The renormalization is investigated of one-loop quantum fluctuations around a constrained instanton in $\phi^4$-theory with negative coupling. It is found that the constraint should be renormalized also. This indicates that in general only renormalizable constraints are permitted.

PACS number 11.10.z

June 27, 2018

1 Introduction

Instantons [1, 2] only exist in conformally invariant field theories because of Derrick's theorem [3]. When conformal invariance is broken, approximate solutions can be used to estimate the path integral of the theory in question [4]; one way of implementing this idea is to introduce a constraint in the theory that explicitly violates conformal invariance [5, 6].

In an earlier publication [7] it was shown that the choice of constraint is more restricted than previously assumed since most constraints do not lead to a finite action. Constrained instantons were explicitly constructed in two instances, $\phi^4$-theory with "wrong" coupling sign, and the Yang-Mills-Higgs theory. In the former case, where an instanton solution was first obtained

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by Lipatov \[2\] in the context of the large-order behaviour of the perturbation series, it was found that there are only two permitted constraints, both involving a cubic coupling, but either in operator form or as a source in the field equation. For the Yang-Mills-Higgs case only a source constraint was determined.

These results indicate that the allowed constraints should have the property of being (i) renormalizable, (ii) gauge invariant (in the case of gauge theories). In order to investigate this conjecture further, we examine in the present paper the quantum fluctuations around a constrained instanton in $\phi^4$-theory with wrong-signed coupling. The Yang-Mills-Higgs system is more interesting physically, but also more complicated. We compute the mass corrections of the one-loop functional determinant, regularized by zeta function regularization \[8\], \[9\], and show that a change of the mass scale introduced through regularization takes place through the customary mass and coupling constant renormalization, if the constraint is enforced by a source term, but with an operator constraint also the constraint coefficient should be renormalized. Thus it is indeed likely that the constraint always should be renormalizable.

The layout of the paper is the following: In sec. 2 the instanton solution of \[7\] is recapitulated, and the mass corrected terms of the classical action relevant for renormalization are constructed; the construction involves an infinite resummation because of infrared divergences. In sec. 3 a heuristic argument on the three one-loop renormalizations of the theory in an instanton background is given. Sec. 4 deals with the quantum fluctuations around the instanton solution in the massless limit, and the mass corrections of the eigenvalues are given in sec. 5. Finally in secs. 6 and 7 the mass corrections of the one-loop functional determinant and their renormalizations are dealt with for the cases of an operator and a source constraint, respectively. Our results are briefly stated in the conclusion, and three appendices deal with the eigenfunctions in the massless case, matrix elements of the mass perturbation terms, and zeta function regularization.

2 Instanton solution in massive $\phi^4$-theory

In the massless scalar $\phi^4$-theory with negative coupling constant the Euclidean Lagrangian is

$$L(\phi) = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{g}{4!}\phi^4.$$  \hspace{1cm} (2.1)
The resulting equation of motion has an instanton solution:

$$\phi_0 = \sqrt{\frac{3}{g}} \frac{4\rho}{\rho^2 + x^2}$$

where the subscript indicates the order in the mass expansion and where $\rho$ is an arbitrary scale parameter. The value of the action at the instanton solution is:

$$S[\phi_0] = \int d^4x L(\phi_0) = \frac{16\pi^2}{g}.$$  

In the presence of a mass term one has to impose a constraint (that also breaks scale invariance) in order to keep the action finite. This may be a source constraint or an operator constraint. In the massive scalar $\phi^4$-theory the appropriate operator constraint is known to be [7]:

$$\int d^4x \phi^3(x) = k\rho$$

with $k$ a constant that to zeroth order has the value, found by insertion of (2.2) into (2.4):

$$k_0 = \frac{96\sqrt{3}}{g}\frac{\pi^2}{g}. \tag{2.5}$$

The presence of the constraint leads to an effective Lagrangian:

$$L_{\text{eff}}(\phi) = L(\phi) + \frac{1}{2}m^2\phi^2 + \bar{\sigma}\phi^3 \tag{2.6}$$

where $\bar{\sigma}$ is chosen such that the field equation has a finite-action instanton solution. The mass parameter is assumed small so the solution for $\bar{\sigma}$ and corrections to the instanton configuration are expressed as a power series in $m$. To second order one finds the field equation:

$$(\partial^2 + \frac{g}{2}\phi^2_0)\phi_2 - m^2\phi_0 - 3\bar{\sigma}_2\phi^2_0 = 0 \tag{2.7}$$

with:

$$\bar{\sigma}_2 = \sqrt{\frac{3g}{6}}m^2\rho(\log \frac{m^2\rho^2}{4} + 2\gamma + 2). \tag{2.8}$$

The corresponding instanton solution is [7]:

$$\phi_{cl} \simeq \phi_0 + \phi_2 = \phi_0 + \phi_{2,a} + \phi_{2,b}, \tag{2.9}$$
where

\[ \phi_{2,a} = \sqrt{\frac{3}{g}} m^2 \rho \left( -\left( \frac{1}{1-u} - 12u \right) \log u + 6u(1-2u)\Phi \left( \frac{1-u}{u} \right) + 12u \right), \]  
\[ \text{(2.10)} \]

with

\[ u = \frac{\rho^2}{\rho^2 + x^2}, \quad \Phi(x) = \int_0^x dx' \frac{\log(1+x')}{x'}, \]  
\[ \text{(2.11)} \]

and

\[ \phi_{2,b} = \sqrt{\frac{3}{g}} m^2 \rho \left( \log \left( \frac{m^2 \rho^2}{4} + 2\gamma - 1 \right) \right). \]  
\[ \text{(2.12)} \]

Here \( \Phi(x) \) is the Spence function \[10\], and \( \phi_{2,b} \) constitutes with \( \phi_0 \) (apart from a proportionality factor) the two first terms of the modified Bessel function \( K_1(m \mid x \mid) \), thus ensuring correct asymptotic behaviour at large values of \( |x| \), where

\[ K_1(m \mid x \mid) \approx \sqrt{\frac{\pi}{m \mid x \mid}} e^{-m|x|}. \]  
\[ \text{(2.13)} \]

To order \( m^2 \) the value of the constant \( k \) is in fact divergent, and a large cutoff \( R \) limiting \( |x| \) must be introduced. Then the following value of \( k_2 \) is found:

\[ k_2 \approx \frac{144}{g} \sqrt{\frac{3}{g}} m^2 \rho^2 \left( \frac{1}{2} \log^2 \frac{R^2}{\rho^2} + \frac{\pi^2}{6} + 1 \right) \]
\[ + (\log \frac{m^2 \rho^2}{4} + 2\gamma - 1)(\log \frac{R^2}{\rho^2} - 1) \]  
\[ \text{(2.14)} \]

by means of (2.13) and the integrals (B.4) and (B.16) in App. B.

The second-order mass corrections to the classical action are also evaluated:

\[ \frac{m^2}{2} \int d^4x \phi_0^2 + \int d^4x (\partial_\mu \phi_0 \partial_\mu \phi_2 - \frac{g}{6} \phi_0^3 \phi_2). \]  
\[ \text{(2.15)} \]

These integrals again diverge logarithmically but the divergences actually cancel out after summation to all orders in the mass parameter. Similar divergences also occur for the one-loop corrections of the action. The resummation necessary to eliminate the divergences of (2.15) is carried out by rewriting the action:

\[ S[\phi] = \int d^4x \left( \frac{1}{2} \partial_\mu (\phi \partial_\mu \phi) + \frac{1}{2} \phi (-\partial^2 \phi - \frac{g}{6} \phi^3 + m^2 \phi^2 + 3\bar{\sigma} \phi^2) \right. \]
\[ + \left. \frac{g}{24} \phi^4 - \frac{3}{2} \bar{\sigma} \phi^3 \right). \]  
\[ \text{(2.16)} \]
Here the second term vanishes for a field configuration that is a solution of the field equation. The first term of (2.16):

$$\frac{1}{2} \int d^4x \partial_\mu (\phi \partial_\mu \phi)$$  

(2.17)

is a surface term and is estimated by means of the partition of the solution of the field equation into leading terms, nextleading terms etc., where the leading terms for large $x$-values \[7\] sum to

$$4\rho \sqrt{\frac{3}{g} \frac{m}{x}} K_1 (m \mid x \mid)$$  

(2.18)

with exponential falloff at large $|x|$. (2.17) thus goes to zero when the integration volume goes to infinity. The same conclusion holds for nextleading etc. terms since they also after resummation have exponential falloff \[7\] and the term (2.17) can be neglected altogether. Thus the value of the classical action at the instanton solution $\phi_{cl}$ is:

$$S[\phi_{cl}] = \int d^4x \left( \frac{g}{24} \phi_{cl}^4 - \frac{3}{2} \bar{\sigma} \phi_{cl}^3 \right)$$  

(2.19)

that to second order is

$$S[\phi_{cl}] \simeq \int d^4x \left( \frac{g}{24} \phi_0^4 + \frac{g}{6} \phi_0^3 \phi_{2,a} + \frac{1}{6} \phi_0^3 (g\phi_{2,b} - 9\bar{\sigma}) \right).$$  

(2.20)

In (2.20) the first term is known from (2.3), and the third term is

$$-\frac{24\pi^2}{g} m^2 \rho^2 (\log \frac{m^2 \rho^2}{4} + 2\gamma + 8).$$  

(2.21)

The second term of (2.20) is by (2.2) and (2.10) as well as (B.4) and (B.16):

$$\frac{g}{6} \int d^4x \phi_0^3 \phi_{2,a} = \frac{16\pi^2}{g} m^2 \rho^2.$$  

(2.22)

The classical action including second order mass corrections is thus

$$S[\phi_{cl}] \simeq \frac{16\pi^2}{g} \left( 1 - \frac{3m^2 \rho^2}{2} (\log \frac{m^2 \rho^2}{4} + 2\gamma + 1) \right).$$  

(2.23)
Mass renormalization should, in the absence of infrared divergences, be carried out upon $\frac{m^2}{2} \int \phi_0^2 d^4x$; after resummation (2.23) should be used instead. The coupling constant renormalization is not troubled by infrared divergences and should be carried out on the mass corrected quartic part of the action before the resummation leading to (2.23), i.e. on:

$$-\frac{g}{6} \int d^4x \phi_2^3 = -\frac{48\pi^2}{g} m^2 \rho^2 (\log \frac{m^2 \rho^2}{4} + 2\gamma + \frac{5}{2})$$

(2.24)

while the renormalization of the constraint coefficient, in the case of an operator constraint, should act on:

$$\bar{\sigma}_2 \int d^4x \phi_0^3 = \bar{\sigma} \frac{96}{g} \sqrt{3} \pi^2 \rho = \frac{48\pi^2}{g} m^2 \rho^2 (\log \frac{m^2 \rho^2}{4} + 2\gamma + 2)$$

(2.25)

in both cases by multiplication with the same coupling constant renormalization factor.

3 Quantum fluctuations

3.1 One-loop path integral

The path integral is evaluated by the Faddeev-Popov procedure, modified to enforce the constraint. The outcome is in the one-loop approximation

$$Z \propto \int d\rho \int d^4z \mu(\rho) e^{-S[\phi_0]} - \delta \bar{\sigma}_2 \int d^4x \phi_0^3 (\det 'M)^{-\frac{1}{2}}$$

(3.1)

with integration over collective coordinates, where $\mu(\rho)$ is the appropriate integral measure, and with the Gaussian fluctuation operator

$$M \simeq M_0 + M_2$$

(3.2)

with

$$M_0 = \partial^2 - \frac{g}{2} \phi_0^2; \ M_2 = m^2 - g\phi_2\phi_0 + 6\bar{\sigma}_2\phi_0.$$  

(3.3)

Here the prime indicates that zero-modes and quasi-zero-modes; in the massless limit there are four translational zero modes and one dilatational zero mode, and after inclusion of mass they get nonzero eigenvalues (the translational zero mode only if the constraint is enforced by a source term). The integral measure $\mu(\rho)$ is found from the normalization factors of the zero modes.
and gets additional mass corrections from the nonzero eigenvalue corrections of these modes. It will not be constructed explicitly since it is irrelevant for the issue of renormalization.

In (3.1) the factor
\[ e^{-\delta^2 \int d^4 x \phi^2} \]
which only should be present for an operator constraint, is generated by having a value of the constant \( \bar{k} \) in the constraint \( \bar{\Phi} \), when applied to the path integral, that is slightly different from \( k_0 \). The factor \( \phi^2 \) is necessary for the renormalization of the constraint coefficient.

A normal coordinate expansion of the quantum fluctuation scalar field \( \delta \phi \) is:
\[ \delta \phi = \sum a_p \phi_p \]
where the sum runs over all normalized field modes including the translational zero modes \( \phi_\mu \) and the dilational quasi-zero mode \( \phi_\rho \); \( a_p \) denotes the corresponding normal coordinates, and the eigenvalue equation is \( \Phi \), \( \Phi \):
\[ M \phi_p = \lambda_p \frac{4 \rho^2}{(\rho^2 + x^2)^2} \phi_p \]
where the nontrivial factor on the right-hand side makes the spectrum discrete and reflects the fact that it conveniently may be obtained by a stereographic projection \( \Phi \). Here the eigenvalue \( \lambda_p \) is dimensionless, and since \( \rho \) sets the instanton scale, the proper dimensionful eigenvalue is \( \frac{\lambda_p \rho^2}{\rho^2} \).

In the eigenvalue equation \( \Phi \) the term \( M_2 \) should be treated as a perturbation, and the equation can then be solved by standard perturbation theory. However, at large distances a direct determination of \( \phi_p \) to all orders is possible. At large values of \( x^2 \) the right-hand side \( \Phi \) vanishes, and it reduces in this limit to the Euclidean Klein-Gordon equation:
\[ (-\partial^2 + m^2) \phi_p \simeq 0 \]
with the normalizable solution
\[ \phi_p \propto \frac{m}{x} K_1(m \cdot x) \]
where the proportionality factor is fixed by comparison with the massless limit.
3.2 Green’s function and renormalization

A rough estimate of the mass correction of the functional determinant, where only terms proportional to \( \log(m^2 \rho^2) \) are considered, is first given by means of Green’s function techniques. From this estimate a heuristic argument on the required renormalizations is made.

Starting from the free massless propagator

\[
D(x) = \frac{1}{4\pi^2 x^2} \tag{3.9}
\]

the determinant is computed perturbatively, with the perturbation

\[
\Delta M = -\frac{g}{2} \phi_0^2 + M_2 \tag{3.10}
\]

where \( M_2 \) is defined in (3.2). The massless propagator is then approximately

\[
G_0(x, x') \simeq D(x - x') + \frac{g}{2} \phi_0^2(x) \frac{1}{16\pi^2} \log \frac{R^2}{(x - x')^2} \quad (3.11)
\]

where \( R \) is the infrared cutoff introduced previously.

The mass correction to the one-loop action is then, also approximately:

\[
\frac{1}{2} \int d^4x M_2(x) G_0(x, x) \simeq \frac{g}{4} \frac{1}{16\pi^2} \log(\Lambda^2 R^2) \int d^4x M_2(x) \phi_0^2(x) \tag{3.12}
\]

with \( \Lambda \) an ultraviolet cutoff. This expression has a contribution from the term \( m^2 \) of (3.2):

\[
-\frac{g^2}{4} \frac{1}{16\pi^2} \log(\Lambda^2 R^2) \int d^4x \phi_2(x) \phi_0^3(x) \simeq -\frac{g}{2} m^2 \rho^2 \log(\Lambda^2 R^2) \log(m^2 \rho^2) \tag{3.13}
\]

a contribution from \(-g\phi_0\phi_2\)

\[
-\frac{g^2}{4} \frac{1}{16\pi^2} \log(\Lambda^2 R^2) \int d^4x \phi_2(x) \phi_0^3(x) \simeq -\frac{9}{2} m^2 \rho^2 \log(\Lambda^2 R^2) \log(m^2 \rho^2) \tag{3.14}
\]
and a term from $6\bar{\sigma}_2\phi_0$:

$$g \frac{1}{4 \cdot 16 \pi^2} \log(\Lambda^2 R^2) 6\bar{\sigma}_2 \int d^4 x \phi_0^3(x) \simeq \frac{9}{2} m^2 \rho^2 \log(\Lambda^2 R^2) \log(m^2 \rho^2)$$

$$\simeq \frac{9}{2} m^2 \rho^2 \log(\Lambda^2 \rho^2) \log(m^2 \rho^2). \quad (3.15)$$

In (3.13), (3.14) and (3.15) the infrared cutoff $\frac{1}{R}$ was replaced by either the instanton scale parameter $\rho$ or the physical mass $m$ in an apparently arbitrary manner; this procedure is justified below in sec. 6 and sec. 7. The two expressions in (3.14) and (3.15) actually cancel, but it is instructive to consider them separately, since they correspond to different renormalizations.

The coupling constant renormalization is easily obtained by this argument also; going to second order in (3.11) one gets the second-order correction to the determinant:

$$- g^2 \frac{1}{16} \frac{1}{16 \pi^2} \log(\Lambda^2 R^2) \int d^4 x \phi_0^4 \simeq - \frac{3}{2} \log(\Lambda^2 R^2). \quad (3.16)$$

The divergent expression (3.13) is eliminated by the customary mass renormalization, which in perturbation theory to lowest order is:

$$m^2_{\text{bare}} \simeq m^2(1 - \frac{g^3}{32 \pi^2} \log \frac{\Lambda^2}{\mu^2}) \quad (3.17)$$

with $\mu$ an arbitrary mass scale (only the logarithmic part is kept).

Replacing in (2.23) the mass with the bare mass according to (3.17) one obtains the double logarithmic term

$$\frac{3}{4} m^2 \rho^2 \log \frac{m^2 \rho^2}{4} \log \frac{\Lambda^2}{\mu^2} \quad (3.18)$$

which exactly cancels the divergence in (3.13).

For the expression (3.14) the infinity is removed by the coupling constant renormalization of ordinary perturbation theory of (2.24):

$$g_{\text{bare}} \simeq g(1 - \frac{3g}{32 \pi^2} \log \frac{\Lambda^2}{\mu^2}) \quad (3.19)$$

Notice that the renormalization is carried out on the coupling in the first version of (2.24) where $g$ is in the numerator.
Finally for (3.15) the infinity is removed by replacing \( \bar{\sigma}_2 \) in (2.25) by the corresponding bare quantity:

\[
\bar{\sigma}_{\text{bare},2} \simeq \bar{\sigma}_2 (1 - \frac{3g}{32\pi^2} \log \frac{\Lambda^2}{\mu^2}) \tag{3.20}
\]

i.e. the cubic coupling is renormalized as the quartic coupling, as in ordinary perturbation theory. In (3.4) this corresponds to the choice:

\[
\delta \bar{\sigma}_2 = -\bar{\sigma}_2 \frac{3g}{32\pi^2} \log \frac{\Lambda^2}{\mu^2}. \tag{3.21}
\]

This loose argument indicates that the cubic operator constraints should be renormalized as an ordinary cubic coupling; that this is indeed the case is verified in detail below in secs. 6 and 7. With a source constraint (3.15) is absent and only mass and coupling constant renormalizations are required.

### 4 The massless limit

In the massless limit the eigenvalue equation (3.6) is explicitly solvable. The eigenfunctions \( \phi_p \) are separated into a radial and an angular part:

\[
\phi_p(x) = \phi_{nlm_1m_2}(x) = (\sqrt{2}\rho)^{-1} P_{lm_1m_2}(\Omega) u^{l+1}(1 - u)^l \chi_{nl}(u) \tag{4.1}
\]

where the angular part is an \( O(4) \) spherical harmonic \( P_{lm_1m_2}(\Omega) \) while the radial function \( \chi_{nl}(u) \) can be expressed in terms of a Gegenbauer polynomial; it is given explicitly in (A.1) and dealt with in detail in App. A. The angular eigenfunctions \( P_{lm_1m_2}(\Omega) \) are normalized on the unit three-sphere:

\[
\int_{S^3} d\Omega P_{lm_1m_2}P_{l'm_1'm_2'} = \delta_{ll'}\delta_{m_1m_1'}\delta_{m_2m_2'}. \tag{4.2}
\]

The eigenfunctions are required to be normalized according to:

\[
\int d^4x \frac{4\rho^2}{(x^2 + \rho^2)^2} \phi_{nlm_1m_2}\phi_{n'l'm_1'm_2'} = \delta_{ll'}\delta_{nn'}\delta_{m_1m_1'}\delta_{m_2m_2'}. \tag{4.3}
\]

The eigenvalues are

\[
\lambda_{nl} = (n + 2l + 4)(n + 2l - 1). \tag{4.4}
\]
The degeneracy is \((2l + 1)^2\). The corresponding eigenvalues in the absence of an instanton are
\[
\lambda_{nl} |_{\text{free}} = (n + 2l + 1)(n + 2l + 2).
\tag{4.5}
\]
The eigenfunctions are unmodified. The functional determinant in the massless limit was computed in Lipatov’s original paper \cite{2} from \eqref{4.4}.

There are five zero modes. The translational zero modes have \(n = 0, l = \frac{1}{2}\) and the dilatation zero mode has \(n = 1, l = 0\). For \(n = l = 0\) an unstable mode occurs. The existence of the unstable mode can be inferred already from the zeroth order field equation which is reformulated
\[
- (\partial^2 + \frac{g}{2} \phi_0^2) \phi_0 = - \frac{16 \rho^2}{(\rho^2 + x^2)^2} \phi_0.
\tag{4.6}
\]
The eigenfunction of the unstable mode is thus proportional to \(\phi_0\). The value of \(\lambda_{00}\) can as shown by Lipatov \cite{2} in the context of the estimate of large-order perturbation theory can be taken as 4 instead of \(-4\).

5 Mass corrections of the eigenvalues

5.1 Statement of the problem

The lowest-order mass corrections to the eigenvalues found in the previous section are now computed. The perturbed Gaussian operator is given in \eqref{3.2}. The perturbed eigenvalue problem reduces to first order in \(m^2\) to
\[
M_0 \phi_{nl,2} + M_2 \phi_{nl,0} = \frac{4 \rho^2}{(\rho^2 + x^2)^2} (\lambda_{nl,0} \phi_{nl,2} + \lambda_{nl,2} \phi_{nl,0})
\tag{5.1}
\]
whence:
\[
\lambda_{nl,2} = - \int d^4x \partial_\mu (\phi_{nl,0} \tilde{\partial}_\mu \phi_{nl,2}) + \int d^4x M_2 \phi_{nl,0}^2
\tag{5.2}
\]
for eigenfunctions \(\phi_{nl,0}\) normalized according to \eqref{4.3}. Here we have suppressed the quantum numbers \(m_1\) and \(m_2\) but indicated the order in the mass expansion for the eigenfunctions. The first term in \eqref{5.2} is a surface term at infinity that is nonvanishing if \(\phi_{nl,0}\) goes as \(\frac{1}{r^l}\) and \(\phi_{nl,2}\) has a constant term and a term that grows logarithmically; this is the case for all \(l = 0\)-modes including the dilatational quasi-zero mode.
5.2 The mass term correction

With \( M_2 \to m^2 \) the second term of (5.2) is by (B.3):

\[
\int d^4xm^2\phi_{nl,0}^2 = \frac{m^2\rho^2}{4} < nl || \frac{1}{u^2} || nl > \\
= \frac{m^2\rho^2}{4} \frac{2n + 4l + 3(n + 2l + 1)(n + 2l + 2)}{l(l + 1)}
\]  

(5.3)

where the notation is explained in connection with (B.1).

For \( l = 0 \) a logarithmic divergence occurs in this matrix element. This is seen by use of (B.5), with the integration volume a large sphere with radius \( R \):

\[
\frac{m^2\rho^2}{4} < n0 || u^{-2} || n0 > \\
= \frac{m^2\rho^2}{4}(2n + 3)(1 + (n + 1)(n + 2)(\log \frac{R^2}{\rho^2} - 3 - 2 \sum_{k=1}^{n} \frac{1}{k + 1}))
\]  

(5.4)

that diverges logarithmically at \( R \to \infty \). The infrared divergence is eliminated by using as a starting point the asymptotic expression for the \( m = 0 \) eigenfunction at large values of \( x^2 \) found from (4.1):

\[
\phi_{n0,0} \simeq \sqrt{\frac{(2n + 3)\Gamma(n + 3) \rho}{4\pi^2n!}} \sqrt{x^2} 
\]  

(5.5)

whence is obtained the following asymptotic expression of the mass corrected eigenfunction necessary to obtain the correct Bessel function according to (3.8):

\[
\phi_{n0,2} \simeq \frac{1}{4} \rho m^2 \sqrt{\frac{(2n + 3)\Gamma(n + 3) \rho}{4\pi^2n!}}(\log \frac{m^2x^2}{4} + 2\gamma - 1).
\]  

(5.6)

This leads to a correction to the eigenvalue from the surface term of (5.2) with the value

\[
- \frac{m^2\rho^2}{4}(2n + 3)(n + 1)(n + 2)(\log \frac{m^2R^2}{4} + 2\gamma).
\]  

(5.7)

Adding this expression to (5.4) one finds:

\[
- \frac{m^2\rho^2}{4}(2n + 3)(-1 + (n + 1)(n + 2)(\log \frac{m^2\rho^2}{4} + 2\gamma + \frac{3}{2} + 2 \sum_{k=1}^{n} \frac{1}{k + 1}))
\]  

(5.8)
where the infrared divergences have cancelled as expected.

In the free-field case (absence of instanton) one has

\[ M_2 \rightarrow M_{2,\text{free}} = m^2 \]  \hspace{1cm} (5.9)

and the eigenfunctions in the massless limit are the same as in the presence of an instanton. Thus the total eigenvalue corrections are in this case (5.3) for \( l \neq 0 \) and (5.8) for \( l = 0 \).

### 5.3 The remaining matrix element

The last term of (5.2) has the remaining term:

\[
\int d^4 x (M_2 - m^2) \phi_{nl,0}^2 = \frac{m^2 \rho^2}{4} < nl \| \frac{M_2 - m^2}{m^2 u^2} \| nl > \\
= \frac{m^2 \rho^2}{4} < nl \| (-12u^{-1}[-(\frac{1}{1-u}) - 12u] \log u \\
+ 6u(1 - 2u)\Phi(\frac{1-u}{u}) + 12u - 3) > . \hspace{1cm} (5.10)
\]

The matrix elements are found in Appendix B by rather laborious calculations, and the relevant results are given in (B.2), (B.6), (B.13) and (B.25), leading to the result:

\[
\int d^4 x (M_2 - m^2) \phi_{nl,0}^2 \\
= -3m^2 \rho^2 \frac{2n + 4l + 3}{2l + 1} \left( 2 \sum_{k=1}^{n+2l+1} \frac{1}{k + 2l + 1} + \frac{1}{2l + 1} - 3 \right) \\
+ 18m^2 \rho^2 \left( \frac{2l + 1}{n + 2l + 2} \sum_{k=0}^{n+2l+1} \frac{1}{k + 2l + 2} + \frac{2l + 1}{n + 2l + 1} \sum_{k=0}^{n+2l} \frac{1}{k + 2l + 1} \\
+ \frac{1}{n + 4l + 3} - \frac{1}{n + 4l + 2} - 2 \right). \hspace{1cm} (5.11)
\]

The sum of (5.3) and (5.11) reduces to 0 in the case \( n = 0, l = \frac{1}{2} \), which corresponds to the translational zero modes. This is expected, since the constraint is an operator constraint and thus respects translational invariance. For the dilatational zero mode with \( n = 1, l = 0 \) the eigenvalue correction is:

\[- \frac{15m^2 \rho^2}{2} (\log \frac{m^2 \rho^2}{4} + 2\gamma + 4) \neq 0. \hspace{1cm} (5.12)\]

The dilatational zero mode thus becomes a quasi-zero mode because of quantum fluctuations.
6 Mass correction of functional determinant

The mass corrected functional determinant in (3.1) is:

$$(\det(M_0 + M_2))^{-\frac{1}{2}} \simeq (\det M_0)^{-\frac{1}{2}} e^{-\frac{1}{2} \text{tr} \frac{M_2}{M_0}}$$  \hspace{1cm} (6.1)$$

where

$$\frac{1}{2} \text{tr} \frac{M_2}{M_0} = \frac{1}{2} \sum_{n=0}^{\infty} \sum_{l=0, n+2l>1}^{\infty} (2l + 1)^2 \frac{\langle nl \| M_2 \| nl \rangle}{(n + 2l + 4)(n + 2l - 1)}$$

$$- \frac{1}{2} \sum_{n=0}^{\infty} \sum_{l=0}^{\infty} (2l + 1)^2 \frac{\langle nl \| M_{2,\text{free}} \| nl \rangle}{(n + 2l + 1)(n + 2l + 2)}$$  \hspace{1cm} (6.2)$$

thus is the mass correction to the one-loop action. Here the free-field contribution was subtracted. Also the unstable mode is disregarded here and henceforth since a single mode does not affect the renormalizations.

The contribution to (6.2) from the mass term of $M_2$ is according to (5.3) for $l \neq 0$:

$$\frac{m^2 \rho^2}{16} \left( \sum_{n=0}^{\infty} \sum_{l=0, n+2l>1}^{\infty} \frac{(2l + 1)(n + 2l + 1)(n + 2l + 2)(2n + 4l + 3)}{(n + 2l + 4)(n + 2l - 1)l(l + 1)} \right)$$

$$- \sum_{n=0}^{\infty} \sum_{l=\frac{1}{2}}^{\infty} \frac{(2l + 1)(2n + 4l + 3)}{l(l + 1)}$$  \hspace{1cm} (6.3)$$

This expression is not well defined at $l = 0$ where it according to (5.8) is replaced by:

$$- \frac{m^2 \rho^2}{8} \sum_{n=2}^{\infty} \frac{2n + 3}{(n + 4)(n - 1)} (-1 + (n + 1)(n + 2)(\log \frac{m^2 \rho^2}{4} + 2\gamma + \frac{3}{2} + 2 \sum_{k=1}^{n} \frac{1}{k + 1}))$$

$$+ \frac{m^2 \rho^2}{8} \sum_{n=0}^{\infty} \frac{2n + 3}{(n + 1)(n + 2)} (-1 + (n + 1)(n + 2)(\log \frac{m^2 \rho^2}{4} + 2\gamma + \frac{3}{2} + 2 \sum_{k=1}^{n} \frac{1}{k + 1})).$$  \hspace{1cm} (6.4)$$

Introducing the summation variable $s = n + 2l + \frac{3}{2}$ and adding (6.3) and (6.4) one gets:

$$- \frac{m^2 \rho^2}{4} (B_{\frac{3}{2}} - B_{\frac{1}{2}})$$  \hspace{1cm} (6.5)$$
with

\[ B_\phi = \sum_{s=\phi+1}^{\infty} \frac{s}{s^2 - \phi^2} (-2 + (s^2 - \frac{1}{4})(\log \frac{m^2 \rho^2}{4} + 2\gamma + 2)). \quad (6.6) \]

The function \( B_\phi \) is zeta function regulated \[8\] \[9\] in terms of the function \( Z_\phi(\epsilon) \) defined by

\[ Z_\phi(\epsilon) = \sum_{s=\phi+1}^{\infty} s(s^2 - \phi^2)^{-\epsilon}. \quad (6.7) \]

Replacing in (6.6) the denominator with the same quantity to the power \( 1+\epsilon \) one gets:

\[ B_\epsilon = (\log \frac{m^2 \rho^2}{4} + 2\gamma + 2)Z_\phi(\epsilon) \]

\[ + (-2 + (\phi^2 - \frac{1}{4})(\log \frac{m^2 \rho^2}{4} + 2\gamma + 2))Z_\phi(1 + \epsilon). \quad (6.8) \]

\( B_\phi \) itself is determined from this expression and (C.8):

\[ B_\phi = (\log \frac{m^2 \rho^2}{4} + 2\gamma + 2)Z_\phi(0) \]

\[ + (-2 + (\phi^2 - \frac{1}{4})(\log \frac{m^2 \rho^2}{4} + 2\gamma + 2))\lim_{\epsilon \to 0}(\log(\mu^2 \rho^2)\epsilon Z_\phi(1 + \epsilon) + \frac{\partial}{\partial \epsilon} \epsilon Z_\phi(1 + \epsilon)) \quad (6.9) \]

that by (C.11) and (C.12) is

\[ B_\phi = -(\log \frac{m^2 \rho^2}{4} + 2\gamma + 2)(\frac{1}{12} + \frac{\phi}{2}) \]

\[ + (-1 + \frac{1}{2}(\phi^2 - \frac{1}{4})(\log \frac{m^2 \rho^2}{4} + 2\gamma + 2))((\log(\mu^2 \rho^2) + 2\gamma - \sum_{s=1}^{2\phi} \frac{1}{s}). \quad (6.10) \]

The part of the mass correction to the one-loop action arising from the term \( m^2 \) in \( M_2 \) is hence

\[ -\frac{m^2 \rho^2}{4}(B_\phi^2 - B_1^2) = \frac{m^2 \rho^2}{4}(\log \frac{m^2 \rho^2}{4} + 2\gamma + 2)(1 - 3(\log(\mu^2 \rho^2) + 2\gamma - \frac{137}{60}) \quad (6.11) \]
where terms not containing $\log \mu$ somewhat inconsequentially have been retained. The double logarithmic term of (6.11) is

$$-\frac{3}{4}m^2 \rho^2 \log \frac{m^2 \rho^2}{4} \log (\mu^2 \rho^2)$$

(6.12)
in agreement with (3.13), with $\Lambda$ replaced by $\mu$. Carrying out in (2.23) a mass renormalization by the replacement (cf. (3.17)):

$$m^2 \rightarrow m^2 (1 - \frac{g}{32\pi^2} \log \frac{\mu^2}{\mu'^2})$$

(6.13)

with $\mu'$ a new mass parameter, one gets the additional term:

$$\frac{3}{4} \log \frac{\mu'^2}{\mu^2} (\log \frac{m^2 \rho^2}{4} + 2\gamma + 2)$$

(6.14)

which when added to (6.11) replaces $\mu^2$ with $\mu'^2$. Thus it has been shown that the mass renormalization (3.17) or (6.13) applies for the whole one-loop mass correction to the action.

The remaining terms of (5.11), which are related to the coupling constant renormalization and the renormalization of the constraint coefficient, are now inserted into (6.2). By means of the relations

$$\sum_{l=0}^{s-\frac{3}{4}} \sum_{k=1}^{s-\frac{3}{4}} \frac{2l + 1}{2l + 1 + k} = \frac{1}{2} (s - \frac{1}{2})^2$$

(6.15)

and

$$\sum_{l=0}^{s-\frac{3}{4}} \sum_{k=1}^{s-\frac{3}{4}} \frac{(2l + 1)^3}{2l + 1 + k} = \frac{1}{3} s(s - \frac{1}{2})^2 (s + \frac{1}{2}) - \frac{1}{8} (s^2 - \frac{1}{4})^2$$

(6.16)

one obtains the simple expression:

$$-\frac{3m^2 \rho^2}{4} \sum_{s=\frac{1}{2}}^{\infty} \frac{s(s^2 - \frac{1}{4})}{s^2 - \frac{25}{4}}.$$ 

(6.17)

Here zeta function regularization is again applied, with the denominator replaced by the same quantity to the power $1 + \epsilon$ according to the definition (6.7), converting (6.17) into

$$-\frac{3m^2 \rho^2}{4} (Z_{\frac{3}{2}}(\epsilon) + 6Z_{\frac{5}{2}}(1 + \epsilon)).$$

(6.18)
Carrying out the evaluation by means of (C.8) combined with (C.11) and (C.12) one obtains the following additional contribution to the one-loop action:

\[-\frac{3m^2 \rho^2}{4} \left( -\frac{4}{3} + 3(\log(\mu^2 \rho^2) + 2\gamma - \frac{137}{60}) \right).\]  \hfill (6.19)

The dependence on \(\mu\) of (6.19) is removed by a coupling constant renormalization, where the sum of (2.24) and (2.25) is multiplied by a factor

\[1 - \frac{3g}{32\pi^2} \log \frac{\mu^2}{\mu'^2}.\]  \hfill (6.20)

This multiplication procedure produces the extra term

\[-\frac{9m^2 \rho^2}{4} \log \frac{\mu^2}{\mu'^2}.\]  \hfill (6.21)

that when added to (6.19) replaces \(\mu^2\) with \(\mu'^2\), demonstrating the consistency of the renormalization procedure sketched in subsection 3.2, which thus has been confirmed in detail for the case of an operator constraint. The case of a source constraint is dealt with in sec. 7.

### 7 Source constraint

The renormalization is different with an operator constraint and with a source constraint since in the latter case the cubic coupling should not be renormalized and the factor (3.4) in the path integral should be replaced by unity. With a source constraint the term \(6\bar{\sigma}_2 \phi_0\) in \(M_2\) is absent, with:

\[6\bar{\sigma}_2 \phi_0 = 12m^2 u(\log \frac{m^2 \rho^2}{4} + 2\gamma + 2).\]  \hfill (7.1)

This induces an eigenvalue correction to be added to those already determined:

\[
\delta_\sigma \lambda_{2,\nu} = -3m^2 \rho^2 (\log \frac{m^2 \rho^2}{4} + 2\gamma + 2) < \nu \| u^{-1} \| \nu > \\
= -3m^2 \rho^2 (\log \frac{m^2 \rho^2}{4} + 2\gamma + 2) \frac{2n + 4l + 3}{2l + 1} \]  \hfill (7.2)
by (B.2). For \( n = 0, l = \frac{1}{2} \) and \( n = 1, l = 0 \) one finds the following correction of the translational and dilatational zero-mode eigenvalues, respectively

\[
\delta_{\sigma} \lambda_{\mu} = -\frac{15}{2} m^2 \rho^2 \left( \log \frac{m^2 \rho^2}{4} + 2\gamma + 2 \right)
\]

and

\[
\delta_{\sigma} \lambda_{\rho} = -15 m^2 \rho^2 \left( \log \frac{m^2 \rho^2}{4} + 2\gamma + 2 \right).
\]

Both eigenvalues are nonzero in this case because the source constraint now also breaks translational invariance.

(7.3) leads to the following correction of the one-loop action:

\[
\frac{1}{2} \sum_{n+2l>1} (2l+1)^2 \frac{\delta_{\sigma} \lambda_{nl}}{\lambda_{nl}}
= -\frac{3m^2 \rho^2}{2} \left( \log \frac{m^2 \rho^2}{4} + 2\gamma + 2 \right) \sum_{n+2l>1} \frac{(2l+1)(2n+4l+3)}{n+2l-1)(n+2l+4)}
\]

which is divergent and must be regularized along with the original determinant.

The sum

\[
\sum_{n+2l>1} \frac{(2l+1)(2n+4l+3)}{n+2l-1)(n+2l+4)} = \sum_{s=-2}^{\infty} \frac{s(s^2 - \frac{1}{4})}{s^2 - \frac{25}{4}}
\]

occurred in (6.17) and was evaluated by zeta function regularization. Using the same procedure here one obtains from (7.5):

\[
-\frac{3m^2 \rho^2}{2} \left( \log \frac{m^2 \rho^2}{4} + 2\gamma + 2 \right) \left( -\frac{4}{3} + 3 \left( \log \left( \frac{m^2 \rho^2}{4} \right) + 2\gamma - \frac{137}{60} \right) \right).
\]

The double logarithmic term of this result, with \( \Lambda \) replaced by \( \mu \), agrees, apart from the sign, with (3.15), as it should, since the cubic coupling no more should be renormalized. However, again the effect of the renormalization goes beyond the double logarithmic terms. This is seen from (2.25), which by multiplication with (6.20) produces the extra term:

\[
-\frac{9m^2 \rho^2}{2} \log \frac{\mu^2}{\mu^2} \left( \log \frac{m^2 \rho^2}{4} + 2\gamma + 2 \right).
\]
with the same dependence on $\mu$ as (7.7). Thus, with a source constraint only the mass and coupling constant should be renormalized, in contrast to the case of an operator constraint, where a renormalization of the constraint coefficient is also necessary and is possible because of the factor (3.4) in the path integral.

8 Conclusion

Our results indicate that constraint terms in the path integral should be renormalized along with the Lagrangian. This seems to suggest that non-renormalizable constraints are not permitted, thus leading to a further restriction on the choice of constraint, which in [7] was shown to be restricted by the requirement of a finite classical action.

Only the scalar $\phi^4$-theory was considered, and thus one should be cautious by carrying over the result to e.g. the standard model or supersymmetric gauge theories [11], where gauge invariance or supersymmetry may cause divergences to cancel.

A Eigenfunctions

Normalized eigenfunctions of (3.6) are:

$$
\chi_{nl}(u) = \frac{1}{\Gamma(2l+2)} \sqrt{\frac{(2n + 4l + 3)\Gamma(n + 4l + 3)}{n!}}
\frac{2F_1(-n, n + 4l + 3; 2l + 2; u)}{2l + 1}.
$$

(A.1)

Here $2F_1(-n, n + 4l + 3; 2l + 2; u)$ are hypergeometric functions (Jacobi polynomials) [12]. The functions $\chi_{nl}(u)$ are normalized with respect to the integral measure $(u(1-u))^{2l+1}$:

$$
\int_0^1 du (u(1-u))^{2l+1} \chi_{nl}(u)^2 = 1.
$$

(A.2)

The Jacobi polynomials are given by a Rodrigues formula and also have a convenient series representation:

$$
2F_1(-n, n + 4l + 3; 2l + 2; u)
$$
\[
\begin{align*}
&= \frac{\Gamma(2l + 2)}{\Gamma(n + 2l + 2)} (u(1-u))^{-2l-1} \frac{d^n}{du^n}(u(1-u))^{n+2l+1} \\
&= \frac{\Gamma(2l + 2)}{\Gamma(n + 4l + 3)} \sum_{k=0}^{n} (-u)^k \frac{n!}{k!(n-k)!} \frac{\Gamma(n + 4l + 3 + k)}{\Gamma(2l + 2 + k)}.
\end{align*}
\]
(A.3)

The integral (A.2), as well as the integrals in Appendix B, are evaluated by combination of the Rodrigues formula and the series representation.

Obviously
\[
_2F_1(-n, n + 4l + 3; 2l + 2; 1 - u) = (-1)^n_2F_1(-n, n + 4l + 3; 2l + 2; u).
\]
(A.4)

This symmetry reflects the fact that these Jacobi polynomials can be expressed in terms of Gegenbauer polynomials \(C^{2l+\frac{3}{2}}_n(1-2u)\):
\[
_2F_1(-n, n + 4l + 3; 2l + 2; u) = \frac{n! \Gamma(4l + 3)}{\Gamma(n + 4l + 3)} C^{2l+\frac{3}{2}}_n(1-2u).
\]
(A.5)

The Jacobi polynomials obey the recursion relation
\[
\begin{align*}
&(2n + 4l + 3)(1 - 2u) \_2F_1(-n, n + 4l + 3; 2l + 2; u) \\
&= (n + 4l + 3) \_2F_1(-n - 1, n + 4l + 4; 2l + 2; u) \\
&\quad + n \_2F_1(-n + 1, n + 4l + 2; 2l + 2; u).
\end{align*}
\]
(A.6)

By differentiation of a Jacobi polynomial is obtained
\[
\begin{align*}
\frac{d^k}{du^k} \_2F_1(-n, n + 4l + 3; 2l + 2; u) \\
&= (-1)^k \frac{n! \Gamma(2l + 2) \Gamma(n + 4l + 3 + k)}{(n-k)! \Gamma(2l + 2 + k) \Gamma(n + 4l + 3)} \_2F_1(-n + k, n + 4l + 3 + k; 2l + 2 + k; u).
\end{align*}
\]
(A.7)

We also record a number of related relations:
\[
\begin{align*}
\frac{d^n}{du^n} u^{-1} \_2F_1(-n, n + 4l + 3; 2l + 2; u) &= (-1)^n n! u^{-n-1},
\end{align*}
\]
(A.8)
\[
\begin{align*}
\frac{d^n}{du^n} u^{-2} \_2F_1(-n, n + 4l + 3; 2l + 2; u) \\
&= (-1)^n (n + 1)! u^{-n-2} - (-1)^n n! \frac{(n + 4l + 3)}{2l + 2} u^{-n-1},
\end{align*}
\]
(A.9)
\[
\frac{d^{n}}{du^{n}} \log u_{2}F_{1}(-n, n + 4l + 3; 2l + 2; u) \\
= (-1)^{n}n! \frac{\Gamma(2l + 2)}{\Gamma(n + 4l + 3)} \left[ (\log u + \sum_{k=1}^{n} \frac{1}{k}) \frac{\Gamma(2n + 4l + 3)}{\Gamma(n + 2l + 2)} \\
- \sum_{k=0}^{n-1} u^{k-n} \frac{1}{n-k} \frac{\Gamma(n + 4l + 3 + k)}{\Gamma(2l + 2 + k)} \right], \quad (A.10)
\]

\[
\frac{d^{n-1}}{du^{n-1}} \log u_{2}F_{1}(-n, n + 4l + 3; 2l + 2; u) \\
= (-1)^{n}n! \frac{\Gamma(2l + 2)}{\Gamma(n + 4l + 3)} \left[ (\log u - \frac{1}{2}) + u \sum_{k=2}^{n} \frac{1}{k} - \frac{1}{2} \sum_{k=1}^{n-1} \frac{1}{k} \frac{\Gamma(2n + 4l + 3)}{\Gamma(n + 2l + 2)} \\
+ \sum_{k=0}^{n-2} u^{k-n+1} \frac{1}{(n-k)(n-k-1)} \frac{\Gamma(n + 4l + 3 + k)}{\Gamma(2l + 2 + k)} \right] \quad (A.11)
\]

and

\[
\frac{d^{n+1}}{du^{n+1}} \log u_{2}F_{1}(-n, n + 4l + 3; 2l + 2; u) \\
= (-1)^{n}n! \frac{\Gamma(2l + 2)}{\Gamma(n + 4l + 3)} \sum_{k=0}^{n} u^{k-n-1} \frac{\Gamma(n + 4l + 3 + k)}{\Gamma(2l + 2 + k)}. \quad (A.12)
\]

B Integrals

Matrix elements involving the functions \(\chi_{nl}(u)\) are evaluated here, with the notation:

\[
< nl || F(u) || nl > = \int_{0}^{1} du (u(1-u))^{2l+1} F(u)\chi_{nl}^{2}(u). \quad (B.1)
\]

By means of (A.8) one finds:

\[
< nl || u^{-1} || nl > = \frac{2n + 4l + 3}{2l + 1} \quad (B.2)
\]

and similarly by means of (A.10):

\[
< nl || u^{-2} || nl > = \frac{(n + 2l + 1)(n + 2l + 2)(2n + 4l + 3)}{2l(l+1)(2l+1)}. \quad (B.3)
\]
This expression is only well-defined for $l \neq 0$. For $l = 0$ also boundary terms arise through partial integrations. Thus a lower cutoff in the $u$-integration $u_{\text{min}} \approx \frac{R^2}{\rho^2}$ is introduced corresponding to the integral in coordinate space being restricted to a large sphere of radius $R$. Using also

$$\int_0^1 du u^{p-1}(1 - u)^{q-1} \log u = -B(p, q) \sum_{k=0}^{q-1} \frac{1}{p + k} \quad (B.4)$$

with $B(p, q)$ Euler’s beta function, one finds the total matrix element in this case:

$$< n0 \parallel u^{-2} \parallel n0 > \quad = \quad 2n + 3 + (2n + 3)(n + 2)(\log \frac{R^2}{\rho^2} - \frac{3}{2} - 2 \sum_{k=1}^{n} \frac{1}{k + 1}). \quad (B.5)$$

Also the result

$$< nl \parallel \log u \parallel nl >= \quad - \sum_{k=0}^{n+2l+1} \frac{1}{n + 2l + 2 + k} - \sum_{k=0}^{n-1} \frac{1}{n + 4l + 3 + k} \quad (B.6)$$

follows from (A.10) and (B.4).

A more complicated matrix element is

$$< nl \parallel \frac{\log u}{u(1 - u)} \parallel nl > \quad = \quad (-1)^n \frac{(2n + 4l + 3)\Gamma(n + 4l + 3)}{\Gamma(2l + 2)\Gamma(n + 2l + 2)} \int_0^1 du(u(1 - u))^{n+2l+1}$$

$$\quad \sum_{k=0}^{n} \frac{1}{k!(n - k)!} \left( \frac{d^k}{du^k} \frac{\log u}{u(1 - u)} \right) \frac{d^{n-k}}{du^{n-k}}F_1(-n, n + 4l + 3; 2l + 2; u)) \quad (B.7)$$

The following useful relation is readily proved by induction:

$$\frac{d^n}{du^n} \frac{\log u}{u(1 - u)} = n! \log u \left( \frac{(-1)^n}{u^{n+1}} + \frac{1}{(1 - u)^{n+1}} \right)$$

$$\quad + n! (-1)^n \sum_{k=1}^{n} \frac{1}{k u^k} \sum_{s=1}^{n-k+1} \frac{(-1)^s}{w^{n+2-k-s}(1 - u)^s}. \quad (B.8)$$
The contribution from the part of (B.7) with a logarithmic integrand is:

\[ (-1)^n \frac{(2n+4l+3)\Gamma(n+4l+3)}{n!\Gamma(2l+2)\Gamma(n+2l+2)} \int_0^1 du(u(1-u))^{n+2l+1} \log u \frac{d^n}{du^n}(u(1-u))^{-1} \]

\[ _2F_1(-n, n+4l+3; 2l+2; u) \]

\[ = -\frac{2n+4l+3}{2l+1} \left( \sum_{k=0}^{n+2l+1} \frac{1}{2l+1+k} + \sum_{k=0}^{2l} \frac{1}{n+2l+2+k} \right). \]  

(B.9)

by (A.8) and (B.4). The nonlogarithmic part of the integrand of (B.7) yields by (A.7) and (B.8)

\[ \frac{2n+4l+3}{2l+1} \sum_{k=1}^{n} \frac{1}{2l+k+1} \]  

(B.10)

where also the following algebraic identities were used:

\[ \sum_{p=0}^{n} (-1)^p \frac{k!}{p!(k-p)!} \frac{\Gamma(n+2l-k+p+s)}{\Gamma(n+2l+2-k+p)} (2n+4l+2-k+p) \]

\[ = \begin{cases} \frac{(n+2l-k)!k!}{(n+2l)!} & \text{for } s = 1 \\ 0 & \text{for } s \geq 2 \end{cases} \]  

(B.11)

and

\[ \sum_{k=1}^{n} \frac{n(n+2l-k)!}{(n-k)!(n+2l+1)!} \sum_{r=1}^{k} \frac{1}{r} = \frac{1}{2l+1} \sum_{k=1}^{n} \frac{1}{2l+1+k}. \]  

(B.12)

Adding (B.9) and (B.10) one obtains:

\[ < nl \bigg| \log \frac{u}{u(1-u)} \bigg| \bigg| nl > = -\frac{2n+4l+3}{2l+1} \left( 2 \sum_{k=1}^{n+2l+1} \frac{1}{2l+1+k} + \frac{1}{2l+1} \right). \]  

(B.13)

The last matrix element needed is

\[ < nl \bigg| (u - \frac{1}{2})\Phi(\frac{1-u}{u}) \bigg| \bigg| nl > \]

\[ = -(-1)^n \frac{\Gamma(n+4l+3)}{2n!\Gamma(2l+2)\Gamma(n+2l+2)} \int_0^1 du(u(1-u))^{n+2l+1} \frac{d^n}{du^n}(u(1-u)) \Phi(\frac{1-u}{u}) \]

\[ (((n+4l+3)_{2}F_1(-n-1, n+4l+4; 2l+2; u)

\[ + n_2F_1(-n+1, n+4l+2; 2l+2; u)) \]  

(B.14)
where the recursion relation (A.6) was used.

(B.14) has the Spence function part:

\[
(−1)^n \frac{\Gamma(n + 4l + 4)}{2n!\Gamma(2l + 2)\Gamma(n + 2l + 2)} \int_0^1 du(u(1 - u))^{n+2l+1} \Phi\left(\frac{1 - u}{u}\right)
\]

\[
\frac{d^n}{du^n} _2F_1(-n - 1, n + 4l + 4; 2l + 2; u)
\]

\[
= - \frac{n + 1}{2n + 4l + 4} \sum_{k=0}^{n+2l+1} \frac{1}{n + 2l + 2 + k}
\]

(B.15)
evaluated by means of (A.7) and

\[
\int_0^1 du u^{-1} (1 - u)^{q-1} \Phi\left(\frac{1 - u}{u}\right) = B(p, q) (\zeta(2, p) + \sum_{r=1}^{q-1} \sum_{s=0}^{r-1} \frac{1}{p + s})
\]

(B.16)

where \(\zeta(2, p) = \sum_{s=p}^{\infty} \frac{1}{s^p}\) is a generalized zeta function.

The logarithmic term of (B.14) is by (A.4):

\[
(−1)^{n+1} \frac{\Gamma(n + 4l + 4)}{2n!\Gamma(2l + 2)\Gamma(n + 2l + 2)} \int_0^1 du(u(1 - u))^{n+2l+1} \log(u(1 - u))
\]

\[
\frac{d^n}{du^n} \log u _2F_1(-n - 1, n + 4l + 4; 2l + 2; u)
\]

\[- \log u \frac{d^n}{du^n} _2F_1(-n - 1, n + 4l + 4; 2l + 2; u)
\]

\[+(−1)^{n+1} \frac{\Gamma(n + 4l + 3)}{2(n - 1)!\Gamma(2l + 2)\Gamma(n + 2l + 2)} \int_0^1 du(u(1 - u))^{n+2l+1} \log(u(1 - u))
\]

\[
\frac{d^n}{du^n} \log u _2F_1(-n + 1, n + 4l + 2; 2l + 2; u)
\]

\[- \log u \frac{d^n}{du^n} _2F_1(-n + 1, n + 4l + 2; 2l + 2; u))
\]

(B.17)

Here each term is evaluated separately by means of (A.11), (A.12) and (B.4).

The first term of (B.14) is

\[
\frac{n + 1}{2n + 4l + 4} \sum_{k=0}^{n+1} \frac{1}{n + 4l + 4 + k} - \frac{1}{2l + 2 + k}
\]

\[+ \frac{1}{2} \sum_{k=0}^{n+1} \frac{1}{2l + 2 + k} - \sum_{k=0}^{n+2l+1} \frac{1}{n + 2l + 2 + k} + \sum_{k=0}^{n+2l+1} \frac{1}{n + 2l + 2 + k}
\]

(B.18)
and the second term of (B.17) is

\[
- \frac{1}{2} \sum_{k=0}^{n} \frac{1}{2l + 1 + k} + \frac{n + 4l + 2}{2n + 4l + 2} \sum_{k=0}^{n-1} \left( \frac{1}{2l + 1 + k} - \frac{1}{n + 4l + 2 + k} \right) + \frac{n + 4l + 2}{4(n + 2l + 1)^2}
- \sum_{q=0}^{2l} \frac{1}{n + 2l + 2 + q} - 1 + \frac{n + 4l + 2}{2n + 4l + 2} \sum_{q=0}^{n+2l} \frac{1}{n + 2l + 2 + q}
\]  

(B.19)

by some algebraic manipulations.

Finally (B.14) has the following contribution from the nonlogarithmic part of the integrand according to (A.7) and (B.8):

\[
(-1)^n \frac{\Gamma(n + 4l + 3)}{2\Gamma(2l + 2)\Gamma(n + 2l + 2)} \int_0^1 du(u(1-u))^{n+2l+1} \sum_{k=1}^{n+2l} \frac{(-1)^k}{k(n-k)!} \\
\sum_{r=1}^{k} \sum_{s=1}^{k-r} \frac{1}{u^{k+1-s}(1-u)^s} du - \sum_{r=1}^{k-r} \sum_{s=1}^{k-r} (-1)^s \Gamma(n + 2l + 2 - s) \\
\left( \frac{(n+1)}{p!} \frac{1}{(k+1-p)!} \frac{\Gamma(n + 2l + 1 - k + p + s)}{\Gamma(n - k + 2l + 2 + p)} (2n - k + 4l + 3 + p) \\
+ (n + 4l + 2) \frac{1}{p!} \frac{1}{(k-1-p)!} \frac{\Gamma(n + 2l + 1 - k + p + s)}{\Gamma(n - k + 2l + 2 + p)} (2n - k + 4l + 2 + p) \right). 
\]

(B.20)

Here the following algebraic identities are used:

\[
\sum_{p=0}^{k+1} (-1)^p \frac{(k+1)!}{p!(k+1-p)!} \frac{\Gamma(n + 2l + 1 - k + p + s)}{\Gamma(n + 2l + 2 - k + p)} (2n - k + 4l + 3 + p) = 0
\]

(B.21)

and

\[
\sum_{p=0}^{k-1} (-1)^p \frac{1}{p!(k-1-p)!} \frac{\Gamma(n + 2l + 1 - k + p + s)}{\Gamma(n - k + 2l + 2 + p)} \frac{1}{2n - k + 4l + 2 + p} \\
= (-1)^{s-1} \frac{\Gamma(n + 2l + 1)}{\Gamma(n + 2l + 2 - s)} \frac{\Gamma(2n + 4l + 2 - k)}{\Gamma(2n + 4l + 2)}. 
\]

(B.22)
Thus (B.20) is:

\[
- \frac{n + 4l + 2}{2(n + 2l + 1)(2n + 4l + 1)!} \sum_{k=1}^{n} \frac{n!(2n + 4l + 1 - k)!}{k(n-k)!} \frac{1}{r} (k-r) \sum_{r=1}^{k-1} \frac{1}{r} (k-r) \\
= - \sum_{k=1}^{n} \frac{1}{n + 4l + 2 + k} + 1 - \frac{n + 4l + 2}{2n + 4l + 2}
\]

(B.23)

by (B.12) and the identity

\[
\sum_{k=1}^{n} \frac{n!(2n + 4l + 1 - k)!}{(n-k)!(2n + 4l + 2)!} = \frac{1}{n + 4l + 2}.
\]

(B.24)

Adding (B.15), (B.18), (B.19) and (B.23) one finally obtains:

\[
< nl \, || \, (u - \frac{1}{2}) \Phi(\frac{1-u}{u}) \, || \, nl > \\
= - \frac{n + 1}{2n + 4l + 4} \sum_{k=0}^{n+2l+1} \frac{1}{2l + 2 + k} - \sum_{k=0}^{n-1} \frac{1}{n + 4l + 4 + k} + \frac{1}{4l + 2} - \frac{1}{n + 4l + 2} \\
+ \sum_{k=0}^{n-1} \frac{1}{2l + 2 + k} - \frac{n}{2n + 4l + 2} \sum_{k=0}^{n+2l} \frac{1}{2l + 1 + k} - \sum_{k=0}^{n-1} \frac{1}{n + 4l + 3 + k}.
\]

(B.25)

\section*{C Zeta function regularization}

\subsection*{C.1 The zeta function}

The Riemann zeta function is [12]

\[
\zeta(s) = \sum_{k=1}^{\infty} k^{-s} = \frac{1}{\Gamma(s)} \int_{0}^{\infty} dt \frac{t^{s-1}}{e^t - 1}.
\]

(C.1)

The zeta function behaves near \( \epsilon = 0 \) according to:

\[
\epsilon \zeta(1 + 2\epsilon) \simeq \frac{1}{2} + \gamma \epsilon
\]

(C.2)

where \( \gamma \) is Euler’s constant. The zeta function as defined in (C.1) is only well-defined for \( s > 1 \), but can be analytically continued to the whole complex plane.
The generalized zeta function is with \( a > 0, s > 1 \):

\[
\zeta(s, a) = \sum_{k=0}^{\infty} (k + a)^{-s} = \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} \frac{e^{t(1-a)}}{e^t - 1} dt \tag{C.3}
\]

It also also has an analytic continuation in the variable \( s \) and obeys for all values of \( s \) the functional equation

\[
\zeta(s, a) = \zeta(s, a + m) + \sum_{k=0}^{m-1} (k + a)^{-s}. \tag{C.4}
\]

### C.2 Zeta function regularization

For a general self-adjoint operator \( \Delta \) with eigenvalues \( \lambda \) a generalized zeta function is formed [8], [9]:

\[
\zeta_\Delta(\epsilon) = \sum \lambda^{-\epsilon} \tag{C.5}
\]

and in the absence of zero modes the determinant of \( \Delta \) is defined by:

\[
\log \det \Delta = - \lim_{\epsilon \rightarrow 0} \frac{\partial}{\partial \epsilon} \mu^{2\epsilon} \zeta_\Delta(\epsilon) = - \log(\mu^2) \zeta(0) - \zeta'(0). \tag{C.6}
\]

with \( \mu \) an arbitrary mass scale (supposing \( \Delta \) has dimension mass squared).

When \( \Delta \) is perturbed the eigenvalue \( \lambda \) is changed by an amount \( \delta \lambda \). Hence the zeta function is changed by the amount

\[
\delta \zeta_\Delta(\epsilon) = - \epsilon \sum \delta \lambda \lambda^{-1-\epsilon} \tag{C.7}
\]

and

\[
\delta \log \det \Delta = \log(\mu^2) \lim_{\epsilon \rightarrow 0} \epsilon \sum \delta \lambda \lambda^{-1-\epsilon} + \lim_{\epsilon \rightarrow 0} \frac{\partial}{\partial \epsilon} \epsilon \sum \delta \lambda \lambda^{-1-\epsilon}. \tag{C.8}
\]

### C.3 The function \( Z_\phi(\epsilon) \)

Zeta function regularization involves the function \( Z_\phi(\epsilon) \) defined in (6.7); it has a useful representation in terms of a Feynman parameter \( \alpha \):

\[
Z_\phi(\epsilon) = \frac{1}{\Gamma^2(\epsilon)} \int_0^{\infty} dt t^{2\epsilon-1} \int_0^1 d\alpha (\alpha(1-\alpha))^{\epsilon-1} e^{-2t\phi(1-\alpha)} \left( \frac{2\epsilon - 1}{t} - \phi(1-2\alpha) \right). \tag{C.9}
\]
By a power series expansion of the exponential and use of (C.1) one expresses $Z_{\phi}(\epsilon)$ as an infinite sum of Riemann zeta functions:

$$
Z_{\phi}(\epsilon) = \zeta(2\epsilon - 1) - \phi(2\epsilon - 1)\zeta(2\epsilon) + \sum_{n=2}^{\infty} \frac{(-2\phi)^n}{(n-2)!} \left( \frac{(\epsilon + n - 1)(2\epsilon - 1)}{n(n-1)} \right) + \frac{1}{2} \frac{\Gamma(\epsilon + n - 1)}{\Gamma(\epsilon)} \frac{1}{2\epsilon + n - 1} \zeta(2\epsilon + n - 1)
$$

whence

$$
Z_{\phi}(0) = -\frac{1}{12} - \frac{\phi}{2},\; Z_{\phi}(-1) = \frac{1}{120} - \frac{\phi^2}{6}.
$$

(C.10)

(C.11)

From (C.10) and (C.1)-(C.4) follows for $\epsilon \simeq 0$:

$$
\epsilon Z_{\phi}(1 + \epsilon) \simeq \frac{1}{2} + \epsilon (\gamma - \frac{1}{2} \sum_{s=1}^{\infty} \frac{1}{s^2}).
$$

(C.12)

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