A $K$-QUADRILATERAL COSINE CHARACTERIZATION OF ALEKSANDROV SPACES OF CURVATURE BOUNDED ABOVE

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Abstract. In this note, we extend the main results of our paper on quasilinearization and curvature of Aleksandrov spaces of curvature $\leq 0$ to curvature bounds other than 0. For non-zero $K$, we employ the previously introduced notion of the $K$-quadrilateral cosine, which is the cosine under parallel transport in model $K$-space, and which is denoted by $\cos q_K$. Our principal result states that a geodesically connected metric space (of diameter not greater than $\pi/(2\sqrt{K})$ if $K > 0$) is an $\mathcal{R}_K$ domain (otherwise known as a CAT ($K$) space) if and only if always $\cos q_K \leq 1$ or always $\cos q_K \geq -1$. (We prove that in such spaces always $\cos q_K \leq 1$ is equivalent to always $\cos q_K \geq -1$). As a corollary, we give necessary and sufficient conditions for a Cauchy complete semimetric space to be a complete $\mathcal{R}_K$ domain. We show that in our theorem the diameter hypothesis for positive $K$ is sharp and we prove an extremal theorem when $|\cos q_K|$ attains an upper bound of 1. We derive from our main theorem and our previous result for $K = 0$ a complete solution of Gromov’s curvature problem in the context of Aleksandrov spaces of curvature bounded above. Then we establish the $K$-Euler’s inequality and the extremal theorem for equality in the $K$-Euler’s inequality in an $\mathcal{R}_K$ domain.

1. Introduction

Classes of Riemannian metrics that satisfy uniform sectional curvature bounds often arise in geometry. In his fundamental papers [1] and [2], Aleksandrov presented the upper and lower curvature conditions for a geodesically connected metric space, i.e., a metric space in which any two points can be joined by a shortest. In particular, Aleksandrov introduced the notion of an $\mathcal{R}_K$ domain, also known as a CAT ($K$) space, a geodesically connected metric space of curvature $\leq K$ in the sense of Aleksandrov, in which shortests depend continuously on their end points and in which the perimeter of every geodesic triangle is less than $2\pi/\sqrt{K}$ if $K > 0$.

In this note, we present a deeper metric analysis of Aleksandrov’s upper boundedness curvature condition by extending the main results of our paper [6] for $K = 0$ to the case of non-zero $K$. There are striking differences in our approach to non-zero $K$ that require different methods. Our results are not local; hence the lack of linearity in the model space presents substantial conceptual and technical problems.

Our main result in [6] states that a geodesically connected metric space $(\mathcal{M}, \rho)$ is an $\mathcal{R}_0$ domain if and only if for every two ordered pairs of distinct points $\overline{AP}$ and $\overline{BQ}$ =
(B, Q) in \( \mathcal{M} \), called (non-zero) bound vectors, their quadrilateral cosine, \( \cosq(A\vec{P}, B\vec{Q}) \), satisfies the following inequality

\[
\cosq(A\vec{P}, B\vec{Q}) = \frac{\rho^2(A, Q) + \rho^2(B, P) - \rho^2(A, B) - \rho^2(P, Q)}{2 \rho(A, P) \rho(B, Q)} \leq 1.
\]

The quadrilateral cosine was introduced in [19] under the name of function \( h \) and was used to construct the generalized Sasaki metric on the set of tangent elements of a metric space and to obtain a pure metric characterization of Riemannian spaces [19], [20].

The generalization of \( \cosq \) to non-zero \( K \) is not straightforward. Let \( K \neq 0 \) and \( \hat{K} = \sqrt{|K|} \). In what follows, \( \hat{K} = K = \sqrt{|K|} \) if \( K > 0 \) and \( \hat{K} = i\sqrt{-K} \) if \( K < 0 \). The following definition is equivalent to Definition 3.2 in [4].

**Definition 1.1.** Let \( (M, \rho) \) be a metric space and \( A, P, B, Q \in \mathcal{M} \) be such that \( A \neq P, B \neq Q \). If \( K > 0 \), we assume that \( \rho(A, P) = x, \rho(B, Q) = y, \rho(A, B) = a, \rho(P, Q) = b, \rho(P, B) = d \) and \( \rho(A, Q) = f \), as shown in Fig. 1.1. Then the \( K \)-quadrilateral cosine \( \cosq_K(A\vec{P}, B\vec{Q}) \) is defined by

\[
\cosq_K(A\vec{P}, B\vec{Q}) = \frac{\cos \hat{K} b + \cos \hat{K} x \cos \hat{K} y}{\sin \hat{K} x \sin \hat{K} y} - \frac{(\cos \hat{K} x + \cos \hat{K} d)(\cos \hat{K} y + \cos \hat{K} f)}{(1 + \cos \hat{K} a) \sin \hat{K} x \sin \hat{K} y}.
\]

In particular, if \( K > 0 \), then

\[
\cosq_K(A\vec{P}, B\vec{Q}) = \frac{\cos \hat{K} b + \cos \hat{K} x \cos \hat{K} y}{\sin \hat{K} x \sin \hat{K} y} - \frac{(\cos \hat{K} x + \cos \hat{K} d)(\cos \hat{K} y + \cos \hat{K} f)}{(1 + \cos \hat{K} a) \sin \hat{K} x \sin \hat{K} y},
\]

and if \( K < 0 \), then

\[
\cosq_K(A\vec{P}, B\vec{Q}) = \frac{(\cosh \hat{K} x + \cosh \hat{K} d)(\cosh \hat{K} y + \cosh \hat{K} f)}{(1 + \cosh \hat{K} a) \sinh \hat{K} x \sinh \hat{K} y} - \frac{\cosh \hat{K} b + \cosh \hat{K} x \cosh \hat{K} y}{\sinh \hat{K} x \sinh \hat{K} y}.
\]
If $K = 0$, we set $\cosq_0 \left( \overrightarrow{A\mathcal{P}}, \overrightarrow{B\mathcal{Q}} \right) = \cosq \left( \overrightarrow{A\mathcal{P}}, \overrightarrow{B\mathcal{Q}} \right)$.

We introduce the following conditions for a metric space $(\mathcal{M}, \rho)$:

(i) The upper four point $\cosq_K$ condition: $\cosq_K \left( \overrightarrow{A\mathcal{P}}, \overrightarrow{B\mathcal{Q}} \right) \leq 1$ for every pair of non-zero bound vectors $\overrightarrow{A\mathcal{P}}$ and $\overrightarrow{B\mathcal{Q}}$ in $\mathcal{M}$ and such that $\rho(A, P), \rho(B, Q)$ and $\rho(A, B) < \pi/\sqrt{K}$ when $K > 0$.

(ii) The lower four point $\cosq_K$ condition: $\cosq_K \left( \overrightarrow{A\mathcal{P}}, \overrightarrow{B\mathcal{Q}} \right) \geq -1$ for every pair of non-zero bound vectors $\overrightarrow{A\mathcal{P}}$ and $\overrightarrow{B\mathcal{Q}}$ in $\mathcal{M}$ and such that $\rho(A, P), \rho(B, Q)$ and $\rho(A, B) < \pi/\sqrt{K}$ when $K > 0$.

We say that $(\mathcal{M}, \rho)$ satisfies the one-sided four point $\cosq_K$ condition if it satisfies either the upper four point $\cosq_K$ condition or the lower four point $\cosq_K$ condition.

Our present main result is given by the following

**Theorem 1.1.** Let $K \neq 0$ and let $(\mathcal{M}, \rho)$ be a geodesically connected metric space such that $\text{diam} (\mathcal{M}) \leq \pi / \left(2\sqrt{K}\right)$ when $K > 0$. Then $(\mathcal{M}, \rho)$ is an $\mathbb{R}_K$ domain with the same diameter restriction if and only if $(\mathcal{M}, \rho)$ satisfies the one-sided $\cosq_K$ condition.

**Remark 1.1.** As Example 4.1 shows, the restriction on the diameter of $(\mathcal{M}, \rho)$ for positive $K$ cannot be dropped and the diameter bound in the hypothesis of Theorem 1.1 is sharp.

**Remark 1.2.** A normed vector space of curvature $\leq K$ in the sense of Aleksandrov is an inner product space. Hence, we can complement the results of the paper by Schoenberg [23] by deriving from Theorem 1.1 that a normed vector space is an inner product space if and only if it satisfies the one-sided $\cosq_K$ condition for some positive $K$.

Recall that the $K$-plane $\mathbb{S}_K$ is the Euclidean plane if $K = 0$, the open hemisphere of radius $1/\sqrt{K}$ if $K > 0$ and the hyperbolic plane of curvature $K$ if $K < 0$. The definition of $K$-space $\mathbb{S}_K^3$ is similar.

**Remark 1.3.** Notice that in a domain of $\mathbb{S}_K^3$ of diameter less than $\pi / \left(2\sqrt{K}\right)$ if $K > 0$, the $K$-quadrilateral cosine of a pair of non-zero bound vectors $\overrightarrow{A\mathcal{P}}$ and $\overrightarrow{B\mathcal{Q}}$ equals the cosine of the angle between the vector $\exp_{B^{-1}}^A \left(Q\right)$ and the vector $\exp_{A^{-1}}^B \left(P\right)$ under the Levi-Civita parallel translation from the point $A$ to the point $B$ along the shortest joining these two points (see, Sec. 5). Moreover, in $\mathbb{S}_K^3$, for every $K$, $\cosq_K$ always can be interpreted as a cosine of an angle (Corollary 4.3). By Theorems 1.1 and 4.1 in a geodesically connected metric space satisfying the diameter restriction of Theorem 1.1 $\cosq_K$ can be interpreted as the cosine of an angle when and only when the metric space is an $\mathbb{R}_K$ domain.

If $K = 0$, the upper four point $\cosq_K$ condition is immediately equivalent to the lower four point $\cosq_K$ condition [6, Introduction]. According to Examples 5.1 and 5.2 in a general metric space, this is not true anymore for non-zero $K$. However, we derive from Theorem 1.1 and Theorem 4.1 of Sec. 4 the following:

**Corollary 1.1.** Let $K \neq 0$ and let $(\mathcal{M}, \rho)$ be a geodesically connected metric space such that $\text{diam} (\mathcal{M}) \leq \pi / \left(2\sqrt{K}\right)$ when $K > 0$. Then $(\mathcal{M}, \rho)$ satisfies the upper four point $\cosq_K$ condition if and only if $(\mathcal{M}, \rho)$ satisfies the lower four point $\cosq_K$ condition.
Recall that a polygonal curve $APQB$ in a Riemannian space is called a Levi-Civita parallelogramoid if the distances between $A$ and $P$ and $B$ and $Q$ are equal, and the vectors $\exp_A^{-1}(P)$ and $\exp_B^{-1}(Q)$ are parallel along a shortest joining $A$ to $B$. We say that a polygonal curve $APQB$ in $S_K$ is a Levi-Civita trapezoid if either the vectors $\exp_A^{-1}(P)$ and $\exp_B^{-1}(Q)$ are parallel along the shortest $AB$ or the vectors $\exp_A^{-1}(P)$ and $-\exp_B^{-1}(Q)$ are parallel along the shortest $AB$. A convex domain in $S_K$ enclosed by a Levi-Civita trapezoid is called a Levi-Civita trapezoidal domain. In particular, the set of points of a shortest in $S_K$ is a degenerate Levi-Civita trapezoidal domain. The following theorem generalizing [4, Theorem 15] and [5, Theorem 6.2] describes the extremal cases when $\cos q_K$ takes values 1 or $-1$.

**Theorem 1.2.** Let $K \neq 0$ and let $(M, \rho)$ be a geodesically connected metric space such that $diam(M) < \pi/\left(2\sqrt{K}\right)$ when $K > 0$. If $(M, \rho)$ satisfies the one-sided four point $\cos q_K$ condition, and for a pair of non-zero bound vectors $\overrightarrow{AP}$ and $\overrightarrow{BQ}$ in $M$, $|\cos q_K(\overrightarrow{AP},\overrightarrow{BQ})| = 1$, then the convex hull of the quadruple $\{A, P, Q, B\}$ is isometric to a Levi-Civita trapezoidal domain in $S_K$.

By Example [2.1] Theorem 1.2 need not be true if $diam(M) = \pi/\left(2\sqrt{K}\right)$ when $K > 0$.

Recall that a semimetric space is a distance space with a positive definite and symmetric distance. A semimetric space $(M, \rho)$ is said to be weakly convex if, for every $A, B \in M$, there is $\lambda \in (0, 1)$, such that, for every $\varepsilon > 0$, there is $C_\varepsilon \in M$ satisfying the inequalities $|\rho(A, C_\varepsilon) - \lambda \rho(A, B)| < \varepsilon$ and $|\rho(B, C_\varepsilon) - (1 - \lambda) \rho(A, B)| < \varepsilon$. Cauchy sequences in a semimetric space and the diameter of a semimetric space are defined in the same way as in a metric space. Finally, notice that the upper and the lower four point $\cos q_K$ conditions can also be stated for semimetric spaces. We derive from Theorem [1.1] and Menger’s theorem [3, Theorem 1.1] the following extension of [6, Theorem 5] to non-zero $K$:

**Theorem 1.3.** Let $K \neq 0$ and let $(M, \rho)$ be a semimetric space such that $diam(M) \leq \pi/\left(2\sqrt{K}\right)$ when $K > 0$. Then $(M, \rho)$ is a complete $\mathbb{K}$-domain with the same diameter restriction if and only if the following conditions are satisfied:

(a) $(M, \rho)$ is weakly convex.

(b) Each Cauchy sequence in $(M, \rho)$ has a limit.

(c) $(M, \rho)$ satisfies the one-sided four point $\cos q_K$ condition.

In his book [17], Gromov offered a method to define classes of metric spaces corresponding to Riemannian manifolds with prescribed curvature restrictions by introducing global and local $K$-curvature classes. Let $r \in \mathbb{N}$ and $M_r$ denote the set of all symmetric $r \times r$ matrices with zero diagonal entries and non-negative entries otherwise. Let $\mathcal{X}$ be a set and $d : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}$ be a non-negative function such that $d(P, Q) = d(Q, P)$ and $d(P, Q) = 0$ if and only if $P = Q$, for all $P, Q \in \mathcal{X}$. Then $K_r(\mathcal{X})$ consists of all matrices $A = (a_{ij})$ in $M_r$ such that for every $A \in K_r(\mathcal{X})$ there is an $r$-tuple $\{P_1, P_2, ..., P_r\} \subseteq \mathcal{X}$ satisfying $a_{ij} = d(P_i, P_j)$, $i, j = 1, 2, ..., r$. A subset $K \subseteq M_r$ defines the (global) $K$-curvature class as follows. The $K$-curvature class consists of all $(\mathcal{X}, d)$ such that $K_r(\mathcal{X}) \subseteq K$. Gromov’s curvature problem is the problem of a meaningful geometric description of $K$-curvature classes ([17], Section 1.19.+, Curvature Problem).
In [6] Theorem 8] we gave a solution of Gromov’s curvature problem in the context of $\mathbb{R}_0$ domains and therefore for Aleksandrov spaces of non-positive curvature. In this note, we obtain a complete solution of Gromov’s curvature problem in the context of $\mathbb{R}_K$ domains and Aleksandrov spaces of curvature $\leq K$ by solving Gromov’s curvature problem for non-zero $K$ as a corollary of Theorems 1.1 and 1.3.

Let $M_G$ be the set of all geodesically connected metric spaces and $M_S$ denote the set of all semimetric spaces satisfying conditions (a) and (b) of Theorem 1.3. For $\kappa > 0$, let $K^+$ ($\kappa^2$) denote the set of all matrices $A = (a_{ij}) \in M_4$ such that

\[
(\cos \kappa a_{23} + \cos \kappa a_{12} \cos \kappa a_{34}) (1 + \cos \kappa a_{14}) - \\
(\cos \kappa a_{12} + \cos \kappa a_{24} \cos \kappa a_{34} + \cos \kappa a_{13}) \leq \\
\sin \kappa a_{12} \sin \kappa a_{34} (1 + \cos \kappa a_{14})
\]

and $a_{12}, a_{13}, a_{14}, a_{23}, a_{24}, a_{34} \leq \pi/(2\kappa)$. For $K^-$ ($\kappa^2$), multiply the left-hand side of the above inequality by $(-1)$. In a similar way, we define $K^+ (-\kappa^2)$ as the set of all matrices $A = (a_{ij}) \in M_4$ such that

\[
(\cosh \kappa a_{12} + \cosh \kappa a_{24}) (\cosh \kappa a_{34} + \cosh \kappa a_{13}) - \\
(\cosh \kappa a_{23} + \cosh \kappa a_{12} \cosh \kappa a_{34}) (1 + \cosh \kappa a_{14}) \leq \\
\sinh \kappa a_{12} \sinh \kappa a_{34} (1 + \cosh \kappa a_{14})
\]

and for $K^- (-\kappa^2)$, multiply the left-hand side of the above inequality by $(-1)$.

**Theorem 1.4.** Let $\kappa > 0$ and $K = \kappa^2$ if $K > 0$ and $K = -\kappa^2$ if $K < 0$. Then

(i) $(X, \rho) \in M_G$ (respectively $(X, \rho) \in M_S$) is in the global $K^+$ ($\kappa^2$)-curvature class if and only if $(X, \rho)$ is an $\mathbb{R}_K$ domain (respectively complete $\mathbb{R}_K$ domain) of diameter not greater than $\pi/(2\kappa)$.

(ii) $(X, \rho) \in M_G$ (respectively $(X, \rho) \in M_S$) is in the global $K^+ (-\kappa^2)$-curvature class if and only if $(X, \rho)$ is an $\mathbb{R}_K$ domain (respectively complete $\mathbb{R}_K$ domain).

**Remark 1.4.** In particular, $(X, \rho) \in M_G$ is in the local $K^\pm (\pm\kappa^2)$-curvature class if and only if $(X, \rho)$ is an Aleksandrov space of curvature $\leq K$ where $K = \pm\kappa^2$.

**Remark 1.5.** For an alternative proof of one of our main theorems [6 Theorem 6] solving Gromov’s curvature problem in the context of $\mathbb{R}_0$-domains, see [22].

In Sec. 9 we generalize the familiar Euler’s equality [14, Corollary 4] to non-zero $K$. Hence, we can extend the quadrilateral inequality condition (also known as Enflo’s 2-roundness condition [13]) to the case of non-zero $K$.

The $K$-quadrilateral (or $K$-Euler) inequality condition: for every quadruple of points $\{A, B, C, D\}$ in a metric space $(M, \rho)$,

(i) if $K > 0$, then

\[
\cos \kappa \rho(A, B) + \cos \kappa \rho(B, C) + \cos \kappa \rho(C, D) + \cos \kappa \rho(D, A) \\
\leq 4 \cos \kappa \frac{\rho(A, C)}{2} \cos \kappa \frac{\rho(B, D)}{2},
\]

and
(ii) if $K < 0$, then
\[
cosh \kappa \rho(A, B) + \cosh \kappa \rho(B, C) + \cosh \kappa \rho(C, D) + \cosh \kappa \rho(D, A) \\
\geq 4 \cosh \kappa \frac{\rho(A, C)}{2} \cosh \kappa \frac{\rho(B, D)}{2}.
\]

According to Theorem 6 in [6], a geodesically connected metric space is an $\mathbb{R}_0$ domain if and only if it satisfies the 0-quadrilateral inequality condition. In Sec. 9 we prove that the $K$-quadrilateral inequality condition holds in an $\mathbb{R}_K$ domain for non-zero $K$. We do not know if the converse is true.

In [18], Lafont and Prassidis established the 0-quadrilateral inequality in $\mathbb{R}_0$ domains. In [15] (also, see the correction in [16]) Foertsch, Lytchak and Schroeder considered a weaker Ptolemaic condition and showed that while each $\mathbb{R}_0$ domain is Ptolemaic, the converse may not be true.

Sec. 2 is a short review of Aleksandrov spaces of curvature bounded above. In Sec. 3, we prove that $|\cosq_K| \leq 1$ in $K$-space. Sec. 4 presents the proof of $|\cosq_K| \leq 1$ in an $\mathbb{R}_K$ domain of diameter not greater than $\pi/\sqrt{2K}$ if $K > 0$. We show that, in contrast to $\mathbb{S}^3_K$, the diameter restriction cannot be dropped for an $\mathbb{R}_K$ domain. In Sec. 5 we present counterexamples showing that in a non-geodesically connected metric space the upper four point $\cosq_K$ condition need not be equivalent to the lower four point $\cosq_K$ condition. Sec. 6 contains the proof of our main result–Theorem 1.1. In this section, we assume that $(\mathcal{M}, \rho)$ is a geodesically connected metric space (of diameter not greater than $\pi/\sqrt{2K}$ if $K > 0$) satisfying the one-sided four point $\cosq_K$ condition. In Sec. 6.2 we prove that in $(\mathcal{M}, \rho)$ shortests depend continuously on their end points; in particular, any pair of points can be joined by a unique shortest. Hence, by Theorem 9 in [2, § 3], the global angle comparison in $(\mathcal{M}, \rho)$ will follow from the local angle comparison, i.e., locally, each vertex angle of a geodesic triangle $T$ is not greater than the corresponding angle of the isometric copy of $T$ in the $K$-plane. In Section 6.3, we derive the main auxiliary estimate–the cross-diagonal estimate. In Section 6.4, the cross-diagonal estimate lemma is used to derive our major estimate of Sec. 6 the growth estimate lemma. In Sec. 6.5 we show that the growth estimate lemma implies that in $(\mathcal{M}, \rho)$, between any pair of shortests starting at a common point $A$, the proportional angle exists, that is, the limit of $\angle_{K,X_t,Y_t}$ as $t \to 0^+$ exists if $\rho(X_t, A) / \rho(Y_t, A) = \text{const}$ (for the notation, see Sec. 2 and Fig. 6.3). In Sec. 6.6 following the method of our proof of Proposition 20 in [4], we derive from existence of proportional angles and growth estimate lemma that in $(\mathcal{M}, \rho)$, between any pair of shortests emanating from a common point, Aleksandrov’s angle exists. Existence of Aleksandrov’s angle and growth estimate lemma enables us to prove the local angle comparison and thereby the global angle comparison (Sec. 6.7). In Sec. 7 we consider an extremal case when $|\cosq_K| = 1$. In Sec. 8 we extend our main result to complete weakly convex semimetric spaces satisfying the one-sided four point $\cosq_K$ condition. In Sec. 9 we derive $K$-Euler’s inequality for $\mathbb{R}_K$ domains and discuss the extremal case of equality in $K$-Euler’s inequality. In Sec. 10 we show that for an individual quadruple in a metric space, the one-sided four point $\cosq_K$ conditions are weaker than previously introduced curvature conditions.

2. **ALEKSANDROV’S UPPER CURVATURE CONDITION**

In this section, we recall some basic definitions of Aleksandrov geometry.
Let $(\mathcal{M}, \rho)$ be a metric space and $\mathcal{L}$ be a curve in $\mathcal{M}$. We denote by $\ell_\rho(\mathcal{L})$ the length of $\mathcal{L}$ in the metric $\rho$. A rectifiable curve $\mathcal{L}$ joining $P$ to $Q$ is called a shortest, or minimal geodesic (joining $P$ to $Q$) if $\rho(P, Q) = \ell_\rho(\mathcal{L})$. If $\mathcal{L}$ is a shortest joining $P$ to $Q$, then often we denote the shortest $\mathcal{L}$ by $\mathcal{P}Q$ if there is no possible ambiguity, and the distance between its end points (or, in general, between a pair of points in $\mathcal{M}$) $P$ and $Q$ by $\rho(P, Q)$. A subset $\mathcal{U}$ of a metric space is said to be convex if every pair of points $P, Q \in \mathcal{U}$ can be joined by a shortest and all shortests joining $P$ to $Q$ are contained in $\mathcal{U}$.

A configuration consisting of three distinct points $A, B, C \in \mathcal{M}$ (vertices) and three shortests $AB, BC$ and $AC$ (sides) is called a (geodesic) triangle $T = ABC$. The perimeter $p(T)$ of a triangle $T = ABC$ (or, in general, of a triple of points $T = \{A, B, C\}$ in $\mathcal{M}$) is the sum $AB + BC + AC$. The isometric copy in the $K$-plane of the triangle $T$ is the triangle $T^K = A^K B^K C^K$ in $\mathcal{S}_K$ having the same side lengths as $T$: $AB = A^K B^K$, $AC = A^K C^K$ and $BC = B^K C^K$ (if $K > 0$ we require that $p(T) < 2\pi/\sqrt{K}$). We let $\angle_K ABC$ denote the angle $\angle B^K A^K C^K$. The area $\sigma(ABC)$ of the triangle $ABC$ is the area of the euclidean triangle $A^0 B^0 C^0$.

Let $\mathcal{L}$ and $\mathcal{N}$ be two shortest arcs with a common starting point $O$ in a metric space $(\mathcal{M}, \rho)$. Let $X \in \mathcal{L} \setminus \{O\}$ and $Y \in \mathcal{N} \setminus \{O\}$. Set $x = OX$, $y = OY$ and $\angle_K(x, y) = \angle_K XOY$. The upper and lower angles between the curves $\mathcal{L}$ and $\mathcal{N}$ are defined by

$$\bar{\angle}(\mathcal{L}, \mathcal{N}) = \lim_{x \to 0^+, y \to 0^+} \angle_K(x, y) \quad \text{and} \quad \underline{\angle}(\mathcal{L}, \mathcal{N}) = \lim_{x \to 0^+, y \to 0^+} \angle_K(x, y).$$

It is known that the above definitions do not depend on $K$. We say that the angle $\angle(\mathcal{L}, \mathcal{N})$ between $\mathcal{L}$ and $\mathcal{N}$ exists if $\bar{\angle}(\mathcal{L}, \mathcal{N}) = \underline{\angle}(\mathcal{L}, \mathcal{N})$.

The (upper) $K$-excess $\delta_K(T)$ of the triangle $T$ is defined by

$$\delta_K(T) = (\underline{\angle}ABC + \underline{\angle}ACB + \underline{\angle}BAC) - (\angle_K ABC + \angle_K ACB + \angle_K BAC).$$

An $\mathcal{R}_K$ domain (otherwise known as a CAT($K$) space) is a metric space with the following properties:

(i) $\mathcal{R}_K$ is convex (that is, $\mathcal{R}_K$ is geodesically connected).

(ii) If $K > 0$, then the perimeter of every triangle in $\mathcal{R}_K$ is less than $2\pi/\sqrt{K}$.

(iii) Each triangle $T$ in $\mathcal{R}_K$ has non-positive $K$-excess $\delta_K(T)$.

We remark that by (ii), $\text{diam}(\mathcal{R}_K) < \pi/\sqrt{K}$ when $K > 0$.

Another name for an $\mathcal{R}_K$ domain is a CAT($K$) space; we will use Aleksandrov’s original notation (see, [1] and [2]). A metric space $(\mathcal{M}, \rho)$ is a space of curvature $\leq K$ in the sense of Aleksandrov if each point of $\mathcal{M}$ is contained in some neighborhood that is an $\mathcal{R}_K$ domain. For more information on Aleksandrov spaces of curvature $\leq K$, see [1], [2], [8] and [10].

We will find useful the following theorem of Reshetnyak [21].

Let $\mathcal{L}$ be a closed rectifiable curve in a metric space $(\mathcal{M}, \rho)$ such that $\ell_\rho(\mathcal{L}) < 2\pi/\sqrt{K}$ if $K > 0$. Let $\mathcal{V}$ be a convex domain in $\mathcal{S}_K$ with the bounding curve $\mathcal{N}$. We say that $\mathcal{V}$ majorizes the curve $\mathcal{L}$ if there is a non-expanding mapping of the domain $\mathcal{V}$ into $\mathcal{M}$ that maps $\mathcal{N}$ onto $\mathcal{L}$ and preserves arc length. The domain $\mathcal{V}$ is called the majorant for $\mathcal{L}$.

**Reshetnyak’s majorization theorem:** In an $\mathcal{R}_K$ domain, for every rectifiable closed curve $\mathcal{L}$ (whose length is less than $2\pi/\sqrt{K}$ when $K > 0$), there is a convex domain in $\mathcal{S}_K$ that majorizes $\mathcal{L}$.

Let $(A_1, A_2, ..., A_n)$ be an $n$-tuple of distinct points in $(\mathcal{M}, \rho)$. Suppose that for every $j \in \{1, 2, ..., n - 1\}$, the points $A_j$ and $A_{j+1}$ can be joined by a shortest $\mathcal{L}_j = A_jA_{j+1}$. 

...
the midpoint of the shortest arc $AB$ defined as the cosine $\cos q K = \cos(q)$.

In our notation, we always assume that the vertices of $N$ are labeled so that $A_j A_{j+1} = A'_j A'_{j+1}$ for every $j = 1, 2, \ldots, n$, where $A_{n+1} = A_1$ and $A'_{n+1} = A'_1$.

If $L$ is a polygonal curve $A_1 A_2 \ldots A_n$ of length $l$ in a metric space, then the $arc length parametrization of $L$ relative to $A_1$ is an arc length parametrization of $L$, $g_{al} = g_{al,l} : [0, l] \to M$, such that the length of the arc of $L$ with the end points at $A_1$ and $g_{al}(s)$ is equal to $s$ in $[0, l]$. The $reduced parametrization of $L$ relative to $A$ is the mapping $g_r = g_{r,l} : [0, 1] \to M$ given by $g_r(t) = g_{al}(tl)$ for every $t \in [0, 1]$. If $l_0 > 0$, then the $l_0$-arc length proportional parametrization of $L$ is the mapping $g_{l_0,pr} = g_{l_0,pr,l} : [0, l_0] \to M$ given by $g_{l_0,pr}(u) = g_{al}(ul/l_0)$.

Let $(M, \rho)$ be a geodesically connected metric space and $F \subseteq M$ be a non-empty set. For a pair of points $P, Q \in (M, \rho)$, we let $G[P, Q]$ denote the set of points each of which belongs to a shortest joining the points $P$ and $Q$. We define $G[F]$ by $G[F] = \bigcup_{P,Q \in F} G[P, Q]$. Next, denote $F$ by $G^0[F]$ and $G^\alpha[F]$ by $G^\alpha[F]$. Then the $geodesic convex hull of $F$ is defined as $G\mathcal{C}[F] = \bigcup_{n=0}^{\infty} G^n[F]$.

3. $K$-Quadrilateral Cosine in $K$-Space

In this section, we prove that $|\cos q_K| \le 1$ in $S^3_K$.

Let $K \ne 0$. Let $\{A, B, P, Q\}$ be a quadruple of distinct points in $S^3_K$. Let $O$ be the midpoint of the shortest arc $AB$. If $PO < \pi/(2\sqrt{K})$ when $K > 0$, we can use the following constructive interpretation of $\cos q_K$ in $S^3_K$. Indeed, let $P'$ be the point symmetric to the point $P$ relative to $O$, that is, $O$ is the midpoint of the shortest arc $PP'$, as illustrated in Fig. 3.1. Then $\xi = \exp^1_{A} (P)$ is (Levi-Civita) parallel along $AB$ to the vector $\xi'' = -\xi'$, where $\xi' = \exp_{B}^{-1}(P')$. Let $\zeta = \exp_{B}^{-1}(Q)$. In [5, Lemma 3.1], we showed that $\cos q_K \left( \overrightarrow{AP}, \overrightarrow{BQ} \right) = -\cos \angle P' BQ = \cos \angle (\zeta, \xi'')$. Hence, for the $K$-quadrilateral

\[ \begin{align*}
\xi & \\
\xi' & \\
P & \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \ Quad
cosine in $S^3_K$, we always have

$$
\left| \cos_K(\vec{AP}, \vec{BQ}) \right| \leq 1
$$

as long as $PO < \pi / \left( 2\sqrt{K} \right)$ when $K > 0$.

Next, we show that the restriction $PO < \pi / \left( 2\sqrt{K} \right)$ for positive $K$ can be dropped for $S^3_K$ itself. We begin with the following simple corollary of the spherical cosine formula.

**Lemma 3.1.** Let $K > 0$ and $T = ABC$ be a non-degenerate triangle in $S^3_K$. Let $M \in AB \setminus \{A, B\}$. Set $a = BC$, $b = AC$, $c = AB$, $l = MC$ and $t = AM/c$, as shown in the Fig. 3.2. Then

$$
\cos Kl = \frac{\cos ka \sin kt + \cos kb \sin k(1 - t)c}{\sin kc}.
$$

In particular, if $M$ is the midpoint of the shortest $AB$, we obtain a familiar spherical Bruhat-Tits equality:

$$
\cos Kl = \frac{\cos ka + \cos kb}{2 \cos \frac{\pi}{2k}}
$$

(for $K = 0$, see Bruhat-Tits inequality in [11], Lemma 3.2.1).

By $K$-concavity in $R_K$ [2, §3, Theorem 2], we also have the following

**Corollary 3.1.** Let $K > 0$ and $T = ABC$ be a non-degenerate triangle in $R_K$ and $M \in AB \setminus \{A, B\}$. Set $a = BC$, $b = AC$, $c = AB$, $l = MC$ and $t = AM/c$. Then

$$
\cos Kl \geq \frac{\cos ka \sin kt + \cos kb \sin k(1 - t)c}{\sin kc}.
$$

**Corollary 3.2.** Let $K > 0$ and $T = ABC$ be a non-degenerate triangle in $R_K$ and $M \in AB \setminus \{A, B\}$. Let $AC, BC \leq \pi / (2k)$. Then $CM \leq \pi / (2k)$. In addition, if either $AC$ or $BC$ is less than $\pi / (2k)$, then $CM < \pi / (2k)$.

Finally, we show that $\cos_K$ remains the same in the half-sphere after cutting the lengths of bound vectors in half.

**Lemma 3.2.** Let $K > 0$ and $\vec{AP}, \vec{BQ}$ be a pair of non-zero bound vectors in $S^3_K$. Let $M_1$ and $M_2$ be the midpoints of the shortest $AP$ and $BQ$, respectively. Then

$$
\cos_K(\vec{AP}, \vec{BQ}) = \cos_K(\vec{AM_1}, \vec{BM_2}).
$$
Proof. We have:

$$\cosq_k \left( \overrightarrow{AM_1}, \overrightarrow{BM_2} \right) = \frac{\cos \kappa b' + \cos \frac{\kappa x}{2} \cos \frac{\kappa y}{2}}{\sin \frac{\kappa x}{2} \sin \frac{\kappa y}{2}} - \left( \cos \frac{\kappa x}{2} + \cos \kappa d' \right) \left( \cos \frac{\kappa y}{2} + \cos \kappa f' \right) \frac{(1 + \cos \kappa a) \sin \frac{\kappa x}{2} \sin \frac{\kappa y}{2}}{4 \cos^2 \frac{\kappa x}{2} \cos^2 \frac{\kappa y}{2}},$$

where the notation is given in Fig. 3.3. By the Bruhat-Tits equality (Lemma 3.1),

![Figure 3.3. Sketch for Lemma 3.2](image)

$$\cos \kappa f' = \frac{\cos \kappa a + \cos \kappa f}{2 \cos \frac{\kappa x}{2}} \text{ (triangle } ABQ),$$

$$\cos \kappa g = \frac{\cos \kappa b + \cos \kappa d}{2 \cos \frac{\kappa y}{2}}, \quad g = PM_2 \text{ (triangle } PQB),$$

$$\cos \kappa b' = \frac{\cos \kappa g + \cos \kappa f'}{2 \cos \frac{\kappa x}{2}} \text{ (triangle } APM_2),$$

whence $$\cos \kappa b' = \frac{(\cos \kappa a + \cos \kappa b + \cos \kappa d + \cos \kappa f)}{4 \cos^2 \frac{\kappa x}{2} \cos^2 \frac{\kappa y}{2}}.$$ Hence,

$$\cosq_k \left( \overrightarrow{AM_1}, \overrightarrow{BM_2} \right) = \frac{(1 + \cos \kappa a) \left[ \cos \kappa a + \cos \kappa b + \cos \kappa d + \cos \kappa f + \right.}{4 \cos^2 \frac{\kappa x}{2} \cos^2 \frac{\kappa y}{2}} - \left( \frac{2 \cos^2 \frac{\kappa x}{2} + \cos \kappa a + \cos \kappa d}{2 \cos^2 \frac{\kappa y}{2} + \cos \kappa a + \cos \kappa f} \right) \left[(1 + \cos \kappa a) \sin \kappa x \sin \kappa y \right]^{-1} \times \left[ (1 + \cos \kappa a) \cos \kappa a + \cos \kappa b + \cos \kappa d + \cos \kappa f + 1 + \cos \kappa x + \cos \kappa y + \cos \kappa x \cos \kappa y \right] \times \left[ (1 + \cos \kappa a) \sin \kappa x \sin \kappa y \right].$$
After elementary but tedious simplifications of the last expression, we get:
\[
\cos_k^\perp (\overrightarrow{AM_1}, \overrightarrow{BM_2}) = \left[(1 + \cos \kappa a) \sin \kappa x \sin \kappa y\right]^{-1} \left[ \cos \kappa b + \cos \kappa a \cos \kappa b + \cos \kappa a \cos \kappa x \cos \kappa y \right.
\left. - \cos \kappa x \cos \kappa f - \cos \kappa y \cos \kappa d - \cos \kappa f \cos \kappa d \right] - 1
\]

as needed.

Let \( K > 0 \). Recall that \( \text{diam}(\mathcal{R}_K) < \frac{\pi}{\sqrt{K}} \). By Lemma 3.2, there is no restriction in assuming that \( AP \) and \( BQ \) are as small as we wish. Hence, without loss of generality, we can assume that \( PO < \frac{\pi}{2\sqrt{K}} \) (see Fig. 3.1). So, we get the following

**Corollary 3.3.** Let \( K \neq 0 \). Then for every pair of non-zero bound vectors \( \overrightarrow{AP} \) and \( \overrightarrow{BQ} \) in \( \mathbb{S}_K^3 \),

\[
\left| \cos_k^\perp (\overrightarrow{AP}, \overrightarrow{BQ}) \right| \leq 1.
\]

### 4. \( K \)-Quadrilateral Cosine in an \( \mathcal{R}_K \) Domain

The main goal of this section is to show that \( \left| \cos_k^\perp \right| \leq 1 \) in an \( \mathcal{R}_K \) domain of diameter not greater than \( \frac{\pi}{2\sqrt{K}} \) if \( K > 0 \). In addition, for \( K > 0 \) we present examples of \( \mathcal{R}_K \) domains of diameter greater than \( \frac{\pi}{2\sqrt{K}} \) and arbitrarily close to \( \frac{\pi}{2\sqrt{K}} \) for which \( \left| \cos_k^\perp \right| \leq 1 \) does not hold.

The following theorem is a minor generalization of Theorem 4.2 in [5].

**Theorem 4.1.** Let \( K \neq 0 \) and let \( \Omega = \{A, P, B, Q\} \) be a quadruple of points in an \( \mathcal{R}_K \) domain such that \( A \neq P, B \neq Q \) and \( \text{diam}(\Omega) \leq \frac{\pi}{2\sqrt{K}} \) if \( K > 0 \). Then

\[
\left| \cos_k^\perp (\overrightarrow{AP}, \overrightarrow{BQ}) \right| \leq 1.
\]

**Proof.** It is sufficient to consider the case of positive \( K \). If \( \text{diam}(\Omega) < \frac{\pi}{2\sqrt{K}} \), then by [5, Theorem 4.2], \( \cos_k^\perp (\overrightarrow{AP}, \overrightarrow{BQ}) \geq -1 \). For the reader’s convenience, we include some omitted details in [5] of the proof of the inequality \( \cos_k^\perp (\overrightarrow{AP}, \overrightarrow{BQ}) \leq 1 \). Consider the closed polygonal curve \( \mathcal{L} = \overrightarrow{APQB} \), as shown in Fig. 1.1. We will follow the part of the proof of Reshetnyak’s Lemma 2 in [21] corresponding to the case of \( K \)-fans consisting of two triangles in \( \mathbb{S}_K \) (a special case of Reshetnyak’s majorization theorem). Namely, under the hypothesis of Theorem 4.1, in addition to the existence of a convex domain \( \mathcal{V} \subseteq \mathbb{S}_K \) majorizing the polygonal curve \( \mathcal{L} \), Reshetnyak’s proof also implies that the domain \( \mathcal{V} \) can be selected so that

\[
d = PB \leq d' = P' B' < \frac{\pi}{2\sqrt{K}}, \quad f = AQ \leq f' = A' Q' < \frac{\pi}{\sqrt{K}}
\]

where \( \mathcal{L}' = \overrightarrow{A' P' Q' B'} \) is the bounding curve of \( \mathcal{V} \). Indeed, as shown in the proof of Lemma 2 in [21], there is a quadrangular domain \( \mathcal{J} \) in \( \mathbb{S}_K \) bounded by a quadrangle \( \mathcal{L}' = \overrightarrow{A' P' Q' B'} \) such that

\[
AP = \overrightarrow{A' P'}, \ AB = \overrightarrow{A' B'}, \ PB = \overrightarrow{P' B'} \text{ and } PQ = \overrightarrow{P' Q'}, \ BQ = \overrightarrow{B' Q'}.
\]
If $\mathcal{F}$ is convex, then we put $\mathcal{F} = \mathcal{V}$ and we have $d = \bar{d} = \bar{P}'\bar{B}' < \pi / \left( 2\sqrt{K} \right)$ and (as shown in Reshetnyak’s proof) $f = AQ \leq \bar{f} = \bar{A}'\bar{Q}' < \pi / \sqrt{K}$. Now suppose that the quadrangular domain $\mathcal{F}$ is not convex. Then either the angle of the quadrangle $\mathcal{E}'$ at its vertex $\bar{P}'$ is greater than $\pi$ or the angle at its vertex $\bar{B}'$ is greater than $\pi$. For definiteness, suppose that the angle of $\mathcal{E}'$ at $\bar{P}'$ is greater than $\pi$, as shown in Fig. 4.1. Let $\mathcal{V} \subseteq S_K$ be the domain bounded by the triangle $A'Q'B'$ obtained from the polygonal curve $\mathcal{E}'$ by rectifying the arc $A'\bar{P}'\bar{Q}'$. Then by Lemma 2, §3], $d = \bar{d}' < d' = P'B'$ and $f \leq A'Q' < \pi / \sqrt{K}$ (as shown in Reshetnyak’s proof). By Corollary 3.2, $d' < \pi / \left( 2\sqrt{K} \right)$; so, inequalities (4.1) hold true.

By (4.1) and because $\overline{\text{diam} (\mathfrak{R}_K)} < \pi / \left( 2\sqrt{K} \right)$, we see that the difference of the products

\[
(\cos \kappa x + \cos \kappa d) (\cos \kappa y + \cos \kappa f) - (\cos \kappa x + \cos \kappa d')(\cos \kappa y + \cos \kappa f')
\]

\[
= \cos \kappa x (\cos \kappa f - \cos \kappa f') + \cos \kappa y (\cos \kappa d - \cos \kappa d') + \cos \kappa f (\cos \kappa d - \cos \kappa d') + \cos \kappa d' (\cos \kappa f - \cos \kappa f')
\]

is non-negative. So, $\cosq_K \left( \overline{A'P', BQ} \right) \leq \cosq_K \left( \overline{A'\bar{P}', B\bar{Q}'} \right)$ follows. By Corollary 3.3, $\cosq_K \left( \overline{A'P', BQ} \right) \leq 1$, as needed.

Now, we consider the case when $\overline{\text{diam} (\mathfrak{Q})} = \pi / \left( 2\sqrt{K} \right)$. If $\varepsilon > 0$ is sufficiently small, by invoking Corollary 3.2, it is not difficult to select points $A_\varepsilon, P_\varepsilon, B_\varepsilon$ and $Q_\varepsilon$ in $\mathfrak{R}_K$ such that the distances $AA_\varepsilon, PP_\varepsilon, BB_\varepsilon$ and $QQ_\varepsilon$ do not exceed $\varepsilon$ and such that $\text{diam} \{ A_\varepsilon, P_\varepsilon, B_\varepsilon, Q_\varepsilon \} < \pi / (2\kappa)$. One of such configurations is shown in Fig. 4.2. From the first part of the proof, we see that $\left| \cosq_K \left( A_\varepsilon \bar{P}, B_\varepsilon \bar{Q} \right) \right| \leq 1$ for every small positive $\varepsilon$. Hence, by passing to the limit as $\varepsilon \to 0+$, we get $\left| \cosq_K \left( \overline{A\bar{P}, B\bar{Q}} \right) \right| \leq 1$, as claimed. □

The following example shows that for positive $K$, the restriction on the diameter of $\mathfrak{R}_K$ cannot be dropped and the diameter bound in Theorem 4.1 is sharp. For simplicity, we consider $K = 1$.

**Example 4.1.** Let $\varepsilon > 0$. Consider the $T$-shaped graph $(M_\varepsilon, \rho_\varepsilon)$ obtained by gluing a segment of straight line $AO$ of length $\pi / 4 + \varepsilon$ to the middle $O$ of another segment of

Figure 4.1. Sketch for Theorem 4.1.
Figure 4.2. \( \text{diam}(\Omega) = \pi / \left(2\sqrt{K}\right) \)

Figure 4.3. Sketch for Example 4.1

A straight line \( BQ \) of length \( \pi/2 + 2\varepsilon \), as shown in Fig. 4.3. It is readily seen that \((M_\varepsilon, \rho_\varepsilon)\) is an \( \mathbb{R}_0 \) domain and that the perimeter of every triangle in \((M_\varepsilon, \rho_\varepsilon)\) is less than \(2\pi\) for small positive \(\varepsilon\). Hence, \((M_\varepsilon, \rho_\varepsilon)\) is also an \( \mathbb{R}_1 \) domain. Notice that \( \text{diam}(M_\varepsilon) = \pi/2 + 2\varepsilon \).

Let \( P \in AO \setminus \{A, O\} \) be such that \( AP = \varepsilon \).

(a)\[
\cos_{q_1} \left( \overrightarrow{BQ}, \overrightarrow{AP} \right) = \frac{\cos \left( \frac{\pi}{2} + \varepsilon \right) + \cos \left( \frac{\pi}{2} + 2\varepsilon \right) \cos \varepsilon}{\sin \left( \frac{\pi}{2} + 2\varepsilon \right) \sin \varepsilon} - \frac{2 \cos \left( \frac{\pi}{2} + 2\varepsilon \right) [\cos \varepsilon + \cos \left( \frac{\pi}{2} + \varepsilon \right)]}{[1 + \cos \left( \frac{\pi}{2} + 2\varepsilon \right)] \sin \left( \frac{\pi}{2} + 2\varepsilon \right) \sin \varepsilon} \leq \frac{1 + \sin 2\varepsilon}{1 - \sin 2\varepsilon} > 1
\]
for every $\varepsilon \in (0, \pi/4)$ and therefore for small positive $\varepsilon$.

(b) In a similar way,

$$\cos_q^1(BQ, PA) = \cos \left( \frac{\pi}{2} + 2\varepsilon \right) + \cos \left( \frac{\pi}{2} + \varepsilon \right) \cos \varepsilon$$

$$\sin \left( \frac{\pi}{2} + 2\varepsilon \right) \sin \varepsilon$$

$$\left[ \cos \left( \frac{\pi}{2} + 2\varepsilon \right) + \cos \left( \frac{\pi}{2} + \varepsilon \right) \right] \left[ \cos \varepsilon + \cos \left( \frac{\pi}{2} + 2\varepsilon \right) \right]$$

$$\left[ 1 + \cos \left( \frac{\pi}{2} + \varepsilon \right) \right] \sin \left( \frac{\pi}{2} + 2\varepsilon \right) \sin \varepsilon$$

$$\frac{(1 + \sin 2\varepsilon) \cos \varepsilon}{(1 - \sin \varepsilon) \cos 2\varepsilon} < -1$$

for every $\varepsilon \in (0, \pi/4)$ and therefore for small positive $\varepsilon$.

So, for small positive $\varepsilon$, the metric space $(M_\varepsilon, \rho_\varepsilon)$ is an $\mathbb{R}_1$ domain, the diameter of $(M_\varepsilon, \rho_\varepsilon)$ is greater than $\pi/2$ and

$$\lim_{\varepsilon \to 0^+} \text{diam}(M_\varepsilon) = \pi/2,$$

whereas $\cos_q^1$ takes values greater than 1 and less than $-1$.

5. Testing $\cos_k^1$. Counterexamples

We begin with the discussion of testing a metric space for the one-sided four point $\cos_q^1$ condition. We present counterexamples showing that in general the upper four point $\cos_q^1$ condition is different from the lower four point $\cos_q^1$ condition.

Let $K \in \mathbb{R}$ and let $Q = \{A, P, B, Q\}$ be a quadruple of distinct points in a metric space $(M, \rho)$ such that the perimeter of every triple $\{A, B, C\}$ in $Q$ is less than $2\pi/\sqrt{K}$ when $K > 0$. For every triple $X, Y, Z \in Q$, the absolute value of the $K$-quadrilateral cosine between any pair of non-zero bound vectors with heads and tails in the triple $\{X, Y, Z\}$ always does not exceed one. Indeed, each such triple can be embedded isometrically into $S_K$; hence, by Corollary 3.3, $|\cos_q^1|$ does not exceed 1 for every pair of such bound vectors.

So, by recalling that $\cos_q^1$ is symmetric, we need consider only the following 12 main cases given in Table 5.1 where the two non-zero bound vectors have no point in common.

| Case | I | II | III | IV | V | VI |
|------|---|----|-----|----|---|----|
| cosq1 | $\bar{A}P, BQ$ | $\bar{A}P, QB$ | $AB, PQ$ | $AB, Q\bar{A}$ | $AQ, PB$ | $AQ, BP$ |
| Case | VII | VIII | IX | X | XI | XII |
| cosq1 | $P\bar{A}, BQ$ | $PA, QB$ | $PB, QA$ | $P\bar{Q}, BA$ | $B\bar{A}, Q\bar{P}$ | $BP, QA$ |

TABLE 5.1. Twelve main cases

The following examples show that the upper and the lower four point $\cos_q^1$ conditions are not equivalent for non-zero $K$. For simplicity, we consider $K = \pm 1$. Adjustment for arbitrary non-zero $K$ is straightforward.

Example 5.1 ($K = 1$). (a) The lower four point $\cos_q^1$ condition holds, whereas the upper four point $\cos_q^1$ condition fails. Consider the $T$-shaped graph obtained by gluing a segment of straight line $AP$ of length $\pi/4 + 0.1$ to the middle $P$ of another segment of straight line $BQ$ of length $\pi/2 + 0.2$, as shown in Fig. 5.1. Let $M = \{A, P, B, Q\}$ with the induced metric $\rho$. All 12 main (approximate) values of $\cos_q^1$ for the four point metric
A characterization of Aleksandrov spaces of curvature \( \leq K \)

Figure 5.1. Sketch for Example 5.1, part (a)

space \((\mathcal{M}, \rho)\) are given in Table 5.2

| Case | I   | II   | III  | IV   | V    | VI   |
|------|-----|------|------|------|------|------|
| \(\cos q_1\) | 1.496 | 1.496 | -0.58 | 1.496 | -0.58 | 1.496 |

Table 5.2. Example 5.1 part (a)

(b) The upper four point \(\cos q_1\) condition holds, whereas the lower four point \(\cos q_1\) condition fails. Consider the quadruple \(\Omega = \{A, P, B, Q\}\) in \(\mathbb{S}_1\) with the metric \(\rho_{\mathbb{S}_1}\) such that the point \(P\) is symmetric to the point \(Q\) w.r.t. the midpoint of the shortest \(AB\). All 6 distances between the pairs of points of \(\Omega\) are shown in Fig. 5.2 with \(\varepsilon = 0\). Then \(\cos q_1(\overrightarrow{AP}, \overrightarrow{BQ}) = -1\). Now we change the metric \(\rho_{\mathbb{S}_1}\) by increasing the distance between \(P\) and \(Q\) by a positive \(\varepsilon\) and leaving all other distances the same. If \(\varepsilon\) is sufficiently small, then the new distance \(\rho_{\varepsilon}\) is a metric. For \(\varepsilon = 0.1\), all 12 main (approximate) values of \(\cos q_1\) for the four point metric space \((\Omega, \rho_{0.1})\) are given in Table 5.3.

Figure 5.2. Sketch for Example 5.1 part (b)
**Example 5.2** \((K = -1)\). We use the same approach to construction of counterexamples for \(K = -1\) as in part \((b)\) of Example 5.1. Let \(\Omega = \{A, P, B, Q\}\) be a four element set.

\(a\) The lower four point \(\cosq_{-1}\) condition holds, whereas the upper four point \(\cosq_{-1}\) condition fails. The 6 (symmetric) distances between the pairs of points in \(\Omega\) are given by

\[
\begin{align*}
\rho(A, P) &= \rho(B, Q) = 1, \\
\rho(A, B) &= 2, \\
\rho(P, Q) &= 2.697 \text{ and } \rho(A, Q) = \rho(B, P) = 2.44.
\end{align*}
\]

All 12 main (approximate) values of \(\cosq_{-1}\) for the four point metric space \((\Omega, \rho)\) are given in Table 5.4.

\[
\begin{array}{c|c|c|c|c|c|c}
\text{Case} & I & II & III & IV & V & VI \\
\hline
\cosq_{-1} & 1.0347 & -0.8133 & 0.7395 & -0.9998 & 0.4534 & -0.9133 \\
\hline
\text{Case} & VII & VIII & IX & X & XI & XII \\
\text{cosq}_{-1} & -0.8133 & 0.1465 & -0.9511 & -0.9998 & 0.7495 & 0.4534 \\
\end{array}
\]

Table 5.4. Example 5.2 part (a)

\(b\) The upper four point \(\cosq_{-1}\) condition holds, whereas the lower four point \(\cosq_{-1}\) condition fails. The 6 distances between the pairs of points in \(\Omega\) are given by

\[
\begin{align*}
\rho(A, P) &= \rho(B, Q) = 1, \\
\rho(A, B) &= 2, \\
\rho(P, Q) &= 3.027 \text{ and } \rho(A, Q) = \rho(B, P) = 2.43.
\end{align*}
\]

All 12 main (approximate) values of \(\cosq_{-1}\) for the four point metric space \((\Omega, \rho)\) are given in Table 5.5.

\[
\begin{array}{c|c|c|c|c|c|c}
\text{Case} & I & II & III & IV & V & VI \\
\hline
\cosq_{-1} & -1.184 & 0.922 & 0.522 & -0.944 & 0.807 & -1.008 \\
\hline
\text{Case} & VII & VIII & IX & X & XI & XII \\
\text{cosq}_{-1} & 0.922 & -1.077 & -1.003 & -0.944 & 0.522 & 0.807 \\
\end{array}
\]

Table 5.5. Example 5.2 part (b)
6. Proof of Theorem 1.1

6.1. Sketch of the proof. Let \((\mathcal{M}, \rho)\) be a geodesically connected metric space (of diameter not greater than \(\pi/2\sqrt{K}\) for positive \(K\)) satisfying the one-sided four point \(\cosq_K\) condition for non-zero \(K\). Theorem 1.1 is proved once we establish the angle comparison: for every geodesic triangle \(T = ABC\) in \((\mathcal{M}, \rho)\), \(\angle ABC \leq \angle K ABC, \angle BAC \leq \angle K BAC\) and \(\angle ACB \leq \angle K ACB\). We begin by proving Lemma 6.1 stating that shortests in \((\mathcal{M}, \rho)\) depend continuously on their endpoints. One of Aleksandrov’s theorem and Lemma 6.1 enable us to reduce the derivation of the global angle comparison estimate to the proof of the local angle comparison. The cross-diagonal estimate lemma (Lemma 6.2) is one of the main steps in the proof of the major growth estimate lemma (Lemma 6.3). Both of these estimates are derived from the one-sided four point \(\cosq_K\) condition. We employ the growth estimate to prove “almost monotonicity” of the angles \(\alpha(t)\) (Corollary 6.3) and existence of proportional angles (Corollary 6.4), an important auxiliary step in proving the existence of Aleksandrov angles (Proposition 6.1). Now we have all necessary means needed for derivation of the local angle comparison inequality. We begin with the identity corresponding to the growth estimate in \(\mathbb{S}_K\) (Proposition 6.2). We consider a sufficiently small geodesic triangle \(T = ABC\) in \((\mathcal{M}, \rho)\). Existence of Aleksandrov angles gives us the freedom of selecting the points in shortests \(AB\) and \(AC\) respectively approaching to the vertex \(A\) in a special way. For every small positive \(t\), we select \(\hat{X}_t \in AB\) and \(\hat{Y}_t \in AC\), \(\hat{X}_t, \hat{Y}_t \to A\) as \(t \to 0^+\) (see Sec. 6.7) so that \(\angle K \hat{X}_t^K \hat{X}_t^K \hat{Y}_t^K\) and \(\angle K \hat{X}_t^K \hat{A} \hat{Y}_t^K\) converge as \(t \to 0^+\) (Lemma 6.5). Hence, it is possible to pass to the limit in the growth estimate as \(t \to 0^+\). The limit form of the growth estimate and the identity of Proposition 6.2 enables us to derive the local angle comparison estimate (Proposition 6.3).

6.2. Continuity and uniqueness of shortests. The main result of this section is the following.

Lemma 6.1. Let \(K \neq 0\) and let \((\mathcal{M}, \rho)\) be a metric space such that \(\text{diam} (\mathcal{M}) \leq \pi / \left(2\sqrt{K}\right)\) when \(K > 0\). Let \(L = AB\) be a shortest and \(L_n = A_n B_n\) be a sequence of shortests in \((\mathcal{M}, \rho)\) such that \(\lim_{n \to \infty} A_n = A\) and \(\lim_{n \to \infty} B_n = B\). Let \(g_r\) be the reduced parametrization of \(L\) relative to \(A\) and \(g_{r,n}\) be the reduced parametrization of \(L_n\) relative to \(A_n, n = 1, 2, \ldots\) (see Sec. 2). If \((\mathcal{M}, \rho)\) satisfies the one-sided four point \(\cosq_K\) condition, then the sequence \(\left(g_{r,n}\right)_{n=1}^\infty\) converges uniformly to \(g_r\) on the closed interval \([0, 1]\).

Proof. Let \(L = AB\) and \(L_n = A_n B_n\) \(n = 1, 2, \ldots\). We can assume that \(l = l_\rho (L) > 0\) and \(l_n = l_\rho (L_n) > 0\) for every \(n\). For \(t \in (0, 1)\), set \(P = g_r (t)\), \(P_n = g_{r,n} (t)\) and \(\delta = \lim_{n \to \infty} \rho P_n\), see Fig. 6.1.

I. Let \((\mathcal{M}, \rho)\) satisfy the upper four point \(\cosq_K\) condition. Consider the non-zero bound vectors \(\hat{A} \hat{P}\) and \(\hat{P}_n \hat{B}_n\). By the upper four point \(\cosq_K\) condition,

\[
\cosq_K \left(\hat{A} \hat{P}, \hat{P}_n \hat{B}_n\right) = \frac{\cos \hat{P} B_n + \cos \hat{A} P \cos \hat{P} B_n}{\sin \hat{A} P \sin \hat{P} B_n} - \frac{(\cos \hat{P} B_n + \cos \hat{P} P_n \cos \hat{P} B_n)}{(1 + \cos \hat{P} P_n) \sin \hat{A} P \sin \hat{P} B_n}
\]
A cosq_{K}\text{-characterization of Aleksandrov spaces of curvature } \leq K

does not exceed 1. By letting \( n \rightarrow \infty \), we get

\[
\lim_{n \rightarrow \infty} \cos q_K \left( \overrightarrow{AP}, \overrightarrow{B_n P_n} \right) = \frac{\cos \hat{\kappa} (1-t) l + \cos \hat{\kappa} t \cos \hat{\kappa} (1-t) l}{\sin \hat{\kappa} t \sin \hat{\kappa} (1-t) l} - \frac{(\cos \hat{\kappa} t + \cos \hat{\kappa} \delta) \cos \hat{\kappa} (1-t) l + \cos \hat{\kappa} l}{(1 + \cos \hat{\kappa} t) \sin \hat{\kappa} t \sin \hat{\kappa} (1-t) l} = 1 + \frac{1}{1 + \cos \hat{\kappa} l} \left[ \cos \hat{\kappa} (1-t) l + \cos \hat{\kappa} l \right] \leq 1.
\]

If \( K > 0 \), then

\[
\frac{(1 - \cos \kappa \delta)}{(1 + \cos \kappa t) \sin \kappa t \sin \kappa (1-t) l} \leq 0.
\]

Because \( \text{diam} (\mathcal{M}) \leq \pi / (2\kappa) \), \( \delta = 0 \) follows.

If \( K < 0 \), then

\[
\frac{(\cosh \kappa \delta - 1)}{(1 + \cosh \kappa t) \sinh \kappa t \sinh \kappa (1-t) l} \leq 0,
\]

whence \( \delta = 0 \) follows.

\textbf{II. Let} \((\mathcal{M}, \rho)\) \textbf{satisfy the lower four point cosq}_{K} \textbf{condition.} In a manner similar to I, we get

\[
\lim_{n \rightarrow \infty} \cos q_K \left( \overrightarrow{AP}, \overrightarrow{B_n P_n} \right) = \frac{\cos \hat{\kappa} (1-t) l + \cos \hat{\kappa} t \cos \hat{\kappa} (1-t) l}{\sin \hat{\kappa} t \sin \hat{\kappa} (1-t) l} - \frac{[\cos \hat{\kappa} t + \cos \hat{\kappa} (1-t) l]^2}{(1 + \cos \hat{\kappa} l) \sin \hat{\kappa} t \sin \hat{\kappa} (1-t) l} \geq -1,
\]

whence

\[
\frac{[\cos \hat{\kappa} t + \cos \hat{\kappa} (1-t) l] (1 + \cos \hat{\kappa} l)}{(1 + \cos \hat{\kappa} l) \sin \hat{\kappa} t \sin \hat{\kappa} (1-t) l} - \frac{[\cos \hat{\kappa} t + \cos \hat{\kappa} (1-t) l]^2}{(1 + \cos \hat{\kappa} l) \sin \hat{\kappa} t \sin \hat{\kappa} (1-t) l} \geq -1.
\]
is non-negative. Notice that
\[
[\cos \hat{\kappa} tl + \cos \hat{\kappa} (1 - t) l]^2 = 4 \cos^2 \frac{\hat{\kappa} tl}{2} \cos^2 \frac{\hat{\kappa} (1 - 2t) l}{2} = (1 + \cos \hat{\kappa} l) (1 + \cos \hat{\kappa} (1 - 2t) l).
\]
Hence,
\[
[\cos \hat{\kappa} \delta + \cos \hat{\kappa} (1 - 2t) l] (1 + \cos \hat{\kappa} l) - [\cos \hat{\kappa} tl + \cos \hat{\kappa} (1 - t) l]^2
= (\cos \hat{\kappa} \delta - 1) (1 + \cos \hat{\kappa} l).
\]
So,
\[
\lim_{n \to \infty} \cos q_K \left( \frac{AP_n}{B_nP_n} \right) + 1 = \frac{\cos \hat{\kappa} \delta - 1}{\sin \hat{\kappa} tl \sin \hat{\kappa} (1 - t) l} \geq 0.
\]
If \( K > 0 \), then
\[
\frac{\cos \hat{\kappa} \delta - 1}{\sin \hat{\kappa} tl \sin \hat{\kappa} (1 - t) l} \geq 0,
\]
whence \( \delta = 0 \) follows.
If \( K < 0 \), then
\[
\frac{\cosh \hat{\kappa} \delta - 1}{\sinh \hat{\kappa} tl \sinh \hat{\kappa} (1 - t) l} \leq 0
\]
whence \( \delta = 0 \).

By I and II, \( g_{r,n} (t) \) converges pointwise to \( g_r (t) \) for every \( t \in [0, 1] \) as \( n \to \infty \). It is not difficult to see that sequence \( (g_{r,n})_{n=1}^\infty \) also converges uniformly to \( g_r \) on the closed interval \([0, 1]\).

The proof of Lemma \( \ref{Lemma6.1} \) is complete. \( \square \)

**Corollary 6.1.** Let \( K \neq 0 \) and let \((M, \rho)\) be a metric space such that \( \operatorname{diam} (M) \) is not greater than \( \pi/\left(2\sqrt{|K|}\right) \) when \( K > 0 \). If \((M, \rho)\) satisfies the one-sided four point \( \cos q_K \) condition, then every pair of points in \( M \) can be joined by at most one shortest.

6.3. **Cross-diagonal estimate lemma.** Let \((M, \rho)\) be a metric space. Let \( A, B, C \) be three distinct points in \( M \), \( 0 < \bar{m} \leq \bar{m} < +\infty \) and \( s, t \in (0, 1] \) satisfying the following conditions:

- M1. The points \( A \) and \( B \) can be joined by a shortest \( \mathcal{L} \) and the points \( A \) and \( C \) can be joined by a shortest \( \mathcal{N} \).
- M2. If \( K > 0 \), then \( AB, AC \leq \pi/2\sqrt{|K|} \). \( \bar{m} \).
- M3. \( \bar{m} \leq s/t \leq \bar{m} \).

From now on, we will use the following notation:

- \( X_s = g_{r, \mathcal{L}} (s) \), \( Y_t = g_{r, \mathcal{N}} (t) \), \( s, t \in (0, 1] \).
- \( x = AB \), \( y = AC \), \( z = BC \), \( d_{s,t} = BY_t \), \( f_{s,t} = CX_s \), \( z_{s,t} = X_s Y_t \),

as illustrated in Fig. 6.2 and we put \( \lambda = \max \{x, y\} \), \( \eta = x/y \), \( \xi = \lambda \max \{s, t\} \). Also, for \( K \in \mathbb{R} \), set

\[
\alpha_K (s, t) = \angle_K X_s Y_t, \quad \beta_K (s, t) = \angle_K A X_s Y_t, \quad \gamma_K (s, t) = \angle_K A Y_t X_s.
\]

If \( p > 0 \), we write \( \varphi (s, t) = O (\xi^p) \) when there is a constant \( C > 0 \) such that \(|\varphi (s, t)| \leq C \xi^p \) for sufficiently small \( s \) and \( t \). If \( C \) is a constant depending on \( M_1, M_2, ..., M_k \), i.e., \( C = C (M_1, M_2, ..., M_k) \), then we write \( \varphi (s, t) = O_{M_1, ..., M_k} (\xi^p) \).
Figure 6.2. Sketch for the cross-diagonal lemma

If $T = ABC$ is a triangle in $\mathbb{S}_1$, then $f_{s,t}$ is not less than the length of the orthogonal projection of the shortest $X_sC$ onto the shortest $AC$. So, if $O_s$ is the orthogonal projection of the point $X_s$ onto the shortest $AC$, then $f_{s,t} \geq y - AO_s$. It is not difficult to see that $AO_s$ approximately equals $(sx) \cos \alpha_0 (s,t)$. Hence, approximately, $f_{s,t}$ is bounded below by $y - (sx) \cos \alpha_0 (s,t)$. The following lemma states a similar estimate for a triangle $T = ABC$ in a metric space satisfying the one-sided four point $\cosq K$ condition.

**Lemma 6.2.** Let $K \neq 0$ and $0 < m \leq \overline{m} < +\infty$. Let $A, B, C$ be three distinct points in a metric space $(\mathcal{M}, \rho)$ and $s, t \in (0, 1]$ satisfying M1-M3. Suppose that $(\mathcal{M}, \rho)$ satisfies the one-sided four point $\cosq K$ condition.

(i) If $K > 0$, then

\[
\cos \kappa f_{s,t} \leq \cos \kappa y + \kappa (sx) \sin \kappa y \cos \alpha_0 (s,t) + O(\xi^2),
\]

\[
\cos \kappa d_{s,t} \leq \cos \kappa x + \kappa (ty) \sin \kappa x \cos \alpha_0 (s,t) + O(\xi^2).
\]

(ii) If $K < 0$, then

\[
\cosh \kappa f_{s,t} \geq \cosh \kappa y - \kappa (sx) \sinh \kappa y \cos \alpha_0 (s,t) + O(\xi^2),
\]

\[
\cosh \kappa d_{s,t} \geq \cosh \kappa x - \kappa (ty) \sinh \kappa x \cos \alpha_0 (s,t) + O(\xi^2),
\]

where $O(\xi^2) = O_{\lambda, \eta, m, \overline{m}, K} (\xi^2)$.

**Proof.** I. Let $(\mathcal{M}, \rho)$ satisfy the upper four point $\cosq K$ condition. For the sake of brevity, set $h_{s,t} = \cosq K (X_sC, AY_t)$. Then
Then, by writing \( \cos \kappa f \) and if \( K < 0 \), it is not difficult to derive the inequalities of (i) and (ii) of the lemma for \( f_{s,t} \).

After lengthy but routine simplifications and using the upper four point \( \cos q_K \) condition, we get:

\[
h_{s,t} = \frac{\hat{h}(t) \sin \hat{y} - \hat{r}^2 (s) (t) \cos \hat{y} \cos \alpha_0 (s,t) + O (\xi^3)}{\sin \hat{r} f_{s,t}} = \frac{\sin \hat{y} - \hat{r} (s) \cos \hat{y} \cos \alpha_0 (s,t)}{\sin \hat{r} f_{s,t}} + O (\xi^2) \leq 1.
\]

Set \( \mu = (\cos \hat{y} + \cos \hat{r} f_{s,t}) / 2 \). By the triangle inequality, \( |f_{s,t} - y| \leq s x \). Hence, \( \mu = \cos \hat{y} + O (\xi) \) follows and we have:

\[
h_{s,t} = \frac{\sin \hat{y} - \hat{r} (s) \cos \hat{y} \cos \alpha_0 (s,t)}{\sin \hat{r} f_{s,t}} + O (\xi^2) \leq 1.
\]

So, if \( K > 0 \), we get:

\[
\sin \kappa y - \kappa (s) \cos \kappa y \cos \alpha_0 (s,t) \leq \sin \kappa f_{s,t} + O (\xi^2).
\]

and if \( K < 0 \), we get:

\[
\sinh \kappa y - \kappa (s) \cosh \kappa y \cos \alpha_0 (s,t) \leq \sinh \kappa f_{s,t} + O (\xi^2).
\]

Now, by writing \( \cos \kappa f_{s,t} = \sqrt{1 - \sin^2 \kappa f_{s,t}} \) if \( K > 0 \) and \( \cosh \kappa f_{s,t} = \sqrt{1 + \sinh^2 \kappa f_{s,t}} \) if \( K < 0 \), it is not difficult to derive the inequalities of (i) and (ii) of the lemma for \( f_{s,t} \).

II. Let \((M, \rho)\) satisfy the lower four point \( \cos q_K \) condition. Set

\[
g_{s,t} = \cos q_K \left( X_s \tilde{C}, Y_t \tilde{A} \right).
\]

Then

\[
g_{s,t} = \frac{(1 + \cos \hat{r} z_{s,t}) (\cos \hat{y} + \cos \hat{r} f_{s,t} \cos \hat{y}) (1 + \cos \hat{y} + \cos \hat{r} f_{s,t} \cos \hat{y})}{(1 + \cos \hat{r} z_{s,t}) \sin \hat{r} f_{s,t} \sin \hat{y}} - \frac{[\cos \hat{r} f_{s,t} + \cos \hat{r} (1 - t) y] (\cos \hat{y} + \cos \hat{r} x)}{(1 + \cos \hat{r} z_{s,t}) \sin \hat{r} f_{s,t} \sin \hat{y}} \geq -1.
\]
Let $I$ denote the numerator of $g_{s,t}$. We have:

$$I = \left[ 2 - \frac{1}{2} \kappa^2 z_{s,t}^2 + O(\xi^4) \right] \left\{ \cos \hat{\kappa} y + \cos \hat{\kappa} f_{s,t} \left[ 1 - \frac{1}{2} \kappa^2 (ty)^2 + O(\xi^3) \right] \right\}$$

$$- \left[ \cos \hat{\kappa} f_{s,t} + \cos \hat{\kappa} y + \kappa (ty) \sin \hat{\kappa} y - \frac{1}{2} \kappa^2 (ty)^2 \cos \hat{\kappa} y + O(\xi^3) \right] \times$$

$$\left[ 2 - \frac{1}{2} \kappa^2 (ty)^2 - \frac{1}{2} \kappa^2 (sx)^2 + O(\xi^4) \right].$$

After elementary simplifications, we get

$$I = -2 \kappa (ty) \sin \hat{\kappa} y + \kappa^2 (\cos \hat{\kappa} y + \cos \hat{\kappa} f_{s,t}) (sx) (ty) \cos \alpha_0 (s,t)$$

$$+ \kappa (ty)^2 (\cos \hat{\kappa} y - \cos \hat{\kappa} f_{s,t}) + O(\xi^3).$$

By the triangle inequality, $|y - f_{s,t}| \leq sx$. Hence, $\cos \hat{\kappa} y - \cos \hat{\kappa} f_{s,t} = O(\xi)$. So,

$$I = -2 \kappa (ty) \sin \hat{\kappa} y + \kappa^2 (\cos \hat{\kappa} y + \cos \hat{\kappa} f_{s,t}) (sx) (ty) \cos \alpha_0 (s,t) + O(\xi^3).$$

Hence,

$$g_{s,t} = \frac{-2 \kappa (ty) \sin \hat{\kappa} y + \kappa^2 (\cos \hat{\kappa} y + \cos \hat{\kappa} f_{s,t}) (sx) (ty) \cos \alpha_0 (s,t) + O(\xi^3)}{2 \left[ 1 + O(\xi^2) \right] \kappa (ty) \sin \hat{\kappa} f_{s,t}}$$

$$= \frac{- \sin \hat{\kappa} y + \kappa \cos \hat{\kappa} y + \cos \hat{\kappa} f_{s,t} (sx) \cos \alpha_0 (s,t)}{\sin \hat{\kappa} f_{s,t}} + O(\xi^2) \geq -1,$$

which implies $|g_{s,t}| \leq 1$. Hence, the inequalities of (i) and (ii) for $f_{s,t}$ follow.

Derivation of the inequalities of parts (i) and (ii) for $d_{s,t}$ is similar.

The proof of the cross-diagonal lemma is complete. \hfill \square

### 6.4. Growth estimate lemma

We keep the notation of Sec. 6.3. To illustrate the estimates of Lemma 6.3, consider a geodesic triangle $T = ABC$ in $S_1$ (for the notation, see Fig. 6.2). Let $z_\perp$ denote the length of the orthogonal projection of the shortest $BC$ onto the (possibly extended) shortest $X_\perp Y_\perp$. For small $x$ and $y$, we can treat the triangle $T$ as approximately Euclidean triangle. Then it is not difficult to see that $z_\perp$ is approximately equal to $x \cos \beta_0 (s,t) + y \cos \gamma_0 (s,t)$. So, for small $x$ and $y$, the length $z$ is approximately bounded below by $x \cos \beta_0 (s,t) + y \cos \gamma_0 (s,t)$. Lemma 6.3 establishes similar estimates for metric spaces satisfying the one-sided four point cosqK condition.

**Lemma 6.3.** Let $K \neq 0$ and $0 < m \leq \overline{m} < +\infty$. Let $A, B, C$ be three distinct points in a metric space $(M, \rho)$ and $s, t \in (0, 1]$ satisfying M1-M3 of Sec. 6.3. In addition, suppose that $(M, \rho)$ satisfies the one-sided four point cosqK condition. Let $A \subseteq (0, 1] \times (0, 1]$ be such that $(0, 0)$ is an accumulation point of the set $A$ and $0 < m \leq z_{s,t} / (sx)$ for every $(s, t) \in A$.

(i) If $K > 0$, then for every $(s, t) \in A$,

$$\sin \kappa y \cos \gamma_0 (s,t) + \frac{\cos \kappa y + \cos \kappa z}{1 + \cos \kappa x} \sin \kappa x \cos \beta_0 (s,t) \leq \sin \kappa z + O(\xi),$$

(ii) If $K < 0$, then for every $(s, t) \in A$,

$$\sinh \kappa y \cos \gamma_0 (s,t) + \frac{\cosh \kappa y + \cosh \kappa z}{1 + \cosh \kappa x} \sinh \kappa x \cos \beta_0 (s,t) \leq \sinh \kappa z + O(\xi),$$

where $O(\xi) = O_{\lambda, \eta, m, \overline{m}, K} (\xi)$. 

Proof. We consider $(s, t) \in \mathcal{A}$.

I. Let $(\mathcal{M}, \rho)$ satisfy the upper four point $\cos q_K$ condition. Set

$$p_{s,t} = \cos q_K \left( \overline{X_sY_t, BC} \right),$$

see Fig. 6.2. Then

$$p_{s,t} = \frac{\cos \hat{\kappa} (1 - t) y + \cos \hat{\kappa} z_{s,t} \cos \hat{\kappa} z}{\sin \hat{\kappa} z_{s,t} \sin \hat{\kappa} z}$$

$$- \frac{(\cos \hat{\kappa} z_{s,t} + \cos \hat{\kappa} d_{s,t}) (\cos \hat{\kappa} z + \cos \hat{\kappa} f_{s,t})}{[1 + \cos \hat{\kappa} (1 - s)] x \sin \hat{\kappa} z_{s,t} \sin \hat{\kappa} z}$$

$$= \frac{\cos \hat{\kappa} y + \hat{\kappa} (ty) \sin \hat{\kappa} y + O (\xi^2) + \cos \hat{\kappa} z [1 + O (\xi^2) - O (\kappa)]}{\sin \hat{\kappa} z_{s,t} \sin \hat{\kappa} z}$$

$$\times (1 + \cos \hat{\kappa} d_{s,t}) (\cos \hat{\kappa} z + \cos \hat{\kappa} f_{s,t}) + O (\xi^2) \times$$

$$\left[ 1 + O (\xi^2) \right] \times \left[ \frac{1}{1 + \cos \hat{\kappa} x} - \frac{\hat{\kappa} (sx) \sin \hat{\kappa} x}{[1 + \cos \hat{\kappa} x]^2} + O (\xi^2) \right].$$

For the sake of brevity, set $\mu = \cos \hat{\kappa} z + \cos \hat{\kappa} y$ and $\nu = 1 + \cos \hat{\kappa} x$. Let $K > 0$. By part (i) of the cross-diagonal estimate lemma (Lemma 6.2),

$$p_{s,t} \geq \frac{\mu + \kappa (ty) \sin \kappa y + O (\xi^2)}{\sin \kappa z_{s,t} \sin \kappa z}$$

$$\frac{\nu + \kappa (ty) \sin \kappa x \cos \alpha_0 (s, t)}{\sin \kappa z_{s,t} \sin \kappa z} \times \frac{1 - \kappa (sx) \sin \kappa x + O (\xi^2)}{\nu}.$$

After elementary simplifications and using the upper four point $\cos q_K$ condition, we get:

$$1 \geq p_{s,t} \geq \frac{(ty) \sin \kappa y - (sx) \sin \kappa x \cos \alpha_0 (s, t)}{\kappa z_{s,t} \sin \kappa z}$$

$$+ \frac{\mu (sx) \sin \kappa x - (ty) \sin \kappa x \cos \alpha_0 (s, t)}{\kappa z_{s,t} \sin \kappa z} + O (\xi).$$

By recalling that $\cos \alpha_0 (s, t) = \left[ (sx)^2 + (ty)^2 - z_{s,t}^2 \right] / [2 (ty) (sx)]$, we readily see that

$$(ty) \sin \kappa y - (sx) \sin \kappa x \cos \alpha_0 (s, t) = z_{s,t} \sin \kappa y \cos \gamma_0 (s, t),$$

$$(sx) \sin \kappa x - (ty) \sin \kappa x \cos \alpha_0 (s, t) = z_{s,t} \sin \kappa x \cos \beta_0 (s, t).$$

Finally, we get:

$$\frac{\sin \kappa y \cos \gamma_0 (s, t) + \cos \kappa y + \cos \kappa z}{1 + \cos \kappa z} \sin \kappa x \cos \beta_0 (s, t) \leq 1 + O (\xi),$$

and the inequality of part (i) follows. The case of negative $K$ is similar and we leave it to the reader.

II. Let $(\mathcal{M}, \rho)$ satisfy the lower four point $\cos q_K$ condition. Set

$$q_{s,t} = \cos q_K \left( \overline{Y_sX_t, BC} \right).$$
Then

$$q_{s,t} = \frac{\cos \hat{\kappa} f_{s,t} + \cos \hat{\kappa} \hat{z}_{s,t} \cos \hat{\kappa} z}{\sin \hat{\kappa} z_{s,t} \sin \hat{\kappa} z} - \left[ \cos \hat{\kappa} z_{s,t} + \cos \hat{\kappa} (1 - s) x \right] \left[ \cos \hat{\kappa} z + \cos \hat{\kappa} (1 - t) y \right]$$

\[
= (1 + \cos \hat{\kappa} d_{s,t}) \left\{ [\cos \hat{\kappa} f_{s,t} + \cos \hat{\kappa} z + O(\xi^2)] - [1 + \cos \hat{\kappa} x + \hat{\kappa} (sx) \sin \hat{\kappa} x + O(\xi^2)] \times [\cos \hat{\kappa} z + \cos \hat{\kappa} y + \hat{\kappa} (ty) \sin \hat{\kappa} y + O(\xi^2)] \right\} \\
\times \left[ (1 + \cos \hat{\kappa} d_{s,t}) \sin \hat{\kappa} z_{s,t} \sin \hat{\kappa} z \right]^{-1},
\]

where we keep the notation \( \mu = \cos \hat{\kappa} y + \cos \hat{\kappa} z \) and \( \nu = 1 + \cos \hat{\kappa} x \). By invoking the triangle inequality, we see that \( \cos \hat{\kappa} d_{s,t} = \cos \hat{\kappa} x + O(\xi) \), whence \( 1/(1 + \cos \hat{\kappa} d_{s,t}) = 1/\nu + O(\xi) \). So, we get:

$$q_{s,t} = \frac{I}{\nu \hat{\kappa} z_{s,t} \sin \hat{\kappa} z} \left[ 1 + O(\xi) \right],$$

where

$$I = (1 + \cos \hat{\kappa} d_{s,t}) \left( \cos \hat{\kappa} f_{s,t} + \cos \hat{\kappa} z \right) - [\mu + \hat{\kappa} (ty) \sin \hat{\kappa} y] \left[ \nu + \hat{\kappa} (sx) \sin \hat{\kappa} x \right] + O(\xi^2).$$

Let \( K > 0 \). By the cross-diagonal estimate lemma,

$$I \leq I' = [\nu + \kappa (ty) \sin \kappa x \cos \alpha_0 (s, t)] \times [\mu + \kappa (sx) \sin \kappa y \cos \alpha_0 (s, t)]$$

\[- [\mu + \kappa (ty) \sin \kappa y] \left[ \nu + \kappa (sx) \sin \kappa x \right] + O(\xi^2)\]

\[= \kappa \left[ - \nu \sin \kappa y [(ty) - (sx)] \cos \alpha_0 (s, t) \right] - \mu \sin \kappa x [(sx) - (ty)] \cos \alpha_0 (s, t) + O(\xi^2),\]

whence by invoking the lower four point \( \cos 4_K \) condition, we have:

$$\frac{-\nu \sin \kappa y [(ty) - (sx)] \cos \alpha_0 (s, t) - \mu \sin \kappa x [(sx) - (ty)] \cos \alpha_0 (s, t)}{\nu \hat{\kappa} z_{s,t} \sin \hat{\kappa} z} \geq q_{s,t} \geq -1 + O(\xi),$$

which is equivalent to inequality \([6.3]\). Hence, the inequality of part (i) of the lemma follows. The case of negative \( K \) is similar.

The proof of the growth estimate lemma is complete. \( \square \)

It is well-known that \( \alpha_K (s, t) - \alpha_0 (s, t), \beta_K (s, t) - \beta_0 (s, t) \) and \( \gamma_K (s, t) - \gamma_0 (s, t) \) are \( O(\sigma (AX, Y_t)) = O(\xi^2). \) Hence, by recalling that \( \alpha_0 (s, t) + \beta_0 (s, t) + \gamma_0 (s, t) = \pi \), we get the following
Corollary 6.2. Under the hypotheses of the growth estimate lemma (Lemma 6.3), the following inequalities hold:
(i) If $K > 0$, then for every $(s, t) \in A$,
\[
\frac{\cos \kappa y + \cos \kappa z}{1 + \cos \kappa x} \sin \kappa x \cos \beta_K (s, t) \\
- \sin \kappa y \left[ \cos (\alpha_K (s, t) + \beta_K (s, t)) \right] \\
\leq \sin \kappa z + O (\xi),
\]
(ii) If $K < 0$, then for every $(s, t) \in A$,
\[
\frac{\cosh \kappa y + \cosh \kappa z}{1 + \cosh \kappa x} \sinh \kappa x \cos \beta_K (s, t) \\
- \sinh \kappa y \left[ \cos (\alpha_K (s, t) + \beta_K (s, t)) \right] \\
\leq \sinh \kappa z + O (\xi),
\]
where $O (\xi) = O_{\lambda, \eta, m, \overline{m}, K} (\xi)$.

6.5. Existence of proportional angles. Let $(\mathcal{M}, \rho)$ be a metric space, let $\mathcal{L} = \mathcal{AB}$ and $\mathcal{N} = \mathcal{AC}$ be shortests in $(\mathcal{M}, \rho)$, starting at a common point $A \in \mathcal{M}$. Let $K \in \mathbb{R}$ and $t \in (0, 1]$. Set $\alpha_K (t) = \alpha_K (t, t)$, $\beta_K (t) = \beta_K (t, t)$, $\gamma_K (t) = \gamma_K (t, t)$ and $z_t = z_{t, t}$.

In this section, we derive from the growth estimate lemma that the proportional angle $\lim_{t \to 0^+} \alpha_0 (t)$ exists. We begin with the following

Lemma 6.4. Let $K \neq 0$, $m > 0$ and $(\mathcal{M}, \rho)$ be a metric space satisfying the one-sided four point $\cos q_K$ condition. Also, suppose that $\text{diam} (\mathcal{M}) \leq \pi / \left( 2\sqrt{K} \right)$ when $K > 0$. Let $\mathcal{L} = \mathcal{AB}$, $\mathcal{N} = \mathcal{AC}$ be shortests in $(\mathcal{M}, \rho)$ starting at a common point $A \in \mathcal{M}$ and $t \in (0, 1]$. If $0 < m \leq z_t / t$ for $0 < t < \varepsilon$ for some $\varepsilon \in (0, 1)$, then there is $\varepsilon' \in (0, \varepsilon]$ such that for every $\tau \in (0, t^2) \cap (0, \varepsilon')$, the following inequality holds:
\[
z_t \leq \frac{\tau}{t} \left( z_t + \mu t^2 \right),
\]
where $\mu = \mu (\lambda, \eta, m, K) > 0$.

Proof. The notation of the lemma is illustrated in Fig. 6.3. Let $0 < \tau < t^2 \leq t \leq 1$. In the growth estimate lemma, take $x := tx$ and $y := ty$. Then $\xi := \xi_t = \max \left\{ tx, ty \right\}$, $\tau = \tau t \lambda = \tau \lambda$. Hence, $O (\xi_t) = O (\tau)$. By the growth estimate lemma applied to the shortests $\mathcal{AX}_t$ and $\mathcal{AY}_t$, if $K > 0$, then
\[
(6.4) \sin \kappa t \cos \gamma_0 (\tau) + \frac{\cos \kappa t y + \cos \kappa z t}{1 + \cos \kappa x} \sin \kappa t \cos \beta_0 (\tau) \\
\leq \sin \kappa z t + O (\tau),
\]
and if $K < 0$, then
\[
\sinh \kappa t y \cos \gamma_0 (\tau) + \frac{\cosh \kappa t y + \cosh \kappa z t}{1 + \cosh \kappa x} \sinh \kappa t \cos \beta_0 (\tau) \\
\leq \sinh \kappa z t + O (\tau),
\]
for every $t \in (0, \varepsilon)$, where $O (\tau) = O_{\lambda, \eta, m, K} (\tau)$. 
Let $K > 0$. Then we can rewrite (6.4) in the following form:

\[ \kappa(ty) \left[ 1 + O(t^2) \right] \cos \gamma_0(\tau) + \left[ 1 + O(t^2) \right] \kappa(tx) \cos \beta_0(\tau) \leq \kappa z_t + O(t^3) + O(\tau) = \kappa z_t + O(t^2), \]

whence,

\[ (6.5) \quad y \cos \gamma_0(\tau) + x \cos \beta_0(\tau) \leq \frac{z_t + O(t^2)}{t}. \]

Let $\eta_{\tau} = z_{\tau}/\tau$. Recall that

\[ \cos \gamma_0(\tau) = \frac{\tau^2 y^2 + z_t^2 - \tau^2 x^2}{2\tau y z_t} = \frac{y^2 + \eta_{\tau}^2 - x^2}{2y\eta_{\tau}}, \]

\[ \cos \beta_0(\tau) = \frac{x^2 + z_t^2 - \tau^2 y^2}{2\tau x z_t} = \frac{x^2 + \eta_{\tau}^2 - y^2}{2x\eta_{\tau}}. \]

Hence, by (6.5),

\[ \eta_{\tau} \leq \frac{z_t + O(t^2)}{t}, \]

and the claim of the lemma for positive $K$ follows. The case of negative $K$ is similar.

The proof of Lemma 6.4 is complete. \hfill \Box

By Lemma 6.4,

\[ \cos \alpha_0(\tau) = \frac{t^2 x^2 + t^2 y^2 - \frac{t^2}{2xy} z_t^2}{2t^2 xy} \geq \frac{t^2 x^2 + t^2 y^2 - (z_t + \mu t^2)^2}{2t^2 xy} \]

\[ = \cos \alpha_0(t) - \frac{\mu z_t}{xy} - \frac{\mu^2 t^2}{2xy}. \]

By the triangle inequality, $z_t \leq (x + y) t$. So, we have the following

**Corollary 6.3.** Under the hypothesis of Lemma 6.4, the following inequality holds:

\[ \cos \alpha_0(\tau) \geq \cos \alpha_0(t) - \mu' t, \]

where $\mu' = \mu'(\lambda, \eta, m, K) > 0$. 

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure6_3.png}
\caption{Sketch for Lemma 6.4}
\end{figure}
Corollary 6.4. Let $K \neq 0$ and $(\mathcal{M}, \rho)$ be a metric space satisfying the one-sided four point $\cos q_{\mathcal{M}}$ condition. Also, suppose that $\text{diam} (\mathcal{M}) \leq \pi / \left(2\sqrt{K}\right)$ when $K > 0$. Let $\mathcal{L} = AB$ and $\mathcal{N} = AC$ be shortests in $(\mathcal{M}, \rho)$ starting at a common point $A \in \mathcal{M}$. Then $\lim_{t \to 0^+} \alpha_0 (t)$ exists.

Proof. Let $\overline{\alpha}_0 = \lim_{t \to 0^+} \alpha_0 (t)$ and $\underline{\alpha}_0 = \lim_{t \to 0^+} \alpha_0 (t)$. Then there are sequences $(t_n)_{n=1}^{\infty}$ and $(\tau_n)_{n=1}^{\infty}$ in $(0, 1]$ convergent to zero such that $\overline{\alpha}_0 = \lim_{n \to \infty} \alpha_0 (\tau_n)$ and $\underline{\alpha}_0 = \lim_{n \to \infty} \alpha_0 (t_n)$.

There is no restriction in assuming that $\tau_n < t_n^n$ for every $n \in \mathbb{N}$. We consider the following cases.

I. $\lim_{n \to \infty} z_{\tau_n}/\tau_n = 0$. Then

$$\cos \alpha_0 (\tau_n) = \frac{x^2 + y^2 - (z_{\tau_n}/\tau_n)^2}{2xy} \to \frac{x^2 + y^2}{2xy} \quad \text{as} \quad n \to \infty.$$ 

By the triangle inequality, $z_{\tau_n}/\tau_n \geq |x - y|$, whence $x = y$, and we have:

$$\lim_{n \to \infty} \cos \alpha_0 (\tau_n) = 1.$$ 

Hence, $\overline{\alpha}_0 = 0$, and

$$\lim_{t \to 0^+} \alpha_0 (t) = 0$$

follows.

II. By Corollary \[6.3\], $\cos \alpha_0 (\tau_n) \geq \cos \alpha_0 (t_n) + O (t_n)$ for every $n \in \mathbb{N}$. So, by passing to the limit as $n \to \infty$ in both sides of the last inequality, we get the inequality $\overline{\alpha}_0 \leq \underline{\alpha}_0$. This completes the proof of Corollary 6.4.

6.6. Existence of angle.

Proposition 6.1. Let $(\mathcal{M}, \rho)$ be a metric space satisfying the one-sided four point $\cos q_{\mathcal{M}}$ condition. Then between any pair of shortests $\mathcal{L}$ and $\mathcal{N}$ in $(\mathcal{M}, \rho)$, starting at a common point $P \in \mathcal{M}$, there exists Aleksandrov’s angle.

Proof. Set $\alpha_{av} = \left( \overline{\angle} (\mathcal{L}, \mathcal{N}) + \underline{\angle} (\mathcal{L}, \mathcal{N}) \right) / 2$. If $\overline{\angle} (\mathcal{L}, \mathcal{N}) = 0$ or $\underline{\angle} (\mathcal{L}, \mathcal{N}) = \pi$, we are done. So, we can assume that $\sin \alpha_{av} > 0$. Contrary to the claim of the proposition, suppose that $\overline{\angle} (\mathcal{L}, \mathcal{N}) - \underline{\angle} (\mathcal{L}, \mathcal{N}) = \varepsilon_0 > 0$.

I. In Step 1 of the proof of Proposition 20 in [6], we showed that for every $0 < \varepsilon < \varepsilon_0$, there are points $\widetilde{X}, \widetilde{X} \in \mathcal{L} \setminus \{P\}$ and $Y, \widetilde{Y} \in \mathcal{N} \setminus \{P\}$, or $\widetilde{X}, \widetilde{X} \in \mathcal{L} \setminus \{P\}$ and $Y, \widetilde{Y} \in \mathcal{L} \setminus \{P\}$ such that the following conditions are satisfied (for simplicity, we drop $\varepsilon$ from our notation for these points):

(i) $\widetilde{X}$ is contained between $X$ and $P$, and $Y$ is contained between $\widetilde{Y}$ and $P$, as illustrated in Figure 6.4 and the points $X, \widetilde{X}, \widetilde{Y}$ and $Y$ can be selected arbitrary close to the point $P$.

(ii) $0 \leq \gamma'' = \angle_0 \overline{XY} \leq \angle (\mathcal{L}, \mathcal{N}) + \varepsilon/4$.

(iii) $\gamma' = \angle_0 \overline{XY} > \angle (\mathcal{L}, \mathcal{N}) - \varepsilon/4$.

(iv) $0 \leq \overline{\gamma} = \angle_0 \overline{XY} < \angle (\mathcal{L}, \mathcal{N}) + \varepsilon/4$.

(v) $\overline{\angle} = \angle_0 \overline{XY} > \angle (\mathcal{L}, \mathcal{N}) - \varepsilon/4$.

(vi) $x/\overline{x} = \overline{y}/y$, where $\overline{x} = P \widetilde{X}$ and $y = PY$.

With little effort, the proof of (i)-(vi) for $K = 0$ in [6] can be extended to non-zero $K$. Indeed, by the definition of the lower angle, for every $\eta > 0$, there is $t_\eta \in (0, 1)$ and $\xi, \zeta \in (0, t_\eta)$ such that

$$\angle_0 (\xi, \zeta) < \angle (\mathcal{L}, \mathcal{N}) + \eta.$$
By Corollary 6.3
\[ \cos \angle_0 (\tau \xi, \tau \zeta) \geq \cos \angle_0 (\xi, \zeta) - \mu' t, \]
where \( t > 0 \) is sufficiently small and \( 0 < \tau < t^2 \). So,
\[ \cos \angle_0 (\tau \xi, \tau \zeta) \geq \cos (\angle (\mathcal{L}, \mathcal{N}) + \eta) - \mu' t. \]
Hence, given \( \varepsilon \in (0, \varepsilon_0) \), there is \( t' \in (0, 1) \) such that the following inequality holds:
\[ \angle_0 (\tau \xi, \tau \zeta) \leq \angle (\mathcal{L}, \mathcal{N}) + \frac{\varepsilon}{4}, \ 0 < \tau < t'. \]

After this point, the proof of (i)-(vi) is the same as in Step I of the proof of Proposition 20 in [6].

Let \( \hat{\gamma} = \max \{ \gamma'', \hat{\gamma} \} \). By (ii) and (iv), for sufficiently small positive \( \varepsilon \), the following inequalities hold:
\[ \text{(vii)} \quad \hat{\gamma} \leq \angle (\mathcal{L}, \mathcal{N}) + \frac{\varepsilon}{4} < \pi. \]
Now consider \( I = 2 \cos \hat{\gamma} - [\cos \gamma + \cos \gamma'] \). By (iii) and (v),
\[ I \geq \cos \left( \angle (\mathcal{L}, \mathcal{N}) + \frac{\varepsilon}{4} \right) - \cos \left( \angle (\mathcal{L}, \mathcal{N}) - \frac{\varepsilon}{4} \right) \]
\[ + \cos (\angle (\mathcal{L}, \mathcal{N}) + \frac{\varepsilon}{4}) - \cos (\angle (\mathcal{L}, \mathcal{N}) - \frac{\varepsilon}{4}) = \]
\[ \frac{2 \sin \frac{\varepsilon}{2}}{2} \sin \alpha_{av} - 2 \sin \frac{\varepsilon}{4} \sin \angle (\mathcal{L}, \mathcal{N}) \]
\[ > 2 \sin \frac{\varepsilon_0}{4} \sin \alpha_{av} - 2 \sin \frac{\varepsilon}{4} \sin \angle (\mathcal{L}, \mathcal{N}). \]
Hence, for small positive \( \varepsilon \), the inequality
\[ \text{(6.6)} \quad I = 2 \cos \hat{\gamma} - [\cos \gamma + \cos \gamma'] > \sin \frac{\varepsilon_0}{4} \sin \alpha_{av} > 0 \]
follows.

By Corollary 6.1 there is no restriction in assuming that \( X \neq \widetilde{Y} \) and \( \widetilde{X} \neq Y \). In what follows \( t = \frac{\varepsilon}{x}. \)

**II. Let** \( (\mathcal{M}, \rho) \) **satisfy the upper four point** \( \cos q_{\mathcal{K}} \) **condition.** Set
\[ p = \cos q_{\mathcal{K}} \left( \longrightarrow_{XY} \longrightarrow_{\widetilde{XY}} \right). \]
Let \( f = \bar{XY} \) and \( d = XY \), as shown in Fig. 6.4 Then

\[
p = \frac{\cos \kappa (1 - t) \bar{y} + \cos \kappa a \cos \kappa b}{\sin \kappa a \sin \kappa b} - \frac{(\cos \kappa a + \cos \kappa f) (\cos \kappa b + \cos \kappa d)}{[1 + \cos \kappa (1 - t) x] \sin \kappa a \sin \kappa b}
\]

\[
= \frac{\cos \kappa \bar{y} + \kappa (\nu y) \sin \kappa \bar{y} + \cos \kappa a + \mathcal{O} (\lambda^2 t^2)}{\sin \kappa a \sin \kappa b} - \frac{(\cos \kappa a + \cos \kappa f) [1 + \cos \kappa d + \mathcal{O} (\lambda^2 t^2)] \times}{\sin \kappa a \sin \kappa b}
\]

\[
\left[ \frac{1}{1 + \cos \kappa x} - \frac{\kappa (\nu y) \sin \kappa a}{(1 + \cos \kappa x)^2} + \mathcal{O} (\lambda^2 t^2) \right].
\]

Let \( K > 0 \). Set \( \gamma' = \angle \kappa \bar{XY} \) and \( \pi_K = \angle \kappa \bar{XY} \). By the spherical cosine formula, \( \cos \kappa f = \cos \kappa \bar{y} + \sin \kappa x \sin \kappa \bar{y} \cos \gamma' \). Recall that \( \gamma_K - \gamma = \mathcal{O} \left( \sigma \left( \bar{XY} \right) \right) = \mathcal{O} (\lambda t) \), whence cos \( \gamma_K = \cos \gamma + \mathcal{O} (\lambda t) \). So, we get:

\[
(6.7) \quad \cos \kappa f = \cos \kappa \bar{y} + \kappa (\nu y) \sin \kappa \bar{y} \cos \gamma' + \mathcal{O} (\lambda^2 t^2).
\]

In a similar way,

\[
(6.8) \quad \cos \kappa d = \cos \kappa x + \kappa (\nu y) \sin \kappa x \cos \gamma + \mathcal{O} (\lambda^2 t^2).
\]

For the sake of brevity, set \( \mu = \cos \kappa a + \cos \kappa \bar{y} \) and \( \nu = 1 + \cos \kappa x \). By \( 6.7 \), \( 6.8 \) and by invoking the upper four point cos\( \kappa_K \) condition, we get:

\[
p = \frac{\mu + \kappa (\nu y) \sin \kappa \bar{y} + \mathcal{O} (\lambda^2 t^2)}{\sin \kappa a \sin \kappa b} - \left[ \mu + \kappa (\nu y) \sin \kappa \bar{y} \cos \gamma' + \mathcal{O} (\lambda^2 t^2) \right] \times
\]

\[
\left[ \nu + \kappa (\nu y) \sin \kappa x \cos \gamma + \mathcal{O} (\lambda^2 t^2) \right] \left[ 1 - \frac{\kappa (\nu y) \sin \kappa x}{\nu} + \mathcal{O} (\lambda^2 t^2) \right]
\]

\[
= \kappa t \frac{\sin \kappa \bar{y} (\bar{y} - x \cos \gamma') + \mu y \sin \kappa x (x - \bar{y} \cos \gamma) + \mathcal{O} (\lambda^2 t)}{\sin \kappa a \sin \kappa b} \leq 1.
\]

Now we approximate \( 6.9 \) w.r.t. \( x \) and \( \bar{y} \):

\[
p = \kappa t \frac{\bar{y} (\bar{y} - x \cos \gamma') + x (x - \bar{y} \cos \gamma) + \mathcal{O} (t^2) + \mathcal{O} (\lambda^4)}{\sin \kappa a \sin \kappa b}
\]

\[
= \kappa ^2 t \frac{x^2 + \bar{y}^2 - x \bar{y} (\cos \gamma + \cos \gamma') + \mathcal{O} (t^2) + \mathcal{O} (\lambda^4)}{\sin \kappa a \sin \kappa b}.
\]

Let \( A = x^2 + \bar{y}^2 - x \bar{y} (\cos \gamma + \cos \gamma') + \mathcal{O} (t^2) + \mathcal{O} (\lambda^4) \) and \( B = x^2 + \bar{y}^2 - 2x \bar{y} \cos \gamma \). Notice that by \( 6.6 \),

\[
A > B + x \bar{y} \sin \frac{\varepsilon_0}{4} \sin \alpha \lambda + \mathcal{O} (t^2) + \mathcal{O} (\lambda^4)
\]

\[
> B + \frac{1}{2} x \bar{y} \sin \frac{\varepsilon_0}{4} \sin \alpha \lambda \geq \frac{1}{2} x \bar{y} \sin \frac{\varepsilon_0}{4} \sin \alpha \lambda > 0
\]

for sufficiently small \( \lambda \) and \( t \). Set

\[
a' = \sqrt{x^2 + \bar{y}^2 - 2x \bar{y} \cos \gamma} \quad \text{and} \quad b' = t \sqrt{x^2 + \bar{y}^2 - 2x \bar{y} \cos \gamma}.
\]
Because \( \tilde{\gamma}, \tilde{\gamma}'' \leq \tilde{\gamma} \), we readily see that \( a \leq a' \) and \( b \leq b' \). Hence,

\[
p \geq k^2 t \frac{A}{\sin \kappa a' \sin \kappa b'} = t \frac{A}{a'b'} \left[ 1 + \mathcal{O} (\lambda^2) \right] = \frac{x^2 + y^2 - xy (\cos \gamma + \cos \gamma') + \mathcal{O} (t \lambda^2) + \mathcal{O} (\lambda^4)}{x^2 + y^2 - 2xy \cos \tilde{\gamma}}.
\]

So, by invoking the upper four point \( \cos q_{K} \) condition, \([6.6]\) and \([6.10]\), for sufficiently small \( \lambda \) and \( t \), we get:

\[
1 < 1 + \frac{xy \sin \frac{\pi}{2} \sin \alpha_{av}}{x^2 + y^2 - 2xy \cos \gamma} \leq \frac{x^2 + y^2 - xy (\cos \gamma + \cos \gamma') + \mathcal{O} (t \lambda^2) + \mathcal{O} (\lambda^4)}{x^2 + y^2 - 2xy \cos \tilde{\gamma}} \leq p \leq 1,
\]

a contradiction. The case of negative \( K \) is similar.

**III. Let** \((M, \rho)\) **satisfy the lower four point** \( \cos q_{K} \) **condition.** Set

\[
q = \cos q_{K} \left( \overrightarrow{XY}, \overrightarrow{X} \right).
\]

We have:

\[
q = \frac{\cos \hat{r}f + \cos \hat{r}a \cos \hat{r}b}{\sin \hat{r}a \sin \hat{r}b} \left[ \cos \hat{r}a + \cos \hat{r} (1 - t) \tilde{y} [\cos \hat{r}b + \cos \hat{r} (1 - t) x].
\right]
\]

Approximating \( q \) relative to \( t \), we get \( q = I (J \sin \hat{r}a \sin \hat{r}b) \), where

\[
I = \left[ \cos \hat{r}f + \cos \hat{r}a + \mathcal{O} (\lambda^2 t^2) \right] [1 + \cos \hat{r}d] - \\
[\mu + \hat{r}t \tilde{y} \sin \hat{r}y + \mathcal{O} (\lambda^2 t^2)] [\nu + \hat{r} (tx) \sin \hat{r}x + \mathcal{O} (\lambda^2 t^2)],
\]

\[
J = (1 + \cos \hat{r}d),
\]

and where we set \( \mu = \cos \hat{r}a + \cos \hat{r} \tilde{y} \) and \( \nu = 1 + \cos \hat{r}x \).

Let \( K > 0 \). By recalling \([6.7]\) and \([6.8]\), we get:

\[
I = [\mu + \kappa (tx) \sin \tilde{y} \cos \gamma'] [\nu + \kappa (tx) \sin k \xi x] \cos \gamma] - [\mu + k t \tilde{y} \sin \hat{r} \tilde{y}] \times [\nu + \kappa (tx) \sin \kappa x] + \mathcal{O} (\lambda^2 t^2), J = (1 + \cos \hat{r}d).
\]

After simplifications, we have:

\[
I = -kt [\nu \sin k \tilde{y} (\tilde{y} - x \cos \gamma') + \mu \sin k \xi x (x - \tilde{y} \cos \gamma) + \mathcal{O} (\lambda^2 t^2)].
\]

By \([6.8]\), \( J^{-1} = \nu^{-1} \left[ 1 + \mathcal{O} (\lambda^2 t^2) \right] \). By the lower four point \( \cos q_{K} \) condition,

\[
-q = kt \frac{\sin k \tilde{y} (\tilde{y} - x \cos \gamma') + \nu \sin k \xi x (x - \tilde{y} \cos \gamma) + \mathcal{O} (\lambda^2 t)}{\sin k \xi x k \tilde{y} + \mathcal{O} (\lambda^4)} \leq 1.
\]

So, we derived from the lower four point \( \cos q \) condition inequality \([6.9]\). Hence, by using the arguments of part II, we see that the lower four point \( \cos q \) condition also implies existence of Aleksandrov’s angle. The case of negative \( K \) is similar.

The proof of Proposition \([6.1]\) is complete. \( \Box \)
6.7. **Angle comparison theorem.** We begin with the following identity in the $K$-plane:

**Proposition 6.2.** Let $K \neq 0$ and $T = ABC$ be a triangle in $S_K$. Set $x = AB, y = AC, z = BC, (x, y, z > 0), \alpha = \angle BAC$ and $\beta = \angle ABC$, as illustrated in Fig. 6.5. Then

\[
\sin \hat{k}z = \frac{\cos \hat{k}y + \cos \hat{k}z}{1 + \cos \hat{k}x} \sin \hat{k}x \cos \beta - \sin \hat{k}y \cos (\alpha + \beta).
\]

In particular, if $K > 0$, then

\[
\sin kz = \frac{\cos ky + \cos kx}{1 + \cos kx} \sin kx \cos \beta - \sin ky \cos (\alpha + \beta)
\]

and if $K < 0$, then

\[
\sinh kz = \frac{\cosh ky + \cosh kx}{1 + \cosh kx} \sinh kx \cos \beta - \sinh ky \cos (\alpha + \beta).
\]

**Figure 6.5.** Sketch for Proposition 6.2

**Proof.** The following cases are possible:

(i) $C$ is between $A$ and $B$. Then $\alpha = \beta = 0, x > y$ and $z = x - y$.

(ii) $A$ is between $B$ and $C$. Then $\alpha = \pi, \beta = 0$ and $z = x + y$.

(iii) $B$ is between $A$ and $C$. Then $\alpha = 0, \beta = \pi, y > x$ and $z = y - x$.

(iv) $T$ is a non-degenerate triangle. Then $\alpha, \beta \in (0, \pi)$.

For example, in case (i), the verification of (6.11) reduces to the direct verification of the elementary trigonometric identity

\[
\sin \hat{k} (x - y) = \cos \hat{k}y + \cos \hat{k} (x - y) \sin \hat{k}x - \sin \hat{k}y.
\]

Cases (ii) and (iii) are similar.

Now we consider case (iv). Let

\[
I = \frac{\sin \hat{k}y}{\sin \beta} \sin (\alpha + \beta) = \frac{\sin \hat{k}y \sin \alpha \cos \beta}{\sin \beta} + \sin \hat{k}y \cos \alpha.
\]

By the sine formula in $S_K$,

\[
\sin \hat{k}y \sin \alpha \cos \beta = \sin \hat{k}z \cos \beta.
\]

By the cosine formula in $S_K$,

\[
\cos \beta = \frac{\cos \hat{k}y - \cos \hat{k}x \cos \hat{k}z}{\sin kx \sin kz},
\]
whence
\[
\frac{\sin \hat{ky}}{\sin \beta} \sin \alpha \cos \beta = \frac{\cos \hat{ky} - \cos \hat{kx} \cos \hat{kz}}{\sin \hat{kx}}.
\]
Again, by the cosine formula in $S_K$,
\[
\cos \alpha = \frac{\cos \hat{kz} - \cos \hat{kx} \cos \hat{ky}}{\sin \hat{kx} \sin \hat{ky}},
\]
whence
\[
\sin \hat{ky} \cos \alpha = \frac{\cos \hat{kz} - \cos \hat{kx} \cos \hat{ky}}{\sin \hat{kx}}.
\]
So,
\[
I = \left(1 - \cos \hat{kx}\right) \left(\cos \hat{ky} + \cos \hat{kz}\right) \left(\frac{\hat{ky}}{\sin \hat{kx}}\right) = \frac{\cos \hat{ky} + \cos \hat{kz}}{1 + \cos \hat{kx}} \sin \hat{kx},
\]
whence
\[
\frac{\cos \hat{ky} + \cos \hat{kz}}{1 + \cos \hat{kx}} \sin \hat{kx} \cos \beta = \frac{\sin \hat{ky}}{\sin \beta} \sin (\alpha + \beta) \cos \beta.
\]
Hence, if $J$ denotes the right-hand side of (6.11), then
\[
J = \frac{\sin \hat{ky}}{\sin \beta} \sin (\alpha + \beta) \cos \beta - \sin \hat{ky} \cos (\alpha + \beta).
\]
Recall that by the sine formula in $S_K$, $\sin \hat{ky} = \sin \hat{kz} \sin \beta / \sin \alpha$. So,
\[
J = \frac{\sin \hat{kz}}{\sin \alpha} \left[\sin (\alpha + \beta) \cos \beta - \cos (\alpha + \beta) \sin \beta\right] = \sin \hat{kz},
\]
as needed.

The proof of Proposition 6.2 is complete. \qed

Let $K \neq 0$ and let $\{A, B, C\}$ be a triple of distinct points in a metric space $(M, \rho)$ of diameter less than $\pi/2\sqrt{K}$ if $K > 0$. In what follows, we assume that the points $A$ and $B$ can be joined by a shortest $L = AB$ and the points $A$ and $C$ can be joined by a shortest $N = AC$. By Proposition 6.1 there exists an angle $\alpha$ between the shortest $L$ and $N$. In what follows, we assume that $0 < \alpha < \pi$. Set $x = AB$ and $y = AC$.

To state our next lemma, we need the following notation. Let $K' \in \{0, K\}$. Consider a geodesic triangle $T' = \tilde{A}K' \tilde{B}K' \tilde{C}K'$ in $S_{K'}$ such that $\tilde{A}K' \tilde{B}K' = x, \tilde{A}K' \tilde{C}K' = y$ and $\alpha = \angle \tilde{B}K' \tilde{A}K' \tilde{C}K'$. If $K' = K$, set
\[
\tilde{A}K' = \tilde{A}, \tilde{B}K' = \tilde{B}, \tilde{C}K' = \tilde{C}, \tilde{B}C = \tilde{z} \text{ and } \tilde{\beta} = \angle \tilde{A}\tilde{B}\tilde{C},
\]
as illustrated in Fig. 6.6. Suppose that for $t \in (0, 1]$, points $\hat{X}_t \in L \setminus \{A\}$ and $\hat{Y}_t \in N \setminus \{A\}$ (in the metric space $(M, \rho)$) have been selected. Consider the Euclidean triangle $\hat{Y}_t^0 = \hat{A}^0 \hat{X}_t^0 \hat{Y}_t^0$ such that $A\hat{X}_t = \hat{A}^0 \hat{X}_t^0, A\hat{Y}_t = \hat{A}^0 \hat{Y}_t^0$ and $\angle \hat{X}_t^0 \hat{A}^0 \hat{Y}_t^0 = \alpha$. We claim that given small $t \in (0, 1]$, there is $s_t \in (0, 1]$ such that $A\hat{X}_t = s_t x, A\hat{Y}_t = ty$ (and $\angle \hat{X}_t^0 \hat{A}^0 \hat{Y}_t^0 = \alpha$), then $\angle \hat{A}^0 \hat{X}_t^0 \hat{Y}_t^0 = \tilde{\beta}$, as illustrated in Fig. 6.7. Indeed, if $\alpha = \pi$, then $\tilde{\beta} = 0$. Set $s_t = t$, and we are done. Now let $\alpha \in (0, \pi)$. First, we remark that $\alpha + \tilde{\beta} < \pi$. It is sufficient to consider $K > 0$. Let $\delta = \angle \hat{A}\hat{C}\hat{B}$. Because $y, \tilde{z} < \pi/2\sqrt{K}$, we can extend the shortests $\tilde{C}\tilde{A}$ and $\tilde{C}\tilde{B}$ to the shortests $\tilde{C}A'$ and $\tilde{C}B'$ of the lengths $\pi/2\sqrt{K}$. Consider the spherical triangle
Figure 6.6. Sketch for Lemma 6.5

Figure 6.7. Definition of $s_t$

$\mathcal{T}' = \tilde{\mathcal{C}}A'B'$. We have: $\angle \tilde{\mathcal{C}}A'B' = \angle \tilde{\mathcal{C}}B'A' = \pi/2$. Hence, by recalling the Gauss-Bonnet theorem, we see that

$$\delta + \alpha + \tilde{\beta} < \delta + \frac{\pi}{2} + \frac{\pi}{2},$$

whence $\alpha + \tilde{\beta} < \pi$ follows. In particular, $\alpha \in (0, \pi)$, and setting $\tilde{\gamma} = \pi - \alpha - \tilde{\beta}$, we see that $\tilde{\gamma} \in (0, \pi)$. Hence, we select $s_t = ty \sin \tilde{\gamma}/(x \sin \tilde{\beta})$.

Finally, set

$$\tilde{\alpha}_K'(t) = \angle_{K'} \tilde{X}_t K' \tilde{Y}_t K', \tilde{\beta}_K'(t) = \angle_{K'} A K' \tilde{X}_t K' \tilde{Y}_t K',$$

$$\tilde{\gamma}_K'(t) = \angle_{K'} A K' \tilde{Y}_t K' \tilde{X}_t K'$$

and $z(t) = \tilde{X}_t \tilde{Y}_t$,

as shown in Fig. 6.8

**Lemma 6.5.** Let $K \neq 0$. If $0 < \alpha \leq \pi$, then

$$\lim_{t \to 0^+} \tilde{\beta}_K(t) = \tilde{\beta}$$

(for the notation, see Fig. 6.6 and Fig. 6.8 for $K' = K'$).

**Proof.** I. Let $\alpha = \pi$; then $\tilde{\beta} = 0$. We have: $\lim_{t \to 0^+} \tilde{\alpha}_0(t) = \pi$, whence $\lim_{t \to 0^+} \tilde{\beta}_0(t) = 0$.

Because $\tilde{\beta}_0(t) - \tilde{\beta}_K(t) = O \left( t^2 \right)$, we have: $\lim_{t \to 0^+} \tilde{\beta}_K(t) = 0$, as needed.
II. Now let \( \alpha \in (0, \pi) \). Then \( \tilde{\beta}, \tilde{\gamma} \in (0, \pi) \), see Fig. 6.7. By the Euclidean sine formula applied to the triangle \( \widetilde{X}_t^0 \widetilde{A}^0 \widetilde{Y}_t^0 \),
\[
\sin \tilde{\beta} = \frac{ty \sin \alpha}{\tilde{z}_0(t)}.
\]
By the Euclidean sine formula applied to the triangle \( \widetilde{X}_t^0 \widetilde{A}^0 \widetilde{Y}_t^0 \) (see Fig. 6.8 for \( K' = 0 \)),
\[
\sin \tilde{\beta}_0(t) = \frac{ty \sin \tilde{\alpha}_0(t)}{z(t)}.
\]
So, by recalling Proposition 6.1 and because \( \tilde{\beta}_0(t) - \tilde{\beta}_K(t) = \mathcal{O}(t^2) \), all we have to do is to show that \( \lim_{t \to +0} t/z(t) = \lim_{t \to +0} t/\tilde{z}_0(t) \) (in fact, \( t/\tilde{z}_0(t) = \text{const} \)). Indeed, by the Euclidean cosine formula applied to the triangle \( \widetilde{X}_t^0 \widetilde{A}^0 \widetilde{Y}_t^0 \) and \( \widetilde{X}_t^0 \widetilde{A}^0 \widetilde{Y}_t^0 \), and by recalling that \( s_t = ty \sin \tilde{\gamma}/(x \sin \tilde{\beta}) \), we get:
\[
\frac{t}{\tilde{z}_0(t)} = \frac{\sin \tilde{\beta}}{y \sqrt{\left(\sin \tilde{\beta} - \sin \tilde{\gamma}\right)^2 + 4 \sin \tilde{\beta} \sin \tilde{\gamma} \sin^2 \frac{\alpha}{2}}},
\]
\[
(6.12) \quad \frac{t}{z(t)} = \frac{\sin \tilde{\beta}}{y \sqrt{\left(\sin \tilde{\beta} - \sin \tilde{\gamma}\right)^2 + 4 \sin \tilde{\beta} \sin \tilde{\gamma} \sin^2 \frac{\tilde{\alpha}(t)}{2}}},
\]
By Proposition 6.1, \( \lim_{t \to +0} \tilde{\alpha}_0(t) = \alpha \). Also recall that \( \alpha, \tilde{\beta}, \tilde{\gamma} \in (0, \pi) \). Hence, \( \lim_{t \to +0} t/z(t) \) and \( \lim_{t \to +0} t/\tilde{z}_0(t) \) exist and they are equal.

The proof of Lemma 6.5 is complete. \( \square \)

Proposition 6.3. Let \( K \neq 0 \) and let \( \{A, B, C\} \) be a triple of distinct points in a metric space \( (\mathcal{M}, \rho) \) such that the points \( A \) and \( B \) can be joined by a shortest \( \mathcal{L} = AB \) and the points \( A \) and \( C \) can be joined by a shortest \( \mathcal{N} = AC \), and \( AB, AC \leq \pi/\left(6\sqrt{K}\right) \) if \( K > 0 \). If \( (\mathcal{M}, \rho) \) satisfies the one-sided four point \( \cos \alpha \) condition, then \( \angle BAC \leq \angle K BAC \).

Remark 6.1. In the hypothesis of Proposition 6.3 we do not require that \( (\mathcal{M}, \rho) \) be a geodesically connected metric space. Also, the bound on \( AB \) and \( AC \) is not sharp.

Proof. Let \( \alpha = \angle BAC \) and \( \alpha_K = \angle K BAC \). There is no restriction in assuming that \( \alpha \in [0, \pi] \). To prove the inequality \( \alpha \leq \alpha_K \), we consider a geodesic triangle \( \tilde{T}^K_t \) such that \( A^K \tilde{B}^K_t = x, A^K \tilde{C}^K_t = y \) and \( \angle \tilde{B}^K_t \tilde{A}^K \tilde{C}^K_t = \tilde{\alpha}_K(t) \). Set \( \tilde{z}_K(t) = \tilde{B}^K_t \tilde{C}^K_t \), as...
illustrated in Fig. 6.9. It is readily seen that \( \alpha \leq \alpha_K \) if and only if \( \tilde{z} = \lim_{t \to 0^+} \hat{\beta}_K(t) \leq z \).

![Figure 6.9. Sketch for Proposition 6.3](image)

For the notation, see Fig. 6.6 and Fig. 6.8 for \( K' = K \). So, our goal is to derive the inequality \( \tilde{z} \leq z \).

By Proposition 6.1, \( \alpha = \lim_{t \to 0^+} \hat{\alpha}_K(t) \). It is readily seen that if \( \alpha = \pi \), then \( z(t)/t = x + y \), i.e., it is bounded above and below by positive constants. Let \( \alpha \in (0, \pi) \). Because \( \hat{\alpha}_0(t) \to \alpha \) as \( t \to 0^+ \),

\[
\sin \frac{\hat{\alpha}_0(t)}{2} \geq \frac{1}{2} \sin \frac{\alpha}{2}
\]

for small \( t \). Then by recalling (6.12), it is not difficult to see that

\[
\frac{t}{z(t)} \leq \frac{1}{2y \sqrt{\sin \gamma \sin \frac{\alpha(t)}{2}}} \leq \frac{1}{y \sqrt{\sin \gamma \sin \frac{\alpha}{2}}} < +\infty.
\]

So, the hypotheses of Corollary 6.2 are satisfied.

Let \( K > 0 \). By Corollary 6.2,

\[
\begin{align*}
&\frac{\cos \kappa y + \cos \kappa z}{1 + \cos \kappa x} \sin \kappa x \cos \hat{\beta}_K(t) - \sin \kappa y \cos \left( \hat{\alpha}_K(t) + \hat{\beta}_K(t) \right) \\
&\leq \sin \kappa z + O(t),
\end{align*}
\]

By Proposition 6.1, \( \lim_{t \to 0^+} \hat{\alpha}_K(t) = \alpha \) and by Lemma 6.5, \( \lim_{t \to 0^+} \hat{\beta}_K(t) = \beta \). Let \( K > 0 \). By letting \( t \to 0^+ \), we get

\[
\sin \kappa z \geq \frac{\cos \kappa z}{1 + \cos \kappa x} \sin \kappa x \cos \beta
\]

\[
\geq \frac{\cos \kappa y}{1 + \cos \kappa x} \sin \kappa x \cos \beta - \sin \kappa y \cos \left( \alpha + \beta \right),
\]

By Proposition 6.2,

\[
\sin \kappa z \geq \frac{\cos \kappa z}{1 + \cos \kappa x} \sin \kappa x \cos \beta = \frac{\cos \kappa y}{1 + \cos \kappa x} \sin \kappa x \cos \beta - \sin \kappa y \cos \left( \alpha + \beta \right),
\]
whence
\[ \sin \kappa z - \frac{\cos \kappa z}{1 + \cos \kappa x} \sin \kappa x \cos \beta \geq \sin \kappa \bar{z} - \frac{\cos \kappa \bar{z}}{1 + \cos \kappa x} \sin \kappa x \cos \beta. \]  
(6.13)

By the triangle inequality, \( z, \bar{z} \leq \pi / (3\kappa) \). By Corollary 6.1, there is no restriction in assuming that \( z > 0 \). So, we can also assume that \( \bar{z} \) is also positive. Consider the function
\[ f(u) = \sin \kappa u - \frac{\cos \kappa u}{1 + \cos \kappa x} \sin \kappa x \cos \beta, \quad u \in (0, \pi / (3\kappa)]. \]

It is readily seen that \( f(u) \) is a strictly increasing function if \( u \in (0, \pi / (3\kappa)] \). So, the inequality \( \bar{z} \leq z \) for positive \( K \) follows from inequality (6.13), as needed.

In a similar way, for \( K < 0 \), we have:
\[ \sinh \kappa z - \frac{\cosh \kappa z}{1 + \cosh \kappa x} \sinh \kappa x \cos \beta \geq \sinh \kappa \bar{z} - \frac{\cosh \kappa \bar{z}}{1 + \cosh \kappa x} \sinh \kappa x \cos \beta. \]  
(6.14)

It is easy to see that the function
\[ g(u) = \sinh \kappa u - \frac{\cosh \kappa u}{1 + \cosh \kappa x} \sinh \kappa x \cos \beta, \quad u \in (0, +\infty) \]
is an increasing function if \( u \in (0, +\infty) \). Hence, (6.14) implies the inequality \( \bar{z} \leq z \) for negative \( K \), as claimed.

The proof of Proposition 6.3 is complete. \( \Box \)

**Corollary 6.5.** Let \( K > 0 \) and let \((\mathcal{M}, \rho)\) be a geodesically connected metric space such that \( \text{diam} (\mathcal{M}) \leq \pi / \left( 2\sqrt{K} \right) \) when \( K > 0 \). If \((\mathcal{M}, \rho)\) satisfies the one-sided four point \( \cos q_{K} \) condition, then it is an \( \mathbb{R}_{K} \) domain with the same diameter restriction.

**Proof.** Theorem 9 in [2, §3] states that a metric space \((\mathcal{M}, \rho)\) such that
(i) \((\mathcal{M}, \rho)\) is geodesically connected,
(ii) the perimeter of every geodesic triangle in \((\mathcal{M}, \rho)\) is less than \( 2\pi / \sqrt{K'} \) if \( K' > 0 \),
(iii) every point of \((\mathcal{M}, \rho)\) has a neighborhood which is an \( \mathbb{R}_{K'} \) domain,
(iv) shortests in \((\mathcal{M}, \rho)\) depend continuously on their end points
is an \( \mathbb{R}_{K'} \) domain.

By the hypothesis of Corollary 6.5 (i) and (ii) for \( K' = K \) are satisfied; (iii) for \( K' = K \) is satisfied by Proposition 6.3 and (iv) is satisfied by Lemma 6.1. Hence, \((\mathcal{M}, \rho)\) is an \( \mathbb{R}_{K} \) domain. \( \Box \)

Finally, Theorem 1.1 follows from Theorem 4.1 Proposition 6.3 \((K < 0)\) and Corollary 6.5 \((K > 0)\).

### 7. Proof of Theorem 1.2

In this section, we consider an extremal case of Theorem 1.1 when \( |\cos q_{K}| = 1 \). We will need a rigidity lemma on geodesic convex hulls of quadruples.

In [2, §4, Theorem 6], Aleksandrov established the following rigidity result: if \( \mathcal{T} = ABC \) is a triangle in an \( \mathbb{R}_{K} \) domain and \( \angle ABC = \angle_{K} ABC \), then \( BX = B^{K}X^{K} \) for every \( X \in \mathcal{A}C \) and \( X^{K} \in \mathbb{R}^{K}C^{K} \) such that \( AX = A^{K}X^{K} \). Aleksandrov’s proof also implies
the converse: if $BX_0 = B^KX_0^K$ for at least one point $X_0 \in \mathcal{AC} \setminus \{A, C\}$, then $\angle ABC = \angle K ABC$. In \cite{10} Proposition 2.9, Bridson and Haefliger slightly improved Aleksandrov’s theorem by proving isometry of the convex hulls of the triangles (see also (1) and (2) of Sec. 2.10 in \cite{10}). The following rigidity lemma is close to Aleksandrov’s rigidity theorem in its spirit and in the method of the proof. For completeness, we include the rigidity lemma and its proof.

**Lemma 7.1.** Let $K \in \mathbb{R}$ and let $\Omega = \{A, P, Q, B\}$ be a quadruple of distinct points in an $\mathbb{R}_K$ domain. Let $\mathcal{R}$ be a convex quadrangle in $\mathbb{S}_K$ bounded by the closed polygonal curve $L' = A'P'Q'B'A'$ with the vertices at $A'$, $P'$, $Q'$ and $B'$. Suppose that there is an isometry $f$ from $\Omega$ onto the quadruple $\Omega' = \{A', P', Q', B'\}$ such that $f(A) = A'$, $f(P) = P'$, $f(Q) = Q'$ and $f(B) = B'$. Then the geodesic convex hull of $\Omega$ is isometric to $\mathcal{R}$.

**Proof.** The proof will be done in a series of steps.

Let $\mathcal{L}$ be a polygonal curve $A_1A_2 \ldots A_n$ in $\mathbb{R}_K$ and $\mathcal{L}'$ be a polygonal curve $A'_1A'_2 \ldots A'_n$ in $\mathbb{S}_K$ such that $A_jA_{j+1} = A'_jA'_{j+1}$ for every $j \in \{1, 2, \ldots , n-1\}$. Let $g_{al, L}, g_{al, L'}$ denote the arc length parametrizations of $L$ and $L'$ relative to $A_1$ and $A'_1$, respectively (for the notation, see Sec. 2). Define $\varphi_{L, L'} : \mathcal{L} \to \mathcal{L}'$ as follows. If $X' \in \mathcal{L}'$ and $X' = g_{al, L'}(t_0)$, then set $X = \varphi_{L, L'}(X') = g_{al, L}(t_0) \in L$.

I. Let $\mathcal{T} = ABC$ be a geodesic triangle in $\mathbb{R}_K$ of perimeter less than $2\pi/\sqrt{K}$ if $K > 0$ and let $\mathcal{T}' = A'B'C'$ be its isometric copy in $\mathbb{S}_K$. If $X$ is a point on the side $AB$, then $X'$ denotes the point on the side $A'B'$ such that $BX = B'X'$. The point $Y' \in B'C'$ corresponding to a point $Y \in BC$ is defined in a similar way. We begin with the following corollary of \cite{10}, Proposition 2.9 and (1), (2) of Sec. 2.10: The convex hull $\mathcal{G}[A, B, C]$ in $\mathbb{R}_K$ is isometric to the convex hull $\mathcal{G}[A', B', C']$ in $\mathbb{S}_K$ if and only if there is $X \in AB \setminus \{B\}$ and $Y \in BC \setminus \{B\}$ such that $XY = XY'$ where either $X \neq A$ or $Y \neq C$.

II. Let $O'$ be the point of intersection of the shortests $A'Q'$ and $B'P'$. Let $u = A'O'$ and $v = O'Q'$. Because $AQ = A'Q'$, we can select $O \in AQ$ such that $AO = u$ and $OQ = v$. By the triangle inequality, $PB \leq PO + OB$. By $K$-concavity (Theorem 2 in \cite{2} § 3),

$$PB \leq PO + OB \leq P'O' + O'B' = P'B' = PB,$$

whence $PB = PO + OB$ follows. By the uniqueness property of shortests in $\mathbb{R}_K$, the polygonal curve $\mathcal{P}OB$ coincides with the shortest $\mathcal{P}B$. We also have: $PO = P'O'$ and $OB = O'B'$.

III. Consider the closed polygonal curves

$$L = \mathcal{P}QBAP$$

and

$$L' = \mathcal{P}'Q'B'A'P',$$

and set $\varphi_{P'} = \varphi_{L, L'}$.

IIIa. Let $E' \in B'Q'$. Set $E = \varphi_{P'}(E')$. Then $AE = A'E'$. Indeed, consider triangle $AQB$. By II, $BO = B'O'$. Then by I, $AE = A'E'$, as needed. In a similar way, all distances from a point of $\Omega$ to a point on one of the shortests $AB, AP, QB$ and $AB$ are the same as the corresponding distances in $\mathbb{S}_K$.

IIIb. Now, let $E' \in A'P'$ (we can assume that $E' \neq P'$), $F' \in \mathcal{P}'Q'$, $E = \varphi_{P'}(E')$ and $F = \varphi_{P'}(F')$. Consider the triangle $E'Q'F'$. Let $G' \in \mathcal{P}'E' \setminus \{E', F'\}$ and $G = \varphi_{P'}(G')$. By IIIa, $QG = Q'G'$, whence by I, $EF = E'F'$ follows.

IIIc. Next, let $E' \in A'P'$, $F' \in B'Q'$, $E = \varphi_{P'}(E')$ and $F = \varphi_{P'}(F')$. Let $O'$ be the point of intersection of the shortest $A'Q'$ and $E'B'$. Recall that by IIIa, $E'Q' = EQ$ and $EB = E'B'$. There is $O \in EB$ such that $EO = E'O'$ and $OB = O'B'$. By employing
arguments similar to those of II, we see that \( OQ = O'Q' \). Hence, by I, applied to triangle \( BEQ \), we have: \( EF = E'F' \).

So, by III, \( \varphi_{P'} \) is an isometry in from \( L' \) onto \( L \).

**IV.** The isometry \( \varphi_{P'} \) from \( L' \) onto \( L \) can be extended to an isometry from \( \mathcal{R} \) into \( \mathcal{G}C[\mathcal{D}] \). Indeed, let \( X', Y' \in \mathcal{R} \). For definiteness, suppose that there are \( D' \in \mathcal{A}'B' \) and \( F' \in \mathcal{B}'Q' \) such that \( X' \in \mathcal{P}'D' \) and \( Y' \in \mathcal{P}'F' \). Let \( D = \varphi_{P'}(D') \) and \( F = \varphi_{P'}(F') \). By III, \( P'D' = PD \) and \( P'F' = PF \). Hence, we can select \( X \in PD \) such that \( P'X = PX \) and \( X'D' = XD' \). Point \( Y \in PD' \) is selected in a similar way so that \( P'Y' = PY \), as illustrated in Fig. 7.1. Set \( \varphi_{P'}(X') = X \) and \( \varphi_{P'}(Y') = Y \). We claim that \( X'Y' = XY \).

**Figure 7.1. Sketch for part IV of Lemma 7.1**

Indeed, by III, \( D'F' = DF \). Let \( E' \) be the point of intersection of the shortest \( \mathcal{P}'B' \) and \( \mathcal{D}'F' \). Because \( DF = D'E' \), we can select \( E \in DF \) such that \( DE = D'E' \) and \( EF = E'F' \). By using arguments similar to those of II, we see that \( PE = P'E' \). Hence, by I, \( \mathcal{G}C[\mathcal{D}'] \) is isometric to \( \mathcal{G}C[\mathcal{D}] \), and \( XY = X'Y' \) follows. Thus, \( \varphi_{P'} \) is an isometry from \( \mathcal{R} \) into \( \mathcal{G}C[\mathcal{D}] \).

**V.** \( \varphi_{P'} \) is a surjection. Because \( \mathcal{R} \) is convex it is sufficient to prove the following claim \( \mathcal{P}(n) \): the isometry \( \varphi_{P'} \) from \( \mathcal{G}^n[\mathcal{D}'] \) into \( \mathcal{G}^n[\mathcal{D}] \) is a surjection for every \( n = 0, 1, 2, \ldots \). Indeed, clearly \( \mathcal{P}(0) \) is true. Suppose that \( \mathcal{P}(n) \) is true. Let \( Z \in \mathcal{G}^{n+1}[\mathcal{D}] \). Then, there are \( X, Y \in \mathcal{G}^n[\mathcal{D}] \) such that \( Z = XY \). Because \( \varphi_{P'} : \mathcal{G}^n[\mathcal{D}'] \rightarrow \mathcal{G}^n[\mathcal{D}] \) is a bijection, there are (unique) \( X' = \varphi_{P'}^{-1}(X), Y' = \varphi_{P'}^{-1}(Y) \) satisfying \( XY = X'Y' \). Without loss of generality, we can assume that there are \( D' \in \mathcal{A}'B' \) and \( F' \in \mathcal{B}'Q' \) such that \( X' \in \mathcal{P}'D' \) and \( Y' \in \mathcal{P}'F' \). Then, by the definition of \( \varphi_{P'} \), we see that \( X \in \mathcal{P}D, Y \in \mathcal{P}F \), where \( D = \varphi_{P'}(D') \) and \( F = \varphi_{P'}(F') \), and \( P'D' = PD, P'F' = PF \), as illustrated in Fig. 7.2. For definiteness, suppose that \( Z' \in \mathcal{P}\mathcal{G}^n \setminus \{P, G\} \) where \( G \in \mathcal{B}'Q' \). Set \( G = \varphi_{P'}(G') \). Then \( PG = P'G' \). By using arguments of II, we see that the polygonal curve \( PZG \) coincides with the shortest joining \( P \) to \( G \). Hence, \( Z = \varphi_{P'}(Z') \). Thus, \( \varphi_{P'} : \mathcal{G}^{n+1}[\mathcal{D}'] \rightarrow \mathcal{G}^{n+1}[\mathcal{D}] \) is a surjection.

The proof of Lemma 7.1 is complete.

Finally, we complete the proof of Theorem 1.2. By Theorem 1.1 \( (M, \rho) \) is an \( \mathcal{R}_K \) domain. Let \( \cosq_K\left(\overrightarrow{AP}, \overrightarrow{BQ}\right) = 1 \). Because \( \operatorname{diam}(A, P, Q, B) < \pi/(2\sqrt{K}) \) if \( K > 0 \), we have: \( AP + PQ + BQ + AB < 2\pi/\sqrt{K} \). Hence, \( \mathcal{A} \) is applicable to the closed curve \( \mathcal{L} = \mathcal{A}PQB \). So, as in the proof of Theorem 4.1, consider the closed polygonal curve \( \mathcal{L} \) and a convex domain \( \mathcal{V} \subseteq \mathcal{S}_K(\partial \mathcal{V} = \mathcal{L} = \mathcal{A}'P'Q'B', \mathcal{A}') \).
majorizing the curve $L$ and satisfying (4.1). Then, as we showed in the proof of Theorem 4.1

$$\cos_K \left( \overrightarrow{AP}, \overrightarrow{BQ} \right) \leq \cos_K \left( \overrightarrow{A'P'}, \overrightarrow{B'Q'} \right) \leq 1.$$ 

If either $d = PB < d' = P'B'$, or $f = AQ < f' = A'Q'$, then $1 = \cos_K \left( \overrightarrow{AP}, \overrightarrow{BQ} \right) < \cos_K \left( \overrightarrow{A'P'}, \overrightarrow{B'Q'} \right)$, a contradiction. So, $f = f'$ and $d = d'$ follows.

Let $\cos_K \left( \overrightarrow{AP}, \overrightarrow{BQ} \right) = -1$. By the hypothesis, Reshetnyak’s majorization theorem is applicable to the closed curve $\mathcal{N} = AQBP$. The reader should follow the proof of Theorem 4.2 in [5] to arrive at the same conclusion $f = f'$ and $d = d'$.

So, if $\left| \cos_K \left( \overrightarrow{AP}, \overrightarrow{BQ} \right) \right| = 1$, then the quadruple $\{A, P, B, Q\}$ in $(\mathcal{M}, \rho)$ is isometric to the quadruple $\{A', P', B', Q'\}$ in $S_K$. Hence, the statement of Theorem 1.2 follows from Lemma 7.1.

**Example 7.1.** Theorem 1.2 need not be true if we allow $\text{diam} (\mathcal{M}) = \pi/2$. Indeed, consider the metric space $(\mathcal{M}, \rho) = (\mathcal{M}_\varepsilon, \rho_\varepsilon)$ of Example 4.1 for $\varepsilon = 0$. Notice that $(\mathcal{M}, \rho)$ is an $\mathbb{R}_1$ domain, $\text{diam} (\mathcal{M}) = \pi/2$ and $\cos_{11} \left( \overrightarrow{PO}, \overrightarrow{BQ} \right) = 1$, whereas $\mathcal{G}C [\{B, Q, O, P\}] = \mathcal{M}$ cannot be isometric to a convex domain in the half-sphere $S_1$.

**8. Proof of Theorem 1.3**

In this section, we extend Theorem 1.1 to complete weakly convex semimetric spaces satisfying the one-sided four point $\cos_K$ condition. We begin with the following

**Lemma 8.1.** Let $K \neq 0$ and let $(\mathcal{M}, \rho)$ be a semimetric space such that $p(T) < 2\pi/\sqrt{K}$ if $K > 0$ for every triple of distinct points $T = \{A, B, C\}$ in $\mathcal{M}$. If $(\mathcal{M}, \rho)$ satisfies the one-sided four point $\cos_K$ condition, then $(\mathcal{M}, \rho)$ is a metric space.

**Proof.** Set $a = BC$, $b = AC$ and $c = AB$. We have to prove the triangle inequality for $\rho$.

**I.** Let $(\mathcal{M}, \rho)$ satisfy the upper four point $\cos_K$ condition. Then

$$\cos_K \left( \overrightarrow{CA}, \overrightarrow{CB} \right) = \frac{\cos \hat{c} - \cos \hat{a} \cos \hat{b}}{\sin \hat{a} \sin \hat{b}} \leq 1,$$
whence
\[ \frac{\cos \hat{\kappa}c - \cos \hat{\kappa}(b-a)}{\sin \hat{\kappa}a \sin \hat{\kappa}b} \leq 0. \] (8.1)

If \( K > 0 \), we get: \( \cos \kappa c - \cos \kappa (b-a) \leq 0 \), whence \( b \leq a + c \). If \( K < 0 \), we get \( \cosh \kappa c - \cosh \kappa (b-a) \geq 0 \), whence \( b \leq a + c \). Verification of remaining triangle inequalities for \( T \) is similar.

\[ \text{II. Let } (\mathcal{M}, \rho) \text{ satisfy the lower four point } \cos K \text{ condition. Then} \]
\[ \cos K \left( \overline{CA}, \overline{BC} \right) = \frac{\cos \hat{\kappa}a \cos \hat{\kappa}b - \cos \hat{\kappa}c}{\sin \hat{\kappa}a \sin \hat{\kappa}b} \geq -1, \]

whence, (8.1) follows. As in 1, this implies the triangle inequality for \( \rho \).

The proof of Lemma 8.1 is complete. \( \square \)

By Lemma 8.1, \( (\mathcal{M}, \rho) \) is a metric space. Next, we show that \( (\mathcal{M}, \rho) \) is a (complete) geodesically connected metric space. Let \( A, B \in \mathcal{M}, A \neq B \). By weak convexity, there is \( \lambda \in (0,1) \) such that, for every \( n = 1, 2, \ldots \), there is a point \( C_n \in \mathcal{M} \) satisfying
\[ |\rho(A, C_n) - \lambda \rho(A, B)| < 1/n \text{ and } |\rho(B, C_n) - (1-\lambda) \rho(A, B)| < 1/n. \]

We claim that \( \{C_n\}_{n=1,2,\ldots} \) is a Cauchy sequence. The proof uses no new ideas beside those of the proof of Lemma 6.1. Indeed, in the proof of Lemma 6.1 for \( m \neq n \), take \( A_n := A, B_n := B, P := C_n, P_n := C_m \), see Fig. 6.1. Set \( l = AB \), and \( \delta_{m,n} = \lim_{m,n \to \infty} \overline{C_n C_m} \)
\[ AC_n - \lambda AB = \varepsilon_n' \to 0 \text{ as } n \to \infty, \]
\[ BC_n - (1-\lambda) AB = \varepsilon_n'' \to 0 \text{ as } n \to \infty. \]

If \( (\mathcal{M}, \rho) \) satisfies the upper four point \( \cos K \) condition, then by (6.1),
\[ \lim_{m,n \to \infty} \cos K \left( \overline{AC_n} \right) = 1 + \left[ 1 - \cos (\hat{\kappa} \delta_{m,n}) \right] \frac{\cos \left( \hat{\kappa} (1-\lambda) l \right) + \cos \left( \hat{\kappa} l \right)}{(1 + \cos (\hat{\kappa} l)) (\sin (\hat{\kappa} l)) \sin (\hat{\kappa} (1-\lambda) l)} \leq 1, \]
whence, as in the proof of Lemma 6.1, \( \delta_{m,n} = 0 \) follows. The case of the lower four point \( \cos K \) condition is similar. Thus, we showed that \( \{C_n\}_{n=1,2,\ldots} \) is a Cauchy sequence. By part (b) of the hypothesis of Theorem 1.3, \( (\mathcal{M}, \rho) \) is a complete metric space. Hence, the sequence \( \{C_n\}_{n=1,2,\ldots} \) converges to a point \( C \in \mathcal{M} \) such that \( AC = \lambda AB \) and \( BC = (1-\lambda) AB \). We readily see that \( AB = AC + CB \). So, every pair \( A, B \) of distinct points of \( \mathcal{M} \) has a point \( C \) between them. By Menger’s theorem [9], Theorem 14.1, a complete convex metric space is geodesically connected. Finally, Theorem 1.3 follows from Theorem 1.1.

The proof of Theorem 1.3 is complete.

9. \( K \)-quadrilateral inequality condition

In this section, we derive \( K \)-Euler’s inequality, a generalization of a familiar Euler’s inequality [14 Corollary 4] (also known as Enflo’s 2-roundness condition [13]) for \( \mathbb{R}_K \) domains for non-zero \( K \), and study the case of equality in \( K \)-Euler’s inequality.
9.1. $K$-Euler inequality in $S_K$.

**Proposition 9.1.** Let $K \neq 0$ and let $R$ be a convex quadrangular domain in $S_K$ bounded by a closed polygonal curve $L = ABCDA$. Let $O_1$ be the midpoint of the shortest diagonal $BD$ and $O_2$ be the midpoint of the shortest diagonal $AC$. Set

\[
a = AB, \quad b = BC, \quad c = CD, \quad d = AD, \quad e = BD, \quad f = AC \text{ and } g = O_1O_2,
\]

as illustrated in Fig. 9.1. Then the following equality, called $K$-Euler’s equality holds:

\[
\cos \kappa_a + \cos \kappa_b + \cos \kappa_c + \cos \kappa_d = 4 \cos \frac{\kappa e}{2} \cos \frac{\kappa f}{2} \cos \kappa g.
\]

In particular, if $K > 0$, then

\[
\cos \kappa a + \cos \kappa b + \cos \kappa c + \cos \kappa d = 4 \cos \frac{\kappa e}{2} \cos \frac{\kappa f}{2} \cos \kappa g,
\]

and if $K < 0$, then

\[
cosh \kappa a + \cosh \kappa b + \cosh \kappa c + \cosh \kappa d = 4 \cosh \frac{\kappa e}{2} \cosh \frac{\kappa f}{2} \cosh \kappa g.
\]

**Proof.** Let $O$ be the point of intersection of the shortests $BD$ and $AC$. Set $x = BO$, $y = DO$, $z = AO$, $w = OC$. There is no restriction in assuming that $x \geq y$ and $w \geq z$. Set $\alpha = \angle BOC$. By the cosine formula in $S_K$,

\[
\begin{align*}
\cos \kappa a &= \cos \kappa x \cos \kappa z - \sin \kappa x \sin \kappa z \cos \alpha, \\
\cos \kappa b &= \cos \kappa x \cos \kappa w + \sin \kappa x \sin \kappa w \cos \alpha, \\
\cos \kappa c &= \cos \kappa y \cos \kappa w - \sin \kappa y \sin \kappa w \cos \alpha, \\
\cos \kappa d &= \cos \kappa y \cos \kappa z + \sin \kappa y \sin \kappa z \cos \alpha,
\end{align*}
\]

whence

\[
\begin{align*}
\cos \kappa a + \cos \kappa b + \cos \kappa c + \cos \kappa d \\
= \cos \kappa x \cos \kappa z + \cos \kappa x \cos \kappa w + \cos \kappa w \cos \kappa y + \cos \kappa y \cos \kappa z \\
(- \sin \kappa x \sin \kappa z + \sin \kappa x \sin \kappa w - \sin \kappa y \sin \kappa w + \sin \kappa y \sin \kappa z) \cos \alpha.
\end{align*}
\]
Notice that
\[
\cos \hat{\kappa}x \cos \hat{\kappa}z + \cos \hat{\kappa}x \cos \hat{\kappa}w + \cos \hat{\kappa}w \cos \hat{\kappa}y + \cos \hat{\kappa}y \cos \hat{\kappa}z
= 4 \cos \frac{\hat{\kappa}z + w}{2} \cos \frac{\hat{\kappa}w - z}{2} \cos \frac{\hat{\kappa}x + y}{2} \cos \frac{\hat{\kappa}x - y}{2},
\]
and
\[
- \sin \hat{\kappa}x \sin \hat{\kappa}z + \sin \hat{\kappa}x \sin \hat{\kappa}w - \sin \hat{\kappa}y \sin \hat{\kappa}w + \sin \hat{\kappa}y \sin \hat{\kappa}z
= 4 \cos \frac{\hat{\kappa}e}{2} \cos \frac{\hat{\kappa}f}{2} \cos \frac{\hat{\kappa}x - y}{2} \cos \frac{\hat{\kappa}w - z}{2}.
\]
We have:
\[
OO_1 = x - \frac{x + y}{2} = \frac{x - y}{2}, \quad OO_2 = w - \frac{z + w}{2} = \frac{w - z}{2}.
\]
So, by the cosine formula in \(S_K\),
\[
\cos \hat{\kappa}a + \cos \hat{\kappa}b + \cos \hat{\kappa}c + \cos \hat{\kappa}d
= 4 \cos \frac{\hat{\kappa}e}{2} \cos \frac{\hat{\kappa}f}{2} \left( \cos \frac{\hat{\kappa}w - z}{2} \cos \frac{\hat{\kappa}x - y}{2} + \sin \frac{\hat{\kappa}w - z}{2} \sin \frac{\hat{\kappa}x - y}{2} \cos \alpha \right)
= 4 \cos \frac{\hat{\kappa}e}{2} \cos \frac{\hat{\kappa}f}{2} \cos \hat{\kappa}g,
\]
as needed.

The proof of Proposition \(9.1\) is complete. \(\square\)

**Corollary 9.1** (\(K\)-Euler's inequality in \(S_K\)). Under the hypothesis of Proposition \(9.1\) the following inequalities hold:

(a) If \(K > 0\), then
\[
\cos \kappa a + \cos \kappa b + \cos \kappa c + \cos \kappa d \leq 4 \cos \frac{\kappa e}{2} \cos \frac{\kappa f}{2},
\]
(b) If \(K < 0\), then
\[
\cosh \kappa a + \cosh \kappa b + \cosh \kappa c + \cosh \kappa d \geq 4 \cosh \frac{\kappa e}{2} \cosh \frac{\kappa f}{2}.
\]

9.2. \(K\)-Euler's inequality in \(R_K\).

**Theorem 9.1.** Let \(K \neq 0\) and \(\Omega = \{A, B, C, D\}\) be a quadruple of distinct points in an \(R_K\) domain. If \(AB + BC + CD + AD < 2\pi/\sqrt{K}\) if \(K > 0\), then the following inequalities (called \(K\)-Euler's, or \(K\)-quadrilateral inequalities) hold:

(a) If \(K > 0\), then
\[
(9.1) \quad \cos \kappa a + \cos \kappa b + \cos \kappa c + \cos \kappa d \leq 4 \cos \frac{\kappa e}{2} \cos \frac{\kappa f}{2},
\]
(b) If \(K < 0\), then
\[
(9.2) \quad \cosh \kappa a + \cosh \kappa b + \cosh \kappa c + \cosh \kappa d \geq 4 \cosh \frac{\kappa e}{2} \cosh \frac{\kappa f}{2},
\]
where we use the notation of Sec. 9.1.
Proof. Consider the closed polygonal curve $\mathcal{L} = ABCDA$ in $\mathbb{R}_K$. We are given that the length of $\mathcal{L}$ is less than $2\pi/\sqrt{K}$ if $K > 0$. By Reshetnyak’s majorization theorem, there is a convex domain $\mathcal{V} \subseteq \mathbb{S}_K$ bounded by a polygonal curve $\mathcal{L}' = A'B'C'D'A'$ such that

\[
\begin{align*}
a &= AB = a' = A'B', & b &= BC = b' = B'C', \\
c &= CD = c' = C'D', & d &= AD = d' = A'D', \\
e &= BD \leq e' = B'D' & f &= AC \leq f' = A'C'.
\end{align*}
\]

Let $K > 0$. By invoking Corollary 9.1, we get:

\[
\cos \kappa a + \cos \kappa b + \cos \kappa c + \cos \kappa d = \cos \kappa a' + \cos \kappa b' + \cos \kappa c' + \cos \kappa d' \leq 4 \cos \kappa \frac{e'}{2} \cos \kappa \frac{f'}{2} \leq 4 \cos \kappa \frac{e}{2} \cos \kappa \frac{f}{2},
\]

as needed. The case of negative $K$ is treated in a similar way.

The proof of Theorem 9.1 is complete. \qed

9.3. Extremal theorem for $K$-Euler’s inequality. The following theorem extends the second part of Theorem 6 in [6] to the case of non-zero $K$.

Theorem 9.2. Let $K \neq 0$ and $\Omega = \{A, B, C, D\}$ be a quadruple of distinct points in an $\mathbb{R}_K$ domain. Suppose that $AB + BC + CD + AD < 2\pi/\sqrt{K}$ if $K > 0$. Then the equality sign in $K$-Euler’s inequality (9.7) for positive $K$ and in (9.2) for negative $K$ holds if and only if the geodesic convex hull of $\Omega$ is isometric to a parallelogramoidal domain $\mathcal{V}$ in $\mathbb{S}_K$, i.e., a segment of straight line or a closed domain bounded by a closed polygonal curve $\mathcal{L}' = A'B'C'D'A'$ such that $\cos \kappa_K \left( \overrightarrow{A'D'}, \overrightarrow{C'B'} \right) = -1$ and $x = y$ (and thereby, $a = b$).

Proof. We can assume that $\Omega$ is not isometric to a quadruple of points in $\mathbb{R}$. Set $a = AB$, $y = BC$, $b = CD$, $x = AD$, $d = AC$ and $f = BD$.

Let $K > 0$.

I. Suppose that

\[
\cos \kappa a + \cos \kappa b + \cos \kappa c + \cos \kappa y = 4 \cos \kappa \frac{f}{2} \cos \kappa \frac{d}{2}.
\]

By Reshetnyak’s majorization theorem, there is a convex domain $\mathcal{V}$ bounded by closed polygonal curve $\mathcal{L}' = A'B'C'D'$ in $\mathbb{S}_K$ majorizing the closed polygonal curve $\mathcal{L} = ABCD$. Let $d' = A'C'$ and $f' = B'D'$, as illustrated in Fig. 9.2. We have: $f \leq f'$ and $d \leq d'$, whence by recalling Corollary 9.1 we get:

\[
\cos \kappa a + \cos \kappa b + \cos \kappa c + \cos \kappa y = 4 \cos \kappa \frac{f}{2} \cos \kappa \frac{d}{2} \geq 4 \cos \kappa \frac{f'}{2} \cos \kappa \frac{d'}{2} \geq
\]

\[
\cos \kappa a + \cos \kappa b + \cos \kappa c + \cos \kappa y.
\]

Hence, by Proposition 9.1

\[
4 \cos \kappa \frac{f'}{2} \cos \kappa \frac{d'}{2} \cos \kappa g' = 4 \cos \kappa \frac{f'}{2} \cos \kappa \frac{d'}{2},
\]

where $g'$ is the distance between the midpoints of the shortests $A'C'$ and $B'D'$. So, $g' = 0$, that is, the shortests $A'C'$ and $B'D'$ intersect at their common midpoint $O'$. By recalling the geometric interpretation of $\cos \kappa_K$ in $\mathbb{S}_K$ in Sec. 3 (see Fig. 3.1) where $A := A'$, $B := C'$,
Figure 9.2. Sketch for Theorem 9.2

\[ P := D', P' := B' \text{ and } O := O' \], we readily see that \( \cos q_K \left( \overrightarrow{A'D'}, \overrightarrow{C'B'} \right) = -1 \), \( x = y \) and \( a = b \).

Now we show that \( f = f' \) and \( d = d' \). Indeed, recall that \( f \leq f' \) and \( d \leq d' \). If, say, \( f < f' \), then because \( f, d, f', d' \in (0, \pi) \),

\[ \cos \kappa a + \cos \kappa b + \cos kx + \cos ky = 4 \cos \kappa \frac{f'}{2} \cos \kappa \frac{d'}{2} < 4 \cos \kappa \frac{f}{2} \cos \kappa \frac{d}{2}, \]

a contradiction because of (9.3). Hence, the quadruple \( Q = \{A, B, C, D\} \) is isometric to the quadruple \( Q' = \{A', B', C', D'\} \), whence, by Lemma 7.1, \( GC[Q] \) is isometric to the parallelogramoidal domain \( V \), as claimed.

II. Let \( f \) be an isometry from \( GC[Q] \) onto a parallelogramoidal domain \( V \) in \( \mathbb{S}_K \) bounded by the closed polygonal curve \( L' = A'B'C'D' \) such that \( f(A) = A' \), \( f(B) = B' \), \( f(C) = C' \) and \( f(D) = D' \). As we mentioned in I, the shortests \( AC' \) and \( BD' \) intersect at the common midpoint \( O' \), i.e., \( g' = 0 \). Hence, by Proposition 9.1

\[ \cos \kappa a + \cos \kappa b + \cos kx + \cos ky = 4 \cos \kappa \frac{f'}{2} \cos \kappa \frac{d'}{2} = 4 \cos \kappa \frac{f}{2} \cos \kappa \frac{d}{2}, \]

as needed.

The case of negative \( K \) is similar.

The proof of Theorem 9.2 is complete. \( \Box \)

10. Remarks

In Sec. 7, part I, Example 21 in [6], we showed that, for an individual quadruple of points, the four point condition need not imply 0-concavity, Berestovskii’s embeddability condition or Reshetnyak’s majorization condition for \( K = 0 \). It is not difficult to construct a similar example for non-zero \( K \).

Example 10.1. Let \( \Omega = \{A, B, C, O\} \) be a four element set. The six (symmetric) distances between the pairs of points in \( \Omega \) are given by

\[ \rho(A, B) = 0.8, \quad \rho(B, C) = 1, \quad \rho(C, O) = 0.95, \]
\[ \rho(A, O) = 0.4, \quad \rho(B, O) = 0.4 \text{ and } \rho(A, C) = 1. \]

It is easy to see that \( \rho \) is a metric. If we take \( A := A', P := B, B := O \) and \( Q := C \), then in the notation of Sec. 9 all 12 main (approximate) values of \( \cos q_1 \) and \( \cos q_{-1} \) for the four point metric space \( (M, \rho) \) are given in Tables 10.1 and 10.2.
### Table 10.1. Example \[10.1\] $K = 1$

| Case  | I    | II   | III   | IV    | V     | VI    |
|-------|------|------|-------|-------|-------|-------|
| $\cos q_1$ | 0.0012 | 0.2048 | 0.2865 | 0.6466 | -0.2865 | 0.2841 |

| Case  | VII  | VIII | IX    | X     | XI    | XII   |
|-------|------|------|-------|-------|-------|-------|
| $\cos q_1$ | 0.0012 | 0.2048 | 0.6466 | 0.2841 | -0.4756 | -0.4756 |

### Table 10.2. Example \[10.1\] $K = -1$

| Case  | I    | II   | III   | IV    | V     | VI    |
|-------|------|------|-------|-------|-------|-------|
| $\cos q_{-1}$ | -0.0106 | -0.1647 | 0.6208 | 0.3287 | -0.6208 | 0.6406 |

| Case  | VII  | VIII | IX    | X     | XI    | XII   |
|-------|------|------|-------|-------|-------|-------|
| $\cos q_{-1}$ | -0.0106 | -0.1647 | 0.3287 | 0.6406 | -0.4887 | -0.4887 |

Hence, $(\Omega, \rho)$ satisfies the upper four point $\cos q_K$ condition and the lower four point $\cos q_K$ condition for $K = \pm 1$. Notice, that $\Omega$ is a triangular quadruple: $O$ is between $A$ and $B$. The quadruple $\Omega$ is not a non-rectilinear quadruple satisfying Case A in \[7\], as is required in Theorem 5 in \[7\] Sec. 3. Let $T'_1 = A'_1 B'_1 C'_1$ be a triangle in $S_1$ and $T'_{-1} = A'_1 B'_1 C'_1$ in $S_{-1}$ be such that the triple $\{A, B, C\}$ is isometric to $\{A'_1, B'_1, C'_1\}$ and $\{A'_1, B'_1, C'_1\}$. Let $O'_1$ be the midpoint of the shortest $A'_1 B'_1$ and $O'_{-1}$ be the midpoint of the shortest $A'_1 B'_{-1}$. By Lemma \[3.7\] and similar formula for $K = -1$, both approximate values for $C'_1 O'_1$ and $C'_{-1} O'_{-1}$ are easy to calculate:

- $C'_1 O'_1 = \arccos \left( \frac{\cos 1}{\cos 0.4} \right) \approx 0.9439 < 0.95 = CO$ and
- $C'_{-1} O'_{-1} = \arccosh \left( \frac{\cosh 1}{\cosh 0.4} \right) \approx 0.8944 < 0.95 = CO$.

Thus, the $K$-concavity condition fails for the triangular quadruple $\Omega$, and, as a corollary, both Berestovskii’s embeddability condition and Reshetnyak’s majorization condition for $K = \pm 1$ fail.

In (c) of Part I in \[6\] Sec. 7, we erroneously omitted the condition that the triangular quadruple cannot be rectilinear and it cannot satisfy case A in \[7\]. We thank Professor Berestovskii for pointing this out in a personal communication.

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