Exact renormalization group flow equations
for free energies and \(N\)-point functions
in uniform external fields

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Abstract

We project the Wilson/Polchinski renormalization group equation onto its uniform external field dependent effective free energy and connected Green’s functions. The result is a hierarchy of equations which admits a choice of “natural” truncation and closure schemes for nonperturbative approximate solution. In this way approximation schemes can be generated which avoid power series expansions in either fields or momenta. When following one closure scheme the lowest order equation is the mean field approximation, while another closure scheme gives the “local potential approximation.” Extension of these closure schemes to higher orders leads to interesting new questions regarding truncation schemes and the convergence of nonperturbative approximations. One scheme, based on a novel “momentum cluster decomposition” of the connected Green’s functions, seems to offer new possibilities for accurate nonperturbative successive approximation.

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1 Introduction

In recent years there has been a resurgence of interest in applying Wilson’s exact renormalization group (ERG) equation [1] to problems in quantum field theory. Starting with Polchinski [2], who derived Wilson’s ERG equation in a context familiar to field theorists, it was first used to provide improved, simplified proofs of perturbative renormalizability for many field theoretic systems [3]. More recently, others have used this equation, or its Legendre transformed version [4, 5], to study, in various nonperturbative approximations, the qualitative, and in some cases quantitative, behaviors of many different systems, including scalar [6, 7] and vector [8, 9] field theories, gauge theories [10], fermionic systems [11], and many phenomenological issues such as top quark and higgs masses [12], bound states [13], and various aspects of QCD [14]. Recent reviews may be found in [6, 15, 16, 17].

The challenge of nonperturbative approximations has been to reduce the infinite dimensional space of couplings generated by the ERG equation to some computationally manageable size. This is often initially done simply by truncating the space to a few operators (usually the relevant and marginal operators of the canonical theory), and useful qualitative and even quantitative results are obtained by this approach. Improved approximations can then be sought by enlarging the space of effective interactions in some systematic way with the hope that results will converge towards the exact result. However, without a small parameter, as in perturbation theory, to give some measure of control over the degree of approximation, the degree of “improvement” is difficult to evaluate. In addition, additional nonphysical solutions are often generated as the finite space of couplings is enlarged [11, 18, 19], making it often difficult to identify with any a priori justification the physical solution sought.

Another approach, which from the start includes an infinite number of interactions, is the derivative expansion [20, 21, 22], essentially a functional power series expansion of the Wilson effective action in powers of momenta coupling the Fourier transformed field variables. In this way all powers of the field are included at each level of approximation. Its lowest order of approximation, known as the “local potential approximation” (LPA) [4, 23, 24], is quite successful in providing qualitative and even reasonably good quantitative information regarding phase diagrams and critical exponents. It appears to generate no spurious solutions and permit the entire space of candidate renormalized theories to be searched for solutions in a simple, systematic way [13, 19, 21, 25]. Its second order
of approximation gives impressive accuracy when compared with other approaches (\(\epsilon\)-expansion, exact, and high temperature expansion) \[20, 21\], and has offered the hope of leading to a practical method of successive approximation.

A third approach, which has not been much used outside of perturbation theory, is complementary to the derivative expansion in that it involves an expansion of the Wilson effective action in powers of the field operators \[4\], the expansion coefficients being the (zero-field) connected Green’s functions of the theory. Thus, all powers of momenta are included at each stage of approximation. This approach has provided an alternative route to the \(\epsilon\)-expansion for scalar field theories \[26\] as well as, in the Legendre transformed version, a promising initial nonperturbative exploration of bound state questions \[13\].

In all of the above approaches the question of convergence of the approximation method involved is difficult to address \textit{a priori} \[5, 21\], and although the approximations often improve when taken to their \textit{next} higher order (usually the only one practical to compute, so far), there are indications that this good fortune cannot continue indefinitely. In \[19\] Morris studies the convergence of an expansion of the LPA in powers of the field variable. He demonstrates that the exact solution of the LPA equation has singularities in the complex plane that prevent the convergence of the power series expansion. If the presence of such singularities is a general feature of the ERG, one can expect it to prevent convergence of any expansion that involves a power series in the field operators. This would eliminate the hopes of convergence of the first and third methods discussed above. The derivative expansion also appears to have its own problem with convergence. In \[27\] Dunne provides an exact solution for the effective potential for QED\(_{2+1}\) in a particular inhomogeneous external magnetic field. From this he derives an all orders derivative expansion and shows that it is asymptotic. We take this to suggest that power series expansions in either fields or momenta are unable to provide a convergent approximation scheme for the ERG equation.

It is interesting to contrast this situation with that of lattice real-space RG approximation schemes \[28\], where a reliable (apparently convergent) method of successive approximation seems to exist. One orders the space of interactions in terms of the interaction’s relative degree of locality, \(i.e.,\) nearest neighbor, next nearest neighbor, \(etc.,\) and truncates accordingly. This locality property does not seem to be readily applicable to momentum-space formulations; it certainly has no relation to the power series approximations discussed above, as all powers of both the field and momenta are typical components of local lattice interactions.
In this paper, motivated by the above considerations, we attempt to construct a method of successive approximation for Wilson/Polchinski’s ERG equation for a scalar field theory in $D$ dimensions that is not based on power series expansions. In particular, each order of approximation involves all powers of the field and (beyond the lowest order) all powers of momenta. In the final analysis it becomes essentially an expansion in the number of momenta coupling the Fourier transformed field variables, though it is, strictly speaking, in the space of connected Green’s functions rather than the space of Wilson effective interactions that this expansion takes place. It is accomplished by projecting the ERG equation onto its uniform external field dependent effective free energy and connected Green’s functions, creating an infinite hierarchy of partial differential equations (PDEs) to be truncated and solved numerically. A set of “fluctuation relations” (derived below) allows a choice of truncation schemes to be investigated. One of these schemes, based on a novel “momentum cluster decomposition” of the connected Green’s functions, appears to have many advantages over the standard approach. Although it still remains to be numerically investigated, it appears to be a likely candidate for a convergent method of successive approximation as well as offering new possibilities for extending the power of the renormalization group approach in many directions.

The remainder of this paper proceeds as follows. In Section 2 we introduce our system and notation and give a summary of the essential steps of the derivation of the Polchinski ERG equation and its Legendre transformed version. In Section 3 we derive a rescaled version of the Polchinski equation that is particularly well suited for the analysis that follows. We also show that it exactly reproduces Wilson’s rescaled ERG equation. In Section 4 we derive a formalism for expressing the connected Green’s functions in terms of the solution of the rescaled ERG equation. We demonstrate that our rescaling procedure gives the correct scaling laws for connected Green’s functions. We also derive the set of “fluctuation relations” which will prove important in our later sections. In Section 5 we derive flow equations for the connected Green’s functions and present an analysis that suggests that the “obvious” truncation strategy will not be a convergent method of successive approximation. In Section 6 we present a new solution strategy based upon the “fluctuation relations” presented in Section 4. We define new “$N$-point momentum clusters” and derive “momentum cluster flow equations” based upon a “momentum cluster decomposition” of the original connected Green’s ($N$-point) functions. We compare these new equations with the equations of Section 5 and discuss the apparent advantages of this new formulation. In Section 7 we offer some concluding comments.
One final note: Although in this paper we treat the calculation of the Gibbs free energy and connected Green’s functions via the Wilson/Pollchinski ERG equation, our entire approach can be equally applied to the calculation of the effective potential and one-particle-irreducible (1PI) vertex functions via the Legendre transformed ERG equation \([4, 5]\). To reflect this more general utility we use the more general terminology “N-point functions” in the title and section headings of this paper.

2 Flow Equations

Our system is a self-interacting scalar field, \(\phi(x)\), coupled to an external source, \(J(x)\), in \(D\) Euclidean dimensions. We obtain the connected Green’s functions from the \(IR\)-cutoff, \(UV\)-regulated generating functional \(W_{\Lambda_0}(J)\) evaluated via the Wilson/Pollchinski renormalization group equation. The derivation of this equation and of its Legendre transformed counterpart is, by now, fairly standard \([4, 13]\). For completeness, and to establish our notation, we will simply summarize, following Ellwanger \([13]\), the essential steps.

The bare theory is regularized in the \(UV\) via a cutoff \(\Lambda_0\) in the propagator. We further introduce a running infrared cutoff, \(\Lambda\), and corresponding propagator

\[
P_{\Lambda_0}(q^2) \equiv (R_{\Lambda_0}(q^2))^{-1} = \frac{K_{\Lambda_0}(q^2) - K_{\Lambda}(q^2)}{q^2 + m^2}
\]

with

\[
K_{\Lambda}(q^2) \to 1 \quad \text{for} \quad q^2 \ll \Lambda^2
\]

\[
K_{\Lambda}(q^2) \to 0 \quad \text{for} \quad q^2 \gg \Lambda^2.
\]

The generating functional for cutoff connected Green’s functions can be represented as the functional integral

\[
\exp(-W_{\Lambda_0}[J]) = \int D\phi \exp\left\{-\frac{1}{2}(\phi, R_{\Lambda_0}^\dagger \phi) - S_{\text{int}}^{\Lambda_0}[\phi] + (J, \phi)\right\},
\]

where \((J, \phi) \equiv \int_q J_q \phi_{-q}, J_q \equiv \frac{1}{(2\pi)^D} \int d^Dq, \) and \(\phi_q\) and \(J_q\) are the Fourier transforms of \(\phi(x)\) and \(J(x)\) respectively. Here the kinetic term of the bare action is represented in terms of the inverse propagator \(R_{\Lambda_0}^\dagger(q^2)\), with the remaining part of the bare action denoted by \(S_{\text{int}}^{\Lambda_0}[\phi]\).

The result of the functional integration in (3) may be formally represented by

\[
e^{-W_{\Lambda_0}[J]} = e^{P_{\Lambda_0}[J]} e^{P_{\Lambda_0}^0} e^{-S_{\text{int}}^{\Lambda_0}[\phi]} \bigg|_{\phi=P_{\Lambda_0}^0 J}
\]
with
\[ D^\Lambda_0 = \frac{1}{2} \left( P^\Lambda_0 \frac{\delta}{\delta \phi} \frac{\delta}{\delta \phi} \right). \]

We define an effective interaction \( S_{\text{int}}[\phi, \Lambda] \) by
\[ e^{-S_{\text{int}}[\phi, \Lambda]} = e^{D^\Lambda_0} e^{-S^\Lambda_0[\phi]}. \]

We obtain Polchinski’s flow equation for \( S_{\text{int}}[\phi, \Lambda] \) by differentiating equation (6) with respect to \( \Lambda \):
\[ \frac{\partial S_{\text{int}}}{\partial \Lambda} = \frac{1}{2} \int_q \frac{\partial P^\Lambda_0(q^2)}{\partial \Lambda} \left\{ \frac{\delta^2 S_{\text{int}}}{\delta \phi_q \delta \phi_{-q}} - \frac{\delta S_{\text{int}} \delta S_{\text{int}}}{\delta \phi_q \delta \phi_{-q}} \right\}. \]

Our initial condition is \( S_{\text{int}}[\phi, \Lambda_0] = S^\Lambda_0[\phi] \).

The generating functional for cutoff connected Green’s functions, \( W^\Lambda_0[J] \), is related to \( S_{\text{int}}[\phi, \Lambda] \) through
\[ W^\Lambda_0[J] = S_{\text{int}}[P^\Lambda_0 J, \Lambda] - \frac{1}{2} (J, P^\Lambda_0 J). \]

It satisfies the flow equation
\[ \frac{\partial W^\Lambda_0}{\partial \Lambda} = -\frac{1}{2} \int_q \frac{\partial R^\Lambda_0(q^2)}{\partial \Lambda} \left\{ \frac{\delta^2 W^\Lambda_0}{\delta J_q \delta J_{-q}} - \frac{\delta W^\Lambda_0 \delta W^\Lambda_0}{\delta J_q \delta J_{-q}} \right\}. \]

The cutoff effective action, \( \Gamma^\Lambda_0 \), is given by the Legendre transform of \( W^\Lambda_0[J] \),
\[ \Gamma^\Lambda_0[\varphi] = W^\Lambda_0[J] + (J, \varphi), \]
where \( \varphi_q \equiv \delta W^\Lambda_0[J]/\delta J_{-q} \). If one further splits off a bare kinetic part of \( \Gamma^\Lambda_0[\varphi] \),
\[ \Gamma^\Lambda_0[\varphi] = \frac{1}{2} (\varphi, R^\Lambda_0 \varphi) + \tilde{\Gamma}^\Lambda_0[\varphi], \]
one obtains a flow equation for \( \tilde{\Gamma}^\Lambda_0[\varphi] \), the generator of 1PI vertex functions, of the form
\[ \frac{\partial \tilde{\Gamma}^\Lambda_0}{\partial \Lambda} = \frac{1}{2} \int_q \frac{\partial R^\Lambda_0(q^2)}{\partial \Lambda} \left\{ R^\Lambda_0(q^2) + \frac{\delta^2 \tilde{\Gamma}^\Lambda_0}{\delta \varphi_q \delta \varphi_{-q}} \right\}^{-1}, \]
with initial value
\[ \tilde{\Gamma}^\Lambda_0[\varphi] = S^\Lambda_0[\phi]\bigg|_{\varphi=\phi}. \]

It is interesting to note that unlike the flow equations for \( S_{\text{int}} \) and \( \Gamma^\Lambda_0 \), which admit well posed initial value problems in terms of the bare action \( S^\Lambda_0[\phi] \), the flow equation for \( W^\Lambda_0 \) does not. This is because the substitution \( \phi = P^\Lambda_0 J \) used in eqs. (4) and (8) gives \( W^\Lambda_0[J] = 0 \) at \( P^\Lambda_0 = 0 \). For this reason we use eqs. (6) and (8) rather than eq. (9) for the calculation of \( W^\Lambda_0[J] \).
3 Rescaling

We recall that Wilson’s original renormalization group program consists of two steps. The first step, performed by the derivation given above, integrates out the high momentum components of the field, generating an equivalent effective action with reduced cutoff. The second step rescales the momenta back to the original cutoff scale and rescales the fields so that the resulting equations are cutoff independent and have a fixed point solution corresponding to the scale-invariant, massless renormalized theory. Bell and Wilson have shown [29] that for “linear” renormalization group equations, such as Wilson/Polchinski above, the field rescaling must be appropriately carried out in terms of the scaling dimension of the field or the resulting equations will not flow to a fixed point. The scale dependence of the cutoff function \( P_\Lambda^\Lambda_0(q^2) \) must also be chosen correctly. Finally, we also transform to dimensionless variables, which, while not really essential, is customary at this stage.

We parametrize our effective momentum scale by

\[
t = \log \frac{|\Lambda_0|}{\Lambda},
\]

and our dimensionless rescaled variables are

\[
q' = \frac{q}{\Lambda} \tag{15}
\]

and

\[
\phi'(q') = \left(\frac{\Lambda}{\Lambda_0}\right)^{D-d_J} \phi_q \Lambda_0^{D-d_\phi} \tag{16}
\]

where \( d_J \) and \( d_\phi \) are the scaling dimensions of \( J(x) \) and \( \phi(x) \) respectively,

\[
d_J = \frac{1}{2} (D + 2 - \eta), \tag{17}
\]

and

\[
d_\phi = \frac{1}{2} (D - 2 + \eta), \tag{18}
\]

where \( \eta \) is the anomalous dimension. A good choice for \( P_\Lambda^\Lambda_0(q^2) \) is

\[
P_\Lambda^\Lambda_0(q^2) = \Lambda_0^{-2} \left[ \bar{P}(q^2/\Lambda_0^2) - \left(\frac{\Lambda}{\Lambda_0}\right)^{2-\eta} \bar{P}(q^2/\Lambda^2) \right], \tag{19}
\]

where

\[
\bar{P}(q^2/\Lambda^2) \to 1 \quad \text{for} \quad q^2 \ll \Lambda^2
\]
$$\tilde{P}(q^2/\Lambda^2) \to 0 \quad \text{for} \quad q^2 \gg \Lambda^2. \quad (20)$$

Finally, by defining

$$S_{\text{int}}'[\phi', t] \equiv S_{\text{int}}[\phi(\phi'), \Lambda(t)], \quad (21)$$

we get

$$\frac{\partial S_{\text{int}}'}{\partial t} = D S_{\text{int}}' - \int_q \phi'_q \left[ \frac{1}{2} (D + 2 - \eta) + q' \cdot \nabla_q' \right] \frac{\delta S_{\text{int}}'}{\delta \phi'_q}$$

$$- \int_{q'} A(q') \left\{ \frac{\delta S_{\text{int}}'}{\delta \phi'_q} \frac{\delta S_{\text{int}}'}{\delta \phi'_{-q'}} - \frac{\delta^2 S_{\text{int}}'}{\delta \phi'_q \delta \phi'_{-q'}} \right\}, \quad (22)$$

where

$$A(q) = \left( 1 - \frac{\eta}{2} \right) \tilde{P}(q^2) - q^2 \frac{d\tilde{P}(q^2)}{dq^2}, \quad (23)$$

and the prime on $\nabla'_q$ means that it ignores the momentum conservation delta functions in $\delta S_{\text{int}}'/\delta \phi'_q$.

In the next section we will show that our choice of rescaling and cutoff functions gives the correct scaling laws at the critical point. Here we note that if we choose $\tilde{P}(q^2) = e^{-2q^2}$, change variables to $\sigma'_q = e^{\phi} \phi'_q$, and define

$$\mathcal{H}[\sigma', t] \equiv -S_{\text{int}}'[\phi'(\sigma'), t] - \frac{1}{2} \int_{q} \sigma'_q \sigma'_{-q}, \quad (24)$$

we get, after an integration by parts and neglecting all primes,

$$\frac{\partial \mathcal{H}}{\partial t} = \int_{q} \left( \frac{D}{2} \sigma_q + q \cdot \nabla_q \sigma_q \right) \frac{\delta \mathcal{H}}{\delta \sigma_q}$$

$$+ \int_{q'} (1 - \frac{\eta}{2} + 2q^2) \left\{ \frac{\delta \mathcal{H}}{\delta \sigma_q} \frac{\delta \mathcal{H}}{\delta \sigma_{-q}} + \frac{\delta^2 \mathcal{H}}{\delta \sigma_q \delta \sigma_{-q}} \right\}, \quad (25)$$

which is exactly Wilson’s equation [1]. Eq. (22) is more useful for our purposes, however, and we will continue to work with it in what follows.

### 4 Scaling Laws and Fluctuation Relations for $N$-point Functions

In this section we derive a number of relations for connected Green’s functions based on their representation in terms of the $t \to \infty$ ($\Lambda \to 0$) solution to the ERG eq. (22). The UV-regularized connected Green’s functions are determined from the generating functional

$$W_0^{\Lambda_0}[J] = \lim_{\Lambda \to 0} \left\{ S_{\text{int}}[P^{\Lambda_0}, \Lambda] - \frac{1}{2} (J, P^{\Lambda_0} J) \right\}, \quad (26)$$
following eq. (8). Let

\[
S'_{\text{int}}[\phi', t] = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{q_1'} \cdots \int_{q_n'} S'_n(q_1', \ldots, q_n', t) \phi'_{q_1'} \cdots \phi'_{q_n'} \delta^D(q_1' + \cdots + q_n'),
\]

(27)

where the above is written in terms of our dimensionless (primed) variables. Rewriting this in terms of our original (unprimed) variables, we get

\[
S_{\text{int}}[\phi, \Lambda] = \Lambda^D \sum_{n=0}^{\infty} \frac{1}{n!} \int_{q_1} \cdots \int_{q_n} S'_n(q_1 \Lambda^{-1}, \ldots, q_n \Lambda^{-1}, t)
\]

\[
\times \Lambda^{-nd} \Lambda_0^{-n(d_j - d)} \phi_{q_1} \cdots \phi_{q_n} \delta^D(q_1 + \cdots + q_n).
\]

(28)

If we now substitute \( \phi_q = P_{\Lambda_0}^\Lambda(q^2)J_q \), we will then be able to study the limit \( \Lambda \to 0 \) of eq. (26). However, because \( P_0^{\Lambda_0}(q^2)J_q = \Lambda_0^{\eta-2} \bar{P}(q^2/\Lambda_0^2)J_q \) is finite we can substitute it directly in (26) to get

\[
W_{\Lambda_0}[J] = \lim_{\Lambda \to 0} \Lambda^D \sum_{n=0}^{\infty} \frac{1}{n!} \int_{q_1} \cdots \int_{q_n} S'_n(q_1 \Lambda^{-1}, \ldots, q_n \Lambda^{-1}, t) \Lambda^{-nd_j}
\]

\[
\times \bar{P}(q^2/\Lambda_0^2) \cdots \bar{P}(q_n^2/\Lambda_0^2)J_{q_1} \cdots J_{q_n} \delta^D(q_1 + \cdots + q_n)
\]

\[- \frac{1}{2} \int_q \Lambda_0^{\eta-2} \bar{P}(q^2/\Lambda_0^2)J_q J_{-q}.
\]

(29)

In the same spirit we can now take the limit \( \Lambda_0 \to \infty \), where \( \Lambda_0 \) appears explicitly, to get the infinite cutoff generating functional

\[
W_0^\infty[J] = \lim_{\Lambda_0 \to \infty} \lim_{\Lambda \to 0} \Lambda^D \sum_{n=0}^{\infty} \frac{1}{n!} \int_{q_1} \cdots \int_{q_n} S'_n(q_1 \Lambda^{-1}, \ldots, q_n \Lambda^{-1}, t)
\]

\[
\times \Lambda^{-nd_j} J_{q_1} \cdots J_{q_n} \delta^D(q_1 + \cdots + q_n),
\]

(30)

where the remaining limit \( \Lambda_0 \to \infty \) must be accompanied by appropriate fine tuning of the relevant bare couplings to produce an acceptable renormalized theory [1, 25, 17].

The infinite cutoff connected Green's functions in a uniform external field \( j \) are

\[
G_k(q_1, \ldots, q_k, j) \delta^D(q_1 + \cdots + q_k) = - \frac{\delta^k}{\delta J_{q_1} \cdots \delta J_{q_k}} W_0^\infty[J] \bigg|_{J_q=(2\pi)^D \delta \delta^D(q_i)},
\]

(31)

where \( F_G(j) \equiv -G_0(j) \) is the Gibbs free energy, the Legendre transform of the effective potential. Thus,

\[
G_k(q_1, \ldots, q_k, j)
\]

\[
= - \lim_{\Lambda_0 \to \infty} \lim_{\Lambda \to 0} \Lambda^D \sum_{n=k}^{\infty} \frac{1}{(n-k)!} S'_n(q_1 \Lambda^{-1}, \ldots, q_k \Lambda^{-1}, 0, \ldots, 0, t)
\]

\[
\times \Lambda^{-kd_j} [j \Lambda^{-d_j}]_{n-k},
\]

(32)
where the $G_k$ and $S'_n$ are symmetric functions of their momentum arguments. If we define

$$
\tilde{G}_k(q_1, \ldots, q_k, x, t)\delta^D(q_1 + \cdots + q_k) \equiv -\frac{\delta^k}{\delta\phi'_{q_1} \cdots \delta\phi'_{q_k}} S'_{\text{int}}[\phi', t] \bigg|_{\phi'_q = (2\pi)^D x^D(q_i)} ,
$$

(33)

giving

$$
\tilde{G}_k(q_1, \ldots, q_k, x, t) = -\sum_{n=k}^{\infty} \frac{1}{(n-k)!} S'_n(q_1, \ldots, q_k, 0, \ldots, 0, t) x^{n-k} ,
$$

(34)

then

$$
G_k(q_1, \ldots, q_k, j) = \lim_{\Lambda_0 \to \infty} \lim_{\Lambda \to 0} \Lambda^{D-kd_j} \tilde{G}_k(q_1\Lambda^{-1}, \ldots, q_k\Lambda^{-1}, j\Lambda^{-d_j}, t) .
$$

(35)

This equation allows us to derive fixed-point scaling laws and asymptotic behaviors as follows.

Let $S'_{\text{int}}^*$ be a fixed-point solution of eq. (24): $\partial S'_{\text{int}}^*/\partial t = 0$. The corresponding $G_k^*$ are

$$
\tilde{G}_k^*(q_1, \ldots, q_k, x) \delta^D(q_1 + \cdots + q_k) = -\frac{\delta^k}{\delta\phi'_{q_1} \cdots \delta\phi'_{q_k}} S'_{\text{int}}^*[\phi'] \bigg|_{\phi'_q = (2\pi)^D x^D(q_i)} .
$$

(36)

In evaluating eq. (33) for such fixed-point solutions, the absence of $t$-dependence allows the limit $\Lambda_0 \to \infty$ to be trivially taken, yielding

$$
G_k^*(q_1, \ldots, q_k, j) = \lim_{\Lambda \to 0} \Lambda^{D-kd_j} \tilde{G}_k^*(q_1\Lambda^{-1}, \ldots, q_k\Lambda^{-1}, j\Lambda^{-d_j}) .
$$

(37)

For this equation to have a nontrivial limit we must have $\tilde{G}_k^* \propto \Lambda^{D-kd_j}$ as $\Lambda \to 0$. We will consider this limit for two cases. First, for the case where $q_i = 1, \ldots, q_k \neq 0$, we let $x_i = q_i\Lambda^{-1}$ and $y = j\Lambda^{-d_j}$. Then the $z_i \equiv x_i y^{-1/d_j} = q_i j^{-1/d_j}$ are independent of $\Lambda$, and

$$
x_1^{-d_j+\frac{D}{2}} \cdots x_k^{-d_j+\frac{D}{2}} = \Lambda^{-D+kd_j} q_1^{-d_j+\frac{D}{2}} \cdots q_k^{-d_j+\frac{D}{2}} .
$$

(38)

So for the limit $\Lambda \to 0$ to exist we must have

$$
\tilde{G}_k^*(x_1, \ldots, x_k, y) \to \mathcal{G}_k(z_1, \ldots, z_k)x_1^{-d_j+\frac{D}{2}} \cdots x_k^{-d_j+\frac{D}{2}} ,
$$

(39)

where $\mathcal{G}_k$ is determined by solving the ERG equations. Then

$$
G_k^*(q_1, \ldots, q_k, j) = \mathcal{G}_k(z_1, \ldots, z_k)q_1^{-d_j+\frac{D}{2}} \cdots q_k^{-d_j+\frac{D}{2}} .
$$

(40)

This is the general (hyper)scaling law for $G_k$ on the critical isotherm when $q_i = 1, \ldots, q_k \neq 0$. For $k = 2$, writing $G_2(q, -q, j)$ as $G_2(q, j)$, we have $G_2^*(q, j) = \mathcal{G}_2(q/j^{\eta - 2})q^\eta - 2$. As this expression generally is used to define the anomalous dimension $\eta$, its derivation
here validates our choice of rescaling operations and cutoff functions. For the second case, where \( j \) is finite and \( q_i = 1, \ldots, k = 0 \), we must have

\[
\tilde{G}_k^*(0, \ldots, 0, y) \rightarrow \text{constant} \cdot y^{-k + \frac{D}{dJ}}.
\]

(41)

Then

\[
G_k^*(0, \ldots, 0, j) = \text{constant} \cdot j^{-k + \frac{D}{dJ}}.
\]

(42)

From the definition of \( \tilde{G}_k \) we also have

\[
\tilde{G}_k(q_1, \ldots, q_{k-n}, 0, \ldots, 0, x, t) = \frac{\partial^n}{\partial x^n} \tilde{G}_{k-n}(q_1, \ldots, q_{k-n}, x, t),
\]

(43)

which immediately gives

\[
G_k(q_1, \ldots, q_{k-n}, 0, \ldots, 0, j) = \frac{\partial^n}{\partial j^n} G_{k-n}(q_1, \ldots, q_{k-n}, j).
\]

(44)

These relations will be important in what follows. They are a generalization of the so-called “fluctuation-dissipation theorem” or “linear response theorem” of statistical mechanics, which relates fluctuations in a thermodynamic average to a corresponding susceptibility (linear response) [30, 31]. The case most often encountered in statistical mechanics is for \( k = n = 2 \), which gives the well-known relation between the 2-point cumulant and order parameter susceptibility,

\[
G_2(0, j) = \partial \varphi_0 / \partial j,
\]

(45)

where \( \varphi_0 = \partial G_0 / \partial j \). We will refer to relations (43) and (44) as fluctuation relations in what follows.

### 5 Flow Equations for \( N \)-point Functions

The flow equations for the \( \tilde{G}_k \) can now be derived by applying the definition (33) to the flow equation (22). Using (33) we can write

\[
\delta^D(q_1 + \cdots + q_k) \frac{\partial \tilde{G}_k}{\partial t} = -\left( \frac{\delta^k}{\delta \phi'_{q_1} \cdots \delta \phi'_{q_k}} \frac{\partial S'_{\text{int}}}{\partial t} \right)_{\phi'_n = (2\pi)^{D-1} \delta^D(q_i)}.
\]

(46)

Then, by substituting the r.h.s. of eq. (22) for \( \partial S'_{\text{int}} / \partial t \) and carrying out the indicated operations, we get (suppressing \( q, x, \) and \( t \) arguments when not necessary for clarity)

\[
\frac{\partial \tilde{G}_0}{\partial t} = D \tilde{G}_0 - d_j x \frac{\partial \tilde{G}_0}{\partial x} + A(0) \left( \frac{\partial \tilde{G}_0}{\partial x} \right)^2 + \int_q A(q) \tilde{G}_2(q)
\]

(47)
\[
\frac{\partial \tilde{G}_{k>0}}{\partial t} = (D - kd_j)\tilde{G}_k - d_j x \frac{\partial \tilde{G}_k}{\partial x} - \sum_{i=1}^{k} q_i \cdot \nabla_{q_i} \tilde{G}_k
\]
\[
+ \sum_{j=0}^{k} \sum_{I_j \subset Z_k} \left\{ A(q_{i_1} + \cdots + q_{i_j})\tilde{G}_{j+1}(-q_{i_1} - \cdots - q_{i_j}, q_{i_1}, \ldots, q_{i_j}) \right. \\
\times \tilde{G}_{k-j+1}(-q_{i_{j+1}} - \cdots - q_{i_k}, q_{i_{j+1}}, \ldots, q_{i_k}) \left. \right\} \\
+ \int_{q'} A(q')\tilde{G}_{k+2}(q', -q', q_1, \ldots, q_k),
\]
(47)

where \(I_j\) stands for the set of indices \(\{i_1, \ldots, i_j\}\), \(Z_k\) stands for the set of consecutive integers \(\{1, \ldots, k\}\), \(\sum_{I_j \subset Z_k}\) stands for the sum over the \(\binom{k}{j}\) different sets, \(I_j\), of indices chosen from the set \(Z_k\), and the set \(\{i_{j+1}, \ldots, i_k\}\) is the complementary set \(Z_k - I_j\). We note that in the equation for \(\partial \tilde{G}_k/\partial t\) we have \(q_1 + \cdots + q_k = 0\) from the \(\delta\)-function of (33).

Using this together with eq. (43) we can rewrite eqs. (47) as
\[
\frac{\partial \tilde{G}_0}{\partial t} = D\tilde{G}_0 - d_j x \frac{\partial \tilde{G}_0}{\partial x} + A(0) \left( \frac{\partial \tilde{G}_0}{\partial x} \right)^2 + \int_{q} A(q)\tilde{G}_2(q)
\]
\[
\frac{\partial \tilde{G}_{k>0}}{\partial t} = (D - kd_j)\tilde{G}_k + \left( 2A(0) \frac{\partial \tilde{G}_0}{\partial x} - d_j x \right) \frac{\partial \tilde{G}_k}{\partial x} - \sum_{i=1}^{k} q_i \cdot \nabla_{q_i} \tilde{G}_k
\]
\[
+ \sum_{j=1}^{k-1} \sum_{I_j \subset Z_k} \left\{ A(q_{i_1} + \cdots + q_{i_j})\tilde{G}_{j+1}(-q_{i_1} - \cdots - q_{i_j}, q_{i_1}, \ldots, q_{i_j}) \right. \\
\times \tilde{G}_{k-j+1}(-q_{i_{j+1}} - \cdots - q_{i_k}, q_{i_{j+1}}, \ldots, q_{i_k}) \left. \right\} \\
+ \int_{q'} A(q')\tilde{G}_{k+2}(q', -q', q_1, \ldots, q_k),
\]
(48)

where we have explicitly pulled out the first and last terms from the double sum.

The first four equations are
\[
\frac{\partial \tilde{G}_0}{\partial t} = D\tilde{G}_0 - d_j x \frac{\partial \tilde{G}_0}{\partial x} + A(0) \left( \frac{\partial \tilde{G}_0}{\partial x} \right)^2 + \int_{q} A(q)\tilde{G}_2(q)
\]
\[
\frac{\partial \tilde{G}_2}{\partial t} = (D - 2d_j)\tilde{G}_2 + \left( 2A(0) \frac{\partial \tilde{G}_0}{\partial x} - d_j x \right) \frac{\partial \tilde{G}_2}{\partial x} - q \cdot \nabla_{q} \tilde{G}_2
\]
\[
+ 2A(q)\tilde{G}_2 + \int_{q'} A(q')\tilde{G}_4(q', -q', q, -q)
\]
\[
\frac{\partial \tilde{G}_3}{\partial t} = (D - 3d_j)\tilde{G}_3 + \left( 2A(0) \frac{\partial \tilde{G}_0}{\partial x} - d_j x \right) \frac{\partial \tilde{G}_3}{\partial x} - \sum_{i=1}^{3} q_i \cdot \nabla_{q_i} \tilde{G}_3
\]
\[
+ 2 \left[ \sum_{i=1}^{3} A(q_i)\tilde{G}_2(q_i) \right] \tilde{G}_3(q_1, q_2, q_3) + \int_{q'} A(q')\tilde{G}_5(q', -q', q_1, q_2, q_3)
\]
\[
\frac{\partial \tilde{G}_4}{\partial t} = (D - 4d_j)\tilde{G}_4 + \left( 2A(0) \frac{\partial \tilde{G}_0}{\partial x} - d_j x \right) \frac{\partial \tilde{G}_4}{\partial x} - \sum_{i=1}^{4} q_i \cdot \nabla_{q_i} \tilde{G}_4
\]

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\[
+ 2 \left[ \sum_{i=1}^{4} A(q_i) \tilde{G}_2(q_i) \right] \tilde{G}_4(q_1, q_2, q_3, q_4) \\
+ \sum_{I_2 \subset \Z_4} A(q_{i_1} + q_{i_2}) \tilde{G}_3(-q_{i_1} - q_{i_2}, q_{i_1}, q_{i_2}) \tilde{G}_3(-q_{i_3} - q_{i_4}, q_{i_3}, q_{i_4}) \\
+ \int_{q' A(q') \tilde{G}_6(q', -q', q_1, \ldots, q_4) . \right] (49)
\]

We note that \( \tilde{G}_1(q, x, t) = \tilde{G}_1(0, x, t) = \partial \tilde{G}_0(x, t) / \partial x \) due to the \( \delta^D(q) \) in the definition of \( \tilde{G}_1(q, x, t) \), so we don’t need to write a separate equation for it. Also, in response to the \( \delta \)-function with \( \tilde{G}_2 \), we write \( \tilde{G}_2(q, -q, x, t) \) as \( \tilde{G}_2(q, x, t) \).

A natural procedure for solving the above hierarchy of equations is to truncate at a certain level by setting all higher order connected Green’s functions equal to zero. To lowest order we get an equation for \( \tilde{G}_0 \) that has been studied long ago by Green, Gunton, and coworkers \([32, 33, 34]\). Green \([32]\) derived this equation via a steepest descent approximation to Wilson’s functional integral form of his ERG equation \([1]\). He showed that, with appropriate rescaling to ensure analyticity, mean field exponents are obtained \([32]\). It was later shown that the equations reproduce more of the mean field phenomenology, including metastable and unstable branches in the free energy \([34]\) and a spinodal fixed point \([33]\). The higher order equations thus reflect systematic corrections to mean field behavior when successively higher order correlations are taken into account.

A perspective on the higher order equations may be gained by realizing that they have already been studied for the zero-field case \((x = 0)\) in \( D = 4 - \epsilon \) dimensions by Green and Shukla \([26]\). Because the equations are first-order quasi-linear they can be solved exactly in \( \epsilon \)-expansion, reproducing the standard results for the critical exponents. The \( \epsilon \)-expansion provides a natural truncation scheme for the hierarchy of equations, as \( \tilde{G}_{2n} \) is of order \( \epsilon^{n-1} \) for \( n \geq 2 \). As the \( \epsilon \)-expansion is known to be asymptotic, we do not expect similar truncation schemes without the benefit of \( \epsilon \) as a small parameter to yield any improvement in convergence. Nor, due to continuity of the equations in \( x \), do we expect different convergence properties for finite values of \( x \). For this reason we do not expect the obvious truncation scheme to yield a convergent method of successive approximation. This is not to say, however, that low orders of approximation could not be useful.

### 6 Momentum Cluster Flow Equations

It is possible to improve upon the above truncation scheme by noting that, according to the fluctuation relations \([13]\), each \( \tilde{G}_k \) contains derivatives of \( \tilde{G}_{k'} < k \) within it. Thus the
truncation scheme discussed above needlessly throws away vital information. A truncation scheme that retains this information from higher order correlations can be constructed as follows.

Due to the momentum conserving $\delta$-function in the definition of $\tilde{G}_k$, $\tilde{G}_k$ is a function of only $(k-1)$ independent momenta, and we will write it henceforth as $\tilde{G}_k(q_1, \ldots, q_{k-1}, x, t)$, with the understanding that the omitted $q_k$ takes the value $-q_1 - \cdots - q_{k-1}$. Defining the projection operator $P_{q_i} \tilde{G}_k(q_1, \ldots, q_{k-1}, x, t) \equiv \tilde{G}_k(q_1, \ldots, q_i = 0, \ldots, q_{k-1}, x, t)$, we define, for $k \geq 2$,

$$\hat{G}_{k\geq2}(q_1, \ldots, q_{k-1}, x, t) \equiv \left[ \prod_{i=1}^{k-1} (1 - P_{q_i}) \right] \tilde{G}_k(q_1, \ldots, q_{k-1}, x, t).$$  \hspace{1cm} (50)

We also define $\hat{G}_0(x, t) \equiv \tilde{G}_0(x, t)$ and $\hat{G}_1(x, t) \equiv 0$. Clearly, $\hat{G}_{k\geq2} = 0$ if any of its $(k-1)$ momenta are zero. We call $\hat{G}_k$ the $k$-point momentum cluster because all of its momentum variables are present in each term of its power series expansion. By expanding the product and making use of the fluctuation relations (43) we obtain

$$\hat{G}_{k\geq2}(q_1, \ldots, q_{k-1}, x, t) = k - 2 \sum_{n=0}^{k-2} \sum_{I_{k-1-n} \subset z_{k-1}} \partial^n \hat{G}_{k-n}(q_{i_1}, \ldots, q_{i_{k-1-n}}, x, t) + (-1)^{k-1} \frac{\partial^k}{\partial x^k} \tilde{G}_0.$$  \hspace{1cm} (51)

In a similar manner, using the identity $1 = P_{q_i} + (1 - P_{q_i})$, we find the inverse relation

$$\tilde{G}_{k\geq2}(q_1, \ldots, q_{k-1}, x, t) = k - 2 \sum_{n=0}^{k-2} \sum_{I_{k-1-n} \subset z_{k-1}} \partial^n \tilde{G}_{k-n}(q_{i_1}, \ldots, q_{i_{k-1-n}}, x, t) + \frac{\partial^k}{\partial x^k} \hat{G}_0.$$  \hspace{1cm} (52)

We call eq. (52) the momentum cluster decomposition of $\tilde{G}_k$.

Using these relations together with the flow equations for the $\tilde{G}_k$ (48) we can write flow equations for the $\hat{G}_k$ as follows. Again we suppress unnecessary arguments for the sake of clarity. Define $F_k[\tilde{G}_0, \ldots, \tilde{G}_k]$ by writing the flow equations (48) as

$$\frac{\partial \tilde{G}_k}{\partial t} = F_k[\tilde{G}_0, \ldots, \tilde{G}_k] + \int_{q'} A(q') \tilde{G}_{k+2}(q', -q', q_1, \ldots, q_{k-1}).$$  \hspace{1cm} (53)

Then

$$\frac{\partial \tilde{G}_0}{\partial t} = F_0[\tilde{G}_0] + \int_{q} A(q) \tilde{G}_2(q).$$
\[
\frac{\partial \hat{G}_{k \geq 2}}{\partial t} = \sum_{n=0}^{k-1} (-1)^n \sum_{i \in \mathbb{Z}_{k-1}}^{i_{k-1-n}} \frac{\partial^n}{\partial x^n} F_{k-n}[\tilde{G}_0, \ldots, \tilde{G}_{k-n}] + I_A \frac{\partial^2 \hat{G}_k}{\partial x^2} + \int_{q'} A(q') \hat{G}_{k+2}(q', -q', q_1, \ldots, q_{k-1}),
\]

where \( I_A \equiv \int_q A(q) \) and \( F_1[\tilde{G}_0, \tilde{G}_1] = \partial F_0[\tilde{G}_0]/\partial x \). To complete the derivation of the flow equations for \( \hat{G}_k \) we must reexpress the arguments of \( F_{k-n} \) in terms of \( \hat{G}_{k' \leq k-n} \) via eq. (52), but already in the above form we can see our most significant results, which we will describe shortly.

The first four momentum cluster flow equations are

\[
\begin{align*}
\frac{\partial \hat{G}_0}{\partial t} &= D \hat{G}_0 - d_j x \frac{\partial \hat{G}_0}{\partial x} + A(0) \left( \frac{\partial \hat{G}_0}{\partial x} \right)^2 + I_A \frac{\partial^2 \hat{G}_0}{\partial x^2} + \int_q A(q) \hat{G}_2(q) \\
\frac{\partial \hat{G}_2}{\partial t} &= (D - 2d_j) \hat{G}_2 + \left( 2A(0) \frac{\partial \hat{G}_0}{\partial x} - d_j x \right) \frac{\partial \hat{G}_2}{\partial x} - q \cdot \nabla_q \hat{G}_2 - 2A(0) \left( \frac{\partial \hat{G}_0}{\partial x} \right)^2 \\
&\quad + 2A(q) \left( \hat{G}_2 + \frac{\partial \hat{G}_0}{\partial x} \right)^2 + I_A \frac{\partial^2 \hat{G}_2}{\partial x^2} + \int_{q'} A(q') \hat{G}_4(q', -q', q) \\
\frac{\partial \hat{G}_3}{\partial t} &= (D - 3d_j) \hat{G}_3 + \left( 2A(0) \frac{\partial \hat{G}_0}{\partial x} - d_j x \right) \frac{\partial \hat{G}_3}{\partial x} - 2 \sum_{i=1}^2 q_i \cdot \nabla_{q_i} \hat{G}_3 \\
&\quad + 2A(0) \frac{\partial^2 \hat{G}_0}{\partial x^2} \left[ \frac{\partial^2 \hat{G}_3}{\partial x^2} - \sum_{i=1}^2 \frac{\partial^2 \hat{G}_2(q_i)}{\partial x^2} \right] \\
&\quad + 2 \sum_{\{i,j\} \in C_2} A(q_i) \left[ \hat{G}_2(q_i) + \frac{\partial \hat{G}_0}{\partial x^2} \right] \left[ \hat{G}_3(q_1, q_2) + \frac{\partial \hat{G}_2(q_2)}{\partial x} - \frac{\partial \hat{G}_2(q_1)}{\partial x} - \frac{\partial \hat{G}_0}{\partial x} \right] \\
&\quad + 2A(q_1 + q_2) \left[ \hat{G}_2(q_1 + q_2) + \frac{\partial \hat{G}_0}{\partial x^2} \right] \left[ \hat{G}_3(q_1, q_2) + \sum_{i=1}^2 \frac{\partial \hat{G}_2(q_i)}{\partial x} + \frac{\partial \hat{G}_0}{\partial x} \right] \\
&\quad + I_A \frac{\partial^2 \hat{G}_3}{\partial x^2} + \int_{q'} A(q') \hat{G}_5(q', -q', q_1, q_2) \\
\frac{\partial \hat{G}_4}{\partial t} &= (D - 4d_j) \hat{G}_4 + \left( 2A(0) \frac{\partial \hat{G}_0}{\partial x} - d_j x \right) \frac{\partial \hat{G}_4}{\partial x} - 3 \sum_{i=1}^3 q_i \cdot \nabla_{q_i} \hat{G}_4 \\
&\quad - 2A(0) \left[ \frac{\partial \hat{G}_0}{\partial x^2} \sum_{\{i,j,k\} \in C_3} \frac{\partial}{\partial x} \hat{G}_3(q_j, q_k) - \frac{\partial \hat{G}_0}{\partial x^2} \sum_{i=1}^3 \frac{\partial \hat{G}_2(q_i)}{\partial x} + \frac{\partial \hat{G}_0}{\partial x^2} \frac{\partial \hat{G}_0}{\partial x^2} \right] \\
&\quad + 2 \sum_{\{i,j,k\} \in C_3} A(q_i) \left[ \hat{G}_2(q_i) + \frac{\partial \hat{G}_0}{\partial x^2} \right] \left[ \hat{G}_4(q_1, q_2, q_3) \\
&\quad + \frac{\partial \hat{G}_3(q_j, q_k)}{\partial x} - \frac{\partial \hat{G}_3(q_j, q_i)}{\partial x} - \frac{\partial \hat{G}_3(q_i, q_k)}{\partial x} + \frac{\partial \hat{G}_2(q_i)}{\partial x^2} + \frac{\partial \hat{G}_0}{\partial x^2} \right].
\end{align*}
\]
\[- \left[ \frac{\partial^2 \tilde{G}_2(q_i)}{\partial x^2} + \frac{\partial^3 \tilde{G}_2(q_i)}{\partial x^3} \right] \left[ \tilde{G}_3(q_i, q_j) + \tilde{G}_3(q_i, q_k) + \frac{\partial \tilde{G}_2(q_j)}{\partial x} + \frac{\partial \tilde{G}_2(q_k)}{\partial x} \right] \right] \\
+ 2 \sum_{i,j,k} A(q_j + q_k) \left\{ \frac{\partial \tilde{G}_3(q_j, q_k)}{\partial x} + \frac{\partial^2 \tilde{G}_2(q_j)}{\partial x^2} + \frac{\partial^3 \tilde{G}_1(q_j, q_k)}{\partial x^3} \right\} \\
\times \left[ \tilde{G}_3(q_j + q_k, q_i) + \frac{\partial \tilde{G}_2(q_j)}{\partial x} + \frac{\partial^2 \tilde{G}_1(q_j, q_k)}{\partial x^2} + \frac{\partial^3 \tilde{G}_0(q_j, q_k)}{\partial x^3} \right] \\
+ 2 A(q_1 + q_2 + q_3) \left[ \tilde{G}_2(q_1 + q_2 + q_3) + \frac{\partial^2 \tilde{G}_1(q_1, q_2, q_3)}{\partial x^2} \right] \left[ \tilde{G}_4(q_1, q_2, q_3) \right] \\
+ \sum_{i,j,k} \frac{\partial \tilde{G}_3(q_j, q_k)}{\partial x} + \sum_{i=1}^{3} \frac{\partial^2 \tilde{G}_2(q_i)}{\partial x^2} + \frac{\partial^3 \tilde{G}_0}{\partial x^3} \right \} \\
+ I_A \frac{\partial^2 \tilde{G}_4}{\partial x^2} + \int_{q'} A(q') \tilde{G}_6(q', q, q_1, q_2, q_3), \tag{55} \]

where \( C_k \) stands for the cyclic permutations of \( \{1, \ldots, k\} \).

The above equations display many features which appear to give them distinct advantages over the standard \( N \)-point flow equations \( [19] \). By comparing eqs. (55) with eqs. (49) we see that the number of terms in the equation for \( \partial \tilde{G}_k / \partial t \) increases dramatically with \( k \) relative to its “unclustered” counterpart \( [19] \). This is also directly apparent in comparing eqs. (54) and eqs. (53). The new terms are those which are discarded in the standard truncation approach to the flow equations \( [24] \), and their presence in \( (54) \) is one measure of the improvement of the momentum cluster approach. Because more terms are “rescued” as \( k \) increases this may offer hope for convergence of the approximation procedure.

A more direct measure of the amount of information retained by the two sets of equations after truncation is obtained by comparing their integral terms, which contain the contribution from higher order functions which is lost when the equations are truncated. Their relative importance can be estimated by noting that, according to \( [39] \), the \( \tilde{G}_k^* \) go to zero as an inverse power of the \( q_i=1,\ldots,k \) as the \( q_i \) go to infinity. As they generally are nonzero for \( q_i = 0 \), we can assume their weight to be largest in the region near the origin and hence make a major contribution to the integral term in eqs. (53). This is because the function \( A(q) \) is typically concentrated near the origin as well, generally chosen to behave as \( e^{-aq^2} \) for large \( q \). In contrast, the functions \( \tilde{G}_k \) have had, by definition, their zero point values subtracted away, leaving functions that are concentrated away from the
origin, tending to nonzero values at infinity. These functions will therefore make a much
closer contribution to the integrals of eqs. (54) and, hopefully, be much less missed upon
truncation.

Another significant difference between the two sets of equations is that the flow
equations for $\hat{G}_k$ are parabolic in $x$ rather than first order, due to the presence of the
$I_A \partial^2 \hat{G}_k / \partial x^2$ term in the equation for $\partial \hat{G}_k / \partial t$. This causes a dramatic change in their
solution structure relative to the flow equations for the $\tilde{G}_k$. This is already apparent in
that the new lowest order of approximation is the LPA approximation. This represents
a significant improvement over its mean field counterpart of eqs. (49), both in terms of
a more accurate calculation of critical exponents and description of universality classes
and in terms of (presumably) avoiding the thermodynamically unstable branches of the
mean field free energy. If the higher order equations of (55) represent as much of an
improvement over their eq. (19) counterparts as the LPA approximation is over the mean
field approximation, we can expect these equations to prove very useful.

Another consequence of the parabolic nature of the flow equations (55) is that they
cannot be solved directly in the $x = 0$ limit, as could eqs. (19). Thus, the full $x$-dependence
of the flow equations will play an essential role in their approximation behavior.

Of course, while the above points suggest the possibility of much improved approxi-
mations based on truncations of the momentum cluster flow equations, the convergence of
the truncation scheme remains an open question. Yet, a hopeful indication that conver-
gence may not be out of reach lies in the fact that the momentum cluster flow equations
are essentially an expansion of the ERG equation in terms of the number of momenta
which couple the Fourier transformed field variables in the effective action. In a lattice
real-space formulation this corresponds to an expansion of the effective lattice action in
terms of local interactions ordered according to the number of interacting sites (of arbi-
trary separation). Thus, we begin to make contact, perhaps with improvements, with the
apparently convergent approximation scheme discussed in our Introduction.

7 Concluding Discussion

We close with some observations about future directions. The prospect for solving the
lower order equations numerically looks quite good, as algorithms and approaches for
solving PDEs are readily available \[35\] and have already begun to be applied to the
solution of approximate RG equations \[3, 13, 36\]. These approaches should put solution
of the $\hat{G}_4$ equation within reach, at least in so far as it contributes to the equation for $\hat{G}_2$, as for that purpose only $\hat{G}_4(q_1, -q_1, q_2, x, t)$ is required. Thus, at this level of truncation, only four variables, $q_1^2, q_2^2, (q_1 + q_2)^2$, and $x$, are needed to solve for the fixed-point solution and critical exponents. To determine the full momentum dependence of $\hat{G}_4$, or the higher order corrections needed for the empirical study of convergence, will probably require further approximation [13].

Questions of convergence aside, the momentum cluster approach provides a possible advantage over the usual formulation in that it works directly with constant-field connected Green's functions rather than the interactions contributing to the RG effective action. Since the field is constant, it has no momentum index, considerably simplifying the description of momentum dependence when the approach is extended to vector models and gauge theories. This suggests that equations analogous to the $\hat{G}_2$ equation may be a useful starting approximation for the investigation of the space of renormalizable gauge theories, just as the LPA approximation was for scalar and vector models [8, 25].

The momentum cluster approach can also be applied to the Legendre transformed ERG equation [12], [4, 5] and possibly to sharp-cutoff versions of the equation [5, 37, 38], as well. In this paper we limited ourselves to the Wilson/Polchinski version because it had the simplest form of nonlinear terms. More generally, one might consider whether the momentum cluster approach might have other applications, as many areas of physics make use of hierarchies of equations that begin with the mean field approximation and involve corrections from higher order correlations. If these equations can be represented in a generating functional format, the same fluctuation relations and momentum cluster decomposition might well apply, giving access to a new hierarchy of equations that no longer begins at a mean field level of approximation.

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