Variable-Length Coding with Stop-Feedback for the Common-Message Broadcast Channel in the Nonasymptotic Regime

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Abstract

We investigate the maximum coding rate for a given average blocklength and error probability over a $K$-user discrete memoryless broadcast channel for the scenario where a common message is transmitted using variable-length stop-feedback codes. For the point-to-point case, Polyanskiy et al. (2011) demonstrated that variable-length coding combined with stop-feedback significantly increase the speed of convergence of the maximum coding rate to capacity. This speed-up manifests itself in the absence of a square-root penalty in the asymptotic expansion of the maximum coding rate for large blocklengths, i.e., zero dispersion. In this paper, we present nonasymptotic achievability and converse bounds on the maximum coding rate of the common-message $K$-user discrete memoryless broadcast channel, which strengthen and generalize the ones reported in Trillingsgaard et al. (2015) for the two-user case. An asymptotic analysis of these bounds reveals that zero dispersion cannot be achieved for certain common-message broadcast channels (e.g., the binary symmetric broadcast channel). Furthermore, we identify conditions under which our converse and achievability bounds are tight up to the second order. Through numerical evaluations, we illustrate that our second-order expansions approximate accurately the maximum coding rate and that the speed of convergence to capacity is indeed slower than for the point-to-point case.

I. INTRODUCTION

We consider the setup in which an encoder wishes to convey a common message over a discrete memoryless broadcast channel with feedback to $K$ decoders. Similarly to the single-decoder case, full feedback (i.e., instantaneous feedback of the received symbols) combined with fixed-blocklength codes does not improve capacity, which is given by [2, p. 126]

$$C = \sup_{P} \min_{k \in \{1, \ldots, K\}} I(P, W_k).$$

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Here, \(W_1, \ldots, W_k\) denote the channels to the decoders \(1, \ldots, K\), respectively, and the supremum is over all input distributions \(P\). For the case of no feedback, the common-message broadcast channel is equivalent to a compound channel, and the speed at which \(C\) is approached as the blocklength \(n\) increases is of the order \(1/\sqrt{n}\) (see [3]), which is the same as in the single-decoder (point-to-point) case [4].

For point-to-point channels, although feedback does not increase capacity, it improves dramatically the error exponent, provided that variable-length codes are used. This was demonstrated by Burnashev who found that the error exponent for this setting is given by [5]

\[
E(R) = \frac{\tilde{C}_1}{C} (\tilde{C} - R)
\]

for all rates \(0 < R < \tilde{C}\). Here, \(\tilde{C}\) denotes the channel capacity for the point-to-point case and \(\tilde{C}_1\) denotes the maximum relative entropy between conditional output distributions. Yamamoto and Itoh [6] proposed a two-phase scheme that attains the error exponent in (2); furthermore, Berlin et al. [7] provides an alternative and simpler converse proof to (2), which parallels the two-phase scheme proposed in [6].

In this paper, we shall be concerned with the scenario in which the feedback channel is used only to stop transmissions (stop/decision feedback). Following [8], we shall refer to variable-length coding schemes relying on stop feedback as variable-length stop-feedback (VLSF) codes. It was shown in [8], [9], [10] that the error exponent

\[
E(R) = \tilde{C} - R
\]

is achievable using only VLSF codes. However, no converse result is known.

In the fixed-error regime, Polyanskiy et al. [8] found that the speed at which the maximum coding rate converges to capacity is significantly improved in the presence of full feedback and variable-length codes. Specifically, they showed that

\[
\frac{1}{\ell} \log \tilde{M}_f(\ell, \epsilon) = \frac{\tilde{C}}{1 - \epsilon} - O\left(\frac{\log \ell}{\ell}\right)
\]

where \(\ell\) denotes the average blocklength (average transmission time) and \(\tilde{M}_f(\ell, \epsilon)\) is the maximum number of codewords that can be transmitted with average transmission time \(\ell\) and average error probability \(\epsilon\) in the point-to-point case. One sees from (4) that no square-root penalty occurs (zero dispersion), which implies a fast convergence to the asymptotic limit. This fast convergence is demonstrated numerically in [8] by means of nonasymptotic bounds. It turns out that full feedback is not necessary to obtain zero dispersion. Indeed, one can achieve (4) with VLSF codes. In other words, stop feedback is enough to achieve zero dispersion. However, to the best of our knowledge, there are no nonasymptotic converse bounds for the case of stop feedback that are tighter than the corresponding ones for full feedback.

The zero-dispersion result (4) does not extend to the case of common-message discrete memoryless broadcast channels (CM-DMBC) considered in this paper when only stop feedback is available. Specifically, we showed in [11] that the dispersion of this channel can be positive when VLSF codes are used. More precisely, there exist CM-DMBCs for which the second term in the asymptotic expansion of the maximum coding rate is of order \(1/\sqrt{\ell}\) (cf. (4)). Our analysis in [11] is limited to the two-user case and relies on the restrictive assumption that there exists a unique input distribution \(P^*\) that simultaneously maximizes \(I(P, W_1)\) and \(I(P, W_2)\). Furthermore, the upper and lower bounds on the maximum coding rate provided in [11] do not match up to the second order. In this paper, we refine the results obtained in [11] and extend them to a broader class of common-message broadcast channels.
We adopt the convention that if we let \( f \) with \( k \) and the index \( N \) constant. Its value may change at each occurrence. Finally, \( n \) and the support of a probability mass function \( P \) simultaneously maximizes \( I(P, W_1) \) and \( I(P, W_2) \) (the case treated in [11]). Numerical evaluations of our nonasymptotic achievability and converse bounds reveal that the asymptotic expansion of \( M_d^*(\ell, \epsilon) \) obtained in this paper provides an accurate approximation for \( M_d^*(\ell, \epsilon) \).

**Notation:** We denote the \( n \)-dimensional all-zero vector and the \( n \)-dimensional all-one vector by \( \mathbf{0}_n \) and \( \mathbf{1}_n \), respectively. Vectors are denoted by boldface letters (e.g., \( x \)), while their entries are denoted by roman letters (e.g., \( x_i \)). The length of a vector is denoted by \( \text{len}(\cdot) \) and the Euclidean norm by \( \|x\| \). For a differentiable function \( f(\cdot) \), we let \( f'(\cdot) \) denote its derivative. Upper case, lower case, and calligraphic letters indicate random variables (RV), deterministic quantities, and sets, respectively. The cardinality of a set is denoted by \(|\cdot|\) (e.g., \(|A|\)). We let \( x_m^n \) denote the tuple \( (x_m, \ldots, x_n) \). When \( m = 1 \), the subscript is sometimes omitted. We denote the set of probability distributions on \( A \) by \( \mathcal{P}(A) \) and the support of a probability mass function \( P \) by \( \text{supp}(P) \). For a RV \( X \) with probability distribution \( P \), we let \( P^n \) denote the joint probability distribution of the vector \( [X_1, \ldots, X_n] \), where \( \{X_i\} \) are independently and identically distributed (i.i.d.) according to \( P \). The probability density function of a standard Gaussian RV is denoted by \( \phi(\cdot) \). Furthermore, \( \Phi(x) \equiv 1 - Q(x) \) is its cumulative distribution function, with \( Q(\cdot) \) being the \( Q \) function. We let \( x^+ \) denote \( \max(0, x) \). Throughout the paper, \( \log(\cdot) \) is the base \( e \) logarithm and the index \( k \) belongs always to the set \( K \equiv \{1, \ldots, K\} \), although this is sometimes not explicitly mentioned. We adopt the convention that \( \sum_{i=j}^{j-1} a_i = 0 \) for all \( a_i \) and for all integers \( j \). We use “\( \epsilon \)” to denote a finite nonnegative constant. Its value may change at each occurrence. Finally, \( \mathbb{N} \) denotes the set of positive integers, \( \mathbb{Z}_+ \equiv \mathbb{N} \cup \{0\} \), the symbol \( \mathbb{R} \) indicate the set of real numbers, and \( \mathbb{R}_0^+ \) denotes the set \( \left\{ x \in \mathbb{R}^n : \sum_{i=1}^n x_i = 0 \right\} \).

**II. System Model**

A CM-DMBC with \( K \) decoders consists of a finite-cardinality input alphabet \( \mathcal{X} \), and finite-cardinality output alphabets \( \{\mathcal{Y}_k\} \), along with \( K \) stochastic matrices \( \{W_k\} \), where \( W_k(y_k|x) \) denotes the probability that \( y_k \in \mathcal{Y}_k \) is observed at decoder \( k \) given the channel input \( x \in \mathcal{X} \). We assume, without loss of generality, that \( \mathcal{X} = \{1, \ldots, |\mathcal{X}|\} \).
The outputs at time $t$ are assumed to be conditionally independent given the input, i.e.,

$$P_{Y_1,t,\ldots,Y_{K,t}|X_t}(y_{1,t},\ldots,y_{K,t}|x_t) \triangleq \prod_k W_k(y_{k,t}|x_t). \quad (5)$$

The assumption (5) comes without loss of generality if one is interested in infinite-blocklength analysis with vanishing error probability. However, this ceases to be true in the fixed-error-probability setup considered in the present paper. Let $\mathcal{P}(\mathcal{X})$ be the set of all probability distributions on $\mathcal{X}$. Let also $P \times W_k : (x, y_k) \mapsto P(x)W_k(y_k|x)$ denote the joint distribution of input and output at decoder $k$. Finally, let $PW_k : y_k \mapsto \sum_{x \in \mathcal{X}} P(x)W_k(y_k|x)$ denote the induced marginal distribution on $\mathcal{Y}_k$. For every $P \in \mathcal{P}(\mathcal{X})$ and $n \in \mathbb{N}$, the information density is defined as

$$i_{P,W_k}(x^n;y^n_k) \triangleq \sum_{i=1}^n \log \frac{W_k(y_{k,i}|x_i)}{PW_k(y_{k,i})}. \quad (6)$$

We let $I_k(P) \triangleq \mathbb{E}_{P \times W_k}[i_{P,W_k}(X;Y_k)]$ be the mutual information,

$$V_k(P) \triangleq \mathbb{E}[\text{Var}_{P \times W_k}[i_{P,W_k}(X;Y_k)|X]] \quad (7)$$

be the conditional information variance, and $T_k(P) \triangleq \mathbb{E}_{P \times W_k}[|i_{P,W_k}(X;Y_k) - I_k(P)|^3]$ be the third absolute moment of the information density.

The capacity of the CM-DMBC is given by (1), where the supremum is over all probability distributions $P \in \mathcal{P}(\mathcal{X})$. We restrict ourselves to the case where the supremum in (1) is attained by a unique probability distribution $P^*$. The corresponding (unique) capacity-achieving output distribution for decoder $k$ is denoted by $P^*_k$. Furthermore, the individual capacities of each of the discrete memoryless component channels $\{W_k\}$ are denoted by

$$C_k \triangleq \sup_{P \in \mathcal{P}(\mathcal{X})} I_k(P). \quad (8)$$

Finally, we let $V_k \triangleq V_k(P^*)$ and let $\nabla I_k(v)$ denote the directional derivative of the mutual information $I_k(P)$ along the direction $v \in \mathbb{R}_0^{[\mathcal{X}]}$ at the point $P^*$

$$\nabla I_k(v) \triangleq \sum_{x \in \mathcal{X}} v_x D(W_k(\cdot|x)||P^*_k). \quad (9)$$

Here, $D(\cdot||\cdot)$ denotes the Kullback-Liebler divergence.

In addition to (5) and to the uniqueness of $P^*$, we shall also assume that the channel laws $\{W_k\}$ satisfy the following conditions:

1) $I_k(P^*) = C$ for every $k \in \mathcal{K}$.
2) $V_k(P^*) > 0$ for every $k \in \mathcal{K}$.
3) $P^*(x) > 0$ for all $x \in \mathcal{X}$.

The first condition is not critical, and it is added only to simplify the statement of our results. Indeed, the nonasymptotic bounds we shall present in Theorem 1 and 2 hold also for the case when $I_k(P^*) > C$ for some $k$. Furthermore, our positive dispersion result (Theorem 4) holds also when the first condition is violated, provided that there exist at least two component channels $k_1$ and $k_2$ such that $I_{k_1}(P^*) = I_{k_2}(P^*) = C$. This is because the decoders whose component channels satisfy $I_k(P^*) > C$ feed back their stop signals much earlier than the remaining decoders and therefore do not contribute to the asymptotic expansion of $\log M^*_0(\ell, \epsilon)$. If $I_k(P^*) = C$ only for a single component channel, then zero dispersion can be attained.

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We are now ready to formally define a VLSF code for the CM-DMBC.

**Definition 1:** An \((\ell, M, \epsilon)\)-VLSF code for the CM-DMBC consists of:

1) A RV \(U \in \mathcal{U}\), with \(|\mathcal{U}| \leq K + 1\), which is known at both the encoder and the decoders.
2) A sequence of encoders \(f_n : \mathcal{U} \times M \to X\), each one mapping the message \(J\), drawn uniformly at random from the set \(\mathcal{M} \triangleq \{1, \ldots, M\}\), to the channel input \(X_n = f_n(U, J)\).
3) Nonnegative integer-valued RVs \(\tau_1, \ldots, \tau_K\) that are stopping times with respect to the filtrations \(\mathcal{F}_{k,n} \triangleq \sigma\{U, Y_{n}^{n}\}\) and satisfy
   \[
   E\left[\max_{k} \tau_k\right] \leq \ell. \tag{10}
   \]
4) A sequence of decoders \(g_{k,n} : \mathcal{U} \times Y_{n}^{n} \to M\) satisfying
   \[
   \Pr[J \neq g_{k,\tau_k}(U, Y_{\tau_k}^{n})] \leq \epsilon, \quad k \in K. \tag{11}
   \]

The maximum number of codewords with average length \(\ell\) and error probability not exceeding \(\epsilon\) is denoted by
\[
M^*_{sf}(\ell, \epsilon) \triangleq \max\{M : \exists(\ell, M, \epsilon)\text{-VLSF code}\}. \tag{12}
\]

**Remark 1:** VLSF codes require a feedback link from each decoder to the encoder. This feedback consists of a 1-bit “stop signal” per decoder which is sent by decoder \(k\) at time \(\tau_k\). The encoder continuously transmits until all decoders have fed back a stop signal. Hence, the blocklength is \(\max_{k} \tau_k\). Note also that, differently from the full-feedback case, the encoder output at time \(n\) depends on the message and on the common randomness \(U\), but does not depend on the past output signals \(\{Y_{n-1}^{n}\}\). It is also independent of the stop signals received before time \(n\).

**Remark 2:** The RV \(U\) serves as common randomness between the transmitter and all receivers, and enables the use of randomized codes [12]. To establish the bound on the cardinality of \(U\) provided in Definition 1, one can proceed as in [8, Th. 19] and use Caratheodory theorem to show that \(|\mathcal{U}| \leq K + 2\). This bound can be further improved to \(|\mathcal{U}| \leq K + 1\) by using the Fenchel-Eggleston theorem [13, p. 35] in place of Caratheodory theorem.

### III. MAIN RESULTS

#### A. Nonasymptotic Achievability Bound

We provide below a \(K\)-user generalization of the nonasymptotic achievability bound reported in [11, Th. 1].

**Theorem 1:** Fix a probability distribution \(P_{X^{\infty}}\) on \(X^{\infty}\). Let \(\gamma \geq 0\) and \(0 \leq q \leq 1\) be arbitrary scalars. Let the stopping times \(\tau_k\) and \(\bar{\tau}_k\), \(k \in \mathcal{K}\), be defined as follows:
\[
\tau_k \triangleq \inf\{n \geq 0 : i_{P_{X^{\infty}}, W_{\epsilon}}(X^n; Y_{k}^{n}) \geq \gamma\} \tag{13}
\]
\[
\bar{\tau}_k \triangleq \inf\{n \geq 0 : i_{P_{X^{\infty}}, W_{\epsilon}}(\bar{X}^n; Y_{k}^{n}) \geq \gamma\}. \tag{14}
\]

Here, the joint probability distribution of \((X^n, \bar{X}^n, Y_{1}^{n}, \ldots, Y_{K}^{n})\) is
\[
P_{X^n, \bar{X}^n, Y_{1}^{n}, \ldots, Y_{K}^{n}}(x^n, \bar{x}^n, y_{1}^{n}, \ldots, y_{K}^{n}) = P_{Y_{1}^{n}, \ldots, Y_{K}^{n}|X^n}(y_{1}^{n}, \ldots, y_{K}^{n}|x^{n})P_{X^n}(x^n)P_{\bar{X}^n}(\bar{x}^n). \tag{15}
\]

\(^1\)Note that there is a typo in [11, Eq. (12)]: a maximization over \(k\) is missing.
For every $M$, there exists an $(\ell, M, \varepsilon)$-VLSF code such that
\[
\ell \leq (1 - q) \mathbb{E} \left[ \max_k \tau_k \right]
\]  
\[
\varepsilon \leq \max_k \left\{ q + (1 - q)(M - 1) \mathbb{P}[\tau_k \geq \gamma_k] \right\}.
\]}

**Remark 3:** The constant $q$ in Theorem 1 is used to enable time-sharing. With probability $q$, the decoders send simultaneously stop signals to the encoder at time 0. The common randomness $U$ can be used to enable this weak form of cooperation among the decoders.

**Remark 4:** Following the same steps as in \cite[Eqs. (111)–(118)], we can further upper-bound $\varepsilon$ in (17) as follows:
\[
\varepsilon \leq q + (1 - q)(M - 1) \exp \{-\gamma\}.
\]

This bound is easier to evaluate numerically and to analyze asymptotically.

**Proof:** The proof of Theorem 1 follows closely the proof of \cite[Th. 3]{[8]}. See Appendix I for details.

**B. Nonasymptotic Converse Bound**

Let $\mathcal{Y}_k$ denote all possible sequences (of arbitrary length) of symbols from $\mathcal{Y}$, i.e., $\mathcal{Y}_k = \{\emptyset\} \cup \bigcup_{n=1}^{\infty} \mathcal{Y}_k^n$, where $\emptyset$ stands for the vector of length 0. A subset $\mathcal{Y}_k$ of $\mathcal{Y}_k$ is called complete prefix-free if and only if, for every $y \in \mathcal{Y}_k^{\infty}$, there exists a unique $\bar{y} \in \mathcal{Y}_k$ such that $\bar{y}$ is a prefix to $y$. Let $Q_{k}^{(\infty)}$ be an arbitrary probability measure on $\mathcal{Y}_k^{\infty}$ and define the mapping $Q_k : \mathcal{Y}_k \mapsto [0, 1]$ as follows:
\[
Q_k(\bar{y}) = \sum_{y \in \mathcal{Y}_k^{\infty}} Q_k^{(\infty)}(y), \quad \bar{y} \in \mathcal{Y}_k.
\]

We shall use the convention that $[y_1, \cdots, y_{\text{len}(\bar{y})}] = \emptyset$ when $\text{len}(\bar{y}) = 0$. For every complete prefix-free subset $\mathcal{Y}_k \subset \mathcal{Y}_k$, we observe that $Q_k(\cdot)$ defines a probability measure on $\mathcal{Y}_k$. Indeed,
\[
1 = \sum_{y \in \mathcal{Y}_k^{\infty}} Q_k^{(\infty)}(y) = \sum_{y \in \mathcal{Y}_k} \sum_{y' \in \mathcal{Y}_k^{\infty}} Q_k^{(\infty)}(y) = \sum_{y \in \mathcal{Y}_k} Q_k(y).
\]

Based on $Q_k(\cdot)$, we define the “mismatched” information density
\[
i_k(x^n; y_k^n) = \log \frac{P_{x^n|y_k^n}(y_k^n|x^n)}{Q_k(y_k^n)}
\]
for $x^n \in \mathcal{X}^n$, $y_k^n \in \mathcal{Y}^n$, and $n \in \mathbb{N}$ with the convention that $i_k([\cdot]; \emptyset) = 0$.

We are now ready to state our converse bound.

**Theorem 2:** For an arbitrary probability measure $Q_{k}^{(\infty)}$ on $\mathcal{Y}_k^{\infty}$, and arbitrary $M \in \mathbb{N}$, $t \in \mathbb{Z}_+$, $\eta > 0$, and $\varepsilon_k \in (0, 1)$, $k \in \mathcal{K}$, define the following function:
\[
L_t(\varepsilon_1, \cdots, \varepsilon_K) = \max_{x \in \mathcal{X}^t} \prod_k \min \left\{ 1, \mathbb{P} \left[ \max_{0 \leq n \leq t} i_k(x^n; y^n_k) \geq \log M + \log \eta \right] + \varepsilon_k \right\}.
\]

Here, the vector $x^n$ contains the first $n$ entries of $x^t$, and $y_k^n \sim P_{Y_k^n|X^n=x^n}$. Then, every $(\ell, M, \varepsilon)$-VLSF code must satisfy
\[
\ell \geq \min_{P_U \in \mathcal{P}(U), \varepsilon_k \in [0, 1]} \mathbb{E}_U \left[ \sum_{t=0}^{\infty} \left( 1 - L_t(\varepsilon_1^{(U)}, \cdots, \varepsilon_K^{(U)}) \right) \right].
\]

**Proof:** See Section VI.

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For the case when \{W_k\} are identical and symmetric,\(^2\) we have the following particularization of Theorem 2.

**Corollary 3:** For arbitrary \(M \in \mathbb{N}, t \in \mathbb{Z}_+, \eta > 0\), and an arbitrary sequence \(x \in \mathcal{X}^\infty\), let

\[
v_t = \mathbb{P}\left[ \max_{0 \leq n \leq t} i_{P^*, W_1}(x^n; Y_1^n) \geq \log M + \log \eta \right]
\]

where \(Y_k^n \sim P_{Y_1^n | X^n = x^n}\). When \(W_1 = \cdots = W_K\) and \(W_1\) is symmetric, every \((\ell, M, \epsilon)\)-VLSF code must satisfy

\[
\ell \geq \min_{P_{U \in} \mathcal{P}(U)} \mathbb{E}_U \left[ \sum_{t=0}^{\infty} \left( 1 - \min \left\{ 1, v_t + \epsilon(U) \right\} \right)^K \right].
\]

**Proof:** We apply the converse bound in Theorem 2 with \(Q_{i}^{(\infty)} = \cdots = Q_{K}^{(\infty)}\). Furthermore, we choose \(Q_{1}^{(\infty)}\) as a product distribution with marginal \(Q_1(y) = 1/|V_1|\) for all \(y \in V_1\). By (19), we have that \(Q_k(y_k) = |V_k|^{-\text{len}(y_k)}\).

Since the capacity-achieving output distribution of a symmetric channel is uniform [14, Eq. (7.22)], we conclude that

\[
i_k(x^n; Y_k^n) \sim i_{P^*, W_1}(x^n, Y_1^n)
\]

for all \(k \in K\). One can verify that, when the channel is symmetric, the probability distribution of the information density \(i_{P^*, W_1}(x^n, Y_1^n)\) does not depend on \(x^n\). This allows us to drop the maximization over \(x^n\) in (22). Finally, to express the minimization problem (23) in the form given in (25), we note that, for every \([\epsilon_1, \cdots, \epsilon_K] \in [0, 1]^K\), we have

\[
\left( \prod_{k=1}^{K} \min \left\{ 1, v_t + \epsilon_k \right\} \right)^{1/K} \leq \frac{1}{K} \sum_{k=1}^{K} \min \left\{ 1, v_t + \epsilon_k \right\} \leq \min \left\{ 1, v_t + \frac{1}{K} \sum_{k=1}^{K} \epsilon_k \right\}.
\]

Here, (27) follows because the geometric mean is no larger than the arithmetic mean and (28) follows from Jensen’s inequality. \(\blacksquare\)

### C. Asymptotic Expansion

Analyzing (16), (18) and (23) in the limit \(\ell \rightarrow \infty\), we obtain the following asymptotic characterization of \(M_{\ell}^{\star}(\ell, \epsilon)\).

**Theorem 4:** Let \(V = (\prod_k V_k)^{1/K}\) and \(g_k = \sqrt{V_k/V}\) and assume, without loss of generality, that \(C_1 \geq \cdots \geq C_K\). For every CM-DMBC satisfying

\[
\frac{1}{C_i} + \frac{i}{C_K} > \frac{i}{C}, \quad i \in \{1, \cdots, K - 1\}
\]

and for every \(\epsilon \in (0, 1)\), we have\(^3\)

\[
\frac{C\ell}{1 - \epsilon} - \sqrt{\frac{V\ell}{1 - \epsilon}} \Xi_a + o(\sqrt{\ell}) \leq \log M_{\ell}^{\star}(\ell, \epsilon) \leq \frac{C\ell}{1 - \epsilon} - \sqrt{\frac{V\ell}{1 - \epsilon}} \Xi_c + o(\sqrt{\ell}).
\]

Here,

\[
\Xi_a \triangleq \min_{v \in \mathcal{X}_a^{\ell}} \mathbb{E}_k \left[ \max_{k} \nabla I_k(v) + g_k Z_k \right]
\]

\[
\Xi_c \triangleq \min_{v \in \mathcal{X}_c^{\ell}} \mathbb{E}_k \left[ \max_{k} \nabla I_k(v) + g_k Z_k \right]
\]

\(^2\)A channel is symmetric if the rows and columns of the stochastic channel matrix are permutations of each other [14, p. 189].

\(^3\)The subscripts “a” and “c” in \(\Xi_a\) and \(\Xi_c\) stand for achievability and converse, respectively.
with \( Z_k \overset{i.i.d.}{\sim} \mathcal{N}(0, 1) \) and
\[
\Xi_c \triangleq \mathbb{E}\left[ \max_k H_k \right]
\]
with the RVs \( \{H_k\} \) being independent and with cumulative distribution functions
\[
F_{H_k}(w) \triangleq \Phi\left( \frac{w + \nabla I_k(\hat{v}(w))}{\vartheta_k} \right).
\]
The function \( \hat{v}(\cdot) \) is defined as follows:
\[
\hat{v}(w) \triangleq \arg \max_{v \in \mathbb{R}^{|X|}} \min_{0} \prod_k \Phi\left( \frac{w + \nabla I_k(v)}{\vartheta_k} \right).
\]

**Proof:** The converse bound in (30) is proved in Section VII and the achievability bound in (30) is proved in Section VIII.

**Remark 5:** The condition (29) is needed only for the converse part.

**Remark 6:** When \( K = 2 \), the condition (29) reduces to
\[
\frac{1}{C_1} + \frac{1}{C_2} > \frac{1}{C_1}.
\]
Then one can achieve a rate larger than \( C \):
\[
\max_{\alpha \in [0, 1]} \min \left\{ \alpha C_1, (1 - \alpha) C_2 \right\} > \frac{1}{C}.
\]

Theorem 4 does not hold for the special case \( \frac{1}{C_1} + \frac{1}{C_2} = \frac{1}{C} \). In this case, since time-sharing is capacity achieving, one can sequentially transmit to the \( K \) decoders and achieve zero dispersion.

**Remark 7:** As we shall show next, the constants \( \Xi_a \) and \( \Xi_c \) defined in (31) and (32), respectively, satisfy \( \Xi_a \geq \Xi_c > 0 \). This implies that the second-order term in the asymptotic expansion of \( \log M_{sf}^{\ell, \epsilon}(\ell, \epsilon) \) is positive for every \( \epsilon \in (0, 1) \) (see (30)). This strengthens [11, Th. 3]. Verifying that \( \Xi_a \geq \Xi_c \) will also allow us to shed light on the reason behind the gap between the achievability and converse bound. To do so, it is convenient to rewrite \( \Xi_a \) as follows:
\[
\Xi_a = \min \lim_{n \to \infty} \mathbb{E}\left[ \max_k (\nabla I_k(v) + n + \vartheta_k Z_k)^+ - n \right].
\]
To obtain (36), we used that for every set \( \{X_k\} \) of RVs with bounded second moment, \( \max_k (X_k + n)^+ - n \) converges in distribution to \( \max_k X_k \) as \( n \to \infty \). Furthermore,
\[
\sup_n \mathbb{E}\left[ \left| \max_k (X_k + n)^+ - n \right|^2 \right] \leq \sup_n \mathbb{E}\left[ \max_k ((X_k + n)^+ - n)^2 \right]
\]
\[
\leq \sum_k \sup_n \mathbb{E}\left[ ((X_k + n)^+ - n)^2 \right]
\]
\[
\leq \sum_k \mathbb{E}[X_k^2]
\]
\[
< \infty.
\]

Hence, we can invoke the corollary in [15, p. 361] to conclude that
\[
\lim_{n \to \infty} \mathbb{E}\left[ \max_k (X_k^+ + n)^+ - n \right] = \mathbb{E}\left[ \max_k X_k \right].
\]

\(^4\)If the maximizer in (34) is not unique, \( \tilde{v}(w) \) is chosen arbitrarily from the set of maximizers.
This implies (36). Next, we further bound (36) through the following chain of inequalities

\[
\Xi_a = \min_{v \in \mathbb{R}^n_0} \lim_{n \to \infty} \int_0^\infty \left(1 - \prod_k \Phi \left(\frac{w - n - \nabla I_k(v)}{\theta_k}\right)\right) \, dw - n
\]

\[
\geq \lim_{n \to \infty} \int_0^\infty \left(1 - \prod_k \Phi \left(\frac{w - n + \nabla I_k(\psi(w) - n)}{\theta_k}\right)\right) \, dw - n
\]

\[
= \lim_{n \to \infty} \mathbb{E} \left[\max_k (H_k + n)^+ - n\right]
\]

\[
= \mathbb{E} \left[\max_k H_k\right]
\]

\[
= \Xi_c.
\]

To obtain (42), we used that for every nonnegative RV \(X\), we have \(\int_0^\infty (1 - \mathbb{P}[X < x]) \, dx = \mathbb{E}[X]\); we also used that the \(\{Z_k\}\) are i.i.d. Gaussian and that for every real-valued RV \(T\) and every \(w \geq 0\), we have that \(\mathbb{P}[T < w] = \mathbb{P}[T < w]\). The inequality (43) follows from (34). Finally, (44) and (45) follow from steps similar to the ones leading to (42) and (36), respectively.

The inequality (43) reveals the origin of the gap between \(\Xi_a\) and \(\Xi_c\). The constant \(\Xi_a\) is obtained by evaluating the achievability bound in Theorem 1 for an i.i.d. process \(X^\infty\). Instead, in the computation of \(\Xi_c\), we find the input distribution that maximizes \(\mathbb{P}[\max_k \tau_k \leq t]\) for each \(t\). The resulting process is not i.i.d.

To prove that \(\Xi_c > 0\), we proceed as follows:

\[
\Xi_c = \lim_{n \to \infty} \int_0^\infty \left(1 - \prod_k \Phi \left(\frac{w - n + \nabla I_k(\psi(w) - n)}{\theta_k}\right)\right) \, dw - n
\]

\[
\geq \lim_{n \to \infty} \int_0^\infty \left(1 - \min_k \Phi \left(\frac{w - n + \nabla I_k(\psi(w) - n)}{\theta_k}\right)\right) \, dw - n
\]

\[
\geq \lim_{n \to \infty} \min_k \int_0^\infty \left(1 - \Phi \left(\frac{w - n}{\theta_k}\right)\right) \, dw - n
\]

\[
= \lim_{n \to \infty} \min_k \mathbb{E} \left[\left(\theta_k Z_k + n\right)^+ - n\right] = \min_k \mathbb{E}[Z_k] = 0.
\]

Here, (47) follows from (43); the inequality in (48) holds because \(\prod_k a_k < \min_k a_k\) for all \(a_k \in (0, 1)\); finally, (49) follows because \(\min_k \nabla I_k(\psi(w)) \leq 0\) for all \(w \in \mathbb{R}\). Indeed, \(\nabla I_k(\cdot)\) is the directional derivative of \(I_k(P)\) computed at the unique capacity-achieving input distribution \(P^*\).

There are cases where \(\Xi_a = \Xi_c\), and hence (30) provides a complete second-order characterization of \(\log M^*_d(\ell, \epsilon)\). This happens when \(0_{|X|}\) maximizes (34). This occurs for example when \(P^*\) simultaneously maximizes \(I_k(P)\) for all \(k \in \mathcal{K}\). In the following corollary, we provide necessary and sufficient conditions for Theorem 4 to yield an asymptotic expansion that is tight up to the second order.

**Corollary 5:** We have that \(\Xi_a = \Xi_c\) and, hence,

\[
\log M_d^*(\ell, \epsilon) = \frac{C\ell}{1 - \epsilon} - \sqrt{\frac{V\ell}{1 - \epsilon}} \mathbb{E} \left[\max_k Z_k\right] + o(\sqrt{\ell})
\]

if and only if the following holds

1. The capacity-achieving input distribution \(P^*\) simultaneously maximizes \(I_k(P)\) for all \(k \in \mathcal{K}\), or
2. \(V_1 = \cdots = V_K\) and

\[
\sum_k \nabla I_k(e_1 (i)) = 0, \quad i \in \{1, \cdots, |X|\}.
\]
Here, $e_{|X|}(i)$ denotes the $|X|$-dimensional vector whose $i$th entry is equal to one and whose remaining entries are equal to zero.

**Proof:** We shall prove that $\Xi_\lambda = \Xi_c$ under the stated conditions by characterizing $\hat{v}(\cdot)$ in (34). When the component channels $W_1, \cdots, W_K$ have the same capacity-achieving input distribution, we have $\nabla I_k(v) = 0$ for all $v \in \mathbb{R}_{0}^{|X|}$ and all $k \in \mathcal{K}$. Hence, $0_{|X|}$ is a maximizer of (34) which implies that (43) holds with equality.

Consider now the case that $\nabla I_k(v) \neq 0$ for some $v \in \mathbb{R}_{0}^{|X|}$ and some $k \in \mathcal{K}$. Let $\text{ker}(\nabla I_k)$ denote the kernel of $\nabla I_k$, and let $D_\perp = \mathbb{R}_{0}^{|X|} \cap \bigcap_{k=1}^{K} \text{ker}(\nabla I_k)$ and $D = \mathbb{R}_{0}^{|X|}\setminus D_\perp$. We note that $D \neq \emptyset$ since, by assumption, there exists a $v \in \mathbb{R}_{0}^{|X|}$ and a $k \in \mathcal{K}$ such that $\nabla I_k(v) \neq 0$. Let the dimension of the linear subspace $D$ be $m$ and let $\mathbb{D} \in \mathbb{R}^{|X| \times m}$ be an $|X|$-by-$m$ matrix with columns spanning the linear subspace $D$. Now, define

$$\hat{v}_D(w) \triangleq \arg \max_{v \in \mathbb{R}^m} \prod_k \Phi \left( \frac{w + \nabla I_k(\mathbb{D}v)}{\varrho_k} \right)$$

and

$$\hat{v}_D \triangleq \arg \min_{v \in \mathbb{R}^m} \lim_{n \to \infty} \int_{0}^{\infty} \left( 1 - \prod_k \Phi \left( \frac{w - n - \nabla I_k(\mathbb{D}v)}{\varrho_k} \right) \right) dw - n.$$  \hspace{1cm} (54)

We note that $\hat{v}_D(w)$ is continuous in $w$ and that log-concavity of the objective function in (54) implies that $\hat{v}_D(w)$ is the unique maximizer of the optimization problem. Moreover, $\mathbb{D}\hat{v}_D(w)$ is a maximizer of (34) and $\mathbb{D}\hat{v}_D$ is a minimizer in (42). This implies that the steps (42)--(46) can be equivalently written as

$$\Xi_\lambda = \min_{v \in \mathbb{R}_{0}^{|X|}} \lim_{n \to \infty} \int_{0}^{\infty} \left( 1 - \prod_k \Phi \left( \frac{w - n - \nabla I_k(v)}{\varrho_k} \right) \right) dw - n$$ \hspace{1cm} (56)

$$= \lim_{n \to \infty} \int_{0}^{\infty} \left( 1 - \prod_k \Phi \left( \frac{w - n - \nabla I_k(\mathbb{D}v)}{\varrho_k} \right) \right) dw - n$$ \hspace{1cm} (57)

$$\geq \lim_{n \to \infty} \int_{0}^{\infty} \left( 1 - \prod_k \Phi \left( \frac{w - n + \nabla I_k(\mathbb{D}\hat{v}_D(w - n))}{\varrho_k} \right) \right) dw - n$$ \hspace{1cm} (58)

$$= \lim_{n \to \infty} \int_{0}^{\infty} \left( 1 - \prod_k \Phi \left( \frac{w - n + \nabla I_k(\hat{v}(w - n))}{\varrho_k} \right) \right) dw - n$$ \hspace{1cm} (59)

$$= \Xi_c.$$ \hspace{1cm} (60)

Since $\hat{v}_D(w)$ is continuous and is the unique maximizer of (54), the inequality in (58) holds with equality if and only if $\hat{v}_D(w) = a$ for some vector $a \in \mathbb{R}^m$ that does not depend on $w$.

Suppose that $\hat{v}_D(w) = a$. The objective function in (54) is positive, strictly log-concave in $v_D \in \mathbb{R}^m$, and tends to zero as $\|v_D\| \to \infty$. Thus, the unique maximum is at the unique stationary point, which can be found by differentiating the logarithm of the objective function in (54) in each of the $m$ dimensions and by equating it to zero. This yields

$$\sum_k \psi_\lambda \left( \frac{w + \nabla I_k(\mathbb{D}a)}{\varrho_k} \right) \nabla I_k(\mathbb{D}e_m(i)) = 0, \quad i \in \{1, \cdots, m\}, w \in \mathbb{R}$$ \hspace{1cm} (61)

where $\psi(w) \triangleq \phi(w)/\Phi(w)$. It follows that (61) cannot be satisfied for every $w \in \mathbb{R}$ unless $a = 0_m$ and $\varrho_1 = \cdots = \varrho_K$. In this case (61) reduces to (53).

For broadcast channels that do not satisfy the conditions of Corollary 5, we can tighten the left-hand side of (30) by using an input distribution that is not stationary memoryless. This yields the following theorem.

**Theorem 6:** Let $V \triangleq (\prod_k V_k)^{1/K}$ and $\varrho_k \triangleq \sqrt{V_k/V}$. Fix a differentiable function $\bar{v} : \mathbb{R} \to \mathbb{R}_{0}^{[X]}$ such that

$$P^* + C\bar{v}'(w) \in \mathcal{P}(X)$$ \hspace{1cm} (62)
for all $w \in \mathbb{R}$. Additionally, define
\[
E_k(s) \triangleq C - I_k(P^* + C \psi'(s)) + C \nabla I_k(\psi'(s))
\] (63)
and assume that
\[
\int_{-\infty}^{\infty} E_k(s) \, ds < \infty
\] (64)
and that
\[
|E'_k(s)| < \infty.
\] (65)

Then, for every CM-DMBC, we have
\[
\log M^*_a(s, \epsilon) \geq C_\ell 1^{-\epsilon - \sqrt{V_\ell 1^{-\epsilon} \Xi_a - o(\sqrt{\ell})}}.
\] (66)
Here,
\[
\Xi_a \triangleq \mathbb{E} \left[ \max_k H_k \right]
\] (67)
where the independent RVs $\{H_k\}$ are defined by the cumulative distribution functions
\[
F_{H_k}(w) \triangleq \Phi \left( \frac{1}{\varrho_k} \left( w + \nabla I_k(\psi(w)) - \int_{-\infty}^{w} \frac{E_k(s)}{C} \, ds \right) \right)
\] (68)

Proof: See Section VIII.

If one sets $\psi(\cdot)$ in Theorem 6 equal to $\hat{\psi}(\cdot)$ in (34), the resulting gap between $\Xi_c$ and $\Xi_a$ is caused only by the “error” term $E_k(s)$. Interestingly, there are channels beyond the ones for which Corollary 5 applies for which $E_k(s) = 0$ and, hence, a complete second-order characterization of $\log M^*_a(s, \epsilon)$ is available. The next corollary describes a class of channels for which this is the case.

Corollary 7: Let $X_1, \ldots, X_R$ be disjoint sets and let $\mathcal{X} = \bigcup_{r=1}^{R} X_r$. Moreover, for $k \in \mathcal{K}$ and $r \in \{1, \cdots, R\}$, let $W_{k,r}$ be a channel from $X_r$ to $\mathcal{Y}_k$ with capacity-achieving input distribution $P^*_r$ (independent of $k$), capacity-achieving output distribution $P^*_Y_k$ (independent of $r$), and capacity $C_{k,r}$. Define for all $x \in \mathcal{X}$ and $y \in \mathcal{Y}_k$ the channel $W_k(y|x) = W_{k,r(x)}(y|x)$, where the function $r : \mathcal{X} \mapsto \{1, \cdots, R\}$ is such that $x \in X_{r(x)}$. Assume that $C_1 \geq \cdots \geq C_K$ and that
\[
\frac{1}{C_i} + \frac{i}{C_K} > \frac{i}{C}, \quad i \in \{1, \cdots, K-1\}.
\] (69)
Define
\[
\beta(w) \triangleq \arg \max_{\beta \in \mathbb{R}^R_0} \prod_k \Phi \left( \frac{1}{\varrho_k} \left( w + \sum_{r=1}^{R} \beta_r C_{k,r} \right) \right)
\] (70)
and assume that
\[
P^*(x) + CP^*_Y(x) \beta'(x)(w) \in [0, 1]
\] (71)
for every $w \in \mathbb{R}$. Then, for every $\epsilon \in (0, 1),
\[
\log M^*_a(s, \epsilon) = \frac{C_\ell}{1-\epsilon} - \frac{\sqrt{\frac{VT}{1-\epsilon} \Xi_c}}{\epsilon} + o(\sqrt{\ell})
\] (72)
where $\Xi_c$ is defined in (32).
Proof: We shall first evaluate the mutual information and the directional derivative of the mutual information at the following input distribution:

\[ P_\alpha \triangleq [\alpha_1 P^*_1, \alpha_2 P^*_2, \ldots, \alpha_R P^*_R]. \]  

(73)

Here, \( \alpha \) is a nonnegative vector with \( \sum_{r=1}^R \alpha_r = 1 \). One can verify that the capacity \( C \) is achieved using time sharing and that the capacity-achieving input distribution \( P^* \) must have the form given by (73). Thus, we choose \( \alpha^* \) so that \( P_{\alpha^*} = P^* \).

For all nonnegative vectors \( \alpha \) with \( \sum_{r=1}^R \alpha_r = 1 \), the mutual information \( I_k(P_\alpha) \) is given by

\[ I_k(P_\alpha) = \sum_{x \in X} \sum_{y \in Y_k} P_\alpha(x) W_k(y|x) \log \frac{W_k(y|x)}{P^*_k(y)} \]

(74)

\[ = \sum_{r=1}^R \alpha_r \sum_{x \in X} \sum_{y \in Y_k} P^*_r(x) W_k(y|x) \log \frac{W_k(y|x)}{P^*_k(y)} \]

(75)

\[ = \sum_{r=1}^R \alpha_r C_{k,r}. \]  

(76)

In (76), we used that the channels \( W_{k,r} \) have the same capacity-achieving output distribution for \( r \in \{1, \ldots, R\} \).

Next, we compute the directional derivative of the mutual information at \( P_{\alpha^*} \) along the direction \( v \in \mathbb{R}_0^{\left| X \right|} \):

\[ \nabla I_k(v) = \sum_{i=1}^R \sum_{x \in X} v_x D(W_{k,r}(\cdot|x)|P^*_k) \]

(77)

\[ = \sum_{r=1}^R \left( \sum_{x \in X} v_x \right) C_{k,r}. \]  

(78)

Here, (78) follows from the assumption \( P^*(x) > 0 \), which implies that \( P^*_r(x) > 0 \) for \( x \in X \) and \( r \in \{1, \ldots, R\} \). In turn, this implies that \( D(W_{k,r}(\cdot|x)|P^*_k) = C_{k,r} \) (see, e.g., [16, Th. 4.5.1]). By substituting (78) in (33), we obtain

\[ F_{H_k}(w) \triangleq \Phi \left( \frac{1}{\varrho_k} \left( w + \sum_{r=1}^R \beta_r(w) C_{k,r} \right) \right) \]  

(79)

where the function \( \beta : \mathbb{R} \rightarrow \mathbb{R}_0^R \) is given by (70).

Next, we shall prove that we can achieve (72) using Theorem 6. Define the function \( \psi : \mathbb{R} \rightarrow \mathbb{R}^{\left| X \right|} \) as follows:

\[ \tilde{v}_x(w) = P^{*}_{r(x)}(w) \beta_r(x)(w), \quad x \in X. \]  

(80)

Note that \( \psi(w) \) maximizes (34). Indeed,

\[ \max_{v \in \mathbb{R}_0^{\left| X \right|}} \prod_k \Phi \left( \frac{w + \nabla I_k(v)}{\varrho_k} \right) = \max_{v \in \mathbb{R}_0^{\left| X \right|}} \prod_k \Phi \left( \frac{1}{\varrho_k} \left( w + \sum_{r=1}^R \left( \sum_{x \in X} v_x \right) C_{k,r} \right) \right) \]

(81)

\[ = \max_{\beta \in \mathbb{R}_0^R} \prod_k \Phi \left( \frac{1}{\varrho_k} \left( w + \sum_{r=1}^R \sum_{x \in X} P^*_{r(x)}(x) \beta_{r(x)}(w) C_{k,r} \right) \right) \]

(82)

\[ = \prod_k \Phi \left( \frac{1}{\varrho_k} \left( w + \sum_{x \in X} P^*_{r(x)}(x) \beta_{r(x)}(w) C_{k,r} \right) \right) \]

(83)

\[ = \prod_k \Phi \left( \frac{1}{\varrho_k} \left( w + \nabla I_k(\psi(w)) \right) \right). \]  

(84)
Here, (81) follows from (78) and (83) follows from the definition of \( \beta(w) \) in (70). Note that the definition of \( \Psi(w) \) implies that \( P^* + C \Psi'(w) = [(\alpha_1^* + C\beta_1^*(w))P_1^*, \ldots, (\alpha_n^* + C\beta_n^*(w))P_n^*] \), which is a probability distribution by the condition in (71) and has the form (73). Next, we have that

\[
E_k(s) = C - I_k(P^* + C \Psi'(s)) + C \nabla I_k(\Psi'(s))
\]

\[
= C - n \sum_{\nu=1}^{d} (\alpha^\nu + C\beta^\nu(w))C_k,\nu + C \nabla \sum_{\nu=1}^{d} \beta^\nu(s)C_k,\nu
\]

\[
= 0.
\]

(85)

(86)

(87)

Here, (86) follows from (76) and (83). Since \( E_k(s) = 0 \), we have that \( H_k \) has the same distribution as \( \hat{H}_k \) for \( k \in K \). Furthermore, \( \int_{-\infty}^{\infty} E_k(s) ds = 0 \) and \( |E_k'(s)| < \infty \) for every \( s \in \mathbb{R} \). The conditions in Theorem 6 are satisfied. This implies that (72) is indeed achievable.

The following lemma shows that there exist nontrivial channels that satisfy the conditions of Corollary 7.

**Lemma 8:** Let \( R = 2 \), \( K = 2 \), and \( \Delta_1 \triangleq C_{11} - C_{12} > 0 \) \( C_{21} - C_{22} \triangleq \Delta_2 \). Let also

\[
D \triangleq -\frac{\Delta_1 \varrho_2^2 + \Delta_2 \varrho_1^2}{\Delta_1 \varrho_2^2 + \Delta_2 \varrho_1^2}.
\]

Then the condition \( P^*(x) + C P_{r(x)}^n(x)\beta_{r(x)}^*(w) \in [0,1] \) holds for every \( x \in \mathcal{X} \) and every \( w \in \mathbb{R} \) provided that

\[
P^*(x) + (-1)^{r(x)+1} C P^*_r(x) D \in [0,1], \quad x \in \mathcal{X}
\]

and

\[
\left( \frac{\Delta_1}{\varrho_1} + \frac{\Delta_2}{\varrho_2} \right) (\varrho_2 - \varrho_1) \geq 0.
\]

**Proof:** See Appendix II.

**IV. NUMERICAL EXAMPLES**

**A. Binary Symmetric Channels**

Let \( W_1 \) and \( W_2 \) be two BSCs, each with crossover probability \( \delta \). Note that \( W_1 \) and \( W_2 \) are symmetric [14, p. 185] and have the same capacity-achieving input distribution. We evaluate the bounds presented in Theorem 1, Corollary 3, and Corollary 5 for the CM-DMBC having \( W_1 \) and \( W_2 \) as its component channels. The bounds are depicted in Fig. 1 for the case \( \delta = 0.11 \) and \( \epsilon = 10^{-3} \). The capacity-achieving input distribution \( P^* \) of the individual BSCs is \( \text{Bern}(1/2) \), their capacity is given by \( 1 - H_b(\delta) \), where \( H_b(\cdot) \) denotes the binary entropy function, and the directional derivatives \( \nabla I_k(\cdot) \) of the mutual information at \( P^* \) are zero. Furthermore, for \( Y_k^n \sim P_{Y_k^n|X_k^n=x^n} \), the information densities \( i_{P^*,W_k}(x^n;Y_k^n) \) satisfy

\[
i_{P^*,W_k}(x^n;Y_k^n) \sim n \log(2 - 2\delta) + \log \left( \frac{\delta}{1-\delta} \right) \sum_{j=1}^{n} Z_{k,j}
\]

(91)

where \( Z_{k,j} \stackrel{iid}{\sim} \text{Bern}(\delta) \). We observe that the distribution of the information density in (91) is independent of \( x^n \).

The converse bound in the figure is obtained from Corollary 3, where the value of \( \eta \) is optimized numerically. The achievability bound is obtained from Theorem 1 for the choice \( P_X = (P^*)^\infty \). To evaluate the bound, we use that \( \tau_1 \) and \( \tau_2 \) are i.i.d. RVs. This allows us to compute \( E[\max\{\tau_1, \tau_2\}] \) by evaluating \( \sum_{t=0}^{\infty} (1 - F_{\tau_1}(t)^2) \) where \( F_{\tau_1}(\cdot) \) is the cumulative distribution function of \( \tau_1 \). To estimate (17), we use the change of measure technique (see [8, p. 4911]).
Fig. 1. Comparison between the achievability bound (16)-(17), the converse bound (25), and the normal approximation (52) for $\epsilon = 10^{-3}$ and $K = 2$ decoders. The component channels in the CM-DMBC are BSCs with crossover probability 0.11. The normal approximation corresponds to the asymptotic expansion in (52) with the $o(\cdot)$ term neglected. The blue curve labeled “achievability (single)” corresponds to the single-user achievability bound in [8, Th. 3] evaluated for a BSC with crossover probability 0.11 and $\epsilon = 10^{-3}$.

Fig. 2. Asymmetric channels $\{W_k\}$ that obey the conditions in Corollary 7 for the channel parameters $\delta_{11} = 0.01, \delta_{12} = 0.40, \delta_{21} = 0.15$, and $\delta_{22} = 0.10$. The channels consist of two BSCs with common outputs.
We observe that, in the two-user case, the speed of convergence to the asymptotic limit is indeed slower than for the single-user case (the curve marked “achievability (single)” in Fig. 1, which is the point-to-point achievability bound reported in [8, Th. 3]). In particular, for \( \ell \geq 1000 \) and \( K = 2 \), our converse bound is strictly below the single-user achievability bound, which implies that the maximum coding rate for the two-user case is strictly smaller than that for the single-user case. Additionally, the speed of convergence becomes slower as the number of users increases.

**B. Asymmetric Channels**

Next, we illustrate through an example that Theorem 6 indeed improves over Theorem 4. We consider the CM-DMBC depicted in Fig. 2; we shall also assume that \( \delta_{11} = 0.01, \delta_{12} = 0.40, \delta_{21} = 0.15, \) and \( \delta_{22} = 0.10 \). The two component channels, \( W_1 \) and \( W_2 \), are two BSCs with common outputs. Let now \( \mathcal{X}_1 = \{1, 2\}, \mathcal{X}_2 = \{3, 4\}, \) and \( W_{k,r} \) be a BSC with crossover probability \( \delta_{k,r} \) for \( k \in \{1, 2\} \) and \( r \in \{1, 2\} \). One can verify that the condition \( \mathbb{P}^*\beta_{\mathcal{X}(x)}(w) \in [0, 1] \) for \( x \in \mathcal{X} \) and \( w \in \mathbb{R} \) in Corollary 7 is satisfied using Lemma 8. Therefore, the asymptotic expansion in (72) holds, i.e., the converse bound in Theorem 4 is tight up to second order. The same is not true for the achievability bound in Theorem 4. Indeed, by computing (31) and (32), we find that \( \Xi_2 = 0.3175 \).

**V. Conclusion**

In this paper, we considered the \( K \)-user CM-DMBC for the scenario where variable-length stop-feedback codes are used. We presented achievability and converse bounds on the maximum coding rate \( \frac{1}{\ell} \log M^*_d(\ell, \epsilon) \) for a fixed average blocklength \( \ell \) and average error probability \( \epsilon \). The main novelty in our analysis is the converse bound, which relies on a nonstandard application of the metaconverse theorem [4, Th. 27] to the variable-length setup. The achievability bound follows instead from a straightforward generalization of [8, Th. 3]. An asymptotic analysis of our bounds in the limit \( \ell \to \infty \) reveals that, under mild technical conditions, the second-order asymptotic expansion of \( \log M^*_d(\ell, \epsilon) \) contains a square-root penalty. We provided upper and lower bounds on the second-order term in the asymptotic expansion of \( \log M^*_d(\ell, \epsilon) \) and derived necessary and sufficient conditions for our bounds on the second-order term to match. This occurs for example for the case when the component channels are two identical BSCs. For this case, we provide numerical evidence that the convergence to the asymptotic limit of the maximum coding rate is indeed slower than in the point-to-point case. Furthermore, our numerical results show that the first two terms in the asymptotic expansion of \( \log M^*_d(\ell, \epsilon) \) approximate \( \log M^*_d(\ell, \epsilon) \) accordingly.

**VI. Proof of Theorem 2**

Fix \( M \) and \( \epsilon \). To establish Theorem 2, we derive a lower bound on the average blocklength \( \ell \) that holds for all VLSF codes having \( M \) codewords and error probability no larger than \( \epsilon \). Since, by definition (see (10)),

\[
\ell \geq \mathbb{E}\left[\max_k \tau_k\right] \tag{92}
\]

\[
= \sum_{t=0}^{\infty} \left(1 - \mathbb{P}\left[\max_k \tau_k \leq t\right]\right) \tag{93}
\]

we can lower-bound \( \ell \) by upper-bounding \( \mathbb{P}[\max_k \tau_k \leq t] \) for every \( t \in \mathbb{Z}_+ \).
We shall also need the following probability measure on $\mathcal{X}$ defined in Section III-B, defines a probability measure on $\mathcal{X}^\infty$ induced by the encoder given $U = u$. Define for $u \in \mathcal{U}$ the set $\bar{Y}_k(u) \triangleq \{ y^n \in Y_k : \varphi(u, y^n) = 1 \}$. Note that $Y_k^\tau_u \in \bar{Y}_k(u)$ and that $\bar{Y}_k(u)$ is a complete prefix-free subset of the set $\bar{Y}_k$ defined at the beginning of Section VI. Thus, $Q_k$, defined in Section III-B, defines a probability measure on $\bar{Y}_k(u)$. It will turn out useful to define the following probability measure on $\mathcal{X}^\infty \times \bar{Y}_1(u) \times \cdots \times \bar{Y}_K(u)$

$$P^{(u)}_{\mathcal{X},Y_1,\ldots,Y_K}(x, \bar{y}_1, \ldots, \bar{y}_K) \triangleq P^{(u)}_{X}(x) \prod_{k=1}^{K} W_k(\bar{y}_k|x_k).$$

(94)

We shall also need the following probability measure on $\mathcal{X}^\infty \times \bar{Y}_1(u) \times \cdots \times \bar{Y}_2(u)$

$$Q^{(u)}_{\mathcal{X},Y_1,\ldots,Y_K}(x, \bar{y}_1, \ldots, \bar{y}_K) \triangleq P^{(u)}_{X}(x) \prod_{k} Q_k(\bar{y}_k).$$

(95)

For notational convenience, we shall indicate the two probability measures in (94) and (95) simply as $P^{(u)}$ and $Q^{(u)}$, respectively. For each decoder $k$, the average error probability is equal to $\epsilon_k(u)$ under $P^{(u)}$, and it is no larger than $1 - 1/M$ under $Q^{(u)}$. Hence, using the meta-converse theorem [4, Th. 27] and the inequality [4, Eq. (102)], we conclude that, for every $\gamma_k(u) > 0$ and for every $k \in K$,

$$\log M \leq \log \gamma_k(u) - \log \left[ P^{(u)} \left[ i_k(u)(x; \bar{Y}_k) \leq \log \gamma_k(u) \right] - \epsilon_k(u) \right] + .$$

(96)

Here,

$$i_k(u)(x; \bar{y}_k) \triangleq \log \frac{P^{(u)}_{X,Y_k}(x, \bar{y}_k)}{Q^{(u)}_{X,Y_k}(x, \bar{y}_k)} = \log \frac{\prod_{i=1}^{\text{len}(\bar{y}_k)} W_k(\bar{y}_{k,i}|x_i)}{Q_k(\bar{y}_k)} = i_k(x; \bar{y}_k), \quad x \in \mathcal{X}^\infty, \bar{y}_k \in \bar{Y}_k(u)$$

(97)

(98)

where $i_k(x; \bar{y}_k)$ was defined in (21). Note that $i_k(x; \bar{y}_k)$ depends on $x \in \mathcal{X}^\infty$ only through its first $\text{len}(\bar{y}_k)$ entries. Set now

$$\gamma_k(u) \triangleq \sup \left\{ \nu \in \mathbb{R} : P^{(u)}[i_k(X; \bar{Y}_k) \leq \log \nu] \leq \epsilon_k(u) + \eta \right\}$$

(99)

where $\eta > 0$ is arbitrary. Using (99), we conclude that

$$P^{(u)}[i_k(X; \bar{Y}_k) \leq \log \gamma_k(u)] \leq \epsilon_k(u) + \eta \leq P^{(u)}[i_k(X; \bar{Y}_k) \leq \log \gamma_k(u)].$$

(100)

Substituting the right-hand side (RHS) of (100) in (96), we obtain

$$\log M \leq \log \gamma_k(u) - \log \left( P^{(u)}[i_k(X; \bar{Y}_k) \leq \log \gamma_k(u)] - \epsilon_k(u) \right)$$

(101)

$$\leq \log \gamma_k(u) - \log \eta.$$  

(102)

Set now $\lambda \triangleq \log(\eta M)$. If follows from (102) and (100) that

$$P^{(u)}[i_k(X; \bar{Y}_k) < \lambda] \leq P^{(u)}[i_k(X; \bar{Y}_k) \leq \log \gamma_k(u)]$$

$$\leq \epsilon_k(u) + \eta.$$  

(103)

(104)
The inequality (104) implies that there exist positive $\varepsilon_k(u)(\bar{x})$, for all $\bar{x} \in \mathcal{X}^\infty$, and satisfying $\mathbb{E}_X[\varepsilon_k(u)(X)] \leq \varepsilon_k(u) + \eta$ such that
\[
P^{(u)}[i_k(X; \bar{Y}_k) < \lambda|X = \bar{x}] \leq \varepsilon_k(u)(\bar{x})
\] (105)
for every $\bar{x} \in \mathcal{X}^\infty$.

Next, we maximize $P[\max_k \tau_k \leq t|U = u, X = \bar{x}]$ over all stopping times $\tau_k$ satisfying (105) for every $u \in U$ and $\bar{x} \in \mathcal{X}^\infty$. Since the stopping times $\tau_k$ are conditionally independent given $U = u$ and $X = \bar{x}$, we have
\[
P^{(u)}[\max_k \tau_k \leq t|U = u, X = \bar{x}]
= \prod_k \mathbb{P}^{(u)}(\text{len}(\bar{Y}_k) \leq t|X = \bar{x})
\] (106)
\[
= \prod_k \left(\mathbb{P}^{(u)}\left[\max_{0 \leq n \leq \text{len}(\bar{Y}_k)} i_k(X; Y^n_k) \geq \lambda, \text{len}(\bar{Y}_k) \leq t|X = \bar{x}\right]
+ \mathbb{P}^{(u)}\left[\max_{0 \leq n \leq \text{len}(\bar{Y}_k)} i_k(X; Y^n_k) < \lambda, \text{len}(\bar{Y}_k) \leq t|X = \bar{x}\right]\right)
\] (107)
\[
\leq \prod_k \min\left\{1, \mathbb{P}^{(u)}\left[\max_{0 \leq n \leq \min\{t, \text{len}(\bar{Y}_k)\}} i_k(X; Y^n_k) \geq \lambda|X = \bar{x}\right]
+ \mathbb{P}^{(u)}\left[\max_{0 \leq n \leq \text{len}(\bar{Y}_k)} i_k(X; Y^n_k) < \lambda|X = \bar{x}\right]\right\}
\] (108)
\[
\leq \prod_k \min\left\{1, \mathbb{P}^{(u)}\left[\max_{0 \leq n \leq \min\{t, \text{len}(\bar{Y}_k)\}} i_k(X; Y^n_k) \geq \lambda|X = \bar{x}\right]
+ \mathbb{P}^{(u)}[i_k(X; \bar{Y}_k) < \lambda|X = \bar{x}]
\right\}
\] (109)
\[
\leq \prod_k \min\left\{1, \mathbb{P}^{(u)}\left[\max_{0 \leq n \leq \min\{t, \text{len}(\bar{Y}_k)\}} i_k(x^n; Y^n_k) \geq \lambda|X = \bar{x}\right]
+ \varepsilon_k(u)(\bar{x})\right\}
\] (110)
\[
\leq \max_{x \in \mathcal{X}} \prod_k \min\left\{1, \mathbb{P}\left[\max_{0 \leq n \leq t} i_k(x^n; Y^n_k) \geq \lambda\right] + \varepsilon_k(u)(\bar{x})\right\}
= L_t(\varepsilon_1^{(u)}(\bar{x}), \ldots, \varepsilon_K^{(u)}(\bar{x}))
\] (112)

Here, (110) follows from (105); in (111), we let $Y^n_k$ be distributed according to $P_{Y^n_k|X^n=x^n}$; finally, (112) follows from (22). Substituting (112) in (93), we conclude that
\[
\mathbb{E}\left[\max_k \tau_k|U = u, X = \bar{x}\right] \geq \sum_{t=0}^{\infty} \left(1 - L_t(\varepsilon_1^{(u)}(\bar{x}), \ldots, \varepsilon_K^{(u)}(\bar{x}))\right).
\] (113)

Hence, $\mathbb{E}[\max_k \tau_k]$ can be lower-bounded as follows:
\[
\mathbb{E}\left[\max_k \tau_k\right] \geq \min_{P_U, \varepsilon_k^{(u)}(\bar{x}) \in [0, 1]} \mathbb{E}\left[\sum_{t=0}^{\infty} \left(1 - L_t(\varepsilon_1^{(u)}(X), \ldots, \varepsilon_K^{(u)}(X))\right)\right]
\] (114)
\[
= \min_{P_U, \varepsilon_k^{(u)}(\bar{x}) \in [0, 1]} \mathbb{E}\left[\sum_{t=0}^{\infty} \left(1 - L_t(\varepsilon_1^{(u)}, \ldots, \varepsilon_K^{(u)})\right)\right].
\] (115)

Here, we have set $\mathcal{U} = U \times \mathcal{X}^\infty$. The bound in (115) requires the minimization over all probability distributions $P_U$ and over all variables $\varepsilon_k^{(u)}$ for $\bar{u} \in \text{supp}(P_U)$. 

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We next simplify the minimization problem in (115) by showing that the minimum is also attained under the additional constraint that $|\mathcal{I}| \leq K + 1$. Define the $(K + 1)$-dimensional region
\begin{equation}
\mathcal{R} \triangleq \left\{ (\varepsilon_1, \cdots, \varepsilon_K, L) \in [0, 1]^K \times \mathbb{R}_+ : L \geq \sum_{i=0}^{\infty} (1 - L_t (\varepsilon_1, \cdots, \varepsilon_K)) \right\}.
\end{equation}
Furthermore, let $\mathcal{R}_{\text{convex}}$ be the convex hull of $\mathcal{R}$. Suppose that $p_0$ lies on the lower convex envelope of $\mathcal{R}_{\text{convex}}$. Then we can write $p_0$ as a convex combination of $I \in \mathbb{N}$ points in $\mathcal{R}$:
\begin{equation}
p_0 = \sum_{i=1}^{I} \alpha_i p_i
\end{equation}
where $p_i \in \mathcal{R}$, $\alpha_i > 0$, and $\sum_{i=1}^{I} \alpha_i = 1$. Since $p_0$ is a boundary point of $\mathcal{R}_{\text{convex}}$, there exists a supporting hyperplane $\{ p \in \mathbb{R}^{K+1} : a^T p = a^T p_0 \}$ for some $a \neq 0_{K+1}$, $a \in \mathbb{R}^{K+1}$, with the property that $a^T p \leq a^T p_0$ for every $p \in \mathcal{R}_{\text{convex}}$ [17, pp. 50–51]. Now we note that the points $\{ p_i \}, i \in \{ 1, \cdots, I \}$, must be on the supporting hyperplane $\{ p \in \mathbb{R}^{K+1} : a^T p = a^T p_0 \}$ for every $i \in \{ 1, \cdots, I \}$. Indeed, suppose on the contrary that $a^T p_i < a^T p_0$ for some $i \in \{ 1, \cdots, I \}$. Then we have a contradiction:
\begin{equation}
a^T p_0 = \sum_{i=1}^{I} \alpha_i a^T p_i < \sum_{i=1}^{I} \alpha_i a^T p_0 = a^T p_0.
\end{equation}
Now, define the region
\begin{equation}
\mathcal{R}_0 \triangleq \mathcal{R} \cap \left\{ p \in \mathbb{R}^{K+1} : a^T p = a^T p_0 \text{ and } p_{K+1} \leq \max_{1 \leq i \leq I} p_{i,K+1} \right\}
\end{equation}
and the convex hull $\mathcal{R}_{0,\text{convex}}$ of $\mathcal{R}_0$. Here, $p_{i,K+1}$ denotes the $(K+1)$th entry of $p_i$. The region $\mathcal{R}_0$ is closed because $L_t(\cdot)$ is continuous in $\{ \varepsilon_k^{(u)} \}$ and the region is bounded in $\mathbb{R}^{K+1}$. This implies that $\mathcal{R}_0$ is compact. Moreover, $p_0 \in \mathcal{R}_{0,\text{convex}}$ because $p_1, \cdots, p_I \in \mathcal{R}_0$, and $\mathcal{R}_0$ lies in a $K$-dimensional affine subspace of $\mathbb{R}^{K+1}$. Hence, Carathéodory theorem [14, Th. 15.3.5] implies that $p_0$ can be written as a convex combination of at most $K + 1$ points in $\mathcal{R}_0$. But since $\mathcal{R}_0 \subseteq \mathcal{R}$, we can also write $p_0$ as a convex combination of at most $K + 1$ points in $\mathcal{R}$.

Now, observe that the point
\begin{equation}
\left( \mathbb{E} \left[ \varepsilon_1^{(U)} \right], \cdots, \mathbb{E} \left[ \varepsilon_K^{(U)} \right], \mathbb{E} \left[ \sum_{i=0}^{\infty} (1 - L_t (\varepsilon_1, \cdots, \varepsilon_K)) \right] \right)
\end{equation}
evaluated for the RV $\mathcal{U}$ supported on the set $\mathcal{U}$ and distributed as $P_{\mathcal{U}}$, and for the $\{ \varepsilon_k^{(U)} \}$ that minimize (115), is a boundary point of $\mathcal{R}_{\text{convex}}$. By the above argument, we conclude that (115) is equal to
\begin{equation}
\min_{P_{\mathcal{U}} \in \mathcal{P}(\mathcal{U}), \varepsilon_k^{(U)} \in (0,1]} : \mathbb{E} \left[ \sum_{i=0}^{\infty} (1 - L_t (\varepsilon_1, \cdots, \varepsilon_K)) \right]
\end{equation}
where the RV $\mathcal{U}$ is supported on the set $\mathcal{U}$ with $|\mathcal{U}| \leq K + 1$.

VII. PROOF OF THEOREM 4 (CONVERSE)

Fix a family of $(\ell, M, \epsilon)$-VLSF codes parameterized by the blocklength $\ell$. We shall assume that $\liminf_{\ell \to \infty} \log(M)/\ell > 0$, that is, $M$ grows at least exponentially with $\ell$. If this does not occur, then the rightmost inequality in (30) holds trivially. To establish the desired result, we analyze the nonasymptotic converse bound (23) in the limit $\log M \to \infty$. We shall set $\eta = (\log M)^{-1}$ and choose the auxiliary distributions $\{Q_k^{(\infty)}\}$ as follows. Let $x^t, t \in \mathbb{N}$, be an arbitrary $t$-dimensional vector in $\mathcal{X}^t$ and let $P_{x^t} \in \mathcal{P}(\mathcal{X})$ denote its type [18, Def. 2.1].
Furthermore, let $Q_k(\infty)$ be the product distribution on $Y_k(\infty)$ generated by the marginal distribution $P_{x^t}W_k$. Finally, let $\mathcal{P}_t(\mathcal{X}) \subseteq \mathcal{P}(\mathcal{X})$ be the set of types of $t$-dimensional sequences. We choose $Q_k(\infty)$ as follows:

\begin{equation}
Q_k(\infty)(y) = \sum_{t=1}^{[\frac{2}{C} \log M]} \sum_{P_{x^t} \in \mathcal{P}_t(\mathcal{X})} \frac{1}{[\frac{2}{C} \log M]|P_t(\mathcal{X})|} Q_k(\infty)^x(y).
\end{equation}

Here, $C$ is the channel capacity given in (1). To keep notation compact, we set

\begin{equation}
\bar{\nu}_t(x^n; y^n) \triangleq i_{P_{x^t}W_k}(x^n; y^n)
\end{equation}

which is defined for $n \leq t$ and for every $x^n \in \mathcal{X}^n$ and $y^n \in \mathcal{Y}_k^n$.

We shall next summarize the key steps of the proof. These steps are analyzed in details in Sections VII-A–VII-D.

**Step 1:** We obtain an upper bound on $L_t(\epsilon)$ that does not involve any maximization over $n$ (the inner maximization in (22)). Specifically, we show in Section VII-A that, whenever $t \leq [\frac{2}{C} \log M]$, we can dispose of this inner maximization by adding an error term of order $1/\log M$:

\begin{equation}
L_t(\epsilon) \leq \max_{x^t \in \mathcal{X}^t} \prod_{k} \min\left\{1, P[\bar{\nu}_k(x'^t; Y_k^t) \geq \lambda] + \epsilon_k \right\} + \frac{2K - 1}{\lambda}.
\end{equation}

Here,

\begin{equation}
\lambda \triangleq \log M - 2 \log \log M - |\mathcal{X}| \log\left(\frac{2}{C} \log M + 1\right).
\end{equation}

and we denoted $L_t(\epsilon_1, \cdots, \epsilon_K)$ by $L_t(\epsilon)$ with $\epsilon = [\epsilon_1, \cdots, \epsilon_K]$.

**Step 2:** We use (124) to lower-bound the RHS of (23). Since (124) holds only for $t \leq [\frac{2}{C} \log M]$, we must truncate the infinite sum in (23) as follows:

\begin{equation}
\sum_{t=0}^{\infty} (1 - L_t(\epsilon)) \geq \sum_{t=0}^{\beta_\epsilon} (1 - L_t(\epsilon))
\end{equation}

where $\beta_\epsilon \in \mathbb{N}$ is given by

\begin{equation}
\beta_\epsilon \triangleq \left\lfloor \frac{\lambda}{C} + \sqrt{\frac{\lambda V}{C^3 \nu_\epsilon}} \right\rfloor.
\end{equation}

Here, $V$ is defined in Theorem 4 and $\nu_\epsilon$ is the solution of

\begin{equation}
\prod_{k} [Q(-g_k \nu_\epsilon) + (1 - 2\delta_1)\nu_\epsilon + \delta_1] = 1
\end{equation}

with $\delta_1 \in (0, 1/2)$ being an arbitrary constant that does not depend on $\lambda$. The role of $\delta_1$ is to ensure that $\nu_\epsilon$ is bounded from above and from below for all $\epsilon \in [0, 1]^K$. We note that $\beta_\epsilon \leq \frac{2}{C} \log M$ for sufficiently large $M$ and for every $\epsilon \in [0, 1]^K$. Note that we define $\beta_\epsilon$ as in (127) instead of setting it equal $2/C \log M$ in order to control the error term originating from central limit theorem as we shall see later (see (251)). Hence, since $\beta_\epsilon \leq \frac{2}{C} \log M$, we can use (124) to further lower-bound (126). Note that $M \to \infty$ implies $\lambda \to \infty$.

**Step 3:** Next, we characterize the asymptotic behavior of the upper bound (124) in the limit $M \to \infty$. This will be used to provide an asymptotic lower bound on the RHS of (126). It turns out convenient to subdivide the interval $[0, \beta_\epsilon]$ into $K + 2$ subintervals and to perform a different asymptotic analysis on each of the subintervals.
Specifically, we set $[0, \beta_{\varepsilon}] = \bigcup_{i=1}^{K+1} T_i$ where $T_i = [t_i, t_{i+1})$, $i \in \{0, \ldots, K\}$, and $T_{K+1} = [t_{K+1}, \beta_{\varepsilon}]$ with $t_0 = 0$ and
\[
t_i \triangleq \left\lfloor \frac{\lambda}{C_i} - \sqrt{\frac{V \lambda}{C_i}} \log \lambda \right\rfloor, \quad i \in \{1, \ldots, K\}
\]
\[
t_{K+1} \triangleq \left\lfloor \frac{\lambda}{C} - \sqrt{\frac{V \lambda}{C}} \log \lambda \right\rfloor.
\]
Recall that since $C_1 \geq C_2 \geq \cdots \geq C_K \geq C$ by assumption, we have that $t_0 \leq t_1 \leq \cdots \leq t_{K+1}$. Additionally, for sufficiently large $M$, we also have that $t_{K+1} < \beta_{\varepsilon}$.

In the first $K + 1$ subintervals, we upper-bound (124) by means of a large-deviation analysis based on Hoeffding’s inequality (see Section VII-B). In the last interval, our upper bound relies on Chebyshev’s inequality and the Berry-Esseen central limit theorem (see Section VII-C). These bounds are used to further lower-bound the RHS of (126), as illustrated in Fig. 3.

We next summarize the asymptotic behavior of the bounds obtained in Sections VII-B–VII-C.

To do so, it is convenient to introduce some notation that will allow us to keep our expressions compact. First, let $\rho > 0$ and $\delta_2 \in (0, 1)$ be constants. For reasons that will become apparent later, we need $\rho$ to satisfy
\[
\frac{1}{C_i} + \frac{i}{C_K} - \frac{(i+1)\rho}{C_1} > \frac{i}{C}, \quad i \in \{1, \ldots, K-1\}.
\]
Note that the assumption (29) ensures that one can find a $\rho$ that satisfies (131). Let
\[
d_0 \triangleq \frac{1}{C} - \frac{1}{C_K} + \frac{\rho}{C_1}
\]
\[
d_1 \triangleq \frac{1}{C_i}(1 - \rho)
\]
\[
d_i \triangleq \frac{1}{C_i} - \frac{1}{C_{i-1}}, \quad i \in \{2, \ldots, K\}
\]
and define, for $\varepsilon \in [0, 1]^K$, the function\footnote{We use the convention that $\prod_{k=0}^{\infty} a_k = 1$ for an arbitrary sequence $\{a_k\}$.}
\[
f(\varepsilon) \triangleq \frac{1}{C} - d_0 \left( \max_k \varepsilon_k \right) - \sum_{i=1}^{K} d_i \left( \prod_{k \in \{1, \ldots, K\}} \varepsilon_k \right).
\]
This function is continuous in $\varepsilon \in [0, 1]^K$ and satisfies $f(0_K) = 1/C$ and $f(1_K) = 0$. We shall also need the function $g_{\delta_1, \delta_2}(\varepsilon)$ given in (255) (its exact expression is not important for the level of detail provided in this section).

This function is continuous in $\varepsilon \in [0, 1]^K$, $\delta_1 > 0$, and $\delta_2 > 0$, and has the following limits:
\[
\lim_{\delta_1 \to 0, \delta_2 \to 0} g_{\delta_1, \delta_2}(0_K) = \sqrt{\frac{V}{C^3}} \mathbb{E} \left[ \max_k H_k \right]
\]
\[
\lim_{\delta_1 \to 0, \delta_2 \to 0} g_{\delta_1, \delta_2}(1_K) = 0.
\]
In Section VII-B, we prove the following asymptotic bound, which holds for every $\varepsilon \in [0, 1]^K$ and for sufficiently large $\lambda$:
\[
\sum_{t=0}^{t_{K+1}-1} (1 - L_t(\varepsilon)) \geq \lambda f(\varepsilon) - \sqrt{\frac{V \lambda}{C^3}} \log(\lambda) \left( 1 - \max_k \varepsilon_k \right) - \varepsilon.
\]
Fig. 3. Our approach to lower-bounding (126). For the case $K = 2$ and $\varepsilon = (0.05, 0.10)$, the range of $t$ is divided into four subintervals: $[0, t_1)$, $[t_1, t_2)$, $[t_2, t_3)$, and $[t_3, \beta \varepsilon)$. The area of the blue shaded region depicts $\sum_{t=0}^{t_{3-1}} (1 - L_t(\varepsilon))$ while the area of the red shaded region depicts $\sum_{t=3}^{\beta \varepsilon} (1 - L_t(\varepsilon))$. The plotted curve represents a nonasymptotic upper bound on $L_t(\varepsilon)$ that is provided in Section VII-B and Section VII-C.

This bound holds for every $\varepsilon \in [0, 1]^K$. Furthermore, in Section VII-C, we provide the following asymptotic bound:

$$\sum_{t=1}^{t_{K+1}} (1 - L_t(\varepsilon)) \geq \sqrt{\lambda g_{\delta_1, \delta_2}(\varepsilon)} + \sqrt{\frac{\lambda V_C}{\varepsilon}} \log(\lambda) \left(1 - \max_k \varepsilon_k\right) + O(\log \lambda).$$

(139)

Here, the $O(\log \lambda)$ term is uniform in $\varepsilon \in [0, 1]^K$.

By combining (126), (138), and (139), we obtain for all $\varepsilon \in [0, 1]^K$ and for all sufficiently large $\lambda$

$$\sum_{t=0}^{\infty} (1 - L_t(\varepsilon)) \geq \sum_{t=0}^{\beta_{\varepsilon}} (1 - L_t(\varepsilon)) \geq \lambda f(\varepsilon) - \sqrt{\frac{\lambda V_C}{\varepsilon}} \log(\lambda)(1 - \max_k \varepsilon_k) - \varepsilon$$

$$+ \sqrt{\lambda g_{\delta_1, \delta_2}(\varepsilon)} + \sqrt{\frac{\lambda V_C}{\varepsilon}} \log(\lambda) \left(1 - \max_k \varepsilon_k\right) + O(\log \lambda)$$

(141)

$$= \lambda f(\varepsilon) + \sqrt{\lambda g_{\delta_1, \delta_2}(\varepsilon)} + O(\log \lambda).$$

(142)

Again, we note that the $O(\log \lambda)$ term in (142) is uniform in $\varepsilon$.

**Step 4:** We are left with solving the minimization in (23). Specifically, we need to evaluate $\min \left\{ \lambda f(\varepsilon^{(U)}) + \sqrt{\lambda g_{\delta_1, \delta_2}(\varepsilon^{(U)})} \right\}$, where the minimization is over all $\{\varepsilon^{(u)}\}_{u \in U}$ and all probability distributions $P_U \in \mathcal{P}(U)$ subject to $E_U[\varepsilon^{(u)}] \leq \epsilon + (\log M)^{-1}$. To do so, we rely on [4, Lem. 63] which is repeated here for convenience.

**Lemma 9 ([4, Lem. 63]):** Let $D$ be a compact metric space. Suppose $f : D \mapsto \mathbb{R}$ and $g : D \mapsto \mathbb{R}$ are continuous. Define

$$f^* \triangleq \max_{x \in D} f(x)$$

(143)

and

$$g^* \triangleq \sup_{x : f(x) = f^*} g(x).$$

(144)
Then,
\[
\max_{x \in D} \left[ nf(x) + \sqrt{n} g(x) \right] = n f^* + \sqrt{n} g^* + o(\sqrt{n}).
\] (145)

As a first step, we show that (see Section VII-D)
\[
\min_{\mu \in \mathcal{P}(U), \epsilon[U] \in [0,1]^K; \mathbb{E}_U \left[ \epsilon_k \right] \leq \epsilon + \epsilon \log M} \mathbb{E}_U \left[ f(\epsilon[U]) \right] = \frac{1 - \epsilon - \epsilon \log M}{C}.
\] (146)

The set of minimizers of the left-hand side (LHS) of (146) is given by
\[
\mathcal{G} = \left\{ (\mu, (\epsilon[U])) : \mathbb{E}_U \left[ \epsilon_k \right] = \epsilon + \epsilon \log M \right\}.
\]

Next, it follows from (147) that
\[
\min_{(\mu, (\epsilon[U])) \in \mathcal{G}} \mathbb{E}_U \left[ g_{\delta_1, \delta_2}(\epsilon[U]) \right]
= \min_{(\mu, (\epsilon[U])) \in \mathcal{G}} \sum_{u \in U, \mu(u) > 0} \mu(u) g_{\delta_1, \delta_2}(\epsilon[u])
= \min_{(\mu, (\epsilon[U])) \in \mathcal{G}} \left[ g_{\delta_1, \delta_2}(0_K) \sum_{\mu(u) > 0, \epsilon[u] = 0_K} \mu(u) + g_{\delta_1, \delta_2}(1_K) \sum_{\mu(u) > 0, \epsilon[u] = 1_K} \mu(u) \right]
= g_{\delta_1, \delta_2}(0_K) \left( 1 - \epsilon - \epsilon \log M \right) + g_{\delta_1, \delta_2}(1_K) \left( \epsilon + \epsilon \log M \right).
\] (150)

By (23), we have that
\[
\ell \geq \min_{\mu \in \mathcal{P}(U), \epsilon[U] \in [0,1]^K; \mathbb{E}_U \left[ \epsilon_k \right] \leq \epsilon + \epsilon \log M} \mathbb{E}_U \left[ f(\epsilon[U]) + \sqrt{m} g_{\delta_1, \delta_2}(\epsilon[U]) \right] + \mathcal{O}(\log \lambda)
\] (151)
where the $\mathcal{O}(\log \lambda)$ term is uniform in $\epsilon[U]$. We observe that the set of minimizers in (147) and the Euclidean norm form a compact metric space. It follows from Lemma 9 that
\[
\ell \geq \min_{\mu \in \mathcal{P}(U), \epsilon[U] \in [0,1]^K; \mathbb{E}_U \left[ \epsilon_k \right] \leq \epsilon + \epsilon \log M} \mathbb{E}_U \left[ f(\epsilon[U]) + \sqrt{m} g_{\delta_1, \delta_2}(\epsilon[U]) \right] + \mathcal{O}(\log \lambda)
\] (152)
\[
= \frac{\lambda(1 - \epsilon - \epsilon \log M)}{C} + \sqrt{m} \min_{(\mu, (\epsilon[U])) \in \mathcal{G}} \mathbb{E}_U \left[ g_{\delta_1, \delta_2}(\epsilon[U]) \right] + \mathcal{O}(\log \lambda)
\] (153)
\[
= \frac{\lambda(1 - \epsilon - \epsilon \log M)}{C} + \sqrt{m} g_{\delta_1, \delta_2}(0_K) \left( 1 - \epsilon - \frac{1}{\log M} \right) + \sqrt{m} g_{\delta_1, \delta_2}(1_K) \left( \epsilon + \epsilon \log M \right) + \mathcal{O}(\log \lambda)
\] (154)
\[
= \frac{\lambda(1 - \epsilon)}{C} + \sqrt{m} g_{\delta_1, \delta_2}(0_K) (1 - \epsilon) + \sqrt{m} g_{\delta_1, \delta_2}(1_K) \epsilon + \mathcal{O}(\log \lambda).
\] (155)

Here, (154) follows from (146) and in (155) we have used that $\lambda \log M = \mathcal{O}(1)$. Recall that $g_{\delta_1, \delta_2}(0_K)$ and $g_{\delta_1, \delta_2}(1_K)$ are continuous in $\delta_1 > 0$ and $\delta_2 > 0$, and that we have the limits (136) and (137). Hence, by choosing $\delta_1$ and $\delta_2$ arbitrarily small, we obtain the inequality
\[
\ell \geq \frac{\lambda(1 - \epsilon)}{C} + (1 - \epsilon) \max_{k} H_k + \mathcal{O}(\sqrt{m}).
\] (156)
Here, recall that \( \{H_k\} \) have cumulative distribution function given in \( (33) \). Finally, using the definition of \( \lambda \) in \( (125) \), we conclude that

\[
\log M \leq \frac{\ell C}{1 - \epsilon} - \sqrt{\frac{TV}{1 - \epsilon}} \max_k H_k + o(\sqrt{\ell})
\]

which establishes the desired result.

A. Disposing of the maximum in \( (22) \)

By \( (19) \), we have that, for all \( \bar{y} \in \mathcal{Y}_k \),

\[
Q_k(\bar{y}) = \sum_{y \in \mathcal{Y}_k^\infty} \left[ \sum_{i=1}^{\lfloor \frac{\sqrt{2} \log M}{\log M} \rfloor} \sum_{P \in \mathcal{P}_\ell(\mathcal{X})} \frac{1}{|\mathcal{P}_\ell(\mathcal{X})|} Q_{k,\mathcal{X}}^{(\infty)}(y) \right] P_{x_1} W_x(k).
\]

Let now \( \ell \) be an integer no larger than \( \frac{\sqrt{2}}{C} \log M \). Using that \( Q_k \) is a convex combination of measures on \( \mathcal{Y}_k \), we obtain the following relation between \( i_k(x^t; y^t_k) \) and \( i_{P_x, w_k}(x^t; y^t_k) \)

\[
i_k(x^t; y^t_k) = \log \frac{W_k^{(t)}(y^t_k|x^t)}{Q_k(y^t_k)} \leq \log \frac{W_k^{(t)}(y^t_k|x^t)}{|P_{x_1} W_x(k)| P_{x_1} W_x^{(t)}(y_k)} \leq i_{P_x, w_k}(x^t; y^t_k) + |\mathcal{X}| \log \left( \frac{2}{C} \log M + 1 \right).
\]

In \( (161) \), \( P_{x_1} W_x^{(t)}(y_k) \triangleq (P_{x_1} W_x)^t(y_k) \) denotes the product distributions induced on \( \mathcal{Y}_k^t \) by the probability distribution \( (P_{x_1})^t \). The inequality in \( (161) \) follows because the logarithm is monotonically increasing and because \( \{P_{x_1} W_x^{(t)}(y_k)\} \) are nonnegative. Finally, \( (162) \) follows because \( |\mathcal{P}_\ell(\mathcal{X})| \leq (t + 1)^{\left| \mathcal{X} \right|} \) [14, Th. 11.1.1]. We can now upper-bound \( L_t(\epsilon) \) in \( (22) \) for \( t \leq \frac{\sqrt{2}}{C} \log M \), where \( \epsilon \triangleq [\epsilon_1, \cdots, \epsilon_K] \), as follows. Let \( \hat{\lambda} \) be defined as

\[
\hat{\lambda} \triangleq \log M - \log \log M - |\mathcal{X}| \log \left( \frac{2}{C} \log M + 1 \right).
\]

Then,

\[
L_t(\epsilon) = \max_{x^t \in \mathcal{X}} \prod_k \min \left\{ 1, \mathbb{P} \left[ \max_{0 \leq n \leq \ell} i_k(x^n; Y^n_k) \geq \log M + \log \eta \right] + \epsilon_k \right\} \leq \max_{x^t \in \mathcal{X}} \prod_k \min \left\{ 1, \mathbb{P} \left[ \max_{0 \leq n \leq \ell} i_{P_x, w_k}(x^n; Y^n_k) \geq \lambda \right] + \epsilon_k \right\} = \max_{x^t \in \mathcal{X}} \prod_k \min \left\{ 1, \mathbb{P} \left[ \max_{0 \leq n \leq \ell} i_k(x^n; Y^n_k) \geq \lambda \right] + \epsilon_k \right\}.
\]

In \( (165) \), we have used \( (162) \) and that \( \eta = (\log M)^{-1} \), and in \( (166) \), we have used \( (123) \).
We dispose of the inner maximization in (166) through the following steps. Fix a constant \( \nu \geq 0 \). Then

\[
P\left[ \max_{0 \leq n \leq t} \tilde{\iota}_k(x^n; Y^n_k) \geq \bar{\lambda} \right] \\
= P\left[ \bigcup_{n=0}^{t} \left\{ \tilde{\iota}_k(x^n; Y^n_k) \geq \bar{\lambda} \right\} \right] \\
\leq P\left[ \tilde{\iota}_k(x^t; Y^t_k) \geq \bar{\lambda} \right] \cup \bigcup_{n=0}^{t-1} \left\{ \tilde{\iota}_k(x^n; Y^n_k) \geq \bar{\lambda} - \nu \right\} \\
\leq P\left[ \tilde{\iota}_k(x^t; Y^t_k) \geq \bar{\lambda} - \nu \right] + P\left[ \bigcup_{n=1}^{t} \left\{ \tilde{\iota}_k(x^n; Y^n_k) \leq -\nu \right\} \right]. \tag{167}
\]

In (168), we denoted the last \( t-n \) entries of \( Y^n_k \) by \( Y^n_{k,n+1} \). The inequality in (168) holds because \( \tilde{\iota}_k(x^n; Y^n_k) \geq \bar{\lambda} - \nu \) and \( \tilde{\iota}_k(x^n; Y^n_k) \leq -\nu \) imply that \( \tilde{\iota}_k(x^n) = \tilde{\iota}_k(x^n; Y^n_k) \). In (170), we have used the union bound. Define now

\[
\bar{\tau}_k(x^t, y^t) \triangleq \max\left\{ \{0\} \cup \{1 \leq n \leq t : \tilde{\iota}_k(x^n; y^n) \leq -\nu \} \right\}. \tag{171}
\]

This definition implies the following: if \( \bar{\tau}(x^t, y^t) = n \), then for all sequences \( y^t \in Y^t_k \) such that \( y^n_k = y^n \), we also have that \( \bar{\tau}_k(x^t, y^t) = n \).

Let \( \bar{y}^t_k \in Y^t_k \) be arbitrary vectors for \( k \in K \). We can now upper-bound the second term in (170) as follows

\[
P\left[ \bigcup_{n=1}^{t} \left\{ \tilde{\iota}_k(x^n; Y^n_k) \leq -\nu \right\} \right] \\
= \sum_{n=1}^{t} \sum_{y^t \in Y^t_k} W_k^n(y^t|x^n) \\
= \sum_{n=1}^{t} \sum_{y^t \in Y^t_k} \prod_{i=n}^{t} P_{x_i} W_k(y_i) \\
\leq \exp(-\nu) \sum_{n=1}^{t} \left( \sum_{y^t \in Y^t_k} \left( \prod_{i=n}^{t} P_{x_i} W_k(y_i) \right) \right) W^{-n-1} (y^n-1|x^n-1) \\
= \exp(-\nu) \sum_{n=1}^{t} \left( \prod_{i=n}^{t} P_{x_i} W_k(y_i) \right) \\
= \exp(-\nu) \sum_{n=1}^{t} \left( \prod_{i=n}^{t} P_{x_i} W_k^n(\bar{\tau}(x^t, y^t)) \right) \\
\leq \exp(-\nu) \sum_{n=1}^{t} P_{x_i} W_k^n(\bar{\tau}(x^t, y^t)) = n. \tag{172}
\]

Here, (174) holds because of (123) and because \( \tilde{\iota}_k(x^n; y^n_k) \leq -\nu \) for every \( y^n_k \) such that \( \bar{\tau}_k(x^t, y^t) = n \); (175) holds because \( \{y^n_k\}_{k \in K} \) are fixed arbitrary vectors and because all sequences \( y^t \) whose last \( t-n+1 \) entries coincide with the ones of \( y^t \) also satisfy \( \bar{\tau}_k(x^t, y^t) = n \).

By substituting (177) into (170), by choosing \( \nu = \log \log M \), and by recalling the definition of \( \lambda \) in (125), we conclude that

\[
P\left[ \max_{0 \leq n \leq t} \tilde{\iota}_k(x^n; Y^n_k) \geq \bar{\lambda} \right] \leq P\left[ \tilde{\iota}_k(x^n; Y^n_k) \geq \bar{\lambda} \right] + \frac{1}{\log M}. \tag{178}
\]
It follows from (166), from (178), and from the inequality $(\log M)^{-1} \leq 1/\lambda$ that for all $t \leq \frac{2}{\lambda} \log M$, we have

$$L_t(\epsilon) \leq \max_{x \in \mathcal{X}} \prod_k \min\left\{1, \mathbb{P}[i_k(x'; Y_k^t) \geq \lambda] + \frac{1}{\log M} + \epsilon_k\right\}$$

$$\leq \max_{x \in \mathcal{X}} \prod_k \left(\min\{1, \mathbb{P}[i_k(x'; Y_k^t) \geq \lambda] + \epsilon_k\} + \frac{1}{\lambda}\right)$$

$$\leq \max_{x \in \mathcal{X}} \prod_k \min\{1, \mathbb{P}[i_k(x'; Y_k^t) \geq \lambda] + \epsilon_k\} + \frac{2^K - 1}{\lambda}. \quad (181)$$

In (181), we have expanded the product in (180) into $2^K$ terms and used that $\min\{1, \mathbb{P}[i_k(x'; Y^t_k) \geq \lambda]\} \leq 1$. This establishes (124).

**B. Large-deviation analysis**

To apply Hoeffding’s inequality for all $t$ within the first $K + 1$ intervals, we use that

$$b \triangleq \max_{k} \max_{x \in \mathcal{X}} \max_{y \in \mathcal{Y}_k(x)} |i_k(x, y)| \quad (182)$$

is finite. Here, $\mathcal{Y}_k(x)$ denotes the support of $W_k(\cdot; x)$. We shall first treat the first $K$ intervals and shortly return to the interval $T_K$ for which we need the additional property that $\min_k I_k(P) \leq C$ for every $P \in \mathcal{P}(\mathcal{X})$. We first obtain the following large-deviation bound, which holds for all $t \in T_i, i \in \{0, \cdots, K - 1\}$ and $k \in \{i + 1, \cdots, K\}$:

$$\max_{x' \in \mathcal{X}} \mathbb{P}[i_k(x'; Y^t_k) \geq \lambda] = \max_{x' \in \mathcal{X}} \mathbb{P}\left[\frac{i_k(x'; Y^t_k)}{t} - I_k(P_{x'}) \geq \frac{\lambda}{t} - I_k(P_{x'})\right]$$

$$\leq \max_{x' \in \mathcal{X}} \exp\left(-\frac{1}{2b^2} \left(\frac{\lambda - I_k(P_{x'})}{\sqrt{t}}\right)^2\right) \quad (184)$$

$$= \max_{x' \in \mathcal{X}} \exp\left(-\frac{1}{2b^2} \left(\frac{\lambda - \frac{t}{t+1} I_k(P_{x'})}{\sqrt{t}}\right)^2\right)$$

$$\leq \max_{x' \in \mathcal{X}} \exp\left(-\frac{1}{2b^2} \left(\frac{\lambda - \frac{t}{t+1} I_k(P_{x'})}{\sqrt{t+1}}\right)^2\right)$$

$$\leq \exp\left(-\frac{\lambda}{2b^2} \left(\frac{\lambda - \frac{t}{t+1} C_{i+1}}{\sqrt{t+1}}\right)^2\right)$$

$$\leq \exp\left(-\frac{\lambda}{2b^2} \left(\frac{\lambda - \frac{t}{t+1} C_{i+1} \sqrt{V/\lambda}}{\sqrt{t+1}}\right)^2\right)$$

$$\leq \exp\left(-\frac{\lambda}{2b^2} \left(\frac{C_{i+1} \sqrt{V/\lambda} \log \lambda}{\lambda / C_{i+1} - \sqrt{V/\lambda} \log \lambda}\right)^2\right)$$

$$= \exp\left(-\frac{\lambda}{2b^2} \left(\frac{C_{i+1} \sqrt{V/\lambda} \log \lambda}{\lambda / C_{i+1} - \sqrt{V/\lambda} \log \lambda}\right)^2\right)$$

$$\leq \exp\left(-\frac{\lambda}{2b^2} \left(\frac{C_{i+1} \sqrt{V/\lambda} \log \lambda}{\lambda / C_{i+1} - \sqrt{V/\lambda} \log \lambda}\right)^2\right)$$

$$= \exp\left(-\frac{\lambda}{2b^2} \left(\frac{C_{i+1} \sqrt{V/\lambda} \log \lambda}{\lambda / C_{i+1} - \sqrt{V/\lambda} \log \lambda}\right)^2\right)$$

$$\leq \exp\left(-\frac{\lambda}{2b^2} \left(\frac{C_{i+1} \sqrt{V/\lambda} \log \lambda}{\lambda / C_{i+1} - \sqrt{V/\lambda} \log \lambda}\right)^2\right)$$

$$\leq \exp\left(-\frac{\lambda}{2b^2} \left(\frac{C_{i+1} \sqrt{V/\lambda} \log \lambda}{\lambda / C_{i+1} - \sqrt{V/\lambda} \log \lambda}\right)^2\right)$$

$$\leq \exp\left(-\frac{\lambda}{2b^2} \left(\frac{C_{i+1} \sqrt{V/\lambda} \log \lambda}{\lambda / C_{i+1} - \sqrt{V/\lambda} \log \lambda}\right)^2\right)$$

$$= \left(\frac{\lambda}{1 - \epsilon_i \log \lambda}\right). \quad (193)$$
Here, (184) follows from Hoeffding’s inequality [19, Th. 2] and from (182), (186) follows because \((\lambda - tI_k(P_st)) / \sqrt{t}\) for a fixed distribution \(P_{st}\), is a nonincreasing function of \(t\) and because \(t < t_{i+1}\), (187) follows because \(I_k(P_{st})\) is uniformly upper-bounded by \(C_{i+1}\) for \(k \in \{i + 1, \cdots, K\}\) (recall that we assumed \(C_1 \geq \cdots \geq C_K\)), (188) follows from (129), and (189)–(193) follow from algebraic manipulations.

Next, we consider the interval \(T_K\). Fix a probability distribution \(P \in \mathcal{P}(X)\), and let

\[
\kappa(P) \triangleq \arg \min_k I_k(P).
\]

Note that \(I_{\kappa(P)}(P) \leq C\) for every \(P \in \mathcal{P}(X)\). Let also \(c_{1K} \triangleq V/(2b^2)\). We have the following bound for all \(t \in T_K\)

\[
\max_{x^t \in \mathcal{X}^t} \min_k \mathbb{P}[\tilde{I}_k(x^t; Y^t_k) \geq \lambda] \leq \max_{x^t \in \mathcal{X}^t} \min_k \exp \left(-\frac{1}{2b^2} \left(\lambda - t_{K+1} I_k(P_{st})\right)^2 / \sqrt{t_{K+1}}\right) \leq \max_{x^t \in \mathcal{X}^t} \exp \left(-\frac{1}{2b^2} \left(\lambda - t_{K+1} c_{1K}\right)^2 / \sqrt{t_{K+1}}\right) \leq \left(\frac{1}{\lambda}\right) c_{1K} \log \lambda.
\]

Here, (195) follows from steps similar to (183)–(186), (197) holds because \(I_{\kappa(P)}(P) \leq C\) for every \(P \in \mathcal{P}(X)\), and (198) follows from steps similar to (187)–(193). Using (193), we conclude that for all \(i \in \{0, \cdots, K - 1\}\)

\[
\sum_{t=i}^{t_{i+1}-1} L_t(\varepsilon) \leq \sum_{t=i}^{t_{i+1}-1} \max_{x^t \in \mathcal{X}^t} \prod_k \min \left\{1, \mathbb{P}[\tilde{I}_k(x^t; Y^t_k) \geq \lambda] + \varepsilon_k\right\} + \mathcal{O}(1)
\]

\[
\leq \sum_{t=i}^{t_{i+1}-1} \max_{x^t \in \mathcal{X}^t} \left(\mathbb{P}[\tilde{I}_k(x^t; Y^t_k) \geq \lambda] + \varepsilon_k\right) + \mathcal{O}(1)
\]

\[
\leq \left(t_{i+1} - t_i\right) \prod_{k \in \{i+1, \cdots, K\}} \varepsilon_k + \mathcal{O}(1)
\]

as \(\lambda \to \infty\). Here, (202) follows because \((t_{i+1} - t_i)\lambda^{-c_{1K} \log \lambda} \leq c_{1K} \lambda^{-c_{1K} \log \lambda + 1} = o(1)\) as \(\lambda \to \infty\). Similarly, it follows from (198) that

\[
\sum_{t=t_K}^{t_{K+1}-1} L_t(\varepsilon) \leq \sum_{t=t_K}^{t_{K+1}-1} \max_{x^t \in \mathcal{X}^t} \prod_k \min \left\{1, \mathbb{P}[\tilde{I}_k(x^t; Y^t_k) \geq \lambda] + \varepsilon_k\right\} + \mathcal{O}(1)
\]

\[
\leq \sum_{t=t_K}^{t_{K+1}-1} \max_{x^t \in \mathcal{X}^t} \left(\max_k \mathbb{P}[\tilde{I}_k(x^t; Y^t_k) \geq \lambda] + \varepsilon_k\right) + \mathcal{O}(1)
\]

\[
\leq \left(t_{K+1} - t_K\right) \left(\lambda^{-c_{1K} \log \lambda} + \max_k \varepsilon_k\right) + \mathcal{O}(1)
\]

\[
\leq \left(t_{K+1} - t_K\right) \max_k \varepsilon_k + \mathcal{O}(1)
\]
as \( \lambda \to \infty \). Here, (204) follows because \( \prod_k \min\{1, a_k\} \leq \min_k a_k \) for all nonnegative constants \( \{a_k\} \), (205) holds because \( \min_k \{a_k + \varepsilon_k\} \leq \min_k a_k + \max_k \varepsilon_k \), and (206) follows from (198). Finally, by combining (202) and (207), we obtain
\[
\sum_{t=0}^{t_K} L_t(\varepsilon) \leq \sum_{i=0}^{K} \left[ (t_i - t_{i-1}) \prod_{k \in \{i, \ldots, K\}} \varepsilon_k \right] + (t_{K+1} - t_K) \max_k \varepsilon_k + O(1)
\]  
(208)

\[
= \left( \frac{\lambda}{C_1} - \sqrt{\frac{\lambda V}{C^3}} \log \lambda \right) \sum_{k=1}^{K} \prod_{k \in \{i, \ldots, K\}} \varepsilon_k
\]
(209)

\[
\leq \left( \frac{\lambda}{C_1} - \lambda \rho \right) \sum_{k=1}^{K} \prod_{k \in \{i, \ldots, K\}} \varepsilon_k
\]

\[
+ \left( \frac{\lambda}{C_1} - \frac{\lambda}{C_K} - \sqrt{\frac{\lambda V}{C^3}} \log \lambda \right) \max_k \varepsilon_k + O(1)
\]  
(210)

\[
= \sum_{i=1}^{K} \left[ d_i \prod_{k \in \{i, \ldots, K\}} \varepsilon_k \right] + \left( d_0 - \sqrt{\frac{\lambda V}{C^3}} \log \lambda \right) \max_k \varepsilon_k + O(1)
\]  
(211)

Here, (210) follows by adding to (209) the term \( \max_k \varepsilon_k - \prod_k \varepsilon_k \left( \lambda \rho / C_1 - \sqrt{\lambda V / C^3} \right) \), which is positive for all sufficiently large \( \lambda \); and (211) follows by the definition of the \( \{d_i\}_{i \in \{0, \ldots, K\}} \) (see (132)–(134)). Finally, (138) follows from (211) and by the definition of \( f(\varepsilon) \) in (135).

C. Central-limit analysis

Within the interval \( [t_{K+1}, \beta \varepsilon] \), we use Chebyshev’s inequality and the Berry-Esseen central limit theorem to obtain a bound on (124). Fix a constant \( \delta_2 \in (0, 1) \) and let \( \mathcal{A} \) be a compact convex neighborhood of \( P^* \) such that for all \( P \in \mathcal{A} \), we have both \( V_k(P) > 0 \) and
\[
\left| \frac{V_k(P)}{I_k(P)^{3/2}} - \sqrt{\frac{V_k}{C^3}} \right| \leq \sqrt{\frac{V_k}{C^3}} \delta_2.
\]  
(212)

The existence of such a set \( \mathcal{A} \) follows from the continuity of \( I_k(\cdot) \) and \( V_k(\cdot) \) at \( P^* \). By the definition of \( \mathcal{A} \), it follows that
\[
I_k(P)(P) < C - \delta_3
\]  
(213)

for every \( P \notin \mathcal{A} \) and for some \( C > \delta_3 > 0 \). This is a consequence of the uniqueness of \( P^* \). The inequality (212) enables us to approximate \( \sqrt{V_k(P) / I_k(P)^{3/2}} \) by \( \sqrt{V_k/C^3} \) (which does not depend on \( P \)) as long as \( P \) belongs to the set \( \mathcal{A} \). In other words, we can eliminate the dependency on \( P \) by introducing an error term proportional to \( \delta_2 \), which can be made arbitrarily small.

We shall use the following upper bound on \( L_1(\varepsilon) \):
\[
L_1(\varepsilon)
\]
\[
\leq \max_{x^i \in \mathcal{X}} \prod_k \min_k \left\{ 1, \mathbb{P}[\tilde{t}_k(x^i; Y_k^i) \geq \lambda] + \varepsilon_k \right\} + \frac{2^K - 1}{\lambda}
\]  
(214)

\[
\leq \max_k \left\{ \max_{x^i \in \mathcal{X}} \min_k \mathbb{P}[\tilde{t}_k(x^i; Y_k^i) \geq \lambda] + \max_k \varepsilon_k, \max_{x^i \in \mathcal{X}} \prod_k \mathbb{P}[\tilde{t}_k(x^i; Y_k^i) \geq \lambda] + \varepsilon_k \right\} + \frac{2^K - 1}{\lambda}
\]  
(215)
Here, (215) follows because $\prod_k (a_k + b_k) \leq \min_k (a_k + b_k) \leq \min_k a_k + \max_k b_k$ for all constants \{a_k\} and \{b_k\).

For all $x^t \in \mathcal{X}^t$ for which $P_{x^t} \not\in A$, we use Chebyshev’s inequality to obtain the estimate
\[
P[\tilde{u}_k(x^t; Y_{k}^t) \geq \lambda] \leq \frac{\text{t}_v(P_{x^t})}{(\lambda - \mu(P_{x^t}))^2} \quad \text{if } \lambda > \text{t}_I(P_{x^t})
\]
\[
\leq \frac{V_{\max t}}{(\lambda - tC + t\delta_3)^2} \quad \text{otherwise.}
\]

It follows from (213) and from the condition $t \leq \beta_\varepsilon$ (see definition of $\beta_\varepsilon$ in (127)) that $\lambda > \text{t}_I(P) \geq \lambda$ for every $P \not\in A$ and for all sufficiently large $\lambda$ (recall the definition of $\kappa(\cdot)$ in (194)). Using (216), we obtain the following upper bound on the first term on the RHS of (215), which holds for $t \leq [t, K+1, \beta_\varepsilon]$ and for all sufficiently large $\lambda$:
\[
\frac{\max_{x^t \in \mathcal{X}^t} \min_{P_{x^t} \notin A} \text{P}[\tilde{u}_k(x^t; Y_{k}^t) \geq \lambda]}{\max_{x^t \in \mathcal{X}^t}} \leq \frac{V_{\max t}}{(\lambda - tC + t\delta_3)^2}
\]
\[
\leq \frac{2V_{\max \lambda}}{(\lambda\delta_3/C - \varepsilon \sqrt{\lambda} \log \lambda - \varepsilon)^2}
\]
\[
\leq \frac{1}{\lambda} 4V_{\max C^2} \delta_3 \leq \frac{\delta_3}{\varepsilon_c < 28}.
\]
Here, (218) follows because there exists a constant $V_{\max} > 0$ such that $V_{\max} \leq V_{\max}$ for every $P \in \mathcal{P}(\mathcal{X})$ [20, p. 7048] and because of (213); (219) follows because $t \leq \beta_\varepsilon$ and $\delta_3 < C$, which imply that, for all sufficiently large $\lambda$,
\[
\lambda - tC + t\delta_3 \geq \lambda - (C - \delta_3)\beta_\varepsilon
\]
\[
= \lambda - (C - \delta_3)(\lambda/C + \varepsilon \sqrt{\lambda} \log \lambda)
\]
\[
= \lambda\delta_3/C - \varepsilon \sqrt{\lambda} \log \lambda - \varepsilon > 0.
\]
Finally, (220) holds for all sufficiently large $\lambda$. We see that (220) can be made arbitrarily close to zero by choosing $\lambda$ sufficiently large. Now, we continue the chain of inequalities in (215) as follows:
\[
L_t(\varepsilon) \leq \max \left\{ \frac{c_2}{\lambda} \max_{k} \varepsilon_k, \max_{P_{x^t} \in A} \prod_k (\mathbb{P}[\tilde{u}_k(x^t; Y_{k}^t) \geq \lambda] + \varepsilon_k) \right\} \frac{2K - 1}{\lambda} \]
\[
\leq \max \left\{ \frac{c_2}{\lambda}, \max_{P_{x^t} \in A} \prod_k (\mathbb{P}[\tilde{u}_k(x^t; Y_{k}^t) \geq \lambda] + \varepsilon_k) - \prod_k \varepsilon_k \right\} + \max_k \varepsilon_k + \frac{2K - 1}{\lambda}
\]
\[
\leq \max_{P_{x^t} \in A} \prod_k (\mathbb{P}[\tilde{u}_k(x^t; Y_{k}^t) \geq \lambda] + \varepsilon_k) - \prod_k \varepsilon_k + \max_k \varepsilon_k + \frac{1}{\lambda} \left( \frac{2K - 1}{\varepsilon_c} \right).
\]
Here, in (225), we used that $\max_k \varepsilon_k \geq \prod_k \varepsilon_k$ because $\varepsilon_k \in [0, 1]$ for all $k \in K$. Note that the upper bound (226) allows us to consider only the $x^t \in \mathcal{X}^t$ for which $P_{x^t} \in \mathcal{P}(\mathcal{X})$. This in turn allows us to make use of (212). Specifically, for all $x^t \in \mathcal{X}^t$ for which $P_{x^t} \in A$, the Berry-Esseen central limit theorem [21, Th. V.2.3] yields the
following estimate:

\[ \mathbb{P}[\hat{t}_k(x'; Y'_k) \geq \lambda] \leq Q \left( \frac{\lambda - t \hat{I}_k(P_{x'})}{\sqrt{V_k(P_{x'})}} \right) + \frac{6tT_k(P_{x'})}{(tV_k(P_{x'}))^{3/2}} \]

\[ \leq Q \left( \frac{\lambda - t \hat{I}_k(P_{x'})}{\sqrt{V_k(P_{x'})}} \right) + \frac{1}{\sqrt{2Ct}} \frac{6\sqrt{2C} \max_{P \in A} T_k(P)}{\min_{P \in A} V_k(P)^{3/2}} \]

\[ \leq Q \left( \frac{\lambda / I_k(P_{x'}) - t}{\sqrt{V_k(P_{x'})/I_k(P_{x'})^2}} \right) + \frac{c_4}{\sqrt{2Ct}} \]

\[ \leq Q \left( \min_{\nu_k \in (-1,1)} \frac{\lambda / I_k(P_{x'}) - t}{\sqrt{V_k/C^3} (1 + \delta_2 \nu_k)} \right) + \frac{c_4}{\sqrt{\lambda}} \]  

In (228), \( c_4 \) is a well-defined positive constant because \( T_k(P) < \epsilon \) uniformly [4, Lem. 46] and because the condition \( V_k(P) > 0 \) for \( P \in A \) combined with the compactness of \( A \) imply that \( \min_{P \in A} V_k(P) \) is well-defined and positive; (229) follows by the inequality (proven in Appendix IV)

\[ \frac{a - tb}{\sqrt{t}} \leq \frac{a - tb}{a/b} \]  

which holds for all positive \( a, b \) and \( t \); and (230), which holds for all sufficiently large \( \lambda \), follows from (212) (recall that \( P_{x'} \in A \), which is equivalent to

\[ \sqrt{\frac{V_k}{C^3} (1 - \delta_2)} \leq \sqrt{\frac{V_k(P)}{I_k(P)^3}} \leq \sqrt{\frac{V_k}{C^3} (1 + \delta_2)}. \]

In (230), the role of \( \nu_k \) is to select the upper or the lower bound in (232). The choice depends on the sign of \( (\lambda / I_k(P_{x'}) - t) \). The bound (230) implies that

\[ \prod_k (\mathbb{P}[\hat{t}_k(x'; Y'_k) \geq \lambda] + \varepsilon_k) \leq \prod_k \left( Q \left( \min_{\nu_k \in (-1,1)} \frac{\lambda / I_k(P_{x'}) - t}{\sqrt{V_k/C^3} (1 + \delta_2 \nu_k)} \right) + \frac{c_5}{\sqrt{\lambda}} \right) \]

where \( c_5 \equiv (2^K - 1)c_4 \).

We shall next eliminate the dependency of the first term of the RHS of (233) on \( x' \) by further upper-bounding this term. Let \( P \in A \); we have

\[ Q \left( \min_{\nu_k \in (-1,1)} \frac{\lambda / I_k(P_{x'}) - t}{\sqrt{V_k/C^3} (1 + \delta_2 \nu_k)} \right) \leq Q \left( \min_{\nu_k \in (-1,1)} \frac{\lambda / (C + \nabla I_k(P - P^*)) - t}{\sqrt{V_k/C^3} (1 + \delta_2 \nu_k)} \right) \]

\[ \leq Q \left( \min_{\nu_k \in (-1,1)} \frac{\lambda / (C + \nabla I_k(P - P^*)) - t}{\sqrt{V_k/C^3} (1 + \delta_2 \nu_k)} \right). \]

Here, (234) follows because \( I_k(P) \) is concave in \( P \) and because the \( Q \) function is monotonically decreasing; (235) follows from the inequality \( \frac{a + \Delta}{b + \Delta} \geq \frac{a}{b} - \frac{\Delta}{b} \epsilon \) which holds for all \( a > 0, b > 0, \) and \( b + c > 0 \). Indeed, \( C + \nabla I_k(P - P^*) > 0 \) for sufficiently small \( \delta_2 \) since \( P \in A \). Using (235) in (233), we obtain

\[ \prod_k (\mathbb{P}[\hat{t}_k(x'; Y'_k) \geq \lambda] + \varepsilon_k) \leq \prod_k \left( Q \left( \min_{\nu_k \in (-1,1)} \frac{\lambda / (C + \nabla I_k(P_{x'}) - t)}{\sqrt{V_k/C^3} (1 + \delta_2 \nu_k)} \right) + \varepsilon_k \right) + \frac{c_5}{\sqrt{\lambda}} \]

\[ \leq \max_{P \in A} \prod_k \left( Q \left( \min_{\nu_k \in (-1,1)} \frac{\lambda / (C - t)}{\sqrt{V_k/C^3} (1 + \delta_2 \nu_k)} \right) + \varepsilon_k \right) + \frac{c_5}{\sqrt{\lambda}} \]

\[ \leq \max_{\nu \in \mathbb{R}^{|X|}} \prod_k \left( Q \left( \min_{\nu_k \in (-1,1)} \frac{\lambda / (C - t)}{\sqrt{V_k/C^3} (1 + \delta_2 \nu_k)} \right) + \varepsilon_k \right) + \frac{c_5}{\sqrt{\lambda}}. \]
Here, in (238), we used that $\varrho_k = \sqrt{v_k/V}$. Let now $\{H_{z_k,k}\}$ be i.i.d. RVs with cumulative distribution function
\[ F_{H_{z,k}}(w) \triangleq Q\left( \min_{\nu_k \in \{-1,1\}} \frac{-w - \nabla I_k(\tilde{v}_{z_k}(w))}{\varrho_k(1 + \delta_2 v_k)} \right) \] (239)
where\(^6\)
\[ \tilde{v}_{z_k}(w) \triangleq \arg \max_{\nu \in \mathbb{R}_{\{0\}}^{[1]}} \prod_k \left( Q\left( \min_{\nu_k \in \{-1,1\}} \frac{-w - \nabla I_k(\nu)}{\varrho_k(1 + \delta_2 v_k)} \right) + \varepsilon_k \right). \] (240)

We also denote by
\[ \mathcal{R} \triangleq \{ \tilde{K} : \tilde{K} \subseteq \mathcal{K} \} \setminus \{ \emptyset \} \] (241)
the set of all nonempty subsets of $\mathcal{K}$. Using these definitions and (238), we have that for every $x^t \in \mathcal{X}^t$ such that $P_{z^t} \in \mathcal{A},^7$
\[ L_t(\nu) \leq \max_{x'^t \in \mathcal{X}^t} \prod_k \left( P\left[ \tilde{r}_k(x'^t; Y_k^t) \geq \lambda \right] + \varepsilon_k \right) - \prod_k \varepsilon_k + \max_k \varepsilon_k + \frac{c_3}{\lambda} + 1 \] (242)
\[ \leq \prod_k \left( F_{H_{z,k}}\left( -\frac{\lambda/C - t}{\sqrt{AV/C^3}} \right) + \varepsilon_k \right) - \prod_k \varepsilon_k + \max_k \varepsilon_k + \frac{2c_5}{\sqrt{\lambda}} + 1 \] (243)
\[ = \sum_{K \in \mathcal{R}} \left( \prod_{k \in \mathcal{K} \setminus \{K\}} \varepsilon_k \right) \prod_{k \in K} F_{H_{z,k}}\left( -\frac{\lambda/C - t}{\sqrt{AV/C^3}} \right) + \max_k \varepsilon_k + \frac{c_6}{\sqrt{\lambda}}. \] (244)

Here, (243), which holds for sufficiently large $\lambda$, follows from (238) and (239); (244) follows from (241) and by expanding the product in the first term of (243) into $2^K$ terms.

We now evaluate (244). For every nonempty subset $\tilde{K} \subseteq \mathcal{K}$ and for sufficiently large $\lambda$, we have
\[ \sum_{t=t_{K+1}}^{\beta_x} \prod_{k \in \tilde{K}} F_{H_{z,k}}\left( -\frac{\lambda/C - t}{\sqrt{AV/C^3}} \right) \leq \int_{-\infty}^{\beta_x} \prod_{k \in \tilde{K}} F_{H_{z,k}}\left( -\frac{\lambda/C - t}{\sqrt{AV/C^3}} \right) \, dt + 1 \] (245)
\[ = \int_{-\infty}^{\beta_x} P\left[ \max_{k \in \tilde{K}} \left\{ H_{z_k,k} + \frac{\lambda/C - t}{\sqrt{AV/C^3}} \right\} \leq 0 \right] \, dt + 1 \] (246)
\[ = \int_{-\infty}^{\beta_x} P\left[ \max_{k \in \tilde{K}} \left\{ \frac{\lambda}{C} + \sqrt{AV/C^3} H_{z_k,k} \right\} \leq t \right] \, dt + 1 \] (247)
\[ = \beta_x - E\left[ \min\left\{ \beta_x, \max_{k \in \tilde{K}} \left\{ \frac{\lambda}{C} + \sqrt{AV/C^3} H_{z_k,k} \right\} \right\} \right] + 1 \] (248)
\[ = \sqrt{\frac{AV}{C^3}} \left( \nu_e - E\left[ \min\left\{ \nu_e, \max_{k \in \tilde{K}} H_{z_k,k} \right\} \right] \right) + 1. \] (249)

In (245), we have used that $F_{H_{z,k}}(\cdot)$ is a monotonically increasing function upper-bounded by one; (248) holds because for every RV $X$,
\[ E[\min\{a, X\}] = a - \int_{-\infty}^{a} P[ X \leq t ] \, dt. \] (250)

Finally, (249) follows from (127).

\(^6\)If the maximizer of (240) is not unique, $\tilde{v}_{z_k}(w)$ is chosen arbitrarily from the set of maximizers.

\(^7\)In (244), we use the convention that $\prod_{k \in \emptyset} a_k = 1$ for every $a_k \in \mathbb{R}$. 

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Next, we substitute (249) into (244) and obtain a lower bound on \( \sum_{t=t_{K+1}}^{\beta_t} (1 - L_t(\epsilon)) \), which holds for all sufficiently large \( \lambda \) and for all \( \epsilon \in [0, 1]^K \):

\[
\sum_{t=t_{K+1}}^{\beta_t} (1 - L_t(\epsilon)) \geq \sqrt{\frac{AV}{C^3}} (\log \lambda + \nu_\epsilon) (1 - \max_k \epsilon_k) - \sum_{t=t_{K+1}}^{\beta_t} \left[ \sum_{K \in \mathcal{H}} \left( \prod_{k \in K \setminus \{k\}} \epsilon_k \right) \left( \prod_{k \in K} F_{H_{\delta_2,k}} \left( -\frac{\lambda/C - t}{\sqrt{AV/C^3}} \right) \right) \right] - 2K - \sqrt{\frac{V}{C^3}} c_6 (\log \lambda + \nu_\epsilon)
\]

(251)

\[
= \sqrt{\frac{AV}{C^3}} (\log \lambda + \nu_\epsilon) (1 - \max_k \epsilon_k) - \sum_{K \in \mathcal{H}} \left( \prod_{k \in K \setminus \{k\}} \epsilon_k \right) \left( \sum_{t=t_{K+1}}^{\beta_t} \prod_{k \in K} F_{H_{\delta_2,k}} \left( -\frac{\lambda/C - t}{\sqrt{AV/C^3}} \right) \right) - c_7 \log \lambda
\]

(252)

\[
\geq \sqrt{\frac{AV}{C^3}} (\log \lambda + \nu_\epsilon) (1 - \max_k \epsilon_k) - \sum_{K \in \mathcal{H}} \left( \prod_{k \in K \setminus \{k\}} \epsilon_k \right) \left( \nu_\epsilon - E \left[ \min_{k \in K} \left( \nu_\epsilon, \max_{H_{\delta_2,k}} \right) \right] \right) - c_7 \log \lambda
\]

(253)

\[
\geq \sqrt{AV} g_{\delta_1,\delta_2}(\epsilon) + \sqrt{\frac{AV}{C^3}} (1 - \max_k \epsilon_k) \log \lambda - c_7 \log \lambda
\]

(254)

where

\[
g_{\delta_1,\delta_2}(\epsilon) \triangleq \sqrt{\frac{V}{C^3}} \left( \nu_\epsilon (1 - \max_k \epsilon_k) - \sum_{K \in \mathcal{H}} \left( \prod_{k \in K \setminus \{k\}} \epsilon_k \right) \left( \nu_\epsilon - E \left[ \min_{k \in K} \left( \nu_\epsilon, \max_{H_{\delta_2,k}} \right) \right] \right) \right). \tag{255}
\]

Here, (251) follows from (244); (252) follows by interchanging the order of summations. We also used that \( \nu_\epsilon \) is bounded from above, which implies that there exists a constant \( c_7 \) that does not depend on \( \epsilon \in [0, 1]^K \) such that

\[
\sqrt{\frac{V}{C^3}} c_6 (\nu_\epsilon + \log \lambda) + 2K \leq c_7 \log \lambda
\]

for all sufficiently large \( \lambda \) (recall that \(|\mathcal{H}| = 2^K - 1\)); in (253), we used (249). This establishes (139).

D. Optimization over \( f(\epsilon) \)

We observe that (131) implies

\[
\sum_{j=1}^{i} (d_j - d_0) > 0, \quad \text{for } i \in \{1, \ldots, K-1\}. \tag{256}
\]

In turn, these inequalities imply that there exist constants \( \{\zeta_i\}_{i=1}^{K-2} \) such that

\[
d_i - d_0 + \zeta_{i-1} - \zeta_i > 0, \quad \text{for } i \in \{1, \ldots, K-1\}. \tag{257}
\]

Here, we have set \( \zeta_0 \triangleq \zeta_{K-1} \triangleq 0 \) for convenience. A proof of this claim can be found in Lemma 12 in Appendix IV.

We use (257) to show that \( f(\epsilon) \) is lower-bounded by an affine function as follows:

\[
f(\epsilon) \geq \frac{1}{C} - \sum_{i=1}^{K} d_i \min_{k \in \{i, \ldots, K\}} \epsilon_k - d_0 \max_k \epsilon_k \tag{258}
\]

\[
= \frac{1}{C} - \sum_{i=1}^{K-1} (d_i - d_0) \min_{k \in \{i, \ldots, K\}} \epsilon_k - d_0 \left( \max_k \epsilon_k + \sum_{i=1}^{K-1} \min_{k \in \{i, \ldots, K\}} \epsilon_k \right) - d_{K} \epsilon_{K} \tag{259}
\]

\[
\geq \frac{1}{C} - \sum_{i=1}^{K-1} (d_i - d_0 + \zeta_{i-1} - \zeta_i) \min_{k \in \{i, \ldots, K\}} \epsilon_k - d_0 \left( \epsilon_1 + \cdots + \epsilon_{K} \right) - d_{K} \epsilon_{K} \tag{260}
\]
\[
\geq \frac{1}{C} - \sum_{i=1}^{K-1} (d_i - d_0 + \xi_i - \xi) \frac{\varepsilon_i + \cdots + \varepsilon_K}{K-i+1} - d_0 (\varepsilon_1 + \cdots + \varepsilon_K) - d_K \varepsilon_K. \tag{261}
\]

Here, (258) holds because \( \prod_{k \in \{i, \ldots, K\}} \varepsilon_k \leq \min_{k \in \{i, \ldots, K\}} \varepsilon_k \); in (260), we used that \( \min_{k \in \{i, \ldots, K\}} \varepsilon_k \leq \min_{k \in \{i+1, \ldots, K\}} \varepsilon_k \) for \( i \in \{1, \ldots, K-1\} \), which implies that
\[
\sum_{i=1}^{K-1} (\xi_{i-1} - \xi_i) \min_{k \in \{i, \ldots, K\}} \varepsilon_k \geq \sum_{i=1}^{K-1} \xi_{i-1} \min_{k \in \{i, \ldots, K\}} \varepsilon_k - \sum_{i=1}^{K-1} \xi_i \min_{k \in \{i+1, \ldots, K\}} \varepsilon_k \tag{262}
\]
\[
= \sum_{i=1}^{K-1} \xi_{i-1} \min_{k \in \{i, \ldots, K\}} \varepsilon_k - \sum_{i=1}^{K-1} \xi_i \min_{k \in \{i+1, \ldots, K\}} \varepsilon_k \tag{263}
\]
\[
= \xi_0 \varepsilon_1 - \xi_{K-1} \varepsilon_K = 0. \tag{264}
\]

Furthermore, (261) holds because \( \min_{k \in \{i, \ldots, K\}} \varepsilon_k \leq (\varepsilon_i + \cdots + \varepsilon_K)/(K-i+1) \). It follows from (257) that \( (d_i - d_0 + \xi_i - \xi) \) is positive for all \( i \in \{1, \ldots, K-1\} \). Hence, the inequality (261) holds with equality if and only if \( \varepsilon_1 = \cdots = \varepsilon_K \). This implies that a necessary and sufficient condition for the chain of inequalities (258)–(261) to hold with equality is that \( \varepsilon_1 = \cdots = \varepsilon_K \) and that \( \varepsilon_1 \in (0, 1) \). This implies that
\[
\min_{P_U, \varepsilon(x) \in [0, 1]^K: \mathbb{E}_U[f(e^{(U)})] = (1 - \varepsilon - (\log M)^{-1})f(0) + (\varepsilon + (\log M)^{-1})f(1)} \mathbb{E}_U \left[ f(e^{(U)}) \right] = (1 - \varepsilon - (\log M)^{-1})f(0) + (\varepsilon + (\log M)^{-1})f(1) \tag{265}
\]
which is equivalent to (146).

**VIII. PROOF OF THEOREM 4 (ACHIEVABILITY) AND OF THEOREM 6**

First, we prove Theorem 6. Then, we show that the achievability part of Theorem 4 follows as a special case of Theorem 6. To establish Theorem 6, we make use of the following lemma.

**Lemma 10:** Under the conditions of Theorem 6, there exists a joint probability distribution \( P_{X^n} \) on \( X^n \) such that the stopping times \( \{\tau_k(\gamma)\} \)
\[
\tau_k(\gamma) \triangleq \inf \{ n \geq 0 : i_{P_{X^n}, W_k} (X^n; Y^n_k) \geq \gamma \} \tag{266}
\]
satisfy
\[
\mathbb{E} \left[ \max_k \tau_k(\gamma) \right] \leq \frac{\gamma}{C} + \sqrt{\frac{2V}{C^3}} \mathbb{E} \left[ \max_k H_k \right] + o(\sqrt{\gamma}). \tag{267}
\]

Here, the independent RVs \( \{H_k\} \) have cumulative distribution functions given in (68) and \( P_{X^n} \) in (266) denotes the joint probability distribution of the first \( n \) entries of \( X^n \sim P_{X^n} \).

**Proof:** See Appendix III.

Define now the function
\[
g(x) \triangleq \sqrt{\frac{2V}{C^3}} \mathbb{E} \left[ \max_k H_k \right] \tag{268}
\]
and let \( P_{X^n} \) be distributed according to Lemma 10. In view of Theorem 1, let for all integers \( \ell > 0 \) and for all \( \delta > 0 \)
\[
\gamma_{\ell} \triangleq C \left( \ell - (1 + \delta)g(C\ell) \right) \tag{269}
\]
\[
q_{\ell} \triangleq \frac{\ell}{\ell - 1} \tag{270}
\]
\[
M_{\ell} \triangleq \left\lfloor \exp (\gamma_{\ell} - \log \ell) \right\rfloor. \tag{271}
\]
Then, we have (cf. (18))

\[ q_{\ell} + (1 - q_{\ell})(M_{\ell} - 1) \exp \{-\gamma_{\ell}\} \leq \frac{\ell e - 1}{\ell - 1} + \frac{\ell(1 - \epsilon)}{\ell - 1} \leq \epsilon. \tag{272} \]

Additionally, suppose that there exists an integer \( \ell_0 \geq 0 \) such that, for all \( \ell > \ell_0 \),

\[ \mathbb{E}\left[ \max_k \tau_k(\gamma_{\ell}) \right] \leq \bar{\ell} \tag{273} \]

and \( M_{\ell} \geq 2. \) Then, for \( \ell \geq \ell_0 \), we have

\[ (1 - q_{\ell})\mathbb{E}\left[ \max_k \tau_k(\gamma_{\ell}) \right] \leq \frac{\ell(1 - \epsilon)}{\ell - 1} \leq \ell_{\ell}. \tag{274} \]

By invoking Theorem 1 with \( \bar{q} = q_{\ell}, \gamma = \gamma_{\ell}, \) and \( M = M_{\ell}, \) and by using (18) along with the inequalities (272) and (274), we conclude that there exists a sequence of \( (\ell_{\ell}, M_{\ell}, \epsilon) \)-VLSF codes for all \( \ell \geq \ell_0. \) Consequently, we have that for all \( \ell \geq \ell_0, \)

\[ \log M_{\ell}(\ell_{\ell}, \epsilon) \geq \log M_{\ell} \geq C \left( \bar{\ell} - (1 + \delta)g(\bar{C}\ell) \right) - \log \bar{\ell} - 1 \tag{275} \]

\[ = \frac{C\ell}{1 - \epsilon} - (1 + \delta) \sqrt{V_{\ell}\ell} \mathbb{E}\left[ \max_k \bar{H}_k \right] + \mathcal{O}(1). \tag{276} \]

Here, in (276), we used that \( \log([x]) \geq \log(x - 1) \geq \log x - 1 \) for \( x \geq 2; \) furthermore, (277) follows because

\[ \ell_{\ell} = \frac{(\delta)^2(1 - \epsilon)}{\ell - 1} \leq \bar{\ell}(1 - \epsilon) + o(1). \tag{277} \]

Since we can choose \( \delta \) arbitrarily small, we conclude that (277) implies (66).

**Proof of (273):** By Lemma 10, there exists an integer \( \ell_0 \) such that, for all \( \ell \geq \ell_0 \), we have

\[ \mathbb{E}\left[ \max_k \tau_k(\gamma_{\ell}) \right] \leq \frac{7\ell}{C} + (1 + \delta)g(\gamma_{\ell}) \tag{278} \]

\[ = \bar{\ell} - (1 + \delta)g(C\ell) + (1 + \delta)g(C\ell - C(1 + \delta)g(C\ell)) \tag{279} \]

\[ \leq \bar{\ell}. \tag{280} \]

Here, (280) follows by the definition of \( \gamma_{\ell} \) in (269), and (281) holds because \( g(x) \) is nonnegative and nondecreasing, which implies that \( g(C\ell) - g(C\ell - C(1 + \delta)g(C\ell)) \geq 0. \)

**Proof of the achievability part of Theorem 4:** The achievability bound in Theorem 4 follows by setting \( \Psi(w) \) in Theorem 6 equal to the constant vector

\[ \Psi_{\text{const}} \triangleq - \arg \min_{w \in \mathbb{R}^{|X|}} \mathbb{E}\left[ \max_k \nabla I_k(v) + \theta_k Z_k \right] \tag{282} \]

where \( Z_k \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0, 1). \) This implies that \( \Psi'(w) = 0_{|X|}. \) Hence, \( E^k(s) = 0. \) In this case, we have that for every \( w \in \mathbb{R} \)

\[ F_{\hat{H}_k}(w) = \Phi\left( \frac{w + \nabla I_k(\Psi_{\text{const}})}{\theta_k} \right) \tag{283} \]

\[ = \mathbb{P}[\nabla I_k(\Psi_{\text{const}}) + \theta_k Z_k \leq w]. \tag{284} \]

Hence, the \( \{\hat{H}_k\} \) have the same distribution as \( \{-\nabla I_k(\Psi_{\text{const}}) + \theta_k Z_k\}. \) The achievability part of Theorem 4 is established by noting that

\[ \mathbb{E}\left[ \max_k \hat{H}_k \right] = \mathbb{E}\left[ \max_k \{-\nabla I_k(\Psi_{\text{const}}) + \theta_k Z_k\} \right] = \min_{w \in \mathbb{R}^{|X|}} \mathbb{E}\left[ \max_k \nabla I_k(v) + \theta_k Z_k \right]. \tag{285} \]
APPENDIX I
PROOF OF THEOREM 1

The proof follows closely [8, Th.3]. Let $S$ be a Bernoulli RV with $P[S = 1] = q$ and let its probability mass function be given by $P_S$. We start by specifying $U, f_n, \{g_{k,n}\}_{k \in K}, \{\tau^*_k\}_{k \in K}$. The RV $U$ has the following domain and probability mass function

$$U = \{1, 2\} \times X^\infty \times \cdots \times X^\infty$$

(286)

$$P_U = P_S \times P_{X^\infty} \times \cdots \times P_{X^\infty}. \quad \text{(287)}$$

As in [8], the realization $u$ of $U = u$ defines a codebook $\{C^{(u)}_{j,n}\}$ consisting of $M$ infinite dimensional codewords $C^{(u)}_j \in X^\infty$, $j \in M$. Differently from [8], it also defines the RV $S$, which we shall use as a source of additional common randomness among the $K$ decoders. The encoder operates as follows:

$$f_n(u, j) \triangleq C^{(u)}_{j,n}, \quad u \in U, j \in M. \quad \text{(288)}$$

Here, $C^{(u)}_{j,n}$ denotes the $n$th entry of $C^{(u)}_j$. To keep the notation compact, we shall omit the superscript $(u)$ in the remaining part of the proof. At time $n$, decoder $k$ computes the information densities

$$S_{k,n}(j) \triangleq i_{P_{X_n,W_k}}(C^n_j; Y^n_k), \quad j \in M \quad \text{(289)}$$

where the vector $C^n_j$ contains the first $n$ entries of $C_j$. Define the stopping times

$$\tau_k(j) \triangleq \inf\{n \geq 0 : S_{k,n}(j) \geq \gamma\} \quad \text{(290)}$$

and let $\tau^*_k$ be the time at which decoder $k$ makes the final decision:

$$\tau^*_k \triangleq \min_{j \in M} \tau_k(j). \quad \text{(291)}$$

The output of decoder $k$ at time $\tau^*_k$ is

$$g_{k,\tau^*_k}(U, Y_{\tau^*_k}) \triangleq \max\{j \in M : \tau_k(j) = \tau^*_k\}. \quad \text{(292)}$$

When $S = 1$, we have $\tau^*_k = 0$, and hence the decoder $g_{k,\tau^*_k}(U, Y_{\tau^*_k})$ outputs $M$. The average blocklength is then given by

$$\mathbb{E}\left[\max_k \tau^*_k\right] = (1 - q)\mathbb{E}\left[\max_k \tau^*_k | S = 0\right] \quad \text{(293)}$$

$$\leq (1 - q) \frac{1}{M} \sum_{j=1}^M \mathbb{E}\left[\max_k \tau_k(j) | J = j, S = 0\right] \quad \text{(294)}$$

$$= (1 - q)\mathbb{E}[\max \tau_k(1) | J = 1, S = 0] \quad \text{(295)}$$

$$= (1 - q)\mathbb{E}\left[\max \tau_k\right] \quad \text{(296)}$$
where the expectation is over $J, Y_1^\infty, Y_2^\infty,$ and $U.$ Here, (294) follows from (291); (295) follows from symmetry; and (296) follows from the definition of $\tau_k$ in (13). For the error probability, we have that

$$P\left[g_{k,\tau_k^*}(U, Y_k^\tau) \neq J\right] \leq q + (1 - q)P\left[g_{k,\tau_k^*}(U, Y_k^\tau) \neq J | S = 0 \right]$$

(297)

$$\leq q + (1 - q)P\left[g_{k,\tau_k^*}(U, Y_k^\tau) \neq J | J = 1, S = 0 \right]$$

(298)

$$\leq q + (1 - q)P[\tau_k(1) \geq \tau_k^* | S = 0]$$

(299)

$$= q + (1 - q)P[\bigcup_{j=2}^M \{\tau_k(1) \geq \tau_k(j)\} | S = 0]$$

(300)

$$\leq q + (1 - q)(M - 1)P[\tau_k(1) \geq \tau_k(2) | S = 0]$$

(301)

$$= q + (1 - q)(M - 1)P[\tau_k \geq \tau_k^*]$$

(302)

Here, (298) follows from (292) and (302) follows by noting that, given $J = 1,$ the RVs $(S_{k,n}(1), S_{k,n}(2), \tau_k(1), \tau_k(2))$ (where the RVs $\{S_{k,n}(j)\}$ are defined in (289)) have the same joint distribution as $(i_{P_{X^n}, W^n_k}(X^n; Y_k^n), i_{P_{X^n}, W^n_k}(X^n; Y_k^n), \tau_k, \tau_k).$ We conclude the proof by noting that, by Definition 1, the tuple $(U, f_n, \{g_{k,n}\}, \{\tau_k^*\})$ defines an $(\ell, M, \epsilon)$-VLSF code satisfying (16) and (17).

**Appendix II**

**Proof of Lemma 8**

Our objective is to show that, if the conditions in Lemma 8 are satisfied, then $\beta(w)$ given in (70) satisfies

$$P^*(x) + CP_{r(x)}^*(x)\beta'(x)(w) \in [0, 1]$$

(303)

for every $x \in \mathcal{X}$ and every $w \in \mathbb{R}.$ First, given $w \in \mathbb{R},$ we shall analyze the function

$$v(w) \triangleq \arg \max_{v \in \mathbb{R}} \prod_{k=1}^2 \Phi\left(\frac{1}{\varrho_k} (w + v\Delta_k)\right).$$

(304)

The objective function in (304) is differentiable everywhere in $v.$ Furthermore, it is strictly log-concave in $v$ because $\Phi(\cdot)$ is strictly log-concave, and it tends to 0 as $|v| \to \infty$ (recall that $\Delta_1 > 0$ and $\Delta_2 < 0$). This implies that (304) has exactly one maximum which is also the unique stationary point.

By (70), we have that $\beta(w) = \begin{bmatrix} v(w), -v(w) \end{bmatrix}^T.$ Therefore (303) is equivalent to

$$P^*(x) + (-1)^{r(x)+1} CP_{r(x)}^* v'(w) \in [0, 1]$$

(305)

for every $x \in \mathcal{X}$ and every $w \in \mathbb{R}.$ We characterize $v'(w)$ in two steps. First, we show that $v'(w) > D$ if $\Delta_1 / q_1 + \Delta_2 / q_2 > 0,$ that $v'(w) < D$ if $\Delta_1 / q_1 + \Delta_2 / q_2 < 0,$ and that $v'(w) = D$ if $\Delta_1 / q_1 + \Delta_2 / q_2 = 0.$ Next, we show that $v'(\cdot)$ is monotonic. Specifically, we demonstrate that (90) implies that $v'(w) < 0$ if $\Delta_1 / q_1 + \Delta_2 / q_2 > 0$ and $v'(w) > 0$ if $\Delta_1 / q_1 + \Delta_2 / q_2 < 0.$ By combining the two steps, we find that $\Delta_1 / q_1 + \Delta_2 / q_2 > 0$ implies $D < v'(w) < 0$ and that $\Delta_1 / q_1 + \Delta_2 / q_2 < 0$ implies $0 < v'(w) < D.$ This argument establishes (89) because $P^*(x) \in [0, 1]$ for all $x \in \mathcal{X}.$
First step: Let \( \psi(x) \triangleq \phi(x)/\Phi(x) \). We characterize the stationary point of the objective function in (304) by taking its logarithm, by differentiating it with respect to \( v \), and by equating the resulting expression to zero:

\[
\sum_{k=1}^{2} \psi' \left( \frac{w + v \Delta_k}{\theta_k} \right) \frac{\Delta_k}{\theta_k} = 0. \tag{306}
\]

One can readily verify that (306) has exactly one solution, which we denote by \( v(w) \) to emphasize its dependence on \( w \). Our objective is to characterize \( v'(w) \). Hence, we differentiate both sides of (306) with respect to \( w \) to obtain an implicit equation for \( v'(w) \):

\[
\sum_{k=1}^{2} \psi' \left( \frac{w + v \Delta_k}{\theta_k} \right) \frac{\Delta_k}{\theta_k} \left( 1 + v'(w) \Delta_k \right) = 0. \tag{307}
\]

Let \( \bar{v}(w, v) \) be the solution to the following equation in \( z \)

\[
\sum_{k=1}^{2} \psi' \left( \frac{w + v \Delta_k}{\theta_k} \right) \frac{\Delta_k}{\theta_k} \left( 1 + z \Delta_k \right) = 0. \tag{308}
\]

Note that we must have \( v'(w) = \bar{v}(w, \psi(w)) \). Solving (308) for \( z \), we obtain that

\[
\bar{v}(w, v) = -\frac{2}{\theta_k} \sum_{k=1}^{2} \frac{\Delta_k}{\theta_k} \psi' \left( \frac{w + v \Delta_k}{\theta_k} \right) \tag{309}
\]

for every \( w \in \mathbb{R} \) and every \( v \in \mathbb{R} \). Since \( \psi'(-) \in (-1, 0) \), since \( \psi'(x) \) is an increasing function in \( x \) (this result is proven in Lemma 13(a)-(b) in Appendix IV), and since \( \Delta_1 > 0 > \Delta_2 \), we conclude that \( \bar{v}(w, v) \) is an increasing function of \( v \) for fixed \( w \).

We proceed by noting that the following equation in \( \zeta \)

\[
\frac{w + \zeta w \Delta_1}{\theta_1} = \frac{w + \zeta w \Delta_2}{\theta_2} \tag{310}
\]

is solved by

\[
\zeta \triangleq \frac{\theta_1 - \theta_2}{\Delta_1 \theta_2 - \Delta_2 \theta_1}. \tag{311}
\]

For the case \( \Delta_1/\theta_1 + \Delta_2/\theta_2 = 0 \), we observe that that \( \zeta = D \) (recall that \( D \) is defined in (88)) and that \( v(w) = Dw \) solves (306). Next, consider the case \( \Delta_1/\theta_1 + \Delta_2/\theta_2 > 0 \). Define

\[
\kappa_w(a) \triangleq 2 \sum_{k=1}^{2} \psi' \left( \frac{w + a \Delta_k}{\theta_k} \right) \frac{\Delta_k}{\theta_k}. \tag{312}
\]

Note that \( \kappa_w(a) \) is a decreasing function in \( a \) because \( \psi'(-) \) is a decreasing function (proved in Lemma 13(a)) and because \( \Delta_1 > 0 > \Delta_2 \). Additionally, (306) implies that \( \kappa_w(v(w)) = 0 \). Now, we use (310), the positivity of \( \psi(-) \), and that \( \Delta_1/\theta_1 + \Delta_2/\theta_2 > 0 \), to conclude that

\[
\kappa_w(\zeta w) = 2 \sum_{k=1}^{2} \psi' \left( \frac{w + \zeta w \Delta_k}{\theta_k} \right) \frac{\Delta_k}{\theta_k} = \psi' \left( \frac{w + \zeta w \Delta_1}{\theta_1} \right) \left( \frac{\Delta_1}{\theta_1} + \frac{\Delta_2}{\theta_2} \right) > 0 = \kappa_w(v(w)). \tag{313}
\]

Since \( \kappa_w(\cdot) \) is a decreasing function, (313) implies that \( v(w) > \zeta w \). Thus, using that \( \bar{v}(w, v) \) is increasing in \( v \) for fixed \( w \), we conclude that

\[
v'(w) = \bar{v}(w, v(w)) > \bar{v}(w, \zeta w) = D. \tag{314}
\]

Following a similar line of reasoning, one can show that \( \Delta_1/\theta_1 + \Delta_2/\theta_2 < 0 \) implies \( v'(w) < D \).
**Second step:** Next, we show that $v(w)$ is monotonic. First, note that the solutions $v^*$ and $w^*$ of the optimization problem

$$
\max_{v, w} \left\{ e^{-w\alpha} \prod_k \Phi \left( \frac{1}{\theta_k} (w + v \Delta_k) \right) \right\}
$$

where $\alpha > 0$, satisfy $v^* = v(w^*)$. The objective function in (315) is differentiable everywhere, strictly log-concave in $w$ and $v$, and it tends to zero as $|v| \to \infty$ or $|w| \to \infty$. These properties imply that (315) has exactly one maximum, which is also the unique stationary point. We let $v^*(\alpha)$ and $w^*(\alpha)$ denote the solution of (315). Then, for each $\tilde{w} \in \mathbb{R}$, there exists a constant $\tilde{\alpha} > 0$ such that $v(\tilde{w}) = v^*(\tilde{\alpha})$ and $\tilde{w} = w^*(\tilde{\alpha})$. Hence, the set of points $\{(v(w), w)\}_{w \in \mathbb{R}}$ is equivalently parameterized by set of points $\{(v^*(\alpha), w^*(\alpha))\}_{\alpha > 0}$. We also point out that $w^*(\alpha)$ is nonincreasing in $\alpha$. By taking the logarithm of the objective function (315) and equating it to zero, we obtain the following stationarity conditions for $v^*(\alpha)$ and $w^*(\alpha)$

\[
\sum_k \psi \left( \frac{w^*(\alpha) + v^*(\alpha) \Delta_k}{\theta_k} \right) \frac{1}{\theta_k} = \alpha \quad (316)
\]

\[
\sum_k \psi \left( \frac{w^*(\alpha) + v^*(\alpha) \Delta_k}{\theta_k} \right) \frac{\Delta_k}{\theta_k} = 0. \quad (317)
\]

Solving (316) and (317) for $v^*(\alpha)$ and $w^*(\alpha)$, we find that

\[
v^*(\alpha) = \frac{1}{\Delta_1 - \Delta_2} \left( \frac{\alpha q_1}{1 - \Delta_1/\Delta_2} \right) - \frac{\alpha q_2}{1 - \Delta_2/\Delta_1}
\]

\[
w^*(\alpha) = \frac{\Delta_1 \Delta_2}{\Delta_2 - \Delta_1} \left( \frac{\alpha q_1}{1 - \Delta_1/\Delta_2} \right) - \frac{\alpha q_2}{1 - \Delta_2/\Delta_1}. \quad (318)
\]

Note that $\Delta_1 - \Delta_2 > 0$ due to the assumption $\Delta_1 > 0 > \Delta_2$. Since $w^*(\alpha)$ is a nonincreasing function of $\alpha$, our objective is to show that $v^*(\alpha)$ is either nonincreasing or nondecreasing in $\alpha$. In particular, if $v^*(\alpha)$ is nonincreasing then $v(w)$ must be nondecreasing and vice versa. Let $g(x) \triangleq -\psi'(\psi^{-1}(x))$ for $x > 0$. By taking the derivative of (318) with respect to $\alpha$, we obtain

\[
\frac{\partial v^*}{\partial \alpha} = \frac{1}{\Delta_1 - \Delta_2} \left( \frac{\alpha q_1}{g \left( \frac{\alpha q_1}{1 - \Delta_1/\Delta_2} \right)} - \frac{\alpha q_2}{g \left( \frac{\alpha q_2}{1 - \Delta_2/\Delta_1} \right)} \right). \quad (320)
\]

Note that $g(x)$ is a positive function, which satisfies the following property: $\beta g(x) \geq g(\beta x)$ for $\beta \geq 1$ and $x > 0$ (this is proved in Lemma 13(c) in Appendix IV). This implies that, when $\beta > 1$,

\[
\frac{g(\beta x)}{g(x)} < \beta. \quad (321)
\]

Similarly, when $0 < \beta < 1$, one readily finds that (321) implies

\[
\frac{g(\beta x)}{g(x)} > \frac{1}{\beta}. \quad (322)
\]

To show monotonicity of $v^*(\alpha)$, we analyze the sign of (320). Let

\[
h(\alpha) \triangleq \frac{q_2^2 (1 - \Delta_1/\Delta_2) g \left( \frac{\alpha q_1}{1 - \Delta_1/\Delta_2} \right)}{q_2^2 (1 - \Delta_2/\Delta_1) g \left( \frac{\alpha q_2}{1 - \Delta_2/\Delta_1} \right)} - \frac{\Delta_1 g \left( \frac{\alpha q_1}{1 - \Delta_1/\Delta_2} \right)}{\Delta_2 g \left( \frac{\alpha q_2}{1 - \Delta_2/\Delta_1} \right)}. \quad (323)
\]

Note that $h(\alpha) > 1$ implies $\partial v^*/\partial \alpha > 0$. Furthermore, $h(\alpha) < 1$ implies $\partial v^*/\partial \alpha < 0$. 



Consider the case $\Delta_1/g_1 + \Delta_2/g_2 > 0$. By (90), we must have that $g_2 \geq g_1$. But this implies that
\begin{equation}
 h(\alpha) > \frac{g_2}{g_1} \left( \frac{\alpha g_1}{1 - \Delta_1/\Delta_2} \right) \frac{\alpha g_1}{\Delta_2} \tag{325}
\end{equation}
\begin{equation}
 \geq -\frac{g_2 g_2 \Delta_1}{g_1 g_1 \Delta_2} \tag{326}
\end{equation}
\begin{equation}
 \geq \frac{g_2}{g_1} \tag{327}
\end{equation}
\begin{equation}
 \geq 1. \tag{328}
\end{equation}
Here, (325) follows from $\Delta_1 g_2/(\Delta_2 g_1) < -1$; (326) follows from (322) with $x = \alpha g_2/(1 - \Delta_2/\Delta_1) > 0$ and $\beta = -g_1 \Delta_2/(g_2 \Delta_1) \in (0, 1)$, (327) holds because $\Delta_1 g_2/(\Delta_2 g_1) < -1$, and (328) holds because $g_2/g_1 \geq 1$.

Using a similar line of reasoning, we obtain for the case $\Delta_1/g_1 + \Delta_2/g_2 < 0$ that
\begin{equation}
 h(\alpha) < 1. \tag{329}
\end{equation}
Using that $h(\alpha) > 1$ implies $\partial v^*/\partial \alpha > 0$ and that $h(\alpha) < 1$ implies $\partial v^*/\partial \alpha < 0$, we conclude that $D < v'(w) < 0$ if $\Delta_1/g_1 + \Delta_2/g_2 > 0$ and that $D > v'(w) > 0$ if $\Delta_1/g_1 + \Delta_2/g_2 < 0$.

**APPENDIX III**

**PROOF OF LEMMA 10**

To prove Lemma 10, we shall first construct a suitable nonstationary joint probability distribution $P_X$ on $\mathcal{X}^\infty$. Then we shall set
\begin{equation}
 \beta_- = \frac{\gamma}{C} - \sqrt{\frac{\gamma V}{C^3} \log \gamma} \quad \text{and} \quad \beta_+ = \frac{\gamma}{C} + \sqrt{\frac{\gamma V}{C^3} \log \gamma} \tag{330}
\end{equation}
and compute an asymptotic upper bound on
\begin{equation}
 \mathbb{E} \left[ \max_k \tau_k \right] = \sum_{t=1}^{\infty} \left( 1 - \mathbb{P} \left[ \max_k \tau_k \leq t \right] \right) \tag{331}
\end{equation}
\begin{equation}
 \leq \beta_- + \sum_{t=\beta_-+1}^{\beta_+} \left( 1 - \mathbb{P} \left[ \max_k \tau_k \leq t \right] \right) + \sum_{t=\beta_++1}^{\infty} \left( 1 - \mathbb{P} \left[ \max_k \tau_k \leq t \right] \right) \tag{332}
\end{equation}
that matches (267) in the limit $\gamma \rightarrow \infty$. This is done by obtaining lower bounds on $\mathbb{P} \left[ \max_k \tau_k \leq t \right]$ for $t \geq \beta_- + 1$.

Similarly as in Section VII, it turns out convenient to treat the two subintervals $[\beta_- + 1, \beta_+]$ and $[\beta_+ + 1, \infty)$ differently. In the former subinterval, our main tool is a multivariate version of the Berry-Esseen central limit theorem for sums of independent RVs. In the latter subinterval, we apply Hoeffding’s inequality. To compute the desired lower bound on $\mathbb{P} \left[ \max_k \tau_k \leq t \right]$, we shall use the following relation between $\tau_k$ and $i_{P^*_{X^t},W_k}(X^t;Y_k^t)$, which follows from the definition of $\tau_k$ in (266):
\begin{equation}
 \mathbb{P} \left[ \max_k \tau_k \leq t \right] \geq \mathbb{P} \left[ \min_k i_{P^*_{X^t},W_k}(X^t;Y_k^t) \geq \gamma \right]. \tag{333}
\end{equation}

We next summarize the key steps of the proof. Details are provided in Appendix III-A–III-D.
Step I: First, we specify the probability distribution $P_{X^n}$ on $X^n$ for which (267) holds. Let
\[
w(t) \triangleq \frac{t - \gamma/C}{\sqrt{\gamma V/C^3}} \tag{334}\]
\[
P^{(1)}(\gamma) \triangleq P^* + \sqrt{VC/\gamma V(w(\beta_-))} \tag{335}\]
\[
P^{(2)}(w) \triangleq P^* + CV'(w) \tag{336}\]
\[
P^{(3)}(\gamma) \triangleq P^* + \sqrt{VC/\gamma V(w(\beta_+))} \tag{337}\]
and let $P^{(1)}(\gamma) \in \mathcal{P}_{\beta_-}(X)$ be the type that minimizes $\|P^{(1)}(\gamma) - P^{(1)}(\gamma)\|$ (recall that $\mathcal{P}_{\beta_-}(X)$ denotes the set of types of $n$-dimensional sequences and that $\|\cdot\|$ denotes the Euclidean distance). For all sufficiently large $\gamma$, $P^{(1)}(\gamma)$ and $P^{(3)}(\gamma)$ are legitimate probability distributions, and (62) implies that $P^{(2)}(w)$ is a valid probability distribution as well. The probability distribution $P_{X^n}$ is specified as follows. We let the distribution $P_{X^n}$ of $X^n$ be uniform over the set of all codewords of type $P^{(1)}(\gamma)$. For $t \in [\beta_- + 1, \beta_+]$, the RVs $\{X_t\}$ are generated independently according to $P^{(2)}(w(t))$ and, for $t \geq \beta_+ + 1$, the RVs $\{X_t\}$ are generated independently according to $P^{(3)}(\gamma)$. For notational convenience, the dependency of $P^{(1)}(\gamma)$ and $P^{(3)}(\gamma)$ on $\gamma$ is omitted in the remainder of the proof.

We need $X^{\beta_-}$ to be of constant composition because the capacity-achieving input distributions of the components channels $\{W_k\}$ are generally not given by $P^*$. The above choice of the distribution of $X^{\beta_-}$ thereby parallels the achievability proof of the asymptotic expansion of the maximum coding rate for compound DMCs for the fixed blocklength case [3].

We note that $\|P^{(1)} - P^{(1)}\|_1 \leq O(1/\gamma)$ as $\gamma \to \infty$. Furthermore, differentiability of $I_k(\cdot)$ implies that
\[
I_k(P^{(1)}) = I_k(P^{(1)}) + O\left(\frac{1}{\gamma}\right). \tag{338}\]
Additionally, since $X^{\beta_-}$ are of constant composition, we cannot write $i_{P^{(1)}_{X^{\beta_-}}, W^{\beta_-}}(X^{\beta_-};Y^{\beta_-})$ as a sum of independent RVs since $P_{Y^{\beta_-}}$ is not a product distribution. Hence, to dispose of the dependency among the RVs $X^{\beta_-}$, we use both the inequality [22, Eq. 4.49]
\[
P^{(1)}_{Y^{\beta_-}}(y) = P_{X^{\beta_-}, W^{\beta_-}}(X^{\beta_-};Y^{\beta_-}) \leq |\mathcal{P}_{\beta_-}(X)|(P^{(1)}_{W_k})^{\beta_-}(y) \tag{339}\]
which holds for all $y \in \mathcal{Y}^{\beta_-}$, and the inequality $|\mathcal{P}_{n}(X)| \leq (n + 1)^{|X|}$ [14, Th. 11.1.1] to conclude that, for all $t \geq \beta_-,$
\[
i_{P^{(1)}_{X^n}, W^n_k}(x_t; y'_k) = \frac{\log W_k^{\beta_-}(y_k^{\beta_-} | x_k^{\beta_-})}{P^{(1)}_{y_k^{\beta_-} | x_k^{\beta_-}}} + \sum_{n=\beta_- + 1}^{t} i_{P_{X^n}, W_k}(x_n; y_k) \geq \frac{\log \prod_{n=1}^{t} W_k(y_k | x_k)\log(\beta_- + 1) + \sum_{n=\beta_- + 1}^{t} i_{P_{X^n}, W_k}(x_n; y_k)}{|\mathcal{P}_{\beta_-}(X)|(P^{(1)}_{W_k})^{\beta_-}(y_k^{\beta_-})} \geq \sum_{n=1}^{t} i_{P_{X^n}, W_k}(x_n; y_k) - |X| \log(\beta_- + 1). \tag{340}\]
It follows from (333) and (342) that
\[
\mathbb{P}\left[\max_k \tau_k \leq t\right] \geq \mathbb{P}\left[\min_k \left(\sum_{n=1}^{t} i_{P_{X^n}, W_k}(X_n; Y_k) - |X| \log(\beta_- + 1)\right) \geq \gamma\right]. \tag{343}\]
We also note that the marginal probability distribution of $X_t$, for $t \leq \beta_-$, is given by $P^{(1)}$. 

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**Step 2:** We shall next estimate the expected value of \( \sum_{i=1}^{t} I_k(P_{X_i}) \). This is needed to lower-bound \( \min_{i} iP_{X_i}, w_{i}(X_i, Y_i) \geq \gamma \) using Hoeffding’s inequality for \( t \leq \beta_{-} \) and a multivariate version of the Berry-Esseen central limit theorem for the case \( t \geq \beta_{-} + 1 \). We first treat the case \( t \in [\beta_{-} + 1, \beta_{+}] \). We will return to the case \( t \in [\beta_{+} + 1, \infty) \) shortly. For \( t \in [\beta_{-} + 1, \beta_{+}] \), we have that

\[
\sum_{i=1}^{t} I_k(P_{X_i}) = \sum_{i=\beta_{-} + 1}^{t} I_k(P^{(2)}(w(i))) + \beta_{-} I_k(P^{(1)})
\]

(344)

\[
= \sum_{i=\beta_{-} + 1}^{t} \left[ C - E_k(w(i)) + C\nabla I_k(\nabla'(w(i))) \right] + \beta_{-} \left( C + \sqrt{\frac{VC}{\gamma}} \nabla I_k(\nabla(w(\beta_{-}))) \right)
\]

(345)

Here, in (344) we used that the marginal distribution of \( P_{X_i} \) for \( i \in [1, \beta_{-}] \) is given by \( P^{(1)} \). To obtain (345), we used (338) and that \( \beta_{-} = O(\gamma) \). Furthermore, we performed a Taylor-expansion of \( I_k(P^{(1)}) \) around \( P^{*} \), and used that \( E_k(s) = C - I_k(P^{(2)}(s)) + C\nabla I_k(\nabla'(s)) \) (recall the definition of \( E_k(s) \) in (63) and of \( P^{(2)} \) in (336)). We note that \( C\nabla I_k(\nabla'(s)) = \nabla I_k(C\nabla'(s)) \) because \( \nabla I_k(\cdot) \) is linear. To simplify (345), we shall apply the following two asymptotic expansions, which are proven in Appendix III-A and Appendix III-B, respectively:

\[
C\nabla I_k(\nabla'(w(t))) = t \int \frac{VC}{\gamma} \nabla I_k(\nabla(w(t))) - (t - 1) \int \frac{VC}{\gamma} \nabla I_k(\nabla(w(t - 1))) + O\left( \frac{\log \gamma}{\sqrt{\gamma}} \right)
\]

(346)

and

\[
\sum_{i=\beta_{-} + 1}^{t} E_k(w(i)) = \sqrt{\frac{V}{C^3}} \int_{w(\beta_{-})}^{w(t)} E_k(s) ds + O(\log \gamma)
\]

(347)

as \( \gamma \rightarrow \infty \) for all \( t \in [\beta_{-} + 1, \beta_{+}] \). By substituting (346) and (347) into (345), we obtain

\[
\sum_{i=1}^{t} I_k(P_{X_i}) = \sum_{i=\beta_{-} + 1}^{t} \left[ C + i \int \frac{VC}{\gamma} \nabla I_k(\nabla(w(i))) - (i - 1) \int \frac{VC}{\gamma} \nabla I_k(\nabla(w(i - 1))) \right]
\]

\[
+ \beta_{-} \left( C + \sqrt{\frac{VC}{\gamma}} \nabla I_k(\nabla(w(\beta_{-}))) \right) - \sqrt{\frac{V}{C^3}} \int_{w(\beta_{-})}^{w(t)} E_k(s) ds
\]

\[
+ O(\log \gamma) + (t - \beta_{-}) O\left( \frac{\log \gamma}{\sqrt{\gamma}} \right)
\]

(348)

\[
= t \left( C + \sqrt{\frac{VC}{\gamma}} \nabla I_k(\nabla(w(t))) \right) - \sqrt{\frac{V}{C^3}} \int_{w(\beta_{-})}^{w(t)} E_k(s) ds + O(\log^2 \gamma).
\]

(349)

Here, (349) follows because \( (t - \beta_{-}) \leq (\beta_{+} - \beta_{-}) = O(\sqrt{\gamma} \log \gamma) \).

We now move to the case \( t \in [\beta_{+} + 1, \infty) \), for which, proceeding as in (344)–(349), we obtain that

\[
\sum_{i=1}^{t} I_k(P_{X_i}) = \beta_{+} \left( C + \sqrt{\frac{VC}{\gamma}} \nabla I_k(\nabla(w(\beta_{+}))) \right) - \sqrt{\frac{V}{C^3}} \int_{w(\beta_{-})}^{w(\beta_{+})} E_k(s) ds
\]

\[
+ \sum_{i=\beta_{+} + 1}^{t} I_k(P^{(3)}) + O(\log^2 \gamma)
\]

(350)

\[
= t \left( C + O\left( \frac{\log \gamma}{\sqrt{\gamma}} \right) \right).
\]

(351)

Here, (350) follows from (349) and (351) follows because \( \int_{-\infty}^{\infty} E_k(s) ds < \infty \) (see (64)) and because \( I_k(P^{(3)}) = C + O(1/\sqrt{\gamma}) \). We have also used that \( \nabla I_k(\nabla'(w(\beta_{+}))) = O(\log \gamma) \), which follows because \( \nabla' \) is bounded by (62) and from \( w(\beta_{+}) = O(\log \gamma) \).
Step 3: We now use (349) and (351) to compute the second and the third term in (332). By applying Hoeffding’s inequality and then using (351), we demonstrate in Appendix III-C that

\[ \sum_{t=\beta_+ + 1}^{\beta_- - 1} \left( 1 - P \left[ \max_k \tau_k \leq t \right] \right) = o(1) \]  

(352)
as \gamma \to \infty$. Hence, third term in (332) vanishes as \( \gamma \to \infty \).

Next, we analyze the second term in (332), which require bounds on \( P \left[ \min_k i_{P_{X^k}, W_k^t}(X^k, Y^t_k) \geq \gamma \right] \) for \( t \in [\beta_- + 1, \beta_+] \). Let \( \delta \) be an arbitrary positive constant. In Appendix III-D, we show using (349) and a multivariate version of the Berry-Esseen central limit theorem for sums of independent RVs [22, Th. 1.8], [23, Th 1.3] that

\[ \mathbb{P} \left[ \min_k i_{P_{X^k}, W_k}(X^k, Y^t_k) \geq \gamma \right] \geq \prod_k F_{H_{\delta,k}}(w(t)) + O \left( \frac{\log^2 \gamma}{\sqrt{\gamma}} \right) \]  

(353)
as \gamma \to \infty$. Here, the \( O(\cdot) \) term is uniform in \( t \in [\beta_- + 1, \beta_+] \). The RVs \( \{H_{\delta,k}\} \) have the cumulative distribution function

\[ F_{H_{\delta,k}}(w) \triangleq \Phi \left( \frac{1}{\delta_k} \min_{w_s \in (-1,1)} \frac{w + \nabla I_k(\Psi(w)) - \frac{1}{\beta} \int_{-\infty}^{w} E_k(s)ds}{1 - \partial v_k} \right). \]  

(354)

Thus, we have that

\[ \sum_{t=\beta_-}^{\beta_+} \mathbb{P} \left[ \max_k \tau_k \leq t \right] \geq \sum_{t=\beta_- + 1}^{\beta_+} \prod_k F_{H_{\delta,k}} \left( -\frac{\gamma/C - t}{\sqrt{\gamma/V/C^d}} \right) + O(\log^3 \gamma) \]  

(355)

(356)

(357)

(358)

(359)

Here, (355) follows from (334), from (353), and because \( (\beta_+ - \beta_-) = O(\sqrt{\log \gamma}) \); (356) follows because \( F_{H_{\delta,k}}(w) \) is a nondecreasing function in \( w \) upper-bounded by one; and (359) follows because, for a continuous RV \( X \) with probability density function \( p_X(x) \) and \( b \geq a \), we have that

\[ \int_a^b \mathbb{P}[X \leq x] \, dx = \int_a^b \int_{-\infty}^x p_X(s) \, ds \, dx \]  

(360)

(361)

(362)

(363)

(364)
In (362), we used Tonelli’s theorem [24, Th. 15.8]. Now, since $\delta$ can be chosen arbitrarily small, we obtain from (359) the asymptotic bound

$$
\sum_{t=\beta_-+1}^{\beta_+} \mathbb{P}\left[\max_k \tau_k \leq t\right] \geq \mathbb{E}\left[\min\left\{\beta_+ - \beta_- - \max_k \left(\frac{\gamma}{C} + \sqrt{\frac{\gamma V}{C^3}} \bar{H}_k\right)\right\}\right] + o(\sqrt{\gamma})
$$

(365)

$$
= \beta_+ - \mathbb{E}\left[\max_k \left(\frac{\gamma}{C} + \sqrt{\frac{\gamma V}{C^3}} \bar{H}_k\right)\right] + o(\sqrt{\gamma})
$$

(366)

$$
= \beta_+ - \frac{\gamma}{C} - \sqrt{\frac{\gamma V}{C^3}} \mathbb{E}\left[\max_k \bar{H}_k\right] + o(\sqrt{\gamma})
$$

(367)

$$
= \beta_+ - \frac{\gamma}{C} - \sqrt{\frac{\gamma V}{C^3}} \max_k \bar{H}_k + o(\sqrt{\gamma}).
$$

(368)

Recall that the $\{\bar{H}_k\}$ have cumulative distribution function given in (68).

Finally, substituting (352) and (368) in (332), we obtain the desired result (267).

A. Proof of (346)

We prove (346), for $t \in [\beta_- + 1, \beta_+]$, through the following chain of equalities

$$
t \sqrt{\frac{VC}{\gamma}} \nabla I_k(\bar{\psi}(w(t))) - (t - 1) \sqrt{\frac{VC}{\gamma}} \nabla I_k(\bar{\psi}(w(t - 1)))
$$

\begin{align*}
= & \nabla I_k \left( \left(\frac{\gamma}{C} + \sqrt{\frac{\gamma V}{C^3}} w(t)\right) \nabla \bar{\psi}(w(t)) - \left(\frac{\gamma}{C} + \sqrt{\frac{\gamma V}{C^3}} w(t - 1)\right) \sqrt{\frac{VC}{\gamma}} \bar{\psi}'(w(t) - \sqrt{\frac{C^3}{\gamma V}}) \right) \\
= & \nabla I_k \left( \left(\frac{\gamma}{C} + \sqrt{\frac{\gamma V}{C^3}} w(t)\right) \nabla \bar{\psi}(w(t)) \\
& \quad - \left(\frac{\gamma}{C} + \sqrt{\frac{\gamma V}{C^3}} w(t - 1)\right) \sqrt{\frac{VC}{\gamma}} \left(\bar{\psi}(w(t)) - \sqrt{\frac{C^3}{\gamma V}} \bar{\psi}'(w(t))\right) \right) + o\left(\frac{1}{\sqrt{\gamma}}\right) \\
= & \nabla I_k \left( \sqrt{\frac{VC}{\gamma}} \bar{\psi}(w(t)) + \left(\frac{\gamma}{C} + \sqrt{\frac{\gamma V}{C^3}} w(t) - 1\right) \sqrt{\frac{VC}{\gamma}} \bar{\psi}'(w(t)) \right) + o\left(\frac{1}{\sqrt{\gamma}}\right) \\
= & \sqrt{\frac{VC}{\gamma}} \nabla I_k(\bar{\psi}(w(t))) + \left(\sqrt{\frac{VC}{\gamma}} w(t) - \frac{C^2}{\gamma}\right) \nabla I_k(\bar{\psi}'(w(t))) + C \nabla I_k(\bar{\psi}'(w(t))) + o\left(\frac{1}{\sqrt{\gamma}}\right) \\
= & C \nabla I_k(\bar{\psi}'(w(t))) + o\left(\frac{\log \gamma}{\sqrt{\gamma}}\right)
\end{align*}

(369)

(370)

(371)

(372)

(373)

as $\gamma \to \infty$. Here, (369) follows because $\gamma/C + \sqrt{\gamma V/C^3} w(t) = t$ (see (334)); moreover, (370) follows from a first-order Taylor expansion of $\bar{\psi}(\cdot)$ around $w(t)$, and (373) follows because $\bar{\psi}'(\cdot)$ is bounded and because $t \in [\beta_- + 1, \beta_+]$ implies that $w(t) = O(\log \gamma)$.

B. Proof of (347)

For $t \in [\beta_- + 1, \beta_+]$, we obtain (347) through the following steps:

$$
\int_{w(\beta_-)}^{w(t)} E_k(s) ds = \sum_{i=\beta_-+1}^{t} \int_{w(i-1)}^{w(i)} E_k(s) ds
$$

(374)

$$
= \sum_{i=\beta_-+1}^{t} [w(i) - w(i - 1)] E_k(s_i)
$$

(375)
\[ \sum_{i=\beta_{-}+1}^{t} [w(i) - w(i - 1)] (E_k(w(i)) - [w(i) - s_i]E'_k(s'_i)) \] (376)

\[ \sum_{i=\beta_{-}+1}^{t} [w(i) - w(i - 1)]E_k(w(i)) - \sum_{i=\beta_{-}+1}^{t} (w(i) - w(i - 1))(w(i) - s_i)E'_k(s'_i) \] (377)

\[ \sqrt{\frac{C^3}{\gamma V}} \sum_{i=\beta_{-}+1}^{t} E_k(w(i)) + \mathcal{O}\left(\frac{1}{\gamma}\right) \sum_{i=\beta_{-}+1}^{t} E'_k(s'_i) \] (378)

\[ \sqrt{\frac{C^3}{\gamma V}} \sum_{i=\beta_{-}+1}^{t} E_k(w(i)) + \mathcal{O}\left(\frac{\log \gamma}{\sqrt{\gamma}}\right) \] (379)

as \( \gamma \to \infty \). Here, (375) follows from the mean value theorem for definite integrals [24, Th. 7.30] which implies that there exist constants \( s_i \in (w(i-1), w(i)) \) satisfying (375); the equality (376) follows from the mean value theorem [24, Th. 5.11], which guarantees the existence of constants \( s'_i \in (s_i, w(i)) \) such that (376) is satisfied; (378) follows because \( w(i) - w(i - 1) = \sqrt{C^3/(\gamma V)} \); and (379) holds because \( t \in [\beta_{-} + 1, \beta_{+}] \), which implies that \( (t - \beta_{-}) = \mathcal{O}(\sqrt{\gamma} \log \gamma) \), and because \( \{E'_k(w)\} \) are bounded (see (65)).

C. Proof of (352)

We shall first apply Hoeffding’s inequality to obtain an upper bound on \( 1 - \mathbb{P}\left[ \max_k \tau_k \leq t \right] \) that holds for \( t \in [\beta_{+} + 1, \infty) \). To obtain (352), we then sum this upper bound over all integers larger than \( \beta_{+} \).

Observe that (351) implies that there exists a constant \( c_1 > 0 \) such that, for all sufficiently large \( \gamma \) and for all \( k \in \mathcal{K} \), we have

\[ \sum_{i=1}^{t} I_k(P_{X_i}) - |\mathcal{X}| \log(\beta_{-} + 1) \geq t \left( C - \frac{c_1 \log \gamma}{\sqrt{\gamma}} \right). \] (380)

Choose an arbitrary \( \tilde{x} \in \mathcal{X}^{\beta_{-}} \) of type \( P^{(1)} \). We proceed as follows:

\[ 1 - \mathbb{P}\left[ \max_k \tau_k \leq t \right] \leq 1 - \mathbb{P}\left[ \min_k \left\{ \sum_{n=1}^{t} i_{P_{X_n},W_k}(X_n;Y_{k,n}) - |\mathcal{X}| \log(\beta_{-} + 1) \right\} \geq \gamma \right] \] (381)

\[ \leq \sum_k \mathbb{P}\left[ \sum_{n=1}^{t} i_{P_{X_n},W_k}(X_n;Y_{k,n}) - |\mathcal{X}| \log(\beta_{-} + 1) \leq \gamma \right] \] (382)

\[ \leq \sum_k \exp\left( -e^\left\{ \sum_{n=1}^{t} i_{P_{X_n},W_k}(X_n;Y_{k,n}) - |\mathcal{X}| \log(\beta_{-} + 1) \right\} \right) \] (383)

\[ \leq \sum_k \exp\left( -e^\left\{ \frac{t(C - c_1 \log(\gamma)/\sqrt{\gamma})}{\sqrt{\gamma}} \right\} \right). \] (384)

Here, (381) follows from (343); (382) follows from the union bound; (383) follows because the distribution of \( \sum_{n=1}^{t} i_{P_{X_n},W_k}(X_n;Y_{k,n}) \) depends only on \( X^{\beta_{-}} \) through its type, since \( X^{\beta_{-}} \) is of constant composition; (384) follows from Hoeffding’s inequality [19, Th. 2], and because \( \{i_{P_{X_n},W_k}(X_n;Y_{k,n})\}_{n=1}^{t} \) are conditionally independent.
given \(X^2\); and (385), for sufficiently large \(\gamma\), holds because \(tC > \gamma\) and because of (380). Consequently, we have
\[
\sum_{t=\beta_+ + 1}^{\infty} \left(1 - P\left[\max_k \tau_k \leq t\right]\right)
\leq K \sum_{t=\beta_+}^{\infty} \exp\left(-e^C \left(\frac{(C - c_1 \log(\gamma) / \sqrt{\gamma}) - \gamma}{\sqrt{\gamma}}\right)^2\right)
\leq K \sum_{t=\beta_+}^{\infty} \sum_{i=1}^{\infty} \exp\left(-e^C \left(\frac{(C - c_1 \log(\gamma) / \sqrt{\gamma}) - \gamma}{\sqrt{\gamma}}\right)^2\right)
\leq e^C \sqrt{\gamma} \log(\gamma) \sum_{i=1}^{\infty} \exp\left(-e^C (i \log \gamma - e)^2\right)
\leq e^C \sqrt{\gamma} \log(\gamma) \sum_{i=1}^{\infty} \exp\left(-e^C (i \log \gamma)^2\right)
= e^C \sqrt{\gamma} \log(\gamma) \sum_{i=1}^{\infty} \exp(-e^C \log^2 \gamma)^i
\leq e^C \sqrt{\gamma} \log(\gamma) \sum_{i=1}^{\infty} \exp(-e^C \log^2 \gamma)^i
= e^C \sqrt{\gamma} \log(\gamma) \frac{\exp(-e^C \log^2 \gamma)}{1 - \exp(-e^C \log^2 \gamma)} = o(1)
\] as \(\gamma \to \infty\). Here, (386) follows by (385), (388) follows because \(\exp\left(-e^C (C - c_1 \log(\gamma) / \sqrt{\gamma}) - \gamma / \sqrt{\gamma}\right)^2\) decreases in \(t\) for sufficiently large \(\gamma\), and both (389) and (390) hold for sufficiently large \(\gamma\).

D. Proof of (353)

We shall apply a multivariate version of the Berry-Esseen central limit theorem for sums of independent RVs to 
\[
\sum_{t=1}^{n} \sum_{k=1}^{\beta_+} \exp\left(-e^C \left(\frac{(C - c_1 \log(\gamma) / \sqrt{\gamma}) - \gamma}{\sqrt{\gamma}}\right)^2\right)
\leq e^C \sqrt{\gamma} \log(\gamma) \sum_{i=1}^{\infty} \exp\left(-e^C (i \log \gamma - e)^2\right)
\leq e^C \sqrt{\gamma} \log(\gamma) \sum_{i=1}^{\infty} \exp\left(-e^C (i \log \gamma)^2\right)
= e^C \sqrt{\gamma} \log(\gamma) \sum_{i=1}^{\infty} \exp(-e^C \log^2 \gamma)^i
\leq e^C \sqrt{\gamma} \log(\gamma) \sum_{i=1}^{\infty} \exp(-e^C \log^2 \gamma)^i
= e^C \sqrt{\gamma} \log(\gamma) \frac{\exp(-e^C \log^2 \gamma)}{1 - \exp(-e^C \log^2 \gamma)} = o(1)
\]

Although \(V_t^1\) depends on \(\gamma\), we omit denoting this explicitly to make the notation more convenient. Then, for \(t \in [\beta_- + 1, \beta_+]\), we have that
\[
\begin{align*}
\text{Var} \left[ \sum_{n=1}^{t} i_{P_n, W_k}(X_n; Y_{k,n}) \right] &= \sum_{n=\beta_- + 1}^{t} U_k(P_{X_n}) \\
&= E \left[ \text{Var} \left[ \sum_{n=1}^{t} i_{P_n, W_k}(X_n; Y_{k,n}) \right] \right] + \text{Var} \left[ E \left[ \sum_{n=1}^{t} i_{P_n, W_k}(X_n; Y_{k,n}) \right] \right] \\
&= \sum_{n=1}^{\beta_-} E \left[ \text{Var} \left[ i_{P_n, W_k}(X_n; Y_{k,n}) \right] \right] + \sum_{n=1}^{\beta_+} E \left[ \text{Var} \left[ i_{P_n, W_k}(X_n; Y_{k,n}) \right] \right] \\
&= \beta_- V_k(P^{(1)}).
\end{align*}
\]
Here, (396) follows from the law of total variance and from (394), (397) follows because \( \{i_{P(1)}, W_{k}(X; Y, n)\}_{n=1}^{\infty} \) are conditionally independent given \( X^{\beta_{-}} \), and (398) follows since the marginal distribution of \( X_{n} \), for \( n \in [1, \beta_{-}] \), is given by \( P^{(1)} \) and since \( X^{\beta_{-}} \) is of constant composition.

Recall that \( \tilde{x} \in X^{\beta_{-}} \) has the type \( P^{(1)} \). By using (342) and by invoking the multivariate version of the Berry-Esseen central limit theorem for sums of independent RVs reported in [22, Th. 1.8] and [23, Th 1.3], we obtain, for \( t \in [\beta_{-} + 1, \beta_{+}] \), the estimate

\[
\Pr \left[ \min_{k} \{\sum_{n=1}^{t} i_{P_{n}}(X; W_{k}(X; Y, n))\} \geq \gamma + \|X\| \log(\beta_{-} + 1) \right] \geq \Pr \left[ \min_{k} \left\{ \sum_{n=1}^{t} i_{P_{n}}(X; Y, n) \right\} \geq \gamma + \|X\| \log(\beta_{-} + 1) \right]
\]

(399)

\[
= \Pr \left[ \min_{k} \left\{ \sum_{n=1}^{t} i_{P_{n}}(X; Y, n) \right\} \geq \gamma + \|X\| \log(\beta_{-} + 1) \right] = \gamma \\sqrt{\frac{\gamma}{\gamma_{-}}}
\]

(400)

\[
\geq \prod_{k} Q \left( \gamma + \|X\| \log(\beta_{-} + 1) - \sum_{n=1}^{t} I_{k}(P_{X_{n}}) \right) + \frac{\gamma}{\sqrt{\gamma_{-}}}
\]

(401)

\[
= \prod_{k} Q \left( \gamma - \sum_{n=1}^{t} I_{k}(P_{X_{n}}) \right) + O \left( \log \frac{\gamma}{\sqrt{\gamma_{-}}} \right).
\]

(402)

Here, (400) follows because the distribution of \( \sum_{n=1}^{t} i_{P_{n}}(X; W_{k}(X; Y, n)) \) depends only on \( X^{\beta_{-}} \) through its type since \( X^{\beta_{-}} \) is of constant composition and (401) follows, in addition to the central limit theorem, from (395), from (398), because \( \{i_{P_{n}}, W_{k}(X; Y, n)\}_{n \in [1, \beta_{-}], k \in K} \) are conditional independent given \( X^{\beta_{-}} \) and because the \( \{T_{k}()\} \) are uniformly upper-bounded [4, Lem. 46]. Furthermore, we obtained (402) by performing a first-order Taylor expansion of the \( Q \) function around \( \left( \gamma - \sum_{n=1}^{t} I_{k}(P_{X_{n}}) \right)/\sqrt{V_{k}^{(1)}} \).

Next, we approximate \( V_{k}^{(1)} \) in (402) by \( V_{k} \) defined in (7) in a sense we shall make precise shortly. Recall that \( \delta \) is an arbitrarily positive constant. Then, for sufficiently large \( \gamma \), we have

\[
\sqrt{\frac{\gamma}{\gamma_{-}}} \leq \sqrt{\frac{\gamma}{\gamma_{-}}}
\]

(403)

for every \( t \in [\beta_{-} + 1, \beta_{+}] \) (recall that \( P_{X_{n}}^{(1)} \) and \( V_{k} \) both depend on \( \gamma \)). This follows because \( U_{k}(P) \) and \( I_{k}(P) \) are upper-bounded by

\[
U_{k} = \text{max}_{k \in K} U_{k}(P) < \infty
\]

(404)

and by \( C_{k} \), respectively, and lower-bounded by 0. Hence, we have that

\[
\sqrt{\frac{\gamma}{\gamma_{-}}} \leq \sqrt{\frac{\gamma}{\gamma_{-}}}
\]

(405)

which converges to \( \sqrt{\frac{\gamma}{\gamma_{-}}} \) as \( \gamma \to \infty \). This convergence follows from (338), because \( t/\beta_{-} \to 1 \), and because

\[
\sqrt{\frac{\gamma}{\gamma_{-}}} \to \sqrt{\frac{\gamma}{\gamma_{-}}} \text{ as } \gamma \to \infty \text{. Likewise, we have}
\]

\[
\sqrt{\frac{\gamma}{\gamma_{-}}} \leq \sqrt{\frac{\gamma}{\gamma_{-}}}
\]

(406)

which also converges to \( \sqrt{\frac{\gamma}{\gamma_{-}}} \text{ as } \gamma \to \infty \). Consequently, these arguments imply that (403) is satisfied for sufficiently large 2. Similarly to the asymptotic analysis of the converse bound in Section VII-C, (403) allows us to approximate \( \sqrt{\frac{\gamma}{\gamma_{-}}} \leq \sqrt{V_{k}^{(1)}}(1/\sqrt{\gamma_{-}}) \) by \( \sqrt{V_{k}/\gamma_{-}} \text{ and, hence, eliminate the dependency on } \gamma \).
In order to further bound (402) for \( t \in [\beta_- + 1, \beta_+] \), we shall now use (403) together with the inequality (proved in Appendix IV)

\[
\frac{\gamma - \xi a}{\sqrt{\xi b}} \leq \frac{\gamma - \xi a}{\sqrt{\gamma b/a}} + \sqrt{\frac{b}{a} \left( \frac{\gamma - \xi a}{\sqrt{\xi b}} \right)^2}
\]

which holds for all positive \( a, b, \xi, \) and \( \gamma \). This yields

\[
\mathbb{P} \left[ \min_k i_{P^{[\gamma]}, W_k}(X^t_i; Y^t_k) \geq \gamma \right] \geq \prod_k Q \left( \frac{\gamma - \frac{1}{t} \sum_{n=1}^t I_k(P_{X_n})}{\sqrt{V_i^t / \left( \frac{1}{t} \sum_{n=1}^t I_k(P_{X_n}) \right)^2}} + \sqrt{\frac{V_i^t}{\gamma \sum_{n=1}^t I_k(P_{X_n})}} \left( \frac{1}{t} \sum_{n=1}^t I_k(P_{X_n}) \right)^2 \right) + O \left( \frac{\log^2 \gamma}{\sqrt{\gamma}} \right)
\]

(407)

Here, (401) follows because \( \sqrt{V_i^t} \). Moreover, (414) follows from (349), and (415) follows because

\[
\mathbb{P} \left[ \min_k i_{P^{[\gamma]}, W_k}(X^t_i; Y^t_k) \geq \gamma \right] \geq \prod_k Q \left( \frac{\gamma - \frac{1}{t} \sum_{n=1}^t I_k(P_{X_n})}{\sqrt{V_i^t / \left( \frac{1}{t} \sum_{n=1}^t I_k(P_{X_n}) \right)^2}} + \sqrt{\frac{V_i^t}{\gamma \sum_{n=1}^t I_k(P_{X_n})}} \left( \frac{1}{t} \sum_{n=1}^t I_k(P_{X_n}) \right)^2 \right) + O \left( \frac{\log^2 \gamma}{\sqrt{\gamma}} \right)
\]

(410)

Here, (410) holds for sufficiently large \( \gamma \), (411) follows from (403), and (412) follows because the derivative of the \( Q \) function is bounded. Finally, we substitute (349) into (412):

\[
\mathbb{P} \left[ \min_k i_{P^{[\gamma]}, W_k}(X^t_i; Y^t_k) \geq \gamma \right] \geq \prod_k Q \left[ \frac{\gamma - \frac{1}{t} \sum_{n=1}^t I_k(P_{X_n}) - C}{\sqrt{V_i^t / C^3(1 - \delta_2 \nu_k)}} + \frac{\log^2 \gamma}{\sqrt{\gamma}} \right) + O \left( \frac{\log^2 \gamma}{\sqrt{\gamma}} \right)
\]

(413)

\[
\mathbb{P} \left[ \min_k i_{P^{[\gamma]}, W_k}(X^t_i; Y^t_k) \geq \gamma \right] \geq \prod_k Q \left[ \frac{\frac{\gamma}{t} - \frac{1}{t} \sum_{n=1}^t I_k(P_{X_n}) - C}{\sqrt{V_i^t / C^3(1 - \delta_2 \nu_k)}} + \frac{\log^2 \gamma}{\sqrt{\gamma}} \right) + O \left( \frac{\log^2 \gamma}{\sqrt{\gamma}} \right)
\]

(414)

\[
\mathbb{P} \left[ \min_k i_{P^{[\gamma]}, W_k}(X^t_i; Y^t_k) \geq \gamma \right] \geq \prod_k Q \left[ \frac{\frac{\gamma}{t} - \frac{1}{t} \sum_{n=1}^t I_k(P_{X_n}) - C}{\sqrt{V_i^t / C^3(1 - \delta_2 \nu_k)}} + \frac{\log^2 \gamma}{\sqrt{\gamma}} \right) + O \left( \frac{\log^2 \gamma}{\sqrt{\gamma}} \right)
\]

(415)

Here, (413) follows because \( \frac{1}{t} \sum_{n=1}^t I_k(P_{X_n}) = C + O(1/\sqrt{\gamma}) \). Furthermore, we operated a Taylor expansion of \( \gamma / x \) around \( x = C \), which implies that there exists a positive constant \( c_2 \) such that the following inequality holds for all sufficiently large \( \gamma \):

\[
\frac{\gamma}{t} \sum_{n=1}^t I_k(P_{X_n}) \leq \frac{\gamma}{C} - \frac{\gamma}{C^2} \left( \frac{1}{t} \sum_{n=1}^t I_k(P_{X_n}) - C \right) + c_2 \gamma \left( \frac{1}{t} \sum_{n=1}^t I_k(P_{X_n}) - C \right)^2
\]

(417)

Moreover, (414) follows from (349), and (415) follows because

\[
\sqrt{\gamma} \left| \frac{1}{C} - \frac{\gamma}{tC^2} \right| = O(\log \gamma).
\]

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To prove (419), we used that $t \in [\beta_-, \beta_+]$ and we operated a first-order Taylor expansion of the $Q$ function, and absorbed the remainder term in the $O(\log^2(\gamma)/\sqrt{\gamma})$ term. Finally, (416) follows from (347) and by the definition of the cumulative distribution functions of the RVs $\{\tilde{H}_{\delta,k}\}$:

$$F_{\tilde{H}_{\delta,k}}(w) \triangleq \Phi \left( \frac{1}{\theta_k} \min_{v_k \in (-1,1)} \frac{w + \nabla I_k(\Psi(w)) - \frac{1}{\theta_k} \int_{-\infty}^{w} E_k(s)ds}{1 - \delta_k u_k} \right).$$

(420)

**APPENDIX IV**

**BASIC LEMMAS**

**Lemma 11:** Fix arbitrary $x \in \mathbb{R}$, $a > 0$, $b > 0$, and $\lambda > 0$. Suppose that $\xi$ is the unique solution to the equation

$$\frac{\lambda - \xi a}{\sqrt{\lambda b^2 a^3}} = x.$$  \hspace{1cm} (421)

Then we have:

$$0 \leq \xi - \left( \frac{\lambda}{a} - x \sqrt{\frac{\lambda b}{a^3}} \right) \leq \frac{b}{a^3} x^2.$$  \hspace{1cm} (422)

The inequalities in (422) are equivalent to

$$\frac{\lambda - \xi a}{\sqrt{\lambda b / a}} \leq x \leq \frac{\lambda - \xi a}{\sqrt{\lambda b / a}} + \sqrt{\frac{b}{a^3} \lambda x^2}.$$  \hspace{1cm} (423)

**Proof:** For all $x \in \mathbb{R}$, we have that

$$\xi = \frac{\lambda}{a} + \frac{b}{2a^3} x^2 - x \sqrt{\frac{b^2}{4a^4} x^2 + \frac{b}{a^3} \lambda}.$$  \hspace{1cm} (424)

When $x \geq 0$,

$$\frac{\lambda}{a} - x \sqrt{\frac{\lambda b}{a^3}} \leq \frac{\lambda}{a} + \frac{b}{2a^3} x^2 - x \sqrt{\frac{b^2}{4a^4} x^2 + \frac{b}{a^3} \lambda} = \xi$$

$$\leq \frac{\lambda}{a} + \frac{b}{2a^3} x^2 - x \sqrt{\frac{b}{a^3} \lambda x^2}.$$  \hspace{1cm} (425)

Furthermore, when $x \leq 0$,

$$\frac{\lambda}{a} + \frac{b}{2a^3} x^2 - x \sqrt{\frac{b}{a^3} \lambda x^2} \leq \frac{\lambda}{a} + \frac{b}{2a^3} x^2 - x \sqrt{\frac{b^2}{4a^4} x^2 + \frac{b}{a^3} \lambda} = \xi$$

$$\leq \frac{\lambda}{a} + \frac{b}{a^3} x^2 - x \sqrt{\frac{b}{a^3} \lambda}.$$  \hspace{1cm} (426)

This establishes (423). $\blacksquare$

**Lemma 12:** Fix an integer $K \geq 2$. Let $\{x_j\}_{j=1}^{K-1}$ be constants such that

$$\sum_{j=1}^{i} x_j > 0 \quad \text{for } i \in \{1, \cdots, K-1\}. \hspace{1cm} (427)$$

Then, there exist positive constants $\{\zeta_i\}_{i=1}^{K-2}$ such that

$$x_i + \zeta_{i-1} - \zeta_i > 0 \quad \text{for } i \in \{1, \cdots, K-1\}. \hspace{1cm} (428)$$

In (428), we set $\zeta_0 \triangleq \zeta_{K-1} \triangleq 0$.

**Proof:** The lemma is obviously satisfied when $K = 2$. Next, we consider the case $K \geq 3$. Define

$$\delta \triangleq \min_{i \in \{1, \cdots, K-1\}} \frac{x_1 + \cdots + x_i}{K-1}$$  \hspace{1cm} (429)
and let \( \zeta_i = x_1 + \cdots + x_i - i\delta \) for \( i \in \{1, \cdots, K-2\} \). Note that \( \zeta_i \) is positive for \( i \in \{1, \cdots, K-2\} \). Then, we establish (428) for \( i \in \{1, \cdots, K-2\} \) as follows

\[
x_i + \zeta_{i-1} - \zeta_i = x_i + (x_1 + \cdots + x_i - (i-1)\delta) - (x_1 + \cdots + x_i - i\delta)
\]

\[
= \delta
\]

(431)

\[
> 0.
\]

(432)

Here, (432) follows from (427) and (429). To prove (428) for \( i = K-1 \), we proceed as follows

\[
x_{K-1} + \zeta_{K-2} - \zeta_{K-1} = x_1 + \cdots + x_{K-1} - (K-2)\delta
\]

\[
\geq x_1 + \cdots + x_{K-1} - \frac{K-2}{K-1}(x_1 + \cdots + x_{K-1})
\]

(433)

(434)

\[
> 0.
\]

(435)

**Lemma 13:** Define \( \psi(x) \triangleq \phi(x)/\Phi(x) \). Then the following holds:

a) \( \psi'(x) \in (-1, 0) \) for all \( x \in \mathbb{R} \),

b) \( \psi''(x) \) is positive for all \( x \in \mathbb{R} \),

c) \( \beta \psi'(\psi^{-1}(x)) < \psi'(\psi^{-1}(\beta x)) \) for all \( x > 0 \) and \( \beta > 1 \).

**Proof:** Define \( \nu(x) \triangleq \psi(-x) \). Then,

\[
\frac{1}{\nu(x)} = e^{\frac{x}{2}} \int_{x}^{\infty} e^{-\frac{u}{2}} du.
\]

(436)

This quantity is known as Mill’s ratio [25]. It follows from [25, Eq. (3)] that \( \nu'(x) \in (0, 1) \), which implies that \( \psi'(x) \in (-1, 0) \). Similarly, (b) follows from [25, Eq. (4)], which states that \( \nu''(x) > 0 \), thereby implying that \( \psi''(x) > 0 \).

To establish (c), we use the identity

\[
\psi'(\psi^{-1}(x)) = -\psi(\psi^{-1}(x))(\psi(\psi^{-1}(x)) + \psi^{-1}(x))
\]

(437)

\[
= -x(x + \psi^{-1}(x)).
\]

(438)

This implies that

\[
\psi'(\psi^{-1}(\beta x)) - \beta \psi'(\psi^{-1}(x)) = -\beta x(\beta x + \psi^{-1}(\beta x)) + \beta x(x + \psi^{-1}(x))
\]

(439)

\[
= \beta x \left[x + \psi^{-1}(x) - \beta x - \psi^{-1}(\beta x)\right]
\]

(440)

\[
> 0.
\]

(441)

Here, (439) follows from (438) and (441) follows because \( x + \psi^{-1}(x) \) is a decreasing function.

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