ON THE GAUSS-KUZMIN-LÉVY PROBLEM FOR NEAREST INTEGER CONTINUED FRACTIONS

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Abstract. This note provides an effective bound in the Gauss-Kuzmin-Lévy problem for some Gauss type shifts associated with nearest integer continued fractions, acting on the interval $I_0 = [0, \frac{1}{2}]$ or $I_0 = [-\frac{1}{2}, \frac{1}{2}]$. We prove asymptotic formulas $\lambda(T^{-n}I) = \mu(I)(\lambda(I_0) + O(q^n))$ for such transformations $T$, where $\lambda$ is the Lebesgue measure on $\mathbb{R}$, $\mu$ the normalized $T$-invariant Lebesgue absolutely continuous measure, $I$ subinterval in $I_0$, and $q = 0.288$ is smaller than the Wirsing constant $q_W = 0.303663\ldots$

1. Introduction

The regular continued fraction establishes a one-to-one correspondence between the set of infinite words with letters in the alphabet $\mathbb{N}$ and the set $[0, 1] \setminus \mathbb{Q}$:

$$(a_1, a_2, a_3, \ldots) \mapsto \frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \cdots$$

The Gauss shift $G$ acts on $\mathbb{N}$ by $G(a_1, a_2, a_3, \ldots) = (a_2, a_3, a_4, \ldots)$, and on $[0, 1]$ by $G(x) = \{\frac{1}{x}\}$ if $x \neq 0$ and $G(0) = 0$. Gauss discovered that the probability measure $d\mu = \frac{dx}{(1+x)\log 2}$ is $G$-invariant on $[0, 1]$ and stated in his diary (October 25, 1800) that

$$\lambda(G^{-n}[0, x]) \sim \mu([0, x]) = \frac{\log(1+x)}{\log 2}, \quad \forall x \in [0, 1] \text{ as } n \to \infty,$$  \hspace{1cm} (1)

where $\lambda$ denotes the Lebesgue measure on $\mathbb{R}$. In a 1812 letter to Laplace ([3], see also Appendix III of [24]), Gauss raised the problem of providing an effective version of (1) and estimate the error

$$E_n(x) = \lambda(G^{-n}[0, x]) - \mu([0, x]), \quad x \in [0, 1], \text{ } n \to \infty.$$ The problem was thoroughly investigated much later, with significant contributions by Kuzmin [9] and Lévy [10]. Kuzmin proved that, uniformly in $x$, $E_n(x) = O(q^{\sqrt{n}})$ for some $q \in (0, 1)$, while Lévy proved that $E_n(x) = O(q^n)$ with $q < 0.7$. The breakthrough result of Wirsing [26] proved that

$$E_n(x) = \psi(x)q_W^n + O(q_1^n),$$  \hspace{1cm} (2)

with $q_W = 0.303663\ldots$ denoting the Wirsing (optimal) constant, $0 < q_1 < q_W$ and $\psi$ some real analytic function on $[0, 1]$. The spectral approach due to Babenko [2] and Mayer-Roepstorff [11] provided a complete solution to the problem, showing that the restriction of the Perron-Frobenius operator of $G$ to some Hardy space on the right half-plane $\text{Re } z > -\frac{1}{2}$ is similar to a self-adjoint trace-class operator with explicit kernel, and thus the expression of $E_n(x)$ in (2) can be completed to the eigenfunction expansion of this compact operator. A detailed discussion of the Gauss problem with complete proofs can be found in the monograph [6].

It is natural to study the analogue of the Gauss problem for other classes of continued fractions. This note takes an elementary look at the situation of the nearest integer continued fraction (NICF), originally considered in Minnigerode’s work on the Pell equation [12] and furthered by Hurwitz [4]. NICF provides a better rate of approximation than the regular continued fraction. Actually, each
nearest integer convergent of an irrational number is a regular continued fraction convergent of that number \[1, 25]. Other Diophantine approximation properties, such as analogues of Vahlen’s theorem, were studied in \([7, 23]\). Analogues of the Gauss problem for other types of continued fractions have been recently studied in \([20, 21, 22]\).

In this paper we denote \(G = \sqrt{\frac{5}{2}+1},\) \(g = \sqrt{\frac{5}{2}-1},\) and employ the equalities \(G - 1 = g,\) \(G + 1 = G^2\) and \((2 - G)(G + 1) = 1.\)

The NICF can appear in various guises. We will consider three possible situations, as follows:

(A) The folded NICF map \(T : [0, \frac{1}{2}] \to [0, \frac{1}{2}]\) defined by \(T(0) = 0\) and, for \(x \neq 0,\) by

\[
T(x) = \begin{cases} 
\frac{1}{x} - \frac{1}{x + 1} & \text{if } 2k \leq x \leq \frac{2}{2k-1}, \ k \geq 3 \\
\frac{1}{x} - 2 & \text{if } \frac{2}{2k-1} \leq x \leq \frac{1}{2} 
\end{cases}
\] (3)

is continuous on \((0, \frac{1}{2})\). For every \(x \in [0, \frac{1}{2}] \setminus \mathbb{Q}\), let \(a_1 = a_1(x) := \lfloor x + \frac{1}{2} \rfloor \geq 2,\) \(e_1 = e_1(x) := \text{sign}(\frac{1}{2} - a_1) \in \{\pm 1\}\). They satisfy \(a_1 + e_1 \geq 2\). Note that \(T(x) = e_1(\frac{1}{2} - a_1) = |\frac{1}{2} - a_1|, \ \forall x \in [0, \frac{1}{2}] \setminus \mathbb{Q}\).

Taking \(a_i = a_i(x) := a_i(T^{i-1}(x)),\) \(e_i = e_i(x) := e_i(T^{i-1}(x))\) if \(i \geq 2\), every irrational \(x \in [0, \frac{1}{2}]\) is represented as

\[
x = \frac{1}{a_1} + \frac{e_1}{a_2} + \frac{e_2}{a_3} + \cdots = [(a_1, e_1), (a_2, e_2), (a_3, e_3), \ldots],
\]

with \(a_i \geq 2,\) \(e_i \in \{\pm 1\}\) and \(a_i + e_i \geq 2.\)

The map \(T\) is called a Gauss type shift because it acts on \([0, \frac{1}{2}] \setminus \mathbb{Q}\) by shifting the digits \((a_i, e_i):\)

\[
T([(a_1, e_1), (a_2, e_2), \ldots]) = [(a_2, e_2), (a_3, e_3), \ldots].
\]

According to Lemma 1 below, the probability measure

\[
d\mu = \frac{1}{\log G} \left( \frac{1}{G + x} + \frac{1}{G + 1 - x} \right) dx
\]

is \(T\)-invariant.

(B) The odd map \(T_o : [-\frac{1}{2}, \frac{1}{2}] \to [-\frac{1}{2}, \frac{1}{2}],\) investigated by Nakada, Ito and Tanaka \([15]\) and defined by \(T_o(0) = 0\) and, for \(x \neq 0,\) by

\[
T_o(x) = \frac{1}{x} - \frac{1}{x + 1} = \begin{cases} 
\frac{1}{x} - k \text{sgn } x & \text{if } \frac{2}{2k+1} < |x| \leq \frac{2}{2k-1}, \ k \geq 3 \\
\frac{1}{x} - 2 \text{sgn } x & \text{if } \frac{2}{2k-1} \leq |x| \leq \frac{1}{2} 
\end{cases}
\] (4)

represents the Gauss shift associated with the continued fraction expansion

\[
x = \frac{1}{b_1} + \frac{1}{b_2} + \frac{1}{b_3} + \cdots = [b_1, b_2, b_3, \ldots]
\]

of irrationals in \([-\frac{1}{2}, \frac{1}{2}]\), with digits \(b_i = b_i(x) \in \mathbb{Z}\) given by \(b_1 := \lfloor \frac{1}{2} + \frac{1}{2} \rfloor,\) \(b_i := b_i(T_o^{i-1}(x))\) if \(i \geq 2.\) Then \(|b_i| \geq 2, b_i = 2 \Rightarrow b_{i+1} \geq 2,\) and \(b_i = -2 \Rightarrow b_{i+1} \leq -2.\) Indeed, it is plain that \(T_o(x) = \frac{1}{x} - b_1, \ \forall x \in [-\frac{1}{2}, \frac{1}{2}] \setminus \mathbb{Q},\) and \(T_o([b_1, b_2, \ldots]) = [b_2, b_3, \ldots].\) As shown in \([15]\), the probability measure

\[
d\mu_o = \frac{1}{2\log G} \left( \frac{1}{G + |x|} + \frac{1}{G + 1 - |x|} \right) dx
\]

is \(T_o\)-invariant.

The identity \(-\frac{1}{b_1} - \frac{1}{b_2} - \frac{1}{b_3} - \cdots = -\frac{1}{-b_1} + \frac{1}{b_2} + \frac{1}{-b_3} + \frac{1}{b_4} + \cdots\) shows that \(T_o\) can also be viewed as the Gauss shift generated by the NICF expansion \(b_0 = \frac{1}{b_1} - \frac{1}{b_2} - \cdots\) considered in \([1, 4, 25].\)
The even map $T_e : [-\frac{1}{2}, \frac{1}{2}] \rightarrow [-\frac{1}{2}, \frac{1}{2}]$, considered by Rieger \[18, 19\] and defined by $T_e(0) = 0$, and, for $x \neq 0$, by

$$T_e(x) := \frac{1}{|x|} - \left\lfloor \frac{1}{|x|} + \frac{1}{2} \right\rfloor = \begin{cases} \frac{1}{|x|} - k & \text{if } \frac{2k+1}{2} < |x| \leq \frac{2k}{2}, k \geq 3 \\ \frac{1}{|x|} - 2 & \text{if } \frac{2}{5} < |x| \leq \frac{1}{2} \end{cases} = T_e(-x),$$

generates the NICF expansion

$$x = \left\lfloor \frac{e_1}{a_1} \right\rfloor + \left\lfloor \frac{e_2}{a_2} \right\rfloor + \left\lfloor \frac{e_3}{a_3} \right\rfloor + \cdots =: [(a_1, e_1), (a_2, e_2), (a_3, e_3), \ldots]$$

of irrationals in $[-\frac{1}{2}, \frac{1}{2}]$, with digits $a_1 = a_1(x) := \lfloor \frac{1}{|x|} + \frac{1}{2} \rfloor$, $e_1 = e_1(x) := \text{sign}(\frac{1}{|x|} - a_1)$, $a_i := a_1(T_e^{-1}(x))$, $e_i := e_1(T_e^{-1}(x))$ if $i \geq 2$ satisfying $a_i \geq 2$, $e_i \in \{\pm 1\}$, $a_i + e_{i+1} \geq 2$. This NICF expansion is also considered in \[7, 16, 23\].

We have $T_e(x) = \frac{a_1}{x} - a_1 = \frac{1}{|x|} - a_1$ and $T_e([(a_1, e_1), (a_2, e_2), \ldots]) = [(a_2, e_2), (a_3, e_3), \ldots]$, so $T_e$ is the Gauss shift associated with this NICF expansion.

The map $T_e$ coincides with Nakada’s map $f_{1/2}$ \[14\]. In particular, the probability measure

$$d\mu_e = h_e(x)dx, \quad h_e(x) = \frac{1}{ \log G} \begin{cases} \frac{1}{G+x} & \text{if } 0 < x < \frac{1}{2} \\ \frac{1}{G+x+1} & \text{if } -\frac{1}{2} < x < 0 \end{cases}$$

is $T_e$-invariant.

The main result of this note provides quantitative estimates for the analogue of the Gauss-Kuzmin-Lévy problem in the situations of the Gauss type shifts $T$, $T_o$ and $T_e$, as follows:

**Theorem 1.**

(i) With $q = 0.288$, for every Borel set $E \subseteq [0, \frac{1}{2}]$,

$$\lambda(T^{-n}E) = \frac{1}{2} \mu(E) + O(\mu(E)q^n).$$

(ii) With $q = 0.288$, for every Borel set $E \subseteq [-\frac{1}{2}, \frac{1}{2}]$,

$$\lambda(T_o^{-n}E) = \mu_o(E) + O(\mu_o(E)q^n).$$
(iii) With \( q = 0.234 \), for every Borel set \( E \subseteq [-\frac{1}{2}, \frac{1}{2}] \),

\[
\lambda(T^{-n}E) = \mu_e(E) + O(\mu_e(E)q^n).
\]

The estimate in (ii) improves upon \( q = g^2 \approx 0.382 \) obtained in [15] Thm.2.1(ii)]. The estimate in (iii) improves upon \( q = \frac{2}{3} \) obtained in [18]. Note that \( q = 0.288 \) is smaller than the Wirsing constant \( q_W = 0.3036 \ldots \).

To prove Theorem 1, we perform an elementary analysis of the Perron-Frobenius operators associated to the transformations \( T \) and \( T_e \) with respect to their invariant Lebesgue absolutely continuous measures along the line of [18].

In [5] and [17], the authors investigated a problem similar to (iii). However, their transition operator \( U \) coincides with the Perron-Frobenius operator associated to the dual of the NICF Gauss map, rather than the NICF Gauss map itself. This dual is the folded Hurwitz transformation \( S \), which acts on \([0, g]\) by \( S(0) = 0 \) and

\[
S(x) = \left| \frac{1}{x} - i \right| \quad \text{if} \quad \frac{1}{i+g} < x \leq \frac{1}{i+g-1}, \quad i \geq 2,
\]

with \( S \)-invariant probability measure

\[
d\nu = k(x)dx, \quad k(x) = \frac{1}{\log G} \begin{cases} \frac{1}{2+x} + \frac{1}{2-x} & \text{if} \quad x \in [0, g^2) \\ \frac{1}{2+x} & \text{if} \quad x \in [g^2, g]. \end{cases}
\]

We also provide some estimates on the rate of mixing of the map \( T \).

**Corollary 1.** With \( q = 0.288 \), for any Borel set \( E \subseteq [0, \frac{1}{2}] \) and any \( T \)-cylinder \( F \),

\[
\mu(T^{-n}E \cap F) = \mu(E)\mu(F) + O_F(q^n).
\]

**Corollary 2.** With \( q = 0.288 \), for any Borel set \( E \subseteq [-\frac{1}{2}, \frac{1}{2}] \) symmetric with respect to the origin, and any \( T_e \)-cylinder \( F \),

\[
\mu_o(T_o^{-n}E \cap F) = \mu_o(E)\mu_o(F) + O_F(q^n).
\]

Corollary 2 was proved with \( q = g^2 \) in [15] Thm.2.1(iii)] without assuming \( E \) symmetric.

An analogue of Corollaries 1 and 2 will be discussed at the end of Section 3.

2. The Folded NICF Map \( T \) and the Nakada-Ito-Tanaka Map \( T_o \)

The folded NICF can be obtained as a particular example of a folded Japanese continued fraction, investigated by Moussa, Cassa and Marmi [13]. The following lemma follows by taking \( \alpha = \frac{1}{2} \) in [13 Thm.15], or it can be verified directly through a plain calculation.

**Lemma 1.** The probability measure \( d\mu = Ch(x)dx \), with

\[
h(x) := \frac{1}{G+x} + \frac{1}{G+1-x}, \quad C = \frac{1}{\log G},
\]

is \( T \)-invariant.

Denote

\[
W \defeq \{(k,1): k \geq 2\} \cup \{(k,-1): k \geq 3\}, \quad w_{(k,e)}(y) \defeq \frac{1}{k+ey}.
\]

Following [8 Sect.2.3], the Perron-Frobenius (Ruelle) operator \( P = \widehat{T}_\lambda \) of \( T \) with respect to the Lebesgue measure \( \lambda \) acts on \( L^1([0, \frac{1}{2}], \lambda) \) by

\[
(Pf)(y) = \sum_{x \in T^{-1}y} f(x(y))g(x)(y) = \sum_{(k,e) \in W} w_{(k,e)}(y)f(w_{(k,e)}(y))
\]

\[
= \sum_{k \geq 2} w_{(k,1)}^2(y)f(w_{(k,1)}(y)) + \sum_{k \geq 3} w_{(k,-1)}^2(y)f(w_{(k,-1)}(y)). \tag{5}
\]
The Perron-Frobenius (transfer) operator $U := \hat{T}_\mu$ of $T$ with respect to the invariant measure $\mu$ acts on $L^1([0, \frac{1}{2}], \mu)$ by

$$U = M_H P M_H^{-1} = M_H P M_h,$$

where $M_H$ denotes the operator of multiplication by $H := \frac{1}{2}$. Since $\mu$ is a $T$-invariant measure, one has $U1 = 1$. One can also consider $U$ as the transpose (dual) of the Koopman operator defined by $K_T f := f \circ T$.

The equalities (7) and (8) show that $U$'s weights $P_{(k,e)}$ satisfy

$$\sum_{(k,e) \in W} P_{(k,e)}(y) = 1 \quad \text{and} \quad \sum_{(k,e) \in W} P'_{(k,e)}(y) = 0, \quad \forall y \in [0, \frac{1}{2}].$$

The first identity in (6) allows us to write

$$(Uf)(y) = \sum_{(k,e) \in W} P_{(k,e)}(y) f(w_{(k,e)}(y)),$$

with

$$P_{(k,e)}(y) = H(y) \left( \frac{1}{k + G - 2 + ey} - \frac{1}{k + G - 1 + ey} \right) \geq 0.$$

The equalities (7) and (8) show that $U(C[0, \frac{1}{2}]) \subseteq C[0, \frac{1}{2}]$ and $U(C^1[0, \frac{1}{2}]) \subseteq C^1[0, \frac{1}{2}]$.

Since $U1 = 1$, the weights $P_{(k,e)}$ satisfy

$$\sum_{(k,e) \in W} P_{(k,e)}(y) = 1 \quad \text{and} \quad \sum_{(k,e) \in W} P'_{(k,e)}(y) = 0, \quad \forall y \in [0, \frac{1}{2}].$$

Inserting $A = k + ey$, $B = G - 2 = -\frac{1}{G^2}$, which satisfy $\frac{1}{A} - \frac{1}{B+1} = -G^3$ and $\frac{1}{B} + \frac{1}{B+1} = -1$, in the identity

$$\frac{1}{A^2} \left( \frac{1}{A+B} - \frac{1}{A+B+1} \right) = \left( \frac{1}{B} - \frac{1}{B+1} \right) \frac{1}{A^2} - \left( \frac{1}{B^2} - \frac{1}{(B+1)^2} \right) \frac{1}{A},$$

we infer

$$\frac{P_{(k,e)}(y)}{(k+ey)^2} = H(y) \left( -\frac{G^3}{(k+ey)^2} - \frac{G^3}{k+G-2+ey} + \frac{G^3}{k+G-1+ey} - \frac{G^2}{k+G-2+ey} \right)$$

$$= H(y) \left( -\frac{G^3}{(k+ey)^2} + \frac{G^3}{(k+ey)(k+G-2+ey)} \right.$$

$$\left. + \frac{G^2}{k+G-2+ey} - \frac{1}{k+G-1+ey} \right).$$

**Proposition 1.** $\|(Uf)'\|_\infty \leq 0.288 \|f'\|_\infty$, $\forall f \in C^1[0, \frac{1}{2}]$.

**Proof.** Employing (7), (9), the Mean Value Theorem, and $|w_{(k,e)}(y) - \frac{1}{4}| \leq \frac{1}{4}$, we can write

$$|(Uf)'(y)| \leq \sum_{(k,e) \in W} \frac{P_{(k,e)}(y)}{(k+ey)^2} |f'(w_{(k,e)}(y))| + \sum_{(k,e) \in W} |P'_{(k,e)}(y)(f(w_{(k,e)}(y)) - f(\frac{1}{4}))|$$

$$\leq (S_I(y) + S_{II}(y))\|f'\|_\infty, \quad \forall y \in [0, \frac{1}{2}],$$
Note that $\Phi$ and compute

leads to

The identity

leads to

Note that $\Phi_1(0^+) = \frac{\pi^2}{3} - \frac{9}{4}$.

We also have

Combining (10) with the last two equations above and using $H(y) = \frac{(G+y)(G+1-y)}{G^3}$ we find

where

Numerically, Mathematica gives

To bound $S_{II}(y)$, we write $P_{(k,e)} = L_{(k,e)} - L_{(k+1,e)}$, with

and compute

$$P'_{(k,e)}(y) = \frac{1 - 2y}{G^3} \left( \frac{1}{k + ey + G - 2} - \frac{1}{k + ey + G - 1} \right) - \frac{e(G^3 + y - y^2)}{G^3} \left( \frac{1}{(k + ey + G - 2)^2} - \frac{1}{(k + ey + G - 1)^2} \right).$$
Summing over \((k, e) \in W\), we find
\[
S_{11}(y) \leq \frac{1 - 2y}{4G^3} \left( \frac{1}{G + y} + \frac{1}{G + 1 - y} \right) + \frac{(G + y)(G + 1 - y)}{4G^3} \left( \frac{1}{(G + y)^2} + \frac{1}{(G + 1 - y)^2} \right)
\]
\[
= \frac{1 - 2y}{4(G + y)(G + 1 - y)} + \frac{(G + y)(G + 1 - y)}{4G^3} \left( \frac{1}{(G + y)^2} + \frac{1}{(G + 1 - y)^2} \right)
\]
\[
< 0.191, \quad \forall y \in [0, \frac{1}{2}].
\]
We infer \(S_1(y) + S_{11}(y) < 0.097 + 0.191 = 0.288, \forall y \in [0, \frac{1}{2}]\). \(\square\)

**Proof of (i) and (ii) in Theorem 1.** (i) Consider \(\gamma_n := U^nH \in C^1[0, \frac{1}{2}]\). With \(h\) as in Lemma 1, \(H := \frac{1}{h}\), and taking \(dv := h(x)dx\), we have
\[
\lambda(T^{-n}E) = \int_E U^nH \, dv = \int_E \gamma_nh \, d\lambda.
\]
Proposition 1 shows that
\[
\|\gamma'_n\|_{\infty} \leq \|H'\|_{\infty} q^n < q^n, \quad \forall n \geq 1.
\]
The Mean Value Theorem then yields
\[
|\gamma_n(x) - \gamma_n(0)| \leq q^n x \quad \forall x \in [0, \frac{1}{2}],
\]
or equivalently
\[
|\gamma_n(x)h(x) - \gamma_n(0)h(x)| \leq q^n x h(x) \leq q^n, \quad \forall x \in [0, \frac{1}{2}], \forall n \geq 1.
\]
Therefore, we have
\[
\frac{1}{2} = \lambda(T^{-n}[0, \frac{1}{2}]) = \int_0^{1/2} \gamma_nh \, d\lambda = \gamma_n(0) \int_0^{1/2} h \, d\lambda + O(q^n)
\]
\[
\implies \gamma_n(0) = \frac{1}{2 \int_0^{1/2} h \, d\lambda} + O(q^n)
\]
\[\text{(13)}\]
\[
\gamma_n(x)h(x) = \frac{h(x)}{2 \int_0^{1/2} h \, d\lambda} + O(q^n)
\]
\[\text{(12)}\]
\[
\lambda(T^{-n}E) = \frac{\int_E h \, d\lambda}{2 \int_0^{1/2} h \, d\lambda} + O(\mu(E)q^n) = \frac{1}{2} \mu(E) + O(\mu(E)q^n).
\]
In the last line above we also used \(\lambda \ll \mu \ll \lambda\).

(ii) The probability measure \(\mu_o\) from the introduction is \(T_o\)-invariant. Furthermore, the measure \(\mu\) is equal to two times the push-forward of \(\mu_o\) under the map \(\|\|: [-\frac{1}{2}, 0] \rightarrow [0, \frac{1}{2}]\).

Consider a Borel set \(E \subseteq [0, \frac{1}{2}]\). We have \(T(x) = |T_o(x)|, \forall x \in [0, \frac{1}{2}]\), so \(T = |T_o|_{[0,1/2]}\) and \(T^n = |T_o^n|_{[0,1/2]}\), \(\forall n \geq 1\). Each map \(T_o^n\) is odd, so \(T_o^{-n}(-E) = -T_o^{-n}E\). This entails
\[
T^{-n}E = \{x \in [0, \frac{1}{2}]: T_o^n(x) \in E \cup (-E)\} = (T_o^{-n}E \cup (-T_o^{-n}E)) \cap [0, \frac{1}{2}].
\]
The conclusion follows from
\[
\lambda(T^{-n}E) = \lambda(T_o^{-n}E \cap [0, \frac{1}{2}]) + \lambda((-T_o^{-n}E) \cap [0, \frac{1}{2}])
\]
\[
= \lambda(T_o^{-n}E \cap [0, \frac{1}{2}]) + \lambda(T_o^{-n}E \cap [-\frac{1}{2}, 0]) = \lambda(T_o^{-n}E)
\]
and Theorem 1 (i), using also \(\mu(E) = 2\mu_o(E)\).

When \(E \subseteq [-\frac{1}{2}, 0]\), we use again \(T_o^{-n}(E) = -T_o^{-n}(-E)\) and \(\mu_o(-E) = \mu_o(E)\). \(\square\)
The $T$-cylinders are given by $\Delta_{[a_1, e_1]} := \{ \frac{1}{a_1 + e_1 y} : y \in [0, \frac{1}{2}] \}$ and when $r \geq 2$ by

$$\Delta_{[a_1, e_1, \ldots, a_r, e_r]} = \bigcap_{i=1}^{r} T^{-(i-1)} \Delta_{[a_i, e_i]} = \left\{ \frac{1}{a_1} + \frac{e_1}{a_2} + \cdots + \frac{e_{r-1}}{a_r + e_r} : y \in [0, \frac{1}{2}] \right\}.$$ 

**Proof of Corollary 1.** Let $F = \Delta_{[a_1, e_1, \ldots, a_r, e_r]}$. We will estimate

$$\mu(T^{-n}E \cap F) = \int_{T^{-n}E} \chi_F d\mu = \int_E U^n \chi_F d\mu.$$ 

From $\chi_F \circ w(k, e) = \delta(k, e), \chi_{\Delta_{[a_2, e_2, \ldots, a_r, e_r]}}$ and equality [7] we infer

$$U\chi_F = P_{(a_1, e_1)} : \chi_{\Delta_{[a_2, e_2, \ldots, a_r, e_r]}}.$$ 

and finally,

$$U^n \chi_F = \prod_{i=1}^{r} P_{(a_i, e_i)} \circ w_{(a_{i+1}, e_{i+1})} \cdots \circ w_{(a_r, e_r)} =: C_F \in C^1[0, \frac{1}{2}],$$

where the term corresponding to $i = r$ is just $P_{(a_r, e_r)}(y)$.

Proposition 1 and equality [15] entail $\|U^n \chi_F\|_{\infty} = \|(U^{n-r} C_F)\|_{\infty} \ll_F q^n$, $\forall n \geq r$, with $q = 0.288$. Applying the Mean Value Theorem we get

$$\|U^n \chi_F \cdot h - (U^n \chi_F)(0) \cdot h\|_{\infty} = \|U^{n-r} C_F \cdot h - (U^{n-r} C_F)(0) \cdot h\|_{\infty} \leq \|U^{n-r} C_F - (U^{n-r} C_F)(0)\|_{\infty} \ll_F q^n, \quad \forall n \geq r.$$ 

Integrating on $[0, \frac{1}{2}]$ this yields

$$\mu(F) = \int_0^{1/2} U^n \chi_F d\mu = (U^n \chi_F)(0) + O_F(q^n), \quad \forall n \geq r.$$ 

Plugging this back in [16] we find

$$(U^n \chi_F)(x) = \mu(F) + O_F(q^n), \quad \forall n \geq r, \quad \forall x \in [0, \frac{1}{2}].$$

Integrating on $E$ and employing [14] we reach the desired conclusion.

The $T_o$-cylinders are given by $\Delta_{[b_1]} := \{ \frac{1}{b_1 + y} : y \in [-\frac{1}{2}, \frac{1}{2}] \}$ and when $r \geq 2$ by $\Delta_{[b_1, \ldots, b_r]} = \Delta_{[b_1]} \cap T_{-1} \Delta_{[b_2]} \cap \cdots \cap T_{-1}^{(r-1)} \Delta_{[b_r]}$.

**Proof of Corollary 2.** Writing $E = E_+ \cup (-E_+)$ with $E_+ \subseteq [0, \frac{1}{2}]$, we have $\mu_o(E) = \mu(E_+) = 2\mu_o(E_+)$. Every $T_o$-cylinder is either a $T$-cylinder or the union of two $T$-cylinders, so we can assume that $F$ is a $T$-cylinder. We have either $F \subseteq (0, \frac{1}{2}]$ or $F \subseteq [-\frac{1}{2}, 0)$.

Assume first $F \subseteq (0, \frac{1}{2}]$. The equality

$$T_{-n}^o E \cap F = (T_{-n}^o E_+ \cap F) \cup ((-T_{-n}^o E_+) \cap F) = T_{-n}^o E_+ \cap F$$

and Corollary 1 entail

$$\mu_o(T_{-n}^o E \cap F) = \mu_o(T_{-n}^o E_+ \cap F) = 2\mu(T_{-n}^o E_+ \cap F) = 2\mu(E_+) \mu(F) + O_F(q^n) = \mu(E) \mu(F) + O_F(q^n).$$

When $F \subseteq [-\frac{1}{2}, 0)$, we employ $\mu_o(T_{-n}^o E \cap F) = \mu_o(-T_{-n}^o E \cap F) = \mu_o(T_{-n}^o E \cap (-F))$. 

$\square$
Conjugating the map $T_e$ by $J : [-\frac{1}{2}, \frac{1}{2}] \rightarrow [0, 1]$, $J(x) := \begin{cases} x & \text{if } 0 \leq x < \frac{1}{2} \\ x + 1 & \text{if } -\frac{1}{2} \leq x \leq 0 \end{cases}$ with $J^{-1}(y) = \begin{cases} y & \text{if } 0 \leq y < \frac{1}{2} \\ y - 1 & \text{if } \frac{1}{2} \leq y \leq 1 \end{cases}$, one gets $\tilde{T}_e = JT_eJ^{-1} : [0, 1] \rightarrow [0, 1]$, which satisfies $0 < y < \frac{1}{2} \Rightarrow \tilde{T}_e(y) = JT_eJ^{-1}(y) = JT_e(y) = \begin{cases} \frac{1}{y} - k & \text{if } \frac{2}{2k+1} < y < \frac{1}{k}, k \geq 2 \\ \frac{1}{y} - k + 1 & \text{if } \frac{1}{k} < y < \frac{2}{2k+1}, k \geq 3 \end{cases}$.

$\frac{1}{2} < y < 1 \Rightarrow \tilde{T}_e(y) = JT_eJ^{-1}(y) = JT_e(y - 1) = JT_e(1 - y) = JT_eJ^{-1}(1 - y) = \tilde{T}_e(1 - y)$.

Observe that $\tilde{T}_e(x) = G(x) = \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor$, $\forall x \in (0, \frac{1}{2}]$.

The push-forward probability measure $\tilde{\mu}_e = J_\ast \mu_e = \tilde{h}_e d\lambda$, $\tilde{h}_e(x) = \frac{1}{(G+x) \log G}$, is $\tilde{T}_e$-invariant. We also have $\lambda(S) = \lambda(JS)$ for every Borel set $S \subseteq [-\frac{1}{2}, \frac{1}{2}]$.

The Perron-Frobenius operator $\tilde{P} = (\tilde{T}_e)_\lambda$ associated with the transformation $\tilde{T}_e$ and the Lebesgue measure is given by

$$(\tilde{P}f)(y) = \sum_{x \in \tilde{T}_e^{-1}(y)} f(x(y))|x'(y)| = \sum_{k=2}^{\infty} \left( f\left( \frac{1}{y+k} \right) + f\left( 1 - \frac{1}{y+k} \right) \right) \frac{1}{(y+k)^2}, \quad y \in [0, 1].$$

It satisfies $\tilde{P \tilde{h}_e} = \tilde{h}_e$, emphasizing that $\tilde{\mu}_e$ is $\tilde{T}_e$-invariant. Set $\tilde{H}_e = \frac{1}{\tilde{h}_e}$. The Perron-Frobenius operator $\tilde{U} := (\tilde{T}_e)_{\tilde{\mu}_e}$ associated with the $\tilde{T}_e$-invariant measure $\tilde{\mu}_e$, given by $\tilde{U} = M_{\tilde{H}_e} \tilde{P} M_{\tilde{h}_e}$, is
Lemma 3. \[ \| f \|_\infty \leq \frac{1}{1346} \| f' \|_\infty, \forall f \in C^1[0, 1]. \]

Proof. We have \[ |(S_I f)(x)| \leq \Phi(x) \| f' \|_\infty, \] where \( \Phi = \Phi_2 + \Phi_3 + \Phi_4 + \Phi_5, \) with

\[
\begin{align*}
\Phi_2(x) &= \frac{A_2(x) + B_2(x)}{(2 + x)^2} = \frac{1}{(2 + x)^2(G + 1 + x)}, \\
\Phi_3(x) &= \frac{A_3(x) + B_3(x)}{(3 + x)^2} = \frac{G + x}{(3 + x)^2(G + 1 + x)(G + 2 + x)}, \\
\Phi_4(x) &= \frac{A_4(x) + B_4(x)}{(4 + x)^2} = \frac{G + x}{(4 + x)^2(G + 2 + x)(G + 3 + x)}, \\
\Phi_5(x) &= \frac{1}{(5 + x)^2} \sum_{k=5}^{\infty} A_k(x) + B_k(x) = \frac{G + x}{(5 + x)^2(G + 3 + x)}. 
\end{align*}
\]

The function \( \Phi \) is decreasing on \( [0, 1] \) with \( \| \Phi \|_\infty \leq \Phi(0) < 0.1346. \)

Lemma 2. \[ \| S_I f \|_\infty \leq 0.1346 \| f' \|_\infty, \forall f \in C^1[0, 1]. \]

We write \((\tilde{U} f)' = S_{II} f + S_{II} f,'\) with

\[
\begin{align*}
(S_{II} f)(x) &= \sum_{k=2}^{\infty} A_k' f \left( \frac{1}{k + x} \right) + B_k' f \left( 1 - \frac{1}{k + x} \right), \\
\end{align*}
\]

\( \forall f \in C^1[0, 1], x \in [0, 1]. \)
Proof. Compute
\[
A_k'(x) = \left( \frac{1}{k+x} - \frac{1}{k+x+G-1} \right) \left( 1 - (G+x) \left( \frac{1}{k+x} + \frac{1}{k+x+G-1} \right) \right),
\]
\[
B_k'(x) = \left( \frac{1}{k+x+G-2} - \frac{1}{k+x} \right) \left( 1 - (G+x) \left( \frac{1}{k+x+G-2} + \frac{1}{k+x} \right) \right),
\]
\[
A_k'(x) + B_k'(x) = \frac{1}{k+x+G-2} - \frac{1}{k+x+G-1} - (G+x) \left( \frac{1}{(k+x+G-2)^2} - \frac{1}{(k+x+G-1)^2} \right).
\]
Using the second identity in (19) we can write
\[
(S_{II}f)(x) = \sum_{k=2}^{\infty} A_k(x) \left( f \left( \frac{1}{k+x} \right) - f \left( \frac{1}{2} \right) \right) + B_k(x) \left( f \left( \frac{1}{k+x} \right) - f \left( \frac{1}{2} \right) \right).
\]
In conjunction with the Mean Value Theorem and \(|\frac{1}{2+x} - \frac{1}{2}| \leq \frac{1}{6}, |\frac{1}{3+x} - \frac{1}{2}| \leq \frac{1}{4}, |\frac{1}{4+x} - \frac{1}{2}| \leq \frac{3}{10}, |\frac{1}{k+x} - \frac{1}{2}| \leq \frac{1}{2}, k \geq 5\), this yields
\[
|(S_{II}f)(x)| \leq \Psi(x)\|f'\|_{\infty},
\]
with \(\Psi = \Psi_2 + \Psi_3 + \Psi_4 + \Psi_5\), where
\[
\Psi_2 = \frac{|A_2'| + |B_2'|}{6}, \quad \Psi_3 = \frac{|A_3'| + |B_3'|}{4}, \quad \Psi_4 = \frac{3(|A_4'| + |B_4'|)}{10}, \quad \Psi_5 = \frac{1}{2} \sum_{k=5}^{\infty} |A_k'| + |B_k'|.
\]
When \(k \geq 5\) we have \(A_k' > 0\) and \(B_k' > 0\) on \([0, 1]\). The above expression for \(A_k' + B_k'\) allows us to compute
\[
\Psi_5(x) = \frac{1}{2} \sum_{k=5}^{\infty} A_k'(x) + B_k'(x) = \frac{1}{2} \left( \frac{G+x}{2(G+3+x)^2} - \frac{G+x}{2(G+3+x)^2} \right)
\]
\[
= \frac{3}{2(G+3+x)^2} \leq \frac{3}{2(G+3)^2} < 0.0704.
\]
On the other hand we have \(A_2' < 0\) and \(B_2' < 0\) on \([0, 1]\), leading to
\[
\Psi_2(x) = -\frac{A_2'(x) + B_2'(x)}{6} = \frac{1}{6} \left( \frac{1}{G+1+x} - \frac{1}{G+x} + (G+x) \left( \frac{1}{(G+x)^2} - \frac{1}{(G+1+x)^2} \right) \right)
\]
\[
= \frac{1}{6(G+1+x)^2} \leq \frac{1}{6(G+1)^2} = \frac{(2-G)^2}{6} < 0.0244.
\]
Numerically, we see that \(\Psi_3(x) \leq \frac{1}{2}(|A_3'(1)| + |B_3'(1)|) < 0.0019\) and \(\Psi_4(x) \leq \frac{3}{10}(|A_4'(0)| + |B_4'(0)|) < 0.0025\). Thus \(\|S_{II}f\|_{\infty} \leq 0.0992\|f'\|_{\infty}\). \(\square\)

Corollary 3. \(\|(\bar{T}f)'\|_{\infty} \leq 0.234\|f'\|_{\infty}, \forall f \in C^1[0, 1].\)

The proof of the following asymptotic formula follows ad litteram the proof of Theorem 1 (i).

Proposition 2. With \(q = 0.234\), for every Borel set \(E \subseteq [0, 1]\),
\[
\lambda(T^{-n}E) = \bar{\mu}_e(E) + O(\bar{\mu}_e(E)q^n).
\]
Let $E \subseteq [-\frac{1}{2}, \frac{1}{2}]$ be a Borel set and $\tilde{E} := JE$. The equality $JT_e = \tilde{T}_e J$ yields $JT_e^{-n}E = \tilde{T}_e^{-n}JE$. Theorem 1 (iii) now follows from $\lambda(JS) = \lambda(S)$ for every Borel set $S \subseteq [-\frac{1}{2}, \frac{1}{2}]$, Proposition 2, $\mu_e(\tilde{E}) = \mu_e(E)$, and

$$\lambda(T_e^{-n}E) = \lambda(J^{-1}\tilde{T}_e^{-n}JE) = \lambda(\tilde{T}_e^{-n}E) = \mu_e(E) + O(\mu_e(E)q^n).$$

The $T_e$-cylinders are given (up to null sets) by $\Delta^e_{[(a_1,1)]} = \pm(\frac{2}{2a_1+1}, \frac{2}{2a_1-1})$, $a_1 \geq 3$, $\Delta^e_{[(2,1)]} = \pm(\frac{3}{2}, \frac{3}{2})$, and when $r \geq 2$ by $\Delta^e_{[(a_1,e_1),...,(a_r,e_r)]} = \Delta^e_{[(a_1,e_1)]} \cap T_e^{-1}\Delta^e_{[(a_2,e_2)]} \cap \ldots \cap T_e^{-(r-1)}\Delta^e_{[(a_r,e_r)]}$.

The $\tilde{T}_e$-cylinders are given by $\Delta^e_{[(a_1,1)]} = \pm(\frac{1}{a_1+1}, \frac{1}{a_1-1})$, $\Delta^e_{[(a_1,1)]} = 1 - \Delta^e_{[(a_1,1)]}$, $a_1 \geq 2$, and when $r \geq 2$ by $\tilde{\Delta}^e_{[(a_1,e_1),...,(a_r,e_r)]} = \Delta^e_{[(a_1,e_1)]} \cap \tilde{T}_e^{-1}\Delta^e_{[(a_2,e_2)]} \cap \ldots \cap \tilde{T}_e^{-(r-1)}\Delta^e_{[(a_r,e_r)]}$.

Note that $J$ does not map $T_e$-cylinders into $\tilde{T}_e$-cylinders.

Formula (17) shows in particular that $\tilde{U}$ acts as

$$(\tilde{U}f)(x) = \sum_{k \geq 2, e=\pm 1} \tilde{P}_{(k,e)}(x) f(\tilde{w}_{(k,e)}(x)), $$

where $\tilde{w}_{(k+1)}(x) = \frac{1}{x+k}$, $\tilde{w}_{(-k)}(x) = 1 - \frac{1}{x+k}$ map the interval $[0, 1]$ onto the rank one cylinders $\tilde{\Delta}^e_{[(k,1)]}$ respectively. As a result, the argument in the proof of Corollary 1 applies, entailing

$$\mu_e(\tilde{T}_e^{-n}\tilde{E} \cap \tilde{F}) = \mu_e(\tilde{E})\tilde{\mu}_e(\tilde{F}) + O_F(q^n),$$

for any Borel set $\tilde{E} \subseteq [0, 1]$ and any $\tilde{T}_e$-cylinder $\tilde{F}$, with $q = 0.234$.

**Corollary 4.** With $q = 0.234$, for any Borel set $E \subseteq [-\frac{1}{2}, \frac{1}{2}]$ and $F = J^{-1}\tilde{F} \subseteq [-\frac{1}{2}, \frac{1}{2}]$, $\tilde{F}$ $\tilde{T}_e$-cylinder,

$$\mu_e(T_e^{-n}E \cap F) = \mu_e(E)\mu_e(F) + O_F(q^n).$$

**Proof.** Employing $\mu_e(S) = \tilde{\mu}_e(JS)$ and $JT_e^{-n}S = \tilde{T}_e^{-n}JS$ for every Borel set $S \subseteq [-\frac{1}{2}, \frac{1}{2}]$, and equation (20) with $\tilde{E} := JE$, we find

$$\mu_e(T_e^{-n}E \cap F) = \tilde{\mu}_e(JT_e^{-n}E \cap JF) = \tilde{\mu}_e(\tilde{T}_e^{-n}\tilde{E} \cap \tilde{F}) = \tilde{\mu}_e(\tilde{E})\tilde{\mu}_e(\tilde{F}) + O_F(q^n) = \mu_e(E)\mu_e(F) + O_F(q^n),$$

which concludes the proof.\qed

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