Advances on Matroid Secretary Problems: Free Order Model and Laminar Case

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Abstract. The best-known conjecture in the context of matroid secretary problems claims the existence of an \(O(1)\)-approximation applicable to any matroid. Whereas this conjecture remains open, modified forms of it were shown to be true, when assuming that the assignment of weights to the secretaries is not adversarial but uniformly at random \([20,18]\). However, so far, no variant of the matroid secretary problem with adversarial weight assignment is known that admits an \(O(1)\)-approximation. We address this point by presenting a 9-approximation for the free order model, a model suggested shortly after the introduction of the matroid secretary problem, and for which no \(O(1)\)-approximation was known so far. The free order model is a relaxed version of the original matroid secretary problem, with the only difference that one can choose the order in which secretaries are interviewed.

Furthermore, we consider the classical matroid secretary problem for the special case of laminar matroids. Only recently, a \(O(1)\)-approximation has been found for this case, using a clever but rather involved method and analysis \([12]\) that leads to a \(16000/3\)-approximation. This is arguably the most involved special case of the matroid secretary problem for which an \(O(1)\)-approximation is known. We present a considerably simpler and stronger \(3\sqrt{3}e \approx 14.12\)-approximation, based on reducing the problem to a matroid secretary problem on a partition matroid.

1 Introduction

The secretary problem is a classical online selection problem of unclear origin \([6,8,9,10,16]\). In its original form, the task is to choose the best out of \(n\)

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Secretaries, also called elements or items. Secretaries arrive (or are interviewed) one by one in random order. As soon as a secretary arrives, it can be ranked against all previously seen secretaries. Then, before the next one arrives, one has to decide irrevocably whether to choose the current secretary or not. There is a classical algorithm that selects the best secretary with probability $1/e^6$, and this is known to be asymptotically optimal. In its initial form, the secretary problem was essentially a stopping time problem, and not surprisingly, it mainly attracted the interest of probabilists.

Recently, secretary problems enjoyed a revival, and various generalizations were studied. These developments are strongly motivated by a close connection to online mechanism design, where a good is sold to agents arriving online. Here, the agents correspond to the secretaries and they reveal prices that they are willing to pay in exchange for goods. This leads to secretary problems where more than one secretary can be chosen. The most canonical generalization asks to hire $k$ out of $n$ secretaries, each revealing a non-negative weight upon arrival, and the goal is to hire a maximum weight subset of $k$ secretaries. This interesting variant was introduced and studied by Kleinberg, who presented a $(1 - O(1/\sqrt{k}))$-approximation for this setting. However, in many applications, additional constraints have to be imposed on the elements that can be chosen. A very general class of constrained secretary problems, where the chosen elements have to form an independent set of a given matroid $M = (N, \mathcal{I})$, was introduced by Babaioff, Immorlica and Kleinberg. This setting, now generally termed matroid secretary problem, covers at the same time many interesting cases and has a rich structure that can be exploited to design strong approximation algorithms.

To give a concrete example of a matroid secretary problem, and to motivate some of our results, consider the following connection problem. Given is an undirected graph $G = (V, E)$, representing a communication network, with non-negative edge-capacities $c : E \rightarrow \mathbb{N}$ and a server $r \in V$. Clients, which are the equivalent of candidates in the secretary problem, reside at vertices of the graph and are interested to connect to the server $r$ via a unit-capacity path. The number of clients and their locations are known. Each client has a price that she is willing to pay to connect to the server. These prices are unknown and no assumptions are made on them except for being non-negative. Clients then reveal themselves one by one in random order, announcing their price. Whenever a client reveals herself, the network operator has to decide irrevocably before the next client appears whether to serve this client and receive the announced price. The goal is to choose a maximum weight subset of clients that can be served simultaneously without exceeding the given capacities $c$. It is well-known that the constraints imposed by the limited capacity on the clients that can be chosen is a special type of matroid constraint, namely a gammoid constraint.

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1. A matroid $M = (N, \mathcal{I})$ consists of a finite set $N$, called the ground set, and a non-empty family $\mathcal{I} \subseteq 2^N$ of subsets of $N$, called independent sets, satisfying: (i) $I \in \mathcal{I}, J \subseteq I \Rightarrow J \in \mathcal{I}$, and (ii) $I, J \in \mathcal{I}, |I| > |J| \Rightarrow \exists f \in I \setminus J$ with $J \cup \{f\} \in \mathcal{I}$. For more information on matroids we refer the reader to.