A simple explanation for the “shuffling phenomenon” for lozenge tilings of dented hexagons

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Abstract

In a recent paper, Lai and Rohatgi proved a “shuffling theorem” for lozenge tilings of a hexagon with “dents” (i.e., missing triangles). Here, we shall point out that this follows immediately from the enumeration of Gelfand–Tsetlin patterns with given bottom row. This observation is also contained in a recent preprint of Byun.

Introduction

In [3, Theorems 2.1 and 2.3], Lai and Rohatgi observed that there is a nice factorization of the quotient of enumerations of lozenge tilings for two “hexagons with dents”, where the hexagons only differ (in some sense to be explained

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below) by a “shuffling of the dents”, and asked for a combinatorial explanation.

The purpose of this note is to show that this observation follows from the formula for the enumeration of Gelfand–Tsetlin patterns with prescribed bottom row. I am grateful to Mihai Ciucu for pointing out to me that this observation is also contained in a recent preprint of Byun [1].

We will briefly repeat the (well–known) background, mainly by presenting pictures which illustrate the ideas.

Lozenge tilings of regions in the triangular lattice

The triangular lattice may be viewed as the tiling of the plane \( \mathbb{R}^2 \) by congruent equilateral triangles. A lozenge (or rhombus) is the union of two such triangles which are adjacent (i.e., sharing an edge). A region in the triangular lattice is simply a finite subset of triangles, and a lozenge tiling of such region \( R \) is a partition of \( R \) whose blocks are lozenges (i.e., a family of disjoint lozenges whose union equals \( R \)). The meaning of this dry set–theoretic definition becomes clear when looking at some pictures; see Figure 1.

As usual in combinatorics, the first question is the enumeration of such lozenge tilings: It turns out that there is a wealth of interesting formulas for regions of “hexagonal shape”; see Figure 1 again.

When visualizing the triangular lattice in the obvious way (see Figure 1), it is immediately clear that there are precisely two possible orientations of the equilateral triangles, namely upwards–pointing and downwards–pointing, and that every lozenge consists of an upwards–pointing and a downwards–pointing triangle, and that there are precisely three possible orientations of such lozenges, namely vertical, left–tilted and right–tilted.

So it is a necessary condition for the existence of lozenge tilings for some region that the number of upwards–pointing triangles equals the number of downwards–pointing triangles in this region (whence the hexagon shown in Figure 1 has no lozenge tiling).
The picture shows a hexagonal region in the triangular lattice, where we started to construct a lozenge tiling at the upper left corner of the hexagon: The three lozenges represent the three possible orientations (vertical, left–tilted and right–tilted). The horizontal diagonal (shown as a dashed line) bisects the hexagon in an upper and lower half. The upper half consists of 6 rows of triangles, all of which start and end with an upwards–pointing triangle, and the lower half consists of 4 rows of triangles, all of which start and end with a downwards–pointing triangle.

Figure 1: Hexagonal region with partial lozenge tiling.

When considering some hexagonal shape with a horizontal diagonal (i.e., a diagonal parallel to the base of the hexagon, see Figure 1), we may consider the bisection of the hexagon in an upper half and a lower half: The upper half consists of rows of adjacent triangles where upwards–pointing and downwards–pointing triangles alternate, and the same is true for the lower half. In the upper half, these rows of triangles start and end with an upwards–pointing triangle. In the lower half, these rows of triangles start and end with a downwards–pointing triangle. Clearly, the lower half appears as upper half when reflected at the horizontal diagonal, and vice versa: We shall call either of these halves a half–hexagon.
Row–wise construction of lozenge tilings

We start the construction of a lozenge tiling for some hexagon in the top row of the upper half: If such row contains \( n \) upwards–pointing triangles, then there are precisely \( n - 1 \) downwards–pointing triangles; and covering the latter by lozenges (which are necessarily left–tilted or right–tilted) leaves precisely one remaining upwards–pointing triangle, which then clearly must be covered by a vertical lozenge (see Figure 2).

If we number the upwards–pointing triangles in rows from the left, then we may encode this partial lozenge tiling by the number of this vertical lozenge. This lozenge protrudes to the next row where it occupies one downwards–pointing triangle, while the remaining downwards–pointing triangles there must be covered by left–tilted or right–tilted lozenges, leaving precisely two upwards–pointing triangle which must be covered by vertical lozenges (see Figure 2 again).

Gelfand–Tsetlin patterns

Clearly we can continue this process of “row by row” construction of partial lozenge tilings as long as we are in the upper half of the hexagon, and the partial tiling obtained by this construction is uniquely encoded by a triangular array of natural numbers

\[
\begin{array}{ccccccc}
& & u_{1,1} & & \\
& u_{2,1} & & u_{2,2} & & \\
& & u_{3,1} & & u_{3,2} & & u_{3,3} \\
& u_{4,1} & & u_{4,2} & & u_{4,3} & & u_{4,4} \\
\cdots & & \cdots & & \cdots & & \cdots & & \cdots
\end{array}
\]

where \( u_{i,j} \) indicates the number of the \( j \)–th upwards–pointing triangle (counted from the left) covered by a vertical lozenge in the \( i \)–th row of triangles (counted from the top row). The entries in row \( i - 1 \) are in the following sense “interlaced” with the entries in row \( i \):

\[
\begin{align*}
& u_{i,1} \leq u_{i-1,1} < u_{i,2} \leq u_{i-1,2} < u_{i,3} \leq \cdots u_{i-1,i-1} < u_{i,i}.
\end{align*}
\]
The pictures show the simple idea of constructing a lozenge tiling of some hexagon “row by row”. Assume the top row of some hexagon has length 6. We may choose upwards–pointing triangle 4 to be the one which should be covered by a vertical lozenge. With this choice, the tiling of the remaining top row is uniquely determined: The upper left pictures shows this top row with the numbering of the upwards–pointing triangles, and the upper right picture shows the (unique) lozenge tiling determined by the position (4) of the vertical lozenge, which protrudes to the next row.

In this next row, we have to choose two upwards–pointing triangles which are covered by vertical lozenges, such that the protruding lozenge from the row above is located between them: The lower pictures show a possible choice of positions for such upwards–pointing triangles (2 and 6) and the lozenge tiling uniquely determined by this choice. Note that we may encode this partial lozenge tiling by the array of numbers ((4), (2, 6)).

Figure 2: Row–wise construction of lozenge tilings.
Such arrays of natural numbers are called *Gelfand–Tsetlin patterns*; see, for instance, [2], where these patterns are considered in reverse order (i.e., from bottom to top row) and called *semi–strict Gelfand patterns*. We shall also use the (space saving) one–line notation

\[
((u_{1,1}), (u_{2,1}, u_{2,2}), (u_{3,1}, u_{3,2}, u_{3,3}), (u_{4,1}, u_{4,2}, u_{4,3}, u_{4,4}), \ldots)
\]

for Gelfand–Tsetlin patterns in the following.

Write \(U_n = \{u_{n,1} < u_{n,2} < \ldots u_{n,n}\}\) for the (ordered) set of the numbers in the \(n\)-the row, and denote the number of Gelfand–Tsetlin patterns with bottom row \(U_n\) as \(GTP(U_n)\). From (1) we immediately obtain:

\[
GTP(U_n) = \sum_{u_{n-1,1}=u_{n,1}}^{u_{n,2}-1} \sum_{u_{n-1,2}=u_{n,2}}^{u_{n,3}-1} \cdots \sum_{u_{n-1,n-1}=u_{n,n-1}}^{u_{n,n}-1} GTP(U_{n-1}) .
\]

(2)

From (2) and the initial condition \(GTP(U_1) \equiv 1\) the following product formula is easily derived (see Cohn, Larsen and Propp [2, Proposition 2.1]):

\[
GTP(U_n) = \prod_{1 \leq i < j \leq n} \frac{u_j - u_i}{j - i} .
\]

(3)

**Putting together two half–hexagons**

If we have two Gelfand–Tsetlin patterns *with the same bottom row*, they encode lozenge tilings for half–hexagons which we can *put together* to obtain a tiling of the “whole” hexagon (see Figure 3). Note that the hexagon thus obtained is *not uniquely determined* by the Gelfand–Tsetlin patterns: The only condition it must obviously fulfil is that the *length* of its horizontal diagonal must be greater or equal than the last element in the bottom row of the Gelfand–Tsetlin patterns.

**Dented hexagons**

For two *arbitrary* Gelfand–Tsetlin patterns (with possibly *different* bottom rows), we may also put together the lozenge tilings of corresponding half–hexagons (see Figure 4): In general, this procedure will not give a hexagon,
The left picture shows two lozenge tilings of a half-hexagon with the same positions of vertical lozenges in the bottom row of triangles (one of the halves is reflected, so its bottom row now appears as top row). These are encoded by Gelfand–Tsetlin patterns (((4), (2, 6), (1, 4, 7), (1, 3, 6, 8)) and ((3), (3, 6), (2, 5, 7), (1, 3, 6, 8)) having the same bottom row.

Clearly, such halves can be combined to a lozenge tiling of the whole hexagon, shown in the right picture.

Figure 3: Putting together two half-hexagons.
Putting together the lozenge tilings encoded by the “incompatible” Gelfand–Tsetlin patterns

\[(4), (3, 6), (1, 5, 8), (1, 3, 5, 11), (1, 3, 4, 7, 9, 12), (1, 2, 4, 6, 8, 10, 13), (1, 2, 4, 5, 8, 10, 11, 14)\]

and

\[(5), (3, 7), (2, 7, 9), (1, 5, 9, 10), (1, 4, 9, 10, 11), (1, 4, 9, 10, 11, 12), (1, 4, 9, 10, 11, 12, 14)\]

(with different bottom rows) is possible if we omit the lozenges bisected by the horizontal diagonal; the result of this operation is a lozenge tiling of the dented hexagon shown in the right picture, where the “dents” (i.e., the missing triangles adjacent to the horizontal triangles) are coloured white. (This is the example presented in [3, Figure 2.1].)

Figure 4: Lozenge tiling of a dented hexagon.
but if we simply omit the protruding vertical lozenges in the bottom rows, we obtain a lozenge tiling of a hexagon with missing triangles (adjacent to the horizontal diagonal). We shall call such region a dented hexagon; by construction it is determined by the bottom rows of the two Gelfand–Tsetlin patterns (but not uniquely: As before, the length of the horizontal diagonal must be greater or equal than the maximal entry in both bottom rows).

Denote the bottom rows of the Gelfand–Tsetlin patterns of the upper and lower half of the dented hexagon by $U = \{u_1 < u_2 < \cdots < u_m\}$ and $L = \{d_1 < d_2 < \cdots < d_n\}$, respectively. Then clearly the number of lozenge tilings of this dented hexagon equals $GTP(U) \cdot GTP(L)$.

**Simple observations**

For two finite sets of natural numbers $A$ and $B$ define

$$P(A, B) := \prod_{(a, b) \in A \times B} (b - a)$$

and observe that we may rewrite (3) as follows:

$$GTP(U) = \frac{P(U, U)}{2! \cdot 3! \cdots (|U| - 1)!}.$$ 

Set $C = U \cap L$, $\overline{U} := U \setminus C$ and $\overline{L} := L \setminus C$. Clearly, we have

$$P(U, U) = P(\overline{U}, \overline{U}) \cdot P(\overline{U}, C) \cdot P(C, \overline{U}) \cdot P(C, C),$$

and the analogous statement holds true for $P(L, L)$. Hence the product $GTP(U) \cdot GTP(L)$ equals

$$\frac{P(\overline{U}, \overline{U}) \cdot P(\overline{L}, \overline{L}) \cdot P(\overline{U} \cup \overline{L}, C) \cdot P(C, \overline{U} \cup \overline{L}) \cdot P(C, C)^2}{(2! \cdot 3! \cdots (|U| - 1)!)(2! \cdot 3! \cdots (|L| - 1)!)}.$$ 

(4)

Now fix some arbitrary subset

$$S \subseteq \overline{U} \cup \overline{L}.$$
By the “shuffling” of dents corresponding to such set $S$ we mean that all elements in $S$ belonging to $U$ are moved to $L$ and vice versa; i.e., the sets $U'$ and $L'$ obtained from this “shuffling” are

$$U' = (U \setminus S) \cup (L \cap S),$$
$$L' = (L \setminus S) \cup (U \cap S).$$

Clearly, such “shuffling” does not change the set $\overline{U} \cup \overline{L}$:

$$\overline{U'} \cup \overline{L'} = \overline{U} \cup \overline{L}.$$

So from (4) we obtain

$$\frac{GTP(U) \cdot GTP(L)}{GTP(U') \cdot GTP(L')} = \frac{P(U, \overline{U}) \cdot P(L, \overline{L}) \cdot (2! \cdots (|U'| - 1)! \cdot (2! \cdots (|L'| - 1)!)}{P(U', \overline{U'}) \cdot P(L', \overline{L'}) \cdot (2! \cdots (|U| - 1)! \cdot (2! \cdots (|L| - 1)!)}.$$  

Lai and Rohatgi’s “shuffling phenomenon”

Now fix some arbitrary subset $V \subseteq C = U \cap L$ and consider a dented hexagon where the following triangles adjacent to the horizontal diagonal have been removed:

- upwards–pointing triangles at positions in $U \setminus V$
- and downwards–pointing triangles at positions in $L \setminus V$.

Every lozenge tiling of this dented hexagon must contain $|V|$ vertical lozenges beaded along the horizontal diagonal, at positions outside the set $F := (U \cup L) \setminus V$.

We may consider the restriction that the positions of the $|V|$ vertical lozenges must be chosen from some finite subset $B \subset \mathbb{N} \setminus F$ of natural numbers and immediately obtain the following formula for the number of such “restricted” lozenge tilings of a dented hexagon given by $U$, $L$ and $B$ (again, such hexagon
is not unique: The length of its horizontal diagonal must be greater or equal than \( \max (U \cup L \cup B) \):

\[
\frac{P (\overline{U}, \overline{U}) \cdot P (\overline{L}, \overline{L})}{(2! \cdots (|U| - 1)!)(2! \cdots (|L| - 1)!)} \cdot \sum_{V' \subseteq B, |V'| = |V|} \frac{P (\overline{U} \cup \overline{L}, X) \cdot P (X, \overline{U} \cup \overline{L}) \cdot P (X, X)^2}{P (U \cup L, X)}
\]

where the sets \( X \) appearing in the summands are defined as

\[
X = ((U \cap L) \setminus V) \cup V'.
\]

Since the “shuffling” corresponding to some set \( S \subseteq \overline{U} \cup \overline{L} \) does neither change the set \( U \cap L \) nor the set \( \overline{U} \cup \overline{L} \), the sum in (5) is invariant under such “shuffling”: This explains the observations formulated as Theorems 2.1 and 2.3 in [3].

References

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