NONLOCAL ADHESION MODELS FOR TWO TYPE CANCER ON MULTIDIMENSIONAL BOUNDED DOMAIN

JAEOOK AHN, MYEONGJU CHAE, AND JIHOON LEE

Abstract. Cell-cell adhesion is an inherently nonlocal phenomenon. Numerous partial differential equation models with nonlocal term have been recently presented to describe this phenomenon, yet the mathematical properties of nonlocal adhesion model are not well understood. Here we consider two kinds of nonlocal cell-cell adhesion model satisfying no-flux conditions in a multidimensional bounded domain. We show global-in-time well-posedness of the solution to this model and obtain the uniform boundedness of solution.

1. Introduction

In this paper we consider the cell to cell nonlocal adhesion models on multidimensional bounded domain. Cellular adhesion are fundamental features of multicellular organisms e.g. embryogenesis, wound healing and self-organization. The regulation of cellular adhesion is critical in cancer formations as well.

Mathematical modelings of cancer invasion have been widely studied for decades; the formation and movement of tumour cells are governed by random diffusion, aggregations and reactions which are often logistic types. In the most of cancer migration models aggregations arise from the cell to cell adhesion, the cell to extra cellular matrix (ECM) adhesion, the movement of cells in response to stimuli of diffusing chemicals (chemotaxis), the movement of cells in response to non-diffusing environmental factor such as ECM (haptotaxis). There has been numerous mathematical analysis on a parabolic or parabolic-elliptic-ODE system consisting of diffusion, aggregation and reaction for describing tumour dynamics; local and global well-posedness, blow-up, asymptotic behavior etc. on the whole space or space with boundary. See and references therein.

In this paper we study on a two population cancer model focused on two aspects that are nonlocal cell to cell adhesion on multidimensional bounded domain and the epithelial-mesenchymal transition (EMT) type reaction. The first successful continuum model of cellular adhesion was proposed by Armstrong et al. in , where they introduced a nonlocal integral term to describe the adhesive forces between cells. The basic single population model on the real line, for instance, is derived to be

\[ \partial_t u = u_{xx} - \alpha \left( u(x,t) \int_{-R}^{R} g(u(x+r))\omega(r)dr \right) \]

where \( \alpha \) is the strength of cell adhesion, \( g(u) \) describes the nature of the adhesion force, \( \omega(r) \) is a function describing the direction and magnitude of the adhesion force, and \( R \) the sensing radius of the cell. In the same paper two cell population version of the model on two-dimensions was also proposed with numerical simulations supporting the active cell-sorting process from the randomly distributed mixture. The model was the first reproduce Steinberg’s cell segregating experiment that are classical result in developmental biology.

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The regulation of cellular adhesion is critical in cancer formations as well. Naturally (1.1) has been extensively used to model cancer cell invasion and developmental processes [5, 7, 11, 20, 23, 24, 35, 34, 37]. Cauchy problems were studied in [5, 27, 28, 37], among which [28] obtained the general results on local and global well-posedness of classical solutions to multidimensional version of (1.1) in $\mathbb{R}^n$.

On the other hand, there have been only a few study on the adhesion model posed on the bounded domain. In fact all the works referred above avoid boundary or leave the sensing domain unspecified. As far as we know, Hillen and Butenschö [27] was the first that considered the well-posedness of the initial-boundary problem of (1.1). They derived various types of adhesion repulsive and no-flux boundary conditions and proved both the local and global well-posedness for the resulting equation. In this paper we extend the work to multidimensional bounded domain with two kinds of boundary conditions satisfying the total flux zero condition. It seems more realistic to consider the cell adhesion in bounded domain since cell-boundary interactions indeed exist.

As is mentioned in [27], there are another class of non-local models so called aggregation equation, where the non-local term has a singular interaction kernel in general. The aggregation equations arise as a gradient flow of a potential, and the well-posedness and blow-up features of the equations have been extensively studied so far [9, 10] e.g.. A recent study of the equation on the bounded domain can be found in [22, 44].

Let $u(x,t)$ and $v(x,t)$ denote the early stage and the late stage cancer cell population densities respectively at spatial location $x \in \Omega$ and time $t$, where $\Omega \subset \mathbb{R}^n$, $n \geq 2$, is an open bounded domain. Following the simplifying assumptions in [6], we assume that the cell adhesion force is linearly proportional to population density and that both cells have the same sampling radius $R$; we set the nonlocal adhesion term between two population as

$$
K[u,v](x,t) = \int_{E(x)} [M_{11}u(x+y,t) + M_{12}v(x+y,t)] \omega(y)dy,
$$

$$
S[u,v](x,t) = \int_{E(x)} [M_{21}u(x+y,t) + M_{22}v(x+y,t)] \omega(y)dy,
$$

$$
E(x) : \text{sensing domain varying on } x \text{ specified later}, \omega := (\omega_1, \cdots, \omega_n) \text{ for } \omega_i \text{ bounded.}
$$

The positive constants $M_{11}$ and $M_{22}$ represent the self-adhesive strength of populations $u$ and $v$, respectively, and the positive constants $M_{12}$ and $M_{21}$ represent the cross-adhesive strengths between the populations.

In the below we set the two initial-boundary problems on $u(x,t)$ and $v(x,t)$ according to boundary conditions.

I. Nonlocal Robin boundary condition

Let $\Omega$ be a $C^2$ smooth bounded open domain and $K[u,v], S[u,v]$ be defined as in (1.2). We have

$$
\begin{align*}
\partial_t u - \Delta u &= -\nabla \cdot (uK[u,v]) - mu + \frac{\lambda}{k}u(k-(u+v)), & x \in \Omega, \ t > 0, \\
\partial_t v - \Delta v &= -\nabla \cdot (vS[u,v]) + mu + \frac{\mu}{k}v(k-(u+v)), & x \in \Omega, \ t > 0, \\
\partial_t u = uK[u,v] \cdot \nu, \quad \partial_t v = vS[u,v] \cdot \nu, & x \in \partial \Omega, \ t > 0, \\
u(x,0) = u_0(x), \ v(x,0) = v_0(x), & x \in \Omega,
\end{align*}
$$

(1.3)
where \( \nu(\cdot) \) is the \( C^1 \)-smooth unit outward normal vector field to \( \partial \Omega \). The positive constants \( m, k, \lambda, \) and \( \mu \) denote the mutation rate, the carrying capacity, the growth rate of \( u \), and the growth rate of \( v \). The sensing domain \( E(x) \) is given by

\[
E(x) = \{ y \in \mathbb{R}^n \mid x + y \in \Omega, \ |y| < R \}.
\]

Moreover we assume

\[
(1.5) \quad \omega = (\omega_1, \cdots, \omega_n) \text{ for } \omega_i \text{ bounded and smooth.}
\]

The terms \(-mu\) and \(+mu\) describe that early stage cancer cells can mutate into later stage cells at a constant rate \( m \) \[11\]. If we interpret \( u \) and \( v \) as the density of primary epithelial tumor cells and cancer stem cell respectively, \( \pm mu \) stands for the epithelial -mesenchymal transition (EMT) that primary epithelial tumor cells undergo to acquire the ability to migrate into surroundings \[45\]. The logistic competition terms \( u(k - (u + v)) \) and \( v(k - (u + v)) \) describe the production of \( u \) and \( v \) are according to logistic law and they compete for free space. The two species cancer model with haptotactic invasion undergoing EMT is analyzed in \[10, 25\]. For more details on the model we refer to references therein.

Note that we require the total flux zero on the boundary in \(1.3\), which leads to a nonlocal boundary condition of Robin type,

\[
(1.6) \quad \partial_n u - uK[u, v] \cdot \nu = 0, \quad \partial_n v - vS[u, v] \cdot \nu = 0, \quad x \in \partial \Omega, \ t > 0.
\]

The EMT term suggests the later stage population to dominate the total cell population if the transition rate \( m \) is large;

\[
(1.7) \quad u \equiv 0, \quad v \equiv k.
\]

If we seek the constant solution with the total flux zero \(1.6\), it must satisfy

\[
(1.8) \quad \partial_t u = 0, \quad \partial_t v = 0 \quad x \in \partial \Omega,
\]

\[
(1.9) \quad K[u, v] = 0, \quad S[u, v] = 0 \quad x \in \partial \Omega.
\]

The first condition is the usual Neumann zero condition on the solutions \( u \) and \( v \), however, \(1.9\) is the condition imposed to the nonlocal operators \( K \) and \( S \). We call this case independent following \[27\] or zero-zero flux condition. An independent case allows the constant solution \( (u, v) = (0, k) \) which is one of two non-negative constant solutions. The other one is \( (0, 0) \).\[1\] The linear stability analysis for the related one dimensional model is performed in \[12\].

We shall study the independent case with an explicit example of the sensing domain \( E(x) \) satisfying \( |E(x)| = 0 \) as \( x \) approaches to \( \partial \Omega \) by which the condition \(1.9\) is assured. Note that the volume of \(1.4\) cannot vanish on the boundary. We formulate the second initial-boundary problems under \(1.8\)-\(1.9\) as follows.

**II. Zero-zero flux condition**

Let \( \Omega \) be the open ball of radius \( L \), \( B_L(0) \), and \( K[u, v], S[u, v] \) be defined as in \(1.2\). We have

\[
(1.10) \quad \begin{aligned}
\partial_t u - \Delta u &= -\nabla \cdot (uK[u, v]) - mu + \frac{\lambda}{k} u(k - (u + v)), \quad x \in \Omega, \ t > 0, \\
\partial_t v - \Delta v &= -\nabla \cdot (vS[u, v]) + mu + \frac{\mu}{k} v(k - (u + v)), \quad x \in \Omega, \ t > 0, \\
\partial_n u &= 0, \quad \partial_n v = 0, \quad K[u, v] = 0, \quad S[u, v] = 0, \quad x \in \partial \Omega, \ t > 0, \\
u(x, 0) &= u_0(x), \quad v(x, 0) = v_0(x), \quad x \in \Omega,
\end{aligned}
\]

\[1\] There is the other steady state, \( u^* = -\frac{k}{\mu} v^* \), \( v^* = \frac{k(1 - \frac{\mu}{\lambda})}{\mu k} \), which are of different signs, hence unrealistic.
where $m, k, \mu, \lambda$ are same as before in case I, and $\nu(\cdot)$ denotes the unit outward normal vector to $\partial \Omega$. The sensing domain $E(x)$ is given for $0 < R < L$ by

\[(1.11) \quad E(x) = B_R(0) \quad \text{for} \quad |x| < L - R, \quad E(x) = B_{L-|x|}(x) \quad \text{for} \quad L - R \leq |x| < L,\]

and we assume

\[(1.12) \quad \omega(x) := \frac{x}{|x|} w(x)\]

for $w \in C_0^\infty(\Omega)$ non-negative. In (1.11) we choose $E(x)$ satisfying the following property as simple as possible; when $x$ is away from the boundary of the domain, it is $B_0(R)$ as was set for two dimensional model in [6]. When $x$ is close to the boundary, it shrinks to a smaller region such that $x + r \in \Omega$ for $r \in E(x)$ and $|E(x)| = 0$ as $x$ reaches to the boundary. The choice of varying integration domain $E(x)$ affects the extent of regularity of the adhesion terms $K[u, v]$ and $S[u, v]$. In Lemma 2 and Lemma 7, we only have $K[u, v]$ and $S[u, v]$ as lipschitz continuos functions however smooth $u, v, \omega$ are.  

In case I the nonlocal nonlinear Robin type boundary condition as well as the restricted regularity of $K[u, v]$ and $S[u, v]$ cause difficulty in constructing local-in-time strong solutions directly by iterations. For the local well-posedness we take two steps; we first construct the generalized solution relying on semi-group theory for parabolic boundary value problem found in Amann’s seminal works [2, 3] e.g.. In particular we introduce a certain extension of the unit outward normal vector field, and employ the generalized variation-of-constants formula for this case. See Section 2.1 and Section 2.2 for details. Similar construction can be also found in [17, 18]. We can show the generalized solutions are indeed strong and satisfy the maximal regularity estimates employing the result of Denk-Hieber-Prüss [19]. The global well-posedness follows from several a priori estimates and the Moser-Alikakos type estimate.

In the zero-zero case the diffusive flux and the adhesion flux are independently zero on the boundary. The global well-posedness is obtained in a standard way, though $K[u, v]$ and $S[u, v]$ are less regular than the adhesion terms in the nonlocal robin type boundary case due to shrinking sensing domain $E(x)$. As is mentioned earlier, this case allows the constant steady state $(0, 0)$ and $(0, k)$. We provide a linear stability analysis in Appendix, where $(0, 0)$ is found linearly unstable, and $(0, k)$ linearly asymptotically stable if $m > \mu$. Also we find a Lyapunov type inequality hold when $v$ has no adhesion term and $\mu > \lambda$. See Remark 1.

We are ready to state the main results in this paper. We first provide the global strong solvability for case I.

**Theorem 1.** Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be an open bounded domain with $C^2$ boundary $\partial \Omega$. Suppose that the non-negative initial conditions $u_0$ and $v_0$ belong to $W^{2,p}(\Omega)$, $n < p < \infty$ and satisfy the compatibility condition

\[\partial_t u_0 - u_0 K[u_0, v_0] \cdot \nu = 0, \quad \partial_t v_0 - v_0 S[u_0, v_0] \cdot \nu = 0, \quad x \in \partial \Omega.\]

Then, (1.3)-(1.5) admits a unique non-negative strong solution $(u, v)$ such that

\[u, v \in C([0, t); W^{1,p}(\Omega)) \cap W^{1,p}(0, t; L^p(\Omega)) \cap L^p(0, t; W^{2,p}(\Omega)), \quad t > 0.\]

Moreover, the solution $(u, v)$ has a boundedness property

\[\sup_{t > 0} (\|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{L^\infty(\Omega)}) \leq C.\]

We next state the global strong solvability for case II.

\[\text{Footnote 2: For case II, we could change the shrinking rate and shape of } E(x) \text{ as } x \text{ approaches to the boundary so that the regularity of adhesion terms is possibly worse.} \]
Theorem 2. Let $Ω \subset \mathbb{R}^n$, $n \geq 2$ be an open ball of radius $L$, $B_L(0)$. Suppose that the non-negative initial datas $u_0$ and $v_0$ belong to $W^{2,p}(Ω)$, $n < p < \infty$, and satisfy the compatibility condition
\[ \partial_ν u_0 = \partial_ν v_0 = 0, \quad x \in \partial Ω. \]
Then, (1.10)–(1.12) admits a unique non-negative strong solution $(u, v)$ such that
\[ u, v \in C([0, t); W^{1,p}(Ω)) \cap W^{1,2,p}(0, t; L^p(Ω)) \cap L^p(0, t; W^{2,p}(Ω)), \quad t > 0. \]
Moreover, the solution $(u, v)$ has a boundedness property
\[ \sup_{t>0}(\|u(\cdot, t)\|_{L^∞(Ω)} + \|v(\cdot, t)\|_{L^∞(Ω)}) \leq C. \]

The paper is organized as follows. We prove Theorem 1 in Section 2 and Theorem 2 in Section 3. Both sections starts from proving preliminary results on the adhesion term $K[u, v]$, and $S[u, v]$. The local well-posedness of (1.10)–(1.12) in Lemma 8 and the uniform boundedness in Lemma 11. Finally a linear stability analysis for the zero-zero case is provided in Appendix.

2. Nonlocal Robin boundary case

2.1. Preliminary. We first consider a lemma on the extension $N$ of the normal vector field $ν$ into the domain $Ω$. The extension $N$ is used in Section 2.2 to interpret our nonlocal Robin boundary value problems as the inhomogeneous Neumann boundary value problems. The lemma holds for a bounded domain with $C^k$-smooth boundary for any integer $k \geq 2$ without difficulty. Before stating the lemma, we remind the definition of $C^k$-boundary in [21, Appendix C.1].

Definition 1. Let $U \subset \mathbb{R}^n$ be an open bounded domain, and $k \in \{1, 2, \ldots\}$. We call $∂U$ is $C^k$ if for each point $x^0 \in ∂U$ there exist $r > 0$ and a $C^k$ function $ξ : \mathbb{R}^{n-1} \to \mathbb{R}$ such that upon an orthogonal change of coordinates we have
\[ U \cap B_r(x^0) = \{ x \in B_r(x^0) \mid x_n > ξ(x_1, \ldots, x_{n-1}) \} \]

Lemma 1. Let $Ω \subset \mathbb{R}^n$, $n \geq 2$, be an open bounded domain with $C^2$ boundary $∂Ω$. Then, there exists at least one vector field $N \in C^1(Ω)$, a continuous extension of the unit outward normal vector field $ν$, such that $N = ν$ on $∂Ω$.

Proof. For $x^0 \in ∂Ω$ there is $r, ξ$, and a relabeled coordinate $(x_1, \ldots, x_n)$ as in the definition such that $∂U \cap B_r(x^0) = \{ x \in B_r(x^0) \mid ξ(x_1, \ldots, x_{n-1}) - x_n = 0 \}$. The outward normal vector field $ν(x)$ is well defined by
\[ ν(x) = \frac{(\nabla ξ(x), -1)}{\sqrt{1 + |\nabla ξ(x)|^2}} \]
where $x = (x_1, x_n)$.

Let us consider the function $E : ∂Ω \times \mathbb{R} \to \mathbb{R}^n$ given by
\[ E(x, t) = x - tν(x). \]

Applying the tubular neighborhood theorem (Theorem 10.19 [31]) there exist $ρ > 0$ such that $E$ is the $C^1$-diffeomorphism on $V := \{ (x, t) \in ∂Ω \times \mathbb{R} \mid |t| < ρ \}$ and $T_ρ := \{ E(x, t) | x \in ∂Ω, 0 < t < ρ \}$ is in $Ω$. We find an smooth extension of $ν$ on $Ω$ as follows. For $y \in T_ρ$ there is
\[ ι(E(x, t)) \text{ is called an tubular neighborhood of } ∂Ω, \]
\[ 3 \text{We follow a definition of } C^1 \text{-function on a submanifold } ∂Ω \times \mathbb{R} \text{ embedded in } \mathbb{R}^n \times \mathbb{R}. \]
the unique \((x, t) \in \partial \Omega \times \mathbb{R}\) such that \(y = x - tv(x)\), since \(E\) is an one to one. Note that the mapping \(y \to (x, t)\) is \(E^{-1}\), hence \(C^1\). We define a continuous extension of \(\nu\) by

\[
N(y) = \begin{cases} 
\frac{\nu - \pi_2 \circ E^{-1}(y)}{\rho} & \text{for } y \in T_\rho, \\
0 & \text{for } y \in \Omega \setminus T_\rho.
\end{cases}
\]

\(N\) being \(C^1\) on \(T_\rho\) follows from the construction; Define \(\pi_1 : \partial \Omega \times \mathbb{R} \to \partial \Omega\) to be the projection of the first slot, and \(\pi_2\) of the second slot. Then we can wirte

\[
N(y) = \frac{\rho - \pi_2 \circ E^{-1}(y)}{\rho} \nu(\pi_1 \circ E^{-1}(y))
\]

with \(\nu(x) = \frac{(\nabla \xi(x), 1)}{1 + |\nabla \xi(x)|^2}\) upon an orthogonal change of coordinates. Finally smoothing out \(N\) in \(T_\rho \setminus T_\|\), we obtain a smooth extension \(\mathcal{N}\) of \(\nu\) on \(\overline{\Omega}\).

Before closing the section we prepare \(W^{1, \infty}(\Omega)\) estimate for the non-local terms \(K\) and \(S\).

**Lemma 2.** Let \(\Omega \subset \mathbb{R}^n, n \geq 2\), be an open bounded domain with \(C^2\) boundary \(\partial \Omega\). Suppose that \(f, g \in L^1(\Omega)\). Assume further that \(K[f, g], S[f, g], E, \) and \(\omega\) are given by \(1.2\), and \((1.4)\)–\((1.5)\). Then, there exists a constant \(C > 0\) satisfying

\[
\|K[f, g]\|_{W^{1, \infty}(\Omega)} + \|S[f, g]\|_{W^{1, \infty}(\Omega)} \leq C (\|f\|_{L^1(\Omega)} + \|g\|_{L^1(\Omega)}).
\]

**Proof.** It suffices to prove the Lipschitz continuity of

\[
\mathcal{I}[f](x) = \int_{E(x)} f(x + y) \omega(y) dy,
\]

where \(E(x)\) and \(\omega\) are introduced in \((1.4)\)–\((1.5)\). By change of variable, \(\mathcal{I}[f](x)\) can be written as

\[
\mathcal{I}[f](x) = \int_{V_x} f(z) \omega(z - x) dz, \quad V_x = \{z \in \Omega \mid |z - x| < R\}.
\]

Choose any two points \(x, y \in \Omega\) and let \(h = y - x\). We have

\[
\mathcal{I}[f](x + h) - \mathcal{I}[f](x) = \int_{V_{x+h}} f(z) \omega(z - x - h) dz - \int_{V_x} f(z) \omega(z - x) dz
\]

\[
= \int_{V_{x+h}} f(z) [\omega(z - x - h) - \omega(z - x)] dz + \int_{V_x} f(z) \omega(z - x) dz - \int_{V_x} f(z) \omega(z - x) dz.
\]

We estimate

\[
\left| \int_{V_{x+h}} f(z) [\omega(z - x - h) - \omega(z - x)] dz \right| \leq C|h| \|f\|_{L^1(\Omega)} \|\omega\|_{W^{1, \infty}(\Omega)}
\]

For the the other terms, we note that \((V_{x+h} \cup V_x) \setminus (V_{x+h} \cap V_x)\) is a subset of \(B_R(x + h) \cup B_R(x) \setminus B_R(x + h) \cap B_R(x)\) in \(\mathbb{R}^n\) and the volume of the latter is bounded by \(Ch\) with a uniform constant \(C\) if \(y = x + h\) are in \(B_R(x)\). We have

\[
\left| \int_{V_{x+h}} f(z) \omega(z - x) dz - \int_{V_x} f(z) \omega(z - x) dz \right| \leq \|f\|_{L^1(\Omega)} \|\omega\|_{L^\infty(\Omega)} \int_{(V_{x+h} \cup V_x) \setminus (V_{x+h} \cap V_x)} 1 dz
\]

\[
\leq C|h|^{n-1} \|f\|_{L^1(\Omega)} \|\omega\|_{L^\infty(\Omega)},
\]

hence it follows that

\[
|\mathcal{I}[f](x + h) - \mathcal{I}[f](x)| \leq C|h|^{n-1} \|f\|_{L^1(\Omega)} \|\omega\|_{W^{1, \infty}(\Omega)}.
\]

This completes the proof. \(\square\)
2.2. Local well-posedness and blow-up criteria. To prove a local well-posedness of (1.3)–(1.5), we shall employ a generalized variation-of-constants formula (2.6) for Neumann type parabolic boundary value problem established by Amann ([2, 3] e.g.). In the below we introduce the interpolation scale of spaces $E_d$ and the set of linear operators $A_d$ needed for writing down the formula. We work with $E_d$ and $A_d$ for $-1 < d, \delta < 0$, while they can be defined for $d, \delta > -1$.

We denote the boundary trace operator by $\gamma$. Note that
\begin{equation}
\gamma \in \mathcal{L}(W^{1,p}(\Omega), W^{1-\frac{1}{p},p}(\partial \Omega)) \text{ for } 1 < p < \infty.
\end{equation}

We also denote the boundary trace of normal derivative by $B$;
\[ Bf := \gamma \nabla f \cdot \nu \text{ for } f \in W^{s,p}(\Omega), 1 + \frac{1}{p} < s \leq 2, 1 < p < \infty. \]

Consider the sectorial operator $A := (I - \Delta)|_{D(A)}$ with its domain
\[ D(A) := \{ f \in W^{2,p}(\Omega) \mid Bf = 0 \text{ on } \partial \Omega \} \text{ for } 1 < p < \infty. \]

Note that $\inf \text{Re } \Sigma(A) > 0$, where $\Sigma(A)$ is the spectrum of $A$. In [3, Section 4] the pair $(A, B)$ is found to satisfy the condition of normally elliptic problem, which enables one to construct an interpolation scale of spaces. Let
\[ W^s_B := \begin{cases} 
\{ f \in W^{s,p}(\Omega) \mid Bf = 0 \}, & 1 + \frac{1}{p} < s \leq 2, \\
W^{s,p}(\Omega), & -1 + \frac{1}{p} < s < 1 + \frac{1}{p}, \\
(W^{-s,p'}(\Omega))', & -2 + \frac{1}{p} < s \leq -1 + \frac{1}{p},
\end{cases} \]

and take $E_0 = L^p(\Omega) = W^{0,p}_B$, $E_1 = W^{2,p}_B$. Following [2, Section 6], we construct an interpolation scale of spaces
\[ E_\theta := (E_0, E_1)_{\theta,p} = W^{2\theta,p}_B \]

for $2\theta \in (0, 2) \setminus \{1, 1 + \frac{1}{p}\}$, where $\langle \cdot, \cdot \rangle_{\theta,p}$ denotes the real interpolation functor. Introducing a completion of the normed space $(E_0, \|A^{-1} \cdot \|_{E_0})$, which is denoted by $E_{-1}$, we can inductively extend the definition of $E_{k+\theta}$ and $A_{k+\theta}$ for $-1 < k + \theta < \infty$, $k = -1, 0, 1, \cdots$ (see [3, (6.4)]). Then, we have a family of operators
\[ A_{k+\theta} \in \mathcal{L}(E_{k+1+\theta}, E_{k+\theta}) \]

such that $-A_{k+\theta}$ is the infinitesimal generator of an analytic semigroup
\[ \{ e^{-tA_{k+\theta}} \mid t \geq 0 \} \text{ on } E_{k+\theta}, \]

and $A_{k+\theta}$ is a $W^{2(k+\theta),p}$ realization of $A$ for $-1 < k + \theta < \infty$, $k = -1, 0, 1, \cdots$, $0 < \theta < 1$, and $1 < p < \infty$. Let us specify
\begin{equation}
A_{\alpha-1} = W^{2(\alpha-1),p}_B \text{ realization of } A \text{ for } \alpha \in \left( 1, 1 + \frac{1}{p} \right) \text{ with } n < p < \infty.
\end{equation}

The semigroup $e^{-tA_{\alpha-1}}$ satisfies the smoothing estimate ([15, Lemma 3.1]):
If $1 < p < \infty$, $1 < \beta < 2\alpha < 1 + \frac{1}{p}$, $f \in W^{2\alpha-2,p}_B$, then there exist positive constants $\sigma = \sigma(\beta) < 1$, $\kappa < 1$ and $C(\alpha, \beta, p)$ such that
\begin{equation}
\| e^{-tA_{\alpha-1}} f \|_{W^{\beta,p}(\Omega)} \leq Ct^{-\sigma} e^{-\kappa t} \| f \|_{W^{2\alpha-2,p}_B}, \quad t > 0.
\end{equation}

$^4 A_\beta$ is a $W^{s,p}$- realization of $A$ if $A = A_\beta$ in $D(A)$ and the range of $A_\beta$ is in $W^{s,p}$. 
We use (2.3) in the proof of Lemma 3 to control the nonlinear terms. Lastly, we define $B^c$ by the continuous extension of $(B|_{\text{Ker}(I - \Delta)})^{-1}$ to $W^{2\alpha - 1 - \frac{2 - p}{p}}(\partial \Omega)$. Note that

$$(2.4) \quad B^c \in \mathcal{L}(W^{2\alpha - 1 - \frac{2 - p}{p}}(\partial \Omega), W^{2\alpha,p}(\Omega))$$

for $\alpha, p$ of same range in (2.2).

Let $\mathcal{N} \in C^1(\overline{\Omega})$ be a fixed vector field satisfying $\mathcal{N} = \nu$ on $\partial \Omega$ constructed in Lemma 1. Now we consider the inhomogeneous Neumann boundary value problems for (1.3)–(1.5):

$$(2.5) \quad \partial_t u + (I - \Delta) u = g_1, \quad \partial_t v + (I - \Delta) v = g_2, \quad x \in \Omega, \quad t \leq T,$$

$$(2.6) \quad \partial_n u = h_1, \quad \partial_n v = h_2, \quad x \in \partial \Omega, \quad t \leq T,$$

$$(2.7) \quad u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x), \quad x \in \partial \Omega,$$

and its generalized variation-of-constants formulas:

$$(2.8) \quad u = e^{-tA_{\alpha - 1}}u_0 + \int_0^t e^{-(t - \tau)A_{\alpha - 1}}(g_1(\tau) + A_{\alpha - 1}B^c \gamma h_1(\tau))d\tau,$$

$$(2.9) \quad v = e^{-tA_{\alpha - 1}}v_0 + \int_0^t e^{-(t - \tau)A_{\alpha - 1}}(g_2(\tau) + A_{\alpha - 1}B^c \gamma h_2(\tau))d\tau,$$

where

$$(2.10) \quad g_1 := -\nabla \cdot (u \mathcal{K}[u, v]) + (1 - m)u + \frac{\lambda}{k}u(k - (u + v)),$$

$$(2.11) \quad g_2 := -\nabla \cdot (v \mathcal{S}[u, v]) + mu + v + \frac{\mu}{k}v(k - (u + v)),$$

$$(2.12) \quad h_1 := u\mathcal{K}[u, v] \cdot \mathcal{N}, \quad h_2 := v\mathcal{S}[u, v] \cdot \mathcal{N}.$$

The formal argument to write (2.5) into (2.6) is presented in [3, (11.16)–(11.20)].

**Definition 2.** Let $\Omega, \mathcal{K}, \mathcal{S}$ and $\omega$ be given as for Theorem 4. Assume $u_0$ and $v_0$ are functions belonging to $W^{2,p}(\Omega), p > n$. We call $u, v$ in $L^\infty(0, T; W^{1,p}(\Omega))$ satisfying the integral equation (2.6), (2.7) a generalized solution of (2.5) for $T > 0$.

What follows we show that (2.5) has a pair of unique local-in-time generalized solution, which coincides with a pair of strong solution satisfying maximal regularity estimates.

**Lemma 3.** Let $\Omega, \mathcal{K}, \mathcal{S}$ and $\omega$ be given as same for Theorem 4. Assume that $u_0$ and $v_0$ are non-negative functions belong to $W^{2,p}(\Omega), p > n$, and satisfy the compatibility condition

$$\partial_n u_0 = u_0\mathcal{K}[u_0, v_0] \cdot \nu, \quad \partial_n v_0 = v_0\mathcal{S}[u_0, v_0] \cdot \nu, \quad x \in \partial \Omega.$$

Then, there exists the maximal time of existence, $T_{\text{max}} \leq \infty$, such that a pair of unique non-negative strong solution $(u, v)$ of (2.5) exists and satisfies

$$(2.13) \quad u, v \in C([0, t]; W^{1,p}(\Omega)) \cap W^{1,p}(0, t; L^p(\Omega)) \cap L^p(0, t; W^{2,p}(\Omega)), \quad t < T_{\text{max}}.$$

Moreover, if $T_{\text{max}} < \infty$, then

$$(2.14) \quad \lim_{t \to T_{\text{max}}} (\|u(\cdot, t)\|_{W^{1,p}(\Omega)} + \|v(\cdot, t)\|_{W^{1,p}(\Omega)}) = \infty.$$

**Proof.** We divide the proof into three parts. We first obtain a unique generalized solution $(u, v)$, and then show that it is indeed a strong solution. In the last step, the non-negativity of solutions is shown.

**Step 1** (Generalized solution) Let $n < p < \infty$. With positive constants $T < 1$ and $R_0$ to be specified below, we introduce the Banach space $X_T := C([0, T]; W^{1,p}(\Omega))$ and its closed convex subset $S_T \subset X_T,$

$$S_T := \{ f \in X_T \mid \|f\|_{L^\infty(0, T; W^{1,p}(\Omega))} \leq R_0 \}.$$
Let $u, v \in S_T$, $2\alpha \in (1, 1 + \frac{1}{p})$ and $0 < t < T$. As in (2.6)--(2.7), we consider

$$
\Phi_1(u, v) := e^{-tA_{\alpha-1}}u_0 + \int_0^t e^{-(t-\tau)A_{\alpha-1}}(g_1(\tau) + A_{\alpha-1}B^c\gamma h_1(\tau))d\tau,
$$

$$
\Phi_2(u, v) := e^{-tA_{\alpha-1}}v_0 + \int_0^t e^{-(t-\tau)A_{\alpha-1}}(g_2(\tau) + A_{\alpha-1}B^c\gamma h_2(\tau))d\tau,
$$

where $\gamma$, $A_{\alpha-1}$, and $B^c$ are as previously defined with (2.1), (2.2) and (2.3).

We now show $\Phi_1(u, v), \Phi_2(u, v) \in S_T$. Let $1 < \beta < 2\alpha$. Using (2.3) and $W^{\beta, p}(\Omega) \hookrightarrow W^{1, p}(\Omega)$, we compute

$$
\|\Phi_1(u, v)(t)\|_{W^{1, p}(\Omega)}
\leq \|e^{-tA_{\alpha-1}}u_0\|_{W^{1, p}(\Omega)} + \int_0^t \|e^{-(t-\tau)A_{\alpha-1}}(g_1(\tau) + A_{\alpha-1}B^c\gamma h_1(\tau))\|_{W^{1, p}(\Omega)} d\tau
\leq M_1 \|u_0\|_{W^{1, p}(\Omega)} + C \int_0^t \|e^{-(t-\tau)A_{\alpha-1}}(g_1(\tau) + A_{\alpha-1}B^c\gamma h_1(\tau))\|_{W^{\beta, p}(\Omega)} d\tau
\leq M_1 \|u_0\|_{W^{1, p}(\Omega)} + C \int_0^t e^{-\kappa(t-\tau)(t - \tau)^{-\sigma}}(\|g_1(\tau)\|_{W^{2\alpha-2, p}B} + \|A_{\alpha-1}B^c\gamma h_1(\tau)\|_{W^{2\alpha-2, p}B})d\tau.
$$

Note that $A_{\alpha-1}B^c\gamma$ is well defined due to Lemma 2 and

$$
W^{1, \frac{1}{p}}(\partial \Omega) \hookrightarrow W^{2\alpha-1-\frac{1}{p}, p}(\partial \Omega), \quad A_{\alpha-1}B^c \in \mathcal{L}(W^{2\alpha-1-\frac{1}{p}, p}(\partial \Omega), W^{2\alpha-2, p}_B).
$$

Using $L^p(\Omega) = W^{0, p}_B(\Omega)$ and $W^{1, p}(\Omega) \hookrightarrow L^\infty(\Omega)$, we can estimate

$$
\|g_1(\tau)\|_{W^{2\alpha-2, p}_B} \leq C \|g_1(\tau)\|_{L^p(\Omega)} \leq C(R_0 + R_0^2).
$$

Using Lemma 2 and (2.11), we also compute that

$$
\|A_{\alpha-1}B^c\gamma h_1(\tau)\|_{W^{2\alpha-2, p}_B} \leq C \|uK[u, v](\tau) \cdot N\|_{W^{1, p}(\Omega)} \leq CR_0^2.
$$

Then, combining the above computations leads to

$$
\|\Phi_1(u, v)(t)\|_{W^{1, p}(\Omega)} \leq M_1 \|u_0\|_{W^{1, p}(\Omega)} + C_1(R_0 + R_0^2)t^{1-\sigma},
$$

where $C_1$ is a positive constant independent of $R_0$. Analogously to above, we can see that

$$
\|\Phi_2(u, v)(t)\|_{W^{1, p}(\Omega)} \leq M_2 \|v_0\|_{W^{1, p}(\Omega)} + C_2(R_0 + R_0^2)t^{1-\sigma},
$$

where $C_2$ is a positive constant independent of $R_0$. Choosing

$$
R_0 := M_1 \|u_0\|_{W^{1, p}(\Omega)} + M_2 \|v_0\|_{W^{1, p}(\Omega)} + 1,
$$

$$
T < T_1 := \min \left\{1, \frac{1}{(2C_1(R_0 + R_0^2))^{1-\sigma}}, \frac{1}{(2C_2(R_0 + R_0^2))^{1-\sigma}} \right\},
$$

and taking supremum over $0 < t < T$, we have $\Phi_1(u, v), \Phi_2(u, v) \in S_T$.

We next show the mapping $(u, v) \mapsto (\Phi_1, \Phi_2)$ is contraction. Note from (2.9)--(2.10) that

$$
\Phi_1(u, v)(t) - \Phi_1(\tilde{u}, \tilde{v})(t) = \int_0^t e^{-(t-\tau)A_{\alpha-1}}((g_1 - \tilde{g}_1)(\tau) + A_{\alpha-1}B^c\gamma (h_1 - \tilde{h}_1)(\tau))d\tau,
$$

$$
\Phi_2(u, v)(t) - \Phi_2(\tilde{u}, \tilde{v})(t) = \int_0^t e^{-(t-\tau)A_{\alpha-1}}((g_2 - \tilde{g}_2)(\tau) + A_{\alpha-1}B^c\gamma (h_2 - \tilde{h}_2)(\tau))d\tau,
$$

where

$$
\tilde{g}_1 = -\nabla \cdot (\tilde{u}K[\tilde{u}, \tilde{v}]) + (1 - m)\tilde{u} + \frac{\lambda}{k}(\tilde{u})(k - (\tilde{u} + \tilde{v})), \quad \tilde{h}_1 := \tilde{u}K[\tilde{u}, \tilde{v}] \cdot N,
$$

and

$$
\tilde{g}_2 = -\nabla \cdot (\tilde{v}K[\tilde{u}, \tilde{v}]) + (1 - m)\tilde{v} + \frac{\lambda}{k}(\tilde{v})(k - (\tilde{u} + \tilde{v})), \quad \tilde{h}_2 := \tilde{v}K[\tilde{u}, \tilde{v}] \cdot N.
$$
Thus, we have
\[ \frac{\partial}{\partial t} \tilde{S}[\tilde{u}, \tilde{v}] + \nabla \cdot (\tilde{v} \tilde{S}[\tilde{u}, \tilde{v}]) + m \tilde{u} + \tilde{v} + \frac{H}{k} \tilde{v}(k - (\tilde{u} + \tilde{v})), \]
\[ \hat{h}_2 := \tilde{v} \tilde{S}[\tilde{u}, \tilde{v}] \cdot \mathcal{N}. \]

Then, by similar computations as above, we have
\[ \sup_{t \leq T} \|(\Phi_1, \Phi_2)(u, v)(t) - (\Phi_1, \Phi_2)(\tilde{u}, \tilde{v})(t)\|_{W^1,p(\Omega)} \leq C_3(R_0 + 1)T^{-\sigma} \sup_{t \leq T} \|(u, v)(t) - (\tilde{u}, \tilde{v})(t)\|_{W^1,p(\Omega)}, \]
where \( C_3 > 0 \) is a constant independent of \( R_0 \). Taking
\[ T < T_2 := \min \left\{ T_1, \frac{1}{(2C_3(R_0 + 1))^{1/\sigma}} \right\}, \]
we obtain
\[ \sup_{t \leq T} \|(\Phi_1, \Phi_2)(u, v)(t) - (\Phi_1, \Phi_2)(\tilde{u}, \tilde{v})(t)\|_{W^1,p(\Omega)} \leq \frac{1}{2} \sup_{t \leq T} \|(u, v)(t) - (\tilde{u}, \tilde{v})(t)\|_{W^1,p(\Omega)}, \]
i.e., the mapping is contraction. According to the Banach fixed point theorem, this mapping has a fixed point in \( S_T \), denoted again as \((u, v)\). Thus, the generalized solution \((u, v)\) for (2.5) is obtained. By the standard extension argument and the fact that the above choice of \( T \) depends only on \( \|u_0\|_{W^{1,p}(\Omega)} \), and \( \|v_0\|_{W^{1,p}(\Omega)} \), it should be noted that there exists \( T_M \leq \infty \) such that \( u, v \in C([0, T_M]; W^{1,p}(\Omega)) \), and
\[ (2.12) \quad \text{either } T_M = \infty, \text{ or } \lim_{t \to T_M} (\|u(\cdot, t)\|_{W^{1,p}(\Omega)} + \|v(\cdot, t)\|_{W^{1,p}(\Omega)}) = \infty. \]

We next show the uniqueness. Let \( T < T_M \), and let \((u, v)\) and \((\tilde{u}, \tilde{v})\) be two constructed solutions for \( t \leq T \). Analogously to above, we can estimate
\[ \|(u - \tilde{u})(t)\|_{W^{1,p}(\Omega)} \leq \int_0^t e^{-\kappa(t-\tau)}(t - \tau)^{-\sigma}(\|(g_1 - \tilde{g}_1)(\tau)\|_{L^p(\Omega)} + \|A_{a-1}B^\sigma(\gamma(h_1 - \tilde{h}_1)(\tau))\|_{L^p(\Omega)})d\tau, \]
and
\[ \|A_{a-1}B^\sigma(\gamma(h_1 - \tilde{h}_1)(\tau))\|_{L^p(\Omega)} \leq \|(h_1 - \tilde{h}_1)(\tau)\|_{W^{1,p}(\Omega)} \]
\[ \leq C \sup_{t \leq T} \|(u(t))|_{W^{1,p}(\Omega)} + \|v(t))|_{W^{1,p}(\Omega)} + \|\tilde{u}(t))|_{W^{1,p}(\Omega)} + \|\tilde{v}(t))|_{W^{1,p}(\Omega)} \]
\[ \times (\|(u - \tilde{u})(\tau)\|_{W^{1,p}(\Omega)} + \|(v - \tilde{v})(\tau)\|_{W^{1,p}(\Omega)}) \]
\[ \leq C(\|(u - \tilde{u})(\tau)\|_{W^{1,p}(\Omega)} + \|(v - \tilde{v})(\tau))|_{W^{1,p}(\Omega)}), \]
and
\[ \|A_{a-1}B^\sigma(\gamma(h_1 - \tilde{h}_1)(\tau))\|_{L^p(\Omega)} \]
\[ \leq C \sup_{t \leq T} \|(u(t))|_{W^{1,p}(\Omega)} + \|v(t))|_{W^{1,p}(\Omega)} + \|\tilde{u}(t))|_{W^{1,p}(\Omega)} + \|\tilde{v}(t))|_{W^{1,p}(\Omega)} \]
\[ \times (\|(u - \tilde{u})(\tau)\|_{W^{1,p}(\Omega)} + \|(v - \tilde{v})(\tau)\|_{W^{1,p}(\Omega)}) \]
\[ \leq C(\|(u - \tilde{u})(\tau)\|_{W^{1,p}(\Omega)} + \|(v - \tilde{v})(\tau))|_{W^{1,p}(\Omega)}). \]

Thus, we have
\[ \|(u - \tilde{u})(t)\|_{W^{1,p}(\Omega)} \leq \int_0^t e^{-\kappa(t-\tau)}(t - \tau)^{-\sigma}(\|(u - \tilde{u})(\tau)\|_{W^{1,p}(\Omega)} + \|(v - \tilde{v})(\tau)\|_{W^{1,p}(\Omega)})d\tau. \]

Similarly, we also have
\[ \|(v - \tilde{v})(t)\|_{W^{1,p}(\Omega)} \leq \int_0^t e^{-\kappa(t-\tau)}(t - \tau)^{-\sigma}(\|(u - \tilde{u})(\tau)\|_{W^{1,p}(\Omega)} + \|(v - \tilde{v})(\tau)\|_{W^{1,p}(\Omega)})d\tau. \]
Adding above two estimates and using the Grönwall type inequality, \((u, v) = (\bar{u}, \bar{v})\) is obtained for \(t \leq T\). Since \(T < T_M\) is arbitrary, we have the uniqueness of solutions.

**Step 2** (Strong solution) We next consider the regularity of the constructed solution \((u, v)\).

Let \(t \leq T < T_M\). As \(1 < \beta < 2\alpha < 1 + \frac{1}{p}\), we first note that

\[
\|e^{-tA_{\alpha-1}}u_0\|_{W^{\beta,p}(\Omega)} \leq C\|u_0\|_{W^{2,p}(\Omega)}, \quad \|e^{-tA_{\alpha-1}}v_0\|_{W^{\beta,p}(\Omega)} \leq C\|v_0\|_{W^{2,p}(\Omega)}.
\]

If we replace the computations for the initial counterparts in the previous step by (2.13), then we have \(u, v \in C([0, T]; W^{\beta,p}_B)\). Thus, as in [2] (3.5), it can be shown that

\[ u, v \in C^1([0, T]; W^{\beta-2,p}_B) \quad \text{as well as} \quad u, v \in C^{\frac{1}{2}}([0, T]; W^{\beta-1+\frac{1}{p}, p}_B). \]

Therefore, we have \(u, v \in W^{\frac{1}{2}-\frac{1}{p}, p}(0, T; W^{\beta-1,p}(\partial\Omega))\) and, in particular,

\[ h_1, h_2 \in W^{\frac{1}{2}-\frac{1}{p}, p}(0, T; W^{\beta-1,p}(\partial\Omega)). \]

Due to these facts along with \(h_1, h_2 \in L^\infty(0, T; W^{1-\frac{1}{p}, p}(\partial\Omega))\) and \(g_1, g_2 \in L^\infty(0, T; L^p(\Omega))\), applying the maximal regularity theorem [19, Theorem 2.1] to

\[
\begin{align*}
\partial_t f_1 + (I - \Delta) f_1 &= g_1, & \partial_t f_2 + (I - \Delta) f_2 &= g_2, & x \in \Omega, & t \leq T, \\
\partial_x f_1 &= h_1, & \partial_x f_2 &= h_2, & x \in \partial\Omega, & t \leq T, \\
f_1(x, 0) &= u_0(x), & f_2(x, 0) &= v_0(x), & x \in \partial\Omega,
\end{align*}
\]

we have the unique strong solution \((f_1, f_2)\) for (2.13) in the class

\[ f_1, f_2 \in W^{1,p}(0, T; L^p(\Omega)) \cap L^p(0, T; W^{2,p}(\Omega)), \quad T < T_M. \]

Since the strong solution is the generalized solution \((\bar{u}, \bar{v})\), we put (2.14) into the generalized variation-of-constants formulas with respect to \(f_1, f_2\) with the same right hand side terms as in (2.6), (2.7). Then we have \((\tilde{f}_1, \tilde{f}_2) = (u, v)\) due to the uniqueness of the generalized solution and thus

\[ u, v \in W^{1,p}(0, T; L^p(\Omega)) \cap L^p(0, T; W^{2,p}(\Omega)), \quad T < T_M. \]

**Step 3** (Non-negativity) It remains to show the non-negativity of the constructed solution \((u, v)\). Let \(t \leq T < T_M\). Define \(u_- := -\min \{ u, 0 \}\). Multiplying \(u\)-equation in (2.5) by \(u_-\) and integrating over \(\Omega\), using the direct computation and Young’s inequality, we can compute

\[
\frac{1}{2} \frac{d}{dt} \int_\Omega |u_-|^2 + \int_\Omega |\nabla u_-|^2 \leq \frac{1}{2} \int_\Omega |\nabla u_-|^2 + C(1 + \|K[u, u]\|_{L^\infty(\Omega)}^2 + \|u\|_{L^\infty(\Omega)} + \|v\|_{L^\infty(\Omega)}) \int_\Omega |u_-|^2, \quad t \leq T,
\]

where we used Lemma 2 and \(W^{1,p}(\Omega) \hookrightarrow L^\infty(\Omega)\). Then, \(u \geq 0\) can be obtained by using Grönwall’s lemma. Similarly, testing \(v_- := -\min \{ v, 0 \}\) to \(v\)-equation in (2.5) and using Young’s inequality, we observe that

\[
\frac{1}{2} \frac{d}{dt} \int_\Omega |v_-|^2 + \int_\Omega |\nabla v_-|^2 \leq \frac{1}{2} \int_\Omega |\nabla v_-|^2 + C(1 + \|S[u, v]\|_{L^\infty(\Omega)}^2 + \|u\|_{L^\infty(\Omega)} + \|v\|_{L^\infty(\Omega)}) \int_\Omega |v_-|^2 - m \int_\Omega uv_- \leq \frac{1}{2} \int_\Omega |\nabla v_-|^2 + C \int_\Omega |v_-|^2, \quad t \leq T.
\]
Again by Grönwall’s lemma, we have \( v \geq 0 \). Since \( T < T_M \) is arbitrary, we obtain the non-negativity of solutions. Finally we end the proof by taking \( T_{\text{max}} \) as \( T_M \) in (2.12). This completes the proof.

We next prove a refined blow-up criteria.

**Lemma 4.** Let the same assumptions as in Lemma 3 be satisfied. The solution \((u, v)\) of (2.5) given by Lemma 3 satisfies

\[
\text{either } T_{\text{max}} = \infty, \text{ or } \lim_{t \to T_{\text{max}}^-} (\|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{L^\infty(\Omega)}) = \infty.
\]

**Proof.** By (2.8), it suffices to show that if

\[
\|u\|_{L^\infty(0, T_{\text{max}}; L^\infty(\Omega))} + \|v\|_{L^\infty(0, T_{\text{max}}; L^\infty(\Omega))} \leq C, \quad T_{\text{max}} < \infty,
\]
then

\[
\|u\|_{L^\infty(0, T_{\text{max}}; W^{1,p}(\Omega))} + \|v\|_{L^\infty(0, T_{\text{max}}; W^{1,p}(\Omega))} \leq C.
\]

Let \( p > n, 2\alpha \in (1, 1 + \frac{1}{p}), 1 < \beta < 2\alpha \), and let \( \varepsilon > 0 \) be a number such that \( 0 < T_{\text{max}} - \varepsilon < t < T_{\text{max}} \). Using (2.10), (2.11), and (2.17), we compute

\[
\|u(t)\|_{W^{1,p}(\Omega)} \leq M_1 \|u(T_{\text{max}} - \varepsilon)\|_{W^{1,p}(\Omega)} + C \int_{T_{\text{max}} - \varepsilon}^t \|e^{-(t-\tau)A_{\alpha-1}(g_1(\tau) + A_{\alpha-1}B^c\gamma h_1(\tau))}\|_{W^{1,p}(\Omega)} d\tau
\]

\[
\leq M_1 \|u(T_{\text{max}} - \varepsilon)\|_{W^{1,p}(\Omega)} + C \int_{T_{\text{max}} - \varepsilon}^t e^{-(t-\tau)}(t-\tau)^{\alpha-1}(\|g_1(\tau)\|_{W^{2\alpha-2,p}_B} + \|A_{\alpha-1}B^c\gamma h_1(\tau))\|_{W^{2\alpha-2,p}_B}) d\tau,
\]

where \( \gamma \in L(W^{1,p}(\Omega), W^{1-\frac{2}{p},p}(\partial \Omega)) \) is the boundary trace operator, \( B^c \) is the continuous extension of \((B|_{Ker(I-\Delta)}^{-1})^{-1}\) to \( W^{2\alpha-1-\frac{1}{p},p}(\partial \Omega)\), and \( h_1, g_1 \) are given in (2.7). Using \( L^p(\Omega) = W^{1,0}_B \hookrightarrow W^{2\alpha-2,p}_B \), Lemma 2 (2.11), and (2.17), we compute

\[
\|g_1(\tau)\|_{W^{2\alpha-2,p}_B} \leq C \|g_1(\tau)\|_{L^p(\Omega)} + C(\|u(\tau)\|_{L^\infty(\Omega)} + \|u(\tau)\|_{L^\infty(\Omega)} \|v(\tau)\|_{L^\infty(\Omega)})
\]

\[
\leq C + C(\|u(\tau)\|_{L^\infty(\Omega)} \|K[u, v](\tau)\|_{L^\infty(\Omega)} + \|u(\tau)\|_{L^\infty(\Omega)} \|K[u, v](\tau)\|_{W^{1,p}(\Omega)})
\]

and

\[
\|A_{\alpha-1}B^c\gamma h_1(\tau))\|_{W^{2\alpha-2,p}_B} \leq C \|uK[u, v](\tau)\|_{W^{1,p}(\Omega)} \cdot \mathcal{N}|_{W^{1,p}(\Omega)}
\]

\[
\leq C(\|u(\tau)\|_{W^{1,p}(\Omega)} \|K[u, v](\tau)\|_{L^\infty(\Omega)} + \|u(\tau)\|_{L^\infty(\Omega)} \|K[u, v](\tau)\|_{W^{1,p}(\Omega)})
\]

\[
\leq C + C(\|u(\tau)\|_{W^{1,p}(\Omega)} + \|v(\tau)\|_{W^{1,p}(\Omega)}).
\]

It then follows that

\[
\|u(t)\|_{W^{1,p}(\Omega)} \leq M_1 \|u(T_{\text{max}} - \varepsilon)\|_{W^{1,p}(\Omega)} + C_4\varepsilon^{1-\sigma} \sup_{T_{\text{max}} - \varepsilon < t < T_{\text{max}}} (\|u(t)\|_{W^{1,p}(\Omega)} + \|v(t)\|_{W^{1,p}(\Omega)}),
\]
where $C_4$ is a positive constant independent of $\varepsilon$. Analogously to above, we can compute
\[
\|v(t)\|_{W^{1,p}(\Omega)} 
\leq M_2 \|v(T_{\text{max}} - \varepsilon)\|_{W^{1,p}(\Omega)} + C_5 \varepsilon^{1-\sigma} \sup_{T_{\text{max}}-\varepsilon < t < T_{\text{max}}} (\|u(t)\|_{W^{1,p}(\Omega)} + \|v(t)\|_{W^{1,p}(\Omega)}),
\]
where $C_5$ is a positive constant independent of $\varepsilon$. Adding above two inequalities and taking supremum over $T_{\text{max}} - \varepsilon < t < T_{\text{max}}$, we have
\[
\sup_{T_{\text{max}}-\varepsilon < t < T_{\text{max}}} (\|u(t)\|_{W^{1,p}(\Omega)} + \|v(t)\|_{W^{1,p}(\Omega)}) 
\leq M_1 \|u(T_{\text{max}} - \varepsilon)\|_{W^{1,p}(\Omega)} + M_2 \|v(T_{\text{max}} - \varepsilon)\|_{W^{1,p}(\Omega)} + (C_4 + C_5) \varepsilon^{1-\sigma} \sup_{T_{\text{max}}-\varepsilon < t < T_{\text{max}}} (\|u(t)\|_{W^{1,p}(\Omega)} + \|v(t)\|_{W^{1,p}(\Omega)}).
\]
Therefore, taking sufficiently small $\varepsilon$, (2.18) is obtained. This completes the proof. \qed

2.3. A priori estimates. Next, we provide some a priori estimates (Lemma 5 and Lemma 6).

**Lemma 5.** Let the same assumptions as in Lemma 3 be satisfied. The solution $(u, v)$ of (2.5) given by Lemma 3 for $T < T_{\text{max}}$ satisfies
\[
(2.19) \quad \sup_{t \leq T} \int_{\Omega} u(\cdot, t) \leq C(\|u_0\|_{L^1(\Omega)}),
\]
\[
(2.20) \quad \sup_{t \leq T} \int_{\Omega} v(\cdot, t) \leq C(\|u_0\|_{L^1(\Omega)}, \|v_0\|_{L^1(\Omega)}),
\]
\[
(2.21) \quad \sup_{t \leq T} \|\mathcal{K}[u, v](\cdot, t)\|_{L^\infty(\Omega)} \leq C(\|u_0\|_{L^1(\Omega)}, \|v_0\|_{L^1(\Omega)}),
\]
and
\[
(2.22) \quad \sup_{t \leq T} \|\mathcal{S}[u, v](\cdot, t)\|_{L^\infty(\Omega)} \leq C(\|u_0\|_{L^1(\Omega)}, \|v_0\|_{L^1(\Omega)}).
\]

**Proof.** Integrating (1.3)\_1 and (1.3)\_2 over $\Omega$, we obtain
\[
(2.23) \quad \frac{d}{dt} \int_{\Omega} u + \frac{\lambda}{k} \int_{\Omega} u + v = (\lambda - m) \int_{\Omega} u,
\]
and
\[
(2.24) \quad \frac{d}{dt} \int_{\Omega} v + \frac{\mu}{k} \int_{\Omega} u + v = m \int_{\Omega} u + \mu \int_{\Omega} v.
\]
As we have
\[
(\lambda - M) \int_{\Omega} u \leq \frac{\lambda}{k} \int_{\Omega} u^2 + C,
\]
it follows from (2.23) that
\[
y' + y \leq C, \quad y(t) := \int_{\Omega} u(\cdot, t).
\]
Therefore, (2.19) is obtained by standard ODE argument. Similarly, as Young’s inequality gives
\[
(\mu + 1) \int_{\Omega} v \leq \frac{\mu}{k} \int_{\Omega} v^2 + C,
\]
it follows from (2.19) and (2.21) that
\[
\frac{d}{dt} \int_{\Omega} v + \int_{\Omega} v \leq C(\|u_0\|_{L^1(\Omega)}).
\]
Thus, we can also obtain (2.20). Then, (2.21) and (2.22) are direct consequence of (2.19)–(2.20) and (1.5). This completes the proof.

Lemma 6. Let the same assumptions as in Lemma 3 be satisfied. The solution \((u, v)\) of (2.5) given by Lemma 3 for \(T < T_{\text{max}}\) satisfies

\[
\sup_{t \leq T} \| u(\cdot, t) \|_{L^\infty(\Omega)} \leq C(\| u_0 \|_{(L^1 \cap L^\infty)(\Omega)}, \| v_0 \|_{L^1(\Omega)}),
\]

and

\[
\sup_{t \leq T} \| v(\cdot, t) \|_{L^\infty(\Omega)} \leq C(\| u_0 \|_{(L^1 \cap L^\infty)(\Omega)}, \| v_0 \|_{(L^1 \cap L^\infty)(\Omega)}).
\]

Proof. Let \(p > 1\) and \(t \leq T < T_{\text{max}}\). Multiplying (1.3) by \(u^{p-1}\), integrating over \(\Omega\), and using integrating by parts, we have

\[
\frac{1}{p} \frac{d}{dt} \int_{\Omega} u^p + \frac{4(p-1)}{p^2} \int_{\Omega} |\nabla u|^2 + \frac{\lambda}{k} \int_{\Omega} u^p (u + v) = (\lambda - m) \int_{\Omega} u^p + (p-1) \int_{\Omega} u^{p-1} \nabla u \cdot \nabla u_K[u, v].
\]

Using Young’s inequality and (2.21), we can compute the rightmost term as

\[
(p-1) \int_{\Omega} u^{p-1} \nabla u \cdot \nabla u_K[u, v] \leq \frac{p-1}{p^2} \int_{\Omega} |\nabla u|^2 + C(p-1) \int_{\Omega} u^p,
\]

and thus, it follows that

\[
\frac{1}{p} \frac{d}{dt} \int_{\Omega} u^p + \frac{3(p-1)}{p^2} \int_{\Omega} |\nabla u|^2 \leq C_6(p+1) \int_{\Omega} u^p,
\]

where \(C_6 > 0\) is a constant independent of \(p\). Then, (2.25) can be deduced by Moser-Alikakos iteration argument [1]. Indeed, for \(p = p_k := 2^k, k = 1, 2, 3, \ldots\), the last inequality becomes

\[
\frac{d}{dt} \int_{\Omega} u^{p_k} + \frac{3(p_k-1)}{p_k} \int_{\Omega} |\nabla u|^{p_k-1}|u|^{2(p_k-1)} \leq C_6 p_k (p_k + 1) \int_{\Omega} u^{p_k}.
\]

Using the Gagliardo-Nirenberg interpolation inequality and Young’s inequality, we note that

\[
\|u^{p_k-1}\|_{L^2(\Omega)}^2 \leq C \|u^{p_k-1}\|_{L^1(\Omega)}^{\frac{n+2}{n-1}} \|\nabla u^{p_k-1}\|_{L^2(\Omega)}^{\frac{2(n+2)}{n-1}} + C \|u^{p_k-1}\|_{L^1(\Omega)}^{2},
\]

for some constant \(C > 0\) independent of \(p_k\). Plugging into (2.27), due to \(\frac{3(p_k-1)}{p_k} \geq \frac{3}{2}\), we have

\[
\frac{d}{dt} \int_{\Omega} u^{p_k} + 2C_6 p_k (p_k + 1) \int_{\Omega} u^{p_k} \leq C_7 p_k^{\frac{n+2}{n-1} + 2} \left( \int_{\Omega} u^{p_k} \right)^2,
\]

where \(C_7 > 0\) is a constant independent of \(p_k\). We take a sufficiently large constant \(C_8 \geq 1\) independent of \(p_k\) satisfying \(C_8 p_k^{\frac{n+2}{n-1} + 2} \geq C_7 p_k^{\frac{n+2}{n-1} + 2} / (2C_6 p_k (p_k + 1))\) and define

\[
M := \max\{\|u_0\|_{L^1(\Omega)}, \|v_0\|_{L^\infty(\Omega)}, 1\}, \quad \delta_k := C_8 p_k^{\frac{n+2}{n-1}}.
\]

Then, it follows from (2.28) that

\[
\sup_{t \leq T} \int_{\Omega} u^{p_k}(\cdot, t) \leq \max \left\{ M p_k, \delta_k \left( \sup_{t \leq T} \int_{\Omega} u^{p_k-1}(\cdot, t) \right)^2 \right\}.
\]
Note that \( \delta_k \geq 1 \) for all \( k = 1, 2, 3, \ldots \). By an inductive computation, we have
\[
\sup_{t \leq T} \left( \int_{\Omega} v^{pk}(\cdot, t) \right)^{\frac{1}{pk}} \leq \left[ \delta_k \delta_{k-1}^{\alpha} \delta_{k-2}^{\alpha} \cdots \delta_{1}^{\alpha} \right] \left( 1 + \sup_{t \leq T} \| u(\cdot, t) \|_{L^1(\Omega)} \right)^{\frac{1}{pk}} M
\]
(2.29)
\[
\leq C_S \sum_{i=1}^{k} \frac{1}{2} \frac{n(n+2)}{n+1} \sum_{i=1}^{k} \frac{1}{2^i} \left( 1 + \sup_{t \leq T} \| u(\cdot, t) \|_{L^1(\Omega)} \right) M.
\]

Thus, by (2.19) and \( \sum_{i=1}^{k} \frac{1}{2^i} < \infty \), taking the limit \( k \to \infty \), we can obtain (2.25).

Similarly, we can see from (2.19) that
\[
\frac{1}{p} \frac{d}{dt} \int_{\Omega} v^p + \frac{4(p - 1)}{p^2} \int_{\Omega} |\nabla v^\frac{p}{2}|^2 + \frac{\mu}{k} \int_{\Omega} v^p(u + v)
\]
\[
= m \int_{\Omega} uv^{p-1} + \mu \int_{\Omega} v^p + (p - 1) \int_{\Omega} v^{p-1} \nabla v S[u, v].
\]

Using Young’s inequality and (2.22), we note that
\[
(p - 1) \int_{\Omega} v^{p-1} |\nabla v S[u, v]| \leq \frac{p - 1}{p^2} \int_{\Omega} |\nabla v^\frac{p}{2}|^2 + C(p - 1) \int_{\Omega} v^p.
\]

Analogously as above, using (2.25), we have
\[
\frac{d}{dt} \left( \int_{\Omega} v^{pk} + C_9 \right) + 2C_{10}p_k(p_k + 1) \left( \int_{\Omega} v^{pk} + C_9 \right) \leq C_{11}p_k^{\frac{n(n+2)}{n+1} + 2} \left( \int_{\Omega} v^{p_{k-1}} + C_9 \right)^2.
\]
(2.31)

We take a sufficiently large constant \( C_{12} \geq 1 \) independent of \( p_k \) satisfying
\[
C_{12}p_k^{\frac{n(n+2)}{n+1}} \geq C_{11}p_k^{\frac{n(n+2)}{n+1} + 2} / (2C_{10}p_k(p_k + 1))
\]
and define
\[
M := \max\{\|v_0\|_{L^1(\Omega)}, \|v_0\|_{L^\infty(\Omega)}, C_9 + 1\}, \quad \delta_k := C_{12}p_k^{\frac{n(n+2)}{n+1}}.
\]

Then, we obtain from (2.31) that
\[
\sup_{t \leq T} \left( \int_{\Omega} v^{pk}(\cdot, t) + C_9 \right) \leq \max\left\{ (2M)^{p_k}, \delta_k \sup_{t \leq T} \left( \int_{\Omega} v^{pk-1}(\cdot, t) + C_9 \right)^{\frac{1}{pk}} \right\}.
\]

Note that \( \delta_k \geq 1 \) for all \( k = 1, 2, 3, \ldots \). By an inductive computation, we have
\[
\sup_{t \leq T} \left( \int_{\Omega} v^{pk}(\cdot, t) \right)^{\frac{1}{pk}} \leq \left[ \delta_k \delta_{k-1}^{\alpha} \delta_{k-2}^{\alpha} \cdots \delta_{1}^{\alpha} \right] \left( 1 + \sup_{t \leq T} \| v(\cdot, t) \|_{L^1(\Omega)} + C_9 + 1 \right)^{\frac{1}{pk}} 2M
\]
(2.29)
\[
\leq C_{12} \sum_{i=1}^{k} \frac{1}{2} \frac{n(n+2)}{n+1} \sum_{i=1}^{k} \frac{1}{2^i} \left( \sup_{t \leq T} \| v(\cdot, t) \|_{L^1(\Omega)} + C_9 + 1 \right) 2M.
\]

Analogously to (2.29), we can see that
\[
\sup_{t \leq T} \left( \int_{\Omega} v^{pk}(\cdot, t) \right)^{\frac{1}{pk}} \leq \sup_{t \leq T} \left( \int_{\Omega} v^{pk}(\cdot, t) + C_9 \right)^{\frac{1}{pk}}.
\]
Due to (2.20) and \( \sum_{i=1}^{k} \frac{1}{2^i} < \infty \), we have (2.26) by taking the limit \( k \to \infty \). This completes the proof.

**Proof of Theorem 1.** It is a direct consequence of local-in-time existence, uniqueness, non-negativity (Lemma 3), blow-up criteria (Lemma 4), and \textit{a priori} estimates (Lemma 6). This completes the proof.

### 3. Independent Case

#### 3.1. Preliminary.

In the below we provide a \( W^{1,\infty}(\Omega) \) estimate for \( K[f,g] \) and \( S[f,g] \) with a shrinking sensing domain given by (1.11).

**Lemma 7.** Let \( \Omega \) be the open ball of radius \( L \), \( B_L(0) \), and let \( 0 < R < L \). Suppose that \( f, g \in W^{1,p}(\Omega), p > n \). Assume further that \( K[f,g], S[f,g], E, \) and \( \omega \) are given by (1.2), and (1.11)–(1.12). Then, there exists a constant \( C > 0 \) satisfying

\[
\|K[f,g]\|_{L^{\infty}(\Omega)} + \|S[f,g]\|_{L^{\infty}(\Omega)} \leq C(R,\Omega)(\|f\|_{L^{1}(\Omega)} + \|g\|_{L^{1}(\Omega)}),
\]

\[
\|K[f,g]\|_{W^{1,p}(\Omega)} + \|S[f,g]\|_{W^{1,p}(\Omega)} \leq C(R,\Omega)(\|f\|_{W^{1,p}(\Omega)} + \|g\|_{W^{1,p}(\Omega)}).
\]

**Proof.** It is suffice to show

(3.1) \[ \|I[f]\|_{L^{\infty}(\Omega)} \leq C(R,\Omega)\|f\|_{L^{1}(\Omega)}, \]

(3.2) \[ \|I[f]\|_{W^{1,\infty}(\Omega)} \leq C(R,\Omega)\|f\|_{W^{1,p}(\Omega)}, \]

where

\[ I[f](x) = \int_{E(x)} f(x + r) \omega(r) dr. \]

(3.1) is obvious. We prove (3.2) when \( B_L(0) \) is the two dimensional disc in \( \mathbb{R}^2 \) for a computational simplicity. Let us denote the radial coordinate of \( B_L(0) \) by \( (s, \varphi) ; x = (x_1, x_2) = (s \cos \varphi, s \sin \varphi) \). We may assume \( f \) is in \( C^1(B_L(0)) \) since \( f \in W^{1,p}(\Omega) \) has an approximated sequences \( C^\infty(\Omega) \) in \( W^{1,p}(\Omega) \).

When \( |x| < L - R \), we have

(3.3) \[ \partial_{x_1} I[f](x) = \int_{E(x)} f_{x_1}(x + r) \omega(r) dr := I_1^{x_1}(x). \]

When \( L - R < |x| < L \), we use polar coordinates \( (r, \theta) \) on \( B_{L-s}(0) \) to write

\[ I[f](x) = \int_0^{L-s} \int_0^{2\pi} f(x + r) \omega(r) r d\theta dr. \]
In this region we have
\begin{align}
\partial_{x_1} \mathcal{I}[f](x) &= \frac{\partial s}{\partial x_1} \int_0^{L-s} \int_0^{2\pi} f(x + r(\rho \cos \theta, \rho \sin \theta)) \omega((L-s)(\rho \cos \theta, \rho \sin \theta)) d\theta dr \\
&= -\frac{\partial s}{\partial x_1} \int_0^{2\pi} f(x + (L-s)(\rho \cos \theta, \rho \sin \theta)) d\theta \\
&+ \int_{E(x)} f(x + r) \omega(r) dr \\
&:= I_1^s(x)
\end{align}
with \( \frac{\partial s}{\partial x_1} = \cos \varphi \). From the above we have
\[ \|\partial_{x_1} \mathcal{I}[f]\|_{L^\infty(\Omega)} \leq \|f\|_{L^\infty(\Omega)} \|\omega\|_{L^\infty(\Omega)} + \|\omega\|_{L^p(\Omega)} \|f\|_{L^p(\Omega)} \]
for \( \frac{1}{p'} + \frac{1}{p} = 1 \). The same bound holds for \( \|\partial_{x_2} \mathcal{I}[f]\|_{L^\infty(\Omega)} \). Hence we prove
\begin{align}
\|\partial_x \mathcal{I}[f]\|_{L^\infty(\Omega)} &\leq C(R, \Omega) \|f\|_{W^{1,p}(\Omega)},
\end{align}
where \( \partial_x \mathcal{I}[f] \) denotes the pointwise differentiation as above.
For \( \psi \in C_0^\infty(\Omega) \) we have
\[ \int_\Omega \mathcal{I}[f](x) \partial_{x_1} \psi(x) dx = \lim_{\epsilon \to 0} \int_{|x| < L - R - \epsilon} \mathcal{I}[f](x) \partial_{x_1} \psi(x) dx \\
+ \lim_{\epsilon \to 0} \int_{|x| < L - R + \epsilon} \mathcal{I}[f](x) \partial_{x_1} \psi(x) dx \\
= \int_{|x| < L - R} I_1^{in}(x) \psi(x) dx - \int_{L - R < |x| < L} I_1^{o}(x) \psi(x) dx \\
+ \lim_{\epsilon \to 0} \int_{|x| = L - R - \epsilon} \mathcal{I}[f](x) \psi(x) \frac{x_1}{L - R - \epsilon} dx \\
- \lim_{\epsilon \to 0} \int_{|x| = L - R + \epsilon} \mathcal{I}[f](x) \psi(x) \frac{x_1}{L - R + \epsilon} dx \\
= \int_{|x| < L - R} I_1^{in}(x) \psi(x) dx - \int_{L - R < |x| < L} I_1^{o}(x) \psi(x) dx \\
\]
since \( \mathcal{I}[f] \) is continuous on \( |x| = L - R \). Hence
\[ D_{x_1} \mathcal{I}[f](x) := \begin{cases} 
I_1^{in}(x) & |x| < L - R \\
I_1^{o}(x) & L - R < |x| < L 
\end{cases} \]
is the weak derivative of \( \mathcal{I}[f] \) in \( x_1 \) and (3.2) follows due to (3.5). The weak derivative in \( x_2 \) is obtained similarly. This completes the proof. \( \square \)

3.2. Local well-posedness, blow-up criteria, and a priori estimates. To prove a local well-posedness of (1.10)–(1.12), we introduce the sectorial operator \( A := (I - \Delta) |_{D(A)} \) with its domain
\[ D(A) := \{ f \in W^{2,p}(\Omega) \mid Bf := \partial_x f = 0 \text{ on } \partial \Omega \}, \quad 1 < p < \infty. \]
Since \( (A, B) \) is sectorial operator, it generates an analytic semigroup \( \{ e^{-tA} \mid t \geq 0 \} \) on \( L^p(\Omega) \). Note that the fractional powers of \( A \) are well-defined (see [26, Section 1.4]). We denote the domains of fractional powers by
\[ X_\eta^p := D(A^\eta), \quad \eta \in (0, 1). \]
We also note from [26, Theorem 1.6.1] that
\[ X^\eta_p \hookrightarrow W^{\kappa,q}(\Omega) \quad \text{for} \quad \kappa - \frac{n}{q} < 2\eta - \frac{n}{p}, \quad q \geq p. \]

Now we consider the homogenous Neumann boundary value problems for [1.10]–[1.12]:
\[ \begin{align*}
\partial_t u + (I - \Delta) u &= g_1, & u(x,0) &= u_0(x), \\
\partial_t v + (I - \Delta) v &= g_2, & v(x,0) &= v_0(x),
\end{align*} \]
\[ \begin{align*}
\partial_\nu u &= 0, & x &\in \partial \Omega, \quad t \leq T, \\
\partial_\nu v &= 0, & x &\in \partial \Omega, \quad t \leq T,
\end{align*} \]
and its integral representation formulas:
\[ \begin{align*}
u(t) &= e^{-tA}u_0 + \int_0^t e^{-(t-\tau)A}g_1(\tau)d\tau, \\
v(t) &= e^{-tA}v_0 + \int_0^t e^{-(t-\tau)A}g_2(\tau)d\tau,
\end{align*} \]
where
\[ \begin{align*}
g_1 &= -\nabla \cdot (uK[u,v]) + (1 - m)u + \frac{\lambda}{k}(u(k - (u + v)), \\
g_2 &= -\nabla \cdot (vS[u,v]) + mu + v + \frac{\mu}{k}(v(k - (u + v)).
\end{align*} \]

Before stating local-in-time result, let us note the smoothing estimates for \( e^{-tA} \) from [26, Theorem 1.4.3], which will be used in the proof of next lemma:

If \( p > 1, \quad 0 < \eta < 1, \quad f \in L^p(\Omega), \) then there exist positive constants \( \kappa, \) and \( C = C(\eta) \) such that
\[ \left\| e^{-tA}f \right\|_{X^\eta_p} \leq Ce^{-\kappa t}\eta^\eta \left\| f \right\|_{L^p(\Omega)}, \quad t > 0. \]

**Lemma 8.** Let \( \Omega \subset \mathbb{R}^n, \quad n \geq 2, \) be an open ball of radius \( L, \) \( B_R(0). \) Assume that \( u_0 \) and \( v_0 \) are non-negative functions belong to \( W^{2,p}(\Omega), \) \( p > n, \) and satisfy compatibility condition
\[ \begin{align*}
\partial_\nu u_0 &= 0, & \partial_\nu v_0 &= 0, & x &\in \partial \Omega.
\end{align*} \]

Then, there exists the maximal time of existence, \( T_{\max} \leq \infty, \) such that a pair of unique non-negative strong solution \( (u, v) \) of (3.7) exists and satisfies
\[ u, v \in C\left([0, T_{\max}); W^{1,p}(\Omega) \right) \cap W^{1,p}(0, t; L^p(\Omega)) \cap L^p(0, t; W^{2,p}(\Omega)), \quad t < T_{\max}. \]

Moreover, if \( T_{\max} < \infty, \) then
\[ \lim_{t \to T_{\max}} \left( \left\| u(\cdot, t) \right\|_{W^{1,p}(\Omega)} + \left\| v(\cdot, t) \right\|_{W^{1,p}(\Omega)} \right) = \infty. \]

**Proof.** Let \( n < p < \infty. \) With positive constants \( T < 1 \) and \( R_0 \) to be specified below, we introduce the Banach space \( X_T := C\left([0, T]; W^{1,p}(\Omega) \right) \) and its closed convex subset \( S_T \subset X_T, \)
\[ S_T := \left\{ f \in X_T \mid \left\| f \right\|_{L^\infty(0, T; W^{1,p}(\Omega))} \leq R_0 \right\}. \]

Let \( u, v \in S_T. \) As in (3.8)–(3.9), we consider
\[ \begin{align*}
\Phi_1(u, v) := e^{-tA}u_0 + \int_0^t e^{-(t-\tau)A}g_1(\tau)d\tau, \\
\Phi_2(u, v) := e^{-tA}v_0 + \int_0^t e^{-(t-\tau)A}g_2(\tau)d\tau,
\end{align*} \]
where \( g_i \) for \( i = 1, 2 \) are functions given in (3.9).

Firstly, we show \( \Phi_1(u, v), \Phi_2(u, v) \in S_T. \) Let \( \frac{1}{2} < \eta < 1 \) and note from (3.10) that
\[ X^\eta_p \hookrightarrow W^{1,p}(\Omega). \]
Using (3.10) and (3.14), we can compute
\[
\|\Phi_1(u, v)\|_{W^{1,p}(\Omega)} \leq \left| e^{-tA}u_0 \right|_{W^{1,p}(\Omega)} + \int_0^t \left| e^{-(t-\tau)A}g_1(\tau) \right|_{W^{1,p}(\Omega)} d\tau \\
\leq M_3 \|u_0\|_{W^{1,p}(\Omega)} + C \int_0^t \left| e^{-(t-\tau)A}g_1(\tau) \right|_{X^p} d\tau \\
\leq M_3 \|u_0\|_{W^{1,p}(\Omega)} + C \int_0^t e^{-k(t-\tau)}(t-\tau)^{-\eta} \|g_1(\tau)\|_{L^p(\Omega)} d\tau.
\]
By Lemma 7 and \(W^{1,p}(\Omega) \hookrightarrow L^\infty(\Omega)\), we have
\[
\|g_1(\tau)\|_{L^p(\Omega)} = \| - \nabla \cdot (uK[u, v]) + (1-m)u + \frac{\lambda}{k}u(k-(u+v)) \|_{L^p(\Omega)} \leq C(R_0 + R_0^2),
\]
and thus,
\[
\|\Phi_1(u, v)\|_{W^{1,p}(\Omega)} \leq M_3 \|u_0\|_{W^{1,p}(\Omega)} + C_{13}(R_0 + R_0^2)t^{1-\eta},
\]
where \(C_{13}\) is a positive constant independent of \(R_0\). Similarly, we also have
\[
\|\Phi_2(u, v)\|_{W^{1,p}(\Omega)} \leq M_4 \|v_0\|_{W^{1,p}(\Omega)} + C_{14}(R_0 + R_0^2)t^{1-\eta},
\]
where \(C_{14}\) is a positive constant independent of \(R_0\). Taking
\[
R_0 := M_3 \|u_0\|_{W^{1,p}(\Omega)} + M_4 \|v_0\|_{W^{1,p}(\Omega)} + 1,
\]
and
\[
T < T_3 := \min \left\{ 1, \frac{1}{(2C_{13}(R_0 + R_0^2))^{\frac{1}{1-\eta}}}, \frac{1}{(2C_{14}(R_0 + R_0^2))^{\frac{1}{1-\eta}}} \right\},
\]
we obtain \(\Phi_1(u, v), \Phi_2(u, v) \in S_T\).

We next show the mapping \((u, v) \mapsto (\Phi_1, \Phi_2)\) is contraction. Using (3.8), we note that
\[
\Phi_1(u, v) - \Phi_1(\bar{u}, \bar{v}) = \int_0^t e^{-(t-\tau)A}(g_1 - \bar{g}_1)(\tau) d\tau,
\]
\[
\Phi_2(u, v) - \Phi_2(\bar{u}, \bar{v}) = \int_0^t e^{-(t-\tau)A}(g_2 - \bar{g}_2)(\tau) d\tau,
\]
where
\[
\bar{g}_1 = -\nabla \cdot (\bar{u}K[\bar{u}, \bar{v}]) + (1-m)\bar{u} + \frac{\lambda}{k}\bar{u}(k-(\bar{u} + \bar{v})),
\]
\[
\bar{g}_2 = -\nabla \cdot (\bar{v}S[\bar{u}, \bar{v}]) + m\bar{u} + \bar{v} + \frac{\mu}{k}\bar{v}(k-(\bar{u} + \bar{v})).
\]
Then, analogously to the above computations, we can estimate
\[
\|(\Phi_1, \Phi_2)(u, v) - (\Phi_1, \Phi_2)(\bar{u}, \bar{v})\|_{W^{1,p}(\Omega)} \leq C_{15}(R_0 + 1)T^{1-\eta} \|(u, v) - (\bar{u}, \bar{v})\|_{W^{1,p}(\Omega)}.
\]
Taking
\[
T < T_4 := \min \left\{ T_3, \frac{1}{(2C_{15}(R_0 + 1))^{\frac{1}{1-\eta}}} \right\},
\]
we obtain
\[
\|(\Phi_1, \Phi_2)(u, v) - (\Phi_1, \Phi_2)(\bar{u}, \bar{v})\|_{W^{1,p}(\Omega)} \leq \frac{1}{2} \|(u, v) - (\bar{u}, \bar{v})\|_{W^{1,p}(\Omega)},
\]
i.e., the mapping is contraction. According to the Banach fixed point theorem, this mapping has a fixed point in \(S_T\), denoted again as \((u, v)\).
Now, the uniqueness of \((u, v)\), and the blow-up criteria (3.11) can be obtained as in the proof of Lemma 3. By the maximal regularity theorem (see, e.g., Ladyzhenskaya-Solonnikov-Ural’ceva [29, Section 4, Theorem 9.1]), we have
\[ u, v \in L^p(0, T; W^{2,p}(\Omega)) \cap W^{1,p}(0, T; L^p(\Omega)). \]
Then, the non-negativity of solutions is followed as (2.15)–(2.16). This completes the proof. □

Next, we provide a refined blow-up criteria (Lemma 9), and \textit{a priori} estimates (Lemma 10 and Lemma 11). We do not give their proofs to reduce the redundancies, instead, we refer the readers to see the proofs of Lemma 4, Lemma 5, and Lemma 6.

**Lemma 9.** Let the same assumptions as in Lemma 3 be satisfied. The solutions \((u, v)\) of (3.7) given by Lemma 3 satisfies
\[ \text{either } T_{\text{max}} = \infty, \text{ or } \lim_{t \to T_{\text{max}}} (\|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{L^\infty(\Omega)}) = \infty. \]

**Lemma 10.** Let the same assumptions as in Lemma 3 be satisfied. The solution \((u, v)\) of (3.7) given by Lemma 3 for \(T < T_{\text{max}}\) satisfies
\[ \sup_{t \leq T} \int_\Omega u(\cdot, t) \leq C(\|u_0\|_{L^1(\Omega)}), \]
\[ \sup_{t \leq T} \int_\Omega v(\cdot, t) \leq C(\|u_0\|_{L^1(\Omega)}, \|v_0\|_{L^1(\Omega)}), \]
\[ \sup_{t \leq T} \|K[u, v](\cdot, t)\|_{L^\infty(\Omega)} \leq C(\|u_0\|_{L^1(\Omega)}, \|v_0\|_{L^1(\Omega)}), \]
and
\[ \sup_{t \leq T} \|S[u, v](\cdot, t)\|_{L^\infty(\Omega)} \leq C(\|u_0\|_{L^1(\Omega)}, \|v_0\|_{L^1(\Omega)}). \]

**Lemma 11.** Let the same assumptions as in Lemma 3 be satisfied. The solution \((u, v)\) of (3.7) given by Lemma 3 for \(T < T_{\text{max}}\) satisfies
\[ \sup_{t \leq T} \|u(\cdot, t)\|_{L^\infty(\Omega)} \leq C(\|u_0\|_{L^1(\Omega)}, \|v_0\|_{L^1(\Omega)}), \]
and
\[ \sup_{t \leq T} \|v(\cdot, t)\|_{L^\infty(\Omega)} \leq C(\|u_0\|_{(L^1 \cap L^\infty)(\Omega)}, \|v_0\|_{(L^1 \cap L^\infty)(\Omega)}). \]

**Proof of Theorem 2.** It is a direct consequence of local-in-time existence, uniqueness, non-negativity (Lemma 3), blow-up criteria (Lemma 9), and \textit{a priori} estimates (Lemma 11). This completes the proof. □

**Remark 1.** If the adhesive strength of \(v\) is negligible, \(S[u, v] = 0\), and the growth rate of \(u\) is smaller than that of \(v\), \(\lambda < \mu\), then the solution asymptotically tends to constant equilibrium. Indeed, the solution \((u, v)\) of (1.10)–(1.12) with \(\inf_\Omega v_0 > 0\) given by Theorem 2 satisfy
\[ \frac{d}{dt} \left[ \frac{1}{\lambda} \int_\Omega u + \frac{1}{\mu} \int_\Omega v - \frac{k}{\mu} \int_\Omega \log v \right] + \frac{k}{\mu} \int_\Omega \left| \nabla v \right|^2 + \frac{mk}{\mu} \int_\Omega \frac{u}{v} + m \left( \frac{1}{\lambda} - \frac{1}{\mu} \right) \int_\Omega u + \frac{1}{k} \int_\Omega |u - (k-v)|^2 \]
\[ = \frac{k}{\mu} \int_\Omega S[u, v] \frac{\nabla v}{v}. \]
Thus, one can verify that \(\int_\Omega u(\cdot, t) \to 0\) and \(\int_\Omega |u - (k-v)|^2(\cdot, t) \to 0\) as time tends to infinity whenever \(S[u, v] = 0\) and \(\lambda < \mu\). Then, by \(u \geq 0\), we have \((u, v) \to (0, k)\).
Let $\Omega, K, S$ and $\omega$ be given same as for Section 3. We define the operator $F : W^p_B \times W^p_B \to L^2 \times L^2$ by

$$F(u, v) = (F_1(u, v), F_2(u, v)),$$

$$F_1(u, v) = \Delta u - \nabla \cdot (uK[u, v]) + (1 - m)u + \frac{\lambda}{k}u(k - (u + v))$$

$$F_2(u, v) = \Delta v - \nabla \cdot (vS[u, v]) + mu + v + \frac{\mu}{k}v(k - (u + v)).$$

We denote the Gâteaux derivative of $F$ at $U = (u, v)$ by $T_U$.

$$T_U(W) = \lim_{t \to 0} \frac{F(U + tW) - F(U)}{t} = (\delta_W F_1(U), \delta_W F_2(U))$$

where $W = (w, z) \in W^p_B \times W^p_B$. By computation we have

$$\delta_W F_1(0, k) = \Delta w - mw$$

$$\delta_W F_2(0, k) = \Delta z - \nabla \cdot (kS[w, 0] + kS[w, z] + zS[0, k]) + (m - \mu)w - \mu z$$

$$\delta_W F_1(0, 0) = \Delta w - mw$$

$$\delta_W F_2(0, 0) = \Delta z + mw + \mu z.$$

We consider the two linearized equations at $(0, k)$ and $(0, 0)$ respectively with initial data $(w_0, z_0) \in W^p_B \times W^p_B$:

$$\partial_t w = \Delta w - mw$$

$$\partial_t z = \Delta z - \nabla \cdot (kS[w, 0] + kS[w, z] + zS[0, k]) + (m - \mu)w - \mu z,$$

and

$$\partial_t w = \Delta w - mw$$

$$\partial_t z = \Delta z + mw + \mu z.$$

The equations (3.21), (3.22) are decoupled and it is immediate that

$$\|w\|_{W^{1,p}(\Omega)} \leq e^{-mt}\|w_0\|_{W^{1,p}}, \quad p \geq 1$$

from

$$\partial_t(e^{mt}w) = \Delta(e^{mt}w).$$

Let $\tilde{z}$ denote $e^{mt}z$. Multiplying $e^{mt}$ to the $z$-equation of (3.21), we have

$$\partial_t \tilde{z} - \Delta \tilde{z} = -\nabla \cdot (2kS[e^{mt}w, 0] + kS[0, \tilde{z}] + \tilde{z}S[0, k]) + (m - \mu)e^{mt}w.$$

It holds that

$$\frac{d}{dt} \int_\Omega |\tilde{z}| \leq |m - \mu|e^{mt} \int_\Omega |w| \leq |m - \mu|e^{-(m-\mu)t}\|w_0\|_{L^1(\Omega)},$$

which implies

$$\int_\Omega |\tilde{z}| \leq \int_\Omega |z_0| - \frac{|m - \mu|}{m - \mu}(e^{-(m-\mu)t} - 1) \int_\Omega |w_0|$$

and

$$\int_\Omega |z| \leq e^{-mt} \int_\Omega |z_0| + (e^{-mt} - e^{-mt}) \int_\Omega |w_0|,$$

respectively.
where we abuse the notation by \(|m - \mu)/(m - \mu) = 0\) if \(\mu = m\). When \(m > \mu\), it holds that
\[
\int |\tilde{z}| \leq \|z_0\|_{L^1(\Omega)} + \|w_0\|_{L^1(\Omega)}.
\] (3.26)

In what follows we find that the different signs of \(\mp \mu z\) in \((3.21)\) and \((3.22)\) lead that \((0, k)\)
is linearly stable and \((0, 0)\) is linearly unstable as expected.

**Proposition 1.** The linearized equation \((3.21), (3.22)\) have the unique global solution \((w, z)\)
for each in
\[
C([0, t]; W^{1,p}(\Omega)) \cap W^{1,p}(0, t; L^p(\Omega)) \cap L^p(0, t; W^{2,p}(\Omega))
\]
for any \(t > 0\). When \(m > \mu\), the solution \((w, z)\) for \((3.21)\) is asymptotically stable such that
\[
\|z\|_{L^p(\Omega)} \leq e^{-\mu t}\|z_0\|_{L^p(\Omega)}\quad \text{for } p \geq 1.
\] (3.27)
The solution \((w, z)\) for \((3.22)\) grows exponentially in its \(L^1\)-norm if the initial data is non-negative;
\[
\int_{\Omega} |z| \geq e^{\mu t}\int_{\Omega} |z_0|.
\] (3.28)

**Proof.** Due to the a priori estimates \((3.23)\) and \((3.24)\), the global well-posedness part for \((3.21)\) follows from the same argument in the subsection 2.3 or the subsection 3.2. Repeating the argument of Lemma 9 to \((3.21)\) it holds that
\[
\|z\|_{L^\infty(\Omega)} \leq C(\|w_0\|_{L^1 \cap L^\infty(\Omega)}, \|z_0\|_{L^1 \cap L^\infty(\Omega)}).
\] (3.29)
For details see \((3.31)-(3.33)\) for \(\tilde{z}\), where the similar estimates are given. By Lemma 9 it also holds that
\[
\|z\|_{W^{1,p}} \leq C(\|w_0\|_{L^1 \cap L^\infty(\Omega)}, \|z_0\|_{L^1 \cap L^\infty(\Omega)})
\] for any \(p \geq 1\). Let us prove \((3.28)\) first. The solution \((w, z)\) remains non-negative and we have
\[
\int_{\Omega} w = e^{-\mu t}\int_{\Omega} w_0,
\]
\[
d\frac{d}{dt}(e^{\mu t}\int_{\Omega} z) = m e^{-(\mu + m)t}\int_{\Omega} w_0.
\]
Integrating the second equation, we have \((3.28)\).

For \((3.27)\) we proceed as in Lemma 6. Multiplying \(|\tilde{z}|^{p-2}\tilde{z}\) into \((3.24)\) for \(p \geq 2\), we have
\[
\frac{1}{p} \frac{d}{dt} \int_{\Omega} |z|^p + \frac{4(p-1)}{p^2} \int_{\Omega} |\nabla \tilde{z}|^2 = \int_{\Omega} |\tilde{z}|^{p-2}\tilde{z}\nabla \cdot (2kS[e^{\mu t}w, 0] + kS[0, \tilde{z}] + \tilde{z}S[0, k])
\]
\[
+ (m - \mu) \int_{\Omega} e^{\mu t}w|\tilde{z}|^{p-2}\tilde{z}.
\] (3.32)

By Lemma 6, \((3.23), (3.30)\) and using \(m > \mu\), we have
\[
\|\nabla \cdot S[e^{\mu t}w, 0]\|_{L^\infty(\Omega)} \leq C\|e^{\mu t}w\|_{W^{1,q}(\Omega)} \leq C\|w_0\|_{W^{1,q}(\Omega)}(q > n)
\]
\[
\|S[0, \tilde{z}]\|_{L^\infty(\Omega)} \leq C\|\tilde{z}\|_{L^1(\Omega)} \leq C(\|w_0\|_{L^1}, \|z_0\|_{L^1})
\]
\[
\|\nabla S[0, k]\|_{L^\infty(\Omega)} \leq C
\]
and estimate the right hand side of (3.31) as follows,
\[
\int_{\Omega} |\tilde{z}|^{p-2} \tilde{z} \cdot (2kS[e^{it} w, 0]) + (m - \mu) \int_{\Omega} e^{it} w |\tilde{z}|^{p-2} \tilde{z} \leq C \int_{\Omega} |\tilde{z}|^{p-1}
\]
\[
\int_{\Omega} |\tilde{z}|^{p-2} \tilde{z} \cdot kS[0, \tilde{z}] \leq \frac{p - 1}{p^2} \int_{\Omega} |\nabla S|_{L^p}^2 + C(p - 1) \int |\tilde{z}|^{p-2} \|S[0, \tilde{z}]_{L^\infty(\Omega)}^2
\]
\[
\leq \frac{p - 1}{p^2} \int_{\Omega} |\nabla S|_{L^p}^2 + C \frac{p - 1}{p} \left( |\tilde{z}| + (p - 2) \int_{\Omega} |\tilde{z}|^p \right),
\]
\[
\int_{\Omega} |\tilde{z}|^{p-2} \tilde{z} \cdot (\tilde{z}S[0, k]) \leq \frac{1}{p^2} \int_{\Omega} |\nabla S|_{L^p}^2 + C \int_{\Omega} |\tilde{z}|^p \|S[0, k]\|_{L^\infty(\Omega)}.
\]

Summing up, we have
\[
\frac{1}{p} \frac{d}{dt} \int_{\Omega} |\tilde{z}|^p + \frac{3(p - 1)}{p^2} \int_{\Omega} |\nabla S|_{L^p}^2 \leq C + C \int_{\Omega} |\tilde{z}|^p, \quad p \geq 2,
\]
where \( C \) is an uniform constant depending on \( \|w_0\|_{L^1(\Omega)} \), \( \|z_0\|_{L^1(\Omega)} \), and given constants \( \mu, m, k \) etc.. As was derived from (2.30) for \( v \) in Lemma 6 it holds that
\[
\sup_{0 \leq t \leq T} \|\tilde{z}\|_{L^{pk}(\Omega)} \leq C(\|z_0\|_{L^1(\Omega)} + \|z_0\|_{L^1(\Omega)} + C^p) \quad p_k = 2^k, \quad k = 0, 1, \ldots
\]
and
\[
(3.33) \quad \sup_{0 \leq t \leq T} \|\tilde{z}\|_{L^\infty(\Omega)} \leq C(\|w_0\|_{L^1\cap L^\infty(\Omega)} + \|z_0\|_{L^1\cap L^\infty(\Omega)}).
\]
That implies (3.27).

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