Asymptotics with respect to the spectral parameter and Neumann series of Bessel functions for solutions of the one-dimensional Schrödinger equation

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Abstract
A representation for a solution \( u(\omega, x) \) of the equation \(-u'' + q(x)u = \omega^2 u\), satisfying the initial conditions \( u(\omega, 0) = 1, u'(\omega, 0) = i\omega \), is derived in the form

\[
u(\omega, x) = e^{i\omega x} \left( 1 + \frac{u_1(x)}{\omega} + \frac{u_2(x)}{\omega^2} \right) + \frac{e^{-i\omega x} u_3(x)}{\omega^2} - \frac{1}{\omega^2} \sum_{n=0}^{\infty} \frac{i^n \alpha_n(x) j_n(\omega x)}{\omega^2}, \tag{1}\]

where \( u_m(x), m = 1, 2, 3 \) are given in a closed form, \( j_n \) stands for a spherical Bessel function of order \( n \) and the coefficients \( \alpha_n \) are calculated by a recurrent integration procedure. The following estimate is proved

\[|u(\omega, x) - u_N(\omega, x)| \leq \frac{1}{|\omega|^2} \varepsilon_N(x) \sqrt{\frac{\sinh(2 \Im \omega x)}{4 \Im \omega}} \]

for any \( \omega \in \mathbb{C}\setminus\{0\} \), where \( u_N(\omega, x) \) is an approximate solution given by truncating the series in (1) and \( \varepsilon_N(x) \) is a nonnegative function tending to zero for all \( x \) belonging to a finite interval of interest. In particular, for \( \omega \in \mathbb{R}\setminus\{0\} \) the estimate has the form

\[|u(\omega, x) - u_N(\omega, x)| \leq \frac{1}{|\omega|^2} \varepsilon_N(x). \]

A numerical illustration of application of the new representation for computing the solution \( u(\omega, x) \) on large sets of values of the spectral parameter \( \omega \) with an accuracy nondeteriorating (and even improving) when \( \omega \to \pm \infty \) is given.

1 Introduction
The equation

\[-u'' + qu = \omega^2 u \]

is considered on a finite interval \((0, b), \omega \in \mathbb{C}\) with \( q \in C^1[0, b] \) being a complex valued function. The asymptotics with respect to the spectral parameter \( \omega \) of solutions of (2) is, of course, a well studied topic exposed in a number of classical books such as [5], [14], [15] and [16]. A typical result establishes the existence for \( q \in W_2^m[-b, b] \) of a solution in the form

\[y(\omega, x) = e^{i\omega x} \left( u_0(x) + \frac{u_1(x)}{2i\omega} + \ldots + \frac{u_n(x)}{(2i\omega)^n} + \frac{u_{n+1}(\omega, x)}{(2i\omega)^{n+1}}\right)\]

with some additional information on \( u_{n+1}(\omega, x) \) and formulas for computing \( u_0, \ldots, u_n \) (see, e.g., [14], Section 1.4). However, in such constructions the solutions \( y(\omega, x) \) are considered which are
not necessarily entire functions with respect to \( \omega \). Moreover, it is often difficult to derive the initial conditions fulfilled by \( y(\omega, x) \), that is to identify such solutions. Whenever such identification is possible, it represents an additional useful result (as, e.g., in [5, p. 35]).

In the present work we are interested in the solution \( u(\omega, x) \) of (2) satisfying the initial conditions

\[ u(\omega, 0) = 1, \quad u'(\omega, 0) = i\omega. \]

It is entire with respect to \( \omega \) and admits the representation

\[ u(\omega, x) = e^{i\omega x} + \int_{-x}^{x} K(x, t)e^{i\omega t} dt \]  \hspace{1cm} (3)

where \( K \) is known as a transmutation (or transformation) kernel [14, Chapter I].

In spite of a vast bibliography dedicated to the transmutation operators (see, e.g., [2], [4], [13], [14], [19], [21]), only recently, in [7] a general representation for the kernel \( K \) was derived in the form of a Fourier-Legendre series with respect to the variable \( t \) and explicit formulas requiring a recurrent integration for its coefficients as functions of the variable \( x \). More precisely, the kernel \( K \) was constructed in [7] in the form

\[ K(x, t) = \sum_{n=0}^{\infty} \beta_n(x) P_n \left( \frac{t}{x} \right) \]  \hspace{1cm} (4)

where \( P_n \) stands for a Legendre polynomial of order \( n \) and \( \{ \beta_n \} \) are the coefficients computed following a recurrent integration procedure.

Substitution of (4) into (3) leads to an interesting representation of the solution \( u(\omega, x) \) in the form of a Neumann series of Bessel functions (NSBF) (we refer to [22] and [23] for more information on this kind of series),

\[ u(\omega, x) = e^{i\omega x} + \sum_{n=0}^{\infty} i^n \beta_n(x) j_n(\omega x) \]  \hspace{1cm} (5)

where \( j_n \) stands for a spherical Bessel function of order \( n \).

An important feature of the representation (5) reveals itself when considering an approximate solution \( u_N(\omega, x) = e^{i\omega x} + \sum_{n=0}^{N} i^n \beta_n(x) j_n(\omega x) \). Let, for simplicity, \( \omega \in \mathbb{R} \). Then there exists a nonnegative function \( \varepsilon_N(x) \) tending to zero for all \( x \in [0, b] \) and such that

\[ |u(\omega, x) - u_N(\omega, x)| \leq \varepsilon_N(x) \]

for all \( \omega \in \mathbb{R} \). That is, the approximate solution \( u_N(\omega, x) \) approximates the exact \( u(\omega, x) \) equally well for small and for large values of \( \omega \). This uniformness of approximation is preserved for complex \( \omega \) meanwhile \( \omega \) belongs to some strip \( |\text{Im} \omega| < C \) in the complex plane.

This unique feature of the representation (5) is a direct consequence of the fact that it was obtained from the transmutation operator (3), and as was shown in [7], it allows one to compute in no time the solution \( u(\omega, x) \) on a large set of values of the spectral parameter \( \omega \) with the same accuracy.

A natural question then arises whether a representation of \( u(\omega, x) \) can be obtained that would admit an estimate of the form

\[ |u(\omega, x) - u_N(\omega, x)| \leq \frac{\varepsilon_N(x)}{|\omega|^k} \]  \hspace{1cm} (6)
for all \( \omega \in \mathbb{R} \setminus \{0\} \) and for some \( k > 0 \), that is an estimate even improving for large values of \( \omega \).

In the present paper we show that this result is indeed possible, it is based again on the use of the transmutation operator. We show how it can be obtained for an arbitrary \( k \) but for the sake of simplicity restrict our consideration to \( k = 2 \). The starting point of this work is a suggestion from [14] p. 51 which nonetheless is followed by the sentence: “It is however difficult to find the explicit form of the coefficients and the remainder of this expansion.” We derive a closed form of the coefficients mentioned and construct an NSBF representation for the remainder. The coefficients of the NSBF are calculated by a recurrent integration procedure.

The estimate (6) is illustrated by a numerical example.

2 Preliminaries: the transmutation operator, the SPPS and NSBF representations for the solutions of the one-dimensional Schrödinger equation

Let \( q \in C^1[0, b] \). Consider the equation

\[-u'' + qu = \omega^2 u \quad \text{on} \ (0, b), \quad \omega \in \mathbb{C}, \tag{7}\]

and its solution \( u(\omega, x) \) satisfying the initial conditions

\[u(\omega, 0) = 1, \quad u'(\omega, 0) = i\omega.\]

It is well known that there exists a function \( K(x, t) \) defined on the domain \( 0 \leq |t| \leq x \leq b \) and twice continuously differentiable with respect to each of the arguments (see, e.g., [14, Chapter 1]) such that

\[u(\omega, x) = e^{i\omega x} + \int_{-x}^{x} K(x, t)e^{i\omega t}dt \quad \text{for all} \ \omega \in \mathbb{C}, \tag{8}\]

and

\[K(x, x) = \frac{Q(x)}{2} \quad \text{and} \quad K(x, -x) = 0 \tag{9}\]

for all \( x \in [0, b] \), where \( Q(x) := \int_{0}^{x} q(s)ds. \)

Note that from the integral equation for the function \( K \) [14] Chapter 1

\[K(x, t) = \frac{1}{2}Q \left( \frac{x + t}{2} \right) + \int_{0}^{\frac{x+t}{2}} \int_{0}^{\frac{x-t}{2}} q(\alpha + \beta)K(\alpha + \beta, \alpha - \beta)\,d\beta\,d\alpha \tag{10}\]

one easily obtains [10] the equalities

\[K_2(x, x) := \frac{q(x)}{4} - \frac{Q^2(x)}{8} \quad \text{and} \quad K_2(x, -x) = \frac{q(0)}{4}. \tag{11}\]

Here and below by \( K_2(x, t) \) we denote the partial derivative with respect to the second variable.

Throughout the paper we suppose that \( f_0 \) is a solution of the equation

\[f'' - qf = 0 \tag{12}\]

satisfying the initial conditions

\[f_0(0) = 1, \quad f_0'(0) = 0.\]
Consider two sequences of recursive integrals (see [6], [8])

\[ X^{(0)}(x) \equiv 1, \quad X^{(n)}(x) = n \int_0^x X^{(n-1)}(s) \left( f_0^2(s) \right)^{(-1)^n} ds, \quad n = 1, 2, \ldots \]  

(13)

and

\[ \tilde{X}^{(0)} \equiv 1, \quad \tilde{X}^{(n)}(x) = n \int_0^x \tilde{X}^{(n-1)}(s) \left( f_0^2(s) \right)^{(-1)^{n-1}} ds, \quad n = 1, 2, \ldots \]  

(14)

Definition 1 The family of functions \( \{\varphi_k\}_{k=0}^{\infty} \) constructed according to the rule

\[ \varphi_k(x) = \begin{cases} f_0(x)X^{(k)}(x), & k \text{ odd}, \\ f_0(x)\tilde{X}^{(k)}(x), & k \text{ even} \end{cases} \]  

(15)

is called the system of formal powers associated with \( f_0 \).

Remark 2 If \( f_0 \) has zeros some of the recurrent integrals (13) or (14) may not exist, although even in that case the formal powers (15) are well defined. It is convenient to construct them in the following way. Take a nonvanishing solution \( f \) of (12) such that \( f(0) = 1 \). Such solution always exists, see [8, Remark 5] or [3], and for real-valued potential \( q \) can be easily constructed explicitly. Indeed, one can take \( f = f_0 + if_1 \) where \( f_1 \) is a solution of (12) satisfying \( f_1(0) = 0, f'_1(0) = 1 \). Then (see [2, Proposition 4.7])

\[ \varphi_k = \begin{cases} \Phi_k, & k \text{ odd}, \\ \Phi_k - f_0(x) \frac{f'(0)}{k+1} \Phi_{k+1}, & k \text{ even}, \end{cases} \]

where \( \Phi_k \) are formal powers associated with \( f \).

Let us recall two series representations of the solution \( u(\omega, x) \) which are used in the present paper.

Theorem 3 (Spectral Parameter Power Series, [6]) The solution \( u(\omega, x) \) has the form

\[ u(\omega, x) = \sum_{n=0}^{\infty} \frac{(i\omega)^n \varphi_n(x)}{n!}. \]

The series converges uniformly with respect to \( x \) on \([0, b]\) and uniformly with respect to \( \omega \) on any compact subset of the complex plane of the variable \( \omega \).

Theorem 4 (Neumann series of Bessel functions, [7]) The solution \( u(\omega, x) \) admits the representation

\[ u(\omega, x) = e^{i\omega x} + \sum_{n=0}^{\infty} i^n \beta_n(x) j_n(\omega x) \]  

(16)

where \( j_n \) stands for the spherical Bessel function of order \( n \), the series converges uniformly with respect to \( x \) on \([0, b]\) and converges uniformly with respect to \( \omega \) on any compact subset of the complex plane of the variable \( \omega \), the coefficients \( \beta_n \) have the form

\[ \beta_n(x) = \frac{2n + 1}{2} \left( \sum_{k=0}^{n} \frac{l_{k,n} \varphi_k(x)}{x^k} - 1 \right) \]

where \( l_{k,n} \) is the coefficient of \( x^k \) in the Legendre polynomial of order \( n \), that is \( P_n(x) = \sum_{k=0}^{n} l_{k,n} x^k \).
3 A representation for the solutions of the one-dimensional Schrödinger equation

Proposition 5 The solution \( u(\omega, x) \) admits the representation

\[
 u(\omega, x) = e^{i\omega x} \left( 1 + \frac{Q(x)}{2i\omega} + \frac{1}{\omega^2} \left( \frac{q(x)}{4} - \frac{Q^2(x)}{8} \right) \right) - \frac{q(0)}{4} e^{-i\omega x} - \frac{1}{\omega^2} \int_{-x}^{x} K_{22}(x, t) e^{i\omega t} dt \quad \text{for all} \; \omega \in \mathbb{C}. \tag{17}
\]

Proof. Multiplication of (8) by \( i\omega \) and integration by parts with the aid of (9) gives us the equality

\[
 i\omega u(\omega, x) = i\omega e^{i\omega x} + \int_{-x}^{x} K(x, t) \frac{de^{i\omega t}}{dt} dt = i\omega e^{i\omega x} + \frac{Q(x)}{2} e^{i\omega x} - \int_{-x}^{x} K_2(x, t) e^{i\omega t} dt. \tag{18}
\]

Consider

\[
 i\omega \int_{-x}^{x} K_2(x, t) e^{i\omega t} dt = \int_{-x}^{x} K_2(x, t) \frac{de^{i\omega t}}{dt} dt = K_2(x, x) e^{i\omega x} - K_2(x, -x) e^{-i\omega x} - \int_{-x}^{x} K_{22}(x, t) e^{i\omega t} dt.
\]

Using (11) we obtain

\[
 \int_{-x}^{x} K_2(x, t) e^{i\omega t} dt = \frac{1}{i\omega} \left( e^{i\omega x} \left( \frac{q(x)}{4} - \frac{Q^2(x)}{8} \right) - \frac{q(0)e^{-i\omega x}}{4} - \int_{-x}^{x} K_{22}(x, t) e^{i\omega t} dt \right). \]

Substitution of this expression into (18) gives us (17). \( \blacksquare \)

Remark 6 By the Riemann-Lebesgue lemma on the decrease at infinity of the Fourier transform of an \( L_1(-\infty, \infty) \)-function, the integral \( \int_{-x}^{x} K_{22}(x, t) e^{i\omega t} dt \) tends to zero when \( \omega \to \pm \infty \), and thus from (17) the asymptotic equality follows

\[
 u(\omega, x) = e^{i\omega x} \left( 1 + \frac{Q(x)}{2i\omega} + \frac{1}{\omega^2} \left( \frac{q(x)}{4} - \frac{Q^2(x)}{8} \right) \right) - \frac{q(0)e^{-i\omega x}}{4\omega^2} + o \left( \frac{1}{\omega^2} \right)
\]

when \( \omega \to \pm \infty \).

Remark 7 Repeating this integration by parts procedure one can obtain representations involving higher order terms of \( \frac{1}{\omega} \) (and hence estimates \( \| \) with \( k > 2 \)). Explicit expressions for the terms \( K_{2,2}(x, x) \) and \( K_{2,2}(x, -x) \) resulting from these integrations by parts can be derived repeatedly differentiating the integral equation (10) with respect to \( t \).

4 Fourier-Legendre series expansion of the kernel \( K_{22} \)

The aim of this section is to derive a representation of \( K_{22} \) in the form

\[
 K_{22}(x, t) = \sum_{n=0}^{\infty} \alpha_n(x) \frac{P_n \left( \frac{t}{x} \right)}{x} \tag{19}
\]
where $P_n$ stands for the Legendre polynomial of order $n$, and $\alpha_n$ are to be found. Note that for every $x$ fixed the function $K_{22}(x, \cdot)$ is continuous and hence admits a Fourier-Legendre representation of the form (19) which is convergent in the sense of $L_2$-norm and hence

$$\varepsilon_N(x) := \|K_{22}(x, \cdot) - K_{22,N}(x, \cdot)\|_{L_2(-x,x)} \to 0, \quad N \to \infty$$

(20)

for all $x \in [0, b]$ where $K_{22,N}(x, \cdot) := \sum_{n=0}^N \alpha_n(x) P_n \left( \frac{t}{x} \right)$. Some more precise decay rate estimates for the function $\varepsilon_N(x)$ may be obtained similarly to [7, Theorem 3.3] and [11, Proposition A.2].

If $q \in C^2[0, b]$ and hence $K_{22}(x, \cdot) \in C^1[-b, b]$, the series in (19) converges uniformly with respect to $t \in [-x, x]$ (see, e.g., [20]).

As a first step let us prove the following equalities. Denote

$$k_n(x) := \int_{-x}^x K_{22}(x, t)n^{th} dt, \quad n = 0, 1, 2, \ldots.$$

**Proposition 8** The following equalities are valid

$$k_0(x) = \frac{q(x)}{4} - \frac{Q^2(x)}{8} - \frac{q(0)}{4},$$

$$k_1(x) = \left( \frac{q(x)}{4} - \frac{Q^2(x)}{8} + \frac{q(0)}{4} \right) x - \frac{Q(x)}{2},$$

$$k_n(x) = n(n-1) \left( \varphi_{n-2}(x) - x^{n-2} \right) + \left( \frac{q(x)}{4} - \frac{Q^2(x)}{8} - \frac{(-1)^n q(0)}{4} \right) x^n - \frac{nQ(x)x^{n-1}}{2},$$

$n = 2, 3, \ldots$.

**Proof.** Multiply (17) by $(i\omega)^2$ and make use of the SPPS representation of the solution $u(\omega, x)$. Then

$$\sum_{n=0}^{\infty} \frac{(i\omega)^{n+2}}{n!} \varphi_n(x) = \sum_{n=0}^{\infty} \frac{(i\omega)^{n+2} x^n}{n!} + \frac{Q(x)}{2} \sum_{n=0}^{\infty} \frac{(i\omega)^{n+1} x^n}{n!}$$

$$- \left( \frac{q(x)}{4} - \frac{Q^2(x)}{8} \right) \sum_{n=0}^{\infty} \frac{(i\omega)^n x^n}{n!} + \frac{q(0)}{4} \sum_{n=0}^{\infty} \frac{(-i\omega)^n x^n}{n!}$$

$$+ \sum_{n=0}^{\infty} \frac{(-i\omega)^n}{n!} \int_{-x}^x K_{22}(x, t)n^{th} dt.$$

Equating terms at equal powers of $\omega$ we obtain the required equalities. 

From (19) we immediately obtain a formula for the coefficients

$$\alpha_n(x) = \frac{2n+1}{2} \int_{-x}^x K_{22}(x, t)n^{th} dt = \frac{2n+1}{2} \sum_{m=0}^{n} l_{m,n} \frac{k_m(x)}{x^m}.$$

In particular, introducing the notation

$$q_{\pm}(x) := \frac{q(x)}{4} - \frac{Q^2(x)}{8} \pm \frac{q(0)}{4}$$

6
we can write
\[
\alpha_0(x) = \frac{q_-(x)}{2}, \quad \alpha_1(x) = \frac{3}{2} \left( q_+(x) - \frac{Q(x)}{2x} \right), \quad (21) \\
\alpha_2(x) = \frac{5}{2} \left( q_-(x) + \frac{3}{x^2} (\varphi_0(x) - 1) - \frac{3Q(x)}{2x} \right), \quad (22) \\
\alpha_3(x) = \frac{7}{2} \left( q_+(x) + \frac{15}{x^3} (\varphi_1(x) - x) - \frac{3Q(x)}{2x} \right). \quad (23)
\]

In Section \[8\] we derive a formula relating the coefficients \(\alpha_n\) with \(\beta_n\), which together with the equalities \(21\), \(23\) and a recurrent integration procedure for computation of the coefficients \(\beta_n\) from \[7\] leads to a convenient numerical algorithm for computing the coefficients \(\alpha_n\).

We finish this section by the following observation, useful for controlling the approximation accuracy.

**Remark 9** Second differentiation of the integral equation for \(K\) in analogy with \(11\) leads to the equalities
\[
K_{22}(x,x) = \frac{1}{8} \left( q'(x) - q(x)Q(x) - \int_0^x q^2(s)ds + \frac{Q^3(x)}{6} \right)
\]
and
\[
K_{22}(x,-x) = \frac{1}{8} \left( q'(0) + q(0)Q(x) \right).
\]

They can be used for a simple and efficient checking of approximation accuracy. Indeed, from \(19\) we have
\[
K_{22}(x,x) = \frac{1}{x} \sum_{n=0}^{\infty} \alpha_n(x) \quad \text{and} \quad K_{22}(x,-x) = \frac{1}{x} \sum_{n=0}^{\infty} (-1)^n \alpha_n(x)
\]
and hence the magnitudes
\[
e_1(x) := \left| \frac{1}{8} \left( q'(x) - q(x)Q(x) - \int_0^x q^2(s)ds + \frac{Q^3(x)}{6} \right) - \frac{1}{x} \sum_{n=0}^{N} \alpha_n(x) \right|
\]
and
\[
e_2(x) := \left| \frac{1}{8} \left( q'(0) + q(0)Q(x) \right) - \frac{1}{x} \sum_{n=0}^{N} (-1)^n \alpha_n(x) \right|
\]
characterize the accuracy of approximation of the kernel \(K_{22}(x,t)\) by a partial sum of the series \(19\).

**5 Representation of the solution \(u(\omega, x)\)**

**Theorem 10** The solution \(u(\omega, x)\) admits the following representation
\[
u(\omega, x) = e^{i\omega x} \left( 1 + \frac{Q(x)}{2i\omega} + \frac{q(x)}{4} - \frac{Q^2(x)}{8} \right) - \frac{q(0)}{4} \frac{e^{-i\omega x}}{\omega^2} - \frac{1}{\omega^2} \sum_{n=0}^{\infty} i^n \alpha_n(x) j_n(\omega x), \quad (24)
\]
where the series converges for any \(x\) on the segment \([0, b]\) and converges uniformly on any compact subset of the complex plane with respect to \(\omega\). Moreover, the following estimate is valid
\[
|u(\omega, x) - u_N(\omega, x)| \leq \frac{1}{|\omega|^2} \varepsilon_N(x) \sqrt{\frac{\sinh(2 \Im \omega x)}{\Im \omega}} \quad \text{for any } \omega \in \mathbb{C} \setminus \{0\} \quad (25)
\]
where

\[ u_N(\omega, x) := e^{i\omega x} \left( 1 + \frac{Q(x)}{2i\omega} + \frac{1}{\omega^2} \left( \frac{q(x)}{4} - \frac{Q^2(x)}{8} \right) \right) - \frac{q(0)}{4} e^{-i\omega x} - \frac{1}{\omega^2} \sum_{n=0}^{N} i^n \alpha_n(x) j_n(\omega x) \]

and \( \varepsilon_N \) is defined by (20).

In particular, for all \( \omega \in \mathbb{R} \setminus \{0\} \) the estimate has the form \( |u(\omega, x) - u_N(\omega, x)| \leq \frac{1}{|\omega|^2} \varepsilon_N(x) \sqrt{2\omega} \).

**Proof.** The representation (24) is obtained by substituting the series (19) into (17) and evaluating the arising integrals with the aid of formula 2.17.7 from [18, pp. 433]. To obtain the estimate (25) consider

\[ |u(\omega, x) - u_N(\omega, x)| = \frac{1}{|\omega|^2} \left| \int_{-x}^{x} (K_{22}(x,t) - K_{22,N}(x,t)) e^{i\omega t} dt \right| \leq \frac{1}{|\omega|^2} \varepsilon_N(x) \| e^{i\omega t} \|_{L_2(-x,x)} \]

where the Cauchy–Bunyakovsky–Schwarz inequality was applied. Calculation of \( \| e^{i\omega t} \|_{L_2(-x,x)} \) leads to (25) from which the convergence with respect to \( x \) follows.

The convergence of the series with respect to \( \omega \) can be established using the fact that for every \( x \) the series represent a Neumann series of Bessel functions (see, e.g., [22] and [23]) of the entire function with respect to \( \omega \),

\[ e^{i\omega x} \left( \omega^2 + \frac{\omega Q(x)}{2i} + \frac{1}{2} \left( \frac{q(x)}{4} - \frac{Q^2(x)}{8} \right) \right) - \frac{q(0)}{4} e^{-i\omega x} - \omega^2 u(\omega, x) . \]

Since the radius of convergence of the Neumann series coincides [22, 524–526] with the radius of convergence of its associated power series (obtained from the SPPS representation), the series in (24) converges uniformly on any compact subset of the complex plane of the variable \( \omega \). 

**6 A relation between the coefficients \( \alpha_n \) and \( \beta_n \)**

In [7] a convenient recurrent integration procedure for computing the coefficients \( \{\beta_n\} \) of the NSBF representation (16) was derived. In order to make use of it for computing the coefficients \( \{\alpha_n\} \) we establish a relation between these two systems of functions.

**Proposition 11** The following equality is valid

\[ \alpha_n = (2n - 1)(2n + 1) \left( \frac{\beta_{n-2}}{x^2} + \frac{2\alpha_{n-2}}{(2n - 5)(2n - 1)} - \frac{\alpha_{n-4}}{(2n - 7)(2n - 5)} \right) \quad (26) \]

for \( n = 4, 5, \ldots \).

**Proof.** Equating the expressions (16) and (24) we obtain the equality

\[ e^{i\omega x} \left( \frac{Q(x)}{2i\omega} + \frac{1}{\omega^2} \left( \frac{q(x)}{4} - \frac{Q^2(x)}{8} \right) \right) - \frac{q(0)}{4} e^{-i\omega x} - \frac{1}{\omega^2} \sum_{n=0}^{\infty} i^n \alpha_n(x) j_n(\omega x) = \sum_{n=0}^{\infty} i^n \beta_n(x) j_n(\omega x) . \]

On the left-hand side the following identity can be used

\[ \frac{j_n(z)}{z^2} = \frac{j_{n-2}(z)}{(2n - 1)(2n + 1)} + \frac{2j_n(z)}{(2n - 1)(2n + 3)} + \frac{j_{n+2}(z)}{(2n + 1)(2n + 3)} . \]
Thus,
\[
e^{i\omega x} \left( \frac{Q(x)}{2i\omega} + \frac{1}{\omega^2} \left( \frac{q(x)}{4} - \frac{Q^2(x)}{8} \right) \right) - \frac{q(0)}{4} e^{-i\omega x} \frac{1}{\omega^2} x^2 \sum_{n=0}^{\infty} \bar{n}^n \alpha_n(x) \left( \frac{j_{n-2}(\omega x)}{(2n-1)(2n+1)} + \frac{2j_n(\omega x)}{(2n-1)(2n+3)} + \frac{j_{n+2}(\omega x)}{(2n+1)(2n+3)} \right)
\]
\[
= \sum_{n=0}^{\infty} \bar{n}^n \beta_n(x) j_n(\omega x). \quad (27)
\]

Recall the orthogonality property of the spherical Bessel functions (see, e.g., \[1\,\text{p. 732}\])
\[
\int_{-\infty}^{\infty} j_m(z) j_n(z) \, dz = \begin{cases} 
0, & m \neq n, \ m + n \geq 0, \\
\frac{\pi}{2n+1}, & m = n.
\end{cases} \quad (28)
\]

Notice that \( \frac{\omega x}{z} = j_{-1}(z) + ij_{0}(z) \) and \( -\frac{x}{z} = -j_{-1}(z) - ij_{1}(z) \) and hence (27) can be written in the form
\[
\frac{xQ(x)}{2i} (j_{-1}(\omega x) + ij_{0}(\omega x)) - x^2 \left( \frac{q(x)}{4} - \frac{Q^2(x)}{8} \right) (j_{-1}(\omega x) - ij_{-1}(\omega x) + j_{0}(\omega x) - ij_{1}(\omega x))
\]
\[
+ \frac{q(0)x^2}{4} (j_{-2}(\omega x) + ij_{-1}(\omega x) + j_{0}(\omega x) + ij_{1}(\omega x))
\]
\[
- x^2 \sum_{n=0}^{\infty} \bar{n}^n \alpha_n(x) \left( \frac{j_{n-2}(\omega x)}{(2n-1)(2n+1)} + \frac{2j_n(\omega x)}{(2n-1)(2n+3)} + \frac{j_{n+2}(\omega x)}{(2n+1)(2n+3)} \right)
\]
\[
= \sum_{n=0}^{\infty} \bar{n}^n \beta_n(x) j_n(\omega x). \quad \text{(27)}
\]

Multiplication of this equality by \( j_m(\omega x), \ m \geq 2 \) and integration with respect to \( \omega \), with the aid of (28), leads to the equality
\[
\beta_m(x) = x^2 \left( \frac{\alpha_{m+2}(x)}{(2m+3)(2m+5)} - \frac{2\alpha_m(x)}{(2m-1)(2m+3)} + \frac{\alpha_{m-2}(x)}{(2m-3)(2m-1)} \right)
\]
which gives us (26). ■

7 Numerical illustration

A numerical approach based on the NSBF representation \[16\] for solving equation (7) for a wide range of values of the spectral parameter \( \omega \) as well as related spectral problems was presented in \[7\]. It is an efficient, simple and practical method which apart from its numerical advantages is available for an unaided programming by researchers which does not require any sophisticated numerical technique usually encoded in purely numerical solvers. The aim of this section is to show that another NSBF representation \[24\] derived in the present work can give even more accurate results when the number of the computed terms is limited.

**Example 12** Consider the following spectral problem (the first Paine problem, \[17\], see also \[7, Example 7.2\])
\[
\begin{align*}
-u'' + e^x u &= \lambda u, \\ u(0, \lambda) &= u(\pi, \lambda) = 0.
\end{align*}
\]
We solve this problem numerically using partial sums from (16) and (24) and referring the reader to [7] where relevant details on numerical aspects can be found. Thus, an approximate solution of the Sturm-Liouville problem is computed using the approximate solutions

$$\tilde{u}_N(\pm \omega, x) = e^{\pm i\omega x} + \sum_{n=0}^{N} i^n \beta_n(x) j_n(\pm \omega x)$$  \hspace{1cm} (29)$$

and

$$u_N(\pm \omega, x) := e^{\pm i\omega x} \left(1 \pm \frac{Q(x)}{2i\omega} + \frac{1}{\omega^2} \left(\frac{q(x)}{4} - \frac{Q^2(x)}{8}\right)\right) - \frac{q(0)e^{\mp i\omega x}}{4\omega^2} - \frac{1}{\omega^2} \sum_{n=0}^{N+2} i^n \alpha_n(x) j_n(\pm \omega x).$$  \hspace{1cm} (30)$$

The upper bound of summation $N + 2$ in (30) in comparison with $N$ in (29) is due to the fact that $N$ computed coefficients $\beta_n$ are transformed into $N + 2$ computed coefficients $\alpha_n$ (see (26)).

For relatively small $N$ the eigenvalues are approximated more accurately by the solution obtained from (30). This can be appreciated on Figs. 1 (a) and (b) where the absolute error of the first 460 eigenvalues computed is presented for $N = 5$ and $N = 15$ respectively. However for larger $N$ the difference in the accuracy practically disappears, as is illustrated by Fig. 1 (c) where the same comparison is made for $N = 25$. Both approximations deliver excellent numerical results in fractions of a second on a usual computer.

We emphasize that the eigenvalues of the problem behave asymptotically [12] as $\sim \left(n + \frac{Q(\pi)}{2\pi n}\right)^2$, however the values computed by both our algorithms are still closer to the exact one. For example, the exact value of $\lambda_{460}$ is $211607.047634847$ (rounded up to the presented digits), the asymptotic expression gives $211607.047660$, while the approximation based on (30) delivered the value $211607.047634847$. Moreover, due to the “largeness” of the higher eigenvalues the errors of approximate values computed using either of NSBF representations are limited by machine precision (while absolute errors are of order $10^{-9}$, relative errors are less than $10^{-15}$).

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Figure 1: Comparison of the absolute errors of approximation of the first 460 eigenvalues computed using partial sums (29) (in blue - x- lines) and (30) (in red - *- lines). (a) with $N = 5$, (b) with $N = 15$ and (c) with $N = 25$. 