Automorphisms of Salem degree 22 on supersingular K3 surfaces of higher Artin invariant

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We give a short proof that every supersingular K3 surface (except possibly in characteristic 2 with Artin invariant $\sigma = 10$) has an automorphism of Salem degree 22. In particular an infinite subgroup of the automorphism group does not lift to characteristic zero. The proof relies on the case $\sigma = 1$ and the cone conjecture for K3 surfaces.

1. Introduction

A Salem number is a real algebraic integer $\lambda > 1$ which is conjugate to $1/\lambda$ and all whose other conjugates lie on the unit circle. Its minimal polynomial is called a Salem polynomial. Salem numbers arise naturally in algebraic geometry as follows: If $X$ is a projective surface over an algebraically closed field $k$ and $f: X \to X$ an automorphism, then the characteristic polynomial

$$\chi(f^*|H^2_{\text{ét}}(X, \mathbb{Q}_\ell(1)))$$

factors as a product of cyclotomic polynomials and at most one Salem polynomial $s(x)$ [5]. We call the degree of the Salem factor $s(x)$ the Salem degree of $f$. Let $H$ be an ample polarization of $X$. Since the order of $f^*$ is finite on

$$(f^{*k}(H) \mid k \in \mathbb{Z})^\perp \subseteq H^2_{\text{ét}}(X, \mathbb{Q}_\ell(1))$$

by [5], we get that $\ker s(f^*|H^2_{\text{ét}}(X, \mathbb{Q}_\ell(1)))$ is contained in the ($\ell$-adic) Néron-Severi group $\text{NS}(X) \otimes \mathbb{Q}_\ell$ of $X$. In particular, we can bound the Salem degree of an automorphism by the Picard number $\rho(X)$. For a K3 surface $X$ it is at most $\rho(X) \leq h^{1,1}(X) = 20$ in characteristic 0 by Lefschetz' Theorem on (1,1)-classes. However, in positive characteristic supersingular K3 surfaces have $\rho(X) = 22$. Indeed:
Theorem 1.1. \([1\,1\,4\,11\,12]\) The supersingular K3 surface \(X/k,\, k = \mathbb{k},\) with char \(k > 0,\) of Artin invariant one has an automorphism of Salem degree 22.

Note that the characteristic polynomial of \(f^*\) is stable under (good) specialization by standard comparison theorems. This observation leads to the interesting feature that an automorphism of Salem degree 22 is not geometrically liftable to characteristic zero (see [3] for details). This is in sharp contrast to the case of non-supersingular K3 surfaces in odd characteristic. There one can always lift a finite index subgroup of the automorphism group to characteristic zero (cf. [6, Thm. 3.2]).

Supersingular K3 surfaces are classified by their Artin invariant \(1 \leq \sigma \leq 10.\) For fixed Artin invariant \(\sigma\) they form a family of dimension \(\sigma - 1,\) while the supersingular K3 surface of Artin invariant \(\sigma = 1\) is unique (cf. [8, 9]). The main purpose of this note is to extend Theorem 1.1 to all supersingular K3 surfaces.

Theorem 1.2 (Main Theorem). Let \(Y/k\) be a supersingular K3 surface over an algebraically closed field such that the crystalline Torelli theorem holds for \(Y.\) Then \(Y\) has an automorphism of Salem degree 22.

Remark 1.3. Set \(p = \text{char } k\) and \(\sigma = \sigma(Y).\) The crystalline Torelli is proven for \(p > 3\) in [8, Thm. 1] and for \(p = 2\) and \(\sigma < 10\) and for \(p = 3\) and \(\sigma < 6\) (at the end of [9]). For \(p = 3\) the main theorem is proved in [12]. Hence the only open case left is \(p = 2\) and \(\sigma = 10.\) The main step in the proof is a reduction to Theorem 1.1.

In a recent preprint [14] Yu gives an independent proof of the main theorem for \(p > 3\) using genus one fibrations. However, I believe the new proof to be of independent interest, as it is shorter and characteristic free. In particular the result for \(p = 2, \sigma > 1\) is new.

2. Preliminaries

A lattice \(L\) is a finitely generated free abelian group equipped with a non-degenerate, integer valued bilinear form. It is called even if \(x^2 \in 2\mathbb{Z}\) for all \(x \in L.\) The dual lattice is \(L^\vee = \{x \in L \otimes \mathbb{Q} : x.L \subseteq \mathbb{Z}\}\) and the discriminant group \(L^\vee/L\) of an even lattice \(L\) is equipped with the quadratic form

\[q : L^\vee/L \to \mathbb{Q}/2\mathbb{Z},\quad \bar{x} \mapsto x^2 \mod 2\mathbb{Z}.\]
A supersingular K3 lattice $N$ is an even lattice of signature $(1,21)$ and discriminant group $N^\vee/N \cong \mathbb{F}_p^{2\sigma}$. If $p = 2$, we require furthermore that it is of type I, i.e. $x^2 \in \mathbb{Z}$ for $x \in N^\vee$. Such a lattice is determined up to isometry by $p$ and $\sigma$ (cf. [9, sect. 1]). Let $X$ be a K3 surface defined over an algebraically closed field $k$ of characteristic $p$. Recall that $X$ is said to be supersingular if

$$\rho(X) = \text{rk} \text{ NS}(X) = 22.$$ 

Then the Néron-Severi lattice $\text{NS}(X)$ is a supersingular K3 lattice for $p = \text{char } k$ and $1 \leq \sigma \leq 10$ (cf. [9, sect. 8]). We call $\sigma$ the Artin invariant of $X$.

For the readers’ convenience we give a proof of the following well known

**Lemma 2.1.** There is an embedding $N_{p,\sigma} \hookrightarrow N_{p',\sigma'}$ of supersingular K3 lattices if and only if $p = p'$ and $\sigma' \leq \sigma$.

**Proof.** The only if part follows from the fact that if $A \subset B$ are two lattices of the same rank, then

$$\det A = [B : A]^2 \det B.$$ 

In this situation

$$A \hookrightarrow B \hookrightarrow B^\vee \hookrightarrow A^\vee$$

and $B/A$ is a totally isotropic subspace of $A^\vee/A$. Now, if $A$ is 2-elementary of type I, then, since $B^\vee \subset A^\vee$, so is $B$. Let $p \neq 2$. Then the quadratic space $N_{p,10}^\vee/N_{p,10} \cong \mathbb{F}_p^{18}$ contains an isotropic line since it is of dimension greater than two. As above this line corresponds to an overlattice $N$ of $N_{p,10}$ which is hyperbolic and $|N^\vee/N| = p^{18}$. Since subquotients of vector spaces are vector spaces, we see that $N^\vee/N \cong \mathbb{F}_p^{18}$. Then $N \cong N_{p,9}$ is in fact a supersingular K3 lattice. Continuing in the same way, we get a chain of overlattices

$$N_{p,10} \subseteq N_{p,9} \subseteq \cdots \subseteq N_{p,1}.$$ 

Note that the process stops at $\sigma = 1$ since there is no isotropic line in the discriminant group. This is in accordance with the fact that there is no even unimodular lattice of signature $(1,21)$. For $p = 2$ the discriminant form is isomorphic to a direct sum of forms of type $q(x,y) = x^2 + xy + y^2 \mod 2\mathbb{Z}$ and the existence of an isotropic vector follows as long as there are at least two summands, i.e., $\sigma > 1$. Since everything is contained in $N_{p,10}^\vee$, the constructed lattices stay of type I.
Let $L$ be an even lattice of signature $(1, n)$ and denote by $O^+(L)$ the subgroup of isometries preserving the two connected components of the positive cone. Set

$$V_L = \{ x \in L \otimes \mathbb{R} \mid x^2 > 0 \text{ and } \forall r \in L \text{ with } r^2 = -2 : (r, x) \neq 0 \} .$$

According to [8, Proposition 1.10], the set $V_L$ is open and each of its connected components meets $L \subset L \otimes \mathbb{R}$. These connected components of $V_L$ are called chambers of $V_L$. Each point $r$ of length $-2$ induces an orthogonal reflection

$$\delta_r : L \to L \quad x \mapsto x + \langle x, r \rangle r$$

along the hyperplane $r^\perp$. The Weyl group $W(L) \subseteq O(L)$ is the group generated by all orthogonal reflections along a $(-2)$-hyperplane. It acts transitively on the set of chambers.

If $L = \text{NS}(X)$ for a K3 surface $X$, then one of the chambers is the ample cone. Its closure is the nef cone $\text{Nef}(X)$. Classes of smooth rational curves are called nodal. By adjunction they are of square $(-2)$ and they are exactly the rays of the effective cone. Note that if $r^2 = -2$, then by Riemann-Roch either $r$ or $-r$ is effective but they are not necessarily nodal.

**Theorem 2.2 (Cone conjecture).** [7, Thm. 6.1] Let $X$ be a K3 surface over an algebraically closed field $k$. If $X$ is supersingular suppose that crystalline Torelli holds for $X$. Let $\Gamma(X) \subseteq O^+(\text{NS}(X))$ be the subgroup preserving the nef cone. Then $\Gamma(X) \cong O^+(\text{NS}(X))/W(\text{NS}(X))$ and

1) The natural map $\text{Aut}(X) \to \Gamma(X)$ has finite kernel and cokernel.

2) The group $\text{Aut}(X)$ is finitely generated.

3) The action of $\text{Aut}(X)$ on $\text{Nef}(X)$ has a rational polyhedral fundamental domain.

4) The set of orbits of $\text{Aut}(X)$ in the nodal classes of $X$ is finite.

Over $\mathbb{C}$ the theorem follows from the strong Torelli theorem by work of Sterk [13, Thm. 01.]. Then, for K3 surfaces of finite height in arbitrary characteristic one can lift $X, \text{NS}(X)$ and a finite index subgroup of $\text{Aut}(X)$ to characteristic zero and apply the cone theorem there. For supersingular K3 surfaces one has to use the crystalline Torelli Theorem. In this case $\text{Aut}(X) \to \Gamma$ is injective and its image contains the finite index subgroup $\ker(\Gamma \to O(\text{NS}^\vee/\text{NS}))$. 
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Lemma 2.3. [10, p. 169] If $\lambda$ is a Salem number of degree $d$ then $\lambda^n, n \in \mathbb{N}$ is a Salem number of the same degree.

Proof. Denote the Galois conjugates of $\lambda = \lambda_1$ by $\lambda_i i = 1, \ldots, n$. Then the Galois conjugates of $\lambda_i^n$ are the $\lambda_i^n$. In particular $\lambda^n$ is a Salem number. It remains to check that its conjugates are all distinct. Suppose that $\lambda^n = \lambda_k^n$. After applying a Galois conjugation we may assume that $i = 1$. In particular, $1 < \lambda_1^n = \lambda_k^n$. Now, $|\lambda_k| > 1$ is the unique conjugate of absolute value greater one, i.e. $k = 1$. □

Corollary 2.4. The maximum occurring Salem degree of an automorphism of a K3 surface $X$ over an algebraically closed field depends only on the isometry class of $\text{NS}(X)$, given that the cone conjecture holds for $X$.

Proof. Since any power of a Salem number of degree $d$ remains a Salem number of this degree, we may pass to a finite index subgroup. Combining this with part (1) of the cone conjecture, we get that the maximum occurring Salem degree of an automorphism of $X$ depends only on $\Gamma(X)$. Now, $\Gamma(X)$ depends up to conjugation by an element of the Weyl group only on the isometry class of $\text{NS}(X)$. In particular, the maximal Salem degree of an automorphism of $X$ depends only on $\text{NS}(X)$. □

3. Proof of the main theorem

Lemma 3.1. Let $N \subseteq L$ be two lattices of the same rank and $G \subseteq O(L)$ a subgroup. Then

$$[G : O(N) \cap G] < \infty$$

where we view $O(N)$ and $O(L)$ as subgroups of $O(N \otimes \mathbb{R})$.

Proof. Since the ranks coincide, the index $n = [L : N]$ is finite and

$$nL \subseteq N \subseteq L.$$  

Any isometry of $L$ preserves $nL$ hence we get a map

$$\varphi : G \to \text{Aut}(L/nL).$$

Set $K = \ker \varphi$, which is a finite index subgroup of $G$. To see that $K \subseteq O(N)$ as well, recall that an isometry $f$ of $O(nL)$ extends to $O(N)$ iff $f(N/nL) = N/nL$. Indeed, $f|L/nL = id|L/nL$ for $f \in K$, by definition. □
The following is a generalization of [14, Thm. 1.2] where the existence of at least one elliptic fibration on $X$ with infinite automorphism group is assumed. We can drop this condition.

**Theorem 3.2.** Let $X/k, Y/k'$ be two K3 surfaces over algebraically closed fields $k, k'$ satisfying the cone conjecture. Suppose that $\rho(X) = \rho(Y)$ and that there is an isometric embedding

$$\iota : \text{NS}(Y) \hookrightarrow \text{NS}(X).$$

Then $s\deg(X) \leq s\deg(Y)$ where

$$s\deg(X) = \max\{\text{Salem degree of } f \mid f \in \text{Aut}(X)\}.$$

**Proof.** Denote by Nef$(X)$ and Nef$(Y)$ the nef cones of $X$ and $Y$. Any chamber of the positive cone of NS$(X)$ is contained in the image of a unique chamber of the positive cone of NS$(Y)$. Since the Weyl group acts transitively on the chambers, we can find an element $\delta \in W(\text{NS}(X))$ of the Weyl group such that Nef$(X) \subset \iota'_\delta(\text{Nef}(Y))$ where $\iota' = \delta \circ \iota$. To ease notation we identify NS$(Y)$ with its image under $\iota'$. By the preceding Lemma $[\Gamma(X) : \Gamma(X) \cap O(\text{NS}(Y))]$ is finite, and since Nef$(X) \subseteq \text{Nef}(Y)$, we get that $\Gamma(X) \cap O(\text{NS}(Y)) \subseteq \Gamma(Y)$. Now, by the cone Theorem 2.2 and the proof of Corollary 2.4

$$s\deg(X) = s\deg(\Gamma(X)) = s\deg(\Gamma(X) \cap O(\text{NS}(Y)))$$

$$\leq s\deg(\Gamma(Y)) = s\deg(Y).$$

□

**Proof of Theorem.** If $X/k$ and $Y/k$ are supersingular K3 surfaces with $\sigma(X) \leq \sigma(Y)$, then NS$(Y) \hookrightarrow$ NS$(X)$ by Lemma 3.1. Combining the $\sigma = 1$ case (Thm. 1.1) and the previous theorem we get that $22 = s\deg(X) \leq s\deg(Y) \leq 22$. □

The converse inequality in Theorem 3.2 is false in general. See [14, rmk. 7.3] for examples.

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References

[1] J. Blanc and S. Cantat, Dynamical degrees of birational transformations of projective surfaces, J. Amer. Math. Soc. 29 (2016), no. 2, 415–471.

[2] S. Brandhorst, Dynamics on supersingular K3 surfaces and automorphisms of Salem degree 22, Nagoya Math. J. (2016), 1–15.

[3] H. Esnault and K. Oguiso, Non-liftability of automorphism groups of a K3 surface in positive characteristic, Math. Ann. 363 (2015), no. 3-4, 1187–1206.

[4] H. Esnault, K. Oguiso, and X. Yu, Automorphisms of elliptic K3 surfaces and Salem numbers of maximal degree, Alg. Geom. 3 (2016), no. 4, 496–507.

[5] H. Esnault and V. Srinivas, Algebraic versus topological entropy for surfaces over finite fields, Osaka J. Math. 50 (2013), no. 3, 827–846.

[6] J. Jang, A Lifting of an Automorphism of a K3 Surface over Odd Characteristic, Int. Math. Res. Notices (2016).

[7] M. Lieblich and D. Maulik, A note on the cone conjecture for K3 surfaces in positive characteristic, (2011).

[8] A. Ogus, A crystalline Torelli theorem for supersingular K3 surfaces, in: Arithmetic and geometry, Vol. II, Vol. 36 of Progr. Math., 361–394, Birkhäuser Boston, Boston, MA (1983).

[9] A. N. Rudakov and I. R. Shafarevich, Surfaces of type K3 over fields of finite characteristic, in: Current problems in mathematics, Vol. 18, 115–207, Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Informatsii, Moscow (1981).

[10] R. Salem, Power series with integral coefficients, Duke Math. J. 12 (1945), 153–172.

[11] M. Schütt, Dynamics on supersingular K3 surfaces, Comment. Math. Helv. (2016), 705–719.

[12] I. Shimada, Automorphisms of supersingular K3 surfaces and Salem polynomials, Exp. Math. 25 (2016), no. 4, 389–398.

[13] H. Sterk, Finiteness results for algebraic K3 surfaces, Math. Z. 189 (1985), no. 4, 507–513.
[14] X. Yu, *Elliptic fibrations on K3 surfaces and Salem numbers of maximal degree*, J. Math. Soc. Japan **70** (2018), no. 3, 1151–1163. DOI: 10.2969/jmsj/75907590.

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