DOMINO TILINGS OF THE EXPANDED AZTEC DIAMOND

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Abstract. The expanded Aztec diamond is a generalized version of the Aztec diamond, with an arbitrary number of long columns and long rows in the middle. In this paper, we count the number of domino tilings of the expanded Aztec diamond. The exact number of domino tilings is given by recurrence relations of state matrices by virtue of the state matrix recursion algorithm, recently developed by the author to solve various two-dimensional regular lattice model enumeration problems.

1. Introduction

In both combinatorial mathematics and statistical mechanics, domino tiling of the Aztec diamond is an important subject. The Aztec diamond of order \( n \) consists of all lattice squares that lie completely inside the diamond shaped region \( \{(x, y) : |x| + |y| \leq n + 1\} \). The Aztec diamond theorem from the excellent article of Elkies, Kuperberg, Larsen and Propp [1] states that the number of domino tilings of the Aztec diamond of order \( n \) is equal to \( 2^n(n+1)/2 \). From the statistical mechanics viewpoint, tilings of large Aztec diamonds exhibit a striking feature. The Arctic circle theorem proved by Jockusch, Propp and Shor [4] says that a random domino tiling of a large Aztec diamond tends to be frozen outside a certain circle.

![Figure 1. The Aztec diamond of order 3, the augmented Aztec diamond of order 3, and their domino tilings](image)

The augmented Aztec diamond looks much like the Aztec diamond, except that there are three long columns in the middle instead of two. See Figure 1. The number of domino tilings of the augmented Aztec diamond of order \( n \) was computed by Sachs and Zernitz [14] as \( \sum_{k=0}^{n} \binom{n}{k} \cdot \binom{n+k}{k} \), known as the Delannoy numbers. Notice that the former number is much larger than the later. Indeed, the number of domino tilings of a region is very sensitive to boundary conditions [6, 7]. More interesting patterns related to

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the Aztec diamond allowing some squares removed have been deeply studied and Propp proposed a survey of these works [13].

In this paper, we consider a generalized region of the Aztec diamond which has an arbitrary number of long columns and long rows in the middle. The expanded \((p, q)\)-Aztec diamond of order \(n\), denoted by \(AD_{(p, q; n)}\), is defined as the union of \(2n(n + p + q + 1) + pq\) unit squares, arranged in bilaterally symmetric fashion as a stack of \(2n + q\) rows of squares, the rows having lengths \(p + 2\), \(p + 4\), \ldots, \(2n + p - 2\), \(2n + p\), \ldots, \(2n + p\), \(2n + p - 2\), \ldots, \(p + 2\), as drawn in Figure 2. Let \(\alpha_{(p, q; n)}\) denote the number of domino tilings of \(AD_{(p, q; n)}\). Note that \(\alpha_{(p, q; n)} = 0\) for odd \(pq\) because \(AD_{(p, q; n)}\) consists of odd number of squares.

![Figure 2. The expanded Aztec diamond AD(3,2;4) and a domino tiling](image)

Recently several important enumeration problems regarding various two-dimensional regular lattice models are solved by means of the state matrix recursion algorithm, introduced by the author. This algorithm provides recursive matrix-relations to enumerate monomer and dimer coverings and independent vertex sets known as the Merrifield–Simmons index. These problems have been major outstanding unsolved combinatorial problems, and this algorithm shows considerable promise for further two-dimensional lattice model enumeration studies. See [8, 9, 12] for more details.

Using the state matrix recursion algorithm, we present a recursive formula producing the exact number of \(\alpha_{(p, q; n)}\). Throughout the paper, \(\mathbb{O}\) is a zero-matrix with an appropriate size, and \(A^t\) is the transpose of a matrix \(A\).

**Theorem 1.** The number \(\alpha_{(p, q; n)}\) of domino tilings of the \((p, q)\)-Aztec diamond of order \(n\) is the \((1, 1)\)-entry of the following \(2^p \times 2^p\) matrix

\[
\prod_{k=1}^{n} \left[ \begin{array}{cc} A_{p+2k-2} & \mathbb{O} \\ \mathbb{O} & A_{p+2k-1} \end{array} \right] \cdot (C_{p+2n})^q \cdot \left( \prod_{k=1}^{n} \left[ \begin{array}{cc} A_{p+2k-2} & \mathbb{O} \\ \mathbb{O} & A_{p+2k-1} \end{array} \right] \right)^t,
\]

where the \(2^{k-1} \times 2^k\) matrix \(A_k\) and the \(2^k \times 2^k\) matrix \(C_k\) are defined by

\[
A_k = \left[ \begin{array}{cc} A_{k-2} & \mathbb{O} \\ \mathbb{O} & A_{k-1} \end{array} \right] \quad \text{and} \quad C_k = \left[ \begin{array}{cc} C_{k-2} & \mathbb{O} \\ \mathbb{O} & C_{k-1} \end{array} \right].
\]
with seed matrices \( A_1 = [0 \ 1], \ A_2 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \ C_0 = [1] \) and \( C_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \). Here, \( \begin{bmatrix} 1 & 0 \end{bmatrix} \) is used for the undefined matrix \( \begin{bmatrix} A_0 & 0 \\ 0 & 0 \end{bmatrix} \) when \( p = 0 \) and \( k = 1 \).

We adjust the main scheme of the state matrix recursion algorithm introduced in [9] to solve Theorem 1 in Sections 2–4 as three stages.

2. Stage 1. Conversion to domino mosaics

First stage is dedicated to the installation of the mosaic system for domino tilings on the expanded Aztec diamond region. A mosaic system was invented by Lomonaco and Kauffman to give a precise and workable definition of quantum knots representing an actual physical quantum system [5]. Later, the author et al. have developed a state matrix argument for knot mosaic enumeration in a series of papers [2, 3, 10, 11]. This argument has been developed further into the state matrix recursion algorithm by which we enumerate monomer–dimer coverings on the square lattice [9]. We follow the notion and terminology in the paper with some modifications.

In this paper, we consider the four mosaic tiles \( T_1, T_2, T_3 \) and \( T_4 \) illustrated in Figure 3. Their side edges are labeled with two letters a and b as follows: letter ‘a’ if it is not touched by a thick arc on the tile, and letter ‘b’ for otherwise.

![Figure 3. Four mosaic tiles labeled with two letters](image)

For non-negative integers \( p, q \) and \( n \), a \((p, q; n)\)-mosaic is an array \( M = (M_{ij}) \) of those tiles placed on \( AD(p,q;n) \), where \( M_{ij} \) denotes the mosaic tile placed at the \( i \)th column from left to right and the \( j \)th row from bottom to top. So, it consists of \( 2n + p \) columns (or \( 2n + q \) rows) of various length. We are mainly interested in mosaics whose tiles match each other properly to represent domino tilings. For this purpose we consider the following rules.

**Adjacency rule:** Abutting edges of adjacent mosaic tiles in a mosaic are labeled with the same letter.

**Boundary state requirement:** All boundary edges in a mosaic are labeled with letter a.

As illustrated in Figure 4, every domino tiling of \( AD(p,q;n) \) can be converted into a \((p, q; n)\)-mosaic which satisfies the two rules. In this mosaic, \( T_1 \) and \( T_4 \) (or, \( T_2 \) and \( T_3 \)) can be adjoined along the edges labeled b to produce a dimer.
A mosaic is said to be *suitably adjacent* if any pair of mosaic tiles sharing an edge satisfies the adjacency rule. A suitably adjacent \((p, q; n)\)-mosaic is called a *domino* \((p, q; n)\)-mosaic if it additionally satisfies the boundary state requirement. The following one-to-one conversion arises naturally.

**One-to-one conversion:** There is a one-to-one correspondence between domino tilings of \(AD(p, q; n)\) and domino \((p, q; n)\)-mosaics.

3. **Stage 2. State matrix recursion formula**

Now we introduce several types of state matrices for suitably adjacent mosaics.

3.1. **Bar state matrices.** Consider a suitably adjacent \(m\times1\)-mosaic \(M\) for \(1 \leq m \leq 2n+p\), representing a row of length \(m\) in \(AD(p, q; n)\), which is called a *bar mosaic*. A *state* is a finite sequence of two letters \(a\) and \(b\). The \(l\)-state \(s_l(M)\) (\(r\)-state \(s_r(M)\)) indicates the state on the left (right, respectively) boundary edge. The \(b\)-state \(s_b(M)\) (\(t\)-state \(s_t(M)\)) indicates the \(m\)-tuple of states, reading off those on the bottom (top, respectively) boundary edges from right to left along the arrows in Figure 5. The state \(aa\cdots a\) is called *trivial*.

![Figure 5. A suitably adjacent bar 6×1-mosaic with four state indications: \(s_l = a\), \(s_r = b\), \(s_b = aaaaab\) and \(s_t = aaabba\).](image)

Length \(m\) bar mosaics have possibly \(2^m\) kinds of \(b\)- and \(t\)-states, called *bar states*. We arrange all bar states in the lexicographic order, called the *ab-order*. For \(1 \leq i \leq 2^m\), let \(e_i^m\) denote the \(i\)th bar state of length \(m\).
Given a triple \((s_r, s_b, s_l)\) of \(r\)-, \(b\)- and \(t\)-states, let \(\mu_{(s_r, s_b, s_l)}\) denote the number of all suitably adjacent bar mosaics \(M\) such that \(s_r(M) = s_r\), \(s_b(M) = s_b\), \(s_l(M) = s_l\) and \(s_1(M) = a\). The last triviality condition for \(s_1(M)\) is necessary for the left boundary state requirement.

Bar state matrix \(X_m\) \((X = A, B)\) for suitably adjacent bar \(m \times m\)-mosaics is a \(2^m \times 2^m\) matrix \((m_{ij})\) defined by

\[
m_{ij} = \mu_{(x, e^m_i, e^m_j)}
\]

where \(x = a, b\), respectively. One can observe that information on suitably adjacent bar \(m \times m\)-mosaics is completely encoded in two bar state matrices \(A_m\) and \(B_m\). Notice that each entry should be either 0 or 1.

**Lemma 2** (Bar state matrix recursion lemma). Bar state matrices \(A_m\) and \(B_m\) are obtained by the recurrence relations:

\[
A_k = \begin{bmatrix} B_{k-1} & A_{k-1} \\ A_{k-1} & 0 \end{bmatrix} \quad \text{and} \quad B_k = \begin{bmatrix} A_{k-1} & 0 \\ 0 & 0 \end{bmatrix}
\]

with seed matrices \(A_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\) and \(B_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\).

Note that we may start with matrices \(A_0 = [1]\) and \(B_0 = [0]\) instead of \(A_1\) and \(B_1\). This lemma is a simple version of the bar state matrix recursion lemma [9, Lemma 7], applying \(z = 0\). Here we restate the proof for the completeness of the paper.

**Proof.** We use induction on \(k\). A straightforward observation on the three mosaic tiles \(T_2\), \(T_3\) and \(T_4\) establishes the lemma for \(k = 1\). For example, \((2, 1)\)-entry of \(A_1\) is

\[
\mu_{(a, e_2, e_1)} = \mu_{(a, b, a)} = 1
\]

since only \(T_3\) satisfies the requirement.

Assume that bar state matrices \(A_{k-1}\) and \(B_{k-1}\) satisfy the statement. Now we consider \(A_k\) for one case. Partition this matrix of size \(2^k \times 2^k\) into four block submatrices of size \(2^{(k-1)} \times 2^{(k-1)}\), and consider the 12-submatrix of \(A_k\) i.e., the \((1, 2)\)-component in the \(2 \times 2\) array of four blocks.

The entries of the 12-submatrix have the numbers \(\mu_{(a, s_b, s_l)}\) where \(s_b\) and \(s_l\) are bar states of length \(k\), starting with letters \(a\) and \(b\), respectively because of the ab-order. A suitably adjacent bar \(1 \times k\)-mosaic corresponding to these triples \((a, s_b, s_l)\) must have unique tile \(T_2\) at the rightmost, and so its second rightmost tile must have \(r\)-state \(a\) by the adjacency rule. Thus the 12-submatrix of \(A_k\) is \(A_{k-1}\).

Using the same argument, we derive Table 1 presenting all possible eight cases as we desired. This completes the proof. \(\Box\)

### 3.2. Three types of bar state matrices.

Now we categorize each \(j\)th row of \(AD(p, q; n)\) into three types: lower bar mosaics for \(j = 1, \ldots, n\), central bar mosaics for \(j = n+1, \ldots, n+q\), and upper bar mosaics for \(j = n+q+1, \ldots, 2n+q\) as in Figure 6. To be a row of a domino \((p, q; n)\)-mosaic, each bar mosaic has trivial \(l\)- and \(r\)-states \(a\), and furthermore each lower (or upper) bar mosaic has \(b\)-state (or \(t\)-state, respectively) of the form whose the first and the last.
Submatrix for \(\langle s_r, s_h, s_l \rangle\) & Rightmost tile & Submatrix \\
11-submatrix \(\langle a, a\cdots, a\cdots \rangle\) & \(T_1\) & \(B_{k-1}\) \\
\(A_k\) 12-submatrix \(\langle a, a\cdots, b\cdots \rangle\) & \(T_2\) & \(A_{k-1}\) \\
21-submatrix \(\langle a, b\cdots, a\cdots \rangle\) & \(T_3\) & \(A_{k-1}\) \\
\(B_k\) 11-submatrix \(\langle b, a\cdots, a\cdots \rangle\) & \(T_4\) & \(A_{k-1}\) \\
The other four cases & None & \(\emptyset\) \\

| TABLE 1. Eight submatrices of \(A_k\) and \(B_k\)|

letters associated to the rightmost and the leftmost tiles, respectively, are a, while each central bar mosaic has no further condition.

Figure 6. Three types of suitably adjacent bar mosaics.

We refine the definition of a bar state matrix according to this category. Central bar state matrix \(C_m\) for suitably adjacent central bar mosaics of length \(m\) is a \(2^m \times 2^m\) matrix \((x_{ij})\) where 
\[
x_{ij} = \mu(\langle a, \epsilon^m_i, \epsilon^m_j \rangle).
\]
Obviously \(C_m\) is just bar state matrix \(A_m\). Lower bar state matrix \(L_m\) for suitably adjacent lower bar mosaics of length \(m\) is a \(2^{m-2} \times 2^m\) matrix \((x_{ij})\) where 
\[
x_{ij} = \mu(\langle a, \epsilon^{m-2}_i, \epsilon^{m}_j \rangle).
\]
Here \(\epsilon^{m-2}_i\) in \(s_b\) indicates the \(i\)th \(b\)-state among \(2^{m-2}\) states after ignoring the leftmost and the rightmost tiles whose two bottom edges have fixed state a. Similarly Upper bar state matrix \(U_m\) for suitably adjacent upper bar mosaics of length \(m\) is a \(2^m \times 2^{m-2}\) matrix \((x_{ij})\) where 
\[
x_{ij} = \mu(\langle a, \epsilon^{m}_i, \epsilon^{m-2}_j \rangle).
\]

Lemma 3. Central bar state matrix \(C_m\) is obtained from the recurrence relation, for \(k = 2, \ldots, m,\)

\[
C_k = \begin{bmatrix}
C_{k-2} & \emptyset \\
\emptyset & C_{k-1}
\end{bmatrix}
\]

starting with \(C_0 = [1]\) and \(C_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.\)
Proof. $C_m$ is just bar state matrix $A_m$ in Lemma 2. Note that the two recurrence relations in the lemma easily merge into one with new seed matrices $C_0$ and $C_1$. □

**Lemma 4.** The lower bar state matrix is

$$L_m = \begin{bmatrix} A_{m-2} & 0 \\ 0 & A_{m-1} \end{bmatrix}$$

where $A_{m-1}$ and $A_{m-2}$ are obtained from the recurrence relation, for $k = 3, \ldots, m-1$,

$$A_k = \begin{bmatrix} A_{k-2} & 0 \\ 0 & A_{k-1} \end{bmatrix}$$

starting with $A_1 = [0 \ 1]$ and $A_2 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$. Here, $[1 \ 0]$ is used for the undefined matrix $\begin{bmatrix} A_0 & 0 \\ 0 & 0 \end{bmatrix}$ when $m = 2$.

Furthermore the upper bar state matrix is the transpose of the lower bar state matrix, i.e.,

$$U_m = (L_m)^t.$$  

Proof. Lower bar state matrix $L_m$ is slightly differ from the central bar state matrix so as the first and the last letters of each $m$-tuple $b$-state must be deleted because of the ignorance of the rightmost and the leftmost tiles, respectively. This means that we only have to select the rows of $A_m$ in Lemma 2 representing $b$-states whose first and last letters are a. Therefore we instead use $A_1 = [0 \ 1]$ and $A_2 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$ (according to the ignorance of the last letter associated to the leftmost tile), and $L_m = [B_{m-1} \ A_{m-1}]$ (according to the ignorance of the first letter associated to the rightmost tile) only when $k = 1, m$. Note that, when $m = 2$, we have to use $[1 \ 0]$ instead of $\begin{bmatrix} A_0 & 0 \\ 0 & 0 \end{bmatrix}$ since $L_2 = [B_1 \ A_1]$.

Upper bar state matrix $U_m$ is just the transpose of $L_m$ because of the symmetricity between a lower bar mosaic and an upper bar mosaic of the same length. This is just exchanging $b$-states and $t$-states. □

3.3. State matrices. State matrix $N_m$ for suitably adjacent mosaics consisting of the $m$ consecutive rows on $AD_{(p,q;n)}$ from the bottom is a $2^p \times 2^l$ matrix $(n_{ij})$ where $l$ is the length of the topmost $m$th row (or, if it is one of the upper bar mosaics, $l = m+2$) and $n_{ij}$ is the number of such suitably adjacent mosaics whose the bottommost bar mosaic has $s_b = ae^p_i$, the topmost bar mosaic has $s_t = e^m_j$ (if it is one of the upper bar mosaics, $s_t = ae^m_j$), and all the other boundary edges have state a as the bottom one in Figure 7.

**Lemma 5.**

$$N_{2n+q} = \prod_{k=1}^{n} L_{p+2k} \cdot (C_{p+2n})^q \cdot \prod_{k=1}^{n} U_{p+2n+2-2k}$$

Proof. It is enough showing that for $m = 1, \ldots, 2n+q$, $N_m$ is the multiplication of the related $m$ lower, central or upper bar state matrices associated to each row.
Use induction on $m$. Obviously $N_1 = L_{p+2}$. Assume that $N_m$ satisfies the statement. Consider a suitably adjacent mosaic consisting of the $m+1$ rows from the bottom. Split it into a suitably adjacent mosaic consisting of $m$ bar mosaics and a suitably adjacent bar mosaic by tearing off the topmost row. According to the adjacency rule, the $t$-state of the lower mosaic and the $b$-state of the topmost bar mosaic on the abutting horizontal edges must coincide as shown in Figure 7. Let $N_m = (n_{ij}), N_{m+1} = (n'_{ij})$ and the bar state matrix for the topmost bar mosaic be $(a_{ij})$.

![Attaching bar mosaics](image)

Figure 7. Attaching bar mosaics

Let $l$ be the length of the abutting horizontal edges, two shaded parts in middle of the picture. Given an $s$th state among $2^l$ states, $n_{is} \cdot a_{sj}$ indicates the number of suitably adjacent mosaics consisting of $m+1$ bar mosaics where $s_b = ae_p^s a$, $s_t = e_j^m$ (or, if it is one of the upper bar mosaics, $s_t = ae_j^m a$), $m$th bar mosaic has $s_t = e_j^m$ (or, if it is one of the upper bar mosaics, $s_t = ae_j^m a$), and all the other boundary edges have state $a$. Since all $2^l$ kinds of states arise as states of these $l$ abutting horizontal edges, we get

$$n_{ij}' = \sum_{s=1}^{2^l} n_{is} \cdot a_{sj},$$

so $N_{m+1}$ is the multiplication of the $m+1$ bar state matrices related to the $m+1$ consecutive rows. □

### 4. Stage 3. State matrix analyzing

**Proof of Theorem 1** Each $(1,1)$-entry of $N_{2n+q}$ is the number of suitably adjacent mosaics on $AD_{(p,q;n)}$ such that the bottommost and the topmost bar mosaics have trivial $b$- and $t$-states, respectively, and all the other boundary edges have state $a$. The point is that this satisfies the boundary state requirement to represent domino $(p, q; n)$-mosaics. Thus we get the equality

$$\alpha_{(p,q;n)} = (1,1)\text{-entry of } N_{2n+q}.$$  

This combined with Lemmas completes the proof. □
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