Large-scale conformal rigidity in dimension three

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1 Introduction

Let $X$ be a noncompact smooth manifold. Two complete Riemannian metrics on $X$ in a given conformal class can have very different asymptotic geometries. For instance, starting with the Euclidean plane $\mathbb{E}^2$ with polar coordinates $(r, \theta)$, multiplying the Euclidean metric $g_0 = dr^2 + r^2 d\theta^2$ by a function equal to $1/r^2$ outside a compact neighborhood of the origin, one obtains a complete Riemannian metric $g$ which is quasi-isometric to a half-line. The metrics $g$ and $g_0$ are in the same conformal class, but they are not quasi-isometric. In fact, they have different asymptotic dimensions.

One can ask whether something similar can happen to two finitely generated groups $\Gamma$ and $\Gamma_0$: can they be 'coarsely quasi-conformal' in some sense and yet not quasi-isometric?

To give a precise meaning to this question, we will choose Riemannian manifolds $X_0, X$ that are geometric models for $\Gamma_0, \Gamma$ and consider conformal mappings between $X_0$ and $X$. First let us recall some definitions. Two metric spaces $(X_1, d_1)$ and $(X_2, d_2)$ are quasi-isometric if there are constants $\lambda \geq 1$ and $C \geq 0$ and a map $f : X_1 \to X_2$ satisfying :

$$\lambda^{-1} d_1(x, x') - C \leq d_2(f(x), f(x')) \leq \lambda d_1(x, x') + C \quad \forall x, x' \in X_1$$

$$\forall y \in X_2, \exists x \in X_1, d(f(x), y) \leq C.$$
If a group $\Gamma$ is finitely generated by a subset $S$, one can make $\Gamma$ into a metric space by means of the word metric associated to $S$. The quasi-isometry class of this metric does not depend on the choice of $S$. Thus one can omit the mention of $S$ when discussing quasi-isometries between groups and metric spaces.

Let $\Gamma$ be a group and $X$ a metric space. A geometric action of $\Gamma$ on $X$ is a proper, cocompact action of $\Gamma$ on $X$ by isometries. We are interested in the case where $X$ is a Riemannian manifold. The Riemannian manifolds we consider will always be complete and of bounded geometry. For the purpose of this paper, it is convenient to take the following definition: a Riemannian manifold $X$ has bounded geometry if there is a number $\epsilon > 0$ and a compact Riemannian manifold $Y$ such that all balls of radius $\epsilon$ in $X$ can be isometrically embedded into $Y$. It is conformally flat if every point has a neighborhood conformal to a ball in Euclidean space.

A special case of the “fundamental observation of geometric group theory” ([3, Proposition 8.19]) is that if a group $\Gamma$ acts geometrically on a complete Riemannian manifold $X$, then $\Gamma$ is finitely generated and quasi-isometric to $X$. Furthermore, such a $X$ is clearly of bounded geometry.

We can now state our problem more formally: let $\Gamma_0$ be a finitely generated group acting geometrically on a complete Riemannian manifold $X_0$. The ‘coarse quasi-conformal’ deformations we are looking for are pairs $(\Gamma, X)$ where $\Gamma$ is a finitely generated group and $X$ a complete Riemannian manifold of bounded geometry quasi-isometric to $\Gamma$ and conformal to $X_0$. We regard a deformation as trivial if all spaces involved are quasi-isometric. Since the existence of nontrivial deformations depends only on $X_0$ and not on $\Gamma_0$, we make the following definition.

**Definition.** Let $X_0$ be a complete Riemannian manifold which admits a geometric group action. We say that $X_0$ is large-scale conformally rigid if every finitely generated group quasi-isometric to a complete Riemannian manifold of bounded geometry conformal to $X_0$ is in fact quasi-isometric to $X_0$.

Our main result is the following rigidity theorem:

**Theorem 1.1.** Let $X_0$ be a complete Riemannian 3-manifold which admits a geometric group action. Assume that $X_0$ is conformally flat and homeomorphic to $\mathbb{R}^3$. Then $X_0$ is large-scale conformally rigid.

(In fact we prove a little more; see the last section for a discussion of this.)
Theorem 1.1 applies to three among Thurston’s eight 3-dimensional geometries, namely $E^3$, $H^3$ and $H^2 \times \mathbb{R}$. Groups which are quasi-isometric to those geometries are known [9, 4, 18]. In particular, we obtain the following characterization of groups acting geometrically on $E^3$ and $H^3$:

**Corollary 1.2.** Let $\Gamma$ be a finitely generated group. Then $\Gamma$ admits a geometric action on $E^3$ (resp. $H^3$) if and only if it is quasi-isometric to some complete Riemannian manifold of bounded geometry conformal to $E^3$ (resp. $H^3$).

The proof of Theorem 1.1 splits into two cases, according to whether the conformal structure of $X_0$ is ‘parabolic’ or ‘hyperbolic’. The necessary background is reviewed in Section 2. The parabolic case of Theorem 1.1 is proved in Section 3 using results from Section 2 and arguments from coarse topology. The hyperbolic case is tackled in Section 4. Various remarks on generalizations and open questions are gathered in Section 5.

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**Notation.** When $A$ is a subset of a finitely generated group or a Riemannian manifold, we denote by $|A|$ its “volume”, i.e. : if $A$ is finite (resp. a curve, resp. a surface, resp. a domain with nonempty interior), $|A|$ denotes the cardinal of $A$ (resp. its length, resp. its area, resp. its volume.)

We systematically denote by $d$ the distance function of a metric space. A metric ball (resp. sphere) around a point $x$ of radius $r$ is denoted by $B(x, r)$ (resp. $S(x, r)$).

## 2 Discrete groups and $p$-parabolicity

### 2.1 A review of $p$-parabolicity

Throughout this subsection we fix an integer $p \geq 2$.

**Definition.** Let $X$ be a Riemannian manifold. The $p$-capacity of a compact subset $K \subset X$ is defined by

$$\text{cap}_p(K) = \inf_u \int_X |\nabla u|^p \, d\text{vol},$$

where the infimum is taken over all compactly supported smooth functions $u$ such that $u(x) \geq 1$ for every $x \in K$. 

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The manifold $X$ is \textit{$p$-parabolic} if $\text{cap}_p(K) = 0$ for every compact $K \subset X$. Otherwise it is \textit{$p$-hyperbolic}.

Our interest in this notion comes from the following fact: if $X_0$ and $X$ are Riemannian $p$-manifolds conformal to each other, then $X_0$ is $p$-parabolic iff $X$ is $p$-parabolic. Moreover, $p$-parabolicity is a quasi-isometry invariant of complete Riemannian manifolds of bounded geometry [13, 11]. However we will not need this result: we are interested in manifolds that are quasi-isometric to groups, and in this case the quasi-isometry invariance is a consequence of a characterization in terms of growth functions and isoperimetric inequalities (Theorem 2.1 below.)

\textbf{Remark.} 2-parabolicity is equivalent to the recurrence of the Brownian motion, or to the existence of a Green function for the Laplace-Beltrami operator. The relevance of these ideas to large-scale conformal rigidity in dimension 2 was observed by G. Mess [16]. For more information and references for the general case, see [8].

2.2 Growth and isoperimetry

Let $\Gamma$ be a finitely generated group and $S$ a finite generating subset of $\Gamma$. For all $\Omega \subset \Gamma$ we set

$$\partial \Omega := \{ g \in \Omega \mid \exists g' \in \Gamma - \Omega, \ d_S(g, g') = 1 \}.$$ 

In the following definitions, $\Gamma$ is a finitely generated group and $X$ a Riemannian manifold.

The \textit{growth function} of $\Gamma$ (resp. $X$) is the function $r \mapsto |B(\ast, r)|$, where $\ast$ is a basepoint. We say that $\Gamma$ (resp. $X$) has \textit{superpolynomial growth} if for each $D > 0$ there exists $C_D > 0$ such that $|B(\ast, r)| \geq C_D n^D$ for all $r$. We say that $\Gamma$ (resp. $X$) has \textit{polynomial growth of exponent} $D \in \mathbb{N}$ if there is $C > 0$ such that $C^{-1} r^D \leq |B(\ast, r)| \leq C r^D$ for all $r$.

An \textit{isoperimetric function} for $\Gamma$ (resp. $X$) is a function $I : [0, +\infty) \to [0, +\infty)$ such that the inequality

$$I(|\Omega|) \leq |\partial \Omega|$$

holds for every finite subset of $\Gamma$ (resp. every bounded domain in $X$ with sufficiently smooth boundary).
The isoperimetric dimension of $\Gamma$ (resp. $X$) is the supremum of the set of $D \geq 0$ for which there is a constant $C > 0$ such that the function $v \mapsto C v^{(D-1)/D}$ is an isoperimetric function.

A theorem of Gromov [9] says that the growth function of a group is either superpolynomial or polynomial; in the latter case, the group is virtually nilpotent and the exponent of growth can be computed from the ranks of quotients in the lower central series [2].

The isoperimetric dimension and the asymptotic behavior of the growth function of $\Gamma$ do not depend on the choice of the generating set $S$; in fact they are quasi-isometry invariants of groups and complete Riemannian manifolds of bounded geometry [12].

The following theorem follows from various results scattered in the literature and does not seem to have been stated before in this generality. Since we think it is of independent interest, we give a more complete statement than we shall actually need. The main ingredients are due to Gromov and Varopoulos.

**Theorem 2.1.** Let $\Gamma$ be a finitely generated group and $X$ a complete noncompact Riemannian manifold of bounded geometry quasi-isometric to $\Gamma$. The following are equivalent:

i. $X$ is $p$-parabolic;

ii. The isoperimetric dimension of $\Gamma$ (or $X$) is at most $p$;

iii. $\Gamma$ is virtually nilpotent of growth exponent at most $p$.

**Proof.** We begin by proving that (i) implies (ii). By [8, section 3], the $p$-parabolicity of $X$ implies that for every isoperimetric function $I$ the following holds:

$$\int_{\mathbb{R}^+} \frac{dv}{I^{p-1}(v)} = \infty.$$  

Assuming by contradiction that there exists $D > p$ such that $v \mapsto C v^{(D-1)/D}$ is an isoperimetric function for $X$, we get:

$$\int_{\mathbb{R}^+} \frac{dv}{C v^{\frac{(D-1)p}{D(p-1)}}} = \infty.$$  

This contradicts the fact that $D > p$.  

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Let us turn to the proof that (ii) implies (iii). If (iii) does not hold, then Gromov’s Theorem [9] implies that the growth of \( \Gamma \) is superpolynomial or polynomial of exponent at least \( p + 1 \). By Varopoulos’s inequality [6, Thm 1], the isoperimetric dimension of \( \Gamma \) is at least \( p + 1 \), so (ii) does not hold.

Finally we show (iii) implies (i). If \( \Gamma \) has polynomial growth of exponent at most \( p \), then the same is true for \( X \), i.e. for all \( x \in X \) there is a constant \( C \) such that \( |B(x, r)| \leq Cr^p \) for large \( r \). By [8, section 3], to prove \( p \)-parabolicity it is enough to check that

\[
\int_{a}^{\infty} \left( \frac{r}{|B(x, r)|} \right)^{1/(p-1)} \, dt = \infty.
\]

From the upper bound on \( |B(x, r)| \) we deduce

\[
\frac{r}{|B(x, r)|} \geq \frac{1}{Cr^{p-1}},
\]

which implies the divergence of the above integral.

We shall be mostly interested in the case \( p = 3 \), so we state for future reference a corollary to Theorem 2.1:

**Corollary 2.2.** Let \( X \) be a noncompact complete Riemannian manifold of bounded geometry and \( \Gamma \) a finitely generated group quasi-isometric to \( X \). Then \( X \) is 3-parabolic if and only if \( \Gamma \) is virtually \( \mathbb{Z}^n \) with \( 1 \leq n \leq 3 \).

**Proof.** The “if” part follows from the implication (iii) \( \implies \) (i) in Theorem 2.1. The “only if” part follows from the implication (i) \( \implies \) (iii) in the same theorem plus the formula for the exponent of growth of a nilpotent group [2]. \( \square \)

## 3 The 3-parabolic case

The goal of this section is to prove Theorem 1.1 in the case where \( X_0 \) is 3-parabolic. In fact we will prove a stronger statement:

**Theorem 3.1.** Let \( X, X_0 \) be complete Riemannian manifolds of bounded geometry homeomorphic to \( \mathbb{R}^3 \) and conformal to each other. Let \( \Gamma_0, \Gamma \) be finitely generated groups quasi-isometric to resp. \( X, X_0 \). If \( X_0 \) is 3-parabolic, then both \( \Gamma_0 \) and \( \Gamma \) are virtually \( \mathbb{Z}^3 \). In particular, they are quasi-isometric to each other.
Our main tools are Corollary 2.2 and the following topological rigidity result:

**Proposition 3.2.** Let $Y$ be a complete Riemannian manifold of bounded geometry homeomorphic to $\mathbb{R}^3$. Then $Y$ is not quasi-isometric to $\mathbb{E}^2$.

**Proof of Theorem 3.1 assuming Proposition 3.2.** By Corollary 2.2, $\Gamma_0$ is virtually $\mathbb{Z}$, $\mathbb{Z}^2$ or $\mathbb{Z}^3$. We must rule out the first two cases. If $\Gamma_0$ were virtually $\mathbb{Z}$, it would have two ends. This contradicts the hypothesis that $X_0$ is homeomorphic to $\mathbb{R}^3$, because the number of ends is a quasi-isometry invariant for groups and complete Riemannian manifolds (cf. [3, Proposition 8.29]). The possibility that $\Gamma_0$ be virtually $\mathbb{Z}^2$ is prohibited by Theorem 3.2. Hence $\Gamma_0$ is virtually $\mathbb{Z}^3$.

Since 3-parabolicity is conformally invariant in dimension 3, the same arguments apply to $X$, so $\Gamma$ is virtually $\mathbb{Z}^3$. In particular, $\Gamma, \Gamma_0, X, X_0$ are all quasi-isometric to $\mathbb{E}^3$. \hfill $\square$

The end of this section is devoted to the proof of Proposition 3.2. We first give the idea of the proof, which is fairly simple.

Seeking a contradiction, we assume that there is a quasi-isometry $f : \mathbb{E}^2 \rightarrow Y$ and fix a coarse inverse $\bar{f} : Y \rightarrow \mathbb{E}^2$. We want to exploit the fact that $Y$ is simply-connected at infinity and $\mathbb{E}^2$ is not. With this in mind, take a large round circle $c_1$ in $\mathbb{E}^2$. Its image by $f$ is a quasi-circle in $Y$; since $Y$ is a geodesic space, we can approximate it by a true topological circle $c_2$. Now $Y$ is simply-connected at infinity, so $c_2$ can be filled by a disc $D_2$ near infinity. Since $\mathbb{E}^2$ is uniformly simply-connected, the quasi-disc $\bar{f}(D_2)$ can be approximated by a topological disc, which stays near infinity and is homotopic near infinity to $c_1$. This contradicts the fact that $\mathbb{E}^2$ is not simply-connected at infinity.

Note that our hypotheses do not imply that $Y$ is uniformly simply-connected, so we must be a bit careful. Before we delve into the detailed proof, we need two straightforward lemmas based on the uniform 1-connectedness of $\mathbb{E}^2$.

**Lemma 3.3.** Let $\bar{f} : Y \rightarrow \mathbb{E}^2$ be a $(\lambda, C)$-quasi-isometry. There exists $\delta$ depending only on $\lambda$ and $C$ such that for any continuous map $D_2 : D^2 \rightarrow Y$, there is a continuous map $D_1 : D^2 \rightarrow \mathbb{E}^2$ such that $d(\bar{f}(D_2(u)), D_1(u)) \leq \delta$ for all $u \in D_2$.  

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Proof. Choose a triangulation of $D^2$ such that the image by $D_2$ of each 2-simplex lies in a ball of radius 1. The map $D_1$ is constructed by induction over the skeleta of this triangulation. □

Lemma 3.4 (cf. [15, Lemma 8.6]).
For every $D > 0$ there exists $\epsilon(D)$ such that any continuous map $h : S^1 \times \{0, 1\} \to \mathbb{E}^2$ satisfying $d(h(t, 0), h(t, 1)) \leq D$ for all $t \in S^1$ can be extended to a continuous map $h : S^1 \times I \to \mathbb{E}^2$ such that $\text{diam}(h(t \times I)) \leq \epsilon(D)$ for all $t \in S^1$. □

Proof of Proposition 3.2. Let $f : \mathbb{E}^2 \to Y$ and $\bar{f} : Y \to \mathbb{E}^2$ be $(\lambda, C)$-quasi-isometries such that $d(\bar{f}(f(x)), x) \leq C$ for all $x \in \mathbb{E}^2$ and $d(f(\bar{f}(y)), y) \leq C$ for all $y \in Y$.

Choose a point $x_0 \in \mathbb{E}^2$ and let $c_1 : S^1 \to \mathbb{E}^2$ be an embedding whose image is the circle around $x_0$ of radius $R$, where $R$ is a large constant to be determined. The map $f \circ c_1$ avoids the ball of radius $\lambda^{-1}R - C$ around $f(x_0)$. There is a continuous map $c_2 : S^1 \to Y$ such that $d(f(c_1(t)), c_2(t)) \leq 2(C+1)$ for all $t$. Thus the image of $c_2$ avoids the ball of radius $\lambda^{-1}R - 3C - 2$ around $f(x_0)$.

We want to fill $c_2$ with a continuous map $D_2 : D^2 \to Y$ which is “far off”. Since $Y$ is homeomorphic to $\mathbb{R}^3$, it is simply-connected at infinity, so for any $R' \geq 0$ we can choose $R$ so that every loop in $Y - B(f(x_0), \lambda^{-1}R - 3C - 2)$ can be filled in $Y - B(f(x_0), R')$. We will see later how to choose $R'$, and therefore $R$.

Applying Lemma 3.3, we get a constant $\delta = \delta(\lambda, C)$ such that there is a continuous map $D_1 : D^2 \to \mathbb{E}^2$ satisfying $d(D_1(u), \bar{f}(D_2(u))) \leq \delta$ for all $u$. We want to apply Lemma 3.4 with $h(\cdot, 0) = c_1$ and $h(\cdot, 1) = \partial D_1$. Chasing through the inequalities, we find that the hypothesis of this lemma is fulfilled with $D = 2(C+1)\lambda + 2C + \delta$. Choose $R'$ large enough so that $\epsilon(D) \leq \lambda^{-1}R' - C - 1$ and $\lambda^{-1}R' - C \geq \delta + 10$. Then Lemma 3.4 implies that $c_1$ and $\partial D_1$ are homotopic in the complement of $x_0$. Furthermore, $D_1$ misses $x_0$. Hence $c_1$ is null-homotopic in the complement of $x_0$, which is a contradiction. □

4 The 3-hyperbolic case

The goal of this section is to prove Theorem 1.1 in the case where $X_0$ is 3-hyperbolic. In the first two subsections, we develop some preliminary ma-
terial. The proof itself is given in subsection 4.3.

4.1 Half-minima and Bloch principle

Definition. Let $X$ be a metric space and $h : X \to (0, +\infty)$ be a function. A half-minimum for $h$ is a point $x \in X$ such that $h(y) \geq \frac{1}{2}h(x)$ for every $y \in B(x, \frac{1}{2}\sqrt{h(x)})$.

Lemma 4.1 ([15, Lemma 7.3]). Let $X$ be a complete metric space and $h : X \to (0, +\infty)$ a function which is locally bounded away from zero. Let $x$ be a point of $X$ such that $h(x) < \frac{1}{2}$. Then there exists $x' \in B(x, 2)$ such that $h(x') \leq h(x)$ and $x'$ is a half-minimum. □

The following result generalizes [15, Lemma 7.4].

Theorem 4.2. Let $k \geq 2$ be an integer. Let $(X, g)$ and $(X_0, g_0)$ be complete conformally flat Riemannian $k$-manifolds of bounded geometry. Suppose that $(X_0, g_0)$ has a cocompact group of isometries and is $k$-hyperbolic. Let $c : X \to X_0$ be a conformal embedding. Define a function $\mu : X \to (0, +\infty)$ by setting $g = \mu^2 c^* g_{hyp}$. Then there is a constant $\mu_0 > 0$ such that $\mu(x) \geq \mu_0$ for all $x$.

Proof. Seeking a contradiction, suppose that there is a sequence $x_n \in X$ such that $\mu(x_n)$ goes to $0$. By Lemma 4.1 applied with $h = \mu$, there is no loss of generality in assuming that each $x_n$ is a half-minimum.

Since $X$ is conformally flat and of bounded geometry, there exist constants $r, \lambda$ and for each $n$ a conformal chart $\phi_n : B_{E^k}(0, r) \to X$ such that $\phi_n(0) = x_n$, $\|D\phi_n(0)\| = 1$ and $\sup_{a \in B_{E^k}(0, r)} \|D\phi_n(a)\| \leq 1/2\lambda$.

Set $B_n := B_{E^k}(0, \lambda/\sqrt{\mu(x_n)})$. Define a mapping $z_n : B_n \to E^k$ by $z_n(a) := \mu(x_n)a$. For large $n$ we have $\lambda \sqrt{\mu(x_n)} \leq r$, so the image of $z_n$ lies in $B_{E^k}(0, r)$. By hypothesis, we can for each $n$ postcompose $c$ with an isometry of $X_0$ so that the resulting map $c_n : X \to X_0$ sends $x_n$ into a compact set $K$ independent of $n$. This map $c_n$ is conformal.

Finally we set $f_n = c_n \circ \phi_n \circ z_n$. The goal is to find a converging subsequence of $f_n$ and look at the limit to get a contradiction.

For this we need to estimate $\sup \|Df_n\|$ from above. First we see that $\|Dz_n(a)\| = \mu(x_n)$ and $\|D\phi_n(z_n(a))\| \leq 1/2\lambda$ for all $a \in B_n$. Thus, if $a \in B_n$, then $\phi_n(z_n(a)) \in B(x_n, \frac{1}{2}\sqrt{\mu(x_n)})$ and the half-minimum property says that $\mu(\phi_n(z_n(a))) \geq \frac{1}{2}\mu(x_n)$. Now $c_n$ is conformal with dilatation $1/\mu$, so $\|Dc_n(\phi_n(z_n(a)))\| \leq 2/\mu(x_n)$. We deduce:
$$\|Df_n(a)\| \leq \|Dc_n(\phi_n(z_n(a)))\| \cdot \|D\phi_n(z_n(a))\| \cdot \|Dz_n(a)\|$$
$$\leq \frac{2}{\mu(x_n)} \cdot \frac{1}{2\lambda} \cdot \mu(x_n)$$
$$\leq \frac{1}{\lambda}.$$ 

As a consequence, for fixed $n$, the sequence $\{\tilde{f}_p\}_{p \geq n}$ obtained by restricting each $f_p$ to $B_n$ is equicontinuous on $B_n$, and for every $p$ we get $\tilde{f}_p(0) \in K$ and $\|D\tilde{f}_n(0)\| = 1$. By Ascoli's Theorem, $\{\tilde{f}_p\}_{p \geq n}$ subconverges. By diagonal extraction we get a sequence of conformal mappings $g_n : B_n \to X_0$ which converges uniformly on compact subsets to a mapping $g : E^k \to X_0$. By general properties of quasiconformal mappings, $g$ is 1-quasiconformal or constant.

Now $X_0$ is $k$-hyperbolic and $E^k$ is $k$-parabolic. Hence by [5, Proposition 5.1], there is no quasiconformal mapping from $E^k$ to $X_0$. This implies that our map $g$ is constant.

Let us write $h_n$ for the restriction of $g_n$ to the unit ball. Since the convergence is uniform on compact subsets, for large $n$ the image of $h_n$ is contained in the image of a conformal chart $\phi$ for $X_0$. Now $\phi^{-1} \circ h_n$ is a sequence of conformal maps between domains in $E^k$. If $k = 2$, such maps are holomorphic; if $k \geq 3$ they are restrictions of Möbius transformations by Liouville’s Theorem. In any case the condition $\|D\tilde{f}_n(0)\| = 1$ gives a uniform lower bound on the derivatives of $h_n$ at 0, which leads to a contradiction. 

4.2 Area and diameter estimates

**Lemma 4.3.** Let $X$ be a geodesic space quasi-isometric to a finitely generated group. There exists a function $f_1 : [0, +\infty) \to [0, +\infty)$ such that for every bounded subset $\Omega \subset X$ we have $\text{diam} \Omega \leq f_1(\text{diam} \delta \Omega)$.

**Proof.** In the Cayley graph of a finitely generated group, each point is contained in a (biinfinite) geodesic. Since $X$ is quasi-isometric to a finitely generated group, it has the corresponding “quasi” property: there exist constants $\lambda, C \geq 0$ such that for every $x \in X$, there is a $(\lambda, C)$-quasi-geodesic $\alpha : \mathbb{R} \to X$ such that $d(x, \alpha(0)) \leq C$. Since $X$ is geodesic, we can assume without loss of generality that $\alpha$ is continuous.
Let $\Omega \subset X$ be a bounded subset and $x$ be a point of $\Omega$. Consider a quasi-geodesic $\alpha$ satisfying the above properties. If $\alpha(0) \notin \text{Int} \, \Omega$, connect $x$ to $\alpha(0)$ by a geodesic segment. This segment has to cross the frontier of $\Omega$, so $d(x, \delta \Omega) \leq C$.

Otherwise $\alpha$ meets $\delta \Omega$ for at least one negative time $t_1$ and one positive time $t_2$. Without loss of generality, suppose that $|t_1| \geq t_2$. Then $d(\alpha(0), \delta \Omega) \leq \lambda t_2 + C$. Moreover,

$$t_2 \leq t_2 - t_1 \leq \lambda d(\alpha(t_1), \alpha(t_2)) + C \leq \lambda \text{diam}(\delta \Omega) + C,$$

hence

$$d(x, \delta \Omega) \leq C + d(\alpha(0), \delta \Omega) \leq \lambda(\lambda \text{diam}(\delta \Omega) + C) + 2C.$$

Therefore, this inequality holds for each $x \in \Omega$. We conclude by setting $f_1(r) := 3(\lambda^2 r + (\lambda + 2)C)$.

**Lemma 4.4.** Let $X$ be a Riemannian 3-manifold quasi-isometric to a finitely generated group. Suppose that $H_2(X) = 0$. Then there exists a function $f_2 : [0, \infty) \to [0, \infty)$ such that for every compact $K \subset X$ and every continuous map $s : S^2 \to X - K$ that does not represent the trivial homology class in $X - K$, we have $\text{diam} \, K \leq f_2(\text{diam}(s(S^2)))$.

**Proof.** To start off, thicken the image of $s$ into a codimension 0 submanifold $Y$ contained in $X - K$ and such that $\text{diam} \, Y \leq \text{diam}(s(S^2)) + 1$. Then represent the class of $s$ in $H_2(Y)$ by a system of embedded surfaces. Since $[s] \neq 0 \in H_2(X - K)$, one of these surfaces, say $F$, is not trivial in $H_2(X - K)$. Since $H_2(X) = 0$, $F$ bounds a compact submanifold $\Omega$.

Now $K \subset \Omega$ and $\text{diam}(\delta \Omega) = \text{diam}(F) \leq \text{diam}(S^2) + 1$, so we conclude by applying Lemma 4.3. \qed

**Proposition 4.5.** For all $A, \epsilon > 0$ there exists $L = L(A, \epsilon)$ such that if $S$ is a Riemannian 2-sphere of area at most $A$ and $\gamma \subset S$ is an embedded curve, then there is a system $\{\xi_1, \ldots, \xi_n\}$ of pairwise disjoint embedded curves that cobound a submanifold $U$ such that $\gamma \subset U \subset N(\gamma, L)$ and $|\xi_i| \leq \epsilon$ for all $i$. 

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Proof. Without loss of generality assume that $|\gamma| \geq \epsilon$. By Loewner's Theorem (see e.g. [10]), there is a constant $L'$ such that any Riemannian annulus of area at most $2A$ and whose boundary components are a distance at least $L'$ has systole at most $\epsilon$ (recall that the systole is the infimum of lengths of noncontractible curves). Set $L := L' + \epsilon$.

Let us give some definitions and notation. Let $\Xi$ be a system of curves embedded in $N(\gamma, L) - \gamma$. We denote by $E(\Xi)$ the set of points $x \in S - N(\gamma, L)$ such that there exists $\xi \in \Xi$ which separates $x$ from $\gamma$. We shall say that $\Xi$ is embedded if its elements are pairwise disjoint.

The conclusion of Propposition 4.5 can be reformulated as follows: there exists an embedded system $\Xi$ satisfying conditions (i) and (ii) below:

i. $|\xi| \leq \epsilon$ for all $\xi \in \Xi$;
ii. $E(\Xi) = S - N(\gamma, L)$.

Here is our strategy: first we find a (possibly not embedded) system satisfying (i) and (ii). Then if it is not embedded, we show how to do surgery on it to produce a system with fewer self-intersections still satisfying (i) and (ii).

**Lemma 4.6.** There is a system $\Xi_0 = \{\xi'_0, \ldots, \xi'_m\}$ satisfying conditions (i) and (ii).

**Proof.** For a generic choice of $L'$, $N(\gamma, L')$ is a planar surface. Let $\eta_0, \ldots, \eta_m$ be its boundary components. Let $D_1, \ldots, D_0$ be discs such that $\partial D_i = \eta_i$ and $D_i \cap \gamma = \emptyset$.

Fix $i$ between 0 and $m$. Let $Y_i$ be the annulus cobounded by $\gamma$ and $\eta_i$. Since $d(\gamma, \eta_i) = L'$, there is an essential curve $\xi'_i \subset Y_i$ of length at most $\epsilon$. Since $\xi'_i$ is essential, it cannot lie in one of the $D_j$’s. Hence it is contained in $N(\gamma, L)$. This ensures that the system $\{\xi'_i\}$ satisfies (ii). By construction it also satisfies (i). \qed

Before proceeding, we need one more piece of notation. Let $\Xi$ be an embedded system of curves and $\eta$ an embedded curve in $S - \gamma$ in general position with respect to $\Xi$. Let $\text{sing}(\Xi, \eta)$ denote the cardinal of $\eta \cap \Xi$. In particular, $\text{sing}(\Xi, \eta) = 0$ iff $\Xi \cup \{\eta\}$ is embedded.

Assume that $\text{sing}(\Xi, \eta) > 0$. A well-known lemma (cf. [17]) ensures that there is a bigon between $\eta$ and some $\xi \in \Xi$, i.e. a disc $D \subset S - \gamma$ whose boundary is the union of two arcs $\alpha_1, \alpha_2$ with $\alpha_1 \subset \eta$, $\alpha_2 \subset \xi$, and such that
Int $D$ meets neither $\eta$ nor $\Xi$. We say that $\alpha_1$ (resp. $\alpha_2$) is *exterior* to $D$ if the curve $\eta$ (resp. $\xi$) surrounds $D$ (i.e. $D$ is contained in the unique disc bounded by this curve). Otherwise we say $\alpha_1$ (resp. $\alpha_2$) is *interior*.

![Figure 1: Various types of bigons.](image)

We will always use this notation, i.e. if the bigon is called $D$, we call $\alpha_1$ the arc lying in $\eta$ and $\alpha_2$ the arc lying in a element of $\Xi$. This allows to distinguish several types of bigons: a bigon $D$ is called *exterior* if (with the same notation as above) both arcs $\alpha_1, \alpha_2$ are interior to $D$, *interior* if $\alpha_1, \alpha_2$ are exterior to $D$, and *mixed* otherwise. Furthermore, a mixed bigon $D$ is of *type 1* if $\alpha_1$ is the interior arc, and of *type 2* otherwise. Two mixed bigons are *paired* if they are both of type 1 and involve the same element of $\Xi$.

In the example illustrated by Figure 1, the curve $\eta$ has five bigons of intersection with $\Xi = \{\xi_1, \xi_2\}$; $D_1$ is interior, $D_2$ is exterior, $D_3, D'_3$ form a pair of mixed bigons of type 1, and $D_4$ is mixed of type 2.
**Lemma 4.7.** Let \( \Xi \) be an embedded system and \( \eta \) be an embedded curve in \( S - \gamma \) in general position with respect to \( \Xi \). Suppose that \( \Xi \cup \{ \eta \} \) satisfies condition (i). Then there is an embedded system \( \Xi' \) satisfying (i) and such that \( E(\Xi') \supset E(\Xi) \cup E(\eta) \).

**Proof.** The proof is by induction on \( \text{sing}(\Xi, \eta) \). If this number is 0, we can just set \( \Xi' := \Xi \cup \{ \eta \} \). Otherwise we will show how to use the induction hypothesis by applying to \( (\Xi, \eta) \) one or more of the three operations described below.

**(T_0)** Let \( \xi \) be an element of \( \Xi \) lying in the interior of a disc \( D \subset S - \gamma \) bounded by \( \eta \) or by another element of \( \Xi \). The operation \( T_0 \) consists in keeping \( \eta \) unchanged and removing \( \xi \) of \( \Xi \).

We say that \( (\Xi, \eta) \) is reduced if \( T_0 \) cannot be applied to it. Clearly, \( T_0 \) does not change condition (i) nor \( E(\Xi) \cup E(\eta) \) and never increases \( \text{sing}(\Xi, \eta) \). Hence we can always assume that \( (\Xi, \eta) \) is reduced.

**(T_1)** Let \( D \) be a bigon bounded by arcs \( \alpha_1 \subset \eta \) and \( \alpha_2 \subset \xi \). Let \( \beta \) be the closure of \( \eta - \alpha_1 \). Replace \( \eta \) by a curve obtained from \( \beta \cup \alpha_2 \) by a small isotopy that removes the intersections in the neighborhood of \( \alpha_2 \). We call this operation *pushing \( \eta \) through \( D \).*

Operation \( T_1 \) decreases \( \text{sing}(\Xi, \eta) \) by 2. If \( |\alpha_1| \geq |\alpha_2| \), it respects condition (i) (taking the isotopy sufficiently small); furthermore, \( E(\Xi) \cup E(\eta) \) can go down only if the bigon is mixed and \( \alpha_1 \) is the exterior arc.

Symmetrically we define *pushing \( \xi \) through \( D \).*

**(T_2)** Add to \( \Xi \) a curve disjoint from \( \Xi \), contained in a bigon \( D \) and obtained from \( \partial D \) by a small isotopy.

Note that \( E(\Xi) \cup E(\eta) \) never goes down when \( T_2 \) is applied. Condition (i) is preserved provided that \( |\partial D| \leq \epsilon \).

To deal with mixed bigons, we need the following lemma:

**Sublemma 4.8.** Let \( \Xi \) be an embedded system of curves and \( \eta \) an embedded curve in general position with respect to \( \Xi \). Suppose that \( (\Xi, \eta) \) is reduced and there are no interior bigons. Then either \( (\Xi, \eta) \) is embedded, or there are paired bigons.

**Proof.** Assume that \( (\Xi, \eta) \) is not embedded. Choose \( \xi \in \Xi \) such that \( \xi \cap \eta \neq \emptyset \). Let \( D_\xi \) be the disc bounded by \( \xi \). Then \( \eta \cap D_\xi \) consists of one or more arcs. Let \( \beta \) be one of these arcs. On each side of \( \beta \) choose an outermost arc \( \beta_1 \)
Figure 2: The three moves $T_0$-$T_2$.

(resp. $\beta_2$). Then each $\beta_i$ cobounds a bigon $D_i$ with a subarc of $\xi$. Each $D_i$ is interior to $\xi$. Since by hypothesis there are no interior bigons, $D_1$ and $D_2$ must be mixed of type 1, hence paired.

We turn to the proof of Lemma 4.7. Take a non-embedded pair $(\Xi, \eta)$ fulfilling the hypotheses of this lemma and assume the result holds for all pairs $(\Xi', \eta')$ such that $\text{sing}(\Xi', \eta') \leq \text{sing}(\Xi, \eta)$. By applying move $T_0$ zero or more times, we can assume that $(\Xi, \eta)$ is reduced. If there is an exterior bigon or an interior bigon, we can assume that $(\Xi, \eta)$ is reduced. If there is an exterior bigon or an interior bigon, we can perform move $T_1$, pushing the bigger arc through the smaller one, and apply the induction hypothesis to the resulting configuration.
If all bigons are mixed, then by Sublemma 4.8 there are some paired bigons. Let us write $\alpha_1 \cup \alpha_2$ and $\alpha'_1 \cup \alpha'_2$ for their boundaries, with the usual conventions. We have $|\alpha_1| + |\alpha'_1| \leq \varepsilon$ and $|\alpha_2| + |\alpha'_2| \leq \varepsilon$. Hence $|\alpha_1| + |\alpha_2| + |\alpha'_1| + |\alpha'_2| \leq 2\varepsilon$, and we may assume without loss of generality that $|\alpha_1| + |\alpha_2| \leq \varepsilon$ and $|\alpha_1| \geq |\alpha_2|$. Then we apply $T_2$ followed by $T_1$. This finishes the proof of Lemma 4.7.

At last we prove Proposition 4.5. Consider the system $\Xi_0 = \{\xi'_0, \ldots, \xi'_m\}$ given by Lemma 4.6. Applying Lemma 4.7 with $\Xi = \{\xi'_0\}$ and $\eta = \xi'_1$, we get an embedded system $\Xi_1$ satisfying (i) and such that $E(\Xi_1) \supset E(\{\xi'_0, \xi'_1\})$. Then we apply Lemma 4.7 successively for each $i$ from 2 to $m$, putting $\Xi = \Xi_{i-1}$ and $\eta = \xi'_i$. The outcome of each step is an embedded system $\Xi_i$ satisfying (i) such that $E(\Xi_i) \supset E(\{\xi'_0, \ldots, \xi'_i\})$. Hence $\Xi_m$ is embedded and satisfies (i) and (ii). This completes the proof of Lemma 4.5.

### 4.3 Proof of the 3-hyperbolic case

Let $(X, g)$ and $(X_0, g_0)$ be Riemannian 3-manifolds satisfying the hypotheses of Theorem 1.1. Assume that $X_0$ (and hence $X$) is 3-hyperbolic. Let $c : (X, g) \to (X_0, g_0)$ be a conformal diffeomorphism. Let $\mu : X \to (0, +\infty)$ denote the function defined by $g = \mu^2 c^* g_0$. We sometimes still denote (abusively) by $\mu$ the function $\mu \circ c^{-1}$. We shall prove:

**Proposition 4.9.** There are constants $r_0, \mu_1 > 0$ such that for every $x \in X_0$ the following holds:

$$\text{diam } c^{-1}(B(x, r_0/2)) \leq \mu_1.$$

To see why this implies Theorem 1.1, we note that Theorem 4.2 implies that $d(x, y) \geq \mu_0 d(c(x), c(y))$ for some constant $\mu_0 > 0$ and all $x, y \in X$. Since our goal is to prove that $X$ is quasi-isometric to $X_0$, we need an upper bound for $d(x, y)$ in terms of $d(c(x), c(y))$. For all $x, y \in X$, choose a geodesic arc $\gamma$ connecting $c(x)$ to $c(y)$. We can cover $\gamma$ by $n$ balls of radius $r_0/2$ with $n$ bounded above by a linear function of $d(c(x), c(y))$. Hence by Proposition 4.9, $d(x, y)$ is bounded by a linear function of $d(c(x), c(y))$. This proves that $c$ is a quasi-isometry, which completes the proof of Theorem 1.1 in the 3-hyperbolic case.

The remainder of this paper is devoted to the proof of Proposition 4.9. First of all, we gather in the next lemma some immediate consequences of the bounded geometry hypothesis on $X_0$. 

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Lemma 4.10. There exist positive constants $R_0, \lambda_1, \lambda_2, \lambda_3, N_1$ such that for every $x \in X_0$ and every $\rho \leq R_0$ we have:

i. $|S(x, \rho)| \leq \lambda_2^2 \rho^2$.

ii. For all $x_1, \ldots, x_n \in S(x, \rho)$ such that if $i \neq j$, then $d(x_i, x_j) \geq 9\rho/5$, we have $n \leq N_1$.

iii. For each parallel circle $\gamma$ on $S(x, \rho)$ of latitude at most $\pi/5$, we have $\text{diam}(\gamma) \geq \lambda_2 \rho$.

iv. Let $\xi$ be a curve on $S(x, \rho)$ of length bounded above by $\lambda_3 \rho$. Then exactly one of the two discs bounded by $\xi$ on $S(x, \rho)$ has diameter bounded above by $3\lambda_2 \rho/4$.

Proof. Choose for $R_0$ a lower bound for the injectivity radius of $X_0$. The restriction of the exponential map at any point $x$ to the ball of radius $R_0$ around the origin is a bilipschitz embedding with uniform Lipschitz constant.

In assertion (iii), the word “parallel” refers to the image by the exponential map at $x$ of a parallel for the standard spherical coordinates in $\mathbb{R}^3$; the word “latitude” is to be interpreted in the same sense. Since $R_0$ is less than the injectivity radius at $x$, $\gamma$ is indeed a topological circle.

Lemma 4.11. i. There is a constant $\lambda_4 > 0$ such that for every $\nu > 0$, every $x \in X_0$ and every $r \leq R_0$, if $|c^{-1}(B(x, r))| \leq \nu$, then there exists $\rho \in [\nu r/10, r]$ such that $|c^{-1}(S(x, \rho))| \leq \lambda_4 \nu^{2/3}$.

ii. For every $A > 0$ there exists $L_1 = L_1(A)$ such that for every $x \in X_0$ and every $\rho \leq R_0$, if $|c^{-1}S(x, \rho)| \leq A$, then there is a curve $\gamma \subset c^{-1}S(x, \rho)$ satisfying $|\gamma| \leq L_1$ and $\text{diam}\, c(\gamma) \geq \lambda_2 \rho$.

Proof. Set

$$\lambda_4 := \left( \frac{\lambda_1}{\ln(10/9)} \right)^{2/3} + 1.$$

If (i) does not hold, then by Hölder’s inequality the following is true for all $\rho \in [\nu r/10, r]$:

$$\lambda_4^{3/2} \nu \leq \left( \int_{S(x, \rho)} \mu^2 \, d\text{vol} \right)^{3/2} \leq |S(x, \rho)|^{1/2} \cdot \int_{S(x, \rho)} \mu^3 \, d\text{vol} \leq \lambda_1 \rho \cdot \int_{S(x, \rho)} \mu^3 \, d\text{vol}.$$
Dividing by $\rho$ and integrating between $9r/10$ and $r$, we get:
\[
\lambda_4^{3/2} \nu \cdot \int_{9r/10}^{r} \frac{d\rho}{\rho} \leq \lambda_1 |c^{-1}B(x,r)| \leq \lambda_1 \nu,
\]
so
\[
\lambda_4^{3/2} \ln(10/9) \leq \lambda_1,
\]
which contradicts the choice of $\lambda_4$.

The proof of (ii) is similar, using 4.10(iii) and the Cauchy-Schwarz inequality in spherical coordinates. $\square$

Given $x \in X_0$ and $\nu > 0$, we let $r(x, \nu)$ denote the infimum of numbers $\rho > 0$ such that $|c^{-1}(B(x, \rho))| \geq \nu$. For fixed $\nu$, the function $x \mapsto r(x, \nu)$ may not be continuous, but it is locally bounded away from zero, so we can apply Lemma 4.1 to it.

**Lemma 4.12.** There is a constant $\nu_0 > 0$ such that $\inf \{r(x, \nu_0) \mid x \in X_0\} > 0$.

**Proof.** First we reduce this lemma to the following claim:

**Claim.** For every $\nu > 0$, if $\inf \{r(x, \nu) \mid x \in X_0\} = 0$, then there is a domain $\Omega \subset X$ such that $|\Omega| \geq \nu$ and $|\partial \Omega| \leq N_1 \lambda_4 \nu^{2/3}$.

Let us prove by contradiction that this claim implies Lemma 4.12. Let $\nu_i \to +\infty$ be a sequence such that $\inf \{r(x, \nu_i) \mid x \in X_0\} = 0$. The claim supplies a sequence of domains $\Omega_i \subset X$ satisfying $|\Omega_i| \to +\infty$ and
\[
\frac{|\partial \Omega_i|}{|\Omega_i|^{2/3}} \leq N_1 \lambda_4.
\]

It follows that for all $D > 3$,
\[
\frac{|\partial \Omega_i|}{|\Omega_i|^{(D-1)/D}} \to 0,
\]
which shows that $X$ has isoperimetric dimension at most 3. By Theorem 2.1, $X$ is 3-parabolic. This contradicts the conformal invariance of 3-parabolicity.

The next task is to prove the claim. Fix $\nu > 0$ and apply Lemma 4.1 to $x \mapsto r(x, \nu)$. This gives a point $x_\nu \in X_0$ satisfying $r(x_\nu, \nu) < R_0$ and such that for any $x \in X_0$, if $d(x, x_\nu) \leq \frac{1}{2} \sqrt{r(x_\nu, \nu)}$ then $r(x, \nu) \geq \frac{1}{2} r(x_\nu, \nu)$. For simplicity, let us write $r_\nu$ for $r(x_\nu, \nu)$. Without loss of generality assume that
Since that, in particular we did not use the hypothesis of conformal
Proposition 4.9, and hence of Theorem 1.1, is now complete.

\[ r_\nu < \frac{1}{4} \sqrt{r_\nu}. \]

Then for every \( x \in B(x_\nu, 2r_\nu) \) and every \( r \leq r_\nu \) the inequality
\[ |c^{-1}(B(x, r))| \leq \nu \] holds.

By Lemma 4.11(i) applied with \( r = r_\nu \), we can associate to each point
\( x \in X_0 \) such that \( d(x, x_\nu) = r_\nu \) a number \( \rho(x) \in [9r_\nu/10, r_\nu] \) satisfying
\[ |c^{-1}(\partial B(x, \rho(x)))| \leq \lambda_4 \nu^{2/3}. \] Let \( \{x_1, \ldots, x_n\} \) be a set of points of \( S(x_\nu, r_\nu) \)
with \( n \) minimal such that the metric balls \( B(x_i, \rho(x_i)) \) cover \( S(x_\nu, r_\nu) \). Set
\( \Omega := c^{-1}(B(x_\nu, r_\nu) \cup \bigcup_i B(x_i, \rho(x_i))) \). Then \( \Omega \) contains \( c^{-1}(B(x_\nu, r_\nu)) \), so
\[ |\Omega| \geq \nu. \] Since \( n \) is minimal, the balls \( B(x_i, \rho(x_i))/3 \) are pairwise disjoint.
Since \( \rho(x_i) \geq 9r_\nu/10 \) and \( r_\nu < R_0 \), Lemma 4.10(ii) gives \( n \leq N_1 \). It follows
that \( |\partial \Omega| \leq n \lambda_4 \nu^{2/3} \leq N_1 \lambda_4 \nu^{2/3} \).

Set \( r_0 := \min(R_0, \inf\{r(x, \nu_0) \mid x \in X_0\}) \). Applying 4.11(i) with \( \nu = \nu_0 \)
and \( r = r_0 \), we obtain a function \( \rho : X_0 \to [9r_0/10, r_0] \) satisfying
\[ |c^{-1}(S(x, \rho(x)))| \leq \lambda_4 \nu_0^{2/3}. \]

Define \( A_0 := \lambda_4 \nu_0^{2/3} \). Fix a point \( x \in X_0 \) and consider the metric sphere
\( S = S(x, \rho(x)) \). By construction, \( c^{-1}(S) \) is not null-homotopic in the
complement of \( c^{-1}(B(x, r_0/2)) \); furthermore, its area is bounded above by \( A_0 \). If
we had a uniform upper bound of the diameter of \( S \) (as opposed to the area)
we could apply Lemma 4.3. Since we do not have such a bound, we are going
to use Proposition 4.5 and Lemma 4.4.

Lemma 4.11(ii) provides a constant \( L_1 = L_1(A_0) \) and an embedded curve
\( \gamma \subset c^{-1}(S) \) such that \( |\gamma| \leq L_1 \) and \( \text{diam } c(\gamma) \geq \lambda_2 \rho(x) \). Let us apply
Lemma 4.5 with \( A = A_0 \) and \( \epsilon = \inf\{\text{inj}(X), 9\mu_0 \lambda_3 r_0/10\} \). Each \( \xi \) bounds a
small disc \( D_i \) of diameter at most \( 9\mu_0 \lambda_3 r_0/10 \). It follows that \( \text{diam } c(D_i) \leq 9\lambda_3 r_0/10 \leq \lambda_3 \rho(x) \).
Therefore, of the two discs bounded by \( c(\xi) \) on \( S \), the
small one is homotopic (with fixed boundary) to \( c(D_i) \) in the complement of
\( B(x, r_0/2) \).

Hence \( U \cup \bigcup_i D_i \) is a (possibly not embedded) 2-sphere homotopic to
\( c^{-1}(S) \), of diameter at most \( C_4 + 2(L + 9\mu_0 \pi r_0/100) \). To conclude, we apply
Lemma 4.4 and set \( \mu_1 := f_2(C_4 + 2(L + 9\mu_0 \pi r_0/100)) \). The proof of
Proposition 4.9, and hence of Theorem 1.1, is now complete.

5 Final remarks

We already remarked that in the 3-parabolic case we proved a stronger state-
ment (Theorem 3.1). In particular we did not use the hypothesis of conformal
flatness. Likewise, in the 3-hyperbolic case we did not use the hypothesis that $X_0 \cong \mathbb{R}^3$. Hence we have actually proved:

**Theorem 5.1.** Every complete, conformally flat, 3-hyperbolic Riemannian 3-manifold which admits a geometric group action is large-scale conformally rigid.

In another direction, the methods of this paper can be used to show:

**Theorem 5.2.** Every complete Riemannian 2-manifold which admits a geometric group action is large-scale conformally rigid.

For $\mathbb{E}^2$ and $\mathbb{H}^2$ this result is essentially proved in [16]. The method used there for the hyperbolic plane does not seem to extend to nonsimply-connected hyperbolic surfaces. Partial results along the same lines (using other techniques closer to those of the present paper) were obtained in [15].

One may ask whether our hypothesis that all manifolds involved be of bounded geometry is necessary. Our definition of bounded geometry is stronger than those usually found in the literature, i.e. uniform bounds on Ricci curvature, sectional curvature, and/or injectivity radius. Some results (e.g. those of section 2) hold under weaker assumptions. Note however that any manifold with sectional curvature bounded in absolute value and injectivity radius bounded away from zero is quasi-isometric to a manifold of bounded geometry in our sense. To see this, construct a triangulation with controlled geometry, take the regular piecewise Euclidean metric associated to this triangulation, and smoothen out consistently the singularities (cf. [15, 1]). For this reason, issues of minimal hypotheses were ignored in this paper.

Theorem 5.1 applies to all 3-hyperbolic manifolds which are regular covers of closed 3-manifolds that admit conformally flat structures. This is a large class of 3-manifolds, which includes e.g. lots of open hyperbolic manifolds, but also many others (see [14] and the references therein).

Consider Thurston’s eight geometries. By Theorem 2.1, the 3-parabolic ones are $S^3$, $S^2 \times \mathbb{R}$, and $\mathbb{E}^3$. They are all large-scale conformally rigid (this is trivial for the first two, and follows from Theorem 1.1 for the third one). Our main theorem applies to $H^3$ and $H^2 \times \mathbb{R}$. It does not apply to $\widetilde{SL}_2(\mathbb{R})$, but since there exist closed $\widetilde{SL}_2(\mathbb{R})$-manifolds with conformally flat metrics, any group quasi-isometric to $\widetilde{SL}_2(\mathbb{R})$ is also quasi-isometric to some complete conformally flat manifold to which Theorem 1.1 applies. This raises
an obvious question: is the large-scale conformal rigidity property invariant by quasi-isometry?

Our result does not give anything for fundamental groups of closed Nil and Sol manifolds, since they do not admit flat conformal structures [7]. One can ask whether Nil and Sol are large-scale conformally rigid. The arguments of the present paper seem to suggest a positive answer.

References

[1] O. Attie. Quasi-isometry classification of some manifolds of bounded geometry. Math. Z., 216(4):501–527, 1994.

[2] H. Bass. The degree of polynomial growth of finitely generated nilpotent groups. Proc. London Math. Soc., 25:603–614, 1982.

[3] M. R. Bridson and A. Haefliger. Metric spaces of non-positive curvature. Springer-Verlag, Berlin, 1999.

[4] J. W. Cannon and D. Cooper. A characterization of cocompact hyperbolic and finite-volume hyperbolic groups in dimension three. Trans. Amer. Math. Soc., 330(1):419–431, 1992.

[5] T. Coulhon, I. Holopainen, and L. Saloff-Coste. Harnack inequality and hyperbolicity for subelliptic p-Laplacians with applications to Picard type theorems. Preprint, 2000.

[6] T. Coulhon and L. Saloff-Coste. Isopérimétrie pour les groupes et les variétés. Revista Math. Iberoamericana, 9:293–314, 1993.

[7] W. M. Goldman. Conformally flat manifolds with nilpotent holonomy and the uniformization problem for 3-manifolds. Trans. Amer. Math. Soc., 278(2):573–583, 1983.

[8] A. Grigor’yan. Isoperimetric inequalities and capacities on Riemannian manifolds. In The Mažya anniversary collection, Vol. 1 (Rostock, 1998), pages 139–153. Birkhäuser, Basel, 1999.

[9] M. Gromov. Groups of polynomial growth and expanding maps. Publ. Math. IHES, 53:53–73, 1981.
[10] M. Gromov, J. Lafontaine, and P. Pansu. *Structures métriques pour les variétés riemanniennes*. CEDIC, Paris, 1981.

[11] I. Holopainen. Rough isometries and $p$-harmonic functions with finite Dirichlet integral. *Rev. Mat. Iberoamericana*, 10(1):143–176, 1994.

[12] M. Kanai. Rough isometries and combinatorial approximations of geometries of non-compact Riemannian manifolds. *J. Math. Soc. Japan*, 37(3):391–413, 1985.

[13] M. Kanai. Rough isometries and the parabolicity of Riemannian manifolds. *J. Math. Soc. Japan*, 38(2):227–238, 1986.

[14] M. È. Kapovich. Flat conformal structures on 3-manifolds (survey). In *Proceedings of the International Conference on Algebra, Part 1 (Novosibirsk, 1989)*, pages 551–570, Providence, RI, 1992. Amer. Math. Soc.

[15] S. Maillot. Quasi-isometries of groups, graphs and surfaces. *Comment. Math. Helv.*, 76(1):29–60, 2001.

[16] G. Mess. The Seifert conjecture and groups which are coarse quasiisometric to planes. Preprint.

[17] V. Poenaru, F. Laudenbach, and A. Fathi. *Travaux de Thurston sur les surfaces*. Société Mathématique de France, Paris, 1979.

[18] E. G. Rieffel. Groups quasi-isometric to $\mathbb{H}^2 \times \mathbb{R}$. *J. London Math. Soc. (2)*, 64(1):44–60, 2001.