Parametrized Invariance for Infinite State Processes

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Abstract. We study the uniform verification problem for infinite state processes, which consists of proving that the parallel composition of an arbitrary number of processes satisfies a temporal property. Our practical motivation is to build a general framework for the temporal verification of concurrent datatypes.

The contribution of this paper is a general method for the verification of safety properties of parametrized programs that manipulate complex local and global data, including mutable state in the heap. This method is based on the separation between two concerns: (1) the interaction between executing threads—handled by novel parametrized invariance rules—and the data being manipulated—handled by specialized decision procedures. The proof rules discharge automatically a finite collection of verification conditions (VCs), the number depending only on the size of the program description and the specification, but not on the number of processes in any given instance or on the kind of data manipulated. Moreover, all VCs are quantifier free, which eases the development of decision procedures for complex data-types on top of off-the-shelf SMT solvers.

We discuss the practical verification (of shape and also functional correctness properties) of a concurrent list implementation based on the method presented in this paper. Our tool also uses VCs using a decision procedure for a theory of list layouts in the heap built on top of state-of-the-art SMT solvers.

1 Introduction

In this paper we present a general method to verify concurrent software which is run by an arbitrary number of threads that manipulate complex data, including infinite local and shared state. Our solution consists of a method that cleanly separates two concerns: (1) the data, handled by specialized decision procedures; and (2) the concurrent thread interactions which is handled by novel proof rules, that we call parametrized invariance. The method of parametrized invariance tackles, for safety properties, the uniform verification problem for parametrized systems with infinite state processes:

Given a parametrized system $S[N] : P(1) \parallel P(2) \parallel \ldots \parallel P(N)$ and a property $\varphi$, establish whether $S[N] \models \varphi$ for all instances $N \geq 1$. 
In this paper we restrict to safety properties. Our method is a generalization of the inductive invariance rule for temporal deductive verification [23], in which each verification condition corresponds to a small-step (a single transition) in the execution of a system. For non-parametrized systems, there is always a finite number of transitions, so one can generate one VC per transition. However, in parametrized systems, the number of transitions depends on the concrete number of processes in each particular instantiation.

The main contribution of this paper is the principle or parametrized invariance, presented as proof rules that capture the effect of single steps of threads involved in the property and extra arbitrary threads. The parametrized invariance rules automatically discharge a finite number of VCs, whose validity imply the correctness for all system instantiations. For simplicity we present the rules for fully symmetric systems (in which thread identifiers are only compared with equality) and show that all VCs generated are quantifier-free (as long as transition relations and specifications are quantifier-free, which is the case is conventional system descriptions).

For many data-types one can use directly SMT solvers [15,25], or specialized decision procedures built on top. We show here how to use the decision procedure for a quantifier-free theory of single linked list layouts with locks [27] to verify fine-grained concurrent list implementation. Other powerful logics and tools for building similar decision procedures include [20,22].

Related Work. The problem of uniform verification of parametrized systems has received a lot of attention in recent years. This problem is, in general, undecidable [3], even for finite state components. There are two general ways to overcome this limitation: deductive proof methods as the one we propose here, and (necessarily incomplete) algorithmic approaches.

Most algorithmic methods are restricted to finite state processes [7,8,11] to obtain decidability. Examples are synchronous communication processes [13,16]; systems with only conjunctive guards or only disjunctive guards [11]; implicit induction [12]; network invariants [21]; etc. A related technique, used in parametrized model checking, is symmetry reduction [9,14]. A very powerful method is invisible invariants [4,26,29], which works by generating invariants on small instantiations and generalizing these to parametrized invariants. However, this method is so far also restricted to finite state processes.

A different tradition of automatic (incomplete) approaches is based on abstracting control and data altogether, for example representing configurations as words from a regular language [1,2,19,24]. Other approaches use abstraction, like thread quantification [5] and environment abstraction [10], based on similar principles as the full symmetry presented here, but relying on building specific abstract domains that abstract symbolic states instead of using SMT solvers.

In contrast with these methods, the verification framework we present here can handle infinite data. The price to pay is, of course, automation because one needs to provide some support invariants. We see our line of research as complementary to the lines mentioned above. We start from a general method and investigate how to improve automation as opposed to start from a restricted
automatic technique and improve its applicability. The VCs we generate can still be verified automatically as long as there are decision procedures for the data that the program manipulates.

Our target application is the verification of concurrent datatypes [18], where the main difficulty arises from the mix of unstructured unbounded concurrency and heap manipulation. Unstructured refers to programs that are not structured in sections protected by locks but that allow a more liberal pattern of shared memory accesses. Unbounded refers to the lack of bound on the number of running threads. Concurrent datatypes can be modeled naturally as fully symmetric parametrized systems, where each thread executes in parallel a client of the datatype. Temporal deductive methods [23], like ours, are very powerful to reason about (structured or unstructured) concurrency.

The rest of the paper is structured as follows. Section 2 includes the preliminaries. Section 3 introduces the parametrized invariance rule. Section 4 contains the examples, a description of our tool and empirical evaluation results. Finally, Section 5 concludes.

2 Running Example and Preliminaries

Running Example. We will use as a running example a concurrent datatype that implements a set [18] using fine-grain locks, shown in Fig. 2. Appendix A contains simpler and more detailed examples of infinite state mutual exclusion protocols. Lock-coupling concurrent lists implement sets by maintaining an ordered list of non-repeating elements. Each node in the list stores an element, a pointer to the next node in the list and a lock used to protect concurrent accesses. To search an element, a thread advances through the list acquiring a lock before visiting a node. This lock is only released after the lock of the next node has been acquired. Concurrent lists also maintain two sentinel nodes, head and tail, with phantom values representing the lowest and highest possible values, $-\infty$ and $+\infty$ respectively. Sentinel nodes are not modified at runtime. We define two "ghost" variables that aid the verification: reg, a set of addresses that contains the set of address pointing to nodes in the list; and elems, a set of elements we use to keep track of elements contained in the list. Ghost variables are compiled away and are only used in the verification process. In Fig. 2 ghost variables and code appear inside a box. As lock-coupling lists implement sets, three main operations are provided: (a) SEARCH: finds an element in the list; (b) INSERT: adds a new element to the list; and (c) REMOVE: deletes an element in the list. For verification purposes, it is common to define the most general client MGC

```
procedure MGC
  Elem e
  begin
  1: while true do
  2:   e := havocListElem()
  3:   nondet call SEARCH(e)
      or call INSERT(e)
      or call REMOVE(e)
  4: end while
  end procedure
```

Fig. 1: Most General Client
shown in Fig. 1. Each process in the parametrized system runs MGC choosing non-deterministically a method and its parameters.

**Preliminaries.** Our verification task starts from a program, and a safety property described as a state predicate. A system is correct if all states in all the traces of the transition system that models the set of executions of the program satisfy the safety property.

A transition system is a tuple $\mathcal{S} : (V, \Theta, T)$ where $V$ is a finite set of (typed) variables, $\Theta$ is a first-order assertion over the variables which describes the possible initial states, and $T$ is a finite set of transitions. We model program data using multi-sorted first order logic. A signature $\Sigma : (S, F, P)$ consists of a set of sorts $S$ (corresponding to the types of the data that the program manipulates), a set $F$ of function symbols, and a set $P$ of predicate symbols. We use $\Sigma_{\text{prog}}$ for the signature of the datatypes in a given program and $T_{\text{prog}}$ for the theory that allows to reason about formulas in $\Sigma_{\text{prog}}$. A state is an interpretation of $V$ that assigns a value of the corresponding type to each program variable. A transition

![Fig. 2: Lock-coupling single linked list implementation](image-url)
To show that $S$ satisfies $\Box \varphi$:

- **B1.** $\Theta \rightarrow \varphi$
- **B2.** $\varphi \land \tau \rightarrow \varphi'$ for all $\tau$

\[ \Box \varphi \]

(a) The basic invariance rule $\text{b-inv}$

To show that $S$ satisfies $\square p$, find $q$ with:

- **I1.** $\Theta \rightarrow q$
- **I2.** $q \land \tau \rightarrow q'$ for all $\tau$
- **I3.** $q \rightarrow \varphi$

\[ \square \varphi \]

(b) The invariance rule $\text{inv}$

Fig. 3: Rules $\text{b-inv}$ and $\text{inv}$ for non-parametrized systems.

is represented by a logical relation $\tau(s, s')$ that describes the relation between the values of the variables in a state $s$ and a successor state $s'$. A *run* of $S$ is an infinite sequence $s_0 \tau_0 s_1 \tau_1 s_2 \ldots$ of states and transitions such that (a) the first state is initial: $s_0 \models \Theta$; (b) all steps are legal: $\tau_i(s_i, s_{i+1})$, that is, $\tau_i$ is taken at $s_i$, leading to state $s_{i+1}$.

A system $S$ satisfies a safety property $\Box \varphi$, which we write $S \models \Box \varphi$, whenever all runs of $S$ satisfy $\varphi$ at all states. For non-parametrized systems, invariants can be proved using the classical invariance rules [23], shown in Fig. 3. The basic rule $\text{b-inv}$ establishes that if the candidate invariant $\varphi$ holds initially and is preserved by every transition then $\varphi$ is indeed an invariant. Rule $\text{inv}$ uses an intermediate strengthening invariant $q$. If $q$ implies $\varphi$ and $q$ is an invariant, then $\varphi$ is also an invariant. For non-parametrized systems, the premises in these rules discharge a number of verification conditions linear in the number of transitions. To use these invariance rules for parametrized systems, one either needs to use quantification or discharge an unbounded number of VCs, depending on the number of processes.

**Parametrized Concurrent Programs.** Parametrized programs consist of the parallel execution of process running the same program (the extension to an unbounded number of processes each running a program from a finite collection is trivial). We assume asynchronous interleaving semantics for parallel composition.

A program is described as a sequence of statements, each assigned to a program location in the range $\text{Loc} : 1 \ldots L$. Each instruction can manipulate a collection of typed variables partitioned into $V_{\text{globals}}$, the set of *global* variables, and $V_{\text{local}}$, the set of *local* variables. There is one special local variable $\text{pc}$ of sort $\text{Loc}$ that stores the program counter of each thread. For example, for the program in Fig. 2 $T_{\text{prog}}$ is the combination of TLL3 (the theory of single-linked lists in the heap with locks [27]), combined with finite discrete values (for program locations). In transition relations we use a primed variable $v'$ to denote the value of variable $v$ after a transition is taken.

A parametrized program $P$ is associated with a parametrized system $S$, a collection of transition systems $S[M]$, one for each number of running threads. We use $[M]$ to denote the set $\{0, \ldots, M - 1\}$ of concrete thread identifiers. For each $M$, there is a system $S[M] : \langle V, \Theta, T \rangle$ consisting of:
- The set $V$ of variables is $V_{\text{global}} \cup \{v[k]\} \cup \{pc[k]\}$ where there is one $v[k]$ for each $v \in V_{\text{local}}$ and for each $k \in [M]$, and one $pc[k]$ for each $k \in [M]$.
- An initial condition $\Theta$, which is described by two predicates $\Theta_\tau$ (that only refers to variables from $V_{\text{global}}$) and $\Theta_t$ (that can refer to variables in $V_{\text{local}}$). Given a thread identifier $a \in [M]$ for a concrete system $S[M]$, $\Theta_t[a]$ is the initial condition for thread $a$, obtained by replacing $v[a]$ for every occurrence of $v$ in $\Theta_t$.
- $T$ contains a transition $\tau_t[a]$ for each location and thread $a$ in $[M]$ obtained from $\tau_t$ by replacing every occurrence of $v$ by $v[a]$, and of $v'$ by $v'[a]$.

We use $V^t$ to denote all variables of sort $t$ in set $V$.

Example 1. Consider the lock-coupling list program in Fig. 2. The instance of this program consisting of two running threads contains the following variables:

$$V = \{\text{head}, \text{tail}, \text{reg}, \text{elems}, e[0], \text{prev}[0], \text{curr}[0], \text{aux}[0], \text{found}[0],$$

$$e[1], \text{prev}[1], \text{curr}[1], \text{aux}[1], \text{found}[1]\}$$

There are 118 transitions in MGC[2], 59 transitions for each thread, one for each line in the program. For non-parametrized systems, like MGC[2], we use the predicate $\text{pres}$ in transition relations to list the variables that are not changed by the transition. That is $\text{pres}(\text{head}, \text{tail})$ is simply a short for $\text{head}' = \text{head} \land \text{tail}' = \text{tail}$.

3 Parametrized Formulas and Parametrized Proof Rules

We show in this paper how to specify and prove invariant properties of parametrized systems. Unlike in [26], we generate quantifier-free verification conditions, enabling the development of decision procedures for complex datatypes.

To model thread ids we introduce the sort $\text{tid}$ interpreted as an unbounded discrete set. The signature $\Sigma_{\text{tid}}$ contains only $=$ and $\neq$, and no constructor. We enrich $T_{\text{prog}}$ using the theory of arrays $T_{\text{A}}$ (see [6]) with indices from $\text{tid}$ and elements ranging over sorts $t$ from the local variables of $T_{\text{prog}}$. For each local variable $v$ of type $t$ in the program, we add a global variable $a_v$ of sort $\text{array}(t)$, including $a_{pc}$ for the program counter $pc$. The expression $a_v(k)$ denotes the element of sort $t$ stored in array $a_v$ at position given by expression $k$ of sort $\text{tid}$. The expression $a_v(k \leftarrow e)$ corresponds to an array update, and denotes the array that results from $a_v$ by replacing the element at position $k$ with $e$. To simplify notation, we use $v(k)$ for $a_v(k)$, and $v(k \leftarrow e)$ for $a_v(k \leftarrow e)$. Note how $v[0]$ is different from $v(k)$: the term $v[0]$ is an atomic term in $V$ (for a concrete system $S[M]$) referring to the local program variable $v$ of a concrete thread with id 0. On the other hand, $v(k)$ is a non-atomic term built using the signature of arrays, where $k$ is a variable (logical variable, not program variable) of sort $\text{tid}$.

Variables of sort $\text{tid}$ indexing arrays play a special role, so we classify formulas depending on the sets of variables used. The parametrized set of program variables with index variables $X$ of sort $\text{tid}$ is:

$$V_{\text{param}}(X) = V_{\text{global}} \cup \{a_v \mid v \in V_{\text{local}}\} \cup \{a_{pc}\} \cup X$$
We use $T$ for the union of theories $T_{\text{prog}}$, $T_{\text{tid}}$, and $T_A$. $F_T(X)$ is the set of first-order formulas constructed using predicates and symbols from $T$ and variables from $V_{\text{param}}(X)$. Given a tid variable $k$ and a program statement, we construct the parametrized transition relation as before, but using array reads and updates (to position $k$) instead of concrete local variable reads and updates. For parametrized formulas, the predicate $\text{pres}$ is defined with array extensional equality for unmodified local variables.

We similarly define the parametrized initial condition for a given set of thread identifiers $X$ as:

$$\Theta(X) : \Theta_g \land \bigwedge_{k \in X} \Theta_l(k)$$

where $\Theta_l(k)$ is obtained by replacing every local variable $v$ in $\Theta_l$ by $v(k)$.

A parametrized formula $\varphi(\overline{k})$ with free variables $\overline{k} = (k_0, \ldots, k_n)$ of sort tid is a formula from $F_T(\{k_0, \ldots, k_n\})$. Note, in particular, how parametrized formulas cannot refer to any constant thread identifier. We use $\text{Var}(\varphi)$ for the set of free tid variables in $\varphi$.

Given a concrete number of threads $N$, a concretization of expression $p(\overline{k})$ is characterized by a substitution $\alpha : \overline{k} \rightarrow [N]$ that assigns to each variable in $\overline{k}$ a unique constant thread identifier in the instance system $S[N]$. The application of $\alpha$ for expressions $p$ is defined inductively, where the base cases are:

$$\alpha(v(k_i)) \mapsto v[\alpha(k_i)]$$

$$\alpha(w = v[k_i \leftarrow e]) \mapsto (w[\alpha(k_i)] = e \land \bigwedge_{a \in N \setminus \alpha(k_i)} w[a] = v[a])$$

Essentially, a concretization provides the state predicate for system $S[N]$ that results from $p(\overline{k})$ by instantiating $\overline{k}$ according to $\alpha$.

We can formulate the uniform verification problem in terms of concretizations. Given a parametrized system $S$, a universal safety property of the form $\forall \overline{k}. \Box p(\overline{k})$ holds whenever for every $N$ and substitution $\alpha : \overline{k} \rightarrow [N]$, the concrete closed system $S[N] \models \Box \alpha(p(\overline{k}))$. In this case we simply write $S \models \Box p$ and say that $p$ is a parametrized invariant of $S$.

A naïve approach to prove parametrized inductive invariants is to enumerate all instances and repeatedly use rule INV for each one. However, this approach requires proving an unbounded number of verification conditions because one (potentially different) VC is discharged per transition and thread in every instantiated closed system.

**Parametrized Proof Rules.** We introduce here specialized proof rules for parametrized systems, which allow to prove parametrized invariants discharging only a finite number of verification conditions. Rule P-INV in Fig. 4 presents the basic parametrized invariance rule. Premise P1 guarantees that the initial condition holds for all instantiations. Premise P2 guarantees that $\varphi$ is preserved under transitions of the threads referred in the formula, and P3 guarantees that $\varphi$ is preserved under transitions of any other thread. P1 discharges only one
To show that \( S \) satisfies \( \square \varphi(\overline{k}) \), with \( \overline{k} = \text{Var}(\varphi) \):

- **P1.** \( \Theta(\overline{k}) \rightarrow \varphi \)
- **P2.** \( \varphi \land \tau^{(i)} \rightarrow \varphi' \) for all \( \tau \) and all \( i \in \overline{k} \)
- **P3.** \( \varphi \land (\bigwedge_{x \in \overline{k}} \neg x \land \tau^{(j)} \rightarrow \varphi') \) for all \( \tau \) and one fresh \( j \notin \overline{k} \)

\[ \square \varphi \]

Fig. 4: The parametrized invariance rule p-inv

verification condition, \( P2 \) discharges one VC per transition in the system and per index variable in the formula \( \varphi \). Finally, \( P3 \) generates one extra VC per transition in the system. All these VCs are quantifier-free provided that \( \varphi \) is quantifier-free. The following theorem justifies the introduction of rule p-inv:

**Theorem 1 (Soundness).** Let \( S \) be a parametrized system and \( \square \varphi \) a parametrized safety property. If \( P1, P2 \) and \( P3 \) hold, then \( S \models \square \varphi \).

**Proof.** (sketch) The proof proceeds by contradiction, assuming that the premises hold but \( S \not\models \square \varphi \). There must be an \( N \) and a concretization \( \alpha \) for which \( S[N] \not\models \square \alpha(\varphi) \). Hence, by soundness of the inv rule for closed systems, there must be a premise of inv that is not valid. By cases, one uses the counter-model of the offending premise to build a counter-model of the corresponding premise in p-inv. \( \square \)

There are cases in which premise \( P3 \) cannot be proved, even if \( \varphi \) is initial and preserved by all transitions of all threads. The reason is that, in the antecedent of \( P3 \), \( \varphi \) does not refer to the fresh arbitrary thread introduced. In other words, p-inv tries to prove the property for an arbitrary process at all reachable system states without assuming anything about any other thread. It is sound, however, to assume in the pre-state and for all processes the property one intends to prove. The notion of support allows to strengthen the antecedent to refer to all threads involved in the verification condition, including the fresh new thread.

**Definition 1 (support).** Let \( \psi \) be a parametrized formula (the support) and let \( (A \rightarrow B) \) be a parametrized formula with \( \text{Var}(A \rightarrow B) = X \). We say that \( \psi \) supports \( (A \rightarrow B) \), whenever \( \left[ (\bigwedge_{\sigma \in S} \psi \sigma \land A) \rightarrow B \right] \) is valid, where \( S \) is a subset of the partial substitutions \( \text{Var}(\psi) \rightarrow X \).

We use \( \psi \triangleright (A \rightarrow B) \) as a short for \( (\bigwedge_{\sigma \in S} \psi \sigma \land A) \rightarrow B \). We can strengthen premise \( P3 \) with self-support, so \( \varphi \) can be assumed (in the pre-state) for every thread, in particular for the fresh thread that takes the transition:

- **P3’.** \( \varphi \triangleright (\bigwedge_{x \in \overline{k}} \neg x \land \tau^{(j)} \rightarrow \varphi’) \) for all \( \tau \) and one fresh \( j \notin \overline{k} \)
To show that $S$ satisfies $\Box \varphi(k)$. Find $\psi$ with:

| Step | Invariant |
|------|-----------|
| S0.  | $\Box \psi$ |
| S1.  | $\Theta \rightarrow \varphi$ |
| S2.  | $\psi, \varphi \supset \tau(i) \rightarrow \varphi'$ for all $\tau$ and all $i \in k$ |
| S3.  | $\psi, \varphi \supset \bigwedge_{x \in \Sigma} j \neq x \land \tau(j) \rightarrow \varphi'$ for all $\tau$ and one fresh $j \notin \bar{k}$ |

$$\Box \varphi$$

Fig. 5: The general strengthening parametrized invariance rule sp-inv.

For example, let $\varphi(i)$ be a candidate invariant with one thread variable (an index 1 invariant candidate). Premise P3’ is $(\varphi \supset (j \neq i \land \tau(j) \rightarrow \varphi'(i)))$, or equivalently

$$(\varphi(j) \land \varphi(i) \land j \neq i \land \tau(j)) \rightarrow \varphi'(i).$$

Note how $\varphi(j)$ in the antecedent is the result of instantiating $\varphi$ for the fresh thread $j$. Rule p-inv can fail to prove invariants if they are not inductive. As for closed systems, one needs to strengthen invariants. However, it is not necessary the case that by conjoining the candidate and its strengthening one obtains a p-inv inductive invariant. Instead, one needs to use a previously proved invariant as support to consider also freshly introduced process identifiers. This idea is captured by rule sp-inv in Fig. 5.

**Theorem 2.** Let $S$ be a parametrized system and $\Box \varphi$ a parametrized safety property. If $S_0$, $S_1$, $S_2$ and $S_3$ hold, then $S \vDash \Box \varphi$.

**Graph Proof Rules** We now introduce a final specialized proof rule for parametrized systems. When using sp-inv, S0 requires to start from an already proved invariant. However, in some cases invariants mutually depend on each other. For example, in the proof of shape preservation of concurrent single-linked list programs, like the one in Fig. 2, one requires that the pointers curr and prev used in the list traversal do not alias. This fact depends on the list having at all program states the shape of a non-cyclic list. A correct but naive solution would be to write down all necessary conditions as a single formula and prove it invariant using P-inv. Unfortunately, this approach does not scale when using sophisticated decision procedures for infinite memory. A more efficient approach consists on building the proof modularly, splitting the invariant into meaningful subformulas to be used when required. Modularity motivates the introduction of g-inv, a rule for proof graphs shown in Fig. 6. This rule handles cases in which invariants that mutually dependent on each other need to be verified.

A proof graph $(V, E)$ has candidate invariants as nodes. An edge between two nodes indicates that in order to prove the formula pointed by the edge it is useful to use the formula at the origin of the edge as support. As a particular case, a formula with no incident edges is inductive and can be shown with P-inv.
To show that \( S \) satisfies \( \square \varphi \) find a proof graph \((V, E)\) with \( \varphi \in V \) such that:

\[
\begin{align*}
G1. & \quad \Theta \rightarrow \psi \quad \text{for all } \psi \in V \\
G2. & \quad \Phi, \psi \quad \tau^{(k)} \rightarrow \psi' \quad \text{for all } \psi \in V, \text{forall } \tau, \\
& \quad \text{and all } k \in \text{Var}(\psi), \quad \text{and } \Phi = \{ \psi_i \mid (\psi_i, \psi) \in E \} \\
G3. & \quad \Phi, \psi \quad \bigwedge_{x \in \psi} k \neq x \land \tau^{(k)} \rightarrow \varphi' \quad \text{for all } \psi \in V, \text{forall } \tau, \\
& \quad \text{and } \Phi = \{ \psi_i \mid (\psi_i, \psi) \in E \} \text{one fresh } k \notin v = \text{Var}(\psi),
\end{align*}
\]

\( \square \varphi \)

Fig. 6: The graph parametrized invariance rule \( g\text{-inv} \).

**Theorem 3.** Let \( S \) be a parametrized system and \((V, E)\) a proof graph. If \( G1, G2, \) and \( G3 \) hold, then \( S \models \square \psi \) for all \( \psi \in V \).

**Proof.** By contradiction assume that some formula in \( V \) is not an invariant. Then, consider a shortest path to a violation in any concrete system \( S[M] \). Let \( \psi \in V \) be the violated formula. By \( G1 \), the path cannot be empty because \( G1 \) implies initiation of all formulas in \( V \) for all concrete system instances. Hence, the offending state \( s \) violating \( \psi \) has a predecessor state \( s_{\text{pre}} \) in the path, which by assumption, satisfies all formulas in \( V \), and in particular all formulas in \( \{ \psi_i \mid (\psi_i, \psi) \in E \} \) i.e., with outgoing edges incident to \( \psi \). Premises \( G2 \) and \( G3 \), guarantee that the execution step from \( s_{\text{pre}} \) to \( s \) guarantees \( \psi \) in \( s \), which is a contradiction. \( \square \)

We now show not that for fully symmetric systems, the dependencies with arrays in the parametrized formulas can be eliminated preserving validity, generating formulas that decision procedures can reason about.

**Theorem 4 (Concretization).** Let \( \varphi(k) \) be with \( |k| = n \). Then \( \varphi(k) \) is valid if and only if \( \bigwedge_{\alpha \in A} \alpha(\varphi) \) is valid where \( A \) is the set of all possible assignments of variables in \( \text{Var}(\varphi) \) to \([n]\).

For example, if one intends to prove that \( p(i) \) is inductive, the concretization theorem allows to reduce P3 in P-inv to \((p[0] \land \tau[1] \rightarrow p'[0])\), where \( p[0] \) is a short for \( \alpha(p(i)) \) with \( \alpha : i \rightarrow 0 \). This formula involves no arrays. Similarly, to show \( \square p(i) \) with support invariant \( q(j) \), rule S3 can be reduced to:

\[
q[0] \land q[1] \land p[0] \land p[1] \land \tau[1] \rightarrow p'[0]
\]

In practice, the concretization can be performed upfront before discharging the verification condition to the SMT-Solver, or handled using the theory of uninterpreted functions and let the solver perform the search and propagation.
4 Implementation and Empirical Evaluation

We illustrate the use of our parametrized invariance rules proving list shape preservation and some functional properties about set representation of the concurrent list implementation presented in Fig. 2. We also show mutual exclusion for some infinite state protocols that use integers and sets of integers (see the appendix for details).

The proof rules are implemented in the temporal theorem prover tool Leap, under development at the IMDEA Software Institute\(^3\). Leap parses a temporal specification and a program description in a C-like language. Leap automatically generates VCs applying the parametrized invariance rules presented in this paper. The validity of each VC is then verified using a suitable decision procedure (DP) for each theory.

We compare here three decision procedures built on top the SMT solvers Z3 and Yices: (1) a simple DP that can reason only about program locations, and considers all other predicates as uninterpreted; (2) a DP based on TLL3 capable of reasoning about single-linked lists layouts in the heap with locks to aid in the verification of fine-grain locking algorithms; and (3) a DP that reasons about program locations, integers and finite sets of integers with minimum and maximum functions (for the mutual exclusion protocols). The last two decision procedures and their implementation are based on small model theorems. The satisfiability of a quantifier free formula is reduced to the search for a model (up to a sufficiently large size). Leap also implements some heuristic optimizations (called tactics) like attempting first to use a simpler decision procedure or instantiating support lazily. This speeds the solvers in many valid instances by reducing the formulas obtained by partial assignments in the application of rules SP-INV or G-INV.

List Preservation and Set Representation for Concurrent Lists. We prove that the program in Fig. 2 satisfies: (1) list shape preservation; and (2) the list implements a set, whose elements correspond to those stored in \textit{elems}. The theory TLL3 (see \cite{27}) allows to reason about addresses, elements, locks, sets, order, cells (i.e., list nodes), memory and list reachability. A cell is a struct containing an element, a pointer to next node in the list and lock to protect the cell. A lock is associated with operations \textit{lock} and \textit{unlock} to acquire and release. The memory (\textit{heap}) is modeled as an array of cells indexed by addresses. The specification is:

\[
\varphi_{\text{lst}} = \begin{cases} 
\text{null} \in \text{reg} \land \text{reg} = \text{addr}2\text{set}(\text{heap}, \text{head}) \land \text{head} \neq \text{tail} & \lor (L1) \\
\text{heap}[\text{tail}].\text{next} = \text{null} \land \text{tail} \neq \text{null} \land \text{head} \neq \text{null} & \lor (L2) \\
\text{heap}[\text{head}].\text{data} = -\infty \land \text{heap}[\text{tail}].\text{data} = +\infty & \lor (L3) \\
\text{elems} = \text{set}2\text{elem}\text{set}(\text{heap}, \text{reg}) \land \text{Ordered}(\text{heap}, \text{head}, \text{tail}) & (L4)
\end{cases}
\]

Formula \(\varphi_{\text{lst}}\) is 0-index since it only constrains global variables. (L1) establishes that \text{null} belongs to \text{reg} and that \text{reg} is exactly the set of addresses reachable in

\(^3\) Available at \url{http://software.imdea.org/leap}
the heap starting from head, which ensures that the list is acyclic. (L2) and (L3) express some sanity properties of the sentinel nodes head and tail. Finally, (L4) establishes that \textit{elems} is the set of elements in cells referenced by addresses in \textit{reg}, and that the list is ordered. The main specification is \textit{list}, defined as $\square \phi_{\text{list}}$.

Using p-inv, LEAP can establish that \textit{list} holds initially, but fails to prove that \textit{list} is preserved by all transitions. The use of decision procedures for proving VCs allows to obtain counter-examples as models of an execution step that leads to a violation of the desired invariant. LEAP parses the counterexample (model) returned by the SMT solver, which is usually very small, involves only few threads and allows to understand the missing intermediate facts. In practice, these models allow to write easily the support invariants. We introduce some support invariants that allow to prove \textit{list}.

Invariant \textit{region}(i) describes that local variables \textit{prev}, \textit{curr} and \textit{aux} point to cells within the region of the list \textit{reg}, and that these variables cannot be null or point to head or tail. The formula \textit{region} is 1-index (because it needs to refer to local variables of a single thread). Invariant \textit{next}(i) captures the relative position in the list of the cells pointed by head and tail and local variables \textit{prev}, \textit{curr} and \textit{aux}. This invariant is needed for (L2). To prove (L3) and (L4) we need to show that order is preserved. We introduce \textit{order}(i), which captures the increasing order between the data in cells pointed by \textit{curr}, \textit{prev} and \textit{aux} and by the searched, inserted or removed element $e$. Invariant \textit{lock}(i) captures those program locations at which a thread owns a cell in the heap by an acquired lock. Finally, \textit{disj}(i, j), defined as $\square \phi_{\text{dis}}(i, j)$ encodes that the calls to \textit{malloc} by different threads return different addresses:

$$\varphi_{\text{dis}}(i, j) \doteq (i \neq j \land pc(i) = 33,34 \land pc(j) = 33,34) \rightarrow \text{aux}_I(i) \neq \text{aux}_I(j)$$

Other properties verified for the concurrent list are functional like specifications. Invariant \textit{funSchLinear}(i) establishes that the result of \textit{Search} matches with the presence of the searched element $e$ at \textit{Search}'s linearization point; \textit{funSchInsert}(i) states that if a search is successful then $e$ was inserted earlier in the history; and \textit{funSchRemove}(i) captures the fact that if the search is unsuccessful then either $e$ was never inserted or it was removed, and it was not present at the linearization point of \textit{Search}. The invariants \textit{funRemove}(i), \textit{funInsert}(i) and \textit{funSearch}(i) consider the case in which one thread handles different elements than all other threads. In this case, the specification is similar to a sequential functional specification (an element is found if and only if it is in the list, an element is not present after removal and an element is present after insertion).

\textbf{Infinite State Mutual Exclusion Protocols.} We also report the proof of mutual exclusion of some simple infinite state protocols that use tickets. The first protocol uses two global integer variables, one to store the next available ticket, and another to represent the minimum ticket present. The decision procedure used is Presburger arithmetic. The second protocol stores the tickets in a global set of integers, and queries for the minimum element in the set. The decision procedure used is Presburger Arithmetic combined with finite sets of integers with minimum.
| formula info | Full supp | Supp | Offend | Tactics |
|--------------|-----------|------|--------|---------|
| #solved vc  | time(s)   | time(s) | time(s) | time(s) |
| index | #vc | pos | dp | list | TO | TO | TO | 12.85 |
| order | 1 | 121 | 62 | 50 | 998.35 | 7.56 | 2.69 | 1.20 |
| lock | 1 | 121 | 76 | 45 | 778.15 | 4.82 | 1.44 | 0.50 |
| next | 1 | 121 | 60 | 61 | TO | TO | 26.58 | 1.76 |
| region | 1 | 121 | 95 | 26 | TO | TO | 85.27 | 25.67 |
| disj | 2 | 181 | 177 | 4 | 121.74 | 1.29 | 1.29 | 0.22 |
| funSearch | 1 | 208 | 198 | 10 | TO | TO | 6.14 | 4.55 |
| funInsert | 1 | 208 | 200 | 8 | TO | TO | 2.04 | 0.51 |
| funRemove | 1 | 208 | 200 | 8 | TO | TO | 2.73 | 1.56 |
| funSchLinear | 1 | 121 | 97 | 24 | TO | TO | 82.13 | 4.63 |
| funSchInsert | 1 | 121 | 93 | 28 | TO | TO | 80.20 | 5.00 |
| funSchRemove | 1 | 121 | 93 | 28 | TO | TO | 110.84 | 5.49 |
| mutex | 2 | 28 | 26 | 2 | 0.32 | 0.23 | 0.10 | 0.01 |
| minticket | 1 | 19 | 18 | 1 | 0.04 | 0.04 | 0.01 | 0.01 |
| notsame | 2 | 28 | 26 | 2 | 0.13 | 0.13 | 0.10 | 0.02 |
| activelow | 1 | 19 | 17 | 2 | 0.01 | 0.01 | 0.01 | 0.01 |
| mutexS | 2 | 28 | 26 | 2 | 0.44 | 0.38 | 0.14 | 0.04 |
| minticketS | 1 | 19 | 18 | 1 | 0.31 | 0.18 | 0.08 | 0.01 |
| notsameS | 2 | 28 | 26 | 2 | 0.14 | 0.13 | 0.10 | 0.02 |
| activelowS | 1 | 19 | 17 | 2 | 0.02 | 0.02 | 0.02 | 0.01 |

Fig. 7: VCs proved using each decision procedure and running times.

Fig. 8 shows the proof graph encoding the proof of list. LEAP can read proof graphs and apply g-inv. Fig. 7 contains the results of this empirical evaluation, executed on a computer with a 2.8 GHz processor and 8GB of memory. Each row reports the results for a single invariant. The first four columns show the index of the formula, the total number of generated VCs, the number of VCs proved by position, and the remaining VCs. The next four columns show the total running time using the specialized decision procedures with different tactics: “Full supp” corresponds to instantiating all support invariants for all VCs; “Supp” corresponds to instantiate only the necessary support; “Offend” corresponds to only using support in potentially offending transitions; “Tactics” reports the running time needed using some basic tactics like lazy instantiation and formula normalization and propagation. TO represents a timeout of 30 minutes. Our results indicate that, in practice, tactics are essential for efficiency when handling non-trivial examples such as concurrent lists. Even though our decision procedures have room for improvements, these results suggest that trying to compute an over-approximation of the reachable state space for complicated algorithms by iteratively computing formulas is not likely to be feasible for complicated heap manipulating programs.
5 Concluding Remarks

This paper has introduced a temporal deductive technique for the uniform verification problem of safety properties of infinite state processes, in particular for the verification of concurrent datatypes that manipulate data in the heap. Our proof rules automatically discharge a finite collection of verification conditions, which depend on the program description and the diameter of the formula to prove, but not on the number of threads in a particular instance. Each VC describes a small-step in the execution of all corresponding instances. The VCs are quantifier-free as long as the formulas are quantifier free. We use the theory of arrays [6] to encode the local variables of a system with an arbitrary number of threads, but the dependencies with arrays can be eliminated, under the assumption of full symmetry. It is immediate to extend our framework to a finite family of process classes, for example to model client/server systems.

Future work includes invariant generation to simplify or even automate proofs. We are studying how to extend the decision procedures with the calculation of weakest precondition formulas (like [20]) and its use for parametrized systems effectively to infer invariants, possibly from the target invariant. We are also studying how to extend the “invisible invariant” approach [20] to processes that manipulate infinite state, by instantiating small systems with a few threads and limiting the exploration to only states where data is limited in size as well. All candidate invariants produced must then be verified with the proof rules presented here for the general system.

We are also extending our previous work on abstract interpretation-based invariant generation for parametrized systems [28] to handle complex datatypes. Our work in [28] was restricted to numerical domains.

Finally, another approach that we are currently investigating is to use the proof rules presented here to enable a Horn-Clause Verification engine [17] to automatically generate parametrized invariants guided by the invariant candidate goal. Our preliminary results are very promising but out of the scope of this paper.

From a theoretical viewpoint the rule sp-inv is complete (all invariants can be proved by support inductive invariants), but the proof of completeness is rather technical and is also out of the scope of this paper.

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A Infinite State Mutual Exclusion Examples

Example: A Parametrized Mutual Exclusion Algorithm. Consider the program in Fig. 9(b) which implements mutual exclusion using a simple ticket-based protocol. Each thread that wants to access the critical section at line 5, acquires a unique increasing number (ticket) and announces its intention to enter the critical section by adding the ticket to a shared global set of tickets. Then, the thread waits until its ticket becomes the lowest value in the set before entering the critical section. After a thread leaves the critical section it removes its ticket from the set. SetMutExc uses two global variables: avail, of type Int, which stores the shared counter; and bag, of type Set(Int), which stores the set of all threads that are trying to access the critical section. For any instance (number of threads) the concrete system is an infinite state program, since the available ticket is ever increasing. Program IntMutExc in Fig. 9(a) is similar except that it stores the minimum value in a global variable of type Int.

Example 2. Consider program SetMutExc in Fig. 9(b). The instance consisting of two running threads, SetMutExc[2], contains the following variables:

\[
V = \{ \text{avail}, \text{bag}, \text{ticket}[0], \text{ticket}[1], \text{pc}[0], \text{pc}[1] \}
\]

Global variable avail has type Int, and global variable bag has type Set(Int). The instances of local variable ticket for threads 0 and 1, ticket[0] and ticket[1], have type Int. The program counters pc[0] and pc[1] have type Loc = \{1 \ldots 6\}. The initial condition of SetMutExc[2] specifies that:

\[
\begin{align*}
\Theta_g : \text{avail} & = 0 \land \text{bag} = \emptyset \\
\Theta_l[0] : \text{ticket}[0] & = 0 \land \text{pc}[0] = 1 \\
\Theta_l[1] : \text{ticket}[1] & = 0 \land \text{pc}[1] = 1
\end{align*}
\]

(a) IntMutExc, using two counters

(b) SetMutExc, using a set of integers

Fig. 9: Two implementations of a ticket based mutual exclusion protocol
There are fourteen transitions in \( \text{SETMutExc}^2 \), seven transitions for each thread: \( \tau_1[0] \ldots \tau_7[0] \) and \( \tau_1[1] \ldots \tau_7[1] \). The transitions corresponding to thread 0 are:

\[
\begin{align*}
\tau_1[0] : pc[0] = 1 & \land pc'[0] = 2 \land \pres(V \setminus \{pc[0]\}) \\
\tau_2[0] : pc[0] = 2 & \land pc'[0] = 3 \land \pres(V \setminus \{pc[0]\}) \\
\tau_3[0] : pc[0] = 3 & \land pc'[0] = 4 \land \pres(V \setminus \{pc[0]\}) \\
\tau_4[0] : pc[0] = 4 & \land pc'[0] = 5 \land \pres(V \setminus \{pc[0]\}) \\
\tau_5[0] : pc[0] = 5 & \land pc'[0] = 6 \land \pres(V \setminus \{pc[0]\}) \\
\tau_6[0] : pc[0] = 6 & \land pc'[0] = 7 \land \pres(V \setminus \{pc[0]\}) \\
\tau_7[0] : pc[0] = 7 & \land pc'[0] = 1 \land \pres(V \setminus \{pc[0]\}) \\
\end{align*}
\]

The transitions for thread 1 are analogous. The predicate \( \pres \) summarizes the preservation of variables’ values. For example, in \( \text{SETMutExc}^2 \), the predicate \( \pres(V \setminus \{pc[0]\}) \) is simply:

\[
\begin{align*}
\text{avail}' = \text{avail} \land \text{ticket}'[0] = \text{ticket}[0] \land \text{pc}'[1] = \text{pc}[1] \land \text{ticket}'[1] = \text{ticket}[1].
\end{align*}
\]

### B Empirical Evaluation: Mutual Exclusion

**Mutual Exclusion for \text{INTMutExc}:** For the programs described in Fig.9 we use \( \text{active}(k) \) for \( (pc(k) = 4, 5, 6) \) and \( \text{critical}(k) \) for \( (pc(k) = 5, 6) \). Mutual exclusion is specified as:

\[
\begin{align*}
\text{mutex}(i, j) & \equiv \Box(i \neq j \rightarrow \neg(\text{critical}(i) \land \text{critical}(j)))
\end{align*}
\]

Using the P-INV rule to prove \( \text{mutex} \) fails for \( \tau_4^{(i)} \), described as:

\[
\begin{align*}
\text{mutex}(i, j) \land \\
& \left( \begin{array}{l}
\text{pc}(i) = 4 \land \text{pc}' = \text{pc}\{i \leftarrow 5\} \\
\text{ticket}(i) = \text{min} \\
\pres(\text{avail}, \text{min}, \text{ticket}(i), \text{ticket}(j))
\end{array} \right) \rightarrow \text{mutex}'(i, j)
\end{align*}
\]

The SMT Solver reports two counter models:

1. \( pc(j) = 5 \land \text{min} = 1 \land \text{avail} = 2 \land \text{ticket}(i) = 1 \land \text{ticket}(j) = 3 \)
2. \( pc(j) = 5 \land \text{min} = 1 \land \text{avail} = 2 \land \text{ticket}(i) = 1 \land \text{ticket}(j) = 1 \)

The decision procedure builds models that show that the VC is not valid. Hence, \( \text{mutex} \) is not inductive. The formula \( \text{mutex}(i, j) \) does not encode two important aspects of the program. First, if a thread is in the critical section, then it owns
the minimum announced ticket (unlike in counter-model 1). Second, the same ticket cannot be given to two different threads (unlike in counter-model 2). Two new auxiliary support invariants encode these facts:

\[
\begin{align*}
\text{minticket}(i) & \equiv \Box (\text{critical}(i) \rightarrow \text{min} = \text{ticket}(i)) \\
\text{notsame}(i, j) & \equiv \Box (i \neq j \land \text{active}(i) \land \text{active}(j) \rightarrow \text{ticket}(i) \neq \text{ticket}(j))
\end{align*}
\]

Now, \text{mutex} can be verified using \text{sp-inv} with \text{minticket} and \text{notsame} as support. Unfortunately, \text{minticket} is not inductive. The solver reports that if two different threads \(i\) and \(j\) are in the critical section with the same ticket and \(\tau_j^{(j)}\) is taken, then \text{minticket}(i) does not hold any longer. Hence, we need \text{notsame} as support for \text{minticket}. However, \text{notsame} in not inductive either. In this case, the offending transition is \(\tau_3\) when an existing ticket is reused. The following invariant precludes that case:

\[
\text{activelow}(i) \equiv \Box (\text{active}(i) \rightarrow \text{ticket}(i) < \text{avail})
\]

The candidate \text{activelow} is inductive (provable using \text{p-inv}) and supports \text{notsame}.

**Mutual Exclusion for \text{SETMutExc}**: We proceed in a similar way. The invariants \text{mutexS}, \text{notsameS} and \text{activelowS} are identically to \text{mutex}, \text{notsame} and \text{activelow}, but \text{minticketS} is defined as follows:

\[
\text{minticketS}(i) \equiv \Box (\text{critical}(i) \rightarrow \text{bag.min} = \text{ticket}(i))
\]

Similarly, \text{minticketS} and \text{notsameS} support \text{mutexS}, but this time, \text{minticketS} requires \text{activelowS} in addition to \text{notsameS} as support. The extra support is needed to encode that a thread taking transition \(\tau_3\) adds to \text{bag} a value strictly greater than any other previously assigned ticket. Finally, \text{notsameS} relies on \text{activelowS}, which again, is inductive.

Fig. 10 shows the proof graphs used for the empirical evaluation reported in Fig. 7 in Section 4.

**C Fully Symmetric Parallelism**

Even though the parametrized rules \text{p-inv} and \text{sp-inv} are sound for all parametrized systems, these rules are particularly useful for symmetric systems.
Intuitively, a parametrized transition system $S[M]$ is symmetric whenever the roles of thread ids are interchangeable, in the sense that swapping two thread ids in a given run produces another legal run that satisfies the corresponding temporal properties (with the ids swapped in the property as well). This notion of symmetry is semantic, but there are simple syntactic characteristics of programs that immediately guarantee symmetry. For example, if the only comparisons between thread identifiers in the program are for equality and inequality, then the system is fully symmetric. In this section, we introduce a semantic notion of symmetry and identify syntactic restrictions on programs that guarantee this notion of symmetry.

We show now some basic properties of fully symmetric systems. The essential semantic element to capture symmetry is a function $\pi_{ij}^t$ for each sort $t$, that defines the effect in elements of $t$ of swapping threads $i$ and $j$. For most of the sorts, like int, bool and Loc this function is simply the identity, because thread identifiers do not interfere with values of these types. For $\text{tid}$, $\pi_{ij}^{\text{tid}}$ is:

$$
\pi_{ij}^{\text{tid}}(e) = \begin{cases} i & \text{if } e = j \\
 j & \text{if } e = i \\
 e & \text{otherwise}
\end{cases}
$$

For sorts that involve thread identifiers (if present in the program), like containers, sets, registers, etc storing elements of sort $\text{tid}$ one can easily define these semantic maps.

Then, to characterize the effect in a run of a system of swapping two thread ids, we define the following maps:

- a model transformation map $\pi_{ij}^M$, which given a first-order model of the theories involved, characterizes the transformed model over the same domains.
- a syntax transformation map $\pi_{ij}^E$, that allows to transform terms and predicates. For variables of sort $\text{tid}$, the actual value is assigned in a concrete interpretation, so the swap between the ids is delegated to the interpreted function $\text{swap}$ added to the theory of thread identifiers:

$$
\text{swap}^M(i, j, k) = \begin{cases} i & \text{if } k = j \\
 j & \text{if } k = i \\
 k & \text{otherwise}
\end{cases}
$$

- from $\pi_{ij}^M$ and $\pi_{ij}^E$, we define the state transformation $\pi_{ij}^S$, that gives the program state obtained by swapping thread identifiers. Essentially, the valuation given to a transformed variable is the transformation of the value given to the original variable.

- finally, $\pi_{ij}^T$ that allows to obtain the transition identifier that corresponds to a given transition when the roles of two threads are exchanged.

Formally, the semantic maps $\pi_{ij}^M$, $\pi_{ij}^E$, $\pi_{ij}^S$ and $\pi_{ij}^T$ are
\[
\begin{array}{l}
\pi_{ij}^M(e : \text{tid}^M) = \begin{cases}
  i & \text{if } e = j \\
  j & \text{if } e = i \\
  e & \text{if } e \neq i, j
\end{cases} \\
\pi_{ij}^M(e : t^M) = \pi_{ij}^M(e) & \text{if } t \neq \text{tid} \\
\pi_{ij}^M(f^M) = \lambda x. \pi_{ij}^M(f^M(\pi_{ij}^M x)) \\
\pi_{ij}^M(P^M) = \{(\pi_{ij}^M x_1, \ldots, \pi_{ij}^M x_k) | P^M(x_1, \ldots, x_k)\}
\end{array}
\]
\[
\begin{array}{l}
\pi_{ij}^E(k : \text{tid}) = \begin{cases}
  i & k = j \\
  j & k = i \\
  k & k \neq i, j
\end{cases} \\
\pi_{ij}^E(v : \text{tid}) = \text{swap}(i, j, v) \\
\pi_{ij}^E(v[k] : \text{tid}) = \text{swap}(i, j, v[\pi_{ij}^E(k)]) \\
\pi_{ij}^E(c : t) = c \\
\pi_{ij}^E(v : t) = v \\
\pi_{ij}^E(v[k] : t) = v[\pi_{ij}^E(k)] \\
\pi_{ij}(f(t_1 \ldots t_n)) = f(\pi_{ij}(t_1) \ldots \pi_{ij}(t_n)) \\
\pi_{ij}^E(P(t_1 \ldots t_n)) = P(\pi_{ij}^E(t_1) \ldots \pi_{ij}^E(t_n))
\end{array}
\]
\[
\pi_{ij}^T(t)[k] = \begin{cases}
  \tau_i[i] & \text{if } k = j \\
  \tau_i[j] & \text{if } k = i \\
  \tau_i[k] & \text{if } k \neq i, j
\end{cases}
\]
\[
\pi_{ij}^S(s)(v) = \pi_{ij}^M(s(\pi_{ij}^E(v)))
\]

The essential building block used to define these transformation maps is a swapping function \(\pi_{ij}^t\) for each sort \(t\), that maps elements in a model of the sort \(t\) to the transformed elements in \(t\). This function characterizes the effect that swapping \(i\) and \(j\) has on elements of \(t\). For most of the sorts, like \(\text{int}\), \(\text{bool}\) and \(\text{Loc}\) this function is simply the identity, because thread identifiers are not related to values of these types. For sorts that involve thread identifiers, like set of threads \(\text{settid}\), for example, one can define:

\[
\pi_{ij}^{\text{settid}}(S) = (S \setminus \{i, j\}) \cup (S \setminus \{i, j\}) \cup (\{j \text{ if } i \in S\} \cup \{i \text{ if } j \in S\})
\]

Similar transformations can easily be defined for containers, registers, etc containing elements of sort \(\text{tid}\). To guarantee full symmetry all basic transformations \(\pi_{ij}^t\) must satisfy:

\[
\pi_{ij}^t \circ \pi_{ij}^t = \text{id}^t. \quad (2)
\]

From (2) it follows that \(\pi_{ij}^M\) satisfies \(\pi_{ij}^M \circ \pi_{ij}^M = \pi_{ij}^M \circ \pi_{ij}^M = \text{id}^M\).

For local program variables, the index is known (it is part of the variable name), so the transformation gives the name of the transformed variable. However, for variables of sort \(\text{tid}\), the actual value is assigned in a concrete interpretation, so the swap between the ids is delegated to the interpreted function \(\text{swap}\) added to the theory of thread identifiers. Note that for every first-order signature \(\pi_{ij}^E\) is uniquely determined. The following commutativity condition is a health condition on the transformation functions \(\pi_{ij}^M\) and \(\pi_{ij}^E\), where \([[]]\) is an interpretation map (that gives a model in the appropriate domain to each term and a truth value to each predicate):

\[
[[\pi_{ij}^E]] = \pi_{ij}^M[[t]] \quad [[\pi_{ij}^E P]] = \pi_{ij}^M[[P]] \quad (3)
\]
To show that $S$ satisfies $\Box \varphi(j)$. Find $\psi(k)$ with:

$S1. \quad S2. \quad S3. \quad U1. \quad U2. \quad U3. \quad S4. \quad \bigwedge_{\sigma \in S} \psi \sigma \land \varphi \land \tau^{(i)} \rightarrow \varphi'$ for all $\tau, i \in j$, and

$S = \text{Arr}(k, k \cup j)$

$S5. \quad \bigwedge_{i \in j} i \neq x \land \bigwedge_{\sigma \in S} \psi \sigma \land \varphi \land \tau^{(i)} \rightarrow \varphi'$ for all $\tau, i \notin j$, and

$S = \text{Arr}(k, k \cup j \cup \{i\})$

Fig. 11: The parametrized strengthening invariance rule $\text{sp-inv}$

This condition ensures that the interpretation obtained after transforming expression $e$, corresponds to the model transformation of the interpretation of $e$. Finally, note in the definition of $\pi^S_{ij}$ that first $\pi^E_{ij}$ exchanges $v[i]$ into $v[j]$, then the interpretation $s$ is used, and finally then the result is transformed according to the model transformation function $\pi^M_{ij}$.

Now we are ready to define the condition for a system to be symmetric.

**Definition 2 (Fully-Symmetric System).** A parametrized system $S$ is fully-symmetric whenever for all $M$, and for all $i, j \in [M]$, the following hold for all states $s$ and $s'$, transition $\tau$ and predicate $P$:

1. $s \models \Theta$ if and only if $(\pi^S_{ij} s) \models \Theta$.
2. $\tau(s, s')$ holds if and only if $(\pi^T_{ij} \tau)(\pi^S_{ij} s, \pi^S_{ij} s')$ holds.
3. $s \models P$ if and only if $(\pi^S_{ij} s) \models (\pi^P_{ij} P)$.

Full symmetry allows to reason about a particular thread, and conclude the properties for arbitrary threads.

**Lemma 1.** Let $S$ be a fully-symmetric system, $\varphi(k)$ be a parametrized formula with free variables $k : \{k_0 \ldots k_n\}$, and $N$ an arbitrary size.

$$S \models \Box \varphi(k) \iff S[N] \models \bigwedge_{i_0, \ldots, i_n \in [n]} (\Box \varphi[i_0, \ldots, i_n])$$

It is important to note that the range of the concrete indices is $[n]$, independent of the number of running threads $N$.

**Corollary 1.** For every fully symmetric $S$ and property $\varphi(k)$

$$S \models \Box \varphi(k) \iff \text{for every } N, S[N] \models \Box \varphi[0]$$

The previous results justify the version of the strengthening invariance rule $\text{sp-inv}$ in Fig. 11 where $\text{Arr}(k, j)$ is the set of substitutions of the form $\sigma : k \rightarrow j$.

Finally, for fully symmetric systems:
Theorem 5 (Concretization). Let $\varphi(k)$ be with $|k| = n$. Then:

$$\varphi(k) \text{ is valid } \iff \bigwedge_{\alpha \in A} \alpha(\varphi) \text{ is valid}$$

where $A = \text{Arr}(k, [n])$ is the set of concretizations of variables in $\text{Var}(\varphi)$.

For example, if one intends to prove that $p(i)$ is inductive, the concretization theorem allows to reduce $P3$ in $p$-INV to:

$$p[0] \land \tau[1] \rightarrow p'[0] \tag{4}$$

where $p[0]$ is a short for $\alpha(p(i))$ with $\alpha : i \rightarrow 0$. Formula (4) involves no arrays. Similarly, to show $\Box p(i)$ with support invariant $q(j)$, rule S5 can be reduced to:

$$q[0] \land q[1] \land p[0] \land \tau[1] \rightarrow p'[0]$$

In practice, the concretization is not performed upfront before discharging the verification condition to the SMT-Solver. Our use of arrays to encode parameterized formulas can be handled using the theory of uninterpreted functions and let the solver perform the search and propagation.

D Detailed Invariants for Case Studies

We prove that the program in Fig. 2 satisfies: (1) list shape preservation; and (2) the list implements a set, whose elements correspond to those stored in $\text{elems}$. The theory $\text{TLL3}$ allows to reason about addresses, elements, locks, sets, order, cells (i.e., list nodes), memory and reachability. A cell is a struct containing an element, a pointer to next node in the list and lock to protect the cell. A lock is associated with operations $\text{lock}$ and $\text{unlock}$ to acquire and release. The memory ($\text{heap}$) is modeled as an array of cells indexed by addresses. The specification is:

$$\varphi_{\text{lst}} := \begin{cases} 
\text{null} \in \text{reg} \land \text{reg} = \text{addr2set}(\text{heap}, \text{head}) \land \text{head} \neq \text{tail} \land \text{heap[tail].next} = \text{null} \land \text{tail} \neq \text{null} \land \text{head} \neq \text{null} \land \text{heap[head].data} = -\infty \land \text{heap[tail].data} = +\infty \land \text{elems} = \text{set2elemset}\left(\text{heap, reg}\right) \land \text{Ordered}\left(\text{heap, head, tail}\right) & \quad (L1) \\
\end{cases}$$

Formula $\varphi_{\text{lst}}$ is 0-index since it only constrains global variables. (L1) establishes that $\text{null}$ belongs to $\text{reg}$ and that $\text{reg}$ is exactly the set of addresses reachable in the $\text{heap}$ starting from $\text{head}$, which ensures that the list is acyclic. (L2) and (L3) express some sanity properties of the sentinel nodes $\text{head}$ and $\text{tail}$. Finally, (L4) establishes that $\text{elems}$ is the set of elements in cells referenced by addresses in $\text{reg}$, and that the list is ordered. The main specification is $\text{list}$, defined as $\Box \varphi_{\text{lst}}$.

\text{LEAP} can establish that $\text{list}$ holds initially, but fails to prove that $\text{list}$ is preserved by all transitions (i.e., $\text{list}$ is not a parametrized invariant), so support invariants are required. To prove (L1) the support invariant $\varphi_{\text{reg}}$ captures how
addresses are added and removed from \( \text{reg} \) in the program. Local variable \( v \) in procedure \text{MGC}, \text{SEARCH}, \text{INSERT} \) and \text{REMOVE}, is denoted by \( v_C, v_S, v_l \) and \( v_R \) respectively:

\[
\varphi_{\text{reg}(i)} = \begin{cases} 
\{ \text{head}, \text{tail}, \text{null} \} \subseteq \text{reg} \land \text{tail} \neq \text{null} \land \text{head} \neq \text{tail} & \land \\
\text{pc}(i) = 24..39 \rightarrow \text{prev}_1(i) \in \text{reg} \land \text{pc}(i) = 26..40 \rightarrow \text{curr}_1(i) \in \text{reg} & \land \\
\text{pc}(i) = 33..34 \rightarrow \neg \text{aux}_1(i) \in \text{reg} \land \text{pc}(i) = 30 \rightarrow \text{aux}_1(i) \in \text{reg} & \land \\
\text{pc}(i) = 43..57 \rightarrow (\text{prev}_R(i) \cap \{ \text{tail}, \text{null} \} = \emptyset \land \text{prev}_R(i) \in \text{reg}) & \land \\
\text{pc}(i) = 45..55 \rightarrow (\text{curr}_R(i) \neq \text{null} \land \text{curr}_R(i) \in \text{reg}) & \land \\
\text{pc}(i) = 49 \rightarrow \text{aux}_R(i) \in \text{reg} & \land 
\end{cases}
\]

Formula \( \varphi_{\text{reg}} \) is 1-index and determines which addresses belong to \( \text{reg} \) depending on each program location. Invariant \( \text{region}(i) \) is defined as \( \square \varphi_{\text{reg}(i)} \). Invariant \( \text{next}(i) \) captures the relative position in the list of the cells pointed by \( \text{head} \) and \( \text{tail} \) and local variables \( \text{prev}, \text{curr} \) and \( \text{aux} \). The details of can be found in the appendix. Invariant \( \text{next}(i) \) is needed for (L2). To prove (L3) and (L4) we need to show that order is preserved. We express this constraint with formula \( \varphi_{\text{ord}} \):

\[
\varphi_{\text{ord}(i)} = \begin{cases} 
\text{heap}[^{\text{head}}].data = -\infty \land \text{heap}[^{\text{tail}}].data = +\infty & \land \\
\text{pc}(i) = 3..7 \rightarrow e_C(i) \notin \{ \pm \infty \} \land \text{pc}(i) = 8..21 \rightarrow e_S(i) \notin \{ \pm \infty \} & \land \\
\text{pc}(i) = 23..40 \rightarrow e_1(i) \notin \{ \pm \infty \} \land \text{pc}(i) = 42..58 \rightarrow e_R(i) \notin \{ \pm \infty \} & \land \\
\text{pc}(i) = 26..40 \rightarrow \text{heap}[^{\text{curr}_1(i)}].data \leq +\infty & \land \\
\text{pc}(i) = 24..39 \rightarrow \text{heap}[^{\text{prev}_1(i)}].data \leq +\infty & \land \\
\text{pc}(i) = 28..31 \rightarrow \text{heap}[^{\text{curr}_1(i)}].data < e_1(i) & \land \\
\text{pc}(i) = 24..37 \rightarrow \text{heap}[^{\text{prev}_1(i)}].data < e_1(i) & \land \\
\text{pc}(i) = 35..37 \rightarrow e_1(i) < \text{heap}[^{\text{curr}_1(i)}].data & \land \\
\text{pc}(i) = 33..34 \rightarrow \text{heap}[^{\text{aux}_1(i)}].data = e_1(i) & \land \\
\text{pc}(i) = 54..55 \rightarrow \text{heap}[^{\text{curr}_R(i)}].data = e_R(i) & \land 
\end{cases}
\]

and define invariant \( \text{order}(i) \) as \( \square \varphi_{\text{ord}(i)} \). Invariant \( \text{lock} \) captures those program locations at which a thread owns a cell in the heap:

\[
\varphi_{\text{lock}(i)} = \begin{cases} 
\text{pc}(i) = 25..39 \rightarrow \text{heap}[^{\text{prev}_1(i)}].\text{lockid} = i & \land \\
\text{pc}(i) = 27..31, 34..40 \rightarrow \text{heap}[^{\text{curr}_1(i)}].\text{lockid} = i & \land \\
\text{pc}(i) = 30 \rightarrow \text{heap}[^{\text{aux}_1(i)}].\text{lockid} = i & \land \\
\text{pc}(i) = 44..57 \rightarrow \text{heap}[^{\text{prev}_R(i)}].\text{lockid} = i & \land \\
\text{pc}(i) = 46..50, 53..58 \rightarrow \text{heap}[^{\text{curr}_R(i)}].\text{lockid} = i & \land \\
\text{pc}(i) = 49 \rightarrow \text{heap}[^{\text{aux}_R(i)}].\text{lockid} = i & \land 
\end{cases}
\]

Finally, formula \( \varphi_{\text{dis}} \) encodes that two different threads calls to \text{malloc} return two different addresses:

\[
\varphi_{\text{dis}(i, j)} = (i \neq j \land \text{pc}(i) = 33..34 \land \text{pc}(j) = 33..34) \rightarrow \text{aux}_1(i) \neq \text{aux}_1(j)
\]
In this case, \( \text{dis}(i, j) \), defined as \( \square \phi_{\text{dis}}(i, j) \) is a 2-index invariant.

In practice, when proving concurrent datatypes these candidate invariants are easily spotted using the information obtained from an unsuccessfully attempt of LEAP to prove a particular VC. LEAP parses the counterexample (model) returned by the SMT solver, which is usually very small, involves few threads and allows to easily understand the missing intermediate facts. Fig.8 shows the proof graph for the verification of concurrent lock-coupling lists. In the graphs, a dashed arrow from \( \varphi \) to \( \psi \) denotes that \( \varphi \) is used as support for \( \psi \). LEAP parses proof graphs as input and applies G-INV when necessary. Additionally, graphs can specify program locations for which to apply a particular formula as support, which greatly speeds proof checks. Fig. 7 shows the results of this empirical evaluation, executed on a computer with a 2.8 GHz processor and 8GB memory. The columns show the index of the formula; the total number of generated VCs; the number of VCs verified using the position based DP; the number of VCs verified using the specialized DP and, finally, the total running time in seconds required to verify all VCs. We benchmark the times in four different scenarios using different tactics. The first scenario (FS) uses SP-INV with full support, that is, all invariant candidates are used as support. The second scenario (FA) considers only full assignments when generating support. The third scenario (FA-SS) involves full assignments in addition to discarding superfluous support information. The last column reflects the forth scenario, using proof graphs. We use OM to represent out-of-memory failure. These results show that, in practice, tactics are essential for efficiency when handling non-trivial examples such as concurrent lists.

## E Proof Graph for Concurrent Lock-coupling Lists

We present now the full proof graph for this concurrent lock-coupling single-linked lists implementation. We use the following notation to represent a proof-graph:

\[
\rightarrow \text{inv} \ [11:P:{sup1, sup2, sup3};
\quad 12:P:{sup4, sup5}]
\{ \text{SMP : pretactic | posttactic} \}
\]

where \text{inv} is the invariant candidate to be verified. Next, between brackets it is possible to specify invariant support for specific program locations. This argument is optional. Required support invariants are provided as a list of support rules, separated by \( ; \). Each support rule consists on a location, a possible premise identifier to localize the support generation on a specific invariant rule premise and a list of invariants to be used as support. Finally, it is possible to describe the method used to compute the domain bounds for the small model property, as well as tactics to be used in support generation and formula simplification, separated by a \( | \).

Fig.12 shows the proof graph for the current lock-coupling list example.
Fig. 12: Proof graph representation for concurrent lock-coupling lists