Resolution Graphs of Some Surface Singularities, II.
(Generalized Iomdin Series)

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Abstract. In this article, we determine the resolution graph of the hypersurface singularity \(\{ f + g^k = 0 \}, 0\), where \((f, g) : (\mathbb{C}^3, 0) \to (\mathbb{C}^2, 0)\) is an ICIS, and \(k\) is a sufficiently large integer. All these graphs are coordinated by an “universal bi-colored graph” \(\Gamma_C\) associated with the ICIS \((f, g)\). Its definition is rather involved, and in concrete examples it is difficult to compute it. Nevertheless, we present a large number of examples. This is very helpful in the exemplification of its properties as well. Then we present our main construction which provides the dual resolution graph of the series \(\{ f + g^k = 0 \}\) from the graph \(\Gamma_C\) and the integer \(k\). This is formulated in a purely combinatorial algorithm. The result is a highly non-trivial generalization of the cyclic covering case of \((\mathbb{C}^2, 0)\).

Introduction.

The present article (in the sequel: Part II) is a natural continuation of [15] (called Part I) to such an extent that we use and cite the notations and results of Part I without reviewing them. (Note that the sections are numbered continuously through Part I and II.)

Consider an isolated complete intersection singularity (ICIS) \(\Phi = (f, g) : (\mathbb{C}^3, 0) \to (\mathbb{C}^2, 0)\) such that \(f\) has an 1-dimensional singular locus. Then construct the so called “generalized Iomdin series” \(f + g^k\) (for \(k \geq k_0\)) (see Section 4, i.e. the first section of Part II). Our goal is to describe the resolution graph \(\Gamma(X_k)\) of \((X_k, 0) := (\{ f + g^k = 0 \}, 0)\).

The Iomdin series is a generalization of the cyclic covering case \(f(x, y) + z^k\), where \(f\) is an isolated plane curve singularity. But the differences (both technical and of principle) are huge. For example, while in the case of the cyclic coverings, the relevant monodromy representations are easy (being representations of \(\mathbb{Z}\)), the monodromy representation of an ICIS is incomparably more complicated. Moreover, in the cyclic case there is a global Galois action, which disappears in the general case.

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However, the construction of $\Gamma(X_k)$ will show strong similarities with the case of cyclic coverings: we will construct from the ICIS $\Phi$ a universal graph $\Gamma C$ which will coordinate the whole series of graphs $\{\Gamma(X_k)\}_k$. Already the existence of such a graph is really remarkable.

The construction of $\Gamma C$ is rather involved. We will show that all the geometric information needed for the construction of the graphs $\{\Gamma(X_k)\}_k$ is contained in a small neighborhood of a special curve arrangement $C$, situated in the embedded resolution of the local divisor ($\{f g = 0\}, 0) \subset (\mathbb{C}^3, 0)$. $\Gamma C$ is exactly the dual graph of $C$. We will decorate this graph with different weights, which codify the topology of the irreducible components of $C$, and also the local behavior of the functions $f$ and $g$ near $C$. In Section 5 we present this construction, some examples and also some properties of $\Gamma C$. E.g., we will prove that from $\Gamma C$ one can recover the resolution graph of ($\{f = 0\}, 0$), and also the equisingular type of the transversal singularities of $\text{Sing}\{f = 0\}$.

In Section 6 we present the algorithm which provides, in a purely combinatorial way, the graph $\Gamma(X_k)$ from $\Gamma C$ and the integer $k$ (for $k \geq k_0$). The validity of the algorithm is based on the following important fact. The graph $\Gamma(X_k)$ appears as a cyclic covering graph of $\Gamma C$ (modified with some Hirzebruch–Jung strings). The weights of $\Gamma C$ codify all the local data from a neighborhood of $C$, but a priori it is not clear that from this data one can recover the necessary global information in order to identify the covering of $\Gamma(X_k)$. But, using the classification results of Section 1 (Part I), and the properties of $\Gamma C$, we show that the covering data of the covering can be determined from the weights of $\Gamma C$, and with the determined covering data there is a unique covering graph. This basically guarantees the vanishing of the global invariants.

The graph $\Gamma C$ can also be connected with some other invariants of the Iomdin series (for details, see (4.10) and (6.16)).

4. Preliminaries about generalized Iomdin series

Basics of ICIS.

4.1. In this subsection we review the basic notions related to isolated complete intersection singularities (ICIS). For details, see [8]. In order to keep the notation uniform, we only present the case $(\mathbb{C}^3, 0) \rightarrow (\mathbb{C}^2, 0)$.

Consider an analytic germ $\Phi = (f, g) : (\mathbb{C}^3, 0) \rightarrow (\mathbb{C}^2, 0)$ which defines an ICIS. This means that if $I \subset \mathcal{O}_{\mathbb{C}^3, 0}$ denotes the ideal generated by $f$ and $g$ and the $2 \times 2$ minors of the Jacobian matrix $(d\Phi)$, then $\dim \mathcal{O}_{\mathbb{C}^3, 0}/I < \infty$. In other words, the scheme-theoretical intersection $\Phi^{-1}(0) = \{f = 0\}\cap\{g = 0\}$ has only an isolated singularity at the origin. In particular, $(\text{Sing}\{f = 0\})\cap\{g = 0\} = \{0\}$ and $\{f = 0\}$ intersects $\{g = 0\}$ in the complement of the origin transversally in a smooth (punctured) curve.

The critical locus ($C_{\Phi}, 0$) of $\Phi$ is the set of points of $(\mathbb{C}^3, 0)$, where $\Phi$ is not a submersion. Its image ($\Delta_{\Phi}, 0) := \Phi(C_{\Phi}, 0) \subset (\mathbb{C}^2, 0)$ is called the discriminant locus of $\Phi$. In the sequel, we denote by $(c, d)$ the local coordinates of $(\mathbb{C}^2, 0)$.

4.2. The Milnor fibration. Fix a germ $\Phi$ as above. Then there exist a sufficiently small closed ball $B_0^3 \subset \mathbb{C}^3$ (centered at the origin with radius $\epsilon > 0$) and $D_0^2 \subset \mathbb{C}^2$ with radius $0 < \eta << \epsilon$ such that:
• the set $(\Phi^{-1}(0) \setminus \{0\}) \cap B^3_\varepsilon$ is non-singular;
• $\partial B^3_\varepsilon$ intersects $\Phi^{-1}(0)$ transversally for all $0 < \varepsilon \leq \epsilon$.
• $C_\Phi \cap \Phi^{-1}(D^2_\eta) \cap \partial B^3_\varepsilon = \emptyset$; and the restriction of $\Phi$ to $\Phi^{-1}(D^2_\eta) \cap \partial B^3_\varepsilon$ is a submersion.

4.4. Theorem. (i) $\Phi : \Phi^{-1}(D^2_\eta) \cap B^3_\varepsilon \rightarrow D^2_\eta$ with the above properties is called a “good” representative of the ICIS $\Phi$. For such a good representative, let $\Sigma_\Phi$ denote the intersection $C_\Phi \cap \Phi^{-1}(D^2_\eta)$ and $\Delta_\Phi = \Phi(\Sigma_\Phi)$.

With these notations we have the following additional properties (cf. 2.8 in [8]):

4.4. Theorem. (i) $\Phi : \frac{B^3_\eta}{\Phi^{-1}(D^2_\eta)} \rightarrow D^2_\eta$ is proper. The analytic sets $\Sigma_\Phi$ and $\Delta_\Phi$ are 1-dimensional, and the restriction $\Phi|_{\Sigma_\Phi} : \Sigma_\Phi \rightarrow \Delta_\Phi$ is proper with finite fibres.
(ii) $\Phi : (\Phi^{-1}(D^2_\eta - \Delta_\Phi) \cap B^3_\varepsilon, \Phi^{-1}(D^2_\eta - \Delta_\Phi) \cap \partial B^3_\varepsilon) \rightarrow D^2_\eta - \Delta_\Phi$ is a locally trivial $C^\infty$-fibration of a pair of spaces.

4.5. Definition. A fibre $F_{c,d} = \Phi^{-1}(c,d) \cap B^3_\varepsilon$, for $(c,d) \in D^2_\eta - \Delta_\Phi$, is called “a Milnor fibre” and the fibration itself is referred to as “the Milnor fibration”. For any fixed base point $b_0 = (c_0,d_0) \subset D^2_\eta - \Delta_\Phi$, one has the natural geometric monodromy representation: $m_{geom} : \pi_1(D^2_\eta - \Delta_\Phi, b_0) \rightarrow Diff^\infty(F_{b_0})/isotopy$.

In general, it is extremely difficult to determine this representation. Actually, even the induced algebraic monodromy representation $m_{alg} : \pi_1(D^2_\eta - \Delta_\Phi, b_0) \rightarrow \text{Aut}\, H_\ast(F_{b_0}, \mathbb{Z})$ is a very complicated object.

4.6. Let $\Delta_\Phi = \Delta_1 \cup \ldots \cup \Delta_t$ denote the decomposition of $\Delta_\Phi$ into irreducible components. Let $\Sigma_{i,1}, \ldots, \Sigma_{i,s_i}$ be the irreducible decomposition of $\Phi^{-1}(\Delta_i)\cap \Sigma_\Phi$ for $i = 1, \ldots, t$. Clearly, $\Sigma_\Phi = \bigcup_{i=1}^t \bigcup_{j=1}^{s_i} \Sigma_{ij}$ is a decomposition of $\Sigma_\Phi$ into irreducible components.

Assume that $f$ has a 1-dimensional singular locus. Clearly, $\text{Sing}\{f = 0\} \subset \Sigma_\Phi$ and $\Phi(\text{Sing}\{f = 0\}) = \{c = 0\}$ is an irreducible component of the discriminant $\Delta_\Phi$. By convention, we denote this component by $\Delta_1$. Then $\Sigma_{1,1}, \ldots, \Sigma_{1,s_1}$ are exactly the irreducible components of $(\text{Sing}\{f = 0\}, 0)$. Sometimes, we will use the simplified notations $\Sigma_{1,j} = \Sigma_j$ and $s_1 = s$. Part (i) of the above theorem guarantees that the restriction $\Phi : \Sigma_j \rightarrow \Delta_1$ is a branched covering for any $1 \leq j \leq s$. Let $d_j$ denote its degree.

4.7. Definition. Transversal singularity types. Fix an irreducible component $\Sigma_j$ of $\text{Sing}\{f = 0\}$, $j \in \{1, \ldots, s\}$. Take a point $q \in \Sigma_j - \{0\}$ and the germ $(H, q)$ of a generic smooth transversal slice to $\Sigma_j$ at $q$. The intersection $(\{f = 0\} \cap H, q)$ determines an isolated plane curve singularity $(T\Sigma_j, q) \subset (H, q)$. Its topological (or equisingular) type does not depend on the choice of $q$ and $(H, q)$. It is called the transversal singularity of $\text{Sing}\{f = 0\}$ corresponding to the branch $\Sigma_j$. 
Generalized Iomdin series.

The generalized Iomdin series is a special family of topological series of composed hypersurface singularities. For the general theory of the topological series, the interested reader can consult [18, 13, 9, 10]. In this subsection we restrict ourselves to the definition of the special case of the Iomdin series.

4.8. Definition. Consider a germ \( f : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0) \) with a 1-dimensional singular locus. Then fix another germ \( g : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0) \), such that the pair \( \Phi = (f, g) : (\mathbb{C}^3, 0) \to (\mathbb{C}^2, 0) \) forms an ICIS. Then the family of singularities \( f + g^k : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0) \) constitutes the generalized Iomdin series of \( f \), associated with \( g \), where \( k \) is any integer larger than a bound \( k_0 = k_{0, \Phi} \). (The bound depends only on \( \Phi \), and will be determined in the sequel).

Now, we provide a more precise description in terms of the resolution graphs of \( f \) and \( f + g^k \). First notice that the germs \( f \) and \( f + g^k \) are composed singularities. Indeed, \( f = \Phi \circ P \), where \( P : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0) \) is the plane curve singularity \((c, d) \to c \). Moreover, \( f + g^k = \Phi \circ P_k \) with the same \( \Phi \), but \( P_k \) given by \((c, d) \to c + d^k \). Similarly as above, let \( \Delta_\Phi \) be the discriminant locus of \( \Phi \) with the distinguished component \( \Delta_1 = \{c = 0\} \). A possible embedded resolution graph of the divisor \((P^{-1}(0) \cup \Delta_\Phi, 0) \subset (\mathbb{C}^2, 0) \) can be visualized schematically as:

\[
\begin{align*}
&\text{t - 1 arrows corresponding to the strict transforms of } \Delta_2, \ldots, \Delta_t \\
\end{align*}
\]

Above, the multiplicities \( m \) are the vanishing orders of the (lift of the) germ \( P \) along the corresponding components of the total transform of \( P^{-1}(0) \cup \Delta_\Phi \).

By definition, the germ \( f + g^k \) belongs to the generalized Iomdin series of \( f \), associated with \( g \), if \( k \geq l + 1 \). This means that a possible (schematic) embedded resolution graph of \( P_k^{-1}(0) \cup \Delta_\Phi \) is:

\[
\begin{align*}
&\text{t - 1 arrows corresponding to the strict transforms of } \Delta_2, \ldots, \Delta_t \\
\end{align*}
\]

where the multiplicities \( m \) are the vanishing orders of the germ \( P_k \) along the corresponding components of the total transform of \( P_k^{-1}(0) \cup \Delta_\Phi \). (Hence \( k_{0, \Phi} = l + 1 \).)

4.9. Example. Let \( f = x^n + y^n + xyz \) with \( n \geq 3 \) and \( g = z \). Then \( \Phi = (f, g) : (\mathbb{C}^3, 0) \to (\mathbb{C}^2, 0) \) is an ICIS. The critical locus of \( \Phi \) is \( \Sigma_\Phi = \{x = y = 0\} \cup \bigcup_{j=1}^{n-1}\{x = \xi_0^jy, z = -n\xi_0^{j(n-1)}y^{n-2}\} \), where \( \xi_0 = e^{2\pi i/n} \). Moreover, \( \{x = y = 0\} = \text{Sing}(f^{-1}(0)) \). Then \( \Phi(\{x = y = 0\}) = \{c = 0\} \) and \( \Phi(\{x = \xi_0^jy, z = \}) \)
\[-n_0^{j(n-1)y^{n-2}} = \{c = (2 - n)y^n, \quad d = -n_0^{j(n-1)y^{n-2}} \} \text{ for } j = 1, ..., n, \]

but some of these components may coincide. Indeed, let \( l = \gcd(n, n - 2) \). Note that \( l = 1 \) or \( l = 2 \).

If \( l = 2 \), i.e. if \( n = 2m \) for some \( m \geq 1 \), then \( \Delta_\Phi = \Delta_1 \cup \Delta_2 \cup \Delta_3 \), where \( \Delta_1 = \{c = 0\} \), \( \Delta_2 = \{c^{m-1} = Kd^m\} \), \( \Delta_3 = \{c^{m-1} = -Kd^m\} \) (for some constant \( K \neq 0 \)).

The resolution graph of \( P^{-1}(0) \cup \Delta_\Phi = \Delta_1 \cup \Delta_2 \cup \Delta_3 \) is

The reader is invited to construct the corresponding diagrams in the case \( l = 1 \).

In this second case \( \Delta_\Phi \) has two components: \( \Delta_1 = \{c = 0\} \) and \( \Delta_2 = \{c^{2m-1} = K'd^{2m+1}\} \) (for some constant \( K' \neq 0 \)). Note that for both \( l = 1 \) and \( l = 2 \) one has \( d_1 = 1 \).

4.10. Remark. Notice that all the germs \( f + g^k \) \((k \geq k_0)\) define isolated hypersurface singularities. The name of the series originates from the work of I. N. Iomdin \([5]\), who studied the case when \( g \) was a generic linear form (called the “classical” case).

There is a rich literature dealing with the invariants of the germs of the (classical or generalized) Iomdin series. Some results compare the invariants of \( f + g^k \) with some invariants of \( \Phi \). For example, using the polar curve method, one can determine the difference of the Euler-characteristic of the Milnor fibers of \( f + g^k \) and of the ICIS \( \Phi \) using the multiplicity of \( \Delta_\Phi \) (for details, see some of the papers of Lê Dũng Tráng and B. Teissier, e.g. \([20]\)).

Other results compare some invariant of \( f + g^k \) with the corresponding invariant of the non-isolated singularity \( f \). Correction term formulae \( i(f + g^k) - i(f) \) were obtained for many “additive” invariants \( i \): in the classical case by Iomdin for the case when \( i \) denotes the Euler characteristic of the corresponding Milnor fibers \([5]\), by D. Siersma when \( i \) denotes the zeta function \([19]\), by M. Saito when \( i \) is the spectrum \([17]\), and by A. Némethi when \( i \) is the signature \([11, 12]\). In order to explain why it is more convenient to compute the correction term \( i(f + g^k) - i(f) \),
rather than the individual terms $i(f + g^k)$ and $i(f)$, we introduce the following object. Fix a good representative of $\Phi$, and assume that $\Delta_1 = \{c = 0\}$.

4.11. Definition. For any integer $M > 0$, we define the “wedge set” of $\Delta_1 \subset D^2$ by

$$W_{\eta,M} = \{(c,d) \in D^2_\eta \mid 0 < |c| < |d|^M\}.$$ 

Now, it is easy to verify that for a sufficiently large integer $M >> 0$ (and with $\eta$ shrunk, if necessary) $W_{\eta,M} \subset D^2_\eta \setminus \Delta_\Phi$. In particular, $\Phi$ is a locally trivial fibration over $W_{\eta,M}$.

The point is that in order to determine the individual terms $i(f + g^k)$, one needs, in general, to understand the usually rather complicated monodromy representation associated with $\Phi$. On the other hand, for some “nice” (in some sense “additive”) invariants (such as the invariants listed above), the correction term $i(f + g^k) - i(f)$ depends only on the behavior of $\Phi$ above $\Delta_1$ and on the representation associated with the fibration above a wedge set. But this representation is much simpler than the original monodromy representation, since $\pi_1(W_{\eta,M}) = \mathbb{Z}^2$. In fact, this is exactly the main idea in the construction of the series: the geometry of the limit object is changed in a small tubular neighborhood of a knot. Here the knot is $\Delta_1 \cap \partial D^2$, and its tubular neighborhood is $W_{\eta,M} \cap \partial D^2$.

Recall that the aim of this work is to compute another invariant: the resolution invariants (such as the invariants listed above), the correction term depends only on the behavior of $\Phi$ over the closure of a wedge set. Moreover, our aim is to obtain the “whole” invariant, not “just” a correction term. Nevertheless, the wedge set will still play a crucial role: the resolution graphs of all the germs $(X_k,0)$ depend only on the behavior of $\Phi$ over the closure of a wedge set. Moreover, the construction will provide a “correction term” as well.

Our method is completely different (although obviously not absolutely independent) from the polar curve method, or from the usual constructions applied in the cases $i(f + g^k) - i(f)$. The discussion of some connections is the subject of another paper [16], where e.g. the relationship between the resolution graphs of the singularities $(X_k,0)$ and the monodromy representation over $W_{\eta,M}$ will be developed.

5. A graph associated with the ICIS $\Phi$.

A special curve arrangement.

5.1. – The embedded resolution of $f^{-1}(0) \cup g^{-1}(0) \subset \mathbb{C}^3$ and its stratification. Consider an ICIS $\Phi = (f,g) : (\mathbb{C}^3,0) \to (\mathbb{C}^2,0)$ and let $(D,0) \subset (\mathbb{C}^3,0)$ denote the local divisor $(D,0) := ((fg)^{-1}(0),0)$.

Fix an embedded resolution $r : V^{emb} \to U$ of $(D,0) \subset (\mathbb{C}^3,0)$. This means that $U$ is a small representative of $(\mathbb{C}^3,0)$, $V^{emb}$ is a smooth analytic 3-manifold and $r$ is a proper map such that:

(a) $r|_{V^{emb} \setminus r^{-1}(Sing(f^{-1}(0)) \cup Sing(g^{-1}(0)))}$ is a biholomorphic isomorphism onto its image;

(b) the total transform $D := r^{-1}(D)$ is a normal crossing divisor.

Recall that by the general theory, $r$ is isomorphic over the complement of $Sing(D)$. In the above definition, $Sing(f^{-1}(0)) \cup Sing(g^{-1}(0))$ is a smaller set than
5.2. Definitions. To each $p \in D$ assign the integer $k \in \{1, 2, 3\}$, if in the above local equation there are exactly $k$ non-zero integers among $l, m, n$. Denote the stratum corresponding to the number $k$ by $D^{(k)}$.

Write $D$ as the union of the following three subsets: $D_c = r^{-1}([f = 0, g \neq 0])$, $D_d = r^{-1}([g = 0, f \neq 0])$ and $D_0 = r^{-1}([g = 0, f = 0])$.

A point $p \in D^{(1)}$ is said to be of type $*$ (where $* = c$ or $d$ or $0$), if $p \in D_c$. Similarly, an irreducible component $B$ of the total transform $D$ is of type $*$ if its generic point is of type $*$.

A point $p \in D^{(2)}$ is said to be of type $*_1 - *_2$, if $p$ is an intersection point of two local divisors of type $*_1$ and $*_2$. An irreducible curve $C \subset D^{(2)}$ is of type $*_1 - *_2$ if its generic point is of that type. There is a similar definition for the points of the strata $D^{(3)}$.

In the sequel $\Phi$ denotes a fixed good representative of the ICIS, $r$ is an embedded resolution as above with the compatibility $\bar{U} = \Phi^{-1}(D^{(c)}_r) \cap B$. Define the composition $\Phi = \Phi \circ r$. Then the restriction $r^{-1}(\Phi^{-1}(D^{(c)}_r - \Delta) \cap B) \to D^{(c)}_r - \Delta$ of $\Phi$ is a locally trivial $C^\infty$-fibration, which is obviously equivalent with the fibration induced by $\Phi$ over $D^{(c)}_r - \Delta$. Its fibre $\tilde{F}_{c,d} = \Phi^{-1}(c,d)$ is called “a lifted Milnor fibre”.

5.3. The curve $C$. Notice that in general (and also in most of the concrete examples) it is rather difficult to find a resolution $r$. Moreover, the codification of its exceptional divisors and the corresponding intersections and normal bundles can be a rather difficult problem. Nevertheless, what we need from this resolution is only a 1–dimensional object, the special curve arrangement:

$C = (D_c \cap D_0) \cup (D_c \cap D_d)$.

In order to understand the special role $C$ plays in the geometry of the series, notice the following (see also the comments in 4.10–4.11). Since $\{f + g^k = 0\} = \Phi^{-1}([c+d^k = 0])$ and $\{c+d^k = 0\} \subset W_{n,M}$ (for all $k > M$), all the information that we need is in the space above $W_{n,M}$. But $\Phi^{-1}(W_{n,M})$ contains the 2–dimensional exceptional divisor $D_0 = \Phi^{-1}(0)$ as well (a fact which is not very encouraging). However, as it turns out, the relevant information is already contained in a smaller set, the closure of $\Phi^{-1}(W_{n,M})$. The intersection of this set with $D_0$ is exactly $C$ (see below). This shows that from the resolution $r$ we only need a small tubular neighbourhood of $C$.

5.4. Let $C = C_1 \cup \cdots \cup C_l$ be the decomposition of $C$ into irreducible components. By construction, each $C_i$ is either a smooth curve or it has some self–intersection (double point) singularities. Let $T(C_i)$ denote a fixed small open tubular neighbourhood of $C_i$ in $V^{emb} \ (1 \leq i \leq l)$, and write $T(C) = \cup_{i=1}^l T(C_i)$.
5.5. Theorem. – First characterization of $\mathcal{C}$. For any open (tubular) neighbourhood $T(\mathcal{C})$, there exist a sufficiently large integer $M$ and a sufficiently small $\eta$ such that

$$\Phi^{-1}(W_{n,M}) \subset T(\mathcal{C}).$$

This follows from the identity: $\Phi^{-1}(W_{n,M}) \cap \Phi^{-1}(0) = \mathcal{C}$.

Proof. The identity follows by local verifications. As an exemplification, we show that if $p \in D^{(1)}$ is of type 0, then $p \notin \Phi^{-1}(W_{n,M})$. Indeed, in a local coordinate neighbourhood $U_p$ of $p$, $\Phi|_{U_p} = (w^m, w^n)$ for some integers $\mu, n > 0$. Assume that $p \in \Phi^{-1}(W_{n,M})$. Then there exists a sequence $\{z_j\}_{j=1}^{\infty}$ in $\Phi^{-1}(W_{n,M}) \cap U_p$ for which $\lim_{j \to \infty} z_j = p$. Hence, if $(u_j, v_j, w_j)$ are the local coordinates of $z_j$, then $(u_j, v_j, w_j) \to 0$. Moreover, since $z_j \in \Phi^{-1}(W_{n,M})$ it follows that $\Phi(z_j) \in W_{n,M}$, therefore $|u_j|^m < |u_j|^\mu$. This contradicts $M\mu - m > 0$, for $M > 0$ sufficiently large. All the other local verifications are left to the reader.

The inclusion follows from the identity and the properness of $\Phi$. \hfill \blacklozenge

5.6. Corollary. a) For any open (tubular) neighbourhood $T(\mathcal{C}) \subset V^{\text{emb}}$, there exist a sufficiently large $M$ and a sufficiently small $\eta$ such that for any $(c, d) \in W_{n,M}$ the “lifted Milnor fibre” $\overline{F}_{c,d}$ is in $T(\mathcal{C})$.

b) For any $p \in \mathcal{C}$ and local neighbourhood $U_p$ of $p$, there exist a sufficiently large $M$ and a sufficiently small $\eta$ such that for any $(c, d) \in W_{n,M}$ one has $U_p \cap \overline{F}_{c,d} \neq \emptyset$.

5.7. Corollary. The curve $\mathcal{C}$ is connected.

Proof. Use (5.6) and the fact that $F_{c,d}$ (or equivalently, $\overline{F}_{c,d}$) is connected. \hfill \blacklozenge

5.8. Theorem. – Second characterization of $\mathcal{C}$. Consider the punctured disc $D_\eta = \{c = 0, d \neq 0\} \cap D^2_\eta \subset \mathbb{C}^2$. Then

$$\Phi^{-1}(D_\eta) \cap \Phi^{-1}(0) = \mathcal{C}.$$ 

Its proof is similar to that of (5.5). The analog of (5.6) is:

5.9. Corollary. a) For any open tubular neighbourhood $T(\mathcal{C}) \subset V^{\text{emb}}$, if $|d|$ is sufficiently small then $\Phi^{-1}(0, d) \subset T(\mathcal{C})$.

b) For any $p \in \mathcal{C}$ and neighbourhood $U_p$ of $p$, if $|d|$ is sufficiently small then $U_p \cap \Phi^{-1}(0, d) \neq \emptyset$.

The weighted dual graph of the curve arrangement $\mathcal{C}$.

5.10. For each irreducible component $C_i$ of $\mathcal{C}$, there are exactly two irreducible components $B_1$ and $B_2$ of the total transform $D$ for which $C_i$ is a component of $B_1 \cap B_2$. By the definition of $\mathcal{C}$, we can assume that $B_1$ is of type $c$ and $B_2$ is of type $d$ or $d$. Let $m_{f,B_i}$ (respectively of $g \circ r$) be the vanishing order (or multiplicity) of $f$ or $g \circ r$ along $B_i$ ($i = 1, 2$). Then $m_{g,B_i} = 0$, $m_{f,B_2} > 0$, $m_{g,B_1} > 0$, and $m_{f,B_2} > 0$. Notice that from the ordered triple $(m_{f,B_1}; m_{f,B_2}, m_{g,B_1})$ one can recover the local equations for $f \circ r|_{U_p}$ and $g \circ r|_{U_p}$ in a small coordinate neighbourhood $U_p$ of any point $p \in C_i \cap D^{(2)}$. 
The components $C_i$ of $\mathcal{C}$ are either compact (projective) or non-compact. The compact components are exactly those which are contained in $r^{-1}(0)$.

The non-compact components form the strict transform of $\{f = g = 0\}$. In particular, they are of type $c - d$. Moreover, since $(f, g)$ is an ICIS, the multiplicity of $f$ (resp. of $g$) along the strict transform of $\{f = 0\}$ (resp. of $\{g = 0\}$) is one. Therefore, the ordered triple $(m_{f,B_1}; m_{f,B_2}, m_{g,B_2})$ assigned to such an irreducible curve is $(1; 0, 1)$.

Now we are ready to define the weighted dual graph $\Gamma_C$ of $\mathcal{C}$. Similarly to the embedded resolution graphs of germs defined on normal surface singularities (cf. 2.1–2.4), the graph $\Gamma_C$ has vertices $\mathcal{V}$ and edges $\mathcal{E}$.

5.11. Definition. The set of vertices $\mathcal{V}$ consists of two disjoint subsets $\mathcal{W}$ and $\mathcal{A}$, where $\mathcal{W}$ denotes the non-arrowhead vertices, and $\mathcal{A}$ denotes the arrowhead vertices. Let the non-arrowhead vertices correspond to the compact irreducible curves, and the arrowhead vertices to the non-compact irreducible curves of $\mathcal{C}$.

For any two curves $C_i, C_j \subset \mathcal{C}$, which correspond to vertices $v_i, v_j \in \mathcal{V}$, if $C_i$ and $C_j$ intersect in $l$ points, then let the vertices $v_i$ and $v_j$ be connected by $l$ edges $e_k = \{(v_i, v_j)_k\}_{k=1,...,l}$ in $\Gamma_C$. Moreover, if a compact component $C_i \subset \mathcal{C}$, corresponding to a vertex $v_i \in \mathcal{W}$, intersects itself, then let each self-intersection point determine a loop (as a special edge) $(v_i, v_i)$ in the graph $\Gamma_C$. The edges are not directed, i.e. $(v_i, v_j)_k = (v_j, v_i)_k$.

Let the graph $\Gamma_C$ be decorated as follows:

- Let each non-arrowhead vertex $v_i \in \mathcal{W}$ have two weights assigned to it: the ordered triple of integers $(m_{f,B_1}; m_{f,B_2}, m_{g,B_2})$ assigned to the irreducible component $C_i$ corresponding to the vertex $v_i$, and the genus $g_i$ of (the normalization of) $C_i$.

- Let each arrowhead vertex have a single weight, the ordered triple $(1; 0, 1)$.

- Let each edge have a weight $\in \{1, 2\}$ determined as follows. By definition, any edge $e$ corresponds to an intersection point $p \in \mathcal{D}^{(3)}$, hence it is the intersection point of three local irreducible components of $\mathcal{D}$. The corresponding types of these components are $c - * - **$ or $c - c - *$, with $* \neq c$. In the first case let the weight on the edge $e$ be 1, in the second case 2.

5.12. Summary of notation for $\Gamma_C$, and local equations.

$(m; n, v)$

Vertices.
- Case 1. For all $w \in \mathcal{W}$, $[g]$ means that the corresponding irreducible curve $C_w \subset \mathcal{C}$ is compact and its genus is $g$. When $C_w$ is rational, $[g]$ will be omitted.

  Furthermore, there is a local coordinate neighbourhood $U_p$ of any point $p \in C_w \cap \mathcal{D}^{(2)}$ such that the local irreducible components $U_p \cap \mathcal{D} = B_1^{(2)} \cup B_2^{(2)}$ of $\mathcal{D}$ have representation $B_1^{(2)} = \{u = 0\}$, $B_2^{(2)} = \{v = 0\}$, and

$$f \circ r|_{U_p} = u^m v^n, \quad g \circ r|_{U_p} = v^\nu$$

with $m, \nu > 0; \ n \geq 0$.

Obviously, if $C_w$ is of type $c - d$ then $n = 0$, and if $C_w$ is of type $c - 0$ then $n > 0$. 

Case 2. \[ (1;0,1) \] denotes an arrowhead \( v \in \mathcal{A} \).

The curve \( C_v \) corresponding to \( v \) is non-compact. In a neighbourhood \( U_p \) of a generic point \( p \in C_v \cap \mathcal{D}^{(2)} \) we have \( U_p \cap \mathcal{D} = B_1^1 \cup B_2^1 \) with \( B_1^1 = \{ u = 0 \} \), \( B_2^1 = \{ v = 0 \} \) and

\[
f \circ r|_{U_p} = u, \quad g \circ r|_{U_p} = v.
\]

**Edges.** An edge \( e \) corresponds to an intersection point \( p \in C_i \cap C_j \) (or a self-intersection of \( C_i \) if \( i = j \)). In a local coordinate neighbourhood of \( p \), the irreducible components of \( U_p \cap \mathcal{D} = B_1^1 \cup B_2^1 \cup B_3^1 \) are given by \( B_1^1 = \{ u = 0 \} \), \( B_2^1 = \{ v = 0 \} \), \( B_3^1 = \{ w = 0 \} \).

Case 1. \[ (m; n, \lambda) \] \[ (m'; \nu) \]

means that the unique component of type \( c \) is \( B_1^1 \), and

\[
f \circ r|_{U_p} = u^m v^n w^l, \quad g \circ r|_{U_p} = v^\nu w^\lambda.
\]

In fact, \( n = l = 0 \) means a \( (c - d - d) \)–type intersection, while \( n > 0, l > 0 \) a \( (c - d - 0) \)–type intersection.

If \( v_2 \) is replaced by an arrowhead, then the corresponding edge is \[ (1; 0, 1) \]

The local equations \( f \circ r|_{U_p} \) and \( g \circ r|_{U_p} \) are as above with \( m = \lambda = 1 \) and \( l = 0 \).

Case 2. \[ (m; n, \nu) \] \[ (m'; n, \nu) \]

means that there are two components of type \( c \), namely \( B_1^1 \) and \( B_2^1 \), and

\[
f \circ r|_{U_p} = u^m v^\nu w^n, \quad g \circ r|_{U_p} = v^\nu.
\]

In fact, \( l > 0 \) gives a \( (c - c - 0) \)–type intersection, while \( l = 0 \) a \( (c - c - d) \)–type intersection.

If \( v_2 \) is replaced by an arrowhead, then the corresponding edge is \[ (1; 0, 1) \]

The local equations \( f \circ r|_{U_p} \) and \( g \circ r|_{U_p} \) are as above with \( m' = \nu = 1 \) and \( n = 0 \).

5.13. A compatibility property of the decorations.

Consider the edge:

\[ (m; a, b) \xrightarrow{x} (n; c, d) \]

a.) if \( m \neq n \), then \( (a, b) = (c, d) \) and \( x = 2 \);

b.) if \( (a, b) \neq (c, d) \), then \( m = n \) and \( x = 1 \).

In particular, in the cases (a)-(b) above, the weight of the edge is determined by the weights of the vertices. In all these cases we omit the corresponding edge-decoration.
Examples.

5.14. Preliminary remarks.

1. We emphasize again that in general it is rather long and difficult to find a resolution \( r \). In the literature there are very explicit resolution algorithms, but they are rather involved, and in general very slow. Therefore, in almost all of our examples, we used a sequence of some ad hoc blow ups (following the naive principle: “blow up the worst singular locus”) with the hope to obtain a more or less small configuration. The computations are extremely long, and are not given here. (They were, in fact, done with the help of Mathematica.) Hence, we admit that for the reader the verification of some of the examples listed in the body of the paper can be a really difficult job.

2. Since the resolution \( r \) is not unique, different resolutions \( r \) might result in different graphs \( \Gamma_C \). Moreover, contrary to the case of resolution graphs of normal surface singularities, in the case of \( \Gamma_C \) we cannot claim (at this moment) the existence of a unique minimal graph. On the other hand, the properties of the graph \( \Gamma_C \) (see the next subsection) suggest that \( \Gamma_C \) inherits a lot of properties of the resolution graphs of normal surface singularities. In particular, we expect the existence of some kind of calculus of graphs, which would provide a method to identify all the possible graphs associated with the same ICIS \( \Phi \).

5.15. Let \( f(x, y, z) = x^n + y^n + xyz^{n-2} \) and \( g(x, y, z) = z \). A possible \( \Gamma_C \) is:

More generally, if \( f \) is given by a homogeneous polynomial, and \( g \) is a generic linear function, then the construction of \( \Gamma_C \) is not difficult. Indeed, first blow up the origin. Then consider the intersection curve \( C \) of the strict transform \( St \) of \( \{ f = 0 \} \) with the exceptional divisor \( E \). In the neighbourhood of a singular point \( p \) of \( C \), \( (St, p) \) has a natural product structure \( (C, p) \times (D, 0) \), where \( (D, 0) \) denotes the disc. Then any embedded resolution of \( (C, p) \subset (E, p) \) provides an embedded resolution of \( (St, p) \) if the quadratic transformations of the plane are replaced by the corresponding blow ups along 1–dimensional axes in the direction \( (D, 0) \).

5.16. Let \( f(x, y, z) = x^2 + y^2 \) and \( g(x, y, z) = x^n + y^n + z^n \). A possible \( \Gamma_C \) is:
The graph $\Gamma_C$ is not unique, but depends on the embedded resolution $r$. Another valid graph $\Gamma_C$ for the same pair $(f, g)$ is (cf. also with 5.22):

5.17. Let $f(x, y, z) = x^2 - y^2$ and $g(x, y, z) = y^2 + z^3$. A possible $\Gamma_C$ is:

5.18. Consider $f(x, y, z) = x^3 + y^2 + xyz$ and $g(x, y, z) = z$. A possible $\Gamma_C$ is:

5.19. Consider $f(x, y, z) = x^2 y^2 + z^2(x + y)$ and $g(x, y, z) = x + y + z$. A possible $\Gamma_C$ is:

5.20. Consider $f(x, y, z) = f'(x, y)$ and $g(x, y, z) = z$, where $f' : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ is an isolated plane curve singularity. Replacing the quadratic transformations of the plane by blowing ups along 1-dimensional axis (the local $z$-axis), one obtains the following. Let $\Gamma(\mathbb{C}^2, f')$ denote the minimal embedded resolution graph of the plane curve singularity $f' : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$. Then a possible dual graph $\Gamma_C$ can be obtained from $\Gamma(\mathbb{C}^2, f')$ using the following conversion:
a non-arrowhead vertex -\(e\) should be replaced by \((m;0,1)\)

an arrowhead vertex \(-\cdot(1)\) should be replaced by \(-\cdot(1;0,1)\)

All edges of \(\Gamma_C\) have weight 2 and all vertices \(w \in W\) of \(\Gamma_C\) have genus 0.

For example, let \(f(x, y, z) = f'(x, y) = x^2 - y^3\) and \(g(x, y, z) = z\). Then we have:

\[\Gamma(C^2, f'):\]

\[
\begin{array}{cccc}
(2) & (6) & (3) \\
\end{array}
\]

\[\Gamma_C:\]

\[
\begin{array}{cccc}
(2;0,1) & (6;0,1) & (3;0,1) \\
\end{array}
\]

Some properties of the weighted dual graph \(\Gamma_C\).

5.21. – The first partition of \(\Gamma_C\). The vertices of the graph \(\Gamma_C\) can be divided into two disjoint sets \(V(\Gamma_C) = V^1(\Gamma_C) \cup V^2(\Gamma_C)\), where \(V^1(\Gamma_C)\) (respectively \(V^2(\Gamma_C)\)) consists of the vertices with weight \((m; n, \nu)\) where \(m = 1\) (respectively \(m \geq 2\)). We will use similar notations for \(W(\Gamma_C)\) and \(A(\Gamma_C)\).

The graph \(\Gamma_1^C\). Consider the maximal subgraph \(\Gamma_1^C\) of \(\Gamma_C\) which is spanned by the vertices \(v \in V^1(\Gamma_C)\) and has no edges of weight 2 (cf. 5.22). Notice that all the arrowheads of \(\Gamma_C\) are arrowheads of \(\Gamma_1^C\) as well, i.e. \(W(\Gamma_1^C) = W^1(\Gamma_C)\) and \(A(\Gamma_1^C) = A(\Gamma_C)\).

If an arrowhead \(v\) is supported by an edge of weight 2, then \(v\) becomes an arrowhead of \(\Gamma_1^C\) without any supporting edge. This arrowhead vertex \(v\) (as a completely isolated vertex) forms a connected component of \(\Gamma_1^C\).

The graph \(\Gamma_2^C\). The “complementary” subgraph \(\Gamma_2^C\) is constructed in three steps. First, consider the maximal subgraph \(\Gamma_2^C\) of \(\Gamma_C\) spanned by the vertices \(v \in V^2(\Gamma_C)\). Second, in order to obtain \(\Gamma_2^C\), we add some arrowheads (and edges which support these arrowheads) as follows. Consider an edge \(e\) of weight 2 with endpoints \(v_2 \in W^2(\Gamma_C)\) and \(v_1 \in V^1(\Gamma_C)\). For such an edge, regardless of \(v_1\) being an arrowhead or not, perform the following transformation:

\[
\begin{array}{cccc}
(m; n, \nu) & 2 & (m; n, \nu) & 2 \\
| g_1 \| & | g_1 \| & | g_1 \| & | g_1 \| \\
\end{array}
\]

\[
\begin{array}{cccc}
v_2 & v_1 & v_2 & v_1 \\
\end{array}
\]

Thus \(\mathcal{E}(\Gamma_2^C) = \mathcal{E}(\Gamma_2^C) \cup \{\text{edges of type } e\}, \mathcal{W}(\Gamma_2^C) = V^2(\Gamma_C) = W^2(\Gamma_C)\) and \(#A(\Gamma_2^C) = #\{\text{edges of type } e\}\).
Finally, consider the set of edges in $\Gamma_C$ of weight 2 with both endpoints in $V^1(\Gamma_C)$ (these edges were deleted in the construction of $\Gamma_C^1$). Any edge of this type is transformed into a double arrow:

\[(l; n, \nu) \xrightarrow{2} (l; n, \nu)\]

(In general, this type of graph codifies the minimal embedded resolution of a plane curve singularity of type $A_1$.) (Cf. also with 5.22)

Then, by definition, the graph $\Gamma_C^2$ is the union of $\Gamma_C^2$ and the collection of the double arrows considered above. Obviously, each double arrow forms a connected component of $\Gamma_C^2$.

5.22. Example. – The case of 2–edges (first case): the double arrows.
Consider an edge of $\Gamma_C$ with weight 2 whose both endpoints are weighted by $(1, n, \nu)$ (and arbitrary second weights $[g]$ and $[g']$). This edge corresponds to a point $p \in D^{(3)}$ of type $c-c-\ast$, with $\ast \in \{0, d\}$. Moreover, since the first multiplicity of the triplet $(1; n, \nu)$ is 1, both $c$–type local irreducible components $B^1_l$ and $B^2_l$ of $D$ at $p$ are part of the strict transform of $\{f = 0\}$ (cf. 5.12). Their intersection corresponds to the strict transform of an irreducible component $\Sigma_l$ of $Sing\{f = 0\}$ with transversal type $A_1$, which has not been blown up during the resolution procedure $r$.

Performing an additional blow-up along this intersection, we obtain a new embedded resolution $r'$, whose special curve arrangement $C'$ will have an additional rational curve. The dual graph $\Gamma_C$ is changed to the dual graph $\Gamma_C'$ via the transformation:

\[
\begin{array}{c}
\Gamma_C \\
(l; n, \nu) & (l; n, \nu) \\
[g] & [g'] & [g] & [g']
\end{array}
\quad \xrightarrow{2} \quad
\begin{array}{c}
\Gamma_C' \\
(l; n, \nu) & (2; n, \nu) & (l; n, \nu) \\
[g] & [g'] & [g]
\end{array}
\]

Therefore, it may be assumed that $\Gamma_C$ has no edges of this type.

Now, we discuss the properties of $\Gamma_C$ and its subgraphs $\Gamma_C^1$ and $\Gamma_C^2$. An immediate consequence of corollary (5.7) is the following.

5.23. Proposition. The graph $\Gamma_C$ is connected.

5.24. Remark. Neither $\Gamma_C^1$ nor $\Gamma_C^2$ is connected, in general. In fact, the number of connected components of $\Gamma_C^1$ is exactly the number of irreducible components of $\{f = 0\}$ (cf. 5.27), and the number of connected components of $\Gamma_C^2$ is exactly the number of irreducible components of $Sing\{f = 0\}$ (cf. 5.32). These two facts can easily be seen on our examples. E.g.: in the case of (5.16), the graph $\Gamma_C^1$ has two components; and in the case of (5.19) $\Gamma_C^2$ has two components.

5.25. – The garph $\Gamma_C^1$. In order to characterize the subgraph $\Gamma_C^1$ one more notation is needed. Consider the germ $((f = 0), 0)$ and let $n : \{f = 0\}^{\text{norm}} \to \{f = 0\}$ be its normalization. Notice that $\{f = 0\}^{\text{norm}}$ is the disjoint union of some germs of normal surface singularities, each germ being the normalization of one of the irreducible components of $((f = 0), 0)$. In particular, their number is exactly the number of irreducible components of $((f = 0), 0)$ (see Part I, 2.6).
5.26. Definition. Consider the subgraph $\Gamma^1_C$ of $\Gamma_C$. For any vertex $v \in \mathcal{V}(\Gamma^1_C)$, replace its weight $(1; n, \nu)$ by the weight $(\nu)$. For any non-arrowhead vertex $w \in \mathcal{W}(\Gamma^1_C)$, keep its weight $[g]$. Think about this modified weighted graph as a weighted embedded resolution graph of a germ of an analytic function defined on a normal surface singularity, and about the weights $(\nu)$ as the multiplicities of the corresponding function (cf. Part I. 2.2). Calculate “self-intersection numbers” $e_w$ ($w \in \mathcal{W}$) for each non-arrowhead vertex via the relation (Part I, 2.32). Let the resulting weighted graph be denoted by $G^1_C$. By convention, a dual resolution graph, which consists of only one non-arrowhead vertex with weight $(1)$ (i.e. it has no other vertices, and no edges), codifies the dual embedded resolution graph of a smooth germ defined on a smooth surface, which has not been blown up during the resolution.

5.27. Theorem. The graph $G^1_C$ is a possible embedded resolution graph of $\{f = 0\}^{\text{norm}} \xrightarrow{\text{non}} (\mathbb{C}, 0)$. In particular, the number of connected components of $G^1_C$ (which is the same as the number of connected components of $\Gamma^1_C$) coincides with the number of irreducible components of the germ $\{f = 0\}$, which restriction of $g \circ \text{non}$ defines a smooth map-germ (and this smooth component of $\{f = 0\}^{\text{norm}}$ has not been modified during the resolution).

(Obviously, the subgraph $\Gamma^1_C$ might contain cycles and/or non-arrowhead vertices with $g > 0$.)

Proof. Consider the embedded resolution $r$ and denote by $St$ the strict transform of $\{f = 0\}$. Then each irreducible component $St_i$ of $St$ is smooth and the restriction $r|_{St_i} : St_i \to \{f = 0\}_i$ is a resolution of the corresponding component $\{f = 0\}_i$ of $\{f = 0\}$, which factors through the normalization of $\{f = 0\}_i$. Moreover, $g \circ r : St_i \to \mathbb{C}$ defines a normal crossing divisor on $St_i$.

Finally, notice that the components $\{St_i\}$ are the only $c$–type irreducible components of $D$ on which the vanishing order $m_f$ of $f$ is one. The remaining part of the proof consists of verifying some local equations and compatibilities of the corresponding graphs, which is left to the reader. $\diamond$

Now we start to analyze the properties of the subgraph $\Gamma^2_C$.

5.28. – Finer partitions. Corresponding to the partition $\mathcal{V}^1(\Gamma_C) \cup \mathcal{V}^2(\Gamma_C)$ of $\mathcal{V}(\Gamma_C)$, define $C^k$ as the union of those irreducible components of $C$ which correspond to vertices from $\mathcal{V}^k(\Gamma_C)$ ($k = 1, 2$). Obviously, $C = C^1 \cup C^2$ and $C^1 \cap C^2$ is a set of points.

Now, we define a finer partition of $\mathcal{V}^2(\Gamma_C)$. To any irreducible component $\Sigma_j$ of $\text{Sing}\{f = 0\}$, $1 \leq j \leq s$ (cf. 4.6), assign the set $B_{\Sigma_j}$ containing those irreducible components $B$ of the total transform $D$ which are of type $c$ and are projected via $r$ onto $\Sigma_j$. Then define $C_{\Sigma_j} \subset C$ as the union of those irreducible components $C$ of $C$ for which $C \subset B$ for some $B \in B_{\Sigma_j}$.
5.29. Lemma. \( \bigcup_{j=1}^{s} C_{\Sigma_j} = C^2 \) and \( C_{\Sigma_i} \cap C_{\Sigma_j} = \emptyset \), for any \( i \neq j \), \( i, j \in \{1, \ldots, s\} \).

Proof. The first identity follows from the fact that the set of \( c \)-type irreducible components of \( D \), along which \( f \) is vanishing with order strictly greater than 1, are exactly those components which project via \( r \) onto some component of \( \text{Sing}\{f = 0\} \).

For the second part, take a point \( p \in C_{\Sigma_i} \cap C_{\Sigma_j} \). Clearly, \( p \in D^{(3)} \) and it is of type \( c - c - * \) with \( * \neq c \). Consider the three local coordinate axes (as in the local description 5.12). One of them is contained in a component \( C \subset C_{\Sigma_i} \), another in a component \( C' \subset C_{\Sigma_j} \). Denote the third local coordinate axis by \( C^* \). Since it is contained in a component \( B \in B_{\Sigma_i} \), it projects via \( r \) onto \( \Sigma_i \). But this is true also for \( j \), hence \( r(C^*) \subset \Sigma_i \cap \Sigma_j = \{0\} \). But along \( C^* \) the function \( g \) is not constant, which contradicts \( r(C^*) = \{0\} \). \( \Box \)

In the next lemma, \( T(C_{\Sigma_j}) \) denotes the tubular neighbourhood \( \bigcup T(C_i) \) (cf. 5.4), where the union is taken over the components \( C_i \subset C_{\Sigma_j} \).

5.30. Lemma. a) For any open (tubular) neighbourhood \( T(C_{\Sigma_j}) \subset V^{emb} \), there exist a sufficiently small \( \gamma > 0 \) such that for any point \( q \in \Sigma_j - \{0\} \subset \mathbb{C}^3 \) with \( |q| < \gamma, r^{-1}(q) \subset T(C_{\Sigma_j}) \).

b) For any \( p \in C_{\Sigma_j} \) and local neighbourhood \( U_p \) of \( p \), there exist a sufficiently small \( \gamma > 0 \) such that for any point \( q \in \Sigma_j - \{0\} \) with \( |q| < \gamma, U_p \cap r^{-1}(q) \neq \emptyset \).

Proof. The proof is similar to the proof of (5.6) and (5.9) (actually, it can also be deduced from (5.9)). \( \Box \)

5.31. Corollary. \( C_{\Sigma_j} \) is connected for all \( j \in \{1, \ldots, s\} \).

Proof. For any \( q \in \Sigma_j - \{0\} \), the fiber \( r^{-1}(q) \) is connected by Zariski’s Connectivity Theorem (see e.g. [4]). Then use (5.30). \( \Box \)

Therefore, \( C^2 \) can be written as a disjoint union of the connected curves \( C_{\Sigma_j} \) (1 \( \leq j \leq s \)). An immediate consequence of this is the following.

5.32. Proposition. There is a one-to-one correspondence between the connected components of \( \Gamma^2_{\Sigma} \) and the irreducible components of \( \text{Sing}\{f = 0\}, 0 \). The set of non–arrowhead vertices in a connected component of \( \Gamma^2_{\Sigma} \) corresponds exactly to the irreducible components of one of the curves \( C_{\Sigma_j} \).

(Remark. Notice that in the case of “double arrow components” of \( \Gamma^2_{\Sigma} \), corresponding to the component \( \Sigma_j \) (cf. 5.22), one has \( C_{\Sigma_j} = \emptyset \).)

Later we will show that the minimal embedded resolution graph of the transversal singularity of a connected component \( \Sigma_j \) can be reconstructed from the corresponding connected components of \( \Gamma^2_{\Sigma} \) (cf. 5.43 and 5.51). But for this we need some further preparation.

5.33. – Even finer partitions. Fix a connected component \( \Gamma^2_{\Sigma_j} \) of \( \Gamma^2_{\Sigma} \) corresponding to the irreducible component \( \Sigma_j \) (1 \( \leq j \leq s \)). Its non–arrowhead vertices are denoted by \( W(\Gamma^2_{\Sigma_j}) \). We define on \( W(\Gamma^2_{\Sigma_j}) \) the following equivalence relation. First, we say that \( w_1 \sim w_2 \) if \( w_1 \) and \( w_2 \) are connected by an edge of weight 1, then we extend \( \sim \) to an equivalence relation. The equivalence classes \( \{K_i\}_i = \{K_i(\Gamma^2_{\Sigma_j})\}_i \) determine a partition of \( W(\Gamma^2_{\Sigma_j}) \).
Let $\Gamma(K_i)$ be the maximal subgraph of $\Gamma_{C,j}^2$ supported by the non-arrowhead vertices $K_i$. For the moment, we keep all the decorations of the corresponding vertices of $\Gamma(K_i)$. This graph is connected, it has no arrowheads, and its edges are all weighted by 1. (If from $\Gamma_{C,j}^2$ all the arrowheads and the edges of weight 2 are deleted, then the disjoint union $\cup_k \Gamma(K_i)$ is obtained.)

**5.34. Example.** Consider the example given in (5.56). The next figure illustrates the partition of $\Gamma_{C,1}^2$:

![Graph](image)

The graphs $\Gamma(K_i)$ are:

![Graph](image)

**5.35.** For a fixed equivalence class $K_i = \{w_{i_1}, w_{i_2}, \ldots, w_{i_t}\}$, consider the corresponding irreducible curves $C_{i_1}, C_{i_2}, \ldots, C_{i_t}$ of $C$. By construction, there exists an irreducible component $B(K_i) \in B_{\Sigma}^C$, which contains all of them. Moreover, the union $\mathcal{C}(K_i) := C_{i_1} \cup \cdots \cup C_{i_t}$ is a connected curve. Let $T(K_i)$ denote a small tubular neighborhood of $\mathcal{C}(K_i)$ in $B(K_i)$.

Finally, consider the weights $(m_{ik}, n_{ik}, \nu_{ik})$ associated with $w_{ik}$ (or with $C_{ik}$) $(1 \leq k \leq t)$. Again, by the definition of the partition, $m_{i_1} = \cdots = m_{i_t}$. We denote this integer by $m(K_i)$; it is exactly the vanishing order of $f \circ r$ along $B(K_i)$. On the other hand, the local equations show that the restriction $g \circ r|_{T(K_i)} : T(K_i) \rightarrow \mathbb{C}$ provides the principal divisor $(g \circ r)^{-1}(0) = \sum_k \nu_{ik} C_{ik}$ in $T(K_i)$.

Therefore, the divisor (multiple curve) $\sum_k \nu_{ik} C_{ik}$ in $T(K_i)$ can be interpreted as a central fiber of a proper analytic map (here $T(K_i)$ can be changed to $(g \circ r)|_{B(K_i)}^{-1}$(small disc)).

It is convenient to introduce the integer $\nu(K_i) := \gcd(\nu_{i_1}, \ldots, \nu_{i_t})$.

**5.36. Lemma.** The generic fiber of $(g \circ r)|_{T(K_i)}$ is a disjoint union of rational curves.

**Proof.** Fix $d \neq 0$ with $|d|$ sufficiently small. Then $g^{-1}(d)$ intersects $\Sigma_j$ in $d_j$ points $\{q_j\}_{i_1}$ (cf. 4.6). Then $((g \circ r)|_{T(K_i)})^{-1}(d) = \cup_i (r^{-1}(q_i) \cap T(K_i))$. But $r^{-1}(q_i)$ is a fiber of an embedded resolution of the transversal singularity associated with $\Sigma_j$, hence each irreducible curve in $r^{-1}(q_i)$ is rational. \[\diamond\]

**5.37.** General properties of morphisms whose generic fiber is rational. First we recall the following fact (cf. [3], page 554):

If $S$ is a minimal smooth surface, $D$ a disc in $\mathbb{C}$, and $\pi : S \rightarrow D$ any proper holomorphic map whose generic fiber $\pi^{-1}(d)$ ($0 \neq d \in D$) is irreducible and rational, then $\pi$ is a (trivial) $\mathbb{P}^1$-bundle over $D$.

Now, assume that $S$ is not minimal, and the generic fiber of $\pi : S \rightarrow D$ is a disjoint union of (say, $N$) rational curves. Then by the Stein Factorization theorem (see e.g. [4], page 280), (and shrinking $D$ if necessary), there exists a map
b : D' → D given by z → z^N, and π' : S → D' such that π = b ∘ π', and the generic fiber of π' is irreducible and rational. Since the central fibers of π and π' are the same, it follows from the above fact that the central fiber of π can be blown down successively until an irreducible rational curve is obtained.

This discussion has the following consequences:

5.38. Proposition. – Properties of the graph Γ(K_1).
   a) The graph Γ(K_1) is a tree with all g_{w_k} = 0. In particular, all the irreducible components of C^2 are rational curves.
   b) From the integers \{v_{i_k}\}_{k=1}^t one can deduce the self–intersections C_{i_k} of the curves C_{i_k} in B(K_1) as follows. First notice that the intersection matrix (C_{i_k} · C_{i_k})_{k,k'} (where the intersections are considered in B(K_1)) is a negative semi–definite matrix with rank t − 1, and the “central divisor” \sum_k v_{i_k} C_{i_k} is one element of its kernel (cf. [2], page 90). The intersections C_{i_k} · C_{i_{k'}} for \( i_k \neq i_{k'} \) can be read from the graph Γ(K_1) considered as a dual graph. The self–intersections can be determined from the relations (\sum_k v_{i_k} C_{i_k} · C_{i_k}) = 0 (cf. also with Part I, 2.32). In particular, if t = 1, then C_{i_1} = 0. If t ≥ 2, then the graph (curve configuration) is not minimal; if we blow down successively all the (−1)–curves we obtain a rational curve with self–intersection zero. This gives a complete classification of the shapes of the possible graphs Γ(K_1) (decorated with the set of integers \{v_{i_k}\}_k).
   c) The number of irreducible (equivalently, connected) components of the generic fiber of (g ∘ r)|T(K_1) is \( \nu(K_1) = g.c.d.(v_1, \ldots, v_t) \).
   The fact that each irreducible component of the generic fiber is rational can be translated into the relation:
   \[
   2 \cdot \nu(K_1) = \sum_{k=1}^t v_{i_k} (2 - \delta_{w_k}),
   \]
   where \( \delta_{w_k} \) is the number of vertices adjacent to \( w_k \) in Γ(K_1).

5.39. Example. Consider the example (5.34) (based on 5.56). Then the partition \{Γ(K_1)_1\}_1 of Γ^2_{C_1} decorated with the integers \( (v_{i_k})_k \) and the self–intersection numbers \( C^2_{i_k} \) is the following:

```
(1) (1) (1) (1) (1) (1) (2) (1)
-1 -1 0 -1 -2 -1 0 -2
```

5.40. Now we return to the connected component Γ^2_{C,j} of Γ^2_{C} corresponding to Σ_j (1 ≤ j ≤ s). In (5.33–5.39) we introduced the partition \{Γ(K_1)_l\}_l. The arrowheads of Γ^2_{C,j}, as well as the edges connecting different classes \{Γ(K_1)_l\}_l in Γ^2_{C} are 2–edges. Their detailed presentation is the subject of the next paragraph.

5.41. Example. – The case of 2–edges (second case). Consider an edge e of Γ_C of weight 2 having endpoints v_1 and v_2 such that v_2 is weighted with the triple \( (m'; n, ν) \) with \( m' ≥ 2 \). Consider the local coordinates \( (u, v, w) \) as in (5.12), with \( B'_1 = \{u = 0\} \), \( B'_2 = \{v = 0\} \) and \( B'_3 = \{w = 0\} \).
   First assume that the first entry of the weight \( (m; n, ν) \) of v_1 also satisfies \( m ≥ 2 \), i.e. both v_1 and v_2 are non–arrowheads of Γ^2_{C,j}. Then \( C_{v_1} \cap U_p \) and \( C_{v_2} \cap U_p \) are the intersections \( B'_1 \cap B'_3 \) and \( B'_2 \cap B'_3 \). Their intersection point p is codified in the
edge $e$. From the local equation $g \circ r|U_p = w''$, we obtain that in the deformation $g \circ r|B_1^1 \cup B_2^1 \to \mathbb{C}$ the intersection point $p$ splits into $\nu$ points.

If the first entry $m$ of the weight $(m; n, \nu)$ of $v_i$ is $m = 1$, then the discussion is similar.

In both cases, set $C^*_e := B_1^1 \cap B_2^1$. Then $r|C^*_e : C^*_e \to \Sigma_j$ is finite. Denote its degree by $d(e)$. Obviously $\text{deg}(r|C^*_e) \cdot \text{deg}(g|\Sigma_j) = \text{deg}(g \circ r|C^*_e) = \nu$. Sometimes it is more convenient (and precise) to use the notation $\nu(e)$ for $\nu$.

With these notations:

$$d(e) \cdot d_j = \nu(e).$$

Now we are able to characterize the structure of the subgraph $\Gamma_{j,j}^2$. The final goal is to reconstruct the embedded resolution graph of the transversal singularity associated with the branch $\Sigma_j$ from the weighted graph $\Gamma_{j,j}^2$. First we start with a construction. The reader is invited to review the theory of cyclic covering of graphs (Section 1, Part I).

5.42. – A covering construction from $\Gamma_{j,j}^2$. Fix an irreducible component $\Sigma_j$ of $\text{Sing}(f = 0)$ (1 $\leq j \leq s$), let $T\Sigma_j$ be the (equisingular type of the) transversal singularity associated with $\Sigma_j$ (cf. 4.7). Recall that $\text{deg}(g|\Sigma_j) = d_j$.

If $(H, q)$ is a transversal slice as in (4.7), then $r$ above $(H, q)$ determines a resolution of the transversal plane curve singularity $(H \cap \{f = 0\}, q) \subset (H, q)$. Its weighted dual embedded resolution graph (for a definition, see Part I, 2.2) is denoted by $G(T\Sigma_j)$. Since in local coordinates it is easier to work with the pullback of $g$, it is convenient to replace the single point $q \in \Sigma_j - \{0\}$ by the collection of $d_j$ points $g^{-1}(d) \cap \Sigma_j$ (where $|d|$ is small and non-zero). The dual weighted graph associated with the curves situated above these points consists of exactly $d_j$ identical copies of $G(T\Sigma_j)$, and it is denoted by $d_j \cdot G(T\Sigma_j)$.

Comparing the curves $r^{-1}(g^{-1}(d) \cap \Sigma_j)$ and $C_{\Sigma_j}$ by the corresponding local equations, and using the results of (5.38), especially part (e), and (5.41), we obtain a cyclic covering of graphs

$$p : d_j \cdot G(T\Sigma_j) \to \{\text{a base graph}\}_{j},$$

where the base graph and the covering data can be determined from $\Gamma_{j,j}^2$. This is given in the next paragraphs.

The base graph will be denoted by $\Gamma_{j,j}^2 / \sim$. It is obtained from $\Gamma_{j,j}^2$ by collapsing it along edges of weight 1.

More precisely, each subgraph $\Gamma(K_i)$ is replaced by a non–arrowhead vertex. If two subgraphs $\Gamma(K_i)$ and $\Gamma(K_l)$ are connected by $k$ 2–edges in $\Gamma_{j,j}^2$, then the corresponding vertices of $\Gamma_{j,j}^2 / \sim$ are connected by $k$ edges. (In fact, from 5.50 will follow that $k \leq 1$.) If the non–arrowhead vertices of $\Gamma(K_i)$ support $k$ arrowheads altogether, then on the corresponding non–arrowhead vertex of $\Gamma_{j,j}^2 / \sim$ one has exactly $k$ arrowheads.

Since $\Gamma_{j,j}^2$ is connected, it is obvious that $\Gamma_{j,j}^2 / \sim$ is connected as well.

The covering data of $p : d_j \cdot G(T\Sigma_j) \to \Gamma_{j,j}^2 / \sim$.

First we recall that the covering data of a graph $\Gamma$ is a collection of positive integers $\{n_i\}_{e \in \Delta(\Gamma)}$ and $\{n_e\}_{e \in E(\Gamma)}$, such that for each edge $e \in E(\Gamma)$ with endpoints $v_1$ and $v_2$, $n_e = d_e \cdot \text{l.c.m}(n_{v_1}, n_{v_2})$ for some integer $d_e$. For details, see Section 1 (Part I), or at least (1.8).
Now, we define a covering data for $\Gamma^2_{C,j}/\sim$. It is provided by the third entries $\nu$ of the weights $(m;n,\nu)$ of the vertices of $\Gamma^2_{C,j}$.

For any non–arrowhead vertex $w$ of $\Gamma^2_{C,j}/\sim$, which corresponds to $K_i$ in the above construction, set $n_w := \nu(K_i)$. For any arrowhead vertex $v$ of $\Gamma^2_{C,j}/\sim$, which corresponds to an arrowhead of $\Gamma^2_{C,j}$ with weight $(1;n,\nu)$, set $n_v := \nu$. For any edge of $\Gamma^2_{C,j}/\sim$, which comes from a 2–edge $e$ of $\Gamma^2_{C,j}$ with endpoints with weight $(\pm;n,\nu)$, set $n_e := \nu(\pm \nu(e))$.

For simplicity (and suggestively) the covering data (which in Section 1 (Part I) is denoted by $(n,d)$), will here be denoted by $\nu$.

The degeneration of $r^{-1}(g^{-1}(d)\cap \Sigma_j)$ into $C_{\Sigma_j}$ provides the (already announced) result:

5.43. Theorem. – Characterization of the transversal singularities (first part). For any $1 \leq j \leq s$, there exists an embedded resolution graph $G(T\Sigma_j)$ of the transversal singularity associated with the branch $\Sigma_j$, and a cyclic covering of graphs

$$p : d_j \cdot G(T\Sigma_j) \to \Gamma^2_{C,j}/\sim$$

with covering data $\nu$ (described above) and with the compatibility of the arrowheads:

$A(d_j \cdot G(T\Sigma_j)) = p^{-1}(A(\Gamma^2_{C,j}/\sim))$ (cf. 1.25 (1)).

In particular, by (Part I, 1.23), (and from the fact that $G(T\Sigma_j)$ is a tree, being the graph of a plane curve singularity), one obtains that $\Gamma^2_{C,j}/\sim$ is a connected tree.

5.44. Remark. It is not easy to find concrete examples, when the covering data $\nu$ is not trivial (i.e. when the integers $n_v$ and $n_e$ are not all the same). Example (5.19) shows such an example (see 5.55 for the detailed discussion).

5.45. The above theorem (5.43) has some important consequences. Before we formulate the first, we invite the reader to verify the following fact.

If $\Gamma$ is a tree, and $p : G \to \Gamma$ is a cyclic covering of $\Gamma$ with covering data $\{n_v\}_{v \in V(\Gamma)}$, then the number of connected components of $G$ is $\text{g.c.d.}\{n_v | v \in V(\Gamma)\}$.

This implied in (5.43), where $G(T\Sigma_j)$ and $\Gamma^2_{C,j}/\sim$ are connected, gives:

5.46. Corollary.

(a) $d_j = \text{g.c.d.}\{n_w | w \in W(\Gamma^2_{C,j})\}$, where $(m_w;n_w,\nu_w)$ is the weight of $C_w$.

(b) Let $E_j$ be the set of 2–edges of $\Gamma_C$ whose endpoints are non–arrowhead vertices such that one of the endpoints is in $W(\Gamma^2_{C,j})$, the other in $W(\Gamma^2_{C,j})$. (By construction, $E_j$ is exactly the index set of the arrowhead vertices of $\Gamma^2_{C,j}$.) Let $\#(T\Sigma_j)$ be the number of irreducible components of the transversal singularity $T\Sigma_j$. Then:

$$d_j \cdot \#(T\Sigma_j) = \sum_{e \in E_j} \nu(e).$$

5.47. Example. In the example (5.19), $\text{Sing}\{f = 0\}$ has two components. For both values $j \in \{1,2\}$, the transversal type of $\Sigma_j$ is $A_1$, $\#(T\Sigma_j) = 2$, $d_j = 1$, $\#E_j = 1$, and $\nu(e) = 2$ for $e \in E_j$. 


5.48. Example. If \( \#(T\Sigma_j) \neq 1 \), then even if \( \Gamma_C^1 \) and all \( \Gamma_C^2, j \)'s are trees, it is possible that \( \Gamma_C \) has some cycles. For example, if \( f = x^3 + y^2 + xyz \) and \( g = z \), then \( \text{Sing}\{f = 0\} = \{x = y = 0\} \) and has transversal type \( A_1 \). \( \Gamma_C^1 \) is a tree (representing a surface singularity of type \( A_2 \)), \( \Gamma_C^2, j \) is a tree, but \( \Gamma_C \) is:

\[
\begin{array}{c}
(2;0,1) \quad (2;4,1) \quad (1;10,3) \quad (1;0,1) \\
\end{array}
\]

5.49. Remark. During the construction of \( \Gamma_C^1 \) we deleted all 2–edges (with at least one end–point in \( W(\Gamma_C^1) \)), but these 2–edges have a geometric meaning in the case of \( \Gamma_C^1 \) as well. Take such an edge \( e \) with endpoints \( v_1 \) and \( v_2 \). If \( v_1 \in W(\Gamma_C^1) \) but \( v_2 \notin W(\Gamma_C^1) \), then add to \( \Gamma_C^1 \) an arrowhead supported by \( v_1 \). If both \( v_1 \) and \( v_2 \) are in \( W(\Gamma_C^1) \), then add to \( \Gamma_C^1 \) two arrowheads, one of them supported by \( v_1 \), the other by \( v_2 \). Let \( S \) denote the collection of these arrowheads. Then \( S \) represents (in a similar way as the arrows of any dual graph) the strict transforms of the singular locus \( \text{Sing}\{f = 0\} \) in the resolution \( r|_{\coprod St_i} : \coprod St_i \to \{f = 0\} \) (cf. 5.27).

In particular, \( \sum_j \#(T\Sigma_j) \geq \#S \).

In general the inequality can be strict. For example, in the case (5.19) (cf. also with 5.47), \( \sum_j \#(T\Sigma_j) = 4 \) but \( \#S = 2 \). In this case, the strict transform \( C_j^* \) of \( \Sigma_j \) is irreducible, even if the transversal type of \( \Sigma_j \) has two components (but \( \text{deg}(C_j^* \to \Sigma_j) = 2 \)).

Now, we continue the list of consequences of theorem (5.43). By theorem (5.38) \( \Gamma(K_l) \) is a tree for each \( l \). Since \( \Gamma_C^2 / \sim \) is a tree as well, one gets:

5.50. Corollary. a) \( \Gamma_C^2, j \) is a connected tree.

b) Moreover, since \( \Gamma_C^2, j / \sim \) is a tree, corresponding to the/any covering data \( \nu \), (by Part I, 1.19) there is only one cyclic graph covering \( p : G \to \Gamma_C^2, j / \sim \) (up to isomorphism). Therefore, the shape of the graph \( G(T\Sigma_j) \) is completely determined by the weighted graph \( \Gamma_C^2, j \).

In the sequel we discuss the compatibility of the decorations: we will show that one can recover all the decorations of \( G(T\Sigma_j) \) from the decorations of \( \Gamma_C^2, j \).

First we recall (cf. Part I, 2.2) that the embedded resolution graph \( G(T\Sigma_j) \) has three type of decorations: (a) each vertex \( v \in \mathcal{V}(G(T\Sigma_j)) \) has a multiplicity \( (m_v) \); (b) each vertex \( w \in \mathcal{W}(G(T\Sigma_j)) \) has a self-intersection \( e_w \); (c) each vertex \( w \in \mathcal{W}(G(T\Sigma_j)) \) has a genus \([g_w] \).

5.51. Theorem. – Characterization of the transversal singularities (second part). For any \( w \in \mathcal{W}(G(T\Sigma_j)) \) corresponding to \( K_l \), \( m_w \) is exactly \( m(K_l) \); and for any arrowhead \( v \in \mathcal{A}(G(T\Sigma_j)) \): \( m_v = 1 \).

By (Part I, 2.32), each self-intersection number \( e_w \) can be determined from the set of multiplicities \( \{m_v\}_{v \in V} \). Finally, \( g_w = 0 \) for each \( w \).

In particular, the weighted dual (embedded) resolution graph \( G(T\Sigma_j) \) of the transversal singularity associated with \( \Sigma_j \) can be completely determined from the weighted graph \( \Gamma_C^2, j \).
5.52. Remark. As we already emphasized in (5.38 b–c), the collection of integers \( \{ \nu_w \} (w \in W(\Gamma^2_{C,j})) \) satisfies serious compatibility restrictions. Moreover, since in the cyclic covering \( d_j \cdot G(\Sigma_j) \to \Gamma^2_{C,j}/\sim \) the covering graph \( d_j \cdot G(\Sigma_j) \) has no cycles, this imposes some additional restrictions for the integers \( \{ \nu_w \} (w \in W(\Gamma^2_{C,j})) \).

The results of this section and theorem (Part I, 1.20) culminate in the following corollary which is crucial for the algorithm presented in the next section.

5.53. Proposition. Up to isomorphism of cyclic covering of graphs (with a fixed covering data), there is only one cyclic covering of the graph \( \Gamma_C \) provided that the covering data satisfies \( n_v = 1 \) for any \( v \in V^1(\Gamma_C) \).

Illustrations of properties of the graph \( \Gamma_C \).

5.54. Example. Consider the ICIS \( (f, g) \), where \( f = x^2 - y^2 \) and \( g = y^2 + z^3 \) (cf. 5.17). Figure 1 shows the decomposition of a possible \( \Gamma_c \) into \( \Gamma^1_C \) and \( \Gamma^2_C \). The graph \( \Gamma^1_C \) has two components \( \Gamma^1_{c,1} \) and \( \Gamma^1_{c,2} \) corresponding to the two smooth components of the (normalization) of \( \{ f = 0 \} \). The corresponding resolution graphs \( G^1_{c,1} \) and \( G^1_{c,2} \) provided by (5.26) represent both an A_2 plane curve singularity corresponding to the restriction of \( g \) to the irreducible components of \( \{ f = 0 \} \). The transversal singularity of \( \text{Sing}\{ f = 0 \} \) is of type A_1 with graph \( G(T\Sigma_1) \).

5.55. Example. Consider the example (5.19) (cf. also 5.47 and 5.49). Then \( \text{Sing}\{ f = 0 \} \) has two components. For both \( j \in \{ 1, 2 \} \) fixed, \( d_j = 1 \) and there is only one equivalence class \( K_1 \). Therefore, \( \Gamma^2_{C,j}/\sim \) consists of only one non-arrowhead which supports exactly one arrowhead. But \( G(T\Sigma_j) \) has one non-arrowhead which supports two arrowheads (representing an A_1 singularity). Therefore, in this case the covering \( G(T\Sigma_j) \to \Gamma^2_{C,j}/\sim \) is not a product (or identity).

5.56. Example. Consider the ICIS \( (f, g) \), where \( f = x^3y^7 - z^4 \) and \( g = x + y + z \). In this example, the reader can verify that the example provides non-trivial partitions \( \{ K_i \} \). The case \( j = 1 \) is explained in (5.34) and (5.39). Figure 2 shows the graph \( G^1_C \), as well as \( G(T\Sigma_1) \) and \( G(T\Sigma_2) \) (corresponding to the two transversal singularities), each of which is obtained from a possible \( \Gamma_C \).

5.57. Example. Consider the pair \( f = y^3 + (x^2 - z^4)^2 \) and \( g = z \). Using the same notations as before, Figure 3 shows the graph \( G^2_C \), as well as \( G(T\Sigma_1) \) and \( G(T\Sigma_2) \) (corresponding to the two transversal singularities), each of which is obtained from a possible \( \Gamma_C \).
the subgraph $\Gamma_{C,1}^1$  

$\Gamma_{C}^2$  

the subgraph $\Gamma_{C,2}^1$

$G(T\Sigma_1)$

$G_{C,1}^1$  

$G_{C,2}^1$

Figure 1
the subgraph $\Gamma^2_{\mathcal{C},1}$

the subgraph $\Gamma^2_{\mathcal{C},2}$

the subgraph $\Gamma^{\ell}_{\mathcal{C}}$

$G(T\Sigma_1)$

$G(T\Sigma_2)$

$G^l_{\mathcal{C}}$

Figure 2
the subgraph $\Gamma_{C,1}^2$

the subgraph $\Gamma_{C,2}^2$

the subgraph $\Gamma_C^1$

$G(T\Sigma_1)$

$G(T\Sigma_2)$

$G_{C_1}^1$

Figure 3
6. The resolution graph of \( \{ f + g^k = 0 \} \).

Consider a pair of germs \( f, g : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0) \), such that \( f \) has a 1-dimensional singular locus and the pair \( \Phi = (f, g) \) forms an ICIS. Let \( k > 0 \) denote a sufficiently large integer (cf. 4.8). Then

\[
(X_k, 0) := \{ f + g^k = 0 \}, 0 \subset (\mathbb{C}^3, 0)
\]

is an isolated hypersurface singularity. The algorithm presented in this section provides the minimal resolution graph \( \Gamma(X_k) \) of \( (X_k, 0) \).

Actually, our algorithm gives even more: we construct the embedded resolution graph \( \Gamma(X_k, g) \) of \( g|_{X_k} : (X_k, 0) \to (\mathbb{C}, 0) \). In particular, the graph will have three types of decorations: multiplicities, genera and self-intersections, and also arrowheads representing the strict transforms of \( \{ g|_{X_k} = 0 \} \) decorated with multiplicities as well. If we delete from the weighted graph \( \Gamma(X_k, g) \) the arrowheads and multiplicities then we obtain the wanted resolution graph \( \Gamma(X_k) \) of \( (X_k, 0) \).

6.1. – A short preview of the algorithm. The algorithm starts with the construction of the weighted dual graph \( \Gamma_C \) associated with an embedded resolution \( r \) of \( (D, 0) := \{fg = 0\}, 0 \subset (\mathbb{C}^3, 0) \) (cf. 5.1). The graph \( \Gamma_C \) is not unique, it depends on the embedded resolution \( r \). However, the algorithm works for any dual graph \( \Gamma_C \).

Obviously, in order to obtain \( \Gamma_C \), the optimal is to have a detailed description of the resolution \( r \). But, let us emphasize again, in order to identify the necessary information for \( \Gamma_C \) from the resolution, only a small part of the resolution is needed: some information from a small neighborhood of the special curve arrangement \( \mathcal{C} \).

This construction is clarified completely in (5.3 and 5.11). In the sequel we assume that a weighted graph \( \Gamma_C \) have already been constructed. Then our algorithm provides, in a purely combinatorial way, the weighted dual embedded resolution graph \( \Gamma(X_k, g) \) from the weighted dual graph \( \Gamma_C \) and the integer \( k \).

The reader is invited to review the definitions regarding cyclic covering of graphs (Section 1, Part I), and its different “variations” (Part I, 1.25). In the construction, the resolution graph (string) of a Hirzebruch-Jung singularity plays a distinguished role. For the definition of these strings and their decorations, see (Part I, 2.11).

The algorithm states that the graph \( \Gamma(X_k, g) \) is a cyclic graph–covering of \( \Gamma_C \) with the additional modification which replaces each edge by a string (cf. Section 1 (Part I) and 1.25; and compare also with the construction of cyclic coverings in Section 3 and 2.9-2.11 (Part I). The point is that for the covering data of this graph–covering, one can apply proposition (5.53), which guarantees that up to an isomorphism, there is only one covering graph (with this covering data). In particular, the graph–covering is completely determined from the covering data, and from the information about the inserted strings. Having this in mind, the steps of the algorithm are self-explanatory:

**Step 1.** provides the covering data of the vertices, and also determines some of the (equivariant) decorations of these vertices.
Step 2. provides the covering data of edges, and the description of the inserted strings with their complete decorations.

Step 3. provides the missing decorations of the vertices constructed in Step 1. After this step we obtain a possible embedded resolution graph $\Gamma(X_k, g)$ of $g | X_k : (X_k, 0) \rightarrow (\mathbb{C}, 0)$.

Step 4. is the procedure of deleting all the arrowheads and multiplicities of $\Gamma(X_k, g)$ (and successively blowing down all the rational $(-1)$–curves) in order to obtain the (minimal) resolution graph of $(X_k, 0)$.

6.2. Definition. – “Legs” and “stars”. Before we start the detailed presentation of the algorithm, we introduce another notion, which helps the presentation of the algorithm.

Fix a non–arrowhead vertex $v$ of $\Gamma_C$ with weights $(m; n, \nu)$ and $[g]$. We define the notion of star of $v$, which keeps track of all the edges adjacent to $v$, preserving the weights $(n; c, d)$ of the other endpoint of the edges, and unifying the different cases represented by loops and edges connecting non–arrowheads or arrowheads. The “star of $v$” consists of the vertex $v$ and different “legs”. A leg has the following form:

$$
\begin{align*}
(m; n, \nu) & \quad x \quad (n; c, d) \\
\{g\} & \quad v
\end{align*}
$$

Here $x \in \{1, 2\}$ and the decorations satisfy the same compatibility conditions as the edges (5.13). Sometimes we say that a leg as above “is supported by the vertex $v$.”

Now, consider again the graph $\Gamma_C$ and the fixed non–arrowhead vertex $v$. The collection of legs supported by $v$ is defined as follows.

Any edge, with weight 1 and decoration as in (5.12 Edges, Case 1), with one of its endpoints $v_1 = v$, provides a leg supported by $v$ with decorations $x = 1$ and $(n; c, d) = (m; l, \lambda)$. If the corresponding edge supports an arrowhead in $\Gamma_C$ then automatically $(n; c, d) = (m; l, \lambda) = (1; 0, 1)$ (but otherwise, at the level of legs, we do not distinguish between the other end of the edge being an arrowhead or not).

Similarly, any edge, with weight 2 and decoration as in (5.12 Edges, Case 2), with one of its endpoints $v_1 = v$, provides a leg supported by $v$ with decorations $x = 2$ and $(n; c, d) = (m'; n, \nu)$. Again, if the corresponding edge supports an arrowhead in $\Gamma_C$ then automatically $(n; c, d) = (m'; n, \nu) = (1; 0, 1)$.

Any loop supported by $v$ and weighted by $x$, provides two legs supported by $v$, both decorated by the same $x$ and $(n; c, d) = (m; n, \nu)$.

The collection of all the legs constructed in this way forms the star of $v$. Obviously, if a non–arrowhead vertex of $\Gamma_C$ is connected to $v$ by more than one edge then each edge contributes a leg. For a general picture of a star, see the next paragraph.

Another way to define the star of $v$ is the following. Consider the topological realization of $\Gamma_C$. Then a small neighborhood of the point which represents $v$ is the star of $v$. (Obviously, we have to add also the natural decorations.)
The algorithm.

6.3. Step 1. – The covering data of the vertices.
In order to construct the graph $\Gamma(X_k, g)$ as a covering graph of $\Gamma_C$ (modified with strings, and decorated as an embedded resolution graph, see the discussion above) we need the covering data. The first step provides the integers $\{n_v\}_{v \in V(\Gamma_C)}$.

**Case 1.** Consider a non-arrowhead vertex $v$ of $\Gamma_C$ decorated by $(m; n, \nu)$ and $[g]$. Consider its star (see the previous paragraph):

Let $s$ (respectively $t$) be the number of legs weighted by $x = 1$ (respectively by $x = 2$).

Above the vertex $v$ of $\Gamma_C$ put $n_v$ non-arrowhead vertices, where

$$n_v = \gcd(m, k\nu - n, k\nu_1 - n_1, ..., k\nu_s - n_s, m_1, ..., m_t).$$

Put on each of these non–arrowhead vertices the same decoration: “the multiplicity” ($\tilde{m}$):

$$\tilde{m} = \frac{mn}{\gcd(m, nk - n)},$$

and the genus $[\tilde{g}]$ determined by the formula:

$$n_v \cdot (2 - 2\tilde{g}) = (2 - 2g - s - t) \cdot \gcd(m, k\nu - n) + \sum_{i=1}^s \gcd(m, k\nu_i - n_i) + \sum_{j=1}^t \gcd(m, k\nu - m_j).$$

(In Step 3, the “self–intersection” of each vertex above $v$ will also be provided.)

**Case 2.** Consider an arrowhead vertex $v$ of $\Gamma_C$. Above the vertex $v$, in the graph $\Gamma(X_k, g)$, put exactly one arrowhead vertex; and let $n_v = 1$. Let the “multiplicity” of this arrowhead be 1, i.e. the arrowhead of $\Gamma(X_k, g)$ is:

6.4. Step 2. – The covering data of edges and the type of the inserted strings.
This step provides the integers $\{n_e\}_{e \in E(\Gamma_C)}$.

**Case 1.** Consider an edge $e$ in $\Gamma_C$, with weight 1 (cf. with 5.12 Edges, Case 1):

Define:

$$n_e = \gcd(m, k\nu - n, k\nu - l).$$

Notice that Step 1 guarantees that in $\Gamma(X_k, g)$ there is a column of $n_{v_2}$ (resp. $n_{v_2}$) vertices above the vertex $v_1$ (resp. $v_2$) of $\Gamma_C$. Moreover, $n_{v_1}$ and $n_{v_2}$ both divide $n_e$. 
Then, above the edge \( e \) insert (cyclically) in \( \Gamma(X_k, g) \) exactly \( n_e \) strings of type

\[
Str \left( \frac{k\nu - n}{n_e}, \frac{k\lambda - l}{n_e}; \frac{m}{n_e}; 0, \nu; 0 \right).
\]

If the right vertex \( v_2 \) is replaced by an arrowhead, then the edge \( e \) is:

\[
(1; n, 0) \quad \xrightarrow{l_{g\ell}} \quad (1; 0, 1)
\]

Therefore, in the above formulae \( m = 1, n_e = 1 \) and the string represents a nonsingular surface, i.e. it is the empty graph. Therefore, above such an edge \( e \) put a single edge (which supports that arrowhead of \( \Gamma(X_k, g) \) which covers the corresponding arrowhead of \( \Gamma_C \)).

**Case 2.** Consider an edge \( e \) in \( \Gamma_C \), with weight 2 (cf. 5.12 Edges, Case 2):

\[
\begin{array}{c}
(m; n, 0) \quad \xrightarrow{l_{g\ell}} \quad (m'; n, 0)
\end{array}
\]

For such an edge, define:

\[ n_e = \text{gcd}(m, m', k\nu - n). \]

Similarly as above, Step 1 guarantees that in \( \Gamma(X_k, g) \) there is a column of \( n_{v_1} \) (resp. \( n_{v_2} \)) vertices above the vertex \( v_1 \) (resp. \( v_2 \)) of \( \Gamma_C \), and \( n_{v_1} \) and \( n_{v_2} \) both divide \( n_e \).

Then, above the edge \( e \) insert (cyclically) in \( \Gamma(X_k, g) \) exactly \( n_e \) strings of type

\[
Str \left( \frac{m}{n_e}, \frac{m'}{n_e}; \frac{k\nu - n}{n_e}; 0, 0; \nu \right).
\]

If the right vertex \( v_2 \) is replaced by an arrowhead, then the edge \( e \) is:

\[
(1; 0, 1) \quad \xrightarrow{l_{g\ell}} \quad (1; 0, 1)
\]

Therefore, in the above formulae \( m' = 1, n = 0, \nu = 1 \) and \( n_e = 1 \). In this case, in general, the string is not empty, it is of type \( Str(m, 1; k|0, 0; 1) = Str(m, 1; k) \). In particular, above the edge \( e \) (which supports an arrowhead of \( \Gamma_C \)) insert a string of type \( Str(m, 1; k) \) (whose right end supports that arrowhead of \( \Gamma(X_k, g) \) which covers the corresponding arrowhead of \( \Gamma_C \)).

The reader is invited to consult (Part I, 1.25) and Step 3 of the algorithm presented in (Part I, Section 3 and 2.9-2.11) where the insertion of the strings is explained in more details.

**6.5. The covering data revisited.** Notice that the set of integers \( \{n_v\}_{v \in V(\Gamma_C)} \) and \( \{n_e\}_{e \in E(\Gamma_C)} \) satisfies the axioms of a covering data. Moreover, if \( v \in V^1(\Gamma_C) \) then \( m = 1 \) hence \( n_v = 1 \). Therefore, by proposition (5.53), there is only one cyclic graph-covering of \( \Gamma_C \) with this covering data. If in this unique graph we replace the edges with the corresponding strings (as explained above) we obtain \( \Gamma(X_k, g) \).
6.6. Step 3. – The missing decorations. The first two steps provide a graph with some decorations: all the multiplicities, all the genera, and some of the self-intersections. More precisely, the self-intersections of the vertices constructed in Step 1 are missing. In this step compute all these numbers using (Part I, 2.32).

This finishes completely the construction of (a possible) embedded resolution graph $\Gamma(X_k, g)$ of the germ $g|X_k : (X_k, 0) \to (\mathbb{C}, 0)$.

6.7. Step 4. In order to obtain the resolution graph $\Gamma(X_k)$ of $(X_k, 0)$, the arrows and multiplicities of $\Gamma(X_k, g)$ have to be deleted. If the resulting graph is not minimal, then blowing down successively the rational curves with self-intersection $-1$ will provide a minimal one.

Examples.

6.8. Notation. In order to help the reader distinguish the vertices of $\Gamma(X_k, g)$ which cover some vertices of $\Gamma_C$ (i.e. vertices constructed in Step 1) from the vertices of the strings inserted in Step 2, we indicate the latter by $\bullet$. Obviously, in the final graph we make no distinction between them.

In the next examples, we give two graphs, the first is the result of steps 1-3, the second is the result of step 4.

6.9. Example. Let $f(x, y, z) = x^2 - y^3$ and $g(x, y, z) = z$. A possible dual graph $\Gamma_C$ is shown in Example 5.20.

Set $k = 5$. The algorithm provides the minimal resolution graph of $(\{x^2 + y^3 + z^5 = 0\}, 0)$. Applying Steps 1-3, one obtains:

```
(1)  -2  (2)
    /    |
   /     |
   /      |
  -2      (5)
    /    |
   /     |
   /      |
  -2      (4)
    /    |
   /     |
   /      |
  -2      (3)
    /    |
   /     |
   /      |
  -2      (2)
```

Step 4 gives:

```
(1)  -2  (2)
    /    |
   /     |
   /      |
  -2      (5)
    /    |
   /     |
   /      |
  -2      (4)
    /    |
   /     |
   /      |
  -2      (3)
```

This is exactly the expected graph of the surface singularity $E_8$.

More generally, the above algorithm contains as a particular case the construction of the resolution graph of surface singularities of type $(\{f(x, y) + z^N = 0\}, 0)$ (cf. Part I, 3.12), via the construction (5.20) of $\Gamma_C$. 
6.10. Example. Let \( f(x, y, z) = x^3 + y^2 + xyz \) and \( g(x, y, z) = z \). A possible dual graph \( \Gamma_C \) is shown in Example 5.18. Let \( k = 9 \), then the algorithm provides the resolution of the cusp singularity \( \{(x^3 + y^2 + z^9 + xyz), 0\} \).

![Graph](image1.png)

6.11. Example. Consider again \( f(x, y, z) = x^3 + y^2 + xyz \) and \( g(x, y, z) = z \), as in the previous example, but take \( k = 12 \). Then the algorithm gives the minimal resolution graph of the cusp singularity \( \{(x^3 + y^2 + z^{12} + xyz), 0\} \).

![Graph](image2.png)

The interested reader can verify that for arbitrary \( k \) (and the same \( \Phi \) and \( \Gamma_C \) as above), the graph \( \Gamma(X_k) \) is a cyclic graph as above with one \(-3\) curve and \( k - 7 \) curves with self-intersection \(-2\).

6.12. Example. Let \( f(x, y, z) = x^3 + y^3 + xyz \) and \( g(x, y, z) = z \). A possible dual graph \( \Gamma_C \) is:

![Graph](image3.png)

Set \( k = 6 \). Then the minimal resolution graph of the cusp singularity \( \{(x^3 + y^3 + z^6 + xyz), 0\} \), obtained in the same steps, is:

![Graph](image4.png)

6.13. Example. Consider again \( f(x, y, z) = x^3 + y^3 + xyz \) and \( g(x, y, z) = z \) as in the previous example. Set \( k = 8 \), then the graph of \( \{(x^3 + y^3 + z^8 + xyz), 0\} \) is:

![Graph](image5.png)
Again, one can verify that for arbitrary \(k\) (and the same \(\Phi\) and \(\Gamma_C\) as above), the graph \(\Gamma(X_k)\) is a cyclic graph as above with one \(-5\) curve and \(k-4\) curves with self-intersection \(-2\).

**6.14. Example.** Let \(f(x, y, z) = y^3 + (x^2 - z^4)^2\) and \(g(x, y, z) = z\). A possible dual graph \(\Gamma_C\) is shown in Example (5.57). Set \(k = 15\). The embedded resolution graph of \(z\) defined on \(\{(y^3 + (x^2 - z^4)^2 + z^{15} = 0)\}, 0)\) is:

Applying Step 4, one gets the resolution graph of \(\{(y^3 + (x^2 - z^4)^2 + z^{15} = 0)\}, 0)\):

Note that this result can be checked by using the algorithm described in (Part I, 3.12), with \(f(x, y) = (x^2 - y^4)^2 + y^{15}\) and \(N = 3\).
Proof of the algorithm and final remarks.

6.15. – Outline of the proof. In this subsection we present the main steps of the proof of the algorithm, we emphasize the key points, and leave all the details to the reader.

The fundamental fact is that the lift of the singularity \((X_k, 0)\) in \(V^{\text{emb}}\), via the embedded resolution \(r : (V^{\text{emb}}, D) \to (\mathbb{C}^3, D)\), has a distinguished position. Namely, let the strict transform of \(X_k\) be denoted by \(X_k^{\text{st}}\). Then for sufficiently large \(k\)

\[
X_k^{\text{st}} \cap D = C \quad \text{and} \quad \text{Sing}(X_k^{\text{st}}) \subset C.
\]

Moreover, along an irreducible component of the intersection \(X_k^{\text{st}} \cap D^{(2)}\), the surface \(X_k^{\text{st}}\) is either smooth or forms an equisingular family of plane curve singularities. Therefore, if \(X_k^{\text{st}}\) is normalized, the singularities along the intersection \(X_k^{\text{st}} \cap D^{(2)}\) are smoothed. The resulting normal surface \(\tilde{X}_k^{\text{st}}\) has singularities only above points of \(X_k^{\text{st}} \cap D^{(3)}\). Moreover, the singularities of the strict transform \(X_k^{\text{st}}\) above the points \(X_k^{\text{st}} \cap D^{(3)}\), have local equations of type \(\{w^\alpha = u^\beta v^\gamma\}\), with \(\beta, \gamma \geq 1\) and \(\alpha > 1\) (in order to find these integers, use the local equations (5.12)). Thus, after normalization, these singularities are of Hirzebruch-Jung type and their resolution is well-known (cf. Part I, 2.11). A smooth surface \(X_k^{\text{res}}\), which provides a resolution of the original isolated surface singularity \((X_k, 0)\), can be obtained by resolving these Hirzebruch–Jung singularities.

The construction is summarized in the next diagram:

\[
X_k^{\text{res}} \xrightarrow{R} \tilde{X}_k^{\text{st}} \xrightarrow{n} X_k^{\text{st}} \xrightarrow{r|} (X_k, 0),
\]

where \(r|\) is the restriction of the fixed embedded resolution \(r\); \(n\) is the normalization of \(X_k^{\text{st}}\) and \(R\) is the map resolving the Hirzebruch-Jung singularities of the surface \(\tilde{X}_k^{\text{st}}\).

The graph constructed using the algorithm of the previous section is exactly the dual resolution graph of the resolution \((X_k^{\text{res}}, r| \circ n \circ R)\). Note that by the above discussion the exceptional curves of the resolution \(X_k^{\text{res}}\) are of two type. Some exceptional curves are irreducible components of the lift \(n^{-1}(C)\) and these are lifted further isomorphically by \(R\). Other exceptional curves appear as a result of resolving Hirzebruch-Jung strings.

The crucial fact in the proof is that all the information necessary to recover the dual embedded resolution graph can be obtained by local multiplicity (vanishing order) computations. This claim needs some explanation.

The first technical difficulty (the “easy one”) arises because of the fact that the dual resolution graph includes the self-intersection numbers of the exceptional curves as weights. These numbers are global invariants (characteristic classes) and in general, their computation is difficult. But if we consider the germ of an analytic function defined on \(X_k\), then in any embedded resolution graph of it, all the self–intersections can be computed from the multiplicities of the germ using (Part I, 2.32). This is exactly the motivation to consider the embedded resolution of the restriction of \(g\) to \((X_k, 0)\) instead of the resolution of \((X_k, 0)\). It is not difficult to verify that the procedure outlined above in fact gives the embedded resolution of
$g|X_k: (X_k, 0) \to (\mathbb{C}, 0)$, and all the vanishing orders can be determined by local computations.

The second (globalization) problem is more serious. First we would like to remind the reader about the result regarding the resolution graph of cyclic covering. Fix a normal surface singularity $(X, 0)$ and a germ $f$ defined on $(X, 0)$, and consider the $N$–cyclic covering $X_{f,N}$ of $(X, 0)$ branched along $\{f = 0\}$. Then in general, it is impossible to recover the resolution graph of $X_{f,N}$ from the embedded resolution graph of $f$ and the integer $N$. Even if the embedded resolution graph of $f$ and the integer $N$ carry all the needed local information about the covering and multiplicities, the global information about monodromies, which determines the type of the graph–covering (i.e. the covering data) and also the shape of the covering graph, is missing. In this case of cyclic coverings, this global information can be codified in the universal covering graph of the embedded resolution graph of $f$. On the other hand, if the embedded resolution graph of $f$ has some additional properties (e.g. if it is a tree and $g_w = 0$ for all vertices $w \in W$), then the necessary global information can be recovered from the local one. (For more details, see Part I of the paper.)

In the case of the generalized Iomdin series we can ask the same question: is the global information (i.e. the number of irreducible components of $n^{-1}(C)$ in $\tilde{X}_k^g$ situated above an irreducible component $C$ of $C$ in $X_k^g$, or the global configuration of the intersection of these components) codified in the local data? The answer is yes! This follows from the results of Section 5, where it is proven that the covering data can be determined from $\Gamma_C$. Moreover, proposition (5.53) (cf. also Part I, 1.20) guarantees that with this covering data there is only one graph–covering. The fact which makes these claims true is that that part of $C$ (namely $C^2$), above which the covering is not trivial, contains only rational curves, and the dual graph of these curves has no cycles.

Once it is clarified that no global information is needed, everything is reduced to local computations. They are left to the reader.

6.16. Final remarks.

1. If $F_h$ denotes the Milnor fiber of a hypersurface singularity $h: (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$, and $\chi$ the topological Euler–characteristic, then one has the following formula:

$$\chi(F_{f+g}) - \chi(F_f) = k \cdot \sum_{j=1}^{s} d_j \mu_j,$$

where $\mu_j$ denoted the Milnor number of the transversal singularity $T\Sigma_j$ associated with $\Sigma_j$ $(1 \leq j \leq s)$. This formula for the “classical case” (when $g$ is a generic linear form) was proved by Iomdin [5], for the general case, see [10, 9].

Notice that both integers $d_j$ and $\nu_j$ can be determined from the weighted graph $\Gamma_C$. Indeed, $d_j$ is given in (5.46). On the other hand, theorems (5.43) and (5.51) provide the embedded resolution graph of $T\Sigma_j$ which via a result of A’Campo’s [1] provides $\mu_j$.

In particular, $\Gamma_C$ determines completely the correction term $i(f + g^k) - i(f)$ in the case of $i(h) = \chi(F_h)$ (cf. 4.10).
2. Notice that for any vertex \( w \in W(\Gamma^1_C) \) the covering data is \( n_w = 1 \), and also for any edge \( e \in E(\Gamma^1_C) \) one has \( n_e = 1 \). Moreover, the type of the inserted string corresponding to \( e \) is trivial. Therefore, above \( \Gamma^1_C \) the covering is trivial, i.e. for any \( k \), the graph (above) \( \Gamma^1_C \) stays as a stable part of the resulting graph \( \Gamma(X_k, g) \) or \( \Gamma(X_k) \).

On the other hand, the graph above \( \Gamma^2_C \) is increasing as \( k \) becomes larger. More precisely, the number of vertices above a vertex \( w \in W(\Gamma^2_C) \) is a periodical function in \( k \). Similarly, the 1–edges of \( \Gamma^2_C \) are replaced by strings which behave periodically when \( k \) runs over the integers \( k \geq k_0 \). But the 2-edges of \( \Gamma^2_C \) are replaced by strings whose “size” increases infinitely as \( k \to \infty \). Basically, the union of these increasing spots, associated with the subgraphs \( \Gamma^2_C,j \ (1 \leq j \leq s) \), correspond to the “correction term” \( i(f + g^k) - i(f) \), where \( i \) denotes the resolution graph (cf. also with 4.10).

3. Not all the decorations of the weighted graph \( \Gamma_C \) were used in the above algorithm. Nevertheless, we believe that all of them have important geometrical significance (we will explain this in the forthcoming paper [16], where we plan to discuss the relationship of \( \Gamma_C \) with other invariants).

4. Any fault–finder reader can object the following. We constructed our resolution graph \( \Gamma(X_k, g) \) from the graph \( \Gamma_C \) whose construction is based on some properties of the embedded resolution \( r \), which is a more complicated object than any resolution of \( (X_k, 0) \). This is perfectly true!

But, if we consider our results as qualitative results, then already the existence of a graph \( \Gamma_C \) which coordinates all the resolutions graphs \( \{\Gamma(X_k, g)\}_k \) in a purely combinatorial way (as it is explained in the algorithm), and has all the properties listed in Section 5, (and conjecturally provides all the “correction terms” \( i(f + g^k) - i(f) \) for different invariants \( i \) such as the zeta function, signature, spectral pairs, etc.), is really remarkable.

5. In this paper the case of the Iomdin series was discussed, but all the results can be generalized to the case of an arbitrary series of composed singularities.

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