THE ABSTRACT TITCHMARSH-WEYL $M$-FUNCTION FOR ADJOINT OPERATOR PAIRS AND ITS RELATION TO THE SPECTRUM

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Abstract. In the setting of adjoint pairs of operators we consider the question: to what extent does the Weyl $M$-function see the same singularities as the resolvent of a certain restriction $A_B$ of the maximal operator? We obtain results showing that it is possible to describe explicitly certain spaces $S$ and $\tilde{S}$ such that the resolvent bordered by projections onto these subspaces is analytic everywhere that the $M$-function is analytic. We present three examples – one involving a Hain-Lüst type operator, one involving a perturbed Friedrichs operator and one involving a simple ordinary differential operators on a half line – which together indicate that the abstract results are probably best possible.

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1. Introduction

In the theory of inverse problems for Schrödinger operators on a half line, 

$$-y'' + q(x) y = \lambda y, \quad x \in (0, \infty), \tag{1.1}$$

it has been well known since the work of Borg [4], of Marchenko [23] and of Gelfand and Levitan [9] that the function $q$ is uniquely determined by the Titchmarsh-Weyl function for the problem. Here $q$ is assumed to be real valued and integrable over any finite sub-interval of $[0, \infty)$ and to give rise to a so-called limit point case at infinity: that is, one requires only a boundary condition at the origin, and no boundary condition at infinity, in order to obtain a selfadjoint operator associated with the expression on the left hand side of (1.1).

The Titchmarsh Weyl function $M(\lambda)$ for this problem can be regarded as a Dirichlet to Neumann map for the problem. Suppose that we define a ‘maximal’ operator $A^*$ by

$$D(A^*) = \{ y \in L^2(0, \infty) \mid -y'' + qy \in L^2(0, \infty) \},$$

$$A^* y = -y'' + qy,$$

where $y''$ is to be understood in the sense of weak derivatives; also define some ‘boundary’ operators $\Gamma_1$ and $\Gamma_2$ on $D(A^*)$ by

$$\Gamma_1 y = y(0), \quad \Gamma_2 y = -y'(0).$$

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Then the Titchmarsh Weyl function may be defined by the expression

\[ M(\lambda) = \Gamma_2 \left( \Gamma_1 \big|_{\ker(A^* - \lambda I)} \right)^{-1} , \]

or equivalently

\[ M(\lambda)y(0) = -y'(0) \quad \text{when} \quad -y'' + qy = \lambda y \quad \text{and} \quad y \in L^2(0, \infty). \]

If we let \( A_D \) denote the ‘Dirichlet restriction’ of \( A^* \), that is the restriction of \( A^* \) to \( D(A_D) = D(A^*) \cap \ker(\Gamma_1) \),

then the \( M \)-function is easily seen to be well defined for \( \lambda \not\in \sigma(A_D) \). One may show that \( (A_D - \lambda)^{-1} \) has the same poles as \( M(\lambda) \), and the famous Weyl Kodaira formula relates the spectral measure \( \rho \) of \( A_D \) to \( M \):

\[ d\rho(k) = \frac{1}{\pi} w - \lim_{\epsilon \to 0} \Im M(k + i\epsilon) dk. \]

In short, complete information about the original operator is encoded in \( M \).

For PDEs, similar inverse results are also available. For Schrödinger operators on smooth domains with smooth potentials, for instance, the Dirichlet-to-Neumann map \( M(\lambda) \) determines the potential uniquely. Moreover in this PDE case it is not necessary to know \( M(\lambda) \) as a function of \( \lambda \); it suffices to know it for one value of \( \lambda \) for which it is well defined. For more general classes of PDEs there are many results guaranteeing that the coefficients can be recovered up to some explicit transformations. See Isakov [14] for a review of inverse problems for elliptic PDEs.

In this paper we consider similar questions in the totally abstract setting of boundary triples (cf. Section 2 for the definition). As shown in the papers by Krein, Langer and Textorius [16, 17, 18] on extensions of symmetric operators, under an assumption of complete nonselfadjointness of the underlying symmetric minimal operator, the maximal operator is determined up to unitary equivalence by the \( M \)-function. Moreover, recently Ryzhov [27] has shown that under the same assumptions and an additional invertibility condition imposed on the Dirichlet restriction \( A_D \), the operators \( A_D \) and \( \Gamma_2 A_D^{-1} \) are determined by the difference \( M(z) - M(0) \) up to unitary equivalence.

For the non-symmetric case, the authors considered in [7] the question of behaviour of the abstract \( M \)-function(s) near the boundary of the essential spectrum and asked: to what extent does the \( M \)-function see the same singularities as the resolvent of a certain restriction \( A_B \) of the maximal operator?

In this paper we obtain results showing that it is possible to describe explicitly certain spaces \( S \) and \( \tilde{S} \) such that the bordered resolvent \( P_S (A_B - \lambda I)^{-1} P_S \), in which the \( P \) are orthogonal projections onto the spaces indicated, is analytic everywhere that \( M(\lambda) \) is analytic. The spaces \( S \) and \( \tilde{S} \) are, in general, not closed. However we present three examples – one involving a Hain-Lüb type operator, one involving a perturbed Friedrichs operator and one involving simple ordinary differential operators on a half line – which together indicate that the abstract results in Section 3 are probably best possible. As a result we conclude that the abstract approach to inverse problems may yield rather limited results unless further hypotheses are introduced which reflect properties of problems involving concrete ordinary and partial differential expressions.

We should mention that since their introduction by Vishik [28] for second order elliptic operators and Lyantze and Storozh [19] for adjoint pairs of abstract
operators, boundary triplets have been widely used to characterise extensions of operators and investigate spectral properties using Weyl M-functions. An extension of the theory to relations can be found in the work of Malamud and Mogilevskii \cite{21,22}. For related recent results, particularly in the context of PDEs, we refer to the works of Alpay and Behrndt \cite{4}, Behrndt and Langer \cite{3}, Brown, Grubb, Wood \cite{6}, Gesztesy and Mitrea \cite{10,11,12} and also to Posilicano \cite{24,25} and Post \cite{26}.

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2. Background theory of boundary triples and Weyl functions

Throughout this article we will make the following assumptions:

1. $A$, $\hat{A}$ are closed, densely defined operators in a Hilbert space $H$.
2. $A$ and $\hat{A}$ are an adjoint pair, i.e. $A^* \supseteq \hat{A}$ and $\hat{A}^* \supseteq A$.

**Proposition 2.1.** [19] (Lyantze, Storozh '83). For each adjoint pair of closed densely defined operators on $H$, there exist “boundary spaces” $\mathcal{H}$, $\mathcal{K}$ and “boundary operators”

\[ \Gamma_1 : D(\hat{A}^*) \to \mathcal{H}, \quad \Gamma_2 : D(\hat{A}^*) \to \mathcal{K}, \quad \tilde{\Gamma}_1 : D(A^*) \to \mathcal{K} \quad \text{and} \quad \tilde{\Gamma}_2 : D(A^*) \to \mathcal{H} \]

such that for $u \in D(\hat{A}^*)$ and $v \in D(A^*)$ we have an abstract Green formula

\[ (\hat{A}^*u, v)_H - (u, A^*v)_H = (\Gamma_1 u, \tilde{\Gamma}_2 v)_H - (\Gamma_2 u, \tilde{\Gamma}_1 v)_K. \]

The boundary operators $\Gamma_1, \Gamma_2, \tilde{\Gamma}_1$ and $\tilde{\Gamma}_2$ are bounded with respect to the graph norm. The pair $(\Gamma_1, \Gamma_2)$ is surjective onto $\mathcal{H} \times \mathcal{K}$ and $(\tilde{\Gamma}_1, \tilde{\Gamma}_2)$ is surjective onto $\mathcal{K} \times \mathcal{H}$. Moreover, we have

\[ D(A) = D(\hat{A}^*) \cap \ker \Gamma_1 \cap \ker \Gamma_2 \quad \text{and} \quad D(\hat{A}) = D(A^*) \cap \ker \tilde{\Gamma}_1 \cap \ker \tilde{\Gamma}_2. \]

The collection $\{ \mathcal{H} \oplus \mathcal{K}, (\Gamma_1, \Gamma_2), (\tilde{\Gamma}_1, \tilde{\Gamma}_2) \}$ is called a boundary triplet for the adjoint pair $A, \hat{A}$.

Malamud and Mogilevskii \cite{21,22} use this setting to define Weyl $M$-functions and $\gamma$-fields associated with boundary triplets and to obtain Kreĭn formulæ for the resolvents. We now summarize some results, using however a slightly different setting taken from \cite{7} in which the boundary conditions and Weyl function contain an additional operator $B \in \mathcal{L}(\mathcal{K}, \mathcal{H})$.

**Definition 2.2.** Let $B \in \mathcal{L}(\mathcal{K}, \mathcal{H})$ and $\hat{B} \in \mathcal{L}(\mathcal{H}, \mathcal{K})$. We define extensions of $A$ and $\hat{A}$ (respectively) by

\[ A_B := \hat{A}^*|_{\ker(\Gamma_1 - B\Gamma_2)} \quad \text{and} \quad \hat{A}_B := A^*|_{\ker(\tilde{\Gamma}_1 - \hat{B}\tilde{\Gamma}_2)}. \]

In the following, we assume $\rho(A_B) \neq \emptyset$, in particular $A_B$ is a closed operator. For $\lambda \in \rho(A_B)$, we define the $M$-function via

\[ M_B(\lambda) : \ker(\Gamma_1 - B\Gamma_2) \to \mathcal{K}, \quad M_B(\lambda)(\Gamma_1 - B\Gamma_2)u = \Gamma_2 u \quad \text{for all} \quad u \in \ker(\hat{A}^* - \lambda) \]

and for $\lambda \in \rho(\hat{A}_B)$, we define

\[ \hat{M}_B(\lambda) : \ker(\tilde{\Gamma}_1 - \hat{B}\tilde{\Gamma}_2) \to \mathcal{H}, \quad \hat{M}_B(\lambda)(\tilde{\Gamma}_1 - \hat{B}\tilde{\Gamma}_2)v = \tilde{\Gamma}_2 v \quad \text{for all} \quad v \in \ker(A^* - \lambda). \]

It is easy to prove that $M_B(\lambda)$ and $\hat{M}_B(\lambda)$ are well defined for $\lambda \in \rho(A_B)$ and $\lambda \in \rho(\hat{A}_B)$ respectively.
Definition 2.3. (Solution Operator) For $\lambda \in \rho(A_B)$, we define the linear operator $S_{\lambda,B} : \text{Ran}(\Gamma_1 - B\Gamma_2) \to \ker(\hat{A}^* - \lambda)$ by
\[
(\hat{A}^* - \lambda)S_{\lambda,B}f = 0, \quad (\Gamma_1 - B\Gamma_2)S_{\lambda,B}f = f,
\]
i.e. $S_{\lambda,B} = \left((\Gamma_1 - B\Gamma_2)|_{\ker(\hat{A}^* - \lambda)}\right)^{-1}$.

Since we shall use solution operators quite extensively in the sequel, we include the proof of the following lemma, for completeness.

Lemma 2.4. $S_{\lambda,B}$ is well-defined for $\lambda \in \rho(A_B)$. Moreover for each $f \in \text{Ran}(\Gamma_1 - B\Gamma_2)$ the map from $\rho(A_B) \to H$ given by $\lambda \mapsto S_{\lambda,B}f$ is analytic.

Proof. For $f \in \text{Ran}(\Gamma_1 - B\Gamma_2)$, choose any $w \in D(\hat{A}^*)$ such that $(\Gamma_1 - B\Gamma_2)w = f$. Let $v = -(A_B - \lambda)^{-1}(\hat{A}^* - \lambda)w$. Then $v + w \in \ker(\hat{A}^* - \lambda)$ and $(\Gamma_1 - B\Gamma_2)(v + w) = (\Gamma_1 - B\Gamma_2)w = f$, so a solution to \eqref{eq:2.3} exists and is given by
\[
S_{\lambda,B}f = \left(I - (A_B - \lambda)^{-1}(\hat{A}^* - \lambda)\right)w
\]
for any $w \in D(\hat{A}^*)$ such that $(\Gamma_1 - B\Gamma_2)w = f$. Moreover $S_{\lambda,B}f$ is well defined because the solution to \eqref{eq:2.3} is unique. For suppose $u_1$ and $u_2$ are two solutions. Then $(u_1 - u_2) \in \ker(\hat{A}^* - \lambda) \cap \ker(\Gamma_1 - B\Gamma_2)$, so $u_1 - u_2 \in D(A_B)$ and $(A_B - \lambda)(u_1 - u_2) = 0$. As $\lambda \in \rho(A_B)$, $u_1 = u_2$. The analyticity of $S_{\lambda,B}$ as a function of $\lambda$ is immediate from \eqref{eq:2.4} using the fact that the choice of $w$ does not depend on $\lambda$. \qed

Corollary 2.5. Under the hypotheses of Lemma 2.4.\footnote{Corollary}
\[
S_{\lambda,B} = S_{\lambda_0,B} + (\lambda - \lambda_0)(A_B - \lambda)^{-1}S_{\lambda_0,B}.
\]

Proof. Fix $\lambda_0 \in \rho(A_B)$ and choose $w = S_{\lambda_0,B}f$. Then
\[
S_{\lambda,B}f = \left(S_{\lambda_0,B} - (A_B - \lambda)^{-1}(\hat{A}^* - \lambda)S_{\lambda_0,B}\right)f = S_{\lambda_0,B}f + (\lambda - \lambda_0)(A_B - \lambda)^{-1}S_{\lambda_0,B}f. \qed
\]

Note that the identity \eqref{eq:2.5} may be regarded as a Hilbert identity for the difference of resolvents corresponding to different boundary conditions.

To be able to study spectral properties of the operator $A_B$ via the $M$-function, we need to relate the $M$-function to the resolvent. This can be done in the following way:

Theorem 2.6. \footnote{Theorem} (1) Let $\lambda, \lambda_0 \in \rho(A_B)$. Then on $\text{Ran}(\Gamma_1 - B\Gamma_2)$
\[
M_B(\lambda) = \Gamma_2 \left(I + (\lambda - \lambda_0)(A_B - \lambda)^{-1}S_{\lambda_0,B}\right)S_{\lambda_0,B}
\]
\[
= \Gamma_2(A_B - \lambda_0)(A_B - \lambda)^{-1}S_{\lambda_0,B}.
\]

(2) Let $B, C \in \mathcal{L}(\mathcal{K}, \mathcal{H})$, $\lambda \in \rho(A_B) \cap \rho(A_C)$. Then
\[
(A_B - \lambda)^{-1} = (A_C - \lambda)^{-1} - S_{\lambda,C}(I + (B - C)M_B(\lambda))(\Gamma_1 - B\Gamma_2)(A_C - \lambda)^{-1}
\]
\[
= (A_C - \lambda)^{-1} - S_{\lambda,C}(I + (B - C)M_B(\lambda))(C - B)\Gamma_2(A_C - \lambda)^{-1}.
\]
Proof. Part (1) is just Proposition 4.6 from [7], while part (2) is a slight improvement to Theorem 4.7 of the same paper. We include the proof of (2) for completeness: Let \( u \in H \). Set \( v := ((A_B - \lambda)^{-1} - (A_C - \lambda)^{-1})u \). Since \( v \in \ker(\hat{A}^* - \lambda) \), we have \( M_B(\lambda)(\Gamma_1 - B\Gamma_2)v = \Gamma_2v \). Then

\[
(\Gamma_1 - CT_2)v = \left[\Gamma_1 - B\Gamma_2 + (B - C)M_B(\lambda)(\Gamma_1 - B\Gamma_2)\right]v \\
= (I + (B - C)M_B(\lambda))(\Gamma_1 - B\Gamma_2)v \\
= -(I + (B - C)M_B(\lambda))(\Gamma_1 - B\Gamma_2)(A_C - \lambda)^{-1}u.
\]

Set \( f := -(I + (B - C)M_B(\lambda))(\Gamma_1 - B\Gamma_2)(A_C - \lambda)^{-1}u \). Then by the above calculation, \( f \in \text{Ran}(\Gamma_1 - CT_2) \) and \( S_{\lambda,C}f = v = ((A_B - \lambda)^{-1} - (A_C - \lambda)^{-1})u \). Therefore,

\[
(A_B - \lambda)^{-1} = (A_C - \lambda)^{-1} - S_{\lambda,C}(I + (B - C)M_B(\lambda))(\Gamma_1 - B\Gamma_2)(A_C - \lambda)^{-1}.
\]

\( \square \)

3. How much of an operator can its Weyl function determine?

In this section we wish to know how much of the spectrum of an operator can be seen by its Weyl function. In the symmetric case, complete non-selfadjointness of the minimal operator \( A \) is required to recover the operator (up to unitary equivalence) from the Weyl function (see e.g. [27]). Motivated by this, we fix \( \mu_0 \notin \sigma(A_B) \) and define the spaces

\[
S = \text{Span}_{\delta \notin \sigma(A_B)}(A_B - \delta I)^{-1}\text{Ran}(S_{\mu_0,B}),
\]

\[
T = \text{Span}_{\mu \notin \sigma(A_B)}\text{Ran}(S_{\mu,B}),
\]

where \( S_{\mu,B} = \left((\Gamma_1 - B\Gamma_2)|_{\ker(A^* - \mu I)}\right)^{-1} \) is the solution operator. Here \( \text{Span} \) denotes the set of finite linear combinations of vectors from the sets indicated.

The spaces \( S \) depend on the choice of \( \mu_0 \), but this dependence will not be indicated explicitly. Moreover the closures of \( S \) do not depend on \( \mu_0 \), as the following lemma shows.

**Lemma 3.1.** Suppose that there exists a sequence \( (z_n) \in \mathbb{C} \) which tends to infinity and is such that the family of operators \( (z_n(A_B - z_n I)^{-1})_{n \in \mathbb{N}} \) is bounded. Then

\[
\overline{S} = \overline{T}.
\]

**Proof.** From the hypothesis that the operators \( z_n(A_B - z_n I)^{-1} \) are bounded it follows that the operators \( A_B(A_B - z_n I)^{-1} = I + z_n(A_B - z_n I)^{-1} \) are uniformly bounded. Let \( \phi \in H \) be arbitrary. Given \( \epsilon > 0 \) exploit the density of \( D(A_B) \) in \( H \) to choose \( \psi \in D(A_B) \) such that \( \|\phi - \psi\| < \epsilon \). Now because \( \psi \in D(A_B) \), it follows that \( A_B(A_B - z_n I)^{-1}\psi = (A_B - z_n I)^{-1}A_B\psi \) and so

\[
\|A_B(A_B - z_n I)^{-1}\psi\| \leq \|A_B\psi\|\frac{\|z_n(A_B - z_n I)^{-1}\|}{|z_n|} \to 0 \quad (n \to \infty).
\]

Hence for all sufficiently large \( n \), \( \|A_B(A_B - z_n I)^{-1}\psi\| < \epsilon \). But we know that the operators \( A_B(A_B - z_n I)^{-1} \) are uniformly bounded, so for all sufficiently large \( n \)

\[
\|A_B(A_B - z_n I)^{-1}\phi\| \leq \|A_B(A_B - z_n I)^{-1}\|\|\phi - \psi\| + \|A_B(A_B - z_n I)^{-1}\psi\| < C\epsilon + \epsilon
\]
for some $C > 0$. Hence
\[ \|A_B(A_B - z_n I)^{-1}\phi\| \to 0 \quad (n \to \infty) \]
for each fixed $\phi \in H$. Similar arguments may be found in, e.g., [3 Lemma II.3.4].

Let $\mu_0$ be as in the definition of $\mathcal{S}$ and let $\phi = S_{\mu_0, B} f$ for some $f$ in the boundary space. Evidently $\|A_B(A_B - z_n I)^{-1}\phi\| \to 0$ and so
\[ -z_n(A_B - z_n I)^{-1}S_{\mu_0, B} f \to S_{\mu_0, B} f. \]

It follows from the definition of $\mathcal{S}$ that $S_{\mu_0, B} f \in \mathcal{F}$. Now if $\mu$ is another point in the resolvent set of $A_B$ then the identity
\[ S_{\mu, B} = S_{\mu_0, B} + (\mu - \mu_0)(A_B - \mu)^{-1}S_{\mu_0, B} \]
from Corollary 2.5 immediately shows that $S_{\mu, B} f$ lies in $\mathcal{F}$ also. It follows that $\mathcal{T} \subseteq \mathcal{F}$ and hence $\mathcal{T} \subseteq \mathcal{S}$.

Next we show that if $f$ lies in the boundary space and $\mu, \delta$ do not lie in $\sigma(A_B)$ then $(A_B - \delta)^{-1}S_{\mu, B} f$ lies in $\mathcal{T}$. For this we again use the formula (2.5) which gives, for $\delta \neq \mu$,
\[ (A_B - \delta)^{-1}S_{\mu, B} f = \frac{1}{\delta - \mu}(S_{\delta, B} f - S_{\mu, B} f); \]
the right hand side of this expression obviously lies in $\mathcal{T}$. Taking the limit as $\mu \to \delta$ it follows that $(A_B - \delta)^{-1}S_{\delta, B} f$ lies in $\mathcal{T}$. Thus $\mathcal{S} \subseteq \mathcal{T}$ and $\mathcal{F} \subseteq \mathcal{T}$. □

Remark 3.2. In fact with some mild additional assumptions one may show that $\mathcal{F}$ is generically independent of $B$ (as well as of $\mu_0$), using the identity
\[ S_{\mu, C}(I - (C - B)\Gamma_2 S_{\mu_0, B}) = S_{\mu_0, B} \]
from Proposition 4.5 of [7].

Remark 3.3. The hypothesis that one can choose $(z_n)$ tending to infinity such that $(z_n(A_B - z_n I)^{-1})_{n \in \mathbb{N}}$ is bounded holds in the case when the numerical range $\omega(A_B)$ is contained in a half plane, for in this case the $z_n$ can be chosen so that
\[ \frac{z_n}{\text{dist}(z_n, \omega(A_B))} \]
is uniformly bounded in $n$.

Lemma 3.4. The space $\mathcal{F}$ is a regular invariant space of the resolvent of the operator $A_B$: that is, $(A_B - \mu I)^{-1}\mathcal{F} = \mathcal{F}$ for all $\mu \in \rho(A_B)$.

Proof. We start by showing that $(A_B - \mu I)^{-1}\mathcal{S} \subseteq \mathcal{S}$ for all $\mu \in \rho(A_B)$. Choose $f$ of the form
\[ f = \sum_{j=1}^{N} (A_B - \delta_j I)^{-1}S_{\mu_0, B} f_j \]
for some functions $f_j$ in $\mathcal{H}$, and note that such $f$ are dense in $\mathcal{F}$. It follows from the resolvent identity
\[ (A_B - \mu I)^{-1}(A_B - \delta I)^{-1} = \frac{1}{\mu - \delta}((A_B - \mu I)^{-1} - (A_B - \delta I)^{-1}) \]
that $(A_B - \mu I)^{-1} f$ also admits a representation of the form (3.3); thus $(A_B - \mu I)^{-1} f$ also lies in $\mathcal{S}$, giving $(A_B - \mu I)^{-1}\mathcal{S} \subseteq \mathcal{S}$. 


Now suppose that \( f \) lies in \( \mathfrak{S} \). We can write \( f = \lim_{N \to \infty} f_N \) where \( f_N \) has the form

\[
f_N = \sum_{j=1}^{N}(A_B - \delta_{j,N} I)^{-1}S_{\mu_0,B}f_{j,N}
\]

and so

\[
f_N = (A_B - \mu I)^{-1} \sum_{j=1}^{N}(A_B - \mu I)(A_B - \delta_{j,N} I)^{-1}S_{\mu_0,B}f_{j,N}
= (A_B - \mu I)^{-1} \sum_{j=1}^{N}\{S_{\mu_0,B}f_{j,N} + (\delta_{j,N} - \mu)(A_B - \delta_{j,N} I)^{-1}S_{\mu_0,B}f_{j,N}\}.
\]

Now the term \( \sum_{j=1}^{N}S_{\mu_0,B}f_{j,N} \) lies in the space \( \mathcal{T} \) of \((3.2)\), which is contained in \( \mathfrak{S} \) by Lemma 3.1. Thus \( f_N \) has the form \((A_B - \mu I)^{-1}h_N\) for some \( h_N \in \mathfrak{S} \). Hence \( f \) lies in \((A_B - \mu I)^{-1}\mathfrak{S}\), in other words \( \mathfrak{S} \subseteq (A_B - \mu I)^{-1}\mathfrak{S} \). This completes the proof. \( \square \)

Corresponding to the spaces \( \mathcal{S} \) and \( \mathcal{T} \) we define, from the formally adjoint \( \beta \) operators, the spaces

\[
\hat{\mathcal{S}} = \text{Span}\{\delta_{\overline{\sigma}(\lambda_B^*)}(\hat{A}_B^* - \delta I)^{-1}\text{Ran}(\hat{\bar{S}}_{\mu,B^*})\},
\]

\[
\hat{\mathcal{T}} = \text{Span}\{\delta_{\overline{\sigma}(\lambda_B^*)}\text{Ran}(\hat{\bar{S}}_{\mu,B^*})\},
\]

where \( \hat{S}_{\mu,B^*} = \left( (\hat{\Gamma}_1 - B^*\hat{\Gamma}_2)|_{\text{ker}(A^* - \mu I)} \right)^{-1} \) is the corresponding solution operator.

Once again, it may be shown that \( \hat{\mathcal{S}} = \hat{\mathcal{T}} \) and so \( \hat{\mathcal{S}} \) does not depend on \( \hat{\mu} \).

We have so far defined the Weyl function \( M_B(\cdot) \) on \( \rho(A_B) \) where it is an analytic function. In what follows we will call a point \( \lambda_0 \in \mathbb{C} \) a point of analyticity of \( M_B \) if all analytic continuations of \( M_B \) coincide in a neighbourhood of \( \lambda_0 \).

**Theorem 3.5.** Suppose that a point \( \lambda_0 \) is a point of analyticity of \( M_B \) and is also a limit point of points of analyticity of \( \lambda \mapsto (A_B - \lambda I)^{-1} \) – that is, \( \lambda \in \rho(A_B) \). Let \( \mathcal{S} \) be as in (2.1) and, for positive integers \( N \) and \( M \), let \( P_{N,\mathcal{S}} \) and \( P_{M,\hat{\mathcal{S}}} \) denote projections onto any \( N \) and \( M \)-dimensional subspaces of \( \mathcal{S} \) and \( \hat{\mathcal{S}} \) respectively. Then \( P_{M,\hat{\mathcal{S}}}(A_B - \lambda I)^{-1}P_{N,\mathcal{S}} \) is analytic at \( \lambda = \lambda_0 \). A similar result holds when one uses projections \( P_{N,T} \) and \( P_{M,\hat{T}} \) onto finite-dimensional subspaces of \( \mathcal{T} \) and \( \hat{T} \).

**Proof.** Let \( f \in \text{Ran}(\Gamma_1 - B\Gamma_2) \) and let \( F = S_{\mu,B}f \) for \( \mu \in \rho(A_B) \). Then for each \( \lambda \in \mathbb{C} \),

\[
(\hat{A}^* - \lambda I)F = (\mu - \lambda)F = (\mu - \lambda)S_{\mu,B}f.
\]

From the resolvent identity (3.4) it follows that for \( \lambda, \delta \in \rho(A_B) \),

\[
(A_B - \lambda I)^{-1}(A_B - \delta I)^{-1}(\hat{A}^* - \lambda I)F
= \frac{\mu - \lambda}{\lambda - \delta} \left\{(A_B - \lambda I)^{-1}S_{\mu,B}f - (A_B - \delta I)^{-1}S_{\mu,B}f\right\}.
\]

\(^1\)In fact we showed in \([\text{7}]\) that \( \hat{A}_{B^*} \) is the adjoint of \( A_B \).
and hence, replacing \((\hat{A}^* - \lambda)F\) on the left hand side by \((\mu - \lambda)S_{\mu,B}f\) and the first copy of \(S_{\mu,B}f\) on the right hand side by \((\mu - \lambda)^{-1}(\hat{A}^* - \lambda)F\),
\[
(A_B - \lambda I)^{-1}[(A_B - \delta I)^{-1}S_{\mu,B}f] = \frac{1}{(\mu - \lambda)(\lambda - \delta)}(A_B - \lambda)^{-1}(\hat{A}^* - \lambda)F - \frac{(A_B - \delta I)^{-1}}{\lambda - \delta}S_{\mu,B}f.
\]
Let \(v \in D(A^*)\) and recall that \((\Gamma_1 - B\Gamma_2)F = f\). The remainder of our proof relies heavily on the identity
\[
(F - (A_B - \lambda)^{-1}(\hat{A}^* - \lambda)F, (A^* - \lambda)F) = -((f, \tilde{\Gamma}_2v)_\mathcal{H} + (M_B(\lambda)f, (\tilde{\Gamma}_1 - B^*\tilde{\Gamma}_2)v)_\mathcal{K}
\]
which is eqn. (5.1) in [7]. Note that on the right hand side of this equation, the only \(\lambda\)-dependent term is \(M_B(\lambda)\). Using this identity yields
\[
((A_B - \lambda I)^{-1}[(A_B - \delta I)^{-1}S_{\mu,B}f], (A^* - \lambda)F) = \frac{1}{(\mu - \lambda)(\lambda - \delta)} \left\{ (F, (A^* - \lambda)F) + (f, \tilde{\Gamma}_2v)_\mathcal{H} - (M_B(\lambda)f, (\tilde{\Gamma}_1 - B^*\tilde{\Gamma}_2)v)_\mathcal{K} \right\}
\]
\[
- \frac{1}{\mu - \lambda} ((A_B - \delta I)^{-1}S_{\mu,B}f, (A^* - \lambda)F)
\]
If we now select \(N\) points \(\delta_j\) in the resolvent set of \(A_B\) and \(N\) functions \(f_j\) in \(\text{Ran}(\Gamma_1 - B\Gamma_2)\), and define
\[
\Phi := \sum_{j=1}^{N} (A_B - \delta_j I)^{-1}S_{\mu,B}f_j \in \mathcal{S}, \quad \Psi := \sum_{j=1}^{N} \frac{S_{\mu,B}f_j}{\lambda - \delta_j} \in \mathcal{T},
\]
\[
\Theta := \sum_{j=1}^{N} (A_B - \delta_j I)^{-1}S_{\mu,B}f_j \in \mathcal{S}, \quad \phi := \sum_{j=1}^{N} f_j,
\]
then we obtain, upon summing the identities (3.9) with \(\delta \mapsto \delta_j\) and \(f \mapsto f_j\),
\[
((A_B - \lambda I)^{-1}\Phi, (A^* - \lambda)F) = - (\Theta, (A^* - \lambda)F) + \frac{1}{\mu - \lambda} \left\{ (\Psi, (A^* - \lambda)F) + (\phi, \tilde{\Gamma}_2v)_\mathcal{H} - (M_B(\lambda)\phi, (\tilde{\Gamma}_1 - B^*\tilde{\Gamma}_2)v)_\mathcal{K} \right\}
\]
We have thus developed from \((3.8)\) an expression in which \((A^* - \lambda)F\) has been replaced by the arbitrary element \(\Phi\) of any finite-dimensional subspace of \(\mathcal{S}\). From the right hand side of the expression \((3.10)\), since \(M_B(\lambda)\) is analytic at \(\lambda_0\), and since none of the \(\delta_j\) is equal to \(\lambda_0\), it follows that \((A_B - \lambda I)^{-1}\Phi, (A^* - \lambda)F\) is analytic at \(\lambda_0\). Now the term \((A^* - \lambda)F\) may also be turned into an arbitrary element \(\Phi\) of any finite-dimensional subspace of \(\mathcal{S}\) by similar reasoning, and so \((A_B - \lambda I)^{-1}\Phi, \Phi\) is analytic at \(\lambda_0\).

The reasoning is similar but slightly simpler when working with elements of \(\mathcal{T}\). \(\square\)

In the case of isolated spectral points this theorem can be generalized as follows.

**Theorem 3.6.** Suppose that a point \(\lambda_0\) is a point of analyticity of \(M_B\) and that \(\lambda_0\) is at worst an isolated singularity of \((A_B - \lambda I)^{-1}\) and suppose that the resolvent set \(\rho(\hat{A}_{B^*})\) has finitely many connected components. Let \(P_{\mathcal{S}}\) and \(P_{\mathcal{T}}\) denote orthogonal
projections onto the closures of $\mathcal{S}$ and $\tilde{\mathcal{S}}$ respectively. Then $P_{\mathcal{S}}(A_B - \lambda I)^{-1}P_{\tilde{\mathcal{S}}}$ is analytic at $\lambda = \lambda_0$.

Proof. Assume that $\lambda \not\in \rho(A_B)$ otherwise the statement is trivial. In eqn. (3.10) take $v = (\tilde{\mathcal{S}}\tilde{\mu},B^*)g$ for any $g \in \mathrm{Ran}(\tilde{\Gamma}_1 - B^*\tilde{\Gamma}_2)$ and any $\tilde{\mu}$ not in the spectrum of $\tilde{A}_B^*$. Then $(A^* - \overline{\lambda})v = (\tilde{\mu} - \overline{\lambda})(\tilde{S}\tilde{\mu},B^*)g$ and so from (3.10),

$$(A_B - \lambda I)^{-1}\Phi, (\tilde{S}\tilde{\mu},B^*)g = -\frac{1}{\mu - \lambda} \left( \Theta, (A^* - \overline{\lambda})v \right)$$

$$+ \frac{1}{(\mu - \lambda)(\mu - \overline{\lambda})} \left\{ (\Psi, (A^* - \overline{\lambda})v) + (\phi, \tilde{\Gamma}_2 v)_{\mathcal{H}} - (M_B(\lambda)\phi, (\tilde{\Gamma}_1 - B^*\tilde{\Gamma}_2)v)_{\mathcal{K}} \right\}$$

Since $\tilde{\mu}$ lies in the resolvent set of $\tilde{A}_B^* = (A_B)^*$ we know that $\tilde{\mu} \neq \overline{\lambda}_0$. Let $\Gamma$ be any smooth closed contour surrounding $\lambda_0$, not enclosing $\mu$ or $\tilde{\mu}$ and bounded away from the spectrum of $A_B$. Integrating (3.11) around $\Gamma$ yields

$$\int_\Gamma (A_B - \lambda I)^{-1}\Phi, (\tilde{S}\tilde{\mu},B^*)g \ d\lambda = 0.$$ 

It follows that for any $\hat{\Phi}$ having a representation of the form

$$\hat{\Phi} = \sum_{j=1}^M (\tilde{S}_{\mu_j,B^*})g_j$$

in which the points $\mu_j$ lie outside $\Gamma$, we have

$$\int_\Gamma (A_B - \lambda I)^{-1}\Phi, \hat{\Phi} \ d\lambda = 0.$$ 

Consider now a general $\hat{\Phi}$ in $\mathcal{S} = \mathcal{T}$. Given $\epsilon > 0$, such a $\hat{\Phi}$ can be approximated to accuracy $\epsilon$ by $\hat{\Phi}_\epsilon$ of the form

$$\hat{\Phi}_\epsilon = \sum_{j=1}^M (\tilde{S}_{\mu_j,\epsilon,B^*})g_j$$

in which the points $\mu_j,\epsilon$ could, however, lie inside $\Gamma$. However the solution operator $\tilde{S}_{\tilde{\mu},B^*}$ is analytic for $\tilde{\mu}$ in the resolvent set $\rho(\tilde{A}_B^*)$. If the curve $\Gamma$ is chosen in a sufficiently small neighbourhood of $\lambda_0$ then its image under complex conjugation, denoted $\overline{\Gamma}$, lies in a single connected component of the resolvent set $\rho(\tilde{A}_B^*)$. Denote this connected component by $\mathcal{U}$ and choose any open set $O$ in $\mathcal{U}$ outside $\overline{\Gamma}$. The values of the analytic function $\tilde{\mu} \mapsto \tilde{S}_{\tilde{\mu},B^*}$ at any point in $\mathcal{U}$ (and hence, in particular, at any points $\tilde{\mu}_j,\epsilon$ inside $\overline{\Gamma}$) are uniquely determined by the values of this function in $O$, so it must be possible to approximate $\hat{\Phi}_\epsilon$ of the form (3.12) to accuracy $\epsilon$ by approximations of the form

$$\hat{\Phi}_\epsilon = \sum_{j=1}^K (\tilde{S}_{\zeta_j,\epsilon,B^*})h_j$$

in which the points $\zeta_j,\epsilon$ either lie in $O$ or in a completely different component of the resolvent set $\rho(\tilde{A}_B^*)$. We have $\|\hat{\Phi} - \hat{\Phi}_\epsilon\| < 2\epsilon$ and we also have, from (3.13),

$$\int_\Gamma (A_B - \lambda I)^{-1}\Phi, \hat{\Phi}_\epsilon \ d\lambda = 0.$$
Since the vectors \((A_B - \lambda I)^{-1}\Phi\) are uniformly bounded on \(\Gamma\), which does not intersect the spectrum of \(A_B\), we can take limits in \(\epsilon\) and obtain

\[
(3.17) \quad \int_\Gamma \left( (A_B - \lambda I)^{-1} \Phi, \tilde{\Phi} \right) d\lambda = 0
\]

for all \(\Phi \in \mathcal{S}, \tilde{\Phi} \in \mathcal{S}\). The result is now immediate from Morera’s theorem. \(\square\)

4. A FIRST-ORDER EXAMPLE

An obvious question arising from the previous section is whether or not the result of Theorem 3.5 remains true if one omits projections onto finite dimensional subspaces: if \(M_B(\lambda)\) is analytic at some point which is a non-isolated spectral point of \(A_B\), is \(\tilde{M}_B^{-1}(A_B - \lambda I)^{-1}\tilde{P}_S\) also analytic at this point? A simple example shows that this result is false.

Consider in \(L^2(0, \infty)\) the operator \(A = \tilde{A}\) given by \(D(A) = H^1_0(0, \infty)\) with

\[
(4.1) \quad Af = \frac{df}{dx}.
\]

The operator \(A\) is maximal symmetric and \(D(A^*) = H^1(0, \infty)\). Define the boundary spaces \(\mathcal{H} = \mathbb{C}, \mathcal{K} = \{0\}\), and boundary value operators \(\Gamma_1, \Gamma_2, \tilde{\Gamma}_1, \tilde{\Gamma}_2\) by

\[
(4.2) \quad \Gamma_1 f = if(0), \quad \Gamma_2 f = f(0).
\]

\[
(4.3) \quad \tilde{\Gamma}_1 f = 0, \quad \tilde{\Gamma}_2 f = 0.
\]

It is easy to see that the pairs \((\Gamma_1, \Gamma_2)\) and \((\tilde{\Gamma}_1, \tilde{\Gamma}_2)\) are surjective and a simple integration shows that

\[
(A^* f, g) - (f, A^* g) = if(0)g(0) = \Gamma_1 f\tilde{\Gamma}_2 g - \Gamma_2 f\tilde{\Gamma}_1 g.
\]

Because \(\tilde{\Gamma}_1\) and \(\Gamma_2\) are trivial it follows immediately from the definitions that

\[
M_B(\lambda) = 0; \quad \tilde{M}_B(\lambda) = -1/\tilde{B}.
\]

Moreover, \(\sigma(A_B) = \mathbb{C}^+\).

Now we consider the space \(\mathcal{T}^\perp\), for simplicity in the case \(B = 0\). For \(\Im(\mu) < 0\) a typical element of \(\mathcal{T}\) has the form \(y_\mu = S_{\mu,0} f\) and therefore satisfies \(iy'_\mu = \mu y_\mu\) with \(y_\mu(0) = f\); in other words, for some complex number \(f\),

\[
y_\mu(x) = f \exp(-i\mu x).
\]

Now suppose that \(u \in \mathcal{T}^\perp\). Then \((u, y_\mu) = 0\) for all \(\Im(\mu) < 0\). This means

\[
\int_0^\infty u(x) \exp(i\mu x)dx = 0
\]

for all \(\Im(\mu) < 0\). Setting \(\mu = \omega - ir, r > 0\), we deduce that for all \(\omega \in \mathbb{R}\)

\[
\int_0^\infty u(x) \exp(-rx) \exp(i\omega x)dx = 0.
\]

From inverse Fourier transformation this implies that \(u(x) \exp(-rx) = 0\) for all \(x\) and hence \(u(x) \equiv 0\). Thus we have proved that for this example,

\[
\mathcal{T} = \mathcal{T}^\perp = L^2(0, \infty)
\]
and so \((A_B - \lambda I)^{-1}\) is not reduced by the bordering projection operators \(P_T\) and \(P_{\tilde{T}}\). It follows that for this example, the set of singular points of the bordered resolvent is strictly greater than the set of singular points of \(M_B(\lambda)\).

5. A Hain-Lüst type example

In this section we consider a block operator matrix example in which \(P_{\tilde{T}}(A_B - \lambda I)^{-1}P_{\tilde{T}}\) has exactly the same singularities as \(M_B(\lambda)\), even though some of these singularities are not isolated. In other words, for the example which we present here, a stronger result holds than those available in Theorems 3.5 and 3.6. It is not yet clear to us what special properties of this example mean that, unlike for the example of Section 4, better results hold here than those in Theorems 3.5 and 3.6.

Let
\[ (5.1) \quad \tilde{A}^* = \left( \begin{array}{cc} -\frac{d^2}{dx^2} + q(x) & w(x) \\ w(x) & u(x) \end{array} \right), \]
where \(q, u\) and \(w\) are complex-valued \(L^\infty\)-functions, and the domain of the operator is given by
\[ (5.2) \quad D(\tilde{A}^*) = H^2(0, 1) \times L^2(0, 1). \]
Also let
\[ (5.3) \quad A^* = \left( \begin{array}{cc} -\frac{d^2}{dx^2} + q(x) & w(x) \\ w(x) & u(x) \end{array} \right), \quad \text{with } D(A^*) = D(\tilde{A}^*). \]
It is then easy to see that
\[ (5.4) \quad \langle \tilde{A}^* \left( \begin{array}{c} y \\ z \end{array} \right), \left( \begin{array}{c} f \\ g \end{array} \right) \rangle - \langle \left( \begin{array}{c} y \\ z \end{array} \right), A^* \left( \begin{array}{c} f \\ g \end{array} \right) \rangle = \langle \Gamma_1 \left( \begin{array}{c} y \\ z \end{array} \right), \Gamma_2 \left( \begin{array}{c} f \\ g \end{array} \right) \rangle - \langle \Gamma_2 \left( \begin{array}{c} y \\ z \end{array} \right), \Gamma_1 \left( \begin{array}{c} f \\ g \end{array} \right) \rangle, \]
where
\[ \Gamma_1 \left( \begin{array}{c} y \\ z \end{array} \right) = \left( \begin{array}{c} -y'(1) \\ y'(0) \end{array} \right), \quad \Gamma_2 \left( \begin{array}{c} y \\ z \end{array} \right) = \left( \begin{array}{c} y(1) \\ y(0) \end{array} \right). \]
Consider the operator
\[ (5.5) \quad A_{\alpha \beta} := \tilde{A}^* \bigg|_{\ker(\Gamma_1 - B\Gamma_2)}, \]
where, for simplicity,
\[ (5.6) \quad B = \left( \begin{array}{cc} \cot \beta & 0 \\ 0 & -\cot \alpha \end{array} \right). \]
It is known \[2\] that
\[ \sigma_{\text{ess}}(A_{\alpha \beta}) = \text{essran}(u) := \{ z \in \mathbb{C} \mid \forall \epsilon > 0, \text{ meas } \{ x \in [0, 1] \mid |u(x) - z| < \epsilon \} > 0 \}. \]
This result is independent of the choice of boundary conditions. The measure used is Lebesgue. Note also that \(\sigma(A_{\alpha \beta})\) is not the whole of \(\mathbb{C}\) for essentially bounded \(q, u\) and \(w\). For future use we also define the set
\[ \mathcal{W} = \{ x \in [0, 1] \mid w(x) \neq 0 \}. \]
The function \(w\) is defined only almost everywhere, but this is sufficient to define \(\mathcal{W}\) up to a set of measure zero, which can be neglected.
We now calculate the function \( M(\lambda) = \begin{pmatrix} m_{11}(\lambda) & m_{12}(\lambda) \\ m_{21}(\lambda) & m_{22}(\lambda) \end{pmatrix} \) such that

\[
M(\lambda)(\Gamma_1 - B\Gamma_2) \begin{pmatrix} y \\ z \end{pmatrix} = \Gamma_2 \begin{pmatrix} y \\ z \end{pmatrix}
\]

for \( \begin{pmatrix} y \\ z \end{pmatrix} \in \ker(\tilde{A}^* - \lambda) \). In our calculation we assume that \( \lambda \notin \sigma_{ess}(A_{\alpha\beta}) \). The condition \( \begin{pmatrix} y \\ z \end{pmatrix} \in \ker(\tilde{A}^* - \lambda) \) yields the equations

\[
-y'' + (q - \lambda) y + wz = 0; \quad wy + (u - \lambda)z = 0
\]

which, in particular, give

\[
(5.7) - y'' + (q - \lambda)y + \frac{w^2}{\lambda - u}y = 0.
\]

The linear space \( \ker(\tilde{A}^* - \lambda) \) is thus spanned by the functions

\[
\begin{pmatrix} y_1(1) \\ wy_1(1)/(\lambda - u) \end{pmatrix}
\]

and

\[
\begin{pmatrix} y_2(1) \\ wy_2(1)/(\lambda - u) \end{pmatrix}
\]

where \( y_1 \) and \( y_2 \) are solutions of the initial value problems consisting of the differential equation \((5.7)\) equipped with initial conditions

\[
(5.8) \quad y_1(0) = \cos \alpha, \quad y'_1(0) = \sin \alpha,
\]

\[
(5.9) \quad y_2(0) = -\sin \alpha, \quad y'_2(0) = \cos \alpha,
\]

where \( \alpha \) is as in \(5.6\). A straightforward calculation shows that

\[
(5.10) \quad m_{11}(\lambda) = -\frac{y_2(1,\lambda)}{y_2'(1,\lambda) + \cot \beta y_2(1,\lambda)},
\]

\[
(5.11) \quad m_{21}(\lambda) = m_{12}(\lambda) = \frac{\sin \alpha}{y_2'(1,\lambda) + \cot \beta y_2(1,\lambda)},
\]

\[
(5.12) \quad m_{22}(\lambda) = \sin \alpha \cos \alpha + \sin^2 \alpha \begin{pmatrix} y'_1(1,\lambda) + \cot \beta y_1(1,\lambda) \\ y'_2(1,\lambda) + \cot \beta y_2(1,\lambda) \end{pmatrix}.
\]

As an aside, notice that all these expressions contain a denominator \( y_2'(1,\lambda) + \cot \beta y_2(1,\lambda) \) and that \( \lambda \notin \text{essran}(u_{|\mathcal{W}}) \) is an eigenvalue precisely when this denominator is zero.

Remark 5.1. For \( \lambda \in \mathbb{C} \setminus \text{essran}(u_{|\mathcal{W}}) \), the coefficient \( w(x)^2/(u(x) - \lambda) \) in \(5.7\) is analytic as a function of \( \lambda \). Therefore, the solutions \( y_1 \) and \( y_2 \) are analytic in \( \lambda \). The \( M \)-function may have an isolated pole at some point \( \lambda \) if \( y_2'(1,\lambda) + \cot \beta y_2(1,\lambda) \) happens to be zero; such a pole will be an eigenvalue of the operator \( A_{\alpha\beta} \) and may or may not be embedded in the essential spectrum of the operator.

As a consequence of this remark, the \( M \)-function can be analytic at points in the essential range of \( u \), as long as those points are outside the essential range of \( u_{|\mathcal{W}} \):
Lemma 5.2. Apart from poles at eigenvalues of $A_{\alpha\beta}$, the M-function $M(\lambda)$ is analytic in the set $\mathbb{C} \setminus \text{essran}(u|_\mathcal{W})$.

We now turn our attention to the behaviour of the resolvent $(A_{\alpha\beta} - \lambda I)^{-1}$ on the spaces $\mathcal{T}$ and $\overline{\mathcal{T}}$.

Theorem 5.3.

(5.13) \[ \mathfrak{F} = \mathcal{T} \subseteq \left( \begin{array}{c} L^2(0,1) \\ L^2(\mathcal{W}) \end{array} \right). \]

Moreover if $M_B(\lambda)$ is analytic at a point $\lambda$ not in essran$(u|_\mathcal{W})$ then

(5.14) \[ \left( \begin{array}{c} y \\ z \end{array} \right) := (A_{\alpha\beta} - \lambda I)^{-1} \left( \begin{array}{c} f \\ g \end{array} \right) \]

admits analytic continuation for any $f \in L^2(0,1)$ and $g \in L^2(\mathcal{W})$.

Proof. Suppose that $(f_1, f_2) \in \mathbb{C}^2$ and that $\mu \in \rho(A_{\alpha\beta})$. Since $\mu$ does not lie in the essential spectrum, it does not lie in the essential range of $u$, so $1/(u - \mu)$ is essentially bounded. Consider the functions $y_\mu$, $z_\mu$ defined by

\[ \left( \begin{array}{c} y_\mu \\ z_\mu \end{array} \right) = S_{\mu, B} \left( \begin{array}{c} f_1 \\ f_2 \end{array} \right); \]

eliminating $z_\mu$ from these equations using

(5.15) \[ z_\mu = \frac{w y_\mu}{u - \mu} \]

we find that $y_\mu$ satisfies the ODE

(5.16) \[ -y_\mu'' + (q - \mu)y_\mu + \left( \frac{w^2}{\mu - u} \right) y_\mu = 0 \]

with boundary conditions $y_\mu'(1) + \cot(\beta)y_\mu(1) = -f_1$ and $y_\mu'(0) + \cot(\alpha)y_\mu(0) = f_2$.

The boundary value problem for $y_\mu$ is uniquely solvable because $\mu \in \rho(A_{\alpha\beta})$ and so $y_\mu \in L^2(0,1)$. It follows from (5.15) that $z_\mu \in L^2(\mathcal{W})$. This proves the inclusion (5.13).

We decompose the space

(5.17) \[ \left( \begin{array}{c} L^2(0,1) \\ L^2(0,1) \end{array} \right) = \left( \begin{array}{c} L^2(0,1) \\ L^2(\mathcal{W}) \end{array} \right) \oplus \left( \begin{array}{c} 0 \\ L^2(\mathcal{W}^c) \end{array} \right) \]

where $\mathcal{W}^c = [0,1] \setminus \mathcal{W}$. Denote $H_1 = \left( \begin{array}{c} L^2(0,1) \\ L^2(\mathcal{W}) \end{array} \right)$ and $H_2 = \left( \begin{array}{c} 0 \\ L^2(\mathcal{W}^c) \end{array} \right)$.

We shall now show that these are reducing subspaces for the operator $A_{\alpha\beta}$. It is clear that if $\left( \begin{array}{c} h \\ g \end{array} \right) \in D(A_{\alpha\beta})$ then the projections of $\left( \begin{array}{c} h \\ g \end{array} \right)$ onto $H_1$ and $H_2$ will also lie in the domain of the operator as $H_2 \subseteq D(A_{\alpha\beta})$. The conditions $A_{\alpha\beta} P_{H_i} \left( \begin{array}{c} h \\ g \end{array} \right) \in H_i$ when $\left( \begin{array}{c} h \\ g \end{array} \right) \in D(A_{\alpha\beta})$ for $i = 1, 2$ are a simple calculation.

Here $P_i$ denotes the orthogonal projection onto $H_i$.

We have $\sigma_{\text{ess}}(A_{\alpha\beta}|_{H_1}) = \text{essran}(u|_\mathcal{W})$. By Remark 5.1 any eigenvalue of the operator $A_{\alpha\beta}|_{H_1}$ will be a pole of $M_B(\lambda)$. Hence, if $M_B(\lambda)$ is analytic at a point $\lambda$ not in essran$(u|_\mathcal{W})$, we have that $\lambda \in \rho(A_{\alpha\beta}|_{H_1})$ and for any $\left( \begin{array}{c} f \\ g \end{array} \right) \in H_1$, $(A_{\alpha\beta} - \lambda I)^{-1} \left( \begin{array}{c} f \\ g \end{array} \right)$ admits analytic continuation. \qed
As an immediate corollary of this theorem we have

**Corollary 5.4.** For \( \lambda \notin \text{essran} (u|_W) \) the bordered resolvent \( P_S(A_{\alpha\beta} - \lambda I)^{-1}P_S \) is analytic precisely where \( M_B(\lambda) \) is analytic.

**Proof.** Since \( (A_{\alpha\beta} - \lambda I)^{-1}|_{H_1} \) is analytic on the space \( H_1 \) which is larger than \( S \) by Theorem 5.3 it is immediate that the bordered resolvent is analytic wherever \( M_B(\cdot) \) is analytic. The fact that \( M_B(\cdot) \) is analytic whenever the bordered resolvent is analytic follows from (3.10). \( \square \)

**Remark 5.5.** Generically one expects that \( M_B(\cdot) \) will not be analytic at points in \( \text{essran} (u|_W) \). The analyticity or otherwise depends on the analyticity or otherwise of solutions of the ODE (5.7).

It is worth mentioning also the following result.

**Proposition 5.6.** Let \( \lambda \) be any fixed point in the resolvent set \( \rho(A_{\alpha\beta}) \). Then

\[
(A_{\alpha\beta} - \lambda I)^{-1} \begin{pmatrix} L^2(0,1) \\ L^2(W) \end{pmatrix} = \begin{pmatrix} L^2(0,1) \\ L^2(W) \end{pmatrix}.
\]

**Proof.** The domain of the differential expression

\[-\frac{d^2}{dx^2} + q - \lambda - \frac{w^2}{u - \lambda}\]
equipped with boundary conditions \( y(1) + \cot(\beta)y(1) = 0 \) and \( y'(0) + \cot(\alpha)y(0) = 0 \), is dense in \( L^2(0,1) \). Thus any function in \( L^2(0,1) \) can be approximated to arbitrary accuracy by a solution \( y \) of a boundary value problem

\[-\frac{d^2}{dx^2} + q - \lambda - \frac{w^2}{u - \lambda})y = h \in L^2(0,1) \quad y(1) + \cot(\beta)y(1) = 0 = y'(0) + \cot(\alpha)y(0)\]

for a suitably chosen \( h \). Having fixed such \( y \) and \( h \), then for any \( z \in L^2(W) \) we may define \( g \) to satisfy

\[z = \frac{1}{u - \lambda}(g - wy)\]

and clearly have \( g \in L^2(W) \). Finally we define \( f \in L^2(0,1) \) by \( f = h + wg/(u - \lambda) \) so that \( f \in L^2(0,1) \) and \( h = f - wg/(u - \lambda) \). We thus have

\[
(-\frac{d^2}{dx^2} + q - \lambda - \frac{w^2}{u - \lambda})y = f - wg/(u - \lambda), \quad z = \frac{1}{u - \lambda}(g - wy).
\]

This is equivalent to (5.14). We have therefore approximated an arbitrary element of \( \begin{pmatrix} L^2(0,1) \\ L^2(W) \end{pmatrix} \) by a function in \( (A_{\alpha\beta} - \lambda I)^{-1} \begin{pmatrix} L^2(0,1) \\ L^2(W) \end{pmatrix} \). To get the opposite inclusion consider

\[
\begin{pmatrix} y \\ z \end{pmatrix} = (A_{\alpha\beta} - \lambda I)^{-1} \begin{pmatrix} f \\ g \end{pmatrix}
\]
in which \( g \in L^2(W) \). We need to show that \( z \in L^2(W) \) also. The expression for \( z \) is given in (5.19): evidently \( wy \in L^2(W) \) and \( g \in L^2(W) \) so the result is immediate. \( \square \)
6. A perturbed multiplication operator in $L^2(\mathbb{R})$

The results of the foregoing sections show that there are often wide gaps between what may be true at an abstract level about the relationship between resolvents and $M$-functions, and what may be achievable in concrete examples.

In light of these gaps, in this section we consider boundary triplets and Weyl $M$-functions for a simple Friedrichs model with a singular perturbation. Our purpose is to show even more unexpected and counter-intuitive results. For example, in [7, Section 4] it is shown that isolated eigenvalues of an operator correspond to isolated poles of the associated $M$-function assuming unique continuation holds, i.e.

$$\ker(\tilde{A}^* - \lambda) \cap \ker(\Gamma_1) \cap \ker(\Gamma_2) = \{0\},$$

while [22, Proposition 5.2] shows this result under the assumption that the point under consideration is in the resolvent set of an extension of the minimal operator. In this section, we shall find that these hypotheses which have seemed reasonable in the development of an abstract theory of boundary triplets are not satisfied by a rather simple example. As a consequence, the relationship between the $M$-function and the spectrum of the operator becomes more interesting.

We consider in $L^2(\mathbb{R})$ the operator $A$ with domain given by

$$D(A) = \left\{ f \in L^2(\mathbb{R}) \mid xf(x) \in L^2(\mathbb{R}), \lim_{R \to \infty} \int_{-R}^{R} f(x)dx \text{ exists and is zero} \right\},$$

given by the expression

$$(Af)(x) = xf(x) + \langle f, \phi \rangle \psi(x),$$

where $\phi, \psi$ are in $L^2(\mathbb{R})$. Observe that since the constant function $1$ does not lie in $L^2(\mathbb{R})$ the domain of $A$ is dense in $L^2(\mathbb{R})$.

Formally, the expression $xf(x) + \langle f, \phi \rangle \psi(x)$ is equivalent, by Fourier transformation, to a sum of a first order differential operator and an inner product (integral) term acting on the Fourier transform $\hat{f}$. The condition $\int_{\mathbb{R}} f = 0$ is equivalent to a ‘boundary’ condition $\hat{f}(0) = 0$.

**Lemma 6.1.** The adjoint of $A$ is given on the domain

$$D(A^*) = \left\{ f \in L^2(\mathbb{R}) \mid \exists c_f \in \mathbb{C} : xf(x) - c_f 1 \in L^2(\mathbb{R}) \right\},$$

by the formula

$$(A^*f)(x) = xf(x) - c_f 1 + \langle f, \psi \rangle \phi.$$

**Proof.** Suppose that $f \mapsto \langle Af, g \rangle$ is a bounded linear functional on $D(A)$. A direct calculation shows that

$$\langle Af, g \rangle = \int_{\mathbb{R}} f(x)(xg(x) + \langle g, \psi \rangle \phi(x))dx.$$

(Note that the integral is convergent since $xf(x) \in L^2(\mathbb{R})$ and $g \in L^2(\mathbb{R})$.) In view of the constraint $\int_{\mathbb{R}} f = 0$ and the density of $D(A)$ in $L^2(\mathbb{R})$, the $L^2(\mathbb{R})$-boundedness of this functional implies that for some constant $c_g$,

$$xf + \langle g, \psi \rangle \phi = c_g 1 + h$$

for some $h \in L^2(\mathbb{R})$. Since $\phi \in L^2(\mathbb{R})$ this implies that $xg - c_g 1 \in L^2(\mathbb{R})$ and so

$$\langle Af, g \rangle = \langle f, xg - c_g 1 + \langle g, \psi \rangle \phi \rangle.$$
The density of $D(A)$ in $L^2(\mathbb{R})$ now gives $A^*g = xg - c_g 1 + \langle g, \psi \rangle \phi$.

**Remark 6.2.** For $f$ sufficiently well behaved at infinity, the constant $c_f$ appearing in Lemma 6.1 is given by

$$c_f = \lim_{x \to \infty} xf(x).$$

For later reference we can calculate the deficiency indices of $A$. To this end we may neglect the finite rank term $\langle \cdot, \psi \rangle \phi$ and calculate the set of $u$ such that

$$xu(x) - c_u 1 = \pm iu.$$

This yields $u = c_u \frac{1}{x \mp i}$; the factor $c_u$ is a normalization. A simple calculation shows that $c_u(x \mp i)^{-1} = 1$ and so we may choose

$$u(x) = (x \mp i)^{-1}$$

as the deficiency elements, showing that $A$ has deficiency indices $(1, 1)$.

We now introduce ‘boundary value’ operators $\Gamma_1$ and $\Gamma_2$ on $D(A^*)$ as follows:

$$(6.5) \quad \Gamma_1 u = \int_{\mathbb{R}} (u(x) - c_u 1 \text{sign}(x) (x^2 + 1)^{-1/2}) \, dx, \quad \Gamma_2 u = c_u.$$

We make the following observations.

**Lemma 6.3.** The operators $\Gamma_1$ and $\Gamma_2$ are bounded relative to $A^*$.

**Proof.** Observe that

$$c_u = -A^* u + xu + \langle u, \psi \rangle \phi.$$

Multiply both sides by the characteristic function $\chi_{(0,1)}$ of the interval $(0, 1)$, then take $L^2$-norms, to obtain

$$|c_u| \leq \|A^* u\| + \|u\| + \|u\| \|\psi\| \|\phi\|$$

which shows that $\Gamma_2$ is bounded relative to $A^*$. Similarly, an elementary calculation shows that

$$\Gamma_1 u = \int_{\mathbb{R}} \left\{ (\sqrt{x^2 + 1} \text{sign}(x) - x) u(x) + (xu(x) - c_u 1) \right\} \frac{\text{sign}(x)}{\sqrt{x^2 + 1}} \, dx;$$

since $(\sqrt{x^2 + 1} \text{sign}(x) - x) \in L^2(\mathbb{R})$, this shows that $\Gamma_1$ is bounded relative to $A^*$.

**Lemma 6.4.** The following ‘Green’s identity’ holds:

$$(6.6) \quad \langle A^* f, g \rangle - \langle f, A^* g \rangle = \Gamma_1 f \Gamma_2 g - \Gamma_2 f \Gamma_1 g + \langle f, \psi \rangle \langle \phi, g \rangle - \langle f, \phi \rangle \langle \psi, g \rangle.$$

Consequently, in the case when $\phi = \psi$, the operators $A^*|_{\ker(\Gamma_1 - B\Gamma_2)}$ are selfadjoint for any real number $B$.

**Proof.** The identity (6.6) is a simple calculation. In the case when $\phi = \psi$ the operator $A$ is symmetric and the selfadjointness of the extensions $A^*|_{\ker(\Gamma_1 - B\Gamma_2)}$ is a well known result from theory of boundary value spaces: see, e.g., Gorbachuk and Gorbachuk [13].

In the case when $\phi \neq \psi$, the terms $\langle f, \psi \rangle \langle \phi, g \rangle - \langle f, \phi \rangle \langle \psi, g \rangle$ on the right hand side of (6.6) arise from the fact that $A^*$ is not an extension of $A$. In order to eliminate
these terms we follow the formalism of Lyantze and Storozh [19] and introduce an operator $\tilde{A}$ in which $\phi$ and $\psi$ are swapped:

\[
D(\tilde{A}) = \left\{ f \in L^2(\mathbb{R}) \mid xf(x) \in L^2(\mathbb{R}), \lim_{R \to \infty} \int_{-R}^{R} f(x)dx = 0 \right\},
\]

(6.7) \[
(\tilde{A}f)(x) = xf(x) + \langle f, \psi \rangle \phi.
\]

In view of Lemma 6.1 we immediately see that $D(\tilde{A}^*) = D(A^*)$ and that

(6.8) \[
\tilde{A}^*f = xf(x) - cf1 + \langle f, \phi \rangle \psi.
\]

Thus $\tilde{A}^*$ is an extension of $A$, $A^*$ is an extension of $\tilde{A}$, and the following result is easily proved.

**Lemma 6.5.**

(6.10) \[
A = \tilde{A}^* \bigg|_{\ker(\Gamma_1) \cap \ker(\Gamma_2)} ; \quad \tilde{A} = A^* \bigg|_{\ker(\Gamma_1) \cap \ker(\Gamma_2)} ;
\]

moreover, the Green’s formula (6.6) can be modified to

(6.11) \[
\langle A^*f, g \rangle - \langle f, \tilde{A}^*g \rangle = \Gamma_1f\Gamma_2g - \Gamma_2f\Gamma_1g.
\]

This is a slight simplification of the situation in [19] as only two boundary operators are required, rather than four.

For any fixed complex number $B$ and suitable $\lambda \in \mathbb{C}$, by the ‘Weyl function $M_B(\lambda)$’ we shall mean the map

(6.12) \[
M_B(\lambda) := \Gamma_2 \left( (\Gamma_1 - B\Gamma_2) \big|_{\ker(\tilde{A}^* - \lambda)} \right)^{-1}.
\]

We now calculate $M_B(\lambda)$. Suppose that $\Im \lambda \neq 0$ and that $f \in \ker(\tilde{A}^* - \lambda I)$. Then

\[
f \in \mathbb{C}^\mathbb{R} \quad \text{and simple algebra yields}
\]

(6.13) \[
f = \frac{cf - \langle f, \phi \rangle \psi}{x - \lambda}.
\]

Taking inner products with $\phi$ and recalling that $\Gamma_2f = cf$ yields

(6.14) \[
\langle f, \phi \rangle D(\lambda) = \Gamma_2f \langle (x - \lambda)^{-1}, \phi \rangle
\]

where $D(\lambda) = 1 + \int_{\mathbb{R}} (x - \lambda)^{-1} \psi \phi dx$. Substituting back into (6.13) yields

(6.15) \[
f = \Gamma_2f \left[ \frac{1}{x - \lambda} - \frac{\langle (x - \lambda)^{-1}, \phi \rangle}{D(\lambda)} \frac{\psi}{x - \lambda} \right],
\]

It follows upon calculating the relevant integrals that

(6.16) \[
\Gamma_1f = \left[ \text{sign}(\Im \lambda) \pi i + \frac{\langle (x - \lambda)^{-1}, \psi \rangle (x - \lambda)^{-1}, \phi \rangle}{D(\lambda)} \right] \Gamma_2f,
\]

and so

(6.17) \[
M_B(\lambda) = \left[ \text{sign}(\Im \lambda) \pi i + \frac{\langle (x - \lambda)^{-1}, \psi \rangle (x - \lambda)^{-1}, \phi \rangle}{D(\lambda)} - B \right]^{-1}.
\]
Remark 6.6. If \( D(\lambda) \) is nonzero then a local unique continuation property holds:
\[
(6.18) \quad f \in \ker(\tilde{A}^* - \lambda) \cap \ker(\Gamma_1) \cap \ker(\Gamma_2) = 0 \implies f = 0.
\]
To see this observe that from (6.15) we see that \( \Gamma_2 f = 0 \) implies \( f = 0 \), giving
unique continuation a fortiori.

Remark 6.7. Generically, the \( M \)-function \( M_B(\lambda) \) ‘sees’ the whole essential spectrum: the term \( \text{sign}(\Im(\lambda))\pi i \) has a discontinuity across the real axis which one cannot expect to be cancelled by the other terms, except possibly on a set of measure zero.

Example. If \( \phi \) and \( \psi \) both lie in the Hardy space \( H^2 \) (see Koosis [15] for definitions and properties of Hardy spaces) then the inner product \( \langle (x - \lambda)^{-1}, \phi \rangle \) is zero for \( \Im \lambda > 0 \) and the inner product \( \langle (x - \lambda)^{-1}, \psi \rangle \) is zero for \( \Im \lambda < 0 \). In this case \( M_B(\lambda) \) has no poles and is given by
\[
M_B(\lambda) = (\text{sign}(\Im(\lambda))\pi i - B)^{-1}.
\]
If \( B = \pi i \) then the entire upper half plane is filled with eigenvalues of the operator \( \tilde{A}^*|_{\ker(\Gamma_1 - B \Gamma_2)} \); if \( B = -\pi i \) then it is the lower half plane which is entirely filled with eigenvalues.

Example. We construct an example with a particularly interesting property: an eigenvalue which is not a pole of the \( M \)-function.
Consider the case where \( \phi \) and \( \psi \) both lie in \( H^2 \); fix \( \lambda_0 \) and, by choice of \( \phi \) and \( \psi \), arrange that \( D(\lambda_0) = 0 \). Avoid the pathological cases where eigenvalues fill the entire upper or lower half planes by choosing \( B = 0 \); we have
\[
M_0(\lambda) = \frac{1}{\pi i} \text{sign}(\Im(\lambda)).
\]
Consider the function
\[
u(x) = \frac{\psi(x)}{x - \lambda_0}.
\]
Since \( D(\lambda_0) = 0 \) it follows that \( \langle u, \phi \rangle = -1 \). Moreover it is easy to check that \( \Gamma_2 \nu = c_u = 0 \). It is now easy to check that \( u \) satisfies
\[
(xu(x) + \langle u, \phi \rangle \psi = \frac{\lambda_0 \psi}{x - \lambda_0}
\]
and so \( u \) is an eigenfunction of \( \tilde{A}^*|_{\ker(\Gamma_2)} \) with eigenvalue \( \lambda_0 \). However \( \lambda_0 \) is not a pole of \( M_0^{-1}(\lambda) \), in apparent contradiction to the results in [22] and [7] mentioned at the beginning of this section.

Which hypotheses have failed?

If \( \Im \lambda_0 < 0 \) then observe that \( \Gamma_1 u = \langle (x - \lambda_0)^{-1}, \psi \rangle = 0 \), so the eigenfunction \( u \) belongs to the domain of the minimal operator \( A \), and hence to the domain of every extension: thus unique continuation fails, so there is no contradiction to the theorems in [7]. The failure of unique continuation implies that there is no extension of \( A \) for which \( \lambda_0 \) lies in the resolvent set, and so there is no contradiction to the results in [22] either.

If \( \Im \lambda_0 > 0 \) then although \( \lambda_0 \) is no longer an eigenvalue for every extension, it nevertheless lies in the spectrum of every extension. To see this, attempt to solve
\[
(x - \lambda)u - \Gamma_1 u + \langle u, \phi \rangle \psi = f,
\]
\[(\Gamma_1 - C\Gamma_2)u = 0,\]

with \(3\lambda > 0\). Taking the inner products of both sides and remembering that \(\langle (x - \lambda)^{-1}, \phi \rangle = 0\) in the upper half plane we obtain

\[\langle u, \phi \rangle = \frac{f(x - \lambda_0, \phi) - \langle u, \phi \rangle}{x - \lambda_0, \phi} = 0.\]

At \(\lambda = \lambda_0\) we have \(\langle \psi(x - \lambda_0), \phi \rangle = -1\) since \(D(\lambda_0) = 0\) and so we obtain

\[(6.19) \langle \frac{f}{x - \lambda_0}, \phi \rangle = 0.\]

Thus the problem can only be solved for \(f\) satisfying the condition \(6.19\) and so \(\lambda_0\) lies in the spectrum of every extension of \(\tilde{A}^*\). This gives a further reason why we would not expect \(\lambda_0\) to be a pole of any \(M\)-function.

**Example.** In the case \(\phi = \psi \in H^2_0\) the operators \(\tilde{A}^*|_{\ker(\Gamma_1 - B\Gamma_2)}\) are selfadjoint for real \(B\). The functions \(M_B(\lambda)\) still cannot ‘see’ \(\phi\) and \(\psi\), however, being given by

\[M_B(\lambda) = (\text{sign}(\Im \lambda) \pi i - B)^{-1}.\]

Any eigenvalues of the operator will obviously be real and will be imbedded in the essential spectrum. If \(\lambda_0 \in \mathbb{R}\) and \(\psi(\lambda_0) = 0\) and

\[\int_{\mathbb{R}} |\psi(x)|^2 \, dx = -1,\]

which can always be arranged, then \(\lambda_0\) will be an eigenvalue with eigenfunction \(\psi/(x - \lambda_0)\). The operator will not be unitarily equivalent to the unperturbed operator, which has no eigenvalues. This is not surprising as the eigenfunction here belongs to the minimal operator, which therefore fails to be completely non-selfadjoint.

There is therefore no contradiction to the results of Krein, Langer and Textorius \[16 \, 17 \, 18\] and Ryzhov \[27\] which state that if the minimal operator is completely non-selfadjoint then the maximal operator is determined up to unitary equivalence by the \(M\)-function.

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