The standard MS renormalization prescription is inadequate for dealing with multiscale problems. To illustrate this, we consider the computation of the effective potential in the Higgs-Yukawa model. It is argued that the most natural way to deal with this problem is to introduce a 2-scale renormalization group. We review various ways of implementing this idea and consider to what extent they fit in with the notion of heavy particle decoupling.

1 Introduction

Let us consider a very simple problem in perturbative quantum field theory, the computation of the effective potential in the four-dimensional Higgs-Yukawa model defined by the Lagrangian

\[ L = \frac{1}{2} (\partial_\mu \phi)^2 - \frac{1}{2} m^2 \phi^2 - \frac{\lambda}{24} \phi^4 + \bar{\psi} i \gamma \psi + g \bar{\psi} \phi \psi + \Lambda. \]

Here \( \Lambda \) is a “cosmological constant” term which enters non-trivially into the renormalization group equation for the effective potential and is well known how to perform a loopwise perturbative expansion of the effective potential \( V(\phi) = V^{(0)}(\phi) + \hbar V^{(1)}(\phi) + \hbar^2 V^{(2)}(\phi) + \ldots \). Using dimensional regularization together with (modified) minimal subtraction gives

\[ V^{(0)}(\phi) = \frac{\lambda}{24} \phi^4 + \frac{1}{2} m^2 \phi^2 + \Lambda, \]

\[ V^{(1)}(\phi) = \frac{(m^2 + \frac{1}{2} \lambda \phi^2)^2}{4(4\pi)^2} \left[ \log \frac{m^2 + \frac{1}{2} \lambda \phi^2}{\mu^2} - \frac{3}{2} \right] - \frac{g^4 \phi^4}{(4\pi)^2} \left[ \log \frac{g^2 \phi^2}{\mu^2} - \frac{3}{2} \right]. \]

Notice the logarithmic terms in the one-loop potential. We have a \( \log \frac{m^2 + \frac{1}{2} \lambda \phi^2}{\mu^2} \) due to the “Higgs” loop and \( \log \frac{g^2 \phi^2}{\mu^2} \) associated with the fermionic contribution to the one-loop potential. The two-loop potential, \( V^{(2)} \), is quadratic in these logarithms, and in general the \( n \)-loop potential is a \( n \)th order polynomial in
the two logarithms. Thus, for believable perturbation theory one must not only have “small” couplings $h\lambda$, $hg^2$, but the two logarithms must also be small. As was explained a long time ago by Coleman and Weinberg (CW) one must make a ($\phi$-dependent) choice of $\mu$ such that the logarithms are not too large. To relate the renormalized parameters at different scales one uses the renormalization group (RG). The CW procedure of RG “improving” the potential is equivalent to a resummation of the large logarithms in the perturbation series.

However, it is not too difficult to see that if $m^2 + \frac{1}{2}\lambda \phi^2 >> g^2 \phi^2$ (the heavy Higgs case) or $g^2 \phi^2 >> m^2 + \frac{1}{2}\lambda \phi^2$ (a heavy fermion) there is no choice of $\mu$ that will simultaneously render both logarithms small. Thus, we are only able to implement the CW method when $m^2 + \lambda \phi^2 \sim g^2 \phi^2$, i.e. when we have essentially a one-scale problem.

### 2 Multiscale renormalization

We have seen that in the MS scheme the RG equation is not powerful enough to deal with multiscale problems. In a single scale problem one is able to track the relevant scale with the MS RG scale $\mu$, whereas in a 2-scale model it is not possible to track two scales with a single RG scale. In order to deal with the 2-scale case it seems natural to seek a 2-scale version of MS. This 2-scale scheme, should be as similar to MS as possible, but with two RG scales $\kappa_1$ and $\kappa_2$ instead of one MS scale $\mu$. Here $\kappa_1$ and $\kappa_2$ should “track” the Higgs and fermionic scale, respectively. Using such a 2-scale scheme one should be able to sum up the two logarithms in the perturbation series. For attempts to deal with this problem while retaining a single scale RG see refs. 4.

How do we define a 2-scale subtraction scheme? In fact, a multiscale renormalization scheme has already been proposed by Einhorn and Jones (EJ) which has some of the properties we seek. To motivate their idea, let us look at the bare Lagrangian for our Higgs-Yukawa problem written in terms of the usual MS renormalized parameters.

\[
\mathcal{L}_{\text{Bare}} = \frac{1}{2}Z_{\phi}(\partial_\mu \phi)^2 - \frac{1}{2}Z_{\phi}Z_{m^2}m^2\phi^2 - \frac{1}{24}Z_{\phi}^2Z_{\lambda}\mu^4\lambda\phi^4
+ Z_\psi \bar{\psi}i\gamma_\mu \psi + Z_\psi Z_{\phi} \frac{1}{2}Z_{\phi} \frac{1}{2}Z_{\psi} \lambda\phi^4 + \Lambda + (Z_{\lambda} - 1)\mu^{-d}m^4\lambda^{-1},
\]  

where $\epsilon = 4 - d$ is the dimensional continuation parameter, and all the $Z$ factors have the form $Z = 1 + \text{pole terms only}$. Notice that the MS RG scale $\mu$ enters eqn. 3 in three places. The EJ idea was simply to replace the three
occurrences of μ in (3) with three independent RG scales \( \kappa_1, \kappa_2, \kappa_3 \), so that

\[
\lambda_{\text{Bare}} = \kappa_1' Z \lambda, \quad g_{\text{Bare}} = \kappa_2' Z g, \quad \Lambda_{\text{Bare}} = \Lambda + \kappa_3^{-1}(Z \Lambda - 1)m^4 \lambda^{-1}.
\] (4)

As in standard MS, the \( Z \) factors are defined by the requirement that the effective action is finite when written in terms of the renormalized parameters and the restriction that the \( Z \) factors have the form \( Z = 1 + \text{pole terms only} \). Note that the \( Z \) factors will not be the same as the MS \( Z \) factors (except where \( \kappa_1 = \kappa_2 = \kappa_3 \)). In the EJ scheme, the \( Z \)'s will contain logarithms of the RG scale ratios.

We now have three separate RG equations associated with the independent variations of the three RG scales. We also have three sets of beta functions

\[
i \beta_{\lambda} = \kappa_i \frac{d}{d\kappa_i} \lambda \quad i = 1, 2, 3.
\] (5)

and similarly for the other parameters. It is straightforward to compute the one-loop beta functions in the EJ scheme:

\[
2 \beta_g = \frac{5 \hbar g^3}{3(4\pi)^2}, \quad 1 \beta_g = 3 \beta_g = 0,
\]

\[
1 \beta_{\lambda} = \frac{\hbar}{(4\pi)^2}(3 \lambda^2 + 48 g^4), \quad 2 \beta_{\lambda} = \frac{\hbar}{(4\pi)^2}(8 \lambda g^2 - 96 g^4), \quad 3 \beta_{\lambda} = 0.
\] (6)

One may be tempted now to turn these one-loop RG functions into running couplings via eqs. (5). However, if one were to compute the two-loop beta functions, one would find terms proportional to \( \log \frac{\kappa_1}{\kappa_2} \), and in general the \( n \)-loop RG functions contain \( \log^{n-1} \frac{\kappa_1}{\kappa_2} \) terms (as well as lower powers of the logarithm). Therefore, unlike in standard MS we cannot trust the perturbative RG functions. So if we still wish to use the EJ scheme we must somehow perform a large logarithms expansion on the beta functions themselves.

Another problem with the EJ prescription is that although it has two RG scales (three if you include \( \kappa_3 \) which is only relevant to the running cosmological constant) \( \kappa_1 \) and \( \kappa_2 \), they do not seem to “track” the Higgs and fermionic scales, respectively. If such a tracking were present we would expect the one-loop beta function for \( \lambda \) to have the form

\[
1 \beta_{\lambda} = \frac{3 \hbar \lambda^2}{(4\pi)^2}, \quad 2 \beta_{\lambda} = \frac{\hbar}{(4\pi)^2}(8 \lambda g^2 - 48 g^4).
\] (7)

That is the contributions to \( 1 \beta_{\lambda} \) and \( 2 \beta_{\lambda} \) can be identified with contributions from the Higgs and fermion loop, respectively. So although the EJ proposal
is very interesting, it is not quite what we were looking for. However, it may still be that some (possibly quite simple) modification of the EJ scheme does the job. Another possibility would be to construct a multiscale version of the Callan-Symanzik equation.

Although we are unable to define such a modified EJ scheme we can exploit the fact that any multiscale scheme must be related to the standard MS prescription by a finite renormalization. That is if we have a scheme with two RG scales \( \kappa_1 \) and \( \kappa_2 \) then we must have

\[
\begin{align*}
g_{\text{MS}} &= F_g(g, \lambda; \kappa_1, \kappa_2, \mu) \\
\lambda_{\text{MS}} &= F_\lambda(g, \lambda; \kappa_1, \kappa_2, \mu) \\
m^2_{\text{MS}} &= m^2 F_m (g, \lambda; \kappa_1, \kappa_2, \mu),
\end{align*}
\]

with similar relations for \( \Lambda, \phi \) and \( \psi \). Here, the MS parameters \( g_{\text{MS}}, \lambda_{\text{MS}}, \) etc. at scale \( \mu \) may be regarded as “bare” ones as opposed to the new “renormalized” 2-scale parameters \( g, \lambda, \) etc. The (finite) \( F \) functions are chosen so that:

i) The effective action \( \Gamma \), when expressed in terms of the new parameters should be independent of the MS scale \( \mu \).

ii) When \( \kappa_1 = \kappa_2 \) the 2-scale scheme should coincide with MS at that scale.

There are an infinite number of 2-scale schemes (ie. \( F \) functions) satisfying conditions i) and ii). Each of these schemes will have different beta functions. Let us now restrict ourselves to schemes with the correct one-loop tracking behaviour. That is we assume that the one-loop beta functions for \( \lambda \) are as in eqn. (7). This tracking assumption also fixes the one-loop RG functions for \( m^2, \Lambda, \phi \) and \( \psi \). The tracking assumption does not fix the one-loop beta functions for \( g \); all we can say is that

\[
\begin{align*}
1\beta_g &= \frac{\epsilon_1 h g^3}{(4\pi)^2}, \\
2\beta_g &= \frac{\epsilon_2 h g^3}{(4\pi)^2},
\end{align*}
\]

where \( \epsilon_1 + \epsilon_2 = \frac{5}{3} \).

A problem with the EJ scheme was the occurrence of logarithmic terms in the higher loop RG functions. Is it possible to devise a 2-scale scheme where the beta functions have no such logarithms? The answer to this question is no, and so anyone wishing to generalize the EJ scheme must face the problem of resumming logs in the beta functions themselves. To see this consider the two RG equation for the effective potential

\[
\mathcal{D}_i V = 0, \quad \mathcal{D}_i = \kappa_i \frac{\partial}{\partial k_i} + i\beta_g \frac{\partial}{\partial g} + i\beta_\lambda \frac{\partial}{\partial \lambda} + i\beta_{m^2} \frac{\partial}{\partial m^2} + i\beta_\Lambda \frac{\partial}{\partial \Lambda} - i\beta_\phi \frac{\partial}{\partial \phi},
\]

\( ^c \) We assume that the transformation has a trivial dependence on \( m^2 \).
where \( i = 1, 2 \) and the summation convention was not used in the last equation.

We have the integrability condition

\[
[D_1, D_2] = 0.
\]  

(11)

The point is that the absence of logs in the RG functions, i.e. \([\kappa_i \partial / \partial \kappa_i, D_j] = 0\) is incompatible with the integrability condition eqn. (11).

However, it is still possible to arrange for one of the two sets of beta functions to be independent of \( \kappa_1 / \kappa_2 \), e.g. we can take the first set of beta functions to be independent of \( \kappa_1 / \kappa_2 \). Alternatively, we can take the second set of RG functions (tracking the fermionic scale) to be independent of \( \kappa_1 / \kappa_2 \). Whichever of these prescriptions we adopt, we can then use the integrability condition to resum the logarithms in the other set of beta functions. Some detailed calculations of this type (though in a different model) have been given in ref. 7.

3 Decoupling and Conclusions

We have argued that it is possible to construct a 2-scale scheme with appropriate tracking at one-loop where one of the two sets of RG functions is independent of the RG scales. Let us consider the case where we require that the first set of beta functions (tracking the Higgs scale) is independent of \( \kappa_1 / \kappa_2 \). Then at leading order the first set of beta functions are

\[
1\beta_{\lambda} = \frac{3h\lambda^2}{(4\pi)^2}, \quad 1\beta_{m^2} = \frac{h\lambda m^2}{(4\pi)^2}, \quad 1\beta_{\Lambda} = \frac{hm^4}{2(4\pi)^2}, \quad 1\beta_{\phi} = 0, \quad 1\beta_{g} = \frac{\epsilon_1 h g^4}{(4\pi)^2}.
\]

(12)

Clearly, these beta functions are just the usual one-loop beta functions for pure \( \phi^4 \) theory (provided we make the choice \( \epsilon_1 = 0 \)). The second set of beta functions depend on \( \kappa_1 / \kappa_2 \); this dependence can be computed via the integrability condition (11). Now if the fermion is much heavier than the Higgs scale we would expect that the beta functions for the low energy theory would be exactly those given by eqn. (12), since in this case we expect to observe a decoupling of the heavy fermion. Thus, it seems that the condition that the first set of beta functions is independent of the RG scales is appropriate for the heavy fermion case \((g^2 \phi^2 >> m^2 + \frac{1}{4} \lambda \phi^2)\). Similarly, one can argue that the alternative possibility of requiring that the second set of beta functions is independent of \( \kappa_1 / \kappa_2 \) is suited to the heavy Higgs case \((m^2 + \frac{1}{4} \lambda \phi^2 >> g^2 \phi^2)\).

Note that although in the previous section we were unable to fix the values of \( \epsilon_1 \) and \( \epsilon_2 \) via a one-loop tracking condition, an explicit calculation shows that the final improved potential only depends on \( \epsilon_1 \) and \( \epsilon_2 \) through the combination \( \epsilon_1 + \epsilon_2 = \frac{5}{3} \).
From decoupling arguments we expect the case where the first set of beta functions has no logarithms is suited to the heavy fermion case. We believe that this approach would correctly interpolate between the heavy fermion case and the single scale regime \( (m^2 + \frac{1}{2} \lambda \phi^2 \sim g^2 \phi^2) \), since by construction our two scale scheme collapses to MS for \( \kappa_1 = \kappa_2 \). There is no reason to expect that this prescription would be valid in the heavy Higgs case (for this we would use the alternative possibility where the second set of beta functions is independent of the RG scales). Thus, it seems that we can deal with both the heavy fermion and heavy Higgs problem, but these require a separate treatment. It remains an open question whether it is possible to devise a simple EJ type scheme which can interpolate all the way from the heavy fermion to the heavy Higgs sector.

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