Reliability analysis of discrete multi-state systems based on survival signature

H Liang¹, J Mi¹, ²*, L Bai¹ and Y Cheng¹

¹ School of Automation Engineering, University of Electronic Science and Technology of China, Chengdu, China
² Center for System Reliability and Safety, University of Electronic Science and Technology of China, Chengdu, China

E-mail: jinhuami@uestc.edu.cn

Abstract. System signature has been widely utilized to facilitate reliability assessment of coherent systems and networks consisting of a single type of components, which enables us to separate aspects of components’ failure time and system structure. Recently, survival signature fulfills the similar work, extended to the implementation of systems with multiple types of components. However, when the system has several states, the relevant work is limited to complete the reliability assessment by the survival signature. This paper proposed a novel pattern of survival signature for the multi-state system with multiple types of components and the presented approach is executed to demonstrate the validity via a numerical case with 2-type and 3-state components.

1. Introduction

System signature [1,2] is a powerful tool for quantifying the reliability of coherent systems or networks that consist of independent and identically distributed (iid) or exchangeable components with respect to their random failure times, which is efficient in analysing the system with components of one type. The main advantage is the capability of fully separating system structure from the components probabilistic failure time distributions and computing the reliability with once calculation. As for the drawback, it only adapts to the system consisting of single type components, which limits its practical implementation. In the real-world scenarios, generalizing the system signature for a system with multiple types of components is not feasible, because it’s complicated to consider the rankings of order statistics from different probability distributions [3]. As the increasing demand for reliability analysis of multi-type component systems, survival signatures are introduced by Coolen and Coolen-Maturi [4] in 2012, in order to assess the system consisting of multiple types components by signatures. Feng et al. [5] proposed a simulation method based on the survival signature, taking account of the imprecision of component specifications and quantifying the importance by the relative importance index. Aslett et al. [3] presented the application of survival signature for the system with multiple types of components from a Bayesian perspective.

However, the aforementioned applications assume that the system is in a binary state: working and failure. In fact, during a long working time, components and subsystems of the equipment often suffer from continuous degradation, instead of instant failure. Therefore, adding several intermediate states between working and completely failed states, which is termed as a multi-state system (MSS) [6–11], can capture more multi-state nature of sophisticated engineering systems. There are a large number
of methods for MSS reliability modeling. Whereas implementing the survival signature to the MSS has just begun. Eryilmaz et al. [12] proposed the survival signature for a multi-state consecutive k-out-of-n system to acquire a representation for the survival function of the time spent by a system in a specific state or above. Then, Liu et al. [11] extended the survival signature from the binary system to a certain class of MSSs in both discrete and continuous cases with multi-state components in the stress-strength model. These works have not inducted an ideal solution for the general multi-state system with multiple types of components. Hence, in this paper, we aim to present a novel pathway to extend the generalized survival signature for the reliability analysis of MSSs.

The rest of this paper is roll out as follows: Section 2 introduces the definitions and exact expressions of a multi-state survival signature. Furthermore, in Section 3, the multi-state survival signature is implemented in a numerical example and the results are demonstrated by the Bayesian network (BN) method. Finally, a brief conclusion is given in Section 4.

2. Definitions and assumptions

2.1 Generalized survival signature

For a system with iid components, the system signature would separate the structure from the probabilistic model. As an improved concept, survival signature [4, 13–15] eliminates the one-type restriction of system signature, where the lifetime distributions of components of different types are independent. Assume that a system is composed of m components with the number of types $K \geq 2$. And the state vector of components could be defined as $x = \left( x_1^k, x_2^k, ..., x_m^k \right)$ for $k=1,2,...,K$, with subvector $x^k = (x_1^k, x_2^k, ..., x_m^k) \in \{0,1\}^m$, where $x_i^k = 0$ if component $i$ of type $k$ is in the failure state and $x_i^k = 1$ if not. The structure function is written as $\phi$. Similar to $x_i^k$, $\phi=1$ if system functions while $\phi=0$ if not. Moreover, $m_k$ represents the number of components of type $k$ $(k = 1,2,...,K)$, which satisfies $\sum_{k=1}^{K} m_k = m$. Then $l_k = 0,1,...,m_k$ indicates the number of components of type $k$ in the working state and $\Phi(l_1,l_2,...,l_K)$, namely survival signature, denotes the probability that system functions given $l_k$ of $m_k$ components work. Based on the assumptions mentioned before, the survival signature could be written as

$$
\Phi(l_1,l_2,...,l_K) = \left[ \prod_{k=1}^{K} \left( \frac{m_k}{l_k} \right)^{l_k} \right] \times \sum_{x \in S_{l_1,...,l_K}} \phi(x)
$$

(1)

where $S_{l_1,...,l_K}$ denotes the set of all state vectors. When $l_k$ of $m_k$ components work, there are $\left( \frac{m_k}{l_k} \right)$ state vectors. Due to the independency of each type of components, with the type $(l_1,l_2,...,l_K)$, the possible forms of system structure are $\prod_{k=1}^{K} \left( \frac{m_k}{l_k} \right)$. Hence, $\phi(x)$ could describe the working probability of each form of the system structure.

Assume that the failure time probability distribution of each type of components is $F_k(t)$, and components with the same type have identical CDF. $C_i^k \in \{0,1,...,m_k\}$ is the number of $k$ components working at time $t$. Then:
\[
P\left(\bigcap_{k=1}^{K} \{C_i = l_k\}\right) = \prod_{k=1}^{K} P\left(C_i = l_k\right)
\]

(2)

Acquired the survival signature and the survival probability of each given state vector, the survival function of the system with \(K\) types of components can be written as equation (3),

\[
P(T_s > t) = \sum_{l_1,\ldots,l_K} \Phi(l_1,\ldots,l_K) P\left(\bigcap_{k=1}^{K} \{C_i = l_k\}\right)
\]

(3)

\[
= \sum_{l_1,\ldots,l_K} \sum_{l_1,\ldots,l_K} \left[ \Phi(l_1,\ldots,l_K) \prod_{k=1}^{K} P\left(C_i = l_k\right) \right]
\]

\[
= \sum_{l_1,\ldots,l_K} \sum_{l_1,\ldots,l_K} \left[ \Phi(l_1,\ldots,l_K) \prod_{k=1}^{K} \left( m_k \left[ F_k(t) \right]^{m_k - l_k} [1 - F_k(t)]^{l_k} \right) \right]
\]

It can be clear from the formulate that the survival signature can separate the structure of the system from the failure time distribution of its components and just need to be calculated once for the reliability analysis, which shares the same advantage of the system signature. Furthermore, it extends the application for the system with multiple types of components.

2.2 Multi-state survival signature

The multi-state survival signature is an extension of the conventional method. The derivation process is basically unchanged. But we notice that the variables like \(\Phi\), \(l_k\) and subvector \(\chi^k\) could not represent the characteristic of multi-state. Hence, the definitions of these variables and vectors should be rewritten referring to principles of state transition. First, state vectors are unchanged, but subvectors should be rewritten as \(\chi^k = \left(x_1^k, x_2^k, \ldots, x_{l_k}^k\right) \in \{0,1,2,...,N\}^{m_k}\). Then, let \(\tilde{t}_{(k,n)}\) indicate the number of survival components of type \(k\) with state \(n\) to replace the \(l_i\). Due to the multi-state characteristics of the system and elements, survival signature \(\Phi\), should also be redefined as \(\Phi_{(k)}\), namely multi-state survival signature, which represents the probability of the system with state \(s\). Furthermore, let \(\phi_s \in \{0,1,2,...,N\}\) denote the state of the system in a given state vector \(\chi\). For instance, \(\phi_s = 2\) describes that the system stays at state 2 and similarly, \(\phi_s = 1\) shows the system in state 1. Since each component would only have one certain state at a time, the number of state vectors with precisely given \(\tilde{t}_{(k,n)}\) could be represented as

\[
Com_k = \left\{ \frac{m_k}{\tilde{t}_{(k,0)}}, \frac{m_k}{\tilde{t}_{(k,1)}}, \ldots, \frac{m_k}{\tilde{t}_{(k,N-1)}} \right\}
\]

(4)

Referring to the equation (4), the multi-state survival signature can be rewritten as:
\[
\Phi_s(\bar{t}_{(0)}, \ldots, \bar{t}_{(K,N)}) = \frac{\sum_{\substack{\mathcal{L} \subset \{1, \ldots, K\} \setminus \{s\}}} \phi_s(x)}{\prod_{k=1}^K \text{Com}_k}
\]

\[
= \sum_{\substack{\mathcal{L} \subset \{1, \ldots, K\} \setminus \{s\}}} \phi_s(x) \left[ \prod_{k=1}^K m_k \left( \frac{m_k - \bar{t}_{(k,0)}}{\bar{t}_{(k,1)}} \right) \left( \frac{m_k - \bar{t}_{(k,0)} - \bar{t}_{(k,1)}}{\bar{t}_{(k,2)}} \right) \cdots \left( \frac{m_k - \cdots - \bar{t}_{(k,0)}}{\bar{t}_{(k,N)}} \right) \right]^{-1}
\]

Next, assume the \( F_{(k,n)}(t) \) represents the CDF of the type \( k \) component staying at state \( n \), then for \( \bar{t}_{(k,n)} \in \{0,1,\ldots,m_k\}, k=1,2,\ldots,K \),

\[
P_s \left( \bigcap_{n=0}^{N-1} \bigcap_{k=1}^K \{ C_i^k = \bar{t}_{(k,n)} \} \right)
= \prod_{k=1}^K \prod_{n=0}^N P_s \left( C_i^k = \bar{t}_{(k,n)} \right)
\]

where the \( P_s \left( C_i^k = \bar{t}_{(k,n)} \right) \) could be written as

\[
P_s \left( C_i^k = \bar{t}_{(k,n)} \right) = \begin{cases} 
\left( \frac{m_k}{l_{(k,0)}} \right) \left( F_{(k,0)}(t) \right)^{m_k} \left( 1 - F_{(k,0)}(t) \right)^{m_k}, j = 0 \\
\left( \frac{m_k}{l_{(k,0)}} \right) \left( 1 - \sum_{j=0}^{n-1} \bar{t}_{(k,n)} \right) \left( F_{(k,n)}(t) \right)^{m_k} \sum_{j=0}^{n-1} \left( 1 - F_{(k,n)}(t) \right)^{m_k}, n > 0 
\end{cases}
\]

Consequently, referring to the equation (4)-(7), the probability of the system in the state \( s \) could be expressed as

\[
P_s(T_s > t) = \sum_{l_{(1)}>0} \cdots \sum_{l_{(K)}>0} \sum_{l_{(k)}>0} \Phi_s(\bar{t}_{(1)}, \ldots, \bar{t}_{(K,N)}) \prod_{n=0}^{N-1} \prod_{k=1}^K P_s \left( C_i^k = \bar{t}_{(k,n)} \right)
\]

\[
= \sum_{l_{(1)}>0} \cdots \sum_{l_{(K)}>0} \sum_{l_{(k)}>0} \Phi_s(\bar{t}_{(1)}, \ldots, \bar{t}_{(K,N)}) \prod_{n=0}^{N-1} \prod_{k=1}^K P_s \left( C_i^k = \bar{t}_{(k,n)} \right)
\]

3. Numerical case

3.1. Application in a 2-type and 3-state system

As shown in Figure 1, a system composed of components of 2 types with the performance assumed to be 3 states, where state 2 and 0 are the best state and the failure state, respectively. There are 5 components in the system where the component 1, 3, and 4 are type 1, others are type 2. The probability of each state is listed in Table 1 and we set 5 independent groups to induce the reliability. In the practical scenario, the values always come from the experience of experts and statistical results of test data. Components of the same type have iid failure times and each type of components is independent in failure times.

As the time passing, states of the system or elements would degrade from state 2 to 0. The transition of system states follows principles as below:

(1) When at least one complete failure event of components occurred in all the paths between the
input and output of the system, the state of the system should be 0.

(2) When whole components involved in a path are in state 2, the state of the system should be 2.

(3) When the above assumptions are not achieved, the system state should be 1.

**Table 1.** Probability of diverse types and states. To validate the accuracy of the method, we set five independent and irrelevant sets, which is equivalent to five different tests.

| Type 1 | Type 2 |
|--------|--------|
| State 2 | State 1 | State 0 | State 2 | State 1 | State 0 |
| 0.8 | 0.1 | 0.1 | 0.7 | 0.1 | 0.2 |
| 0.87 | 0.12 | 0.01 | 0.83 | 0.15 | 0.02 |
| 0.98 | 0.01 | 0.01 | 0.94 | 0.03 | 0.03 |
| 0.95 | 0.04 | 0.01 | 0.75 | 0.2 | 0.05 |
| 0.86 | 0.11 | 0.03 | 0.88 | 0.11 | 0.01 |

**Figure 1.** A system with 2 types of components.

**Table 2.** Multi-state survival signature of the numerical case.

| Type 1 | Type 2 | System |
|--------|--------|--------|
| State 2 | State 1 | State 0 | State 2 | State 1 | State 0 | State 2 | State 1 | State 0 |
| 0 | 0 | 3 | 0 | 2 | 0 | 0 | 1 |
| … | … | … | … | … | … | … | … |
| 2 | 0 | 1 | 2 | 0 | 0 | 1 | 0 | 0 |
| 2 | 1 | 0 | 0 | 0 | 2 | 0.333 | 0.667 | 0 |
| … | … | … | … | … | … | … | … |
| 3 | 0 | 0 | 2 | 0 | 0 | 1 | 0 | 0 |

Referring to the derivation method of the multi-state survival signature and the state transition mechanism, crisp probability values of each system final state are calculated and shown in Table 3.

**Table 3.** Probability of each state computed by survival signature.

| State 2 | State 1 | State 0 |
|---------|---------|---------|
| 0.8702 | 0.0874 | 0.0424 |
| 0.9521 | 0.0475 | 0.0004 |
| 0.9976 | 0.0018 | 0.0006 |
| 0.9734 | 0.0247 | 0.0019 |
| 0.9631 | 0.0363 | 0.0006 |

3.2. Validation
In order to verify the correctness of results from the multi-state survival function, the BN is selected to make a comparison, which describes the relationship of failure events by the directed acyclic graph (DAG) as well as the conditional probability table (CPT). It’s first proposed by Pearl [16], achieving significant development in the system reliability and safety analysis [17–19].

The DAG of the system is plotted as Figure 2, where \( C_n (1 \leq n \leq 5) \) describes components of type \( n \), and the value 0–2 is treated as the states. The derivation algorithm of the CPT is shown in Algorithm 1, where \( C_{14}, C_{25}, C_{12}, \) and \( C_{45} \) represent the subsystems composed of relevant components.

![Figure 2. A DAG based on the system of Figure 1.](image)

**Algorithm 1**

```plaintext
for i ← 1 to cptRowsNumber
    C3 ← cpt(i,3);
    C12 ← max(cpt(i,1),cpt(i,2));
    C45 ← max(cpt(i,4),cpt(i,5));
    C14 ← min(cpt(i,1),cpt(i,4));
    C25 ← min(cpt(i,2),cpt(i,5));
    if C3=2
        cpt (i,6) ← min(C12,C45);
    if C3=1
        if max(C14,C25)=2
            cpt (i,6) ← 2;
        else
            if min(C12,C45)=0
                cpt (i,6) ← 0;
            else
                cpt (i,6) ← 1;
        if C3=0
            cpt (i,6) ← max(C14,C25);
```

Analyzed by BN [20], we obtain the probability of state 0–2 of node OUT shown in Table 4. It is clear that the value is completely consistent with Table 3. Hence, it successfully demonstrates that the multi-state survival signature is feasible for the 3-state and 2-types MSS reliability assessment.

**Table 4.** Probability of each state validated by the BN inference.

| State 2 | State 1 | State 0 | Consistent with Table 3 |
|---------|---------|---------|-------------------------|
| 0.8702  | 0.0874  | 0.0424  | True                    |
4. Conclusion
The survival signature has been an efficient strategy for performing reliability evaluation of complex systems with multiple component types. However, it is applicable only in the binary system. For the sake of deriving the reliability of MSSs, this paper proposed a method to extend the survival signature. Compared with the conventional survival signature of binary state, it redefined some variables and formulas to represent the diverse states. For instance, the state vector is changed to show multiple states $0$–$N$ and the number of state vectors is calculated from equation (4) rather than the simple multiplication of equation (1), because each component could be only in one state at a time. Moreover, in order to prove the feasibility, we compare the results computed from the multi-state survival signature and BN through a 3-state and 2-type numerical example. Consequently, consistent calculation results verify the correctness of the method.

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