RATIONAL CONNECTEDNESS AND ORDER OF NON-DEGENERATE MEROMORPHIC MAPS FROM $\mathbb{C}^n$

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Abstract. We show that an $n$-dimensional compact Kähler manifold $X$ admitting a non-degenerate meromorphic map $f : \mathbb{C}^n \to X$ of order $\rho_f < 2$ is rationally connected.

1. Introduction/Summary

The purpose of this paper is the following result.

Theorem 1.1. Let $X$ be a compact Kähler manifold of fixed dimension $n$. Let $f : \mathbb{C}^n \dashrightarrow X$ be a non-degenerate meromorphic map of order $\rho_f < 2$ (see the next section for the definition of $\rho_f$).

Then $X$ is rationally connected, hence projective.

This result belongs to a series of similar statements relating the existence and growth of maps from $\mathbb{C}^n$ to algebro-geometric properties of the target space $X$.

These statements are better expressed by introducing the following invariant $\rho(X)$, which suggests many questions, some of which are raised in the last section:

Definition. Let $X$ be an $n$-dimension connected compact complex manifold. Define $\rho(X) := \inf \{ \rho_f | f : \mathbb{C}^m \dashrightarrow X, m \geq n \}$, $f$ non-degenerate.

It is understood that $\rho(X) \in [0, +\infty]$, and that $\rho_f = +\infty$ if there exists no non-degenerate meromorphic map $f : \mathbb{C}^n \to X$.

The invariant $\rho$ is easily seen to be bimeromorphic, preserved by finite étale covers, and ‘increasing’ (ie: $\rho(X) \geq \rho(Y)$ if there exists a dominant meromorphic map $g : X \to Y$).

It is obvious that $\rho(X) = 0$ if $X$ is unirational.

Kobayashi and Ochiai proved that the existence of a non-degenerate map from $\mathbb{C}^n$ to a projective manifold $X$ implies that $X$ is not of general type ([10]). Thus $\rho(X) = +\infty$ if $X$ is of general type. It is proved in [4], more generally, that $X$ is ‘special’ if there exists a non-degenerate map

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1
meromorphic map $f: \mathbb{C}^n \to X$. In particular, $X$ must be special if $ho(X) < +\infty$.

If $X$ is Kähler and $K_X$ is pseudo-effective of numerical dimension $\nu \in \{0, 1, \ldots, n\}$, then $\rho(X) \geq (1 - \frac{\nu}{n})^{-1} \geq 2$ (6). This implies the previous result of Kobayashi-Ochiai (since $X$ is of general type if and only if $\nu = n$). Using [2], it also implies that if $X$ is projective, then $X$ is uniruled if $\rho(X) < 2$.

If $h^0(X, \text{Sym}^k(\Omega^p_X)) \neq 0$, for some $k, p > 0$, then $\rho(X) \geq 2$ (12). The Kähler condition is not required here. If, in addition, $X$ is assumed to be a Kähler surface, then the condition $h^0(X, \text{Sym}^k(\Omega^p_X)) = 0 \forall p, k$ implies that $X$ is Kähler. Thus, if $X$ is a Kähler surface and if $\rho(X) < 2$, then $X$ must be rational (12). On the other hand, some Hopf (thus non-rational, non-Kähler) surfaces have $\rho(X) \leq 1$ (12).

The present result generalizes these two results from [6], [12], avoiding the deep methods of [6]. The estimate $\rho_f < 2$ is optimal, since for an Abelian variety $A$ we have $\rho_e = 2$ for the universal covering map $\tau: \mathbb{C}^g \to A$.

All these results provide lower bounds for $\rho(X)$ deduced from the geometry of $X$. Producing upper bounds for $\rho(X)$ (ie: the existence of non degenerate $f'$s) turns out to be a completely open topic, except in the trivial case of $X$ unirational or a torus.

The case of rationally connected vs unirational manifolds when $n \geq 3$ is of great interest. For example: what is $\rho(X)$ if $X$ is a ‘general’ smooth quartic in $\mathbb{P}^4$? If $\rho(X) > 0$, then $X$ is not unirational. More generally: are there projective (necessarily rationally connected) $X$ such that $\rho(X) \in ]0, 2[$? See the last section for some more questions.

2. Characteristic function and order of a non-degenerate meromorphic map

We start with some preparations. Let $X$ be a compact complex manifold and let $f: \mathbb{C}^n \to X$ be a meromorphic map. $f$ is said to be (differentiably) “non-degenerate” if there is a point $p \in \mathbb{C}^n$ such that $f$ is holomorphic at $p$ and such that $(Df)_p: T_p \mathbb{C}^n \to T_{f(p)}X$ is surjective.

Let $\alpha := \frac{dd^c}{\pi} ||z||^2$ on $\mathbb{C}^n$, and let $\omega$ be a positive $(1, 1)$-form on $X$. The characteristic function of $f$ is defined as:

$$T_f(r; \omega) = \int_1^r \frac{dt}{t^{2n-1}} \int_{B_t} (f^* \omega) \wedge \alpha^{n-1}.$$ 

Here $B_t = \{ z \in \mathbb{C}^n : ||z|| < t \}$. Observe that the integral over $B_t$ is well-defined even if $f$ is only meromorphic, not necessarily holomorphic. (To see this, note that locally $\omega$ can be dominated by a sum $\sum_i \alpha_i \wedge \bar{\alpha}_i$ where the $\alpha_i$ are holomorphic 1-forms. The holomorphy of the $\alpha_i$ implies that
If $\omega$ and $\tilde{\omega}$ are any two positive $(1,1)$-forms on $X$, then (by the compactness of $X$) there are constants $C_1, C_2 > 0$ such that

$$C_1 \omega < \tilde{\omega} < C_2 \omega$$

and consequently

$$C_1 T_f(r, \omega) < T_f(r, \tilde{\omega}) < C_2 T_f(r, \omega) \quad \forall r > 1.$$  

The “order” $\rho_f$ is defined as

$$\rho_f = \limsup_{r \to \infty} \frac{\log T_f(r, \omega)}{\log r}$$

where $\omega$ is a positive $(1,1)$-form on $X$. Due to the inequalities (2.1) this number $\rho_f \in \mathbb{R} \cup \{+\infty\}$ is independent of the choice of the positive $(1,1)$-form $\omega$ used in the definition of $T_f(r, \omega)$.

The well-known Crofton’s formula (see e.g. [8]) implies that $T_f(r)$ equals the average of $T_{f|L}(r)$ taken over all complex lines $L \subset \mathbb{C}^n$. This permits to extend many properties of characteristic functions from the case of entire curves to the higher-dimensional domains.

For instance, let $\tau : X' \to X$ be a bimeromorphic holomorphic map between compact complex manifolds. Then $\tilde{f} = \tau^{-1} \circ f : \mathbb{C}^n \to X'$ is again a non-degenerate meromorphic map, and $\rho_{\tilde{f}} = \rho_f$.

Now let $\alpha : X \to Y$ be a dominant holomorphic map. Then it follows directly from the definitions that $T_{\alpha \circ f}(r) \leq T_f(r)$ (if $\omega_X \geq \alpha^*(\omega_Y)$) and consequently $\rho_{\alpha \circ f} \leq \rho_f$.

The order is easily seen to behave nicely with respect to products. Let $X = X_1 \times X_2$, let $\omega_i$ be positive $(1,1)$-forms in $X_i$ and let $f_i : \mathbb{C}^{n_i} \to X_i$ be non-degenerate meromorphic maps. Define $\omega = \pi_1^* \omega_1 + \pi_2^* \omega_2$, $n = n_1 + n_2$ and $f : \mathbb{C}^n \to X$ as $f(v_1, v_2) = (f_1(v_1), f_2(v_2))$.

Then $T_f(r) = T_{f_1}(r) + T_{f_2}(r)$. This implies $\rho_f = \max\{\rho_{f_1}, \rho_{f_2}\}$, because for any $t_1, t_2 > 0$ we have

$$\max_i \log t_i \leq \log(t_1 + t_2) \leq \log(2 \max_i t_i) = \log 2 + \log \max_i t_i.$$  

3. **Rational connectedness**

As usual, a compact complex manifold $X$ (usually Kähler) is called “rationally connected” (short: RC) if every two points can be linked by a (possibly singular) irreducible rational curve. (Equivalently in the Kähler case: can be linked by a chain of rational curves). Unirational manifolds are RC. There are unirational threefolds which are not rational, e.g. smooth cubic hypersurfaces in $\mathbb{P}^4$. However, it is not known whether there exist RC manifolds which are not unirational, although
it is expected that these should exist (the case of ‘general’ quartics in \( \mathbb{P}_4 \) being one of the first open cases).

Rationally connected compact Kähler manifolds are projective (since \( h^{2,0} = 0 \), using Kodaira’s criterion).

Let \( X \) be a compact connected Kähler manifold. Then there exists an “almost holomorphic” rational dominant map \( \rho : X \to Y \) (called the “RC-reduction”, or “rational quotient” [3], or the “MRC-fibration” [11] if \( X \) is projective), such that the fibers are RC, and maximum with this property.

When \( X \) is projective, it is known (by [9]) that the base \( Y \) is not uniruled. In fact, [9] shows that if \( f : X \to Y \) is a surjective meromorphic map with fibres and base \( Y \) which are both RC, then \( X \) is RC if it is compact Kähler (remark first that \( h^{2,0}(X) = 0 \), and that \( X \) is thus projective). The base \( Y \) of the RC-reduction \( \rho : X \to Y \) is not uniruled also when \( X \) is compact Kähler. Let indeed \( r : Y \to Z \) be the RC-reduction of \( Y \). The fibres of \( r \circ \rho : X \to Z \) are thus RC, and so \( Y = Z \), which means that \( Y \) is not uniruled.

Due to [2] a projective manifold is uniruled if and only if \( K_X \) is not pseudoeffective. Based on this, another recent criterion is the following (see [5]):

Let \( X \) be a compact Kähler manifold. Then \( X \) is rationally connected if and only if there is no pseudoeffective invertible subsheaf \( F \subset \Omega_X^p \) (for some \( p \in \mathbb{N} \)).

The proof consists in observing that \( X \) is not RC precisely if its RC-reduction \( \rho : X \to Y \) has \( \dim(Y) = p > 0 \). Define then \( F := \rho^*(K_Y) \), which is pseudo-effective since \( Y \) is not uniruled.

It is conjectured that a compact Kähler (or equivalently: projective) manifold \( X \) is RC if (and only if) there is no \( \mathbb{Q} \)-effective (instead of pseudoeffective) invertible subsheaf \( F \subset \Omega_X^p \) (for some \( p \in \mathbb{N} \)). By means of the RC-reduction as above, this conjecture is equivalent to the ‘non-vanishing conjecture’, claiming that if \( K_X \) is pseudo-effective, it is \( \mathbb{Q} \)-effective.

4. PSEUDEFFECTIVE LINE BUNDLES

A singular hermitian metric \( h \) on a complex line bundle is given in the form \( \| h = e^{-\phi} \|_s \) where \( s \) is the standard metric for some local holomorphic trivialization and \( \phi \) is a \( L^1_{\text{loc}} \)-function.

The \( L^1_{\text{loc}} \)-condition ensures that the curvature \( \Theta = dd^c \log(e^{-\phi}) = -dd^c \phi \) makes sense (in the sense of currents) and represents the Chern class of the line bundle.
A line bundle $L$ on a compact complex manifold is called “pseudo-
effective” iff there is a singular hermitian metric $h$ with semipositive
curvature $\Theta_h \geq 0$. This condition means that the metric is locally
given via a weight function $e^{-\phi}$ with $\phi$ plurisubharmonic. If $s$ is a holomor-
phic section in $F$, this implies that $-\log ||s||_h$ is plurisubharmonic.

5. Measuring the derivative

Let $V, W$ be complex vector spaces equipped with hermitian inner
products. The norm $||F||$ of a complex linear map $F : V \to W$ is
defined by: $||F||^2 = \text{trace}(F^* \circ F)$ where $F^*$ denotes the adjoint of $F$.
If $A$ is the matrix describing $F$ with respect to orthonormal bases on
$V$ and $W$, then

$$||F||^2 = \sum_{i,j} |A_{ij}|^2.$$ 

We continue with a local observation. Let $U, V$ be open subsets in
$\mathbb{C}^n$, let $\alpha = dd^c||z||^2 = \sum_j i \cdot dz_j \wedge d\bar{z}_j$ be the Kähler form for the
euclidean metric and let $f : U \to V$ be a holomorphic map.

Then

$$f^* \alpha \wedge \alpha^{n-1} = \frac{1}{n} ||Df||^2 \alpha^n$$

where

$$||Df||^2 = \text{trace} ((Df)^* \circ (Df)) = \sum_{j,k} \left| \frac{\partial f_j}{\partial z_k} \right|^2$$

**Proposition 5.1.** Let $U \subset \mathbb{C}^n$ be an open subset, let $X$ be a compact
Kähler manifold equipped with a hermitian metric $h$ and a positive
$(1, 1)$-form $\omega$ and let $f : U \to X$ be a holomorphic map.

Then there is a constant $C > 0$ such that

$$f^* \omega \wedge \alpha^{n-1} \geq C||Df||^2 \alpha^n.$$ 

Here $||Df||$ is calculated with respect to $h$.

**Proof.** We cover $X$ with finitely many open subsets $V_k$ such that each
$V_k$ admits an embedding $j_k : V_k \to \mathbb{C}^n$ and each $V_k$ contains a relatively
compact open subset $W_k \subset V_k$ such that $X = \cup_k W_k$. Let $h_k$ denote
the hermitian metric on $V_k$ induced by the euclidean metric via its
embedding in $\mathbb{C}^n$. We choose $C_1 > 0$ such that $h_k \leq C_1 h$ everywhere on
each $W_k$ and $C_2 > 0$ such that $\omega \geq C_2 j_k^* \alpha$ on each $W_k$. Then the claim
(with $C = C_1 C_2$) follows from the preceding local observation. \qed
We show:

**Theorem 6.1.** Let $X$ be a compact Kähler manifold. Let $f : \mathbb{C}^n \to X$ be a non-degenerate meromorphic map of order $\rho_f < 2$.

Then $X$ is projective and rationally connected.

**Proof.** Assume by contradiction that $X$ is not rationally connected. There is then a pseudoeffective invertible subsheaf $\mathcal{F} \subset \Omega^p_X$ (for some $p \in \mathbb{N}$). We fix a hermitian metric $h$ on $X$. The hermitian metric on $T_X$ induces a hermitian metric on $\Omega^p_X$, by abuse of language also denoted by $h$. The injection of sheaves $\mathcal{F} \hookrightarrow \Omega^p_X$ corresponds to a non-zero vector bundle homomorphism $\xi_0 : F \to \Omega^p_X$, where $F$ is a pseudoeffective line bundle. Let $g$ denote a smooth hermitian metric on $F$ with $g \leq h|_F$ and let $g_0$ denote a singular hermitian metric on $F$ such that $\Theta_{g_0} \geq 0$, i.e., with positive curvature current. Then there is an upper semicontinuous function $\phi : X \to \mathbb{R}$ such that $g_0 = e^{-\phi} g$. Since $\phi$ is upper semicontinuous, and $X$ is compact, it is bounded from above: $M := \sup_{x \in X} \phi(x) < \infty$.

The meromorphic map $f : \mathbb{C}^n \to X$ and the vector bundle homomorphism $\xi_0 : F \to \Omega^p_X$ induce vector bundle homomorphisms

$$f^* F \xrightarrow{\xi} f^* \Omega^p_X \xrightarrow{Df^*} \Omega^p_{\mathbb{C}^n}.$$ 

on $\mathbb{C}^n \setminus I(f)$ (with $I(f)$ denoting the indeterminacy set of the meromorphic map $f$.) We are interested in a lower bound for $||Df|| = ||(Df)^*||$, calculated with respect to the metric induced by $h$ resp. the euclidean metric on $f^* \Omega^p_X$ resp. $\Omega^p_{\mathbb{C}^n}$. Let $\beta := (Df^*) \circ \xi$. Since we assumed $g \leq h|_F$, we have $||\beta||_g \leq ||Df||$ where $||\beta||_g \leq ||Df||$ denotes the norm with respect to $g$ on $F$ and the euclidean metric on $\Omega^p_{\mathbb{C}^n}$. Let $||\beta||_{g_0}$ denote the norm with respect to $g_0$ on $F$.

By using the standard trivialization of $\Omega^p_{\mathbb{C}^n}$, the bundle morphism $\beta : f^* \Omega^p_{\mathbb{C}^n}$ corresponds to a vector valued section on the dual bundle $f^* F^*$. Now $\Theta_{g_0} \geq 0$ implies that $\log ||s||_{g_0}$ is plurisubharmonic for every holomorphic section $s$ in $f^* F^*$.

Hence $\log ||\beta||_{g_0}$ is a plurisubharmonic function on $\mathbb{C}^n \setminus I(f)$ where $I(f)$ denotes the indeterminacy set of the meromorphic map $f$. Plurisubharmonic functions extend through closed analytic subsets of codimension at least two. Hence $\log ||\beta||_{g_0}$ extends to a plurisubharmonic function $\zeta_0$ defined on the whole $\mathbb{C}^n$.

We observe that $||\beta||_{g_0} = e^\phi ||\beta||_g$, because $g_0 = e^{-\phi} g$. 

By the definition of the constant $M$, we have $||\beta||_{g_0} \leq e^M ||\beta||_g$, implying
$$\zeta_0 - M \leq \log ||\beta||_g \leq \log ||Df||$$
Define $\zeta = \exp(\zeta_0 - M)$. The plurisubharmonicity of $\zeta_0$ implies the plurisubharmonicity of $\zeta$. Thus we obtain the existence of a plurisubharmonic function $\zeta$ on $\mathbb{C}^n$ such that $||Df|| \geq \zeta$.

Using proposition 5.1 we may deduce that $f^* \omega \wedge \alpha^{n-1} \geq \zeta \alpha^n$. It follows that
$$T_f(r; \omega) \geq \int_1^r \frac{dt}{t^{2n-1}} \int_{B_t} \zeta \alpha^n$$
Since moving the origin, i.e., replacing $f$ by $f \circ \tau$ where $\tau$ denotes a translation, does not affect the order $\rho_f$, we may assume that $c = \zeta(0, \ldots, 0) > 0$. Using the sub-mean value property of plurisubharmonic functions it follows that
$$T_f(r; \omega) \geq c \int_1^r \frac{dt}{t^{2n-1}} \int_{B_t} \alpha^n = c \nu \int_1^r \frac{dt}{t^{2n-1}} t^{2n} = c \nu \frac{r^2}{2} + O(1)$$
where $\nu$ denote the volume of the unit ball. Therefore
$$\rho_f = \limsup \frac{\log T_f(r)}{\log r} \geq \limsup \frac{\log(r^2)}{\log r} = 2.$$

\[ \Box \]

\section*{7. Non-Kähler manifolds}

Our result is not valid for non-Kähler manifolds. In fact, there are Hopf surfaces $X$ admitting a non-degenerate holomorphic map $f : \mathbb{C}^n \to X$ of order $\rho_f = 1$ (see [12]). Of course, these Hopf surfaces are non-Kähler and not rationally connected; they do not contain any rational curve at all.

More precisely, in [12] the following is proved:

\textbf{Theorem 7.1.} Let $\lambda \in \mathbb{C}$ with $|\lambda| > 1$ and let $\sim$ denote the equivalence relation on $\mathbb{C}^2 \setminus \{(0, 0)\}$ given by
$$v \sim w \iff \exists k : v = \lambda^k w.$$  
Let $X = \mathbb{C}^2 \setminus \{(0, 0)\}/\sim$ and let $f : \mathbb{C}^2 \to X$ be the map induced by $(z, w) \mapsto (z, 1 + zw)$.
Then $\rho_f = 1$.

This result has been generalized by T. Amemiya to the class of Hopf surfaces defined by an equivalence $(z, w) \sim (\lambda^k z, \mu^k w) \ (k \in \mathbb{Z})$ where $\lambda$ may be different from $\mu$ (but $|\lambda|, |\mu| > 1$).
8. Questions

The following questions are not expected to have necessarily positive answers.

Let $X$ be an $n$-dimensional compact Kähler manifold, $f : \mathbb{C}^m \to X$ meromorphic non-degenerate.

1. Is $X$ unirational if $\rho_f = 0$? Is $X$ unirational if $\rho(X) = 0$?

   (It should be remarked that $\rho_f = 0$ for every algebraic map, but the condition $\rho_f$ is substantially weaker than algebraicity, as seen by appropriate power series in one variable.)

2. If there exists such an $f : \mathbb{C}^m \to X$, can it be chosen so that $\rho_f < +\infty$? In other words: if there exists an $f$ as above, is $\rho(X) < +\infty$?

3. If $\rho(X) < +\infty$, does there exist some $f : \mathbb{C}^m \to X$ with $\rho_f = \rho(X)$?

4. If $X$ is rationally connected, does there exist a non-degenerate meromorphic map $f : \mathbb{C}^m \to X$? Is then $\rho(X) < +\infty$?

5. If $X$ is RC, and if there exists a non-degenerate $f : \mathbb{C}^m \to X$, is $X$ unirational? (ie: can $f$ be chosen algebraic?). A positive answer would imply that there exists no $X$ with $\rho(X) \in ]0,2[$.

The questions 3 and 4 were raised for $n = 3$ in [4], question 9.5.

6. Is the estimate $\rho(X) \geq 2(1 - \frac{\nu}{n})^{-1}$ in [6] optimal if $K_X$ is pseudoeffective with $\nu(X) =: \nu$? In other words: does there exists $X_n$ with $\nu(X) = \nu$ (or better: with $\kappa(X) = \nu$) and with $\rho(X) \geq 2(1 - \frac{\nu}{n})^{-1}$ for any $n > 0$ and $\nu \in \{0,1,\ldots,n\}$?

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