REELING & WRITHING

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Abstract. We suggest interpreting a certain braided monoidal topological category of finite subsets of $\mathbb{C}$ as an Archimedean specialization of a category of Kapranov-Smirnov $\mathbb{F}_t[t]$ - modules, with the classical Burau representation of the plane mapping-class groups as a monoidal functor to the symmetric monoidal category of $\mathbb{Z}[t, t^{-1}]$ - modules. The writhe of a mapping class (or of a braid, or more generally of a framed oriented tangle) can then be regarded as a kind of topological charge, defining an $A_\infty$ subcategory with morphisms of writhe zero, underlying the Mahowald-Hopkins interpretation [1] of the integral Eilenberg - Mac Lane spectrum $HZ$ as a Thom spectrum.

An appendix kindly supplied by Dale Rolfsen plugs an apparent hole in the literature of configuration spaces.

Acknowledgements and thanks

I owe deep thanks to Jon Beardsley, Lars Hesselholt and Mona Merling for many conversations about this material, and more emphatically to Dale Rolfsen and Valentin Zakharevich for help climbing out of various holes I had dug myself into during the preparation of this note. Of course none of them are responsible for remaining fumbles and howlers.

Some notation and conventions

A continuous action $\alpha : G \times X \to X$ of a topological group on a topological space defines a topological groupoid $[X/G]$ with space $X$ of objects and $X \times G$ of morphisms. I will call such categories transformation groupoids, using terminology going back to Weyl. Presenting this category as a simplicial space is a kind of bar construction, with the homotopy resolution or Borel quotient

$$|[X/G]| = X//G \simeq EG \times_G X \to \text{pt} \times_G X = X/G$$

as its geometric realization. Thus $|[*/G]| := BG$ is a canonical choice of classifying space for $G$; note that this construction respects Cartesian product.

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§ 1 Braids and mapping class groups

1.1 The complex regular representation

\[ \Sigma_n \times \mathbb{C}^n \ni \tau \times (z_i) \mapsto (z_{\tau(i)}) \in \mathbb{C}^n \]

(1 \leq i \leq n) of the group \( \Sigma_n \) of permutations of \( n \) things sends the ‘thick diagonal’ \( \Delta^n \subset \mathbb{C}^n \) (consisting of vectors \( z \) with a repeated entry, i.e. \( z_i = z_k \) for some \( i \neq k \)) to itself. Its complement is therefore also invariant, and the quotient

\[ (\mathbb{C}^n - \Delta^n)/\Sigma_n := \text{Config}^n(\mathbb{C}) \]

can be interpreted as the space of subsets of \( \mathbb{C} \) of cardinality \( n \). Let

\[ \text{Config}^*(\mathbb{C}) = \coprod_{n \geq 0} \text{Config}^n(\mathbb{C}) , \]

(with \( \text{Config}^0(\mathbb{C}) := \{\emptyset\} \)) be the space of finite subsets of \( \mathbb{C} \). It will be convenient to let \( \{z\} \) denote the unordered set \( \{z_i\} \) of coordinates of \( z = (z_1, \ldots, z_n) \).

Let \( \mathbb{D} \) be the topological group of diffeomorphisms \( \phi \) of the Riemann sphere \( \mathbb{C}_+ \) which leave fixed a neighborhood of the point + at infinity; \( \phi \mapsto \phi_t(z) := t \cdot \phi(t^{-1}z) \) (0 < \( t < \infty \)) contracts \( \mathbb{D} \) equivariantly to a point. The group \( \mathbb{D} \) acts diagonally (i.e. by \( \phi(z) = (\phi(z_i)) \) on \( \mathbb{C}^n \), compatibly with the action of \( \Sigma_n \), thus defining an action of \( \mathbb{D} \) on \( \text{Config}^n(\mathbb{C}) \). The resulting topological groupoid

\[ \mathcal{C} := [\text{Config}^*(\mathbb{C})/\mathbb{D}] \]

can be interpreted as the category of finite subsets of \( \mathbb{C} \), with compactly supported diffeomorphisms as its maps\(^1\). When \( \{z\}, \{z'\} \in \mathcal{C} \) both have cardinality \( n \),

\[ \mathcal{C}(\{z\}, \{z'\}) := \{ \phi \in \mathbb{D} | \phi(\{z\}) = \{z'\} \} \]

is the space of morphisms in \( \mathcal{C} \) from \( \{z\} \) to \( \{z'\} \); it is otherwise empty.

Let \( \mathbb{D}(\{z\}) := \mathcal{C}(\{z\}, \{z\}) = \{ \phi \in \mathbb{D} | \phi(\{z\}) = \{z\} \} \) be the isotropy subgroup of a finite subset of \( \mathbb{C} \); if \( \{n\} = \{1, \ldots, n\} \), I will write \( \mathbb{D}_n \) for this subgroup. \( \mathbb{D} \) then acts freely and transitively on \( \text{Config}^n(\mathbb{C}) \), defining a diffeomorphism

\[ \phi \mapsto \phi(\{n\}) : \mathbb{D}/\mathbb{D}_n \to \text{Config}^n(\mathbb{C}) . \]

The functor

\[ \mathcal{C} := \coprod_{n \geq 0} [\mathbb{D}/\mathbb{D}_n] \to \coprod_{n \geq 0} [(\mathbb{D}/\mathbb{D}_n)/\mathbb{D}] = \mathcal{C} \]

induces a homotopy-equivalence of geometric realizations, and I will argue below that \( \mathcal{C} \) has some technical advantages as a model for \( \mathcal{C} \).

1.2 Since \( \mathbb{D} \) is contractible, the geometric realization

\[ |\mathcal{C}| \simeq \text{Config}^*(\mathbb{C}) \]

\(^1\)If I understood what (\( \infty, 1 \))-categories are, I might suggest that \( \mathcal{C} \) is an example.
of the category $\mathcal{C}$ is homotopy equivalent to the configuration space itself, and I take it as well-known that the latter space is homotopy equivalent to the geometric realization of the braid monoid
$$\mathcal{B} := \bigcoprod_{n \geq 0} [*/\mathbb{B}_n].$$

A corollary is that $\mathbb{D}_n$ is homotopically trivial \cite{4,6}, with $\pi_0 \mathbb{D}_n$ \textit{(i.e.} the mapping-class group for the disk marked with $n$ distinct interior points) isomorphic to $\mathbb{B}_n$. I believe this isomorphism sends Artin’s generating twist $\sigma_i$ to the isotopy class of a Dehn half-twist along the line from $i$ to $i + 1$.

A quarter-century ago Joyal and Street defined braided monoidal categories, and exhibited $\mathcal{B}$ as the free such thing on one generator \cite{10}. The composition
$$\mathcal{C} = \bigcoprod_{n \geq 0}[\mathbb{D}/\mathbb{D}_n] \xrightarrow{\pi_0} \bigcoprod_{n \geq 0} [*/\mathbb{B}_n] = \mathcal{B}$$
identifies $\mathcal{B}$ as a kind of homotopy-theoretic skeleton of $\mathcal{C}$, and I would like to believe that this suffices to make $\mathcal{C}$ into a braided monoidal ($\infty, 1$)-category. Classical work of Boardman and Vogt on spaces of rectangles in $\mathbb{C}$ suggests an alternative path to a similar conclusion.

\textbf{A memorable fancy} suggests thinking of this equivalence as an example of what physicists call a duality, with braids interpreted as something like particle trajectories, and diffeomorphisms of the plane as delocalized wave-like phenomena . . .

\textbf{Remark:} A braid, in Artin’s presentation, can be regarded as a framed oriented tangle, and Joyal and Street \cite{11} have identified the category of such things as a (ribbon or tortile) braided monoidal category with (what mathematicians call) duality. Such tangles will recur below.

\section*{2 The Burau representations}

\subsection*{2.1} Let
$$\tilde{\phi} : (\mathbb{C}_+ - \{z\}, +) \to (\mathbb{C}_+ - \{z'\}, +)$$
be the diffeomorphism of punctured Riemann spheres associated to $\phi \in \mathcal{C}(\{z\}, \{z'\})$. The commutative diagram
$$\xymatrix{
\pi_1(\mathbb{C}_+ - \{z\}, +) \ar[r]^-{\pi_1(\tilde{\phi})} \ar[d]^H & \pi_1(\mathbb{C}_+ - \{z'\}, +) \ar[d]^H \\
H_1(\mathbb{C}_+ - \{z\}; \mathbb{Z}) \ar[r]^-{H_1(\tilde{\phi})} \ar[d]^\text{tr} & H_1(\mathbb{C}_+ - \{z'\}; \mathbb{Z}) \ar[d]^\text{tr}' \\
\mathbb{Z} & \mathbb{Z}
}$$
(with the top vertical homomorphisms defined by Hurewicz, and trace-

homomorphisms

\[ \text{tr} : H_1(C_+ - \{z\}, \mathbb{Z}) \cong \mathbb{Z}^n \to \mathbb{Z} \]

sending the class of a small loop around \( z \) to its winding number) defines an equivariant lift

\[
\begin{array}{c}
B(\{z\}, \ast) \xrightarrow{B(\phi)} B(\{z'\}, \ast) \\
\downarrow \mathbb{Z} \quad \downarrow \mathbb{Z} \\
(C_+ - \{z\}, +) \xrightarrow{\phi} (C_+ - \{z'\}, +)
\end{array}
\]

of \( \phi \) to a map between infinite cyclic (Burau) covers defined by the homomorphisms \( \text{tr} \circ H \).

**Lemma** If \( \phi \in \mathcal{C}(\{z\}, \{z'\}) \) and \( \phi' \in \mathcal{C}(\{z'\}, \{z''\}) \) then

\[
B(\phi') \circ B(\phi) = B(\phi' \circ \phi)
\]
as maps equivariant with respect to the action of \( \mathbb{Z} \) by deck-transformations. □

The homology group

\[ H_1(B(\{z\}); \mathbb{Z}) \in (\Lambda - \text{Mod}) \]
is a free module of rank \( n - 1 \) over the deck-transformation group ring \( \Lambda := \mathbb{Z}[\mathbb{Z}] = \mathbb{Z}[t, t^{-1}] \).

**Corollary** If \( \phi \in \mathcal{C}(\{z\}, \{z'\}) \) then

\[
\beta(\phi) = \det H_1(B(\phi)) : \Lambda^{\top} H_1(B(\{z\})) \to \Lambda^{\top} H_1(B(\{z'\}))
\]
defines an element of the Picard group

\[ \text{Pic}(\Lambda) \cong \{ \pm t^w \mid w \in \mathbb{Z} \} \]
of \( \Lambda \).

**2.2** The functor

\[ \beta : \mathcal{C} \to \mathcal{C} \ni \{z\} \mapsto H_1(B(\{z\}); \mathbb{Z}) \in (\Lambda - \text{Mod}) \]
(or, alternately, its factorization through \( \mathcal{B} \)) defines the reduced Burau representation of the braid group (or of the mapping-class group for \( \mathcal{C} \) with marked points). There is a more subtle unreduced Burau representation \( \beta \) which blows up the marked points; it is related to \( \beta \) much as the regular representation of the symmetric group is related to its reduced version.

It sends an element of \( \text{Config}^n(\mathbb{C}) \) to a free \( \Lambda \)-module of rank \( n \), which differs from the reduced module by a rank one module and a copy of the reduced regular representation of \( \Sigma_n \). With the following exception I will work below, out of ignorance, with the reduced Burau representation:
Claim: The unreduced Burau representation defines a braided monoidal functor

\[
\begin{array}{c}
\mathbb{C} \\
\downarrow \beta \\
\mathbb{B}
\end{array} \rightarrow (\Lambda - \text{Mod})
\]

(with values in a symmetric monoidal category), expressed on Artin generators by

\[
\beta(\sigma_i) := 
\begin{bmatrix}
1_{i-1} & 0 & 0 \\
0 & b & 0 \\
0 & 0 & 1_{n-1-i}
\end{bmatrix} \in M_n(\Lambda),
\]

where \(1_k \in M_k(\mathbb{Z})\) is the \(k \times k\) identity matrix, and

\[
b := \begin{bmatrix} 1 - t & t \\ 1 & 0 \end{bmatrix}.
\]

2.3 Recall that abelianization sends a braid \(b\), regarded as a word in Artin’s generators, to the sum of the exponents appearing in that presentation; or, more geometrically, to the number of overcrossings less the number of undercrossings in a blackboard presentation. It is therefore not invariant under Reidemeister moves of type I, but is an invariant of oriented (framed) tangles, and of the oriented link defined by the braid closure, as will be seen below. Evidently

\[
\det \beta(b) = (-t)^w(b),
\]

which can be shown to be equal to \(\det \beta(b)\) as well. I will refer to this invariant here as the writhe of the braid (or of the corresponding mapping class) \([1,6]\).

Definition: The closely related discriminant

\[
\Delta_n : \text{Config}^n(\mathbb{C}) \ni \{z\} \mapsto (-1)^n \prod_{i \neq j} (z_i - z_j) \in \mathbb{C}^\times
\]

may be recalled from high school. I am indebted to Dale Rolfsen [email, 4/8/018] for a proof that its induced map

\[
w = \Delta_\ast : \mathbb{B}_n = \pi_1(\text{Config}^n(\mathbb{C}), \{n\}) \rightarrow \pi_1(\mathbb{C}^\times, 1) = \mathbb{Z}
\]

on fundamental groups is the writhe homomorphism; neither of us can find this anywhere in the literature, and he has kindly allowed me to include his argument in an appendix below.
If $D^n_0$ denotes the kernel of $w \circ \pi_0 : D_n \to \mathbb{Z}$, ie the closed subgroup of diffeomorphisms of writhe zero in $D_n$, then we have a commutative diagram

$$
\begin{array}{ccc}
C^0 / \mathbb{Z} & \rightarrow & C \\
\downarrow & & \downarrow \pi_0 \\
B^0 & \rightarrow & B \\
\end{array}
\begin{array}{ccc}
det \beta & \rightarrow & [\ast / \mathbb{Z}] \\
\end{array}
\begin{array}{ccc}
= & & \\
\end{array}
\begin{array}{ccc}
\prod_{n \geq 0} [\ast / E^0_n] := B^0 \\
\end{array}
$$

where

$$
\prod_{n \geq 0} [\ast / E^0_n] := B^0
$$

is the dewrithed category $[1 \S 4]$ of $B$. Similarly,

$$
C := \prod_{n \geq 0} [D / D^n_0].
$$

Note that the fiber product

$$
\begin{array}{ccc}
\widetilde{\text{Config}}^n(\mathbb{C}) & \longrightarrow & \text{Config}^n(\mathbb{C}) \\
\downarrow & & \downarrow w \\
\mathbb{C} & \longrightarrow & \mathbb{C}^\times \\
\end{array}
$$

(with $e(x) = \exp(2\pi i x)$) defines a classifying space for $B^0_n$; this corrects an error in [1 §4.4], where $\Delta_{n*}$ is mistakenly taken to equal $n(n-1)w$.

**Remark** It will not be needed here, but it is interesting that work of Jones [9 §7], going back via Conway and Squiers to Burau, shows that

$$
det(1 - \beta(b)) = (-t^{1/2})^{m(b)} \cdot (-1)^{n-1}[n](t) \cdot \nabla(b^*) \in \tilde{\Lambda},
$$

or, more impressionistically,

$$
det(\beta(b)^{1/2} - \beta(b)^{-1/2}) = (-1)^{m(b)/2}[n](t) \cdot \nabla(b^*),
$$

where $b^*$ is the braid closure of $b$, $\nabla$ is Conway’s normalized Alexander polynomial of a link,

$$
[n](t) = \frac{t^{n/2} - t^{-n/2}}{t^{1/2} - t^{-1/2}}
$$

is a Gaussian $q$-combinatorial function, and

$$
\tilde{\Lambda} = \mathbb{Z}[t^{1/2}, t^{-1/2}] \supset \Lambda
$$

is a ring with involution $t^{1/2} \mapsto -t^{-1/2}$.

**2.3** The (ribbon) category $\mathcal{F}$ of (isotopy classes of) oriented framed tangles can be generated by elements of six types, two corresponding to the twists $\sigma^\pm 1$ generating the braid groups, the rest corresponding to ‘caps’ and ‘cups’,
each with two possible orientations. Extending \( \mathfrak{m} \) to suitably presented tangles by assigning the value zero to these caps and cups – i.e. by defining the writhe of a tangle as the number of overcrossings minus the number of undercrossings - is visibly compatible with the fifteen relations between these generators \([7,13,15]\), defining a commutative diagram

\[
\begin{array}{c}
\mathcal{B} \\
\downarrow \mathfrak{m} \\
\star/\mathbb{Z}
\end{array} \rightarrow \begin{array}{c}
\mathcal{T}
\end{array}
\]

of braided monoidal functors, cf [3 §5].

§ III \( \mathbb{F}_1[t] \)-modules

3.1 In their groundbreaking work on the hypothetical field \( \mathbb{F}_1 \) with one element, Kapranov and Smirnov suggested regarding \( \mathcal{B}_n \) as an analog of the group \( \text{Gl}_n(\mathbb{F}_1[t]) \), and provided evidence [12 §1.2] for this using a hypothetical base-change morphism \( \mathbb{F}_1[t] \to \mathbb{F}_q[t] \). The definitions above might be similarly interpreted as an Archimedean version of their construction, i.e. that some category of \( \mathbb{F}_1[t,t^{-1}] \)-modules and isomorphisms maps to \( \mathcal{C} \) via a base-change map

\[ \mathbb{F}_1[t,t^{-1}] \to \mathbb{Z}[t,t^{-1}] \]

underlying the Burau representation. Note, however, that \( \Lambda \) - modules form a symmetric monoidal category, whereas \( \mathcal{C} \) is only braided monoidal; this raises issues perhaps related to statistics of topological quantum field theories.\(^3\)

3.2 This document is a draft follow-up to [1], which is concerned with the group completion \( \Omega^2 S^3 \) of the braid monoid (and its universal cover \( \Omega^2 S^3(3) \), similarly related to \( \mathcal{B}_0 \)), in connection with the Mahowald - Hopkins construction of the integral Eilenberg-Mac Lane spectrum as a generalized Thom spectrum. The intent of this note is to suggest that we can reasonably think of that cover as the group completion of a monoidal (\( A_{\infty} \) but not braided) topological category (\( \mathcal{B}_0 \) or \( \mathcal{C}_0 \) or perhaps some version of \( \mathcal{T} \)) with finite subsets of \( \mathcal{C} \) as objects, and compactly-supported diffeomorphisms with writhe zero as its maps. More picturesquely, we might think of such objects as collections of particles moving in the plane subject to some kind of constraint which prevents kinks, i.e. Reidemeister moves of type I.

\(^2\)The author is a friend of \( \mathbb{F}_1 \) but is by no means an expert. He is aware of considerable interesting work (e.g. [5,14, 17, ...]) on categories of \( \mathbb{F}_1 \)-algebras, but \( \mathbb{F}_1 \)-modules seem to have received less attention.
3.3 Aside from such issues of imagination, it may be worth noting that the (unreduced) Burau representation

$$\beta : \prod_{n \geq 0} [*/\mathbb{B}_n] \rightarrow \prod_{n \geq 0} [*/\text{Gl}_n(\Lambda)]$$

suggests the existence of a potentially interesting map

$$\Omega^2 S^3 \rightarrow \Omega^\infty K_{\text{alg}}(\Lambda) \simeq \Omega^\infty K_{\text{alg}}(\mathbb{Z}) \times \Omega^{-1+\infty} K_{\text{alg}}(\mathbb{Z})$$

of (two-fold ?) loop spaces, with the last equivalence being a consequence of what was called (before Quillen) the ‘fundamental theorem of algebraic K-theory’ [18 Ch V]. On homotopy groups this defines a homomorphism

$$\pi_i(S^3) \rightarrow K_{1-2}(\mathbb{Z}) \oplus K_{1-3}(\mathbb{Z})$$

e.g.

$$\mathbb{Z}_2^2 \times \mathbb{Z}_{84} \rightarrow ?0 \oplus \mathbb{Z}_{1008}$$

when \( i = 14 \); thanks to L Hesselholt for correcting (among other things) my arithmetic. I don’t know if this map factors through the Hurewicz homomorphism from stable homotopy (\( \pi_{11}(S^0) = \mathbb{Z}_{504} \ldots \)).

§ Appendix, by Dale Rolfsen: The discriminant and the writhe

**Proposition:** For \( \beta \in \mathbb{B}_n \), \( \Delta \ast (\beta) = w(\beta) \), where \( w(\beta) \) is the writhe of \( \beta \), equal to the sum of exponents of \( \beta \) written in the Artin generators; at least, up to sign.

**Proof:** We warm up with \( n = 2 \): take \( \{1, 2\} \) as the basepoint of \( \text{Config}^2(\mathbb{C}) \). Consider \( \beta = \sigma_1 \). This is represented by a 180° rotation, so \( \beta \in \text{Config}^2\mathbb{C} \) is represented by the loop

$$z_1(t) = \frac{1}{2}(3 - e^{i\pi t}), \quad z_2(t) = \frac{1}{2}(3 + e^{i\pi t})$$

with \( 0 \leq t \leq 1 \), so \( z_1(0) = z(1) = 1 \), \( z_1(1) = z_2(0) = 2 \). Then

$$\Delta(z_1(t), z_2(t)) = -(z_1(t) - z_2(t))(z_2(t) - z_1(t)) = e^{i\pi t} \cdot e^{i\pi t} = e^{2i\pi t} .$$

This is a loop in \( \mathbb{C}^\times \) based at 1 and represents a generator of \( \pi_1(\mathbb{C}^\times, 1) \cong \mathbb{Z} \), which we write additively. So each factor of \( \sigma_1 \) contributes \(+1\) to \( \Delta_\ast(\beta) \), and \( \sigma_1^{-1} \) contributes \(-1\). This is the formula for the writhe, (\( n = 2 \)). \( \Box \)

For the general case \( \Delta : \text{Config}^n\mathbb{C} \rightarrow \mathbb{C}^\times \) we take \( \{1, \ldots, n\} \in \text{Config}^n\mathbb{C} \) as basepoint. Note that

$$\Delta(z_1, \ldots, z_n) = \pm \prod_{i < k} (z_i - z_k)^2 ,$$

where the sign depends only on \( n \): it’s \((-1)^{\binom{n}{2}}\). For convenience, normalize and let

$$\Gamma(z_1, \ldots, z_n) = \frac{\pm \Delta(z_1, \ldots, z_n)}{|\Delta(z_1, \ldots, z_n)|}$$
with the same ± sign as before, and note that, since they differ by a (variable) real (or negative real), we have
\[ \Gamma_* = \Delta_* : \pi_1 \text{Config}^n \mathbb{C} \to \pi_1 \mathbb{C}^\times \]
(though basepoints may differ).

Noting that if \( \{z_1(t), \ldots, z_n(t)\}, \ 0 \leq t \leq 1 \) is a loop in \( \text{Config}^n \mathbb{C} \) based at \( \{1, \ldots, n\} \) representing \( \beta \), and using
\[ \Gamma(z_1, \ldots, z_n) = \prod_{i<k} \frac{(z_i - z_k)^2}{|z_i - z_k|^2} \]
then \( \Gamma(z_1(t), \ldots, z_n(t)) \) is a loop in \( \mathbb{C}^\times \) based at +1. Moreover, if we define
\[ v_{i,k}(t) := \frac{(z_i - z_k)^2}{|z_i - z_k|^2} \]
we see that \( v_{i,k}(t) \) is also a loop based at +1 if \( i < k \).

**Lemma:** Suppose \( f_m : X \to \mathbb{C}^\times \) are maps from a topological space \( X \) to the nonzero complexes, and that \( x_0 \in X \) satisfies \( f_m(x_0) = 1, \forall m = 1, \ldots, M \).
Let \( f(x) = f_1(x) \cdots f_M(x) \) be the pointwise product of these complex-valued functions; then \( f_* \) and \( f_m* \) are all homomorphisms \( \pi_1(X, x_0) \to \pi_1(\mathbb{C}^\times, 1) \), and for \( \alpha \in \pi_1(X, x_0) \), we have
\[ f_*(\alpha) = f_1*(\alpha) + \cdots + f_M*(\alpha) \in \mathbb{Z} \cong \pi_1(\mathbb{C}^\times, 1) \]
as additive group.

**Beweis:** Klar. □

Now consider a generator \( \sigma_j \) of \( \pi_1(\text{Config}^n \mathbb{C}) \). It’s represented by a loop \( z_k(t) = k \) if \( k \neq j \),
\[ z_j(t) = \frac{1}{2}((2j+1) - e^{i\pi t}), \quad z_{j+1}(t) = \frac{1}{2}((2j+1) + e^{i\pi t}), \]
\( (0 \leq t \leq 1) \). Recall that \( v_{i,k} \) is a loop in \( \mathbb{C}^\times \) based at 1, with induced homomorphism \( v_{i,k*} \). As when \( n = 2 \), we have that the loop
\[ v_{j,j+1}(t) := \frac{(z_{j+1} - z_j)^2}{|z_{j+1} - z_j|^2} \]
is a generator of \( \mathbb{Z} \).

We just showed that \( v_{j,j+1*}(\sigma_j) = 1 \).

**Claim:** If \( i < k \) and \( \{i, k\} \neq \{j, j + 1\} \), then \( v_{i,k*}(\sigma_j) = 0 \).

It follows from this that
\[ \Delta_*(\sigma_j) = \Gamma_*(\sigma_j) = \sum_{i<k} v_{i,k*}(\sigma_j) = 1. \]
It remains to prove the claim: that for \( j \neq i < k \neq j + 1 \), \( v_{i,k}(\sigma_j) = 0 \). The easy case is that if \( \{i, k\} \cap \{j, j + 1\} = \emptyset \), then \( v_{i,k}(t) = 1 \) is constant.

Consider the case \( i < j = k \). We see that the unit complex number \( \frac{z_k - z_i}{|z_k - z_i|} \) is restricted to have modulus certainly contained in \((-\pi/4, +\pi/4)\). Therefore \( v_{i,k}(\sigma_j(t)) \), which is the square of this, has real part always positive. Such a loop, based at 1, is nullhomotopic.

Similar calculations hold for the other cases,

\[ \square \]

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References

1. J Beardsley, J Morava, Toward a Galois theory of the integers over the sphere spectrum, in the Connes Festschrift, https://arxiv.org/abs/1710.05992
2. S Bigelow, A Cattabriga, V Florens, Alexander representation of tangles, https://arxiv.org/abs/1203.4590
3. ——, A diagrammatic Alexander invariant of tangles, https://arxiv.org/abs/1203.5457
4. J Birman, T Brendel, Braids: a survey, https://arxiv.org/abs/math/0409205
5. A Connes, C Consani, On the notion of geometry over \( \mathbb{F}_1 \), https://arxiv.org/abs/0809.2926
6. C Earle, A Schatz, Teichmüller theory for surfaces with boundary, J. Diff. Geo. 4 (1970) 169 - 185
7. P Freyd, D Yetter, Braided compact closed categories with applications to low-dimensional topology, Adv. Math. 77 (1989) 156 - 182
8. FB Fuller, The writhing number of a space curve, PNAS 68 (1971) 815 – 819
9. VFR Jones, Hecke algebra representations of braid groups and link polynomials, Annals of Math 126 (1987) 335 – 388
10. A Joyal, R Street, Braided tensor categories, Adv. Math. 102 (1993) 20 - 78
11. ——, ——, Tortile Yang-Baxter operators in tensor categories. J. Pure Appl. Algebra 71 (1991) 43 - 51
12. M Kapranov, A Smirnov, Cohomology determinants and reciprocity laws: number field case, available at e.g. http://cage.ugent.be/~kthas/Fun/index.php/kapranovsmirnov.html
13. C Kassel, Quantum groups. Springer Graduate Texts 155 (1995)
14. M Marcolli, Y Manin, Homotopy types and geometries below Spec \( \mathbb{Z} \), arXiv:1805:10801
15. Mei Chee Shum, Tortile tensor categories, J. Pure Appl. Algebra 93 (1994) 57 - 110
16. C Squiers, The Burau representation is unitary, Proc. AMS 90 (1984) 199 – 202
17. B Toen, M Vaquie, Under Spec \( \mathbb{Z} \), arXiv:math/0509684
18. C Weibel, The K-book, AMS Graduate Studies in Mathematics 145 (2013)