WEIGHTED POINCARÉ INEQUALITIES FOR NONLOCAL DIRICHLET FORMS

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Abstract. Let $V$ be a locally bounded measurable function such that $e^{-V}$ is bounded and belongs to $L^1(dx)$, and let $\mu_V(dx) := C_V e^{-V(x)} dx$ be a probability measure. We present the criterion for the weighted Poincaré inequality of the non-local Dirichlet form

$$D_{\rho,V}(f,f) := \int \int (f(y) - f(x))^2 \rho(|x-y|) \, dy \, \mu_V(dx)$$

on $L^2(\mu_V)$. Taking $\rho(r) = e^{-\delta r^{-\alpha}}$ with $0 < \alpha < 2$ and $\delta \geq 0$, we get some conclusions for general fractional Dirichlet forms, which can be regarded as a complement of our recent work [13], and an improvement of the main result in [8]. In this especial setting, concentration of measure for the standard Poincaré inequality is also derived.

Our technique is based on the Lyapunov conditions for the associated truncated Dirichlet form, and it is considerably efficient for the weighted Poincaré inequality of the following non-local Dirichlet form

$$D_{\psi,V}(f,f) := \int \int (f(y) - f(x))^2 \psi(|x-y|) e^{-V(y)} dy e^{-V(x)} dx$$

on $L^2(\mu_{2V})$, which is associated with symmetric Markov processes under Girsanov transform of pure jump type.

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1. Introduction and Main Results

1.1. Background for Functional Inequalities of Fractional Dirichlet Forms.
For $\alpha \in (0,2)$, let $\mu_\alpha$ be a rotationally symmetric stable infinite divisible probability distribution, such that

$$\hat{\mu}_\alpha(\xi) := \int e^{ix \cdot \xi} \mu_\alpha(dx) = e^{-\frac{1}{\alpha} |\xi|^\alpha}, \quad \xi \in \mathbb{R}^d.$$ 

For any $f \in C_0^\infty(\mathbb{R}^d)$, the set of smooth functions with bounded derivatives of every order, define

$$D_\alpha(f,f) := \int \int \frac{(f(y) - f(x))^2}{|y-x|^{d+\alpha}} \, dy \, \mu_\alpha(dx).$$

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Then, \((D_\alpha, C^\infty_b(\mathbb{R}^d))\) can be extended to a *non-local Dirichlet form* associated with the operator

\[
L_\alpha = -(-\Delta)^{\alpha/2} - x \cdot \nabla, \quad \alpha \in (0, 2),
\]

which is the infinitesimal generator of an Ornstein-Uhlenbeck process driven by symmetric \(\alpha\)-stable Lévy processes. Poincaré inequalities for \((D_\alpha, C^\infty_b(\mathbb{R}^d))\) were studied in [9, Theorem 1.3 and Corolalry 1.4].

In (1.1) the singular kernel \(|y - x|^{-(d+\alpha)} \, dy\) is the Lévy measure associated with \(\mu_\alpha\), which is a strong constraint to study functional inequalities for general non-local Dirichlet forms. The first breakthrough in this direction was established in [8] in virtue of the methods from harmonic analysis. The main result in [8] (see [8, Theorem 1.2]) states that, if \(e^{-V} \in L^1(dx) \cap C^2(\mathbb{R}^d)\) such that for some constant \(\varepsilon > 0\),

\[
(1.2) \quad \frac{(1 - \varepsilon)|\nabla V|^2}{2} - \Delta V \to \infty, \quad x \to \infty,
\]

then there exist two positive constants \(\delta\) and \(C_0\) such that for all \(f \in C^\infty_b(\mathbb{R}^d)\) with \(\int f(x) e^{-V(x)} \, dx = 0\),

\[
(1.3) \quad \int f^2(x)(1 + |\nabla V(x)|^\alpha) e^{-V(x)} \, dx \leq C_0 \int\int \frac{(f(y) - f(x))^2}{|y - x|^{d+\alpha}} e^{-\delta|y - x|} \, dy e^{-V(x)} \, dx.
\]

On the other hand, as a generalization of (1.1), recently explicit and sharp criteria of Poincaré type (i.e., Poincaré, super Poincaré and weak Poincaré) inequalities have been presented in [13] for the following general fractional Dirichlet form

\[
(1.4) \quad D_{\alpha,V}(f, f) := \int\int \frac{(f(y) - f(x))^2}{|y - x|^{d+\alpha}} \, dy \, \mu_V(dx), \quad d \geq 1, \alpha \in (0, 2),
\]

where \(V\) is a Borel measurable function on \(\mathbb{R}^d\) such that \(e^{-V} \in L^1(dx)\), and \(\mu_V(dx) = \frac{1}{\int e^{-V(x)} \, dx} e^{-V(x)} \, dx\). According to the paragraph below [8, Remark 1.3], (1.4) is natural in the sense that: we should regard the measure \(|y - x|^{-(d+\alpha)} \, dy\) as the Lévy measure, and \(\mu_V(dx)\) as the ambient measure. Namely, \(D_{\alpha,V}\) does get rid of the constraint in \(D_\alpha\), and it should be a typical example in study functional inequalities for non-local Dirichlet forms.

To move further, we briefly recall the results developed in [13]. Let \(e^{-V} \in L^1(dx) \cap C^2(\mathbb{R}^d)\) satisfying some regular assumptions. [13, Theorem 1.1 (1) and (2)] shows that, if

\[
\liminf_{|x| \to \infty} \frac{e^{V(x)}}{|x|^{d+\alpha}} > 0,
\]

then there is a constant \(C_1 > 0\) such that for all \(f \in C^\infty_b(\mathbb{R}^d)\),

\[
(1.5) \quad \mu_V(f - \mu_V(f))^2 \leq C_1 D_{\alpha,V}(f, f);
\]

if

\[
\liminf_{|x| \to \infty} \frac{e^{V(x)}}{|x|^{d+\alpha}} = \infty,
\]
then the following super-Poincaré inequality
\[ \mu_V(f^2) \leq r D_{a,V}(f,f) + \beta(r \mu_V(|f|)^2), \quad r > 0 \]
holds for some non-increasing function \( \beta \) and all \( f \in C_0^\infty(\mathbb{R}^d) \).

Note that the exponentially decaying factor \( e^{-\delta|y-x|} \) and the weighted function \( 1 + |\nabla V(x)|^\alpha \) in (1.3) indicate that the functional inequality (1.3) is stronger than the expected Poincaré inequality (1.5) for fractional Dirichlet form \( D_{a,V} \), see [8, Remark 1.4]. Therefore, the work of [13] does not extend [8], and there still exists a gap between [13] and [8]. That is just the motivation of our present paper.

1.2. Weighted Poincaré Inequalities for \( D_{a,V,\delta} \) with \( \delta > 0 \): Improvement of the Work in [8]. We first introduce some notations. Let \( V \) be a locally bounded measurable function on \( \mathbb{R}^d \) such that \( e^{-V} \) is bounded and \( e^{-V} \in L^1(dx) \). Define a probability measure \( \mu_V \) as follows

\[ \mu_V(dx) = \frac{1}{\int e^{-V(x)} \, dx} e^{-V(x)} \, dx. \]  

For any \( \delta \geq 0 \) and \( f \in C_0^\infty(\mathbb{R}^d) \), set

\[ D_{a,V,\delta}(f,f) := \int \int \frac{(f(y) - f(x))^2}{|y - x|^{d+\alpha}} e^{-\delta|y-x|} \, dy \mu_V(dx). \]

In particular, when \( \delta = 0 \), \( D_{a,V,\delta} = D_{a,V} \). We say that the weighted Poincaré inequality holds for \( D_{a,V,\delta} \), if there exist a positive weighted function \( \tilde{\omega} \) and a constant \( C > 0 \) such that for all \( f \in C_0^\infty(\mathbb{R}^d) \),

\[ \int (f - \mu_V(f))^2 \tilde{\omega} d\mu_V \leq CD_{a,V,\delta}(f,f). \]

Even though it is known that in the context of local Dirichlet forms some super Poincaré inequalities can imply weighted Poincaré inequalities (e.g. see [11]), to the best of our knowledge there is no literature about such relation for non-local Dirichlet form \( D_{a,V,\delta} \), even for \( D_{a,V} \). Instead of studying this topic, the purpose of this paper is to establish the weighted Poincaré inequalities for non-local Dirichlet form \( D_{a,V,\delta} \) directly.

The main result is as following,

**Theorem 1.1.** Suppose that for some constants \( \delta > 0 \), \( \alpha \in (0,2) \) and \( \alpha_0 \in (0,1) \),

\[ \limsup_{|x| \to \infty} \left( \sup_{|z| > |x|} e^{-V(z)} e^{\delta|x-z|^\alpha} \right) = 0. \]

Then, there exists a constant \( C_1 > 0 \) such that the following weighted Poincaré inequality

\[ \int (f(x) - \mu_V(f))^2 \frac{e^{V(x)-\delta|x|}}{(1+|x|)^{d+\alpha}} \mu_V(dx) \leq C_1 D_{a,V,\delta}(f,f) \]

holds for all \( f \in C_0^\infty(\mathbb{R}^d) \).

To see that Theorem 1.1 improves [8, Theorem 1.2], we consider the following example.
Example 1.2. (1) For $\delta > 0$, let $V(x) = \varepsilon (1 + |x|^2)^{1/2}$ with some $\varepsilon > \delta$. Then, (1.7) is fulfilled, and so the corresponding weighted Poincaré inequality (1.8) holds. Note that, (1.2) is not satisfied for $V(x) = \varepsilon (1 + |x|^2)^{1/2}$ with any $\varepsilon > 0$.

(2) Let $V(x) = 1 + |x|^2$. Then, Theorem 1.1 implies that for any $\delta > 0$ there exists a constant $c_1 > 0$ such that for all $f \in C_b^\infty (\mathbb{R}^d)$ with $\int f(x)e^{-V(x)} \, dx = 0$,

$$
\int f^2(x) \exp \left( \frac{1}{2} (1 + |x|^2) \right) e^{-V(x)} \, dx \leq c_1 \int \int \frac{(f(y) - f(x))^2}{|y - x|^{d+\alpha}} e^{-\delta|y-x|} \, dy e^{-V(x)} \, dx.
$$

However, in this setting [8, Theorem 1.2] only implies that there exist two constants $\delta, c_0 > 0$ such that for all $f \in C_b^\infty (\mathbb{R}^d)$ with $\int f(x)e^{-V(x)} \, dx = 0$,

$$
\int f^2(x)(1 + |x|^n) e^{-V(x)} \, dx \leq c_0 \int \int \frac{(f(y) - f(x))^2}{|y - x|^{d+\alpha}} e^{-\delta|y-x|} \, dy e^{-V(x)} \, dx.
$$

As a direct consequence of Theorem 1.1, we know that for $\delta > 0$ and $\alpha \in (0, 2)$, if

$$
(1.9) \quad \lim_{|x| \to \infty} \frac{e^{V(x)}}{|x|^{d+\alpha} e^{\delta|x|}} > 0,
$$

then (1.8) holds, which implies the standard Poincaré inequality:

$$
(1.10) \quad \mu_V \left( f - \mu_V(f) \right)^2 \leq C_2 D_{\alpha,V,\delta}(f, f) \quad \text{for all } f \in C_b^\infty (\mathbb{R}^d).
$$

In order to show that (1.9) is qualitatively sharp, we will study the concentration of measure for the Poincaré inequality (1.10) of $D_{\alpha,V,\delta}$ with $\delta > 0$.

Proposition 1.3. Let $\delta > 0$ and $0 < \alpha < 2$, and let $\mu_V$ be a probability measure defined by (1.6). Suppose that there is a constant $C_2 > 0$ such that the Poincaré inequality (1.10) holds for such $\mu_V$ and $D_{\alpha,V,\delta}$. Then, there is a constant $\lambda_0 > 0$ such that

$$
\int e^{\lambda_0 |x|} \mu_V(dx) < \infty.
$$

1.3. Weighted Poincaré Inequalities for $D_{\alpha,V}$: Completeness of the Work in [13]. Let $D_{\alpha,V}$ be the bilinear form defined by (1.4). We have the following result.

Theorem 1.4. Let $\alpha \in (0, 2)$. If for some constant $\alpha_0 \in (0, \alpha/2)$,

$$
(1.11) \quad \limsup_{|x| \to \infty} \left[ \left( \sup_{|z| \geq |x|} e^{-V(z)} \right) |x|^{d+\alpha - \alpha_0} \right] = 0,
$$

then there exists a constant $C_2 > 0$ such that the following weighted Poincaré inequality

$$
(1.12) \quad \int \left( f(x) - \mu_V(f) \right)^2 \frac{e^{V(x)}}{(1 + |x|)^{d+\alpha}} \mu_V(dx) \leq C_2 D_{\alpha,V}(f, f)
$$

holds for all $f \in C_b^\infty (\mathbb{R}^d)$.

The weighted function in the weighted Poincaré inequality (1.12) is

$$
\omega(x) := \frac{e^{V(x)}}{(1 + |x|)^{(d+\alpha)}}.
$$
This function is optimal in the sense that, the inequality (1.12) fails if we replace \( \omega(x) \) by a positive function \( \omega^*(x) \), which satisfies that

\[
\liminf_{|x| \to \infty} \frac{\omega^*(x)}{\omega(x)} = \infty.
\]

Theorem 1.4 can be seen as a complement of [13, Theorem 1.1], where explicit criteria are presented for fractional Dirichlet form \( D_{\alpha,V} \) to satisfy Poincaré, super Poincaré and weak Poincaré inequalities.

We first mention that the weighted Poincaré inequality (1.12) can be satisfied for some probability measures, which do not fulfill the true Poincaré inequality.

**Example 1.5.** For \( \epsilon > 0 \), let \( \mu_\epsilon(dx) = C_\epsilon(1 + |x|^2)^{-(d+\epsilon)/2} \) be a probability measure, where \( C_\epsilon \) is a normalizing constant. According to Theorem 1.4, if \( \epsilon > \alpha/2 \), there exists a constant \( c_3 > 0 \) such that the following weighted Poincaré inequality

\[
\int (f(x) - \mu_\epsilon(f))^2 \frac{1}{(1 + |x|)^{d/\alpha}} \mu_\epsilon(dx) \leq c_3 \int \int \frac{(f(y) - f(x))^2}{|y - x|^{d+\alpha}} dy \mu_\epsilon(dx)
\]

holds for all \( f \in C_b^\infty(\mathbb{R}^d) \). However, by [13, Corollary 1.2 (1)], we know that the following Poincaré inequality

\[
\int (f(x) - \mu_\epsilon(f))^2 \mu_\epsilon(dx) \leq c_4 \int \int \frac{(f(y) - f(x))^2}{|y - x|^{d+\alpha}} dy \mu_\epsilon(dx), \quad f \in C_b^\infty(\mathbb{R}^d)
\]

does not hold for any \( \epsilon \in (\alpha/2, \alpha) \).

The next result shows that the weighted Poincaré inequality for \( D_{\alpha,V} \) with continuous weighted function, which tends to infinite when \( |x| \) tends to infinite, indeed implies the super Poincaré inequality. For any \( r > 0 \), define

\[
h(r) := \inf_{|x| \leq r} e^{V(x)}, \quad H(r) := \sup_{|x| \leq r} e^{V(x)}.
\]

**Proposition 1.6.** Let \( \mu_V \) be a probability measure given by (1.6), and \( \omega \) be a positive continuous function on \( \mathbb{R}^d \) such that \( \lim_{|x| \to \infty} \omega(x) = \infty \). Suppose that there is a constant \( C_0 > 0 \) such that the following weighted Poincaré inequality holds

\[
(1.13) \quad \int (f(x) - \mu_V(f))^2 \omega(x) \mu_V(dx) \leq C_0 D_{\alpha,V}(f,f), \quad f \in C_b^\infty(\mathbb{R}^d).
\]

Then the following super Poincaré inequality

\[
(1.14) \quad \mu_V(f^2) \leq r D_{\alpha,V}(f,f) + \beta(r) \mu_V(|f|^2), \quad r > 0, \quad f \in C_b^\infty(\mathbb{R}^d)
\]

holds with

\[
\beta(r) = \inf \left\{ C_1 H(t)^{2+d/\alpha} h(t)^{-1-d/\alpha} (1 + s^{-d/\alpha}) : \frac{2C_0}{\inf_{|x| \geq t} \omega(x)} + s \leq r, t > 1, s > 0 \right\}.
\]

In particular, there are \( r_0 > 0 \) small enough and a constant \( C_2 > 0 \) such that for all \( 0 < r \leq r_0 \),

\[
\beta(r) \leq C_2 \left( 1 + r^{-d/\alpha} (h \circ \kappa(4C_0 r^{-1}))^{-1-d/\alpha} \left( H \circ \kappa(4C_0 r^{-1}) \right)^{2+d/\alpha} \right),
\]

where

\[
\kappa(r) := \inf \{ s > 0 : \inf_{|x| \geq s} \omega(x) \geq r \}.
\]
Let \( V(x) = \frac{d+\varepsilon}{2} \log(1 + |x|^2) \) with \( \varepsilon > \alpha \), and
\[
\mu_V(dx) = C_V e^{-V(x)} dx = C_V (1 + |x|^2)^{(d+\varepsilon)/2} dx
\]
be the corresponding probability measure. According to Proposition 1.6 and the weighted Poincaré inequality obtained in Example 1.5, we know that the super Poincaré inequality (1.14) holds for such \( \mu_V \) and \( D_{\alpha,V} \) with
\[
\beta(r) = c \left( 1 + r \frac{(d+\varepsilon)/(2\alpha + d)}{(\varepsilon - \alpha)} \right).
\]
This estimate for the rate function \( \beta \) is exactly the same as that in [13, Corollary 1.2 (2)], which indicates that the estimate above is optimal. However, due to the non-local property, we do not know whether the super Poincaré inequality implies the weighted Poincaré inequality for the Dirichlet form \( D_{\alpha,V} \), although it is true for local Dirichlet forms, e.g. see [11].

There exist a lot of works for weighted Poincaré type inequalities for local Dirichlet forms, e.g. see [2, 4]. The difference between the main results in those cited papers and Theorem 1.4 is that, the weighted function of weighted Poincaré inequalities in [2, Theorem 3.1] and [4, Theorem 2.1] is inside the associated Dirichlet form, but the weighted function of the inequality (1.12) here appears in the variation term (i.e. the left hand side of the inequality). The following proposition shows that the weighted Poincaré inequality (1.12) implies more information, which may indicate that weighted Poincaré inequalities of the form (1.12) are more suitable to study for non-local Dirichlet forms.

**Proposition 1.7.** Let the function \( V \) satisfying
\[
\liminf_{|x| \to \infty} \frac{e^{V(x)}}{|x|^{d+\alpha}} > 0,
\]
and let \( \omega : \mathbb{R}^d \to \mathbb{R}_+ \) be a continuous and positive function. Then there exists a constant \( C_2(\omega) > 0 \) such that the following weighted Poincaré inequality holds for all \( f \in C_0^\infty(\mathbb{R}^d) \).
\[
\int (f(x) - \mu_V(f))^2 e^{V(x)} (1 + |x|)^{d+\alpha} \mu_V(dx) \leq C_2(\omega) \int \omega(x) \int \frac{(f(y) - f(x))^2}{|y - x|^{d+\alpha}} dy \mu_V(dx)
\]
Under (1.15), the function \( V \) satisfies (1.11), which implies that the inequality (1.12) holds. According to (1.16), in this situation we can improve the inequality (1.12) by adding a weighted function \( \omega \), which may tend to 0 in any rate as \( |x| \to \infty \), inside the non-local Dirichlet form \( D_{\alpha,V} \). However, in the context of local Dirichlet forms, to obtain such weighted Poincaré inequality we need to put some restrictive conditions on the rate of decay for the weighted function \( \omega \).

\footnote{To deduce (1.16) from (1.12), one may take \( C_2(\omega) = \sup_{x \in \mathbb{R}^d} \omega(x)^{-1} \). However, as mentioned above the weighted function \( \omega \) may tend to 0 as \( |x| \to \infty \), and so in this case \( C_2(\omega) \) is infinite, which does not work. We will see below that the proof of Proposition 1.7 is not trivial.}

\footnote{One can easily check this point by using the criteria about Poincaré inequalities for one dimensional diffusion processes, e.g. see [5, Table 1.4, Page 15].}
the difference is as follows: applying \( f \in C^\infty_0(\mathbb{R}^d) \) with support contained in the set \( \{ x \in \mathbb{R}^d : |x| > r \} \) into the weighted local Dirichlet form
\[
\tilde{D}_\omega(f, f) := \int \omega(x)|\nabla f(x)|^2 \, \mu_V(dx),
\]
we find that \( \tilde{D}_\omega(f, f) \) only depends on the value of \( \omega \) in domain \( \{ x \in \mathbb{R}^d : |x| > r \} \); while for the weighted non-local Dirichlet form
\[
D_\omega(f, f) := \int \omega(x) \int \frac{(f(y) - f(x))^2}{|y - x|^{d+\alpha}} \, dy \, \mu_V(dx),
\]
\( D_\omega(f, f) \) depends on the value of \( \omega \) in \( \mathbb{R}^d \), not only in \( \{ x \in \mathbb{R}^d : |x| > r \} \).

The remaining part of this paper is organized as follows. In the next section we recall results on non-local Dirichlet forms and their generator, which apply all the examples to be studied in our paper. Section 3 is devoted to general theory on the existence of weighted Poincaré inequalities for non-local Dirichlet forms via Lyapunov-type conditions of the associated truncated Dirichlet forms, which shall be interesting of itself. In Section 4, we establish efficient Lyapunov conditions for the truncated Dirichlet form associated with original Dirichlet form. This, along with the results in Sections 2 and 3, gives us weighted Poincaré inequalities for general non-local Dirichlet forms (see Theorem 4.1), which immediately yield Theorems 1.1 and 1.4. The proofs of Proposition 1.3, Proposition 1.6 and Proposition 1.7 are also included here. In Section 5, we will state that our approach to Theorem 4.1 also yields the criterion for weighted Poincaré inequalities for Dirichlet forms associated with symmetric Markov processes under Girsanov transform of pure pump type. We also consider the corresponding concentration of measure, which indicates that the inequalities we derived above are optimal in some sense.

2. Characterization of Operators Associated with Non-local Dirichlet Forms

2.1. Non-local Dirichlet Forms in Terms of Generators. Let \( C^\infty_0(\mathbb{R}^d) \) be the set of smooth functions with compact support on \( \mathbb{R}^d \). Let \( V \) be a locally bounded measurable function on \( \mathbb{R}^d \) such that \( \int e^{-V(x)} \, dx < \infty \), and \( j \) be a measurable function on \( \mathbb{R}^{2d} \setminus \{(x, y) \in \mathbb{R}^{2d} : x = y\} \) such that \( j(x, y) \geq 0 \) and \( j(x, y) = j(y, x) \). Let \( \mu_V(dx) = C_V e^{-V(x)} \, dx \) be a probability measure on \( (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d)) \) with a normalizing constant
\[
C_V := \frac{1}{\int e^{-V(x)} \, dx}.
\]
Consider
\[
D_{j,V}(f, g) := \frac{1}{2} \int \int (f(y) - f(x))(g(y) - g(x)) j(x, y) \, \mu_V(dy) \, \mu_V(dx),
\]
\[\mathcal{D}(D_{j,V}) := \left\{ f \in L^2(\mu_V) : D_{j,V}(f, f) < \infty \right\}.
\]
Note that the kernel \( j(x, y) \) is only defined on the set \( \{(x, y) \in \mathbb{R}^{2d} : x \neq y\} \). Since \( \{(x, y) \in \mathbb{R}^{2d} : x = y\} \) is a zero-measure set under \( \mu_V(dx) \mu_V(dy) \), we can still write the integral domain as \( \mathbb{R}^d \times \mathbb{R}^d \) in the expression above for \( D_{j,V} \).
Suppose that for any \( \varepsilon > 0 \), the function
\[
x \mapsto \int_{\{ |x - y| > \varepsilon \}} j(x, y) \mu_V(dy)
\]
is locally integrable with respect to \( \mu_V(dx) \). Then, according to the proof of [7, Example 1.2.4], \( (D_{j,V}, \mathcal{D}(D_{j,V})) \) is a symmetric Dirichlet form on \( L^2(\mu_V) \) in the wide sense; namely, the set \( \mathcal{D}(D_{j,V}) \) is not necessarily dense in \( L^2(\mu_V) \), c.f. see [7, Chapter 1.3].

**Theorem 2.1.** The following three statements are satisfied.

(1) If
\[
I_1(x) := \int (1 \wedge |x - y|^2) j(x, y) \mu_V(dy) \in L^1_{loc}(\mu_V),
\]
then \( C_c^\infty(\mathbb{R}^d) \subset \mathcal{D}(D_{j,V}) \); if
\[
I_1(x) := \int (1 \wedge |x - y|^2) j(x, y) \mu_V(dy) \in L^1(\mu_V),
\]
then \( C_c^\infty(\mathbb{R}^d) \subset \mathcal{D}(D_{j,V}) \). In particular, in both cases \( (D_{j,V}, \mathcal{D}(D_{j,V})) \) is a Dirichlet form on \( L^2(\mu_V) \).

(2) For any \( x \in \mathbb{R}^d \), set
\[
I_2(x) := \int_{\{|z| \leq 1\}} |z| \left| j(x, x + z)e^{-V(x+z)} - j(x, x - z)e^{-V(x-z)} \right| dz.
\]
Suppose that
\[
I_2(x) := \int_{\{|z| \leq 1\}} |z| \left| j(x, x + z)e^{-V(x+z)} - j(x, x - z)e^{-V(x-z)} \right| dz.
\]
the function \( x \mapsto I_i(x) \) is locally bounded for \( i = 1, 2 \).

Then, for any \( f, g \in C_c^\infty(\mathbb{R}^d) \),
\[
D_{j,V}(f, g) = - \int g L_{j,V} f d\mu_V,
\]
where
\[
L_{j,V} f(x) = C_V \left[ \int (f(x + z) - f(x) - \nabla f(x) \cdot z \mathbf{1}_{\{|z| \leq 1\}}) j(x, x + z)e^{-V(x+z)} dz \right. \\
+ \left. \frac{1}{2} \nabla f(x) \cdot \int_{\{|z| \leq 1\}} z \left( j(x, x + z)e^{-V(x+z)} - j(x, x - z)e^{-V(x-z)} \right) dz \right].
\]

(3) Suppose that (2.19) holds, and there is a constant \( r_0 > 0 \) such that for each \( r \geq r_0 \),
\[
I_{3,r}(x) := \mathbf{1}_{(B(0,r))^c}(x) \int_{\{|z| + |x| \leq r\}} j(x, x + z)e^{-V(x+z)} dz \in L^2(\mu_V).
\]
Then for each \( f \in C_c^\infty(\mathbb{R}^d) \), \( L_{j,V} f \in L^2(\mu_V) \).

**Proof.** (a) For any \( f \in C_c^\infty(\mathbb{R}^d) \), choose \( r_2, r_1 \) large enough such that \( r_2 > r_1 + 1 \) and \( \text{supp}(f) \subseteq B(0, r_1) \). Set
\[
c_0(f) := \max \left\{ \sup_{x \in \mathbb{R}^d} |\nabla f(x)|^2, 4 \sup_{x \in \mathbb{R}^d} |f(x)|^2 \right\}.
\]
Then, by the symmetric property that \( j(x, y) = j(y, x) \),

\[
D_{j, V}(f, f)
= \iint_{B(0, r_2) \times B(0, r_2)} (f(x) - f(y))^2 j(x, y) \mu_V(dy) \mu_V(dx)
+ \iint_{B(0, r_2) \times B(0, r_2)^c} f(x)^2 j(x, y) \mu_V(dy) \mu_V(dx)
+ \iint_{B(0, r_2)^c \times B(0, r_2)} f(y)^2 j(x, y) \mu_V(dy) \mu_V(dx)
= \iint_{B(0, r_2) \times B(0, r_2)} (f(x) - f(y))^2 j(x, y) \mu_V(dy) \mu_V(dx)
+ 2 \iint_{B(0, r_2) \times B(0, r_2)^c} f(x)^2 j(x, y) \mu_V(dy) \mu_V(dx)
\leq \sup_{x \in \mathbb{R}^d} |\nabla f(x)|^2 \iint_{B(0, r_2) \times B(0, r_2)} |x - y|^2 j(x, y) \mu_V(dy) \mu_V(dx)
+ 4 \sup_{x \in \mathbb{R}^d} |f(x)|^2 \iint_{B(0, r_1) \times B(0, r_2)^c} j(x, y) \mu_V(dy) \mu_V(dx)
\leq c_0(f) \left[ \int_{B(0, r_2)} \int_{|x - y| \in 2r_2} |x - y|^2 j(x, y) \mu_V(dy) \mu_V(dx)
+ \int_{B(0, r_1)} \int_{|x - y| \in r_2 - r_1} j(x, y) \mu_V(dy) \mu_V(dx) \right]
\leq c_0(f) \left[ \int_{B(0, r_2)} \int (|x - y|^2 \wedge (4r_2^2)) j(x, y) \mu_V(dy) \mu_V(dx)
+ \int_{B(0, r_1)} \int (|x - y|^2 \wedge (r_2 - r_1)^2) j(x, y) \mu_V(dy) \mu_V(dx) \right]
\leq 2c_0(f) \int_{B(0, r_2)} \int (|x - y|^2 \wedge (4r_2^2)) j(x, y) \mu_V(dy) \mu_V(dx)
\leq 8r_2^2 c_0(f) \int_{B(0, r_2)} \int (|x - y|^2 \wedge 1) j(x, y) \mu_V(dy) \mu_V(dx).
\]

This, along with (2.17), yields the first conclusion of part (1).

For each \( f \in C_b^\infty(\mathbb{R}^d) \), we still set

\[
c_0(f) := \max \left\{ \sup_{x \in \mathbb{R}^d} | \nabla f(x)|^2, \ 4 \sup_{x \in \mathbb{R}^d} |f(x)|^2 \right\}.
\]

Then, by the mean value theorem, for any \( x, y \in \mathbb{R}^d \),

\[
|f(x) - f(y)|^2 \leq c_0(f) \left( 1 \wedge |x - y|^2 \right).
\]

Hence, \( D_{j, V}(f, f) < \infty \), if (2.18) holds. This proves the second desired assertion of part (1).

(b) The proof of part (2) essentially follows from that of [15, Theorem 1.2]. For the sake of completeness, here we present the proof in a different and simple way. We first note that under (2.19), (2.17) is satisfied, and so \( C_c^\infty(\mathbb{R}^d) \subset \mathcal{D}(D_{j, V}) \). For
each \( f \in C_c^\infty(\mathbb{R}^d) \), by (2.20) and the mean value theorem, we have
\[
L_{j,V} f(x) \leq c_1 (I_1(x) + I_2(x))
\]
for some constant \( c_1 := c_1(f) > 0 \). Hence, (2.19) implies that \( L_{j,V} f \) is well defined and locally bounded.

Next, for each \( \varepsilon \in (0,1) \) and \( f \in C_c^\infty(\mathbb{R}^d) \), define
\[
L_{j,V,\varepsilon} f(x) := C_V \int_{\{|z|>\varepsilon\}} \left( f(x + z) - f(x) - \nabla f(x) \cdot z \mathbb{1}_{\{|z| \leq 1\}} \right) j(x, x + z) e^{-V(x+z)} dz \\
+ \frac{1}{2} \nabla f(x) \cdot \int_{\{\varepsilon<|z| \leq 1\}} z \left( j(x, x + z) e^{-V(x+z)} - j(x, x - z) e^{-V(x-z)} \right) dz
\]
\[
= C_V \int_{\{|z|>\varepsilon\}} (f(x + z) - f(x)) j(x, x + z) e^{-V(x+z)} dz \\
- \frac{C_V}{2} \nabla f(x) \cdot \int_{\{\varepsilon<|z| \leq 1\}} z \left( j(x, x + z) e^{-V(x+z)} + j(x, x - z) e^{-V(x-z)} \right) dz
= : L_{1,\varepsilon} f(x) + L_{2,\varepsilon} f(x).
\]
Since for any \( x \in \mathbb{R}^d \) and \( z \in \mathbb{R}^d \) with \( |z| \geq \varepsilon \),
\[
j(x, x + z) \leq \left( \frac{|z|^2}{\varepsilon^2} \wedge 1 \right) j(x, x + z),
\]
the condition \( I_1(x) \) is locally bounded implies that \( L_{i,\varepsilon} f \), for \( i = 1, 2 \) and any \( \varepsilon \in (0,1) \), are well defined and locally bounded. By the change of variable from \( z \) to \(-z\) and the symmetric property of \( j(x,y) \), we have \( L_{2,\varepsilon} f(x) = 0 \) for all \( x \in \mathbb{R}^d \).
That is, \( L_{j,V,\varepsilon} = L_{1,\varepsilon} \). Hence, for each \( f, g \in C_c^\infty(\mathbb{R}^d) \),
\[
- \int L_{j,V,\varepsilon} f(x) g(x) \mu_V(dx) = - \int L_{1,\varepsilon} f(x) g(x) \mu_V(dx)
= - \int \int_{\{|x-y|>\varepsilon\}} (f(y) - f(x)) g(x) j(x,y) \mu_V(dy) \mu_V(dx).
\]
Changing the position of \( x \) and \( y \), it holds that
\[
- \int L_{j,V,\varepsilon} f(x) g(x) \mu_V(dx) = \int \int_{\{|x-y|>\varepsilon\}} (f(y) - f(x)) g(y) j(x,y) \mu_V(dy) \mu_V(dx),
\]
Therefore, combining two equalities above, we have
\[
- \int L_{j,V,\varepsilon} f(x) g(x) \mu_V(dx) = \frac{1}{2} \int \int_{\{|x-y|>\varepsilon\}} (f(y) - f(x)) (g(y) - g(x)) j(x,y) \mu_V(dy) \mu_V(dx).
\] (2.23)
By the mean value theorem and (2.19), in the support of \( g \) the function \( L_{j,V,\varepsilon} f(x) \) is uniformly bounded for any \( \varepsilon \in (0,1) \). Thus, the dominated convergence theorem yields that
\[
\lim_{\varepsilon \to 0} \int L_{j,V,\varepsilon} f(x) g(x) \mu_V(dx) = \int L_{j,V} f(x) g(x) \mu_V(dx)
\]
Consider the following form
\[ \lim_{\varepsilon \to 0} \frac{1}{2} \int_{|x-y|>\varepsilon} (f(y) - f(x))(g(y) - g(x)) j(x,y) \mu_V(dy) \mu_V(dx) = D_{j,V}(f,g). \]
Then, letting \( \varepsilon \to 0 \) in (2.23), we prove the conclusion of part (2).

(c) For the part (3), the proof is almost the same as that of [15, Lemma 2.1], but we can get an improvement of the conclusion by a minor modification. In fact, for each \( f \in C_c^\infty(\mathbb{R}^d) \), there is a constant \( r > r_0 \) such that \( \text{supp}(f) \subseteq B(0,r) \). As mentioned in the proof of part (2) above, under (2.19) the function \( L_{j,V} f \) is locally bounded, and so \( \|1_{B(0,r)}L_{j,V}f\|_{L^2(\mu_V)} < \infty \). On the other hand, since for any \( |x| > r, f(x) = 0 \) and \( \nabla f(x) = 0 \), it follows from (2.20) that
\[
\left| 1_{(B(0,2r))^c}(x)L_{j,V}f(x) \right| = \left| C_V 1_{(B(0,2r))^c}(x) \int f(x+z) j(x,x+z)e^{-V(x+z)} \, dz \right| \\
\leq C_V \|f\|_\infty 1_{(B(0,2r))^c}(x) \int 1_{B(0,r)}(x+z) j(x,x+z)e^{-V(x+z)} \, dz \\
= C_V \|f\|_\infty 1_{(B(0,2r))^c}(x) \int_{|z+x| \leq r} j(x,x+z)e^{-V(x+z)} \, dz,
\]
Then, by (2.21), we get that for any \( r > r_0 \), \( \|1_{(B(0,2r))^c}L_{j,V}f\|_{L^2(\mu_V)} < \infty \). This completes the proof. \( \square \)

Under (2.17), let \( (D_{j,V},\mathcal{E}(D_{j,V})) \) be the closure of \( (D_{j,V},C_c^\infty(\mathbb{R}^d)) \) under norm \( \| \cdot \|_{D_{j,V},1} \) on \( L^2(\mu_V) \), where \( \|f\|_{D_{j,V},1} := (\|f\|_{L^2(\mu_V)}^2 + D_{j,V}(f,f))^{1/2} \) for \( f \in C_c^\infty(\mathbb{R}^d) \). Then, the bilinear form \( (D_{j,V},\mathcal{E}(D_{j,V})) \) becomes a regular Dirichlet form on \( L^2(\mu_V) \). It holds that \( \mathcal{E}(D_{j,V}) \subseteq \mathcal{D}(D_{j,V}) \). However, those two domains may be different, and the Dirichlet form \( (D_{j,V},\mathcal{D}(D_{j,V})) \) may not be regular in generally. On the other hand, we note that under (2.19) the operator \( L_{j,V} \) does not necessarily map \( C_c^\infty(\mathbb{R}^d) \) into \( L^2(\mu_V) \), though it is well defined in the sense of pointwise on \( C_c^\infty(\mathbb{R}^d) \). If moreover (2.21) holds, then the Friedrich extension of \( (L_{j,V},C_c^\infty(\mathbb{R}^d)) \) is a self-joint operator, which is the infinitesimal generator of the Dirichlet form \( (D_{j,V},\mathcal{E}(D_{j,V})) \).

2.2. Examples. In this part, we will present several examples as an application of Theorem 2.1. In all the examples, let \( V \) be a locally bounded function on \( \mathbb{R}^d \) such that \( \int e^{-V(x)} \, dx < \infty \) and \( e^{-V} \) is bounded in \( \mathbb{R}^d \). Let \( \mu_V(dx) = C_V e^{-V(x)} \, dx \) be a probability measure on \( (\mathbb{R}^d,\mathcal{B}(\mathbb{R}^d)) \).

Example 2.2. Let \( \rho \) be a positive measurable function on \( \mathbb{R}_+ := (0,\infty) \) such that
\[
\int_{(0,\infty)} \rho(r)(1 \wedge r^2)r^{d-1} \, dr < \infty.
\]
Consider the following form
\[
D_{\rho,V}(f,g) := \frac{1}{2} \iint (f(y) - f(x))(g(y) - g(x)) \rho(|x-y|) \, dy \mu_V(dx),
\]
\[
\mathcal{D}(D_{\rho,V}) := \left\{ f \in L^2(\mu_V) : D_{\rho,V}(f,f) < \infty \right\}.
\]
Then, \((D_{\rho,V}, \mathcal{D}(D_{\rho,V}))\) is a symmetric Dirichlet form on \(L^2(\mu_V)\) such that \(C_b^\infty(\mathbb{R}^d) \subset \mathcal{D}(D_{\rho,V})\). Moreover, if \(e^{-V} \in C_b^1(\mathbb{R}^d)\), then for any \(f, g \in C_c^\infty(\mathbb{R}^d)\),

\[
D_{\rho,V}(f,g) = -\int g L_{\rho,V} f \, d\mu_V,
\]

where

\[
L_{\rho,V} f(x) = \frac{1}{2} \left[ \int \left( f(x + z) - f(x) - \nabla f(x) \cdot z \mathbf{1}_{\{|z| \leq 1\}} \right) \rho(|z|) \left( e^{V(x) - V(x+z)} + 1 \right) \, dz 
+ \nabla f(x) \cdot \int_{\{|z| \leq 1\}} z \rho(|z|) \left( e^{V(x) - V(x+z)} - 1 \right) \, dz \right].
\]

Additionally, if for \(r\) big enough,

\[
\int_{|x| \geq 2r} \left( \sup_{x \in \mathbb{R}^d : |x| \geq |x| - r} \rho(|z|) \right)^2 e^{V(x)} \, dx < \infty,
\]

then \(L_{\rho,V}\) maps \(C_c^\infty(\mathbb{R}^d)\) into \(L^2(\mu_V)\).

**Proof.** By changing the position of \(x, y\) and the symmetric property of \(D_{\rho,V}\), it is easy to see that for any \(f, g \in C_c^\infty(\mathbb{R}^d)\),

\[
D_{\rho,V}(f,g) = \frac{1}{2} \iint \left( f(y) - f(x) \right) \left( g(y) - g(x) \right) j(x,y) \mu_V(dy) \mu_V(dx),
\]

where

\[
j(x,y) = \frac{1}{2C_V} \rho(|x-y|) (e^{V(x)} + e^{V(y)}).
\]

First, for all \(x \in \mathbb{R}^d\),

\[
I_1(x) = \frac{1}{2} \int \left( 1 \wedge |x - y|^2 \right) \rho(|x-y|) \left( e^{V(x) - V(y)} + 1 \right) \, dy,
\]

so

\[
\int I_1(x) \mu_V(dx) = \frac{C_V}{2} \left[ \iint \left( 1 \wedge |x - y|^2 \right) \rho(|x-y|) e^{-V(y)} \, dy \, dx 
+ \iint \left( 1 \wedge |x - y|^2 \right) \rho(|x-y|) e^{-V(y)} \, dy \, dx \right] = C_V \int \left( 1 \wedge |x - y|^2 \right) \rho(|x-y|) \, dy e^{-V(x)} \, dx
\]

Then, according to (2.24) and \(e^{-V} \in L^1(dx)\),

\[
\int I_1(x) \mu_V(dx) < \infty,
\]

i.e. (2.18) is true. According to Theorem 2.1 (1), \((D_{\rho,V}, \mathcal{D}(D_{\rho,V}))\) is a symmetric Dirichlet form on \(L^2(\mu_V)\) such that \(C_b^\infty(\mathbb{R}^d) \subset \mathcal{D}(D_{\rho,V})\).

Second, since \(e^{-V}\) is bounded and \(V\) is locally bounded, by (2.28) and (2.24), it is easy to check \(I_1(x)\) is locally bounded. On the other hand,

\[
I_2(x) = \frac{1}{2C_V} \int_{\{|z| \leq 1\}} |z| \rho(|z|) \left| e^{V(x) - V(x+z)} - e^{V(x) - V(x-z)} \right| \, dz,
\]
Due to the fact $e^{-V} \in C_b^1 (\mathbb{R}^d)$, it follows from (2.24) and the mean value theorem that $I_2(x)$ is locally bounded. Thus, (2.19) holds. Therefore, according to Theorem 2.1 (2), we know that for any $f, g \in C_c^\infty (\mathbb{R}^d)$,

$$D_{\rho,V} (f,g) = - \int g L_{\rho,V} f \, d\mu_V,$$

where

$$L_{\rho,V} f(x) = \frac{1}{2} \left[ \int \left( f(x + z) - f(x) - \nabla f(x) \cdot z \mathbf{1}_{\{|z| \leq 1\}} \right) \rho(|z|) \left( e^{V(x) - V(x + z)} + 1 \right) \, dz + \frac{1}{2} \nabla f(x) \cdot \int_{\{|z| \leq 1\}} z \rho(|z|) \left( e^{V(x) - V(x + z)} - e^{V(x) - V(x - z)} \right) \, dz \right].$$

Combining it with the fact

$$\int_{\{|z| \leq 1\}} z \rho(|z|) \left( e^{V(x) - V(x + z)} - e^{V(x) - V(x - z)} \right) \, dz$$

yields the required assertion (2.26).

Finally, for $r$ large enough, we have

$$I_{3,r}(x) = \frac{1}{2C_V} \mathbf{1}_{(B(0,2r))^c}(x) \int_{\{|x+z| \leq r\}} \rho(|z|) \left( 1 + e^{V(x) - V(x + z)} \right) \, dz,$$

According to (2.24) and the facts that $e^{-V}$ is bounded and for any $|x| \geq 2r$ and $|x + z| \leq r$,

$$|z| \geq |x| - |x + z| \geq |x| - r \geq r,$$

we get that for $r > 0$ large enough, there exist $c_i := c_i (r) > 0$ ($i = 1, 2, 3$) such that

$$I_{3,r}(x) \leq c_1 \mathbf{1}_{(B(0,2r))^c}(x) \left[ \int_{\{|z| \geq r\}} \rho(|z|) \, dz + e^{V(x)} \left( \int_{\{|x+z| \leq r\}} e^{-V(x + z)} \, dz \right) \left( \sup_{|x+z| \leq r} \rho(|z|) \right) \right]$$

$$\leq c_2 \mathbf{1}_{(B(0,2r))^c}(x) \left[ 1 + e^{V(x)} \left( \int_{\{|z| \leq r\}} e^{-V(x)} \, dz \right) \left( \sup_{|z| \leq r} \rho(|z|) \right) \right]$$

$$\leq c_3 \mathbf{1}_{(B(0,2r))^c}(x) \left[ 1 + e^{V(x)} \left( \sup_{|z| \geq r} \rho(|z|) \right) \right],$$

which, along with (2.27), yields that $I_{3,r} \in L^2 (\mu_V)$. Hence, by Theorem 2.1 (3), we know that $L_{\rho,V} f \in L^2 (\mu_V)$ for every $f \in C_c^\infty (\mathbb{R}^d)$. \qed
Example 2.3. Let $\psi$ be a positive measurable function on $(0, \infty)$ satisfying
\begin{equation}
\int \psi(r)(1 \wedge r^2)r^{d-1}dr < \infty. \tag{2.29}
\end{equation}
Consider the following form
\begin{equation}
D_{\psi,V}(f,g) := \frac{1}{2}\int \left((f(y) - f(x))(g(y) - g(x)) \times \psi(|x-y|)e^{-V(y)}dy e^{-V(x)}dx,
\end{equation}
\begin{equation}
\mathcal{D}(D_{\psi,V}) := \left\{ f \in L^2(\mu_{2V}) : D_{\psi,V}(f,f) < \infty \right\}. \tag{2.30}
\end{equation}
Then, $(D_{\psi,V}, \mathcal{D}(D_{\psi,V}))$ is a symmetric Dirichlet form on $L^2(\mu_{2V})$ such that $C^\infty_b(\mathbb{R}^d) \subset \mathcal{D}(D_{\psi,V})$. Moreover, if $e^{-V} \in C^1_c(\mathbb{R}^d)$, then for any $f, g \in C^\infty_c(\mathbb{R}^d)$,
\begin{equation}
D_{\psi,V}(f,g) = -\int gL_{\psi,V}f d\mu_{2V},
\end{equation}
where
\begin{equation}
L_{\psi,V}f(x) = \frac{1}{C_{2V}}\left[ \int (f(x+z) - f(x) - \nabla f(x) \cdot z\mathbb{1}_{\{|z| \leq 1\}})\psi(|z|)e^{V(x)-V(x+z)}dz + \nabla f(x) \cdot \int_{\{|z| \leq 1\}} z\psi(|z|)(e^{V(x)-V(x+z)} - 1)dz \right]. \tag{2.31}
\end{equation}
Additionally, if for $r$ big enough,
\begin{equation}
\int_{\{|x| > r\}} \left( \sup_{|z| > |x|-r} \psi(|z|) \right)^2 dx < \infty, \tag{2.32}
\end{equation}
then $L_{\psi,V}f \in L^2(\mu_{2V})$ for each $f \in C^\infty_c(\mathbb{R}^d)$.

Proof. It is easy to see that for any $f, g \in C^\infty_c(\mathbb{R}^d)$,
\begin{equation}
D_{\psi,V}(f,g) = \frac{1}{2}\int \left( (f(x) - f(y))(g(x) - g(y))j(x,y) \mu_{2V}(dy) \mu_{2V}(dx),
\end{equation}
where
\begin{equation}
j(x,y) = \frac{1}{C_{2V}^2}\psi(|x-y|)(e^{V(x)+V(y)}).
\end{equation}
Then,
\begin{equation}
I_1(x) = \frac{1}{C_{2V}^2}\int (1 \wedge |x-y|^2)\psi(|x-y|)e^{V(x)-V(y)}dy,
\end{equation}
\begin{equation}
I_2(x) = \frac{1}{C_{2V}^2}\int_{\{|z| \leq 1\}} |z|\psi(|z|)\left|e^{V(x)-V(x+z)} - e^{V(x)-V(x-z)} \right|dz,
\end{equation}
Since $e^{-V}$ is bounded, $e^{-V} \in L^1(dx)$ and $V$ is locally bounded, it follows from (2.29) that (2.18) holds and $I_1(x)$ is locally bounded. If $e^{-V} \in C^1_b(\mathbb{R}^d)$, by (2.29) and the mean value theorem, we can check that $I_2(x)$ is also locally bounded. According to Theorem 2.1 (1) and (2), we know that $(D_{\psi,V}, \mathcal{D}(D_{\psi,V}))$ is a symmetric Dirichlet form on $L^2(\mu_{2V})$ such that $C^\infty_b(\mathbb{R}^d) \subset \mathcal{D}(D_{\psi,V})$, and for any $f, g \in C^\infty_c(\mathbb{R}^d)$,
\begin{equation}
D_{\psi,V}(f,g) = -\int gL_{\psi,V}f d\mu_{2V},
\end{equation}
where
\[ L_{\psi,V} f(x) = \frac{1}{C_{2V}} \left[ \int (f(x + z) - f(x) - \nabla f(x) \cdot z \mathbb{1}_{\{|z| \leq 1\}}) \psi(|z|) e^{V(x) - V(x+z)} \, dz 
+ \frac{1}{2} \nabla f(x) \cdot \int_{\{|z| \leq 1\}} z \psi(|z|) \left( e^{V(x) - V(x+z)} - e^{V(x) - V(x-z)} \right) \, dz \right]. \]

Combining this with the fact that
\[
\int_{\{|z| \leq 1\}} z \psi(|z|) \left( e^{V(x) - V(x+z)} - e^{V(x) - V(x-z)} \right) \, dz = \int_{\{|z| \leq 1\}} z \psi(|z|) \left( (e^{V(x) - V(x+z)} - 1) - (e^{V(x) - V(x-z)} - 1) \right) \, dz
= 2 \int_{\{|z| \leq 1\}} z \psi(|z|) (e^{V(x) - V(x+z)} - 1) \, dz
\]
we get the required expression (2.31).

For \( r \) big enough, we have
\[
I_{3,r}(x) = \frac{1}{C_{2V}} \mathbb{1}_{(B(0,2r))'}(x) \int_{\{|z| \leq r\}} \psi(|z|) (e^{V(x) - V(x+z)}) \, dz,
\]
By the direct computation as in Example 2.2,
\[
I_{3,r}(x) \leq c_1 \mathbb{1}_{(B(0,2r))'}(x) \left[ e^{V(x)} \left( \int_{\{|z| \leq r\}} e^{-V(z)} \, dz \right) \left( \sup_{|z| \geq |x| - r} \psi(|z|) \right) \right]
\leq c_2 \mathbb{1}_{(B(0,2r))'}(x) \left[ e^{V(x)} \left( \sup_{|z| \geq |x| - r} \psi(|z|) \right) \right]
\]
holds for some constants \( c_i := c_i(r) \) \( (i = 1, 2) \). This, along with (2.32), implies that
\( I_{3,r} \in L^2(\mu_{2V}) \). According to Theorem 2.1 (3), we know that \( L_{\psi,V} f \in L^2(\mu_{2V}) \) for every \( f \in C_c^\infty(\mathbb{R}^d) \). \( \square \)

In the following example, we consider the same form \( D_{\psi,V}(f, g) \) as that in Example 2.3 on the space \( L^2(\mu_V) \), not \( L^2(\mu_{2V}) \).

**Example 2.4.** Let \( \psi \) be the same function as that in Example 2.3. Consider the form \( D_{\psi,V}(f, g) \) defined by (2.30), but with the following domain
\[
\mathcal{D}_2(D_{\psi,V}) := \left\{ f \in L^2(\mu_V) : D_{\psi,V}(f, f) < \infty \right\}.
\]
Then, the bilinear form \( (D_{\psi,V}, \mathcal{D}_2(D_{\psi,V})) \) is a symmetric Dirichlet form on \( L^2(\mu_V) \) such that \( C^\infty_b(\mathbb{R}^d) \subset \mathcal{D}_2(D_{\psi,V}) \). Furthermore, if \( e^{-V} \in C^1_b(\mathbb{R}^d) \), then for any \( f, g \in C_c^\infty(\mathbb{R}^d) \),
\[
D_{\psi,V}(f, g) = - \int g L_{\psi,V,2} f \, d\mu_V,
\]
where
\[
L_{\psi,V,2} f(x) = C_V \left[ \int \left( f(x + z) - f(x) - \nabla f(x) \cdot z \mathbb{1}_{\{|z| \leq 1\}} \right) \psi(|z|) e^{-V(x+z)} \, dz 
+ \nabla f(x) \cdot \int_{\{|z| \leq 1\}} z \psi(|z|) \left( e^{-V(x+z)} - e^{-V(x)} \right) \, dz \right].
\]
Moreover, for each \( f \in C^\infty_c(\mathbb{R}^d) \), \( L_{\psi,V,2} f \in L^2(\mu_V) \).
Proof. It is easy to see that for any $f, g \in C_c^\infty(\mathbb{R}^d)$,
\[
D_{\psi,V}(f, g) = \frac{1}{2} \iint (f(x) - f(y))(g(x) - g(y)) j(x, y) \mu_V(dy) \mu_V(dx),
\]
where
\[
j(x, y) = \psi(|x - y|).
\]
Following the computation in Example 2.2, we can check that under (2.29) and the condition $e^{-V} \in C_b^1(\mathbb{R}^d)$, the statements (2.18) and (2.19) hold. Then, applying Theorem 2.1 (1) and (2), we know that $(D_{\psi,V}, \mathcal{D}(D_{\psi,V}))$ is a symmetric Dirichlet form on $L^2(\mu_V)$ such that $C_b^\infty(\mathbb{R}^d) \subset \mathcal{D}(D_{\psi,V})$, and for any $f, g \in C_c^\infty(\mathbb{R}^d)$,
\[
D_{\psi,V}(f, g) = -\int g L_{\psi,V} f d\mu_V,
\]
where
\[
L_{\psi,V} f(x) &= C_V \left[ \int \left( f(x + z) - f(x) - \nabla f(x) \cdot z 1_{\{|z|\leq 1\}} \right) \psi(|z|) e^{-V(x+z)} dz \\
&\quad + \nabla f(x) \cdot \int_{\{|z|\leq 1\}} z \psi(|z|) (e^{-V(x+z)} - e^{-V(x)}) dz \right].
\]
On the other hand, by direct computation,
\[
I_{3,r}(x) = 1_{(B(0,2r))'}(x) \int_{\{|x+z|\leq r\}} \psi(|z|) e^{-V(x+z)} dz,
\]
which is bounded with respect to $x$, and hence $I_{3,r} \in L^2(\mu_V)$. It follows from Theorem 2.1 (3) that the operator $L_{\psi,V} f d\mu_V$ maps $C_c^\infty(\mathbb{R}^d)$ into $L^2(\mu_V)$. 

In order to drive weighted functional inequalities in the next two sections, we consider the following examples about truncated Dirichlet forms. The proof is similar to that of Examples 2.2 and 2.3, and we omit the details here.

**Example 2.5. (Truncated Dirichlet Form for $(D_{\rho,V}, \mathcal{D}(D_{\rho,V}))$)** Let $\rho$ be the same function as that in Example 2.2 such that (2.24) is satisfied. For all $r > 0$, define $\hat{\rho}(r) := \rho(r) 1_{\{r > 1\}}$. Consider the following truncated form
\[
\hat{D}_{\rho,V}(f, g) := \frac{1}{2} \iint (f(y) - f(x))(g(y) - g(x)) \hat{\rho}(|x - y|) dy \mu_V(dx) \\
= \frac{1}{2} \iint_{\{|x-y|>1\}} (f(y) - f(x))(g(y) - g(x)) \rho(|x - y|) dy \mu_V(dx),
\]
\[
\mathcal{D}(\hat{D}_{\rho,V}) := \left\{ f \in L^2(\mu_V) : \hat{D}_{\rho,V}(f, f) < \infty \right\}.
\]
Then, $(\hat{D}_{\rho,V}, \mathcal{D}(\hat{D}_{\rho,V}))$ is a symmetric Dirichlet form on $L^2(\mu_V)$ such that $C_b^\infty(\mathbb{R}^d) \subset \mathcal{D}(\hat{D}_{\rho,V})$, and for any $f, g \in C_c^\infty(\mathbb{R}^d)$,
\[
(2.33) \quad \hat{D}_{\rho,V}(f, g) = -\int g \hat{L}_{\rho,V} f d\mu_V,
\]
where
\[
\hat{L}_{\rho,V} f(x) = \frac{1}{2} \int_{\{|z|>1\}} (f(x+z) - f(x)) \rho(|z|) (e^{V(x)} e^{-V(x+z)} + 1) dz.
\]
Furthermore, if (2.27) holds, then $\hat{L}_{\rho,V} f \in L^2(\mu_V)$ for each $f \in C_c^\infty(\mathbb{R}^d)$. 

Remark 2.6. For the truncated Dirichlet form \((\tilde{D}_{\rho,V}, \mathcal{D}(\tilde{D}_{\rho,V}))\), in order to let (2.33) hold, we do not need the condition \(e^{-V} \in C^\alpha_0(\mathbb{R}^d)\) as that in Example 2.2. The reason is as follows: for truncated Dirichlet form \((\tilde{D}_{\rho,V}, \mathcal{D}(\tilde{D}_{\rho,V}))\), \(I_1(x) = 0\), and so it suffices to check that \(I_1(x)\) belongs to \(L^1(\mu_V)\) and is locally bounded. Hence, according to the proof of Example 2.2, the condition \(e^{-V} \in C^\alpha_0(\mathbb{R}^d)\) is not necessary, and the assumption that \(e^{-V}\) is bounded and \(V\) is locally bounded is enough to ensure the function \(I_1(x)\) belongs to \(L^1(\mu_V)\) and is locally bounded.

Example 2.7. (Truncated Dirichlet Form for \((D_{\psi,V}, \mathcal{D}(D_{\psi,V}))\)) Let \(\psi\) be the same function as that in Example 2.3 such that (2.29) holds. For any \(r > 0\), define \(\hat{\psi}(r) := \psi(r) \mathbb{1}_{\{r > 1\}}\) and a truncated form as follows

\[
\tilde{D}_{\psi,V}(f,g) := \frac{1}{2} \int \int (f(y) - f(x))(g(y) - g(x)) \hat{\psi}(|x-y|) e^{-V(y)} dy e^{-V(x)} dx
\]

\[
= \frac{1}{2} \int \int_{\{|x-y| > 1\}} (f(y) - f(x))(g(y) - g(x)) \hat{\psi}(|x-y|) e^{-V(y)} dy e^{-V(x)} dx,
\]

\[\mathcal{D}(\tilde{D}_{\psi,V}) := \left\{ f \in L^2(\mu_{2V}) : \tilde{D}_{\psi,V}(f,f) < \infty \right\}.
\]

Then, \((\tilde{D}_{\psi,V}, \mathcal{D}(\tilde{D}_{\psi,V}))\) is a symmetric Dirichlet form on \(L^2(\mu_{2V})\), which satisfies that \(C^\infty_c(\mathbb{R}^d) \subset \mathcal{D}(\tilde{D}_{\psi,V})\), and for any \(f, g \in C^\infty_c(\mathbb{R}^d)\),

\[
\tilde{D}_{\psi,V}(f,g) = -\int g \tilde{L}_{\psi,V} f d\mu_{2V}.
\]

Here,

\[
\tilde{L}_{\psi,V} f(x) = \frac{1}{C_{2V}} \int_{\{|z| > 1\}} (f(x+z) - f(x)) \hat{\psi}(|z|) e^{V(x-V(x+z))} dz,
\]

In particular, if (2.32) holds, then \(\tilde{L}_{\psi,V} f \in L^2(\mu_{2V})\) for each \(f \in C^\infty_c(\mathbb{R}^d)\).

Remark 2.8. The Dirichlet form in Example 2.3 is associated with symmetric Markov processes under Girsanov transform of pure jump type, see [6, Theorem 3.4] and [10, Theorem 3.1]. The Dirichlet form in Example 2.4 was first introduced in [15] to study the symmetric property of Lévy type operators. In particular, if \(\psi(r) = r^{-(d+\alpha)}\) for constant \(\alpha \in (0,2)\) in Example 2.4, then the Dirichlet form \((D_{\psi,V}, \mathcal{D}(D_{\psi,V}))\) has a non-local expression of the Gagliardo semi-norms for fractional Sobolev spaces \(W^{\alpha/2,2}(\mathbb{R}^d)\) of order \(\alpha/2\), see [8]. Different from Examples 2.3 and 2.4, the Dirichlet form appearing in Example 2.2 is of the “not symmetric” expression.

3. Weighted Poincaré Inequalities for General Non-local Dirichlet Forms via Lyapunov Conditions

Let \(j\) be a nonnegative and symmetric jump kernel on \(\mathbb{R}^{2d} \setminus \{(x, y) \in \mathbb{R}^d : x = y\}\), and \(\mu_V\) be a probability measure on \((\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))\) such that (2.18) and (2.19) hold. Let \((D_{j,V}, \mathcal{E}(D_{j,V}))\) be the regular Dirichlet form given in the paragraph below the proof of Theorem 2.1. In order to consider weighted Poincaré inequalities for \((D_{j,V}, \mathcal{E}(D_{j,V}))\), we start with the truncated Dirichlet form \((\tilde{D}_{j,V}, \mathcal{E}(\tilde{D}_{j,V}))\) corresponding to \((D_{j,V}, \mathcal{E}(D_{j,V}))\); namely, for any \(f, g \in C^\infty_c(\mathbb{R}^d)\),

\[
\tilde{D}_{j,V}(f,g) := \frac{1}{2} \int \int_{\{|x-y| > 1\}} (f(x) - f(y))(g(x) - g(y)) j(x,y) \mu_V(dy) \mu_V(dx),
\]
and $\mathcal{E}(\hat{D}_{j,V})$ is the closure of $C_b^\infty(\mathbb{R}^d)$ under the norm
\[
\|f\|_{\hat{D}_{j,V}} := \left(\|f\|_{L^2(\mu_V)}^2 + \hat{D}_{j,V}(f,f)\right)^{1/2}.
\]

According to the proof of Theorem 2.1 and the paragraph below it (also see Example 2.5), for any $f, g \in C_c^\infty(\mathbb{R}^d)$,
\[
\hat{D}_{j,V}(f,g) = -\int g\hat{L}_{j,V}f\,d\mu_V,
\]
where
\[
\hat{L}_{j,V}f(x) = \int_{\{|x-y|>1\}} (f(y) - f(x))j(x,y)\mu_V(dy).
\]

Denote by $\hat{\Gamma}_{j,V}(f,g)$ the carré de champ of $(\hat{L}_{j,V}, C_c^\infty(\mathbb{R}^d))$, i.e., for any $f, g \in C_c^\infty(\mathbb{R}^d)$,
\[
\hat{\Gamma}_{j,V}(f,g)(x) := \hat{L}_{j,V}(fg)(x) - g(x)\hat{L}_{j,V}f(x) - f(x)\hat{L}_{j,V}g(x).
\]
It is easy to check that for each $f, g \in C_c^\infty(\mathbb{R}^d)$,
\[
\hat{\Gamma}_{j,V}(f,g)(x) = \int_{\{|x-y|>1\}} (f(x) - f(y))(g(x) - g(y))j(x,y)\mu_V(dy).
\]
Furthermore, due to (2.18) and (2.19), we know that for every $f, g \in C_c^\infty(\mathbb{R}^d)$, $\hat{L}_{j,V}f$ and $\hat{\Gamma}_{j,V}(f,g)$ are well defined and locally bounded.

In order to apply Lyapunov functions (which usually are unbounded) to $\hat{L}_{j,V}$, we need the following bigger domain associated with $\hat{L}_{j,V}$ and $\hat{\Gamma}_{j,V}$:
\[
\mathcal{E}_{j,V} := \left\{ \phi \in C^\infty(\mathbb{R}^d) : x \mapsto \int_{\{|x-y|>1\}} |\phi(y)|j(x,y)\mu_V(dy) \text{ is locally bounded}\right\}.
\]

First, by (3.34), (3.35), (3.36) and some direct computation, we have the following

**Lemma 3.1.** For each $\phi \in \mathcal{E}_{j,V}$ and $f \in C_c^\infty(\mathbb{R}^d)$, both $\hat{L}_{j,V}\phi$ and $\hat{\Gamma}_{j,V}(f,\phi)$ are pointwise well defined by (3.34) and (3.35), respectively; moreover, both of them are locally bounded, and
\[
\hat{\Gamma}_{j,V}(f,\phi)(x) = \int_{\{|x-y|>1\}} (f(x) - f(y))(\phi(x) - \phi(y))j(x,y)\mu_V(dy)
\]

Next, we shall use Lyapunov type conditions for the operator $\hat{L}_{j,V}$. These conditions are known to yield functional inequalities for Markov processes, e.g. Poincaré inequalities for diffusion processes, see [1], and super-Poincaré inequalities for symmetric Markov processes, see [3]. The following lemma further shows that Lyapunov type conditions imply functional inequalities for non-local symmetric Dirichlet forms. The proof follows some method from [16, Theorem 2.1].

**Proposition 3.2.** Suppose that there exist two positive functions $\phi \in \mathcal{E}_{j,V}$, $h \in C^\infty(\mathbb{R}^d)$, and two constants $b, r > 0$ such that for all $x \in \mathbb{R}^d$,
\[
\hat{L}_{j,V}\phi(x) \leq -h(x) + b1_{B(0,r)}(x).
\]

(3.37)
Then for any \( f \in C_\infty^c(\mathbb{R}^d) \),
\begin{equation}
(3.38) \quad \int f^2 h\phi^{-1} \, d\mu_V \leq \hat{D}_{j,V}(f, f) + b \int_{B(0,r)} f^2 \phi^{-1} \, d\mu_V.
\end{equation}

Moreover, if there is a constant \( c > 0 \) such that \( \phi \geq c \), then the inequality (3.38) still holds for every \( f \in C_b^\infty(\mathbb{R}^d) \).

**Proof.** (a) We first consider the case for \( f \in C_\infty^c(\mathbb{R}^d) \). By Theorem 2.1 and Lemma 3.1, for any \( f \in C_\infty^c(\mathbb{R}^d) \),
\begin{align*}
\hat{D}_{j,V}(f, f) &= - \int f \hat{L}_{j,V} f \, d\mu_V = - \int f \hat{L}_{j,V} \left( \frac{f}{\phi} \right) \, d\mu_V \\
&= - \int \left( \frac{f^2}{\phi} \hat{L}_{j,V} \phi + f \phi \hat{L}_{j,V} \left( \frac{f}{\phi} \right) + f \hat{\Gamma}_{j,V} \left( \frac{f}{\phi} \right) \right) \, d\mu_V \\
&= - \int f^2 \frac{\hat{L}_{j,V} \phi}{\phi} \, d\mu_V + \left[ \hat{D}_{j,V} \left( f \phi, \frac{f}{\phi} \right) - f \hat{\Gamma}_{j,V} \left( \frac{f}{\phi}, \phi \right) \right] \, d\mu_V \\
&= - \int f^2 \frac{\hat{L}_{j,V} \phi}{\phi} \, d\mu_V + \left( \frac{1}{2} \int \hat{\Gamma}_{j,V} \left( f \phi, \frac{f}{\phi} \right) \, d\mu_V - \int f \hat{\Gamma}_{j,V} \left( \frac{f}{\phi}, \phi \right) \, d\mu_V \right) \\
&= \hat{J}_1(f) + \hat{J}_2(f),
\end{align*}

where in the equalities above we have used the facts that for \( f \in C_\infty^c(\mathbb{R}^d) \), \( f\phi^{-1} \in C_\infty^c(\mathbb{R}^d) \), and \( \hat{\Gamma}_{j,V} \left( \phi^{-1}, \phi \right) \) is well defined and locally bounded, thanks to \( \phi \in \mathcal{C}_j,V \).

Next, we will claim that \( \hat{J}_2(f) \geq 0 \). If this holds, then we get
\begin{equation}
(3.39) \quad \hat{D}_{j,V}(f, f) \geq - \int f^2 \frac{\hat{L}_{j,V} \phi}{\phi} \, d\mu_V,
\end{equation}

which, along with (3.37), immediately yields the inequality (3.38) for every \( f \in C_\infty^c(\mathbb{R}^d) \).

To prove \( \hat{J}_2(f) \geq 0 \), it suffices to verify that for all \( f \in C_\infty^c(\mathbb{R}^d) \),
\begin{equation*}
\hat{J}_3(f) := 2 \hat{J}_2(f \phi) = \int \hat{\Gamma}_{j,V} \left( f \phi^2, f \right) \, d\mu_V - 2 \int f \phi \hat{\Gamma}_{j,V} \left( f, \phi \right) \, d\mu_V \geq 0.
\end{equation*}

Note that
\begin{align*}
\hat{J}_3(f) &= \int_{\{ |x-y| > 1 \}} (f(x) - f(y)) (f(x) \phi^2(x) - f(y) \phi^2(y)) j(x, y) \, d\mu_V(dy) \, d\mu_V(dx) \\
&\quad - 2 \int_{\{ |x-y| > 1 \}} f(x) \phi(x) (f(x) - f(y)) (\phi(x) - \phi(y)) j(x, y) \, d\mu_V(dy) \, d\mu_V(dx) \\
&= \int_{\{ |x-y| > 1 \}} (f(x) - f(y)) \\
&\quad \times \left( f(x) \phi^2(x) - f(y) \phi^2(y) - 2 f(x) \phi(x) (\phi(x) - \phi(y)) \right) j(x, y) \, d\mu_V(dy) \, d\mu_V(dx) \\
&= \int_{\{ |x-y| > 1 \}} (f(x) - f(y)) \\
&\quad \times \left( - f(x) \phi^2(x) - f(y) \phi^2(y) + 2 f(x) \phi(x) \phi(y) \right) j(x, y) \, d\mu_V(dy) \, d\mu_V(dx),
\end{align*}
In particular, due to (3.36), every item above is finite, and the case $\infty - \infty$ will not happen.

By the symmetric property that $j(x, y) \mu_V(dy) \mu_V(dx) = j(y, x) \mu_V(dy) \mu_V(dx)$, we have

\[
\hat{J}_3(f) = \int \int_{\{|x-y|>1\}} (f(y) - f(x)) \times \left( - f(y)\phi^2(y) - f(x)\phi^2(x) + 2f(y)\phi(y)\phi(x) \right) j(x, y) \mu_V(dy) \mu_V(dx)
\]

\[
= \int \int_{\{|x-y|>1\}} (f(x) - f(y)) \times \left( f(y)\phi^2(y) + f(x)\phi^2(x) - 2f(y)\phi(y)\phi(x) \right) j(x, y) \mu_V(dy) \mu_V(dx).
\]

Adding the two equalities above, we arrive at

\[
\hat{J}_3(f) = 2 \int \int_{\{|x-y|>1\}} (f(x) - f(y))^2 \phi(y)\phi(x) j(x, y) \mu_V(dy) \mu_V(dx) \geq 0.
\]

This proves (3.38) for every $f \in C_c^\infty(\mathbb{R}^d)$.

(b) For every $f \in C_c^\infty(\mathbb{R}^d)$, there is a sequence of functions $\{f_n\}_{n=1}^{\infty} \subseteq C_c^\infty(\mathbb{R}^d)$ such that

\[
\lim_{n \to \infty} f_n(x) = f(x), \quad \sup_n \|f_n\|_\infty < \infty, \quad \sup_n \|\nabla f_n\|_\infty < \infty.
\]

Let $F_n(x, y) := (f_n(x) - f_n(y))^2$. We have

\[
\hat{D}_{j,V}(f_n, f_n) = \frac{1}{2} \int \int_{\{|x-y|>1\}} F_n(x, y) j(x, y) \mu_V(dx) \mu_V(dy).
\]

Note that (3.40) implies that $\sup_n |F_n(x, y)| < \infty$. Thus, by (2.18) and the dominated convergence theorem, we get

\[
\lim_{n \to \infty} \hat{D}_{j,V}(f_n, f_n) = \hat{D}_{j,V}(f, f)
\]

Since $\phi \geq c > 0$, also due to the dominated convergence theorem,

\[
\lim_{n \to \infty} \int f_n^2 \phi^{-1} d\mu_V = \int f^2 \phi^{-1} d\mu_V.
\]

On the other hand, thanks to the Fatou lemma,

\[
\int f^2 h\phi^{-1} d\mu_V \leq \liminf_{n \to \infty} \int f_n^2 h\phi^{-1} d\mu_V.
\]

Since (3.38) holds for each $f_n$, letting $n$ tend to infinity and using the estimates above, we can show that (3.38) holds for $f \in C_c^\infty(\mathbb{R}^d)$.

\[\Box\]

**Remark 3.3.** (1) If $\phi \geq c > 0$, then, by taking $f = 1$ in (3.38), we have $\int h\phi^{-1} d\mu_V < \infty$.

(2) In the proof of Proposition 3.2 above, the main step is to show (3.39), which formally is the same as the conclusion of [3, Lemma2.12]. However, here we can not apply [3, Lemma2.12] directly, since the generator $(\hat{L}_{j,V}, C_c^\infty(\mathbb{R}^d))$ associated with $\hat{D}_{j,V}$ is not necessarily self-joint in $L^2(\mu_V)$, see the paragraph below the proof of Theorem 2.1.
Another ingredient is the following local Poincaré inequality:

**Lemma 3.4.** For any \( r > 0 \) and any \( f \in C_b^\infty(\mathbb{R}^d) \),
\[
(3.41) \quad \int_{B(0,r)} f^2 \, d\mu_V \leq \kappa_r D_{j,V}(f, f) + \mu_V(B(0, r))^{-1} \left( \int_{B(0,r)} f \, d\mu_V \right)^2,
\]
where
\[
(3.42) \quad \kappa_r = \frac{1}{\mu_V(B(0,r))^2} \sup_{x \in B(0,r)} \int_{B(0,r)} j(x,y)^{-1} \, \mu_V(dy).
\]

**Proof.** Note that, (3.41) is equivalent to
\[
\int_{B(0,r)} \left( f(x) - \frac{1}{\mu_V(B(0,r))} \int_{B(0,r)} f(x) \, \mu_V(dx) \right)^2 \, \mu_V(dx) \leq \kappa_r D_{j,V}(f, f).
\]
For any \( f \in C_b^\infty(\mathbb{R}^d) \), by the Cauchy-Schwarz inequality,
\[
\int_{B(0,r)} \left( f(x) - \frac{1}{\mu_V(B(0,r))} \int_{B(0,r)} f(x) \, \mu_V(dx) \right)^2 \, \mu_V(dx)
= \int_{B(0,r)} \left( \frac{1}{\mu_V(B(0,r))} \int_{B(0,r)} (f(x) - f(y)) \, \mu_V(dy) \right)^2 \, \mu_V(dx)
\leq \frac{1}{\mu_V(B(0,r))^2} \int_{B(0,r)} \left( \int_{B(0,r)} (f(x) - f(y))^2 j(x,y) \, \mu_V(dy) \right) \mu_V(dx)
\leq \frac{1}{\mu_V(B(0,r))^2} \sup_{x \in B(0,r)} \left( \int_{B(0,r)} j(x,y)^{-1} \, \mu_V(dy) \right) \mu_V(dx)
\times \int_{B(0,r)} \left( \int_{B(0,r)} (f(x) - f(y))^2 j(x,y) \, \mu_V(dy) \right) \mu_V(dx)
\leq \kappa_r D_{j,V}(f, f).
\]
The proof is complete. \( \Box \)

**Remark 3.5.** The item \( \int_{B(0,r)} j(x,y)^{-1} \, \mu_V(dy) \), and so the constant \( \kappa_r \), may not always be finite. For instance, if \( j(x,y) = 0 \) for any \( x, y \in \mathbb{R}^d \) with \( |x - y| \leq 1 \), then \( \int_{B(0,r)} j(x,y)^{-1} \, \mu_V(dy) = \infty \) for all \( x \in \mathbb{R}^d \) with \( |x| < r - 1 \).

Having Proposition 3.2 and Lemma 3.4 at hand, we are in position to present the main result in this section.

**Theorem 3.6.** Suppose that there exist \( \phi \in \mathcal{C}^{1,1}_j \) with \( \phi \geq 1 \), \( h \in C^\infty(\mathbb{R}^d) \) with \( h > 0 \) and two constants \( b, r_0 > 0 \) such that for all \( x \in \mathbb{R}^d \),
\[
(3.43) \quad \check{L}_{j,V}\phi(x) \leq -h(x) + b \mathbb{1}_{B(0, r_0)}(x).
\]
If \( \mu_V(\phi^{-1}) < \infty \) and the constant \( \kappa_\cdot \) defined by (3.42) satisfies \( \kappa_r < \infty \) for each \( r > 1 \), then there exists a constant \( C_1 > 0 \) such that for any \( f \in C_b^\infty(\mathbb{R}^d) \) with \( \int f \, d\mu_V = 0 \),
\[
(3.44) \quad \int f^2 \phi^{-1} \, d\mu_V \leq C_1 D_{j,V}(f, f).
\]
Remark 3.7. If \( \inf_{|x| \geq \eta}(h\phi^{-1})(x) > 0 \) for \( \eta \) large enough, then \( \sup_{|x| \geq \eta}(h^{-1}\phi)(x) < \infty \), which yields that \( \mu_V(h^{-1}\phi) < \infty \). In this case, the weighted Poincaré inequality (3.44) is stronger than the usual Poincaré inequality.

Proof of Theorem 3.6. According to (3.43) and Proposition 3.2, for any \( f \in \mathcal{C}_b^\infty(\mathbb{R}^d) \) with \( \int f \, d\mu_V = 0 \),

\[
\int f^2 h\phi^{-1} \, d\mu_V \leq D_{j,V}(f,f) + b \int_{B(0,r_0)} f^2 \phi^{-1} \, d\mu_V,
\]

where we have used the fact that \( \hat{D}_{j,V}(f,f) \leq D_{j,V}(f,f) \). Since \( \phi \geq 1 \), by the local Poincaré inequality (3.41), for any \( r \geq r_0 \),

\[
\int_{B(0,r)} f^2 \phi^{-1} \, d\mu_V \leq \int_{B(0,r)} f^2 \, d\mu_V \\
\leq \int_{B(0,r)} f^2 \, d\mu_V \\
\leq \kappa_r D_{j,V}(f,f) + \frac{1}{\mu_V(B(0,r))} \left( \int_{B(0,r)} f \, d\mu_V \right)^2 \\
= \kappa_r D_{j,V}(f,f) + \frac{1}{\mu_V(B(0,r))} \left( \int_{B(0,r)} f \, d\mu_V \right)^2,
\]

where in the equality above we have used the fact that

\[
\int_{B(0,r)} f \, d\mu_V = -\int_{B(0,r)^c} f \, d\mu_V.
\]

Using the Cauchy-Schwarz inequality, we find

\[
\left( \int_{B(0,r)^c} f \, d\mu_V \right)^2 \leq \left( \int_{B(0,r)^c} f^2 h\phi^{-1} \, d\mu_V \right) \left( \int_{B(0,r)^c} \phi h^{-1} \, d\mu_V \right),
\]

Therefore, for any \( f \in \mathcal{C}_b^\infty(\mathbb{R}^d) \) with \( \int f \, d\mu_V = 0 \) and \( r \geq r_0 \),

\[
\int f^2 h\phi^{-1} \, d\mu_V \leq (1 + b\kappa_r) D_{j,V}(f,f) + \frac{b \int_{B(0,r)^c} \phi h^{-1} \, d\mu_V}{\mu_V(B(0,r))} \int f^2 \phi^{-1} \, d\mu_V.
\]

Since \( \mu_V(\phi h^{-1}) < \infty \), we can choose \( r_1 \geq r_0 \) large enough such that

\[
\frac{b \int_{B(0,r_1)} \phi h^{-1} \, d\mu_V}{\mu_V(B(0,r_1))} \leq 1/2,
\]

which gives us the inequality (3.44) with \( C_1 = 2(1 + b\kappa_{r_1}) \).

Remark 3.8. In this section, we have showed the method on how to deduce functional inequalities for \( D_{j,V} \) from the Lyapunov conditions for the generator associated with truncated Dirichlet form \( \hat{D}_{j,V} \). In fact, we can also use the Lyapunov conditions for \( D_{j,V} \) itself (see [13]), but more technical conditions in the definition (3.36) for the class \( \hat{\mathcal{C}}_{j,V} \) are required. We will see in the next two sections that the Lyapunov conditions for truncated Dirichlet form is efficient in a number of applications.
4. Weighted Poincaré Inequalities for Non-local Dirichlet Forms

4.1. Weighted Poincaré Inequalities for Non-local Dirichlet Forms $D_{\rho,V}$

Throughout this section, we assume that $\rho : \mathbb{R}_+ := (0, \infty) \to \mathbb{R}_+$ satisfies (2.24), $\mu_V(dx) = C_V e^{-V(x)} dx$ is a probability measure such that $e^{-V}$ is bounded and $V$ is locally bounded, and $D_{\rho,V}$ is the bilinear form defined by (2.25). The following statement presents the criterion about weighted Poincaré inequalities for $D_{\rho,V}$.

**Theorem 4.1.** Suppose $\rho(r) : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function satisfying (2.24),

\[
\int_{\{r \leq 1\}} r^{d-1} \rho(r)^{-1} \, dr < \infty,
\]

and

\[
\int_{\{|x| > 1\}} \frac{e^{-2V(x)} \gamma(|x|)}{\gamma(|x|)} \, dx < \infty,
\]

where for $r > 0$,

\[
\gamma(r) := \inf_{0 < s \leq r+1} \rho(s).
\]

If there exists some constant $0 < \alpha_0 < 1$ such that

\[
\int_{\{r > 1\}} r^{d+\alpha_0-1} \rho(r) \, dr < \infty
\]

and

\[
\limsup_{|x| \to \infty} \sup_{|x| \geq |z|} \frac{e^{-V(z)}}{\gamma(|x|)|x|^\alpha_0} = 0,
\]

then there exists a constant $C_1 > 0$ such that the following weighted Poincaré inequality

\[
\left( \int (f(x) - \mu_V(f))^2 e^{V(x)} \gamma(|x|) \mu_V(dx) \right)^2 e^{V(x)} \gamma(|x|) \mu_V(dx) \leq C_1 D_{\rho,V}(f, f)
\]

holds for all $f \in C_0^\infty(\mathbb{R}^d)$.

To prove Theorem 4.1, we begin with the truncated form $\hat{D}_{\rho,V}$ associated with $D_{\rho,V}$. In particular, according to Example 2.5, we have

**Lemma 4.2.** For any $f, g \in C_c^\infty(\mathbb{R}^d)$,

\[- \int g(\hat{L}_{\rho,V} f) \, d\mu_V = \hat{D}_{\rho,V}(f, g),\]

where

\[
\hat{L}_{\rho,V} f(x) = \frac{1}{2} \int_{\{|z| > 1\}} \left( f(x + z) - f(x) \right) \rho(|z|) \left( e^{V(x)} - e^{V(x+z)} + 1 \right) \, dz.
\]

Next, we shall study Lyapunov type conditions for the operator $\hat{L}_{\rho,V}$. The crucial step is the proper choice of the Lyapunov function, see e.g. [14]. The following lemma is motivated by the proof of [13, Lemma 3.8], which was used to prove super Poincaré inequalities and Poincaré inequalities for fractional Dirichlet forms $D_{\alpha,V}$ (see [13, Theorem 3.6]).
Lemma 4.3. Let $\alpha_0$ be the constant in Theorem 4.1, and $\phi \in C^\infty(\mathbb{R}^d)$ be a function such that $\phi \geq 1$ and $\phi(x) = 1 + |x|^\alpha$ for $|x| > 1$. Suppose that (4.47) and (4.48) hold. Then $\hat{L}_{\rho,V}\phi(x)$ is well defined by (4.49) and locally bounded; moreover, there exist $r_0$, $C_1$ and $C_2 > 0$ such that

$$\hat{L}_{\rho,V}\phi(x) \leq -C_1 e^V(x)\gamma(|x|)\phi(x) + C_2 \mathbb{I}_{B(0,r_0)}(x),$$

where $\gamma(r) := \inf_{0 < s \leq r+1} \rho(s)$ as that in Theorem 4.1.

Proof. Throughout the proof, all the constants $c_i$ ($i \geq 0$) do not depend on $x$. Let $c_1 := \sup_{|x| \leq 1} \phi(x)$. We claim that

$$x \mapsto \int_{\{|x-y| > 1\}} |\phi(y)|\rho(|x-y|) \, dy$$

is locally bounded. In fact,

$$\int_{\{|x-y| > 1\}} |\phi(y)|\rho(|x-y|) \, dy \leq \int_{\{|x-y| > 1\}} (c_1 + 1 + |y|^{\alpha_0})\rho(|x-y|) \, dy$$

$$\leq \int_{\{|x-y| > 1\}} (c_1 + 1 + |x|^{\alpha_0} + |x-y|^{\alpha_0})\rho(|x-y|) \, dy$$

$$\leq c_2 (1 + |x|^{\alpha_0}),$$

where in the second inequality we have used the fact that

$$|x + y|^{\alpha_0} \leq (|x| + |y|)^{\alpha_0} \leq |x|^{\alpha_0} + |y|^{\alpha_0}, \quad \alpha_0 \in (0, 1), x, y \in \mathbb{R}^d,$$

and the last inequality follows from (4.47). Hence, the claim is true, and this yields the first conclusion of the Lemma.

In order to complete the proof, we only need to verify (4.50) for large values of $|x|$. For $|x|$ large enough,

$$\int_{\{|z| > 1\}} (\phi(x+z) - \phi(x)) \rho(|z|) \, dz \leq \int_{\{|z| > 1\}} (c_1 + |x+z|^{\alpha_0} - |x|^{\alpha_0}) \rho(|z|) \, dz$$

$$\leq \int_{\{|z| > 1\}} (c_1 + |z|^{\alpha_0}) \rho(|z|) \, dz$$

$$= c_3 < \infty.$$

Moreover, for $|x|$ large enough,

$$\int_{\{|z| > 1\}} (\phi(x+z) - \phi(x)) e^{V(x)-V(x+z)} \rho(|z|) \, dz$$

$$\leq e^{V(x)} \int_{\{|z| \leq 1\}} (c_1 - |x|^{\alpha_0}) e^{-V(x+z)}\rho(|z|) \, dz$$

$$+ e^{V(x)} \int_{\{|z| > 1, |x+z| \leq |x|\} \cup \{|z| > 1, |x+z| > |x|\}} (|x+z|^{\alpha_0} - |x|^{\alpha_0}) e^{-V(x+z)}\rho(|z|) \, dz$$

$$+ e^{V(x)} \int_{\{|z| > 1, |x+z| > |x|\}} (|x+z|^{\alpha_0} - |x|^{\alpha_0}) e^{-V(x+z)}\rho(|z|) \, dz$$

$$\leq e^{V(x)} \int_{\{|z| \leq 1\}} (c_1 - |x|^{\alpha_0}) e^{-V(x+z)}\rho(|z|) \, dz$$

$$+ e^{V(x)} \int_{\{|z| > 1, |x+z| > |x|\}} |z|^{\alpha_0} e^{-V(x+z)}\rho(|z|) \, dz$$
Here, in the first inequality we split the integral domain and use the facts that for $|x| \leq 1$ and $|x| > 1$ it is easy to check that under (4.47), $\varphi$ is well defined and locally bounded as follows:

$$\varphi(x) = \begin{cases} \frac{1}{2} \rho(|x|) (e^{V(x)} + e^{V(y)}) & |x| \leq 1 \\ e^{V(x)} & |x| > 1 \end{cases}$$

and

$$\varphi(x) = \begin{cases} \frac{1}{2} \rho(|x|) (e^{V(x)} + e^{V(y)}) & |x| \leq 1 \\ e^{V(x)} & |x| > 1 \end{cases}$$

in the second inequality we drop the second term in the first inequality since it is negative; in the fifth inequality we use (4.47) and the estimate that

$$\inf_{x:|x+z|\leq 1} \rho(|z|) \geq \inf_{z:|z|\leq |x|+1} \rho(|z|) = \gamma(|x|);$$

and the last inequality follows from (4.48).

Combining all the estimates above and using (4.49), we get the inequality (4.50) for $|x|$ large enough. This completes the proof. \hfill \Box

Now we give the proof of Theorem 4.1.

**Proof of Theorem 4.1.** From Example 2.2,

$$j(x,y) = \frac{1}{2C_V} \rho(|x-y|) (e^{V(x)} + e^{V(y)}),$$

where

$$C_V = \frac{1}{\int e^{V(x)} dx}.$$ 

By (3.36), we define the class of functions $C_{\rho,V}$ for $D_{\rho,V}$ as follows

$$C_{\rho,V} := \left\{ f \in C^\infty(\mathbb{R}^d) : x \mapsto \int_{|x-y| > 1} |f(y)| \rho(|x-y|) \left( 1 + e^{V(x-V(y))} \right) dy \right\}$$

is well defined and locally bounded.

Let $\phi \geq 1$ be the test function in Lemma 4.3. According to the proof of Lemma 4.3, it is easy to check that under (4.47), $\phi \in C_{\rho,V}$.

Due to (4.45), and the facts that $\rho(r)$ is positive and continuous on $r \in [1, \infty)$ and $V$ is locally bounded, $j(x,y)^{-1}$ is integrable in $B(0,2r) \times B(0,2r)$. Hence, for each $r > 0$, $\kappa_r$ defined by (3.42) is finite.
By Lemma 4.3, there exist \( r_0, C_1 \) and \( C_2 > 0 \) such that
\[
\hat{L}_{\rho,V} \phi(x) \leq -C_1 e^{V(x)} \gamma(|x|) \phi(x) + C_2 \mathds{1}_{B(0,r_0)}(x).
\]
That is, (3.43) holds with \( h \).

Note that (4.46) implies \( \mu_V(\phi h^{-1}) < \infty \). Therefore, the desired assertion follows from Theorem 3.6.

4.2. Weighted Poincaré Inequalities for \( D_{\alpha,V,\delta} \) with \( \delta > 0 \). We are now in a position to present the proof of Theorem 1.1.

Proof of Theorem 1.1. Choose \( \rho(r) = e^{-\delta r} r^{-(d+\alpha)} \) with \( \alpha \in (0,2) \) and \( \delta > 0 \) in Theorem 4.1. We know that (4.45) and (4.47) hold. It follows from (1.7) that (4.48) is satisfied. On the other hand, under (1.7), we know that for \( |x| \) large enough,
\[
e^{-V(x)} \leq \sup_{|x| \geq |z|} e^{-V(z)} \leq e^{-\delta|x|} |x|^{-d+\alpha_0}.
\]
Therefore, there exists a constant \( c_1 > 0 \) such that
\[
\int_{\{|x| \geq 1\}} |x|^{d+\alpha} e^{-2V(x)+\delta|x|} d\mu(x) \leq c_1 \int_{\{|x| \geq 1\}} |x|^{-d+2\alpha_0} e^{-\delta|x|} d\mu(x) < \infty,
\]
and so (4.46) also holds. Combining all the conclusions above, we get the required assertion.

Next, we turn to the proof of Proposition 1.3.

Proof of Proposition 1.3. For any \( n \geq 1 \), define \( g_n(x) := e^{\lambda(|x|/n)} \), where \( \lambda > 0 \) is a constant to be determined later. Clearly, \( g_n \) is a Lipschitz continuous and bounded function. By the approximation procedure in the proof of Proposition 3.2, we can apply the function \( g_n \) into the inequality (1.10). Thus,
\[
\int g_n^2(x) \mu_V(dx) \leq C_2 \int \int \frac{(g_n(x) - g_n(y))^2}{|y - x|^{d+\alpha}} e^{-\delta|y-x|} d\mu_V(dx) \mu_V(dx) + \left( \int g_n(x) \mu_V(dx) \right)^2.
\]
(4.51)

First, it holds for each \( N > 1 \) that
\[
\int \frac{(g_n(x) - g_n(y))^2}{|y - x|^{d+\alpha}} e^{-\delta|y-x|} dy \leq \int_{\{|y-x| \leq N\}} \frac{(e^{\lambda(|x|/n)} - e^{\lambda(|y|/n)})^2}{|y - x|^{d+\alpha}} e^{-\delta|y-x|} dy + \int_{\{|y-x| > N\}} \frac{(e^{\lambda(|x|/n)} - e^{\lambda(|y|/n)})^2}{|y - x|^{d+\alpha}} e^{-\delta|y-x|} dy
\]
\[= : J_{1,N}(x) + J_{2,N}(x).
\]
By the mean value theorem and the facts that for any \( x, y \in \mathbb{R}^d, n \geq 1 \),
\[
| |x| \wedge n - |y| \wedge n | \leq |x - y|,
\]
and
\[
\int \frac{1}{|z|^{d+\alpha-2}} e^{-\delta|z|} dz < \infty,
\]
holds for some constant $c_1 > 0$ independent of $n$, $\lambda$ and $N$. On the other hand, since

$$|x| \wedge n - |y| \wedge n \leq |x - y|,$$

it holds that

$$|y| \wedge n \leq |x| \wedge n + |x - y|.$$

Hence, choosing $\lambda \in (0, \delta/4)$, we obtain that for any $x \in \mathbb{R}^d$

$$J_{2,N}(x) \leq 2 \int_{|x-y|>N} \left( e^{2\lambda (|x| \wedge n)} + e^{2\lambda (|y| \wedge n)} \right) e^{-\delta |y-x|} \, dy$$

$$\leq 2 \int_{|x-y|>N} \frac{e^{2\lambda (|x| \wedge n)} + e^{2\lambda (|y| \wedge n)} e^{2\lambda |x-y|}}{|x-y|^{d+\alpha}} e^{-\delta |y-x|} \, dy$$

$$\leq 2 e^{2\lambda (|x| \wedge n)} \int_{|z|>|N|} \frac{1 + e^{\delta |z|/2}}{|z|^{d+\alpha}} e^{-\delta |z|} \, dz$$

$$\leq 2k(N) e^{2\lambda (|x| \wedge n)},$$

where

$$k(N) := \int_{|z|>|N|} \frac{1 + e^{\delta |z|/2}}{|z|^{d+\alpha}} e^{-\delta |z|} \, dz \leq k(1) < \infty.$$

Therefore,

$$\int \int \frac{(g_n(x) - g_n(y))^2}{|y-x|^{d+\alpha}} e^{-\delta |y-x|} \, dy \, \mu_V(dx) \leq (c_1 e^{2\lambda N} + 2k(N)) \int e^{2\lambda (|x| \wedge n)} \, \mu_V(dx).$$

Second, for any $n \geq 1$ and $\lambda > 0$, set

$$l_n(\lambda) := \int g_n^2(x) \, \mu_V(dx) = \int e^{2\lambda (|x| \wedge n)} \, \mu_V(dx).$$

Then, combining all the estimates above with (4.51), for each $\lambda \in (0, \delta/4)$,

$$l_n(\lambda) \leq C_2(c_1 \lambda e^{2\lambda N} + 2k(N)) l_n(\lambda) + l_n^2(\lambda/2).$$

Furthermore, using the Cauchy-Schwarz inequality, for any $R > 1$, we have

$$l_n^2(\lambda/2) \leq \left( e^{\lambda R} + \int_{|x|>R} e^{\lambda (|x| \wedge n)} \, \mu_V(dx) \right)^2 \leq 2e^{2\lambda R} + 2p(R) l_n(\lambda),$$

where $p(R) := \mu_V(|x| > R)$. Therefore, for each $N$, $R > 0$ and $\lambda \in (0, \delta/4)$,

$$l_n(\lambda) \leq C_2(c_1 \lambda e^{2\lambda N} + 2k(N)) l_n(\lambda) + 2e^{2\lambda R},$$

Now, we fix $R_0$ and $N_0 > 0$ large enough such that $p(R_0) < 1/8$ and $C_2k(N_0) < 1/8$, and then take $\lambda_0 \in (0, \delta/4)$ small enough such that $C_2c_1 \lambda_0 e^{2\lambda_0 N_0} < 1/4$. Then, by (4.53), we arrive at

$$l_n(\lambda_0) \leq 8e^{2\lambda_0 R_0}.$$

Letting $n \to \infty$, we obtain the desired assertion. \qed
4.3. Weighted Poincaré Inequalities for $D_{\alpha,V}$.

In the subsection, we will present the proofs of Theorem 1.4, Propositions 1.6 and 1.7.

**Proof of Theorem 1.4.** (a) Choose $\rho(r) = r^{-(d+\alpha)}$ with $\alpha \in (0, 2)$ in Theorem 1.1. It is easy to see that (4.45), (4.47) and (4.48) hold. On the other hand, under (1.11), we know that for $|x|$ large enough,

$$e^{-V(x)} \leq \sup_{|z| > |x|} e^{-V(z)} \leq |x|^{-d-\alpha+\alpha_0} ,$$

and so there is a constant $c_2 > 0$ such that for all $x \in \mathbb{R}^d$ with $|x| \geq 1$,

$$e^{-V(x)} \leq c_2 |x|^{-d-\alpha+\alpha_0} .$$

Therefore, since $\alpha_0 \in (0, \alpha/2)$,

$$\int_{\{|x| > 1\}} |x|^{d+\alpha} e^{-2V(x)} \, dx \leq c_2 \int_{\{|x| > 1\}} |x|^{-d-\alpha+2\alpha_0} \, dx < \infty .$$

That is, (4.46) also holds. The first required assertion follows from all the conclusions above.

(b) Now, we will verify the second assertion. Suppose that the inequality (1.12) holds with the weighted function $\omega^*(x)$. Then, for all $f \in C_b^\infty(\mathbb{R}^d)$,

$$\int (f(x) - \mu_V(f))^2 \omega^*(x) \mu_V(dx) \leq C_3 D_{\alpha,V}(f, f).$$

For any $n \geq 1$, choose a smooth function $f_n : \mathbb{R}^d \to [0, 1]$ such that

$$f_n(x) = \begin{cases} 0, & |x| \leq n; \\ 1, & |x| > 2n, \end{cases}$$

and $|\nabla f_n(x)| \leq 2n^{-1}$ for all $x \in \mathbb{R}^d$. Therefore, for all $x \in \mathbb{R}^d$,

$$\Gamma(f_n)(x) := \int \frac{(f_n(y) - f_n(x))^2}{|y - x|^{d+\alpha}} \, dy$$

$$\leq \frac{4}{n^2} \int_{\{|y-x| \leq n\}} \frac{1}{|y - x|^{d+\alpha-2}} \, dy + \int_{\{|y-x| \geq n\}} \frac{1}{|y - x|^{d+\alpha}} \, dy$$

$$\leq c_1 n^{-\alpha},$$

and

$$D_{\alpha,V}(f_n, f_n) = \mu_V(\Gamma(f_n)) \leq c_1 n^{-\alpha},$$

where $c_1$ is a constant independent of $n$.

Then, there exists a constant $c_2 > 0$ independent of $n$, such that for any $n$ large enough,

$$\int (f_n(x) - \mu_V(f_n))^2 \omega^*(x) \mu_V(dx)$$

$$\geq \left( \inf_{|\omega(x)-2n} \frac{\omega^*(x)}{\omega(x)} \right) \int_{\{|x| \geq 2n\}} (f_n(x) - \mu_V(f_n))^2 \omega(x) \mu_V(dx)$$

$$\geq \left( \inf_{|\omega(x)-2n} \frac{\omega^*(x)}{\omega(x)} \right) \left( 1 - \int_{\{|x| \geq n\}} \mu_V(dx) \right)^2 \int_{\{|x| \geq 2n\}} \omega(x) \mu_V(dx)$$

$$\geq c_2 \left( \inf_{|\omega(x)-2n} \frac{\omega^*(x)}{\omega(x)} \right) n^{-\alpha},$$

28 XIN CHEN JIAN WANG
where in the second inequality we have used the fact that
\[ 0 \leq \left(1 - \int_{|x|\geq n} \mu_V(dx)\right) \leq (1 - \mu_V(f_n)). \]

Thus, applying $f_n$ into (4.54), we get that there exists some constant $c_3 > 0$ independent of $n$, such that for $n$ large enough,
\[ \left( \inf_{|x|>2n} \frac{\omega^*(x)}{\omega(x)} \right)^n \leq c_3 n^{-a}. \]

Since
\[ \lim_{n \to \infty} \frac{\omega^*(x)}{\omega(x)} = \infty, \]
there is a contradiction, and hence the conclusion is proved. \(\square\)

**Proof of Proposition 1.6.** We first claim that if the inequality (1.13) holds, then $\mu_V(\omega) < \infty$. In fact, choose a function $g \in C_c^\infty(\mathbb{R}^d)$ such that $g(x) = 0$ for every $|x| \geq 1$ and $\mu_V(g) = 1$. Then, applying this test function $g$ into (1.13), we have
\[
\int_{|x|>t} \omega(x) \mu_V(dx) \leq \int (g(x) - \mu_V(g))^2 \omega(x) \mu_V(dx) \\
\leq C_D \omega_V(g,g) \\
< \infty,
\]
thanks to the fact that $g \in C_c^\infty(\mathbb{R}^d) \subset \mathcal{D}(D_{\alpha,V})$. Since the weighted function $\omega$ is continuous, it is bounded on $\{x \in \mathbb{R}^d : |x| \leq 1\}$, and hence $\int_{|x| \leq 1} \omega(x) \mu_V(dx) < \infty$. Combining both estimates above, we prove the desired claim.

For any $t > 1$ and $f \in C_c^\infty(\mathbb{R}^d)$, by (1.13), we have
\[
\int_{|x|>t} f^2(x) \mu_V(dx) \leq \frac{1}{\inf_{|x|>t} \omega(x)} \int f^2(x) \omega(x) \mu_V(dx) \\
\leq \frac{2}{\inf_{|x|>t} \omega(x)} \int (f(x) - \mu_V(f))^2 \omega(x) \mu_V(dx) \\
+ \frac{2}{\inf_{|x|>t} \omega(x)} \int \mu_V^-(f) \omega(x) \mu_V(dx) \\
\leq \frac{2}{\inf_{|x|>t} \omega(x)} \left( C_D \omega_V(f,f) + \mu_V(\omega) \mu_V^2(|f|) \right),
\]
where in the second inequality we have used the fact that for any $a, b \in \mathbb{R}$,
\[ a^2 \leq 2(a - b)^2 + 2b^2. \]
Since $\lim_{|x| \to \infty} \omega(x) = \infty$, we can obtain that there exists a constant $c_1 > 0$ such that for any $t > 1$,
\[
\int_{|x|>t} f^2(x) \mu_V(dx) \leq \frac{2C_D}{\inf_{|x|>t} \omega(x)} \omega_V(f,f) + c_1 \mu_V(|f|)^2.
\]
On the other hand, according to [13, Lemma 3.1], for any \( t > 1 \), the following local super Poincaré inequality
\[
\int_{\{|x|\leq t\}} f^2(x) \mu_V(dx) \leq s D_{\alpha,V}(f,f) + \frac{c_2 H(t)^{2+d/\alpha}}{h(t)^{1+d/\alpha}} (1 + s^{-d/\alpha}) \mu_V(|f|)^2, \quad s > 0
\]
holds with some constant \( c_2 > 0 \) independent of \( t \).

Combining both estimates above, we get that there is a constant \( c_3 > 0 \) such that for each \( t > 1 \) and \( f \in C_b^\infty(\mathbb{R}^d) \),
\[
\mu_V(f^2) \leq \left( \frac{2C_0}{\inf_{|x|\geq t} \omega(x) + s} \right) D_{\alpha,V}(f,f) + \frac{c_3 H(t)^{2+d/\alpha}}{h(t)^{1+d/\alpha}} (1 + s^{-d/\alpha}) \mu_V(|f|)^2, \quad s > 0.
\]
(4.56)

This, along with the assumption that \( \lim_{|x|\to\infty} \omega(x) = \infty \), yields the first required assertion.

For any \( 0 < r < 4C_0 \left( \inf_{x \in \mathbb{R}^d} \omega(x) \right)^{-1} \), we can choose \( t > 1 \) large enough such that
\[
2C_0 \left( \inf_{|x|\geq t} \omega(x) \right)^{-1} \leq r/2, \quad \text{e.g.} \quad t = \kappa(4C_0/r). \]
Then, the second desired assertion follows by taking \( s = r/2 \) and \( t = \kappa(4C_0/r) \) in the definition of rate function \( \beta(r) \).

Proof of Proposition 1.7. Since \( \omega \) is positive and continuous on \( \mathbb{R}^d \), for any \( r > 0 \), \( \inf_{|x|\leq r} \omega(x) > 0 \). For any \( x \in \mathbb{R}^d \), set \( \omega^*(x) = \inf_{|z|\leq |x|} \omega(z) \). Then, \( \omega^*(x) \leq \omega(x) \) for all \( x \in \mathbb{R}^d \), and so it suffices to prove (1.16) holds for such weighted function \( \omega^* \).

It is easy to check that, under (1.15) the function \( V \) satisfies all the conditions in Theorem 1.4. Therefore, the inequality (1.12) holds. If \( \lim_{|x|\to\infty} \omega^*(x) > 0 \), then we can choose a constant \( C_0 > 0 \) such that \( \omega^*(x) \geq C_0 \) for all \( x \in \mathbb{R}^d \). Hence, (1.12) implies that (1.16) holds for such weighted function \( \omega^* \).

In the following, we assume that \( \omega^* \) is a positive function on \( \mathbb{R}^d \) such that
\[
\lim_{|x|\to\infty} \omega^*(x) = 0.
\]
(4.57)

For any \( r > 1 \), which will be determined later, define a function \( \omega_r^*(x) \) as follows
\[
\omega_r^*(x) := \begin{cases} 
1, & \text{if } |x| \leq r, \\
\omega^*(x), & \text{if } |x| > r.
\end{cases}
\]
Clearly, there is a constant \( c_1 := c_1(r) > 0 \) such that \( \omega_r^*(x) \leq c_1 \omega^*(x) \) for each \( x \in \mathbb{R}^d \). Therefore, it is sufficient to prove (1.16) holds for weighted function \( \omega_r^* \) with some \( r > 0 \).

Let \( V^*(x) := V(x) - \log \omega^*_r(x) \) (we omit the index \( r \) in \( V^* \) for simplicity). By (1.15) and (4.57), we can check that the function \( V^* \) satisfies all the conditions in Theorem 1.4. Thus, there exists a constant \( c_2 := c_2(r) > 0 \) such that the following weighted Poincaré inequality
\[
\int \left( f(x) - \mu_{V^*}(f) \right)^2 \frac{1}{(1 + |x|)^{d+\alpha}} dx \leq c_2 \int \int \frac{(f(y) - f(x))^2}{|y - x|^{d+\alpha}} dy \mu_{V^*}(dx)
\]
holds for all \( f \in C_b^\infty(\mathbb{R}^d) \), where
\[
\mu_{V^*}(dx) = \frac{1}{\int e^{-V^*(x)} dx} e^{-V^*(x)} dx =: C_{V^*} e^{-V^*(x)} dx.
\]
(4.58)
Since \( r > 1 \) and \( w^*_r(x) = 1 \) for all \( |x| \leq 1 \),

\[
C_{V^*} = \frac{1}{\int e^{-V(x)} \, dx} \leq \frac{1}{\int_{|x| \leq 1} e^{-V(x)} \, dx} =: C_1 < \infty,
\]

where \( C_1 > 0 \) is a constant independent of \( r \). According to (4.57), we can fix \( r \) large enough such that \( \sup_{x \in \mathbb{R}^d} |w^*_r(x)| \leq 1 \). Hence, there exists a constant \( C_2 > 0 \) independent of \( r \), such that for every \( f \in C_b^\infty(\mathbb{R}^d) \) with \( \int f \, d\mu_V = 0 \),

\[
\left( \int f(x) \mu_{V^*}(dx) \right)^2 = \left( C_{V^*} \int_{|x| \leq r} f(x)e^{-V(x)} \, dx + C_{V^*} \int_{|x| > r} f(x)w^*(x)e^{-V(x)} \, dx \right)^2
\]

\[
= \left( C_{V^*} \int_{|x| > r} f(x)e^{-V(x)} \, dx + C_{V^*} \int_{|x| > r} f(x)w^*(x)e^{-V(x)} \, dx \right)^2
\]

\[
\leq 2C_1^2 \left[ \left( \int_{|x| > r} f(x)e^{-V(x)} \, dx \right)^2 + \left( \int_{|x| > r} f(x)w^*(x)e^{-V(x)} \, dx \right)^2 \right]
\]

\[
\leq C_2 \left[ \mu_V(|x| > r) \int f^2(x) \mu_V(dx)
\right]
\]

\[
+ \left( \int_{|x| > r} \omega^2(x) \mu_V(dx) \right) \left( \int f^2(x) \mu_V(dx) \right)
\]

\[
\leq 2C_2 \mu_V(|x| > r) \int f^2(x) \mu_V(dx),
\]

where the second equality follows from

\[
\int_{|x| \leq r} f(x)e^{-V(x)} \, dx = \int_{|x| > r} f(x)e^{-V(x)} \, dx,
\]

thanks to \( \int f \, d\mu_V = 0 \); in the first inequality we have used (4.59); in the second inequality we applied the Cauchy-Schwarz inequality; and the last inequality follows from the fact that \( \sup_{x \in \mathbb{R}^d} |w^*_r(x)| \leq 1 \).

From now on, all the constants \( C_i \) are independent of \( r \). For each \( f \in C_b^\infty(\mathbb{R}^d) \) with \( \int f \, d\mu_V = 0 \),

\[
\int f^2(x) \frac{e^{V(x)}}{(1 + |x|)^{d+\alpha}} \mu_V(dx)
\]

\[
\leq C_3 \left( \int f(x) - \int f \, d\mu_{V^*} + \int f \, d\mu_{V^*} \right)^2 \frac{1}{(1 + |x|)^{d+\alpha}} \, dx
\]

\[
\leq C_4 \left( \int f(x) - \int f \, d\mu_{V^*} \right)^2 \frac{1}{(1 + |x|)^{d+\alpha}} \, dx + C_4 \left( \int f(x) \mu_{V^*}(dx) \right)^2
\]

\[
\leq C_4 \left( \int f(x) - \int f \, d\mu_{V^*} \right)^2 \frac{1}{(1 + |x|)^{d+\alpha}} \, dx + C_5 \mu_V(|x| > r) \int f^2(x) \mu_V(dx)
\]

\[
\leq C_4 \left( \int f(x) - \int f \, d\mu_{V^*} \right)^2 \frac{1}{(1 + |x|)^{d+\alpha}} \, dx
\]

\[
+ C_6 \mu_V(|x| > r) \int f^2(x) \frac{e^{V(x)}}{(1 + |x|)^{d+\alpha}} \mu_V(dx),
\]
Theorem 5.1. Assume that $C_0 \mu_V(|x| > r) < 1/2$. Then, for each $f \in C_b^\infty(\mathbb{R}^d)$ with $\int f \, d\mu_V = 0$,
\[
\int f^2(x) \frac{e^{V(x)}}{(1 + |x|)^{d+\alpha}} \mu_V(dx) \leq 2C_4 \int \left( f^2(x) - \int f \, d\mu_V \right)^2 \frac{1}{(1 + |x|)^{d+\alpha}} \, dx.
\]
This, along with (4.58), yields that for each $f \in C_b^\infty(\mathbb{R}^d)$ with $\int f \, d\mu_V = 0$,
\[
\int f^2(x) \frac{e^{V(x)}}{(1 + |x|)^{d+\alpha}} \mu_V(dx) \leq c_3 \int \omega_r^*(x) \int \frac{(f(y) - f(x))^2}{|y - x|^{d+\alpha}} \, dy \, d\mu_V(dx).
\]
This proves the desired assertion. \hfill \Box

5. Weighted Poincaré Inequalities for Non-local Dirichlet Forms Associated with Symmetric Markov Processes under Girsanov Transform of Pure Jump Type

5.1. Weighted Poincaré Inequalities for Dirichlet Forms $D_{\psi,V}$. In this section, we aim to state that the technique to yield Theorem 4.1 also gives us the criterion for weighted Poincaré inequalities of the Dirichlet form given in Example 2.3. In the following, let $V$ be a locally bounded function on $\mathbb{R}^d$ such that $\int e^{-V(x)} \, dx < \infty$ and $e^{-V}$ is bounded. Let
\begin{equation}
\mu_{2V}(dx) := \frac{1}{\int e^{-2V(x)} \, dx} e^{-2V(x)} \, dx
\end{equation}
be a probability measure. Given a measurable function $\psi(r) : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ satisfying (2.29), we define for each $f, g \in C_b^\infty(\mathbb{R}^d)$,
\[
D_{\psi,V}(f,g) := \frac{1}{2} \iint (f(y) - f(x))(g(y) - g(x))\psi(|x-y|) e^{-V(x+y)} \, dy \, e^{-V(x)} \, dx.
\]

**Theorem 5.1.** Assume that $\psi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a continuous function such that
\[
\int_{\{|r| < 1\}} r^{d-1} \psi(r)^{-1} \, dr < \infty,
\]
and
\[
\int_{\{|x| > 1\}} e^{-3V(x)} \, dx < \infty,
\]
where for $r > 0$,
\[
\gamma(r) := \inf_{0 < s < r+1} \psi(s).
\]
If there is some constant $0 < \alpha_0 < 1$ such that
\[
\int_{\{|r| \geq 1\}} r^{d+\alpha_0-1} \psi(r) \, dr < \infty,
\]
and
\[
\limsup_{|x| \to \infty} \sup_{|z| \leq |x|} \frac{e^{-V(z)}}{\gamma(|x|)|x|^\alpha_0} = 0,
\]
then there exists a constant $C_1 > 0$ such that the following weighted Poincaré inequality
\[
\int (f(x) - \mu_{2V}(f))^2 e^{V(x)} \gamma(|x|) \mu_{2V}(dx) \leq C_1 D_{\psi,V}(f,f)
\]
holds for all \( f \in C_b^\infty(\mathbb{R}^d) \).

The proof of Theorem 5.1 is similar to that of Theorem 4.1. It is based on the expression for the generator of the associated truncated Dirichlet form in Example 2.7, which enables us to take the same Lyapunov function as that in the Lemma 4.3. We omit the details here.

5.2. The Case that: \( \psi(r) = e^{-\delta r}r^{-(d+\alpha)} \) with \( \delta > 0 \) and \( 0 < \alpha < 2 \). Taking \( \psi(r) = e^{-\delta r}r^{-(d+\alpha)} \) with \( \delta > 0 \) and \( 0 < \alpha < 2 \) in Theorem 5.1, we have the following statement.

Corollary 5.2. Suppose that for some constants \( \delta > 0 \) and \( \alpha \in (0, 1) \),

\[
\limsup_{|x| \to \infty} \left[ \left( \sup_{|z| \geq |x|} e^{\delta |x|} |x|^{d+\alpha} \right) e^{-V(x)} \right] = 0.
\]

Then, there exists a constant \( C_0 > 0 \) such that the following weighted Poincaré inequality

\[
\int (f(x) - \mu_{2V}(f))^2 \frac{e^{V(x)-\delta|x|}}{(1+|x|)^{d+\alpha}} \mu_{2V}(dx) \leq C_0 \int \int \frac{(f(y) - f(x))^2}{|y-x|^{d+\alpha}} e^{-\delta|y-x|} e^{-V(y)} dy e^{-V(x)} dx
\]

(5.62)

holds for all \( f \in C_b^\infty(\mathbb{R}^d) \).

According to Corollary 5.2, we know that if

\[
\liminf_{|x| \to \infty} \frac{e^{V(x)}}{|x|^{d+\alpha} e^{\delta|x|}} > 0,
\]

then (5.62) holds, which implies the standard Poincaré inequality, see (5.63) below. On the other hand, we can prove the following statement, which indicates the concentration of measure for such Poincaré inequality. Roughly speaking, in this setting, under some regular assumptions on \( V \), the exponential integrability of the distance function is also necessary for this inequality.

Proposition 5.3. Let \( \delta > 0 \) and \( \alpha \in (0, 2) \), and let \( \mu_{2V} \) be a probability measure defined by (5.61) such that \( V \in C^1(\mathbb{R}^d) \) and \( \sup_{x \in \mathbb{R}^d} |\nabla V(x)| < \delta \). If there is a constant \( C_1 > 0 \) such that the following inequality

\[
\int (f(x) - \mu_{2V}(f))^2 \mu_{2V}(dx) \leq C_1 \int \int \frac{(f(y) - f(x))^2}{|y-x|^{d+\alpha}} e^{-\delta|y-x|} e^{-V(y)} dy e^{-V(x)} dx
\]

(5.63)

holds for all \( f \in C_b^\infty(\mathbb{R}^d) \), then there exists a constant \( \lambda_0 > 0 \) such that

\[
\int e^{\lambda_0|x|} \mu_{2V}(dx) < \infty.
\]

Proof. For any \( n \geq 1 \), define \( g_n(x) := e^{\lambda(|x|/n)} \), where \( \lambda > 0 \) is a constant to be determined later. As the same reason as that in the proof of Proposition 1.3, we
can apply $g_n$ into (5.63), and get that
\[
\int g_n^2(x) \mu_{2V}(dx) \leq C_1 \int \int \frac{(g_n(y) - g_n(x))^2}{|y - x|^{d+\alpha}} e^{-\delta|y-x|} e^{-V(y)} dy e^{-V(x)} dx 
+ \left( \int g_n(x) \mu_{2V}(dx) \right)^2.
\]
(5.64)

In the following, for $\lambda > 0$, set
\[
l_n(\lambda) := \int e^{2\lambda(|x|^{\lambda})} \mu_{2V}(dx) = \int g_n^2(x) \mu_{2V}(dx).
\]
For each $N > 1$ and all $x \in \mathbb{R}^d$,
\[
\int \frac{(g_n(x) - g_n(y))^2}{|y - x|^{d+\alpha}} e^{-\delta|y-x|} e^{-V(y)} dy 
\leq \int_{\{|x-y| \leq N\}} \frac{(e^{\lambda(|x|^{\lambda})} - e^{\lambda(|y|^{\lambda})})^2}{|y - x|^{d+\alpha}} e^{-\delta|y-x|} e^{-V(y)} dy 
+ \int_{\{|x-y| > N\}} \frac{(e^{\lambda(|x|^{\lambda})} - e^{\lambda(|y|^{\lambda})})^2}{|y - x|^{d+\alpha}} e^{-\delta|y-x|} e^{-V(y)} dy 
= : J_{1,N}(x) + J_{2,N}(x).
\]
From now on, the constant $C$ will be changed in different lines, but does not depend on $n$, $N$, $\lambda$ or $R$. Let $b := \sup_{x \in \mathbb{R}^d} \{\nabla V(x)\}$. As in the proof of Proposition 1.3, by the mean value theorem,
\[
J_{1,N}(x) \leq \lambda e^{2\lambda(|x|^{\lambda}) + 2\lambda N} \int_{\{|x-y| \leq N\}} \frac{|y - x|^2}{|y - x|^{d+\alpha}} e^{-\delta|y-x|} e^{-V(y)} dy 
\leq \lambda e^{2\lambda(|x|^{\lambda}) + 2\lambda N} e^{-V(x) + b N} \int_{\{|x-y| \leq N\}} \frac{|y - x|^2}{|y - x|^{d+\alpha}} e^{-\delta|y-x|} dy 
\leq C\lambda e^{2\lambda(|x|^{\lambda}) + 2\lambda N} e^{-V(x) + b N} 
= C\lambda e^{2\lambda N + b N} e^{2\lambda(|x|^{\lambda}) - V(x)},
\]
where in the second inequality we have used the fact that
\[-V(y) = -V(x) + (V(x) - V(y)) \leq -V(x) + |V(x) - V(y)| \leq -V(x) + b|x - y|.
\]
Hence,
\[
\int J_{1,N}(x) e^{-V(x)} dx \leq C\lambda e^{2\lambda N + b N} \int e^{2\lambda(|x|^{\lambda})} \mu_{2V}(dx) = C\lambda e^{2\lambda N + b N} l_n(\lambda).
\]
On the other hand, by symmetric property,
\[
\int J_{2,N}(x) e^{-V(x)} dx 
\leq 2 \int \int \frac{e^{2\lambda(|x|^{\lambda})} + e^{2\lambda(|y|^{\lambda})}}{|x - y|^{d+\alpha}} e^{-\delta|x-y|} e^{-V(y)} dy e^{-V(x)} dx 
\leq 4 \int \int \frac{e^{2\lambda(|x|^{\lambda})}}{|x - y|^{d+\alpha}} e^{-\delta|x-y|} e^{-V(y)} dy e^{-V(x)} dx.
\]
(5.65)
Since \(-V(y) \leq -V(x) + |V(x) - V(y)| \leq -V(x) + b|x - y|\), it holds that
\[
\int_{\{x - y > N\}} \frac{e^{-\delta|x-y|}}{|x - y|^{d+\alpha}} e^{-V(y)} \, dy \leq e^{-V(x)} \int_{\{x - y > N\}} \frac{e^{-\delta|x-y|}}{|x - y|^{d+\alpha}} e^{b|x-y|} \, dy
= k(N)e^{-V(x)},
\]
where
\[
k(N) := \int_{\{x - y > N\}} \frac{e^{-(\delta-b)|z|}}{|z|^{d+\alpha}} \, dz \leq k(1) < \infty,
\]
also thanks to the fact that \(b < \delta\). Combining this with (5.65), we find
\[
\int J_{2N}(x)e^{-V(x)} \, dx \leq Ck(N) \int e^{2\lambda(|x|^\lambda)} e^{-2V(x)} \, dx \leq Ck(N)I_n(\lambda).
\]

According to all the estimates above and (5.64), we have
\[
I_n(\lambda) \leq C(\lambda e^{2\lambda N + bN} + k(N))I_n(\lambda) + l_n^2(\lambda/2).
\]
As the same way in the proof of (4.52), for any \(R > 1\), it holds that
\[
l_n^2(\lambda/2) \leq 2e^{2\lambda R} + 2p(R)l_n(\lambda),
\]
where \(p(R) := p_{2\lambda}(|x| > R)\). Then,
\[
l_n(\lambda) \leq C(\lambda e^{2\lambda N + bN} + k(N) + p(R))I_n(\lambda) + 2e^{2\lambda R}.
\]

Now, we first fix \(R_0\) and \(N_0 > 0\) large enough such that \(Cp(R_0) < 1/4\) and \(Ck(N_0) < 1/4\), then choose a constant \(\lambda_0 > 0\) such that \(C\lambda_0 e^{2\lambda_0 N_0 + bN_0} < 1/4\). We can finally get that
\[
l_n(\lambda_0) \leq 8e^{2\lambda_0 R_0}.
\]
Letting \(n\) tends to \(\infty\), we can prove the conclusion. \(\square\)

**Remark 5.4.** Let \(\delta > 0\) be the constant in the Poincaré inequality (5.63). For any \(\varepsilon > d\), let \(\mu_{2\varepsilon}(dx) = C_{\varepsilon}(1 + |x|)^{-2\varepsilon} \, dx\) be a probability measure. We will claim that the Poincaré inequality (5.63) does not hold for \(\mu_{2\varepsilon}\) with any \(\varepsilon > d\). Indeed, for any \(l \geq 1\) and \(\varepsilon > d\), define a probability measure
\[
\mu_{l,2\varepsilon}(dx) = C_{l,\varepsilon}(l + |x|^2)^{-\varepsilon} \, dx =: C_{l,\varepsilon}e^{-2V_{l,\varepsilon}(x)} \, dx.
\]
We can choose \(l_0\) large enough such that \(\sup_{x \in \mathbb{R}^d} |\nabla V_{l_0,\varepsilon}(x)| < \delta\). Thus, according to Proposition 5.3, the Poincaré inequality (5.63) does not hold for \(\mu_{l_0,2\varepsilon}\). Then, the desired claim follows from that fact that there is a constant \(C := C(l_0) > 1\) such that
\[
\frac{1}{C} \mu_{l_0,2\varepsilon} \leq \mu_{2\varepsilon} \leq C\mu_{l_0,2\varepsilon}.
\]
Similarly, we also can show that the Poincaré inequality (5.63) does not hold for \(\mu_{2\beta}(dx) = C_\beta e^{-(1+|x|^\beta)} \, dx\) with any \(0 < \beta < 1\).
5.3. The Case that: \( \psi(r) = r^{-(d+\alpha)} \) with \( 0 < \alpha < 2 \). Letting \( \psi(r) = r^{-(d+\alpha)} \) with \( 0 < \alpha < 2 \) in Theorem 5.1, we have the following statement.

**Corollary 5.5.** If for some constants \( \alpha \in (0, 2) \) and \( \alpha_0 \in (0, \alpha \land 1) \),

\[
\limsup_{|x| \to \infty} \left( \sup_{|z| \geq |x|} e^{-V(z)} \right) |x|^{d+\alpha-\alpha_0} = 0,
\]

then there exists a constant \( C_0 > 0 \) such that the following weighted Poincaré inequality

\[
\int (f(x) - \mu_{2V}(f))^2 \frac{e^{V(x)}}{(1 + |x|)^{d+\alpha}} \mu_{2V}(dx) \leq C_0 \int \int \frac{(f(y) - f(x))^2}{|y - x|^{d+\alpha}} e^{-V(y)} dy e^{-V(x)} dx
\]

holds for all \( f \in C_b^\infty(\mathbb{R}^d) \).

To show that the inequality (5.66) is optimal, we consider the following result. First, for each \( \varepsilon > 0 \), let

\[
V_\varepsilon(x) := \frac{1}{2} \log \left( 1 + |x|^2 \right)^{d+\varepsilon},
\]

and

\[
\mu_{2V_\varepsilon}(dx) := C_\varepsilon e^{-2V_\varepsilon(x)} dx = \frac{C_\varepsilon}{(1 + |x|^{d+\varepsilon})} dx,
\]

where \( C_\varepsilon \) is the normalizing constant.

**Proposition 5.6.** The following Poincaré inequality

\[
\int (f(x) - \mu_{2V_\varepsilon}(f))^2 \mu_{2V_\varepsilon}(dx) \leq C_1 \int \int \frac{(f(y) - f(x))^2}{|y - x|^{d+\alpha}} e^{-V_\varepsilon(y)} dy e^{-V_\varepsilon(x)} dx \quad \text{for all } f \in C_b^\infty(\mathbb{R}^d)
\]

holds some constant \( C_1 > 0 \) if and only if

\[
\varepsilon \geq \alpha.
\]

Moreover, for the constant \( \beta \in \mathbb{R} \) and the probability measure \( \mu_{2V_\varepsilon} \) with \( \varepsilon \geq \alpha \), the following weighted Poincaré inequality

\[
\int (f(x) - \mu_{2V_\varepsilon}(f))^2 (1 + |x|^{\beta}) \mu_{2V_\varepsilon}(dx) \leq C_2 \int \int \frac{(f(y) - f(x))^2}{|y - x|^{d+\alpha}} e^{-V_\varepsilon(y)} dy e^{-V_\varepsilon(x)} dx \quad \text{for all } f \in C_b^\infty(\mathbb{R}^d)
\]

holds for some constant \( C_2 > 0 \) if and only if

\[
\beta \leq \varepsilon - \alpha.
\]

**Proof.** (a) According to Corollary 5.5, if \( \varepsilon \geq \alpha \), then the inequality (5.67) holds for \( \mu_{2V_\varepsilon} \). Next, it suffices to verify that (5.67) does not hold for \( \mu_{2V_\varepsilon} \) with \( \varepsilon < \alpha \). For any \( n > 1 \), choose a smooth function \( f_n : \mathbb{R}^d \to [0, 1] \) such that

\[
f_n(x) = \begin{cases} 0, & \text{if } |x| \leq 3n; \\ 1, & \text{if } |x| > 4n, \end{cases}
\]
and \( \sup_{x \in \mathbb{R}^d} |\nabla f(x)| \leq \frac{2}{n} \). Suppose (5.67) holds for some probability measure \( \mu_{2V_\epsilon} \) with \( \epsilon < \alpha \). Then,

\[
\int \left( f_n(x) - \int f_n(x) \mu_{2V_\epsilon}(dx) \right)^2 \mu_{2V_\epsilon}(dx) \leq C \int \frac{(f_n(y) - f_n(x))^2}{|y - x|^{d+\alpha}} e^{-V_\epsilon(y)} dy e^{-V_\epsilon(x)} dx,
\]

(5.69)

In the following, set

\[
\Gamma(f_n)(x) := \int \frac{(f_n(y) - f_n(x))^2}{|y - x|^{d+\alpha}} e^{-V_\epsilon(y)} dy.
\]

The constant \( C > 0 \) will be changed in different line, but does not depend on \( n \). When \( |x| \leq 2n \), \( |f_n(x) - f_n(y)| \neq 0 \) only if \( |x - y| > n \) and \( |y| > 3n \). Then,

\[
\Gamma(f_n)(x) \leq C \int_{\{ |x-y| > n, |y| > 3n \}} \frac{1}{|y - x|^{d+\alpha}} \frac{1}{(1 + |y|)^{d+\epsilon}} dy \leq \frac{C}{n^{d+\alpha+\epsilon}}.
\]

For \( 2n < |x| \leq 5n \), it holds that \( \{ y : |x - y| \leq n \} \subseteq \{ y : |y| \geq n \} \), and so,

\[
\Gamma(f_n)(x) \leq C \left[ \int_{\{ |x-y| \leq n, |y| \geq n \}} \frac{(f_n(y) - f_n(x))^2}{|y - x|^{d+\alpha}} e^{-V_\epsilon(y)} dy 
+ \int_{\{ |x-y| > n \}} \frac{(f_n(y) - f_n(x))^2}{|y - x|^{d+\alpha}} e^{-V_\epsilon(y)} dy \right]
\leq C \left[ \frac{1}{n^2} \int_{\{ |x-y| \leq n, |y| \geq n \}} \frac{|x-y|^2}{|y - x|^{d+\alpha}} e^{-V_\epsilon(y)} dy 
+ \int_{\{ |x-y| > n \}} \frac{1}{|y - x|^{d+\alpha}} e^{-V_\epsilon(y)} dy \right]
\leq C \left[ \frac{1}{n^{d+2\alpha+\epsilon}} \int_{\{ |x-y| \leq n \}} \frac{|x-y|^2}{|y - x|^{d+\alpha}} dy + \frac{1}{n^{d+\alpha}} \int e^{-V_\epsilon(y)} dy \right]
\leq C \frac{1}{n^{d+\alpha}}.
\]

If \( |x| > 5n \), then \( |f_n(x) - f_n(y)| \neq 0 \) only for \( |x - y| > n \), and hence

\[
\Gamma(f_n)(x) \leq C \int_{\{ |x-y| > n \}} \frac{1}{|y - x|^{d+\alpha}} e^{-V_\epsilon(y)} dy \leq \frac{C}{n^{d+\alpha}} \int e^{-V_\epsilon(y)} dy \leq \frac{C}{n^{d+\alpha}}.
\]

Combining all the estimates above, we get

\[
\int \Gamma(f_n)(x) e^{-V_\epsilon(x)} dx
= \int_{\{ |x| \leq 2n \}} \Gamma(f_n)(x) e^{-V_\epsilon(x)} dx 
+ \int_{\{ 2n < |x| \leq 5n \}} \Gamma(f_n)(x) e^{-V_\epsilon(x)} dx 
+ \int_{\{ |x| > 5n \}} \Gamma(f_n)(x) e^{-V_\epsilon(x)} dx 
\leq \frac{C}{n^{d+\alpha+\epsilon}} \int e^{-V_\epsilon(x)} dx 
+ \frac{C}{n^{d+\alpha}} \int_{\{ |x| > 2n \}} e^{-V_\epsilon(x)} dx
\leq \frac{C}{n^{d+\alpha+\epsilon}}.
\]
On the other hand, for $n$ large enough, following the proof of (4.55), we have
\[
\int \left( f_n(x) - \int f_n(x) \mu_{2V}(dx) \right)^2 \mu_{2V}(dx) \\
\geq C \left( 1 - \int \mu_{2V}(dx) \right)^2 \int_{|x| > 3n} e^{-2V(x)} dx \\
\geq \frac{C}{n^{d+2\varepsilon}}.
\]
Therefore, according to (5.69), it holds for $n$ large enough that
\[
\frac{1}{n^{d+2\varepsilon}} \leq \frac{C}{n^{d+\alpha+\varepsilon}}.
\]
Since $\varepsilon < \alpha$, there is a contradiction, and hence (5.67) does not hold for $\mu_{2V,\varepsilon}$ with $\varepsilon < \alpha$. We have proved the first conclusion.

(b) For $\varepsilon \geq \alpha$, by Corollary 5.5, we can check that (5.68) holds with $\beta \leq \varepsilon - \alpha$. On the other hand, one can follow the argument above to verify that (5.68) does not hold with $\beta > \varepsilon - \alpha$. This finished the proof. □

To compare the Dirichlet forms given in Examples 2.3 and 2.4, we will show that the corresponding Poincaré inequality for the Dirichlet form given in Example 2.4 does not hold for a large class of probability measures.

**Proposition 5.7.** Let $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ be a positive function satisfying (2.29). Let $\mu_V$ be a probability measure defined by (1.6) such that $V$ is a locally bounded function, $e^{-V} \in L^1(dx)$, $e^{-V}$ is bounded and $\lim_{|x| \to \infty} e^{-V(x)} = 0$. If there is a constant $C_1 > 0$ such that the following Poincaré inequality
\[
\int (f(x) - \mu_V(f))^2 \mu_V(dx) \leq C_1 \int \int (f(y) - f(x))^2 \psi(|x - y|) \mu_V(dy) \mu_V(dx)
\]
holds for all $f \in C^\infty_b(\mathbb{R}^d)$, then for any $\lambda > 0$,
\[
\int e^{\lambda|x|} \mu_V(dx) < \infty.
\]
Furthermore, for any $r > 0$, set $q(r) := \sup_{|x| > r} e^{-V(x)}$. Then, there exist $C_2, C_3 > 0$ such that
\[
\int \left( \frac{1}{q(C_2|x|)} \right)^{C_3|x|} \mu_V(dx) < \infty.
\]

**Remark 5.8.** (1) According to (5.71), the probability measures
\[
\mu_\varepsilon(dx) = C_\varepsilon (1 + |x|)^{-d-\varepsilon} dx
\]
for any $\varepsilon > 0$ or
\[
\mu_\beta(dx) = C_\beta e^{-(1+|x|^\beta)} dx
\]
for any $\beta > 0$ do not satisfy the Poincaré inequality (5.70).

(2) As seen above, the conclusions about the concentration of measure for functional inequalities of Dirichlet forms given in Examples 2.2 and 2.3 highly depend on the function $\rho$ or $\psi$ in these Dirichlet forms. However, here for the Dirichlet form
given in Example 2.4, the property of the concentration of measure for Poincaré inequality is independent of $\psi$.

**Proof of Proposition 5.7.** Similar to the proof above, the constant $C$ will be changed in different lines, but does not depend on $n$, $\lambda$, $M$, $N$ or $R$. For each $\lambda > 0$ and $n \geq 1$, set

$$g_n(x) := e^{\lambda(|x| \wedge n)}.$$  

Suppose that the inequality (5.70) holds. Then, we may and do apply $g_n$ into (5.70) to get that

$$\int g_n^2(x) \mu_V(dx) \leq C_1 \int (g_n(y) - g_n(x))^2 \psi(|x - y|) \mu_V(dy) \mu_V(dx) + \left( \int g_n(x) \mu_V(dx) \right)^2. \tag{5.72}$$

For any $n \geq 1$ and $\lambda > 0$, set

$$l_n(\lambda) := \int e^{2\lambda(|x| \wedge n)} \mu_V(dx) = \int g_n^2(x) \mu_V(dx).$$

For each $N > 1$, we have

$$\int (g_n(x) - g_n(y))^2 \psi(|x - y|) e^{-V(y)} dy \leq \int_{\{|x-y| \leq N\}} (g_n(x) - g_n(y))^2 \psi(|x - y|) e^{-V(y)} dy$$

$$+ \int_{\{|x-y| > N\}} (g_n(x) - g_n(y))^2 \psi(|x - y|) e^{-V(y)} dy$$

$$= : J_{1,N}(x) + J_{2,N}(x).$$

For $r > 0$, set

$$a(r) := \int_{\{|z| \leq r\}} |z|^2 \psi(|z|) \, dz$$

which is well defined due to (2.29).

First, by using the mean value theorem, we have

$$J_{1,N}(x) \leq \lambda e^{2\lambda N} e^{2\lambda (|x| \wedge n)} \int_{\{|x-y| \leq N\}} |x - y|^2 \psi(|x - y|) e^{-V(y)} dy$$

$$\leq \lambda a(N) e^{2\lambda N} e^{2\lambda (|x| \wedge n)},$$

where we have used the fact that $e^{-V}$ is bounded. For any $r > 0$, define

$$q(r) := \sup_{x \in \mathbb{R}^d : |x| > r} e^{-V(x)}.$$
Then, for $M > N$ large enough,
\[
\int J_{1,N}(x) \mu_V(dx) \\
= \int_{\{|x|\leq M\}} J_{1,N}(x) \mu_V(dx) + \int_{\{|x|> M\}} J_{1,N}(x) \mu_V(dx) \\
\leq C\lambda e^{2\lambda N}a(N) \int_{\{|x|\leq M\}} e^{2\lambda |x/N|} \mu_V(dx) \\
+ \lambda e^{2\lambda N} \left[ \int_{\{|x|> M\}} e^{2\lambda |x/N|} \left( \int_{\{|x-y|\leq N, |y|> M-N\}} |x-y|^2 \psi(|x-y|) e^{-V(y)}dy \right) e^{-V(x)} dx \right] \\
\leq C\lambda e^{2\lambda (M+N)}a(N) + C\lambda e^{2\lambda N}a(N)q(M-N)l_n(\lambda),
\]
where in the first inequality we have used the fact that if $|x| > M$ and $M > N$ large enough, then
\[
\{y : |x-y| \leq N\} \subseteq \{y : |x-y| \leq N, |y| > M-N\}.
\]
On the other hand, by the symmetric property and also the fact that $e^{-V}$ is bounded,
\[
\int J_{2,N}(x) \mu_V(dx) \leq 2 \int_{\{|x-y|> N\}} \left( e^{2\lambda |x/N|} + e^{2\lambda |y/N|} \right) \psi(|x-y|) \mu_V(dy) \mu_V(dx) \\
\leq 4 \int_{\{|x-y|> N\}} e^{2\lambda |x/N|} \psi(|x-y|) \mu_V(dy) \mu_V(dx) \\
\leq Ck(N)l_n(\lambda),
\]
where
\[
k(N) := \int_{\{|x|> N\}} \psi(|x|) dz.
\]
Combining all the estimates above with (5.72),
\[
l_n(\lambda) \leq C\left( \lambda e^{2\lambda N}a(N)q(M-N) + k(N) \right) l_n(\lambda) + l_n^2(\lambda/2) + C\lambda e^{2\lambda (M+N)}a(N).
\]
By (4.52), we get for each $R > 1$,
\[
l_n(\lambda) \leq C\left( \lambda e^{2\lambda N}a(N)q(M-N) + k(N) + p(R) \right) l_n(\lambda) + C\left( \lambda e^{2\lambda (M+N)}a(N) + e^{2\lambda R} \right),
\]
where $p(R) := \mu_V(|x| > R)$.

Next, we first choose the constants $R_0$ and $N_0 > 0$ large enough such that
\[
C(k(N_0) + p(R_0)) < 1/2.
\]
Since $\lim_{r\to\infty} q(r) = 0$, for all $\lambda > 0$, we can take $M_0 := M_0(\lambda) > 0$ large enough such that
\[
C\lambda e^{2\lambda N_0}a(N_0)q(M_0 - N_0) \leq 1/4,
\]
e.g.
\[
M_0 = N_0 + q^{-1}\left( \frac{e^{2\lambda N_0}}{4C(\lambda \lor 1)a(N_0)} \right),
\]
where
\[
q^{-1}(r) := \inf\{s : q(s) \leq r\},
\]
and we use the convention that $q^{-1}(r) := 0$ if $r > q(0)$. Then, we have
\[
l_n(\lambda) \leq C\left( \lambda e^{2\lambda (M_0+N_0)}a(N_0) + e^{2\lambda R_0} \right).
\]
Letting $n \to \infty$, we get $\int e^{\lambda|x|} \mu_V(dx) < \infty$ for every $\lambda > 0$. This proves the first required assertion.

Actually, according to the arguments above, there are constants $c_1, c_2 > 0$ such that
\begin{equation}
\int e^{\lambda|x|} \mu_V(dx) < c_1 e^{c_1(\lambda + \lambda q^{-1}(e^{-c_2 \lambda}))}.
\end{equation}
Now, we will follow the proof of [12, Theorem 3.3.21]. Define
\[ w(\lambda) := e^{-(c_1 + 1)\lambda - c_1 \lambda q^{-1}(e^{-c_2 \lambda})} \]
and
\[ h(r) := \int_1^{+\infty} e^{r\lambda} w(\lambda) d\lambda. \]
Then, by Fubini theorem and (5.74), we have
\begin{equation}
\int h(|x|) \mu_V(dx) = \int_1^{+\infty} \int e^{\lambda|x|} \mu_V(dx) w(\lambda) d\lambda < \infty.
\end{equation}

For a fixed $0 < \varepsilon < 1$ and for $r$ large enough, define
\[ \lambda_0(r) := -\frac{1}{c_2} \log \left[ q \left( \frac{\varepsilon r}{c_1} \right) \right]. \]
Thus,
\[ e^{-c_2 \lambda_0(r)} = q \left( \frac{\varepsilon r}{c_1} \right). \]
According to the definition (5.73) of $q^{-1}$, for $r$ large enough
\[ c_1 q^{-1}(e^{-c_2 \lambda_0(r)}) \leq \varepsilon r. \]
Hence, for any $\lambda \leq \lambda_0(r)$ and $r$ large enough,
\[ w(\lambda) \geq e^{-[(1+c_1)+\varepsilon r] \lambda}, \]
and so, there exist some constants $c_3, c_4 > 0$ such that for $r$ large enough,
\begin{align*}
h(r) &\geq \int_1^{\lambda_0(r)} \exp \left\{ [(1 - \varepsilon)r - (1 + c_1)] \lambda \right\} d\lambda \\
&\geq c_3 e^{c_4 r \lambda_0(r)} \\
&= c_3 \exp \left\{ - \frac{c_4 r}{c_2} \log \left[ q \left( \frac{\varepsilon r}{c_1} \right) \right] \right\}.
\end{align*}
Combining this with (5.75), we have finished the proof of the second assertion. \qed

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