SLOW EXPONENTIAL GROWTH REPRESENTATIONS OF $\text{SP}(n, 1)$ AT THE EDGE OF COWLING’S STRIP

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Abstract. We obtain a slow exponential growth estimate for the spherical principal series representation $\rho_s$ of the Lie group $\text{Sp}(n, 1)$ at the edge $(\text{Re}(s) = 1)$ of Cowling’s strip ($|\text{Re}(s)| < 1$) on the Sobolev space $H^\alpha(G/P)$ when $\alpha$ is the critical value $Q/2 = 2n + 1$. As a corollary, we obtain a slow exponential growth estimate for the homotopy $\rho_s$ ($s \in [0, 1]$) of the spherical principal series which is required for the first author’s program for proving the Baum–Connes conjecture with coefficients for $\text{Sp}(n, 1)$.

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Introduction

Let $G = \text{SO}_0(n, 1)$, resp. $\text{SU}(n, 1)$, resp. $\text{Sp}(n, 1)$, resp. $F_{4(-20)}$, a simple Lie group of real rank one. Consider its spherical principal series representation $\rho_s(g)$ on $C^\infty(G/P)$ for $s \in \mathbb{C}$ where $P$ is a minimal parabolic subgroup of $G$. Let us normalize the parameters $s$ so that $\rho_{it}(g)$ is unitary for $t \in \mathbb{R}$ with respect to the canonical $L^2$-norm on $G/P$ and so that it contains the trivial sub-representation at $s = 1$. More specifically, we have

$$\rho_s(g) = \lambda_{it}^{Q/2(1-s)} \rho(g)$$

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on $C^\infty(G/P)$ where $\rho(g)$ is the left-translation by $g$, $\lambda_g$ is a cocycle, and $Q = n - 1, 2n, 4n + 2, 22$ respectively.

The picture of the unitary dual is rather different between the case of $SO_0(n, 1)$ and $SU(n, 1)$, which have the Haagerup property (or Gromov’s a-T-menability), and the case of $Sp(n, 1)$ and $F_{4(-20)}$ which have Kazhdan’s property (T) (see [6, Chapter 5]). The complementary series (i.e. the $\rho_s$, $s$ real, which are unitary for a suitable Hilbert scalar product) is defined for $-1 < s < 1$ in the first case, whereas in the second case a gap appears between the trivial representation and the complementary series, which ranges for $-\frac{2n-1}{2n+1} < s < \frac{2n-1}{2n+1}$ for $Sp(n, 1)$, $-\frac{5}{11} < s < \frac{5}{11}$ for $F_{4(-20)}$ (see [9]).

However, it was pointed out in the 1980’s by Michael Cowling that if one replaces “unitary” by “uniformly bounded”, the difference disappears (see [4]). Namely, the representation $\rho_s$ for $s$ in the strip $-1 < \text{Re}(s) < 1$ becomes a uniformly bounded representation of $G$ on a Hilbert space, if we consider an appropriate Sobolev space. Here, a representation is uniformly bounded in a sense that the operator norm $||\rho_s(g)||$ of $\rho_s(g)$ is bounded by a constant $C > 0$ which is independent of $g$ in $G$.

Analysis on Heisenberg groups plays an important role as a nilpotent subgroup $N$ of $G$ provides the so-called open picture of the principal series representation $\rho_s$. For $-1 < \text{Re}(s) < 1$, the representation $\rho_s$ is equivalently represented on the homogeneous Sobolev space $\mathcal{H}^\alpha(N)$ on $N$ with respect to a sub-Laplacian on $N$. This picture has an advantage that it uses the canonical norm for which $\rho_s(g)$ is unitary for $g$ in $P$. It then suffices to show that a single element $w$ in $G$ is bounded to see $\rho_s$ is uniformly bounded. On the other hand, for $s$ outside the strip $-1 < \text{Re}(s) < 1$, $w$ is not bounded with respect to such a norm and hence the representation itself would not be well-defined.

In the compact picture, Cowling’s result reads as follows: the representation $\rho_s$ on the Sobolev space $\mathcal{H}^\alpha(G/P)$ for $s$ in the strip $-1 < \text{Re}(s) < 1$ is uniformly bounded for $\alpha = (Q/2)s$ (see [1]). Here, the Sobolev space $\mathcal{H}^\alpha(G/P)$ is defined as the completion of $C^\infty(G/P)$ with respect to the Euclidean norm $|| (1 + \Delta_E)^{\alpha/2} \xi ||_{L^2(G/P)}$ where $\Delta_E$ is a K-invariant sub-Laplacian on $G/P$ for a maximal compact subgroup $K$ of $G$. The compact picture has an advantage that the representation $\rho_s$ on $\mathcal{H}^\alpha(G/P)$ is well-defined (bounded) for all $s$, in particular for $s = 1$.

In the first author’s program for proving the Baum–Connes conjecture with coefficients for $Sp(n, 1)$ [7], it is crucial to consider the growth of the operator norm $||\rho_s(g)||_{\mathcal{H}^\alpha(G/P) \to \mathcal{H}^\alpha(G/P)}$ of the representation $\rho_s(g)$ on $\mathcal{H}^\alpha(G/P)$ for $\alpha = (Q/2)s$ and $0 \leq s \leq 1$ as $s$ approaches to 1 (see [7, Section 8.3]). In particular, we would like to show that the homotopy $\rho_s(g)$ ($s \in [0, 1]$) of representations has slow exponential growth in a sense that for any $\epsilon > 0$, there is $C = C(\epsilon) > 0$ such that

$$||\rho_s(g)||_{\mathcal{H}^\alpha(G/P) \to \mathcal{H}^\alpha(G/P)} \leq Ce^{l(g)}$$

for all $s \in [0, 1]$ and for all $g$ in $G$. Here, $l(g)$ is a $K$-bi-invariant length function on $G$ defined as $l(g) = d_{G/K}(gK, K)$, i.e. $l(ka_kk') = |t|$ for any $k, k' \in K$ and $t \in \mathbb{R}$.

With an application to this problem in mind, Astengo, Cowling and Di Blasio obtained a similar type of estimates for $\rho_s(g)$ on $\mathcal{H}^\alpha(G/P)$. They showed that [11, Theorem 5.1] for any $\alpha \in (-Q/2, Q/2)$ fixed, there is $C > 0$ such that

$$||\rho_s(g)||_{\mathcal{H}^\alpha(G/P) \to \mathcal{H}^\alpha(G/P)} \leq Ce^{l((Q/2)\text{Re}(s) - \alpha)l(g)}$$

for all $s \in \mathbb{C}$ and for all $g$ in $G$. 
Although the above estimate suggests that the Sobolev space $\mathcal{H}^{Q/2}(G/P)$ should provide a Hilbert space for which $\rho_s(g)$ has slow exponential growth when $\Re(s) = 1$, the estimate itself is not enough to conclude that the slow exponential growth of the homotopy $\rho_s(g)$ ($s \in [0, 1]$). Because of this, we shall study the growth of the representation $\rho_s(g)$ on $\mathcal{H}^\alpha(G/P)$ when $\alpha$ is the critical value $\alpha = Q/2$. The following is our main result:

**Theorem A.** (See Theorem 2.1) For any $\epsilon > 0$, there is $C = C(\epsilon) > 0$ such that for all $s$ satisfying $\Re(s) = 1$, we have the following upper-bound for the operator norm $||\rho_s(g)||_{\mathcal{H}^{Q/2}(G/P) \rightarrow \mathcal{H}^{Q/2}(G/P)}$ of $\rho_s(g)$ on $\mathcal{H}^{Q/2}(G/P)$ for all $g$ in $G$:

$$||\rho_s(g)||_{\mathcal{H}^{Q/2}(G/P) \rightarrow \mathcal{H}^{Q/2}(G/P)} \leq Ce^{\epsilon\ell(g)}(1 + |\Im(s)|)^{Q/2}.$$ 

As a corollary, by a simple application of complex interpolation, we show the desired slow exponential growth estimate for the homotopy $\rho_s(g)$ ($s \in [0, 1]$).

**Corollary B.** (See Corollary 2.2) For any $\epsilon > 0$, there is $C = C(\epsilon) > 0$ such that for any $s \in [0, 1]$, we have for all $g$ in $G$,

$$||\rho_s(g)||_{\mathcal{H}^{(Q/2)}(G/P) \rightarrow \mathcal{H}^{(Q/2)}(G/P)} \leq Ce^{\epsilon\ell(g)}.$$ 

We end our introduction by explaining the current status of the first author’s program for proving the Baum–Connes conjecture with coefficients for $\text{Sp}(n, 1)$ [7]. In [7], a BGG-cycle $(H, \pi, F)$ for the Kasparov’s ring $R(G) = KK^G(\mathbb{C}, \mathbb{C})$ was constructed. Let us call its class $\gamma_r = [H, \pi, F]$ in $R(G)$. In [7], the remaining problems were to prove the following:

1. The element $\gamma_r$ is equal to the gamma element $\gamma$ in $R(G)$ [7] Conjecture 1;
2. The element $\gamma_r$ is equal to the identity $1_G$ in $R_\epsilon(G)$ for any $\epsilon > 0$ [7] Conjecture 2).

See [7] Section 1] for the definition of the gamma element $\gamma$. It suffices to say that it is constructed for all almost connected groups and that the construction is based on the de-Rham complex on the Riemannian symmetric space $G/K$, whereas the element $\gamma_r$ is based on the BGG-complex on the spherical variety $G/P$. The ring $R_\epsilon(G)$ in the second item is defined in the same way as $R(G)$ except that the representations of $G$ on Hilbert spaces for defining cycles $(H, \pi, F)$ may have exponential growth $||\pi(g)|| \leq Ce^\epsilon$ (to be precise, one has to fix the parameter $C_\epsilon > 0$ for each $\epsilon$). As explained in [7], the validity of the two items imply the Baum–Connes conjecture with coefficients for $\text{Sp}(n, 1)$.

Our main result (Corollary 2.2) implies that the second item indeed holds.

**Theorem C.** The item (2) holds.

Therefore, we are left with the problem (1), which is essentially reduced to a problem of showing the compactness of the commutator $[S_0, f]$ for any continuous function $f$ on the disk $G/K \cup G/P$ and for a certain bounded operator $S_0$, which is essentially a Poisson transform (see [7] Section 7.4).

Here, it is perhaps worth to recall the following hypothesis made by Kasparov in [8].

**Hypothesis** ([8 Section 5.11]). For any almost connected group $G$, the restriction to a maximal compact subgroup $K$ determines an isomorphism $R_r(G) \cong R(K)$. 

Here, the ring $R_r(G)$ is defined in the in the same way as $R(G)$ except the representations of $G$ have to be weakly contained in the left-regular representations. There is a canonical map $R_r(G) \to R(G)$ which is an isomorphism precisely when $G$ is $K$-amenable and is never surjective for $G$ with Kazhdan’s property (T). If we denote the support of the gamma element by $\gamma R(G)$, there is a natural map

$$\gamma R(G) \to R_r(G).$$

Since we already know $\gamma R(G) \cong R(K)$, the hypothesis of Kasparov is equivalent to saying this map is an isomorphism. If this is the case, the item (1), $\gamma_r = \gamma$, would follow since both are elements in $R_r(G)$ that have the property that its restriction to $R(K)$ is the identity $1_K$. In other words, the validity of the hypothesis would immediately imply the item (1), and hence the Baum–Connes conjecture with coefficients for $Sp(n, 1)$. However, the hypothesis would be strictly harder to prove than the item (1), let alone we are not sure whether it is true in general.

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1. Preliminaries

Let $F = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$ be the field of real numbers, complex numbers or quaternions. For $z \in F$, we define

$$|z|^2 = z^*z, \quad \text{Re}(z) = \frac{z + z^*}{2}, \quad \text{Im}(z) = \frac{z - z^*}{2}.$$  

We also write $\text{Im}(F) \subset F$ to be the image of $\text{Im}( )$ on $F$.

A sesquilinear form $q$ on a right vector space $F^{n+1}$ over $F$ is given by

$$q(z, w) = -\bar{z}_0w_0 + \sum_{j=1}^{\text{Im}(\mathbb{H})} \bar{z}_j w_j$$

for $z, w$ in $F^{n+1}$.

Let $O(q)$ be the group of $(n + 1) \times (n + 1)$ matrices over $F$ which act on $F^{n+1}$ from left and preserve the quadratic form $q$.

The Lie group $SO_0(n, 1)$ is the connected component of the identity of $O(q)$ for $F = \mathbb{R}$, $SU(n, 1)$ is $O(q) \cap SL(n + 1, \mathbb{C})$ for $F = \mathbb{C}$ and $Sp(n, 1)$ is $O(q)$ for $F = \mathbb{H}$.

In this paper, we consider $G = Sp(n, 1)$ ($n \geq 2$) and thus $F = \mathbb{H}$ but all the results have obvious analogues for $SO_0(n, 1)$ and $SU(n, 1)$. The closed subgroup $K$ of $G$ that preserves the canonical Euclidean metric on $F^{n+1}$ is a maximal compact subgroup of $G$. 


The Lie group $G$ naturally acts on the projective space $\mathbb{P}(\mathbb{R}^{n+1})$ over $\mathbb{R}$ and we have
\[
G \cdot [1, 0, \cdots, 0]^T = \{ [z_0, z_1, \cdots, z_n]^T \in \mathbb{P}(\mathbb{R}^{n+1}) \mid \sum_{1 \leq j \leq n} |z_j|^2 < |z_0|^2 \} = \{ [1, z_1, \cdots, z_n]^T \in \mathbb{P}(\mathbb{R}^{n+1}) \mid \sum_{1 \leq j \leq n} |z_j|^2 < 1 \}.
\]

The isotropy subgroup of $G$ at the point $[1, 0, \cdots, 0]^T$ is $K$. In this way, $G/K$ can be viewed as the disk $\mathbb{D}^{4n}$ in $\mathbb{R}^n$.

The boundary of $G/K$ in $\mathbb{P}(\mathbb{R}^{n+1})$ is
\[
G \cdot [1, 0, \cdots, 0, 1]^T = \{ [z_0, z_1, \cdots, z_n]^T \in \mathbb{P}(\mathbb{R}^{n+1}) \mid \sum_{1 \leq j \leq n} |z_j|^2 = |z_0|^2 \} = \{ [1, z_1, \cdots, z_n]^T \in \mathbb{P}(\mathbb{R}^{n+1}) \mid \sum_{1 \leq j \leq n} |z_j|^2 = 1 \}.
\]

The isotropy subgroup $P$ of $G$ at the point $[1, 0, \cdots, 0, 1]^T$ is a minimal parabolic subgroup of $G$. In this way, $G/P$ can be viewed as the sphere $S^{4n-1}$ in $\mathbb{R}^n$.

Let $A$ be a closed subgroup of $G$ defined as
\[
A = \{ a_t \in G \mid t \in \mathbb{R} \}, \quad a_t = \begin{bmatrix} \cosh t & 0 & \sinh t \\ 0 & 1 & 0 \\ \sinh t & 0 & \cosh t \end{bmatrix} = U \begin{bmatrix} e^{-t} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^t \end{bmatrix} U^{-1},
\]
where each matrix has an $(n-1) \times (n-1)$ matrix in the middle entry and
\[
U = U^* = U^{-1} = \begin{bmatrix} -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}.
\]

We have $KA^+K$ decomposition
\[
G = KA^+K
\]
where $A^+$ consists of $a_t$ for $t \geq 0$. Let $M$ be the centralizer of $A$ in $K$. Consider the restricted root space decomposition of $\mathfrak{g}$ with respect to $a$:
\[
\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{m} \oplus \mathfrak{n} \oplus \mathfrak{n}^\ast.
\]

We have
\[
n = n_1 \oplus n_2,
\]
\[
n_1 = \{ U \begin{bmatrix} 0 & 0 & 0 \\ -X/\sqrt{2} & 0 & 0 \\ 0 & 0 & X/\sqrt{2} \end{bmatrix} U^{-1} = \begin{bmatrix} 0 & X^*/2 & 0 \\ X/2 & 0 & -X/2 \\ 0 & X^*/2 & 0 \end{bmatrix} \in \mathfrak{g} \mid X \in \mathbb{R}^{n-1} \},
\]
\[
n_2 = \{ U \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -Z & 0 & 0 \end{bmatrix} U^{-1} = \begin{bmatrix} Z/2 & 0 & -Z/2 \\ 0 & 0 & 0 \\ Z/2 & 0 & -Z/2 \end{bmatrix} \in \mathfrak{g} \mid Z \in \text{Im}(\mathbb{R}) \},
\]
and
\[
\mathfrak{n}^\ast = \mathfrak{n}_1 \oplus \mathfrak{n}_2,
\]
We shall identify elements in \( \pi \) with \( g \) as \( g = \exp(x) \) and we take a Haar measure given by the measure on \( \pi \)

\[
(1.1) \quad \pi_1 = \{ U \begin{bmatrix} 0 & X^*/\sqrt{2} & 0 \\ 0 & 0 & -X/\sqrt{2} \\ 0 & 0 & 0 \end{bmatrix} U^{-1} = \begin{bmatrix} 0 & -X^*/2 & 0 \\ -X/2 & 0 & -X/2 \\ 0 & X^*/2 & 0 \end{bmatrix} \in g \mid X \in \mathbb{F}^{n-1} \},
\]

\[
(1.2) \quad \pi_2 = \{ U \begin{bmatrix} 0 & 0 & -Z \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} U^{-1} = \begin{bmatrix} Z/2 & 0 & Z/2 \\ 0 & 0 & 0 \\ -Z/2 & 0 & -Z/2 \end{bmatrix} \in g \mid Z \in \text{Im}(F) \}.
\]

With respect to these expressions, we shall use the coordinates \((X, Z) \in \mathbb{F}^{n-1} \oplus \text{Im}(F)\) to express elements in \( \pi \). In these coordinates, the Lie bracket on \( \pi \) is

\[
[(X_1, Z_1), (X_2, Z_2)] = \langle 0, \text{Im}(X_1^*X_2) \rangle.
\]

The exponential map \( \exp: \pi \to N \subset G \) (as well as \( \exp: \pi \to N \)) is a diffeomorphism. For \((X, Z)\) in \( \pi \), we have

\[
\exp(X, Z) = \begin{bmatrix} 1 + X^*X/8 + Z/2 & -X^*/2 & X^*X/8 + Z/2 \\ -X/2 & 1 & -X/2 \\ -X^*X/8 - Z/2 & X^*/2 & 1 - X^*X/8 - Z/2 \end{bmatrix} \in \overline{N} \subset G.
\]

We shall identify \( \overline{N} = \mathbb{F}^{n-1} \oplus \text{Im}(F) \) and equip it with the standard Euclidean metric. The group \( \overline{N} \) is unimodular and we fix a Haar measure given by \( d\pi = 2^\dim(\pi) dX dZ \) on \( \overline{N} \). The point \( 0 \) corresponds to the sphere \( S^{4n-1} \subset \mathbb{F}^{n} \).

We have \( KAN \) decomposition

\[
G = KAN,
\]

Langlands decomposition

\[
P = MAN,
\]

and Bruhat decomposition

\[
G = PW P \sqcup P
\]

where \( w \in K \) is any representative of the non-trivial element in the Weyl group \( N_K(A)/Z_K(A) \).

Let \( o = [1, 0, \ldots, 0, 1]^T \) in \( G/P \subset \mathbb{F}^{(p^{n+1})} \). Recall that the minimal parabolic subgroup \( P \) is the isotropy subgroup of \( G \) at \( o \). The Cayley transform

\[
C: \overline{N} \to G/P
\]

is defined by sending \( n \) to \( n \cdot o = nP \) in \( G/P \). The Cayley transform is a diffeomorphism from \( \overline{N} \) onto an open dense subset \( G/P - \{ -o \} \) where \( -o = [1, 0, \ldots, 0, -1]^T = w \cdot o \).

Recall that \( G/P \) is naturally viewed as the sphere \( S^{4n-1} \subset \mathbb{F}^{n} \) and the point \( o \) corresponds to the point \( (0, \ldots, 0, 1) \) in the sphere \( S^{4n-1} \).

With this identification of \( G/P \subset \mathbb{F}^{(p^{n+1})} \) and \( S^{4n-1} \subset \mathbb{F}^{n} \), we have for \((X, Z) \in \mathbb{F}^{n-1} \oplus \text{Im}(F) = \pi\)

\[
C(\exp(X, Z)) = \exp(X, Z) \cdot o = \left( \begin{array}{c} -X \\ 1 - X^*X/4 - Z \end{array} \right) \left( \begin{array}{c} 1 + X^*X/4 + Z \end{array} \right)^{-1}
\]

\[
= \left( \begin{array}{c} -X(1 + X^*X/4 - Z) \\ 1 - (X^*X)^2/16 - |Z|^2 - 2Z \end{array} \right) \left( \begin{array}{c} (1 + X^*X/4)^2 + |Z|^2 \end{array} \right)^{-1}
\]
in $S^{4n-1} \subset \mathbb{F}^n$. Here, the first row represents a vector in $\mathbb{F}^n$ and the second row represents a vector in $\mathbb{F}$. Setting

$$||X||^2 = X^*X, \quad B(X, Z) = (1 + ||X||^2/4)^2 + |Z|^2,$$

we have

$$(1.4) \quad C(\exp(X, Z)) = B(X, Z)^{-1} \left( -X(1 + ||X||^2/4 - Z) \right).$$

This is essentially the same formula as the one in [1]. In [1], it is defined from $N$ to $G/P - \{0\}$.

The Riemannian symmetric space $G/K \subset \mathbb{P}(\mathbb{F}^{n+1})$ has a canonical $G$-invariant metric for which the distance between $eK = [1, 0, \cdots, 0]^T$ and $a_tK = [\cosh t, 0, \cdots, 0, \sinh t]^T = [1, 0, \cdots, 0, \tanh t]^T$ is $|t|$ for $t \in \mathbb{R}$.

For $x, y$ in $G/K$, the Busemann cocycle $\gamma_{x,y}$ is a smooth function on $G/P$ defined as

$$\gamma_{x,y}(z) = \lim_{z \to z'} \left( d_{G/K}(z', y) - d_{G/K}(z', x) \right) = \log \left| \frac{q(y, z)q(x, x)^{1/2}}{q(x, z)q(y, y)^{1/2}} \right|$$

for $z \in G/P$ (see [3 Proposition 3.1.1]).

Let $d\mu_x$ be the normalized $K_x$ invariant measure on $G/P$ for the isotropy group $K_x$ at $x$ of the $G$-action on $G/K$. We have [3 Proposition 3.1.2]

$$d\mu_y = e^{-Q\gamma_{x,y}} d\mu_x$$

for $Q = \dim(\mathfrak{n}_1) + 2\dim(\mathfrak{n}_2) = 4(n-1) + 2(3) = 4n + 2$.

We simply write $0$ for the fixed origin $eK$ of $G/K$. Let

$$(1.5) \quad \lambda_g = e^{-\gamma_{0,0}}$$

so that

$$d\mu_{g'0} = \lambda_g^Q d\mu_0.$$ We have for $a_t \in A$ and for $z = [1, z_1, \cdots, z_n]^T$ in $G/P$,

$$\gamma_{0,a_t0}(z) = \log |1 - (\tanh t)z_n| - (1/2) \log (1 - (\tanh t)^2) = \log |\cosh t - (\sinh t)z_n|$$

and

$$(1.6) \quad \lambda_{a_t} = e^{-\gamma_{0,a_t0}} = |1 - (\tanh t)z_n|^{-1} (\cosh t)^{-1} = |\cosh t - (\sinh t)z_n|^{-1}.$$

We now briefly recall from [7], the definition of a $G$-equivariant sub-bundle $E$ of the tangent-bundle $T(G/P)$ and its associated sub-Laplacian $\Delta_E$. Denote by $P_x$ the isotropy subgroup of $G$ at $x$ in $G/P$. The fiber of the cotangent bundle $T^*(G/P)$ at $x$ is naturally identified as $(\mathfrak{g}/\mathfrak{p}_x)^*$. The latter space is naturally identified as the annihilator $\mathfrak{p}_x^\perp$ of $\mathfrak{p}_x$ in $\mathfrak{g}$ with respect to the Killing form on $\mathfrak{g}$. The space $\mathfrak{p}_x^\perp$ coincides with the nilpotent radical $\mathfrak{n}_x$ of $\mathfrak{p}_x$ which is a 2-step nilpotent Lie algebra with center $\mathfrak{z}_x = [\mathfrak{n}_x, \mathfrak{n}_x]$. A $G$-equivariant sub-bundle $F$ of $T^*(G/P)$ is defined so that $A$-equivariant sub-bundle $E$ of the tangent-bundle $T(G/P)$ is defined as the annihilator $F^\perp$ of $F$.

Let $d_E^*$ be the differential operator from $\mathcal{C}^{\infty}(G/P)$ to the section $\Gamma(E^*)$ of $E^*$, defined as the composition of the de-Rham differential operator and the restriction map from $\Gamma(T^*(G/P))$ to $\Gamma(E^*)$. The adjoint $d_E^*$ is defined with respect to the standard $K$-invariant metric on $G/P$. The
Proof. For any $\alpha$ in $\mathbb{R}$, we have the following upper-bound for the operator norm $\Re F$ or any $\alpha$ in $\mathbb{R}$.

For any $\alpha$ in $\mathbb{R}$, let $\mathcal{H}^\alpha(G/P)$ be the Sobolev space defined by the completion of $C^\infty(G/P)$ by the norm

$$||\xi||_{\mathcal{H}^\alpha(G/P)} = ||(1 + \Delta_E)^{\alpha/2}\xi||_{L^2(G/P, d\mu_0)}$$

for $\xi$ in $C^\infty(G/P)$.

2. Main result

As in [7, Section 8.1], for $0 \leq \Re(s) \leq 1$, let

$$\pi_s(g) = (1 + \Delta_E)^{(Q/4)s}\lambda_g^{(Q/2)(1-s)}\rho(g)(1 + \Delta_E)^{-(Q/4)s}$$

defined on the Hilbert space $\mathcal{H}^0(G/P) = L^2(G/P, d\mu_0)$. Here, $\rho(g)$ for $g$ in $G$ denotes the left-translation action on the functions on $G/P$ and the cocycle $\lambda_g$ is as in [1.5].

Note,

$$\pi_0(g) = \lambda_g^{Q/2}\rho(g)$$

is unitary on $L^2(G/P, d\mu_0)$. This is because

$$\rho(g)d\mu_0 = (g^{-1})^*d\mu_0 = d\mu_{g0} = \lambda_g^Q d\mu_0$$

so we have

$$\langle f_1, f_2 \rangle_{\mathcal{H}^0} = \int_{G/P} \overline{f_1}(g)\overline{f_2}d\mu_0 = \int_{G/P} \rho(g)(\overline{f_1}f_2)\lambda_g^Q d\mu_0 = \langle \pi_0(g)f_1, \pi_0(g)f_2 \rangle.$$

Similarly, if $\Re(s) = 0$, $\pi_s(g)$ is unitary on $L^2(G/P, d\mu_0)$.

The following is our main result.

**Theorem 2.1.** For any $\epsilon > 0$, there is $C = C(\epsilon) > 0$ such that for all $s$ satisfying $\Re(s) = 1$, we have the following upper-bound for the operator norm $||\pi_s(g)||$ of $\pi_s(g)$ on $\mathcal{H}^0(G/P)$ for all $g$ in $G$:

$$||\pi_s(g)|| \leq Ce^\epsilon l(g)(1 + ||\Im(s)||)^{Q/2}.$$ 

Here, $l(g)$ is a $K$-bi-invariant length function on $G$ defined as $l(g) = d_{G/K}(gK, K)$, i.e. $l(ka, k') = |t|$ for any $k, k'$ in $K$ and $t$ in $\mathbb{R}$.

**Corollary 2.2.** For any $\epsilon > 0$, there is $C = C(\epsilon) > 0$ such that for any $t \in [0, 1]$, we have for all $g$ in $G$:

$$||\pi_t(g)|| \leq Ce^{\epsilon l(g)}.$$

**Proof.** For any $g$ in $G$ and for any $K$-finite functions $u$ and $v$ in $L^2(G/P)$, we consider a holomorphic function

$$s \mapsto \langle \pi_s(g)u, v \rangle$$
for $s$ in the strip $0 \leq \Re(s) \leq 1$. For each fixed $g$ and $u, v$, it is not hard to see that this holomorphic function is bounded on the strip. Now fix any positive constant $A > 0$ and consider a holomorphic function

$$s \mapsto \langle \pi_s(g) u, v \rangle e^{As^2}$$

on the strip. For each $g$ and $u, v$ fixed, by the three lines theorem, we have

$$\sup_{\Re(s) = t} |\langle \pi_s(g) u, v \rangle e^{As^2}| \leq \left( \sup_{\Re(s) = 0} |\langle \pi_s(g) u, v \rangle e^{As^2}| \right)^{1-t} \left( \sup_{\Re(s) = 1} |\langle \pi_s(g) u, v \rangle e^{As^2}| \right)^{t}$$

for any $0 \leq t \leq 1$. We have

$$|\langle \pi_t(g) u, v \rangle| e^{At^2} \leq \sup_{\Re(s) = t} |\langle \pi_s(g) u, v \rangle e^{As^2}|.$$ 

For $\Re(s) = 0$, $\pi_s(g)$ is unitary so we have

$$\sup_{\Re(s) = 0} |\langle \pi_s(g) u, v \rangle e^{As^2}| \leq ||u|| ||v||.$$ 

For $\Re(s) = 1$, by Theorem 2.1

$$\sup_{\Re(s) = 1} |\langle \pi_s(g) u, v \rangle e^{As^2}| \leq C e^{e^{I(g)} ||u|| ||v||} \sup_{b \in \mathbb{R}} (1 + |b|)^{Q/2} e^{A(1+ib)^2}$$

$$\leq C' e^{e^{I(g)}} ||u|| ||v||$$

where

$$C' = C \sup_{b \in \mathbb{R}} (1 + |b|)^{Q/2} e^{A(1+ib)^2} < +\infty.$$ 

Combining all these, we get

$$\sup_{0 \leq t \leq 1} |\langle \pi_t(g) u, v \rangle| \leq C' e^{e^{I(g)}} ||u|| ||v||.$$ 

The constant $C'$ does not depend on $g, u$ and $v$ so we are done. \[\square\]

### 3. Some Reduction

We begin our proof of Theorem 2.1. We will eventually reduce this problem to a certain technical estimate of functions. First of all,

**Proposition 3.1.** For $s = 1$, we have

$$||\pi_1(g)|| \leq C (1 + I(g))$$

for $C > 0$ independent of $g$ in $G$, i.e. $\pi_1(g)$ is of linear growth.

**Proof.** Recall

$$\pi_1(g) = (1 + \Delta_E)^{(Q/4)} \rho(g) (1 + \Delta_E)^{-(Q/4)}.$$ 

The left-translation $\rho(g)$ defines a bounded representation of $G$ on the Sobolev space $\mathcal{H}^\alpha(G/P)$. Our assertion is that

$$||\rho(g)||_{\mathcal{H}^{Q/2}(G/P) \rightarrow \mathcal{H}^{Q/2}(G/P)} \leq C (1 + I(g)).$$ 

This follows from the following two lemmas. \[\square\]
Lemma 3.2. Define a new Hilbert space norm \( || \cdot ||' \) on \( C^\infty(G/P) \) by
\[
||\xi||'^2 = ||\xi||^2_{L^2(G/P)} + ||(1 + \Delta_E)^{-Q/4-1/2} d_E \xi||^2_{\Gamma L^2(E^*)}.
\]
The norm \( || \cdot ||' \) and the Sobolev norm \( || \cdot ||_{H^{Q/2}(G/P)} \) are equivalent. Here, \( \Gamma L^2(E^*) \) is the Hilbert space completion of \( \Gamma(E^*) \) with respect to the \( K \)-invariant metric and a \( K \)-invariant sub-Laplacian \( \Delta_E \) on \( \Gamma L^2(E^*) \) is defined as \( \Delta_E = \nabla_E^* \nabla_E \) where a differential operator \( \nabla_E : \Gamma(E^*) \to \Gamma(E^* \otimes E^*) \) is the composition of a fixed \( K \)-invariant connection \( \nabla \) on \( E^* \) and the restriction map from \( T^*(G/P) \) to \( E^* \).

Proof. Both operators
\[
(1 + \Delta_E)^{-Q/4} (1 + d_E^* (1 + \Delta_E)^{Q/2} d_E)^{1/2}, \quad (1 + d_E^* (1 + \Delta_E)^{Q/2} d_E)^{-1/2} (1 + \Delta_E)^{Q/4}
\]
on \( C^\infty(G/P) \) have weighted order zero in a sense of [7, Section 4.1], and thus are bounded on \( L^2(G/P, d\mu_0) \). These two operators give an equivalence between \( H^{Q/2}(G/P) \) and the completion of \( C^\infty(G/P) \) with respect to \( || \cdot ||' \). \( \square \)

Lemma 3.3. The left-translation \( \rho(g) \) defines a bounded representation on the completion of \( C^\infty(G/P) \) with respect to the new norm \( || \cdot ||' \) and it has linear growth.

Proof. Denote by \( W' \), the completion of \( C^\infty(G/P) \) with respect to \( || \cdot ||' \). Note that \( W' \) contains the trivial sub-representation \( \mathbb{C} 1_{G/P} \) spanned by the constant functions. Our claim follows from the fact that the induced representation on the quotient space \( W_0 = W'/\mathbb{C} 1_{G/P} \) is uniformly bounded. This fact was proved in [10, Corollary 4.6] by observing that \( W_0 \) is naturally viewed as a sub-representation of the representation of \( G \) on (the completion of) \( \Gamma(E^*) \) with respect to the norm \( ||(1 + \Delta_E)^{-Q/4-1/2} \xi|| \) for \( \xi \in \Gamma(E^*) \). This representation of \( G \) on \( \Gamma(E^*) \) is a principal series representation and it is uniformly bounded with respect to the given norm (See [10, Theorem 4.5]). \( \square \)

Let \( s = 1 + ib \) for \( b \in \mathbb{R} \). We have
\[
\pi_s(g) = (1 + \Delta_E)^{(Q/4)ib} (1 + \Delta_E)^{(Q/4)\lambda_g^{-1}(Q/2)(ib)} \rho(g) (1 + \Delta_E)^{-1(Q/4)} (1 + \Delta_E)^{-1(Q/4)ib},
\]
so
\[
||\pi_s(g)|| = ||(1 + \Delta_E)^{(Q/4)\lambda_g^{-1}(Q/2)(ib)} \rho(g) (1 + \Delta_E)^{-1(Q/4)}||
= ||(1 + \Delta_E)^{(Q/4)\lambda_g^{-1}(Q/2)(ib)} (1 + \Delta_E)^{-1(Q/4)\pi_1(g)}||.
\]
In view of Proposition 3.1 using \( G = K A^* K \), in order to prove Theorem 2.1, we just need to show, for any \( \epsilon > 0 \), the existence of \( C = C(\epsilon) > 0 \) such that for \( t \geq 0 \) and for \( b \in \mathbb{R} \),
\[
||((1 + \Delta_E)^{(Q/4)\lambda_g^{-1}(Q/2)(ib)} (1 + \Delta_E)^{-1(Q/4)}|| \leq C e^{\epsilon t} (1 + |b|)^Q/2.
\]

Let \( \chi \) be a smooth function on \( G/P \) which is 1 near the point \( o = [1, 0, \cdots, 0, 1]^T \) and vanishes outside a small neighborhood of \( o \).

Lemma 3.4. There is \( C > 0 \) such that for \( t \geq 0 \) and for \( b \in \mathbb{R} \),
\[
||((1 + \Delta_E)^{(Q/4)} (1 - \chi) \lambda_a^{-1}(Q/2)(ib) (1 + \Delta_E)^{-1(Q/4)}|| \leq C (1 + |b|)^Q/2.
\]
Proof. Recall from (1.6)
\[ \lambda_{a_t} = |1 - (\tanh t)z_n|^{-1} (\cosh t)^{-1} \]
so
\[ \lambda_{a_t}^{-1/2} = |1 - (\tanh t)z_n|^{-1/2} (\cosh t)^{-1/2}. \]
Note that the second factor is a constant function on \( G/P \) of modulus one and that
\[ (1 - \chi) |1 - (\tanh t)z_n|^{-1/2} \]
smoothly converges as \( t \to +\infty \) to the smooth function
\[ (1 - \chi) |1 - z_n|^{-1/2} \]
on \( G/P \). The claim follows from these. \( \square \)

Thus, we reduced our proof of Theorem 2.1 to the following.

**Lemma 3.5.** Let \( \chi \) be a smooth function on \( G/P \) which is supported near the point \( o \) (\( z_n = 1 \)). Consider the multiplication operator
\[ \chi \lambda_{a_t}^{-1/2} : \mathcal{H}^{Q/2}(G/P) \to \mathcal{H}^{Q/2}(G/P). \]
For any \( \epsilon > 0 \), there is \( C = C(\epsilon) > 0 \) such that for \( t \geq 0 \) and for \( b \in \mathbb{R} \),
\[ ||\chi \lambda_{a_t}^{-1/2}||_{\mathcal{H}^{Q/2}(G/P) \to \mathcal{H}^{Q/2}(G/P)} \leq Ce^{\epsilon t} (1 + |b|)^{Q/2}. \]

We will prove Lemma 3.5 using a local model \( \overline{N} \cong N \) of \( G/P \) around the point \( o \) via the Cayley transform \( C \).

Fix an orthonormal basis \( \{E_j\}_{1 \leq j \leq \dim(\overline{N})} \) of \( \overline{\pi}_1 \). We identify \( \overline{\pi} \) as the left-invariant vector field on \( \overline{N} \). The sub-Laplacian \( \Delta_{\overline{\pi}} \) on \( \overline{N} \) is defined by
\[ \Delta_{\overline{\pi}} = -\sum_{j=1}^{\dim(\overline{N})} E_j^2 \]
on the space \( C^\infty_c(\overline{N}) \) of compactly supported smooth functions on \( \overline{N} \). It defines an essentially self-adjoint, positive operator on \( L^2(\overline{N}) \). For any \( \alpha \in \mathbb{R} \), we define the homogeneous Sobolev space \( \mathcal{H}^\alpha(\overline{N}) \) as the completion of \( C^\infty_c(\overline{N}) \) by the norm
\[ ||\xi||_{\mathcal{H}^\alpha(\overline{N})} = ||\Delta_{\overline{\pi}}^{\alpha/2} \xi||_{L^2(\overline{N})}. \]
For a technical purpose, we also define the non-homogenous Sobolev space \( \mathcal{H}^\alpha(\overline{N}) \) as the completion of \( C^\infty_c(\overline{N}) \) by the norm
\[ ||\xi||_{\mathcal{H}^\alpha(\overline{N})} = ||(1 + \Delta_{\overline{\pi}})^{\alpha/2} \xi||_{L^2(\overline{N})}. \]

Multiplication by any compactly supported smooth function on \( \overline{N} \) defines a bounded operator on \( \mathcal{H}^\alpha(\overline{N}) \) for any \( \alpha \) [5; Corollary 4.15]. It is also bounded on the homogeneous Sobolev space \( \mathcal{H}^\alpha(\overline{N}) \) for \( -Q/2 < \alpha < Q/2 \) but not necessarily for \( |\alpha| \geq Q/2 \).

For any open subset \( U \) of \( \overline{N} \), we define \( \mathcal{H}^\alpha(U) \) and \( \mathcal{H}^\alpha(\overline{N}) \) as the closures of \( C^\infty(U) \) in \( \mathcal{H}^\alpha(\overline{N}) \) and \( \mathcal{H}^\alpha(\overline{N}) \) respectively. Similarly, for any open subset \( U_{G/P} \) of \( G/P \), we define \( \mathcal{H}^\alpha(U_{G/P}) \) as the closure of \( C^\infty(U_{G/P}) \) in \( \mathcal{H}^\alpha(G/P) \).
We note two things. First, if $U \subset \overline{N}$ is relatively compact, the inclusion $\mathcal{H}^\alpha(U)$ to $\mathcal{H}^\alpha(U)$ is an isomorphism for all $\alpha \geq 0$. This is because the sub-Laplacian $\Delta_{\overline{\n}}$ is locally, bounded away from zero. Secondly, for any $\alpha$, locally but not globally, the Cayley transform $C : \overline{N} \to G/P$ induces an isomorphism on Sobolev spaces $\mathcal{H}^\alpha$. That is, the composition by the Cayley transform $C$ induces an isomorphism from $\mathcal{H}^\alpha(U_{G/P})$ to $\mathcal{H}^\alpha(U_{\overline{\n}})$ for any relatively compact open subset $U_{\overline{\n}}$ and for $U_{G/P} = CU_{\overline{\n}}$. We only use this fact for $\alpha = Q/2$.

Thus, Lemma 3.7 follows from the following claim. Notice that we are using the non-homogenous Sobolev space in this statement.

**Claim 3.6.** Let $\chi$ be a smooth compactly supported function on $\overline{N}$. For any $\epsilon > 0$, there is $C = C(\epsilon) > 0$ such that for $t \geq 0$ and for $b \in \mathbb{R}$, the multiplication operator
\[
\chi(\lambda_{\alpha,t} \circ C)^{-Q/2}(ib) : \mathcal{H}^{Q/2}(\overline{N}) \to \mathcal{H}^{Q/2}(\overline{N})
\]

has norm bound
\[
||\chi(\lambda_{\alpha,t} \circ C)^{-Q/2}(ib)||_{\mathcal{H}^{Q/2}(\overline{N}) \to \mathcal{H}^{Q/2}(\overline{N})} \leq Ce^{\epsilon t}(1 + |b|)^{Q/2}.
\]

In order to prove this claim, we give an explicit expression of $\lambda_{\alpha,t} \circ C$ on $\overline{N}$. From now on, when there is no confusion, we identify $\overline{N}$ with $\overline{\n}$ via the exponential map and use the coordinates $(X, Z) \in \mathbb{R}^{n-1} \oplus \text{Im}(\mathbb{F})$ for $\overline{\n}$ to express the element $\exp(X,Z)$ in $\overline{N}$ as well.

**Lemma 3.7.** (c.f. [1 Corollary 3.9]) For any $t \in \mathbb{R}$, for any $(X, Z) \in \mathbb{R}^{n-1} \oplus \text{Im}(\mathbb{F})$, we have
\[
(\lambda_{\alpha,t} \circ C)(X, Z) = e^{-t} B(X, Z)^{1/2} B_t(X, Z)^{1/2},
\]
where
\[
B(X, Z) = (1 + ||X||^2/4)^2 + |Z|^2, \quad B_t(X, Z) = (e^{-2t} + ||X||^2/4)^2 + |Z|^2.
\]

**Proof.** Recall that
\[
d\mu_{\alpha,0} = (a_{-t})^* d\mu_0 = \lambda_{\alpha,t}^O d\mu_0.
\]
That is, $\lambda_{\alpha,t}^O$ is the Jacobian $J_{a_{-t}}$ of the map $a_{-t}$ on $G/P$ (with respect to $d\mu_0$). Let $\delta_t : \overline{N} \to \overline{N}$ be the map $(X, Z) \to (e^t X, e^{2t} Z)$. We have
\[
\lambda_{\alpha,t} = \delta_t
\]
on $\overline{N}$. By the chain rule, we have
\[
\lambda_{\alpha,t}^O \circ C = J_{a_{-t}} \circ C = J_C \circ \delta_t \cdot J_{\delta_t} \cdot J_C^{-1}
\]
on $\overline{N}$ where $J_C$ and $J_{\delta_t}$ are the Jacobian of the map $C$ and $\delta_t$ respectively. We have
\[
J_{\delta_t} = e^{Qt}
\]
and
\[
J_C = B(X, Z)^{-Q/2}
\]
(see the end of Section 2.3 of [1]). Thus,
\[
\lambda_{\alpha,t}^O \circ C = J_C \circ \delta_t \cdot J_{\delta_t} \cdot J_C^{-1} = e^{Qt} \left( \frac{B(X, Z)}{B(e^t X, e^{2t} Z)} \right)^{Q/2}.
\]
We get
\[ \lambda_{a_t} \circ C = e^{t} \frac{B(X, Z)^{1/2}}{B(e^{t}X, e^{2t}Z)^{1/2}} = e^{-t} \frac{B(X, Z)^{1/2}}{B_{t}(X, Z)^{1/2}}. \]

\[ \square \]

**Remark 3.8.** Recall
\[ \lambda_{a_t} = e^{-y_0 a_t^{0}} = | \cosh t - \sinh t |^{-1} = | 1 - (\tanh t) z_n |^{-1} (\cosh t)^{-1}. \]

The link with Lemma [3.7] is the following. For \( z_n = z_n \circ C(X, Z) \), using (1.4),
\begin{align*}
z_n &= B^{-1}(1 - |X|^4/16 - |Z|^2 - 2Z) = B^{-1}(-N^2 + 2N - |Z|^2 - 2Z) \\
&= -1 + 2B^{-1}N - 2B^{-1}Z
\end{align*}
where \( N = 1 + |X|^2/4 \) and \( B = N^2 + |Z|^2 \). An easy calculation yields (denote \( r = \tanh t \))
\[ \lambda_{a_t}^{-2} \circ C = (1 - r^{-2})^{-1} | 1 - rz_n |^2 = \frac{1 + r}{1 - r} (1 - \frac{4r}{1 + r} B^{-1} N + \frac{4r^2}{(1 + r)^2} B^{-1}). \]

On the other hand
\[ B_t = (|X|^2/4 + e^{-2t})^2 + |Z|^2 = (N - \frac{2r}{1 + r})^2 + |Z|^2 = B - \frac{4r}{1 + r} N + \frac{4r^2}{(1 + r)^2}, \]
so that \( \lambda_{a_t}^{-2} \circ C = e^{2t} B^{-1} B_t \) which coincides with the above formula.

In the light of the formula in Lemma [3.7] we have
\[ (\lambda_{a_t} \circ C)^{-Q/2(i)b} = e^{Q/2(t)(ib)} B(X, Z)^{-Q/4(ib)} B_{t}(X, Z)^{(Q/4)ib}. \]

Note that the first factor \( e^{Q/2(t)(ib)} \) is a constant function on \( \overline{N} \) of modulus one. We reduced Claim 3.6 and hence Theorem 2.1 to the following.

**Lemma 3.9.** Let \( \chi \) be a smooth compactly supported function on \( \overline{N} \). For any \( \epsilon > 0 \), there is \( C = C(\epsilon) > 0 \) such that for \( t \geq 0 \) and for \( b \in \mathbb{R} \), the multiplication operator
\[ \chi B_t(X, Z)^{ib} B(X, Z)^{-ib} : \mathcal{H}^{Q/2}(\overline{N}) \to \mathcal{H}^{Q/2}(\overline{N}) \]
has norm bound
\[ ||\chi B_t(X, Z)^{ib} B(X, Z)^{-ib}||_{\mathcal{H}^{Q/2}(\overline{N}) \to \mathcal{H}^{Q/2}(\overline{N})} \leq Ce^{\epsilon t} (1 + |b|)^{Q/2}. \]

From now on, we focus on proving the lemma 3.9. We have the following easy estimates for \( B_t(X, Z) \):
\begin{enumerate}
\item \( B_t(X, Z) \geq e^{-4t} \),
\item \( B_t(X, Z) \geq |X|^4/16 + |Z|^2 = N^4 \),
\end{enumerate}
where
\[ N = (|X|^4/16 + |Z|^2)^{1/4} \]
is a homogeneous gauge on \( \overline{N} \).
4. Proof of Lemma 3.9

Let us follow some convention from [1]: for a positive integer $k \in \mathbb{Z}$, $\mathcal{D}^k$ denotes the set of all differential operators of the form

$$E_{j_1}E_{j_2}\cdots E_{j_k},$$

where $1 \leq j_1, j_2, \cdots, j_k \leq \text{dim}(\pi_1)$. The following result from [1] will be the key. Note that the homogeneous Sobolev space is used in this statement.

**Lemma 4.1.** [1, Theorem 3.6] Let $0 \leq \alpha \leq \beta < Q/2$. Let $m$ be a smooth function on $\overline{N} - \{0\}$. Suppose that on $\overline{N} - \{0\}$,

$$|D^j m| \leq C_j N^{-d-j}$$

holds for any $0 \leq j \leq k = \lceil \alpha \rceil$, $D^j \in \mathcal{D}^j$ and $d = \beta - \alpha$. Then,

$$||m||_{\mathcal{H}^\alpha(\overline{N})} \leq C(\alpha, \beta)(C_0 + C_1 + \cdots + C_k)$$

where the constant $C(\alpha, \beta)$ only depends on $\alpha$ and $\beta$.

**Proof.** This is what is proven in [1, Theorem 3.6] and used in the proof of [1, Corollary 3.9]. For the detail of its proof, see the proof of [2, Theorem 7]. □

The following technical estimate for the derivatives of $B_t$ very roughly means $B_t$ behaves like a homogeneous function of degree 4.

**Lemma 4.2.** Let $U$ be a relatively compact open subset of $\overline{N}$. For any positive integer $s \geq 0$, there is $C = C(s) > 0$ so that for any $D^s \in \mathcal{D}^s$ and for any $(X, Z)$ in $U$,

$$|D^s(B_t^{ib})|(X, Z) \leq C \frac{1}{B_t(X, Z)^{s/4}}(1 + |b|)^s$$

for all $t \geq 0$ and $b \in \mathbb{R}$.

**Proof.** The proof will be given in the next (last) section. □

Now we use Lemma 4.1 together with Lemma 4.2 to prove Lemma 3.9, that is we show that for any smooth compactly supported function $\chi$ on $\overline{N}$ and for any $\epsilon > 0$, the multiplication operator

$$\chi B_t^{ib} B^{-ib} : \mathcal{H}^{Q/2}(\overline{N}) \to \mathcal{H}^{Q/2}(\overline{N})$$

has norm bound

$$||\chi B_t^{ib} B^{-ib}||_{\mathcal{H}^{Q/2}(\overline{N})} \to \mathcal{H}^{Q/2}(\overline{N}) \leq C e^{e t} (1 + |b|)^{Q/2},$$

for $C = C(\epsilon) > 0$ independent of $t \geq 0$ and $b \in \mathbb{R}$.

Note that for $\xi$ in $\mathcal{H}^{Q/2}(\overline{N})$, the following two norms

$$||\xi||_{\mathcal{H}^{Q/2}(\overline{N})}, \quad ||\xi||_{\mathcal{H}^{Q/2}(\overline{N})} + \sum_{j=1}^{\text{dim}(\pi_1)} ||E_j \xi||_{\mathcal{H}^{Q/2-1}(\overline{N})}$$

are equivalent. Using this, we see that it is enough to show that for any compactly supported smooth function $\chi$ and for any $\epsilon > 0$, there is $C = C(\epsilon) > 0$ so that for any $t \geq 0$ and $b \in \mathbb{R}$,

(4.1) $||\chi B_t^{ib} B^{-ib}||_{\mathcal{H}^{Q/2-1}(\overline{N})} \to \mathcal{H}^{Q/2-1}(\overline{N}) \leq C e^{e t} (1 + |b|)^{Q/2}$,
\[(4.2) \quad \|\chi E_j(B_i^{ib} B^{-ib})\|_{\mathcal{H}^{Q/2}(\mathcal{N})} \leq C e^{\epsilon t} (1 + |b|)^{Q/2}.\]

By Lemma 4.2 on a ball \(U\) which contains the support of \(\chi\),
\[|D^s(B_i^{ib})| \leq C_s B_t^{-s/4} (1 + |b|)^s \leq C_s (1 + |b|)^s N^{-s}.\]

It follows that there is \(C > 0\) such that for \(0 \leq s \leq Q/2 - 1\),
\[|D^s(\chi B_i^{ib} B^{-ib})| \leq C (1 + |b|)^s N^{-s}.\]

By Lemma 4.1, it follows that for some \(C > 0\),
\[\|\chi B_i^{ib} B^{-ib}\|_{\mathcal{H}^{Q/2}(\mathcal{N})} \leq C (1 + |b|)^{Q/2-1} \leq C e^{\epsilon t} (1 + |b|)^{Q/2}\]
for all \(t \geq 0\) and \(b \in \mathbb{R}\). Since \(\chi\) is compactly supported, we can replace the homogeneous Sobolev space \(\mathcal{H}^\alpha\) by the non-homogeneous Sobolev space \(\mathcal{H}^\alpha\) in this inequality. This proves (4.1).

As for (4.2), first note that for any compactly supported smooth function \(\chi_0\),
\[\chi_0 : \mathcal{H}^{Q/2}(\mathcal{N}) \to \mathcal{H}^{Q/2-\epsilon}(\mathcal{N})\]
is bounded for any \(\epsilon > 0\). Taking \(\chi_0\) such that \(\chi_0 \chi = \chi\), it is enough to bound the norm
\[\chi E_j(B_i^{ib} B^{-ib}) : \mathcal{H}^{Q/2-\epsilon}(\mathcal{N}) \to \mathcal{H}^{Q/2-1}(\mathcal{N}).\]

Again, we are using that \(\chi\) is compactly supported to replace the homogeneous Sobolev space \(\mathcal{H}^\alpha\) by the non-homogeneous Sobolev space \(\mathcal{H}^\alpha\).

Let \(\beta = Q/2 - \epsilon\), \(\alpha = Q/2 - 1\) and \(d = \beta - \alpha = 1 - \epsilon\). By Lemma 4.2 we have on a ball \(U\) which contains the support of \(\chi\),
\[|D^s(E_j(B_i^{ib}))| = |D^{s+1} B_i^{ib}| \leq C_{s+1} B_t^{-(s+1)/4} (1 + |b|)^{s+1}\]
\[\leq C_{s+1} B_t^{-\epsilon/4} B_t^{-(s+1-\epsilon)/4} (1 + |b|)^{s+1} \leq C_{s+1} B_t^{-\epsilon/4} N^{-d-s} (1 + |b|)^{s+1}.\]

Moreover, we have
\[B_t^{-\epsilon/4} \leq e^{\epsilon t}\]
for \(t \geq 0\). It follows that there is \(C > 0\) such that for \(0 \leq s \leq Q/2 - 1\),
\[|D^s(\chi E_j(B_i^{ib} B^{-ib}))| \leq C e^{\epsilon t} N^{-d-s} (1 + |b|)^{s+1}.\]

Hence, by Lemma 4.1, it follows that for some \(C > 0\),
\[\|\chi E_j(B_i^{ib} B^{-ib})\|_{\mathcal{H}^{Q/2-\epsilon}(\mathcal{N})} \to \mathcal{H}^{Q/2-1}(\mathcal{N}) \leq C (1 + |b|)^{Q/2} e^{\epsilon t}.\]

This gives (4.2). This proves Lemma 3.9 and hence Theorem 2.1 modulo the technical estimate, Lemma 4.2.
5. Technical Estimate

We give a proof of Lemma 4.2. We fix a relatively compact open set $U$ of $\mathcal{N}$ and let $C_0 > 0$ so that

$$B_t(X, Z) = (e^{-2t} + ||X||^2/4)^2 + |Z|^2 \leq C_0$$

for all $t \geq 0$.

We need a small lemma. Again, this roughly means $B_t$ behaves like a homogeneous function of degree 4.

**Lemma 5.1.** On $\mathcal{N}$, we have

1. $|D^1 B_t| \leq C_1 B_t^{3/4}$,
2. $|D^2 B_t| \leq C_2 B_t^{2/4}$,
3. $|D^3 B_t| \leq C_3 B_t^{1/4}$,
4. $|D^4 B_t| \leq C_4$.
5. $D^j B_t = 0$ $(j \geq 5)$,

for some constants $C_j \geq 0$ independent of $t \geq 0$ and of $D^j \in \mathcal{D}^j$.

**Proof.** Let us write

$$B_t = e^{-3t} + e^{-2t}||X||^2/2 + P_4$$

where $P_4 = N^4 = ||X||^4/16 + |Z|^2$. We have

$$D^1 B_t = e^{-2t}D^1(||X||^2)/2 + P_3$$
$$D^2 B_t = e^{-2t}D^2(||X||^2)/2 + P_2$$
$$D^3 B_t = P_1$$
$$D^4 B_t = P_0$$

where $P_d$ is a homogenous polynomial of degree $d$. Note that for each such $P_d$, we have

$$|P_d| \leq C'_d N^d \leq C'_d B_t^{d/4}$$

for some $C'_d > 0$. We also have

$$|e^{-2t}D^1(||X||^2)/2| \leq 2B_t^{3/4}$$
$$|e^{-2t}D^2(||X||^2)/2| \leq B_t^{2/4}.$$}

The first one follows from

$$B_t^3 \geq (e^{-3t} + ||X||^4/16)^3 \geq e^{-8t}||X||^4/16.$$

The second one follows from

$$B_t \geq e^{-3t}.$$

The claim follows from these. □
Proof of Lemma 4.2. For any $s > 0$, we need to show that for some $C = C(s) > 0$, on $U$,\[ |D^s(B_t^{ib})| \leq C \frac{1}{(B_t)^{s/4}} (1 + |b|)^s.\] For $s = 0$, this is trivial. For $s = 1$, we have, on $U$,\[ |D^1(B_t^{ib})| = |D^1 e^{ib \log(B_t)}| = \frac{|D^1(B_t)|}{B_t} |b| \leq C_1 \frac{B_t^{3/4}}{B_t} (1 + |b|) = C_1 B_t^{-1/4} (1 + |b|).\] The general case for $s \geq 1$ follows from that $D^s e^{ib \log(B_t)}$ is a finite sum of the product\[ (ib)^l e^{ib \log(B_t)} \prod_{1 \leq j \leq l} (D^{k_j} \log(B_t))\] where $k_1 + \cdots + k_l = s$ ($k_j \geq 1$) and $D^{k_j} \in \mathfrak{d}^{k_j}$ and that on $U$,\[ |D^j(\log(B_t))| \leq C'_j (B_t)^{-j/4} \ (j \geq 1).\] We can see the latter from that $D^s(\log(B_t))$ is a finite sum of the product\[ (B_t)^{-m_1 + \cdots + m_s} \prod_{1 \leq j \leq s} (D^j B_t)^{m_j}\] where $m_1 + 2m_2 + \cdots + sm_s = s$ and $D^j \in \mathfrak{d}^{j}$ and that\[ \left| \frac{D^j B_t}{B_t} \right| \leq C_j (B_t)^{-j/4},\] that is for $j \geq 1$,
\[ |D^j B_t| \leq C_j (B_t)^{1-j/4}.\] The last one follows from Lemma 5.1. In this argument, we used that $B_t \leq C_0$ on $U$ thus \[ B_t^{-s_1} \leq C_0^{s_2-s_1} B_t^{-s_2}\] on $U$ for $0 \leq s_1 \leq s_2$.\[ \square \]

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