An analogue of the Heisenberg uncertainty relation in prequantum classical field theory

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Abstract

Prequantum classical statistical field theory (PCSFT) is a model that provides the possibility of representing averages of quantum observables, including correlations of observables on subsystems of a composite system, as averages with respect to fluctuations of classical random fields. PCSFT is a classical model of wave type. For example, ‘electron’ is described by electronic field. In contrast to quantum mechanics (QM), this field is a real physical field and not a field of probabilities. An important point is that the prequantum field of, for example, an electron contains the irreducible contribution of the background field vacuum fluctuations. In principle, the traditional QM-formalism can be considered as a special regularization procedure: subtraction of averages with respect to vacuum fluctuations. In this paper, we derive a classical analogue of the Heisenberg–Robertson inequality for dispersions of functionals of classical (prequantum) fields. The PCSFT Robertson-like inequality provides a restriction on the product of classical dispersions. However, this restriction is not so rigid as in QM.

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1. Introduction

At the very beginning of the history of quantum mechanics (QM), the idea that QM is simply a special model of wave mechanics was quite popular. It was supported by the discovery of the ‘quantum wave equation’ by Schrödinger and by the association, with each particle, of its wavelength, the De Broglie wavelength. However, in spite of increased success in ‘technical applications’¹, quantum wave mechanics was ideologically inconsistent. Both Schrödinger and De Broglie should have given up in face of the difficulties in the interpretation of ‘quantum waves’ as real physical waves. Although both Schrödinger and De Broglie did not like the Copenhagen interpretation, they were not able to present a consistent ‘physical waves approach’ to QM.² Finally, the interpretation of quantum waves as waves of probability (proposed by Born) became commonly accepted.

One of the main problems of the ‘physical wave interpretation’ was the impossibility of describing a composite system by waves defined on the physical space, X = \(R^3\). The wave function of a composite system is defined on the space \(X_m = R^{3m}\), where \(m\) is the number of subsystems. Pauli wrote that one may consider physical waves, but they will be defined on unphysical space. In particular, this problem was the main reason why Schrödinger accepted the probabilistic interpretation of the wave function: see [1–5] for recent debates.

Nevertheless, after 80 years of dominance of the Copenhagen interpretation, nowadays various wave models are very popular in attempts to go beyond QM. At the present time, the most successful models are stochastic electrodynamics (SED), see, e.g., [6–11], and semiclassical theory, see e.g., [12–14]. Some ideas of SED are quite similar to the ideas that will be discussed in this paper.

¹ Concrete spectra were found by using Schrödinger’s representation.
² In particular, De Broglie’s double solution model was not so attractive even for its creator. He was happy with Bohmian mechanics, but the latter has its own difficulties, e.g. nonlocality.
A crucial point of SED is the assumption of the presence of the so-called background field (zero point field, field of quantum fluctuations). This idea is very physical. Everybody will agree that the ‘totally empty space’ is only a mathematical idealization. With SED there are no ‘free quantum particles’. It is impossible to isolate, e.g., an electron from the zero point field. The dynamics of quantum systems is a motion in the sea of quantum fluctuations. The latter produce new dynamical and statistical effects known as quantum effects. SED these effects are purely classical. The mystery of these effects is in the use of incomplete information, cf Einstein [15–17], ignoring the zero point field, cf SED.

Before going to my own wave model, I would like to present motivations to go beyond QM, to create prequantum models emerging from QM. One of the standard questions after my talks is ‘Why put effort into such an activity? QM works very well!’

The main reason, see Einstein [15–17], is the creation of a finer description of micro-processes than given by the wave function. Such a description is based on new parameters (‘hidden variables’) providing a possibility for the monitoring of an individual quantum system. We call this project ‘the great Einstein dream’. At the very beginning of the quantum epoch, Einstein dreamed of a kind of Hamiltonian dynamics for quantum particles. It seems that he never gave up and dreamed of a deterministic prequantum model until the last days of his life. However, in the 1930s he concentrated his efforts on showing that quantum randomness is reducible to classical randomness, i.e. quantum statistics can be reproduced in the classical probabilistic framework; see his correspondence with Schrödinger [18].

We know that even in the classical world, deterministic dynamics is not so common. The theory of stochastic processes (including stochastic differential equations) is widely used in classical physics and other domains of science. Personally I believe that the great Einstein’s dream was wrong. However, Einstein’s dream of reduced quantum randomness, i.e. the possibility of describing behavior of quantum systems by classical stochastics, seems to be true.

Although it is too early to predict experimental consequences of the realization of ‘the reduced Einstein dream’, ideological consequences are evident. We would create a harmonic and unified picture of physical reality: QM and classical statistical mechanics would be described by the same probabilistic model.

My model, prequantum classical statistical field theory (PCSFT) [19–26], is a realization of ‘the reduced Einstein dream’, cf SED and semiclassical model, cf Nelson’s stochastic mechanics [27] and its generalization by Davidson [28, 29], cf also the tomographic approach of Man’ko et al. [30–34] reproducing all quantum statistical predictions by operating with classical probability.

To simplify mathematical presentation, we assume that all operators under consideration are bounded and, moreover, that they are of the trace class (the so-called nuclear operators as per the terminology of functional analysis).

2. Correspondence between the terminologies of QM and PCSFT

In [19–26], it was proposed to describe an ensemble of ‘quantum particles’ prepared in a state (may be mixed) given (in QM-formalism) by the density operator \( \rho \) by a random classical field, whose covariance operator coincides with \( \rho \) (and mean value equals zero). It seems that Gaussian fields provide the best matching with QM. However, it became evident only in the process of generalization of PCSFT to composite systems [35]. For a quantum observable given by a symmetric linear operator \( \hat{A} \), we introduce the corresponding classical variable

\[ f_{\hat{A}}(\phi) = \langle \hat{A}\phi, \phi \rangle, \]

the quadratic functional of classical fields. In the real physical model, the argument \( \phi \) varies in a complex Hilbert space \( H = L_2(\mathbb{R}^d) \) of square-summable (complex valued) functions. In the mathematical formalism, we proceed with an arbitrary complex Hilbert space \( H \). To escape mathematical difficulties related to the theory of Gaussian measures on infinite-dimensional spaces, the reader can confine considerations to the case of finite-dimensional spaces, i.e. \( H = \mathbb{C}^n \), where \( \mathbb{C} \) is the field of complex numbers.

The basic mathematical formula [26] coupling QM-average and PCSFT-average is

\[ \int_H f_{\hat{A}}(\phi) d\mu_\rho(\phi) = \text{Tr}_{\rho} \hat{A} = \langle \hat{A} \rangle_{\rho}, \]

where \( \mu_\rho \) is a probability measure with the covariance operator \( \rho \). By using the language of probability theory, we can write this equality as

\[ E_{\mu_\rho} f_{\hat{A}} = \langle \hat{A} \rangle_{\rho}, \]

where \( E_{\mu_\rho} \) is the classical probabilistic expectation. Although in our previous papers [19–26] this formula was proved only for symmetric operators, it is easy to check that it is also valid for an arbitrary (bounded) linear operator. We will use this mathematical fact at the very end of the paper.

The next natural question is on coupling of the PCFT- and QM-dispersions and, hence, on an analogue of Heisenberg’s uncertainty relation in PCSFT. ‘Old PCSFT’ [19–26] did not provide a reasonable coupling between the classical and quantum dispersions. Heisenberg-like inequalities were not found, and the role of noncommutativity was unclear.

We clarify this problem. Let \( \psi \in H, ||\psi|| = 1 \), be a pure quantum state. In QM, it is represented by the density operator

\[ \rho_\psi = \psi \psi^\dagger. \]

Consider a Gaussian random variable \( \phi(\omega) \) valued in \( H \) and having the covariance operator \( \rho_\psi \). It can be represented as \( \phi(\omega) = \xi(\omega)\psi \), where \( \xi \) takes its values in the field of complex numbers \( \mathbb{C} \), it is a scalar random variable, and it has zero mean value and dispersion 1. Then

\[ E(u, \phi(\omega))(\phi(\omega), v) = \langle \rho_\psi u \mid v \rangle. \]

Let operator \( \hat{A} \) be self-adjoint and bounded. To simplify considerations, assume that its QM-average \( \langle \hat{A} \rangle_\psi = 0 \). By (2)
Let operator \( E_{\mu \nu} f_A = 0 \). Thus \( \sigma_{\mu \nu}^2 (f_A) = E_{\mu \nu} f_A^2 \). Hence, the PCSFT-dispersion has no coupling with the QM-dispersion \( \sigma_{\mu \nu}^2 (\hat{A}) = (\hat{A})_\psi^2 \) (we recall that it was assumed that \( (\hat{A})_\psi = 0 \)).

Hence, the ‘old PCSFT’ [19–26] provides matching of averages, but not dispersions. It was a problem. Recently, PCSFT was successfully generalized to the composite quantum system; see [35]. Surprisingly, one can proceed without the tensor product state space. ‘Quantum waves’ for composite systems can be described by the Cartesian product of Hilbert spaces (similar to the classical description of a few particles). One of the main reasons (at least for Schrödinger and Pauli) for supporting the probabilistic interpretation of particles was perturbation by the unit operator \( I \). The latter is the covariance operator of white noise. For a pure state \( \psi \), we set

\[
D_{\rho \psi} = D_\psi.
\]

From this viewpoint, QM is a special mathematical formalism designed to eliminate the effects of vacuum fluctuations of white noise type. It is a natural formalism to describe observations performed on the random background. The contribution of this background should be subtracted. We have for trace class operator \( \hat{A} \)

\[
E_{\mu \nu} f_A = \text{Tr } \rho \hat{A} + \text{Tr } \hat{A},
\]

i.e.,

\[
(\hat{A})_\rho = E_{\mu \nu} f_A - \text{Tr } \hat{A}.
\]

or at least formally (there are some mathematical difficulties in the case of the infinite-dimensional Hilbert space),

\[
(\hat{A})_\rho = E_{\mu \nu} f_A - E_{\mu \nu} f_A.
\]

The QM formalism can be interpreted as a rather special regularization procedure. If \( \text{Tr } \hat{A} = \infty \), then the PCSFT average (with respect to \( \mu ) \) is not defined: the Gaussian integral diverges. Of course, this effect is a consequence of the infinite dimension of the state space. However, the QM formalism provides its regularization.

Coupling (5) between QM and PCSFT averages is not so straightforward as (2). However, as we will see, equality (5) will provide coupling between the QM and PCSFT dispersions.

We remark that correspondence (4) appeared originally due to purely mathematical reasons. To construct a positively defined operator, on the Cartesian product of state spaces of subsystems of a composite system [35], one should modify (3) to (4) even for each subsystem. However, this modification has a natural physical interpretation. The background field of white noise type should be taken into account. Its contribution was missed in ‘old PCSFT’ [19–26]. New PCSFT taking into account so to say vacuum fluctuations became even closer to SED. Although in reality this white noise exists, its contribution can be eliminated from all experimental averages, since both ‘quantum systems’ and measurement devices are located in the ‘vacuum thermostat’. The formalism of QM eliminates from all classical (prequantum) averages, since both ‘quantum systems’ and measurement averages, since both ‘quantum systems’ and measurement devices are located in the ‘vacuum thermostat’. It is a good place to come back to the question of the possible consequences of creation of PCSFT. In particular, it will start to play a role when experimental technology will approach such a degree of precision that individual vacuum fluctuations would be visible. At that level it would not be more possible just to subtract averages with respect to these fluctuations from all answers. The boundary of possible application of the formalism of QM will be approached.

3. Coupling of dispersions

We confine our considerations to pure states. Let \( \hat{A} \) be symmetric. Its dispersion in a pure state \( \psi \) is defined by

\[
\sigma_{\psi}^2 (\hat{A}) = (\hat{A} - (\hat{A})_\psi)^2 = (\hat{A})_\psi^2 - (\hat{A})_\psi^2.
\]

Let \( \xi \) be a classical random variable. Its dispersion is given by \( \sigma_{\psi}^2 (\xi) = (P_\psi (\xi) - \text{E} P_\psi)^2 = P_\psi^2 - \text{E} P_\psi^2 \), where \( P \) is a probability measure.

**Lemma 1.** Let operator \( \hat{A} \) be symmetric and let \( \psi \) be a pure state. Then

\[
\text{Tr} (D_\psi \hat{A})^2 = 2 \langle \hat{A}^2 \rangle_\psi + \langle (\hat{A})_\psi^2 \rangle_\psi,
\]

where \( \rho_\psi = \psi \otimes \psi \) and operator \( D_\psi \) is defined by (4).

**Proof.** We have (for an orthonormal basis)

\[
\sum_k (D_\psi \hat{A} \hat{e}_k \mid \hat{A} \hat{D}_\psi \hat{e}_k) = \sum_k \langle (I + \rho_\psi) \hat{A} \hat{e}_k \mid \hat{A} (I + \rho_\psi) \hat{e}_k \rangle
\]

\[
= \sum_k \langle \hat{A} \hat{e}_k \mid \hat{A} \hat{e}_k \rangle + \sum_k \langle \hat{A} \rho_\psi \hat{e}_k \mid \hat{A} \rho_\psi \hat{e}_k \rangle
\]

\[
+ \sum_k \langle \rho_\psi \hat{A} \hat{e}_k \mid \hat{A} \rho_\psi \hat{e}_k \rangle
\]

\[
= \text{Tr} \hat{A}^2 + \sum_k \langle \hat{A} \hat{e}_k \mid \hat{e}_k \mid \hat{A} \hat{e}_k \rangle + \sum_k \langle (\hat{A} \hat{e}_k \mid \hat{e}_k \mid \hat{A} \hat{e}_k) \rangle
\]

\[
+ \sum_k \langle (\hat{A} \hat{e}_k \mid \hat{e}_k \mid \hat{A} \hat{e}_k) \rangle
\]

\[
= \text{Tr} \hat{A}^2 + 2 \langle \hat{A}^2 \rangle_\psi + \langle (\hat{A})_\psi^2 \rangle_\psi
\]

\[
= \text{Tr} \hat{A}^2 + 2 \langle \hat{A}^2 \rangle_\psi + \langle (\hat{A})_\psi^2 \rangle_\psi.
\]
We recall once again that each pure quantum state $\psi$ determines the density operator $\rho_\psi$. The latter determines the covariance operator $D_\psi$ by (4) of the Gaussian measure $\mu_{D_\psi}$. To simplify the notation, we will use the symbol $\mu_\psi$ for this measure. Thus $\mu_\psi$ is the prequantum Gaussian distribution corresponding to the pure quantum state $\psi$. It describes the prequantum random field of ‘quantum system coupled to vacuum thermostat’. We remark that, in particular, our activity is translation of the operator language of the traditional quantum formalism to the language of traditional probability theory.\[\square\]

We will use the following result on the Gaussian integral of the product of two quadratic forms on the complex Hilbert space [35]:

**Lemma 2.** Let $\mu$ be a Gaussian measure with the covariance operator $D$ and let $\hat{A}_i$ be self-adjoint operators, where $i = 1, 2$. Then

$$E_{\mu} f_{\hat{A}_1} f_{\hat{A}_2} = \int_H f_{\hat{A}_1}(\phi) f_{\hat{A}_2}(\phi) d\mu(\phi) = \text{Tr} D \hat{A}_1 \text{Tr} D \hat{A}_2 + \text{Tr} D \hat{A}_2 D \hat{A}_1. \tag{7}$$

**Theorem.** Let the conditions of lemma 1 hold. Then

$$E_{\mu_\psi} f_{\hat{A}}^2 = (E_{\mu_\psi} f_{\hat{A}})^2 + \text{Tr} \hat{A}^2 + 2(\hat{A}^2)_{\psi} + (\langle \hat{A} \rangle_{\psi})^2. \tag{8}$$

By using (8):

$$\sigma_{\mu_\psi}^2(f_{\hat{A}}) = \text{Tr} \hat{A}^2 + 2\sigma_{\psi}^2(\hat{A}) + 3(\langle A \rangle_{\psi})^2. \tag{9}$$

We remark that the classical dispersion $\sigma_{\mu_\psi}^2(f_{\hat{A}})$ is always larger than the quantum dispersion $\sigma_{\psi}^2(\hat{A})$—since in the last one, we ignore the dispersion of vacuum fluctuations. Moreover, the classical dispersion is larger than the dispersion produced by vacuum fluctuations, which is given by $\text{Tr} \hat{A}^2$. Really, we have

$$E_{\mu_\psi} f_{\hat{A}}^2 = (\text{Tr} \hat{A}^2 + \text{Tr} \hat{A}^2).$$

By taking into account that $E_{\mu_\psi} f_{\hat{A}} = \text{Tr} \hat{A}$, we obtain

$$\sigma_{\mu_\psi}^2(f_{\hat{A}}) = \text{Tr} \hat{A}^2.$$

Let now $\langle \hat{A} \rangle_{\psi} = 0$. Then

$$\sigma_{\psi}^2(\hat{A}) = \frac{1}{2}[\sigma_{\mu_\psi}^2(f_{\hat{A}}) - \text{Tr} \hat{A}^2]$$

\[\equiv \frac{1}{2}[\sigma_{\mu_\psi}^2(f_{\hat{A}}) - \sigma_{\mu_\psi}^2(f_{\hat{A}})] = \Gamma_{\mu_\psi}(f_{\hat{A}}).\]

Thus the QM dispersion can be obtained as regularization, shift by $\text{Tr} \hat{A}^2$, of the classical dispersion (as to the factor $1/2$)—by ignoring the contribution of vacuum fluctuations. We remark that the equality of the quantum dispersion to zero is nothing other than the reduction of the classical dispersion to the dispersion of vacuum fluctuations—the dispersion of the irreducible background noise.

Consider two quantum observables, symmetric operators $\hat{A}_1$ and $\hat{A}_2$; in general, noncommutative: $[\hat{A}_1, \hat{A}_2] \neq 0$. Take a pure quantum state $\psi$. We can always assume that these observables have zero averages in this state:

$$\langle \hat{A}_i \rangle_{\psi} = \langle \hat{A}_i | \psi \rangle = 0, \quad i = 1, 2. \tag{10}$$

If $\langle \hat{A}_i \rangle_{\psi} \neq 0$, we just consider shifted observables:

$$\hat{C}_i = \hat{A}_i - \langle \hat{A}_i \rangle_{\psi} I. \tag{11}$$

Since the QM dispersions satisfy the Schrödinger–Robertson inequality:\[3\]

$$\sigma_{\psi}^2(\hat{A}_1) + \sigma_{\psi}^2(\hat{A}_2) \geq \frac{1}{4} [\langle [\hat{A}_1, \hat{A}_2] \rangle_{\psi}]^2, \tag{12}$$

the dispersions of the corresponding PCSFT variables shifted by the background white noise satisfy its classical counterpart:

$$\Gamma_{\mu_\psi}(f_{\hat{A}_1}) + \Gamma_{\mu_\psi}(f_{\hat{A}_2}) \geq \frac{1}{4} [\langle [\hat{A}_1, \hat{A}_2] \rangle_{\psi}]^2. \tag{13}$$

We now represent even the right-hand side as a classical (PCSFT) average. Set

$$\hat{K} = [\hat{A}_1, \hat{A}_2].$$

It is a skew-symmetric operator. By applying (1) and taking into account the remark that this formula is valid not only for quantum observables, but even for arbitrary linear operators, we obtain

$$E_{\mu_\psi} f_K = \text{Tr} \rho_\psi \hat{K} + \text{Tr} \hat{K}.$$

Thus

$$\langle \hat{K} \rangle_{\psi} = E_{\mu_\psi} f_K - \text{Tr} \hat{K}.$$

Inequality (13) can be written in purely classical terms:

$$\Gamma_{\mu_\psi}(f_{\hat{A}_1}) \Gamma_{\mu_\psi}(f_{\hat{A}_2}) \geq \frac{1}{4} [E_{\mu_\psi} f_K - \text{Tr} \hat{K}]^2. \tag{14}$$

Thus noncommutativity has its trace even in PCSFT. In contrast to QM, the Schrödinger–Robertson PCSFT-inequality does not have drastic consequences. The main difference is the presence of the shift of the dispersion. This shift that is, by the way, produced by the contribution of vacuum fluctuations makes the restriction on the product of the dispersions not so rigid as it is in QM. However, we note that even the conventional Schrödinger–Robertson inequality has a statistical interpretation, which has nothing to do with incompatibility of the observables (in the sense of impossibility of joint measurement). This statistical interpretation is due to Margenau and Ballentine; see [37–39].

4. Conclusion

The prequantum field theory contains a counterpart of the Heisenberg uncertainty relation—the Schrödinger–Robertson-like inequality (14). This classical analogue of the Schrödinger–Robertson inequality provides an estimate from below not for the dispersions but for the shifted (by contribution of vacuum fluctuations) dispersions. Such an estimate is not so rigid as the one provided by the Schrödinger–Robertson inequality for the quantum dispersions. It may be that paradoxical consequences of the Heisenberg’s uncertainty principle were induced by neglecting the mentioned shift.\[3\] For position and momentum, it was found by Schrödinger and Robertson; see the full story in [36].
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