A THEORY OF META-FACTORIZATION

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ABSTRACT

We introduce meta-factorization, a theory that describes matrix decompositions as solutions of linear matrix equations: the projector and the reconstruction equation. Meta-factorization reconstructs known factorizations, reveals their internal structures, and allows for introducing modifications, as illustrated with SVD, QR, and UTV factorizations. The prospect of meta-factorization also provides insights into computational aspects of generalized matrix inverses and randomized linear algebra algorithms. The relations between the Moore-Penrose pseudoinverse, generalized Nyström method, the CUR decomposition, and outer-product decomposition are revealed as an illustration. Finally, meta-factorization offers hints on the structure of new factorizations and provides the potential of creating them.

Keywords Matrix factorization; Generalized inverses; Low-rank matrix approximation; Randomized algorithms

1 Introduction

Matrix factorization decomposes a matrix into a product of other matrices to reveal essential properties of the original matrix. We introduce meta-factorization, a procedure that further factorizes the factors to reveal details of the original decomposition and control its internal structure. The main asset of meta-factorization is the projector equation, which reveals hidden projectors of matrix decomposition and describes the mutual interactions of the factors needed to reconstruct the original matrix. The reconstruction equation then governs the reconstruction.

Identifying the hidden projectors is one of the key benefits of meta-factorization. Once their internal structure is revealed, the projector equation keeps the reconstruction abilities of the projectors intact while allowing for modifications to be introduced when necessary. The original matrix is safely mapped onto its column and row space. As a result, meta-factorization develops a nontrivial perspective in which matrix factorizations emerge as solutions to linear matrix equations.
This prospect becomes useful when studying matrix decompositions and designing matrix algorithms, including randomized linear algebra and low-rank approximation algorithms. By following the rules of meta-factorization, we can replace selected factors of a given decomposition with their numerically attractive equivalents, which control the performance and stability of the design. Finally, meta-factorization provides hints on the structure of new factorizations and offers the potential to create them.

1.1 Related work

Matrix factorizations play a critical role in mathematics, science, and engineering. They reveal essential matrix properties, such as spectrum, rank, or fundamental subspaces, which in turn characterize solutions to many fundamental problems in physics, signal processing, control engineering, artificial intelligence, and data science, see, e.g., Strang [30, 29]. Matrix factorizations also correspond to matrix algorithms, which is clearly illustrated by Golub and Van Loan [8], Stewart [28], Demmel [5], or Trefethen [32, 33]. Moreover, Edelman and Jeong [6] show that there are still new families of matrix factorizations to be analyzed together with their unifying algebraic structure. The search for new matrix factorizations has been fueling linear algebra for years.

From a mathematical point of view, meta-factorization is related to the theory of generalized matrix inverses. The equations of meta-factorization lead to solutions involving pseudoinverses or define them in special cases. For the related fundamental results, see Penrose [24] and Langenhop [15]. Matsaglia and Styan [21] and Ben-Israel and Greville [1] for the comprehensive exposition. Meta-factorization refers to computational aspects of generalized inverses as well. In particular, it describes explicit formulas for the Moore-Penrose pseudoinverse. Such formulas, critical for the accuracy and stability of computations, have been summarized by Ben-Israel and Greville in [1]. See also James [13] for an introduction, and Strang and Drucker [31] for a fresh perspective. One more relation should be mentioned when a big picture of matrix factorizations is considered. Namely, there is a connection between the reconstruction equation of meta-factorization and the quadratic matrix equation generating the Lie groups. The latter, studied by Edelman and Jeong in [6], gives rise to families of matrix factorizations. Under the appropriate assumptions, it can be interpreted as the meta-factorization equation.

The results presented in this paper were initially inspired by the work of Sorensen and Embree [27], with their technique for deriving the column and row space projectors for the CUR decomposition. This idea in the form of a projector equation is included in the process of meta-factorization. Also, as we will see, the generalized form of the projector equation may pave the way to possible new factorizations.

Designing projectors has been of great interest in the field of randomized linear algebra, which is visible in the works of Halko et al. [10], Mahoney [16], Martinsson and Tropp [18, 20], Shabat et al. [26], and Kannan and Vempala [14]. Recently, Nakatsukasa [22] showed that many algorithms of randomized linear algebra could be written in the form of the generalized Nyström’s method, a two-sided projection defined by sampling (sketch) matrices. Meta-factorization gives rise to the generalized Nyström’s method. It also explains the structure of the CUR decomposition studied by Mahoney and Drineas in [17], Wang and Zhang [34], Sorensen and Embree [27], Hamm and Huang [11], and predicted by Penrose in [25]. The outer-product decomposition resulting from the the rank-reduction process introduced by Wedderburn [35], Egerváry [7], Householder [12], Cline and Funderlic [4], and Chu et al. [3], is also described by the meta-factorization equations. Finally, Nakatsukasa and Tropp in [25] show remarkable results of
redesigning projectors in GMRES and Rayleigh-Ritz methods. In particular, they predict the benefits of moving into non-orthogonal linear algebra, a move meta-factorization fully supports.

1.2 Outline

The paper is divided into four sections. Section 2 introduces the concept of meta-factorization. First, it demonstrates how matrix factors interact to reconstruct the original matrix. Next, it describes the nature of those interactions in the form of a projection equation and defines the equations of meta-factorizations. Finally, it shows that solutions of the equations define matrix factorizations. Section 3 shows how meta-factorization works. The theory is applied to reconstruct the singular value decomposition (SVD), examine the internal structure of column-pivoted QR decomposition (CPQR), and develop a family of UTV factorizations. Section 4 discusses additional benefits of meta-factorization. It presents a relationship of meta-factorization with the theory of generalized matrix inverses, reconstructs the generalized Nyström’s method for computing low-rank matrix approximations, and shows its relationship with the CUR decomposition. Finally, further research directions are proposed in the Appendix A.

Notation. Capital bold letters denote matrices, \( W^* \) denotes the conjugate transpose of \( W \), and \( W^+ \) denotes the (Moore-Penrose inverse) pseudoinverse of \( W \). The \( k \times k \) identity matrix is denoted as \( I_k \). Given ordered subindex sets \( I \) and \( J \), \( A(I, J) \) denotes the submatrix of \( A \) containing rows and columns of \( A \) indexed by \( I \) and \( J \), with \( A(:, J) \) extracting the columns of \( A \) indexed by \( J \). With \( k \) being a positive integer, \( 1:k \) denotes the ordered set \( (1, \ldots, k) \). Finally, \( C(A) \) denotes the column space and \( C(A^*) \) denotes the row space for \( A \). An orthonormal \( m \times n \) matrix \( U \) has orthonormal columns: \( U^*U = I_n \). A unitary matrix is a square orthonormal matrix.

2 Meta-factorization

This section introduces the theoretical foundations of the proposed concept of meta-factorization. First, by investigating the internal structure of certain matrix factorizations, we demonstrate how the factors interact with each other to reconstruct the original matrix. Next, we derive the conditions describing the nature of those interactions. Finally, the concept of meta-factorization is formulated.

2.1 Factorizations and hidden projections

The matrix factorizations we study in this paper are of the following form:

\[
A = F \quad G \quad H^*;
\]

where an arbitrary \( m \) by \( n \) matrix \( A \) is decomposed here into the product of an \( m \) by \( k \) matrix \( F \), a square \( k \) by \( k \) matrix \( G \), and an \( n \) by \( k \) matrix \( H \). These factors may reveal essential matrix properties, such as spectrum, rank, or four fundamental subspaces. The most important factorizations, such as the singular value decomposition (SVD), the column-pivoted QR (CPQR) decomposition, the eigenvalue decomposition (ED), the LU decomposition, or the CUR decomposition, are all described by this pattern.

\(^1\)A Matlab code supplement is available here: \texttt{https://github.com/mkarpowi/mft}
Matrix $\mathbf{F}$ defines a basis for the column space, whereas $\mathbf{H}$ defines a basis for the row space of $\mathbf{A}$. Those two bases can be derived directly from the columns and rows of the original matrix, for example as a result of (random) sampling, elimination, or orthogonalizing transformations. Matrix $\mathbf{G}$ turns out to be a more challenging one. It encodes the essential properties of $\mathbf{A}$ isolated by $\mathbf{F}$ and $\mathbf{H}$ in a way that guarantees their reconstruction in the original matrix.

The key question is how to determine the mixing matrix $\mathbf{G}$ when $\mathbf{F}$ and $\mathbf{H}$ are given? As mentioned, the best way to learn more about a given matrix is to factor it into a particular mixture of other matrices. Let us apply the same trick here: let us factor the mixing matrix $\mathbf{G}$.

A straightforward way to calculate $\mathbf{G}$ as a product of other matrices is to invert (1):

$$
\mathbf{G} = \mathbf{F}^+ \mathbf{A} (\mathbf{H}^*)^+.
$$

However, this operation does not reveal everything there is to reveal. Fortunately, there is a more promising way. Instead of inverting $\mathbf{F}$ and $\mathbf{H}$, we can introduce new input and output transformations, $\mathbf{Y}$ and $\mathbf{X}$, and assume that

$$
\mathbf{G} = \mathbf{Y}^* \mathbf{A} \mathbf{X},
$$

where $\mathbf{Y}$ is $(k \times m)$, $\mathbf{A}$ is $(m \times n)$, and $\mathbf{X}$ is $(n \times k)$.

The key insight follows from substituting (3) into (1):

$$
\mathbf{A} = \mathbf{F} \mathbf{Y}^* \mathbf{X} \mathbf{H}^*.
$$

The original decomposition (1) is a reconstruction formula in disguise. Matrix factorization reconstructs the original matrix by projecting it onto its own column space and row space. For that to be possible it defines two projectors, the column space projector $\mathbf{F} \mathbf{Y}^*$ and the row space projector $\mathbf{X} \mathbf{H}^*$. Those projectors are determined by $\mathbf{F}$ and $\mathbf{H}$, but also by $\mathbf{Y}$ and $\mathbf{X}$, the factors encoding the properties of $\mathbf{A}$ in $\mathbf{G}$. Therefore, to calculate the mixing matrix when $\mathbf{F}$ and $\mathbf{H}$ are known, it is necessary to find $\mathbf{Y}$ and $\mathbf{X}$ that define projectors for the column space and the row space of $\mathbf{A}$. That is the idea behind meta-factorization.

2.2 The projector equation

As we have seen, matrix factorization reconstructs the original matrix by projecting it onto its own column and row space. Matrix decomposition given by (1) is a reconstruction formula (4) in disguise. The idea behind meta-factorization is to control (or shape) the internal structure of that reconstruction formula. This, however, can only be accomplished provided that the properties of the projection matrices remain intact. We formulate that constraint in the form of a projector equation.

Suppose that $\mathbf{F}$ and $\mathbf{H}$ are arbitrary matrices. The projector equation defines $\mathbf{Y}$ and $\mathbf{X}$ such that

$$
\mathbf{P} = \mathbf{F} \mathbf{Y}^* \quad \text{and} \quad \mathbf{R} = \mathbf{X} \mathbf{H}^*.
$$

are idempotent matrices, i.e., $\mathbf{P}^2 = \mathbf{P}$ and $\mathbf{R}^2 = \mathbf{R}$.

**Theorem 1.** If $\mathbf{F} \mathbf{Y}^*$ and $\mathbf{X} \mathbf{H}^*$ are idempotent rank-$k$ matrices, then they satisfy the projector equation,

$$
\mathbf{Y}^* \mathbf{F} = \mathbf{H}^* \mathbf{X} = \mathbf{I}_k.
$$

Conversely, if the the projector equation holds, then $\mathbf{F} \mathbf{Y}^*$ and $\mathbf{X} \mathbf{H}^*$ are idempotent rank-$k$ matrices.
Proof. Consider $P = FY^*$ and $R = XH^*$. If (6) holds, then $P^2 = FY^*FY^* = FY^* = P$. By assumption, $\text{rank}(F) \leq k$, so we have

$$\text{rank}(FY^*) \leq \text{rank}(F) \leq k = \text{rank}(Y^*F) \leq \text{rank}(F) = \text{rank}(FY^*F) \leq \text{rank}(FY^*),$$

(7)

so $\text{rank}(P) = k$. Similarly, $R^2 = XH^*XH^* = XH^* = R$, and $\text{rank}(R) = k$, by the same arguments.

Conversely, if $FY^*$ is idempotent rank-$k$ matrix, then

$$F(Y^*F - I_k)Y^* = 0$$

and $\text{rank}(F) = \text{rank}(Y^*) = k$. Indeed, suppose $F$ has linearly dependent columns and the equation holds for a nonzero matrix $(Y^*F - I_k)Y^*$. But then $FY^*$ is not a rank-$k$ matrix, which contradicts the assumption. The same argument holds for $Y^*$. Therefore, we conclude that (6) holds.

Similarly, if $XH^*$ is idempotent rank-$k$, then

$$X(H^*X - I_k)H^* = 0.$$  (9)

Since $\text{rank}(X) = \text{rank}(H^*) = k$, we conclude that (6) holds.

We shall call equation (6) the projector equation. The equation is fundamental and paves the way to our meta-factorization results. It connects $Y^*$ with $F$ and $X$ with $H^*$ to make them projectors, which is a key step of the meta-factorization procedure.

The equation has been often used in linear algebra. For example, it appears in the already mentioned works of Langenhop [15], Ben-Israel and Greville [1] on the generalized inverses of matrices, and in the recent work of Sorensen and Embree [27] on the CUR decomposition. However, it seems that its role has not been extensively explored in the context being considered. The projector equation can also be generalized, in which case it may give rise to new factorizations, as predicted in Appendix A.

To appreciate the role of the projector equation, we must characterize its solutions.

**Theorem 2.** The projector equation (6) admits solutions:

$$Y^* = (B^*F)^+B^* \quad \text{and} \quad X = D(H^*D)^+$$  (10)

for any $B$ and $D$ such that:

$$\text{rank}(B^*F) = \text{rank}(H^*D) = k = \min\{m, n\}.\quad (11)$$

Furthermore, $Y^*$ and $X$ satisfy the generalized matrix inverse equations:

$$FY^*F = F \quad \text{and} \quad H^*XH^* = H^*.$$  (12)

Finally, when $B = F$ and $D = H$, then $Y^* = F^+$ and $X = (H^*)^+$.

**Proof.** Substituting (10) into (6), we obtain

$$Y^*F = (B^*F)^+B^*F = I_k = H^*D(H^*D)^+ = H^*X.$$  (13)

It follows that $Y^*$ and $X$ satisfy the generalized inverse equations (12).
To prove that \( Y^* = F^+ \) when \( B = F \), we show that in the general case,

\[
(F^*F)^+F^* = F^+.
\]

(14)

Let \( W = (F^*F)^+F^* \). We must verify that \( W \) satisfies the Penrose equations \([23]\) for a pseudoinverse:

\[
\begin{align*}
FWF &= F, \\
WFW &= W, \\
(FW)^* &= FW, \\
(WF)^* &= WF.
\end{align*}
\]

(15)

First, consider \( F^*F \) and observe that by the properties of a pseudoinverse,

\[
F^*F = F^*(F^*F)^+F^*F = F^*FWF,
\]

(16)

which implies that

\[
F^*F(WF - I_k) = 0.
\]

(17)

Multiplying on the left by \((WF - I_k)^*\) and using the properties of transposition gives \((FWF - F)^*(FWF - F) = 0\). Therefore,

\[
FWF = F
\]

(18)

and the first Penrose equation is satisfied.

Next, consider the equation,

\[
(F^*F)^+F^*F(F^*F)^+ = (F^*F)^+.
\]

(19)

Multiplying on the right by \( F^* \), we conclude that

\[
WFW = W
\]

(20)

Therefore, the second Penrose equation is satisfied, as well.

The last two Penrose equations follow from the properties of transpositions. Namely, we have

\[
FW = (FW)^* \quad \text{and} \quad WF = (WF)^*
\]

(21)

and we conclude that

\[
W = F^+.
\]

(22)

By similar arguments,

\[
(H^*)^+ = H(H^*H)^+
\]

(23)

and the result holds with \( D = H \). \( \Box \)

We have demonstrated that matrices of the form given by (10) solve the projector equation and give rise to orthogonal projectors when \( B = F \) and \( D = H \). It remains to show that they also define oblique projectors. In other words, \( Y^* \) and \( X \) must satisfy the generalized inverse equations even when \( B^*F \) and \( H^*D \) are not full rank matrices.
Theorem 3. Suppose that
\[ Y^* = (B^*F)^+B^* \quad \text{and} \quad X = D(H^*D)^+, \]
where \( B \) and \( D \) satisfy condition:
\[ \text{rank}(B^*F) = \text{rank}(H^*D) = k. \]
Then
\[ FY^*F = F \quad \text{and} \quad H^*XH^* = H^*. \]

Proof. It follows from the properties of pseudoinverse that:
\[ B^*F(B^*F)^+B^*F - B^*F = 0. \]
Multiplying on the left by \((B^*F)^+B^*F - I_k)^*\), yields
\[ (B^*F)^+B^*F - (F(B^*F)^+B^*F - F) = 0. \]
Notice that \( B^*F \) is not a full rank matrix, so we cannot expect \((B^*F)^+B^*F\) to be equal to \( I_k \). Therefore, for the equation to hold for every \( B \), we must have
\[ F(B^*F)^+B^*F = FY^*F = F. \]
Similarly, consider the equation
\[ H^*D(H^*D)^+H^*D - H^*D = (H^*D(H^*D)^+ - I_k)H^*D = 0. \]
Multiplying on the right by \((H^*D(H^*D)^+ - I_k)^*\), yields
\[ (H^*D(H^*D)^+ - I_k)H^*D(H^*D)^+ - I_k^* \]
\[ = (H^*D(H^*D)^+H^* - H^*)D^*(H^*D(H^*D)^+ - I_k)^* \]
\[ = (H^*D(H^*D)^+H^* - H^*)(H^*D(H^*D)^+D^* - D^*)^* = 0. \]
Again, since \( H^*D(H^*D)^+ \) is not equal to \( I_k \) in the general case considered and the equation must hold for every \( D \), we conclude that
\[ H^*D(H^*D)^+H^* = H^*XH^* = H^*. \]

In short, the projector equation defines necessary and sufficient conditions for \( FY^* \) and \( XH^* \) to be idempotent rank-\( k \) matrices. The form of the solutions \( Y^* \) and \( X \), given by \([10]\), can be generalized to make \( FY^* \) and \( XH^* \) oblique projectors (idempotent matrices).

2.3 The reconstruction equation and meta-factorization procedure

We are now ready to introduce the concept of meta-factorization, a procedure for decomposing the factorization’s factors into the product of matrices of the desired structure.
When $F$ and $H$ are given, the projector equation defines matrices $Y$ and $X$ that satisfy the constraints of the matrix reconstruction problem. It keeps the properties of the projection matrices, $FY^*$ and $XH^*$, intact. Therefore, we must also enforce the projections onto the required subspaces. Indeed, notice that if the projector equation holds, $A$ is transformed into some $M = (FY^*)A(XH^*)$.

But when $F$ and $H$ are arbitrary, we cannot expect $M$ to approximate $A$. For the reconstruction to be successful it is also necessary to demand that $F$ and $H$ define the column space and row space of $A$. That brings us to our main result. Theorem 4 introduces the idea of meta-factorization exploiting the projector equation to find and tune the desired projector matrices.

**Theorem 4.** The reconstruction equation

$$A = FY^*AXH^*$$

holds for a rank-$k$ matrix $A$, when $Y^*$ and $X$ solve the projector equation (6) with $F$ and $H$ representing bases for the column and row space of $A$.

**Proof.** The result follows from the elementary properties of projections. By Theorem 1, $P = FY^*$ is a projection. By the assumption made, we also have $C(A) = C(P)$. Take any $a \in C(A) = \{y \in \mathbb{R}^m \mid y = Px\}$. It follows that

$$Pa = P(Px) = Px = a.$$  \hspace{1cm} (35)

The same argument holds for $R = XH^*$. For any $b \in C(A^*) = \{w \in \mathbb{R}^n \mid w = R^*v\}$, we have

$$R^*b = b.$$  \hspace{1cm} (36)

As a result, $PA = A$ and $R^*A^* = A^*$, which gives $PAR = A$. \qed

Basic meta-factorization proceeds according to the following steps (illustrated with Matlab code):

**Step 1:** Given a matrix $A$, select basis $F$ for the column space $C(A)$ and basis $H$ for the row space $C(A^*)$.

**Step 2:** Solve the projector equation $Y^*F = H^*X = I_k$ to find the input basis $X$ and the output basis $Y$:

$$[QY,RY] = \text{qr}(B'*F,0);$$  \hspace{1cm} % If needed, select $B$ and $D$,

$$[QX,RX] = \text{qr}(H'*D,0);$$  \hspace{1cm} % otherwise, set $B = F$ and $D = H$.

$$Y = (RY \setminus (QY'*B'))';$$  \hspace{1cm} % If rank$(B'*F) = \text{rank}(H'*D) = k$,

$$X = (D/RX)*QX';$$  \hspace{1cm} % calculate $Y$ and $X$.

**Step 3:** Calculate the mixing matrix $G = Y^*AX$ and (if needed) perform its factorization to obtain the desired decomposition $A = FGH^*$:

$$G = Y^*A*X;$$  \hspace{1cm} % If needed, perform additional factorization of $G$, e.g.,

$$\% [U,S,V] = \text{svd}(G), [Q,R] = \text{qr}(G) \text{ or } [L,U] = \text{lu}(G), \text{ etc.}$$

$$Ar = F*G*H';$$
The procedure finds solutions of the meta-factorization equations,

\[
\begin{align*}
G &= Y^* A X, \\
A &= F G H^*,
\end{align*}
\]  

(37)

Matrices \(F\) and \(H\) are the parameters of the procedure, and their role is to define bases for the column space and the row space of \(A\). Given \(F\) and \(H\), matrices \(X\), \(Y\), and \(G\) are determined in a way that allows \(A\) to be reconstructed from \(G\). As a result, \(A\) is decomposed into the product of \(F\), \(G\), and \(H\), and \(G\) into the product of \(Y\), \(X\) and \(A\). Finally, factorization of the mixing matrix \(G\), i.e., meta-factorization, shapes the structure of projectors and their numerical properties.

Notice that the proof of Theorem 5 only requires idempotent \(FY^*\) and \(XH^*\). We can take advantage of that fact and use Theorem 3 to formulate a more general result.

**Corollary 1.** The reconstruction equation holds, if

\[
Y^* = (B^*F)^+ B^* \quad \text{and} \quad X = D(H^*D)^+, \tag{38}
\]

where \(B\) and \(D\) satisfy condition:

\[
\text{rank}(B^*F) = \text{rank}(H^*D) = k, \tag{39}
\]

and \(F\) and \(H\) represent bases for the column and row space of \(A\).

We can further adjust the meta-factorization procedure to admit low-rank bases as well. Given a rank-\(k\) matrix \(A\) and some positive \(r \leq k\), we can find the required rank-\(r\) projectors by solving the projector equation (6) with \(I_r\) for the low-rank bases \(F_r\) and \(H_r\). Setting \(B = F_r\) and \(D = H_r\) is preferred in this case. The procedure also admits designing one projector only, in which case it is enough to work with either \(F\) or \(H\).

### 2.4 Factorizations as solutions of linear matrix equations

Finally, this section develops the prospect of meta-factorization that describes matrix factorizations as solutions of linear matrix equations.

In his seminal work on the general inverse for matrices, Penrose characterized solutions to linear matrix equations. The statement of the theorem is presented below in the notation of the present paper.

**Theorem 5** (Penrose [24]). A necessary and sufficient condition for the equation

\[
FGH^* = A \tag{40}
\]

to have a solution \(G\) is

\[
FF^+ A(H^*)^+ H^* = A, \tag{41}
\]

in which case the general solution is

\[
G = F^+ A(H^*)^+ + W - F^+ FWH^* (H^*)^+, \tag{42}
\]

where \(W\) is arbitrary.
The proof presented by Penrose exploits the properties of pseudoinverses, introduced in the very same paper. Meta-factorization, exploiting the properties of projections, shows that another perspective can be taken in which solutions to the addressed matrix equation become matrix factorizations. Indeed, when we interpret (40) as a description of a matrix factorization, then equation (41) defines that matrix factorization by the properly constructed projection matrices. The latter equation is clearly a special case of the reconstruction equation (34) of Theorem 4. In other words, matrix factorization given by (40) is a solution of the meta-factorization equations (34).

**Theorem 6.** Given $A$, $F$ and $H$, a necessary and sufficient condition for the equation

$$FGH^* = A$$

(43)
to have a solution $G$ is

$$FY^*AXH^* = A,$$

(44)
where

$$Y^* = (B^*F)^+B^* \quad \text{and} \quad X = D(H^*D)^+,$$

(45)
and $\text{rank}(B^*F) = k = \text{rank}(H^*D)$. The general solution is

$$G = Y^*AX + W - Y^*FWH^*X,$$

(46)
where $W$ is arbitrary.

**Proof.** If $G$ satisfies (43), then $C(F) = C(A)$ and $C(H^*) = C(A^*)$. By Theorem 3 and Corollary 1 we have

$$A = F(B^*F)^+B^*AD(H^*D)^+H^*,$$

(47)
where $G = (B^*F)^+B^*AD(H^*D)^+ = Y^*AX$.

Conversely, if (44) holds, then $G = (B^*F)^+B^*AD(H^*D)^+$ is a particular solution of (43). The general solution must also satisfy the homogeneous equation, $FGH^* = 0$. Consider

$$G = W - Y^*FWH^*X.$$  

(48)
By Theorem 2 and 3 we have

$$FGH^* = FWH^* - FY^*FWH^*XH^*$$  

$$= FWH^* - FWH^* = 0.$$  

(49)
Therefore, the general solution to (43) is given by (46). \qed

Notice that when the projector equation holds, the solution of the homogeneous equation is $G = 0$. Otherwise, when the projector equation is not satisfied, $G$ is rectangular and depends on $W$.

As a special case of the result above we get the characterization of solutions of the vector equations.

**Corollary 2.** If $c \in C(A)$, then the general solution of the vector equation $Ax = c$ is

$$x = Y^*c + (I - Y^*A)y,$$

(50)
where $Y^* = (B^*A)^+B^*$ and $\text{rank}(B^*F) = k$, and $y$ is arbitrary.

In the following section we show how the meta-factorization procedure works and how certain well-known matrix factorizations satisfy its equations.
3 How meta-factorization describes known factorizations

Meta-factorization explores the structure of known factorizations, thereby revealing their properties and providing hints for possible modifications. These insights may become useful for designing numerical algorithms, as they provide ways to control the properties of numerical algorithms, including performance, stability, and complexity. The key benefit of meta-factorization is that it allows for modifying the factors of factorizations while preserving the ability of the factors’ product to reconstruct the original matrix.

This section illustrates how meta-factorization can be applied to describe and modify selected factorizations. First, it reconstructs the SVD. Next, it derives the form of the mixing matrix for the CPQR. Finally, by performing meta-factorization of the mixing matrix for QR-based decompositions, it develops a family of UTV-like factorizations.

3.1 Singular value decomposition (SVD)

Let us begin by illustrating how meta-factorization procedure works. We apply it to reconstruct the reduced SVD.

For every rectangular \( m \times n \) matrix \( A \), the SVD shows that \( A \) is isometric to the singular value matrix

\[
S = U^* A V. \tag{51}
\]

Matrices \( U \) and \( V \) are complex unitary, and \( S \) is a rectangular diagonal matrix with (real non-negative) singular values on the diagonal. The fundamental subspaces for \( A \), namely, the column space, the row space and the corresponding nullspaces, as well as the rank of \( A \), all emerge as a result of this special factorization.

The reduced singular value decomposition factors a rank-\( k \) matrix \( A \) into the following product:

\[
A = U(:, 1:k) S(1:k, 1:k) V(:, 1:k)^*. \tag{52}
\]

We can reconstruct the reduced singular value decomposition by following the steps of meta-factorization. The goal is to build the diagonal mixing matrix \( G = S(1:k, 1:k) = \mbox{diag}(\sigma_1, \ldots, \sigma_k) \).

The first step is to select bases for \( C(A) \) and \( C(A^*) \). As is well known, the required choice of bases is

\[
F = U(:, 1:k) \quad \text{and} \quad H^* = V(:, 1:k)^*, \tag{53}
\]

where \( U(:, 1:k) \) and \( V(:, 1:k) \) hold orthonormal eigenvectors corresponding to the nonzero eigenvalues of \( AA^* \) and \( A^* A \). More generally, \( U \) and \( V \) are defined by the following (symmetric) eigenvalue decompositions,

\[
(A A^*) U = U (S S^*) \quad \text{and} \quad (A^* A) V = V (S^* S). \tag{54}
\]

The structure of the SVD decomposition is imposed by the choice of \( F \) and \( H \). In fact, this particular choice makes \( S \) diagonal. The columns of \( H = [v_1 | \cdots | v_k] \) are connected with the columns of \( F = [u_1 | \cdots | u_k] \) by the equations

\[
A v_i = \sigma_i u_i \quad \text{and} \quad A^* u_i = \sigma_i v_i \quad \text{for} \quad i = 1, \ldots, k. \tag{55}
\]

Thus, by orthonormality of \( U \) (and \( V \)), we have

\[
u_i^* A v_i = \sigma_i, \quad i = 1, \ldots, k, \tag{56}
\]

\[
u_i^* A v_j = 0, \quad i \neq j. \tag{57}
\]
The second step of meta-factorization should confirm this fundamental fact and, indeed, it does. Solving the projector equation (56) for $Y$ and $X$, we obtain

$$Y^* = U(:, 1:k)^* \quad \text{and} \quad X = V(:, 1:k).$$

(58)

These are precisely the matrices that diagonalize $A$ and make it isometric to the diagonal singular value matrix. This is demonstrated in the final third step of the meta-factorization procedure, which yields

$$G = S(1:k, 1:k) = U(:, 1:k)^* A V(:, 1:k) = \text{diag}(\sigma_1, \ldots, \sigma_k).$$

(59)

As desired, we have reconstructed the reduced SVD for $A$, namely,

$$A = U(:, 1:k) U(:, 1:k)^* A V(:, 1:k) V(:, 1:k)^*.$$

(60)

### 3.2 Column-Pivoted QR decomposition (CPQR)

We now show how meta-factorization reveals the form of the mixing matrix $G$ for the CPQR.

The CPQR decomposition factors any rectangular $m \times n$ matrix $A$ into the product of an orthogonal $m \times m$ matrix $Q$, an upper rectangular $m \times n$ matrix $R$ and a rectangular $n \times n$ permutation matrix $\Pi$:

$$A = Q R \Pi^*.$$

(61)

The CPQR also reveals matrix rank; see also Chan, Gu and Eisenstat [2, 9] for more details. Given a rank-$k$ matrix $A$, we obtain

$$R = \begin{bmatrix} R_{11} & R_{12} \\ 0 & 0 \end{bmatrix},$$

(62)

where $R_{11}$ is a nonsingular upper triangular $k \times k$ matrix. The thin QR decomposition is then given by

$$A = Q(:, 1:k) R(1:k, :) \Pi^*.$$

(63)

To get additional insight into the structure of the decomposition, we can perform its meta-factorization. Suppose that $Q$, $R$ and $\Pi$ are known for $A$. The goal is to examine the form of the decomposition’s mixing matrix $G$.

Define $F$ and $H$ by the factors of the thin QR decomposition,

$$F = Q(:, 1:k) \quad \text{and} \quad H^* = R(1:k, :) \Pi^*.$$

(64)

Solving the projector equation yields

$$G = Q(:, 1:k)^* A (R(1:k, :) \Pi^*)^+.$$

(65)

However, by the properties of the thin QR decomposition,

$$Q(:, 1:k)^* A = R(1:k, :) \Pi^*.$$

(66)

The mixing matrix for the CPQR decomposition is, therefore,

$$G = I_k.$$

(67)
Notice that in the light of this result, equation (65) can be viewed as a special type of diagonalization or similarity-type relation.

By the properties of pseudoinverse, the final step of the meta-factorization yields the column-pivoted QR decomposition,
\[
A = Q(:, 1:k)GR(1:k, :)\Pi^* \quad \text{(68)}
\]
as expected.

3.3 UTV decompositions

We have already seen how meta-factorization describes selected factorizations, the SVD and the CPQR. This section shows how it can be applied to design matrix factorizations. The key step involved is to perform factorization of the mixing matrix \( G \) encoding the properties of \( A \). We demonstrate this approach by reconstructing the UTV decomposition and then introducing its modifications.

As Golub et al. and Stewart presented in [8, 28], matrix \( A \) can be factored into a product of two unitary matrices, \( U \) and \( V \), and an upper-triangular or lower-triangular mixing matrix \( T \). The resulting UTV decomposition is a generalization of the SVD and the CPQR decomposition. The SVD is the case with diagonal \( T \) and the CPQR is the case with \( V \) being the permutation matrix.

To reconstruct the UTV decomposition, we only need to consider the row space projector. Define the row space basis as
\[
H^* = Q_r(:, 1:k)^* \quad \text{(69)}
\]
The projector equation yields
\[
X = Q_r(:, 1:k) \quad \text{(70)}
\]
As a result, we reach the decomposition
\[
G = AQ_r(:, 1:k) \quad \text{and} \quad A = AQ_r(:, 1:k)Q_r(:, 1:k)^* \quad \text{(71)}
\]
Extending the obtained reconstruction formula for \( A \) to include the remaining columns of \( Q_r \) yields
\[
A = AQ_r(:, 1:k)Q_r(:, 1:k)^* \\
= [AQ_r(:, 1:k) \ AQ_r(:, (k+1):n)] [Q_r(:, 1:k)^*] [Q_r(:, (k+1):n)^*] \\
= \begin{bmatrix} G & AQ_r(:, (k+1):n) \end{bmatrix} Q_r^* \quad \text{(72)}
\]
This is the factorization we wish to transform into the UTV decomposition. The key step is to perform factorization (meta-factorization) of the mixing matrix \( G \). In this case, the required factors are provided by the SVD. Namely, we get
\[
G = AQ_r(:, 1:k) = USV^* \quad \text{(73)}
\]
\[
\text{The UTV decomposition with upper-triangular } T \text{ is also known as the URV decomposition, whereas the one with lower-triangular } T \text{ is also known as the ULV decomposition.}
\]

13
Inserting (73) into (72) and factoring out \( \bar{U} \) and \( \bar{V}^* \), we obtain the desired UTV decomposition:

\[
A = \bar{U} \left[ \begin{array}{cc} \bar{S} & \bar{U}^* A Q_r(:,(k+1):n) \end{array} \right] \left[ \begin{array}{cc} \bar{V}^* & 0 \\ 0 & I_{n-k} \end{array} \right] Q_r^* = UTV^*,
\]

\[
U = \bar{U}, \quad T = \left[ \begin{array}{cc} \bar{S} & \bar{U}^* A Q_r(:,(k+1):n) \end{array} \right], \quad V = Q_r \left[ \begin{array}{cc} \bar{V} & 0 \\ 0 & I_{n-k} \end{array} \right].
\]

Notice that \( U \) and \( V \) are indeed unitary (square and orthonormal), and \( T \) is upper-triangular. For more details on computing the UTV decomposition see Martinsson et al. [19].

We can now turn to the more general case of factorizations with two sided projectors. Consider the following CPQR decompositions:

\[
A = Q_r R_c \Pi_c^* \quad \text{and} \quad A^* = Q_r R_r \Pi_r^*.
\]

Selecting \( Q_c \) to represent the column space and \( Q_r \) to represent the row space for \( A \), we have

\[
F = Q_c(:,1:k) \quad \text{and} \quad H = Q_r(:,1:k),
\]

and solving the projector equation, we get

\[
Y^* = Q_c(:,1:k)^* \quad \text{and} \quad X = Q_r(:,1:k).
\]

From that, we immediately obtain the desired meta-factorization equations,

\[
G = Q_c(:,1:k)^* A Q_r(:,1:k), \quad A = Q_c(:,1:k) Q_c(:,1:k)^* A Q_r(:,1:k) Q_r(:,1:k)^*.
\]

Since the SVD-defining condition (55) does not hold in this case, the mixing matrix \( G \) is not diagonal anymore. However, it does store enough information to reconstruct \( A \) in the system of coordinates given by \( F \) and \( H \). This is the observation we can exploit. Furthermore, we can drop the requirement for both matrices, \( U \) and \( V \), to be unitary. The result is the design of UTV-like factorizations.

The key step is factoring the mixing matrix. Consider again performing the SVD of \( G \):

\[
G = Q_c(:,1:k)^* A Q_r(:,1:k) = \bar{U} \bar{S} \bar{V}^*.
\]

Introducing that result into (74) defines the following factorization:

\[
A = \bar{U} \left[ \begin{array}{cc} \bar{S} & \bar{U}^* Q_c(:,1:k)^* A Q_r(:,(k+1):n) \end{array} \right] \left[ \begin{array}{cc} \bar{V}^* & 0 \\ 0 & I_{n-k} \end{array} \right] Q_r^* = UTV^*,
\]

\[
U = Q_c(:,1:k) \bar{U}, \quad T = \left[ \begin{array}{cc} \bar{S} & \bar{U}^* Q_c(:,1:k)^* A Q_r(:,(k+1):n) \end{array} \right], \quad V = Q_r \left[ \begin{array}{cc} \bar{V} & 0 \\ 0 & I_{n-k} \end{array} \right].
\]

The following Matlab code illustrates the design:
Notice that $U$ is not unitary anymore. If $A$ is a full column-rank matrix, then $Q_c(:,1:k)$ is rectangular and
\[ UU^* = Q_c(:,1:k)Q_c(:,1:k)^*, \] (81)
which is not an identity matrix in general.

By following the same technique of meta-factorization, we can design another UTV-like decomposition. Instead of the SVD, performing the CPQR decomposition of the mixing matrix yields
\[ G = \tilde{Q}\tilde{R}\tilde{\Pi}^*. \] (82)
The desired factorization can now be computed as
\[ A = Q_c(:,1:k)\tilde{Q}\tilde{R}\tilde{\Pi}^*Q_r(:,1:k)^* = UTV^*, \] (83)
\[ U = Q_c(:,1:k)\tilde{Q}, \]
\[ T = \tilde{R}, \]
\[ V = Q_r(:,1:k)\tilde{\Pi}. \]
In this case, $U$ and $V$ have orthogonal columns and $T$ is upper-triangular. The Matlab code illustrates the design:

Similarly, to obtain the decomposition with the lower-triangular mixing matrix $T$, consider the permutated LU decomposition of the mixing matrix. This gives rise to the following equality,
\[ G = \tilde{\Pi}\tilde{L}\tilde{U}. \] (84)
By following the same procedure as before, we obtain

\[ A = Q_c(:, 1:k)\tilde{\Pi}^*\tilde{L}\tilde{U}Q_r(:, 1:k)^* = UTV^*, \]
\[ U = Q_c(:, 1:k)\tilde{\Pi}^*, \]
\[ T = L, \]
\[ V = Q_r(:, 1:k)\tilde{U}^*. \]  

As before, the Matlab code illustrates the design:

```matlab
F = Qc(:,1:k); B = F;
H = Qr(:,1:k); D = H;
[QY,RY] = qr(B'*F,0); [QX,RX] = qr(H'*D,0);
Y = (RY\(QY'*B'))'; X = (D/RX)*QX';
G = Y'*A*X;
[Lbar,Ubar,Pbar] = lu(G);
U = Qc(:,1:k)*Pbar';
T = Lbar;
V = Qr(:,1:k)*Ubar';
Ar = U*T*V';
```

In that case, \( T \) is lower-triangular by design. However, only \( U \) has orthogonal columns. That is not the case with \( V \) due to the upper-triangular factor given by \( \tilde{U} \).

This section demonstrated that meta-factorization reconstructs known factorizations, reveals their internal structures, and allows for introducing modifications. Let us see if there is more to the application of presented concept.

### 4 Additional benefits of meta-factorization

This section takes the prospect of meta-factorization to investigate a relationship of meta-factorization with the theory of generalized matrix inverses, reconstructs the generalized Nyström’s method for computing randomized low-rank matrix approximations, and shows the role of the CUR decomposition.

#### 4.1 Pseudoinverse and its formulas

It can be noticed that meta-factorization equations resemble the first and second Penrose equation characterizing the pseudoinverse of a matrix, which suggests a relationship of meta-factorization with the theory of generalized matrix inverses. This section provides some basic observations on this matter. First, it shows that meta-factorization equations and Penrose equations are equivalent for square matrices. Second, it shows how meta-factorization can be used to derive explicit formulas for a pseudoinverse.

The Moore-Penrose pseudoinverse satisfies the two equations

\[ A A^+ A = A \quad \text{and} \quad A^+ A A^+ = A^+. \]  

These become meta-factorization equations (under the generalized assumptions of Theorem\[3\]). Indeed, taking

\[ F = A \quad \text{and} \quad H^* = A \]  

(87)
immediately results in the mixing matrix:
\[ G = A^+ A A^+ = A^+, \]  
where \( Y^* = X = A^+ \). Therefore, we also have
\[ A = A G A = A A^+ A. \]  

We can now use meta-factorization to describe the internal structure of pseudoinverse.

To illustrate the general idea, let us follow James [13], Strang and Drucker [31], and start with the CR decomposition,
\[ A = C R. \]  
When \( A \) is a rank-\( k \) matrix, it can be factored into the product of the \( m \times k \) matrix \( C \), containing the first \( k \) independent columns of \( A \), and the \( k \times n \) matrix \( R \), containing \( k \) nonzero rows of the row reduced echelon form of \( A \). Since both \( C \) and \( R \) have full-rank \( k \), the pseudoinverse of \( A \) is
\[ A^+ = R^+ C^+ = R^+ (R R^*)^{-1} (C^* C)^{-1} C^*. \]  
However, we also have
\[ (R R^*)^{-1} (C^* C)^{-1} = (C^* C R R^*)^{-1} = (C^* A R^*)^{-1}. \]  
Therefore, we obtain the following formula for the pseudoinverse:
\[ A^+ = R^* (C^* A R^*)^{-1} C^*. \]  

We can now derive that formula by following the steps of meta-factorization, thereby revealing the internal structure of a pseudoinverse. Selecting
\[ F = R^* \quad \text{and} \quad H = C, \]  
and solving the projector equation for \( Y \) and \( X \), yields
\[ Y^* = (R R^*)^{-1} R \quad \text{and} \quad X = C (C^* C)^{-1}. \]  
This brings us to the decomposition
\[ G = (R R^*)^{-1} R A^+ C (C^* C)^{-1} \quad \text{and} \quad A^+ = R^* (R R^*)^{-1} R A^+ C (C^* C)^{-1} C^*. \]  
To show that this is equivalent to \([23]\), it is enough to notice that
\[ A^+ = R^+ C^+. \]  
Indeed, we have
\[ A^+ = R^* (R R^*)^{-1} R A^+ C (C^* C)^{-1} C^* \]
\[ = R^* (R R^*)^{-1} R R^+ C^+ C (C^* C)^{-1} C^* \]
\[ = R^* (R R^*)^{-1} (C^* C)^{-1} C^* \]
\[ = R^* (C^* A R^*)^{-1} C^*, \]  
as required. Thus, the mixing matrix for the pseudoinverse is given by
\[ G = (C^* A R^*)^{-1}. \]  

The following result, presented by Ben-Israel and Greville in [31], shows that formula \([23]\) is a special case of the general formula derived by MacDuffee.

17
Theorem 7 (MacDuffee [1]). If a complex $m \times n$ matrix $A$ of rank $k > 0$ has a full-rank factorization

$$A = BD,$$  \hspace{1cm} (100)

then

$$A^+ = D^*(B^*AD^*)^{-1}B^*.$$  \hspace{1cm} (101)

Let us derive the formula by following the steps of meta-factorization. By Theorem 2, orthogonal projections corresponding to

$$F = D^* \quad \text{and} \quad H^* = B^*$$  \hspace{1cm} (102)

are given by

$$Y^* = (D^*)^+ \quad \text{and} \quad X = (B^*)^+.$$  \hspace{1cm} (103)

Therefore, the mixing matrix for a pseudoinverse is

$$G = (D^*)^+A^+(B^*)^+ = (B^*AD^*)^+.$$  \hspace{1cm} (104)

Since $B^*AD^*$ is nonsingular as a product of nonsingular matrices, we have

$$A^+ = D^*(B^*AD^*)^{-1}B^*,$$  \hspace{1cm} (105)

giving the desired explicit formula for a pseudoinverse.

4.2 Generalized Nyström’s method, the CUR and the outer-product decomposition

Nakatsukasa [22] showed that many algorithms of randomized linear algebra can be written in the form of the generalized Nyström’s method. Matrix $A$ can be approximated by matrix $A_r$ being a two-sided projection defined by two sampling (sketch) matrices, an $n \times k$ matrix $\Omega_c$ and an $m \times k$ matrix $\Omega_r$:

$$A_r = A\Omega_c(\Omega_r^*A\Omega_c)^+\Omega_r^*A.$$  \hspace{1cm} (106)

There are two projection matrices involved. The first one,

$$P = A\Omega_c(\Omega_r^*A\Omega_c)^+\Omega_r^*,$$  \hspace{1cm} (107)

defines an oblique projection onto the column space of $A\Omega_c$. The second,

$$R = \Omega_c(\Omega_r^*A\Omega_c)^+\Omega_r^*A,$$  \hspace{1cm} (108)

defines an oblique projection onto the row space of $\Omega_r^*A$.

The generalized Nyström’s method can be derived from the meta-factorization procedure. The goal is to design the oblique projectors. By Theorem 2,

$$F = A\Omega_c \quad \text{and} \quad H^* = \Omega_r^*A$$  \hspace{1cm} (109)

$$B = \Omega_r \quad \text{and} \quad D = \Omega_c$$  \hspace{1cm} (110)

yield the following solutions to the projector equation:

$$Y^* = (\Omega_r^*A\Omega_c)^+\Omega_r^* \quad \text{and} \quad X = \Omega_c(\Omega_r^*A\Omega_c)^+. $$  \hspace{1cm} (111)
As a result, the mixing matrix becomes:

$$G = (\Omega_r^* A \Omega_c)^+ \Omega_r^* A \Omega_c (\Omega_r^* A \Omega_c)^+ = (\Omega_r^* A \Omega_c)^+,$$

(112)

which gives rise to the generalized Nyström’s method:

$$A_r = F G H^*$$

$$= A \Omega_c (\Omega_r^* A \Omega_c)^+ \Omega_r^* A.$$

(113)

As pointed out by Nakatsukasa [22], the preferred way to compute $A_r$ is to perform QR factorization $\Omega_r^* A \Omega_c = QR$ and then take

$$A_r = (A \Omega_c R^{-1})(Q^* \Omega_r^* A),$$

(114)

which completes the third step of the meta-factorization procedure. The approximation error can be substantially reduced with the row-space oversampling. The Matlab code below illustrates the idea,

```matlab
D = randn(n,k); B = randn(m,ceil(2*k)); % row-space oversampling
F = A*D; H = A'*B;
[Q,R] = qr(B'*A*D,0);
Ar = (F/R)*(Q'*H');
```

The generalized Nyström’s method is closely related to the outer-product decomposition introduced by Wedderburn [35], Egerváry [7], Householder [12], Cline and Funderlic [4], and Chu et al. [3]. Suppose that $F$, $G$ and $H$ are given. Then,

$$\text{rank}(A - F G H^*) = \text{rank}(A) - \text{rank}(F G H^*)$$

(115)

if and only if there exist matrices $\Omega_c$ and $\Omega_r$ such that

$$F = A \Omega_c \quad \text{and} \quad G^{-1} = \Omega_r^* A \Omega_c \quad \text{and} \quad H = A^* \Omega_r.$$

(116)

Therefore, when the conditions above hold, there exists a finite rank-reducing process which decomposes $A$ into a sum of low-rank matrices. The result is the outer-product decomposition. To be more specific, consider the Wedderburn rank-one reduction process,

$$A_1 = A,$$

$$A_{r+1} = A_r - g_r^{-1} A_r u_r v_r^* A_r,$$

(117)

$$g_r = v_r^* A_r u_r \neq 0 \quad \text{for} \quad r = 1, \ldots, k - 1.$$

The process terminates after $k$ iterations for any rank-$k$ matrix and gives rise to the factorization of the following form:

$$A = F G H^* = [A_1 u_1, \ldots, A_k u_k] \begin{bmatrix} g_1 \\
& \ddots \\
g_k \end{bmatrix}^{-1} \begin{bmatrix} v_1^* A_1 \\
& \ddots \\
v_k^* A_k \end{bmatrix} = A \Omega_c (\Omega_r^* A \Omega_c)^{-1} \Omega_r^* A.$$

(118)

We have reached the following conclusion. The generalized Nyström’s method produces high-quality low-rank approximations of $A$ by exploiting the structure of the outer-product factorization and relaxing the rank-reduction conditions, defined by (116), with random sketch matrices $\Omega_c$ and $\Omega_r$. In both cases, the factors of the decompositions satisfy the equations of meta-factorization.
We close the section with one final observation. It regards the relation between the generalized Nyström’s method and the CUR factorization. Recall that the CUR factorization provides an approximation:

$$A \approx CUR,$$

(119)

where $C = A(:, J)$ is a subset of representative columns of $A$, $R = A(I, :)$ is a subset of representative rows of $A$, and the mixing matrix is $U$. The properties of the approximation depend critically on the rank of $U$ and the selection of columns and rows, $J$ and $I$, as demonstrated by Sorensen and Embree [27], Wang and Zhang [34], Martinsson and Tropp [18], Hamm and Huang [11].

To reconstruct the CUR decomposition from the meta-factorization equations, consider the following special case of sampling matrices,

$$\bar{\Omega}_c = I_n(:, J) \quad \text{and} \quad \bar{\Omega}_r = I_m(:, I).$$

(120)

Matrix

$$F = A\bar{\Omega}_c = C$$

(121)

contains the original columns of $A$. Similarly, matrix

$$H^* = \Omega^*_r A = R$$

(122)

contains the original rows of $A$. For a feasible matrix $B$, we then obtain

$$Y^* = (B^* A\bar{\Omega}_c)^+ B^*,$$

(123)

and for a feasible matrix $D$, we obtain

$$X = D(\bar{\Omega}_r^* AD)^+.$$  

(124)

By the rules of meta-factorization, the corresponding mixing matrix is

$$G = (B^* A\bar{\Omega}_c)^+ B^* AD(\bar{\Omega}_r^* AD)^+$$

$$= (B^* C)^+ B^* AD(RD)^+.$$  

(125)

Selecting

$$B = A\bar{\Omega}_c = C \quad \text{and} \quad D = A^* \bar{\Omega}_r = R^*$$

(126)

immediately leads to the following well-known form of the CUR decomposition,

$$A = C(C^* C)^+ C^* AR^* (RR^*)^+ R$$

$$= CC^+ AR^+ R.$$  

(127)

We have reconstructed the CUR decomposition from the meta-factorization equations.

We can now explain the relation between the CUR factorization and generalized Nyström’s method. It is defined by the structure of the hidden projectors. The projectors defined by (126) give rise to the CUR decomposition, whereas the projectors defined by (110) give rise to the generalized Nyström’s method.

The following Matlab code illustrates the concepts for a random (naive) selection of columns and rows:
In = eye(n); Im = eye(m);
I = randperm(m,k); J = randperm(n,k);
Q = In(:,J); P = Im(:,I);
F = A*Q; H = A'*P;
B = P; D = Q; \% Nystrom-like method, or
B = F; D = H; \% CUR (with naive sampling)

[QY,RY] = qr(B'*F,0); [QX,RX] = qr(H'*D,0); \% If rank(B'*F) = rank(H'*D) = k,
Y = (RY\(QY'*B'))'; X = (D/RX)*QX'; \% calculate Y and X.
Y = (pinv(B'*F)*B')'; X = D*pinv(H'*D); \% Otherwise, consider using pinv()
G = Y'*A*X;
Ar = F*G*H';

The examples of this section provide valuable insights into the world of matrix factorizations and reveal relations between well-known matrix structures and randomized algorithms.

5 Summary

The concept of meta-factorization develops a perspective in which matrix factorizations become solutions of linear matrix equations. This perspective leads to exciting insights when applied in the study of matrix algorithms. It allows for reconstructing known factorizations, investigating their internal structures, and introducing modifications on demand. Furthermore, once the prospect of meta-factorization is considered, it is possible to notice analogies between theories previously hidden from view. That is how we have gained insights investigating the generalized matrix inverses and randomized linear algebra algorithms. Also, there is much more to be studied about that idea and further twists are yet to be discovered.

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A Side note on generalizations

The projector equation plays one of the leading roles in the developed theory of meta-factorization. Therefore, it is tempting to ask if it can be generalized and what would be the outcomes of such a generalization? One potential answer is included here, possibly for future investigation.
Consider a process of recursive meta-factorization defined by a composition of meta-factorization equations. For the fixed column space and row space bases, \( F \) and \( H^* \), the reconstruction formula for \( A \) becomes

\[
A = (FY^*) (FY^*) \cdots (FY^*)A(XH^*) \cdots (XH^*)(XH^*)
\]

\[
= (FY^*)^N A(XH^*)^N
\]

\[
= F(Y^*F)^{N-1}Y^*AX(H^*X)^{N-1}H^*.
\]

(128)

Suppose that we generalize the projector equation, so that we have:

\[
Y^*F = Z_c \quad \text{and} \quad H^*X = Z_r \quad \text{and} \quad Z_c^N = Z_r^N = I_k.
\]

(129)

Observe now that, given rank-\( k \) matrices \( F \) and \( H^* \), the generalized projector equations admit the following solutions:

\[
Y^* = Z_c F^+ \quad \text{and} \quad X = (H^*)^+ Z_r.
\]

(130)

We can now see that as a result we get:

\[
(FY^*)^N = FZ_c^{N-1}Y^* = FZ_c^N F^+ = FF^+
\]

(131)

and:

\[
(XH^*)^N = XZ_r^{N-1}H^* = (H^*)^+ Z_r^N H^* = (H^*)^+ H^*.
\]

(132)

In other words, for \( p = 0, 1, 2, \ldots \) we obtain:

\[
A = (FY^*)^{NP} A(XH^*)^{NP} = FF^+ A(H^*)^+ H^*.
\]

(133)

Therefore, any matrix factorization:

\[
A = FGH^*
\]

(134)

becomes periodic (with period \( N \)), when:

\[
G = Z_c F^+ A(H^*)^+ Z_r \quad \text{and} \quad Z_c^N = Z_r^N = I_k.
\]

(135)

This is one of the potential spots to investigate the internal structure of new factorizations\(^3\). Meta-factorization provides hints on the structure and offers the potential of creating new ones.

**References**

[1] Adi Ben-Israel and Thomas N.E. Greville. *Generalized inverses: theory and applications*. Springer Science & Business Media, 2003.

[2] Tony F. Chan. Rank-revealing QR factorizations. *Linear algebra and its applications*, 88:67–82, 1987.

[3] Moody T Chu, Robert E Funderlic, and Gene H Golub. A rank–one reduction formula and its applications to matrix factorizations. *SIAM review*, 37(4):512–530, 1995.

[4] Randall E Cline and Robert E Funderlic. The rank of a difference of matrices and associated generalized inverses. *Linear Algebra and its Applications*, 24:185–215, 1979.

\(^3\)For more details and numerical demonstration see: [https://github.com/mkarpowi/mft](https://github.com/mkarpowi/mft)
[5] James W. Demmel. *Applied Numerical Linear Algebra*. SIAM, 1997.

[6] Alan Edelman and Sungwoo Jeong. Fifty three matrix factorizations: A systematic approach. *arXiv:2104.08669*, 2021.

[7] Eugen Egerváry. On rank-diminishing operations and their applications to the solution of linear equations. *Zeitschrift für angewandte Mathematik und Physik ZAMP*, 11(5):376–386, 1960.

[8] Gene H Golub and Charles F Van Loan. *Matrix Computations*. JHU Press, 2013.

[9] Ming Gu and Stanley C Eisenstat. Efficient algorithms for computing a strong rank-revealing QR factorization. *SIAM Journal on Scientific Computing*, 17(4):848–869, 1996.

[10] Nathan Halko, Per-Gunnar Martinsson, and Joel A Tropp. Finding structure with randomness: Probabilistic algorithms for constructing approximate matrix decompositions. *SIAM Review*, 53(2):217–288, 2011.

[11] Keaton Hamm and Longxiu Huang. Perspectives on CUR decompositions. *Applied and Computational Harmonic Analysis*, 48(3):1088–1099, 2020.

[12] Alston Scott Householder. *The Theory of Matrices in Numerical Analysis*. Blaisdell, 1964.

[13] M James. The generalised inverse. *The Mathematical Gazette*, 62(420):109–114, 1978.

[14] Ravindran Kannan and Santosh Vempala. Randomized algorithms in numerical linear algebra. *Acta Numerica*, 26:95, 2017.

[15] Carl E. Langenhop. On generalized inverses of matrices. *SIAM Journal on Applied Mathematics*, 15(5):1239–1246, 1967.

[16] Michael W Mahoney. Lecture notes on randomized linear algebra. *arXiv:1608.04481*, 2016.

[17] Michael W Mahoney and Petros Drineas. CUR matrix decompositions for improved data analysis. *Proceedings of the National Academy of Sciences*, 106(3):697–702, 2009.

[18] Per-Gunnar Martinsson. Randomized methods for matrix computations. *The Mathematics of Data*, 25:187–231, 2019.

[19] Per-Gunnar Martinsson, Gregorio Quintana-Orti, and Nathan Heavner. randUTV: A blocked randomized algorithm for computing a rank-revealing UTV factorization. *ACM Transactions on Mathematical Software*, 45(1):1–26, 2019.

[20] Per-Gunnar Martinsson and Joel Tropp. Randomized numerical linear algebra: Foundations & algorithms. *arXiv:2002.01387*, 2020.

[21] George Matsaglia and George PH Styan. Equalities and inequalities for ranks of matrices. *Linear and multilinear Algebra*, 2(3):269–292, 1974.

[22] Yuji Nakatsukasa. Fast and stable randomized low-rank matrix approximation. *arXiv:2009.11392*, 2020.

[23] Yuji Nakatsukasa and Joel A. Tropp. Fast & accurate randomized algorithms for linear systems and eigenvalue problems. *arXiv:2111.00113*, 2021.

[24] Roger Penrose. A generalized inverse for matrices. In *Mathematical proceedings of the Cambridge Philosophical Society*, volume 51, pages 406–413. Cambridge University Press, 1955.
[25] Roger Penrose. On best approximate solutions of linear matrix equations. In *Mathematical Proceedings of the Cambridge Philosophical Society*, volume 52, pages 17–19. Cambridge University Press, 1956.

[26] Gil Shabat, Yaniv Shmueli, Yariv Aizenbud, and Amir Averbuch. Randomized LU decomposition. *Applied and Computational Harmonic Analysis*, 44(2):246–272, 2018.

[27] Danny C. Sorensen and Mark Embree. A DEIM induced CUR factorization. *SIAM Journal on Scientific Computing*, 38(3):A1454–A1482, 2016.

[28] G. W. Stewart. *Matrix Algorithms: Basic Decompositions*. SIAM, 1998.

[29] Gilbert Strang. *Computational Science and Engineering*. Wellesley-Cambridge Press, Wellesley, MA, 2007.

[30] Gilbert Strang. *Linear Algebra and Learning from Data*. Wellesley-Cambridge Press Cambridge, 2019.

[31] Gilbert Strang and Daniel Drucker. Three matrix factorizations from the steps of elimination. *In preparation*, 2021.

[32] Alex Townsend and Lloyd N. Trefethen. Continuous analogues of matrix factorizations. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences*, 471(2173), 2015.

[33] Lloyd N. Trefethen and David Bau III. *Numerical Linear Algebra*. SIAM, 1997.

[34] Shusen Wang and Zhihua Zhang. Improving CUR matrix decomposition and the Nyström approximation via adaptive sampling. *The Journal of Machine Learning Research*, 14(1):2729–2769, 2013.

[35] Joseph Henry Maclagan Wedderburn. *Lectures on matrices*, volume 17. American Mathematical Soc., 1934.