Kinematic Numerators and a Double-Copy Formula for $\mathcal{N} = 4$
Super-Yang-Mills Residues

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Abstract

Recent work by Cachazo, He, and Yuan shows that connected prescription residues obey the
global identities of $\mathcal{N} = 4$ super-Yang-Mills amplitudes. In particular, they obey the Bern-
Carrasco-Johansson (BCJ) amplitude identities. Here we offer a new way of interpreting this
result via objects that we call residue numerators. These objects behave like the kinematic numer-
ators introduced by BCJ except that they are associated with individual residues. In particular,
these new objects satisfy a double-copy formula relating them to the residues appearing in recently-
discovered analogs of the connected prescription integrals for $\mathcal{N} = 8$ supergravity. Along the way,
we show that the BCJ amplitude identities are equivalent to the consistency condition that allows
kinematic numerators to be expressed as amplitudes using a generalized inverse.
I. INTRODUCTION

There has been much recent progress in calculating scattering amplitudes in gauge and gravity theories. Among the many advances is the discovery by Bern, Carrasco, and Johansson (BCJ) of a duality between color and kinematics [1]. In this duality, kinematic numerators of diagrams obey relations similar to the Jacobi identities obeyed by color factors. One consequence of these relations is that color-ordered tree-level partial amplitudes obey a set of nontrivial identities, known as BCJ amplitude identities. These amplitude identities are even more constraining than the Kleiss-Kuijf identities [2], and so reduce the number of partial amplitudes required to determine a full scattering amplitude. Color-kinematic duality has been conjectured to hold for any number of loops or legs in a wide variety of theories, including pure Yang-Mills theories and their supersymmetric extensions. While no proof exists at loop level, a variety of nontrivial constructions exist [3–10], and its original tree-level formulation has been fully proven.

One remarkable feature of color-kinematic duality is that one can use it to construct gravity amplitudes directly from gauge-theory amplitudes. To do so one replaces the color factors in a gauge-theory amplitude with corresponding kinematic numerators that obey the duality. This gives what is known as the double-copy form of gravity amplitudes [1, 4]. At tree level, this encodes the Kawai-Lewellen-Tye (KLT) relations [11] between gauge and gravity amplitudes [12]. The extension of the double-copy formula to loop level [4] requires first constructing loop-level Yang-Mills amplitudes in a form where the duality is manifest. Then, as at tree level, one obtains gravity amplitudes simply by replacing color factors with corresponding kinematic-numerator.

A parallel direction of research originated from writing the tree-level scattering amplitude of $\mathcal{N} = 4$ super-Yang-Mills with the Roiban, Spradlin, Volovich, and Witten (RSVW) twistor string formula [13–17]. (This is also known as the “connected prescription”.) The RSVW formula expresses the scattering amplitude as an integral over a moduli space of curves in $\mathbb{CP}^{3|4}$ supertwistor space, effectively reducing the entire scattering amplitude calculation to solving an algebraic system of equations.

This method for determining tree-level scattering amplitudes as integrals was extended to $\mathcal{N} = 8$ supergravity, first in specific cases [18–20], and later in general [21–24]. Like the RSVW formula, the integrals for $\mathcal{N} = 8$ supergravity can be interpreted as contour
integrand, and hence as sums of residues. The formula appearing in Ref. [21] was originally a conjecture, in part because the formula required the KLT relations to hold for the residues in the same way that it holds for the amplitudes; this “KLT orthogonality conjecture” has now been proven [25].

Very recently, $\mathcal{N} = 4$ super-Yang-Mills amplitudes have been constructed using the new-found “scattering equations” [26], shown to have a color-kinematic structure, and have been explicitly related to color-dual numerators [27]. Because both the scattering-equation-based amplitudes and the kinematic numerator decomposition of amplitudes yield identical scattering amplitudes, it is not surprising that the former can be written in terms of the latter. On the other hand, the discovery that global relations, such as the KLT relations, hold at the residue level is intriguing because it suggests there is a Jacobi-like numerator structure for the residues as well.

Additional hints of a Jacobi-like numerator structure at the level of residues appeared in a proof of the BCJ amplitude identities in $\mathcal{N} = 4$ super-Yang-Mills [28]. The proof uses the explicit structure of the RSVW integrand to prove the BCJ relations. Further, Ref. [28] discusses how the method of the proof indicates that the BCJ amplitude identities hold at the level of the residues themselves, analogously to the KLT orthogonality conjecture. Given that the original BCJ amplitude relations were originally derived starting from considering color-dual numerators, the reappearance of amplitude relations for residues strongly hints at an analogous set of numerators for the residues.

Following these hints, we define objects called residue numerators. By construction, the residue numerators are analogous to the kinematic numerators of partial amplitudes, except that they hold for RSVW residues. We use the KLT orthogonality conjecture to show that the residue numerators obey both a double-copy formula and an orthogonality condition. We also expound on the observation of Ref. [28] to verify that RSVW residues do indeed satisfy the BCJ relations. To do all this, we work in the linear algebra formalism of Refs. [29, 30]. To formally prove our results in this formalism, we prove a conjecture of Ref. [30], establishing that the BCJ amplitude identities are equivalent to a consistency condition equation.

The structure of the paper is as follows. Section II outlines the necessary background material: the linear algebra formalism, the RSVW formula and its gravitational generalizations, and the concepts of color-kinematic duality and double copy. Section II also contains some novel material. Subsection II A 1 presents a proof that the BCJ amplitude identities
are equivalent to a system of constraint equations in the linear algebra formalism. In subsection II.B.1 we explicitly demonstrate that the BCJ amplitude identities (and hence the constraint equations) apply to RSVW residues, as was noted in [28]. Section II defines residue numerators, the central objects of this paper, and proves their double-copy formula. Finally, Section IV provides a conclusion and discusses future directions.

II. BACKGROUND MATERIAL AND LEMMAS

A. Linear Algebra

A convenient way of structuring discussions of kinematic numerators is the linear algebra approach pioneered in Ref. [29] and extended in Ref. [30] (whose notation we largely adopt). This formalism makes generalized gauge invariance manifest, and, as we shall demonstrate, also reinterprets the BCJ amplitude identities as algebraic consistency conditions.

To motivate this approach, recall that Yang-Mills scattering amplitudes in four dimensions can be written in a so-called Del Duca-Dixon-Maltoni (DDM) decomposition [31]. In this form, the full \( n \)-particle amplitude at tree level is written as

\[
A_n = g^{n-2} \sum_{\tau \in S_{n-2}} c_\tau A_n(1, \tau(2), \ldots, \tau(n-1), n),
\]

where the coupling constant is \( g \), the notation \( \tau \in S_{n-2} \) indicates that the sum runs over permutations \( \tau \) of the particle labels \( 2, \ldots, n-1 \), the \( A_n \) are color-ordered partial amplitudes, and the \( c_\tau \) are color factors\(^1\) of cubic diagrams. Cubic diagrams are diagrams with only trivalent vertices, which conserve color and momentum at each vertex. Any diagrams containing higher-point contact terms are absorbed into cubic diagrams with the same color factor, with missing propagators \( P \) introduced by multiplying by \( 1 = \frac{P}{P} \). While there is no known Lagrangian from which this decomposition can be directly generated by Feynman rules, these trivalent diagrams are a useful way of reorganizing the usual sum over Feynman diagrams.

Another decomposition of the tree level amplitude, which we will refer to as the BCJ decomposition, is

\[
A_n = g^{n-2} \sum_{i=1}^{(2n-5)!!} \frac{c_i n_i}{D_i},
\]

\(^1\) These are a product of group-theory structure constants; see [31] for details.
where the sum is now over the unique set of \((2n - 5)!!\) cubic diagrams, with color factors \(c_i\), products of propagators \(D_i\), and so-called “kinematic numerators” \(n_i\). These last objects are functions only of the external momenta and helicities, and are not uniquely defined. This is because a generalized gauge transformation \(n_i \mapsto n_i + \Delta_i\), for functions \(\Delta_i\) that obey
\[
\sum_{i=1}^{(2n-5)!!} \frac{c_i \Delta_i}{D_i} = 0,
\]
will leave the BCJ decomposition Eq. (2) invariant. The notion of a generalized gauge transformation will turn out to be nicely expressed in the linear algebra formalism.

We can relate the DDM and BCJ decompositions in a useful way. It was shown in Ref. \[31\] that the \((n - 2)!\) color factors \(c_\tau\) form a basis of the space of color factors of cubic diagrams. This is possible because the Jacobi relations of the structure constants induce linear relations among the color factors. In other words, any of the \((2n - 5)!!\) color factors \(c_i\) that appear in the decomposition Eq. (2) can be written as
\[
c_i = \sum_{\tau \in S_{n-2}} W_{i\tau} c_\tau \tag{4}
\]
where \(W_{i\tau}\) is a \((2n - 5)!! \times (n - 2)!\) matrix that encodes the Jacobi relations among the color factors. Our notation expresses sums over permutations (as in the DDM decomposition) with \(\tau\) and \(\omega\), and sums over cubic diagrams by Latin indices \(i, j\).

Color-kinematic duality states that there exists a set of color-dual numerators \(n_i\) that obey the exact same Jacobi relations as the color factors \(c_i\). In other words, for the same matrix \(W_{i\tau}\) defined above in Eq. (4), we can write
\[
n_i = \sum_{\tau \in S_{n-2}} W_{i\tau} n_\tau \tag{5}
\]
for some set of \((n - 2)!\) numerators \(n_\tau\). Substituting Eq. (4) and Eq. (5) into the BCJ decomposition, we find
\[
\mathcal{A}_n = g^{n-2} \sum_{i=1}^{(2n-5)!!} \sum_{\tau, \omega \in S_{n-2}}^{c_\tau n_\omega} \frac{c_\tau n_\omega}{D_i} W_{i\tau} W_{i\omega},
\]
\[
= g^{n-2} \sum_{\tau, \omega \in S_{n-2}} c_\tau n_\omega F_{\tau\omega},
\]
where \(F_{\tau\omega}\) is an \((n - 2)! \times (n - 2)!\) symmetric matrix with products of inverse propagators as entries:
\[
F_{\tau\omega} \equiv \sum_{i=1}^{(2n-5)!!} \frac{W_{i\tau} W_{i\omega}}{D_i}. \tag{7}
\]
The matrix $F_{\tau \omega}$ is a convenient way of simultaneously encoding both the color and numerator Jacobi relations in the basis of partial amplitudes.

Equating Eq. (6) to the DDM decomposition and matching coefficients of the $c_\tau$, we have the identity

$$A_n(1, \tau(2), \ldots, \tau(n-1), n) = \sum_{\omega \in S_{n-2}} F_{\tau \omega} n_\omega.$$  \hspace{1cm} (8)

This can be thought of in matrix notation as a system of linear equations

$$FN = A$$  \hspace{1cm} (9)

in the $(n-2)!$-dimensional space of partial amplitudes spanned by Kleiss-Kuijf basis amplitudes and indexed by $\tau \in S_{n-2}$, and $N$ is a column vector of numerators. Ideally we could invert this formula to get an expression for the numerators in terms of the partial amplitudes, but this is impossible because $F$ is singular. This is no surprise: the invariance of the full amplitude under generalized gauge transformations Eq. (3) ensures that the numerators are not unique, so $F$ cannot be invertible.

Therefore $F$ has a nontrivial kernel. To circumvent this problem, Ref. [30] suggested using the machinery of generalized inverses (also called pseudoinverses). A generalized inverse is a matrix $F^+$ satisfying $FF^+F = F$, and it can be shown that such an $F^+$ always exists, but is not unique. Generalized inverses are useful because of the following theorem [32]: a solution to $FN = A$ exists if and only if the consistency condition

$$FF^+A = A$$  \hspace{1cm} (10)

holds for some generalized inverse $F^+$. The general solution is then given by

$$N = F^+A + (I - F^+F) v$$  \hspace{1cm} (11)

for an arbitrary vector $v$ that parametrizes the kernel of $F$.

Notice that $I - F^+F$ is a projection operator onto the kernel of $F$, since, by the definition of $F^+$, $F(I - F^+F) = F - FF^+F = 0$. The consistency condition Eq. (10) has been conjectured to be equivalent to the BCJ amplitude identities [30], and we prove this conjecture below. Note, however, that because the existence of color-dual numerators has been proven at tree level [33, 34], we know that the consistency condition is satisfied thanks to the “if and only if” logic. Our proof that the consistency condition Eq. (10) is equivalent to the BCJ amplitude identities gives an alternative proof of the existence of color dual numerators, one that will extend to the residue numerators defined in Section 3.
1. Equivalence of the BCJ Amplitude Identities and the Consistency Condition

Our goal is to show that $FF^+A = A$ is the same as the BCJ amplitude identities, which we shall write formally as $SA = 0$, where $S$ is a matrix that forms the linear combination of amplitudes appearing in the BCJ identities. Since we chose the $A$ in $FF^+A$ to be a vector of amplitudes in the Kleiss-Kuijf basis, we have to be careful to choose the matrix $S$ so that it acts on the same vector of Kleiss-Kuijf amplitudes to produce the BCJ amplitude identities. One particularly nice choice for $S$ that accomplishes this is a matrix representation of the momentum kernel [34–37], given explicitly by $S_{\tau\omega} = S[\tau(2), \tau(3), \ldots, \tau(n-1) | \omega(2), \omega(3), \ldots, \omega(n-1)]$. This matrix has polynomials of Mandelstam variables as entries, and many of its properties are discussed in Ref. [34].

There are two assumptions that will go into our proof. These assumptions are both widely believed and have survived extensive low-point checks, but remain unproven for general $n$. The first is that the basis of $(n-3)!$ amplitudes appears to be minimal [38–40]. In other words, there are not further relations that will reduce the number of independent amplitudes below the $(n-3)!$ BCJ-independent amplitudes. The second assumption is that Eq. (9) only has solutions for $A$ in the basis spanned by the $(n-3)!$ BCJ-independent amplitudes. It is known that the span of these amplitudes is sufficient for a wide class of theories, including both Yang-Mills theories and the colored trivalent scalar theories discussed in [39]. Our assumption is that this sufficiency holds for any theory that possesses color-kinematic duality, and so this sufficiency also holds for any theories for which $F$ can be defined.

We demonstrate that the two equations $FF^+A = A$ and $SA = 0$ impose the same constraints on the elements of $A$ using a simple dimension counting argument. As mentioned above, the BCJ amplitude identities are known to reduce the $(n-2)!$ Kleiss-Kuijf independent amplitudes to $(n-3)!$ independent amplitudes [1], [38–40], i.e. the rank of $S$ is $(n-2)! - (n-3)!$. Our two assumptions imply that for generic momenta, the solution space of Eq. (9) necessarily has dimension $(n-3)!$ i.e. rank $F = (n-3)!$. But rank($FF^+)$ = rank($F)$ by a theorem of linear algebra [32], so the image of $FF^+$ has dimension $(n-3)!$. Then the solution space of $FF^+A = A$ has dimension at most $(n-3)!$, since it must be contained in the image of $FF^+$. But then by the assumption that the $(n-3)!$ basis is minimal, the solution space of $FF^+A = A$ must have dimension equal to $(n-3)!$. The basis vectors of this space must therefore be BCJ basis amplitudes, up to at
most a linear transformation. Therefore the operators $FF^+$ and $S + I$ must be equal up to this linear transformation, proving the result. Explicitly, we write $Q (FF^+ - I) = S$ for some $Q \in GL((n - 3)!)$ embedded in an $(n - 2)! \times (n - 2)!$ matrix with all other entries zero, and $I$ the $(n - 2)! \times (n - 2)!$ identity matrix.

This may be understood geometrically. If the BCJ amplitudes form the true minimal basis of color-ordered amplitudes, then the larger space spanned by Kleiss-Kuijf amplitudes must be constrained to the smaller space spanned by the BCJ amplitudes. We illustrate this for the four-point case in Fig.\textsuperscript{1}. In this simplest example, there are $(4 - 2)! = 2$ amplitudes in the Kleiss-Kuijf basis, and $(4 - 3)! = 1$ amplitude in the BCJ basis. If the Kleiss-Kuijf basis were minimal, then the vector

$$A = (A(1, 2, 3, 4), A(1, 3, 2, 4))$$

in the plane $\mathbb{C}^2$ would fully determine all partial amplitudes. The four-point BCJ basis linearly relates the two elements of the vector $A$ by

$$A(1, 3, 2, 4) = \frac{u}{s} A(1, 2, 3, 4) \quad \text{or} \quad A(1, 2, 3, 4) = \frac{s}{u} A(1, 3, 2, 4)$$

where either equation is valid, and amounts to choosing either $A(1, 2, 3, 4)$ or $A(1, 3, 2, 4)$ as a basis amplitude. This is equivalent to projecting to one axis or the other in the Fig.\textsuperscript{1}. Since the Kleiss-Kuijf vectors $A$ describe physically valid partial amplitudes, they cannot lie at an arbitrary point in the plane, but must instead lie on the BCJ line. For the four-point case, both operators $FF^+$ and $S + I$ are rank one, and act on the Kleiss-Kuijf vector of amplitudes. This means both operators necessarily map to a point along the BCJ line. The linear transformation $Q \in GL((4 - 3)!)$ in this case is just a constant that translates a point along the BCJ line, but such movement does not alter the relation between the amplitudes. The next-highest-point case, $n = 5$, has $(5 - 2)! = 6$ amplitudes in the Kleiss-Kuijf basis and $(5 - 3)! = 2$ amplitudes in the BCJ basis; geometrically the $n = 5$ case corresponds to a $\mathbb{C}^6$ hyperplane for the Kleiss-Kuijf basis with all points actually lying on the $\mathbb{C}^2$ plane spanned by the BCJ basis amplitudes. This same line of geometric reasoning supports the rank-counting argument for all $n$.

\textsuperscript{2} We say “plane” for $\mathbb{C}^2$ and “line” for $\mathbb{C}$ to highlight the geometry.
FIG. 1. Reduction of the Kleiss-Kuijf amplitude basis to the BCJ amplitude basis for \( n = 4 \). In the figure, \( A(1, 2, 3, 4) \equiv A_{23} \) and \( A(1, 3, 2, 4) \equiv A_{32} \). Because the BCJ basis is the minimal basis, any Kleiss-Kuijf amplitude vector actually lies on the “BCJ line”. Both \( A_{23} \) and \( A_{32} \) are complex numbers, indicated by the \( \mathbb{C} \) labels on the axes. The “rescaling by \( Q \)” arrows indicate the \( GL(1) \) freedom that rescales the point \( A \) along the BCJ line.

B. RSVW Formula and Residues

As mentioned in the introduction, significant work has been done on the special case of \( \mathcal{N} = 4 \) super-Yang-Mills. In particular, the aforementioned RSVW formula that gives all tree level partial amplitudes is

\[
A_n(1, 2, \ldots, n) = \int \frac{d^{2n} \sigma}{\text{vol} GL(2) (12) (23) \cdots (n1)} \prod_{\alpha=1}^{k} \delta^2 \left( C_{\alpha \alpha} \tilde{\lambda}_\alpha \right) \delta^{0|4} \left( C_{\alpha \alpha} \tilde{\eta}_\alpha \right) \\
\times \int d^2 \rho_\alpha \prod_{b=1}^{n} \delta^2 \left( \rho_\beta C_{\beta \beta} - \lambda_b \right),
\]

(14)

where the \( C_{\alpha \alpha} \) are \( k \times n \) matrices parametrized by \( \sigma \), as discussed in the Grassmannian formulations of Refs. [41–43], for particles in the \( R \)-charge sector given by \( k \). The minors \( (12), (23), \) etc. are minors of \( C_{\alpha \alpha} \), and are thus functions of the \( \sigma \). The delta functions enforce the conditions that the spinor helicity variables \( \lambda \) and \( \tilde{\lambda} \) (along with \( \tilde{\eta} \)) are appropriately orthogonal, and thus that overall supermomentum is conserved.

Notice that both the delta functions and the measure are invariant under permutations
of the particle labels. This means we can write
\[
A_n(1, \tau(2), \ldots, \tau(n-1), n) = \int \frac{d^{2n}\sigma}{\text{vol} \, GL(2)} L_\tau \prod_{a=1}^{k} \delta^2 \left( C_{\alpha a} \tilde{\lambda}_a \right) \delta^{04} \left( C_{\alpha a} \tilde{\eta}_a \right) \times \int d^2 \rho_\alpha \prod_{b=1}^{n} \delta^2 \left( \rho_\beta C_{\beta b} - \lambda_b \right),
\]
where
\[
L_\tau \equiv \frac{1}{(1 \tau(2)) \cdots (\tau(n-1) \ n) \ (n1)}.
\]
This representation has an important consequence. To understand the action of \( F \) on \( A_\tau \), we only need to understand how the inverse minor factor \( L_\tau \) will be affected, since \( L_\tau \) is the only factor in \( A_\tau \) that depends on the particle label ordering. We are in particular interested in how the consistency condition \( FF^+ A = A \) manifests itself in this setting. By the permutation invariance discussed above, \( FF^+ A = A \) holds provided (in matrix notation)
\[
FF^+ L = L
\]
on the support of the delta functions. One goal in the remainder of this section will be to establish this fact.

The proof employs two lemmas. The first lemma is the claim that \( FF^+ A = A \) and \( SA = 0 \) are algebraically equivalent, which was shown in Subsection [11B.1]. The second lemma states that the BCJ amplitude identities descend to the level of residues. Mathematically, this is the statement that \( SA = 0 \) implies \( SR_\tau = 0 \) for all RSVW residues \( R_\tau \) (which will be defined below). This was originally noted in Ref. [28]. Because Ref. [28] proves that \( SL = 0 \) (in our notation) on the support of the delta functions, the residue consistency condition Eq. (17) will follow from these two lemmas. Explicitly, we can rewrite \( SL = 0 \) as
\[
Q \left( FF^+ - I \right) L = 0.
\]
Multiplying by \( Q^{-1} \) and rearranging gives Eq. (17).

1. **Proof of the BCJ Amplitude Identities for RSVW Residues**

Our argument proceeds by invoking several equivalent forms of Eq. (15) found in Section 3.1 of Ref. [42]. We begin by noting that in supertwistor space, Eq. (15) is
\[
A_n(1, \tau(2), \ldots, \tau(n-1), n) = \int \frac{d^{2n}\sigma}{\text{vol} \, GL(2)} L_\tau \prod_{a=1}^{k} \delta^{44} \left( C_{\alpha a} W^{\alpha a} \right),
\]
with \( L_\tau \) still defined as in Eq. (16). Here the \( C_{\alpha a} \) are functions of \( \sigma \) as given by the Veronese map discussed in Ref. [42]. This can be cleverly rewritten as

\[
A_n (1, \tau (2), \ldots, \tau (n - 1), n) = \int \frac{d^{k \times n} C_{\alpha a}}{\text{vol} \ GL (k)} G_\tau (C) \prod_{a=1}^{k} \delta^{4|4} (C_{\alpha a} \mathcal{W}^{\alpha a})
\]  

(20)

where

\[
G_\tau (C) = \int \frac{d^n \sigma \ d^{k \times k} M}{\text{vol} \ GL (2)} L_\tau \prod_{a=1}^{k} \prod_{a=1}^{n} \delta \left( C_{\alpha a} - M_\beta^{\beta a} \right),
\]

(21)

with \( M_\beta^{\beta a} \) a set of \( k \times k \) matrix integration variables. This form makes the integral look more like the Grassmannian formulation of Ref. [41], but more importantly for our purposes, it allows us to recast it into the form of a contour integral. The idea is that there are \( (k - 2) (n - k - 2) \) delta functions beyond those that fix the kinematics[^3]. These extra delta functions are factors of \( G_\tau (C) \). Following the discussions in Refs. [17, 42, 44], we can reinterpret integrating against one of these delta functions as instead integrating around a contour that encloses a pole located at the argument of the delta function. Therefore, using the notation of Ref. [42], the integral can be recast as

\[
A_n (1, \tau (2), \ldots, \tau (n - 1), n) = \int_{S_{1} = \ldots = S_{m} = 0} \frac{d^{k \times n} C_{\alpha a}}{\text{vol} \ GL (k)} H_\tau (C) \prod_{a=1}^{k} \prod_{a=1}^{m} \delta \left( C_{\alpha a} - M_\beta^{\beta a} \right).
\]

(22)

There are \( m = (k - 2) (n - k - 2) \) functions \( S (C) \), called Veronese operators, and these contain the locations of the poles. As the notation suggests, \( H_\tau \) contains the (integrated) minor factor \( L_\tau \), since \( L_\tau \) was part of \( G_\tau \). Therefore \( H_\tau \) depends on \( \tau \), while none of the Veronese operators do.

A concrete example is the \( n = 6, k = 3 \) Yang-Mills amplitude. In this case, there is \( m = 1 \) function \( S (C) \) that determines the correct contour in the complex plane. Gauge fixing in \( GL (3) \) and overall momentum conservation results in only one remaining complex integration variable \( c \), and so the calculation reduces to a standard contour integral in \( \mathbb{C} \). It may be shown in this \( n = 6, k = 3 \) case, that \( S (c) \) is quartic for arbitrary momenta [17, 42]. The correct contour for calculating the amplitude must enclose the four roots \( c_1, c_2, c_3, \) and \( c_4 \) of \( S (c) \), as indicated in Fig. 2. The remaining three points \( \tilde{c}_1, \tilde{c}_2, \tilde{c}_3 \) in the \( c \)-plane correspond to the poles of the function \( H (c) \).

[^3]: In other words, delta functions other than \( \prod_{a=1}^{k} \delta^{4|4} (C_{\alpha a} \mathcal{W}^{\alpha a}) \).
FIG. 2. The integral for the $n = 6, k = 3$ Yang-Mills amplitude has one integration variable not fixed by gauge choice or momentum conservation, and so may be calculated as a standard contour integral of a complex variable $c \in \mathbb{C}$. The four poles $c_1, c_2, c_3$, and $c_4$ correspond to the four roots of $S(c)$, and the three remaining poles $\tilde{c}_1, \tilde{c}_2$, and $\tilde{c}_3$ correspond to the poles of the function $H(c)$. This figure is meant only as a guide; the actual location of the poles changes for different external momenta.

For general $n$ and $k$, we can then use the global residue theorem\(^4\) to write the amplitude as a sum of residues (where we have absorbed factors of $2\pi i$):

$$A_n (1, \tau (2), \ldots, \tau (n - 1), n) = \sum_r R_r (1, \tau (2), \ldots, \tau (n - 1), n).$$

Letting $C_r$ denote the location of residue $R_r$, the residues have the general form (with the $\tau$-dependence explicit)

$$R_{r, \tau} = \frac{H_\tau (C_r)}{T (C_r)}.$$

Here $T (C)$ is a function consisting of the product of the nonvanishing Veronese operators and containing the gauge fixing. Notice that $T$ is independent of $\tau$.

We can now apply the matrix $S$ to each residue $R_r$. Doing so, we have

$$S_{\tau \omega} R_{r, \omega} = \frac{1}{T (C_r)} S_{\tau \omega} H_\omega (C_r).$$

Remembering that $H_\tau$ is precisely $G_\tau$ with some integration variables fixed by delta functions and the remaining delta functions stripped off, we see that $H_\tau (C_r)$ is proportional to $L_{\tau}$.

\(^4\) This is necessary because we are dealing with a multidimensional contour integral. See [41] for details.

\(^5\) We cannot simply say the $C_r$ are the zeros of the Veronese operators, due to issues at high multiplicity arising from so-called composite residues. A more complete discussions of these issues can be found in [41] [42].
evaluated on the support of all of the delta functions. But, as proved in Ref. [28], $S L_r = 0$ on the support of the delta functions in the RSVW formula. Therefore $S_{r\omega} H_{\omega} (C_r) = 0$, and so $S$ annihilates the residues $R_r$ individually, exactly as we wished to show.

C. Gravity

We are ultimately interested in the application of the techniques developed so far to analyze gravity. We therefore discuss how the material of the previous two sections can be generalized to gravity.

The first concept is the double-copy formula. This states that tree level gravitational amplitudes can be written as

$$M_n = \left(\frac{\kappa}{2}\right)^{n-2} \sum_{i=1}^{(2n-5)!!} \frac{n_i \tilde{n}_i}{D_i},$$

where the $n_i$ and $\tilde{n}_i$ are both sets of kinematic numerators, possibly from different Yang-Mills theories (such as with varying amounts of supersymmetry), at least one set of which is color-dual. The gravitational coupling constant is $\kappa$. This formula was first proposed in Ref. [4], and was proven in Ref. [12] using color-kinematic duality and the BCFW recursion relations. Our ultimate goal is to derive an analogous formula valid at the level of residues.

The second concept defines exactly what the residues look like on the gravitational side. The Cachazo-Geyer formula for gravitational amplitudes was proposed in Ref. [21):

$$M_n = \int \frac{d^{2n} \sigma}{\text{vol} GL(2)} \frac{H_n}{J_n} \prod_{\alpha=1}^{k} \delta^2 \left(\lambda_a \lambda_a - \lambda_b\right) \delta^{0|8} \left(\eta_a \eta_a\right) \int d^2 \rho_b \prod_{b=1}^{n} \delta^2 \left(\rho_\beta \left(C_{\beta b} - \lambda_b\right)\right),$$

which looks exactly the same as the RSVW formula Eq. (14) except for the four extra supersymmetries and the replacement of the inverse minor factor with $H_n / J_n$. The exact definition of $H_n / J_n$ is not important for our purposes, but it is also a function of the minors of $C_{aa}$. All of the discussion involving writing the RSVW formula as a contour integral applies to this formula as well. Indeed, since the delta functions in both formulas are the same, the residues occur at the exact same points, and the same contours may be used. This is crucial, as it allows us to put RSVW residues in one-to-one correspondence with the residues of this formula. Explicitly, we write

$$M_n = \sum_r R^G_r,$$
with the index \( r \) matching the RSVW residue index \( r \).

The formula Eq. (27) was conjectured in Ref. [21]; its proof depended on the resolution of a conjecture called the KLT orthogonality conjecture. This conjecture was proven in Ref. [45]. In addition to putting Eq. (27) on a solid foundation, it will play an important role in proving the residue analog of Eq. (26). We will give a formal statement of the conjecture at that time.

III. RESIDUE NUMERATORS

The lemmas proved in the previous section have important implications. In particular Eq. (17), the consistency condition for residues, guarantees the existence of what we have christened residue numerators, defined by

\[
N_r \equiv F^+ R_r + (I - F^+ F) v, \tag{29}
\]

again for arbitrary \( v \). These numerators obey all of the properties of full amplitude numerators, essentially by definition. In particular, \( R_r = FN_r \).

Recall that the gravity amplitude \( M_n \) can be written in terms of Yang-Mills partial amplitudes using the KLT relations, which in our notation take the form [30]

\[
M_n = \left( \frac{\kappa}{2} \right)^{n-2} A^T F^+ \tilde{A}. \tag{30}
\]

\( A \) and \( \tilde{A} \) are the partial amplitudes associated with two (possibly different) Yang-Mills theories. We can now address our central question: if we combine two \( \mathcal{N} = 4 \) super-Yang-Mills theories to get \( \mathcal{N} = 8 \) supergravity, can we replace everything by residues and still get the same result? In other words, we conjecture

\[
R^G_r \equiv \left( \frac{\kappa}{2} \right)^{n-2} R^T_r F^+ \tilde{R}_r. \tag{31}
\]

To test this hypothesis, we substitute the expressions for \( M_n \) and \( A_n \) as sums of residues into the KLT relations Eq. (30). This yields

\[
\sum_r R^G_r = \sum_{r, \tilde{r}} R^T_r F^+ \tilde{R}_{\tilde{r}}
= \sum_r R^T_r F^+ \tilde{R}_r + \sum_{r \neq \tilde{r}} R^T_r F^+ \tilde{R}_{\tilde{r}}, \tag{32}
\]
where we have separated out the cross terms in the second line. If our conjecture is true, it must imply

$$\sum_{r \neq \tilde{r}} R_{r}^{\dagger} F^{+} \tilde{R}_{r} = 0.$$  \hspace{1cm} (33)

This is the aforementioned KLT orthogonality conjecture, and it was recently proved in Ref. [45].

Now we have

$$\sum_{r} R_{r}^{G} = (\frac{\kappa}{2})^{n-2} \sum_{r} R_{r}^{T} F^{+} \tilde{R}_{r}. \hspace{1cm} (34)$$

However, this is insufficient to show the two sides are equal term-by-term. To show this stronger statement, we need to go back to the derivation of Eq. (27) found in Ref. [21]. In particular, the residues $R_{r}^{G}$ and $R_{r}$ are dependent only on the integrands of the RSVW and Cachazo-Geyer formulas. After inserting two copies of the RSVW formula into the KLT relations, the use of KLT orthogonality reduces the integral to an integral over a single set of the RSVW variables (rather than one set for $A$ and one for $\tilde{A}$). Therefore the integrands, not just the integrals, are in fact equal. Since the integrands are evaluated at exactly the same set of points in determining the residues, this implies that Eq. (34) holds term-by-term, proving $R_{r}^{G} = (\kappa/2)^{n-2} R_{r}^{T} F^{+} \tilde{R}_{r}$.

We can now recast this in terms of residue numerators. Specifically, we can write

$$R_{r}^{G} = (\frac{\kappa}{2})^{n-2} R_{r}^{T} F^{+} \tilde{R}_{r}$$

$$= (\frac{\kappa}{2})^{n-2} R_{r}^{T} F F^{+} \tilde{R}_{r}$$

$$= (\frac{\kappa}{2})^{n-2} (F^{+})^{T} R_{r}^{T} F F^{+} \tilde{R}_{r}$$

$$= (\frac{\kappa}{2})^{n-2} N_{r}^{T} F \tilde{N}_{r}, \hspace{1cm} (35)$$

where in going to the second line we have used the fact that the residues obey the consistency condition $\tilde{R}_{r} = F F^{+} \tilde{R}_{r}$, and in the last line we have used the fact that for $F$ symmetric, $(F^{+})^{T}$ is also a generalized inverse. But the last line is precisely the double-copy formula with ordinary kinematic numerators replaced by residue numerators, and the gravitational amplitude replaced by the corresponding gravitational residue!

Notice that this argument holds in reverse. Assuming the double-copy formula for residue numerators, we can reverse the logic in the equations leading to Eq. (35) and derive the

\[ \text{by a generalized gauge transformation, but this is irrelevant for our purposes.} \]
KLT relations for RSVW residues. The course of this argument uses “residue numerator orthogonality”

$$\sum_{\hat{r} \neq r} N^T_{\hat{r}} F N_{\hat{r}} = 0, \quad (36)$$

which follows from the residue numerator double-copy formula, to prove KLT orthogonality.

As a simple consistency check, we have numerically verified this result at six points ($n = 6$ is the smallest $n$ for which nontrivial residues occur in the connected prescription[17]).

This equivalency between the KLT relations and the double-copy formula harks back to the equivalency at the amplitude level [30], and we expect it will be important for similar reasons. In particular, the double-copy formulation has two major advantages over the KLT relations. First, the double-copy formula is much cleaner, making the “gravity is the square of gauge theory” adage more transparent. Second, the KLT relations are restricted to tree level, while the double-copy formula is conjectured to apply to all loop orders. This suggests that the residue numerators, while only now defined at tree level, might have loop-level analogs.

We have now shown that not only do the BCJ amplitude identities and the KLT relations descend from the full amplitude to their residues, but so do the concepts of kinematic numerators and the double-copy formula. In the same vein, we emphasize that it is equivalent to use residue numerators as a starting point, and derive the residue relations, much as the original BCJ amplitude identities were originally derived from numerators.

### IV. CONCLUSION AND FUTURE WORK

We have introduced residue numerators. These objects serve to give a BCJ decomposition of RSVW residues. They have the property that under replacement of a color factor by a residue numerator we obtain gravity residues, and they obey an orthogonality condition equivalent to the KLT orthogonality condition [25].

We have confirmed all three points in a specific case ($n = 6$). To generalize this result, we proved two lemmas. First we proved that the consistency condition of the generalized inverse is equivalent to the BCJ amplitude identities by counting the rank of the matrices $F$ and $F F^+$, largely in the spirit of the arguments suggested in Ref. [30]. By replacing amplitudes with RSVW residues, $A \rightarrow R_r$, and $N \rightarrow N_r$ in expressions for amplitudes, our proof implies that RSVW residues obey BCJ amplitude identities if and only if there
exist decompositions of RSVW residues into residue numerators. It is possible to explicitly construct such residue numerator solutions [27, 46], so we only needed to verify that RSVW residues obey amplitude relations in general. We did this by demonstrating that RSVW residues contain permutation-dependent factors which vanish in the BCJ amplitude relation, as discussed in Ref. [28]. This was the second lemma that we proved.

From these two lemmas, our main result followed. We showed that the new proof of KLT orthogonality for RSVW residues implies a residue numerator orthogonality condition. Conversely, from the ab initio assumption that both RSVW and gravity residues could be decomposed into residue numerators, we arrived at a residue numerator orthogonality condition that implies the KLT orthogonality condition.

We expect that residue numerators will offer insight into a variety of topics that are currently phrased in terms of amplitudes. At tree-level, the interplay of combinatorics and linear algebra that go into constructing $F$ are suggestive of the positive Grassmannian [43] and amplituhedron [47, 48]. The numerator Jacobi relations may be relevant to lifting the theory described by the amplituhedron out of the planar limit, since numerator Jacobi identities relate planar and nonplanar diagrams. (See, for example, Ref. [5].) Also at tree level, we suspect a direct link between residue numerators and the recently constructed scattering equations [45], especially in light of the even more recent decomposition of the scattering equations in terms of kinematic numerators [27]. It would also be interesting to explore the role of residue numerators in theories like ABJM, in which even the tree-level connection between color-kinematic duality and BCJ amplitude identities is less well understood [49, 50].

Numerator decompositions have proven to be a powerful way to evaluate loop-level contributions to amplitudes as well. The decomposition of residues into numerators performed here may offer insight into improved ways of constructing loop integrands. We expect that reexamining loop-level amplitudes where known color-dual numerators are available will offer insight into how residue numerators might be applied in loop calculations. There are good starting points in the literature pursuing such systematization at tree level [51], at loop level with color-dual numerators [9], and at higher-loop level without color-dual numerators [52, 53]. Recent work [3] also explicitly illustrates a color-dual numerator construction for pure Yang-Mills at one loop and two loops, providing nontrivial evidence that numerator representations extend to nonsupersymmetric theories. We expect that residue numerators
might be applicable to theories with less supersymmetry.

That said, the extension of the RSVW formula (and more recently the scattering equations) to loop level has been fraught with difficulties [54, 55]. It has been difficult to understand generalized gauge invariance at loop level [30], as the invariance applies at the level of the integrand, and must therefore be extended to include terms that vanish upon integration over loop momenta. We hope the algebraic simplicity of residue numerators will help these barriers be overcome, but at present there is not an obvious path forward.

Finally, the existence of residue numerators reemphasizes the role of numerators in color-kinematic duality. In the same way that color-kinematic duality underlies the BCJ amplitude identities, we have demonstrated that these residue numerators imply KLT and BCJ relations as well as the KLT orthogonality relations between RSVW residues.

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Here we present one possible way of defining numerator residues. We first argue that six-point next-to-maximal-helicity-violating amplitudes (NMHV) present the simplest, non-trivial appearance of residue numerators. Borrowing from Ref. [17], an amplitude of any helicity may be written as an integral in \((k-2)(n-k-2)\) integration variables \(c_i\):

\[
A_n = \int d^{(k-2)(n-k-2)}c_i a_n (c_i) ,
\]

where \(k\) is the number of negative-helicity gluons in the scattering process. By definition, an amplitude with \(k=2\) is called “MHV”. Any amplitude with \(k=K+2\) is known as an \(N^K\)MHV amplitude. The function \(a_n (c_i)\) in Eq. 37 is unimportant for the current explanation. These complex integrals are exactly the ones that produce residues and corresponding residue numerators. We require that \(k > 2\) (since \(k \leq 2\) results in no integration according to Eq. 37). The simplest case is then \(k = 3\). We further look for the case where there is only one complex integration parameter:

\[
(k-2)(n-k-2) = 1 \Rightarrow n = 6.
\]

Thus \(n = 6, k = 3\) amplitudes offer the first, simplest opportunity for the appearance of residue numerators.

While there is likely illuminating structure hiding in the functional form of the integrands, we here ignore such details in favor of a broader view. The expression Eq. 37 for \(n = 6, k = 3\) says that any amplitude may be expressed as

\[
A_n (L,h) = \int dc a_n (L,h,c) ,
\]

for a momentum label configuration \(L \in \mathcal{P} (\{1,2,\ldots,n\})\) and helicity configuration \(h = \{h_1,h_2,\ldots,h_n\}\) (the \(L\) and \(h\) were suppressed in Eq. 37). For general \(n\) and \(k\), \(a_n (c_i)\) contains delta functions in \(c_i\). In the case \(n = 6, k = 3\), the argument of the delta function is quartic in the complex variable \(c\), and so the integral may be reexpressed as a contour integral enclosing exactly the four roots of the argument of the delta function. More explicitly if

\[
a_n (L,H,c) = a_n (L,H,c) \delta (S (c)) ,
\]

then

\[
S (c) = \kappa (c-c_1) (c-c_2) (c-c_3) (c-c_4)
\]
for an overall constant $\kappa$. Converting the amplitude into a complex integral and ignoring factors of $2\pi i$ which cancel out in the final result:

$$A_n (L, h) = \oint_{S(c)} dc \frac{a_n (L, H, c)}{S(c)}$$

$$= \sum_{i=1}^{4} \text{Res}_{c=c_i} \left( \frac{a_n (L, H, c)}{S(c)} \right)$$

$$\equiv \sum_{r=1}^{4} R_r.$$  \hspace{1cm} (42)

We now define residue numerators by expressing color-dual numerators in terms of amplitudes, replacing each amplitude with a residue of that amplitude, and indexing the resulting residue numerator accordingly. Schematically:

$$n = f (A) \implies n_r = f (R_r), \quad A = \sum_r R_r.$$  \hspace{1cm} (43)

There are several methods of determining the function $f (A)$; here we present one possible method. We determine residue numerators in the $n = 3, k = 6$ case by comparing two different expressions for the gravity amplitudes. The first is the KLT expression at six-point:

$$M (123456) = -i \left( \frac{\kappa}{2} \right)^{6-2} \sum_{\tau \in S_3} s_{1\tau(2)} s_{\tau(4)5} \tilde{A} (1, \tau (2), \tau (3), \tau (4), 5, 6) \times$$

$$\left( s_{\tau(3)5} A (\tau (2), 1, 5, \tau (3), \tau (4), 6) + \right.$$  

$$\left. + (s_{\tau(3)\tau(4)} + s_{\tau(3)5}) A (\tau (2), 1, 5, \tau (4), \tau (3), 6) \right), \hspace{1cm} (44)$$

where $S_3$ is the set of all permutations of $\{2, 3, 4\}$. The second is the numerator decomposition of the gravity amplitude \[\text{[12]}\]:

$$M (123456) = i\kappa^{6-2} \sum_{\tau \in S_4} n_{1\tau(2)\tau(3)\tau(4)\tau(5)6} \tilde{A} (1, \tau (2), \tau (3), \tau (4), \tau (5), 6), \hspace{1cm} (45)$$

where $S_4$ is the set of all permutations of $\{2, 3, 4, 5\}$. Equating the two expressions for $M (123456)$ given in Eq. 44 and Eq. 45 yields expressions for the $(n-2)!$ numerators $n_{1\tau(2)\tau(3)\tau(4)\tau(5)6}$:

$$n_{1\tau(2)\tau(3)\tau(4)\tau(5)6} = -2^4 s_{1\tau(2)} s_{\tau(4)5} \left( s_{\tau(3)5} A (\tau (2), 1, 5, \tau (3), \tau (4), 6) \right.$$  

$$\left. + (s_{\tau(3)\tau(4)} + s_{\tau(3)5}) A (\tau (2), 1, 5, \tau (4), \tau (3), 6) \right), \hspace{1cm} (46)$$
\[ n_{1\tau(2)\tau(3)\tau(4)\tau(5)6} = 0 \text{ (for } \tau(5) \neq 5) . \] (47)

The residue numerators are then constructed by replacing

\[ A(\tau(1), \tau(2), \tau(3), \tau(4), \tau(5), \tau(6)) \rightarrow R_{r}(\tau(1), \tau(2), \tau(3), \tau(4), \tau(5), \tau(6)), \] (48)

where Eq. 23 holds. Explicitly:

\[
n_{r,1\tau(2)\tau(3)\tau(4)56} = -2^{-4}s_{1\tau(2)}s_{\tau(3)5} R_{r}(\tau(2), 1, 5, \tau(3), \tau(4), 6) \\
+ (s_{\tau(3)\tau(4)} + s_{\tau(3)5}) R_{r}(\tau(2), 1, 5, \tau(4), \tau(3), 6).
\] (49)

This approach may seem circular since the residues are used to define the residue numerators. In the end, however, the residue numerators are nothing more than complex numbers \( n_{r,1\tau(2)\tau(3)\tau(4)\tau(5)6} \in \mathbb{C} \) that serve as the numerators for the residues of amplitudes, and the manner of determining those complex numbers is irrelevant.
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