Abstract

We consider an impact of hadronic light-by-light scattering on the muonium hyperfine structure. A shift of the hyperfine interval $\Delta \nu(Mu)_{HLBL}$ is calculated with the light-by-light scattering approximated by exchange of pseudoscalar and pseudovector mesons. Constraints from the operator product expansion in QCD are used to fix parameters of the model similar to the one used earlier for the hadronic light-by-light scattering in calculations of the muon anomalous magnetic moment. The pseudovector exchange is dominant in the resulting shift, $\Delta \nu(Mu)_{HLBL} = -0.0065(10)$ Hz. Although the effect is tiny it is useful in understanding the level of hadronic uncertainties.
1 Introduction

Pure leptonic objects, such as free electron and muon or leptonic bound systems, positronium and muonium, are of specific interest because they allow ab initio calculations with a high accuracy. There is no effect of strong interactions in the leading terms and in a number of terms in perturbative series. Still hadronic effects enter through higher loops in electromagnetic and electroweak interactions.

The most important leptonic property affected by hadronic effects is the anomalous magnetic moment of a muon, $a_\mu = (g_\mu - 2)/2$, where the main hadronic contribution comes from the vacuum polarization (HVP), see Fig. 1. The hadronic contribution is quite small $\Delta a_\mu \text{(HVP)} \simeq 7 \times 10^{-8} \simeq 7 \times 10^{-5} a_\mu$, but nevertheless it is much larger than the experimental error in the $a_\mu$ measurement [2],

$$a_\mu^{\text{exp}} = 116\,592\,080(63) \times 10^{-11},$$

as well as the uncertainty of the QED calculations [1] and the electroweak contribution. The HVP contribution is obtained with sufficient accuracy by applying data from $e^+e^-$ annihilation into hadrons.

At this level of accuracy one need to take into account higher order hadronic effects and, in particular, the virtual light-by-light scattering (HLBL), see Fig. 2. In contrast to HVP there is no direct experimental input for determining HLBL so one should rely on a theoretical model.

Figure 1: The hadronic vacuum polarization contribution to $a_\mu$

Figure 2: The light-by-light scattering contribution to $a_\mu$
Two relevant theoretical parameters are the smallness of the chiral symmetry breaking, $m_\pi^2/m_\rho^2 \ll 1$, and the large number of colors, $N_c \gg 1$. The first parameter enters a powerlike, $1/m_\pi^2$ chiral enhancement for the charged pion loop in HLBL while the large $N_c$ limit implies dominance of meson exchanges, see Fig. 3 where mesons $M$ include neutral pion and heavier $C$-even neutral mesons.

In a number of papers dwelt on the problem it was shown that the chirally enhanced two-pion contribution is significantly smaller than the color enhanced one [3–7]. The model for light-by-light scattering developed in [7] is based also on QCD constraints which follow from operator product expansion at large photon virtualities. Together with the neutral pion the exchange of pseudovector mesons plays major role in the model.

In the present paper we consider an impact of the hadronic light-by-light scattering on another ‘pure leptonic’ quantity, namely, to the muonium hyperfine splitting (HFS), see Fig. 4. The hadronic effects in muonium are of somewhat less practical importance since there has been no experimental progress for years [8]. However, the accuracy in the former experiment was limited by statistics due to low muon

\[ = \sum_{M \text{ & permutations}} \]

Figure 3: Meson exchanges in the light-by-light scattering. Summation goes over interchanges of photons and over $C$-even neutral mesons.

Figure 4: The hadronic light-by-light scattering in the electron-muon interaction
flux. At present, better muon sources are available, e.g., at the Paul Scherrer Institut, and more accurate results are in principle possible. To start preparation for a new experiment one has to clearly understand the ultimate limit of the theoretical accuracy.

In principle, pure QED calculations are *ab initio* calculations and can be done with any accuracy (which does not mean that they can be done easy—see, reviews [9, 10] for the present status). However, the very involvement of the hadronic effects sets a certain limit of accuracy. As well as in the case of $a_\mu$ one has to calculate the HVP contribution in the leading order [11, 12], see Fig.5 and the next-to-leading term [12, 13].

The HVP contribution can be found from experimental data on the $e^+e^-$ annihilation into hadrons. The HLBL contribution is of the same order as the next-to-leading HVP contributions [12] and cannot be derived from existing scattering and annihilation data. So we extend the model of Ref. [7] for the hadronic light-by-light scattering to apply it to the muonium HFS.

An interesting feature of this application is that the dominant contribution comes from the “vertical” exchange by pseudovector mesons, see Fig.6. The reason for this dominance is that the pseudovector exchange shown in Fig.6 is the most relevant one.
for the spin-spin interaction of the electron and muon which determines the HFS. By contrast, a similar exchange of a neutral pion vanishes in the scattering amplitude when the electron and muon are at rest.

The pion and pseudovector cross-channel “horizontal” exchange, see Fig.7 are

\[
\begin{align*}
\text{e} & \quad + \quad \pi , a \\
\text{µ}
\end{align*}
\]

Figure 7: The “horizontal” exchange of pseudoscalar \(\pi\) and pseudovector mesons \(a\) in the \(e\mu\) scattering

also accounted in the model. This contribution is numerically smaller than the “vertical” one. Thus, the situation in HFS is opposite to that for the \(a_\mu\), where the pseudoscalar exchanges dominate.

Another interesting point is that the chirally enhanced charge pion loop in the blob of Fig.4 does not contribute to spin-dependent part of the scattering amplitude. Indeed, the quantum numbers \(J^P\) of exchange should be \(1^+\) as for pseudovector mesons. However, such quantum numbers are not allowed for the pair \(\pi^+\pi^-\). Thus, in contrast to \(a_\mu\), an ambiguous charged pion loop does not enter the muonium HFS.

In the next section we introduce general expressions for the HLBL effect in HFS. In Sec. 3 we present calculations of the pseudovector exchange, and in Sec. 4 we consider the pseudoscalar exchange. In the last section we summarize the results.

2 Generalities

Let us start with some general expressions. The muonium HFS is determined by the spin-dependent part of the forward \(e^-\mu^+\) scattering in the low-velocity limit. It is convenient to start with the \(e^-\mu^-\) amplitude and then make the charge conjugation for the muon.

The spin-dependent part of the forward \(e^- (p) + \mu^- (r) \to e^- (p) + \mu^- (r)\) scattering associated with HLBL can be presented as

\[
M^{\text{spin}}(e^- \mu^- \to e^- \mu^-) = A \bar{u}^{(e)} \gamma^\sigma \gamma_5 u^{(e)} \bar{u}^{(\mu)} \gamma^\sigma \gamma_5 u^{(\mu)} \to -4m_e m_\mu A \vec{\sigma}_e \vec{\sigma}_\mu , \quad (2)
\]

where \(u^{(e)}, u^{(\mu)}\) are Dirac spinors describing the electron and muon (we are using relativistic normalization and units \(\hbar = c = 1\)) and we took the nonrelativistic limit.
in the last expression. The transition from $\mu^-$ to $\mu^+$ does not change the result because of the positive $C$-parity of the axial current $\bar{u}(\mu)\gamma_\rho\gamma_5 u(\mu)$.

The above amplitude leads to the following addition in the $e^-\mu^+$ Hamiltonian,

$$\Delta H^{\text{HFS}} = A \delta^3(\vec{r}) \vec{\sigma}_e \vec{\sigma}_\mu.$$  (3)

This should be compared with the leading term for the $s$-wave HFS Hamiltonian,

$$H^{\text{HFS}} = \frac{2\pi\alpha}{3m_e m_\mu} \delta^3(\vec{r}) \vec{\sigma}_e \vec{\sigma}_\mu,$$  (4)

which gives for the HFS interval (Fermi energy $E_F$)

$$h\nu^{\text{HFS}} = E_F = \frac{8\alpha^4 m_e^2}{3 m_\mu} \left(\frac{m_\mu}{m_e + m_\mu}\right)^3 \simeq h \cdot 4.459 \times 10^9 \text{ Hz}.$$  (5)

The shift in the splitting due to HLBL is

$$h\Delta\nu^{\text{HLBL}} = \Delta E^{\text{HLBL}} = E_F \frac{3m_e m_\mu}{2\pi\alpha} A.$$  (6)

The amplitude $A$ is defined by the diagram in Fig. 4.

$$A = \frac{4\alpha^2}{3} \int \frac{d^4k d^4q}{(2\pi)^6} \frac{M_{\mu\mu'^\nu'\nu}(k, q)}{(k^2)^3(q^2)^3} \epsilon^{\nu\rho\mu\delta} k_\rho e^{\nu'\rho'\mu'\delta'} q_{\delta'} \left(g_{\delta\delta'} - \frac{r_{\delta\delta'}}{m_\mu^2}\right).$$  (7)

Here $M_{\mu\mu'^\nu'\nu}(k, q)$ is the the amplitude of the forward scattering of two virtual photons,

$$\gamma^\ast(\mu, k) + \gamma^\ast(\nu, q) \rightarrow \gamma^\ast(\mu', k) + \gamma^\ast(\nu', q),$$  (8)

(we mark their polarization indices and momenta), $r = \{m_\mu, 0\}$ is the 4-momentum of the muon at rest, and we neglected by the electron mass. In many cases, in particular for pseudovector exchanges, one can neglect by the muon mass as well. Then the expression in Eq. (7) simplifies further,

$$A = \alpha^2 \int \frac{d^4k d^4q}{(2\pi)^6} \frac{M_{\mu\mu'^\nu'\nu}(k, q)}{(k^2)^3(q^2)^3} \epsilon^{\nu\rho\mu\delta} k_\rho e^{\nu'\rho'\mu'\delta'} q_{\delta'}.$$  (9)

### 3 Pseudovector exchange

To calculate the “vertical” pseudovector exchange, see Fig. 6 let us start by introducing the effective vertex for lepton interaction with the pseudovector meson $a$,

$$h_a a_\rho \bar{l} \gamma^\rho \gamma_5 l, \quad l = e, \mu.$$  (10)
where $h_a$ is the coupling constant and $a_\rho$ is the polarization of the axial meson. Implying the same $h_a$ for the electron and muon (corrections due to their mass difference are small and can be accounted for) we get the pseudovector contribution to the amplitude $A$ of the forward $e\mu$ scattering, see Eq. (2),

$$A_{\text{vert}}^{\text{PV}} = - \frac{h_a^2}{m_a^2},$$ (11)

where $m_a$ is the pseudovector meson mass. Note the negative sign which follows from unitarity (see Ref. [14] for a detailed discussion of the sign). Hence even before explicit calculation we know that the pseudovector exchange correction to HFS is negative.

### 3.1 Coupling of pseudovector mesons to photons and leptons

The next step is to calculate $h_a$. To fix the $a\gamma^*\gamma^*$ vertex one can use that at large virtualities the operator product expansion relates product of two electromagnetic currents to the axial current [15], see also [7],

$$\int d^4 x d^4 y e^{-i q_1 x - i q_2 y} T \{ j_\mu(x) j_\nu(y) \} = \int d^4 z e^{i q_3 z} \frac{2 \epsilon_{\mu\nu\delta\rho} \hat{q}_\delta}{q^2} j_5^\rho(z) + \cdots. \quad (12)$$

Here

$$j_5^\rho = q \hat{Q}^2 \gamma^\rho \gamma_5 q$$ (13)

is the axial current, where different flavors enter with weights proportional to squares of their electric charges, $q_3 = q_1 + q_2$ and $\hat{q} = (q_1 - q_2)/2 \approx q_1 \approx -q_2$.

The $a\gamma^*\gamma^*$ vertex which satisfies this constraint at large $q^2$ and regular at small $q$ can be chosen in the form

$$V_{\rho\mu} a^\rho = \frac{i e^2 \langle a | j_5^\rho(0) | 0 \rangle}{(q_1^2 - m_v^2)(q_2^2 - m_v^2)} \left[ q_2^2 \epsilon_{\mu\nu\delta\rho} q_1^\delta + (q_1 \leftrightarrow q_2, \mu \leftrightarrow \nu) \right]. \quad (14)$$

Here $\langle a | j_5^\rho(0) | 0 \rangle$ is the matrix element between vacuum and the outgoing axial meson with 4-momentum $q_3 = q_1 + q_2$ and polarization $a^\rho$ (photon momenta $q_1$ and $q_2$ momenta are taken as incoming). The form factor parameter $m_v$ is the mass of the appropriate vector meson. Of course, this form of the vertex is model-dependent. This refers not only to the above form of $q_1^2, q_2^2$ dependence but also to choosing a particular structure, one of three possible structures for the vertex. The choice (14) picks up the structure which survives in asymptotics.
The lepton interaction with $a$ can be calculated then from the triangle diagram (the upper and lower blocks in Fig.6). Taken all external momenta to be vanishing and neglecting by lepton mass we get

\[ V = -2e^4 \langle a|j_5^a(0)|0\rangle \epsilon_{\mu\nu\delta\rho} \int \frac{d^4q}{(2\pi)^4} \frac{q^\delta}{q^2(q^2 - m_v^2)^2} \bar{l} \gamma^\mu \frac{1}{q} \gamma^\nu l \]

\[ = -\frac{3\alpha^2}{m_v^2} \langle a|j_5^a(0)|0\rangle \bar{l} \gamma_\rho \gamma_5 l. \]  

(15)

It gives the result for the pseudovector coupling to leptons in terms of the vector mass $m_v$ and the matrix element of the axial current between the vacuum and pseudovector meson.

Actually there are three electrically neutral pseudovector mesons $a^{(k)}$ which differ by their features under flavor SU(3). Therefore it is convenient to present the axial current $j_5^a = \bar{q} \hat{Q}^2 \gamma_\rho \gamma_5 q$ as a linear combination of axial currents with the same SU(3) quantum numbers as the mesons $a^{(k)}$. In particular, we can introduce the isovector, $j_5^{(3)} = \bar{q} \lambda_3 \gamma_\rho \gamma_5 q$, hypercharge, $j_5^{(8)} = \bar{q} \lambda_8 \gamma_\rho \gamma_5 q$, and the SU(3) singlet, $j_5^{(0)} = \bar{q} \gamma_\rho \gamma_5 q$, and write

\[ j_5^a = \sum_{k=3,8,0} \frac{\text{Tr} [\lambda_k \hat{Q}^2]}{\text{Tr} [\lambda_k]} j_5^{(k)} a^a_\rho, \]  

(16)

where $\lambda_0$ is the unity matrix. Accounting for mixing of the hypercharge and singlet pseudovector mesons is simply done by substituting $\lambda_8$ and $\lambda_0$ by appropriate linear combinations.

Thus, we get for the coupling $h_a^{(k)}$ of the meson $a^{(k)}$ to leptons (see Eq. (10) for definition)

\[ h_a^{(k)} = -\frac{3\alpha^2}{m_v^2} \frac{\text{Tr} [\lambda_k \hat{Q}^2]}{\text{Tr} [\lambda_k]} f_a^{(k)} a^a_\rho, \]  

(17)

where $f_a^{(k)}$ is the coupling of the meson $a^{(k)}$ to the corresponding axial current,

\[ \langle a^{(k)}|j_5^{(k)}|0\rangle = f_a^{(k)} a^a_\rho. \]  

(18)

### 3.2 Coupling of pseudovector mesons to axial currents

The value of $f_a^{(k)}$ can be fixed from consideration of the transition of the axial current, $j_5^{(k)}$, into two photons. We consider a special kinematics when one of those photons is soft with momentum $k \to 0$ and polarization $\epsilon^\nu$ and another is virtual, carrying the
same momentum $q$ as the axial current. The transition amplitude can be represented as

$$ T^{(k)}_{\mu\nu} e^\nu = i \langle 0 | \int d^4 z \ e^{i q z} T \{ f^{(k)}_\alpha(z) e j_\mu(0) \} | \gamma \rangle. $$

(19)

Generically, as it is shown in [16], the transition $T^{(k)}_{\mu\nu}$ can be written in terms of two Lorentz invariant amplitudes, $w^{(k)}_L(q^2)$ and $w^{(k)}_T(q^2)$,

$$ T^{(k)}_{\mu\nu} e^\nu = - \frac{i e^2 N_c \text{Tr} [\lambda_k \tilde{Q}^2]}{4\pi^2} \left\{ w^{(k)}_L(q^2) q_\rho q_\sigma \tilde{f}_{\sigma\mu} + w^{(k)}_T(q^2) \left( -q^2 \tilde{f}_{\mu\rho} + q_\mu q_\sigma \tilde{f}_{\sigma\rho} - q_\rho q_\sigma \tilde{f}_{\sigma\mu} \right) \right\}, $$

(20)

where $\tilde{f}_{\mu\rho} = \epsilon_{\mu\rho\gamma\sigma} k^\gamma e^\sigma$.

In perturbation theory, $w^{(k)}_L(q^2)$ are computed from triangle diagrams with two vector currents and an axial current. For massless quarks, we have

$$ w^{(k)}_L(q^2) = 2 w^{(k)}_T(q^2) = - \frac{2}{q^2}. $$

(21)

An appearance of the longitudinal part for the axial current which classically is conserved is a signal of the famous Adler-Bell-Jackiw axial anomaly [17]. The pole at $q^2 = 0$ in $w^{(k)}_L(q^2)$ is associated with propagation of massless Goldstone particles, the pion in case of $w^{(3)}_L$.

There is no perturbative corrections to these functions in the chiral limit. Moreover, the longitudinal functions $w^{(3,8)}_L$ protected even against nonperturbative corrections. It is not the case for transversal functions $w^{(k)}_T$ where the pole should be shifted from zero to vector and pseudovector masses. A particular model which account for this shift suggested in [16] has the form (in the chiral limit),

$$ w^{(k)}_T(q^2) = \frac{1}{m_a^2 - m_v^2} \left[ \frac{m_a^2}{m_a^2 - q^2} - \frac{m_v^2}{m_v^2 - q^2} \right], $$

(22)

where $m_{a,v}$ denote masses of pseudovector and vector mesons in the given channel $k$.

Equation (19) implies the following expression for the residue of the pole at $q^2 = m_a^2$,

$$ \lim_{q^2 \to m_a^2} (q^2 - m_a^2) T^{(k)}_{\mu\nu} e^\nu = f^{(k)}_a V_{\mu\nu} (q_1 = q, q_2 = k) e^\nu. $$

(23)

Comparing this with the residue from Eqs. (20) and (22) we get the result for $f^{(3)}_a$. In particular for $f^{(3)}_a$ we have

$$ \left[ f^{(3)}_a \right]^2 = \frac{N_c m_v^4}{2\pi^2}. $$

(24)
An independent way to find $f_a^{(3)}$ is to use Weinberg’s sum rules to relate it with the $\rho$ coupling to electromagnetic current, $\langle \rho | j_\mu | 0 \rangle = (m_\rho^2/g_\rho) \rho_\mu$,

$$
\left[ f_a^{(3)} \right]^2 = \left( \frac{2m_\rho^2}{g_\rho} \right)^2 .
$$

Then Eq. (24) implies an interesting relation

$$
g^2_\rho \frac{4\pi}{N_c} = 2\pi, \tag{26}
$$

reasonably good phenomenologically. This can be also compared with the QCD sum rule result [18], $g^2_\rho/(4\pi) = 2\pi/e$, where $e$, the base of natural logarithm, enters instead of $N_c$ – a pretty good approximation for $N_c = 3$.

### 3.3 Pseudovector exchange results

Combining Eqs. (11), (17) and (25) we get for the exchange by the isovector $a_1(1260)$ meson,

$$
A_{a_1}^{\text{vert}} = -\frac{3}{8} \frac{\alpha^4}{\pi^2} \frac{1}{m_{a_1}^2} .
$$

For the isoscalar pseudovector mesons $f_1(1285)$ and $f_1^*(1420)$ we assume the ideal mixing, similar to $\omega$ and $\phi$. It means that $f_1$ has the $(\bar{u}u + \bar{d}d)/\sqrt{2}$ structure and $f_1^*$ is $\bar{s}s$; this assumption is consistent with experimental data for decays of these resonances. In terms of the relevant axial currents the linear combinations $(\gamma_0/3) - (\gamma_8/\sqrt{3})$ and $(2\gamma_0/3) + (\gamma_8/\sqrt{3})$ enter correspondingly. Then, similarly to the $a_1$ exchange, we get

$$
A_{f_1}^{\text{vert}} = -\frac{25}{24} \frac{\alpha^4}{\pi^2} \frac{1}{m_{f_1}^2}, \quad A_{f_1^*}^{\text{vert}} = -\frac{1}{12} \frac{\alpha^4}{\pi^2} \frac{1}{m_{f_1^*}^2} .
$$

Altogether the “vertical” exchange by pseudovector mesons produces

$$
\Delta E_{\text{PV}}^{\text{vert}} = -\left( \frac{\alpha}{\pi} \right)^3 \frac{m_\mu m_\rho}{m_{a_1}^2} E_F \left[ \frac{9}{16} + \frac{25}{16} \frac{m_{f_1}^2}{m_{f_1^*}^2} + \frac{1}{8} \frac{m_{a_1}^2}{m_{f_1}^2} \right] = \frac{1}{16} \frac{m_{a_1}^2}{m_{f_1}^2} E_F \cdot 2.16 = h \cdot (-0.0041 \text{ Hz}) .
$$

Now let us add up the “horizontal” pseudovector exchanges shown in Fig.7. In the limit of heavy pseudovector mass this exchange differs from the “vertical” one just by
averaging over angles and constitutes 1/3 of the “vertical” exchange. Accounting for the finite pseudovector mass we found an extra to 1/3 suppression of the “horizontal” exchange by the factor 0.614,

$$\Delta E^{PV}_{\text{horiz}} = - \left( \frac{\alpha}{\pi} \right)^3 \frac{m_e m_\mu}{m_{a_1}^2} E_F \cdot \frac{0.614}{3} = h \cdot (-0.00084 \text{ Hz}).$$  \hspace{1cm} (30)$$

Thus, the total for pseudovector exchange is

$$\Delta E^{PV} = - \left( \frac{\alpha}{\pi} \right)^3 \frac{m_e m_\mu}{m_{a_1}^2} E_F \cdot 2.6 = h \cdot (-0.0049 \text{ Hz}).$$ \hspace{1cm} (31)$$

Note that we limit ourselves by exchanges of pseudovectors with the lowest mass in each flavor channel. Exchanges by higher $1^+$ excitation contribute in the same direction but probably are numerically suppressed.

## 4 Pseudoscalar exchange

Let us start with the $\pi^0 \gamma^* \gamma^*$ vertex,

$$V_{\mu \nu} = c_{\pi \gamma \gamma} F_{\pi \gamma \gamma^*} (k^2, q^2) \epsilon_{\mu \rho \sigma \delta} k^\rho q^\sigma \epsilon_{\nu \rho' \sigma' \delta'} k'^\rho q'^\sigma'. \hspace{1cm} (32)$$

Here $k$ and $q$ are photon momenta, $\mu$ and $\nu$ are their polarization indices, the constant $c_{\pi \gamma \gamma}$ is fixed by the width of $\pi^0 \rightarrow \gamma \gamma$ decay and $F_{\pi \gamma \gamma^*} (k^2, q^2)$ is the form factor of the transition, $F_{\pi \gamma \gamma^*} (0,0) = 1$. Theoretical expression for $c_{\pi \gamma \gamma}$

$$c_{\pi \gamma \gamma} = \frac{\alpha N_c}{3 \pi F_\pi} \hspace{1cm} (33)$$

follows from the Adler-Bell-Jackiw anomaly [17]. Indeed, it could be read off from the residue of the pole in the longitudinal part in Eq. (20).

The pion exchange gives then the following expression for the forward scattering of two virtual photons:

$$M^{\text{pion}}_{\mu \nu', \mu'} = c_{\pi \gamma \gamma}^2 [F_{\pi \gamma \gamma^*} (k^2, q^2)]^2 \frac{\epsilon_{\mu \rho \sigma} k^\rho q^\sigma \epsilon_{\nu' \rho' \sigma'} k'^\rho q'^\sigma'}{m_\pi^2 - \left( k + q \right)^2} + \left( \mu \leftrightarrow \mu', k \rightarrow -k \right). \hspace{1cm} (34)$$

The third permutation involving $k \leftrightarrow -q$ vanishes for the forward scattering. It means an absence of the “vertical” exchange for the pion mentioned earlier once atomic momenta are neglected.\footnote{The higher order contributions to muonium HFS due to the “vertical” pion exchange together with additional photon are suppressed by extra small factors such as $\alpha$ and $m_e/m_\pi$.}
Now we have to substitute $M^{\text{pion}}_{\mu
u\mu'\nu'}$ to Eq. (1) and integrate over $k$ and $q$. By power counting at large momenta it is simple to see that in absence of the form factor $F_{\pi\gamma\gamma^*}(k^2, q^2)$ the integral logarithmically diverges. The form factor provides a convergence above momenta of order of $m_\rho$, while its infrared convergence is regulated by pion and muon masses. The $\ln(m_\rho/m_\pi)$ term can be determined analytically,

$$
\Delta E^\text{log}_\pi = -\left(\frac{\alpha}{\pi}\right)^3 \frac{m_em_\mu}{(4\pi F_\pi^*)^2} E_F \cdot \frac{9}{8} \ln \frac{m_\rho}{m_\pi} = h \cdot (-0.0042 \text{ Hz}). \quad (35)
$$

For numerical estimates we use $m_\pi = 135$ MeV, $m_\rho = 775$ MeV, $F_\pi = 92$ MeV. The logarithm is not that big, $\ln(m_\rho/m_\pi) = 1.75$ so a numerical integration with a certain model for the form factor is needed.

For the form factor

$$
F_{\pi\gamma\gamma^*}(k^2, q^2) = \frac{m_\rho^4}{(m_\rho^2 - k^2)(m_\rho^2 - q^2)} \quad (36)
$$

numerical integration gives

$$
\Delta E_\pi = -\left(\frac{\alpha}{\pi}\right)^3 \frac{m_em_\mu}{(4\pi F_\pi^*)^2} E_F \cdot 0.61 = h \cdot (-0.0014 \text{ Hz}). \quad (37)
$$

The suppression of the logarithmic result can be approximated by substitution

$$
\ln \frac{m_\rho}{m_\pi} \rightarrow \left(\ln \frac{m_\rho}{m_\pi} - 1.2\right) \quad (38)
$$
in Eq. (35).

The result (37) can be compared with the earlier calculation of the pion contribution by Faustov and Martynenko [19]. They found $\Delta E_\pi = h \cdot (+0.0011 \text{ Hz})$ for the same form factor (36). While the magnitude is close we differ in the sign. Note that Faustov and Martynenko also considered change of HFS due to effect of HLBL pion exchange on $a_\mu$, the effect we are not considering.

Strictly speaking the form factor (36) violates the QCD constraints. It follows from the OPE expansion (12) that at $q^2 = k^2$ the form factor should decrease as $1/q^2$ at large Euclidean momenta, not as $1/q^4$ as in Eq. (36). So we made numerical integration with the form factor which satisfies the above mentioned constraint as well as other theoretical and experimental limitations [5] (see also [7]),

$$
F_{\pi\gamma\gamma^*}(k^2, q^2) = \frac{m_\rho^4 M_2^4 - (4\pi^2 F_\pi^2/N_c) \left[ q_1^2 q_2^2 (q_1^2 + q_2^2) + h_2 \cdot q_1^2 q_2^2 + h_5 \cdot (q_1^2 + q_2^2) \right]}{(q_1^2 - m_\rho^2)(q_1^2 - M_2^2)(q_2^2 - m_\rho^2)(q_2^2 - M_2^2)}, \quad (39)
$$
where $M_2 = 1465\text{ MeV}$, $h_5 = 6.93\text{ GeV}^4$, $h_2 = -10\text{ GeV}^2$. The result of integration turns out to be very close to the one in Eq. [37], the difference is insignificant.

Calculations for the other pseudoscalars, $\eta(547)$ and $\eta'(958)$, can be done in a similar fashion. We use their experimental two-photon width to determine the two-photon couplings and simple vector-dominance form factors for off-shell photons with $m_\rho = m_\omega = 775\text{ MeV}$ and $m_\phi = 1020\text{ MeV},$

\[
F_{\eta\gamma\gamma^*}(k^2, q^2) = \frac{5}{3} \frac{m_\rho^4}{(m_\rho^2 - k^2)(m_\rho^2 - q^2)} - \frac{2}{3} \frac{m_\phi^4}{(m_\phi^2 - k^2)(m_\phi^2 - q^2)},
\]

\[
F_{\eta'\gamma\gamma^*}(k^2, q^2) = \frac{5}{6} \frac{m_\rho^4}{(m_\rho^2 - k^2)(m_\rho^2 - q^2)} + \frac{1}{6} \frac{m_\phi^4}{(m_\phi^2 - k^2)(m_\phi^2 - q^2)},
\]

based on the octet and singlet quark structure of $\eta$ and $\eta'$. The results of numerical integration are

\[
\Delta E_\eta = -\left(\frac{\alpha}{\pi}\right)^3 \frac{m_e m_\mu}{(4\pi F_\pi)^2} E_F \cdot 0.063 = h \cdot (-0.00014 \text{ Hz}),
\]

\[
\Delta E_{\eta'} = -\left(\frac{\alpha}{\pi}\right)^3 \frac{m_e m_\mu}{(4\pi F_\pi)^2} E_F \cdot 0.046 = h \cdot (-0.00010 \text{ Hz}).
\]

This can be compared with calculations by Faustov and Martynenko [19], they obtained 0.0002 Hz for $\eta$ and 0.0001 Hz for $\eta'$. Again, we have a sign difference.

Thus, the total for pseudoscalar exchanges is

\[
\Delta E_{\text{PS}} = -\left(\frac{\alpha}{\pi}\right)^3 \frac{m_e m_\mu}{(4\pi F_\pi)^2} E_F \cdot 0.72 = h \cdot (-0.0016 \text{ Hz}{}).
\]

5 Summary

Collecting the results (31), (42) for pseudovector and pseudoscalar exchanges we come to

\[
\Delta E_{\text{HLBL}} = -\left(\frac{\alpha}{\pi}\right)^3 m_e m_\mu E_F \left[ \frac{2.6}{m_{a_1}^2} + \frac{0.72}{(4\pi F_\pi)^2} \right] = h \cdot (-0.0049 \text{ Hz} - 0.0016 \text{ Hz}) = h \cdot (-0.0065 \text{ Hz}).
\]

The main contribution is due to pseudovector exchange, the “vertical” one in Fig. 6. It is fivefold larger than “horizontal” pseudovector exchange and threefold larger than the “horizontal” pion exchange.
What is the accuracy of the result? We mentioned in Introduction an absence of the charged pion loop associated with chiral enhancement. This makes the result more reliable. Looking on variations of parameters such as coupling of the pseudovectors to axial currents we would estimate the uncertainty of the model for the dominant pseudovector exchange as 10%. Staying on the conservative side we ascribe a total uncertainty to be about 25% of the pseudovector "vertical" contribution, i.e., about 0.001 Hz. Thus, our final result is

\[ h\Delta \nu_{\text{HLBL}} = \Delta E_{\text{HLBL}} = -0.0065(10) \text{ Hz}. \]

We see that the HLBL correction is tiny and rather unobservable. However, it shows the level of limitations on theoretical accuracy which comes from hadrons. In our study we also obtained a few relations for couplings of pseudovector mesons to photons, leptons and axial currents which can be applied to variety of processes.

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