Relativistic Quantum Coulomb Law

Yury M. Zinoviev*

Steklov Mathematical Institute, Gubkin St. 8, 119991, Moscow, Russia, e-mail: zinoviev@mi.ras.ru

Abstract. The relativistic quantum mechanics equations for the electromagnetic interaction are proposed.

1 Introduction

The celestial mechanics is based on the gravity law discovered by Newton (1687). Cavendish (1773) proved by experiment that the force of interaction between the electric charged bodies is inversely proportional to the square of distance. This discovery was left unpublished and later was repeated by de Coulomb (1785). The electrodynamics equations were formulated by Maxwell (1873). The analysis of these equations led Lorenz (1904), Poincaré (1905, 1906), Einstein (1905) and Minkowski (1908) to the creation of the theory of relativity.

Due to the paper [1] the Maxwell equations are completely defined by the relativistic Coulomb law. The relativistic Coulomb law equations for two charged bodies are solved in the paper [2] for the case when one body moves freely. The results [2] may be applied to the study of the hydrogen atom. The light electron moves around the heavy proton. The heavy proton moves freely. The hydrogen spectrum is discrete and does not correspond to the results of the paper [2]. In order to study this problem we need to apply the quantum electrodynamics. The following citation from the book ([3], Chapter 4) describes the situation in the quantum electrodynamics:

"But there is one additional problem that is characteristic of the theory of quantum electrodynamics itself, which took twenty years to overcome. It has to do with ideal electrons and photons and the numbers $n$ and $j$.

"If electrons were ideal, and went from point to point in space-time only by the direct path, then there would be no problem: $n$ would be the mass of an electron (which we can determine by observation), and $j$ would simply be its "charge" (the amplitude for the electron to couple with a photon). It can also be determined by experiment.

"But no such ideal electron exist. The mass we observed in the laboratory is that of a real electron, which emits and absorbs its own photons from time to time, and therefore depends on the amplitude for coupling, $j$. And the charge we observe is between a real electron and a real photon - which can form an electron - positron pair from time to time - and therefore depends on $E(A$ to $B)$, which involves $n$. Since the mass and charge of an

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electron are affected by these and all other alternatives, the experimentally measured mass, \(m\), and experimentally measured charge, \(e\), of the electron are different from the numbers we use in our calculations, \(n\) and \(j\).

"If there were a definite mathematical connection between \(n\) and \(j\) on the one hand, and \(m\) and \(e\) on the other, there would still be no problem: we would simply calculate what values of \(n\) and \(j\) we need to start with in order to end up with the observed values, \(m\) and \(e\). (If our calculations didn’t agree with \(m\) and \(e\), we would jiggle the original \(n\) and \(j\) around until they did.)"

"Let’s see how we actually calculate \(m\). We write a series of terms that is something like the series we saw for the magnetic moment of the electron: the first term has no couplings - just \(E(A \text{ to } B)\) - and represents an ideal electron going directly from point to point in space - time. The second term has two couplings and represents a photon being emitted and absorbed. Then come terms with four, six, and eight couplings, and so on.

"When calculating terms with couplings, we must consider (as always) all the possible points where couplings can occur, right down to cases where the two coupling points are on top of each other - with zero distance between them. The problem is, when we try to calculate all the way down to zero distance, the equation blows up in our face and gives meaningless answers - things like infinity. This caused a lot of trouble when the theory of quantum electrodynamics first came out. People were getting infinity for every problem they tried to calculate! (One should be able to go down to zero distance in order to be mathematically consistent, but that’s there is no \(n\) or \(j\) that makes any sense; that’s where the trouble is.)"

"Well, instead of including all possible coupling points down to a distance of zero, if one stops the calculation when the distance between coupling points is very small - say, \(10^{-30}\) centimeters, billions and billions of times smaller than anything observable in experiment (presently \(10^{-16}\) centimeters) - then there are definite values for \(n\) and \(j\) that we can use so that the calculated mass comes our to match the \(m\) observed in experiments, and the calculated charge matches the observed charge, \(e\). Now, here’s the catch: if somebody else comes and stops their calculation at a different distance - say, \(10^{-40}\) centimeters - their values for \(n\) and \(j\) needed to get the same \(m\) and \(e\) come out different!"

"Twenty years later, in 1949, Hans Bethe and Victor Weisskopf noticed something: if two people who stopped at different distances to determine \(n\) and \(j\) from the same \(m\) and \(e\) then calculated the answer to some other problem - each using the appropriate but different values for \(n\) and \(j\) - when all the arrows from all the terms were included, their answers to this other problem came out nearly the same! In fact, the closer to zero distance that the calculations for \(n\) and \(j\) were stopped, the better the final answers for the other problem would agree! Schwinger, Tomonaga, and I independently invented ways to make definite calculations to confirm that it is true (we got prizes for that). People could finally calculate with the theory of quantum electrodynamics!

"So it appears that the only thing that depend on the small distance between coupling points are the values for \(n\) and \(j\) - theoretical numbers that are not directly observable anyway; everything else, which can be observed, seems not to be affected.

"The shell game that we play to find \(n\) and \(j\) is technically called "renormalization". But no matter how clever the word, it is what I would call a dippy process! Having to resort to such hocus - pocus has prevented us from proving that the theory of quantum electrodynamics is mathematically self - consistent. It’s surprising that the theory still hasn’t been proved self - consistent one way or the other by now; I suspect that renormalization is
not mathematically legitimate. What is certain is that we do not have a good mathematical way to describe the theory of quantum electrodynamics: such a bunch of words to describe the connection between \( n \) and \( j \) and \( m \) and \( e \) is not good mathematics.”

This paper is devoted to the relativistic quantum mechanics. The equations for the electromagnetic interaction are proposed.

## 2 Lorentz group

The theory of relativity has the mathematical foundation. It is possible to add and multiply the complex numbers. Let us consider the complex \( 2 \times 2 \) - matrices

\[
A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}.
\]

(2.1)

The \( 2 \times 2 \) - matrix

\[
A^* = \begin{pmatrix} \bar{A}_{11} & \bar{A}_{21} \\ \bar{A}_{12} & \bar{A}_{22} \end{pmatrix}
\]

(2.2)

is called Hermitian adjoint matrix. If \( A^* = A \), then the matrix (2.1) is Hermitian. Let us consider the basis of Hermitian \( 2 \times 2 \) - matrices

\[
\sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

(2.3)

Any matrix (2.1) has the form

\[
\sum_{\mu=0}^{3} z^\mu \sigma^\mu
\]

(2.4)

where \( z^0,..,z^3 \) are the complex numbers. It is possible to add and multiply the matrices (2.1). These matrices form the eight dimensional space of the matrices (2.4) which is an algebra. The following multiplication rules are valid

\[
\sigma^\mu \sigma^\mu = \sigma^0, \quad \sigma^0 \sigma^\mu = \sigma^\mu \sigma^0 = \sigma^\mu, \quad \mu = 0,..,3;
\]

\[
\sigma^{k_1} \sigma^{k_2} = \sum_{k_3=1}^{3} \epsilon^{k_1 k_2 k_3} i \sigma^{k_3}, \quad k_1, k_2 = 1,2,3, \quad k_1 \neq k_2
\]

(2.5)

where the antisymmetric tensor \( \epsilon^{k_1 k_2 k_3} \) has the normalization \( \epsilon^{123} = 1 \). Hence for the real numbers \( x^0,..,x^3 \) the matrices

\[
\tilde{x} = x^0 \sigma^0 + i \sum_{k=1}^{3} x^k \sigma^k
\]

(2.6)

form an algebra. The matrix (2.6) is called quaternion. The quaternion algebra was invented by Hamilton (1843). The algebra of the matrices (2.6) is the non - commutative extension of the complex numbers field. The real numbers are the real diagonal \( 2 \times 2 \) - matrices \( x^0 \sigma^0 \). The pure imaginary numbers are the real antisymmetric \( 2 \times 2 \) - matrices \( ix^2 \sigma^2 \). The determinant of a quaternion (2.6)

\[
\det \tilde{x} = \sum_{\mu=0}^{3} (x^\mu)^2
\]

(2.7)
is called the Euclidean metric. Any matrix (2.1) satisfying the equation

\[ A^* = (\det A)A^{-1} \] (2.8)

has the form (2.6) and is a quaternion. The equation (2.8) implies also that the matrices (2.6) form an algebra. The matrices (2.6) with determinant equal to 1 satisfy the equations \( A^* A = \sigma^0 \), \( \det A = 1 \) and form the group \( SU(2) \). The matrices (2.1) with determinant equal to 1 form the group \( SL(2, \mathbb{C}) \). The group \( SU(2) \) is the maximal compact subgroup of the group \( SL(2, \mathbb{C}) \). We identify a vector \( x^\mu, \mu = 0, \ldots, 3 \), from the four dimensional Euclidean space and a quaternion (2.6). The unit sphere in the four dimensional Euclidean space is isomorphic to the group \( SU(2) \). A rotation of the four dimensional Euclidean space is a matrix product

\[ R(A, B)(\tilde{x}) = A\tilde{x}B \] (2.9)

where the matrices \( A, B \in SU(2) \). The metric (2.7) is not changed under any rotation (2.9).

In the quaternion (2.6) the coordinate \( x^0 \) is a real number and the coordinates \( ix^k, k = 1, 2, 3 \), are the pure imaginary numbers. Let us consider the four dimensional space of Hermitian matrices

\[ \tilde{x} = \sum_{\mu=0}^{3} x^\mu \sigma^\mu \] (2.10)

where the coordinates \( x^0, \ldots, x^3 \) are the real numbers. It is possible to add \( \tilde{x} + \tilde{y} \) and to multiply \( 1/2(\tilde{x}\tilde{y} + \tilde{y}\tilde{x}) \) these matrices. The determinant of a matrix (2.10) is

\[ \det \tilde{x} = (x^0)^2 - |x|^2 = (x, x) = \sum_{\mu, \nu=0}^{3} \eta_{\mu\nu} x^\mu x^\nu. \] (2.11)

Here the diagonal matrix \( \eta_{\mu\nu} = \eta^{\mu\nu} \), \( \eta_{00} = -\eta_{11} = -\eta_{22} = -\eta_{33} = 1 \). The Minkowski metric (2.11) was introduced by Poincaré (1906). For the matrices \( A, B \in SL(2, \mathbb{C}) \) we consider a transformation

\[ L(A, B)(\tilde{x}) = A\tilde{x}B \] (2.12)

similar to a rotation (2.9). A transformation (2.12) does not change a determinant (2.11). A matrix (2.12) is Hermitian if

\[ B^* \tilde{x} A^* = A\tilde{x}B. \] (2.13)

A matrix \( A \) is invertible since its determinant is equal to 1. Hence the relation (2.13) implies

\[ A^{-1}B^* \tilde{x} = \tilde{x}B(A^*)^{-1}. \] (2.14)

By inserting the matrix \( \tilde{x} = \sigma^0 \) into the equality (2.14) we have \( A^{-1}B^* = B(A^*)^{-1} \). The Hermitian matrix \( B(A^*)^{-1} \) commutes with any Hermitian matrix. Hence

\[ B(A^*)^{-1} = \lambda \sigma^0 \] (2.15)

where \( \lambda \) is a real number. The number \( \lambda = \pm 1 \) since the determinants of the matrices \( A, B \) are equal to 1. It is easy to verify for any matrix (2.1)

\[ \text{tr}(L(A, \lambda A^*)(\sigma^0)) = \lambda \sum_{i,j=1}^{2} |A_{ij}|^2. \] (2.16)

The number (2.16) is the double coefficient at the matrix \( \sigma^0 \) in the decomposition (2.10) for the matrix \( L(A, \lambda A^*)(\sigma^0) \). For \( \lambda = 1 \) the number (2.16) is positive and the time direction is not changed. The group of the transformations \( L(A, A^*), A \in SL(2, \mathbb{C}) \), is called the Lorentz group.
3 Relativistic quantum laws

For a complex $2 \times 2$ - matrix (2.1) we define the following $2 \times 2$ - matrices

$$A^T = \begin{pmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{pmatrix}, \quad \tilde{A} = \begin{pmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{pmatrix}.$$ \hspace{1cm} (3.1)

Let us describe the irreducible representations of the group $SU(2)$. We consider the half - integers $l \in \frac{1}{2}\mathbb{Z}_+$, i.e. $l = 0, 1/2, 1, 3/2, ...$. We define the representation of the group $SU(2)$ on the space of the polynomials with degree less than or equal to $2l$

$$T_l(A)\phi(z) = (A_{12}z + A_{22})^{2l}\phi\left(\frac{A_{11}z + A_{21}}{A_{12}z + A_{22}}\right). \hspace{1cm} (3.2)$$

We consider a half - integer $n = -l, -l + 1, ..., l - 1, l$ and choose the polynomial basis

$$\psi_n(z) = ((l - n)!(l + n)!)^{-1/2}z^{l-n}.$$ \hspace{1cm} (3.3)

The definitions (3.2), (3.3) imply

$$T_l(A)\psi_n(z) = \sum_{m=-l}^{l} \psi_m(z)t^l_{mn}(A), \hspace{1cm} (3.4)$$

$$t^l_{mn}(A) = \frac{((l - m)!(l + m)!(l - n)!(l + n)!)^{1/2}}{\sum_{j=-\infty}^{\infty} \frac{A^l_{11}A^l_{12}A^{m-n+j}_{21}A^{l+n-j}_{22}}{\Gamma(j + 1)\Gamma(l - m - j + 1)\Gamma(m - n + j + 1)\Gamma(l + n - j + 1)}}.$$ \hspace{1cm} (3.5)

where $\Gamma(z)$ is the gamma - function. The function $(\Gamma(z))^{-1}$ equals zero for $z = 0, -1, -2, ...$. Therefore the series (3.5) is a polynomial.

The relation (3.2) defines a representation of the group $SU(2)$. Thus the polynomial (3.5) defines a representation of the group $SU(2)$

$$t^l_{mn}(AB) = \sum_{k=-l}^{l} t^l_{mk}(A)t^l_{kn}(B). \hspace{1cm} (3.6)$$

This $(2l + 1)$ - dimensional representation is irreducible ([4], Chapter III, Section 2.3). The relations (3.5), (3.6) have an analytic continuation to all matrices (2.1).

By making the change $j \rightarrow j + n - m$ of the summation variable in the equality (3.5) we have

$$t^l_{mn}(A) = t^l_{nm}(A^T). \hspace{1cm} (3.7)$$

The polynomial (3.5) is homogeneous of the matrix elements (2.1). Its degree is $2l$. The sum (2.5) contains the only non - zero term. Since the definition (3.5) implies

$$t^l_{mn}(\sigma^0) = \delta_{mn}, \hspace{1cm} (3.8)$$
the relations (2.5), (3.6) imply
\[
\sum_{p=-l}^{l} \sum_{\bar{p}=-i}^{i} \left( \sum_{\nu=0}^{3} \gamma_{m}^{\nu} t_{m}^{\nu} (\sigma^{\nu}) \left( -i \frac{\partial}{\partial \sigma^{\nu}} \right) \right) \left( \sum_{\nu=0}^{3} \eta_{\nu}^{\nu} t_{p m}^{\nu} (\sigma^{\nu}) \eta_{p m}^{\nu} (\sigma^{\nu}) \left(-i \frac{\partial}{\partial \sigma^{\nu}} \right) \right) = \\
\sum_{1 \leq k_1 < k_2 \leq 3} \sum_{k_3=1}^{3} t_{m n}^{i} (i \sigma^{k_3}) t_{m n}^{i} (i \sigma^{k_3}) \left( \frac{\partial^2}{\partial x^{k_1} \partial x^{k_2}} \right) - \delta_{m n} \delta_{\bar{m} \bar{n}} (\partial_x, \partial_x), \\
(\partial_x, \partial_x) = \sum_{\nu=0}^{3} \eta_{\nu}^{\nu} \left( \frac{\partial}{\partial x^{\nu}} \right)^2. \tag{3.9}
\]

For an odd integer \(2l + 2\bar{l}\) the relation (3.9) has the form
\[
\sum_{p=-l}^{l} \sum_{\bar{p}=-i}^{i} \left( \sum_{\nu=0}^{3} \gamma_{m}^{\nu} t_{m}^{\nu} (\sigma^{\nu}) \left( -i \frac{\partial}{\partial \sigma^{\nu}} \right) \right) \left( \sum_{\nu=0}^{3} \eta_{\nu}^{\nu} t_{p m}^{\nu} (\sigma^{\nu}) \eta_{p m}^{\nu} (\sigma^{\nu}) \left(-i \frac{\partial}{\partial \sigma^{\nu}} \right) \right) = \\
- \delta_{m n} \delta_{\bar{m} \bar{n}} (\partial_x, \partial_x). \tag{3.10}
\]

By making use of the relation (3.10) for the odd integer \(2l + 2\bar{l}\) the Lorentz invariant equation
\[-(\partial_x, \partial_x) - \mu^2) \phi_{mn} (x) = 0, \quad m = -l, -l + 1, ..., -1, l, \quad \bar{m} = -\bar{l}, -\bar{l} + 1, ..., \bar{l} - 1, \bar{l} \tag{3.11}
\]
may be rewritten as the system of the linear equations
\[
\sum_{n=2l+2}^{4l+2} \sum_{\bar{n}=1}^{2l+1} \sum_{\nu=0}^{3} \eta_{\nu}^{\nu} t_{m-1,n-3l-2} (\sigma^{\nu}) t_{\bar{m}-1,\bar{n}-1} (\sigma^{\nu}) \left(-i \frac{\partial}{\partial x^{\nu}} \right) \psi_{mn} (x) + \psi_{\bar{m} \bar{n}} (x) = 0, \\
m = 1, ..., 2l + 1, \quad \bar{m} = 1, ..., 2\bar{l} + 1; \\
\sum_{n=1}^{2l+1} \sum_{\bar{n}=1}^{2l+1} \sum_{\nu=0}^{3} t_{m-3l-2,n-1} (\sigma^{\nu}) t_{\bar{m}-1,\bar{n}-1} (\sigma^{\nu}) \left(-i \frac{\partial}{\partial x^{\nu}} \right) \psi_{mn} (x) + \mu^2 \psi_{\bar{m} \bar{n}} (x) = 0, \\
m = 2l + 2, ..., 4l + 2, \quad \bar{m} = 1, ..., 2\bar{l} + 1. \tag{3.12}
\]

Let us define the \(((4l + 2)(2\bar{l} + 1)) \times ((4l + 2)(2\bar{l} + 1))\) - matrices
\[
(\alpha_{l,i}^{\nu} (\mu^{2}))_{m_{i} n_{i}}, \quad \mu^{2} \delta_{m_{i} n_{i}} \delta_{\bar{m}_{i} \bar{n}_{i}}, \quad m, n = 1, ..., 2l + 1, \quad \bar{m}, \bar{n} = 1, ..., 2\bar{l} + 1; \\
(\alpha_{l,i}^{\nu} (\mu^{2}))_{m_{i} n_{i}}, \quad \delta_{m_{i} n_{i}} \delta_{\bar{m}_{i} \bar{n}_{i}}, \quad m, n = 2l + 2, ..., 4l + 2, \quad \bar{m}, \bar{n} = 1, ..., 2\bar{l} + 1; \\
(\beta_{l,i}^{\nu} (\mu^{2}))_{m_{i} n_{i}}, \quad \mu^{2} \delta_{m_{i} n_{i}} \delta_{\bar{m}_{i} \bar{n}_{i}}, \quad m, n = 2l + 2, ..., 4l + 2, \quad \bar{m}, \bar{n} = 1, ..., 2\bar{l} + 1; \\
(\gamma_{l,i}^{\nu} (\sigma^{0}))_{m_{i} n_{i}}, \quad \eta_{\nu}^{\nu} t_{m-i-1,n-3l-2} (\sigma^{\nu}) t_{\bar{m}-1,\bar{n}-1} (\sigma^{\nu}), \\
m = 1, ..., 2l + 1, \quad \bar{m}, \bar{n} = 1, ..., 2\bar{l} + 1, \quad \nu = 0, ..., 3; \\
(\gamma_{l,i}^{\nu} (\sigma^{0}))_{m_{i} n_{i}}, \quad t_{m-3l-2,n-1} (\sigma^{\nu}) t_{\bar{m}-1,\bar{n}-1} (\sigma^{\nu}), \\
m = 2l + 2, ..., 4l + 2, \quad n = 1, ..., 2l + 1, \quad \bar{m}, \bar{n} = 1, ..., 2\bar{l} + 1, \quad \nu = 0, ..., 3. \tag{3.13}
\]

The other matrix elements are equal to zero. If an integer \(2l + 2\bar{l}\) is odd, then an integer \((2l + 1)(2\bar{l} + 1)\) is even and an integer \((4l + 2)(2\bar{l} + 1)\) has the form \(4k\) where \(k\) is an integer.
By making use of the definitions (3.13) we can rewrite the equations (3.12) as

\[
\sum_{n=1}^{d+2} \sum_{\hat{n}=1}^{2l+1} \left( \sum_{\nu=0}^{3} (\gamma_{l,i}^{\nu}(\sigma^0))_{m\hat{m},n\hat{n}} \left(-i \frac{\partial}{\partial x^\nu}\right) + (\beta_{l,i}(\mu^2))_{m\hat{m},n\hat{n}}\right) \psi_{n\hat{n}}(x) = 0. \tag{3.14}
\]

The definitions (3.13) imply

\[
\alpha_{l,i}(\mu)\beta_{l,i}(\mu^2) = \mu \beta_{l,i}(\mu), \quad \alpha_{l,i}(\mu)\gamma_{l,i}^{\nu}(\sigma^0) = \gamma_{l,i}^{\nu}(\sigma^0)\beta_{l,i}(\mu). \tag{3.15}
\]

In view of the relations (3.15) the action of the matrix \(\alpha_{l,i}(\mu)\) on the equation (3.14) yields

\[
\sum_{n=1}^{d+2} \sum_{\hat{n}=1}^{2l+1} \sum_{\nu=0}^{3} (\gamma_{l,i}^{\nu}(\sigma^0))_{m\hat{m},n\hat{n}} \left(-i \frac{\partial}{\partial x^\nu}\right) \xi_{n\hat{n}}(x) + \mu \xi_{m\hat{m}}(x) = 0,
\]

\[
\xi_{m\hat{m}}(x) = \sum_{n=1}^{d+2} \sum_{\hat{n}=1}^{2l+1} (\beta_{l,i}(\mu))_{m\hat{m},n\hat{n}} \psi_{n\hat{n}}(x) = 0. \tag{3.16}
\]

For \(\mu > 0\) the transformation given by the second relation (3.16) is the isomorphism. The definition (3.5) implies

\[
t^q_{\hat{m}n}(A) = A_{m+\frac{q}{2},n+\frac{q}{2}}. \tag{3.17}
\]

Due to the relations (3.13), (3.17) the equation (3.16) for \(l = \frac{1}{2}, \hat{l} = 0\) coincides with the Dirac equation ([5], equation (1-41)).

The relations (2.10), (2.12) define the representation of the group \(SL(2, \mathbb{C})\) in the Lorentz group

\[
\sum_{\mu,\nu=0}^{3} \Lambda^{\mu}_{\nu}(A)x^\nu \sigma^\mu = L(A, A^*)(\hat{x}) = A\tilde{x}A^*. \tag{3.18}
\]

Let us define \((4l + 2)(2\hat{l} + 1)\) - dimensional representation of the group \(SL(2, \mathbb{C})\)

\[
(S_{l,i}(A))_{m\hat{m},n\hat{n}} = t^l_{m-l-1,n-l-1}(A)t^l_{n-l-1,m-l-1}(\tilde{A}),
\]

\[
m, n = 1, \ldots, 2l + 1, \quad \hat{m}, \hat{n} = 1, \ldots, 2\hat{l} + 1;
\]

\[
(S_{l,i}(A))_{\hat{m}m,\hat{n}n} = t^l_{m-3l-2,n-3l-2}(A^*^{-1})t^l_{n-3l-2,m-3l-2}(\tilde{A}^T^{-1}),
\]

\[
m, n = 2l + 2, \ldots, 4l + 2, \quad \hat{m}, \hat{n} = 1, \ldots, 2\hat{l} + 1. \tag{3.19}
\]

The other matrix elements are equal to zero. The definitions (3.13), (3.19) imply

\[
S_{l,i}(A)\alpha_{l,i}(\mu^2)S_{l,i}(A^{-1}) = \alpha_{l,i}(\mu^2), \quad S_{l,i}(A)\beta_{l,i}(\mu^2)S_{l,i}(A^{-1}) = \beta_{l,i}(\mu^2). \tag{3.20}
\]

Let the functions \(\psi_{n\hat{n}}(x), m = 1, \ldots, 4l + 2, \hat{m} = 1, \ldots, 2\hat{l} + 1\) be the solutions of the equation (3.14). The relations (3.20) imply that the functions

\[
\xi_{m\hat{m}}(x) = \sum_{n=1}^{d+2} \sum_{\hat{n}=1}^{2l+1} (S_{l,i}(A))_{m\hat{m},n\hat{n}} \psi_{n\hat{n}} \left(\sum_{\nu=0}^{3} \Lambda^{\mu}_{\nu}(A^{-1})x^\nu\right) \tag{3.21}
\]

are the solutions of the equation

\[
\sum_{n=1}^{d+2} \sum_{\hat{n}=1}^{2l+1} \left( \sum_{\nu=0}^{3} (\gamma_{l,i}^{\nu}(A))_{m\hat{m},n\hat{n}} \left(-i \frac{\partial}{\partial x^\nu}\right) + (\beta_{l,i}(\mu^2))_{m\hat{m},n\hat{n}}\right) \xi_{n\hat{n}}(x) = 0, \tag{3.22}
\]

\[
\xi_{n\hat{n}}(x) = \sum_{n=1}^{d+2} \sum_{\hat{n}=1}^{2l+1} (\beta_{l,i}(\mu))_{m\hat{m},n\hat{n}} \psi_{n\hat{n}}(x) = 0.
\]
\[ \gamma_{i,i}^{\mu}(A) = \sum_{\nu=0}^{3} \Lambda_{\nu}^{\mu}(A) S_{i,i}^{\nu}(A) \gamma_{i,i}^{\nu}(\sigma^{0}) S_{i,i}^{\nu}(A^{-1}) \]  
(3.23)

for any matrix \( A \in SL(2, \mathbb{C}) \). The definition (3.23) implies

\[ \gamma_{i,i}^{\mu}(AB) = \sum_{\nu=0}^{3} \Lambda_{\nu}^{\mu}(A) S_{i,i}^{\nu}(A) \gamma_{i,i}^{\nu}(B) S_{i,i}^{\nu}(A^{-1}) \]  
(3.24)

for any matrices \( A, B \in SL(2, \mathbb{C}) \). By changing the coordinate system we change the matrix \( \gamma_{i,i}^{\nu}(\sigma^{0}) \) in the equation (3.14) for the matrix (3.23). The solutions of the equation (3.14) transform to the solutions (3.21) of the equation (3.22). It is valid for all half - integers \( l, \hat{l} \in \mathbb{Z} \). Due to ([5], relation (1 - 43))

\[ \gamma_{\frac{1}{2},0}^{\mu}(A) = \gamma_{\frac{1}{2},0}^{\mu}(\sigma^{0}) \]  
(3.25)

for any matrix \( A \in SL(2, \mathbb{C}) \). Hence the equation (3.14) for \( l = \frac{1}{2}, \hat{l} = 0 \) is covariant under the group \( SL(2, \mathbb{C}) \).

In view of the definitions (3.13), (3.19), (3.23) the equation (3.22) is equivalent to the system of two equations

\[ \sum_{n=2l+2}^{4l+2} \sum_{\hat{n}=1}^{2l+1} \sum_{\nu=1}^{3} (\gamma_{i,i}^{\nu}(A))_{mn,\hat{m}} \left( -i \frac{\partial}{\partial x^{\nu}} \right) \xi_{m\hat{m}}(x) + \xi_{mn}(x) = 0, \]

\[ m = 1, \ldots, 2l+1, \; \hat{m} = 1, \ldots, 2\hat{l}+1; \]

\[ \sum_{n=1}^{2l+1} \sum_{\hat{n}=1}^{2\hat{l}+1} \sum_{\nu=0}^{3} (\gamma_{i,i}^{\nu}(A))_{mn,\hat{m}} \left( -i \frac{\partial}{\partial x^{\nu}} \right) \xi_{n\hat{m}}(x) + \mu^{2} \xi_{m\hat{m}}(x) = 0, \]

\[ m = 2l+2, \ldots, 4l+2, \; \hat{m} = 1, \ldots, 2\hat{l}+1. \]  
(3.26)

The system of two equations (3.26) is equivalent to the equation

\[ \sum_{n=2l+2}^{4l+2} \sum_{\hat{n}=1}^{2l+1} \sum_{\nu=0}^{3} (\gamma_{i,i}^{\nu}(A))_{mn,\hat{m}} \frac{\partial}{\partial x^{\nu}} \left( \sum_{\nu=0}^{3} (\gamma_{i,i}^{\nu}(A))_{pp,\hat{n}} \frac{\partial}{\partial x^{\nu}} \right) \xi_{n\hat{m}}(x) + \mu^{2} \xi_{m\hat{m}}(x) = 0, \]

\[ m, n = 2l+2, \ldots, 4l+2, \; \hat{m}, \hat{n} = 1, \ldots, 2\hat{l}+1. \]  
(3.27)

Let us prove that for an odd integer \( 2l + 2\hat{l} \)

\[ \sum_{p=1}^{2l+1} \sum_{\nu=0}^{3} (\gamma_{i,i}^{\nu}(A))_{mn,pp} \frac{\partial}{\partial x^{\nu}} \left( \sum_{\nu=0}^{3} (\gamma_{i,i}^{\nu}(A))_{pp,\hat{n}} \frac{\partial}{\partial x^{\nu}} \right) = (\partial_{x}, \partial_{x}) \delta_{mn} \delta_{\hat{m} \hat{n}}, \]

\[ m, n = 2l+2, \ldots, 4l+2, \; \hat{m}, \hat{n} = 1, \ldots, 2\hat{l}+1. \]  
(3.28)

We denote

\[ \frac{\partial}{\partial y^{\nu}} = \sum_{\nu=0}^{3} \Lambda_{\nu}^{\mu}(A) \frac{\partial}{\partial x^{\nu}}. \]  
(3.29)

The matrix \( \Lambda_{\nu}^{\mu}(A) \) belongs to the Lorentz group and the definition (3.29) implies

\[ \sum_{\mu=0}^{3} \eta_{\mu}^{\mu} \left( \frac{\partial}{\partial y^{\mu}} \right)^{2} = (\partial_{x}, \partial_{x}). \]  
(3.30)
The definitions (3.13), (3.19), (3.23) and the relations (3.10), (3.30) imply the equality (3.28). Therefore for an odd integer $2l + 2\hat{l}$ the equation (3.22) is equivalent to the equation (3.11).

The relations (3.13) imply
\[
(\alpha_{\hat{i},\hat{l}}(\mu^2)\beta_{\hat{i},\hat{l}}(\mu^2))_{m\bar{n},n\bar{n}} = \mu^2\delta_{m\bar{n}}\delta_{n\bar{n}}, \quad m, n = 1, \ldots, 4\hat{l} + 2, \quad \hat{m}, \hat{n} = 1, \ldots, 2\hat{l} + 1.
\] (3.31)

In view of the second relation (3.15) and the relations (3.20), (3.23), (3.31) the action of the matrix $\gamma_{\hat{i},\hat{l}}^0(\sigma^0)\alpha_{\hat{i},\hat{l}}(\mu^2)$ on the equation (3.14) yields
\[
\sum_{n=1}^{4\hat{l}+2} \sum_{\hat{n}=1}^{2\hat{l}+1} \sum_{\nu=0}^{3} \left\{ \sum_{l=1}^{n} \left( \gamma_{\hat{i},\hat{l}}^0(\sigma^0)\gamma_{\hat{l},\hat{i}}^\nu(\sigma^0)\beta_{\hat{i},\hat{l}}(\mu^2) \right)_{m\bar{n},n\bar{n}} \left( -i\frac{\partial}{\partial x^\nu} \right) + \mu^2(\gamma_{\hat{i},\hat{l}}^0(\sigma^0))_{m\bar{n},n\bar{n}} \right\} \psi_{\bar{n}\bar{n}}(x) = 0.
\] (3.32)

The relations (2.5), (3.6), (3.8), (3.13) imply
\[
((\gamma_{\hat{i},\hat{l}}^0(\sigma^0))^2)_{m\bar{n},n\bar{n}} = \delta_{m\bar{n}}\delta_{n\bar{n}}, \quad m, n = 1, \ldots, 4\hat{l} + 2, \quad \hat{m}, \hat{n} = 1, \ldots, 2\hat{l} + 1;
\]
\[
(\gamma_{\hat{i},\hat{l}}^0(\sigma^0)\gamma_{\hat{l},\hat{i}}^k(\sigma^0))_{m\bar{n},n\bar{n}} = t^l_{m-1,n-1}(\sigma^k)t^l_{m-1,n-1}(\sigma^k),
\]
\[
m, n = 1, \ldots, 2\hat{l} + 1, \quad \hat{m}, \hat{n} = 1, \ldots, 2\hat{l} + 1, \quad k = 1, 2, 3;
\]
\[
(\gamma_{\hat{i},\hat{l}}^0(\sigma^0)\gamma_{\hat{l},\hat{i}}^k(\sigma^0))_{m\bar{n},n\bar{n}} = -t^l_{m-3l-2,n-3l-2}(\sigma^k)t^l_{m-3l-2,n-3l-2}(\sigma^k),
\]
\[
m, n = 2\hat{l} + 2, \ldots, 4\hat{l} + 2, \quad \hat{m}, \hat{n} = 1, \ldots, 2\hat{l} + 1, \quad k = 1, 2, 3.
\] (3.33)

The other matrix elements are equal to zero. The coefficients of the polynomial (3.5) are real. Hence in view of the relations (3.7), (3.13), (3.33) the matrices $\gamma_{\hat{i},\hat{l}}^0(\sigma^0), \gamma_{\hat{i},\hat{l}}^\nu(\sigma^0)\gamma_{\hat{l},\hat{i}}^k(\sigma^0)\beta_{\hat{i},\hat{l}}(\mu^2), \nu = 0, \ldots, 3$, are Hermitian. Due to the first relation (3.33) we have $(\gamma_{\hat{i},\hat{l}}^0(\sigma^0))^2\beta_{\hat{i},\hat{l}}(\mu^2) = \beta_{\hat{i},\hat{l}}(\mu^2)$. Let the functions $\xi_{m\bar{n}}(x)$ have the form (3.21). Now the equation (3.32) implies that the integral
\[
\int d^3x \sum_{m=1}^{4\hat{l}+2} \sum_{\hat{n}=1}^{2\hat{l}+1} (\beta_{\hat{i},\hat{l}}(\mu^2))_{m\bar{n},n\bar{n}} \sum_{n=1}^{4\hat{l}+2} \sum_{\bar{n}=1}^{2\hat{l}+1} (S_{\hat{i},\hat{l}}(A^{-1}))_{m\bar{n},n\bar{n}} \xi_{\bar{n}} \left( \sum_{\nu=0}^{3} \Lambda_{\nu}(A) x^\nu \right) \left( \sum_{\nu=0}^{3} \Lambda_{\nu}(A) x^\nu \right)
\] (3.34)
is independent of the variable $x^0$ for $x^0 > 0$. The integrand (3.34) is called the probability density of a solution of the equation (3.22). For $A = \sigma^0$ the integrand (3.34) coincides with the usual probability density for the function (3.16). In the quantum mechanics the fixed probability density defines Hilbert space where any Hamiltonian acts. The integral (3.34) depends on the parameter $\mu^2$ in the equation (3.22). We do not expect the solutions of the equations with interaction to have the independent of time integrals similar to (3.34). This mathematical assumption does not seem physically reasonable. We deal with the asymptotic solutions of the equations with interaction in an experiment. We expect the solutions of the equations with interaction to coincide asymptotically with the products of the solutions of the equation (3.22). The probability density of the last solutions is given by the integrand (3.34).

Let the functions $\xi_{m\bar{n}}(x)$ be the solutions of the equation (3.22). Let us introduce the distributions
\[
f_{m\bar{n}}(x) = \theta(x^0)\xi_{m\bar{n}}(x), \quad \theta(x) = \begin{cases} 1, & x \geq 0, \\ 0, & x < 0. \end{cases}
\] (3.35)

The equation (3.22) implies
\[
\sum_{n=1}^{4\hat{l}+2} \sum_{\hat{n}=1}^{2\hat{l}+1} \left( \sum_{\nu=0}^{3} (\gamma_{\hat{i},\hat{l}}^\nu(A))_{m\bar{n},n\bar{n}} \left( -i\frac{\partial}{\partial x^\nu} \right) + (\beta_{\hat{i},\hat{l}}(\mu^2))_{m\bar{n},n\bar{n}} \right) f_{\bar{n}}(x) = -i\delta(x^0)f^0_{m\bar{n}}(0, x), \quad x = (x^1, x^2, x^3) \in \mathbb{R}^3,
\] (3.36)
Let a support of a distribution \( e_{\mu_1^2, ..., \mu_2^n}(x) \in S'(\mathbb{R}^4) \) lie in the closed upper light cone. Let a distribution \( e_{\mu_1^2, ..., \mu_2^n}(x) \) satisfy the equation
\[
\left( \prod_{i=1}^{n} (- (\partial_x, \partial_x) - \mu_i^2) \right) e_{\mu_1^2, ..., \mu_2^n}(x) = \delta(x). \tag{3.38}
\]
By changing the differential operator \(- (\partial_x, \partial_x)\) for the differential operator
\[
\prod_{i=1}^{n} (- (\partial_x, \partial_x) - \mu_i^2)
\]
in the proof of Lemma 3 from the paper [1] we obtain the uniqueness of the distribution \( e_{\mu_1^2, ..., \mu_2^n}(x) \). Due to ([6], Section 30)
\[
e_0(x) = -(2\pi)^{-1} \theta(x^0)(x(x, x)), \quad e_{0,0}(x) = (8\pi)^{-1} \theta(x^0)\theta((x, x)). \tag{3.39}
\]
The second definition (3.35) implies
\[
(\partial_x, \partial_x)(\theta(x^0)\theta((x, x))(x, x)^n) = 4n(n + 1)\theta(x^0)\theta((x, x))(x, x)^n - 1, \quad n = 1, 2, ..., \tag{3.40}
\]
Due to the relations (3.39), (3.40) the distribution \( e_{0, ..., 0}(x) \) with \( n \) zeros has the form
\[
e_{0, ..., 0}(x) = (-1)^n(2\pi 4n^{-1}(n - 2)!(n - 1)!)^{-1} \theta(x^0)\theta((x, x))(x, x)^{n-2}, \quad n = 2, 3, ... \tag{3.41}
\]
Let us prove
\[
e_{\mu_1^2, ..., \mu_2^n}(x) = \lim_{\epsilon \to +0} (2\pi)^{-4} \int d^4 p \exp \{-i(p, x)\} \prod_{j=1}^{n} ((p^0 + i\epsilon)^2 - |p|^2 - \mu_j^2)^{-1},
\]
\[
(x, y) = x^0 y^0 - \sum_{k=1}^{3} x^k y^k \tag{3.42}
\]
The integral (3.42) is the solution of the equation (3.38). By making the shift of the integration path in the right - hand side of the equality (3.42) we obtain that the distribution (3.42) is equal to zero for \( x^0 < 0 \). The distribution (3.42) is Lorentz invariant. Hence its support lies in the closed upper light cone. Now the uniqueness of the distribution (3.42) implies the equality (3.42).
For an odd integer \( 2l + 2\hat{l} \) the relation
\[
\sum_{p=1}^{4l+2} \sum_{\hat{p}=1}^{2\hat{l}+1} \left( \sum_{\nu=0}^{3} (\gamma^\nu_{l,\tilde{i}}(A))_{\mu_\nu, p\hat{p}} \left( -i \partial_{x^\nu} \right) + (\beta_{l, \nu}(\mu^2))_{\nu, p\hat{p}} \right) \times
\]
\[
\left( \sum_{\nu=0}^{3} (\gamma^\nu_{l,\tilde{i}}(A))_{\nu, \tilde{m}\tilde{n}} \left( -i \partial_{x^\nu} \right) - (\alpha_{l, \nu}(\mu^2))_{\nu, \tilde{m}\tilde{n}} \right) = -(\partial_x, \partial_x) - \mu^2) \delta_{\mu\nu} \delta_{\nu\hat{m}, \hat{n}},
\]
m, n = 1, ..., 4l + 2, \quad \hat{m}, \hat{n} = 1, ..., 2\hat{l} + 1, \tag{3.43}
similar to the relation (3.28) is valid. The relations (3.38), (3.43) imply that the solution of the equation (3.36) has the form

\[ f_{\hat{m}n}(x) = \sum_{n=1}^{4l+2} \sum_{\hat{n}=1}^{2\hat{l}+1} \left( \sum_{\nu=0}^{3} (\gamma_{\hat{l},\hat{n}}^{\nu}(A))_{m\hat{m},\hat{n}\hat{n}} \left( -i \frac{\partial}{\partial x^{\nu}} - (\alpha_{\hat{l},\hat{n}}(\mu^{2})\gamma_{\hat{l},\hat{n}}^{0}(A))_{m\hat{m},\hat{n}\hat{n}} \right) \times \left( -i \int d^{4}ye^{\mu_2}(x-y)\delta(y^{0})f_{\hat{n}\hat{n}}(+0,y) \right), \quad m = 1, \ldots, 4l + 2, \hat{m} = 1, \ldots, 2\hat{l} + 1. \]  

(3.44)

We suppose that the smooth function \( f_{\hat{n}\hat{n}}(+0, x) \) is rapidly decreasing at the infinity. For the solution of the equation (3.22) in the domain \( x^{0} < 0 \) it is sufficient to use the distribution \(-e^{\mu_2}(-x)\) in the relation (3.44). By shifting the integration path in the integral (3.42) we have

\[ \int d^{4}y e^{\mu_2}(x-y)\delta(y^{0})f_{\hat{n}\hat{n}}(+0, y) = (2\pi)^{-4} \int d^{4}p \exp\{x^{0} - i(p, x)\}((p^{0} + i)^{2} - |p|^{2} - \mu^{2})^{-1} \hat{f}_{\hat{m}n}(+0, \cdot)(p), \]

\[ \hat{f}_{\hat{m}n}(+0, \cdot)(p) = \int d^{3}x \exp\{-i \sum_{k=1}^{3} p^{k} x^{k}\} f_{\hat{m}n}(+0, x). \]  

(3.45)

The integral with respect to \( p^{0} \) may be easily calculated. For \( x^{0} > 0 \) and \( \hat{f}_{\hat{m}n}(+0, \cdot)(p) = f_{\hat{m}n}\delta(p - q) \) the functions (3.44), (3.45) are not the eigenfunctions of the differential operator \(-i\partial/\partial x^{0}\) and are the eigenfunctions of the differential operator \((-i\partial/\partial x^{0})^{2}\) (see the Dirac discussion of the negative energy electrons).

Let us introduce the interaction coefficients into the equation (3.36). Let \( j, k \) be the permutation of the numbers 1, 2. We construct the equation for \( j \) particle. Let us multiply the equations (3.36) for the particles 1 and 2. We change the differential operator

\[ \left( -i \frac{\partial}{\partial x_{1}^{\mu}} \right) \left( -i \frac{\partial}{\partial x_{2}^{\nu}} \right) \]  

(3.46)

for the differential operator

\[ \left( -i \frac{\partial}{\partial x_{1}^{\mu}} \right) \left( -i \frac{\partial}{\partial x_{2}^{\nu}} \right) + A_{\nu_1\nu_2}^{(jk)}(x_{j} - x_{k}). \]  

(3.47)

The differential operator (3.47) should transform like the differential operator (3.46). Therefore for any matrix \( A \in SL(2, \mathbb{C}) \)

\[ A_{\nu_1\nu_2}^{(jk)} \left( \sum_{\mu=0}^{3} \Lambda_{\mu}^{\lambda}(A^{-1})x^{\mu} \right) = \sum_{\mu_1, \mu_2 = 0}^{3} \Lambda_{\nu_1}^{\mu_1}(A)\Lambda_{\nu_2}^{\mu_2}(A)A_{\mu_1\mu_2}^{(jk)}(x). \]  

(3.48)

The interaction coefficients in the relativistic Coulomb law ([1], relations (2.15), (2.16), (2.23)) are defined by the trajectory of another particle. A particle has no trajectory in the quantum mechanics. We integrate the obtained equation with respect to the variable \( x_{k} \)

\[ \sum_{n_{s} = 1, \ldots, 4l_{s} + 2, s = 1, 2} \sum_{\hat{n}_{s} = 1, \ldots, 2\hat{l}_{s} + 1, s = 1, 2} \int d^{4}x_{k} \]  

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The first and the second terms in the left-hand side of the equation (3.49) correspond to the multiplied equations (3.36) for the particles 1 and 2. The third term in the left-hand side of the equation (3.49) corresponds to the interaction. The equation (3.49) transforms like two equations (3.36).

We will construct the interaction coefficients $A_{h}^{(4)}(x)$ using the Clebsch-Gordan coefficients. Due to ([4], Chapter III, Section 8.3) we have for a matrix $A \in SU(2)$

$$t_{m_{1}n_{1}}^{l_{1}}(A)t_{m_{2}n_{2}}^{l_{2}}(A) = \sum_{l_{3} \in 1/2\mathbb{Z}_{+}} \sum_{m_{3}n_{3} = -l_{3}} C(l_{1}, l_{2}, l_{3}; m_{1}, m_{2}, m_{3})C(l_{1}, l_{2}, l_{3}; n_{1}, n_{2}, n_{3})t_{m_{3}n_{3}}^{l_{3}}(A)$$

(3.50)

The Clebsch-Gordan coefficient $C(l_{1}, l_{2}, l_{3}; m_{1}, m_{2}, m_{3})$ is not zero only if $m_{3} = m_{1} + m_{2}$ and the half-integers $l_{1}, l_{2}, l_{3} \in 1/2\mathbb{Z}_{+}$ satisfy the triangle condition: the half-integer $l_{3}$ is one of the half-integers $|l_{1} - l_{2}|, |l_{1} - l_{2}| + 1, ..., l_{1} + l_{2} - 1, l_{1} + l_{2}$. Let the half-integers $l_{1}, l_{2}, l_{3} \in 1/2\mathbb{Z}_{+}$ satisfy the triangle condition. Let the half-integers $m_{i} = -l_{i}, -l_{i} + 1, ..., l_{i} - 1, l_{i}, i = 1, 2, 3, m_{3} = m_{1} + m_{2}$. Then due to ([4], Chapter III, Section 8.3)

$$C(l_{1}, l_{2}, l_{3}; m_{1}, m_{2}, m_{3}) = (-1)^{l_{1} - l_{3} + m_{2}(2l_{3} + 1)^{1/2}} \times \left( \frac{(l_{1} + l_{2} - l_{3})!(l_{1} + l_{3} - l_{2})!(l_{2} + l_{3} - l_{1})!(l_{3} - m_{3})!(l_{3} + m_{3})!}{(l_{1} + l_{2} + l_{3} + 1)!(l_{1} - m_{1})!(l_{1} + m_{1})!(l_{2} - m_{2})!(l_{2} + m_{2})!} \right)^{1/2} \times \sum_{j=0}^{l_{2} + l_{3} - l_{1}} \frac{(-1)^{j}(l_{1} + m_{1} + j)!(l_{2} + l_{3} - m_{1} - j)!}{j!\Gamma(l_{3} - m_{3} - j + 1)\Gamma(l_{1} - l_{2} + m_{3} + j + 1)(l_{2} + l_{3} - l_{1} - j)!}.$$  

(3.51)

Let $dA$ be the normalized Haar measure on the group $SU(2)$. Due to ([4], Chapter III, Section 8.3)

$$C(l_{1}, l_{2}, l_{3}; m_{1}, m_{2}, m_{3})C(l_{1}, l_{2}, l_{3}; n_{1}, n_{2}, n_{3}) = (2l_{3} + 1) \int_{SU(2)} dA t_{m_{1}n_{1}}^{l_{1}}(A)t_{m_{2}n_{2}}^{l_{2}}(A)t_{m_{3}n_{3}}^{l_{3}}(A).$$

(3.52)

The coefficients of the polynomial (3.5) are real. By using the relations (3.7) and $A^{*} = A^{-1}$ we can rewrite the equality (3.52) as

$$C(l_{1}, l_{2}, l_{3}; m_{1}, m_{2}, m_{3})C(l_{1}, l_{2}, l_{3}; n_{1}, n_{2}, n_{3}) = (2l_{3} + 1) \int_{SU(2)} dA t_{m_{1}n_{1}}^{l_{1}}(A)t_{m_{2}n_{2}}^{l_{2}}(A)t_{m_{3}n_{3}}^{l_{3}}(A^{-1}).$$

(3.53)

If the half-integers $l_{1}, l_{2}, l_{3} \in 1/2\mathbb{Z}_{+}$ satisfy the triangle condition, then due to ([4], Chapter III, Section 8.3) we have

$$C(l_{1}, l_{2}, l_{3}; l_{1} - l_{2}, l_{1} - l_{2}) = \left( \frac{(2l_{3} + 1)(2l_{1})!(2l_{2})!}{(l_{1} + l_{2} - l_{3})!(l_{1} + l_{2} + l_{3} + 1)!} \right)^{1/2}.$$  

(3.54)
Let us choose the half - integers \( n_1 = l_1, n_2 = -l_2 \) in the equality (3.53). Then the relations (3.6), (3.53), (3.54) and the invariance of the Haar measure \( dA \) imply

\[
\sum_{n_1 = -l_1}^{l_1} \sum_{n_2 = -l_2}^{l_2} t_{m_1 n_1}^l (A) t_{m_2 n_2}^l (A) C(l_1, l_2, l_3; n_1, n_2, m_3) = \sum_{n_3 = -l_3}^{l_3} C(l_1, l_2, l_3; m_1, m_2, n_3) t_{n_3 m_3}^l (A).
\]

(3.55)

The substitution of the matrix \( A^T \) into the equality (3.55) and the equality (3.7) yield

\[
\sum_{n_1 = -l_1}^{l_1} \sum_{n_2 = -l_2}^{l_2} t_{n_1 m_1}^l (A) t_{n_2 m_2}^l (A) C(l_1, l_2, l_3; n_1, n_2, m_3) = \sum_{n_3 = -l_3}^{l_3} C(l_1, l_2, l_3; m_1, m_2, n_3) t_{m_3 n_3}^l (A).
\]

(3.56)

The relations (3.55), (3.56) have an analytic continuation to the group \( SL(2, \mathbb{C}) \).

By making use of the matrices (2.3) as the coefficients we define \( 2 \times 2 \)-matrix

\[
\tilde{\partial}_x = \sum_{\mu=0}^{3} \eta^{\mu \nu} \sigma^{\mu} \frac{\partial}{\partial x^{\nu}}.
\]

(3.57)

We insert the matrix (3.57) into the polynomial (3.5) and obtain the differential operator \( t_{mn}^l (\tilde{\partial}_x) \). Due to ([5], relation (1 - 18)) for any matrix \( A \in SL(2, \mathbb{C}) \)

\[
\sigma^2 A^T \sigma^2 = A^{-1}.
\]

(3.58)

Similar to the relativistic Coulomb law ([1], relations (2.15), (2.16), (2.23)) we construct the interaction coefficients \( A^{(jk)}_{\nu_1 \nu_2} (x) \) from the derivatives of the distributions (3.39)

\[
A^{(jk)}_{\nu_1 \nu_2} (x) = K_{jk;0} Q^0_{\nu_1 \nu_2} (\tilde{\partial}_x) e_0(x) + K_{jk;1} Q^1_{\nu_1 \nu_2} (\tilde{\partial}_x) e_{0,0}(x),
\]

\[
Q^l_{\nu_1 \nu_2} (\tilde{\partial}_x) = \sum_{m_1, m_2, \tilde{m}_1, \tilde{m}_2} \sum_{l = -l_1, -l_1+1, \ldots, -l_1} C(\nu_{\tilde{m}_1}, \frac{1}{2}; l; m_1, m_2, m_{12}) \times
\]

\[
C(\nu_{\tilde{m}_2}, \frac{1}{2}; l; \tilde{m}_1, \tilde{m}_2, \tilde{m}_{12}) \left( \prod_{k=1}^{2} t_{m_k \tilde{m}_k} (\sigma^2 \sigma^{\nu_k} \sigma^2) \right) t_{m_{12} \tilde{m}_{12}}^l (\tilde{\partial}_x).
\]

(3.59)

Here \( K_{jk;l}, \ l = 0, 1, \) are the interaction constants. In view of the relations (3.18), (3.55) - (3.58) the interaction coefficients (3.59) satisfy the covariance relation (3.48). Due to the triangle condition the Clebsch - Gordan coefficient \( C(\frac{1}{2}, \frac{1}{2}; l; m_1, m_2, m_{12}) \) is not zero for the integers \( l = 0, 1 \) only. The degree of the homogeneous polynomial (3.5) is \( 2l \). Therefore the relation (3.41) implies that the support of the distribution \( t_{m_{12} \tilde{m}_{12}}^l (\tilde{\partial}_x) e_{0,0}(x) \) lies in the boundary of the upper light cone and the support of the distribution \( t_{m_{12} \tilde{m}_{12}}^l (\tilde{\partial}_x) e_{0,0,0}(x) \) lies in all closed upper light cone. Let us prove

\[
Q^0_{\nu_1 \nu_2} (\tilde{\partial}_x) = \eta_{\nu_1 \nu_2}.
\]

(3.60)
The definition (3.5) implies $\eta^0_{00}(A) = 1$. Therefore the left - hand side of the equality (3.60) does not depend on the differential operator (3.57). The definition (2.3) and the relations (3.17), (3.51) imply
\[ C\left(\frac{1}{2}, \frac{1}{2}, 0; m_1, m_2, 0\right) = -i2^{-\frac{1}{2}}t^{\frac{1}{2}}_{m_1m_2}(\sigma^2). \] (3.61)

In view of the definition (2.3) $(\sigma^2)^T = -\sigma^2$. The substitution of the relation (3.61) into the definition (3.59) yields
\[ Q^0_{\nu_1\nu_2}(\tilde{\partial}_x) = \frac{1}{2} \sum_{m = -\frac{1}{2},\frac{1}{2}} t^{\frac{1}{2}}_{mm}(\sigma^\nu_1\sigma^2(\sigma^\nu_1)^T\sigma^2). \] (3.62)

Due to ([5], Section 1 - 3)
\[ \sigma^2(\sigma^\nu)^T\sigma^2 = \eta_{\mu\nu}\sigma^\nu, \ \nu = 0, ..., 3. \] (3.63)

The relations (2.3), (2.5), (3.17), (3.62), (3.63) imply the relation (3.60).

For a vector $p \in \mathbb{R}^4$, $(p, p) > 0$, we define a matrix $(p, p)^{-1/2}\tilde{p} \in SL(2, \mathbb{C})$. In view of the relations (2.5) it satisfies the equation
\[ ((p, p)^{-1/2}\tilde{p})^{-1} = (p, p)^{-1/2}\sum_{\mu = 0}^3 \eta_{\mu\mu}p^\mu\sigma^\mu. \]

This relation and the relations (3.55), (3.58), (3.59) imply
\[ Q^1_{\nu_1\nu_2}((p, p)^{-1/2}\tilde{p}) = \sum_{m_1, m_2, \tilde{m}_1, \tilde{m}_2 = -\frac{1}{2}, \frac{1}{2}} \sum_{m_1m_2 = 1, 0, 1} C\left(\frac{1}{2}, \frac{1}{2}, 1; m_1, m_2, m_{12}\right) \times \]
\[ \sum_{k = 1}^2 \tilde{t}^{\frac{1}{2}}_{m_k\tilde{m}_k} \left(\frac{3}{2}(p, p)^{-1/2}\left(\sum_{\mu = 0}^3 \eta_{\mu\mu}p^\mu\sigma^\mu\right)\sigma^\nu\sigma^2\right). \] (3.64)

In view of the relations (3.8), (3.61) the substitution of the matrix $A = \sigma^0$ into the relation (3.60) yields
\[ \sum_{m_1m_2 = 1, 0, 1} C\left(\frac{1}{2}, \frac{1}{2}, 1; m_1, m_2, m_{12}\right)C\left(\frac{1}{2}, \frac{1}{2}, 1; \tilde{m}_1, \tilde{m}_2, m_{12}\right) = \]
\[ \delta_{m_1\tilde{m}_1}\delta_{m_2\tilde{m}_2} + \frac{i}{2}t^{\frac{1}{2}}_{m_1m_2}(\sigma^2)t^{\frac{1}{2}}_{\tilde{m}_1\tilde{m}_2}(\sigma^2). \] (3.65)

The relations (2.3), (2.5), (3.17), (3.58), (3.63) - (3.65) imply
\[ Q^1_{\nu_1\nu_2}((p, p)^{-1/2}\tilde{p}) = 4(p, p)^{-1}\eta_{\nu_1\nu_1}\eta_{\nu_2\nu_2}p^\nu p^{\nu} - \eta_{\nu_1\nu_2}. \]

This relation and the definitions (3.5), (3.57), (3.59) imply
\[ Q^1_{\nu_1\nu_2}(\tilde{\partial}_x) = 4\frac{\partial}{\partial x^{\nu_1}}\frac{\partial}{\partial x^{\nu_2}} - \eta_{\nu_1\nu_2}(\partial_x, \partial_x). \] (3.66)

The substitution of the relations (3.60), (3.66) into the definitions (3.42), (3.59) yields
\[ A^{(jk)}_{\nu_1\nu_2}(x) = \lim_{\epsilon \to 0} (K_{jk;0} + K_{jk;1})(2\pi)^{-4}\int d^4p \exp\{-i(p, x)\} \times \]
\[ ((p^0 + i\epsilon)^2 - |p|^2)^{-1}\left(\eta_{\nu_1\nu_2} = \frac{4K_{jk;1}}{K_{jk;0} + K_{jk;1}} \frac{p_{\nu_1}p_{\nu_2}}{(p^0 + i\epsilon)^2 - |p|^2}\right). \] (3.67)
Due to ([7], Chapter III, relation (3.58)) the propagation function of the vector particles in the Yang–Mills theory is

\[
D_{\mu \nu}^{ab}(x) = \lim_{\epsilon \to +0} -\delta^{ab}(2\pi)^{-4} \int d^4p \exp\{-i(p, x)\} \times \\
((p, p) + i\epsilon)^{-1} \left( \eta_{\mu \nu} - (1 - \alpha) \frac{p_{\mu}p_{\nu}}{(p, p) + i\epsilon} \right). \tag{3.68}
\]

The number \( \alpha \) is the consequence of the gauge condition. The choices \( \alpha = 1 \) and \( \alpha = 0 \) are called the gauge conditions of Feynman and Landau. The distribution (3.68) differs from the distribution (3.67) in the rule of going around the poles in the integral.

In the quantum electrodynamics ([8], Lecture 24) the equation

\[
\sum_{n_1, n_2 = 1}^{4} \left\{ \prod_{s=1}^{2} \left( \sum_{\nu=0}^{3} (\gamma_{s}^{\nu} (\sigma^0))_{m_0, n_0} \left( i \frac{\partial}{\partial x^{\nu}} \right) - \mu s \delta_{m, n} \right) \right\} \psi_{n_1 n_2, p_1 p_2} (x_1, x_2) + \sum_{\nu_1, \nu_2 = 0}^{3} \eta_{\nu_1 \nu_2} K_0 D_0^\nu (x_1 - x_2) \left( \prod_{s=1}^{2} (\gamma_{s}^\nu (\sigma^0))_{m_0, n_0} \right) \psi_{n_1 n_2, p_1 p_2} (x_1, x_2) = i \prod_{s=1}^{2} (\delta(x_s) \delta_{m, p_s})
\]

\[
D_0^\nu (x) = \lim_{\epsilon \to +0} -2(2\pi)^{-3} \int d^4p \exp\{-i(p, x)\} ((p, p) + i\epsilon)^{-1} \tag{3.69}
\]
is studied. In view of the relation (3.42) the distribution \(- (4\pi)^{-1} D_0^\nu (x)\) differs from the distribution \(e_0(x)\) in the rule of going around the poles in the integral. \( D_0^\nu (x_1 - x_1) = D_0^\nu (x_2 - x_1) \neq e_0(x_1 - x_2) \). Therefore the equation (3.49), (3.59), (3.60), \( K_{jk; 1} = 0 \) without integration with respect to the variable \( x_k \) has the logical contradiction. The right-hand sides of the equations (3.49) and (3.69) are similar if at the initial moment both particles are concentrated at the origin of coordinates.

We have considered up to now that the interaction propagates at the speed of light. Let the interaction propagate at the speed less or equal to the speed of light. Hence the interaction coefficients (3.59) may be changed in the following way

\[
A_{\nu_1 \nu_2}^{(jk)} (x) = \sum_{l_1, l_2 \in 1/2\mathbb{Z}^+} \int d\lambda_1 \cdots d\lambda_{l_2 + 1} K_{jk; l_2} (\lambda_1, \ldots, \lambda_{l_2 + 1}) Q_{\nu_1 \nu_2} (\tilde{\omega} x) e^\gamma \ldots e^{\gamma_{l_2 + 1}} (x). \tag{3.70}
\]
The distributions \( K_{jk; l_2} (\lambda_1, \ldots, \lambda_{l_2 + 1}) \) have the compact supports. In view of the relations (3.18), (3.55) - (3.58) the interaction coefficients (3.70) satisfy the covariance relation (3.48).

Let us consider the equation (3.49), (3.70), \( j = 1, k = 2 \) for the odd integers \( 2l_1 + 2l_2 \), \( 2l_2 + 2l_2 \). Let the functions \((f_2)_{m_2 \hat{m}_2} (x_2)\) be the solutions of the equation (3.36). The heavy second particle moves freely. The functions \((f_2)_{m_2 \hat{m}_2} (x_2)\) are given by the relations (3.44), (3.45). Let the initial function \((f_2)_{m_2 \hat{m}_2} (+0, x_2)\) be such that the integral of the function (3.37) is not equal to zero for some numbers \( m_2, \hat{m}_2 \). In view of the relation (3.44) the equation (3.49), \( j = 1, k = 2 \) implies

\[
\sum_{n_1 = 1}^{4l_1 + 2} \sum_{\hat{n}_1 = 1}^{2l_1 + 1} \left( \sum_{\nu=0}^{3} (\gamma_{l_1, \hat{l}_1}^{\nu} (A))_{m_1 \hat{m}_1, n_1 \hat{n}_1} \left( -i \frac{\partial}{\partial x^{\nu}_1} \right) + (\beta_{l_1, \hat{l}_1} (\nu^2))_{m_1 \hat{m}_1, n_1 \hat{n}_1} \right) (f_1)_{n_1 \hat{n}_1} (x_1) + \\
i \left( \int d^3y_2 (f_2^0)_{m_2 \hat{m}_2} (+0, y_2) \right)^{-1} \sum_{\nu_1, \nu_2 = 0}^{3} \int d^4x_2 A_{\nu_1 \nu_2}^{(12)} (x_1 - x_2) \times \\
\left( \prod_{k=1,2} \left( \sum_{n_k = 1}^{4l_k + 2} \sum_{\hat{n}_k = 1}^{2l_k + 1} (\gamma_{l_k, \hat{l}_k}^{\nu_k} (A))_{m_k \hat{m}_k, n_k \hat{n}_k} (f_k)_{n_k \hat{n}_k} (x_k) \right) \right) = -i \delta(x_1^0) (f_1^0)_{m_2 \hat{m}_2} (+0, x_1), \tag{3.71}
\]
equation (3.71) in the form solutions of the equation (3.71) for the functions (

retarded potential of the second particle. We have inserted into the equation (3.71) the $K$ (3.39) into the equation (3.59), (3.60), (3.71),

The equation (3.71) has the simple physical meaning: the substitution of the first relation

In view of the relations (3.38), (3.43) the substitution of the function (3.73) into the equation

Similar to the solutions (3.44) of the equation (3.36) we look for the solutions of the

The equation (3.71) has the simple physical meaning: the substitution of the first relation

The relations (3.42), (3.70) allow to rewrite the equation (3.74) in the form

\[
(g_1)_{m_1n_1}(x_1) - \lim_{\epsilon \to 0} \sum_{n_s = 1, \ldots, 4l_s + 1} \sum_{s = 1, 2} \sum_{\nu_1, \nu_2 = 0} \int d^4 x_1 d^4 x_3 d^4 x_4
\]

\[
\int d^3 y_2 (f^0_2)_{m_2n_2}(+0, y_2)^{-1} (P^\nu_2)_{m_2n_2}(0, y_2) \delta(x_2) (f^0_1)_{m_1n_1}(+0, x_2).
\]
\[ (\prod_{j=1}^{l_2+1} (p_j^0 - p_j^2 + i\epsilon)^2 - |p_1 - p_2|^2 - \lambda_j^2)^{-1} Q_{\nu_1\nu_2}^{l_2} (-i(\tilde{p}_1 - \tilde{p}_2)) \times \]
\[ (P_{1}^{\nu_1})_{m_1\hat{n}_1,m_1\hat{n}_1} (\eta_0 P_{2}^{0}) (\tilde{g}_1)_{n_1\hat{n}_1} (p_2) \left( \int d^3 y_2 (f_{2}^0)_{m_2\hat{m}_2} (+0, y_2) \right)^{-1} \times \]
\[ (P_{2}^{\nu_2})_{m_2\hat{m}_2,n_2\hat{n}_2} (\eta_0 (p_1^0 - p_2^0) (\tilde{f}_2^0)_{n_2\hat{n}_2} (+0, \cdot) (p_1 - p_2) = -i(2\pi)^{-4} \lim_{\epsilon \to +0} \int d^4 p_1 \exp \{-i(x_1, p_1)\} ((p_0^0 + i\epsilon)^2 - |p_1|^2 - \mu_1^2)^{-1} (f_1^0)_{m_1\hat{m}_1} (+0, \cdot) (p_1), \]

\[ (\hat{g}_1)_{n_1\hat{n}_1} (p_1) = \int d^4 x_1 \exp \{i(x_1, p_1)\} (g_1)_{n_1\hat{n}_1} (x_1). \quad (3.75) \]

Let us consider the series

\[ (g_1)_{m_1^{(s)} m_1^{(s)}} (x_1) = -i(2\pi)^{-4} \lim_{\epsilon \to +0} \int d^4 p_1 \exp \{-i(x_1, p_1)\} \times \]
\[ ((p_1^0 + i\epsilon)^2 - |p_1|^2 - \mu_1^2)^{-1} (f_1^0)_{m_1^{(s)} m_1^{(s)}} (-0, \cdot) (p_1) \]
\[ -i(2\pi)^{-4} \lim_{\epsilon \to +0} \sum_{k=1}^{\infty} \sum_{m_{1k}} \sum_{\mu_{1k}} \sum_{m_{2k}} \sum_{\mu_{2k}} \sum_{s=1}^{2l_2+2} \sum_{\lambda_{1k}} \sum_{\lambda_{2k}} \left( \prod_{j=1}^{l_2+1} \right) (p_{s+1}^0 + i\epsilon)^2 - |p_s - p_{s+1}|^2 - \mu_1^2)^{-1} \times \]
\[ \int d\lambda_1 \cdots d\lambda_{l_2+1} i(2\pi)^{-8} K_{12l_2} (\lambda_1, \ldots, \lambda_{l_2+1}) \times \]
\[ ((p_s^0 + i\epsilon)^2 - |p_s|^2 - \mu_1^2)^{-1} ((p_{s+1}^0 - i\epsilon)^2 - |p_s - p_{s+1}|^2 - \mu_1^2)^{-1} \times \]
\[ \left( \prod_{j=1}^{l_2+1} (p_{s+1}^0 + i\epsilon)^2 - |p_s - p_{s+1}|^2 - \lambda_j^2)^{-1} \right) Q_{\nu_1\nu_2}^{l_2} (-i(\tilde{p}_s - \tilde{p}_{s+1})) \times \]
\[ (P_{1}^{\nu_1})_{m_1^{(s)} m_1^{(s)}} (\eta_0 (p_{s+1}^0 - p_{s+1}^0) (\tilde{f}_1^0)_{n_2\hat{n}_2} (+0, \cdot) (p_{s+1}). \quad (3.76) \]

If all integrals in the right-hand side of the equality (3.76) exist and the series (3.76) is convergent, this series is the solution of the equation (3.75). The function of the variables \( p_{s+1}^0 \) in \( k \) term of the series (3.76) is holomorphic in the domain \( \text{Imp}_{k+1}^0 > \text{Imp}_{k}^0 > \cdots > \text{Imp}_{1}^0 > 0 \). We suppose that the integrals with respect to the variables \( p_{s+1}^0 \) are absolutely convergent and it is possible to make the shifts \( p_{s+1}^0 \to p_{s+1}^0 + i(k + 2 - s) \), \( s = 1, \ldots, k+1, \)

\[ (g_1)_{m_1^{(s)} m_1^{(s)}} (x_1) = -i(2\pi)^{-4} \int d^4 p_1 \exp \{x_1^0 - i(x_1, p_1)\} ((p_1^0 + i\epsilon)^2 - |p_1|^2 - \mu_1^2)^{-1} \times \]
\[ \left( \prod_{j=1}^{l_2+1} \right) (p_{s+1}^0 + i\epsilon)^2 - |p_s - p_{s+1}|^2 - \lambda_j^2)^{-1} \right) Q_{\nu_1\nu_2}^{l_2} (-i(\tilde{p}_s - \tilde{p}_{s+1})) \times \]

\[ (P_{1}^{\nu_1})_{m_1^{(s)} m_1^{(s)}} (\eta_0 (p_{s+1}^0 - p_{s+1}^0) (\tilde{f}_1^0)_{n_2\hat{n}_2} (+0, \cdot) (p_{s+1}). \quad (3.76) \]
The inequalities (3.78), (3.79) imply the inequality

\[ ((p_s^0 + i(k + 2 - s))^2 - |p_s| - \mu_2^{-1})^{-1}((p_s^0 - p_{s+1}^0 + i)^2 - |p_s - p_{s+1}| - \mu_1^{-1}) \times \int d\lambda_1 \cdots d\lambda_{l_{12}+1} i(2\pi)^{-8} K_{12,l_{12}}(\lambda_1, \ldots, \lambda_{l_{12}+1}) \times \]

\[ \left( \prod_{j=1}^{l_{12}+1} ((p_s^0 - p_{s+1}^0 + i)^2 - |p_s - p_{s+1}| - \lambda_j^2)^{-1} \right) Q_{\nu_{1,2}}^{l_{12}}(-i(\bar{p}_s - \bar{p}_{s+1}) + \sigma^0) \times \]

\[ (P_1^{\nu_{1}})_{m_1^{(s)},m_1^{(s+1)}}(\eta_{\nu} p_{s+1}^\nu + i \eta_{\nu_0}(k + 1 - s)) \left( \int d^2 y_2 (f_0^0)_{m_2 n_2}(+0, y_2) \right)^{-1} \times \]

\[ (P_2^{\nu_{2}})_{m_2 n_2}(\eta_{\nu} (p_s^\nu - p_{s+1}^\nu) + i \eta_{\nu_0}(f_0^0)_{n_2 n_2}(+0, \cdot)(p_s - p_{s+1})). \quad (3.77) \]

Let us prove that all integrals in the right-hand side of the equality (3.77) are absolutely convergent if the functions \((f_k^0)_{m_k n_k}(+0, \cdot)(p_k), k = 1, 2\), are rapidly decreasing at infinity. Similarly we can prove that it is possible to make the shifts \(p_s^0 \to p_s^0 + i(k + 2 - s), s = 1, \ldots, k + 1\), in the \(k\) term of the series (3.76).

Let us estimate the quadratic polynomial of the variable \((p^0)^2\)

\[ |(p^0 + im)^2 - |p|^2 - \mu|^2|^2 = ((p^0)^2 - |p|^2 - \mu^2 + m^2)^2 + 4m^2(|p|^2 + \mu^2), m = 1, 2, \ldots. \]

For \(|p|^2 + \mu^2 \geq m^2\)

\[ |(p^0 + im)^2 - |p|^2 - \mu|^2|^2 \geq 4m^2(|p|^2 + \mu^2). \quad (3.78) \]

For \(|p|^2 + \mu^2 \leq m^2\)

\[ |(p^0 + im)^2 - |p|^2 - \mu|^2|^2 \geq (|p|^2 + \mu^2 + m^2)^2. \quad (3.79) \]

The inequalities (3.78), (3.79) imply

\[ |(p^0 + im)^2 - |p|^2 - \mu|^2|^2 \leq m^{-2}, \quad (3.80) \]

\[ |p||p^0 + im|^2 - |p|^2 - \mu|^2|^2 \leq (2m)^{-1}. \quad (3.81) \]

By making use of the equality

\[ |(p^0 + im)^2 - |p|^2 - \mu|^2|^2 = ((p^0)^2 - |p|^2 - \mu^2 - m^2)^2 + 4m^2(p^0)^2 \quad (3.82) \]

we obtain the inequality

\[ |p^0||p^0 + im|^2 - |p|^2 - \mu|^2|^2 \leq (2m)^{-1}. \quad (3.83) \]

The inequality (3.80), the equality (3.82) and the equality

\[ (p^0)^2 = ((p^0)^2 - |p|^2 - \mu^2 - m^2) + (|p|^2 + \mu^2 + m^2) \]

imply the inequality

\[ (1 + (p^0)^2)|p^0 + im|^2 - |p|^2 - \mu|^2|^2 \leq m^{-2}(|p|^2 + \mu^2 + 2m^2 + 1). \quad (3.84) \]
The inequalities (3.80), (3.81), (3.83) imply
\[
|\langle P^\nu \rangle_{m_1^{(s)} m_1^{(1)}} (\eta_{\nu \nu} P^\nu + i \eta_{\nu 0} (k + 1 - s))| \times
|\langle P^\nu \rangle_{m_1^{(s)} m_1^{(1)}} (\eta_{\nu \nu} P^\nu + i \eta_{\nu 0} (k + 1 - s))| - |p_0^0 + i(k + 1 - s)|^2 - |p_0^0 + i(k + 1 - s)|^2 - \mu_0^2|^{-1} \leq (k + 1 - s)^{-1} D^1_{m_1^{(s)} m_1^{(1)}} (\eta_{\nu \nu} P^\nu + i \eta_{\nu 0} (k + 1 - s)) \times
\]
where the positive matrix elements \( D^1_{m_1^{(s)} m_1^{(1)}} \), \( D^2_{m_2^{(s)} m_2^{(1)}} \) do not depend on the vectors \( p_s, s = 1, \ldots, k + 1 \).

The degree of the homogeneous polynomial (3.5) is equal to 2l. Hence the inequalities (3.80), (3.84) imply
\[
|Q^\nu_{\nu_1 \nu_2} (-i(\tilde{p}_s - \tilde{p}_{s+1}) + \sigma^0)| \int d\lambda_1 \cdots d\lambda_{l_{12}+1} K_{12} \times
\]
where the positive matrix elements \( D^3_{\nu_1 \nu_2} \) do not depend on the vectors \( p_s, s = 1, \ldots, k + 1 \).

The inequalities (3.85), (3.86) imply the estimation of the series (3.77)
\[
|\langle g_1 \rangle_{m_1^{(s)} m_1^{(1)}} (x)| \leq (2\pi)^{-1} \int d^4 p_1 \exp\{x_0^0\}(p_0^0 + i)^2 - |p_1|^2 - \mu_0^2|^{-1} \times
\]
where the positive matrix elements \( D^4_{\nu_1 \nu_2} \) are rapidly decreasing at the infinity. The series (3.77) is absolutely convergent. It is the solution of the equation (3.75).

The Cauchy inequality implies
\[
|p_1|^2 \leq \left( \sum_{s=1}^k |p_s - p_{s+1}| + |p_k+1| \right)^2 \leq (k + 1) \left( \sum_{s=1}^k |p_s - p_{s+1}|^2 + |p_k+1|^2 \right). \quad (3.88)
\]

In view of the inequalities (3.84), (3.88) all integrals in the right - hand side of the inequality (3.87) exist if the functions \( (f^0_{\nu_1})_{m_1^{(s)} m_1^{(1)}} (0, \cdot) (p_1) \), \( k = 1, 2 \), are rapidly decreasing at the infinity. The series (3.77) is absolutely convergent. It is the solution of the equation (3.75).

Let us consider the interaction equations for three particles. Let \( \tau \) be a permutation of the numbers 1, 2, 3. Let us construct the interaction equation for \( \tau(1) \) particle
\[
\sum_{n_1=1, \ldots, l_1+2} \sum_{n_2=1, \ldots, l_2+1} \sum_{n_3=1, \ldots, l_3+1} \int d^4 x_{\tau(2)} d^4 x_{\tau(3)}
\]
\[
\left\{ \prod_{s=1}^{3} \left( \sum_{\nu=0}^{3} (\gamma_{t,s}^{\nu}(A))_{m_{s}\bar{m}_{s},n_{s}\bar{n}_{s}} \left(-i \frac{\partial}{\partial n_{s}}\right) + (\beta_{t,i}^{(2)}(\mu_{s}^{2}))_{m_{s}\bar{m}_{s},n_{s}\bar{n}_{s}} \right) (f_{s})_{n_{s}\bar{n}_{s}}(x_{s}) \right\} + \sum_{j,k=2, j \neq k, k_{1}, k_{2}, k_{3}=0}^{3} A_{\nu_{1}\nu_{2}\nu_{3}}^{(1)}(x_{1} - x_{2}) (f_{s})_{n_{s}\bar{n}_{s}}(x_{s}) \times \left( \sum_{\nu=0}^{3} (\gamma_{t,s}^{\nu}(A))_{m_{s}\bar{m}_{s},n_{s}\bar{n}_{s}} \left(-i \frac{\partial}{\partial n_{s}}\right) + (\beta_{t,i}^{(2)}(\mu_{s}^{2}))_{m_{s}\bar{m}_{s},n_{s}\bar{n}_{s}} \right) \times \sum_{s=1}^{3} (f_{s})_{n_{s}\bar{n}_{s}}(x_{s}) = 0 \quad (3.89)
\]

where the potentials \( A_{\nu_{1}\nu_{2}\nu_{3}}^{(1)}(x, x_{2}) \) have the form (3.70) and the potentials \( A_{\nu_{1}\nu_{2}\nu_{3}}^{(1)}(x_{1}, x_{2}) \) satisfy the covariance relation

\[
A_{\nu_{1}\nu_{2}\nu_{3}}^{(1)} \left( \sum_{\mu_{s}=0}^{3} A_{\mu_{s}}^{(A^{-1})} x_{s}^{\mu_{s}}, s = 1, 2 \right) = \sum_{\mu_{1}, \mu_{2}, \mu_{3}=0}^{3} A_{\mu_{1}\mu_{2}\mu_{3}}^{(1)}(x_{1}, x_{2}) \sum_{s=1}^{3} A_{\mu_{s}}^{(1)}(A) \quad (3.90)
\]

for any matrix \( A \in SL(2, \mathbb{C}) \). The equation (3.89) transforms like three equations (3.36).

For any numbers \( m, n \in \mathbb{Z} \) we define the generalized Clebsch - Gordan coefficient

\[
C(l_{1}, \ldots, l_{m+2}; l_{m+3}, \ldots, l_{m+n+4}; j_{1}, \ldots, j_{m+n+1}; m_{1}, \ldots, m_{m+2}; m_{m+3}, \ldots, m_{m+n+4}) = \sum_{k_{s} = -j_{s}, \ldots, -j_{s} + \bar{l}_{s}, \ldots, j_{s} - l_{s}, s = 1, \ldots, m+n+1} \left( \prod_{s=1}^{m+n+4} C(j_{s}, l_{s+2}, j_{s+1}; k_{s}, m_{s+2}, k_{s+1}) \right) \times C(l_{m+n+4}, l_{m+n+3}; j_{m+n+1}; m_{m+n+4}, m_{m+n+3}, k_{m+n+1}) \quad (3.91)
\]

where the half - integers \( l_{1}, \ldots, l_{m+n+4}, j_{1}, \ldots, j_{m+n+1} \in 1/2\mathbb{Z} \) and \( m_{s} = -l_{s}, -l_{s} + 1, \ldots, l_{s} - 1, l_{s}, \) \( i = 1, \ldots, m + n + 4 \). The definition (3.91) and the relations (3.55), (3.56) imply for any matrix \( A \in SL(2, \mathbb{C}) \)

\[
C(l_{1}, \ldots, l_{m+2}; l_{m+3}, \ldots, l_{m+n+4}; j_{1}, \ldots, j_{m+n+1}; m_{1}, \ldots, m_{m+2}; m_{m+3}, \ldots, m_{m+n+4}) = \sum_{n_{s} = -l_{s}, \ldots, -l_{s} + 1, l_{s} - 1, l_{s}, s = 1, \ldots, m+n+1} \left( \prod_{i=1}^{m+n+4} l_{s}^{i} (A) \right) \times C(l_{1}, \ldots, l_{m+2}; l_{m+3}, \ldots, l_{m+n+4}; j_{1}, \ldots, j_{m+n+1}; m_{1}, \ldots, m_{m+2}; n_{m+3}, \ldots, n_{m+n+4}), \quad (3.92)
\]

\[
C(l_{1}, \ldots, l_{m+2}; l_{m+3}, \ldots, l_{m+n+4}; j_{1}, \ldots, j_{m+n+1}; m_{1}, \ldots, m_{m+2}; m_{m+3}, \ldots, m_{m+n+4}) = \sum_{n_{s} = -l_{s}, \ldots, -l_{s} + 1, l_{s} - 1, l_{s}, s = 1, \ldots, m+n+1} \left( \prod_{i=1}^{m+n+4} l_{s}^{i} (A) \right) \times C(l_{1}, \ldots, l_{m+2}; l_{m+3}, \ldots, l_{m+n+4}; j_{1}, \ldots, j_{m+n+1}; n_{1}, \ldots, n_{m+2}; m_{m+3}, \ldots, n_{m+n+4}), \quad (3.93)
\]

\[
C(l_{1}, \ldots, l_{m+2}; l_{m+3}, \ldots, l_{m+n+4}; j_{1}, \ldots, j_{m+n+1}; m_{1}, \ldots, m_{m+2}; n_{m+3}, \ldots, n_{m+n+4}), \quad (3.93)
\]
We define the potential

$$A_{\nu_1\nu_2\nu_3}^{(r(1))}(x_1, x_2) = \sum_{l_{12},l_{13},j_1,j_2 \in 1/2\mathbb{Z}} \sum_{m_{1s},\bar{m}_{1s}} = -l_{1s}, -l_{1s} + 1, \ldots, l_{1s} - 1, l_{1s}, s = 2, 3$$

$$C(\frac{1}{2}, \frac{1}{2}; l_{12}, l_{13}; j_1, j_2; m_1, m_2, m_3; l_{12}, l_{13}) \times$$

$$C(\frac{1}{2}, \frac{1}{2}; l_{12}, l_{13}; j_1, j_2; m_1, m_2, m_3; l_{12}, l_{13}) \left( \prod_{s=1}^{3} t_{m_s \bar{m}_s}^1 (\sigma^2 \sigma^\nu \sigma^2) \right) \times$$

$$\int d\lambda_1 \cdots d\lambda_{l_{12} + \frac{3}{2}} d\kappa_1 \cdots d\kappa_{l_{13} + \frac{3}{2}} K_{\nu_1 \nu_2 \nu_3}^{(r)}(\lambda_1, \ldots, \lambda_{l_{12} + \frac{3}{2}}; \kappa_1, \ldots, \kappa_{l_{13} + \frac{3}{2}}) \times$$

$$t_{m_1 \bar{m}_1}^{l_{12}} (\tilde{\partial}_{x_1}) e_{\lambda_1}^2, \ldots, \lambda_{l_{12}}^2 (x_1) t_{m_3 \bar{m}_3}^{l_{13}} (\tilde{\partial}_{x_2}) e_{\kappa_1}^2, \ldots, \kappa_{l_{13}}^2 (x_2)$$

(3.94)

where \([a]\) is the integral part of a real number \(a\). In view of the relations (3.18), (3.42), (3.57), (3.58), (3.92), (3.93) the potentials (3.94) satisfy the covariance relation (3.90). We suppose that the distributions \(K_{\nu_1 \nu_2 \nu_3}^{(r)}(\lambda_1, \ldots, \lambda_{l_{12} + \frac{3}{2}}; \kappa_1, \ldots, \kappa_{l_{13} + \frac{3}{2}})\) have the compact supports and are not zero for the finite number of the values \(l_{12}, l_{13}\) only. The definition (3.91) and the triangle condition imply that the generalized Clebsch - Gordan coefficients are not zero for the finite number of the values of half - integers \(j_1, j_2\) only.

Let \(\tau(1)\) and \(\tau(2)\) particles interact by means of the zero mass particles

$$K_{\nu_1 \nu_2 \nu_3}^{(r)}(\lambda_1, \ldots, \lambda_{l_{12} + \frac{3}{2}}; \kappa_1, \ldots, \kappa_{l_{13} + \frac{3}{2}}) =$$

$$K_{\nu_1 \nu_2 \nu_3}^{(r)}(\lambda_1, \ldots, \lambda_{l_{12} + \frac{3}{2}}; \kappa_1, \ldots, \kappa_{l_{13} + \frac{3}{2}}) \prod_{s=1}^{[l_{12} + \frac{3}{2}]} \delta(\lambda_s).$$

(3.95)

The degree of the homogeneous polynomial (3.5) is equal to \(2l\). In view of the relations (3.39), (3.41) the distribution

$$t_{m_1 \bar{m}_1}^{l_{12}} (\tilde{\partial}_{x}) e_{0, \ldots, 0} (x)$$

(3.96)

with \([l_{12} + \frac{3}{2}]\) zeros has a support in the boundary of the upper light cone. For \(l_{12} = 0\) the integer \([l_{12} + \frac{3}{2}] = 1\) and the distribution (3.96) coincides with the first distribution (3.39). If the integer \(2l_{12} > 1\) is even, then the integer \([2l_{12} + \frac{3}{2}] - 4 = 2l_{12} - 3\). If the integer \(2l_{12}\) is odd, then the integer \(2[l_{12} + \frac{3}{2}] - 4 = 2l_{12} - 1\). In view of the relations (3.39), (3.41) the distribution (3.96) with \([l_{12} + \frac{3}{2}] + 1\) zeros has a support in all closed upper light cone. If the half - integer \(l_{12} = 0\), then the integer \([l_{12} + \frac{3}{2}] + 1 = 2\) and the second distribution (3.39) has the support in all closed upper light cone. If the integer \(2l_{12} > 1\) is even, then the integer \(2[l_{12} + \frac{3}{2}] - 2 = 2l_{12}\). If the integer \(2l_{12}\) is odd, then the integer \(2[l_{12} + \frac{3}{2}] - 2 = 2l_{12} + 1\).

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