VANISHING VISCOSITY LIMIT OF THE ROTATING SHALLOW WATER EQUATIONS WITH FAR FIELD VACUUM

ZHIGANG WANG*

School of Mathematics and Statistics, Fuyang Normal College
Fuyang 236037, China
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Abstract. In this paper, we consider the Cauchy problem of the rotating shallow water equations, which has height-dependent viscosities, arbitrarily large initial data and far field vacuum. Firstly, we establish the existence of the unique local regular solution, whose life span is uniformly positive as the viscosity coefficients vanish. Secondly, we prove the well-posedness of the regular solution for the inviscid flow. Finally, we show the convergence rate of the regular solution from the viscous flow to the inviscid flow in $L^\infty((0,T);H^{s'})$ for any $s' \in [2,3)$ with a rate of \( \epsilon^{1-s'}/3 \).

1. Introduction. In this paper, we consider the Cauchy problem of the rotating shallow water equations, which have height-dependent viscosities, arbitrarily large initial data and far field vacuum. The shallow water equations simulate the evolution of an incompressible fluid in regard to gravitational and rotational accelerations. It is also considered as an important extension of the two-dimensional compressible Navier-Stokes equations with additional rotating force, and the solutions present many types of motion. In general, the rotating shallow water equations with the viscous flow have the form

\[
\begin{aligned}
&h_t + \text{div}(hu) = 0, \\
&(hu)_t + \text{div}(hu \otimes u) + gh \nabla h + fh u^\perp = \text{div} \mathbb{T},
\end{aligned}
\]

where \( x = (x_1, x_2) \in \mathbb{R}^2, \ t \geq 0, \ h \) is the height of the fluid surface, \( u = (u^{(1)}, u^{(2)}) \) is the horizontal velocity field, \( u^\perp = (-u^{(2)}, u^{(1)}) \), \( g > 0 \) is the gravity constant, \( f > 0 \) is the Coriolis frequency, and \( \mathbb{T} \) denotes the viscosity stress tensor with the following form

\[
\mathbb{T} = \mu(h)(\nabla u + (\nabla u)^\top) + \lambda(h)\text{div}u\mathbb{I}_2,
\]

where \( \mathbb{I}_2 \) is the 2 \times 2 identity matrix, \( \mu(h) = \epsilon \alpha h \) is the shear viscosity coefficient, \( \lambda(h) = \epsilon \beta h, \mu(h) + \lambda(h) \) is the bulk viscosity coefficient, \( \alpha \) and \( \beta \) are both constants satisfying

\[
\alpha > 0, \quad \alpha + \beta \geq 0.
\]
In addition, when \( \epsilon = 0 \), the system (1) is degenerate to the rotating shallow water equations with the inviscid flow:

\[
\begin{aligned}
    & h_t + \text{div}(hu) = 0, \\
    & (hu)_t + \text{div}(hu \otimes u) + gh \nabla h + fhu^\perp = 0.
\end{aligned}
\]  

(4)

Without loss of generality, we can assume that \( g = 1 \), \( f = 1 \) and \( \epsilon \in (0, 1] \).

We only consider the local classical solution satisfying the initial data

\[
(h, u)|_{t=0} = (h_0(x), u_0(x)),
\]  

(5)

and the far field behavior

\[
(h, u) \to (0, 0) \quad \text{as} \quad |x| \to +\infty, \quad t > 0.
\]  

(6)

Throughout this paper, we will adopt the following simplified notations for Sobolev spaces:

\[
    \|f\|_s = \|f\|_{H^s(\mathbb{R}^2)}, \quad \|f\|_p = \|f\|_{L^p(\mathbb{R}^2)}, \quad \|f\|_{m,p} = \|f\|_{W^{m,p}(\mathbb{R}^2)},
\]

\[
    D^{k,r} = \{ f \in L^1_{\text{loc}}(\mathbb{R}^2) : |f|_{D^k,r} = |\nabla^k f|^r < +\infty \}, \quad \|f\|_{C^k} = \|f\|_{C^k(\mathbb{R}^2)},
\]

\[
    D^k = D^{k,2}, \quad \|f\|_{X,Y} = \|f\|_X + \|f\|_Y, \quad \int_{\mathbb{R}^2} f \, dx = \int f \, dx.
\]

There is a great deal of work studying the shallow water equations. Ton \[18\] used Lagrangian coordinates and Hölder space estimates to study the local existence and uniqueness of classical solutions to the Cauchy-Dirichlet problem. Kloeden \[10\] and Sundbye \[16, 17\] used Sobolev space estimates and the energy method to prove the global well-posedness of classical solutions to the Cauchy-Dirichlet problem and also the Cauchy problem when the initial data are a small perturbation. Wang-Xu \[19\] obtained local solutions and global solutions for small initial data \( h_0 - \bar{h}_0, u_0 \in H^{2+r}(\mathbb{R}^2) \) with \( \bar{h}_0, s > 0 \). Later, Chen-Miao-Zhang \[4\] improved their results to the initial data \( h_0 - \bar{h}_0 \in \dot{B}^{1}_{2,1} \cap \dot{B}^{1}_{2,1}, u_0 \in \dot{B}^{1}_{2,1} \) with \( h \geq \bar{h}_0 > 0 \). Hao-Hsiao-Li \[9\] established the global existence of strong solutions in the space of Besov type when the initial data are close to a positive constant equilibrium state. For arbitrarily large initial data, the global existence of weak solutions in bounded domain with periodic boundary conditions was obtained by Bresch-Desjardins \[1, 2\] and Bresch-Desjardins-Lin \[3\], and the global existence of weak entropy solution for the one-dimensional initial boundary value problem was proved by Li-Li-Xin \[11\]. Later, Guo-Jiu-Xin \[8\] obtained the similar result in \[11\] for the multi-dimensional spherically symmetric weak solutions. Duan-Zheng-Luo \[7\] established a local existence theory of strong solutions for the rotating shallow water equations with constant viscosity coefficients. When viscosity coefficients are height-dependent, Li-Pan-Zhu \[12, 14, 20, 21\] proposed a new quantity \( \nabla \rho/\rho \), which should belong to space \( L^6 \cap D^1 \cap D^2 \), to obtain the local existence of classical solutions with far vacuum. Some other results on degenerate viscosities and initial vacuum can be seen in \[13, 15\]. Recently, via introducing a symmetric structure for the quantity \( \nabla \rho/\rho \) and a “quasi-symmetric hyperbolic”-“elliptic” coupled structure for \( (\rho, u) \), Ding-Zhu \[6\] showed the vanishing viscosity limit of the Navier-Stokes equations to the Euler equations for the compressible fluid with far field vacuum. The motivation of this paper is to establish the vanishing viscosity limit of the rotating shallow water equations from the viscous flow to the inviscid flow.
This paper is organized as follows. In §2, we introduce some definitions of regular solutions and state our main results. In §3, we reformulate problem (1)-(5)-(6) into two coupled symmetric systems, and prove the existence and uniqueness of the regular solution to this reformulated systems. Based on this result, we can obtain the well-posedness of problem (1)-(5)-(6). In §4, we establish the existence of a unique regular solution for problem (4)-(5)-(6). Finally, we show the convergence rate of the regular solution to the rotating shallow water equations from the viscous flow to the inviscid flow in §5.

2. Definition and main results. In this section, we introduce some definitions of regular solutions and state our main results.

Firstly, we denote some notations. For any $2 \times 2$ matrices $M = (m_{ij})_{2 \times 2}$ and $A_i (i = 1, 2)$, vectors $U = (U_1, U_2)$ and $V = (V_1, V_2)$, we set

$$A = (A_1, A_2), \quad \text{div} A = \partial_1 A_1 + \partial_2 A_2, \quad U \cdot MV = \sum_{i,j=1}^{2} U_i m_{ij} V_j,$$

where $\partial_i = \partial_i x (i = 1, 2)$.

We now introduce some definitions of regular solutions. Under the assumption that $h > 0$, (1) can be rewritten into

$$u_t + u \cdot \nabla u + 2\sqrt{h} \nabla \sqrt{h} + \epsilon L(u) + u^\perp = \epsilon \left(\nabla h / h\right) \cdot Q(u),$$

where the so-called Lamé operator $L$ and operator $Q$ are given by

$$\begin{align*}
L(u) &= -\alpha \Delta u - (\alpha + \beta) \nabla \text{div} u, \\
Q(u) &= \alpha (\nabla u + (\nabla u)^T) + \beta \text{div} u I_2.
\end{align*}$$

It is well-known that the quantity $\nabla h / h$ is very important for the analysis on velocity. In order to solve the essential obstacle, Li-Pan-Zhu [12] first found that the quantity $\psi = \nabla \sqrt{h} / \sqrt{h}$ satisfies the following equation

$$\psi_t + \sum_{i=1}^{2} B_i(U) \partial_i \psi + B(U) \psi + \nabla \text{div} u = 0,$$

where $U = (\sqrt{h}, u)$, $B = (\nabla u)^T$, $B_i = (a^{(i)}_{ij})$ and

$$a^{(i)}_{ij} = \begin{cases} u^{(i)}, & i = j, \\ 0, & i \neq j. \end{cases}$$

By (9) and (1), we can obtain two coupled symmetric systems: the symmetric hyperbolic system for the quantity $\psi$ and the symmetric hyperbolic-parabolic coupled system for the quantities $(\sqrt{h}, u)$, see (16). Thus, we can introduce a proper class of solutions called regular solutions to problem (1)-(5)-(6).

Definition 2.1 (Regular solution of problem (1)-(5)-(6)). Assume that $T > 0$ be a finite constant. A solution $(h, u)$ to Cauchy problem (1)-(5)-(6) is called a regular solution in $[0, T] \times \mathbb{R}^2$ if $(h, u)$ satisfies

(A1) $(h, u)$ satisfies Cauchy problem (1)-(5)-(6) in the sense of distributions;

(B1) $h > 0, \quad \inf h = 0, \quad \sqrt{h} \in C([0, T]; H^3(\mathbb{R}^2));$
Definition 2.2 (Regular solution of problem (4)-(5)-(6)). Assume that $T > 0$ be a finite constant. A solution $(h, u)$ to Cauchy problem (4)-(5)-(6) is called a regular solution in $[0, T] \times \mathbb{R}^2$ if $(h, u)$ satisfies

(A2) $(h, u)$ satisfies Cauchy problem (4)-(5)-(6) in the sense of distributions;

(B2) $h > 0$, $\inf h = 0$, $\sqrt{h} \in C([0, T]; H^3(\mathbb{R}^2))$;

(C2) $u \in C([0, T]; H^3(\mathbb{R}^2))$;

(D2) $\lim_{|x| \to \infty} \left( u_t + u \cdot \nabla u + u^\perp \right) = 0$.

The condition (B1) and (B2) imply that the regular solutions do not contain vacuum in any local point, but have vacuum at infinity. When the density approaches to vacuum, the velocity is determined by the condition (E1) and (D2), which is a physically reasonable way.

Before considering the limit of the regular solution from the viscous flow to the inviscid flow, we first prove the following uniform local-in-time well-posedness of problem (1)-(5)-(6) independent of $\epsilon$.

Theorem 2.3. If the initial data $(h_0, u_0)$ satisfy

$$h_0 > 0, \quad (\sqrt{h_0}, u_0) \in H^3, \quad \nabla h_0 / h_0 \in L^6 \cap D^1 \cap D^2,$$

then there exists a time $T_* > 0$ independent of $\epsilon$, such that there is a unique regular solution $(h, u)$ on $[0, T_*] \times \mathbb{R}^2$ to problem (1)-(5)-(6).

Moreover, there exists a constant $E_0$ (depending only on $T_*, \alpha, \beta, h_0, u_0$) such that

$$\sup_{0 \leq t \leq T_*} \left( \left\| (\sqrt{h}, u)(t) \right\|_{L^3}^2 + \left\| \nabla h/h(t) \right\|_{L^2 \cap D^1}^2 + \epsilon \left\| \nabla h/h(t) \right\|_{D^2}^2 \right)$$

$$+ \epsilon \int_0^{T_*} \| u(t) \|_{D^1}^2 \, dt \leq E_0^2. \tag{11}$$

Based on the uniform existence in Theorem 2.3, we can establish the following existence of the regular solution for problem (4)-(5)-(6).

Theorem 2.4. If the initial data $(h_0, u_0)$ satisfy (10), then there exists a time $T_* > 0$, such that there is a unique regular solution $(h, u)$ on $[0, T_*] \times \mathbb{R}^2$ to problem (4)-(5)-(6).

Moreover, we can show the following vanishing viscosity convergence of the regular solution form the viscous flow to the inviscid flow.

Theorem 2.5. Assume that $(h^\epsilon, u^\epsilon)$ is the regular solution to problem (1)-(5)-(6) obtained in Theorem 2.3 and $(h, u)$ is the regular solution to problem (4)-(5)-(6) obtained in Theorem 2.4. If

$$(h^\epsilon, u^\epsilon)|_{\epsilon=0} = (h, u)|_{\epsilon=0} = (h_0, u_0) \tag{12}$$

holds, then $(h^\epsilon, u^\epsilon)$ converges strongly to $(h, u)$ as $\epsilon \to 0$ in $L^p(0, T_*; L^q(\mathbb{R}^2))$ for any $1 \leq p, q \leq \infty$.
satisfy (10), then when \(\epsilon \to 0\), \((h^\prime, u^\prime)\) converges to \((h, u)\) in the following sense

\[
\lim_{\epsilon \to 0} \sup_{0 \leq t \leq T_*} \left( \left\| (\sqrt{h^\prime} - \sqrt{h}(t)) \right\|_{s'} + \left\| (u^\prime - u)(t) \right\|_{s'} \right) = 0, \tag{13}
\]

for any constant \(s' \in [0, 3)\). Moreover, we also have

\[
\sup_{0 \leq t \leq T_*} \left( \left\| (h^\prime - h)(t) \right\|_1 + \left\| (u^\prime - u)(t) \right\|_1 \right) \leq C \epsilon, \tag{14}
\]

\[
\sup_{0 \leq t \leq T_*} \left( \left\| (h^\prime - h)(t) \right\|_{D^2} + \left\| (u^\prime - u)(t) \right\|_{D^2} \right) \leq C \sqrt{\epsilon},
\]

where \(C > 0\) is a constant depending only on the quantities \(T_*, \alpha, \beta, h_0, u_0\).

3. Well-posedness of regular solutions for the viscous flow. In this section, we first reformulate the rotating shallow water equations with the viscous flow into two coupled symmetric systems, and prove the existence and uniqueness of the regular solution to this reformulated system, whose life span is uniformly positive as the viscosity coefficients go to zero. Based on the well-posedness of this reformulated system, we establish the well-posedness of regular solutions for the viscous flow (Theorem 2.3).

3.1. Reformulation. Set

\[
\phi = \sqrt{h}, \psi = \nabla h \cdot h = 2 \nabla \phi / \phi = (\psi^{(1)}, \psi^{(2)})^\top, \text{ and } U = (\phi, u), \tag{15}
\]

we can rewrite system (1) into the following nonlinear symmetric form:

\[
\begin{align*}
\psi_t + \sum_{l=1}^{2} B_l(U) \partial_t \psi + B(U) \psi + \nabla \text{div}u &= 0, \\
A_0 U_t + \sum_{l=1}^{2} A_l(U) \partial_t U &= -\epsilon F(u) + \epsilon G(\psi, u) - H(u),
\end{align*}
\tag{16}
\]

where

\[
A_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad A_1 = \begin{pmatrix} u^{(1)} & \phi & 0 \\ \phi & u^{(1)} & 0 \\ 0 & 0 & u^{(1)} \end{pmatrix}, \quad A_2 = \begin{pmatrix} u^{(2)} & 0 & \phi \\ 0 & u^{(2)} & 0 \\ \phi & 0 & u^{(2)} \end{pmatrix},
\]

\[
F(u) = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{4} L(u) \end{pmatrix}, G(\psi, u) = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{4} \psi \cdot Q(u) \end{pmatrix}, H(u) = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{4} u^\perp \end{pmatrix}.
\]

In addition, \((\psi, \phi, u)\) satisfies the following initial data

\[
(\psi, \phi, u)(t, x)|_{t=0} = \left( \nabla h_0 / h_0, \sqrt{h_0}, u_0 \right)(x), \quad x \in \mathbb{R}^2, \tag{17}
\]

and the following far field behavior:

\[
(\psi, \phi, u) \to (0, 0, 0) \quad \text{as} \quad |x| \to +\infty, \quad t > 0. \tag{18}
\]

In order to prove Theorem 2.3, we first establish the existence and some uniform estimates of regular solutions for the reformulated problem (16)-(18):

**Theorem 3.1.** If the initial data \((\psi_0, h_0, u_0)\) satisfy

\[
\psi_0 \in L^6 \cap D^1 \cap D^2, \quad h_0 > 0, \quad (\sqrt{h_0}, u_0) \in H^3, \tag{19}
\]
then there exists a time $T_* > 0$ independent of $\epsilon$, and a unique classical solution $(\psi, \phi, u)$ to problem (16)-(18) satisfying
\begin{align*}
\psi &\in C([0, T_*]; L^6 \cap D^1 \cap D^2), \quad \psi_t \in C([0, T_*]; H^1), \\
\phi &\in C([0, T_*]; H^3), \quad \phi_t \in C([0, T_*]; H^2), \\
u &\in C([0, T_*]; H^3) \cap L^2([0, T_*]; H^4), \quad \nu_t \in C([0, T_*]; H^1) \cap L^2([0, T_*]; D^2).
\end{align*}

Moreover, there exists a constant $E_0$ depending only on the quantities $T_*, \alpha, \beta, h_0, u_0$ such that
\begin{align*}
\sup_{0 \leq t \leq T_*} \left( \|\psi(t)\|_{L^6 \cap D^1}^2 + \epsilon \|\psi(t)\|_{D^2}^2 + \|\phi(t)\|_{H^3}^2 + \|u(t)\|_{H^1}^2 \right) + \epsilon \int_0^{T_*} \|u(t)\|_{D^2}^2 dt \leq E_0^2.
\end{align*}

3.2. Linearization. By the linearization for problem (16)-(18), we can obtain the following linearized problem
\begin{align*}
\begin{cases}
\psi_t + \sum_{l=1}^2 B_l(V) \partial_t \psi + B(V) \psi + \nabla \text{div} v = 0, \\
A_0 u_t + \sum_{l=1}^2 A_l(V) \partial_t u = -\epsilon F(u) + \epsilon G(\psi, v) - H(v), \\
(\psi, u)|_{t=0} = (\psi_0, u_0), \quad x \in \mathbb{R}^2, \\
(\psi, u) \to (0, 0) \quad \text{as} \ |x| \to \infty, \ t > 0,
\end{cases}
\end{align*}

where $V = (V_1, V_2, V_3) = (\varphi, v)$, $\varphi$ is a known scalar function and $v = (v^{(1)}, v^{(2)}) \in \mathbb{R}^2$ is a known vector satisfying $(\varphi, v)(t = 0, x) = (\phi_0(x), u_0(x))$, and
\begin{align*}
\varphi &> 0, \quad \varphi \in C([0, T]; H^3), \quad \varphi_t \in C([0, T]; H^2), \\
v &\in C([0, T]; H^3) \cap L^2([0, T]; H^4), \quad v_t \in C([0, T]; H^1) \cap L^2([0, T]; H^2).
\end{align*}

We now prove the global existence of classical solutions to the linear problem (22).

Lemma 3.2. Suppose that the initial data $(\psi_0, u_0)$ satisfy (19). Then for any $T > 0$, there exists a unique classical solution $(\psi, U)$ to problem (22) satisfying
\begin{align*}
\psi &\in C([0, T]; L^6 \cap D^1 \cap D^2), \quad \psi_t \in C([0, T]; H^1), \\
\phi &\in C([0, T]; H^3), \quad \phi_t \in C([0, T]; H^2), \\
u &\in C([0, T]; H^3) \cap L^2([0, T]; H^4), \quad \nu_t \in C([0, T]; H^1) \cap L^2([0, T]; H^2).
\end{align*}

The proof of this lemma could be obtained by the standard hyperbolic and parabolic theory, see [5]. Here we omit its details.

3.3. The a priori estimates independent of $\epsilon$. In this subsection, we will establish a priori estimates for $(\psi, U)$, and in particular, some of the estimates are independent of $\epsilon$.

We first chose a constant $T > 0$ and a positive constant $c_0$ large enough such that
\begin{align*}
1 + \|\psi_0\|_{L^6 \cap D^1 \cap D^2} + \|\phi_0\|_3 + \|u_0\|_3 \leq c_0,
\end{align*}

\begin{align*}
1 + \|\psi_0\|_{L^6 \cap D^1 \cap D^2} + \|\phi_0\|_3 + \|u_0\|_3 \leq c_0,
\end{align*}
and
\[
\sup_{0 \leq t \leq T^*} \left( \|\varphi(t)\|_3^3 + \|v(t)\|_3^2 \right) + \int_0^{T^*} \epsilon|\nabla^4 v(t)|_2^2 dt \leq c_1^2, \tag{26}
\]
for some fixed constants \(T^* \in (0, T)\) and \(c_1\) satisfying \(1 \leq c_0 \leq c_1\). Here, \(T^*\) and \(c_1\) depend only on the quantities \(T_*, \alpha, \beta, c_0\) and the initial data \(h_0, u_0\), and are determined later, see (39).

In the following, we will prove a series of uniform local (in time) estimates given in Lemmas 3.3–3.4. Hereinafter, we denote a generic positive constant by \(C \geq 1\), which is only dependent of the quantities \(T_*, \alpha, \beta, c_0\) and may be different from line to line.

From [6], we know that the function \(\psi\) satisfies the following estimates.

\textbf{Lemma 3.3.} Let \((\psi, \phi, u)\) be the unique classical solution to problem (22) on \([0, T] \times \mathbb{R}^2\). Then
\[
|\psi(t)|_0^2 + |\psi(t)|_{L^2}^2 + \epsilon|\psi(t)|_{L^2}^2 + \epsilon^\alpha_\epsilon |\psi(t)|_\infty^2 \leq C c_0^2,
\]
for \(0 \leq t \leq T_1 \equiv \min(T^*, (1 + c_1)^{-1}).

Next, we prove a priori estimates for \(U\), some of which are independent of \(\epsilon\).

\textbf{Lemma 3.4.} Let \((\psi, \phi, u)\) be the unique classical solution to problem (22) on \([0, T] \times \mathbb{R}^2\). Then
\[
\|U(t)\|_3^2 + \int_0^t \epsilon|u|_{L^2}^2 ds \leq C c_0^2,
\]
for \(0 \leq t \leq T_2 \equiv \min(T^*, (1 + c_1^4))^{-1}).

\textbf{Proof.} Applying \(\partial_x^2\) to (22), multiplying by \(2\partial_x^2 U\) and integrating over \(\mathbb{R}^2\), we have
\[
\frac{d}{dt} \int \partial_x^2 U \cdot A_0 \partial_x^2 U dx + \frac{1}{2} \int \left( |\nabla \partial_x^2 u|^2 + (\alpha + \beta) |\text{div} \partial_x^2 u|^2 \right) dx
\]
\[
= \int \text{div} A |\partial_x^2 U|^2 dx - 2 \sum_{i=1}^2 \left( \partial_x^2 (A_i \partial_i U) - A_i \partial_x^2 \partial_i U \right) \cdot \partial_x^2 U dx
\]
\[
+ \frac{1}{2} \int \partial_x^2 (\psi \cdot Q(v)) \cdot \partial_x^2 u dx - \frac{1}{2} \int \partial_x^2 v \cdot \partial_x^2 u dx \equiv I_1 + I_2.
\]

We consider the terms on the right-hand side of (29) as \(|\zeta| \leq 3\). For the term \(I_1\), we have
\[
I_1 = \int \text{div} A |\partial_x^2 U|^2 dx \leq C |\text{div} A|_{L^\infty} |\partial_x^2 U|_{L^2}^2 \leq C c_1 |\partial_x^2 U|_{L^2}^2 \quad \text{for} \quad |\zeta| \leq 3.
\]
For the term \(I_2\), we have
\[
I_2 = -2 \sum_{i=1}^2 \int (\partial_x^2 (A_i \partial_i U) - A_i \partial_x^2 \partial_i U) \cdot \partial_x^2 U dx
\]
\[
\leq C |\nabla V|_{L^\infty} |\nabla U|_{L^2}^2 \leq C c_1 |\nabla U|_{L^2}^2 \quad \text{when} \quad |\zeta| = 1;
\]
\[
I_2 \leq C (|\nabla V|_{L^\infty} |\nabla^2 U|_{L^2} + |\nabla^2 V|_{L^\infty} |\nabla^2 U|_{L^2}) \leq C c_1 |\nabla U|_{L^2}^2 \quad \text{when} \quad |\zeta| = 2;
\]
\[
I_2 \leq C (|\nabla V|_{L^\infty} |\nabla^3 U|_{L^2} + |\nabla^3 V|_{L^\infty} |\nabla^3 U|_{L^2}) \leq C c_1 |\nabla U|_{L^2}^2 \quad \text{when} \quad |\zeta| = 3.
\]
For the term $I_3$, we have
\[
I_3 = \frac{1}{2} \epsilon \int \frac{\partial_x^2 \left( \psi \cdot Q(v) \right)}{\partial_x^2} \cdot \partial_x^2 u \, dx
\]
\[
\leq C\epsilon \left( |\psi| |\nabla^2 v|_3 + |\nabla \psi|_3 |\nabla^2 v|_6 \right) \leq Cc^3 |\nabla u|_2 \quad \text{when} \quad |\zeta| = 1; \quad (32)
\]
\[
I_3 \leq C\epsilon \left( |\psi|_\infty |\nabla^2 v|_2 + |\nabla \psi|_3 |\nabla^2 v|_6 + |\nabla^2 v|_2 |\nabla v|_\infty \right) \leq Cc^3 |\nabla^2 u|_2 \quad \text{when} \quad |\zeta| = 2;
\]
On the other hand, when $|\zeta| = 3$, we need to consider the term $I_3$ carefully. Due to
\[
\partial_x^2 \left( \psi \cdot Q(V) \right) = \psi \cdot \partial_x^2 Q(V) + \sum_{i=1}^{3} C_{i1} \partial_x^{\zeta_i} \psi \cdot \partial_x^{\zeta_i} Q(V)
\]
\[
+ \sum_{i=1}^{3} C_{i2} \partial_x^{\zeta_i} \psi \cdot \partial_x^{\zeta_i} Q(V) + \partial_x^2 \psi \cdot Q(V),
\]
where $\zeta = \zeta^1 + \zeta^2 + \zeta^3$, $\zeta^i \in \mathbb{R}^2$ is a multi-index satisfying $|\zeta^i| = 1$ ($i = 1, 2, 3$), we have
\[
I_{32} = \frac{1}{2} \epsilon \int \sum_{i=1}^{3} C_{i1} \left( \partial_x^{\zeta_i} \psi \partial_x^{\zeta_i} Q(V) \right) \cdot \partial_x^2 u \, dx
\]
\[
\leq C\epsilon \left( |\nabla \psi|_3 |\nabla^3 v|_3 \right) \leq C\epsilon \leq C\epsilon \left( |\nabla^3 u|_2 \right)^\frac{3}{2}
\]
\[
\leq \frac{\epsilon \alpha}{16} |\nabla^4 u|_2^2 + C|\nabla^3 u|_2 + Cc^4,
\]
\[
I_{33} = \frac{1}{2} \epsilon \int \sum_{i=1}^{3} C_{i2} \left( \partial_x^{\zeta_i} \psi \partial_x^{\zeta_i} Q(V) \right) \cdot \partial_x^2 u \, dx
\]
\[
\leq C\epsilon |\nabla^2 \psi|_2 |\nabla^3 v|_3 \leq \frac{\epsilon \alpha}{16} |\nabla^4 u|_2^2 + C|\nabla^3 u|_2 + Cc^4,
\]
where we have used Lemma 3.2. Via integration by parts, it is easy to see that
\[
I_{31} = \frac{1}{2} \epsilon \int \left( \psi \cdot \partial_x^2 Q(V) \right) \cdot \partial_x^2 u \, dx
\]
\[
\leq C\epsilon \left( |\nabla \psi|_3 |\nabla^3 v|_2 \right) \leq \frac{\epsilon \alpha}{16} |\nabla^4 u|_2^2 + C|\nabla^3 u|_2 + Cc^4,
\]
and
\[
I_{34} = \frac{1}{2} \epsilon \int \left( \partial_x^2 \psi \cdot Q(V) \right) \cdot \partial_x^2 u \, dx
\]
\[
\leq C\epsilon |\nabla^2 \psi|_2 \left( |\nabla^3 v|_3 \right) \leq \frac{\epsilon \alpha}{16} |\nabla^4 u|_2^2 + C|\nabla^3 u|_2 + Cc^4.
\]
For the term $I_4$, we have
\[
I_4 = -\frac{1}{2} \int \partial_x^2 v^4 \cdot \partial_x^2 u \, dx \leq C|\partial_x v|_2 |\partial_x^2 u|_2 \leq |\partial_x^2 U|_2^2 + Cc^2.
\]
From (29) and (30)–(36), along with the Gronwall’s inequality, we have
\[
\|U(t)\|_2^2 + \epsilon \int_0^t |u|_D^2 \, ds \leq \left( \|U_0\|_2^2 + Cc^2 t \right) \exp \left( Ct \right) \leq Cc_0^2,
\]
for $0 \leq t \leq T_2$. 

Step 2. We consider the estimates for $U_1$. From (22), we have
\[ \|\phi_1\|_2 \leq C(\|\varphi\|_2 + \|\phi_0\|_2) \leq C_1, \]
\[ \|u_1\|_2 \leq C(\|\varphi\|_2 + \|\phi_0\|_2 + \|\psi\|_\infty \leq C_1, \]
\[ \|u_1\|_{D_2} \leq C(\|\varphi\|_2 + \|\phi_0\|_2 + \|\psi\|_\infty \leq C_1, \]
\[ + C\|\nabla^2 u_1\|_2 + C(\|\varphi\|_2 + \|\phi_0\|_2 + \|\psi\|_\infty \leq C_1, \]
and
\[ \|u_1\|_{D_2} \leq C(\|\varphi\|_2 + \|\phi_0\|_2 + \|\psi\|_\infty \leq C_1, \]
\[ + C\|\nabla^2 u_1\|_2 + C(\|\varphi\|_2 + \|\phi_0\|_2 + \|\psi\|_\infty \leq C_1, \]
which immediately implies that for $0 \leq t \leq T_2$
\[ \int_0^t \|u_1\|_{D_2}^2 dt \leq C \int_0^t (\epsilon^2 + c_1) dt \leq C_2. \]
Therefore, the proof of this lemma is completed. \qed

From Lemmas 3.3--3.4, for $0 \leq t \leq T_2$, we have
\[ \sup_{0 \leq t \leq T_2} \left( \|\phi\|_{L^\infty \cap D_1}^2 + \|U\|_3^2 + \|\psi\|_{D_2}^2 \right)(t) + \int_0^{T_2} \epsilon \|u\|_{D_1}^2 dt \leq C_2, \]
\[ \sup_{0 \leq t \leq T_2} \left( \|U\|_3^2 + \|\phi\|_{D_2}^2 + \|\psi\|_{D_2}^2 \right)(t) + \int_0^{T_2} \epsilon \|u\|_{D_2}^2 dt \leq C_4. \]
Therefore, if we define the constants $c_1$ and $T^*$ by
\[ c_1 \equiv C_2, \quad \text{and} \quad T^* \equiv \min(T, (1 + c_1)^{-1}), \]
then we deduce that
\[ \sup_{0 \leq t \leq T^*} \left( \|\phi\|_{L^\infty \cap D_1}^2 + \|U\|_3^2 + \|\psi\|_{D_2}^2 \right)(t) + \int_0^{T^*} \epsilon \|u\|_{D_1}^2 dt \leq c_1, \]
\[ \sup_{0 \leq t \leq T^*} \left( \|U\|_3^2 + \|\phi\|_{D_2}^2 + \|\psi\|_{D_2}^2 \right)(t) + \int_0^{T^*} \epsilon \|u\|_{D_2}^2 dt \leq C_1. \]

3.4. Proof of Theorem 3.1. We will use the classical iteration scheme and the existence results in §3.3.3 to prove Theorem 3.1. As shown in §3.3, we chose the constants $c_0$, $c_1$ and suppose that
\[ 1 + \|\phi_0\|_{L^\infty \cap D_1 \cap D_2} + \|\phi_0, u_0\|_3 \leq c_0. \]
Assume that $\phi^0(t, x) \in C([0, T^*]; H^3)$ and $u^0(t, x) \in C([0, T^*]; H^3) \cap L^2([0, T^*]; H^4)$ is the solution of the following Cauchy problem:
\[
\begin{cases}
X_t - Y \cdot \nabla X = 0 & \text{in } (0, +\infty) \times \mathbb{R}^2, \\
Y_t - \nabla Y = 0 & \text{in } (0, +\infty) \times \mathbb{R}^2, \\
(X, Y)(0, x) = (\phi_0(x), u_0(x)) & \text{in } \mathbb{R}^2, \\
(X, Y)(t, x) & \text{as } |x| \to \infty, t > 0.
\end{cases}
\]
Therefore, there exists a time $T^{**} \in (0, T^*]$ such that $(\phi^0, u^0)$ satisfies
\[ \sup_{0 \leq t \leq T^{**}} \left( \|\phi^0(t)\|_3^2 + \|u^0(t)\|_3^2 \right) + \epsilon \int_0^{T^{**}} \|u^0(t)\|_{D_1}^2 dt \leq c_1. \]
Our proof is divided into two steps.

**Step 1.** Existence of classical solutions. Set $V = (\phi^0, u^0)$, we can obtain a classical solution of problem (22) denoted by $(\psi^1, \phi^1, u^1)$. Inductively, we can construct approximate sequences $(\psi^{k+1}, U^{k+1} = (\phi^{k+1}, u^{k+1}))$, which are the solutions of the following problem:

$$
\begin{align*}
\psi^{k+1} &+ \sum_{l=1}^{2} B_l(U^k) \partial_t \psi^{k+1} + B(U^k) \psi^{k+1} + \nabla \text{div} u^k = 0, \\
A_0 U_t^{k+1} &+ \sum_{l=1}^{2} A_l(U^k) \partial_t U^{k+1} = -\epsilon F(u^{k+1}) + \epsilon G(\psi^{k+1}, u^k) - H(u^k), \\
(\psi^{k+1}, U^{k+1})(t = 0, x) &= (\psi^0_k(x), U^k_0(x)), \quad x \in \mathbb{R}^2, \\
(\psi^{k+1}, U^{k+1})(t, x) &\to (0, 0) \quad \text{as} \quad |x| \to \infty, \quad t > 0.
\end{align*}
$$

(42)

It is obvious that $(\psi^k, \phi^k, u^k) \ (k = 1, 2, \ldots)$ satisfy the uniform estimates in (40) with respect to $k \geq 1$. We next prove that the whole sequence $(\psi^k, \phi^k, u^k)$ converges strongly to a limit $(\psi, \phi, u)$ in some sense. Set

$$
\bar{\psi}^{k+1} = \psi^{k+1} - \psi^k, \quad \bar{\phi}^{k+1} = \phi^{k+1} - \phi^k, \quad \bar{u}^{k+1} = u^{k+1} - u^k, \quad \overline{U}^{k+1} = (\bar{\phi}^{k+1}, \bar{u}^{k+1}),
$$

then from (42), we have

$$
\begin{align*}
\bar{\psi}_t^{k+1} &+ \sum_{l=1}^{2} B_l(u^k) \partial_t \bar{\psi}^{k+1} + B(u^k) \bar{\psi}^{k+1} + \nabla \text{div} \bar{u}^k = \Upsilon_1^k + \Upsilon_2^k, \\
A_0 \overline{U}_t^{k+1} &+ \sum_{l=1}^{2} A_l(U^k) \partial_t \overline{U}^{k+1} + \epsilon F(\bar{u}^{k+1}) + H(\bar{u}^{k+1}) \\
&= -2 \sum_{l=1}^{2} A_l(\overline{U}^k) \partial_t U^k + \epsilon G(\bar{\psi}^{k+1}, u^{k-1}) + \epsilon G(\psi^{k+1}, \bar{u}^k),
\end{align*}
$$

(43)

where $\Upsilon_1^k$ and $\Upsilon_2^k$ are defined by

$$
\Upsilon_1^k = -2 \sum_{l=1}^{2} (B_l(u^k) - B_l(u^{k-1})) \partial_t \psi^k, \quad \Upsilon_2^k = -(B(u^k) - B(u^{k-1})) \psi^k.
$$

Firstly, multiplying (43)$_1$ by $2\bar{\psi}^{k+1}$ and integrating over $\mathbb{R}^2$, we have

$$
\frac{d}{dt} |\bar{\psi}^{k+1}|^2 \leq \left( \sum_{l=1}^{2} |\partial_t B_l(u^k)|_\infty + 2|B(u^k)|_\infty \right) |\bar{\psi}^{k+1}|^2 \\
+ 2(|\Upsilon_1^k|_2 + |\Upsilon_2^k|_2 + |\nabla^2 \bar{u}^k|_2) |\bar{\psi}^{k+1}|_2 \leq C \eta^{-1} |\bar{\psi}^{k+1}|^2 + \eta |\bar{u}^k|^2,
$$

(44)

where $\eta \in (0, 1/10)$ is a positive constant and will be determined later.

Secondly, multiplying (43)$_2$ by $2\overline{U}^{k+1}$ and integrating over $\mathbb{R}^2$, we can derive that

$$
\frac{d}{dt} \int \overline{U}^{k+1} \cdot A_0 \overline{U}^{k+1} \, dx + \frac{1}{2} \int \left( \alpha |\nabla \bar{u}^{k+1}|^2 + (\alpha + \beta) |\text{div} \bar{u}^{k+1}|^2 \right) \, dx,
$$
We can observe that the time derivative of the energy is bounded by

\[ \frac{d}{dt} \int \text{div}(A(U^k_0)) |\nabla U^{k+1}|^2 \, dx - 2 \sum_{l=1}^2 \int \left( A_l(U^k_0) \partial_l U^k_0 \right) \cdot \nabla U^{k+1} \, dx \\
+ \frac{1}{2} \epsilon \int \left( \nabla \psi^{k+1}_0 \cdot Q(u^{k-1}_0) + \psi^{k+1}_0 \cdot Q(u^{k+1}_0) \right) \cdot \nabla U^{k+1} \, dx - \frac{1}{2} \int (\tilde{U}^{k+1}_0)^{-1} \tilde{U}^{k+1} \, dx \]  

Finally, applying the operator \( \partial^c \) \((\zeta = 1)\) to (43), multiplying by \( 2 \partial^c \nabla U^{k+1} \) and integrating over \( \mathbb{R}^2 \), via integration by parts, we can obtain

\begin{align*}
\frac{d}{dt} \int \partial^c \nabla U^{k+1} \cdot A_0 \partial^c U^{k+1} \, dx + \frac{1}{2} \epsilon \int \left( |\nabla \partial^c_0 U^{k+1}|^2 + (\alpha + \beta) |\text{div} \partial^c_0 U^{k+1}|^2 \right) \, dx \\
= \int \text{div}(A(U^k_0)) \partial^c_0 |\nabla U^{k+1}|^2 \, dx - 2 \sum_{l=1}^2 \int \partial^c_0 \left( A_l(U^k_0) \partial_l U^k_0 \right) \cdot \partial^c_0 \nabla U^{k+1} \, dx \\
+ \int 2 \sum_{l=1}^2 \left( A_l(U^k_0) \partial_l \partial^c_0 U^{k+1} - \partial^c_0 \left( A_l(U^k_0) \partial_l U^k_0 \right) \right) \cdot \partial^c_0 \nabla U^{k+1} \, dx \\
+ \frac{1}{2} \epsilon \int \partial^c_0 \left( (\nabla)^4 + \epsilon \psi^{k+1}_0 \cdot Q(u^{k-1}_0) + \epsilon \psi^{k+1}_0 \cdot Q(u^{k+1}_0) \right) \cdot \partial^c_0 \nabla U^{k+1} \, dx
\end{align*}

Combining (45)-(46) and Young’s inequality, it is easy to get

\begin{align*}
\frac{d}{dt} \|\nabla U^{k+1}\|_1^2 + \epsilon \alpha \|\nabla u^{k+1}\|_1^2 \\
\leq \Theta^k_n(t) \|\nabla U^{k+1}\|_1^2 + \Theta^k_2(t) \|\psi^{k+1}\|_1^2 + \Theta^k_3(t) \epsilon \|\psi^{k+1}\|_2^2 + C \eta \|\nabla U^k\|_1^2 + C \epsilon \|\nabla^2 u^k\|_2^2
\end{align*}

and

\[ \int_0^t \left( \Theta^k_n(s) + \Theta^k_2(s) + \Theta^k_3(s) \right) ds \leq C + C \eta t, \quad \text{for} \ t \in (0, T^*) . \]

Finally, let

\[ \Gamma^{k+1}(t) = \sup_{0 \leq s \leq t} \epsilon \|\psi^{k+1}(s)\|_2^2 + \sup_{0 \leq s \leq t} \|\psi^{k+1}(s)\|_1^2 + \sup_{0 \leq s \leq t} \|\nabla U^{k+1}(s)\|_1^2 . \]

According to (44), (47) and Gronwall’s inequality, we get

\[ \Gamma^{k+1}(t) + \epsilon \alpha \int_0^t \|\nabla U^{k+1}\|_1^2 ds \\
\leq C \eta \int_0^t \|\nabla^2 u^k\|_2^2 ds + t \sup_{0 \leq s \leq t} \Gamma^k(t) \exp\left( C + C \eta t \right) . \]

We choose \( \eta > 0 \) and \( T_* \in (0, T^*) \) small enough such that

\[ C \eta \exp C \leq \min \left( \frac{1}{4}, \frac{\alpha}{4} \right) , \quad \text{and} \ \exp(C \eta T_*) \leq 2 . \]

We can observe that the time \( T_* > 0 \) is independent of \( \epsilon \).
Then, we can derive that
\[
\sum_{k=1}^{\infty} \left( \Gamma^{k+1}(T_*) + \int_0^{T_*} \epsilon \alpha \| \nabla u^{k+1} \|_1^2 ds \right) \leq C < +\infty.
\]
By virtue of
\[
\lim_{k \to +\infty} |\tilde{\psi}^{k+1}|_6 \leq C \lim_{k \to +\infty} \left( |\tilde{\psi}^{k+1}|_3 |\tilde{\psi}^{k+1}|_2^{3/4} \right) \leq C \lim_{k \to +\infty} |\tilde{u}^{k+1}|_2^{3/4} = 0,
\]
we can deduce that the whole sequence \((\psi^k, \phi^k, u^k)\) converges to a limit \((\psi, \phi, u)\) in the following strong sense:
\[
\psi^k \to \psi \text{ in } L^\infty([0, T_*]; L^6(\mathbb{R}^2)), \\
\phi^k \to \phi \text{ in } L^\infty([0, T_*]; H^1(\mathbb{R}^2)), \\
u^k \to u \text{ in } L^\infty([0, T_*]; H^1(\mathbb{R}^2)) \cap L^2([0, T_*]; D^2(\mathbb{R}^2)).
\]
By the weak lower semi-continuity of norms, it is easy to see that \((\psi, \phi, u)\) satisfies the estimates in (40). Therefore, \((\psi, \phi, u)\) is a weak solution of problem (16)-(18) with the following regularities:
\[
\psi \in L^\infty([0, T_*]; L^6 \cap D^1 \cap D^2), \quad \partial_t \psi^{(j)} = \partial_i \psi^{(i)} (i, j = 1, 2), \\
\psi_1 \in L^\infty([0, T_*]; H^1), \quad \phi \in L^\infty([0, T_*]; H^2), \quad \phi_1 \in L^\infty([0, T_*]; H^3), \\
u_1 \in C([0, T_*]; H^2) \cap L^2([0, T_*]; H^4), \quad u_1 \in L^\infty([0, T_*]; H^1) \cap L^2([0, T_*]; D^2).
\]

**Step 2.** Uniqueness of classical solutions. Assume that \((\psi_1, \phi_1, u_1)\) and \((\psi_2, \phi_2, u_2)\) are two solutions for Cauchy problem (16)-(18). Set
\[
\tilde{\psi} = \psi_1 - \psi_2, \quad \tilde{\phi} = \phi_1 - \phi_2, \quad \tilde{u} = u_1 - u_2, \quad \tilde{U} = U_1 - U_2.
\]
then, \((\tilde{\psi}, \tilde{\phi}, \tilde{u})\) satisfies the following problem
\[
\begin{aligned}
\tilde{\psi}_t + \sum_{l=1}^{2} B_l(u_1) \partial_t \tilde{\psi} + B(u_1) \tilde{\psi} + \nabla \text{div} \tilde{u} &= \tilde{Y}_1 + \tilde{Y}_2, \\
A_0 \tilde{U}_t + \sum_{l=1}^{2} A_l(U_1) \partial_t \tilde{U} + \epsilon F(\tilde{u}) + H(\tilde{u}) &= -2 \sum_{l=1}^{2} A_l(U) \partial_t U_2 + \epsilon G(\tilde{\psi}, u_2) + \epsilon G(\psi_1, \tilde{u}), \\
(\tilde{\psi}, \tilde{U})(t = 0, x) &= (0, 0), \quad x \in \mathbb{R}^2,
\end{aligned}
\]
where
\[
\tilde{Y}_1 = -\sum_{l=1}^{2} (B_l(u_1) - B_l(u_2)) \partial_t \psi_2, \quad \tilde{Y}_2 = -(B(u_1) - B(u_2)) \psi_2.
\]
Set \(\Phi(t) = \|\tilde{\psi}(t)\|_2^2 + \|\tilde{\phi}(t)\|_1^2 + \|\tilde{u}(t)\|_2^2\). Similar as the derivation of (44) and (47), it is easy to show that
\[
\frac{d}{dt} \Phi(t) + C \| \nabla \tilde{u}(t) \|_1^2 \leq G(t) \Phi(t),
\]
where \(\int_0^t G(s) ds \leq C\) for \(0 \leq t \leq T_*\). By the Gronwall's inequality, we can conclude that \(\tilde{\psi} = \tilde{\phi} = \tilde{u} = 0\), then we complete the proof of uniqueness.
3.5. Proof of Theorem 2.3. From Theorem 3.1, we can obtain a unique classical solution \((\psi, \phi, u)\) to problem (16)-(18) satisfying (20), which means that
\[
(\sqrt{h} = \phi, u) \in C^1([0, T^\ast] \times \mathbb{R}^2), \quad (\nabla u, \nabla h/h) \in C([0, T^\ast] \times \mathbb{R}^2).
\]
Thus, we have
\[
h = \phi^2 \in C^1([0, T^\ast] \times \mathbb{R}^2).
\]
By (16), it is easy to conclude that \((h, u)\) satisfies the equation (1) in the sense of distribution.
By (1), we have
\[
h(t, x) = h_0(W(0, t, x)) \exp \left( \int_0^t \text{div} u(s, W(s, t, x)) ds \right), \quad (52)
\]
where \(W \in C^1([0, T_\ast] \times [0, T_\ast] \times \mathbb{R}^2)\) is a solution of the Cauchy problem
\[
\begin{aligned}
\frac{d}{dt} W(t, s, x) &= u(t, W(t, s, x)), \quad 0 \leq t \leq T_\ast, \\
W(s, s, x) &= x, \quad 0 \leq s \leq T_\ast \ x \in \mathbb{R}^2.
\end{aligned} \quad (53)
\]
Since \(h_0 > 0\), by we can also conclude
\[
h(t, x) > 0 \text{ for } \forall (t, x) \in [0, T_\ast] \times \mathbb{R}^2.
\]
Therefore, problem (1)-(5)-(6) has a unique regular solution.

4. Well-posedness of regular solutions for the inviscid flow. In this section, we establish the existence and uniqueness of regular solutions to the rotating shallow water equation with the inviscid flow (Theorem 2.4) based on the uniform estimates in Theorem 2.3.

4.1. Existence. From Theorem 2.3, we know that for any \(\epsilon(0 < \epsilon < 1)\) there exists a unique regular solution of problem (1)-(5)-(6) on \([0, T^\ast] \times \mathbb{R}^2\) denoted by \((h^\epsilon, u^\epsilon)\), which satisfies
\[
\sup_{0 \leq t \leq T^\ast} \left( \left\| (\sqrt{h^\epsilon}, u^\epsilon) \right\|_{L^3(B^R)}^3 + \left\| \nabla h^\epsilon/h^\epsilon \right\|_{L^2(B^R)}^2 + \epsilon \left\| \nabla h^\epsilon/h^\epsilon \right\|_{L^2(B^R)}^2 \right) + \int_0^{T^\ast} \epsilon \left| u^\epsilon \right|_{L^4(B^R)}^4 dt \leq C,
\]
\[
\sup_{0 \leq t \leq T^\ast} \left( \left\| (\sqrt{h^\epsilon}, u^\epsilon) \right\|_{L^3(B^R)}^3 \right) + \int_0^{T^\ast} \epsilon \left| u^\epsilon \right|_{L^4(B^R)}^4 dt \leq C,
\]
(54)
where \(T^\ast\) and \(C\) is independent of \(\epsilon\).

Thanks to the uniform estimates (54), we can derive that as \(\epsilon \to 0\) there exists a subsequence of solutions
\[
(\sqrt{h^\epsilon}, u^\epsilon) \text{ converges to a limit } (\sqrt{h}, u) \text{ in weak or weak}^* \text{ sense}. \quad (55)
\]
By virtue of Lemma 2.4 in [6], for any \(R > 0\) we can also obtain a subsequence of \((h^\epsilon, u^\epsilon)\) satisfy
\[
(\sqrt{h^\epsilon}, u^\epsilon) \to (\sqrt{h}, u) \text{ in } C([0, T^\ast]; H^1(B_R)), \quad (56)
\]
where \(B_R\) is a ball with radius \(R\) centered at origin. Using the lower semi-continuity of norms, (55) and (56), we can derive that \((h, u)\) satisfies
\[
\sup_{0 \leq t \leq T^\ast} \left( \left\| (\sqrt{h}, u) \right\|_{L^3(B^R)}^3 \right) \leq C. \quad (57)
\]
Moreover, we can also conclude that \( U = (\sqrt{h}, u) \) satisfies the following system in the sense of distribution:

\[
A_0 U_t + \sum_{i=1}^2 A_i(U) \partial_t U + H(u^\perp) = 0. 
\]

(58)

Using (57) and (58), it is easy to see that

\[
(\sqrt{h}, u)_t \in C([0, T]; H^2(\mathbb{R}^2)) 
\]

(59)

Thus, we have

\[
(\sqrt{h} = \phi, u) \in C^1([0, T^\ast] \times \mathbb{R}^2), \quad \nabla u \in C([0, T^\ast] \times \mathbb{R}^2).
\]

Similarly as the proof of Theorem 2.3, we can conclude \( (h, u) \) is a regular solution of problem (4)-(5)-(6).

4.2. **Uniqueness.** Assume that \((h_1, u_1)\) and \((h_2, u_2)\) are two solutions of problem (4)-(5)-(6). Set \( U_1 = (\sqrt{h_1}, u_1), \) \( U_2 = (\sqrt{h_2}, u_2), \) \( \mathbf{u} = u_1 - u_2 \) and \( \mathbf{U} = U_1 - U_2, \) then we have

\[
A_0 \mathbf{U}_t + \sum_{i=1}^2 A_i(U_1) \partial_t \mathbf{U} + \mathbf{H}(\mathbf{u}^\perp) = - \sum_{i=1}^2 A_i(U_2) \partial_t U_2. 
\]

(60)

By the energy estimate, it is easy to show that

\[
\frac{d}{dt} ||\mathbf{U}(t)||_1 \leq C ||\mathbf{U}(t)||_1,
\]

(61)

which implies that \( \mathbf{U} = 0 \). Therefore, the proof of the uniqueness is completed.

5. **Vanishing viscosity limit.** In this section, we will show the convergence rate of the regular solution to the rotating shallow water equations from the viscous flow to the inviscid flow (Theorem 2.5).

Assume that \( (h^\epsilon, u^\epsilon) \) is the regular solution on \([0, T_\ast] \times \mathbb{R}^2\) to problem (1)-(5)-(6) obtained in Theorem 2.3 and \( (h, u) \) is the regular solution on \([0, T_\ast] \times \mathbb{R}^2\) to problem (4)-(5)-(6) obtained in Theorem 2.4. Here, the life span \( T_\ast \) is independent of \( \epsilon \), and \( (h^\epsilon, u^\epsilon)|_{t=0} = (h, u)|_{t=0} = (h_0, u_0) \).

Set \( \psi^\epsilon = \nabla h^\epsilon / h^\epsilon \). From Theorem 2.3 and Theorem 2.4, we have

\[
\sup_{0 \leq t \leq T_\ast} \left( ||\psi^\epsilon(t)||_{L^2 \cap D^1}^2 + ||\psi^\epsilon(t)/2^2 + ||U^\epsilon(t)||_3^2 \right) + \epsilon \int_0^{T_\ast} ||\nabla^4 u^\epsilon||^2 dt \leq C, 
\]

(62)

and

\[
\sup_{0 \leq t \leq T_\ast} \left( ||h(t)||_3^2 + ||u(t)||_3^2 \right) \leq C. 
\]

(63)

Thus, it is easy to derive that \( \mathbf{U}^\epsilon = U^\epsilon - U \) satisfies

\[
\begin{align*}
A_0 \mathbf{U}_t^\epsilon + \sum_{i=1}^2 A_i(U^\epsilon) \partial_t \mathbf{U}^\epsilon &= - \sum_{i=1}^2 A_i(U^\epsilon) \partial_t U - \epsilon F(u^\epsilon) - H(u^\epsilon) + \epsilon G(\psi^\epsilon, u^\epsilon), \\
\mathbf{U}^\epsilon(t, x)|_{t=0} &= (0, 0).
\end{align*}
\]

(64)

Now, we will prove some necessary estimates for \( \mathbf{U}^\epsilon \) in order to establish the vanishing viscosity limit from \( U^\epsilon \) to \( U \) when \( \epsilon \to 0 \).

Firstly, we give the estimate of \( \mathbf{U}^\epsilon \) in \( L^2 \) space.
Lemma 5.1. If $\bar{U}^\epsilon$ satisfies (64), then
\[ |\bar{U}^\epsilon(t)|_2^2 \leq C\epsilon^2 \quad \text{for} \quad 0 \leq t \leq T_*. \]  
(65)

Proof. Multiplying (64) by $2\bar{U}^\epsilon$, and integrating by parts on $\mathbb{R}^2$, thanks to (62)-(63), we can derive that
\[
\frac{d}{dt} \int U^\epsilon \cdot A_0 U^\epsilon \, dx = \int \text{div}A(U^\epsilon)|U^\epsilon|_2^2 \, dx - 2 \sum_{l=1}^{2} \int U^\epsilon \cdot A_l(U^\epsilon) \partial_l U \, dx \\
- 2\epsilon \int L(u^\epsilon) \cdot \bar{u}^\epsilon \, dx - 2 \int (\bar{u}^\epsilon)^\perp \cdot \bar{u}^\epsilon \, dx + 2\epsilon \int (\psi^\epsilon \cdot Q(u^\epsilon)) \cdot \bar{u}^\epsilon \, dx \\
\leq C \left( |\nabla U^\epsilon|_\infty |U^\epsilon|_2^2 + |\nabla U|_\infty |U^\epsilon|_2^2 + \epsilon|\nabla^2 u^\epsilon|_2 |U^\epsilon|_2 \\
+ \epsilon |\psi^\epsilon|_6 |\nabla u^\epsilon|_3 |U^\epsilon|_2 \right) \leq C(|U^\epsilon|_2^2 + \epsilon^2).
\]

By Gronwall’s inequality, we have
\[ |\bar{U}^\epsilon(t)|_2^2 \leq C\epsilon^2 t \exp(Ct) \leq C\epsilon^2 \quad \text{for} \quad 0 \leq t \leq T_*. \]

Secondly, we prove the estimate of $|\partial_\zeta \bar{U}^\epsilon|_2^2$ when $|\zeta| = 1$.

Lemma 5.2. If $\bar{U}^\epsilon$ satisfies (64), then
\[ |\bar{U}^\epsilon(t)|_{2,1}^2 \leq C\epsilon^2 \quad \text{for} \quad 0 \leq t \leq T_*. \]

Proof. Applying the operator $\partial_\zeta^2$ to (64), multiplying by $2\partial_\zeta \bar{U}^\epsilon$ and integrating by parts, we can derive that
\[
\frac{d}{dt} \int \partial_\zeta^2 \bar{U}^\epsilon \cdot A_0 \partial_\zeta^2 \bar{U}^\epsilon \, dx = -2 \sum_{l=1}^{2} \int \partial_\zeta (U^\epsilon) \partial_l \partial_\zeta \bar{U}^\epsilon \cdot \partial_\zeta \bar{U}^\epsilon \, dx \\
- 2 \sum_{l=1}^{2} \int \partial_\zeta \bar{U}^\epsilon \cdot \left( \partial_\zeta^2 (A_l(U^\epsilon) \partial_l U^\epsilon) - A_l(U^\epsilon) \partial_l \partial_\zeta \bar{U}^\epsilon \right) \, dx \\
- 2 \int \left( \sum_{l=1}^{2} \partial_\zeta \bar{U}^\epsilon \cdot \partial_\zeta^2 (A_l(U^\epsilon) \partial_l U^\epsilon) + \epsilon \partial_\zeta^2 L(u^\epsilon) \cdot \partial_\zeta \bar{u}^\epsilon \right) \, dx \\
+ 2\epsilon \int \partial_\zeta^2 (\psi^\epsilon \cdot Q(u^\epsilon)) \cdot \partial_\zeta \bar{u}^\epsilon \, dx \equiv \sum_{i=5}^{9} I_i.
\]

When $|\zeta| = 1$, it is easy to see that
\[
I_5 = -2 \sum_{l=1}^{2} \int A_l(U^\epsilon) \partial_l \partial_\zeta \bar{U}^\epsilon \cdot \partial_\zeta \bar{U}^\epsilon \, dx \leq C|\nabla U^\epsilon|_\infty |\partial_\zeta \bar{U}^\epsilon|_2^2 \leq C|\partial_\zeta \bar{U}^\epsilon|_2^2,
\]
\[
I_6 = -2 \sum_{l=1}^{2} \int \partial_\zeta \bar{U}^\epsilon \cdot \left( \partial_\zeta^2 (A_l(U^\epsilon) \partial_l U^\epsilon) - A_l(U^\epsilon) \partial_l \partial_\zeta \bar{U}^\epsilon \right) \, dx \\
\leq C|\nabla U^\epsilon|_\infty |\nabla \bar{U}^\epsilon|_2^2 \leq C|\nabla \bar{U}^\epsilon|_2^2,
\]
Lemma 5.3. If $\mathbf{U}^t$ satisfies (64), then
$$|\mathbf{U}^t(t)|_{D_2}^2 \leq C\epsilon \quad \text{for} \quad 0 \leq t \leq T_*.$$  

Proof. As $|\zeta| = 2$, we can derive that

$$I_5 = -2 \sum_{l=1}^{2} \int A_l(U^t) \partial_t \partial^2_x \mathbf{U}^t \cdot \partial^2_x \mathbf{U}^t \, dx \leq C|\partial^2_x \mathbf{U}^t|_{2}^2 \leq C|\mathbf{U}^t|_{D_2}^2,$$

$$I_6 = -2 \sum_{l=1}^{2} \int \left( \partial^2_x (A_l(U^t) \partial_t \mathbf{U}^t) - A_l(U^t) \partial_t \partial^2_x \mathbf{U}^t \right) \cdot \partial^2_x \mathbf{U}^t \, dx \leq C(|\nabla^2 U^t|_{6} |\nabla^2 \mathbf{U}^t|_{3} + |\nabla U^t|_{\infty} |\nabla^2 \mathbf{U}^t|_{2}) \leq C(\|\nabla^2 \mathbf{U}^t\|_{2}^2 + |\nabla U^t|_{\infty} |\nabla^2 \mathbf{U}^t|_{2}^2 \leq C(|\nabla^2 \mathbf{U}^t|_{2}^2 + \epsilon^2),$$

$$I_7 = -2 \sum_{l=1}^{2} \int \partial^2_x \mathbf{U}^t \cdot \partial^2_x (A_l(U^t) \partial_t U^t) \, dx \leq C(|\partial^2_x \mathbf{U}^t|_{2}^2 + |\partial^2_x \mathbf{U}^t|_{2} |\partial^2_x \mathbf{U}^t|_{2}) \leq C(|\mathbf{U}^t|_{D_2}^2 + \epsilon),$$

$$I_8 = 2\epsilon \int \partial^2_x \left( \partial^2_x (\alpha \Delta \mathbf{u}^t + (\alpha + \beta)\nabla \mathbf{d} \mathbf{u}^t) \right) \cdot \partial^2_x \mathbf{u}^t \, dx \leq C\epsilon |\partial^2_x \mathbf{u}^t|_{2} |\nabla^4 \mathbf{u}^t|_{2} \leq C(|\partial^2_x \mathbf{u}^t|_{2}^2 + \epsilon^2 |\nabla^4 \mathbf{u}^t|_{2}^2),$$

$$I_9 = 2\epsilon \int \partial^2_x \left( \psi^t \cdot Q(\mathbf{u}^t) \right) \cdot \partial^2_x \mathbf{u}^t \, dx \leq C\epsilon |\partial^2_x \mathbf{u}^t|_{2} |\nabla^4 \mathbf{u}^t|_{\infty} + |\nabla \psi^t|_{6} |\nabla^2 \mathbf{u}^t|_{3} + |\psi^t|_{\infty} |\nabla^4 \mathbf{u}^t|_{2} \leq C(|\partial^2_x \mathbf{U}^t|_{2}^2 + \epsilon).$$

Using (66) with the estimates (68) and Gronwall’s inequality, we can conclude that

$$|\mathbf{U}^t(t)|_{D_2}^2 \leq \exp(\epsilon t) \int_0^t C(\epsilon + \epsilon^2 |\nabla^4 \mathbf{u}^t|_{2}^2) \, ds \leq C\epsilon \quad \text{for} \quad 0 \leq t \leq T_*.$$

□
Finally, we can also get the estimate of $\overline{U}'$ in $H^{s'}(s' < 3)$ space.

**Lemma 5.4.** If $\overline{U}'$ satisfies (64), then

$$||\overline{U}'(t)||^2_{H^{s'}} \leq C\epsilon^{2(1-\frac{s'}{3})} \exp \left( C \left( 1 - \frac{3}{s'} \right) \right), \quad \text{for} \quad 0 \leq t \leq T_*.$$

**Proof.** By Lemma 2.5 in [6], (62), (63) and (65), it is easy to see that

$$||\overline{U}'||_{s'} \leq ||\overline{U}'||_{0}^{1-\frac{s'}{3}} ||\overline{U}'||_{3}^{\frac{s'}{3}} \leq C\epsilon^{1-\frac{s'}{3}} \exp \left( C \left( 1 - \frac{3}{s'} \right) \right), \quad s' < 3. \quad (69)$$

Based on Lemmas 5.1-5.4, it is easy to obtain Theorem 2.5.

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_E-mail address:_ zgw_11220163.com