Solving the AdS/CFT Y-system

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(in preparation)
Integrability in AdS/CFT

- Integrable planar superconformal 4D N=4 SYM and 3D N=8 Chern-Simons... (non-BPS, summing genuine 4D Feynman diagrams!)
- Based on AdS/CFT duality to very special 2D superstring $\sigma$-models on AdS-background
- **Y-system** (for planar AdS$_5$/CFT$_4$, AdS$_4$/CFT$_3$, ...) calculates exact anomalous dimensions of all local operators at any coupling
- **Y-system** is an infinite set of functional or integral nonlinear eqs.

$$Y_{a,s} \left( u + \frac{i}{2} \right) Y_{a,s} \left( u - \frac{i}{2} \right) = \frac{1 + Y_{a,s+1}}{1 + \frac{1}{Y_{a+1,s}}} \frac{1 + Y_{a,s-1}}{1 + \frac{1}{Y_{a+1,s}}}$$

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- Problem: how to transform Y-system into a finite system of non-linear integral equations (FiNLIE) using its Hirota discrete integrable dynamics and analyticity properties in spectral parameter?
Worm up: SU(2) x SU(2) principal chiral field at finite $L$

\[ \mathcal{A} = \frac{\sqrt{\lambda}}{4\pi} \int d^2 x \ Tr \left( g^{-1} \partial_\mu g(x) \right)^2, \quad g \in SU(2). \]

- **Y-system**

\[ Y_s(u + \frac{i}{2}) Y_s(u - \frac{i}{2}) = \left[ 1 + Y_{s+1}(u) \right] \left[ 1 + Y_{s-1}(u) \right] \]

- **Energy**

\[ E(Lm) = -\frac{1}{2} \int_{-\infty}^{\infty} du \cosh(\pi u) \log(1 + Y_0(u)) \]

- **Large volume**

\[ Y_s(u) \approx C_s e^{-\delta_{s,0} mL \cosh \pi u}, \quad mL \to \infty \]
Y-system and Hirota relation

Parametrize: \[ Y_s(u) = \frac{T_{s+1}(u)T_{s-1}(u)}{\Phi(u + is/2)\Phi(u - is/2)} \]

Hirota equation:
\[ T_s(u + i/2)T_s(u - i/2) - T_{s-1}(u)T_{s+1}(u) = \Phi(u + is/2)\Phi(u - is/2) \]
Determinant solution of Hirota eq.

\[ T_s(u) = h(u + is/2) \begin{vmatrix} Q(u + is^{+1} / 2) & R(u + is^{+1} / 2) \\ \bar{Q}(u - is^{-1} / 2) & \bar{R}(u - is^{-1} / 2) \end{vmatrix} \]

\[ \Phi(u) = h(u + i/2) \begin{vmatrix} R(u) & Q(u) \\ R(u + i) & Q(u + i) \end{vmatrix} \quad h(u + i) = h(u) \]

Gauge transformations

\[ T_s(u) \rightarrow g\left(u + i \frac{s}{2}\right) \bar{g}\left(u - i \frac{s}{2}\right) T_s(u) \]

\[ \Phi(u) \rightarrow g(u - i/2)g(u + i/2)\Phi(u) \]

\[ Q(u) \rightarrow g(u - i/2)Q(u) \]

Leaves \( Y \)'s invariant!
Lax pair and Baxter relation

Krichever, Lipan, Wiegmann, Zabrodin

- Solution: linear Lax pair (discrete integrable dynamics!)

\[
T_{s+1}(u)Q(u + is/2) - T_s(u - i/2)Q(u + i(s + 2)/2) = \Phi(u + i(s + 1)/2)\overline{Q}(u - i(s + 2)/2)
\]

Complex conjugate

- Baxter relations

\[
T_1(u) = T_0(u - i/2)\frac{Q(u + i)}{Q(u)} + \Phi(u)\frac{\overline{Q}(u - i)}{Q(u)}
\]

\[
T_{-1}(u) = T_0(u + i/2)\frac{Q(u)}{Q(u + i)} - \Phi(u)\frac{\overline{Q}(u)}{Q(u + i)}
\]

- Wing exchange symmetry:

\[
T_s \leftrightarrow T_{-s} , \quad \Phi \leftrightarrow -\overline{\Phi} , \quad Q^+ \leftrightarrow \overline{Q}^- , \quad \overline{Q}^- \leftrightarrow Q^+
\]

- \(Q(R)\) are interpreted (in different gauges) as Baxter functions for right (left) wing
Analyticity and ground state solution $Q=1$

- Baxter eq.
- "Jump" eq.

Solution: we assume $T_0(u), \Phi(u)=T_0(u+i/2+i0)$ and $T_{-1}(u)$:

\[
T_1(u) = T_0(u - i/2) + T_0(u + i/2)
\]

\[
T_{-1}(u) = T_0(u + i/2 - i0) - T_0(u + i/2 + i0)
\]

relates $T_0$ and $\Phi$ to $T_{-1}(u)$ through analyticity:

\[
T_0 = 1 - \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{dv T_{-1}(v)}{u-v-i/2} + \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{dv T_{-1}(v)}{u-v+i/2}
\]

- $T_s$ have no singularities on real axis

- TBA eq. for $Y_0$, imposing for vacuum $Y_1 = Y_{-1}$ get non-linear integral eq. for $T_{-1}$

\[
\log Y_0 = -L \cosh \pi u + 2r \log [1 + Y_1], \quad r = \frac{1}{2 \cosh \pi u}
\]

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SU(2) PCF numerics (using Hirota solution): Energy versus size for various states

EL/2π

For vacuum and mass gap in accordance with Balog, Hegedus’00’02
Inspiring example: \( SU(N)_L \times SU(N)_R \) principal chiral field

\[
S = \frac{\sqrt{\lambda}}{4\pi} \int d^2x \, \text{Tr} \left( g^{-1} \partial_\mu g(x) \right)^2, \quad g \in SU(N).
\]

- **Y-system** \( \Rightarrow \) **Hirota** dynamics in a in \((a,s)\) plane.

\[
Y_{a,s} \left( u + \frac{i}{2} \right) Y_{a,s} \left( u - \frac{i}{2} \right) = \frac{1 + Y_{a,s+1}}{1 + \frac{1}{Y_{a+1,s}}} \frac{1 + Y_{a,s-1}}{1 + \frac{1}{Y_{a-1,s}}}
\]

- Relation of Y-system to T-system (Hirota equation)

\[
T_{a,s}(u + \frac{i}{2})T_{a,s}(u - \frac{i}{2}) = T_{a,s-1}(u) T_{a,s+1}(u) + T_{a+1,s}(u) T_{a-1,s}(u)
\]

- Gauge symmetry

\[
T_{a,s} \rightarrow g_1^{[a+s]} g_2^{[a-s]} g_3^{-[a-s]} g_4^{-[a+s]} T_{a,s}
\]
General wronskian solution in a band

- Parametrized through $N$ functions of $u$: $q_1, q_2, \cdots, q_N$

- Representation in terms of exterior forms:
\[
q \equiv q_j \theta^j, \quad p \equiv p_j \theta^j, \quad \theta^1 \wedge \theta^2 \wedge \cdots \wedge \theta^N = 1
\]

Gromov, V. K., Leurent, Volin

- $k$-forms:
\[
q(k) \equiv q[k-1] \wedge q[k-3] \cdots \wedge q[1-k]
\]

- Solution in $N$-band:
\[
T_{a,s} = q_{(a)}^{[s]} \wedge p_{(N-a)}^{[-s]}
\]
Analyticity properties and solution

- Finite volume solution: finite system of NLIE: parametrization fixing the analytic structure:

\[ q_k(u) = P_k(u) + \int_{-\infty}^{\infty} \frac{dv \, f_k(v)}{2\pi u - v}, \quad \text{Im}(u) < 0, \]

- From reality:

\[ p_k(u) = \bar{q}_k(u) \]

- N-1 spectral densities \( f_k(u) \) (for L ↔ R symmetric states):

\[ Y_{a,s} = Y_{a,-s} \]

\[ \log Y_{a,0} = -Lm_a \sinh(2\pi Nu) + K_{a,a'} \log \left[ \frac{(1 + Y_{a',1})^2}{(1 + Y_{a'+1,0})(1 + Y_{a'-1,0})} \right], \quad \text{where} \quad Y_{a,s} = \frac{T_{a,s+1}T_{a,s-1}}{T_{a+1,s}T_{a-1,s}} \]

- Solved numerically by iterations
SU(3) PCF numerics: Energy versus size for vacuum and mass gap

\[ \frac{E L}{2\pi} \]

V.K., Leurent’09

\[ \frac{N^2 - 1}{12} \]
AdS/CFT Y-system and asymptotics

\[ \frac{Y_{a,s}^+ Y_{a,s}^-}{Y_{a+1,s} Y_{a+1,s}} = \frac{1 + Y_{a,s}^+}{1 + Y_{a+1,s}} \frac{1 + Y_{a,s}-1}{1 + Y_{a+1,s}} \]

- Large L asymptotics:

\[ Y_{a,s}(u) \simeq C_{a,s} e^{\delta_{s,0}} L i p_{a}(u) \]

- Momentum of elementary excitation appears as a zero mode of discrete D’Alembert operator in the l.h.s.

We should fix it and find the corresponding energy as well.
Dispersion relation in physical and crossing channels

- Exact one particle dispersion relation: \( \epsilon^2 = 1 + \lambda \sin^2 \frac{p}{2} \)

- Bound states (fusion) \( \epsilon^2_a = a^2 + \lambda \sin^2 \frac{pa}{2} \)

- Changing physical L-circle to cross channel R-circle

\[ \epsilon^2_a = a^2 + \lambda \sin^2 \frac{pa}{2} \quad \rightarrow \quad -\epsilon^2_a = a^2 - \lambda \sinh^2 \frac{\tilde{p}a}{2} \]

- Parametrization for the dispersion relation by Zhukovsky map:

\[
\begin{align*}
p_a(u) &= \frac{1}{i} \log \frac{z(u + ia/2)}{z(u - ia/2)} \\
\epsilon_a(u) &= 2i\sqrt{\lambda} [z(u - ia/2) - z(u + ia/2)] + 1
\end{align*}
\]

- From physical to crossing kinematics: continuation through the cut

\[ z = \frac{1}{2\sqrt{\lambda}} \left( u + \sqrt{u - 2\sqrt{\lambda} \sqrt{u + 2\sqrt{\lambda}}} \right) \quad \rightarrow \quad \tilde{z} = \frac{1}{2\sqrt{\lambda}} \left( u + i\sqrt{4\lambda - u^2} \right) \]

cuts in complex \( u \)-plane
Y-system for excited states of AdS/CFT at finite size

\[ Y_{a,s}(u) \]

\[ Y_{a,s}\left(\frac{u + i}{2}\right) Y_{a,s}\left(\frac{u - i}{2}\right) = \frac{1 + Y_{a,s+1}}{1 + \frac{1}{Y_{a+1,s}}} \frac{1 + Y_{a,s-1}}{1 + \frac{1}{Y_{a+1,s}}} \]

- Complicated analyticity structure in \( u \) dictated by non-relativistic dispersion
- Extra equation (remnant of classical \( Z_4 \) monodromy):
  \[ Y_{2,\pm 2}(u+i0) \cdot Y_{1,\pm 1}(u-i0) = 1 \]

- Energy:
  (anomalous dimension)
  \[ \Delta - \Delta_0 = f_{\text{min}} = \int \frac{du}{2\pi i} \partial_u \tilde{\epsilon}_a \log (1 + Y_{a,0}) \]

- \( u_j \) obey the exact Bethe eq.:
  \[ Y_{1,0}(u_j) + 1 = 0 \]
Konishi operator $\text{Tr } [\mathcal{D}, Z]^2$ : numerics from $Y$-system

$\Delta_{Konishi} = 2 + 12\lambda - 48\lambda^2 + 336\lambda^3 + 96(-26 + 6\zeta(3) - 15\zeta(5))\lambda^4 - 96(-158 - 72\zeta(3) + 54\zeta(3)^2 + 90\zeta(5) - 315\zeta(7))\lambda^5 + O(\lambda^6)$

$= 2! \text{ From quasiclassics}$
Gromov, Shenderovich, Serban, Volin
Roiban, Tseytlin
Masuccato, Valilio

$\Delta_{Konishi} = 2\lambda^{1/4} + 1.994\cdot\lambda^{-1/4} + O(\lambda^{-3/4})$

$Y$-system numerics
Gromov, V.K., Vieira

$Y$-system passes all known tests

millions of 4D Feynman graphs!
Y-system and Hirota eq.: discrete integrable dynamics

- Relation of Y-system to T-system (Hirota equation) (the Master Equation of Integrability!)

\[
Y_{a,s} = \frac{T_{a,s+1} T_{a,s-1}}{T_{a+1,s} T_{a-1,s}}
\]

\[
T_{a,s}(u + \frac{i}{2}) T_{a,s}(u - \frac{i}{2}) = T_{a,s-1}(u) T_{a,s+1}(u) + T_{a+1,s}(u) T_{a-1,s}(u)
\]

Hirota eq. in T-hook for AdS/CFT

Gromov, V.K., Vieira

Discrete classical integrable dynamics!
Quasiclassical solution of AdS/CFT Y-system

- Classical limit: highly excited long strings/operators, strong coupling:
  \[ L \sim \sqrt{\lambda} \sim u \rightarrow \infty \]

- Explicit u-shift in Hirota eq. dropped (only slow parametric dependence)

  \[ T_{a,s} \left( u + \frac{i}{2\sqrt{\lambda}} \right) T_{a,s} \left( u - \frac{i}{2\sqrt{\lambda}} \right) = T_{a,s-1}(u) T_{a,s+1}(u) + T_{a+1,s}(u) T_{a-1,s}(u) \]

(Quasi)classical solution - \( \text{psu}(2,2|4) \) character of classical monodromy matrix in Metsaev-Tseytlin superstring sigma-model

- Its eigenvalues (quasi-momenta) encode conservation lows

- An important property: \( \mathbb{Z}_4 \) symmetry of the coset and related monodromy for the eigenvalues (quasimomenta)

Gromov, V.K., Tsuboi

Zakharov, Mikhailov
Bena, Roiban, Polchinski

V.K., Marshakov, Minahan, Zarembo
Beisert, V.K., Sakai, Zarembo
(Super-)group theoretical origins

- A curious property of $gl(N|M)$ representations with rectangular Young tableaux:

- For characters – simplified Hirota eq.:

- Boundary conditions for Hirota eq.:

- Solution of Hirota for any irrep: Jacobi-Trudi formula for $GL(K|M)$ characters:

$$\det_{1 \leq i, j \leq a} T_{a,s-i+j}[g], \quad g \in GL(K|M).$$
Construction of solution for AdS/CFT T-system

- There are three different analytically friendly gauges for T-functions for right, left and upper bands.
- They have certain analyticity strips:
  - Right (left) band: only two cuts
    \[ T_{0,\pm s} = 1, \quad T_{1,\pm s} \in \mathcal{A}_s, \quad T_{2,\pm s} \in \mathcal{A}_{s-1} \]
  - Upper band
    \[ T_{a,0} \in \mathcal{A}_{a+1}, \quad T_{a,\pm 1} \in \mathcal{A}_a, \quad T_{a,\pm 2} \in \mathcal{A}_{a-1} \]
    \[ T_{n,2} = T_{2,n} \quad \text{ - from representation theory} \]
    \[ T^+_0,0 = T^-_{0,0} \quad \text{ - from quantum unimodularity of monodromy matrix} \]
- Relating right (left) and upper bands
  - \( \mathbb{Z}_4 \) symmetry of coset: \( T_{a,s} \) can be analytically continued in labels \( a,s \)
    \[ \hat{T}_{a,s} = (-1)^s \hat{T}_{-a,s}, \quad \hat{T}_{a,s} = (-1)^a \hat{T}_{a,-s} \]
Wronskian solution for the right (left) band

- Specifying it for N=2 we get:

\[ \hat{T}_{0,s} = 1, \quad \hat{T}_{1,s} = q^{[+s]} \wedge q^{[-s]}, \quad \hat{T}_{2,s} = \hat{T}_{1,1}^{[+s]} \hat{T}_{1,1}^{[-s]} . \]

- Using the \( \mathbb{Z}_4 \) symmetry and after a certain gauge transformation

\[ \hat{T}_{1,s} = \hat{h}^{[+s]} \hat{h}^{[-s]} \hat{T}_{1,s}, \quad \hat{T}_{2,s} = \hat{h}^{[+s+1]} \hat{h}^{[+s-1]} \hat{h}^{[-s+1]} \hat{h}^{[-s-1]} \hat{T}_{2,s} \]

we find a representation similar to SU(2) PCF (but with two short cuts)

\[ \hat{T}_{0,s} = 1, \]
\[ \hat{T}_{1,s} = q^{[+s]} - q^{[-s]}, \quad q = -iu + \frac{1}{2\pi i} \int_{-2g}^{2g} \frac{\rho(v)dv}{u - v}, \]
\[ \hat{T}_{2,s} = \hat{T}_{1,1}^{[+s]} \hat{T}_{1,1}^{[-s]} \]

\[ q_1 = q, \quad q_2 = 1 \]
Wronskian solution for the upper band

- Specifying it for $N=2$ we get:
  $$\hat{T}_{a,s} = q_{(2-s)}^{[+a]} \wedge p_{(2+s)}^{[-a]}.$$

  $$q_{(2)} = \frac{q^+ \wedge q^-}{q_0}, \quad q_{(3)} = \frac{q^{++} \wedge q \wedge q^{--}}{q_0^+ q_0^-}, \quad q_{(4)} = \frac{q^{[+3]} \wedge q^+ \wedge q^- \wedge q^{[-3]}}{q_0^{++} q_0^- q_0^-}.$$

- From reality, $\mathbb{Z}_4$ symmetry and asymptotic properties at large $L$ and considering only left-right symmetric states $\hat{T}_{a,-s} = \hat{T}_{a,s}$ we find

  $$\hat{T}_{a,\pm 1} = q_{1}^{[+a]} q_{2}^{[-a]} + q_{2}^{[+a]} q_{1}^{[-a]} + q_{3}^{[+a]} q_{4}^{[-a]} + q_{4}^{[+a]} q_{3}^{[-a]},$$

  $$\hat{T}_{a,0} = q_{12}^{[+a]} q_{12}^{[-a]} + q_{34}^{[+a]} q_{34}^{[-a]} - q_{14}^{[+a]} q_{14}^{[-a]} - q_{23}^{[+a]} q_{23}^{[-a]} - q_{13}^{[+a]} q_{24}^{[-a]} - q_{24}^{[+a]} q_{13}^{[-a]}.$$

- We choose to parameterize $T$'s through $q_1, q_2, q_{12}$

- Remarkably, if $q_1, q_2$ are analytic in the upper half plane, and $q_{12}$ analytic above $-i/2 + \mathbb{R}$ then all $T$-functions have the right analyticity strips.

- We can parameterize in terms of 2 spectral densities $\rho_2, \rho_3$ and find from analyticity of a closed system of equations on them - FiNLIE
Closing FiNLIE: spectral densities and sawing together 3 bands

- We can parameterize in terms of 2 spectral densities $\rho_2, \rho_3$
  
  $q_1 = \sqrt{U} f^+ f^-, \quad q_2 = \sqrt{U} f^+ f^- W, \quad q_{12} = f^2 V$

Example:

$$U = x^\mu(u) \int_{-\infty}^{+\infty} \frac{dv \rho_3(v)}{2\pi i (u - v)}, \quad \text{Im } u > 0$$

- From analyticity - closed system of equations on densities $\rho, \rho_2, \rho_3$

FiNLIE!
Numerical solution of FiNLIE

- Writing our FiNLIE as $X = f(X), \quad X = (\rho, \rho_2, \rho_3, \ldots)$

we attempted to solve it on Mathematica by iterations: $X_{n+1} = f(X_n)$

- The coincidence with earlier results from the infinite Y-system (TBA) is very satisfactory!
Comments

- The Bethe roots characterizing a state are encoded into zeros of some $q$-functions (in particular $q_{12}$).

- The energy of a state can be extracted from the asymptotics

$$T_{0,0}(u) \sim u^{2E}, \quad u \to \infty$$
Conclusions

• Y-system obeys integrable Hirota dynamics – can be reduced to a finite system of non-linear integral eqs (FiNLIE).

• Our FiNLIE is based on a few natural, or even physical analyticity assumptions. Not based on TBA but proven to have the same solution

• Certainly our FiNLIE is still perfectible.

• Hirota dynamics provides a general method of solving quantum $\varnothing$-models on a finite space circle.

• Possible mathematical subject: “sigma model”-like solution of Hirota equations and the associated Riemann-Hilbert problems for general $(a,s)$ boundary conditions.
END