REAL AND COMPLEX K-PLANES IN CONVEX HYPERSURFACES

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Abstract. It is shown that that the rank of the second fundamental form (resp. the Levi form) of a \(C^2\)-smooth convex hypersurface \(M\) in \(\mathbb{R}^{n+1}\) (resp. \(\mathbb{C}^{n+1}\)) does not exceed an integer constant \(k < n\) near a point \(p \in M\), then through any point \(q \in M\) near \(p\) there exists a real (resp. complex) \((n-k)\)-dimensional plane that locally lies on \(M\).

It is a classical result in the differential geometry that any developable surface \(M\) in \(\mathbb{R}^3\) (i.e. with zero Gaussian curvature) is a part of a complete ruled surface (i.e. through every point of \(M\) there exists a straight line that lies on \(M\)). Note that second fundamental form of such an \(M\) has rank 0 or 1 at any point. A similar result holds in higher dimensions (cf. [3, Lemma 2]):

(R) If the rank of the second fundamental form of a \(C^2\)-smooth hypersurface \(M\) in \(\mathbb{R}^{n+1}\) is a constant \(k < n\) near a point \(p \in M\), then \(M\) is locally generated by \((n-k)\)-dimensional planes. (In particular, if \(k = 0\), then \(M\) is locally a hyperplane.)

This result has a complex version (see [4, Theorem 6.1, Corollary 5.2]):

(C) If the rank of the Levi form of a \(C^2\)-smooth real hypersurface \(M\) in \(\mathbb{C}^{n+1}\) is a constant \(k < n\) near a point \(p \in M\), then \(M\) is locally foliated by complex \((n-k)\)-dimensional manifolds. Moreover, if \(k = 0\) (i.e. \(M\) is Levi-flat) and \(M\) is real analytic, then \(M\) is locally biholomorphic to a complex hyperplane.

On the other hand, in both cases (real and complex), almost nothing is known if the rank is not maximal and non-constant.

The aim of this note is consider the last case when the hypersurface \(M\) is convex, i.e. \(M\) is a part of the boundary of convex domain.

Proposition 1. The rank of the second fundamental form (resp. the Levi form) of a \(C^2\)-smooth convex hypersurface \(M\) in \(\mathbb{R}^{n+1}\) (resp. \(\mathbb{C}^{n+1}\)) does not exceed an integer constant \(k < n\) near a point \(p \in M\) if and...
only if through any point \( q \in M \) near \( p \) there exists a real (resp. complex) \((n - k)\)-dimensional plane that locally lies on \( M \).

**Remark.** If \( k = 0 \) in the complex case, then \( M \) is locally linearly equivalent to the Cartesian product of \( \mathbb{C}^n \) and a planar domain (see [2, Theorem 1]).

**Proof.** If the respective real (complex) \((n - k)\)-dimensional plane exists for a point \( q \in M \), then the non-negativity of the second fundamental form (the Levi form) at \( q \) easily implies that the rank of the form at \( q \) does not exceed \( k \).

For the converse, let first consider the complex case.

It is enough to show that through \( p \) there exists a complex line that locally lies on \( M \). Then, considering the intersection of \( M \) with the orthogonal complement of this line, we may proceed by induction on \( n \) to find \( n - k \) orthogonal complex lines locally lying on \( M \). The convexity of \( M \) easily implies that the \((n - k)\)-dimensional planes, spanned by these lines, locally lies on \( M \). Finally, note that the same holds for any point \( q \in M \) near \( p \) (since may replace \( p \) by \( q \)).

Assume that there does not exist such a line. It is claimed in [6, p. 310] and proved in [5, Theorem 6] that \( p \) is a local holomorphic peak point for one of the sides, say \( M^+ \), of \( M \) near \( p \) (the convex one). By [1, Corollary 2], \( p \) is a limit of strictly pseudoconvex point of \( M^+ \) which is a contradiction to the rank assumption.

The proof in the real case is similar. Recall that a point \( q \in M \) is called exposed if there exists a real hyperplane that intersects \( M \) in \( p \) alone (i.e. \( p \) is a linear peak point). It is enough to combine two facts:
- the set of exposed points is dense in the set of extreme points (see [7]);
- the set of strictly convex points of \( M^+ \) (all the eigenvalues of the second fundamental form are positive) is dense in the set of exposed points.

The last fact can be shown following, for example, the proof of [1, Theorem].

**Remark.** The 'only if' part \( (\rightarrow) \) of Proposition [1] remains true if we replace convexity by real-analyticity. Indeed, if \( c_M(q) \) denotes the rank of the Levi form of \( M \) at \( q \in M \) and \( \tilde{c}_M(p) = \limsup_{q \to p} c_M(q) \), then, by (C), through any \( q \) near \( p \) with \( c_M(q) = \tilde{c}_M(p) \) there exists a complex line \((n - c_q)\)-dimensional complex plane that locally lies on \( M \). Then one may find a \((n - c_q)\)-dimensional complex plane with infinite order of contact with \( M \) at \( p \). Since \( M \) is real-analytic, we conclude that this
plane lies on \( M \) near \( p \). The real case follows analogously by using (R) instead of (C).

**References**

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