Comments on non-Gaussian density perturbations and the production of primordial black holes

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Abstract. We review the basic arguments for the likelihood of non-Gaussian density perturbations in inflation models with primordial black hole (PBH) production. We discuss our derived distributions of field fluctuations and their implications, specifically commenting on the fine-tuning problem. We also discuss how the derived distributions may be affected when linked to metric perturbations. While linking the metric perturbations to field fluctuations in a nonlinear way may be important for determining exact probability distributions, the correct mapping is not self-evident. The calculation of P. Ivanov, which yields a skew positive distribution, is based on an ansatz for the behavior of the nonlinear metric perturbation. We note that the "natural" generalization of the gauge-invariant formalism favored by Bond and Salopek yields an effective linear link between the distribution of field fluctuations and metric perturbations during inflation.

1. Primordial black holes and non-Gaussianity

Any (realistic) PBH-producing model of inflation must yield a power spectrum that is consistent with COBE on large scales ($\delta = \delta \rho / \rho \sim 10^{-5}$), while at the same time giving $\delta \sim 0.01$ on some small scale. Such models are possible, however, as we explain below and in detail [1], the large-amplitude small-scale power can lead to non-Gaussian fluctuation statistics.

Let $\phi$ represent the inflaton. During inflation, quantum fluctuations in the inflaton, $\delta \phi_Q \sim H$, generate density perturbations. The density fluctuations at horizon crossing map to the following expression as the field rolls down the potential: $\delta \sim H^2 / \dot{\phi}$, where $\dot{\phi}$ is the time derivative of $\phi$. If the classical trajectory of $\phi$ is unaffected by quantum fluctuations, the fluctuations evolve linearly and Gaussian statistics prevail. This is true in most models of inflation. Over a Hubble time, the change in $\phi$ due to its classical trajectory is $\Delta \phi_{CL} \sim \dot{\phi} H^{-1}$. Gaussian statistics are then likely provided $\delta \phi_Q / \Delta \phi_{CL} \ll 1$. But notice the following relation: $\delta \sim H^2 / \dot{\phi} \sim \delta \phi_Q / \Delta \phi_{CL}$. The Gaussian criterion breaks down as the amplitude of $\delta$ gets large. Therefore, if we do have some inflation model that produces the $\delta \sim 0.01$ needed for PBH formation, we may expect non-Gaussian statistics to be important.

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Figure 1. Our calculated distribution of field fluctuations (solid line) compared to a Gaussian with the same mean and standard deviation (dashed line).

Figure 2. The initial mass fraction of PBHs produced, $\beta$, versus standard deviation fluctuations needed for PBH formation. The dashed line is derived from a Gaussian distribution and the solid line results from the non-Gaussian distribution shown in Figure 1.

2. Toy model results

In order to test the qualitative ideas presented in the previous section, we have constructed several (toy) PBH-producing inflation models and used stochastic inflation calculations to find the associated fluctuation statistics [1]. For example, Figure 1 shows a typical probability distribution of fluctuations from one of our models together with a Gaussian distribution with the same standard deviation. Note that the Gaussian over produces large fluctuations relative to the derived distribution. See Ref. [2] for a review.

One important consequence of this non-Gaussian result is that it intensifies the fine-tuning problem in PBH formation. The fine-tuning problem arises because the height of the peak in power, $\delta(M)$, (where $M$ is the mass of PBHs produced) must be fine-tuned to an extremely precise value in order to obtain a cosmologically relevant fraction of PBHs without over-production.

Limits on PBH abundances are often quoted in terms of the initial mass fraction of PBHs $\beta$ [1]. Primordial black holes form when $\delta \gtrsim \delta_{PBH} \sim 1/3$. Therefore, if $P(\delta)$ is the distribution of fluctuations on a mass scale $M$ with an rms deviation $\delta(M)$, the value of $\beta$ is determined by integrating over the high-$\delta$ tail of the distribution, $(\delta \gtrsim \delta_{PBH})$. The result will depend on the ratio, $\delta_{PBH}/\delta(M)$ (i.e. the size of a PBH-producing fluctuation relative to the rms deviation).

Figure 2 shows $\beta$ plotted against this ratio for a Gaussian distribution (dashed line) and
the non-Gaussian case (solid line) plotted in Figure 1. The horizontal lines show the range of cosmologically interesting values of $\beta$ over all relevant PBH masses. We see that the value of $\beta$ changes drastically for small changes in the relevant ratio for both distributions, but is even worse for the non-Gaussian expectation.

For the non-Gaussian case, $\delta_{PBH}/\delta(M)$ must fall between 2 and 3 in order to be cosmologically relevant at any mass scale. This precision must occur within a spike in power that is $\sim 4$ orders of magnitude above the COBE normalization. The upshot is that any inflation model that attempts to construct cosmologically relevant PBH formation must be tuned to at least $10^{-4}$, and if we restrict ourselves to any specific mass scale, the required tuning becomes much worse.

### 3. Nonlinear metric perturbations

In the standard linear treatment of inflationary fluctuations $\delta \phi \propto \delta$. So, to first order we may infer the probability distribution of density fluctuations by deriving the distribution of $\phi$ fluctuations. However, as mentioned in Ref. [1], since the density fluctuations are large, any nonlinearity in the connection between metric perturbations and field fluctuations may change the distribution shape. One should formally understand this relationship by coupling field fluctuations to the Einstein equations during inflation.

This problem was explicitly noted by Ivanov [3], who presented a PBH-producing model using a flat plateau feature in the inflaton potential. Because the region is exactly flat, the Ivanov model yields a near Gaussian distribution of field fluctuations or, equivalently, a near Gaussian distribution in the stochastic time, $\delta t$, that the inflaton spends in the flat region of the potential (the region associated with PBH production).

Ivanov chooses to relate metric perturbations to this stochastic time delay using the ansatz: $h = \exp(H\delta t) - 1$, where $H$ is the value of the Hubble parameter associated with the flat region of the potential. In the limit of small $h$, this relation reduces to the standard gauge invariant quantity. The proposed exponential relation between $\delta t$ and $h$, when combined with the near Gaussian distribution in $\delta t$, yields a positively skewed distribution of $h$. Note, however, that this result is based entirely on the ansatz. The relation favored by Bond and Salopek [4] is $h = H\delta t$, which produces a linear transformation and, thus, an effectively Gaussian distribution in $h$.

Understanding the nature of the mapping between derived $\delta \phi$ statistics and $\delta$ statistics could be important in determining the exact nature of fluctuation distributions associated with PBHs. However, since the standard treatment yields a linear relationship, the exponential relation discussed above should, perhaps, be viewed as an extreme possibility. We are currently attempting to clarify this issue, but advocate the linear mapping as a first approximation until a more strongly motivated relation can be found.
References

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