the crucial factor $1/2$ in front of the $\text{tr}R^2$ terms. This is in complete analogy with the appearance of nonvanishing sources in the internal part of the Bianchi identity which led us to introduce the nonzero mode.

3 Relation to Five-Dimensional Supergravity Theories

As we have argued in the previous section, the five-dimensional effective action (2.22) should have $\mathcal{N} = 1$, $D = 5$ supersymmetry in the bulk and $\mathcal{N} = 1$, $D = 4$ supersymmetry on the boundary. In this section, we will rewrite the action in a supersymmetric form. This will allow us to complete the action (2.23) to include fermionic terms and give the supersymmetry transformations. One thing we will not do is complete the supersymmetry transformations to include the bulk and boundary couplings, but we assume a consistent completion is possible, as in eleven dimensions.

Of particular interest is the presence of potential terms in the bulk theory. Such terms are forbidden unless the theory is gauged; that is, unless some of the fields are charged under Abelian gauge fields $A_\alpha$. In order to identify the supersymmetry structure of the theory in hand, we derived, in appendix B of ref. [5], the general form of gauged $D = 5$, $\mathcal{N} = 1$ supergravity with charged hypermultiplets, borrowing heavily from the work of Günaydin et al. [52, 53] and Sierra [54], and from the general theory of gauged $D = 4$, $\mathcal{N} = 2$ supergravity as given, for instance, in [55].

Let us start by giving the $\mathcal{N} = 1$ structure of the four-dimensional boundary theory. As discussed above, we have a set of chiral multiplets with scalar components $C^{i\alpha}$, together with vector multiplets with gauge fields $A^{(i)}_{\mu}$. (The vectors live on both boundaries, but the chiral matter lives only on the $E_6$ boundary.) In addition, the scalars from the bulk $(A, \sigma)$ and $(b^i, A_{11}^i)$
also form chiral multiplets. From the form of the theory on the boundaries we can give explicitly the functions determining the $\mathcal{N} = 1$ theory. We have already given the form of the superpotential and the $D$-term on the $E_6$ boundary in equations (2.31) and (2.32). It is also easy to read off the Kähler potential on the $E_6$ boundary and the gauge kinetic functions on either fixed plane. We find, without care to correct normalizations,

$$K = G_{i\bar{j}} C^{i\bar{p}} \bar{C}^{\bar{p}} f^{(n)} = V + i \sigma$$

(3.1)

The appearance of $\sigma$ in the gauge kinetic function is not immediately apparent from the action (2.22). However, it is easy to show that on making the dualization of $C_{\alpha\beta\gamma}$ to $\sigma$, which is described in more detail below, the magnetic source in the Bianchi identity (2.24) for $C_{\alpha\beta\gamma}$, becomes an electric source for $\sigma$. The result is that the gauge kinetic terms in the boundary action are modified to

$$-\frac{1}{16 \pi G_{\text{UT}}} \sum_{n=1}^{2} \int_{\mathcal{M}_H^{(n)}} \sqrt{-g} \left[ V \text{Tr} F_{\mu\nu}^{(n)} F^{(n)\mu\nu} - \frac{\sigma}{2} \epsilon^{\mu\nu\rho\sigma} \text{Tr} F_{\mu\nu}^{(n)} F^{(n)}_{\rho\sigma} \right]$$

(3.2)

One notes that the expressions (3.1) include dependence on the bulk fields $b^i$ and $V$, evaluated on the appropriate boundary. Further, we are considering the bulk multiplets as parameters, as their dynamics comes from bulk kinetic terms.

Now let us turn to the bulk theory. Our goal will be to identify the action (2.23) with the bosonic part of the general gauged theory discussed in appendix B of ref. [5]. The gauged theory is characterized by a special Riemannian manifold $\mathcal{M}_V$ describing the vector multiplet sigma-model, a quaternionic manifold $\mathcal{M}_H$ describing the hypermultiplet sigma-model, and a set of Killing vectors and prepotentials on $\mathcal{M}_H$. These are the structures we must identify in the action (2.23).

We start by concentrating on the hypermultiplet structure. We have argued that, after dualizing the three-form potential $C_{\alpha\beta\gamma}$ to a scalar $\sigma$, the fields $(V, \sigma, \xi, \bar{\xi})$ represent the scalar components of a hypermultiplet. Concentrating on the kinetic terms let us make the dualization explicit. We find

$$G_{\alpha\beta\gamma\delta} = \frac{1}{\sqrt{2}} V^{-2} \epsilon_{\alpha\beta\gamma\delta} \left\{ \partial_{\alpha} \sigma - i \left( \xi \partial_{\gamma} \bar{\xi} - \bar{\xi} \partial_{\gamma} \xi \right) - 2 \epsilon(x^{11}) \alpha_i A^i \right\}.$$  

(3.3)

The kinetic terms can then be written in the form

$$h_{uv} D_{\alpha} q^u D^\alpha q^v$$

(3.4)

where $q^u = (V, \sigma, \xi, \bar{\xi})^u$ and

$$D_{\alpha} q^u = (\partial_{\alpha} V, \partial_{\alpha} \sigma - 2 \epsilon(x^{11}) \alpha_i A^i_{\alpha}, \partial_{\alpha} \xi, \partial_{\alpha} \bar{\xi})^u$$

(3.5)

and the metric is given by

$$h_{uv} dq^u dq^v = \frac{1}{4 V^2} dV^2 + \frac{1}{4 V^2} \left[ d\sigma + i(\xi d\bar{\xi} - \bar{\xi} d\xi) \right]^2 + \frac{1}{V} d\xi d\bar{\xi}.$$

(3.6)
This reproduces the well-known result that the universal multiplet classically parameterizes the quaternionic space $\mathcal{M}_H = SU(2,1)/U(2)$ \cite{56}.

In what follows, we would like to have an explicit realization of the quaternionic structure of $\mathcal{M}_H$. A review of quaternionic geometry is given in appendix B of ref. \cite{5}. We will now give expressions for the quantities defined there, following a discussion given in \cite{57}. Since we have a single hypermultiplet, the holonomy of $\mathcal{M}_H$ should be $SU(2) \times Sp(2) = SU(2) \times SU(2)$. To distinguish these, we will refer to the first factor as $SU(2)$ and the second as $Sp(2)$. Defining the symplectic matrix $\Omega_{ab}$ such that $\Omega_{12} = -1$, we have the vielbein

$$V^{Aa} = \frac{1}{\sqrt{2}} \begin{pmatrix} u & \bar{v} \\ v & -\bar{u} \end{pmatrix}^{Aa} \quad (3.7)$$

where we have introduced the one-forms

$$u = \frac{d\sigma}{\sqrt{V}}, \quad v = \frac{1}{2V} (dV + id\sigma + \xi d\bar{\xi} - \bar{\xi} d\xi) \quad (3.8)$$

and their complex conjugates $\bar{u}$ and $\bar{v}$. We find that the $SU(2)$ connection is given by

$$\omega^{A}_{\ B} = \begin{pmatrix} \frac{1}{4} (v - \bar{v}) & -u \\ \bar{u} & -\frac{1}{4} (v - \bar{v}) \end{pmatrix}^A_B \quad (3.9)$$

while the $Sp(2)$ connection is

$$\Delta^a_{\ b} = \begin{pmatrix} -\frac{3}{4} (v - \bar{v}) & 0 \\ 0 & \frac{3}{4} (v - \bar{v}) \end{pmatrix}^a_b. \quad (3.10)$$

The triplet of Kähler forms is given by

$$K^{A}_{\ B} = \begin{pmatrix} \frac{1}{2} (u \wedge \bar{u} - v \wedge \bar{v}) & u \wedge \bar{v} \\ v \wedge \bar{u} & -\frac{1}{2} (u \wedge \bar{u} - v \wedge \bar{v}) \end{pmatrix}^A_B. \quad (3.11)$$

With these definitions, one finds that the coset space $SU(2,1)/U(2)$, satisfies the conditions for a quaternionic manifold.

So far our discussion has ignored the most important aspect of the hypermultiplet sigma-model. We note that the kinetic terms in (3.4) were in terms of a modified derivative (3.5), which included the gauge fields $A_i^\alpha$. It appears that the hypermultiplet is charged under a $U(1)$ symmetry. Comparing with our discussion of gauged supergravity given in appendix B of ref. \cite{5}, we see that this is indeed the case. The coset space $\mathcal{M}_H$ admits an Abelian isometry generated by the Killing vector

$$k = \partial_\sigma = iV^{-1} (\partial_v - \partial_{\bar{u}}). \quad (3.12)$$
In general, we can write the modified derivative (3.5) in the covariant form

\[ D_\alpha q^u = \partial_\alpha q^u + gA^i_\alpha k^u_i \]  

(3.13)

with

\[ gk^u_i = -2\epsilon(x^{11})\alpha_i k^u = -2i\epsilon(x^{11})\alpha_i V^{-1}(\partial_u - \partial_\bar{u}) \]  

(3.14)

(Note that the gauge coupling is absorbed in \(\alpha_i\).) For consistency, the \(k^u_i\) should be writable in terms of a triplet of prepotentials. This is indeed the case and we find the prepotentials

\[ gP^A_iB = \left( \begin{array}{cc} -\frac{i}{4}i\epsilon(x^{11})\alpha_i V^{-1} & 0 \\ 0 & \frac{i}{4}i\epsilon(x^{11})\alpha_i V^{-1} \end{array} \right)^A_B. \]  

(3.15)

Thus it appears that the \(\sigma\)-component of the hypermultiplet is charged under each Abelian gauge field \(A^i_\alpha\), with a charge proportional to \(\alpha_i\). In particular, we can write the covariant derivative as

\[ D_\alpha \sigma = \partial_\alpha \sigma + \frac{1}{4\sqrt{2\pi}} \left( \frac{4\pi}{\kappa_5} \right)^{2/3} \alpha_{\text{GUT}} \epsilon(x^{11})\beta_i A^i_\alpha \]  

(3.16)

where \(\beta_i\) are integers characterizing the first Pontrjagin class of the Calabi-Yau.

If this interpretation is correct, the rest of the action should coincide with the general form for gauged supergravity given in appendix B of ref. [5]. It is clear that the vector multiplets are already in the correct form. Comparing the bosonic action (2.23) with the general form given in (B.25) of ref. [5], we see that the gravitational and vector kinetic terms exactly match. (In the appendix of ref. [5], we have set the five-dimensional gravitational coupling \(\nu/\kappa^2\) to unity.) The structure of the metric \(G_{ij}\) is identical, as is the appearance of Chern-Simons couplings. The compactification gives an interpretation of the numbers \(d_{ijk}\) in the Kähler potential (B.7) of ref. [5] and (2.21). They are the Calabi-Yau intersection numbers.

The final check of this identification is to calculate the form of the potential. We have in general, from (B.29) of ref. [5],

\[ g^2 V = -2g^2G_{ij}\text{tr}P^i\text{tr}P^j + 4g^2b_ib_j\text{tr}P^i\text{tr}P^j + \frac{g^2}{2}b^ib^j\text{h}_{uv}k^u_i k^v_j \]  

\[ = \frac{1}{4}V^{-2}G^{ij}\alpha_i\alpha_j, \]  

(3.17)

exactly matching the derived potential.

Thus, we can conclude that the bulk effective action is described by a set of Abelian vector multiplets coupled to a single charged hypermultiplet. The vector sigma-model manifold \(\mathcal{M}_V\) has the general form described in appendix B of ref. [5], but now the \(d_{ijk}\) in the Kähler potential have the interpretation as Calabi-Yau intersection numbers. The hypermultiplet manifold \(\mathcal{M}_H\) is the coset space \(SU(2,1)/U(2)\). A \(U(1)\) isometry, corresponding to the shift symmetry of the dualized
three-form, is gauged. The charge of the hypermultiplet scalar field under each Abelian vector field \( A^i_\alpha \) is given by \( \alpha_i \).

The appearance of gauged supergravity when non-zero modes are included has been seen before in the context of type II compactifications on Calabi-Yau manifolds to four-dimensions \([58, 59]\). It is natural to ask why this gauging arises. The appearance of a potential term is easy to interpret. We have included a non-zero four-form field strength \( G_{IJKL} \) on four-cycles of the Calabi-Yau. These contribute an energy proportional to the square of the field strength. For fixed total charge \( \alpha_i \) (the integral of \( G \) over a cycle), the energy is reduced the larger the four-cycle. Thus it is no longer true that all points in Calabi-Yau moduli space have the same energy. As an example we see that the potential naturally drives the Calabi-Yau to large volume, minimizing the \( G^2 \) energy.

From the five-dimensional point of view, once we have a potential term, the theory must be gauged if it is to remain supersymmetric. We see that it is the dual of the five-dimensional three-form which is gauged. This arises because of the Chern-Simons term in eleven dimensions. Turning on non-zero modes, this term acts as an electric source for the five-dimensional three-form, though dependent on the gauge fields \( A^i \). Dualizing, the invariance \( \sigma \rightarrow \sigma + \text{const} \) is a reflection of an absence of local electric charge. Thus it not surprising that the effect of the electric Chern-Simons terms is to modify this to a local gauge symmetry. We note that from this argument it can only ever be the five-dimensional three-form which becomes gauged by non-zero modes, whatever particular compactification to a \( N = 1 \) five-dimensional theory is considered.

We end this section by giving the the specific form of the fermionic supersymmetry variations. These are calculated using the general forms given in (B.30), (B.31) and (B.32) of appendix B in ref. \( \text{[5]} \), together with the explicit expressions for the vielbein, connections, Killing vectors and prepotentials given above. We find

\[
\delta \psi^A = \nabla_\alpha \epsilon^A + \frac{\sqrt{2} i}{8} \left( \gamma^\beta_\alpha \gamma^\gamma_\beta - 4 \delta_\alpha^\beta \gamma^\gamma_\beta \right) b_i \mathcal{F}^i_{\beta\gamma} \epsilon^A - P^{A}_{\alpha \beta} B \epsilon^B \\
- \frac{\sqrt{2}}{12} V^{-1} b^i \alpha_i \epsilon^A \left( \tau_3 A^i B \epsilon^B \right)
\]

(3.18a)

\[
\delta \lambda^x = b_i^x \left( -\frac{1}{2} \gamma^\alpha_\beta \partial_\alpha b^j \epsilon^A - \frac{1}{2\sqrt{2}} \gamma^\alpha_\beta \mathcal{F}^i_{\alpha\beta} \epsilon^A - i \frac{V^{-1} \alpha_i \epsilon^A}{2\sqrt{2}} \right)
\]

(3.18b)

\[
\delta \zeta^a = -i Q^{A}_{\alpha \beta} \gamma^a B \epsilon^B - i \sqrt{2} b^i \alpha_i V^{-1} \epsilon^A \left( \tau_3 B \epsilon^B \right)
\]

(3.18c)

where \( \tau_i \) with \( i = 1, 2, 3 \) are the Pauli spin matrices and we have the matrices

\[
P^{A}_{\alpha \beta} = \left( \begin{array}{cc} V^{-1/2} \partial_\alpha \xi & \frac{\sqrt{2}}{96} V \epsilon_{\alpha \beta \gamma \delta a} G^{\beta \gamma \delta a} \\ -V^{-1/2} \partial_\alpha \bar{\xi} & -\frac{\sqrt{2}}{96} V \epsilon_{\alpha \beta \gamma \delta a} G^{\beta \gamma \delta \bar{a}} \end{array} \right)
\]

(3.19)

\[
Q^{A}_{\alpha \beta} = \left( \begin{array}{cc} \frac{\sqrt{2}}{48} V \epsilon_{\alpha \beta \gamma \delta a} G^{\beta \gamma \delta a} - \frac{1}{2} V^{-1} \partial_\alpha V & V^{-1/2} \partial_\alpha \xi \\ V^{-1/2} \partial_\alpha \bar{\xi} & \frac{\sqrt{2}}{48} V \epsilon_{\alpha \beta \gamma \delta a} G^{\beta \gamma \delta a} + \frac{1}{2} V^{-1} \partial_\alpha V \end{array} \right)
\]
4 The Domain Wall Solution

In this section, we would like to find the simplest BPS solutions of the five–dimensional theory, including the coupling to the potential terms induced by the nonzero mode. As we will see, these solutions provide the appropriate background for a reduction to four dimensions and can therefore be viewed as the “vacua” of the theory. After a general derivation of the solutions, we will discuss several limiting cases of interest.

4.1 The General Solution

Let us first simplify the discussion somewhat by concentrating on the fields which are essential. Since we would like to find solutions that couple to the bulk potential terms we should certainly keep the hypermultiplet scalar $V$ (the Calabi–Yau breathing mode) and the vector multiplet scalars $b^i$ (the shape moduli). It turns out that those fields plus the five–dimensional metric are already sufficient. The action (2.22) can be consistently truncated to this reduced field content leading to

$$2\kappa_5^2 S_5 = -\int_{M_5} \sqrt{-g} \left[ R + G_{ij} \partial^i b^j + \frac{1}{2} V^{-2} \partial_\alpha V \partial^\alpha V + \frac{1}{2} V^{-2} G^{ij} \alpha_i \alpha_j + \lambda (\mathcal{K} - 6) \right]$$

$$+ 2\sqrt{2} \int_{M_4^{(1)}} \sqrt{-g} V^{-1} \alpha_i b^i - 2\sqrt{2} \int_{M_4^{(2)}} \sqrt{-g} V^{-1} \alpha_i b^i .$$

(4.1)

Note that we have explicitly added the Lagrange multiplier term which ensures the constraint (2.30) on $b^i$. For a finite Calabi–Yau volume $V$, that is, for an uncompactified internal space, the potential terms in this action do not vanish and, hence, flat space is not a solution of the theory. Therefore, the question arises of what the “vacuum” state of the theory is. A clue is provided by the fact that cosmological–type potentials in $D$ dimensions generally couple to $D - 2$ branes. This is well known from the eight–brane [60] which appears as a solution of the massive extension of type IIA supergravity [61] in ten dimensions. There, the eight–brane couples to a cosmological–type potential which consists of a single “cosmological” constant multiplied by a certain power of the dilaton. A way to understand to appearance of an eight–brane in this context is to dualize the cosmological constant to a nine–form antisymmetric tensor field which, according to the usual counting, should couple to an $8 + 1$–dimensional extended object. A systematic study of $D - 2$ brane solutions in various dimensions using a generalized Scherk–Schwarz reduction can be found in ref. [62]. The present case is somewhat more complicated in that it involves $h^{1,1}$ scalar fields (as opposed to just the dilaton) and, correspondingly, $h^{1,1}$ constants $\alpha_i$ (as opposed to just one cosmological constant). Still, we can take a lead from the massive IIA example and dualize each of the constants $\alpha_i$ to a four–form antisymmetric tensor field. This would leave us with a theory that contains $h^{1,1}$ such antisymmetric tensor fields and, hence, a corresponding number of different types of three–branes that couple to those. The constants $\alpha_i$ can then be identified as the charges of these different types of three–branes. Since those constants are fixed in terms of the underlying
theory (and are generically nonzero) one cannot really look for a “pure” solution which carries only one type of charge. Instead, what we are looking for is a multi–charged three–brane which is a mixture of the various different types as specified by the charges $\alpha_i$. Clearly, the transverse space for a three–brane in five–dimensions is just one–dimensional. Given that the boundary source terms necessarily introduce dependence on the $x^{11}$ coordinate, this one–dimensional space can only be in the direction of the orbifold.

From the above remarks it is now clear that the proper Ansatz for the type of solutions we are looking for is given by

\[ ds^2_5 = a(y)^2 dx^\mu dx^\nu \eta_{\mu\nu} + b(y)^2 dy^2 \]
\[ V = V(y) \]
\[ b^i = b^i(y) , \]

where we use $y = x^{11}$ from now on. The equations of motion derived from the action (4.1) still contains the Lagrange multiplier $\lambda$. It can be eliminated using eqs. (A.19)–(A.21) from appendix A of ref. [5]. A solution to the resulting equations of the form (4.2) is still somewhat hard to find, essentially due to the complication caused by the inclusion of all $(1,1)$ moduli and the associated Kähler structure. The trick is to express the solution in terms of certain functions $f^i = f^i(y)$ which are only implicitly defined rather than trying to find fully explicit formulae. It turns out that those functions are fixed by the equations

\[ d_{ijk} f^i f^j f^k = H_i, \quad H_i = 2\sqrt{2k}\alpha_i |y| + k_i \]

where $k$ and $k_i$ are arbitrary constants. Then the solution can be written as

\[ V = \left( \frac{1}{6} d_{ijk} f^i f^j f^k \right)^2 \]
\[ a = \tilde{k} V^{1/6} \]
\[ b = k V^{2/3} \]
\[ b^i = V^{-1/6} f^i \]

where $\tilde{k}$ is another arbitrary constant. We should check that this solution is indeed a BPS state of the theory; that is, that it preserves four of the eight supercharges. For the reduced field content,
the supersymmetry transformations (3.18) lead to the following Killing spinor equations

\[ \delta \psi^A_\mu = 0 : \gamma_\mu \left( \frac{a'}{a} \gamma_{11} \epsilon^A - \sqrt{2} b^i \alpha_i \epsilon(y) \tau_3 A_B \epsilon^B \right) = 0 \quad (4.5a) \]

\[ \delta \psi^A_{11} = 0 : \epsilon' - \sqrt{2} b^i \alpha_i \gamma_{11} \epsilon(y) \tau_3 A_B \epsilon^B = 0 \quad (4.5b) \]

\[ \delta \lambda^{xA} = 0 : b^i \gamma_{11} \epsilon^A + \frac{b}{\sqrt{2} V} \left( 2 b^j \alpha_j b^i \right) \epsilon(y) \tau_3 A_B \epsilon^B = 0 \quad (4.5c) \]

\[ \delta \zeta^a = 0 : V' \gamma_{11} \epsilon^A - \sqrt{2} b^i \alpha_i \epsilon(y) \tau_3 A_B \epsilon^B = 0 \quad , \quad (4.5d) \]

where the prime denotes the derivative with respect to \( y \). These equations are satisfied for the solution (4.5) if the spinor \( \epsilon^A \) takes the form

\[ \epsilon^A = a^{1/2} \epsilon_0^A , \quad \gamma_{11} \epsilon_0^A = (\tau_3)^A B \epsilon_0^B , \quad (4.6) \]

where \( \epsilon_0^A \) is a constant spinor. As a result, the solution preserves indeed four supercharges.

As can be seen from eq. (4.3) the solution is described in terms of \( h^{1,1} \) linear functions \( H_i \). This follows the general pattern of \( p \)-brane solutions coupled to \( n \) different charges which can be expressed in terms of \( n \) harmonic functions on the transverse space. In our case the number of charges \( \alpha_i \) is precisely \( h^{1,1} \) and the transverse space is just one–dimensional leading to linear functions. Generally, elementary brane solutions have singularities at the location of the branes which have to be supported by brane worldvolume theories. The pure bulk theory does not impose any restrictions on the number and locations of these singularities. Correspondingly, if we would just consider the bulk part of the action (4.1) we could place an arbitrary number of parallel three–branes anywhere on the orbifold. However, the theory (4.1) involves two four–dimensional boundary actions which provide source terms that should be matched. This is possible, in the present case, because the height of the boundary potentials in (4.1) is set by the three–brane charges \( \alpha_i \). If we decide that the solution should have no further singularities other than those matched by the two boundaries we arrive at the specific form of the harmonic functions \( H_i \) in eq. (4.3). In fact, we have

\[ H_i'' = 4 \sqrt{2} k \alpha_i (\delta(y) - \delta(y - \pi \rho)) \quad , \quad (4.7) \]

indicating sources at the orbifold planes \( y = 0, \pi \rho \). Recall that we have restricted the range of \( y \) to \( y \in [-\pi \rho, \pi \rho] \) with the endpoints identified. This explains the second delta–function at \( y = \pi \rho \) in the above equation.

In conclusion, the solution (4.5) represents a multi–charged double domain wall (three–brane) solution with the two walls located at the orbifold planes. It preserves four–dimensional Poincaré invariance as well as four of the eight supercharges and has therefore the correct properties to make contact with four–dimensional \( \mathcal{N} = 1 \) supergravity. More precisely, those theories should arise as a dimensional reduction of the five–dimensional theory on the domain wall background. In this
sense, the solution (4.5) can be viewed as the vacuum state of the five–dimensional theory. From the perspective of the four–dimensional theory the domain wall solution plays an interesting rôle. It is oriented precisely in the four uncompactified dimensions and carries the physical gauge and gauge matter fields. Therefore, at low energy four–dimensional space–time gets identified with the three–brane worldvolume. In this sense, our Universe lives on the worldvolume of a three–brane. Finally, we would like to discuss some physically relevant limiting examples of the general solution.

4.2 Universal Solution

In ref. [4] we have presented a related three–brane solution which was less general in that it involved the universal Calabi–Yau modulus \( V \) only. Clearly, we should be able to recover this solution from eq. (4.5) if we consider the specific case \( h^{1,1} = 1 \). Then we have \( d_{111} = 6 \) and it follows from eq. (4.3) that

\[
f^1 = \left( \frac{\sqrt{2}}{3} k \alpha_1 |y| + k_1 \right)^{1/2}.
\]

(4.8)

Inserting this into eq. (4.5) provides us with the explicit solution in this case which is given by

\[
a = a_0 H^{1/2} \\
b = b_0 H^2 \\
V = b_0 H^3.
\]

The constant \( a_0, b_0 \) and \( c_0 \) are related to the integration constants in eq. (4.3) by

\[
a_0 = \tilde{k} k^{1/2}, \quad b_0 = k^3, \quad c_0 = \frac{k_1}{k}.
\]

(4.10)

Eq. (4.9) is indeed exactly the solution that was found in ref. [4]. It still represents a double domain wall. However, in contrast to the general solution it couples to one charge \( \alpha = \alpha_1 \) only. Geometrically, it describes a variation of the five–dimensional metric and the Calabi–Yau volume across the orbifold. The form of the solution (4.9) is typical for brane solutions that couple to one charge and, in fact, fits into the general scheme of domain walls in various dimensions [62].

One may ask if a structure as simple as the above universal solution is, in some way, also part of the general solution (4.5) even if \( h^{1,1} > 1 \). To see that this is indeed the case, we define constants \( \tilde{\alpha}^i \) and \( \alpha \) by

\[
d_{ijk} \tilde{\alpha}^j \tilde{\alpha}^k = \frac{2}{3} \alpha_i, \quad \alpha = 9 \left( \frac{1}{6} d_{ijk} \tilde{\alpha}^i \tilde{\alpha}^j \tilde{\alpha}^k \right)^{2/3}.
\]

(4.11)

In addition, we choose the following special values for the integration constants \( k_i \) in eq. (4.3)

\[
k_i = 6 k c_0 \frac{\alpha_i}{\alpha}.
\]

(4.12)
where \( c_0 \) is an arbitrary constant. Thanks to this specific choice, we can easily solve (4.3) for \( f^i \). Inserting the result into eq. (4.5) gives the explicit solution
\[
\begin{align*}
    a &= a_0 H^{1/2} \\
    b &= b_0 H^2, \quad H = \frac{\sqrt{2}}{3} \alpha |y| + c_0 \\
    V &= b_0 H^3 \\
    b^i &= 3 \alpha^{-1/2} \bar{\alpha}^i .
\end{align*}
\] (4.13)

As before, \( a_0 \) and \( b_0 \) are constants expressed in terms of the integration constants in (4.5) as
\[
    a_0 = \tilde{k} k^{1/2}, \quad b_0 = k^3 .
\] (4.14)

Hence, for arbitrary values of \( h^{1,1} \), we have identified a special case of the general solution (4.3) where the fields \( a, b \) and \( V \) behave in exactly the same way as in the universal solution (4.9). The charge \( \alpha \) which appears in this special solution is now a complicated function of the various charges \( \alpha_i \) in the way defined by eq. (4.11). In addition, the shape moduli \( b^i \) are constant. Consequently, for this special solution the metric and the Calabi–Yau volume vary as in the universal solution while the shape of the Calabi–Yau space is fixed.

### 4.3 Another Simple Example

A nontrivial example where the domain wall solution can be obtained explicitly is provided by
\[
    h^{1,1} = 3 , \quad d_{123} = 1 ,
\] (4.15)

and \( d_{ijk} = 0 \) otherwise. The Kähler potential is then given by
\[
    \mathcal{K} = 6 b^1 b^2 b^3 .
\] (4.16)

In a four–dimensional effective theory the real fields \( b^i \) are promoted to complex scalars. Then the Kähler potential (4.16) is associated with the coset space \([SU(1,1)/U(1)]^3 \) and describes the STU–model. Due to the simple structure of intersection numbers eq. (4.3) can be easily solved for the functions \( f_i \) resulting in
\[
    f^i = (H_1 H_2 H_3)^{1/2} H_i^{-1} .
\] (4.17)

Inserting into eq. (4.5) then gives the explicit solution
\[
\begin{align*}
    V &= (H_1 H_2 H_3)^{-1} \\
    a &= \tilde{k} (H_1 H_2 H_3)^{-1/6} \quad H_i = 2 \sqrt{2} k \alpha_i |y| + k_i \\
    b &= k (H_1 H_2 H_3)^{-2/3} \\
    b^i &= (H_1 H_2 H_3)^{2/3} H_i^{-1}
\end{align*}
\] (4.18)

for \( i = 1, 2, 3 \). As before \( k, \tilde{k} \) and \( k_i \) denote constants.
Lecture 2: Non-Standard Embedding and Five–Branes

To make contact with low-energy physics, one of the central issues in string theory has been to find vacua leading to chiral four-dimensional theories with $\mathcal{N} = 1$ supersymmetry. In recent years, the new understanding of the non-perturbative behavior of string theory has broadened the scope for approaching these issues. Specifically, the inclusion of brane states, that is, vacua with non-trivial form-fields, increases the class of possible backgrounds giving a chiral $\mathcal{N} = 1$ theory in four dimensions, and has raised the possibility of gauge interactions arising from the brane world-volume theory itself.

In this second lecture, we will consider a class of eleven-dimensional M–theory vacua based on the strongly coupled limit of the $E_8 \times E_8$ heterotic string, as described by Hořava and Witten [1, 2]. At low energy, these are compactifications of eleven-dimensional supergravity on an $S^1/Z_2$ orbifold, with $E_8$ gauge fields at each of the two orbifold fixed planes. Following Witten [3], we can further compactify on a Calabi–Yau three-fold to give a chiral $\mathcal{N} = 1$ theory in four-dimensions. Essentially, all the early discussions of the low-energy properties of compactifications [4]–[65] were limited to the standard embedding, where the Calabi–Yau spin connection is embedded in one of the $E_8$ gauge groups. In [6], we considered the general configuration leading to $\mathcal{N} = 1$ supersymmetry, where, first, we allowed for general gauge bundles, and, second, included five-branes, states which are essentially non-perturbative in heterotic string theory. The possibility of such generalizations was first put forward by Witten [3]. Recently, non–standard embedding gauge threshold corrections of orbifold models have been computed in ref. [80] and have, in the large radius limit, been compared to the expressions calculated from Hořava–Witten theory. Gauge thresholds of non-standard embeddings in the strongly coupled limit have also been discussed in [43]. A toy model of gauge fields coming from five-branes close to the orbifold planes has been presented in [65]. In this lecture, we review the results of [6].

The $\mathcal{N} = 1$ vacua we will discuss have the following structure. One starts with the spacetime $M_{11} = S^1/Z_2 \times X \times M_4$, where $X$ is a Calabi–Yau three-fold and $M_4$ is flat Minkowski space. As in the weakly coupled limit, to preserve the four supercharges, arbitrary holomorphic $E_8$ gauge bundles over $X$ (satisfying the Donaldson–Uhlenbeck–Yau condition) are allowed on each plane. In particular, there is no requirement that the spin-connection of the Calabi–Yau space be embedded in the gauge connection of one of the $E_8$ bundles. This generalization is what is meant by non-standard embedding, and has a long history in the phenomenology of weakly coupled strings (for early discussions see refs. [66, 67, 49]). In addition, one can add five-branes, located at points throughout the orbifold interval. The five-branes will preserve some supersymmetry, provided the branes are wrapped on holomorphic two-cycles within $X$ and otherwise span the flat Minkowski space $M_4$ [3].

Both the gauge fields and the five-branes are magnetic sources for the four-form field strength.
of the bulk supergravity, and so excite a non-zero $G$ within the compact $S^1/Z_2 \times X$ space. This has two effects. First, since the space is compact, there can be no net magnetic charge, for there is nowhere for the flux to “escape”. Thus, there is a cohomological condition that the sum of the sources must be zero. Secondly, the non-zero form field enters the Killing spinor equation and so, to preserve supersymmetry, the background geometry must have a compensating distortion. This leads to a perturbative expansion of the supersymmetric background. Such an expansion is familiar in non-standard embeddings in the weakly coupled heterotic string [66, 67, 49]. In the strongly coupled limit, it appears even for the standard embedding. From this point of view, the generalization to include non-standard embedding and five-branes is very natural.

Having found the vacuum as a perturbative solution, one is then interested in the form of the low-energy theory of the massless excitations around this compactification. It is well known that, in the standard embedding, to match the low-energy Newton constant and grand unified parameters, one needs to take a Calabi–Yau manifold of size comparable to the eleven-dimensional Planck length, with the orbifold an order of magnitude or so larger. Thus, it is natural to consider effective actions both in five dimensions, where only $X$ is compactified, and four, which is appropriate to momenta below the orbifold scale. For the standard embedding, the four-dimensional action has been calculated to leading non-trivial order [24, 13]. Although the expansion is completely non-perturbative, it turns out that, to this order, the form of the effective action is identical to the large radius Calabi–Yau limit of the one-loop effective action calculated in the weak limit. There are threshold corrections in the gauge couplings as well as in the matter field Kähler potential. In five dimensions, because of the non-zero mode of $G$, the theory is a form of gauged supergravity in the bulk, coupled to gauge theories on the fixed planes [4, 5]. There is no homogeneous background solution but, rather, the correct vacuum is a BPS domain wall solution, supported by sources on the fixed planes and a potential in the bulk.

Calculating the modifications to the low-energy effective actions due to non-standard embedding and five-branes will be the main point of this lecture. Our results can be summarized as follows. In section two, we discuss the expansion of the background solution, the cohomology condition on the five-brane and orbifold magnetic sources and the constraints on the zeroth-order background to preserve supersymmetry. We then give the solution to first order. Expanding in terms of eigenfunctions on the Calabi–Yau three-fold, we show that the main contribution comes from the massless modes. Sections three and four discuss the low-energy actions in the case of non-standard embedding and inclusion of five-branes respectively. This requires an analysis of the theory on the five-brane world-volume, which is given in section 4.2. In summary, we find

- For non-standard embeddings, in the absence of five-branes, the five-dimensional action has the same form as in the standard embedding case both in the bulk and on the orbifold planes. However, the values of the gauge coupling parameters, related to the gauging of the bulk
supergravity, depend on the form of the non-standard embedding.

- The non-standard embedding allows many different breaking patterns for the $E_8$ groups. In particular, it is no longer necessary that the visible sector be broken to $E_6$. Rather, more general gauge groups $G^{(1)}, G^{(2)} \subset E_8$ and corresponding gauge matter can occur on the respective orbifold planes.

- In the presence of five branes, the form of the bulk five-dimensional action between any pair of neighboring branes is the same as in the case of standard embedding. The four-dimensional fixed-plane theories also have the same form and couplings to the bulk fields. However, there are additional four-dimensional theories, arrayed throughout the orbifold and again coupling to the bulk fields, which arise from the five-brane world-volume degrees of freedom.

- In the conventional picture, the five-brane worldvolume theories provide new hidden sectors. Generically, the theory for a single five-brane is $\mathcal{N} = 1$ supersymmetric with $gU(1)$ vector multiplets, together with a universal chiral multiplet and a set of chiral fields parameterizing the moduli space of holomorphic genus $g$ two-cycles in $X$. This gauge group can be enhanced when five-branes overlap or when the embedding of a single fivebrane degenerates. In general, the total rank of the gauge group remains unchanged.

- The presence of five-branes also allows for new types of $E_8 \times E_8$ breaking patterns, beyond those associated with non-standard embeddings alone. This is because the presence of five-brane sources leads to a wider range of solutions satisfying the zero cohomology condition.

- Reducing to four dimensions, the effective action is modified with respect to the standard embedding case. For pure non-standard embeddings, both the gauge and Kähler threshold corrections are identical in form to those of the standard embedding. However, the presence of the five-branes significantly modifies these corrections so that, for instance, both $E_8$ sectors can get threshold corrections of the same sign.

The new threshold corrections due to the five-branes have no analog in the weakly coupled limit since, first, the branes are non-perturbative and, second, the corrections depend on the positions of the five-branes across the orbifold, moduli which simply do not exist in the weakly coupled limit. Similarly, the appearance of new gauge groups due to five-branes is a non-perturbative effect. Finally, we note, it appears that there is a constraint on the total rank of the full gauge group from orbifold fixed planes and five-branes, which arises from positivity constraints in the magnetic charge cohomology condition.
5 Vacua with Non-Standard Embedding and Five-Branes

In this section, we are going to construct generalized heterotic M–theory vacua appropriate for a reduction of the theory to $\mathcal{N} = 1$ supergravity theories in both five and four dimensions. To lowest order (in the sense explained below), these vacua have the usual space-time structure $M_{11} = S^1/Z_2 \times X \times M_4$ where $X$ is a Calabi–Yau three-fold and $M_4$ is four-dimensional Minkowski space. As compared to the vacua constructed to date, we will allow for two generalizations. First, we will not restrict ourselves to embedding the Calabi–Yau spin connection into a subgroup $SU(3) \subset E_8$ but, rather, allow for general (supersymmetry preserving) gauge field sources on the orbifold hyperplanes. Secondly, we will allow for the presence of five-branes that stretch across $M_4$ and wrap around a holomorphic curve in $X$. As we will see, the inclusion of five-branes makes it much easier to satisfy the necessary constraints. Therefore, their inclusion is essential for a complete discussion non-standard embeddings, and leads to a considerable increase in the number of such vacua.

5.1 Expansion Parameters

Before we proceed to the actual computation, let us explain the types of corrections to the lowest order background that one expects. For the weakly coupled heterotic string, it is well known that non-standard embeddings lead to corrections to the Calabi–Yau background. They can be computed perturbatively $[66, 67, 49]$ as a series in

$$\epsilon_W = \frac{\alpha'}{v_{10}^{1/3}}$$

(5.1)

where $v_{10}$ is the Calabi–Yau volume measured in terms of the ten-dimensional Einstein frame metric. At larger string coupling, one also gets contributions from string loops. Thus the full solution is a double expansion involving both $\epsilon_W$ and the string coupling constant.

On the other hand, in the strongly coupled heterotic string, it has been shown that, even in the case of the standard embedding, there are corrections originating from the localization of the gauge fields to the ten-dimensional orbifold planes $[3, 13]$. Again, these corrections can be organized in a double expansion. However, one now uses a parameterization appropriate to the strongly coupled theory. The 11-dimensional Hořava–Witten effective action has an expansion in $\kappa$, the 11-dimensional Newton constant. For the compactification on $S^1/Z_2 \times X$, there are two other scales, the size of the orbifold interval $\pi \rho$ and the volume $v$ of the Calabi–Yau threefold, each measured in the 11-dimensional metric. Solving the equations of motion and supersymmetry conditions for the action to order $\kappa^{2/3}$, one finds the correction to the background is a double expansion, linear, at this order, in the parameter

$$\epsilon_S = \left(\frac{\kappa}{4\pi} \right)^{2/3} \frac{\pi \rho}{v^{2/3}}$$

(5.2)
but to all orders in

\[ \epsilon_R = \frac{v^{1/6}}{\pi \rho}. \] (5.3)

It is natural to use the same expansion for the background with non-standard embedding and the inclusion of five-branes. As we will show explicitly, the solution to the order \( \kappa^{2/3} \) can be obtained as an expansion in eigenfunctions of the Calabi–Yau Laplacian. It turns out that the zero-eigenvalue, or “massless”, terms in this expansion are precisely of order \( \epsilon_S \), while the massive terms are of order \( \epsilon_R \epsilon_S \). Therefore, although one could expect corrections to arbitrary order in \( \epsilon_R \), to leading order in \( \epsilon_S \) only the zeroth-order and linear terms in \( \epsilon_R \) contribute.

Clearly, for the above expansion to be valid both \( \epsilon_S \) and \( \epsilon_R \) should be small. Let us briefly discuss the situation at the physical point, that is, at the values of \( \kappa \), \( v \) and \( \rho \) that lead to the appropriate values for the four-dimensional Newton constant and the grand unification coupling parameter and scale. There, both the 11–dimensional Planck length \( \kappa^{2/9} \), as well as the Calabi–Yau radius \( v^{1/6} \), are of the order \( 10^{-16} \) GeV\(^{-1} \) while the orbifold radius is an order of magnitude or so larger. Inserting this into eq. (5.2) and (5.3) shows that \( \epsilon_S \) is of order one \([10]\) while \( \epsilon_R \) is an order of magnitude or so smaller. At the physical point, therefore, we have

\[ \epsilon_R \ll \epsilon_S = O(1). \] (5.4)

Consequently, neglecting higher-order terms in \( \epsilon_S \) might not provide a good approximation at the physical point. It is, however, the best one can do at the moment given that M–theory on \( S^1/Z_2 \) is only known as an effective theory to order \( \kappa^{2/3} \). On the other hand, in fact, higher-order terms in \( \epsilon_R \) should be strongly suppressed and can be safely neglected.

It is interesting to note how this strong coupling expansion is related to the weak coupling expansion with non-standard embedding. Writing \( \epsilon_W \) in terms of 11-dimensional quantities, one finds

\[ \epsilon_W = \left( \frac{\kappa}{4\pi} \right)^{2/3} \frac{1}{\pi^2 \rho v^{1/3}} \] (5.5)

and hence

\[ \epsilon_W = \frac{1}{\pi} \epsilon_R \epsilon_S. \] (5.6)

Let us try to make this relation plausible. In the weak coupling limit, the orbifold becomes small. Hence, one expects to extract the weak coupling part of the full background by performing an orbifold average. We recall that the massive terms in the full background are of order \( \epsilon_R \epsilon_S \). In addition, we will find that those massive modes decay exponentially as one moves away from the orbifold planes, at a rate set by the Calabi–Yau radius \( v^{1/6} \). Therefore, when performing the average, one picks up another factor of \( \epsilon_R \) leading to \( \epsilon_R^2 \epsilon_S \) as the order of the averaged massive
terms. This is in perfect agreement with the expectation, \((5.6)\), from the weakly coupled heterotic string.

### 5.2 Basic Equations and Zeroth-Order Background

The M–theory vacuum is given in the 11-dimensional limit by specifying the metric \(g_{IJ}\) and the three-form \(C_{IJK}\) with field strength \(G_{IJKL} = 24 \partial[J \Gamma_{KLM}]\). To the order \(\kappa^{2/3}\), the set of equations to be solved consists of the Killing spinor equation

\[
\delta \Psi_I = D_I \eta + \sqrt{2} \frac{288}{8} (\Gamma_{IJKLM} - 8g_{IJ} \Gamma_{KLM}) G^{JKLM} \eta = 0 ,
\]

for a Majorana spinor \(\eta\), the equation of motion for \(G\)

\[
D_I G^{IJKL} = 0
\]

and the Bianchi identity

\[
(dG)_{IJJ\bar{K}L} = 2\sqrt{2} \pi \left( \frac{\kappa}{4\pi} \right)^{2/3} \left[ J^{(0)}(x^{11}) + J^{(N+1)}(x^{11} - \pi \rho) + \sum_{n=1}^{N} \frac{1}{2} J^{(n)}(x^{11} - x_n) + \delta(x^{11} + x_n) \right] _{IJKL} . \tag{5.9}
\]

Here the sources \(J^{(0)}\) and \(J^{(N+1)}\) on the orbifold planes are as usual given by

\[
J^{(0)} = -\frac{1}{8\pi^2} \left( \text{tr} F^{(1)} \wedge F^{(1)} - \frac{1}{2} \text{tr} R \wedge R \right) |_{x^{11}=0} ,
\]

\[
J^{(N+1)} = -\frac{1}{8\pi^2} \left( \text{tr} F^{(2)} \wedge F^{(2)} - \frac{1}{2} \text{tr} R \wedge R \right) |_{x^{11}=\pi \rho} . \tag{5.10}
\]

We have also introduced \(N\) additional sources \(J^{(n)}\), \(n = 1, \ldots, N\). They come from \(N\) five-branes located at \(x^{11} = x_1, \ldots, x_N\) where \(0 \leq x_1 \leq \cdots \leq x_N \leq \pi \rho\) (see fig. 1). Note that each five-brane at \(x^{11} = x_n\) has to be paired with a mirror five-brane at \(x^{11} = -x_n\) with the same source since the Bianchi identity must be even under the \(Z_2\) orbifold symmetry. Our normalization is such that the total source of each pair is \(J^{(n)}\). The structure of these five-brane sources will be discussed below.

We are interested in finding solutions of these equations that preserve 3 + 1-dimensional Poincaré invariance and admit a Killing spinor \(\eta\) corresponding to four preserved supercharges and, hence, \(N = 1\) supersymmetry in four dimensions.

---

\(^4\)There is no such comparison for the massless modes as they correspond to trivial integration constants on the weakly coupled side which can be absorbed into a redefinition of the moduli. This will be explained in detail later on.

\(^5\)Here we are using the normalization given in ref. [2]. Conrad [19] has argued that the correct normalization is smaller. In that case, the coefficient of the right-hand side of the Bianchi identity \((5.9)\) and eqns. \((5.20)\) and \((5.26)\) below are all multiplied by \(2^{-1/3}\). Furthermore, the definition of \(\epsilon_S\) in eqn. \((5.2)\) should also be multiplied by \(2^{-1/3}\).
The usual procedure to find such solutions is to solve the equations perturbatively. One starts by choosing a space $S^1/Z_2 \times X \times M_4$, where $X$ is a Calabi–Yau three-fold with a Ricci-flat metric $g_{AB}$, admitting a Killing spinor $\eta^{(\text{CY})}$. To lowest order, the solution, denoted in the following by $(0)$, is then given by

$$d\!s^{(0)}{}^2 \equiv g^{(0)}_{IJ} dx^I dx^J = \eta_{\mu \nu} dx^\mu dx^\nu + g_{AB} dx^A dx^B + (dx^{11})^2$$

$$G^{(0)}_{IJKL} = 0$$

$$\eta^{(0)} = \eta^{(\text{CY})}.$$  \hfill (5.11)

Note that it is consistent, to this order, to set the antisymmetric tensor field to zero since the sources in the Bianchi identity are proportional to $\kappa^{2/3}$ and, hence, first order in $\epsilon_S$.

One must also ensure that the theories on the orbifold planes preserve supersymmetry. This leads to the familiar constraint, following from the vanishing of the supersymmetry variation of the gauginos, that

$$\Gamma^{IJ} F^{(1)}_{IJ} \eta|_{x^{11}=0} = \Gamma^{IJ} F^{(2)}_{IJ} \eta|_{x^{11}=\pi\rho} = 0.$$  \hfill (5.12)

As discussed in [49], this implies that each $E_8$ gauge field is a holomorphic gauge bundle over the Calabi–Yau three-fold, satisfying the Donaldson–Uhlenbeck–Yau condition. The holomorphicity implies that $F^{(1)}_{AB}$ and $F^{(2)}_{AB}$ are (1,1)-forms. It follows that, since $R_{AB}$ for a Calabi–Yau three-fold is also a (1,1)-form, the orbifold sources $J^{(0)}$ and $J^{(N+1)}$, defined by eq. (5.10), are closed (2,2)-forms.

For the five-brane world-volume theory to be supersymmetric, the branes must be embedded in the Calabi–Yau space in a particular way [3]. To preserve Lorentz invariance in $M_4$, they must
span the 3 + 1-dimensional uncompactified space. The remaining spatial dimensions must then be wrapped on a two-cycle in the Calabi–Yau space. The condition of supersymmetry implies that the cycle is a holomorphic curve $\mathcal{C}_2$ [3, 68, 69]. As we will show in section 4.2, in such a situation, we preserve four supercharges on the five-brane worldvolume corresponding to $\mathcal{N} = 1$ supersymmetry in four dimensions. Since the five-branes are magnetic sources for $G$, they enter the right-hand side of the Bianchi identity (5.9) as source terms, which should be localized on the five-brane world-volumes. The delta function in $x^{11}$ gives the localization in the orbifold direction, while the four-forms $J^{(n)}$ must give the localization of the $n$-th five-brane on the two-cycle $\mathcal{C}_2^{(n)}$. Explicitly, for any two-cycle $\mathcal{C}_2$, one can introduce a delta-function four-form $\delta(\mathcal{C}_2)$, defined in the usual way, such that for any two-form $\chi$,

$$\int_X \chi \wedge \delta(\mathcal{C}_2) = \int_{\mathcal{C}_2} \chi, \quad (5.13)$$

so that $\delta(\mathcal{C}_2)$ is localized on $\mathcal{C}_2$. In general, we would expect that $J^{(n)}$ is proportional to $\delta(\mathcal{C}_2^{(n)})$. In fact, the correct normalization of the five-brane magnetic charge [70, 50] implies that the two are equal, that is

$$J^{(n)} = \delta(\mathcal{C}_2^{(n)}). \quad (5.14)$$

Since the cycles are holomorphic, $J^{(n)}$, like the orbifold sources, are closed (2,2)-forms.

There is one further condition which the five-branes and the fields on the orbifold planes must satisfy. This is a cohomology condition on the Bianchi identity [3]. Consider integrating the identity (5.9) over a five-cycle which spans the orbifold interval together with an arbitrary four-cycle $\mathcal{C}_4$ in the Calabi–Yau three-fold. Since $dG$ is exact, this integral must vanish. Physically this is the statement that there can be no net charge in a compact space, since there is nowhere for the flux to “escape”. Performing the integral over the orbifold, we derive, using (5.9), the condition

$$-\frac{1}{8\pi^2} \int_{\mathcal{C}_4} \text{tr} F^{(1)} \wedge F^{(1)} - \frac{1}{8\pi^2} \int_{\mathcal{C}_4} \text{tr} F^{(2)} \wedge F^{(2)} + \frac{1}{8\pi^2} \int_{\mathcal{C}_4} \text{tr} R \wedge R + \sum_{n=1}^{N} \int_{\mathcal{C}_4} J^{(n)} = 0. \quad (5.15)$$

Hence, the net magnetic charge over $\mathcal{C}_4$ is zero. Equivalently, this implies that the sum of the sources must be cohomologically trivial, that is

$$\left[ \sum_{n=0}^{N+1} J^{(n)} \right] = 0. \quad (5.16)$$

Let us now return to the normalization of the five-brane charges. We note that in equation (5.15) the first three terms are all integers. They are topological invariants, giving the instanton numbers (second Chern numbers) of the two $E_8$ bundles and the instanton number (first Pontrjagin number) of the tangent bundle of the Calabi–Yau three-fold. Hence, the above constraint shows that $n_5(\mathcal{C}_4) = \text{third term}$.
Figure 2: Intersection of a five-brane wrapped on the holomorphic cycle $C_2^{(n)}$ and a four-cycle $C_4$. In this example the five-brane contributes two units of magnetic charge on $C_4$.

$$\sum_{n=1}^{N} \int_{C_4} J^{(n)}$$ must also be an integer. In fact, with the normalization given in eqn. (5.14), each $\int_{C_4} J^{(n)}$ is an integer. It is also a topological invariant, giving the intersection number $[71]$ of the $n$-th brane, on the two-cycle $C_2^{(n)}$, with the four-cycle $C_4$. This can be understood as follows (see fig. 2). The two cycles naturally intersect at points in the Calabi–Yau manifold. Thus in $C_4$, the five-brane appears as a set of point-like magnetic charges located at each intersection. The net contribution of the five-brane to the magnetic charge on $C_4$ is then the sum of the point charges, which is precisely the intersection number. Given the normalization of (5.14), each intersection contributes one unit of magnetic charge. We also note that, for a holomorphic curve $C_4$, since $C_2^{(n)}$ is holomorphic, it is a theorem [71] that the intersection number is always positive. This is related to the fact that only five-branes and not anti-five-branes are allowed if we are to preserve supersymmetry. In summary, the main point is that the normalization of the five-brane charge is such that each five-brane intersection with $C_4$ and each gauge instanton on the orbifold plane carry the same amount of magnetic charge [71, 50].

We can then rewrite the cohomology condition (5.15) on a particular holomorphic four-cycle $C_4$ as

$$n_1(C_4) + n_2(C_4) + n_5(C_4) = n_R(C_4)$$  \hspace{1cm} (5.17)

which states that the sum of the number of instantons on the two $E_8$ bundles and the sum of the intersection numbers of each five-brane with the four-cycle $C_4$, must equal the instanton number for the Calabi–Yau tangent bundle, a number which is fixed once the Calabi–Yau geometry is chosen.
In summary, we see that to define the zeroth-order background we must specify the following data

- a Calabi–Yau three-fold $X$,
- two holomorphic vector bundles over $X$, one for each fixed plane, satisfying the Donaldson–Uhlenbeck–Yau condition. In general, there is no constraint that these bundles correspond to the embedding of the spin-connection in the gauge connection,
- a set of five-branes, each spanning the uncompactified $3 + 1$ dimensional space and wrapping a holomorphic two-cycle in the Calabi–Yau space,
- the sum of the five-branes magnetic charges and the instanton numbers from the gauge bundles, must equal the tangent space instanton number of $X$, as in equation (5.17).

We can then proceed to calculate the first-order corrections to the background.

### 5.3 First-Order Background

As an expansion in $\epsilon$, we write the bulk fields and the Killing spinor as

$$
\begin{align*}
g_{IJ} &= g^{(0)}_{IJ} + g^{(1)}_{IJ} \\
C_{IJK} &= C^{(0)}_{IJK} + C^{(1)}_{IJK} \\
\eta &= \eta^{(0)} + \eta^{(1)}.
\end{align*}
$$

where the index $(0)$ refers to the uncorrected background, given in (5.11), and the index $(1)$ to the corrections to first order in $\epsilon$.

Expanding to this order in $\epsilon$, we get for the Killing spinor equation (5.7)

$$
\delta \Psi_I = D^{(0)}_I \eta^{(1)} - \frac{1}{8} \left( D^{(0)}_J g^{(1)}_{KI} - D^{(0)}_K g^{(1)}_{JI} \right) \Gamma^{JK} \eta^{(0)}
+ \frac{\sqrt{2}}{288} \left( \Gamma_{IJKLM} - 8 g^{(0)}_{IJ} \Gamma_{KLM} \right) G^{(1)JKLM} \eta^{(0)} = 0 \tag{5.19}
$$

and for the equation of motion for $G$ (5.8) and the Bianchi identity (5.9)

$$
D^{(0)}_I G^{(1)IJKL} = 0
$$

and

$$
(dG^{(1)})_{11IJKL} = 2 \sqrt{2} \pi \left( \frac{\kappa}{4\pi} \right)^{2/3} \left[ J^{(0)}(x^{11}) + J^{(N+1)}(x^{11} - \pi \rho) \right.
+ \frac{1}{2} \sum_{n=1}^{N} J^{(n)}(x^{11} - x_n) + \delta(x^{11} + x_n) \bigg]_{IJKL} \tag{5.20}
$$

First, we note that the only nonvanishing components of the antisymmetric tensor $G^{(1)}$ are $G^{(1)}_{abcd}$ and $G^{(1)}_{abc11}$. This follows from the Bianchi identity for $G^{(1)}$ in eq. (5.20) and the fact that all sources
$J^{(n)}$ are $(2,2)$ forms. For $G^{(1)}$ of this form, the Killing spinor equation has been analyzed in ref. [3]. It has been shown in that paper that the corrections first order in $\epsilon_S$ to the metric and Killing spinor should have the structure

$$g^{(1)}_{\mu\nu} = b\eta_{\mu\nu}, \quad g^{(1)}_{AB} = h_{AB}, \quad g^{(1)}_{11,11} = \gamma, \quad \eta^{(1)} = \psi\eta^{(0)}$$  \hspace{1cm} (5.21)

with orbifold and Calabi–Yau dependent functions $b$, $h_{AB}$, $\gamma$ and $\psi$. Furthermore, in [3] a consistent set of differential equations has been derived from eq. (5.19) which determines $b$, $h_{AB}$, $\gamma$ and $\psi$ in terms of $G^{(1)}$. An explicit solution for these differential equations in terms of the dual antisymmetric tensor $B$ defined by

$$\mathcal{H} = dB = *G^{(1)}$$  \hspace{1cm} (5.22)

was presented in ref. [13]. In the following, we adopt the harmonic gauge, $d^*B = 0$. Then, since the sources in the Bianchi identity (5.20) are $(2,2)$ forms, the only nonvanishing components of $B$ are

$$B_{\mu\nu\rho\sigma a\bar{b}} = \epsilon_{\mu\nu\rho\sigma} B_{ab}$$  \hspace{1cm} (5.23)

with $B_{a\bar{b}}$ a $(1,1)$ form on the Calabi–Yau space. Using the results of ref. [13], the Killing spinor equation (5.19) is solved by

$$h_{a\bar{b}} = \sqrt{2}i \left( B_{a\bar{b}} - \frac{1}{3} \omega_{a\bar{b}} B \right)$$

$$b = \frac{\sqrt{7}}{6} B$$

$$\gamma = -\frac{\sqrt{2}}{3} B$$

$$\psi = -\frac{\sqrt{2}}{24} B$$

$$G^{(1)}_{ABCD} = \frac{1}{2} \epsilon^{ABCDEF} \partial_{11} B^{EF}$$

$$G^{(1)}_{ABC11} = \frac{1}{2} \epsilon^{ABCDEF} \partial^D B^{EF}$$

where $B = \omega^{AB} B_{AB}$ and $\omega_{a\bar{b}} = -ig_{a\bar{b}}$ is the Kähler form. We have, therefore, explicitly expressed the complete background in terms of the $(1,1)$ form $B_{a\bar{b}}$. All that remains then is to determine this $(1,1)$ form, which can be done following the methods given in ref. [13]. In the harmonic gauge, which implies

$$D_A^{(0)} B^{AB} = 0,$$  \hspace{1cm} (5.25)
\( B_{AB} \) is determined from eq. (5.20) by solving

\[
(\Delta_X + D_{11}^2) B_{AB} = 2\sqrt{2} \pi \left( \frac{K}{4\pi} \right)^{2/3} \left[ *X J^{(0)}(x^{11}) + *X J^{(N+1)}(x^{11} - \pi \rho) \right. \\
\left. + \frac{1}{2} \sum_{n=1}^{N} *X J^{(n)}(\delta(x^{11} - x_n) + \delta(x^{11} + x_n)) \right]_{AB}.
\]  

(5.26)

where \( \Delta_X \) is the Laplacian and \(*_X\) the Hodge star operator on the Calabi–Yau space. Essentially, this is the equation for a potential between a set of charged plates positioned through the orbifold interval at the fixed planes and the five-brane locations. The charge is not uniform over the Calabi–Yau space. To find a solution, following ref. [13] we introduce eigenmodes \( \pi_{i\bar{a}b} \) of this Laplacian with eigenvalues \(-\lambda_i^2\) so that

\[
\Delta_X \pi_{i\bar{a}b} = -\lambda_i^2 \pi_{i\bar{a}b}.
\]  

(5.27)

Generically, \( \lambda_i \) is of order \( v^{-1/6} \). The metric on the space of eigenmodes

\[
G_{ij} = \frac{1}{2v} \int_X \pi_i \wedge (\ast \pi_j)
\]  

(5.28)

is used to raise and lower \( i \)-type indices. Particularly relevant are the massless modes with \( \lambda_i = 0 \), which are precisely the \( h^{1,1} \) harmonic \((1,1)\) forms of the Calabi–Yau space. We will also denote these harmonic \((1,1)\) forms by \( \omega_{iAB} \). In the following, in order to distinguish between massless and massive modes, we will use indices \( i_0, j_0, k_0, \ldots = 1, \ldots, h^{1,1} \) for the former and indices \( \hat{i}, \hat{j}, \hat{k}, \ldots \) for the latter, while we continue to use \( i, j, k, \ldots \) for all modes. Let us now expand the sources in terms of the eigenfunctions as

\[
_*X J^{(n)} = \frac{1}{2v^{2/3}} \sum_i \beta_i^{(n)} \pi_i
\]  

(5.29)

where

\[
\beta_i^{(n)} = \frac{1}{v^{1/3}} \int_X \pi_i \wedge J^{(n)}.
\]  

(5.30)

If we introduce four-cycles \( C_{4i0} \) dual to the harmonic \((1,1)\) forms \( \omega_{i0} \), we can write for the massless modes

\[
\beta_i^{(n)} = \int_{C_{4i0}} J^{(n)}.
\]  

(5.31)

Specifically, it follows from (5.11) that \( \beta_i^{(0)} \) and \( \beta_i^{(N+1)} \) represent the instanton numbers of the gauge fields on the orbifold planes minus half the instanton number of the tangent bundle and, hence, it would appear, are in general half-integer. However, since \( M_{11} \) must be a spin manifold (since it must admit spinors), the tangent bundle instanton number must be divisible by two \([50]\) and so
\( \beta_{i_0}^{(0)} \) and \( \beta_{i_0}^{(N+1)} \) are, in fact, integer. Furthermore, \( \beta_{i_0}^{(n)} \), \( n = 1, \ldots, N \) are the five-brane charges, given by the intersection number of each five-brane with the cycle \( C_{i_0} \), and are also integers. Let us also expand \( B_{AB} \) in terms of eigenfunctions as

\[
B_{AB} = \sum_i b_i \pi_{AB} \tag{5.32}
\]

Then inserting this expansion, together with the expression (5.29) for the sources, into eq. (5.26), it is straightforward to obtain

\[
\left( \partial_{11}^2 - \lambda_i^2 \right) b_i = \frac{\sqrt{2} \epsilon_S}{\rho} \left[ \beta_i^{(0)} \delta(x^{11}) + \beta_i^{(N+1)} \delta(x^{11} - \pi \rho) 
\right.
\]

\[
+ \frac{1}{2} \sum_{n=1}^{N} \beta_i^{(n)} \left( \delta(x^{11} - x_n) + \delta(x^{11} + x_n) \right) \right] \tag{5.33}
\]

It is then easy to solve these equation to give an explicit solution for the massive and massless modes. We note that the size of the sources is set by \( \epsilon_S / \rho \) which, from eq. (5.2), is independent of the size of the orbifold. We first solve eq. (5.33) for the massive modes, that is, for \( \lambda_i \neq 0 \). In terms of the normalized orbifold coordinates

\[
z = \frac{x^{11}}{\pi \rho}, \quad z_n = \frac{x_n}{\pi \rho}, \quad n = 1, \ldots, N, \tag{5.34}
\]

\( z_0 = 0 \) and \( z_{N+1} = 1 \), we find

\[
b_i = \frac{\pi \epsilon_S}{\sqrt{2}} \delta_i \left[ \left( \sum_{m=0}^{n} c_{i,m} \beta_i^{(n)} \right) \sinh(\delta_i^{-1} |z|) 
\right.
\]

\[
+ \left( \sum_{m=n+1}^{N+1} s_{i,m} \beta_i^{(m)} - \frac{c_{i,N+1}}{s_{i,N+1}} \sum_{m=0}^{N+1} c_{i,m} \beta_i^{(m)} \right) \cosh(\delta_i^{-1} |z|) \right] \tag{5.35}
\]

in the interval

\[ z_n \leq |z| \leq z_{n+1}, \]

for fixed \( n \), where \( n = 0, \ldots, N \). Here we have defined

\[
\delta_i = \frac{1}{\pi \rho \lambda_i}, \quad c_{i,n} = \cosh(\delta_i^{-1} z_n), \quad s_{i,n} = \sinh(\delta_i^{-1} z_n). \tag{5.36}
\]

Note that, since the eigenvalues \( \lambda_i \) are of order \( v^{-1/6} \), the quantities \( \delta_i \) defined above are of order \( \epsilon_R \). Therefore, as already stated, the size of the massive modes is set by \( \epsilon_R \epsilon_S \).

We now turn to the massless modes. First note that, in order to have a solution of (5.33), we must have

\[
\sum_{n=0}^{N+1} \beta_{i_0}^{(n)} = 0. \tag{5.37}
\]
However, from the definition (5.31), we see that this is, of course, none other than the cohomology condition (5.16) described above, and so is indeed satisfied. Integrating eq. (5.33) for \( \lambda_i = 0 \) we then find

\[
b_{i0} = \frac{\pi \epsilon_S}{\sqrt{2}} \left[ \sum_{m=0}^{n} \beta_{i0}^{(m)} (|z| - z_m) - \frac{1}{2} \sum_{m=0}^{N+1} (z_m^2 - 2z_m) \beta_{i0}^{(m)} \right] (5.38)
\]

in the interval

\[z_n \leq |z| \leq z_{n+1},\]

for fixed \( n \), where \( n = 0, \ldots, N \). As already discussed, the massless modes are of order \( \epsilon_S \) and, unlike for the massive modes, no additional factor of \( \epsilon_R \) appears.

It is important to note that there could have been an arbitrary constant in the zero-mode solutions. However, such a constant can always be absorbed into a redefinition of the Calabi–Yau zero modes or, correspondingly, the low energy fields. Consequently, in the solution (5.38) we have fixed the constant by taking the orbifold average of the solution to be zero. This will be important later in deriving low-energy effective actions.

Before we discuss the implications of these equations in detail, let us summarize our results. We have constructed heterotic M–theory backgrounds with non-standard embeddings including the presence of bulk five-branes. We started with a standard Calabi–Yau background with gauge fields and five-branes to lowest order and showed that corrections to it can be computed in a double expansion in \( \epsilon_S \) and \( \epsilon_R \). Explicitly, we have solved the problem to linear order in \( \epsilon_S \) and to all orders in \( \epsilon_R \). We found the massive modes to be of order \( \epsilon_R \epsilon_S \) while the massless modes are of order \( \epsilon_S \). Therefore, although one could have expected corrections of arbitrary power in \( \epsilon_R \), we only find zeroth- and first-order contributions at the linear level in \( \epsilon_S \). Concentrating on the leading order massless modes, in each interval between two five-branes, \( z_n \leq |z| \leq z_{n+1} \), the massless modes vary linearly with a slope proportional to the total charge \( \sum_{m=0}^{n} \beta_{i0}^{(m)} \) to the left of the interval. (Note that the total charge to the right of the interval has the same magnitude but opposite sign due to eq. (5.37).) At the five-brane locations, the linear pieces match continuously but with kinks which lead to the delta-function sources when the second derivative is computed. (A specific example is given in section 4.1, see fig. [3].) Similar kinks appear for the massive modes which, however, vary in a more complicated way between each pair of five-branes.

6 Backgrounds Without Five-Branes

In this section, we will restrict the previous general solutions to the case of pure non-standard embedding without additional five-branes and discuss some properties of such backgrounds and the resulting low-energy effective actions in both four and five dimensions.
6.1 Properties of the Background

To specialize to the case without five-branes, we set \( N = 0 \) and recall that \( z_0 = 0 \) and \( z_1 = 1 \). Also, the vanishing cohomology condition \( (5.37) \) implies that we have only one independent charge

\[
\beta_{i0} \equiv \beta_{i0}^{(0)} = -\beta_{i0}^{(1)}
\]

per mode. Using this information to simplify eq. \( (5.38) \), we find for the massless modes

\[
b_{i0} = \frac{\pi \epsilon S}{\sqrt{2}} \beta_{i0} \left( |z| - \frac{1}{2} \right).
\]

(6.2)

In the same way, we obtain from eq. \( (5.35) \) for the massive modes

\[
b_i = \frac{\pi \epsilon S}{\sqrt{2}} \delta_i \left[ (\beta_i^{(0)} - \beta_i^{(1)}) \frac{\sinh(\delta_i^{-1}(|z| - 1/2))}{2 \cosh(\delta_i^{-1}/2)} - (\beta_i^{(0)} + \beta_i^{(1)}) \frac{\cosh(\delta_i^{-1}(|z| - 1/2))}{2 \sinh(\delta_i^{-1}/2)} \right].
\]

(6.3)

Note that, unlike for the massless modes, here we have no relation between the coefficients \( \beta_i^{(0)} \) and \( \beta_i^{(1)} \). Let us compare these results to the case of the standard embedding \[13\]. We see that the massless modes solution is, in fact, completely unchanged in form from the the standard embedding case, though the parameter \( \beta_{i0} \) can be different. This is a direct consequence of the cohomology condition \( (5.37) \) which, for the simple case without five-branes, tells us that the instanton numbers on the two orbifold planes always have to be equal and opposite. There is no similar condition for the massive modes and we therefore expect a difference from the standard embedding case. Indeed, the standard embedding case is obtained from eq. \( (6.3) \) by setting \( \beta_i^{(0)} + \beta_i^{(1)} = 0 \) so that the second term vanishes. As was noticed in ref. \[13\], the first term in eq. \( (6.3) \) vanishes at the middle of the interval \( z = 1/2 \) for all modes. Hence, for the standard embedding, at this point the space-time background receives no correction and, in particular, the Calabi–Yau space is undeformed.

We see that the second term in eq. \( (6.3) \) does not share this property. Therefore, for non-standard embeddings, there is generically no point on the orbifold where the space-time remains uncorrected.

Furthermore, we see that the massive modes depend on the combination \( \delta_i^{-1} z \) only. Therefore, in terms of the normalized orbifold coordinate \( z \) (the orbifold coordinate \( x^{11} \)), the massive modes indeed fall off exponentially with a scale set by \( \delta_i \) (by \( v^{1/6} \)). In fact, as might be expected, we see that this part of the solution is essentially independent of the size of the orbifold. Averaging the above expression for the massive modes over the orbifold, one should pick up the corresponding weak coupling correction. Clearly, as a consequence of the exponential fall-off, the averaging procedure leads to an additional suppression by \( \epsilon_R \). Given that the order of a heavy mode is \( \epsilon_R \epsilon_S \), we conclude that its average is of the order \( \epsilon_W^2 \). According to eq. \( (5.3) \), this is just \( \epsilon_W \) and, hence, the expected weak coupling expansion parameter.
6.2 Low-Energy Effective Actions

What are the implications of the above results for the low-energy effective action? Since the orbifold is expected to be larger than the Calabi–Yau radius, it is natural to first reduce to a five-dimensional effective theory consisting of the usual $3 + 1$ space-time dimensions and the orbifold and, subsequently, reduce this theory further down to four dimensions. First, we should explain how a background appropriate for a reduction to $\mathcal{N} = 1$ supersymmetry in four dimensions can be used to derive a sensible $\mathcal{N} = 1$ theory in five dimensions \cite{13}. The point is that, as we have seen, the background can be split into massless and massive eigenmodes. Reducing from eleven to five dimensions on an undeformed Calabi–Yau background, these correspond to massless moduli fields and heavy Kaluza–Klein modes. Working to linear order in $\epsilon_S$, the heavy modes completely decouple from the massless modes and so can essentially be dropped. The background then appears as a particular solution to the five-dimensional effective action, where the moduli depend non-trivially on the orbifold direction. Thus, in summary, to derive the correct five-dimensional action, we need only keep the massless modes in a reduction on an undeformed Calabi–Yau space. However, a similar procedure is not possible for the topologically non-trivial components $G^{(1)}_{\alpha\beta\gamma\delta}$ of the antisymmetric tensor field strength. Such a configuration of the internal field strength is not a modulus, but rather a non-zero mode. As a consequence, the proper five-dimensional theory is obtained as a reduction on an undeformed Calabi–Yau background but including non-zero modes for $G$. It is these non-zero modes which introduce all the interesting structure into the theory, notably, that in the bulk we have a gauged supergravity and that the theory admits no homogeneous vacuum. In the case at hand, the precise structure of the non-zero mode can be directly read off from the background as presented.

Let us now briefly review the results of such a reduction for the standard embedding as presented in ref. \cite{4, 5} and discussed in Lecture 1. It was found that the five-dimensional effective action consists of a gauged $\mathcal{N} = 1$ bulk supergravity theory with $h^{1,1} - 1$ vector multiplets and $h^{2,1} + 1$ hypermultiplets coupled to four-dimensional $\mathcal{N} = 1$ boundary theories. The field content of the orbifold plane at $x^{11} = 0$ consists of an $E_6$ gauge multiplet and $h^{1,1}$ and $h^{2,1}$ chiral multiplets, while the plane at $x^{11} = \pi \rho$ carries $E_8$ gauge multiplets only. The gauging of the bulk supergravity is with respect to a $U(1)$ isometry in the universal hypermultiplet coset space with the gauge field being a certain linear combination of the graviphoton and the vector fields in the vector multiplets. The gauging also leads to a bulk potential for the $(1,1)$ moduli. In addition, there are potentials for the $(1,1)$ moduli confined to the orbifold planes which have opposite strength. As we have mentioned, the characteristic features of this theory, such as the gauging and the existence of the potentials, can be traced back to the existence of the non-zero mode. Furthermore, the vacuum

\footnote{By $\mathcal{N} = 1$ in five dimensions we mean a theory with eight supercharges. In four dimensions, $\mathcal{N} = 1$ means a theory with four supercharges.}

38
solution of this five-dimensional theory, appropriate for a reduction to four dimensions, was found to be a double BPS domain wall with the two worldvolumes stretched across the orbifold planes.

Which of the above features generalize to non-standard embeddings? The spectrum of zero mode fields in the bulk will, of course, be unchanged. Due to the nonstandard embedding, we can have more general gauge multiplets with groups \( G^{(1)}, G^{(2)} \subset E_8 \) on the orbifold planes and also corresponding observable and hidden sector matter transforming under these groups. We are interested in the effective action up to linear order in \( \epsilon_s \). It is clear that, as above, to this order, the massive part of the background completely decouples from the low-energy effective action since the massless and massive eigenfunction on the Calabi–Yau space are orthogonal \(^{[13]}\). Hence, the form of the effective action to linear order in \( \epsilon_s \) is completely determined by the massless part of the background. On the other hand, due to the cohomology condition \(^{[5.37]}\), the form of the massless part of the background corrections is same as in the standard embedding case, as we have just shown. Hence, in deriving the five-dimensional effective action for non-standard embedding, we use the same non-zero mode in the reduction as for the standard embedding. This will lead to gauging and bulk and boundary potentials exactly as in the standard embedding case.

Let us explain these last facts in some more detail. First, we identify the non-zero mode of \( G \) in the case of non-standard embedding. Inserting the mode \(^{[6.2]}\) into the expansion for \( B_{AB} \), eq. \(^{[5.32]}\), we can use eq. \(^{[5.24]}\) to compute the four-form field strength \( G^{(1)} \). While the massless part of \( G^{(1)}_{ABC11} \) vanishes, we find for the massless part of \( G^{(1)}_{ABCD} \)

\[
G^{(1)} = \frac{1}{2V} \omega_i \alpha_i^0 \epsilon(x^{11})
\]

where \( V \) is the Calabi–Yau volume modulus defined by

\[
V = \frac{1}{2\pi \rho v} \int_{X \times S^1/Z_2} \sqrt{|g|}
\]

and we have introduced the parameter

\[
\alpha_i^0 = \frac{\sqrt{2} \epsilon_S}{\rho} \beta_i^0.
\]

to conform with the notation of \(^{[4.5]}\). Furthermore, \( \epsilon(x^{11}) \) is the stepfunction which is +1 for positive \( x^{11} \) and −1 otherwise. Eq. \(^{[6.4]}\) is precisely the non-zero mode we have mentioned above. Note that \( V \) measures the orbifold average of the Calabi–Yau volume in units of \( v \). In general, the parameters \( \alpha_i^0 \) depend on the choice of both the tangent and the gauge bundles. Explicitly, from eqs. \(^{[5.10]}\), \(^{[5.31]}\) and the cohomology condition \(^{[5.37]}\), we have, for general embeddings,

\[
\alpha_i^0 = -\frac{\epsilon_S}{4\sqrt{2}\pi^2 \rho} \int_{C_{4i0}} \left( \text{tr} F^{(1)} \wedge F^{(1)} - \frac{1}{2} \text{tr} R \wedge R \right) = \frac{\epsilon_S}{4\sqrt{2}\pi^2 \rho} \int_{C_{4i0}} \left( \text{tr} F^{(2)} \wedge F^{(2)} - \frac{1}{2} \text{tr} R \wedge R \right).
\]
In the case of the standard embedding, the tangent bundle and one of the $E_8$ gauge bundles are identified, while the other gauge bundle is taken to be trivial, so that this reduces to
\[ \alpha_{i_0} = -\frac{\epsilon_S}{8\sqrt{2\pi^2\rho}} \int_{C_{i_0}} \text{tr}R \wedge R . \] (6.8)

This is the relation given in ref. [3]. The point is that the expression for the non-zero mode (6.4) has the same form for both standard and non-standard embeddings. All that changes are the values of the parameters $\alpha_{i_0}$.

Now let us demonstrate how the gauging of the bulk supergravity arises in the case of non-standard embedding. Consider the five-dimensional three-form zero–mode $C_5$, with field strength $G_5$, and the part of the 11–dimensional three-form that leads to the $h^{1,1}$ vector fields $A^{i_0}$, namely $C = A^{i_0} \wedge \omega_{i_0}$. Inserting these two fields, together with the non-zero mode (6.4), into the Chern–Simons term in the eleven-dimensional supergravity action [5] leads to
\[ \int_{M_{11}} C \wedge G \wedge G \sim \int_{M_5} \epsilon(x^{11}) \alpha_{i_0} A^{i_0} \wedge G_5 . \] (6.9)

The three-form $C_5$ can be dualized to a scalar in five dimensions, which becomes one of the four scalars $q^u$ in the universal hypermultiplet. Then, the above term directly causes the gauging of the isometry in the hypermultiplet coset space that corresponds to the axionic shift in the dual scalar. The gauging is with respect to the linear combination $\alpha_{i_0} A^{i_0}$. Explicitly, we find [5] that the universal hypermultiplet kinetic term is of the form
\[ \int_{M_5} \sqrt{-g} h_{uv} D_\alpha q^u D^\alpha q^v \] (6.10)

with the covariant derivative
\[ D_\alpha q^u = \partial_\alpha q^u + \epsilon(a^{11}) \alpha_{i_0} A^{i_0} k^u \] (6.11)

where $k^u$ is a Killing vector in the hypermultiplet sigma-model manifold, pointing in the direction of the axionic shift. We see that, since the non-zero mode (6.4) had the same form for both standard and non-standard embeddings, the gauging of the supergravity also has the same form. The only difference is in the values of the charges $\alpha_{i_0}$.

Similarly, the bulk potential should have the same form in the standard and non-standard embedding cases. Inserting the non-zero mode (6.4) into the kinetic term $G \wedge *G$ of the four-form field strength in the eleven-dimensional supergravity action leads to a bulk potential for the volume modulus $V$ and the other $(1,1)$ moduli. More precisely, one finds
\[ \int_{M_{11}} G^{(1)} \wedge *G^{(1)} \sim \int_{M_5} \sqrt{-g} V^{-2} \alpha_{i_0} \alpha_{j_0} \tilde{G}^{i_0 j_0} \] (6.12)

where
\[ \tilde{G}^{i_0 j_0} = V^{2/3} G^{i_0 j_0} \] (6.13)
is a renormalized metric that depends on the Calabi–Yau shape moduli (see ref. [3] for details). Note that it follows from supersymmetry that such a potential must arise when an isometry of the universal hypermultiplet sigma-model manifold is gauged.

The potentials on the orbifold planes arise from the ten-dimensional actions on the planes, with the internal gauge fields and curvature inserted. Using identities of the form

\[ \int_X \omega \wedge \text{tr} R \wedge R \sim \int_X \sqrt{-g} \text{tr} R^2 \]  

we find

\[ \sum_{n=1}^{2} \int_{M_{10}^{(n)}} \sqrt{-g} \left( \text{tr}(F^{(n)})^2 - \frac{1}{2} \text{tr} R^2 \right) \sim \int_{M_{4}^{(1)}} \sqrt{-g} V^{-1} \alpha_{i_0} b_{i_0} + \int_{M_{4}^{(2)}} \sqrt{-g} V^{-1} \alpha_{i_0} b_{i_0} \]  

(6.15)

where \( b_{i_0} \) are the Kähler shape moduli defined by the expansion of the Kähler form \( \omega = V^{1/3} b_{i_0} \omega_{i_0} \).

As for the standard embedding case, the potentials come out with opposite strength, again a consequence of the cohomology condition \( [3] \), \( \beta_{i_0}^{(0)} = -\beta_{i_0}^{(1)} \).

In summary, we conclude that the five-dimensional effective action derived in ref. [4, 5] for the standard embedding is, in fact, much more general and applies, with appropriate adjustment of the boundary field content and the charges \( \alpha_{i_0} \), to any Calabi–Yau-based non-standard embedding without additional five-branes. Furthermore, the double domain wall vacuum solution of the five-dimensional theory is unchanged, since it does not depend on the field content on the orbifold planes.

The four-dimensional theory is obtained as a reduction on this domain wall. Hence, the four-dimensional effective action will be unchanged in the case of non-standard embeddings without five-branes, except for the possibility of more general gauge groups and matter multiplets. One further new feature, in the case of non-standard embedding, is the possibility of gauge matter on the hidden orbifold plane. In this case, the threshold-like correction to the matter part of the Kähler potential will be different for observable and hidden sectors in the same way the gauge kinetic functions of the two sectors differ.

To be more concrete, let us consider the universal case with moduli \( S \) and \( T \), gauge fields of \( G^{(1)} \times G^{(2)} \subset E_8 \times E_8 \) and corresponding gauge matter \( C^{(1)} \) and \( C^{(2)} \), transforming under \( G^{(1)} \) and \( G^{(2)} \), respectively. Then, we have for the Kähler potential and the gauge kinetic functions

\[
\begin{align*}
K &= -\log(S + \bar{S}) - 3 \log(T + \bar{T}) + Z_1 |C^{(1)}|^2 + Z_2 |C^{(2)}|^2 \\
Z_1 &= \frac{3}{T + \bar{T}} + \frac{\pi \epsilon S \beta}{S + \bar{S}} \\
Z_2 &= \frac{3}{T + \bar{T}} - \frac{\pi \epsilon S \beta}{S + \bar{S}} \\
f^{(1)} &= S + \pi \epsilon S \beta T \\
f^{(2)} &= S - \pi \epsilon S \beta T
\end{align*}
\]  

(6.16)
where $\beta$ is the single instanton charge, of the type defined in eqn. (6.1), corresponding to the universal Kähler deformation. For vacua based on the standard embedding, it was pointed out in ref. [3] that, if $\beta > 0$ so that the smaller of the two couplings corresponds to the observable sector, then, fitting this to the grand unification coupling, the larger coupling is of order one at the “physical” point. Hence, gaugino condensation in the hidden sector appears to be a likely scenario. We have just shown that, in fact, this statement continues to apply to all Calabi–Yau based non-standard embedding vacua without additional bulk five-branes, provided $\beta > 0$, since the gauge kinetic functions are completely unchanged. Gaugino condensation, therefore, appears to be a generic possibility for such vacua.

7 Backgrounds with Five-Branes

Let us now turn to the much more interesting case of non-standard embeddings with five-branes in the bulk. We will concentrate on the massless modes, since, as above, it is these modes which will determine the low-energy action.

7.1 Properties of the Background

The general solution (5.38) for the massless modes shows a linear behaviour for each interval between two five-branes. The slope, however, varies from interval to interval in a way controlled by the five-brane charges. The same statement applies to the variation of geometrical quantities, like the Calabi–Yau volume, across the orbifold. Let us consider an example for a certain massless mode $b$. Four five-branes with charges $(\beta^{(1)}, \beta^{(2)}, \beta^{(3)}, \beta^{(4)}) = (1, 1, 1, 1)$ are positioned at $(z_1, z_2, z_3, z_4) = (0.2, 0.6, 0.8, 0.8)$. Note that the third and fourth five-brane are coincident. The instanton numbers on the orbifold planes are chosen to be $(\beta^{(0)}, \beta^{(4)}) = (-1, -3)$. Note that the total charge sums up to zero as required by the cohomology constraint (5.37). The orbifold dependence of $(\sqrt{2}/\pi\epsilon_S)b$ is depicted in fig. 3. It is clear that the additional five-brane charges introduce much more freedom as compared to the case without five-branes. For example, while in the latter case one always has $b(0) = -b(1)$ leading to equal, but opposite, gauge threshold corrections, the example in fig. 3 shows that $b(0), b(1) > 0$ is possible. One, therefore, expects the thresholds in the low-energy gauge kinetic functions to change. This will be analyzed in a moment. Another interesting phenomenon in the above example is that the mode is constant between the first and second five-brane. This is a direct consequence of our choice of the charges which sum up to zero both to the left and the right of this interval. If such a property is arranged for all massless modes, the Calabi–Yau volume remains exactly constant throughout this interval.
Figure 3: Orbifold dependence of a massless mode \((\sqrt{2}/\pi \epsilon_S)b\) for four five-branes at \((z_1, z_2, z_3, z_4) = (0.2, 0.6, 0.8, 0.8)\) with charges \((\beta^{(1)}, \beta^{(2)}, \beta^{(3)}, \beta^{(4)}) = (1, 1, 1, 1)\) and instanton numbers \((\beta^{(0)}, \beta^{(4)}) = (-1, -3)\).

### 7.2 Five-Branes on Calabi–Yau Two-Cycles

The inclusion of five-branes not only generalizes the types of background one can consider, but also introduces new degrees of freedom into the theory, namely, the dynamical fields on the five-branes themselves. In this section, we will consider what low-energy fields survive on one of the five-branes when it is wrapped around a two-cycle in the Calabi–Yau three-fold.

In general, the fields on a single five-brane are as follows \cite{72, 73}. The simplest are the bosonic coordinates \(X^I\) describing the embedding of the brane into 11-dimensional spacetime. The additional bosonic field is a world-volume two-form potential \(B\) with field strength \(H = dB\) satisfying a generalized self-duality condition. For small fluctuations, the duality condition simplifies to the conventional constraint \(H = *H\). These degrees of freedom are paired with spacetime fermions \(\theta\), leading to a Green–Schwarz type action, with manifest spacetime supersymmetry and local kappa-symmetry \cite{74, 75}. (As usual, including the self-dual field in the action is difficult, but is possible by either including an auxiliary field or abandoning a covariant formulation.) For a five-brane in flat space, one can choose a gauge such that the dynamical fields fall into a six-dimensional massless tensor multiplet with \((0, 2)\) supersymmetry on the brane world-volume \cite{76, 77}. The multiplet has five scalars describing the motion in directions transverse to the five-brane, together with the self-dual tensor \(H\).

For a five-brane embedded in \(S^1/Z_2 \times X \times M_4\), to preserve Lorentz invariance in \(M_4\), 3 + 1 dimensions of the five-brane must be left uncompactified. The remaining two spatial dimensions are
then wrapped on a two-cycle of the Calabi–Yau three-fold. To preserve supersymmetry, the two-cycle must be a holomorphic curve \[3, 68, 69]. Thus, from the point of view of a five-dimensional effective theory on \(S^1/Z_2 \times M_4\), since two of the five-brane directions are compactified, it appears as a flat three-brane (or equivalently domain wall) located at some point \(x^{11} = x\) on the orbifold. Thus, at low energy, the degrees of freedom on the brane must fall into four-dimensional supersymmetric multiplets.

An important question is how much supersymmetry is preserved in the low-energy theory. One way to address this problem is directly from the symmetries of the Green–Schwarz action, following the discussion for similar brane configurations in \[68\]. Locally, the 11-dimensional spacetime \(S^1/Z_2 \times X \times M_4\) admits eight independent Killing spinors \(\eta\), so should be described by a theory with eight supercharges. (Globally, only half of the spinors survive the non-local orbifold quotienting condition \(\Gamma_{11}\eta(-x^{11}) = \eta(x^{11})\), so that, for instance, the eleven-dimensional bulk fields lead to \(N = 1\), not \(N = 2\), supergravity in four dimensions.) The Green–Schwarz form of the five-brane action is then invariant under supertranslations generated by \(\eta\), as well as local kappa-transformations. In general the fermion fields \(\theta\) transform as (see for instance ref. \[77\])

\[
\delta \theta = \eta + P_+ \kappa \tag{7.1}
\]

where \(P_+\) is a projection operator. If the brane configuration is purely bosonic then \(\theta = 0\) and the variation of the bosonic fields is identically zero. Furthermore, if \(H = 0\) then the projection operator takes the simple form

\[
P_\pm = \frac{1}{2} \left(1 \pm \frac{1}{6!} \sqrt{g} \epsilon^{m_1 \cdots m_6} \partial_{m_1} X^{I_1} \cdots \partial_{m_6} X^{I_6} \Gamma_{I_1 \cdots I_6} \right) \tag{7.2}
\]

where \(\sigma^m, m = 0, \ldots, 5\) label the coordinates on the five-brane and \(g\) is the determinant of the induced metric

\[
g_{mn} = \partial_m X^I \partial_n X^J g_{IJ}. \tag{7.3}
\]

If the brane configuration is invariant for some combination of supertranslation \(\eta\) and kappa-transformation, then we say it is supersymmetric. Now \(\kappa\) is a local parameter which can be chosen at will. Since the projection operators satisfy \(P_+ + P_- = 1\), we see that for a solution of \(\delta \theta = 0\), one is required to set \(\kappa = -\eta\), together with imposing the condition

\[
P_- \eta = 0 \tag{7.4}
\]

For a brane wrapped on a two-cycle in the Calabi–Yau space, spanning \(M_4\) and located at \(x^{11} = x\) in the orbifold interval, we can choose the parameterization

\[
X^\mu = \sigma^\mu \quad X^A = X^A(\sigma, \bar{\sigma}) \quad X^{11} = x \tag{7.5}
\]
where $\sigma = \sigma^4 + i\sigma^5$. The condition (7.4) then reads

$$-\frac{i}{\sqrt{g}} \partial X^A \bar{\partial} X^B \Gamma^{(4)} \Gamma_{AB} \eta = \eta$$

(7.6)

where we have introduced the four-dimensional chirality operator $\Gamma^{(4)} = \Gamma_0 \Gamma_1 \Gamma_2 \Gamma_3$. Recalling that on the Calabi–Yau three-fold the Killing spinor satisfies $\Gamma^{\delta} \eta = 0$, it is easy to show that this condition can only be satisfied if the embedding is holomorphic, that is $X^a = X^a(\sigma)$, independent of $\bar{\sigma}$. The condition then further reduces to

$$\Gamma^{(4)} \eta = i \eta$$

(7.7)

which, given that the spinor has definite chirality in eleven dimensions as well as on the Calabi–Yau space, implies that $\Gamma^{11} \eta = \eta$, compatible with the global orbifold quotient condition. Thus, finally, we see that only half of the eight Killing spinors, namely those satisfying (7.7), lead to preserved supersymmetries on the five-brane. Consequently the low-energy four-dimensional theory describing the five-brane dynamics will have $\mathcal{N} = 1$ supersymmetry.

The simplest excitations on the five-brane surviving in the low-energy four-dimensional effective theory are the moduli describing the position of the five-brane in eleven dimensions. There is a single modulus $X^{11}$ giving the position of the brane in the orbifold interval. In addition, there is the moduli space of holomorphic curves $C_2$ in $X$ describing the position of the brane in the Calabi–Yau space. This moduli space is generally complicated, and we will not address its detailed structure here. (As an example, the moduli space of genus one curves in K3 is K3 itself [69].) However, we note that these moduli are scalars in four dimensions, and we expect them to arrange themselves as a set of chiral multiplets, with a complex structure presumably inherited from that of the Calabi–Yau manifold.

Now let us consider the reduction of the self-dual three-form degrees of freedom. (Here we are essentially repeating a discussion given in [78, 79].) The holomorphic curve is a Riemann surface and, so, is characterized by its genus $g$. One recalls that the number of independent harmonic one-forms on a Riemann surface is given by $2g$. In addition, there is the harmonic volume two-form $\Omega$. Thus, if we decompose the five-brane world-volume as $C_2 \times M_4$, we can expand $H$ in zero modes as

$$H = da \wedge \Omega + F^u \wedge \lambda_u + h$$

(7.8)

where $\lambda_u$ are a basis $u = 1, \ldots, 2g$ of harmonic one-forms on $C_2$, while the four-dimensional fields are a scalar $a$, $2g$ $U(1)$ vector fields $F^u = dA^u$ and a three-form field strength $h = db$. However, not all these fields are independent because of the self-duality condition $H = *H$. Rather, one easily concludes that

$$h = *da$$

(7.9)
and, hence, that the four-dimensional scalar $a$ and two-form $b$ describe the same degree of freedom. To analyze the vector fields, we introduce the matrix $T_{uv}$ defined by

$$\star \lambda_u = T_{uv}^v \lambda_v \quad (7.10)$$

If we choose the basis $\lambda_u$ such that the moduli space metric $\int C_2 \lambda_u \wedge (\star \lambda_v)$ is the unit matrix, $T$ is antisymmetric and, of course, $T^2 = -1$. The self-duality constraint implies for the vector fields that

$$F^u = T_{vu}^v \star F^v \quad (7.11)$$

If we choose a basis for $F^u$ such that

$$T = \text{diag} \left( \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right) \quad (7.12)$$

with $g$ two by two blocks on the diagonal, one easily concludes that only $g$ of the $2g$ vector fields are independent. In conclusion, for a genus $g$ curve $C_2$, we have found one scalar and $g U(1)$ vector fields from the two-form on the five-brane worldvolume. The scalar has to pair with another scalar to form a chiral $\mathcal{N} = 1$ multiplet. The only other universal scalar available is the zero mode of the transverse coordinate $X^{11}$ in the orbifold direction.

Thus, in general, the $\mathcal{N} = 1$ low-energy theory of a single five-brane wrapped on a genus $g$ holomorphic curve $C_2$, we have found one scalar and $g U(1)$ vector fields from the two-form on the five-brane worldvolume. The scalar has to pair with another scalar to form a chiral $\mathcal{N} = 1$ multiplet. The only other universal scalar available is the zero mode of the transverse coordinate $X^{11}$ in the orbifold direction.

It is well known that when two regions of the five-brane world-volume in M-theory come into close proximity, new massless states appear \cite{50, 57}. These are associated with membranes stretching between the two nearly overlapping five-brane surfaces. In general, this can lead to enhancement of the gauge symmetry. Let us now consider this possibility, heretofore ignored in our discussion. In general, one can consider two types of brane degeneracy where parts of the five-brane world-volumes are in close proximity. The first, and simplest, is to have $N$ distinct but coincident five-branes, all wrapping the same cycle $C_2$ in the Calabi–Yau space and all located at the same point in the orbifold interval. Here, the new massless states come from membranes stretching between the distinct five-brane world-volumes. The second, and more complicated, situation is where there is a degeneracy of the embedding of a single five-brane, such that parts of the curve $C_2$ become close together in the Calabi–Yau space. In this case, the new states come from membranes stretching between different parts of the same five-brane world-volume. These two situations were studied in \cite{3}. Summarizing the two cases, we found that for $N$ five-branes wrapping the same curve $C_2$ of genus $g$, we expect that the symmetry is enhanced from $N$ copies of $U(1)^9$ to $U(N)^9$. Alternatively
in the second case, even for a single brane, we can get enhancement if the embedding degenerates. In general, $U(1)^g$ enhances to a product of unitary groups such that the total rank is equal to $g$. The maximal enhancement is presumably to $SU(g+1)$, and the other allowed groups correspond to different “Higgsings” of $SU(g+1)$ by fields in the adjoint representation. For example, if $g = 2$, then $SU(3)$ could be broken to either $SU(2) \times U(1)$ or $U(1) \times U(1)$. In all cases, the total rank of the symmetry group is conserved. Finally, we note that in the case where the Calabi–Yau space itself degenerates to become a singular orbifold, and the five-branes are wrapped at the singularity, we could expect more exotic enhancement, in particular, to gauge groups other than unitary groups. In this paper, however, we will restrict ourselves to the case of smooth Calabi–Yau spaces.

7.3 Low Energy Effective Actions

Next, we would like to discuss the five-dimensional effective actions that result from the reduction of Hořava–Witten theory on a background that includes five-branes. It has already been explained in section 3.2 how the vacua without five-branes found in this paper can be used to construct a sensible five-dimensional theory. Essentially the same arguments apply here. We begin with the five-dimensional bulk theory. Clearly, the zero-mode content is unchanged with respect to the case without five-branes. Thus we have $\mathcal{N} = 1$ supergravity coupled to $h^{1,1} - 1$ vector multiplets and $h^{2,1} + 1$ hypermultiplets. What about the gauging of the hypermultiplet coset space? Inserting the massless modes (5.38) into eq. (5.32) and calculating $G^{(1)}$ via eq. (5.24) one finds

$$G^{(1)} = \frac{1}{2V} (\ast \omega_{i_0}) \sum_{m=0}^{n} \alpha^{(m),i_0} \epsilon(z)$$

in the interval

$$z_n \leq |z| \leq z_{n+1}$$

for fixed $n$, where $n = 0, \ldots, N$, and as in eqn. (6.6) we have introduced the parameters

$$\alpha^{(m)}_{i_0} = \frac{\sqrt{2} \epsilon S}{\rho} \beta^{(m)}_{i_0}$$

(7.15)

to conform with the notation of [4, 5]. Hence, we still have a non-zero mode that must be taken into account in the dimensional reduction. Its form, however, depends on the interval one is considering. Consequently, the five-dimensional action again contains a term of the form (6.3), but with $\alpha_{i_0}$ being replaced by $\sum_{m=0}^{n} \alpha^{(m)}_{i_0}$ for the interval $z_n \leq |z| \leq z_{n+1}$. In other words, we have gauging in the bulk between each two five-branes but the gauge charge differs from interval to interval. Since the bulk potential (6.12) is directly related to the gauging, it is subject to a similar replacement of charges. In summary, we conclude that the bulk theory between any pair of neighboring five-branes in the interval $z_n \leq |z| \leq z_{n+1}$ is as given in ref. [4, 5], but with $\alpha_{i_0}$ replaced by $\sum_{m=0}^{n} \alpha^{(m)}_{i_0}$. 

47
We now turn to the orbifold planes. They are described by four-dimensional $\mathcal{N} = 1$ theories at $x^{11} = 0, \pi \rho$ coupled to the bulk. The zero mode spectrum on these planes is, of course, unchanged with respect to the situation without five-branes. It consists of gauge multiplets corresponding to the unbroken gauge groups $G^{(1)}$ and $G^{(2)}$, as dictated by the choice of the internal gauge bundle, and corresponding gauge matter multiplets. The height of the boundary potentials (see eqn. (6.13)) is now set by the charges $\alpha_{i_0}^{(0)}$ and $\alpha_{i_0}^{(N+1)}$ which, due to the presence of additional five-brane charges, are no longer necessarily equal and opposite.

Finally, we should consider the worldvolume theories of the three-branes that originate from wrapping the five-branes around supersymmetric cycles. Applying the results of the previous subsection to each of the $N$ five-branes, we have $N$ additional four-dimensional $\mathcal{N} = 1$ theories at $x^{11} = x_1, \ldots, x_N$ which couple to the five-dimensional bulk. The field content of such a theory at $x^{11} = x_n$ for $n = 1, \ldots, N$ is generically given by $U(1)^{g_n}$ gauge multiplets, where $g_n$ is the genus of the holomorphic curve on which the $n$-th five-brane is wrapped, a universal chiral multiplet and a number of additional chiral multiplets describing the moduli space of the holomorphic curve within the Calabi–Yau manifold. By the mechanisms described at the end of the previous subsection, the $U(1)^{g_n}$ gauge groups can be enhanced to non-Abelian groups. As the simplest example, two five-branes located at $x^{11} = x_n$ and $x^{11} = x_{n+1}$ could be wrapped on the same Calabi–Yau cycle with genus $g_n$. As long as two five-branes are separated in the orbifold, that is, $x_{n+1} \neq x_n$, we have two gauge groups $U(1)^{g_n}$, one group on each brane. However, when the two five-branes coincide, that is, for $x_{n+1} = x_n$, these groups are enhanced to $U(2)^{g_n}$. The precise form of the three-brane world-volume theories should be obtained by a reduction of the five-brane world-volume theory on the holomorphic curves, in a target space background of the undeformed Calabi–Yau space together with the non-zero mode for the four-form field strength. We expect those three-brane theories to have a potential depending on the moduli living on the three-brane and the projection of the bulk moduli to the three-brane world-volume. This expectation is in analogy with the theories on the orbifold planes which, as we have seen, possess such a potential. It has been shown in ref. [4, 5] that those boundary potentials provide the source terms for a BPS double-domain wall solution of the five-dimensional theory in the absence of additional five-branes. This double domain wall is the appropriate background for a further reduction to four dimensions. Again, in analogy, we expect the vacuum of the five-dimensional theory in the presence of five-branes to be a BPS multi-domain wall. More precisely, for $N$ five-branes, we expect $N + 2$ domain walls with two world-volumes stretching across the orbifold planes and the remaining $N$ stretching across the three-brane planes. The rôle of the potentials on the three-brane world-volume theories is to provide the $N$ additional source terms needed to support such a solution.

Let us finally discuss some consequences for the four-dimensional effective theory. Clearly, there is a sector of the theory which has just the conventional field content of four-dimensional $\mathcal{N} = 1$
low-energy supergravities derived from string theory. More precisely, this is $h^{1,1} + h^{2,1}$ chiral matter multiplets containing the moduli, gauge multiplets with gauge group $G^{(1)} \times G^{(2)} \subset E_8 \times E_8$ and corresponding gauge matter. In the presence of five-branes, however, we have additional sectors of the four-dimensional theory leading to additional chiral multiplets containing the five-brane moduli and, even more important, to gauge multiplets with generic gauge group

$$G = \prod_{n=1}^{N} U(1)^{g_n}. \quad (7.16)$$

At specific points in the five-brane moduli space, one expects enhancement to a non-Abelian group $G = G_1 \times \cdots \times G_M$. As explained above, in typical cases, the factors $G_m$ can be $U(n)$ and $SU(n)$ groups. We expect the enhancement to preserve the rank, that is, we have

$$\text{rank}(G) = \sum_{n=1}^{N} g_n. \quad (7.17)$$

We recall that $g_n$ is the genus of the curve on which the $n$-th five-brane is wrapped. As it stands, it appears that the rank could be made arbitrarily large. However, for a given Calabi–Yau space, we expect a constraint on the rank which originates from positivity constraints in the the zero-cohomology condition (5.17). As is, the five-brane sectors and the conventional sector of the theory only interact via the bulk supergravity fields. Therefore, at this point, they are most naturally interpreted as hidden sectors.

We should, however, point out that the presence of five-branes provides considerably more flexibility in the choice of $G^{(1)} \times G^{(2)}$, the “conventional” gauge group that originates from the heterotic $E_8 \times E_8$. This happens because it is much simpler to satisfy the zero cohomology condition (5.17) in the presence of five-branes. Let us give an an example which is illuminating, although not necessary physically relevant. Consider a Calabi–Yau space $X$ with topologically nontrivial $\text{tr} R \wedge R$. In addition, we set both $E_8$ gauge field backgrounds to zero, which implies that the unbroken gauge group is simply $E_8 \times E_8$. Without five-branes, such a background is inconsistent since it is in conflict with the zero-cohomology condition (5.17). However, if for each independent four-cycle $C_{4\text{i}_0}$, we can introduce $N_{\text{i}_0}$ five-branes, each having unit intersection number with the cycle $C_{4\text{i}_0}$, such that

$$N_{\text{i}_0} = -\frac{1}{8\pi^2} \int_{C_{4\text{i}_0}} \text{tr} R \wedge R \quad (7.18)$$

then the zero-cohomology condition is satisfied. Of course, the gauge group will then be enlarged to $E_8 \times E_8 \times G$ where the gauge group $G$ originates from the five-branes, as discussed above.

What about the form of the four-dimensional effective action? We have seen that non-standard embedding without five-branes does not change the form of the effective action with respect to the standard embedding case. This could be understood as a direct consequence of the fact that the
five-dimensional effective theory remains unchanged. Above we have seen, however, that the five-
dimensional effective theory does change in the presence of five-branes. In particular, its vacuum
BPS solution is now a multi-domain wall, as opposed to a double-domain wall in the case without
five-branes. Hence, we expect the four-dimensional theory obtained as a reduction on this multi-
domain wall to change as well. Let us, as an example of this, calculate the gauge kinetic functions
in four dimensions to linear order in $\epsilon_S$. Here, we will not do this using the five-dimensional theory
but, equivalently, reduce directly from eleven to four dimensions. We define the modulus $R$ for the
orbifold radius by

$$R = \frac{1}{2V\pi \rho} \int_{S^1/\mathbb{Z}_2 \times X} \sqrt{\gamma} g.$$  \hfill (7.19)

Note that with this definition, $R$ measures the averaged orbifold size in units of $2\pi \rho$. Let us also
introduce the $(1,1)$ moduli $a^{i_0}$ in the usual way as

$$\omega_{AB} = a^{i_0} \omega_{i_0 AB}.$$  \hfill (7.20)

Then, the real parts of the low energy fields $S$ and $T^i$ are given by

$$\text{Re}(S) = V, \quad \text{Re}(T^{i_0}) = VR^{-1}a^{i_0}.$$  \hfill (7.21)

We stress that with these definitions, $S$ and $T^{i_0}$ have the standard Kähler potential, that is, the
order $\epsilon_S$ corrections to the Kähler potential vanish \cite{13}. The gauge kinetic functions can be directly
read off from the 10–dimensional Yang–Mills actions (6.15). Using the metric from eq. (5.24) with
(5.38), (5.32) inserted and the above definition of the moduli, we find

$$f^{(1)} = S + \pi \epsilon_S T^{i_0} \sum_{n=0}^{N+1} (1 - z_n)^2 \beta_{i_0}^{(n)}$$  \hfill (7.22)

$$f^{(2)} = S + \pi \epsilon_S T^{i_0} \sum_{n=1}^{N+1} z_n^2 \beta_{i_0}^{(n)},$$  \hfill (7.23)

where, in addition, we have the cohomology constraint (5.37). Recall from eq. (6.10) that in the case
without five-branes, the threshold correction on the two orbifold planes are identical but opposite in
sign. Note that here the expressions for these two thresholds are, in fact, different. The possibilty
of such an asymmetry due to five–branes has also been suggested in ref. \cite{80}. If, for example, there
is only one five-brane with charges $\beta_{i_0}^{(1)}$ at $z = z_1$ on the orbifold, we have

$$f^{(1)} - f^{(2)} = 2\pi \epsilon_S T^{i_0} \left[ \beta_{i_0}^{(0)} + (1 - z_1)\beta_{i_0}^{(1)} \right].$$  \hfill (7.24)

We see that the gauge thresholds on the orbifold planes depend on both the position and the charges
of the additional five-branes in the bulk. This gives considerably more freedom than in the case
without five-branes. In particular, for special choices of the charges and the five-brane position,
the difference of the gauge kinetic functions can be small. Thus, for instance, the hidden gauge
coupling at the physical point need not be as large as it was in the case without five-branes.
Lecture 3: Holomorphic Vector Bundles and Non-Perturbative Vacua

As discussed in Lecture 2, the results of [6] indicated the importance of heterotic $M$-theories with non-standard embeddings and non-perturbative vacua, but did not actually construct such theories. This shortcoming was rectified in [7], where explicit constructions were carried out within the context of holomorphic vector bundles on the orbifold planes of heterotic $M$-theory compactified on elliptically fibered Calabi–Yau three–folds which admit a section. The results of [7] rely upon recent mathematical work by Friedman, Morgan and Witten [81], Donagi [82] and Bershadsky, Johansen, Panetev and Sadov [83] who show how to explicitly construct such vector bundles, and on results of [84, 85] who computed the family generation index in this context. Extending these results, we were able to formulate rules for constructing three-family particle physics theories with phenomenologically interesting gauge groups. As expected, the appearance of gauge groups other than the $E_6$ group of the standard embedding, as well as the three-family condition, necessitate the existence of $M5$-branes and, hence, non-perturbative vacua. In [7], we showed how to compute the topological class of these five-branes and, given this class, how to construct the moduli spaces of the associated holomorphic curves. Our results were presented as a set of rules in [7]. In addition, we gave one concrete example of a three-family model with gauge group $SU(5)$, along with its five-brane class and a discussion of the moduli space of that class.

In [8], we greatly enlarged the discussion of the results in [7], deriving in detail the rules presented there. In order to make this work more accessible to physicists, as well as to lay the foundation for the necessary derivations and proofs, we presented brief discussions of 1. elliptically fibered Calabi–Yau three–folds, 2. spectral cover constructions of both $U(n)$ and $SU(n)$ bundles, 3. Chern classes and 4. complex surfaces, specifically del Pezzo, Hirzebruch and Enriques surfaces. Using this background, we explicitly derived the rules for the construction of three-family models based on semi-stable holomorphic vector bundles with structure group $SU(n)$. Specifically, we constructed the form of the five-brane class $[W]$, as well as the constraints imposed on this class by the three-family condition, the restriction that the vector bundle have structure group $SU(n)$ and the requirement that $[W]$ be an effective class. From these considerations, we derived a set of rules. As discussed in that paper, elliptically fibered Calabi–Yau three–folds that admit a section can only have del Pezzo, Hirzebruch, Enriques and blown-up Hirzebruch surfaces as a base. We showed, however, that Enriques surfaces can never lead to effective five-brane curves in vacua with three generations. Therefore, the base $B$ of the elliptic fibration is restricted to be a del Pezzo, Hirzebruch or a blow-up of a Hirzebruch surface. In Appendix B of [8], we presented the generators of all effective classes in $H_2(B, \mathbb{Z})$, as well as the first and second Chern classes $c_1(B)$ and $c_2(B)$, for these allowed bases. Combining the rules with the generators and Chern classes given in Appendix B, we presented a general algorithm for the construction of non-perturbative vacua corresponding to three-family particle physics theories with phenomenologically relevant gauge groups. In this
lecture, we review the results of \[8\], referring the reader frequently to Appendix B of that paper for the necessary details.

8 Holomorphic Gauge Bundles, Five-Branes and Non-Perturbative Vacua

In this section, we will briefly review the generic properties of heterotic $M$–theory vacua appropriate for a reduction of the theory to $\mathcal{N} = 1$ supersymmetric theories in both five and four dimensions. As discussed in Lecture 1, the $M$–theory vacuum is given in eleven dimensions by specifying the metric $g_{IJ}$ and the three-form $C_{IJK}$ with field strength $G_{IJKL} = 24 \partial_{[I}C_{JKL]}$ of the supergravity multiplet. Following Hořava and Witten \[1, 2\] and Witten \[3\], the space-time structure of these vacua, to lowest order in the expansion parameter $\kappa^{2/3}$, will be taken to be

$$M_{11} = M_4 \times S^1/Z_2 \times X$$

(8.1)

where $M_4$ is four-dimensional Minkowski space, $S^1/Z_2$ is a one-dimensional orbifold and $X$ is a smooth Calabi–Yau three-fold. The vacuum space-time structure becomes more complicated at the next order in $\kappa^{2/3}$, but, as discussed in the previous two lectures, this metric “deformation”, which has been the subject of a number of papers \[3, 10, 13\], can be viewed as arising as the static vacuum of the five-dimensional effective theory \[4, 5\] and, hence, need not concern us here.

The $Z_2$ orbifold projection necessitates the introduction, on each of the two ten-dimensional orbifold fixed planes, of an $\mathcal{N} = 1$, $E_8$ Yang–Mills supermultiplet which is required for anomaly cancellation. On each plane, the gauge field structure of these vacua, called the gauge bundle, must be a solution of the hermitian Yang–Mills equations for an $E_8$-valued connection in order to be compatible with four preserved supercharges in four dimensions. Equivalently, as shown by Donaldson, Uhlenbeck and Yau \[86, 87\], each gauge bundle must be a semi-stable, holomorphic bundle with the structure group being the complexification $E_{8C}$ of $E_8$. In the following, we will denote both groups by $E_8$, letting context dictate which group is being referred to. (In general, we will denote any group $G$ and its complexification $G_C$ simply as $G$). These semi-stable, holomorphic gauge bundles are, a priori, allowed to be arbitrary in all other respects. In particular, there is no requirement that the spin-connection of the Calabi–Yau three-fold be embedded into an $SU(3)$ subgroup of the gauge connection of one of the $E_8$ bundles, the so-called standard embedding. This generalization to arbitrary semi-stable holomorphic gauge bundles is what is referred to as non-standard embedding. The terms standard and non-standard embedding are historical and somewhat irrelevant in the context of $M$-theory, where no choice of embedding can ever set the entire three-form to zero. For this reason, we will avoid those terms and simply refer to arbitrary semi-stable holomorphic $E_8$ gauge bundles. Since these bundles can be chosen arbitrarily, it is
clear that we can restrict the transition functions to be elements of any subgroup $G$ of $E_8$, such as $G = U(n)$, $SU(n)$ or $Sp(n)$. We will refer to the restricted bundle as a semi-stable, holomorphic $G$ bundle, or simply as a $G$ bundle. It is clear that the $G_1$ bundle on one orbifold plane and the $G_2$ bundle on the other plane need not, generically, have the same subgroups $G_1$ and $G_2$ of $E_8$. We will denote the semi-stable holomorphic gauge bundle on the $i$-th orbifold plane by $V_i$ and the associated structure group by $G_i$.

In addition, as discussed in [6, 7], we will allow for the presence of five-branes located at points throughout the orbifold interval. The five-branes will preserve $\mathcal{N} = 1$ supersymmetry provided they are wrapped on holomorphic two-cycles within $X$ and otherwise span the flat Minkowski space $M_4$. The inclusion of five-branes is essential for a complete discussion of $M$-theory vacua. The reason for this is that, given a Calabi–Yau three-fold background, the presence of five-branes allows one to construct large numbers of gauge bundles that would otherwise be disallowed [6, 7].

The requirements of gauge and gravitational anomaly cancellation on the two orbifold fixed planes, as well as anomaly cancellation on each five-brane worldvolume, places a further very strong constraint on, and relationship between, the space-time manifold, the gauge bundles and the five-brane structure of the vacuum. Specifically, anomaly cancellation necessitates the addition of four-forms sources to the four-form field strength Bianchi identity. As discussed in Lecture 2, the modified Bianchi identity is given by

\[
(dG)_{11\bar{I}\bar{J}\bar{K}\bar{L}} = 2\sqrt{2}\pi \left( \frac{K}{4\pi} \right)^{3/2} \left[ J^{(0)}(x^{11}) + J^{(N+1)}(x^{11} - \pi \rho) + \sum_{n=1}^{N} J^{(n)}(x^{11} - x_n + \delta(x^{11} + x_n)) \right]_{\bar{I}\bar{J}\bar{K}\bar{L}}
\]  

The sources $J^{(0)}$ and $J^{(N+1)}$ on the orbifold planes are

\[
J^{(0)} = -\frac{1}{8\pi} (tr F^{(1)} \wedge F^{(1)} - \frac{1}{2} tr R \wedge R)|_{x^{11}=0}
\]  

and

\[
J^{(N+1)} = -\frac{1}{8\pi^2} (tr F^{(2)} \wedge F^{(2)} - \frac{1}{2} tr R \wedge R)|_{x^{11}=\pi \rho}
\]

respectively. By $tr$ we mean $\frac{1}{30}$-th of the trace over the generators in the 248 representation of $E_8$. The two-form $F^{(i)}$ is the field strength of a connection on the gauge bundle $V_i$ of the $i$-th orbifold plane and $R$ is the curvature two-form on the Calabi-Yau three-fold. We have also introduced $N$ additional sources $J^{(n)}$, where $n = 1, \ldots, N$. These arise from $N$ five-branes located at $x^{11} = x_1, \ldots, x_N$ where $0 \leq x_1 \leq \cdots \leq x_N \leq \pi \rho$. Note that each five-brane at $x = x_n$ has to be paired with a mirror five-brane at $\bar{x} = -x_n$ with the same source since the Bianchi identity must be even under the $Z_2$ orbifold symmetry. Our normalization is chosen so that the total source of each pair is $J^{(n)}$.
Non-zero source terms on the right hand side of the Bianchi identity (8.2) preclude the simultaneous vanishing of all components of the three-form $C_{IJK}$. The result of this is that, to next order in the Hořava–Witten expansion parameter $\kappa^{2/3}$, the space-time of the supersymmetry preserving vacua gets “deformed” away from that given in expression (8.1). As discussed above, this deformation of the vacuum need not concern us here. In this lecture, we will focus on yet another aspect of Bianchi identity (8.2), a topological condition that constrains the cohomology of the vacuum. This constraint is found as follows. Consider integrating the Bianchi identity (8.2) over any five-cycle which spans the orbifold interval together with an arbitrary four-cycle $C_4$ in the Calabi-Yau three-fold. Since $dG$ is exact, this integral must vanish. Physically, this is the statement that there can be no net charge in a compact space, since there is nowhere for the flux to “escape”. Performing the integral over the orbifold interval, we derive, using (8.2), that

$$\sum_{n=0}^{N+1} \int_{C_4} J^{(n)} = 0$$  \hspace{1cm} (8.5)

Hence, the total magnetic charge over $C_4$ vanishes. Since this is true for an arbitrary four-cycle $C_4$ in the Calabi-Yau three-fold, it follows that the sum of the sources must be cohomologically trivial. That is

$$[\sum_{n=0}^{N+1} J^{(n)}] = 0$$  \hspace{1cm} (8.6)

The physical meaning of this expression becomes more transparent if we rewrite it using equations (8.3) and (8.4). Using these expressions, equation (8.6) becomes

$$-[\frac{1}{8\pi^2} tr F^{(1)} \wedge F^{(1)}] - [\frac{1}{8\pi^2} tr F^{(2)} \wedge F^{(2)}] + [\frac{1}{8\pi^2} tr R \wedge R] + \sum_{n=1}^{N} [J^{(n)}] = 0$$  \hspace{1cm} (8.7)

It is useful to recall that the second Chern class of an arbitrary $G$ bundle $V$, thought of as an $E_8$ sub-bundle, is defined to be

$$c_2(V) = -\frac{1}{2 \cdot 8\pi^2} tr F \wedge F$$  \hspace{1cm} (8.8)

Similarly, the second Chern class of the tangent bundle of the Calabi-Yau manifold $X$ is given by

$$c_2(TX) = -\frac{1}{2 \cdot 8\pi^2} tr_6 R \wedge R$$  \hspace{1cm} (8.9)

where $tr_6$ implies that the trace is taken over the vector representation of $SO(6) \supset SU(3)$, that is, the usual tangent space representation. It follows that expression (8.7) can be written as

$$c_2(V_1) + c_2(V_2) + [W] = c_2(TX)$$  \hspace{1cm} (8.10)

where

$$[W] = \sum_{n=1}^{N} [J^{(n)}]$$  \hspace{1cm} (8.11)
is the four-form cohomology class associated with the five-branes. This is a fundamental constraint imposed on the vacuum structure. We will explore this cohomology condition in great detail in this lecture. Note that integrating this constraint over an arbitrary four-cycle $C_4$ yields the expression

$$n_1(C_4) + n_2(C_4) + n_5(C_4) = n_R(C_4)$$

which states that the sum of the number of gauge instantons on the two orbifold planes, plus the sum of the five-brane magnetic charges, must equal the instanton number for the Calabi-Yau tangent bundle, a number which is fixed once the Calabi-Yau three-fold is chosen.

To summarize, we are considering vacuum states of $M$–theory with the following structure.

- Space-time is taken to have the form
  $$M_{11} = M_4 \times S^1/Z_2 \times X$$
  where $X$ is a Calabi-Yau three-fold.

- There is a semi-stable holomorphic gauge bundle $V_i$ with fiber group $G_i \subset E_8$ over the Calabi-Yau three-fold on the $i$-th orbifold fixed plane for $i = 1, 2$. The structure groups $G_1$ and $G_2$ of the two bundles can be any subgroups of $E_8$ and need not be the same.

- We allow for the presence of five-branes in the vacuum, which are wrapped on holomorphic two-cycles within $X$.

- The Calabi-Yau three-fold, the gauge bundles and the five-branes are subject to the cohomological constraint
  $$c_2(V_1) + c_2(V_2) + [W] = c_2(TX)$$

where $c_2(V_i)$ and $c_2(TX)$ are the second Chern classes of the gauge bundle $V_i$ and the tangent bundle $TX$ respectively and $[W]$ is the class associated with the five-branes.

Vacua of this type will be referred to as non-perturbative heterotic M-theory vacua.

The discussion given in this section is completely generic, in that it applies to any Calabi-Yau three-fold and any gauge bundles that can be constructed over it. However, realistic particle physics theories require the explicit construction of these gauge bundles. In the following, we will review the formalism for the construction of semi-stable holomorphic gauge bundles with fiber groups $G_1$ and $G_2$ over the two orbifold fixed planes. For specificity, we will restrict the structure groups to be

$$G_i = U(n_i) \quad \text{or} \quad SU(n_i)$$
for \( i = 1, 2 \). Our explicit bundle constructions will be achieved over the restricted, but rich, set of elliptically fibered Calabi-Yau three-folds which admit a section. Such three-folds have been extensively discussed within the context of duality between M- and F-theory. Independently of this usage, however, elliptically fibered Calabi-Yau three-folds with a section are known to be the simplest Calabi-Yau spaces on which one can explicitly construct bundles, compute Chern classes, moduli spaces and so on. This latter property makes them a compelling choice for the construction of concrete particle physics theories. Having constructed the bundles, one can explicitly calculate the gauge bundle Chern classes \( c_2(V_i) \) for \( i = 1, 2 \), as well as the tangent bundle Chern class \( c_2(TX) \). Having done so, one can then find the class \([W]\) of the five-branes using the cohomology condition (8.10). That is, we will present a formalism in which the entire structure of non-perturbative M-theory vacua can be calculated.

As will be discussed in detail below, having constructed a non-perturbative vacuum, we can compute the number of low energy families and the Yang-Mills gauge group associated with that vacuum. We will show that, because of the flexibility introduced by the presence of five-branes, we will easily construct non-perturbative vacua with three-families. Similarly, one easily finds phenomenologically interesting gauge groups, such as \( E_6, SU(5) \) and \( SO(10) \), as the \( E_8 \) subgroups commutant with the \( G \)-bundle structure groups, such as \( SU(3), SU(4) \) and \( SU(5) \) respectively, on the observable orbifold fixed plane. In addition, using the cohomology constraint (8.10), one can explicitly determine the cohomology class \([W]\) of the five-branes for a specific vacuum. Hence, one can compute the holomorphic curve associated with the five-branes exactly and determine all of its geometrical attributes. These include its the number of its irreducible components, which in turn tells us how many independent five-branes appear in five-dimensions, and its genus, which will tell us the minimal gauge group on the five-brane worldvolume when dimensionally reduced on the holomorphic curve. Furthermore, we are, in general, able to compute the entire moduli space of the holomorphic curve. This can tell us about gauge group enhancement on the five-brane worldvolume, for example. In [7], we discussed the generic properties of the holomorphic curves associated with five-branes. We presented a more detailed discussion in [9].

9 Elliptically Fibered Calabi–Yau Three-Folds

As discussed previously, we will consider non-perturbative vacua where the Calabi–Yau three–fold is an elliptic fibration which admits a section. In this section, we give an introduction to these spaces, summarizing the properties we will need in order to explicitly compute important aspects of the vacua.

An elliptically fibered Calabi–Yau three–fold \( X \) consists of a base \( B \), which is a complex two–
surface, and an analytic map

$$\pi : X \longrightarrow B$$

(9.1)

with the property that for a generic point \( b \in B \), the fiber

$$E_b = \pi^{-1}(b)$$

(9.2)

is an elliptic curve. That is, \( E_b \) is a Riemann surface of genus one. In addition, we will require that there exist a global section, denoted \( \sigma \), defined to be an analytic map

$$\sigma : B \longrightarrow X$$

(9.3)

that assigns to every point \( b \in B \) the zero element \( \sigma(b) = p \in E_b \) discussed below. The requirement that the elliptic fibration have a section is crucial for duality to \( F \)-theory and to make contact with the Chern class formulas in [81]. However, this assumption does not seem fundamentally essential and we will explore bundles without sections in future work [88]. The Calabi–Yau three-fold must be a complex Kähler manifold. This implies that the base is itself a complex manifold, while we have already assumed that the fiber is a Riemann surface and so has a complex structure. Furthermore, the fibration must be holomorphic, that is, it must have holomorphic transition functions. Finally, the condition that the Calabi–Yau three-fold has vanishing first Chern class puts a further constraint on the types of fibration allowed.

Let us start by briefly summarizing the properties of an elliptic curve \( E \). It is a genus one Riemann surface and so can be embedded in the two-dimensional complex projective space \( \mathbb{CP}^2 \). A simple way to do this is by using the homogeneous Weierstrass equation

$$zy^2 = 4x^3 - g_2xz^2 - g_3z^3$$

(9.4)

where \( x, y \) and \( z \) are complex homogeneous coordinates on \( \mathbb{CP}^2 \). It follows that we identify \((\lambda x, \lambda y, \lambda z)\) with \((x, y, z)\) for any non-zero complex number \( \lambda \). The parameters \( g_2 \) and \( g_3 \) encode the different complex structures one can put on the torus. Provided \( z \neq 0 \), we can rescale to affine coordinates where \( z = 1 \). We then see, viewed as a map from \( x \) to \( y \), that there are two branch cuts in the \( x \)-plane, linking \( x = \infty \) and the three roots of the cubic equation \( 4x^3 - g_2x - g_3 = 0 \). When any two of these points coincide, the elliptic curve becomes singular. This corresponds to one of the cycles in the torus shrinking to zero. Such singular behaviour is characterized by the discriminant

$$\Delta = g_2^3 - 27g_3^2$$

(9.5)

vanishing. Finally, we note that the complex structure provides a natural notion of addition of points on the elliptic curve. The torus can also be considered as the complex plane modulo a
discrete group of translations. Addition of points in the complex plane then induces a natural notion of addition of points on the torus. Translated to the Weierstrass equation, the identity element corresponds to the point where \( x/z \) and \( y/z \) become infinite. Thus, in affine coordinates, the element \( p \in E \) is the point \( x = y = \infty \). This can be scaled elsewhere in non-affine coordinates, such as to \( x = z = 0, \ y = 1 \).

The elliptic fibration is defined by giving the elliptic curve \( E \) over each point in the base \( B \). If we assume the fibration has a global section, and in this lecture we do, then on each coordinate patch this requires giving the parameters \( g_2 \) and \( g_3 \) in the Weierstrass equation as functions on the base. Globally, \( g_2 \) and \( g_3 \) will be sections of appropriate line bundles on \( B \). In fact, specifying the type of an elliptic fibration over \( B \) is equivalent to specifying a line bundle on \( B \). Given the elliptic fibration \( \pi : X \longrightarrow B \), we define \( \mathcal{L} \) as the line bundle on \( B \) whose fiber at \( b \in B \) is the cotangent line \( T_p(E_b) \) to the elliptic curve at the origin. That is, \( \mathcal{L} \) is the conormal bundle to the section \( \sigma(B) \) in \( X \). Conversely, given \( \mathcal{L} \), we take \( x \) and \( y \) to scale as sections of \( \mathcal{L}^2 \) and \( \mathcal{L}^3 \) respectively, which means that \( g_2 \) and \( g_3 \) should be sections of \( \mathcal{L}^4 \) and \( \mathcal{L}^6 \). By \( \mathcal{L}^i \) we mean the tensor product of the line bundle \( \mathcal{L} \) with itself \( i \) times. In conclusion, we see that the elliptic fibration is characterized by a line bundle \( \mathcal{L} \) over the base \( B \) together with a choice of sections \( g_2 \) and \( g_3 \) of \( \mathcal{L}^4 \) and \( \mathcal{L}^6 \).

Note that the set of points in the base over which the fibration becomes singular is given by the vanishing of the discriminant \( \Delta = g_3^2 - 27g_2^3 \). It follows from the above discussion that \( \Delta \) is a section of the line bundle \( \mathcal{L}^{12} \). The zeros of \( \Delta \) then naturally define a divisor, which in this case is a complex curve, in the base. Since \( \Delta \) is a section of \( \mathcal{L}^{12} \), the cohomology class of the discriminant curve is 12 times the cohomology class of the divisors defined by sections of \( \mathcal{L} \).

Finally, we come to the important condition that on a Calabi–Yau three–fold \( X \) the first Chern class of the tangent bundle \( T_X \) must vanish. The canonical bundle \( K_X \) is the line bundle constructed as the determinant of the holomorphic cotangent bundle of \( X \). The condition that

\[
c_1(T_X) = 0 \tag{9.6}
\]

implies that \( K_X = \mathcal{O} \), where \( \mathcal{O} \) is the trivial bundle. This, in turn, puts a constraint on \( \mathcal{L} \). To see this, note that the adjunction formula tells us that, since \( B \) is a divisor of \( X \), the canonical bundle \( K_B \) of \( B \) is given by

\[
K_B = K_X|_B \otimes N_{B/X} \tag{9.7}
\]

where \( N_{B/X} \) is the normal bundle of \( B \) in \( X \). From the above discussion, we know that

\[
N_{B/X}^{-1} = \mathcal{L}, \quad K_X|_B = \mathcal{O} \tag{9.8}
\]

Inserting this into (9.7), and switching to additive notation tells us that

\[
\mathcal{L} = K_B^{-1} \tag{9.9}
\]
This condition means that $K_B^{-4}$ and $K_B^{-6}$ must have sections $g_2$ and $g_3$ respectively. Furthermore, the Calabi–Yau property imposes restrictions on how the curves where these sections vanish are allowed to intersect. It is possible to classify the surfaces on which $K_B^{-4}$ and $K_B^{-6}$ have such sections. These are found to be the del Pezzo, Hirzebruch and Enriques surfaces, as well as blow-ups of Hirzebruch surfaces. In this lecture, we will discuss the first three possibilities in detail.

As noted previously, in order to discuss the anomaly cancellation condition, we will need the second Chern class of the holomorphic tangent bundle of $X$. Friedman, Morgan and Witten show that it can be written in terms of the Chern classes of the holomorphic tangent bundle of $B$ as

$$c_2(TX) = c_2(B) + 11c_1(B)^2 + 12\sigma c_1(B)$$

(9.10)

where the wedge product is understood, $c_1(B)$ and $c_2(B)$ are the first and second Chern classes of $B$ respectively and $\sigma$ is the two-form Poincare dual to the global section. We have used the fact that

$$c_1(L) = c_1(K_B^{-1}) = c_1(B)$$

(9.11)

in writing (9.10).

10 Spectral Cover Constructions

In this section, we follow the construction of holomorphic bundles on elliptically fibered Calabi–Yau manifolds presented in \[81, 82, 83\]. The idea is to understand the bundle structure on a given elliptic fiber and then to patch these bundles together over the base. The authors in \[81, 82, 83\] discuss a number of techniques for constructing bundles with different gauge groups. Here we will restrict ourselves to $U(n)$ and $SU(n)$ sub-bundles of $E_8$. These are sufficient to give suitable phenomenological gauge groups. This restriction allows us to consider only the simplest of the different constructions, namely that via spectral covers. In this section, we will summarize the spectral cover construction, concentrating on the properties necessary for an explicit discussion of non-perturbative vacua. We note that for structure groups $G \neq U(n)$ or $SU(n)$, the construction of bundles is more complicated than the construction of rank $n$ vector bundles presented here.

As we have already mentioned, the condition of supersymmetry requires that the $E_8$ gauge bundles admit a field strength satisfying the hermitian Yang–Mills equations. Donaldson, Uhlenbeck and Yau have shown that this is equivalent to the topological requirement that the associated bundle be semi-stable, with transition functions in the complexification of the gauge group. Since we are considering $U(n)$ and $SU(n)$ sub-bundles, this means $U(n)_C = GL(n, \mathbb{C})$ and $SU(n)_C = SL(n, \mathbb{C})$ respectively. The spectral cover construction is given in terms of this latter
formulation of the supersymmetry condition. Note that the distinction between semi-stable and stable bundles corresponds to whether the hermitian Yang-Mills field strength is reducible or not. This refers to whether, globally, it can be diagonalized into parts coming from different subgroups of the full gauge group. More precisely, it refers to whether or not the holonomy commutes with more that just the center of the group. Usually, a generic solution of the hermitian Yang-Mills equations corresponds to a stable bundle. However, on some spaces, for instance on an elliptic curve, the generic case is semi-stable.

**U(n) and SU(n) Bundles Over An Elliptic Curve**

We begin by considering semi-stable bundles on a single elliptic curve $E$. A theorem of Looijenga [90] states that the moduli space of such bundles for any simply-connected group of rank $r$ is an $r$-dimensional complex weighted projective space. For the simply-connected group $SU(n)$, this moduli space is the projective space $CP^{n-1}$. $U(n)$ is not simply-connected. $U(n)$ bundles have a discrete integer invariant, their degree or first Chern class, which we denote by $d$. Let $k$ be the greatest common divisor of $d$ and $n$. It can be shown that the moduli space of a $U(n)$ bundle of degree $d$ over a single elliptic curve $E$ is the $k$-th symmetric product of $E$, denoted by $E[^{[k]}]$. In this lecture, we will restrict our discussion to $U(n)$ bundles of degree zero. For these bundles, the moduli space is $E[^{[n]}]$.

A holomorphic $U(n)_C = GL(n, C)$ bundle $V$ over an elliptic curve $E$ is a rank $n$ complex vector bundle. As discussed earlier, we will denote $U(n)_C$ simply as $U(n)$, letting context dictate which group is being referred to. To define the bundle, we need to specify the holonomy; that is, how the bundle twists as one moves around in the elliptic curve. The holonomy is a map from the fundamental group $\pi_1$ of the elliptic curve into the gauge group. Since the fundamental group of the torus is Abelian, the holonomy must map into the maximal torus of the gauge group. This means we can diagonalize all the transition functions, so that $V$ becomes the direct sum of line bundles

$$ V = \mathcal{N}_1 \oplus \cdots \oplus \mathcal{N}_n $$

Furthermore, the Weyl group permutes the diagonal elements, so that $V$ only determines the ordering of the $\mathcal{N}_i$ up to permutations. To reduce from a $U(n)$ bundle to an $SU(n)$ bundle, one imposes the additional condition that the determinant of the transition functions be taken to be unity. This implies that the product

$$ \mathcal{N}_1 \otimes \cdots \otimes \mathcal{N}_n = \mathcal{O} $$

where $\mathcal{O}$ is the trivial bundle on $E$.

The semi-stable condition implies that the line bundles $\mathcal{N}_i$ are of the same degree, which can be taken to be zero. We can understand this from the hermitian Yang–Mills equations. On a Riemann
surface, these equations imply that the field strength is actually zero. Thus, the first Chern class of each of the bundles $\mathcal{N}_i$ must vanish, or equivalently be of degree zero. On an elliptic curve, this condition means that there is a unique point $Q_i$ on $E$ such that there is a meromorphic section of $\mathcal{N}_i$ which vanishes only at $Q_i$ and has a simple pole only at the origin $p$. We can write this as

$$\mathcal{N}_i = \mathcal{O}(Q_i) \otimes \mathcal{O}(p)^{-1}$$

(10.3)

If one further restricts the structure group to be $SU(n)$, then condition (10.2) translates into the requirement that

$$\sum_{i=1}^{n} (Q_i - p) = 0$$

(10.4)

where one uses the natural addition of points on $E$ discussed above.

Thus, on a given elliptic curve, giving a semi-stable $U(n)$ bundle is equivalent to giving an unordered (because of the Weyl symmetry) $n$-tuple of points on the curve. An $SU(n)$ bundle has the further restriction that $\sum_i (Q_i - p) = 0$. For an $SU(n)$ bundle, these points can be represented very explicitly as roots of an equation in the Weierstrass coordinates describing the elliptic curve. In affine coordinates, where $z = 1$, we write

$$s = a_0 + a_2 x + a_3 y + a_4 x^2 + a_5 x^2 y + \cdots + a_n x^{n/2}$$

(10.5)

(If $n$ is odd the last term is $a_n x^{(n-3)/2} y$.) Solving the equation $s = 0$, together with the Weierstrass equation (hence the appearance of only linear terms in $y$ in $s$), gives $n$ roots corresponding to the $n$ points $Q_i$, where one can show that $\sum_i (Q_i - p) = 0$ as required. One notes that the roots are determined by the coefficients $a_i$ only up to an overall scale factor. Thus the moduli space of roots $Q_i$ is the projective space $\mathbb{CP}^{n-1}$ as anticipated, with the coefficients $a_i$ acting as homogeneous coordinates.

In summary, semi-stable $U(n)$ bundles on an elliptic curve are described by an unordered $n$-tuple of points $Q_i$ on the elliptic curve. $SU(n)$ bundles have the additional condition that $\sum_i (Q_i - p) = 0$. In the $SU(n)$ case, these points can be realized as roots of the equation $s = 0$ and give a moduli space of bundles which is simply $\mathbb{CP}^{n-1}$, as mentioned above.

**The Spectral Cover and the Line Bundle $\mathcal{N}$**

Given that a bundle on an elliptic curve is described by the $n$-tuple $Q_i$, it seems reasonable that a bundle on an elliptic fibration determines how the $n$ points vary as one moves around the base $B$. The set of all the $n$ points over the base is called the spectral cover $C$ and is an $n$-fold cover of $B$ with $\pi_C : C \longrightarrow B$. The spectral cover alone does not contain enough information to allow us to construct the bundle $V$. To do this, one must specify an additional line bundle, denoted by $\mathcal{N}$, on
the spectral cover $C$. One obtains $\mathcal{N}$, given the vector bundle $V$, as follows. Consider the elliptic fiber $E_b$ at any point $b \in B$. It follows from the previous section that

$$V|_{E_b} = \mathcal{N}_{1b} \oplus \ldots \oplus \mathcal{N}_{nb}$$

(10.6)

where $\mathcal{N}_{ib}$ for $i = 1, \ldots, n$ are line bundles on $E_b$. In particular, we get a decomposition of the fiber $V_{\sigma}(b)$ of $V$ at $p = \sigma(b)$. Let $V|_B$ be the restriction of $V$ to the base $B$ embedded in $X$ via the section $\sigma$. We have just shown that the $n$-dimensional fibers of $V|_B$ come equipped with a decomposition into a sum of lines. As point $b$ moves around the base $B$, these $n$ lines move in one to one correspondence with the $n$ points $Q_i$ above $b$. This data specifies a unique line bundle $\mathcal{N}$ on $C$ such that the direct image $\pi_{C*}\mathcal{N}$ is $V|_B$ with its given decomposition. The direct image $\pi_{C*}\mathcal{N}$ is a vector bundle on $B$ whose fiber at a generic point $b$, where the inverse image $\pi^{-1}_C(b)$ consists of the $n$ distinct points $Q_i$, is the direct sum of the $n$ lines $\mathcal{N}|_{Q_i}$.

**Construction of Bundles**

We are now in a position to construct the rank $n$ vector bundle starting with the spectral data $[81, 82, 83]$. The spectral data consists of the spectral cover $C \subset X$ together with the line bundle $\mathcal{N}$ on $C$. The spectral cover is a divisor (hypersurface) $C \subset X$ which is of degree $n$ over the base $B$; that is, the restriction $\pi_C : C \to B$ of the elliptic fibration is an $n$-sheeted branched cover. Equivalently, the cohomology class of $C$ in $H^2(X, \mathbb{Z})$ must be of the form

$$[C] = n\sigma + \eta$$

(10.7)

where $\eta$ is a class in $H^2(B, \mathbb{Z})$ and $\sigma$ is the section. This is equivalent to saying that the line bundle $\mathcal{O}_X(C)$ on $X$ determined by $C$, whose sections are meromorphic functions on $X$ with simple poles along $C$, is given by

$$\mathcal{O}_X(C) = \mathcal{O}_X(n\sigma) \otimes \mathcal{M}$$

(10.8)

where $\mathcal{M}$ is some line bundle on $X$ whose restriction to each fiber $E_b$ is of degree zero. Written in this formulation

$$\eta = c_1(\mathcal{M})$$

(10.9)

The line bundle $\mathcal{N}$ is, at this point, completely arbitrary.

Given this data, one can construct a rank $n$ vector bundle $V$ on $X$. It is easy to describe the restriction $V|_B$ of $V$ to the base $B$. It is simply the direct image $V|_B = \pi_{C*}\mathcal{N}$. It is also easy to describe the restriction of $V$ to a general elliptic fiber $E_b$. Let $C \cap E_b = \pi_C^{-1}(b) = Q_1 + \ldots + Q_n$

---

\[\text{When } C \text{ is singular, } \mathcal{N} \text{ may be more generally a rank-1 torsion free sheaf on } C. \text{ For non-singular } C \text{ this is the same as a line bundle.}\]
and $\sigma \cap E_b = p$. Then each $Q_i$ determines a line bundle $N_i$ of degree zero on $E_b$ whose sections are the meromorphic functions on $E_b$ with first order poles at $Q_i$ which vanish at $p$. The restriction $V_{|E_b}$ is then the sum of the $N_i$. Now the main point is that there is a unique vector bundle $V$ on $X$ with these specified restrictions to the base and the fibers.

To describe the entire vector bundle $V$, we use the Poincare bundle $P$. This is a line bundle on the fiber product $X \times_B X'$. Here $X'$ is the “dual fibration” to $X$. In general, this is another elliptic fibration which is locally, but not globally, isomorphic to $X$. However, when $X$ has a section (which we assume), then $X$ and $X'$ are globally isomorphic, so we can identify them if we wish. (Actually, the spectral cover $C$ lives most naturally as a hypersurface in the dual $X'$, not in $X$. When we described it above as living in $X$, we were implicitly using the identification of $X$ and $X'$.) The fiber product $X \times_B X'$ is four-dimensional. It is fibered over $B$, the fiber over $b \in B$ being the ordinary product $E_b \times E'_b$ of the two fibers. Now, the Poincare bundle $P$ is determined by the following two properties: (1) its restriction $P\mid_{E_b \times x}$ to a fiber $E_b \times x$, for $x \in E'_b$, is the line bundle on $E_b$ determined by $x$ while (2) its restriction to $\sigma \times_B X'$ is the trivial bundle. Explicitly, $P$ can be given by the bundle whose sections are meromorphic functions on $X \times_B X'$ with first order poles on $D$ and which vanish on $\sigma \times_B X'$ and on $X \times_B \sigma'$. That is

$$P = \mathcal{O}_{X \times_B X'}(D - \sigma \times_B X' - X \times_B \sigma') \otimes K_B$$

where $D$ is the diagonal divisor representing the graph of the isomorphism $X \to X'$.

Using this Poincare bundle, we can finally describe the entire vector bundle $V$ in terms of the spectral data. It is given by

$$V = p_1^*(p_2^*N \otimes P)$$

Here $p_1, p_2$ are the two projections of the fiber product $X \times_B C$ onto the two factors $X$ and $C$. The two properties of the Poincare bundle guarantee that the restrictions of this $V$ to the base and the fibers indeed agree with the intuitive versions of $V_B$ and $V_{|E_b}$ given above.

In general, this procedure produces $U(n)$ bundles. In order to get $SU(n)$ bundles, two additional conditions must hold. First, the condition that the line bundle $\mathcal{M}$ in equation [10.8] has degree zero on each fiber $E_b$ must be strengthened to require that the restriction of $\mathcal{M}$ to $E_b$ is the trivial bundle. Hence, $\mathcal{M}$ is the pullback to $X$ of a line bundle on $B$ which, for simplicity, we also denote by $\mathcal{M}$. This guarantees that the restrictions to the fibers $V_{|E_b}$ are $SU(n)$ bundles. The second condition is that $V_{|B}$ must be an $SU(n)$ bundle as well. That is, the line bundle $\mathcal{N}$ on $C$ is such that the first Chern class $c_1$ of the resulting bundle $V$ vanishes. This condition, and its ramifications, will be discussed in the next section.

$U(n)$ vector bundles on the orbifold planes of heterotic $M$-theory are always sub-bundles of an $E_8$ vector bundle. As such, issues arise concerning their stability or semi-stability which are
important and require considerable analysis. Furthermore, the associated Chern classes require an extended analysis to compute. For these reasons, we will limit our discussion to $SU(n)$ bundles, which are easier to study.

**Chern Classes and Restrictions on the Bundle**

As discussed above, the global condition that the bundle be $SU(n)$ is that

$$c_1(V) = (1/2\pi)\text{tr}F = 0 \quad (10.12)$$

This condition is clearly true since, for structure group $SU(n)$, the trace must vanish. A formula for $c_1(V)$ can be extracted from the discussion in Friedman, Morgan and Witten [81]. One finds that

$$c_1(V) = \pi_{C*} \left( c_1(N) + \frac{1}{2} c_1(C) - \frac{1}{2} \pi^* c_1(B) \right) \quad (10.13)$$

where $c_1(B)$ means the first Chern class of the tangent bundle of $B$ considered as a complex vector bundle, and similarly for $C$, while $\pi_C$ is the projection from the spectral cover onto $B$; that is, $\pi_C : C \to B$. The operators $\pi_C^*$ and $\pi_{C*}$ are the pull-back and push-forward of cohomology classes between $B$ and $C$. The condition that $c_1(V)$ is zero then implies that

$$c_1(N) = -\frac{1}{2} c_1(C) + \frac{1}{2} \pi_{C*} c_1(B) + \gamma \quad (10.14)$$

where $\gamma$ is some cohomology class satisfying the equation

$$\pi_{C*} \gamma = 0 \quad (10.15)$$

The general solution for $\gamma$ constructed from cohomology classes is

$$\gamma = \lambda \left( n\sigma - \pi_C^* \eta + n\pi_{C*} c_1(B) \right) \quad (10.16)$$

where $\lambda$ is a rational number and $\sigma$ is the global section of the elliptic fibration. Appropriate values for $\lambda$ will emerge shortly. From (10.7) we recall that $c_1(C)$, which is given by

$$c_1(C) = -n\sigma - \pi_C^* \eta \quad (10.17)$$

Combining the last three equations yields

$$c_1(N) = n \left( \frac{1}{2} + \lambda \right) \sigma + \left( \frac{1}{2} - \lambda \right) \pi_{C*} \eta + \left( \frac{1}{2} + n\lambda \right) \pi_{C*} c_1(B) \quad (10.18)$$

Essentially, this means that the bundle $\mathcal{N}$ is completely determined in terms of the elliptic fibration and $\mathcal{M}$. It is important to note, however, that there is not always a solution for $\mathcal{N}$. The reason for this is that $c_1(\mathcal{N})$ must be integer, a condition that puts a substantial constraint on the
allowed bundles. To see this, note that the section is a horizontal divisor, having unit intersection number with the elliptic fiber. On the other hand, the quantities \( \pi^* c_1(B) \) and \( \pi^* \eta \) are vertical, corresponding to curves in the base lifted to the fiber and so have zero intersection number with the fiber. Therefore, we cannot choose \( \eta \) to cancel \( \sigma \) and, hence, the coefficient of \( \sigma \) must, by itself, be an integer. This implies that a consistent bundle \( \mathcal{N} \) will exist if either

\[
\text{n is odd, } \quad \lambda = m + \frac{1}{2} \tag{10.19}
\]
or

\[
\text{n is even, } \quad \lambda = m, \quad \eta = c_1(B) \text{mod} 2 \tag{10.20}
\]

where \( m \) is an integer. Here, the \( \eta = c_1(B) \text{mod} 2 \) condition means that \( \eta \) and \( c_1(B) \) differ by an even element of \( H^2(B, \mathbb{Z}) \). Note that when \( n \) is even, we cannot choose \( \eta \) arbitrarily. For \( n \) odd, condition (10.19) is necessary and sufficient. For \( n \) even, condition (10.20) is sufficient for the existence of a consistent line bundle \( \mathcal{N} \). It is also sufficient for the examples we consider in this lecture, and it is the only class of solutions which is easy to describe in general. However, other solutions do exist. We could, for example, take \( n = 4 \), \( \lambda = \frac{1}{4} \) and \( \eta = 2c_1(B) \text{mod} 4 \).

Finally, we can give the explicit Chern classes for the \( SU(n) \) vector bundle \( V \). Friedman, Morgan and Witten calculate \( c_1(V) \) and \( c_2(V) \), while Curio and Andreas [84, 85] have found \( c_3(V) \). The results are

\[
c_1(V) = 0 \tag{10.21}
\]

\[
c_2(V) = \eta \sigma - \frac{1}{24} c_1(B)^2 (n^3 - n) + \frac{1}{2} \left( \lambda^2 - \frac{1}{4} \right) n \eta (\eta - nc_1(B)) \tag{10.22}
\]

\[
c_3(V) = 2\lambda \sigma \eta (\eta - nc_1(B)) \tag{10.23}
\]

where the wedge product is understood.

### 11 Summary of Elliptic Fibrations and Bundles

The previous two sections are somewhat abstract. For the sake of clarity, we will here summarize those results which are directly relevant to constructing physically acceptable non-perturbative vacua.

- An elliptically fibered Calabi–Yau three-fold is composed of a two-fold base \( B \) and elliptic curves \( E_b \) fibered over each point \( b \in B \). In this lecture, we consider only those elliptic fibrations that admit a global section \( \sigma \).

- The elliptic fibration is characterized by a single line bundle \( \mathcal{L} \) over \( B \). The vanishing of the first Chern class of the canonical bundle \( K_X \) of the Calabi–Yau three-fold \( X \) implies that

\[
\mathcal{L} = K_B^{-1} \tag{11.1}
\]
where $K_B$ is the canonical bundle of the base $B$.

- From the previous condition, it follows that the base $B$ is restricted to del Pezzo, Hirzebruch and Enriques surfaces, as well as blow-ups of Hirzebruch surfaces.

- The second Chern class of the holomorphic tangent bundle of $X$ is given by

$$c_2(TX) = c_2(B) + 11c_1(B)^2 + 12\sigma c_1(B)$$

(11.2)

where $c_1(B)$ and $c_2(B)$ are the first and second Chern classes of $B$.

- A general semi-stable $SU(n)$ gauge bundle $V$ is determined by two line bundles, $\mathcal{M}$ and $\mathcal{N}$. The relevant quantities associated with $\mathcal{M}$ and $\mathcal{N}$ are their first Chern classes

$$\eta = c_1(\mathcal{M})$$

(11.3)

and $c_1(\mathcal{N})$ respectively. The class $c_1(\mathcal{N})$, in addition to depending on $n, \sigma, c_1(B)$ and $\eta$, also contains a complex number $\lambda$.

- The condition that $c_1(\mathcal{N})$ be an integer leads to the constraints on $\eta$ and $\lambda$ given by

$$n \text{ is odd, } \lambda = m + \frac{1}{2}$$

(11.4)

$$n \text{ is even, } \lambda = m, \eta = c_1(B) \mod 2$$

(11.5)

where $m$ is an integer.

- The relevant Chern classes of an $SU(n)$ gauge bundle $V$ are given by

$$c_1(V) = 0$$

(11.6)

$$c_2(V) = \eta \sigma - \frac{1}{24} c_1(B)^2 (n^3 - n) + \frac{1}{2} \left( \lambda^2 - \frac{1}{4} \right) n\eta (\eta - nc_1(B))$$

(11.7)

$$c_3(V) = 2\lambda \sigma \eta (\eta - nc_1(B))$$

(11.8)

How can one use this data to construct realistic particle physics theories? One proceeds as follows.

- Choose a base $B$ from one of the allowed bases; namely, a del Pezzo, Hirzebruch or Enriques surface, or a blow-up of a Hirzebruch surface. The associated Chern classes $c_1(B)$ and $c_2(B)$ can be computed for any of these surfaces.

This allows one to construct the second Chern class of the Calabi-Yau tangent bundle and a significant part of the gauge bundle Chern classes.
• Specify $\eta$ and $\lambda$ subject to the above constraints.

These constraints greatly reduce the number of physically relevant non-perturbative vacua. Given appropriate $\eta$ and $\lambda$, one can completely determine the relevant gauge bundle Chern classes. We will carry this out explicitly in the next section.

12 Effective Curves and Five-Branes

Consider a complex manifold $X$ which is an elliptic fibration over a base $B$. Effective classes are defined, and their physical meaning discussed, in Appendix A of ref. [8]. Let us suppose we have found an effective class in $H_2(B, \mathbb{Z})$. Then, it naturally also lies in an effective homology class in $H_2(X, \mathbb{Z})$ of the elliptic fibration. Note that the fibration structure guarantees that if two curves are in different classes in the base, then they are in different classes in the full manifold $X$. This implies, among other things, that if one finds the effective generating class of the Mori cone of $B$, these classes remain distinct classes of $X$. In addition, there is at least one other effective class that is not associated with the base. This is the class $F$ of the fiber itself. There may also be other such classes, for example, those related to points where the fiber degenerates. However, we will ignore these since they will not appear in the homology classes of the five-branes, our main interest.

The algebraic classes that arise naturally are quadratic polynomials in classes of the line bundles. The only line bundle classes on a general elliptically fibered Calabi–Yau three-fold $X$ are the base $B$ and the divisors $\pi^{-1}(C)$, where $C$ is a curve in $B$. Any quadratic polynomial in these classes can be written as

$$W = W_B + a_F F$$

(12.1)

where $W_B$ is an algebraic homology class in the base manifold $B$ embedded in $X$ and $a_F$ is some integer. Under what conditions is $W$ an effective class? It is clear that $W$ is effective if $W_B$ is an effective class in the base and $a_F \geq 0$. One can also prove that the converse is true in almost all cases. Specifically, we can prove the following. First, the converse is true for any del Pezzo and Enriques surface. Second, the converse is true for a Hirzebruch surface $F_r$, with the exception of when $W_B$ happens to contain the negative section $S$ and $r \geq 3$. In this lecture, for simplicity, we will consider only those cases for which the converse is true. Thus, under this restriction, we have that

$$W \text{ is effective } \iff W_B \text{ is effective in } B \text{ and } a_F \geq 0$$

(12.2)

This reduces the question of finding the effective curves in $X$ to knowing the generating set of effective curves in the base $B$. For the set of base surfaces $B$ we are considering, finding such generators is always possible.
Recall from equation (8.10) that the cohomology class associated with the five-branes is given by

\[ W = c_2(TX) - c_2(V_1) - c_2(V_2) \]  

(12.3)

For simplicity, in this lecture we will allow for arbitrary semi-stable gauge bundles \( V_1 \), which we henceforth call \( V \), on the first orbifold plane, but always take the gauge bundle \( V_2 \) to be trivial. Physically, this corresponds to allowing observable sector gauge groups to be subgroups, such as \( SU(5) \), \( SO(10) \) or \( E_6 \), of \( E_8 \) but leaving the hidden sector \( E_8 \) gauge group unbroken. We do this only for simplicity. Our formalism also allows an analysis of the general case where the hidden sector \( E_8 \) gauge group is broken by a non-trivial bundle \( V_2 \). With this restriction, equation (12.3) simplifies to

\[ W = c_2(TX) - c_2(V) \]  

(12.4)

Inserting the expressions (11.2) and (11.7) for the second Chern classes, we find that

\[ W = W_B + a_f F \]  

(12.5)

where

\[ W_B = \sigma(12c_1(B) - \eta) \]  

(12.6)

is the part of the class associated with the base \( B \) and

\[ a_f = c_2(B) + \left(11 + \frac{n^3 - n}{24}\right) c_1(B)^2 - \frac{1}{2} n \left(\lambda^2 - \frac{1}{4}\right) \eta (\eta - nc_1(B)) \]  

(12.7)

is the part associated with the elliptic fiber.

Now, to make physical sense, five-branes must be wrapped on a curve composed of holomorphic submanifolds of \( X \) and, hence, \( W \) must be an effective class. This physical requirement then implies, using the above theorem, that necessarily

\[ W_B \text{ is effective in } B, \quad a_f \geq 0 \]  

(12.8)

As we will see, this puts a strong constraint on the allowed non-perturbative vacua.

13 Number of Families and Model Building Rules

The first obvious physical criterion for constructing realistic particle physics models is that we should be able to find theories with a small number of families, preferably three. We will see that this is, in fact, easy to do via the bundle constructions on elliptically fibered Calabi–Yau three-folds.
that we are discussing. We start by deriving the three family criterion as discussed, for instance, in Green, Schwarz and Witten [49].

The number of families is related to the number of zero-modes of the Dirac operator in the presence of the gauge bundle on the Calabi–Yau three-fold, since we want to count the number of massless fermions of different chiralities. The original gauginos are in the adjoint representation of $E_8$. In this lecture, we are considering only gauge bundles $V$ with $SU(n)$ fiber groups. To count the number of families, we need to count the number of fields in the matter representations of the low energy gauge group, that is, the subgroup of $E_8$ commutant with $SU(n)$, and their complex conjugates respectively. Explicitly, in this lecture, we will be interested in the following breaking patterns

$$E_8 \supset SU(3) \times E_6 : \quad 248 = (8, 1) \oplus (1, 78) \oplus (3, 27) \oplus (\bar{3}, \bar{27})$$

$$E_8 \supset SU(4) \times SO(10) : \quad 248 = (15, 1) \oplus (1, 45) \oplus (4, 16) \oplus (\bar{4}, \bar{16}) \quad (13.1)$$

$$E_8 \supset SU(5) \times SU(5) : \quad 248 = (24, 1) \oplus (1, 24) \oplus (10, 5) \oplus (\bar{10}, \bar{5}) \oplus (5, 10) \oplus (\bar{5}, 10)$$

Note, however, that the methods presented here will apply to any breaking pattern with an $SU(n)$ subgroup. We see that all the matter representations appear in the fundamental representation of the bundle group $SU(n)$. By definition, the index of the Dirac operator measures the difference in the number of positive and negative chirality spinors, in this case, on the Calabi–Yau three-fold. Since six-dimensional chirality is correlated with four-dimensional chirality, the index gives the number of families. From the fact that all the relevant fields are in the fundamental representation of $SU(n)$, we have that the number of generations is

$$N_{\text{gen}} = \text{index} (V, D) = \int_X \text{td} (X) \text{ch} (V) = \frac{1}{2} \int_X c_3 (V) \quad (13.2)$$

where $\text{td} (X)$ is the Todd class of $X$. For the case of $SU(n)$ bundles on elliptically fibered Calabi–Yau three–folds, one can show, using equation (11.8) above, that the number of families becomes

$$N_{\text{gen}} = \lambda \eta (\eta - nc_1 (B)) \quad (13.3)$$

where we have integrated over the fiber. Hence, to obtain three families the bundle must be constrained so that

$$3 = \lambda \eta (\eta - nc_1 (B)) \quad (13.4)$$

It is useful to express this condition in terms of the class $W_B$ given in equation (12.1) and integrated over the fiber. We find that

$$3 = \lambda \left( W_B^2 - (24 - n)W_B c_1 (B) + 12(12 - n)c_1 (B)^2 \right) \quad (13.5)$$
Furthermore, inserting the three family constraint into (12.7) gives

\[ a_f = c_2(B) + \left( 11 + \frac{1}{24} (n^3 - n) \right) c_1(B)^2 - \frac{3n}{2\lambda} \left( \lambda^2 - \frac{1}{4} \right) \]  

(13.6)

We are now in a position to summarize all the rules and constraints that are required to produce particle physics theories with three families. The conditions obtained in this section are

- The homology class associated with the five-branes is specifically of the form

\[ [W] = W_B + a_f F \]  

(13.7)

where

\[ W_B = \sigma(12c_1(B) - \eta) \]  

(13.8)

\[ a_f = c_2(B) + \left( 11 + \frac{1}{24} (n^3 - n) \right) c_1(B)^2 - \frac{3n}{2\lambda} \left( \lambda^2 - \frac{1}{4} \right) \]  

(13.9)

and \( c_1(B) \) and \( c_2(B) \) are the first and second Chern classes of \( B \).

- The requirement that the five-brane curve be a true submanifold of \( X \) constrains \([W]\) to be an effective class. Therefore, we must guarantee that

\[ W_B \text{ is effective in } B, \quad a_f \geq 0 \text{ integer} \]  

(13.10)

- The condition that the theory have three families imposes the further constraint that

\[ 3 = \lambda \left( W_B^2 - (24 - n)W_Bc_1(B) + 12(12 - n)c_1(B)^2 \right) \]  

(13.11)

To these conditions, we can add the remaining relevant constraint from section 4. It is

- The condition that \( c_1(N) \) be an integer leads to the constraints on \( W_B \) and \( \lambda \) given by

\[ n \text{ is odd, } \quad \lambda = m + \frac{1}{2} \]

\[ n \text{ is even, } \quad \lambda = m, \quad W_B = c_1(B) \mod 2 \]  

(13.12)

where \( m \) is an integer. The \( n \) even condition is sufficient, but not necessary.

Note that in this last condition, the class \( \eta \), which appeared in constraint (11.3), has been replaced by \( W_B \). That this replacement is valid can be seen as follows. For \( n \) odd, there is no constraint on \( \eta \) and, hence, using (13.8), no constraint on \( W_B \). When \( n \) is even, it is sufficient for \( \eta \) to satisfy \( \eta = c_1(B) \mod 2 \). Since \( 12c_1(B) \) is an even element of \( H^2(B, \mathbb{Z}) \), it follows that \( W_B = c_1(B) \mod 2 \).
It is important to note that all quantities and constraints have now been reduced to properties of the base two-fold $B$. Specifically, if we know $c_1(B)$, $c_2(B)$, as well as a set of generators of effective classes in $B$ in which to expand $W_B$, we will be able to exactly specify all appropriate non-perturbative vacua. For the del Pezzo, Hirzebruch, Enriques and blown-up Hirzebruch surfaces, all of these quantities are known.

Finally, from the expressions in (13.1) we find the following rule.

- If we denote by $G$ the structure group of the gauge bundle and by $H$ its commutant subgroup, then

\[
G = SU(3) \implies H = E_6
\]

\[
G = SU(4) \implies H = SO(10)
\] (13.13)

\[
G = SU(5) \implies H = SU(5)
\]

$H$ corresponds to the low energy gauge group of the theory.

Armed with the above rules, we now turn to the explicit construction of phenomenologically relevant non-perturbative vacua.

## 14 Three Family Models

In this section, we will construct two explicit solutions satisfying the above rules. In general, we will look for solutions where the class representing the curve on which the fivebranes wrap is comparatively simple. As discussed above, the allowed base surfaces $B$ of elliptically fibered Calabi–Yau three–folds which admit a section are restricted to be the del Pezzo, Hirzebruch and Enriques surfaces, as well as blow-ups of Hirzebruch surfaces. Relevant properties of del Pezzo, Hirzebruch and Enriques surfaces, including their generators of effective curves, are given in the Appendix B of ref. [8]. However, we now show that Calabi–Yau three–folds of this type with an Enriques base never admit an effective five–brane curve if one requires that there be three families. Recall that the cohomology class of the spectral cover must be of the form

\[
[C] = n\sigma + \eta
\] (14.1)

and this necessarily is an effective class in $X$. We may assume that $C$ does not contain $\sigma(B)$. Otherwise, replace $C$ in the following discussion with its subcover $C'$ obtained by discarding the appropriate multiples of $\sigma(B)$. This implies that the class

\[
\sigma[C] = n\sigma^2 + \sigma\eta
\] (14.2)
must be effective in the base $B$. Let us restrict $B$ to be an Enriques surface. Using the adjunction formula, we find that

$$\sigma^2 = K_B$$  \hspace{1cm} (14.3)

where $K_B$ is the torsion class. Since $nK_B$ vanishes for even $n$, it follows that when $n$ is even

$$\sigma[C] = \sigma \eta$$  \hspace{1cm} (14.4)

Clearly, $\sigma \eta$ is effective, since $\sigma[C]$ is. For $n$ odd, $nK_B = K_B$ and, hence

$$\sigma[C] = K_B + \sigma \eta$$  \hspace{1cm} (14.5)

Using the discussion in Appendix B of [8], one can still conclude that $\sigma \eta$ is either an effective class or it equals $K_B$. From the fact that

$$\sigma c_1(B) = K_B$$  \hspace{1cm} (14.6)

it follows, using equation (13.8), that the five-brane class restricted to the Enriques base is given by

$$W_B = 12K_B - \sigma \eta$$  \hspace{1cm} (14.7)

Since $12K_B$ vanishes, this becomes

$$W_B = -\sigma \eta$$  \hspace{1cm} (14.8)

from which we can conclude that $W_B$ is never effective for non-vanishing class $\sigma \eta$. Since, as explained above, $W_B$ must be effective for the five-branes to be physical, such theories must be discarded. The only possible loop-hole is when $\sigma \eta$ vanishes or equals $K_B$. However, in this case, it follows from (13.3) that

$$N_{gen} = 0$$  \hspace{1cm} (14.9)

which is also physically unacceptable. We conclude that, on general grounds, Calabi–Yau three-folds with an Enriques base never admit effective five–brane curves if one requires that there be three families. For this reason, we henceforth restrict our discussion to the remaining possibilities.

In this lecture, for specificity, the base $B$ will always be chosen to be a del Pezzo surface. We first give two $SU(5)$ examples, each on del Pezzo surfaces; one where the base component, $W_B$, is simple and one where the fiber component has a small coefficient.

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8We thank E. Witten for pointing out to us the likelihood of this conclusion.
Example 1: $B = dP_8$, $H = SU(5)$

We begin by choosing

$$H = SU(5)$$

(14.10)

as the gauge group for our model. Then it follows from (13.13) that we must choose the structure group of the gauge bundle to be

$$G = SU(5)$$

(14.11)

and, hence, $n = 5$. Since $n$ is odd, constraint (13.12) tells us that $\lambda = m + \frac{1}{2}$ for integer $m$. Here we will, for simplicity, choose $m = 1$ and, therefore

$$\lambda = \frac{3}{2}$$

(14.12)

At this point, it is necessary to explicitly choose the base surface, which we take to be

$$B = dP_8$$

(14.13)

It follows from Appendix B of ref. [8] that for the del Pezzo surface $dP_8$, a basis for $H_2(dP_8, \mathbb{Z})$ composed entirely of effective classes is given by $l$ and $E_i$ for $i = 1, \ldots, 8$ where

$$l \cdot l = 1 \quad l \cdot E_i = 0 \quad E_i \cdot E_j = -\delta_{ij}$$

(14.14)

There are other effective classes in $dP_8$ not obtainable as a linear combination of $l$ and $E_i$ with non-negative integer coefficients, but we will not need them in this example. To these we add the fiber class $F$. Furthermore

$$c_1(B) = 3l - \sum_{r=1}^{8} E_i$$

(14.15)

and

$$c_2(B) = 11$$

(14.16)

We now must specify the component of the five-brane class in the base. In this example, we choose

$$W_B = 2E_1 + E_2 + E_3$$

(14.17)

Since $E_1$, $E_2$ and $E_3$ are effective, it follows that $W_B$ is also effective, as it must be. Using the above intersection rules, one can easily show that

$$W_B^2 = -6, \quad W_Bc_1(B) = 4, \quad c_1(B)^2 = 1$$

(14.18)
Using these results, as well as $n = 5$ and $\lambda = \frac{3}{2}$, one can check that
\[
\lambda(W_B^2 - (24 - n)W_Bc_1(B) + 12(12 - n)c_1(B)^2) = 3
\] (14.19)
and, therefore, the three family condition is satisfied. Finally, let us compute the coefficient $a_f$ of $F$. Using the above information, we find that
\[
a_f = c_2(B) + \left(11 + \frac{n^3 - n}{24}\right)c_1(B)^2 - \frac{3n}{2\lambda}\left(\lambda^2 - \frac{1}{4}\right) = 17
\] (14.20)
Since this is a positive integer, it follows from the above discussion that the full five-brane curve $[W]$ is effective in the Calabi–Yau three–fold $X$, as it must be. This completes our construction of this explicit non-perturbative vacuum. It represents a model of particle physics with three families and gauge group $H = SU(5)$, along with explicit five-branes wrapped on a holomorphic curve specified by
\[
[W] = 2E_1 + E_2 + E_3 + 17F
\] (14.21)
The properties of the moduli spaces of five-branes were discussed in [7, 9].

**Example 2:** $B = dP_8$, $H = SO(10)$

As a second example, we choose the gauge group to be
\[
H = SO(10)
\] (14.22)
and, hence, the structure group
\[
G = SU(4)
\] (14.23)
Then $n = 4$. Since $n$ is even, then from constraint (13.12) we must have $\lambda = m$ where $m$ is an integer and $W_B = c_1(B) \mod 2$. Here we will choose $m = -1$ so that
\[
\lambda = -1
\] (14.24)
We will return to the choice of $W_B$ momentarily. In this example, we will take as a base surface
\[
B = dP_8
\] (14.25)
Some of the effective generators and the first and second Chern classes of $dP_8$ were given in the previous example. We now must specify the component of the five-brane class in the base. In this example, we choose
\[
W_B = 2E_1 + 2E_2 + (3l - \sum_{i=1}^{8} E_i)
\] (14.26)
Since $E_1$, $E_2$ and $3l - \sum_{i=1}^{8} E_i$ are effective, it follows that $W_B$ is also effective, as it must be. Furthermore, since

$$c_1(B) = 3l - \sum_{i=1}^{8} E_i \tag{14.27}$$

it follows that

$$W_B = c_1(B) \mod 2 \tag{14.28}$$

since $2E_1 + 2E_2$ is an even element of $H^2(dP_9, \mathbb{Z})$. Using the above intersection rules, one can easily show that

$$W_B^2 = 1, \quad W_B c_1(B) = 5, \quad c_1(B)^2 = 1 \tag{14.29}$$

Using these results, as well as $n = 4$ and $\lambda = -1$, one can check that

$$\lambda (W_B^2 - (24 - n)W_B c_1(B) + 12(12 - n)c_1(B)^2) = 3 \tag{14.30}$$

and, therefore, the three family condition is satisfied. Finally, let us compute the coefficient $a_f$ of $F$. Using the above information, we find that

$$a_f = c_2(B) + \left(11 + \frac{n^3 - n}{24}\right) c_1(B)^2 - \frac{3n}{2\lambda} \left(\lambda^2 - \frac{1}{4}\right) = 29 \tag{14.31}$$

Since this is a positive integer, it follows from the above discussion that the full five-brane curve $[W]$ is effective, as it must be. This completes our construction of this explicit non-perturbative vacuum. It represents a model of particle physics with three families and gauge group $H = SO(10)$, along with explicit five-branes wrapped on a holomorphic curve specified by

$$[W] = 2E_1 + 2E_2 + (3l - \sum_{i=1}^{8} E_i) + 29F \tag{14.32}$$

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