MULTILINEAR FRACTIONAL TYPE OPERATORS
AND THEIR COMMUTATORS ON HARDY SPACES WITH
VARIABLE EXPONENTS

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Abstract. In this article, we show that multilinear fractional type operators are bounded from product Hardy spaces with variable exponents into Lebesgue or Hardy spaces with variable exponents via the atomic decomposition theory. We also study continuity properties of commutators of multilinear fractional type operators on product of certain Hardy spaces with variable exponents. Some of these results are new even for the constant exponents case.

1. Introduction

The study of Hardy spaces began in the early 1900s in the context of Fourier series and complex analysis in one variable. It was not until 1960 when the groundbreaking work in Hardy space theory in $\mathbb{R}^n$ came from Stein, Weiss, Coifman and C. Fefferman in [5, 7, 14]. The classical Hardy space can be characterized by the Littlewood-Paley-Stein square functions, maximal functions and atomic decompositions. Especially, atomic decomposition is a significant tool in harmonic analysis and wavelet analysis for the study of function spaces and the operators acting on these spaces (see Meyer [31] and Coifman-Meyer [6], etc.) Atomic decomposition was first introduced by Coifman [5] in one dimension in 1974 and later was extended to higher dimensions by Latter [26]. As we all know, atomic decompositions of Hardy spaces play an important role in the boundedness of operators on Hardy spaces and it is commonly sufficient to check that atoms are mapped into bounded elements of quasi-Banach spaces.

Another stage in the progress of the theory of Hardy spaces was done by Nakai and Sawano [32] and Cruz-Uribe and Wang [11] recently when they independently considered Hardy spaces with variable exponents. It is quiet different to obtain the boundedness of operators on Hardy spaces with variable exponents. It is not sufficient to show the $H_{p(\cdot)}$-boundedness merely by checking the action of the operators on $H_{p(\cdot)}$-atoms. In the linear theory, the boundedness of some operators on variable Hardy spaces and some variable Hardy-type spaces have been established in [11, 21, 32, 38, 43, 17] as applications of the corresponding atomic decompositions theories.

In more recent years, the study of multilinear operators on Hardy space theory has received increasing attention by many authors, see for example [17, 22, 23]. While
the multilinear operators worked well on the product of Hardy spaces, it is surprising
that these similar results in the setting of variable exponents were unknown for a
long time. The boundedness of some multilinear operators on products of classical
Hardy spaces was investigated by Grafakos and Kalton ([17]) and Li, Xue and Yabuta
([28]). In [42], Tan, Liu and Zhao studied some multilinear operators are bounded
on variable Lebesgue spaces $L^{p(\cdot)}$. However, there are some subtle difficulties in
proving the boundedness results when we deal with the $H^{p(\cdot)}$-norm. The first goal
of this article is to show that multilinear fractional type operators are bounded
on product of Hardy spaces with variable exponents via atomic decompositions theory. We also remark that some
boundedness of many types of multilinear operators on some variable Hardy and
Hardy-type spaces have established in [10, 41, 44].

On the other hand, we study the boundedness of commutators of multilinear frac-
tional type operators. In 1976, Coifman, Rochberg and Weiss ([4]) studied the $L^p$
boundedness of linear commutators generated by the Calderón-Zygmund singular in-
tegral operator and $b \in \text{BMO}$. In 1982, Chanillo ([2]) consider the boundedness of
commutators of fractional integral operators on classical Lebesgue spaces. Similar
to the property of a linear Calderón-Zygmund operator, a linear fractional type op-
erator $I_\alpha$ associated with a BMO function $b$ fails to satisfy the continuity from the
Hardy space $H^p$ into $L^p$ for $p \leq 1$. In 2002, Ding, Lu and Zhang ([13]) proved that
$[b, I_\alpha]$ is continuous from an atomic Hardy space $H^{p(\cdot)}_b$ into $L^p$, where $H^{p(\cdot)}_b$
is a subspace of the Hardy space $H^p$ for $n/(n + 1) < p \leq 1$. In addition, the boundedness of
the commutators of multilinear operators has also been studied already in [3, 31, 36].
Then, Li and Xue ([27]) consider continuity properties for commutators of multilinear
type operators on product of certain Hardy spaces. It is natural to ask whether such
results are also hold in variable exponents setting. The answer is affirmative. The
second purpose in this article is to study the commutators of multilinear fractional
type operators on product of certain Hardy spaces with variable exponents. To do
so, we will introduce a new atomic space with variable exponents, $H^{p(\cdot)}_b$, which is a
subspace of the Hardy space with variable exponents $H^{p(\cdot)}$ and obtain the endpoint
$(H^{p(\cdot)}_b \times \cdots \times H^{p(\cdot)}_b, L^{q(\cdot)})$ boundedness for multilinear fractional type operators.

First we recall the definition of Lebesgue spaces with variable exponent. Note that
the variable exponent spaces, such as the variable Lebesgue spaces and the variable
Sobolev spaces, were studied by a substantial number of researchers (see, for instance,
[9, 25]). For any Lebesgue measurable function $p(\cdot) : \mathbb{R}^n \to (0, \infty]$ and for any
measurable subset $E \subset \mathbb{R}^n$, we denote $p^-(E) = \inf_{x \in E} p(x)$ and $p^+(E) = \sup_{x \in E} p(x)$.
Especially, we denote $p^- = p^-(\mathbb{R}^n)$ and $p^+ = p^+(\mathbb{R}^n)$. Let $p(\cdot) : \mathbb{R}^n \to (0, \infty)$
be a measurable function with $0 < p^- \leq p^+ < \infty$ and $\mathcal{P}^0$ be the set of all these
$p(\cdot)$. Let $\mathcal{P}$ denote the set of all measurable functions $p(\cdot) : \mathbb{R}^n \to [1, \infty)$ such that
$1 < p^- \leq p^+ < \infty$.

**Definition 1.1.** Let $p(\cdot) : \mathbb{R}^n \to (0, \infty]$ be a Lebesgue measurable function. The
variable Lebesgue space $L^{p(\cdot)}$ consists of all Lebesgue measurable functions $f$, for
which the quantity $\int_{\mathbb{R}^n} |\varepsilon f(x)|^{p(x)} \, dx$ is finite for some $\varepsilon > 0$ and

$$
\|f\|_{L^{p(\cdot)}} = \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^n} \left( \frac{|f(x)|}{\lambda} \right)^{p(x)} \, dx \leq 1 \right\}.
$$

The variable Lebesgue spaces were first established by Orlicz [34] in 1931. Two decades later, Nakano [33] first systematically studied modular function spaces which include the variable Lebesgue spaces as specific examples. However, the modern development started with the paper [25] of Kováčik and Rákosník in 1991. As a special case of the theory of Nakano and Luxemburg, we see that $L^{p(\cdot)}$ is a quasinormed space. Especially, when $p^- \geq 1$, $L^{p(\cdot)}$ is a Banach space.

We also recall the following class of exponent function, which can be found in [12]. Let $\mathcal{B}$ be the set of $p(\cdot) \in \mathcal{P}$ such that the Hardy-littlewood maximal operator $M$ is bounded on $L^{p(\cdot)}$. An important subset of $\mathcal{B}$ is LH condition.

In the study of variable exponent function spaces it is common to assume that the exponent function $p(\cdot)$ satisfies LH condition. We say that $p(\cdot) \in LH$, if $p(\cdot)$ satisfies

$$
|p(x) - p(y)| \leq \frac{C}{-\log(|x-y|)}, \quad |x-y| \leq 1/2
$$

and

$$
|p(x) - p(y)| \leq \frac{C}{\log|x| + c}, \quad |y| \geq |x|.
$$

It is well known that $p(\cdot) \in \mathcal{B}$ if $p(\cdot) \in \mathcal{P} \cap LH$. Moreover, example shows that the above LH conditions are necessary in certain sense, see Pick and Růžička ([37]) for more details. Next we also recall the definition of variable Hardy spaces $H^{p(\cdot)}$ as follows.

**Definition 1.2.** ([11] [32]) Let $f \in \mathcal{S}^\prime$, $\psi \in \mathcal{S}$, $p(\cdot) \in \mathcal{P}^0$ and $\psi_t(x) = t^{-n} \psi(t^{-1} x)$, $x \in \mathbb{R}^n$. Denote by $\mathcal{M}$ the grand maximal operator given by $\mathcal{M}f(x) = \sup\{|\psi_t * f(x)| : t > 0, \psi \in \mathcal{F}_N\}$ for any fixed large integer $N$, where $\mathcal{F}_N = \{\varphi \in \mathcal{S} : \int \varphi(x) \, dx = 1, \sum_{|\alpha| \leq N} \sup(1 + |x|)^N |\partial^\alpha \varphi(x)| \leq 1\}$. The variable Hardy space $H^{p(\cdot)}$ is the set of all $f \in \mathcal{S}^\prime$, for which the quantity

$$
\|f\|_{H^{p(\cdot)}} = \|\mathcal{M}f\|_{L^{p(\cdot)}} < \infty.
$$

Throughout this paper, $C$ or $c$ will denote a positive constant that may vary at each occurrence but is independent to the essential variables, and $A \sim B$ means that there are constants $C_1 > 0$ and $C_2 > 0$ independent of the essential variables such that $C_1 B \leq A \leq C_2 B$. Given a measurable set $S \subset \mathbb{R}^n$, $|S|$ denotes the Lebesgue measure and $\chi_S$ means the characteristic function. For a cube $Q$, let $Q^*$ denote with the same center and $2\sqrt{n}$ its side length, i.e. $l(Q^*) = 100\sqrt{n}l(Q)$. The symbols $\mathcal{S}$ and $\mathcal{S}^\prime$ denote the class of Schwartz functions and tempered functions, respectively. As usual, for a function $\psi$ on $\mathbb{R}^n$ and $\psi_t(x) = t^{-n} \psi(t^{-1} x)$. We also use the notations $j \land j' = \min\{j, j'\}$ and $j \lor j' = \max\{j, j'\}$. Moreover, denote by $L_{comp}^\ast$ the set of all $L^2$-functions with compact support. For $L = 0, 1, 2, \ldots$, $\mathcal{P}_L$ denotes the set of all polynomials with degree less than of equal to $L$ and $\mathcal{P}_{-1} \equiv \{0\}$. The spaces $\mathcal{P}^\perp_L$ is the set of all integrable functions $f$ satisfying $\int_{\mathbb{R}^n} (1 + |x|)^L |A(x)| \, dx < \infty$ and
\[ \int_{\mathbb{R}^n} x^\alpha A(x) dx = 0 \] for all multiindices \( \alpha \) such that \( |\alpha| \leq L \). By convention, \( \mathcal{P}_{-1}^\perp \) is the set of all measurable functions. For \( L = -1, 0, 1, \ldots \), define \( L_{\text{comp}}^{q,L} = L_{\text{comp}}^q \cap \mathcal{P}_{-1}^\perp \).

In what follows, we recall the new atoms for Hardy spaces with variable exponents \( H^{p(\cdot)} \), which is introduced in [38]. Define

\[ p_- = p^- \wedge 1, \quad d_{p(\cdot)} \equiv [n/p_- - n] \vee -1 \]

for \( p \in (0, \infty) \). Let \( p(\cdot) : \mathbb{R}^n \to (0, \infty) \), \( 0 < p^- \leq p^+ \leq \infty \). Fix an integer \( d \geq d_{p(\cdot)} \) and \( 1 < q \leq \infty \). A function \( a \) on \( \mathbb{R}^n \) is called a \((p(\cdot), q)\)-atom, if there exists a cube \( Q \) such that \( \text{supp} \ a \subset Q; \|a\|_{L^q} \leq |Q|^{1/q}; \int_{\mathbb{R}^n} a(x)x^\alpha dx = 0 \) for \( |\alpha| \leq d \). Especially, the first two conditions can be replaced by \( |a| \leq \chi_Q \) when \( q = \infty \).

The atomic decomposition of Hardy spaces with variable exponents was first established independently in \([11, 32]\). Recently, the author revisited the atomic decomposition results obtained by Sawano ([38]), which extends and sharp the ones of above papers.

**Theorem 1.1.** [38] Let \( p(\cdot) \in LH \cap \mathcal{P}^0 \) and \( q > (p^+ \vee 1) \). Suppose that \( d \geq d_{p(\cdot)} \) and \( s \in (0, \infty) \). If \( f \in H^{p(\cdot)} \), there exists sequences of \((p(\cdot), \infty)\)-atoms \( \{a_j\} \) and scalars \( \{\lambda_j\} \) such that \( f = \sum_{j=1}^\infty \lambda_j a_j \) in \( H^{p(\cdot)} \cap L^q \) and that

\[ \left\| \left\{ \sum_{j=1}^\infty (\lambda_j \chi_{Q_j})^s \right\} \right\|_{L^{p(\cdot)}} \leq C \|f\|_{H^{p(\cdot)}}. \]

The multilinear fractional type operators are natural generalization of linear ones. Their earliest version was originated on the work of Grafakos ([15]) in 1992, in which he studied the multilinear fractional integral defined by

\[ I_{\alpha}(f)(x) = \int_{\mathbb{R}^n} \frac{1}{|y|^{n-\alpha}} \prod_{i=1}^m f_i(x - \theta_i y) dy, \]

where \( \theta_i (i = 1, \ldots, m) \) are fixed distinct and nonzero real numbers and \( 0 < \alpha < n \). Later on, in 1998, Kenig and Stein ([24]) established the boundedness of another type of multilinear fractional integral \( I_{\alpha,A} \) on product of Lebesgue spaces. \( I_{\alpha,A}(f) \) is defined by

\[ I_{\alpha,A}(f)(x) = \int_{\mathbb{R}^m} \frac{1}{|y_1, \ldots, y_m|^{mn-\alpha}} \prod_{i=1}^m f_i(\ell_i(y_1, \ldots, y_m, x)) dy_i, \]

where \( \ell_i \) is a linear combination of \( y_i \)'s and \( x \) depending on the matrix \( A \). In [29], Lin and Lu obtained \( I_{\alpha,A} \) is bounded from product of Hardy spaces to Lebesgue spaces when \( \ell_i(y_1, \ldots, y_m, x) = x - y_i \). We denote this multilinear fractional type integral operators by \( I_{\alpha} \), namely,

\[ I_{\alpha}(f)(x) = \int_{\mathbb{R}^m} \frac{1}{|y_1, \ldots, y_m|^{mn-\alpha}} \prod_{i=1}^m f_i(x - y_i) dy_i. \]

For convenience, we also denote \( K_{\alpha}(y_1, \ldots, y_m) = \frac{1}{|y_1, \ldots, y_m|^{mn-\alpha}}. \)
For any $1 \leq j \leq m$, we can define the commutator of multilinear operator by
\[
[b, T]_j(f)(x) := bT(f)(x) - T(f_1, \ldots, bf_j, \ldots, f_m)(x),
\]
where $b$ is a locally integral function and $T$ is a multilinear operator.
Then $[b, I^\alpha]_j(f)$ is defined by
\[
[b, I^\alpha]_j(f)(x) = \int_{\mathbb{R}^n} \frac{(b(x) - b(y_j)) \prod_{i=1}^m f_i(y_i)}{(\sum_{i=1}^m |x - y_i|)^{mn-\alpha}} \prod_{i=1}^m dy_i.
\]

In Section 2, we will show that $I^\alpha$ is bounded from product of Hardy spaces with variable exponents to Lebesgue or Hardy spaces with exponents. Then we will introduce the new atomic Hardy spaces with variable exponents $H^p_b(\cdot)$ in section 3. Moreover, we also consider continuity properties for commutators of multilinear type operators on product of the atomic Hardy spaces with variable exponents $H^p_b(\cdot)$.

2. Multilinear fractional type operators on product of Hardy spaces with variable exponents

In this section, we will discuss the boundedness of multilinear fractional type operators on product of Hardy spaces with variable exponents. The results are new even of the classical constant $\prod_{j=1}^m H^{p_j} \to H^q$ boundedness for multilinear fractional type operators. First we introduce some necessary notations and requisite lemmas. The following generalized Hölder inequality on variable Lebesgue spaces can be found in [8] or [42].

Lemma 2.1. Given exponent function $p_i(\cdot) \in \mathcal{P}^0$, define $p(\cdot) \in \mathcal{P}^0$ by
\[
\frac{1}{p(x)} = \sum_{i=1}^m \frac{1}{p_i(x)},
\]
where $i = 1, \ldots, m$. Then for all $f_i \in L^{p_i(\cdot)}$ and $f_1 \cdots f_m \in L^{p(\cdot)}$ and
\[
\| \prod_{i=1}^m f_i \|_{p(\cdot)} \leq C \prod_{i=1}^m \| f_i \|_{p_i(\cdot)}.
\]

Lemma 2.2. ([42]) Let $m \in \mathbb{N}$,
\[
\frac{1}{s(x)} = \sum_{i=1}^m \frac{1}{r_i(x)} - \frac{\alpha}{n}, x \in \mathbb{R}^n,
\]
with $0 < \alpha < mn$, $1 < r_i \leq \infty$. Then
\[
\| I^\alpha(f) \|_{s(\cdot)} \leq C \prod_{i=1}^m \| f_i \|_{r_i(\cdot)}.
\]

We also need the following boundedness of the vector-valued fractional maximal operators on variable Lebesgue spaces whose proof can be found in [13]. Let $0 \leq \alpha < n$, we define
\[
M_\alpha f(x) = \sup_{Q \ni x} \frac{1}{|Q|^{1-\alpha/n}} \int_Q |f(y)| dy.
\]
Lemma 2.3. Let $0 \leq \alpha < n$, $p(\cdot), q(\cdot) \in B$ be such that $p^+ < \frac{n}{\alpha}$ and
\[
\frac{1}{p(x)} - \frac{1}{q(x)} = \frac{\alpha}{n}, \quad x \in \mathbb{R}^n.
\]
If $q(\cdot)(n - \alpha)/n \in B$, then for any $q > 1$, $f = \{f_i\}_{i \in \mathbb{Z}}, f_i \in L_{loc}$, $i \in \mathbb{Z}$
\[
\|\|M_\alpha(f)\|_{L^q(\cdot)}\|_{\mathcal{L}} \leq C\|f\|_{L^p(\cdot)},
\]
where $M_\alpha(f) = \{M_\alpha(f_i)\}_{i \in \mathbb{Z}}$.

Lemma 2.4. Let $p(\cdot) \in \mathcal{P}$, $f \in L^p(\cdot)$ and $g \in L^{p'}(\cdot)$, then $fg$ is integrable on $\mathbb{R}^n$ and
\[
\int_{\mathbb{R}^n} |f(x)g(x)|dx \leq r_p\|f\|_{L^p(\cdot)}\|g\|_{L^{p'}(\cdot)},
\]
where $r_p = 1 + 1/p^- - 1/p^+$. Moreover, for all $g \in L^{p'}(\cdot)$ such that $\|g\|_{L^{p'}(\cdot)} \leq 1$ we get that
\[
\|f\|_{L^p(\cdot)} \leq \sup_g |\int_{\mathbb{R}^n} f(x)g(x)dx| \leq r_p\|f\|_{L^p(\cdot)}.
\]

Our first theorem is the following.

Theorem 2.5. Let $0 < \alpha < n$. Suppose that $p_1(\cdot), \ldots, p_m(\cdot) \in LH \cap \mathcal{P}^0$ and $q(\cdot) \in \mathcal{P}^0$ be Lebesgue measure functions satisfying
\[
(2.1) \quad \frac{1}{p_1(x)} + \cdots + \frac{1}{p_m(x)} - \frac{\alpha}{n} = \frac{1}{q(x)}, \quad x \in \mathbb{R}^n.
\]
Then $I_\alpha$ can be extended to a bounded operator from $\prod_{j=1}^m H^{p_j(\cdot)}$ into $L^q(\cdot)$ and can also be extended to a bounded operator from $\prod_{j=1}^m H^{p_j(\cdot)}$ into $L^q(\cdot)$.

Proof. Observe that $p_j(\cdot) \in LH \cap \mathcal{P}^0$ and choose that $\bar{q} > (p_j^+ \vee 1)$, $j = 1, \cdots, m$. By Theorem 1.1, for each $f_j \in H^{p_j(\cdot)} \cap L^q$, $j = 1, \cdots, m$, $f_j$ admits an atomic decomposition: Suppose that $d_j \geq d_{p_j(\cdot)}$ and $s \in (0, \infty)$. If $f \in H^{p_j(\cdot)}$, there exists sequences of $(p_j(\cdot), \infty)$-atoms $\{a_{j,k}\}$ and scalars $\{\lambda_{j,k}\}$ such that $f_j = \sum_{k=1}^\infty \lambda_{j,k}a_{j,k}$ in $H^{p_j(\cdot)} \cap L^q$ and that
\[
\left\|\left\{\sum_{k=1}^\infty \left(\lambda_{j,k}^sQ_j\right)^{\frac{1}{s}}\right\}^{\frac{1}{s}}\right\|_{L^{p_j(\cdot)}} \leq C\|f\|_{H^{p_j(\cdot)}}.
\]

For the decomposition of $f_j$, $j = 1, \cdots, m$, we write
\[
I_\alpha(f_j)(x) = \sum_{k_1} \cdots \sum_{k_m} \lambda_{1,k_1} \cdots \lambda_{m,k_m} I_\alpha(a_{1,k_1}, \ldots, a_{m,k_m})(x)
\]
in the sense of distributions.

Fixed $k_1, \cdots, k_m$, there are two cases for $x \in \mathbb{R}^n$.

Case 1: $x \in Q_{\cdot, k_1}^* \cap \cdots \cap Q_{\cdot, k_m}^*$.

Case 2: $x \in Q_{\cdot, k_1}^* \cup \cdots \cup Q_{\cdot, k_m}^*$.
Then we have
\[
|I_\alpha(f)(x)| \leq \sum_{k_1} \cdots \sum_{k_m} |\lambda_{1,k_1}| \cdots |\lambda_{m,k_m}| I_\alpha(a_{1,k_1}, \ldots, a_{m,k_m})(x) \chi_{Q^*_1 \cap \cdots \cap Q^*_m, k_m}(x) \\
+ \sum_{k_1} \cdots \sum_{k_m} |\lambda_{1,k_1}| \cdots |\lambda_{m,k_m}| I_\alpha(a_{1,k_1}, \ldots, a_{m,k_m})(x) \chi_{Q^*_1 \cup \cdots \cup Q^*_m, k_m}(x) \\
= I_1(x) + I_2(x).
\]

First, we consider the estimate of \(I_1(x)\). We will show that
\[
\|I_1\|_{L^q(\mathbb{R}^n)} \leq C \prod_{j=1}^m \|f_j\|_{H^{p_j}(\mathbb{R}^n)}.
\]

For fixed \(k_1, \ldots, k_m\), assume that \(Q^*_{1,k_1} \cap \cdots \cap Q^*_m, k_m \neq 0\), otherwise there is nothing to prove. Without loss of generality, suppose that \(Q^*_{1,k_1}\) has the smallest size among all these cubes. We can pick a cube \(G_{k_1, \ldots, k_m}\) such that
\[
Q^*_{1,k_1} \cap \cdots \cap Q^*_m, k_m \subset G_{k_1, \ldots, k_m} \subset G^*_{k_1, \ldots, k_m} \subset Q^{**}_{1,k_1} \cap \cdots \cap Q^{**}_m, k_m
\]

and \(|G_{k_1, \ldots, k_m}| \geq C|Q^*_{1,k_1}|\).

Denote \(H(x) := I_\alpha(a_{1,k_1}, \ldots, a_{m,k_m})(x) \chi_{Q^*_1 \cap \cdots \cap Q^*_m, k_m}(x)\). Obviously,
\[
\text{supp}H(x) \subset Q^*_{1,k_1} \cap \cdots \cap Q^*_m, k_m \subset G_{k_1, \ldots, k_m}.
\]

By Lemma 2.2 since \(I_\alpha\) maps \(L^{r_1} \times L^{\infty} \times \cdots \times L^{\infty}\) into \(L^s\) \((s > 1 \text{ and } \frac{1}{s} = \frac{1}{r_1} - \frac{\alpha}{n})\), we get that
\[
\|H\|_{L^s} \leq \|I_\alpha(a_{1,k_1}, \ldots, a_{m,k_m})\|_{L^s} \\
\leq C \|a_{1,k_1}\|_{L^{r_1}} \|a_{2,k_2}\|_{L^{\infty}} \cdots \|a_{m,k_m}\|_{L^{\infty}} \\
\leq C|Q^*_{1,k_1}|^{\frac{1}{r_1}} \chi_{Q^*_{1,k_1}} \|a_{2,k_2}\|_{L^{\infty}} \|a_{m,k_m}\|_{L^{\infty}} \\
\leq C \chi_{Q^*_{1,k_1}} \cdots \chi_{Q^*_m, k_m} Q^*_{1,k_1}^{\frac{1}{r_1}} \\
\leq C \chi_{Q^*_{1,k_1}} \cdots \chi_{Q^*_m, k_m} |G_{k_1, \ldots, k_m}|^{\frac{1}{r_1}}.
\]

For any \(g \in L^{(q')/(q-\gamma)}\) with \(\|g\|_{L^{(q')/(q-\gamma)}} \leq 1\), by Hölder inequality and (2.16) we find that
\[
\left| \int_{\mathbb{R}^n} H(x)^{q'} g(x) dx \right| \leq \|H^{q'}\|_{L^{s/q-\gamma}} \|g\|_{L^{(s/q-\gamma)}} \\
\leq C \chi_{Q^*_{1,k_1}} \cdots \chi_{Q^*_m, k_m} |G_{k_1, \ldots, k_m}|^{\frac{q'}{s}} \left( \int_{G_{k_1, \ldots, k_m}} |g(x)|^{(s/q-\gamma)'} dx \right)^{\frac{1}{(s/q-\gamma)'}}.
\]
where \((s/q^-)\)' is the conjugate of \(s/q^-\). Thus,

\[
\left| \int_{\mathbb{R}^n} H(x)^q^- g(x) \, dx \right| \\
\leq C_\chi_{Q_1, k_1 \cap \cdots \cap Q_m, k_m} \left| G_{k_1, \ldots, k_m} \right|^{1 + \frac{a_n}{n}} \left( \frac{1}{\left| G_{k_1, \ldots, k_m} \right|} \int_{G_{k_1, \ldots, k_m}} |g(x)|^{(s/q^-)'} \, dx \right)^{\frac{1}{(s/q^-)'}} \\
\leq C_\chi_{Q_1, k_1 \cap \cdots \cap Q_m, k_m} \left| G_{k_1, \ldots, k_m} \right|^{\frac{a_n}{n}} \left( \inf_{x \in \mathbb{R}^n} M(|g|^{(s/q^-)'})(x) \right)^{\frac{1}{(s/q^-)'}} \\
\leq C_\chi_{Q_1, k_1 \cap \cdots \cap Q_m, k_m} \left| G_{k_1, \ldots, k_m} \right|^{\frac{a_n}{n}} \int_{G_{k_1, \ldots, k_m}} \left( M(|g|^{(s/q^-)'})(x) \right)^{\frac{1}{(s/q^-)'}} \, dx.
\]

When \(0 < q^- \leq 1\), using Lemma 2.4 we obtain that

\[
\left| \int_{\mathbb{R}^n} \left( \sum_{k_1} \cdots \sum_{k_m} \prod_{j=1}^m \lambda_{j,k_j} \chi_{Q_1, k_1 \cap \cdots \cap Q_m, k_m} \frac{a_n}{n} \right) g(x) \, dx \right| \\
\leq \sum_{k_1} \cdots \sum_{k_m} \prod_{j=1}^m \lambda_{j,k_j} \chi_{Q_1, k_1 \cap \cdots \cap Q_m, k_m} \frac{a_n}{n} \int_{G_{k_1, \ldots, k_m}} H(x)^q^- g(x) \, dx \\
\leq C \sum_{k_1} \cdots \sum_{k_m} \prod_{j=1}^m \lambda_{j,k_j} \chi_{Q_1, k_1 \cap \cdots \cap Q_m, k_m} \frac{a_n}{n} \int_{G_{k_1, \ldots, k_m}} H(x)^q^- g(x) \, dx \\
\leq \int_{\mathbb{R}^n} \left( \sum_{k_1} \cdots \sum_{k_m} \prod_{j=1}^m \lambda_{j,k_j} \chi_{Q_1, k_1 \cap \cdots \cap Q_m, k_m} \frac{a_n}{n} \right) \left( M(|g|^{(s/q^-)'})(x) \right)^{\frac{1}{(s/q^-)'}} \, dx \\
= \int_{\mathbb{R}^n} \left( \sum_{k_1} \cdots \sum_{k_m} \prod_{j=1}^m \lambda_{j,k_j} \chi_{Q_1, k_1 \cap \cdots \cap Q_m, k_m} \frac{a_n}{n} \right) \left( M(|g|^{(s/q^-)'})(x) \right)^{\frac{1}{(s/q^-)'}} \, dx \\
\leq C \left\| \sum_{k_1} \cdots \sum_{k_m} \prod_{j=1}^m \lambda_{j,k_j} \chi_{G_{k_1, \ldots, k_m}} \frac{a_n}{n} \chi_{G_{k_1, \ldots, k_m}} \right\|_{L^q/(s/q^-)} \\
\times \left( M(|g|^{(s/q^-)'})(x) \right)^{\frac{1}{(s/q^-)'}} \right\|_{L^q/(s/q^-)} \\
\leq C \left\| \sum_{k_1} \cdots \sum_{k_m} \prod_{j=1}^m \lambda_{j,k_j} \chi_{G_{k_1, \ldots, k_m}} \frac{a_n}{n} \chi_{G_{k_1, \ldots, k_m}} \right\|_{L^q/(s/q^-)} \\
\times \left( M(|g|^{(s/q^-)'})(x) \right)^{\frac{1}{(s/q^-)'}} \right\|_{L^q/(s/q^-)}.
\]

Choose \(s\) large enough such that \((q(\cdot)/q^-)'/(s/q^-)' > 1\). Then by Hardy-Littlewood operator \(M\) is bounded on \(L^q/(s/q^-)'\) and \(||g||_{L^q/(s/q^-)'} \leq 1\), we know that

\[
\left( M(|g|^{(s/q^-)'})(x) \right)^{\frac{1}{(s/q^-)'}} \right\|_{L^q/(s/q^-)'(s/q^-)'} \leq C.
\]
Applying Lemma 2.4 again, we get that

$$\left\| \sum_{k_1} \cdots \sum_{k_m} |\lambda_{1,k_1}|^{q^{-}} \cdots |\lambda_{m,k_m}|^{q^{-}} |I_{\alpha}(a_{1,k_1}, \cdots, a_{m,k_m})(x)|^{q^{-}} \chi Q_{1,k_1}^{\alpha} \cdots \chi Q_{m,k_m}^{*} \right\|_{L^{q^{\alpha}}/q^{-}} \leq C \left\| \sum_{k_1} \cdots \sum_{k_m} \prod_{j=1}^{m} |\lambda_{j,k_j}|^{q^{-}} |G_{k_1,\ldots,k_m}|^{\frac{q^{-}}{n}} \chi G_{k_1,\ldots,k_m} \right\|_{L^{q^{\alpha}}/q^{-}}$$

$$\leq C \left\| \prod_{j=1}^{m} \sum_{k_j} |\lambda_{j,k_j}|^{q^{-}} |G_{k_1,\ldots,k_m}|^{\frac{q^{-}}{n}} \chi Q_{j,k_j}^{*} \right\|_{L^{q^{\alpha}}/q^{-}}$$

Denote $\frac{1}{q_j(x)} = \frac{1}{p_j(x)} - \frac{\alpha}{m}$ for any $x \in \mathbb{R}^n$, $j = 1, \ldots, m$. Then $q_j(\cdot) \in \mathcal{P}^0$ and for any $x \in \mathbb{R}^n$

$$\frac{1}{q(x)} = \frac{1}{q_1(x)} + \cdots + \frac{1}{q_m(x)} - \frac{\alpha}{n} = \frac{1}{q_1(x)} + \cdots + \frac{1}{q_m(x)}.$$  

By Lemma 2.1, we obtain (2.4)

$$\left\| \prod_{j=1}^{m} \sum_{k_j} |\lambda_{j,k_j}|^{q^{-}} |Q_{j,k_j}|^{\frac{q^{-}}{mn}} \chi Q_{j,k_j}^{*} \right\|_{L^{q^{\alpha}}/q^{-} L^{q^{\alpha}}/q^{-}} \leq C \left\| \sum_{j=1}^{m} \prod_{k_j} |\lambda_{j,k_j}|^{q^{-}} |Q_{j,k_j}|^{\frac{q^{-}}{mn}} \chi Q_{j,k_j}^{*} \right\|_{L^{q^{\alpha}}/q^{-}}$$

Furthermore, it is easy to verify that

$$|Q_{j,k_j}|^{\frac{q^{-}}{mn}} \chi Q_{j,k_j}^{*} (x) \leq CM_{\alpha q^{-}/2}(\chi Q_{j,k_j}^{*})^{\frac{2}{q^{-}}}(x).$$

Applying Lemma 2.6, then we get that (2.5)

$$\left\| \sum_{k_j} |\lambda_{j,k_j}|^{q^{-}} |Q_{j,k_j}|^{\frac{q^{-}}{mn}} \chi Q_{j,k_j}^{*} \right\|_{L^{q^{\alpha}}/q^{-} L^{q^{\alpha}}/q^{-}} = \left\| \left( \sum_{k_j} |\lambda_{j,k_j}|^{q^{-}} |Q_{j,k_j}|^{\frac{q^{-}}{mn}} \chi Q_{j,k_j}^{*} \right)^{q^{-}} \right\|_{L^{q^{\alpha}}/q^{-}}^{\frac{1}{q^{-}}}$$

$$\leq C \left\| \left( \sum_{k_j} |\lambda_{j,k_j}|^{q^{-}} M_{\alpha q^{-}/2}(\chi Q_{j,k_j}^{*})^{\frac{2}{q^{-}}} \right)^{q^{-}} \right\|_{L^{q^{\alpha}}/q^{-}}^{\frac{1}{q^{-}}} = C \left\| \left( \sum_{k_j} |\lambda_{j,k_j}|^{q^{-}} M_{\alpha q^{-}/2}(\chi Q_{j,k_j}^{*})^{2} \right)^{\frac{q^{-}}{2}} \right\|_{L^{q^{\alpha}}/q^{-}}^{\frac{1}{q^{-}}}$$

$$\leq C \left\| \left( \sum_{k_j} |\lambda_{j,k_j}|^{q^{-}} M_{\alpha q^{-}/2}(\chi Q_{j,k_j}^{*})^{2} \right)^{\frac{q^{-}}{2}} \right\|_{L^{q^{\alpha}}/q^{-}}^{\frac{1}{q^{-}}} \leq C \left\| \left( \sum_{k_j} |\lambda_{j,k_j}|^{q^{-}} \chi Q_{j,k_j}^{*} \right)^{\frac{q^{-}}{2}} \right\|_{L^{q^{\alpha}}/q^{-}}^{\frac{1}{q^{-}}}.$$
Applying Fefferman-Stein vector value inequality on $L^{2p_j(q^-)}$, we get that
\[ \left\| \left( \sum_{j} |\lambda_{j,k_j}|^{q^-(X_{Q_{j,k_j}}^*)} \right)^{\frac{1}{q^-}} \right\|_{L^{2p_j(q^-)}} \leq C \left\| \left( \sum_{j} |\lambda_{j,k_j}|^{q^-} M(\chi_{Q_{j,k_j}})^2 \right)^{\frac{1}{q^-}} \right\|_{L^{2p_j(q^-)}} \leq C \left\| f \right\|_{H^{p_j}(\cdot)} \]
(2.6)

Therefore, when $0 < q^- \leq 1$, by (2.4), (2.5) and (2.6) we have that
\[ \left\| I_1 \right\|_{L^q(\cdot)} = \left\| (I_1)^{q^-} \right\|_{L^{q^-}(\cdot)} \]
\[ \leq C \left\| \prod_{j=1}^{m} \sum_{k_j} |\lambda_{j,k_j}|^{q^-} |Q_{j,k_j}|^{\frac{q^-}{mn}} \chi_{Q_{j,k_j}}^* \right\|_{L^q(\cdot)} \]
\[ \leq C \left\| f \right\|_{H^{p_j}(\cdot)} \]

When $q^- > 1$, repeating similar but more easier argument, we can also get the desired result (2.2). In fact, we only need to replace $p^-$ by 1 in the above proof. We omit the detail.

Secondly, we consider the estimate of $I_2$. Let $A$ be a nonempty subset of $\{1, \ldots, m\}$, and we denote the cardinality of $A$ by $|A|$, then $1 \leq |A| \leq m$. Let $A^c = \{1, \ldots, m\} \setminus A$. If $A = \{1, \ldots, m\}$, we define
\[ (\cap_{j \in A} Q_{j,k_j}^{*,c}) \cap (\cap_{j \in A^c} Q_{j,k_j}^{*,c}) = \cap_{j \in A} Q_{j,k_j}^{*,c}, \]
then
\[ Q_{1,k_1}^{*,c} \cup \cdots \cup Q_{m,k_m}^{*,c} = \cup_{A \subset \{1, \ldots, m\}} ( (\cap_{j \in A} Q_{j,k_j}^{*,c}) \cap (\cap_{j \in A^c} Q_{j,k_j}^{*,c})). \]

Set $E_A = (\cap_{j \in A} Q_{j,k_j}^{*,c}) \cap (\cap_{j \in A^c} Q_{j,k_j}^{*,c})$. For fixed $A$, assume that $Q_{j,k_j}$ is the smallest cubes in the set of cubes $Q_{j,k_j}, j \in A$. Let $z_{j,k_j}$ is the center of the cube $Q_{j,k_j}$.

Denote $K_{\alpha}(x,y_1,\ldots,y_m) = |(x-y_1,\ldots,x-y_m)|^{-mn+\alpha}$. Notice that for all $|\beta| = d + 1, \beta = (\beta_1,\ldots,\beta_m)$
\[ |\partial_{y_1}^{\beta_1} \cdots \partial_{y_m}^{\beta_m} K_{\alpha}(x,y_1,\ldots,y_m)| \leq C |(x-y_1,\ldots,x-y_m)|^{-mn+\alpha-|\beta|}. \]
(2.7)
Since \(a_{j,k_j}\) has zero vanishing moment up to order \(d_j\), using Taylor expansion we get
\[
I_\alpha(a_{1,k_1}, \ldots, a_{m,k_m})(x) = \int_{\mathbb{R}^n} K_\alpha(x,y_1, \ldots, y_m)a_{1,k_1}(y_1) \cdots a_{m,k_m}(y_m)dy
\]
\[
= \int_{\mathbb{R}^n} \prod_{j \neq j^*} a_{j,k_j}(y_j) \int_{\mathbb{R}^n} [K_\alpha(x,y_1, \ldots, y_m) - P^d_{z_j,k_j}(x,y_1, \ldots, y_m)]a_{j,k_j}dy
\]
\[
= \int_{\mathbb{R}^n} \prod_{j \neq j^*} a_{j,k_j}(y_j) \int_{\mathbb{R}^n} \sum_{|\gamma|=d+1} (\partial^\gamma y_j K_\alpha)(x,y_1, \ldots, \xi, \ldots, y_m) \frac{(y_j - z_{j,k_j})^\gamma}{\gamma!}a_j(y_j)dy
\]
for some \(\xi\) on the line segment joining \(y_j\) to \(z_{j,k_j}\), where \(P^d_{z_{j,k_j}}(x,y_1, \ldots, y_m)\) is Taylor polynomial of \(K_\alpha(x,y_1, \ldots, y_m)\). Since \(x \in (Q^*_{j,k_j})^c\), we can easily obtain that \(|x-\xi| \geq \frac{1}{2}|x-z_{j,k_j}|\). Similarly, \(|x-y_j| \geq \frac{1}{2}|x-z_{j,k_j}|\) for \(y_j \in Q_{j,k_j}, j \in A \backslash \{j\}\).

Applying the estimate for the kernel \(K_\alpha\) satisfies \(\text{[2.1]}\) and the size estimates for the new atoms yield
\[
\int_{\mathbb{R}^n} \prod_{j \neq j^*} |a_{j,k_j}(y_j)| \int_{\mathbb{R}^n} \sum_{|\gamma|=d+1} |(\partial^\gamma y_j K_\alpha)(x,y_1, \ldots, \xi, \ldots, y_m)| \frac{|y_j - z_{j,k_j}|^\gamma}{\gamma!} |a_j(y_j)| dy
\]
\[
\leq C \int_{\mathbb{R}^n} \prod_{j \in A} |a_{j,k_j}(y_j)| \int_{\mathbb{R}^n} \sum_{|\gamma|=d+1} \frac{|y_j - z_{j,k_j}|^{d+1}}{|x-\xi| + \sum_{j \neq j^*}|x-y_j|^{mn+d+1-\alpha}} \prod_{j \in A^c} |a_{j,k_j}(y_j)| dy
\]
\[
\leq C \left(\prod_{j \in A} \|a_{j,k_j}\|_{L^1}\right)^\left(\prod_{j \in A^c} \|a_{j,k_j}\|_{L^\infty}\right) \int_{\mathbb{R}^n} \sum_{|\gamma|=d+1} \frac{|y_j - z_{j,k_j}|^{d+1}}{|x-\xi| + \sum_{j \neq j^*}|x-y_j|^{mn+d+1-\alpha-n(|A|)}} d \gamma
\]
\[
\leq C \left(\prod_{j \in A} |Q_{j,k_j}|\right)^\left(\prod_{j \in A^c} \chi_{Q_{j,k_j}}\right) \left(\sum_{j \in A} |x-z_{j,k_j}|^{mn+d+1-\alpha-n(|A|)}\right)^\left(\prod_{j \in A^c} Q_{j,k_j}^{(d+1)/n}\right) \left(\sum_{j \in A} |x-z_{j,k_j}|^{mn+d+1-\alpha-n(|A|)}\right)
\]

Observe that \(x \in \cap_{j \in A} Q_{j,k_j}^c\), then we can find constant \(C\) such that \(|x - z_{j,k_j}| \geq C(|x - z_{j,k_j}| + l(Q_{j,k_j}))\). On the other hand, using the fact that \(x \in \cap_{j \in A^c} Q_{j,k_j}^c\) yields that there exists a constant \(C\) such that \(|x - z_{j,k_j}| \leq Cl(Q_{j,k_j})\) for \(j \in A^c\). Then we
have that
\[
\frac{|Q_{j,k_j}|^{1+\frac{d+1}{n|A|}}}{(|x - z_{j,k_j}| + l(Q_{j,k_j}))^{n+\frac{d+1}{|A|}}} \geq C, \quad \text{for} \quad j \in A^c.
\]

Moreover, since \(Q_{j,k_j}\) is the smallest cube among \(\{Q_{j,l_j}\}_{j \in A}\), we have that
\[
|Q_{j,k_j}| \leq \prod_{j \in A} |Q_{j,l_j}|^{\frac{1}{|A|}}.
\]
Thus,
\[
\left|I_2(a_{1, k_1}, \ldots, a_{m, k_m})(x)\right| \leq C \left( \prod_{j \in A} \frac{|Q_{j,k_j}|^{1+\frac{d+1}{n|A|}}}{(|x - z_{j,k_j}| + l(Q_{j,k_j}))^{n+\frac{d+1}{|A|}}\chi_{Q_{j,k_j}}} \prod_{j \in A^c} |Q_{j,k_j}|^{1+\frac{d+1}{n|A|}} \right)
\]
for all \(x \in E_A\).

Then applying the generalized Hölder’s inequality in variable Lebesgue spaces and Fefferman-Stein inequality in Lemma 2.3, we obtain the estimate
\[
\|I_2\|_{L^q(\cdot)} \leq C \left( \sum_{k_1} \cdots \sum_{k_m} \prod_{j=1}^m |\lambda_{j,k_j}| \sum_{A \subset \{1, \ldots, m\}} \prod_{j \in A} \frac{|Q_{j,k_j}|^{1+\frac{d+1}{n|A|}}}{(|x - z_{j,k_j}| + l(Q_{j,k_j}))^{n+\frac{d+1}{|A|}}\chi_{E_A}} \prod_{j \in A^c} |Q_{j,k_j}|^{1+\frac{d+1}{n|A|}} \right),
\]
for \(x \in \mathbb{R}^n\), denote \(\frac{1}{s_j(x)} = \frac{1}{p_j(x)} - \frac{\alpha}{n|A|}, \ j \in A\). Then \(0 < r(x) < \infty\) and
\[
\frac{1}{q(x)} = \sum_j \frac{1}{p_j(x)} - \frac{\alpha}{n} = \sum_{j \in A} \frac{1}{s_j(x)} + \sum_{j \in A^c} \frac{1}{p_j(x)}.
\]
For convenience, we denote that
\[
U_A = \sum_{k_j} |\lambda_{j,k_j}| \frac{|Q_{j,k_j}|^{1+\frac{d+1}{n|A|}}}{(|x - z_{j,k_j}| + l(Q_{j,k_j}))^{n+\frac{d+1}{|A|}}\chi_{E_A}}.
\]
and

\[ U_{A^c} = \sum_{k_j} |\lambda_{j,k_j}| \frac{|Q_{j,k_j}|^{1 + \frac{d+1}{n+1}}}{(|x - z_{j,k_j}| + l(Q_{j,k_j}))^{n+\frac{d+1}{n+1}}}. \]

Applying (2.9) and the generalized Hölder inequality with variable exponents \( s_j(\cdot) \), \( j \in A \), \( p_j(\cdot) \), \( j \in A^c \) and \( q(\cdot) \) yield that

\[
\|I_2\|_{L^q(\cdot)} \leq C \sum_{A \subseteq \{1, \ldots, m\}} \left\| \prod_{j \in A} U_{A^c} \chi_{E_A} \right\|_{L^q(\cdot)} \leq C \sum_{A \subseteq \{1, \ldots, m\}} \left( \prod_{j \in A} \|U_{A^c} \chi_{E_A}\|_{L^{s_j}(\cdot)} \right) \left( \prod_{j \in A^c} \|U_{A^c} \chi_{E_A}\|_{L^{p_j}(\cdot)} \right).
\]

(2.10)

Denote \( \theta = \frac{n}{n+\frac{d+1}{n+1}} \) and we can choose \( d \) large enough such that \( \theta p_j^- > 1 \) and \( \theta s_j^- > 1 \). Notice that \( \frac{1}{\theta s_j(x)} = \frac{1}{\theta p_j(x)} - \frac{\alpha|A|}{n} \). By Lemma 2.3 we get that

\[
\prod_{j \in A} \|U_{A^c} \chi_{E_A}\|_{L^{s_j}(\cdot)} = \prod_{j \in A} \left\| \sum_{k_j} |\lambda_{j,k_j}| \frac{l(Q_{j,k_j})^{\frac{d+1}{n+1}}}{(|x - z_{j,k_j}| + l(Q_{j,k_j}))^{n+\frac{d+1}{n+1} - \frac{\alpha|A|}{n}}} \right\|_{L^{s_j}(\cdot)} \leq C \prod_{j \in A} \left( \sum_{k_j} |\lambda_{j,k_j}| (M_{\alpha/\theta|A|} \chi_{Q_{j,k_j}})^{1/\theta} \right)^{\frac{1}{\theta}} \leq C \prod_{j \in A} \left( \|f_j\|_{H^{p_j}(\cdot)} \right),
\]

(2.11)

where the first inequality follows from the following claim which can be proved easily: For any \( x \in \mathbb{R}^n \) and \( 0 \leq \alpha < \infty \), there exists a constant \( C \) such that

\[
\frac{r^\alpha}{(r + |x - y|)^n} \leq C(M_{\alpha} \chi_{Q(y,r)})(x),
\]

where \( Q(y,r) \) is a cube centered in \( y \) and \( r \) its side length.

Repeating the similar argument to (2.11) with \( \alpha = 0 \), we get that
Therefore, for any $f_j \in H^{p_j(\cdot)} \cap L^{pj_j+1}$ by the estimates (2.2), (2.10), (2.11) and (2.12) then we have

$$
\|L_{\alpha}(\mathcal{F})\|_{\mathcal{L}(\cdot)} \leq C \prod_{j=1}^{m} \|f_j\|_{H^{p_j(\cdot)}}.
$$

(2.13)

From Remark 4.12 in [32], we have that $H^{p_j(\cdot)} \cap L^{\bar{p}_j}$ is dense in $H^{p_j(\cdot)}$. Thus, by the density argument we prove the first part of Theorem 2.5.

We now consider the boundedness of multilinear fractional type integral operators from product Hardy spaces with variable exponents $\prod_{j=1}^{m} H^{p_j(\cdot)}$ to Hardy spaces with variable exponents $H^{q(\cdot)}$.

We begin with the equivalent definition of variable Hardy spaces. For $\psi \in \mathcal{S}$, write

$$
\psi^j(x) = 2^{jn_0} \psi(2^j x).
$$

We define the discrete maximal function with respect to $\psi$ by

$$
M_{\psi}f(x) = \sup_{j \in \mathbb{Z}} |\psi^j \ast f(x)|.
$$

We choose $\psi \in C^\infty_c$ such that $\chi_{Q(0,1)} \leq \psi \leq \chi_{Q(0,2)}$. Then it is proved in [32] that

$$
\|f\|_{H^{p(\cdot)}} \sim \|M_{\psi}\|_{L^{p(\cdot)}}.
$$

We follow the similar arguments as above. For any $f_j \in H^{p_j(\cdot)} \cap L^{pj_j+1}$, we have

$$
M_{\psi} \circ I_\alpha(f_1, \ldots, f_m)(x) \leq \sum_{k_1, \ldots, k_m} \prod_{j=1}^{m} \lambda_{j,k_j} M_{\psi} \circ I_\alpha(a_{1,k_1}, \ldots, a_{m,k_m})(x),
$$

(2.14)

in $H^{p_j(\cdot)} \cap L^{pj_j+1}$. 

To complete our proof, it now suffices to establish the following result: for any \(0 < s < \infty\)

\[
\left\| \sum_{k_1, \ldots, k_m} \prod_{j=1}^{m} \lambda_{j,k_j} M \circ I_\alpha (a_{1,k_1}, \ldots, a_{m,k_m}) \right\|_{L^{q/\epsilon}} \leq C \prod_{j=1}^{m} \left\| \left( \sum_{k_j} |\lambda_{j,k_j}|^s \chi_{Q_{j,k_j}} \right)^{1/s} \right\|_{L^{p_j/\epsilon}}.
\]  

(2.15)

Thus, in view of (2.14) and (2.15), one obtains the required estimate

\[
\| I_\alpha (\vec{f}) \|_{H^{q/\epsilon}} \leq \| M \circ I_\alpha (\vec{f}) \|_{H^{q/\epsilon}} \leq C \prod_{j=1}^{m} \| f_j \|_{H^{p_j/\epsilon}}.
\]

Now we focus on the proof of (2.15). By the triangle inequality and the maximal operator, we have

\[
\left| \psi^j \ast \sum_{k_1, \ldots, k_m} \prod_{j=1}^{m} \lambda_{j,k_j} I_\alpha (a_{1,k_1}, \ldots, a_{m,k_m}) (x) \right|
\leq \sum_{k_1, \ldots, k_m} \left| \prod_{j=1}^{m} \lambda_{j,k_j} \psi^j \ast I_\alpha (a_{1,k_1}, \ldots, a_{m,k_m}) (x) \right|
\leq \sum_{k_1, \ldots, k_m} \left| \prod_{j=1}^{m} \lambda_{j,k_j} \left( M \circ I_\alpha (a_{1,k_1}, \ldots, a_{m,k_m}) (x) \chi_{G_{k_1}^{**} \ldots k_m} (x) \right) + \left| \psi^j \ast I_\alpha (a_{1,k_1}, \ldots, a_{m,k_m}) (x) \chi_{E \setminus G_{k_1}^{**} \ldots k_m} (x) \right) \right|
\]

By repeating the similar argument using in (2.8) and [32 Theorem 5.2] together with the moment condition of \(I_\alpha (a_1, \ldots, a_m)\), that is,

\[
\int_{\mathbb{R}^n} x^\gamma I_\alpha (a_{1,k_1}, \ldots, a_{m,k_m}) (x) \, dx = 0
\]

for any \(\gamma \in \mathbb{N}^n\) satisfying \(|\gamma| \leq d\) in view of [15], we have we have

\[
\left| \psi^j \ast I_\alpha (a_{1,k_1}, \ldots, a_{m,k_m}) (x) \right| \chi_{E_A} (x)
\leq C \prod_{j \in A} \frac{|Q_{j,k_j}|^{1 + \frac{d+1}{n+A}}}{(|x - z_{j,k_j}| + l(Q_{j,k_j}))^{n + \frac{d+1}{n+A}}} \prod_{j \in A^c} \frac{|Q_{j,k_j}|^{1 + \frac{d+1}{n+A}}}{(|x - z_{j,k_j}| + l(Q_{j,k_j}))^{n + \frac{d+1}{n+A}}}.
\]
Since $I_\alpha$ maps $L^{r_1} \times L^\infty \times \cdots \times L^\infty$ into $L^{s'}$ ($s > 1$ and $\frac{1}{s} = \frac{1}{r_1} - \frac{a}{n}$), applying the boundedness of $M$ on $L^s$ yields that

\[
\| M \circ I_\alpha(a_{1,k_1}, \ldots, a_{m,k_m}) \|_{L^s} \\
\leq \| I_\alpha(a_{1,k_1}, \ldots, a_{m,k_m}) \|_{L^s} \\
\leq C \| a_{1,k_1} \|_{L^{r_1}} \| a_{2,k_2} \|_{L^\infty} \cdots \| a_{m,k_m} \|_{L^\infty} \\
\leq C \chi_{Q_{1,k_1} \cap \cdots \cap Q_{m,k_m}} |G_{k_1, \ldots, k_m}|^{\frac{1}{r_1}}.
\]

The rest of the proof is similar to the proof of (2.13). Thus, this completes the proof of Theorem 2.5. □

3. Commutators of multilinear fractional type operators on certain Hardy spaces with variable exponents

In this section, we will study continuity properties of commutators of multilinear fractional type operators on product of certain Hardy spaces with variable exponents. The results are even new for the linear case in the variable exponents setting. First we introduce a new atomic Hardy space with variable exponent $H_{b}^{p(\cdot)}$.

**Definition 3.1.** Let $b$ be a locally integrable function and $d \gg 1$. It is said that a bounded function $a$ is a $(p(\cdot), b, d, \infty)-$atom if it satisfies

1. $\text{supp} \ a \subset Q = Q(x_0, r)$ for some $r > 0$;
2. $\|a\|_{\infty} \leq \frac{1}{\|\chi_Q\|_{L^{p(\cdot)}}}$;
3. $\int_{\mathbb{R}^n} a(x)x^{\alpha}dx = \int_{\mathbb{R}^n} a(x)b(x)x^{\alpha}dx = 0$ for any $|\alpha| \leq d$.

A temperate distribution $f$ is said to belong to the atomic Hardy space $H_{b}^{p(\cdot)}$, if it can be written as

$$f = \sum_{j=1}^{\infty} \lambda_j a_j,$$

in $S'$--sense, where $a$ is a $(p(\cdot), b, d, \infty)-$atom and

$$\mathcal{A}(\{\lambda_j\}_{j=1}^{\infty}, \{Q_j\}_{j=1}^{\infty}) := \left\| \left\{ \sum_j \left( \frac{\|\lambda_j\|_{\chi_{Q_j}}}{\|\chi_{Q_j}\|_{L^{p(\cdot)}}} \right)^{\frac{1}{p^{-}}} \right\}_{L^{p(\cdot)}} \right\| < \infty.$$

Moreover, we define the quasinorm on $H_{b}^{p(\cdot)}$ by

$$\|f\|_{H_{b}^{p(\cdot)}} = \inf \mathcal{A}(\{\lambda_j\}_{j=1}^{\infty}, \{Q_j\}_{j=1}^{\infty}).$$

where the infimum is taken over all admissible expressions as in $f = \sum_{j=1}^{\infty} \lambda_j a_j$.

Obviously, the new atomic Hardy space with variable exponent $H_{b}^{p(\cdot)}$ is a subspace of the Hardy space with variable exponent $H^{p(\cdot)}$. When $p(\cdot) = \text{constant}$, the spaces $H_{b}^{1}$ and $H_{b}^{p}$ first appeared in [35] and [1]. Now we state our second results.
Theorem 3.1. Let $0 < \alpha < n$ and $b \in BMO$. Suppose that $p_1(\cdot), \ldots, p_m(\cdot) \in LH \cap P^0$ and $q(\cdot) \in P^0$ be Lebesgue measure functions satisfying

\begin{equation}
\frac{1}{p_1(x)} + \cdots + \frac{1}{p_m(x)} - \frac{\alpha}{n} = \frac{1}{q(x)}, \quad x \in \mathbb{R}^n.
\end{equation}

Then for any $1 \leq j \leq m$ the operator $[b, I_{\alpha}]_j$ can be extended to a bounded operator from $\prod_{i=1}^{m} H_{b_i}^{p_i(\cdot)}$ into $L^{q(\cdot)}$ which satisfies the norm estimate

$$
\| [b, I_{\alpha}]_j (\vec{f}) \|_{L^{q(\cdot)}} \leq C \| b \|_{BMO} \prod_{i=1}^{m} \| f_i \|_{H_{b_i}^{p_i(\cdot)}}.
$$

Proof The idea of the proof is similar to Theorem 2.5. We only show the differences. For each $f_i \in H_{b_i}^{p_i(\cdot)}$, $i = 1, \ldots, m$, $f_i = \sum_{k_i=1}^{\infty} \lambda_{i,k_i} a_i$, where $a_i$ is a $(p_i(\cdot), b, d_i, \infty)$-atom $(d_i \geq d_{p_i(\cdot)})$ and

$$
\| f_i \|_{H_{b_i}^{p_i(\cdot)}} = \inf A(\{\lambda_{i,k_i}\}_{i=1}^{\infty}, \{Q_{i,k_i}\}_{i=1}^{\infty}).
$$

For the decomposition of $f_i$, $i = 1, \ldots, m$, we can write

$$
[b, I_{\alpha}]_j (\vec{f}) (x) = \sum_{k_1} \cdots \sum_{k_m} \lambda_{1,k_1} \cdots \lambda_{m,k_m} [b, I_{\alpha}]_j (a_{1,k_1}, \ldots, a_{m,k_m})(x)
$$
in the sense of distributions.

We follow the standard argument in the previous chapter. Fixed $k_1, \ldots, k_m$, there are two cases for $x \in \mathbb{R}^n$.

Case 1: $x \in Q_{1,k_1}^* \cap \cdots \cap Q_{m,k_m}^*$.

Case 2: $x \in Q_{1,k_1}^* \cup \cdots \cup Q_{m,k_m}^*$.

Then we have

$$
\| [b, I_{\alpha}]_j (\vec{f}) (x) \|
\leq \sum_{k_1} \cdots \sum_{k_m} |\lambda_{1,k_1}| \cdots |\lambda_{m,k_m}| \| [b, I_{\alpha}]_j (a_{1,k_1}, \ldots, a_{m,k_m})(x) \| \chi_{Q_{1,k_1}^* \cap \cdots \cap Q_{m,k_m}^*}(x)

+ \sum_{k_1} \cdots \sum_{k_m} |\lambda_{1,k_1}| \cdots |\lambda_{m,k_m}| \| [b, I_{\alpha}]_j (a_{1,k_1}, \ldots, a_{m,k_m})(x) \| \chi_{Q_{1,k_1}^* \cup \cdots \cup Q_{m,k_m}^*}(x)

= J_1(x) + J_2(x).
$$

Now let us discuss the term $J_1(x)$. We will show that

\begin{equation}
\| J_1 \|_{L^{q(\cdot)}} \leq C \| b \|_{BMO} \prod_{j=1}^{m} \| f_j \|_{H_{b_j}^{p_j(\cdot)}}.
\end{equation}

For fixed $k_1, \ldots, k_m$, assume that $Q_{1,k_1}^* \cap \cdots \cap Q_{m,k_m}^* \neq 0$, otherwise there is nothing to prove. Without loss of generality, suppose that $Q_{1,k_1}^*$ has the smallest size among all these cubes. We can pick a cube $G_{k_1, \ldots, k_m}$ such that

$$
Q_{1,k_1}^* \cap \cdots \cap Q_{m,k_m}^* \subset G_{k_1, \ldots, k_m} \subset C_{k_1, \ldots, k_m}^* \subset Q_{1,k_1}^{**} \cap \cdots \cap Q_{m,k_m}^{**}
$$
and $|G_{k_1, \ldots, k_m}| \geq C|Q_{1,k_1}|$. 

Denote $M(x) := \|b, I_\alpha \|_j(a_{1,k_1}, \ldots, a_{m,k_m})(x) \chi_{Q_{i,k_1}^c \cap \cdots \cap Q_{m,k_m}^c}(x)$. Obviously,

$$\text{supp}M(x) \subset Q_{i,k_1}^c \cap \cdots \cap Q_{m,k_m}^c \subset G_{k_1, \ldots, k_m}.$$

By [42, Theorem 1.1], we have that $[b, I_\alpha]_j$ maps $L^1 \times L^\infty \times \cdots \times L^\infty$ into $L^s$ $(s > 1)$ and $\frac{1}{s} = \frac{1}{r_1} - \frac{2}{n}$.

Then we get that

$$\|M\|_{L^s} \leq \|b, I_\alpha \|_j(a_{1,k_1}, \ldots, a_{m,k_m})\|_{L^s} \leq C\|b\|_{BMO}\|a_{1,k_1}\|_{L^{r_1}}\|a_{2,k_2}\|_{L^{r_2}} \cdots \|a_{m,k_m}\|_{L^{r_m}} \leq C\|b\|_{BMO} \prod_{i=1}^m \|\chi_{Q_{i,k_i}}\|_{L^{p_i}(\cdot)}^\frac{1}{p_i}. $$

Repeating similar argument to the proof of (2.2) in Theorem 2.5, we can obtain the desired result (3.2).

Secondly, we consider the estimate of $J_2$. When $x \in \chi_{Q_{1,k_1}^c \cup \cdots \cup Q_{m,k_m}^c}(x)$, the $L^{q(\cdot)}$ norm of $J_2$ is controlled by

$$\left\| \sum_{k_1} \cdots \sum_{k_m} |\lambda_{1,k_1} \cdots \lambda_{m,k_m}|(b(x) - b_{Q,j,k_j})I_\alpha(a_{1,k_1}, \ldots, a_{m,k_m}) \right\|_{L^{q(\cdot)}} + \left\| \sum_{k_1} \cdots \sum_{k_m} |\lambda_{1,k_1} \cdots \lambda_{m,k_m}|I_\alpha(a_{1,k_1}, \ldots, (b - b_{Q,j,k_j})a_{j,k_j}, \ldots, a_{m,k_m}) \right\|_{L^{q(\cdot)}}$$

$$=: J_{21} \|L^{q(\cdot)}\| + \|J_{22}\|_{L^{q(\cdot)}}.$$

Let $A$ be a nonempty subset of $\{1, \ldots, m\}$, and we denote the cardinality of $A$ by $|A|$, then $1 \leq |A| \leq m$. Let $A^c = \{1, \ldots, m\} \backslash A$. If $A = \{1, \ldots, m\}$, we define

$$\big(\cap_{i \in A} Q_{i,k_i}^c \big) \cap \big(\cap_{i \in A^c} Q_{i,k_i}^c \big) = \cap_{i \in A} Q_{j,k_i}^c$$

then

$$Q_{1,k_1}^c \cup \cdots \cup Q_{m,k_m}^c = \bigcup_{A \subset \{1, \ldots, m\}} \big(\cap_{i \in A} Q_{i,k_i}^c \big) \cap \big(\cap_{i \in A^c} Q_{i,k_i}^c \big).$$

Set $E_A = (\cap_{i \in A} Q_{i,k_i}^c) \cap (\cap_{i \in A^c} Q_{i,k_i}^c)$. For fixed $A$, assume that $Q_{i,k_i}$ is the smallest cubes in the set of cubes $Q_{i,k_i}, i \in A$. Let $\tilde{z}_{i,k_i}$ is the center of the cube $Q_{i,k_i}$.

From Definition 3.1 and (2.8), we can easily get that

$$\|I_\alpha(a_{1,k_1}, \ldots, a_{m,k_m})(x)\| \leq C\prod_{i \in A} \|\chi_{Q_{i,k_i}}\|_{L^{p_i}(\cdot)}\left(|x - z_{i,k_i}| + l(Q_{i,k_i})\right)^{n + \frac{d+1}{|A|} - \frac{d}{|A|}} \prod_{i \in A^c} \|\chi_{Q_{i,k_i}}\|_{L^{p_i}(\cdot)}\left(|x - z_{i,k_i}| + l(Q_{i,k_i})\right)^{n + \frac{d+1}{|A|} - \frac{d}{|A|}} =: C\prod_{i \in A} U_i \prod_{i \in A^c} V_i$$

for all $x \in E_A$. 

For \( x \in \mathbb{R}^n \), denote \( \frac{1}{s_i(x)} = \frac{1}{p_i(x)} - \frac{\alpha}{n|A|} \), \( i \in A \). Then \( 0 < r(x) < \infty \) and

\[
\frac{1}{q(x)} = \sum_i \frac{1}{p_i(x)} - \frac{\alpha}{n} = \sum_{i \in A} \frac{1}{s_i(x)} + \sum_{i \in A^c} \frac{1}{p_i(x)}.
\]

We will discuss in two case to estimate \( J_2 \). When \( j \in A \), by generalized Hölder’s inequality in variable Lebesgue spaces we have

\[
\|J_{21}\|_{L^q(\mathbb{R}^n)} \leq \left\| \prod_{k_1} \sum_{k_m} |\lambda_{1,k_1} \cdots \lambda_{m,k_m}| (b(x) - b_{Q_{j,k_1}}) \prod_{i \in A} U_i \prod_{i \in A^c} V_i \chi_{E_A} \right\|_{L^q(\mathbb{R}^n)}
\]

\[
\leq \prod_{i \in A, i \neq j} \left\| \sum_{k_1} |\lambda_{i,k_1}| U_i \chi_{E_A} \right\|_{L^{q_i}(\mathbb{R}^n)} \left\| \sum_{k_j} |\lambda_{j,k_j}| (b(x) - b_{Q_{j,k_j}}) U_j \chi_{E_A} \right\|_{L^{q_j}(\mathbb{R}^n)}
\]

\[
\times \prod_{i \in A^c} \left\| \sum_{k_1} |\lambda_{i,k_1}| V_i \chi_{E_A} \right\|_{L^{q_i}(\mathbb{R}^n)}.
\]

We follow the similar argument in the previous chapter again. Denote \( \theta = \frac{n + \frac{d}{A^1}}{n} \) and we can choose \( d \) large enough such that \( \theta p_j^- > 1 \) and \( \theta s_j^- > 1 \). Notice that

\[
\frac{1}{\theta s_j(x)} = \frac{1}{\theta p_j(x)} - \frac{\alpha |\theta| A}{n}. \]

By Lemma 2.3 we get that

\[
(3.4) \quad \prod_{i \in A, i \neq j} \left\| \sum_{k_1} |\lambda_{i,k_1}| U_i \chi_{E_A} \right\|_{L^{q_i}(\mathbb{R}^n)} \leq C \prod_{i \in A, i \neq j} \left\| \left( \sum_{k_1} |\lambda_{i,k_1}| (M_{\alpha/|\theta| A} \chi_{Q_{i,k_1}})^{\theta} \right)^{1/\theta} \right\|_{L^{\theta s_i}(\mathbb{R}^n)}
\]

\[
\leq \prod_{i \in A, i \neq j} \left\| \left( \sum_{k_1} |\lambda_{i,k_1}| \chi_{Q_{i,k_1}} \right)^{1/\theta} \right\|_{L^{\theta s_i}(\mathbb{R}^n)} \leq C \prod_{i \in A, i \neq j} \|f_j\|_{H^{p_j}(\mathbb{R}^n)}.
\]

Repeating the similar but easier argument to (2.11) with \( \alpha = 0 \), we get that

\[
(3.5) \quad \prod_{i \in A^c} \left\| \sum_{k_1} |\lambda_{i,k_1}| V_i \chi_{E_A} \right\|_{L^{q_i}(\mathbb{R}^n)} \leq C \prod_{i \in A^c} \|f_i\|_{H^{p_i}(\mathbb{R}^n)}.
\]

Next we need to prove

\[
(3.6) \quad \left\| \sum_{k_j} |\lambda_{j,k_j}| (b(x) - b_{Q_{j,k_j}}) U_j \chi_{E_A} \right\|_{L^{q_j}(\mathbb{R}^n)} \leq C \|f_j\|_{H^{p_j}(\mathbb{R}^n)}.
\]

For any constants \( s, s_i > 1, \ i = 1, 2, 3 \), denote that \( \frac{1}{s} = \frac{1}{s_1} + \frac{1}{s_2} = \frac{1}{s_1} + \frac{1}{s_2} - \frac{\alpha}{n|A|} =: \frac{1}{s_3} - \frac{\alpha}{n|A|}. \)
Applying Hölder’s inequality yields that
\[
\left\| (b(x) - b_{Q_{j,k_j}}) U_j \chi_{E_A} \right\|_{L^s} \leq C \left\| (b(x) - b_{Q_{j,k_j}}) U_j \chi_{E_A} \right\|_{L^{s_1}} \left\| U_j \chi_{E_A} \right\|_{L^{s_2}}
\]
\[
\leq C \|b\|_{BMO} |Q|^{1/s_1} \left\| (M_{\alpha/\theta} A) \chi_{Q_{i,k_i}} \right\|_{L^{s_2}}
\]
\[
\leq C \|b\|_{BMO} |Q|^{1/s_1} |Q|^{1/r_2}
\]
\[
\leq C \|b\|_{BMO} |Q|^{1/s_3},
\]
where the second estimate comes from the John-Nirenberg inequality.

Now we resort to the proof of (2.2) to have the desired results (3.6).

Therefore, when \( j \in A \), by the estimates (3.4), (3.5) and (3.6) we have
\[
\left\| J_{21} \right\|_{L^q} \leq C \prod_{i=1}^m \| f_i \|_{H^p_j}.
\]

When \( j \in A^c \), we similarly have
\[
\left\| J_{21} \right\|_{L^q} \leq \prod_{i \in A} \left\| \sum_{k_i} |\lambda_{i,k_i}| U_i \chi_{E_A} \right\|_{L^{s_i}()} \left\| \sum_{k_j} |\lambda_{j,k_j}| (b(x) - b_{Q_{j,k_j}}) V_j \chi_{E_A} \right\|_{L^{p_j}()}
\]
\[
\times \prod_{i \in A^c; i \neq j} \left\| \sum_{k_i} |\lambda_{i,k_i}| V_i \chi_{E_A} \right\|_{L^{p_i}()}.
\]

We only need to observe that
\[
\left\| (b(x) - b_{Q_{j,k_j}}) V_j \chi_{E_A} \right\|_{L^s} \leq C \left\| (b(x) - b_{Q_{j,k_j}}) U_j \chi_{E_A} \right\|_{L^s} \left\| U_j \chi_{E_A} \right\|_{L^\infty}
\]
\[
\leq C \|b\|_{BMO} |Q|^{1/s} \left\| (M \chi_{Q_{i,k_i}})^\theta \right\|_{L^\infty}
\]
\[
\leq C \|b\|_{BMO} |Q|^{1/s}.
\]

Similarly, we can get (3.7).

Let us estimate \( J_{22} \). Observe that \( \| a_{i,k_i} \|_{\infty} \leq \frac{1}{\| \chi_{Q_{i,k_i}} \|_{L^{p_i}()}} \) and \( \int_{\mathbb{R}^n} a_{i,k_i} (x) x^\alpha dx = 0 \) for any \( |\alpha| \leq d \).

As the argument for (2.8), we similarly have that
\[
|I_{\alpha}(a_{1,k_1}, \ldots, (b - b_{Q_{j,k_j}}) a_{j,k_j}, \ldots, a_{m,k_m})| \leq C \|b\|_{BMO} \prod_{i \in A} \left\| \chi_{Q_{i,k_i}} \right\|_{L^{p_i}()} (|x - z_{i,k_i}| + l(Q_{t,k_i}))^{\frac{d+1}{q}} \left( |x - z_{i,k_i}| + l(Q_{t,k_i}) \right)^{\frac{d+1}{q}} \frac{\alpha}{|\alpha|}.
\]

\[
\times \prod_{i \in A^c} \left\| \chi_{Q_{i,k_i}} \right\|_{L^{p_i}()} (|x - z_{i,k_i}| + l(Q_{t,k_i}))^{\frac{d+1}{q}} \left( |x - z_{i,k_i}| + l(Q_{t,k_i}) \right)^{\frac{d+1}{q}}
\]
for all $x \in E_A$. The rest of the proof is same as the above. Then we have

$$
(3.8) \quad \|J_{22}\|_{L^q(\cdot)} \leq C \prod_{i=1}^{m} \|f_i\|_{H^p_b(\cdot)}.
$$

In conclusion, combing 3.2, 3.7 and 3.8 we obtain that

$$
\|[b, I^\alpha_j](\vec{f})\|_{L^q(\cdot)} \leq C \|b\|_{BMO} \prod_{i=1}^{m} \|f_i\|_{H^p_b(\cdot)}.
$$

We completed the proof of Theorem 3.1. \hfill \Box

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