Analysis and Probability over Infinite Extensions of a Local Field, II: A Multiplicative Theory

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Abstract

Let $V$ be a projective limit, with respect to the renormalized norm mappings, of the groups of principal units corresponding to a strictly increasing sequence of finite separable totally and tamely ramified Galois extensions of a local field. We study the structure of the dual group $V'$, introduce and investigate a fractional differentiation operator on $V$, and the corresponding Lévy process.

1 INTRODUCTION

This paper is a continuation of the article [9] (see also [10, 11]), in which we considered an infinite extension $K$ of a local field of zero characteristic assuming that $K$ is a union of an increasing sequence of finite extensions. If $K$ is equipped with a natural inductive limit topology, its strong conjugate $\overline{K}$ is a projective limit with respect to the renormalized trace mappings. A Gaussian measure, a Fourier transform, a fractional differentiation operator, and a cadlag Markov process $X_\alpha$ (an analog of the $\alpha$-stable process) on $K$ were constructed. The semigroup of measures defining $X_\alpha$ is concentrated on a compact (additive) subgroup $S \subset \overline{K}$. Sample paths properties of the part of $X_\alpha$ in $S$ were studied in [11].

All the above constructions were based essentially on algebraic structures related to the additive groups of the field $K$ and its subfields. In this paper we develop a parallel theory based on the multiplicative structures. Note that in analysis over local fields both the approaches are closely connected (see Sections 3.5 and 4.7 in [10]).

Let $k$ be a non-Archimedean local field of an arbitrary characteristic. We consider a strictly increasing sequence of its finite separable Galois extensions

$$k = K_1 \subset K_2 \subset \ldots \subset K_n \subset \ldots .$$

We shall assume that all the extensions (1) are totally and tamely ramified. Denote $m_n = [K_n : K_1], n = 2, 3, \ldots$. It will be convenient to write $m_0 = 0, m_1 = 1$. By our assumptions, the residue field cardinality for each field $K_n$ is the same positive integer $q = p^\kappa$ where $p$ is the characteristic of the residue fields. If $\nu > n$, then $[K_\nu : K_n] = \frac{m_\nu}{m_n}$.

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Let $U_{1,n}$ be the group of principal units of the field $K_n$. Then

$$U_{1,1} \subset U_{1,2} \subset \ldots \subset U_{1,n} \subset \ldots .$$

Under the above assumptions the norm mapping $N_{\nu,n} : K_{\nu} \to K_n$ maps $U_{1,\nu}$ onto $U_{1,n}$ ($\nu > n$); see Chapter 1, §8 in [4]. For any $x \in U_{1,n}$ denote

$$N_{\nu,n}(x) = [N_{\nu,n}(x)]^{m_n/m_{\nu}}.$$

Since $m_n/m_{\nu}$ is prime to $p$, $N_{\nu,n}(x)$ is well-defined and belongs to $U_{1,n}$ ([6], Chapter 1, Corollary (5.5)). It is easy to check that $N_{\nu,n}(x) : U_{1,\nu} \to U_{1,n}$ is an epimorphism, and $N_{\nu,n}(x) = x$ if $x \in U_{1,n}$.

If $l > \nu > n$, then for any $x \in U_{1,l}$

$$N_{\nu,n}(N_{l,\nu}(x)) = [N_{\nu,n}(N_{l,\nu}(x))]^{m_{l,n}/m_{\nu}}.$$

Note that for any positive integer $r$ $N_{\nu,n}(x^r) = [N_{\nu,n}(x)]^r$. If $r$ is prime to $p$, we may substitute $x^{1/r}$ for $x$ and find that $N_{\nu,n}(x) = [N_{\nu,n}(x^{1/r})]^r$, so that $N_{\nu,n}$ commutes with the $r$-th root operation. Therefore $N_{\nu,n} \circ N_{l,\nu} = N_{l,n}$, and we can define the projective limit

$$V = \lim_{\leftarrow} U_{1,n}$$

with respect to the homomorphisms $N_{\nu,n}$.

$V$ is a compact topological group. Its structure is investigated in Sect. 2. $V$ is totally disconnected, its topology is determined by an explicitly written descending chain of open-closed subgroups. The structure of its dual group is studied.

In Sect. 3 we introduce a fractional differentiation operator $D^{\alpha}$ on $V$ (continuous characters on $V$ are its eigenfunctions), and prove that it is a generator of a Markov process $\xi_{\alpha}$ on $V$, an analog of the stable process. Sect. 4 is devoted to properties of $\xi_{\alpha}$. In particular, we find the Hausdorff and packing dimensions of the image of a time interval under $\xi_{\alpha}$. This is based on the explicit calculation of the Lévy measure corresponding to $\xi_{\alpha}$, and employs the general results by Evans [5] on sample paths properties of Lévy processes on Vilenkin groups.

## 2 THE GROUP $V$ AND ITS DUAL

2.1. The group $V$ consists of sequences $x = (x_1, \ldots, x_n, \ldots)$, $x_n \in U_{1,n}$, such that for any $\nu > n$ $N_{\nu,n}(x_\nu) = x_n$, with the component-wise operations. The topology on $V$ is induced by the Tychonoff topology on the direct product $\prod_{n=1}^{\infty} U_{1,n}$. Thus a fundamental system of neighbourhoods of the unit element in $V$ can be obtained by taking finite intersections of the sets

$$\{x = (x_1, \ldots, x_n, \ldots) \in V : |1 - x_n|_n \leq \varepsilon\}$$

with some $n \geq 1$, $0 < \varepsilon < 1$. Here $| \cdot |_n$ is the normalized absolute value on $K_n$. Note that each of the above sets is an open-closed subgroup in $V$ due to the ultra-metric property of the absolute values.
Below we shall use the following property of the mappings $\mathfrak{N}_{\nu,n}$, $\nu > n$. Let

$$U_{l,i} = \{ y \in K_i : |1 - y|_i \leq q^{-l} \}, \quad l \geq 2.$$  

Then

$$\mathfrak{N}_{\nu,n} : U_{\frac{m\nu}{mn}(l-1)+1,\nu} \underset{\text{onto}}{\rightarrow} U_{l,n},$$  

$$\mathfrak{N}_{\nu,n} : U_{\frac{m\nu}{mn}(l-1),\nu} \underset{\text{onto}}{\rightarrow} U_{l-1,n}.$$  

For the mappings $N_{\nu,n}$ this is proved in [13], Chapter V, §6, Corollary 3, where a more general case is considered; the Hasse-Herbrand function $\psi(v)$ of an extension $L/K$, which appears in that corollary, equals $[L : K]v$ for a totally and tamely ramified extension (see [13], Chapter IV, §3). In order to consider the mappings $\mathfrak{N}_{\nu,n}$ it remains to use the $m$-divisibility of the groups $U_{l,n}$ for any $m$ prime to $p$ ([5], Chapter 1, (5.5)).

Let us consider subgroups $V_n \subset V$,

$$V_n = \{ x = (x_1, \ldots, x_n, \ldots) \in V : |1 - x_n|_n \leq q^{-nm_n-1} \}, \quad n = 1, 2, \ldots.$$  

If $x \in V_n$ and $n > i$, then $x_i = \mathfrak{N}_{n,i}(x_n)$, and by (2)

$$|1 - x_i|_i \leq q^{-m_i(n+1)} < q^{-im_i - 1},$$  

so that $x_i \in V_i$. Thus we have a filtration

$$V = V_0 \supset V_1 \supset \ldots \supset V_n \supset \ldots.$$  

The same arguments show that the subgroups $V_n$ form a fundamental system of neighbourhoods of the unit element.

Note that the descending chain (4) can be “lengthened” by including intermediate subgroups so that the resulting chain would be such that the quotient group of two consecutive subgroups is of a prime order. This property is often assumed in the investigation of totally disconnected groups. In particular, this assumption was made in [3]. However it will be more convenient for us to use the chain (4). All the results of [3] remain valid here.

Let $\mu$ be the normalized Haar measure on $V$. Since the mappings $\mathfrak{N}_{\nu,n}$ are surjective, $V$ can be identified with the projective limit of its quotient groups with respect to a descending system of compact subgroups ([2], Chapter III, §7). Then $\mu$ is a projective limit of the normalized Haar measures $\mu_n$ on the groups $U_{1,n}$, $n = 1, 2, \ldots$ ([3], Chapter VII, §1, Sect. 6). This means in particular that

$$\mu(V_n) = \mu_n(U_{nm_n+1,n}).$$  

On the other hand, $\mu_n$ is proportional to the normalized Haar measure $dx_n$ on the additive group of the field $K_n$. Since (see e.g. [11]) \( \int_{U_{1,n}} dx_n = q^{-1} \), we have

$$\mu_n(U_{nm_n+1,n}) = q \int_{|1-x_n|_n \leq q^{-nm_n-1}} = q^{-nm_n}. $$
It is known \[\mathbb{N} \leq \mathbb{M}\] that \(V/V_n\) is a finite group of the order \(M(n)\) where \(\mu (V_n) = [M(n)]^{-1}\). From the above calculations we find that
\[
M(n) = q^{nm_n}.
\]

2.2. By the duality theorem \[\mathbb{N} \leq \mathbb{M}\], the dual discrete group \(V'\) is the inductive limit \(\lim_{\nu \to \infty} U_{1,n}\) with respect to the dual mappings \(\mathcal{N}_{\nu,n} : U'_{1,n} \to U'_{1,\nu} (\nu > n)\) defined as follows. If \(\theta_n \in U_{1,n}^\prime\), that is \(\theta_n\) is a (multiplicative) continuous character of the group \(U_{1,n}\), then \(\mathcal{N}_{\nu,n}(\theta_n)\) is a character of the group \(U_{1,\nu}\), and for any \(x_\nu \in U_{1,\nu}\)
\[
\mathcal{N}_{\nu,n}(\theta_n)(x_\nu) = \theta_n(\mathcal{N}_{\nu,n}(x_\nu)).
\]

By definition (see \[\mathbb{N} \leq \mathbb{M}\]), \(V' = \mathfrak{A}/\mathfrak{B}\) where \(\mathfrak{A}\) is the direct sum \(\bigoplus_{n=1}^{\infty} U'_{1,n}\), \(\mathfrak{B}\) is a subgroup consisting of all elements \((\theta_{i_1}, \ldots, \theta_{i_n})\), \(\theta_{i_j} \in U'_{1,i_j}\), such that for some \(i \geq i_1, \ldots, i_n\)
\[
(\mathcal{N}_{i_1,1} \theta_{i_1}) \cdots (\mathcal{N}_{i_n,1} \theta_{i_n}) = 1;
\]
in our multiplicative notation we identify \((\theta_{i_1}, \ldots, \theta_{i_n})\) with \(\theta_{i_1} \ldots \theta_{i_n}\). Each element of \(V'\) can be written as \(\theta = \theta_{i} \mathfrak{B}\), \(\theta_i \in U'_{1,i}\) for some \(i\). Then the coupling between \(V'\) and \(V\) is given by \(\langle \theta, x \rangle = \langle \theta_i, x_i \rangle_i\), if \(x = (x_1, \ldots, x_i, \ldots)\). Here \(\langle \cdot, \cdot \rangle_i\) is the coupling between \(U'_{1,i}\) and \(U_{1,i}\). The correctness of this definition is proved in \[\mathbb{N} \leq \mathbb{M}\].

In our specific situation we can say a little more about the above definition of \(V'\). Suppose that \(\theta^{(1)}_i \in U'_{1,i}, \theta^{(2)}_j \in U'_{1,j}\), and
\[
\mathcal{N}_{i,1} \theta^{(1)}_i = \mathcal{N}_{j,1} \theta^{(2)}_j; \quad i \leq j \leq n.
\]
The equality (6) means that
\[
\theta^{(1)}_i(\mathcal{N}_{n,i}(x)) = \theta^{(2)}_j(\mathcal{N}_{n,j}(x)) \quad \text{for any } x \in U_{1,n}.
\]
In particular, taking \(x \in U_{1,j} \subset U_{1,n}\) we find that \(\mathcal{N}_{n,j}(x) = x, \mathcal{N}_{n,i}(x) = \mathcal{N}_{j,i}(\mathcal{N}_{n,j}(x)) = \mathcal{N}_{j,i}(x)\), so that
\[
\mathcal{N}_{j,i} \theta^{(1)}_i = \theta^{(2)}_j, \quad \text{if } i < j; \quad \theta^{(1)}_i = \theta^{(2)}_j, \quad \text{if } i = j.
\]

Thus, two representatives of the same coset either coincide (if they belong to the same group \(U'_{1,i}\)), or one of them is obtained by “lifting” another.

In order to classify elements of \(V'\), we shall need the notion of a ramification degree of a multiplicative character of a local field (adapted to our setting). Let \(\theta_n \in U'_{1,n}\). The character \(\theta_n\) is said to have the ramification degree \(1\), if \(\theta_n \equiv 1\), and the ramification degree \(\nu \geq 2\), if \(\theta_n(x_n) = 1\) for any \(x_n \in U_{\nu,n}\), and \(\theta_n(x^0_n) \neq 1\) for some \(x^0_n \in U_{\nu-1,n}\).

It follows from (2) and (3) that if \(\theta_i \in U'_{1,i}\) has the ramification degree \(\nu_i\), and \(n > i\), then \(\mathcal{N}_{n,i} \theta_i\) has the ramification degree \(\nu_n = \frac{m_n}{m_i}(\nu_i - 1) + 1\). We see that
\[
\frac{\nu_n - 1}{m_n} = \frac{\nu_i - 1}{m_i}.
\]
Therefore we may assign to any \( \theta \in V' \) the number
\[
    r(\theta) = \frac{\nu_i - 1}{m_i}
\]
where \( \nu_i \) is the ramification degree of an arbitrary representative of \( \theta \) lying in \( U'_{1,i} \).

2.3. Let \( V_n^\perp \subset V' \) be the annihilator of the subgroup \( V_n \). We have
\[
\{1\} = V_0^\perp \subset V_1^\perp \subset \ldots \subset V_n^\perp \subset \ldots ; \quad \bigcup_{n=0}^{\infty} V_n^\perp = V'.
\]
It is known \([14, 5]\) that \( \text{card} (V_n^\perp) = M(n) \).

**Proposition 1.** The annihilator \( V_n^\perp \) consists of those cosets \( \theta \in V' \) for which \( r(\theta) \leq n \), and there exists a representative \( \theta_n \in \theta, \theta_n \in U'_{1,n} \).

**Proof.** Let us show first of all that a coset \( \theta \in V_n^\perp \) contains a representative \( \theta_n \in U'_{1,n} \). Indeed, if \( \theta_i \in \theta \), and \( i < n \), then the character \( \theta_i \) can be lifted (within the coset \( \theta \)) to the character \( \theta_n(x_n) = \theta_i (\mathcal{M}_{i,n}(x_n)), \) \( x_n \in U_{1,n} \).

Suppose that \( i > n \). Let us define a character \( \theta_n \in V_n' \) as a restriction of \( \theta_i \) to \( U_{1,n} \). We have to check that \( \theta_i \theta_n^{-1} \in \mathfrak{B} \); it is sufficient to verify that \( \theta_i \cdot (\mathcal{M}_{i,n} \theta_n) = 1 \), that is
\[
\theta_i(x_i) [\theta_n (\mathcal{M}_{i,n}(x_i))]^{-1} = 1
\]
for any \( x_i \in U_{1,i} \), or (by the definition of \( \theta_n \)) that for each \( x_i \in U_{1,i} \)
\[
\theta_i \left( \frac{x_i}{\mathcal{M}_{i,n}(x_i)} \right) = 1.
\] (7)

Denote \( \tilde{x}_i = \frac{x_i}{\mathcal{M}_{i,n}(x_i)}, \tilde{x}_l = \mathcal{M}_{i,l}(\tilde{x}_i) \) for \( l < i \) (since \( \mathcal{M}_{i,n}(x_i) \in U_{1,n} \subset U_{1,i} \), the element \( \tilde{x}_i \) belongs to \( U_{1,i} \)). Choosing \( \tilde{x}_{i+1} \in U_{1,i+1} \) in such a way that \( \mathcal{M}_{i+1,i}(\tilde{x}_{i+1}) = \tilde{x}_i \), then taking such \( \tilde{x}_{i+2} \in U_{1,i+2} \) that \( \mathcal{M}_{i+2,i+1}(\tilde{x}_{i+2}) = \tilde{x}_{i+1} \) etc., we obtain an element \( \tilde{x} = (\tilde{x}_1, \ldots, \tilde{x}_n, \ldots, \tilde{x}_i, \ldots) \in V \). We have
\[
\tilde{x}_n = \left\{ \frac{N_{i,n}(x_i)}{[\mathcal{M}_{i,n}(x_i)]^{m_n}} \right\}^{m_n} \stackrel{m_n}{=} 1,
\]
so that \( \tilde{x} \in V_n \), which means that \( 1 = \theta(\tilde{x}) = \theta_i (\tilde{x}_i) \), and we have proved the equality (7).

Thus, \( \theta = \theta_n \cdot \mathfrak{B} \), \( \theta_n \in U'_{1,n} \). If \( r(\theta) > n \), then the ramification degree of the character \( \theta_n \) is greater than \( nm_n + 1 \), which implies the existence of an element \( x \in V_n \), such that \( \theta(x) \neq 1 \) (as in the above reasoning, an element \( x_n \in U_{nm_n+1,n} \) can be prolonged to an element \( (x_1, \ldots, x_{nm_n+1,n}) \in V_n \)). This means that the inequality \( r(\theta) > n \) implies that \( \theta \not\in V_n^\perp \). On the other hand, if \( r(\theta) \leq n \) and \( \theta = \theta_n \cdot \mathfrak{B} \), \( \theta_n \in U'_{1,n} \), then obviously \( \theta \in V_n^\perp \). \( \blacksquare \)
3 Fractional Differentiation Operator

3.1. Let us consider the function $f$ on $V'$ defined as

$$f(\theta) = \varphi_n \text{ for } \theta \in V_n^\perp \setminus V_{n-1}^\perp, \ n = 1, 2, \ldots; \ f(1) = \varphi_0,$$

where $\{\varphi_n\}_{n=0}^{\infty}$ is a given sequence of complex numbers. It follows from (5) that $f \in l_1(V')$, if

$$\sum_{n=0}^{\infty} |\varphi_n| q^{nmn} < \infty,$$

and $f \in l_2(V')$, if $\sum_{n=0}^{\infty} |\varphi_n|^2 q^{nmn} < \infty$.

Let us compute the Fourier transform

$$F(x) = \sum_{\theta \in V'} f(\theta) \theta(x), \ x \in V.$$

We have

$$F(x) = \varphi_0 + \sum_{n=1}^{\infty} \varphi_n \sum_{\theta \in V_n^\perp \setminus V_{n-1}^\perp} \theta(x).$$

If $x \in V \setminus V_1$, then $\sum_{\theta \in V_n^\perp} \theta(x) = 0$ for $n \geq 1$, whence $F(x) = \varphi_0 - \varphi_1$. If $x \in V_l \setminus V_{l+1}, l \geq 1$, then

$$F(x) = \varphi_0 + \sum_{n=1}^{l} \varphi_n \sum_{\theta \in V_n^\perp \setminus V_{n-1}^\perp} 1 + \varphi_{l+1} \left[ \sum_{\theta \in V_{l+1}^\perp} \theta(x) - \sum_{\theta \in V_l^\perp} 1 \right] + \sum_{n=l+2}^{\infty} \varphi_n \sum_{\theta \in V_n^\perp \setminus V_{n-1}^\perp} \theta(x).$$

Since $\sum_{\theta \in V_{l+1}^\perp} \theta(x) = 0$ and $\sum_{\theta \in V_n^\perp \setminus V_{n-1}^\perp} \theta(x) = 0$ for $n \geq l + 2$, and $\sum_{\theta \in V_n^\perp \setminus V_{n-1}^\perp} 1 = q^{nmn}$, we find that

$$F(x) = \varphi_0 + \sum_{n=1}^{l} \varphi_n \left[ q^{nmn} - q^{(n-1)m_{n-1}} \right] - \varphi_{l+1} q^{lm_l}.$$  

This gives after a simple transformation that

$$F(x) = \sum_{n=0}^{l} (\varphi_n - \varphi_{n+1}) q^{nmn}, \ x \in V_l \setminus V_{l+1}, l \geq 0. \quad (8)$$

If $\sum_{n=0}^{\infty} |\varphi_n| q^{nmn} < \infty$, then it follows from (8) that

$$F(1) = \sum_{n=0}^{\infty} (\varphi_n - \varphi_{n+1}) q^{nmn}.$$

3.2. Consider the function $f^{(\alpha)}(\theta), \ \theta \in V'$, corresponding, as above, to the sequence

$$\varphi_n^{(\alpha)} = \begin{cases} q^{\alpha nmn}, & \text{if } n \geq 1, \\ 0, & \text{if } n = 0, \end{cases}$$
where $\alpha < -1$. Let $F^{(\alpha)}$ be the Fourier transform of $f^{(\alpha)}$. Introducing the convolution operator $A^{(\alpha)}u = F^{(\alpha)} \ast u$, $u \in L_2(V)$, we can write

$$(A^{(\alpha)}u) (x) = \int_V F^{(\alpha)}(y)u(xy^{-1})\mu(dy) = \int_V F^{(\alpha)}(y) [u(xy^{-1}) - u(x)] \mu(dy),$$

since

$$\int_V F^{(\alpha)}(y)\mu(dy) = f^{(\alpha)}(1) = \varphi_0^{(\alpha)} = 0.$$

Denote by $D(V)$ the vector space of locally constant complex-valued functions on $V$, that is such functions $u$ that $u(x) = u(y)$ if $xy^{-1} \in V_l$ (the number $l$ depends on $u$ and does not depend on $x$). Since $\bigcup_{n=0}^{\infty} V_n \perp = V'$, any continuous character on $V$ belongs to $D(V)$. Therefore $D(V)$ is dense in the Banach space $C(V)$ of all continuous functions on $V$.

By (8),

$$F^{(\alpha)}(y) = -q^\alpha + \sum_{n=1}^{l} (q^{\alpha mn} - q^{\alpha(n+1)m_{n+1}}) q^{mn}, \quad y \in V_l \setminus V_{l+1},$$

(9) (the sum is missing for $l = 0$). If $u \in D(V)$, then $(A^{(\alpha)}u) (x)$ is an entire function of $\alpha$, and we define the operator $D^{\alpha}u$, $\alpha > 0$, as the analytic continuation of $A^{(\alpha)}u$. Thus,

$$(D^{\alpha}u) (x) = \int_V F^{(\alpha)}(y) [u(xy^{-1}) - u(x)] \mu(dy), \quad \alpha > 0,$$

for any $u \in D(V)$. The expression (9) is valid for $\alpha > 0$ too. Equivalently, $D^{\alpha}$ can be written on $D(V)$ as a pseudo-differential operator with the symbol $f^{(\alpha)}(\theta)$.

**Theorem 1.** (i) The operator $D^{\alpha}$ ($\alpha > 0$) is an essentially selfadjoint operator on $L_2(V)$. Its closure has a purely discrete spectrum consisting of the eigenvalues

$$\varphi_n^{(\alpha)} = \begin{cases} q^{\alpha mn}, & \text{if } n \geq 1, \\ 0, & \text{if } n = 0, \end{cases}$$

corresponding to the eigenspaces $V_n \perp \setminus V_{n-1}$, $n = 0, 1, 2, \ldots$ ($V_{-1} \perp = \emptyset$). In particular, $D^{\alpha}1 = 0$.

(ii) The semigroup $e^{-tD^{\alpha}}$, $t \geq 0$, consists of the integral operators of the form

$$(e^{-tD^{\alpha}}u) (x) = \int_V G^{(\alpha)}(t, xy^{-1})u(y)\mu(dy)$$

where the kernel

$$G^{(\alpha)}(t, z) = \sum_{n=0}^{l} \left[ e^{-t\varphi_n^{(\alpha)}} - e^{-t\varphi_{n+1}^{(\alpha)}} \right] q^{mn}, \quad z \in V_l \setminus V_{l+1}, \quad l \geq 0,$$

10 is positive. The corresponding stochastic process $\xi^{\alpha}$ with independent increments on $V$ is stochastically continuous.
Proof. All the assertions except the last one are immediate consequences of the above constructions. Note that the function \( z \mapsto G_\alpha(t, z) \) is constant on each open-closed set \( V_l \setminus V_{l+1} \), which implies its continuity. Thus \( \xi_\alpha \) has a strong Feller property. The stochastic continuity is equivalent to the \( C_0 \)-property of the semigroup \( e^{-tD_\alpha} \) in \( C(V) \). It is sufficient to prove that \( \|e^{-tD_\alpha}u - u\|_{C(V)} \to 0 \) as \( t \to 0 \), for any \( u \in \mathcal{D}(V) \).

Since \( \int V G_\alpha(t, xy^{-1}) \mu(dy) = 1 \), we find that for any \( u \in \mathcal{D}(V) \)

\[
(e^{-tD_\alpha}u - u)(x) = \int V G_\alpha(t, xy^{-1})[u(y) - u(x)] \mu(dy) = \int V G_\alpha(t, z)[(xz^{-1}) - u(x)] \mu(dz)
\]

\[
= \int_{V \setminus V_l} G_\alpha(t, z)[(xz^{-1}) - u(x)] \mu(dz) = \sum_{j=0}^{l-1} \int_{V_j \setminus V_{j+1}} G_\alpha(t, z)[(xz^{-1}) - u(x)] \mu(dz)
\]

for some \( l \). Now it follows from (10) that

\[
\|e^{-tD_\alpha}u - u\|_{C(V)} \leq \text{const} \cdot \|u\|_{C(V)} \sum_{j=0}^{l-1} \sum_{n=0}^{j} \left[ e^{-t\varphi_n^{(a)}} - e^{-t\varphi_{n+1}^{(a)}} \right] q^{nm_n} \to 0, \quad t \to 0,
\]

as desired. \( \blacksquare \)

4 Sample Path Properties

4.1. It is well known (see e.g. [8]) that for any \( \theta \in V' \)

\[
E \langle \theta, \xi_\alpha(t) \rangle = \exp \left\{ \int_{V} [(\theta, x) - 1] \Pi(t, dx) \right\}
\]

(11)

where \( \Pi \) is the Lévy measure, that is \( \Pi(t, \Gamma) = EN(t, \Gamma), N(t, \Gamma) \) is the number of jumps of \( \xi_\alpha \) on the interval \([0, t)\) belonging to a Borel set \( \Gamma \neq 1 \).

**Proposition 2.** The Lévy measure, \( \Pi(t, \Gamma) \) has the form \( \Pi(t, \Gamma) = tv(\Gamma) \) where \( v(\Gamma) \) is a Borel measure on \( V \setminus \{1\} \) finite outside any open neighbourhood of 1, which possesses the following properties:

(i) \( v \) satisfies the condition of local spherical symmetry \([3]\), that is \( v(dx) = v_n \mu(dx) \) on \( V_n \setminus V_{n+1} \). Specifically,

\[
v_n = q^a + \sum_{l=1}^{n} q^{lm_l} [q^{a(l+1)m_{l+1}} - q^{a l m_l}].
\]

(12)

(ii) The equality

\[
v(V \setminus V_n) = \sum_{j=0}^{n-1} v_j \left[ q^{-jm_j} - q^{-(j+1)m_{j+1}} \right]
\]

(13)
holds.

(iii) The asymptotic relation

$$
\nu(V \setminus V_n) \sim q^{\alpha n \eta}, \quad \text{as } n \to \infty,
$$

is valid.

**Proof.** By the construction of the process $\xi_\alpha$,

$$
E\langle \theta, \xi_\alpha(t) \rangle = \int \langle \theta, z \rangle G_\alpha(t, z) \mu(dz) = e^{-tf(\alpha)(\theta)} = \begin{cases} 
& e^{-tq^{\alpha n \eta}}, \quad \text{if } \theta \in V_n \setminus V_{n-1}, n \geq 1, \\
& 1, \quad \text{if } \theta \equiv 1.
\end{cases}
$$

Since $G_\alpha(t, z) = G_\alpha(t, z^{-1})$, the measure $\Pi$ is also invariant with respect to the inversion $z \mapsto z^{-1}$. Therefore comparing the last inequality with (11) we find that

$$
\int [\langle \theta, x \rangle - 1]\Pi(t, dx) = \begin{cases} 
& -tq^{\alpha n \eta}, \quad \text{if } \theta \in V_n \setminus V_{n-1}, n \geq 1, \\
& 0, \quad \text{if } \theta \equiv 1.
\end{cases}
$$

Let $\Gamma \subset V_n \setminus V_{n+1}$ be a Borel set, $\omega_\Gamma(x)$ the indicator of the set $\Gamma$. We have

$$
\omega_\Gamma(x) = \sum_{\theta \in V'} \omega_\Gamma(\theta) \theta(x), \quad \omega_\Gamma(\theta) = \int_{\Gamma} \langle \theta, x \rangle \mu(dx).
$$

In particular, $\sum_{\theta \in V'} \omega_\Gamma(\theta) = \omega_\Gamma(1) = 0$, so that

$$
\omega_\Gamma(x) = \sum_{\theta \in V'} \omega_\Gamma(\theta)[\theta(x) - 1],
$$

whence

$$
\int_{V} \omega_\Gamma(x) \Pi(t, dx) = -t \sum_{l=1}^{\infty} q^{\alpha lm_l} \sum_{\theta \in V_{l+1}^l \setminus V_{l-1}^l} \int_{\Gamma} \langle \theta, x \rangle \mu(dx) = -t \sum_{l=1}^{n} q^{\alpha lm_l} \text{card} (V_{l+1}^l \setminus V_{l-1}^l) \mu(\Gamma)
$$

\[ - t q^{\alpha(n+1)m_{n+1}} \left( \int_{\Gamma} \left( \sum_{\theta \in V_{n+1}^l} \langle \theta, x \rangle \right) \mu(dx) - \mu(\Gamma) \text{card } V_{n}^l \right) \]

\[ - t \sum_{l=n+2}^{\infty} q^{\alpha lm_l} \left( \int_{\Gamma} \left( \sum_{\theta \in V_{l+1}^l} \langle \theta, x \rangle \right) \mu(dx) - \int_{\Gamma} \left( \sum_{\theta \in V_{l-1}^l} \langle \theta, x \rangle \right) \mu(dx) \right). \]

An element $x \in V_n \setminus V_{n+1}$ can be considered as a non-trivial character of each group $V_{l+1}^l$, $l \geq n + 1$ (since $(V_{l+1}^l)^\perp = V_l$). Therefore $\sum_{\theta \in V_{l+1}^l} \langle \theta, x \rangle = 0$ for $l \geq n + 1$, so that

$$
\int_{V} \omega_\Gamma(x) \Pi(t, dx) = -t \mu(\Gamma) \left\{ \sum_{l=1}^{n} q^{\alpha lm_l} \left[ q^{lm_l} - q^{(l-1)m_{l-1}} \right] - q^{\alpha(n+1)m_{n+1}} \cdot q^{\alpha n \eta} \right\}
$$

\[ = t \mu(\Gamma) \left\{ \sum_{l=1}^{n} q^{lm_l} \left[ q^{\alpha(l+1)m_{l+1}} - q^{\alpha lm_l} \right] + q^{\alpha} \right\}, \]
and we have come to (12), which easily implies (13).

In order to prove (14), we apply the Abel transform to the sum in (12). We have

\[ \nu_n = q^n + q^{nm_n} (q^{(n+1)m_n + 1} - q^{n}) - \sum_{i=1}^{n-1} (q^{(i+1)m_{i+1}} - q^{i}) (q^{(i+1)m_{i+1}} - q^{i}). \]  

(15)

Since \( m_{n+1} \geq 2m_n \), it follows from (15) that

\[ \nu_n \sim q^{nm_n \alpha (n+1)m_n + 1}, \quad n \to \infty. \]  

(16)

Now the asymptotics (14) is a consequence of (13) and (16). Indeed, the right-hand side of (14) is clearly the “leading” term in (13). As for other summands of (13), we find that

\[ \nu_{n-1} q^{-nm_n} (\nu_{n-1} q^{-(n-1)m_{n-1} - nm_n} \leq \text{const} \cdot q^{\alpha m_n} \cdot q^{-\frac{n+1}{2}m_n} = o(q^{\alpha m_n}); \]

\[ \sum_{j=0}^{n-2} \nu_j [q^{-jm_j} - q^{-(j+1)m_{j+1}}] \leq \text{const} \cdot \sum_{j=0}^{n-2} q^{\alpha (j+1)m_{j+1}} \leq \text{const} \cdot (n-1) q^{\frac{n+1}{2}(n-1)m_n} = o(q^{\alpha m_n}) \]

because \( m_n \geq 2^{n-1}. \)

4.2. Using Proposition 2 and the general results by Evans [5], we can find now the Hausdorff and packing dimensions of the image of a time interval under the random mapping \( t \mapsto \xi_\alpha(t) \).

See [5] for the definitions. Below we assume that \( \xi_\alpha(0) = 1 \).

Let \( \pi(n) \) be the first exit time of the process \( \xi_\alpha \) out of the subgroup \( V_n \). Let

\[ Q(n, N) = \mathbb{P} \{ \xi_\alpha(t) \notin V_n \, \forall t \in [\pi(n), \pi(N)) \}, \quad n > N. \]

An essential assumption in [5] is the inequality

\[ \liminf_{n \to \infty} Q(n, N) > 0. \]  

(17)

In specific situations the verification of (17) can be a difficult task; see, for example, [11]. However if the Lévy measure is locally spherically symmetric, as we have here by Proposition 2, then (17) takes place if

\[ \limsup_{n \to \infty} \frac{\nu(V \setminus V_n)}{M(n)} \cdot \frac{M(n-1)}{\nu(V \setminus V_{n-1})} < 1 \]  

(18)

([5], Corollary 2).

It follows from (5) and (14) that the inequality (18) holds if \( \alpha < 1 \).

Denote by \( \dim \) the Hausdorff dimension, and by \( \text{Dim} \) the packing dimension corresponding to the natural metric on \( V \) (see [5]).

**Theorem 2.** If \( \alpha < 1 \), then for each \( t > 0 \) we have that

\[ \dim \xi_\alpha([0, t]) = \text{Dim} \xi_\alpha([0, t]) = \alpha \]  

almost surely.

**Proof.** The equalities (19) follow from (17) and Theorems 8, 10 of [5]. ■

Note that the Hausdorff and packing dimensions of an image of an interval for a stable process on the field \( \mathbb{Q}_p \) of \( p \)-adic numbers were found recently in [1].
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