The Topography of $W_\infty$-type Algebras

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ABSTRACT

We chart out the landscape of $W_\infty$-type algebras using $W_{KP}^{(q)}$—a recently discovered one-parameter deformation of $W_{KP}$. We relate all hitherto known $W_\infty$-type algebras to $W_{KP}^{(q)}$ and its reductions, contractions, and/or truncations at special values of the parameter.

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Introduction and History

Ever since the realization [1] that the Magri bracket [2] for the KdV hierarchy could be identified with the Virasoro algebra, the study of classical integrable systems and $W$-algebras (see [3] and [4] respectively for up-to-date reviews on the subjects) have been inextricably linked to each other. For instance, the realization that the second Gel'fand–Dickey bracket [5] for the Boussinesque hierarchy could be identified with Zamolodchikov's $W_3$, lead Fateev and Lykyanov [6] to construct the generalisations known as $W_n$ by quantizing [7] the second Gel'fand–Dickey bracket for the $n$th-order KdV hierarchy. These are extensions of the Virasoro algebra by primary fields of spins $3, 4, \ldots, n$. This example is prototypical: classical realizations of $W$-algebras appear as hamiltonian structures of integrable hierarchies and their quantization is made possible by quantizing free field realizations defined at the classical level via Miura-type transformations.

In this letter we shall focus on $W$-algebras containing an infinite tower of higher spin generators. For convenience we refer to these algebras as being of the $W_\infty$-type. The study of these algebras was started in [8] where the algebra $w_\infty$ was obtained as a contraction $n \to \infty$ of $W_n$ and was identified with a subalgebra of the Lie algebra of area-preserving diffeomorphisms on the cylinder. Just like any other 2-manifold, the cylinder is symplectic and area-preserving diffeomorphisms are simply symplectomorphisms, and can thus be identified locally with the Poisson algebra of smooth functions. If $z, \xi$ are local Darboux coordinates on the cylinder $S^1 \times \mathbb{R}$, then a basis for the functions is given by $\{z^n \xi^s\}$, where $n \in \mathbb{Z}$ and $s \in \mathbb{N}$. They obey a Lie algebra defined by the fundamental Poisson bracket $\{\xi, z\} = 1$. The algebra $w_\infty$ is the subalgebra generated by those $z^n \xi^s$ with $s \geq 1$. Being a limit $n \to \infty$ of $W_n$ we expect that it contains generators of all spins $s \geq 2$. This is indeed the case: if we let $W_n^{(s)}$ for $s \geq 2$ and $n \in \mathbb{Z}$ denotes the modes of a generator of spin $s$, and we map $W_n^{(s)} \mapsto -z^{n+1-s} \xi^{s-1}$, the induced mode algebra is such that $W^{(2)}$ satisfies a $\text{diff}(S^1)$ algebra under which $W^{(s)}$ is a primary field of spin $s$. This algebra, as it stands, admits a central extension only in the Virasoro sector. The inclusion of central charges in all spin sectors requires a modification of the algebra.

This was achieved in [9] where a deformation $W_\infty$ of $w_\infty$ was found by brute-force imposition of the Jacobi identities within a suitable Ansatz. With some hindsight, the construction of $W_\infty$ appears very naturally as simply the quantization of the functions on the cylinder. The cylinder is symplectomorphic to the cotangent bundle of the circle and can thus be interpreted as the phase space of a physical system whose configuration space is a circle. The Poisson algebra of functions are the classical observables, which upon quantization get mapped to the Lie algebra of differential operators on the circle which
has a unique nontrivial central extension [10] given recently by the Khesin–Kravchenko cocycle [11]. Therefore $W_\infty$ is a subalgebra of the Lie algebra of differential operators on the circle, the mapping being $W^{(s)}_n \mapsto -z^{n+1-s}\partial^{s-1}$ for $n \in \mathbb{Z}$ and $s \geq 1$. In other words, $W_\infty$ corresponds to those differential operators on the circle with no piece of order zero. Adding the zeroth order piece yields an extension of $W_\infty$ by a generator of spin 1. This algebra, which is clearly isomorphic to the algebra of differential operators on the circle, is called $W_{1+\infty}$ and was first constructed in [12]. Its identification with the Lie algebra of differential operators on the circle was carried out in [13] in a different context. This identification makes manifest a nested sequence of subalgebras of $W_{1+\infty}$:

$$W_{1+\infty} \supset W_\infty \supset W_{\infty-2} \supset \cdots \supset W_{\infty-N} \supset \cdots,$$

(1)

where $W_{\infty-N}$ is the algebra of differential operators of the form $\sum_{i>N} a_i \partial^i$. In terms of $W_{1+\infty}$ they correspond to successive truncations from below: keeping only generators of spin $> N$. The central extension for all these algebras can be uniformly described [14] by the Khesin–Kravchenko cocycle and we will return to this point later.

A more conclusive interpretation of $W_{1+\infty}$ in terms of integrable systems was found in [15] where it was identified (albeit without central extension) with the first hamiltonian structure [16] of the KP [17] hierarchy.\footnote{More recently, $W_\infty$ (also without a central extension) has been identified as the algebra of additional symmetries of the KP hierarchy [18].} This suggests that one should try to recover the characteristic nonlinearity of $W$-algebras, which is absent in $W_{1+\infty}$ and its truncations, by looking for the analog in the KP hierarchy of the second Gel’fand–Dickey bracket. This was carried out in [19], rediscovering earlier work of Dickey [20], where we constructed $W_{KP}$: a nonlinear centerless deformation of $W_{1+\infty}$. Independently and at the same time, Wu and Yu constructed in [21] a centerless nonlinear deformation of $W_\infty$ called $\hat{W}_\infty$ and which can be seen as a reduction of $W_{KP}$. Considering the fact that the generalized KdV hierarchies are reductions of KP, it was hoped that $W_{KP}$ (or $\hat{W}_\infty$) would be universal for the $W_n$ algebras in the sense—made precise in [22]—that all $W_n$ algebras could be obtained from it by reduction. Nevertheless no $W_n$ has ever been constructed as a reduction of neither of these two algebras—although, as shown in [23], the classical limits of $W_n$ can all be obtained from the classical limit of $W_{KP}$ by reduction.
This result prompted us to investigate possible deformations of $W_{\text{KP}}$ in hopes of finding the universal $W$-algebra. In [24] we found a one-parameter family of hamiltonian structures for the KP hierarchy and, as a consequence, a one-parameter deformation $W^{(q)}_{\text{KP}}$ of $W_{\text{KP}}$. This one-parameter family relates all hitherto known $W_{\infty}$-type algebras (with or without central extensions) either as reductions, contractions, or truncations and, moreover, it also reduces (for $q = n$) to $W_n$. The purpose of this paper is to exhibit the topography of the $W_{\infty}$-landscape which has emerged as a consequence of the construction of this one-parameter family of $W$-algebras. We hope to convince the reader that $W^{(q)}_{\text{KP}}$ is a valuable and much-needed organizational tool in this field and to this effect we summarize diagrammatically throughout the paper the relations found between different $W_{\infty}$-type algebras.

We organize this paper as follows. In the next section we briefly review the construction of $W^{(q)}_{\text{KP}}$ in terms of an extension of the Adler map to the space of generalized pseudodifferential symbols, and its reduction to $\hat{W}^{(q)}_{\infty}$ by setting the generator of spin 1 to zero. We then go on to analyze the structure of $W^{(q)}_{\text{KP}}$ for $q = N$ a positive integer. We show that $W^{(N)}_{\text{KP}}$ (resp. $\hat{W}^{(N)}_{\infty}$) reduces to the Gel’fand–Dickey algebra $GD_N$ (resp. $W_N$). It also contracts to centerless versions of $W_{1+\infty}$, $W_{\infty}$, and $W_{\infty-N}$. We then show how to recover the centrally extended version of these algebras as particular contractions in which the limit $q \to N$ is approached at the same as the generators are rescaled. The expressions for the limiting Poisson structures involve the Khesin–Kravchenko cocycle. We also obtain a new nonlinear $W$-algebra $W^{\#}_{\infty}$ as a reduction of $W_{1+\infty}$ or, equivalently, as a contraction $q \to 0$ of $\hat{W}^{(q)}_{\infty}$. We then take a tour through the classical limits of these algebras. The $q$-dependence of $W^{(q)}_{\text{KP}}$ (resp. $\hat{W}^{(q)}_{\infty}$) becomes fictitious in this limit and for $q \neq 0$ they yield algebras equivalent to $w_{\text{KP}}$ (resp. $\hat{w}_{\infty}$). As contractions we obtain $w_{1+\infty}$, $w_{\infty}$, as well as its further truncations $w_{\infty-N}$. The classical limit of $W^{\#}_{\infty}$ is a nonlinear algebra inequivalent to $\hat{w}_{\infty}$. Finally, we discuss the covariance properties of $W_{\infty}$-type algebras. Every such algebra admits an action of the algebra of diffeomorphisms of the circle via derivations. For the case of $W_N$ it was proven in [25] that a primary basis for the algebra exists. It is a tacit assumption in the literature that this continues to be the case in $W_{\infty}$-type algebras—albeit at the price of introducing nonlinearities. However we find that this is not the case for $\hat{W}_{\infty}$. We conjecture that a primary basis for the algebra exists as long as the Virasoro central charge is nonzero.
The $\mathbb{W}_{\text{KP}}^{(q)}$ Algebra

All the $\mathbb{W}$-algebras that we will examine in this letter appear as Poisson structures on differential algebras generated by a collection $\{u_i\}$ of infinitely differentiable functions, say, on the circle. In other words, we work with the algebra of polynomials in the $\{u_i\}$ and their derivatives. A $\mathbb{W}$-algebra generated by the $\{u_i\}$ will then take the form of a bracket

$$\{u_i(z), u_j(w)\} = J_{ij}(z) \cdot \delta(z - w),$$

(2)

where $J_{ij}$ are some differential operators whose coefficients are differential polynomials in the $\{u_i\}$. For (2) to define a Poisson bracket we must require antisymmetry and the Jacobi identity. A very elegant way of generating such Poisson structures is via the Adler map in the space of one-dimensional pseudodifferential symbols. We shall construct $\mathbb{W}_{\text{KP}}^{(q)}$ by a slight generalization of this procedure.

By a pseudodifferential symbol we mean a formal Laurent series in a parameter $\xi^{-1}$ of the form $P(z, \xi) = \sum_{i=\text{finite}}^\infty p_i(z) \xi^i$ whose coefficients we take to be smooth functions on the circle. Symbols have a commutative multiplication given by multiplying the Laurent series; but one can also define a composition law (denoted by $\circ$)

$$P(z, \xi) \circ Q(z, \xi) = \sum_{k \geq 0} \frac{1}{k!} \frac{\partial^k P}{\partial \xi^k} \frac{\partial^k Q}{\partial z^k},$$

(3)

making the map

$$P(z, \xi) \mapsto \sum_i p_i(z) \partial^i$$

(4)

from pseudodifferential symbols to pseudodifferential operators into an algebra homomorphism.

Symbol composition is moreover a well-defined operation on arbitrary smooth functions of $z$ and $\xi$. For example, for $a = a(z)$,

$$\log \xi \circ a = a \log \xi - \sum_{j=1}^\infty \frac{(-1)^j}{j} a^{(j)} \xi^{-j},$$

(5)

which shows that the commutator (under symbol composition) with $\log \xi$, denoted by $\text{ad} \log \xi$, is an outer derivation on the algebra of pseudodifferential symbols. Similarly, if $q$ is any complex number, not necessarily an integer, we find

$$\xi^q \circ a = \sum_{j=0}^\infty \binom{q}{j} a^{(j)} \xi^{q-j},$$

(6)

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where we have introduced, for \( q \) any complex number, the generalized binomial coefficients \( \binom{q}{j} \equiv [q]_j/j! \), where \( [q]_j \equiv q(q-1) \cdots (q-j+1) \) is the Pochhammer symbol. Conjugation by \( \xi^q \) is therefore an outer automorphism of the algebra of pseudodifferential symbols, which is the integrated version of \( \text{ad} \log \xi \).

It follows from (6) that (left and right) multiplication by \( \xi^q \) sends pseudodifferential symbols into symbols of the form \( \sum_{j \leq N} p_j(z)\xi^{q+j} \). Let us denote the set of these symbols by \( \mathcal{S}_q \). It is clear that \( \mathcal{S}_q \) is a bimodule over the algebra of pseudodifferential symbols, which for \( q \in \mathbb{Z} \) coincides with the algebra itself. In fact, since \( \mathcal{S}_q = \mathcal{S}_p \) for \( p \equiv q \mod \mathbb{Z} \), we will understand \( \mathcal{S}_q \) from now on as implying that \( q \) is reduced modulo the integers. Moreover, symbol composition induces a multiplication \( \mathcal{S}_p \times \mathcal{S}_q \to \mathcal{S}_{p+q} \), where we add modulo the integers.

We call their union \( \mathcal{S} = \bigcup_q \mathcal{S}_q \), the algebra of generalized pseudodifferential symbols. Furthermore, the algebra \( \mathcal{S}_0 \) of pseudodifferential symbols splits into the direct sum of two subalgebras \( \mathcal{S}_0 = \mathcal{R}_+ \oplus \mathcal{R}_- \), corresponding to the differential and integral symbols respectively. We denote by \( A_\pm \) the projection of \( A \) onto \( \mathcal{R}_\pm \) along \( \mathcal{R}_\mp \).

The construction of \( \mathcal{W}_{\text{KP}}^{(q)} \) now runs as follows. Put the generators \( \{u_i\} \) into the following generalized pseudodifferential symbol:

\[
L = \alpha \xi^q + \sum_{i=1}^{\infty} u_i \xi^{q-i} \in \mathcal{S}_q ,
\]

(7)

where \( \alpha \) is an inessential parameter which we introduce for later convenience. Now let \( X = \sum_{i=1}^{\infty} \xi^{i-q-1} \circ x_i \in \mathcal{S}_{-q} \) and define the following generalization of the Adler map \[26\]:

\[
J^{(q)}(X) = \frac{1}{\alpha} [(L \circ X)_+ \circ L - L \circ (X \circ L)_+] = \frac{1}{\alpha} [L \circ (X \circ L)_- - (L \circ X)_- \circ L] .
\]

(8)

It is clear that \( J^{(q)}(X) \in \mathcal{S}_q \), is linear in \( X \) and has thus the general form \( J^{(q)}(X) = \sum_{i,j=1}^{\infty} (J^{(q)}_{ij} \cdot x_j) \xi^{q-i} \), for some differential operators \( J^{(q)}_{ij} \) defining, via (2), a \( \mathcal{W} \)-algebra generated by the \( \{u_i\} \). The explicit expression for the \( J^{(q)}_{ij} \) can be found in \[24\], from where one can read that this is a nonlinear centrally extended (for generic \( q \)) \( \mathcal{W} \)-algebra of the \( \mathcal{W}_\infty \)-type. Moreover, for \( q = 1 \) it agrees with \( \mathcal{W}_{\text{KP}} \)—hence it is a one-parameter deformation of it. Therefore \( \mathcal{W}_{\text{KP}}^{(q)} \) is a one-parameter family of nonlinear \( \mathcal{W} \)-algebras.
We can obtain a one-parameter deformation of $\hat{W}_\infty$ by reduction. For $q \neq 0$, the constraint $u_1 = 0$ is formally second-class and the Dirac brackets (denoted $\Omega_{ij}^{(q)}$) define a $W$-algebra on the remaining generators $\{u_{i>1}\}$; the generator of spin 2 obeying a Virasoro algebra

$$\Omega_{22}^{(q)} = \frac{\alpha}{12} q(q^2 - 1)\partial^3 + u_2 \partial + \partial u_2 .$$

For $q = 0$, $u_1$ decouples from the algebra and the remaining generators obey a centerless nonlinear $W$-algebra whose linear terms are simply those of $W_\infty$. We denote by $\hat{W}_\infty^{(q)}$ the $W$-algebra defined by $\Omega_{ij}^{(q \neq 0)}$ or by $J_{ij}^{(0)}$ when $q = 0$.

The Structure of $W_{\text{KP}}^{(q)}$ for $q \in \mathbb{N}$

We already saw that $W_{\text{KP}}^{(1)} \cong W_{\text{KP}}$ and also that $\hat{W}_\infty^{(1)} \cong \hat{W}_\infty$. We now investigate the structure of the algebras for $q = N$ a positive integer $\geq 2$. In this case it makes sense to impose the constraint $L_- = 0$. It follows from (8) that $L_- = 0 \Rightarrow J^{(N)}(X)_- = 0$ for all $X$, whence the $\{u_i\}_{i>N}$ generate an ideal $I_N^{-}$ of $W_{\text{KP}}^{(N)}$. Since $W$-algebras are nonlinear, this does not mean that these generators close among themselves: in the nonlinear terms they can appear multiplied by arbitrary generators and a closer look at the explicit form of the algebra shows that in fact they do. A closer look also reveals that there are no central terms in this ideal, whence the linear and the quadratic terms must obey the Jacobi equations separately—since they can both be obtained unperturbed as different contractions of the ideal. In particular, the linear terms define a Lie algebra isomorphic to $W_{\infty-N}^-$ without central extension. In [27], Depireux conjectured the existence of a nonlinear centerless deformation of $W_{\infty-N}^-$ which he calls $\hat{W}_{\infty-N}^-$. It may be that redefining the generators $u_i \mapsto \tilde{u}_i = u_i + p_i(u_{j<i})$, the $W$-subalgebra of $W_{\text{KP}}^{(N)}$ generated by $\{\tilde{u}_{i>N}\}$ could be equivalent to $\hat{W}_{\infty-N}^-$. It is however still an open problem to prove that the algebras conjectured in [27] actually exist.

Since $I_N^{-}$ is an ideal of $W_{\text{KP}}^{(N)}$, the quotient is an algebra. This is the $W$-algebra generated by the first $N$ generators or, equivalently, by $L$ a differential symbol. In other words, this is precisely the Gel’fand–Dickey algebra $GD_N$. The constraint $u_1 = 0$ is again formally second-class and upon reduction, we recover $W_N$.

We could equivalently first reduce to $\hat{W}_\infty^{(N)}$. The same arguments as before imply that $\{u_{i>N}\}$ generate a centerless ideal $\hat{I}_N^{-}$ which again can be shown not to close because of the nonlinear terms. However the linear terms are unchanged from those of $I_N^{-}$ and hence $\hat{I}_N^{-}$ also contracts to $W_{\infty-N}^c=\emptyset$.
For $q \in \mathbb{N}$ it also makes sense to deform the generalized Adler map to obtain the analog of the first Gel’fand–Dickey bracket. This bracket gives rise to a $\mathcal{W}$-algebra which can therefore be obtained as a contraction of $\mathcal{W}^{(N)}_{KP}$ as follows. Let us shift $L$ by a constant $L \mapsto L + \lambda$ and call the resulting Adler map $J^{(N)}_\lambda$. From (8) it follows that

$$J^{(N)}_\lambda(X) = J^{(N)}(X) + \lambda J^{(N)}_\infty(X), \quad (10)$$

where $J^{(N)}_\infty$ is given by

$$J^{(N)}_\infty(X) = [L_+, X_-]_+ - [L_-, X_+]_+. \quad (11)$$

This means that the deformed Poisson bracket contains a piece linear in $\lambda$. Since this shift corresponds to the change of variables $u_N \mapsto u_N + \lambda$ and the Jacobi identity holds for arbitrary $\{u_j\}$, it means that for all $\lambda$ the new bracket satisfies the Jacobi identity. This being a quadratic identity, it means that it will contain pieces of orders 0, 1, and 2 in $\lambda$ which vanish separately. In particular, the terms quadratic in $\lambda$ are the Jacobi identities for the bracket defined by $J^{(N)}_\infty$. It follows from (11) that the resulting $\mathcal{W}$-algebra breaks up as a direct sum of commuting subalgebras generated by the coefficients of the differential and integral parts of $L$, respectively. The $\mathcal{W}$-algebra generated by the coefficients of the differential part is simply the first Gel’fand–Dickey bracket $GD^{(1)}_N$, whereas the one generated by the coefficients of the integral part is precisely $\mathcal{W}_{1+\infty}$ without central extension. For all $\lambda$ the $\mathcal{W}$-algebra defined by $J^{(N)}_\lambda$ is isomorphic to $\mathcal{W}^{(N)}_{KP}$ and, since $J^{(N)}_\infty = \lim_{\lambda \to \infty} \lambda^{-1} J^{(N)}_\lambda$, we deduce that $GD^{(1)}_N \times \mathcal{W}_{1+\infty}$ is a contraction of $\mathcal{W}^{(N)}_{KP}$.

For $N = 1$ this result is simply the statement that the first Hamiltonian structure for the KP hierarchy is isomorphic to a centerless $\mathcal{W}_{1+\infty}$ algebra. For $N > 1$, $\mathcal{W}^{(N)}_{KP}$ agrees with the family of Hamiltonian structures defined by Radul in [28] (see also [29]) and their contraction are simply the Watanabe structures studied in [16].

We can summarize graphically our results for this section in the following commutative diagram of $\mathcal{W}$-algebras:

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2 Strictly speaking $I_N$ and $\hat{I}_N$ are not $\mathcal{W}$-algebras but we include them nevertheless for completeness.
where the horizontal reductions all correspond to setting $u_1 = 0$ and the horizontal contractions correspond to the deformation of the Adler map induced by the shift in $u_N$.

**Some Contractions of $W^{(q)}_{KP}$**

The algebras $W_{1+\infty}$, $W_{\infty}$, and its truncations, obtained by contracting $W^{(N)}_{KP}$ are all centerless. In order to obtain algebras with central extensions it is necessary to perform the contraction at the same time as the limit $q \to N$ is approached. For example, if we take the limit $\alpha \to \infty$ and $q \to 0$ in such a way that $q\alpha = c$ remains constant, an easy calculation reveals the following expression for the limiting Poisson structure

\[
\lim_{q \to 0} J^{(q)}(\xi q \circ X)\xi^{-q} = - [c \log \xi + L_-, X_+]_-, \tag{12}
\]

where $L$ is evaluated at $q = 0$ in the right-hand-side of the equation. The $c$-independent term agrees with the similar term in (11) and as discussed in the previous section gives rise to a centerless $W_{1+\infty}$. The $c$-dependent piece, however, is nothing but the outer derivation on the algebra of pseudodifferential symbols giving rise to the Khesin–Kravchenko 2-cocycle $c_{KK}(P, Q) = \text{Tr} [\log \xi , P] \circ Q$. As shown in [14] this gives precisely the central terms in $W_{1+\infty}$. We let $W_{c1+\infty}$ denote the central extension of $W_{1+\infty}$ given by the Khesin–Kravchenko cocycle.

Suppose we now take $q \to 1$ in such a way that $\alpha(q - 1) = c$ remains constant. Inspection reveals that $J^{(q)}_{11} (q)$ diverges, whence it is necessary to impose the constraint $u_1 = 0$, reducing to $W^{(q)}_{\infty}$. An easy calculation reveals that the limiting Poisson structure is exactly the one given above but where $u_1$ does not appear. Therefore it corresponds to the truncation of $W_{c1+\infty}$ obtained by taking $\{u_{i>1}\}$ or, in other words, $W_{c\infty}$.
This fact generalizes. If we take the limit \( q \to N \) and \( \alpha \to \infty \) such that \( \alpha(q - N) = c \), we find that the central terms in \( W^{(q)}_{\text{KP}} \) for \( i, j \leq N \) all diverge in the limit. We must therefore reduce the algebra by setting them to zero. The resulting algebra—denoted tentatively\(^3\) by \( \hat{W}^{(q)}_{\infty-N} \) is nonlocal for all values of \( q \). However in the limit, the nonlocal (as well as the nonlinear) terms all vanish and we are left with the following expression for the Poisson structure (here \( \varepsilon = q - N \)):

\[
\lim_{\varepsilon \to 0} J^{(q)}(\xi^\varepsilon \circ X)\xi^{-\varepsilon} = - \left( \left[ c \log \xi + (L\xi^{-N})_+ (\xi^N \circ X)_+ \right] \xi^N \right)_-, \tag{13}
\]

where \( L \) in the right hand-side is evaluated at \( q = N \). This Poisson structure is identical to (12) except that only the \( \{u_{i>N}\} \) occur. In other words, it is the subalgebra of \( W^{c}_{1+\infty} \) generated by the \( \{u_{i>N}\} \)—namely, \( W^{c}_{\infty-N} \).

Imposing the constraint \( u_1 = 0 \) on \( W^{c\neq 0}_{1+\infty} \) we obtain a new nonlinear algebra we call \( W^{\#}_{\infty} \). It follows that \( W^{\#}_{\infty} \) is the contraction of \( \hat{W}^{(q)}_{\infty} \) as \( q \to 0 \) and \( \alpha q = c \). One can show that this is a genuinely nonlinear algebra in that there exists no polynomial field redefinition which linearizes it. Moreover it can be shown by inspection of the first few Poisson brackets not to be equivalent to \( \hat{W}^{(q)}_{\infty} \) for any \( q \).

We can again summarize these results diagrammatically:

\[
\begin{array}{c}
\text{\( W^{(q)}_{\text{KP}} \)} \quad \text{\( \hat{W}^{(q)}_{\infty-N} \)} \quad \text{\( W^{c}_{\infty-N} \)} \\
\downarrow \text{reduction} \quad \downarrow \text{reduction} \quad \downarrow \text{truncation} \\
\quad \text{\( W^{(q)}_{\text{KP}} \)} \quad \text{\( \hat{W}^{(q)}_{\infty} \)} \quad \text{\( W^{c}_{\infty} \)} \\
\downarrow \text{reduction} \quad \downarrow \text{reduction} \quad \downarrow \text{contraction} \quad \text{\( q \to 0 \)} \quad \text{\( q \to 0 \)} \\
\quad \text{\( W^{(q)}_{\text{KP}} \)} \quad \text{\( \hat{W}^{(q)}_{\infty} \)} \quad \text{\( W^{c}_{\infty} \)} \\
\downarrow \text{contraction} \quad \downarrow \text{contraction} \quad \downarrow \text{contraction} \quad \text{\( q \to 1 \)} \\
\quad \text{\( W^{(q)}_{\text{KP}} \)} \quad \text{\( \hat{W}^{(q)}_{\infty} \)} \quad \text{\( W^{c}_{\infty} \)} \\
\end{array}
\]

where the horizontal arrows labelled reductions have the same meaning as in the previous diagram.

\(^3\) Our choice of notation notwithstanding, we don’t imply any relation between these nonlocal algebras and the ones conjectured in \([27]\) besides the fact that they are both deformations of \( W_{\infty-N} \).
A Classical $W_\infty$ Tour

As shown in [23], one can obtain classical limits of these Poisson structure by taking the commutative limit of the algebra of pseudodifferential symbols. Rescaling $\xi$, we can introduce in (3) a formal parameter $\hbar$ as follows:

$$
P \circ Q = \sum_{k \geq 0} \frac{\hbar^k}{k!} \frac{\partial^k P}{\partial \xi^k} \frac{\partial^k Q}{\partial z^k}.
$$

The classical limit of any structure is obtained by introducing the parameter $\hbar$ via (14) and keeping only the lowest term in its $\hbar$ expansion. For example, the classical limit of $\circ$ is simply the commutative multiplication of symbols. Analogously, the classical limit of the commutator is given by

$$
\left[ P, Q \right] = \lim_{\hbar \to 0} \hbar^{-1} \left[ P, Q \right] = \frac{\partial P}{\partial \xi} \frac{\partial Q}{\partial z} - \frac{\partial P}{\partial z} \frac{\partial Q}{\partial \xi},
$$

which we recognize as the Poisson bracket relative to a two-dimensional Darboux chart $z, \xi$. The classical limit of the generalized Adler map (8) is thus given by

$$
J_{ij}^{(q)}(X) = [L, X]_+ + L - [L, (XL)_+] = [L, (XL)_-] - [L, X]_+ - L.
$$

Notice that these consist of the terms in $J_{ij}^{(q)}$ with exactly one derivative.

It was shown in [23] that the dependence in $q$ in the classical limit of $W_{KP}^{(q=N)}$ is fictitious and can be eliminated by a polynomial redefinition of variables. In [24] it was shown that this result persists for $q \neq 0$. In particular, this shows that the classical limit of $W_{KP}^{(q)}$ is $w_{KP}$ for all $q \neq 0$ where this algebra is defined in [23]. Analogously, and after the $u_1 = 0$ reduction, the classical limit of $\hat{W}_{\infty}^{(q)}$ is independent of $q$ and yields the reduction of $w_{KP}$ denoted [21] $\hat{w}_{\infty}$.

Further contracting these algebras eliminates the nonlinear terms and we recover $w_{1+\infty}$ and $w_{\infty}$—the classical limits of $W_{1+\infty}$ and $W_{\infty}$, respectively. It is important to remark that $w_{1+\infty}$ appears without the central term in the Poisson bracket of $u_1$. In order to recover this central term we have to perform the contraction more carefully. We reintroduce the parameter $q$ in $w_{KP}$, which in the classical limit is inessential, and we take $q \to 0$ with $\alpha \to \infty$ in such a way that their product remains constant. In this limit we obtain $w_{1+\infty}^{(q)}$.

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4 We use $[,]$ to denote the Poisson bracket to avoid confusion with the Poisson bracket defining the $W$-algebras.
In $w_{1+\infty}^c$ we can impose the constraint $u_1 = 0$ which is formally second-class for $c \neq 0$. The resulting algebra is the classical limit $w_{\infty}^\#$ of $W_{\infty}^\#$. The nonlinearity still persists, but it can be shown that this algebra is not equivalent to $\hat{w}_\infty$, as its spectrum would suggest.

The following commutative diagram of $W$-algebras summarizes some of these correspondences.

The Spectrum of $W_{\infty}$-type Algebras

We end the paper with some comments on the spectrum of $W_{\infty}$-type algebras—that is, how the generators transform under the action of $\text{diff}(S^1)$. In the introduction, the definition of what we mean by $W_{\infty}$-type algebras was purposefully vague. In the literature, the accepted working definition is that of $W$-algebra with one generator for each spin $s \geq 2$ (and possibly also $s = 1$). This usually connotes the existence of a Virasoro (or at least a $\text{diff}(S^1)$) subalgebra and a choice of generators which transform tensorially under the action of $\text{diff}(S^1)$ induced by this subalgebra. In $W_n$—the prototypical example of a $W$-algebra—this is indeed the case, as it is also in its limit algebra $w_{\infty}$. In both cases, the generator of lowest spin $u_2$ satisfies a Virasoro (for $W_n$) or $\text{diff}(S^1)$ (for $w_{\infty}$) algebra and one can define an action of $\text{diff}(S^1)$ as follows $\delta_\varepsilon u_j = J_{j,2} \cdot \varepsilon$. Under this action, the transformation laws of the $W_n$ generators contain terms with more than one derivative acting on $\varepsilon$—whence the generators are not tensorial. Nevertheless, as shown in [25], there is a differential algebra automorphism $u_j \mapsto w_j$ such that for $j > 2$, $w_j$ is a tensor of spin $j$. The proof of this statement exploited the $\text{diff}(S^1)$-covariance of the Lax operator (the differential operator to which $L$ gets sent by (4)) as well as the fact that $u_2$ transforms as a projective connection and can thus be put to zero in a projective chart. For $w_{\infty}$—as for any classical limit of a $W$-algebra—the generators are automatically tensorial since the classical limit of $J_{ij}$ are differential operators of order 1 and therefore in $\delta_\varepsilon u_j$ there cannot appear terms with more than one derivative acting on $\varepsilon$. 

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It is tacitly assumed in the literature that this paradisiacal situation persists for $\mathcal{W}_\infty$-type algebras—albeit at the price of introducing nonlinearities (see footnote in [13]). To the best of our knowledge this has not been proven in any $\mathcal{W}_\infty$-type algebra (except for the classical limits in which there is nothing to prove) and, in fact, it is not even true for $\hat{\mathcal{W}}_\infty$. It is conceivable, however, that if the two properties exploited in [25] for the $\mathcal{W}_n$ case still hold, one could prove the similar result: the main difficulty being now the fact that we are not dealing with differential operators, but with pseudodifferential ones instead. As shown, for example, in [25], the form of the Adler map and the fact that the action of $\text{diff}(S^1)$ is hamiltonian relative to it implies the covariance of the Lax operator. It is therefore natural to conjecture that, for nonvanishing Virasoro central charge, a primary basis exists.

For simplicity, we shall restrict ourselves to discussing those $\mathcal{W}_\infty$-type algebras without the spin 1 generator—namely $\hat{\mathcal{W}}^{(q)}_\infty$, $\mathcal{W}^\#_\infty$, and $\mathcal{W}^c_\infty$. Since $\mathcal{W}^\#_\infty$, $\mathcal{W}^c_\infty$, both arise as contractions of $\hat{\mathcal{W}}^{(q)}_\infty$ for special values of the parameter, one can argue that to prove existence of a primary basis for any of these algebras it is enough to find a primary basis for $\hat{\mathcal{W}}^{(q)}_\infty$ for generic $q$. Treating $q$ as a formal parameter one finds by inspection of the transformation laws under $\text{diff}(S^1)$, that the first few fields are indeed redefinable into primaries. These redefinitions, however, involve adding to the fields differential polynomials whose coefficients are rational functions of $q$—the dependence on $q$ of the denominator occurring only through the Virasoro central charge in (9). As a result, for those values of $q$ for which the central charge vanishes, namely $q = 0, \pm 1$, the redefinition becomes singular and no longer legal. This implies that $\hat{\mathcal{W}}_\infty$ (and also $\hat{\mathcal{W}}^{(0)}_\infty$ and $\hat{\mathcal{W}}^{(-1)}_\infty$) do not have a primary basis. For the contractions, however, things are different. In the contracting limits $q \to 0$ and $q \to 1$, the value of the central charge remains finite and thus in the limit the redefinitions remain nonsingular. Therefore a primary basis for $\hat{\mathcal{W}}^{(q)}_\infty$ would induce a primary basis on such a contraction. For $q = N$, it follows that both $\{u_{j\leq N}\}$ and $\{u_{j>N}\}$ generate $\text{diff}(S^1)$ submodules. It follows from [25] that the $\{u_{3\leq j\leq N}\}$ can be redefined into primaries. Moreover $u_{N+1}$ is always a primary and we have been able to redefine the first few higher $u_j$’s into primaries; although we have not yet elaborated a general proof. It is not clear to what extent covariance of the Lax operator plays a crucial role since, as exemplified by the case of $\hat{\mathcal{W}}_\infty$, when one deals with pseudodifferential operators, there are more ways to make a covariant operator than out of tensors and covariant derivatives.
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