Thermodynamics of the harmonic oscillator: derivation of the Planck blackbody spectrum from pure thermodynamics

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Abstract
In 1893, Wien applied the first two laws of thermodynamics to blackbody radiation and derived his displacement theorem. Believing that the information from thermodynamics had been exhausted, Planck turned to statistical ideas in 1900 in order to provide a physical understanding for his experimental-data-based interpolation giving the Planck spectrum without zero-point radiation. Here we point out that the third law of thermodynamics (which was developed in the early years of the 20th century) introduces additional thermodynamic information regarding thermal radiation. The Planck spectrum for thermal radiation can be derived from purely thermodynamic ideas applied to the classical simple harmonic oscillator, since every radiation mode takes a simple oscillator form. Thermodynamics alone implies the Planck spectrum including zero-point energy without any need for quantum theory or statistical ideas. It is pointed out that the inclusion of zero-point energy involves more natural thermodynamics than its exclusion.

Keywords: blackbody radiation, Planck spectrum, thermodynamics of a harmonic oscillator, third law of thermodynamics, zero-point energy

1. Introduction
At the end of the 19th century, there were only two laws of thermodynamics. The first law introduced heat energy into the ideas of energy conservation. The second law involved the existence of an entropy state function and an absolute temperature scale. In 1893, these two
laws were used by Wien to derive his displacement theorem for the blackbody radiation spectrum [1]. However, Wien’s displacement theorem did not solve the blackbody radiation problem because there still remained an unknown function of a single variable involving frequency divided by absolute temperature. In 1900, when Planck became convinced that the useful information from the laws of thermodynamics had been exhausted, he turned to statistical mechanics to give a physical basis for his interpolation guess for the blackbody radiation spectrum [2]. Planck’s use of statistical ideas led to the beginnings of quantum theory. It is striking that within a decade of Planck’s work, Nernst’s heat theorem associated with the low-temperature specific heats of solids gave the basis for what is now termed the third law of thermodynamics. Although the textbooks of thermodynamics and statistical mechanics regard the third law as related to quantum theory, [3] the third law also fits into classical theory [4]. Indeed it turns out that the use of all three laws of thermodynamics allows a complete derivation of the Planck spectrum based upon purely thermodynamic ideas without any recourse whatsoever to a statistical theory. The basic thermodynamics and mathematics of the analysis is elementary and can be followed by an advanced undergraduate physics student.

2. Outline of the analysis

Our analysis makes use of information about classical harmonic oscillators, the laws of thermodynamics, and applications of analytic function theory. We start by noting that a classical harmonic oscillator in a thermodynamic bath can be regarded as a thermodynamic system with an energy, generalized force, and temperature. The classical oscillator has an adiabatic invariant given by the energy divided by the frequency, and this adiabatic invariant allows us to recognize the generalized force associated with the work done by an adiabatic change in the spring constant (or of the frequency). Thus a classical harmonic oscillator in a heat bath can be treated as a very simple thermodynamic system of two independent variables, energy $U$ and frequency $\omega$. Next we turn to the laws of thermodynamics, starting with the existence of an empirical temperature and then using the first two laws to establish that the entropy $S$ of the oscillator is a function of a single variable involving the energy divided by the frequency, $S = S(U/\omega)$. Next we assume that at high temperature for fixed frequency, the energy of the oscillator becomes directly proportional to the thermodynamic temperature. This thermodynamic assumption corresponds to the experimental observation found for high temperatures that dilute gases follow the ideal gas law and that solids follow the law of Dulong and Petit. The direct proportionality of the oscillator energy $U$ and the thermodynamic temperature $T$ at high temperature allows us to obtain the functional form of the oscillator entropy at large energy. We next assume that the entropy $S(U/\omega)$ is an analytic function of its one variable $U/\omega$, and we apply the third law of thermodynamics to the analytic form. We find that in order to satisfy the third law of thermodynamics, the oscillator must have finite energy at zero thermodynamic temperature. The behavior of the derivatives of the entropy near zero temperature requires singular terms in the entropy as a function of energy. Making use of the singularities and comparing with the assumed high-temperature form obtained earlier, we are able to obtain the Planck spectrum with zero-point energy as the entropy expression consistent with the three laws of thermodynamics. Finally we comment on the classical thermodynamics which we have obtained in relation to the corresponding

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1 See, for example, Kuhn in [1], pp 213–214, or Hermann in [2], pp 128–130.
quantum expressions. The article closes with brief historical remarks regarding the third law of thermodynamics.

3. Harmonic oscillators as thermodynamic systems

Elementary discussions of thermodynamics often refer to gas systems with two independent variables, pressure and volume, whereas the simple harmonic oscillator is usually introduced only in connection with statistical mechanics. Nevertheless, the classical simple harmonic oscillator provides a very simple thermodynamic system of two independent variables, energy $U$ and frequency $\omega$. A mechanical oscillator has an equation of motion $m\ddot{x} = -kx$, an oscillator (angular) frequency $\omega = (k/m)^{1/2}$, and an energy $U = (1/2)mx^2 + (1/2)kx^2$. The average kinetic energy of the oscillator is equal to the average potential energy of the oscillator. It can be shown that the oscillator energy divided by the frequency, $U/\omega = U(m/k)^{1/2}$, is an adiabatic invariant for the mechanical system, so that under a very slow change of the spring constant $k$, the quantity $U(m/k)^{1/2}$ remains unchanged. Therefore we have

$$0 = d[U(m/k)^{1/2}] = dU(m/k)^{1/2} - dk[U(m/k)^{1/2}/2] = dU/\omega - d\omega(U/\omega^2). \quad (1)$$

The expressions involving frequency $\omega$ are simpler in form than those involving the spring constant $k$, and so we will refer to the change in the oscillator frequency $\omega$ rather than to the change in the spring constant $k$. The change in energy $dU$ associated with the change $d\omega$ in the oscillator frequency $\omega$ must be provided by some external agent. Then from equation (1) written as $dU = (U/\omega)d\omega$, the work $dW$ done by the oscillator is just the negative of this external agent’s work, $dW = -(U/\omega)d\omega$, and therefore

$$X = -U/\omega \quad (2)$$

can be regarded as a generalized force associated with the oscillator.

A classical harmonic oscillator can be regarded as coming to equilibrium in a thermal bath and so providing a simple thermodynamic system. Equation (2), connecting $X$, $U$, and $\omega$, may be regarded as an equation of state for the system. We will regard thermal radiation as providing the thermal bath, and each oscillator is treated as a small electric dipole oscillator with a very small electric charge interacting with electromagnetic radiation. At the end of the 19th century, Planck showed that a small electric dipole oscillator came to equilibrium with random radiation when the average energy of the oscillator matched the average energy of the radiation-bath normal modes at the same frequency as that of the oscillator.

4. Zeroth law: empirical temperature for a harmonic oscillator system

We will consider a collection of oscillators at a various mechanical frequencies $\omega$, where for each oscillator the average energy is given by its energy $U$, and the generalized force $X$ is given by $X = -U/\omega$. The zeroth law of thermodynamics introduces the idea of an empirical temperature $\theta(U, \omega)$ associated with the average oscillator energy $U$ and the mechanical frequency $\omega$. If two oscillators of different frequencies $\omega$ and $\omega'$ agree on the empirical temperature, $\theta(U, \omega) = \theta(U', \omega')$, then the two oscillators will be in thermal equilibrium with the same thermal radiation bath and in thermal equilibrium with each other.

2 I have yet to find a single textbook which considers the thermodynamics of the harmonic oscillator aside from its appearance in the application of statistical mechanics.
5. First law of thermodynamics

The first law of thermodynamics introduces the idea that heat energy $\delta Q$ can be included in the conservation law for energy, as $\delta Q = dU + \delta W$. For a harmonic oscillator, this conservation equation takes the form

$$\delta Q = dU - (U/\omega)\delta \omega,$$

where $\delta Q$ corresponds to the heat energy added to the oscillator, $U$ is the internal energy of the oscillator, and $-(U/\omega)\delta \omega$ corresponds to the work done by the oscillator. The internal energy $U(\theta, \omega)$ of an oscillator is a function of the empirical temperature $\theta$ and the frequency $\omega$ of the oscillator. For any oscillator, as more heat energy $\delta Q$ is added to the system at fixed frequency $\omega$, the energy $U$ will increase according to equation (3). Thus the oscillator energy $U(\omega, \theta)$ is an increasing function of the empirical temperature $\theta$.

6. Second law of thermodynamics

For our harmonic oscillator system, the second law of thermodynamics declares that there exists an entropy state function $S(U, \omega)$ which is a function of the average oscillator energy $U$ and of the oscillator frequency $\omega$, such that

$$T dS = dU - (U/\omega)d\omega,$$

where

$$T = (\partial U/\partial S)_{\omega},$$

is a thermodynamic temperature which depends only upon the empirical temperature $\theta$. Here if we introduce the relation (5) into the second-law relation (4) divided by $T$, then we find

$$dS = \frac{1}{T} dU - \frac{U}{T \omega} d\omega = \left( \frac{\partial S}{\partial U} \right)_\omega dU - \frac{U}{\omega} \left( \frac{\partial S}{\partial U} \right)_\omega d\omega.$$

However, if we regard $S$ as a function of $U$ and $\omega$, so that $dS = (\partial S/\partial U)_\omega dU + (\partial S/\partial \omega)_U d\omega$, then the relation (6) requires

$$\left( \frac{\partial S}{\partial \omega} \right)_U = -\frac{U}{\omega} \left( \frac{\partial S}{\partial U} \right)_\omega,$$

which means that $S$ is a function of the single variable $(U/\omega)$. Thus we have

$$\left( \frac{\partial S(U/\omega)}{\partial \omega} \right)_U = -\frac{U}{\omega^2} S'(U/\omega) = -\frac{U}{\omega} \left( \frac{\partial S(U/\omega)}{\partial U} \right)_\omega,$$

where $S'(U/\omega)$ refers to the derivative of $S(U/\omega)$ with respect to its one argument $U/\omega$.

7. Adiabatic lines

A reversible adiabatic process involving a harmonic oscillator involves work being done without the addition of heat. Under such a process, the entropy of the system remains unchanged since $\delta Q = T dS = 0$. Since for a harmonic oscillator thermodynamic system, the entropy $S$ is a function of $U/\omega$, the adiabatic lines will be straight lines through the origin in a plot of average energy $U$ versus $\omega$, or will be horizontal straight lines in an indicator plot of the generalized force $U/\omega = -X$ versus $\omega$. 
8. Wien’s displacement theorem

If we regard the average energy $U$ of an oscillator as a function of frequency and thermodynamic temperature, then we have $dU = (\partial U/\partial T)_{\omega}dT + (\partial U/\partial \omega)_{T}d\omega$. Substituting this expression into the second-law equation (4), we obtain

$$dS = \frac{1}{T}\left(\frac{\partial U}{\partial T}\right)_{\omega}dT + \frac{1}{T}\left[\left(\frac{\partial U}{\partial \omega}\right)_{T} - \frac{U}{\omega}\right]d\omega.$$  

(9)

Now comparing this expression with $dS = (\partial S/\partial T)_{\omega}dT + (\partial S/\partial \omega)_{T}d\omega$, gives

$$\left(\frac{\partial S}{\partial T}\right)_{\omega} = \frac{1}{T}\left(\frac{\partial U}{\partial T}\right)_{\omega} \quad \text{and} \quad \left(\frac{\partial S}{\partial \omega}\right)_{T} = \frac{1}{T}\left[\left(\frac{\partial U}{\partial \omega}\right)_{T} - \frac{U}{\omega}\right].$$  

(10)

Using the agreement of the mixed second-partial derivatives of $S$ with respect to $T$ and $\omega$ in opposite orders, we have

$$\left\{ \frac{\partial}{\partial \omega} \left[ \frac{1}{T} \left( \frac{\partial U}{\partial T}\right)_{\omega} \right] \right\}_{T} = \left\{ \frac{\partial}{\partial T} \left[ \frac{1}{T} \left( \frac{\partial U}{\partial \omega}\right)_{T} - \frac{U}{\omega} \right] \right\}_{\omega}. $$  

(11)

Next simplifying this expression and canceling the mixed second derivatives of $U$ on the two sides, we have

$$\left(\frac{\partial U}{\partial \omega}\right)_{T} = -\frac{T}{\omega} \left(\frac{\partial U}{\partial T}\right)_{\omega} + \frac{U}{\omega},$$  

(12)

or equivalently

$$\frac{U}{\omega} = f(T/\omega),$$  

(13)

where $f(T/\omega)$ is some function of the single variable $T/\omega$. The information in equation (13) corresponds to that of Wien’s displacement theorem [7].

9. Assumption of proportionality of energy $U$ and thermodynamic temperature $T$ at high temperature

We now introduce the thermodynamic assumption that the energy $U$ of the oscillator for fixed frequency $\omega$ is proportional to the thermodynamic temperature $T$ at high temperature

$$T \approx CU + O\left(\frac{\omega}{U}\right).$$  

(14)

This corresponds to assuming that the energy of the oscillator becomes independent of the frequency $\omega$ of the oscillator for fixed frequency $\omega$ and sufficiently high thermodynamic temperature $T$. This same situation is encountered in the thermodynamic measurements regarding gases and solids. For dilute gases, the internal energy becomes proportional to the thermodynamic temperature at high temperature, corresponding to ideal gas behavior when the kinetic energy is dominant over any potential energy between the molecules of the gas. In contrast to this situation for gas molecules, harmonic oscillators always have the average kinetic energy equal to the average potential energy. For a harmonic oscillator, the assumption of internal energy proportional to thermodynamic temperature at high temperature

3 See for example, Reif in [3], p 64.
corresponds to the situation found for solids where the law of Dulong and Petit is found at high temperatures\(^4\). This high-temperature thermodynamic assumption agrees with the equipartition result found from nonrelativistic classical statistical mechanics.

The scale constant \(C\) in equation (14) connects thermodynamic temperature \(T\) and internal energy. The usual choice corresponding to ‘practical units’ is \(C = 1/k_B\) where \(k_B\) is Boltzmann’s constant. However, in most of the analysis to follow, it is more convenient to choose \(C = 1\), corresponding to ‘rational units’, where absolute temperature \(T\) is measured in energy units and entropy \(S\) is a dimensionless pure number\(^5\).

\[
T \to U \left[ 1 + O \left( \frac{\omega^2}{U^2} \right) \right] \text{ for large } U.
\]

(15)

Thus at high temperatures, the energy of a harmonic oscillator becomes independent of the frequency \(\omega\) of the oscillator.

The entropy expression at high temperatures following from equation (15) as

\[
\left( \frac{\partial S}{\partial (U/\omega)} \right)_T = \frac{\omega}{T} = \frac{\omega}{U[1 + O(\omega^2/U^2)]} = \frac{\omega}{U} \left[ 1 + O \left( \frac{\omega^2}{U^2} \right) \right]
\]

(16)

Integrating once, we have

\[
S(U/\omega) = \ln(U/\omega) + \text{const} + O \left( \frac{\omega^2}{U^2} \right)
\]

(17)

10. Assumption of analytic behavior for the entropy function

Consistent with the usual situation in classical physics, the entropy function \(S(U/\omega)\) is expected to be a smooth function of its argument. Specifically, we will assume that the entropy function is analytic in \(U/\omega\) on the positive real axis and can be written in the form

\[
S(U/\omega) = \ln(U/\omega) + \text{const} + \sum_{n=1}^{\infty} \frac{a_n}{(U/\omega)^{n+1}},
\]

(18)

where the \(a_n\) are constants and the sum is convergent along the real axis for all values of \(U/\omega\) larger than some singular value which determines the radius of convergence of the sum about \(U/\omega \to \infty\) in the complex plane\(^6\). This analytic form is consistent with the high-temperature expression in equation (17) which arose from the assumption in equation (14) that the oscillator energy becomes proportional to the thermodynamic temperature at high temperature. Note that the sum in equation (18) involves the power \((U/\omega)^{-(n+1)}\) because the correction in equation (17) is \(O(\omega^2/U^2)\). As \(U/\omega\) becomes large, each of the terms \((U/\omega)^{-(n+1)}\) in the sum goes to zero faster than \(\ln(U/\omega)\) so that \(\ln(U/\omega)\) indeed can dominate at large values of \(U/\omega\).

11. Third law of thermodynamics and zero-point energy

Although the first two laws of thermodynamics hold generally at any temperature, the third law of thermodynamics deals specifically with thermodynamic behavior at low temperatures.

\(^4\) See for example, Ref in \[3\], p 254.

\(^5\) The choice of rational or practical units is discussed, for example, by \[8\], or by Garrod, in \[3\], p 120.

\(^6\) We assume that the sum is holomorphic at \(U/\omega \to \infty\) with a finite radius of convergence. We recall that \(f(z)\) is holomorphic at \(z \to \infty\) if \(f(1/w)\) is holomorphic at \(w \to 0\). See for example \[9\].
Indeed, the third law states that as \( \frac{\partial U}{\partial S} \rvert_{\infty} = T \to 0 \), the values of the entropy fall toward zero, \( S(U/\omega) \to 0 \). Since \( \ln(U/\omega) \to -\infty \) as \( U \to 0 \) for fixed \( \omega \), it is clear that the function given in equation (18) cannot have a radius of convergence with positive and analytic zero of temperature. Zero-point energy for a harmonic oscillator is implicit in the third law of thermodynamics.

### 12. Form of the entropy function

The third law of thermodynamics requires that the entropy \( S \) approaches zero as the temperature \( T \) falls toward zero corresponding to \( U/U_0 \) approaching \( 1 \). We will set \( z = U/U_0 = U/(\omega h/2) \) so that the entropy in equation (18) can be regarded as a function of \( z \) with
\[
S(z) = \ln(z) + \text{Const} + \sum_{n=1}^{\infty} \frac{A_n}{z^{n+1}},
\]
where \( \text{Const} \) and the coefficients \( A_n \) are adjusted from const and the \( a_n \) in equation (18) so as to absorb the constant \( h/2 \) associated with the oscillator’s zero-point energy. Then it follows from equation (19) that
\[
\frac{U_0}{T} \frac{dS}{dz} = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{(n+1)A_n}{z^{n+2}},
\]
and
\[
\frac{d^2S}{dz^2} = -\frac{1}{z^2} + \sum_{n=1}^{\infty} \frac{(n+1)(n+2)A_n}{z^{n+3}},
\]
where \( S(z) \to 0 \) and \( T(U, \omega) \to 0 \) as \( z \to 1 \). It follows from equation (20) that the entropy function must have a divergent derivative \( dS/dz \) as \( z \to 1 \) since \( U_0/T \to \infty \) as \( T \to 0 \) for fixed \( U_0 = \hbar \omega/2 \). Also, if the first derivative of \( S(z) \) has a singularity at \( z = 1 \), then the second derivative must have an even stronger singularity. The weakest singularity would be a logarithmic singularity, such as \( d^2S/dz^2 \sim \ln(z - 1) \), but the next derivative would give a first-order singularity as \( d^3S/dz^3 \sim 1/(z - 1) \). And indeed for these singularities, the function \( S \) itself could have a finite limit at the singularity corresponding to \( S \sim (z - 1) \ln(z - 1) \to 0 \) for \( z \to 1 \).

The known high-temperature behavior places further limits on the low-temperature singular behavior. If we try a first-order singularity at \( z = 1 \) for the second derivative of \( S(z) \), we must
include an additional first-order singularity at some other value $z = \alpha$ in order that the second derivative of $S$ falls off at large $z$ as the second power of $z$ as in equation (21). Thus we can write

\[
\frac{d^2 S}{dz^2} = -\frac{1}{(z-1)(z-\alpha)} + \sum_{n=1}^{\infty} \frac{(n+1)(n+2)b_n}{z^{n+3}}.
\]

(22)

where the $b_n$ are constants, and so preserve the analytic form and high-temperature limit in equation (21), provided that the value of $\alpha$ lies on the real axis with $\alpha < 1$, which is out of the physical region $1 < z$, corresponding to oscillator energies above the zero-point energy.

Integrating once in equation (22), we have

\[
\frac{dS}{dz} = -\frac{1}{1-\alpha}[\ln(z-1) - \ln(z-\alpha)] + \gamma - \sum_{n=1}^{\infty} \frac{(n+1)b_n}{z^{n+2}}
\]

\[
= -\frac{1}{1-\alpha}\left[\ln z + \ln\left(1 - \frac{1}{z}\right) - \ln z - \ln(1 - \frac{\alpha}{z})\right] + \gamma + \sum_{n=1}^{\infty} \frac{(n+1)b_n}{z^{n+2}}
\]

\[
= -\frac{1}{1-\alpha}\left[\frac{1}{z} + \frac{1}{z^2} - \ldots + \frac{\alpha}{z^2} + \ldots\right] + \gamma - \sum_{n=1}^{\infty} \frac{(n+1)b_n}{z^{n+2}}
\]

(23)

where $\gamma$ and the $b_n$ are constants. In order for this expression (23) to go over to the high-temperature form given in equation (20) where the first term in the summation is $z^{-3}$ and there is no constant term, we must have $\alpha = -1$ and $\gamma = 0$. Introducing these values into the first line in equation (23), we obtain

\[
\frac{dS}{dz} = -\frac{1}{2}[\ln(z-1) - \ln(z+1)] - \sum_{n=1}^{\infty} \frac{(n+1)b_n}{z^{n+2}}.
\]

(24)

Now integrating equation (24) gives

\[
S(z) = -\frac{1}{2}[(z-1)\ln(z-1) - (z+1)\ln(z+1)] + \text{const} + \sum_{n=1}^{\infty} \frac{b_n}{z^{n+1}}
\]

\[
= -\frac{1}{2}\left\{[(z-1)\ln z + (z-1)\ln(1 - \frac{1}{z}) - (z+1)\ln z - (z+1)\ln(1 + \frac{1}{z})]\right\}
\]

\[
+ \text{const} + \sum_{n=1}^{\infty} \frac{b_n}{z^{n+1}}
\]

\[
= \ln z - \frac{1}{2}\left\{(z-1)\left(\frac{1}{z} + \frac{1}{z^2} - \ldots\right) - (z+1)\left(\frac{1}{z} - \frac{1}{z^2} + \ldots\right)\right\} + \text{const}'
\]

\[
+ \sum_{n=1}^{\infty} \frac{b_n}{z^{n+1}}
\]

\[
= \ln z + \text{const}'' + \sum_{n=1}^{\infty} \frac{c_n}{z^{n+1}}.
\]

(25)

where $\text{const}'$, $\text{const}''$, and $c_n$ are all constants. The final line in equation (25) has a large-$z$ analytic behavior consistent with that in equation (19).

In our analysis starting with the third law of thermodynamics, we see the need for certain singular terms to appear in the derivatives of $S(z)$, and we can then use integrals to trace back
the implications for their appearance in the function \( S(z) \) itself. However, we must be careful to maintain the assumed analytic behavior at \( z \to \infty \), which will be influenced by the presence of singularities in the complex plane. If we take only the terms required by the singularities at low temperature while still maintaining the high-temperature limit, then we have the expression for the entropy

\[
S(z) = -\frac{1}{2}[(z - 1)\ln(z - 1) - (z + 1)\ln(z + 1)] - \ln 2
\]

(26)

which vanishes as \( z \) approaches 1, \( z \to 1_+ \). This function (26) can be rewritten in the high-temperature form given in equation (19) as

\[
S(z) = -\frac{1}{2}[(z - 1)\ln(z - 1) - (z + 1)\ln(z + 1)] - \ln 2
\]

\[
= -\frac{1}{2}\left\{(z - 1)\ln z + (z - 1)\ln\left(1 - \frac{1}{z}\right) \right. \\
- (z + 1)\ln z - (z + 1)\ln\left(1 + \frac{1}{z}\right) - \ln 2 \\
= \ln z + \frac{1}{2}\left\{(z - 1)\frac{1}{z} + \frac{1}{2z^2} - ... \right. \\
- (z + 1)\left(\frac{1}{z} - \frac{1}{2z^2} + ... \right) \right\} - \ln 2 \\
= \ln z + 1 - \ln 2 - \frac{1}{6z^2} - \frac{1}{20z^4} - \frac{1}{42z^6} - ... \text{ for } 1 < z.
\]

(27)

The derivatives follow as

\[
\frac{dS}{dz} = -\frac{1}{2}[\ln(z - 1) - \ln(z + 1)] = \frac{1}{2}\ln\left(\frac{z + 1}{z - 1}\right)
\]

\[
= \frac{1}{z} + \frac{1}{3z^3} + \frac{1}{5z^5} + ... > 0 \text{ for } 1 < z
\]

(28)

and

\[
\frac{d^2S}{dz^2} = -\frac{1}{2}\left(\frac{1}{z - 1} - \frac{1}{z + 1}\right) = \frac{-1}{z^2 - 1}
\]

\[
= -\frac{1}{z^2} - \frac{1}{z^4} - \frac{1}{z^6} - ... < 0 \text{ for } 1 < z.
\]

(29)

Indeed by repeated differentiation, we can show that \( S(z) \) and all its derivatives are monotonic functions of \( z \) for \( 1 < z \) along the real line. When considered in the complex \( z \)-plane, \( S(z) \) in equation (26) has singularities at \( z = 1 \) and \( z = -1 \) and at no other points in the complex plane. The entropy function \( S(z) \) in equation (26) satisfies all the thermodynamic and analytic-function assumptions of our analysis. Indeed we see the divergence of the summations at \( z = \pm 1 \) when the derivatives of the entropy are expanded in series. The circle of convergence (about infinity)\(^7\) for the series giving the derivatives of the entropy cannot allow any further singularities in the complex plane within the circle of convergence \( |z| > 1 \). It remains conceivable that additional singularities within the region \( |z| < 1 \) may be added without

\(^7\) The complex plane becomes compact with the addition of the point at infinity through identification by stereographic projection onto a sphere. See for example [9], section 2.5, pp 38–44. The circle of convergence about the point at infinity corresponds to values of \( z \) in the complex plane which have \( |z| \) greater than some fixed value.
violating the thermodynamic assumptions of our analysis. However, additional singularities would represent specialized additional structure in the thermodynamic system beyond what is required by the thermodynamics of the simple harmonic oscillator.

13. Obtaining the Planck spectrum

We have obtained the expression for the entropy $S(z) = S[U/(\omega h/2)]$ of a harmonic oscillator in terms of the single variable involving the oscillator energy divided by its zero-point energy. However, it is usual to give the spectrum for blackbody radiation in terms of frequency $\omega$ and thermodynamic temperature $T$. We obtain this form by using equation (28) rewritten in practical units as

$$\frac{\omega h}{k_B T} = \frac{d[S/k_B]}{d[U/(\omega h/2)]} = \frac{1}{2} \ln \left( \frac{U/(\omega h/2) + 1}{U/(\omega h/2) - 1} \right)$$

Taking the exponential, we find

$$\exp\left( \frac{\hbar \omega}{k_B T} \right) = \left( \frac{U/(\omega h/2) + 1}{U/(\omega h/2) - 1} \right).$$

Now solving for $U/(\omega h/2)$, we obtain

$$U/(\omega h/2) = \exp[\hbar \omega/(k_B T)] + 1 \exp[\hbar \omega/(k_B T)] - 1 = \coth\left( \frac{\hbar \omega}{2k_B T} \right)$$

or

$$U = \frac{1}{2} \hbar \omega \coth\left( \frac{\hbar \omega}{2k_B T} \right) = \frac{\hbar \omega}{\exp[\hbar \omega/(k_B T)] - 1} + \frac{1}{2} \hbar \omega,$$

which is the familiar Planck form, including zero-point energy $U(\omega, 0) = (h/2)\omega$. Indeed by the use of purely thermodynamic ideas, we are led to the Planck formula including zero-point radiation for blackbody radiation.

14. Regarding the absence of zero-point energy in the interpolation formula obtained by Planck

14.1. Planck’s thermodynamic analysis

In classical physics, a dipole harmonic oscillator must come to equilibrium with the surrounding radiation bath where the average energy of the oscillator matches the average energy of the bath radiation modes at the same frequency as the oscillator frequency. Thus the harmonic oscillator treated in the present article must be in equilibrium with a thermal bath which includes both zero-point radiation and thermal radiation above the zero-point radiation as given in equation (34).

The unambiguous situation for classical physics is in contrast to the confusion which sometimes occurs in quantum physics. The extrapolation expression based upon experimental data obtained by Planck in 1900 was
and included no zero-point energy. This result without zero-point energy is often repeated in the textbooks [11] as the energy of the radiation modes of blackbody radiation. Indeed at high temperature, the Planck form (35) goes over to

\[ U_p = k_B T - \frac{\hbar \omega}{2} + O \left( \frac{\hbar \omega}{k_B T} \right) \]  

which still subtracts the zero-point energy compared with the equipartition result \( U = k_B T \). This situation is in contrast to the analysis in the present article where the oscillator energy at high temperatures was assumed in equation (15) to go over to the equipartition result \( U = k_B T + O(\hbar \omega / k_B T) \).

Modern quantum mechanics does include a zero-point energy for a quantum harmonic oscillator 8. However, the quantum oscillator does not radiate when it is in its energy eigenstates, and so the association with zero-point energy in the radiation field is not obvious.

One must recall that Planck worked from thermodynamics insofar as requiring the Wien-displacement-theorem form, which depended upon the first two laws of thermodynamics. His blackbody energy expression (35) was obtained as an interpolation (involving experimental data) between the Wien radiation formula, which seemed to fit the high-frequency data well, and the new experimental results of Rubens and Kurlbaum, which suggested a different low-frequency behavior 9. Once he had obtained his interpolation function in equation (35), which provided an excellent fit to the experimental data, he knew that he had to have an entropy expression of the form (see footnote 9)

\[ S(U_p, \omega) = k_B \left[ \frac{U_p}{\hbar \omega} + 1 \right] \ln \left( \frac{U_p}{\hbar \omega} + 1 \right) - \left( \frac{U_p}{\hbar \omega} \right) \ln \left( \frac{U_p}{\hbar \omega} \right) \]  

where \( U_p \) was the thermal energy of Planck’s oscillator in equilibrium with the radiation spectrum measured by the experimentalists. This equation (37) is the same as our equation (26) when we introduce the missing constant \( k_B \) so as to give

\[ S[U/(\omega h / 2)] = k_B \left[ \frac{1}{2} \left( \frac{U}{\hbar \omega / 2} + 1 \right) \ln \left( \frac{U}{\hbar \omega / 2} + 1 \right) \right. \]

\[ - \left. \frac{1}{2} \left( \frac{U}{\hbar \omega / 2} - 1 \right) \ln \left( \frac{U}{\hbar \omega / 2} - 1 \right) \right] - \ln 2 , \]  

and we note the connection between the thermodynamic oscillator energy \( U \) and the thermal energy \( U_p \) used by Planck,

\[ U = U_p + \hbar \omega / 2 . \]  

Planck’s energy \( U_p \) is the thermal energy above the zero-point energy. The Planck form in equation (35) represents a shift in the oscillator energy over to \( U_p \) where the entropy \( S[U_p/(\omega h / 2)] \) goes to zero as the energy \( U_p \) goes to zero.

Equations (37) and (38) give the oscillator entropy \( S(U, \omega) \) as a function of the energy divided by the frequency. However, we can also obtain the entropy \( S(\omega, T) \) as a function of frequency \( \omega \) and thermodynamic temperature \( T \) by noting from equation (32) that \( \omega / T \) is a function of \( U/\omega \) so that equation (31) and the derivative of equation (33) can be combined as

8 See for example, Griffiths in [11], p 46.
9 See, for example, Hermann in [2], pp 12–14.
\[
\frac{dS(\omega/T)}{d(\omega/T)} = \left[ \frac{dS(U/\omega)}{d(U/\omega)} \right] \left[ \frac{d(U/\omega)}{d(\omega/T)} \right] = \left( \frac{\hbar \omega}{2T} \right) \left[ -\frac{\hbar}{2k_B} \coth\left( \frac{\hbar \omega}{2k_B T} \right) - \ln 2 \right].
\] (40)

Then integrating gives
\[
S(\omega/T) = k_B \left\{ -\ln \left[ \sinh \left( \frac{\hbar \omega}{2k_B T} \right) \right] + \frac{\hbar \omega}{2k_B T} \coth \left( \frac{\hbar \omega}{2k_B T} \right) - \ln 2 \right\}
\] (41)

where the constant of integration is chosen so that \( S \to 0 \) as \( T \to 0 \). The zero-point energy makes no contribution to the entropy, and therefore both the Planck form (35) without zero-point energy and the thermodynamic form (34) including zero-point energy give the same entropy dependence upon frequency and temperature for the harmonic oscillator system.

The discrepancy between the forms in equations (34) and (35) reappears in the calculation of the generalized force associated with the harmonic oscillator system. The generalized force \( X \) for the classical simple harmonic oscillator system appears in the fundamental thermodynamic relation \( dQ = TdS = dU + Xd\omega \). From the adiabatic invariant of the mechanical system, we saw that \( X = -U/\omega \) where \( U \) is the mechanical energy of the mechanical oscillator. In thermodynamics, we expect (from \( TdS = dU + Xd\omega \)) that the generalized force is given by
\[
X = \left( \frac{\partial U}{\partial \omega} \right)_S.
\] (42)

Now from equation (38), we see that if \( S \) is held constant, then \( U/\omega \) must be constant, so \( U = \beta \omega \) where \( \beta \) is a constant. But then the generalized force follows from \( \partial(U/\partial \omega)_\beta = \beta = U/\omega \), so that
\[
X = \left( \frac{\partial U}{\partial \omega} \right)_S = -\frac{U}{\omega} = -\frac{U_P + \hbar \omega / 2}{\omega} = -\frac{U_P}{\omega} - \frac{\hbar}{2},
\] (43)

involving the same connection \( X = -U/\omega \) as was given in equation (2). The contribution of the classical zero-point energy is absent in the Planck form (35). Thus the use of the Planck energy \( U_P \), which excludes the zero-point energy, requires the introduction of an additional constant \(-\hbar/2\) at all temperatures in the expression (43) for the generalized force \( X \). Even at low temperature \( T \approx 0 \), where the Planck thermal energy \( U_P \) approaches zero, there is still a non-zero generalized force given by \( X \approx -\hbar/2 \). In classical physics, this force arises from the random radiation associated with zero-point radiation. In quantum theory, this generalized force is a ‘quantum force’ which is different from that associated with thermal energy. As far as classical physics is concerned, the omission of the zero-point energy in the expression (35) is misleading in suggesting that thermodynamic effects vanish as the temperature goes toward absolute zero. The inclusion of an explicit zero-point energy in the system energy, as in equation (34), fits with the thermodynamics of classical physics and is far more natural than its absence.

14.2. Einstein’s energy fluctuations

It is also interesting that the second derivative of the entropy with respect to the energy in equation (29) can be rewritten in the form associated with Einstein’s ideas of thermodynamic energy fluctuations [12, 13]. We can understand the fluctuations by considering one-dimensional waves, for example waves on a long string, where the oscillations are associated with random phases for the waves. Einstein’s analysis corresponds to analyzing the energy fluctuations in a small region of the long string. The energy in the small region can be
regraded as fluctuating about a mean value and can be analyzed in terms of normal modes of oscillation of the small region of the string. If the string is regarded as a thermodynamic system, then the energy fluctuations in a normal mode of oscillation can be related to the entropy of the normal mode. Einstein connected the second derivative of the entropy $S$ with respect to the average energy $U$ in the small region to the mean-square energy fluctuations $\langle \epsilon^2 \rangle$ in the small region giving (per normal mode)

$$\frac{d^2S}{dU^2} = -\frac{k_B}{\langle \epsilon^2 \rangle}.$$  \hfill (44)

The energy fluctuation for waves with random phases on a string can be calculated within classical wave theory as

$$\langle \epsilon^2 \rangle = U^2.$$  \hfill (45)

Einstein substituted equations (45) into (44) and so obtained energy equipartition and the Rayleigh–Jeans spectrum as the results of classical wave theory. However, we have noted that thermodynamic entropy $S$ should be associated not with all the energy $U$ of an oscillator or wave normal mode, but rather only the energy above the zero-point energy. Thus our equation (29) becomes in practical units

$$\frac{d^2S}{dU^2} = -\frac{k_B}{U^2 - U_0^2} = -\frac{k_B}{U^2 - (\hbar \omega/2)^2}.$$  \hfill (46)

and can be regarded as associating the entropy $S$ with the classical energy fluctuations $U^2 - U_0^2$ (of the oscillator or radiation mode) above the classical zero-point energy fluctuations. However, this equation can also be rewritten in terms of the energy $U_P$ corresponding to Planck’s shift of the energy by the amount of the zero-point energy, $U = U_P + U_0 = U_P + \hbar \omega/2$. In the form using the energy $U_P$, the equation becomes

$$\frac{d^2S}{dU^2} = -\frac{k_B}{(U_P + U_0)^2 - U_0^2} = -\frac{k_B}{U_P^2 + 2U_0U_P} = -\frac{k_B}{U_P^2 + \hbar \omega U_P}.$$  \hfill (47)

It is this last form involving $U_P$ which corresponds to the analysis given by Einstein in 1909. Einstein interpreted the fluctuations leading to the Planck spectrum as involving a ‘wave-like’ contribution $U_P^2$ and a ‘particle-like’ (photon) contribution $\hbar \omega U_P$. Connecting the energy $U_P$ to the number $N$ of photons of frequency $\omega$ as $U_P = N\hbar \omega$, the entropy (which follows as in equation (37)) goes to zero as the number $N$ of photons of energy $\hbar \omega$ goes to zero.

14.3. Energy measurements using bolometers versus Casimir forces

Planck and Einstein’s use of the thermal energy $U_P$ without zero-point energy was natural at the turn of the 20th century since the experimental bolometric measurements of thermal radiation involved only the contributions of sources at temperatures above the temperature of the detectors, and thus did not measure zero-point energy. A century later, by Casimir force [15] measurements, [16] it has become possible (within a classical understanding) to measure all the radiation (including the zero-point radiation) surrounding two parallel conducting plates. Although the generalized force $X$ given in equation (43) for a hypothetical simple harmonic oscillator (associated with the energy dependence on the frequency $\omega$) may not be accessible at the atomic level, the generalized force involving random radiation surrounding two conducting parallel plates is indeed measurable [16]. The forces between conducting parallel plates are the radiation-mode analogs of the oscillator generalized force $X$. The Planck form (34) involving photons for the blackbody radiation spectrum, which appears in modern
physics textbooks and in quantum statistical mechanics, [11] does not include zero-point energy and goes to zero as the thermodynamic temperature goes to zero. Thus the photons of the Planck spectrum make no contribution to the Casimir forces between conducting parallel plates at zero temperature. Quantum theory must introduce a separate ‘quantum mechanical’ calculation for the Casimir force at zero temperature, and then adds finite-temperature corrections involving thermal excitations (photons) at non-zero temperature. This is exactly analogous to the need for a ‘quantum mechanical’ generalized force $X$ for an oscillator at zero temperature. Classical theory, which uses the full thermodynamic expression (34) including the zero-point radiation for the energy of a radiation mode, needs only a single calculation to obtain the Casimir force at any temperature [17]. In addition to the need to separate a ‘quantum behavior’ at zero temperature and a thermal contribution (due to photons) at positive temperature, there are also arguments among quantum theorists as to whether or not the Casimir effect at zero temperature should actually be associated with quantum zero-point energy [18]. Within classical physics, there is no ambiguity; Casimir forces are associated with changes in the energy of zero-point radiation [19].

15. Discussion

For over a century now, physicists have presented various theoretical explanations for the Planck spectrum, some based upon quantum theory[10] and some based upon classical theory [20]. However, one may wonder what is the minimal set of assumptions necessary to derive the experimentally observed spectrum. Thermodynamics, which was developed during the 19th century, has provided fundamental ideas of absolute generality. The first two laws of thermodynamics provided the basis for Boltzmann’s derivation of the Stefan–Boltzmann relation $U = a_0 T^4 V$ connecting the thermal energy $U$ with the temperature $T$ and volume $V$ for the thermal radiation in a closed container at thermal equilibrium. These same two laws of thermodynamics, applied to the reversible adiabatic compression of thermal radiation, allowed Wien to derive his displacement theorem in 1893. Planck’s enthusiasm for thermodynamics led him to consider the problem of blackbody radiation at the end of the 19th century; however, he became convinced that the information available from the first two laws of thermodynamics had been exhausted by Wien in obtaining the displacement theorem. Therefore he turned to statistical ideas after his successful experimental-data-based interpolation giving the Planck spectrum (without zero-point radiation). When the third law of thermodynamics was introduced in the first decade of the 20th century, the law was associated with the developing quantum theory. Apparently physicists never looked back to see whether thermodynamics (now including the third law) might provide the crucial missing function in Wien’s displacement theorem.

The thermodynamics of thermal radiation is tightly connected with the thermodynamics of the harmonic oscillator since the behavior of each normal mode of oscillation for radiation takes the simple-harmonic oscillator form. Indeed, it was Planck who first pointed out that a small classical dipole oscillator would come to thermal equilibrium with an average energy which was the same as the average energy of the random-radiation normal modes at the same frequency as the oscillator. Now the thermodynamics of the harmonic oscillator is virtually never discussed in connection with fundamental thermodynamics rather than in connection with statistical mechanics. Yet the thermodynamics of the simple harmonic oscillator takes a very simple form. Fifteen years ago, it was pointed out [7] that the application of the first two

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[10] See, for example, Griffiths in [11], pp 243–244, or Reif in [3], pp 373–378.
laws of thermodynamics to a harmonic oscillator could lead easily to the information corresponding to Wien’s displacement theorem involving the existence of one unknown thermodynamic potential function $\phi(\omega/T)$ depending upon the oscillator frequency $\omega$ divided by the thermodynamic temperature $T$. At that time it was noted that the unknown function $\phi(\omega/T)$ allowed two natural limits which made the oscillator energy $U(\omega, T)$ independent of one of its two variables; the limits corresponded to energy equipartition $U \to k_B T$ at high temperature, and zero-point energy $U \to \hbar \omega/2$ at low temperature. It was then suggested that ‘the smoothest interpolation’ between these two limits led to the Planck spectrum. Although the analysis may be compelling physics, the idea of a ‘smoothest interpolation’ is not standard mathematics.

In the present article, we have attempted to go back to the question of the blackbody spectrum within classical physics from the vantage point of all three laws of thermodynamics, now including the third law which was not used by Wien or Planck at the turn of the 20th century. We have attempted to use simply the ideas of thermodynamics and mathematical function theory to obtain as much information as possible. We find that the third law of thermodynamics forces the idea of zero-point energy for the classical harmonic oscillator. Of course, the idea of a zero-point energy is now associated in the minds of many physicists with quantum theory. However, our work does not require any assumption regarding discrete energy processes. In addition to requiring that an oscillator have a non-zero energy in thermal equilibrium at thermodynamic absolute zero, the ideas of thermodynamics (including the third law) provide sufficiently stringent conditions that one can use them alone to derive the Planck blackbody radiation spectrum including zero-point radiation. Finally, we point out that the inclusion of classical zero-point energy, which is automatic from the thermodynamic point of view, involves more natural force ideas than the Planck expression which omits the zero-point energy.

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