Maharaja Nim

Wythoff’s Queen meets the Knight

Urban Larsson and Johan Wästlund,
Mathematical Sciences,
Chalmers University of Technology and University of Göteborg,
Göteborg, Sweden
urban.larsson@chalmers.se, wastlund@chalmers.se

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Abstract

New combinatorial games are introduced, of which the most pertinent is Maharaja Nim. The rules extend those of the well-known impartial game of Wythoff Nim in which two players take turn in moving a single Queen of Chess on a large board, attempting to be the first to put her in the lower left corner. Here, in addition to the classical rules a player may also move the Queen as the Knight of Chess moves. We prove that the second player’s winning positions of Maharaja Nim are close to the ones of Wythoff Nim, namely they are within a bounded distance to the lines with slope \( \frac{\sqrt{5}+1}{2} \) and \( \frac{\sqrt{5}-1}{2} \) respectively. For a close relative to Maharaja Nim, where the Knight’s jumps are of the form \((2,3)\) and \((3,2)\) (rather than \((1,2)\) and \((2,1)\)), we also demonstrate polynomial time complexity to the decision problem of the outcome of a given position.

1 Maharaja Nim

We introduce a 2-player combinatorial game called Maharaja Nim, an extension of the well-known game of Wythoff Nim \([Wy]\). (The name ‘Maharaja’ is taken from a variation of Chess, ‘The Maharaja and the Sepoys’, \([Fa]\).) Both these games are impartial, that is, the set of options are the same regardless of whose turn it is. For a background on impartial games see \([BCG]\).

Place a Queen (of Chess) on a given position of a large Chess board, with the position in the lower left corner labeled \((0,0)\). In the game of Wythoff Nim, here denoted by \(W\), the two players move the Queen alternately as
it moves in Chess, but with the restriction that, by moving, no coordinate increases, see Figure 1. Only non-negative coordinates are allowed so that the first player to reach the position \((0,0)\) wins.

In Maharaja Nim, denoted by M, the rules are as in Wythoff Nim, except that the Queen is exchanged for a ‘Maharaja’, a piece which may move both as the Queen and the Knight of Chess, again, provided by moving no coordinate increases. See Figure 1.

We say that a position is P if the second player to move has a winning strategy, otherwise N. Also, let \(P_M\) and \(P_W\) denote the set of P-positions of Maharaja Nim and Wythoff Nim respectively. See Figure 2 for a computation of the initial P-positions of the respective games and the Appendix, Section A1 for the corresponding code.

Let \(N\) denote the positive integers and \(\mathbb{N}_0\) the non-negative integers. Let

\[
\phi = \frac{1 + \sqrt{5}}{2}
\]

denote the golden ratio. The well-known winning strategy of Wythoff Nim \([W]\) is

\[
P_W = \{([\lfloor \phi n \rfloor, \lfloor \phi^2 n \rfloor), ([\lfloor \phi^2 n \rfloor, \lfloor \phi n \rfloor]) \mid n \in \mathbb{N}_0\},
\]

From this it follows that there is precisely one P-position of Wythoff Nim in each row and each column of the board (see also \([Le]\)).

The purpose of this paper is to explore the P-positions of Maharaja Nim and some related games. Clearly \((0,0)\) is P. Another trivial observation is that, since the rules of game are symmetric, if \((x,y)\) is P then \((y,x)\) is P. It
is also easy to see that there is at most one P-position in each row and each
column (corresponding to the Rook-type moves). But, in fact, the same
assertion as for Wythoff Nim holds:

**Proposition 1.1.** There is precisely one P-position of Maharaja Nim in
each row and each column of \(N_0 \times N_0\).

**Proof.** Since all Nim-type moves are allowed in Maharaja Nim, there is at
most one P-position in each row and column of \(N_0 \times N_0\). This implies that
there are at most \(k\) P-positions strictly to the left of the \(k^{th}\) column (row).
Each such P-position is an option for at most three N-positions in column
(row) \(k\). This implies that there is a least position in column (row) \(k\) which
has only N-positions as options. By definition this position is P and so, since
\(k\) is an arbitrary index, the result follows.

\[ \blacksquare \]

Figure 2: The initial P-positions of Wythoff Nim and Maharaja Nim respec-
tively.

Another claim holds for both Wythoff Nim and Maharaja Nim. There is
*at most* one P-position on each *diagonal* of the form

\[ \{\{x, x + C\} \mid x \in N_0\}, C \in N_0, \quad (2) \]

(corresponding to the Bishop-type moves). But (1) readily gives that, for
Wythoff Nim, there is *precisely* one P-position on each such diagonal. Even
more is true: If

\[ P_W = \{(a_i, b_i), (b_i, a_i)\}, \quad (3) \]
with \((a_i)\) increasing and for all \(i\), \(a_i \leq b_i\), then for all \(n\),
\[
\{0, 1, \ldots, n\} = \{b_i - a_i \mid i \in \{0, 1, \ldots, n\}\}.
\] (4)

As we will see in Section 2, a somewhat weaker, but crucial, property holds also for Maharaja Nim, but let us now state our main result (see also Figure 3). We let \(O(1)\) denote bounded functions on \(\mathbb{N}_0\).

**Theorem 1.2** (Main Theorem). Each P-position of Maharaja Nim lies on one of the ‘bands’ \(\phi n + O(1)\) or \(\phi^{-1} n + O(1)\), that is, if \((x, y) \in \mathcal{P}_M\), with \(y \geq x\), then \(y - \phi x\) is \(O(1)\).

![Figure 3: To the left, the P-positions of Wythoff Nim lie ‘on’ the lines \(\phi x\) and \(\phi^{-1} x, \ x \geq 0\). The figure to the right illustrates a main result of this paper, that the P-positions of Maharaja Nim are bounded below and above by the ‘bands’, \(\phi x + O(1)\) and \(\phi^{-1} x + O(1)\) ](image)

We give the proof of this result in Section 2. It is quite satisfactory in one sense, but for the two gamesters trying to figure out how to quickly find safe positions, it does not quite suffice. The following question is left open.

**Question 1.** Does Maharaja Nim’s decision problem, to determine whether a given position \((x, y)\), with input length \(\log(xy)\), is \(P\), have polynomial time complexity in \(\log(xy)\) ?

In Section 5 we provide an affirmative answer of this question for a close relative of Maharaja Nim, namely the extension of Wythoff Nim where moves of type (2, 3) and (3, 2) are adjoined (but not (1, 2) or (2, 1)).
result builds upon an analog result, of ‘approximately linear’ P-positions, as that for Maharaja Nim in Theorem \[1.2\]. See also \[FP\], which was the inspiration for some results in this paper, although its methods do not seem to encompass the complexity of Maharaja Nim.

1.1 Complementary sequences and a central lemma

We say that two sequences of positive integers are \textit{complementary} if each positive integer is contained in precisely one of these sequences. In our setting this corresponds to Proposition \[1.1\] together with the claim before \[2\]. In \[FP\] the authors proved the following result.

\textbf{Proposition 1.3} (Fraenkel, Peled). \textit{Suppose \(x\) and \(y\) are complementary and increasing sequences of positive integers. Suppose further that there is a positive real constant, \(\delta\), such that, for all \(n\),}

\[ y_n - x_n = \delta n + O(1). \]  

\textit{Then there are constants, \(1 < \alpha < 2 < \beta\), such that, for all \(n\),}

\[ x_n - \alpha n = O(1) \]  

\textit{and}

\[ y_n - \beta n = O(1). \]

As they have remarked (see also \[HL\]), by simple density estimates one may decide the constants \(\alpha\) and \(\beta\) as functions of \(\delta\). Namely, notice that \[5\] and \[6\] together imply

\[ \beta = \alpha + \delta \]  

and, by complementarity, we must have

\[ \frac{1}{\alpha} + \frac{1}{\beta} = 1. \]  

(Thus \(\alpha\) and \(\beta\) are algebraic numbers if and only if \(\delta\) is.) By this we get the relation

\[ \delta(1 - \alpha) + \alpha = (\alpha - 1)\alpha, \]  

which will turn out to be useful in what will come next, namely we have found a short proof of an extension of their theorem—an extension which is easier to adapt to the circumstances of Maharaja Nim. Let us explain.
If we denote
\[ \mathcal{P}_M = \{(a_n, b_n), (b_n, a_n) \mid n \in \mathbb{N}_0\}, \tag{11} \]
with \((a_n)\) increasing and for all \(n, b_n \geq a_n\), then, for all \(n, b_n\) is uniquely defined by the rules of \(M\). At this point, one might want to observe that, if the \(b\)-sequence would have been increasing (by Figure 2 it is not) then Theorem 1.2 would follow from Proposition 1.3 if one could only establish the following claim: \(b_n - a_n - n\) is \(O(1)\). Namely in (10) \(\delta = 1\) gives \(\alpha = \phi\) in Proposition 1.3. Now, interestingly enough, it turns out that Proposition 1.3 holds without the condition that the \(y\)-sequence is increasing, namely (5) together with an increasing \(x\)-sequence suffices.

**Lemma 1.4** (Central Lemma). *Suppose \(x\) and \(y\) are complementary sequences of positive integers with \(x\) increasing. Suppose further that there is a positive real constant, \(\delta\), such that, for all \(n\),
\[ y_n - x_n = \delta n + O(1). \tag{12} \]
Then there are constants, \(1 < \alpha < 2 < \beta\), such that, for all \(n\),
\[ x_n - \alpha n = O(1) \tag{13} \]
and
\[ y_n - \beta n = O(1) \tag{14} \]

**Proof.** We begin by demonstrating that, for all \(n \in \mathbb{N}\),
\[ x_{n+1} = x_n + O(1), \tag{15} \]
and
\[ y_{n+1} = y_n + O(1). \tag{16} \]
By (12), for all \(k, n \in \mathbb{N}\) we have that
\[ y_{n+k} - y_n = x_{n+k} + \delta (n+k) - x_n - \delta n + O(1), \]
\[ = x_{n+k} - x_n + \delta k + O(1). \tag{17} \]
Since for all \(k, n \in \mathbb{N}\), \(x_{n+k} - x_n \geq k\) and \(\delta > 0\) this means that, for all \(k, n \in \mathbb{N}\),
\[ y_{n+k} \geq y_n - C, \tag{18} \]
where $C$ is some universal positive constant (which may depend on $\delta$). But, with $C$ as in (18), we can find another universal constant $\kappa = \kappa(C) \in \mathbb{N}$ such that, for all $n$,

$$y_{n+\kappa} - y_n \geq \kappa + 2C + 1.$$  \hspace{1cm} (19)

This follows since, in (17), for any $C$, we can find $k = k(C)$ such that, for all $n$, $\delta k + O(1) > 2C$. Any such $k$ suffices as our $\kappa$. On the one hand there can be at most $\kappa - 1$ numbers from the $y$-sequence strictly between $y_n$ and $y_{n+\kappa}$ (with indexes strictly in-between $n$ and $n + \kappa$). On the other hand the inequality (18) gives that there can be at most $C$ numbers from the $y$-sequence with index greater than $n + \kappa$ but less than $y_{n+\kappa}$. It also gives that there can be at most $C$ numbers with index less than $n$ but greater than $y_n$. Therefore, by complementarity and (19), there has to be a number from the $x$-sequence in every interval of length $\kappa + 2C + 1$. Thus the jumps in the $x$-sequence are bounded, which is (15). But then (16) follows from (12) and (15) since

$$y_{n+1} - y_n = x_{n+1} + \delta(n + 1) - x_n - \delta n + O(1)$$
$$= x_{n+1} + \delta - x_n + O(1)$$
$$= O(1).$$

By (19) we may define $m$ as a function of $n$ with

$$x_n = y_m + O(1).$$  \hspace{1cm} (20)

(For example, one can take $m = m(n)$ the least number such that $x_n < y_m$. Then $y_m - x_n$ has to be bounded for otherwise $y_m - y_{m-1}$ is not bounded.) This has two consequences, of which the first one is

$$x_n = n + m + O(1).$$  \hspace{1cm} (21)

This follows since the numbers $1, 2, \ldots, x_n$ are partitioned in $n$ numbers from the $x$-sequence, and the rest, by complementarity, $m + O(1)$ numbers from the $y$-sequence.

The second consequence of (20) is that, by using (12),

$$x_n = x_m + \delta m + O(1).$$  \hspace{1cm} (22)

If $\lim x_n/n$ and $\lim y_n/n$ exist then, clearly they must satisfy (8) and (9) with $\delta$ as in the lemma. Thus, using this definition of $\alpha = \alpha(\delta)$, for all $n$, denote

$$\Delta_n := x_n - \alpha n.$$
We want to use (21) and (22) to express $\Delta_n$ in terms of $\Delta_m$.

Equation (22) expresses $x_n$ in terms of $x_m$ and $m$. Therefore, we wish to combine (21) and (22) to express $n$ in terms of $x_m$ and $m$, that is, we wish to eliminate $x_n$ from (21). If we plug in the expression (22) for $x_n$ in (21) and solve for $n$ we get

$$n = x_m + (\delta - 1)m + O(1).$$

(23)

Combining (22) and (23) gives

$$\Delta_n = x_m + \delta m - \alpha(x_m + (\delta - 1)m) + O(1)$$

$$= (1 - \alpha)x_m + (\delta(1 - \alpha) + \alpha)m + O(1)$$

$$= (1 - \alpha)\Delta_m + O(1),$$

(24)

where the last equality is by (10).

Notice that, by (22), for sufficiently large $n$ we have that $m < n$. Hence we may use strong induction and by (24) conclude that $\Delta_n$ is $O(1)$ which is (13). Then (14) follows from (12). \qed

2 Perfect sectors, a dictionary and the proof of Theorem 1.2

This whole section is devoted to the proof of Theorem 1.2. We begin by proving that there is precisely one P-position of Maharaja Nim on each diagonal of the form in (2). Then we explain how the proof of this result leads to the second part of the theorem, the bounding of the P-positions within the desired ‘bands’ (Figure 3).

A position, say $(x, y)$, is an upper position if it is strictly above the main diagonal, that is if $y > x$. Otherwise it is lower.

We call a $(C, X)$-perfect sector, or simply a perfect sector, all positions strictly above some diagonal of the form in (2) and strictly to the right of column $X$. Suppose that we have computed all P-positions in the columns $1, 2, \ldots, a_{n-1}$ and that, when we erase each upper position from which a player can move to an upper P-position (Figures 4 and 5), then the remaining upper positions strictly to the right of $a_{n-1}$ constitute an $(n-1, a_{n-1})$-perfect sector (Figure 5). Then we say that $a_{n-1}$ is perfect and, in fact, it is easy to see that also property (4) holds for all such $n$. On the other hand, we will see that the converse statement holds if and only if for any such $n$,

$$b_n - a_n = n,$$

(25)
given that the lower P-positions do not interfere. It is crucial to our approach that the first implication can be made stronger to also include (25).

**Lemma 2.1.** Let $n \in \mathbb{N}$ be sufficiently large so that Knight type moves from lower P-positions do not affect the coordinates of upper P-positions and define $(a_i)$ and $(b_i)$ as in (11). Suppose also that

$$\{0, 1, \ldots, n - 1\} = \{b_i - a_i \mid 0 \leq i < n\} \quad (26)$$

holds. Then (25) must hold if and only if $a_{n-1}$ is perfect.

**Proof.** Suppose that (25) does not hold. Then clearly $b_n - a_n > n$. This must be due to a Knight type move from an upper P-position from $(a_{n-1}, b_{n-1})$, that is to position $(a_{n-1} + 1, b_{n-1} + 2)$. Hence $a_{n-1}$ is not perfect. For the other direction, whenever there is no $i < n$ such that $b_i = a_{n-1} + 1$, so that $a_n = a_{n-1} + 1$ excludes a Knight type move as in the previous paragraph and hence assures a perfect sector. $\square$

### 2.1 Constructing Maharaja Nim’s bit-string

We study a bit-string, a sequence of ‘0’s and ‘1’s, where the $i^{th}$ bit equals ‘0’ if and only if there is an upper P-position of Maharaja Nim in column $i$. By Proposition 1.1, if there is no upper P-position in column $i$, there is a lower ditto (the $i^{th}$ bit equals 1).

Suppose that $a_x = n$ is perfect. Then, by symmetry we know some lower P-positions in columns to the right of $n$. The next step is to erase each column in this perfect sector which has a lower P-position, a ‘1’ in the bit-string (see Figure 6) and to, recursively in the non-erased part of the perfect sector, compute new upper P-positions. We do this until we reach the next perfect sector (for the moment assume that this will happen) at say column $n + m$, $m > 0$. Thus, using this notation, we may define a word of length $m$, containing the information of whether the P-position in column $i \in \{n, n + 1, \ldots, n + m - 1\}$ is below or above the main diagonal.

At this point we adjoin this word together with its unique translate to Maharaja Nim’s dictionary. The translate is obtained accordingly: For each P-position in the columns $n$ to $n + m - 1$ define the $i^{th}$ bit in the translation as a ‘1’ if and only if row $k + i$ has an upper P-position and where $k$ is the largest row index strictly below the perfect sector. See also Figure 7 and the next section for examples. Then the translate has length $m + l$, where $l$ denotes the number of ‘0’s in the word.
We then concatenate the translate at the end of the existing bit-string. In this way, provided a next perfect sector will be detected, the bit-string will always grow faster than we read from it. However, there is no immediate guarantee that we will be able to repeat the procedure—that the next word exists—or for that matter that the size of the dictionary will be finite, so that the process may be described by a finite system of words and translates. But, in the coming, we aim to prove that, in fact, the next perfect sector will always (in the sense outlined above) be detected within a ‘period’ of at most 7 P-positions, that is ‘0’ s in the bit-string. As we will see, a complete dictionary needs only (between 9 and) 14 translations.

Let us describe a bit more in detail how the first part of Maharaja’s bit-string is constructed.

### 2.2 A detailed example

Initially there is some interference which does not allow a recursive definition of words and translates, see Figure 2. The first perfect sector beyond the origin is attained when the four first P-positions strictly above the main diagonal has been computed. This happens to the right of column 8. To the right of column 12 a new perfect sector is detected. Thus the first word (left hand side entry) in the dictionary will be ‘00100’, corresponding to the P-positions (8, 13), (9, 16), (10, 7), (11, 19) and (12, 18). (Here there is no interference since the y-coordinate of the first P-position is greater than the x-coordinate of the last P-position.) Let us verify that this word translates...
to ‘100101100’. Notice that the first ‘1’-bit means that the P-position (8, 13) is to the left of the main diagonal—by symmetry this corresponds to a lower P-position in column 13. The second bit is ‘0’. This means that the next upper P-position is in column 14. Then, by rules of game, it has to be at least in row 16, which indeed will be attained, so that the next P-position will be (9, 16). By the rules of game, the rows 14 and 15 cannot have P-positions to the left of the main diagonal, so that a prefix is ‘1001’. Continuing up to the last P-position of this translate, (12, 18), extends the prefix to ‘10010111’. The next upper P-position will be in at least row 22 since the least unused diagonal is $22 - 13 = 9$. After this a new perfect sector will start. This gives the two last ‘0’s in the translate, ‘100101100’, which may now be concatenated at the end of the first part of the bit-string, ‘00100’, so that the new bit-string becomes ‘001001001011000010010011000’. Again, concatenating this translate at the end of the existing string gives ‘001001001011000010010011000’. As we continue to read from the ‘0’ in the seventh position it turns out that, this time, we need to read ‘0010110’ (Figure 7 to the right) to obtain a new perfect sector and also that this word translates to ‘10010011000’. Again, concatenating this translate at the end of the existing string gives ‘001001001011000010010011000’, and so on.
Figure 6: (Step 2) Each column in the perfect sector which corresponds to a lower P-position (a ‘1’ in the bit-string) has been erased.

Figure 7: To the left, the unique (upper) P-positions of Maharaja Nim in the columns 8 to 12 are computed. The corresponding translation is 00100 → 100101100. To the right are the P-positions in the columns 14 to 20 together with the translate 0010110 → 10010011000. (Here we have omitted column 13 with its translation 1 → 0.) See also Figure 2 and Section 2.2.
2.3 Maharaja Nim’s dictionary

The dictionary of M is

\[
1 \rightarrow 0 \\
01 \rightarrow 100 \\
00100 \rightarrow 100101100 \\
00110 \rightarrow 100101000 \\
000100 \rightarrow 10010110100 \\
001110 \rightarrow 100100100 \\
0010110 \rightarrow 10010011000 \\
0000100 \rightarrow 1001001111000100 \\
00000010 \rightarrow 100101101100100 .
\]

By computer simulations we have verified that each one of the words (27) to (35) does appear in Maharaja Nim’s bit-string. We have included the code the Appendix, Section A2. By our method of proof, we have found no way to exclude the latter five, but a guess is that they do not appear. At least they do not appear among the first 20000 bits of the bit-string. The following result gives the first part of the theorem.

**Lemma 2.2** (Completeness Lemma). *When we read from Maharaja Nim’s bit-string each prefix is contained in our extended dictionary of (left hand side) words of Maharaja Nim.*

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Proof. Let us present a list in lexicographic order of all words in our extended dictionary together with the words we need to exclude:

0000000 → 10010110110100
00000010 → 100101101100100
0000011 'to exclude' (a)
0000100 → 100101100111000
0000101 'to exclude' (b)
000011 'to exclude (c)'
0001000 → 100100111100100
00010010 → 1001001111000100
00010011 'to exclude' (d)
000101 'to exclude' (e)
00011 'to exclude' (f)
00100 → 100101100
0010100 → 100100110100
0010101 'to exclude' (i)
0010110 → 10010011000
0010111 'to exclude' (j)
00110 → 10010100
001110 → 100100100
0011110 → 1001000100
0011111 'to exclude' (k)
01 → 100
1 → 0

This list is ‘complete’ in the sense that any bit-string has precisely one of the words on the left hand side as a prefix. This motivates why it suffices to exclude the words ‘to exclude’. For example (a) needs to be excluded since the only word in our list beginning with ‘0000001’ ends with a ‘0’. Neither can we translate words beginning with ‘000001’s and ending with ‘01’ or ‘1’. This motivates why we need to exclude (b) and (c). All left hand side words
in our dictionary beginning with 4 ‘0’s continues with 100, which motivates that (e) and (f) need to be excluded, and so on. We move on to verify that the strings (a) to (k) are not contained in the bit-string.

No translate can contain more than three consecutive ‘0’s. To get a longer string one has to finish off one translate and start a new. The only translate which starts with ‘0’ is ‘0’. Thus, when a sequence of four or more ‘0’s is interrupted it means that a new translate has begun. But all translates that begin with a ‘1’ begins with ‘100’. Thus, a sequence of four or more ‘0’s cannot be followed by ‘11’ or ‘101’. This gives that the exclusion of the words (a), (b), (c), (e) and (f) is correct.

Clearly, the string ‘100’ in (d) has to be the prefix of some translate. Since the next two bits are ‘11’, by the dictionary, this translate has to be ‘100’. But then the next translate has the prefix ‘11’, which is impossible.

For the exclusion of (g) and (h) notice that the only strings of three consecutive ‘0’s that exist within a translate is either at the end or is followed by the string ‘100’. Therefore, a string of three ‘0’s cannot be followed by ‘11’ or ‘101’.

For (i), notice that the sub-string ‘101010’ is not contained in any translate. If it were, it needed to be either at the beginning of a translate, which is impossible (since all of them except ‘0’ begin with ‘100’) or be split between two. The latter is impossible since all translates except ‘0’ ends with ‘00’. In analogy to this, also (j) must be excluded and similarly for (k) since no translate contains 5 consecutive ‘1’s and all translates ends in a ‘0’, but starts with either ‘0’ or ‘10’.

Since the left hand side words have at most 7 ‘0’s we adjoin at most 6 P-positions in a sequence with \( b_n - a_n \) distinct from \( n \). Namely, by Lemma 2.1 when we start a new perfect sector we know that the next P-position will satisfy \( b_n - a_n = n \). The number of bits in a translate is bounded (by 16) so that \( b_n \) can never deviate more than a bounded number of positions from \( a_n + n \). Hence, by Proposition 1.1 the conditions of Lemma 1.4 are satisfied with the \( a \)-sequence as \( x \), the \( b \)-sequence as \( y \) and \( \delta = 1 \). Thus, \( b_n - a_n - n \) is \( O(n) \) (as discussed in the paragraph before Lemma 1.4) this concludes the proof of Theorem 1.2. By inspecting the dictionary one can see that, in fact, for all \( n, -4 \leq b_n - a_n - n \leq 3 \).
3 Dictionary processes and undecidability

Let us briefly discuss a problem related to the method used in this paper. Given a dictionary (of binary words and translations) and a starting string, will the translation process of the bit-string ‘terminate’ or not?

More precisely, let us assume that we have a finite list of words $A = \{A_1, A_2, \ldots, A_m\}$ with translates $B_1, B_2, \ldots, B_m$ respectively, each word being a string of ‘0’s and ‘1’$s$, and where we assume that none of the words in $A$ is a prefix of another. (The latter is a convenient, but not necessary, condition as we will explore further in Section 5.) Namely, as the read head reads from the bit-string, a natural generalization of a prefix free dictionary is to translate precisely the longest word containing a certain prefix.

Take any string $S$ as a starting string (for example $A_1$ but it could be an arbitrary string, not necessarily in the list). A ‘read head’ starts to read $S$ from left to the right and as soon as it finds a string $A_i$ in $A$ it stops, sends a signal to a printer at the other end which concatenates the translation $B_i$ at the end of $S$. Then the read head continues to read from where it ended until it finds the next word in $A$, its translation being concatenated at the end, and so on.

If the read head gets to the end of the string without finding a word in the list $A$, the process stops with the current string as ‘output’. Otherwise, the process continues and gives as output an infinite string.

It follows from E. Post’s tag productions that it is algorithmically undecidable whether our ‘dictionary processes’ stop or not. We give a proof in the Appendix, part B.

4 Approximate linearity, converging dictionaries and polynomial time complexity

There are infinitely many relatives to Maharaja Nim of the form ‘adjoin a finite set of moves to Wythoff Nim’. It is easy to see that Proposition 1.1 and 2 hold also for these type of games. For any given such generalization, is it possible to determine the greatest departure from $n$ for $b_n - a_n$? For example see the games in Figure 8 and 9. Even simpler, is it decidable, whether there is a P-position above some straight line? More precisely:

**Question 2.** Given the moves of Wythoff Nim together with some finite list of moves, that is ordered pairs of integers (in Maharaja Nim the list is $\{(1,2), (2,1)\}$) and a linear inequality in two variables $x$ and $y$, is it decidable whether there is a P-position in the game which satisfies the inequality?
On the one hand it is not even clear if a 'generalized Maharaja Nim' has a finite dictionary in the sense of Section 2. On the other hand the solution of a similar game may or may not depend on the possible outcome of a dictionary process as in Section 2. In fact, in Section 5 we prove that a related dictionary process is successful in giving a polynomial time algorithm for the decision problem of whether a certain position is P. Therefore, let us look into some questions regarding some close relatives of Maharaja Nim.

Figure 8: The initial P-positions (the coordinates are less than 100) of $(k,l)M$ for $(k,l) = (3,5), (4,6), (4,7), (5,8), (6,10)$ and $(7,11)$ respectively. In support of Conjecture 4.1, the ratios of the respective coordinates seem to closely approximate $\phi$ or $1/\phi$. (For $(2,3)M$, see Section 5.)

Figure 9: The initial P-positions (the coordinates are less than 1500) of four extensions of Maharaja Nim where the adjoined moves are $\{(t,2t),(2t,t)\}$ where $t \in \{1,2,\ldots,10\}, \{1,2,\ldots,50\}, \{1,2,\ldots,100\}$ and $\mathbb{N}$ respectively. That is the three first games have a finite number of moves adjoined to Wythoff Nim but the last one has infinitely many. Notice the seemingly emerging 'bounded split' of the (upper) P-positions in the middle two figures, the ratio of the coordinates still seem to be within a bounded distance of $\phi$, but in the last figure, where an infinite number of moves are adjoined the convergence to $\phi$ is destroyed, a fact which is proved in [La], and an 'unbounded split' (as in the rightmost figure) is established, which was recently proved in [La].

To begin with, one might want to pay special attention to the family of extensions of Wythoff Nim, where the adjoined moves are of the form $(k,l)$
and \((l, k), k, l \in \mathbb{N}, k < l\). We call a game in this family \((k, l)\)-Maharaja Nim, \((k, l)\)-M. (Another problem is indicated in Figure 9 and its discussion.) The P-positions are distinct from those of Wythoff Nim, see [La], if and only if \((k, l)\) is a so-called ‘Wythoff pair’ or a ‘dual Wythoff pair’, that is of the form \((\lfloor \phi n \rfloor, \lfloor \phi^2 n \rfloor)\) or \((\lceil \phi n \rceil, \lceil \phi^2 n \rceil)\), \(n \in \mathbb{N}\). Thus, in Maharaja Nim we take the first Wythoff pair \((1, 2) = (\lfloor \phi \rfloor, \lfloor \phi^2 \rfloor)\), whereas in the next section we study \((2, 3)\)-Maharaja Nim, that is we let \((k, l)\) take the values of the first dual Wythoff pair \((2, 3) = (\lceil \phi \rceil, \lceil \phi^2 \rceil)\).

Figure 10: A ‘telescope’ with ‘focus’ \(O(1)\) and ‘reflectors’ along the lines \(\phi n\) and \(n/\phi\) attempts to determine the outcome (P or N) of some position, \((x, y)\) at the top of the picture. As we demonstrate in Section 5 the method is successful for \((2, 3)\)-Maharaja Nim. (It gives the correct value for all extensions of Wythoff Nim with a finite non-terminating converging dictionary). The focus is kept sufficiently wide (a constant) to provide correct translations in each step. The number of steps is linear in \(\log(xy)\).

**Conjecture 4.1.** Let \(k, l \in \mathbb{N}, k < l\). Then each upper P-position \((x, y)\) of \((k, l)\)-M satisfies \(y = \phi x + O(1)\).

Does this conjecture hold for any game of the form ‘a finite number of moves adjoined to Wythoff Nim’?

Suppose that a given game \((k, l)\)M has a finite (non-terminating) dictionary (as for Maharaja Nim) thus, hypothetically, providing an affirmative answer to Conjecture 4.1. Suppose further that the dictionary converges, that is, given an arbitrary string-position, we can, within the distance of a
bounded number of bits, precisely determine when a new word starts. For this particular game, let us sketch a polynomial time algorithm which determines whether a given position \((x, y)\) (with \(\frac{y}{x}\) approximately \(\phi\)) is \(P\), see also Figure 10. Suppose that we have computed an initial (sufficiently large) sequence of the bit-string. We sketch the steps of the decision problem of \((k, l)M\) as follows:

- Back track \((x, y)\) via orthogonal reflections along the lines \(\phi n\) and \(n/\phi\). Here we do not need to use our dictionary, only to put marks at the precise locations of our reflecting points on the lines \(\phi n\) and \(n/\phi\). That is, we get a finite sequence of pairs of the form

\[
(x, \phi x), (x, x/\phi), (x/\phi^2, x/\phi), \ldots, (x/\phi^p, x/\phi^{p-1}),
\]

some \(p \in \mathbb{N}\).

- When we have back tracked as far as to our initial bit-string, the ‘forward’ translations can begin. Suppose that we know that the dictionary converges within \(q\) (which is supposed to be much less than \(x\) and \(y\) bits and that the maximal length of a translate is \(c \leq q\) bits.

- Then it suffices to translate \(< \phi q\) bits in each step. If the first left hand side word begins with, say the bit \(\lfloor x/\phi^p \rfloor - \phi q \leq b_1 \leq \lfloor x/\phi^p \rfloor - q\) we may translate it and be assured to find another left hand side word beginning at a bit \(\lfloor x/\phi^{p-1} \rfloor - \phi q \leq b_2 \leq \lfloor x/\phi^{p-1} \rfloor - q\) and so on. For the final computation of the value of \((x, y)\) it suffices to, given the left hand side word which contains \(x\), compute the \(P\)-positions in some area of size less than \(c \times c\) squares. (Alternatively, given a short dictionary, the list of \(P\)-positions corresponding to each word may be computed beforehand.)

- This procedure takes \(p\) steps where \(\phi^p\) is proportional to \(x + y\).

5 The close relative \((2, 3)\)-Maharaja Nim has polynomial time complexity

The game \((2, 3)\)-Maharaja Nim, \((2, 3)\)-M, is as Maharaja Nim except that, for this game, the Knight’s jumps are of the form \((2, 3)\) and \((3, 2)\) (and not \((1, 2)\) and \((2, 1)\)). In this section we let \((a_1, b_1), (a_2, b_2), \ldots\) denote the upper \(P\)-positions of \((2, 3)\)-M, where \((a_i)\) is increasing. As we have remarked in
Figure 11: The move options from a given position of (2, 3)-Maharaja Nim.

Figure 12: The initial P-positions of (2, 3)-Maharaja Nim together with its initial bit-string.

Section 4 analogs of Proposition 1.1 and (2) hold for (2, 3)-M. Hence $(a_i)$ and $(b_i)$ are complementary.

Since Lemma 2.1 does not hold for (2, 3)-M, for the analysis of this game we use a relaxation of the approach in Section 2. As we saw at the end of that section, the crucial property for approximate linearity to hold is
that the dictionary promised a sufficiently frequent reappearance of property (25). Hence, for a new left hand side word to be translated it is not necessary that we require a perfect sector as defined for (1, 2)-Maharaja to be detected. It turns out that the condition (26) in Lemma 2.1 suffices for our purposes. This almost corresponds to a perfect sector, by which we mean that at most a finite number of positions are deleted from a perfect sector. That is, the requirement is still that (25) and (26) hold simultaneously.

Suppose that the initial P-positions up to column $a_n$ has been coded in a unique (2, 3)-M bit-string, where as before, a ‘1’ (‘0’) in the $i^{th}$ position denotes a lower (upper) P-position in column $i$. That is the read head is about to read the $a_n^{th}$ bit in the string. As in Section 2 by symmetry of P-positions, a finite number of bits follow to the right of the read head’s current position. Then a (new) left hand side word $\omega \neq 1$ (the word ‘1’ is translated to ‘0’) is included to the dictionary if and only if the following two criteria are satisfied. Each one of the numbers $0, 1, \ldots, n-1$ is represented as the difference $b_i - a_i$ of the coordinates of precisely one of the first $n-1$ upper P-positions and $b_n - a_n = n$.

As usual, the translation of $\omega$ is computed and concatenated at the end of the bit-string. The next left hand side word begins by the $a_n^{th}$ column.

5.1 (2, 3)-Maharaja Nim’s dictionary process

Given a finite binary dictionary, we define unique non-prefix free translations by the following rule. Suppose that the read head has finished one translation in the (infinite) binary string $x$ and starts reading at position $n$. Suppose further that it detects the left hand side entries $\omega_1, \ldots, \omega_k$ of the dictionary, reading from position $n$ and onwards, where $\omega_i$ is a prefix of $\omega_{i+1}$ for all $i < k$, so that $\omega_k$ is not the prefix of any other entry. Then it accepts the translation of $\omega_k$ and it is unique if it exists.

Let us illustrate this definition by defining the following (very short) non-prefix free dictionary of (2, 3)-M:

\[
\begin{align*}
0 & \rightarrow 10 \\
1 & \rightarrow 0 \\
01000 & \rightarrow 10001100 \\
01010 & \rightarrow 10001100.
\end{align*}
\]

Since the bit ‘0’ is a prefix of the words ‘01000’ and ‘01010’ we need some external rule to decide which translation to use in the construction of
the bit-string. Suppose that the next bit detected by the read head is ‘0’. Then the translation is as in (41), except if the next four bits are either ‘1000’ or ‘1010’. For these cases the translations are as in (43) and (44) respectively. Notice that, by these translation rules, by (41) and (42), any bit-string has a (longer) translation, and therefore the construction cannot terminate. Before proving that this dictionary is correct, let us provide some initial translations.

Column-wise, the first non-terminal P-positions of (2,3)$M$ are (1,2), (2,1), (3,6), (4,8), (5,7), (6,3), (7,5) and (8,4). These P-positions correspond to the bit-string ‘01000111’ on the $x$-axis and ‘0100011100000’ on the $y$-axis, see Figure 12. That is, we can assume that the first word to be translated starts in position (9,14), which corresponds to the first diagonal (of the form (2)) in the 9$th$ column. It is clear that the initial interference between rows and columns has ended here so, to begin with the read head’s position is ‘0100011100000’. Thus, the first four translations are of the type (41) which produce the bit-string ‘010001110000010101010001100’. Then a type (44) translation follows which produces ‘0100011100000101010001100101010001100’, and so on.

It is easy to see that, given a perfect sector to the right of the column of the read head’s position, each translation in (2,3)$M$’s dictionary is correct. However, since there is no a priori guarantee that a new translation starts at a perfect sector, we need to exclude certain combinations of translations, thus preventing any (2,3)-type move to short-circuit two P-positions. The translations (41) and (43) could potentially interfere with a succeeding translation but (42) and (44) cannot. Precisely, if the word ‘0’ were followed by a ‘1’ and then any of the words beginning with ‘0’, or if the word ‘01000’ were followed by a ‘0’, then the translation rules would be wrong, because of a (2,3)-type “short-circuit” of P-positions. These are all cases that we need to exclude. Let us begin to rule out the latter case.

Claim 1: If the left hand side word ‘01000’ is detected by the read head, then it is succeeded by the left hand side word ‘1’.

Suppose, on the contrary, that the read head reads the left hand side word ‘01000’ followed by a ‘0’. This string, ‘010000’, which we say is part of our original string, must have been translated from the left hand side words ‘x’, ‘0’, ‘1’, ‘1’, ‘1’ (in this order and where $x$ is the left hand side word in either (41), (43) or (44)). But the string ‘00111’ only appears as a translation in (43). Further, the string ‘01000111’ is forced since ‘11000111’ cannot appear, but it cannot be that the read head detected the first five bits ‘01000’ as the
Claim 2: Any sequence of left hand side words beginning with ‘0’, ‘1’ and then some pattern beginning with a ‘0’ is impossible.

We are here concerned with that the read head detects any sequence of left hand side words beginning with ‘0’, ‘1’ and then some sequence ‘0xy’, where x and y represent two bits. It is immediate by the translation rules that we may exclude the cases where xy represents ‘00’ or ‘10’. Namely, for these two cases, by the ‘prefix-rule’ of choosing the longest left hand side word in the dictionary, we would rather have used one of the translations in (43) or (44). Also, the case where xy is ‘11’ may be excluded since the string ‘01011’ does not appear in any combination of the right hand side translates. Thus, it only remains to analyze the case where the two bits are ‘01’. That is, we want to exclude the pattern ‘01001’. By looking at the translations it is obvious that the string ‘1001’ must have been translated from the left hand side words ‘0’, ‘1’ and then a word beginning with a ‘0’. This means that precisely the pattern which we want to exclude has appeared in a previous translation (and thereby also short-circuiting two P-positions in columns strictly to the left of the current position). Thus (using Figure 12 as a base case) strong induction resolves this case.
5.2 Polynomiality

We have proved that the dictionary in (41) to (44) is correct and thereby also that the P-positions of (2,3)-Maharaja Nim lie within a bounded distance of either the ‘line’ $\phi n$ or $\phi^{-1} n$. Next, we will demonstrate that this dictionary gives a polynomial strategy, as outlined in Section 4. For this, it suffices to prove that, given an arbitrary position in the infinite bit-string, by a search within a bounded number of bits we can determine which one of the four given translations is correct.

If the read head reads the pattern ‘11’ then, by the left hand side words in the dictionary and in particular (42), we can conclude that a new word starts by the first ‘1’. Hence we assume that no two consecutive ‘1’s are detected. By analyzing the translations in the dictionary one can see that at most five consecutive ‘0’s can appear. Therefore, we may assume that the read head reads the pattern ‘010’ within a bounded distance, which by previous arguments mean either ‘01000’ or ‘01010’. Both these strings are detected as words, unless the preceding pattern ends with ‘0100’, ‘01’ or ‘0101’. Hence one needs to investigate the following six ambiguous strings:

(a) 010001000,
(b) 0101000,
(c) 010101000,
(d) 010001010,
(e) 0101010,
(f) 010101010.

The pattern ‘10001000’ in (a) cannot have been translated from the string ‘011011’. This follows by viewing the possible combinations of right hand side translates. Hence, the combination of translations comes from first ‘0’, ‘1’ and ‘1’ and then ‘01000’ or ‘01010’. But these combinations are also impossible since they both enforce the impossible pattern “1101”. Hence (a) cannot appear.

The string in (b) must have been translated from ‘0’, ‘0’, ‘1’, ‘1’ which, by (43) and (44) and since all translates end with a ‘0’, implies that the three preceding bits must have been ‘010’. Hence, we can extend the pattern to be translated to ‘0100011’. It is given that the prefix ‘01000’ of this string
cannot be detected as a left hand side word. Therefore, the translation of ‘0100011’ must be ‘10010101000’ which has the prefix ‘1001’.  

(48)

But, by the left hand side words in the dictionary, any string containing (48) must converge between the two ‘0’s. Hence a new word must start as ‘01010’ followed by ‘1’, ‘0’, ‘0’,… Notice that (c) has this string as a suffix and hence it may also be included in the argument. Also, by (48) and by the argument in (a), in any attempt to disprove convergence (d) must be preceded by the pattern ‘01’, but then again, we may analyze (d) as (b).

We are left with the strings (e) and (f). Since a repetition of more than five consecutive patterns ‘01’ implies that more than five consecutive 0s has been translated, which is impossible, we may assume that the repetitions of ‘01’ in (f) has been preceded by either of the patterns ‘10’ or ‘00’ (‘11’ is already ruled out). Again, the first case leads to (48). Notice that (e) can also be included in this argument. For the second case, notice that any string beginning with ‘00001’ converges after the three first ‘0’s, that is a new word must begin with ‘01’, so it suffices to study the string ‘100101010’, which (since the pattern ‘11’ is excluded) has been treated already in (d).

We have proved that, given an arbitrary position in the bit-string, at most a bounded number of preceding bits need to be searched in order to find the correct translation. By Section 4 this convergence gives a polynomial time winning strategy of (2, 3)-Maharaja Nim.

Appendix

A Code

A1 The Maple code corresponding to Figure 2

The below code includes the P-positions of both Wythoff Nim and Maharaja Nim in one and the the same diagram.

restart: with(plots): with(plottools):

N:=50;

theLine1:=CURVES([[0.0,0.0], [evalf(N), evalf(N*(1+sqrt(5))/2)]]):
theLine2:=CURVES([[0.0,0.0], [evalf(N*(1+sqrt(5))/2), evalf(N)]]):

#Compute the P-positions of Wythoff Nim and store as a list of squares.
#0=Not yet computed, 1=P, 2=N.
for i from 0 to N do for j from 0 to N do A[i,j]:=0: od: od:
for i from 0 to N do for j from 0 to N do if A[i,j]=0 then A[i,j]:=1:
for k to N do A[i+k, j]:=2: A[i+k,j+k]:=2: A[i,j+k]:=2: od: fi: od: od:
rectListW:=[]: for i from 0 to N do for j from 0 to N do if A[i,j]=1
then rectListW:=[op(rectListW), [[i,j],[i,j+1],[i+1,j+1],[i+1,j]]]: fi: od: od:

#Draw the P-positions and the two lines with slopes the golden ratio:
display(polygonplot(rectListW, color=red), theLine1, theLine2, axes=none,
scaling=constrained, view=[0..N, 0..N]);

#Compute the P-positions of Maharaja Nim:
for i from 0 to N do for j from 0 to N do A[i,j]:=0: od: od:
for i from 0 to N do for j from 0 to N do if A[i,j]=0 then A[i,j]:=1:
A[i+1,j+2]:=2:
A[i+2,j+1]:=2:
for k to N do A[i+k, j]:=2: A[i+k,j+k]:=2: A[i,j+k]:=2: od: fi: od: od:
rectListM:=[]: for i from 0 to N do
for j from 0 to N do if A[i,j]=1 then rectListM:=[op(rectListM),
[[i+0.2,j+0.2],[i+0.2,j+0.8],[i+0.8,j+0.8],[i+0.8,j+0.2]]]: fi: od: od:
display(polygonplot(rectListM, color=blue), axes=none, scaling=constrained);
display(polygonplot(rectListM, color=blue), polygonplot(rectListW, color=red), theLine1, theLine2, axes=none,
scaling=constrained, view=[0..N, 0..N]);

A2 The Maple code corresponding to Maharaja Nim’s dictionary.
The following code explores whether the first 9 words in Maharaja Nim’s dictionary
suffices.

dictionary:={[[1], [0,1], [0,0,1,0,0], [0,0,1,0,1,0], [0,0,1,1,0],
[0,0,0,1,0,0], [0,0,0,0,1,0,0,1,0], [0,0,0,0,0,1,0,0], [0,0,1,1,1,0]};

translation:=table({[[1]=[0,0], [0,1]=[1,0], [0,0,1,0,0]=[1,0,0,1,0,1,0,0],
[0,0,1,0,1,0]=[1,0,0,1,0,1,0,0], [0,0,1,1,0]=[1,0,0,1,0,1,0,0],
[0,0,0,1,0,0]=[1,0,0,1,0,1,0,0],
[0,0,0,0,1,0,0]=[1,0,0,1,0,1,1,1,0,0,0,1,0,0],
[0,0,0,0,0,1,0,0]=[1,0,0,1,0,1,1,1,0,0,0,1,0,0],
[0,0,0,1,1,0]=[1,0,0,1,0,1,0,0,1,0,0,1,0,0,1,0,0]})

theString:=[0,0,1,0,0]: reader:=0:

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for times to 12000 do foundWord:=false:
for i to 9 do if not foundWord then theWord:=theString[reader+1..reader+i]:
if member(theWord, dictionary) then foundWord:=true:
theString:=[op(theString), op(translation[theWord])]:
reader:=reader+i: fi: fi: od: if not foundWord
then print(reader, theString[reader+1..reader+20]): fi:
if times mod 100 = 0 then print(times, nops(theString)): fi: od:

B An undecidable dictionary process

First we describe a known undecidable problem.
Suppose we have an alphabet consisting of one special symbol S which acts as
"space" or "stop" symbol, and a finite number of other symbols denoted A, B, C,...

We start from a "multiplication table" that describes an operation $x \ast y = z$,
where $x$ and $y$ are arbitrary symbols from the alphabet, and $z$ is a symbol other
than S.
For instance, the table may look like

\[
\begin{array}{cccc}
S & A & B & C \\
S & A & B & B \\
A & C & B & A \\
B & C & C & C \\
C & A & A & B \\
\end{array}
\]

Given such a table, we form a triangular pattern of symbols consisting of rows
starting and ending with S, and where the other symbols are obtained by "multi-
plying" the two symbols above it.
The table in the example gives

\[
S \\
S S \\
S A S \\
S B C S \\
S B C A S \\
S B C A C S \\
S B C A A A S \\
S B C A B B C S \\
\]

and so on.

Naively, one would like to understand how this pattern behaves by looking at
the table. The hope of general understanding of this kind is shattered by the fact
that the behavior of the pattern can simulate any given Turing machine. It follows
that a number of simple questions are generally algorithmically undecidable. We
mention a few such questions which are easily seen to be "equivalent".
If we are given two multiplication tables, do they produce the same pattern of symbols or not? We can examine the tables and find the entries where they differ. If only we can decide whether any of these entries is ever going to be used, we are done. If we fill in the entries where the tables differ with a "new" symbol Z, then in turn the problem becomes equivalent to deciding whether or not a certain symbol of the alphabet is ever going to occur in the pattern. This question in turn is equivalent to deciding whether a partial multiplication table (one with empty places) is "consistent" in the sense of determining a pattern.

In the example above, it is straightforward to see that the table entries B*S and S*C are never going to occur, but that all other entries do. Thus if we change the two entries B*S and S*C to something else, the pattern will still be the same, while if we change any other entry, the resulting pattern will be different. But in general, answering such questions may be as difficult as any mathematical problem. For instance, it is possible to "program" a multiplication table to look for counterexamples to the Goldbach conjecture, so that a certain symbol of the alphabet occurs if and only if there is a large even number which is not the sum of two primes.

Now consider a different type of process. Here we may without loss of generality assume the alphabet to be \{0, 1\}. Suppose we have a given starting string A, and a "dictionary" consisting of "translations" of the form \(x \rightarrow y\), where \(x\) and \(y\) are binary strings. The dictionary is a finite set of such translations \(x_1 \rightarrow y_1, \ldots, x_n \rightarrow y_n\), and to avoid ambiguity, it is required that no \(x_i\) is a prefix of any other. (Although generalized dictionaries such as those in Section 5 are also undecidable for the same reasons as explained here).

A "reader" starts at the left endpoint of the string A, and "reads" until it finds a word \(x_i\). Then a "writer", initially at the right endpoint of A, writes the translation \(y_i\) and concatenates it to the right of A. The reader then continues from where it was interrupted, and reads until it finds the next word etc.

The process may either go on forever, or get stuck by the reader reaching the right endpoint of the string without finding a word in the dictionary. The analogous questions may be asked about this process. Does it terminate or not? Is a certain word ever read? We will show that the "multiplication process" can be encoded as a "dictionary process", thereby showing that in general, the fundamental questions about the dictionary process are undecidable.

Suppose therefore that we are given a multiplication table. We will construct a dictionary that mimics the pattern of symbols arising from the given multiplication table.

First we introduce "metasymbols" that are binary strings representing the symbols of the alphabet. The starting string is going to be SS, and there is one dictionary entry for each entry of the multiplication table. If the table contains, for instance, A*B = C, then there is a translation rule AB \(\rightarrow\) CC.

The idea is that instead of writing a C, the writer writes CC. Eventually, the reader will read the second "half" of the previous symbol together with the first C, and then the second C together with the first "half" of the next symbol, while the
writer produces the corresponding products. To achieve this, the symbol S needs special treatment. We therefore also use the translation rules SS \rightarrow S(S+S)(S+S)S, and for each other symbol A, SA \rightarrow S(S+A)(S+A) and AS \rightarrow (A+S)(A+S)S.

In the example above, we get the dictionary

SS \rightarrow SAAS
SA \rightarrow SB
SB \rightarrow SBB
SC \rightarrow SBB
AS \rightarrow CCS
AA \rightarrow BB
AB \rightarrow AA
AC \rightarrow AA
BS \rightarrow CCS
BA \rightarrow CC
BB \rightarrow CC
BC \rightarrow CC
CS \rightarrow AAS
CA \rightarrow AA
CB \rightarrow BB
CC \rightarrow AA

Starting from the string SS, this produces (and here we have introduced some spacing just to increase readability)

SS SAAS SBBCCS SBBCCAAS SBBCCAACCS SBBCCAAAAAS ...

By leaving out certain rows of the dictionary, we may mimic a partial multiplication table. Therefore the question whether the dictionary process terminates is algorithmically undecidable.

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