COUNTING UNLABELED INTERVAL GRAPHS

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Abstract. We improve the bounds on the number of interval graphs on \( n \) vertices. In particular, denoting by \( I_n \) the quantity in question, we show that 
\[
\log I_n \sim n \log n \quad \text{as} \quad n \to \infty.
\]

A simple undirected graph is an interval graph if it is isomorphic to the intersection graph of a family of intervals on the real line. Several characterizations of interval graphs are known; see [4, Chapter 3] for some of them. Linear time algorithms for recognizing interval graphs are given in [1] and [2].

In this paper, we are interested in counting interval graphs. Let \( I_n \) denote the number of unlabeled interval graphs on \( n \) vertices. (This is the sequence with id A005975 in the On-Line Encyclopedia of Integer Sequences [6].) Initial values of this sequence are given by Hanlon [3]. Answering a question posed by Hanlon [3], Yang and Pippenger [5] proved that the generating function
\[
I(x) = \sum_{n \geq 1} I_n x^n
\]
diverges for any \( x \neq 0 \) and they established the bounds
\[
\frac{n \log n}{3} + O(n) \leq \log I_n \leq n \log n + O(n).
\]

The upper bound in (1) follows from \( I_n \leq (2n - 1)!! = \prod_{j=1}^{n}(2j - 1) \), where the right hand side is the number of matchings on \( 2n \) points. For the lower bound, the authors showed
\[
I_{3k} \geq k!/3^{3k}
\]
by finding an injection from \( S_k \), the set of permutations of length \( k \), to three-colored interval graphs of size \( 3k \).

Using an idea similar to the one in [5], we improve the lower bound in (1) so that the main terms of the lower and upper bounds match. In other words, we find the asymptotic value of \( \log I_n \).

For a set \( S \), we denote by \( (S)_k \) the set of \( k \)-subsets of \( S \).

Theorem 1. As \( n \to \infty \), we have
\[
\log I_n \geq n \log n - 2n \log \log n - O(n).
\]

Proof. We consider certain interval graphs on \( n \) vertices with colored vertices. Let \( k \) be a positive integer smaller than \( n/2 \) and \( \varepsilon \) a positive constant smaller than \( 1/2 \). For \( 1 \leq j \leq k \), let \( B_j \) and \( R_j \) denote the intervals \( [-j - \varepsilon, -j + \varepsilon] \) and \( [j - \varepsilon, j + \varepsilon] \), respectively. These \( 2k \) pairwise-disjoint intervals will make up \( 2k \) vertices in the

2010 Mathematics Subject Classification. Primary 05C30; Secondary 05A16.

Key words and phrases. Interval graphs, counting.

The author is supported by National Science Foundation Fellowship (Award No. 1502650).
graphs we consider. Now let $W$ denote the set of $k^2$ closed intervals with one endpoint in $\{-k, \ldots, -1\}$ and the other in $\{1, \ldots, k\}$. We color $B_1, \ldots, B_k$ with blue, $R_1, \ldots, R_k$ with red, and the $k^2$ intervals in $W$ with white.

Together with $S := \{B_1, \ldots, B_k, R_1, \ldots, R_k\}$, each $\{J_1, \ldots, J_{n-2k}\} \in \binom{W}{n-2k}$ gives an $n$-vertex, three-colored interval graph. For a given $J = \{J_1, \ldots, J_{n-2k}\}$, let $G_J$ denote the colored interval graph whose vertices correspond to $n$ intervals in $S \cup J$, and let $\mathcal{G}$ denote the set of all $G_J$.

Now let $G \in \mathcal{G}$. For a white vertex $w \in G$, the pair $(d_B(w), d_R(w))$, which represents the numbers of blue and red neighbors of $w$, uniquely determine the interval corresponding to $w$; this is the interval $[-d_B(w), d_R(w)]$. In other words, $J$ can be recovered from $G_J$ uniquely. Thus

$$|\mathcal{G}| = \binom{k^2}{n-2k}.$$ Since there are at most $3^n$ ways to color the vertices of an interval graph with blue, red, and white, we have

$$I_n \cdot 3^n \geq |\mathcal{G}| = \binom{k^2}{n-2k} \geq \left(\frac{k^2}{n-2k}\right)^{n-2k} \geq \left(\frac{k^2}{n}\right)^n$$

for any $k < n/2$. Setting $k = \lfloor n/ \log n \rfloor$ and taking the logarithms, we get

$$\log I_n \geq n \log (k^2/n) - O(n) = n \log n - 2n \log \log n - O(n).$$

Remark 2. Yang and Pippenger [5] posed the question whether

$$\log I_n = Cn \log n + O(n)$$

for some $C$ or not. According to Theorem 1, this boils down to getting rid of the $2n \log \log n$ term in (2). Such a result would imply that the exponential generating function

$$J(x) = \sum_{n \geq 1} I_n \frac{x^n}{n!}$$

has a finite radius of convergence. (As noted in [5], the bound $I_n \leq (2n-1)!$ implies that the radius of convergence of $J(x)$ is at least 1/2.) Of course, a strong result would be finding $I_n$ asymptotically.

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