New Braided $T$-Categories over Hopf (co)quasigroups

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ABSTRACT

Let $H$ be a Hopf quasigroup with bijective antipode and let $\text{Aut}_{HQG}(H)$ be the set of all Hopf quasigroup automorphisms of $H$. We introduce a category $\mathcal{YDQ}^H(\alpha, \beta)$ with $\alpha, \beta \in \text{Aut}_{HQG}(H)$ and construct a braided $T$-category $\mathcal{YDQ}(H)$ having all the categories $\mathcal{YDQ}^H(\alpha, \beta)$ as components.

Key words: Hopf quasigroup; braided $T$-category; Quasi $(\alpha, \beta)$-Yetter-Drinfeld category.

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Introduction

To understand the structure and relevant properties of the algebraic 7-sphere, Klim and Majid in [4] introduced the notion of Hopf quasigroups. Which is the linearise of quasigroups, here Hopf quasigroups are not associative but the lack of this property is compensated by some axioms involving the antipode $S$. The concept of Hopf quasigroup is a particular case of the notion of unital coassociative $H$-bialgebra introduced in [5], and Hopf quasigroup includes the example of an enveloping algebra $U(L)$ of a Maltsev algebra (see [4]) as well as the notion of quasigroup algebra $RL$ of an I.P. loop $L$. In particular, Hopf quasigroups unify I.P. loops and Maltsev algebras as the same as Hopf algebras unified groups and Lie algebras.

Turaev in [8, 9] generalized quantum invariants for 3-manifolds to the case of a 3-manifold $M$ endowed with a homotopy class of maps $M \to K(G, 1)$, where $G$ is a group. And braided $T$-categories which are braided monoidal categories in Freyd-Yetter categories of crossed $G$-sets(see in [3]) play a key role of constructing these homotopy invariants.

Based on these two notions and structures, the aim of this paper is to construct classes of new braided $T$-categories over Hopf quasigroups. We first introduce the concept of an

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$(\alpha, \beta)$-Yetter-Drinfeld quasimodule over a Hopf quasigroup, which is a generalization of a Yetter-Drinfeld quasimodule over a Hopf quasigroup (see [1]), then we construct new examples of braided $T$-categories, which generalize the construction of braided $T$-categories over Hopf algebras given by Panaite and Staic (see in [3]).

Let $H$ be a Hopf quasigroup with bijective antipode. We denote by $\text{Aut}_{HQC}(H)$ the set of all Hopf quasigroup automorphisms $\alpha$ of $H$ that satisfying $S \circ \alpha = \alpha \circ S$, and we consider $G$ a certain crossed product group $\text{Aut}_{HQC}(H) \times \text{Aut}_{HQC}(H)$.

In Section 1, we recall definitions and basic results related to Hopf quasi-groups, Yetter-Drinfeld quasimodules over Hopf quasigroups and braided $T$-categories. In Section 2, we introduce a class of new categories $\mathcal{YDQ}^H(\alpha, \beta)$ (see Definition 2.1) of $(\alpha, \beta)$-Yetter-Drinfeld quasimodules associated with $\alpha, \beta \in \text{Aut}_{HQC}(H)$. Then in Section 3, we prove $\mathcal{YDQ}(H)$ is a monoidal category and then construct a class of new braided $T$-categories $\mathcal{YDQ}(H)$ in the sense of Turaev [8].

1. Preliminaries

Throughout, let $k$ be a fixed field. Everything is over $k$ unless otherwise specified. We refer the readers to the book of Sweedler [7] for the relevant concepts on the general theory of Hopf algebras. Let $(C, \Delta)$ be a coalgebra. We use the simplified Sweedler-Heyneman’s notation for $\Delta$ as follows:

$$\Delta(c) = c_1 \otimes c_2,$$

for all $c \in C$.

1.1. Hopf (co)quasigroups.

We first recall that an (inverse property) quasigroup is a Set $Q$ with a product, an identity $e$ and for each $s \in Q$, there is an element $s^{-1} \in Q$ such that

$$s^{-1}(st) = t, \quad (ts)s^{-1} = t, \quad \forall t \in Q.$$

A quasigroup is Moufang if $s(t(sr)) = ((st)s)r$ for all $s, t, r \in Q$. In [4], Klim and Majid linearised these notions to Hopf quasigroups in the same way that Hopf algebras linearises the notions of groups.

A Hopf quasigroup is a unital algebra $H$ (possibly-nonassociative), equipped with algebra maps $\Delta : H \to H \otimes H$ and $\varepsilon : H \to k$ forming a coassociative coalgebra and a map $S : H \to H$ such that

$$S(h_1)(h_2g) = g\varepsilon(h) = h_1(S(h_2)g),$$

$$(gS(h_1))h_2 = g\varepsilon(h) = (gh_1)S(h_2),$$

for all $h_1, h_2, g \in H$. In particular, the antipode $S$ is coassociative, that is, $S(\Delta(h)) = \Delta(S(h))$.
for all $h, g \in H$. In this notation the Hopf quasigroup $H$ is called Moufang if 
\[ h_1(g(h_2 f)) = ((h_1 g)h_2) f, \quad \forall h, g, f \in H. \]
And a Hopf quasigroup $H$ is called flexible if 
\[ h_1(gh) = (h_1 g)h_2, \quad \forall g, h \in H. \]

We know here the conditions of antipode $S$ are stronger than the usual Hopf algebra antipode axioms and then compensate for $H$ nonassociative. For instance, $S$ is antimultiplicative and anticomultiplicative in the sense 
\[ S(hg) = S(g)S(h), \quad \Delta(S(h)) = S(h_2) \otimes S(h_1), \]
for all $g, h \in H$. If we linearise an (inverse property) quasigroup $Q$ to a Hopf quasigroup algebra $kQ$ with grouplike coproduct on elements of $Q$ and linear extension of the product and inverse, and the Hopf quasigroup is Moufang if $Q$ is Moufang.

A (left) Hopf quasimodule (see [2]) over a Hopf quasigroup $H$ is a vector space $M$ equipped with a structure $\cdot : H \otimes M \rightarrow M$ such that 
\[ 1_H \cdot m = m, \]
\[ h_1 \cdot (S(h_2) \cdot m) = S(h_1) \cdot (h_2 \cdot m) = \varepsilon(h) m. \]
for all $h \in H$ and $m \in M$.

Dually we also know the notion of the Hopf coquasigroup, that $H$ is an associative algebra and a counital coalgebra (the comultiplication is possibly noncoassociative), and a map $S : H \rightarrow H$ such that 
\[ S(h_1)h_{21} \otimes h_{22} = 1_H \otimes h = h_1 S(h_{21}) \otimes h_{22}, \]
\[ h_{11} \otimes S(h_{12})h_2 = h \otimes 1_H = h_{11} \otimes h_{12} S(h_2), \]
for all $h \in H$. Similarly we call a Hopf coquasigroup Flexible if $h_1 h_{22} \otimes h_{21} = h_{11} h_2 \otimes h_{12}, \forall h \in H$, and Moufang if $h_1 h_{221} \otimes h_{21} \otimes h_{222} = h_{111} h_{12} \otimes h_{112} \otimes h_2, \forall h \in H$.

1.2. Yetter-Drinfeld quasimodules over a Hopf quasigroup.

Let $H$ be a Hopf quasigroup, in [1], authors gave the notion of left-left Yetter-Drinfeld quasimodule over $H$. Similarly, we say that $M = (M, \cdot, \rho)$ is a left-right Yetter-Drinfeld quasimodule over $H$ if $(M, \cdot)$ is a left $H$-quasimodule and $(M, \rho)$ is a right $H$-comodule which satisfies the following:
\[ (h_2 \cdot m)(0) \otimes (h_2 \cdot m)(1) h_1 = h_1 \cdot m(0) \otimes h_2 m(1), \quad (1.1) \]
\[ m(0) \otimes m(1) (hg) = m(0) \otimes (m(1) h) g, \quad (1.2) \]
\[ m(0) \otimes h (m(1) g) = m(0) \otimes (hm(1)) g, \quad (1.3) \]
for all $h, g \in H$ and $m \in M$. Here we use the notation of right $H$-comodule by $\rho(m) = m(0) \otimes m(1), \forall m \in M$. 

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Let $M$ and $N$ be two left-right Yetter-Drinfeld quasimodules over $H$. We call morphism $f : M \to N$ a left-right Yetter-Drinfeld quasimodule morphism if $f$ is both a left $H$-quasimodule morphism and a right $H$-comodule morphism. We use $\mathcal{H}YDQ^H$ denote the category of left-right Yetter-Drinfeld quasimodules over $H$. Moreover, if we assume $H$ is associative, that $H$ is a Hopf algebra, then conditions (1.2) and (1.3) become trivial. In this case, if $M$ is a left $H$-module we obtain the classical definition of left-right Yetter-Drinfeld module over a Hopf algebra.

1.3. Braided $T$-categories.

A monoidal category $\mathcal{C} = (\mathcal{C}, \mathbb{I}, \otimes, a, l, r)$ is a category $\mathcal{C}$ endowed with a functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ (the tensor product), an object $\mathbb{I} \in \mathcal{C}$ (the tensor unit), and natural isomorphisms $a$ (the associativity constraint), $l$ (the left unit constraint) and $r$ (the right unit constraint), such that for all $U, V, W, X \in \mathcal{C}$, the associativity pentagon $a_{U,V,W} \circ a_{U\otimes V,W,X} = (U \otimes a_{V,W,X}) \circ (a_{U,V} \otimes (U \otimes id_V))$ and $(U \otimes l_V) \circ (r_U \otimes V) = a_{U,V}$ are satisfied. A monoidal category $\mathcal{C}$ is strict when all the constraints are identities.

Let $G$ be a group and let $Aut(\mathcal{C})$ be the group of invertible strict tensor functors from $\mathcal{C}$ to itself. A category $\mathcal{C}$ over $G$ is called a crossed category if it satisfies the following:

- $\mathcal{C}$ is a monoidal category;
- $\mathcal{C}$ is disjoint union of a family of subcategories $\{\mathcal{C}_\alpha\}_{\alpha \in G}$, and for any $U \in \mathcal{C}_\alpha$, $V \in \mathcal{C}_\beta$, $U \otimes V \in \mathcal{C}_{\alpha \beta}$.
- Consider a group homomorphism $\varphi : G \to Aut(\mathcal{C})$, $\beta \mapsto \varphi_\beta$, and assume that $\varphi_\beta(\varphi_\alpha) = \varphi_{\beta \alpha \beta^{-1}}$, for all $\alpha, \beta \in G$. The functors $\varphi_\beta$ are called conjugation isomorphisms.

Furthermore, $\mathcal{C}$ is called strict when it is strict as a monoidal category.

Left index notation: Given $\alpha \in G$ and an object $V \in \mathcal{C}_\alpha$, the functor $\varphi_\alpha$ will be denoted by $V(\cdot)$, as in Turaev [8] or Zunino [10], or even $\alpha(\cdot)$. We use the notation $V^\alpha(\cdot)$ for $\alpha^{-1}(\cdot)$. Then we have $V id_U = id_{VU}$ and $V(g \circ f) = Vg \circ Vf$. Since the conjugation $\varphi : G \to Aut(\mathcal{C})$ is a group homomorphism, for all $V, W \in \mathcal{C}$, we have $V \otimes W(\cdot) = V(W(\cdot))$ and $1(\cdot) = V(1(\cdot)) = V(\cdot) = id_c$. Since, for all $V \in \mathcal{C}$, the functor $V(\cdot)$ is strict, we have $V(f \otimes g) = Vf \otimes Vg$, for any morphisms $f$ and $g$ in $\mathcal{C}$, and $V1 = 1$.

A braiding of a crossed category $\mathcal{C}$ is a family of isomorphisms $(c = c_{U,V} )_{U,V} \in \mathcal{C}$, where $c_{U,V} : U \otimes V \to V \otimes U$ satisfying the following conditions:

1. For any arrow $f \in \mathcal{C}_\alpha(U,U')$ and $g \in \mathcal{C}(V,V')$,

$$((a^g) \otimes f) \circ c_{U,V} = c_{U'V'} \circ (f \otimes g).$$

2. For all $U, V, W \in \mathcal{C}$, we have

$$c_{U \otimes V,W} = a_{U \otimes V,W,U,V} \circ (c_{U,V,W} \otimes id_V) \circ (U \otimes c_{V,W}) \circ a_{U,V,W},$$

$$c_{U,V \otimes W} = a_{U,V \otimes W,U,V} \circ (c_{U,V,W} \otimes id_U) \circ (c_{U,V,W} \otimes id_U) \circ a_{U,V,W},$$

where $a$ is the natural isomorphisms in the tensor category $\mathcal{C}$.

3. For all $U, V \in \mathcal{C}$ and $\beta \in G$,

$$\varphi_\beta(c_{U,V}) = c_{\varphi_\beta(U),\varphi_\beta(V)}.$$
A crossed category endowed with a braiding is called a braided $T$-category.

2. $(\alpha, \beta)$-Yetter-Drinfeld quasimodules over a Hopf quasigroup

In this section, we will define the notion of a Yetter-Drinfeld quasimodule over a Hopf quasigroup that is twisted by two Hopf quasigroup automorphisms as well as the notion of a Hopf quasi-entwining structure and how to obtain such structure from automorphisms of Hopf quasigroups.

In what follows, let $H$ be a Hopf quasigroup with the bijective antipode $S$ and let $\text{Aut}_{HQG}(H)$ denote the set of all automorphisms of a Hopf quasigroup $H$.

**Definition 2.1.** Let $\alpha, \beta \in \text{Aut}_{HQG}(H)$. A left-right $(\alpha, \beta)$-Yetter-Drinfeld quasimodule over $H$ is a vector space $M$, such that $M$ is a left $H$-quasimodule (with notation $h \otimes m \mapsto h \cdot m$) and a right $H$-comodule (with notation $M \rightarrow M \otimes H$, $m \mapsto m(0) \otimes m(1)$) and with the following compatibility condition:

$$\rho(h \cdot m) = h_{21} \cdot m(0) \otimes (\beta(h_{22})m_{(1)})\alpha(S^{-1}(h_1)), \quad (2.1)$$

for all $h \in H$ and $m \in M$. We denote by $H_{\text{YDQ}}^H(\alpha, \beta)$ the category of left-right $(\alpha, \beta)$-Yetter-Drinfeld quasimodules, morphisms being both $H$-linear and $H$-colinear maps.

**Remark.** Note that, $\alpha$ and $\beta$ are bijective, algebra morphisms, coalgebra morphisms and commute with $S$.

**Proposition 2.2.** One has that Eq.(2.1) is equivalent to the following equations:

$$h_1 \cdot m_{(0)} \otimes \beta(h_2)m_{(1)} = (h_2 \cdot m)_{(0)} \otimes (h_2 \cdot m)_{(1)}\alpha(h_1). \quad (2.2)$$

**Proof.** To prove this proposition, we need to use the property of antipode of a Hopf quasigroup. That is, $(gS(h_1))h_2 = (gh_1)S(h_2) = g\varepsilon(h)$, for all $g, h \in H$.

Eq.(2.1)$\Rightarrow$ Eq.(2.2).

We first do calculation as follows:

$$(h_2 \cdot m)_{(0)} \otimes (h_2 \cdot m)_{(1)}\alpha(h_1)$$

$$= (2.1) h_{22} \cdot m_{(0)} \otimes ((\beta(h_{23})m_{(1)})\alpha(S^{-1}(h_{21})))\alpha(h_1)$$

$$= h_{22} \cdot m_{(0)} \otimes ((\beta(h_{23})m_{(1)})\alpha(S^{-1}(h_{21})))\alpha(h_1)$$

$$= h_{2} \cdot m_{(0)} \otimes ((\beta(h_{3})m_{(1)})\varepsilon(h_1)$$

$$= h_{1} \cdot m_{(0)} \otimes \beta(h_{2})m_{(1)}.$$

For $\text{Eq.}(2.2) \Rightarrow \text{Eq.}(2.1)$, we have

$$h_2 \cdot m_{(0)} \otimes (\beta(a_3)m_{(1)})\alpha(S^{-1}(a_1))$$
This finishes the proof. □

**Example 2.3.** For $\beta \in Aut_{HQG}(H)$, define $H_\beta = H$ as vector space over a field $k$, with regular right $H$-comodule structure and left $H$-quasimodule structure given by $h \cdot h' = (\beta(h_2)h')S^{-1}(h_1)$, for all $h, h' \in H$. And more generally, if $\alpha, \beta \in Aut_{HQG}(H)$, define $H_{\alpha, \beta} = H$ with regular right $H$-comodule structure and left $H$-module structure given by $h \cdot h' = (\beta(h_2)h')\alpha(S^{-1}(h_1))$, for all $h, h' \in H$. If $H$ is flexible, then we get $H_\beta \in H \mathcal{YDQ}^H(id, \beta)$ and if we add $H$ is $(\alpha, \beta)$-flexible, that is

$$\alpha(h_1)(g\beta(h_2)) = (\alpha(h_1)g)\beta(h_2),$$

for all $g, h \in H$ and $\alpha, \beta \in Aut_{HQG}(H)$. Then $H_{\alpha, \beta} \in H \mathcal{YDQ}^H(\alpha, \beta)$.

Let $\alpha, \beta \in Aut_{HQG}(H)$. As defined in [6], an $H$-bicomodule algebra $H(\alpha, \beta)$ as follows; $H(\alpha, \beta) = H$ as algebras, with comodule structures

$$H(\alpha, \beta) \to H \otimes H(\alpha, \beta), \quad h \mapsto h_{[-1]} \otimes h_{[0]} = \alpha(h_1) \otimes h_2,$$

$$H(\alpha, \beta) \to H(\alpha, \beta) \otimes H, \quad h \mapsto h_{<0>} \otimes h_{<1>} = h_1 \otimes \beta(h_2).$$

Then we also consider the Yetter-Drinfeld quasimodules like $H(\alpha, \beta) \mathcal{YDQ}^H(H)$.

**Proposition 2.4.** $H \mathcal{YDQ}^H(\alpha, \beta) = H(\alpha, \beta) \mathcal{YDQ}^H(H)$.

**Proof.** Left to readers.

3. A BRAIDED $T$-CATEGORY $\mathcal{YDQ}(H)$

In this section, we will construct a class of new braided $T$-categories $\mathcal{YDQ}(H)$ over any Hopf quasigroup $H$ with bijective antipode. Here for $\alpha \in Aut_{HQG}(H)$, the $H \mathcal{YDQ}^H(\alpha, \beta)$ is the object of $\mathcal{YDQ}(H)$.

Let $M \in H \mathcal{YDQ}^H(\alpha, \beta), N \in H \mathcal{YDQ}^H(\gamma, \delta)$, with $\alpha, \beta, \gamma, \delta \in Aut_{HQG}(H)$.

**Proposition 3.1.** If $M \in H \mathcal{YDQ}^H(\alpha, \beta), N \in H \mathcal{YDQ}^H(\gamma, \delta)$, with $\alpha, \beta, \gamma, \delta \in Aut_{HQG}(H)$, then $M \otimes N \in H \mathcal{YDQ}^H(\alpha \gamma, \delta \gamma^{-1} \beta \gamma)$ with structures as follows:

$$h \cdot (m \otimes n) = \gamma(h_1) \cdot m \otimes \gamma^{-1} \beta \gamma(h_2) \cdot n,$$

$$m \otimes n \mapsto (m_{(0)} \otimes n_{(0)}) \otimes n_{(1)} m_{(1)}.$$ 

for all $m \in M, n \in N$ and $h \in H$. 

\[ \begin{align*} 
(2.2) & \quad (h_3 \cdot m)_{(0)} \otimes ((a_3 \cdot m)_{(1)} \alpha(a_2)) \alpha(S^{-1}(a_1)) \\
& \quad = (a_2 \cdot m)_{(0)} \otimes (a_2 \cdot m)_{(1)} \epsilon(a_1) \\
& \quad = (a \cdot m)_{(0)} \otimes (a \cdot m)_{(1)}. \end{align*} \]
Proof. Let \( h, g \in H \) and \( m \otimes n \in M \otimes N \). We can prove \( 1 \cdot (m \otimes n) = m \otimes n \) and 
\[ h_1 \cdot (S(h_2) \cdot (m \otimes n)) = S(h_1) \cdot (h_2 \cdot (m \otimes n)) = (m \otimes n), \] straightforwardly.

This shows that \( M \otimes N \) is a left \( H \)-quasimodule, the right \( H \)-comodule condition is straightforward to check.

Next, we compute the compatibility condition as follows:

\[
(h_2 \cdot (m \otimes n))(0) \otimes (h_2 \cdot (m \otimes n))(1) \alpha \gamma(h_1) \\
= (\gamma(h_2) \cdot m \otimes \gamma^{-1} \beta \gamma(h_3) \cdot n)(0) \otimes (\gamma(h_2) \cdot m \otimes \gamma^{-1} \beta \gamma(h_3) \cdot n)(1) \alpha \gamma(h_1) \\
= (\gamma(h_1) \cdot m)(0) \otimes (\gamma^{-1} \beta \gamma(h_3) \cdot n)(0) \otimes (\gamma^{-1} \beta \gamma(h_3) \cdot n)(1) \gamma \gamma^{-1} \beta \gamma(h_2) m(1) \\
= (\gamma(h_1) \cdot m)(0) \otimes \gamma^{-1} \beta \gamma(h_2) \cdot n(0) \otimes \delta \gamma^{-1} \beta \gamma(h_3)(n(1)m(1)) \\
= h(m(0) \otimes n(0)) \otimes \delta \gamma^{-1} \beta \gamma(h_2)(n(1)m(1)).
\]

Thus \( M \otimes N \in \mathcal{YDQ}_H^{*}(\alpha \gamma, \beta \gamma^{-1} \beta \gamma) \).

Remark. Note that, if \( M \in \mathcal{YDQ}_H^{*}(\alpha, \beta), \ N \in \mathcal{YDQ}_H^{*}(\gamma, \delta) \) and \( P \in \mathcal{YDQ}_H^{*}(\mu, \nu) \), then \( (M \otimes N) \otimes P = M \otimes (N \otimes P) \) as objects in \( \mathcal{YDQ}_H^{*}(\alpha \gamma \mu, \nu \mu^{-1} \delta \gamma^{-1} \beta \gamma \mu) \).

Denote \( G = Aut_{HQC}(H) \times Aut_{HQC}(H) \) a group with multiplication as follows: for all \( \alpha, \beta, \gamma, \delta \in Aut_{HQC}(H) \),

\[
(\alpha, \beta) * (\gamma, \delta) = (\alpha \gamma, \delta \gamma^{-1} \beta \gamma).
\]

The unit of this group is \((id, id)\) and \((\alpha, \beta)^{-1} = (\alpha^{-1}, \alpha \beta^{-1} \alpha^{-1})\).

The above proposition means that if \( M \in \mathcal{YDQ}_H^{*}(\alpha, \beta) \) and \( N \in \mathcal{YDQ}_H^{*}(\gamma, \delta) \), then \( M \otimes N \in \mathcal{YDQ}_H^{*}(\alpha \gamma, \beta \gamma^{-1} \beta \gamma) \).

**Proposition 3.2.** Let \( N \in \mathcal{YDQ}_H^{*}(\gamma, \delta) \) and \((\alpha, \beta) \in G\). Define \((\alpha, \beta)N = N\) as vector space, with structures: for all \( n \in N \) and \( h \in H \),

\[
h \triangleright n = \gamma^{-1} \beta \gamma \alpha^{-1}(h) \cdot n,
\]

\[
n \mapsto n_{<0>} \otimes n_{<1>} = n(0) \otimes \alpha \beta^{-1}(n(1)).
\]

Then

\[
(\alpha, \beta)N \in \mathcal{YDQ}_H^{*}((\alpha, \beta) * (\gamma, \delta) * (\alpha, \beta)^{-1}),
\]

where \(((\alpha, \beta) * (\gamma, \delta) * (\alpha, \beta)^{-1}) = (\alpha \gamma \alpha^{-1}, \alpha \beta^{-1} \delta \gamma^{-1} \beta \gamma \alpha^{-1})\) as an element in \( G \).

**Proof.** Obviously, the equations above define a quasi-module and a comodule action of \( N \). In what follows, we show the compatibility condition:

\[
(h \triangleright n)_{<0>} \otimes (h \triangleright n)_{<1>} \\
= (\gamma^{-1} \beta \gamma \alpha^{-1}(h) \cdot n)(0) \otimes \alpha \beta^{-1}(\gamma^{-1} \beta \gamma \alpha^{-1}(h) \cdot n)(1) \\
= \gamma^{-1} \beta \gamma \alpha^{-1}(h_2 \cdot n)(0) \otimes \alpha \beta^{-1}(\delta \gamma^{-1} \beta \gamma \alpha^{-1}(h_3 \cdot n)(1)) \gamma(S^{-1}(\gamma^{-1} \beta \gamma \alpha^{-1}(h_1))) \\
= h_2 \triangleright n(0) \otimes (\alpha \beta^{-1} \delta \gamma^{-1} \beta \gamma \alpha^{-1}(h_3 \cdot n_{<1>})(\alpha \gamma \alpha^{-1}(S^{-1}(h_1)))) \\
= h_2 \triangleright n(0) \otimes (\alpha \beta^{-1} \delta \gamma^{-1} \beta \gamma \alpha^{-1}(h_3 \cdot n_{<1>})(\alpha \gamma \alpha^{-1}(S^{-1}(h_1))))
\]
for all $n \in N$ and $h \in H$, that is $(\alpha, \beta)N \in H\mathcal{YDQ}^H(\gamma\alpha^{-1}, \alpha\beta^{-1}\delta\gamma^{-1}\beta\alpha^{-1})$.

**Remark.** Let $M \in H\mathcal{YDQ}^H(\alpha, \beta)$, $N \in H\mathcal{YDQ}^H(\gamma, \delta)$, and $(\mu, \nu) \in G$. Then by the above proposition, we have:

$$(\alpha, \beta) \ast (\mu, \nu) N = (\mu, \nu) (\alpha, \beta)N,$$

as objects in $H\mathcal{YDQ}^H(\alpha\mu\gamma\mu^{-1}\alpha^{-1}, \alpha\beta^{-1}\mu\nu^{-1}\delta\gamma^{-1}\nu\mu^{-1}\beta\gamma\mu^{-1}\alpha^{-1})$ and

$$(\mu, \nu)(M \otimes N) = (\mu, \nu) M \otimes (\mu, \nu) N,$$

as objects in $H\mathcal{YDQ}^H(\mu\alpha\gamma\mu^{-1}, \mu\nu^{-1}\delta\gamma^{-1}\beta\alpha^{-1}\nu\alpha\gamma\mu^{-1})$.

**Proposition 3.3.** Let $M \in H\mathcal{YDQ}^H(\alpha, \beta)$, $N \in H\mathcal{YDQ}^H(\gamma, \delta)$ and $P \in H\mathcal{YDQ}^H(\mu, \nu)$, take $M \otimes N = (\alpha, \beta)N$ as explained in Subsection 1.3. Define a map $c_{M,N} : M \otimes N \to M \otimes N$ by

$$c_{M,N}(m \otimes n) = n(0) \otimes \beta^{-1}(n(1)) \cdot m.$$  \hspace{1cm} (3.3)

for all $m \in M, n \in N$. Then $c_{M,N}$ is both an $H$-module map and an $H$-comodule map, and satisfies the following formulae:

$$c_{M \otimes N, P} = (c_{M, N} \otimes id_N) \circ (id_M \otimes c_{N, P}),$$  \hspace{1cm} (3.4)

$$c_{M, N \otimes P} = (id_M \otimes c_{M, P}) \circ (c_{M, N} \otimes id_P).$$  \hspace{1cm} (3.5)

**Proof.** First, we prove that $c_{M,N}$ is an $H$-module map. Take $h \cdot (m \otimes n) = \gamma(h_1) \cdot m \otimes \gamma^{-1}\beta\gamma(h_2) \cdot n$ as explained in Proposition 3.1.

$$c_{M,N}(h \cdot (m \otimes n)) = c_{M,N}(\gamma(h_1) \cdot m \otimes \gamma^{-1}\beta\gamma(h_2) \cdot n) = (\gamma^{-1}\beta\gamma(h_2) \cdot n(0)) \otimes \beta^{-1}((\gamma^{-1}\beta\gamma(h_2) \cdot n(1)) \cdot (\gamma(h_1) \cdot m) = (\gamma^{-1}\beta\gamma(h_2) \cdot n(0)) \otimes (\beta^{-1}((\gamma^{-1}\beta\gamma(h_2) \cdot n(1))\gamma(h_1))) \cdot m = \gamma^{-1}\beta\gamma(h_3) \cdot n(0) \otimes ((\beta^{-1}\delta\gamma^{-1}\beta\gamma(h_4)\beta^{-1}(n(1)))S^{-1}(\gamma(h_2)))\gamma(h_1) \cdot m = \gamma^{-1}\beta\gamma(h_2) \cdot n(0) \otimes (\beta^{-1}\delta\gamma^{-1}\beta\gamma(h_3)\beta^{-1}(n(1)))\epsilon(h_1) \cdot m = \gamma^{-1}\beta\gamma(h_1) \cdot n(0) \otimes (\beta^{-1}\delta\gamma^{-1}\beta\gamma(h_2)\beta^{-1}(n(1))) \cdot m,$$

on the other side, we have

$$h \cdot c_{M,N}(m \otimes n) = h \cdot (n(0)) \otimes \beta^{-1}(n(1)) \cdot m = \alpha(h_1) \triangleright n(0) \otimes (\beta^{-1}\delta\gamma^{-1}\beta\gamma(h_2)\beta^{-1}(n(1))) \cdot m,$$

similarly we can check that $c_{M,N}$ is an $H$-comodule map.

Finally we will check Eqs.(3.4) and (3.5). Using equations $^{M}(NP) = ^{M \otimes N} P$ and $^{M}(N \otimes P) = ^{M}N \otimes ^{M}P$ we have

$$(c_{M,N} \otimes id_N) \circ (id_M \otimes c_{N,P})(m \otimes n \otimes p)$$
\[
\begin{align*}
&= c_{M,N,P}(m \otimes p(0)) \otimes \delta^{-1}(p(1)) \cdot n \\
&= p(0) \otimes \beta^{-1}(\gamma \delta^{-1}(p(1))) \cdot m \otimes \delta^{-1}(p(1)) \cdot n \\
&= p(0) \otimes \beta^{-1}(\gamma \delta^{-1}(p(1))) \cdot m \otimes \delta^{-1}(p(1)) \cdot n \\
&= p(0) \otimes \gamma \beta^{-1}(\gamma \delta^{-1}(p(1))) \cdot (m \otimes n) \\
&= c_{M \otimes N,P}(m \otimes n \otimes p).
\end{align*}
\]

Similar we can check the equation 3.5, that ends the proof. ■

**Lemma 3.5.** The map \(c_{M,N}\) defined by \(c_{M,N}(m \otimes n) = n(0) \otimes \beta^{-1}(n(1)) \cdot m\) is bijective; with inverse \(c_{M,N}^{-1}(n \otimes m) = \beta^{-1}(S(n(1))) \cdot m \otimes n(0)\).

**Proof.** First, we prove \(c_{M,N} \circ c_{M,N}^{-1} = id\). For all \(m \in M, n \in N\), we have
\[
\begin{align*}
&c_{M,N} \circ c_{M,N}^{-1}(n \otimes m) \\
&= c_{M,N}(\beta^{-1}(S(n(1))) \cdot m \otimes n(0)) \\
&= n(0) \otimes \beta^{-1}(n(0)(1)) \cdot (\beta^{-1}(S(n(1))) \cdot m) \\
&= n(0) \otimes n(0)(1) \cdot (\beta^{-1}(S(n(1))) \cdot m) \\
&= \epsilon(n(1))m \\
&= n \otimes m.
\end{align*}
\]

The fact that \(c_{M,N}^{-1} \circ c_{M,N} = id\) is similar. This completes the proof. ■

Let \(H\) be a Hopf quasigroup and \(G = Aut_{HQG}(H) \times Aut_{HQG}(H)\). Define \(\mathcal{YDQ}(H)\) as the disjoint union of all \(H\mathcal{YDQ}^H(\alpha, \beta)\) with \((\alpha, \beta) \in G\). If we endow \(\mathcal{YDQ}(H)\) with tensor product shown in Proposition 3.1, then \(\mathcal{YDQ}(H)\) becomes a monoidal category.

Define a group homomorphism \(\varphi : G \to Aut(\mathcal{YDQ}(H)), (\alpha, \beta) \mapsto \varphi(\alpha, \beta)\) on components as follows:
\[
\varphi(\alpha, \beta) : H\mathcal{YDQ}^H(\gamma, \delta) \to H\mathcal{YDQ}^H((\alpha, \beta) * (\gamma, \delta) * (\alpha, \beta)^{-1}),
\]
and the functor \(\varphi(\alpha, \beta)\) acts as identity on morphisms.

The braiding in \(\mathcal{YDQ}(H)\) is given by the family \(\{c_{M,N}\}\) in Proposition 3.4. So we get the following main theorem of this article.

**Theorem 3.6.** \(\mathcal{YDQ}(H)\) is a braided \(T\)-category over \(G\).

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