DOUBLE LOOP QUANTUM ENVELOPING ALGEBRAS

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Abstract. In this paper we describe certain homological properties and representations of a two-parameter quantum enveloping algebra $U_{g,h}$ of $\mathfrak{sl}(2)$, where $g, h$ are group-like elements.

1. Introduction

It is well-known that there is a bijective map $L \rightarrow P_L$ from the set of all oriented links $L$ in $\mathbb{R}^3$ to the ring $\mathbb{Z}[g^\pm 1, h^\pm 1]$ of two-variable Laurent polynomials. $P_L$ is called the Jones-Conway polynomial of the link $L$. The Jones-Conway polynomial is an isotopy invariant of oriented links satisfying what knot theorists call “skein relations” (see [11]). Suppose $\mathbb{K}$ is a field with characteristic zero and $q$ is a nonzero element in $\mathbb{K}$ satisfying $q^2 \neq 1$. Let $U_q(\mathfrak{sl}(2))$ be the usual quantum enveloping algebra of the Lie algebra $\mathfrak{sl}(2)$ with generators $E, F, K$. Then the vector space

$$U_{g,h} := \mathbb{K}[g^\pm 1, h^\pm 1] \otimes_{\mathbb{K}} U_q(\mathfrak{sl}(2))$$

has been endowed with a Hopf algebra structure in [13].

We abuse notation and write $g^\pm 1, h^\pm 1, E, F, K^\pm 1$ for $g^\pm 1 \otimes 1, h^\pm 1 \otimes 1, 1 \otimes E, 1 \otimes F, 1 \otimes K^\pm 1$ respectively. In addition, $g^\pm 1, h^\pm 1, K^\pm 1$ are abbreviated to $g, h, K$ respectively. Then $U_{g,h}$ is an algebra over $\mathbb{K}$ generated by $g, g^{-1}, h, h^{-1}, E, F, K, K^{-1}$. These generators satisfy the following relations.

\begin{align*}
K^{-1}K &= KK^{-1} = 1, \quad g^{-1}g = gg^{-1} = 1, \quad h^{-1}h = hh^{-1} = 1, \\
KEK^{-1} &= q^2E, \quad gh = hg, \quad gK = Kg, \quad gE = Eg, \quad hE = Eh, \\
KFK^{-1} &= q^{-2}F, \quad hK = Kh, \quad hF = Fh, \quad gF = Fg, \\
EF - FE &= \frac{K - K^{-1}g^2}{q - q^{-1}}.
\end{align*}

The other operations of the Hopf algebra $U_{g,h}$ are defined as follows:

\begin{align*}
\Delta(E) &= h^{-1} \otimes E + E \otimes hK, \\
\Delta(F) &= K^{-1}hg^2 \otimes F + F \otimes h^{-1},
\end{align*}

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\[ \Delta(K) = K \otimes K, \quad \Delta(K^{-1}) = K^{-1} \otimes K^{-1}, \]
\[ \Delta(a) = a \otimes a, \quad a \in G, \]
where \( G = \{ g^m h^n | m, n \in \mathbb{Z} \} \),
\[ \varepsilon(K) = \varepsilon(K^{-1}) = \varepsilon(a) = 1, \quad a \in G, \]
\[ \varepsilon(E) = \varepsilon(F) = 0, \]
and
\[ S(E) = -EK^{-1}, \quad S(F) = -KFg^{-2}, \]
\[ S(a) = a^{-1}, \quad a \in G, \quad S(K) = K^{-1}, \quad S(K^{-1}) = K. \]

The Hopf algebra \( U_{g,h} \) is a special case of the Hopf algebras defined in [14]. It is isomorphic to the tensor product of \( U_q(\mathfrak{sl}(2)) \) and \( \mathbb{K}[g^{\pm 1}, h^{\pm 1}] \) as algebras. However, the coproduct of \( U_{g,h} \) is not the usual coproduct of the tensor product of two coalgebras. Neither is the antipode.

Homological methods have been used to study Hopf algebras by many authors (see [2], [15] and their references). However, there are few examples of Hopf algebras satisfying a given set of homological properties. In this paper, we describe certain homological properties of the Hopf algebra \( U_{g,h} \) and consequently give an example satisfying some homological properties. Moreover, we study the representation theory of the algebra \( U_{g,h} \). Similar to [7] and [8], we can define some version of the Bernstein-Gelfand-Gelfand (abbreviated as BGG) category \( \mathcal{O} \). Furthermore, we decompose the BGG category \( \mathcal{O} \) into a direct sum of subcategories, which are equivalent to categories of finitely generated modules over some finite-dimensional algebras.

Let us outline the structure of this paper. In Section 2, we study the homological properties of \( U_{g,h} \). We prove that \( U_{g,h} \) is Auslander-regular and Cohen-Macaulay, and the global dimension and Gelfand-Kirillov dimension of \( U_{g,h} \) are equal. We also prove that the center of \( U_{g,h} \) is equal to \( \mathbb{K}[g^{\pm 1}, h^{\pm 1}, C] \), where \( C \) is the Casimir element of \( U_{g,h} \). To study the category \( \mathcal{O} \) in Section 4, we prove that \( U_{g,h} \) has an anti-involution that acts as the identity on all of \( \mathbb{K}[K^{\pm 1}, g^{\pm 1}, h^{\pm 1}] \).

Since there is a finite-dimensional non-semisimple module over the algebra \( \mathbb{K}[g^{\pm 1}, h^{\pm 1}] \), there is a finite-dimensional non-semisimple module over \( U_{g,h} \). In Section 3, we compute the extension group \( \text{Ext}^1(M, M') \) in the case that the nonzero \( q \) is not a root of unity, where \( M, M' \) are finite-dimensional simple modules over \( U_{g,h} \). We prove that the tensor functor \( V \otimes - \) determines an isomorphism from \( \text{Ext}^1(\mathbb{K}_{\alpha}, \mathbb{K}_{\alpha'}) \) to \( \text{Ext}^1(V \otimes \mathbb{K}_{\alpha}, V \otimes \mathbb{K}_{\alpha'}) \) for any finite-dimensional simple \( U_q(\mathfrak{sl}(2)) \)-module \( V \). We also obtain a decomposition theory about the tensor product of two simple \( U_{g,h} \)-modules. From this, we obtain a Hopf subalgebra of the finite dual Hopf algebra \( U_{g,h}^0 \) of \( U_{g,h} \), which is generated by coordinate functions of finite-dimensional simple modules of \( U_{g,h} \).

In Section 4, we briefly discuss the Verma modules of \( U_{g,h} \). The BGG subcategory \( \mathcal{O} \) of the category of representations of \( U_{g,h} \) is introduced and studied. The main results in [8] also hold in the category \( \mathcal{O} \) over the algebra \( U_{g,h} \).
Throughout this paper $\mathbb{K}$ is a fixed algebraically closed field with characteristic zero; $\mathbb{N}$ is the set of natural numbers; $\mathbb{Z}$ is the set of all integers. $\ast^{+1}$ is usually abbreviated to $\ast$. All modules over a ring $R$ are left $R$-modules.

It is worth mentioning that some results of this article are also true if $\mathbb{K}$ is not an algebraically closed field. We always assume that $\mathbb{K}$ is an algebraically closed field for simplicity throughout this paper.

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2. SOME PROPERTIES OF $U_{g,h}$

In this section, we firstly prove that $U_{g,h}$ is a Noetherian domain with a PBW basis. Then we compute the global dimension and Gelfand-Kirillov dimension of $U_{g,h}$. Moreover, we show that $U_{g,h}$ is Auslander regular, Auslander Gorenstein, Cohen-Macaulay and Tdeg-stable. For the undefined terms in this section, we refer the reader to [2] and [3].

**Theorem 2.1** (PBW Theorem). The algebra $U_{g,h}$ is a Noetherian domain. Moreover, it has a PBW basis \{$F^lK^m g^n h^s E^t | l, t \in \mathbb{Z}_{\geq 0}; m, n, s \in \mathbb{Z}$\}.

**Proof.** Let $R = \mathbb{K}[K^{\pm 1}, g^{\pm 1}, h^{\pm 1}]$. Since $R$ is a homomorphic image of the polynomial ring $\mathbb{K}[x_1, x_2, \cdots, x_5, x_6] (\varphi(x_1) = K, \varphi(x_2) = K^{-1}, \varphi(x_3) = g, \varphi(x_4) = g^{-1}, \varphi(x_5) = h, \varphi(x_6) = h^{-1})$, $R$ is a Noetherian ring with a basis \{$K^m g^n h^s | m, n, s \in \mathbb{Z}$\}. It is easy to prove that $R$ is a domain.

Define $\sigma(K^n g^a h^b) = q^{2a} K^n g^a h^b$, $\forall a, b \in \mathbb{Z}$, $\delta(R) \equiv 0$, and extend $\sigma$ by additivity and multiplicativity. It is trivial to check that $\sigma$ is a ring automorphism of $R$, and $\delta : R \to R$ is a $\sigma$-skew derivation. Hence $R' := R[F; \sigma, \delta]$ is a Noetherian domain with a basis \{$K^a g^b h^c F^d | a, b, c \in \mathbb{Z}, d \in \mathbb{Z}_{\geq 0}$\} by [9, Theorem 1.2.9].

Next, define $\sigma'$ on $R'$ via:

$$\sigma'(K^a g^b h^c F^d) = q^{-2a} K^a g^b h^c F^d,$$

(for all integers $d \geq 0$, and $a, b, c \in \mathbb{Z}$), and extend $\sigma'$ by additivity and multiplicativity. One can check that $\sigma'$ is indeed a ring automorphism of $R'$. Define $\delta'$ on $R'$ via

$$\delta'(R) \equiv 0, \quad \delta'(F) = \frac{K - K^{-1} q^2}{q - q^{-1}}.$$

Also extend $\delta'$ to all of $R'$ by additivity and the following equation:

$$\delta'(ab) := \delta'(a)b + \sigma'(a)\delta'(b), \quad \forall a, b \in R'.$$

One can check that $\delta'$ is a $\sigma'$-skew derivation of $R'$. Now by the above results, $U_{g,h} = R'[E; \sigma', \delta']$ is indeed a Noetherian domain, since $R'$ is. Moreover, $U_{g,h}$ has a basis \{$K^a g^b h^c F^d E^t | a, b, c \in \mathbb{Z}, d, t \in \mathbb{Z}_{\geq 0}$\}. Since $K^a g^b h^c F^d E^t = q^{-2ad} F^d K^a g^b h^c E^t$,

$$\{F^l K^m g^n h^s E^t | l, t \in \mathbb{Z}_{\geq 0}; m, n, s \in \mathbb{Z}\}$$

is also a basis of $U_{g,h}$. This basis is called a PBW basis. □
Proposition 2.2. (1) \( U_{g,h} \) is isomorphic to \( \mathbb{K}[g^{\pm 1}, h^{\pm 1}] \otimes U_q(\mathfrak{sl}(2)) \) as algebras;
(2) \( U_{g,h} \) is an Auslander regular, Auslander Gorenstein and Tdeg-stable algebra with Gelfand-Kirillov dimension 5.

Proof. Define \( E' := g^{-1}E, K' := g^{-1}K \). By Theorem 2.1,

\[
\{ F^aK^b g^c h^d E^n | b, c, d \in \mathbb{Z}, a, t \in \mathbb{Z}_{\geq 0} \}
\]

is also a basis of \( U_{g,h} \). Let \( \varphi(E') = 1 \otimes E, \varphi(F) = 1 \otimes F, \varphi(K') = 1 \otimes K, \varphi(g) = g \otimes 1, \varphi(h) = h \otimes 1 \) and \( \varphi \) extends by additivity and multiplicativity. One can check that \( \varphi \) is an epimorphism of algebras from \( U_{g,h} \) to \( \mathbb{K}[g^{\pm 1}, h^{\pm 1}] \otimes U_q(\mathfrak{sl}(2)) \). Similarly, define \( \phi(1 \otimes E) = E', \phi(1 \otimes F) = F, \phi(1 \otimes K) = K', \phi(g \otimes 1) = g, \phi(h \otimes 1) = h, \) and extend \( \phi \) by additivity and multiplicativity. Then \( \phi \) is an epimorphism of algebras from \( \mathbb{K}[g^{\pm 1}, h^{\pm 1}] \otimes U_q(\mathfrak{sl}(2)) \) to \( U_{g,h} \). It is easy to verify that \( \phi \circ \varphi = \text{id} \) and \( \varphi \circ \phi = \text{id} \). So \( \varphi \) is an isomorphism of algebras.

Let us recall that if the global dimension of a Noetherian ring \( A \), denoted by \( \text{gldim}(A) \), is finite, then \( \text{gldim}(A) = \text{injdim}(A) \), the injective dimension of \( A \). From [9, Section 7.1.11], one obtains that the right global dimension of a Noetherian algebra \( A \) is equal to \( \text{gldim}(A) \) as well. In [1], H. Bass proved that if \( A \) is a commutative Noetherian ring with a finite injective dimension, then \( A \) is Auslander-Gorenstein. Thus \( \text{gldim}(\mathbb{K}[g^{\pm 1}, h^{\pm 1}, K^{\pm 1}]) = 3 \), and

\[
\text{gldim} U_{g,h} \leq \text{gldim}(\mathbb{K}[g^{\pm 1}, h^{\pm 1}, K^{\pm 1}]) + 2 = 5
\]

by [9, Theorem 7.5.3]. Hence \( U_{g,h} \) is an Auslander regular and Auslander Gorenstein ring by [3, Theorem 4.2].

Recall that an algebra \( A \) with total quotient algebra \( Q(A) \) is said to be Tdeg-stable if

\[
\text{Tdeg}(Q(A)) = \text{Tdeg}(A) = \text{GKdim}(A),
\]

where \( \text{GKdim}(A) \) is the Gelfand-Kirillov dimension of \( A \). By [15, Example 7.1], \( U_q(\mathfrak{sl}(2)) \) is Tdeg-stable, and \( \text{GKdim}(U_q(\mathfrak{sl}(2))) = 3 \). Since

\[
U_{g,h} \cong \mathbb{K}[g^{\pm 1}, h^{\pm 1}] \otimes U_q(\mathfrak{sl}(2)) \cong U_q(\mathfrak{sl}(2))[g, g^{-1}][h, h^{-1}],
\]

\[
\text{GKdim}(U_{g,h}) = 2 + \text{GKdim}(U_q(\mathfrak{sl}(2))) = 5,
\]

and \( U_{g,h} \) is Tdeg-stable by [15, Theorem 1.1].

Remark 2.3. (1) Since \( U_{g,h} \cong \mathbb{K}[g^{\pm 1}, h^{\pm 1}] \otimes U_q(\mathfrak{sl}(2)) \) as algebras, we call the Hopf algebra \( U_{g,h} \) a double loop quantum enveloping algebra.
(2) Since \( \mathbb{K}[g^{\pm 1}, h^{\pm 1}] \) and \( U_q(\mathfrak{sl}(2)) \) are Hopf algebras, \( \mathbb{K}[g^{\pm 1}, h^{\pm 1}] \otimes U_q(\mathfrak{sl}(2)) \) has a natural Hopf algebra structure. However, as

\[
\Delta(E') = h^{-1}g^{-1} \otimes E' + E' \otimes hK',
\]

and

\[
\Delta(F) = K'^{-1}hg \otimes F + F \otimes h^{-1},
\]

by (1.5) and (1.6), the above isomorphism of algebras is not an isomorphism of Hopf algebras, i.e., \( U_{g,h} \) has a different coproduct than the usual coproduct of the tensor product of the two coalgebras.
Corollary 2.4. Suppose \( q \) is not a root of unity. Then the center of \( U_{g,h} \) is equal to \( \mathbb{K}[g^{\pm 1}, h^{\pm 1}, C] \), where \( C = FE + \frac{qK+q^{-1}K^{-1}}{(q-q^{-1})^2}g^2 \).

Proof. Let \( c' = F'E' + \frac{qK+q^{-1}K^{-1}}{(q-q^{-1})^2} \), where \( K', E', F' \) are the Chevalley generators of \( U_q(\mathfrak{sl}(2)) \). Then the center of \( U_q(\mathfrak{sl}(2)) \) is generated by \( c' \) by [6, Theorem VI.4.8]. Since \( U_{g,h} \simeq \mathbb{K}[g^{\pm 1}, h^{\pm 1}] \otimes U_q(\mathfrak{sl}(2)) \) as algebras by Proposition 2.2, the center of \( U_{g,h} \) is isomorphic to \( \mathbb{K}[g^{\pm 1}, h^{\pm 1}] \otimes \mathbb{K}[c'] \). So the center of \( U_{g,h} \) is equal to \( \mathbb{K}[g^{\pm 1}, h^{\pm 1}, c_1] \), where \( c_1 = g^{-1}FE + \frac{q-1qK+q^{-1}K^{-1}}{(q-q^{-1})^2} \). Hence the center of \( U_{g,h} \) is equal to \( \mathbb{K}[g^{\pm 1}, h^{\pm 1}, C] \), where \( C = FE + \frac{qK+q^{-1}K^{-1}}{(q-q^{-1})^2}g^2 \). □

The element \( C = FE + \frac{qK+q^{-1}K^{-1}}{(q-q^{-1})^2}g^2 \) is called a Casimir element of \( U_{g,h} \).

Proposition 2.5. There exists an anti-involution \( i \) of \( U_{g,h} \) that acts as the identity on all of \( \mathbb{K}[K^{\pm 1}, g^{\pm 1}, h^{\pm 1}] \).

Proof. Let \( i(E) = -KF, i(F) = -EK^{-1}, i(K^{\pm 1}) = K^{\pm 1}, i(g^{\pm 1}) = g^{\pm 1}, i(h^{\pm 1}) = h^{\pm 1} \). Extend \( i \) by additivity and multiplicativity. Then \( i \) is an anti-involution of \( U_{g,h} \), which acts as the identity on all of \( \mathbb{K}[K^{\pm 1}, g^{\pm 1}, h^{\pm 1}] \). □

Suppose \( M \) is a finitely generated module over an algebra \( A \). Then the grade of \( M \), denoted by \( j(M) \), is defined to be

\[ j(M) := \min\{j \geq 0 | \operatorname{Ext}_A^j(M, A) \neq 0 \} \]

Recall that an algebra \( A \) is Cohen-Macaulay if

\[ j(M) + \operatorname{GKdim}(M) = \operatorname{GKdim}(A) \]

for every nonzero finitely generated \( A \)-module \( M \).

Proposition 2.6. The algebra \( U_{g,h} \) is a Cohen-Macaulay algebra with \( \operatorname{gldim} U_{g,h} = 5 \).

Proof. Let \( A = \mathbb{K}[g,h,u,v,K,L][F; \alpha] \), where \( \alpha|_{\mathbb{K}[g,h,u,v]} = \text{id}, \alpha(K) = q^2K, \alpha(L) = q^{-2}L, \alpha(F) = F, \delta(F) = \frac{K-L}{q-q^{-1}}, \) and \( \delta(\mathbb{K}[g,h,u,v,K,L]) = 0 \). Then \( A \) is Auslander-regular and Cohen-Macaulay by [2, Lemma II.9.10]. Since \( U_{g,h} \cong A/(gu - 1, hv - 1, KL - 1) \), \( U_{g,h} \) is Auslander-Gorenstein and Cohen-Macaulay by [2, Lemma II.9.11]. Let \( \mathbb{K} \) be the trivial \( U_{g,h} \)-module defined by \( a \cdot 1 = \varepsilon(a)1 \). Then \( \operatorname{GKdim}(\mathbb{K}) = 0 \) and \( \operatorname{gldim}(U_{g,h}) = 5 \) by [2, Exercise II.9.D]. □

In the presentation for \( U_{g,h} \) given in Section 1, the generators \( K^{\pm 1} \), and the generators \( E, F \) play a different role respectively. Similar to [4], we write down an equitable presentation for \( U_{g,h} \) as follows.

Theorem 2.7. The algebra \( U_{g,h} \) is isomorphic to the unital associative \( \mathbb{K} \)-algebra with generators \( x^{\pm 1}, y, z; u^{\pm 1}, v^{\pm 1} \) and the following relations:

\[ x^{-1}x = xx^{-1} = 1, \quad u^{-1}u = uu^{-1} = 1, \quad v^{-1}v = vv^{-1} = 1, \]  

\[ (2.1) \]
(2.2) \[ ux = xu, \quad uy = yu, \quad uz = zu, \quad uv = vu, \]
(2.3) \[ vx = xv, \quad yv = vy, \quad zv = vz, \]
(2.4) \[ \frac{qxy - q^{-1}yx}{q - q^{-1}} = 1, \]
(2.5) \[ \frac{qzx - q^{-1}xz}{q - q^{-1}} = 1, \]
(2.6) \[ \frac{qyz - q^{-1}zy}{q - q^{-1}} = 1. \]

Proof. Let \( \mathcal{V}_{u,v} \) be the algebra generated by \( x^\pm, y, z, u^\pm, v^\pm \) satisfying the relations from (2.1) to (2.6). Let us define \( \Phi(x^\pm) = g^\mp K^\pm, \Phi(y) = K^{-1}g + F(q - q^{-1}), \Phi(z) = K^{-1}g - K^{-1}Eq(q - q^{-1}), \Phi(u^\pm) = g^\mp, \Phi(h^\pm) = h^\pm, \) and extend \( \Phi \) by additivity and multiplicativity. Then \( \Phi \) is a homomorphism of algebras from \( \mathcal{V}_{u,v} \) to \( U_{g,h} \).

Define \( \Psi(K^\pm) = u^\mp x^\pm, \Psi(F) = \frac{y-x^{-1}}{q-q^{-1}}, \Psi(E) = \frac{1-xz}{(q-q^{-1})yu}, \Psi(g) = u^{-1}, \) and \( \Psi(h) = v. \) We extend \( \Psi \) by additivity and multiplicativity. It is routine to check that \( \Phi \) and \( \Psi \) are isomorphisms of Hopf algebras. Since \( \Phi \Psi \) fixes each of the generators \( E, F, K^\pm, g^\pm, h^\pm \) of \( U_{g,h} \), \( \Phi \Psi = \text{id}. \) Similarly we can check that \( \Psi \Phi = \text{id}. \) So \( \Phi \) is the inverse of \( \Psi. \)

Since \( \mathcal{V}_{u,v} \) is isomorphic to \( U_{g,h} \) as algebras, we can regard \( U_{g,h} \) as an algebra generated by \( x^\pm, u^\pm, v^\pm, y \) and \( z \) with relations (2.1)–(2.6). To make the above algebra isomorphisms \( \Phi, \Psi \) into isomorphisms of Hopf algebras, we only need to define the other operations of the Hopf algebra \( U_{g,h} \) with these new generators as follows:

(2.7) \[ \Delta(x^\pm) = x^\pm \otimes x^\pm, \]
(2.8) \[ \Delta(u^\pm) = u^\pm \otimes u^\pm, \]
(2.9) \[ \Delta(v^\pm) = v^\pm \otimes v^\pm, \]
(2.10) \[ \Delta(y) = x^{-1} \otimes (x^{-1} - v^{-1}) + u^{-1}vx^{-1} \otimes (y - x^{-1}) + y \otimes v^{-1}, \]
(2.11) \[ \Delta(z) = x^{-1} \otimes x^{-1} + uv^{-1}x^{-1} \otimes (z - x^{-1}) + (z - x^{-1}) \otimes v, \]
(2.12) \[ \varepsilon(x^\pm) = \varepsilon(u^\pm) = \varepsilon(v^\pm) = 1, \]
(2.13) \[ \varepsilon(y) = \varepsilon(z) = 1, \]
and

(2.14) \[ S(x^\pm) = x^\mp, \quad S(u^\pm) = u^\mp, \quad S(v^\pm) = v^\mp, \]
(2.15) \[ S(y) = x - x^{-1}y + u, \quad S(z) = x + u^{-1} - u^{-1}xz. \]

Then one can check that the above isomorphisms \( \Phi, \Psi \) are isomorphisms of Hopf algebras. For example, \( \Delta(\Psi(g^\mp K^\pm)) = \Delta(x^\pm) = x^\pm \otimes x^\pm = (\Psi \otimes \Psi)\Delta(g^\mp K^\pm). \)
3. Finite-dimensional representations of $U_{g,h}$

Let $q$ be a nonzero element in an algebraically closed field $\mathbb{K}$ with characteristic zero. Moreover, we assume that $q$ is not a root of unity. The main purpose of this section is to classify all extensions between two finite-dimensional simple $U_{g,h}$-modules. Let us start with a description of the finite-dimensional simple $U_{g,h}$-modules.

For any three elements $\lambda, \alpha, \beta \in \mathbb{K}^\times$ and any $U_{g,h}$-module $V$, let

$$V^{\lambda,\alpha,\beta} = \{ v \in V | Kv = \lambda v, gv = \alpha v, hv = \beta v \}.$$

The $(\lambda, \alpha, \beta)$ is called a weight of $V$ if $V^{\lambda,\alpha,\beta} \neq 0$. A nonzero vector in $V^{\lambda,\alpha,\beta}$ is called a weight vector with weight $(\lambda, \alpha, \beta)$.

The next result is proved by a standard argument.

**Lemma 3.1.** We have $EV^{\lambda,\alpha,\beta} \subseteq V^{q^2\lambda,\alpha,\beta}$ and $FV^{\lambda,\alpha,\beta} \subseteq V^{q^{-2}\lambda,\alpha,\beta}$.

**Definition 3.2.** Let $V$ be a $U_{g,h}$-module and $(\lambda, \alpha, \beta) \in \mathbb{K}^\times$. A nonzero vector $v$ of $V$ is a highest weight vector of weight $(\lambda, \alpha, \beta)$ if $Ev = 0$, $Kv = \lambda v$, $gv = \alpha v$, and $hv = \beta v$.

A $U_{g,h}$-module $V$ is a standard cyclic module with highest weight $(\lambda, \alpha, \beta)$ if it is generated by a highest weight vector $v$ of weight $(\lambda, \alpha, \beta)$.

**Proposition 3.3.** Any nonzero finite-dimensional $U_{g,h}$-module contains a highest weight vector. Moreover, the endomorphisms induced by $E$ and $F$ are nilpotent.

**Proof.** By Lie's theorem, there is a nonzero vector $w \in V$ and $(\mu, \alpha, \beta) \in \mathbb{K}^\times$ such that

$$Kw = \mu w, \quad gw = \alpha w, \quad hw = \beta w.$$

In fact, there is an elementary and more direct proof as follows. Since $\mathbb{K}$ is algebraically closed and $V$ is finite-dimensional, there is a nonzero vector $v \in V$ such that $Kv = \mu v$ for some element $\mu \in \mathbb{K}$. Moreover $\mu \in \mathbb{K}^\times$ as $K$ is invertible. Let

$$V_\mu = \{ v \in V | Kv = \mu v \} \neq 0.$$

Then $V_\mu$ is also a finite-dimensional vector space. For any $v \in V_\mu$, we have

$$K(gv) = g(Kv) = \mu gv.$$

So $gv \in V_\mu$ and $g$ induces a linear transformation on the nonzero finite-dimensional vector space $V_\mu$. There is a nonzero vector $v' \in V_\mu$ such that $gv' = \alpha v'$ for some nonzero element $\alpha \in \mathbb{K}$. Let $V_{\mu, \alpha} = \{ v' \in V_\mu | gv' = \alpha v' \}$. Then $V_{\mu, \alpha}$ is also a nonzero finite-dimensional linear space. Similarly we can prove that $h(V_{\mu, \alpha}) \subseteq V_{\mu, \alpha}$ as $gh = hg$, $hK = Kh$ by (1.2) and (1.3). Hence there exists a nonzero vector $w \in V_{\mu, \alpha}$ and $(\mu, \alpha, \beta) \in \mathbb{K}^\times$ such that

$$Kw = \mu w, \quad gw = \alpha w, \quad hw = \beta w.$$

The proof now follows [6, Proposition VI.3.3], using Lemma 3.1. □

For any positive integer $m$, let $[m] = \frac{q^m - q^{-m}}{q - q^{-1}}$, and $[m]! = [1][2] \cdots [m]$. Similar to the proof of [6, Lemma VI.3.4], we get the following:
Lemma 3.4. Let \( v \) be a highest weight vector of weight \((\lambda, \alpha, \beta)\). Set \( v_p = \frac{1}{[p]} F^p v \) for \( p > 0 \) and \( v_0 = v \). Then
\[
Kv_p = q^{-2p}\lambda v_p, \quad gv_p = \alpha v_p, \quad Fv_{p-1} = [p]v_p, \quad hv_p = \beta v_p
\]
and
\[
Ev_p = \frac{q^{-(p-1)}\lambda - q^{p-1}\lambda^{-1}\alpha^2}{q - q^{-1}}v_{p-1}.
\]

Theorem 3.5. (a) Let \( V \) be a finite-dimensional \( U_{g,h} \)-module generated by a highest weight vector \( v \) of weight \((\lambda, \alpha, \beta)\). Then
(i) \( \lambda = \varepsilon \alpha q^n \), where \( \varepsilon = \pm 1 \) and \( n \) is the integer defined by \( \dim V = n + 1 \).
(ii) Setting \( v_p = \frac{1}{[p]} F^p v \), we have \( v_p = 0 \) for \( p > n \) and in addition the set \( \{ v = v_0, v_1, \ldots, v_n \} \) is a basis of \( V \).
(iii) The operator \( K \) acting on \( V \) is diagonalizable with \((n + 1)\) distinct eigenvalues
\[
\{ \varepsilon \alpha q^n, \varepsilon \alpha q^{n-2}, \ldots, \varepsilon \alpha q^{-n+2}, \varepsilon \alpha q^{-n} \},
\]
and the operators \( g, h \) act on \( V \) by scalars \( \alpha, \beta \) respectively.
(iv) Any other highest weight vector in \( V \) is a scalar multiple of \( v \) and is of weight \((\lambda, \alpha, \beta)\).
(v) The module is simple.
(b) Any simple finite-dimensional \( U_{g,h} \)-module is generated by a highest weight vector. Two finite-dimensional \( U_{g,h} \)-modules generated by highest weight vectors of the same weight are isomorphic.

Proof. The proof follows that of [6, Theorem VI.3.5] or [13, Theorem 3.4]. It is omitted here.

Let us denote the \((n + 1)\)-dimensional simple \( U_{g,h} \)-module generated by a highest weight vector \( v \) of weight \((\varepsilon \alpha q^n, \alpha, \beta)\) in Theorem 3.5 by \( V_{\varepsilon,n,\alpha,\beta} \). Since \( \mathbb{K} \) is an algebraically closed field, the dimension of a simple module over \( \mathbb{K}[g^{\pm1}, h^{\pm1}] \) is equal to one. Any such simple \( \mathbb{K}[g^{\pm1}, h^{\pm1}] \)-module is determined by \( g \cdot 1 = \alpha, h \cdot 1 = \beta \), for \( \alpha, \beta \in \mathbb{K}^{\times} \). This simple module is denoted by \( \mathbb{K}_{\alpha,\beta} := \mathbb{K} \cdot 1 \) in the sequel. The finite-dimensional simple \( U_q(\mathfrak{sl}(2)) \)-modules are characterized in [6, Theorem VI.3.5]. These simple modules are denoted by \( V_{\varepsilon,n} \), where \( \varepsilon = \pm 1 \), and \( n \in \mathbb{Z}_{\geq 0} \). By Proposition 2.2 and [8, Proposition 16.1], every finite-dimensional simple \( U_{g,h} \)-module is isomorphic to \( \mathbb{K}_{\alpha,\beta} \otimes V_{\varepsilon,n} \). It is not difficult to verify that \( \mathbb{K}_{\alpha,\beta} \otimes V_{\varepsilon,n} \) is isomorphic to \( V_{\varepsilon,n,\alpha,\beta} \).

Corollary 3.6 (Clebsch-Gordan Formula). Let \( n \geq m \) be two non-negative integers. There exists an isomorphism of \( U_{g,h} \)-modules
\[
V_{\varepsilon,n,\alpha,\beta} \otimes V_{\varepsilon',m,\alpha',\beta'} \cong V_{\varepsilon\varepsilon',n+m,\alpha\alpha',\beta\beta'} \oplus V_{\varepsilon\varepsilon',n+m-2,\alpha\alpha',\beta\beta'} \oplus \cdots \oplus V_{\varepsilon\varepsilon',n-m,\alpha\alpha',\beta\beta'}.
\]

Proof. Since \( V_{\varepsilon,n,\alpha,\beta} \otimes V_{\varepsilon',m,\alpha',\beta'} \cong \mathbb{K}_{\alpha\alpha',\beta\beta'} \otimes (V_{\varepsilon,n} \otimes V_{\varepsilon',m}) \), and
\[
V_{\varepsilon,n} \otimes V_{\varepsilon',m} \cong V_{\varepsilon\varepsilon',n+m} \oplus V_{\varepsilon\varepsilon',n+m-2} \oplus \cdots \oplus V_{\varepsilon\varepsilon',n-m}
\]
as modules over \( U_q(\mathfrak{sl}(2)) \) by [6, Theorem VII.7.1],
\[
V_{\varepsilon,n,\alpha,\beta} \otimes V_{\varepsilon',m,\alpha',\beta'} \cong V_{\varepsilon\varepsilon',n+m,\alpha\alpha',\beta\beta'} \oplus V_{\varepsilon\varepsilon',n+m-2,\alpha\alpha',\beta\beta'} \oplus \cdots \oplus V_{\varepsilon\varepsilon',n-m,\alpha\alpha',\beta\beta'}.
\]
This completes the proof.
Lemma 3.7. Let $m \in \mathbb{N}$. Then

$$[E, F^m] = [m]F^{m-1}q^{-(m-1)}K - q^{m-1}K^{-1}g^2 \over q - q^{-1}.$$ 

Proof. Let $E', F', K'$ be the Chevalley generators of $U_q(\mathfrak{sl}(2))$. Then

$$[E', F^m] = [m]F^{m-1}q^{-(m-1)}K' - q^{m-1}K'^{-1}$$

by [6, Lemma VI.1.3]. Substituting $Eg^{-1}, g^{-1}K, F$ for $E', K', F'$ in the above identity respectively, we obtain

$$[E, F^m] = [m]F^{m-1}q^{-(m-1)}K - q^{m-1}K^{-1}g^2 \over q - q^{-1}.$$ 

Let $M := \mathbb{K}_{\alpha, \beta} = \mathbb{K} \cdot 1$ be a module over $\mathbb{K}[g^{\pm 1}, h^{\pm 1}]$, where $g \cdot 1 = \alpha$ and $h \cdot 1 = \beta$. About the simple modules over the algebra $\mathbb{K}[g^{\pm 1}, h^{\pm 1}]$, we have the following

**Proposition 3.8.** Given two simple $\mathbb{K}[g^{\pm 1}, h^{\pm 1}]$-modules $M := \mathbb{K}_{\alpha, \beta}$ and $M' := \mathbb{K}_{\alpha', \beta'}$, if $M$ is not isomorphic to $M'$, then $\text{Ext}^n(M', M) = 0$ for all $n \geq 0$; if $M \cong M'$, then

$$\text{Ext}^n(M', M) \cong \begin{cases} \mathbb{K}, & n = 0 \\ \mathbb{K}^2, & n = 1 \\ 0, & n \geq 2. \end{cases}$$

Proof. Denote the algebra $\mathbb{K}[g^{\pm 1}, h^{\pm 1}]$ by $R$. Construct a projective resolution of the simple $R$-module $\mathbb{K}_{\alpha, \beta}$ as follows:

$$0 \longrightarrow R \overset{\varphi_2}{\longrightarrow} R^2 \overset{\varphi_1}{\longrightarrow} R \overset{\varphi_0}{\longrightarrow} \mathbb{K}_{\alpha, \beta} \longrightarrow 0,$$

where

$$\varphi_0(r(g, h)) = r(\alpha, \beta), \quad \varphi_1(r(g, h), s(g, h)) = r(g, h)(g - \alpha) + s(g, h)(h - \beta),$$

and

$$\varphi_2(r(g, h)) = (r(g, h)(h - \beta), -r(g, h)(g - \alpha))$$

for $r(g, h), s(g, h) \in R$. Applying the functor $\text{Hom}_R(-, \mathbb{K}_{\alpha, \beta})$ to the exact sequence (3.2), we obtain the following complex:

$$0 \longrightarrow \mathbb{K} \overset{\varphi_1^*}{\longrightarrow} \mathbb{K}^2 \overset{\varphi_2^*}{\longrightarrow} \mathbb{K} \longrightarrow 0.$$

For any $\theta \in \text{Hom}_R(R, \mathbb{K}_{\alpha, \beta})$,

$$\varphi_1^*(\theta)((1, 0)) = \theta(g - \alpha) = (g - \alpha)\theta(1) = 0, \\
\varphi_1^*(\theta)((0, 1)) = \theta(h - \beta) = (h - \beta)\theta(1) = 0.$$

This means that $\varphi_1^* = 0$. Similarly, one can prove that $\varphi_2^* = 0$. So

$$\text{Ext}^0(\mathbb{K}_{\alpha, \beta}, \mathbb{K}_{\alpha, \beta}) \cong \mathbb{K}, \quad \text{Ext}^1(\mathbb{K}_{\alpha, \beta}, \mathbb{K}_{\alpha, \beta}) \cong \mathbb{K}^2, \\
\text{Ext}^2(\mathbb{K}_{\alpha, \beta}, \mathbb{K}_{\alpha, \beta}) \cong \mathbb{K}, \quad \text{Ext}^n(\mathbb{K}_{\alpha, \beta}, \mathbb{K}_{\alpha, \beta}) = 0$$

for $n \geq 3$. 

If we use the functor $\text{Hom}_R(-, \mathbb{K}_{\alpha, \beta'})$ to replace the functor $\text{Hom}_R(-, \mathbb{K}_{\alpha, \beta})$ in the above proof, we can also obtain the complex (3.3). In this case, we have
\[
\varphi^*_1(\theta)((1,0)) = \alpha' - \alpha, \quad \varphi^*_2(\theta)((0,1)) = \beta' - \beta,
\]
and
\[
\varphi^*_2(\eta)(a) = (\beta' - \beta)a\eta((1,0)) - (\alpha' - \alpha)a\eta((0,1))
\]
for $\theta \in \text{Hom}_R(R, \mathbb{K}_{\alpha, \beta'})$, $\eta \in \text{Hom}_R(R^2, \mathbb{K}_{\alpha, \beta'})$, and $a \in R$. Hence both $\varphi^*_1$ and $\varphi^*_2$ are not zero linear mappings provided that either $\alpha \neq \alpha'$, or $\beta \neq \beta'$. Consequently, the sequence (3.3) is exact in this case. So $\text{Ext}^n(\mathbb{K}_{\alpha, \beta}, \mathbb{K}_{\alpha', \beta'}) = 0$ for $n \geq 0$. \hfill \Box

It is well-known that the group $\text{Ext}^1(M', M)$ can be described by short exact sequences. Next, we describe $\text{Ext}^1(\mathbb{K}_{\alpha, \beta}, \mathbb{K}_{\alpha, \beta})$ by short exact sequences.

Let $0 \rightarrow \mathbb{K}_{\alpha, \beta} \overset{\varphi}{\longrightarrow} N \overset{\psi}{\longrightarrow} \mathbb{K}_{\alpha, \beta} \rightarrow 0$ be an element in $\text{Ext}^1(\mathbb{K}_{\alpha, \beta}, \mathbb{K}_{\alpha, \beta})$. Suppose $\{w_1, w_2\}$ be a basis of $N$ such that $\psi(w_2) = 1$ and $w_1 = \varphi(1)$. Then $gw_1 = \alpha w_1$, $hw_1 = \beta w_1$. Suppose $gw_2 = \alpha w_2 + x w_1$. Then
\[
\alpha \psi(w_2) = \psi(gw_2) = a \psi(w_2).
\]
So $gw_2 = \alpha w_2 + x w_1$. Similarly, we can prove $hw_2 = \beta w_2 + y w_1$. If $\{u_1, u_2\}$ is another basis satisfying $u_1 = \varphi(1) = w_1$ and $\psi(u_2) = 1$, then $u_2 - w_2 = \lambda w_1$ for some $\lambda \in \mathbb{K}$. Thus $gu_2 = \alpha u_2 + x u_1 + \lambda \alpha w_1 = \alpha u_2 + x u_1$. Similarly, we obtain that $hu_2 = \beta u_2 + y u_1$. Hence $x, y$ are independent of the choice of the bases of $N$. So we can use $M_{x, y}$ to denote the module $N$. In the following, we abuse notation and use $M_{x, y}$ to denote the following exact sequence $0 \rightarrow \mathbb{K}_{\alpha, \beta} \overset{\varphi}{\longrightarrow} M_{x, y} \overset{\psi}{\longrightarrow} \mathbb{K}_{\alpha, \beta} \rightarrow 0$ meanwhile.

Let $0 \rightarrow \mathbb{K}_{\alpha, \beta} \overset{\varphi'}{\longrightarrow} M'_{x', y'} \overset{\psi'}{\longrightarrow} \mathbb{K}_{\alpha, \beta} \rightarrow 0$ be another element in $\text{Ext}^1(\mathbb{K}_{\alpha, \beta}, \mathbb{K}_{\alpha, \beta})$, and $\{w'_1, w'_2\}$ be a basis of $M'_{x', y'}$ such that $w'_1 = \varphi'(1)$, $\psi'(w'_2) = 1$ and
\[
\begin{align*}
    gw'_1 &= \alpha w'_1, \\
    gw'_2 &= \alpha w'_2 + x' w'_1, \\
    hw'_1 &= \beta w'_1, \\
    hw'_2 &= \beta w'_2 + y' w'_1.
\end{align*}
\]
Consider the following commutative diagram
\[
\begin{array}{ccccccc}
0 & \longrightarrow & \mathbb{K}_{\alpha, \beta} & \overset{\varphi}{\longrightarrow} & N & \overset{\psi}{\longrightarrow} & \mathbb{K}_{\alpha, \beta} & \longrightarrow & 0 \\
& & \mu_1 \downarrow & & \mu_2 \downarrow & & \mu_3 \downarrow & & \\
0 & \longrightarrow & \mathbb{K}_{\alpha, \beta} & \overset{\varphi'}{\longrightarrow} & N' & \overset{\psi'}{\longrightarrow} & \mathbb{K}_{\alpha, \beta} & \longrightarrow & 0,
\end{array}
\]
where $\mu_i$ are isomorphisms. Then $\mu_2(w_1) = \mu_2(\varphi(1)) = \varphi' \mu_1(1) = \mu_1(1)w'_1$. Suppose $\mu_2(w_2) = aw'_1 + bw'_2$. Then $g\mu_2(w_2) = (a\alpha + bx')w'_1 + b\alpha w'_2$, and
\[
\mu_2(gw_2) = \mu_2(\alpha w_2 + x w_1) = (a\alpha + x \mu_1(1))w'_1 + b\alpha w'_2.
\]
Since $g\mu_2(w_2) = \mu_2(gw_2)$, $bx' = x \mu_1(1)$. Similarly, we have $by' = y \mu_1(1)$. Moreover,
\[
\mu_3(1) = \mu_3(\psi(w_2)) = \psi'\mu_2(w_2) = b.
\]
If $\mu_1(1) = \mu_3(1) = 1$, then $b = 1$ and $(x, y) = (x', y')$. Thus $M_{x, y} = M'_{x', y'}$ as elements in the group $\text{Ext}^1(\mathbb{K}_{\alpha, \beta}, \mathbb{K}_{\alpha, \beta})$ if and only if $(x, y) = (x', y')$. 

From Proposition 3.8, we know that $\text{Ext}^1(\mathbb{K}_{\alpha\beta}, \mathbb{K}_{\alpha\beta})$ is a vector space over $\mathbb{K}$. To describe the operations of the vector space $\text{Ext}^1(\mathbb{K}_{\alpha\beta}, \mathbb{K}_{\alpha\beta})$ in the terms of exact sequences, we use $I$ to denote the ideal of $R = \mathbb{K}[g^{-1}, h^{-1}]$ generated by $g - \alpha$ and $h - \beta$, i.e., $I = R(g - \alpha) + R(h - \beta)$. Let $\xi$ be the embedding homomorphism, and $f$ be the epimorphism of $R$-modules from $R$ to $\mathbb{K}_{\alpha\beta}$, given by

$$f(a(g, h)) = a(\alpha, \beta), \quad a(g, h) \in R.$$ 

Then we have the following exact sequence of $R$-modules:

$$0 \rightarrow I \xrightarrow{\xi} R \xrightarrow{f} \mathbb{K}_{\alpha\beta} \rightarrow 0.$$ 

Applying the functor $\text{Hom}_R(-, \mathbb{K}_{\alpha\beta})$ to the exact sequence (3.4) yields the exact sequence

$$\text{Hom}_R(R, \mathbb{K}_{\alpha\beta}) \xrightarrow{\varphi} \text{Hom}_R(I, \mathbb{K}_{\alpha\beta}) \xrightarrow{\partial} \text{Ext}^1(\mathbb{K}_{\alpha\beta}, \mathbb{K}_{\alpha\beta}) \rightarrow 0.$$ 

For any exact sequence of $R$-modules $0 \rightarrow \mathbb{K}_{\alpha\beta} \varphi M_{x,y} \psi \mathbb{K}_{\alpha\beta} \rightarrow 0$, and a basis $\{w_1, w_2\}$ of $M_{x,y}$ satisfying $\psi(w_2) = 1$, $w_1 = \varphi(1)$, define a homomorphism of $R$-modules

$$\sigma : R \rightarrow M_{x,y}, \quad \sigma(1) = w_2.$$ 

Let $\eta_{x,y}$ be a homomorphism of $R$-modules from $I$ to $\mathbb{K}_{\alpha\beta}$, where

$$\eta_{x,y}(a(g, h)(g - \alpha) + b(g, h)(h - \beta)) = xa(\alpha, \beta) + yb(\alpha, \beta),$$

for $a(g, h), b(g, h) \in R$. Now, we have the following commutative diagram:

$$\begin{array}{ccc}
0 & \xrightarrow{\xi} & I \\
\eta_{x,y} & \downarrow & \sigma \\
0 & \xrightarrow{\varphi} & M_{x,y} \\
\downarrow \id & & \downarrow \psi \\
0 & \xrightarrow{\psi} & \mathbb{K}_{\alpha\beta} \\
& & \downarrow \id \\
0 & & 0
\end{array}$$

It is easy to check that $M_{x,y}$ is the pushout of $\eta_{x,y}$ and $\xi$. If we use $M_{kx,ky}$ for any $k \in \mathbb{K}$ to replace $M_{x,y}$, we get a homomorphism $\eta_{kx,ky}$ from $I$ to $\mathbb{K}_{\alpha\beta}$. Similarly, we have a homomorphism $\eta_{x+x',y+y'}$ from $I$ to $\mathbb{K}_{\alpha\beta}$ by using $M_{x+x',y+y'}$ to replace $M_{x,y}$. From the definition of $\eta_{x,y}$, one obtains the following:

$$\eta_{kx,ky} = k\eta_{x,y}, \quad \eta_{x+x',y+y'} = \eta_{x,y} + \eta_{x',y'}.$$ 

Define

$$M_{x,y} \boxplus M_{x',y'} = M_{x+x',y+y'}, \quad k \boxplus M_{x,y} = M_{kx,ky},$$

for $k \in \mathbb{K}$. Then $\{M_{x,y}|x, y \in \mathbb{K}\}$ becomes a vector space over $\mathbb{K}$. By [12, Theorem 3.4.3], we have a bijection $\Psi_1$ from $\{M_{x,y}|x, y \in \mathbb{K}\}$ to $\text{Ext}^1(\mathbb{K}_{\alpha\beta}, \mathbb{K}_{\alpha\beta})$ such that

$$\Psi_1(M_{x,y}) = \partial(\eta_{x,y}) \in \text{Ext}^1(\mathbb{K}_{\alpha\beta}, \mathbb{K}_{\alpha\beta}).$$

It follows from (3.7) that

$$\Psi_1(M_{kx,ky}) = k\Psi_1(M_{x,y}), \quad \Psi_1(M_{x+x',y+y'}) = \Psi_1(M_{x,y}) + \Psi_1(M_{x',y'}).$$

Thus $\Psi_1$ is an isomorphism of vector spaces.
Proposition 3.9. Let $V_{\varepsilon,n}$ be a simple $U_q(\mathfrak{sl}(2))$-module with a basis $\{v_0, \cdots, v_n\}$ satisfying $E'v_0 = 0, E'v_p = \varepsilon[n - p + 1]v_{p-1}, v_p = \frac{F_p}{[p]}v_0$, for $p = 1, \cdots, n$; $F'v_n = 0, K'v_p = \varepsilon q^{-2p}v_p$ for $p = 0, \cdots, n$, where $E', K', F'$ are Chevalley generators of $U_q(\mathfrak{sl}(2))$. Then

$$V_{\varepsilon,n} \otimes M_{x,y} \in \text{Ext}^1(V_{\varepsilon,n,\alpha,\beta}, V_{\varepsilon,n,\alpha,\beta}),$$
where $M_{x,y} \in \text{Ext}^1(\mathbb{K}_{\alpha,\beta}, \mathbb{K}_{\alpha,\beta})$. The action of $U_{g,h}$ on $V_{\varepsilon,n} \otimes M_{x,y}$ with the basis

$$\{v_0 \otimes w_1, \cdots, v_n \otimes w_1; v_0 \otimes w_2, \cdots, v_n \otimes w_2\}$$

is given by

$$E(v_0 \otimes w_1) = E(v_0 \otimes w_2) = F(v_0 \otimes w_1) = F(v_n \otimes w_2) = 0,$$

$$E(v_p \otimes w_1) = E'(v \otimes gw_1 = \varepsilon[n - p + 1]v_{p-1} \otimes w_1,$$

$$E(v_p \otimes w_2) = \varepsilon \alpha[n - p + 1]v_{p-1} \otimes w_2 + \varepsilon[n - p + 1]v_{p-1} \otimes w_1,$$

for $p = 1, \cdots, n$;

$$v_p \otimes w_i = \frac{F_p}{[p]}v_1 \otimes w_i, \quad F(v_n \otimes w_i) = 0$$

for $p = 1, \cdots, n, i = 1, 2$;

$$K(v_p \otimes w_1) = K'(v_p \otimes gw_1 = \varepsilon q^{-2p}(v_p \otimes w_1),$$

$$K(v_p \otimes w_2) = \varepsilon q^{-2p}(v_p \otimes w_2) + \varepsilon q^{-2p}x(v_p \otimes w_1),$$

for $p = 0, 1, \cdots, n$; and

$$g(v_p \otimes w_1) = \alpha(v_p \otimes w_1), \quad g(v_p \otimes w_2) = \alpha(v_p \otimes w_2) + x(v_p \otimes w_1),$$

$$h(v_p \otimes w_1) = \beta(v_p \otimes w_1), \quad h(v_p \otimes w_2) = \beta(v_p \otimes w_2) + y(v_p \otimes w_1),$$

for $p = 0, 1, \cdots, n$. Moreover, $V_{\varepsilon,n} \otimes -$ is an injective linear mapping from the linear space $\text{Ext}^1(\mathbb{K}_{\alpha,\beta}, \mathbb{K}_{\alpha,\beta})$ to the linear space $\text{Ext}^1(V_{\varepsilon,n,\alpha,\beta}, V_{\varepsilon,n,\alpha,\beta})$.

Proof. We only need to prove that the mapping $V_{\varepsilon,n} \otimes -$ is an injective linear mapping, since it is easy to check the other results. Consider the following commutative diagram

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & V_{\varepsilon,n} \otimes \mathbb{K}_{\alpha,\beta} & \stackrel{\text{id} \otimes \varphi}{\longrightarrow} & V_{\varepsilon,n} \otimes M_{x,y} & \stackrel{\text{id} \otimes \psi}{\longrightarrow} & V_{\varepsilon,n} \otimes \mathbb{K}_{\alpha,\beta} & \longrightarrow & 0 \\
& & \text{id} & \downarrow & \mu & \text{id} & \downarrow & & \\
0 & \longrightarrow & V_{\varepsilon,n} \otimes \mathbb{K}_{\alpha,\beta} & \stackrel{\text{id} \otimes \varphi'}{\longrightarrow} & V_{\varepsilon,n} \otimes M'_{x,y} & \stackrel{\text{id} \otimes \psi'}{\longrightarrow} & V_{\varepsilon,n} \otimes \mathbb{K}_{\alpha,\beta} & \longrightarrow & 0.
\end{array}
\]

Since $(\text{id} \otimes \psi')(v_0 \otimes w_2) = (\text{id} \otimes \varphi)(v_0 \otimes w_2) = v_0 \otimes 1 = (\text{id} \otimes \psi')(v_0 \otimes w'_2)$,

$$\mu(v_0 \otimes w_2) = v_0 \otimes w'_2 + v \otimes w'_1$$

for some $v \in V_{\varepsilon,n}$. Then $g\mu(v_0 \otimes w_2) = \alpha(v_0 \otimes w'_2 + v \otimes w'_1) + x'v_0 \otimes w'_1$, and

$$\mu(g(v_0 \otimes w_2)) = \mu(\alpha v_0 \otimes w_2 + xv_0 \otimes w_1) = \alpha(v_0 \otimes w'_2 + v \otimes w'_1) + xv_0 \otimes w'_1.$$

Since $g\mu(v_0 \otimes w_2) = \mu(g(v_0 \otimes w_2))$, we have $x = x'$. Similarly, we can prove that $y = y'$. So $V_{\varepsilon,n} \otimes -$ induces an injective mapping from $\text{Ext}^1(\mathbb{K}_{\alpha,\beta}, \mathbb{K}_{\alpha,\beta})$ to $\text{Ext}^1(V_{\varepsilon,n,\alpha,\beta}, V_{\varepsilon,n,\alpha,\beta})$.

To prove $V_{\varepsilon,n} \otimes -$ is linear, we choose an exact sequence of $U_q(\mathfrak{sl}(2))$-modules

$$0 \longrightarrow L \longrightarrow P \overset{f}{\longrightarrow} V_{\varepsilon,n} \longrightarrow 0,$$
where $P$ is a finitely generated projective $U_q(\mathfrak{sl}(2))$-module. Then $P \otimes R$ is a projective $U_q$-module, and the kernel $Q$ of $F$ is Ker $f \otimes R + P \otimes I$, where $F$ is a homomorphism from $P \otimes R$ to $V_{\varepsilon,n} \otimes K_{\alpha,\beta}$ given by $F(a \otimes b) = f(a) \otimes b \cdot 1$.

\[ I = R(g - \alpha) + R(h - \beta). \]

Applying the functor $\text{Hom}_{U_q}(\cdot, V_{\varepsilon,n} \otimes K_{\alpha,\beta})$ to the exact sequence

\[ 0 \longrightarrow Q \longrightarrow P \otimes R \xrightarrow{F} V_{\varepsilon,n} \otimes K_{\alpha,\beta} \longrightarrow 0 \]

yields an exact sequence

\[ (3.12) \quad \text{Hom}_{U_q}(U_q, A) \xrightarrow{\tau} \text{Hom}_{U_q}(Q, A) \xrightarrow{\eta} \text{Ext}^1(A, A) \longrightarrow 0, \]

where $A = V_{\varepsilon,n} \otimes K_{\alpha,\beta}$. Define a homomorphism of $U_q$-modules $\sigma : P \otimes R \to V_{\varepsilon,n} \otimes M_{x,y}$ by

\[ \sigma(a \otimes b) = f(a) \otimes b \cdot w_2, \quad a \otimes b \in P \otimes R, \]

and a homomorphism of $U_q$-modules $\zeta_{x,y} : Q \to V_{\varepsilon,n} \otimes K_{\alpha,\beta}$ by

\[ \zeta_{x,y}(a \otimes b) = \begin{cases} 0, & a \otimes b \in \text{Ker } f \otimes L, \\ \eta_{x,y}(b)f(a) \otimes 1, & a \otimes b \in P \otimes (R(g - \alpha) + R(h - \beta)), \end{cases} \]

where $\eta_{x,y}$ is defined by (3.6). Let $\nu$ be the embedding mapping from $Q$ to $P \otimes R$. Then we have the following commutative diagram:

\[ \begin{array}{ccc}
0 & \longrightarrow & Q \\
\zeta_{x,y} & \downarrow & \sigma \\
0 & \longrightarrow & V_{\varepsilon,n} \otimes K_{\alpha,\beta}
\end{array} \xrightarrow{\varphi} \begin{array}{ccc}
P \otimes R & \longrightarrow & V_{\varepsilon,n} \otimes K_{\alpha,\beta} \\
\downarrow & & \downarrow \\
V_{\varepsilon,n} \otimes K_{\alpha,\beta} & \longrightarrow & V_{\varepsilon,n} \otimes K_{\alpha,\beta}
\end{array} \longrightarrow 0. \]

It is easy to check that $V_{\varepsilon,n} \otimes M_{x,y}$ is the pushout of $\zeta_{x,y}$ and $\nu$. If we use $M_{kx,ky}$ for any $k \in K$ to replace $M_{x,y}$, we will obtain a homomorphism $\zeta_{kx,ky}$ from $Q$ to $V_{\varepsilon,n} \otimes K_{\alpha,\beta}$. Similarly, we get a homomorphism $\zeta_{x+x',y+y'}$ from $Q$ to $V_{\varepsilon,n} \otimes K_{\alpha,\beta}$ by using $M_{x+x',y+y'}$ to replace $M_{x,y}$. From the definitions of these mappings and (3.7), we obtain the following

\[ (3.13) \quad \zeta_{kx,ky} = k\zeta_{x,y}, \quad \zeta_{x+x',y+y'} = \zeta_{x,y} + \zeta_{x',y'} \cdot \]

We abuse notation and write $V_{\varepsilon,n} \otimes M_{x,y}$ for the following exact sequence

\[ 0 \longrightarrow V_{\varepsilon,n} \otimes K_{\alpha,\beta} \xrightarrow{1 \otimes \varphi} V_{\varepsilon,n} \otimes M_{x,y} \xrightarrow{1 \otimes \psi} V_{\varepsilon,n} \otimes K_{\alpha,\beta} \longrightarrow 0. \]

Define

\[ (V_{\varepsilon,n} \otimes M_{x,y}) \boxplus (V_{\varepsilon,n} \otimes M_{x',y'}) = V_{\varepsilon,n} \otimes M_{x+x',y+y'} \quad k \boxplus (V_{\varepsilon,n} \otimes M_{x,y}) = V_{\varepsilon,n} \otimes M_{kx,ky}, \]

for $k \in K$. Then $\{V_{\varepsilon,n} \otimes M_{x,y}|x, y \in K\}$ becomes a vector space with the above operations. By [12, Theorem 3.4.3], we have an injective linear mapping $\Psi_2$ of linear spaces from

\[ \{V_{\varepsilon,n} \otimes M_{x,y}|x, y \in K\} \]

to $\text{Ext}^1(V_{\varepsilon,n} \otimes K_{\alpha,\beta}, V_{\varepsilon,n} \otimes K_{\alpha,\beta})$ such that

\[ \Psi_2(V_{\varepsilon,n} \otimes M_{x,y}) = \partial(\zeta_{x,y}) \in \text{Ext}^1(V_{\varepsilon,n} \otimes K_{\alpha,\beta}, V_{\varepsilon,n} \otimes K_{\alpha,\beta}). \]
Therefore,  

\[ \Psi_2(V_{\varepsilon, n} \otimes M_{kx, ky}) = \partial(k\zeta_{x, y}) = k\Psi_2(V_{\varepsilon, n} \otimes M_{x, y}) \]

and

\[ \Psi_2(V_{\varepsilon, n} \otimes M_{x+y', y'+y'}) = \Psi_2(V_{\varepsilon, n} \otimes M_{x, y'}) + \Psi_2(V_{\varepsilon, n} \otimes M_{x', y'}) \]

by (3.13). Since \( \Psi_2 \) is injective and linear,

\[ V_{\varepsilon, n} \otimes k \square M_{x, y} = V_{\varepsilon, n} \otimes M_{kx, ky} = k \square (V_{\varepsilon, n} \otimes M_{x, y}) \]

and

\[ V_{\varepsilon, n} \otimes (M_{x, y} \oplus M_{x', y'}) = V_{\varepsilon, n} \otimes M_{x+x', y+y'} = (V_{\varepsilon, n} \otimes M_{x, y}) \oplus (V_{\varepsilon, n} \otimes M_{x', y'}). \]

By now, we have completed the proof. \( \square \)

We now completely classify all extensions between two finite-dimensional simple \( U_{g,h} \)-modules.

**Theorem 3.10.** Suppose \( q \) is not a root of unity. Given two simple \( \mathbb{K}[g^{\pm 1}, h^{\pm 1}] \)-modules \( \mathbb{K}_{\alpha, \beta}, \mathbb{K}_{\alpha', \beta'} \) and a finite-dimensional simple \( U_q(\mathfrak{sl}(2)) \)-module \( V_{\varepsilon, n} \), the assignment \( V_{\varepsilon, n} \otimes - \) is an isomorphism of vector spaces from \( \text{Ext}^1(\mathbb{K}_{\alpha', \beta'}, \mathbb{K}_{\alpha, \beta}) \) to \( \text{Ext}^1(M', M) \). Here,  

\[ M := V_{\varepsilon, n} \otimes \mathbb{K}_{\alpha, \beta} \cong V_{\varepsilon, n, \alpha, \beta}, \quad M' := V_{\varepsilon, n} \otimes \mathbb{K}_{\alpha', \beta'} \cong V_{\varepsilon, n, \alpha', \beta'}. \]

Moreover, \( \text{Ext}^1(V_{\varepsilon, m, \alpha, \beta}, V_{\varepsilon, n, \alpha', \beta'}) = 0 \) provided that \( (\varepsilon, m, \alpha, \beta) \neq (\varepsilon, n, \alpha', \beta') \).

**Proof.** Let \( C \) be the Casimir element of \( U_{g,h} \) defined in Corollary 2.4,

\[ d_{m,n} = \frac{q^{m+1}}{(q^{m-n} - \varepsilon \varepsilon')(q^{m+n+2} - \varepsilon \varepsilon')} \left(C - \varepsilon' \alpha' q^{n+1} + q^{-n-1}\right), \]

and

\[ a = \begin{cases} \frac{h-\beta'}{\beta-\beta'}, & \text{if } \beta \neq \beta' \\ \frac{q-q^{-1}}{\alpha - \alpha'}, & \text{if } \alpha \neq \alpha' \\ \frac{\varepsilon(q-q^{-1})^2}{\alpha} d_{m,n}, & \text{otherwise.} \end{cases} \]

Then \( a \) is in the center of \( U_{g,h} \) by Corollary 2.4.

Observe that \( V_{\varepsilon, m, \alpha, \beta} \cong V_{\varepsilon, n, \alpha', \beta'} \) if and only if \( \varepsilon = \varepsilon', \alpha = \alpha', \beta = \beta' \), and \( m = n \).

Suppose \( V_{\varepsilon, m, \alpha, \beta} \) is not isomorphic to \( V_{\varepsilon, n, \alpha', \beta'} \), then \( (\varepsilon, m, \alpha, \beta) \neq (\varepsilon, n, \alpha', \beta') \). Let \( v \in V_{\varepsilon, m, \alpha, \beta} \) be a nonzero highest weight vector satisfying

\[ Kv = \varepsilon \alpha q^m v, \quad gv = \alpha v, \quad hv = \beta v, \quad Ev = 0. \]

Then

\[ d_{m,n} v = \frac{q^{m+1}}{q^{m+2} - \varepsilon \varepsilon' q^{m+1} - \varepsilon \varepsilon' q^{m-n+1}} \left(FE + \frac{qK+q^{-1}K^{-1}g^2}{(q-q^{-1})^2} - \varepsilon' \alpha' q^{n+1} + q^{-n-1}\right) v \]

\[ = \frac{\varepsilon \alpha}{(q-q^{-1})^2} v, \]

in the case \( \alpha' = \alpha \). Therefore \( am = m \) for any \( m \in V_{\varepsilon, m, \alpha, \beta} \) by Schur’s Lemma, since \( V_{\varepsilon, m, \alpha, \beta} \) is a simple module and \( a \) induces an endomorphism of \( V_{\varepsilon, m, \alpha, \beta} \). Similarly, we can prove that \( am = 0 \) for any \( m \in V_{\varepsilon, n, \alpha', \beta'} \).
Consider the short exact sequence of $U_{g,h}$-modules

\begin{equation}
\begin{array}{c}
0 \longrightarrow V_{\varepsilon,m,\alpha,\beta} \overset{\phi}{\longrightarrow} V \overset{\varphi}{\twoheadrightarrow} V_{\varepsilon,n,\alpha',\beta'} \longrightarrow 0.
\end{array}
\end{equation}

Since $a\varphi(V) = \varphi(aV) = 0,$

\[\phi(V_{\varepsilon,m,\alpha,\beta}) = \text{Ker } \varphi \supseteq aV \supseteq a\phi(V_{\varepsilon,m,\alpha,\beta}) = \phi(aV_{\varepsilon,m,\alpha,\beta}) = \phi(V_{\varepsilon,m,\alpha,\beta}).\]

So $\phi(V_{\varepsilon,m,\alpha,\beta}) = aV.$ In particular, $a(\varphi v) = av$ for any $v \in V.$ Therefore

\[V = \text{Ker } a \oplus aV = \text{Ker } a \oplus \phi(V_{\varepsilon,m,\alpha,\beta}).\]

Hence the sequence (3.14) is splitting and $\text{Ext}^1(V_{\varepsilon,n,\alpha',\beta'}, V_{\varepsilon,m,\alpha,\beta}) = 0.$

Suppose $(\alpha, \beta) \neq (\alpha', \beta').$ Then $\text{Ext}^1(M', M) = 0$ and $\text{Ext}^1(\mathbb{K}_{\alpha',\beta'}, \mathbb{K}_{\alpha,\beta}) = 0$ by Proposition 3.8. It is trivial that $V_{\varepsilon,n} \otimes -$ is an isomorphism of linear spaces.

Next, we assume that $V_{\varepsilon,n,\alpha',\beta'} \cong V_{\varepsilon,m,\alpha,\beta} \cong V_{\varepsilon,n} \mathbb{K}_{\alpha,\beta}.$ Consider the following exact sequence of $U_{g,h}$-modules

\begin{equation}
\begin{array}{c}
0 \longrightarrow M \overset{\phi}{\longrightarrow} V \overset{\varphi}{\twoheadrightarrow} M \longrightarrow 0.
\end{array}
\end{equation}

Since $U_q(\mathfrak{sl}(2))$ is a subalgebra of $U_{g,h},$ we can regard the exact sequence (3.15) as a sequence of $U_q(\mathfrak{sl}(2))$-modules. Since every finite-dimensional $U_q(\mathfrak{sl}(2))$-module is semisimple, there is a homomorphism $\lambda$ of $U_q(\mathfrak{sl}(2))$-modules from $M$ to $V$ such that $\varphi \lambda = \text{id}_M.$ For any $v \in V,$ we have $v = (v - \lambda\varphi(v)) + \lambda\varphi(v).$ Moreover, $\varphi(v - \lambda\varphi(v)) = 0.$ Hence

\[V = \text{Ker } \varphi \oplus \text{Im } \lambda = \text{Im } \phi \oplus \text{Im } \lambda,\]

where $\text{Im } \lambda \cong V_{\varepsilon,n}$ as $U_q(\mathfrak{sl}(2))$-modules. Let $K' = Kg^{-1}.$ Suppose $u_1, u_2$ are the highest weight vectors of the $U_{g,h}$-module $\text{Im } \phi$ and the $U_q(\mathfrak{sl}(2))$-module $\text{Im } \lambda$ respectively. Then

\[\left\{ \frac{F^i}{[i]!}u_1, \frac{F^i}{[i]!}u_2 | i = 0, \ldots, n \right\}\]

is a basis of $V.$ Moreover,

\[K\varphi(u_2) = g\varphi(K'u_2) = \varepsilon\alpha q^n\varphi(u_2), \quad E\varphi(u_2) = g\varphi(Eg^{-1}u_2) = 0,\]

\[g\varphi(u_2) = \alpha\varphi(u_2), \quad h\varphi(u_2) = \beta\varphi(u_2).\]

So $\varphi(u_2)$ is a highest weight vector of $M.$ Suppose $gu_2 = \sum_{i=0}^{n} a_i \frac{1}{[i]!}F^iu_1 + \sum_{i=0}^{n} x_i \frac{1}{[i]!}F^iu_2.$ Then

\begin{equation}
\begin{array}{c}
\varepsilon q^n gu_2 = gK'u_2 = K'gu_2 = \varepsilon\left( \sum_{i=0}^{n} q^{n-2i}a_i \frac{1}{[i]!}F^iu_1 + \sum_{i=0}^{n} q^{n-2i}x_i \frac{1}{[i]!}F^iu_2.\right).
\end{array}
\end{equation}

Since $q^n \neq 1$ for any positive integer $m,$ we obtain $a_i = x_i = 0,$ $i = 1, 2, \ldots, n$ from (3.16). Hence $gu_2 = a_0u_2 + x_0u_1.$ Moreover, $a_0\varphi(u_2) = \varphi(gu_2) = g\varphi(u_2) = \alpha\varphi(u_2).$ So $a_0 = \alpha.$ Similarly, we can prove that $hu_2 = \beta u_2 + y_0u_1.$ Moreover, by using Lemma 3.7, one can prove that

\begin{equation}
\begin{array}{c}
\begin{cases}
E(u_1) = E(u_2) = F\left(\frac{1}{[n]!}F^n u_1\right) = F\left(\frac{1}{[n]!}F^n u_2\right) = 0, \\
E\left(\frac{1}{[n]!}F^n u_1\right) = \varepsilon[n - p + 1]\alpha F^{p-1} u_1, \\
E\left(\frac{1}{[n]!}F^n u_2\right) = \varepsilon\alpha[n - p + 1]F^{p-1} u_2 + \varepsilon[n - p + 1]x_0 F^{p-1} u_1,
\end{cases}
\end{array}
\end{equation}
for \( p = 1, \ldots, n; \)
\[
(3.18) \quad K\left( \frac{F^p}{[p]!} u_2 \right) = \varepsilon q^{n-2p} \frac{F^p}{[p]!} u_2 + \varepsilon q^{n-2p} x_0 \frac{F^p}{[p]!} u_1,
\]
for \( p = 0, 1, \ldots, n; \) and
\[
(3.19) \quad \begin{cases} 
  g\left( \frac{F^p}{[p]!} u_1 \right) = \alpha \frac{F^p}{[p]!} u_1, & g\left( \frac{F^p}{[p]!} u_2 \right) = \alpha \frac{F^p}{[p]!} u_2 + x_0 \frac{F^p}{[p]!} u_1, \\
  h\left( \frac{F^p}{[p]!} u_1 \right) = \beta \frac{F^p}{[p]!} u_1, & h\left( \frac{F^p}{[p]!} u_2 \right) = \beta \frac{F^p}{[p]!} u_2 + y_0 \frac{F^p}{[p]!} u_1,
\end{cases}
\]
for \( p = 0, 1, \ldots, n. \)

Define \( \tau(\frac{p}{n!} u_j) = v_i \otimes w_j \) for \( i = 0, 1, \ldots, n; j = 1, 2, \) and extend it by linearity. Comparing the relations from (3.8) to (3.11) in Proposition 3.9 with the above relations from (3.17) to (3.19), we know that \( \tau \) is an isomorphism of \( U_{g,h} \)-modules from \( V \) to \( V_{\varepsilon,n} \otimes M_{x_0,y_0}. \) Hence \( V_{\varepsilon,n} \otimes - \) is an isomorphism of linear spaces by Proposition 3.9. \( \square \)

**Remark 3.11.** Since \( \text{Ext}^1(V_{\varepsilon,n}, V_{\varepsilon,n}) = 0 \) and \( \text{Ext}^1(V_{\varepsilon,n} \otimes \mathbb{K}_{\alpha,\beta}, V_{\varepsilon,n} \otimes \mathbb{K}_{\alpha,\beta}) \neq 0, \) the functor \( - \otimes \mathbb{K}_{\alpha,\beta} \) is the zero mapping from \( \text{Ext}^1(V_{\varepsilon,n}, V_{\varepsilon,n}) \) to \( \text{Ext}^1(V_{\varepsilon,n} \otimes \mathbb{K}_{\alpha,\beta}, V_{\varepsilon,n} \otimes \mathbb{K}_{\alpha,\beta}). \) Hence the functor \( - \otimes \mathbb{K}_{\alpha,\beta} \) does not induce an isomorphism.

Since \( U_{g,h} \) is a Hopf algebra, the dual \( M^* \) of any \( U_{g,h} \)-module \( M \) is still a \( U_{g,h} \) module. For \( a \in U_{g,h}, \ f \in M^*, \) the action of \( a \) on \( f \) is given by
\[
(af)(m) := f((Sa)m), \quad m \in M,
\]
where \( S \) is the antipode of \( U_{g,h}. \) Next we describe the dual module of a simple module over \( U_{g,h}. \)

**Theorem 3.12.** The dual module \( V_{\varepsilon,n,\alpha,\beta}^* \) of the simple \( U_{g,h} \)-module \( V_{\varepsilon,n,\alpha,\beta} \) is a simple module, and \( V_{\varepsilon,n,\alpha,\beta}^* \cong V_{\varepsilon,n,\alpha^{-1},\beta^{-1}}. \)

**Proof.** By Theorem 3.5, we can assume that the simple module \( V_{\varepsilon,n,\alpha,\beta} \) has a basis \( \{v_0, \ldots, v_n\} \) with relations:
\[
K v_p = \varepsilon q^{n-2p} \alpha v_p, \quad g v_p = \alpha v_p, \quad h v_p = \beta v_p
\]
for \( p = 0, 1, \ldots, n, \)
\[
F v_n = 0, \quad E v_0 = 0
\]
and
\[
E v_p = \frac{q^{n-(p-1)} \alpha - q^{p-1-n} \alpha}{q - q^{-1}} v_{p-1} = \varepsilon \alpha [n - p + 1] v_{p-1}
\]
for \( p = 1, \ldots, n. \) Let \( \{v_0^*, \ldots, v_n^*\} \) be the dual basis of \( \{v_0, \ldots, v_n\}. \) Then
\[
(E v_n^*)(v_i) = -v_n^*(EK^{-1} v_i) = -q^{2i-n} [n - i + 1] v_n^*(v_{i-1}) = 0
\]
for \( i = 1, \ldots, n, \) and
\[
(E v_n^*)(v_0) = -v_n^*(EK^{-1} v_0) = -\varepsilon^{-1} q^{-n} v_n^*(0) = 0.
\]
Hence \( E(v_n^*) = 0. \) Since
\[
(K v_n^*)(v_i) = v_n^*(K^{-1} v_i) = q^{2i-n} \varepsilon^{-1} v_n^*(v_i) = \delta_{ni} q^n \varepsilon^{-1}
\]
for \( i = 0, 1, \cdots, n \), \( Kv^*_n = \varepsilon \alpha^{-1} q^n v^*_n \). Similarly, that \( gv^*_n = \alpha^{-1} v^*_n \) follows from
\[
(gv^*_n)(v_i) = v^*_n(g^{-1}v_i) = \alpha^{-1} v^*_n(v_i)
\]
for \( i = 0, 1, \cdots, n \), and that \( hv^*_n = \beta^{-1} v^*_n \) follows from
\[
(hv^*_n)(v_i) = v^*_n(h^{-1}v_i) = \beta^{-1} v^*_n(v_i)
\]
for \( i = 0, 1, \cdots, n \). So \( V^*_{\varepsilon,n,\alpha,\beta} \) is a simple \( U_{g,h} \)-module generated by the highest weight vector \( v^*_n \) with weight \((\varepsilon \alpha^{-1} q^n, \alpha^{-1}, \beta^{-1})\). Hence \( V^*_{\varepsilon,n,\alpha,\beta} \cong V^*_{\varepsilon,n,\alpha^{-1},\beta^{-1}} \). \( \square \)

Let \( H \) be a Hopf algebra, and \( H^0 = \{ f \in H^* | \ker f \) contains an ideal I\} such that the dimension of \( H/I \) is finite\}. Then \( H^0 \) is a Hopf algebra, which is called the finite dual Hopf algebra of \( H \). Now let \( M \) be a left module over the Hopf algebra \( U_{g,h} \). For any \( f \in M^* \) and \( v \in M \), define a coordinate function \( c^M_{f,v} \in U_{g,h}^* \)
\[
c^M_{f,v}(x) = f(xv) \quad \text{for } x \in H.
\]
If \( M \) is finite dimensional, then \( c^M_{f,v} \in U_{g,h}^0 \), the finite dual Hopf algebra of \( U_{g,h} \). The coordinate space \( C(M) \) of \( M \) is a linear subspace of \( U_{g,h}^* \), spanned by the coordinate functions \( c^M_{f,v} \) as \( f \) runs over \( M^* \) and \( v \) over \( M \).

**Corollary 3.13.** Let \( A \) be the subalgebra of \( U_{g,h}^0 \) generated by all the coordinate functions of all finite dimensional simple \( U_{g,h} \)-modules. Then \( A \) is a sub-Hopf algebra of \( U_{g,h}^0 \).

**Proof.** Let \( \mathcal{C} \) be the subcategory of the left \( U_{g,h} \)-module category consisting of all finite direct sums of finite dimensional simple \( U_{g,h} \)-modules. Then \( \mathcal{C} \) is closed under tensor products and duals by Corollary 3.6 and Theorem 3.12. Thus \( A \) is a sub-Hopf algebra of \( U_{g,h}^0 \), and is the directed union of the coordinate spaces \( C(V) \) for \( V \in \mathcal{C} \) by [2, Corollary I.7.4]. \( \square \)

Finally, we describe the simple modules over \( U_{g,h} \) when \( q \) is a root of unity. Assume that the order of \( q \) is \( d > 2 \) and define
\[
e = \begin{cases} 
d, & \text{if } d \text{ is odd}, \\
\frac{d}{2}, & \text{otherwise}.
\end{cases}
\]
We will use the notations \( V(\lambda, a, b), V(\lambda, a, 0), \tilde{V}(\pm q^{1-j}, c) \) to denote finite-dimensional simple \( U_q(\mathfrak{sl}(2)) \)-modules. These simple modules have been described in [6, Theorem VI.5.5]. The next results follow from [6, Proposition VI.5.1, Proposition VI.5.2, Theorem VI.5.5] and [8, Proposition 16.1].

**Proposition 3.14.** Suppose \( q \) is a root of unity. Then
\( (1) \) Any simple \( U_{g,h} \)-module of dimension \( e \) is isomorphic to a module of the following list:
\( (i) \ K_{\alpha,\beta} \otimes V(\lambda, a, b), \) where \( K_{\alpha,\beta} = K \cdot 1 \) is a one-dimensional module over \( K[\pm g^1, h^1] \), and \( g \cdot 1 = \alpha, h \cdot 1 = \beta \) for some \( \alpha, \beta \in K^\times \).
\( (ii) \ K_{\alpha,\beta} \otimes V(\lambda, a, 0), \) where \( \lambda \) is not of the form \( \pm q^{i-1} \) for any \( 1 \leq j \leq e-1 \),
\( (iii) \ K_{\alpha,\beta} \otimes \tilde{V}(\pm q^{1-j}, c) \).

\( (2) \) Any simple \( U_{g,h} \)-module of dimension \( n < e - 1 \) is isomorphic to a module of the form \( V_{\varepsilon,n,\alpha,\beta} \), where the structure of \( V_{\varepsilon,n,\alpha,\beta} \) is given by Theorem 3.5.
(3) The dimension of any simple $U_{g,h}$-module is not larger than $e$.

4. **Verma modules and the category $O$**

In this section, we assume that the nonzero element $q \in \mathbb{K}$ is not a root of unity. We will study the BGG subcategory of the category of all left $U_{g,h}$-modules. For the undefined terms in this section, we refer the reader to [8] and [10].

If $M$ is a $U_{g,h}$-module, a maximal weight vector is any nonzero $m \in M$ that is killed by $E$, and is a common eigenvector for $K, g, h$. A standard cyclic module is one which is generated by exactly one maximal weight vector. For each $(a, b, c) \in \mathbb{K} \times \mathbb{K}$, define the Verma module

$$V(a, b, c) := U_{g,h}/I(a, b, c),$$

where $I(a, b, c)$ is the left ideal of $U_{g,h}$ generated by $E, K - a, g - b, h - c$. $V(a, b, c)$ is a free $\mathbb{K}[F]$-module of rank one, by the PBW Theorem 2.1 for $U_{g,h}$. Hence the set $W(V(a, b, c))$ of weights of the Verma module $V(a, b, c)$ is equal to \{(q^{-2n}a, b, c)|n \geq 0\}.

About the extension group $\text{Ext}^1(V(a', b', c'), V(a, b, c))$ of two Verma modules $V(a', b', c')$, $V(a, b, c)$, we have the following:

**Proposition 4.1.** Suppose $V(a, b, c)$ and $V(a', b', c')$ are two Verma modules. Then $\text{Ext}^1((V(a, b, c), V(a, b, c)) \neq 0$ and $\text{Ext}^1((V(a', b', c'), V(a, b, c)) = 0$ if $a, b, c; a', b', c'$ satisfy one of the following conditions.

1. $(b, c) \neq (b', c')$;
2. $(b, c) = (b', c')$, $a \neq a'$ and $aa' \neq q^{-2}b^2$.

**Proof.** Let $M_{x,y} \in \text{Ext}^1(\mathbb{K}_{b,c}, \mathbb{K}_{b,c})$ be the module described in Proposition 3.9, where either $x \neq 0$ or $y \neq 0$. Consider the $U_{g,h}$-module $M = V(ab^{-1}) \otimes M_{x,y}$, where $V(ab^{-1})$ is a Verma module over $U_q(\mathfrak{sl}(2))$ generated by a highest weight vector $v$ with weight $ab^{-1}$. Suppose $w_1, w_2$ is a basis of $M_{x,y}$ such that $gw_1 = bw_1$, $gw_2 = bw_2 + xw_1$, $hw_1 = cw_1$, $hw_2 = cw_2 + yw_1$. Then $K(v \otimes w_1) = a(v \otimes w_1)$ and

$$K(v \otimes w_2) = a(v \otimes w_2) + ab^{-1}x(v \otimes w_1).$$

Therefore the subspace $V_1$ of $M$ generated by

$$\frac{F^n}{n!}v \otimes w_1, \quad n \in \mathbb{Z}_{\geq 0}$$

is a $U_{g,h}$-module, which is isomorphic to $V(a, b, c)$. Moreover $M/V_1$ is also isomorphic to $V(a, b, c)$. Thus $M \in \text{Ext}^1(V(a, b, c), V(a, b, c))$. Suppose $M \cong V(a, b, c) \oplus V(a, b, c)$. Then the actions of $g, h$ on $M$ are given via multiplications by $b, c$ respectively. This is impossible when either $x \neq 0$, or $y \neq 0$. So $M$ is a nonzero element in $\text{Ext}^1(V(a, b, c), V(a, b, c))$.

Now let

$$u = \begin{cases} \frac{g^{-b'}b}{b-b'}, & \text{if } b \neq b' \\ \frac{h^{-c'}}{h-h'}, & \text{if } c \neq c' \\ \frac{(a-a')(q^{-1}g^{-1}h^{-1})^2}{(a-a')(q^{-1}g^{-1}h^{-1})^2} \left(C - \frac{aa'g^{-1}h^{-1}b^2}{(q^{-1}g^{-1}h^{-1})^2}\right), & \text{if } a \neq a', aa' \neq q^{-2}b^2, (b, c) = (b', c'), \end{cases}$$

where $C$, which is given in Corollary 2.4, is the Casimir element of $U_{g,h}$. Then $u$ is in the center of $U_{g,h}$ by Corollary 2.4. Suppose $V(a, b, c)$ and $V(a', b', c')$ are generated by the
Suppose \( V(a,b,c) \) is a Verma module over \( K \). We determine when the Verma module \( V(a,b,c) \) is a simple module \( L(a,b,c) \).

**Remark 4.2.** It is unknown whether \( \text{Ext}^1(V(q^{-2}a^{-1}b^2, b, c), V(a,b,c)) = 0 \) in the case when \( b^2 \neq q^2a^2 \).

The proof of the following proposition is standard (see e.g. [5], [7] or [8]).

**Proposition 4.3.** (1) The Verma module \( V(a,b,c) \) has a unique maximal submodule \( N(a,b,c) \), and the quotient \( V(a,b,c)/N(a,b,c) \) is a simple module \( L(a,b,c) \).

(2) Any standard cyclic module is a quotient of some Verma module.

By [8, Theorem 4.2] and Proposition 2.2, every Verma module over \( U_{g,h} \) is isomorphic to \( V(\lambda) \otimes K_{b,c} \), where \( V(\lambda) \) is a Verma module over \( U_q(\mathfrak{sl}(2)) \). Conversely, \( V(\lambda) \otimes K_{b,c} \) is a Verma \( U_{g,h} \) module if \( V(\lambda) \) is a Verma module over \( U_q(\mathfrak{sl}(2)) \). In the following, we determine when the Verma module \( V(\lambda) \otimes K_{b,c} \) is isomorphic to the Verma module \( V(a,b,c) \), using the isomorphism in Proposition 2.2(1).

**Proposition 4.4.** Suppose \( V(\lambda) \) is a Verma module over \( U_q(\mathfrak{sl}(2)) \) and \( K_{b,c} \) is a simple module over \( K[\hat{g}^{\pm1}, \hat{h}^{\pm1}] \). Then \( V(\lambda) \otimes K_{b,c} \) is a Verma module over \( U_{g,h} \) with the highest weight \( (b\lambda, b, c) \). Conversely, every Verma module \( V(a,b,c) \) over \( U_{g,h} \) is isomorphic to

\[
V(ab^{-1}) \otimes K_{b,c},
\]

where \( V(ab^{-1}) \) is a Verma module over \( U_q(\mathfrak{sl}(2)) \).

Therefore the Verma module \( V(a,b,c) \) is isomorphic to \( V(\lambda) \otimes K_{b',c'} \) if and only if \( (a,b,c) = (b',\lambda', b', c') \).

**Proof.** Suppose \( E', K', F' \) are Chevalley generators of \( U_q(\mathfrak{sl}(2)) \). Let \( V(\lambda) \) be a Verma module over \( U_q(\mathfrak{sl}(2)) \). Then \( V(\lambda) \) has a basis \( \{v_p | p \in \mathbb{Z}_{\geq 0}\} \) satisfying

\[
K'v_p = \lambda q^{-2p}v_p, \quad K'^{-1}v_p = \lambda^{-1}q^{2p}v_p,
\]

\[
E'v_{p+1} = \frac{q^{-p}\lambda - q^p\lambda^{-1}}{q-q^{-1}}v_p, \quad F'v_p = [p+1]v_{p+1}
\]

and \( E'v_0 = 0 \). Since \( U_{g,h} \cong U_q(\mathfrak{sl}(2)) \otimes K[\hat{g}^{\pm1}, \hat{h}^{\pm1}] \), then \( V(\lambda) \otimes K_{b,c} \) is a cyclic module with the highest vector \( v_0 \otimes 1 \), where the action of \( x \otimes y \in U_q(\mathfrak{sl}(2)) \otimes K[\hat{g}^{\pm1}, \hat{h}^{\pm1}] \) on \( v \otimes 1 \in V(\lambda) \otimes K_{b,c} \) is given by

\[
(x \otimes y) \cdot (v \otimes 1) = x \cdot v \otimes y \cdot 1.
\]

The highest weight of \( V(\lambda) \otimes K_{b,c} \) is \((b\lambda, b, c)\). Let \( v = 1 + I(b\lambda, b, c) \) be the highest weight vector of the Verma module \( V(b\lambda, b, c) \). Define a linear map \( f \) from \( V(\lambda) \otimes K_{b,c} \) to \( V(a,b,c) \) by \( f(v_p \otimes 1) = \frac{1}{[p]!}F^p v \). Similar to [6, Proposition VI.3.7], we can prove that \( f \) is a homomorphism of \( U_{g,h} \)-modules. Therefore \( V(\lambda) \otimes K_{b,c} \) is the Verma module with highest weight \((b\lambda, b, c)\) by Proposition 4.3(2).
Conversely, let \( \lambda = a/b^{-1} \). Consider an infinite-dimensional vector space \( V(\lambda) \) with basis \( \{v_i | i \in \mathbb{Z}_{\geq 0}\} \). For \( p \geq 0 \), set

\[
K'v_p = \lambda q^{-2p}v_p, \quad K'^{-1}v_p = \lambda^{-1}q^{2p}v_p,
\]

\[
E'v_{p+1} = \frac{q^{-p}a - q^p\lambda^{-1}}{q - q^{-1}}v_p, \quad F'v_p = [p + 1]v_{p+1}
\]

and \( E'v_0 = 0 \), where \( E', K', F' \) are Chevalley generators of \( U_q(\mathfrak{sl}(2)) \). Then \( V(\lambda) \) is a Verma module over \( U_q(\mathfrak{sl}(2)) \) with the above actions by [6, Lemma VI.3.6]. The highest weight of \( V(\lambda) \otimes K_{b,c} \) is \( (a, b, c) \). Therefore \( V(\lambda) \otimes K_{b,c} \) is isomorphic to the Verma module over \( U_{g,h} \) with highest weight \( (a, b, c) \). \( \Box \)

One of the basic questions about a Verma module is to determine its maximal weight vectors. We now answer this question.

**Theorem 4.5.** Let \( V(a, b, c), V(a', b', c') \) be two Verma modules, where \( a, b, c; a', b', c' \in \mathbb{K}^\times \).

1. If \( V(a, b, c) \) has a maximal weight vector of weight \( q^{-2n}a, b, c \), then it is unique up to scalars and \( a = \varepsilon bq^{n-1} \) with \( n > 0 \).
2. \( \dim_{\mathbb{K}} \text{Hom}_{U_{g,h}}(V(a', b', c'), V(a, b, c)) = 0 \) or \( 1 \) for all \( (a', b', c') \) and \( (a, b, c) \), and all nonzero homomorphisms between two Verma modules are injective.
3. The nonzero submodule of \( V(a, b, c) \) (which is unique if it exists) is precisely of the form

\[
V(q^{-2n}a, b, c) = \mathbb{K}[F]v_{q^{-2n}a,b,c}.
\]

**Proof.** Suppose \( p(F) = (a_nF^n + a_{n-1}F^{n-1} + \cdots + a_0)\bar{1} \) is a maximal weight vector, where \( \bar{1} \) is the maximal weight vector of \( V(a, b, c) \) and \( a_n \neq 0 \). Then

\[
E(p(F)) = [n]q^{n+1}a - q^{n-1}a^{-1}b^2a_nF^{n-1}\bar{1} + \text{(lower degree terms)}\bar{1} = 0,
\]

by Lemma 3.7. This implies \( a = \varepsilon bq^{n-1} \). Moreover \( p(X) = a_nF^n\bar{1} \) and (1) follows.

(2) follows from (1) and the fact that \( \mathbb{K}[F] \) is a principal ideal domain directly.

If \( M \) is a nonzero submodule of \( V(a, b, c) \), then \( M \) contains a vector of the highest possible weight \( (q^{-2n}a, b, c) \). We claim that \( M = V(q^{-2n}a, b, c) = \mathbb{K}[F]v_{q^{-2n}a,b,c} \), where \( v_{q^{-2n}a,b,c} \) is the weight vector in \( M \) with weight \( (q^{-2n}a, b, c) \). The weight vector \( v_{q^{-2n}a,b,c} \) is unique up to scalar by (1). To prove the above claim, we only need to show that \( M \subseteq \mathbb{K}[F]v_{q^{-2n}a,b,c} \).

Suppose, to the contrary, that \( v \in M \) is of the form

\[
v = p(F)v_{q^{-2n}a,b,c} + a_{n-1}F^{n-1}\bar{1} + \cdots + a_1F\bar{1} + a_0\bar{1}.
\]

We may assume that \( p(F) = 0 \) because \( v_{q^{-2n}a,b,c} \in M \). Since \( K^iv \in M \) for any \( i \), \( a_{n-k}F^{n-k}\bar{1} \in M, k = 1, 2, \cdots, n \). This is a contradiction since \( (q^{-2i}a, b, c) \) is not a weight of \( M \) if \( i < n \). \( \Box \)

**Remark 4.6.** (1) If \( \frac{a}{b} \neq \varepsilon q^n \) for any \( n \geq 1 \), then the Verma module is a simple module by Theorem 4.5.
(2) It is well-known that the Verma module $V(\lambda)$ over $U_q(\mathfrak{sl}(2))$ is simple provided that $\lambda \neq \varepsilon q^n$ for any integer $n > 0$, where $\varepsilon = \pm 1$. Since

$$V(\lambda) \otimes \mathbb{K}_{b,c} \cong V(b\lambda, b, c)$$

by Proposition 4.4, $V(\lambda) \otimes \mathbb{K}_{b,c}$ is a simple $U_{g,h}$-module provided that $\lambda \neq \varepsilon q^n$ for any $n$, where $\varepsilon = \pm 1$.

(3) The simple module $L(a, b, c)$ is finite-dimensional if and only if the only maximal submodule $N(a, b, c)$ of $V(a, b, c)$ is equal to $V(q^{-2n}a, b, c)$ and $a = \varepsilon bq^{n-1}$ for some $n \in \mathbb{N}$. In this case, $L(\varepsilon bq^{n-1}, b, c) \cong V_{\varepsilon,n-1,b,c}$, which is given by Theorem 3.5.

Finally, we study the BGG category $\mathcal{O}$, which is defined below.

**Definition 4.7.** The BGG category $\mathcal{O}$ consists of all finitely generated $U_{g,h}$-modules and all homomorphisms of modules with the following properties:

1. The actions of $K, g, h$ are diagonalized with finite-dimensional weight spaces.
2. The $B_+$-action is locally finite, where $B_+$ is the subalgebra generated by $E$, $K^{\pm 1}$, $g^{\pm 1}$, $h^{\pm 1}$.

It is obvious that every Verma module is in $\mathcal{O}$. By Theorem 3.5, all finite-dimensional simple $U_{g,h}$-modules are in $\mathcal{O}$. Any simple module in $\mathcal{O}$ is isomorphic to either a simple Verma module or a finite-dimensional simple module $V_{\varepsilon,n,a,\beta}$ described in Theorem 3.5. In fact, if $M$ is a simple module in $\mathcal{O}$, then $M = V_{g,h}v$ for some common eigenvector $v$ of $K, g, h$. Suppose $Kv = \lambda v$. Then $KE^n v = \lambda^n E^n v$ for any positive integer $n$. Since the action of $E$ is locally finite, there is an $n$ such that $E^n v \neq 0$ and $E^{n+1} v = 0$. Thus $M = U_{g,h}E^n v$ is a standard cyclic $U_{g,h}$-module. So it is a quotient of a Verma module. Hence it is isomorphic to either a simple Verma module or a finite-dimensional simple module $V_{\varepsilon,n,a,\beta}$.

Suppose $0 \to M \to V_{\varepsilon,n,a,\beta} \to M \to 0$ is a nonzero element in $\text{Ext}^1(V_{\varepsilon,n,a,\beta}, V_{\varepsilon,n,a,\beta})$. We remark that this $M$ is not in $\mathcal{O}$ since the actions of $g, h$ on $M$ can not be diagonalized by Proposition 3.8 and Theorem 3.10. Similarly, if $0 \to V(a, b, c) \to M \to V(a, b, c) \to 0$ is a nonzero element in $\text{Ext}^1(V(a, b, c), V(a, b, c))$, then $M$ is not in $\mathcal{O}$.

By using results in [8], we obtain that every finite-dimensional module in $\mathcal{O}$ is semisimple. In the following we give a direct proof of this fact.

**Proposition 4.8.** Every finite-dimensional module in $\mathcal{O}$ is semisimple.

**Proof.** Let $0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n = M$ be a composition series of a finite-dimensional module $M$ for $M \in \mathcal{O}$. We prove that $M$ is semisimple by using induction. If $n = 2$, then we have the following exact sequence

$$0 \to M_1 \to M \to M/M_1 \to 0.$$ 

Suppose the above sequence is not splitting, then the either the action of $g$ or the action of $h$ on $M$ is not semisimple by Theorem 3.10. Thus $M \notin \mathcal{O}$. This contradiction implies that $M$ is semisimple. Suppose $M$ is semisimple in the case when $n = k \geq 2$. Now let $n = k + 1$. Then $M_k = \bigoplus_{i=1}^k S_i$ is a direct sum of simple $U_{g,h}$-modules $S_i$ by the assumption. Now let $S_i = S_1 \oplus \cdots \oplus \tilde{S}_i \oplus \cdots \oplus S_k$, where $\tilde{S}_i$ means that $S_i$ is omitted. Consider the
following commutative diagrams for \( i = 1, 2, \cdots, k \):

\[
\begin{array}{ccccccc}
0 & \longrightarrow & M_k & \overset{\phi}{\longrightarrow} & M & \overset{\pi}{\longrightarrow} & M/M_k & \longrightarrow & 0 \\
\lambda_i & \downarrow & & \pi_i & \downarrow & \text{id} & \downarrow & & \\
0 & \longrightarrow & N_i & \overset{\varphi_i}{\longrightarrow} & M/S_i & \overset{\psi_i}{\longrightarrow} & M/M_k & \longrightarrow & 0,
\end{array}
\]

where \( \phi, \varphi_i \) are embedding mappings, and \( \lambda_i, \pi_i, \pi, \psi_i \) are the canonical projections. Since the bottom exact sequences are splitting by the inductive assumption, there are homomorphisms \( \xi_i : M/S_i \rightarrow N_i \) such that \( \xi_i \varphi_i = \text{id}_{N_i} \). Define \( \xi : M \rightarrow M_k \) via

\[
\xi(m) = \frac{1}{k - 1} \sum_{i=1}^{k} \xi_i \pi_i (m)
\]

for \( m \in M \). Now let \( m = m_1 + \cdots + m_k \in M_k \), where \( m_i \in S_i \). Then

\[
\xi_i \pi_i (m) = \xi_i \pi_i \phi (m) = \xi_i \varphi_i \lambda_i (m) = m - m_i,
\]

and

\[
\xi \phi (m) = \frac{1}{k - 1} \sum_{i=1}^{k} \xi_i \pi_i (m) = m.
\]

This means that the top exact sequence of the above commutative diagrams is splitting. Hence \( M \cong M_k \oplus \text{Ker} \xi \cong M_k \oplus M/M_k \cong S_1 \oplus \cdots \oplus S_k \oplus M/M_k \) is semisimple. \( \square \)

By the PBW Theorem 2.1, the algebra \( U_{g,h} \) has a triangular decomposition \( \mathbb{K}[F] \otimes H \otimes \mathbb{K}[E] \), where \( H = \mathbb{K}[K^\pm, g^\pm, h^\pm] \). In the same way as [8, Definition 11.1], we can define the Harish-Chandra projection \( \xi \) as follows:

\[
\xi := \varepsilon \otimes \text{id} \otimes \varepsilon : U_{g,h} = \mathbb{K}[F] \otimes H \otimes \mathbb{K}[E] \rightarrow H.
\]

Let \( V(a, b, c) \) be a Verma module generated by a nonzero highest weight vector \( v \). Then

(4.1) \[
Cv = \frac{qa + q^{-1}a^{-1}b^2}{(q - q^{-1})^2} v, \quad gv = bv, \quad hv = cv,
\]

where \( C \) is the Casimir element of \( U_{g,h} \). By Corollary 2.4, the center of \( U_{g,h} \) is \( \mathbb{K}[C, g^{\pm 1}, h^{\pm 1}] \). For any element \( z \in \mathbb{K}[C, g^{\pm 1}, h^{\pm 1}] \), \( zv = \xi_{(a,b,c)} (z) v \) for some \( \xi_{(a,b,c)} (z) \in \mathbb{K} \). Then

\[
\xi_{(a,b,c)} \in \text{Hom}_{\text{alg}} (\mathbb{K}[C, g^{\pm 1}, h^{\pm 1}], \mathbb{K}).
\]

We call \( \xi_{(a,b,c)} \) the central character determined by \( V(a, b, c) \).

**Proposition 4.9.** (1) Suppose \( V(a, b, c) \) and \( V(a', b', c') \) are two Verma modules. Then \( \xi_{(a', b', c')} = \xi_{(a,b,c)} \) if and only if

(4.2) \[
(a - a')(aa' - q^{-2}b^2) = 0, \quad b = b', \quad c = c'.
\]

(2) \( \text{Hom}_{U_{g,h}} (V(a, b, c), V(a', b', c')) \neq 0 \) if and only if \( a = \varepsilon q^{-n-1}b \) and \( a' = \varepsilon q^{n-1}b \) for some nonnegative integer \( n \) and \( (b, c) = (b', c') \).
Proof. Let \( v, v' \) be the nonzero highest weight vectors of \( V(a, b, c) \) and \( V(a', b', c') \) respectively. Then \( \xi_{(a', b', c')} = \xi_{(a, b, c)} \) if and only if \( C'v = \xi_{(a, b, c)}(C)v' \), \( gv' = \xi_{(a, b, c)}(g)v' \) and \( hv' = \xi_{(a, b, c)}(h)v' \). Thus (4.2) follows from (4.1).

If there is a nonzero homomorphism \( \varphi \) from \( V(a, b, c) \) to \( V(a', b', c') \), then
\[
\xi_{(a', b', c')} = \xi_{(a, b, c)}.
\]
Thus (4.2) holds. Suppose \( \varphi(v) = (\sum a_i F^i)v' \), where \( a_n \neq 0 \). Since \( \varphi(Kv) = K\varphi(v) \),
\[
0 = \varphi(Ev) = E\varphi(v) = a_n E F^m v' = a_n [n] a' q^{-n+1} - a' q^{-n+1} b^2 q^{-1} F^{n-1} v'.
\]
Hence \( aa' = q^{2n-2} b^2 \). So \( a' = \varepsilon q^{n-1} b \) and \( a = \varepsilon q^{-n-1} b \).

Conversely, notice that \( V(a, b, c) = \mathbb{K}[F]^v \) and \( V(a', b', c') = \mathbb{K}[F]^v' \) are two free \( \mathbb{K}[F] \)-modules. Thus the mapping
\[
\varphi(f(F)v) = f(F) F^m v', \quad f(F) \in \mathbb{K}[F]
\]
is a nonzero linear mapping. Since \( b = b' \) and \( c = c' \), \( \varphi(g f(F)v) = g \varphi(f(F)v) \) and \( \varphi(h f(F)v) = h \varphi(f(F)v) \). It is routine to check that \( \varphi(E f(F)v) = E \varphi(f(F)v) \) and \( \varphi(K f(F)v) = K \varphi(f(F)v) \). So \( \varphi \) is a nonzero homomorphism of \( U_{g,h} \)-modules. \( \square \)

For any \( \nu \in \text{Hom}_{alg}(\mathbb{K}[C, g^\pm, h^\pm], \mathbb{K}) \), define a full subcategory \( \mathcal{O}(\nu) \) of \( \mathcal{O} \) as follows:
\[
\mathcal{O}(\nu) = \{ M \in \mathcal{O} \mid \forall m \in M, z \in \mathbb{K}[C, g^\pm, h^\pm], \exists n \in \mathbb{N} \text{ such that } (z - \nu(z))^n m = 0 \}.
\]
For any \( \nu \in \text{Hom}_{alg}(\mathbb{K}[C, g^\pm, h^\pm], \mathbb{K}) \), suppose \( \nu(C) = \mu \), \( \nu(g) = b \), \( \nu(h) = c \). Then \( b, c \in \mathbb{K}^\times \). Since \( \mathbb{K} \) is an algebraically closed field, there is \( a \in \mathbb{K} \) such that \( \frac{qa + q^{-2a}b^2}{(q - q^{-1})^2} = \mu \).

Therefore the Verma module \( V(a, b, c) \in \mathcal{O}(\nu) \) by (4.1), and \( \mathcal{O}(\nu) \) is not empty. By results in [8, Theorem 11.2], we have the following decomposition of \( \mathcal{O} \).

**Theorem 4.10.** The category \( \mathcal{O} = \bigoplus_{\nu \in \text{Hom}_{alg}(\mathbb{K}[C, g^\pm, h^\pm], \mathbb{K})} \mathcal{O}(\nu) \).

Let \( \mathcal{H} \) be the Harish-Chandra category over \( (U_{g,h}, H) \), which consists of all \( U_{g,h} \)-modules \( M \) with a simultaneous weight space decomposition for \( H = \mathbb{K}[K^\pm, g^\pm, h^\pm] \), and finite-dimensional weight spaces. By Proposition 2.5, \( U_{g,h} \) has an anti-involution \( i \). Thus we can define a duality functor \( F : \mathcal{H} \to \mathcal{H} \) as follows: \( F(M) \) is the vector space spanned by all \( H \)-weight vectors in \( M^* = \text{Hom}_{\mathbb{K}}(M, \mathbb{K}) \). It is a module under the action determined by
\[
\langle a m^*, m \rangle = \langle m^*, i(a)m \rangle
\]
for \( a \in U_{g,h}, m^* \in F(M), m \in M \). By results in [8], \( F \) defines a duality functor \( F : \mathcal{O} \to \mathcal{O}^{op} \). Moreover, \( F(L(a, b, c)) = L(a, b, c), F(V(a, b, c)) \) has the socle \( L(a, b, c) \) and so on.

By Proposition 4.9, \( U_{g,h} \) satisfies the condition (S4) defined in [8]. Therefore it satisfies the conditions (S1), (S2), and (S3) by [8, Proposition 11.3] and [8, Theorem 10.1], where
(S1), (S2) and (S3) are defined in [8]. By [8, Theorem 4.3], we have the following theorem since $\Gamma$ is trivial.

**Theorem 4.11.** Let $\nu \in \text{Hom}_{K}(K^{\pm 1}, g^{\pm 1}, h^{\pm}), K)$ and $\mathcal{O}(\nu)$ have the same meaning as in Theorem 4.10. Then:

1. Each object of the block $\mathcal{O}(\nu)$ has a filtration whose subquotients are quotients of Verma modules.
2. Each block $\mathcal{O}(\nu)$ has enough projective objects.
3. Each block $\mathcal{O}(\nu)$ is a highest weight category, equivalent to the category of finitely generated right modules over a finite-dimensional $K$-algebra.

In particular, BGG Reciprocity holds in $\mathcal{O}$.

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