A Study of Diffusion Processes Generated by Kohn-Laplacian Type Operators*

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Abstract

As a generalization to the heat semigroup on the Heisenberg group, the diffusion semigroup generated by the subelliptic operator

$$L := \frac{1}{2} \sum_{i=1}^{m} X_i^2$$

on $\mathbb{R}^{m+d} := \mathbb{R}^m \times \mathbb{R}^d$ is investigated, where

$$X_i(x, y) = \sum_{k=1}^{m} \sigma_{ki} \partial_{x_k} + \sum_{l=1}^{d} (A_l(x))_i \partial_{y_l}, \quad (x, y) \in \mathbb{R}^{m+d}, 1 \leq i \leq m$$

for $\sigma$ an invertible $m \times m$-matrix and $\{A_l\}_{1 \leq l \leq d}$ some $m \times m$-matrices such that the Hörmander condition holds. We first establish Bismut-type and Driver-type derivative formulae with applications on gradient estimates and the coupling/Liouville properties, which are new even for the heat semigroup on the Heisenberg group; then prove under the present framework some recent results derived for the heat semigroup on the Heisenberg group.

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1 Introduction

In recent years, the heat semigroup generated by the Kohn-Laplacian on the Heisenberg group regularity has been intensively investigated, see \cite{2, 12, 14} for derivative estimates and

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applications, and see [3, 5] (where a more general model was considered) for the generalized curvature conditions and applications.

The first purpose of this paper is to establish Bismut’s derivative formula [6] and Driver’s integration by parts formula [10] for the semigroup generalized by a class of Kohn-Laplacian type operators. These two formulae are crucial for stochastic analysis of diffusion processes and are not explicitly known even for the heat semigroup on the Heisenberg group. Our second aim is to extend some known results derived recently for the heat semigroup on the Heisenberg group to a more general framework of Kohn-Laplacian type operators. These results include the generalized curvature-dimension condition and applications studied in [3, 4, 5], and explicit Poincaré and reverse Poincaré inequalities for the semigroup derived in [2].

Let us first recall the Kohn-Laplacian on the three-dimensional Heisenberg group. Consider the following two vector fields on \( \mathbb{R}^3 \):

\[
X_1(x) = \partial_{x_1} - \frac{x_2}{2} \partial_{x_3}, \quad X_2(x) = \partial_{x_2} + \frac{x_1}{2} \partial_{x_3}, \quad x = (x_1, x_2, x_3) \in \mathbb{R}^3.
\]

Then \( \Delta_K := X_1^2 + X_2^2 \) is called the Kohn-Laplacian. It is crucial in the study of this operator that \([X_1, X_2] = \partial_{x_3}, [X_i, \partial_{x_3}] = 0 (i = 1, 2) \) and \( X_1, X_2 \) are left-invariant under the group action

\[
(x_1, x_2, x_3) \cdot (x_1', x_2', x_3') = (x_1 + x_1', x_2 + x_2', x_3 + x_3' + \frac{1}{2}(x_1 x_2' - x_2 x_1')).
\]

To do stochastic analysis with this operator, let us introduce the associated stochastic differential equation for \((X(t), Y(t)) \in \mathbb{R}^2 \times \mathbb{R} \):

\[
\begin{align*}
dX(t) &= dB(t), \\
dY_l(t) &= \langle AX(t), dB(t) \rangle,
\end{align*}
\]

where \( B(t) \) is the 2-dimensional Brownian motion and \( A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \). Then \((X(t), Y(t)) \) is the diffusion process generated by \( \frac{1}{2} \Delta_K \), and the associated transition semigroup is known as the heat semigroup on the Heisenberg group.

In this paper we consider the following natural extension of this equation for \((X(t), Y(t)) \in \mathbb{R}^m \times \mathbb{R}^d =: \mathbb{R}^{m+d} (m \geq 2, d \geq 1) \):

\[
\begin{align*}
dX(t) &= \sigma dB(t), \\
dY_l(t) &= \langle A_l X(t), dB(t) \rangle, \quad 1 \leq l \leq d,
\end{align*}
\]

where \( B(t) \) is the \( m \)-dimensional Brownian motion on a complete probability space \((\Omega, \mathcal{F}, \mathbb{P})\) with the natural filtration \( \{\mathcal{F}_t\}_{t \geq 0} \), \( \sigma \) is an invertible \( m \times m \)-matrix, and \( (A_l)_{1 \leq l \leq d} \) are \( m \times m \)-matrices. Let

\[
X_i(x, y) = \sum_{k=1}^m \sigma_{ki} \partial_{x_k} + \sum_{l=1}^d (A_l x)_l \partial_{y_l}, \quad (x, y) = (x_1, \ldots, x_m, y_1, \ldots, y_d) \in \mathbb{R}^{m+d}, 1 \leq i \leq m.
\]
Then the solution to (1.1) is the diffusion process generated by

\[ L := \frac{1}{2} \sum_{i=1}^{m} X_i^2. \]

Obviously, for any \( 1 \leq i, j \leq m \) and \( 1 \leq l \leq d \), we have \([X_i, \partial_{y_l}] = 0\) and

\[ [X_i, X_j] = \sum_{l=1}^{d} \{(A_l \sigma)_{ji} - (A_l \sigma)_{ij}\} \partial_{y_l} = \sum_{l=1}^{d} (G_l)_{ji} \partial_{y_l}, \]

where \( G_l := A_l \sigma - \sigma^* A_l^* \). Then the Hörmander condition holds (thus, \( L \) is subelliptic) if and only if

(1.2)  The \( \{m(m - 1)\} \times d\)-matrix \( (M_{(i,j),l})_{1 \leq i,j \leq m; 1 \leq l \leq d} \) has rank \( d \),

where

\[ M_{(i,j),l} := (G_l)_{ij}, \quad 1 \leq i < j \leq m, 1 \leq l \leq d; \]

or equivalently,

(1.3) \[ \sum_{i,j=1}^{m} \sum_{l=1}^{d} (G_l)_{ij} a_l^2 \geq \lambda |a|^2, \quad a = (a_l)_{1 \leq l \leq d} \in \mathbb{R}^d \]

holds for some constant \( \lambda > 0 \).

A simple example for (1.2) or (1.3) to hold is that \( d = m - 1, \sigma = I_{m \times m} \) and

\[ (A_l)_{ij} = \begin{cases} \alpha_l, & \text{if } i = 1, j = l + 1, \\ \beta_l, & \text{if } i = l + 1, j = 1, \\ 0, & \text{otherwise} \end{cases} \]

for \( \alpha_l \neq \beta_l, 1 \leq l \leq d \).

Moreover, let \( \mathbb{R}^{m+d} \) be equipped with the group action

\[ (x, y) \bullet (x', y') = (x + x', y + y' + \langle \sigma^{-1} A x, x' \rangle), \quad (x, y), (x', y') \in \mathbb{R}^{m+d}, \]

where \( \langle \sigma^{-1} A x, x' \rangle := \langle (\sigma^{-1} A_l x, x') \rangle_{1 \leq l \leq d} \in \mathbb{R}^d \). Then \((0, 0)\) is the unique unit element, and the inverse element of \((x, y) \in \mathbb{R}^{m+d}\) is

\[ (x, y)^{-1} := (-x, \langle \sigma^{-1} A x, x \rangle - y). \]

It is easy to see that \( \{X_i\}_{1 \leq i \leq m} \) are left-invariant vector fields under the group structure. Indeed, for any \( f \in C^1(\mathbb{R}^{m+d}) \) and \((u, v) \in \mathbb{R}^{m+d}\), letting \( f_{(u,v)}(z) = f((u, v) \bullet z), z \in \mathbb{R}^{m+d}\), we have

\[ X_i f_{(u,v)}(0, 0) = \sum_{k=1}^{m} \sigma_{ki} \left\{ \partial_{x_k} f_{(u,v)} \right\}(0, 0) = \sum_{k=1}^{m} \sigma_{ki} \left\{ \partial_{x_k} f + \sum_{l=1}^{d} ( (\sigma^*)^{-1} A_l x_l ) \partial_{y_l} f \right\}(u, v) \]

\[ = \left\{ \sum_{k=1}^{m} \sigma_{ki} \partial_{x_k} f + \sum_{l=1}^{d} (A_l x_l \partial_{y_l} f) \right\}(u, v) = (X_i f)(u, v), \quad 1 \leq i \leq m. \]
It is also easy to see that the Lebesgue measure $\mu$ is invariant under the group action.

We will investigate the Markov semigroup $(P_t)_{t \geq 0}$ for the solution to equation (1.1):

$$P_t f(x, y) := \mathbb{E} f(X^x(t), Y^{(x,y)}), \quad (x, y) \in \mathbb{R}^{m+d}, \quad t \geq 0, \quad f \in L^1_{\mathbb{R}}(\mathbb{R}^{m+d}),$$

where $(X^x(t), Y^{(x,y)}(t))_{t \geq 0}$ is the solution to the equation with initial data $(x, y)$. Since $\text{div} X_i = 0, 1 \leq i \leq m$, $P_t$ is symmetric in $L^2(\mu)$.

In Section 2 we investigate Bismut/Driver-type derivative formulae for $P_t$ and applications. In Section 3 we verify Baudoin-Garafalo’s generalized curvature-dimension condition for the operator $L$ so that results derived in [3, 5] apply. Finally, in Section 4 and Section 5 we modify the argument in [2] to derive explicit Poincaré and reverse Poincaré inequalities for $P_t$.

2 Derivative formulae

Recall that $G_l := A_l \sigma - \sigma^* A_l^* (1 \leq l \leq d)$ are skew-symmetric, i.e. $G_l^* = -G_l$. In this section we assume

(A1) $G_l \neq 0$ for all $1 \leq l \leq d$, and there exists a constant $\theta \in [0, 1)$ such that

$$\theta \sum_{l=1}^{d} |G_l u|^2 \geq \sum_{1 \leq l \neq k \leq d} |\langle G_l^* G_k u, u \rangle|, \quad u \in \mathbb{R}^m, 1 \leq l \leq d.$$ 

It is easy to see that (A1) implies the Hörmander condition. Indeed, for any $a = (a_l)_{1 \leq l \leq d} \in \mathbb{R}^d$, (A1) implies that

$$\sum_{1 \leq i, j \leq m} \sum_{l=1}^{d} M_{(i,j),l} a_l \geq \sum_{k,l=1}^{d} \sum_{i,j=1}^{m} (G_l)_{ij} (G_k)_{ij} a_l a_k$$

$$= \sum_{k,l=1}^{d} \text{Tr}(G_l^* G_k) a_k a_l \geq (1 - \theta) \sum_{l=1}^{d} \text{Tr}(G_l^* G_l) a_l^2,$$

so that (1.3) holds for $\lambda := (1 - \theta) \inf_{1 \leq l \leq d} \|G_l\|_{HS}^2 > 0$. A simple example such that (A1) holds is that $\sigma = I_{d \times d}, d = m - 1$ and $A_l$ given in (1.4) with $\alpha_l \neq \beta_l, 1 \leq l \leq d$. In this case we have $G_l^* G_k = 0$ for $l \neq k$, so that (A1) holds for $\theta = 0$.

The main tool in the study is the integration by parts formula of the Malliavin gradient. For fixed $T > 0$, let $(D, \mathcal{G}(D))$ be the Malliavin gradient operator for the Brownian motion $\{B(t)\}_{t \in [0,T]}$, and let $(D^*, \mathcal{G}(D^*))$ be the adjoint operator. For any $F \in \mathcal{G}(D)$, the Malliavin gradient $DF$ is an element in $L^2(\Omega \to \mathbb{H}; \mathbb{P})$, where

$$\mathbb{H} := \{ \beta \in C([0, T]; \mathbb{R}^d) : \int_0^T |b'(t)|^2 dt < \infty \}$$
Theorem 2.1. Assume \( \mathcal{A}_1 \) and let \( T > 0 \) and \( (u, v) \in \mathbb{R}^{m+d} \) be fixed. Let \( h \) and \( \tilde{h} \) be such that \( h(0) = \tilde{h}(0) = 0 \) and

\[
\begin{align*}
   h'(t) &= \frac{\sigma^{-1}u}{T} + \sum_{k=1}^{d} (Q_{T}^{-1} \alpha_{T,u,v})_k G_k \left( B(t) - \frac{1}{T} \int_{0}^{T} B(s)ds \right), \\
   \tilde{h}'(t) &= \frac{\sigma^{-1}u}{T} + \sum_{k=1}^{d} (Q_{T}^{-1} \tilde{\alpha}_{T,u,v})_k G_k \left( B(t) - \frac{1}{T} \int_{0}^{T} B(s)ds \right), \quad t \in [0, T].
\end{align*}
\]

Then:

1. \( h, \tilde{h} \in \mathscr{D}(\mathcal{D}^{*}) \) and for any \( p > 1 \) there exists a constant \( c_p > 0 \) independent of \( (u, v) \in \mathbb{R}^{m+d} \) and \( T > 0 \) such that

\[
\mathbb{E}[D^*h]^p + \mathbb{E}[D^*\tilde{h}]^p \leq \frac{c_p}{T^p} \{ |v|^p + |u|^p (\|X(0)\|^p + T^\frac{p}{2}) \}.
\]

2. For any \( f \in C^1_b(\mathbb{R}^{m+d}) \), \( P_T(\nabla_{(u,v)} f) = \mathbb{E}[f(X(T), Y(T))D^*h] \).

3. For any \( f \in C^1_b(\mathbb{R}^{m+d}) \), \( \nabla_{(u,v)}P_Tf = \mathbb{E}[f(X(T), Y(T))D^*\tilde{h}] \).

To prove Theorem 2.1(1), we need the following lemmas.

Lemma 2.2. Assume \( \mathcal{A}_1 \). Then \( Q_T \) is invertible and for any \( p > 1 \) there exists a constant \( C_p > 0 \) independent of \( T > 0 \) such that

\[
\mathbb{E}\|Q_T^{-1}\|^p \leq \frac{C_p}{T^{2p}}, \quad T > 0,
\]

where \( \| \cdot \| \) is the operator norm.
Proof. Let $\bar{Q}_T = \text{diag} Q_T$; that is, $\bar{Q}_T = (q_{kl}(T)1_{\{l\}}(k))_{1 \leq k, l \leq d}$. By (A1), $\bar{Q}_T$ is invertible and $Q_T \geq (1 - \theta) \bar{Q}_T$. Therefore, it suffices to show that

\begin{equation}
\mathbb{E} q_{ll}(T)^{-p} \leq \frac{C_p}{T^{2p}}, \quad T > 0, 1 \leq l \leq d
\end{equation}

holds for some constant $C_p > 0$ independent of $T > 0$. Let $e_l \in \mathbb{R}^d$ with $|e_l| = 1$ such that $|G_l e_l| = \|G_l\| > 0$. Then

\begin{align*}
q_{ll}(T) &= \int_0^T \left| G_l \left( B(t) - \frac{1}{T} B(s) \right) \right|^2 dt \\
&\geq \int_0^T \left\langle G_l \left( B(t) - \frac{1}{T} B(s) \right), e_l \right\rangle^2 dt \\
&= \|G_l\|^2 \int_0^T \left| b_l(t) - \frac{1}{T} \int_0^T b_l(s) ds \right|^2 dt,
\end{align*}

where

\begin{align*}
b_l(t) := \left\langle B(t), \frac{G_l^* e_l}{\|G_l^* e_l\|} \right\rangle, \quad t \geq 0
\end{align*}

is an one-dimensional Brownian motion. Therefore,

\begin{align*}
\mathbb{E} q_{ll}(T)^{-p} \leq \frac{1}{\|G_l\|^{2p}} \mathbb{E} \left[ \frac{1}{(\int_0^T |b_l(t) - \frac{1}{T} \int_0^T b_l(s) ds|^2 dt)^p} \right].
\end{align*}

Combining this with

\begin{align*}
\int_0^T \left| b_l(t) - \frac{1}{T} \int_0^T b_l(s) ds \right|^2 dt &= \frac{1}{T} \int_{[0,T]^2} |b_l(t) - b_l(s)|^2 dt ds \\
&\geq \frac{1}{T} \int_0^{\frac{T}{2}} ds \int_{\frac{T}{2}}^T |b(t) - b(s)|^2 dt,
\end{align*}

and using the Jensen inequality, we obtain

\begin{align*}
\mathbb{E} q_{ll}(T)^{-p} \leq \frac{3^p}{\|G_l\|^{2p}} \mathbb{E} \left[ \frac{1}{(\int_0^T |b(t) - b(s)|^2 dt)^p} \right].
\end{align*}

According to [19, Lemma 3.3], this implies (2.2) for some constant $C_p$ independent of $T > 0$, and we thus finish the proof.
Lemma 2.3. Assume (A1). Then $Q_T^{-1} \alpha_{T,u,v}, Q_T^{-1} \tilde{\alpha}_{T,u,v} \in \mathcal{D}(D)^{\otimes d}$, and there exists a constant $c > 0$ independent of $T > 0$ such that for any adapted random variable $\beta$ on the Cameron-Martin space $\mathbb{H}$,

$$|D_\beta Q_T^{-1} \alpha_{T,u,v}| + |D_\beta Q_T^{-1} \tilde{\alpha}_{T,u,v}| \leq cT \|Q_T^{-1}\|_\infty \|B\|_\infty \{ |v| + |u|(|X(0)| + \|B\|_\infty) \} + c\|Q_T^{-1}\| \cdot |u| \cdot \|\beta\|_\infty,$$

where $\| \cdot \|_\infty$ is the uniform norm on $C([0,T]; \mathbb{R}^d)$.

Proof. We only prove the desired upper bound for $\|D_\beta Q_T^{-1} \alpha_{T,u,v}\|$, since that for the other term is completely similar. It is easy to see that

$$|\alpha_{T,u,v}| \leq c_1 \{ |v| + |u|(|X(0)| + \|B\|_\infty) \}$$

holds for some constant $c_1 > 0$. Moreover, since $D_\beta B(t) = \beta(t), t \in [0,T]$, it follows from the definitions of $g_{kl}(T)$ and $\alpha_{T,u,v}$ that each components of $Q_T$ and $\alpha_{T,u,v}$ are in $\mathcal{D}(D)$ with

$$|D_\beta g_{kl}(T)| \leq c_2 \|\beta\|_\infty T \|B\|_\infty, \quad |D_\beta \alpha_{T,u,v}| \leq c_2 |u| \cdot \|\beta\|_\infty$$

holding for some constant $c_2 > 0$ and all $1 \leq k, l \leq d$. Combining these with the fact that

$$D_\beta Q_T^{-1} \alpha_{T,u,v} = -Q_T^{-1} \{D_\beta Q_T\} Q_T^{-1} \alpha_{T,u,v} + Q_T^{-1} D_\beta \alpha_{T,u,v},$$

we derive the desired upper bound estimate of $\|D_\beta Q_T^{-1} \alpha_{T,u,v}\|$. \qed

Proof of Theorem 2.1. (1) We only prove for $h$ as that for $\tilde{h}$ is similar. Let $\{e_i\}_{1 \leq i \leq m}$ be the canonical ONB of $\mathbb{R}^m$. Then $a_i := \langle a, e_i \rangle$ is the $i$-th coordinate of $a \in \mathbb{R}^m$. Let

$$h_0(t) = \frac{t}{T} \sigma^{-1} u, \quad h_i(t) = te_i, \quad \beta_k(t) = \int_0^t G_k B(s) ds, \quad 1 \leq i \leq m, 1 \leq k \leq d.$$  

We have

$$h(t) = h_0(t) + \sum_{k=1}^d \left( Q_T^{-1} \alpha_{T,u,v} \right)_k \beta_k(t) - \sum_{i=1}^m \left( \sum_{k=1}^d \frac{Q_T^{-1} \alpha_{T,u,v}}{T} \int_0^T (G_k B(t)) dt \right) h_i(t)$$

and

$$D^* h_0 = \frac{1}{T} \int_0^T \langle \sigma^{-1} u, dB(t) \rangle = \frac{1}{T} \langle \sigma^{-1} u, B(T) \rangle, \quad D^* h_i = B_i(T),$$

$$D^* \beta_k = \int_0^T \langle G_k B(t), dB(t) \rangle, \quad D h_i B(t) = h_i(t) = te_i, \quad 1 \leq i \leq m, 1 \leq k \leq d.$$  

Combining these with Lemma 2.3 and the fundamental identity

$$D^*(F\beta) = FD^* \beta - D_\beta F$$

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for \( F \in \mathcal{D}(D), \beta \in \mathcal{D}(D^*) \) such that \( FD^* \beta - D\beta F \in L^2(\mathbb{P}) \), we conclude that \( h \in \mathcal{D}(D^*) \) and

\[
D^* h = \frac{1}{T} \langle \sigma^{-1} u, B(T) \rangle + \sum_{k=1}^{d} (Q_{-1}^{-1} \alpha_{T,u,v})_k \int_0^T \langle G_k B(t), dB(t) \rangle \\
- \sum_{k=1}^{d} D_{\beta_k} (Q_{-1}^{-1} \alpha_{T,u,v})_k - \sum_{k=1}^{d} \sum_{i=1}^{m} \left( \frac{(Q_{-1}^{-1} \alpha_{T,u,v})_k}{T} \int_0^T (G_k B(t))_i dt \right) B_i(T) \\
+ \sum_{i=1}^{m} \sum_{k=1}^{d} \frac{D_{\beta_k} (Q_{-1}^{-1} \alpha_{T,u,v})_k}{T} \int_0^T (G_k B(t))_i dt + \sum_{i=1}^{m} \sum_{k=1}^{d} \frac{T}{2} (Q_{-1}^{-1} \alpha_{T,u,v})_k (G_k)_{ii}.
\]

Therefore, it is easy to see from Lemma 2.3, (2.3) and \((G_k)_{ii} = 0\) that

\[
|D^* h| \leq \frac{C|u| \cdot \|B\|_\infty}{T} + C\|Q_{-1}^{-1}\| \left\{ |v| + |u|(|X(0)| + \|B\|_\infty) \right\} \sum_{l=1}^{d} \left| \int_0^T \langle G_k B(t), dB(t) \rangle \right| \\
+ CT^2 \|Q_{-1}^{-1}\|^2 \cdot \|B\|_\infty \left\{ |v| + |u|(|X(0)| + \|B\|_\infty) \right\} + CT|u| \cdot \|Q_{-1}^{-1}\| \cdot \|B\|_\infty \\
+ C\|Q_{-1}^{-1}\| \cdot \|B\|_\infty \left\{ |v| + |u|(|X(0)| + \|B\|_\infty) \right\} \\
\leq C|u| \cdot \|B\|_\infty \left( \frac{1}{T} + T \|Q_{-1}^{-1}\| \right) + C\|Q_{-1}^{-1}\| \left\{ |v| + |u|(|X(0)| + \|B\|_\infty) \right\} \\
\times \left( \sum_{l=1}^{d} \left| \int_0^T \langle G_k B(t), dB(t) \rangle \right| + T^2 \|Q_{-1}^{-1}\| \cdot \|B\|_\infty + \|B\|_\infty^2 \right)
\]

holds for some constant \( C > 0 \). Combining this with Lemma 2.2 and the fact that for any \( p > 1 \)

\[
\mathbb{E} \|B\|_\infty^p + \sum_{k=1}^{d} \mathbb{E} \left| \int_0^T \langle G_k B(t), dB(t) \rangle \right|^p \leq c(p) T^p, \quad T > 0
\]

holds for some constant \( c(p) > 0 \), we obtain the desired upper bound of \( \mathbb{E} |D^* h|^p \).

(2) For \( \beta(s) = \sum_{i=1}^{n} \xi_i \beta_i(s) \), where \( \xi_i \) are real-valued random variables and \( \beta_i(s) \) are square-integrable adapted processes on \( \mathbb{R}^d \), define

\[
\int_0^t \langle \beta(s), dB(s) \rangle = \sum_{i=1}^{n} \xi_i \int_0^t \langle \beta_i(s), dB(s) \rangle.
\]

Then it is easy to see from (1.1) that

\[
\begin{cases}
D_h X(t) = \sigma h(t), \quad D_h X(0) = 0, \\
D_h Y_l(t) = \int_0^t (A_l \sigma h(s), dB(s)) + \int_0^t \langle A_l X(s), h'(s) \rangle ds, \quad D_h Y_l(0) = 0, \quad 1 \leq l \leq d.
\end{cases}
\]

In particular,

\[
(2.4) \quad D_h X(T) = \sigma h(T) = u.
\]
Noting that $X(s) = X(0) + \sigma B(s)$ and $h(T) = \sigma^{-1}u$, we obtain

$$D_h Y(t) = \langle A_l \sigma h(T), B(T) \rangle - \int_0^T \langle A_l \sigma h'(t), B(t) \rangle dt + \int_0^T \langle \sigma^* A_l^* h'(t), \sigma^{-1}(X(0) + \sigma B(t)) \rangle dt$$

$$= \langle A_l u, B(T) \rangle + \langle A_l^* \sigma^{-1} u, X(0) \rangle + \int_0^T \langle G_l^* h'(t), B(t) \rangle dt. $$

Combining this with the definition of $h'(t)$ and letting

$$\hat{B}(t) = B(t) - \frac{1}{T} \int_0^T B(s) ds, \quad t \in [0, T],$$

we arrive at

$$D_h Y(t) = \langle A_l u, B(T) \rangle + \langle A_l^* \sigma^{-1} u, X(0) \rangle$$

$$+ \int_0^T \left \langle \sum_{k=1}^d (Q_T^{-1} \alpha_{T,u,v})_k G_l^* G_k \hat{B}(t) + \frac{G_l^* \sigma^{-1} u}{T}, B(t) \right \rangle dt$$

$$= \langle A_l u, B(T) \rangle + \langle A_l^* \sigma^{-1} u, X(0) \rangle + \frac{1}{T} \int_0^T \langle G_l^* \sigma^{-1} u, B(t) \rangle dt$$

$$+ \sum_{k=1}^d (Q_T^{-1} \alpha_{T,u,v})_k \int_0^T \langle G_l^* G_k \hat{B}(t), \hat{B}(t) \rangle dt$$

$$= \langle A_l u, B(T) \rangle + \langle A_l^* \sigma^{-1} u, X(0) \rangle + \frac{1}{T} \int_0^T \langle G_l^* \sigma^{-1} u, B(t) \rangle dt + (\alpha_{T,u,v})_t \hat{v}_t.$$

By (2.4) and (2.5) we obtain

$$D_h (X(T), Y(T)) = (u, v).$$

Therefore, it follows from (2.1) that

$$P_T (\nabla_{(u,v)} f) = \mathbb{E} \langle \nabla f(X(T), Y(T)), (u, v) \rangle = \mathbb{E} \langle \nabla f(X(T), Y(T)), D_h (X(T), Y(T)) \rangle$$

$$= \mathbb{E} \left \langle f(X(T), Y(T)) \right \rangle = \mathbb{E} \left \{ f(X(T), Y(T)) D^* h \right \}.$$

(3) Similarly to (2), we have

$$\begin{cases}
D_h X(t) = \sigma \tilde{h}(t), \\
D_h Y(t) = \int_0^t \langle A_l \sigma \tilde{h}(s), dB(s) \rangle + \int_0^t \langle A_l X(s), \tilde{h}'(s) \rangle ds.
\end{cases}$$

In particular,

$$D_h X(T) = \sigma \tilde{h}(T) = u.$$  

Noting that $X(s) = X(0) + \sigma B(s)$ and $\tilde{h}(T) = \sigma^{-1}u$, as in (2) we obtain

$$D_h Y(T) = \langle A_l u, B(T) \rangle + \langle A_l^* \sigma^{-1} u, X(0) \rangle + \int_0^T \langle G_l^* \tilde{h}'(t), B(t) \rangle dt.$$
Combining this with the definition of $\tilde{h}'(t)$ we arrive at
\begin{equation}
D_h Y_l(T) = \langle A_l u, B(T) \rangle + \langle A_l^{*} \sigma^{-1} u, X(0) \rangle + \frac{1}{T} \int_{0}^{T} (G_{l}^{*} \sigma^{-1} u, B(t)) dt + (\tilde{\alpha}_{l,u,v})_{l} = v_{l} + \langle A_l u, B(T) \rangle.
\end{equation}

Moreover, it is easy to see that
\begin{equation}
\begin{cases}
d\nabla_{(u,v)} X(t) = 0, & \nabla_{(u,v)} X(0) = u, \\
d\nabla_{(u,v)} Y_l(t) = \langle A_l \nabla_{(u,v)} X(t), dB(t) \rangle, & \nabla_{(u,v)} Y_l(0) = v_l, \ 1 \leq l \leq d.
\end{cases}
\end{equation}

Then
\begin{equation}
\nabla_{(u,v)} X(T) = u, \ \nabla_{(u,v)} Y_l(T) = v_l + \int_{0}^{T} \langle A_l u, dB(t) \rangle = v_l + \langle A_l u, B(T) \rangle, \ 1 \leq l \leq d.
\end{equation}

Combining this with (2.6) and (2.7) we obtain
\[ D_h(X(T), Y(T)) = (\nabla_{(u,v)} X(T), \nabla_{(u,v)} Y(T)). \]

Therefore, it follows from (2.1) that
\[ \nabla_{(u,v)} P_T f = \mathbb{E}\langle \nabla f(X(T), Y(T)), \nabla_{(u,v)} Y(T) \rangle = \mathbb{E}\langle \nabla f(X(T), Y(T)), D_h(X(T), Y(T)) \rangle = \mathbb{E}\{ f(X(T), Y(T)) D^{*} \tilde{h} \}. \]

□

As consequence of Theorem 2.1 we have the following estimate (2.10) of $\Gamma(P_t f)$, where
\[ \Gamma(f) := \frac{1}{2} \sum_{i=1}^{m} \langle X_{i} f \rangle^{2} \]
is the energy form associated to $L$. This estimate will imply the coupling property of the diffusion process as well as the Liouville property for the time-space harmonic functions. Recall that the $L$-diffusion process has the coupling property if for any initial points $z, z' \in \mathbb{R}^{m+d}$ one may construct two processes $Z_t, Z'_t$ generated by $L$ starting at $z, z'$ respectively, such that the coupling time $\tau := \inf \{ t \leq 0 : Z_t = Z'_t \} < \infty$. In this case $(Z_t, Z'_t)$ is called a successful coupling of the process. Moreover, a bounded function $u$ on $[0, \infty) \times \mathbb{R}^{m+d}$ is called time-space harmonic associated to $P_t$, if $P_s u(t, \cdot) = u(t-s, \cdot)$ holds for any $t \geq s \geq 0$. In particular, a bounded harmonic function is a time-space harmonic function.

Let $\rho$ be the distance induced by $\Gamma$, i.e.
\begin{equation}
\rho(z, z') = \sup \{|f(z) - f(z')| : f \in C^{1}(\mathbb{R}^{m+d}), \Gamma(f) \leq 1 \}.
\end{equation}

**Corollary 2.4.** For any $p > 1$ there exists a constant $c_p > 0$ such that
\begin{equation}
\sqrt[1/p]{\Gamma(P_t f)} \leq \frac{c_p}{\sqrt{t}} (P_t |f|^p)^{1/p}, \ \ t > 0, f \in \mathcal{B}_p(\mathbb{R}^{m+d}).
\end{equation}

Consequently:
Let $P_t(z, \cdot)$ be the transition probability kernel of $P_t$, and let $\| \cdot \|_{\text{var}}$ be the totally variational norm. There exists a constant $c > 0$ such that

$$
\| P_t(z, \cdot) - P_t(z', \cdot) \|_{\text{var}} \leq \frac{c \rho(z, z')}{\sqrt{t}}, \quad t > 0, z, z' \in \mathbb{R}^{m+d}.
$$

The L-diffusion process has the coupling property.

Any time-space harmonic function associated to $P_t$ has to be constant.

Proof. By an approximation argument, it suffices to prove (1) for $f \in C_0^\infty(\mathbb{R}^{m+d})$. Indeed, for any $z \in \mathbb{R}^{m+d}$, let $e \in \mathbb{R}^{m+d}$ be a unit vector such that $\sqrt{\Gamma(P_t f)(z)} = \nabla_e P_t f(z)$. Since $P_t f \in C_b(\mathbb{R}^{m+d})$ for $f \in \mathcal{B}_b(\mathbb{R}^{m+d})$ and $t > 0$, (2.10) holds at point $z$ provided

$$
(2.11) \quad \frac{P_t f(z + \varepsilon v) - P_t f(v)}{\varepsilon} \leq \frac{c_{p, l}}{\sqrt{t}} \varepsilon \int_0^\varepsilon (P_t |f|^p)^{1/p}(z + s v) ds, \quad \varepsilon \in (0, 1).
$$

Noting that (2.10) with $f \in C_0^\infty(\mathbb{R}^{m+d})$ also implies (2.11) for $f \in C_0^\infty(\mathbb{R}^{m+d})$, and that $C_0^\infty(\mathbb{R}^{m+d})$ is dense in $L^p(P_t(z, \cdot) + P_t(z + \varepsilon e, \cdot) + \int_0^\varepsilon P_t(z + s \varepsilon e, \cdot) ds)$, we conclude that (2.10) for $f \in C_0^\infty(\mathbb{R}^{m+d})$ implies (2.11) for all $f \in \mathcal{B}_b(\mathbb{R}^{m+d})$, and hence also implies (2.10) for all $f \in \mathcal{B}_b(\mathbb{R}^{m+d})$.

Next, by the left-invariant property of $X_t$, it suffices to prove the desired estimate at point $(0, 0) \in \mathbb{R}^{m+d}$. To see this, for any $z \in \mathbb{R}^{m+d}$, let

$$
\ell_z(z') = z \cdot z', \quad z' \in \mathbb{R}^{m+d}.
$$

Since $X_t$ are left-invariant, we have

$$
\Gamma(P_t f)(z) = \Gamma((P_t f) \circ \ell_z)(0, 0) = \Gamma(P_t f \circ \ell_z)(0, 0),
$$

so that the desired estimate at point $(0, 0)$ implies

$$
\Gamma(P_t f)(z) \leq \frac{c_{p, l}}{\sqrt{t}} (P_t |f \circ \ell_z|^p)^{1/p}(0, 0) = \frac{c_{p, l}}{\sqrt{t}} (P_t |f|^p)^{1/p}(z).
$$

Now, we intend to prove (2.10) for $f \in C_0^\infty(\mathbb{R}^{m+d})$ at point $(0, 0)$. In this case, there exists an unit element $u \in \mathbb{R}^m$ such that $\sqrt{\Gamma(P_t f)(0, 0)} = \nabla_{(u,0)} P_t f(0, 0)$. Then, by Theorem 2.1 (1) and (3) with $X(0) = 0$ and using the Hölder inequality, we derive the desired upper bound for $\sqrt{\Gamma(P_t f)(0, 0)}$.

Finally, noting that

$$
\| P_t(z, \cdot) - P_t(z', \cdot) \|_{\text{var}} = 2 \sup_{\| f \|_\infty \leq 1} |P_t f (x) - P_t f (y)| \leq 2 \rho(x, y) \sup_{\| f \|_\infty \leq 1} \sqrt{\| \Gamma(P_t f) \|_\infty},
$$

then (2.10) implies (1). According to [7, Sect 4] (see also [8]), (2) and (3) follow from (1).
3 Generalized curvature-dimension condition and applications

In this section we verify the generalized curvature-dimension condition in the sense of Baudoin-Garofalo [5] such that a number of estimates on the semigroup derived in [5, 3, 4] hold true. It is easy to see that the model we consider has a sub-Riemannian structure with transverse symmetries as in [5, Section 2.3] (see also Section 2.4 in [5] for $\sigma = I_{m \times m}$), so that by Theorems 2.19 and 2.20 in [5] the generalized curvature-dimension condition is valid. We will present the exact generalized curvature-dimension condition by direct computations, and provide some applications concerning gradient estimates, Harnack inequalities and coupling/Liouville properties.

Let

$$c_1(G) = \sum_{i,j=1}^{m} \left( \sum_{l=1}^{d} (G_l)_{ij} \right)^2.$$ 

By the Hörmander condition (1.3), let $c_2(G) > 0$ be the largest constant such that

$$\sum_{1 \leq i < j \leq m} \left( \sum_{l=1}^{d} (G_l)_{ij} a_l \right)^2 \geq c_2(G) \sum_{l=1}^{d} a_l^2$$

holds for all $(a_l)_{1 \leq l \leq d} \in \mathbb{R}^d$. To introduce the generalized curvature-dimension condition, let

$$\Gamma(f, g) = \frac{1}{2} \sum_{i=1}^{m} (X_i f)(X_i g), \quad \Gamma^Z(f, g) = \frac{1}{2} \sum_{l=1}^{d} (\partial y_l f)(\partial y_l g), \quad f, g \in C^1(\mathbb{R}^{m+d})$$

and denote $\Gamma(f) = \Gamma(f, f), \Gamma^Z(f) = \Gamma^Z(f, f)$. Define

$$\Gamma_2(f) = \frac{1}{2} L \Gamma(f, f) - \Gamma(f, Lf), \quad \Gamma^Z_2(f) = \frac{1}{2} L \Gamma^Z(f, f) - \Gamma^Z(f, Lf), \quad f \in C^3(\mathbb{R}^{m+d}).$$

**Theorem 3.1.** Assume the Hörmander condition (1.3). For any $f \in C^3(\mathbb{R}^{m+d})$ and $r > 0$,

$$\Gamma_2(f) + r \Gamma^Z_2(f) \geq \frac{1}{m} (Lf)^2 + \frac{c_2(G) \Gamma^Z(f)}{4} - \frac{c_1(G)}{r} \Gamma(f).$$

**Proof.** Recall that $L = \frac{1}{2} \sum_{i=1}^{m} X_i^2$ and

$$[X_i, X_j] = -\sum_{l=1}^{d} (G_l)_{ij} \partial y_l, \quad [X_i, \partial y_l] = 0, \quad 1 \leq i,j \leq m, 1 \leq l \leq d.$$
Then
\[
\Gamma_2(f) = \frac{1}{8} \sum_{i,j=1}^{m} X_i^2(X_j f)^2 - \frac{1}{4} \sum_{j=1}^{m} \left( X_j \sum_{i=1}^{m} X_i^2 f \right) (X_j f)
\]
\[
= \frac{1}{4} \sum_{i,j=1}^{m} (X_j f) (X_i^2 X_j f) + \frac{1}{4} \sum_{i,j=1}^{m} (X_i X_j f)^2 - \frac{1}{4} \sum_{i,j=1}^{m} (X_j X_i^2 f)(X_j f)
\]
(3.1)
\[
= \frac{1}{4} \sum_{i,j=1}^{m} (X_j f)([X_i, X_j] X_i + X_i [X_i, X_j]) f + \frac{1}{4} \sum_{i,j=1}^{m} (X_i X_j f)^2
\]
\[
= \frac{1}{4} \sum_{i,j=1}^{m} (X_i X_j f)^2 - \frac{1}{2} \sum_{i,j=1}^{m} (X_j f) \left( \sum_{l=1}^{d} (G_l)_{ij} \partial_{y_l} X_i f \right).
\]
Moreover,
\[
\Gamma_2^Z(f) = \frac{1}{8} \sum_{l=1}^{d} \sum_{i=1}^{m} X_i^2 (\partial_{y_l} f)^2 - \frac{1}{4} \sum_{i=1}^{m} \sum_{l=1}^{d} (\partial_{y_l} f)(X_i^2 \partial_{y_l} f) = \frac{1}{4} \sum_{i=1}^{m} \sum_{l=1}^{d} (\partial_{y_l} X_i f)^2.
\]
Combining this with (3.1) and the fact
\[
\frac{1}{4} \sum_{i,j=1}^{m} (X_i X_j f)^2 \geq \frac{(L f)^2}{m} + \frac{1}{4} \sum_{1 \leq i \neq j \leq m} (X_i X_j f)^2,
\]
we obtain
\[
\Gamma_2(f) + r \Gamma_2^Z(f) \geq \frac{(L f)^2}{m} + \frac{1}{4} \sum_{1 \leq i \neq j \leq m} (X_i X_j f)^2 - \frac{1}{4 r} \sum_{i=1}^{m} \left( \sum_{j=1}^{m} \sum_{l=1}^{d} (X_j f)(G_l)_{ij} \right)^2
\]
\[
\geq \frac{(L f)^2}{m} + \frac{1}{4} \sum_{1 \leq i \neq j \leq m} (X_i X_j f)^2 - \frac{c_1(G)}{2r} \Gamma(f, f), \quad r > 0.
\]
Finally, as observed in [5] we have
\[
\sum_{1 \leq i \neq j \leq m} (X_i X_j f)^2 = \sum_{1 \leq i < j \leq m} \{(X_i X_j f)^2 + (X_j X_i f)^2\}
\]
\[
= \frac{1}{2} \sum_{1 \leq i < j \leq m} \{(X_i X_j f + X_j X_i f)^2 + (X_i X_j f - X_j X_i f)^2\}
\]
\[
\geq \frac{1}{2} \sum_{1 \leq i < j \leq m} ([X_i, X_j] f)^2 = \frac{1}{2} \sum_{1 \leq i < j \leq m} \left( \sum_{l=1}^{d} (G_l)_{ij} \partial_{y_l} f \right)^2 \geq c_2(G) \Gamma^Z(f).
\]
Combining this with (3.2) we complete the proof. \(\square\)
As it is easy to see that the commutation condition
\[ \Gamma(f, \Gamma^Z(f, f)) = \Gamma^Z(f, \Gamma(f, f)), \quad f \in C^\infty(\mathbb{R}^{m+d}) \]
holds, the following assertions follow from the curvature-dimension condition presented in Theorem 3.1 where (3) and (4) are known as Li-Yau type gradient estimate and parabolic Harnack inequality (see [15]), and (2) is the dimension-free Harnack inequality initiated by the author in [16], which implies the log-Harnack inequality (3) as observed in [18]. This type of Harnack inequality was also established in [11] on a class of Lie groups. The entropy gradient inequality (1) implying the dimension-free Harnack inequality (2) was first observed in [1].

**Corollary 3.2.** Assume the Hörmander condition [1,3]. For any \( t > 0 \) and positive \( f \in \mathcal{B}_b(\mathbb{R}^{m+d}) \), the following assertions hold:

1. \( \frac{t \Gamma(P_t f)}{P_t f} + \frac{c_2(G) t^2 \Gamma^Z(P_t f)}{4P_t f} \leq \frac{c_2(G) + 8c_1(G)}{c_2(G)} \{ P_t (f \log f) - (P_t f) \log P_t f \} \).

2. \( (P_t f)^p(z) \leq (P_t f^p(z')) \exp \left[ \frac{p(c_2(G) + 8c_1(G))}{4(p-1)c_2(G)t} \rho(z, z')^2 \right], \quad p > 1, z, z' \in \mathbb{R}^{m+d} \).

3. \( P_t \log f(z) \leq \log P_t f(z') + \frac{c_2(G) + 8c_1(G)}{4c_2(G)t} \rho(z, z')^2, \quad z, z' \in \mathbb{R}^{m+d} \).

4. \( \Gamma(\log P_t f) + \frac{c_2(G)t}{6} \Gamma^Z(\log P_t f) \leq \frac{c_2(G) + 6c_1(G)}{c_2(G)} \partial_t \log P_t f + \frac{m(c_2(G) + 6c_1(G))^2}{2c_2(G)^2 t} \).

5. \( P_t f(z) \leq P_{t+s} f(z') \left( \frac{t+s}{t} \right)^{\frac{m(c_2(G) + 6c_1(G))}{2c_2(G)}} \exp \left[ \frac{(c_2(G) + 6c_1(G)) \rho(x, y)^2}{4mc_2(G)s} \right], \quad s, t > 0 \).

6. Let \( p_t \) be the density of \( P_t \) w.r.t. the Lebesgue measure \( \mu \), then there exist two constants \( c_1, c_2 > 0 \) such that
\[
p_t(z, z') \leq \frac{c_1 \exp \left[ -\frac{c_2 \rho(z, z')^2}{t} \right]}{t^{(m+2d)/2}}, \quad t > 0, z, z' \in \mathbb{R}^{m+d}.
\]

**Proof.** (a) Proof of (1)-(5). According to the curvature-dimension condition in Theorem 3.1 [8] Propositions 3.1, 3.4 and [5] Theorems 6.1, 7.1, 8.1, for assertions (1)-(5) it suffices to verify the following conditions:

(i) There exists a sequence \( \{ h_n \} \subset C_0^\infty(\mathbb{R}^{m+d}) \) such that \( h_n \uparrow 1 \) and \( \| \Gamma(h_n) \|_\infty + \| \Gamma^Z(h_n) \|_\infty \to 0 \) as \( n \uparrow \infty \).

(ii) \( \Gamma(f, \Gamma^Z(f, f)) = \Gamma^Z(f, \Gamma(f), f) \in C^\infty(\mathbb{R}^{m+d}) \).

(iii) For any \( f \in C_0^\infty(\mathbb{R}^{m+d}) \) and \( T > 0 \),
\[
\sup_{t \in [0, T]} (\| \Gamma(P_t f) \|_\infty + \| \Gamma^Z(P_t f) \|_\infty) < \infty.
\]
Let \( f \in C^\infty_0([0, \infty)) \) with \( f' \leq 0, f|_{[0,1]} = 1 \) and \( f|_{(2, \infty)} = 0 \). Then (i) holds for \( h_n(z) := f(|z|/n), \ n \geq 1, z \in \mathbb{R}^{m+d} \). Next, (ii) follows from \( [X_i, \partial_y] = 0, \ 1 \leq i \leq m, 1 \leq l \leq d \). Finally, it is easy to see from (2.8) that

\[
|\nabla_{(u,v)} P_t f| = |\mathbb{E}\langle \nabla_{(u,v)}(X(t), Y(t)), \nabla f(X(t), Y(t)) \rangle| \leq c(1 + \sqrt{t})\|\nabla f\|_\infty, \ t \geq 0
\]

holds for some constant \( c > 0 \) and all unit \( (u, v) \in \mathbb{R}^{m+d}, f \in C^\infty_0(\mathbb{R}^{m+d}) \). Then (iii) holds.

(b) Proof of (6). Each of (2) and (5) implies (cf. [15, 13])

\[
= \sup_{|\nabla s| \leq \infty, s \in \mathbb{R}} |s|^2 \leq c(1 + \sqrt{t})\|\nabla s\|_\infty, \ t \geq 0
\]

for some constant \( c > 0 \). It is well-known that this is equivalent to the Nash inequality (see e.g. [9])

\[
\mu(f)^2 \leq C \mu(\Gamma(f)) \frac{m+d}{m+2d} \mu(|f|^2)^{\frac{1}{m+2d}}, \ f \in C^1_0(\mathbb{R}^{m+d})
\]

for some constant \( C > 0 \). This inequality also follows from [17, Corollary 1.2] with \( d_1 = m, d_2 = d \) and \( l_i = 1(1 \leq i \leq d) \).
4 An explicit inverse Poincaré inequality

In this section we aim to derive an explicit $L^2$-estimate on $\Gamma(P_t f)$ as in [2, Section 3], where the heat semigroup on the Heisenberg group is concerned. To this end, we need the following assumption:

\[(A2)\] For any $l, l', l'' \in \{1, \cdots, d\}$, $A^*_l = -A_l$, $\sigma A_l = A_l \sigma$, $A_l A_{l'} = A_{l'} A_l$, and $A_l \sigma, A_l A_{l'} A_{l''} \sigma$ and $A_l \sigma^2 \sigma^*$ are skew-symmetric.

A simple example for this assumption to hold is that $\sigma = I_{m \times m}$ and $\{A_l\}$ are commutative skew-symmetric $m \times m$-matrices.

**Theorem 4.1.** Assume (1.2) and (A2). Then for any $f \in \mathcal{B}_b(\mathbb{R}^{m+d})$ and $t > 0$,

$$\Gamma(P_t f) \leq \frac{m+2}{2t} \{P_t f^2 - (P_t f)^2\}.$$ 

This estimate is equivalent to (2.10) for $p = 2$ with explicit constant $c_p = \left(\frac{m+2}{2}\right)^{\frac{1}{p}}$. To prove this result, we introduce the dilation operator modified from [2],

$$D := \frac{1}{2} \sum_{i=1}^{m} x_i \partial_{x_i} + \sum_{l=1}^{d} y_l \partial_{y_l}$$

and the dual vector fields

$$\hat{X}_i(x, y) := X_i(x, y) - 2 \sum_{l=1}^{d} (A_l x) \partial_{y_l}, \quad 1 \leq i \leq m.$$ 

Simply denote

$$p^0_t(z) = p_t((0, 0), z), \quad z \in \mathbb{R}^{m+d},$$

where $p_t$ is the heat kernel of $P_t$ w.r.t. the Lebesgue measure $\mu$ on $\mathbb{R}^{m+d}$, which exists due to the Hörmander condition.

**Lemma 4.2.** Assume (1.2) and (A2). Then $LD - DL = L$ and $[\hat{X}_i, X_j] = 0$, $1 \leq i, j \leq m$. Consequently, $(tL + D + \frac{m+2}{2})p^0_t = 0$ and $\hat{X}_i P_t = P_t \hat{X}_i$, $1 \leq i \leq m$.

**Proof.** It is easy to see that $[X_i, D] = \frac{1}{2} X_i$, $1 \leq i \leq m$. Then

$$(4.1) \quad LD - DL = \frac{1}{2} \sum_{i=1}^{m} (X_i^2 D - D X_i^2) = \frac{1}{2} \sum_{i=1}^{m} (X_i [X_i, D] + [X_i, D] X_i) = \frac{1}{2} \sum_{i=1}^{m} X_i^2 = L.$$ 

Let $T_t$ be the semigroup generated by $D$. Then $T_t e^t$ is generated by $D + 1$ and due to (4.1) $LD = (D + 1)L$. So, $LT_t = T_s e^t L$, which implies that

$$(4.2) \quad P_t T_s = T_s P_{e^t}, \quad t, s \geq 0.$$
Differentiating both sides w.r.t. $s$ at $s = 0$, we obtain

$$P_t D = DP_t + tP_t L, \quad t \geq 0.$$  

(4.3)

Since $D(0, 0) = 0$, it follows that

$$P_t(tL - D)f(0, 0) = 0, \quad f \in C^\infty_0(\mathbb{R}^{m+d}).$$

Combining this with

$$P_t(tL - D)f(0, 0) = \int_{\mathbb{R}^{m+d}} p^0_t(z)(tL - D)f(z)dz$$

$$= \int_{\mathbb{R}^{m+d}} f(z)\{(tL + D)p^0_t(z) + (\text{div}D)p^0_t(z)\}dz$$

$$= \int_{\mathbb{R}^{m+d}} f(z)\left(tL + D + \frac{m+2}{2}\right)p^0_t(z)dz,$$

we conclude that $(tL + D + \frac{m+2}{2})p^0_t = 0$.

Next, for any $1 \leq i, j \leq m$, we have

$$[\hat{X}_i, X_j] = [X_i, X_j] + 2 \sum_{k=1}^m \sum_{l=1}^d (A_l)_{ik} \sigma_{kj} \partial_{y_l}$$

$$= \sum_{l=1}^d \{(A_l \sigma)_{ji} - (A_l \sigma)_{ij}\} \partial_{y_l} + 2 \sum_{l=1}^d (A_l \sigma)_{ij} \partial_{y_l} = 0$$

since $A_l \sigma$ is skew-symmetric. This implies $\hat{X}_i P_t = P_t \hat{X}_i$ for any $1 \leq i \leq m$. \hfill $\square$

**Lemma 4.3.** Assume (A2) and let $\hat{\Gamma}(f) = \frac{1}{2} \sum_{i=1}^m (\hat{X}_i f)^2$. Then $\hat{\Gamma}(p^0_t) = \Gamma(p^0_t)$.

**Proof.** It is easy to see that at point $(x, y) \in \mathbb{R}^{m+d}$,

$$\hat{\Gamma}(f) = \Gamma(f) - 2 \sum_{i=1}^m (X_if)\left(\sum_{l=1}^d (A_l x)_i \partial_{y_l} f\right) + 2 \sum_{i=1}^m \left(\sum_{l=1}^d (A_l x)_i \partial_{y_l} f\right)^2$$

$$= \Gamma(f) - 2 \sum_{l=1}^d \sum_{k=1}^m (\sigma A_l x)_k \partial_{y_k} f(\partial_{y_l} f) = \Gamma(f) - 2 \sum_{l=1}^d (\partial_{y_l} f) \Theta_l f,$$

where

$$\Theta_l := \sum_{k=1}^m (\sigma A_l x)_k \partial_{x_k}, \quad 1 \leq l \leq d.$$

So, it remains to prove $\Theta_l p^0_t = 0$ for $1 \leq l \leq d$. We prove it by two steps.
(1) \( \Theta_t L = L \Theta_t \). Since \( A_t \sigma = \sigma A_t \), it is easy to see that

\[
[\Theta_t, X_i] = \sum_{t'=1}^{d} \sum_{k=1}^{m} (\sigma A_t x)_k (A_{t'})_{ik} \partial_{y_{t'}} - \sum_{k,j=1}^{m} \sigma_j (\sigma A_t)_{k_j} \partial_{x_k}
\]

\[
= \sum_{t'=1}^{d} (A_{t'} \sigma A_t x)_i \partial_{y_{t'}} - \sum_{k=1}^{m} (\sigma A_t)_{k_l} \partial_{x_k}
\]

\[
= \sum_{t'=1}^{d} (A_{t'} A_t x)_i \partial_{y_{t'}} - \sum_{i=1}^{m} (A_t \sigma^2)_{k_i} \partial_{x_k}.
\]

Then

\[
\sum_{i=1}^{m} (\Theta_t X_i^2 - X_i^2 \Theta_t) = \sum_{i=1}^{m} \{[\Theta_t, X_i] X_i + X_i [\Theta_t, X_i]\}
\]

\[
= 2 \sum_{t'=1}^{d} \sum_{i,k=1}^{m} (A_{t'} A_t x)_i \sigma_{ki} \partial_{x_k} \partial_{y_{t'}} + 2 \sum_{t',t''=1}^{d} \sum_{i=1}^{m} (A_{t'} A_t x)_i (A_{t''} x)_i \partial_{y_{t'}} \partial_{y_{t''}}
\]

\[
- 2 \sum_{i,j=1}^{m} (A_t \sigma^2)_{k_i} \sigma_{ji} \partial_{x_k} \partial_{x_j} - 2 \sum_{i,k=1}^{m} \sum_{t'=1}^{d} (A_t \sigma^2)_{k_i} (A_{t'} x)_i \partial_{x_k} \partial_{y_{t'}}
\]

\[
+ \sum_{t'=1}^{d} \sum_{i,k=1}^{m} \sigma_{ki} (A_{t'} A_t x)_i \partial_{y_{t'}} - \sum_{t'=1}^{d} \sum_{i,k=1}^{m} (A_t \sigma^2)_{k_i} (A_{t'} x)_i \partial_{y_{t'}}
\]

\[
= 2 \sum_{t'=1}^{d} \sum_{k=1}^{m} \{ (\sigma A_{t'} A_t x)_k - (A_t \sigma^2 A_{t'} x)_k \} \partial_{x_k} \partial_{y_{t'}} + 2 \sum_{t',t''=1}^{d} \langle A_{t'} A_t x , A_{t''} x \rangle \partial_{y_{t'}} \partial_{y_{t''}}
\]

\[
- 2 \sum_{i,j=1}^{m} (A_t \sigma^2)^*_{ij} \partial_{x_i} \partial_{x_j} + \sum_{t'=1}^{d} \text{Tr}(\sigma A_{t'} A_t \sigma - A_{t'} A_t \sigma^2) \partial_{y_{t'}}.
\]

Due to (A2), this implies that \( \Theta_t L = L \Theta_t \).

(2) By (1), \( \text{div} \Theta_t = 0 \) and \( \Theta_t (0, 0) = 0 \), for any \( f \in C_0^\infty(\mathbb{R}^{m+d}) \) we have

\[
0 = (\Theta_t P_t f)(0, 0) = (P_t \Theta_t f)(0, 0)
\]

\[
= \int_{\mathbb{R}^{m+d}} p_t^0(z) \Theta_t f(z) dz = - \int_{\mathbb{R}^{m+d}} \{ \Theta_t p_t^0(z) \} f(z) dz.
\]

Therefore, \( \Theta_t p_t^0 = 0 \). \[ \square \]

Lemma 4.4. \( \int_{\mathbb{R}^{m+d}} \Gamma(\log p_t^0, p_t^0)(z) dz = \frac{m+2}{2t}, \ t > 0. \)

Proof. We shall prove the lemma by using an approximation argument. Let \( h \in C_0^\infty([0, \infty)) \) such that \( 0 \leq h \leq 1, h|_{[0,1]} = 1 \) and \( h|_{[2,\infty)} = 0 \). Let \( f_n(z) = h(z/n), n \geq 1 \). Then there is a constant \( C_1 > 0 \) such that

\[
|L f_n|(z) + \Gamma(f_n)(z) + |D f_n|(z) \leq C_1 1_{[n,2n]}(|z|), \ z \in \mathbb{R}^{m+d}.
\]
Moreover, there is a constant $C_2 > 0$ such that $\Gamma(C_2 \log(1 + |\cdot|)) \leq 1$. So, according to (2.9)\[ \rho(0, z) \geq C_2 \log(1 + |z|), \quad z \in \mathbb{R}^{m+d}. \]Combing this with Corollary 3.2(6) we obtain
\[ p_t^0(z) \leq c_1(t) \exp [-c_2(t)\{\log(1 + |z|)\}^2], \quad z \in \mathbb{R}^{m+d} \]
for some constants $c_1(t), c_2(t) > 0$. Since by Lemma 4.2 $(tL + D + \text{div}D)p_t^0 = 0$, for any $n \geq 1$ we have
\[ \int_{\mathbb{R}^{m+d}} \{f_n(\log p_t^0, p_t^0)\}(z)dz = -t \int_{\mathbb{R}^{m+d}} \{f_n(\log p_t^0)\}Lp_t^0 + (\log p_t^0)\Gamma(f_n, p_t^0)\}(z)dz \]
\[ = \int_{\mathbb{R}^{m+d}} \{(f_n \log p_t^0)(D + \text{div}D)p_t^0\}(z)dz - t \int_{\mathbb{R}^{m+d}} (\log p_t^0)(z)\Gamma(f_n, p_t^0)(z)dz \]
\[ = - \int_{\mathbb{R}^{m+d}} D(f_np_t^0)(z)dz + \int_{\mathbb{R}^{m+d}} \{(Df_n)p_t^0 - (p_t^0 \log p_t^0)Df_n - t(\log p_t^0)\Gamma(f_n, p_t^0)\}(z)dz \]
\[ = \frac{m + 2}{2} \int_{\mathbb{R}^{m+d}} (f_np_t^0)(z)dz + \int_{\mathbb{R}^{m+d}} \{(Df_n)p_t^0 - (p_t^0 \log p_t^0)Df_n - t(\log p_t^0)\Gamma(f_n, p_t^0)\}(z)dz \]
\[ \leq \frac{m + 2}{2} + C(t) \int_{\{n \leq |x| \leq 2n\}} \{p_t^0 + |p_t^0 \log p_t^0| + |\log p_t^0|/\Gamma(p_t^0)\}(z)dz \]
for some constant $C(t) > 0$ according to (4.1). Therefore, it suffices to verify
\[ \int_{\mathbb{R}^{m+d}} \{p_t^0 + |p_t^0 \log p_t^0| + |\log p_t^0|/\Gamma(p_t^0)\}(z)dz < \infty \]
so that the desired estimate follows by letting $n \to \infty$. Noting that $p_t^0 = p_t^0 D$, Corollary 2.4 for $p = 2$ and Corollary 3.2(6) yield
\[ \sqrt{\Gamma(p_t^0)} \leq C_1(t)\sqrt{P_{\frac{t}{2}}(p_t^0)^2} \leq C_2(t)\sqrt{p_t^0} \]
for some constants $C_1(t), C_2(t) > 0$. Therefore, (4.6) follows from (4.4) since
\[ p_t^0 + |p_t^0 \log p_t^0| + C_2(t)|\log p_t^0|\sqrt{p_t^0} \leq C_3(t)\{(p_t^0)^2 + (p_t^0)^{\frac{3}{2}}\} \]
holds for some constant $C_3(t) > 0$. \hfill \Box

5 The Poincaré Inequality

In this section we prove the estimate (5.1) below by following the argument in [2] Section 4. This estimate for the heat semigroup on the Heisenberg group was first derived [12] was reproved in [2].

According to (1.2), there exists $\{(i_l, j_l)\}_{1 \leq l \leq d}$ with $1 \leq i_l < j_l \leq m$ such that the matrix
\[ \tilde{M} := (\tilde{M}_{l,l'})_{1 \leq l, l' \leq d} \]
is invertible, where $\tilde{M}_{l,l'} := M(i_l, j_l, l) = (G_{l'})_{i_l i_l}$. Recall that for any $x \in \mathbb{R}^m$ and $1 \leq i \leq m$, $(A.x)_i := ((A_i x)_i)_{1 \leq i \leq d} \in \mathbb{R}^d$. Similarly, we let $(A.\sigma)_{i j} := ((A_i \sigma)_{i j})_{1 \leq i \leq d} \in \mathbb{R}^d$. 

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Theorem 5.1. Assume (A2). Then
\begin{equation}
\Gamma(P_t f) \leq C P_t \Gamma(f), \quad t \geq 0, f \in C^1_b(\mathbb{R}^{m+d})
\end{equation}
holds for
\[ C := 2 + 16 \sum_{i,j=1}^{m} |(\vec{M}^*)^{-1}(A,\sigma)_{ij}|^2 + 32P_1 \left\{ \sup_{1 \leq l \leq d} \{ (\vec{M}^*)^{-1}(A,\sigma)_i \}^2 \Gamma(\log p_1^0) \right\}(0,0) < \infty, \]
where \( p_1^0(z) := p_1((0,0),z) \) and \( (A,\sigma)_i(z) := (A,\sigma)_i(z) \) for \( z = (x',y') \in \mathbb{R}^{m+d} \). Consequently, the Poincaré inequality
\[ P_t f^2 - (P_t f)^2 \leq 2C t P_t \Gamma(f), \quad f \in C^1_b(\mathbb{R}^{m+d}) \]
holds for all \( t > 0 \).

Proof. The desired Poincaré inequality follows immediately from (5.1) by noting that for \( f \in C^\infty_0(\mathbb{R}^{m+d}) \),
\[ \frac{d}{ds} P_s(P_{t-s} f)^2 = 2P_s \Gamma(P_{t-s} f) \leq 2C P_t \Gamma(f), \quad s \in [0,t]. \]
Below we prove (5.1) and the finite of \( C \) respectively, where the proof of (5.1) is modified from [2].

1. We first observe that to prove (5.1) it suffices to confirm
\begin{equation}
\Gamma(P_1 f)(0,0) \leq C P_1 \Gamma(f)(0,0), \quad f \in C^\infty_0(\mathbb{R}^{m+d}).
\end{equation}
Indeed, by the left-invariant property of \( \Gamma \) and \( P_t \), we only need to prove (5.1) at point \( (0,0) \); and by a standard approximation argument as in the proof of Corollary 2.4, we may assume that \( f \in C^\infty_0(\mathbb{R}^{m+d}) \). Finally, for any \( t > 0 \), it follows from (5.2) that
\[ P_t f = P_t T_{-\log t} T_{\log t} = T_{-\log t} P_t T_{\log t} f. \]
Noting that \( T_{s} f(x, y) = f(e^{\frac{s}{m}} x, e^{s} y) \), we have \( X_i T_s = e^{\frac{s}{m}} T_s X_i \). Therefore, if (5.2) holds, then at point \( (0,0) \) we have
\[ \Gamma(P_t f) = \Gamma(T_{-\log t} P_t T_{\log t} f) = \frac{1}{t} T_{-\log t} \Gamma(P_t T_{\log t} f) \leq \frac{C}{t} T_{-\log t} P_t \Gamma(T_{\log t} f) = C T_{-\log t} P_t \Gamma(f) = C P_t \Gamma(f). \]

2. Note that
\[ \sum_{l=1}^{d} \{ (\vec{M}^*)^{-1}(A,\sigma)_i \}^2 [X_{ij}, X_{j}^l] = - \sum_{l,l' = 1}^{d} (\vec{M}^*)^{-1} (A_{l'} x)_i \vec{M}_{l'\nu} \partial_{y_{l'}} \]
\[ = - \sum_{l',l'' = 1}^{d} \{ \vec{M}^* (\vec{M}^*)^{-1} \}_{l''} (A_{l'} x)_i \partial_{y_{l''}} = - \sum_{l' = 1}^{d} (A_{l'} x)_i \partial_{y_{l''}}. \]
Then, by Lemma 4.2 at point \((0,0)\) we have

\[
X_iP_i f = \dot{X}_iP_i f = P_1X_i f = P_1\{X_i - 2 \sum_{l=1}^{d} (A_i x)_i \partial_{y_l}\} f
\]

(5.3)

\[
= P_1(X_i f) - 2P_1\left(\sum_{l=1}^{d} (A_i x)_i \partial_{y_l} f\right)
= P_1(X_i f) + 2\sum_{l=1}^{d} P_1\left(\{(\tilde{M}^*)_i^{-1}(A.x)_i\}_l[X_i, X_l] f\right).
\]

Next, for any \(f \in C_0^\infty\), at point \((0,0)\) we have

\[
P_1\left(\{(\tilde{M}^*)_i^{-1}(A.x)_i\}_l[X_i, X_l] f\right)
= \int_{\mathbb{R}^{m+d}} p^0_i(x, y)\{(\tilde{M}^*)_i^{-1}(A.x)_i\}_l(X_i, X_l - X_j, X_l) f(x, y) dx dy
\]

\[
= \int_{\mathbb{R}^{m+d}} p^0_i(x, y)(X_i, f)(x, y)\left\{(\tilde{M}^*)_i^{-1}(A.x)_i\}_l X_i \log p^0_i(x, y) + (\tilde{M}^*)_i^{-1}(A.S)_{i,j}\right\} dx dy
- \int_{\mathbb{R}^{m+d}} p^0_i(x, y)(X_j, f)(x, y)\left\{(\tilde{M}^*)_i^{-1}(A.x)_i\}_l X_i \log p^0_i(x, y) + (\tilde{M}^*)_i^{-1}(A.S)_{i,j}\right\} dx dy
= P_1\{(X_i, f)\left\{(\tilde{M}^*)_i^{-1}(A.x)_i\}_l X_i \log p^0_i + (\tilde{M}^*)_i^{-1}(A.S)_{i,j}\} \right\}
- P_1\{(X_j, f)\left\{(\tilde{M}^*)_i^{-1}(A.x)_i\}_l X_i \log p^0_i + (\tilde{M}^*)_i^{-1}(A.S)_{i,j}\} \right\}.
\]

Combining this with (5.3) and noting that

\[
\sum_{l=1}^{d} \left[\{(\tilde{M}^*)_i^{-1}(A.x)_i\}_l X_{i,j} \log p^0_i + (\tilde{M}^*)_i^{-1}(A.S)_{i,j}\}_l\right]^2
\leq 2 \sum_{i,j=1}^{m} |(\tilde{M}^*)_i^{-1}(A.S)_{i,j}|^2 + 4 \sup_{1 \leq l \leq d} \{(\tilde{M}^*)_i^{-1}(A.x)_i\}_l \int \Gamma(\log p^0_i)
\]

and the same holds for \(i_l\) in place of \(j_l\), we prove (5.2) at point \((0,0)\).

(3) Obviously, \(C < \infty\) follows from

\[
\int_{\mathbb{R}^{m+d}} |x|^2p^0_i(x, y) \Gamma(\log p^0_i)(x, y) dx dy < \infty.
\]

(5.4) Let \(h \in C^\infty_{\infty}(0, \infty)\) such that \(h|_{[0,1]} = 1\) and \(h|_{[2,\infty)} = 0\). Let \(f_n(x) = |x|^2h(|x|/n)\). Then \(f_n \in C_0^\infty(\mathbb{R}^{m+d}), n \geq 1\). By Corollary 3.2 (4), there exists a constant \(c > 0\) such that

\[
\Gamma(\log p^0_i) \leq c \left(1 + \frac{Lp^0_i}{p^0_i}\right).
\]
Combining these with \((L + D + \frac{m+2}{2})p^0_1 = 0\) according to Lemma 4.2, we arrive at

\[
\int_{\mathbb{R}^{m+d}} f_n(|x|)p^0_1(x, y)\Gamma(\log p^0_1)(x, y)dx dy \\
\leq c \int_{\mathbb{R}^{m+d}} f_n(|x|)p^0_1(x, y)\left(1 + \frac{Lp^0_1}{P^0_1}\right)(x, y)dx dy \\
= c \int_{\mathbb{R}^{m+d}} f_n(|x|)Lp^0_1(x, y)dx dy + cP_1|x|^2(0, 0) \\
= cP_1|x|^2(0, 0) - c \int_{\mathbb{R}^{m+d}} f_n(|x|)\left(D + \frac{m+2}{2}\right)p^0_1(x, y)dx dy \\
= cP_1|x|^2(0, 0) + cP_1(Df_n)(0, 0) \leq cP_1|x|^2(0, 0) + cP_1\left(|x|^2 + \frac{\|h'\|_{\infty}}{n}|x|^3\right)(0, 0).
\]

Letting \(n \to \infty\) and noting that \(P_1|x|^p(0, 0) < \infty\) holds for any \(p \geq 1\), we obtain (5.4). \(\square\)

Finally, we remark that for the heat semigroup on the Heisenberg group the following stronger estimate than (5.1) was proved in [14] (see also [2, Section 5]):

\[
\sqrt{\Gamma(P^1_t f)} \leq CP_1\sqrt{\Gamma(f)}, \quad f \in C^1_b(\mathbb{R}^3), t > 0
\]

for some constant \(C > 0\). This estimate implies the semigroup log-Sobolev inequality. However, in the moment we are not able to prove this type estimate under our more general framework.

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