OPTIMAL CONTROL OF SECOND ORDER DELAY-DISCRETE AND DELAY-DIFFERENTIAL INCLUSIONS WITH STATE CONSTRAINTS

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Abstract. The present paper studies a new class of problems of optimal control theory with state constraints and second order delay-discrete (DSIs) and delay-differential inclusions (DFIs). The basic approach to solving this problem is based on the discretization method. Thus under the regularity condition the necessary and sufficient conditions of optimality for problems with second order delay-discrete and delay-approximate DSIs are investigated. Then by using discrete approximations as a vehicle, in the forms of Euler-Lagrange and Hamiltonian type inclusions the sufficient conditions of optimality for delay-DFIs, including the peculiar transversality ones, are proved. Here our main idea is the use of equivalence relations for subdifferentials of Hamiltonian functions and locally adjoint mappings (LAMs), which allow us to make a bridge between the basic optimality conditions of second order delay-DSIs and delay-discrete-approximate problems. In particular, applications of these results to the second order semilinear optimal control problem are illustrated as well as the optimality conditions for non-delayed problems are derived.

1. Introduction. Discrete and continuous time processes with first order ordinary discrete-differential and partial differential inclusions found wide application in the field of mathematical economics and in problems of control dynamic system optimization and differential games (see [3], [6]-[11], [14]-[21], [23]-[29] and their references). The paper [7] deals with the variational convergence of a sequence of optimal control problems for functional differential state equations with deviating argument. Variational limit problems are found under various conditions of convergence of the input data. It is shown that, upon sufficiently weak assumptions on convergence of the argument deviations, the limit problem can assume a form different from that of the whole sequence. In particular, it can be either an optimal control problem for an integro-differential equation or a purely variational problem. Conditions are found under which the limit problem preserves the form of the original sequence. In the paper [25] are considered evolution inclusions driven by a time dependent subdifferential plus a multivalued perturbation. Are proved

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existence results for the convex and nonconvex valued perturbations, for extremal trajectories (solutions passing from the extreme points of the multivalued perturbation). Are also proved a strong relaxation theorem showing that each solution of the convex problem can be approximated in the supremum norm by extremal solutions. Finally, are presented some examples illustrating these results. The book [3] is concerned with the optimal convex control problem of Bolza in a Banach space. A distinctive feature is a strong emphasis on the connection between theory and application. The main emphasis is put on the characterization of optimal arcs as well as on the synthesis of optimal controllers. Necessary and sufficient conditions of optimality, generalizing the classical Euler-Lagrange equations, are obtained in Sect. 4.1 in terms of the subdifferential of the convex cost integrand. The abstract cases of distributed and boundary controls are treated separately. The paper [24] concerns constrained dynamic optimization problems governed by delay control systems whose dynamic constraints are described by both delay-differential inclusions and linear algebraic equations. The authors are not familiar with any results in these directions for such systems even in the delay-free case. In the first part of the paper are established the value convergence of discrete approximations as well as the strong convergence of optimal arcs in the classical Sobolev space $W^{1,2}$. Then using discrete approximations as a vehicle, are derived necessary optimality conditions for the initial continuous-time systems in both Euler-Lagrange and Hamiltonian forms via basic generalized differential constructions of variational analysis. In the paper [9] are examined functional differential inclusions with memory and state constraints. For the case of time-independent state constraints, are shown that the solution set is $R^3$ under Caratheodory conditions on the orientor field. For the case of time-dependent state constraints are proved two existence theorems. For this second case, the question of whether the solution set is $R^3$ remains open. In the work [11] dynamic optimization problems for differential inclusions on manifolds are considered. A mathematical framework for derivation of optimality conditions for generalized dynamical systems is proposed. By using metric regularity of terminal and dynamic constraints in form of generalized Euler-Lagrange relations and in form of partially convexified Hamiltonian inclusions are obtained optimality conditions. In [8] are provided intrinsic sufficient conditions on a multifunction $F$ and endpoint data $\varphi$ so that the value function associated to the Mayer problem is semiconcave. The paper [6] introduces a new class of variational problems for differential inclusions, motivated by the control of forest fires. The area burned by the fire at time $t > 0$ is modelled as the reachable set for a differential inclusion $x' \in F(x)$, starting from an initial set $R_0$. To block the fire, a wall can be constructed progressively in time, at a given speed. In this paper, is studied the possibility of constructing a wall which completely encircles the fire. Moreover, is derived necessary conditions for an optimal strategy, which minimizes the total area burned by the fire. In the paper [29] are used a new approach to obtain the existence of mild solutions and controllability results, avoiding hypotheses of compactness on the semigroup generated by the linear part and any conditions on the multivalued nonlinearity expressed in terms of measures of noncompactness. Finally, two examples are given to illustrate their theoretical results.

Recently, a number of authors have started investigating boundary value problems and controllability problems for second order DFIs. The problems accompanied with the second order discrete and differential inclusions are more complicated due to the second order derivatives and their discrete analogous. Thus, optimal
control problems with ordinary DSIs and DFIs are one of the area in mathematical theory of optimal processes being intensively developed. More specifically, we deal with similar problems with delay-differential and state constraints of Bolza type. Observe that such problems arise frequently not only in mechanics, aerospace engineering, management sciences, and economics, but also in problems of automatic control, aviovibration, burning in rocket motors, and biophysics (see [15], [23], [24] and references therein).

For second order differential inclusions, the existence of solutions and other qualitative properties has been intensively analyzed in the recent literature (see [2], [12]; [28] and their references).

In paper [2] in a separable Banach space a three point boundary value problem for a second order DFI of the form $x''(t) \in F(t, x(t), x'(t))$, a.e. $t \in [0, 1], x(0) = 0; x(\theta) = x(1)$ are considered. The existence of solutions, when the set-valued mapping $F$ is unbounded-valued and satisfies a pseudo-Lipschitz property are investigated. Then, a Lipschitz case is derived and the associated relaxed problem is studied.

The paper [28] is concerned with the nonlinear boundary value problems for a class of semilinear second order DFIs. Using the tools involving topological transformation and fixed points of the set-valued map, some existence theorems of solutions in the convex case are given.

In the paper [12], second order DFIs with a maximal monotone term and generalized boundary conditions are studied. The nonlinear differential operator need not be necessary homogeneous and incorporates as a special case the one-dimensional $p$-Laplacian. The generalized boundary conditions incorporate as special cases well-known problems such as the Dirichlet (Picard), Neumann and periodic problems. As application to the proven results existence theorems for both “convex” and “non-convex” problems are obtained.

In the second half of the 20th century, many mathematicians in Russia made great contributions to the field of optimal control theory (see [1], [4], [5], [13] and references therein) In the paper [1] using Linear LyapunovKrotov functions are obtained sufficient conditions for strong and global minima for the classical smooth problem of optimal control. In the paper [5] sufficient optimality conditions are proved in the form of a maximum principle for the time-optimal problem of transfer from a set $M_0$ into a set $M_1$, where an object’s behavior is described by the first order differential inclusion $x \in F(t, x)$. It is shown that state constraints may be active. This means that the adjoint function may have points of discontinuity or jumps. In the work [13] sufficient conditions of optimality of systems described by a finite number of common differential equations are derived. On the basis of these conditions a single formalism of solution of variational problems for such systems is constructed and within its boundaries different methods are considered which permit to solve the variational problems completely.

As is pointed out in [26], [27], [30], boundary value problems (BVPs) for higher order differential equations play a very important role in both theory and applications. In recent years, BVPs for second order differential equations have been extensively studied. In particular, fourth order linear differential equations [27] subject to some boundary conditions arise in the mathematical description of some physical systems. For example, mathematical models of deflection of beams [26], [27].
Optimization of higher order differential inclusions was first developed by Mahmudov in [20]-[22]. The paper [20] is mainly concerned with the sufficient conditions of optimality for Cauchy problem (with fixed initial and free endpoint constraints) of third-order DFIs. Some special transversality conditions, which are peculiar to problems including third order derivatives are formulated. It is worthwhile to highlight that optimization problem with higher order (say \( m \)-th-order) DFIs sometimes has its own importance for every \( m \) in the theoretical and practical point of view.

The paper [21] is devoted to a second order polyhedral optimization described by ordinary DSIs and DFIs. The stated second order discrete problem is reduced to the polyhedral minimization problem with polyhedral geometric constraints and in terms of the polyhedral Euler-Lagrange inclusions, necessary and sufficient conditions of optimality are derived. Derivation of the sufficient conditions for the second order polyhedral DFIs is based on the discrete-approximation method.

The paper [29] deals with a Bolza problem of optimal control theory given by second order convex differential inclusions with second order state variable inequality constraints. Necessary and sufficient conditions of optimality including distinctive “transversality” condition are proved in the form of Euler-Lagrange inclusions. Construction of Euler-Lagrange type adjoint inclusions is based on the presence of equivalence relations of locally adjoint mappings.

In our present paper, we discuss a special kind of optimization problem with second order delay-DFIs and delay DFIs in which the constraints are defined by set-valued mappings. To the best of our best knowledge, there is no paper which considers an optimality conditions for these problems. We try to fill this gap in the literature in this paper. In fact, the difficulty in the problems with higher order DFIs is rather to construct the Mahmudov’s higher order adjoint inclusions and the suitable transversality conditions.

The stated problems and obtained optimality conditions in our paper are new. We pursue a twofold goal: to study optimality conditions for delay-DSIs of control systems with respect to discrete approximations and to derive sufficient optimality conditions for second order delay-DFIs. We are not familiar with any results in these directions for such systems even in the nondelay case. The paper conditionally can be divided into four parts and is structured as follows;

In the first part of the paper, an optimal control problem in which the system dynamics are described by a so-called second order delay-DSIs are investigated. In the second part, for transition from problem with delay-DSIs to the problem with delay-discrete-approximate problem the idea of equivalence of LAMs is central. Third part of the paper is devoted to the construction of discrete-approximation problem for second order delay-DFIs. In the fourth part of the paper, optimization of second order delay-DFIs is considered and on the basic results for second order delay-DSIs sufficient conditions of optimality for delay-DFIs are proved.

Thus, in Section 2 for the readers convenience from the monograph of Mahmudov [15] and papers [17]-[20] the necessary notions and results such as LAM properties in finite dimensional Euclidean spaces, Hamiltonian functions, argmaximum sets, locally tents, and set-valued mappings are given, etc. Then the problems for second order delay-DSIs and delay-DFIs are formulated.

In Section 3 the optimality problem for posed second order DSIs are reduced to the mathematical programming problem with finite number of geometric constraints. By constructions of convex analysis under the “regularity” condition necessary and sufficient conditions of optimality for these problems are proved. By
using separation theorems of convex analysis it is shown that in terms of Hamiltonian functions these optimality conditions can be rewritten in a more symmetrical form.

In Section 4 equivalence of LAMs for delay discrete \((PD_h)\) and delay discrete-approximate \((PDA_h)\) problems are proven; both in term of classical subdifferential calculation of composition functions and locally tents are established connection between subdifferential of Hamiltonian functions and LAMs of discrete and discrete-approximate problems.

In Section 5 by using first and second order difference operators with the problem for second order delay-DFIs and state constraints we associate the second order delay-discrete-approximation problem. Here we use the equivalence results of LAMs from Section 4 connecting the main results of \((PD_h)\) and \((PDA_h)\) problems.

In Section 6 we derive sufficient conditions of optimality for second order delay-DFIs. Formulation of these conditions is based on formal limiting procedure in the optimality conditions for second order delay-DSIs. Thus, employing LAM and the discrete approximations method, in the Mahmudov form we derive sufficient optimality conditions for second order delay-DFIs. In fact, by using the functional analysis approach in the convex problems-Arzela-Ascoli type theorem for compactness in corresponding function spaces, uniformly convergence and another functional analysis approaches, substantiate passing the limit it can be justify necessity of these conditions for optimality. But to prove necessary conditions is rather difficult and are separate subject of discussion and omitted.

2. Preliminary studies and problem statement. In this section we recall the key notions of set-valued mappings from the book [15]; let \(\mathbb{R}^n\) be a \(n\)-dimensional Euclidean space, \((x, v)\) be an inner product of elements \(x, v \in \mathbb{R}^n\), \((x, v)\) be a pair of \(x, v\). Lets suppose that \(F: \mathbb{R}^{3n} \Rightarrow \mathbb{R}^n\) is a set-valued mapping from \(\mathbb{R}^{3n}\) into the set of subsets of \(\mathbb{R}^n\). Therefore \(\mathbb{R}^{3n} \Rightarrow \mathbb{R}^n\) is a convex set-valued mapping, if its graph \(gphF = \{(x, u_1, u_2, v) : v \in F(x, u_1, u_2)\}\) is a convex subset of \(\mathbb{R}^{3n}\).

A set-valued mapping \(F\) is called closed if its \(gphF\) is a closed subset in \(\mathbb{R}^{4n}\). The domain of a set-valued mapping \(F\) is denoted by \(domF\) and is defined as \(domF = \{(x, u_1, u_2) : F(x, u_1, u_2) \neq \emptyset\}\). A set-valued mapping \(F\) is convex-valued if \(F(x, u_1, u_2)\) is a convex set for each \((x, u_1, u_2) \in domF\).

A set-valued mapping \(F: \mathbb{R}^{3n} \Rightarrow \mathbb{R}^n\) is said to be upper semicontinuous at \((x^0, u^0_1, u^0_2)\) if for any neighbourhood \(U\) of zero in \(\mathbb{R}^n\) there exists a neighborhood \(V\) of zero in \(\mathbb{R}^{3n}\) such that
\[
F(x, u_1, u_2, \cdot) \subseteq F(x^0, u^0_1, u^0_2) + U, \forall (x, u_1, u_2) \in (x^0, u^0_1, u^0_2) + V.
\]

The Hamiltonian function and argmaximum set corresponding to a set-valued mapping \(F\) are defined by the following relations
\[
H_f(x, u_1, u_2, v^*) = \sup_{v} \{\langle v, v^* \rangle : v \in F(x, u_1, u_2)\}, \quad v^* \in \mathbb{R}, \quad F_{Arg}(x, u_1, u_2; v^*)
\]
\[
= \{v \in F(x, u_1, u_2) : \langle v, v^* \rangle = H_F(x, u_1, u_2, v^*)\},
\]
respectively. For a convex \(F\) we put \(H_F(x, u_1, u_2, v^*) = -\infty\) if \(F(x, u_1, u_2) = \emptyset\). In other terms, \(H_F(x, u_1, u_2, v^*)\) is the support function to the set \(F(x, u_1, u_2)\), evaluated at \(v^*\).

As usual, \(intQ\) denotes the interior of the set \(Q \subset \mathbb{R}^{4n}\) and \(riQ\) denotes the relative interior of a set \(Q\), i.e. the set of interior points of \(Q\) with respect to its affine hull \(Aff Q\). The closure of \(Q\) is denoted by \(clQ\).
A convex cone \( K_Q(x_0), z_0 = (x^0_0, u^0_0, w^0_0, v^0_0) \) is called a cone of tangent directions at a point \( x_0 \in Q \) to the set \( Q \) if from \( \bar{z} = (\bar{x}, \bar{u}_1, \bar{u}_2, \bar{v}) \in K_Q(z_0) \) it follows that \( \bar{z} \) is a tangent vector to the set \( Q \) at a point \( z_0 \in Q \), i.e., there exists such function \( q(\alpha) \in \mathbb{R}^{4n} \) that \( z_0 + \alpha \bar{z} + q(\alpha) \in Q \) for sufficiently small \( \alpha > 0 \) and \( \alpha^{-1}q(\alpha) \to 0 \), as \( \alpha \downarrow 0 \).

For a convex set-valued mapping \( F : \mathbb{R}^{3n} \rightrightarrows \mathbb{R}^n \) a set-valued mapping defined by \( F^*(r) : \mathbb{R}^{3n} \rightrightarrows \mathbb{R}^n \) as follows:

\[
F^*(r; (x, u_1, u_2, v)) = \{ (x^*, u_1^*, u_2^*) : (x^*, u_1^*, u_2^*, -v^*) \in K^*_F(x, u_1, u_2, v) \}
\]

\[
F^*_r(x, u_1, u_2; v) = \text{cone}\left[ gphF - (x, u_1, u_2, v) \right], \quad \forall (x^1, u^1_1, u^1_2, v^1) \in gphF,
\]

is called the LAM to \( F \) at a point \( (x, u_1, u_2, v) \in gphF \), where \( K^* = \{ z^* : \langle \bar{z}, z^* \rangle \geq 0, \forall \bar{z} \in K \} \) denotes the dual cone to the cone \( K \), as usual.

Below we will define the LAM to a set-valued mapping \( F \) by using the Hamiltonian function, associated to \( F \). Thus, the LAM to nonconvex mapping \( F \) is defined as follows:

\[
\begin{align*}
\n & F^* (r; (x, u_1, u_2, v)) = \{ (x^*, u_1^*, u_2^*) : H_F(x^1, u^1_1, u^1_2, v^*) \} \\
\geq & \{ (x^*, x^1 - x) + (u^1_1, u^1_1 - 1) + (u^1_2, u^1_2 - 2) \}, \quad \forall (x^1, u^1_1, u^1_2) \in \mathbb{R}^{3n},
\end{align*}
\]

Clearly, for the convex mapping \( F \) the Hamiltonian function \( H_F(\cdot, \cdot, v^*) \) is concave and the latter definition of LAM coincide with the previous definition of LAM (Theorem 2.1 [15]).

Note that prior to the LAM the notion of coderivative has been introduced for set-valued mappings in terms of the basic normal cone to their graphs by Mordukhovich [23] (however, for the smooth and convex maps the two notions are equivalent).

In the most interesting settings for the theory and applications, coderivatives are nonconvex-valued and hence are not tangentially /derivatively generated. This is the case of the first coderivative for general finite dimensional set-valued mappings for the purpose of applications to optimal control.

We have already seen that the cone of tangent directions involve directions for each of which there exists a function \( q(\alpha) \). But in order to predetermine properties of the set, \( Q \), this is not sufficient. Nevertheless, the following notion of a local tent allow us to predetermine mapping in \( Q \) for nearest tangent directions among themselves.

**Definition 2.1.** A cone of tangent directions \( K_Q(z_0) \) is called local tent if for any \( \bar{z}_0 \in riK_Q(z_0) \) there exists a convex cone \( K \subseteq K_Q(z_0) \) and a continuous function \( \gamma(\cdot) \) defined in the neighborhood of the origin, such that

1. \( \bar{z}_0 \in riK, \quad LinK = LinK_Q(z_0) \), where LinK is the linear span of \( K \),
2. \( \gamma(z) = \bar{z} + r(\bar{z}), \quad r(\bar{z}) \| \bar{z} \|^{-1} \to 0 \) as \( \bar{z} \to 0 \),
3. \( \bar{z}_0 + \gamma(z) \in A, \quad \bar{z} \in K \cap S_\varepsilon(0) \) for some \( \varepsilon > 0 \), where \( S_\varepsilon(0) \) is the ball of radius \( \varepsilon \).

**Definition 2.2.** With respect to [12] \( h(\bar{x}, x) \) is called a convex upper approximation (CUA) of the function \( g : \mathbb{R} \to \mathbb{R}^1 \{ \pm \infty \} \) at a point \( x \in dom g = \{ x : g(x) < +\infty \} \) if \( h(\bar{x}, x) \geq V(\bar{x}, x) \) for all \( \bar{x} \neq 0 \) and \( h(\cdot, x) \) is a convex closed positive homogeneous function, where
Definition 2.3. A set defined as follows

\[ g(x) = \sup_{\alpha \downarrow 0} \frac{1}{\alpha} [g(x + \alpha \bar{x} + r(\alpha))], \quad \alpha^{-1}r(\alpha) \to 0. \]

Here the exterior supremum is taken on all \( r(\alpha) \) such that \( \alpha^{-1}r(\alpha) \to 0 \) as \( \alpha \downarrow 0 \).

The main advantage of this definition is its simplicity. It is known [11] that in the convexity of the given definition coincides with the usual definition of a subdifferential. For various classes of functions, the notion of subdifferential can be defined in different ways and the reader can consult Mordukhovich [23] for related material; several useful subdifferentials of nonsmooth functions such as Clarke’s, Mordukhovich’s and Ioffe’s subdifferentials etc. consist of a main class and additional material; several useful subdifferentials of nonsmooth functions such as Clarke’s, Mordukhovich’s and Ioffe’s subdifferentials etc. consist of a main class of generalized differentials and play a vital role in pure and applied analysis.

A function \( g \) is called a proper function if it does not assume the value \(-\infty\) and is not identically equal to \(+\infty\).

In the first part of the paper is considered the following labelled by \((PD_h)\) optimization problem with delay-DSIs and state constraints:

\[
\begin{align*}
\text{minimize} & \quad \sum_{t=1}^{N-1} g(x_t, t) \\
\text{subject to} & \quad x_{t+2} \in F(x_t, x_{t+1}, x_{t-h}, t), \ t_0, 1, \ldots, N-2, \\
& \quad x_t = \xi_t, \ t = -h, -h+1, \ldots, -1; \ x_0 = \theta_0, \\
& \quad x_t \in \Omega_t, \ t = 1, \ldots, N; \ x_N \in P_N
\end{align*}
\]

where \( g(\cdot, t) \) is a real-valued function, \( g(\cdot, t) : \mathbb{R}^n \to \mathbb{R}^1 \cup \{\pm \infty\} \), \( F(\cdot, t) \) is a time dependent set-valued mapping; \( F(\cdot, t) : \mathbb{R}^{3n} \rightrightarrows \mathbb{R}^n \), \( h > 0 \) are fixed natural numbers, \( \theta_0, \xi_1, \ldots, \xi_t = -h, -h+1, \ldots, -1 \), are fixed vectors, \( \Omega_t \subseteq \mathbb{R}^n, t = 1, \ldots, N; P_N \subseteq \mathbb{R}^n \). A sequence \( \{x_t\}_{t=-h}^N = \{x_t : t = -h, -h+1, \ldots, N\} \) is called a feasible trajectory for the stated problem 1, 2.

In fact, a model of economic dynamics \((PD_h)\) described by discrete inclusions with constant delay is considered; the functioning of some economic system takes place at the discrete times \( t = 0, 1, \ldots, N \) and at time \( t \) one has a resource vector \((x, u_1, u_2) \in \mathbb{R}^3 \) which can be transformed at time \( t + 1 \) to one of the vectors \( v \in F(x, u_1, u_2, t) \). Here is assumed that all possible amounts of resources \((x_t, x_{t+1}, x_{t-h}, t = 0, \ldots, N) \) are connected by \( x_{t+2} \in F(x_t, x_{t+1}, x_{t-h}, t), t = 0, \ldots, N-2; x_t \in \Omega_t, t = 0, \ldots, N; x_N \in P_N \), where \( x_t = \xi_t, x_0 = \theta_0 \), are the vectors of initial resources. Usually, the sum \( \sum_{t=0}^{N-1} g(x_t, t) \) can be interpreted as the total expenditure.

For most of this paper we consider optimization problems with second order delay-DFIs and state constraints of the form, labelled as \((PC_h)\):

\[
\begin{align*}
\text{minimize} & \quad J[x(\cdot)] = \int_0^1 g(x(t), t)dt + \varphi_0(x(1)), \\
\text{subject to} & \quad x''(t) \in F(x(t), x'(t), x(t-h), t), \text{ a.e. } t \in [0, 1], \\
& \quad x(t) = \xi(t), \ t \in [-h, 0], \ x(0) = \theta, \\
& \quad x(t) \in \Omega(t), \ t \in [0, 1], \ x(1) \in P,
\end{align*}
\]
where $F(\cdot, t) : \mathbb{R}^{3n} \rightarrow \mathbb{R}^n$ and $g(\cdot, t)$, $\varphi_0$ are time dependent set-valued mapping and continuous proper functions, respectively, $P \subseteq \mathbb{R}^n$, $\xi(t), t \in [-h, 0)$ is an absolutely continuous initial function, $\theta$ is a fixed vector, $\Omega : [0, 1] \rightarrow \mathbb{R}^n$ is a set-valued mapping. It is required to find a feasible trajectory (arc) $x(t), t \in [-h, 1]$ minimizing the Bolza functional $J[x(t)]$ over a set of feasible trajectories. Here, a feasible trajectory $x(t), t \in [-h, 1]$ satisfies everywhere state constraints in $[0, 1]$, endpoint constraint $x(1) \in P$, almost everywhere (a.e.) the second order delay-DFI (with a possible jump discontinuity at $t = 0$), whose second order derivative in $[0, 1]$ belongs to the standard Lebesgue space $L^2_t([0, 1])$. In more detail, a feasible solution $x(\cdot)$ of $(PC_h)$ is a mapping $x(\cdot) : [-h, 1] \rightarrow \mathbb{R}^n$ satisfying $x''(t) \in F(x(t), x'(t), x(t-h), t)$, a.e. $t \in [0, 1], x(t) = \xi(t)$ for all $t \in [-h, 0), x(t) \in \Omega(t)$ for all $t \in [0, 1], x(1) \in P$ and $x(0) = \theta$ with $x(\cdot) \in AC([-h, 1]) \cap W^{1,2}_t([0, 1])$, where $AC([-h, 1])$ is a space of absolutely continuous functions from $[-h, 1]$ into $\mathbb{R}^n$ and $W^{1,2}_t([0, 1])$ is a Banach space of absolutely continuous functions from $[0, 1]$ into $\mathbb{R}^n$ together with the first order derivatives for which $x''(\cdot) \in L^2_t([0, 1])$. Notice that a Banach space $W^{1,2}_t([0, 1])$ can be equipped with the different equivalent norms.

The problem $(PD_h)$ is convex, if the set-valued mapping $F$, the sets $P_N, \Omega_t, t = 1, \ldots, N$ are convex and $g(\cdot, t)$ is convex proper function.

**Definition 2.4.** We say that for the convex problem $1, 2$ the regularity condition is satisfied if for points $x_t \in \mathbb{R}^n$ one of the following cases is fulfilled:

(i) $(x_t, x_{t+1}, x_{t-h}, x_{t+2}) \in ri(gphF(\cdot, t))(t = 0, \ldots, N - 2),
 x_t \in ri\Omega_t \cap ridomg(\cdot, t), x_N \in riP_N,$
(ii) $(x_t, x_{t+1}, x_{t-h}, x_{t+2}) \in int(gphF(\cdot, t))(t = 0, \ldots, N - 2),
 x_t \in int\Omega_t, t = 1, \ldots, N, x_N \in \Omega_N$ (with the possible exception of one fixed $t$),
and $g(\cdot, t)$ are continuous at $x_t$.

**Hypothesis (H)** Assume that in the problem $(PD_h)$ in the non convex case the set-valued mappings are such that the cones of tangent directions $K_{gphF(\cdot, t)}(\tilde{x}_t, \tilde{x}_{t+1}, \tilde{x}_{t-h}, \tilde{x}_{t+2}), K_{\Omega(\cdot)}(\tilde{x}_t), K_{P_N}(\tilde{x}_N)$ are local cones, where $\tilde{x}_t$ are the points of the optimal trajectory $\{\tilde{x}_t\}_{t\in[-h]}$ of problem $(PD_h)$. Further, the functions $d(\cdot, t)$ admit a CUA $h(\tilde{x}, \tilde{x})$ at the points $\tilde{x}_t$, that is continuous with respect to $\tilde{x}$ and consequently, $\partial g(\tilde{x}_t, t) := \partial h(\tilde{x}, \tilde{x}_t)$ is defined.

3. Necessary and sufficient conditions of optimality for second order delay-DSIs. First part of this section is devoted to optimization of convex problem $(PD_h)$. At once, note that the convexity of $(PD_h)$ is assumed for the sake of simplicity and all results can be generalized to the nonconvex case.

In order to solve this problem, we reduce it to a convex minimization problem and then apply (Theorem 3.4 [15]). Let us introduce the following sets:

$$
M_t = \{w : (x_t, x_{t+1}, x_{t-h}, x_{t+2}) \in gphF(\cdot, t), \ t = 0, 1, \ldots, N - 2
\}
$$

$$
D = \{w : x_t = \xi, t = -h, -h + 1, \ldots, 0\}, \ \xi_0 = \theta_0,
$$

$$
\Omega_t = \{w : x_t \in \Omega_1\}, \ t = 1, \ldots, N, \ \bar{P} = \{w : x_N \in P_N\}, \ (5)
$$

where $w = (x_{-h}, x_{-h+1}, \ldots, x_N) \in \mathbb{R}^{n(h+1+N)}$. 

Then the posed problem is equivalent to the following convex minimization problem

$$
\text{minimize } \varphi(w) = \sum_{i=1}^{N-1} g(x_i, t) \text{ subject to } w \in \left( \bigcap_{t=0}^{N-2} M_t \right) \cap \left( \bigcap_{t=1}^{N} \tilde{\Omega}_t \right) \cap D \cap \tilde{P}. \quad (6)
$$

As is well known [10], [15], [23] non separability of the cones of tangent directions $K_{M_t}(\tilde{w}), K_D(\tilde{w}), K_{\tilde{\Omega}_t}(\tilde{w}), K_P(\tilde{w})$ is one of the important concepts in the theory of extremal problems of the form 6. It should be noted that under the regularity condition (i) or (ii) of Definition 2.4 the dual cone associated with intersection of cones of tangent directions is equal to the algebraic sum of their dual cones. Another speaking by Theorems 1.30 and 3.3 [15] these cones are not separable. The following example shows that the intersection rule does not hold generally without the validity of the regularity condition.

**Example 1.** Suppose we have two closed disks $\Omega_+$ and $\Omega_-$ on the plane $\mathbb{R}^2 = \mathbb{R}^1 \times \mathbb{R}^1$ of radius $r$ around $(0,0)$, and $(0,-r)$, respectively:

$$
\Omega_+ = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + (x_2 - r)^2 \leq r^2\},
$$

$$
\Omega_- = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + (x_2 + r)^2 \leq r^2\}.
$$

Then for $(x_0^1, x_0^2) = (0,0) \in \Omega_+ \cap \Omega_-$ we have $K_{\Omega_+ \cap \Omega_-}(x_0^1, x_0^2) = \{(0,0)\}$, $K_{\Omega_+}(x_0^1, x_0^2) = \mathbb{R}^1 \times \mathbb{R}^1$, $K_{\Omega_-}(x_0^1, x_0^2) = \mathbb{R}^1 \times \mathbb{R}^1$, and $K^{\ast}_{\Omega_+ \cap \Omega_-}(x_0^1, x_0^2)$ is equal to $K^{\ast}_{\Omega_+}(x_0^1, x_0^2) + K^{\ast}_{\Omega_-}(x_0^1, x_0^2)$, while $K^{\ast}_{\Omega_+}(x_0^1, x_0^2) = \{0\} \times \mathbb{R}^1$ and $K^{\ast}_{\Omega_-}(x_0^1, x_0^2) = \{0\} \times \mathbb{R}^1$, where $\mathbb{R}^1 = \{x_2 \geq 0 : x_2 \in \mathbb{R}\}$, $\mathbb{R}^1$ = $\{x_2 \leq 0 : x_2 \in \mathbb{R}\}$. Consequently, $K^{\ast}_{\Omega_+ \cap \Omega_-}(x_0^1, x_0^2) \neq K^{\ast}_{\Omega_+}(x_0^1, x_0^2) + K^{\ast}_{\Omega_-}(x_0^1, x_0^2)$.

As we will see below the regularity condition implies that the necessary condition of optimality for problem $(PD_h)$ with second order delay-DSIs also is sufficient for optimality.

**Theorem 3.1.** Let $F(\cdot, t)$ and $g(\cdot, t)$, $t = 1, \ldots, N-1$ be a convex set-valued mapping and a convex function, respectively. Moreover, let $\{x_t\}_{t=-h}^N$ be a feasible solution of problem $(PD_h)$ with second order delay-DSIs and that $g(\cdot, t)$ is continuous at $x_t, t = 1, \ldots, N - 1$. Then in order for $\{\tilde{x}_t\}_{t=-h}^N$ to be an optimal solution to problem $(PD_h)$ with the initial values $\xi(t = -h, \ldots, -1), x_0 = \theta_0$, state constraints $x_t \in \Omega_t, t = 1, \ldots, N$ and target set $P_N$, it is necessary that there exist a number $\lambda \in \{0,1\}$ and vectors $x^*_t, u^*_t, \eta^*_t, t = 0, \ldots, N$ not all equal to zero, such that

(i) $$(x^*_t - u^*_t - \eta^*_t, u^*_t, \eta^*_t) \in F^\ast(x^*_t, \tilde{x}_t, \tilde{x}_t, \tilde{x}_t, \tilde{x}_t, \tilde{x}_t, \tilde{x}_t, \tilde{x}_t, \tilde{x}_t, \tilde{x}_t) + \{K^{\ast}_{\Omega_+}(\tilde{x}_t) - \lambda \partial g(\tilde{x}_t, t) \times \{0\} \times \{0\}, g(\tilde{x}_t, 0) = \{0\}, x^*_t = 0, t = 0, \ldots, N - 2 - h,$$

(ii) $$(x^*_t - u^*_t, u^*_t, \eta^*_t) \in F^\ast(x^*_t, \tilde{x}_t, \tilde{x}_t, \tilde{x}_t, \tilde{x}_t, \tilde{x}_t, \tilde{x}_t, \tilde{x}_t, \tilde{x}_t, \tilde{x}_t) + \{K^{\ast}_{\Omega_+}(\tilde{x}_t) - \lambda \partial g(\tilde{x}_t, t) \times \{0\} \times \{0\}, t = N - 1 - h, \ldots, N - 2;$$

$$x^*_{N-1} - u^*_{N-1} \in K^{\ast}_{\Omega_{N-1}}(\tilde{x}_{N-1}) - \lambda \partial g(\tilde{x}_{N-1}, N - 1), x^*_N \in K^{\ast}_{\Omega_{N} \cap P_N}(\tilde{x}_N).$$

In addition, under the regularity condition these conditions are necessary for optimality of the trajectory $\{\tilde{x}_t\}_{t=-h}^N$.

**Proof.** Clearly, if $\{\tilde{x}_t\}_{t=-h}^N$ is an optimal trajectory of the problem 1, 2 we claim that $\tilde{w} = (\tilde{x}_{-h}, \tilde{x}_{-h+1}, \ldots, \tilde{x}_N)$ is a solution of the convex mathematical programming problem 6. On the basis of results concerning convex mathematical programming [8], [11], [20] we can prove necessary optimality conditions for problem 6 with geometric constraints. Thus, by continuity of $g(\cdot, t)$ at points of some feasible solution $\{\tilde{x}_t\}_{t=-h}^N$ it follows from Theorem 3.4 [15] that there exist vectors...
$w^*(t) \in K_{M_1}(\tilde{w}), t = 0, \ldots, N - 2; \tilde{w}^* \in K_{D_1}(\tilde{w}); \bar{w}^*(t) \in K_{\bar{D}_1}(\bar{w}), t = 1, \ldots, N, \bar{w}^* \in K_{\bar{D}}(\bar{w}), \bar{w}^* \in \partial \varphi(\bar{w})$ and the number $\lambda \in \{0, 1\}$, not all equal to zero, such that

$$
\tilde{w}^* + \bar{w}^* + \sum_{t=0}^{N-2} w^*(t) + \sum_{t=1}^{N} \bar{w}^*(t) = \lambda \bar{w}^*, \bar{w}^* \in \partial \varphi(\bar{w}).
$$

(7)

In turn, it is not hard to see that

$$
K_{M_1}(w) = \{w^* = (x^*_1, x^*_{t+1}, x^*_{t+h}, x^*_{t+2}) \in K_{\text{ph}}(x_t, x_{t+1}, x_{t-h}, x_{t+2}) \mid x^*_k = 0, k \neq t, t-h, t-1, t+2, t = 0, \ldots, N-2; \}
$$

(8)

$$
K_{\bar{D}_1}(w) = \{w : x^*_N \in K_{\bar{D}_1}(x_N), x^*_t = 0, t < N\};
$$

(9)

$$
K_{\bar{D}_1}(w) = \{w^* : x^*_t \in K_{\bar{D}_1}(x_t), x^*_t = 0, k \neq t, t = 1, \ldots, N, \}
$$

Therefore, it is clear that

$$
w^*(t) = (x^*_h(t), \ldots, x^*_N(t)), x^*_t(t) = 0, k \neq t, t-h, t+1, t+2; t = 0, \ldots, N-2;$$

(10)

$$\bar{w}^*(t) = (\tilde{x}^*_h(t), \tilde{x}^*_{t+1}(t), \ldots, \tilde{x}^*_N(t)), \tilde{x}^*_t(t) = 0, k \neq t; t = 1, \ldots, N;$$

$$\tilde{w}^* = (0, \ldots, 0, \tilde{x}^*_h), \tilde{x}^*_N \in K_{\bar{D}_1}(\tilde{x}_N), \tilde{w} = (\tilde{x}^-_{h}, \ldots, \tilde{x}_N), \tilde{x}^* = \tilde{x}, t = -h, \ldots, 0 (\tilde{x}_0 = \theta_0).$$

In fact, under the regularity conditions (i) or (ii) of Definition 2.4 the dual cone to the cone of intersections of $K_{M_1}(\tilde{w}), K_{\bar{D}_1}(\tilde{w}), K_{\bar{D}_1}(\tilde{w}), K_{\bar{D}}(\tilde{w})$ is equal to the algebraic sum of the dual cones $K_{M_1}(\tilde{w}), K_{\bar{D}_1}(\tilde{w}), K_{\bar{D}_1}(\tilde{w}), K_{\bar{D}}(\tilde{w})$ and by Theorems 1.30 and 3.4 [15] for the convex programming problem 6 the equality 7 is satisfied with $\lambda = 1$, which guarantees sufficiency for optimality. Besides, the continuity conditions of Moreau-Rockafellar Theorem [10], [15], [23] are satisfied.

Let us denote $t$-th component of the vector $\bar{w}^*(t)$ by $[\bar{w}^*(t)]_t$. Then using the structure of the vector $w^*(t)(t = -h, -h+1, \ldots, 1)$ we have

$$
\left[ \sum_{t=0}^{N-2} w^*(t) \right]_t = \begin{cases} 
  x^*_1(t+h), & t = -h, \ldots, -1, \\
  x^*_0(0) + x^*_0(h), & t = 0, \\
  x^*_0(1) + x^*_1(1+h), & t = 1. 
\end{cases}
$$

(11)

On the other hand, it is not hard to conclude that taking $t = 2, \ldots, N$ we get another important relationship for representation of components of the vector $\left[ \sum_{t=0}^{N-2} w^*(t) \right]_t$ as follows:

$$
\left[ \sum_{t=0}^{N-2} w^*(t) \right]_t = \begin{cases} 
  x^*_2(t-2) + x^*_1(t-1) + x^*_1(t+h) + x^*_1(t), & t = 2, \ldots, N - 2, \\
  x^*_0(t-2) + x^*_1(t-1) + x^*_1(t), & t = N - 1, \\
  x^*_N(N-3) + x^*_N(N-2), & t = N, \\
  x^*_N(N-2). & t = N.
\end{cases}
$$

(12)

By similar way for the components of the vector $\left[ \sum_{t=1}^{N} \tilde{w}^*(t) \right]_t$ we deduce that

$$
\left[ \sum_{t=1}^{N} \tilde{w}^*(t) \right]_t = \begin{cases} 
  \tilde{x}^*_t(t) \in K_{\bar{T}_1}(\tilde{x}_t), & t = 1, \ldots, N, \\
  0, & t = -h, \ldots, 0.
\end{cases}
$$

(13)
Moreover, since \( \tilde{w}^* = (0, \ldots, 0, \bar{x}^*_N) \) and \( \hat{w}^* = (\hat{x}^*_{-h}, \ldots, \hat{x}^*_0, 0, \ldots, 0) \), then the sum of the vectors \( \tilde{w}^*, \hat{w}^* \) is

\[
\tilde{w}^* + \hat{w}^* = (\hat{x}^*_{-h}, \ldots, \hat{x}^*_0, 0, \ldots, 0, \bar{x}^*_N)
\]

(14)

where \( \hat{x}^*_k, k = -h, -h + 1, \ldots, 0 \) are arbitrary vectors.

Besides, it follows from the definition of the function \( \varphi(w) \equiv \sum_{t=1}^{N-1} g(x_t, t) \) that the vector \( \tilde{w}^* \) has the form \( \tilde{w}^* = (0, \ldots, 0, \bar{x}^*_1, \ldots, \bar{x}^*_{N-2}, \bar{x}^*_N|N-1, 0) \), where \( \bar{x}^*_t \in \partial g(\hat{x}, t), t = 1, \ldots, N - 1 \). Then taking into account 8-14 by component-wise representation the relation 7 can be written as follows

\[
x^*_t(t - 2) + x^*_t(t - 1) + x^*_t(t + h) + x^*_t(t) + \hat{x}^*_t(t) = \lambda \bar{x}^*_t, \quad t = 2, \ldots, N - 2 - h
\]

\[
x^*_t(t - 2) + x^*_t(t - 1) + x^*_t(t) + \hat{x}^*_t(t) = \lambda \bar{x}^*_t, \quad t = N - 1 - h, \ldots, N - 2.
\]

(15)

By analogy, easily can be checked that

\[
x^*_0(0) + x^*_0(h) + \hat{x}^*_0 = 0, \\
x^*_0(0) + x^*_0(h) = 0, \\
x^*_0(0) + x^*_0(h) + \bar{x}^*_0(1) + \hat{x}^*_0(1) = \lambda \bar{x}^*_1,
\]

\[
x^*_{N-1}(N - 3) + x^*_{N-1}(N - 2) + \bar{x}^*_{N-1}(N - 1) = \lambda \bar{x}^*_{N-1}, \\
x^*_N(N - 2) + \bar{x}^*_N(N) = 0.
\]

(16)

(17)

Clearly, because of arbitrariness of \( \hat{x}^*_t(k = -h, -h + 1, \ldots, 0) \) the first and second relations of 16 always hold. By virtue of 8 and the definition of LAM we can write

\[
(x^*_t(t), x^*_{t+1}(t), x^*_{t-h}(t)) \in F^*(\hat{x}_t, x^*_{t+1}, x^*_{t-h}, t), \quad t = 0, \ldots, N - 2.
\]

(18)

Then from 15 we obtain, that the following splitting in the condition 18 is assured

\[
(\lambda \bar{x}^*_t - x^*_t(t - 2) - x^*_t(t - 1) - x^*_t(t + h) - \bar{x}^*_t(t), x^*_{t+1}(t), x^*_{t-h}(t)), \\
\in F^*(-x^*_{t+2}(t), x^*_t, x^*_{t+1}, x^*_{t-h}, t), \quad t = 0, \ldots, N - 2 - h
\]

(19)

\[
(\lambda \bar{x}^*_t - x^*_t(t - 2) - x^*_t(t - 1) - \bar{x}^*_t(t), x^*_{t+1}(t), x^*_{t-h}(t)), \\
\in F^*(-x^*_t(t+2), x^*_t, x^*_t, x^*_t, x^*_t, t = N - 1 - h, \ldots, N - 2.
\]

(20)

Thus, if in the latter inclusions 19, 20 we introduce the notations

\[
x^*_t(0) \equiv x^*_t(t - 2), \quad u^*_t \equiv x^*_t(t - 1), \quad \hat{x}^*_t \equiv \hat{x}^*_t(t), \quad t = 0, \ldots, N - 2;
\]

\[
\eta^*_t \equiv x^*_t(t + h), \quad t = 0, \ldots, N - 2 - h; \quad \eta^*_t \equiv x^*_{t-h}(t), \quad t = N - 1 - h, \ldots, N - 2
\]

then we have the delay-discrete Euler-Lagrange type conditions (i), (ii) of theorem for \( t = 2, 3, \ldots, N - 2 - h \) and \( t = N - 1 - h, \ldots, N - 2 \), respectively. Moreover, from 16 and 17 for \( t = 0, 1 \) and \( t = N - 1, N \) we have

\[
x^*_0(0) + \hat{x}^*_0 = \lambda \bar{x}^*_0, \quad \bar{x}^*_0 = 0, \\
x^*_1(1) + \eta^*_0 + u^*_0 + \hat{x}^*_0 - \bar{x}^*_1 = \lambda \bar{x}^*_1, \quad \bar{x}^*_1 = 0,
\]

\[
- x^*_{N-1} + u^*_N + \bar{x}^*_N = \lambda \bar{x}^*_{N-1}, \\
- x^*_N + \hat{x}^*_N + x^*_N = 0,
\]

whereas \( x^*_{N-1} - u^*_N \in K^*_{\Omega_{N-1}}(\bar{x}_{N-1}, \lambda) \) and \( x^*_N \in K^*_{\Omega_{N}}(\bar{x}_N) \) can be extended to the case \( t = 0, 1 \). Thus, by virtue of 16, 17, 19, 20 we have proved this theorem.
Lemma 3.3. For each fixed $(gphF, \partial H F)$ the conditions of this theorem can be rewritten in a more symmetrical form. Evidently, this assumption is weaker than the usual closedness of $F(\cdot, t)$ $(gph\ F(\cdot, t)$ is closed). Before all we need the following lemma.

**Lemma 3.2.** Let $F(\cdot, t) : \mathbb{R}^{3n} \rightarrow \mathbb{R}^n$ be convex set-valued mapping and $F(x, u, t), u = (u_1, u_2)$ be closed set for each $(x, u) \in domF(\cdot, t)$ and $H_F(x, u, v^*) = \sup_v\{\langle v, v^* \rangle : v \in F(x, u, t)\}, v^* \in \mathbb{R}^n$.

In particular, if $\bar{v} = 0$, then $\partial v \bar{H}(x, u, 0) = \bar{H}(x, u, 0)$.

**Proof.** If $v \in F_A(x, u; v^*, t)$ or, equivalently, if $v \in F_A(x, u, t)$. and $\langle v, v^* \rangle = H_F(x, u, v^*)$, then $H_F(x, u, v^*) - H_F(x, u, \bar{v}^*) \geq \langle v, v^* \rangle - \langle v, \bar{v}^* \rangle$ or $H_F(x, u, v^*) - H_F(x, u, \bar{v}^*) \geq \langle v, v^* - \bar{v}^* \rangle$. That is $v \in \partial v \bar{H}(x, u, \bar{v}^*)$. Suppose now that $\bar{v} \in \partial v \bar{H}(x, u, \bar{v}^*)$. At first, we prove that $\bar{v} \notin F(x, u, t)$. On the contrary, let $\bar{v} \notin F(x, u, t)$. Then by separation theorems of convex sets (see, for example [15]) there is a vector such that

$$\sup_v\{\langle v, a \rangle : v \in F(x, u, t)\} < \langle \bar{v}, a \rangle.$$ (21)

On the other hand, since supremum of the difference no less than the difference of supremum’s, we have

$$\sup_v\{\langle v, v^* - \bar{v}^* \rangle : v \in F(x, u, t)\} \geq H_F(x, u, v^*) - H_F(x, u, \bar{v}^*) \geq \langle \bar{v}, v^* - \bar{v}^* \rangle.$$

Recall that $v^*$ is an arbitrary vector and so by setting in this inequality $v^* = \bar{v}^* + a$, we have $\sup_v\{\langle v, a \rangle : v \in F(x, u, t)\} \geq \langle \bar{v}, a \rangle$, which contradicts the inequality 21. It follows that $\bar{v} \notin F(x, u, t)$. Then, since $\bar{v} \in \partial v \bar{H}(x, u, \bar{v}^*)$ one has $H_F(x, u, v^*) - \langle v^*, \bar{v}^* \rangle \geq H_F(x, u, \bar{v}^*) - \langle \bar{v}^*, \bar{v}^* \rangle$. By setting here $v^* = 0$, we have $\langle \bar{v}^*, \bar{v}^* \rangle \geq H_F(x, u, \bar{v}^*)$. On the other hand, $\bar{v} \in F(x, u, t)$ and so $\langle \bar{v}^*, \bar{v}^* \rangle \leq H_F(x, u, \bar{v}^*)$. From these obtained two inequalities it follows that $\langle \bar{v}^*, \bar{v}^* \rangle = H_F(x, u, \bar{v}^*)$. This equality together with $\bar{v} \in F(x, u, t)$ means that $\bar{v} \in F_A(x, u, \bar{v}^*, t)$. In particular, if $\bar{v}^* = 0$, then $H_F(x, u, 0) = \langle v, 0 \rangle = 0$ for all $v \in F(x, u, t)$ and so $\partial v \bar{H}(x, u, 0) = F(x, u, t)$. The proof of lemma is ended. \hfill $\square$

Below we prove that an upper semi-continuous set-valued mapping $F(\cdot, t)$ (not necessarily convex) with closed values is closed $(gph\ F(\cdot, t)$ is closed).

**Lemma 3.3.** Let $F(\cdot, t) : \mathbb{R}^{3n} \rightarrow \mathbb{R}^n$ be an upper semi-continuous set-valued mapping and $F(x, u, t)$ be closed set for each $(x, u) \in domF(\cdot, t)$. Then $F(\cdot, t)$ is closed.

**Proof.** We proceed by a contradiction argument; suppose that $(x_k, u_k, v_k) \in gph\ F(\cdot, t)$ is a convergent sequence and $(x_k, u_k, v_k) \rightarrow (x_0, u_0, v_0)$, but $v_0 \notin F(x_0, u_0, t)$. Then there exists an open set $\Lambda$ containing $F(x_0, u_0, t)$ such that $v_0 \notin cl\Lambda$. Recalling that the set-valued mapping $F(\cdot, t)$ is upper semi-continuous it follows that there exists a positive integer $k_0$ such that $v_k \in \Lambda$ for $\forall k > k_0$. Therefore, $v_0 \in cl\Lambda$, a contradiction. \hfill $\square$

**Corollary 3.4.** Let $(PD_h)$ be an optimization problem for the second order delay-DISs satisfying the hypotheses of Theorem 3.1. In addition, let $F(\cdot, t)$ be closed set for each fixed $(x, u_1, u_2)$. Then in order that $\{x_t\}_{t=-h}^N$ be an optimal solution to
problem \((PD_h)\), it is necessary that there exist a number \(\lambda \in \{0, 1\}\) and vectors \(x_t^*, \eta_t, u_t^*, t = 0, \ldots, N\), not all equal to zero, such that

\[
(x_t^* - u_t^* - \eta_{t+h}, u_{t+1}^*, \eta_t^*) \in \partial(x, u_1, u_2)HF(\tilde{x}_t, \tilde{x}_{t+1}, \tilde{x}_{t-h}, x_{t+2}^*) + \{K_{\Omega_h}^*(\tilde{x}_t) - \lambda \partial g(\tilde{x}_t, t)\}
\]

\[
\lambda \partial g(\tilde{x}_0, 0) = 0, \quad x_t^* = 0, \quad t = 0, \ldots, N - 2 - h;
\]

\[
(x_t^* - u_t^* - u_{t+1}^*, \eta_t^*) \in \partial(x, u_1, u_2)HF(\tilde{x}_t, \tilde{x}_{t+1}, \tilde{x}_{t-h}, x_{t+2}^*) + \{K_{\Omega_h}^*(\tilde{x}_t) - \lambda \partial g(\tilde{x}_t, t)\}
\]

\[
\times \{0\} \times \{0\}, \quad x_{N-1}^* - u_{N-1}^* \in K_{\Omega_h}^*(\tilde{x}_{N-1}), \quad x_N^* \in K_{\Omega_h}^* \cap P_h(\tilde{x}_N),
\]

\[
t = N - 1 - h, \ldots, N - 2; \quad \tilde{x}_{t+2} \in \partial(\nu, HF(\tilde{x}_t, \tilde{x}_{t+1}, \tilde{x}_{t-h}, x_{t+2}^*), t = 0, \ldots, N - 2.
\]

In addition, under the regularity condition these conditions are sufficient for optimality.

Proof. By Theorem 2.1 \cite{15} \(F^*(v^*, x, u, v, t) = \partial(x, N_1, L_2)HF(x, u, v^*), u = (u_1, u_2),\) which is nonempty if the argmaximum set \(\mathcal{F}_A(x, u_1, u_2; v^*, t)\) is nonempty. On the other hand, since \(F(. , t)\) is closed for each \((x, u)\), and \(HF(x, u)\) is convex, it follows from Lemma 3.2 that \(\partial_0 HF(x, u, u_2; v^*) = F_0(x, u_1, u_2; v^*, t).\) Thus, taking into account the last two formulas in delay-discrete Euler-Lagrange type conditions (i), (ii) of Theorem 3.1 we obtain the needed result.

\(\square\)

**Theorem 3.5.** Suppose that the Hypothesis hold for the nonconvex problem \((PD_h)\) with second order delay-DSIs, that is set-valued mappings and sets are such that the cones of tangent directions \(K_{\Omega_h}^*(\tilde{x}_t, \tilde{x}_{t+1}, \tilde{x}_{t-h}, \tilde{x}_{t+2}), K_{\Omega_h}^*(\tilde{x}_t), K_{\Omega_h}^*(\tilde{x}_{N-1}), K_{\Omega_h}^*(\tilde{x}_N)\) are local tents, \(g(\cdot, t)\) admit a continuous \(\text{CUA}\) at the points \(\tilde{x}_t\). Then for \(\{\tilde{x}_i\}_{i=0}^N\) to be an optimal trajectory of this nonconvex problem, it is necessary that there exist a number \(\lambda \in \{0, 1\}\) and triple of vectors \(\{x_t^*\}, \{\eta_t^*\}, \{u_t^*\}\), not all equal to zero, satisfying the conditions of Theorem 3.1 in the nonconvex case.

Proof. In this case the Hypothesis \((H)\) ensures the conditions of Theorem 3.25 \cite{15} for the problem 6. Therefore, according to this theorem, we get the necessary condition as in Theorem 3.1 by starting from the relation 7, written out for nonconvex problem.

\(\square\)

4. Equivalence relations of LAMs for delay-DSIs and delay discrete-approximate problems. Let \(L_h\) and \(L_1\) be positive natural numbers such that \(h/L_1 = 1/L_1 = \delta\) Obviously, \(\delta\) is a step on the \(t\)-axis and \(x(t) \equiv x_\delta(t)\) is a grid functions on a uniform grid on \([0, 1]\]. We introduce the following second order difference operators:

\[
\Delta x(t) = \frac{1}{2}[x(t + \delta) - x(t)], \quad \Delta^2 x(t) = \frac{1}{2}[x(t + 2\delta) - \Delta x(t)] = \frac{x(t + 2\delta) - 2x(t + \delta) + x(t)}{\delta^2}.
\]

Note that the delay-DSIs associated with the delay -DFI of the problem \((PC_h)\) is the following inclusion

\[
\Delta^2 x(t) \in F(x(t), \Delta x(t), x(t - h), t), \quad t = 0, \delta, 2\delta, ..., 1 - 2\delta
\]

which can be rewritten in more relevant form

\[
x(t + 2\delta) \in 2x(t + \delta) - x(t) + \delta^2 F(x(t), \Delta x(t), x(t - h), t), \quad t = 0, \delta, 2\delta, ..., 1 - 2\delta.
\]

Denoting \(x(t) \equiv x, x(t + \delta) \equiv u_1, x(t - h) \equiv u_2, x(t + 2\delta) \equiv v\) we define the following auxiliary mapping \(G(x, u_1, u_2, t) = 2u_1 - x + \delta^2 F(x, (u_1 - x)/\delta, u_2, t).\) Obviously, the delay DSIs 22 is equivalent to

\[
x(t + 2\delta) \in G(x(t), x(t + \delta), x(t - h), t), \quad t = 0, \delta, 2\delta, ..., 1 - 2\delta.
\]
According to these delay-DSIs by Theorem 3.1 the optimization conditions will be written in term of LAM $G^*$. Hence, we should express the LAM $G^*$ in terms of LAM $F^*$.

The following two propositions play a crucial role in further results based on the second order delay-discrete approximations.

**Proposition 4.1.** Let $G$ be a convex set-valued function defined as $G(x, u_1 u_2, t) = 2u_1 - x + \delta^2 F(x, (u_1 - x)/\delta, u_2, t)$. Then between the Hamiltonians $H_G$ and $H_F$ there is the following relationship

$$H_G(x, u_1, u_2, v^*) = (2u_1 - x, v^*) + \delta^2 H_F(x, \frac{u_1 - x}{\delta}, u_2, v^*).$$

**Proof.** Evidently, by definition of the Hamiltonians and set-valued mapping $G$ we have

$$H_G(x, u_1, u_2, v^*) = \sup\{v, v^* : v \in G(x, u_1, u_2, t)\} = (2u_1 - x, v^*)$$

$$+ \delta^2 \sup\{v_1, v^* : v_1 \in F(x, \frac{u_1 - x}{\delta}, u_2, t)\} = (2u_1 - x, v^*)$$

$$+ \delta^2 H_F(x, \frac{u_1 - x}{\delta}, u_2, v^*).$$

\[\Box\]

**Proposition 4.2.** For a convex set-valued mappings $F$ and $G$ defined as $G(x, u_1 u_2, t) = 2u_1 - x + \delta^2 F(x, (u_1 - x)/\delta, u_2, t)$ between the subdifferentials of the Hamiltonian functions $\partial_{(x,u_1,u_2)} H_G(x, u_1 u_2, v^*)$ and $\partial_{(x,u_1,u_2)} H_F(x, (u_1 - x)/\delta, u_2, v^*)$ there is the following connection

$$\partial_{(x,u_1,u_2)} H_G(x, u_1 u_2, v^*) = \{-v^*\} \times \{0\} + \delta^2 A^* \partial_{(x,u_1,u_2)} H_F(x, \frac{u_1 - x}{\delta}, u_2, v^*).$$

Here

$$A = \begin{pmatrix} 1 & 0 & 0 \\ -1/\delta & 1/\delta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is $3n \times 3n$ matrix partitioned into submatrices $I$, $-1/\delta, 1/\delta$ and $n \times n$ zero matrix $0$, $I$ is a $n \times n$ identity matrix and $A^*$ is transposes of $A$.

**Proof.** Our aim is to express the subdifferential $\partial_{(x,u_1,u_2)} H_F(x, (u_1 - x)/\delta, u_2, v^*)$ in terms of the subdifferential $\partial_{(x,u_1,u_2)} H_G(x, u_1 u_2, v^*)$. Notice that $H(\cdot, \cdot, v^*)$ is concave and by convention $\partial_{(x,u_1,u_2)} H(x, u_1 u_2, v^*) = \partial_{(x,u_1,u_2)} [-H(x, u_1 u_2, v^*)]$. Let $Q : \mathbb{R}^{3n} \rightarrow \mathbb{R}^1$ be a convex function continuous at a point $(\varphi_1(x, u_1, u_2), \varphi_2(x, u_1, u_2), \varphi_3(x, u_1, u_2))$, where $\varphi_i : \mathbb{R}^{3n} \rightarrow \mathbb{R}^n, i = 1, 2, 3$ are Frechet differentiable functions at a point $(x, u_1, u_2)$. Then it follows from Theorem 2 (Section 4.4 [10]) that the subdifferential $\partial f(x, u_1, u_2)$ of composition function

$$f(x, u_1, u_2) = Q(\varphi_1(x, u_1, u_2), \varphi_2(x, u_1, u_2), \varphi_3(x, u_1, u_2))$$

should be computed by the following formula

$$\partial f(x, u_1, u_2) = A^* \partial Q(\varphi_1(x, u_1, u_2), \varphi_2(x, u_1, u_2), \varphi_3(x, u_1, u_2))$$

where

$$A = \begin{pmatrix} \partial \varphi_1(x, u_1, u_2) / \partial x & \partial \varphi_1(x, u_1, u_2) / \partial u_1 & \partial \varphi_1(x, u_1, u_2) / \partial u_2 \\ \partial \varphi_2(x, u_1, u_2) / \partial x & \partial \varphi_2(x, u_1, u_2) / \partial u_1 & \partial \varphi_2(x, u_1, u_2) / \partial u_2 \\ \partial \varphi_3(x, u_1, u_2) / \partial x & \partial \varphi_3(x, u_1, u_2) / \partial u_1 & \partial \varphi_3(x, u_1, u_2) / \partial u_2 \end{pmatrix}$$

(23)
is $3n \times 3n$ matrix, $\frac{\partial \varphi_i(x, u_1, u_2)}{\partial x}, \frac{\partial \varphi_i(x, u_1, u_2)}{\partial u_1}$, $i = 1, 2, 3, j = 1, 2$ are Jacobi matrices. Setting $\varphi_1(x, u_1, u_2) \equiv x, \varphi_2(x, u_1, u_2) \equiv (u_1 - x)/\delta, \varphi_3(x, u_1, u_2) \equiv u_2$ in 24 it is easy to compute that

$$A = \begin{pmatrix} 1 & 0 & 0 \\ -1/\delta & 1/\delta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then putting $f(x, u_1, u_2) = Q(\varphi_1(x, u_1, u_2), \varphi_2(x, u_1, u_2), \varphi_3(x, u_1, u_2)) = H_F(x, (u_1 - x)/\delta, u_2, v^*)$, $(v^*$ is fixed) by the Moreau-Rockafellar Theorem [8], [12], [21] and formula 23 from Proposition 4.1 we have the desired result.

The following two theorems on equivalence relations of LAMs allow us to make a bridge between the basic optimality conditions of second order delay-discrete and delay-discrete-approximate problems.

**Theorem 4.3.** Assume that $G(\cdot, t) : \mathbb{R}^{3n} \supseteq \mathbb{R}^n$ is a set-valued mapping defined as $G(x, u_1, u_2, t) = 2u_1 - x + \delta^2 F(x,(u_1 - x)/\delta, u_2, t)$. Then the following statements are equivalent

1. $(x^*, u_1^*, u_2^*) \in G^* (v^*; (x, u_1, u_2, v), t)$, $v \in G_A (x, u_1, u_2; v^*, t)$,

2. $\left(\frac{x^* + u_1^* - v^*}{\delta^2}, \frac{u_1^* - 2v^*}{\delta^2}, \frac{u_2^*}{\delta^2}\right) \in F^* (v^*; (x, \frac{u_1 - x}{\delta}, u_2 \frac{v - 2u_1 + x}{\delta^2}), t)$,

$v - 2u_1 + x \in F_A (x, \frac{u_1 - x}{\delta}, u_2; v^*, t), v^* \in \mathbb{R}^n$

where $G_A (x, u_1, u_2; v^*, t)$ is the argmaximum set for mapping $G$,

$$G_A (x, u_1, u_2; v^*, t) = \{ v \in G(x, u_1, u_2, t) : \langle v, v^* \rangle = H_G (x, u_1, u_2, t) \}.$$

**Proof.** At first let us prove $1 \Rightarrow 2$. By Proposition 4.2 it is easy to see that if

$$(x^*, u_1^*, u_2^*) \in \partial_{(x, u_1, u_2)} H_G \left( x, u_1, u_2, v^* \right),$$

(25)

then

$$(A^*)^{-1} \left( \frac{x^* + v^*}{\delta^2}, \frac{u_1^* - 2v^*}{\delta^2}, \frac{u_2^*}{\delta^2} \right) \in \partial_{(x, u_1, u_2)} H_F \left( x, \frac{u_1 - x}{\delta}, u_2, v^* \right).$$

(26)

But it is easy to compute that

$$(A^*)^{-1} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & \delta \cdot I & 0 \\ 0 & 0 & I \end{pmatrix}.$$

Then by substituting $(A^*)^{-1}$ into 26 we derive that

$$\left( \frac{x^* + u_1^* - v^*}{\delta^2}, \frac{u_1^* - 2v^*}{\delta^2}, \frac{u_2^*}{\delta^2} \right) \in \partial_{(x, u_1, u_2)} H_F \left( x, \frac{u_1 - x}{\delta}, u_2, v^* \right).$$

(27)

Now, recall that by Theorem 2.1 [15]

1. $G^* (v^*; (x, u_1, u_2, v), t) = \partial_{(x, u_1, u_2)} H_G \left( x, u_1, u_2, v^* \right), v \in G_A (x, u_1, u_2; v^*, t)$,

2. $F^* \left( v^*; \left( x, \frac{u_1 - x}{\delta}, u_2, \frac{v - 2u_1 + x}{\delta^2} \right), t \right) = \partial_{(x, u_1, u_2)} H_F \left( x, \frac{u_1 - x}{\delta}, u_2, v^* \right),$

$$v - 2u_1 + x \in F_A \left( x, \frac{u_1 - x}{\delta}, u_2; v^*, t \right).$$

(28)

By using these formulas, it remains to express 25 and 26 in term of LAMs. Besides, recall that $v \in (x, u_1, u_2; v^*, t), v - 2u_1 + x/\delta^2 \in F_A (u_1 - x)/\delta, u_2; v^*, t)$ ensure that the LAMs are nonempty at a given point. Thus, taking into account
28 in 25 and 26 we have proved that 1 → 2. The opposite implication 2 → 1 can be proved by analogy.

For a nonconvex problem (PDh) the Theorem 4.3 can be generalised to the case of local tents. Notice that for convex set-valued mapping and sets a local tent exists always [15].

**Theorem 4.4.** Let a set-valued mapping $F(\cdot, t) : \mathbb{R}^{3n} \rightrightarrows \mathbb{R}^n$ be such that the cone of tangent directions $K_{gphF(\cdot, t)}(x, u_1, u_2, v)$ determine a local tent to the mapping $G(x, u_1, u_2, t) = 2u_1 - x + \delta^2 F(x, (u_1 - x)/\delta, u_2, t)$ at a point $(x, u_1, u_2, v) \in gphF(\cdot, t)$. Then, the inclusions (a) and (b) of Theorem 4.3 are equivalent in the nonconvex case.

**Proof.** By hypothesis of theorem there exist functions $r(\bar{z}), r_i(\bar{z}), \bar{z} = (\bar{x}, \bar{u}_1, \bar{u}_2, \bar{v})$, $r_1(\bar{z}) \parallel \bar{z} \parallel^{-1} \to 0 (i = 1, 2, 3)$, $r(\bar{z}) \parallel \bar{z} \parallel^{-1} \to 0$ as $\bar{z} \to 0$, such that

$$v + \bar{v} + r(\bar{z}) \in 2 \{u_1 + \bar{u}_1 + r_2(\bar{z})\} - x - \bar{x} - r_1(\bar{z})$$

$$+ \delta^2 F \left( x + \bar{x} + r_1(\bar{z}), \frac{u_1 + \bar{u}_1 + r_2(\bar{z}) - x - \bar{x} - r_1(\bar{z})}{\delta}, u_2 + \bar{u}_2 + r_3(\bar{z}), t \right)$$

for sufficiently small $\bar{z} \in K$, where $K \subseteq riK_{gphG(\cdot, t)}(z)$, $z = (x, u_1, u_2, v)$ is a convex cone. Dividing by $\delta^2$ the latter relation, we have

$$\frac{v - 2u_1 + x}{\delta^2} + \frac{\bar{v} - 2\bar{u}_1 + \bar{x}}{\delta^2} + \frac{r(\bar{z}) - 2r_2(\bar{z}) + r_1(\bar{z})}{\delta^2}$$

$$\in F \left( x + \bar{x} + r_1(\bar{z}), \frac{u_1 + \bar{u}_1 + r_2(\bar{z}) - x - \bar{x} - r_1(\bar{z})}{\delta}, u_2 + \bar{u}_2 + r_3(\bar{z}), t \right).$$

On the other hand, it is easy to see that the cone $K_{gphF(\cdot, t)}(x, \frac{u_1 - x}{\delta}, u_2, \frac{v - 2u_1 + x}{\delta^2})$ is a local tent to $gphF$ and

$$\left( \frac{\bar{x} - \bar{u}_1 - \bar{x}}{\delta}, \bar{u}_2, \frac{\bar{v} - 2\bar{u}_1 + \bar{x}}{\delta^2} \right) \in K_{gphF(\cdot, t)} \left( x, \frac{u_1 - x}{\delta}, u_2, \frac{v - 2u_1 + x}{\delta^2} \right).$$

(29)

In analogy with the relation 29 in the reverse direction, it can be easily checked that

$$\left( \bar{x}, \bar{u}_1, \bar{u}_2, \bar{v} \right) \in K_{gphF(\cdot, t)}(x, u_1, u_2, v),$$

(30)

where $K_{gphF(\cdot, t)}(x, u_1, u_2, v)$ is a local tent to $gphG(\cdot, t)$. This implies that 29 and 30 are equivalent. Assume, now that

$$(x^*, u_1^*, u_2^*) \in G^*(v^*; (x, u_1, u_2, v), t), v \in G_A(x, u_1, u_2; v^*, t),$$

whereas

$$(\bar{x}, x^*) + (\bar{u}_1, u_1^*) + (\bar{u}_2, u_2^*) - (\bar{v}, v^*) \geq 0, \left( \bar{x}, \bar{u}_1, \bar{u}_2, \bar{v} \right) \in K_{gphF(\cdot, t)}(x, u_1, u_2, v).$$

Taking into account the tangent directions of the cone $K_{gphF(\cdot, t)}(x, (u_1 - x)/\delta, u_2, (v - 2u_1 + x)/\delta^2)$ (see 29) let us rewrite this inequality in the form

$$\langle \bar{x}, \theta_1 \rangle + \left( \frac{\bar{u}_1 + \bar{x}}{\delta}, \theta_2 \right) + \langle \bar{u}_2, \theta_3 \rangle - \left( \frac{\bar{v}, 2\bar{u}_1 + \bar{x}}{\delta^2}, v^* \right) \geq 0.$$

$$\left( \bar{x}, \bar{u}_1 - \bar{x}, \bar{u}_2, \bar{v} - 2\bar{u}_1 + \bar{x} \right) \in K_{gphF(\cdot, t)} \left( x, \frac{u_1 - x}{\delta}, u_2, \frac{v - 2u_1 + x}{\delta^2} \right).$$

(32)

Multiplying 32 by $\delta^2$ after a rearrangement we obtain

$$\langle \bar{x}, \delta^2 \theta_1 - \delta \theta_2 - v^* \rangle + \langle \bar{u}_1, \delta \theta_2 + 2v^* \rangle + \langle \bar{u}_2, \delta \theta_3 \rangle - \langle \bar{v}, v^* \rangle \geq 0.$$
Comparison of this inequality with 31 implies that
\[
\theta_1 = \frac{x^* + u_1^* - v^*}{\delta^2}, \quad \theta_2 = \frac{u_1^* - 2v^*}{\delta}, \quad \theta_3 = \frac{u_2^*}{\delta^2}.
\]
Therefore, from the equivalence relations 31 and 32 we derive that
\[
((x^* + u_1^* - v^*)/\delta^2, (u_1^* - 2v^*)/\delta, u_2^*/\delta^2) \\
\in F^*(v^*; (x,(u_1 - x)/\delta, u_2, (v - 2u_1 + x)/(\delta^2), t).
\]
Consequently, recalling that \(G^*(v^*; (x,u_1,u_2,v),t) \neq \emptyset, v \in G_A(x,u_1,u_2;v^*,t)\) and
\[
F^*(v^*; (x,u_1 - x/\delta, u_2, v - 2u_1 + x)/(\delta^2), t) \neq \emptyset, (v - 2u_1 + x)/\delta^2
\]
\(\in F_A(x,(u_1 - x)/\delta,u_2;v^*,t)\) we ended the proof of theorem.

\[\square\]

An unexpected equivalence relation between the LAMs to a certain class of set-valued mappings was discovered by the following theorem.

**Theorem 4.5.** Let \(F(\cdot,t) : \mathbb{R}^{3n} \to \mathbb{R}^n\) be a set-valued mapping such that the cone \(K_{gphG(\cdot,t)}(x,u_1,v), (x,u_1,v) \in gphG(\cdot,t)\) of tangent directions to the mapping \(G(x,u_1,v) = 2u_1 - x + \delta^2F(x,t)\) determines a local tent. Then, the following inclusions under the condition that \(u_1^* = 2v^*\) are equivalent

(a) \((x^*, u_1^* ) \in G^*(v^*; (x,u_1,v),t), v \in G_A(x,u_1; v^*, t)\),

(b) \(x^*,v^* \in F^*(v^*; (x,v - 2u_1 + x)/(\delta^2), t), v - 2u_1 + x)/(\delta^2) \in F_A(x,v^*,t), v^* \in \mathbb{R}^n\).

**Proof.** By conditions of theorem there exist functions \(r(z), r_i(z), r_i(z) \| z \|^{-1} \to 0, z = (\overline{x}, \overline{u_1}, v)(i = 1, 2), r(z) \| z \|^{-1} \to 0\) as \(z \to 0\), such that
\[
v + \overline{v} + r(z) \in 2(u_1 + \overline{u}_1 + r_2(z)) - x - \overline{x} - r_1(z) + \delta^2F(x + \overline{x} + r_1(z), t)\]
for sufficiently small \(\overline{z} \in K \subseteq r_1K_{gphG(\cdot,t)}(z), z = (x,u_1,v)\). Now rewriting this relation in the form
\[
\frac{v - 2u_1 + x}{\delta^2} + \overline{v} - 2\overline{u}_1 + \overline{x} + \frac{r(z) - 2r_2(z) + r_1(z)}{\delta^2} \in F(x + \overline{x} + r_1(z), t)
\]
as in Theorem 4.4 it is easily to see that the cone \(K_{gphF(\cdot,t)}(z), (v - 2u_1 + x)/(\delta^2)\) is a local tent to \(gphF\). Moreover, the inclusions
\[
(\overline{x}, \overline{v} - 2\overline{u}_1 + \overline{x}) \in K_{gphF(\cdot,t)}(x, v - 2u_1 + x)/(\delta^2) \quad \text{and,} \quad (\overline{x}, \overline{u}_1, \overline{v}) \in K_{gphG(\cdot,t)}(x,u_1,v)
\]
are equivalent. Thus, rewriting the inequality
\[
\langle \overline{x}, x^* \rangle + \langle \overline{u}_1, u_1^* \rangle - \langle \overline{v}, v^* \rangle \geq 0, (\overline{x}, \overline{u}_1, \overline{v}) \in K_{gphF(\cdot,t)}(x, u_1, v)
\]
in the form
\[
\langle \overline{x}, \theta \rangle - \langle \overline{v} - 2\overline{u}_1 + \overline{x}/\delta^2, v^* \rangle \geq 0, \quad (\overline{x}, \overline{v} - 2\overline{u}_1 + \overline{x}) \in K_{gphF(\cdot,t)}(x, v - 2u_1 + x)/(\delta^2)
\]
we observe that \(\theta = (x^* + u_1^* - v^*)/\delta^2\), where \(u_1^* = 2v^*\). This implies
\[
\frac{x^* + v^*}{\delta^2} \in F^*(v^*; (x, v - 2u_1 + x)/(\delta^2), t).
\]
Besides, it is easy to see that \( G^* (v^*; (x, u_1, v), t) \neq \emptyset \) and \( F^* (v^*; (x, (v - 2u_1 + x)/\delta^2), t) \neq \emptyset \) if \( v \in G_A (x, u_1; v^*, t) \) and \((v - 2u_1 + x)/\delta^2 \in F_A (x; v^*, t), \) respectively. The proof of theorem is completed.

5. Necessary and sufficient conditions of optimality for second order delay-discrete-approximate problems. The results derived in this section can be essentially viewed as optimization of second order delay-discrete-approximate inclusions crucial for optimization of second order delay-DFIs. The key idea is to use the equivalence results obtained in Section 4. In term of designations of Section 4 we define the second order discrete-approximation problem associated with 1, 2 as follows

\[
\text{minimize } J_\delta[x(\cdot)] = \sum_{t=0,\ldots,1-2\delta} \delta g(x(t), t) + \varphi_0(x(1-\delta)),
\]

\[
\Delta^2 x(t) \in F(x(t), \Delta x(t), x(t-h), t), \ t = 0, \delta, 2\delta, \ldots, 1 - 2\delta, \quad (33)
\]

\[
x(t) = \xi(t), \ t = -h, -h+\delta, \ldots, -\delta; \ x(0) = \theta; \ x(t) \in \Omega(t), \ t = \delta, \ldots, 1, \ (x(1) \in P).
\]

Using now set-valued mapping \( G(x, u_1, u_2, t) = 2u_1 - x + \delta^2 F(x, (u_1 - x)/\delta, u_2, t) \) we can reduce the problem 33 and 34 to the following form, labelled by \((PDA_h)\):

\[
\text{minimize } J_\delta[x(\cdot)] = \sum_{t=0,\ldots,1-2\delta} \delta g(x(t), t) + \varphi_0(x(1-\delta)),
\]

\[
(PDA_h) \quad x(t + 2\Delta) \in G(x(t), x(t + \Delta), x(t-h), t), \ t = 0, \delta, 2\delta, \ldots, 1 - 2\delta, \quad (35)
\]

\[
x(t) = \xi(t), \ t = -h, -h+\delta, \ldots, -\delta; \ x(0) = \theta; \ x(t) \in \Omega(t), \ t = \delta, \ldots, 1, \ (x(1) \in P). \quad (36)
\]

It follows from Theorems 3.1 and 3.5 that if \( \{\tilde{x}(t) : T = 0, \delta, \ldots, 1\} \) is an optimal trajectory in the problem 35, 36 then there exist a triple of vectors \( \{x^*(t)\}, \{\eta^*(t)\}, \{u^*(t)\} \) and a number \( \lambda = \lambda_\delta \in (0, 1), \) not all zero, such that

\[
(x^*(t) - u^*(t) - \eta^*(t + h), u^*(t + \delta), \eta^*(t)) \in G^*(x^*(t + 2\delta), \tilde{x}(t), \tilde{x}(t + \delta), \tilde{x}(t-h), \tilde{x}(t + 2\delta), t) + \{K^*_{\Omega(t)}(\tilde{x}(t)) - \lambda \delta \partial g(\tilde{x}(t), t)\} \times \{0\} \times \{0\};
\]

\[
\partial g(\tilde{x}(0), 0) = \{0\},
\]

\[
x^*(\delta) = 0, \ t = 0, \delta, 1 - 2\delta - h, \quad (37)
\]

\[
(x^*(t) - u^*(t), \eta^*(t)) \in G^*(x^*(t + 2\delta), \tilde{x}(t), \tilde{x}(t + \delta), \tilde{x}(t-h), \tilde{x}(t + 2\delta), t) + \{K^*_{\Omega(t)}(\tilde{x}(t)) - \lambda \delta \partial g(\tilde{x}(t), t)\} \times \{0\} \times \{0\}; \ t = 1 - \delta - h, \ldots, 1 - 2\delta, \quad (38)
\]

\[
x^*(1 - \delta) - u^*(1 - \delta) \in K^*_{\Omega(t)}(\tilde{x}(1)) - \lambda \partial \varphi_0(x(1 - \delta)), \ x^*(1) \in K^*_{\Omega(t)}(\tilde{x}(1)). \quad (39)
\]

Theorem 5.1. Let \( F(\cdot, t) : \mathbb{R}^{3n} \to \mathbb{R}^n \) be a convex set-valued mapping, \( g(\cdot, t) : \mathbb{R}^n \to \mathbb{R}^1 \cup \{\pm \infty\} \) be a proper convex functions and continuous at the points of some feasible trajectory \( \{x(t)\}, t = 0, h, \ldots, 1. \) Then for optimality of the trajectory \( \{\tilde{x}(t)\} \) in the problem \((PDA_h)\) with second order delay-DSIs, it is necessary that there exist a number \( \lambda = \lambda_\delta \in (0, 1) \) and a triple of vectors \( \{x^*(t)\}, \{\eta^*(t)\}, \{u^*(t)\}, \) not all equal to zero, satisfying the second order delay-approximate Euler-Lagrange
type inclusions (i), (ii) and transversality conditions (iii):

(i) \((\Delta^2 x^*(t) + \Delta \psi^*(t) - \eta^*(t + h), \psi^*(t + \delta), \eta^*(t)) \in F^*(x^*(t + 2\delta); (\tilde{x}(t), \\
\Delta \tilde{x}(t), \tilde{x}(t - h), \Delta^2 \tilde{x}(t), t) + \{ K^*_\Omega(\tilde{x}(t)) - \lambda \partial g(\tilde{x}(t), t) \} \times \{ 0 \} \times \{ 0 \};
\partial g(\tilde{x}(0), 0) = 0, \ t = 0, \ldots, 1 - 2\delta - h; \)

(ii) \((\Delta^2 x^*(t) + \Delta \psi^*(t), \psi^*(t + \delta), \eta^*(t)) \in F^*(x^*(t + 2\delta); (\tilde{x}(t), \\
\Delta \tilde{x}(t), \tilde{x}(t - h), \Delta^2 \tilde{x}(t), t) + \{ K^*_\Omega(\tilde{x}(t)) - \lambda \partial g(\tilde{x}(t), t) \} \times \{ 0 \} \times \{ 0 \};
\ t = 1 - \delta - h, \ldots, 1 - 2\delta, \)

(iii) 
\[-\Delta x^*(1 - \delta) - x^*(t) - \psi^*(1 - \delta) \in K^*_\Omega(1 - \delta)(\tilde{x}(1 - \delta)) - \lambda \partial \phi_0(\tilde{x}(1 - \delta)),
\ x^*(1) \in K^*_\Omega(1) \cap P(\tilde{x}(1)). \]

And under the regularity condition, these conditions are also sufficient for optimality of \{\tilde{x}(t)\}.

Proof. Indeed, by the equivalence relations (a),(b) of Theorem 4.3 the conditions 37, 38 for convex problem take the forms
\[
\left( \frac{x^*(t) - u^*(t) - \eta^*(t + h) + u^*(t + \delta) - x^*(t + 2\delta)}{\delta^2}, \frac{u^*(t + \delta) - 2x^*(t + 2\delta)}{\delta}, \eta^*(t) \right)
\in F^*(x^*(t + 2\delta); (\tilde{x}(t), \Delta \tilde{x}(t), \tilde{x}(t - h), \Delta^2 \tilde{x}(t), t) + \{ K^*_\Omega(\tilde{x}(t)) - \lambda \partial g(\tilde{x}(t), t) \} \times \{ 0 \} \times \{ 0 \}; \partial g(\tilde{x}(0), 0) = 0, \ t = 0, \ldots, 1 - 2\delta - h; \)

\[
\left( \frac{x^*(t) - u^*(t) + u^*(t + \delta) - x^*(t + 2\delta)}{\delta^2}, \frac{u^*(t + \delta) - 2x^*(t + 2\delta)}{\delta}, \eta^*(t) \right)
\in F^*(x^*(t + 2\delta); (\tilde{x}(t), \Delta \tilde{x}(t), \tilde{x}(t - h), \Delta^2 \tilde{x}(t), t) + \{ K^*_\Omega(\tilde{x}(t)) - \lambda \partial g(\tilde{x}(t), t) \} \times \{ 0 \} \times \{ 0 \}; \ t = 1 - \delta - h, \ldots, 1 - 2\delta. \]

We notice only that the LAM is positive homogeneous on the first argument and so in relations 40, 41 \(\delta x^*(t), \delta u^*, \delta \psi^*(t), \eta^*(t)/\delta\) are denoted again by \(x^*(t), u^*, \psi^*(t), \eta^*(t)\), respectively. Moreover, denoting \(\psi^*(t + \delta) = [u^*(t + \delta) - 2x^*(t + 2\delta)]/\delta\) we have \(u^*(t + \delta) = \delta \psi^*(t + \delta) + 2x^*(t + 2\delta)\) and then
\[
\frac{x^*(t) - u^*(t) + u^*(t + \delta) - x^*(t + 2\delta)}{\delta^2} = \Delta^2 x(t) + \Delta \psi^*(t). \]

On the other hand, substituting \(u^*(1 - \delta) = \delta \psi^*(1 - \delta) + 2x^*(1)\) into relation 39 we have \(x^*(1 - \delta) - 2x^*(1) - \delta \psi^*(1 - \delta) \in K^*_\Omega(1 - \delta)(\tilde{x}(1 - \delta)) - \lambda \partial \phi_0(\tilde{x}(1 - \delta)), x^*(1) \in K^*_\Omega(1) \cap P(\tilde{x}(1)),\) which dividing by \(\delta\) yields
\[
- \Delta x^*(1 - \delta) - x^*(1) - \psi^*(1 - \delta) \in K^*_\Omega(1 - \delta)(\tilde{x}(1 - \delta)) - \lambda \partial \phi_0(\tilde{x}(1 - \delta)), \]
\[
x^*(1) \in K^*_\Omega(1) \cap P(\tilde{x}(1)). \]

Taking into account 42 and 43 in the second order delay-discrete Euler-Lagrange type conditions 40 and 41 we have the needed result.

In a similar way as in Theorem 4.4 we can prove the following theorem.

**Theorem 5.2.** If hypothesis \((H)\) hold for the nonconvex problem \((PD_h)\), then for \{\(\tilde{x}(t)\)\} to be an optimal trajectory of this problem, it is necessary that there exist a number \(\lambda \in \{0, 1\}\) and a triple of vectors \{\(x^*(t)\), \{\(\eta^*(t)\), \{\(\psi^*(t)\)\} not all equal to zero, satisfying items (i)-(iii) of Theorem 5.1 in the nonconvex case.

By using Theorem 4.5 we have the following result.
Theorem 5.3. Suppose that in the discrete-approximation problem \((PD_h)\) \(F(x, (u_1 - x)/h, u_2, t) \equiv F(x, t)\), i.e. a mapping \(F\) doesn't depend both on second and third arguments. Then, the adjoint second order Euler-Lagrange type DSIs (i), (ii) and discrete transversality condition (iii) of Theorem 5.1 have the forms

\[
\Delta^2 x^*(t) \in F^* (x^*(t + 2\delta); (\tilde{x}(t), \Delta^2 \tilde{x}(t)), t) + \{K_{\Omega(t)}^*(\tilde{x}(t)) - \lambda \partial g(\tilde{x}(t), t), \}
\]

\[
t = 0, \delta, \ldots, 1 - 2\delta; \quad \partial g(\tilde{x}(0), 0) = \{0\}, x^*(\delta) = 0; -\Delta x^*(1 - \delta) - x^*(1)
\]

\[
\in K_{\Omega(1-\delta)}^*(\tilde{x}(1 - \delta)) - \lambda \partial \varphi_0(\tilde{x}(1 - \delta)), \quad x^*(1) \in K_{\Omega(1)}^* \cap P(\tilde{x}(1)),
\]

respectively.

Proof. By condition (2) of Theorem 4.5 we have

\[
\frac{x^*(t) - u^*(t) + x^*(t + 2\delta)}{\delta^2} \in F^* (x^*(t + 2\delta); (\tilde{x}(t), \Delta^2 \tilde{x}(t)), t)
\]

\[
+ \{K_{\Omega(t)}^*(\tilde{x}(t)) - \lambda \partial g(\tilde{x}(t), t), \quad \partial g(\tilde{x}(0), 0) = \{0\}, x^*(\delta) = 0, t = 0, \ldots, 1 - 2\delta.
\]

Further, the condition \(u^* = 2v^*\) of the Theorem 4.5 means that \(u^*(t + \delta) = 2x^*(t + \delta)\). Hence, substitution of \(u^*(t + \delta) = 2x^*(t + \delta)\) into this relation implies

\[
\frac{x^*(t) - 2x^*(t + \delta) + x^*(t + 2\delta)}{\delta^2} = \Delta^2 x^*(t).
\]

The proof of theorem is completed. \(\square\)

6. Transversality conditions and optimization of second order delay-DFIs.

Construction of sufficient conditions of optimality for problem \((PC_h)\) with second order delay DFIs is based essentially on results from Section 5. By using the obtained results in Section 5, we formulate a sufficient condition of optimality for the continuous problem \((PC_h)\). Indeed, setting \(\lambda = 1\) in conditions (i)-(iii) of Theorem 5.1 (or Theorem 5.2) and by passing to the limit procedure as \(\delta \to 0\), we establish the so-called second order adjoint Euler-Lagrange type delay-DFIs for problem \((PC_h)\). At first, let us formulate an adjoint delay-DFIs for convex problem \((PC_h)\):

(i) \(\left( \frac{d^2 x^*(t)}{dt^2} + \frac{d\psi^*(t)}{dt} - \eta^*(t + h), \psi^*(t), \eta^*(t) \right) \in F^* (x^*(t); (\tilde{x}(t), \tilde{x}'(t), \tilde{x}(t - h), \tilde{x}''(t), t) + \{K_{\Omega(t)}^*(\tilde{x}(t)) - \partial g(\tilde{x}(t), t) \times \{0\} \times \{0\}, a.e. t \in [0, 1 - h), x^*(0) = 0; \)

(ii) \(\left( \frac{d^2 x^*(t)}{dt^2} + \frac{d\psi^*(t)}{dt}, \psi^*(t), \eta^*(t) \right) \in F^* (x^*(t); (\tilde{x}(t), \tilde{x}'(t), \tilde{x}(t - h)\tilde{x}''(t), t) + \{K_{\Omega(t)}^*(\tilde{x}(t)) - \partial g(\tilde{x}(t), t) \times \{0\} \times \{0\}, a.e. t \in [1 - h, 1] \)

and the transversality conditions at point \(t = 1; \)

(iii) \(- \frac{dx^*(1)}{dt} - \psi^*(1) \in K_{\Omega(1)}^* \cap P(\tilde{x}(1)) - \partial \varphi_0(\tilde{x}(1)); \quad x^*(1) = 0\)

where \(K_{\Omega(1)}^* \cap P(\tilde{x}(1))\) is a cone of tangent directions at a point \(\tilde{x}(1) \in \Omega(1) \cap P.\)

Here we assume that \(x^*(t), t \in [0, 1]\) is an absolutely continuous function together with the first order derivatives for which \(x''(\cdot) \in L^1_\Omega[0, 1]\) Moreover \(\psi^*(t), \eta^*(t), t \in [0, 1]\) are absolutely continuous and \(\psi''(\cdot) \in L^1_\Omega[0, 1].\)
The condition guaranteeing nonemptiness of the LAM $F^*$ at a given point is the following

(iv) \[ \frac{d^2 \tilde{x}(t)}{dt^2} \in F_A(\tilde{x}(t), \tilde{x}'(t), \tilde{x}(t-h); x^*(t), t), \text{ a.e. } t \in [0, 1]. \]

It appears that the following assertion is true.

**Theorem 6.1.** Let $g : \mathbb{R}^n \times [0, 1] \to \mathbb{R}^1$ be continuous and convex with respect to $x$ function, and $F(\cdot, t) : \mathbb{R}^3 \to \mathbb{R}^n$ be a convex set-valued mapping. Then for optimality of the arc $\tilde{x}(t)$ to the convex problem $(PC_h)$ with second order delay-DFIs it is sufficient that there exist a triple \{\(x^*(t), \psi^*(t), \eta^*(t)\)\} of absolutely continuous functions, $x^*(t), \psi^*(t), \eta^*(t), t \in [0, 1]$, satisfying a.e. the second order Euler-Lagrange type delay-DFIs (i), (ii), inclusion (iv) and transversality condition (iii).

**Proof.** By Theorem 2. \[ F^*(v^*, (x, u_1, u_2, v), t) = \partial_{x,u_1,u_2}H_F(x, u_1, u_2, v^*), v \in F_A(x, u_1, u_2; v^*, t). \] Then by using the Moreau-Rockafellar Theorem [8], [12], [23] and the convention that $-\partial_x g(\cdot, t) = \partial_x (-g(\cdot, t))$ from conditions (i), (ii) we obtain the second order adjoint DFIs rewritten in term of Hamiltonian function

\[
\begin{align*}
\left( \frac{d^2 x^*(t)}{dt^2} + \frac{d \psi^*(t)}{dt} - \eta^*(t+h), \psi^*(t), \eta^*(t) \right) &\in \partial [H_F(\tilde{x}(t), \tilde{x}'(t), \tilde{x}(t-h), x^*(t))] \\
+ \{K^*_L(t) \tilde{x}(t) - \partial g(\tilde{x}(t), t) \} &\times \{0\} \times \{0\}, \text{ a.e. } t \in [0, 1-h];
\end{align*}
\]

Next, by definition of subdifferential of the Hamiltonian function $H_F$ we rewrite relation 44 in the form:

\[
H_F(x(t), x'(t), x(t-h), x^*(t)) - H_F(\tilde{x}(t), \tilde{x}'(t), \tilde{x}(t-h), x^*(t)) - g(x(t), t) +
\]

\[
g(\tilde{x}(t), t) \leq \left( \frac{d^2 x^*(t)}{dt^2} + \frac{d \psi^*(t)}{dt} - \eta^*(t+h) - \tilde{x}'(t), x(t) - \tilde{x}(t) \right) + \{\psi^*(t), \psi^*(t) - \tilde{x}'(t), (\eta^*(t), x(t-h) - \tilde{x}(t-h)), \tilde{x}(t) \in K^*_L(t) \tilde{x}(t), t \in [0, 1-h].
\]

Since $\tilde{x}^*(t) \in K^*_L(t) \tilde{x}(t))$, by definition of the dual cone, obviously $\langle -\tilde{x}^*(t), x(t) - \tilde{x}(t) \rangle \leq 0$ for all feasible trajectories $x(\cdot)$ and the inequality 46 can be converted to the relation

\[
\begin{align*}
\left( \frac{d^2 x(t)}{dt^2}, x^*(t) \right) - \left( \frac{d^2 \tilde{x}(t)}{dt^2}, x^*(t) \right) - g(x(t), t) + g(\tilde{x}(t), t) \leq \left( \frac{d^2 x^*(t)}{dt^2}, x(t) - \tilde{x}(t) \right) \\
+ \frac{d}{dt} (\psi^*(t), x(t) - \tilde{x}(t)) - \langle \eta^*(t+h), x(t) - \tilde{x}(t) \rangle + \langle \eta^*(t), x(t-h) - \tilde{x}(t-h) \rangle, \\
t \in [0, 1-h].
\end{align*}
\]

In turn, the latter inequality can be rewritten as follows

\[
\begin{align*}
g(x(t), t) - g(\tilde{x}(t), t) &\geq \left( \frac{d^2 x(t)}{dt^2} - \tilde{x}(t), x^*(t) \right) - \left( \frac{d^2 x^*(t)}{dt^2}, x(t) - \tilde{x}(t) \right) \\
- \frac{d}{dt} (\psi^*(t), x(t) - \tilde{x}(t)) + \langle \eta^*(t+h), x(t) - \tilde{x}(t) \rangle - \langle \eta^*(t), x(t-h) - \tilde{x}(t-h) \rangle, \\
t \in [0, 1-h].
\end{align*}
\]
Integrating this inequality over the interval \([0, 1 - \theta]\) and taking into account that \(x(0) = \bar{x}(0) = \theta\) we can write

\[
\int_0^{1-h} \left[ g(x(t), t) - g(\bar{x}(t), t) \right] \geq \int_0^{1-h} \left[ \left\langle \frac{d^2(x(t) - \bar{x}(t))}{dt^2}, x(t) - \bar{x}(1 - h) \right\rangle \right.
\]
\[
\left. - \left\langle \frac{d^2(x(t) - \bar{x}(t))}{dt^2}, \bar{x}(t) \right\rangle \right]\ dt + \langle \psi^*(1 - h), x(1 - h) - \bar{x}(1 - h) \rangle
\]
\[+ \int_0^{1-h} \langle \eta^*(t + h), x(t) - \bar{x}(t) \rangle dt - \int_0^{1-h} \langle \eta^*(t), x(t) - \bar{x}(t) \rangle dt = \int_0^{1-h} \langle \eta^*(t), x(t) - \bar{x}(t) \rangle dt.
\]  

(47)

By similar way, it follows from second order Euler-Lagrange type inclusion 45 that

\[
g(x(t), t) - g(\bar{x}(t), t) \geq \left\langle \frac{d^2(x(t) - \bar{x}(t))}{dt^2}, x(t) - \bar{x}(t) \right\rangle - \left\langle \frac{d^2(x(t) - \bar{x}(t))}{dt^2}, \bar{x}(t) \right\rangle
\]
\[- \frac{d}{dt} \langle \psi^*(1 - h), x(1 - h) - \bar{x}(1 - h) \rangle, t \in [1 - h, 1].
\]

Then an integration of this inequality over the interval \([1 - h, 1]\) give us

\[
\int_1^{1-h} \left[ g(x(t), t) - g(\bar{x}(t), t) \right] \geq \int_1^{1-h} \left[ \left\langle \frac{d^2(x(t) - \bar{x}(t))}{dt^2}, x(t) - \bar{x}(1) \right\rangle \right.
\]
\[
\left. - \left\langle \frac{d^2(x(t) - \bar{x}(t))}{dt^2}, \bar{x}(t) \right\rangle \right]\ dt + \langle \psi^*(1 - h), x(1 - h) - \bar{x}(1 - h) \rangle - \langle \psi^*(1), x(1) - \bar{x}(1) \rangle
\]
\[+ \int_0^{1-h} \langle \eta^*(t), x(t) - \bar{x}(t) \rangle dt - \int_0^{1-h} \langle \eta^*(t), x(t) - \bar{x}(t) \rangle dt.
\]  

(48)

Hence, summing the inequalities 47 and 48 we have

\[
\int_0^{1} \left[ g(x(t), t) - g(\bar{x}(t), t) \right] \geq \int_0^{1} \left[ \left\langle \frac{d^2(x(t) - \bar{x}(t))}{dt^2}, x(t) - \bar{x}(1) \right\rangle \right.
\]
\[
\left. - \left\langle \frac{d^2(x(t) - \bar{x}(t))}{dt^2}, \bar{x}(t) \right\rangle \right]\ dt - \langle \psi^*(1), x(1) - \bar{x}(1) \rangle
\]
\[+ \int_0^{1} \langle \eta^*(t), x(t) - \bar{x}(t) \rangle dt + \int_0^{1} \langle \eta^*(t), x(t) - \bar{x}(t) \rangle dt - \int_0^{1} \langle \eta^*(t), x(t) - \bar{x}(t) \rangle dt.
\]  

(49)

On the other hand, because of the initial condition \(x(t) = \bar{x}(t) = \xi(t), t \in [-h, 0]\) we get \(\int_0^{1} \langle \eta^*(t + h), x(t) - \bar{x}(t) \rangle dt = 0\). Then it is easy to compute the sum of the last three integrals on the right hand side of the inequality 49 as follows:

\[
\int_0^{1} \langle \eta^*(t + h), x(t) - \bar{x}(t) \rangle dt - \int_0^{1} \langle \eta^*(t), x(t) - \bar{x}(t) \rangle dt
\]
\[- \int_0^{1} \langle \eta^*(t), x(t) - \bar{x}(t) \rangle dt = \int_0^{1} \langle \eta^*(t + h), x(t) - \bar{x}(t) \rangle dt
\]
Thus, the inequality 49 can be simplified as follows
\[
\int_0^1 \left[ g(x(t), t) - g(\hat{x}(t), t) \right] \geq \int_0^1 \left[ \langle \frac{d^2(x(t) - \hat{x}(t))}{dt^2}, x^*(t) \rangle - \langle \frac{d^2x^*(t)}{dt^2}, x(t) - \hat{x}(t) \rangle \right] dt
- \langle \frac{dx^*(1)}{dt}, x(1) - \hat{x}(1) \rangle + \langle \frac{dx(0)}{dt}, x(0) - \hat{x}(0) \rangle = -\langle \frac{dx^*(1)}{dt}, x(1) - \hat{x}(1) \rangle.
\] (50)

Now we transform the expression in the square parentheses on the right hand side of 50:
\[
\langle \frac{d^2(x(t) - \hat{x}(t))}{dt^2}, x^*(t) \rangle - \langle \frac{d^2x^*(t)}{dt^2}, x(t) - \hat{x}(t) \rangle = d\langle \frac{d(x(t) - \hat{x}(t))}{dt}, x^*(t) \rangle - d\langle \frac{dx^*(t)}{dt}, x(t) - \hat{x}(t) \rangle.
\]

Then we use again the simplest and most useful particular case \(x(0) = \hat{x}(0) = \theta\) of feasibility solution of \((PC_3)\) and the condition \(x^*(0) = x^*(1) = 0\) of theorem. Then the integral on the right hand side of 50 over the interval \([0, 1]\) can be computed as follows
\[
\int_0^1 \left[ \langle \frac{d^2(x(t) - \hat{x}(t))}{dt^2}, x^*(t) \rangle - \langle \frac{d^2x^*(t)}{dt^2}, x(t) - \hat{x}(t) \rangle \right] dt
= \langle \frac{d(x(1) - \hat{x}(1))}{dt}, x^*(1) \rangle - \langle \frac{dx^*(0)}{dt}, x^*(0) \rangle
- \langle \frac{dx^*(1)}{dt}, x(1) - \hat{x}(1) \rangle + \langle \frac{dx(0)}{dt}, x(0) - \hat{x}(0) \rangle = -\langle \frac{dx^*(1)}{dt}, x(1) - \hat{x}(1) \rangle.
\] (51)

By substituting the expression of the integral 51 into the inequality 50 we deduce that
\[
\int_0^1 \left[ g(x(t), t) - g(\hat{x}(t), t) \right] dt \geq -\langle \frac{dx^*(1)}{dt}, x(1) - \hat{x}(1) \rangle
- \langle \psi^*(1), x(1) - \hat{x}(1) \rangle = -\langle \frac{dx^*(1)}{dt} + \psi^*(1), x(1) - \hat{x}(1) \rangle.
\] (52)

On the other hand by condition (iii) of theorem for all feasible arc \(x(\cdot)\) we obtain
\[
\varphi_0(x(1)) - \varphi_0(\hat{x}(1)) \geq \langle \frac{dx^*(1)}{dt} + \psi^*(1), x(1) - \hat{x}(1) \rangle, \quad \forall x(1) \in K_{\Omega(1)}^{P}(\hat{x}(1)).
\]

Thus, summing this inequality with the inequality 52 we have
\[
\int_0^1 \left[ g(x(t), t) - g(\hat{x}(t), t) \right] dt + \varphi_0(x(1)) - \varphi_0(\hat{x}(1)) \geq 0.
\]

Finally, we obtain that \(J[x(t)] \geq J[\hat{x}(t)], \forall x(t), t \in [0, 1]\), i.e. \(\hat{x}(t), t \in [0, 1]\) is optimal.

Below we prove that if a mapping \(F\) depends only on \(x\), then the adjoint inclusion involves only one conjugate variable, that is, there are no auxiliary adjoint variables \(\eta^*(t), u^*(t)\) in the conjugate second order delay-DFIs. Apparently, this occurs because a mapping \(F\) doesn’t depend on derivatives \(x(t), x(t - h)\).
Corollary 6.2. Suppose that for the problem (PCₙ) with second order delay-DFIs a set-valued mapping \( F \) depends only on \( x \), that is, \( F(\cdot, t) \equiv G(\cdot, t) : \mathbb{R}^{n} \Rightarrow \mathbb{R}^{n} \) and that the conditions of Theorem 6.1 are satisfied. Then the second order Euler-Lagrange type delay-DFIs and transversality condition (iii) of Theorem 6.1 consist of the following

\[
\begin{align*}
(i) \quad & \frac{d^2x^*(t)}{dt^2} \in G^*(x^*(t); (\dot{x}(t), \ddot{x}(t)), t) + \{K^*_{\Omega(t)}(\dot{x}(t)) - \partial g(\dot{x}(t), t)\} \\
& \quad \text{a.e. } t \in [0, 1], \quad x^*(0) = 0, \\
(ii) \quad & -\frac{dx^*(1)}{dt} \in K^*_{\Omega(1) \cap F} + \partial \varphi_0(\dot{x}(1)) - \partial g(\dot{x}(1)), \quad x^*(1) = 0, \\
(iii) \quad & \frac{d^2\tilde{x}(t)}{dt^2} \in G_A(\tilde{x}(t); x^*(t), t), \quad \text{a.e. } t \in [0, 1].
\end{align*}
\]

Proof. Indeed, it remains only to establish the second order Euler-Lagrange type delay-DFIs and transversality condition. This is an immediate consequence of Theorem 5.3; by passing to the formal limit in the conditions of this theorem we have the needed result.

Corollary 6.3. In addition, to assumptions of Theorem 6.1 let \( F \) be a closed set-valued mapping. Then the conditions (i), (ii), (iv) of Theorem 6.1 can be rewritten in term of subdifferentials of Hamiltonian function in the more symmetric form.

Proof. Using Lemma 3.2 we should prove only the validity of the inclusion

\[
\frac{d^2\tilde{x}(t)}{dt^2} \in \partial_* H_F(\tilde{x}(t), \dot{x}(t), \ddot{x}(t-h); x^*(t)), \quad \text{a.e. } t \in [0, 1].
\] (53)

Indeed, by Lemma 3.2 the argmaximal set at a given point is the subdifferential of the Hamiltonian function with respect to \( v^* \) and the inclusion (iv) of Theorem 6.1 coincides with the inclusion 53. Therefore, the assertions of corollary are equivalent to the conditions (i), (ii), (iv) of Theorem 6.1.

Corollary 6.4. For the problem (PCₙ) with non-delayed second order differential inclusions

\[
\begin{align*}
\text{minimize} & \quad J[x(\cdot)] = \int_0^1 g(x(t), t)dt + \varphi_0(x(1)), \\
\text{subject to} & \quad x''(t) \in F_0(x(t), x'(t), t), \quad \text{a.e. } t \in [0, 1], \\
& \quad x(t) \in \Omega(t), \quad x(0) = \theta, \quad t \in [0, 1], \quad x(1) \in P,
\end{align*}
\]

the second order Euler-Lagrange type inclusion has a form

\[
\begin{align*}
\left(\frac{d^2x^*(t)}{dt^2} + \frac{d\psi^*(t)}{dt}, \psi^*(t)\right) \in F_0^*(x^*(t); (\dot{x}(t), \dot{x}'(t), \dot{x}''(t)), t) \\
& + \{K^*_{\Omega(t)}(\dot{x}(t)) - \partial g(\dot{x}(t), t)\} \times \{0\}; \quad \text{a.e. } t \in [0, 1].
\end{align*}
\]

Proof. Indeed, in this case \( F_0(\cdot, t) : \mathbb{R}^{2n} \Rightarrow \mathbb{R}^{n} \) and \( F(x, u_1, u_2, t) = F_0(x, u_1, t), \forall u_2 \in \mathbb{R}^{n} \) is defined as \( F^*(v^*; (x, u_1, u_2, v), t) = F_0^*(v^*; (x, u_1, v), t) \times \{0\} \). It means that \( u^*_2 = 0 \) and consequently, \( \eta^*(t) \equiv 0, \forall t \in [0, 1] \). Then the proof of the corollary follows immediately from the conditions (i), (ii) of Theorem 6.1.

Theorem 6.5. Suppose that \( g : \mathbb{R}^n \times [0, 1] \to \mathbb{R}^1 \) is nonconvex function with respect to \( x \), and \( F \) is a nonconvex set-valued mapping such that \( K_{\partial gF(\cdot, \cdot)}(\tilde{x}(t), \dot{x}(t), \ddot{x}(t-h), x''(t)) \) is a local tent. Besides, suppose that \( K_{\Omega(t)}(\tilde{x}(t)), t \in [0, 1] \) and \( K_{F}(\tilde{x}(1)), \)
\( \ddot{x}(1) \in P \) are local tents. Then for an optimality of the arc \( \ddot{x}(t), t \in [0,1] \), among all feasible solutions in such a nonconvex problem \((PC_n)\), it is sufficient that there exist a triple \( \{x^*(t), \psi^*(t), \eta^*(t)\} \) of absolutely continuous functions \( x^*(t), x^*(t), \psi^*(t), \eta^*(t), t \in [0,1] \), satisfying the conditions of Theorem 6.1 in the nonconvex case:

(i) \( \left( \frac{d^2x^*(t)}{dt^2} + \frac{d\psi^*(t)}{dt} - \eta^*(t + \delta t) + x^*(t), \psi^*(t), \eta^*(t) \right) \in F^*(x^*(t)); (\ddot{x}(t), \dddot{x}(t), t) + \{K^*_\Omega(t)(\ddot{x}(t)) \times \{0\} \times \{0\}, \text{a.e. } t \in [0,1 - h], x^*(0) = 0; \)

(ii) \( \left( \frac{d^2x^*(t)}{dt^2} + \frac{d\psi^*(t)}{dt} + x^*(t), \psi^*(t), \eta^*(t) \right) \in F^*(x^*(t)); (\ddot{x}(t), \dddot{x}(t), \ddot{x}(t), t), t + \{K^*_\Omega(t)(\ddot{x}(t)) \times \{0\} \times \{0\}, \text{a.e. } t \in [1 - h,1]; \)

(iii) \( \frac{d^2\ddot{x}(t)}{dt^2} \in F_A(\ddot{x}(t), \dddot{x}(t), \ddot{x}(t), t), \text{a.e. } t \in [0,1], \)

(iv) \( g(x,t) - g(\ddot{x}(t),t) \geq \langle x^*(t), x - \ddot{x}(t) \rangle, t \in [0,1], \forall x \in \mathbb{R}^n, \)

(v) \( \varphi_0(x) - \varphi_0(\ddot{x}(1)) \geq \left\langle \frac{dx^*(1)}{dt} + \psi^*(1), x - \ddot{x}(1) \right\rangle, \forall x \in \mathbb{R}^n, \ x^*(1) = 0. \)

Proof. By condition (i) of theorem and definition of LAM in the nonconvex case (see Section 2)

\[
\langle \frac{d^2x(t)}{dt^2}, x^*(t) \rangle - \langle \frac{d^2\ddot{x}(t)}{dt^2}, x^*(t) \rangle \leq \langle \frac{d^2x^*(t)}{dt^2}, x(t) - \ddot{x}(t) \rangle + \langle x^*(t), x(t) - \ddot{x}(t) \rangle
\]

\[
+ \frac{d}{dt} \langle \psi^*(t), x(t) - \ddot{x}(t) \rangle - \langle \eta^*(t + \delta t), x(t) - \ddot{x}(t) \rangle + \langle \eta^*(t), x(t) - \ddot{x}(t) \rangle, t \in [0,1 - h]
\]

whereas by the condition (iv) of theorem for \( x = x(t) \) we can write

\[
g(x(t),t) - g(\ddot{x}(t),t) \geq \left\langle \frac{d^2(x(t) - \ddot{x}(t))}{dt^2}, x^*(t) \right\rangle - \left\langle \frac{d^2x^*(t)}{dt^2}, x(t) - \ddot{x}(t) \right\rangle
\]

\[
- \frac{d}{dt} \langle \psi^*(t), x(t) - \ddot{x}(t) \rangle + \langle \eta^*(t + \delta t), x(t) - \ddot{x}(t) \rangle - \langle \eta^*(t), x(t) - \ddot{x}(t) \rangle, t \in [0,1 - h].
\]

By similar way for \( x = x(t) \) we obtain

\[
g(x(t),t) - g(\ddot{x}(t),t) \geq \left\langle \frac{d^2(x(t) - \ddot{x}(t))}{dt^2}, x^*(t) \right\rangle - \left\langle \frac{d^2x^*(t)}{dt^2}, x(t) - \ddot{x}(t) \right\rangle
\]

\[
- \frac{d}{dt} \langle \psi^*(t), x(t) - \ddot{x}(t) \rangle - \langle \eta^*(t), x(t) - \ddot{x}(t) \rangle, t \in [1 - h,1].
\]

In the proof of Theorem 6.1 from the latter inequalities 54, 55 is justified 50 for a nonconvex case. Thus, the furthest proof of theorem is similar to the one for Theorem 6.1.

We note that in the convex case, the conditions (iv), (v) of Theorem 6.5 are equivalent to the conditions \( x^*(t) \in \partial_x g(\ddot{x}(t),t) \) and (iii) of Theorem 6.1, respectively. Then for a convex problem \((PC_n)\) it is easy to see that the previous conditions (i) and (ii) of Theorem 6.1 coincide with the conditions (i) and (ii) of Theorem 6.5, respectively.
In the conclusion of this section, let us consider an example on the problem of so-called linear optimal control problem for the second order delay-differential equations:

\[
\text{minimize } J[x(\cdot)] = \int_0^1 g(x(t), t) dt,
\]

\[(PL_h) \quad x''(t) = A_0 x(t) + A_1 x'(t) + A_2 x(t - h) + Bu(t), \text{ a.e. } t \in [0, 1],
\]

\[x(t) = \xi(t), \ t \in [-h, 0), \ x(0) = \theta, \ x(1) \in P,
\]

where \( g \) is continuously differentiable function in \( x, A_i, i = 0, 1, 2 \) and \( B \) are \( n \times n \) and \( n \times r \) matrices, respectively, \( U \subseteq \mathbb{R}^r \) is a convex closed subset. The problem is of finding corresponding to the controlling parameter \( \tilde{w}(t) \in U \) an arc \( \tilde{x}(t) \), minimizing \( J[x(\cdot)] \) over a set of feasible solutions.

We transform this problem to the following problem with second order delay-DFIs of the form:

\[
\text{minimize } J[x(\cdot)] = \int_0^1 g(x(t), t) dt,
\]

\[x''(t) \in F(x(t), x'(t), x(t-h)), \text{ a.e. } t \in [0, 1],
\]

\[x(t) = \xi(t), \ t \in [-h, 0), \ x(0) = \theta, \ x(1) \in P,
\]

\[F(x, u_1, u_2) = A_0 x + A_1 u_1 + A_2 u_2 + BU,
\]

where an admissible arc \( x(\cdot) \) is absolutely continuous function together with the first order derivatives for which \( x''(\cdot) \in L^2([0, 1]) \).

**Theorem 6.6.** The arc \( \tilde{x}(t) \) corresponding to the controlling parameter \( \tilde{w}(t) \) minimizes \( J[x(\cdot)] \) over a set of feasible solutions in the convex second order delay-differential problem \( (PL_h) \), if there exists an absolutely continuous function \( x^*(t) \) together with the first order derivatives, satisfying the second order an adjoint delay-differential equation, the transversality and Weierstrass-Pontryagin conditions:

\[
\frac{dx^*(t)}{dt} = A_0^* x^*(t) - A_1^* \frac{dx^*(t)}{dt} + A_2^* x^*(t+h) - g'(\tilde{x}(t), t), \text{ a.e. } t \in [0, 1-h);
\]

\[
\frac{d^2 x^*(t)}{dt^2} = A_0^* x^*(t) - A_1^* \frac{dx^*(t)}{dt} - g'(\tilde{x}(t), t), \text{ a.e. } t \in [1-h, 1],
\]

\[- \frac{dx^*(1)}{dt} \in K_p(\tilde{x}(1)); \ x^*(0) = x^*(1) = 0,
\]

\[\langle Bw(t), x^*(t) \rangle = \sup_{w \in U} \{Bu(t), x^*(t) \}, \ t \in [0, 1].
\]

**Proof.** In this problem we are proceeding on the basis of Theorem 6.1. Thus, taking into account that \( F(x, u_1, u_2) = A_0 x + A_1 u_1 + A_2 u_2 + BU \) in the problem 56 it can be easily computed that

\[
H_F(x, u_1 u_2, v^*) = \sup_v \{v, v^* : v \in F(x, u_1, u_2)\}
\]

\[= \sup_v \{\langle A_0 x + A_1 u_1 + A_2 u_2 + Bw, v^* \rangle : v \in F(x, u_1, u_2)\}
\]

\[= \langle x, A_0^* v^* \rangle + \langle u_1, A_1^* v^* \rangle + \langle u_2, A_2^* v^* \rangle + \sup_w \{\langle Bw, v^* \rangle : w \in U\}
\]

where \( A^* \) is adjoint (transposed) matrix of \( A \).
Then by Theorem 2.1 [15] one has

\[
F^* (v^*; (\tilde{x}, \tilde{u}_1, \tilde{u}_2, \tilde{v})) = \begin{cases} 
(A_0^* v^*, A_1^* v^*, A_2^* v^*), & -B^* v^* \in K_U(w), \\
\emptyset, & -B^* v^* \notin K_U(w),
\end{cases} (57)
\]

where \( \tilde{v} = A_0 \tilde{x} + A_1 \tilde{u}_1 + A_2 \tilde{u}_2 + B \tilde{w}, \tilde{w} \in U \). Thus, using (57) and the relations (i), (ii) of Theorem 6.1 we have the following system of Euler-Lagranges type linear adjoint equations:

\[
\frac{d^2 x^* (t)}{dt^2} + \frac{d \psi^* (t)}{dt} - \eta^* (t + h) = A_0^* x^* (t) - g^\prime (\tilde{x}(t), t), \\
\psi^* (t) = A_1^* x^* (t), \eta^* (t) = A_2^* x^* (t), \text{ a.e. } t \in [0, 1 - h); \quad (58)
\]

\[
\frac{d^2 x^* (t)}{dt^2} + \frac{d \psi^* (t)}{dt} = A_0^* x^* (t) - g^\prime (\tilde{x}(t), t), \\
\psi^* (t) = A_1^* x^* (t), \eta^* (t) = A_2^* x^* (t), \text{ a.e. } t \in [1 - h, 1]. \quad (59)
\]

Substituting the expressions for \( \psi^* (t), \eta^* (t) \) into equations (58, 59) we have second order Euler-Lagrange type adjoint delay-differential inclusions (equations):

\[
\frac{d^2 x^* (t)}{dt^2} = A_0^* x^* (t) - A_1^* \frac{dx^* (t)}{dt} + A_2^* x^* (t + h) - g^\prime (\tilde{x}(t), t), \quad \text{a.e. } t \in [0, 1 - h); \quad x^* (0) = 0, \quad (60)
\]

\[
\frac{d^2 x^* (t)}{dt^2} = A_0^* x^* (t) - A_1^* \frac{dx^* (t)}{dt} - g^\prime (\tilde{x}(t), t), \text{ a.e. } t \in [1 - h, 1]. \quad (61)
\]

On the other hand, since \( \psi^* (1) = A_1^* x^* (1) \) and \( x^* (1) = 0 \) it follows that the transversality conditions (iii) of Theorem 6.1 for linear optimal control problem \( (PL_h) \) consist of the following

\[
- \frac{dx^* (1)}{dt} \in K_P^\ast (\tilde{x} (1)); \quad x^* (1) = 0. \quad (62)
\]

Moreover, the Weierstrass-Pontryagin maximum principle [10], [15], [23] of theorem is an immediate consequence of the conditions (iv) of Theorem 6.1 and formula 57. Indeed, the condition \(-B^* v^* \in K_U (w)\) means that \( \sup_{w \in U} \langle Bw, v^* \rangle = \langle Bw, v^* \rangle \) and finally,

\[
\langle B\tilde{w}(t), x^* (t) \rangle = \sup_{w \in U} \langle Bw, x^* (t) \rangle, \quad t \in [0, 1]
\]

Then by this maximum principle and relations 60-62 we have the desired result. The proof is completed.

\[\square\]

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