Euler-Heisenberg Lagrangian with axial gauge

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Augmentations to the Euler-Heisenberg Lagrangian (QED one-loop effective action in homogeneous electromagnetic fields) under a constant axial gauge are examined. Two special configurations admit an exact eigendecomposition, and hence effective action as a spectral sum, of the augmented Dirac operator: one with a magnetic field with chiral chemical potential, and the other with an electric field with spatial axial gauge. An enhancement to Schwinger pair production is found for the latter, which is more fully analyzed using the worldline instanton formalism. There it is found the overall enhancement is due to the spatial axial gauge serving as a negative mass shift. Finally, we remark on the exactly solvable massless case for arbitrary electromagnetic and axial gauges.

I. INTRODUCTION

The Euler-Heisenberg Lagrangian [1, 2] has proved to be a spectacularly successful theoretical tool with which to study a wealth of physical phenomena in quantum electrodynamics (QED) including but not limited to: Schwinger pair production [1, 3], light-by-light scattering [4], and charge renormalization [5]; see [6] for a review. The Lagrangian has been extended to inhomogeneous fields [7], to include a chemical potential [8], and to encompass higher-order loops [9], and remains an active research thread thanks to its all-orders in background coupling construction, imparting a wealth of physics in compact formula. Even so, studies on the Euler-Heisenberg Lagrangian under a background CP-violating axial gauge with electromagnetic field are few despite its importance.

The temporal component of an axial gauge field could be regarded as an axial chemical potential or the time derivative of an axion-like particle, which plays an important role in pseudoscalar inflation of early Universe when coupled to the (hyper) gauge fields via an effective Chern-Simons term. In such a scenario, backreaction from the Schwinger effect could modify the dynamics of axion inflatons and the corresponding physics in cosmology [10] (see also [11] for a brief review and more references therein). There have been some related studies for the Schwinger effect with an axial chemical potential. E.g., in [12], it is found that a nonzero \( A_0^5 \) could assist the pair production. On the contrary, the influence from the spatial component of \( A_0^5 \) on the Schwinger effect is rarely studied due to the lack of phenomenological applications.

However, such a background axial gauge field with the non-vanishing spatial component may be generated from a mechanical strain in the effective theory of Dirac or Weyl semimetals [13]. Moreover, the spatial component of a background axial gauge field may be realized as a vortical field coming from rotation, while the temporal component is associated with the fluid helicity [14]. In phenomenology, the vortical field (or vorticity for brevity) and magnetic field could lead to similar phenomena such as spin polarization or anomalous currents in chiral matter, known as the chiral magnetic effect [15] and chiral vortical effect [16], even though they are intrinsically different objects. It is thus intriguing to further explore how a background axial gauge field as vorticity may affect the Schwinger effect as opposed to its counterpart with magnetic fields that has been widely studied in literature [6, 17].

One may define the Euler-Heisenberg Lagrangian, \( L_{EH}[A, A_5] \) with axial gauge, from the one-loop effective action,

\[
\Gamma[A, A_5] = -i \text{Tr} \ln \left[ i \gamma^\mu (D_\mu + i \gamma_5 A_5^\mu) - m \right] ,
\]

in homogeneous fields for system spacetime volume, \( \mathcal{T} \mathcal{L}^3 \) as \( \Gamma[A, A_5] = \mathcal{T} \mathcal{L}^3 L_{EH}[A, A_5] \). To study the one-loop QED effective action with an axial gauge we make use of the propertime method [3, 18], and evaluate using a spectral decomposition of the fermionic determinant [19, 20], as well as the worldline instanton method [21]. The former method relies on expressing the effective action as a sum over its eigenvalues, and provides exact solutions for, e.g., the case of a homogeneous [22] or Sauter background [19]. Here we exploit two special cases in which exact eigenvalues of the Dirac operator can be found by virtue of the Ritus basis [23]: a magnetic field with chiral chemical potential and an electric field with spatial axial gauge. Note that a novel feature here is that we extend the Ritus basis to an equivalent formulation for electric fields. What is more, is we determine that an enhancement of Schwinger pair production can be found with a spatial axial gauge. To support this finding we study it semi-classically as well using the worldline instanton method [21]; what is novel here as well is that we adopt the method for the phase-space worldline formalism [24]. The worldline instanton method has made...
possible the study of pair production in otherwise complicated backgrounds, not limited to dynamically assisting fields [25], under finite temperature [26], or non-Abelian backgrounds [27]. Euler-Heisenberg Lagrangians have been studied perturbatively in electromagnetic and axial gauge couplings [28], and related worldline approaches have been put forth that include an axial gauge field [29].

We first analyze the linear Dirac operator’s eigenvalues and how they lead to a perturbative in \( O(A_5) \) definition of the effective action in Sec. II. Next we take a deeper look at the Schwinger pair production enhancement for a Schwinger propertime integral;

\[
\sum_{\text{tractions assumed in place of matrix multiplication.}}^{}
\]

We have inserted in a complete set of eigenstates, \( \sum_N \langle \psi_N | \psi_N \rangle = 1 \), and assumed a normalization of \( \text{tr} \int d^4x \langle x | \psi_N \rangle \langle \psi_N | x \rangle = 1 \). The mass \( m \rightarrow m - i \epsilon \) can provide convergence for a Schwinger propertime integral; in this sense the log of operator is understood. By making use of the fact that \( \gamma_5 \) anti-commutes with the operator given in Eq. (3), one may also equally well write

\[
\Gamma[A, A_5] = -i \text{Tr} \ln[\gamma_5 \hat{A}_5 \gamma_5 - m].
\]

Then averaging over both the above and Eq. (4), and through the application of Schwinger propertime [3], we arrive at an alternative expression of the effective action as sum over quadratic eigenvalues as

\[
\Gamma[A, A_5] = \frac{i}{2} \int_0^\infty \frac{dT}{T} \sum_N e^{-i(\lambda N - m)T}. \tag{6}
\]

Then determination of the full effective action has been reduced to the evaluation of eigenvalues, \( \lambda_N \), for \( \langle x | \psi_N \rangle = \psi_N \) where

\[
\hat{A}_5 \psi_N = \lambda_N \psi_N. \tag{7}
\]

An intuitive strategy for the evaluation of Eq. (7) in homogeneous electromagnetic fields is to seek a separable solution wherein modes for the magnetic (electric) field are described by their Landau levels (electric equivalent imaginary Landau levels). However, due to the presence of the axial gauge this separation is not easily implemented; in principal \( A_5^\mu \) cannot be removed by a field redefinition. (This is however possible in the purely massless case; see IV.) Even so, there are two particular background electromagnetic and axial gauge configurations where separable solutions are readily found, one for each set of Landau levels. These cases are:

1. A magnetic field with a chiral chemical potential:
   \[
   A_\mu = \frac{B}{2} (\delta^1_{\mu} x^2 - \delta^2_{\mu} x^1), \quad A_5^\mu = g^{\mu 0} \mu_5. \tag{8}
   \]

2. An electric field with spatial axial gauge:
   \[
   A_\mu = \frac{E}{2} (\delta^1_{\mu} x^3 - \delta^3_{\mu} x^0), \quad A_5^\mu = g^{\mu 1} \omega_5. \tag{9}
   \]

In both cases a Fock-Schwinger gauge has been used, \( A_0 = -\frac{1}{2} F_{\mu \nu} x^\nu \). Let us however stress that the eigenvalues of Eq. (7) are gauge-invariant since the electromagnetic part of the spectrum is determined by \( [D_\mu, D_\nu] = i q F_{\mu \nu} \); one may equally well use another electromagnetic gauge, which is a transform of the above. For case 2, an essential identification of the spacelike axial gauge is its connection to vorticity [14]. This identification proceeds along the lines of vorticity→torsion in spacetime [30]→axial gauge field [31]; we refer the reader to [14] for a derivation along with supplementary details. Note that for vorticity in the local rest frame, \( \omega = 2 \omega_5 \), in Eq. (9). Let us first proceed with the evaluation of the eigenvalues for the magnetic field case next.

### A. Magnetic field with chiral chemical potential

The selection of a magnetic field with chiral chemical potential, Eq. (8), will show benefits from an exactly
solvable setup by virtue of the Ritus basis [23]. Note
our conventions follow those used in [32, 33]. To begin
our discussion let us first introduce magnetic field spin
projection operators:

$$\mathcal{P}_\pm := \frac{1}{2} \left( 1 \pm \sigma^z \text{sgn}(qB) \right). \quad (10)$$

The projection operators satisfy idempotency, complete-
ness, and orthogonality, i.e., \( \mathcal{P}_\pm \mathcal{P}_\mp = 0 \),
\( \mathcal{P}_+ \mathcal{P}_+ = \mathcal{P}_- \mathcal{P}_- = 1 \), and
\( \mathcal{P}_\pm \mathcal{P}_\pm = 0 \). We also have that \( \gamma^1 \mathcal{P}_\pm = \mathcal{P}_\pm \gamma^1 \).
Furthermore, the spin projection operators commute
with terms in the eigenvalue equation containing the chiral
chemical potential. The magnetic field spin projection
operators act to diagonalize the spin factor associated
with the magnetic field as \( qB \sigma_1 \mathcal{P}_\pm = \pm |qB| \mathcal{P}_\pm \).

Before introducing the Ritus basis it is convenient to
highlight the harmonic oscillator parallels of the Dirac
equation in a magnetic field. Notably with the eigenvalue
(Dirac) equation, Eq. (7), in a magnetic field with chiral
chemical potential one has

$$\hat{i} \left[ \gamma^0 (\partial_0 + \gamma_5 \mu_5) - \gamma^1 D^1 - \gamma^2 D^2 + \gamma^3 \partial_3 \right] \psi_N = \lambda_N \psi_N \ ; \quad (11)$$

notice that we may introduce the following creation and
annihilation operators:

$$\hat{a} = \frac{1}{\sqrt{2|qB|}} \left[ iD^1 - \text{sgn}(qB) D^2 \right], \quad (12)$$

$$\hat{a}^\dagger = \frac{1}{\sqrt{2|qB|}} \left[ iD^1 + \text{sgn}(qB) D^2 \right], \quad (13)$$

that satisfy \( [\hat{a}, \hat{a}^\dagger] = 1 \), \( [\hat{a}, \hat{a}] = [\hat{a}^\dagger, \hat{a}^\dagger] = 0 \). A quantum
harmonic oscillator solution is then available, \( \hat{a}^\dagger \hat{a}(x|n) = n(x|n) = n \phi_n \). One may then write for the magnetic
Hamiltonian, \(- (D^1)^2 - (D^2)^2 = (2\hat{\partial}^1 + 1)|qB| \). Then it
immediate follows that the eigenvalues are \((2n + 1)|qB| \)
for \( n = 0, 1, 2, \ldots \), which will lead to the Landau levels.
We can also determine that for the normalized eigenbasis one
must have that \( \phi_n(x) = \sqrt{n} \phi_{n-1} \) and \( \phi_n(x) = \sqrt{n + 1} \phi_{n+1} \).
One may then rewrite for the argument in the eigenvalue
equation

$$- i \gamma^1 D^1 - i \gamma^2 D^2 = - \sqrt{2|qB| \gamma^1} (\hat{a} \mathcal{P}_+ + \hat{a}^\dagger \mathcal{P}_-) \ . \quad (14)$$

Let us now introduce the Ritus basis to evaluate the magnetic
degrees of freedom in the eigenvalue equation. The basis is

$$\mathcal{R}_n(x_{1,2}) = \phi_n(x_{1,2}) \mathcal{P}_+ + \phi_{n-1}(x_{1,2}) \mathcal{P}_- \ , \quad (15)$$

which combines the spin degenerate Landau levels. Note
that a lower truncation of \( \phi_{-1} = 0 \) is assumed. The
Ritus basis acts to diagonalize the quadratic operator as
\(- (D^1)^2 - (D^2)^2 - \sigma^z qB |qB| \mathcal{R}_n(x_{1,2}) = 2n|qB| \mathcal{R}_n(x_{1,2}) \).
Let us also point out that the magnetic eigenvectors also
have a perpendicular canonical momentum dependence
acting to shift the center of the wavepacket; we will in-
troduce the quantum label where necessary. The basis
allows us to evaluate the eigenvalue equation, Eq. (11),
for arbitrary spinor solution, \( z_s \), as

$$\psi_N(x) = e^{-ip^0 x^0 + ip^3 x^3} \mathcal{R}_n(x_{1,2}) z_s \ . \quad (16)$$

Inserting the above into Eq. (11), the following relationship
may be found:

$$\mathcal{R}_n \left[ \gamma^0 (p^0 - \gamma_5 \mu_5) - \sqrt{2|qB| \gamma^1} - \gamma^3 p^3 - \lambda \right] z_s = 0 \ . \quad (17)$$

A non-trivial solution to the above can be had if the ex-
pression in brackets, along with the spinor acting on it,
vanishes. We can see that solutions are simply a modified
Dirac equation in a chiral chemical potential; the chiral
chemical potential lifts the degeneracy of eigenvalues and
now four independent eigenvalues can be found reading

$$\lambda_N = \pm \sqrt{ (p^0)^2 - p_\parallel^2 - \mu_5^2 + 2s|\mu_5|p_\parallel} \ . \quad (18)$$

where \( p_\parallel = \sqrt{2|qB| n + (|p^3|)^2} \) and \( s = \pm 1 \) is an eigen-
value of the matrix \( S = -\gamma_5 \gamma^0 (\sqrt{2|qB| \gamma^1} + \gamma^3 p^3)/p_\parallel \).
In case of the LLL, those expressions reduce to \( \lambda_N = \pm \sqrt{(p^0)^2 - (|p^3| - |\mu_5| s)^2} \) and \( S = -\text{sgn}(p^3)^\gamma_1 \gamma_2 \) that is
the helicity operator. Since we will be summing over the
squared operator there will be an overall degeneracy
associated with squaring each eigenvalue above. Therefore
we can conclude based on the above that the set of good
quantum numbers includes

$$N = \{ p^0, p^3, n, s \} \ . \quad (19)$$

With the above eigenvalues one may determine the effec-
tive action, Eq. (6). It should be noticed that, in addition
to \( n \), there should be another quantum number so that the number of degrees of freedom remains the
same as in the case without the magnetic field. This
quantum number does not appear in the eigenvalue (18),
indicating energy degeneracy. The density of degenerate
states can be counted conveniently by using the Landau
gauge \( A_\mu = \delta_\mu^3 B x^2 \) which can be thought of as the spatial
rotation of the Fock-Schweriger gauge (8) by the use
of the gauge. Then, one finds that one component of the
canonical momentum \( p^c = p^1 \) is a good quantum number.
Therefore, the summation over the quantum numbers is
implemented as

$$\sum_N \to \mathcal{T} \mathcal{L}^2 \int dp^0, 3, z \sum_{n, s}, \quad (20)$$

where \( \mathcal{T} \) and \( \mathcal{L} \) are the system lengths in the tempo-
ral and spatial directions, respectively. The integration
over the modes is unbounded in homogeneous magnetic
fields, and one can determine the physical cutoff by con-
sidering a closed box of Landau modes [20]. This leads
to \( \int dp^c = |qB| \mathcal{L} \). The physics behind this is that the
degeneracy stems from the distribution of cyclotron or-
bits in the transverse plane and the conserved canonical
momentum serves as the center coordinate $\xi$ of the orbits via the relation $\xi = p^r/|qB|$, leading to a constraint $0 \leq p^r \leq |qB|L$. Last, there is an overall factor of 2 to account for the degeneracy of both the positive and negative components of the eigenvalues in $\lambda_{N}$ after squaring both.

Let us next sum over the Landau levels. A virtue of the Ritus basis is that it combines the spin degenerate Landau levels into one value of $n$, and the lowest Landau level is accounted for under $\phi_{-1} = 0$. We can see this in the construction of the basis in Eq. (15), one eigenvector for $n$ and another $n - 1$, and both are independent of one another guaranteed by the spin projectors. The key point here is that we must have a factor of one-half for $n = 0$ in comparison to the higher Landau levels, $n > 0$, since only the $P_+$ factor is present. Gathering everything together, including the volume and momentum integrals, we find:

$$\Gamma(B,\mu_5) = \frac{i}{4\pi}\mathcal{T}\cdot\mathcal{L}^3|qB| \int_{0}^{\infty} \frac{dT}{T} \int \frac{dp_0}{(2\pi)^3} e^{-im^2T} \sum_{\pm \pm 1} \left\{ \sum_{n=1}^{\infty} e^{i(\lambda_{0,\rho,3,..})^2T} + \frac{1}{2} e^{i(\lambda_{0,\rho,3,..})^2T} \right\}. \tag{21}$$

Let us furthermore evaluate the $p^0$ integral and take the sum over $s$ to find

$$\Gamma(B,\mu_5) = \frac{i\mathcal{T}\cdot\mathcal{L}^3|qB|}{(2\pi)^3} \int_{0}^{\infty} \frac{dT}{T} \sqrt{i\pi T} \int dp^3 e^{-i(m^2+s_0^2)T} \times \left\{ \sum_{n=1}^{\infty} e^{-ip_0T} \cos(2\mu_5 p_0T) + \frac{1}{2} e^{-ip_0T} \cos(2\mu_5 p_0T) \right\}. \tag{22}$$

To proceed let us take a power series expansion of the cosine terms expressing the arguments as propertime partials, then integrate over $p^0$, and last summing over the Landau levels as $2\sum_{n=1}^{\infty} e^{-2inx} + 1 = -i \cot x$ to find

$$\Gamma(B,\mu_5) = \frac{\mathcal{T}\cdot\mathcal{L}^3|qB|}{(2\pi)^3} \int_{0}^{\infty} \frac{dT}{T} \sqrt{i\pi T} \frac{1}{T} e^{-i(m^2+s_0^2)T} \times \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} (2\mu_5 T)^{2k} i^k \sqrt{\frac{1}{T}} \cot(|qB|T). \tag{23}$$

To clarify $\partial_0^2 \equiv 1$. In either of the above representations, whether in integral or summation form, the effective action is still exact. However, in the above expression one may perform a convenient perturbation in $\mu_5$ analysis. Let us therefore evaluate Eq. (23) to $O(\mu_5^3)$ to find

$$\Gamma(B,\mu_5) \approx \frac{\mathcal{T}\cdot\mathcal{L}^3|qB|}{8\pi^2} \int_{0}^{\infty} \frac{dT}{T} \sqrt{i\pi T} \frac{1}{T} e^{-i(m^2+s_0^2)T} \times \sum_{k=0}^{3} \frac{(-1)^k}{(2k)!} (2\mu_5 T)^{2k} i^k \sqrt{\frac{1}{T}} \cot(|qB|T). \tag{24}$$

where $c_B \equiv \cot(|qB|T)$.

One can see in Eq. (23), and more readily in Eq. (24) that poles only reside on the real propertime axis. Therefore we can conclude that there should not be an imaginary contribution to the effective action with the presence of $\mu_5$ (and a magnetic field) since all poles can be associated to a power of $c_B$, and hence there would not be Schwinger pair production as in the sole magnetic field case.

Let us evaluate the real part of Eq. (24). First let us explore the case of a weak magnetic field. There is a UV divergence as there is in the case without the chiral chemical potential. Let us treat this with a physical cutoff at a QED scale of $\Lambda^{-2}$ for small Schwinger propertimes. To find the adequate effective representation we first deform the contour to the imaginary propertime axis and perform a change of variables as $T \rightarrow iT$, then introduce the cutoff. We find for small magnetic field such that $\coth(|qB|T) \approx (|qB|T)^{-1}$,

$$\Gamma(B,\mu_5) \approx \frac{\mathcal{T}\cdot\mathcal{L}^3}{8\pi^2} \int_{\Lambda^{-2}}^{\infty} \frac{dT}{T} e^{-i|m^2T|} \left[ 2\mu_5^2 (|qB|^2 - \frac{1}{T^2}) - 1 \right]. \tag{25}$$

The term proportional to $T^{-3}$ is independent of the fields and axial gauge and should be removed by a suitable counterterm in the effective action, i.e., $\Gamma_{\text{reg.}}[A,A_5] = \Gamma[A,A_5] - \Gamma[0,0]$. Let us emphasize that we treat the axial gauge field as a background field. We see there is no $\mu_5^3$ term present in the above; this is because $1 - |qB|\lambda T \approx 0$ here. For later usage let us write out each integral from $T^0$ to $T^{-3}$:

$$\int_{\Lambda^{-2}}^{\infty} dTe^{-m^2T} \approx \frac{1}{m^2}, \tag{26}$$

$$\int_{\Lambda^{-2}}^{\infty} \frac{dT}{T} e^{-m^2T} \approx -\ln\left(\frac{m^2}{\Lambda^2}\right), \tag{27}$$

$$\int_{\Lambda^{-2}}^{\infty} \frac{dT}{T^2} e^{-m^2T} \approx \Lambda^2 + m^2 \ln\left(\frac{m^2}{\Lambda^2}\right), \tag{28}$$

$$\int_{\Lambda^{-2}}^{\infty} \frac{dT}{T^3} e^{-m^2T} \approx \frac{\Lambda^4}{2e^{\frac{m^2}{\Lambda^2}}} \frac{m^2}{2} \left(\Lambda^2 + m^2 \ln\left(\frac{m^2}{\Lambda^2}\right)\right), \tag{29}$$

where we have kept only leading order terms in $\Lambda^2$. Then one can find that

$$\Gamma(0,0) = \frac{\mathcal{T}\cdot\mathcal{L}^3}{8\pi^2} \left\{ -\frac{1}{2} \Lambda^4 + m^2 \left[\Lambda^2 + \frac{m^2}{2} \ln\left(\frac{m^2}{\Lambda^2}\right)\right] \right\}. \tag{30}$$

Moreover we have for weak magnetic fields that

$$\Gamma_{\text{reg.}}(B,\mu_5) = \frac{\mathcal{T}\cdot\mathcal{L}^3\mu_5^2}{4\pi^2} \left[ |qB|^2 - \Lambda^2 - m^2 \ln\left(\frac{m^2}{\Lambda^2}\right) \right]. \tag{31}$$

One can see that Eq. (31) is dominant in $\Lambda^2$. Hence we can tell with the presence of a sole chiral chemical potential the energy of the system is decreased, indicating stability. Also, notice that an augmentation of the quadratic $|qB|^2$ term is present; this suggests that the axial gauge field also augments the Maxwell electromagnetic Lagrangian, affecting photon-propagation [34].
As the chiral chemical potential is decoupled from the mass and electromagnetic field, to extract physically relevant quantities one must define a renormalization scheme. In the presence of a background electromagnetic field, Λ may have explicit qBξ dependence. This identification can more rigorously be seen using a cutoff in momentum space (see [17] for its usage with an electric field) rather than in propertime. Then renormalization occurs with redefined g.

We also examine the case of a strong magnetic field, strong enough such that the lowest Landau level (LLL) is projected. Fortunately, the Landau levels are contained in the various cotangent factors (see Ref. [35] for details), and that the LLL be projected amounts to \(\coth(|qB|T) = 1\). Importantly here since \(c_B \to i\coth(|qB|T)\) under the redefinition, terms which go as \(1 + c_B^2\) vanish in Eq. (24), indicating that to \(\mathcal{O}(\mu_5^2)\) there is no augmentation to the effective action or

\[
\Gamma_{\text{LLL}}(B, \mu_5) = -\frac{T L^3 |qB|}{8\pi^2} \left[ \Lambda^2 + m^2 \ln\left(\frac{m^2}{\Lambda^2}\right) \right].
\]  

(32)

One can understand this process superficially to all orders by examining the \(n = 0\) part of Eq. (21); there it can be determined that \(\mu_5\) can be absorbed into the canonical momentum \(p^3\). However let us caution, such a redefinition is not entirely rigorous due to the UV divergence. The effective action is closely related to the chiral condensate, and thus Eq. (32) is a restatement of magnetic catalysis [36], or that in the presence of a strong magnetic field the chiral condensate takes a nonzero value with arbitrary attractive interaction strength. Having explored the case of a magnetic field with chiral chemical potential let us turn our attention to the case of an electric field with spatial axial gauge.

### B. Electric Field and Spatial Axial Gauge

Let us now address the case of an electric field with spatial axial gauge as given in Eq. (9). With strong similarity to the above case with magnetic field, we derive a Ritus-like basis for the electric field, whose eigenvalues, or electric-like Landau levels, are now imaginary. Much of the discussion here parallels the one in Sec. II A; however with important subtleties. The final form of our effective action can be seen in Eq. (42). First, let us introduce electric field spin projection operators:

\[
P_{\pm} := \frac{1}{2} \left( 1 \pm i\sigma^3 \text{sgn}(qE) \right).
\]  

(33)

They satisfy as before \(P_+ P_\pm = P_\pm, P_+ + P_- = 1\), and \(P_+ P_\mp = 0\). We also have that \(\gamma^0, \gamma^3 P_\pm = P_\pm \gamma^0, \gamma^3\). Furthermore, we have that \(|P_\pm, \gamma^1, \gamma^2| = 0\). The operators have the virtue of projecting eigenvalues of the spin matrices as \(\pm qE \sigma^3 P_\pm = \pm i|qE| P_\pm\).

The eigenvalue equation in the electric field with space-like axial gauge becomes

\[
i\left[ \gamma_0 D^0 + \gamma^1 \left( D_1 - \gamma_5 \omega_5 \right) + \gamma^2 D_2 - \gamma^3 D^3 \right] \psi_N = \lambda_N \psi_N.
\]  

(34)

As before we will find that the above differential equation can be solved through the introduction of creation and annihilation operators:

\[
\bar{a} = \frac{1}{\sqrt{-2i|qE|}} \left[ D^0 - \text{sgn}(qE) D^3 \right],
\]  

(35)

\[
a = \frac{1}{\sqrt{-2i|qE|}} \left[ D^0 + \text{sgn}(qE) D^3 \right],
\]  

(36)

that satisfy \([a, \bar{a}] = 1\) and \([a, a] = [\bar{a}, \bar{a}] = 0\). Note that for the Ritus basis for the electric field \(\bar{a} \neq a\). We now have that \((D^0)^2 - (D^3)^2 = -(2\bar{a}a + 1)i|qE|\), and to determine the eigenspectrum let us note that \(\bar{a}a|n\rangle = n|n\rangle\), \(\bar{a}\bar{a}|n\rangle = (n + 1)\bar{a}|n\rangle\), and \(\bar{a}a|n\rangle = (n - 1)a|n\rangle\). Therefore we can find the eigenspectrum of \((D^0)^2 - (D^3)^2\) as the electric field equivalent of the Landau levels: \(-(2n + 1)i|qE|\bar{n}\in \mathbb{Z}^+\), where notice the eigenvalues are now imaginary. Also as an immediate consequence for normalization we have that \(a_0 = \sqrt{n_0}a_{n-1}\) and \(\bar{a}_0 = \sqrt{n_0 + 1}a_{n+1}\). Finally we can express the linear Dirac operator in the following form:

\[
i\gamma^0 D^0 - i\gamma^3 D^3 = i\sqrt{-2i|qE|}\gamma^0 [a\bar{P}_- + a\bar{P}_+].
\]  

(37)

We may now define the Ritus-like basis for the electric fields as

\[
R_n(x_{0,3}) = \phi_n(x_{0,3})P_+ + \phi_{n-1}(x_{0,3})P_-,
\]  

(38)

where we take that \(\phi_{-1} \equiv 0\). As before this basis diagonalizes the quadratic Dirac operator with electric fields as \(\frac{1}{2} \left( -(D_0)^2 + (D_3)^2 + qEa^0)R_n(x_{0,3}) = 2i|qE|nR_n(x_{0,3}) \right)\). One can then determine a solution to the Dirac equation as

\[
\psi_{p^1, p^2, n, s}(x) = e^{ip^1 x^1 + ip^2 x^2} R_n(x_{0,3}) z_s,
\]  

(39)

for arbitrary spinor \(z_s\). From the eigenvalue equation, Eq. (34), using the above we can now find that the following expression must be met

\[
R_n \left[ i\sqrt{-2i|qE|}\bar{n}\gamma^0 - \gamma^1 (p^1 + \gamma_5 \omega_5) - \gamma^2 p^2 - \lambda_N \right] z_s = 0.
\]  

(40)

A non-trivial solution can be found when the expression in the brackets acting on the spinor vanishes. We find the four independent eigenvalues are

\[
\lambda_N = \pm \sqrt{|p_0^2 - p_1^2 + \omega_5^2 + 2s|\omega_5|p_0^2|},
\]  

(41)

where we write \(p_0^2 = \sqrt{2i|qE|n - p_1^2}\), and once again \(s = \pm 1\).

Using Eq. (6), one may now express the effective action as a sum over the above eigenvalues in the same way as before as

\[
\Gamma(E, \omega_5) = \frac{i}{4\pi} TL^3 |qE| \int_0^\infty \frac{dT}{T} \int \frac{dp^1}{(2\pi)^2} \sum_{s=\pm 1} e^{-im^2T}
\]  

(42)
\[ \times 2 \left\{ \sum_{n=1}^{\infty} e^{i \lambda_{p,1,p,2,n,s}} T + \frac{1}{2} e^{i \lambda_{p,2,p,2,n,s}} T \right\}. \] (42)

The above effective action is exact albeit in integral form. Let us proceed as before with its perturbative in \( \omega_5 \) evaluation. To do so we integrate out \( p \) and sum over \( s \) to find

\[ \Gamma(E, \omega_5) = \frac{\pi L^4 |qE|}{8 \pi^3} \int_0^\infty \frac{dt}{T} \sqrt{T} \int dp^2 e^{-i(m^2 - \omega_5^2)T} \]
\[ \times \left\{ 2 \sum_{n=1}^{\infty} e^{i \lambda_{p,2,p,2,n,s}} T \cos(2 \omega_5 p_n^2 T) + e^{-i \lambda_{p,2,p,2,n,s}} T \cos(2 \omega_5 p_2^2 T) \right\}. \] (43)

Expanding out the cosine terms in a power series whose argument we express as a propertime partial derivative and summing over the effective action by deforming the Schwinger function, and hence one may evaluate the imaginary part of the effective action by deforming the Schwinger propertime contour to the lower half of the complex \( T \) plane and applying the residue theorem. Summing the contributions of \( n \) poles we may write

\[ \text{Im} \Gamma(E, \omega_5) = -\pi \frac{\pi L^4 |qE|}{8 \pi^2} \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \text{res}_k \left( \frac{-i \pi}{|qE|} \right). \] (45)

We do not include the pole at \( T = 0 \) since it is nothing to do with effects of electromagnetic fields or the axial gauge. To evaluate the residue, notice that acting the derivative \( \partial_T \) on the hyperbolic cotangent function gives \( (k + 1) \)-th order poles as the highest-order pole of the integrand. Thus, after application of the Leibniz rule to the derivative operators one may find

\[ \text{res}_k \left( \frac{-i \pi}{|qE|} \right) \]
\[ = \frac{d_k}{k!} \sum_{r=0}^{k} \sum_{s=0}^{k} \text{C}_{r,s} \text{C}_{k-r,s} \lim_{T \to -i \pi / |qE|} \]
\[ \times \partial_T^{k-r-s} \left\{ T^{-3/2} e^{-i(m^2 - \omega_5^2)T} T^{2k} \partial_T^{k-r} \sqrt{T-1} \right\} \]
\[ \times \partial_T^r \left\{ \left( T + \frac{i \pi}{|qE|} \right)^{k+1} \partial_T \text{coth}(|qE|T) \right\}. \] (46)

where \( d_k = (-1)^k (2 \omega_5)^{2k} (-i)^k / [(2k)!] \) and \( \text{C}_{r,s} \) is the binomial coefficient. A key observation here is that only the \( O((T + i \pi / |qE|)^{-1}) \) term in a power series expansion of the hyperbolic cotangent term will ultimately remain after taking the limit. Therefore, under the limit and differential operators we may take that \( \text{coth}(|qE|T) \sim |qE|^{-1} (T + i \pi / |qE|)^{-1} \). Then one can find that

\[ \text{res}_k \left( \frac{-i \pi}{|qE|} \right) = \frac{1}{|qE|} \sum_{r=0}^{k} \sum_{s=0}^{k} \text{C}_{r,s} \text{C}_{k-r,s} \lim_{T \to -i \pi / |qE|} \]
\[ \times k \partial_T^{k-r-s} \left\{ T^{-3/2} e^{-i(m^2 - \omega_5^2)T} T^{2k} \partial_T^{k-r} \sqrt{T-1} \right\}. \] (47)

Then taking the innermost partial derivative, applying the Leibniz rule once more, and taking the limit one can find

\[ \text{res}_k \left( \frac{-i \pi}{|qE|} \right) = \frac{e^{-(m^2 - \omega_5^2)T}}{|qE|} \sum_{r=0}^{k} \sum_{s=0}^{k} \text{C}_{r,s} \text{C}_{k-r,s} \]
\[ \times k \partial_T^{k-r-s} \left\{ T^{-3/2} e^{-i(m^2 - \omega_5^2)T} T^{2k} \partial_T^{k-r} \sqrt{T-1} \right\} \]
\[ \times \left( -i \pi / |qE| \right)^{k+r-2-l}, \] (48)

where

\[ \beta_q = \begin{cases} (q-2)! / (q-2-t)! & q - 2 > q - 2 - l \geq 0 \\ (1-q-1)! / (q-1-t)! & 0 > q - 2 \geq q - 2 - l \end{cases}. \] (49)

Finally after some manipulations we arrive at

\[ \text{res}_k \left( \frac{-i \pi}{|qE|} \right) = - \frac{1}{|qE|} e^{-(m^2 - \omega_5^2)T} \sum_{r=0}^{k} \sum_{s=0}^{k} \alpha_k^{r,s} \]
\[ \times \omega_5^{2k(m^2 - \omega_5^2)^r} \left( \frac{n \pi}{|qE|} \right)^{k+r-2-l}, \] (50)

with the coefficient

\[ \alpha_k^{r,s} = (-1)^{k-r} \text{C}_{k-r,s} \text{C}_{2k-2-r,l}^{l} \beta_{k+r,s}(2k - 2r - 1)!. \] (51)

Applying Eq. (50) with Eqs. (49) and (51) to Eq. (45) leads to an exact in summation form of the imaginary part of the effective action. Remarkably, the exponential suppression is reduced by a nonzero \( \omega_5 \) as \( \exp[-(m^2 - \omega_5^2)T \pi / |qE|] \) in Eq. (50). When \( \omega_5 \rightarrow m \), the exponential suppression completely goes away, and magnitude of the residue is determined by the polynomial at \( r = l \) as

\[ \text{res}_k \left( \frac{-i \pi}{|qE|} \right) = - \sum_{r=0}^{k} \alpha_k^{r,s} \omega_5^{2k} \left( \frac{n \pi}{|qE|} \right)^{k-2}. \] (52)

A resummation to a simple form is not achievable with arbitrary parameters, therefore as before let us examine a perturbative to \( O(\omega_5^2) \) expression. We find

\[ \text{Im} \Gamma(E, \omega_5) = \text{Im} \Gamma(E, 0) + \frac{T L^4 m^2 \omega_5^2}{24 \pi |qE|^2} \sinh^{-2} \left( \frac{\pi m^2}{2 |qE|} \right) \]

\[ \times \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \text{res}_k \left( \frac{-i \pi}{|qE|} \right). \]
that lines of constant $\text{Im}\Gamma(E, \omega_5)$ exist for approximately linear relationship between $\bar{m}$ and $\bar{\omega}_5$, indicating that for a given electric field the same pair production threshold can be met by substituting the mass for the spatial axial gauge. Or in other words, the spatial axial gauge serves to shift the mass, which is shown here for small $\bar{m}$ and $\bar{\omega}_5$, as visible in Eq. (50). One may still find a noticeable enhancement of pair production at higher $\bar{\omega}_5$, however the enhancement appears to polynomial degree, not exponential, in the above perturbative expansion. It is therefore important to further discuss the non-perturbative effect on the exponential found in Eq. (50). For this purpose we use the semi-classical worldline instanton approach in the next section. Doing so will also furthermore provide us with physical insight into the enhancement process.

Before introducing the worldline instanton approach, let us determine the real part of the effective action. Let us proceed as before and write Eq. (44) to $O(\omega_5^4)$ for $c_E \equiv \coth(qE|T|$ as

$$\Gamma(E, \omega_5) \approx \frac{T \mathcal{L}^3 |qE|^2}{8\pi^2} \int_0^\infty dT \left\{ \frac{c_E}{T^2} + 2\omega_5^2 |qE|(1 - c_E^2) \right\} e^{-im^2T}.$$  

(57)

Also like in the previous case with a magnetic field, one can introduce a cutoff in Schwinger propertime at a QED scale of $\Lambda^{-2}$, which is applied after a deformation of the propertime contour to the imaginary axis, and a change of variables, $T \to -iT$. Furthermore, we restrict our attention to the case of weak electric fields, then $\text{cot}(|qE|T) \approx (|qE|T)^{-1}$, and then one can find for the real part of Eq. (57)

$$\text{Re}\Gamma(E, \omega_5) \approx \frac{T \mathcal{L}^3 \omega_5^2}{8\pi^2} \int_{\Lambda^{-2}} \int_0^\infty dT \left\{ 2\omega_5^2 |qE|^2 + \frac{1}{T^2} - 1 \right\} e^{-im^2T}.$$  

(58)

Then using the integrals in Eqs. (26)-(29) one can find for $\text{Re}\Gamma_{\text{reg.}}[A, A_5] = \text{Re}[A, A_5] - \Gamma[0, 0]$ that

$$\text{Re}\Gamma(E, \omega_5)_{\text{reg.}} = \frac{T \mathcal{L}^3 \omega_5^2}{4\pi^2} \left\{ |qE|^2 + \Lambda^2 + m^2 \ln\left(\frac{m^2}{\Lambda^2}\right) \right\}.$$  

(59)

The key difference we see in this expression compared to Eq. (31) is that whereas the chiral chemical potential decreased the system energy, we see here that the spatial axial gauge increases the energy for the dominant $\Lambda^2$ factor indicating an instability. This is to be expected as we have observed above that indeed an instability is present in the form of an imaginary part of the effective action, whereby the spatial axial gauge further increases the imaginary part. $\Gamma[0, 0]$ here is the same as it was for Eq. (30). It is interesting to investigate non-Abelian extensions (see, e.g., Refs. [37]) that may capture some aspects of intertwined dynamics with nonzero topological configurations of the gauge field.
III. WORLDLINE INSTANTON METHOD

Above it was demonstrated that with a spatial axial gauge background the threshold for Schwinger pair production is reduced. Here we analyze its non-perturbative structure using the semi-classical worldline instanton method [21]. This has the added benefit of providing physical insight into the enhancement process. In contrast to conventional approaches used in applying the worldline instanton method, here with the addition of the axial gauge, the quadratic form of the Dirac operator, Eq. (133), is unnecessarily complicated for our purposes. The structure of the propagator and also the causal nature of the propagator is specified by inserting into each the resolution of identity of the expression into a product of infinitesimal proper-time elements, 

\[ \int dx \exp\left\{ iS_\text{W} \right\} \]

where

\[ S_\text{W} := \int_0^T d\tau \left[ -m - p_\mu \dot{x}^\mu - qA_\mu \dot{x}^\mu + \Pi_5 \right] \]

\[ \Pi_5 = \dot{p} + \gamma_5 A_5 \]

Here \( \int dx = \int dx' \int Dx \) denotes a periodic path integral with boundary conditions \( x(0) = x(T) = x' \). Also, note that we have left implicit the small imaginary component resolving the causal nature of the propagator and also providing convergence in the IR into the mass such that \( m \to m - i\epsilon \). Then, one need only split the path-ordered expression into a product of infinitesimal proper-time elements, inserting into each the resolution of identity of the BZ spinors given as

\[ \int d\Omega_\bar{z} z_a \bar{z}_b : = \mathcal{Z}_\eta^{-1} \partial_{\bar{z}_a} \partial_{\bar{z}_b} \mathcal{Z}_\eta \big|_{\eta = 0} = \delta_{ab} \]

where

\[ \mathcal{Z}_\eta = \int d\bar{z} d\bar{z} \exp(-\bar{z} \bar{z} + \bar{z} \eta \bar{z} + \eta \bar{z}) = \mathcal{Z}_0 \exp(\bar{\eta} \eta) \]

with \( \mathcal{Z}_0 = \int d\bar{z} d\bar{z} \exp(-\bar{z} \bar{z}) = \pi^4 \) [40]. After summing the infinitesimal elements, and also taking a change of variables such that \( \tau \to T_\tau \), one can finally find for the effective action the following form:

\[ \Gamma[A, A_5] = i \int_0^\infty \frac{dt}{T} \int Dz D\Omega_\bar{z} \int \frac{Dp}{(2\pi)^4} e^{iS'_\text{W}} \]

\[ S'_\text{W} := \int_0^1 d\tau \left[ -mT - p_\mu \dot{x}^\mu - qA_\mu \dot{x}^\mu + i\bar{z}\dot{z} + T\bar{z}\Pi_5 z \right] \]

the “dots” now represent total derivatives with respective to the integration variable.

In configuration space and in Euclidean spacetime, worldline instantons represent periodic classical solutions found at stationary points in \( x^\mu \) and propertime \( T \), which for simple field configurations are expressible for real coordinates thanks to an inverted time after Wick rotation. Here in Minkowski spacetime we have complex worldline instantons [41]: this can be seen in the fact the stationary points (poles in Euler-Heisenberg Lagrangian) lie on the imaginary propertime axis. An additional unique feature is that we are in phase-space and worldline instantons can be found at stationary points in \( p^\mu, x^\mu, \) and \( T \), whose classical equations of motion of Eq. (65) are

\[ \dot{x}_\mu = T \bar{z} \gamma_\mu z, \]

\[ \dot{p}_\mu = qF_{\mu\nu} \dot{x}^\nu, \]

\[ m = \bar{z} \Pi_5 z. \]

Here, we assume that the axial gauge \( A_5^a \) is independent of \( x^\mu \), but can otherwise take a general configuration. Let us also emphasize that we do not directly treat the configurations presented in Eqs. (8)-(9). Propertime in Eq. (65) acts as a Lagrange multiplier, sending the particle on-shell as can be seen in Eq. (68) above. Let us remark that for non-Abelian systems coherent state variables extend the phase-space leading to Wong’s equations [42] instead of just the Lorentz force. And it is instructive that here too stationary points should lie in the entire phase space, including \( \bar{z} \) and \( z \). However we already have the mass-shell constraint in Eq. (68) that will provide solutions in \( \bar{z} \) and \( z \). We must look for eigenvectors of a Dirac equation in the presence of an axial gauge background. There are four independent solutions,

\[ z \in u_i, v_i \quad \text{for} \quad i = 1, 2 \]

that are closely analogous to ordinary Dirac spinors that satisfy

\[ \Pi_5 z = \lambda z, \]

for eigenvalue, \( \lambda \) to be determined. In this way we find worldline instantons can be found for solutions in Eqs. (66)-(68) that lead to an imaginary part of the effective action; and according to Eq. (64), we have the periodic boundary conditions: \( x^\mu(0) = x^\mu(1), \bar{z}(0) = \bar{z}(1), \) and \( z(0) = z(1) \). Let us remark that the full equations of motion in \( \bar{z} \) and \( z \) read respectively \( \dot{\bar{z}} = -iT\bar{z}\Pi_5 \) and \( \dot{z} = iT\Pi_5 z \). However, they are incompatible with the eigenvalue solution found in Eq. (70), and whose inclusion would only lead to a trivial solution considering the constraint given in Eq. (68). One is left with a choice of whether to expand about stationary points in \( T \) (and evaluate Eq. (68) for \( z \) and \( \bar{z} \)) or in \( z \) and \( \bar{z} \) (and then later evaluate \( T \)); we have selected the former for two reasons: 1. In the case without an axial gauge field it is known that the spinor degrees of freedom do not affect worldline instantons in homogeneous fields. 2. Furthermore, we have
found above (e.g., Eq. (45)), that all poles in the exact expression can be located independent of the axial gauge field, as well as spin degrees of freedom. One caveat of taking the T integral and evaluating for $z$ and $\bar{z}$ can be seen in that what would be off-diagonal terms are neglected in the geometric or Berry [43] phase term, e.g., $\bar{u}_z u_z \sim 0$ for $x \neq u_\mu$. Therefore we can see that our approach is equivalent to the adiabatic theorem wherein each eigenvector persists and no level jumping between eigenvectors may occur. To see this connection, one need only diagonalize the path ordered expression in Eq. (60), giving rise to a geometric phase whose diagonal entries then describe our current setup.

Since we wish to extract the leading order exponential suppression, additionally, we neglect prefactor fluctuations about the worldline instantons [44]. One can then find for worldline instanton with winding number $n$ that

$$\text{Im} \Gamma[A, \bar{A}_5] \approx \sum_{z=u_{\pm}, \bar{u}_{\pm}} \sum_{n=1}^{\infty} e^{iS_{z,n}},$$

where $S_{z,n}$ is the worldline action, Eq. (65), evaluated for the classical worldline instanton solution. Since $T$ is linear in Eq. (65) and moreover since Eq. (68) holds, there is no fluctuation term and its evaluation about a given stationary point only introduces a numerical prefactor.

Let us proceed with the evaluation of the equations of motion. We begin by determining the eigensystem of $\mathcal{H}_5$, shown in Eq. (70). Its quadratic and quartic forms read

$$\mathcal{H}_5^2 = p^2 - A_5^2 + 2\sigma_{\mu\nu} p^\mu A_5^\nu,$$

$$\mathcal{H}_5^4 = 2(p^2 - A_5^2)\mathcal{H}_5^2 - 4[p^2 A_5^2 - (\sigma_{\mu\nu} A_5^\mu)^2] - (p^2 - A_5^2)^2,$$

where we have made use of the fact that $(\sigma_{\mu\nu} A_5^\mu)^2 = p^2 A_5^2 - (\sigma_{\mu\nu} A_5^\mu)^2$. We have also assumed implicit identity elements where appropriate for the (3+1)-dimensional Clifford group. We can see that this is nothing more than the characteristic equation leading to the eigenvalues of $\mathcal{H}_5$ according to Cayley-Hamilton’s theorem with $\mathcal{H}_5 \to \lambda$. Therefore we find the eigenvalues of $\mathcal{H}_5$ are

$$\lambda_{\mp, \pm} = \pm \sqrt{p^2 - A_5^2} \pm \frac{1}{2} \frac{p^2 - A_5^2}{\sqrt{p^2 A_5^2 + (\sigma_{\mu\nu} A_5^\mu)^2}}.$$

One can compare the above to operator eigenvalues that were determined for the magnetic field case in Eq. (18), and the electric field case in Eq. (41). However, we caution that the c-number momentum here represents the kinetic momentum, whereas above it represents a canonical momentum. Nevertheless, one can see the similar structure in $A_5^\mu$ present for all. In the absence of $A_5^\mu$ it can be seen the eigenvalues would describe degenerate $\pm \sqrt{p^2}$. A novelty here is that with the presence of $A_5^\mu$ the degeneracy is lifted, which we will later show impacts on pair production.

Then to determine the eigenvectors of $\mathcal{H}_5$, let us next construct projection operators from the above quartic equation as

$$P_+ = \frac{-i\gamma_{\sigma\mu\nu} p^\mu A_5^\nu}{2\sqrt{-p^2 A_5^2 + (\sigma_{\mu\nu} A_5^\mu)^2}} + \frac{1}{2},$$

$$P_- = \frac{i\gamma_{\sigma\mu\nu} p^\mu A_5^\nu}{2\sqrt{-p^2 A_5^2 + (\sigma_{\mu\nu} A_5^\mu)^2}} + \frac{1}{2}.$$

One can confirm that $P_+ P_- = 0$, $P_+^2 = P_+$, and that $P_- P_- = 1$. The projection operators have the effect of taking $\mathcal{H}_5^\mu P_\pm^\nu = \lambda^2_{\pm} P_\pm^\nu$. Finally let us construct orthonormalized eigenvectors using the projection operators. For the following eigenvectors corresponding to eigenvalue,

$$\mathcal{H}_5 u_{\pm} = \lambda_{\pm} u_{\pm}, \quad \bar{u}_{\pm} \mathcal{H}_5 = \bar{u}_{\pm} \lambda_{\pm},$$

one can determine that

$$u_{\pm} = N^\mu_\pm (\mathcal{H}_5 - \lambda_\pm) P_\pm^\mu \xi_\pm,$$

$$\bar{u}_{\pm} = N^\nu_\mp (\mathcal{H}_5 - \lambda_\pm)^0 P_\mp^\nu \eta_\pm,$$

where $\eta_\pm = u_{\pm}^\dagger \gamma^0$ need not be true and likewise for $\bar{u}_{\pm}$. This is because we will find worldline instantons in Minkowski space are in general complex. Let us also remark that the above reduce to ordinary Dirac spinors in the Weyl representation upon carefully taking the $A_5^\mu \to \pm 0$ limit in a symmetric way. We also take that $\xi_{\pm}^0 = [1, 0, 1, 0], \xi_{\mp}^1 = [0, 1, 0, 1], \eta_{\pm}^0 = [1, 0, -1, 0], \eta_{\mp}^1 = [0, 1, 0, -1]$. The eigenvectors are normalized such that $\bar{u}_{\pm} u_{\pm} = -\bar{v}_{\pm} v_{\pm} = 1$; this entails that

$$(N^\mu_\pm)^2 = [-2\lambda_\pm \gamma^0 T \mp (\mathcal{H}_5 - \lambda_\pm) P_\pm]^{-1},$$

$$(N_{\mp}^\nu)^2 = [-2\lambda_\pm \gamma^0 T \mp (\mathcal{H}_5 - \lambda_\pm) P_\pm]^{-1}. (83)$$

Next, taking partial derivative with respect to $p^\mu$—we denote as $\partial_\mu^p$—in Eqs. (77)-(78) one can determine that the velocities, according to Eq. (66), are

$$\bar{u}_{\pm} \gamma^0 u_{\pm} = \partial_\mu^p \lambda_{\pm}, \quad \bar{v}_{\pm} \gamma^0 v_{\pm} = -\partial_\mu^p \lambda_{\pm}. (85)$$

Note that we acquire a minus sign in $v_{\pm}$ due to their normalization.

$$\partial_\mu^p \lambda_{\pm} = \frac{1}{\lambda_{\pm}} \left\{ p_{\mu} A_5^\mu \frac{A_5^\mu (p_\mu A_5^\mu) - p_\mu A_5^\mu}{\sqrt{-p^2 A_5^2 + (\sigma_{\mu\nu} A_5^\mu)^2}} \right\}. (86)$$

At this point we must emphasize that there are four separate solutions, and with each one in general different values of $p_\mu$ and $x_\mu$ such that the three equations of motion, Eqs. (66)-(68), can be satisfied. To reduce cumbersome notation, we leave this distinction implicit in $p_\mu$ and $x_\mu$. This fact is readily apparent for Eq. (68) in which the
on-shell constraint is taken; the four distinct solutions follow from
\[ \check{u}_\pm \mathbf{1}_5 u_\pm = \lambda_{\pm} = m, \quad \check{v}_\pm \mathbf{1}_5 v_\pm = -\lambda_{\pm} = m. \] (87)

The above then entails that the following conditions be met for each set of solutions:
\[ u_+, v_+ : \sqrt{-p^2 A^2_\beta + (\rho_{\mu} A^\mu_\beta)^2} = \frac{1}{2} [m^2 - p^2 + A^2_\beta], \] (88)

\[ u_-, v_- : \sqrt{-p^2 A^2_\beta + (\rho_{\mu} A^\mu_\beta)^2} = \frac{1}{2} [p^2 - m^2 - A^2_\beta]. \] (89)

With application of the above constraint, we can modify Eq. (85); for example here we turn our attention to just the \( u_+ \) solution, whose velocity may now be written as
\[ \check{u}_+ \gamma_m u_+ =: \frac{1}{m^*} (p_\mu + C_\mu) = \frac{1}{T} \dot{x}_\mu, \] (90)

where we see the momentum has been modified with an augmented mass term as well as a shift:
\[ m^* = \frac{m^2 - p^2 + A^2_\beta}{m^2 - p^2 - A^2_\beta}, \] (91)

\[ C_\mu = \frac{2 \rho_{\nu} A^\nu_\mu A^\mu_\nu}{m^2 - p^2 - A^2_\beta}. \] (92)

One can find similar expressions for \( u_- \) and \( v_\pm \). With the addition of an axial gauge field the usual constraint \( p^2 = m^2 \) no longer applies we will find. The Lorentz force equation, Eq. (67), now becomes for \( u_+ \) solution,
\[ \dot{p}_\mu = \frac{q T}{m^*} F_{\mu \nu} (p^\nu + C^\nu). \] (93)

According to the constraint given in Eq. (88), we can determine that the invariants \( p^2 \) and \( A^\mu_{\nu} p_\mu \) must be constants of motion and independent of propertime. Then we can evaluate the Lorentz force equation as \( p(\tau) = \exp[(q T/m^*) F \tau] k - C \) for \( k = p(0) + C \), or rather
\[ p(\tau) = f_E(\tau) P_E k + f_B(\tau) P_B k - C, \] (94)

\[ f_E(\tau) = \cosh\left(\frac{q \lambda_{E} T \tau}{m^*}\right) + \frac{F}{\lambda_E} \sinh\left(\frac{q \lambda_{E} T \tau}{m^*}\right), \] (95)

\[ f_B(\tau) = \cos\left(\frac{q \lambda_{B} T \tau}{m^*}\right) + \frac{F}{\lambda_B} \sin\left(\frac{q \lambda_{B} T \tau}{m^*}\right). \] (96)

Note that when \( p^2 \) and \( A^\mu_\nu p_\mu \) are constants, so are \( C^\mu \) and \( m^\star \). Thus, \( k^\mu \) is independent of propertime. Here and where appropriate we use the matrix form for tensors and vectors with Lorentz indices, e.g., \( F_{\mu \nu} \equiv F \) and \( p^\mu \equiv p \), where contractions are assumed. An exact solution to the Lorentz force is possible with the use of the projection operators [45]
\[ P_E = \frac{\lambda_B^2 + F^2}{\lambda_B^2 + \lambda_E^2}, \quad P_B = \frac{\lambda_E^2 - F^2}{\lambda_B^2 + \lambda_E^2}, \] (97)

who satisfy \( P_E^2 = P_E, \quad P_B^2 = P_B, \quad P_E P_B = 0, \) and \( P_E + P_B = 1 \), and act to decouple the Lorentz force equation.

Note that we use similar conventions as employed in [46]. The electric and magnetic field eigenvalue strengths are respectively
\[ \lambda_E = \frac{1}{\sqrt{2}} \sqrt{I_{EF}^2 + 4 I_{EF}^2 - I_{EF}^2}, \] (98)

\[ \lambda_B = \frac{1}{\sqrt{2}} \sqrt{I_{EF}^2 + 4 I_{EF}^2 + I_{EF}^2}, \] (99)

where \( I_{EF} = -(1/8) e_{\mu \nu \alpha \beta} F^{\mu \nu} F^{\alpha \beta} = -(1/4) \tilde{F}_{\mu \nu} F^{\mu \nu} \) and \( I_{EF} = (1/2) F_{\mu \nu} F^{\mu \nu} \) are the Lorentz invariants. Then the coordinate solution follows from Eq. (66) as
\[ x(\tau) = \frac{1}{q} F^{-1} [e^{\frac{q}{m^*} F \tau} - 1] k + x(0); \] (100)

one can understand the exponential from Eq. (94). Then we find in order to satisfy the periodicity requirement, \( x(0) = x(1) \), using the above we must have that
\[ (f_E(1) - 1) P_E k = 0, \quad (f_B(1) - 1) P_B k = 0. \] (101)

One may determine a set of stationary points associated with the electric degrees of freedom as we wish to evaluate the effects of pair production; these are located for \( f_E(1) = 1 \) at
\[ T = \frac{2 \pi m^*}{q \lambda_E} \quad \forall n \in \mathbb{Z}^+. \] (102)

One may then select trivial magnetic like solutions through the initial condition \( P_B k = 0 \).

To further search for viable worldline instanton solutions let us examine the constant Lorentz invariant, \( A^\mu_\nu p_\mu \). By taking the propertime total derivative one can find that \( p_\mu F^{\mu \nu} A^\nu_\beta = 0 \) must hold, and hence that \( A^{\mu}_\nu F^\nu_\tau (P E)_{\mu \nu} k^\nu = 0 \) must hold as well. One cannot find a solution in the propertime independent \( k^\mu \) for all propertimes. Therefore, one is left with an orthogonality constraint in the instanton momentum and axial gauge field, or rather \( A^\mu_\nu p_\mu = 0 \); this in turn requires that \( C^\nu = 0 \). Since the magnetic degrees of freedom are trivial the constraint implies that
\[ A^\mu_\nu F^\nu_\tau P_E = 0. \] (103)

To further emphasize this point let us take a Lorentz transformation that will effectively diagonalize our field strength tensor as (where we have made use of the matrix form)
\[ F^\nu_\tau \rightarrow A^{-1} F A = \begin{pmatrix} \lambda_B & \lambda_E \\ -\lambda_B & \lambda_E \end{pmatrix}, \] (104)

then the projection operators have the simple form
\[ P_E^\mu_\nu = P_E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad P_B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \] (105)
Solutions to the momenta follow as
\begin{align}
p^0 &= \cosh\left(\frac{q\lambda ET\tau}{m^*}\right)k^0 + \sinh\left(\frac{q\lambda ET\tau}{m^*}\right)k^3, \\
p^3 &= \cosh\left(\frac{q\lambda ET\tau}{m^*}\right)k^3 + \sinh\left(\frac{q\lambda ET\tau}{m^*}\right)k^0.
\end{align}

Since \(P_\perp k = 0\), we have that \(p^1 = p^2 = 0\). Last, let us assume the axial gauge we are studying is the one after the unique Lorentz transformation leading to the above, i.e., \(A^\perp \to (A^\perp)' = \Lambda^\perp_5 A^\perp\). Then according to the axial gauge as described above, we find that the constraint in Eq. (103) implies that in order to find worldline instanton solutions we must have \(A_5^\perp k^0 - A_3^\perp k^3 = 0\) and \(A_0^\perp k^3 - A_3^\perp k^0 = 0\). However, for \(A_5^\perp \neq 0\) this constraint would ultimately imply that \(k^2 = p^2(0) = 0\), which we will shortly find would give a trivial worldline instanton solution. Therefore, we take \(A^\perp = A^3 = 0\) with only \(A^1\) and \(A^2\) contributing. This leads us to the following definition for the axial gauge field invariant:
\[\omega_5 = \sqrt{-A_5^2};\] (108)
while such a distinction is not in one-to-one correspondence to the definition used for the spectral decomposition in Eq. (9), (in particular since here we have projected the electric eigenvalue strength, \(\lambda_E\), rather than background electric field), the above is useful for comparison purposes. Let us however caution that differences do exist for the setups and assumptions here in contrast to those used in Sec. II. We will find a similar treatment holds for the \(u_+\) and \(v_\pm\) solutions as well leading to Eq. (108) for all cases.

Before further pressing on with the \(u_+\) solution, let us determine the various Berry phase factors for all \(u_\pm\) and \(v_\pm\) under the condition \(p_\mu A_5^\mu = 0\). Let us begin by expressing the Berry phases as
\[\bar{u}_\pm \dot{u}_\pm = \frac{N_{u\mp}^2}{2} \xi_\pm^I B_5^I \xi_\pm^I, \quad \bar{v}_\pm \dot{v}_\pm = -\frac{N_{v\mp}^2}{2} \eta_\pm^I B_5^I \eta_\pm^I,\]
with arguments entirely expressible in terms of commutation relations in
\[B_5^\mu = \left[ P_\mp (\Pi_5 - \lambda_\mp), \frac{d}{d\tau} (\Pi_5 - \lambda_\mp) P_\mp \right], \quad \bar{B}_5^\mu = \left[ P_\mp (\Pi_5 - \lambda_\pm), \frac{d}{d\tau} (\Pi_5 - \lambda_\pm) P_\mp \right].\] (109)

After some lengthy but straightforward manipulations one can find that the above reduce to
\[B_5^\mu = \frac{\lambda_\pm}{p^2 \omega_5^2} \left( \omega_5^2 \lambda_\mp + \left[ \omega_5^2 + \omega_5 \sqrt{p^2} \gamma_5 A_5 \right] \right) \dot{p} \dot{\phi}, \quad \bar{B}_5^\mu = \frac{\lambda_\pm}{p^2 \omega_5^2} \left( \omega_5^2 \lambda_\pm + \left[ \omega_5^2 + \omega_5 \sqrt{p^2} \gamma_5 A_5 \right] \right) \dot{p} \dot{\phi}.\] (110)

However, with application of \(\xi_\pm\) and \(\eta_\pm\) as given above Eq. (83), and application of worldline instanton solutions in momentum space, Eqs. (106)-(107) as well as the fact that \(A_5^0 = A_3^0 = 0\), one can confirm for the \(u_+\) solution, the Berry phase disappears, \(\bar{u}_+ \dot{u}_+ = 0\). In a similar way, once worldline instanton solutions are provided one can also confirm the Berry phases for the remaining spinor solutions also vanish; all together
\[\bar{u}_\pm \dot{u}_\pm = \bar{v}_\pm \dot{v}_\pm = 0.\] (111)

The disappearance of the Berry phase factors for the case of \(A^1 = A^2 \neq 0\) occurs from the decoupling nature of the axial gauge degrees of freedom with those of the instantons. Indeed, from a physical standpoint it was determined that a vanishing Berry phase was present in systems with inversion and time-reversal symmetries present [47], and here the spatial axial gauge does not break parity, and hence the time-reversal symmetry should also be present. Let us furthermore note that even for the case of no axial gauge the Berry phase term is trivial; one can understand this from the familiar quadratic and in coordinate space expression with spin factor, there the spin factor, and hence spin degrees of freedom, does not affect instanton trajectories.

Returning to the evaluation of the \(u_+\) solution, by invoking Cayley-Hamilton’s theorem, Eq. (73), once more (now under the constraint \(A_5^\mu p_\mu = 0\)), and evaluating for the momentum in terms of the eigenvalue, we can determine that two solutions for the momentum invariant are possible:
\[p^2 = (m \pm \omega_5)^2.\] (112)
We need to find which solution is valid for \(u_+\), therefore we insert both solutions for \(p^2\) into the constraint given in Eq. (88). Then using the fact that \(A_5^0 < 0\), we can determine that for the \(u_+\) solution
\[p^2 = (m - \omega_5)^2.\] (113)
Furthermore in order to satisfy Eq. (88), we find that \(m \geq \omega_5\) must also hold. Hence, with the addition of the axial gauge the mass-shell condition has been augmented, in this way using the above, one can find that the effective mass, Eq. (91), has also been augmented,
\[m^* = m - \omega_5,\] (114)
leading to augmented stationary point on the imaginary propertime axis according to Eq. (102). It is in this way that pair production is enhanced. Let us now determine the worldline action for the instanton solutions, and to do so let us select a gauge: we use the Fock-Schwinger gauge, i.e., \(A_\mu = -(1/2) F_{\mu\nu} x^\nu\). Using Eqs. (90) and (93), one could obtain
\[\int_0^1 d\tau p_\mu \dot{x}^\mu = \int_0^1 d\tau p_\mu \dot{x}^\mu / 2 = -\int_0^1 d\tau p_\mu \dot{x}^\mu / 2 \text{ from the integration by parts and cancellation of surface terms. With Eq. (68) and vanishing Berry phases, the remaining term in Eq. (65) becomes } S_W' = -\int_0^1 \dot{x}_\mu \dot{x}^\mu = -\int_0^1 d\tau p_\mu \dot{x}^\mu / 2 = -T \dot{p}^2 / (2m^*) \text{ based on } \dot{p}^2 \text{ is constant.}
\]
Finally using Eqs. (102), (115), and (116), we can find that the action for worldline instanton for the \(u_+\) solution is
\[S_{u+,n} = \frac{n \pi i (m - \omega_5)^2}{q \lambda_E}.\] (117)
The threshold with which the Schwinger mechanism is to overcome has been effectively reduced in mass by the axial gauge. However, we still must evaluate the remaining three other solutions.

Next, we evaluate instanton solutions associated with \( v_+ \). However, notice that the mass-shell constraint leading to Eq. (88) is the same as it was for \( u_+ \). This is not all. In fact, for \( v_+ \) too one can confirm in analogy to Eq. (90) that

\[
\bar{v}_+ \gamma \mu v_+ = \frac{1}{m^*} p_\mu = \frac{1}{T} \bar{x}_\mu ,
\]

which is the same as for the \( u_+ \) case. Furthermore, we find all the equations of motion and constraints are identical to the \( u_+ \) case, thus we lead to

\[
S_{v_+,n} = S_{u_+,n} ,
\]

since the Berry phases vanish for both.

Let us now last evaluate solutions associated with \( u_- \) and \( v_- \). In analogy to \( u_+ \) and \( v_+ \) we find here too that the solutions for either \( u_- \) or \( v_- \) are identical to the other, therefore let us treat \( u_- \). The mass-shell constraint here changes according to Eq. (89). However, because of the constraint, for \( \bar{u}_- \gamma \mu u_- = p_\mu /m^* \) we find that the definition of the effective mass is the same as the one given in Eq. (91); what we find will change is the momentum invariant. Recall that two solutions are possible for the momentum invariant: \( p^2 = (m \pm \omega_5)^2 \). However, here after inserting both into Eq. (89) we now find that

\[
p^2 = (m + \omega_5)^2 ,
\]

c.f., Eq. (115), and hence also

\[
m^* = m + \omega_5 ,
\]

c.f., Eq. (116). Last, here our stationary points are now at Eq. (102) with Eq. (121) above. Again, our equations of motion are identical to before with the only change being the mass-shell constraint leading to the above augmented mass. Therefore, one can confirm in this case that

\[
S_{u_-n} = S_{v_-n} = \frac{n \pi i (m + \omega_5)^2}{q \lambda E} .
\]

Gathering all the solutions one can finally find for the worldline instanton evaluated imaginary part of the effective action

\[
\text{Im} \Gamma(\lambda_E, \omega_5) \approx 2 \sum_{n=1}^{\infty} \left\{ e^{-\frac{n \pi i}{q \lambda E} (m-\omega_5)^2} + e^{-\frac{n \pi i}{q \lambda E} (m+\omega_5)^2} \right\} .
\]

The most important observation in the above is that the exponential quadratic negative mass shift, found before in Eq. (50), persists in the semiclassical approach. Therefore we can see the pair production enhancement from a spatial axial gauge, or vorticity, is a robust feature. The origin of the mass shift can be traced back to the augmented Dirac equation (34). The spatial axial gauge term can be rearranged as an energy shift by the spin-vorticity coupling \( \gamma_5 (\bar{D}^a + i \gamma_5 \gamma_3 \omega_5) \), where \( i \gamma_5 \gamma_3 \) is the spin operator along \( \omega_5 \) applied there. Even though there is a decreased probability of pair production for the \( u_- \) and \( v_- \) cases, the overall pair production is favorably enhanced due to the \( u_+ \) and \( v_+ \) cases.

Let us finally mention that the results here agree–or rather are proportional to, since we are neglecting a prefactor contribution here–with those found in the previous section for \( n = 1 \) to \( \mathcal{O}(\omega_5^2) \), or alternatively in the \( m \gg \omega_5 \) limit. The discrepancy at higher orders may be attributed to fluctuations about the instantons, and or from contributions of off-diagonal terms in the Berry phase. Nevertheless we most importantly see the overall subtraction of the mass in the exponential due to the spatial axial gauge.

IV. MASSLESS EFFECTIVE ACTION WITH AXIAL GAUGE

Above we analyzed the one-loop effective action with massive fields, and alternatively one could arrive at corresponding observables for the nearly massless case by carefully taking the massless limit. The effective action determined from a purely massless case, i.e., a theory which begins with no mass term, in fact need not be the same. Namely, even in the case of no axial gauge, i.e.,

\[
\lim_{m \to 0} \int \mathcal{D}\bar{\psi} \mathcal{D}\psi e^{i \int d^4x \bar{\psi}\{\mathbb{D} - m\}\psi} \\
\neq \int \mathcal{D}\bar{\psi} \mathcal{D}\psi e^{i \int d^4x \bar{\psi}\{\mathbb{D} - m\}\psi} = e^{i \Gamma_{\text{massless}}(A,0)} .
\]

With axial gauge, clarifying the differences between the two effective actions is important in that we will find markedly different physics emerge. Notably, the massless case possesses an exact and simple solution through the Fujikawa method [48], whereas in the massless limit case no such term is present (for homogeneous fields). First let us examine the massive case perturbatively in \( A_5^E \), as doing so provides context for the above perturbative results.

Let us begin by writing down the massive effective action, c.f., Eq. (1) for arbitrary \( A_5^E \) as

\[
e^{i \Gamma(A, A_5^E)} = \int \mathcal{D}\bar{\psi} \mathcal{D}\psi e^{i \int d^4x \bar{\psi}\{\mathbb{D} - m\}\psi} .
\]

Then to define a perturbative expansion about \( A_5^E \), let us look at a formal functional expansion [49] of the fermionic determinant that leads to

\[
\Gamma(A, A_5^E) = \Gamma(A,0) + \sum_{n=1}^{\infty} \frac{i}{n} \text{Tr} \left[ -\frac{1}{i \mathbb{D} - m} \gamma_5 A_5^E \right]^n .
\]
One can then find that to $\mathcal{O}(A_5^3)$

$$
\Gamma[A,A_5] \approx \Gamma[A,0] + i \int d^4x \lim_{x \to y} \text{tr} [\gamma_5 A_5 S^c(x,y)] \\
+ \frac{i}{2} \int d^4x d^4y \text{tr} [\gamma_5 A_5 S^c(x,y) \gamma_5 A_5 S^c(y,x)] ,
$$

(127)

where the electromagnetic dressed propagator satisfies

$$(i\partial_x - m)S^c(x,y) = -\delta^4(x-y).$$

It can be seen that the effective action admits an expansion whereby the dressed propagator interacts with any number of interactions of an external axial gauge field of $\gamma_5 A_5$.

For concreteness let us examine homogeneous electromagnetic fields; here $S^c$ is well-known [3]. Then for the point split in-out propagator in the coincidence limit we take $\lim_{x \to y} S^c(x,y) := \lim_{x \to 0} (1/2)[S^c(x,x+\epsilon) + S^c(x+\epsilon,x)]$, which leads to a vanishing axial current, i.e., $j_5^\mu = \lim_{x \to y} \gamma^\mu \gamma_5 S^c(x,y) = 0$. See Ref. [50] for details. Therefore the $\mathcal{O}(A_5)$ term in the massive effective action, Eq. (127), vanishes. Higher order terms, as evidenced above, do not vanish. And importantly, for the purely massless case we will show that only the $\mathcal{O}(A_5^3)$ term is present.

Using the Fujikawa method [48] one may determine an entirely different perturbative scheme about the axial gauge. Let us take an axial gauge rotation in Eq. (125), with $\psi \to \exp(-i\gamma_5 A_{5\mu} x^\mu)\psi$, then

$$
e^{i\Gamma[A,A_5]} = \int D\bar{\psi} D\psi \exp(2i\text{tr}[\gamma_5 A_{5\mu} x^\mu])$$

$$\times \exp\{i \int d^4x \bar{\psi} [i\not{D} - me^{-2i\gamma_5 A_{5\mu} x^\mu}] \psi\},
$$

(128)

where we remind that the functional trace of $\gamma_5$ requires regulation and is in general non-vanishing, this is more easily recognizable with angle $\theta(x) = A_{5\mu} x^\mu$. The functional trace term, $\text{Tr}[\gamma_5 A_{5\mu} x^\mu]$, leads to the anomalous Chern-Simons term in QED. Let us perform the perturbative expansion as was done before for Eq. (126). One can eventually find to $\mathcal{O}(A_5^3)$ that

$$
\Gamma[A,A_5] = \Gamma[A,0] + 2\text{Tr}[\gamma_5 A_{5\mu} x^\mu]$$

$$+ 2im \int d^4x \text{tr} [i\gamma_5 A_{5\mu} x^\mu + (A_{5\mu} x^\mu)^2] \lim_{x \to y} S^c(x,y)]$$

$$- 2im^2 \int d^4x d^4y \text{tr}[S^c(x,y)\gamma_5 A_{5\mu} y^\mu S^c(y,x)\gamma_5 A_{5\nu} x^\nu].
$$

(129)

Here we have acquired a new perturbative expansion about $A_5$ using the Fujikawa method. At $\mathcal{O}(A_5)$ we can then see that we have a statement of the axial-Ward identity considering Eq. (127). Therefore we can understand for massive fermions in homogeneous fields why the $\mathcal{O}(A_5)$ term vanishes as a consequence of the vanishing axial-Ward identity [50]. Yet, no such vanishing occurs for the purely massless case.

The purely massless effective action for arbitrary field can be read off of Eq. (129), and is

$$
\Gamma_{\text{massless}}[A,A_5] = \Gamma_{\text{massless}}[A,0] + 2\text{Tr}[\gamma_5 A_{5\mu} x^\mu].
$$

(130)

The full massless effective action with an axial gauge potential is augmented only through the anomalous term arising from the fermion determinant. Let us next evaluate the functional trace for arbitrary field. We define the regularized trace through eigenmodes of the quadratic form, $\hat{\Pi}_5$. Even though our starting Lagrangian is augmented with the axial gauge, its zero modes are the same as the case without an axial gauge since the axial gauge can be removed by a redefinition, and therefore it is expected the index theorem [51] for both QED and QED with an axial gauge be the same. Let us demonstrate this identification to $\mathcal{O}(A_5^3)$. To begin we define for $\Lambda \to \infty$ with

$$I_{\text{CS}} \equiv \text{Tr} \theta_5 = \int d^4x \theta \text{tr}[x|\gamma_5 \frac{\Lambda^2}{\not{D} + \not{\Lambda}} |x\rangle ,
$$

(131)

which agrees with a conventional quadratic operator in operator regularization if no axial gauge field were present [52]. Let us cast the functional trace in Schwinger propertime (where we have performed a propertime redefinition) as

$$I_{\text{CS}} = \int d^4x \theta \int_0^\infty dt \text{tr}[x|\gamma_5 \frac{\Lambda^2}{\not{D} + \not{\Lambda}} |x\rangle.
$$

(132)

One may express the quadratic operator as [33]

$$-\hat{\Pi}_5^2 = \not{D}^2 + A_5^2 + i\gamma_5 [D_\mu A_{5\mu}^\nu + 2i\sigma_{\mu\nu} A_{5\mu}^\rho D^\rho].
$$

(133)

Then expanding to $\mathcal{O}(A_5^2)$ and treating only the lowest order terms in the electromagnetic coupling that are non-vanishing after taking the trace we find that

$$I_{\text{CS}} = \int d^4x \theta \text{tr}[x|\gamma_5 \frac{\Lambda^2}{\not{D} + \not{\Lambda}} |x\rangle.
$$

(134)

Therefore we can see that the functional trace is the same as the one in QED and hence familiar manipulations lead to [49]

$$I_{\text{CS}} = - \int d^4x \theta \frac{q^2}{8\pi^2} F_{\mu\nu} F^{\mu\nu}.
$$

(135)

One can see that if we were to have homogeneous fields such a term would trivially vanish due to the linear $x$ under the integrand. However, this does not entail that new physics does not emerge for the purely massless case. Indeed one may calculate the vector current:

$$j_\mu = 2M_{\text{CS}} / \delta A^\mu$$

to find the chiral magnetic effect [15] for $A_5^\mu = \mu_5$:

$$j = \frac{q^2}{2\pi^2} \mu_5 B,
$$

(136)

and also for $A_5 = \omega_5$, one can find corrections to the charge density [53] such that

$$j^0 = \frac{q^2}{2\pi^2} \omega_5 \cdot B.
$$

(137)

However such terms would not be present for the massive case in the massless limit according to Eq. (129) since the Chern-Simons term is matched by the pseudoscalar condensate term via the axial-Ward identity.
Augmentations to the one-loop Euler-Heisenberg Lagrangian with an axial gauge coupling have been examined. For the massive case (as well as massless limit of a massive theory) we confined our attention to two configurations with an exact eigendecomposition: 1. a magnetic field with chiral chemical potential and 2. an electric field with spatial axial gauge field (that has been argued to resemble a vorticity). In a perturbative expansion about $A_5^\mu$ for weak fields in $\Re[\mathcal{L}_{\text{EH}}]$, it was determined in the absence of electromagnetic fields an overall increase (decrease) of the system energy was present for the chiral chemical potential (spatial axial gauge). It was also shown how the axial gauge may alter the Maxwell Lagrangian.

In important feature was observed in $\Im[\mathcal{L}_{\text{EH}}]$ for the case of the electric field with spatial axial gauge in that the spatial axial gauge enhanced the Schwinger pair production. This was shown perturbatively in $O(A_5^\mu)$ through the eigendecomposition approach; an all-orders expression was also derived. Using a semi-classical world-line instanton approach it was determined that the enhancement was possible from a negative mass shift. The enhancement occurs at the exponential level and is therefore thought significant is systems with vorticity.

The massless case (where no mass term is present in the partition function) had an exact solution thanks to the Fujikawa method. It was demonstrated that an axial gauge could be rotated away producing a Chern-Simons term as a corrective factor to the massless Euler-Heisenberg Lagrangian.

An important feature not discussed here is the massive case with simultaneous electric and magnetic fields such that $\mathbf{E} \cdot \mathbf{B} \neq 0$. Although the two configurations examined here have exact eigendecompositions, we cannot study the anomalous features present in the massless case. One should anticipate such features for the massive case, because they are present even without an axial gauge, leading to a description of the axial-Ward identity. Therefore, it is an important extension to this work to analyze the case with $\mathbf{E} \cdot \mathbf{B} \neq 0$ under an axial gauge field.

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