Invariant chiral differential operators
and the $\mathcal{W}_3$ algebra

Andrew R. Linshaw

ABSTRACT. Attached to a vector space $V$ is a vertex algebra $S(V)$ known as the $\beta\gamma$-system or algebra of chiral differential operators on $V$. It is analogous to the Weyl algebra $D(V)$, and is related to $D(V)$ via the Zhu functor. If $G$ is a connected Lie group with Lie algebra $\mathfrak{g}$, and $V$ is a linear $G$-representation, there is an action of the corresponding affine algebra on $S(V)$. The invariant space $S(V)^{\mathfrak{g}[t]}$ is a commutant subalgebra of $S(V)$, and plays the role of the classical invariant ring $D(V)^G$. When $G$ is an abelian Lie group acting diagonally on $V$, we find a finite set of generators for $S(V)^{\mathfrak{g}[t]}$, and show that $S(V)^{\mathfrak{g}[t]}$ is a simple vertex algebra and a member of a Howe pair. The Zamolodchikov $\mathcal{W}_3$ algebra with $c = -2$ plays a fundamental role in the structure of $S(V)^{\mathfrak{g}[t]}$.

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1. Introduction

Let $G$ be a connected, reductive Lie group acting algebraically on a smooth variety $X$. Throughout this paper, our base field will always be $\mathbb{C}$. The ring $D(X)^G$ of invariant
differential operators on $X$ has been much studied in recent years. In the case where $X$ is the homogeneous space $G/K$, $D(X)^G$ was originally studied by Harish-Chandra in order to understand the various function spaces attached to $X$ [8][9]. In general, $D(X)^G$ is not a homomorphic image of the universal enveloping algebra of a Lie algebra, but it is believed that $D(X)^G$ shares many properties of enveloping algebras. For example, the center of $D(X)^G$ is always a polynomial ring [12]. In the case where $G$ is a torus, the structure and representation theory of the rings $D(X)^G$ were studied extensively in [16], but much less is known about $D(X)^G$ when $G$ is nonabelian. The first step in this direction was taken by Schwarz in [17], in which he considered the special but nontrivial case where $G = SL(3)$ and $X$ is the adjoint representation. In this case, he found generators for $D(X)^G$, showed that $D(X)^G$ is an FCR algebra, and classified its finite-dimensional modules.

1.1. A vertex algebra analogue of $D(X)^G$

In [15], Malikov-Schechtman-Vaintrob introduced a sheaf of vertex algebras on any smooth variety $X$ known as the chiral de Rham complex. For an affine open set $V \subset X$, the algebra of sections over $V$ is just a copy of the $bc\beta\gamma$-system $S(V) \otimes \mathcal{E}(V)$, localized over the function ring $\mathcal{O}(V)$. A natural question is whether there exists a subsheaf of “chiral differential operators” on $X$, whose space of sections over $V$ is just the (localized) $\beta\gamma$-system $S(V)$. For general $X$, there is a cohomological obstruction to the existence of such a sheaf, but it does exist in certain special cases such as affine spaces and certain homogeneous spaces [15][7].

In this paper, we focus on the case where $X$ is the affine space $V = \mathbb{C}^n$, and we take $S(V)$ to be our algebra of chiral differential operators on $V$. $S(V)$ is related to $D(V)$ via the Zhu functor, which attaches to every vertex algebra $\mathcal{V}$ an associative algebra $A(\mathcal{V})$ known as the Zhu algebra of $\mathcal{V}$, together with a surjective linear map $\pi_{Zh}: \mathcal{V} \to A(\mathcal{V})$.

If $V$ carries a linear action of a group $G$ with Lie algebra $\mathfrak{g}$, the corresponding representation $\rho: \mathfrak{g} \to \text{End}(V)$ induces a vertex algebra homomorphism

$$\mathcal{O}(\mathfrak{g}, B) \to S(V). \quad (1.1)$$

Here $\mathcal{O}(\mathfrak{g}, B)$ is the current algebra of $\mathfrak{g}$ associated to the bilinear form $B(\xi, \eta) = -\text{Tr}(\rho(\xi)\rho(\eta))$ on $\mathfrak{g}$. Letting $\Theta$ denote the image of $\mathcal{O}(\mathfrak{g}, B)$ inside $S(V)$, the commutant $\text{Com}((\Theta, S(V)))$, which we denote by $S(V)^{\Theta^+}$, is just the invariant space $S(V)^{\Theta^+}$. 

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Accordingly, we call \( S(V)^\Theta^+ \) the algebra of invariant chiral differential operators on \( V \).

There is a commutative diagram

\[
\begin{array}{ccc}
S(V)^\Theta^+ & \hookrightarrow & S(V) \\
\pi \downarrow & & \pi_{Zh} \downarrow \\
D(V)^G & \hookrightarrow & D(V)
\end{array}
\] (1.2)

Here the horizontal maps are inclusions, and the map \( \pi \) on the left is the restriction of the Zhu map on \( S(V) \) to the subalgebra \( S(V)^\Theta^+ \). In general, \( \pi \) is not surjective, and \( D(V)^G \) need not be the Zhu algebra of \( S(V)^\Theta^+ \).

For a general vertex algebra \( V \) and subalgebra \( A \), the commutant \( Com(A, V) \) was introduced by Frenkel-Zhu in [4], generalizing a previous construction in representation theory [10] and conformal field theory [6] known as the coset construction. We regard \( V \) as a module over \( A \) via the left regular action, and we regard \( Com(A, V) \), which we often denote by \( V^A^+ \), as the invariant subalgebra. Finding a set of generators for \( V^A^+ \), or even determining when it is finitely generated as a vertex algebra, is generally a non-trivial problem. It is also natural to study the double commutant \( Com(V^A^+, V) \), which always contains \( A \). If \( A = Com(V^A^+, V) \), we say that \( A \) and \( V^A^+ \) form a Howe pair inside \( V \).

Since

\[
Com(Com(V^A^+, V), V) = V^A^+,
\]
a subalgebra \( B \) is a member of a Howe pair if and only if \( B = V^A^+ \) for some \( A \).

Here are some natural questions one can ask about \( S(V)^\Theta^+ \) and its relationship to \( D(V)^G \).

**Question 1.1.** When is \( S(V)^\Theta^+ \) finitely generated as a vertex algebra? Can we find a set of generators?

**Question 1.2.** When do \( S(V)^\Theta^+ \) and \( \Theta \) form a Howe pair inside \( S(V) \)? In the case where \( G = SL(2) \) and \( V \) is the adjoint module, this question was answered affirmatively.
in [13].

**Question 1.3.** What are the vertex algebra ideals in $S(V)^{\Theta+}$, and when is $S(V)^{\Theta+}$ a simple vertex algebra?

**Question 1.4.** When is $S(V)^{\Theta+}$ a conformal vertex algebra?

**Question 1.5.** When is $\pi : S(V)^{\Theta+} \to D(V)^G$ surjective? More generally, describe $\text{Im}(\pi)$ and $\text{Coker}(\pi)$.

These questions are somewhat outside the realm of classical invariant theory because the Lie algebra $g[t]$ is both infinite-dimensional and non-reductive. Moreover, when $G$ is nonabelian, $S(V)$ need not decompose into a sum of irreducible $\mathcal{O}(g, B)$-modules. The case where $G$ is simple and $V$ is the adjoint module is of particular interest to us, since in this case $S(V)^{\Theta+}$ is a subalgebra of the complex $(\mathcal{W}(g)_{bas}, d)$ which computes the chiral equivariant cohomology of a point [14].

In this paper, we focus on the case where $G$ is an abelian group acting faithfully and diagonalizably on $V$. This is much easier than the general case because $\mathcal{O}(g, B)$ is then a tensor product of Heisenberg vertex algebras, which act completely reducibly on $S(V)$. For any such action, we find a finite set of generators for $S(V)^{\Theta+}$, and show that $S(V)^{\Theta+}$ is a simple vertex algebra. Moreover, $S(V)^{\Theta+}$ and $\Theta$ always form a Howe pair inside $S(V)$. For generic actions, we show that $S(V)^{\Theta+}$ admits a $k$-parameter family of conformal structures where $k = \dim V - \dim g$, and we find a finite set of generators for $\text{Im}(\pi)$. Finally, we show that $\text{Coker}(\pi)$ is always a finitely generated module over $\text{Im}(\pi)$ with generators corresponding to central elements of $D(V)^G$. The Zamolodchikov $\mathcal{W}_3$ algebra of central charge $c = -2$ plays an important role in the structure of $S(V)^{\Theta+}$. Our description relies on the fundamental papers [18] [19] of W. Wang, in which he classified the irreducible modules of $\mathcal{W}_{3,-2}$.

In the case where $G$ is nonabelian, very little is known about the structure of $S(V)^{\Theta+}$, and the representation-theoretic techniques used in the abelian case cannot be expected to
work. In a separate paper, we will use tools from commutative algebra to describe $S(V)_{\Theta^+}$ in the special cases where $G$ is one of the classical Lie groups $SL(n)$, $SO(n)$, or $Sp(2n)$, and $V$ is a direct sum of copies of the standard representation.

One hopes that the vertex algebra point of view can also shed some light on the classical algebras $D(V)^G$. For example, the vertex algebra products on $S(V)$ induce a family of bilinear operations $\ast_k$, $k \geq -1$ on $D(V)^G$, which coincide with classical operations known as transvectants. $D(V)^G$ is generally not simple as an associative algebra, but in the case where $G$ is an abelian group acting diagonalizably on $V$, $D(V)^G$ is always simple as a $*$-algebra in the obvious sense.

1.2. Acknowledgements

I thank B. Lian for helpful conversations and for suggesting the Friedan-Martinec-Shenker bosonization as a tool in studying commutant subalgebras of $S(V)$. I also thank A. Knutson, G. Schwarz, and N. Wallach for helpful discussions about classical invariant theory, especially the theory of invariant differential operators.

2. Invariant differential operators

Fix a basis $\{x_1, \ldots, x_n\}$ for $V$ and a corresponding dual basis $\{x'_1, \ldots, x'_n\}$ for $V^*$. The Weyl algebra $D(V)$ is generated by the linear functions $x'_i$ and the first-order differential operators $\frac{\partial}{\partial x'_i}$, which satisfy $[\frac{\partial}{\partial x'_i}, x'_j] = \delta_{i,j}$. Equip $D(V)$ with the Bernstein filtration

$$D(V)_{(0)} \subset D(V)_{(1)} \subset \cdots,$$

defined by $(x'_1)^{k_1} \cdots (x'_n)^{k_n}(\frac{\partial}{\partial x'_1})^{l_1} \cdots (\frac{\partial}{\partial x'_n})^{l_n} \in D(V)_{(r)}$ if $k_1 + \cdots + k_n + l_1 + \cdots + l_n \leq r$. Given $\omega \in D(V)_{(r)}$ and $\nu \in D(V)_{(s)}$, $[\omega, \nu] \in D(V)_{(r+s-2)}$, so that

$$grD(V) = \bigoplus_{r>0} D(V)_{(r)}/D(V)_{(r-1)} \cong Sym(V \oplus V^*).$$

We say that $deg(\alpha) = d$ if $\alpha \in D(V)_{(d)}$ and $\alpha \notin D(V)_{(d-1)}$. 

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Let $G$ be a connected Lie group with Lie algebra $\mathfrak{g}$, and let $V$ be a linear representation of $G$ via $\rho : G \to Aut(V)$. Then $G$ acts on $\mathcal{D}(V)$ by algebra automorphisms, and induces an action $\rho^* : \mathfrak{g} \to Der(\mathcal{D}(V))$ by derivations of degree zero. Since $G$ is connected, the invariant ring $\mathcal{D}(V)^G$ coincides with $\mathcal{D}(V)^{\mathfrak{g}}$, where

$$\mathcal{D}(V)^{\mathfrak{g}} = \{ \omega \in \mathcal{D}(V) | \rho^*(\xi)(\omega) = 0, \forall \xi \in \mathfrak{g} \}.$$ 

We will usually work with the action of $\mathfrak{g}$ rather than $G$, and for greater flexibility, we do not assume that the $\mathfrak{g}$-action comes from an action of a reductive group $G$.

The action of $\mathfrak{g}$ on $\mathcal{D}(V)$ can be realized by inner derivations: there is a Lie algebra homomorphism

$$\tau : \mathfrak{g} \to \mathcal{D}(V), \quad \xi \mapsto -\sum_{i=1}^{n} x_i^\prime \rho^*(\xi)(\frac{\partial}{\partial x_i^\prime}).$$

(2.3)

$\tau(\xi)$ is just the linear vector field on $V$ generated by $\xi$, so $\xi \in \mathfrak{g}$ acts on $\mathcal{D}(V)$ by $[\tau(\xi), -]$. Clearly $\tau$ extends to a map $\mathfrak{U}\mathfrak{g} \to \mathcal{D}(V)$, and

$$\mathcal{D}(V)^{\mathfrak{g}} = Com(\tau(\mathfrak{U}\mathfrak{g}), \mathcal{D}(V)).$$

Since $\mathfrak{g}$ acts on $\mathcal{D}(V)$ by derivations of degree zero, (2.1) restricts to a filtration $\mathcal{D}(V)_{(0)}^{\mathfrak{g}} \subset \mathcal{D}(V)_{(1)}^{\mathfrak{g}} \subset \cdots$ on $\mathcal{D}(V)^{\mathfrak{g}}$, and $gr(\mathcal{D}(V)^{\mathfrak{g}}) \cong gr(\mathcal{D}(V))^{\mathfrak{g}} \cong Sym(V \oplus V^*)^{\mathfrak{g}}$.

2.1. The case where $\mathfrak{g}$ is abelian

Our main focus is on the case where $\mathfrak{g}$ is the abelian Lie algebra $\mathfrak{C}^m = gl(1) \oplus \cdots \oplus gl(1)$, acting diagonally on $V$. Let $R(V)$ be the $\mathbb{C}$-vector space of all diagonal representations of $\mathfrak{g}$. Given $\rho \in R(V)$ and $\xi \in \mathfrak{g}$, $\rho(\xi)$ is a diagonal matrix with entries $a_1^\xi, \ldots, a_n^\xi$, which we regard as a vector $a^\xi = (a_1^\xi, \ldots, a_n^\xi) \in \mathbb{C}^n$. Let $A(\rho) \subset \mathbb{C}^n$ be the subspace spanned by $\{ \rho(\xi) | \xi \in \mathfrak{g} \}$.

The action of $GL(m)$ on $\mathfrak{g}$ induces a natural action of $GL(m)$ on $R(V)$, defined by

$$(g \cdot \rho)(\xi) = \rho(g^{-1} \cdot \xi)$$

(2.4)

for all $g \in GL(m)$. Clearly $A(\rho) = A(g \cdot \rho)$ for all $g \in GL(m)$. Note that $dim Ker(\rho) = dim Ker(g \cdot \rho)$ for all $g \in GL(m)$, so in particular $GL(m)$ acts on the dense open set
\(R^0(V) = \{\rho \in R(V) | \ker(\rho) = 0\}\). The correspondence \(\rho \mapsto A(\rho)\) identifies \(R^0(V)/GL(m)\) with the Grassmannian \(Gr(m, n)\) of \(m\)-dimensional subspaces of \(\mathbb{C}^n\).

Given \(\rho \in R(V)\), \(\mathcal{D}(V)^g = \mathcal{D}(V)^{g'}\) where \(g' = g/Ker(\rho)\), so we may assume without loss of generality that \(\rho \in R^0(V)\). We denote \(\mathcal{D}(V)^g\) by \(\mathcal{D}(V)^g_{\rho}\) when we need to emphasize the dependence on \(\rho\). Given \(\omega \in \mathcal{D}(V)\), the condition \(\rho^*(\xi)(\omega) = 0\) for all \(\xi \in g\) is equivalent to the condition that \(\rho^*(g \cdot \xi)(\omega) = 0\) for all \(\xi \in g\), so it follows that \(\mathcal{D}(V)^g_{\rho} = \mathcal{D}(V)^g_{g \cdot \rho}\) for all \(g \in GL(m)\). Hence the family of algebras \(\mathcal{D}(V)^g_{\rho}\) is parametrized by the points \(A(\rho) \in Gr(m, n)\).

Fix \(\rho \in R^0(V)\), and choose a basis \(\{\xi^1, \ldots, \xi^m\}\) for \(g\). Let \(a^i = (a^i_1, \ldots, a^i_n) \in \mathbb{C}^n\) be the vectors corresponding to the diagonal matrices \(\rho(\xi^i)\), and let \(A = A(\rho)\) be the subspace spanned by these vectors. The map \(\tau : g \rightarrow \mathcal{D}(V)\) is defined by

\[
\tau(\xi^i) = -\sum_{j=1}^{n} a^i_j x'_j \frac{\partial}{\partial x'_j}.
\] (2.5)

The Euler operators \(\{e_j = x'_j \frac{\partial}{\partial x'_j} | j = 1, \ldots, n\}\) lie in \(\mathcal{D}(V)^g\), and we denote the polynomial algebra \(\mathbb{C}[e_1, \ldots, e_n]\) by \(E\).

For each \(j = 1, \ldots, n\) and \(d \in \mathbb{Z}\), define \(v^d_j \in \mathcal{D}(V)\) by

\[
v^d_j = \begin{cases} 
(\frac{\partial}{\partial x'_j})^{-d} & d < 0 \\
1 & d = 0 \\
(x'_j)^d & d > 0
\end{cases}
\] (2.6)

Let \(\mathbb{Z}^n \subset \mathbb{C}^n\) denote the lattice generated by the standard basis, and for each lattice point \(l = (l_1, \ldots, l_n) \in \mathbb{Z}^n\), define

\[
\omega_l = \prod_{j=1}^{n} v^l_{j}.
\] (2.7)

As a module over \(E\),

\[
\mathcal{D}(V) = \bigoplus_{l \in \mathbb{Z}^n} M_l,
\] (2.8)

where \(M_l\) is the free \(E\)-module generated by \(\omega_l\). Moreover, we have

\[
[e_j, \omega_l] = l_j \omega_l,
\] (2.9)
so the \( Z^n \)-grading (2.8) is just the eigenspace decomposition of \( \mathcal{D}(V) \) under the family of diagonalizable operators \( [e_j, -] \). In particular, (2.9) shows that
\[
\rho^*(\xi^i)(\omega_l) = \langle \tau(\xi^i), \omega_l \rangle = -\langle l, a^i \rangle \omega_l,
\]
where \( \langle \cdot, \cdot \rangle \) denotes the standard inner product on \( \mathbb{C}^n \). Hence \( \omega_l \) lies in \( \mathcal{D}(V)^\Phi \) precisely when \( l \in A^\perp \), so
\[
\mathcal{D}(V)^\Phi = \bigoplus_{l \in A^\perp \cap Z^n} M_l.
\] (2.11)
For generic actions, the lattice \( A^\perp \cap Z^n \) has rank zero, so \( \mathcal{D}(V)^\Phi = M_0 = E \).

Consider the double commutant \( \text{Com}(\mathcal{D}(V)^\Phi, \mathcal{D}(V)) \), which always contains \( T = \tau(\mathcal{U}g) = \mathbb{C}[\tau(\xi_1), \ldots, \tau(\xi_m)] \). Since \( \text{Com}(E, \mathcal{D}(V)) = E \), we have \( \text{Com}(\mathcal{D}(V)^\Phi, \mathcal{D}(V)) = E \) for generic actions.

Suppose next that \( A^\perp \cap Z^n \) has rank \( r \) for some \( 0 < r \leq n - m \). For \( i = 1, \ldots, r \) let \( \{l^i = (l^i_1, \ldots, l^i_n)\} \) be a basis for \( A^\perp \cap Z^n \), and let \( L \) be the \( \mathbb{C} \)-vector space spanned by \( \{l^1, \ldots, l^r\} \). If \( r < n - m \), we can choose vectors \( s^k = (s^k_1, \ldots, s^k_n) \in L^\perp \cap A^\perp \), so that \( \{l^1, \ldots, l^r, s^{r+1}, \ldots, s^{n-m}\} \) is a basis for \( A^\perp \). For \( i = 1, \ldots, r \) and \( k = r + 1, \ldots, n - m \), define differential operators
\[
\phi^i = \sum_{j=1}^n l^i_j e_j, \quad \psi^k = \sum_{j=1}^n s^k_j e_j.
\]
Note that \( \mathbb{C}[e_1, \ldots, e_n] = T \otimes \Psi \otimes \Phi \), where \( \Phi = \mathbb{C}[\phi^1, \ldots, \phi^r] \) and \( \Psi = \mathbb{C}[\psi^{r+1}, \ldots, \psi^{n-m}] \).

**Theorem 2.1.** \( \text{Com}(\mathcal{D}(V)^\Phi, \mathcal{D}(V)) = T \otimes \Psi \). Hence \( \mathcal{D}(V)^\Phi \) and \( T \) form a pair of mutual commutants inside \( \mathcal{D}(V) \) precisely when \( \Psi = \mathbb{C} \), which occurs when \( A^\perp \cap Z^n \) has rank \( n - m \).

**Proof:** By (2.9), for any lattice point \( l \in A^\perp \cap Z^n \), and for \( k = r + 1, \ldots, n - m \) we have
\[
[\psi^k, \omega_l] = \langle s^k, l \rangle \omega_l = 0
\]
since \( s^k \in L^\perp \). It follows that \( \Psi \subset \text{Com}(\mathcal{D}(V)^\Phi, \mathcal{D}(V)) \). Hence \( T \otimes \Psi \subset \text{Com}(\mathcal{D}(V)^\Phi, \mathcal{D}(V)) \). Moreover, since \( [\phi^i, \omega_l] = \langle l^i, l \rangle \omega_l \) and \( \{l^1, \ldots, l^r\} \) form a basis for \( A^\perp \cap Z^n \), it follows that the variables \( \phi^i \) cannot appear in any element \( \omega \in \text{Com}(\mathcal{D}(V)^\Phi, \mathcal{D}(V)) \). \( \square \)

In the case \( \Psi = \mathbb{C} \), we can recover the action \( \rho \) (up to \( GL(m) \)-equivalence) from the algebra \( \mathcal{D}(V)^\Phi \) by taking its commutant inside \( \mathcal{D}(V) \), but otherwise \( \mathcal{D}(V)^\Phi \) does not determine the action.
3. Vertex algebras

We will assume that the reader is familiar with the basic notions in vertex algebra theory. For a list of references, see page 117 of [13]. We briefly describe the examples and constructions that we need, following the notation in [13].

Given a Lie algebra $\mathfrak{g}$ equipped with a symmetric $\mathfrak{g}$-invariant bilinear form $B$, the current algebra $\mathcal{O}(\mathfrak{g}, B)$ is the universal vertex algebra with generators $X^\xi(z), \xi \in \mathfrak{g}$, which satisfy the OPE relations

$$X^\xi(z)X^\eta(w) \sim B(\xi, \eta)(z - w)^{-2} + X^{[\xi, \eta]}(w)(z - w)^{-1}.$$ 

Given a finite-dimensional vector space $V$, the $\beta\gamma$-system, or algebra of chiral differential operators $\mathcal{S}(V)$, was introduced in [5]. It is the unique vertex algebra with generators $\beta^x(z), \gamma^{x'}(z)$ for $x \in V, x' \in V^*$, which satisfy

$$\beta^x(z)\gamma^{x'}(w) \sim \langle x', x \rangle (z - w)^{-1}, \quad \gamma^{x'}(z)\beta^y(w) \sim -\langle x', x \rangle (z - w)^{-1},$$

$$\beta^x(z)\beta^y(w) \sim 0, \quad \gamma^{x'}(z)\gamma^{y'}(w) \sim 0. \quad (3.1)$$

Given $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{C}^n$, $\mathcal{S}(V)$ has a Virasoro element

$$L^\alpha(z) = \sum_{i=1}^n (\alpha_i - 1) : \partial \beta^{x_i}(z)\gamma^{x'_i}(z) : + \alpha_i : \beta^{x_i}(z)\partial \gamma^{x'_i}(z) : \quad (3.2)$$

of central charge $\sum_{i=1}^n (12\alpha_i^2 - 12\alpha_i + 2)$. Here $\{x_1, \ldots, x_n\}$ is any basis for $V$ and $\{x'_1, \ldots, x'_n\}$ is the corresponding dual basis for $V^*$. An OPE calculation shows that $\beta^{x_i}(z), \gamma^{x'_i}(z)$ are primary of conformal weights $\alpha_i, 1 - \alpha_i$, respectively.

$\mathcal{S}(V)$ has an additional $\mathbb{Z}$-grading which we call the $\beta\gamma$-charge. Define

$$v(z) = \sum_{i=1}^n : \beta^{x_i}(z)\gamma^{x'_i}(z) : \quad (3.3)$$

The zeroth Fourier mode $v(0)$ acts diagonalizably on $\mathcal{S}(V)$; the $\beta\gamma$-charge grading is just the eigenspace decomposition of $\mathcal{S}(V)$ under $v(0)$. For $x \in V$ and $x' \in V^*$, $\beta^x(z)$ and $\gamma^{x'}(z)$ have $\beta\gamma$-charges $-1$ and $1$, respectively.
There is also an odd vertex algebra $\mathcal{E}(V)$ known as a $bc$-system, or a semi-infinite exterior algebra, which is generated by $b^x(z)$, $c^{x'}(z)$ for $x \in V$ and $x' \in V^*$, which satisfy

\begin{align*}
    b^x(z)c^{x'}(w) &\sim (x', x)(z - w)^{-1}, & c^{x'}(z)b^x(w) &\sim (x', x)(z - w)^{-1}, \\
    b^x(z)b^y(w) &\sim 0, & c^{x'}(z)c^{y'}(w) &\sim 0.
\end{align*}

$\mathcal{E}(V)$ has an analogous conformal structure $L^\alpha(z)$ for any $\alpha \in \mathbb{C}^n$, and an analogous $\mathbb{Z}$-grading which we call the $bc$-charge. Define

\[ q(z) = -\sum_{i=1}^{n} : b^{x_i}(z)c^{x'_i}(z) :. \tag{3.4} \]

The zeroth Fourier mode $q(0)$ acts diagonalizably on $S(V)$, and the $bc$-charge grading is just the eigenspace decomposition of $\mathcal{E}(V)$ under $q(0)$. Clearly $b^x(z)$ and $c^{x'}(z)$ have $bc$-charges $-1$ and $1$, respectively.

### 3.1. The commutant construction

**Definition 3.1.** Let $\mathcal{V}$ be a vertex algebra, and let $\mathcal{A}$ be a subalgebra. The commutant of $\mathcal{A}$ in $\mathcal{V}$, denoted by $\text{Com}(\mathcal{A}, \mathcal{V})$ or $\mathcal{V}^\mathcal{A}+$, is the subalgebra of vertex operators $v \in \mathcal{V}$ such that $[a(z), v(w)] = 0$ for all $a \in \mathcal{A}$. Equivalently, $a(z) \circ_n v(z) = 0$ for all $a \in \mathcal{A}$ and $n \geq 0$.

We regard $\mathcal{V}$ as a module over $\mathcal{A}$, and we regard $\mathcal{V}^\mathcal{A}+$ as the invariant subalgebra. If $\mathcal{A}$ is a homomorphic image of a current algebra $\mathcal{O}(\mathfrak{g}, B)$, $\mathcal{V}^\mathcal{A}+$ is just the invariant space $\mathcal{V}^g[t]$. We will always assume that $\mathcal{V}$ is equipped with a weight grading, and that $\mathcal{A}$ is a graded subalgebra, so that $\mathcal{V}^\mathcal{A}+$ is also a graded subalgebra of $\mathcal{V}$.

Our main example of this construction comes from a representation $\rho : \mathfrak{g} \to \text{End}(V)$ of a Lie algebra $\mathfrak{g}$. There is an induced vertex algebra homomorphism $\hat{\tau} : \mathcal{O}(\mathfrak{g}, B) \to S(V)$, which is analogous to the map $\tau : \mathfrak{U}\mathfrak{g} \to \mathcal{D}(V)$ given by (2.3). Here $B$ is the bilinear form $B(\xi, \eta) = -\text{Tr}(\rho(\xi)\rho(\eta))$ on $\mathfrak{g}$. In terms of a basis $\{x_1, \ldots, x_n\}$ for $V$ and dual basis $\{x'_1, \ldots x'_n\}$ for $V^*$, $\hat{\tau}$ is defined by

\[ \hat{\tau}(X^\xi(z)) = \theta^\xi(z) = -\sum_{i=1}^{n} : \gamma^{x'_i}(z)\beta^{\rho(\xi)(x_i)}(z) :. \tag{3.5} \]
Definition 3.2. Let $\Theta$ denote the subalgebra $\hat{\tau}(O(g, B)) \subset S(V)$. The commutant algebra $S(V)^{\Theta^+}$ will be called the algebra of invariant chiral differential operators on $V$.

If $S(V)$ is equipped with the conformal structure $L^\alpha$ given by (3.2), $\Theta$ is not a graded subalgebra of $S(V)$ in general. For example, if $g = gl(n)$ and $V = \mathbb{C}^n$, $\Theta$ is graded by weight precisely when $\alpha_1 = \alpha_2 = \cdots = \alpha_n$. However, when $g$ is abelian and its action on $V$ is diagonal, $\theta^\xi(z)$ will be homogeneous of weight one for any $\alpha$. Hence $S(V)^{\Theta^+}$ is also graded by weight, but this grading will depend on the choice of $\alpha$.

3.2. The Zhu functor

Let $V$ be a vertex algebra with weight grading $V = \bigoplus_{n \in \mathbb{Z}} V_n$. In [21], Zhu introduced a functor that attaches to $V$ an associative algebra $A(V)$, together with a surjective linear map $\pi_{Zh} : V \rightarrow A(V)$. For $a \in V_m$ and $b \in V$, we define

$$a \ast b = \text{Res}_z \left( a(z) \frac{(z+1)^m}{z} b \right), \quad (3.6)$$

and extend $\ast$ by linearity to a bilinear operation $V \otimes V \rightarrow V$. Let $O(V)$ denote the subspace of $V$ spanned by elements of the form

$$a \circ b = \text{Res}_z \left( a(z) \frac{(z+1)^m}{z^2} b \right), \quad (3.7)$$

where $a \in V_m$, and let $A(V)$ be the quotient $V/O(V)$, with projection $\pi_{Zh} : V \rightarrow A(V)$. For $a, b \in V$, $a \sim b$ means $a - b \in O(V)$, and $[a]$ denotes the image of $a$ in $A(V)$. A useful fact which is immediate from (3.6) and (3.7) is that for $a \in V_m$,

$$\partial a \sim ma. \quad (3.8)$$

Theorem 3.3. (Zhu) $O(V)$ is a two-sided ideal in $V$ under the product $\ast$, and $(A(V), \ast)$ is an associative algebra with unit $[1]$. The assignment $V \mapsto A(V)$ is functorial. If $I$ is a vertex algebra ideal of $V$, we have

$$A(V/I) \cong A(V)/I, \quad I = \pi_{Zh}(I). \quad (3.9)$$
The main application of the Zhu functor is to study the representation theory of $\mathcal{V}$, or at least reduce it to a more classical problem. Let $M = \bigoplus_{n \geq 0} M_n$ be a module over $\mathcal{V}$ such that for $a \in \mathcal{V}_m$, $a(n)M_k \subset M_{m+k-n-1}$ for all $n \in \mathbb{Z}$. Given $a \in \mathcal{V}_m$, the Fourier mode $a(m-1)$ acts on each $M_k$. The subspace $M_0$ is then a module over $A(\mathcal{V})$ with action $[a] \mapsto a(m-1) \in \text{End}(M_0)$. In fact, $M \mapsto M_0$ provides a one-to-one correspondence between irreducible $\mathbb{Z}_{\geq 0}$-graded $\mathcal{V}$-modules and irreducible $A(\mathcal{V})$-modules.

A vertex algebra $\mathcal{V}$ is said to be strongly generated by a subset $\{v_i(z)\mid i \in I\}$ if $\mathcal{V}$ is spanned by collection of iterated Wick products

$$\{ : \partial^{k_1}v_{i_1}(z) \cdots \partial^{k_m}v_{i_m}(z) : \mid k_1, \ldots, k_m \geq 0 \}.$$ 

**Lemma 3.4.** Suppose that $\mathcal{V}$ is strongly generated by $\{v_i(z)\mid i \in I\}$, which are homogeneous of weights $d_i \geq 0$. Then $A(\mathcal{V})$ is generated as an associative algebra by the collection $\{ \pi_{Zh}(v_i)\mid i \in I \}$.

**Proof:** Let $\mathcal{C}$ be the algebra generated by $\{ \pi_{Zh}(v_i)\mid i \in I \}$. We need to show that for any vertex operator $\omega \in \mathcal{V}$, we have $\pi_{Zh}(\omega) \in \mathcal{C}$. By strong generation, it suffices to prove this when $\omega$ is a monomial of the form

$$: \partial^{k_1}v_{i_1} \cdots \partial^{k_r}v_{i_r} :.$$

We proceed by induction on weight. Suppose first that $\omega$ has weight zero, so that $k_1 = \cdots = k_r = 0$ and $v_{i_1}, \ldots, v_{i_r}$ all have weight zero. Note that $v_{i_1} \circ_n (: v_{i_2} \cdots v_{i_r} :)$ has weight $-n - 1$, and hence vanishes for all $n \geq 0$. It follows from (3.6) that

$$[v_{i_1}] \ast [: v_{i_2} \cdots v_{i_r} :] = [\omega].$$

Continuing in this way, we see that $[\omega] = [v_{i_1}] \ast [v_{i_2}] \ast \cdots \ast [v_{i_r}] \in \mathcal{C}$. Next, assume that $\pi_{Zh}(\omega) \in \mathcal{C}$ whenever $wt(\omega) < n$, and suppose that $\omega = : \partial^{k_1}v_{i_1} \cdots \partial^{k_r}v_{i_r} :$ has weight $n$. We calculate

$$[\partial^{k_1}v_{i_1}] \ast [: \partial^{k_2}v_{i_2} \cdots \partial^{k_r}v_{i_r} :] = [\omega] + \cdots,$$
where \( \cdots \) is a linear combination of terms of the form \( \partial^{k_1} v_{i_1} \circ_k (\partial^{k_2} v_{i_2} \cdots \partial^{k_r} v_{i_r}) \) for \( k \geq 0 \). The vertex operators \( \partial^{k_1} v_{i_1} \circ_k (\partial^{k_2} v_{i_2} \cdots \partial^{k_r} v_{i_r}) \) all have weight \( n - k - 1 \), so by our inductive assumption, \( [\partial^{k_1} v_{i_1} \circ_k (\partial^{k_2} v_{i_2} \cdots \partial^{k_r} v_{i_r})] \in C \). Applying the same argument to the vertex operator \( \partial^{k_2} v_{i_2} \cdots \partial^{k_r} v_{i_r} \) and proceeding by induction on \( r \), we see that \( \omega \equiv [\partial^{k_1} v_{i_1}] \cdots [\partial^{k_n} v_{i_n}] \) modulo \( C \). Finally, by applying (3.8) repeatedly, we see that \( \omega \in C \), as claimed. \( \square \).

Example 3.5. \( \mathcal{V} = \mathcal{O}(g, B) \) where each generator \( X^\xi \) has weight 1. Then \( A(\mathcal{O}(g, B)) \) is generated by \( \{[X^\xi] | \xi \in g \} \), and is isomorphic to the universal enveloping algebra \( Ug \) via \( [X^\xi] \mapsto \xi \).

Example 3.6. Let \( \mathcal{V} = S(V) \) where \( V = \mathbb{C}^n \), and \( S(V) \) is equipped with the conformal structure \( L^\alpha \) given by (3.2). Then \( A(S(V)) \) is generated by \( \{[\gamma^x], [\beta^x] \} \) and is isomorphic to the Weyl algebra \( D(V) \) with generators \( x_i, \partial / \partial x'_i \) via

\[
[\gamma^x_i] \mapsto x'_i, \quad [\beta^x_i] \mapsto \frac{\partial}{\partial x'_i}.
\]

Even though the structure of \( A(S(V)) \) is independent of the choice of \( \alpha \), the Zhu map \( \pi_{Zh} : S(V) \to A(S(V)) \) does depend on \( \alpha \). For example, (3.6) shows that

\[
\pi_{Zh} : (\gamma^x_i, \beta^x_i) = x'_i \frac{\partial}{\partial x'_i} + 1 - \alpha_i.
\] (3.10)

We will be particularly concerned with the interaction between the commutant construction and the Zhu functor. If \( a, b \in \mathcal{V} \) are (super)commuting vertex operators, \([a]\) and \([b]\) are (super)commuting elements of \( A(\mathcal{V}) \). Hence for any subalgebra \( B \subset \mathcal{V} \), we have a commutative diagram

\[
\begin{array}{ccc}
Com(B, \mathcal{V}) & \xrightarrow{\iota} & \mathcal{V} \\
\pi \downarrow & & \pi_{Zh} \downarrow \\
Com(B, A(\mathcal{V})) & \xleftarrow{\iota} & A(\mathcal{V})
\end{array}
\] (3.11)

Here \( B \) denotes the subalgebra \( \pi_{Zh}(B) \subset A(\mathcal{V}) \), and \( Com(B, A(\mathcal{V})) \) denotes the (super)commutant of \( B \) inside \( A(\mathcal{V}) \). The horizontal maps are inclusions, and \( \pi \) is the restriction of the Zhu map on \( \mathcal{V} \) to \( Com(B, \mathcal{V}) \). Clearly \( Im(\pi) \) is a subalgebra of \( Com(B, A(\mathcal{V})) \). A natural problem is to describe \( Im(\pi) \) and \( Coker(\pi) \). In our main example \( \mathcal{V} = S(V) \) and \( A = \Theta \), we have \( \pi_{Zh}(\Theta) = \tau(Ug) \subset D(V) \) and \( Com(\tau(Ug), D(V)) = D(V)^g \), so (3.11) specializes to (1.2).
4. The Friedan-Martinec-Shenker bosonization

4.1. Bosonization of fermions

First we describe the bosonization of fermions and the well-known boson-fermion correspondence due to [3]. Let \( A \) be the Heisenberg algebra with generators \( j(n), n \in \mathbb{Z} \), and \( \kappa \), satisfying \([j(n), j(m)] = n\delta_{n+m,0}\kappa\). The field \( j(z) = \sum_{n \in \mathbb{Z}} j(n) z^{-n-1} \) satisfies the OPE

\[ j(z) j(w) \sim (z - w)^{-2} , \]

and generates a Heisenberg vertex algebra \( \mathcal{H} \) of central charge 1. Define the free bosonic scalar field

\[ \phi(z) = q + j(0) \ln z - \sum_{n \neq 0} \frac{j(n)}{n} x^{-n} , \]

where \( q \) satisfies \([q, j(n)] = \delta_{n,0}\). Clearly \( \partial \phi(z) = j(z) \), and we have the OPE

\[ \phi(z) \phi(w) \sim \ln(z - w) . \]

Given \( \alpha \in \mathbb{C} \), let \( \mathcal{H}_\alpha \) denote the irreducible representation of \( A \) generated by the vacuum vector \( v_\alpha \) satisfying

\[ j(n)v_\alpha = \alpha \delta_{n,0} v_\alpha , \quad n \geq 0 . \quad (4.1) \]

Given \( \eta \in \mathbb{C} \), the operator \( e^{\eta q}(v_\alpha) = v_{\alpha+\eta} \), so \( e^{\eta q} \) maps \( \mathcal{H}_\alpha \rightarrow \mathcal{H}_{\alpha+\eta} \). Define the vertex operator

\[ X_\eta(z) = e^{\eta \phi(z)} = e^{\eta q} z^{\eta} \exp(\eta \sum_{n>0} j(-n) \frac{z^n}{n}) \exp(\eta \sum_{n<0} j(-n) \frac{z^n}{n}) . \]

The \( X_\eta \) satisfy the OPEs

\[ j(z) X_\eta(w) = \eta X_\eta(w)(z - w)^{-1} + \frac{1}{\eta} \partial X_\eta(w) , \]

\[ X_\eta(z) X_\nu(w) = (z - w)^{\eta \nu} : X_\eta(z) X_\nu(w) : . \]

If we take \( \eta = \pm 1 \), the pair of (fermionic) fields \( X_1, X_{-1} \) generate the lattice vertex algebra \( V_L \) associated to the one-dimensional lattice \( L = \mathbb{Z} \). The state space of \( V_L \) is just \( \sum_{n \in \mathbb{Z}} \mathcal{H}_n = \mathcal{H} \otimes_\mathbb{C} L \). It follows that

\[ X_1(z) X_{-1}(w) \sim (z - w)^{-1} , \quad X_{-1}(z) X_1(w) \sim (z - w)^{-1} , \]

\[ X_1(z) X_1(w) \sim 0 , \quad X_{-1}(z) X_{-1}(w) \sim 0 , \]

so the map \( \mathcal{E} \rightarrow V_L \) sending \( b \mapsto X_{-1}, c \mapsto X_1 \) is a vertex algebra isomorphism. Here \( \mathcal{E} \) denotes the \( bc \)-system \( \mathcal{E}(V) \) in the case where \( V \) is one-dimensional.
Next, we describe the bosonization of bosons, following [2]. Recall that $E$ has the grading $E = \oplus_{l \in \mathbb{Z}} E^l$ by $bc$-charge. As in [2], define $N(s) = \sum_{l \in \mathbb{Z}} E^l \otimes H_{i(s+l)}$, which is a module over the vertex algebra $E \otimes V_{L'}$. Here $L'$ is the one-dimensional lattice $i \mathbb{Z}$, and $V_{L'}$ is generated by $X_{\pm i}$. We define a map $\epsilon : S \to E \otimes V_{L'}$ by

$$
\beta \mapsto \partial b \otimes X_{-i}, \quad \gamma \mapsto c \otimes X_i.
$$

(4.2)

It is straightforward to check that (4.2) is a vertex algebra homomorphism, which is injective since $S$ is simple. Moreover Proposition 3 of [2] shows that the image of (4.2) coincides with the kernel of $c(0) : N(s) \to N(s - 1)$. Let $E'$ be the subalgebra of $E$ generated by $c$ and $\partial b$, which coincides with the kernel of $c(0) : E \to E$. It follows that

$$
\epsilon(S) \subset E' \otimes V_{L'}.
$$

(4.3)

5. $W$ algebras

The $W$ algebras are vertex algebras which arise as extended symmetry algebras of two-dimensional conformal field theories. For each integer $n \geq 2$ and $c \in \mathbb{C}$, the algebra $W_{n,c}$ of central charge $c$ is generated by fields of conformal weights $2, 3, \ldots, n$. In the case $n = 2$, $W_{2,c}$ is just the Virasoro algebra of central charge $c$. In contrast to the Virasoro algebra, the generating fields for $W_{n,c}$ for $n \geq 3$ have nonlinear terms in their OPEs, which makes the representation theory of these algebras highly nontrivial. One also considers various limits of $W$ algebras denoted by $W_{1+\infty,c}$ which may be defined as modules over the universal central extension $\hat{D}$ of the Lie algebra $D$ of differential operators on the circle [11].

We will be particularly concerned with the $W_3$ algebra, which was introduced by Zamolodchikov in [20] and studied extensively in [1]. Our discussion is taken directly from [18][19]. First, let $\mathcal{F}(W_3)$ denote the free associative algebra with generators $L_m, W_m$, $m \in \mathbb{Z}$. Let $\hat{\mathcal{F}}(W_3)$ be the completion of $\mathcal{F}(W_3)$ consisting of (possibly) infinite sums of monomials in $\mathcal{F}(W_3)$ such that for each $N > 0$, only finitely many terms depend only
on the variables $L_n, W_n$ for $n \leq N$. For a fixed central charge $c \in \mathbb{C}$, let $\mathfrak{W}_{3,c}$ be the quotient of $\hat{\mathcal{F}}(\mathcal{W}_3)$ by the ideal generated by

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m,-n}, \quad (5.1)$$

$$[L_m, W_n] = (2m-n)W_{m+n}, \quad (5.2)$$

$$[W_m, W_n] = (m-n)\left(\frac{1}{15}(m+n+3)(m+n+2) - \frac{1}{6}(m+2)(n+2)\right)L_{m+n} \quad (5.3)$$

$$+ \frac{16}{22+5c}(m-n)\Lambda_{m+n} + \frac{c}{360}m(m^2-1)(m^2-4)\delta_{m,-n}.$$  

Here

$$\Lambda_m = \sum_{n \leq -2} L_n L_{m-n} + \sum_{n > -2} L_{m-n} L_n - \frac{3}{10}(m+2)(m+3)L_m.$$  

Let

$$\mathcal{W}_{3,c,\pm} = \{L_n, W_n, \pm n > 0\}, \quad \mathcal{W}_{3,c,0} = \{L_0, W_0\}.$$  

The Verma module $\mathcal{M}_c(t, w)$ of highest weight $(t, w)$ is the induced module

$$\mathfrak{W}_{3,c} \otimes \mathcal{W}_{3,c,t} \oplus \mathcal{W}_{3,c,0} \mathbb{C}_{t, w},$$

where $\mathbb{C}_{t, w}$ is the one-dimensional $\mathcal{W}_{3,c,+} \oplus \mathcal{W}_{3,c,0}$-module generated by the vector $v_{t, w}$ such that

$$\mathcal{W}_{3,c,+}(v_{t, w}) = 0, \quad L_0(v_{t, w}) = tv_{t, w}, \quad W_0(v_{t, w}) = wv_{t, w}.$$  

A vector $v \in \mathcal{M}_c(t, w)$ is called singular if $\mathcal{W}_{3,c,+}(v) = 0$. In the case $t = w = 0$, the vectors

$$L_{-1}(v_{0,0}), \quad W_{-1}(v_{0,0}), \quad W_{-2}(v_{0,0}) \quad (5.4)$$

are singular vectors in $\mathcal{M}_c(0, 0)$. The vacuum module $\mathcal{V}\mathcal{W}_{3,c}$ is defined to be the quotient of $\mathcal{M}_c(0, 0)$ by the $\mathfrak{W}_{3,c}$-submodule generated by the vectors (5.4). $\mathcal{V}\mathcal{W}_{3,c}$ has the structure of a vertex algebra which is freely generated by the vertex operators

$$L(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2}, \quad W(z) = \sum_{n \in \mathbb{Z}} W_n z^{-n-3}.$$  

In particular, the vertex operators

$$\{\partial^{i_1}L(z) \cdots \partial^{i_m}L(z)\partial^{j_1}W(z) \cdots \partial^{j_n}W(z)\mid 0 \leq i_1 \leq \cdots \leq i_m, 0 \leq j_1 \leq \cdots \leq j_n\}$$
which correspond to \(i_1! \cdots i_m! j_1! \cdots j_n! L_{-i_1-2} \cdots L_{-i_m-2} W_{-j_1-3} \cdots W_{-j_n-3} v_{0,0}\) under the state-operator correspondence, form a basis for \(\mathcal{VW}_{3,c}\). By Lemma 4.1 of [19], the Zhu algebra \(A(\mathcal{VW}_{3,c})\) is just the polynomial algebra \(\mathbb{C}[l, w]\) where \(l = \pi_{Z_h}(L)\) and \(w = \pi_{Z_h}(W)\).

Let \(I_c\) denote the maximal proper \(\mathfrak{g}\mathcal{W}_{3,c}\)-submodule of \(\mathcal{VW}_{3,c}\), which is a vertex algebra ideal. The quotient \(\mathcal{VW}_{3,c}/I_c\) is a simple vertex algebra which we denote by \(\mathcal{W}_{3,c}\). Let \(I_c = \pi_{Z_h}(I_c)\), which is an ideal of \(\mathbb{C}[l, w]\). By (3.9), we have \(A(\mathcal{W}_{3,c}) = \mathbb{C}[l, w]/I_c\).

Generically, \(I_c = 0\), so that \(\mathcal{VW}_{3,c} = \mathcal{W}_{3,c}\). We will be primarily concerned with the non-generic case \(c = -2\), in which \(I_{-2} \neq 0\). The generators \(L(z), W(z) \in \mathcal{VW}_{3,-2}\) satisfy the following OPEs:

\[
L(z)L(w) \sim -(z-w)^{-4} + 2L(w)(z-w)^{-2} + \partial L(w)(z-w)^{-1}, \quad (5.5)
\]

\[
L(z)W(w) \sim 3W(w)(z-w)^{-2} + \partial W(w)(z-w)^{-1}, \quad (5.6)
\]

\[
W(z)W(w) \sim -\frac{2}{3}(z-w)^{-6} + 2L(w)(z-w)^{-4} + \partial L(w)(z-w)^{-3}
\]

\[
+ (\frac{8}{3} : L(w) L(w) : - \frac{1}{2} \partial^2 L(w))(z-w)^{-2}
\]

\[
+ (\frac{4}{3} \partial ( : L(w) L(w) : ) - \frac{1}{3} \partial^3 L(w))(z-w)^{-1}. \quad (5.7)
\]

The simple vertex algebra \(\mathcal{W}_{3,-2}\) also has generators \(L(z), W(z)\) satisfying (5.5)-(5.7), but \(\mathcal{W}_{3,-2}\) is no longer freely generated.

In order to avoid introducing extra notation, we will not use the change of variables \(\tilde{W}(z) = \frac{1}{2} \sqrt{6} W(z)\) given by Equation 3.13 of [19]. By Lemma 4.3 of [19], the ideal \(I_{-2} \subset \mathbb{C}[l, w]\) is generated (in our variables) by the polynomial

\[
w^2 - \frac{2}{27} l^2 (8l + 1). \quad (5.8)
\]

5.1. The representation theory of \(\mathcal{W}_{3,-2}\)

In [19], W. Wang gave a complete classification of the irreducible modules over the simple vertex algebra \(\mathcal{W}_{3,-2}\). An important ingredient in his classification is the following realization of \(\mathcal{W}_{3,-2}\) as a subalgebra of the Heisenberg algebra \(\mathcal{H}\) with generator \(j(z)\) satisfying \(j(z)j(w) \sim (z-w)^{-2}\). Define

\[
L_{\mathcal{H}} = \frac{1}{2} (: j^2 :) + \partial j, \quad W_{\mathcal{H}} = \frac{2}{3\sqrt{6}} (: j^3 :) + \frac{1}{\sqrt{6}} (: j \partial j :) + \frac{1}{6\sqrt{6}} \partial^2 j. \quad (5.9)
\]
The map $\mathcal{W}_{3,-2} \hookrightarrow \mathcal{H}$ sending $L \mapsto L_{\mathcal{H}}$ and $W \mapsto W_{\mathcal{H}}$ is a vertex algebra homomorphism, so we may regard any $\mathcal{H}$-module as a $\mathcal{W}_{3,-2}$-module. Given $\alpha \in \mathbb{C}$, consider the irreducible $\mathcal{H}$-module $\mathcal{H}_{\alpha}$ defined by (4.1), and let $V_{\alpha}$ denote the irreducible quotient of the $\mathcal{W}_{3,-2}$-submodule of $\mathcal{H}_{\alpha}$ generated by $v_{\alpha}$. It is easily checked that the generator $v_{\alpha}$ is a highest weight vector of $\mathcal{W}_{3,-2}$ with highest weight

$$\left(\frac{1}{2}\alpha(\alpha - 1), \frac{1}{3\sqrt{6}}\alpha(\alpha - 1)(2\alpha - 1)\right).$$

(5.10)

The main result of [19] is that the modules $\{V_{\alpha} \mid \alpha \in \mathbb{C}\}$ account for all the irreducible modules of $\mathcal{W}_{3,-2}$.

6. The commutant algebra $S(V)^{\Theta^+}$ for $g = gl(1)$ and $V = \mathbb{C}$

In this section, we describe $S(V)^{\Theta^+}$ in the case where $g = gl(1)$ and $V = \mathbb{C}$, where the action $\rho : g \to End V$ is by multiplication. Fix a basis $\xi$ of $g$ and a basis $x$ of $V$, such that $\rho(\xi)(x) = x$. Then $S = S(V)$ is generated by $\beta(z) = \beta x(z)$ and $\gamma(z) = \gamma x'(z)$, and the map (2.5) is given by

$$g \to D = D(V), \quad \xi \mapsto -x' \frac{d}{dx}. $$

In this case, $O(g, B)$ is just the Heisenberg algebra $\mathcal{H}$ of central charge $-1$, and the action of $\mathcal{H}$ on $S$ given by (3.5) is

$$\theta(z) = - : \gamma(z)\beta(z) :, $$

(6.1)

which clearly satisfies

$$\theta(z)\theta(w) \sim -(z - w)^{-2}.$$ 

(6.2)

As usual, $\Theta$ will denote the subalgebra of $S$ generated by $\theta(z)$. Since $-\theta(0)$ is the $\beta\gamma$-charge operator, $S^{\Theta^+}$ must lie in the subalgebra $S^0$ of $\beta\gamma$-charge zero.

Let $: \theta^n :$ denote the $n$-fold iterated Wick product of $\theta$ with itself. It is clear from (6.2) that each $: \theta^n :$ lies in $S^0$ but not in $S^{\Theta^+}$. A natural place to look for elements in $S^{\Theta^+}$ is to begin with the operators $: \theta^n :$ and try to “quantum correct” them so that they lie in $S^{\Theta^+}$. As a polynomial in $\beta, \partial \beta, \ldots, \gamma, \partial \gamma, \cdots$, note that

$$: \theta^n : = (-1)^n \beta^n \gamma^n + \nu_n,$$
where \( \nu_n \) has degree at most \( 2n - 2 \). By a quantum correction, we mean an element \( \omega_n \in S \) of polynomial degree at most \( 2n - 2 \), so that \( : \theta^n : + \omega_n \in S^\Theta^+ \).

Clearly \( \theta \) has no such correction \( \omega_1 \), because \( \omega_1 \) would have to be a scalar, in which case \( \theta \circ_1 (\theta + \omega_1) = \theta \circ_1 \theta = -1 \). However, the next lemma shows that we can find such \( \omega_n \) for all \( n \geq 2 \).

**Lemma 6.1.** Let

\[
\omega_2 = : \beta (\partial \gamma) : - : (\partial \beta) \gamma :
\]

\[
\omega_3 = -\frac{9}{2} : \beta^2 \gamma (\partial \gamma) : + \frac{9}{2} : \beta (\partial \beta) \gamma^2 : - \frac{3}{2} : \beta (\partial^2 \gamma) : - \frac{3}{2} : (\partial^2 \beta) \gamma : + 6 : (\partial \beta) (\partial \gamma) :
\]

Then \( : \theta^2 : + \omega_2 \in S^\Theta^+ \) and \( : \theta^3 : + \omega_3 \in S^\Theta^+ \). Since \( : (\theta^n) : \) and \( : (\theta^i :) (\theta^j :) : \) have the same leading term as polynomials in \( \beta, \partial \beta, \ldots, \gamma, \partial \gamma, \ldots \) for \( i + j = n \), it follows that for any \( n \geq 2 \) we can find \( \omega_n \) such that \( : \theta^n : + \omega_n \in S^\Theta^+ \).

**Proof:** This is a straightforward OPE calculation. □

Next, define vertex operators \( L_S, W_S \in S^\Theta^+ \) as follows:

\[
L_S = \frac{1}{2} ( : \theta^2 : + \omega_2 ) = \frac{1}{2} ( : \beta^2 \gamma^2 : ) - : (\partial \beta) \gamma : + : \beta (\partial \gamma) :
\]

\[
W_S = -\sqrt{\frac{2}{27}} ( : \theta^3 : + \omega_3 )
\]

\[
= \sqrt{\frac{2}{27}} ( : \beta^3 \gamma^3 : ) - \sqrt{\frac{3}{2}} ( : \beta (\partial \beta) \gamma^2 : ) + \sqrt{\frac{3}{2}} ( : \beta^2 \gamma (\partial \gamma) : )
\]

\[
+ \sqrt{\frac{1}{6}} ( : (\partial^2 \beta) \gamma : ) - \sqrt{\frac{8}{3}} ( : (\partial \beta) (\partial \gamma) : ) + \sqrt{\frac{1}{6}} ( : \beta (\partial^2 \gamma) : )
\]

Let \( \mathcal{W} \subset S^\Theta^+ \) be the vertex algebra generated by \( L_S, W_S \). An OPE calculation shows that the map

\[
\mathcal{V} W_{3,-2} \rightarrow S^\Theta^+, \quad L \mapsto L_S, \quad W \mapsto W_S
\]

is a vertex algebra homomorphism. Moreover, the ideal \( \mathcal{I}_{-2} \) is annihilated by (6.5), so this map descends to a map

\[
f : W_{3,-2} \hookrightarrow S^\Theta^+.
\]
In fact, (6.6) is related to the realization of $\mathcal{W}_{3,-2}$ as a subalgebra of $\mathcal{H}$ defined earlier. First, under the boson-fermion correspondence,

$$L_\mathcal{H} \mapsto L_\mathcal{E} = :\partial bc: \, ,$$

$$W_\mathcal{H} \mapsto W_\mathcal{E} = \frac{1}{\sqrt{6}} ( : (\partial^2 b)c : - : (\partial b)(\partial c) : ).$$

(6.7)

(6.8)

Next, under the map $\epsilon: \mathcal{S} \to \mathcal{E} \otimes \mathcal{H}$ given by (4.2), we have

$$L_\mathcal{S} \mapsto L_\mathcal{E} \otimes 1, \quad W_\mathcal{S} \mapsto W_\mathcal{E} \otimes 1.$$  

(6.9)

The subalgebra $\mathcal{S}^0$ of $\beta\gamma$-charge zero has a natural set of generators

$$\{ J^i = :\beta(\partial^i\gamma) : , \, i \geq 0 \},$$

and it is well known that $\mathcal{S}^0$ is isomorphic to $\mathcal{W}_{1+\infty,-1}$ [11]. One of the main results of [18] is that $\epsilon: \mathcal{S} \to \mathcal{E} \otimes \mathcal{H}$ restricts to an isomorphism

$$\mathcal{S}^0 \cong \mathcal{A} \otimes \mathcal{H},$$  

(6.10)

where $\mathcal{A} \cong \mathcal{W}_{3,-2}$ is the subalgebra of $\mathcal{E}$ generated by $L_\mathcal{E}$ and $W_\mathcal{E}$. By (6.9), $\epsilon$ maps $\mathcal{W}$ onto $\mathcal{A} \otimes 1$. Similarly, $\epsilon(\theta) = i(1 \otimes j)$, so $\epsilon$ maps $\Theta$ onto $1 \otimes \mathcal{H}$, and $\mathcal{S}^0 = \mathcal{W} \otimes \Theta$.

For each $d \in \mathbb{Z}$, the subspace $\mathcal{S}^d$ of $\beta\gamma$-charge $d$ is a module over $\mathcal{S}^0$, which is in fact irreducible [11][19]. Define $v^d(z) \in \mathcal{S}^d$ by

$$v^d(z) = \begin{cases} 
\beta(z)^{-d} & d < 0 \\
1 & d = 0 \\
\gamma(z)^d & d > 0 
\end{cases}.$$  

(6.11)

Here $\beta(z)^{-d}$ and $\gamma(z)^d$ denote the $d$-fold iterated Wick products $: \beta(z) \cdots \beta(z) :$ and $: \gamma(z) \cdots \gamma(z) :, \,$ respectively. Each $v^d(z)$ is a highest weight vector for the action of $\mathcal{W}_{3,-2}$, and the highest weight of $v^d(z)$ is given by (5.10) with

$$\begin{cases} 
\alpha = d & d \leq 0 \\
\alpha = d + 1 & d > 0 
\end{cases}.$$  

(6.12)

Moreover, $v^d(z)$ is also a highest weight vector for the action of $\mathcal{H}$, so $\mathcal{S}^d$ is generated by $v^d(z)$ as a module over $\mathcal{W}_{3,-2} \otimes \mathcal{H}$.  

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Theorem 6.2. The map \( f : \mathcal{W}_{3,-2} \rightarrow S^{\Theta+} \) given by (6.6) is an isomorphism of vertex algebras. Moreover, \( \text{Com}(S^{\Theta+}, S) = \Theta \). Hence \( \Theta \) and \( S^{\Theta +} \) form a Howe pair inside \( S \).

Proof: Clearly \( S^{\Theta +} \subset S^0 \), and since \( S^0 = \mathcal{W} \otimes \Theta \), we have
\[
S^{\Theta +} = \text{Com}(\Theta, \mathcal{W} \otimes \Theta) = \mathcal{W} \otimes \text{Com}(\Theta, \Theta) = \mathcal{W}.
\]
This proves the first statement. As for the second statement, it is clear from (5.10) and (6.12) that \( \text{Com}(S^{\Theta +}, S) \subset S^0 \). Hence
\[
\text{Com}(S^{\Theta +}, S) = \text{Com}(\mathcal{W}, \mathcal{W} \otimes \Theta) = \Theta \otimes \text{Com}(\mathcal{W}, \mathcal{W}) = \Theta.
\]
\[\square\]

6.1. The map \( \pi : S^{\Theta +} \rightarrow \mathcal{D}^g \)

Equip \( S \) with the conformal structure \( L^\alpha = (\alpha - 1) : \partial \beta(z) \gamma(z) : + \alpha : \beta(z) \partial \gamma(z) : \), and consider the map \( \pi : S^{\Theta +} \rightarrow \mathcal{D}^g \) given by (1.2). In this case, \( \mathcal{D}^g \) is just the polynomial algebra \( \mathbb{C}[e] \), where \( e \) is the Euler operator \( x' \frac{d}{dx} \).

Lemma 6.3. We have
\[
\pi(L_S) = \frac{1}{2}(e^2 + e), \quad \pi(W_S) = \frac{2}{3\sqrt{6}}e^3 + \frac{1}{\sqrt{6}}e^2 + \frac{1}{3\sqrt{6}}e.
\]
(6.13)
In particular, \( \pi(L_S) \) and \( \pi(W_S) \) are independent of the choice of \( \alpha \).

Proof: This is a straightforward computation using (3.6) and the fact that \( \pi_{Zh}(\gamma(z)) = x' \) and \( \pi_{Zh}(\beta(z)) = \frac{d}{dx} \). Note that \( l = \pi(L_S) \) and \( w = \pi(W_S) \) satisfy (5.8). \( \square\)

Corollary 6.4. For any conformal structure \( L^\alpha \) on \( S \) as above, \( \text{Im}(\pi) \) is the subalgebra of \( \mathbb{C}[e] \) generated by \( \pi(L_S) \) and \( \pi(W_S) \). Moreover, \( \text{Coker}(\pi) = \mathbb{C}[e]/\text{Im}(\pi) \) has dimension one, and is spanned by the image of \( e \) in \( \text{Coker}(\pi) \).

Proof: The first statement is immediate from Lemma 3.4, since \( S^{\Theta +} \) is strongly generated by \( L_S \) and \( W_S \) which have weights 2 and 3 respectively. The second statement follows from (3.10) and (6.13), because any polynomial in \( \mathbb{C}[e] \) is equivalent to an element which is homogeneous of degree 1 modulo \( \text{Im}(\pi) \). \( \square\)
7. $S(V)^{θ_+}$ for abelian Lie algebra actions

Fix a basis $\{x_1, \ldots, x_n\}$ for $V$ and dual basis $\{x'_1, \ldots, x'_n\}$ for $V^*$. We regard $S(V)$ as $S_1 \otimes \cdots \otimes S_n$, where $S_j$ is the copy of $S$ generated by $β^{x_j}(z), γ^{x'_j}(z)$. Let $f_j : S → S(V)$ be the obvious map onto the $j$th factor. The subspace $S^0_j$ of $βγ$-charge zero is isomorphic to $W_j \otimes H_j$, where $H_j$ is generated by $θ_j(z) = f_j(θ(z))$, and $W_j$ is generated by $L_j = f_j(L_S)$, $W_j = f_j(W_S)$. Moreover, as a module over $W_j \otimes H_j$, the space $S^d_j$ of $βγ$-charge $d$ is generated by the highest weight vector $v^d_j(z) = f_j(v^d(z))$, which is given by

$$v^d_j(z) = \begin{cases} \beta^{x_j}(z)^{-d} & d < 0 \\ 1 & d = 0 \\ \gamma^{x'_j}(z)^d & d > 0 \end{cases}$$  

(7.1)

We denote by $S'_j$ the linear span of the vectors $\{v^d_j(z)| d \in \mathbb{Z}\}$. Note that for any conformal structure $L^α$ on $S(V)$, the differential operators $v^d_j ∈ D(V)$ defined by (2.6) correspond to $v^d_j(z)$ under the Zhu map. Let $B$ denote the vertex algebra

$$S^0_1 \otimes \cdots \otimes S^0_n \cong (W^1 \otimes H^1) \otimes \cdots \otimes (W^n \otimes H^n).$$

Clearly the space $S(V)'$ consisting of highest-weight vectors for the action of $B$ is just $S'_1 \otimes \cdots \otimes S'_n$. As usual, let $\mathbb{Z}^n ⊂ \mathbb{C}^n$ denote the standard lattice. For each lattice point $l = (l_1, \ldots, l_n) ∈ \mathbb{Z}^n$, define

$$ω_l(z) = :v^{l_1}_1(z) \cdots v^{l_n}_n(z):,$$  

(7.2)

where $v^d_j(z)$ is given by (7.1). For example, in the case $n = 2$ and $l = (2, -3) ∈ \mathbb{Z}^2$, we have

$$ω_l(z) = :v^2_1(z)v^{-3}_2(z): = :γ^{x_1}(z)γ^{x_1}(z)β^{x_2}(z)β^{x_2}(z)β^{x_2}(z):.$$  

For any conformal structure $L^α$ on $S(V)$, $ω_l(z)$ corresponds under the Zhu map to the element $ω_l ∈ D(V)$ given by (2.7).

**Lemma 7.1.** For each $l ∈ \mathbb{Z}^n$, the $B$-module $M_l$ generated by $ω_l(z)$ is irreducible. Moreover, as a module over $B$,

$$S(V) = \bigoplus_{l ∈ \mathbb{Z}^n} M_l.$$  

(7.3)
Proof: This is immediate from the description of $S^d$ as the irreducible $S^0$-module generated by $v_d(z)$, and the fact that $S(V)' = S'_1 \otimes \cdots \otimes S'_n$. □

Note that $\theta^j(z) \circ_0 \omega_l(z) = -l_j \omega_l(z)$, so the $\mathbb{Z}^n$-grading on $S(V)$ above is just the eigenspace decomposition of $S(V)$ under the family of diagonalizable operators $-\theta^j(z)\circ_0$.

For the remainder of this section, $\mathfrak{g}$ will denote the abelian Lie algebra

$$C^m = gl(1) \oplus \cdots \oplus gl(1),$$

and $\rho : \mathfrak{g} \to \text{End}(V)$ will be a faithful, diagonal action. Let $A(\rho) \subset C^n$ be the subspace spanned by $\{\rho(\xi)| \xi \in \mathfrak{g}\}$. As in the classical setting, we denote $S(V)^{\Theta^+}$ by $S(V)^{\Theta^+}_\rho$ when we need to emphasize the dependence on $\rho$. Clearly $S(V)^{\Theta^+}_\rho = S(V)^{\Theta^+}_{\rho g}$ for all $g \in GL(m)$, so the family of algebras $S(V)^{\Theta^+}_\rho$ is parametrized by the points $A(\rho) \in Gr(m,n)$.

Choose a basis $\{\xi^1, \ldots, \xi^m\}$ for $\mathfrak{g}$ such that the corresponding vectors $\rho(\xi^i) = a^i = (a^i_1, \ldots, a^i_n) \in C^n$ form an orthonormal basis for $A = A(\rho)$. Let $\theta^{\xi^i}(z)$ be the vertex operator corresponding to $\rho(\xi^i)$, and let $\Theta$ be the subalgebra of $\mathcal{B}$ generated by $\{\theta^{\xi^i}(z)| i = 1, \ldots, m\}$. By (3.5), we have

$$\theta^{\xi^i}(z) = \sum_{j=1}^{n} a_j \theta^j(z) = -\sum_{j=1}^{n} a_j : \gamma^{x^j}(z) \beta^{x^j}(z) :.$$

Clearly $\theta^{\xi^i}(z)\theta^{\xi^j}(w) \sim -\langle a^i, a^j \rangle (z - w)^{-2} = \delta_{i,j}(z - w)^{-2}$.

If $m < n$, extend the set $\{a^1, \ldots, a^m\}$ to an orthonormal basis for $C^n$ by adjoining vectors $b^i = (b^i_1, \ldots, b^i_n) \in C^n$, for $i = m + 1, \ldots, n$. Let

$$\phi^i(z) = \sum_{j=1}^{n} b^i_j \theta^j(z) = -\sum_{j=1}^{n} b^i_j : \gamma^{x^j}(z) \beta^{x^j}(z) :$$

be the corresponding vertex operators, and let $\Phi$ be the subalgebra of $\mathcal{B}$ generated by $\{\phi^i(z)| i = m + 1, \ldots, n\}$. The OPEs

$$\phi^i(z)\phi^j(w) \sim -\langle b^i, b^j \rangle (z - w)^{-2}, \quad \theta^{\xi^i}(z)\phi^j(w) \sim -\langle a^i, b^j \rangle (z - w)^{-2}$$
show that the \( \phi^i(z) \) pairwise commute and each generates a Heisenberg algebra of central charge \(-1\), and that \( \Phi \subset S(V)^{\Theta^+} \). In particular, we have the decomposition

\[
\mathcal{H}^1 \otimes \cdots \otimes \mathcal{H}^n = \Theta \otimes \Phi.
\]

Next, let \( \mathcal{W} \) denote the subalgebra of \( \mathcal{B} \) generated by \( \{ L^j(z), W^j(z) \mid j = 1, \ldots, n \} \). Theorem 6.2 shows that \( \mathcal{W} \) commutes with both \( \Theta \) and \( \Phi \), so we have the decomposition

\[
\mathcal{B} = \mathcal{W} \otimes \Theta \otimes \Phi.
\]  

(7.4)

In particular, the subalgebra \( \mathcal{B}' = \mathcal{W} \otimes \Phi \) lies in the commutant \( S(V)^{\Theta^+} \). Let \( \mathcal{M}'_l \) denote the \( \mathcal{B}' \)-submodule of \( \mathcal{M}_l \) generated by \( \omega_l(z) \), which is clearly irreducible as a \( \mathcal{B}' \)-module.

In order to describe \( S(V)^{\Theta^+} \), we first describe the larger space \( S(V)^{\Theta>} \) which is annihilated by \( \theta^{\xi_i}(k) \) for \( i = 1, \ldots, m \) and \( k > 0 \). Then \( S(V)^{\Theta^+} \) is just the subspace of \( S(V)^{\Theta>} \) which is annihilated by \( \theta^{\xi_i}(0) \), for \( i = 1, \ldots, m \). It is clear from (7.4) and the irreducibility of \( \mathcal{M}_l \) as a \( \mathcal{B} \)-module that \( S(V)^{\Theta>} \cap \mathcal{M}_l = \mathcal{M}'_l \), so

\[
S(V)^{\Theta>} = \bigoplus_{l \in \mathbb{Z}^n} \mathcal{M}'_l.
\]  

(7.5)

**Theorem 7.2.** As a module over \( \mathcal{B}' \),

\[
S(V)^{\Theta^+} = \bigoplus_{l \in A^+ \cap \mathbb{Z}^n} \mathcal{M}'_l.
\]  

(7.6)

Proof: Let \( \omega(z) \in S(V)^{\Theta^+} \). Since \( \omega \) lies in the larger space \( S(V)^{\Theta>} \) which is a direct sum of irreducible, cyclic \( \mathcal{B}' \)-modules \( \mathcal{M}'_l \) with generators \( \omega_l(z) \), we may assume without loss of generality that \( \omega(z) = \omega_l(z) \) for some \( l \). An OPE calculation shows that

\[
\theta^{\xi_i}(z)\omega_l(w) \sim -(a^i,l)\omega_l(w)(z-w)^{-1}.
\]  

(7.7)

Hence \( \omega_l \in S(V)^{\Theta^+} \) if and only if \( l \) lies in the sublattice \( A^+ \cap \mathbb{Z}^n \). \( \square \)

Our next step is to find a finite generating set for \( S(V)^{\Theta^+} \). Generically, \( A^+ \cap \mathbb{Z}^n \) has rank zero, so \( S(V)^{\Theta^+} = \mathcal{B}' \), which is (strongly) generated by the set

\[
\{ \phi^i(z), L^j(z), W^j(z) \mid i = m + 1, \ldots, n, j = 1, \ldots, n \}.
\]
If $A^+ \cap \mathbb{Z}^n$ has rank $r$ for some $0 < r \leq n - m$, choose a basis $\{l_1, \ldots, l_r\}$ for $A^+ \cap \mathbb{Z}^n$. We claim that for any $l \in A^+ \cap \mathbb{Z}^n$, $\omega_l(z)$ lies in the vertex subalgebra generated by

$$\{\omega_{\ell_1}(z), \ldots, \omega_{\ell_r}(z), \omega_{-\ell_1}(z), \ldots, \omega_{-\ell_r}(z)\}.$$  

It suffices to prove that given lattice points $l = (l_1, \ldots, l_n)$ and $l' = (l'_1, \ldots, l'_n)$ in $\mathbb{Z}^n$, $\omega_{l+l'}(z) = k\omega_l(z) \circ_d \omega_{l'}(z)$ for some $k \neq 0$ and $d \in \mathbb{Z}$.

First, consider the special case where $l = (l_1, 0, \ldots, 0)$ and $l' = (l'_1, 0, \ldots, 0)$. If $l_1l'_1 \geq 0$, we have $\omega_l(z) \circ_{-1} \omega_{l'}(z) = \omega_{l+l'}(z)$. Suppose next that $l_1 < 0$ and $l'_1 > 0$, so that $\omega_l(z) = \beta^{x_1}(z)^{-l_1}$ and $\omega_{l'}(z) = \gamma^{x_1}(z)^{l'_1}$. Let

$$d_1 = \min\{-l_1, l'_1\}, \quad e_1 = \max\{-l_1, l'_1\}, \quad d = d_1 - 1.$$  

An OPE calculation shows that

$$\omega_l(z) \circ_d \omega_{l'}(z) = \frac{e_1!}{(e_1 - d_1)!} \omega_{l+l'}(z),$$

where as usual $0! = 1$. Similarly, if $l_1 > 0$ and $l'_1 < 0$, we take $d_1 = \min\{l_1, -l'_1\}$, $e_1 = \max\{l_1, -l'_1\}$, and $d = d_1 - 1$. We have

$$\omega_l(z) \circ_d \omega_{l'}(z) = -\frac{e_1!}{(e_1 - d_1)!} \omega_{l+l'}(z).$$

Now consider the general case $l = (l_1, \ldots, l_n)$ and $l' = (l'_1, \ldots, l'_n)$. For $j = 1, \ldots, n$, define

$$d_j = \begin{cases} \min\{|l_j|, |l'_j|\}, & l_jl'_j \geq 0, \\ 0, & l_jl'_j < 0 \end{cases}, \quad e_j = \begin{cases} \max\{|l_j|, |l'_j|\}, & l_jl'_j \geq 0, \\ 0, & l_jl'_j < 0 \end{cases},$$

$$k_j = \begin{cases} 0, & l_j \leq 0, \\ d_j, & l_j > 0 \end{cases}, \quad d = -1 + \sum_{j=1}^n d_j.$$  

Using (7.8) and (7.9) repeatedly, we calculate

$$\omega_l(z) \circ_d \omega_{l'}(z) = \left(\prod_{j=1}^n (-1)^{k_j} \frac{e_j!}{(e_j - d_j)!}\right) \omega_{l+l'}(z),$$

which shows that $\omega_{l+l'}(z)$ lies in the vertex algebra generated by $\omega_l(z)$ and $\omega_{l'}(z)$. Thus we have proved
Theorem 7.3. Let \( \{l^1, \ldots, l^n\} \) be a basis for the lattice \( A^\perp \cap \mathbb{Z}^n \), as above. Then \( S(V)^{\Theta+} \) is generated as a vertex algebra by \( \mathcal{B}' \) together with the additional vertex operators

\[
\omega_{l^1}(z), \ldots, \omega_{l^n}(z), \quad \omega_{-l^1}(z), \ldots, \omega_{-l^n}(z).
\]

In particular, \( S(V)^{\Theta+} \) is finitely generated as a vertex algebra.

In the generic case where \( A^\perp \cap \mathbb{Z}^n \neq 0 \) and \( S(V)^{\Theta+} = \mathcal{B}' \), we claim that \( S(V)^{\Theta+} \) has a natural \((n - m)\)-parameter family of conformal structures for which the generators \( \phi^i(z), L^j(z), W^j(z) \) are primary of conformal weights \( 1, 2, 3 \), respectively. Note first that \( \mathcal{W} \) has the conformal structure \( L_{\mathcal{W}}(z) = \sum_{j=1}^n L^j(z) \) of central charge \(-2n\).

It is well known that for \( k \neq 0 \) and \( c \in \mathbb{C} \), the Heisenberg algebra \( \mathcal{H} \) of central charge \( k \) admits a Virasoro element \( L^c(z) = \frac{1}{2\pi} j(z) j(z) + c \partial j(z) \) of central charge \( 1 - 12c^2k \), under which the generator \( j(z) \) is primary of weight one. Hence given \( \lambda = (\lambda_{m+1}, \ldots, \lambda_n) \in \mathbb{C}^{n-m} \) the Heisenberg algebra generated by \( \phi^i(z) \) has a conformal structure

\[
L^\lambda_i(z) = -\frac{1}{2} : \phi^i(z) \phi^i(z) : + \lambda_i \partial \phi^i(z)
\]
of central charge \( 1 + 12\lambda_i^2 \). Since \( \phi^i(z) \) and \( \phi^j(z) \) commute for \( i \neq j \), it follows that \( L^\lambda_i(z) = \sum_{i=m+1}^n L^\lambda_i(z) \) is a conformal structure on \( \Phi \) of central charge \( \sum_{i=m+1}^n 1 + 12\lambda_i^2 \). Finally,

\[
L_{\mathcal{B}'}(z) = L_{\mathcal{W}}(z) \otimes 1 + 1 \otimes L^\lambda_i(z) \in \mathcal{W} \otimes \Phi = \mathcal{B}'
\]
is a conformal structure on \( \mathcal{B}' \) of central charge \(-2n + \sum_{i=m+1}^n 1 + 12\lambda_i^2 \) with the desired properties.

When the lattice \( A^\perp \cap \mathbb{Z}^n \) has positive rank, the vertex algebras \( S(V)^{\Theta+} \) have a very rich structure which depends sensitively on \( A^\perp \cap \mathbb{Z}^n \). In general, the set of generators for \( S(V)^{\Theta+} \) given by Theorem 7.3 will not be a set of strong generators, and the conformal structure \( L_{\mathcal{B}'} \) on \( \mathcal{B}' \) will not extend to a conformal structure on all of \( S(V)^{\Theta+} \).

Theorem 7.4. For any action of \( \mathfrak{g} \) on \( V \), \( \text{Com}(S(V)^{\Theta+}, S(V)) = \Theta \). Hence \( S(V)^{\Theta+} \) and \( \Theta \) form a Howe pair inside \( S(V) \).

Proof: Since \( \mathcal{B}' \subset S(V)^{\Theta+} \), we have \( \Theta \subset \text{Com}(S(V)^{\Theta+}, S(V)) \subset \text{Com}(\mathcal{B}', S(V)) \), so it suffices to show that \( \text{Com}(\mathcal{B}', S(V)) = \Theta \). Recall that \( \mathcal{B}' = \mathcal{W} \otimes \Phi \) and \( \Theta \otimes \Phi = \mathcal{H}^1 \otimes \cdots \otimes \mathcal{H}^n \).
Since $\text{Com}(W, S(V)) = \mathcal{H}$ by Theorem 6.2, it follows that $\text{Com}(W, S(V)) = \Theta \otimes \Phi$. Then

$\text{Com}(B', S(V)) = \text{Com}(\Phi, \text{Com}(W, S(V))) = \text{Com}(\Phi, \Theta \otimes \Phi) = \Theta \otimes \text{Com}(\Phi, \Phi) = \Theta$. □

This result shows that we can always recover the action of $g$ (up to $GL(m)$-equivalence) from $S(V)^{\Theta_+}$, by taking its commutant inside $S(V)$. This stands in contrast to Theorem 2.1, which shows that we can reconstruct the action from $D(V)^g$ only when $A^\perp \cap \mathbb{Z}^n$ has rank $n - m$.

**Theorem 7.5.** For any action of $g$ on $V$, $S(V)^{\Theta_+}$ is a simple vertex algebra.

Proof: Given a non-zero ideal $I \subset S(V)^{\Theta_+}$, we need to show that $1 \in I$. Let $\omega(z)$ be a non-zero element of $I$. Since each $M'_{l}$ is irreducible as a module over $B'$, we may assume without loss of generality that

\[
\omega(z) = \sum_{l \in \mathbb{Z}^n} c_l \omega_l(z) \tag{7.10}
\]

for constants $c_l \in \mathbb{C}$, such that $c_l \neq 0$ for only finitely many values of $l$.

For each lattice point $l = (l_1, \ldots, l_n) \in \mathbb{Z}^n$, both $\omega_l(z)$ and $\omega_{-l}(z)$ have degree $d = \sum_{j=1}^n |l_j|$ as polynomials in the variables $\beta^{x_j}(z)$ and $\gamma^{x'_j}(z)$. Let $d$ be the maximal degree of terms $\omega_l(z)$ appearing in (7.10) with non-zero coefficient $c_l$, and let $l$ be such a lattice point for which $\omega_l(z)$ has degree $d$. An OPE calculation shows that

\[
\omega_{-l}(z) \circ_{d-1} \omega_{l'}(z) = \begin{cases} 
0 & l' \neq l \\
\left(\prod_{j=1}^n (-1)^{k_j |l_j|} \right) 1 & l' = l \end{cases} \tag{7.11}
\]

where $k_j = \min\{0, l_j\}$, for all lattice points $l'$ appearing in (7.10) with non-zero coefficient. It follows from (7.11) that

\[
\frac{1}{c_l \left(\prod_{j=1}^n (-1)^{k_j |l_j|} \right)} \omega_{-l}(z) \circ_{d-1} \omega(z) = 1. \quad \square
\]
The map $\pi : S(V)^{\Theta^+} \rightarrow D(V)^{g}$

Equip $S(V)$ with the conformal structure $L^\alpha$ given by (3.2), for some $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{C}^n$. Suppose first that $A^e \cap \mathbb{Z}^n$ has rank zero, so that $S(V)^{\Theta^+} = B'$, and $D(V)^g = \mathbb{C}[e_1, \ldots, e_n] = E$. Let $\pi : S(V)^{\Theta^+} \rightarrow D(V)^g$ be the map given by (1.2). By Lemma 6.3, for $j = 1, \ldots, n$ we have

$$\pi(L^j(z)) = \frac{1}{2}(e_j^2 + e_j), \quad \pi(W^j(z)) = \frac{2}{3\sqrt{6}}e_j^3 + \frac{1}{\sqrt{6}}e_j^2 + \frac{1}{3\sqrt{6}}e_j.$$

Moreover, (3.10) shows that $\pi(\phi^i(z)) = \langle b^i, \alpha \rangle - \sum_{j=1}^n b^j e_j + 1$. Since $B'$ is strongly generated by $\{\phi^i(z), L^j(z), W^j(z)\} | i = m + 1, \ldots, n, j = 1, \ldots, n$, it follows from Lemma 3.4 that $Im(\pi)$ is generated by the collection

$$\{\pi(\phi^i(z)), \pi(L^j(z)), \pi(W^j(z)) \mid i = m + 1, \ldots, n, j = 1, \ldots, n\}.$$

The map $\pi$ is not surjective, but $Coker(\pi)$ is generated as a module over $Im(\pi)$ by the collection $\{t^\xi_i \mid i = 1, \ldots, m\}$, where $t^\xi_i$ is the image of

$$\pi_{Z\theta}(\theta^\xi(z)) = \langle a^i, \alpha \rangle - \sum_{j=1}^n a^j e_j + 1$$

in $Coker(\pi) = E/\pi(B')$. Unlike the case where $V$ is one-dimensional, $\pi$ depends on the choice of $\alpha$.

Suppose next that the lattice $A^e \cap \mathbb{Z}^n = 0$ has positive rank. Clearly $\pi_{Z\theta}(Mli) = Mli$ for all $l$, so $\pi(Mli) \subset Mli$. This map need not be surjective, but since $Mli$ is the free $E$-module generated by $\omega_l$, and $E/\pi(B')$ is generated as a $\pi(B')$-module by $\{t^\xi_i \mid i = 1, \ldots, m\}$, it follows that each $Mli/\pi(Mli)$ is generated as a $\pi(B')$-module by $\{t^\xi_i \mid i = 1, \ldots, m\}$, where $t^\xi_i$ is the image of $\pi_{Z\theta}(\theta^\xi(z))\omega_l$ in $Mli/\pi(Mli)$.

**Theorem 7.6.** For any action of $g$ on $V$, $Coker(\pi)$ is generated as a module over $Im(\pi)$ by the collection $\{t^\xi_i \mid i = 1, \ldots, m\}$. In particular, $Coker(\pi)$ is a finitely generated module over $Im(\pi)$ with generators corresponding to central elements of $D(V)^g$.

Proof: First, since $\pi(\omega_l(z)) = \omega_l$ for all $l$, it is clear that the generators $t^\xi_i$ of $Mli/\pi(Mli)$ lie in the $Im(\pi)$-module generated by $\{t^\xi_i \mid i = 1, \ldots, m\}$, which proves the first statement. Finally, the fact that the elements $\pi_{Z\theta}(\theta^\xi(z))$ corresponding to $t^\xi_i$ each lie in the center of $D(V)^g$ is immediate from (2.10). 

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7.2. A vertex algebra bundle over the Grassmannian $Gr(m, n)$

As $\rho$ varies over the space $R^0(V)$ of effective actions, recall that $S(V)^{\Theta, +}_\rho$ is uniquely determined by the point $A(\rho) \in Gr(m, n)$. The algebras $S(V)^{\Theta, +}_\rho$ do not form a fiber bundle over $Gr(m, n)$. However, the subspace of $S(V)^{\Theta, +}_\rho$ of degree zero in the $A(\rho)^\perp \cap \mathbb{Z}^n$-grading (7.6) is just $B'_\rho = B'$, and the algebras $B'_\rho$ form a bundle of vertex algebras $E$ over $Gr(m, n)$. The classical analogue of $E$ is not interesting; it is just the trivial bundle whose fiber over each point is the polynomial algebra $E$.

For each $\rho$, recall that $B'_\rho = \mathcal{W}_\rho \otimes \Phi_\rho$, where $\mathcal{W}_\rho$ is generated by $\{L_j(z), W_j(z) \mid j = 1, \ldots, n\}$, and $\Phi_\rho$ is generated by $\{\phi^i(z) \mid i = m + 1, \ldots, n\}$. Since $\mathcal{W}_\rho$ is independent of $\rho$, it gives rise to a trivial subbundle of $E$. As a vector space, note that $\Phi_\rho = \text{Sym}(\bigoplus_{k \geq 1} A(\rho)^\perp_k)$, where $A(\rho)^\perp_k$ is the copy of $A(\rho)^\perp$ spanned by the vectors $\partial^k \phi^i(z)$ for $i = m + 1, \ldots, n$. It follows that the factor $\Phi_\rho$ in the fiber over $A(\rho)$ gives rise to the following subbundle of $E$:

$$\text{Sym}(\bigoplus_{k \geq 1} F_k),$$

(7.12)

where $F_k$ is the quotient of the rank $n$ trivial bundle over $Gr(m, n)$ by the tautological bundle. Since each $F_k$ has weight $k$, the weighted components of the bundle (7.12) are all finite-dimensional. The non-triviality of this bundle is closely related to Theorem 7.4.

8. Vertex algebra operations and transvectants on $D(V)^\Theta$

If we fix a basis $\{x_1, \ldots, x_n\}$ for $V$ and a dual basis $\{x'_1, \ldots, x'_n\}$ for $V^*$, $S(V)$ has a basis consisting of iterated Wick products of the form

$$\mu(z) = : \partial^{k_1} \gamma^{x'_{i_1}}(z) \cdots \partial^{k_r} \gamma^{x'_{i_r}}(z) \partial^{l_1} \beta^{x_{j_1}}(z) \cdots \partial^{l_s} \beta^{x_{j_s}}(z) :.$$

Define gradings degree and level on $S(V)$ as follows:

$$\text{deg}(\mu) = r + s, \quad \text{lev}(\mu) = \sum_{i=1}^{r} k_i + \sum_{j=1}^{s} l_j,$$

and let $S(V)^{(n)}[d]$ denote the subspace of level $n$ and degree $d$. The gradings

$$S(V) = \bigoplus_{n \geq 0} S(V)^{(n)} = \bigoplus_{n, d \geq 0} S(V)^{(n)}[d] = \bigoplus_{d \geq 0} S(V)[d]$$

(8.1)
are clearly independent of our choice of basis on $V$, since an automorphism of $V$ has the effect of replacing $\beta^{x_i}$ and $\gamma^{x'_i}$ with linear combinations of the $\beta^{x_i}$'s and $\gamma^{x'_i}$'s, respectively.

Let $\sigma : \mathcal{D}(V) \to gr\mathcal{D}(V) = Sym(V \oplus V^*)$ be the map

$$x'_{i_1} \cdots x'_{i_r} \frac{\partial}{\partial x'_{j_1}} \cdots \frac{\partial}{\partial x'_{j_s}} \mapsto x'_{i_1} \cdots x'_{i_r} x_{j_1} \cdots x_{j_s},$$

(8.2)

which is a linear isomorphism. Any bilinear product $\ast$ on $Sym(V \oplus V^*)$ corresponds to a bilinear product on $\mathcal{D}(V)$, which we also denote by $\ast$, as follows:

$$\omega \ast \nu = \sigma^{-1}(\sigma(\omega) \ast \sigma(\nu)),$$

for $\omega, \nu \in \mathcal{D}(V)$, Moreover, $\omega_1, \ldots, \omega_k$ generate $\mathcal{D}(V)$ as a ring if and only if $\sigma(\omega_1), \ldots, \sigma(\omega_k)$ generate $Sym(V \oplus V^*)$ as a ring. The map $f : Sym(V \oplus V^*) \to S(V^{(0)})$ given by

$$x'_{i_1} \cdots x'_{i_r} x_{j_1} \cdots x_{j_s}, \mapsto : \gamma^{x'_{i_1}}(z) \cdots \gamma^{x'_{i_r}}(z) \beta^{x_{j_1}}(z) \cdots \beta^{x_{j_s}}(z),$$

(8.3)

is a linear isomorphism, so that $f \circ \sigma : \mathcal{D}(V) \to S(V^{(0)})$ is a linear isomorphism as well.

$S(V^{(0)})$ has a family of bilinear products $\ast_k$ which are induced by the circle products on $S(V)$. Given $\omega(z), \nu(z) \in S(V^{(0)})$, define

$$\omega(z) \ast_k \nu(z) = p(\omega(z) \circ_k \nu(z)),$$

(8.4)

where $p : S(V) \to S(V^{(0)})$ is the projection onto the subspace of level zero. Clearly $\omega(z) \ast_k \nu(z) = 0$ whenever $k < -1$ because $p \circ \partial$ acts by zero on $S(V^{(0)})$. For $k \geq -1$, $\ast_k$ is homogeneous of degree $-2k - 2$.

Via (8.3), we may pull back the products $\ast_k$, $k \geq -1$ to obtain a family of bilinear products on $Sym(V \oplus V^*)$, which we also denote by $\ast_k$. In fact, these products have a classical description. Let

$$\Gamma = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \otimes \frac{\partial}{\partial x'_i} - \frac{\partial}{\partial x'_i} \otimes \frac{\partial}{\partial x_i},$$

(8.5)

and define the $k$th transvectant$^1$ on $Sym(V \oplus V^*)$ by

$$[\cdot, \cdot]_k : Sym(V \oplus V^*) \otimes Sym(V \oplus V^*) \to Sym(V \oplus V^*), \quad [\omega, \nu]_k = m \circ \Gamma^k(\omega \otimes \nu).$$

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I thank N. Wallach for explaining this construction to me.
Here $m$ is the multiplication map sending $\omega \otimes \nu \mapsto \omega \nu$.

**Theorem 8.1.** The product $*^k$ on $\text{Sym}(V \oplus V^*)$ given by (8.4) coincides with the transvectant $[,]_{k+1}$ for $k \geq -1$.

Proof: First consider the case $k = -1$. In this case $[,]_0$ is just ordinary multiplication. Recall the formula $\omega \nu = (\omega \nu)$. Hence given $\omega, \nu \in \text{Sym}(V \oplus V^*)$, we have $[\omega, \nu]_0 = \omega \nu = \omega *_{-1} \nu.$

Next, if $k \geq 0$, it is clear from the definition of the vertex algebra products $\circ_k$ that given $\omega(z), \nu(z) \in \text{Sym}(V)^{(0)}$, $\omega(z) *_k \nu(z)$ is just the sum of all possible contractions of $k + 1$ factors of the form $\beta^{x_i}(z)$ or $\gamma^{x_i}(z)$ appearing in $\omega(z)$ with $k + 1$ factors of the form $\beta^{x_j}(z)$ or $\gamma^{x_j}(z)$ appearing in $\nu(z)$. Here the contraction of $\beta^{x_i}(z)$ with $\gamma^{x_j}(z)$ is $\delta_{i,j}$, and the contraction of $\gamma^{x_i}(z)$ with $\beta^{x_j}(z)$ is $-\delta_{i,j}$. Similarly, it follows from (8.5) that given $\omega, \nu \in \text{Sym}(V \oplus V^*)$, $[\omega, \nu]_{k+1}$ is the sum of all possible contractions of $k + 1$ factors of the form $x_i$ or $x'_i$ appearing in $\omega$ with $k + 1$ factors of the form $x_i$ or $x'_i$ appearing in $\nu$. The contraction of $x_i$ with $x'_j$ is $\delta_{i,j}$ and the contraction of $x'_i$ with $x_j$ is $-\delta_{i,j}$. Since $f : \text{Sym}(V \oplus V^*) \rightarrow \text{Sym}(V)^{(0)}$ is the algebra isomorphism sending $x_i \mapsto \beta^{x_i}(z)$ and $x'_i \mapsto \gamma^{x_i}(z)$, the claim follows. □

Via $\sigma : D(V) \rightarrow \text{Sym}(V \oplus V^*)$ the products $*_k$ on $\text{Sym}(V \oplus V^*)$ pull back to bilinear products on $D(V)$, which we also denote by $*_k$. These products satisfy $\omega *_k \nu \in D(V)_{(r+s-2k-2)}$ for $\omega \in D(V)_{(r)}$ and $s \in D(V)_{(s)}$. It is immediate from Theorem 8.1 that $*_1$ and $*_0$ correspond to the ordinary associative product and bracket on $D(V)$, respectively. Since the circle product $\circ_0$ is a derivation of every $\circ_k$, it follows that $\omega *_0$ is a derivation of $*_k$ for all $\omega \in D(V)$ and $k \geq -1$.

We call $D(V)$ equipped with the products $\{*_k| k \geq -1\}$ a $*$-algebra. A similar construction goes through in other settings as well. For example, given a Lie algebra $\mathfrak{g}$
equipped with a symmetric, invariant bilinear form $B$, $\mathfrak{u}g$ has a $*$-algebra structure (which depends on $B$). Given a $*$-algebra $\mathcal{A}$, we can define $*$-subalgebras, $*$-ideals, quotients, and homomorphisms in the obvious way. If $V$ is a module over a Lie algebra $\mathfrak{g}$, $\mathcal{D}(V)^{\mathfrak{g}}$ is a $*$-subalgebra of $\mathcal{D}(V)$ because the action of $\xi \in \mathfrak{g}$ is given by $[\tau(\xi), -] = \tau(\xi)^*0$ which is a derivation of all the other products.

Given elements $\omega_1, \ldots, \omega_k \in \mathcal{D}(V)^{\mathfrak{g}}$, examples are known where $\omega_1, \ldots, \omega_k$ do not generate $\mathcal{D}(V)^{\mathfrak{g}}$ as a ring, but do generate $\mathcal{D}(V)^{\mathfrak{g}}$ as a $*$-algebra. This phenomenon occurs in our main example, in which $\mathfrak{g}$ is the abelian Lie algebra $C^m$ acting diagonally on $V = C^n$. Recall that $\mathcal{D}(V)^{\mathfrak{g}} = \bigoplus_{l \in A^+ \cap \mathbb{Z}^n} M_l$, where $M_l$ is the free $E$-module generated by $\omega_l$. Suppose that $A^+ \cap \mathbb{Z}^n$ has rank $r$, and let $\{l^i = (l^i_1, \ldots, l^i_n) | i = 1, \ldots, r\}$ be a basis for $A^+ \cap \mathbb{Z}^n$. In general, the collection

$$e_1, \ldots, e_n, \ \omega_{l^1}, \ldots, \omega_{l^r}, \ \omega_{-l^1}, \ldots, \omega_{-l^r} \tag{8.6}$$

is too small to generate $\mathcal{D}(V)^{\mathfrak{g}}$ as a ring.

**Theorem 8.2.** $\mathcal{D}(V)^{\mathfrak{g}}$ is generated as a $*$-algebra by the collection (8.6). Moreover, $\mathcal{D}(V)^{\mathfrak{g}}$ is simple as a $*$-algebra.

Proof: To prove the first statement, it suffices to show that given lattice points $l = (l_1, \ldots, l_n)$ and $l' = (l'_1, \ldots, l'_n)$, $\omega_{l+l'}$ lies in the $*$-algebra generated by $\omega_l$ and $\omega_{l'}$. For $j = 1, \ldots, n$, define

$$d_j = \begin{cases} 0 & l_j l'_j \geq 0 \\ \min\{|l_j|, |l'_j|\}, & l_j l'_j < 0 \end{cases}, \quad e_j = \begin{cases} 0 & l_j l'_j \geq 0 \\ \max\{|l_j|, |l'_j|\}, & l_j l'_j < 0 \end{cases},$$

$$k_j = \begin{cases} 0 & l_j \leq 0 \\ d_j & l_j > 0 \end{cases}, \quad d = -1 + \sum_{j=1}^n d_j.$$

The same calculation as in the proof of Theorem 7.3 shows that

$$\omega_l * d \omega_{l'} = \left( \prod_{j=1}^n (-1)^{k_j} \frac{e_j!}{(e_j - d_j)!} \right) \omega_{l+l'},$$

which shows that $\omega_{l+l'}$ lies in the $*$-algebra generated by $\omega_l$ and $\omega_{l'}$.

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I thank N. Wallach for pointing this out to me.
As for the second statement, the argument is analogous to the proof of Theorem 7.5. Given a non-zero *-ideal $I \subset D(V)^*$, we need to show that $1 \in I$. Let $\omega$ be a non-zero element of $I$. It is easy to check that for $i, j = 1, \ldots, n$, and $l \in A^\perp \cap \mathbb{Z}^n$, we have

$$e_i *_1 e_j = -\delta_{i,j}, \quad e_i *_1 \omega_l = 0$$

By applying the operators $e_i *_1$ for $i = 1, \ldots, n$, we can reduce $\omega$ to the form

$$\sum_{l \in \mathbb{Z}^n} c_l \omega_l$$

for constants $c_l \in \mathbb{C}$, such that $c_l \neq 0$ for only finitely many values of $l$. We may assume without loss of generality that $\omega$ is already of this form. Let $d$ be the maximal degree (in the Bernstein filtration) of terms $\omega_l$ appearing in (8.7) with non-zero coefficient $c_l$, and let $l$ be such a lattice point for which $\omega_l$ has degree $d$. We have

$$\omega_{-l} *_{d-1} \omega_{l'} = \begin{cases} 0 & l' \neq l \\ \left( \prod_{j=1}^n (-1)^{k_j} |l_j|! \right) 1 & l' = l \end{cases}$$

where $k_j = \min\{0, l_j\}$, for all $l'$ appearing in (8.7). Hence

$$\frac{1}{c_l \left( \prod_{j=1}^n (-1)^{k_j} |l_j|! \right)} \omega_{-l} *_{d-1} \omega = 1. \quad \Box$$
References

[1] P. Bouwknegt, J. McCarthy, K. Pilch, The $\mathcal{W}_3$ algebra: Modules, Semi-infinite Cohomology and BV-algebras, Lect. Notes in Phys, New Series Monographs 42, Springer-Verlag, 1996.

[2] B. Feigin and E. Frenkel, Semi-Infinite Weil complex and the Virasoro algebra, Comm. Math. Phys. 137 (1991), 617-639.

[3] I. Frenkel, Two constructions of affine Lie algebras and boson-fermion correspondence in quantum field theory, J. Funct. Anal. 44 (1981) 259-327.

[4] I.B. Frenkel, and Y.C. Zhu, Vertex operator algebras associated to representations of affine and Virasoro algebras, Duke Mathematical Journal, Vol. 66, No. 1, (1992), 123-168.

[5] D. Friedan, E. Martinec, S. Shenker, Conformal invariance, supersymmetry and string theory, Nucl. Phys. B271 (1986) 93-165.

[6] P. Goddard, A. Kent, and D. Olive, Virasoro algebras and coset space models, Phys. Lett B 152 (1985) 88-93.

[7] V. Gorbounov, F. Malikov, V. Schectman, Gerbes of chiral differential operators, Math. Res. Lett. 7 (2000), no. 1, 55–66.

[8] Harish-Chandra, Differential operators on a semisimple Lie algebra, Am. J. Math, Vol. 79, No. 1 (1957) 87-120.

[9] Harish-Chandra, Invariant differential operators and distributions on a semisimple Lie algebra, Am. J. Math. Vol. 86, No. 3 (1964) 534-564.

[10] V. Kac, D. Peterson, Infinite-dimensional Lie algebras, theta functions and modular forms, Adv. Math. 53 (1984) 125-264.

[11] V. Kac, A. Radul, Representation theory of the vertex algebra $\mathcal{W}_{1+\infty}$, Transf. Groups, Vol 1 (1996) 41-70.
[12] F. Knop, A Harish-Chandra homomorphism for reductive group actions, Ann. Math. 140 (1994), 253-288.

[13] B. Lian, A. Linshaw, Howe pairs in the theory of vertex algebras, J. Algebra 317, 111-152 (2007).

[14] B. Lian, and A. Linshaw, Chiral equivariant cohomology I, Adv. Math. 209, 99-161 (2007).

[15] F. Malikov, V. Schectman, and A. Vaintrob, Chiral de Rham complex, Commun. Math. Phys, 204, (1999) 439-473.

[16] I. Musson, M. van den Bergh, Invariants under Tori of Rings of Invariant Operators and Related Topics, Mem. Am. Math. Soc. No. 650 (1998).

[17] G. Schwarz, Finite-dimensional representations of invariant differential operators, J. Algebra 258 (2002) 160-204.

[18] W. Wang, $\mathcal{W}_{1+\infty}$ algebra, $\mathcal{W}_3$ algebra, and Friedan-Martinec-Shenker bosonization, Commun. Math. Phys. 195 (1998), 95–111.

[19] W. Wang, Classification of irreducible modules of $\mathcal{W}_3$ with $c = -2$, Commun. Math. Phys. 195 (1998), 113–128.

[20] A. Zamolodchikov, Infinite additional symmetries in two-dimensional conformal field theory, Theor. Math. Phys. 65 (1985) 1205-1213.

[21] Y. Zhu, Modular invariants of characters of vertex operators, J. Amer. Math. Soc. 9 (1996) 237-302.

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