MORSE INEQUALITIES FOR FOURIER COMPONENTS OF KOHN-ROSSI COHOMOLOGY OF CR COVERING MANIFOLDS WITH $S^1$-ACTION

RUNG-TZUNG HUANG AND GUOKUAN SHAO

Abstract. Let $X$ be a compact connected CR manifold of dimension $2n + 1, n \geq 1$. Let $\tilde{X}$ be a paracompact CR manifold with a transversal CR $S^1$-action, such that there is a discrete group $\Gamma$ acting freely on $\tilde{X}$ having $X = \tilde{X}/\Gamma$. Based on an asymptotic formula for the Fourier components of the heat kernel with respect to the $S^1$-action, we establish the Morse inequalities for Fourier components of reduced $L^2$-Kohn-Rossi cohomology with values in a rigid CR vector bundle over $X$. As a corollary, we obtain the Morse inequalities for Fourier components of Kohn-Rossi cohomology on $X$ which were obtained by Hsiao-Li [15] by using Szegő kernel method.

1. Introduction and statement of the results

Gromov-Henkin-Shubin [10, Theorem 0.2] considered covering manifolds that are strongly pseudoconvex of complex manifolds and analyzed the holomorphic $L^2$-functions on the coverings. Todor-Chiose-Marinescu [20] generalized in a similar manner the Morse inequalities of Siu-Demailly [18, 7] on coverings of complex manifolds. The study of problems on CR manifolds with $S^1$-action becomes active recently, see [5, 11, 12, 14, 15] and the references therein. In particular, Hsiao-Li [15] established the Morse inequalities for Fourier components of Kohn-Rossi cohomology on $X$ by using the Szegő kernel method. Inspired by the results of [10, 15, 20, 18, 7], we establish Morse inequalities for Fourier components of reduced $L^2$-Kohn-Rossi cohomology with values in a rigid CR vector bundle on a covering manifold over a compact connected CR manifold with $S^1$-action. This generalizes the results of [15] to CR covering manifolds with $S^1$-action. We present a proof by the heat kernel method, which is inspired by Bismut’s proof [3, 16] of the holomorphic Morse inequalities. The crucial estimate for Fourier components of the heat kernel of Kohn Laplacians was given in [12].

Now we formulate the main results. We refer to other sections for notations and definitions (see Definition 2.1, 2.2, 2.3, 2.5 and (3.1), (3.25) used here). Let $X$ be a compact connected CR manifold of dimension $2n + 1, n \geq 1$ with a transversal CR $S^1$-action $e^{i\theta}$ on $X$. For $x \in X$, we say that the period of $x$ is $2\pi \ell$, $\ell \in \mathbb{N}$, if $e^{i\theta} \circ x \neq x$, for every $0 < \theta < \frac{2\pi}{\ell}$, and $e^{i\frac{2\pi}{\ell}} \circ x = x$. For each $\ell \in \mathbb{N}$, put

\begin{equation}
X_\ell = \{ x \in X; \text{ the period of } x \text{ is } \frac{2\pi}{\ell} \}
\end{equation}

\begin{equation}
(1.1)
\end{equation}

and let

\begin{equation}
p = \min \{ \ell \in \mathbb{N}; X_\ell \neq \emptyset \}.
\end{equation}

\begin{equation}
(1.2)
\end{equation}

It is well-known that if $X$ is connected, then $X_p$ is an open and dense subset of $X$ (see Duistermaat-Heckman [9]). Assume $X = X_{p_1} \cup X_{p_2} \cup \cdots \cup X_{p_k}$, $p =: p_1 < p_2 < \cdots < p_k$. Set

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$X_{\text{reg}} := X_p$. We call $x \in X_{\text{reg}}$ a regular point of the $S^1$ action. Let $X_{\text{sing}}$ be the complement of $X_{\text{reg}}$.

Let $\tilde{X}$ be a paracompact CR manifold, such that there is a discrete group $\Gamma$ acting freely on $\tilde{X}$ having $X = \tilde{X}/\Gamma$. Let $\pi : \tilde{X} \to X$ be the natural projection with the pull-back map $\pi^* : TX \to T\tilde{X}$. Then $\tilde{X}$ admits a pull-back CR structure $T^{1,0}\tilde{X} := \pi^*T^{1,0}X$ and, hence, a CR manifold. We assume that $\tilde{X}$ admits a transversal CR locally free $S^1$ action, denote by $e^{i\theta}$. We further assume that the map

$$\Gamma \times \tilde{X} \to \tilde{X}, \ (\gamma, \tilde{x}) \mapsto \gamma \circ \tilde{x}, \ \ \forall \gamma \in \Gamma, \ \ \forall \tilde{x} \in \tilde{X}.$$ 

is CR, see (2.3), and

$$e^{i\theta} \circ \gamma \circ \tilde{x} = \gamma \circ e^{i\theta} \circ \tilde{x}, \ \ \forall \gamma \in [0, 2\pi], \ \ \forall \tilde{x} \in \tilde{X}.$$ 

Let $\tilde{E} := \pi^*E$ be the pull-back bundle of a rigid CR vector bundle $E$ over $X$. Then $\tilde{E}$ is a $\Gamma$-invariant rigid CR vector bundle over $\tilde{X}$. We denote by $\tilde{X}_{\text{reg}}$ the set of regular points of the $S^1$-action on $\tilde{X}$. Note that since $\Gamma$ acts on $\tilde{X}$ freely so that $\tilde{X}/\Gamma = X$, hence, we have $\tilde{X}_{\text{reg}}/\Gamma = X_{\text{reg}} = X_p$. We denote by $X(q)$ a subset of $X$ such that

$$X(q) := \{x \in X : \mathcal{L}_x \text{ has exactly } q \text{ negative eigenvalues and } n - q \text{ positive eigenvalues}\}.$$ 

We refer to Section 2 for more details. Our main theorem is the following

**Theorem 1.1.** With the above notations and assumptions, as $m \to \infty$, for $q = 0, 1, \cdots, n$, the $m$-th Fourier components of reduced $L^2$-Kohn-Rossi cohomology (see §2.25) satisfy the following strong Morse inequalities

$$\sum_{j=0}^q (-1)^{q-j} \dim_{\Gamma} \overline{H}_{b(2),m}^j(\tilde{X}, \tilde{E}) \leq \frac{prm^n}{2\pi^{n+1}} \sum_{j=0}^q (-1)^{q-j} \int_{X(j)} |\text{det}(\mathcal{L}_x)| \, dv_{X}(x) + o(m^n), \text{ for } p \mid m,$$

$$\sum_{j=0}^q (-1)^{q-j} \dim_{\Gamma} \overline{H}_{b(2),m}^j(\tilde{X}, \tilde{E}) = o(m^n), \text{ for } p \nmid m.$$  

where $r$ denotes the rank of $\tilde{E}$, $\dim_{\Gamma}$ denotes the Von Neumann dimension (see §2.3 in the below, [16] §3.6.1 or [11] §3) and $\mathcal{L}_x$ is the Levi form at $x \in X$. When $p \mid m$, $q = n$, as $m \to \infty$, we have the asymptotic Riemann-Roch-Hirzebruch theorem

$$\sum_{j=0}^n (-1)^j \dim_{\Gamma} \overline{H}_{b(2),m}^j(\tilde{X}, \tilde{E}) = \frac{prm^n}{2\pi^{n+1}} \sum_{j=0}^n (-1)^j \int_{X(j)} |\text{det}(\mathcal{L}_x)| \, dv_{X}(x) + o(m^n).$$ 

In particular, we get the weak Morse inequalities

$$\dim_{\Gamma} \overline{H}_{b(2),m}^0(\tilde{X}, \tilde{E}) \leq \frac{prm^n}{2\pi^{n+1}} \int_{X(q)} |\text{det}(\mathcal{L}_x)| \, dv_{X}(x) + o(m^n).$$ 

By the standard argument in [16] or [18], we deduce easily the following Grauert-Riemenschneider criterion on coverings of CR manifolds.

**Corollary 1.2.** With the above notations and assumptions in Theorem 1.1, we assume also that $X$ is weakly pseudoconvex and strongly pseudoconvex at a point. Then

$$\dim_{\Gamma} \overline{H}_{b(2),m}^0(\tilde{X}, \tilde{E}) \approx m^n, \text{ for } p \mid m.$$ 

In particular, $\dim_{\Gamma} \overline{H}_{b(2)}^0(\tilde{X}, \tilde{E}) = \infty.$
When $\Gamma = \{ e \}, p = 1$ and $E$ is trivial line bundle, we deduce the following Morse inequalities of Hsiao-Li, see [15] Theorem 2.2 and Theorem 2.5.

**Corollary 1.3.** With the above notations and assumptions, as $m \to \infty$, for $q = 0, 1, \ldots, n$, the $m$-th Fourier components of Kohn-Rossi cohomology satisfy the following strong Morse inequalities,

$$
(1.7) \quad \sum_{j=0}^{q} (-1)^{q-j} \dim H_{b,m}^{j}(X) \leq \frac{m^{n}}{2^{n+1}} \sum_{j=0}^{q} (-1)^{q-j} \int_{X(j)} |\det(\mathcal{L}_{x})| \, dv_{X}(x) + o(m^{n}),
$$

where $\mathcal{L}_{x}$ is the Levi form at $x \in X$. In particular, we get the weak Morse inequalities

$$
(1.8) \quad \dim H_{b,m}^{q}(X) \leq \frac{m^{n}}{2^{n+1}} \int_{X(q)} |\det(\mathcal{L}_{x})| \, dv_{X}(x) + o(m^{n}).
$$

Let $X$ be a compact CR manifold of dimension $2n + 1$, $n \geq 1$. A classical theorem due to Boutet de Monvel [4] asserts that $X$ can be globally CR embedded into $\mathbb{C}^{N}$, for some $N \in \mathbb{N}$, when $X$ is strongly pseudoconvex with dimension $n \geq 5$. Epstein [5] proved that if $X$ is strongly pseudoconvex with dimension $3$ and a global free transversal CR $S^{1}$-action, then $X$ can be embedded into $\mathbb{C}^{N}$ by positive Fourier components of CR functions. Corollary 1.3 guarantees the abundance of positive Fourier components of CR functions to do embedding in general cases (e.g. the $S^{1}$-action can be only locally free). In [15], the authors’ proofs include localization of analytic objects (eigenfunctions, Szegö kernels), Kohn $L^{2}$ estimates and scaling techniques. A more general version of Corollary 1.3 (with $X$ being weakly pseudoconvex) is proved by Cheng-Hsiao-Tsai in [5] Proposition 1.20 and Corollary 1.21 in a different way. By using the Morse inequalities, (1.7) and (1.8), Hsiao-Li [15, Theorem 2.6] proved that there are abundant CR functions on $X$ when $X$ is weakly pseudoconvex and strongly pseudoconvex at a point. Corollary 1.2 generalizes Theorem 2.6 of [15] to CR covering manifolds.

This paper is organized as follows. In Section 2 we introduce some basic notations, terminology and definitions. In Section 3 we study the asymptotic behavior of heat kernels of Kohn Laplacians. Section 4 is devoted to the heat kernel proof of the main theorem.

### 2. Preliminaries

#### 2.1. Some standard notations

We use the following notations: $\mathbb{N} = \{1, 2, \ldots \}$, $\mathbb{N}_{0} = \mathbb{N} \cup \{0 \}$, $\mathbb{R}$ is the set of real numbers, $\mathbb{R}_{+} := \{x \in \mathbb{R}; x > 0 \}$, $\mathbb{R}_{+} := \{x \in \mathbb{R}; x \geq 0 \}$. For a multiindex $\alpha = (\alpha_{1}, \ldots, \alpha_{n}) \in \mathbb{N}_{0}^{n}$ we set $|\alpha| = \alpha_{1} + \cdots + \alpha_{n}$. For $x = (x_{1}, \ldots, x_{n})$ we write

$$
x^\alpha = x_{1}^{\alpha_{1}} \cdots x_{n}^{\alpha_{n}}, \quad \partial_{x_{j}} = \frac{\partial}{\partial x_{j}}, \quad \partial_{x}^{\alpha} = \partial_{x_{1}}^{\alpha_{1}} \cdots \partial_{x_{n}}^{\alpha_{n}} = \frac{\partial|\alpha|}{\partial x^\alpha}.
$$

Let $z = (z_{1}, \ldots, z_{n})$, $z_{j} = x_{2j-1} + ix_{2j}$, $j = 1, \ldots, n$, be coordinates of $\mathbb{C}^{n}$. We write

$$
z^\alpha = z_{1}^{\alpha_{1}} \cdots z_{n}^{\alpha_{n}}, \quad \bar{z}^\alpha = \overline{z_{1}^{\alpha_{1}} \cdots z_{n}^{\alpha_{n}}},
$$

$$
\partial_{z_{j}} = \frac{\partial}{\partial z_{j}} = -\frac{1}{2} \left( \frac{\partial}{\partial x_{2j-1}} - i \frac{\partial}{\partial x_{2j}} \right), \quad \partial_{\bar{z}_{j}} = \frac{\partial}{\partial \bar{z}_{j}} = -\frac{1}{2} \left( \frac{\partial}{\partial x_{2j-1}} + i \frac{\partial}{\partial x_{2j}} \right),
$$

$$
\partial_{\bar{z}} = \frac{\partial^{\alpha}}{\partial z^{\alpha}} = \frac{\partial_{z}^{\alpha}}{\partial z^{\alpha}}, \quad \partial_{\bar{z}} = \frac{\partial_{\bar{z}}^{\alpha}}{\partial \bar{z}^{\alpha}} = \frac{\partial_{\bar{z}}^{\alpha}}{\partial \bar{z}^{\alpha}}.
$$

Let $X$ be a $C^\infty$ orientable paracompact manifold. We let $TX$ and $T^{*}X$ denote the tangent bundle of $X$ and the cotangent bundle of $X$, respectively. The complexified tangent bundle of $X$ and the complexified cotangent bundle of $X$ will be denoted by $\mathbb{C}TX$ and $\mathbb{C}^{*}X$, respectively. We write $\langle \cdot, \cdot \rangle$ to denote the pointwise duality between $T^{*}X$ and $TX$. We extend $\langle \cdot, \cdot \rangle$ bilinearly to $\mathbb{C}^{*}X \times \mathbb{C}TX$. For $u \in \mathbb{C}^{*}X$, $v \in \mathbb{C}TX$, we also write $u(v) := \langle u, v \rangle$. 


Let $Y \subset X$ be an open set. The spaces of smooth sections of $E$ over $Y$ and distribution sections of $E$ over $Y$ will be denoted by $C^\infty(Y, E)$ and $D'(Y, E)$, respectively.

2.2. CR manifolds with $S^1$-action. Let $(X, T^{1,0}X)$ be a compact CR manifold of dimension $2n+1$, $n \geq 1$, where $T^{1,0}X$ is a CR structure of $X$. That is, $T^{1,0}X$ is a subbundle of rank $n$ of the complexified tangent bundle $\mathbb{C}T X$, satisfying $T^{1,0}X \cap T^{0,1}X = \{0\}$, where $T^{0,1}X = \overline{T^{1,0}X}$, and $[\mathcal{V}, \mathcal{V}] \subset \mathcal{V}$, where $\mathcal{V} = C^\infty(X, T^{1,0}X)$. We assume that $X$ admits a $S^1$ action: $S^1 \times X \to X$. We write $e^{i\theta}$ to denote the $S^1$ action. Let $T \in C^\infty(X, TX)$ be the global real vector field induced by the $S^1$ action given by $(Tu)(x) = \frac{\partial}{\partial \theta}(u(e^{i\theta} \circ x))|_{\theta = 0}$, $u \in C^\infty(X)$.

**Definition 2.1.** We say that the $S^1$ action $e^{i\theta}$ is CR if $[T, C^\infty(X, T^{1,0}X)] \subset C^\infty(X, T^{1,0}X)$ and the $S^1$ action is transversal if for each $x \in X$, $\mathcal{C}T(x) \oplus T^{1,0}_x X \oplus T^{0,1}_x X = \mathcal{C}T_x X$. Moreover, we say that the $S^1$ action is locally free if $T \neq 0$ everywhere.

Note that if the $S^1$ action is transversal, then it is locally free. We assume throughout that $(X, T^{1,0}X)$ is a connected CR manifold with a transversal CR $S^1$ action $e^{i\theta}$ and we let $T$ be the global vector field induced by the $S^1$ action. Let $\omega_0 \in C^\infty(X, T^*X)$ be the global real one form determined by $\langle \omega_0, u \rangle = 0$, for every $u \in T^{1,0}X \oplus T^{0,1}X$ and $\langle \omega_0, T \rangle = -1$.

**Definition 2.2.** For $p \in X$, the Levi form $\mathcal{L}_p$ is the Hermitian quadratic form on $T^{1,0}_p X$ given by $\mathcal{L}_p(U, \overline{V}) = -\frac{i}{2}(\partial \omega_0(p), U \wedge \overline{V})$, $U, V \in T^{1,0}_p X$.

**Definition 2.3.** If the Levi form $\mathcal{L}_p$ is positive definite, we say that $X$ is strongly pseudoconvex at $p$. If the Levi form is positive definite at every point of $X$, we say that $X$ is strongly pseudoconvex.

Denote by $T^{*1,0}X$ and $T^{*0,1}X$ the dual bundles of $T^{1,0}X$ and $T^{0,1}X$, respectively. Define the vector bundle of $(0, q)$ forms by $T^{*0,q}X = \Lambda^q(T^{*1,0}X)$. Put $T^{*0,q}X := \bigoplus_{j \in \{0, 1, \ldots, n\}} T^{*0,j}X$. Let $D \subset X$ be an open subset. Let $\Omega^{0,q}(D)$ denote the space of smooth sections of $T^{*0,q}X$ over $D$ and let $\Omega^{0,q}_0(D)$ be the subspace of $\Omega^{0,q}(D)$ whose elements have compact support in $D$. Put

$$\Omega^{0,*}(D) := \bigoplus_{j \in \{0, 1, \ldots, n\}} \Omega^{0,j}(D),$$

$$\Omega^{0,*}_0(D) := \bigoplus_{j \in \{0, 1, \ldots, n\}} \Omega^{0,j}_0(D).$$

Similarly, if $E$ is a vector bundle over $D$, then we let $\Omega^{0,q}(D, E)$ denote the space of smooth sections of $T^{*0,q}X \otimes E$ over $D$ and let $\Omega^{0,q}_0(D, E)$ be the subspace of $\Omega^{0,q}(D, E)$ whose elements have compact support in $D$. Put

$$\Omega^{0,*}(D, E) := \bigoplus_{j \in \{0, 1, \ldots, n\}} \Omega^{0,j}(D, E),$$

$$\Omega^{0,*}_0(D, E) := \bigoplus_{j \in \{0, 1, \ldots, n\}} \Omega^{0,j}_0(D, E).$$

Fix $\theta_0 \in [-\pi, \pi]$, $\theta_0$ small. Let $de^{i\theta_0} : \mathcal{C}T_x X \to \mathcal{C}T_{e^{i\theta_0}x} X$ denote the differential map of $e^{i\theta_0} : X \to X$. By the CR property of the $S^1$ action, we can check that

$$de^{i\theta_0} : T^{1,0}_x X \to T^{1,0}_{e^{i\theta_0}x} X,$$

$$de^{i\theta_0} : T^{0,1}_x X \to T^{0,1}_{e^{i\theta_0}x} X,$$

$$de^{i\theta_0}(T(x)) = T(e^{i\theta_0}x).$$

Let $(e^{i\theta_0})^* : \Lambda^j(\mathcal{C}T^*X) \to \Lambda^j(\mathcal{C}T^*X)$ be the pull-back map by $e^{i\theta_0}$, $j = 0, 1, \ldots, 2n + 1$. From (2.1), it is easy to see that for every $q = 0, 1, \ldots, n$,

$$\left(e^{i\theta_0}\right)^* : T^{*0,q}_{e^{i\theta_0}x} X \to T^{*0,q}_x X.$$
Let $u \in \Omega^{0,q}(X)$. Define

$$Tu := \frac{\partial}{\partial \theta}((e^{i\theta})^*u)|_{\theta=0} \in \Omega^{0,q}(X).$$

For every $\theta \in \mathbb{R}$ and every $u \in C^\infty(X, \Lambda^j(\mathbb{C}T^*X))$, we write $u(e^{i\theta} \circ x) := (e^{i\theta})^*u(x)$.

Let $\overline{\partial}_b : \Omega^{0,q}(X) \rightarrow \Omega^{0,q+1}(X)$ be the tangential Cauchy-Riemann operator. From the CR property of the $S^1$ action, it is straightforward to see that

$$T\overline{\partial}_b = \overline{\partial}_b T \quad \text{on} \quad \Omega^{0,\cdot}(X).$$

**Definition 2.4.** Let $D \subset U$ be an open set. We say that a function $u \in C^\infty(D)$ is rigid if $Tu = 0$. We say that a function $u \in C^\infty(X)$ is Cauchy-Riemann (CR for short) if $\overline{\partial}_b u = 0$. We call $u$ a rigid CR function if $\overline{\partial}_b u = 0$ and $Tu = 0$.

**Definition 2.5.** Let $F$ be a complex vector bundle over $X$. We say that $F$ is rigid (CR) if $X$ can be covered with open sets $U_j$ with trivializing frames $\{f^1_j, f^2_j, \ldots, f^n_j\}$, $j = 1, 2, \ldots$, such that the corresponding transition matrices are rigid (CR). The frames $\{f^1_j, f^2_j, \ldots, f^n_j\}$, $j = 1, 2, \ldots$, are called rigid (CR) frames.

**Definition 2.6.** Let $F$ be a complex rigid vector bundle over $X$ and let $\langle \cdot | \cdot \rangle_F$ be a Hermitian metric on $F$. We say that $\langle \cdot | \cdot \rangle_F$ is a rigid Hermitian metric if for every rigid local frames $f_1, \ldots, f_r$ of $F$, we have $T\langle f_j | f_k \rangle_F = 0$, for every $j, k = 1, 2, \ldots, r$.

It is known that there is a rigid Hermitian metric on any rigid vector bundle $F$ (see Theorem 2.10 in [3] and Theorem 10.5 in [11]). Note that Baouendi-Rothschild-Treves [2] proved that $T^{1,0}X$ is a rigid complex vector bundle over $X$.

From now on, let $E$ be a rigid CR vector bundle over $X$ and we take a rigid Hermitian metric $\langle \cdot | \cdot \rangle_E$ on $E$ and take a rigid Hermitian metric $\langle \cdot | \cdot \rangle$ on $\mathbb{C}T^*X$ such that $T^{1,0}X \perp T^{0,1}X$, $T \perp (T^{2,0}X \oplus T^{0,1}X)$, $\langle T | T \rangle = 1$. The Hermitian metrics on $\mathbb{C}T^*X$ and on $E$ induce Hermitian metrics $\langle \cdot | \cdot \rangle$ and $\langle \cdot | \cdot \rangle_E$ on $T^{0,\cdot}X$ and $T^{0,\cdot}X \otimes E$, respectively. We denote by $d\nu_E = d\nu_X(x)$ the volume form on $X$ induced by the fixed Hermitian metric $\langle \cdot | \cdot \rangle$ on $\mathbb{C}T^*X$. Then we get natural global $L^2$ inner products $\langle \cdot | \cdot \rangle_E$, $\langle \cdot | \cdot \rangle$ on $\Omega^{0,\cdot}(X, E)$ and $\Omega^{0,\cdot}(X)$, respectively. We denote by $L^2(X, T^{0,\cdot}X \otimes E)$ and $L^2(X, T^{0,\cdot}X)$ the completions of $\Omega^{0,\cdot}(X, E)$ and $\Omega^{0,\cdot}(X)$ with respect to $\langle \cdot | \cdot \rangle_E$ and $\langle \cdot | \cdot \rangle$, respectively. Similarly, we denote by $L^2(X, T^{0,\cdot}X \otimes E)$ and $L^2(X, T^{0,\cdot}X)$ the completions of $\Omega^{0,\cdot}(X, E)$ and $\Omega^{0,\cdot}(X)$ with respect to $\langle \cdot | \cdot \rangle_E$ and $\langle \cdot | \cdot \rangle$, respectively. We extend $\langle \cdot | \cdot \rangle_E$ and $\langle \cdot | \cdot \rangle$ to $L^2(X, T^{0,\cdot}X \otimes E)$ and $L^2(X, T^{0,\cdot}X)$ in the standard way, respectively. For $f \in L^2(X, T^{0,\cdot}X \otimes E)$, we denote $\|f\|_E^2 := \langle f \mid f \rangle_E$. Similarly, for $f \in L^2(X, T^{0,\cdot}X)$, we denote $\|f\|_E^2 := (f \mid f)$.

We also write $\overline{\partial}_b$ to denote the tangential Cauchy-Riemann operator acting on forms with values in $E$:

$$\overline{\partial}_b : \Omega^{0,\cdot}(X, E) \rightarrow \Omega^{0,\cdot}(X, E).$$

Since $E$ is rigid, we can also define $Tu$ for every $u \in \Omega^{0,q}(X, E)$ and we have

$$T\overline{\partial}_b = \overline{\partial}_b T \quad \text{on} \quad \Omega^{0,\cdot}(X, E).$$

For every $m \in \mathbb{Z}$, let

$$\Omega^{0,q}_m(X, E) := \{u \in \Omega^{0,q}(X, E); Tu = imu\}, \quad q = 0, 1, 2, \ldots, n,$$

$$\Omega^{0,\cdot}_m(X, E) := \{u \in \Omega^{0,\cdot}(X, E); Tu = imu\}.$$  

For each $m \in \mathbb{Z}$, we denote by $L^2_m(X, T^{0,q}X \otimes E)$ and $L^2_m(X, T^{0,q}X)$ the completions of $\Omega^{0,q}_m(X, E)$ and $\Omega^{0,\cdot}_m(X)$ with respect to $\langle \cdot | \cdot \rangle_E$ and $\langle \cdot | \cdot \rangle$, respectively. Similarly, we denote by $L^2_m(X, T^{0,\cdot}X \otimes E)$ and $L^2_m(X, T^{0,\cdot}X)$ the completions of $\Omega^{0,\cdot}_m(X, E)$ and $\Omega^{0,\cdot}_m(X)$ with respect to $\langle \cdot | \cdot \rangle_E$ and $\langle \cdot | \cdot \rangle$, respectively.
2.3. Covering manifolds, Von Neumann dimension. Let $(X, T^{1,0}X)$ be a compact CR manifold of dimension $2n + 1$, $n \geq 1$. Let $\tilde{X}$ be a paracompact CR manifold, such that there is a discrete group $\Gamma$ acting freely on $\tilde{X}$ having $X = \tilde{X}/\Gamma$. Let $\pi : \tilde{X} \to X$ be the natural projection with the pull-back map $\pi^* : TX \to \tilde{T}X$. Then $\tilde{X}$ admits a pull-back CR structure $\tilde{T}^{1,0}\tilde{X} := \pi^*T^{1,0}X$ and, hence, a CR manifold. We assume that $\tilde{X}$ admits a transversal CR locally free $S^1$ action, denoted by $e^{i\theta}$. We further assume that the map
\[\Gamma \times \tilde{X} \to \tilde{X}, \ (\gamma, \tilde{x}) \mapsto \gamma \circ \tilde{x}, \ \forall \tilde{x} \in \tilde{X}, \ \forall \gamma \in \Gamma.\]
is CR, i.e.
\[\gamma_*(T^{1,0}_{\tilde{x}}\tilde{X}) \subseteq T^{1,0}_{\tilde{\gamma}(\tilde{x})}\tilde{X},\]
and
\[e^{i\theta} \circ \gamma \circ \tilde{x} = \gamma \circ e^{i\theta} \circ \tilde{x}, \ \forall \tilde{x} \in \tilde{X}, \ \forall \theta \in [0, 2\pi[, \ \forall \gamma \in \Gamma.\]

It is easy to see that the $S^1$-action $e^{i\theta}$ on $\tilde{X}$ induces a transversal CR locally free $S^1$ action, also denoted by $e^{i\theta}$. We denote by $\tilde{T} := \pi^*T$ the pull-back one form on $\tilde{X}$, then $T$ is the global real vector field induced by the $S^1$-action on $X$. Let $\tilde{\omega}_0 := \pi^*\omega_0$ be the pull-back one form on $\tilde{X}$, where $\omega_0$ is the global real one form on $X$ as defined in Subsection 2.2. Then, for $\tilde{p} \in \tilde{X}$, the Levi form $\tilde{\mathcal{L}}_\tilde{\omega}$ is the Hermitian quadratic form on $T^{1,0}_{\tilde{p}}\tilde{X}$ given by
\[\tilde{\mathcal{L}}_\tilde{\omega}(\tilde{U}, \tilde{V}) = -\frac{1}{2i}(d\tilde{\omega}_0(\tilde{p}), \tilde{U} \wedge \tilde{V}) = -\frac{1}{2i}(d\omega_0(\pi(\tilde{p})), \pi_*\tilde{U} \wedge \pi_*\tilde{V}),\]
where $\tilde{U}, \tilde{V} \in T^{1,0}_{\tilde{p}}\tilde{X}$.

As usual, let $\Omega^{0,q}(\tilde{X})$ denote the space of smooth sections of $\wedge^q(T^{1,0}\tilde{X})$. We also denote by $\tilde{\partial}_b : \Omega^{0,q}(\tilde{X}) \to \Omega^{0,q+1}(\tilde{X})$ the tangential Cauchy-Riemann operator. Then $\tilde{T}\tilde{\partial}_b = \tilde{\partial}_b\tilde{T}$ on $\Omega^{0,q}(\tilde{X})$. Let $E$ be a rigid CR vector bundle over $X$, then $\tilde{E} := \pi^*E$ is a $\Gamma$-invariant rigid CR vector bundle over $\tilde{X}$. Again let $\Omega^{0,q}(\tilde{X}, \tilde{E})$ denote the space of smooth sections of $\wedge^q(T^{1,0}\tilde{X}) \otimes \tilde{E}$. We again denote by $\tilde{\partial}_b : \Omega^{0,q}(\tilde{X}, \tilde{E}) \to \Omega^{0,q+1}(\tilde{X}, \tilde{E})$ the tangential Cauchy-Riemann operator. Then again $\tilde{T}\tilde{\partial}_b = \tilde{\partial}_b\tilde{T}$ on $\Omega^{0,q}(\tilde{X}, \tilde{E})$. We denote by $L^2(\tilde{X}, T^{1,0}\tilde{X} \otimes \tilde{E})$ and $L^2(\tilde{X}, T^{1,0}\tilde{X})$ the completions of $\Omega^{0,q}(\tilde{X}, \tilde{E})$ and $\Omega^{0,q}(\tilde{X})$ with respect to the corresponding pull-back metrics $(\cdot, \cdot)_{\tilde{E}}$ and $(\cdot, \cdot)$. Similarly, we denote by $L^2(\tilde{X}, T^{1,0}\tilde{X} \otimes \tilde{E})$ and $L^2(\tilde{X}, T^{1,0}\tilde{X})$ the completions of $\Omega^{0,q}(\tilde{X}, \tilde{E})$ and $\Omega^{0,q}(\tilde{X})$ with respect to the corresponding pull-back metrics $(\cdot, \cdot)_{\tilde{E}}$ and $(\cdot, \cdot)$.

As usual, for every $m \in \mathbb{Z}$, let
\[\Omega_m^{0,q}(\tilde{X}, \tilde{E}) := \left\{ u \in \Omega^{0,q}(\tilde{X}, \tilde{E}); \tilde{T}u = imu \right\}, \quad q = 0, 1, 2, \ldots, n,\]
\[\Omega_m^{0,q}(\tilde{X}, \tilde{E}) := \left\{ u \in \Omega^{0,q}(\tilde{X}, \tilde{E}); \tilde{T}u = imu \right\}.\]

For each $m \in \mathbb{Z}$, we denote by $L_m^2(\tilde{X}, T^{1,0}\tilde{X} \otimes \tilde{E})$ and $L_m^2(\tilde{X}, T^{1,0}\tilde{X})$ the completions of $\Omega_m^{0,q}(\tilde{X}, \tilde{E})$ and $\Omega_m^{0,q}(\tilde{X})$ with respect to the corresponding pull-back metrics $(\cdot, \cdot)_{\tilde{E}}$ and $(\cdot, \cdot)$. Similarly, we denote by $L_m^2(\tilde{X}, T^{1,0}\tilde{X} \otimes \tilde{E})$ and $L_m^2(\tilde{X}, T^{1,0}\tilde{X})$ the completions of $\Omega_m^{0,q}(\tilde{X}, \tilde{E})$ and $\Omega_m^{0,q}(\tilde{X})$ with respect to the corresponding pull-back metrics $(\cdot, \cdot)_{\tilde{E}}$ and $(\cdot, \cdot)$.

Recall that $U \subset \tilde{X}$ is called a fundamental domain of the action of $\Gamma$ on $\tilde{X}$ if the following conditions hold:

1. $\tilde{X} = \bigcup_{\gamma \in \Gamma} \gamma(U)$,
2. $\gamma_1(U) \cap \gamma_2(U) = \emptyset$ for $\gamma_1, \gamma_2 \in \Gamma, \gamma_1 \neq \gamma_2$,
3. $U \setminus \tilde{U}$ is of measure 0.
We can take $U$ to be $S^1$-invariant and with the pull-back $S^1$-action $e^{i\theta}$. We construct such a fundamental domain in the following: From the discussion in the proof of [5, Theorem 2.11], we can find local trivializations $W_1, \cdots, W_N$ such that $X = \bigcup_{j=1}^N W_j$ and each $W_j$ is $S^1$-invariant. For each $j$, let $\tilde{W}_j \subset \tilde{X}$ be an $S^1$-invariant open set such that $\pi : \tilde{W}_j \to W_j$ is a diffeomorphism and a CR map with inverse $\phi_j : W_j \to \tilde{W}_j$. Define $U_j = W_j \setminus (\bigcup_{i<j} \tilde{W}_i \cap W_j)$. Then $U := \bigcup_j \phi_j(U_j)$ is the fundamental domain we want.

It is easy to see that

$$L^2(\tilde{X}, \tilde{E}) \simeq L^2 T \otimes L^2(U, \tilde{E}) \simeq L^2 \Gamma \otimes L^2(X, E).$$

We then have a unitary action of $\Gamma$ by left translations on $L^2 \Gamma$ by $t_\gamma \delta_\eta = \delta_{\gamma \eta}$, where $\{\delta_\eta : \eta \in \Gamma\}$ is the orthonormal basis of $L^2 \Gamma$ formed by the delta functions. It induces a unitary action of $\Gamma$ on $L^2(\tilde{X}, \tilde{E})$ by $\gamma \mapsto T_\gamma = t_\gamma \otimes \text{Id}$.

Let us recall the definition of the Von Neumann dimension or $\Gamma$-dimension of a $\Gamma$-module $V \subset L^2(\tilde{X}, T^{*, q} \tilde{X} \otimes \tilde{E})$, see also [10, Definition 3.6.1]. We shall denote by $\mathcal{L}(A)$ the space of bounded operators of the Hilbert space $H$. Let $\mathcal{A}_T \subset \mathcal{L}(L^2 \Gamma)$ be the algebra of operators which commute with all left translations and denote the unit element of $\Gamma$ by $1$. We define $\text{Tr}_T[A] := \langle A \delta_\eta, \delta_\eta \rangle$, $A \in \mathcal{A}_T$. Note that a $\Gamma$-module is a left $\Gamma$-invariant subspace $V \subset L^2 \Gamma$. The orthogonal projection $P_V$ on $V$ is in $\mathcal{A}_T$ for a $\Gamma$-module $V$. Set $\dim_\Gamma V := \text{Tr}_T[P_V]$. Now we replace $L^2 \Gamma$ by $L^2(\tilde{X}, T^{*, q} \tilde{X} \otimes \tilde{E})$. Then to any operator $A \in \mathcal{L}(L^2(\tilde{X}, T^{*, q} \tilde{X} \otimes \tilde{E}))$, we associate operators $a_{\gamma \eta} \in \mathcal{L}(L^2(U, T^{*, q} \tilde{X} \otimes \tilde{E}))$ such that $a_{\gamma \eta}(f)$ is the projection of $A(\delta_{\gamma \eta} \otimes f)$ on $\mathbb{C} \delta_\eta \otimes L^2(U, T^{*, q} \tilde{X} \otimes \tilde{E})$. In addition, if $A \in \mathcal{A}_T$ and $A$ is positive, then $a_{\gamma \eta} = a_{\eta \gamma^{-1} \eta}$ and

$$\text{Tr}_T[A] := \text{Tr}[a_{ee}] \geq 0,$$

is well-defined. The orthogonal projection $P_V$ on $V \subset L^2(\tilde{X}, T^{*, q} \tilde{X} \otimes \tilde{E})$ is in $\mathcal{A}_T$ for a $\Gamma$-module $V$.

Definition 2.7. The Von Neumann dimension or $\Gamma$-dimension of a $\Gamma$-module $V$ is defined by

$$\dim_\Gamma V := \text{Tr}_T[P_V].$$

3. Asymptotic expansion of heat kernels of Kohn Laplacians

In this section, we recall the definition of heat kernels. Then we give a new version of asymptotic expansions of heat kernels of Kohn Laplacians.

3.1. Asymptotics of heat kernels of Kohn Laplacians on a compact CR manifold.

Since $T\bar{\partial}_b = \bar{\partial}_bT$ and $E$ is a rigid CR vector bundle with a rigid Hermitian metric, we have

$$\bar{\partial}_{b,m} := \bar{\partial}_b : \Omega^0_{m^*}(X, E) \to \Omega^0_{m^*}(X, E), \quad \forall m \in \mathbb{Z}.$$

The $m$-th Fourier component of Kohn-Rossi cohomology is given by

$$H^0_{b,m}(X, E) := \frac{\ker \bar{\partial}_b : \Omega^0_{m,q}(X, E) \to \Omega^0_{m,q+1}(X, E)}{\im \bar{\partial}_b : \Omega^0_{m,q-1}(X, E) \to \Omega^0_{m,q}(X, E)}.$$

We also write

$$\overline{\partial}_b^* : \Omega^0_{m^*}(X, E) \to \Omega^0_{m^*}(X, E)$$

to denote the formal adjoint of $\bar{\partial}_b$ with respect to $(\cdot | \cdot)_E$. Since $(\cdot | \cdot)_E$ and $(\cdot | \cdot)$ are rigid, we can check that

$$T\overline{\partial}_b = \overline{\partial}_b T \quad \text{on } \Omega^0_{m^*}(X, E),$$

$$\overline{\partial}_{b,m} := \overline{\partial}_b : \Omega^0_{m^*}(X, E) \to \Omega^0_{m^*}(X, E), \quad \forall m \in \mathbb{Z}.$$
Now, we fix \( m \in \mathbb{Z} \). The \( m \)-th Fourier component of Kohn Laplacian is given by
\[
\Box_{b,m} := (\overline{\partial}_{b,m} + \overline{\partial}_{b,m}^\ast)^2 : \Omega^0_m(X, E) \to \Omega^0_m(X, E).
\]
We extend \( \Box_{b,m} \) to \( L^2_m(X, T^{*0 \cdot}X \otimes E) \) by
\[
\Box_{b,m} : \text{Dom} \Box_{b,m} \subset L^2_m(X, T^{*0 \cdot}X \otimes E) \to L^2_m(X, T^{*0 \cdot}X \otimes E),
\]
where \( \text{Dom} \Box_{b,m} := \{ u \in L^2_m(X, T^{*0 \cdot}X \otimes E); \Box_{b,m} u \in L^2_m(X, T^{*0 \cdot}X \otimes E) \} \), where for any \( u \in L^2_m(X, T^{*0 \cdot}X \otimes E) \), \( \Box_{b,m} u \) is defined in the sense of distributions. We recall the following results (see Section 3 in [5]).

**Theorem 3.1.** The Kohn Laplacian \( \Box_{b,m} \) is self-adjoint, \( \text{Spec} \Box_{b,m} \) is a discrete subset of \([0, \infty[\) and for every \( \nu \in \text{Spec} \Box_{b,m} \), \( \nu \) is an eigenvalue of \( \Box_{b,m} \) with finite multiplicity.

For every \( \nu \in \text{Spec} \Box_{b,m} \), let \( \{ f_1^\nu, \ldots, f_{d_\nu}^\nu \} \) be an orthonormal frame for the eigenspace of \( \Box_{b,m} \) with eigenvalue \( \nu \). The heat kernel \( e^{-t \Box_{b,m}}(x, y) \) is given by
\[
e^{-t \Box_{b,m}}(x, y) = \sum_{\nu \in \text{Spec} \Box_{b,m}} \sum_{j=1}^{d_\nu} e^{-t \nu} f_j^\nu(x) \otimes (f_j^\nu(y))^\dagger,
\]
where \( f_j^\nu(x) \otimes (f_j^\nu(y))^\dagger \) denotes the linear map:
\[
f_j^\nu(x) \otimes (f_j^\nu(y))^\dagger : T^{*0 \cdot}X \otimes E_y \to T^{*0 \cdot}X \otimes E_x,
\]
\[
u(y) \in T^{*0 \cdot}X \otimes E_y \to f_j^\nu(x) \langle u(y) | f_j^\nu(y) \rangle_E \in T^{*0 \cdot}X \otimes E_x.
\]
Let \( e^{-t \Box_{b,m}} : L^2(X, T^{*0 \cdot}X \otimes E) \to L^2_m(X, T^{*0 \cdot}X \otimes E) \) be the continuous operator with distribution kernel \( e^{-t \Box_{b,m}}(x, y) \).

We denote by \( \hat{\mathcal{R}} \) the Hermitian matrix \( \hat{\mathcal{R}} \in \text{End}(T^{1,0}X) \) such that for \( V, W \in T^{1,0}X \),
\[
id\omega_0(V, W) = \langle \hat{\mathcal{R}}V | W \rangle.
\]
Let \( \{ \omega_j \}_{j=1}^n \) be a local orthonormal frame of \( T^{1,0}X \) with dual frame \( \{ \omega^j \}_{j=1}^n \). Set
\[
\gamma_d = -i \sum_{l,j=1}^n d\omega_0(\omega_j, \overline{\omega_l}) \omega^l \wedge t\omega_j,
\]
where \( t\omega_j \) denotes the interior product of \( \omega_j \). Then \( \gamma_d \in \text{End}(T^{*0 \cdot}X) \) and \( -id\omega_0 \) acts as the derivative \( \gamma_d \) on \( T^{*0 \cdot}X \). If we choose \( \{ \omega_j \}_{j=1}^n \) to be an orthonormal basis of \( T^{1,0}X \) such that
\[
\hat{\mathcal{R}}(x) = \text{diag}(a_1(x), \ldots, a_n(x)) \in \text{End}(T^{1,0}_xX),
\]
then
\[
\gamma_d(x) = -\sum_{j=1}^n a_j(x) \omega^j \wedge t\omega_j.
\]
Define \( \text{det} \hat{\mathcal{R}}(x) := a_1(x) \ldots a_n(x) \).

Fix \( x, y \in X \). Let \( d(x, y) \) denote the standard Riemannian distance of \( x \) and \( y \) with respect to the given Hermitian metric. Take \( \zeta \)
\[
o < \zeta < \inf \left\{ \frac{2\pi}{p_k}, \frac{2\pi}{p_r} - \frac{2\pi}{p_{r+1}} \right\}, r = 1, \ldots, k - 1.
\]
For \( x \in X \), put
\[
\hat{d}(x, X_{\text{sing}}) := \inf \left\{ d(x, e^{-i\theta}x); \zeta \leq \theta \leq \frac{2\pi}{p} - \zeta \right\}.
\]

The following result generalizes Theorem 3.1 in [12].
Theorem 3.2. With the above notations and assumptions, for every \( \epsilon > 0 \), there are \( m_0 > 0 \), \( \varepsilon_0 > 0 \) and \( C > 0 \) such that for all \( m \geq m_0 \), we have

\[
\left| e^{-\frac{b m}{4\pi}} h_{\circ b, m}(x, x) - \sum_{s=1}^{p} e^{2\pi s(1-mi)} (2\pi)^{n-1} m^{n} \frac{\det(R) \exp(t_\gamma d)}{\det(1 - \exp(-tR))} (x) \otimes \text{Id}_{E_x} \right|
\]

\[
\leq \epsilon m^n + C m^n t^{-n} e^{-\frac{c_\circ m d(x, x)}{t}}, \quad \forall (t, x) \in \mathbb{R} \times X_{\text{reg}}.
\]

Proof. We use the notations from Section 3 in [12]. Recall that \( \Gamma_m \) is defined in [12] (3.31) (see also (3.29)). For \( x \in X_{\text{reg}} \), we have

\[
\Gamma_m(t, x, x) = \frac{1}{2\pi} \sum_{j=1}^{N} \int_{0}^{2\pi} H_{j,m}(t, x, e^{iu} \circ x) e^{imu} du
\]

\[
= \frac{1}{2\pi} \sum_{s=1}^{p} e^{2\pi s(1-mi)} \sum_{j=1}^{N} \int_{0}^{2\pi} H_{j,m}(t, x, e^{iu} \circ x) e^{imu} du
\]

\[
= \frac{1}{2\pi} \sum_{s=1}^{p} e^{2\pi s(1-mi)} \sum_{j=1}^{N} \int_{u \in [\frac{2\pi}{p} - \zeta]} H_{j,m}(t, x, e^{iu} \circ x) e^{imu} du
\]

\[
= \frac{1}{2\pi} \sum_{s=1}^{p} e^{2\pi s(1-mi)} \sum_{j=1}^{N} \int_{-\zeta}^{\zeta} H_{j,m}(t, x, e^{iu} \circ x) e^{imu} du,
\]

where \( H_{j,m} \) is defined in [12] (3.30) (see also (3.29)). From [12] (3.29), (3.34) and [5] (6.4), there are \( \varepsilon_0 > 0 \) and \( C_0 \) independent of \( j, x, m, t \) such that, for all \( t \in \mathbb{R}_+ \) and for all \( m \in \mathbb{N} \), we have

\[
\left| \frac{1}{2\pi} \int_{u \in [\frac{2\pi}{p} - \zeta]} H_{j,m}(t, x, e^{iu} \circ x) e^{imu} du \right| \leq C_0 m^n t^{-n} e^{-\frac{c_\circ m d(x, x)}{t}}.
\]

Then the proof is completed by applying [12] (3.32), (3.39) and (3.12).

Remark 3.3. It is easy to check that

\[
\sum_{s=1}^{p} e^{2\pi s(1-mi)} = \begin{cases} p & \text{if } m \mid p, \\ 0 & \text{if } p \nmid m. \end{cases}
\]

\[ \square \]

3.2. BRT trivializations. To prove Theorem 3.2 we need some preparations. We first need the following result due to Baouendi-Rothschild-Treves [2].

Theorem 3.4. For every point \( x_0 \in X \), we can find local coordinates \( x = (x_1, \ldots, x_{2n+1}) = (z, \theta) = (z_1, \ldots, z_n, \theta), z_j = x_{2j-1} + ix_{2j}, j = 1, \ldots, n, x_{2n+1} = \theta \), defined in some small neighborhood \( D = \{|z| < \delta, -\varepsilon_0 < \theta < \varepsilon_0\} \) of \( x_0 \), \( \delta > 0 \), \( 0 < \varepsilon_0 < \pi \), such that \( (z(x_0), \theta(x_0)) = (0, 0) \) and

\[
T = \frac{\partial}{\partial \theta}
\]

\[
Z_j = \frac{\partial}{\partial z_j} + i \frac{\partial \varphi}{\partial z_j} (z) \frac{\partial}{\partial \theta}, j = 1, \ldots, n
\]

where \( Z_j(x), j = 1, \ldots, n \), form a basis of \( T_x^{1,0} X \), for each \( x \in D \), and \( \varphi(z) \in C^\infty(D, \mathbb{R}) \) is independent of \( \theta \). We call \( (D, (z, \theta), \varphi) \) BRT trivialization.
By using BRT trivialization, we get another way to define $Tu, \forall u \in \Omega^0,q(X)$. Let $(D, (z, \theta), \varphi)$ be a BRT trivialization. It is clear that
$$\{d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_q}, \, 1 \leq j_1 < \cdots < j_q \leq n\}$$
is a basis for $T^*_{x}^{0,q} X$, for every $x \in D$. Let $u \in \Omega^0,q(X)$. On $D$, we write
$$(3.15)\quad u = \sum_{1 \leq j_1 < \cdots < j_q \leq n} u_{j_1, \ldots, j_q} d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_q}.$$Then, on $D$, we can check that
$$(3.16)\quad Tu = \sum_{1 \leq j_1 < \cdots < j_q \leq n} (Tu_{j_1, \ldots, j_q}) d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_q}$$and $Tu$ is independent of the choice of BRT trivializations. Note that, on BRT trivialization $(D, (z, \theta), \varphi)$, we have
$$(3.17)\quad \overline{\partial}u = \sum_{j=1}^{n} d\bar{z}_j \wedge \left(\frac{\partial}{\partial \overline{z}_j} - i \frac{\partial}{\partial \theta}(z) \frac{\partial}{\partial \theta}\right).$$

3.3. Local heat kernels on BRT trivializations. Until further notice, we fix $m \in \mathbb{Z}$. Let $B := (D, (z, \theta), \varphi)$ be a BRT trivialization. We may assume that $D = U \times \mathbb{C}^n$. Since $E$ is rigid, we can consider $E$ as a holomorphic vector bundle over $U$. We may assume that $E$ is trivial on $U$. Consider a trivial line bundle $L \rightarrow U$ with non-trivial Hermitian fiber metric $|1|^2 L = e^{-2\varphi}$. Let $(L^m, h^L_m) \rightarrow U$ be the $m$-th power of $(L, h^L)$. Let $\Omega^0,q(U, E \otimes L^m)$ and $\Omega^0,q(U, E)$ be the spaces of $(0,q)$ forms on $U$ with values in $E \otimes L^m$ and $E$, respectively, $q = 0, 1, 2, \ldots, n$. Put
$$\Omega^0,q(U, E \otimes L^m) := \oplus_{j \in \{0, 1, \ldots, n\}} \Omega^0,j(U, E \otimes L^m),$$
$$\Omega^0,q(U, E) := \oplus_{j \in \{0, 1, \ldots, n\}} \Omega^0,j(U, E).$$Since $L$ is trivial, from now on, we identify $\Omega^0,q(U, E)$ with $\Omega^0,q(U, E \otimes L^m)$. Since the Hermitian fiber metric $(\cdot \mid \cdot)_E$ is rigid, we can consider $(\cdot \mid \cdot)_E$ as a Hermitian fiber metric on the holomorphic vector bundle $E$ over $U$. Let $(\cdot, \cdot)$ be the Hermitian metric on $\mathbb{C}TU$ given by
$$(\cdot, \cdot) = \left(\frac{\partial}{\partial z_j}, \frac{\partial}{\partial z_k}\right) = \left(\frac{\partial}{\partial \overline{z}_j}, i \frac{\partial}{\partial \theta}(z) \frac{\partial}{\partial \theta}\right) + \left(\frac{\partial}{\partial \theta}(z) \frac{\partial}{\partial \theta}, \frac{\partial}{\partial \overline{z}_k}\right), \quad j, k = 1, 2, \ldots, n.$$$(\cdot, \cdot)$ induces a Hermitian metric on $T^*U := \bigoplus_{j=0}^{n} T^{0, j} U$, where $T^{0,j} U$ is the bundle of $(0,j)$ forms on $U$, $j = 0, 1, \ldots, n$. We shall also denote this induced Hermitian metric on $T^*U$ by $(\cdot, \cdot)$. The Hermitian metrics on $T^*U$ and $E$ induce a Hermitian metric on $T^*U \otimes E$. We shall also denote this induced metric by $(\cdot \mid \cdot)_E$. Let $(\cdot, \cdot)_m$ be the $L^2$ inner product on $\Omega^0,q(U, E \otimes L^m)$ induced by $(\cdot, \cdot)_E$. Similarly, let $(\cdot, \cdot)_0$ be the $L^2$ inner product on $\Omega^0,q(U, E \otimes L^m)$ induced by $(\cdot, \cdot)_0$ and $h^L_m$.

The curvature of $L$ induced by $h^L$ is given by $R^L := 2\overline{\partial} \varphi$. Let $R^L \in \text{End}(T^{1,0}U)$ be the Hermitian matrix given by
$$R^L(W, \overline{Y}) = \langle \hat{R}^L W, Y \rangle, \quad W, Y \in T^{1,0} U.$$Let $\{w_j\}_{j=1}^{n}$ be a local orthonormal frame of $T^{1,0} U$ with dual frame $\{\overline{w}_j\}_{j=1}^{n}$. Set
$$(3.18)\quad \omega_d = - \sum_{l,j} R^L(w_j, \overline{w}_l) \overline{w}_l \wedge (\omega_j),$$where $\omega_j$ denotes the interior product of $\overline{w}_j$.
Let
$$\overline{\partial} : \Omega^0,q(U, E \otimes L^m) \rightarrow \Omega^0,q(U, E \otimes L^m)$$
be the Cauchy-Riemann operator and let
\[ \overline{\partial}^m : \Omega^0\bullet(U, E \otimes L^m) \to \Omega^0\bullet(U, E \otimes L^m) \]
be the formal adjoint of \( \overline{\partial} \) with respect to \( \langle \cdot, \cdot \rangle_m \). Put
\[
(3.19) \quad \Box_{B,m} := (\overline{\partial} + \overline{\partial}^m)^2 : \Omega^0\bullet(U, E \otimes L^m) \to \Omega^0\bullet(U, E \otimes L^m).
\]
We need the following result (see Lemma 5.1 in [5])

**Lemma 3.5.** Let \( u \in \Omega^0\bullet_m(X, E) \). On \( D \), we write \( u(z, \theta) = e^{im\theta \bar{u}(z)} \), \( \tilde{u}(z) \in \Omega^0\bullet(U, E) \). Then,
\[
(3.20) \quad e^{-m\varphi} \Box_{B,m}(e^{m\varphi} \tilde{u}) = e^{-m\varphi} \Box_{B,m}(u).
\]

Let \( z, w \in U \) and let \( T(z, w) \in (T^0\bullet_0 U \otimes E_w) \boxtimes (T^0\bullet_0 U \otimes E_z) \). We write \( |T(z, w)| \) to denote the standard pointwise matrix norm of \( T(z, w) \) induced by \( \langle \cdot, \cdot \rangle \). Let \( \Omega^0\bullet(U, E) \) be the subspace of \( \Omega^0\bullet(U, E) \) whose elements have compact support in \( U \). Let \( dv_U \) be the volume form on \( U \) induced by \( \langle \cdot, \cdot \rangle \). Assume \( T(z, w) \in C^\infty(U \times U, (T^0\bullet_0 U \otimes E_w) \boxtimes (T^0\bullet_0 U \otimes E_z)) \). Let \( u \in \Omega^0\bullet(U, E) \). We define the integral \( \int T(z, w)u(w)dw_U(w) \) in the standard way. Let \( G(t, z, w) \in C^\infty(\mathbb{R}_+ \times U \times U, (T^0\bullet_0 U \otimes E_w) \boxtimes (T^0\bullet_0 U \otimes E_z)) \). We write \( G(t) \) to denote the continuous operator
\[
G(t) : \Omega^0\bullet(U, E) \to \Omega^0\bullet(U, E),
\]
and we write \( G'(t) \) to denote the continuous operator
\[
G'(t) : \Omega^0\bullet(U, E) \to \Omega^0\bullet(U, E),
\]
\[
u \to \int G(t, z, w)u(w)dw_U(w)\]
\[
u \to \int \frac{\partial G(t, z, w)}{\partial t}u(w)dw_U(w).
\]

We consider the heat operator of \( \Box_{B,m} \). By using the standard Dirichlet heat kernel construction (see [9]) and the proofs of Theorem 1.6.1 and Theorem 5.5.9 in [16], we deduce the following

**Theorem 3.6.** There is \( A_{B,m}(t, z, w) \in C^\infty(\mathbb{R}_+ \times U \times U, (T^0\bullet_0 U \otimes E_w) \boxtimes (T^0\bullet_0 U \otimes E_z)) \) such that
\[
\lim_{t \to 0^+} A_{B,m}(t) = I \text{ in } D'(U, T^0\bullet_0 U \otimes E),
\]
\[
(3.21) \quad A'_{B,m}(t)u + \frac{1}{m} A_{B,m}(t)(\Box_{B,m}u) = 0, \forall u \in \Omega^0\bullet(U, E), \forall t > 0,
\]
and \( A_{B,m}(t, z, w) \) satisfies the following:

(I) For every compact set \( K \Subset U \), \( \alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{N}_0 \), there are constants \( C_{\alpha_1, \alpha_2, \beta_1, \beta_2, K} > 0 \) and \( \varepsilon_0 > 0 \) independent of \( t \) and \( m \) such that
\[
\left| \frac{\partial^{\alpha_1} \partial^{\alpha_2} \partial^{\beta_1} \partial^{\beta_2}}{\partial_t^m} \left( A_{B,m}(t, z, w)e^{m(\varphi(w) - \varphi(z))} \right) \right| \leq C_{\alpha_1, \alpha_2, \beta_1, \beta_2, K} \left( \frac{m}{t} \right)^{n+|\alpha_1|+|\alpha_2|+|\beta_1|+|\beta_2|} e^{-m\varepsilon_0 \frac{1}{\beta_1^2}}, \forall (t, z, w) \in \mathbb{R}_+ \times K \times K.
\]

(II) \( A_{B,m}(t, z, w) \) admits an asymptotic expansion:
\[
(3.22) \quad A_{B,m}(t, z, w) = (2\pi)^{-m} \frac{\det(\hat{R}_t)}{\det(1 - \exp(-t\hat{R}_t))} \left( z \otimes \text{Id}_{E_z} + o(m^n) \right)
\]
in \( C^\ell(U, \text{End}(T^0\bullet_0 U \otimes E)) \) locally uniformly on \( \mathbb{R}_+ \times U \), for every \( \ell \in \mathbb{N} \). Here we use the convention that if an eigenvalue \( \alpha_j(z) \) of \( \hat{R}_t(z) \) is zero, then its contribution for \( \frac{\det(\hat{R}_t)}{\det(1 - \exp(-t\hat{R}_t))} \) is \( \frac{1}{t} \).
3.4. $L^2$ Kohn-Rossi cohomology on a covering manifold. Let

$$\widetilde{b}_b : \text{Dom} \widetilde{b}_b \subset L^2(\widetilde{X}, T^{s_0 \bullet} \widetilde{X}) \to L^2(\widetilde{X}, T^{s_0 \bullet} \widetilde{X})$$

be the Gaffney extension of the pull-back Kohn Laplacian on $\widetilde{X}$. By a result of Gaffney, $\widetilde{b}_b$ is a positive self-adjoint operator (see Proposition 3.1.2 in Ma-Marinescu [10]). That is, $\widetilde{b}_b$ is self-adjoint and the spectrum of $\widetilde{b}_b$ is contained in $\mathbb{R}_+$. Now, we fix $m \in \mathbb{Z}$. As in (3.33), we introduce the $m$-th Fourier component of the Kohn Laplacian $\tilde{b}_{b,m}$ on $\Omega^{0,\bullet}_m(\tilde{X}, \tilde{E})$. We can easily see that $\tilde{b}_{b,m}$ is also self-adjoint. By the second isomorphism of (2.10), we can see that, for any $\gamma \in \Gamma$,

$$T_\gamma(\text{Dom}(\tilde{b}_{b,m})) \subset \text{Dom}(\tilde{b}_{b,m}), \quad T_\gamma \tilde{b}_{b,m} = \tilde{b}_{b,m}T_\gamma \quad \text{on} \quad \text{Dom}(\tilde{b}_{b,m}).$$

(3.24) Consider the spectral resolution $E^q_\lambda(\tilde{b}_{b,m})$ of $\tilde{b}_{b,m}$ acting on $L^2(\tilde{X}, T^{s_0 q} \tilde{X} \otimes \tilde{E})$. (See [16, Appendix C.2]). The proof of the following lemma is similar to Lemma 3.6.3 in Ma-Marinescu [10].

**Lemma 3.7.** For any $q = 0, 1, \cdots, n$ and $\lambda \in \mathbb{R}$, then $E^q_\lambda(\tilde{b}_{b,m})$ commutes with $\Gamma$, its Schwartz kernel is smooth and

$$\dim_\Gamma E^q_\lambda(\tilde{b}_{b,m}) < +\infty.$$ 

**Proof.** By (2.10) and (3.24), we can see that, for any $\lambda \in \mathbb{R}$, $E^q_\lambda(\tilde{b}_{b,m})$ commutes with $\Gamma$. We claim that $\tilde{b}_{b,m} - \tilde{T}^2 \equiv \Delta$ is a second order elliptic operator, so is $\Delta - m^2$. Its principal symbol is locally written as

$$\sigma_\Delta(x, \xi) = \sigma_{\tilde{b}_{b,m}}(x, \xi) - \sigma_{\tilde{T}^2}(x, \xi) = \sum_{j=1}^{n} |\sigma_{L_j}(x, \xi)|^2 - \sigma_{\tilde{T}}(x, \xi)^2,$$

where $\xi = (\xi_1, \ldots, \xi_{2n}, \xi_{2n+1})$ and $\{L_j\}$ is an orthonormal basis of $T^{0,1}_x \tilde{X}$. It is well-known that the characteristic manifold of $\tilde{b}_b$ is

$$\Sigma = \{ (x, c\tilde{w}_0(x)) \in T^* \tilde{X} : c \neq 0 \}.$$ 

It means that $\sigma_{\tilde{b}_{b,m}}(x, \xi) > 0$ if and only if $(\xi_1, \ldots, \xi_{2n}) \neq 0$. Meanwhile, in a local BRT coordinate [2], we have $\tilde{T} = \frac{\partial}{\partial \theta}$, then $\sigma_{\tilde{T}} = i\xi_{2n+1}$. That is, $\sigma_{\tilde{T}^2} = -\xi_{2n+1}^2$. Then the claim is proved. By the spectral theorem, cf. [16, Theorem C.2.1], we have $\text{Im}(E_\lambda(\Delta - m^2)) \subset \text{Dom}((\Delta - m^2)^k)$ for $k \in \mathbb{N}$. Using the uniform Sobolev spaces [19, pp. 511-512], it is easy to see that $\text{Im}(E_\lambda(\Delta - m^2)) \subset \Omega^\bullet(\tilde{X}, \tilde{E})$, so that $E_\lambda(\Delta - m^2) : L^2(\tilde{X}, T^{s_0 \bullet} \tilde{X} \otimes \tilde{E}) \to \Omega^\bullet(\tilde{X}, \tilde{E})$ is linear continuous. Hence, $\text{Im} E_\lambda(\tilde{b}_{b,m}) = \text{Im}(E_\lambda(\Delta - m^2)) \cap L^2_{m_2}(\tilde{X}, T^{s_0 \bullet} \tilde{X} \otimes \tilde{E}) \subset \Omega^\bullet(\tilde{X}, \tilde{E})$ and $E_\lambda(\tilde{b}_{b,m}) : L^2_{m_2}(\tilde{X}, T^{s_0 \bullet} \tilde{X} \otimes \tilde{E}) \to \Omega^\bullet(\tilde{X}, \tilde{E})$ is also linear continuous. By Schwartz kernel theorem, the kernel $E_\lambda(\tilde{b}_{b,m})(x, x)$ of $E_\lambda(\tilde{b}_{b,m})$ with respect to $dv_{\tilde{X}}(x)$ is smooth. By [16] (3.6.12),

$$\dim_\Gamma E_\lambda(\tilde{b}_{b,m}) = \int_U \text{Tr}[E_\lambda(\tilde{b}_{b,m})(\tilde{x}, \tilde{x})]dv_{\tilde{X}}(\tilde{x}) < +\infty.$$

□

**Definition 3.8.** (a) The $m$-th Fourier component of the space of harmonic forms $\mathcal{H}_\bullet^\bullet(\tilde{X}, \tilde{E})$ is defined by

$$\mathcal{H}_{b,m}^\bullet(\tilde{X}, \tilde{E}) := \text{Ker}(\tilde{b}_{b,m}) = \{ s \in \text{Dom}(\tilde{b}_{b,m}) : \tilde{b}_{b,m}s = 0 \}.$$
Asymptotics of heat kernels of Kohn Laplacians on a covering manifold. We may assume that, for each $x$, we may suppose that

Theorem 3.9. By (3.26), we the isomorphism

$$\tau \{z \in C_0, \delta \} = 1 \gamma, j$$

Put

$$\gamma, j \{ \gamma, j \} \equiv 1$$

Note that when $\Gamma = \tau \{z \in C_0, \delta \}$, we have

$$\gamma, j \{ \gamma, j \} = 1$$

where $[V]$ denotes the closure of the space $V$.

We can easily obtain the following weak Hodge decomposition

$$L^2_n(\tilde{X}, T^{*\bullet} \tilde{X} \otimes \tilde{E}) = \mathcal{H}^\bullet(\tilde{X}, \tilde{E}) \oplus [\text{Im}(\mathcal{D}_{b,m})] \oplus [\text{Im}(\mathcal{D}_{b,m})]$$

By (3.26), we the the isomorphism

$$\mathcal{H}_{b(2),m}^\bullet(\tilde{X}, \tilde{E}) \cong \mathcal{H}^\bullet_{b(2),m}(\tilde{X}, \tilde{E})$$.

3.5. Asymptotics of heat kernels of Kohn Laplacians on a covering manifold. Assume that $X = D_1 \cup D_2 \cup \cdots \cup D_N$, where $B_j := (D_j, (z, \theta), \varphi_j)$ is a BRT trivialization, for each $j$. We may assume that, for each $j$, $D_j = U_j \times -2 \delta_j \cup \delta_j \subset \mathbb{C}^n \times \mathbb{R}$, $\delta_j > 0$, $\delta_j > 0$, $U_j = \{z \in \mathbb{C}^n; |z| < l_j \}$. For each $j$, put $\hat{D}_j = \hat{U}_j \times \delta_j \tilde{x} \frac{l_j}{2}, \tilde{y} \frac{l_j}{2}, \tilde{w} \frac{l_j}{2}, \tilde{z} \frac{l_j}{2}$, where $\hat{U}_j = \tilde{U}_j \times \delta_j \tilde{x} \frac{l_j}{2}, \tilde{y} \frac{l_j}{2}, \tilde{w} \frac{l_j}{2}, \tilde{z} \frac{l_j}{2}$.

Let $\{\psi_j\}$ be a partition of unity subordinate to $\{\hat{D}_j\}$. Then $\{\tilde{\psi}_{\gamma,j} := \psi_1 \circ \pi\}$ is a partition of unity subordinate to $\{\hat{D}_{\gamma,j}\}$, where $\pi^{-1}(\hat{D}_j) = U_{\gamma,j} \hat{D}_{\gamma,j}$ and $\hat{D}_{\gamma,j}$ and $\hat{D}_{\gamma,j}$ are disjoint for $\gamma_1 \neq \gamma_2$. For each $\gamma \in \Gamma$ and each $j$, we have $\tilde{U}_{\gamma,j} = \tilde{U}_{\gamma,j} \times \delta_j \tilde{x} \frac{l_j}{2}, \tilde{y} \frac{l_j}{2}, \tilde{w} \frac{l_j}{2}, \tilde{z} \frac{l_j}{2}$.

Fix $\gamma \in \Gamma$ and $j = 1, 2, \ldots, N$. Put

$$K_{\gamma,j} = \{z \in \tilde{U}_{\gamma,j}; there is a \theta \in [-\frac{\delta_j}{2}, \frac{\delta_j}{2}] such that \tilde{\psi}_{\gamma,j}(z, \theta) \neq 0\}.$$

Let $\tau_{\gamma,j}(z) \in C^\infty(\tilde{U}_{\gamma,j})$ with $\tau_{\gamma,j} \equiv 1$ on some neighborhood $W_{\gamma,j}$ of $K_{\gamma,j}$. Let $\sigma_{\gamma,j} \in C^\infty([-\frac{\delta_j}{2}, \frac{\delta_j}{2}])$ with $\int \sigma_{\gamma,j}(\theta) d\theta = 1$. Let $\tilde{A}_{B_{\gamma,j},m}(t, z, w) \in C^\infty(\mathbb{R}_+ \times \tilde{U}_{\gamma,j} \times \tilde{U}_{\gamma,j}, (T_w^{*\bullet} \tilde{U}_{\gamma,j} \otimes \tilde{E}_w))$ be as in Theorem 3.6.

Put

$$\tilde{H}_{\gamma,j,m}(t, \tilde{x}, \tilde{y}) = \tilde{\psi}_{\gamma,j}(\tilde{x})e^{-m\varphi_j(z) - im\theta} \tilde{A}_{B_{\gamma,j},m}(t, z, w)e^{m\varphi_j(w)}e^{im\theta} \tau_{\gamma,j}(w) \sigma_{\gamma,j}(\eta),$$

where $\tilde{x} = (z, \theta)$, $\tilde{y} = (w, \eta) \in \mathbb{C}^n \times \mathbb{R}$. Let

$$\tilde{\Gamma}_m(t, \tilde{x}, \tilde{y}) := \frac{1}{2\pi} \sum_{\gamma \in \Gamma} \sum_{j=1}^N \int_0^\pi \tilde{H}_{\gamma,j,m}(t, \tilde{x}, e^{iu} \tilde{y})e^{imu} du.$$

Note that when $\gamma = \{e\}$, $\tilde{\Gamma}_m(t, \tilde{x}, \tilde{y}) = \Gamma_m(t, \pi(\tilde{x}), \pi(\tilde{y}))$ is defined in $[12] (3.31)$.

From Lemma 3.5, off-diagonal estimates of $\tilde{A}_{B_{\gamma,j},m}(t, \tilde{x}, \tilde{y})$ (see (3.22)), we can repeat the proof of Theorem 5.14 in $[5]$ with minor change and deduce that

Theorem 3.9. For every $\ell \in \mathbb{N}$, $\ell \geq 2$, and every $M > 0$, there are $\epsilon_0 > 0$ and $m_0 > 0$ independent of $t$ and $m$ such that for every $m \geq m_0$, we have

$$\left\|e^{-\pi b_{\gamma,j,m}(\tilde{x}, \tilde{y})} - \tilde{\Gamma}_m(t, \tilde{x}, \tilde{y})\right\|_{C^\ell(\tilde{x} \times \tilde{x})} \leq e^{-\frac{t}{\epsilon_0}}, \ \forall t \in (0, M).$$

From Theorem 3.6.4 in $[16]$, we have
Proposition 3.10. For any $t_0 > 0, \epsilon > 0$ and any $\gamma \in \Gamma, j = 1, 2, \cdots, N,$ there exists $C > 0$ such that for any $z \in \tilde{U}_{\gamma,j}, m \in \mathbb{N}, t > t_0,$
\[
\left\| \tilde{A}_{\gamma,j,m}(t, z, z) - A_{\gamma,j,m}(t, \pi(z), \pi(z)) \right\|_{C^0(\tilde{U}_{\gamma,j} \times \tilde{U}_{\gamma,j})} \leq C \exp \left( -\frac{m}{32t}\epsilon \right).
\]

From (3.11) (see (3.31) in [12], 3.28, 3.29), Proposition 3.10 and the fact that $\tilde{\psi}_{\gamma,j} = \psi_j \circ \pi,$ we can easily deduce that

Lemma 3.11. With the above notations and assumptions as in Theorem 3.9, we have
\[
\left\| \tilde{\Gamma}_m(t, \tilde{x}, \tilde{x}) - \Gamma_m(t, \pi(\tilde{x}), \pi(\tilde{x})) \right\|_{C^0(\tilde{X} \times \tilde{X})} \leq C \exp \left( -\frac{m}{t}\epsilon_0 \right).
\]

From Theorem 3.9, Lemma 3.11 and Theorem 3.5 of [12], we have

Theorem 3.12. For every $\ell \in \mathbb{N}, \ell \geq 2,$ and every $M > 0,$ there are $\epsilon_0 > 0$ and $m_0 > 0$ independent of $t$ and $m$ such that for any $\tilde{x} \in \tilde{X}$ and $m \geq m_0,$ we have
\[
\left\| e^{-\frac{t}{m}\tilde{\Delta}_{b,m}}(\tilde{x}, \tilde{x}) - e^{-\frac{t}{m}\Delta_{b,m}}(\pi(\tilde{x}), \pi(\tilde{x})) \right\|_{C^0(\tilde{X} \times \tilde{X})} \leq C \exp \left( -\frac{m}{t}\epsilon_0 \right), \quad \forall t \in (0, M).
\]

By Theorem 3.2 and Theorem 3.12, we have

Theorem 3.13. With the above notations and assumptions, for every $\epsilon > 0,$ there are $m_0 > 0, \epsilon_0 > 0$ and $C > 0$ such that for all $m \geq m_0,$ we have
\[
\left(3.31\right) \quad \left\| e^{-\frac{t}{m}\tilde{\Delta}_{b,m}}(\tilde{x}, \tilde{x}) - \sum_{s=1}^{p} e^{\frac{2\pi i (s-1)m_l}{n} (2\pi)^{n-1} m_l^{-1} m_l^n \frac{\det(\tilde{R})}{\det(1 - \exp(-t\tilde{R}))} \rho(\tilde{x}) \otimes \text{Id}_{E_{\rho(\tilde{x})}}} \right\| \leq C m^n t^{-n} e^{-\epsilon_0 m_d(\rho(\tilde{x}), X_{\text{sing}})^2}, \quad \forall (t, \tilde{x}) \in \mathbb{R}_+ \times \tilde{X}_{\text{reg}}.
\]

Recall that since $\Gamma$ acts on $\tilde{X}$ freely so that $\tilde{X}/\Gamma = X,$ hence, we have $\tilde{X}_{\text{reg}}/\Gamma = X_{\text{reg}}.$

4. Heat kernel proof

In this section, we will present the heat kernel proof of the main theorem.

We denote by $\text{Tr}_{\Gamma,q}$ the $\Gamma$-trace of operators acting on $L^2_m(\tilde{X}, T^{0,\rho} \tilde{X} \otimes \tilde{E}),$ see Subsection 2.3 or [16, Subsection 3.6.1].

Lemma 4.1. For any $t > 0, m \in \mathbb{N}, 0 \leq q \leq n,$ we have
\[
\left(4.1\right) \quad \sum_{j=0}^{q} (-1)^{q-j} \dim_{\Gamma} \tilde{T}_{b,(2),m}^j(\tilde{X}, \tilde{E}) \leq \sum_{j=0}^{q} (-1)^{q-j} \text{Tr}_{\Gamma,j}[\exp(-\frac{t}{m}\tilde{\Delta}_{b,m})],
\]
with equality for $q = n.$

Proof. Let $E_{\lambda}^{1,0}$ be the spectral resolution of $\tilde{\Delta}_{b,m}$ acting on $L^2_m(\tilde{X}, T^{0,\rho} \tilde{X} \otimes \tilde{E}).$ We consider the projectors $E_{\lambda}^{1,0}([\lambda_1, \lambda_2]) = E_{\lambda_1}^{1,0} - E_{\lambda_2}^{1,0},$ where $\lambda_2 > \lambda_1 \geq 0.$ Then, by the Hodge decomposition \cite{22, 28}, $\sum_{j=0}^{q} (-1)^{q-j} E_{\lambda}^{1,0}([\lambda_1, \lambda_2])$ is the projection on the range of $\partial_{b,m} E_{\rho(\tilde{x})}^{1,0}([\lambda_1, \lambda_2])$ and thus a positive operator. Hence the $\Gamma$-invariant measure $\sum_{j=0}^{q} (-1)^{q-j} dE_{\lambda}^{1,0}$ is positive on $\{ \lambda > 0 \}.$ It follows that
\[
\left(4.2\right) \quad R := \int_{\lambda > 0} e^{-\frac{t}{m}\lambda} \sum_{j=0}^{q} (-1)^{q-j} dE_{\lambda}^{1,0} \geq 0,
\]
and $R$ commutes with $\Gamma.$ On the other hand,
\[
\left(4.3\right) \quad \text{Tr}_{\Gamma,j}[\exp(-\frac{t}{m}\tilde{\Delta}_{b,m})] \quad = \quad \dim_{\Gamma} \tilde{T}_{b,(2),m}^j(\tilde{X}, \tilde{E}) + \text{Tr}_{\Gamma} \int_{\lambda > 0} e^{-\frac{t}{m}\lambda} dE_{\lambda}^{1,0}.
\]

By (4.2) and (4.3), we obtain the result. \qed
Let \( \text{Tr}_q[\exp(-\frac{t}{m}\Box_{b,m})] \) be the trace of the operator \( \exp(-\frac{t}{m}\Box_{b,m}) \) acting on \( \Omega^0_m(X, E) \). It is well-known that (see Theorem 8.10 in [17])

\[
(4.4) \quad \text{Tr}_q[\exp(-\frac{t}{m}\Box_{b,m})] = \int_X \text{Tr}_q[\exp(-\frac{t}{m}\Box_{b,m})(x,x)] dv_X(x).
\]

By [16, (3.6.7)] and [16, (3.6.8)], as in (4.4), Proposition 4.2.

We have

\[
(4.5) \quad \text{Tr}_{\Gamma,q} \left[ \exp\left(-\frac{t}{m}\tilde{\Box}_{b,m}\right) \right] = \int_U \text{Tr}_q \left[ e^{-\frac{t}{m}\tilde{\Box}_{b,m}(\tilde{x},\tilde{x})} \right] dv_{\tilde{X}}(\tilde{x}).
\]

Now we are in a position to give the heat kernel proof of the Morse inequalities for the Fourier components of reduced \( L^2 \) Kohn-Rossi cohomology.

**Proof of Theorem 1.1** Denote by \( \text{Tr}_{A^0,q} \) the trace on \( T^{*0,q}X \). The basis for \( T^{*0,q}X \) is

\[
\{ \omega^{j_1} \wedge \cdots \wedge \omega^{j_q} : j_1 < \cdots < j_q \}.
\]

We write for the index \((1, \ldots, q)\)

\[
\exp(t\gamma_d)(\omega^1 \wedge \cdots \wedge \omega^q)
\]

\[
= \prod_{j=1}^q (1 + (e^{-ta_j} - 1)\omega^j \wedge t\omega^j)(\omega^1 \wedge \cdots \wedge \omega^q)
\]

\[
= \sum_{k_1 < \cdots < k_q} c_{k_1 \ldots k_q}(x)\omega^{k_1} \wedge \cdots \wedge \omega^{k_q}.
\]

From direct calculations, we see that

\[
(4.8) \quad c_{1 \ldots q}(x) = \exp(-t \sum_{j=1}^q a_j(x)).
\]

Then we have

\[
(4.9) \quad \text{Tr}_{A^0,q}[\exp(t\gamma_d)] = \sum_{j_1 < \cdots < j_q} \exp(-t \sum_{i=1}^q a_{j_i}(x)).
\]

Hence

\[
\lim_{t \to \infty} \frac{\text{Tr}_{A^0,q}[\exp(t\gamma_d)]}{\det(1 - \exp(-t\mathcal{R}))} = \lim_{t \to \infty} \frac{\sum_{j_1 < \cdots < j_q} \exp(-t \sum_{i=1}^q a_{j_i}(x))}{\prod_{j=1}^n (1 - \exp(-ta_j(x)))} = (-1)^q 1_{X(q)},
\]
where the function $X(q)$ is defined by 1 on $X(q)$, 0 otherwise. As usual, for $\bar{x} \in \widetilde{X}$, $\pi(\bar{x}) = x \in X$. It follows from Theorem 3.13, (4.5) and Lemma 4.1 that

\[
\frac{1}{m^n} \sum_{j=0}^{q} (-1)^{q-j} \text{dim}_F \overline{H}_{b,(2),m}(\widetilde{X}, \widetilde{E}) \leq \frac{1}{m^n} \sum_{j=0}^{q} (-1)^{q-j} \text{Tr}_{\Gamma,q}[\exp(-\frac{t}{m} \mathcal{D}_{b,m})] \\
\leq \frac{1}{m^n} \sum_{j=0}^{q} (-1)^{q-j} \int_U \text{Tr}_{\Gamma,q}[\exp(-\frac{t}{m} \mathcal{D}_{b,m}(\bar{x}, \bar{x}))]d\nu(\bar{x}) \\
= \frac{1}{m^n} \sum_{j=0}^{q} (-1)^{q-j} \int_U \text{Tr}_{\Gamma,q}[\exp(-\frac{t}{m} \mathcal{D}_{b,m}(\bar{x}, \bar{x}))]d\nu(\bar{x}) \\
\leq (2\pi)^{-n-1} \sum_{s=1}^{p} e^{\frac{2\pi(n+1)}{p}} \sum_{j=0}^{q} (-1)^{q-j} \int_X \frac{\det(\dot{\mathcal{R}})\text{Tr}_{\Lambda_d}\exp(t\gamma_d) \otimes \text{Id}_X}{\det(1 - \exp(-t\mathcal{R}))}d\nu_X(x) \\
+ \epsilon \sum_{j=0}^{q} (-1)^{q-j} \text{Vol}(X) + C \sum_{j=0}^{q} (-1)^{q-j} \int_X t^{-n}e^{-\frac{\epsilon m d(x, \text{sing})^2}{t}}d\nu_X(x).
\]

Note that $\epsilon$ is arbitrarily small. By the dominant convergence theorem with $t \to \infty$, we have

\[
\lim_{m \to \infty, p|m} \frac{1}{m^n} \sum_{j=0}^{q} (-1)^{q-j} \text{dim}_F \overline{H}_{b,(2),m}(\widetilde{X}, \widetilde{E}) \leq \frac{pp}{(2\pi)^{n+1}} \sum_{j=0}^{q} (-1)^{q-j} \int_{X(j)} |\det(\mathcal{R})|d\nu_X(x),
\]

\[
\lim_{m \to \infty} \frac{1}{m^n} \sum_{j=0}^{q} (-1)^{q-j} \text{dim}_F \overline{H}_{b,(2),m}(\widetilde{X}, \widetilde{E}) = 0, \quad \text{for } p \nmid m.
\]

From Definition 2.7, (3.6) and (4.12), we finally get

\[
\sum_{j=0}^{q} (-1)^{q-j} \text{dim}_F \overline{H}_{b,(2),m}(\widetilde{X}, \widetilde{E}) \leq \frac{ppm^n}{2\pi^{n+1}} \sum_{j=0}^{q} (-1)^{q-j} \int_{X(j)} |\det(\mathcal{L}_x)|d\nu_X(x) + o(m^n), \quad \text{for } p|m,
\]

\[
\sum_{j=0}^{q} (-1)^{q-j} \text{dim}_F \overline{H}_{b,(2),m}(\widetilde{X}, \widetilde{E}) = o(m^n), \quad \text{for } p \nmid m.
\]

Let $q = n$ in (4.11), by applying Theorem 3.13, we obtain for $p|m$,

\[
\frac{1}{m^n} \sum_{j=0}^{n} (-1)^{n-j} \text{dim}_F \overline{H}_{b,(2),m}(\widetilde{X}, \widetilde{E}) \\
\geq \frac{1}{m^n} \sum_{j=0}^{n} (-1)^{n-j} \int_U \text{Tr}_{\Gamma,j}[\exp(-\frac{t}{m} \mathcal{D}_{b,m}(\bar{x}, \bar{x}))]d\nu_X(\bar{x}) \\
\geq (2\pi)^{-n-1} p \frac{1}{m^n} \sum_{j=0}^{n} (-1)^{n-j} \int_X \frac{\det(\dot{\mathcal{R}})\text{Tr}_{\Lambda_d}\exp(t\gamma_d) \otimes \text{Id}_X}{\det(1 - \exp(-t\mathcal{R}))}d\nu_X(x) \\
- cn\text{Vol}(X) - Cn \int_X t^{-n}e^{-\frac{\epsilon m d(x, \text{sing})^2}{t}}d\nu_X(x).
\]

Note that $\epsilon$ is arbitrarily small. By the dominant convergence theorem with $t \to \infty$, we have

\[
\lim_{m \to \infty, p|m} \frac{1}{m^n} \sum_{j=0}^{n} (-1)^{n-j} \text{dim}_F \overline{H}_{b,(2),m}(\widetilde{X}, \widetilde{E}) \geq \frac{pp}{(2\pi)^{n+1}} \sum_{j=0}^{n} (-1)^{n-j} \int_{X(j)} |\det(\mathcal{R})|d\nu_X(x).
\]
Then
\[
\liminf_{m \to \infty, p|m} \frac{1}{m^n} \sum_{j=0}^{n} (-1)^{n-j} \dim \mathcal{H}_{b(2),m}^j (\tilde{X}, \tilde{E}) = \frac{pr}{(2\pi)^{n+1}} \sum_{j=0}^{n} (-1)^{n-j} \int_{X(j)} |\det(\tilde{\mathcal{R}})| d\nu_X(x).
\]
We finally get
\[
\sum_{j=0}^{n} (-1)^{n-j} \dim \mathcal{H}_{b(2),m}^j (\tilde{X}, \tilde{E}) = \frac{prm^n}{2\pi^{n+1}} \sum_{j=0}^{n} (-1)^{n-j} \int_{X(j)} |\det(\mathcal{L}_x)| d\nu_X(x) + o(m^n) \text{ for } p|m.
\]
Then the proof is completed. \qed

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