ON MIXED JOINT DISCRETE UNIVERSALITY FOR A CLASS OF ZETA-FUNCTIONS

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ABSTRACT. We prove a mixed joint discrete universality theorem for a Matsumoto zeta-function \( \varphi(s) \) (belonging to the Steuding subclass) and a periodic Hurwitz zeta-function \( \zeta(s, \alpha; \mathcal{B}) \). For this purpose, certain independence condition for the parameter \( \alpha \) and the minimal step of discrete shifts of these functions is assumed. This paper is a continuation of authors’ works [12] and [13].

Keywords: discrete shift, joint approximation, linear independence, periodic Hurwitz zeta-function, Matsumoto zeta-function, universality.

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1. INTRODUCTION

In analytic number theory, the problem of so-called mixed joint universality in Voronin’s sense is a very interesting problem since it solves a problem on simultaneous approximation of certain tuples of analytic functions by shifts of tuples consisting of zeta-functions having an Euler product expansion over the set of primes and other zeta-functions without such product. For such type of universality, a very important role is played by the parameters which occur in the definitions of zeta-functions.

The first result on mixed joint universality was obtained by H. Mishou in [21]. He proved that the Riemann zeta-function \( \zeta(s) \) and the Hurwitz zeta-function \( \zeta(s, \alpha) \) with transcendental parameter \( \alpha \) are jointly universal.

Let \( \mathbb{P}, \mathbb{N}, \mathbb{N}_0, \mathbb{Z}, \mathbb{Q}, \mathbb{R} \) and \( \mathbb{C} \) be the sets of all primes, positive integers, non-negative integers, integers, rational numbers, real numbers and complex numbers, respectively. Denote by \( s = \sigma + it \) a complex variable. Recall that the functions \( \zeta(s) \) and \( \zeta(s, \alpha) \), \( 0 < \alpha \leq 1 \), for \( \sigma > 1 \), are defined by

\[
\zeta(s) = \sum_{m=1}^{\infty} \frac{1}{m^s} = \prod_{p \in \mathbb{P}} \left( 1 - \frac{1}{p^s} \right)^{-1} \quad \text{and} \quad \zeta(s, \alpha) = \sum_{m=0}^{\infty} \frac{1}{(m+\alpha)^s},
\]

respectively. Both of them are analytically continued to the whole complex plane, except for a simple pole at the point \( s = 1 \) with residue 1. Note that the Riemann zeta-function has the Euler product expansion, while in general the Hurwitz zeta-function does not have (except the cases \( \alpha = \frac{1}{2}, 1 \)).

For Mishou’s result and further statements, we introduce some notations. Let \( D(a, b) = \{ s \in \mathbb{C} : a < \sigma < b \} \) for any \( a < b \). For every compact set \( K \subset \mathbb{C} \), denote by \( H^c(K) \) the
set of all \(\mathbb{C}\)-valued continuous functions defined on \(K\) and holomorphic in the interior of \(K\). By \(H_0^c(K)\) we denote the subset of \(H^c(K)\), consisting of all elements which are non-vanishing on \(K\).

**Theorem 1** ([21]). Suppose that \(\alpha\) is a transcendental number. Let \(K_1\) and \(K_2\) be compact subsets of the strip \(D(\frac{1}{2}, 1)\) with connected complements. Suppose that \(f_1(s) \in H_0^c(K_1)\) and \(f_2(s) \in H^c(K_2)\). Then, for every \(\varepsilon > 0\),

\[
\liminf_{T \to \infty} \frac{1}{T} \text{meas}\left\{ \tau \in [0, T] : \sup_{s \in K_1} |\zeta(s + i\tau) - f_1(s)| < \varepsilon, \sup_{s \in K_2} |\zeta(s + i\tau, \alpha) - f_2(s)| < \varepsilon \right\} > 0.
\]

Here, as usual, \(\text{meas}\{A\}\) denotes the Lebesgue measure of the measurable set \(A \subset \mathbb{R}\).

Note that J. Sander and J. Steuding [23] proved the same type of universality, but for rational \(\alpha\), by a quite different method.

In [12], we consider the mixed joint universality property for a wide class of zeta-functions consisting of Matsumoto zeta-functions \(\phi(s)\) belonging to the Steuding class \(\tilde{S}\) and periodic Hurwitz zeta-functions \(\zeta(s, \alpha; \mathfrak{B})\).

Recall the definition of the polynomial Euler products \(\tilde{\phi}(s)\) or so-called Matsumoto zeta-functions. (Remark: The function \(\tilde{\phi}(s)\) was introduced by the second author in [18].) For \(m \in \mathbb{N}\), let \(g(m)\) be a positive integer and \(p_m\) the \(m\)th prime number. Moreover, let \(a_{m}^{(j)} \in \mathbb{C}\), and \(f(j, m) \in \mathbb{N}\), \(1 \leq j \leq g(m)\). The function \(\tilde{\phi}(s)\) is defined by the polynomial Euler product

\[
\tilde{\phi}(s) = \prod_{m=1}^{\infty} \prod_{j=1}^{g(m)} \left(1 - a_{m}^{(j)} p_m^{-sf(j,m)}\right)^{-1}.
\]

It is assumed that

\[
g(m) \leq C_1 p_m^\alpha \quad \text{and} \quad |a_{m}^{(j)}| \leq p_m^\beta
\]

with a positive constant \(C_1\) and non-negative constants \(\alpha\) and \(\beta\). In view of (2), the function \(\tilde{\phi}(s)\) converges absolutely for \(\sigma > \alpha + \beta + 1\), and hence, in this region, it can be given by the absolutely convergent Dirichlet series

\[
\tilde{\phi}(s) = \sum_{k=1}^{\infty} \frac{\tilde{c}_k}{k^s}.
\]

The shifted function \(\phi(s)\) is given by

\[
\phi(s) = \sum_{k=1}^{\infty} \frac{\tilde{c}_k}{k^{s+\alpha+\beta}} = \sum_{k=1}^{\infty} \frac{c_k}{k^s}
\]

with \(c_k = k^{-\alpha-\beta}\tilde{c}_k\). For \(\sigma > 1\), the last series in (4) converges absolutely too.

Also, suppose that, for the function \(\phi(s)\), the following assumptions hold (for the details, see [18]):

(a) \(\phi(s)\) can be continued meromorphically to \(\sigma \geq \sigma_0\), where \(\frac{1}{2} \leq \sigma_0 < 1\), and all poles in this region are included in a compact set which has no intersection with the line \(\sigma = \sigma_0\),
(b) $\varphi(\sigma + it) = O(|t|^{C_2})$ for $\sigma \geq \sigma_0$, where $C_2$ is a positive constant,
(c) the mean-value estimate
\[
\int_0^T |\varphi(\sigma_0 + it)|^2 dt = O(T).
\] (5)

It is possible to discuss functional limit theorems for Matsumoto zeta-functions (see Section 2 below), but this framework is too wide to consider the universality property. To investigate the universality, we introduce the Steuding subclass $\tilde{S}$, for which the following slightly more restrictive conditions are required. We say that the function $\varphi(s)$ belongs to the class $\tilde{S}$, if the following conditions are fulfilled:

(i) there exists a Dirichlet series expansion
\[
\varphi(s) = \sum_{m=1}^{\infty} \frac{a(m)}{m^s}
\]
with $a(m) = O(m^\varepsilon)$ for every $\varepsilon > 0$;
(ii) there exists $\sigma_\varphi < 1$ such that $\varphi(s)$ can be meromorphically continued to the half-plane $\sigma > \sigma_\varphi$, and holomorphic except for at most a pole at $s = 1$;
(iii) there exists a constant $c \geq 0$ such that
\[
\varphi(\sigma + it) = O(|t|^{c+\varepsilon})
\]
for every fixed $\sigma > \sigma_\varphi$ and $\varepsilon > 0$;
(iv) there exists the Euler product expansion over prime numbers, i.e.,
\[
\varphi(s) = \prod_{p \in \mathbb{P}} \prod_{j=1}^I \left(1 - \frac{a_j(p)}{p^s}\right)^{-1};
\]
(v) there exists a constant $\kappa > 0$ such that
\[
\lim_{x \to \infty} \frac{1}{\pi(x)} \sum_{p \leq x} |a(p)|^2 = \kappa,
\]
where $\pi(x)$ denotes the number of primes up to $x$, i.e., $p \leq x$.

For $\varphi \in \tilde{S}$, let $\sigma^*$ be the infimum of all $\sigma_1$ for which
\[
\frac{1}{2T} \int_{-T}^{T} |\varphi(\sigma + it)|^2 dt \sim \sum_{m=1}^{\infty} \frac{|a(m)|^2}{m^{2\sigma}}
\]
holds for every $\sigma \geq \sigma_1$. Then it is known that $\frac{1}{2} \leq \sigma^* < 1$. (Remark: The class $\tilde{S}$ was introduced by J. Steuding in [24].)

Now we recall the definition of the periodic Hurwitz zeta-function $\zeta(s, \alpha; \mathcal{B})$ with a fixed parameter $\alpha$, $0 < \alpha \leq 1$. (Remark: The function $\zeta(s, \alpha; \mathcal{B})$ was introduced by A. Javtokas and A. Laurinčikas in [8].) Let $\mathcal{B} = \{b_m : m \in \mathbb{N}_0\}$ be a periodic sequence
of complex numbers (not all zero) with minimal period \( k \in \mathbb{N} \). For \( \sigma > 1 \), the function \( \zeta(s, \alpha; B) \) is defined by

\[
\zeta(s, \alpha; B) = \sum_{m=0}^{\infty} \frac{b_m}{(m+\alpha)^s}.
\]

It is known that

\[
\zeta(s, \alpha; B) = \frac{1}{k^s} \sum_{l=0}^{k-1} b_l \zeta\left(s, \frac{l+\alpha}{k}\right), \quad \sigma > 1.
\] (6)

The last equality gives an analytic continuation of the function \( \zeta(s, \alpha; B) \) to the whole complex plane, except for a possible simple pole at the point \( s = 1 \) with residue

\[
b := \frac{1}{k} \sum_{l=0}^{k-1} b_l.
\]

If \( b = 0 \), then \( \zeta(s, \alpha; B) \) is an entire function.

In [12], we prove the mixed joint universality property of the functions \( \varphi(s) \) and \( \zeta(s, \alpha; B) \).

**Theorem 2** ([12]). Suppose that \( \varphi(s) \in \tilde{S} \), and \( \alpha \) is a transcendental number. Let \( K_1 \) be a compact subset of \( D(\sigma^*, 1) \), and \( K_2 \) be a compact subset of \( D(\frac{1}{2}, 1) \), both with connected complements. Suppose that \( f_1 \in H^c_0(K_1) \) and \( f_2 \in H^c(K_2) \). Then, for every \( \varepsilon > 0 \),

\[
\liminf_{T \to \infty} \frac{1}{T} \text{meas} \left\{ \tau \in [0, T] : \sup_{s \in K_1} |\varphi(s+i\tau) - f_1(s)| < \varepsilon, \sup_{s \in K_2} |\zeta(s+i\tau, \alpha; B) - f_2(s)| < \varepsilon \right\} > 0.
\]

In [13], we obtain a generalization of Theorem 2, in which several periodic Hurwitz zeta-functions are involved.

More interesting and convenient in practical applications is so-called discrete universality of zeta-functions (for example, see [21]). This pushes us to extend our investigations of mixed joint universality for a class of zeta-functions to the discrete case. Recall that, in this case, the pair of analytic functions is approximated by discrete shifts of tuple \( (\varphi(s+ikh), \zeta(s+ikh, \alpha; B)), k \in \mathbb{N}_0 \), where \( h > 0 \) is the minimal step of given arithmetical progression.

The aim of this paper is to prove a mixed joint discrete universality theorem for the collection of the functions \( (\varphi(s), \zeta(s, \alpha; B)) \), i.e., the discrete version of Theorem 2.

For \( h > 0 \), let

\[
L(P, \alpha, h) = \left\{ (\log p : p \in \mathbb{P}), (\log(m+\alpha) : m \in \mathbb{N}_0), \frac{2\pi}{h} \right\}.
\]
Theorem 3. Let $\varphi(s) \in \tilde{S}$, $K_1$, $K_2$, $f_1(s)$ and $f_2(s)$ satisfy the conditions as in Theorem 2. Suppose that the set $L(\mathbb{P}, \alpha, h)$ is linearly independent over $\mathbb{Q}$. Then, for every $\varepsilon > 0$,

$$\liminf_{N \to \infty} \frac{1}{N+1} \left\{ 0 \leq k \leq N : \sup_{s \in K_1} |\varphi(s + ikh) - f_1(s)| < \varepsilon, \sup_{s \in K_2} |\zeta(s + ikh, \alpha; B) - f_2(s)| < \varepsilon \right\} > 0.$$ 

Remark 1. A typical situation when $L(\mathbb{P}, \alpha, h)$ is linearly independent is the case when $\alpha$ and $\exp \left\{ \frac{2\pi}{h} \right\}$ are algebraically independent over $\mathbb{Q}$. The proof of this fact is given in [3].

Now recall some known facts of discrete universality which directly connect with objects under our interests.

Discrete universality property for the Matsumoto zeta-function under the condition that $\exp \left\{ \frac{2\pi k}{h} \right\}$ is irrational for every non-zero integer $k$ was obtained by the first author in [9]. While the discrete universality of the periodic Hurwitz zeta-functions was proved by A. Laurinčikas and R. Macaitienė in [17].

Also, some results on discrete analogue of mixed universality are known. The first attempt in this direction was done by the first author in [10], under the assumption that $\alpha$ is transcendental and $\exp \left\{ \frac{2\pi}{h} \right\}$ is rational. Unfortunately the proof in [10] is incomplete, as mentioned by A. Laurinčikas in 2014 (see [4]). However the argument in [10] gives a correct proof for the modified $L$-functions where all Euler factors corresponding to primes in the set of all prime numbers appearing as a prime factor of $a$ or $b$ such that $q = \exp \left\{ \frac{2\pi}{h} \right\} \in \mathbb{Q}$, $a, b \in \mathbb{Z}$, $(a, b) = 1$, are removed; see Section 5.

In [3] and [4], E. Buivydas and A. Laurinčikas proved the joint mixed discrete universality for the Riemann zeta-function $\zeta(s)$ and Hurwitz zeta-function $\zeta(s, \alpha)$. The first result [3] deals with the case when the minimal steps of arithmetical progressions $h$ for both functions are common, while in the second paper [4], for $\zeta(s)$ and $\zeta(s, \alpha)$, the minimal steps $h_1$ and $h_2$ are different from each other.

It is the purpose of the present paper to give the proof of the joint mixed discrete universality theorem (Theorem 3) for $(\varphi(s), \zeta(s, \alpha; B))$, which generalizes the result from [3], and to clarify the situation in [10].

2. A JOINT MIXED DISCRETE LIMIT THEOREM

The proof of Theorem 3 is based on a joint mixed discrete limit theorem in the sense of weakly convergent probability measures in the space of analytic functions for the Matsumoto zeta-functions $\varphi(s)$ and the periodic Hurwitz zeta-function $\zeta(s, \alpha; B)$ (for the detailed expositions of this method, see [15], [24]), which we prove in this section, using the linear independence of the set $L(\mathbb{P}, \alpha, h)$. In this section, $\varphi(s)$ denotes any general Matsumoto zeta-function.
For further statements, we start with some notations and definitions.

For a set $S$, denote by $\mathcal{B}(S)$ the set of all Borel subset of $S$. Let $\gamma = \{ s \in \mathbb{C} : |s| = 1 \}$.

Define

$$
\Omega_1 = \prod_{p \in \mathbb{P}} \gamma_p \quad \text{and} \quad \Omega_2 = \prod_{m=0}^{\infty} \gamma_m,
$$

where $\gamma_p = \gamma$ for all $p \in \mathbb{P}$, and $\gamma_m = \gamma$ for all $m \in \mathbb{N}_0$. By the Tikhonov theorem (see [14]), the tori $\Omega_1$ and $\Omega_2$ with the product topology and the pointwise multiplication are compact topological groups. Then

$$\Omega := \Omega_1 \times \Omega_2$$

is a compact topological Abelian group too, and we obtain the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$. Here $m_H = m_{1H} \times m_{2H}$ with the probability Haar measures $m_{1H}$ and $m_{2H}$ defined on the spaces $(\Omega_1, \mathcal{B}(\Omega_1))$ and $(\Omega_2, \mathcal{B}(\Omega_2))$, respectively.

Let $\omega_1(p)$ stand for the projection of $\omega_1 \in \Omega_1$ to the coordinate space $\gamma_p$, $p \in \mathbb{P}$, and, for every $m \in \mathbb{N}$, we put

$$
\omega_1(m) = \prod_{j=1}^{r} \omega_1(p_j)^{i_j},
$$

where, by factorizing of $m$ into the primes, $m = p_1^{l_1} \cdots p_r^{l_r}$. Let $\omega_2(m)$ denotes the projection of $\omega_2 \in \Omega_2$ to the coordinate space $\gamma_m$, $m \in \mathbb{N}_0$. Define $\omega = (\omega_1, \omega_2)$ for elements of $\Omega$. For any open subregion $G$ in the complex plane, let $H(G)$ be the space of analytic functions on $G$ equipped with the topology of uniform convergence in compacta.

The function $\varphi(s)$ has only finitely many poles by the condition (a). Denote those poles by $s_1(\varphi), \ldots, s_l(\varphi)$, and define

$$
D_\varphi = \{ s : \sigma > \sigma_0, \sigma \neq \Re s_j(\varphi), 1 \leq j \leq l \}.
$$

Then $\varphi(s)$ and its vertical shift $\varphi(s + ikh)$ are holomorphic in $D_\varphi$. The function $\zeta(s, \alpha; \mathfrak{B})$ can be written as a linear combination of Hurwitz zeta-functions [6], therefore it is entire, or has a simple pole at $s = 1$. Therefore $\zeta(s, \alpha; \mathfrak{B})$ and its vertical shift $\zeta(s + ikh, \alpha; \mathfrak{B})$ are holomorphic in

$$
D_\zeta = \begin{cases} 
\{ s \in \mathbb{C} : \sigma > \frac{1}{2} \} & \text{if } \zeta(s, \alpha; \mathfrak{B}) \text{ is entire,} \\
\{ s : \sigma > \frac{1}{2}, \sigma \neq 1 \} & \text{if } s = 1 \text{ is a pole of } \zeta(s, \alpha; \mathfrak{B}).
\end{cases}
$$

Now, in view of the definitions of $D_\varphi$ and $D_\zeta$, let $D_1$ and $D_2$ be two open subsets of $D_\varphi$ and $D_\zeta$, respectively. Let $H = H(D_1) \times H(D_2)$. On $(\Omega, \mathcal{B}(\Omega), m_H)$, define the $H$-valued random element $Z(s, \omega)$ by the formula

$$
Z(s, \omega) = (\varphi(s_1, \omega_1), \zeta(s_2, \alpha, \omega_2; \mathfrak{B})),
$$

where $s = (s_1, s_2) \in D_1 \times D_2$,

$$
\varphi(s_1, \omega_1) = \sum_{k=1}^{\infty} \frac{c_k \omega_1(k)}{k^{s_1}}.
$$
and
\[ \zeta(s_2, \alpha, \omega_2; \mathcal{B}) = \sum_{m=0}^{\infty} \frac{b_m \omega_2(m)}{(m + \alpha)^{s_2}}, \] (8)
respectively.

Denote by \( P_{\mathcal{Z}} \) the distribution of \( \mathcal{Z}(s, \omega) \) as an \( H \)-valued random element, i.e.,
\[ P_{\mathcal{Z}}(A) = m_H \{ \omega \in \Omega : \mathcal{Z}(s, \omega) \in A \}, \quad A \in \mathcal{B}(H). \]

Let \( N > 0 \). Define the probability measure \( P_N \) on \( H \) by the formula
\[ P_N(A) = \frac{1}{N+1} \# \{ 0 \leq k \leq N : \mathcal{Z}(s + ikh) \in A \}, \quad A \in \mathcal{B}(H), \]
where \( s + ikh = (s_1 + ikh, s_2 + ikh) \) with \( s_1 \in D_1, s_2 \in D_2 \), and
\[ \mathcal{Z}(s) = (\varphi(s_1), \zeta(s_2, \alpha; \mathcal{B})). \]

In the course of the proof of Theorem 3, the first main goal is the following mixed joint discrete limit theorem.

**Theorem 4.** Suppose that the set \( L(\mathbb{P}, \alpha, h) \) is linearly independent over \( \mathbb{Q} \). Then the probability measure \( P_N \) converges weakly to \( P_{\mathcal{Z}} \) as \( N \to \infty \).

We will omit some details of the proof, because the proof follows in the standard way (see, for example, the proof of Theorem 7 of [3]). However, though the following lemma, a mixed joint discrete limit theorem on the torus \( \Omega \), is exactly the same as Lemma 1 of [3], we reproduce the detailed proof, since this result plays a crucial role, and from the proof we can see why the linear independence of \( L(\mathbb{P}, \alpha, h) \) is necessary.

Define
\[ Q_N(A) := \frac{1}{N+1} \# \{ 0 \leq k \leq N : ((p^{-ikh} : p \in \mathbb{P}), ((m + \alpha)^{-ikh} : m \in \mathbb{N}_0)) \in A \}, \quad A \in \mathcal{B}(\Omega). \]

**Lemma 5 ([3]).** Suppose that the set \( L(\mathbb{P}, \alpha, h) \) satisfies the condition of Theorem 3. Then \( Q_N \) converges weakly to the Haar measure \( m_H \) as \( N \to \infty \).

**Proof.** For the proof of Lemma 5, we use the Fourier transformation method (for the details, see [15]). The dual group of \( \Omega \) is isomorphic to the group
\[ G := \left( \bigoplus_{p \in \mathbb{P}} Z_p \right) \bigoplus \left( \bigoplus_{m \in \mathbb{N}_0} Z_m \right) \]
with \( Z_p = \mathbb{Z} \) for all \( p \in \mathbb{P} \) and \( Z_m = \mathbb{Z} \) for all \( m \in \mathbb{N}_0 \). The element of \( G \) is written as \( (k, l) = ((k_p : p \in \mathbb{P}), (l_m : m \in \mathbb{N}_0)) \), where only finite number of integers \( k_p \) and \( l_m \) are non-zero, and acts on \( \Omega \) by
\[ (\omega_1, \omega_2) \to (\omega_1^{k_p} \omega_2^{l_m}) = \prod_{p \in \mathbb{P}} \omega_1^{k_p} \prod_{m \in \mathbb{N}_0} \omega_2^{l_m}(m). \]
Let $g_N(k, l)$, for $(k, l) \in G$, be the Fourier transform of the measure $Q_N(A)$. Then we have

$$g_N(k, l) = \int \Omega \left( \prod_{p \in \mathbb{P}} \omega_1^{k p} \prod_{m \in \mathbb{N}_0} \omega_2^{m} \right) dQ_N.$$ 

Thus, from the definition of $Q_N(A)$,

$$g_N(k, l) = \frac{1}{N+1} \sum_{k=0}^{N} \prod_{p \in \mathbb{P}} p^{-ikp} \prod_{m \in \mathbb{N}_0} (m + \alpha)^{-ikm}$$

$$= \frac{1}{N+1} \sum_{k=0}^{N} \exp \left\{ -ikh \left( \sum_{p \in \mathbb{P}} k_p \log p + \sum_{m \in \mathbb{N}_0} l_m \log (m + \alpha) \right) \right\}. \quad (9)$$

By the assumption of the lemma, the set $L(\mathbb{P}, \alpha, h)$ is linearly independent over $\mathbb{Q}$. Then the set $\{ (\log p : p \in \mathbb{P}) \}, \{ (\log (m + \alpha) : m \in \mathbb{N}_0) \}$ is linearly independent over $\mathbb{Q}$, and

$$\sum_{p \in \mathbb{P}} k_p \log p + \sum_{m \in \mathbb{N}_0} l_m \log (m + \alpha) = 0$$

if and only if $k = 0$ and $l = 0$. Moreover, if $(k, l) \neq (0, 0)$,

$$\exp \left\{ -ih \left( \sum_{p \in \mathbb{P}} k_p \log p + \sum_{m \in \mathbb{N}_0} l_m \log (m + \alpha) \right) \right\} \neq 1. \quad (10)$$

In fact, if (10) is false, then

$$\sum_{p \in \mathbb{P}} k_p \log p + \sum_{m \in \mathbb{N}_0} l_m \log (m + \alpha) = \frac{2 \pi a}{h} \quad (11)$$

with some $a \in \mathbb{Z} \setminus \{0\}$. But this contradicts to the linear independence of the set $L(\mathbb{P}, \alpha, h)$. Therefore, from (9) and (10), we find that

$$g_N(k, l) = \begin{cases} 1, & \text{if } (k, l) = (0, 0), \\ \frac{1}{N+1} \left( 1 - \exp \left\{ -ih \left( \sum_{p \in \mathbb{P}} k_p \log p + \sum_{m \in \mathbb{N}_0} l_m \log (m + \alpha) \right) \right\} \right), & \text{if } (k, l) \neq (0, 0). \end{cases}$$

Hence,

$$\lim_{N \to \infty} g_N(k, l) = \begin{cases} 1, & \text{if } (k, l) = (0, 0), \\ 0, & \text{otherwise}. \end{cases}$$

By a continuity theorem for probability measures on compact groups (see [7]), we obtain the statement of the lemma, i.e., that $Q_N(A)$ converges weakly to $m_H$ as $N \to \infty$. $\square$

Now, using Lemma[5] we may prove a joint mixed discrete limit theorem for absolutely convergent Dirichlet series.

Let, for fixed $\hat{\sigma} > \frac{1}{2}$,

$$\nu_1(m, n) = \exp \left\{ - \left( \frac{m}{n} \right)^{\hat{\sigma}} \right\}, \quad m, n \in \mathbb{N},$$
and
\[ v_2(m, n, \alpha) = \exp \left\{ -\left( \frac{m + \alpha}{n + \alpha} \right) \hat{\sigma} \right\}, \quad m \in \mathbb{N}_0, \quad n \in \mathbb{N}. \]

Define the series
\[ \varphi_n(s) = \sum_{m=1}^{\infty} \frac{c_m v_1(m, n)}{m^s}, \]
\[ \zeta_n(s, \alpha; \mathcal{B}) = \sum_{m=0}^{\infty} \frac{b_m v_2(m, n, \alpha)}{(m + \alpha)^s}, \]
and, for \( \hat{\omega} := (\hat{\omega}_1, \hat{\omega}_2) \in \Omega, \)
\[ \varphi_n(s, \hat{\omega}_1) = \sum_{m=1}^{\infty} \frac{\hat{\omega}_1(m) c_m v_1(m, n)}{m^s}, \]
\[ \zeta_n(s, \alpha, \hat{\omega}_2; \mathcal{B}) = \sum_{m=0}^{\infty} \frac{\hat{\omega}_2(m) b_m v_2(m, n, \alpha)}{(m + \alpha)^s}, \]
respectively. These series are absolutely convergent for \( \sigma > \frac{1}{2}. \)

For brevity, denote
\[ Z_n(s) = (\varphi_n(s_1), \zeta_n(s_2, \alpha; \mathcal{B})) \]
and
\[ \hat{Z}_n(s, \hat{\omega}) = (\varphi_n(s_1, \hat{\omega}_1), \zeta_n(s_2, \alpha, \hat{\omega}_2; \mathcal{B})). \]

Now, on the space \((H, \mathcal{B}(H))\), we consider the weak convergence of the measures
\[ P_{N,n}(A) = \frac{1}{N+1} \# \left\{ 0 \leq k \leq N : Z_n(s +ikh) \in A \right\}, \]
and, for \( \hat{\omega} \in \Omega, \)
\[ \hat{P}_{N,n}(A) = \frac{1}{N+1} \# \left\{ 0 \leq k \leq N : \hat{Z}_n(s +ikh, \hat{\omega}) \in A \right\}. \]

**Lemma 6.** Suppose that the set \( L(P, \alpha, h) \) is linearly independent over \( \mathbb{Q}. \) Then, on \((H, \mathcal{B}(H))\), there exists a probability measure \( P_n \) such that the measures \( P_{N,n} \) and \( \hat{P}_{N,n} \) both converge weakly to \( P_n \) as \( N \to \infty. \)

**Proof.** The proof of the lemma follows analogous to Lemma 2 from [3]. \( \square \)

The next step of the proof is to approximate the tuple \((Z(s), \hat{Z}(s, \hat{\omega}))\) by the tuple \((\hat{Z}_n(s), \hat{Z}_n(s, \hat{\omega}))\). For this purpose, we will use the metric on the space \( H. \) For any open region \( G, \) it is known (see [5] or [15]) that there exists a sequence of compact sets \( \{K_l : l \in \mathbb{N}\} \subset G \) satisfying conditions:

1. \( G = \bigcup_{l=1}^{\infty} K_l, \)
2. \( K_l \subset K_{l+1} \) for any \( l \in \mathbb{N}, \)
3. if \( K \) is a compact set, then \( K \subset K_l \) for some \( l \in \mathbb{N}. \)
For functions \( g_1, g_2 \in H(G) \), define a metric \( \rho_G \) by the formula

\[
\rho_G(g_1, g_2) = \sum_{l=1}^{\infty} \frac{1}{2^l} \sup_{s \in \mathbb{K}_l} |g_1(s) - g_2(s)|
\]

which induces the topology of uniform convergence on compacta. Put \( \rho_1 = \rho_{D_1} \) and \( \rho_2 = \rho_{D_2} \). Define, for \( g_1 = (g_{11}, g_{21}) \) and \( g_2 = (g_{12}, g_{22}) \) from \( H \),

\[
\rho(g_1, g_2) = \max \{ \rho_1(g_{11}, g_{12}), \rho_2(g_{21}, g_{22}) \}.
\]

In such a way, we obtain a metric on the space \( H \) including its topology.

**Lemma 7.** Suppose that the set \( L(\mathbb{P}, \alpha, h) \) is linearly independent over \( \mathbb{Q} \). The equalities

\[
\lim \limsup_{n \to \infty} \frac{1}{N+1} \sum_{k=0}^{N} \rho(Z(s+ikh), Z_n(s+ikh)) = 0 \tag{12}
\]

and, for almost all \( \omega \in \Omega \),

\[
\lim \limsup_{n \to \infty} \frac{1}{N+1} \sum_{k=0}^{N} \rho(Z(s+ikh, \omega), Z_n(s+ikh, \omega)) = 0 \tag{13}
\]

hold.

**Proof.** This can be shown in a way similar to the proofs of Lemmas 3 and 4 of [3], respectively. The main body of the argument in [3], based on an application of Gallagher’s lemma, is going back to the proof of Theorem 4.1 of [17]. We just indicate some different points from the proof in [3] and [17].

The starting point of the proof of (12) is the integral expression

\[
\varphi_n(s) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \varphi(s+z)I_n(z) \frac{dz}{z} \tag{14}
\]

and

\[
\zeta_n(s, \alpha; B) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \zeta(s+z, \alpha; B)I_n(z, \alpha) \frac{dz}{z}, \tag{15}
\]

where \( a > \frac{1}{2} \), and

\[
l_n(z) = \frac{z}{a} \Gamma \left( \frac{z}{a} \right) a^{-z} \quad \text{and} \quad l_n(z, \alpha) = \frac{z}{a} \Gamma \left( \frac{z}{a} \right) (n+\alpha)^z,
\]

respectively. We shift the paths to the left and apply the residue calculus. The case (15) is discussed in [17], where the path is moved to \( \mathcal{R}z = b - \sigma \) with \( \frac{1}{2} < b < 1 \) and \( \sigma > b \). In this case, the relevant poles are only \( z = 0 \) and \( z = 1 - s \). As for (14), we shift the path to \( \mathcal{R}z = \sigma_0 + \delta_0 - \sigma \), where \( \delta_0 \) is a small positive number such that \( \varphi(s) \) is holomorphic in the strip \( \sigma_0 \leq \mathcal{R}s \leq \sigma_0 + \delta_0 \). We encounter all the poles \( z = s_j(\varphi) - 1 \leq j \leq l \), so we have to consider all the residues coming from those poles. But they can be handled by the same method as described in the proof of Theorem 4.1 of [17].
To complete the proof of (12), it is also necessary to show the discrete mean square estimate
\[
\sum_{k=0}^{N} |\varphi(\sigma_0 + \delta_0 + it + ikh)|^2 \ll N(1 + |t|).
\] (16)
This is an analogue of Lemma 4.3 of [17], and can be obtained similarly from (5) and Gallagher’s lemma (Lemma 1.4 of [22]).

As for the proof of (13), we need the “random” version of (5), that is
\[
\int_0^T |\varphi(\sigma + it, \omega_1)|^2 dt = O(T), \quad \sigma > \sigma_0,
\] (17)
for almost all \( \omega_1 \in \Omega_1 \). This is actually a special case of Lemma 10 of [16]. The corresponding mean value result for \( \zeta(s, \alpha, \omega_2; \mathcal{B}) \) has been shown in [8]. Using those mean value results, we can show (13) in the same way as the proof of Lemma 4 of [3]. □

Lemma 7 together with the weak convergence of the measures \( P_N, n \) and \( \hat{P}_N, n \) (Lemma 6) enables us to prove that the probability measure \( P_N \) and one more probability measure defined as
\[
\hat{P}_N(A) = \frac{1}{N+1} \# \{ 0 \leq k \leq N : Z(s +ikh, \omega) \in A \}, \quad A \in \mathcal{B}(H),
\]
both converge weakly to the same probability measure \( P \), i.e., the following statement holds.

**Lemma 8.** Suppose that the set \( L(\mathbb{P}, \alpha, h) \) is linearly independent over \( \mathbb{Q} \). Then, on \( (H, \mathcal{B}(H)) \), there exists a probability measure \( P \) such that the measures \( P_N \) and \( \hat{P}_N \) both converge weakly to \( P \) as \( N \to \infty \).

**Proof.** This lemma can be shown analogously to Lemma 5 from [3]. □

**Proof of Theorem 4** As usual, in the last step of the proof of the functional discrete limit theorem, we show that the limit measure \( P \) in Lemma 8 coincides with \( P_Z \).

Define the measurable measure-preserving transformation \( \Phi_h : \Omega \to \Omega \) on the group \( \Omega \) by \( \Phi_h(\omega) = f_h \omega, \omega \in \Omega \), where \( f_h = \{(p^{-ih} : p \in \mathbb{P}), ((m+\alpha)^{-ih} : m \in \mathbb{N}_0)\} \). Again using (10), we see that \( \{\Phi_h(s)\} \) is a one-parameter group, and is ergodic. This together with the well-known Birkhoff-Khintchine theorem (see [6]) and the weak convergence of \( \hat{P}_N(A) \) gives that \( P(A) = P_Z(A) \) for all \( A \in \mathcal{B}(H) \). For the details, consult the proof of Theorem 7 of [3] or Theorem 6.1 of [17]. □

3. THE SUPPORT OF THE MEASURE \( P_Z \)

To introduce the support of \( P_Z \), we repeat the arguments of Section 4 from [12].

Let \( \varphi \in \tilde{S} \), and \( K_1, K_2, f_1 \) and \( f_2 \) be as in the statement of Theorem 3. Then we can find a real number \( \sigma_0 \) with \( \sigma^* < \sigma_0 < 1 \) and a positive number \( M > 0 \), such that \( K_1 \) is included
in the open rectangle

\[ D_M = \{ s : \sigma_0 < \sigma < 1, |t| < M \}. \]

Since \( \varphi(s) \in \tilde{S} \), the pole of \( \varphi \) is at most at \( s = 1 \), then, in this case, we find that

\[ D_\varphi = \{ s : \sigma > \sigma_0, \sigma \neq 1 \}. \]

Therefore \( D_M \) is an open subset of \( D_\varphi \). Also we can find \( T > 0 \) such that \( K_2 \) belongs to the open rectangle

\[ D_T = \left\{ s : \frac{1}{2} < \sigma < 1, |t| < T \right\}. \]

To obtain the support of the measure \( P_Z \) we will use Theorem 4 with \( D_1 = D_M \) and \( D_2 = D_T \). Let \( S_\varphi \) be the set of all \( f \in H(D_M) \) which is non-vanishing on \( D_M \), or constantly equivalent to 0 on \( D_M \).

**Theorem 9.** Suppose that the set \( L(P, \alpha, h) \) is linearly independent over \( \mathbb{Q} \). The support of the measure \( P_Z \) is the set \( S = S_\varphi \times H(D_T) \).

**Proof.** This is an analogue to Lemma 4.3 of [12] or Theorem 8 from [3]. The fact that \( \varphi \in \tilde{S} \) is essentially used here. \( \square \)

4. PROOF OF THE MIXED JOINT DISCRETE UNIVERSALITY THEOREM

The proof of Theorem 3 follows from Theorems 4 and 9 and the Mergelyan theorem (see [20]) which we state as a lemma.

**Lemma 10** (Mergelyan). Let \( K \subset \mathbb{C} \) be a compact subset with connected complement, and \( f(s) \) be a continuous function on \( K \) which is analytic in the interior of \( K \). Then, for every \( \varepsilon > 0 \), there exists a polynomial \( p(s) \) such that

\[ \sup_{s \in K} |f(s) - p(s)| < \varepsilon. \]

**Proof of Theorem 3** By Lemma 10, there exist polynomials \( p_1(s) \) and \( p_2(s) \) such that

\[ \sup_{s \in K_1} |f_1(s) - e^{p_1(s)}| < \frac{\varepsilon}{2}, \]  

and

\[ \sup_{s \in K_2} |f_2(s) - p_2(s)| < \frac{\varepsilon}{2}. \]

We introduce the set

\[ G = \left\{ (g_1, g_2) \in H : \sup_{s \in K_1} |g_1(s) - e^{p_1(s)}| < \frac{\varepsilon}{2}, \sup_{s \in K_2} |g_2(s) - p_2(s)| < \frac{\varepsilon}{2} \right\}. \]

Then \( G \) is an open set of the space \( H \). In virtue of Theorem 9, it is an open neighbourhood of the element \((e^{p_1(s)}, p_2(s))\) of the support of \( P_Z \). Thus \( P_Z(G) > 0 \). Using Theorem 4
and an equivalent statement of the weak convergence in terms of open sets (see (11)), we obtain
\[
\liminf_{N \to \infty} P_N(G) \geq P_Z(G) > 0.
\]
This and the definitions of \( P_N \) and \( G \) show that
\[
\liminf_{N \to \infty} \frac{1}{N+1} \sum_{0 \leq k \leq N} \sup_{s \in K_1} |\varphi(s + ikh) - e^{p_1(s)}| < \frac{\varepsilon}{2},
\]
\[
\sup_{s \in K_2} |\zeta(s + ikh, \alpha; \mathcal{M}) - p_2(s)| < \frac{\varepsilon}{2} > 0. \quad (20)
\]
From (18) and (19), we deduce that
\[
\left\{ 0 \leq k \leq N : \sup_{s \in K_1} |\varphi(s + ikh) - f_1(s)| < \varepsilon, \sup_{s \in K_2} |\zeta(s + ikh, \alpha; \mathcal{M}) - f_2(s)| < \varepsilon \right\}
\supset \left\{ 0 \leq k \leq N : \sup_{s \in K_1} |\varphi(s + ikh) - e^{p_1(s)}| < \frac{\varepsilon}{2}, \sup_{s \in K_2} |\zeta(s + ikh, \alpha; \mathcal{M}) - p_2(s)| < \frac{\varepsilon}{2} \right\}.
\]
This together with the inequality (20) gives the assertion of the theorem. \( \square \)

5. The case of modified zeta-functions

In Section 1 we mentioned an incomplete point in (10). An inaccuracy is actually included in a former paper (11), whose result is applied to (10). On p. 103 of (11), the same as (10) for \((k, l) \neq (0, 0)\) is claimed under the assumption that \(\alpha\) is transcendental and \(\exp(\frac{2\pi}{h})\) is rational. The same reasoning as in the case of (10) is valid if there is some \(l_m \neq 0\), because from (11) we have
\[
\prod_{p \in \mathbb{P}} p^{k_p} \prod_{m \in \mathbb{N}_0} (m + \alpha)^{l_m} = \left( \exp \left\{ \frac{2\pi}{h} \right\} \right)^a, \quad (21)
\]
which contradicts the assumption. But if all \(l_m = 0\), then (21) does not produce a contradiction. Therefore the results in (11), and hence in (10), is to be amended.

Write \(\exp \left\{ \frac{2\pi}{h} \right\} = \frac{a}{b}, a, b \in \mathbb{Z}, (a, b) = 1\), and denote by \(\mathbb{P}_h\) the set of all primes appearing as prime divisors of \(a\) or \(b\). Instead of \(Q_N(A)\) defined in Section 2 we define \(Q_{N, h}(A)\) by replacing \(\mathbb{P}\) in the definition of \(Q_N(A)\) by \(\mathbb{P} \setminus \mathbb{P}_h\). Let
\[
\Omega_{1h} = \prod_{p \in \mathbb{P} \setminus \mathbb{P}_h} \gamma_p,
\]
and denote the probability Haar measure on \((\Omega_{1h}, \mathcal{B}(\Omega_{1h}))\) by \(m_{1hH}\).

**Lemma 11.** Assume that \(\alpha\) is transcendental and \(\exp(\frac{2\pi}{h})\) is rational. Then \(Q_{N, h}\) converges weakly to the Haar measure \(m_{1hH} = m_{1hH} \times m_{2H}\) on the space \(\Omega_h = \Omega_{1h} \times \Omega_2\) as \(N \to \infty\).

**Proof.** If we replace \(\mathbb{P}\) by \(\mathbb{P} \setminus \mathbb{P}_h\) in (21), then the resulting equality is impossible even if all \(l_m = 0\). Therefore (10) is valid for any \((k, l) \neq (0, 0)\), and so we can mimic the proof of Lemma 5. \( \square \)
This lemma is the corrected version of Lemma 2.1 of [11]. Let \( \chi \) be a Dirichlet character, and define a modified Dirichlet \( L \)-function by

\[
L_h(s, \chi) = \prod_{p \in \mathbb{P} \setminus \mathbb{P}_h} \left( 1 - \frac{\chi(p)}{p^s} \right)^{-1}
\]

Then, using Lemma 11, we can show a mixed joint discrete universality theorem for \( L_h(s, \chi) \) and a periodic Hurwitz zeta-function, by the argument in [10]. This is the corrected version of Theorem 1.7 of [10], which was already mentioned in [19].

It is possible to generalize the above arguments to the class of Matsumoto zeta-functions. We conclude the present paper with the statement of such results.

Define the modified Matsumoto zeta-function by

\[
\tilde{\varphi}_h(s) = \prod_{m \in \mathbb{N} \setminus N_h} \prod_{j=1}^{g(m)} \left( 1 - a_m(p^{-s} f(j,m)) \right)^{-1},
\]

where \( \mathbb{N}_h \) is the set of all \( m \in \mathbb{N} \) such that \( p_m \in \mathbb{P}_h \), and \( \varphi_h(s) = \tilde{\varphi}_h(s + \alpha + \beta) \). The difference between \( \varphi_h(s) \) and \( \varphi(s) \) is only finitely many Euler factors, so their analytic properties are not so different. In particular, if \( \varphi(s) \) satisfies the properties (a), (b) and (c), then so is \( \varphi_h(s) \), too. Therefore, the method developed in the previous sections of the present paper can be applied to \( \varphi_h(s) \). Let

\[
Z_h(s) = (\varphi_h(s_1), \zeta(s_2, \alpha; \mathcal{B})),
\]

\[
Z_h(s, \omega_h) = (\varphi_h(s_1, \omega_{1h}), \zeta(s_2, \alpha, \omega_2; \mathcal{B}))
\]

where \( \omega_h = (\omega_{1h}, \omega_2) \in \Omega_h \). Define \( P_{Z_h} \) and \( P_{N,h} \) analogously to \( P_Z \) and \( P_N \), just replacing \( Z(s, \omega) \) and \( Z(s + ikh) \) by \( Z_h(s, \omega_h) \) and \( Z_h(s + ikh) \), respectively. Then, using Lemma 11 we obtain

**Theorem 12.** Suppose that \( \alpha \) is transcendental and \( \exp \left\{ \frac{2\pi h}{n} \right\} \) is rational. Then \( P_{N,h} \) converges weakly to \( P_{Z_h} \) as \( N \to \infty \).

**Theorem 13.** Let \( \varphi(s) \in \tilde{S}, K_1, K_2, f_1(s) \) and \( f_2(s) \) satisfy the conditions as in Theorem 2. Suppose that \( \alpha \) is transcendental and \( \exp \left\{ \frac{2\pi h}{n} \right\} \) is rational. Then, for every \( \varepsilon > 0 \),

\[
\liminf_{N \to \infty} \frac{1}{N+1} \# \left\{ 0 \leq k \leq N : \sup_{s \in K_1} |\varphi_h(s + ikh) - f_1(s)| < \varepsilon, \sup_{s \in K_2} |\zeta(s + ikh, \alpha; \mathcal{B}) - f_2(s)| < \varepsilon \right\} > 0.
\]

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