REMARKS ON VERY BASIC SLC-TRIVIAL FIBRATIONS

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Abstract. We study very basic slc-trivial fibrations. We show that restricting on any lc center of a very basic slc-trivial fibration, its moduli part is numerically trivial if and only if it is $\mathbb{Q}$-linearly trivial. We then prove that abundance conjecture for very basic slc-trivial fibrations holds true in dimension two when the moduli part is $\mathbb{Q}$-Cartier. As an application, we prove that the log canonical ring of a projective plt pair with Kodaira dimension 3 is finitely generated.

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1. Introduction

We begin with Kodaira’s canonical bundle formula in a generalized form due to Ueno [U], Kawamata [Ka1] and Fujita [Ft2]. Let $f: S \to C$ be an elliptic surface and $J: C \to \mathbb{P}^1$ the $j$-function. Then there are $\mathbb{Q}$-divisors $B_S, B_C$ and $M_C$ such that

\begin{equation}
K_S + B_S = f^*(K_C + B_C + M_C)
\end{equation}

where $(S, B_S)$ and $(C, B_C)$ are log pairs related by adjunction, and

\begin{equation}
\mathcal{O}_C(12M_C) \simeq J^*\mathcal{O}_{\mathbb{P}^1}(1)
\end{equation}

which implies that $M_C$ is semi-ample. These results can be generalized further to the case that $f: X \to Y$ is an elliptic fibration, that is, an algebraic fiber space whose general fiber is an elliptic curve, and the $j$-function extends to a morphism $J: Y \to \mathbb{P}^1$. Then Kodaira’s original canonical bundle formula is a special case where $f$ is minimal and $\dim Y = 1$.

This generalized formula plays a central role in the study of elliptic fibrations. It is natural to expect that some more general kinds of canonical bundle formulas could play a similar role in the study of trivial fibration, that is, a projective surjective morphism $f: (X, B_X) \to Y$ where $(X, B_X)$ has mild singularities and $K_X + B_X$ is $\mathbb{Q}$-linearly trivial over $Y$.

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One of the starting points in this direction is Mori’s work \([\text{M}]\). It is a prototype of the so-called Fujino–Mori canonical bundle formula, established by Fujino and Mori in \([\text{FM}]\) combining Kawamata’s work \([\text{Ka2}]\) and Mori’s unpublished preprint. On the other hand, Ambro started to study some applications of \([\text{Ka2}]\) at around the same time. Then he formulated and studied lc-trivial fibrations in \([\text{A3}]\), which is renamed klt-trivial fibrations (Ambro’s lc-trivial fibrations). Recently, Fujino generalized klt-trivial fibrations and lc-trivial fibrations (Fujino–Gongyo’s lc-trivial fibrations) to those for klt-trivial fibrations formulated in \([\text{FG}]\). In that paper, Fujino and Gongyo further showed how to reduce some problems for lc-trivial fibrations (Fujino–Gongyo’s lc-trivial fibrations) to those for klt-trivial fibrations. As a result, they generalized the notation of the lc-trivial fibrations to those of the klt-trivial fibrations. 

The following definition gives the definitions of klt-trivial fibrations \([\text{A3} \text{ Definition 2.1}]\), lc-trivial fibrations \([\text{FG} \text{ Definition 3.2}]\) and (basic) slc-trivial fibrations \([\text{Fn7} \text{ Definition 4.1}]\) simultaneously. They are slightly different with the original ones. Moreover, we define a special case of (basic) slc-trivial fibrations, which is similar to qlc pairs in \([\text{A1}], \text{Fn6}\).

**Definition 1.1.** A pre-basic slc-trivial (resp. lc-trivial, klt-trivial) fibration \(f: (X, B_X) \rightarrow (Y, D)\) consists of a projective surjective morphism \(f: X \rightarrow Y\) and a simple normal crossing pair \((X, B_X)\) satisfying the following properties:

1. \(Y\) is a normal variety,
2. every stratum of \(X\) is dominant onto \(Y\) and \(f_*\mathcal{O}_X \cong \mathcal{O}_Y\),
3. \((X, B_X)\) is sub-slc (resp. sub-lc, sub-klt) over \(Y\),
4. \(D\) is a \(\mathbb{Q}\)-Cartier \(\mathbb{Q}\)-divisor on \(Y\) such that \(K_X + B_X \sim_\mathbb{Q} f^*D\).

If a pre-basic slc-trivial (resp. lc-trivial, klt-trivial) fibration \(f: (X, B_X) \rightarrow (Y, D)\) satisfies

5. \(\text{rank } f_*\mathcal{O}_X([-(B_X^{\leq 1})]) = 1\),

then it is called a basic slc-trivial (resp. lc-trivial, klt-trivial) fibration; furthermore, if a basic slc-trivial (resp. lc-trivial, klt-trivial) fibration \(f: (X, B_X) \rightarrow (Y, D)\) also satisfies

6. the natural map \(\mathcal{O}_Y \rightarrow f_*\mathcal{O}_X([-(B_X^{\leq 1})])\) is an isomorphism,

then it is called a very basic slc-trivial (resp. lc-trivial, klt-trivial) fibration.

**Remark 1.2.** It is more natural (cf. \([\text{Fn6} \text{ Remark 6.2.3}]\)) to replace condition (4) above by that

\(4')\ \mathcal{D}\ \text{is a } \mathbb{Q}\text{-line bundle on } Y \text{ such that } K_X + B_X \sim_\mathbb{Q} f^*\mathcal{D}\).

Let \(\text{Div}(X)\) be the group of Cartier divisors on \(X\) and \(\text{Pic}(X)\) the Picard group of \(X\). Let

\[\delta_X: \text{Div}(X) \otimes \mathbb{Q} \rightarrow \text{Pic}(X) \otimes \mathbb{Q}\]

be the homomorphism induced by \(A \rightarrow \mathcal{O}_X(A)\) where \(A\) is a Cartier divisor on \(X\). Then \(K_X + B_X \sim_\mathbb{Q} f^*\mathcal{D}\) means that

\[\delta_X(K_X + B_X) = f^*\mathcal{D}\]

in \(\text{Pic}(X) \otimes \mathbb{Q}\). Let \(b = \min\{m \in \mathbb{Z}_{>0} | m(K_F + B_F) \sim 0\}\) where \(F\) is a general fiber of \(f\) and \(K_F + B_F = (K_X + B_X)|_F\). Then we can take a \(\mathbb{Q}\)-Cartier \(\mathbb{Q}\)-divisor \(D\) on \(Y\) and a rational function \(\varphi \in \Gamma(X, K_X^*)\) (see \([\text{Fn7} \text{ Section 6}]\)) such that \(\mathcal{O}_Y(D) = \mathcal{D}\) in \(\text{Pic}(Y) \otimes \mathbb{Q}\) and

\[K_X + B_X + \frac{1}{b}(\varphi) = f^*D\.]
Theorem 1.4. Let $(X, \Delta)$ be a projective plt pair such that $\Delta$ is a $\mathbb{Q}$-divisor. Assume that $\kappa(X, K_X + \Delta) = 3$. Then the log canonical ring

$$R(X, \Delta) = \bigoplus_{m \geq 0} H^0(X, \mathcal{O}_X(|m(K_X + \Delta)|))$$

is a finitely generated $\mathbb{C}$-algebra.

To this end, we study very basic slc-trivial fibrations in details and prepare some more general and technical results. Let $f : (X, B_X) \to (Y, D)$ be a pre-basic slc-trivial fibration. Let $P$ be a prime divisor on $Y$. By shrinking $Y$ around the generic point of $P$, we assume that $P$ is Cartier. Set

$$b_P := \max \{ t \in \mathbb{Q} \mid (X, B_X + tf^*P) \text{ is sub-slc over the generic point of } P \}.$$

Then we put

$$B_Y := \sum_P (1 - b_P)P,$$

where $P$ runs over prime divisors on $Y$. The $\mathbb{Q}$-divisor $B_Y$ is well-defined and is called the discriminant $\mathbb{Q}$-divisor (or discriminant part) of $f : (X, B_X) \to (Y, D)$. We further put

$$M_Y := D - K_Y - B_Y,$$

where $K_Y$ is the canonical divisor of $Y$. Then the $\mathbb{Q}$-divisor $M_Y$ is called the moduli $\mathbb{Q}$-divisor (or moduli part) of $f : (X, B_X) \to (Y, D)$. We call $D = K_Y + B_Y + M_Y$ as above the structure decomposition. See [FL1] for more details.

Conjecture 1.5 ([FL1 Conjecture 1.5]). Let $f : (X, B_X) \to (Y, D)$ be a very basic slc-trivial fibration with the structure decomposition $D = K_Y + B_Y + M_Y$. Then there exists an effective $\mathbb{Q}$-divisor $\Delta \sim_{\mathbb{Q}} B_Y + M_Y$ on $Y$ such that $(Y, \Delta)$ is log canonical.
It is easy to see that Conjecture \ref{conj:1.5} holds true in the case that $M_Y$ is $\mathbb{Q}$-Cartier and $M_Y \equiv 0$ by Corollary \ref{cor:3.2} and \cite{FFL} Theorem 1.3. In this paper, we study the numerically trivial moduli part further and prove the following useful theorem.

**Theorem 1.6 (Theorem \ref{thm:1.4}).** Let $f: (X, B_X) \to (Y, D)$ be a very basic slc-trivial fibration with the structure decomposition $D = K_Y + B_Y + M_Y$. Let $Z$ be a union of lc centers of $f$. If $M_Y$ is $\mathbb{Q}$-Cartier and $M_Y|_Z \equiv 0$, then $M_Y|_Z \sim _\mathbb{Q} 0$.

The lc center of a very basic slc-trivial fibration is defined in Definition \ref{def:3.9}. We emphasize that the union of lc centers $Z$ above is not necessary normal.

If we assume that abundance conjecture for lc pairs holds true, then Conjecture \ref{conj:1.5} implies the following conjecture immediately.

**Conjecture 1.7.** Let $f: (X, B_X) \to (Y, D)$ be a very basic slc-trivial fibration. If $D$ is nef, then $D$ is semi-ample.

Conjecture \ref{conj:1.7} can be viewed as a kind of abundance conjecture for very basic slc-trivial fibrations. It holds true in the case that $\dim Y = 1$ by using \cite{FFL} Corollary 1.4] directly (cf. \cite{FL3} Corollary 5.4] or \cite{L2} Corollary 5.3]). We show that it also holds true in the case that $\dim Y = 2$ with the assumption that $M_Y$ is $\mathbb{Q}$-Cartier, equivalently, $K_Y + B_Y$ is $\mathbb{Q}$-Cartier. More precisely,

**Theorem 1.8.** Let $f: (X, B_X) \to (Y, D)$ be a very basic slc-trivial fibration with the structure decomposition $D = K_Y + B_Y + M_Y$, where $Y$ is a normal surface. If $K_Y + B_Y$ is $\mathbb{Q}$-Cartier and $D$ is nef, then $D$ is semi-ample.

The assumption that $K_Y + B_Y$ is $\mathbb{Q}$-Cartier is very natural in practice. Actually, we can even assume that $Y$ is $\mathbb{Q}$-factorial if we only use Theorem 1.8 to prove Theorem 1.4. It is also worth to mention that we only need Theorem 1.8 for very basic klt-trivial fibrations in the proof of Theorem 1.4. See Section 6 for the details.

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2. Preliminaries

2.1 (Notation and conventions). We work over the complex number field $\mathbb{C}$ throughout this paper. We freely use the basic notation of the minimal model program in \cite{Fn3, Fn4, Fn6}. In this paper, we consider only $\mathbb{Q}$-divisors instead of $\mathbb{R}$-divisors. A scheme is always assumed to be separated and of finite type over $\mathbb{C}$. A variety is a reduced and irreducible scheme. A curve (resp. surface) is a variety of dimension 1 (resp. 2).

Let $D$ be a $\mathbb{Q}$-divisor on an equidimensional scheme $X$, that is, $D$ is a finite formal sum $\sum_i d_i D_i$ where $d_i \in \mathbb{Q}$ and $\{D_i\}_i$ are distinct prime divisors on $X$. We put

$$D^{<1} = \sum_{d_i < 1} d_i D_i, \quad D^{\leq 1} = \sum_{d_i \leq 1} d_i D_i, \quad \text{and} \quad D^{=1} = \sum_{d_i = 1} D_i.$$  

We also put

$$[D] = \sum_i [d_i] D_i, \quad \|D\| = -\|D\|, \quad \text{and} \quad \{D\} = D - [D],$$

where $[d_i]$ is the integer defined by $d_i \leq [d_i] < d_i + 1$. A $\mathbb{Q}$-divisor $D$ is called a sub boundary if $D = D^{\leq 1}$ holds. If a sub boundary $D$ is also effective ($d_i \geq 0$ for every $i$),
then $D$ is called a boundary. Let $D_1$ and $D_2$ be two $\mathbb{Q}$-divisors on $X$. We denote $D_1 \leq D_2$ (resp. $D_1 < D_2$) if $D_2 - D_1$ is effective (resp. strictly effective). Assume that $f: X \to Y$ is a surjective morphism onto a normal variety $Y$. Then

$$D^v = \sum_{f(D_i) \subset Y} d_i D_i \quad \text{and} \quad D^h = D - D^v$$

are called the vertical part and horizontal part of $D$ with respect to $f: X \to Y$ respectively.

A $\mathbb{Q}$-Cartier divisor $D$ on $X$ is an element of $\Gamma(X, K_X^*/\mathcal{O}_X^*) \otimes \mathbb{Q}$. Let $D_1$ and $D_2$ be two $\mathbb{Q}$-Cartier divisors on $X$. Then we write $D_1 \sim_\mathbb{Q} D_2$ if there exists a positive integer $m$ such that $mD_1 \sim mD_2$, that is, $mD_1$ is linearly equivalent to $mD_2$. Let $f: X \to Y$ be a surjective morphism onto a normal variety $Y$. Then we write $D_1 \sim_\mathbb{Q}, f D_2$ if there exists a $\mathbb{Q}$-Cartier divisor $B$ on $Y$ such that $D_1 - D_2 \sim_\mathbb{Q} f^*B$. If $P$ is a prime divisor on $Y$, then its generic point is Cartier. We denote as $f^*P_\eta$ the closure of the pulling back of the generic point of $P$. If $P = \sum a_i P_i$ is a $\mathbb{Q}$-divisor, then $f^*P_\eta := \sum a_i f^*P_{i,\eta}$.

2.2 (Singularities of pairs). A pair $(X, \Delta)$ consists of a normal variety $X$ and a $\mathbb{Q}$-divisor $\Delta$ on $X$ such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. Let $f: Y \to X$ be a projective birational morphism from a normal variety $Y$. Then we can write

$$K_Y = f^*(K_X + \Delta) + \sum_E a(E, X, \Delta) E$$

with

$$f_* \left( \sum_E a(E, X, \Delta) E \right) = -\Delta,$$

where $E$ runs over prime divisors on $Y$. We call $a(E, X, \Delta)$ the discrepancy of $E$ with respect to $(X, \Delta)$. Note that we can define the discrepancy $a(E, X, \Delta)$ for any prime divisor $E$ over $X$ by taking a suitable resolution of singularities of $X$. If $a(E, X, \Delta) \geq -1$ (resp. $a(E, X, \Delta) > -1$) for every prime divisor $E$ over $X$, then $(X, \Delta)$ is called sub log canonical (sub-lc for short) or sub Kawamata log terminal (sub-klt for short) respectively. If $a(E, X, \Delta) > -1$ for every exceptional divisor $E$ over $X$, then $(X, \Delta)$ is called sub purely log terminal (sub-plt for short).

If $\Delta$ is effective, then a pair $(X, \Delta)$ is called lc (resp. klt, plt) if it is sub-lc (resp. sub-klt, sub-plt). A divisorial log terminal (dlt for short) pair is a limit of klt pairs in the sense of [KM] Proposition 2.43] (see [KM] Definition 2.37 and Proposition 2.40 for precise definitions).

Let $(X, \Delta)$ be a sub-lc pair. If there exist a projective birational morphism $f: Y \to X$ from a normal variety $Y$ and a prime divisor $E$ on $Y$ with $a(E, X, \Delta) = -1$, then $f(E)$ is called an lc center of $(X, \Delta)$.

Let $X$ be a reduced equidimensional scheme which satisfies Serre’s $S_2$ condition and is normal crossing in codimension one. Let $\Delta$ be an effective $\mathbb{Q}$-divisor on $X$ such that no irreducible component of Supp $\Delta$ is contained in the singular locus of $X$ and $K_X + \Delta$ is $\mathbb{Q}$-Cartier. We say that $(X, \Delta)$ is a sub semi log canonical (sub-slc for short) pair if $(X^v, \Delta_{X^v})$ is sub log canonical, where $\nu: X^v \to X$ is the normalization of $X$ and $K_{X^v} + \Delta_{X^v} = \nu^*(K_X + \Delta)$, that is, $\Delta_{X^v}$ is the sum of the inverse image of $\Delta$ and the conductor of $X$. We say that $(X, \Delta)$ is semi log canonical (slc for short) if $\Delta$ is effective.

An slc center of a sub-slc pair $(X, \Delta)$ is the $\nu$-image of an lc center of $(X^v, \Delta_{X^v})$. An slc stratum of $(X, \Delta)$ means either an slc center of $(X, \Delta)$ or an irreducible component of $X$. For more details of semi log canonical pairs, see [En5] [Ko1] [Ko3].
2.3 (Kodaira dimension). Let $D$ be a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on a normal projective variety $X$. Let $m_0$ be a positive integer such that $m_0D$ is a Cartier divisor. Let
\[ \Phi_{|mm_0D|} : X \dasharrow \mathbb{P}^{\dim |mm_0D|} \]
be the rational map given by the complete linear system $|mm_0D|$ for a positive integer $m$. Note that $\Phi_{|mm_0D|}(X)$ denotes the closure of the image of the rational map $\Phi_{|mm_0D|}$. We put
\[ \kappa(X, D) := \max_m \dim \Phi_{|mm_0D|}(X) \]
if $|mm_0D| \neq \emptyset$ for some $m$ and $\kappa(X, D) := -\infty$ otherwise. We call $\kappa(X, D)$ (\(\kappa(D)\) for short if there is no danger of confusion) the Iitaka dimension of $D$, and $\kappa(X, K_X + \Delta)$ the log Kodaira dimension of $(X, \Delta)$ when $(X, \Delta)$ is a projective lc pair. We simply call it Kodaira dimension for short in this paper.

We say that $D$ is big if $\kappa(X, D) = \dim X$, and that a projective lc pair $(X, \Delta)$ is big if $\kappa(X, K_X + \Delta) = \dim X$. We say that $D$ is nef if $D \cdot C \geq 0$ for every curve $C$ on $X$. For a general $\mathbb{Q}$-divisor $D$, instead of using $b$-divisor (cf. \cite{Fujita7, Definition 2.10}, \cite{C, 2.3.2}), we say that $D$ is nef if there exist a projective birational morphism $f : Y \to X$ from a normal variety $Y$ and a nef $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor $B$ on $Y$ such that $D = f_*B$.

2.4 (Nef dimension). Let $D$ be a nef $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on a normal projective variety $X$. By \cite{Beauville88} Theorem 2.1 (see also \cite{Fujita94, 2.6} and \cite{Aka83, Section 2}), there exists an almost holomorphic, dominant rational map $f : X \dasharrow Y$ with connected fibers, called a reduction map associated to $D$ such that

(i) $D$ is numerically trivial on all compact fibers $F$ of $f$ with $\dim F = \dim X - \dim Y$,  
(ii) if $x \in X$ is a very general point and $C$ is an irreducible curve through $x$ with $\dim f(C) > 0$, then $D \cdot C > 0$.

The map $f$ is unique up to birational equivalence of $Y$. We call $n(X, D) := \dim Y$ the nef dimension of $D$ (\(n(D)\) for short if there is no danger of confusion). It is well known that $n(X, D) \geq 0$ and $\kappa(X, D) \leq n(X, D) \leq \dim X$.

In Section 3, we will prove Theorem \cite{Liu} case by case according to the nef dimension $n(D)$ instead of the Iitaka dimension $\kappa(D)$. The advantage is to avoid proving the non-vanishing theorem for the moduli part directly.

3. On very basic slc-trivial fibrations

Throughout this section, we fix the notation that $f : (X, B_X) \to (Y, D)$ is a very basic slc-trivial fibration with the structure decomposition $D = K_Y + B_Y + M_Y$. Two sheaves are isomorphic in codimension $k$ means they are isomorphic outside a subset of codimension $k + 1$.

First, we show some properties about the discriminant part $B_Y$ and the moduli part $M_Y$. The following proposition follows from \cite{Flamini-Lazarsfeld} Lemma 5.1 and \cite{Liu2} Lemma 5.1 directly.

Proposition 3.1. $B_Y$ is a boundary.

The effectiveness of $B_Y$ follows from condition (6) of Definition \cite{Liu1}. This property is one of the reasons to consider very basic slc-trivial fibrations. Moreover, it is easy to check that in this case, the pair $(Y, B_Y + M_Y)$ is a generalized lc pair in the sense of \cite{Beauville-Zuo} Definition 4.1]. Combining Remark \cite{Liu3} we see that very basic slc-trivial fibrations sit between normal qlc pairs and generalized lc pairs.

Corollary 3.2. If $K_Y + B_Y$ is $\mathbb{Q}$-Cartier, then $(Y, B_Y)$ is log canonical.
Proof. By assumption and [En7 Theorem 5.1], $(Y, B_Y)$ is sub log canonical. Thus it is log canonical by Proposition 3.1.

We immediately get the following proposition about the moduli part by [En7 Theorem 1.2].

Proposition 3.3. $M_Y$ is nef. That is, there exist a birational morphism $p : Y' \to Y$ and a nef $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor $M_{Y'}$ on $Y'$ such that $M_Y = p_*M_{Y'}$.

Remark 3.4. In Kodaira’s canonical bundle formula for elliptic fibrations, the moduli part is semi-ample by [12]. It is natural to wonder the abundance of the moduli part in general cases. In [Ka1 Theorem 2], Kawamata essentially proved that the moduli part of a basic klt-trivial fibration is nef. After Fujino learned [Ka1], he discussed the semi-ampleness of the moduli part for certain algebraic fiber spaces in [Fn2]. Then he further showed that the moduli part of a basic lc-trivial fibration is nef. In [A3], Ambro proved that the moduli part of a basic lc-trivial fibration is b-nef and abundant under some assumptions, and is semi-ample when dim $Y = 1$. Then Fujino–Gongyo generalized these results for basic lc-trivial fibrations in [FG]. Soon after Fujino posted the b-semi-ampleness conjecture for basic slc-trivial fibrations in his recent work [Fn7], Fujino–Fujisawa–Liu [FFL] proved that b-semi-ampleness conjecture holds true in dim $Y = 1$. It is worth to mention that there are effective versions of the abundance of the moduli part. In this direction, see [PS, Fl] and the references therein.

Next, we prepare some general lemmas for very basic slc-trivial fibrations. The first lemma is quite similar to [FL2, Lemma 2.2].

Lemma 3.5. Let $f : (X, B_X) \to (Y, D)$ be a very basic slc-trivial fibration with $K_X + B_X + \frac{1}{b}(\varphi) = f^*D$. Let $g : Y \to Y'$ be a proper surjective morphism between varieties such that $g_*\mathcal{O}_Y \simeq \mathcal{O}_{Y'}$. Assume that $D = g^*D' + E$ holds for some $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor $D'$ on $Y'$, and $rE$ is an effective Cartier divisor for some integer $r \geq 1$ such that $\mathcal{O}_{Y'} \simeq g_*\mathcal{O}_Y(rE)$. Then $g \circ f : (X, B_X - f^*E) \to (Y', D')$ is an induced very basic slc-trivial fibration.

Proof. Since $Y$ is normal and $g_*\mathcal{O}_Y \simeq \mathcal{O}_{Y'}$, $Y'$ is also normal. Consider the induced morphism $g \circ f : X \to Y'$ with

$$K_X + B_X - f^*E + \frac{1}{b}(\varphi) = f^*g^*D'.$$

Let $B'_X = B_X - f^*E$. It is easy to check conditions (1)–(4) of Definition 3.1 one by one. For condition (6), which implies condition (5), we have that $B'_{X}^{< 1} + B_{X}^{= 1} = B_{X}^{< 1} + B_{X}^{= 1} - f^*E$ and $B_{X}^{= 1} \leq B_{X}^{= 1}$. Therefore,

$$\mathcal{O}_{Y'} \hookrightarrow g_*f_*\mathcal{O}_X([--(B_X^{< 1})]) \hookrightarrow g_*f_*\mathcal{O}_X([--(B_X^{< 1}) + f^*E]) \hookrightarrow g_*\mathcal{O}_Y(rE) \simeq \mathcal{O}_{Y'}.$$

That is, $\mathcal{O}_{Y'} \hookrightarrow g_*f_*\mathcal{O}_X([--(B_X^{< 1})])$ is an isomorphism.

Remark 3.6. Note that the morphism $g : Y \to Y'$ above is not necessary birational. In this paper, we only use Lemma 3.5 in the case that either $g$ is birational or $E = 0$.

The second lemma is similar to [L1 Lemma 2.10]. Let $f : (X, B_X) \to (Y, D)$ be a very basic slc-trivial fibration with the structure decomposition $D = K_Y + B_Y + M_Y$. Let $p : Y' \to Y$ be a birational morphism between normal varieties. By [En7 Section 4, Theorem 1.2 and 1.7], there is an induced (not necessary very) basic slc-trivial fibration...
as \( f': (X', B_{X'}) \to (Y', D') \) such that \( D' = p^*D \) and the following diagram commutes:

\[
\begin{array}{ccc}
(X', B_{X'}) & \xrightarrow{q} & (X, B_X) \\
\downarrow{f'} & & \downarrow{f} \\
Y' & \xrightarrow{p} & Y
\end{array}
\]

(3.1)

In particular, there is also a structure decomposition \( D' = K_{Y'} + B_{Y'} + M_{Y'} \). In general, \( B_{Y'} \) is not effective, but is a sub boundary and \( B_{Y'}^{≤0} \) is \( p \)-exceptional by Proposition 3.1 and [Fe7] Theorem 1.7. Let \( P \) be a prime divisor on \( Y' \) and \( t_P = \text{mult}_P B_{Y'} \). Then, \( K_{X'} + B_{X'} + (1 - t_P)f^*P \) is sub-slc over the generic point of \( P \) by [Fe7] (4.5). That is, for any prime divisor \( Q \in \text{Supp}(f^*P) \), there are two rational numbers \( a_Q := \text{mult}_Q B_{X'} \) and \( b_Q := \text{mult}_Q f^*P \) such that

\[ a_Q + (1 - t_P)b_Q \leq 1. \]

Moreover, \( a_Q = 1 \) implies that \( t_P = 1 \). Since the generic point of \( P \) is Cartier, \( b_Q \) is a positive integer for every \( Q \). Therefore,

\[
0 \leq b_Q - 1 \leq -a_Q + t_Pb_Q.
\]

Let \( R = f^*(B_{Y'}^{≤0}) \). Then by [B2], we can check that \( -(B_{X'}^{≤1}) + R \) is effective. Therefore, there is a natural injective morphism

\[ \alpha: \mathcal{O}_{Y'} \hookrightarrow f'_*\mathcal{O}_{X'}(-(B_{X'}^{≤1}) + R). \]

The second lemma shows that \( \alpha \) is an isomorphism. More precisely,

**Lemma 3.7.** Notation as above. Then

\[
\alpha: \mathcal{O}_{Y'} \xrightarrow{\sim} f'_*\mathcal{O}_{X'}(-(B_{X'}^{≤1}) + R)
\]

is an isomorphism. In particular, if \( B_{Y'} \) is effective, then \( f': (X', B_{X'}) \to (Y', D') \) is a very basic slc-trivial fibration.

**Proof.** Since \( Y' \) is normal and \( f'_*\mathcal{O}_{X'}(-(B_{X'}^{≤1}) + R) \) is torsion-free, it suffices to prove that \( \alpha \) is an isomorphism in codimension one. Let \( P \) be a prime divisor on \( Y' \) not contained in \( \text{Exc}(P) \). Then

\[ f'_*\mathcal{O}_{X'}(-(B_{X'}^{≤1}) + R) \cong f'_*\mathcal{O}_{X'}(-(B_{X'}^{≤1})) \]

at the generic point of \( P \). Since \( p: P \to p(P) \) is an isomorphism at the generic points, \( \alpha \) is an isomorphism at the generic point of \( P \) by pulling back the isomorphism

\[ \mathcal{O}_{Y'} \cong f_*\mathcal{O}_{X'}(-(B_{X'}^{≤1})) \]

at the generic point of \( p(P) \). Let \( P \) be a prime divisor in \( \text{Exc}(P) \) and \( t_P = \text{mult}_P B_{Y'} \). Then there exist a prime divisor \( Q \in \text{Supp}(f^*P) \), a rational number \( a := \text{mult}_Q B_{X'} \) and a positive integer \( b := \text{mult}_Q f^*P \) such that \( a + (1 - t_P)b = 1 \). Equivalently,

\[ [-a + t_Pb] = b - 1. \]

That is, \( -(B_{X'}^{≤1}) + R \not\supseteq f^*P \) over the generic point of \( P \). Then

\[ \mathcal{O}_{Y'} \hookrightarrow f'_*\mathcal{O}_{X'}(-(B_{X'}^{≤1}) + R) \subsetneq \mathcal{O}_{Y'}(P) \]

at the generic point of \( P \). It follows that \( \mathcal{O}_{Y'} \cong f'_*\mathcal{O}_{X'}(-(B_{X'}^{≤1}) + R) \) at the generic point of \( P \). Therefore, \( \alpha \) is isomorphic at the generic point of any prime divisor on \( Y' \) and is indeed an isomorphism. In particular, if \( B_{Y'} \) is effective, then \( R = 0 \) and \( \mathcal{O}_{Y'} \cong f'_*\mathcal{O}_{X'}(-(B_{X'}^{≤1})) \). By definition, \( f': (X', B_{X'}) \to (Y', D') \) is a very basic slc-trivial fibration. \( \square \)
There are several useful corollaries of Lemma 3.7. One of them is that, this lemma enable us to modify $Y$ by a higher model $Y'$ (a birational morphism $p: Y' \to Y$) keeping the discriminant part being a boundary. In particular, we have the following corollary.

**Corollary 3.8.** Let $f: (X, B_X) \to (Y, D)$ be a very basic slc-trivial fibration with the structure decomposition $D = K_Y + B_Y + M_Y$. Assume that $K_Y + B_Y$ is $\mathbb{Q}$-Cartier. Let $p: Y' \to Y$ be a $\mathbb{Q}$-factorial dlt blow-up such that $K_{Y'} + B = p^*(K_Y + B_Y)$ and $(Y', B)$ is a dlt pair. Then there is an induced very basic slc-trivial fibration $f': (X', B_{X'}) \to (Y', D')$.

**Proof.** By Lemma 3.7, it suffices to show that $B$ coincides with $B_{Y'}$ in the structure decomposition of $p^*D = K_{Y'} + B_{Y'} + M_{Y'}$. Note that $M_{Y'}$ is a nef $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor and $M_Y = p_*M_{Y'}$ is also a nef $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor by assumption. By negative lemma (cf. [KM] Lemma 3.39), we have that $M_{Y'} + E = p^*M_Y$ for some effective $p$-exceptional $\mathbb{Q}$-divisor $E$ on $Y'$. Then

$$K_{Y'} + B + p^*M_Y = p^*D = K_{Y'} + B_{Y'} + M_{Y'} = K_{Y'} + B_{Y'} - E + p^*M_Y,$$

which implies that $B + E = B_{Y'}$. Since $p$ is a dlt blow-up, $E$ is supported by $B = 1$. It follows that $E = 0$ since $B_{Y'}$ is a sub boundary. In particular, $M_{Y'} = p^*M_Y$ and $B = B_{Y'}$. \qed

In the rest of this section, we use Lemma 3.7 to generalize the connectedness properties in [L2] Section 4.

**Definition 3.9.** Let $f: (X, B_X) \to (Y, D)$ be a basic slc-trivial fibration. Then an lc center $Z$ of $f$ is an image of some slc center of $(X, B_X)$ such that $Z \subseteq Y$.

**Remark 3.10.** As showed in Remark 1.3, after some modifications, a normal qlc pair is a very basic slc-trivial fibration. Then the definition of qlc centers (cf. [En6] Definition 6.2.8) coincides with the lc centers defined above. Note also that a crepant log resolution $f: (Y', B_{Y'}) \to (Y, B_Y)$ of an lc pair $(Y, B_Y)$ is also a very basic lc-trivial fibration. In this case, the definition of lc centers in Section 2.2 coincides with the lc centers of $f$.

Since every stratum of $X$ is dominant onto $Y$, the slc center of $(X, B_X)$ mapping onto an lc center $Z$ of $f$ must be contained in Supp $B_X^{\equiv 1}$. By further blowing ups, we always assume that every maximal slc center of $(X, B_X)$ mapping into $Z$ is a smooth component of $B_X^{\equiv 1}$ and the union of them is a simple normal crossing divisor on $X$.

**Corollary 3.11 (Connectedness).** Assume that there exists a commutative diagram

$$(X', B_{X'}) \xrightarrow{q} (X, B_X)$$

$$(T', B_{T'}) \xrightarrow{q'} (T, B_T)$$

with the same notation as in 3.1. Let $Z$ be a union of lc centers of $f$ and $Z'$ be the union of all lc centers of $f'$ mapping into $Z$. Assume that $Z'$ is a simple normal crossing divisor on $Y'$. We also assume that the union of strata of $B_X^{\equiv 1}$ mapping into $Z$ is a simple normal crossing divisor on $X$, denote it as $T$, and that the union of strata of $B_X^{\equiv 1}$ mapping into $Z'$ is a simple normal crossing divisor on $X'$, denote it as $T'$. Put $K_T + B_T := (K_X + B_X)|_T$ and $K_{T'} + B_{T'} := (K_{X'} + B_{X'})|_{T'}$. Then there exists a commutative diagram

$$(T', B_{T'}) \xrightarrow{q'} (T, B_T)$$

$$(Y', B_{Y'}) \xrightarrow{p} (Y, B_Y)$$

with the same notation as in 3.1. Let $Z$ be a union of lc centers of $f$ and $Z'$ be the union of all lc centers of $f'$ mapping into $Z$. Assume that $Z'$ is a simple normal crossing divisor on $Y'$. We also assume that the union of strata of $B_X^{\equiv 1}$ mapping into $Z$ is a simple normal crossing divisor on $X$, denote it as $T$, and that the union of strata of $B_X^{\equiv 1}$ mapping into $Z'$ is a simple normal crossing divisor on $X'$, denote it as $T'$. Put $K_T + B_T := (K_X + B_X)|_T$ and $K_{T'} + B_{T'} := (K_{X'} + B_{X'})|_{T'}$. Then there exists a commutative diagram

$$Z' \xrightarrow{p} Z$$
such that $\mathcal{O}_Z \hookrightarrow (f|_T)_*\mathcal{O}_T$ is an isomorphism and $\mathcal{O}_{Z'} \hookrightarrow (f'|_{T'})_*\mathcal{O}_{T'}$ is an isomorphism at every generic points of $Z'$.

Proof. The proof is similar to [Fl3] Lemma 4.1 and Corollary 4.2 and [L2] Lemma 4.1. First, we show that $\mathcal{O}_Z \simeq (f|_T)_*\mathcal{O}_T$. Consider the following short exact sequence

$$0 \to \mathcal{O}_X(A - T) \to \mathcal{O}_X(A) \to \mathcal{O}_T(A|_T) \to 0$$

where $A = [-(B_X^{\leq 1})]$ and $A|_T = [-(B_X^{\leq 1})|_T] = [-(B_T^{\leq 1})]$. Then we obtain the long exact sequence

$$0 \to f_*\mathcal{O}_X(A - T) \to f_*\mathcal{O}_X(A) \to f_*\mathcal{O}_T(A|_T) \to \cdots$$

Note that

$$A - T - (K_X + \{B_X\} + B_X^{\leq 1} - T) = -(K_X + B_X) \sim_{Q,f} 0.$$ By [Fm6] Theorem 5.6.3, every nonzero local section of the sheaf $R^1f_*\mathcal{O}_X(A - T)$ contains in its support the $f$-image of some slc stratum of $(X, \{B_X\} + B_X^{\leq 1} - T)$. By the construction of $T$, no slc stratum of $(X, \{B_X\} + B_X^{\leq 1} - T)$ are mapped into $Z$ by $f$. On the other hand, the support of $f_*\mathcal{O}_T(A|_T)$ is contained in $Z = f(T)$. Therefore, the connecting homomorphism $\delta$ is a zero map. Thus we get a short exact sequence

$$0 \to f_*\mathcal{O}_X(A - T) \to \mathcal{O}_Y \to f_*\mathcal{O}_T(A|_T) \to 0.$$ Since $f_*\mathcal{O}_X(A - T)$ is contained in $\mathcal{O}_Y$ and $f(T) = Z$, we have $f_*\mathcal{O}_X(A - T) = \mathcal{I}_Z$, where $\mathcal{I}_Z$ is the defining ideal sheaf of $Z$ on $Y$. Thus, by the above short exact sequence, we obtain that the natural map $\mathcal{O}_Z \to f_*\mathcal{O}_T(A|_T)$ is an isomorphism. It follows that the natural map $\mathcal{O}_Z \to (f|_T)_*\mathcal{O}_T$ is an isomorphism.

Next, we show that $\mathcal{O}_{Z'} \hookrightarrow (f'|_{T'})_*\mathcal{O}_{T'}$ is an isomorphism at every generic points of $Z'$. Consider the following short exact sequence

$$0 \to \mathcal{O}_X'(A' - T') \to \mathcal{O}_X'(A') \to \mathcal{O}_{T'}(A'|_{T'}) \to 0$$

where $A' = [-(B_X^{\leq 1}) + R]$. Then we obtain the long exact sequence

$$0 \to f'_*\mathcal{O}_X'(A' - T') \to f'_*\mathcal{O}_X'(A') \to f'_*\mathcal{O}_{T'}(A'|_{T'}) \to \cdots$$

Note that

$$A' - T' - (K_X + \{B_X^{\leq 1} - R\} + B_X^{\leq 1} - T') = -(K_X + B_X) + R \sim_{Q,f'} f'^* (B_Y^{\leq 0})_{\eta'}.$$ Similarly as above, $\delta'$ is a zero map in codimension one of $Y'$. By Lemma 3.7 and the assumption that $Z'$ is a simple normal crossing divisor, we get a short exact sequence

$$0 \to \mathcal{I}_{Z'} \to \mathcal{O}_{Y'} \to f'_*\mathcal{O}_{T'}(A'|_{T'}) \to 0$$
at every generic points of $Z'$. That is, $\mathcal{O}_{Z'} \hookrightarrow f'_*\mathcal{O}_{T'}(A'|_{T'})$ is an isomorphism outside a codimension one subset of $Z'$. It follows that $\mathcal{O}_{Z'} \hookrightarrow f'_*\mathcal{O}_{T'}$ is an isomorphism at every generic points of $Z'$.

The following corollary is a generalisation of [L2] Corollary 4.2.

**Corollary 3.12.** Notation as in Corollary 3.11. Assume further that $Z''$ is a normal component of $Z'$. Let $T_i$ (resp. $T'_i$) be the union of irreducible components of $T'$ dominant onto (resp. mapping into) $Z''$. Put $K_{T_i} + B_{T_i} = (K_X + B_X)|_{T_i}$ by adjunction for $i = 1, 2$. Then

$$\text{rank}(f'|_{T_i})_*\mathcal{O}_{T_i}([-(B_{T_i}^{\leq 1})]) = 1 \quad \text{and} \quad \mathcal{O}_{Z''} \simeq (f'|_{T_i})_*\mathcal{O}_{T_i}.$$
In particular, \( f'|_{T_1} : (T_1, B_{T_1}) \rightarrow (Z'', D'|_{Z''}) \) is a basic slc-trivial fibration.

**Proof.** Consider the following commutative diagram:

\[
\begin{array}{ccc}
T_1 & \xrightarrow{\iota} & T_2 \\
\downarrow{g} & & \downarrow{f'} \\
\tilde{Z} & \xrightarrow{p} & Z''
\end{array}
\]

where \( \iota : T_1 \rightarrow T_2 \) is the natural closed immersion and

\[
T_1 \xrightarrow{g} \tilde{Z} \xrightarrow{p} Z''
\]

is the Stein factorization of \( f' \circ \iota : T_1 \rightarrow Z'' \). By Corollary 3.11, \( \mathcal{O}_{Z''} \twoheadrightarrow (f'|_{T_2})_* \mathcal{O}_{T_2} \) is an isomorphism at the generic point of \( Z'' \). Therefore, \( \iota : T_1 \rightarrow T_2 \) is an isomorphism over the generic point of \( Z'' \). In particular, \( p \) is birational. By [FL1, Claim 1], \( p \) is the normalization. Since \( Z'' \) is normal, \( p \) is an isomorphism. It follows that \( \mathcal{O}_{Z''} \cong (f'|_{T_1})_* \mathcal{O}_{T_1} \). Note that

\[
(f'|_{T_1})_* \mathcal{O}_{T_1} (\lceil -(B_{T_1}^{\leq 1}) \rceil)
\]

is torsion-free, and isomorphic to \( (f'|_{T_2})_* \mathcal{O}_{T_2} (\lceil -(B_{T_2}^{\leq 1}) \rceil) \) at the generic point of \( Z'' \) by the same proof of [FL1 Claim 2] or [Fn7, Lemma 10.4]. It follows that

\[
\text{rank}(f'|_{T_1})_* \mathcal{O}_{T_1} (\lceil -(B_{T_1}^{\leq 1}) \rceil) = 1.
\]

By Definition 1.1, \( f'|_{T_1} : (T_1, B_{T_1}) \rightarrow (Z'', D'|_{Z''}) \) is a basic slc-trivial fibration. □

**Remark 3.13.** In general, \( (f'|_{T_1})_* \mathcal{O}_{T_1} (\lceil -(B_{T_1}^{\leq 1}) \rceil) \) is not isomorphic to \( \mathcal{O}_{Z''} \). This is because the pushforward of \( R \) on \( Z'' \) provides a gap between these two sheaves. Therefore, \( f'|_{T_1} \) is not necessary a very basic slc-trivial fibration.

**Remark 3.14.** In Corollary 3.12, the natural commutative diagram

\[
\begin{array}{ccc}
\mathcal{O}_{X'} & \xrightarrow{j} & \mathcal{O}_{T_1} \\
\downarrow{f'} & & \downarrow{f'|_{T_1}} \\
\mathcal{O}_{X'} & \xrightarrow{f'_*} & \mathcal{O}_{T_1}
\end{array}
\]

induces a commutative diagram:

\[
\begin{array}{ccc}
\mathcal{O}_{X'} & \xrightarrow{j} & \mathcal{O}_{T_1} \\
\downarrow{f'} & & \downarrow{f'|_{T_1}} \\
\mathcal{O}_{Y'} & \xrightarrow{j} & \mathcal{O}_{Z''}
\end{array}
\]

Assume that a \( \mathbb{Q} \)-line bundle \( \varphi = 0 \) in Pic(\( X' \)) \( \otimes \mathbb{Q} \). Then \( \varphi|_{T_1} = 0 \) in Pic(\( T_1 \)) \( \otimes \mathbb{Q} \). By above commutative diagram, \( f'_*(\varphi) := \frac{1}{r}f'_*(r\varphi) = 0 \) in Pic(\( Y' \)) \( \otimes \mathbb{Q} \) and

\[
f'_*(\varphi)|_{Z''} = (f'|_{T_1})_*(\varphi|_{T_1}) = 0
\]

in Pic(\( Z'' \)) \( \otimes \mathbb{Q} \) for some integer \( r \) such that \( r\varphi \) is a line bundle.
4. The numerically trivial moduli part

Let \( f: (X, B_X) \to (Y, D) \) be a very basic slc-trivial fibration with the structure decomposition \( D = K_Y + B_Y + M_Y \). Assume that \( M_Y \) is \( \mathbb{Q} \)-Cartier, equivalently, \( K_Y + B_Y \) is \( \mathbb{Q} \)-Cartier. By Corollary 3.12, \( (Y, B_Y) \) is log canonical. Let \( Z \) be a union of lc centers of \( f \). Then we have the following lemma, which is a generalisation of [L2, Lemma 5.4 and Corollary 5.6].

**Theorem 4.1.** If \( M_Y|_Z \equiv 0 \), then \( M_Y|_Z \sim_\mathbb{Q} 0 \).

For simplicity, we can assume that \( Z \) is connected. Take a log resolution \( g: Y' \to Y \) such that \( \text{Supp} \, g^{-1}(Z) \) is a simple normal crossing divisor on \( Y' \). Let \( D' = g^*D \) and \( f': (X', B_{X'}) \to (Y', D') \) be the induced basic slc-trivial fibration with the structure decomposition \( D' = K_{Y'} + B_{Y'} + M_{Y'} \). Let \( Z' \) be the union of all lc centers of \( f' \) mapping into \( Z \). Note that \( Z' \) is contained in \( B_{Y'}^{-1} \). By Corollary 3.12, there is an induced basic slc-trivial fibration on each smooth component of \( Z' \). Using connectedness lemma on \( g: Y' \to Y \), \( Z' \) is connected and \( g_*\mathcal{O}_{Z'} = \mathcal{O}_Z \). As in Corollary 3.8, \( M_{Y'} + E = g^*M_Y \) where \( E \) is an effective \( g \)-exceptional \( \mathbb{Q} \)-divisor supported by \( B_{Y'}^{-1} \). By assumption, \( M_{Y'}|_{Z'} + E|_{Z'} \equiv 0 \), where \( E|_{Z'} \) is an effective \( \mathbb{Q} \)-divisor on \( Z' \). Since \( M_{Y'}|_{Z'} \) is nef, it follows that \( E|_{Z'} = 0 \) and \( M_{Y'}|_{Z'} \equiv 0 \). If we can show that \( M_{Y'}|_{Z'} \sim_\mathbb{Q} 0 \), that is, \( g^*\mathcal{O}_Z(rM_Y) \cong \mathcal{O}_Z(rM_{Y'}) \cong \mathcal{O}_{Z'} \) for some positive integer \( r \), then by projection formula,

\[
\mathcal{O}_Z \cong g_*\mathcal{O}_{Z'} \cong g_*g^*\mathcal{O}_Z(rM_Y) \cong g_*\mathcal{O}_{Z} \otimes \mathcal{O}_Z(rM_Y) \cong \mathcal{O}_Z(rM_Y).
\]

That is, \( M_Y|_Z \sim_\mathbb{Q} 0 \). Therefore, in the rest of this section, we will take log resolutions as high as we want by keeping Corollary 3.12 in mind, and prove Theorem 4.1 from two different points of view. Note that these two approaches are not totally separated.

4.1. The viewpoint of trivial fibrations. We can further assume that the morphism \( f: (X, B_X) \to (Y, D) \) satisfies the following conditions (a)–(g), stated also in [Fn7, Proposition 6.1] and [FFL, Section 5]:

(a) \( Y \) is smooth and \( X \) is simple normal crossing,
(b) \( f \) is a projective surjective morphism,
(c) \( \Sigma_X \) and \( \Sigma_Y \) are simple normal crossing divisors on \( X \) and \( Y \) respectively,
(d) \( B_X \) and \( B_Y \), \( M_Y \) are supported by \( \Sigma_X \) and \( \Sigma_Y \) respectively,
(e) every stratum of \( (X, \Sigma_X^b) \) is smooth over \( Y^\ast := Y \setminus \Sigma_Y \),
(f) \( f^{-1}(\Sigma_Y) \subset \Sigma_X \), \( f(\Sigma_X^b) \subset \Sigma_Y \),
(g) \( (B_X^b)^{-1} \) is Cartier.

Let \( Z := B_Y^{-1} = \sum Z_i \), where \( Z_i \) is an irreducible component of \( Z \) for every \( i \). Let \( T_i \) be the union of components of \( B_X^{-1} \) dominant onto \( Z_i \). As stated in Remark 1.2, it is less confusing by using \( \mathbb{Q} \)-line bundle. That is, we have an isomorphism in \( \text{Pic}(X) \otimes \mathbb{Q} \)

\[
\varphi: f^*\mathcal{D} = f^*\mathcal{O}_Y(K_Y + B_Y + M_Y) \to \mathcal{O}_X(K_X + B_X).
\]

Note that we can view \( \varphi \) as a \( \mathbb{Q} \)-line bundle \( = (f^*\mathcal{D})^* \otimes \mathcal{O}_X(K_X + B_X) \) such that \( \varphi = 0 \) in \( \text{Pic}(X) \otimes \mathbb{Q} \) (see Remark 3.14). By restricting sheaves on \( T_i \), we have an induced isomorphism

\[
\varphi_i: f^*\mathcal{D}_i \to \mathcal{O}_{T_i}(K_{T_i} + B_{T_i})
\]

where \( \varphi_i = \varphi|_{T_i} \) and \( \mathcal{D}_i = \mathcal{D}|_{Z_i} \), for every \( i \). By Corollary 3.12, \( f|_{T_i}: (T_i, B_{T_i}) \to (Z_i, \mathcal{D}_i) \) is a basic slc-trivial fibration with the structure decomposition

\[
\mathcal{D}_i = \mathcal{O}_{Z_i}(K_{Z_i} + B_{Z_i} + M_{Z_i})
\]

for every \( i \). Let \( \tau := f_*(\varphi) \) and \( \tau_i := (f|_{T_i})_*(\varphi_i) \) as in Remark 3.14.
Lemma 4.2. Assume that $M_Y|_{Z_i} ≡ 0$. Then $\tau_i \otimes O_{Z_i}(M_{Z_i}) = (\tau \otimes O_Y(M_Y))|_{Z_i}$.

Proof. By Remark 3.14, $(\tau \otimes O_Y(M_Y))|_{Z_i} = \tau_i \otimes O_{Z_i}(M_Y)$. Therefore, it suffices to show that $O_{Z_i}(M_Y) = O_{Z_i}(M_{Z_i})$. By adjunction, we have

$$O_{Z_i}(K_{Z_i} + B_{Z_i} + M_{Z_i}) = O_Y(K_Y + B_Y + M_Y)|_{Z_i} = O_{Z_i}(K_{Z_i} + B_Y - Z_i + M_Y).$$

Let $B_i = (B_Y - Z_i)|_{Z_i}$. Note that $\text{Supp} B_Y$ is simple normal crossing on $Y$ and $\text{Supp} B_i$ is simple normal crossing on $Z_i$. Let $P$ be a prime divisor in $\text{Supp}(B_Y - Z_i)$. Then $P_i := P|_{Z_i}$ is a union of prime divisors. Let $t_P$ be a number such that $(T_i, B_{T_i} + t_P(f|_{T_i})^* P_i)$ is sub-slc over the generic points of $P_i$. Then by inversion of adjunction, $(X, B_X + t_P f^* P)$ is sub-slc at around the generic points of $P_i$. This implies that

$$B_i \leq B_{Z_i}$$

by the definition of the discriminant part. Therefore, there is a natural injective morphism

$$\alpha : O_{Z_i}(M_{Z_i}) \hookrightarrow O_{Z_i}(M_Y).$$

Since $M_Y|_{Z_i} ≡ 0$ and $M_{Z_i}$ is nef, $\alpha$ has to be an identity.

Remark 4.3. Note that the symbol “=” for two sheaves in Lemma 4.2 and the rest of this paper does not only mean equal in the $\mathbb{Q}$-Picard group, but also mean that the defining Cartier divisors in both sides are exactly the same, equivalently, the principle divisor of the difference is given by some constant function.

Remark 4.4. In general, another direction that $B_{Z_i} \leq B_i$ is not necessary true. This is because the discriminant part $B_Y$ is defined by the slc threshold over the generic point of $P$ on $Y$, which is not necessary the slc threshold over $Z_i \cap P$. But in this case, we can take further blowing ups at $Z_i \cap P$ and consider the new relationship between $B_{Z_i}$ and $B_i$. We won’t go into these details in this paper since Lemma 4.2 is sufficient.

By the inheriting nature showed in Lemma 4.2, we give the following definition:

Definition 4.5. Notation as above. $\tau \otimes O_Y(M_Y)$ is called the canonical moduli part of $f$ and denote it as $M_Y^\tau$.

Remark 4.6. Note that the canonical moduli part is defined on a sufficiently high model of $(Y, D)$. Actually, it is more convenient to define it as a b-divisor (cf. [Fn7, Definition 2.10]) by considering $\tau$ as a b-divisor.

By [Fn7, Lemma 7.3, Theorem 8.1], for every $i$, there exist a finite surjective morphism $\pi_i : W_i \rightarrow Z_i$ (unipotent reduction) and an induced pre-basic slc-Trivial fibration $f_i^* : (T_i', B_{T_i'}) \rightarrow (W_i, \pi_i^* D_i)$ such that

$$\pi_i^* \varphi_i : f_i^* \pi_i^* D_i \rightarrow O_{T_i'}(K_{T_i'} + B_{T_i'})$$

is an induced natural isomorphism. Moreover, by [Fn7, Proposition 6.3, Theorem 8.1], the $\mathbb{Q}$-divisor $M_{W_i} := \pi_i^* M_{Z_i}$ is Cartier and $\pi_i^* D_i = O_{W_i}(K_{W_i} + B_{W_i} + M_{W_i})$ is the structure decomposition. Let $N_i^\tau$ be the eigensheaf defined in [Fn7 (6.11)]. By the Claim in the proof of [Fn7, Proposition 6.3], we have that

Lemma 4.7 ([Fn7, Proposition 6.3]). $\pi_i^* M_{Z_i}^\tau = M_{W_i}^\tau = N_i^\tau$.

By assumption and Lemma 4.2, $M_{W_i}^\tau ≡ 0$. Then in this case, we have that

Lemma 4.8. $N_{1}^{r_1} = O_{W_i}$ for some positive integer $r_i$. 


Proof. From the proof of [FFL, Theorem 1.3], we see that $\mathcal{N}_i$ is a canonical extension of a local subsystem of a polarizable variation of $\mathbb{Q}$-Hodge structures (see also Lemma 4.10). By [D, Corollaire (4.2.8) (iii) b)], there is a positive integer $r_i$ such that $\mathcal{N}_i^{\otimes r_i}$ is constant. It follows that $\mathcal{N}_i^{\otimes r_i} = O_{W_i}$. Or in another way, we can cut $W_i$ by general hyperplanes and reduce to the case that $W_i$ is a curve. More precisely, for a general hyperplane $H_i \subset W_i$, 

$$\mathcal{N}_i^{\otimes r_i}|_{H_i} = O_{H_i} \iff \mathcal{N}_i^{\otimes r_i} = O_{W_i}$$

since $H_i$ is general. So it suffices to prove the lemma for $\dim W_i = 1$, which is [L2, Lemma 5.4].

Combining Lemma 4.7 and Lemma 4.8, we immediately get that

Corollary 4.9. There exists a positive integer $r_i$ such that $(M_{Z_i}^{c})^{\otimes r_i} = O_{Z_i}$.

Then by Lemma 4.2 and Corollary 4.9 there exists a positive integer $r$ such that

$$(M_{Y}^{c})^{\otimes r}|_{Z} = O_{Z}$$

for every $i$. Therefore, $(M_{Y}^{c})^{\otimes r}|_{Z} = O_{Z}$ by patching together. This proves Theorem 4.1.

4.2. The viewpoint of period maps. Let $f: (X, B_X) \to (Y, D)$ be a basic slc-trivial fibration satisfying the conditions (a)–(g) in Subsection 4.1 and $\pi: W \to Y$ be the unipotent reduction with the induced pre-basic slc-trivial fibration $h: (V, B_V) \to (W, \pi^* D)$. Let $\Sigma_W := \pi^{-1}(\Sigma_Y)$ and $\Sigma_V := h^{-1}(\Sigma_W)$. Replacing $h$ by a cyclic cover of the generic fiber of $h$ (denote the cyclic cover group as $G$) and a further resolution (see [Fn7, Section 6]), we get that

$$\mathcal{V}_0 := R^{\dim V - \dim W}(h|_{V_0})_* t_0^! Q_{V_0 \setminus (B_{V_0}^i)} = 1$$

underlies a graded polarizable variation of $\mathbb{Q}$-mixed Hodge structure on $W_0$ which is unipotent at around $\Sigma_W$, where $V_0 = V \setminus \Sigma_V$, $W_0 = W \setminus \Sigma_W$ and $B_{V_0} = B_V|_{V_0}$. By [FF, Theorem 7.1 and 7.3],

$$h_* \omega_{V/W}(B_{V}^{1}) \simeq \text{Gr}_F^{0}(\mathcal{V}^*)$$

where $\mathcal{V}$ is the canonical extension of $V_0$. By [Fn7, Proposition 6.3], $h_* \omega_{V/W}(B_{V}^{1})$ has an eigensheaf $\mathcal{N} \simeq O_W(M_{\mathcal{V}})$. Using Definition 4.5 as in Lemma 4.7, we actually have that $\mathcal{N} = M_{\mathcal{V}}^c$. Moreover,

Lemma 4.10. On the open set $W_0$, the canonical moduli part

$$M_{W_0}^{c} = M_{W}|_{W_0} \subset F^0 \text{Gr}_t^W((\mathcal{V}_0)^*)$$

is a direct summand of the lowest piece of the Hodge filtration of $\text{Gr}_t^W((\mathcal{V}_0)^*)$ for some integer $l$.

Proof. The same as Step 4 in the proof of [FFL, Theorem 1.3], it suffices to show that the action of the cyclic cover group $G$ on $\text{Gr}_F^0((\mathcal{V}_0)^*)$ preserves its weight filtration. Note that the action of $G$ on $\text{Gr}_F^0((\mathcal{V}_0)^*)$ does not commute the stalks of different points of $W_0$. That is, it suffices to show that the action of $G$ preserves the weight filtration of $\text{Gr}_F^0((\mathcal{V}_0)^*)|_w$ on each point $w \in W_0$. Since $h$ is log smooth over $W_0$, by [FF, Remark 4.7], we can assume that $w \in W_0 = \Delta^*$ and $W = \Delta$ (cf. [FF, Lemma 4.10 and 4.12]). By [FFL, Lemma 4.6], we get what we want.

Let $\mathcal{H} = \text{Gr}_t^W((\mathcal{V}_0)^*)_{\text{prim}}$ be the polarizable variation of $\mathbb{Q}$-Hodge structure containing $M_{W_0}$. Note then that the Hodge filtration on $\mathcal{H}$ is (cf. [FF, Remark 3.15] and [FFL, Remark 4.7]):

$$0 \subset F^0 \subset F^{-1} \subset \cdots \subset F^{-p} \subset \cdots \subset F^{-n+1} \subset F^{-n} = \mathcal{H}.$$
By the first Riemann bilinear relation of its $\mathbb{Q}$-polarization $Q$, $\mathcal{H}$ is determined by

$$0 \subset F^0 \subset \cdots \subset F^{-m}, \quad m = \left\lceil \frac{n-1}{2} \right\rceil.$$ 

Fix a particular $\mathbb{Q}$-Hodge structure $H_o$ corresponding to some point $o \in W_0$. We drop the subscript of $H_o$ if there is no confusion and denote $H_C$ (resp. $H_{\mathbb{R}}, H_{\mathbb{Z}}$) as the $\mathbb{C}$-structure (resp. $\mathbb{R}$-structure, integral lattice) of $H_o$. Let $G(k, H_C)$ be a Grassmannian that parameterizes $k$-dimensional subspaces of $H_C$. Consider the set $\hat{D}$ of all Hodge filtrations of weight $n$ with a sequence of fixed positive integers $k^i = \dim F^i$ and satisfying the first Riemann bilinear relation. Then there is a projective embedding (cf. [G2, Proposition 8.2])

$$\iota: \hat{D} \to G(k^0, H_C) \times \cdots \times G(k^{-m}, H_C).$$

It follows that $\hat{D}$ is a complete and projective algebraic variety. Let $p_0: G(k^0, H_C) \times \cdots \times G(k^{-m}, H_C) \to G(k^0, H_C)$ be the first projection. We denote

$$\tilde{D}_0 := p_0 \circ \iota(\hat{D})$$

as the image of the first projection of $\hat{D}$.

By Lemma 4.10 the polarization $Q$ induces a bilinear form on $\mathcal{M}_W^c \oplus \bar{\mathcal{M}}_W^c$ such that

\begin{align*}
(4.2) & \quad Q(u, u) = 0 \quad \text{if } u \in \mathcal{M}_w^c \text{ or } \bar{u} \in \mathcal{M}_w^c, \\
(4.3) & \quad Q(Cu, \bar{u}) > 0 \quad \text{if } u \neq 0 \in \mathcal{M}_w^c
\end{align*}

for every point $w \in W_0$, where $\mathcal{M}_w^c = \mathcal{M}_W^c|_w$ and $C$ is the Weil operator defined by $Cu = i^nu$ for $u \in \mathcal{M}_w^c$. Note that (4.2) is corresponding to the first Riemann bilinear relation and (4.3) is corresponding to the second Riemann bilinear relation. Let $\ell_0 := \mathcal{M}_w^c$ be the 1-dimensional subspace of $H_C$. Then a candidate definition of the period domain for the canonical moduli part is the following:

**Definition 4.11.** The period domain of the canonical moduli part (or classifying space of the canonical moduli part) is defined as

$$D^c = \{ \ell \in G(1, H_C) | \ell \subset S \text{ for some } S \in \tilde{D}_0, \ Q(C\ell, \bar{\ell}) > 0 \}$$

and

$$\tilde{D}^c = \{ \ell \in G(1, H_C) | \ell \subset S \text{ for some } S \in \tilde{D}_0 \}$$

as the compact dual of $D^c$.

Obviously, $\tilde{D}^c$ contains in $\{ \ell \in G(1, H_C) | Q(\ell, \ell) = 0 \}$, which is a quadric hypersurface of $G(1, H_C) \cong \mathbb{P}(H_C)$. Let $D_1$ be the set of flags $\ell \subset F^0 \subset H_C$ and $\iota_1: D_1 \to G(1, H_C) \times G(k^0, H_C)$ be the projective embedding. Then $\hat{D}^c = p_1(\iota_1^{-1}p_2^{-1}(\tilde{D}_0))$ where $p_1$ and $p_2$ are the first and second projections of $\iota_1$, which implies that $\hat{D}^c$ is a projective algebraic variety.

The orthogonal group of the bilinear form $Q$ is a linear algebraic group, defined over $\mathbb{Q}$. Let

$$G_C = \{ g \in \text{GL}(H_C) \mid Q(gu, gv) = Q(u, v) \text{ for all } u, v \in H_C \}$$

be the $\mathbb{C}$-rational points and

$$G_\mathbb{R} = \{ g \in \text{GL}(H_\mathbb{R}) \mid Q(gu, gv) = Q(u, v) \text{ for all } u, v \in H_\mathbb{R} \}$$

the $\mathbb{R}$-rational points respectively. The same as the classical discussions under the condition that $\ell \subset S$ for some $S \in \tilde{D}_0$ (cf. [GL1, Theorem 1.26]), $G_C$ (resp. $G_\mathbb{R}$) acts on $\hat{D}^c$ (resp. $D^c$)
transitively. Therefore, $\tilde{D}^c$ is nonsingular and $D^c \subset \tilde{D}^c$ is open in the Hausdorff topology of $\tilde{D}^c$ inheriting the structure of complex manifold. Moreover,

$$\tilde{D}^c \cong G_C/B$$

where $B = \{g \in G_C \mid g\ell_0 = \ell_0\}$

and

$$D^c \cong G_R/(G_R \cap B)$$

where the embedding $D^c \subset \tilde{D}^c$ corresponds to the inclusion $G_R/(G_R \cap B) \subset G_C/B$. Since $Q$ is assumed to take rational values on the lattice $X_Z$,

$$G_Z = \{g \in \text{GL}(H_R) \mid gH_Z = H_Z\}$$

lies in $G_R$ as an arithmetic subgroup. The action of $G_Z$ on $D^c$ is properly discontinuous. Therefore, for any subgroup $\Gamma \subset G_Z$, the quotient of the complex structure of $D^c$ by $\Gamma$ turns $D^c/\Gamma$ into a complex analytic variety. Now we are ready to define the period map induced by the canonical moduli part:

**Definition 4.12.** Induced by above data, the natural holomorphic mapping

$$p^c: W_0 \to D^c/\Gamma$$

is called the period map of the canonical moduli part.

If $h$ is a basic slc-trivial fibration, or more generally, if rank $h_*\omega_{V/W}(B^{-1}_V) = 1$, then $\tilde{D}^c = \tilde{D}_0$ and $p^c = p_0 \circ \Phi$ where $\Phi$ is the usual period map and $p_0$ is the first projection. But even in this case, $\tilde{D}^c = \tilde{D}_0$ is not necessary a Hermitian symmetric domain. For example, $h$ is a family of Calabi–Yau 3-folds. So we don’t have the Baily–Borel–Satake compactification of $D^c/\Gamma$ in general. Moreover, $D^c/\Gamma$ is not necessary algebraic.

On the other hand, in case $\Gamma$ is arithmetic, Kato–Nakayama–Usui [KU] have constructed an analogue of the toroidal compactification in the horizontal directions of $D/\Gamma$. Recently, Bakker–Brunebarbe–Tsimerman [BBT] have proved that $\Phi(W_0)$ is quasi-projective and Green–Griffiths–Laza–Robles [GGLR] tried to strengthen the results of [BBT] by the construction of a canonical projective compactification of $\Phi(W_0)$. These evidents and efforts enlighten us to consider the compactification of $p^c(W_0)$ by patching strata of period maps of the canonical moduli part. Anyway, we post the conjecture below following from [G3] Conjecture 10.5] and [GGLR] Conjecture 1.2.2]:

**Conjecture 4.13.** Notation as above. Let $P_0 = p^c(W_0)$. Then there exists a projective compactification $P$ of $P_0$ such that

(i) $P$ admits a structure of a normal complex analytic variety.

(ii) There exists an analytic extension $p_0^c: W \to P$.

(iii) There exists an integer $r$ such that $(M^c_W)^{\otimes r} = (p_0^c)^*(\mathcal{O}_P^{an}(1))$.

We prove Conjecture 4.13 for a special case.

**Theorem 4.14.** If $M^c_W \equiv 0$, then Conjecture 4.13 holds true.

**Proof.** As in Lemma 4.8 and 4.10, $M^c_{W_0}$ is a local subsystem of $\mathcal{H}$. This is equivalent to say that $P_0 = p^c(W_0)$ is a point of $D^c/\Gamma$ and $(M^c_{W_0})^{\otimes r} = (p_0^c)^*(\mathcal{L}|_P)$ for some integer $r$ where $\mathcal{L}$ is the universal subbundle. Obviously $P_0 = P$ is a bounded symmetric domain and we can trivially extend $p^c$ to $p_0^c: W \to P$. In particular, by taking the canonical extension we have

$$(M^c_W)^{\otimes r} = (p_0^c)^*(\mathcal{O}_P^{an})$$

where $\mathcal{O}_P^{an} = \mathcal{O}_P^{an}(1) = (\mathcal{L}|_P) = \mathbb{C}(P)$. \qed
Theorem 4.14 allows us to patch different $\mathcal{M}_{W_i} \equiv 0$ on $W_i$ together into a union $W' := \cup W_i$ if there is an identity

$$M_{W_i}|_{W_i \cap W_j} = M_{W_j}|_{W_i \cap W_j}$$

(4.4)

for any $i, j$. Note that $W'$ is not necessary equidimensional. Acturally, (4.4) implies an identity of the images of period maps $P_i$ and $P_j$ for any $i, j$. Assume that $W'$ is connected. Then there is only one point $P = P_i = P_j$ for any $i, j$ and the period maps $(p^c_i)_i$ can be patched together giving $p^c_i: W' \to P$ such that $(p^c_i)^*\mathcal{O}^an_{P_i} = (M_{W_i})^@_{P_i}$ for some integer $r_i$ for every $i$. This gives another pointview of (4.4). Then to prove Theorem 4.1 the rest is the same as Subsection 4.1 by descending the global sections of $(p^c_i)^*\mathcal{O}^an_P$ onto $Z$.

**Remark 4.15.** Conjecture 4.13 holds true directly implies that $b$-semi-ampleness conjecture (see [FMT1] Conjecture 1.4]) holds true. We hope that the rich structures given by arithmetic and representation theory on the period domain could shed some lights on the abundance of the moduli part.

**Remark 4.16.** Conjecture 4.13 also holds true if $\check{D}^c = \check{D}_0$ and $D^c$ is in the classical case (cf. [GGLR] Remark 1.1.2]), that is, $D^c$ is a Hermitian symmetric domain and geometrically corresponding to period maps for abelian varieties or K3-type objects (e.g. K3’s, hyper-Kähler manifolds, cubic 4-folds). For example, if $h$ is a family of K3 surfaces, then the projective compactification $P$ contains in the first projection of the Baily–Borel–Satake compactification of $D/\Gamma$ and the semi-ampleness of the canonical moduli part is given by [FMT2, Theorem 1.2].

5. Proof of Theorem 1.8

We discuss various cases according to the nef dimension $n(D)$. Note that $(Y, B_Y)$ is log canonical by Corollary 3.2 and the moduli $\mathbb{Q}$-divisor $M_Y$ is nef and $\mathbb{Q}$-Cartier by Proposition 3.3.

**Case 1.** Assume that $n(D) = 0$. That is, $D = K_Y + B_Y + M_Y \equiv 0$. We run the $(K_Y + B_Y)$-MMP (cf. [FMT1] Theorem 1.1]) which is a sequence of extremal contractions and terminates at a model $(Y^*, B_Y^*)$ such that one of the following holds:

1. $K_Y^* + B_Y^*$ is semi-ample.
2. There is a morphism onto a curve $g: Y^* \to C$ such that $-(K_Y^* + B_Y^*)$ is $g$-ample.
3. $-(K_Y^* + B_Y^*)$ is ample.

Let $\pi: Y \to Y^*$ be the composition of the sequence of extremal contractions.

First, we assume that $K_Y^* + B_Y^*$ is semi-ample. Since $D$ is $\mathbb{Q}$-Cartier and $D \equiv 0$, there is a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor $D^*$ such that $D = \pi^*D^*$ by the cone and contraction theorem (cf. [FMT3] Theorem 3.2)). In particular, $D^* \equiv 0$ on $Y^*$. Let $M_Y^* = D^* - K_Y^* - B_Y^*$. Then it is easy to see that $M_Y^* = \pi_*M_Y$ is nef. It follows that $K_Y^* + B_Y^* \equiv M_Y^* \equiv 0$. The first part $K_Y^* + B_Y^* \equiv 0$ implies that $K_Y^* + B_Y^* \sim_\mathbb{Q} 0$ since it is semi-ample. As in Corollary 3.3, we have $M_Y + E = \pi^*M_Y^* \equiv 0$ for some effective $\pi$-exceptional $\mathbb{Q}$-divisor $E$. It follows that $E = 0$ and $\pi = \text{id}$. Then by [FMT1] Theorem 1.3], $D^* \sim_\mathbb{Q} M_Y^* \sim_\mathbb{Q} 0$.

Next, we assume that there is a morphism onto a curve $g: Y^* \to C$ such that $-(K_Y^* + B_Y^*)$ is $g$-ample. By the cone and contraction theorem again, there is a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor $D_C$ such that $D = \pi^*g^*(D_C)$. By Lemma 3.3, $g \circ \pi \circ f: (X, B_X) \to (C, D_C)$ is an induced very basic slc-trivial fibration such that $D_C \equiv 0$. Let $0 \equiv D_C = K_C + B_C + M_C$.
be the structure decomposition. Then $C$ is either $\mathbb{P}^1$ or a smooth elliptic curve since $B_C$ is effective and $M_C$ is nef. If $C = \mathbb{P}^1$, then it is obvious that $D_C \sim_\mathbb{Q} 0$ and thus $D \sim_\mathbb{Q} 0$. If $C$ is a smooth elliptic curve, then $B_C = 0$ and $D_C \sim M_C \equiv 0$. By [FFL, Theorem 1.3] again, $M_C \sim_\mathbb{Q} 0$ and thus $D \sim_\mathbb{Q} 0$.

Finally, we assume that there is a morphism onto a point $g: Y^* \to P$ such that $-(K_Y + B_Y)$ is ample. Then by the cone and contraction theorem again, $D \sim_\mathbb{Q} 0$.

Anyway, we prove that if $n(D) = 0$, then $D \sim_\mathbb{Q} 0$.

**Case 2.** Assume that $n(D) = 1$. By [BS, Proposition 2.11 and 2.4.4], there exist a morphism $g: Y \to C$ mapping onto a smooth curve $C$ and an ample $\mathbb{Q}$-divisor $A$ on $C$ such that $D \equiv g^* A$. Let $F$ be a general fiber of $g$. Then

$$0 \equiv D|_F = (K_Y + B_Y + M_Y)|_F = K_F + B_F + M_Y|_F$$

where $B_F = B_Y|_F$ is effective and $M_Y|_F$ is nef. It follows that either $\deg K_F < 0$ or $\deg K_F = 0$. In the former case, $F \equiv \mathbb{P}^1$ and thus $D|_F \sim_\mathbb{Q} 0$. In the latter case, $F$ is a smooth elliptic curve, $B_F = 0$ and $M_Y|_F \equiv 0$. Let $T = f^* F$ on $X$. Then $f: (X, B_X + T) \to (Y, D + F)$ with

$$K_X + T + B_X \sim_\mathbb{Q} f^*(K_Y + F + B_Y + M_Y)$$

is an induced very basic slc-trivial fibration since $\text{Supp} B_Y \cap F = \emptyset$. Note that $F$ is an lc center of the induced $f$. By Theorem 4.1, $M_Y|_F \sim_\mathbb{Q} 0$. It follows that $D|_F \sim_\mathbb{Q} 0$. Therefore, we have $\kappa(F, D|_F) = 0$ in both cases. By the canonical bundle formula (cf. [M, Theorem 1.12]), there exists a $D_C$ on $C$ such that

$$D \sim_\mathbb{Q} g^* D_C + E$$

where $E$ is an effective $\mathbb{Q}$-divisor supported by some special fibers of $g$. Note that $D_C$ is ample since $D_C \equiv A$ by pushforward. Then $1 = n(D) \geq \kappa(D) \geq \kappa(g^* D_C) = \kappa(D_C) = 1$. That is, $\kappa(D) = n(D) = 1$. By [FT1, Theorem 4.1], $D$ is semi-ample.

**Case 3.** Assume that $n(D) = 2$. If $D$ is big, then $D$ is semi-ample by a suitable dlt blow-up as in Corollary 3.8 and [L2, Corollary 5.6]. If $M_Y$ is big, then $2D - (K_Y + B_Y) = D + M_Y$ is nef and big, and $D$ is also semi-ample by Corollary 3.8 and [L2, Remark 5.7]. Therefore, we further assume that $D$ and $M_Y$ are not big. That is, $D^2 = M_Y^2 = 0$. We follow the discussions in [A2, Section 6] and [FN4, Section 6] whose results heavily depended on [S1] and [FT1].

Let $\nu: V \to Y$ be a Sakai minimal resolution with respect to $D$ (see [FT1, Definition 3.3] for example). We put $K_V + B_V = \nu^*(K_Y + B_Y)$, $M_V = \nu^* M_Y$ and $D_V = \nu^* D$. Note then that $D_V^2 = M_V^2 = 0$ and $D$ is semi-ample if and only if $D_V$ is semi-ample. Note also that there is an induced very basic slc-trivial fibration on $V$ by Lemma 3.7. Therefore, there is no harm to replace $Y$ by its Sakai minimal resolution $V$. Then the same as the proof of [A2, Theorem 0.3], $K_V + t D$ is nef and $(K_V + t D)^2 \geq 0$ for some $t$. Then either $K_V + t D$ is big or $K_V^2 = K_V \cdot D = D^2 = 0$. In the former case,

$$a D - (K_V + B_V) = (a - 1) D + M_Y = (a - 2) D + K_V + B_V + 2M_Y$$

for any $a > 2 + t$. Then $a D - (K_V + B_V)$ is nef and big since $(a - 1) D + M_Y$ is nef and $K_V + (a - 2) D + B_V + 2M_Y$ is big. Therefore, $D$ is semi-ample by [L2, Remark 5.7] again. In the latter case, we have $B_V \cdot D = M_Y \cdot D = 0$ since $D$ is nef and $\mathbb{Q}$-linearly equivalent to an effective divisor by [A2, Proposition 6.1 (4)]. Since $D + M_Y$ has maximal nef dimension (cf. [A2, Remark 2.4]), $M_Y \cdot K_Y = (D + M_Y) \cdot K_Y \geq 0$ by [A2, Proposition 6.1 (2)]. It
follows that $M_Y \cdot K_Y = M_Y \cdot B_Y = 0$. Using [A2 Proposition 6.1] again, we have that $\kappa(D) = \kappa(M_Y) = 0$.

We further assume that $H^1(Y, \mathcal{O}_Y) = 0$. Then by [S1 Proposition 4] (see also [A2 Theorem 6.2] or the proof of [Ft1, Theorem 6.3]), $Y$ is a degenerate del Pezzo surface and there exist positive rational numbers $a$, $b$ and $c$ such that $b + c > a$ and

$$K_Y \sim -aT, \quad B_Y \sim bT, \quad M_Y \sim cT, \quad D \sim (b + c - a)T$$

where $T$ is indecomposable of canonical type in the sense of Mumford (cf. [Ft1, Lemma 6.3] or [Ft1, Lemma 5.5]). We will derive a contradiction assuming $T \neq 0$. Note that $(Y, B_Y)$ is log canonical. Assume that $\text{Supp}T$ is smooth rational curves with self intersection $-2$. By adding a multiple of $T$ to $B_Y$, we can assume that $(Y, B_Y)$ is lc but not klt. Replacing by a further blowing up if necessary (though not Sakai minimal anymore), we can further assume that some smooth rational curve $C \subset \text{Supp}T$ is an lc center of $(Y, B_Y)$. We run the $(K_Y + B_Y - C + M_Y)$-MMP. Let $C' \subset \text{Supp}(B_Y - C)$ be a smooth rational curve such that $C' \cdot C > 0$. Then $C'$ stays in an extremal ray and $(K_Y + B_Y - C + M_Y) \cdot C' = (D - C) \cdot C' = -C \cdot C' < 0$. Therefore, we get a morphism $\pi_1: Y \to Y_1$ contracting $C'$ with $D_1 = \pi_1^*D, M_{Y_1} = \pi_1^*M_Y, T_1 = \pi_1^*T$ and $C_1 = \pi_1^*C$. Since $D \cdot C' = M_Y \cdot C' = T \cdot C' = 0$, we have that $D = \pi_1^*D_1, M_Y = \pi_1^*M_{Y_1}$ and $T = \pi_1^*T_1$ by the cone and contraction theorem. In particular, $T_1$ is a curve of fibre type in the sense of [S1 Section 2] (not necessary indecomposable of canonical type) and $D_1 \sim_{\mathbb{Q}} M_{Y_1} \sim_{\mathbb{Q}} T_1$. Repeatedly, we finally get a minimal model $(Y^*, B_{Y*})$ with that $D^* = \pi_*D, B^* = \pi_*B_Y, M^* = \pi_*M_Y, T^* = \pi_*T$ and $C^* = \pi_*C$, where $\pi: Y \to Y^*$ is the composition of extremal contractions. Note that $(T^*)^2 = 0$ and $T^*$ is irreducible. Hence

$$D^* \sim_{\mathbb{Q}} M^* \sim_{\mathbb{Q}} T^* \sim_{\mathbb{Q}} C^* = B^*.$$ 

By Lemma [3.5] there is an induced very basic slc-trivial fibration on $(Y^*, D^*)$ and $C^*$ is an lc center. By $M^* \cdot C^* = 0$ and Theorem [4.1] $C^*|_{C^*}$ and $T^*|_{T^*}$ are torsions. It is a contradiction since $T^*|_{T^*}$ is not a torsion by [S1 Proposition 5].

Finally we assume that $H^1(Y, \mathcal{O}_Y) \neq 0$. Then $Y$ is an irrational ruled surface over an elliptic curve $E$, of type $II_0$ or $II_1^*$ in Sakai’s classification table [S2] (see also [A2 Theorem 6.2 (ii)]). Let $\alpha: Y \to E$ be the Albanese fibration. Note then that $D = d_0E_0 + d_1E_1$ where $E_0$ and $E_1$ are disjoint sections of $\alpha$ in type $II_0$ and $D = d_0E_0$ in type $II_1^*$. In both cases, $B_Y$ is supported by $E_0$ and $E_1$. As above, we add a multiple of $D$ and assume that $(Y, B_Y)$ is log canonical and $C$ is an lc center of $(Y, B_Y)$ supported by $E_0$ or $E_1$. By Lemma [3.5] and Theorem [4.1] again, there is an induced very basic slc-trivial fibration on $C$ and $C|_C$ is a torsion. But the same as [Ft1 (5.9)], $C|_C$ is not a torsion. Again, we get a contradiction.

6. Minimal model program for very basic slc-trivial fibrations

At the beginning of this section, we should note that the proof of [FL3, Section 7] (see also [L2 Section 6]) works for any plt pairs with $\kappa(K_X + \Delta) < \dim X$ once we assume that Conjecture [L7] holds true in lower dimensions (playing the same role as [FL3 Corollary 5.4] in the proof of [FL3 Theorem 1.2]) and the termination of the minimal model program for very basic slc-trivial fibrations (playing the same role as [FL3, Section 6] in the proof of [FL3 Theorem 1.2]). Since we have Theorem [1.8], we can prove Theorem [1.3] immediately once we establish the termination of the minimal model program for very basic slc-trivial fibrations in dimension three. Therefore, in the rest of this section, we try to explain the minimal model program for very basic slc-trivial fibrations in some general settings.
**Definition 6.1.** Let $f: (X, B_X) \to (Y, D)$ be a very basic slc-trivial fibration with the structure decomposition $D = K_Y + B_Y + M_Y$. The commutative diagram

$$
\begin{array}{c}
Y \xrightarrow{p} Y^+ \\
\downarrow g \quad \quad \quad \quad \quad \downarrow g^+ \\
Z \xrightarrow{f} Y^+
\end{array}
$$

is called a **flip with respect to $D$** or **D-flip** for short, if

1. $D^+$ is $\mathbb{Q}$-Cartier, where $D^+$ is the strict transform of $D$ on $Y^+$;
2. $\text{Ex}(g)$ has codimension at least two in $Y$ and $-D$ is $g$-ample;
3. $\text{Ex}(g^+)$ has codimension at least two in $Y^+$ and $D^+$ is $g^+$-ample.

We show that $D$-flips perform in the framework of very basic slc-trivial fibrations.

**Lemma 6.2.** Let $f: (X, B_X) \to (Y, D)$ be a very basic slc-trivial fibration with the structure decomposition $D = K_Y + B_Y + M_Y$. Let

$$
\begin{array}{c}
Y' \xrightarrow{p} Y^+ \\
\downarrow \alpha \quad \quad \quad \quad \quad \downarrow \beta \\
Y \xrightarrow{g} Z \xrightarrow{g^+} Y^+
\end{array}
$$

be a $D$-flip. Then there is an induced very basic slc-trivial fibration $f^+: (X^+, B_{X^+}) \to (Y^+, D^+)$.

**Proof.** Take a common resolution of $p$ such that

$$
\begin{array}{c}
Y' \xrightarrow{p} Y^+ \\
\downarrow \alpha \quad \quad \quad \quad \quad \downarrow \beta \\
Y \xrightarrow{g} Z \xrightarrow{g^+} Y^+
\end{array}
$$

commutes over $Z$. Let $K_{Y'} = \alpha^* D + E_1$ and $K_{Y'} = \beta^* D^+ + E_2$. Let $h = g \circ \alpha = g^+ \circ \beta$. Then $E_1 - E_2$ is $h$-nef and exceptional since $g_*(D) = g^+(D^+)$. By negative lemma (cf. [KM, Lemma 3.39]), $E_2 - E_1$ is effective. Let $E_1 + \alpha^* M_Y = E^+ - E^-$ where $E^+$ and $E^-$ are both effective. Then $K_{Y'} = \alpha^* D + E_1 = \alpha^*(K_Y + B_Y) + E^+ - E^-$. By [En7, Theorem 1.7], $E^-$ is a boundary. Therefore, we can run the relative minimal model program with respect to $K_{Y'} + E^- + \alpha^* M_Y \sim_{\mathbb{Q}, \alpha} E^+$ over $Y$ and contract the divisor $E^+$ exactly. Replacing $Y'$ by the relative minimal model, we can assume that $B_{Y'}$ (the pushforward of $E^-$) is effective. Note that

$$
K_{Y'} + E^- + \alpha^* M_Y \sim_{\mathbb{Q}, \beta} E^+ + (E_2 - E_1)
$$

over $Y^+$. Therefore, by viewing the contraction of $E^+$ as steps of the relative minimal model program over $Y^+$, we can see that $\beta$ is still a morphism since $E_2 - E_1$ is effective. Then there is an induced very basic slc-trivial fibration on $Y'$ with $D' = \alpha^* D = \beta^* D^+ + (E_2 - E_1)$ by Lemma 3.7 and an induced very basic slc-trivial fibration on $Y^+$ with $D^+$ by Lemma 3.5.

In practice, the $D$-flips can be controlled by flips of log canonical pairs. Assume that $Y$ is $\mathbb{Q}$-factorial and $R$ is a $D$-negative extremal ray determining a small contraction. Then $D \cdot R = (K_Y + B_Y + M_Y) \cdot R < 0$. Since $M_Y$ is nef, we have that $(K_Y + B_Y) \cdot R < 0$. Take a $(K_Y + B_Y)$-flip (cf. [KM, Definition 3.33]) and get a small contraction $g$ such that $(K_{Y^+} + B_{Y^+})$ is $g^+$-ample. Then the strict transform $D^+ = K_{Y^+} + B_{Y^+} + M_{Y^+}$ is also $g^+$-ample since the strict transform $M_{Y^+}$ ($M_{Y^+} := \beta_*(\alpha^* M_Y)$ as in above proof) is also $g^+$-ample.
nef. Since flips of log canonical pairs terminate in dimension three, we immediately have the following lemmas by above discussions:

**Lemma 6.3.** \(D\)-flips terminate in dimension three.

**Lemma 6.4.** We can run the \(D\)-minimal model program beginning from a very basic slc-trivial fibration \(f: (X, B_X) \to (Y, D)\) where \(Y\) is a \(\mathbb{Q}\)-factorial threefold and \(D\) is big, ending up with a very basic slc-trivial fibration \(f^*: (X^*, B_{X^*}) \to (Y^*, D^*)\) such that \(D^*\) is nef and big. Furthermore, if \((Y, B_Y)\) is plt (resp. dlt), then so is \((Y^*, B_{Y^*})\).

Now we are ready to prove Theorem 1.4:

**Sketch of proof.** We reduce to prove that the ring \(R(Y, D)\) is finitely generated for a very basic lc-trivial fibration \(f: (X, B_X) \to (Y, D)\) where \(Y\) is a \(\mathbb{Q}\)-factorial threefold, \((Y, B_Y)\) is plt and \(D\) is big. By Lemma 6.4, we can further assume that \(D\) is nef. Then it suffices to prove that \(D\) is semi-ample. By Kawamata–Shokurov basepoint-free theorem (cf. [FL3, Lemma 4.3]), it suffices to prove that \(D|_{B_Y^1}\) is semi-ample where \(B_Y^1\) is normal. By connectedness lemma (see [FL3, Corollary 4.2] or Corollary 3.12), there is an induced very basic klt-trivial fibration on \(B_Y^1\). By Theorem 1.8, \(D|_{B_Y^1}\) is semi-ample. This is what we want. \(\square\)

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