GEOMETRIC PROPERTIES OF SOME BANACH ALGEBRAS RELATED TO THE FOURIER ALGEBRA ON LOCALLY COMPACT GROUPS.

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ABSTRACT. Let $A_p(G)$ denote the Figa-Talamanca-Herz Banach Algebra of the locally compact group $G$, thus $A_2(G)$ is the Fourier Algebra of $G$. If $G$ is commutative then $A_2(G) = L^1(\mathcal{G})^\wedge$. Let $A_p^r(G) = A_p \cap L^r(G)$ with norm $\|u\|_{A_p^r} = \|u\|_{A_p} + \|u\|_{L^r}$.

We investigate for which $p, r, G$ do the Banach algebras $A_p^r(G)$ have the Banach space geometric properties: The Radon-Nikodym Property (RNP), the Schur Property (SP) or the Dunford-Pettis Property (DPP).

The results are new even if $G = \mathbb{R}$ (the real line) or $G = \mathbb{Z}$ (the additive integers).

INTRODUCTION. Let $G$ be a locally compact group and let $A_p(G)$ denote the Figa-Talamanca-Herz Banach algebra of $G$, as defined in [Hz1], thus generated by $L^p(\mathcal{G})$ for $1 < p < \infty$, and $1/p + 1/p' = 1$, see sequel. Hence $A_2(G)$ is the Fourier algebra of $G$ as defined and studied by Eymard in [Ey1]. If $G$ is abelian then $A_2(G) = L^1(\mathcal{G})^\wedge$.

Denote $A_p^r(G) = A_p \cap L^r(G)$, for $1 \leq r \leq \infty$, equipped with the norm $\|u\|_{A_p^r} = \|u\|_{A_p} + \|u\|_{L^r}$. If $r = \infty$ let $A_p^\infty(G) = A_p(G)$.

If $G$ is abelian then, $A_p^r(G) = L^1(\mathcal{G})^\wedge \cap L^r(G)$ with the norm $\|u\| = \|f\|_{L^1(\mathcal{G})} + \|\hat{f}\|_{L^r(G)}$ if $u = \hat{f}$.

The study of these Banach Algebras started in a beautiful paper of Larsen Liu and Wang [LLW] in the abelian case, and continued in [La1], [La2], [Gr1]-[Gr5] ...

etc.

Let $X$ be a Banach space. Then

$X$ has the Schur Property (SP) if weak convergent sequences are norm convergent.

$X$ has the Dunford-Pettis Property (DPP) if whenever $(x_n), (x'_n)$ are weakly null sequences in $X$ and $X^*$ respectively, then, $\lim(x_n, x'_n) = 0$.

Clearly the SP implies the DPP, [Di].

The Banach space $\ell^{1}$ is a dual Banach space which has the SP, hence the DPP. But for any measure space, $L^{1}(\mu)$, has the DPP, [Di] p.19, yet it does not have the SP if $\mu$ is non atomic.

$X$ has the Krein-Milman (KMP) property if any closed convex bounded subset $B$ is the norm closed convex hull of its extreme points (ext $B$)

$X$ has the Radon Nikodym property (RNP) if every such $B$ is the norm closed convex hull of its strongly exposed points (strexp $B$), see sequel or [DU1] p.190 and p.218.

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2010 Mathematics Subject classification. Primary 43A15, 46J10, 43A25, 46B22. Secondary 46J20, 43A30, 43A80, 22A30. Key words and phrases: Fourier Algebra, Radon-Nikodym property, weakly amenable, locally compact groups.
Points in $\text{stexp}(B)$ are points of $\text{ext}(B)$ that have beautiful smoothness properties. In particular they are weak to norm continuity points of $B$ and are peak points of $B$, see sequel.

Quoting Jerry Uhl: “A Banach space has the RNP if it’s unit ball “wants to be weakly compact, but just cannot make it”.

Denote by $PM_p(G) = A_p(G)^*$, and by $PF_p(G)$, the norm closure in $PM_p(G)$ of $L^1(G)$, (as a space of left convolutors on $L_p(G)$). Let $W_p(G) = PF_p(G)^*$

Then $W_p(G)$ is a Banach algebra of bounded continuous functions on $G$ containing the ideal $A_p(G)$, see Cowling [Co1].

Let $W^r_p(G) = W_p \cap L^r(G)$, with the sum norm.

If $G$ is abelian and $p = 2$ then $W^2_2(G) = M(\hat{G})^\wedge$, where $M(G)$ is the space of bounded Borel measures on $G$.

Our Main Result on the SP and the DPP, is the

**THEOREM A:** Let $G$ be a noncompact locally compact group.

1. $\forall 1 < r \leq \infty$, $A^r_p(G)$ does not have the SP. 
   Yet $A^1_p(G)$ has the SP and the RNP, if $G$ is discrete.

2. If $G$ is weakly amenable then $\forall 1 < r \leq p'$, $A^r_p(G)$ does not have even the DPP yet it has the RNP.

The above result is of interest even if $G = R$ or $G = Z$.

**CONJECTURE:** If $G$ is a connected semisimple Lie group with finite center then $A^2_2(G)$ has the DPP for any $r > 2$.

**QUESTION:** Does this hold if $G = Z$ or $G = R$?

It has been proved by W. Braun, in an unpublished preprint [Br], that if $G$ is amenable, then $A^1_p(G)$ is a dual Banach space with the RNP.

This is improved in our Main Result on the RNP in the:

**THEOREM B:** Let $G$ be a weakly amenable (see sequel) locally compact group. Then

1. $\forall 1 \leq r \leq p'$, $A^r_p(G) = W^r_p(G)$ and

   $A^r_p(G)$ is a dual Banach algebra with the RNP.

   If $G$ is unimodular, this is the case $\forall 1 \leq r \leq \max(p, p')$.

2. If $G$ is a noncompact connected semisimple Lie group with finite center then $\forall r > 2$, $A^r_2(G)$, does not have the RNP and is not a dual Banach space.

The above is new and of interest even if $G = R$ or $G = Z$.

**QUESTION:** Does (2) hold true if $G = Z$, or $G = R$?

**THEOREM C:** Assume that $G$ is second countable and weakly amenable, and $1 < p < \infty$.

If for some $t \leq \infty$, $A^t_p(G)$ has the RNP, then so does $A^r_p(G), \forall 1 \leq r \leq t$

Fell groups, see sequel or [B], are non compact groups for which $A^2_2(G)$ has the RNP. Hence $A^t_p(G)$ has the RNP for all $1 \leq t \leq \infty$, for such $G$.

This paper was inspired by the important paper of F. Lust-Piquard [L], where the RNP and SP were investigated for $PM_p(E)$ for nowhere dense compact sets $E$, for abelian $G$. See also T. Miao and P.F. Mah [MM].
NOTATIONS AND DEFINITIONS: Denote as in [Hz1]

(3) \( A_p(G) = \left\{ u = \sum u_n * v_n'; u_n \in L^{p'}, v_n \in L^p, \Sigma \| u_n\|_{L^{p'}} \| v_n\|_{L^p} < \infty \right\} \)

where the norm of \( u \in A_p \) is the infimum of the last sum over all the representations of \( u \) as above.

Let \( C_0(G)|C_c(G) \) denote the continuous functions which tend to \( 0 \) at \( \infty \), with compact support, with norm \( \| u \|_\infty = \sup \{ |u(x)| : x \in G \} \).

The group \( G \) is weakly amenable if \( A_2(G) \) has an approximate identity \( \{ v_\alpha \} \) bounded in the norm of \( B_2(G) \), the space of Herz-Schur multipliers, see, [Ey2] (or [DCH], [Gr3]).

Any closed subgroup \( G \) of any finite extension of the general Lorenz group \( SO_0(n,1) \), for \( n > 1 \), hence the free group on \( n > 1 \) generators is weakly amenable but not amenable. For this and much more see [DCH].

Definition: Let \( B \) be a bounded subset of the Banach space \( X \) and \( b \in B \). \( b \) is a strongly exposed point of \( B \) (and \( \text{streq}(B) \) denotes the set of all such), if \( \exists b^* \in X^* \) such that:

\[
\Re b^*(x) < \Re b^*(b), \forall x \in B \text{ and } x \neq b, \text{ and} 
\]

(4) \( \Re b^*(x_n) \rightarrow \Re b^*(b) \) for \( x_n \in B \) implies \( \| x_n - b \| \rightarrow 0 \). (see [DU1] p.158)

Hence in order to Test an Algorithm for some \( b \) in \( \text{streq}(B) \) it is Enough to Test it on One Particular Element of \( X^* \).

2. MAIN RESULTS

(1) THE RNP CASE.

We first improve results in [Gr3], [Gr4], by removing the unimodularity of the group in the next

PROPOSITION 1: Let \( G \) be a locally compact group. If \( p = 2 \), or if \( G \) is weakly amenable and \( 1 < p < \infty \), then

(1) \( (\ast) W_p \cap L^r(G) = A_p \cap L^r(G) \), \( \forall 1 \leq r \leq p' \).

and \( A^*_p(G) \) is a dual Banach space.

If \( G \) is unimodular then this holds for \( \forall 1 \leq r \leq \max(p,p') \).

(2) If \( G = SL(2,R) \) and \( p = 2 \) then \( (\ast) \) does not hold for any \( r > 2 \), and \( A^*_2(G) \) is not a dual space for \( r > 2 \), (see Prop.3).

REMARK: The interval \([1,p']\) is the best one can do even for \( G=Z \) and \( p=2 \) as proved in [HZ], (see [Gr4] p.4379, or [LiR]).

PROOF: By weak amenability, for all \( 1 < p < \infty \), the \( W_p \) norm restricted to \( A_p \) is equivalent to the \( A_p \) norm, (Gr4 Corollary 3.7.). If \( p = 2 \) then Kaplansky’s density theorem will yield the same result.

Now with the notations of [Gr4] Thm. 2.1. p.4379, if \( e_\alpha \in C_c(G) \) is an approximate identity for \( L_1(G) \), such that each \( e_\alpha \) is the “square of a special operator”, a la Fendler [Fe] p.129, we have, by the Lemma, loc. cit. that

(5) \( \| e_\alpha * w - w \|_{W_{p'}} \rightarrow 0 \) \( \forall w \in W_{p'} \) a fortiori \( \forall w \in W_{p'} \cap L^{p'\vee} \).

But, since \( e_\alpha \in C_c(G) \), we have for such \( w \), that

(6) \( e_\alpha * w \in L^p \cap L^{p\vee} \subset A_{p'} \), thus \( e_\alpha * w \) is a Cauchy sequence in \( A_{p'} \).

Hence \( w \in A_{p'} \). It follows that \( W_{p'} \cap L^{p\vee} = A_{p'} \cap L^{p\vee} \).
However by [CoI] p.91, \( W_p = W_{p^\vee} \), \( A_p = A_{p^\vee} \). Hence

\[
(7) \quad W_p \cap L^{p'} = A_p \cap L^{p'}, \quad \forall 1 < p < \infty.
\]

But \( W_p \) contains only bounded functions, hence

\[
(8) \quad \forall r \leq p', W_p \cap L^r = W_p \cap L^{p'} \cap L^r = A_p \cap L^{p'} \cap L^r = A_p \cap L^r. \quad \text{Thus}
\]

\[
(9) \quad (i) \quad W_p \cap L^r = A_p \cap L^r, \quad \forall r \leq p'.
\]

If \( G \) is unimodular then, since \((W_p \cap L^r)^\vee = (A_p \cap L^r)^\vee\), it follows that \( W_p' \cap L^r = A_p' \cap L^r, \forall r \leq p', \) which holds for all \( 1 < p' < \infty \).

Replace now \( p' \) by \( p \), then \( W_p \cap L^r = A_p \cap L^r, \forall r \leq p \).

The above implies the unimodular case.

By Theorem 2.2 of [Gr5] \( W_p(G) \cap L^r(G) \) is a dual Banach space for all \( 1 < p < \infty \) and \( 1 < r \leq \infty \), and all locally compact groups \( G \). This proves (*) The proof of (ii) and Theorem B requires the next results. \( \square \)

**LEMMA 2:** Let \( G \) be a locally compact group. Assume that \( A_p(G) \) has an approximate identity \( u_\alpha \) such that \( \sup \| u_\alpha \|_\infty \leq B < \infty \). Then

1. \( A_p \cap C_c \) is norm dense in \( A_p' \) and
2. If \( G \) is second countable then \( A_p' \) is norm separable.

**PROOF:**

(1) Let \( e_\alpha \in A_p \cap C_c \) satisfy \( \| e_\alpha - u_\alpha \|_{A_p} \to 0 \) and

\[
(10) \quad \| e_\alpha - u_\alpha \|_{A_p} \leq 1, \forall \alpha. \quad \text{Then} \quad \| e_\alpha \|_\infty \leq \| e_\alpha - u_\alpha \|_\infty + \| u_\alpha \|_\infty \leq \| u_\alpha \|_\infty \leq 1 + B. \quad \text{Hence}
\]

\[
(11) \quad \| e_\alpha v - v \|_{A_p} \leq \| (e_\alpha - u_\alpha) v \|_{A_p} + \| u_\alpha v - v \|_{A_p} \to 0, \forall v \in A_p. \quad \text{But if} \quad w \in A_p^r \quad \text{and} \quad K \subset G \quad \text{is compact such that} \quad \int_{G \sim K} |w|^r \, dx < \epsilon \quad \text{then}
\]

\[
(12) \quad \int_{G \sim K} |(e_\alpha - 1)w|^r \leq \int_{G \sim K} (2 + B)|w|^r \leq (2 + B)\epsilon. \quad \text{But} \quad \int_K |(e_\alpha - 1)w|^r \to \infty.
\]

It thus follows that \( \| e_\alpha w - w \|_{A_p^r} \to 0. \quad \text{But} \quad e_\alpha w \in A_p \cap C_c. \)

(2) \( A_p(G) \) is norm separable, hence so is \( A_p[K] = \{ u \in A_p(G); \text{spt } u \subset K \} \), where \( K \subset G \). Let \( A_p^r[K] = \{ u \in A_p^r(G); \text{spt } u \subset K \} \). If \( K \) is compact then the identity \( I : A_p^r[K] \to A_p[K] \) is 1-1, onto and continuous, hence it is bicontinuous. Hence \( A_p^r[K] \) is separable. Let now \( K_n \subset \text{int } K_{n+1} \subset G \), be compact (int denotes interior), such that \( \cup K_n = G \). It is hence enough to show that \( \cup A_p^r[K_n] \) is norm dense in \( A_p^r(G) \).

By (a) we know that \( A_p \cap C_c \) is norm dense in \( A_p^r(G) \). But if \( v \in A_p^r(G) \) has compact support \( S \) then \( S \subset K_j \) for some \( j \), hence \( v \in A_p^r[K_j] \). Thus \( \cup A_p^r[K_n] \) is norm dense in \( A_p^r(G) \). \( \square \)

**REMARK:** We do not know if, \( A_p \cap C_c(G) \) is norm dense in \( A_p^r(G) \) even for \( G = SL(2, R) \triangle R^2 \), if \( p = 2 \) and any \( r \). As shown in [Do], \( A_2(G) \) does not have an approximate identity, bounded in the multiplier norm.

**COROLLARY 3:** Let \( G \) be a second countable locally compact group. If \( G \) is weakly amenable then \( \forall 1 \leq r \leq p' \), \( A_p'(G) \) is a separable dual Banach algebra and thus has the RNP.

If \( G \) is unimodular, this is the case for \( 1 \leq r \leq \max(p, p') \).
REMARK: Weak amenability, namely the existence in $A_2$ of an approximate identity norm bounded in $B_2$ depends only on $p = 2$, yet the result holds for all $p$. Since by Furutaâ€™s Thm. 2.4 in [Fu], $B_2 \subset B_p$ contractively, see also [Gr] p.23. The $B_p$ norm dominates the multiplier norm by [Fu].

PROOF: $A_p^r(G)$ is a dual Banach space $\forall 1 \leq r \leq p'$. But since $G$ is weakly amenable $\forall 1 < p < \infty$, $A_p(G)$, has a multiplier norm, bounded approximate identity, by the Remark above. It thus follows by the Lemma above, that $A_p^r(G)$ is norm separable. But separable dual Banach spaces have the RNP by [DU] p.218. □

The second countability of $G$ is removed in the main result of this section, namely

THEOREM B: Let $G$ be a weakly amenable locally compact group and $1 < p < \infty$. Then

1) $\forall 1 \leq r \leq p'$, $A_p^r(G) = W_p^r(G)$ and $A_p^r(G)$ is a dual Banach algebra with the RNP.
   If $G$ is unimodular, this is the case $\forall 1 \leq r \leq \max(p, p')$.
2) If $G$ is a noncompact connected semisimple Lie group with finite center then $\forall r > 2$, $A_p^2(G)$ does not have the RNP and is not a dual Banach algebra.

PROOF: (1) By [DU] it is enough to prove that every separable subspace of $A_p^r(G)$ has the RNP. Based on the above Corollary follow the proof of Theorem 3.1 on p.22-24 of [Gr] and [Gr] p.4381.

   (2) By a deep result of Cowling [Co], $A_2^r(G) = A_2(G)$ if $r > 2$. Assume that $A_2(G)$ has the RNP. Then the regular representation is the direct sum of irreducible unitary representations, by K. Taylorâ€™s Thm. 4.1. Denote by $\hat{G}$, the set of all such. Then by [Dix] 14.1.2, 14.3.2, $\hat{G}$ contains only square integrable representations. Now, by Lipsman, [Li] p.412–413, $\hat{G}$ induces the discrete topology on $\hat{G}$ (this being the set of all square integrable representations). But by the Corollary on p.228 of [Fell], the topology of $\hat{G}$ is second countable, since $G$ is such. Hence so is that of $\hat{G}$, which is in addition discrete and thus is countable. But by Baggettâ€™s [B] Prop. 2.2, a connected semisimple Lie group whose reduced dual is countable is compact.

   $A_2(G)$ is separable. If it was a dual space it would have the RNP, see [DU].

   We note that (2) has been proved for $SL(2, R)$ in [Gr], by using the support of its Plancherell measure. □

REMARK: Any closed subgroup of any finite extension of the general Lorenz group $SO_0(n, 1)$ for $n > 1$, hence the free group of $n > 1$ generators (a nonamenable group), is a weakly amenable group.

This group is a noncompact connected simple Lie group, see [DCH] p.474 for this and much more.

(II) INTERVALS WITH THE RNP.

We will show that if $G$ is second countable and weakly amenable then $\forall 1 < p < \infty$, $A_p^r(G)$ having the RNP for $t = s$ implies that it has for all $1 \leq t \leq s$, where $s = \infty$ is allowed.

Definition: Let $X, Y$ be Banach spaces and $T : X \rightarrow Y$ be a bounded linear operator. $T$ is a semi-embedding if it is one to one and it maps the closed unit ball in $X$ into a closed set in $Y$. If such $T$ exists we say that $X$ semi embeds in $Y$. 
THEOREM (H.P. Rosenthal): A separable Banach space has the RNP if it semi-embeds in a Banach space with the RNP. See [DU2] p.160 or [Rö], [LPP]. We will use of the above Theorem, to prove the main Theorem C. We need the following:

LEMMA 4: If \( r < s \) then the identity \( I : A_p^r(G) \to A_p^s(G) \) is a semi-embedding, for any \( s \leq \infty \) (If \( s = \infty \), \( A_p^\infty(G) = A_p(G) \)).

PROOF: Denote by \( B_r \) the closed unit ball of \( A_p^r(G) \). Let \( v_n \in B_r \) satisfy that
\[
\|v_n - w\|_{A_p^r} = \|v_n - w\|_{A_p} + \|v_n - w\|_{L^r} \to 0, \quad \text{for some } w \in A_p^s(G).
\]
If \( s = \infty \) only \( \|v_n - w\|_{A_p} \) appears). Clearly \( |v_n(x)| \to |w(x)| \), \( \forall x \in G \). And by Fatou’s Lemma we have \( \int |w|^r \, dx \leq \liminf \int |v_n|^r \, dx \leq 1 \). Thus \( w \in A_p^r \). But
\[
1 \geq \limsup \left( \|v_n\|_{A_p} + \|v_n\|_{L^r} \right) \geq \lim \|v_n\|_{A_p} + \liminf \|v_n\|_{L^r} \geq \|w\|_{A_p} + \|w\|_{L^r}.
\]
Thus \( w \in B_r \). \( \square \)

THEOREM C: Assume that \( G \) is second countable and weakly amenable. If for some \( t \leq \infty \), \( A_p^{∞}(G) \) has the RNP, then so does \( A_p^r(G) \), \( \forall 1 \leq r \leq t \). In particular, if \( A_p(G) \) has the RNP then \( A_p^r(G) \) has the RNP for all \( 1 \leq r < \infty \).

PROOF: Apply Rosenthal’s Theorem and the above Lemma 4, and note that by Lemma 1, \( A_p^r(G) \) is norm separable, since \( \|u\|_{\infty} \leq \|u\|_{B_r} \leq \|u\|_{B_2} \). \( \square \)

REMARKS: (1) A group \( G \) with completely reducible regular representation is called in [T] an [AR] group. \( G \) is such iff \( A_2(G) \) has the RNP, as proved by Keith Taylor [T]. A noncompact [AR] group is called a Fell group in [B]. Larry Baggett and Keith Taylor construct in [BT] p.596 (iii) an example of a connected nonunimodular Lie group \( G = G_3 \) such that \( A_2(G) \neq W_2 \cap C_0(G) \) and such that \( G \) is a Fell group. The above implies that \( A_2^r(G) \) has the RNP, for all \( 1 \leq r \leq \infty \) for the above and any Fell group. [BT] includes examples of Fell groups which are connected Lie groups and which are (i) solvable, (ii) amenable nonsolvable, (iii) nonamenable, (iv) non-TypeI. All of which are not unimodular. See also [MP].

(2) Assume that for arbitrary \( G \), \( A_2^r(G) \) having the RNP for some \( s > 2 \) implies the equality \( W_2^s(G) = A_2^s(G) \). It would then follow for \( G = Z \), that \( A_2^s(Z) \) does not have the RNP for this \( s \). This is implied by the fact that \( A_2^s(Z) \neq W_2^r(Z) \), \( \forall s > 2 \) as proved in [HZ]. Hence there would be no need to take \( G = SL(2, R) \) in the remark above, and \( Z \) would suffice.

QUESTION: If \( G \) is noncompact abelian then \( A_2(G) \) does not have the RNP, since its closed unit ball has no extreme points see [DU1] p.219. Yet, \( A_2^r(G) \), \( \forall 1 \leq r \leq 2 \) does have the RNP, by Theorem B.

For such \( G \) (or even for \( G = Z \) and \( p = 2 \)), does \( A_2^r(G) \) fail to have the RNP for \( r > 2 \)?

(III) THE SCHUR AND THE DUNFORD-PETTIS PROPERTY.

The following result clarifies the DPP case, for discrete \( G \) and \( 1 \leq r \leq \max\{p, p'\} \).

PROPOSITION 5: Let \( G \) be any discrete group. Then

(1) For any \( 1 < r \leq \max\{p, p'\} \), \( A_p^r(G) \) fails the DPP and a fortiori fails the SP, yet has the RNP.

(2) \( A_p^r(G) \) has the SP and the RNP.

PROOF: (1) By Theorem 7 of [GR3] \( A_p^r(G) = \ell^r(G) \), if \( 1 \leq r \leq \max\{p, p'\} \).
Assume in addition that $r > 1$. Then, $\ell^r, \ell^r'$, considered over the positive integers, are isometric to subspaces of $\ell^r(G), \ell^r'(G)$, respectively. Let $x_n = (0, 0, \ldots, 1, 0, 0, \ldots)$, where 1 appears in the $n$-th place, considered as an element of $\ell^r$, and let $x_n^*$ be defined exactly as $x_n$, but considered as an element of $\ell^r'$. Then $x_n, x_n^*$ are weakly null sequences in $\ell^r, \ell^r'$, respectively, yet $(x_n, x_n^*) = 1$. It follows that $A_{p}^r(G) = \ell^r(G)$, if $1 < r \leq \max\{p, p'\}$, fails the DPP, a fortiori the SP. These have the RNP, since they are reflexive Banach spaces.

(2) If $r = 1$ then $\ell^1(G)$ has the SP and the RNP.

**QUESTION:** It is not clear to us if $A_{p}^r(G)$ has the DPP if $r > \max\{p, p'\}$ in case $G$ is discrete, or even if $G = Z$ and $p = 2$.

If $G$ is abelian *non compact* then $A_2(G)$ does not have the SP, since $\hat{G}$ is non discrete. This is substantially improved in the main result, namely

**THEOREM A:** Let $G$ be a noncompact locally compact group.

1. $\forall 1 < r \leq \infty, A_{p}^r(G)$ does not have the SP
   Yet $A_{p}^1(G)$ has the SP if $G$ is discrete.
2. If $G$ is weakly amenable then $\forall 1 < r \leq p', A_{p}^r(G)$ does not have the DPP
   yet it has the RNP.

**CONJECTURE:** If $G$ is a connected semisimple Lie group with finite center then $A_{p}^2(G)$ has the DPP for any $r > 2$.

For a proof, we need the following results.

**LEMMA 6:** Let $G$ be a locally compact group, and $1 < p < \infty$.

If $u_n \in L^p$ and $u_n \to 0$ weakly in $L^p$, then

$$\forall v \in L^p, u_n * v^\vee \to 0 \text{ weakly } = \sigma(A_p, PM_p) \text{ in } A_p.$$ 

**PROOF:** It is enough to prove that

$$\forall \Phi \in PM_p, u \in L^{p'}, v \in L^p \quad (\Phi, u * v^\vee) = (\Phi * v, u).$$

Since then $(\Phi, u_n * v^\vee) = (\Phi * v, u_n) \to 0$, since $\Phi * v \in L^p, \forall v \in L^p$. Now (i) is an old result of Eymard [Ey3], a paper, not easily available. Here is a proof based on [Hz1].

Any $\Phi \in PM_p$ is in the ultrastrong closure of $PF_p$ in $PM_p$. Let $\Phi \in PM_p$ and $u \in L^{p'}, v \in L^p$. Let $w_\alpha \in L^1, w_\alpha \to \Phi$, ultrastrongly. Thus $\|w_\alpha * h - \Phi * h\| \to 0, \forall h \in L^p$. But then $(w_\alpha, s) \to (\Phi, s), \forall s \in A_p$, since $w_\alpha \to \Phi$, ultraweakly $= \sigma(PM_p, A_p)$.

Hence

$$\Phi, u * v^\vee) = (w_\alpha, u * v^\vee) = (w_\alpha, v, u) \to (\Phi * v, u). \quad \Box$$

Recall that $\ell_\xi u(x) = u(\alpha x)$ if $u$ is a function on $G$.

**LEMMA 7:** (i) Let $G$ be $\sigma$ compact and $V = V^{-1}$ be a neighborhood of $\epsilon$ such that $V$ is compact. Let $g_1 \to \infty$ and $u_i = \ell_{g_i}(1_V * 1_V)$. Then $\forall 1 < r \leq \infty$, $\{u_i\}$ is weakly convergent, but not norm convergent in $A_p^r(G)$.

(ii) If $G$ is any noncompact group, then $A_p^r(G)$ does not have the SP $\forall 1 < r \leq \infty$.

**PROOF:** Let $u_i = \ell_{g_i}(1_V * 1_V)$. Then $\ell_{g_i}1_V \to 0$ weakly in $L^{p'}$. Since, if $f \in L^{p'}$ and $\epsilon > 0$, let $K \subset G$ be compact such that $\left(\int_{K} ||f||^{p'}\right)^{1/p'} < \epsilon$ where $F = G \sim K$.  


Let $k$ satisfy that if $i > k$ then $g_i^{-1}K \subset F$. Then

\begin{equation}
\int f_\ell_g,1_V \leq \epsilon \lambda(V)^{1/p} \text{ if } i > k.
\end{equation}

By the above Lemmas $u_i \to 0$ weakly in $A_p$, and clearly weakly in $L^r, \forall 1 < r < \infty$. (If $r = \infty$, $A_p^\prime = A_p^\prime$). Thus $u_i \to 0$, weakly in $A_p^\prime$, $\forall 1 < r < \infty$, by Cor. 6 in \cite{Gr3}. But $1_V * 1_V(x) = \lambda (xV \cap V) = \lambda(V)$, if $x = 1$. Hence

\begin{equation}
\|u_i\|_{A^\prime_p} \geq \|u_i\|_\infty = \lambda(V) > 0.
\end{equation}

Thus $\{u_i\}$ is not norm convergent in $A_p^\prime(G)$. If $G$ is not $\sigma$ compact, let $H$ be a $\sigma$ compact open subgroup. Consider the two Banach algebras $A = A_p^\prime(H)$ and $B = \{1_H u ; u \in A_p^\prime(G)\}$ as a closed subalgebra of $A_p^\prime(G)$. Both these are semisimple, since $A_p(H)$ and $A_p(G)$ are such, see \cite{H1} Proposition 5. Define $T : B \to A$, by $T(1_H u) = v$. Then $T$ is a 1-1 onto algebraic isomorphism. By \cite{Ru} Thm. 11.10, both $T^{-1}$ and $T$ are continuous. Since $H$ is $\sigma$ compact let $\{u_i\} \subset A$ be a sequence which converges weakly, but not in norm, in $A$. Then $\{T^{-1} u_i\}$, satisfies the same in $B$ hence in $A_p^\prime(G)$. \hfill \Box

**PROOF OF THEOREM A:** (i) is proved in the above Lemma.

(ii) If $1 \leq r \leq p^\prime$ and $G$ is weakly amenable then $A_p^\prime(G)$ is a dual Banach space with the RNP. Thus $A_p^\prime(G) = B^*$ for some Banach space $B$, and $B$ does not contain $\ell^1$ by \cite{Ha}. If $A_p^\prime(G)$ has the DPP then so does $B$, by \cite{Dil} Cor. 2. But then $B^*$ has the SP by \cite{Dil} Thm. 3, p.23, which contradicts (i), if $r > 1$.

**CONJECTURE:** If $G$ is a connected semisimple real Lie group then $A_p^\prime(G)$ for $r > 2$, has the DPP.

**References**

[B] L. Baggett, A separable group having discrete dual is compact. J. Functional Anal. 10 (1972), 131–148.

[BT] Larry Baggett and Keith Taylor, Groups with Completely Reducible Regular Representation. Proc. Amer. Math. Soc. 72 (1978), 593–600.

[Br] W. Braun, Einige Bemerkungen Zu $S_0(G)$ und $A^\prime(G)$ int $L^1(G)$. Preprint.

[BrF] W. Braun and Hans G. Feichtinger, Banach Spaces of Distributions Having Two Module Structures. J. Funct. Analysis. 51 (1983), 174–212.

[CI] Cho-Ho Chu and Bruno Iochum, The Dunford-Pettis property in $C^*$ algebras. Studia Mathematica 97 (1) (1990), 59–64.

[Co1] Michael Cowling, An Application of Littlewood-Paley Theory in Harmonic Analysis. Math. Ann. 241 (1979), 83–96.

[Co2] Michael Cowling, The Kunze-Stein phenomenon. Ann. of Math. 106 (1978), 209–234.

[CoHa] Michael Cowling and Uffe Haagerup, Completely bounded multipliers of the Fourier algebra of a simple Lie group of real rank one. Invent. Math. 96 (1989), 507–549.

[DCH] J. deCanniere and U. Haagerup, Multipliers of the Fourier algebra of some simple Lie groups and their discrete subgroups. Amer. J. Math. 107 (1985), 455–500.

[DU1] J. Diestel and J.J. Uhl, Jr. Vector Measures. Math. Surveys. Amer. Math. Soc. 1977.
[DU2] J. Diestel and J.J. Uhl, Jr. Progress in Vector measures – 1977-83. Measure Theory and Applications. LNM. 1033. Springer. 1983.

[Di] Joe Diestel, A Survey of Results Related to the Dunford-Pettis Property. Contemporary Mathematics Vol. 2, 1980. Amer. Math. Soc.

[Dix] Jaques Dixmier, Les C*-algèbres et leurs représentations. Paris. Gautiers-Villars. 1969.

[Do] B. Dorofaeff, The Fourier Algebra of $SL(2, R) \wedge R_n, n > 1$ has no multiplier bounded approximate unit. Math. Ann. 297 (1993), 707–724.

[Ey1] P. Eymard, L’algebre de Fourier d’un groupe localement compacte. Bull. Soc. Math. France. 92 (1964), 181–236.

[Ey2] P. Eymard, Algebre $A_p$ et convoluteurs de $L_p$. Lecture Notes in Math. No. 180 Springer (1971), 364–381.

[Ey3] P. Eymard, Publ. Inst. Elie. Cartan. (1968-1969).

[Fell] J.M.J. Fell, $C^*$ Algebras with Smooth Dual. Ill. J. Math. 4 (1960), 221–230.

[Fe] Gero Fendler, An $L_p$ version of a theorem of D.A. Raikov. Ann. Inst. Fourier, Grenoble. 35 (1985), 125–135.

[Fu] Koji Furuta, Algebras $A_p$ and $B_p$ and amenability of locally compact groups. Hokkaido Math. J. 20 (1991), 579–591.

[Gr1] Edmond E. Granirer, An Application of the Radon Nikodym Property in Harmonic Analysis. Bull. U.M.I. (5) 18-B (1981), 663–671.

[Gr2] —— Amenability and semisimplicity for second duals of quotients of the Fourier Algebra $A(G)$. J. Austral. Math. Soc. (Series A) 63 (1997), 289–296.

[Gr3] —— The Figa-Talamanca-Herz-Lebesgue Banach Algebras $A^r_p(G) = A_p \cap L^r(G)$. Math. Proc. Camb. Phil. Soc. 140 (2006), 401–416.

[Gr4] —— The Radon-Nikodym Property for some Banach Algebras related to the Fourier Algebras. Proc. Amer. Math. Soc. 139 (2011), 4377–4384.

[Gr5] —— Weakly Amenable Groups and the RNP for some Banach Algebras related to the Fourier Algebras. Coll. Math. 130 (2013), 19–26.

[Ha] Richard Haydon, Some more characterisations of Banach spaces containing $\ell^1$. Math. Proc. Camb. Phil. Soc. 80 (1976), 269–276.

[HZ] Edwin Hewitt and Herbert Zuckerman, Singular measures with absolutely continuous convolution squares. Proc. Camb. Phil. Soc. 62 (1966), 399–420.

[Hz1] C. Herz, Harmonic Synthesis for Subgroups. Ann. Inst. Fourier, Grenoble. 23 (1973), 91–123.

[Hz2] C. Herz, The theory of $p$ spaces with an application to convolution operators. Trans. Amer. Math. Soc. 154 (1971), 69–82.

[LLW] Ron Larsen, Ten-sun Liu, Ju-kwei Wang, On functions with Fourier Transforms in $L_p$. Mich. Math. J. 11 (1964), 369–378.

[La1] Hang-Chin Lai, On some properties of $A^p(G)$ algebras. Proc. Japan Acad. 45 (1969), 572–576.

[La2] Hang-Chin Lai, A remark on $A^p(G)$ algebras. Proc. Japan Acad. 46 (1970), 58–63.

[Li] Ronald L. Lipsman, The Dual Topology for the Principal and Discrete Series of Semisimple Groups. Trans. Amer. Math. Soc. 152 (1970), 399–417.

[LPP] H.P. Lotz, N.T. Peck, and H. Porta, Semi-embeddings of Banach Spaces. Proc. Edinburgh Math. Soc. 22 (1979), 233–240.

[LiR] Teng-sun Liu and Arnoud van Rooij, Sums and Intersections of Normed Linear Spaces. Math. Nachrichten. 42 (1969), 29–42.
[Ped] G.K. Pedersen, $C^*$ Algebras and their automorphism groups. Academic Press. (1979).

[Lu] Francoise Lust-Piquard, Means on $CV_p(G)$-Subspaces of $CV_p(G)$ with RNP and Schur Property. Ann. Inst. Fourier, Grenoble. 39, 4 (1989), 969–1006.

[MM] Peter F. Mah and Tianxuan Miao, Extreme Points of the Unit Ball of the Fourier-Stiltjes Algebra. Proc. Amer. Math. Soc. 128 (1999), 1097–1103.

[MP] G. Mauceri and M.A. Picardello, Noncompact unimodular groups with purely atomic Plancherel measures. Proc. Amer. Math. Soc. 78 (1980), 77–84.

[Ri] N.W. Rickert, Convolutions of $L_p$ functions. Proc. Amer. Math. Soc. 18 (1967), 762–763. MR0216301 (35:7136).

[Ro] H.P. Rosenthal, Convolution by a Biased Coin. The Altgelt Book 1975/76. University of Illinois Functional Analysis Seminar.

[Ru] Walter Rudin, Functional Analysis. McGraw-Hill Book Company. 1973.

[Sa] Sadahiro Saeki, The $L_p$ conjecture and Young’s Inequality. Ill. J. Math. 34 (1990), 614–627.

[T] Keith Taylor, Geometry of the Fourier Algebras and Locally Compact Groups with Atomic Unitary Representations. Math. Ann. 262 (1983), 183–190.

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