QUANTUM GAUGE TRANSFORMATIONS
AND BRAIDED STRUCTURE ON
QUANTUM PRINCIPAL BUNDLES

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Abstract. It is shown that every quantum principal bundle is braided, in
the sense that there exists an intrinsic braid operator twisting the functions
on the bundle. A detailed algebraic analysis of this operator is performed.
In particular, it turns out that the braiding admits a natural extension to
the level of arbitrary differential forms on the bundle. Applications of the
formalism to the study of quantum gauge transformations are presented. If
the structure group is classical then the braiding is fully compatible with the
bundle structure. This gives a possibility of a direct construction of the quan-
tum gauge bundle together with its braided quantum group structure and the
Corresponding differential calculus. In the general case, the braiding is not
completely compatible with the bundle structure, however the construction
gives a gauge coalgebra containing all the informations about quantum gauge
transformations.

1. Introduction

In this paper we shall prove that every quantum principal bundle is equipped
with a natural braided structure. This algebraic structure will be analyzed in
details, from various viewpoints. A particular attention will be given to the study of
relations with quantum gauge transformations, and compatibility with differential
calculus on quantum principal bundles. Our considerations are based on a general
theory of quantum principal bundles, developed in [3, 4].

Quantum principal bundles are noncommutative-geometric counterparts of
classical principal bundles. Quantum groups play the role of structure groups, and
the bundle and the base are appropriate quantum objects.

In the next section we shall start from a quantum principal bundle $P$ over a
quantum space $M$, and introduce a braid operator $\sigma_M : \mathcal{B} \otimes_M \mathcal{B} \to \mathcal{B} \otimes_M \mathcal{B}$. Here
$\mathcal{B}$ is a $*$-algebra representing $P$ and the tensor product is over the base space
algebra. The map $\sigma_M$ is algebraically expressed in terms of the right action map
$F : \mathcal{B} \to \mathcal{B} \otimes \mathcal{A}$ and the translation map $\tau : \mathcal{A} \to \mathcal{B} \otimes_M \mathcal{B}$, where $\mathcal{A}$ is a Hopf
$*$-algebra representing the structure group $G$. We shall then analyze interrelations
between $\sigma_M$ and maps forming the structure of a quantum principal bundle on $P$.

We shall prove that $\sigma_M$ is fully compatible with the product and the $*$-structure
on $\mathcal{B}$. On the other hand, it turns out that $\sigma_M$ is only partially compatible with
the action $F$ and the translation $\tau$.

Section 3 is devoted to the study of relations between the braiding and differential
calculi on the bundle. It turns out that for an arbitrary differential structure $\Omega(P)$
on the bundle there exists a natural extension $\tilde{\sigma}_M : \Omega(P) \otimes_M \Omega(P) \to \Omega(P) \otimes_M \Omega(P)$
of the operator $\sigma_M$, where now the tensor product is over the graded-differential
* -algebra $\Omega(M)$, playing the role of differential forms on the base in the general theory \[4\].

The extended operator is constructed with the help of the graded-differential extension $\hat{\tau}: \Gamma^\wedge \rightarrow \Omega(P) \otimes_M \Omega(P)$ of the translation map, and the pull-back map $\hat{F}: \Omega(P) \rightarrow \Omega(P) \otimes \Gamma^\wedge$. Here $\Gamma$ is a bicovariant \[12\] first-order *-calculus over $G$ and $\Gamma^\wedge$ is its differential envelope, representing the higher-order calculus. It is also possible to assume that the complete calculus on $G$ is based on the associated braided exterior algebra \[12\].

We prove that $\hat{\sigma}_M$ is completely compatible with the product and the *-structure on $\Omega(P)$, as well as with the differential map $d: \Omega(P) \rightarrow \Omega(P)$. However $\hat{\sigma}_M$ is only partially compatible with $\hat{F}$ and $\hat{\tau}$, because of the incompatibility appearing already at the level of spaces.

Then we pass to the applications of the developed techniques to the study of quantum gauge transformations. We shall first consider a special case when the structure group $G$ is classical. The classicality assumption is equivalent to the disappearance of mentioned incompatibility between $\sigma_M$ and $\tau, F$. This is further equivalent to the involutivity of $\sigma_M$.

Quantum principal bundles with classical structure groups give interesting examples \[7\] for the study of quantum phenomena, but still incorporating without any change various classical concepts. In ‘completely classical’ geometry the operator $\sigma_M$ reduces to the standard transposition.

The full compatibility between $\sigma_M$ and the bundle structure opens a possibility of constructing in the framework of the formalism the complete quantum gauge bundle $\mathfrak{ad}(P)$, which is generally understandable as a braided quantum group \[10\] over the quantum space $M$. If the structure group is classical, this bundle is represented by the *-algebra $\mathcal{L} \subseteq \mathcal{B} \otimes_M \mathcal{B}$ consisting of the elements invariant under the corresponding action of $G$ on $\mathcal{B} \otimes_M \mathcal{B}$.

The compatibility naturally extends to the level of differential structures if we assume that the calculus on $G$ is classical, too. Then we can construct the appropriate differential calculus on $\mathfrak{ad}(P)$. In classical geometry, the operator $\hat{\sigma}_M$ reduces to the standard graded-transposition.

In the last section we shall discuss quantum gauge bundles for general structure groups. In this general context the full quantum gauge bundle algebra $\mathcal{G}(P)$ can be constructed \[8\] by applying the structural analysis coming from Tannaka-Krein duality theory \[3\]. The present formalism gives a *-$V$-bimodule coalgebra $\mathcal{L}$, together with a natural fiber-preserving action map $\Delta: \mathcal{B} \rightarrow \mathcal{L} \otimes_M \mathcal{B}$. This is sufficient for the study of the transformation properties. It turns out \[8\] that the coalgebra $\mathcal{L}$ is naturally embeddable into the gauge bundle algebra $\mathcal{G}(P)$, via a coalgebra map $\iota: \mathcal{L} \rightarrow \mathcal{G}(P)$.

Essentially the same picture holds at the graded-differential level. Finally, we shall present some explicit calculations with connection forms, including gauge transformations of connections, covariant derivative and curvature.

2. Canonical Braiding

Let us consider a compact \[11\] matrix quantum group $G$, represented by a Hopf *-algebra $\mathcal{A}$. The elements of $\mathcal{A}$ play the role of polynomial functions on $G$. We shall denote by $\phi: \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{A}$ the coproduct, and by $\kappa: \mathcal{A} \rightarrow \mathcal{A}$ and $\epsilon: \mathcal{A} \rightarrow \mathbb{C}$ the antipode and the counit map respectively.
Let $M$ be a quantum space described by a $*$-algebra $\mathcal{V}$. Let $P = (\mathcal{B}, i, F)$ be a quantum principal $G$-bundle over $M$. Here $\mathcal{B}$ is a $*$-algebra representing the quantum space $P$, while $F: \mathcal{B} \to \mathcal{B} \otimes \mathcal{A}$ and $i: \mathcal{V} \to \mathcal{B}$ are $*$-homomorphisms playing the role of the dualized action of $G$ on $P$ and the dualized projection of $P$ on $M$ respectively.

The map $i$ is an isomorphism between $\mathcal{V}$ and the $F$-fixed-point subalgebra of $\mathcal{B}$. In what follows, we shall identify $\mathcal{V}$ with its image $\text{im}(i) \subseteq \mathcal{B}$. The algebra $\mathcal{B}$ is a bimodule over $\mathcal{V}$, in a natural manner. We shall use the symbol $\otimes$ for the tensor products over the algebra $\mathcal{V}$.

Let $X: \mathcal{B} \otimes_M \mathcal{B} \to \mathcal{B} \otimes \mathcal{A}$ be a $\mathcal{V}$-bimodule homomorphism given by

$$X(b \otimes q) = bF(q).$$

The freeness property of the action $F$ is equivalent to the surjectivity of this map. As explained in [6], the map $X$ is actually bijective. In other words, $P$ is interpretable as a Hopf-Galois extension.

Let $\tau: \mathcal{A} \to \mathcal{B} \otimes_M \mathcal{B}$ be the associated translation map [1]. This is a linear map given by inverting $X$ on $\mathcal{A}$, that is

$$\tau(a) = X^{-1}(1 \otimes a).$$

Throughout the various computations of this paper, we shall use a symbolic notation

$$\tau(a) = [a]_1 \otimes [a]_2.$$

According to [1] and the analysis of [3]-Subsection 6.6, the following identities hold:

$$\tau(a)^* = \tau[\kappa(a)]^*$$

$$[a]_1[a]_2 = \epsilon(a)1$$

$$(\text{id} \otimes F)\tau(a) = \tau(a^{(1)}) \otimes a^{(2)}$$

$$\tau(ac) = [c]_1[a]_1 \otimes [a]_2[c]_2$$

$$(F \otimes \text{id})\tau(a) = [a^{(2)}]_1 \otimes \kappa(a^{(1)}) \otimes [a^{(2)}]_2.$$
(iii) The product map \( \mu_M \) is compatible with the braid \( \sigma_M \), in the sense that

\[
\begin{align*}
\sigma_M (\mu_M \otimes \text{id}) &= (\text{id} \otimes \mu_M)(\sigma_M \otimes \text{id})(\text{id} \otimes \sigma_M) \\
\sigma_M (\text{id} \otimes \mu_M) &= (\mu_M \otimes \text{id})(\sigma_M \otimes \text{id})(\sigma_M \otimes \text{id}).
\end{align*}
\]

(iv) The algebra \( \mathcal{B} \) is \( \sigma_M \)-commutative. In other words

\[
\mu_M \sigma_M = \mu_M.
\]

\textbf{Proof.} A direct computation gives

\[
\begin{align*}
\sigma_M^{-1}(q \otimes b) &= \sum_k \sigma_M \{ [\kappa^{-1}(c_k)]_1 \otimes [\kappa^{-1}(c_k)]_2 q \sigma_k \} \\
&= \sum_k [\kappa^{-1}(c_k^{(2)})]^1 [\kappa^{-1}(c_k^{(2)})]^2 q \sigma_k [c_k^{(1)}]_1 \otimes [c_k^{(1)}]_2 = \sum_k q \sigma_k [c_k]_1 \otimes [c_k]_2 = q \otimes b,
\end{align*}
\]

and similarly

\[
\begin{align*}
\sigma_M^{-1} \sigma_M (b \otimes q) &= \sum_k \sigma_M \{ b_k q [c_k]_1 \otimes [c_k]_2 \} \\
&= \sum_k [\kappa^{-1}(c_k^{(2)})]^1 [\kappa^{-1}(c_k^{(2)})]^2 b_k q [c_k^{(1)}]_1 [c_k^{(1)}]_2 \\
&= \sum_k [\kappa^{-1}(c_k)]_1 [\kappa^{-1}(c_k)]_2 b_k q = b \otimes q.
\end{align*}
\]

We have applied equalities

\[
\begin{align*}
\sum_k b_k [c_k]_1 \otimes [c_k]_2 &= 1 \otimes b \\
\sum_k \tau \kappa^{-1}(c_k) b_k &= b \otimes 1.
\end{align*}
\]

Let us prove the compatibility relations between \( \sigma_M \) and \( \mu_M \). We compute

\[
\begin{align*}
\sigma_M (b q \otimes r) &= \sum_{k l} b_k q_l [r]_{1} [c_k]_1 \otimes [c_k]_2 [d_l]_2 \\
&= \sum_l (\text{id} \otimes \mu_M)(\sigma_M \otimes \text{id})(b \otimes q_l [d_l]_1 \otimes [d_l]_2) \\
&= (\text{id} \otimes \mu_M)(\sigma_M \otimes \text{id})(\text{id} \otimes \sigma_M)(b \otimes q \otimes r),
\end{align*}
\]

where \( \sum_l q_l \otimes d_l = F(q) \). Similarly,

\[
\begin{align*}
\sigma_M (b \otimes qr) &= \sum_k b_k q_r [c_k]_1 \otimes [c_k]_2 = \sum_k b_k q [c_k^{(1)}]_1 [c_k^{(1)}]_2 r [c_k^{(2)}]_1 \otimes [c_k^{(2)}]_2 \\
&= \sum_k (\mu_M \otimes \text{id})(\text{id} \otimes \sigma_M)(b_k q [c_k]_1 \otimes [c_k]_2 \otimes r) \\
&= (\mu_M \otimes \text{id})(\text{id} \otimes \sigma_M)(\sigma_M \otimes \text{id})(b \otimes q \otimes r).
\end{align*}
\]
Lemma 2.2. We have

\[ \sigma_M \ast = \ast \sigma_M^{-1}. \]
Proof. A direct computation gives

\[(\ast \sigma_M)(q \otimes b) = \sum_k \ast (b_k^* q^*[c_k^*]_1 \otimes [c_k^*]_2) = \sum_k [c_k^*]_2 \otimes [c_k^*]_1 q b_k\]

\[= \sum_k [\kappa^{-1}(c_k)]_1 \otimes [\kappa^{-1}(c_k)]_2 q b_k = \sigma_M^{-1}(q \otimes b),\]

where \(\sum_k b_k \otimes c_k = F(b)\).

As a direct consequence of the previous lemma, it follows that the formula

\[\tag{2.13} \quad (b \otimes q)^* = \sigma_M(q^* \otimes b^*)\]

defines a new \(*\)-involution which is a \(*\)-structure on the algebra \(B_2\). In a similar way, we can introduce the \(*\)-structure on the higher tensor powers \(B_n\). Let us denote by \(\chi\) the operators of the standard transposition.

**Lemma 2.3.** (i) The map \(\mu_M : B_2 \to B\) is a \(*\)-homomorphism.

(ii) The map \(\tau : A \to B_2\) is a \(*\)-homomorphism, and we have

\[\tag{2.14} \quad \sigma_M \tau(a) = \tau \kappa(a)\]

for each \(a \in A\). The following compatibility property holds:

\[\tag{2.15} \quad (\sigma_M \otimes \id)(\id \otimes \sigma_M)(\tau \otimes \id) = (\id \otimes \tau) \chi.\]

**Proof.** The hermicity and multiplicativity of \(\mu_M\) directly follow from the \(\sigma_M\)-symmetricity property. We have

\[\sigma_M \tau(a) = \sigma_M([a]_1 \otimes [a]_2) = [a^{(2)}]_1 [a^{(2)}]_2 [\kappa(a^{(1)})]_1 \otimes [\kappa(a^{(1)})]_2 = \tau \kappa(a).\]

Furthermore,

\[(\sigma_M \otimes \id)(\id \otimes \sigma_M)(\tau \otimes \id)(a \otimes b) = (\sigma_M \otimes \id)
\left(\left[a^{(1)}\right]_1 \otimes [a^{(1)}]_2^* b \left[a^{(2)}\right]_1 \otimes [a^{(2)}]_2^*\right) = [a^{(2)}]_1 [a^{(2)}]_2 b [a^{(3)}]_1 [\kappa(a^{(1)})]_1 \otimes [\kappa(a^{(1)})]_2 \otimes [a^{(3)}]_2 = b \otimes [a]_1 \otimes [a]_2,\]

which proves (2.15). Let us check the multiplicativity of \(\tau\). Elementary transformations give

\[\tau(a) \tau(c) = [a]_1 \sigma_M([a]_2 \otimes [c]_1)[c]_2 = [a^{(1)}]_1 [a^{(1)}]_2 [c]_1 [a^{(2)}]_1 \otimes [a^{(2)}]_2 [c]_2 = [c]_1 [a]_1 \otimes [a]_2 \tau(ac).\]

Finally, applying the definition of the \(*\)-structure in \(B_2\) we obtain

\[\tau(a)^* = \sigma_M([\kappa(a)^*]_1 \otimes [\kappa(a)^*]_2) = \tau \kappa([\kappa(a)^*]) = \tau(a^*),\]

and we conclude that \(\tau\) is a \(*\)-homomorphism. \(\square\)

**Lemma 2.4.** We have

\[\tag{2.16} \quad (\sigma_M \otimes \id)(\id \otimes \chi)(F \otimes \id) = (\id \otimes F) \sigma_M.\]

**Proof.** We compute

\[(\sigma_M \otimes \id)(\id \otimes \chi)(F \otimes \id)(b \otimes q) = \sum_k \sigma_M(b_k \otimes q) \otimes c_k = \sum_k b_k q [c_k^{(1)}]_1 \otimes [c_k^{(1)}]_2 \otimes c_k^{(2)} \]

\[= \sum_k b_k q [c_k]_1 \otimes F[c_k]_2 = (\id \otimes F) \sigma_M(b \otimes q). \square\]
For each \( n \geq 2 \) let us denote by \( F_n : B_n \to B_n \otimes A \) a natural action defined as the \( n \)-fold direct product of \( F \) with itself. The maps \( F_n \) are \(*\)-\( \mathcal{V} \)-bimodule homomorphisms.

Let \( \mathcal{L} \subseteq B_2 \) be a \( \mathcal{V} \)-submodule consisting of \( F_2 \)-invariant elements. This space is closed under the standard conjugation, however generally it will be not closed under the \( \sigma_M \)-induced conjugation. Furthermore, we have a natural projection map \( p_{\mathcal{L}} : B_2 \to \mathcal{L} \), given by

\[
p_{\mathcal{L}} = (\text{id} \otimes h)F_2
\]

where \( h : A \to \mathbb{C} \) is the Haar measure \([11]\) of \( G \). This map is \( \mathcal{V} \)-linear and in particular \( \mathcal{L} \) is a direct summand in the bimodule \( B_2 \). On the other hand, \( \mathcal{L} \) is generally not a subalgebra of \( B_2 \).

Let us now consider the map \( \delta_3 : B \to B_3 \) given by

\[
\delta_3(b) = (\text{id} \otimes \tau)F(b) = \sum_k b_k \otimes [c_k]_1 \otimes [c_k]_2.
\]

**Lemma 2.5.** The map \( \delta_3 \) is a \(*\)-homomorphism and we have

\[
\delta_3(B) \subseteq \mathcal{L} \otimes_M B.
\]

**Proof.** Multiplicativity and hermicity of \( \delta_3 \) directly follow from (2.15). Let us check the above inclusion. We have

\[
(F_2 \otimes \text{id})\delta_3(b) = \sum_k F_2(b_k \otimes [c_k]_1) \otimes c_k = \sum_k b_k \otimes [c_k]_1 \otimes [c_k]_2 = \sum_k b_k \otimes [c_k]_1 \otimes 1 \otimes [c_k]_2;
\]

and we conclude that (2.13) holds. \( \square \)

Now let us denote by \( \Delta : B \to \mathcal{L} \otimes_M B \) the map \( \delta_3 \) with the restricted codomain. The diagram

\[
\begin{array}{ccc}
\mathcal{L} \otimes_M B \otimes A & \xleftarrow{\Delta \otimes \text{id}} & B \otimes A \\
\text{id} \otimes F & \nearrow F \\
\mathcal{L} \otimes_M B & \xleftarrow{\Delta} & B
\end{array}
\]

is commutative. Indeed,

\[
(id^2 \otimes F)\delta_3 = (id^2 \otimes F)(id \otimes \tau)F = (id \otimes \tau \otimes id)(id \otimes \phi)F = (id \otimes \tau \otimes id)(F \otimes id)F = (\delta_3 \otimes id)F.
\]

**Lemma 2.6.** The following identities hold:

\[
\mu_M^2(id^2 \otimes \sigma_M)(\delta_3 \otimes id) = \mu_M \otimes 1
\]

(2.21)

\[
\mu_M^2(\sigma_M \otimes id^2)(\delta_3 \otimes id) = 1 \otimes \mu_M.
\]

(2.22)
Proof. A direct computation gives
\[
\mu^2_M(\text{id}^2 \otimes \sigma_M)(\delta_3(b) \otimes q) = \sum_k (b_k \otimes [c_k]_1)\sigma_M([c_k]_2 \otimes q) = \sum_k b_k (\text{id} \otimes \mu_M)(\sigma_M \otimes \text{id})(\tau(c_k) \otimes q) = \sum_k b_k q \otimes [c_k]_1[c_k]_2 = bq \otimes 1.
\]

Similarly, we find
\[
\mu^2_M(\sigma_M \otimes \text{id}^2)(\delta_3(b) \otimes q) = \sum_k \sigma_M(b_k \otimes [c_k]_1)([c_k]_2 \otimes q) = \sum_k \{b_k[c_k]_1^2[c_k]_1 \otimes [c_k]_2^1\}([c_k]_2 \otimes q) = \sum_k b_k[c_k]_1 \otimes [c_k]_2 q = 1 \otimes bq.
\]
We have applied (2.17) and the \(\sigma_M\)-commutativity of \(B\).

Since the product \(\mu_M\) intertwines \(F_2\) and \(F\), it follows that
\[
(\mu_M(L)) = \mathcal{V}.
\]

We shall denote by \(\epsilon_M : \mathcal{L} \to \mathcal{V}\) the corresponding restriction map. The map \(\epsilon_M\) is hermitian, relative to the standard \(*\)-involution on \(B_2\). The intertwining property (2.20) implies
\[
(\Delta \otimes \text{id})(\mathcal{L}) \subseteq \mathcal{L} \otimes_M \mathcal{L}.
\]

Let \(\phi_M : \mathcal{L} \to \mathcal{L} \otimes_M \mathcal{L}\) be the corresponding restriction map.

**Proposition 2.7.** The following identities hold:
\[
\begin{align*}
(\epsilon_M \otimes \text{id})\phi_M &= (\text{id} \otimes \epsilon_M)\phi_M = \text{id} \\
(\epsilon_M \otimes \text{id})\Delta &= \text{id} \\
(\epsilon_M \otimes \text{id})\Delta &= (\phi_M \otimes \text{id})\Delta \\
(\phi_M \otimes \text{id})\phi_M &= (\text{id} \otimes \phi_M)\phi_M.
\end{align*}
\]

Proof. We have
\[
(\epsilon_M \otimes \text{id})\Delta(b) = \sum_k b_k[c_k]_1 \otimes [c_k]_2 = \sum_k b_k[c_k]_1[c_k]_2 = b = \sum_k b_k \otimes [c_k]_1[c_k]_2,
\]
which proves (2.25)-(2.26). Furthermore,
\[
(\phi_M \otimes \text{id})\Delta(b) = \sum_k b_k \otimes \tau(c_k^{(1)}) \otimes \tau(c_k^{(2)}) = \sum_k b_k \otimes [c_k]_1[\tau([c_k]_2)],
\]
which proves (2.27)-(2.28).

From equalities (2.7) and (2.13) it follows that the map \(X : B_2 \to B \otimes A\) is a \(*\)-isomorphism. Actually, the multiplicativity property of \(X\), together with the definition (2.11) of the product in \(B_2\), completely characterizes the braiding \(\sigma_M\).
Moreover, we can define a sequence of $^\ast$-isomorphisms $X_n: B_n \to B \otimes \{ A \otimes^n \}$ by equalities

$$X_1 = X \quad X_{n+1}(b \otimes q_n) = (X \otimes \text{id}^n)(b \otimes X_n(q_n)).$$

We can also introduce generalized translation maps $\tau_n: A \otimes^n \to B_{n+1}$ by taking the restricted inverse of $X_n$. It follows that

$$\tau_n(a_1 \otimes \cdots \otimes a_n) = (\text{id} \otimes \mu_M \otimes \cdots \otimes \mu_M \otimes \text{id})(\tau(a_1) \otimes \cdots \otimes \tau(a_n)).$$

3. Extensions to Differential Structures

In this section we are going to prove that the braid operator $\sigma_M$ is naturally compatible with an arbitrary differential calculus on a quantum principal bundle $P$. All constructions of the previous section will be incorporated at the level of graded-differential structures.

Let $\Gamma$ be a given bicovariant first-order $^\ast$-calculus over $G$. Let us denote by $\wp_\Gamma$ and $\ell_\Gamma: \Gamma \to A \otimes \Gamma$ be the corresponding right and left action maps. We shall denote by $\Gamma_{\text{inv}}$ the space of left-invariant elements of $\Gamma$. The formula

$$\pi(a) = \kappa(a^{(1)})d(a^{(2)})$$

defines a projection map $\pi: A \to \Gamma_{\text{inv}}$. Let $\varpi: \Gamma_{\text{inv}} \to \Gamma_{\text{inv}} \otimes A$ be the corresponding adjoint action. This map coincides with the restriction of the right action map on the left-invariant elements. Also, it is given by projecting the adjoint action of $G$ onto $\Gamma_{\text{inv}}$. Explicitly,

$$\varpi\pi = (\pi \otimes \text{id})\text{ad}, \quad \text{ad}(a) = a^{(2)} \otimes \kappa(a^{(1)})a^{(3)}.$$ 

We shall assume that a higher-order calculus on $G$ is described by the universal envelope $\text{Appendix B}$ of $\Gamma$. The same considerations are applicable to the braided exterior algebra $\text{Appendix B}$ associated to $\Gamma$, relative to the canonical braid operator $\sigma: \Gamma_{\text{inv}} \otimes \Gamma_{\text{inv}} \otimes A$ be the corresponding adjoint action. This map explicitly given by

$$\sigma(\eta \otimes \vartheta) = \sum_k \vartheta_k \otimes (\eta \circ c_k),$$

where $\sum_k \vartheta_k \otimes c_k = \varpi(\vartheta)$ and $\circ: \Gamma_{\text{inv}} \otimes A \to \Gamma_{\text{inv}}$ is the canonical right $A$-module structure, defined by

$$\pi(a) \circ b = \pi(ab) - \epsilon(a)\pi(b).$$

The Hopf $^\ast$-algebra structure is naturally liftable from $A$ to a graded differential Hopf $^\ast$-algebra structure on $\Gamma^\wedge$. In particular, we have

$$\hat{\vartheta}(\vartheta) = \ell_\Gamma(\vartheta) + \wp_\Gamma(\vartheta)$$

$$\epsilon(\Gamma) = \{0\} \quad \kappa[\pi(a)] = -\pi(a^{(2)})\kappa(a^{(1)})a^{(3)}$$

for each $\vartheta \in \Gamma$ and $a \in A$. Here $\hat{\vartheta}: \Gamma^\wedge \to \Gamma^\wedge \otimes \Gamma^\wedge$ is the extended coproduct map, and we have denoted by the same symbols the extended counit and the antipode.

Let us consider a quantum principal bundle $P = (B, i, F)$ and let $\Omega(P)$ be a graded-differential $^\ast$-algebra representing a differential calculus on $P$, in the framework of the general theory $\text{Appendix B}$. By definition, $\Omega(P)$ is generated as a differential algebra by $B = \Omega^0(P)$ and there exists a (necessarily unique) graded-differential
homomorphism \( \hat{F} : \Omega(P) \to \Omega(P) \otimes \Gamma^\wedge \) extending the right action map \( F \). This map is hermitian, and also satisfies

\[
(id \otimes \hat{\varphi}) \hat{F} = (\hat{F} \otimes id) \hat{F}.
\]

Let \( \mathfrak{hor}(P) \subseteq \Omega(P) \) be a graded *-algebra representing horizontal forms. Horizontal forms are characterized by having trivial differential properties along vertical fibers, in other words

\[
\mathfrak{hor}(P) = \hat{F}^{-1}(\Omega(P) \otimes A).
\]

Finally, a graded-differential *-algebra \( \Omega(M) \subseteq \mathfrak{hor}(P) \) describing a differential calculus on the base \( M \) is defined as a \( \hat{F} \)-fixed point subalgebra of \( \Omega(P) \).

There exists a natural right action map \( F^\wedge : \Omega(P) \to \Omega(P) \otimes A \), defined as the *-homomorphism obtained from \( \hat{F} \) by eliminating from the image the summands with strictly positive second degrees. It turns out that \( \mathfrak{hor}(P) \) is \( F^\wedge \)-invariant.

We shall use the symbol \( \otimes_M \) to denote a tensor product over the algebra \( \Omega(M) \). The map \( \chi \) admits a natural extension \( \hat{X} : \Omega(P) \otimes_M \Omega(M) \to \Omega(P) \otimes \Gamma^\wedge \), given by

\[
\hat{X}(\varphi \otimes w) = \varphi \hat{F}(w).
\]

As explained in [6], this map is also bijective. This fact allows us to extend the translation map \( \tau : A \to B \otimes_M B \) to the level of differential algebras, by inverting \( \hat{X} \) on \( \Gamma^\wedge \). In such a way we obtain a map \( \tilde{\tau} : \Gamma^\wedge \to \Omega(P) \otimes_M \Omega(P) \). We shall use the same symbolic notation \( \tilde{\tau}(\vartheta) = [\vartheta]_1 \otimes [\vartheta]_2 \) for this extended map.

Let us define \( \Omega(M) \)-bimodules \( \mathcal{W}_n \) as \( n \)-fold tensor products

\[
\mathcal{W}_n = \Omega(P) \otimes_M \cdots \otimes_M \Omega(P), \tag{3.3}
\]

where \( n \geq 2 \). The differential map \( d : \Omega(P) \to \Omega(P) \) and the *-involution are naturally extendible from \( \Omega(P) \) to the spaces \( \mathcal{W}_n \). Furthermore, there exist the natural action maps \( \hat{F}^\wedge_n : \mathcal{W}_n \to \mathcal{W}_n \otimes \Gamma^\wedge \) obtained by taking the graded-direct products of the action \( \hat{F} \) with itself. The maps \( \hat{F}^\wedge_n \) are hermitian, and intertwine the corresponding differentials.

We shall denote by \( \chi \) the standard graded-twist operations. The graded adjoint action map \( \text{ad} : \Gamma^\wedge \to \Gamma^\wedge \otimes \Gamma^\wedge \) is given by

\[
\text{ad}(\vartheta) = \chi \left\{ \kappa(\vartheta^{(1)}) \otimes \vartheta^{(2)} \right\} \vartheta^{(3)}.
\]

The map \( \tilde{\tau} : \Gamma \to \mathcal{W}_2 \) possesses the following main algebraic properties:

\[
(id \otimes \hat{F}) \tilde{\tau}(\vartheta) = \tilde{\tau}(\vartheta^{(1)}) \otimes \vartheta^{(2)} \tag{3.4}
\]

\[
\tilde{\tau}(\vartheta)^* = \tilde{\tau}(\kappa(\vartheta)^*) \tag{3.5}
\]

\[
\tilde{\tau}[d(\vartheta)] = d\tilde{\tau}(\vartheta) \tag{3.6}
\]

\[
[\vartheta]_1 [\vartheta]_2 = \epsilon(\vartheta) [\vartheta]_1 \tag{3.7}
\]

where \( \mu_M : \mathcal{W}_2 \to \Omega(P) \) is the product map. It turns out that the elements from \( \text{im}(\tilde{\tau}) \) graded-commute with differential forms from \( \Omega(M) \).

Let us consider a canonical filtration \( \mathcal{F} = \left\{ \Omega_k(P) ; k \geq 0 \right\} \) of the algebra \( \Omega(P) \), induced by the action \( \hat{F} \) and the grading on \( \Gamma^\wedge \). Let \( \tilde{\sigma}_M : \mathcal{W}_2 \to \mathcal{W}_2 \) be a bimodule homomorphism defined by

\[
\tilde{\sigma}_M(w \otimes u) = \sum_{\alpha} w_{\alpha}(\mu_M \otimes id)(id \otimes \tilde{\tau}) \chi \left\{ \vartheta_{\alpha} \otimes u \right\}, \tag{3.8}
\]
where \( \sum w_\alpha \otimes \partial_\alpha = \tilde{F}(w) \). By definition, \( \tilde{\sigma}_M \) extends \( \sigma_M : \mathcal{B}_2 \to \mathcal{B}_2 \). The basic properties of this map are collected in the following

**Proposition 3.1.** *(i)* The map \( \tilde{\sigma}_M \) is bijective, and its inverse is given by

\[
\tilde{\sigma}_M^{-1}(u \otimes w) = \sum_{\alpha} (\text{id} \otimes \mu_M)(\tilde{\tau}^{-1} \otimes \text{id}) \chi \{ u w_\alpha \otimes \partial_\alpha \}.
\]

Furthermore, \( \tilde{\sigma}_M \) is compatible with the filtration \( \mathcal{F} \), in the sense that

\[
\tilde{\sigma}_M[\Omega_k(P) \otimes_M \Omega(P)] = [\Omega(P) \otimes_M \Omega_k(P)],
\]

for each \( k \in \mathbb{N} \cup \{0\} \). In particular \( \tilde{\sigma}_M : \text{hor}(P) \otimes_M \Omega(P) \leftrightarrow \Omega(P) \otimes_M \text{hor}(P) \).

(ii) The map \( \tilde{\sigma}_M \) satisfies the braid equation

\[
(\tilde{\sigma}_M \otimes \text{id})(\text{id} \otimes \tilde{\sigma}_M)(\tilde{\sigma}_M \otimes \text{id}) = (\text{id} \otimes \tilde{\sigma}_M)(\tilde{\sigma}_M \otimes \text{id})(\text{id} \otimes \tilde{\sigma}_M).
\]

(iii) The product map \( \mu_M : \mathcal{W}_2 \to \Omega(P) \) is compatible with \( \tilde{\sigma}_M \), in a natural manner:

\[
\tilde{\sigma}_M(\mu_M \otimes \text{id}) = (\text{id} \otimes \mu_M)(\tilde{\sigma}_M \otimes \text{id})(\text{id} \otimes \tilde{\sigma}_M),
\]

\[
\tilde{\sigma}_M(\text{id} \otimes \mu_M) = (\mu_M \otimes \text{id})(\text{id} \otimes \tilde{\sigma}_M)(\tilde{\sigma}_M \otimes \text{id}).
\]

(iv) The algebra \( \Omega(P) \) is \( \tilde{\sigma}_M \)-commutative. In other words

\[
\mu_M \tilde{\sigma}_M = \mu_M.
\]

We also have

\[
* \tilde{\sigma}_M = \tilde{\sigma}_M^{-1} *.
\]

(v) The braiding \( \tilde{\sigma}_M \) commutes with the differential \( d : \mathcal{W}_2 \to \mathcal{W}_2 \).

**Proof.** A direct computation gives

\[
d\tilde{\sigma}_M(w \otimes u) = \sum_{\alpha} d(w_\alpha)(\mu_M \otimes \text{id})(\text{id} \otimes \tilde{\tau})\chi(\partial_\alpha \otimes u)
\]

\[
+ (-1)^{\partial w} \sum_{\alpha} w_\alpha(\mu_M \otimes \text{id})(\text{id} \otimes \tilde{\tau})\chi(\partial_\alpha \otimes du)
\]

\[
+ \sum_{\alpha} (-1)^{\partial w} w_\alpha(\mu_M \otimes \text{id})(\text{id} \otimes \tilde{\tau})\chi(d\partial_\alpha \otimes u)
\]

\[
= \tilde{\sigma}_M(dw \otimes u) + (-1)^{\partial w} \tilde{\sigma}_M(w \otimes du) = \tilde{\sigma}_Md(w \otimes u).
\]

Property (3.10) directly follows from definitions of \( \tilde{\sigma}_M \) and the filtration \( \mathcal{F} \). The rest of the proof is essentially the same as for the operator \( \sigma_M : \mathcal{B}_2 \to \mathcal{B}_2 \), the only additional moment is the appearance of graded-twists at the appropriate places. \( \square \)

With the help of the operator \( \tilde{\sigma}_M \) we can naturally introduce the \(*\)-algebra structure on the \( \Omega(M)\)-bimodules \( \mathcal{W}_n \). Property (v) implies that all differentials \( d : \mathcal{W}_n \to \mathcal{W}_n \) are hermitian antiderivations.

Let us consider an \( \Omega(M)\)-submodule \( \tilde{\mathcal{L}} \subseteq \mathcal{W}_2 \) consisting of all \( \tilde{F}_2 \)-invariant elements. The space is naturally graded, and we have \( \tilde{\mathcal{L}}^0 = \mathcal{L} \). Furthermore, \( \tilde{\mathcal{L}} \) is closed under the standard conjugation and the action of the differential map. However, \( \tilde{\mathcal{L}} \) is generally not a subalgebra of \( \mathcal{W}_2 \). The product map induces a bimodule
homomorphism $\epsilon_M : \hat{\mathcal{L}} \to \Omega(M)$. This map satisfies
\begin{align}
\epsilon_M \ast & = \ast \epsilon_M \\
\epsilon_M d & = d \epsilon_M.
\end{align}

The map $\Delta : \mathcal{B} \to \mathcal{L} \otimes_M \mathcal{B}$ admits a natural extension $\hat{\Delta} : \Omega(P) \to \hat{\mathcal{L}} \otimes_M \Omega(P)$ given by
\begin{equation}
\hat{\Delta}(w) = \sum_{\alpha} w_{\alpha} \otimes [\vartheta_{\alpha}]_1 \otimes [\vartheta_{\alpha}]_2.
\end{equation}
As a map between $\Omega(P)$ and $\mathcal{W}^3$, this map is a $\ast$-homomorphism. The map $\hat{\Delta}$ preserves the action $\hat{\mathcal{F}}$. In other words, the diagram
\begin{equation}
\begin{array}{c}
\hat{\mathcal{L}} \otimes_M \Omega(P) \otimes \Gamma^\wedge & \xrightarrow{\hat{\Delta} \otimes \text{id}} & \Omega(P) \otimes \Gamma^\wedge \\
\xrightarrow{\text{id} \otimes \hat{\mathcal{F}}} & & \xleftarrow{\hat{\Delta}} \\
\hat{\mathcal{L}} \otimes_M \Omega(P) & & \Omega(P)
\end{array}
\end{equation}
is commutative. The formula
\begin{equation}
\sum_{w \in n} \hat{\phi}_M(w \otimes u) = \hat{\Delta}(w) \otimes u
\end{equation}
determines a differential bimodule homomorphism $\hat{\phi}_M : \hat{\mathcal{L}} \to \hat{\mathcal{L}} \otimes_M \hat{\mathcal{L}}$.

**Proposition 3.2.** The following identities hold:
\begin{align}
(\epsilon_M \otimes \text{id}) \hat{\phi}_M & = (\text{id} \otimes \epsilon_M) \hat{\phi}_M = \text{id} \\
(\epsilon_M \otimes \text{id}) \hat{\Delta} & = \text{id} \\
(\text{id} \otimes \hat{\Delta}) \hat{\Delta} & = (\phi_M \otimes \text{id}) \hat{\Delta} \\
(\hat{\phi}_M \otimes \text{id}) \hat{\phi}_M & = (\text{id} \otimes \hat{\phi}_M) \hat{\phi}_M.
\end{align}
In other words, $\hat{\mathcal{L}}$ is a counital differential coalgebra over $\Omega(M)$ naturally acting on the calculus $\Omega(P)$.

Let us also mention that all the maps $X_n$ admit natural extensions $\hat{X}_n : \mathcal{W}_{n+1} \leftrightarrow \Omega(P) \otimes \underbrace{\Gamma^\wedge \otimes \cdots \otimes \Gamma^\wedge}_n$ which are isomorphisms of graded-differential $\ast$-algebras. The corresponding partial inverses $\hat{\tau}_n$ are given by
\begin{equation}
\hat{\tau}_n(\vartheta_1 \otimes \cdots \otimes \vartheta_n) = (\text{id} \otimes \underbrace{\mu_M \otimes \cdots \otimes \mu_M}_{n} \otimes \text{id})(\hat{\tau}(\vartheta_1) \otimes \cdots \otimes \hat{\tau}(\vartheta_n)).
\end{equation}

4. **Quantum Gauge Bundles for Classical Structure Groups**

4.1. **The Level of Spaces**

As we have seen in the previous sections, the braid operators $\sigma_M$ and $\hat{\sigma}_M$ are fully compatible with the product maps $\mu_M$, the $\ast$-structure and the differential $d : \Omega(P) \to \Omega(P)$. Interestingly, $\sigma_M$ is not completely compatible with the action map $\mathcal{F} : \mathcal{B} \to \mathcal{B} \otimes \mathcal{A}$. It turns out that the full compatibility between $\sigma_M$ and $\mathcal{F}$
holds only in a special case when $G$ is a classical Lie group. Moreover, differential extensions $\tilde{\sigma}_M$ and $\tilde{F}$ will be compatible only if the calculus on such a classical structure group $G$ is assumed to be classical, too.

**Proposition 4.1.** Let $G$ be a compact matrix quantum group, and $P = (B, i, F)$ a quantum principal $G$-bundle over a quantum space $M$. Then the following conditions are equivalent:

(i) The algebra $\mathcal{A}$ is commutative. In other words, $G$ is a classical compact Lie group.

(ii) The following equality holds

\[(\text{id} \otimes \chi)(F \otimes \text{id})\sigma_M = (\sigma_M \otimes \text{id})(\text{id} \otimes F).
\]

(iii) The following equality holds

\[(\tau \otimes \text{id})\chi = (\text{id} \otimes \sigma_M)(\sigma_M \otimes \text{id})(\text{id} \otimes \tau).
\]

(iv) The map $\sigma_M$ is involutive.

**Proof.** Let us first check equalities (4.1)–(4.2). Direct transformations give

\[(\text{id} \otimes \sigma_M)(\sigma_M \otimes \text{id})(b \otimes \tau(a)) = \sum_k (\text{id} \otimes \sigma_M)[b_k[a]_1 \otimes [c_k]_2] \otimes [a]_2
\]

\[= \sum_k b_k[a]_1[\epsilon^{(1)}_k]_1 \otimes [\epsilon^{(2)}_k]_2 \otimes [\epsilon^{(2)}_k]_2 \longrightarrow \sum_k b_k \otimes a \otimes c_k,
\]

where $\sum_k b_k \otimes c_k = F(b)$ and we have applied the transformation $X_2$ at the end. Similarly,

\[\tau(a) \otimes b = [a]_1 \otimes [a]_2 \rightarrow \sum_k b_k \otimes ac_k.
\]

Therefore, equality (4.3) holds if and only if $\mathcal{A}$ is commutative. Furthermore,

\[(\text{id} \otimes \chi)(F \otimes \text{id})\sigma_M(b \otimes q) = \sum_k (\text{id} \otimes \chi)(F \otimes \text{id})[b_k q[c_k]_1 \otimes [c_k]_2]
\]

\[= \sum_{kl} b_k q_k[c_k]_1 \otimes [c_k]_2 \otimes c_k \rightarrow \sum_{kl} b_k q_k \otimes c_k \otimes c_k
\]

where $\sum_k q_k \otimes d_k = F(q)$. On the other hand,

\[(\sigma_M \otimes \text{id})(\text{id} \otimes F)(b \otimes q) = \sum_{kl} b_k q_k[c_k]_1 \otimes [c_k]_2 \otimes d_k \rightarrow \sum_{kl} b_k q_k \otimes c_k \otimes d_k.
\]

The obtained expressions coincide if and only if $\mathcal{A}$ is commutative. Finally, let us analyze the question of the involutivity of $\sigma_M$. We compute

\[\sigma_M^2(b \otimes q) = \sum_k (\sigma_M(b_k q[c_k]_1 \otimes [c_k]_2)) = \sum_{kl} b_k q_k[c_k]_1 \otimes [c_k]_2 \tau[c_k]_1 \otimes d_k \kappa(c_k)
\]

\[= \sum_{kl} b_k q_k \tau[c_k]_1 \otimes d_k \kappa(c_k)
\]

Therefore, $\sigma_M$ will be involutive if and only if

\[\kappa(a^{(1)})ca^{(2)} = e(a):c
\]

for each $c, a \in \mathcal{A}$. This is further equivalent to the commutativity of $\mathcal{A}$.  \(\square\)
It is worth noticing that for involutive braids $\sigma_M$ equalities (4.1)–(4.2) are equivalent to (2.16)–(2.15) respectively. Throughout the rest of this section, we shall assume that $G$ is a classical compact Lie group. The bundle $P$ and the space $M$ are arbitrary.

**Proposition 4.2.** The diagram

\[
\begin{array}{ccc}
B_2 & \xrightarrow{F_2} & B_2 \otimes A \\
\sigma_M & \downarrow & \sigma_M \otimes \text{id} \\
B_2 & \xrightarrow{F_2} & B_2 \otimes A
\end{array}
\]

is commutative. In particular, the action $F_2: B_2 \to B_2 \otimes A$ is a $\ast$-homomorphism and $L$ is a $\ast$-subalgebra of $B_2$.

**Proof.** We compute

\[
F_2\sigma_M (b \otimes q) = \sum_k F_2(b_k q [c_k^1]_1 \otimes [c_k^2]_2) = \sum_{kl} b_k q [c_k^{(3)}]_1 \otimes [c_k^{(3)}]_2 \otimes c_k^{(1)} d_l \kappa(c_k^{(2)}) c_k^{(4)} = \sum_{kl} b_k q [c_k^{(1)}]_1 \otimes [c_k^{(1)}]_2 \otimes c_k d_l = \sum_{kl} \sigma_M (b_k \otimes d_l) \otimes c_k d_l.
\]

The algebra $L$ is covariant relative to the action of the braid operator $\sigma_M$, in the sense that

\[
\begin{align*}
(\sigma_M \otimes \text{id})(\text{id} \otimes \sigma_M)(L \otimes M B) &= B \otimes_M L \\
(\text{id} \otimes \sigma_M)(\sigma_M \otimes \text{id})(B \otimes_M L) &= L \otimes_M B.
\end{align*}
\]

There exists the induced braiding $\Sigma: L \otimes_M L \to L \otimes_M L$, given by the restriction of

\[
\Sigma = (\text{id} \otimes \sigma_M \otimes \text{id})(\sigma_M \otimes \sigma_M)(\text{id} \otimes \sigma_M \otimes \text{id}): B_4 \to B_4.
\]

In particular, we have a natural $\ast$-algebra structure in all tensor products involving $L$ and $B_k$.

It turns out that the braiding $\Sigma$ together with the coalgebra structure defines a braided quantum group $\mathbb{L}$ structure on $L$. Indeed, the map $\phi_M: L \to L \otimes_M L$ is a $\ast$-homomorphism. Furthermore,

\[
\begin{align*}
\Sigma &= \Sigma^{-1} \\
\ast \Sigma &= \Sigma\ast
\end{align*}
\]

as follows from the established full compatibility between the maps $F$ and $\tau$, and the braiding $\sigma_M$.

From diagram (4.3) it follows that $L$ is $\sigma_M$-invariant. Let $\kappa_M: L \to L$ be the corresponding restriction map.

**Proposition 4.3.** Endowed with maps $\{\kappa_M, \epsilon_M, \phi_M, \Sigma\}$ the $\ast$-algebra $L$ becomes a braided-Hopf $\ast$-algebra over the quantum space $M$.

**Proof.** It is sufficient to check the antipode axiom. However this is a direct consequence of Lemma 2.6. \qed
Geometrically speaking, \( L \) represents a quantum gauge bundle \( \mathfrak{a}(P) \) associated to \( P \). The elements of \( L \) are interpretable as ‘smooth functions’ on the quantum space \( \mathfrak{a}(P) \). The inclusion \( V \ni f \mapsto f \otimes 1 \in L \) is interpretable as the dualized fibering of \( \mathfrak{a}(P) \) over \( M \), and \( \mathfrak{a}(P) \) is actually a ‘bundle of groups’. This is formalized in the introduced group structure. In particular, the map \( \epsilon_M \) is interpretable as the unit section of the bundle \( \mathfrak{a}(P) \). The homomorphism \( \Delta \) is the dualized left fiberwise bundle action of \( \mathfrak{a}(P) \) on \( P \). The diagram (2.20) expresses the idea that gauge transformations preserve the structure of the bundle. In classical geometry \( \mathfrak{a}(P) \) will be the standard quantum gauge bundle, and actual gauge transformations are naturally interpretable as smooth sections of \( \mathfrak{a}(P) \).

Following the classical geometry, it is natural to define gauge transformations as unital \( V \)-linear *-homomorphisms \( \gamma : L \to V \). In our context it is necessary to impose an additional compatibility condition

(4.5) \[
\gamma(\rho) b = \sum_j b_j \gamma_j(\rho),
\]

where \( \sum_j b_j \otimes \rho_j = (\sigma_M \otimes \text{id})(\text{id} \otimes \sigma_M)(\rho \otimes b) \).

Let \( C(P) \) be the set of all maps \( \gamma \) of the described type. This set is a group, in a natural manner. The product and the inverse are given by

(4.6) \[
\gamma \gamma' = (\gamma \otimes \gamma') \phi_M, \quad \gamma^{-1} = \gamma \kappa^{-1}_M,
\]

while the unit element is given by the counit map.

The elements of \( C(P) \) naturally act on \( P \), by *-automorphisms of \( B \) given by the formula

(4.7) \[
\gamma \cdot b = (\gamma \otimes \text{id}) \Delta(b).
\]

The following equalities hold

(4.8) \[
(\gamma \gamma') \cdot b = \gamma \cdot (\gamma' \cdot b) \quad F(\gamma \cdot b) = \sum_k (\gamma \cdot b_k) \otimes c_k.
\]

The group \( C(P) \) is generally insufficient to cover all symmetry properties inherent in the action \( \Delta \), because \( \mathfrak{a}(P) \) is a quantum object not reducible to a classical group. The presence of the braiding \( \Sigma \) reflects the quantum nature of \( P \) and \( \mathfrak{a}(P) \).

4.2. Differential Structures

The whole reasoning from the previous subsection can be incorporated at the graded-differential level. Let us assume that \( \Gamma \) is the classical module of differential 1-forms on \( G \). In this case \( \Gamma^\wedge \) gives the standard higher-order calculus on \( G \). Let \( \Omega(P) \) be an arbitrary differential calculus on the bundle \( P \). We have then

Proposition 4.4. The following identities hold:

(4.9) \[
(\text{id} \otimes \chi)(\hat{F} \otimes \text{id})\hat{\sigma}_M = (\hat{\sigma}_M \otimes \text{id})(\text{id} \otimes \hat{F})
\]

(4.10) \[
\hat{\sigma}_M = \hat{\sigma}^{-1}_M
\]

(4.11) \[
\hat{F}_2 \hat{\sigma}_M = (\hat{\sigma}_M \otimes \text{id})\hat{F}_2
\]

(4.12) \[
(\hat{\tau} \otimes \text{id})\chi = (\text{id} \otimes \hat{\sigma}_M)(\hat{\sigma}_M \otimes \text{id})(\text{id} \otimes \hat{\tau}).
\]

We see that \( \hat{\tau} \) and \( \hat{F} \) are completely compatible with the braiding \( \hat{\sigma}_M \). In particular, the space \( \hat{L} \) is a graded-differential *-subalgebra of \( \mathcal{W}_2 \). It is fully
covariant under the action of $\hat{\sigma}_M$ and there exists a natural induced braiding $\hat{\Sigma}: \hat{L} \otimes_M \hat{L} \rightarrow \hat{L} \otimes_M \hat{L}$. We have
\begin{equation}
\begin{aligned}
\hat{d}\hat{\Sigma} &= \hat{\Sigma}d \\
\hat{\Sigma}^{-1} &= (\hat{\Sigma} \otimes \hat{\Sigma})(\hat{\phi}_M \otimes \hat{\phi}_M)
\end{aligned}
\end{equation}

Furthermore, $\hat{\phi}_M: \hat{L} \rightarrow \hat{L} \otimes_M \hat{L}$ is a *-homomorphism. The operator $\hat{\sigma}_M$ reduces in the space $\hat{L}$. Let $\kappa_M: \hat{L} \rightarrow \hat{L}$ be the corresponding restriction map.

**Proposition 4.5.** Endowed with a system of maps $\{\hat{\phi}_M, \epsilon_M, \kappa_M, \hat{\Sigma}\}$, the differential *-algebra $\hat{L}$ becomes a differential braided-Hopf *-algebra over $M$.

This gives a natural differential calculus on the quantum gauge bundle. It is worth mentioning that the differential algebra $\hat{L}$ is not necessarily compatible with the classical gauge group $C(P)$. This is because transformations $\gamma: \hat{L} \rightarrow V$ from $C(P)$ are not automatically extendible to the appropriate maps on $\hat{L}$. This extendibility property can be understood as an additional condition on the calculus $\Omega(P)$ over the bundle.

## 5. Quantum Gauge Bundles for General Structure Groups

Let $G$ be an arbitrary compact matrix quantum group, equipped with a differential calculus $\Gamma$. Let us consider a quantum principal bundle $P = (B, i, F)$ over $M$, equipped with a differential structure $\Omega(P)$.

The construction of quantum gauge bundles associated to general quantum principal bundles $P$ can be performed [8] applying the methods developed in [5]. As explained in [8], the structure of $P$ is completely encoded in a system of intertwiner $\mathcal{V}$-bimodules $E_u = \text{Mor}(u, F)$, associated to finite-dimensional representations $u: H_u \rightarrow H_u \otimes A$ of $G$. It turns out that the structure of the gauge bundle $\mathfrak{a}(P)$ is expressible in terms of the $\mathcal{V}$-bimodules $\mathcal{G}_u = E_u \otimes_M E_u$ interpretable as consisting of right $\mathcal{V}$-linear homomorphisms of $E_u$.

We have the following natural decompositions
\begin{equation}
\begin{aligned}
\mathcal{B} &= \sum_{\alpha \in \mathcal{T}} \oplus \mathcal{B}^\alpha \\
\mathcal{L} &= \sum_{\alpha \in \mathcal{T}} \oplus \mathcal{G}_\alpha
\end{aligned}
\end{equation}
of $\mathcal{V}$-bimodules. Here $\mathcal{T}$ is a complete set of mutually inequivalent irreducible representations of $G$. The spaces $\mathcal{B}^\alpha \leftrightarrow \mathcal{E}_\alpha \otimes H_\alpha$ are the multiple irreducible bimodules corresponding to the action $F$.

To obtain the full quantum gauge bundle $\mathfrak{a}(P)$, it is necessary to take into account all possible bimodules $\mathcal{G}_u$, and to factorize through appropriate compatibility relations. In such a way we obtain a braided quantum group $\hat{\mathcal{G}}(P)$, together with a *-coalgebra inclusion $\iota: \mathcal{L} \rightarrow \mathcal{G}(P)$. This map admits a graded-differential extension $\iota: \hat{\mathcal{L}} \rightarrow \hat{\mathcal{G}}(P)$, where $\hat{\mathcal{G}}(P)$ is a graded-differential *-algebra describing a complete calculus [8] on $\mathfrak{a}(P)$. In various interesting (sufficiently regular) special cases the map $\iota$ will be bijective. This includes, in particular, locally-trivial quantum principal bundles over classical smooth manifolds [8 8].
However, for the study of all the phenomena related to the action of quantum gauge transformations on the bundle $P$, the coalgebra structure on $L$ and $\hat{L}$ is sufficient. We shall analyze in this section the transformation properties of basic entities of the general formalism \cite{4}, under the action of quantum gauge transformations.

We have a natural decomposition $\Gamma^\wedge \otimes \leftrightarrow A \otimes \Gamma^\wedge \otimes \hat{\otimes}$, where the left-invariant part $\Gamma^\wedge \otimes \hat{\otimes} \subseteq \Gamma^\wedge$ is the quadratic algebra explicitly given by

$$
\Gamma^\wedge \otimes S^\wedge = \text{gen}(S^\wedge^2), \quad S^\wedge^2 = \{ \pi(a^{(1)} \otimes \pi(a^{(2)}); a \in R) \},
$$

and $R \subseteq \text{ker}(\epsilon)$ is the right $A$-ideal canonically corresponding \cite{12} to $\Gamma$. In particular, it follows that the map $\hat{\tau}: \Gamma^\wedge \rightarrow \mathcal{W}_2$ is completely determined by its restriction on $\Gamma^\wedge \otimes \hat{\otimes}$. It turns out that the map $\hat{\tau}$ is effectively computed in terms of $\tau$, and the connection forms.

By definition \cite{4}, connections on the bundle $P$ are first-order hermitian linear maps $\omega: \Gamma^\wedge \otimes \rightarrow \Omega(P)$ satisfying the identity

$$
\hat{F}\omega(\vartheta) = \sum_k \omega(\vartheta_k) \otimes c_k + 1 \otimes \vartheta,
$$

where $\sum_k \vartheta_k \otimes c_k = \varpi(\vartheta)$. If we now fix a grade-preserving splitting

$$
\Gamma^\wedge \otimes \leftrightarrow S^\wedge \otimes \Gamma^\wedge \otimes \hat{\otimes},
$$

compatible with the *-structure and the action $\varpi$, then every connection induces a natural decomposition

$$
\Omega(P) \leftrightarrow \mathcal{A}(P) \otimes \Gamma^\wedge \otimes \hat{\otimes} \otimes \mathcal{H}(P).
$$

This decomposition plays a fundamental role in various considerations involving the algebra $\Omega(P)$. Let $\omega$ be an arbitrary connection on $P$.

**Lemma 5.1.** The following identities hold:

$$
\begin{align*}
\Delta \tau(a) &= \tau(a^{(1)}) \omega \pi(a^{(2)}) - \omega \pi(a^{(1)}) \tau(a^{(2)}) \\
\hat{\tau}(\vartheta) &= 1 \otimes \omega(\vartheta) - \sum_k \omega(\vartheta_k) \tau(c_k).
\end{align*}
$$

**Proof.** Using the basic transformation property of connections we find

$$
\hat{X} \left( \tau(a^{(1)}) \omega \pi(a^{(2)}) - \omega \pi(a^{(1)}) \tau(a^{(2)}) \right) = (1 \otimes a^{(1)}) \left( \omega \pi(a^{(2)}) \otimes \kappa(a^{(2)}) a^{(4)} \right) + 1 \otimes a^{(1)} \pi(a^{(2)}) - \omega \pi(a^{(1)}) \otimes a^{(2)}
$$

$$
= 1 \otimes a^{(1)} \pi(a^{(2)}) = 1 \otimes d(a).
$$

Similarly, we obtain

$$
\hat{X} \left( 1 \otimes \omega(\vartheta) - \sum_k \omega(\vartheta_k) \tau(c_k) \right) = 1 \otimes \vartheta + \sum_k \omega(\vartheta_k) \otimes c_k - \sum_k \omega(\vartheta_k) X \tau(c_k) = 1 \otimes \vartheta,
$$

which proves (5.3).
Now applying (5.3) and the definition of $\tilde{\sigma}_M$ we obtain the following expression for the braiding between connections and arbitrary differential forms:

\[(5.4)\quad \tilde{\sigma}_M(\omega(\vartheta) \otimes \psi) = \sum_k \omega(\vartheta_k) \psi \tau(c_k) - (-1)^{\partial \psi} \sum_k \psi \omega(\vartheta_k) \tau(c_k) + (-1)^{\partial \psi} \psi \otimes \omega(\vartheta).\]

Furthermore, the action of quantum gauge transformations on connections is given by

\[(5.5)\quad \tilde{\Delta}[\omega(\vartheta)] = \sum_k \omega(\vartheta_k) \otimes \tau(c_k) + 1 \otimes \tau(\vartheta),\]

as directly follows from the definition of the map $\tilde{\Delta}: \Omega(P) \to \mathcal{L} \otimes M \Omega(P)$, and the basic transformation property of connections.

Now let us analyze the transformation of the covariant derivative and the curvature map \[4\]. By definition, the curvature of $\omega$ is a linear map $R_\omega: \Gamma_{\text{inv}} \to \mathfrak{hor}(P)$ given by the structure equation

\[R_\omega = d\omega - \langle \omega, \omega \rangle\]

where $\langle \rangle$ are the brackets associated to the embedded differential $\delta: \Gamma_{\text{inv}} \to \Gamma_{\text{inv}} \otimes 2$, coming from the splitting (5.1). More precisely,

\[\langle \omega, \omega \rangle(\vartheta) = \sum_k \omega(\vartheta_k^1) \omega(\vartheta_k^2) \quad \sum_k \vartheta_k^1 \otimes \vartheta_k^2 = \delta(\vartheta).\]

Furthermore, the covariant derivative is a linear map $D_\omega: \mathfrak{hor}(P) \to \mathfrak{hor}(P)$ given by

\[D_\omega(\varphi) = d(\varphi) - (-1)^{\partial \varphi} \sum_k \varphi_k \omega \pi(c_k)\]

where $F^\wedge(\varphi) = \sum_k \varphi_k \otimes c_k$.

**Lemma 5.2.** The transformation properties of the curvature and covariant derivative are given by

\[(5.6)\quad \tilde{\Delta}R_\omega(\vartheta) = \sum_k \varsigma(c_k) R_\omega(\vartheta_k)\]
\[(5.7)\quad \tilde{\Delta}D_\omega(\varphi) = \sum_k \varsigma(c_k) D_\omega(\varphi_k),\]

where $\varsigma(a) = [\kappa^{-1}(a^{(1)})]^1 \otimes \tau(a^{(2)})[\kappa^{-1}(a^{(1)})]^2$.

**Proof.** Using the definition of of $\tilde{\Delta}$, and the covariance of the operators $R_\omega$ and $D_\omega$ we obtain

\[(5.8)\quad \tilde{\Delta}R_\omega(\vartheta) = \sum_k R_\omega(\vartheta_k) \otimes \tau(c_k)\]
\[(5.9)\quad \tilde{\Delta}D_\omega(\varphi) = \sum_k D_\omega(\varphi_k) \otimes \tau(c_k).\]
On the other hand,
\[
\sum_k R_\omega(\vartheta_k) \otimes \tau(c_k) = \sum_k [\kappa^{-1}(c_k^{(1)})]_1 \otimes [\kappa^{-1}(c_k^{(1)})]_2 R_\omega(\vartheta_k) \tau(c_k^{(2)})
\]
\[
= \sum_k [\kappa^{-1}(c_k^{(1)})]_1 \otimes \tau(c_k^{(2)})[\kappa^{-1}(c_k^{(1)})]_2 R_\omega(\vartheta_k).
\]
Similarly, using the covariance of \(D_\omega\), we conclude that equality (5.7) holds.

In the definition of the quantum gauge transformation map \(\Delta\) and \(\hat{\Delta}\) we have not used the braid operators \(\sigma_M\) and \(\tilde{\sigma}_M\). On the other hand, \(\sigma_M\) induces a differential *-algebra structure on \(W_3\) such that \(\hat{\Delta}\) is a differential *-homomorphism. This property of \(\hat{\Delta}\) allows us to transform composed algebraic expressions, by transforming separately their constitutive elements.

As a concrete illustration, let us derive formulas (5.8)–(5.9) by a direct computation, starting from the transformation of connections (5.5). We have
\[
\hat{\Delta}(\omega,\omega)(\vartheta) = \sum_k (\omega,\omega)(\vartheta_k) \otimes \tau(c_k) - \sum_k \omega(\vartheta_k) \otimes \tau(dc_k) + 1 \otimes d\hat{\tau}(\vartheta),
\]
where we have used the identity
\[
\sigma \delta - \delta = (\text{id} \otimes \pi)\omega = c^\top,
\]
and the property
\[
(\hat{\tau}, \hat{\tau})(\vartheta) = \sum_j \hat{\tau}(\vartheta_j^1)\hat{\tau}(\vartheta_j^2) = d\tau(\vartheta).
\]
Therefore,
\[
\hat{\Delta} R_\omega(\vartheta) = \sum_k d\omega(\vartheta_k) \otimes \tau(c_k) - \sum_k \omega(\vartheta_k) \otimes \tau(dc_k) + 1 \otimes d\hat{\tau}(\vartheta) - \hat{\Delta}(\omega,\omega)(\vartheta)
\]
\[
= \sum_k d\omega(\vartheta_k) \otimes \tau(c_k) - \sum_k (\omega,\omega)(\vartheta_k) \otimes \tau(c_k) = \sum_k R_\omega(\vartheta_k) \otimes \tau(c_k).
\]
Furthermore, applying (5.2) we obtain
\[
\hat{\Delta} D_\omega(\varphi) = \hat{\Delta} d(\varphi) - (-1)^{\partial \varphi} \sum_k (\varphi_k \otimes \tau(c_k^{(1)})) \hat{\tau} \pi(c_k^{(2)})
\]
\[
- (-1)^{\partial \varphi} \sum_k (\varphi_k \otimes \tau(c_k^{(3)})) \{\omega \pi(c_k^{(3)}) \otimes \tau(c_k^{(4)})\}
\]
\[
= \sum_k d(\varphi_k) \otimes \tau(c_k) + (-1)^{\partial \varphi} \varphi_k \otimes d\tau(c_k) - (-1)^{\partial \varphi} \sum_k \varphi_k \otimes \hat{\tau} d(c_k)
\]
\[
- (-1)^{\partial \varphi} \sum_k \varphi_k \omega \pi(c_k^{(1)}) \otimes \tau(c_k^{(2)}) = \sum_k D_\omega(\varphi_k) \otimes \tau(c_k),
\]
and equality (5.9) is proven.
References

[1] Brzezinski T: Translation Map in Quantum Principal Bundles, Preprint hep-th/9407145
[2] Connes A: Noncommutative Geometry, Academic Press (1994)
[3] Đurđević M: Geometry of Quantum Principal Bundles I, Commun Math Phys 175 (3) 457–521 (1996)
[4] Đurđević M: Geometry of Quantum Principal Bundles II, Preprint QmmP 4/93, Belgrade University, Serbia; Extended version: Preprint, Instituto de Matematicas, UNAM, México
[5] Đurđević M: Quantum Principal Bundles & Tannaka-Krein Duality Theory, Rep Math Phys (to appear, 1996)
[6] Đurđević M: Quantum Principal Bundles as Hopf-Galois Extensions, Preprint, Instituto de Matematicas, UNAM, México (1995)
[7] Đurđević M: Classical Spinor Structures on Quantum Spaces, Clifford Algebras and Spinor Structures, Special Volume, 365–377, Kluwer (1995)
[8] Đurđević M: Quantum Gauge Bundles, in preparation (1996)
[9] Đurđević M: Quantum Principal Bundles and Corresponding Gauge Theories, Preprint, Instituto de Matematicas, UNAM, México
[10] Majid S: Braided Groups J Pure Applied Algebra 86 187–221 (1993)
    Majid S: Beyond Supersymmetry and Quantum Symmetry (An Introduction to Braided Groups and Braided Matrices) Quantum Groups, Integrable Statistical Models and Knot Theory (World Sci) 231–282 (1993)
    Majid S: Algebras and Hopf Algebras in Braided Categories Advances in Hopf Algebras (Marcel Dekker, 1993)
[11] Woronowicz S L: Compact matrix pseudogroups, Commun Math Phys 111 613–665 (1987)
[12] Woronowicz S L: Differential Calculus on Compact Matrix Pseudogroups (Quantum Groups) Commun Math Phys 122 125–170 (1989)