Self-organized critical behavior: the evolution of frozen spin networks model in quantum gravity

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In quantum gravity, we study the evolution of a two-dimensional planar open frozen spin network, in which the color (i.e. the twice spin of an edge) labeling edge changes but the underlying graph remains fixed. The mainly considered evolution rule, the random edge model, is depending on choosing an edge randomly and changing the color of it by an even integer. Since the change of color generally violate the gauge invariance conditions imposed on the system, detailed propagation rule is needed and it can be defined in many ways. Here, we provided one new propagation rule, in which the involved even integer is not a constant one as in previous works, but changeable with certain probability. In random edge model, we do find the evolution of the system under the propagation rule exhibits power-law behavior, which is suggestive of the self-organized criticality (SOC), and it is the first time to verify the SOC behavior in such evolution model for the frozen spin network. Furthermore, the increase of the average color of the spin network in time can show the nature of inflation for the universe.

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I. INTRODUCTION

Self-organized critical phenomenon, is exhibited by a dynamic system that displays spatial/temporal scale invariance characteristic of the critical point of phase transition, but without the need of external fine tuning of control parameters [1]. Many such examples can be found in natural and social systems, including granular materials, earthquakes, biological evolution and stock markets [1, 2].

Recently, the self-organized criticality (SOC) also became an attractive problem in quantum gravity [3, 4, 5]. Rovelli and Smolin [6] pointed out that the space has discrete structure at Plank scale in canonical quantum gravity [7, 8, 9], and then it was proposed by Markopoulou et al. [3] that the discrete space should be a SOC system for the lack of an external fine tuning to the universe’s parameters. The proposal was studied further by Borissov and Gupta [4], and then by Ansari and Smolin [5] more recently. For other researches on critical phenomenon in quantum gravity see refs. [10, 11].

The research on the SOC in quantum gravity [4, 5] was based on the frozen spin network, a simplified case of loop quantum gravity, in which the embedding graph is fixed and the spin that labels the edge evolves. Furthermore, the identities of the gauge invariance imposed at vertices is not chosen as the conditions on states, but a dynamical process in which the system evolves to gauge invariance state. The evolution rules adopted by Borissov and Gupta [4] did not lead to the SOC. Then Ansari and Smolin [5] developed a new set of evolution rules and observed the SOC. The authors mainly applied the classical statistical mechanics on spin networks instead of quantum mechanics, and the quantum SOC is still a significant issue to be studied.

In this paper, we developed a set of more realistic propagation rules based on the ideas of refs. [4, 5], and studied the SOC in loop quantum gravity. In Sec. II the model of spin network adopted in this paper is briefly reviewed. Following that, the two main classes of evolution rules for the frozen spin networks are described in Sec. III. One is the random edge model, which involves changing the color (i.e. the twice spin of an edge) of a randomly chosen edge by an even integer. The other one is the random vertex model, which involves choosing a vertex at random and evolving the colors of its three incident edges by an even integer. Since the changes of color generally lead to the violation of the gauge invariance conditions, detailed propagation rules should be defined for the spin network to regain the gauge invariance. The propagation rules can be defined in different ways. In Sec. IV we presented a new possible propagation rule, in which the change of the edge color is an even integer chosen with a certain probability distribution, but no longer a fixed value as in previous works. We found that the system under our propagation rule exhibits the SOC in the random edge model for the first time. Summary and discussions can be found in Sec. V.

II. THE SPIN NETWORK

As introduced by Rovelli and Smolin [12], and then by Baez [13], in canonical quantum gravity, a spin network state is defined by a closed graph γ which is constructed by a finite number of oriented edges $e_1, e_2, \cdots$ incident
at vertices \(v_1, v_2, \ldots\). A vertex with \(p\) incident edges is called a \(p\)-valent vertex or a vertex of valence \(p\). The edge \(e_i\) is labeled by an irreducible representation \(j_i\) (i.e. spin) of SU(2). The color of edge \(e_i\) is defined as its twice spin, \(c_i = 2j_i\).

For simplicity, we consider only the planar trivalent spin network which is dual to a colored triangulation of the space \([13]\). Firstly, we take a triangular space and divide it into \(N\) sub-triangles to perform the triangulation. The result of the triangulation is exhibited by the solid lines in Fig. 1. Then the dual trivalent spin network is constructed by connecting the centers of each sub-triangle to the centers of its adjacent sub-triangles, and its boundary is given by the dual to the segments of the edges of the original triangle space. The dual spin network is shown in Fig. 1 as the dashed lines.

The length of a side in the triangulated network and the color of its dual edge in the spin network have the following relationship, \(2l_{\text{side}} = l_{\text{Plank}} \cdot c_{\text{edge}}\). Since the lengths of the three sides of a triangle obey the triangle inequalities, the colors of edges incident at a vertex in the dual spin network is constrained by the following *gauge invariance condition*: (1) the triangle inequalities are satisfied and (2) the sum of the colors of a vertex’s incident edges should be an even integer. Suppose \(a, b\) and \(c\) are the colors of edges incident at a vertex and they are all positive integers, the gauge invariance condition for a vertex can be expressed as

\[
a + b \geq c, b + c \geq a, c + a \geq b; \quad (1)
\]

\[
a + b + c = \text{even}. \quad (2)
\]

If the gauge invariance condition (eqs. 1, 2) is not satisfied for a vertex, that vertex is called a gauge-non-invariance (GNI) vertex. The spin network reaches a gauge invariance state when the gauge invariance condition is satisfied at all vertices.

The spin network evolves from one gauge invariance state to another one by adding or subtracting a loop flux. The evolving rules fall into two main classes \([4, 5]\): One is the random vertex class, which involves choosing a random vertex and simultaneously changing the colors of its three incident edges by an even integer \(\Delta c\). The other one is the random edge class in which one edge is chosen randomly and its color is changed by an even number \(\Delta c\). The two evolution rules will be detailedly discussed in Sec. III respectively.

The spin network studied hereafter is constructed using the method described above (Fig. 1). It is a trivalent spin network in a two-dimensional space with open boundary. Moreover, during the evolution of the spin network, the underlying graph is fixed and only the color of the edges can be changed. Such model is also referred to as the *frozen spin network*. In the initial state, random integers between 1 and \(c_{\text{max}}\), where \(c_{\text{max}}\) is a given maximum value of the initial color, are assigned to each edge and the spin network is required to be gauge invariant. For an easier comparison with the previous works, \(c_{\text{max}}\) is chosen to be 30 as in Ref. \([\text{3}])\) in the following parts unless noted specifically. But we can see in Sec. IV that the values of \(c_{\text{max}}\) have negligible influence on the SOC behavior.

III. THE RANDOM EDGE MODEL AND THE RANDOM VERTEX MODEL

The random edge model: In \([4]\), Borissov and Gupta established their evolution rule on a two-dimensional planar spin network. (1) Initialize the graph. (2) Choose an edge at random and change its color by \(\Delta c = -2\). This will generally break the gauge invariance condition of the two vertices linked by that edge. (3) To restore the gauge invariance state, the system evolves with a propagation rule, which can be defined in many different ways (see \([4, 5]\) for examples). When the gauge invariance is restored, a new edge is selected and step (3) is repeated.

The random vertex model: After initializing the spin network, the propagation rule of the random vertex model, according to Ansari and Smolin in \([5]\), consists of the following steps: (1) Choose a vertex randomly and subtract 2 from the color of each of its edges; (2) Check the gauge condition (1, 2) at all vertices. If the gauge invariance does not hold for a vertex, then add \(\Delta c' = +2\) to the color of each of its edges. Continue until the gauge invariance is restored, and then repeat steps (1) and (2).

The random edge model was studied in Ref. \([4]\), and the SOC was not observed. Ref. \([5]\) studied the random vertex model and found to display the SOC behavior.

In the following, we will perform the work of our detailed propagation rule in the two main models respectively. Our propagation rule can verify the SOC behavior in random edge model, but not in random vertex model. Therefore we will focus on the discussion of the random edge model in the next section.
IV. OUR PROPAGATION RULE AND THE RESULTS

Reviewing the random edge model, one may find that the main steps in the evolution of the spin network can be summarized as the following: First, construct and initialize the graph. Then, (1) randomly choose one edge and subtract the color of it by $\Delta c$ [17]. (2) If the gauge invariance does not hold for a vertex, randomly choose one of its incident edges and add $\Delta c'$ to its color. This is repeated until the graph is in gauge invariance state again. The set of consecutive updates of vertices in recovering the gauge invariance is an avalanche, and the number of updated vertices is the size of the avalanche.

The area of the avalanche has a different definition: it is the number of vertices involved in one avalanche, no matter how many times they are updated. (3) Repeat steps (1) and (2) for a large number of times to see the behavior of the spin network in a long time and hunt for the propagation rules to lead the distribution of size of avalanches become scale variant.

In [5], the authors initialized the network by assigning to each edge a random even integer between 10 and 30. Here we extend the initialization to a more general condition. The initial colors are randomly selected from the integers between 1 and $c_{\text{max}}$. Furthermore, the $\Delta c$ involved in step (1) is not fixed as 2 any more, but an even integer randomly chosen between 2 and $c_{\text{max}}$.

The more important generalization is in step (2). Since the vertex is not isolated, but affected by the adjacent vertices in general, the $\Delta c'$ involved in the update of a GNI vertex is suggested to be a variable dependent on the original color of the vertex and the colors of its adjacent vertices jointly. Assume the GNI vertex in study, $v_i$, has three adjacent vertices $v_1$, $v_2$ and $v_3$ [18], the sums of color of the vertices' three incident edges are $c_1$, $c_1$, $c_2$ and $c_3$ respectively. During the update, the color of each of the incident edges on vertex $v_i$ is changed by $\Delta c'_i$. $c'_i$ is an even integer chosen between 2 and a maximum value, $\Delta c'_{i,\text{max}}$, with a probability distribution,

$$p = \frac{\exp(-\beta \Delta c'_i)}{\sum_{\Delta c'_i} \exp(-\beta \Delta c'_i)},$$

where

$$\beta = \frac{1}{|(c_1 + c_2 + c_3)/3 - c_i|}$$

and

$$\Delta c'_i = 2, 4, \cdots, \Delta c'_{i,\text{max}}.$$

When $(c_1 + c_2 + c_3)/3 = c_i$, we take $p = 1$ when $\Delta c'_i = 2$ and $p = 0$ for other values of $\Delta c'_i$.

In the present study, $\Delta c'_{i,\text{max}}$ is set to be equal to the maximum value of the initialized color, $c_{\text{max}}$ (or $c_{\text{max}} - 1$ if $c_{\text{max}}$ is an odd number). Obviously, $\Delta c'_{i,\text{max}}$ can have other suggestions, but for the sake of brevity, the other suggestions will not be discussed here.

We did the simulation for $10^7$ steps on a spin network with 102400 vertices. The results of a typical run are shown in Figs. 2 and 3.

As shown in Fig. 2, the distribution of the size of avalanches is found to be

$$P(s) \propto s^{-\alpha},$$

with the power-law exponent, $\alpha \approx 2.06$ (in [5], $\alpha = 3.3$). Moreover, the power-law exponent, $\alpha$, is slightly affected by the finite-size effect. For instance, $\alpha \approx 2.17$ for the spin network with vertices 10000 in Fig. 1.

Strictly speaking, the area of avalanches is also need to be a power-law distribution in a SOC model. The log-log subplot in Fig. 2 indicates a good power-law relation of area (corresponding to the distribution of size of avalanche in Fig. 2). The concurrence of the power-law relations of the size and the area in their distributions supports that the evolution process under the propagation rule is self-similar, which means that the system is a SOC system.

Besides the distribution of the size and area of avalanche, the fraction of the flat triangles (abbr. FFT) and the average color (abbr. Avge) of network are also of interests and studied in the literatures [1, 3].

A flat triangle is dual to a flat vertex which has special edge colors making one of the three conditions in Eq. 11 saturated. The flat triangle plays a pivotal role in the evolution of spin network. Its next evolution by the addition or subtraction of loop flux will likely lead to a gauge-non-invariant state and cause an avalanche of recovering gauge invariant state. Especially, when the system is in the critical state, the fraction of the number of the flat triangle in time should be a fixed value [3]. Such result can be found in Fig. 5 and 3. In this
Interestingly, the distributions of size (area) of avalanche and color network is not independent from the initial maximum that an asymptotic system may also display SOC [5, 16]. The linear expansion of the universe described by the spin side of the triangulated space, we are also observing in [5] is mainly caused by the limited size of the spin model, the fraction of flat triangles rapidly drops from a finite value (∼0.165) and saturates around a fixed value, about 0.026.

The average color of the network in time fits well onto a straight line after the first relaxation period in Fig. 3(b), in agreement with the result in [3] (through the repeated work of [4] on the spin network with 361, 10000 and 102400 vertices respectively, we find out that the strong oscillation of the Avgc of the network presented in [3] is mainly caused by the limited size of the spin network). Since the color is proportional to the length of the side of the triangulated space, we are also observing the linear expansion of the universe described by the spin network model. It is another evidence of the statement that an asymptotic system may also display SOC [5, 16].

In our detailed evolving rule, the evolution of the spin network is not independent from the initial maximum color \( c_{\text{max}} \). We did simulations for \( 10^7 \) steps on a spin network with 10000 vertices for \( c_{\text{max}} = 10, 30, 100, 1000 \) and 10000, respectively. The distributions of size and area of avalanche for each typical run are shown in Fig. 4. Interestingly, the distributions of size (area) of avalanche for each \( c_{\text{max}} \) very well overlap on each other. In other words, \( c_{\text{max}} \) have no obvious effect on the SOC behavior in this model.

V. SUMMARY AND DISCUSSION

Based on the work of Borissov and Gupta [4], and the work of Ansari and Smolin [5], we studied the evolution of a frozen spin network, in which the colors of the edges (namely their twice spins) change but the underlying graph remains fixed. Two different classes of evolution rules are considered. One class (the random edge model) involves choosing an edge randomly and changing its color by an even integer. The other class (the random vertex model) involves choosing a vertex at random and changing the colors of its three incident edges by an even integer. Since the changes of colors generally lead to the violation of the gauge invariance conditions, a detailed propagation rule is defined for the spin network to regain the gauge invariance during the evolution. The possible propagation rules can be defined in different ways. In the previous works [4, 5], the value of the color change during the evolution of the spin network is fixed. In the present work, considering the fact that the vertices are interdependent, we suggested that the change of color is a variable dependent on the colors of the vertex itself and the adjacent vertices. To be more specific, the change of color is an even integer chosen between 2 and \( c_{\text{max}} \) (the maximum color appearing in the initial state), with a probability distribution dependent on the colors of the considered vertex and its neighboring vertices. We applied this rule to the two models mentioned above respectively, in the framework of a two-dimensional planar open spin network. The random edge model under our
propagation rule exhibits the SOC behavior. But we had not found evidence of SOC in the random vertex model. Furthermore, the increase of the average color of the spin network in time exhibits the inflation of universe.

In summary, we studied the loop quantum gravity using a statistical physics method. The results show that a quantum gravity system (under the limit of low energy) is equivalent to a non-equilibrium statistical system whose temporal evolution exhibits the self-organized criticality. Moreover, our research shows that the size of a gravity system (which also can be regarded as a universe) expands in time. Even though the accelerated expansion of the universe can not be proved yet in the paper, we expect that it can be observed under more realistic and accurate propagation rules. Our work is only the inchoate investigations of the SOC in quantum gravity. Further works along this line, such as the dynamics of spin network, generalization to higher dimensional space and different propagation rules, will hopefully help us gain more insight into the self-organization criticality in quantum gravity.

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[17] In our paper, the randomly chosen edge should not have color less than $\Delta c$, otherwise it will be skipped with no action taken.
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