ON TIMELIKE BERTRAND CURVES IN MINKOWSKI
3-SPACE

ALI UÇUM* AND KAZIM İLARSLAN

Abstract. In this paper, we study the timelike Bertrand curves in
Minkowski 3-space. Since the principal normal vector of a timelike
curve is spacelike, the Bertrand mate curve of this curve can be a
timelike curve, a spacelike curve with spacelike principal normal or
a Cartan null curve, respectively. Thus, by considering these three
cases, we get the necessary and sufficient conditions for a timelike
curve to be a Bertrand curve. Also we give the related examples.

1. Introduction

A classical problem in Differential Geometry raised by Saint-Venant
in 1845 ([14]) led to discovery of Bertrand curves in 1850 ([3]). A Bertrand
curve is a curve in the Euclidean space such that its principal normal is
the principal normal of the second curve. J. Bertrand proved that a nec-
essary and sufficient condition for the existence of such a second curve
is required in fact a linear relationship calculated with constant coeffi-
cients should exist between the first and second curvatures of the given
original curve. In other words, if we denote first and second curvatures
of a given curve by \( k_1 \) and \( k_2 \) respectively, we have \( \lambda k_1 + \mu k_2 = 1 \), \( \lambda, \mu \in \mathbb{R} \). Since 1850, after the paper of Bertrand, the pairs of curves like
this have been called Conjugate Bertrand Curves, or more commonly
Bertrand Curves (see [8]).

The study of this kind of curves has been extended to many other
ambient spaces. In [10], Pears studied this problem for curves in the
\( n \)-dimensional Euclidean space \( \mathbb{E}^n \), \( n > 3 \), and showed that a Bertrand
curve in \( \mathbb{E}^n \) must belong to a three-dimensional subspace \( \mathbb{E}^3 \subset \mathbb{E}^n \). This

*Corresponding author
result is restated by Matsuda and Yorozu [9]. They proved that there was not any special Bertrand curves in $\mathbb{E}^n$ ($n > 3$) and defined a new kind, which is called $(1, 3)$-type Bertrand curves in 4-dimensional Euclidean space. Bertrand curves and their characterizations were studied by many researchers in Minkowski 3-space and Minkowski space-time (see [1], [2], [6], [7], [11]) as well as in Euclidean space. In addition, in [12] and [13], the authors studied $(1, 3)$-type Bertrand curves in semi-Euclidean 4-space with index 2.

In the present paper, we study the timelike Bertrand curves in Minkowski 3-space. Since the principal normal vector of a timelike curve is spacelike, the Bertrand mate curve of this curve can be a timelike curve, a spacelike curve with spacelike principal normal or a Cartan null curve, respectively. Thus, by considering these three cases, we get the necessary and sufficient conditions for a timelike curve to be a Bertrand curve. Also we give the related examples.

2. Preliminaries

The Minkowski space $\mathbb{E}_1^3$ is the 3-dimensional real vector space $\mathbb{R}^3$ equipped with the indefinite flat metric given by

$$g = -dx_1^2 + dx_2^2 + dx_3^2,$$

where $(x_1, x_2, x_3)$ is a rectangular coordinate system of $\mathbb{R}^3$. Recall that a vector $v \in \mathbb{E}_1^3 \setminus \{0\}$ can be spacelike if $g(v, v) > 0$, timelike if $g(v, v) < 0$ and null (lightlike) if $g(v, v) = 0$ and $v \neq 0$. In particular, the vector $v = 0$ is spacelike. The norm of a vector $v$ is given by $||v|| = \sqrt{|g(v, v)|}$, and two vectors $v$ and $w$ are said to be orthogonal, if $g(v, w) = 0$. An arbitrary curve $\alpha(s)$ in $\mathbb{E}_1^3$, can locally be spacelike, timelike or null (lightlike), if all its velocity vectors $\alpha'(s)$ are respectively spacelike, timelike or null ([8]). A spacelike curve in $\mathbb{E}_1^3$ is called pseudo null curve if its principal normal vector $N$ is null [4]. A null curve $\alpha$ is said to be parameterized by pseudo-arc $s$ if $g(\alpha ''(s), \alpha ''(s)) = 1$. A spacelike or a timelike curve $\alpha$ is said to be parameterized by arc-length $s$ if $g(\alpha'(s), \alpha'(s)) = \pm 1$ ([4]).

Let $\{T, N, B\}$ be the moving Frenet frame along a curve $\alpha$ in $\mathbb{E}_1^3$, consisting of the tangent, the principal normal and the binormal vector fields, respectively. Depending on the causal character of $\alpha$, the Frenet equations have the following forms.
Case I. If $\alpha$ is a non-null curve, the Frenet equations are given by ([8]):

$$
\begin{bmatrix}
T' \\
N' \\
B'
\end{bmatrix} = \begin{bmatrix}
0 & \epsilon_2 k_1 & 0 \\
-\epsilon_1 k_1 & 0 & \epsilon_3 k_2 \\
0 & -\epsilon_2 k_2 & 0
\end{bmatrix} \begin{bmatrix}
T \\
N \\
B
\end{bmatrix},
$$

where $k_1$ and $k_2$ are the first and the second curvature of the curve respectively. Moreover, the following conditions hold:

$$
g(T, T) = \epsilon_1 = \pm 1, \quad g(N, N) = \epsilon_2 = \pm 1, \quad g(B, B) = \epsilon_3 = \pm 1$$

and

$$
g(T, N) = g(T, B) = g(N, B) = 0.
$$

Case II. If $\alpha$ is a null curve, the Frenet equations are given by ([4])

$$
\begin{bmatrix}
T' \\
N' \\
B'
\end{bmatrix} = \begin{bmatrix}
0 & k_1 & 0 \\
k_2 & 0 & -k_1 \\
0 & -k_2 & 0
\end{bmatrix} \begin{bmatrix}
T \\
N \\
B
\end{bmatrix},
$$

where the first curvature $k_1 = 0$ if $\alpha$ is straight line, or $k_1 = 1$ in all other cases. In particular, the following conditions hold:

$$
g(T, T) = g(B, B) = g(T, N) = g(N, B) = 0, \quad g(N, N) = g(T, B) = 1.
$$

Case III. If $\alpha$ is a pseudo null curve, the Frenet formulas have the form ([5])

$$
\begin{bmatrix}
T' \\
N' \\
B'
\end{bmatrix} = \begin{bmatrix}
0 & k_1 & 0 \\
0 & k_2 & 0 \\
-k_1 & 0 & -k_2
\end{bmatrix} \begin{bmatrix}
T \\
N \\
B
\end{bmatrix},
$$

where the first curvature $k_1 = 0$ if $\alpha$ is straight line, or $k_1 = 1$ in all other cases. In particular, the following conditions hold:

$$
g(N, N) = g(B, B) = g(T, N) = g(T, B) = 0, \quad g(T, T) = g(N, B) = 1.
$$

3. Timelike Bertrand curves in Minkowski 3-space $\mathbb{E}_1^3$

In this section, we consider the timelike Bertrand curves in $\mathbb{E}_1^3$. We get the necessary and sufficient conditions for the timelike curves to be Bertrand curves in $\mathbb{E}_1^3$ and we also give the related examples.
Definition 3.1. A timelike curve $\alpha : I \to \mathbb{E}^3_1$ with $\kappa_1(s) \neq 0$ is a Bertrand curve if there is a curve $\alpha^* : I^* \to \mathbb{E}^3_1$ such that the principal normal vectors of $\alpha(s)$ and $\alpha^*(s^*)$ at $s \in I$, $s^* \in I^*$ are equal. In this case, $\alpha^*(s^*)$ is called the Bertrand mate of $\alpha(s)$.

Let $\beta : I \to \mathbb{E}^3_1$ be a timelike Bertrand curve in $\mathbb{E}^3_1$ with the Frenet frame $\{T, N, B\}$ and the curvatures $\kappa_1, \kappa_2$. Let $\beta^* : I \to \mathbb{E}^3_1$ be a Bertrand mate curve of $\beta$ with the Frenet frame $\{T^*, N^*, B^*\}$ and the curvatures $\kappa_1^*, \kappa_2^*$.

Theorem 3.2. Let $\beta : I \subset \mathbb{R} \to \mathbb{E}^3_1$ be a unit speed timelike curve with the non-zero curvatures $\kappa_1, \kappa_2$. Then the curve $\beta$ is a Bertrand curve with Bertrand mate $\beta^*$ if and only if one of the following conditions holds:

(i) there exist constant real numbers $\lambda$ and $h$ satisfying

$$h^2 > 1, \quad 1 + \lambda \kappa_1 = h \lambda \kappa_2, \quad h \kappa_1 - \kappa_2 \neq 0, \quad h \kappa_2 - \kappa_1 \neq 0.$$  
In this case, $\beta^*$ is a timelike curve in $\mathbb{E}^3_1$.

(ii) there exist constant real numbers $\lambda$ and $h$ satisfying

$$h^2 < 1, \quad 1 + \lambda \kappa_1 = h \lambda \kappa_2, \quad h \kappa_1 - \kappa_2 \neq 0, \quad h \kappa_2 - \kappa_1 \neq 0.$$  
In this case, $\beta^*$ is a spacelike curve with spacelike principal normal in $\mathbb{E}^3_1$.

Proof. Assume that $\beta$ is a timelike Bertrand curve parametrized by arc-length $s$ with non-zero curvatures $\kappa_1, \kappa_2$ and the curve $\beta^*$ is the Bertrand mate curve of the curve $\beta$ parametrized by with arc-length or pseudo arc $s^*$.

(i) Let $\beta^*$ be a timelike curve. The proof of this case can be similarly done to the theorem in [15].

(ii) Let $\beta^*$ be a spacelike curve with spacelike principal normal. Then, we can write the curve $\beta^*$ as

$$\beta^*(s^*) = \beta^*(f(s)) = \beta(s) + \lambda(s)N(s)$$

for all $s \in I$ where $\lambda(s)$ is $C^\infty-$function on $I$. Differentiating (6) with respect to $s$ and using (1), we get

$$T^* f' = (1 + \lambda \kappa_1)T + \lambda' N + \lambda \kappa_2 B.$$  
By taking the scalar product of (7) with $N$, we have

$$\lambda' = 0.$$  

(8)
Substituting (8) in (7), we find
\begin{equation}
(9) \quad T^* f' = (1 + \lambda \kappa_1) T + \lambda \kappa_2 B.
\end{equation}
By taking the scalar product of (9) with itself, we obtain
\begin{equation}
(10) \quad \left( f' \right)^2 = -(1 + \lambda \kappa_1)^2 + (\lambda \kappa_2)^2.
\end{equation}
If we denote
\begin{equation}
\delta = \frac{1 + \lambda \kappa_1}{f'} \quad \text{and} \quad \gamma = \frac{\lambda \kappa_2}{f'},
\end{equation}
we get
\begin{equation}
(12) \quad T^* = \delta T + \gamma B_1.
\end{equation}
Differentiating (12) with respect to $s$ and using (1), we find
\begin{equation}
(13) \quad f' \kappa_1^* N^* = \delta' T + (\delta \kappa_1 - \gamma \kappa_2) N + \gamma' B.
\end{equation}
By taking the scalar product of (13) with itself, we get
\begin{equation}
(14) \quad \delta' = 0 \quad \text{and} \quad \gamma' = 0.
\end{equation}
Since $\gamma \neq 0$, we have $1 + \lambda \kappa_1 = h \lambda \kappa_2$ where $h = \delta / \gamma$. Substituting (14) in (13), we find
\begin{equation}
(15) \quad f' \kappa_1^* N^* = (\delta \kappa_1 - \gamma \kappa_2) N
\end{equation}
By taking the scalar product of (15) with itself, using (10) and (11), we have
\begin{equation}
(16) \quad \left( f' \right)^2 \left( \kappa_1^* \right)^2 = \frac{(h \kappa_1 - \kappa_2)^2}{1 - h^2}
\end{equation}
where $h \kappa_1 - \kappa_2 \neq 0$ and $h^2 < 1$. If we put $v = (\delta \kappa_1 - \gamma \kappa_2) / f' \kappa_1^*$, we get
\begin{equation}
(17) \quad N^* = v N.
\end{equation}
Differentiating (17) with respect to $s$ and using (1), we find
\begin{equation}
(18) \quad -f' \kappa_2^* B^* = v \kappa_1 T + v \kappa_2 B + f' \kappa_1^* T^*
\end{equation}
where $v' = 0$. Rewriting (18) by using (9), we get
\begin{equation}
(19) \quad -f' \kappa_2^* B^* = P(s) T + Q(s) B
\end{equation}
where
\begin{align*}
P(s) &= \frac{\lambda \kappa_2 (h \kappa_1 - \kappa_2)}{(f')^2 \kappa_1^* (1 - h^2)} (\kappa_1 - h \kappa_2), \\
Q(s) &= \frac{\lambda \kappa_2 (h \kappa_1 - \kappa_2) h}{(f')^2 \kappa_1^* (1 - h^2)} (\kappa_1 - h \kappa_2)
\end{align*}
which implies that $h \kappa_2 - \kappa_1 \neq 0$. 

Conversely, assume that $\beta$ is a timelike curve parametrized by arc-length $s$ with non-zero curvatures $\kappa_1, \kappa_2$, and the conditions of (4) hold for constant real numbers $\lambda$ and $h$. Then, we can define a curve $\beta^*$ as

$$\beta^*(s^*) = \beta(s) + \lambda N(s).$$

Differentiating (20) with respect to $s$ and using (1), we find

$$\frac{d\beta^*}{ds} = \lambda \kappa_2 (hT + B)$$

which leads to that

$$f' = \sqrt{g\left(\frac{d\beta^*}{ds}, \frac{d\beta^*}{ds}\right)} = m_1 \lambda \kappa_2 \sqrt{1 - h^2}$$

where $m_1 = \pm 1$ such that $m_1 \lambda \kappa_2 > 0$. Rewriting (21), we obtain

$$T^* = \frac{m_1}{\sqrt{1 - h^2}} (hT + B), \quad g(T^*, T^*) = 1.$$  

Differentiating (23) with respect to $s$ and using (1), we get

$$\frac{dT^*}{ds} = \frac{m_1(h\kappa_1 - \kappa_2)}{f'(1 - h^2)} N$$

which causes that

$$\kappa^*_1 = \left\| \frac{dT^*}{ds^*} \right\| = \frac{m_2(h\kappa_1 - \kappa_2)}{f'(1 - h^2)}$$

where $m_2 = \pm 1$ such that $m_2(h\kappa_1 - \kappa_2) > 0$. Now, we can find $N^*$ as

$$N^* = m_1 m_2 N, \quad g(N^*, N^*) = 1.$$  

Differentiating (26) with respect to $s$, using (23) and (24), we get

$$\frac{dN^*}{ds^*} + \kappa^*_1 T^* = \frac{m_1 m_2 (h\kappa_1 - h\kappa_2)}{f'(1 - h^2)} (T + hB)$$

which bring about that

$$\kappa^*_2 = \frac{m_3 (h\kappa_1 - h\kappa_2)}{f'(1 - h^2)},$$

where $m_3 = \pm 1$ such that $m_3(h\kappa_1 - h\kappa_2) > 0$. Lastly, we define $B^*$ as

$$B^* = \frac{m_1 m_2 m_3}{\sqrt{1 - h^2}} (T + hB), \quad g(B^*, B^*) = -1.$$

Then $\beta^*$ is a spacelike curve with spacelike principal normal and the Bertrand mate curve of $\beta$. Thus $\beta$ is a Bertrand curve.
Theorem 3.3. Let $\beta : I \subset \mathbb{R} \rightarrow \mathbb{E}^3_1$ be a unit speed timelike curve with the non-zero curvatures $\kappa_1, \kappa_2$ and $\beta^* : I \subset \mathbb{R} \rightarrow \mathbb{E}^3_1$ be a Cartan null curve with curvatures $\kappa_1^*, \kappa_2^*$. If the curve $\beta^*$ is a Bertrand mate curve of the curve $\beta$, then there exist constant real numbers $\lambda$ and $h = \pm 1$ satisfying $1 + \lambda\kappa_1 = h\lambda\kappa_2$ and $h\kappa_1 - \kappa_2 \neq 0$.

Proof. Assume that $\beta$ is a timelike Bertrand curve parametrized by arc-length $s$ with non-zero curvatures $\kappa_1, \kappa_2$ and the curve $\beta^*$ is the Cartan null Bertrand mate curve of the curve $\beta$ parametrized by with pseudo arc $s^*$ with curvatures $\kappa_1^* = 1, \kappa_2^*$. Then, we can write the curve $\beta^*$ as

$$\beta^*(s^*) = \beta^*(f(s)) = \beta(s) + \lambda(s)N(s)$$

for all $s \in I$ where $\lambda(s)$ is $C^\infty$ function on $I$. Using (1) and (2), differentiating (28) with respect to $s$, we get

$$T^*f' = (1 + \lambda\kappa_1)T + \lambda'N + \lambda\kappa_2B.$$ 

By taking the scalar product of (29) with $N$, we have

$$\lambda' = 0.$$ 

Substituting (30) in (29), we find

$$T^*f' = (1 + \lambda\kappa_1)T + \lambda\kappa_2B.$$ 

By taking the scalar product of (9) with itself, we obtain

$$1 + \lambda\kappa_1 = h\lambda\kappa_2$$

which implies that $1 + \lambda\kappa_1 = h\lambda\kappa_2$ where $h = \pm 1$. Rewriting (31) by using (32), we get

$$T^*f' = \lambda\kappa_2(hT + B).$$ 

Putting $v = \lambda\kappa_2/f'$ and differentiating (33) with respect to $s$ by using (1), we find

$$f'N^* = a(h\kappa_1 - \kappa_2)N$$

which means that $h\kappa_1 - \kappa_2 \neq 0$. 

Theorem 3.4. Let $\beta : I \subset \mathbb{R} \rightarrow \mathbb{E}^3_1$ be a unit speed timelike curve with non-zero constant curvatures $\kappa_1, \kappa_2$ and $\beta^* : I \subset \mathbb{R} \rightarrow \mathbb{E}^3_1$ be a Cartan null curve with curvatures $\kappa_1^* = 1, \kappa_2^*$. Then the curve $\beta^*$ is a Bertrand mate curve of the curve $\beta$ if and only if there exist constant real numbers $\lambda$ and $h = \pm 1$ satisfying $1 + \lambda\kappa_1 = h\lambda\kappa_2$ and $h\kappa_1 - \kappa_2 \neq 0$. 

Proof. Assume that $\beta$ is a timelike Bertrand curve parametrized by arc-length $s$ with non-zero constant curvatures $\kappa_1, \kappa_2$ and the curve $\beta^*$ is the Cartan null Bertrand mate curve of the curve $\beta$ parametrized by with pseudo arc $s^*$ with curvatures $\kappa_1^* = 1, \kappa_2^*$. Then from above theorem, there exist constant real numbers $\lambda$ and $h = \pm 1$ satisfying $1 + \lambda \kappa_1 = h \lambda \kappa_2$ and $h \kappa_1 - \kappa_2 \neq 0$.

Conversely, assume that $\beta$ is a timelike curve parametrized by arc-length $s$ with non-zero constant curvatures $\kappa_1, \kappa_2$ and there exist constant real numbers $\lambda$ and $h = \pm 1$ satisfying $1 + \lambda \kappa_1 = h \lambda \kappa_2$ and $h \kappa_1 - \kappa_2 \neq 0$. Then, we can define a curve $\beta^*$ as
\begin{equation}
\beta^*(s^*) = \beta(s) + \lambda N(s).
\end{equation}
Differentiating (35) with respect to $s$ and using (1), we find
\begin{equation}
\frac{d\beta^*}{ds} = \lambda \kappa_2 (hT + B).
\end{equation}
Differentiating (35) with respect to $s$ and using (1), we find
\begin{equation}
\frac{d^2\beta^*}{ds^2} = \lambda \kappa_2 (h \kappa_1 - \kappa_2) N
\end{equation}
which leads to that
\begin{equation}
f' = \left( \left| g \left( \frac{d\beta^*}{ds}, \frac{d\beta^*}{ds} \right) \right| \right)^{1/4} = \sqrt{m_1 \lambda \kappa_2 (h \kappa_1 - \kappa_2)}
\end{equation}
where $m_1 = \pm 1$ such that $m_1 \lambda \kappa_2 (h \kappa_1 - \kappa_2) > 0$. Rewriting (36) and (37), we obtain
\begin{align}
T^* &= \frac{\lambda \kappa_2}{\sqrt{m_1 \lambda \kappa_2 (h \kappa_1 - \kappa_2)}} (hT + B), \quad g(T^*, T^*) = 0, \\
N^* &= m_1 N, \quad g(N^*, N^*) = 1 \quad \text{and} \quad \kappa_1^* = 1.
\end{align}
We know that $\kappa_2^* = -\frac{1}{2} g \left( \frac{dN^*}{ds^*}, \frac{dN^*}{ds^*} \right)$. Thus we have
\begin{equation}
\kappa_2^* = \frac{\kappa_1^2 - \kappa_2^2}{2m_1 \lambda \kappa_2 (h \kappa_1 - \kappa_2)}.
\end{equation}
Lastly, we can define $B^*$ as
\begin{equation}
B^* = \kappa_2^* T^* - \frac{dN^*}{ds^*} = \frac{-\lambda \kappa_2 h (h \kappa_1 - \kappa_2)^2}{2m_1 \lambda \kappa_2 (h \kappa_1 - \kappa_2)^{3/2}} (T - hB), \quad g(B^*, B^*) = 0.
\end{equation}
Then $\beta^*$ is a Cartan null curve and the Bertrand mate curve of $\beta$. Thus $\beta$ is a Bertrand curve.
Example 1. Let us consider a timelike curve in $E^3_1$ with the equation
\[
\beta(s) = \left( \sqrt{2} \sinh s, \sqrt{2} \cosh s, s \right)
\]
with the Frenet Frame
\[
T(s) = \left( \sqrt{2} \cosh s, \sqrt{2} \sinh s, 1 \right),
\]
\[
N(s) = (\sinh s, \cosh s, 0),
\]
\[
B_1(s) = \left( \cosh s, \sinh s, \sqrt{2} \right)
\]
and the curvatures $\kappa_1(s) = \sqrt{2}$ and $\kappa_2(s) = -1$. If we take $h = \sqrt{2}$ and $\lambda = -1/2\sqrt{2}$ in (i) of theorem 3.2, then we get the curve $\beta^*$ as follows:
\[
\beta^*(s) = \beta(s) - \frac{1}{2\sqrt{2}} N(s) = \left( \frac{3}{2\sqrt{2}} \sinh s, \frac{3}{2\sqrt{2}} \cosh s, s \right)
\]
By straight calculations, we get
\[
T^*(s) = \left( 3 \cosh s, 3 \sinh s, 2\sqrt{2} \right),
\]
\[
N^*(s) = (\sinh s, \cosh s, 0),
\]
\[
B^*_1(s) = \left( 2\sqrt{2} \cosh s, 2\sqrt{2} \sinh s, 3 \right)
\]
and $\kappa^*_1(s) = 6\sqrt{2}$, $\kappa^*_2(s) = -8$. It can be easily seen that the curve $\beta^*$ is a timelike Bertrand mate curve of the curve $\beta$.

Example 2. For the same timelike curve $\beta$ in Example 1, if we take $h = \sqrt{2}/2$ and $\lambda = -\sqrt{2}/3$ in (ii) of theorem 3.2, then we get the curve $\beta^*$ as follows:
\[
\beta^*(s) = \beta(s) - \frac{\sqrt{2}}{3} N(s) = \left( \frac{2\sqrt{2}}{3} \sinh s, \frac{2\sqrt{2}}{3} \cosh s, s \right)
\]
By straight calculations, we get
\[
T^*(s) = \left( 2\sqrt{2} \cosh s, 2\sqrt{2} \sinh s, 3 \right),
\]
\[
N^*(s) = (\sinh s, \cosh s, 0),
\]
\[
B^*_1(s) = \left( 3 \cosh s, 3 \sinh s, 2\sqrt{2} \right)
\]
and $\kappa^*_1(s) = 6\sqrt{2}$, $\kappa^*_2(s) = -9$. It can be easily seen that the curve $\beta^*$ is a spacelike Bertrand mate curve of the curve $\beta$. 

Example 3. For the same timelike curve $\beta$ in Example 1, if we take $\lambda = 1 - \sqrt{2}$ in theorem 3.4, then we get the curve $\beta^*$ as follows:

$$
\beta^* (s) = \beta (s) + \left( 1 - \sqrt{2} \right) N (s) = (\sinh s, \cosh s, s)
$$

By straight calculations, we get

$$
T^* (s) = (\cosh s, \sinh s, 1),
$$
$$
N^* (s) = (\sinh s, \cosh s, 0),
$$
$$
B^* (s) = \left( -\frac{\cosh s}{2}, -\frac{\sinh s}{2}, 1 \right)
$$

and $\kappa_1^* (s) = 1$, $\kappa_2^* (s) = 1/2$. It can be easily seen that the curve $\beta^*$ is a Cartan null Bertrand mate curve of the curve $\beta$.

Acknowledgement 1. The authors express thanks to the referees for their valuable suggestions. The first author would like to thank TUBITAK (The Scientific and Technological Research Council of Turkey) for their financial supports during his PhD studies.

References

[1] H. Balgetir, M. Bektaş and M. Ergüt, Bertrand curves for non-null curves in 3-dimensional Lorentzian space, Hadronic J. 27(2) (2004), 229-236.
[2] H. Balgetir, M. Bektaş and J. Inoguchi, Null Bertrand curves in Minkowski 3-space and their characterizations, Note Mat. 23(1) (2004/05), 7-13.
[3] J. M. Bertrand, Mémoire sur la théorie des courbes à double courbure, Comptes Rendus 36 (1850).
[4] W. B. Bonnor, Null curves in a Minkowski space-time, Tensor 20 (1969), 229-242.
[5] W. B. Bonnor, Curves with null normals in Minkowski space-time, A random walk in relativity and cosmology, Wiley Easten Limited (1985), 33-47.
[6] N. Ekmekci and K. İlarslan, On Bertrand curves and their characterization, Differ. Geom. Dyn. Syst. 3(2) (2001), 17-24.
[7] D. H. Jin, Null Bertrand curves in a Lorentz manifold, J. Korea Soc. Math. Educ. Ser. B Pure Appl. Math. 15(3) (2008), 209-215.
[8] W. Kühnel, Differential geometry: curves-surfaces-manifolds, Braunschweig, Wiesbaden, 1999.
[9] H. Matsuda and S. Yorozu, Notes on Bertrand curves, Yokohama Math. J. 50(1-2) (2003), 41-58.
[10] L. R. Pears, Bertrand curves in Riemannian space, J. London Math. Soc. 1-10(2) (1935), 180-183.
[11] A. Uçum, O. Keçilioğlu and K. İlarslan, Generalized Bertrand curves with timelike $(1,3)$-normal plane in Minkowski space-time, Kuwait J. Sci. 42(3) (2015), 10-27.
On timelike Bertrand Curves

[12] A. Uçum, O. Keçilioğlu and K. İlarslan, Generalized pseudo null Bertrand curves in semi-Euclidean 4-space with index 2, Rend. Circ. Mat. Palermo (2016), DOI 10.1007/s12215-016-0246-x.

[13] A. Uçum, K. İlarslan and M. Sakaki, On (1,3)-Cartan null Bertrand curves in Semi-Euclidean 4-Space with index 2, J. Geom. (2015), DOI 10.1007/s00022-015-0290-2.

[14] B. Saint Venant, Mémoire sur les lignes courbes non planes, Journal de l’Ecole Polytechnique 18 (1845), 1-76.

[15] C. Xu, X. Cao and P. Zhu, Bertrand curves and Razzaboni surfaces in Minkowski 3-space, Bull. Korean Math. Soc. 52(2) (2015), 377-394.

Ali Uçum
Kirikkale University, Faculty of Sciences and Arts, Department of Mathematics, Kirikkale-Turkey.
E-mail: aliucum05@gmail.com

Kazım İlarslan
Kirikkale University, Faculty of Sciences and Arts, Department of Mathematics, Kirikkale-Turkey.
E-mail: kilarslan@yahoo.com