SINGULARITY OF THE VARIETIES OF REPRESENTATIONS
OF LATTICES IN SOLVABLE LIE GROUPS

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Abstract. For a lattice $\Gamma$ of a simply connected solvable Lie group $G$, we
describe the analytic germ in the variety of representations of $\Gamma$ at the trivial
representation as an analytic germ which is linearly embedded in the analytic
germ associated with the nilpotent Lie algebra determined by $G$. By this
description, under certain assumption, we study the singularity of the analytic
germ in the variety of representations of $\Gamma$ at the trivial representation by
using the Kuranishi space construction. By the similar technique, we also
study deformations of holomorphic structures of trivial vector bundles over
complex parallelizable solvmanifolds.

1. INTRODUCTION

Let $X$ be an analytic germ in $\mathbb{C}^n$ at the origin defined by analytic equations
$$f_1(z) = 0, \ldots, f_k(z) = 0.$$  
We say that $X$ is cut out by polynomial equations of degree at most $\nu$ if
$$f_1(z), \ldots, f_k(z)$$
are polynomial functions of degree at most $\nu$ with trivial linear terms. We say
that an analytic germ $Y$ is linearly embedded in $X$ if for a subspace $V \subset \mathbb{C}^n$ $Y$ is
equivalent to an analytic germ in $V$ at the origin defined by analytic equations
$$f_1(z) = 0, \ldots, f_k(z) = 0, \ z \in V.$$  
If $X$ is cut out by polynomial equations of degree at most $\nu$ and $Y$ is linearly
embedded in $X$, then $Y$ is also cut out by polynomial equations of degree at most $\nu$.

Let $\Gamma$ be a finitely generated group, $A$ a linear algebraic group with a Lie algebra $\mathfrak{a}$ and $R(\Gamma, A)$ the set of homomorphisms $\Gamma \to A$. Then $R(\Gamma, A)$ can be considered
as an affine algebraic variety. For a representation $\rho \in R(\Gamma, A)$ we are interested in
the analytic germ $(R(\Gamma, A), \rho)$. The singularity of the analytic germ $(R(\Gamma, A), \rho)$ is
considered as obstructions of deformations of $\rho$.

If $\Gamma$ is the fundamental group of a manifold $M$, we can geometrically describe
the analytic germ $(R(\Gamma, A), \rho)$ by using the deformation theory of differential graded
Lie algebra (shortly DGLA) developed by Goldman and Millson \[6\], \[7\]. By such
technique and the Hodge theory of local systems over Kähler manifolds studied by
Simpson \[14\], if $\Gamma$ is a Kähler group (i.e. a group which can be the fundamental
group of a compact Kähler manifold ), then for a semisimple representation $\rho \to$
$GL_m(\mathbb{C})$, the analytic germ $(R(\Gamma, GL_m(\mathbb{C})), \rho)$ is cut out by polynomial equations of degree at most 2.

However in general the analytic germ $(R(\Gamma, A), \rho)$ is not cut out by polynomial equations of degree at most 2. In \[6\], Goldman and Millson observed that for a lattice $\Gamma$ in the three dimensional real Heisenberg group, the analytic germ $(R(\Gamma, A), 1)$ at the trivial representation 1 is equivalent to a cubic cone. In this paper we consider certain class of groups which contains this example.

Let $\Gamma$ be a lattice in a simply connected solvable Lie group $G$. Then the solvmanifold $G/\Gamma$ is an aspherical manifold with the fundamental group $\Gamma$. The purpose of this paper is to study the analytic germ $(R(\Gamma, A), 1)$ at the trivial representation 1.

For a manifold $M$ the analytic germ at the trivial representation of the fundamental group of $M$ can be studied by the differential graded algebra (shortly DGA) $A^*(M)$ of the differential forms on $M$. The following result is known.

**Theorem 1.1** (\[6\], \[7\], \[3\]). Let $M$ be a compact manifold with the fundamental group $\Gamma$. Suppose that we have a finite dimensional sub-DGA $C^* \subset A^*(M) \otimes \mathbb{C}$ such that the inclusion induces a cohomology isomorphism and $C^0 = \mathbb{C}$. Then the analytic germ $(R(\Gamma, A), 1)$ at the trivial representation 1 is equivalent to the analytic germ $(F(C^*, a), 0)$ at the origine 0 for the affine variety

$$F(C^*, a) = \left\{ \omega \in C^* \otimes a : d\omega + \frac{1}{2} [\omega, \omega] = 0 \right\}.$$

Consider a solvmanifold $G/\Gamma$, Lie algebra $\mathfrak{g}$ of $G$ and the cochain complex $\mathfrak{g}^*$ which is regarded as a differential graded algebra of left-$G$-invariant forms on $G/\Gamma$. Suppose that $G$ is completely solvable. In \[8\] Hattori proved that the inclusion $\mathfrak{g}^* \subset A^*(G/\Gamma)$ induces a cohomology isomorphism. By Theorem 1.1 and Hattori’s theorem, in \[4\], Dimca and Papadima remarked that the analytic germ $(R(\Gamma, A), 1)$ is equivalent to the analytic germ $(F(\mathfrak{g}^*, a), 0)$ at the origine 0. However, for a general solvmanifold $G/\Gamma$, the inclusion $\mathfrak{g}^* \subset A^*(G/\Gamma)$ does not induces a cohomology isomorphism.

In this paper, we consider general solvmanifolds. Let $\mathfrak{g}$ be a solvable Lie algebra. Then we can define the nilpotent Lie algebra $\mathfrak{a}$ called nilshadow of $\mathfrak{g}$ which is uniquely determined by $\mathfrak{g}$ as \[9\].

**Theorem 1.2** (\[9\]). Let $G$ be a simply connected solvable Lie group with a lattice $\Gamma$ and $\mathfrak{g}$ the Lie algebra of $G$. We consider the nilshadow $\mathfrak{a}$ of $\mathfrak{g}$. Then we have a sub-DGA $A^*_\mathfrak{a} \subset A^*(G/\Gamma) \otimes \mathbb{C}$ such that:

- The inclusion $A^*_\mathfrak{a} \subset A^*(G/\Gamma) \otimes \mathbb{C}$ induces a cohomology isomorphism.
- $A^*_\mathfrak{a}$ can be regarded as a sub-DGA of $\mathfrak{u}^* \otimes \mathbb{C}$.

See Section 2 for the constructions of the nilshadow and DGA $A^*_\mathfrak{a}$. By this theorem and Theorem 1.1 the analytic germ $(R(\Gamma, A), 1)$ is equivalent to the analytic germ $(F(A^*_\mathfrak{a}, a), 0)$. By the second assertion of the theorem we have the following theorem.

**Theorem 1.3.** Let $\Gamma$ be a lattice in a simply connected solvable Lie group $G$ and $\mathfrak{g}$ the Lie algebra of $G$. Let $\mathfrak{a}$ be a linear algebraic group with the Lie algebra $\mathfrak{a}$. Consider the nilshadow $\mathfrak{a}$ of $\mathfrak{g}$. Then the analytic germ $(R(\Gamma, A), 1)$ at the trivial representation 1 is linearly embedded in the analytic germ $(F(\mathfrak{u}^*, a), 0)$ at the
abelian and consists of semi-simple elements. Let \( \bar{g} \)

Then we have

\[
\text{ad}
\]

where \((\text{ad})\)

of a choice of a subvector space

have

\[
\text{ad}
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of \( g \) of

\( \nu \)-step naturally graded nilpotent Lie algebra and

\( g \) a Lie algebra. Then the analytic germ \( (F(\bigwedge n^*, g), 0) \) is cut out by polynomial equations of degree at most \( \nu + 1 \).

By this proposition and Theorem 1.3 we have the following theorem.

Theorem 1.5. Let \( \Gamma \) be a lattice in a simply connected solvable Lie group \( G \) and \( g \) the Lie algebra of \( G \). Let \( A \) be a linear algebraic group with a Lie algebra \( a \). Consider the nilshadow \( u \) of \( g \). We suppose that the Lie algebra \( u \) is \( \nu \) step naturally graded. Then the analytic germ \( (R(\Gamma, A), 1) \) at the trivial representation \( 1 \) is cut out by polynomial equations of degree at most \( \nu + 1 \).

Let \( n \) be a two-step nilpotent Lie algebra. For any complement \( a^{(1)} \) of \( n^{(2)} \) in \( n \), we have \( n = a^{(1)} \oplus n^{(2)} \) and \( [a^{(1)}, a^{(1)}] \subset n^{(2)} \) and so a two-step nilpotent Lie algebra \( n \) is naturally graded. Hence as an application of Theorem 1.5 we have the following Corollary

Corollary 1.6. Let \( \Gamma \) be a lattice in a simply connected solvable Lie group \( G \) and \( g \) the Lie algebra of \( G \). Let \( A \) be a linear algebraic group with a Lie algebra \( a \). Consider the nilshadow \( u \) of \( g \). We suppose that the Lie algebra \( u \) is two-step naturally graded. Then the analytic germ \( (R(\Gamma, A), 1) \) at the trivial representation \( 1 \) is cut out by polynomial equations of degree at most 3.

2. Nilshadows and cohomology of solvmanifolds

Let \( g \) be a solvable \( K \)-Lie algebra for \( K = \mathbb{R} \) or \( \mathbb{C} \). Let \( n \) be the nilradical of \( g \). There exists a subvector space (not necessarily Lie algebra) \( V \) of \( g \) so that \( g = V \oplus n \) as the direct sum of vector spaces and for any \( A, B \in V (\text{ad}_A)_n(B) = 0 \) where \((\text{ad}_A)_n \) is the semi-simple part of \( \text{ad}_A \) (see [5 Proposition III.1.1]). We define the map \( \text{ad}_A : g \rightarrow D(g) \) as \( \text{ad}_A + X = (\text{ad}_A)_n \) for \( A \in V \) and \( X \in n \). Then we have \([\text{ad}_A(g), \text{ad}_A(g)] = 0 \) and \( \text{ad}_A \) is linear (see [5 Proposition III.1.1]). Since we have \([g, g] \subset n \), the map \( \text{ad}_A : g \rightarrow D(g) \) is a representation and the image \( \text{ad}_A(g) \) is abelian and consists of semi-simple elements. Let \( \bar{g} = \text{Im} \text{ad}_A \times g \) and

\[
u = \{X - \text{ad}_X \in \bar{g} | X \in g\}.
\]

Then we have \([g, \bar{g}] \subset n \subset u \) and \( u \) is the nilradical of \( \bar{g} \) (see [5]). Hence we have \( \bar{g} = \text{Im} \text{ad}_A \times u \). It is known that the structure of the Lie algebra \( u \) is independent of a choice of a subvector space \( V \) (see [5 Corollary III.3.6]).
Lemma 2.1. ([9] Lemma 2.2) Suppose \( g = \mathbb{R}^k \ltimes \phi n \) such that \( \phi \) is a semi-simple action and \( n \) is nilpotent. Then the nilshadow \( u \) of \( g \) is the direct sum \( \mathbb{R}^k \oplus n \).

Let \( G \) be a simply connected solvable Lie group with the \( \mathbb{R} \)-Lie algebra \( g \). We denote by \( Ad_s : G \to \text{Aut}(g) \) the extension of \( ad_s \). Then \( Ad_s(G) \) is diagonalizable. Let \( X_1, \ldots, X_n \) be a basis of \( g \otimes \mathbb{C} \) such that \( Ad_s \) is represented by diagonal matrices. Then we have \( Ad_sX_i = \alpha_i(g)X_i \) for characters \( \alpha_i \) of \( G \). Let \( x_1, \ldots, x_n \) be the dual basis of \( X_1, \ldots, X_n \).

We suppose \( G \) has a lattice \( \Gamma \). Then we consider the sub-DGA \( A^*_\Gamma \) of the de Rham complex \( A^*(G/\Gamma) \otimes \mathbb{C} \) which is given by

\[
A^*_\Gamma = \left\langle \alpha_I x_I \mid I \subset \{1, \ldots, n\}, \ (\alpha_I)|_{\Gamma} = 1 \right\rangle.
\]

where for a multi-index \( I = \{i_1, \ldots, i_p\} \) we write \( x_I = x_{i_1} \wedge \cdots \wedge x_{i_p} \), and \( \alpha_I = \alpha_{i_1} \cdots \alpha_{i_p} \).

Theorem 2.2. ([9] Corollary 7.6) Let \( G \) be a simply connected solvable Lie group with a lattice \( \Gamma \). Then we have:

- The inclusion \( A^*_\Gamma \subset A^*(G/\Gamma) \otimes \mathbb{C} \) induces a cohomology isomorphism.
- \( A^*_\Gamma \) can be regarded as a sub-DGA of \( \wedge u^* \otimes \mathbb{C} \).

We explain the second assertion more precisely. We consider the subspace \( \bar{u} = \langle \alpha_1^{-1}X_1, \ldots, \alpha_n^{-1}X_n \rangle \) of the space of complex valued vector fields on \( G \). Then \( \bar{u} = \langle \alpha_1^{-1}X_1, \ldots, \alpha_n^{-1}X_n \rangle \) is a Lie sub-algebra of the Lie algebra of vector fields and the map

\[
\bar{u} \ni \alpha_i^{-1}X_i \mapsto X_i - \text{ad}_{sX_i} \in u \otimes \mathbb{C}
\]

is a Lie algebra isomorphism where \( u \) is the nilshadow of \( g \) (see [9] Proof of Lemma 5.2).

Example 1. Let \( g \) be a 4-dimensional Lie algebra such that

- \( g = \langle T, X, Y, Z \rangle \)
- \([T, X] = X, [T, Y] = -Y, [X, Y] = Z\).

Then we have the splitting \( g = \langle T \rangle \ltimes \langle X, Y, Z \rangle \) such that \( \langle X, Y, Z \rangle \) is the three dimensional real Heisenberg Lie algebra \( h(3) \) and the action of \( \langle T \rangle \) is semi-simple. Hence by Lemma 2.1, the nilshadow \( u \) of \( g \) is given by \( u = \mathbb{R} \oplus h(3) \). Hence as similar to [9] Example 9.1, the analytic germ \( (F(\wedge u^*, a), 0) \) is equivalent to a cubic cone.

Consider the simply connected solvable Lie group \( G \) whose Lie algebra is \( g \). Then \( G \) has a lattice \( \Gamma \) [13]. We can easily show that the DGA \( A^*(G/\Gamma) \) is formal and hence the analytic germ \( (R(\Gamma, A), \mathbf{1}) \) at the trivial representation \( \mathbf{1} \) is cut out by polynomial equations of degree at most 2. Hence \( (R(\Gamma, A), \mathbf{1}) \) is linearly embedded in the analytic germ \( (F(\wedge u^*, a), 0) \) but its singularity is different from \( (F(\wedge u^*, a), 0) \).

By Lemma 2.1, we give one more corollary of Theorem 2.2.

Corollary 3. Let \( g = \mathbb{R}^k \ltimes \phi n \) such that \( \phi \) is a semi-simple action and \( n \) is \( \nu \)-step naturally graded nilpotent Lie algebra. Consider the simply connected solvable Lie group \( G \) whose Lie algebra is \( g \). Suppose \( G \) has a lattice \( \Gamma \). Then the analytic germ \( (R(\Gamma, A), \mathbf{1}) \) at the trivial representation \( \mathbf{1} \) is cut out by polynomial equations of degree at most \( \nu + 1 \).
3. Proof of Proposition 1.4

3.1. Finite-dimensional DGAs of Poincaré duality type. Let $A^*$ be a finite-dimensional graded commutative $\mathbb{C}$-algebra.

**Definition 3.1** ([10]). $A^*$ is of Poincaré duality type (PD-type) if the following conditions hold:

- $A^{*<0} = 0$ and $A^0 = \mathbb{C}1$ where 1 is the identity element of $A^*$.
- For some positive integer $n$, $A^{*>n} = 0$ and $A^n = \mathbb{C}v$ for $v \neq 0$.
- For any $0 < i < n$ the bi-linear map $A^i \times A^{n-i} \ni (\alpha, \beta) \mapsto \alpha \cdot \beta \in A^n$ is non-degenerate.

Suppose $A^*$ is of PD-type. Let $h$ be a Hermitian metric on $A^*$ which is compatible with the grading. Take $v \in A^n$ such that $h(v, v) = 1$. Define the $\mathbb{C}$-anti-linear map $\bar{\omega} : A^i \rightarrow A^{n-i} = \alpha \cdot \bar{\beta} = h(\alpha, \beta)v$.

**Definition 3.2** ([10]). A finite-dimensional DGA $(A^*, d)$ is of PD-type if the following conditions hold:

- $A^*$ is a finite-dimensional graded $\mathbb{C}$-algebra of PD-type.
- $dA^{n-1} = 0$ and $dA^0 = 0$.

Let $(A^*, d)$ be a finite-dimensional DGA of PD-type. Denote $d^* = -\bar{\omega}d\bar{\omega}$.

**Lemma 3.3** ([10]). We have $h(d\alpha, \beta) = h(\alpha, d^*\beta)$ for $\alpha \in A^{i-1}$ and $\beta \in A^i$.

Define $\Delta = dd^* + d^*d$. and $\mathcal{H}^*(A) = \ker \Delta$. By Lemma 3.3 and finiteness of the dimension of $A^*$, we can easily show the following lemma.

**Lemma 3.4** ([10]). We have the Hodge decomposition

$$A^r = \mathcal{H}^r(A) \oplus \Delta(A^r) = \mathcal{H}^r(A) \oplus d(A^{r-1}) \oplus d^*(A^{r+1}).$$

By this decomposition, the inclusion $\mathcal{H}^*(A) \subset A^*$ induces an isomorphism

$$\mathcal{H}^p(A) \cong H^p(A)$$

of vector spaces.

We denote by $H$ the projection $H : A^r \rightarrow \mathcal{H}^r(A)$ and define the operator $G$ as the composition $\Delta^{-1}_{\Delta(A^p)} \circ (id - H)$. Let $\beta : A^* \rightarrow dA^{r-1}$ be the projection for the decomposition

$$A^r = \mathcal{H}^r(A) \oplus d(A^{r-1}) \oplus d^*(A^{r+1}).$$

The restriction map $d : d^*(A^r) \rightarrow d(A^{r-1})$ is an isomorphism. Take the inverse $d^{-1} : d(A^{r-1}) \rightarrow d^*(A^r)$. Consider the map $d^*G : A^* \rightarrow A^{r-1}$. Then for $\omega \in \mathcal{H}^r(A)$, $d^*x \in d^*(A^r)$ and $d^*y \in d^*(A^{r+1})$, we have

$$d^*G(\omega + dd^*x + d^*y) = d^*(dd^*)^{-1}dd^*x = d^*x.$$

Hence we have $d^*G = d^{-1} \circ \beta$.

3.2. Kuranishi spaces of finite-dimensional DGLAs. Let $L^*$ be a finite-dimensional DGLA with a differential $d$. Consider the splitting $d(L^p) \rightarrow L^p$ for the short exact sequence

$$0 \rightarrow \ker d_{|L^p} \rightarrow L^p \xrightarrow{d} d(L^p) \rightarrow 0$$

and the splitting $H^p(L^*) \rightarrow \ker d_{|L^p}$ for the short exact sequence

$$0 \rightarrow d(L^p-1) \rightarrow \ker d_{|L^p} \rightarrow H^p(L^*) \rightarrow 0.$$
Denote by $A^p$ and $H^p$ the images of the splittings $d(L^p) \rightarrow L^p$ and $H^p(L^*) \rightarrow \ker d_{L^p}$ respectively. Then we have

$$L^p = H^p \oplus d(L^{p-1}) \oplus A^p.$$ 

Consider the projections $\beta^* : L^* \rightarrow d(L^{*-1})$, $H : L^* \rightarrow H^*$ and $\alpha^* : L^* \rightarrow A^*$. Since the restriction $d : A^p \rightarrow d(L^p)$ is an isomorphism, we have the inverse $d^{-1} : d(L^p) \rightarrow A^p$ of $d : A^p \rightarrow d(L^p)$. We define $\delta = d^{-1} \circ \beta : L^{p+1} \rightarrow L^p$. Define the map $F : L^1 \rightarrow L^1$ as

$$F(\zeta) = \zeta + \frac{1}{2} \delta[m, \zeta].$$

Then by the inverse function theorem, on a small ball $B$ in $L^1$ $F$ is an analytic diffeomorphism. Then the Kuranishi space $K(L^*)$ is defined by

$$K(L^*) = \{ \eta \in F(B) \cap H^1 : H([F^{-1}(\eta), F^{-1}(\eta)]) = 0 \}.$$ 

It is known that the analytic germ $(K(L^*), 0)$ is equivalent to the germ at the origine for the variety

$$\left\{ \zeta \in L^1 : d\zeta + \frac{1}{2}[\zeta, \zeta] = 0, \delta\zeta = 0 \right\}$$

(see [7 Theorem 2.6]). In particular, if $d(L^0) = 0$, then $K(L^*)$ is equivalent to the germ at the origine for the variety

$$\left\{ \zeta \in L^1 : d\zeta + \frac{1}{2}[\zeta, \zeta] = 0 \right\}.$$ 

Take a basis $\zeta_1, \ldots, \zeta_m$ of $H^1$. For parameters $t = (t_i)$, we consider the formal power series $\phi(t) = \sum r \phi_r(t)$ with values in $L^1$ given inductively by $\phi_1(t) = \sum t_i \zeta_j$ and

$$\phi_r(t) = -\frac{1}{2} \sum_{s=1}^{r-1} [\delta(\phi_s(t), \phi_{r-s}(t))].$$

Then $F^{-1}$ is given by $\phi_1(t) \mapsto \phi(t)$ and the Kuranishi space $K(L^*)$ is an analytic germ in $\mathbb{C}^m$ at the origin defined by equations

$$H([\phi(t), \phi(t)]) = 0.$$ 

Let $A$ be a finite-dimensional DGA of PD-type and $\mathfrak{g}$ a Lie algebra. Then we consider the DGLA $A^* \otimes \mathfrak{g}$. Then we have the Hodge decomposition

$$A^* \otimes \mathfrak{g} = H^p(A) \otimes \mathfrak{g} \oplus d(A^{p-1}) \otimes \mathfrak{g} \oplus d^* (A^{p+1}) \otimes \mathfrak{g}$$

as above with $\delta = d^* G \otimes \text{id}$. Take a basis $\zeta_1, \ldots, \zeta_m$ of $H^1(A^*) \otimes \mathfrak{g}$. For parameters $t = (t_i)$, we consider the formal power series $\phi(t) = \sum r \phi_r(t)$ with values in $A^1 \otimes \mathfrak{g}$ given inductively by $\phi_1(t) = \sum t_i \zeta_j$ and

$$\phi_r(t) = -\frac{1}{2} \sum_{s=1}^{r-1} d^* G \otimes \text{id} \phi_s(t), \phi_{r-s}(t).$$

By the above argument we have the following lemma

**Lemma 3.5.** The analytic germ $(F(A^*, \mathfrak{g}), 0)$ is equivalent to an analytic germ in $\mathbb{C}^m$ at the origin defined by equations

$$H([\phi(t), \phi(t)]) = 0.$$
3.3. Nilpotent Lie algebras. Let \( n \) be a \( \nu \)-step nilpotent \( K \)-Lie algebra for \( K = \mathbb{R} \) or \( \mathbb{C} \). Consider the lower central series

\[
n = n^{(1)} \supset n^{(2)} \supset \cdots \supset n^{(\nu)} \supset n^{(\nu+1)} = \{0\}
\]

where \( n^{(i+1)} = [n, n^{(i)}] \). Take a subspace \( a^{(i)} \) such that \( n^{(i)} = n^{(i+1)} \oplus a^{(i)} \). We have

\[
n = a^{(1)} \oplus a^{(2)} \oplus \cdots \oplus a^{(\nu)}.
\]

Consider the dual spaces \( n^* \) and \( a^{(i)*} \) of \( n \) and \( a^{(i)} \) respectively. We consider the cochain complex \( \Lambda n^* \) of the Lie algebra with the differential \( d \). Then \( \Lambda n^* \) is a finite-dimensional DGA of PD-type. We have

\[
\Lambda n^* = (\Lambda a^{(1)*}) \wedge \cdots \wedge (\Lambda a^{(\nu)*}).
\]

We have

\[
H^1(n) = \ker d_{\Lambda^1 n^*} = a^{(1)*}.
\]

**Lemma 3.6.**

\[
\ker d_{\Lambda^2 n^*} \subset \bigoplus_{i+j \leq \nu+1, i \leq j} a^{(i)*} \wedge a^{(j)*}.
\]

**Proof.** Let \( \sigma \in \ker d_{\Lambda^2 n^*} \). For a positive integer \( k < \nu \), we say that \( \sigma \) is \( k \)-decomposable if we have a decomposition

\[
\sigma = \sigma_1 + \sigma_2 + \sigma_3
\]

such that:

- \( \sigma_1 \in \bigoplus_{i+j \leq \nu+1, i \leq j, k < j} a^{(i)*} \wedge a^{(j)*} \).
- \( \sigma_2 \in \bigoplus_{i \leq k} a^{(i)*} \wedge a^{(k)*} \).
- \( \sigma_3 \in \bigoplus_{i \leq j, j < k} a^{(i)*} \wedge a^{(j)*} \).

If \( k \leq \frac{\nu+1}{2} \), then we have

\[
\sigma \in \bigoplus_{i+j \leq \nu+1, i \leq j} a^{(i)*} \wedge a^{(j)*}.
\]

Consider the case \( \frac{\nu+1}{2} < k \). For \( X, Y \in n \) and \( Z \in n^{(k)} \), we have \( \sigma_1([X, Y], Z) = 0 \), \( \sigma_2(X, [Y, Z]) = 0 \), \( \sigma_2(Y, [X, Z]) = 0 \), \( \sigma_3([X, Y], Z) = 0 \), \( \sigma_3(X, [Y, Z]) = 0 \) and \( \sigma_3(Y, [X, Z]) = 0 \). By \( d\sigma = 0 \), we have

\[
\sigma_2([X, Y], Z) = \sigma_1(X, [Y, Z]) - \sigma_1(Y, [X, Z]).
\]

Taking \( X \in n \) and \( Y \in n^{(l-1)} \) such that \( \nu + 1 < k + l \), we have

\[
\sigma_2([X, Y], Z) = 0.
\]

Hence for \( W \in n^{(l)} \) and \( Z \in n^{(k)} \) such that \( \nu + 1 < k + l \), we have

\[
\sigma_2(W, Z) = 0.
\]

Thus we have

\[
\sigma_2 \in \bigoplus_{i+k \leq \nu+1, i \leq k} a^{(i)*} \wedge a^{(k)*}.
\]
Hence taking \( \sigma_1' = \sigma_1 + \sigma_2 \) and \( \sigma_3 = \sigma_2' + \sigma_3' \) such that
\[
\sigma_2' \in \bigoplus_{i \leq k-1} a^{(i)*} \wedge a^{(k-1)*}
\]
and
\[
\sigma_3' \in \bigoplus_{i \leq j, j < k-1} a^{(i)*} \wedge a^{(j)*},
\]
by the decomposition \( \sigma = \sigma_1' + \sigma_2' + \sigma_3' \), \( \sigma \) is \((k-1)\)-decomposable. Thus we can say that if \( \sigma \) is \( k \)-decomposable and \( \frac{n+1}{2} < k - l - 1 \) for an integer \( l \), then \( \sigma \) is also \((k-l)\)-decomposable. Take \( l \) such that \( k - l \leq \frac{n+1}{2} \). Then we can say
\[
\sigma \in \bigoplus_{i+j \leq \nu+1, i \leq j} a^{(i)*} \wedge a^{(j)*}.
\]

Hence it is sufficient to show the above decomposition of \( \sigma \) for \( k = \nu - 1 \). This was shown in [1, Lemma 2.8]. Hence the Lemma follows.

\[ \square \]

It is known that \([n^i, n^j] \subset n^{i+j}\) (see [2]) and hence we have
\[
d\left( a^{(k)*} \right) \subset \bigoplus_{i+j \leq k, i \leq j} a^{(i)*} \wedge a^{(j)*}.
\]

**Definition 3.7.** A nilpotent Lie algebra \( n \) is called naturally graded if we can choose subspaces \( a_i \) such that \([a_i, a_j] \subset a_{i+j}\)

If \( n \) is naturally graded, then we have
\[
d\left( a^{(k)*} \right) \subset W_k
\]
where \( W_k = \bigoplus_{i+j=k, i \leq j} a^{(i)*} \wedge a^{(j)*} \).

Let \( g \) be a Hermitian metric on \( n \) such that the sum
\[
n = a^{(1)} \oplus a^{(2)} \oplus \cdots \oplus a^{(\nu)}
\]
is an orthogonal direct sum. Then \( g \) give a Hermitian metric on the finite-dimensional DGA \( \wedge n^* \) of PD-type. Consider the decomposition
\[
\wedge n^* = H^r(\langle \wedge n^* \rangle) \oplus d(\wedge n^*) \oplus d^s(\wedge n^*).
\]

Then
\[
\wedge n^* = W_1 \oplus W_2 \oplus \cdots \oplus W_{2\nu}
\]
is an orthogonal direct sum and we have \( d^{-1} \circ \beta(W_k) \subset a^{(k)*} \) by \( d(a^{(k)*}) \subset W_k \).

**Proposition 3.8.** Let \( n \) be a \( \nu \)-step naturally graded nilpotent Lie algebra and \( g \) a Lie algebra. Then the analytic germ \((F(\wedge u^*), g), 0)\) is cut out by polynomial equations of degree at most \( \nu + 1 \).

**Proof.** Take a basis \( \zeta_1, \ldots, \zeta_\nu \) of \( H^1(\langle \wedge u^* \rangle) \otimes g \). For parameters \( t = (t_\nu) \), we consider the formal power series \( \phi(t) = \sum_r \phi_r(t) \) with values in \( L^1 \) given inductively by \( \phi_1(t) = \sum t_\nu \zeta_\nu \) and
\[
\phi_r(t) = \frac{1}{2} \sum_{s=1}^{r-1} \delta(\phi_s(t), \phi_{r-s}(t)).
\]
By Lemma 3.5, the analytic germ \((F(\wedge u^*, g), 0)\) is equivalent to the analytic germ in \(\mathbb{C}^n\) at the origine defined by equations

\[ H([\phi(t), \phi(t)]) = 0 \]

where \(H : \wedge n^* \otimes g \to \mathcal{H}^*(\wedge u^*) \otimes g\) is the projection.

We have

\[ [a^{(j)*} \otimes g, a^{(j)*} \otimes g] \subset W_{i+j} \otimes g. \]

By \(d^* G(W_k) = d^{-1} \circ \beta(W_k) \subset a^{(k)*}\), we have

\[ d^* G \otimes \text{id}([a^{(i)*} \otimes g, a^{(j)*} \otimes g]) \subset a^{(i+j)*} \otimes g. \]

This implies \(\phi_c(t) \in a^{(r)*} \otimes g\) and we have

\[ \phi(t) = \phi_1(t) + \cdots + \phi_c(t). \]

By Lemma 3.5 we have \(\mathcal{H}^2(\wedge n^*) \subset \ker d_{\wedge^2 n^*} \subset \bigoplus_{l \leq \nu+1} W_l\) and hence

\[ H(\wedge n^* \otimes g) \subset \ker d_{\wedge^2 n^*} \otimes g \subset \bigoplus_{l \leq \nu+1} W_l \otimes g. \]

Since we have \([\phi_i(t), \phi_j(t)] \in W_{i+j} \otimes g\) by \(\phi_c(t) \in a^{(r)*} \otimes g\), we have \(H[\phi_i(t), \phi_j(t)] = 0\) for \(\nu + 1 < i + j\). Hence \(H[\phi(t), \phi(t)] = 0\) are polynomial equations of degree at most \(\nu + 1\).

\[ \square \]

4. Complex analogy

4.1. Complex parallelizable solvmanifolds. Let \(G\) be a simply connected \(n\)-dimensional complex solvable Lie group. Consider the Lie algebra \(g_{1,0}\) (resp. \(g_{0,1}\)) of the left-invariant holomorphic (resp. anti-holomorphic) vector fields on \(G\). Let \(N\) be the nilradical of \(G\). We can take a simply connected complex nilpotent subgroup \(C \subset G\) such that \(G = C \cdot N\) (see [3]). Since \(C\) is nilpotent, the map

\[ C \ni c \mapsto (\text{Ad}_c) \in \text{Aut}(g_{1,0}) \]

is a homomorphism where \((\text{Ad}_c)\) is the semi-simple part of \(\text{Ad}_s\).

We have a basis \(X_1, \ldots, X_n\) of \(g_{1,0}\) such that

\[ (\text{Ad}_c) = \text{diag}(\alpha_1(c), \ldots, \alpha_n(c)) \]

for \(c \in C\). Let \(x_1, \ldots, x_n\) be the basis of \(g_{1,0}^*\) which is dual to \(X_1, \ldots, X_n\).

**Theorem 4.1.** ([11] Corollary 6.2 and Remark 5]) Suppose \(G\) has a lattice \(\Gamma\). Let \(B_{\Gamma}^*\) be the subcomplex of \((A^{0,*}(G/\Gamma), \tilde{\partial})\) defined as

\[ B_{\Gamma}^* = \left\{ \alpha_I x_I \mid \left( \frac{\alpha_I}{\alpha_J} \right)_{I \neq J} = 1 \right\} \]

where for a multi-index \(I = \{i_1, \ldots, i_p\}\) we write \(x_I = x_{i_1} \wedge \cdots \wedge x_{i_p}\), and \(\alpha_I = \alpha_{i_1} \cdots \alpha_{i_p}\). Consider the nilshadow \(u\) of the \(C\)-Lie algebra \(g\). Then we have:

- The inclusion \(B_{\Gamma}^* \subset A^{0,*}(G/\Gamma)\) induces a cohomology isomorphism.
- \(B_{\Gamma}^*\) can be regarded as a sub-DGA of \(\wedge u^*\).
It is known that a simply connected solvable Lie group $G$ admitting a lattice $\Gamma$ is unimodular. Hence we have $\alpha_1 \cdots \alpha_n = 1$. For a multi-index $I \subset \{1, \ldots, n\}$ and its complement $I' = \{1, \ldots, n\} - I$, if $\left(\frac{\alpha_I}{\alpha_{I'}}\right)|_I = 1$ then $\left(\frac{\alpha_{I'}}{\alpha_I}\right)|_{I'} = 1$. Thus the DGA $B^*_\Gamma$ as in Theorem 4.1 is of PD-type.

4.2. Deformations of holomorphic vector bundles. For a compact complex manifold $(M, J)$ and a holomorphic vector bundle $E$ over $M$, consider

$$L^* = A^{0,*}(M, \text{End}(E))$$

the differential graded Lie algebra of differential forms of $(0, *)$-type with values in the holomorphic vector bundle $\text{End}(E)$ with the Dolbeault operator induced by the holomorphic structure on $E$. Then the Kuranishi space $K(L^*)$ represents the deformation functor for deformations of holomorphic structures on $E$ (see [6] and [7]).

Let $G$ be a simply connected $n$-dimensional complex solvable Lie group with a lattice $\Gamma$. For the Lie algebra $\mathfrak{gl}_n(\mathbb{C})$ of complex valued $n \times n$ matrices, we consider the DGLA $L^* = A^{0,*}(G/\Gamma) \otimes \mathfrak{gl}_n(\mathbb{C})$. Then the Kuranishi space $K(L^*)$ represents the deformation functor for deformations of holomorphic structures on $G/\Gamma \times \mathbb{C}^n$ near the trivial holomorphic structure. As an analytic germ, the Kuranishi space $K(L^*)$ is an invariant under quasi-isomorphisms between analytic DGLAs. Hence by Theorem 4.1, considering the DGLA $L^* = B^*_\Gamma \otimes \mathfrak{gl}_n(\mathbb{C})$, the analytic germ $K(L^*)$ is equivalent to $K(L^*)$. As Section 3.2, the analytic germ $K(L^*)$ is equivalent to the analytic germ $(F(B^*_\Gamma, \mathfrak{gl}_n(\mathbb{C})), 0)$. Hence we have the following theorems.

**Theorem 4.2.** Let $G$ be a simply connected complex solvable Lie group with a lattice $\Gamma$ and $\mathfrak{g}$ the $\mathbb{C}$-Lie algebra of $G$. We consider the nilshadow $u$ of $\mathfrak{g}$. Then the analytic germ which represents the deformation functor for deformations of holomorphic structures on $G/\Gamma \times \mathbb{C}^n$ near the trivial holomorphic structure is linearly embedded in the analytic germ $(F(\wedge u^\ast, \mathfrak{gl}_n(\mathbb{C})), 0)$.

Moreover we suppose that the Lie algebra $u$ is $\nu$-step naturally graded. Then such analytic germ is cut out by polynomial equations of degree at most $\nu + 1$.

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