ON THE BLOW-UP BEHAVIOR OF SOLUTIONS TO A SYSTEM WITH FRACTIONAL POWER

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ABSTRACT

The aim of this research is to study the nonexistence of global weak solutions for systems of fractional power. The method relies on a suitable choice of test function.

Keywords: Nonexistence Global, Critical Exponent

1. INTRODUCTION

Since the articles of Fujita (1966) and OSEU (1968), critical exponents have attracted the attention of a sizable number of researchers. Fore valuable surveys of Fujita type theorems for equations as well as for systems of reaction-diffusion equations we refer to Levine (1990); Samarskii et al. (1995); Bandle and Brunner (1998) and Levine (1990). Escobedo and Herrero (1991) considered the system:

\[
\begin{align*}
\frac{\partial u}{\partial t} &= \delta \Delta u + u + v^p, \quad v = \Delta v + v^q, \quad t > 0, x \in \mathbb{R}^n \\
\frac{\partial u}{\partial t} &= u(0, x) = u_0(x) \geq 0, \quad v(0, x) = v_0(x) \geq 0 \\
\end{align*}
\]

(S)

with \( \delta = 1, p, q > 0 \), they derived global existence and blow-up results for (S). They showed that all positive solutions of (S) blow-up in finite time for:

\[
\frac{pq}{2} > 1 \text{ and } \frac{n}{2} \leq \max \left( \frac{p+q}{pq}, \frac{1}{pq} \right) + 1
\]

Then in Fila et al. (1994) extended the results to the case where \( 0 \leq \delta \leq 1 \). They use the same technique as in (Escobedo and Herrero, 1991) and a property satisfied by the heat kernel. In a recent paper, Igbida and Kirane (2002) considered, with respect to the nonexistence of global solutions, the more general system:

\[
\begin{align*}
\frac{\partial u_i}{\partial t} &= (-\Delta)(a_{ij}u) = h(t, x) |v|^p \\
\frac{\partial v_i}{\partial t} &= (\Delta)(a_{ij}v) = k(t, x) |w|^q \\
\frac{\partial w_i}{\partial t} &= (-\Delta)(a_{ij}w) = l(t, x) |u|^r \\
\end{align*}
\]

(1K)

for \( p, q, r > 0, a_{ij} \) measurable, positive and bounded functions. The conditions on \( h, k \) and \( l \) required are:

\[
\begin{align*}
\frac{h(R^2, R^2y)}{R^2} &\leq O(R^\mu) \\
\frac{k(R^2, R^2y)}{R^2} &\leq O(R^\kappa) \\
\frac{l(R^2, R^2y)}{R^2} &\leq O(R^\lambda)
\end{align*}
\]

Let us note in passing that Renclawowicz studied the completely coupled Fujita-type system:

\[
\begin{align*}
\frac{\partial u_i}{\partial t} &= \Delta u + v^p \quad (t, x) \in Q \\
\frac{\partial v_i}{\partial t} &= \Delta v + w^q \quad (t, x) \in Q \\
\frac{\partial w_i}{\partial t} &= \Delta w + u^r \quad (t, x) \in Q
\end{align*}
\]

(1R)

with \( p, q, r > 0, n \geq 0 \) and nonnegative bounded continuous initial values. She proved that, if \( pqr \leq 1 \), then any solution is global, while when \( pqr > 1 \) and:

\[
\frac{n}{2} \leq \frac{1}{pqr} \max (1 + p + pq, 1 + q + qr, 1 + r + rp)
\]
then every nontrivial solution exhibits a finite blow-up time. She also uses Fujita’s method. Renclawowicz (2000), she extended her study to a diagonal system of \( n \) equations. The aim of this study is to establish new results on the blowing-up of solutions to systems of the following type Equation 1:

\[
\begin{align*}
0 &< h(R^2, T) = O(R^n) \\
0 &< k(R^2, T) = O(R^s) \\
0 &< l(R^2, T) = O(R^t)
\end{align*}
\]

1.1. Main Results

Let \( L^1_{\text{loc}} ((0, T) \times IR^n, 1 \ dt \ dx) \) be the set of all functions \( u: (0, T) \times IR^n \rightarrow IR \) such that \( \int_K |u|^p \ dt \ dx < \infty \) for any compact \( K \subset (0, T) \times IR^n \).

**Definition 2.1:**

The 3-tuple \((u, v, w)\) such that \( u \in C([0, T]; L^1_{\text{loc}} (IR^n) \cap C([0, T]; L^1_{\text{loc}} ((0, T) \times IR^n, 1 \ dt \ dx)), \) \( v \in C([0, T]; L^1_{\text{loc}} (IR^n) \cap C([0, T]; L^1_{\text{loc}} ((0, T) \times IR^n, k \ dt \ dx)) \) is called a solution to system (1) if:

\[
\begin{align*}
\int_0^T h(|v|^p) \ dz = -\int_{\mathbb{R}^n} u\xi(0) - \int_0^T |v|^p a_{ij} u(-\Delta)^{-\frac{\beta}{2}} \xi \\
\int_0^T k(|w|^p) \ dz = -\int_{\mathbb{R}^n} v\xi(0) - \int_0^T |w|^p a_{ij} v(-\Delta)^{-\frac{\beta}{2}} \xi \\
\int_0^T |u|^p \ dz = -\int_{\mathbb{R}^n} w\xi(0) - \int_0^T |w|^p a_{ij} w(-\Delta)^{-\frac{\beta}{2}} \xi
\end{align*}
\]

where \( \beta \in (0, 2) \) and \( \beta \in 2 \). For any nonnegative test function \( \xi \in C^0_c (IR^n \times IR^n) \) with \( \xi(T, x) = 0 \) if \( T = +\infty \), we say that \((u, v, w)\) is a global weak solution.

Here, we require that the initial data are such that a local solution exists. Our result is.

**Theorem 2.1:**

Let \((u, v, w)\) be a solution of (1) such that \( u_0, v_0, w_0 \geq 0 \). Let \( p, q, r > 1 \) and:
Then every nontrivial solution of (1) blows up in finite time.

**Proof**

The proof is by contradiction. Let \( (u, v, w) \) be a global solution of (1) with \( pqr > 1 \) and suppose (K) is satisfied. Let \( \zeta \) be a nonnegative test function such that:

\[
\begin{align*}
\zeta - \Delta \zeta &< \infty \\
\zeta - \Delta \zeta &< \infty \\
\zeta - \Delta \zeta &< \infty
\end{align*}
\]

As \( u(0, x), v(0, x), w(0, x) \geq 0 \), using (H) we have, for \( \zeta \geq 0 \) Equation 3-5:

\[
\begin{align*}
\int_0^1 |v|^r \zeta &\leq \int_0^1 \|u\|_\zeta,|1+\|\alpha\|_\infty,\int_0^1 \|u\|(-\Delta)^{\frac{a_1}{2}} \zeta \\
\int_0^1 |w|^r \zeta &\leq \int_0^1 \|v\|_\zeta,|1+\|\alpha\|_\infty,\int_0^1 \|u\|(-\Delta)^{\frac{a_2}{2}} \\
\int_0^1 |u|^r \zeta &\leq \int_0^1 \|w\|_\zeta,|1+\|\alpha\|_\infty,\int_0^1 \|u\|-\Delta^{\frac{a_3}{2}} \zeta + \|a\|_\infty \\
\int_0^1 |x|^r \zeta &\leq \int_0^1 \|w\|_\zeta,|1+\|\alpha\|_\infty,\int_0^1 \|u\|(-\Delta)^{\frac{a_4}{2}} \zeta
\end{align*}
\]

where, \( \|a\|_\infty = \max_{i,t} |a_{i,t}| \). To estimate \( \int_0^1 \|u\| \zeta \), we observe that it can be written as:

\[
\int_0^1 |u|^2 \zeta \leq \int_0^1 |u| \left( |(\zeta,_{\zeta})| |(\zeta,_{\kappa})| \right) \]

using Holder inequality, we have with \( \frac{1}{r} + \frac{1}{r} = 1 \)

Equation 6 and 7:

\[
\begin{align*}
\int_0^1 |u|^2 \zeta &\leq \left( \int_0^1 |u|^r \zeta \right)^\frac{1}{r} \left( \int_0^1 \|u\| \zeta \right)^\frac{1}{r} \\
\int_0^1 \|x\|(-\Delta)^{\frac{a_3}{2}} \zeta &\leq \left( \int_0^1 \|u\| \zeta \right)^\frac{1}{r} \left( \int_0^1 \|x\|(-\Delta)^{\frac{a_3}{2}} \zeta \right)^\frac{1}{r}
\end{align*}
\]

Next, we set:

\[
\begin{align*}
A_{\alpha} - a = \left( \int_0^1 \|u\| \zeta \right)^\frac{1}{r} + C \left( \int_0^1 \|u\| \zeta \right)^\frac{1}{r}
\end{align*}
\]

And:

\[
X = \left( \int_0^1 \|u\| \zeta \right)^\frac{1}{r} \quad \text{and} \quad Y = \left( \int_0^1 \|u\| \zeta \right)^\frac{1}{r}
\]

Then using (6) and (7) in (3), we obtain Equation 8:

\[
Y^p \leq X A_{\alpha}^{\frac{1}{r}}
\]

We also have:

\[
\int_0^1 \|k\| \zeta \leq \left( \int_0^1 \|w\| \zeta \right)^\frac{1}{r} A_{\beta}^{\frac{1}{r}} + C \left( \int_0^1 \|u\| \zeta \right)^\frac{1}{r} A_{\alpha}^{(2)}
\]

And:

\[
\begin{align*}
\int_0^1 \|k\| \zeta &\leq \int_0^1 \|w\| \zeta \left( \int_0^1 \|u\| \zeta \right)^\frac{1}{r} A_{\gamma}^{(1)} + C \left( \int_0^1 \|u\| \zeta \right)^\frac{1}{r} A_{\gamma}^{(2)}
\end{align*}
\]

If we set \( Z = \left( \int_0^1 \|k\| \zeta \right)^\frac{1}{r} \) then we can write Equation 9 and 10:
\[ Z^k \leq Y \ A_{\beta a_k}^{p, h} + C \ X \ A_{\beta a_k r_l}^{(2)} \]  \hspace{1cm} (9)

\[ X' \geq Z \ A_{\gamma a_k}^{q, k} + C \ X \ A_{\gamma a_k r_l}^{(2)} + C \ Y \ A_{\gamma a_k p, h}^{(2)} \]  \hspace{1cm} (10)

So Equation 11 and 12:

\[ Z^m \leq C \ Y^p \left[ A_{\beta a_k}^{p, h} \right] + C \ X^l \left[ A_{\beta a_k r_l}^{(2)} \right] \]  \hspace{1cm} (11)

\[ X^m \geq C \ Z^p \left[ A_{\gamma a_k}^{q, k} \right] + C \ X^l \left[ A_{\gamma a_k r_l}^{(2)} \right] \]  \hspace{1cm} (12)

Inserting (11) in (12), we get Equation 13:

\[ X^m \leq C \ Y^p \left[ A_{\gamma a_k}^{q, k} \right] + C \ X^l \left[ A_{\gamma a_k r_l}^{(2)} \right] \]  \hspace{1cm} (13)

Using (8) in (13), we obtain:

\[ X^m \leq C \ X^p \left[ A_{\gamma a_k}^{q, k} \right] + C \ X^l \left[ A_{\gamma a_k r_l}^{(2)} \right] \]  \hspace{1cm} (14)

With:

\[ a = C \left[ A_{\gamma a_k}^{q, k} \right], \quad b = C \left[ A_{\gamma a_k r_l}^{(2)} \right] \]

which we write as Equation 14:

\[ X^m \leq a + bX^m + cX^m + dX^m \]  \hspace{1cm} (15)

\[ cX^m \leq \varepsilon X^m + C(\varepsilon) b^m; \quad m_2 = \frac{pq - 1}{p - 1}, \quad m_1 = 1 \]  \hspace{1cm} (16)

\[ dX^m \leq \varepsilon X^m + C(\varepsilon) d^m; \quad m_3 = \frac{pq - 1}{p - 1}, \quad m_1 = 1 \]  \hspace{1cm} (17)

\[ C(\varepsilon) \] has a different meaning in 15, 16 and 17. Taking \( \varepsilon \) small enough and using 15-17 in 14, we obtain Equation 18:

\[ (1 - 3\varepsilon)X^m \leq a + C(\varepsilon) b^m + c^m + d^m \]  \hspace{1cm} (18)

Next, we consider \( \phi \in C^2(\mathbb{R}; \mathbb{R}^+) \) such that:

\[ \phi(r) = \begin{cases} 1 & \text{for } r \leq 1 \\ 0 & \text{for } r \geq 2 \end{cases} \]

and \( 0 \leq \phi \leq 1 \) for any \( r > 0 \). If we set:

\[ \zeta(t, x) = \phi \left( \frac{t + \| \mathbf{r} \|^2}{R^2} \right), \quad R > 0 \]

and take \( \theta \) large enough, we ensure the validity of the requirement (I) at the beginning of the proof. At this stage, we introduce the scaled variables:

\[ \tau = \frac{r}{R}, \quad y = x R^{-1} \]

We have:

\[ a \leq C R^m, \quad b \leq C R^m, \quad c \leq C R^m, \quad d \leq C R^m \]

Where:

\[ s_a = \frac{\alpha - \alpha_{a_k} - \frac{\lambda}{r} + \frac{(2n)(r + 1)}{r}}{\lambda} \]

\[ s_b = \frac{\gamma - \alpha_{a_k} - \frac{k}{q} + \frac{(2n)(q + 1)}{q}}{p} \]

\[ s_c = \frac{\beta - \alpha_{a_k} - \frac{u}{p} + \frac{(2n)(q + 1)}{p}}{p} \]

\[ s_d = \frac{\beta - \alpha_{a_k} - \frac{\lambda}{r} + \frac{(2n)(r - 1)}{r}}{r} \]
\[ s_c = \frac{p}{r} \left[ \gamma \frac{\lambda}{r} - \alpha_r + \frac{(2 + n)(r + 1)}{r} \right] \]
\[ s_d = q \frac{p q - p \alpha_s - \mu + (2 + n)(p + 1)}{r} + q \left[ \alpha - \alpha_r - \frac{\lambda}{r} \left( \frac{2 + n}{r} \right) \right] \]

Now, we require:
\[ s_c \leq 0, \quad s_d \leq 0, \quad s_e \leq 0, \quad s_d \leq 0 \]

which are, respectively, equivalent to:
\[ n \leq \min \left\{ \frac{r \left[ \mu + \alpha_s - \alpha_r + p \alpha_s - \beta - 2q + k + q \alpha_s - q \gamma \right]}{p q r - 1}, \quad \frac{r \left[ \alpha_r + k \sigma - \beta - \gamma \alpha_s - q \gamma \right] + \lambda + 2}{r - 1}, \quad \frac{r \left[ \mu + \alpha_s - \alpha_r + p \alpha_s - \gamma - 2 \right]}{p r - 1} \right\} \]

We have two cases:

- Either \( s_a < 0, \quad s_b < 0, \quad s_c < 0 \) and \( s_d < 0 \). In this case, we let \( R \rightarrow \infty \) in (18) to obtain:
  \[ \lim_{R \rightarrow \infty} X_{p q r - 1}^{-1} = 0 \]
  Hence \( u = 0 \); this implies via (8) that \( v = 0 \) and finally \( w = 0 \) from (9). That is a contradiction

- Or \( s_a < 0 \) or \( s_b < 0 \) or \( s_c < 0 \) or \( s_d < 0 \), i.e., at least one of the exponents is not zero. In this case, we get:
  \[ \lim_{R \rightarrow \infty} X_{p q r - 1}^{-1} \leq C < \infty \]

So:
\[ \lim_{k \rightarrow -1} \int_{\Omega_k} |h| \xi = 0, \quad \text{where} \quad \Omega_k = \{(t, x) : R^2 \leq t + |x|^2 \leq 2R^2 \} \]

Now we write (8) in the form:
\[ \int_{\Omega_k} |h| \xi \leq \left( \int_{\Omega_k} |h| \xi \right) A_{v, u}^{(1, i)} \]

where:
\[ \sigma_1 = \left[ \alpha - \alpha_r - \frac{\lambda}{r} \left( \frac{2 + n}{r} \right) \right] + \left[ p q - \kappa - q \alpha_r + (2 + n)(q - 1) + \left[ p \beta - \mu - p \alpha_r + (2 + n)(p - 1) \right] \right] \]
\[ \sigma_2 = \left[ \beta - \alpha_r - \frac{\mu}{p} \left( \frac{2 + n}{p} \right) \right] + \left[ q \alpha_r - \lambda - q \alpha_r + (2 + n)(r - 1) + \left[ q \beta - \mu - p \alpha_r + (2 + n)(q - 1) \right] \right] \]
\[ \sigma_3 = \left[ \gamma - \alpha_r - \frac{k}{q} \left( \frac{2 + n}{q} \right) \right] + \left[ r \beta - \mu - p \alpha_r + (2 + n)(p - 1) + \left[ r \alpha_r - \lambda - r \alpha_r + (2 + n)(r - 1) \right] \right] \]

and let \( R \rightarrow \infty \). The right-hand side goes to zero while the left-hand side is assumed to be positive. A contradiction.

**Remark 2.1**

When the system (1) is diagonal (\( a_{21} = a_{31} = a_{32} = 0 \)), the inequalities (8-10) become:
\[ Y^r \leq A_{a, a}^{(1, i)} \quad Z^r \leq Y A_{b, a}^{(1, i)} \quad X^r \leq Z A_{b, a}^{(1, i)} \]

which combined leads to:
\[ X_{p q r - 1}^{-1} \leq \left[ A_{a, a}^{(1, i)} \right]^{p q r} \left[ A_{b, a}^{(1, i)} \right] \]
\[ Y_{p q r - 1}^{-1} \leq \left[ A_{b, a}^{(1, i)} \right]^{p q r} \left[ A_{a, a}^{(1, i)} \right] \]
\[ Z_{p q r - 1}^{-1} \leq \left[ A_{b, a}^{(1, i)} \right]^{p q r} \left[ A_{b, a}^{(1, i)} \right] \]

Now, if we use the scaled variables, we obtain:
\[ X_{p q r - 1}^{-1} \leq R^a, \quad Y_{p q r - 1}^{-1} \leq R^b, \quad Z_{p q r - 1}^{-1} \leq R^c \]

where:
\[ A_{v, u}^{(1, i)} = \left[ \begin{array}{cccc} \alpha - \alpha & - \frac{\lambda}{r} \frac{2 + n}{r} & p q - \kappa - q \alpha_r & (2 + n)(q - 1) \\ p \beta - \mu - p \alpha_r & (2 + n)(p - 1) \\ q \alpha_r - \lambda - q \alpha_r & (2 + n)(r - 1) \\ r \beta - \mu - p \alpha_r & (2 + n)(p - 1) \end{array} \right] \]
The choice of $\sigma_1 \leq 0$, $\sigma_2 \leq 0$ and $\sigma_3 \leq 0$ leads to:

$$
\begin{align*}
\lambda + 2 + p & \left( \mu + \alpha - \alpha + \frac{p\alpha_i - p\beta - 2pq + p\rho + p\kappa - pq\gamma}{pqr - 1} \right) \\
\mu + 2 + q & \left( \alpha + \kappa - \beta + q(\alpha_i - \gamma + \lambda) + r\left( \alpha_i - \alpha - 2 \right) \right) \\
\kappa + 2 + q & \left( \lambda + \alpha_i - \gamma + r(\alpha_i - \alpha + \mu - \rho + \kappa - 2p) \right)
\end{align*}
$$

Now, if we take the case studied by (Igibida and Kirane, 2002): $a = \beta = \gamma = 0$ and $a_1 = a_3 = a_6 = 2$, we obtain the same results as they did:

$$
n \leq \min \left\{ \frac{\lambda + 2 + p(\mu + 2 + p(\kappa + 2))}{pqr - 1}, \frac{\mu + 2 + q(\lambda + 2 + (\kappa + 2))}{pqr - 1}, \frac{\kappa + 2 + q(r(\mu + 2) + \lambda + 2)}{pqr - 1} \right\}
$$

If we take the case studied by (Renclawowicz, 1998): $a = \beta = \gamma = \lambda = \mu = k = 0$ and $a_1 = a_3 = a_6 = 2$, we obtain the same results as she did:

$$
n \leq \frac{2}{pqr - 1} \min \{pq + p + 1, rp + r + 1, qr + q + 1\}
$$

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