µ–τ interchange symmetry and lepton mixing

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We focus on the use of a µ–τ interchange symmetry to explain features of lepton mixing, especially maximal atmospheric neutrino mixing. We review two models which achieve this goal and are based on the seesaw mechanism and on the soft breaking of the family-lepton-number symmetries. We also note that that symmetry may be embedded in a generalized CP symmetry. We show that, in the context of some of our models, arguments of naturalness may be used for explaining the smallness of the mass ratio m_µ/m_τ.

1 Introduction

Experiments on neutrino oscillations have greatly enlarged our knowledge on neutrino masses and lepton mixing [1]. The lepton mixing matrix is quite different from the quark mixing (CKM) matrix [1], since it features two large mixing angles—the angles θ_{12} and θ_{23} for the solar and atmospheric neutrino oscillations, respectively—while the third mixing angle, θ_{13}, which is relevant for instance in the oscillations of reactor neutrinos, is small [2, 3].

Much effort has been devoted for quite some time to model building for neutrino masses and lepton mixing, with focus on possible underlying flavour symmetries; for recent reviews see, for instance, refs. [4] and [5] for the model-building and group-theoretical, respectively, aspects of that problem. In the basis where the charged-lepton mass matrix is diagonal, a µ–τ interchange symmetry [6] ν_µL ↔ ν_τ L in the neutrino mass terms ensures a form of the (effective) neutrino mass matrix \( M_\nu \) which simultaneously gives \( \theta_{23} = 45^\circ \) and \( \theta_{13} = 0 \). A stronger restriction on \( M_\nu \) leads to tri-bimaximal mixing [7] in which, in addition to the predictions of µ–τ symmetry, one also has \( \sin^2 \theta_{12} = 1/3 \); however, the models which aim at predicting tri-bimaximal mixing are, in our opinion, very much contrived. On the other hand, as we shall show in this review, less ambitious models which aim at including only the µ–τ symmetry are relatively simple.

Recently, evidence of \( \theta_{13} \neq 0 \) has been found [3]; this reduces the µ–τ interchange symmetry to just an approximate symmetry. Several strategies have been proposed in this context:

- Some terms in the Lagrangian may violate explicitly the µ–τ symmetry. This has been discussed for instance in ref. [8]. We shall not pursue that possibility any further in this review.
- Radiative corrections to \( M_\nu \) induced by the renormalization-group running may disturb the µ–τ interchange symmetry. In this mechanism, the µ–τ symmetry is assumed to be exact in \( M_\nu \) at some high energy scale, e.g. at the seesaw scale [9]; the renormalization-group evolution of the neutrino mass operators from that high scale to the electroweak scale produces a mixing among those operators which, after insertion of the vacuum expectation values (VEVs), adds a µ–τ-antisymmetric contribution to the original µ–τ-symmetric part of \( M_\nu \) [10]. Note that this mechanism must be active in any realistic model, since scalars odd under the µ–τ interchange symmetry are necessarily present in order to achieve \( m_\mu \neq m_\tau \). As noted in ref. [10], however, this mechanism requires a quasi-degenerate neutrino mass spectrum in order to produce a sizeable \( \theta_{12} \); such a quasi-degenerate spectrum is on the verge of being ruled out by cosmological arguments [11].

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• One may replace the $\mu$–$\tau$ symmetry by a generalized CP symmetry which includes $\mu$–$\tau$ interchange. This mechanism \cite{11,12}, which will be discussed in this review, leaves $\theta_{13}$ free and predicts instead maximal CP violation via the CKM-type phase $\delta$ of the lepton mixing matrix.

In the construction of $\mu$–$\tau$-symmetric models, the most straightforward mechanism for achieving small neutrino masses is the seesaw mechanism \cite{9}. In a minimal version of that mechanism one may content oneself with two right-handed neutrino gauge singlets, as was done for instance in refs. \cite{13,14}. In this review, however, we shall always use three heavy neutrino singlets, and also assume the existence of only three light neutrinos.

For completeness we want to mention here some aspects which have been investigated in the framework of the $\mu$–$\tau$ interchange symmetry but will not be discussed in this review:

• The $\mu$–$\tau$ symmetry may be incorporated in grand unified models based on the gauge group $SU(5)$ \cite{15,16}.

• The $\mu$–$\tau$ symmetry may be extended to the quark sector, thus promoting it to a universal flavour symmetry \cite{17}.

• Leptogenesis can successfully be implemented in $\mu$–$\tau$-symmetric models \cite{18}.

• The $\mu$–$\tau$ symmetry may be supplemented by further conditions, for instance by texture zeros, in order to enhance the predictivity of a model \cite{19}.

This review is organized as follows. After section 2, which contains some elementary notation, we proceed in section 3 to expose the consequences for lepton mixing of a $\mu$–$\tau$-symmetric neutrino mass matrix $M_\nu$, and of a $M_\nu$-symmetric under a generalized CP transformation involving the $\mu$–$\tau$ interchange. Sections 4 and 5 contain specific models which lead to the neutrino mass matrices studied in section 3. Section 6 is devoted to the addition, to the models of sections 4 and 5, of an extra symmetry $K$ which allows one to explain the smallness of the ratio $m_\mu/m_\tau$ as the natural result of the small soft breaking of $K$. Section 7 shortly summarizes the contents of this review.

2 Notation

Let $M_\nu$ denote the light-neutrino (effective) Majorana mass matrix in the weak basis where the charged-lepton mass matrix is diagonal. Thus,

$$L_{\text{lepton mass}} = - \left( \begin{array}{ccc} e_L, & \bar{\mu}_L, & \bar{\tau}_L \end{array} \right) \left( \begin{array}{ccc} m_e & 0 & 0 \\ 0 & m_\mu & 0 \\ 0 & 0 & m_\tau \end{array} \right) \left( \begin{array}{c} e_R \\ \mu_R \\ \tau_R \end{array} \right) + \frac{1}{2} \left( \begin{array}{ccc} \nu_{eL}^T & \nu_{\mu L}^T & \nu_{\tau L}^T \end{array} \right) M_\nu C^{-1} \left( \begin{array}{c} \nu_{eL} \\ \nu_{\mu L} \\ \nu_{\tau L} \end{array} \right) + \text{H.c.}, \quad (1)$$

where $C$ is the Dirac–Pauli charge-conjugation matrix. The matrix $M_\nu$ is symmetric, hence it can be bi-diagonalized by a single unitary matrix $U$ via

$$U^T M_\nu U = \text{diag} (m_1, m_2, m_3) \equiv \tilde{m}, \quad (2)$$

where the $m_j$ ($j = 1, 2, 3$) are non-negative real. Equation (2) means that

$$M_\nu c_j = m_j c_j^*, \quad (3)$$

(no sum over $j$ is assumed), where the $c_j$ denote the three columns of $U = (c_1, c_2, c_3)$. 

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If we consider the charged-current Lagrangian

\[ \mathcal{L}_{\text{lepton-W}} = \frac{g}{\sqrt{2}} W^-_{\mu} \left( \bar{e}_L, \ \bar{\mu}_L, \ \bar{\tau}_L \right) \gamma^\mu \begin{pmatrix} \nu_{\mu L} \\ \nu_{\tau L} \end{pmatrix} + \text{H.c.} \]

\[ = \frac{g}{\sqrt{2}} W^-_{\mu} \left( \bar{e}_L, \ \bar{\mu}_L, \ \bar{\tau}_L \right) \gamma^\mu U \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix} + \text{H.c.}, \quad (4) \]

we see that \( U \) is, indeed, the lepton mixing (PMNS) matrix.

3 \( \mu-\tau \) interchange symmetry

A \( \mu-\tau \)-symmetric \( \mathcal{M}_\nu \) is, by definition, of the form

\[ \mathcal{M}_\nu = \begin{pmatrix} x & y & y \\ y & z & w \\ y & w & z \end{pmatrix}, \quad (5) \]

where \( x, y, z, \) and \( w \) are in general complex. The mass matrix \( \mathcal{M}_\nu \) of equation (5) can be characterized as being the general solution of the equation \( S \mathcal{M}_\nu S = \mathcal{M}_\nu \) \( \quad \text{(6)} \)

with \( S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \) \( \quad \text{(7)} \)

Notice that the Lagrangian can never enjoy full \( \mu-\tau \) interchange symmetry, since the masses of the muon (\( m_\mu \)) and tau (\( m_\tau \)) charged leptons differ (\( m_\mu \neq m_\tau \)); thus, a \( \mu-\tau \) interchange symmetry exists, at most, in the neutrino mass matrix.

Since the \( \mu-\tau \) interchange symmetry, viz. the matrix \( S \) of equation (7), is a \( \mathbb{Z}_2 \) symmetry, it has eigenvalues \( \pm 1 \). The matrix in equation (5) corresponds to the eigenvalue +1, \( \text{cf.} \) equation (6). Corresponding to the eigenvalue −1, one might consider \( \text{[21, 22]} \) a \( \mu-\tau \)-antisymmetric

\[ \mathcal{M}_\nu = \begin{pmatrix} 0 & y & -y \\ y & z & 0 \\ -y & 0 & -z \end{pmatrix}, \quad (8) \]

which is the solution of \( S \mathcal{M}_\nu S = -\mathcal{M}_\nu \). However, the matrix in equation (8) leads not only to maximal atmospheric mixing but also to maximal solar mixing and to two degenerate neutrinos \( \text{[22]} \) and can therefore, at best, serve as a first approximation for the true \( \mathcal{M}_\nu \). We shall not consider equation (8) any further in this review.

Because of the freedom of rephasing the lepton fields in equations (1) and (4), the matrix \( \mathcal{M}_\nu \) in equation (5) may be generalized and we need only require

\[ |(\mathcal{M}_\nu)_{\mu\mu}| = |(\mathcal{M}_\nu)_{\tau\tau}|, \]

\[ |(\mathcal{M}_\nu)_{e\mu}| = |(\mathcal{M}_\nu)_{e\tau}|, \]

\[ \text{arg} [(\mathcal{M}_\nu)_{ee} (\mathcal{M}_\nu)_{\mu\mu} (\mathcal{M}_\nu)_{e\mu}]^2 = \text{arg} [(\mathcal{M}_\nu)_{ee} (\mathcal{M}_\nu)_{\tau\tau} (\mathcal{M}_\nu)_{e\tau}]^2 \]

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in order to obtain a $\mu\tau$-invariant $\mathcal{M}_\nu$. Equations (9) constitute two constraints on the moduli and one constraint on a phase of the matrix elements of $\mathcal{M}_\nu$. We expect an identical number of constraints on the lepton-mixing observables to follow from the three conditions (9).

Indeed, the characteristic feature of the matrix $\mathcal{M}_\nu$ in equation (5) is that it has an eigenvector $(0, 1, -1)$ corresponding to the eigenvalue $z - w$. Let $\varphi = \text{arg}(z - w)$, then

$$
\mathcal{M}_\nu = \begin{pmatrix}
0 & e^{-i\varphi/2} \\
e^{-i\varphi/2} & 0
\end{pmatrix} = |z - w| \begin{pmatrix}
0 & e^{i\varphi/2} \\
e^{i\varphi/2} & 0
\end{pmatrix}.
$$

This is an instance of equation (3). One thus finds that a $\mu\tau$-symmetric $\mathcal{M}_\nu$ leads to one of the columns of the PMNS matrix being $(0, e^{-i\varphi/2}, -e^{-i\varphi/2})^T$. Phenomenologically, this is realistic only if that column is the third one, i.e. the one corresponding to the neutrino ($\nu_3$) which has a mass ($m_3$) most afar from the other two. Indeed, phenomenologically $U_{e3}$ is very close to zero but $U_{e1}$ and $U_{e2}$ are certainly non-zero.

We thus conclude that a $\mu\tau$-symmetric $\mathcal{M}_\nu$ leads to

$$
U_{e3} = 0, \quad \theta_{23} = 45^\circ; \tag{11,12}
$$

the condition (12) is equivalent to $|U_{\mu3}| = |U_{\tau3}|$, viz. maximal atmospheric neutrino mixing. Since the condition (11) entails that the Dirac phase $\delta$ is meaningless and may be removed from $U$, equations (11) and (12) in fact constitute two constraints on the moduli and one constraint on a phase of the PMNS matrix elements, as expected.

The matrix $\mathcal{M}_\nu$ in equation (5) does not impose any constraint on the neutrino masses. One has, from equation (10),

$$
m_3 = |z - w|. \tag{13}
$$

The other two masses may be obtained by considering the determinant and the trace of $\mathcal{M}_\nu\mathcal{M}_\nu^\dagger$: \n
$$
m_1 m_2 = |x(z + w) - 2y^2|, \quad m_1^2 + m_2^2 = |x|^2 + |z + w|^2 + 4|y|^2. \tag{14,15}
$$

The solar mixing angle is given by (23)

$$
\tan 2\theta_{12} = \frac{2\sqrt{2}|x^*y + y^*(z + w)|}{|z + w|^2 - |x|^2}. \tag{16}
$$

Note that tri-bimaximal mixing, i.e. $\tan 2\theta_{12} = 2\sqrt{2}$, ensues if some additional symmetry enforces $|x^*y + y^*(z + w)| = \sqrt{|x|^2 - |z + w|^2}$. This is the case, in particular, if the $\mu\tau$-symmetric $\mathcal{M}_\nu$ has the additional property that the sums of its matrix elements over each row are all equal, viz. $x + y = z + w$.

Besides $\mu\tau$ interchange, one may consider the superposition of the $\mu\tau$ interchange on a CP transformation, i.e. one may require invariance of the mass Lagrangian in the second line of equation (11) under the generalized CP transformation (11,12)

$$
\nu_{eL} \rightarrow i\gamma^0 C \bar{\nu}_{eL}^T, \quad \nu_{\mu L} \rightarrow i\gamma^0 C \bar{\nu}_{\tau L}^T, \quad \nu_{\tau L} \rightarrow i\gamma^0 C \bar{\nu}_{\mu L}^T. \tag{17}
$$

That invariance requirement leads to (12,20)

$$
\mathcal{M}_\nu = \begin{pmatrix}
a & r & r^* \\
r & s & b \\
r^* & b & s^*
\end{pmatrix}, \tag{18}
$$
where \( r \) and \( s \) are in general complex but \( a \) and \( b \) are real. The \( \mathcal{M}_\nu \) in equation (13) is the general solution of the equation

\[
S \mathcal{M}_\nu S = \mathcal{M}_\nu^*.
\]

The neutrino mass matrix of equation (18) may also be obtained from an \( A_4 \) model [24] without invoking invariance under the CP transformation of equation (17).

What are the predictions the mass matrix of equation (18)? We first use equation (2) to express that matrix as \( \mathcal{M}_\nu = U^* \hat{m} U \). This allows one to reformulate equation (19) as

\[
X \hat{m} = \hat{m}^* X^*,
\]

with

\[
X = U^* S U^*.
\]

The matrix \( X \) is symmetric and unitary. Equation (20) has the following real and imaginary components:

\[
(m_i - m_j) \text{Re} X_{ij} = 0,
\]

(22)

\[
(m_i + m_j) \text{Im} X_{ij} = 0.
\]

(23)

Since both the solar and the atmospheric neutrino mass-squared differences are non-zero, \( \hat{m} \) is non-degenerate. Consequently, from equation (22), \( \text{Re} X_{ij} = 0 \) if \( i \neq j \). Moreover, the neutrino masses \( m_j \) are non-negative and at most one of them may vanish. Consequently, from equation (23), \( \text{Im} X_{ij} = 0 \) if \( i \neq j \). Thus, \( X \) is diagonal. Since \( X \) is unitary, its diagonal elements are phases. Because of equation (23), \( X_{jj} = \pm 1 \) whenever \( m_j \) is non-zero. If \( m_j = 0 \), then \( X_{jj} \) is an arbitrary phase; however, in that case one can absorb a factor \((X_{jj})^{-1/2}\) into the neutrino field \( \nu_j \), whereupon \( X_{jj} \) becomes equal to 1. Now, we rewrite equation (21) as

\[
S U^* = U X.
\]

Equation (24) determines the following forms for the columns \( c_j \) of \( U \) [12]:

\[
X_{jj} = 1 \Rightarrow c_j = \begin{pmatrix} u_j \\ w_j \\ w_j^* \end{pmatrix},
\]

\[
X_{jj} = -1 \Rightarrow c_j = \begin{pmatrix} i u_j \\ w_j \\ -w_j^* \end{pmatrix},
\]

(25)

with real \( u_j \). One reads off from equations (25) that [11, 12]

\[
|U_{\mu j}| = |U_{\tau j}| \quad \text{for} \quad j = 1, 2, 3.
\]

(26)

Inserting this condition into the standard parameterization of \( U \) yields the predictions

\[
\theta_{23} = 45^\circ,
\]

(27)

\[
\sin \theta_{13} \cos \delta = 0.
\]

(28)

Moreover, equations (25) show that the relative phases among \( U_{e1}, U_{e2}, \) and \( U_{e3} \) are multiples of 90°; this means that the Majorana phases, which arise for instance in the computation of the effective mass relevant for neutrinoless \( \beta \beta \) decay, have values either zero or \( \pi \).
4 Softly broken lepton numbers and $\mu-\tau$ interchange symmetry

In this section we discuss a scenario where the $\mu-\tau$ interchange symmetry is embedded in a model with softly broken family lepton numbers \cite{23}. Let $\mathbb{Z}_2^{(\mu\tau)}$ denote the cyclic group of order two generated by the $\mu-\tau$ interchange symmetry. The model of ref. \cite{23} is a (type I) seesaw model in which three right-handed heavy neutrino singlets $\nu_{aR}$ ($\alpha = e, \mu, \tau$) are added to the Standard Model multiplets. Moreover, as we will see shortly, the model needs three Higgs doublets $\phi_j$. The symmetries of the model are the following:

i) The $U(1)_{L_{\alpha}}$ associated with the family lepton numbers $L_{\alpha}$.

\begin{align}
\mathbb{Z}_2^{(\mu\tau)} : \quad & D_{\mu L} \leftrightarrow D_{\tau L}, \quad \mu_R \leftrightarrow \tau_R, \quad \nu_{\mu R} \leftrightarrow \nu_{\tau R}, \quad \phi_3 \rightarrow -\phi_3. \\
\mathbb{Z}_2^{(\text{aux})} : \quad & \nu_{e R}, \quad \nu_{\mu R}, \quad \nu_{\tau R}, \quad \phi_1, \quad e_R \text{ change sign}. 
\end{align}

(29)

(30)

In our notation, $D_{\alpha L}$ denotes the left-handed-lepton gauge-$SU(2)$ doublets and $\alpha_R$ denotes the right-handed charged-lepton $SU(2)$ singlets. The symmetry $\mathbb{Z}_2^{(\mu\tau)}$ transposes the muon and tau family and is spontaneously broken by the VEV of $\phi_3$; as pointed out earlier, that Higgs doublet is necessary in order to obtain $m_\mu \neq m_\tau$. The symmetry $\mathbb{Z}_2^{(\text{aux})}$, which is spontaneously broken by the VEV of $\phi_1$, is auxiliary and forbids the Yukawa coupling of $\phi_3$ to the $\nu_{aR}$; this is imperative for keeping the neutrino mass matrix $\mu-\tau$-symmetric at tree level.

The above symmetries and multiplets uniquely determine the Yukawa Lagrangian as

\begin{align}
\mathcal{L}_Y &= -y_1 D_{e L} \nu_{e R} \tilde{\phi}_1 - y_2 \left( D_{\mu L} \nu_{\mu R} + D_{\tau L} \nu_{\tau R} \right) \tilde{\phi}_1 \\
&\quad - y_3 D_{e L} \nu_{e R} \tilde{\phi}_3 - y_4 \left( D_{\mu L} \mu_R + D_{\tau L} \tau_R \right) \phi_2 - y_5 \left( D_{\mu L} \mu_R - D_{\tau L} \tau_R \right) \phi_3 + \text{H.c.} 
\end{align}

(31)

Due to the symmetries $U(1)_{L_{\alpha}}$ the Yukawa couplings are flavour-diagonal. There must be a source of non-trivial lepton mixing and the obvious choice for this purpose are the Majorana mass terms of the $\nu_{aR}$. These mass terms have dimension three, therefore there will be soft breaking of the family lepton numbers\footnote{The same mechanism was used in ref. \cite{23} for the purpose of breaking $L_e - L_\mu - L_\tau$.}.

The mass Lagrangian of the $\nu_{aR}$ is given by

\begin{align}
\mathcal{L}_{\text{Maj}} &= \frac{1}{2} \begin{pmatrix} \nu^T_{e R}, & \nu^T_{\mu R}, & \nu^T_{\tau R} \end{pmatrix} C^{-1} M_R^\dagger \begin{pmatrix} \nu_{e R} \\ \nu_{\mu R} \\ \nu_{\tau R} \end{pmatrix} + \text{H.c.}
\end{align}

(32)

In this model $M_R$ is the only source of lepton mixing. Although $\mathcal{L}_{\text{Maj}}$ breaks the family lepton numbers softly, it has to preserve the $\mu-\tau$ symmetry. Therefore, the mass matrix $M_R$ must be $\mu-\tau$-symmetric. Consequently, the only source of $\mathbb{Z}_2^{(\mu\tau)}$ breaking is the VEV of $\phi_3$.

From equation (31) we read off that the neutrino Dirac mass matrix has the $\mu-\tau$-symmetric form $M_D = \text{diag} (c, d, d)$. With the seesaw formula we obtain $\mathcal{M}_\nu$ as

\begin{align}
\mathcal{M}_\nu &= -M_D^T M_R^{-1} M_D. 
\end{align}

(33)

Since both $M_D$ and $M_R$ are $\mu-\tau$-symmetric, the same applies to $\mathcal{M}_\nu$, which, therefore, has the form displayed in equation (5). Moreover, since the charged-lepton mass matrix is diagonal, the lepton mixing matrix is obtained by diagonalizing $\mathcal{M}_\nu$ according to equation (2).

The symmetries on which the model is based generate a non-Abelian symmetry group because the $U(1)_{L_{\alpha}}$ with $\alpha = \mu, \tau$ do not commute with $\mathbb{Z}_2^{(\mu\tau)}$. It is easy to show that the non-Abelian core of the symmetry group is $O(2)$ \cite{15}, with $U(1)_{(L_\mu - L_\tau)/2}$ and $\mathbb{Z}_2^{(\mu\tau)}$ corresponding, respectively, to the $U(1)$ and the reflection contained in $O(2)$. In the present model, the Higgs doublets transform according to one-dimensional irreducible representations of $O(2)$. For a viable model with Higgs doublets in a two-dimensional irreducible representation of $O(2)$, see ref. \cite{26}.

An interesting property of the model in this section is that lepton-flavour-changing processes have finite one-loop amplitudes \cite{27}. The reason is that flavour violation occurs only via soft $L_{\alpha}$-breaking terms.

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5 A model based on a generalized CP transformation

In this section we modify the model of the previous one in order to obtain a model which reproduces the neutrino mass matrix of equation (18). The new model has exactly the same multiplets as the one of section 4. The symmetries $U(1)_{L_\alpha}$ and $\mathbb{Z}_2^{(aux)}$ also apply. The new feature is that $\mathbb{M}_2^{(\mu \tau)}$ is replaced by the non-standard (generalized) CP transformation $[11, 12]$

\[
\begin{align*}
D_{\alpha L} & \rightarrow iS_{\alpha \beta} \gamma^0 C \bar{D}_{\beta L}^T, \\
\nu_{\alpha R} & \rightarrow iS_{\alpha \beta} \gamma^0 \nu_{\beta R}^T, \\
\alpha_R & \rightarrow iS_{\alpha \gamma} \gamma^0 \bar{C} \beta_R^T, \\
\phi_{1,2} & \rightarrow \phi_{1,2}^*, \\
\phi_3 & \rightarrow -\phi_3,
\end{align*}
\]  

(34)

where $\alpha, \beta = e, \mu, \tau$ and $S$ is the matrix in equation (7). The resulting Yukawa Lagrangian is very similar, but not quite identical, to the one of the previous model, cf. equation (31):

\[
L_Y' = -y_1 \bar{D}_e \nu_e R \tilde{\phi}_1 - (y_2 \bar{D}_\mu \nu_\mu R + y_2^* \bar{D}_\tau \nu_\tau R) \tilde{\phi}_1 - y_3 \bar{D}_e e R \phi_1 - (y_4 \bar{D}_\mu \mu R + y_4^* \bar{D}_\tau \tau R) \phi_2 - (y_5 \bar{D}_\mu \mu R - y_5^* \bar{D}_\tau \tau R) \phi_3 + \text{H.c.} \]  

(35)

The coupling constants $y_1$ and $y_3$ are real whereas $y_2, y_4,$ and $y_5$ are in general complex.

The breaking of the symmetries proceeds in the same way as in the previous model. In particular, the family lepton numbers are broken softly by the Majorana mass terms of the $\nu_{\alpha R}$. Because of the non-standard CP transformation of equation (34), the mass matrix $M_R$ in equation (32) is of the same form of the $M_\nu$ in equation (18). Moreover, assuming, without loss of generality, that the VEV of $\phi_1$ is real, we read off from the Lagrangian of equation (35) that $M_D = \text{diag}(c, d, d^*)$ with real $c$. Therefore, $M_D$ fulfills $SM_D S^\dagger = M_\nu^\dagger$. Since by construction the same is true for $M_R$, the neutrino mass matrix resulting from the present model is the one in equation (18).

6 On the small ratio of muon to tau mass

In models with $\mu$–$\tau$ interchange symmetry one would naively expect the muon and tau charged-lepton masses to be of the same order of magnitude. However, this is not so, since $m_\mu / m_\tau \approx 1/18$. Moreover, assuming, without loss of generality, that the VEV of $\phi_1$ is real, we read off from the Lagrangian of equation (35) that $M_D = \text{diag}(c, d, d^*)$ with real $c$. Therefore, $M_D$ fulfills $SM_D S = M_\nu^\dagger$. Since by construction the same is true for $M_R$, the neutrino mass matrix resulting from the present model is the one in equation (18).

Let us consider the muon and tau masses in the models of refs. [23] and [12]. Denoting the VEV of $\phi_j^0$ ($j = 1, 2, 3$) by $v_j / \sqrt{2}$, the muon and tau masses are given by

\[
m_\mu = \frac{1}{\sqrt{2}} |y_4 v_2 + y_5 v_3|, \]  

(36)

\[
m_\tau = \frac{1}{\sqrt{2}} |y_4 v_2 - y_5 v_3|, \]  

(37)

respectively, in the model of ref. [23], cf. equation (31), while in the model of ref. [12] those masses are given by

\[
m_\mu = \frac{1}{\sqrt{2}} |y_4 v_2 + y_5 v_3|, \]  

(38)

\[
m_\tau = \frac{1}{\sqrt{2}} |y_4^* v_2 - y_5^* v_3|. \]  

(39)

Clearly, one must use finetuning for obtaining $m_\mu \ll m_\tau$. Moreover, as one can read off from the formulas above, that finetuning is rather awkward since one has to choose two products of unrelated quantities—one
Yukawa coupling and one VEV—such that those two products nearly cancel in \( m_\mu \). In order to soften the amount of finetuning, we have proposed in refs. [12, 23] to add to the models the following symmetry:

\[
K : \quad \mu_R \to -\mu_R, \quad \phi_2 \leftrightarrow \phi_3.
\]  

(40)

This symmetry leads to

\[
y_4 = -y_5.
\]  

(41)

in both the Lagrangians of equations (31) and (35). In this way the finetuning is confined to the VEVs, since

\[
m_\mu \sim \left| \frac{v_2 - v_3}{v_2 + v_3} \right|.
\]  

(42)

The symmetry \( K \) has the nice property that it can be directly implemented in both models; its effect is simply to reduce the number of free parameters.

The symmetry \( K \) also affects the scalar potential. With \( K \) the minimum of the potential features \( v_2 = v_3 \), under certain conditions on the coupling constants [23]. Therefore, \( K \) leads to \( m_\mu = 0 \) and has to be broken. The idea is to break it softly, through terms of dimension two in the scalar potential. With this mechanism, \( m_\mu \ll m_\tau \) can be justified, at least in a technically natural way, since it will be the consequence of a small soft breaking.

There are some differences between the scalar potentials \( V \) and \( V' \) of the models of refs. [23] and [12], respectively. We first treat the simpler case of the model of ref. [23]. The symmetries \( \mathbb{Z}_2^{(tr)} \) in equation (29) and \( \mathbb{Z}_2^{(aux)} \) in equation (30) require that in every term of \( V \) each scalar doublet \( \phi_i \) occurs an even number of times. Requiring in addition invariance under the symmetry \( K \) in equation (40), one finds the potential

\[
V_\phi = -\mu_1 \phi_1^\dagger \phi_1 - \mu_2 \left( \phi_2^\dagger \phi_2 + \phi_3^\dagger \phi_3 \right) + \lambda_1 \left( \phi_1^\dagger \phi_1 \right)^2 + \lambda_2 \left( \phi_2^\dagger \phi_2 \right)^2 + \left( \phi_3^\dagger \phi_3 \right)^2
\]

\[
+ \lambda_3 \phi_1^\dagger \phi_1 \left( \phi_2^\dagger \phi_2 + \phi_3^\dagger \phi_3 \right) + \lambda_4 \phi_2^\dagger \phi_2 \phi_3^\dagger \phi_3
\]

\[
+ \lambda_5 \left( \phi_1^\dagger \phi_2 \phi_3^\dagger \phi_1 + \phi_1^\dagger \phi_3 \phi_2^\dagger \phi_1 \right) + \lambda_6 \phi_2^\dagger \phi_3 \phi_3^\dagger \phi_2
\]

\[
+ \lambda_7 \left( \phi_2^\dagger \phi_3 \right)^2 + \text{H.c.} + \left\{ \lambda_8 \left( \phi_1^\dagger \phi_2 \right)^2 + \left( \phi_1^\dagger \phi_3 \right)^2 \right\} + \text{H.c.}.
\]  

(43)

All the coupling constants in \( V_\phi \), except for \( \lambda_8 \), are real. The term which conserves \( \mathbb{Z}_2^{(tr)} \) and \( \mathbb{Z}_2^{(aux)} \) but breaks \( K \) softly is unique:

\[
V_{\text{soft}} = \mu_{\text{soft}} \left( \phi_1^\dagger \phi_2 - \phi_3^\dagger \phi_3 \right).
\]  

(44)

The full potential is \( V = V_\phi + V_{\text{soft}} \). It can be shown [23] that \( \bar{\lambda} \equiv -2\lambda_2 + \lambda_4 + \lambda_6 + 2\lambda_7 < 0 \) and \( \lambda_7 < 0 \) are sufficient conditions for the minimum of \( V_\phi \) to be at \( v_2 = v_3 \). It is then straightforward to find the approximate relation

\[
\frac{m_\mu}{m_\tau} \approx \left| \frac{2\mu_{\text{soft}}}{\bar{\lambda} \left( |v_2|^2 + |v_3|^2 \right)} \right|.
\]  

(45)

which explicitly displays that a small \( \mu_{\text{soft}} \) leads to a small \( m_\mu/m_\tau \).

Replacing \( \mathbb{Z}_2^{(tr)} \) by the generalized CP transformation of equation (34) allows two extra terms in the scalar potential:

\[
V_9 = \lambda_9 \left( i\phi_1^\dagger \phi_2 \phi_3^\dagger \phi_1 + \text{H.c.} \right),
\]  

(46)

\[
V_{10} = \lambda_{10} \left( \phi_2^\dagger \phi_2 - \phi_3^\dagger \phi_3 \right) \left( i\phi_1^\dagger \phi_3 + \text{H.c.} \right),
\]  

(47)

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with real $\lambda_0$ and $\lambda_{10}$. Moreover, in equation (43) the coupling constant $\lambda_8$ must now be real. To complicate matters, besides the soft $K$-breaking term in equation (44), a second soft-breaking term is allowed, which is given by

$$V''_{\text{soft}} = \mu'_{\text{soft}} \left(i \phi_2^\dagger \phi_3 + \text{H.c.}\right). \quad (48)$$

The full potential under discussion is then $V'' = V_0 + V_9 + V_{10} + V''_{\text{soft}}$. Though it is now more involved, it can still be demonstrated that a non-zero ratio $m_\mu/m_\tau$ arises at first order in the soft $K$-breaking parameters $\mu_{\text{soft}}$ and $\mu'_{\text{soft}}$. One and the same mechanism is operative in both models.

7 Summary

In this review we have seen that a $\mu-\tau$-symmetric effective light-neutrino Majorana mass matrix $M_\nu$ leads to the predictions $\theta_{23} = \pi/4$ and $U_{e3} = 0$. Since the second prediction now seems to be in disagreement with phenomenology, one may alternatively employ a $M_\nu$ symmetric under a CP transformation involving the $\mu-\tau$ interchange symmetry; in that case, $\theta_{23} = \pi/4$ and $\cos \delta = 0$ ensue. One has in both cases, at the tree level, maximal atmospheric neutrino mixing ($\theta_{23} = 45^\circ$), a prediction which is in full agreement with experiment.

We have proceeded to show that both effective $M_\nu$‘s may be obtained from simple models based on the type I seesaw mechanism and on family-lepton-number symmetries which are softly broken in the dimension-three Majorana mass matrix $M_R$ of the right-handed gauge-singlet neutrinos. Those models may be furnished with an extra symmetry $K$, whose soft breaking through dimension-two terms in the scalar potential allows one to explain, in a natural fashion, the small but non-zero ratio of the muon and tau charged-lepton masses.

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References

[1] K. Nakamura et al. (Particle Data Group), Review of particle physics, J. Phys. G: Nucl. Part. Phys. 37, 075021 (2010).
[2] M. Maltoni, T. Schwetz, M.A. Törnola, and J.W.F. Valle, Status of global fits to neutrino oscillations, New J. Phys. 6, 122 (2004) [hep-ph/0405172];
G.L. Fogli, E. Lisi, A. Marrone, and A. Palazzo, Global analysis of three-flavor neutrino masses and mixings, Prog. Part. Nucl. Phys. 57, 742 (2006) [hep-ph/0506083];
G.L. Fogli, E. Lisi, A. Marrone, A. Palazzo, and A.M. Rotunno, Hints of $\theta_{13} > 0$ from global neutrino data analysis, Phys. Rev. Lett. 101, 141801 (2008) [arXiv:0806.1649];
T. Schwetz, M. Törnola, and J.W.F. Valle, Three-flavour neutrino oscillation update, New J. Phys. 10, 113011 (2008) [arXiv:0808.2016];
M.C. González-García, M. Maltoni, and J. Salvado, Updated global fit to three neutrino mixing: status of the hints of $\theta_{13} > 0$, J. High Energy Phys. 04, 056 (2010) [arXiv:1001.4524];
T. Schwetz, M. Törnola, and J.W.F. Valle, Global neutrino data and recent reactor fluxes: status of three-flavor oscillation parameters, New J. Phys. 13, 063004 (2011) [arXiv:1103.0754];
[3] G.L. Fogli, E. Lisi, A. Marrone, A. Palazzo, and A.M. Rotunno, Evidence of $\theta_{13} > 0$ from global neutrino data analysis, Phys. Rev. D 84, 053007 (2011) [arXiv:1105.6028];
T. Schwetz, M. Törnola, and J.W.F. Valle, Where we are on $\theta_{13}$: addendum to ‘Global neutrino data and recent reactor fluxes: status of three-flavor oscillation parameters’, New J. Phys. 13, 109401 (2011) [arXiv:1108.1376];
F.P. An et al. (Daya Bay Coll.), Observation of electron-antineutrino disappearance, [arXiv:1203.1609];
S.-B. Kim et al. (RENO Coll.), Observation of reactor electron antineutrino disappearance in the RENO experiment, [arXiv:1204.0626];
D.V. Forero, M. Törnola, and J.W.F. Valle, Global status of neutrino oscillations after recent reactor measurements, [arXiv:1205.4018].
G.L. Fogli, E. Lisi, A. Marrone, D. Montanino, A. Palazzo, and A.M. Rotunno, *Global analysis of neutrino masses, mixings and phases: entering the era of leptonic CP violation searches*, arXiv:1205.5254

G. Altarelli and F. Feruglio, *Discrete flavor symmetries and models of neutrino mixing*, Rev. Mod. Phys. **82**, 2701 (2010) [arXiv:1002.0211];

A. Yu. Smirnov, *Discrete symmetries and models of flavor mixing*, J. Phys. Conf. Ser. **335**, 012006 (2011) [arXiv:1103.3461];

H. Ishimori, T. Kobayashi, H. Okada, Y. Shimizu, and M. Tanimoto, *Non-Abelian discrete symmetries in particle physics*, Prog. Theor. Phys. Suppl. **183**, 1 (2010) [arXiv:1003.3552];

W. Grimus and P.O. Ludl, *Finite flavour groups of fermions*, arXiv:1110.6376.

T. Fukuyama and H. Nishiura, *Mass matrix of Majorana neutrinos*, hep-ph/9702253

R.N. Mohapatra and S. Nussinov, *Bimaximal neutrino mixing and neutrino mass matrix*, Phys. Rev. D **60**, 013002 (1999) [hep-ph/9809415];

E. Ma and M. Raidal, *Neutrino mass, muon anomalous magnetic moment, and lepton flavor nonconservation*, Phys. Rev. Lett. **87**, 011802 (2001); *erratum ibid.**87**, 159901 (2001) [hep-ph/0102255];

C.S. Lam, *A 2–3 symmetry in neutrino oscillations*, Phys. Lett. B **507**, 214 (2001) [hep-ph/0104116];

K.R.S. Balaji, W. Grimus, and T. Schwetz, *The solar LMA neutrino oscillation solution in the Zee model*, Phys. Lett. B **508**, 301 (2001) [hep-ph/0104035];

E. Ma, *The all-purpose neutrino mass matrix*, Phys. Rev. D **66**, 117301 (2002) [hep-ph/0207352];

C.S. Lam, *A supersymmetric contribution to the neutrino mass matrix and breaking of μ–τ symmetry*, Phys. Rev. D **74**, 017701 (2006) [hep-ph/0603206];

S. Luo and Z.-Z. Xing, *Friedberg–Lee symmetry breaking and its prediction for θ13*, Phys. Lett. B **646**, 242 (2007) [hep-ph/0611360];

Y. Koide and H. Nishiura, *How can CP violation in the neutrino sector be large in a 2 ↔ 3 symmetric model?*, Int. J. Mod. Phys. A **25**, 3661 (2010) [arXiv:0911.2279].

P. Minkowski, *μ → eγ at a rate of one out of 109 muon decays?*, Phys. Lett. B **67**, 421 (1977);

T. Yanagida, *Horizontal symmetry and masses of neutrinos*, in: *Proceedings of the workshop on unified theory and baryon number in the universe*, edited by O. Sawata and A. Sugamoto, KEK report 79-18, Tsukuba, Japan (1979);

S.L. Glashow, *The future of elementary particle physics*, in: *Quarks and leptons, proceedings of the advanced study institute* (Cargèse, Corsica, 1979), edited by J.-L. Basdevant et al. (Plenum, New York, 1981);

M. Gell-Mann, P. Ramond, and R. Slansky, *Complex spinors and unified theories*, in: *Supergravity*, edited by D.Z. Freedman and F. van Nieuwenhuizen (North Holland, Amsterdam, 1979);

R.N. Mohapatra and G. Senjanović, *Neutrino mass and spontaneous parity violation*, Phys. Rev. Lett. **44**, 912 (1980).

W. Grimus and L. Lavoura, *Renormalization of the neutrino mass operators in the multi-Higgs-doublet Standard Model*, Eur. Phys. J. C **39**, 219 (2005) [hep-ph/0409231].

P.F. Harrison and W.G. Scott, *μ–τ reflection symmetry in lepton mixing and neutrino oscillations*, Phys. Lett. B **547**, 219 (2002) [hep-ph/0210197];

W. Grimus and L. Lavoura, *A non-standard CP transformation leading to maximal atmospheric neutrino mixing*, Phys. Lett. B **579**, 113 (2004) [hep-ph/0305309];

T. Baba and M. Yasač, *Leptonic CP violation induced by approximately μ–τ symmetric seesaw mechanism*, Phys. Rev. D **77**, 075008 (2008) [arXiv:0710.2713];

H.-J. He and F.-R. Yin, *Common origin of μ–τ and CP breaking in neutrino seesaw, baryon asymmetry, and hidden flavor symmetry*, Phys. Rev. D **84**, 053009 (2011) [arXiv:1104.2654];

W. Grimus and L. Lavoura, *Maximal atmospheric neutrino mixing in an SU(5) model*, Eur. Phys. J. C **28**, 123 (2003) [hep-ph/0211334];

R.N. Mohapatra, S. Nasri, and H.-B. Yu, *Grand unification of μ–τ symmetry*, Phys. Lett. B **636**, 114 (2006) [hep-ph/0603020].

Y. Koide, *Universal texture of quark and lepton mass matrices with an extended flavor 2 ↔ 3 symmetry*, Phys. Rev. D **69**, 093001 (2004) [hep-ph/0312207];

K. Matsuda and H. Nishiura, *Broken flavor 2 ↔ 3 symmetry and phenomenological approach for universal...
quark and lepton mass matrices, Phys. Rev. D 73, 013008 (2006) [hep-ph/0511338].
H. Nishiura, K. Matsuda, and T. Fukuyama, Quark mixing from matrix model with flavor 2 ↔ 3, Int. J. Mod. Phys. A 23, 4557 (2008) [arXiv:0804.4515].

[18] W. Grimus and L. Lavoura, Leptogenesis in seesaw models with a two-fold degenerate neutrino Dirac mass matrix, J. Phys. G: Nucl. Part. Phys. 30, 1073 (2004) [hep-ph/0311362];
R.N. Mohapatra and S. Nasri, Leptogenesis and \( \mu-\tau \) symmetry, Phys. Rev. D 71, 033001 (2005) [hep-ph/0410369];
Y.H. Ahn, S.K. Kang, C.S. Kim, and J. Lee, Phased breaking of \( \mu-\tau \) symmetry and leptogenesis, Phys. Rev. D 73, 093005 (2006) [hep-ph/0602160];
Y.H. Ahn, C.S. Kim, S.K. Kang, and J. Lee, \( \mu-\tau \) symmetry and radiatively generated leptogenesis, Phys. Rev. D 75, 013012 (2007) [hep-ph/0610007].

[19] W. Grimus and L. Lavoura, A discrete symmetry for maximal atmospheric neutrino mixing, Phys. Lett. B 572, 189 (2003) [hep-ph/0305046];
B. Adhikary, A. Ghosal, and Probir Roy, \( \mu\tau \) symmetry, tribimaximal mixing and four zero neutrino Yukawa textures, J. High Energy Phys. 10, 040 (2009) [arXiv:0908.2686].

[20] W. Grimus and L. Lavoura, Models of maximal atmospheric neutrino mixing, Acta Phys. Polon. B 34, 5393 (2003) [hep-ph/0310050].

[21] T. Kitabayashi and M. Yasu`e, \( \mu - \tau \) symmetry and maximal CP violation, Phys. Lett. B 621, 133 (2005) [hep-ph/0504212];
I. Aizawa, T. Kitabayashi, and M. Yasu`e, Determination of neutrino mass texture for maximal CP violation, Nucl. Phys. B 728, 220 (2005) [hep-ph/0507332];
T. Kitabayashi and M. Yasu`e, A new type of complex neutrino mass texture and \( \mu - \tau \) symmetry, Phys. Rev. D 73, 015002 (2006) [hep-ph/0510132].

[22] W. Grimus, S. Kaneko, L. Lavoura, H. Sawanaka, and M. Tanimoto, \( \mu-\tau \) antisymmetry and neutrino mass matrices, J. High Energy Phys. 01, 110 (2006) [hep-ph/0510326].

[23] W. Grimus and L. Lavoura, Softly broken lepton numbers and maximal neutrino mixing, J. High Energy Phys. 07, 045 (2001) [hep-ph/0105212];
W. Grimus and L. Lavoura, Softly broken lepton numbers: an approach to maximal neutrino mixing, Acta Phys. Polon. B 32, 3719 (2001) [hep-ph/0110041].

[24] K.S. Babu, E. Ma, and J.W.F. Valle, Underlying \( A_4 \) symmetry for the neutrino mass matrix and the quark mixing matrix, Phys. Lett. B 552, 207 (2003) [hep-ph/0206292];
E. Ma, Plato’s fire and the neutrino mass matrix, Mod. Phys. Lett. A 17, 2361 (2002) [hep-ph/0211393].

[25] L. Lavoura and W. Grimus, Seesaw model with softly broken \( L_\alpha - L_\mu - L_\tau \), J. High Energy Phys. 09, 007 (2000) [hep-ph/0008020].

[26] W. Grimus, L. Lavoura, and D. Neubauer, A light pseudoscalar in a model with lepton family symmetry \( O(2) \), J. High Energy Phys. 07, 051 (2008) [arXiv:0805.1175].

[27] W. Grimus and L. Lavoura, Soft lepton-flavor violation in a multi-Higgs-doublet seesaw model, Phys. Rev. D 66, 014016 (2002) [hep-ph/0204070].

[28] W. Grimus and L. Lavoura, Maximal atmospheric neutrino mixing and the small ratio of muon to tau mass, J. Phys. G: Nucl. Part. Phys. 30, 73 (2004) [hep-ph/0309050].