We have previously (Gomez and Schubert (2011)) defined synchronization as a relation between the times at which a pair of events can happen, and introduced an algebra that covers all possible relations for such pairs. In this work we introduce the synchronization matrix, to make it easier to calculate the properties and results of $N$ event synchronizations, such as are commonly encountered in parallel execution of multiple processes. The synchronization matrix leads to the definition of $N$-event synchronization algebras as specific extensions to the original algebra. We derive general properties of such synchronization, and we are able to analyze effects of synchronization on the phase space of parallel execution introduced in Gomez E (2017).

Index Terms—synchronization; entropy; Boolean algebra
I. ALGEBRA OF SYNCHRONIZATION

Real time advances monotonically; to achieve a particular ordering between events $a$ and $b$, it is sufficient to block $a$ or $b$ or both and release them when the relation is satisfied. Our Boolean algebra of synchronization Gomez and Schubert (2011) and Gomez and Schubert (2010) describes all possible relations between the times associated with any number of event pairs. In our previous work, we described how a set of synchronizations can imply relations that are not explicit in the code, leading to unanticipated results such as deadlock. At that time we handled N-event synchronizations from a graph of connections. In this work we give systematic methods to describe and analyze any number of events. These methods become more relevant after Gomez E (2017). In that work, we showed a phase space for parallel execution and its relation to execution entropy, N-event analysis is required to predict the effect of synchronization on execution phase space and estimate the work required.

A. Algebra of synchronization for event pairs

Synchronization is often described in the context of operating systems (see Tanenbaum and Woodhull (1997), Tanenbaum (2001), Stallings (2005), and Silberschatz, Galvin, and Wylie (2005),) usually through examples of synchronization mechanisms. The literature on parallel computing such as Foster (1995), Jordan and Alaghband (2003), and L. Ridgway Scott (2005) has mostly also treated synchronization through specific example. Apt and Olderog (1997) establish a more theoretical approach to synchronization between parallel processes, using the semantics of process waiting. However they also do not give a general definition of synchronization.

In Gomez and Schubert (2011) (2010) we introduced a formal definition of synchronization as a relation between sets of allowable times for two events. This led to a Boolean algebra of synchronization based on subset relations between event pairs. It follows that a synchronization between multiple events can be described in terms of a collection of event pairs, and we used it in that paper to prove some properties of semaphores and deadlock.

We associate each event with a real number denoting the time at which it occurs. There are precisely 6 possible relations between ordered pairs of numbers - adding 1 for union of all relations and 0 for the empty set, we have the set of binary relations , together with operators union $\cup$, intersection $\cap$, and complement $\sim$ we have the algebra of synchronization. The algebra is complete for event pairs, since there are no other mathematical relations between two numbers.

Let $s \in S$, then the ordered tuple $s(A, B)$ is the set of all pairs of times $A$ and $B$ that satisfy $A \leq B$. In figure I.1 we show how $S$ is displayed as a graph, with 1 and 0 at opposite vertices. That graph is a lattice (see Gill (1976), Birkhoff and Lane (1999), each node of the graph except TOP=1 and BOTTOM=0 is on multiple paths from TOP to BOTTOM, and it is easily verified that, for any pair of nodes on the same path, the node closer to TOP is a superset of the node closer to BOTTOM. It is simple to verify that the set $\{1, \leq, \geq, \neq, >, <, 0\}$ including all the nodes in the graph, when interpreted as labels denoting sets of ordered pairs, is closed under union, intersection and complement, and is therefore an algebra of subsets. Immediately this means that $S = \{1, \leq, \geq, \neq, >, <, 0\}, \cup, \cap, \sim$ (the set of relations with operations union, intersection and complement) is a Boolean algebra, isomorphic to $B^3$ because it has $2^3 \text{ elements}$. The lattice for a Boolean algebra $B^n$ is an n-dimensional cube or hypercube (see Gill (1976). See figure E.1.

As a Boolean algebra, the entire body of rules developed for such algebras (for example DeMorgan’s law) is immediately applicable to $S$.

The elements of $S$ represent the relation between two sets of points - for instance $A > B$ means that a selected point in set $A$ is greater than every point in set $B$; so $B$ is bounded from above; correspondingly any point we select in $B$ is less
than every point in \( A \). \( A = B \) means that the range of both sets is the same, it is always possible to select matching points in \( B \) and \( A \). A further implication: Since \( S \) is an algebra of pairs ordered by relations between numbers, we have some relations that are antisymmetric - \( \{ \geq, \leq, \neq, >, < \} \) so we add a mirror \( \mu \) unary operator. It is immediate that \( S \) is closed under \( \mu \), and \( \mu(a(\cap | \cup | ~)b) \Rightarrow \mu(a)(\cup | \cap | ~)\mu(b) \) - that is, \( \mu \) is associative with the standard Boolean operators.

Summarizing, we have:

\[
s \in \{ 1, \geq, \leq, \neq, >, <, 0 \},
\]

\( S \) is the set of number pairs \((t_1, t_2)\) where \( t_1 \) and \( t_2 \) is the relation between times \( t_i \).

\( S \) is an algebra over the set of ordered pairs of numbers, complete and closed under the given operations (union, intersection, complement and mirror), is Boolean algebra of subsets isomorphic to \( B^3 \) (see Gomez and Schubert (2011) and Gomez and Schubert (2010)). When considering >2 events, we will refer to this algebra as \( S_2 \) and its extension to \( n \) events as \( S_n \). We have:

- \( \cup, \cap \) (union, intersection) are associative and commutative over each other, by properties of subset algebras.
- \( \sim, \mu \) (complement, mirror) are commutative (complement is the same relation as not)
- \( \sim, \mu \) are associative over \( \cap, \cup \)
- deMorgan’s laws: \( \sim(A \cup B) \Rightarrow \sim A \cap \sim B \) and \( \sim(A \cap B) \Rightarrow \sim\sim A \cup \sim B \)

We will use the symbols in \( S \) as label for a synchronization defined by the given symbol, as well as in numeric expressions.

Consider now the effects of synchronization on a single event. It is evident from \( S \) that a single event can be bounded from above, below, above and below or unbounded. This gives a Boolean algebra of boundedness as a sub-algebra of \( S \) isomorphic to \( B^2 \), this is described in Gomez and Schubert (2011) and Gomez and Schubert (2010) and furthered detailed here. Formal development of N-event algebra, the synchronization matrix and other properties is developed below.

1) Sub-algebras of \( S_2 \): We identify 2 sub-algebras, \( L_1, \geq, \leq, =, L_0 = \neq, >, <, 0 \). \( L_1 \) includes the \( = \) relation, \( L_0 \) does not. Every relation in \( L_1 \) includes points on the boundary between sets, whereas \( L_0 \) does not include points on the boundary. We can view \( L_1 \) as expressing relations between closed sets, and \( L_0 \) as relations between non-overlapping open sets. Immediately we have that \( \cup, \cap \) are closed in each of the sub-algebras, as is mirror \( \mu \) which just reverses the order. The \( ~ \) (not) relation from \( S_2 \) shifts between \( L_1 \) and \( L_0 \) since \( ~\neq \) is the same as \( ; \) we take complement to be the same as mirror in the sub-algebras. (see Gomez and Schubert (2010)).

B. Properties of the binary synchronization algebra

1) Algebra of bounded event sets \( \beta \): In order to deal with synchronizations involving multiple events, we need to account for cases in which a single event synchronizes with more than one other event. We describe how synchronization imposes boundaries on events to analyze this. Since in \( S_2 \) is an ordered pair of events \((t_1, t_2)\), where each \( t_i \) is a set of numbers corresponding to allowed event times, synchronization imposes a boundary on membership in each set. We use notation LH and RH to denote sets textually to the left and to the right of a synchronization operator, respectively:

1) No boundaries: \( 1, \neq \) do not impose any upper or lower bounds on sets; in either case the range of allowed values extends to positive and negative infinity. \( \neq \) excludes a point (or continuous subset of points) corresponding to an event in RH from the set of points in LH (and vice-versa). The excluded point(s) are bounded above and
below, but do not impose upper or lower bounds on sets RH and LH.

2) Bounded above and below: =, 0 impose upper and lower bounds on both LH and RH. The bounds on the LH set are the same as on RH, in the case of 0 synchronization, upper and lower bounds may coincide allowing no points in either set, but it is possible for two events to bind a third - \( a > x > b \) or \( a \geq x \geq b \). The first case yields 0, but does not have to be empty if \( a > b \), this makes \( x \) an open set not including its boundaries. The second case yields =, the set \( x \) includes boundaries. A mixed case is still 0 since the relation between \( a \) and \( b \) satisfies \( a \cap b = \phi \).

3) Bounded above or below: \( \geq, > \) or \( \leq, < \) impose a single upper (lower) bound, differing only in that \( \geq, \leq \) include the boundary value in both RH and LH sets, and \( >, < \) exclude the boundary from both.

We get 8 cases, 4 isomorphic to \( L_0 \): \( B_0 \Rightarrow L_0 \) (excluding boundaries) and 4 isomorphic to \( L_1 \): \( B_1 \Rightarrow L_1 \) (including boundaries). Their combination gives the full algebra \( \beta \Rightarrow S_2 \). As before, the union of an element in \( B_0 \) with an element in \( B_1 \) yields an element in \( B_1 \), and intersection yields an element in \( B_0 \). \( \beta \) is complete; it expresses all possible boundaries that can be imposed on an event by a synchronization with one other event. We now consider boundaries imposed by multiple events.

Remark 1. Notation: Since algebras \( \beta, B_0, B_1 \) are isomorphic to \( S, L_0, L_1 \) it is convenient to use the same notation. Given sets \( t_1, t_2 \) representing events, with a synchronization \( s_{12} \), then the set \( t_1 \) has boundary \( s_{12} \) with respect to itself, and \( t_2 \) has boundary \( s_{21} = \mu(s_{12}) \). For example, if \( s_{12} \) is \( > \), then \( t_1 \) is bounded below by \( t_2 \) and the boundary is \( > \), the same label as the synchronization.

Theorem 2. Every synchronization between \( n > 2 \) events can be described using binary synchronizations in the algebra \( S_2 \) between the given events.

Proof: Base case: events \( e_1, e_2 \) with synchronization \( s_{12} \in S_2 \), completely defines the relation between the events (proved in Gomez and Schubert (2011)).

Induction: Given \( n \) events, add event \( n+1 \) , then \( \forall i < n \) add pairs \( (e_i, e_{n+1}) \) with synchronization \( s_{i,n+1} \), which completely defines the relation between the added event and each other event. Since relations between every pair in \( n \) are completely described in \( S_2 \), and this is also true for every relation between the original \( n \) events and event \( n+1 \) is completely described in \( S_2 \), then every relation between a set of \( n+1 \) events is described in \( S_2 \).

II. SYNCHRONIZATION MATRIX AND EXTENSION TO \( N \) EVENTS

We introduce a matrix notation that makes it easier to describe synchronizations of more than 2 events. With \( > 2 \) events, we can get implied synchronizations - for example consider 3 events such that: \( e_1 > e_2 \) and \( e_2 > e_3 \). It follows that \( e_1 > e_3 \) which is not coded, but is implied. Implied synchronizations happen because we have transitivity for some synchronizations. We will describe transitivity, and show an algorithm that uses it to compute closure of a synchronization matrix, which includes all the implied synchronizations and gives us the boundedness condition on each event.

A. Synchronization matrix

Definition 3. Binary Synchronization Matrix: \[
\begin{bmatrix}
s_{11} & s_{12} \\
s_{21} & s_{22}
\end{bmatrix}
\]
The \( s_{ij} \) are in the synchronization algebra \( S_2 \), \( s_{ij} = \mu(s_{ij}) \) and diagonal elements \( s_{ii} \) are defined as 1 since they relate an event to itself.

Following are the synchronization matrices for algebra \( S_2 \):

\[
M_2 = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{bmatrix}
\]

The upper right element \( s_{12} \) is the synchronization between ordered event pair (1,2). The lower left \( s_{21} \) represents the event pair (2,1) which reverses (1,2), so the corresponding synchronization is the mirror, \( s_{21} = \mu(s_{12}) \). Each matrix is fully determined by the element above the diagonal, so \( M_2 \) has the same relations as \( S_2 \).

Remark 4. It would be convenient to set the elements \( s_{ii} \) to their boundedness in \( \beta \). We do not do this; the resulting matrices in \( M_n, n > 2 \) would not be closed under \( \cap, \cup \) because diagonal element boundedness is always defined by intersection, but the off-diagonal elements \( s_{ij} \) would be subject to \( \cap \) or \( \cup \). Assume matrices \( A \) and \( B \) with diagonal elements in \( \beta \). For 3 events, we have: \( a_{ii} = a_{ij} \cap a_{ik} \), \( b_{ii} = b_{ij} \cap b_{ik} \). Closure implies \( a_{ii} \cup b_{ii} = (a_{ij} \cap a_{ik}) \cup (b_{ij} \cap b_{ik}) \) and \( a_{ii} \cap b_{ii} = (a_{ij} \cap a_{ik}) \cap (b_{ij} \cap b_{ik}) \), but these four statements taken together would violate de Morgan’s laws.

We now define element-wise operations on the matrix:
Definition 5. Operations on $M_2$ follow from the operations on $S_2$; given $m_i, m_j \in M_2$ with $o \in \text{Op}=\{\cap, \cup, \sim, \mu\}$ applied on each element of a matrix $M \in M_2$:

- If $o \in \text{Op}$ is unary, then $o[M] \Leftrightarrow \forall i,j, o(m_{i,j})$
- If $o \in \text{Op}$ is binary, then $o[p,q] \Leftrightarrow \forall i,j, o(p_{i,j}, q_{i,j})$

We show $M_2$ is closed under these operations:

Lemma 6. The set $M_2$ is closed under Op $= o \in \{\cap, \cup, \sim, \mu\}$

Proof: Binary operators on matrices act on individual elements, and are closed in $S$. Mirror ($\mu$) symmetry is preserved across the diagonal because synchronization matrices all have $s_{ij} = \mu(s_{ji})$ and $\mu(a \circ b) = \mu(a) \circ \mu(b)$ for $o \in S$ (associative property)

B. Synchronization matrix for $n>2$

An $n$ event synchronization matrix is specified by the set of all synchronizations between event pairs (theorem 2). This can be represented as a synchronization matrix. We now construct an algebra $S_n$ for $n$ events, with elements $s_{ij}$ denoting relations:

Definition 7. $n$-element synchronization matrix. Using the basic layout of the binary synchronization matrix $S_2$ Number the events $i, j \in (1...n)$, this gives us $n^2$ ordered pairs of events $(i, j)$ . For an $n \times n$ matrix $M$, we map each event pair to element $m_{ij} \Leftrightarrow (i, j)$. Set the diagonal elements $m_{ii} = 1$.

We are left with $(n^2 - n)$ off-diagonal event pairs. However, every element $s_{ij}$ above the diagonal determines a corresponding element $s_{ji} = \mu(s_{ij})$ below the diagonal, so only half the event pairs are independent. Therefore

Lemma 8. The $p = (n^2 - n)/2$ elements above the diagonal fully determine the matrix. Since each of the entries above the diagonal must be an element of $S_2$ , there are precisely $8p = 2^{3p}$ different $n \times n$ synchronization matrices of $n$ events.

Corollary 9. The algebra of synchronization $S_n$ is isomorphic to Boolean $B^{3p}$, since it is a complete algebra of subsets with $2^{3p}$ elements.

C. Atoms

A Boolean algebra $B^n$ has $2^n$ elements, represented graphically as an $n$-dimensional hypercube of side 1. As $n$ increases, it becomes impractical to list all the elements of the algebra, but we can find $n$ elements $a_i$ which we call atoms, such that $a_i \in B^n$ and $a_i \cap a_j = 0$ for $i \neq j$. With $\cup$ as addition, the $n$ atoms are an additive basis for $B^n$ (Birkhoff and Lane 1951, Jacobson 1976) For example, $S_2$ is isomorphic to $B^3$, and its 3 atoms are $\{1, <, >\}$.

We visualize the $n$ atoms in the lattice hypercube as the $n$ arcs $(0,i)$, equivalent to $n$ directions in a Cartesian coordinate system. It is evident that we can reach any point by in the hypercube from any other by taking $\leq n$ steps along arcs parallel to the atoms, equivalent to $\cup$ of $n$ atoms

To construct the synchronization matrices for the 3$p$ atoms of $S_n$ : Take the 0 matrix in $S_n$. (diagonal elements are 1, off-diagonal elements are 0). Each above diagonal element $m_{i,j}$ with $i < j$ gives 3 atoms, by setting it to each of the 3 atoms of $S_2$, set below-diagonal $m_{j,i}$ to $\mu(m_{i,j})$. Repeat for each element above the diagonal.

- Calculation of boundedness

We have defined an algebra of boundedness on events in section I-B1. Since these apply to single events, we have no need to extend them for synchronization of more events. However, the bounds on any specific event may include restrictions of implied synchronizations, so we need to take this into account.

We compute the boundedness on event $i$ as follows:

Theorem 10. The boundedness condition on an event $i$ in a synchronization matrix is $1 \cap s \cap m_{ij}$.

Proof: Immediate for $s \in S_2$ (see I-B1) because $1 \cap s = s$, every $m_{ij}$ is an additional (and) restriction on the bounds of event $i$, $m_{ij}$ in row $i$ includes all pairs $(i, j)$

Corollary 11. The boundedness of an event in a synchronization matrix is in the boundedness algebra $\beta$ (see I-B1).

Proof: Let $p$ be the number of elements above the diagonal (see lemma 8), the set of $2^{3p}$ (see lemma 8) $n$ element synchronization matrices (definition 7) forms the algebra of synchronization $S_n$ for $n$ events. Since $S_n$ is a closed and complete algebra of subsets, it is Boolean.

The algebras of synchronization $S_2$ and boundedness $\beta$ are closed under binary operators $\cap, \cup$ and unary operators $\sim, \mu$ then applying these element-wise to $N \times N$ synchronization matrices yields another $N \times N$ matrix. This matrix is a synchronization matrix because $S_2$ is Boolean and closed under $\sim, \mu$ and so preserve the symmetry between upper and lower diagonal areas.

By theorem 2 we have that an $n$-event synchronization is composed of event pairs.

By lemma 10 we have that the diagonal elements are determined by the synchronizations in the same row.

From section II we have that elements below the diagonal are determined by mirror of above diagonal entries.
The $nxn$ matrix has $n^2$ entries, but only the elements above the diagonal are independent. Subtracting $n$ diagonal elements and dividing by 2 to get half the off-diagonals, we have $p = (n^2 - n)/2$ independent elements in the synchronization matrix.

There are $2^3$ elements in algebra $S_2; S_n$ with $p$ independent elements gives $2^{3p}$ combinations of values

By induction on lemmas [6] and [8] the $n$ element synchronization matrices are closed under set and mirror operators, and the set of such matrices is complete because the algebra $S$ (sec. I-A) is complete.

We have that boundedness of each event in a matrix $S_n$ is in $\beta$, so it is represented by a vector $B$ of size $n$, such that each $b_i$ represents the boundedness on event $e_{ii}$ in the synchronization matrix.

For a specific $e_{ii}$, the synchronizations that explicitly apply are $s_{ij}, j \neq i$ (row $i$ and $s_{ji}, j \neq i$ (column $i$); since column $i$ is the $\mu$ of row $i$ (by definition of a synchronization matrix) we need only consider one of the two; we choose to use synchronizations on the same row for readability. Synchronizations on row $i$ have the form $s_{ij}$ imposing a restriction on event $j$. Therefore the effect on $i$ is $\mu(s_{ij})$ and the effect of all the synchronizations in row $i$ on $e_{ii}$ is $1 \cap \prod_j \mu(s_{ij})$, where $\cap$ takes the expected meaning of applying all the restrictions on the row to the same event. If we need an or condition it can be specified by union of synchronization matrices.

D. Transitivity:

We need boundedness to define transitivity, because matching boundaries transmit the effect of a synchronization. For example, $a > b > c$ transmits the boundary imposed by $> b$ from $a$ to $c$. In $a > b < c$ the boundaries established by $>$ and $<$ do not match and no relation is enforced between $a$ and $c$.

We distinguish the following cases, in order of evaluation:

1) if $e_{ik}, e_{kj} \in \{\geq, >\}$ and $e_{kj} \in \{\leq, <\}$ - or mirror of these : lower (upper if $\mu(e_{ij})$ and $\mu(e_{kj})$) bound is transmitted, but does not enforce any relation between events on either side, so $e_{ij} = 1$.

2) if $e_{ik}, e_{kj} \in \{\geq, >\}$ (alternately both in $\{\leq, <\}$ : $e_{ij} = e_{ik} \cap e_{kj}$. The boundary is transmitted, the edge point is in the set only if included by both $e_{ik}, e_{kj}$.

3) if either $e_{ik}, e_{kj}$ is $=$ : upper and lower boundaries transmitted and included, if $e_{ik}$ is $=$ then $e_{ij} = e_{kj}$, else if $e_{kj}$ is $=$ then $e_{ij} = e_{ik}$.

4) if either $e_{ik}, e_{kj} \in \{1, \neq\}$ : no bounds are transmitted, $e_{ij} = 1$.

III. Closure and semantics

A synchronization matrix for $n$ events may be defined using explicitly declared synchronizations; these relations may imply other relations not explicitly written into the matrix, including deadlock (if a relation between different events resolves to 0). The closure of the synchronization matrix computes all the relations enforced by the declared synchronizations, and may be extended by writing the boundedness condition on each event into the diagonal. Therefore it gives us the meaning of the synchronization.

Given: a a list of binary synchronizations $L$ between a set of $n$ events. To produce the closure of the synchronization matrix:

1) Initialize every element $= 1$.

2) Set every defined synchronization relation $s_{ij}$ by an appropriate element $s \in S$.

3) For every $s$ entered in step 2 between events $(i, j)$ $s_{ij} = s \cap s_{ij}$ and $s_{ji} = \mu(s_{ij}) \cap s_{ji}$ (if either $s_{ij}$ or $s_{ji}$ was not initially 1, verify that $s_{ij} = \mu(s_{ji})$, if it is not, replace $s_{ij} = \mu(s_{ji}) \cap s_{ij}$ and check $s_{ji}$ again).

4) For every pair of indices $(i, j), i \neq j$ : calculate the transitive value $t = s_{ik} s_{kj}$ for all $k \neq i, j$, and set $s_{ij} = t \cap s_{ij}$, repeat until no change in any $s_{ij}$. By construction the result is a synchronization matrix, because elements on the diagonal are 1, every element $(i, j), i \neq j$ is set to a synchronization in $S$, and every $s_{ij} = \mu(s_{ji})$.

5) Set $e_{ii} = 1 \cap \prod s_{ik} :$ each element in trace is the effect of sync applied from the left onto the diagonal, mirror the off-diagonal elements because index $ij$ on row $i$ denotes the action of $i$ on $j$ and we want the action of $j$ on $i$.

6) If either $e_{ik}, e_{kj}$ is 0 : we have an impossible condition so $e_{ij} = 0$.

7) The diagonal is computed as described in [II-C]

Remark 12. In calculating transitivity in step 4, we need to consider the implementation of a critical section $\neq$, for example as a semaphore. The $\neq$ relation is not transitive; and the original semaphore definition by Dijkstra [1965] does not deadlock, but in practice a queue is frequently attached to hold processes that try to access the critical section when it is busy. For example, if a process $p_1$ enters a semaphore, a process $p_2$ that requests the same code is placed in a queue. Rather than $p_1 \neq p_2$ we actually get $p_1 < p_2$. We give a detailed discussion in Gomez and Schubert [2010]. We show that fairness condition established by a queue allows semaphores to deadlock some of the time. For practical computations of closure we may choose to deal with $\neq$ as either $>$ or
<, depending on which alternative is transitive, since the queue can lead to deadlock unlike the pure ≠. We are able to establish the possibility of deadlock in the closure, at the cost of displaying only a worst case of the synchronization.

Although closure is a useful and compact description of a synchronization, it does not retain all the properties of a synchronization matrix. In the definition of a synchronization matrix the diagonal elements are 1, meaning that an event does not restrict itself. Boundedness is not preserved by element-wise union and intersection operations on the closure matrix (although operations are correct for the off-diagonal elements). When combining closure matrices using the standard definitions in the algebra, we need to recalculate the diagonal of the resulting matrices.

To consider “what does a synchronization mean?” we need to move past the relations that synchronization imposes to the events that are ordered by them. The events themselves are actions that are dependent in some way on time - for example, state changes, start or end of a process, receipt or sending a signal, whatever. In general we can assign names to these events for our convenience. Names of events, even though they may be arbitrary, are not to be regarded as simple interchangeable labels, they are added as an extra property of the event (for example, the name π references a particular real number, we may refer to π by another label or alias, but the name would lose its usefulness if it could designate different numeric values). We can label a set of n events using numbers (1 . . . n), or any other set of symbols, and use these labels to reference a row in a synchronization matrix, with the understanding that we can now speak of changing the position of an event in a synchronization matrix as a re-labeling.

The actual index label is not itself significant, but synchronizations we specify on a row of a matrix are affected by the relation between the index of two events. For example, if an event labeled A must happen before B, then the time relation between them is A < B. However, if event B is in a synchronization matrix at row 1, and event A is at row 2, then the synchronization m_{12} describes A < B. Re-labeling m_{11} = A and m_{22} = B would reverse (operator μ) the relation between m_{11} and m_{22} while preserving the meaning A < B. For n > 2, such re-labeling may affect the synchronizations between re-labeled events and all other events in the matrix. Function eventswap (specified in smat-u.sce) describes the changes in the synchronization matrix that preserve the event order while changing the event indices:

```c
function [T]=eventswap(M,i,j)
// input - a synchronization matrix and two event numbers
// switches event positions in matrix, rewrites rows and columns
// output - T represents same sync as M, but events are labeled differently
size=M(1,1);
T=[];
for row=1:size
    T(row,:)=M(row,i);
end
M(T);
for col=1:size
    T(:,col)=M(i,col);
end
```

Eventswap reads the original matrix and then writes the swapped rows and columns into the copy. By definition of a synchronization matrix, all synchronization relations on the same row/column involve the event selected on the diagonal. Therefore order of operations in eventswap does not matter.

**Claim 13.** Given a synchronization matrix representing N events and their relations, any permutation performed by a sequence of eventswap actions represents the same synchronization since it preserves the relations between the actual events. From this we conclude that any permutation of events performed through a succession of eventswap preserves the relations between actual events and therefore the meaning of the synchronization remains the same.

### A. Phase space

In Gomez and Schubert (2010) and Gomez and Schubert (2011) we described execution in terms of basic blocks and a control flow graph (CFG), a standard concept in compiler theory (see Aho, Sethi, and Ullman (1985) and Apt and Olderog (1997)). A state parallel execution can be identified as a list of blocks that are executing concurrently at a given time t, and possible successor states are given by the CFG. A synchronization is defined by a set of events each occurring at a different block and having a particular time relation with events at other blocks. For N processes, this gave us a phase space as an N-sided hypercube, with each side labeled with code block numbers. A state would be an ordered N tuple of block labels, and as such would uniquely identify a point in the phase space hypercube.

To relate Algebra of Synchronization hi described here and in Gomez and Schubert (2010) and Gomez and Schubert (2011) to the phase space hypercube, we note that numeric relations restrict the set of possible states. For example, A > B describes sets such that every number in A is greater than every number in B. If the numbers A and B represent block numbers executed concurrently by two processes, this relation
blocks states $a_i \in A, b_j \in B$ with $a_i$ earlier than $b_j$. From the control flow graph we can predict successor states to both $a_i$ and $b_i$ that are in blocked regions of the phase space. We can see the effect of forced waits in the execution graphs in Gomez and Schubert (2010) and Gomez and Schubert (2011), as dense clumps of states resulting from processes blocked from advancing by a synchronization condition.

We can see what happens to each specific event in a synchronization from the boundedness condition of events on the diagonal of the synchronization matrix (subsection II-C). In continuing work, we will define how each boundedness condition will reduce the accessible states from each event on the diagonal and how this will affect the phase space. This in turn will be used to predict the effect on phase space hypervolume, entropy, time and work costs of synchronization, and verify predictions experimentally.

IV. OPEN PROBLEMS

A. Equivalence

We claim that two synchronization matrices are equivalent IFF they have the same closure - do we need a separate theorem to prove this? If the closure algorithm is correct, then the proof is easy, because closure gets all the implied synchronizations. We may need to consider semaphores as implemented in combination with queues, however (and possibly other cases in which a relation changes with time).

An implication of EVENTSWAP, is that any permutation of events in the synchronization matrix is equivalent to every other permutation. This property should extend to every matrix that has the same closure, since EVENTSWAP preserves the boundedness of events being swapped.

B. Separability

When can we say that a set of individual binary synchronizations constitute a collective synchronization? Suppose we have a set of events that occur in a loop which runs in parallel on multiple processes. Further suppose there are multiple synchronizations involving different processes in the loop. We could always represent every relation in the loop with a single synchronization matrix, but this could hide insight about what is logically happening and what is taking more time in execution. Understanding what the code is supposed to do, we may be able to divide synchronizations into different sets that are logically independent, so they could be representable by different synchronization matrices, and possibly abstracted into collective functions.

We do not know if an algorithm exists that would allow us to identify sets of related synchronizations.

C. Optimization

It seems (Gomez, Schubert, and Cai (2016)) that the major time cost of synchronization is in the synchronization waits and entropy reduction, rather than in the signals or messages required to implement a synchronization. Nevertheless we would like to know what is the simplest synchronization code that would have a given effect on execution. We claim that the meaning of synchronization is given by the closure of the synchronization matrix, and we also know that different code can lead to the same closure (due to implied synchronization).

Therefore: given a the closure of a particular synchronization matrix, what is the simplest matrix that produces the same closure?

An allied question: what do we mean by simplest? Smallest number of binary synchronizations required is the simplest, but do we need to narrow down to something like smallest number of atoms used?

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