Quantum Corrections to the Stochastic Gravitational Wave Background

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Abstract

We study 1-loop corrections to the primordial stochastic background of gravitational waves produced during inflation. While in single-clock, at the leading order in slow-roll, quantum corrections keep the amplitude scale-free this is not the case when the pattern of symmetry breaking is different. In particular, when spatial diffeomorphisms are also broken during inflation as for solid inflation, a log-running in the external momentum is generated. We relate the appearance of a log-running to the spontaneous breaking of dilatation invariance. The running could be instrumental to distinguish single-clock from alternative models of inflation in future high sensitivity CMB polarisation and GWs experiments.
1 Introduction

In the last couple of decades cosmology has become sharper and sharper and the ΛCDM model is tested with great accuracy [1]. Within the inflationary paradigm, the primordial perturbations have a quantum origin and become "classical" only after horizon exit. Given that, it is useful to test our ability to compute and predict genuine quantum corrections to cosmological correlation functions. Such a program was pioneered by Weinberg [2] by using the IN-IN formalism. Loop effects are notoriously subtle to compute in de Sitter and quasi-de Sitter spacetime due to gauge and infrared effects; see for instance [3] for a review and a some early reference. In particular, it was found [4, 5] that in single field inflation, the scalar and tensor 2-point functions get a sizeable correction of order log(\(k\)) where \(k\) is the comoving external momentum at the leading order in slow-roll. Such a result was confuted in [6, 7, 8], arguing that after a proper implementation of the regularisation procedure actually the correction log(\(k\)) to scalar power spectrum is turn into a log(\(H\)) one and thus no “running” is present. Here we reconsider the matter, focusing on the tensor power spectrum exploring also scenarios beyond single field inflation where the symmetry breaking pattern is different. As a matter of fact there is a deep connection between the residual symmetry group of the inflationary background and the presence of a log(\(k\)) running induced by quantum corrections. During inflation the spacetime is close to a de Sitter spacetime that has SO(4, 1) as isometry group. The underlying conformal symmetry was used to derive [9, 10, 11, 12] the consistency relation in single-field inflation [13, 14, 15], to study systematically non-Gaussianity in the tensor sector [16] and scalar sector [17] for spectator fields. The idea is that though in general dilation invariance is broken by vacuum expectation value (VEV) of the inflaton, it remains unbroken in combination with a suitable internal symmetry of the inflaton sector. This is the case in single-clock inflation but when the symmetry breaking pattern is different the running may appear. Such a running, besides its theoretical interest, could be instrumental to improve our chances for a direct detection of the primordial stochastic background of gravitational waves. It should also be stressed that resurgence of log(\(k\)) running also casts some doubts on the physical meaning of the result at very small \(k\) where perturbativity is at stake.

2 Single Field Inflation

Consider single field inflation based on a real scalar field \(\Phi\) minimally coupled to gravity described by the action

\[
S = M_{pl}^2 \int d^4x \sqrt{-g} \left[ \frac{R}{2} + P(X, \Phi) \right], \quad X = -\frac{1}{2} g^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi. \tag{2.1}
\]

In the spatially flat gauge, perturbations can be taken as

\[
\begin{align*}
\bar{ds}^2 &= (N_i N^i - N^2) dt^2 + N_i dx^i dt + h_{ij} dx^i dx^j; \\
h_{ij} &= a^2 (e^\gamma)_{ij}, \quad \Phi = \phi(t) + \pi(x); \tag{2.2}
\end{align*}
\]

where \(\gamma_{ij}\) are transverse and traceless tensor perturbations describing gravitational waves. The background metric is approximately close to de Sitter (dS) spacetime for which\(^1\)

\[
a_{dS} = -\frac{1}{H t}, \quad t \in (-\infty, 0]. \tag{2.3}
\]

Here \(H\) is the Hubble rate in physical time (i.e. the time independent one in a pure dS background), which is related to the time dependent one expressed in conformal time \(\mathcal{H}\) by\(^2\)

\[
\mathcal{H} = a H = a' / a \tag{2.4}
\]

\(^1\)We use conformal time \(t\).
\(^2\)The prime denotes derivative with respect to the conformal time \(t\).
When $\phi' \neq 0$, a departure from a pure dS background is controlled by the slow-roll parameter

$$\epsilon = \frac{(H^2 - H')}{H^2} = \frac{P_X \phi'^2}{2H^2},$$

(2.5)

while the background field equations gives

$$a^2 P + P_X \phi'^2 - 3H^2 = 0.$$  

(2.6)

Thus the kinetic part of the action for the inflaton is subdominant with $\phi'$ small and $\dot{X} = \phi'^2/(2a^2)$ almost constant in a quasi dS regime. Indeed, the background equations are satisfied up to corrections of order $\epsilon^2$ by taking

$$\phi = \alpha \log(a), \quad a = (-H t)^{-1-\epsilon}.$$  

(2.7)

For instance taking $P(X, \Phi) = X^n - V(\Phi)$, with $n > 0$, one can check that $\alpha \sim \epsilon^{1/(2n)}$. The background dS spacetime has 10 Killing vectors associated to the generators of $SO(4,1)$; however, the inflaton classical configuration typically breaks spontaneously $SO(4,1)$ down to spatial rotations and translation that are linearly realised. The spontaneous broken dS symmetry is behind the so-called consistency relations [13] among correlation functions in the soft limit as shown in [14, 15, 18, 10, 11]. Dilatations and special conformal transformations of $SO(4,1)$ are non-linearly realised and the underlying symmetry does not show up as invariance of correlation functions as for rotations. Sometimes even dilatations can be linearly realised. Suppose now that $P$ has shift symmetry

$$\Phi \rightarrow \Phi + c.$$  

(2.8)

Such a symmetry is consistent with slow-roll $P$, according with the derivatives of $P$ with respect to $\Phi$ have to be small. The background value of $\Phi$ is not invariant under dilatation; indeed, neglecting small slow-roll corrections, we have that

$$\phi \rightarrow \phi - \alpha \log \lambda.$$  

(2.9)

However the effect of dilatation (2.4) on $\phi$ can be compensated by a shift transformation of $\Phi$ by taking $c = \epsilon \alpha H^{-1} \log \lambda$. The combined action of a dilatation followed by a shift symmetry leaves the background invariant. Thus, under a dilatation we have for $\pi$

$$\Phi'(x') = \Phi(x) \Rightarrow \phi(t) - \frac{\alpha}{H} \log(a) + \pi'(x') = \phi(t) + \pi(x)$$

$$\Rightarrow \pi'(x') = \pi(x) + \frac{\alpha}{H} \log(a);$$

(2.10)

while in the “diagonal” combination of a dilatation followed by a shift symmetry (dshift), the nonlinear part of the trasformation is absent

$$\pi'(x') = \pi(x)$$

(2.11)

and the dshift transformation is linearly realised. The same is true for curvature perturbation of the 3D hypersurface orthogonal to $\partial_\mu \Phi$ which is given by the gauge invariant quantity $R$ that for the metric in the flat gauge (2.2) is given by

$$R = \frac{H^2}{\phi'} \pi.$$  

(2.12)

The quantity $R$ is almost scale free constant on superhorizon scales and it provides the seed for primordial perturbation. The quadratic Lagrangian for the field fluctuations $R$ reads in Fourier space

$$L_2 = a^2 M_{pl}^2 \epsilon \left[ \frac{1}{c_s^2} R'^2 - k^2 R^2 \right],$$  

(2.13)

3With a bar we denote the background values of a quantity.
with
\[ c_s^2 = \frac{a^2 P_X}{a^2 P_X + P_{XX} \phi'^2}. \tag{2.14} \]

The cubic interaction with gravity is described by
\[ L_{TSS} = M_{pl}^2 a^2 \epsilon \gamma_{ij} \partial_i \mathcal{R} \partial_j \mathcal{R}. \tag{2.15} \]

The Lagrangian \( L_2 + L_{TSS} + L_{gw} \), where \( L_{gw} \) describes the free dynamics of gravitons, is invariant under a dshift and such a symmetry is linearly realised up to small slow-roll corrections, with both \( \mathcal{R} \) and \( \gamma_{ij} \) being dimensionless and with zero weight.

The bottom line is that the tensor 2-point function is constrained to be dilation invariant and has to be of the following form (see appendix A)
\[ \langle \gamma_{ij}(t, k_1) \gamma^{ij}(t, k_2) \rangle = (2\pi)^3 \delta^{(3)}(k_1 + k_2) \frac{F(k_1 a^{-1})}{k_1^3}, \quad |k_1| = k_1, \tag{2.16} \]

where \( F \) is an arbitrary function. Notice that (2.16) is exact up to slow-roll correction and represents the power spectrum of stochastic gravitational waves produced during inflation, that is one of the key prediction of inflationary models. The form (2.16) is incompatible with the presence of a \( \log(k/\mu) \) dependence induced by loop corrections. Indeed, on superhorizon scales \( F \) becomes a constant. The actual 1-loop computation given in appendix (C) confirms that for \( t \to 0 \), no running is present and one finds
\[ P^{(\gamma)} = \frac{H^2}{4 \pi^2 M_{pl}^2} \left[ 1 + \frac{H^2}{M_{pl}^2} \left( 41 + 5 c_s^4 - 118 c_s^2 \right) \frac{\log \left( \frac{H}{\mu} \right)}{240 c_s^7} \right], \tag{2.17} \]

where \( \mu \) is a physical scale introduced by dimensional regularisation. Similar results were obtained in [19, 20]. In the above expression we have kept only non-analytic contributions, the divergent part are subtracted by suitable counter-terms. The curvature scale acts as a natural infrared cutoff that prevents the correction to become too large for small \( k \). Notice that the correction is local and analytical only on superhorizon scales and thus there is no genuine local counter-term able to cancel it at all scales. Finally, as a technical note, as pointed out in [6], for the cancelation it is important to correct the modes wavefunctions when using dimensional regularisation in de Sitter, in contrast with what happens in Minkowski spacetime.

## 3 Beyond Single-Field: Fluids and Solids

Things are different for what concerns dilation symmetry when we go beyond single-field inflation. An interesting laboratory is inflation driven by a generic self-gravitating medium [21, 22, 23] originally introduced as infrared modification of gravity [24, 25] to explain dark energy [26, 27]. In particular, let us consider solid inflation [28] based on three scalar fields \( \varphi^a, a = 1, 2, 3 \) with a shift symmetry [29, 27]
\[ \varphi^a \to \varphi^a + c^a, \tag{3.1} \]

and \( SO(3) \) internal symmetry
\[ \varphi^a \to \varphi^a = \mathcal{R}^a_b \varphi^b, \quad a, b = 1, 2, 3, \quad \mathcal{R}^i \mathcal{R} = 1. \tag{3.2} \]

Among the spacetime scalars shift symmetric operators with a single derivative of \( \varphi^a \)
\[ B^{ab} = g^{\mu\nu} \partial_\mu \varphi^a \partial_\nu \varphi^b, \quad a, b = 1, 2, 3, \tag{3.3} \]
one can extract 3 operators invariant under internal \( SO(3) \) rotations (3.2)
\[ b = \sqrt{\text{Det}[B]}, \quad \tau_X = \text{Tr}[B], \quad \tau_Y = \frac{\text{Tr}[B^2]}{\tau_X^2}, \quad \tau_Z = \frac{\text{Tr}[B^3]}{\tau_X^3}. \tag{3.4} \]
Thus, we arrive at the action
\[ S = \frac{M_{pl}^2}{2} \int d^4x \sqrt{-g} \left( R + M_{pl}^2 \int d^4x \sqrt{-g} U(b, \tau_Y, \tau_Z) \right). \tag{3.5} \]

Notice that the case \( U(b) \) describes an adiabatic fluid\(^4\)\cite{26,27}. The scalar fields have the VEV
\[ \bar{\varphi}^i = x^i. \tag{3.6} \]

The existence of a spatially homogeneous background is allowed by the presence of global symmetries of the scalar field action. Consider a special multi-field model of inflation based on the “vacuum” configuration (3.6), which has a residual global “diagonal” \(ISO(3) \) symmetry. Indeed, a global spatial rotation \( R \) can be absorbed by a corresponding inverse internal transformation of \( \varphi^a \) and the same is true for a global translation \( x^i \rightarrow x^i + c^i \) thanks to the shift symmetry (3.1).

Let us focus on the scalar part by setting \( \varphi^i = \partial_i \pi_L \), notice that \( \pi_L \) has dimension \(-2\). By looking at the quadratic Lagrangian for the fluctuations one gets in the flat gauge
\[ L_2 = \frac{1}{2} a^2 M_{pl}^2 \mathcal{H}^2 \epsilon \left[ 4 (\partial_i \pi_L)^2 - 4 c_2^2 (\Delta \pi_L)^2 \right] + \frac{a^2 M_{pl}^2}{4} \left[ (\gamma_{ij} \gamma_{ij} - M_2 a^2 \gamma_{ij} \gamma_{ij} + \partial_{\kappa} \gamma_{ij} \partial_{\kappa} \gamma_{ij}) \right], \tag{3.7} \]
where \( \Delta = \partial_i^2 \) and \( c_2^2 \) is an effective sound speed defined by \( c_2^2 = -1 + 4 c_2^2 / 3 \); the parameter \( c_2 \) is expressed in terms of the second derivatives of \( P \) and controls the presence anisotropic stress in the energy momentum tensor which characterises a solid; one could also verify that it is nothing else that the sound speed of the transverse phonons. The curvature perturbation is related to \( \pi_L \) in the flat gauge by
\[ \zeta = \frac{k^2}{3} \pi_L. \tag{3.8} \]

Notice that the form of (3.7) is very similar to the case of a single field inflation with a notable exception: by dimensional reasons, there is an overall \( \mathcal{H}^2 \) due to the fact that \( \pi_L \) has dimension \(-2\). Such a change is due to the different breaking pattern: while in single field 4-dimensional diffeomorphisms are spontaneously broken down to 3-dimensional spatial diffeomorphisms, for solids time diffeomorphisms are unbroken while the spatial ones are broken. While the Lagrangian \( L_2 \) for solid inflation together with the TSS interaction part is still invariant under dilation by assigning to \( \pi_L \) weight \(-2\), the argument used for \( P(X, \Phi) \) does not work anymore. The nature of the breaking is such that shift symmetry is unable to compensate the effect of a dilation. The change of the background configuration (3.6) by a dilatation is
\[ \varphi^i \rightarrow \lambda \bar{x}^i \tag{3.9} \]
which is \( x \)-dependent and cannot be removed by (3.1). Another way to see this is the following argument: we can try to impose some internal symmetry in order to make the background configuration invariant under a dilation symmetry; what we get is that a necessary condition is that the symmetry required, consistent with the background is
\[ \pi_L \rightarrow \pi_L + c \cdot x \tag{3.10} \]
that is nothing else but the galilean shift also discussed in \cite{9} that is not well defined in a curved background; therefore we expect the 2-point function be not invariant under dilation, since it probes scales comparable with \( \mathcal{H} \). In addition, because of the presence of three scalar fields the arguments typical of single field according with the presence of the inflaton is equivalent to a universal change of the time coordinate \cite{31} does not apply to a self-gravitating media of the form (3.5). On a more practical side, while mode for the field \( \pi \) in the interaction picture in the single-field case is a Bessel function of order \( 3/2 \) in the case of solid the mode of \( \pi_L \) is a Bessel function of order \( 5/2 \). Indeed,

\(^4\)Actually, for a fluid the slow-roll regime does not exist.
the fact that both graviton modes and $\pi$ have modes proportional to Bessel function of order $3/2$ is instrumental in the cancellation of the log running. A direct computation gives

$$P(\gamma) = \frac{H^2}{4\pi^2 M_{pl}^4} \left[ 1 + \frac{H^2}{M_{pl}^2} F(c_L) \log \left( \frac{k}{\mu} \right) \right]$$

which indeed features a log running in the external momentum $k$. We stress again that we are working at the leading order in slow-roll and the correction has pure quantum origin. This last point has to be carefully considered when higher-order corrections in perturbation theory are studied by analysing the classical equations of motion in a quasi de Sitter space, this is sometimes referred as gravitational wave (GW) backreaction and the results in the literature disagree even in single field models [32, 33, 34]. This topic and the role of quantum effects in a dS space during the inflation-radiation transition will be the subject of future work.

For the complete form of the TSS interacting Hamiltonian and the $F(c_L)$ function see Appendix D. Finally, the fluid limit of the solid action [35] is rather interesting as a consistency check. In a such limit only the operator $b$ is present in (3.5) and the internal symmetry is enhanced to internal volume preserving diffeomorphisms [29, 24, 30, 27] and out of the three TSS interactions terms [28, 21], only a term of the form

$$L_{TSS} = M_{pl}^2 a_s^4 \tilde{A} \gamma_{ij} \partial_i \zeta' \partial_j \zeta'$$

(3.12)
survives, with $\zeta' \propto \partial^2 R/H$.

As a matter of fact, the combination of modes (Hankel functions of order 5/2) and time derivatives of the vertex is such that it is completely equivalent to the vertex in (2.15) with $\nu = 3/2$ and thus the cancellation mechanism still works as it should be. The reason behind such non-trivial relation is that a perfect irrotational fluid described in terms of three scalar field by the Lagrangian $U(b)$ is related to the description in terms of a single scalar by the Lagrangian $P(X)$ given in section 2.

4 Discussion and Conclusions

The fact that in single field inflation there is no running in the external momentum makes the quantum correction rather uneventful in the sense that the amplitude of the power spectrum stays basically scale-free at the leading order in slow-roll. Scale dependence is reintroduced at 1-loop only by looking at the tiny effect of different modes exiting the horizon at slightly different times, when slow-roll corrections to the scale factor $a$ are taken into account (see [20]). Future CMB polarisation experiments can hardly measure the simple dependence on the renormalisation scale. An enhancement by a factor $N$ in the amplitude can be obtained by considering $N$ scalar spectator fields and a non-trivial sound speed. Indeed the amplitude (2.17) scales with the sound speed as $c_s^{-7}$ and an even stronger in the case of a supersolid [23]. Of course, the value of $c_s$ cannot be pushed toward too small values without jeopardising the validity of the derivative expansion in the effective theory of single field inflation and boosting the level of non-Gaussianity on the verge of conflicting with existing observations [35].

When one moves away from the symmetry breaking pattern of single field inflation the perspective is different. A genuine running with the external momentum $k$ is indeed present even at the leading order in slow-roll and actually without any $\epsilon$ suppression. The running is in principle significative enough to be used to distinguish the breaking pattern (3.6) from the one of single-field inflation. On the down side, the result (3.11) is problematic in the limit $k \to 0$ and thus not infrared safe. Of course our analysis of 1-loop effect is not exhaustive, further future investigations will establish whether the running is one of many quirks of de Sitter spacetime or a clean physical prediction.

Things are different for future GW experiments. Even if suppressed by slowroll parameters, the completely different log-scale dependence may give a signature beyond the standard and suppressed contribution to the stochastic GW background. In this case, we suggest that the presence of a non trivial phonon sound speed (present in single field models with a non-canonical kinetic term) could

\[\text{The equivalence comes from the identity } \frac{d}{dx} \left[ x^\nu H_\nu^{(1)}(x) \right] = x^\nu H_{\nu-1}^{(1)}(x).\]
play an essential role in turning the *red tilted* behaviour (see [36, 37, 38, 39]) in a *blue tilted* one. A related point is the observation that the coefficient of the log in (2.17), a sort of beta function, is a monotonic function of $c_s$ that changes sign becoming negative for $c_s \gtrsim 0.6$. However, to find the complete form of this beta function, an analysis at the second order in slowroll expansion is mandatory during the dimensional regularisation procedure as argued in [20]. This will be the object of a future work.

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## A Dilatation invariance

Take a generic field that transforms as

$$
\Phi(x) \rightarrow \Phi'(x') = \mathcal{F}(\Phi)(x); 
$$

(A.1)

then the n-point function

$$
G_n(x_1, \cdots, x_n) = \langle \Phi(x_1) \cdots \Phi(x_n) \rangle
$$

(A.2)

is such that

$$
G_n(x'_1, \cdots, x'_n) = \langle \mathcal{F}(\Phi)(x_1) \cdots \mathcal{F}(\Phi)(x_n) \rangle.
$$

(A.3)

In particular for a field that is a scalar (namely a dimensionless scalar field) under the symmetry

$$
\Phi'(x') = \Phi(x) 
$$

(A.4)

and the correlation functions are invariant

$$
G_n(x_1, \cdots, x_n) = G_n(x'_1, \cdots, x'_n).
$$

(A.5)

Let us discuss the properties of a dimensionless scalar under dilatation. We are often interested in the spatial Fourier transform of correlators; for instance

$$
G_2(t_1, x_1; t_2, x_2) = \int \frac{d^3k_1}{(2\pi)^3} \frac{d^3k_2}{(2\pi)^3} e^{i k_1 \cdot x_1 + i k_2 \cdot x_2} \tilde{G}(t_1, k_1; t_2, k_2).
$$

(A.6)

In Fourier space (A.5) reads

$$
\tilde{G}(t_1, k_1; t_2, k_2) = \lambda^{-6} \tilde{G}(\lambda t_1, \lambda k_1; \lambda t_2, \lambda k_2).
$$

(A.7)

Let us focus on equal time correlators: $t_1 = t_2 = t$ and by using spatial translation and rotational invariance of the dS metric one cast a generic 2-point functions as

$$
\tilde{G}(t, k_1; k_2) \equiv \tilde{G}(t, k_1, k_2) = (2\pi)^3 \delta^{(3)}(k_1 + k_2) F(k_1, t), \quad |k_1| = k_1.
$$

(A.8)

By imposing (A.7) to the 2-point function we have that

$$
\lambda^{-3} F(\lambda^{-1} k_1, \lambda t) = F(k_1, t);
$$

(A.9)

which gives (A.10)

$$
\tilde{G}(t, k_1, k_2) = (2\pi)^3 \delta^{(3)}(k_1 + k_2) \frac{F(k_1 a_{0S}^{-1})}{k_1^3}.
$$

(A.10)
B \textbf{The In-In Formalism}

Quantum correlators during inflation are typically computed by using the in-in formalism expanding perturbatively the time evolution operator. A generic hermitian operator $W$ in the Heisenberg picture is written as perturbative expansion in the interaction picture

$$W(t) = U(t)^\dagger W_I(t) U(t) = W_I(t) + \sum_{n=1}^{\infty} W_I^{(n)}(t), \quad U(t) = T \exp \left[-i \int_{t_0}^{t} dt' H_I(t') \right]; \quad (B.1)$$

where $W_I(t)$ and $H_I$ are the operator $W$ and the relevant interaction Hamiltonian are both in the interaction picture. For the 1-loop corrections we will need the second order term of the expansion that reads

$$W^{(2)}(t) = \int_{t_0}^{t} dt_2 \int_{t_0}^{t} dt_1 H_I(t_2) W_I(t_1) H_I(t_1) - \int_{t_0}^{t} dt_2 \int_{t_0}^{t_2} dt_1 H_I(t_1) H_I(t_2) W_I(t)$$

$$- \int_{t_0}^{t} dt_2 \int_{t_0}^{t_2} dt_1 W_I(t_1) H_I(t_2) H_I(t_1). \quad (B.2)$$

In dS in conformal time $t_0 \to -\infty$. Besides UV divergencies due to virtual particles, in dS when the limit $t_0 \to -\infty$ is taken additional troubles appear and a regularisation procedure is needed. An option is to take $t_0 = \tilde{t} (1 + i \epsilon)$ with $\tilde{t} \to -\infty$, to select the Bunch-Davies vacuum. Thus

$$(W(t)) = 2 \text{Re} [\langle A(t) \rangle] + (B(t));$$

$$B(t) = \int_{t_0}^{t} dt_2 \int_{t_0}^{t} dt_1 H_I(t_2) W_I(t_1) H_I(t_1), \quad A(t) = - \int_{t_0}^{t} dt_2 \int_{t_0}^{t_2} dt_1 W_I(t_1) H_I(t_2) H_I(t_1). \quad (B.3)$$

It should be stressed that the above regularisation, due to the complex nature of $t_0$, forbids to write down $W^{(n)}$ as a multiple commutator a la Weinberg. In order to do that one should consider an adiabatic turn-off of the interaction, see for instance

$$H_I(t) \to \hat{H}_I(t) = H_I(t) e^{\epsilon t}, \quad \epsilon \in \mathbb{R}, \quad (B.4)$$

to keep $U$ unitary. Thus, one can write

$$W_I^{(2)}(t) = i^2 \int_{t_0}^{t} dt_2 \int_{t_0}^{t_2} dt_1 \left[ \hat{H}_I(t_1), \left[ \hat{H}_I(t_2), W_I(t) \right] \right]. \quad (B.5)$$

While the form $[B.3]$ leads to simpler time integrals, the Weinberg form is more compact and allows to read off more easily the late time behavior.

C \textbf{Single Field Inflation}

To discuss the main features of the quantum corrections to the tensor power spectrum consider single clock inflation based on a real scalar field $\Phi$ described by action $\[21\]$. In the interaction picture we have the following expressions for the fields

$$\mathcal{R}(t, \mathbf{x}) = \int \frac{d^3 k}{(2\pi)^3} e^{i \mathbf{k} \cdot \mathbf{x}} \left[ a(\mathbf{k}) f_k(t) + a^\dagger(\mathbf{k}) f_k(t)^* \right] = \int \frac{d^3 k}{(2\pi)^3} e^{i \mathbf{k} \cdot \mathbf{x}} \mathcal{R}_k(t);$$

$$\gamma_{ij}(t, \mathbf{x}) = \sum_{s=1}^{2} \int \frac{d^3 k}{(2\pi)^3} e^{i \mathbf{k} \cdot \mathbf{x}} \left[ b(\mathbf{k})_s \epsilon_{ij} e_s h_k(t) + b^\dagger(\mathbf{k})_s \epsilon^*_{ij} e_s h_k(t) \right] = \int \frac{d^3 k}{(2\pi)^3} e^{i \mathbf{k} \cdot \mathbf{x}} \gamma_{ij}(t). \quad (C.1)$$

\footnote{Of course the limit $\epsilon \to 0$ has to be taken afterwards.}
In $3 + \delta$ spatial dimensions, the wavefunctions for the scalar and tensor modes are given by

$$f_k(t) = \frac{A_f}{k^{3/2}} (-Ht)^{\delta/2} (-c_s kt)^{3/2} H_{(3+\delta)/2}^{1}(-c_s kt)$$

(C.2)

$$h_k(t) = \frac{A_s}{k^{3/2}} (-Ht)^{\delta/2} (-kt)^{3/2} H_{(3+\delta)/2}^{1}(-kt)$$

(C.3)

By expanding at first order in $\delta$ we get

$$f_k(t) = \sqrt{\frac{2}{\pi}} \frac{A_f}{k^{3/2}} e^{-ic_s kt} (-i + c_s kt) + \delta \frac{A_f}{k^{3/2}} \sqrt{2\pi} e^{-ic_s kt} [(-i + c_s kt) \log(-Ht) + u(c_s, k, t)]$$

(C.4)

$$h_k(t) = \sqrt{\frac{2}{\pi}} \frac{A_s}{k^{3/2}} e^{-ikt} (-i + kt) + \delta \frac{A_s}{k^{3/2}} \sqrt{2\pi} e^{-ikt} [(-i + kt) \log(-Ht) + u(c_s, k, t)]$$

(C.5)

where

$$u(c_s, k, t) = 2e^{2ic_s kt} (c_s kt + i)(\text{Ei}(-2ic_s kt) - i\pi) - i(4 + \pi(c_s kt - i))$$

(C.6)

with $\text{Ei}(z)$ exponential integral function. The actual 1-loop computation leading to (2.17) is described in appendix E.

### D Solid Inflation: 1-Loop

The relevant cubic TSS vertex for solid inflation is in Fourier space (see [21] for more details)

$$\mathcal{L} = \epsilon M_{pl}^2 H^2 a^4 \delta_{ij} (k) \left[ V_1 \hat{k}^i \hat{k}^j + V_2 \left( \hat{k}' \cdot \hat{k}'' \right) \hat{k}'^i \hat{k}''^j \right] h_k \zeta_{k'} \zeta_{k''},$$

(D.1)

where we have denoted a unit vector with an hat. The coefficients $V_1$ and $V_2$ can be written in terms of the solid Lagrangian $U(b, \gamma, \tau_2)$ and its derivatives, see [21].

Performing the momentum integrals by using dimensional regularisation we get (3.11), where

$$\mathcal{F}(c_L) = -\frac{1}{6531840 c_L^4} \left[ (83181 - 4938 c_L^2 + 25 c_L^4 - 40320 \gamma_E) V_1^2 \right.$$

$$+ 2 (83181 - 13968 c_L^2 + 355 c_L^4 - 40320 \gamma_E) V_1 V_2$$

$$\left. + (83181 - 22998 c_L^2 + 1105 c_L^4 - 40320 \gamma_E) V_2^2 \right].$$

(D.2)

Actually, by expanding the action (3.5) at the cubic order [21], there exists another TSS of the form

$$\mathcal{L} = \epsilon M_{pl}^2 H^2 a^4 V_3 \delta_{ij} (k) \frac{\hat{k}^i \hat{k}''^j}{k' k''} h_k \zeta_{k'} \zeta_{k''}.$$

(D.3)

The above vertex can be brought into the same form of (2.15), with the scalar and tensor modes have the same index for the Hankel function 3/2 as argued in section 3 and no logarithmic running in the external momentum is produced.

However, one may ask if a mixed term coming from (D.1) and (D.3) could compete with (3.11): after a lengthy computation one can see that the mixed terms proportional to $V_1 V_3$ and $V_2 V_3$ are suppressed by extra powers of $c_L$, and therefore are negligible for reasonable values of the (non trivial) effective sound speed $c_L$.

### E Dimensional regularisation: beyond single field Inflation

The structure of the logarithmic running coming from the loop calculation can be inferred without the need of evaluating explicitly the momentum integrals by extending the argument given in [6] for
the case of standard single field inflation with some mathematical assumptions. For TSS interactions, the contribution we need to compute is of the form

\[ \left. \langle \hat{W}(t) \rangle \right|_{1-L} = -2 \text{Re} \left[ \int_{-\infty}^{t} \text{d}t_2 \int_{-\infty}^{t_2} \text{d}t_1 \left( H_l^{(3)}(t_1) H_l^{(3)}(t_2) \hat{W}(t) \right) \right] + \int_{-\infty}^{t} \text{d}t_2 \int_{-\infty}^{t} \text{d}t_1 \left( H_l^{(3)}(t_1) \hat{W}(t) H_l^{(3)}(t_2) \right), \tag{E.1} \]

where \( H_l \) is the interaction Hamiltonian. A regularisation procedure is necessary in order to perform consistently the internal momentum integral, which is divergent. By using dimensional regularisation, we have to move to \( 3+\delta \) spatial dimensions and being in a curved spacetime background also the form of the wavefunctions is modified; the index of the Hankel function depends on the spatial dimension. This means that, expanding around \( \delta = 0 \), the new modes are those in (C.2) and (C.3), plus a term proportional to \( \delta \). The contribution coming from the ‘unperturbed’ modes will be of the form

\[ P_0^{(2)}(k, p_1, p_2) = \int \text{d}t \int \text{d}t_1 \int \text{d}t_2 \frac{d^{3+\delta} p_1}{\mu^\delta} \frac{d^{3+\delta} p_2}{\mu^\delta} \delta^{(3+\delta)}(\vec{k} + \vec{p}_1 + \vec{p}_2) f(k, p_1, p_2), \tag{E.2} \]

where \( \mu \) is a physical renormalisation scale.

By dimensional analysis we can conclude that

\[ \int \frac{d^{3+\delta} p_1}{\mu^\delta} \frac{d^{3+\delta} p_2}{\mu^\delta} \delta^{(3+\delta)}(\vec{k} + \vec{p}_1 + \vec{p}_2) f(k, p_1, p_2) = k^{-3} \left( \frac{k}{\mu} \right)^\delta F(\delta), \tag{E.3} \]

where \( F \) is a dimensionless analytic function of \( \delta \) with a pole of order one in \( \delta = 0 \)

\[ F(\delta) = \frac{F_0}{\delta} + F_1 + \mathcal{O}(\delta). \tag{E.4} \]

Notice that we are interested in the late time behaviour, with \( k \) superhorizon. In this limit, we suppose that no residual time dependence is left in the 2-point function; this is the case when non-adiabatic contribution can be neglected. Therefore, expanding around \( \delta = 0 \), the correction can be written as

\[ P_0^{(2)}(k) = k^{-3} \left[ F_0 \log \left( \frac{k}{\mu} \right) + \Lambda \right], \tag{E.5} \]

where \( \Lambda \) is a divergent constant. The coefficient \( F_0 \) can be easily obtained by evaluating the LHS of (E.3) and then by differentiating both the sides of the equation a suitable number of times. However, this is not the end of the story, since there are now the terms proportional to \( \delta \) in the corrected wavefunctions; one can verify that they will always be of the form

\[ \delta \frac{f_k}{\delta \delta} = \delta \left[ \frac{1}{2} f_k|_{\delta=0} \log(-H t) + u(c_s, k, t) \right]. \tag{E.6} \]

Let us discuss briefly why contributions coming from the \( u(c_s, k, t) \) functions will not provide any logarithmic correction after the integrations are performed. Schematically:

\[ P_{u}^{(2)}(k) = k^{-3} \delta \left( \frac{k}{\mu} \right)^\delta \tilde{F}(\delta). \tag{E.7} \]

Therefore those terms could give contributions only if \( \tilde{F}(\delta) \) has a double pole in \( \delta \). Still, these corrections to the wavefunctions do not increase the degree of divergence of \( \tilde{F}(\delta) \), at least in our cases of interest, so that they will give no contribution to the loop correction.

The results are different for the terms proportional to \( \log(-H t) \); although it could be quite involved, the calculation can be performed exactly, and the structure results to be

\[ P_{d}^{(2)}(k) = k^{-3} \delta \left( \frac{k}{\mu} \right)^\delta G(\delta) \quad \text{where} \quad G(\delta) = \frac{G_0}{\delta} + G_1 + \mathcal{O}(\delta). \tag{E.8} \]
Therefore the total correction will be given by putting together (E.5) and (E.8)

\[ P^{(2)}(k) = k^{-3} \left[ F_0 \log \left( \frac{k}{\mu} \right) + G_0 + \Lambda \right] \]  

(E.9)

with \( \Lambda \) again a divergent constant. The important message is that \( G_0 \) could contribute to the loop correction only if it has some logarithmic dependence on \( k \), otherwise it will be absorbed into \( \Lambda \).

Let us now consider a generic Lagrangian in Fourier space of the form

\[ L = M_{pl}^2 a_{dS}^{2\nu-1} \left( f'^2 - k^2 f^2 \right) + L_{TSS}, \quad L_{TSS} = A M_{pl}^2 a_{dS}^{2\nu-1} D(k, k', k'') h_k f_{k'} f_{k''}; \]  

(E.10)

where \( A \) is a constant and \( f \) is some scalar quantity gauge invariant and constant at superhorizon scales (e.g. \( R \) for standard single field inflation, \( \zeta \) for solid inflation). The \( D \) function encodes all the information about the structure of the spatial derivatives acting on the fields contracted with the polarisation tensor. By solving the linear equation of motion for \( f \), the mode corresponding to the Bunch-Davies vacuum has the form

\[ f_k(t) = \frac{i \pi}{2 \nu} \Gamma(\nu) \frac{A_f}{k^{3/2}} x'^{\nu} H^{(1)}_\nu(x), \quad x = -c_s k t, \]  

(E.11)

where \( A_f \) is the scale invariant amplitude of the power spectrum of \( f \). For instance, in single field \( \nu = 3/2 \) while for a solid \( \nu = 5/2 \). If we use dimensional regularisation, then one can verify that for each vertex there is a contribution

\[ a^{2\nu-1}(t_{1,2}) \left[ 1 - \left( \nu - \frac{1}{2} \right) \delta \log(-H t_{1,2}) \right]; \quad (E.12) \]

and for each (internal) wavefunction a correction coming from

\[ (-H t_{1,2})^{\nu+\delta/2} = (-H t_{1,2})^\nu \left[ 1 + \frac{\delta}{2} \log(-H t_{1,2}) \right]. \quad (E.13) \]

Note that is not necessary to regularize external fields, since every logarithm coming from this functions will be canceled by the corresponding counterterm which renormalises the interaction, see \([6]\). If we write the tree level integrand as \( \mathcal{I}_0 \), then we have to compute the integral of

\[ \mathcal{I} = \mathcal{I}_0 \left[ 1 + \delta \left( 2 - \nu \right) \sum_{i=1}^2 \log(-H t_i) \right]. \quad (E.14) \]

The observation made in \([6]\) is that the contribution coming from the logarithmic corrections of the wavefunctions is simply a multiplicative factor in front of the tree level result, namely

\[ \frac{1}{2} \int dt_1 dt_2 \log(H^2 t_{1,2}) \mathcal{I}_0 = \log \left( \frac{H}{k} \right) \int dt_1 dt_2 \mathcal{I}_0. \quad (E.15) \]

Following this idea, what we have in the general case is

\[ \delta \left( 2 - \nu \right) \int dt_1 dt_2 \log(H^2 t_{1,2}) \mathcal{I}_0 = 2(2 - \nu) \delta \log \left( \frac{H}{k} \right) \int dt_1 dt_2 \mathcal{I}_0. \quad (E.16) \]

As usual, the tree level contribution is

\[ \int dt_1 dt_2 \mathcal{I}_0 = \frac{1}{k^3} \left( \frac{k}{\mu} \right)^\delta F(\delta); \quad (E.17) \]

thus, the total result is given by

\[ P^{(2)} = \frac{F_0}{k^3} \left[ \log \left( \frac{k}{\mu} \right) + 2(2 - \nu) \log \left( \frac{H}{k} \right) \right]. \quad (E.18) \]

As a result, in the case of single-field inflation or when a scalar spectator field is present \( \nu = 3/2 \) and no running in \( k \) is present. On the contrary, for a solid where \( \nu = 5/2 \), or more in general when \( \nu \neq 3/2 \) the cancellation does not occur anymore.
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