SOME NOTES ON IMPROVING UPON THE JAMES-STEIN ESTIMATOR

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We consider estimation of a multivariate normal mean vector under sum of squared error loss. We propose a new class of smooth estimators parameterized by $\alpha$ dominating the James-Stein estimator. The estimator for $\alpha = 1$ corresponds to the generalized Bayes estimator with respect to the harmonic prior. When $\alpha$ goes to infinity, the estimator converges to the James-Stein positive-part estimator. Thus the class of our estimators is a bridge between the admissible estimator ($\alpha = 1$) and the inadmissible estimator ($\alpha = \infty$). Although the estimators have quasi-admissibility which is a weaker optimality than admissibility, the problem of determining whether or not the estimator for $\alpha > 1$ admissible is still open.

1. Introduction. Let $X$ be a random variable having $p$-variate normal distribution $N_p(\theta, I_p)$. Then we consider the problem of estimating the mean vector $\theta$ by $\delta(X)$ relative to quadratic loss. Therefore every estimator is evaluated based on the risk function

$$R(\theta, \delta) = E_\theta \left[ \| \delta(X) - \theta \|^2 \right] = \int_{\mathbb{R}^p} \frac{\| \delta(x) - \theta \|^2}{(2\pi)^{p/2}} \exp \left( -\frac{\| x - \theta \|^2}{2} \right) dx.$$ 

The usual estimator $X$, with the constant risk $p$, is minimax for any dimension $p$. It is also admissible when $p = 1$ and 2, as shown in [1] and [13], respectively. [13] showed, however, that when $p \geq 3$, there exists an estimator dominating $X$ among a class of equivariant estimators relative to the orthogonal transformation group which have the form

(1.1) \[ \delta_\phi(X) = (1 - \phi(\|X\|^2)/\|X\|^2)X. \]

[7] succeeded in giving an explicit form of an estimator improving on $X$ as

$$\delta_{JS}(X) = (1 - (p - 2)/\|X\|^2)X,$$

which is called the James-Stein estimator. More generally, a large class of better estimators than $X$ has been proposed in the literature. The strongest tool for this is [13]'s identity as follows.

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Lemma 1.1. If \( Y \sim N(\mu, 1) \) and \( h(y) \) is any differentiable function such that \( E[|h'(Y)|] < \infty \), then
\[
E[h(Y)(Y - \mu)] = E[h'(Y)].
\]

By using the identity (1.2), the risk function of the estimator of the form \( \delta_g(X) = X + g(X) = (X_1 + g_1(X), \ldots, X_p + g_p(X))' \) is written as
\[
R(\theta, \delta_g) = E_\theta \left[ \|\delta_g(X) - \theta\|^2 \right]
= E_\theta \left[ \|X - \theta\|^2 \right] + E_\theta \left[ \|g(X)\|^2 \right] + 2 \sum_{i=1}^p E_\theta \left[ (X - \theta)'g(X) \right]
= p + E_\theta \left[ \|g(X)\|^2 \right] + 2 \sum_{i=1}^p E_\theta \left[ \frac{\partial}{\partial x_i} g_i(X) \right],
\]
where \( g_i(x) \) is assumed to be differentiable and \( E\left|\frac{\partial}{\partial x_i} g_i(x)\right| < \infty \) for \( i = 1, \ldots, p \). Since a statistic \( \hat{R}(\delta(X)) \) given by
\[
\phi(w) (2(p - 2) - \phi(w)) / w + 4\phi'(w) \geq 0,
\]
for any \( w = \|x\|^2 \). The inequality (1.5) is, for example, satisfied if \( \phi(w) \) is monotone nondecreasing and within \([0, 2(p - 2)]\) for any \( w \geq 0 \).

Since \( \phi(w) = c \) for \( 0 < c < 2(p - 2) \) satisfies the inequality (1.5),
\[
(1 - c/\|X\|^2)X
\]
with $0 < c < 2(p - 2)$ dominates $X$. The estimator $\delta_{JS}$, (1.6) with $c = p - 2$, is the best estimator among a class of estimators (1.6) because the risk function of (1.6) is given by

$$p + c(c - 2(p - 2))E[\|X\|^{-2}]$$

and hence minimized by $c = p - 2$.

It is however noted that when $\|x\|^2 < p - 2$, the James-Stein estimator yields an over-shrinkage and changes the sign of each component of $X$. The James-Stein positive-part estimator

$$\delta_{JS}^+(X) = \max(0, 1 - (p - 2)/\|X\|^2)X,$$

eliminates this drawback and dominates the James-Stein estimator. We notice here that the technique for proving the inadmissibility of $\delta_{JS}$ is not from the Stein identity or the unbiased estimator of risk given in (1.3). The risk difference between $\delta_{JS}$ and $\delta_\phi$ is given by

$$R(\delta_{JS}, \theta) - R(\delta_\phi, \theta) = E \left[ -\frac{(\phi(\|X\|^2) - p + 2)^2}{\|X\|^2} + 4\phi'(\|X\|^2) \right],$$

but $\phi_{JS}^+(w) = \min(w, p - 2)$ which makes the James-Stein positive-part estimator does not satisfy the inequality

$$(1.7) \quad -(\phi(w) - p + 2)^2 + 4w\phi'(w) \geq 0$$

for any $w \geq 0$. In Section 2, we will show that there is no $\phi$ which satisfies (1.7) for any $w \geq 0$, that is, the Stein identity itself is not useful for finding estimators dominating $\delta_{JS}$. We call such optimality for the James-Stein estimator quasi-admissibility. We will explain the concept and give a sufficient condition for quasi-admissibility in Section 2.

In spite of such difficulty, some estimators which dominate the James-Stein estimator have been given by several authors. [11] and [6] considered the class of estimators of forms $\delta_{LK}(X) = (1 - \phi_{LK}(\|X\|^2)/\|X\|^2)X$ where

$$\phi_{LK}(w) = p - 2 - \sum_{i=1}^{n} a_i w^{-b_i},$$

where $a_i \geq 0$ for any $i$ and $0 < b_1 < b_2 < \cdots < b_n$. For example when $n = 1$, they both showed that, $\delta_{LK}(X)$ for $0 < b_1 < 4^{-1}(p - 2)$ and $a_1 = 2b_1 2^{b_1} \Gamma(p/2 - b_1 - 1) / \Gamma(p/2 - 2b_1 - 1)$ is superior to the James-Stein
estimator. [10] gave two estimators which shrink toward the ball with center 0, \( \delta^i_{KT}(X) = (1 - \phi^i_{KT}(\|X\|^2)/\|X\|^2)X \) for \( i = 1, 2 \) where
\[
\phi^i_{KT}(w) = \begin{cases} 
0 & w \leq r^2 \\
p - 2 - \sum_{i=1}^{p-2} (r/w^{1/2})^i & w > r^2,
\end{cases}
\]
\[
\phi^2_{KT}(w) = \begin{cases} 
0 & w \leq \{(p - 1)/(p - 2)\}^2 r^2 \\
p - 2 - r/(w^{1/2} - r) & w > \{(p - 1)/(p - 2)\}^2 r^2.
\end{cases}
\]
They showed that when \( r \) is sufficiently small, these two estimators dominate the James-Stein estimator. However, these estimators are not appealing since they are inadmissible. In our setting, the estimation of a multivariate normal mean, [3] showed that any admissible estimator should be generalized Bayes. Since the shrinkage factor \( (1 - \phi^i_{LK}(w)/w) \) becomes negative for some \( w \) and \( \phi^i_{KT} \) for \( i = 1, 2 \) fail to be analytic, neither \( \delta_{LK} \) nor \( \delta^i_{KT} \) can be generalized Bayes or admissible.

In general, when we propose an estimator \( (\delta_*, \text{say}) \) dominating a certain inadmissible estimator, it is extremely important to find it among admissible estimators. If not, a more difficult problem (finding an estimator improving on \( \delta_* \)) just occurs. To the best of our knowledge, the sole admissible estimator dominating the James-Stein estimator is [8]'s estimator \( \delta_{K}(X) = (1 - \phi_{K}(\|X\|^2)/\|X\|^2)X \) where
\[
(1.8) \quad \phi_{K}(w) = p - 2 - 2 \frac{\exp(-w/2)}{\int_0^1 \lambda^{p/2-2} \exp(-w\lambda/2) d\lambda}.
\]
The estimator is generalized Bayes with respect to the harmonic prior density \( \|\theta\|^{2-p} \) which was originally suggested by [14]. The only fault is, however, that it does not improve upon \( \delta_{JS}(X) \) at \( \|\theta\| = 0 \). (See Section 3 for the detail.) Shrinkage estimators like (1.1) make use of the vague prior information that \( \|\theta\| \) is close to 0. It goes without saying that we would like to get the significant improvement of risk when the prior information is accurate. Though \( \delta_{K}(X) \) is an admissible generalized Bayes estimator and thus smooth, it has no improvement on \( \delta_{JS}(X) \) at the origin \( \|\theta\| = 0 \). On the other hand, \( \delta^+_{JS}(X) \) improves on \( \delta_{JS}(X) \) significantly at the origin, but it is not analytic and is thus inadmissible by [3]'s complete class theorem. Therefore a more challenging open problem is to find admissible estimators dominating the James-Stein estimator especially at \( \|\theta\| = 0 \). In this paper, we will consider a class of estimators
\[
(1.9) \quad \delta_{a}(X) = \frac{\int_0^1 (1 - \lambda) \lambda^{(p/2-1)-1} \exp(-\|X\|^2\alpha\lambda/2) d\lambda}{\int_0^1 \lambda^{(p/2-1)-1} \exp(-\|X\|^2\alpha\lambda/2) d\lambda} X
\]
for $\alpha \geq 1$. In Section 3, we show that $\delta_\alpha$ with $\alpha \geq 1$ improves on the James-Stein estimator and that it has strictly risk improvement at $\|\theta\| = 0$. Furthermore we see that $\delta_\alpha$ approaches $\delta_{JS}^+$ as $\alpha$ goes to $\infty$. Since $\delta_\alpha$ with $\alpha = 1$ corresponds to $\delta_K$, the class of $\delta_\alpha$ with $\alpha \geq 1$ is a bridge between $\delta_K$ which is admissible and $\delta_{JS}^+$ which is inadmissible. Although we show that $\delta_\alpha$ with $\alpha > 1$ is quasi-admissible, which is introduced in Section 2, we have no idea on its admissibility at this stage.

2. Quasi-admissibility. In this section, we introduce the concept of quasi-admissibility and give a sufficient condition for quasi-admissibility. We deal with a reasonable class of estimators which have the form

$$\delta_m = X + \nabla \log m(\|X\|^2)$$

where $\nabla$ is a differential operator ($\partial / \partial x_1, \ldots, \partial / \partial x_p$)' and $m$ is a positive function. If $m(w) = w^{-c}$ for $c > 0$, (2.1) becomes the James-Stein type estimator (1.6). Any (generalized) Bayes estimator with respect to spherical symmetric measure $\pi$ should also have the form (2.1) because it is written as

$$\frac{\int_{R^p} \theta \exp(-\|x - \theta\|^2/2) \pi(d\theta)}{\int_{R^p} \exp(-\|x - \theta\|^2/2) \pi(d\theta)} = x + \frac{\int_{R^p} (\theta - x) \exp(-\|x - \theta\|^2/2) \pi(d\theta)}{\int_{R^p} \exp(-\|x - \theta\|^2/2) \pi(d\theta)}$$

$$= x + \nabla \frac{\int_{R^p} \exp(-\|x - \theta\|^2/2) \pi(d\theta)}{\int_{R^p} \exp(-\|x - \theta\|^2/2) \pi(d\theta)}$$

$$= x + \nabla \log \int_{R^p} \exp(-\|x - \theta\|^2/2) \pi(d\theta)$$

$$= x + \nabla \log m_\pi(\|x\|^2)$$

where $m_\pi(\|x\|^2) = \int_{R^p} \exp(-\|x - \theta\|^2/2) \nu(d\theta)$. (2) and (3) called an estimator of the form (2.1) pseudo-Bayes and quasi-Bayes, respectively. If, for given $m$, there exists a nonnegative measure $\nu$ which satisfies

$$m(\|x\|^2) = \int_{R^p} \exp(-\|\theta - x\|^2/2) \nu(d\theta),$$

the estimator of the form (2.1) is truly generalized Bayes. However it is often difficult to determine whether or not $m$ has such an exact integral form.
Suppose that \( \delta_{m,k}(X) = \delta_m(X) + k(\|X\|^2)X \) is a competitor of \( \delta_m \).
Then substituting \( g(x) = \nabla \log m(\|x\|^2) = 2xm'(\|x\|^2)/m(\|x\|^2) \) and \( g(x) = 2xm'(\|x\|^2)/m(\|x\|^2) + k(\|x\|^2)x \) in (1.3) respectively, we have the unbiased estimators of risk as

\[
(2.3) \hat{R}(\delta_m) = p - 4w \left( \frac{m'(w)}{m(w)} \right)^2 + 4p \frac{m'(w)}{m(w)} + 8w \frac{m''(w)}{m(w)}
\]

for \( w = \|x\|^2 \) and \( \hat{R}(\delta_{m,k}) = \hat{R}(\delta_m) + \Delta(m,mk) \) where

\[
(2.4) \Delta(m,mk) = 4wh'(w) M(w) + 2wk(w) + 4wh^2(w) + wk^2(w).
\]

If there exists \( k \) such that \( \Delta(m,mk) \leq 0 \) for any \( w \geq 0 \) with strict inequality for some \( w \), that implies that \( \delta_m \) is inadmissible, that is, \( R(\theta,\delta_{mk}) \leq R(\theta,\delta_m) \) for all \( \theta \) with strict inequality for some \( \theta \). If there does not exist such \( k \), \( \delta_m \) is said to be quasi-admissible. Hence quasi-admissibility is a weaker optimality than admissibility. Now we state a sufficient condition for quasi-admissibility. The idea is originally from [4], but the paper is not so accessible. See also [12], where quasi-admissibility is called permissibility.

**Theorem 2.1.** The estimator of the form (2.1) is quasi-admissible if

\[
\int_0^1 w^{-p/2}m(w)^{-1}dw = \infty \quad \text{as well as} \quad \int_1^\infty w^{-p/2}m(w)^{-1}dw = \infty.
\]

**Proof.** We have only to show that \( \hat{R}(\delta_{m,k}) \leq \hat{R}(\delta_m) \), that is, \( \Delta(m,mk) \leq 0 \) implies \( k(w) \equiv 0 \). Let \( M(w) = w^{p/2}m(w) \) and \( h(w) = M(w)k(w) \). Then we have

\[
\Delta(m,mk) = 4w \frac{h'(w)}{M(w)} + w \frac{h^2(w)}{M^2(w)} = 4wh^2(w) M(w) \left( - \frac{d}{dw} \left\{ \frac{1}{h(w)} \right\} + \frac{1}{4M(w)} \right) .
\]

First we show that \( h(w) \geq 0 \) for all \( w \geq 0 \). Suppose to the contrary that \( h(w) < 0 \) for some \( w_0 \). Then \( h(w) < 0 \) for all \( w \geq w_0 \) since \( h'(w) \) should be negative. For all \( w > w_0 \), the inequality

\[
(2.5) \frac{d}{dw} \left( \frac{1}{h(w)} \right) \geq \frac{1}{4M(w)}
\]

should be satisfied. Integrating both sides of (2.5) from \( w_0 \) to \( w^* \) leads to

\[
\frac{1}{h(w^*)} - \frac{1}{h(w_0)} \geq \frac{1}{4} \int_{w_0}^{w^*} M^{-1}(t)dt .
\]
As \( w^* \to \infty \), the right-hand side of above inequality tends to infinity, and this provides a contradiction since the left-hand side is less than \(-1/h(w_0)\). Thus we have \( g(w) \geq 0 \) for all \( w \).

Similarly we can show that \( g(w) \leq 0 \) for all \( w \). It follows that \( h(w) \) is zero for all \( w \), which implies that \( k(w) \equiv 0 \) for all \( w \). This completes the proof.

Combining Theorem 2.1 and \([3]\)'s sufficient condition for admissibility, we see that a quasi-admissible estimator of the form (2.1) is admissible if it is truly generalized Bayes, that is, there exists a nonnegative measure \( \nu \) such that

\[
m(||x||^2) = \int_{\mathbb{R}^p} \exp(-||x - \theta||^2/2) \nu(d\theta).
\]

However, for given \( m \), it is often quite difficult to determine whether \( m \) has such an integral form. Furthermore, even if we find that \( m \) does not have an integral form like (2.6), that is, the estimator is inadmissible, it is generally very difficult to find an estimator dominating the inadmissible estimator.

The function \( m \) for the James-Stein estimator is \( m(w) = w^2 - p \), which satisfies the assumptions of Theorem 2.1.

**Corollary 2.1.** The James-Stein estimator is quasi-admissible.

In this case, it is not difficult to find an estimator dominating \( \delta_{JS} \) by taking positive part. But it is not easy to find a large class of estimators dominating the James-Stein estimator. In Section 3, we introduce an elegant sufficient condition for domination over \( \delta_{JS} \) proposed by \([9]\).

**3. A class of quasi-admissible estimators improving upon the James-Stein estimator.** In this section, we introduce \([9]\)'s sufficient condition for improving upon the James-Stein estimator and propose a class of smooth quasi-admissible estimators satisfying it.

\([9]\) showed that if \( \lim_{w \to \infty} \phi(w) = p-2 \) then the difference of risk functions between \( \delta_{JS} \) and \( \delta_{\phi} \) can be written as

\[
R(\theta, \delta_{JS}) - R(\theta, \delta_{\phi}) = 2 \int_{0}^{\infty} \phi'(w) \left( \phi(w) - (p - 2) + \frac{2f(w, \lambda)}{\int_{0}^{w} y^{-1} f_p(y, \lambda)dy} \right) \times \int_{0}^{w} y^{-1} f_p(y, \lambda)dydw,
\]

(3.1)
where $\lambda = \|\theta\|^2$ and $f_p(x; \lambda)$ denotes a density of a non-central chi-square distribution with $p$ degrees of freedom and non-centrality parameter $\lambda$.

Moreover by the inequality
\[
\frac{f_p(w; \lambda)}{\int_0^w y^{-1} f_p(y; \lambda) dy} \geq \frac{f_p(w)}{\int_0^w y^{-1} f_p(y) dy},
\]
where $f_p(y) = f_p(y; 0)$, which can be shown by the correlation inequality, we have
\[
R(\theta, \delta_{JS}) - R(\theta, \delta_\phi)
\geq 2 \int_0^\infty \phi'(w) (\phi(w) - \phi_0(w)) \left( \int_0^w y^{-1} f_p(y, \lambda) dy \right) dw,
\]
where
\[
\phi_0(w) = p - 2 + 2\frac{f_p(w)}{\int_0^w y^{-1} f_p(y) dy}.
\]
Since $\phi_0 = \phi_K$ given in (1.8) by an integration by parts and $\lim_{w \to \infty} \phi_K(w) = p - 2$, we have the following result.

**Theorem 3.1** ([9]). If $\phi(w)$ is nondecreasing and within $[\phi_K(w), p - 2]$ for any $w \geq 0$, then $\delta_\phi(X)$ of form (1.1) dominates the James-Stein estimator.

The assumption of the theorem above is satisfied by $\phi_K$ and $\phi_{JS}^+$. By (3.1), we see that the risk difference at $\|\theta\| = 0$ between $\delta_{JS}$ and $\delta_\phi$, the limit of which is $p - 2$, is given by
\[
(3.2) \quad 2 \int_0^\infty \phi'(w) (\phi(w) - \phi_K(w)) \left( \int_0^w y^{-1} f_p(y) dy \right) dw.
\]
Hence $\delta_K$ does not improve upon $\delta_{JS}(X)$ at $\|\theta\| = 0$, although $\delta_K(X)$ is an admissible generalized Bayes estimator and thus smooth. On the other hand, $\delta_{JS}^+(X)$ improves on $\delta_{JS}(X)$ significantly at the origin, but it is not analytic and is thus inadmissible by [3]'s complete class theorem. Therefore a more challenging problem is to find admissible estimators dominating the James-Stein estimator especially at $\|\theta\| = 0$.

In this paper, we propose a class of estimators
\[
(3.3) \quad \delta_\alpha(X) = \left( 1 - \phi_\alpha(\|X\|^2)/\|X\|^2 \right) X
\]
where

$$\phi_\alpha(w) = \frac{\int_0^1 \lambda^\alpha \exp(-w \lambda/2) d\lambda}{\int_0^1 \lambda^\alpha \exp(-w \lambda/2) d\lambda}$$

$$= p - 2 - \frac{2 \exp(-w \lambda/2)}{\alpha \int_0^1 \lambda^\alpha \exp(-w \lambda/2) d\lambda}$$

$$= p - 2 - \frac{2}{\alpha \int_0^1 (1 - \lambda)^{\alpha(p/2-1)-1} \exp(w \lambda/2) d\lambda}.$$  

(3.4)

The main theorem of this paper is as follows.

**Theorem 3.2.**

1. \(\delta_\alpha(X)\) dominates \(\delta_{JS}(X)\) for \(\alpha \geq 1\).

2. The risk of \(\delta_\alpha(X)\) for \(\alpha > 1\) at \(\|\theta\| = 0\) is strictly less than the risk of the James-Stein estimator at \(\|\theta\| = 0\).

3. \(\delta_\alpha(X)\) approaches the positive-part James-Stein estimator when \(\alpha\) tends to infinity, that is,

$$\lim_{\alpha \to \infty} \delta_\alpha(X) = \delta_{JS}^+(X).$$

Clearly \(\delta_1(X) = \delta_K(X)\). The class of \(\delta_\alpha\) with \(\alpha \geq 1\) is a bridge between \(\delta_K\) which is admissible and \(\delta_{JS}^+\) which is inadmissible.

**Proof.** [part 1] We shall verify that \(\phi_\alpha(w)\) for \(\alpha \geq 1\) satisfies assumptions in Theorem 3.1. Applying the Taylor expansion to a part of (3.4), we have

$$\frac{\alpha}{2} \int_0^1 (1 - \lambda)^{\alpha(p/2-1)-1} \exp(w \lambda/2) d\lambda$$

$$= \sum_{i=0}^{\infty} w^i \prod_{j=0}^{i} (p - 2 + 2j/\alpha)^{-1} = \psi(\alpha, w) \quad (say.)$$

As \(\psi(\alpha, w)\) is increasing in \(w\), \(\phi_\alpha(w)\) is increasing in \(w\). As \(\lim_{w \to \infty} \psi(\alpha, w) = \infty\), for any \(\alpha \geq 1\), it is clear that \(\lim_{w \to \infty} \phi_\alpha(w) = p - 2\). In order to show that \(\phi_\alpha(w) \geq \phi_K(w) = \phi_1(w)\) for \(\alpha \geq 1\), we have only to check that \(\psi(\alpha, w)\) is increasing in \(\alpha\). It is easily verified because the coefficient of each term of \(\psi(\alpha, w)\) is increasing in \(\alpha\). We have thus proved the theorem.

[part 2] Since \(\phi_\alpha\) for \(\alpha > 1\) is strictly greater than \(\phi_K\) and strictly increasing in \(w\), the risk difference of \(\delta_\alpha\) for \(\alpha > 1\) and \(\delta_{JS}^+\) at \(\|\theta\| = 0\), which is given in (3.2), is strictly positive.
[part 3] Since \( \psi(\alpha, w) \) is increasing in \( \alpha \), it converges to
\[
(p - 2)^{-1} \sum_{i=0}^{\infty} \left( \frac{w}{p - 2} \right)^i
\]
by the monotone convergence theorem when \( \alpha \) goes to infinity. Considering two cases: \( w < (\geq)p - 2 \), we obtain \( \lim_{\alpha \to \infty} \phi_\alpha(w) = w \) if \( w < p - 2 \); = \( p - 2 \) otherwise. This completes the proof. \( \square \)

The estimator \( \delta_\alpha \) is expressed as \( X + \nabla \log m_\alpha(\|X\|^2) \) where
\[
m_\alpha(w) = \left\{ \int_0^1 \lambda^{\alpha(p/2 - 1) - 1} \exp \left( -\frac{\alpha w}{2} \lambda \right) d\lambda \right\}^{1/\alpha}.
\]
Since \( m_\alpha(w) \sim w^{2-p} \) for sufficiently large \( w \) by Tauberian’s theorem and \( 0 < m_\alpha(0) < \infty \), we have the following result by Theorem 2.1.

**Corollary 3.1.** \( \delta_\alpha(X) \) is quasi-admissible for \( p \geq 3 \).

Needless to say, we are extremely interested in determining whether or not \( \delta_\alpha(X) \) for \( \alpha > 1 \) is admissible. Since \( \delta_\alpha(X) \) with \( \alpha > 1 \) is quasi-admissible, it is admissible if it is generalized Bayes, that is, there exists a measure \( \nu \) which satisfies
\[
\int_{\mathbb{R}^p} \exp(-\|\theta - x\|^2/2)\nu(d\theta) = \left( \int_0^1 \lambda^{\alpha(p/2 - 1) - 1} \exp\left( -\alpha \|X\|^2/2\right) d\lambda \right)^{1/\alpha}.
\]
I have no idea on the way to construct such a measure \( \nu \) so far. Even if we find that there is no \( \nu \), which implies \( \delta_\alpha \) is inadmissible, it is very difficult to find an estimator dominating \( \delta_\alpha \) for \( \alpha > 1 \).

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