Theory of low-energy behaviors in topological s-wave pairing superconductors

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Abstract

We construct a low-energy effective theory of topological s-wave pairing superconductors, focusing on the mean-field model of superconductor Cu$_2$Bi$_2$Se$_3$. Our approach is second-order perturbation with respect to the inverse of the mass (i.e., large-mass expansion) in the Dirac-type electron dispersion from topological insulator Bi$_2$Se$_3$. Since the Dirac-type dispersion with a large mass describes non-relativistic electrons, the large-mass expansion corresponds to a low-energy theory with respect to the original setup. We show that the effective gap function has not only a p-wave-like component as the primary contribution, but also an s-wave-like one as higher-order corrections. The mixture of p- and s-wave explains the numerical results [Phys. Rev. B 89 (2014) 214506] of the non-magnetic impurity effects.

Keywords: Topological superconductors, Dirac-type dispersion, Large-mass expansion, Mixture of p- and s-wave states

1. Introduction

Superconductor Cu$_2$Bi$_2$Se$_3$ and the related compounds [1][2][3][4] attract a great deal of attention in condensed matter physics since they are strong candidates for bulk topological superconductors. To answer whether Cu$_2$Bi$_2$Se$_3$ is a genuine topological superconductor, various experimental studies are performed, including point-contact spectroscopy [5][6], magnetization curves [7], scanning tunneling spectroscopy [8], and Knight-shift measurements [9].

Impurity effects lead to a definite way to discriminating unconventional features of superconductors [10]. The effects in Cu$_2$Bi$_2$Se$_3$ are of particular interest. Nagai et al. [11] numerically studied the non-magnetic impurity effects in the mean-field model of Cu$_2$Bi$_2$Se$_3$ [5] (i.e., topological s-wave pairing superconductivity), by a self-consistent T-matrix approach. Since the model allows the presence of surface gapless modes [12], one expects fragile behaviors against non-magnetic impurities. In contrast to the intuitive assertion, the sensitivity is variable, depending on the mass in the Dirac-type dispersion relation from topological insulator Bi$_2$Se$_3$. When the mass is large, in-gap states in the density of states occurs; the superconducting state is not robust against non-magnetic impurities. In contrast, when the mass is small, there is no in-gap state.

In this paper, we study an effective theory to understand the unconventional features of the mean-field model of Cu$_2$Bi$_2$Se$_3$, motivated by the numerical results by Nagai et al. [11]. To answer how the robustness disappears depending on the mass term, we derive a low-energy effective theory in a large-mass limit. Using the second-order perturbation with respect to the inverse of the mass with a basis transformation to take higher-order corrections, we show that the effective superconducting gap function is described by a mixture of p- and s-wave-like components, as seen in Eq. (16). The latter is smaller than the former in the large-mass limit. Therefore, we obtain the effective description of the system, supporting the previous numerical calculations about the non-magnetic impurity effects. The effective theory is useful for revealing the elementary properties of the topological s-wave pairing superconductors, from a gap-function-type point of view.

2. Model

The mean-field Hamiltonian is $\hat{H}_{MF} = (1/2) \sum_\mathbf{k} \hat{\Psi}_\mathbf{k}^\dagger H(\mathbf{k}) \hat{\Psi}_\mathbf{k}$, with the Bogoliubov-de Gennes (BdG) Hamiltonian [5]

$$H(\mathbf{k}) = \frac{1 + \tau^3}{2} \otimes h_0(\mathbf{k}) + \frac{1 - \tau^3}{2} \otimes [-h_0(-\mathbf{k})]^\dagger + [\tau^2 \otimes \Delta + \text{(h.c.)}].$$

(1)

The 8-component column vector $\hat{\Psi}_\mathbf{k}$ has the electron annihilation operators $(\hat{c}_\mathbf{k,\alpha})$ in the upper 4-component
inversion is given by (effective theory is to focus on the mass term, transformation property of $\Pi = \Delta$ (Eq. (2)). The superconducting pairing potential $\Delta$ has no $k$-dependence. The model corresponds to a topological $s$-wave pairing superconductor. Fu and Berg proposed Eq. (1), based on short-range charge-density interaction.

The normal-electron Hamiltonian in Eq. (1) is

$$h_0(k) = -\mu + M_0 \gamma^0 + \sum_{i=1}^3 d_i(k) \gamma^i \gamma^i,$$  

with chemical potential $\mu$ and spin-orbit couplings $(d_1, d_2, d_3) = (A k_x, A k_y, A k_z)$. The $4 \times 4$ matrices $\gamma^i$ ($\mu = 0$, 1, 2, 3) satisfy the Clifford algebra: $\gamma^i \gamma^j + \gamma^j \gamma^i = 4 \eta^{ij}$, with $\eta^{01} = 1$, $\eta^{12} = -1$, and $\eta^{00} = 0$ for $\mu = 0$. We describe the orbit and the spin degrees of freedom in the system. Using the orbital $2 \times 2$ Pauli matrices $d^i$ and the spin $2 \times 2$ Pauli ones $s^i$, we have $\gamma^0 = \sigma^3 \otimes 1$ and $\gamma^i = i \sigma^i \otimes s^i$. The role of $\gamma^0$ is of particular importance to the system since it is related to spatial inversion; we find that $(\gamma^0)^* h_0(k) \gamma^0 = h_0(-k)$. Calculating the eigenvalues of $h_0$, we find that the normal electrons have the massive Dirac-type dispersion relation, $\epsilon^\pm(k) = -\mu \pm [M_0^2 + A^2 (k_x^2 + k_y^2) + A^2 k_z^2]^{1/2}$. Without changing $\epsilon^\pm(k)$, the anisotropy along $z$-axis in Eq. (2) can be taken by a different way.

We focus on the odd-parity fully-gapped superconducting order in this paper. The pairing potential is

$$\Delta = i \Delta_{odd} \gamma^0 \gamma^2 = i \Delta_{odd} \sigma^1 \otimes i s^2,$$  

with a complex constant $\Delta_{odd}$. Using the notations by Sasaki et al. [3], we find that $\Delta_{11}^\pm = -\Delta_{21}^\pm = i \Delta_{odd}$. $\Delta_{11}^2 = -\Delta_{21}^2 = i \Delta_{odd}$, and the other elements vanish. The transformation property of $\Delta$ with respect to the spatial inversion is given by $(\gamma^0)^* \Delta \gamma^0 = -\Delta$. When the odd-parity gap appears, the BdG Hamiltonian does not have a definite property with respect to the spatial inversion and rotation in the Nambu space [13,15].

$$[(i \gamma^1 \otimes 1) \Pi]^2 H(k) [(i \gamma^1 \otimes 1) \Pi] = H(-k),$$  

with $\Pi = (1/2)(1 + \tau^1) \otimes \gamma^0 + (1/2)(1 - \tau^1) \otimes (\gamma^0)^*$. This relation is essential for characterizing the model in terms of invariants [13,15].

A key point in our construction of a low-energy effective theory is to focus on the mass term, $M_0 \gamma^0$ in Eq. (2) [11]. The use of a dimensionless quantity is convenient for our arguments,

$$\beta = \frac{\bar{k}_F}{|M_0|} = \sqrt{\frac{\mu^2}{M_0^2} - 1},$$  

with the Fermi momentum $\bar{k}_F = (1/|\mu|)(\mu^2 - M_0^2)^{1/2}$. When $\beta$ is small, the system is in a large-mass (non-relativistic) region. The Fermi surface is determined by a larger eigenvalue of Eq. (2), i.e., $0 = \epsilon^\tau(k_F)$, with $\mu > |M_0|$. We obtain $\bar{k}_F$ for the spherical Fermi surface, redefining the momentum; $(A' / A) k_z \rightarrow k_z$.

Focusing on $\beta$ is effective for understanding the system. The stable superconducting order is ruled by the value of $(1 + \beta^2)^{-1/2} (= |M_0|/|\mu|)$, within the linearized gap equation [13]. The value of $\beta$ is also relevant to the response properties to impurities of an odd-parity superconducting gap with a point node [16].

Before closing this section, we summarize the properties of Eq. (1) when $M_0 = 0$ (i.e., $\beta \rightarrow \infty$). Using the chiral symmetry [13], we find that $\gamma^2 \gamma^3 h_0 - h_0 \gamma^2 = 0$ and $\gamma^2 \Delta - \Delta \gamma^2 = 0$, with $\gamma^2 = i \gamma^1 \gamma^0 \gamma^3 = \sigma^1 \otimes 1$. Hence, exchanging the order between $\tau$ (Nambu) and $\sigma^i$ (orbital), we have $H(k)_{M_0=0} = [(1 + \tau^1)/2] \otimes H^0(k) + [(1 - \tau^1)/2] \otimes H^2(k)$, with

$$H^{[R,L]}(k) = \frac{1 + \tau^3}{2} \otimes [-\mu + \kappa_{R,L} A(\bar{k} \cdot s)]$$

$$+ \frac{1 - \tau^3}{2} \otimes [\mu + \kappa_{R,L} A(\bar{k} \cdot s)]^*$$

$$+ [\tau^+ \otimes \kappa_{R,L} (i \Delta_{odd} i s^2 + (h.c.))],$$

where $\bar{k} = [k_x, k_y, (A' / A) k_z]$ and $(A, \kappa_{R,L}) = (1, -1)$. In each block, the normal part is transformed into a diagonal form by $[((1 + \tau^1)/2) \otimes U_k + [(1 - \tau^1)/2] \otimes (-i s^2 U_k)]$, with $U_k^\dagger (\bar{k} \cdot s) U_k = |\bar{k}| \text{diag}(1, -1)$. To show this statement, a relation of the Pauli matrices is employed; $s^1 s'^2 = -s^2$. After the basis transformation, the superconducting part is independent of $\bar{k}$ since $U_k^\dagger i s^2 (-i s^2 U_k) = 1$. We mention that the system with $M_0 = 0$ is different from the conventional $s$-wave superconductors; the odd-parity property of the pairing potential appears as a sign change between the right- and left-handed blocks [11]. However, the gap function is considered to be that of an $s$-wave state as long as the BdG Hamiltonian is decoupled into the blocks with different kinds of chirality. Therefore, we may say that the system with $M_0 = 0$ reduces to an $s$-wave superconducting model [11,13].

The character of this $s$-wave-like gap function implies that the odd-parity state with a large $\beta$ is robust against non-magnetic impurities. Nagai et al. [11] numerically found no occurrence of in-gap states under
non-magnetic impurities when \( \beta > 1 \). Michaeli and Fu [17] proposed the protection of the odd-parity superconducting state against non-magnetic impurities in terms of spin-orbit locking effects. The effects are predominant when \( \beta \) (i.e., \( A \)) is large; in a relativistic region this mechanism is reasonable.

### 3. Results: Derivation of low-energy effective theory

Now, we construct a low-energy effective theory when \( \beta \sim 0 \) (i.e., large-mass expansion). Our approach is similar to the arguments in semiconductor-superconductor junction systems [18], but we take higher-order corrections. The corrections would be primary, taking a large \( \beta \). The arguments at the end of Section 2 show that an s-wave character is manifest in the system when \( \beta \rightarrow \infty \). Thus, our approach is useful for understanding how this s-wave character disappears when \( \beta \rightarrow 0 \). Throughout this section, we exchange the order between the Nambu space and the orbital degrees of freedom.

#### 3.1. Effective theory without higher-order corrections

We show the low-energy effective theory without higher-order corrections [11]. The arguments can straightforwardly extend to the case with corrections.

When \( \beta \approx 0 \), the mass term in Eq. (2) is predominant. Let us seek the perturbation terms in Eq. (1). In the normal part, the spin-orbit couplings are regarded as the perturbation since the magnitude of these terms is characterized by \( \hbar k_F \). Moreover, in a weak-coupling superconductor, the contributions from the normal potentials are smaller than those from the normal electrons. Thus, the pairing potential in the mean-field Hamiltonian should be the perturbation. Therefore, we find that \( H(k) = H_0(k) + V(k) \), with the free part \( H_0 = [(1+\sigma^3)/2] \otimes H_0^2 + [(1-\sigma^3)/2] \otimes H_0^2 \) and the perturbation \( V = \sigma^3 \otimes V^+ \), where

\[
H_0^s(k) = (-\mu \pm \mu_0) s^1 \otimes 1,
\]

\[
V^+(k) = \frac{1+r^3}{2} \otimes \mathcal{A}(\vec{k} \cdot s) + \frac{1-r^3}{2} \otimes \mathcal{A}(\vec{k} \cdot s)^* + [\tau^+ \otimes (i\Delta_{odd})] s^2 + (h.c.)].
\]

Since the perturbation contains the spin-orbit couplings and the pairing potential, the perturbation expansion is valid within \( |\mu_0| \gg \hbar k_F, |\Delta_{odd}| \).

To perform the perturbation systematically, we use the orthogonal projectors \( \mathcal{P} \) and \( \mathcal{Q} \), defined as

\[
\mathcal{P} = \frac{1+\sigma^3}{2} \otimes 1 \otimes 1, \quad \mathcal{Q} = \frac{1-\sigma^3}{2} \otimes 1 \otimes 1.
\]

Throughout this section, we focus on a positive \( M_0 \). In this case, the subspace given by \( \mathcal{P} \) is our target space. When \( M_0 \) is negative, we can derive the effective theory exchanging the role between \( \mathcal{P} \) and \( \mathcal{Q} \). Using the second-order Brillouin-Wigner perturbation approach [20], we obtain the effective Hamiltonian,

\[
H_{\text{eff}}(k) = \mathcal{P} H_0(k) \mathcal{P} + \sum_{m=2}^{\infty} [\mathcal{P} V(k) \mathcal{Q} R^m(k)] [\mathcal{Q} V(k) \mathcal{P}], \quad (10)
\]

with \( (E_0^m - \mathcal{Q} H_0(k) \mathcal{Q} R^m = Q \) and \( E_0^+ = \pm (-\mu + M_0) \). A similar technique is applied to deriving a low-energy effective theory in a different model of superconducting topological insulator [19]. After straightforward calculations, we find that \( H_{\text{eff}}(k) = [(1+\sigma^3)/2] \otimes H_{\text{eff}}^+(k) \), where

\[
H_{\text{eff}}^+(k) = \frac{1+r^3}{2} \otimes h_{\text{eff},0}(k)
\]

\[
+ \frac{1-r^3}{2} \otimes [-h_{\text{eff},0}(-k)]^*
\]

\[
+ [\tau^+ \otimes \Delta_{\text{eff},0}(k)] + (h.c.). \quad (11)
\]

In the vicinity of the Fermi surface (i.e., \( \mu \approx M_0 \)), we obtain \( h_{\text{eff},0}(k) \approx (1/2M_0) 2[(\bar{\Delta} \bar{k})^2 - |\Delta_{\text{odd}}|^2] \) and

\[
\Delta_{\text{eff},0}(k) \approx 2\bar{\beta} (i\Delta_{\text{odd}}) [\bar{d}(k) \cdot s] i\bar{s}^2, \quad (12)
\]

with \( \bar{d} = \bar{k}/k_F \). Thus, we find that the effective gap function corresponds to a p-wave-like state [11].

#### 3.2. Mixture of p- and s-wave components

Let us show the low-energy effective theory with higher-order corrections. We take a positive \( M_0 \) again; the target subspace is specified by \( \mathcal{P} \) in Eq. (2). A straightforward way to taking the corrections is to add higher-order expansion terms to Eq. (10), but this approach would be messy. We use an alternative way, i.e., a second-order perturbation approach with a basis transformation. Let us concisely summarize our approach. A unitary transformation is first applied to Eq. (10), to obtain a better basis in perturbation. Then, in this transformed basis the second-order perturbation equivalent to that in Section 3.1 is performed. Thus, the corrections from the basis transformation would appear in Eq. (11).

Using a basis transformation, the BdG Hamiltonian is partially diagonalized. The perturbation in the transformed basis would lead to better expansion. To focus on the pairing potential leads to a clue of finding a proper basis. We find that this term commutes with the mass term in Eq. (1). The proof is done by the same
technique of showing relation (4). Thus, the basis transformation should include rotation in the Nambu space. We propose the basis transformation given by a unitary matrix with a small angle $\beta$,

$$S_0 = \exp\left(\frac{ij\beta}{2} \sigma^3 \otimes [\chi_{\text{odd}}\sigma^+ + (\text{h.c.})] \otimes s^2 \right),$$ (13)

with $\chi_{\text{odd}} = -\frac{\Delta_{\text{odd}}}{|\Delta_{\text{odd}}|}$. We denote the unitary-transformed BdG Hamiltonian as $H_\beta(k)$. The free part and the perturbation are, respectively, expressed by $H_{0,\beta}(k) = S_\beta H_0(k) S_\beta^\dagger$ and $V_\beta(k) = S_\beta V(k) S_\beta^\dagger$. To perform small-$\beta$ expansion systematically in the unitary-transformed formulae, we focus on an algebraic relation in Eq. (13). Let us rewrite $S_\beta = \exp[-i\beta/2]W$, with $W = -\sigma^3 \otimes [\chi_{\text{odd}}\sigma^+ + (\text{h.c.})] \otimes s^2$. We find that $W$ is a Hermite matrix and does not contain any small parameters (i.e., $\beta$ and $|\Delta_{\text{odd}}|/|M_0|$). Since $\chi_{\text{odd}} = \frac{\Delta_{\text{odd}}}{|\Delta_{\text{odd}}|}$, we have $W^2 = 1$. Therefore, we find that $S_\beta = \cos[(\beta/2)] - iW \sin[(\beta/2)] = 1 - (i\beta/2)W + O(\beta^2)$. It indicates that for a matrix $M$ the unitary-transformed one is $M_\beta = M + (i\beta/2)[M, W] + O(\beta^2)$.

We explicitly write down the formulae of $H_{0,\beta}(k)$ and $V_\beta(k)$. First, we focus on the free part. Using a similar manner to Eq. (7), we obtain $H_{0,\beta}(k) = [(1 + \sigma^3)/2] \otimes H_{0,\beta}^\sigma + [(1 - \sigma^3)/2] \otimes H_{0,\beta}^p$. We find that

$$H_{0,\beta}^\sigma(k) = (-\mu + M_0)\sigma^3 \otimes 1 + \beta\Delta^{(\pm)}(k)\sigma^+ \otimes i\sigma^2 + (\text{h.c.}) + O(\beta^2),$$ (14)

with $\Delta^{(\pm)}(k) = -i(-\mu + M_0)\chi_{\text{odd}}$. We find that $\Delta^{(\pm)}(k)$ vanishes in the vicinity of the Fermi surface since $\mu \approx M_0$. As a result, we consider that $H_{0,\beta}^\sigma \approx H_0^\sigma$ in the perturbation expansion. From this point of view, the present basis transformation minimally deforms the original issue. Next, we examine the transformed perturbation. We find that

$$V_\beta(k) = \sigma^3 \otimes V^+(k) + (-\beta)\Delta^{(\pm)}(k)\sigma^+ \otimes i\sigma^2 \otimes 1 + (\text{h.c.}) + O(\beta^2).$$ (15)

The second term is an important effect caused by the basis transformation. This term is relevant to an $s$-wave behavior in the large-mass limit.

Now, we show the low-energy effective Hamiltonian. We repeat the same discussion as in Section 5.1 with the free part $H_{0,\beta}$ and the perturbation $V_\beta$. The calculation details are shown in Appendix A. Performing the second-order perturbation expansion, we obtain the effective Hamiltonian in the subspace specified by $\mathcal{P}$. The Hamiltonian $H_{\beta}^\sigma(k)$ in this subspace is attainable, replacing $h_{\text{eff},0}$ and $\Delta_{\text{eff},0}$ in Eq. (11), respectively, with $h_{\text{eff},\beta}$ and $\Delta_{\text{eff},\beta}$. In the vicinity of the Fermi surface, we find that $h_{\text{eff},\beta}(k) \approx h_{\text{eff},0}(k)$. The effective gap is

$$\Delta_{\text{eff},\beta}(k) \approx 2\beta(i\Delta_{\text{odd}})\left(\hat{d}(k) \cdot s + \frac{|\Delta_{\text{odd}}|}{M_0}\right)i\sigma^2. \quad (16)$$

The first term is equal to Eq. (12) and describes a $p$-wave-like state, whereas the second term is an $s$-wave-like state. The physical origin of the $p$-wave component is a cross term between the spin-orbit couplings and the superconducting gap. The second term comes from a higher-order correction in the perturbation term derived by the basis transformation. Thus, in a large-mass limit, the system is effectively expressed by a mixture of the $p$-wave and the $s$-wave components. The primary contribution is the $p$-wave one in this limit. However, when the second term is not negligible, the system can behaves as an $s$-wave superconducting state.

### 3.3. Link of effective theory with odd-parity pairing potential

We argue a link of the effective gap with the pairing potential given by Eq. (3), from a parity point of view. One approach is to make a transformation of spatial inversion in the projected subspace appeared in the perturbation analysis. The resultant formula might lead to a transformation property of $\Delta_{\text{eff},\beta}$, similar to that of $\Delta$ in Section 5. For this purpose, we implement the projector $\mathcal{P}$ on the spatial-inversion transformation in the full BdG formulation. According to the calculations in Appendix B, we find that the projected spatial inversion is represented by the identity matrix. Thus, this transformation leads to no information on the transformation property of the effective gap.

We take an alternative approach of finding a connection of the effective theory with the full BdG formulation. Let us compare the effective theory with that in the other projected subspace. We change a kind of the projectors: $\mathcal{P} = [(1 - \sigma^3)/2] \otimes 1 \otimes 1$ and $\mathcal{Q} = [(1 + \sigma^3)/2] \otimes 1 \otimes 1$. We have two setups relevant to this choice. One is the case of $\mu < 0$ with $M_0 > 0$, while the other is that of $\mu > 0$ with $M_0 < 0$. We take the former in this paper. This setup means that the chemical potential intersets with a lower energy band of the normal-electron Dirac dispersions; the Fermi surface is defined by $\mathcal{P} = c^-(k)$, leading to $\mu = -M_0 + O(\beta^2)$. The derivation of the corresponding effective theory is parallel to the previous case. The calculations are shown in Appendix A. We obtain the effective gap

$$\Delta_{\text{eff},\beta}(k) \approx -2\beta(i\Delta_{\text{odd}})\left(\hat{d}(k) \cdot s - \frac{|\Delta_{\text{odd}}|}{M_0}\right)i\sigma^2. \quad (17)$$
We note that the other setup (i.e., $\mu > 0$ and $M_0 < 0$) leads to Eq. (17), up to an overall phase. We find again that the first term has momentum dependence (i.e., $p$-wave-like component), while the second term is independent of momentum (i.e., $s$-wave-like component). The difference from Eq. (16) is the relative sign between the two components, except for a global phase. The fact that the spatial inversion in the full BdG formalism is ruled by the orbital-space Pauli matrix $\sigma^3$ causes the discrepancy of the effective gaps.

The above arguments are contrast to an effective theory with an even-parity (i.e., conventional $s$-wave) pairing potential. An even-parity gap can be written by \[ \Delta = \Delta_{\text{even}}i\gamma^0\gamma^2 = -i\Delta_{\text{even}}I \otimes i\gamma^2, \quad (18) \]

with a complex constant $\Delta_{\text{even}}$. We find that in the orbital space a trivial (i.e., identity) matrix appears. The e

Corrections, we showed that in the low-energy theory, within $M_0 > 0$, the orbital-space Pauli matrix $\sigma^3$ indicates that our calculations are performed after the basis transformation given by Eq. (13). When $\beta = 0$, this expression is equal to the second term in Eq. (10). The intermediate matrix $R^{m}_{\beta}$ represents the propergator via virtual states in the perturbation expansion, given by \( (E_{0,\beta}^{m} - QH_{0,\beta}Q)R^{m}_{\beta} = Q \). Two distinct eigenvalues of $P^\dagger H_{0,\beta}Q$ are denoted by $E_{0,\beta}^{m}$. Each of them is four-fold degenerate. Each matrix element of $P^\dagger H_{0,\beta}Q$ is regarded as transition amplitude from a subspace defined by $Q$ to that by $P$.

We argue a way of constructing $R^{m}_{\beta}$. First, we study the case when $\mu$ is positive. The Fermi surface is defined by $0 = \epsilon^\uparrow(k)$. This is the same setting as that in the main text. The projectors $P$ and $Q$ are given by Eq. (9). According to the arguments on Eq. (13) in Section 3, we find that $P^\dagger H_{0,\beta}Q = [(1 + \sigma^3)/2] \otimes 1 \otimes 1$ since the even-parity pairing potential is invariant under any transformation in the orbital space. To sum up, the odd-parity property of the pairing potential in the full BdG Hamiltonian appears as an inter-component sign difference in the effective gap depending on a kind of the projectors into a low-energy space.

4. Conclusion

We built up a low-energy effective theory, focusing on a model of superconductor Cu$_3$Bi$_2$Se$_3$, motivated by the numerical results of the non-magnetic impurity effects. Using the second-order perturbation with respect to the mass in the Dirac-type electron dispersion and performing a basis transformation to take higher-order corrections, we showed that in the low-energy effective theory the effective superconducting gap is described by a mixture of a $p$-wave component and an $s$-wave component. We stress that the latter is smaller than the former in the large-mass limit. Thus, we obtained the effective description of the system, supporting the previous numerical calculations about the non-magnetic impurity effects. An interesting future work is to clarify how the $p$- and $s$-wave components in the effective gap contribute to the topological invariant in this model.

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Appendix A. Details of perturbation calculations

We show the perturbation calculations to derive the low-energy effective theory, within $M_0 > 0$. In the Appendix we use the same arrangement of matrices as that in Section 3; the orbital space is first written, next the Nambu space is put, and finally the spin space is placed. We focus on the case of performing basis transformation \(~\text{Eq. (13)}~\). We are going to calculate \[ H_{1,\beta}(k) = \sum_{m=\pm} [P^\dagger V_{\beta}(k)Q]R^{m}_{\beta}(k)(QV_{\beta}(k)P). \quad (A.1) \]

The subcript $\beta$ indicates that our calculations are performed after the basis transformation given by Eq. (13). When $\beta = 0$, this expression is equal to the second term in Eq. (10). The intermediate matrix $R^{m}_{\beta}$ represents the propergator via virtual states in the perturbation expansion, given by \( (E_{0,\beta}^{m} - QH_{0,\beta}Q)R^{m}_{\beta} = Q \). Two distinct eigenvalues of $P^\dagger H_{0,\beta}P$ are denoted by $E_{0,\beta}^{m}$. Each of them is four-fold degenerate. Each matrix element of $P^\dagger H_{0,\beta}Q$ is regarded as transition amplitude from a subspace defined by $Q$ to that by $P$.

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Second, we examine $R^{m}_{\beta}$ when $\mu$ is negative. This setting corresponds to the case when the Fermi surface intersects with a lower energy of the normal Hamiltonian; $0 = \epsilon^\downarrow(k)$, indicating that $\mu = -M_0 + O(\beta^2)$. Thus, the predominant part of the normal Hamiltonian comes from

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the subspace characterized by the negative eigenvalue of $\sigma^3$. Therefore, the projectors in Eq. (A.1) need to be set by $\mathcal{P} = [(1 + \sigma^3) / 2] \otimes 1 \otimes 1$ and $Q = [(1 + \sigma^3) / 2] \otimes 1 \otimes 1$. The arguments in $\mu > 0$ with swapping from the role of $\mathcal{P}$ to that of $Q$ lead to

$$ R^\mu_{\beta} |_{\mu \geq 0} = \frac{-1}{2M_0} [1 + \sigma^3] \otimes \tau^3 \otimes 1 + O(\beta^2). \quad (A.3) $$

The calculations of $\mathcal{P}V_{\beta}(k)Q$ are straightforwardly performed, using the algebraic properties of the orbital-space Pauli matrices. When $\mu > 0$, we have

$$ \mathcal{P}V_{\beta}Q |_{\mu \geq 0} = \sigma^+ \otimes (V^+ - i \phi 1 \otimes 1) + O(\beta^2), \quad (A.4) $$

with $\sigma^+ = (\sigma^1 + i \sigma^2) / 2$ and $\phi = (-\beta) \Delta_{\text{odd}}$. In contrast, when $\mu < 0$, the formula is

$$ \mathcal{P}V_{\beta}Q |_{\mu < 0} = \sigma^- \otimes (V^- + i \phi 1 \otimes 1) + O(\beta^2), \quad (A.5) $$

with $\sigma^- = (\sigma^1)^\dagger$. It is worth for pointing out the relative sign between the first and second terms in part of the Nambu-spin space (i.e., inside the parentheses). The difference of the signs between Eqs. (A.4) and (A.5) comes from that of the projectors. Thus, a character in the orbital space alters the transition amplitude in the perturbation expansion.

Now, we show the explicit formulæ of $H_{1, \beta}$, using the expressions of $R^\mu_{\beta}$ and $\mathcal{P}V_{\beta}Q$. When $\mu$ is positive, we find that

$$ H_{1, \beta} |_{\mu \geq 0} = \frac{1 + \sigma^3}{2} \otimes 1 \otimes \left[ V^+(\tau^3 \otimes 1) V^+ - i \phi (\tau^3 \otimes 1, \tau^3 - 1) \right] + O(\beta^2). \quad (A.6) $$

In contrast, when $\mu$ is negative, we have

$$ H_{1, \beta} |_{\mu < 0} = \frac{1 - \sigma^3}{2} \otimes 1 \otimes \left[ V^-(\tau^3 \otimes 1) V^- + i \phi (\tau^3 \otimes 1, \tau^3 - 1) \right] + O(\beta^2). \quad (A.7) $$

In part of the Nambu-spin space, the first term, $V^+(\tau^3 \otimes 1) V^+$ contains an effective gap with momentum dependence, whereas the second term, $+i\phi (\tau^3 \otimes 1, \tau^3 - 1)$ leads to an effective gap without momentum dependence.

**Appendix B. Spatial inversion in a projected space**

We show an approach of formulating spatial inversion in a projected space. We take the projected space specified by $[(1 + \sigma^3) / 2] \otimes 1 \otimes 1$. In the Appendix we use the same arrangement of matrices as that in Section 3.

The transformation matrix of spatial inversion in the Nambu space is expressed by $\Pi = \sigma^3 \otimes 1 \otimes 1$ [See the text below Eq. (5)]. We find that the spatial inversion leads to $\Pi^* \Omega \Pi$ for a matrix $\Omega$ in the Nambu space, such as the BdG Hamiltonian. We first perform the basis transformation given by Eq. (13) on $\Pi$; $\Pi_{\beta} = S_{\beta} \Pi S_{\beta}^\dagger$. We write $S_{\beta} = \exp[-i\beta / 2W]$, similar to the arguments on Eq. (13) in Section 3. Since $[\Pi, W] = 0$, this transformation has no effect on $\Pi$; $\Pi_{\beta} = \Pi$. Next, we expand $\Pi$ in terms of the projectors given by Eq. (9), using the resolution of unity, $\mathcal{P} + Q = 1 \otimes 1 \otimes 1$. Since $[\Pi, \mathcal{P}] = [\Pi, Q] = 0$ and $\mathcal{P}Q = Q\mathcal{P} = 0$, we obtain $\Pi_{\beta} = \Pi_{\beta} \mathcal{P} + Q \Pi_{\beta} Q$. This formula is rewritten by

$$ \Pi_{\beta} = \frac{1 + \sigma^3}{2} \otimes \Pi_{\beta}^0 + \frac{1 - \sigma^3}{2} \otimes \Pi_{\beta}^0. \quad (B.1) $$

The matrix $\Pi_{\beta}^0$ acts on $H_{1, \beta}$. It indicates that in the projected space the spatial inversion leads to $H_{1, \beta}^0 (k) \rightarrow (\Pi_{\beta}^0)^* H_{1, \beta}^0 (k) (\Pi_{\beta}^0)$. A straightforward calculation of $\Pi_{\beta}^0$ leads to the fact that $\Pi_{\beta}^0$ are the identity matrix.

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