ON THE $\sigma_2$-CURVATURE AND VOLUME OF COMPACT MANIFOLDS

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Abstract. In this work we are interested in studying deformations of the $\sigma_2$-curvature and the volume. For closed manifolds, we relate critical points of the total $\sigma_2$-curvature functional to the $\sigma_2$-Einstein metrics and, as a consequence of results of M. J. Gursky and J. A. Viaclovsky [24] and Z. Hu and H. Li [26], we obtain a sufficient and necessary condition for a critical metric to be Einstein. Moreover, we show a volume comparison result for Einstein manifolds with respect to $\sigma_2$-curvature which shows that the volume can be controlled by the $\sigma_2$-curvature under certain conditions. Next, for compact manifold with nonempty boundary, we study variational properties of the volume functional restricted to the space of metrics with constant $\sigma_2$-curvature and with fixed induced metric on the boundary. We characterize the critical points to this functional as the solutions of an equation and show that in space forms they are geodesic balls. Studying second order properties of the volume functional we show that there is a variation for which geodesic balls are indeed local minima in a natural direction.

1. Introduction

Given a compact smooth Riemannian manifold $(M^n, g)$ of dimension $n \geq 3$, there exists an orthogonal decomposition of the curvature tensor $Rm_g$ which is given by $Rm_g = W_g \odot g$, where $\odot$ is the Kulkarni-Nomizu product, $W_g$ is the Weyl tensor and $A_g$ is the Schouten tensor defined as

$$A_g = \frac{1}{n-2} \left(Ric_g - \frac{R_g}{2(n-1)}g\right).$$

(1.1)

Here $Ric_g$ and $R_g$ are the Ricci and scalar curvature of the metric $g$, respectively, (e.g., see [6,16]). For $k \in \{1, \ldots, n\}$, the $\sigma_k$-curvature is defined as

$$\sigma_k(g) := \sum_{1 \leq i_1 < \cdots < i_k \leq n} \lambda_{i_1} \cdots \lambda_{i_k},$$

where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of $A_g$; that is, $\sigma_k(g)$ is the $k$-th elementary symmetric function of the eigenvalues of $A_g$.

The $\sigma_k$ equation is always elliptic for $k = 1$. The $\sigma_1$-curvature is the scalar curvature, up to a dimensional constant, and it was extensively studied along the years. For $k \geq 2$ the picture is quite different. In fact, to assure the ellipticity of the equation we need some additional assumption. For example, a sufficient condition for this is that $g$ is positive or negative $k$-admissible. By definition, a metric $g$ on $M$ is said to be positive $k$-admissible if it belongs to the $k$-th cone $\Gamma^+_k$; this means that a metric $g$ belongs to $\Gamma^+_k$ if and only if $\sigma_i(g) > 0$ for all $i = 1, \ldots, k$. The negative cone $\Gamma^-_k$ is defined similarly. See [39, Section 6] and [40, Section 3].
In the last few decades much attention has been played to the study of the \( \sigma_2 \)-curvature, see for example [14, 24, 25, 36, 37, 39] and the references therein.

We are interested to study conformal and non conformal deformations of the \( \sigma_2 \)-curvature. We regard the \( \sigma_2 \)-curvature as a nonlinear map on the space \( \mathcal{M} \) of all Riemannian metrics on \( M \),

\[
\sigma_2 : \mathcal{M} \rightarrow C^\infty(M); \quad g \mapsto \sigma_2(g).
\]

See (2.1) for an explicit expression of the \( \sigma_2 \)-curvature in terms of the Ricci and scalar curvature.

In order to understand the behavior of this map we consider its linearization at a metric \( g \) given by \( \Lambda_g : S_2(M) \rightarrow C^\infty(M) \) and also its \( L^2 \)-formal adjoint, denoted by \( \Lambda_g^* : C^\infty(M) \rightarrow S_2(M) \). Here \( S_2(M) \) is the space of symmetric 2-tensors on \( M \). It was proved in [37] that \( \Lambda_g^*(1) \) give us a 2-tensor canonically associated with the \( \sigma_2 \)-curvature (see Section 2). In this way we say that a metric \( g \) is \( \sigma_2 \)-Einstein if \( \Lambda_g^*(1) = \kappa g \) for some constant \( \kappa \).

J. A. Viaclovsky [39] has considered the following functional, defined in \( \mathcal{M} \),

\[
\mathcal{F}_k(g) = \int_M \sigma_k(g)dv_g. \tag{1.2}
\]

He has proved that for \( k \neq n/2 \) and for \( (M, g) \) locally conformally flat in case \( k \geq 3 \), then the Euler-Lagrange equation of \( \mathcal{F}_k \) restricted to a conformal class of a metric \( g \) is \( \sigma_k(g) = \text{constant} \). Later S. Brendle and J. A. Viaclovsky [8] extended this result for \( k = n/2 \). Also, it is well known that critical metrics of \( \mathcal{F}_1 \) restricted to the space of unit volume metrics are exactly Einstein metrics [6, Theorem 4.21]. For \( k > 1 \), the problem is more intriguing since the Euler-Lagrange equations are fourth order equations in the metric. In fact, T. P. Branson and A. R. Gover [7] have studied the functional \( \mathcal{F}_k \) in a conformal class. While the equation \( \sigma_1(g) = \text{constant} \) is always variational, they have shown that for \( k \in \{3, \ldots, n\} \) the equation \( \sigma_k(g) = \text{constant} \) is variational if and only if the manifold is locally conformally flat. However, in dimension three it holds an interesting result proved by M. J. Gursky and J. A. Viaclovsky [24].

**Theorem 1.1** (M. J. Gursky, J. A. Viaclovsky). Let \( (M^3, g) \) be a closed manifold of dimension three. Then a metric \( g \) with \( \mathcal{F}_2(g) \geq 0 \) is critical for \( \mathcal{F}_2 \) restricted to the space of unit volume metrics if and only if \( g \) is Einstein and therefore has constant sectional curvature.

They observed that the condition \( \mathcal{F}_2(g) \geq 0 \) is necessary, since we can write the \( \sigma_2 \)-curvature as \( \sigma_2(g) = -\frac{1}{2} |\text{Ric}_g|^2 + \frac{1}{48} R_g^2 \), where \( \text{Ric}_g = \text{Ric} - \frac{1}{4} R_g g \) denotes the trace free Ricci tensor. Also, they have proved that there exists a Berger metric on \( S^3 \) which is a critical metric with \( \mathcal{F}_2(g) < 0 \). Later, Z. Hu and H. Li [26] generalized this result for \( n > 4 \), in the case that the metric is locally conformally flat. G. Catino, P. Mastrolia and D. D. Monticelli [13] have extended this result to the noncompact setting. They obtained that a complete critical metric for \( \mathcal{F}_2 \) with non-negative scalar curvature is flat. The non-negativity condition on the scalar curvature cannot be removed.

We remark that Theorem 1.1 cannot hold in four-dimensional manifolds on account of the specificity of this dimension. Indeed, in this dimension we have the Gauss-Bonnet-Chern Formula

\[
\int_M \left( \sigma_2(g) + \frac{|W_g|^2}{4} \right) dv_g = 8\pi^2 \chi(M), \tag{1.3}
\]

where \( \chi(M) \) denotes the Euler characteristic of \( M \). This formula serves as a “bridge” between the topological and geometric information, which implies that under conformal change of metrics, since \( |W_g|^2 dv_g \) is point-wisely conformally invariant, the integral \( \int_M \sigma_2(g) dv_g \) is conformally invariant.

As a consequence of the results by M. J. Gursky and J. A. Viaclovsky [24] and Z. Hu and H. Li [26] we obtain our first result, which reads as follows.
Theorem A. Let \((M^n, g)\) be a closed Riemannian manifold of dimension \(n \geq 3\), \(n \neq 4\), which is locally conformally flat for \(n \geq 5\). Then \((M^n, g)\) is a \(\sigma_2\)-Einstein manifold with \(\sigma_2(g) \geq 0\) if and only if \((M^n, g)\) is an Einstein manifold.

We will see that \(\sigma_2\)-Einstein metrics are critical metrics of the volume functional, justifying our interest. Moreover, we present applications of Theorem A in the second variation formula of Viaclovsky’s functional \((1.2)\) for \(k = 2\). To this end, we compute the formula for the second derivative of the \(\sigma_2\)-curvature (Proposition 3.4). It should be mentioned that this formula is of independent interest and we hope that can be useful in other contexts.

We recall that a closed Einstein manifold \((M^n, g)\) with dimension \(n \geq 3\) is said to be stable if the Einstein operator

\[
\Delta^g_E := \nabla^* \nabla - 2 \tilde{R} : \Gamma (S^2 M) \to \Gamma (S^2 M)
\]

restricted to \(S^{TT}_2 (M) := \{ h \in S_2 (M) : \delta_g h = 0, tr_g h = 0 \} \) is nonnegative, i.e., if there exists \(\lambda \geq 0\) such that \(\langle \Delta^g_E h, h \rangle \geq \lambda \| h \|^2\), for any \(h \in S^{TT}_2 (M)\). If \(\lambda > 0\), then the metric is said strictly stable, see [6, Definition 4.63]. Otherwise, \(g\) is said unstable. Here \(\tilde{R}(h)_{ij} = g^{kl} g^{st} R_{kij} h_{lt}\) for any \(h \in S_2 (M)\). Stability of Einstein manifolds plays a fundamental role in the study of Einstein manifolds, see for instance [6] and references therein. The space \(S^{TT}_2 (M)\) is often called the space of transverse-traceless tensors, or TT tensors for short.

We present a geometric characterization of \(\sigma_2\)-curvature showing a volume comparison theorem for metrics close to strictly stable Einstein metrics. It should be mentioned that a similar question was first addressed in the context of scalar curvature and \(Q\)-curvature by W. Yuan [42] and Y. Lin and W. Yuan in [31], respectively.

Theorem B. For \(n \geq 3\), suppose \((M^n, g_0)\) is an \(n\)-dimensional closed strictly stable Einstein manifold with Ricci curvature \(Ric_{g_0} = (n - 1) \lambda g_0\), where \(\lambda > 0\) is a constant. Then there exists a constant \(\varepsilon_0 > 0\) such that for any metric \(g\) on \(M\) satisfying

\[
\sigma_2(g) \geq \sigma_2(g_0) \quad \text{and} \quad \| g - g_0 \|_{C^2} < \varepsilon_0,
\]

we have the following volume comparison

\[
V(g) \leq V(g_0),
\]

with the equality holding if and only if \(g\) is isometric to \(g_0\).

Since the round sphere is strictly stable (see for instance [28, Section 3.1]), we obtain the following immediate result.

Corollary 1.2. Let \((S^n, g_0)\) be the round sphere, with \(n \geq 3\). Then there exists a constant \(\varepsilon_0 > 0\) such that for any metric \(g\) on \(S^n\) satisfying

\[
\sigma_2(g) \geq \frac{n(n - 1)}{8} \quad \text{and} \quad \| g - g_0 \|_{C^2} < \varepsilon_0,
\]

we have

\[
V(S^n, g) \leq V(S^n, g_0),
\]

with the equality holding if and only if \(g\) is isometric to \(g_0\).

After we finished this paper, we learned that Y. Fang, Y. He and J. Zhong [21] proved independently the same comparison theorem. They found examples showing that the strictly stable assumption cannot be dropped in Theorem B. Also they studied the Ricci-flat case.

Motivated by the characterization of Einstein metrics as critical points of the volume functional restricted to the space of metrics with constant scalar curvature \(-1\) on a closed manifold, P. Miao and L.-F. Tam [32] have shown necessary and sufficient conditions to a metric be a critical point
of the volume functional restricted to the space of all Riemannian metric with constant scalar curvature and, if the manifold possesses a boundary, the induced metric on the boundary is fixed. The condition is related with the \( L^2 \)-formal adjoint of the linearization of the scalar curvature. In fact, in the literature these critical points are the so-called V-static metrics. Such a concept was introduced by P. Miao and L.-F. Tam [32] and are useful as an attempt to better understand the interplay between scalar curvature and volume. Later, in [33], they classified all Einstein or conformally flat metrics which are critical points for the volume functional restricted to the above space. See also the works [2, 4, 19, 32, 33] and references therein.

Before we state our next result, let us define the following self-adjoint operator in the \( L^2 \)-norm \( T_g : C_0^\infty(M) \to C_0^\infty(M) \), which is related with the conformal change of the \( \sigma_2 \)-curvature (2.6), as

\[
T_g(u) := \frac{1}{2} \langle T_1, \nabla^2 u \rangle + 2u\sigma_2(g) = \frac{1}{2} \text{div} (T_1(\nabla u)) + 2u\sigma_2(g),
\]

where \( C_0^\infty(M) \) is the space of smooth functions defined in \( M \) which are equal to zero on the boundary, \( T_1 \) is the first Newton transformation associated with \( A_g \) and it is given by

\[
T_1 = \frac{1}{n-2} \left( \frac{1}{2} R_{g} - \text{Ric}_g \right).
\]

We notice that by the contracted second Bianchi identity, \( T_1 \) is divergence free. It is well known that if \( g \) is a positive 2-admissible metric, i.e., it belongs to the positive elliptic convex 2-cone, \( \Gamma_2^+ := \{ g \in \mathcal{M}; \sigma_1(g) > 0 \text{ and } \sigma_2(g) > 0 \} \), then \( T_1 \) is positive definite (See [9, 23] and [25, Proposition 2.1]). This implies that the operator \( T_g \) is elliptic. Moreover, if \( g \in \Gamma_2^+ \), then \( T_1 \geq 0 \).

Let \( \mathcal{M}^K \) be the space of metrics with constant \( \sigma_2 \)-curvature \( K \neq 0 \). The critical points of the volume functional restricted to \( \mathcal{M}^K \) are precisely stationary points of \( F_2 \) (see (1.2)) restricted to \( \mathcal{M}^K \). It was proved in [10, Theorem 6.2], under some assumptions, that a metric \( g \) is critical point of the volume functional restricted to the space of metrics with constant \( \sigma_2 \)-curvature if and only if there is a function \( f \in C^\infty(M) \) such that \( \Lambda_\gamma^g(f) = g \). More generally, J. S. Case, Y.-J. Lin and W. Yuan [10] have defined a conformally variational invariant (CVI) as a conformally Riemannian scalar invariant which is homogeneous and has a conformal primitive. One recall that CVIs generalizes concepts as scalar curvature, Branson’s Q-curvature and \( \sigma_2 \)-curvature. They extended to the CVI context, for closed manifold, some well known results for scalar curvature (see also [11]).

Inspired by the above discussion and the V-static works, we also establish a similar result for the \( \sigma_2 \)-curvature on compact manifolds with boundary whose proof contrasts sharply with the closed case due to the nature of the issue. We investigate critical points of the volume functional on the space of metrics which have constant \( \sigma_2 \)-curvature \( K \) and fixed metric \( \gamma \) on the boundary, i.e,

\[
\mathcal{M}^K_\gamma = \{ g \in \mathcal{M}; \sigma_2(g) = K \text{ and } g|_{\partial M} = \gamma \}.
\]

In addition, we show that these critical points give a class of manifolds with very nice properties.

**Theorem C.** Let \( (M^n, g) \) be a compact Riemannian manifold of dimension \( n \geq 3 \) with nonempty boundary. Let \( g \) be a 2-admissible metric such that the first Dirichlet eigenvalue of \( -\Delta_g \) is positive. Then, \( g \) is a critical point of the volume functional \( V : \mathcal{M}^K_\gamma \to \mathbb{R} \) if and only if there exists a smooth function \( f \) on \( M \) such that \( f = 0 \) on \( \partial M \) and

\[
\Lambda_\gamma^g(f) = g \text{ in } M.
\]

We can see Theorem C as a counterpart to the manifold with boundary, at least for the \( \sigma_2 \)-curvature. In fact, as expected in [10, Remark 6.3], when \( f|_{\partial M} \) is not necessarily zero, we obtain the Proposition 5.3, which gives us an interesting relation involving the boundary curvature associated to the \( \sigma_2 \)-curvature defined in [15].
Metrics satisfying equation (1.7) are special, because they have many interesting properties. For instance, they have constant \( \sigma_2 \)-curvature and critical metrics in space forms are geodesic balls (see Theorem 6.4). More generally, in Einstein manifolds (we will see in Section 6) the equation (1.7) is closely related with \( V \)-static metric (or Miao-Tam critical metric as first denoted in [3]). For more results see [2, 3, 5, 33, 42].

In this paper we also study second order properties for the volume functional on \( \mathcal{M}_{\gamma}^K \) and we show that there is a variation for which geodesic balls are indeed local minima for the volume functional in a natural direction. We also observe that it is not difficult modify our proof for a manifold without boundary. We point out that such result is new even in the closed case.

**Theorem D.** Let \( \Omega \) be a geodesic ball with compact closure in \( \mathbb{S}^n_+ \) and \( \mathbb{H}^n \), and let \( g \) be the standard metric on \( \Omega \). Let \( \gamma = g|_{T\partial \Omega} \) and \( K \) be the constant equal the \( \sigma_2 \)-curvature of \( g \).

(a) If \( \Omega \subset \mathbb{S}^n_+ \), there exists a smooth path \( \{g(t)\} \) in \( \mathcal{M}_{\gamma}^K \) such that \( V'(0) = 0 \) and \( V''(0) > 0 \).

(b) If \( \Omega \subset \mathbb{H}^n \), given \( p \in M \) there exists a geodesic ball centered at \( p \) with radius \( \delta > 0 \), which depends on \( p \) and \( (\Omega, g) \), and a smooth path \( \{g(t)\} \) in \( \mathcal{M}_{\gamma}^K \) such that \( V'(0) = 0 \) and \( V''(0) > 0 \).

Here \( V(t) \) is the volume of \( (\Omega, g(t)) \). In particular, for the above models the volume of the standard metric \( g \) are strict local minimum along those variations.

The organization of the paper is as follows. In Section 2, we give some preliminaries and an overview about deformations of \( \sigma_2 \)-curvature. In Section 3, we study critical metrics in closed manifolds and prove Theorem A. Moreover, we also study second order properties of the \( \sigma_2 \)-curvature in order to compute the second variation of the functional (1.2) for \( k = 2 \). In Section 4, we find some variational formulae in order to prove Theorem B. We study variational properties of the volume functional, constraint to the space of metrics of constant \( \sigma_2 \)-curvature with a prescribed boundary metric in Section 5, moreover we prove Theorem C. In Section 6, we give some examples of functions satisfying (1.7), we also show that the only domains in the space forms \( \mathbb{H}^n \) or \( \mathbb{S}^n \), on which the canonical metrics are critical points, are geodesic balls. In Section 7, we compute the second variation of the volume functional at critical points in \( \mathcal{M}_{\gamma}^K \). Then we calculate the second variation formula for the volume functional in order to prove Theorem D.

## 2. Background

Let \( (M^n, g) \) be a Riemannian manifold of dimension \( n \geq 3 \), with or without boundary. The \( \sigma_k \)-curvature, which we denote by \( \sigma_k(g) \), is defined as the second elementary symmetric function of the eigenvalue of the Schouten tensor, see (1.1) for its definition. Since \( \sigma_1(g) = \text{tr}_g A_g \) and \( \sigma_2(g) = \frac{1}{2}((\text{tr}_g A_g)^2 - |A_g|^2) \) we note that \( \sigma_1(g) = R_g/(2(n-1)) \) and

\[
\sigma_2(g) = \frac{1}{2(n-2)^2} \left( \frac{n}{4(n-1)} R_g^2 - |\text{Ric}_g|^2 \right) = \frac{1}{2(n-2)^2} \left( \frac{(n-2)^2}{4n(n-1)} R_g^2 - |\tilde{\text{Ric}}_g|^2 \right),
\]

where \( \tilde{\text{Ric}}_g \) is the trace free Ricci tensor. We consider the \( \sigma_2 \)-curvature as a map \( \sigma_2 : \mathcal{M} \rightarrow C^\infty(M) \), where \( \mathcal{M} \) is the space of all Riemannian metrics on \( M \) and \( C^\infty(M) \) is the space of all smooth functions on \( M \). It was proved in [37] that the linearization of the \( \sigma_2 \)-curvature map is the map \( \Lambda_g : S_2(M) \rightarrow C^\infty(M) \) given by

\[
c(n)\Lambda_g(h) = \langle \text{Ric}_g, -\Delta_g^h h + \nabla^2 \text{tr}_g h + 2\delta^g \delta h \rangle - \frac{n}{2(n-1)} R_g \left( \Delta_g \text{tr}_g h - \delta^g h + \langle \text{Ric}, h \rangle \right), \quad (2.2)
\]
where \( c(n) = 2(n - 2)^2 \), and the \( L^2 \)-formal adjoint of \( \Lambda_g \) is the map \( \Lambda_g^* : C^\infty(M) \to S_2(M) \) given by

\[
c(n)\Lambda_g^*(f) = -\Delta_g^2(fRic_g) + \delta^2(fRic_g)g + 2\delta^*\delta(fRic_g) - \frac{n}{2(n-1)}(\Delta_g(fR_g)g - \nabla^2(fR_g) + fR_gRic_g).
\]

Here \( S_2(M) \) is the space of symmetric 2-tensors on \( M \), \( \delta = \text{div} \), \( \delta^* \) is the \( L^2 \)-formal adjoint of \( \delta \) which is given by \( (\delta^*\alpha)_{ij} = \frac{1}{2}(\nabla_i\alpha_j + \nabla_j\alpha_i) \) for all 1-tensors \( \alpha \), \( \hat{R}(h)_{ij} = g^{kl}g^{st}R_{klij}h_{st} \) for all \( h \in S_2(M) \), and \( \Delta_g^2 \) is the Einstein operator defined in (1.4). This implies that

\[
tr_g\Lambda_g^*(f) = -\mathcal{T}_g(f) \quad \text{and} \quad \delta\Lambda_g^*(f) = \frac{1}{2}f d\sigma_2(g),
\]

where \( \mathcal{T}_g(u) \) is defined in (1.5). When \( g \) is an Einstein metric, it holds

\[
\sigma_2(g) = \frac{1}{8n(n-1)}R_g^2 \quad \text{and} \quad \mathcal{T}_g = \frac{1}{4n}R_g\Delta_g + \frac{1}{4n(1)}R_g^2.
\]

One recalls the \( L^2 \)-formal adjoint of the linearization of scalar curvature at a given metric \( g \) which is given by \( L^*_g(f) = \nabla^2 f - g\Delta_g f - fRic_g \) (see for instance [22]). We observe that we can recover the Ricci tensor and the scalar curvature from \( L^*_g \), since \( L^*_g(1) = -Ric_g \) and \( tr_g L^*_g(1) = -R_g \). In a complete analogy we can define a symmetric two-tensor by \( \Lambda_g^*(1) \). Moreover, we can recover the \( \sigma_2 \)-curvature taking the trace of \( \Lambda_g^*(1) \). In fact, this is a consequence of (1.5) and (2.4), or of the following lemma, which the proof is a direct computation.

**Lemma 2.1.** For any metric \( g \) we have

\[
2(n - 2)^2\Lambda_g^*(1) = -\Delta_g^2\hat{R}ic_g + \frac{n-2}{2n(n-1)}(\Delta_g R_g)g - \frac{n-2}{2(n-1)}\nabla^2 R_g - \frac{(n-2)^2}{2n(n-1)}R_g\hat{R}ic_g - \frac{2}{n}\hat{R}ic_g^2g - \frac{4(n-2)^2}{n}\sigma_2(g)g.
\]

Note that \( \mathcal{T}_g \) is self-adjoint. Then for any \( v \in C_0^\infty(M) \) (the space of all smooth functions with compact support in \( M \)), using (2.4) it holds

\[
\int_M v\Lambda_g(u g) dv_g = -\int_M u\mathcal{T}_g(v) dv_g = -\int_M v\mathcal{T}_g(u) dv_g,
\]

which implies the following result.

**Lemma 2.2.** For any \( u \in C^\infty(M) \) we have \( \Lambda_g(ug) = -\mathcal{T}_g(u) \).

Finally, given \( u \in C^\infty(M) \), the \( \sigma_2 \)-curvature of the metric \( g_0 \) and of the conformal metric \( g = e^{2u}g_0 \) are related by the equation (see [36]),

\[
-\mathcal{T}_{g_0}(u) + \left(2u + \frac{1}{2}\right)\sigma_2(g_0) - \frac{1}{2}\sigma_2(g)e^{4u} + \mathcal{I}_{g_0}(u) = 0,
\]

where

\[
\mathcal{I}_g(u) = \frac{1}{4}\left((\Delta_g u)^2 - |\nabla_g u|^2 + \langle \nabla_g u, \nabla_g u \rangle \right) + \frac{(n-3)}{4} |\nabla_g u|^2 \Delta_g u - \frac{n(n-4)}{16} |\nabla_g u|^4 - \frac{1}{2(n-2)}Ric_g(\nabla_g u, \nabla_g u) - \frac{(n-4)}{8(n-2)}R_g |\nabla_g u|^2.
\]
3. Critical Metrics in Closed Manifolds

In this section we consider a closed Riemannian manifold \((M^n, g)\) of dimension \(n \geq 3\) satisfying the equation

\[
\Lambda^*_g(f) = \kappa g. \tag{3.1}
\]

for some constant \(\kappa\) and some smooth function \(f\). An interesting question is if it is possible to classify the Riemannian manifolds \((M, g)\) which satisfy (3.1) for some smooth function \(f\). This question does not seem to be easy to deal with due to the complexity of the equation.

3.1. Proof of Theorem A. We start this subsection with the following preliminary result.

**Theorem 3.1.** Let \((M^n, g)\) be a connected closed Riemannian manifold of dimension \(n \geq 3\). Suppose there exists a non trivial smooth function \(f \in C^\infty(M)\) satisfying (3.1).

(a) If \(g\) is an admissible metric, then the \(\sigma_2\)-curvature is constant.

(b) If \(f\) is a nonzero constant \(c\), then the \(\sigma_2\)-curvature is constant equal to \(-\frac{nk}{2}\).

**Proof.** By (2.4) and (3.1) we obtain \(0 = \delta \lambda^*_g(f) = \frac{1}{4} f d\sigma_2(g)\). Suppose there exists \(p \in M\) with \(f(p) = 0\) and \(d\sigma_2(g) \neq 0\) at \(p\). By taking derivatives, we can see that \(\nabla^m f(p) = 0\) for all \(m \geq 1\). Moreover, note that by (2.4) the function \(f\) satisfies

\[
\mathcal{T}_g(f) = -nk, \tag{3.2}
\]

where \(\mathcal{T}_g\) is defined in (1.5). Since \(g\) is an admissible metric, then (3.2) is an elliptic equation. By results in [1] and [17] we can conclude that \(f\) vanishes identically in \(M\). But this is a contradiction. Therefore, \(d\sigma_2(g)\) vanishes in \(M\) and thus \(\sigma_2(g)\) is constant.

Now, if \(f\) is a nonzero constant \(c\), by (2.4) we obtain \(nk = tr_g \Lambda^*_g(c) = -2\sigma_2(g)c\).

When in (3.1) the function \(f\) is constant we obtain the following result, which is a consequence of the results in [24, 26].

**Proof of Theorem A.** Direct computations show the sufficiency. Therefore, we need only to prove the "only if" part. Then, suppose that \(g\) is a \(\sigma_2\)-Einstein metric with \(\sigma_2(g) \geq 0\). By Theorem 3.1 \(\sigma_2(g)\) is constant and we can write \(\Lambda^*_g(1) = -\frac{2}{n} \sigma_2(g)g\).

Now, define the Riemannian functional \(\tilde{F}_2 : \mathcal{M} \rightarrow \mathbb{R}\), where \(\mathcal{M}\) is the space of all Riemannian metrics in \(M\), given by

\[
\tilde{F}_2(g) = V(g)^{\frac{4-n}{n}} \int_M \sigma_2(g) dv_g, \tag{3.3}
\]

where \(V(g)\) is the volume of \((M, g)\). For a general \(k\) the functional \(\tilde{F}_k\) associated to the \(\sigma_k\)-curvature was introduced and discussed in [39].

Note that \(\tilde{F}_2(\lambda g) = \tilde{F}_2(g)\) for all positive constant \(\lambda\). Also, the first variation of \(\tilde{F}_2\) is given by

\[
D \tilde{F}_2(g)(h) = V(g)^{\frac{4-n}{n}} \int_M \left( \Lambda^*_g(h) + \frac{1}{2} \sigma_2(g) tr_g h \right) dv_g + \frac{4-n}{2n} V(g)^{\frac{4-2n}{n}} \int_M \sigma_2(g) dv_g \int_M tr_g h dv_g = V(g)^{\frac{4-n}{n}} \int_M \left( \Lambda^*_g(1) + \frac{2}{n} \sigma_2(g) g, h \right) dv_g,
\]

where in the second equality we used that \(\Lambda^*_g\) is the \(L^2\)-formal adjoint of \(\Lambda_g\) and that \(\sigma_2(g)\) is constant. This implies that a \(\sigma_2\)-Einstein metric is a critical point for the functional \(\tilde{F}_2\). Therefore, the result follows by [24, Theorem 1.1] and [26, Theorem B].
We remark that the case $n = 4$ in Theorem A cannot be treated on account of the specificity of this dimension. Indeed, for a locally conformally flat manifold the Weyl tensor vanishes. Taking derivatives of the Gauss-Bonnet-Chern Formula (1.3), for all symmetric 2-tensor $h$, we obtain

$$0 = \int_M \left( \Lambda_g(h) + \frac{1}{2} \sigma_2(g) \text{tr}_g h \right) dv_g = \int_M \left( \Lambda_g^*(1) + \frac{1}{2} \sigma_2(g) g, h \right) dv_g.$$  

This implies that $\Lambda_g^*(1) = -\frac{1}{2} \sigma_2(g) g$. Therefore, all Riemannian manifold of dimension 4 is $\sigma_2$-Einstein. This is a counterpart to Einstein manifold where all Riemannian manifold of dimension 2 is Einstein.

The proof of Theorem A tells us that a $\sigma_2$-Einstein metric is a critical point for the functional $\hat{F}_2$ (see (3.3)). From this and the example given in [26, Section 6] and [24, Remark 7.1], we obtain the existence of $\sigma_2$-Einstein metrics with $\sigma_2(g) < 0$ which are not Einstein.

Restricting the functional $F_2(g) = \int_M \sigma_2(g) dv_g$ to a certain space of metrics, it is possible to give conditions for its critical metrics to be hyperbolic, see Theorem 1.2 of [24], which holds as long as Theorem B of [26] holds. Arguing similarly as in Theorem A we have

**Theorem 3.2.** Let $(M^n, g)$ be a closed Riemannian manifold of dimension $n \geq 3$, $n \neq 4$, which is locally conformally flat for $n \geq 5$. If $(M, g)$ is a $\sigma_2$-Einstein manifold with $\sigma_2(g) > 0$ and scalar curvature $R_g < 0$, then $(M^n, g)$ is hyperbolic.

### 3.2. Second Variation of $F_2$.

In this subsection, we will compute the second variation of the functional $\hat{F}_2$ at critical points in $\mathcal{M}$. First, we give a formula for the second derivative of the $\sigma_2$-curvature. We remark that our notation convention for the curvature tensor gives

$$R_{ijkl} = \kappa(g_{il}g_{jk} - g_{ik}g_{jl})$$  

(3.4)

in the case that $g$ has constant sectional curvature $\kappa$. First, we remember the second derivative of the scalar curvature, which can be found in [32, Lemma 3].

**Lemma 3.3.** Let $\{g(t)\}$ be a smooth path of smooth metrics with $g(0) = g$. Let $R(t)$ be the scalar curvature of $g(t)$. Then

$$R''(0) = \Delta g |h|^2 + 2(h, \nabla^2 \text{tr}_g h) + 4(\nabla \delta h, h) - 2 \left| \delta h + \frac{1}{2} d\text{tr}_g h \right|^2 - \frac{1}{2} |\nabla h|^2$$

$$+ 2(\tilde{R}(h), h) - g^{pq}g^{ij} g^{st} \nabla_p h_{is} \nabla_t h_{jq} + DR_g(h').$$

where $h = g'(0)$ and $h' = g''(0)$, the covariant derivative, the curvature tensor are with respect to $g$, and $DR_g$ is the linearization of the scalar curvature at $g$.

Using this lemma we find the second derivative of the $\sigma_2$-curvature.
Proposition 3.4. Let \( \{g(t)\} \) be a smooth path of metrics with \( g(0) = g \), \( h = g'(0) \) and \( h' = g''(0) \). Let \( \sigma_2(t) \) be the \( \sigma_2 \)-curvature of \( g(t) \). Then

\[
\begin{align*}
\sigma_2''(0) &= -\frac{1}{2} \left| \Delta_E^2 h - \nabla^2 \text{tr} g h - 2\delta^* (\delta h) \right|^2 - \langle Ric_g \circ h, 2\Delta_E^2 h - \nabla^2 \text{tr} g h - 2\delta^* (\delta h) - Ric_g \circ h \rangle \\
&\quad - \langle Ric_g, h, \nabla^2 h \rangle - 4g^{pq} \text{g}^{pq} h_{im} \nabla_p \nabla_q h_{ij} + 2g^{pq} g^{st} \nabla_p h_{sj}(\nabla_i h_{il} - \nabla_i h_{qi}) + \nabla^2 h_{ij}^2 \\
&\quad - g^{sl} h_{sj}(\nabla_i (\delta h)_l - \nabla_l (\delta h)_i) + \frac{1}{2} g^{kl} (\nabla_j h_{il} - 3\nabla_i h_{ij}) \left( (\delta h)_k + \frac{1}{2} \nabla_k (\text{tr} g h) \right) \\
&\quad - 2\delta \left( (\delta h + \frac{1}{2} \nabla \text{tr} g h) \right) + \langle h, \nabla^2 h \rangle - g^{pq} g^{st} \nabla_p h_{si} \nabla_s h_{jq} + g^{st} g^{pq} h_{sj} h_{mi} R_{ijkt} \\
&\quad + \frac{n}{2(n-1)} \left( (\Delta g h - \delta^2 h + \langle Ric, h \rangle)^2 + R_g \left( \Delta g |h|_g^2 + 2\langle h, \nabla^2 \text{tr} g h \rangle - \frac{1}{2} |\nabla h|^2 \right) \\
&\quad + 4\langle \nabla \delta h, h \rangle - 2 \left| \delta h + \frac{1}{2} \nabla \text{tr} g h \right|^2 - g^{pq} g^{ij} g^{st} \nabla_p h_{is} \nabla_s h_{jq} + 2\langle R(h), h \rangle \right) \right) + c_n \Lambda_g(h'),
\end{align*}
\]

where \( c_n = 2(n-2)^2 \), all covariant derivative are with respect to \( g \), \( R_{ijkt} \) is the curvature tensor of \( g \), and \( \Lambda_g \) is the linearization of the \( \sigma_2 \)-curvature map given by (2.2).

Proof. For any fixed point \( p \), let \( \{x_i\} \) be a normal coordinate chart at \( p \) with respect to \( g(0) = g \). In these coordinates the Christoffel symbols are equal to zero at \( p \). It is well known the following variational formulae

\[
\begin{align*}
\frac{\partial}{\partial t} \Gamma^k_{ij} &= \frac{1}{2} g^{kl} (\nabla_i h_{jl} + \nabla_j h_{il} - \nabla_l h_{ij}), \\
\frac{\partial}{\partial t} R_{p_{ij}} &= \frac{1}{2} g^{im} (\nabla_p \nabla_j h_{im} - \nabla_p \nabla_m h_{ij} - \nabla_i \nabla_j h_{pm} + \nabla_m \nabla_i h_{pj}) \\
&\quad + g^{ar} R_{ipjr} h_{am} + g^{ar} R_{ipmr} h_{ja} - g^{ar} R_{impr} h_{aj} - g^{ar} R_{imjr} h_{pa}, \\
Ric' &= -\frac{1}{2} \left( \Delta g h + \nabla^2 \text{tr} g h + 2\delta^* (\delta h) + 2\hat{R}(h) - Ric \circ h - \hat{R} \circ h \circ Ric_g \right), \\
R_g' &= -\Delta g \text{tr} g h + \delta^2 h - \langle h, Ric_g \rangle.
\end{align*}
\]

From now on, all derivatives are taken at \( t = 0 \). By (2.2) we have

\[
\begin{align*}
\sigma_2''(0) &= \langle 2Ric_g \circ h - Ric_g', \Delta_E^2 h - \nabla^2 \text{tr} g h - 2\delta^* (\delta h) \rangle - \langle Ric_g, (\Delta_E^2 h - \nabla^2 \text{tr} g h - 2\delta^* (\delta h))' \rangle \\
&\quad + \frac{n}{2(n-1)} \left( (\Delta g \text{tr} g h - \delta^2 h + \langle Ric, h \rangle)^2 + R_g R_g''(0) \right).
\end{align*}
\]

First, we have \( \hat{R}(h)'_{ij} = \hat{R}(h')_{ij} - 2\hat{R}(h \circ h)_{ij} + g^{pq} g^{st} R_{p_{ij}p_{st}} h_{qt} \) and \( R_{p_{ij}p_{st}} = g^{lm} h_{st} R_{p_{ij}p_{lm}p_{st}} + g_{st} \frac{\partial}{\partial t} R_{p_{ij}} \), which implies that

\[
\langle Ric_g, \hat{R}(h)' \rangle = \left\langle Ric_g, \hat{R}(h') - \hat{R}(h \circ h) + g^{pq} g^{im} h_{qt} \left( \nabla_p \nabla_j h_{im} - \frac{1}{2} (\nabla_p h_{mij} + \nabla_i h_{pmj}) \right) \right\rangle.
\]

Using the variations formula for the geometric quantities above, the Ricci Identity and that the Ricci tensor is a symmetric tensor, we obtain

\[
\langle Ric_g, (\nabla^2 \text{tr} g h)' \rangle = \left\langle Ric_g, \nabla^2 \text{tr} g h' - \nabla^2 |h|_g^2 - g^{ij} \left( \nabla_j h_{il} - \frac{1}{2} \nabla i h_{ij} \right) \right\rangle \nabla_k (\text{tr} g h),
\]
\[\langle Ric_g, (\Delta g h_{ij})' \rangle = \langle Ric_g, \Delta g h_{ij} - \langle h, \nabla^2 h_{ij} \rangle - g^{pq} (2\nabla_p (\Gamma^s_{qi})' h_{sj}) + (\Gamma^s_{pq})' \nabla_s h_{ij} + 2(\Gamma^s_{pi})' \nabla_q h_{sj} \rangle \]

\[= \langle Ric_g, \Delta g h' - \langle h, \nabla^2 h_{ij} \rangle - h \circ (\Delta g h) - 2g^{pq} g^{st} \nabla_p h_{sj} (\nabla_q h_{it} + \nabla_i h_{qt} - \nabla_i h_{qi}) \]

\[+ g^{st} (\nabla_i (\delta h)_t - \nabla_t (\delta h)_i) + g^{st} \left( (\delta h)_t + \frac{1}{2} \nabla_t tr g h \right) \nabla_s h_{ij} \]

\[- (h \circ h, Ric_g \circ Ric_g) + g^{ia} g^{ib} g^{mt} h_{sj} h_{mi} R_{ab} R_{it}, \]

\[\langle Ric_g, \delta^s \delta (h') \rangle = -g^{ia} g^{ib} R_{ab} (g^{pq} \nabla_j \nabla_p h_{iq})' = \langle Ric_g, \delta^s \delta (h') \rangle \]

\[+ g^{ia} g^{ib} g^{pq} R_{ab} h_{tm} \nabla_j \nabla_p h_{iq} + \langle Ric_g, g^{pq} (\nabla_j (\Gamma^s_{pi})' h_{sq} + (\Gamma^s_{pq})' h_{is}) \nabla_i (\delta h)_t - \nabla_t (\delta h)_i) + \frac{1}{2} \nabla_t tr g h \rangle \nabla_s h_{ij} \]

Therefore, the result follows by these evolution equations, Lemma 3.3 and (3.5).

A direct consequence is the following.

**Corollary 3.5.** Suppose \((M^n, g)\) has constant sectional curvature \(\kappa\). Let \(\{g(t)\}\) be a smooth path of metrics with \(g(0) = g, h = g'(0)\) and \(h' = g''(0)\). Let \(\sigma_2(t)\) be the \(\sigma_2\)-curvature of \(g(t)\). Then

\[\sigma_2''(0) = \Lambda_g (h') + \frac{1}{2(n-2)^2} I,\]

where

\[I = -\frac{1}{2} |h + \nabla^2 tr g h + 2\delta^s \delta (h) + 2\kappa ((tr g h) g - h)|^2 + (n - 2)[(tr g h)^2 - |h|^2] \]

\[+ \frac{(n - 2)^2}{2} \kappa (\Delta |h|^2 - g^{ij} g^{pq} \nabla_p h_{sj} \nabla_i h_{qi}) + \frac{n}{2(n-1)} (\Delta tr g h - \delta^2 h + \kappa (n-1) tr g h)^2 \]

\[+ \kappa (h, (n^2 - 3n + 3) \nabla^2 tr g h - 2(n^2 - n + 1) \delta \delta h) - (n^2 + 2n - 2) \kappa \left( \delta h + \frac{1}{2} \nabla tr g h \right)^2 \]

\[+ 2(n - 1) \kappa \left( \delta (h \circ (\delta h + \frac{1}{2} \nabla tr g h)) - \left( \delta h + \frac{1}{2} \nabla tr g h, \frac{1}{2} \nabla tr g h \right) \right) - \frac{(n - 2)^2}{4} \kappa \nabla h|^2.\]

Here all covariant derivatives are with respect to \(g\) and \(\Lambda_g\) is the linearization of the \(\sigma_2\)-curvature map given by (2.2).

As an application of Theorem A we have the following simplification of the second variation formula of the functional total \(\sigma_2\)-curvature.

**Proposition 3.6.** Let \((M^n, g)\) be a closed Riemannian manifold of dimension \(n \geq 3, n \neq 4\), which is locally conformally flat for \(n \geq 5\). Suppose \(g\) is \(\sigma_2\)-Einstein with \(\sigma_2(g) = \frac{n(n-1)}{8} k^2\) and unit volume. Then the second derivative of \(F_2\) at \(g\) restricted to the space of metrics with unit
volume $\mathcal{M}_1$ in the direction $h \in T_g \mathcal{M}_1$ is given by

$$
\mathcal{F}_2''(h) = \frac{1}{2} \int_M \Lambda_g(h) tr_g h \, dv_g - \frac{1}{4(n-2)^2} \int_M \left| \Delta h + \nabla^2 tr_g h + 2\delta^* \delta(h) + 2\kappa((tr_g h)g - h) \right|^2 \, dv_g \\
+ \kappa^2 \int_M ((tr_g h)^2 - |h|^2) \, dv_g + \frac{\kappa}{4} \int_M \left( \Delta |h|^2 - g^{ij} g^{pq} g^{st} \nabla_p h_{sj} \nabla_t h_{qi} \right) \, dv_g \\
+ \frac{\kappa}{4(n-1)(n-2)^2} \int_M \left( \Delta tr_g h - \delta^2 h + \kappa(n-1)tr_g h \right)^2 \, dv_g \\
+ \frac{\kappa}{2(n-2)^2} \int_M \left( h, (n^2 - 3n + 3) \nabla^2 tr_g h - 2(n^2 - n + 1) \delta^* \delta h \right) \, dv_g \\
- \frac{(n^2 + 2n - 2)}{2(n-2)^2} \int_M |\delta h + \frac{1}{2} \nabla tr_g h|^2 \, dv_g - \frac{(n-1)}{4} \kappa^2 \left( \int_M |h|^2 \, dv_g - \frac{1}{2} \int_M (tr_g h)^2 \, dv_g \right) \\
- \frac{(n-1)}{(n-2)^2} \kappa \int_M \left( \delta h + \frac{1}{2} \nabla tr_g h, \frac{1}{2} \nabla tr_g h \right) \, dv_g - \frac{\kappa}{8} \int_M |\nabla h|^2 \, dv_g.
$$

Proof. Consider a one parameter deformation $g(t)$ of $g$ in $\mathcal{M}_1$ with $g(0) = h$. Since $(M, g)$ is a $\sigma_2$-Einstein manifold, we have that $\sigma_2(g)$ is constant (Theorem 3.1). Thus

$$
d^2 \left| \int_M \sigma_2(g(t)) \, dv_g \right| = \int_M \sigma_2''(g) \, dv_g + \frac{1}{2} \int_M \Lambda_g(h) tr_g h \, dv_g,
$$

where we have used that the volume of $g(t)$ is unit. Moreover, for $n \geq 3$, $n \neq 4$, and assuming local conformal flatness for $n \geq 5$, a $\sigma_2$-Einstein metric is Einstein by Theorem A. Since

$$
0 = \left. \frac{d^2}{dt^2} \right|_{t=0} V(M, g(t)) = \frac{1}{2} \int_M \left[ tr_g h' + (1/2) (tr_g h)^2 - |h|_g^2 \right] \, dv_g,
$$

where $h' = g''(0)$, then the result follows from Corollary 3.5 and the fact that $(M, g)$ is a $\sigma_2$-Einstein manifold, which implies

$$
\int_M \Lambda_g(h') \, dv_g = \int_M \langle h', \Lambda_g(1) \rangle \, dv_g = -\frac{(n-1)}{4} \kappa^2 \int_M tr_g h' \, dv_g.
$$

\[ \square \]

Considering a manifold with constant sectional curvature, we show that there always exists an infinite-dimensional subspace of tensors in $S_{2,g}^{TT}(M)$ on which the second variation of $\mathcal{F}_2$ at $t = 0$ is negative definite. Remember from introduction that $S_{2,g}^{TT}(M) := \{ h \in S_2(M) : \delta g = 0, tr_g h = 0 \}$.

Corollary 3.7. Let the setting be as in Proposition 3.6. If the sectional curvature $\kappa$ is a positive constant, then restricted to variations $g(t)$ with $h = g'(0) \in S_{2,g}^{TT}(M)$, then $\mathcal{F}_2''(h) < 0$.

Proof. By Proposition 3.6, for $h \in S_{2,g}^{TT}(M)$ we have

$$
\mathcal{F}_2''(h) = -\frac{n+1}{4} \kappa^2 \int_M |h|^2 \, dv_g - \frac{1}{4(n-2)^2} \int_M |\Delta g h - 2\kappa| h|^2 \, dv_g \\
- \kappa \int_M |\nabla g h|^2 \, dv_g - \frac{\kappa}{4} \int_M g^{ij} g^{pq} g^{st} \nabla_p h_{sj} \nabla_t h_{qi} \, dv_g.
$$
On the other hand,
\[
\int_M g^{ij} g^{pq} g^{sl} \nabla_p h_{sj} \nabla_l h_{qi} = - \int_M g^{ij} g^{pq} g^{sl} h_{sj} \nabla_p \nabla_l h_{qi} = \int_M g^{ij} g^{pq} g^{sl} (-h_{sj} \nabla_l \nabla_p h_{qi} + g^{mt} h_{sj} R_{pqlt} h_{mi} + g^{mt} h_{sj} R_{ptlhqn})
\]
\[
= -n \kappa \int_M |h|^2 + \int_M (\kappa (tr_g h)^2 - 2(h, \nabla g \nabla h) + |\nabla g h|^2).
\]

Hence the result follows. \(\square\)

We remark that if \(\kappa = 0\), then the second variation is strictly negative except for parallel \(h\).

Since in a compact and hyperbolic \(n\)-manifold the smallest eigenvalue of the rough Laplacian on \(S^{TT}_{2,n}(M)\) is at least \(n\), we have the following result.

**Corollary 3.8.** Let the setting be as in Proposition 3.6. If \((M, g)\) is a hyperbolic manifold, then it is locally strictly maximizing for \(\mathcal{F}_2\) with respect to variations in \(S^{TT}_{2,n}(M)\).

### 4. Volume comparison

The goal of this section is to prove the Theorem B, which was motivated by results due to W. Yuan [42] and Y. J. Lin and W. Yuan [31] for the scalar curvature and \(Q\)-curvature context, respectively. Roughly speaking, this theorem says that we cannot increase the volume of an \(\sigma_2\)-Einstein manifold increasing the \(\sigma_2\)-curvature. First we prove some variational formulae, which will be necessary to obtain the volume comparison.

#### 4.1. Some Variational Formulae.

Let \((M^n, g_0)\) be a closed manifold and consider the functional \(\mathcal{E}_{g_0} : M \rightarrow \mathbb{R}\) given by
\[
\mathcal{E}_{g_0}(g) = V(g)^{\frac{n}{2}} \int_M \sigma_2(g) dv_{g_0},
\]
where the volume form \(dv_{g_0}\) does not depend on \(g\). Remember from the introduction that \(M\) is the space of all Riemannian metrics on \(M\). This implies that \(\mathcal{E}_{g_0}(\lambda g) = \mathcal{E}_{g_0}(g)\) for all real number \(\lambda > 0\), since the volume form does not depend on \(g\). Define the 2-symmetric tensor \(B_{g_0}\) as
\[
B_{g_0} := -\frac{1}{2} \Lambda^*_g(1). \quad (4.1)
\]

By (2.4) and (1.5), this tensor satisfies \(tr_{g_0} B_{g_0} = \sigma_2(g_0)\) and \(div_{g_0} B_{g_0} = \frac{1}{3} \sigma_2(g_0)\). Its trace free part will be denoted by \(\tilde{B}_{g_0} := B_{g_0} - \frac{1}{n} \sigma_2(g_0) g_0\). To simplify the notation, we will use the convention that \(\prime\) and \(\prime\prime\) stand for first and second variations with respect to a certain \(h \in S_2(M)\), respectively. For a \(\sigma_2\)-Einstein metric \(g_0\), a direct computation gives us
\[
\mathcal{E}_{g_0}'(g_0) = -2 V(g_0)^{\frac{n}{2}} \int_M \left( \tilde{B}_{g_0}, h \right) dv_{g_0}.
\]

This implies the following lemma.

**Lemma 4.1.** A \(\sigma_2\)-Einstein manifold \((M, g_0)\) is a critical point to the functional \(\mathcal{E}_{g_0}\).

Before we find the second variation of \(\mathcal{E}_{g_0}\) let us prove the next lemma.

**Lemma 4.2.** For any \(\sigma_2\)-Einstein metric \(g\) we have
\[
\int_M \sigma_2''(g) dv_g = -2 \int_M \left( \langle \tilde{B}_g', h \rangle - \frac{1}{n} \sigma_2(g) |h|^2 + \left( \frac{n+4}{4n} \Lambda_g(h) + \frac{n-2}{2n^2} \sigma_2(g)(tr_g h) \right) (tr_g h) \right) dv_g.
\]
Proof. For any metric $g$ we have
\[
\int_M \sigma_2^2(g) dv_g = \left( \int_M \sigma_2(g) dv_g \right)^{\prime\prime} - 2 \int_M \sigma_2(g)(dv_g)^{\prime} - \int_M \sigma_2(g)(dv_g)'' = - 2 \left( \int_M (B_g, h) dv_g \right)
\]
Using that $\hat{B}_g = 0$, for a $\sigma_2$-Einstein metric $g$, and $\hat{B}_g = B_g - \frac{1}{n} \Lambda_g(h)g - \frac{1}{n} \sigma_2(g)h$, taking the derivative of both sides in $tr_g B_g = \sigma_2(g)$, we find $tr_g \hat{B}_g = 0$. Thus, considering that $h = \hat{h} + \frac{1}{n} (tr_g h) g$, with $tr_g h = 0$, we conclude our lemma.

\[\square\]

**Proposition 4.3.** The second variation of $\mathcal{E}_{g_0}$ at a $\sigma_2$-Einstein metric $g_0$ is given by
\[
V(g_0)^{\prime\prime} \mathcal{E}_{g_0}''(g_0) = -2 \int_M \langle (D\hat{B}_g)(\hat{h}), \hat{h} \rangle dv_{g_0} - \frac{n+4}{2n} \int_M \Lambda_{g_0}(h)(tr_{g_0} h) dv_{g_0}
- \frac{2n}{n} \int_M \langle tr_{g_0}(\hat{B}_g)^*(\hat{h})), (tr_{g_0} h) \rangle dv_{g_0} + \frac{n+4}{2n^2} \int_M (tr_{g_0} h - tr_{g_0} h) T (tr_{g_0} h - tr_{g_0} h) dv_{g_0}.
\]

Proof. Since $\hat{B}_g = \frac{1}{n} \sigma_2(g_0) g_0$, we obtain
\[
\mathcal{E}_{g_0}''(g_0) = V(g_0)^{\prime\prime} \mathcal{E}_{g_0}''(g_0) = \frac{4}{n} \int_M \sigma_2^2(g) dv_{g_0} - \frac{8}{n^2} \sigma_2(g_0) V(g_0)^{\prime\prime} - \frac{1}{n} \left( \int_M tr_{g_0} h dv_{g_0} \right)^2 + \frac{1}{n} \left( \int_M tr_{g_0} h dv_{g_0} \right)^2 + \int_M \left( \frac{n-2}{n} (tr_{g_0} h)^2 - 2 \hat{h}_{g_0}^2 \right) dv_{g_0}.
\]
By Lemma 4.2, we have
\[
V(g_0)^{\prime\prime} \mathcal{E}_{g_0}''(g_0) = -2 \int_M \langle \hat{B}_g, \hat{h} \rangle dv_{g_0} - \frac{n+4}{4} \Lambda_{g_0}(h)(tr_{g_0} h) dv_{g_0}
- \frac{n+4}{2n^2} \sigma_2(g_0) V(g_0)^{\prime\prime} - \frac{1}{n} \int_M \Lambda_{g_0}(h)(tr_{g_0} h) dv_{g_0} - \frac{n+4}{2n^2} \sigma_2(g_0) V(g_0)^{\prime\prime} - \left( \int_M tr_{g_0} h dv_{g_0} \right)^2
- \frac{n+4}{2n^2} \int_M (tr_{g_0} h) (tr_{g_0} (\Lambda_{g_0}^*(tr_{g_0} h))) dv_{g_0}.
\]
Using that $\hat{B}_g = D\hat{B}_g(h)$ and $h = \hat{h} + \frac{1}{n} (tr_{g_0} h) g_0$, with $tr_{g_0} \hat{h} = 0$, we find
\[
-2 \int_M \langle \hat{B}_g, \hat{h} \rangle dv_{g_0} = -2 \int_M \langle (D\hat{B}_g)(\hat{h}), \hat{h} \rangle dv_{g_0} - \frac{2}{n} \int_M \left( tr_{g_0} h (tr_{g_0} (\hat{B}_g)^*(\hat{h}))) (tr_{g_0} h) dv_{g_0}.
\]
Also, by (1.5) and (2.4), we obtain
\[
- \frac{1}{2} \int_M (tr_{g_0} h) (tr_{g_0} (\Lambda_{g_0}^*(tr_{g_0} h))) dv_{g_0} - \sigma_2(g_0) V(g_0)^{-1} \left( \int_M tr_{g_0} h dv_{g_0} \right)^2
= \frac{1}{2} \int_M (tr_{g_0} h) T_{g_0}(tr_{g_0} h) dv_{g_0} - \sigma_2(g_0) V(g_0)(tr_{g_0} h)^2 = \frac{1}{2} \int_M (tr_{g_0} h - tr_{g_0} h) T (tr_{g_0} h - tr_{g_0} h) dv_{g_0},
\]
where \( \text{tr}_{g_0} h = V(g_0)^{-1} \int_M \text{tr}_{g_0} h \, dv_{g_0} \). From this and the previous formulas we obtain the result. □

**Proposition 4.4.** Let \( g \) be an Einstein metric. Then for any \( \tilde{h} \in S^T_{2, g}(M) \), it holds

\[
DB_g(\tilde{h}) = \frac{1}{4(n-2)^2} \left( \Delta^g_E + 2(n-2)^2 \sigma_2^g(\cdot) \right) (\Delta^g_E \tilde{h}).
\]

**Proof.** Since \( g \) is Einstein, then \( \tilde{Ric}_g \equiv 0 \) and \( R_g \) is constant. Also, \( \tilde{h} \in S^T_{2, g}(M) \) implies that \( R_g' = -\Delta_g(tr_g \tilde{g}) + 2 \Delta \tilde{h} - \langle Ric_g, \tilde{h} \rangle = 0 \) and \( \sigma_2^g(\cdot) = 0 \) (see (2.1)). By Lemma 2.1 we obtain

\[
4(n-2)^2 \tilde{B}_g = \Delta^g_E \tilde{Ric}_g - \frac{n-2}{2n(n-1)} (\Delta_g R_g) g + \frac{n-2}{2n(n-1)} \nabla^2 R_g + \frac{(n-2)^2}{2n(n-1)} R_g \tilde{Ric}_g + \frac{4}{n} |\tilde{Ric}_g|^2 g.
\]

Thus the result follows by taking derivatives and using Lemma 3.2 in [31], which gives us \( D\tilde{Ric}_g(\tilde{h}) = \frac{1}{4} \Delta^g_E \tilde{h} \).

We notice that the definition (1.4) of the Einstein operator differs from [31, Definition 1.6] and [30, Definition 1.6] by a sign. Using (2.5) we have the following.

**Corollary 4.5.** Let \( g \) be an Einstein metric. Then \( DB_g \) is a self-adjoint operator on \( S^T_{2, g}(M) \). In addition, if \( g \) is a strictly stable Einstein metric, then \( DB_g \) is positive.

**Proposition 4.6.** Let \( g \) be a Riemannian metric. For \( h \in S^T_{2, g}(M) \oplus (C^\infty(M) \cdot g) \) we have

\[
\Lambda_g(h) = \frac{1}{2(n-2)} \text{div} \left( \langle \tilde{Ric}_g, \Delta_g h \rangle - \langle h, \Delta_g \tilde{Ric}_g \rangle + \frac{n-2}{2(n-1)} \tilde{h}(\nabla R_g, \cdot) \right) - 2 \left( \tilde{B}_g, \tilde{h} \right) - \frac{1}{n} T(tr_g h),
\]

where \( \tilde{h} \) is the trace free part of \( h \).

**Proof.** By Lemma 2.2 we get \( \Lambda_g \left( \frac{1}{n} (tr_g h) g \right) = -\frac{1}{n} T(tr_g h) \). Using (2.2) and Lemma 2.1 we obtain

\[
\begin{align*}
\Lambda_g(h) + 2 \left( \tilde{B}_g, \tilde{h} \right) &= -\frac{1}{2(n-2)^2} \left( \tilde{Ric}_g, \Delta^g_E \tilde{h} + \frac{(n-2)^2}{2n(n-1)} R_g \tilde{h} \right) \\
&+ \frac{1}{2(n-2)^2} \left( \Delta^g_E \tilde{Ric}_g + \frac{n-2}{2(n-1)} \nabla^2 R_g + \frac{(n-2)^2}{2n(n-1)} R_g \tilde{Ric}_g, \tilde{h} \right) \\
&= \frac{1}{2(n-2)^2} \text{div} \left( \langle \tilde{Ric}_g, \Delta_g h \rangle - \langle h, \Delta_g \tilde{Ric}_g \rangle + \frac{n-2}{2(n-1)} \tilde{h}(\nabla R_g, \cdot) \right) .
\end{align*}
\]

□

**Corollary 4.7.** Let \( (M, g_0) \) be an Einstein manifold. For \( h \in S^T_{2, g_0}(M) \oplus (C^\infty(M) \cdot g_0) \) we have

\[
V(g_0)^{-\frac{1}{n}} e''_{g_0}(g) = \frac{n+4}{2n^2} \int_M (tr_{g_0} h - tr_{g_0} \tilde{h}) T_{g_0} (tr_{g_0} h - tr_{g_0} \tilde{h}) \, dv_{g_0} - 2 \int_M \langle (DB_{g_0}) \tilde{h}, \tilde{h} \rangle \, dv_{g_0},
\]

where \( \tilde{h} \) is the trace free part of \( h \).

**Proof.** By Corollary 4.5, \( DB_{g_0} \) is a self-adjoint operator on \( S^T_{2, g_0}(M) \). Since \( B_{g_0} = 0 \) when \( g_0 \) is Einstein, and \( tr_g B_g = 0 \), for any metric \( g \), then differentiating \( tr_g B_g = 0 \) with respect to \( g \) we get \( tr_{g_0} \left( DB_{g_0}(\tilde{h}) \right) = \langle B_{g_0}, \tilde{h} \rangle \). This implies that \( tr_{g_0} \left( (DB_{g_0})^* \tilde{h} \right) = tr_{g_0} \left( DB_{g_0}(\tilde{h}) \right) = \langle B_{g_0}, \tilde{h} \rangle = 0 \). Also, by Proposition 4.6, we obtain that \( \Lambda_{g_0}(\tilde{h}) = 0 \). The result follows by Proposition 4.3. □
Proposition 4.8. Let \((M, g_0)\) be a strictly stable Einstein manifold with \(\text{Ric}_{g_0} = (n - 1)\lambda g_0\), where \(\lambda \geq 0\) is a constant. Then \(g_0\) is a critical point of \(\mathcal{E}_{g_0}\) and \(D^2\mathcal{E}_{g_0}(h, h) \leq 0\) for any \(h \in S^T_{2, g_0}(M) \oplus (C^\infty(M) \cdot g_0)\). Moreover, the equality holds if and only if 

(a) \(h \in C^\infty(M) \cdot g_0\), for \(\lambda = 0\).

(b) \(h \in (\mathbb{R} \oplus E_\lambda)g_0\), for \(\lambda > 0\) and \((M, g_0)\) is isometric to the round sphere with radius \(\frac{1}{\sqrt{\lambda}}\).

(c) \(h \in \mathbb{R}g_0\), for \(\lambda > 0\) and \((M, g_0)\) is not isometric to the round sphere, up to rescaling.

Here \(E_\lambda := \{ u \in C^\infty(S^n(\frac{1}{\sqrt{\lambda}})) : \Delta_{S^n(\frac{1}{\sqrt{\lambda}})} u + n\lambda u = 0\}\).

Proof. The metric \(g_0\) is a critical point of \(\mathcal{E}_{g_0}\) by Lemma 4.1, since an Einstein metric is a \(\sigma_2\)-Einstein metric. By (2.5) we have \(\sigma_2(g) \geq 0\) and \(\mathcal{T}_{g_0} = \frac{n-1}{n} \lambda (\Delta_{g_0} + n\lambda)\). By the Lichnerowicz-Obata’s Theorem (see [35, Chapter 3], for instance) we get that the first eigenvalue of \(\mathcal{T}_{g_0}\) is greater than equal to \(n\lambda\). By Proposition 4.4 and Corollary 4.7 we obtain the first part of the result.

Since \(g_0\) is strictly stable, by Corollary 4.7 the equality \(D^2\mathcal{E}_{g_0}(h, h) = 0\) implies that \(h = fg\), for some \(f \in C^\infty(M)\), and \(\lambda \int_M (f - \bar{f})(\Delta_{g_0} + n\lambda)(f - \bar{f})d\nu_{g_0} = 0\). This and the Lichnerowicz-Obata’s Theorem implies the result. \(\square\)

4.2. Volume comparison with respect to \(\sigma_2\)-curvature. Now, with the variational formulae obtained in the previous section we will prove the Theorem B. The proof is motivated by the results related to the volume comparison to the scalar curvature and \(Q\)-curvature contained in [42] and [31], respectively. In this way, we will not provide all details of the proof, which can be found in these references.

We remark that it is well known, see for example [6, Lemma 4.57] or [41], that if \((M^n, g_0)\) is a closed Einstein manifold, but not the standard sphere, then we have the direct sum decomposition

\[S_2(M) = \text{Im} \delta^* \oplus (C^\infty(M) \cdot g_0) \oplus S^T_{2, g_0}(M).\]

For the standard sphere \(S^n(\frac{1}{\sqrt{\lambda}})\) of radius \(\frac{1}{\sqrt{\lambda}}\), the same result is true if the factor \(C^\infty(M)\) is replaced by the \(L^2\)-orthogonal space to the first order spherical harmonics, i.e., by the space \(E_\lambda^\perp\), where \(E_\lambda := \{ u \in C^\infty(S^n(\frac{1}{\sqrt{\lambda}})) : \Delta_{S^n(\frac{1}{\sqrt{\lambda}})} u + n\lambda u = 0\}\).

A local slice \(S_{g_0}\) is a set of equivalence classes of metrics near \(g_0\) modulo diffeomorphisms. For any closed Einstein manifold \((M, g_0)\), there exists a local slice \(S_{g_0}\) through \(g_0\) in the space of all Riemannian metrics \(\mathcal{M}\). This means that for a fixed real number \(p > n\), there exists \(\varepsilon > 0\) such that for any metric \(g \in \mathcal{M}\) with \(\|g - g_0\|_{W^{2, p}(M, g_0)} < \varepsilon\), there exists a diffeomorphism \(\varphi\) with \(\varphi^* g \in S_{g_0}\), see [42, Theorem 5.6] for details.

The next result is fundamental to the proof of the volume comparison result, which is a slight modification of [30, Proposition 5.8] and [31, Proposition 5.7]. We prove it by completeness.

Proposition 4.9. Let \((M^n, g_0)\) be a strictly stable Einstein manifold satisfying \(\text{Ric}_{g_0} = (n - 1)\lambda g_0\), with \(\lambda > 0\). Then there exists a local slice \(S_{g_0}\) through \(g_0\) and a neighborhood \(U_{g_0}\) of \(g_0\) in \(S_{g_0}\), such that any metric \(g \in U_{g_0}\) satisfying \(\mathcal{E}_{g_0} |_{S_{g_0}(g)}^\perp \geq \mathcal{E}_{g_0} |_{S_{g_0}(g_0)}^\perp\) implies that \(g = c^2 g_0\) for some constant \(c > 0\).

Proof. Using the Ebin-Palais Slice Theorem (see [30, Theorem 5.6] and [31, Theorem 5.3]) there exists a local slice \(S_{g_0}\) through \(g_0\), such that \(S_2(M) = T_{g_0} S_{g_0} \oplus (T_{g_0} S_{g_0})^\perp\), where

- \(T_{g_0} S_{g_0} := S^T_{2, g_0}(M) \oplus (C^\infty(M) \cdot g_0)\) and \((T_{g_0} S_{g_0})^\perp = \{ \delta^* X : \langle X, \nabla_{g_0} u \rangle_{L^2} = 0, \forall u \in C^\infty(M) \}\)

when \((M, g_0)\) is not isometric to the round sphere, up to rescaling.

- \(T_{g_0} S_{g_0} := S^T_{2, g_0}(M) \oplus (E_{\lambda}^\perp \cdot g_0)\), \((T_{g_0} S_{g_0})^\perp = \{ \delta^* X : \langle X, \nabla_{g_0} u \rangle_{L^2} = 0, \forall u \in E_{\lambda}^\perp \}\)

when \((M, g_0)\) is isometric to the round sphere with radius \(\frac{1}{\sqrt{\lambda}}\). Here, \(E_\lambda = \{ u \in C^\infty(S^n(1/\sqrt{\lambda})) : \Delta u + n\lambda u = 0\}\) is the space of first eigenfunctions for the spherical metric.
From Proposition 4.8 we conclude that $g_0$ is a critical point of $\mathcal{E}_{g_0}|_{\mathcal{S}_{g_0}}$ with $D^2\mathcal{E}_{g_0}(h, h) \leq 0$ for all $h \in T_{g_0}\mathcal{S}_{g_0}$. Since $g_0$ is a strictly stable Einstein metric, by [27, Corollary 3.4] we conclude that $g_0$ is rigid, in the sense that there is a neighborhood $U_{g_0}$ of $g_0$ such that an Einstein metric $g \in U_{g_0}$ is of constant sectional curvature, which implies that $g = cg_0$ for some positive constant $c > 0$. Define

$$Q_{g_0} := \{ g \in U_{g_0} \cap \mathcal{S}_{g_0} : g \text{ is Einstein} \} = \{ g \in U_{g_0} \cap \mathcal{S}_{g_0} : g = cg_0, \text{ with } c > 0 \text{ constant} \}.$$ 

In particular, the tangent space of $Q_{g_0}$ at $g_0$ is given by $T_{g_0}Q_{g_0} = \mathbb{R}g_0$ and its $L^2$-orthogonal complement $C_{g_0}$ in $T_{g_0}\mathcal{S}_{g_0}$ is given by $C_{g_0} := \{ h \in T_{g_0}\mathcal{S}_{g_0} : \int_M tr h dv_{g_0} = 0 \}$, since a 2-tensor $h \in T_{g_0}\mathcal{S}_{g_0}$ can be written as $h = h + \frac{1}{n}(tr h)g_0$. By Proposition 4.8 we obtain that $D^2\mathcal{E}_{g_0}(h, h) < 0$ for all $h \in C_{g_0}$.

Using a similar argument as in [30, Proposition 5.8] and [31, Proposition 5.7], we define a weak Riemannian structure\(^1\) on the local slice $\mathcal{S}_{g_0}$.

$$(h, h)_g := \int_M \left[ \langle h, h \rangle_{\bar{g}} + \langle \nabla \bar{g} h, \nabla \bar{g} h \rangle_{\bar{g}} \right] dv_{\bar{g}} = \int_M \langle (1 - \Delta \bar{g}) h, h \rangle_{\bar{g}} dv_{\bar{g}} \tag{4.2}$$

on $\mathcal{S}_{g_0}$. We observe that it has a smooth connection by [20]. Define a vector field $Z$ on $\mathcal{S}_{g_0}$ as

$$Z(\bar{g}) := V(\bar{g}) \frac{2}{n} \left( \Lambda^*_{\bar{g}}(f_{\bar{g}}) + \frac{2}{n} \bar{g} V(\bar{g}) - \frac{\sqrt{-1}}{n^2} \mathcal{E}_{g_0}(\bar{g}) \right),$$

where $f_{\bar{g}}$ is a smooth positive function on $M$ satisfying $dv_{\bar{g}} = f_{\bar{g}} dv_{\bar{g}}$. Since $g_0$ is Einstein, we have $Z(g_0) = 0$. Note that in the scalar product (4.2) the gradient of $\mathcal{E}_{g_0}|_{\mathcal{S}_{g_0}}$ at $\bar{g}$ is given by

$$Y(\bar{g}) = P_{\bar{g}} \left( (1 - \Delta \bar{g})^{-1} (Z(\bar{g})) \right),$$

where $P_{\bar{g}} : S_2(M) \to T_{g_0}\mathcal{S}_{g_0}$ is the orthogonal projection to $T_{g_0}\mathcal{S}_{g_0}$. This means that for any $h \in T_{g_0}\mathcal{S}_{g_0}$ it holds $D \left( \mathcal{E}_{g_0}|_{\mathcal{S}_{g_0}} \right)_{\bar{g}}(h) = \langle h, Y(\bar{g}) \rangle_{\bar{g}}$. This implies that for any $h = \tilde{h} + \frac{1}{n}(tr g_0,h)g_0 \in C_{g_0}$ we have $D^2 \left( \mathcal{E}_{g_0}|_{\mathcal{S}_{g_0}} \right)_{\bar{g}}(\tilde{h}, h) = \langle h, DY(\bar{g}) \rangle_{\bar{g}}$. Since $D^2\mathcal{E}_{g_0}|_{\mathcal{S}_{g_0}}(h, h) < 0$ on $C_{g_0}$, we have that $DY_{g_0} : C_{g_0} \to C_{g_0}$ is an isomorphism.

Using [22, Lemma 5] we can find a neighborhood $U_{g_0} \subseteq \mathcal{S}_{g_0}$ such that any metric $g \in U_{g_0}$ satisfying $\mathcal{E}_{g_0}(\varphi^* g) \geq \mathcal{E}_{g_0}(g_0)$ implies that $g \in Q_{g_0}$. From the definition of $Q_{g_0}$ we conclude that $g = c^2 g_0$ for some positive constant $c > 0$.

\[\square\]

Now we are ready to prove a proof of Theorem B.

**Proof of Theorem B.** By Ebin-Palais Slice Theorem [30, Theorem 5.6] we can find a sufficiently small constant $\varepsilon_0 > 0$ such that for any metric $g$ satisfying $\| g - g_0 \|_{C^2} < \varepsilon_0$, there exists a diffeomorphism $\varphi$ such that $\varphi^* g \in U_{g_0} \subseteq \mathcal{S}_{g_0}$, where $U_{g_0}$ is given by Proposition 4.9.

Assume that $\sigma_2(g) \geq \sigma_2(g_0)$, $\| g - g_0 \|_{C^2} < \varepsilon_0$ and

$$V(g) \geq V(g_0) \tag{4.3}$$

for some Riemannian metric $g$ on $M$. Since there exists a diffeomorphism $\varphi$ such that $\varphi^* g \in U_{g_0}$ and $\mathcal{E}_{g_0}|_{\mathcal{S}_{g_0}}(\varphi^* g) = \mathcal{E}_{g_0}|_{\mathcal{S}_{g_0}}(g) \geq \mathcal{E}_{g_0}|_{\mathcal{S}_{g_0}}(g_0)$, where we used (4.3) and that $\sigma_2(g_0)$ is constant. By Proposition 4.9, we have that $\varphi^* g = c^2 g_0$ for some constant $c > 0$. Observe that (4.3) becomes

$$V(g) = V(\varphi^* g) = c^n V(g_0) \geq V(g_0).$$

Thus $c \geq 1$. On the other hand, $\sigma_2(g_0) = \sigma_2(\varphi^* g_0) \leq \sigma_2(\varphi^* g) = c^{-4} \sigma_2(g_0)$, which implies that $c \leq 1$. Hence, $\varphi^* g = g_0$ and the result follows. \[\square\]

\(^1\)The term weak is due the fact that (4.2) defines in each tangent space a topology weaker than the current.
5. Variational Characterization of Critical Metrics of the Volume Functional

In Section 3 we have investigated critical points of the volume functional in a manifold without boundary. In this section we study variational properties of the volume functional constrained to the space of metrics of constant \( \sigma_2 \)-curvature with a prescribed boundary metric.

Let \( M^n \) be a connected, compact manifold of dimension \( n \geq 3 \) with smooth nonempty boundary \( \partial M \) and a fixed boundary metric \( \gamma \). Let \( \mathcal{M}_\gamma \subset \mathcal{M} \) be the space of metrics on \( M \) with induced metric on \( \partial M \) given by \( \gamma \). Let \( K \) be a constant and \( \mathcal{M}^K_\gamma \) be the space of metrics \( g \in \mathcal{M}_\gamma \) which have constant \( \sigma_2 \)-curvature \( K \). Let \( S^{k,2}(M) \) be the space of \( W^{k,2} \) symmetric 2-tensors on \( M \), with \( k > n/2 + 2 \). Thus each \( h \in S^{k,2}(M) \) is \( C^{2,\alpha} \) up to the boundary. By [22, Lemma 1] we get that \( \sigma_2 : \mathcal{M}_\gamma \to W^{k-2,2}(M) \) is smooth, where \( W^{k-2,2}(M) \) is the space of \( W^{k-2,2} \) functions on \( M \). Let \( C^{\infty}_0(M) \) be the space of smooth function which vanishes in \( \partial M \).

**Lemma 5.1.** Let \( g_0 \in \mathcal{M}^K_\gamma \) be a 2-admissible metric on \( M \) such that the first Dirichlet eigenvalue of the operator \( -T_{g_0} \) is positive. Then there exists a neighborhood \( U \subset \mathcal{M} \) of \( g_0 \) and an unique smooth function \( \Phi : U \to C^{\infty}_0(M) \) such that for every \( g \in U \) it holds \( e^{2\Phi(g)}g \in \mathcal{M}^K_\gamma \).

**Proof.** Consider the smooth map \( \mathcal{F} : \mathcal{M} \times C^{\infty}_0(M) \to C^{\infty}(M) \) given by

\[
\mathcal{F}(g, u) = \sigma_2(e^{2u}g).
\]

Differentiating with respect to \( u \) we get

\[
D_2\mathcal{F}(g_0, 0)(v) = 2\Lambda_{g_0}(v, g_0) = -2T_{g_0}(v),
\]

where we used Lemma 2.2 and \( T_{g_0} \) is defined in (1.5). Let \( f \in C^{\infty}(M) \) and consider the following boundary value problem

\[
\begin{align*}
-\mathcal{T}_{g_0}(u) &= f & \text{in } M \\
u &= 0 & \text{on } \partial M.
\end{align*}
\]

(5.1)

By hypothesis \( g_0 \) is an admissible metric, which implies that (5.1) is an elliptic equation and has a unique solution by the Fredholm alternative, see [34, Theorem 2.2.4]. Thus, the operator \( D_2\mathcal{F}(g_0, 0) : C^{\infty}_0(M) \to C^{\infty}(M) \) is an isomorphism. The Implicit Function Theorem for Banach spaces [29, Theorem 5.9] implies that there exists a neighborhood \( U \subset \mathcal{M} \) of \( g_0 \) and an unique smooth function \( \Phi : U \to C^{\infty}_0(M) \) such that \( \mathcal{F}(g, \Phi(g)) = K \), for all \( g \in U \), i.e. \( e^{2\Phi(g)}g \in \mathcal{M}^K_\gamma \), for every \( g \in U \subset \mathcal{M} \). \( \square \)

Next, we consider the volume functional \( V : \mathcal{M}_\gamma \to \mathbb{R} \), whose first variation (see Proposition 1.186 of [6]) is given by

\[
DV_g(h) = \frac{1}{2} \int \text{tr}_g hdv_g.
\]

(5.2)

We are interested in critical points of \( V \) restricted to \( \mathcal{M}^K_\gamma \).

**Theorem 5.2** (Theorem C). Let \( g \in \mathcal{M}^K_\gamma \) be a 2-admissible metric such that the first Dirichlet eigenvalue of \( -T_g \) is positive. Then, \( g \) is a critical point of the volume functional in \( \mathcal{M}^K_\gamma \) if and only if there exists a smooth function \( f \) on \( M \) such that

\[
\begin{align*}
\Lambda_g^*(f) &= g & \text{in } M \\
f &= 0 & \text{on } \partial M.
\end{align*}
\]

(5.3)

**Proof.** Suppose that \( g \) is a critical point of \( V \) in \( \mathcal{M}^K_\gamma \). Since \( g \) is a 2-admissible metric and the first eigenvalue of \( -T_g \) is positive, it follows by the Fredholm alternative [34, Theorem 2.2.4] that there exists an unique function \( f \) on \( M \) satisfying the following equation

\[
\begin{align*}
-\mathcal{T}_g(f) &= -n & \text{in } M \\
f &= 0 & \text{on } \partial M.
\end{align*}
\]

(5.4)
We will prove that \( f \) satisfies the equation (5.3). Let \( h \) be a smooth symmetric 2-tensor such that \( h|_{\partial M} \equiv 0 \). For small \(|\ell|\) we have that \( g(\ell) = g + th \) is a smooth metric in \( \mathcal{M}_\gamma \). Define 
\[
u(t) = \Phi(g + th),\]
where \( \Phi \) is given by Lemma 5.1. By the uniqueness we obtain that \( u(0) \equiv 0 \) and \( u' \equiv 0 \) in \( \partial M \), since \( \sigma_2(g) = K \) and \( u \equiv 0 \) in \( \partial M \). Thus, if \( \bar{g}(t) = e^{2u(t)}g(t) \), then
\[
sigma_2(\bar{g}(t)) = K. \tag{5.5}
\]
Note that \( \bar{g}'(0) = 2u'(0)g + h \). Since \( \bar{g}(t) \) and \( g(t) \) are conformal metrics, using (5.5), their \( \sigma_2 \)-curvature are related by
\[
-\mathcal{I}_g(t)(u) + \left( 2u(t) + \frac{1}{2} \right) \sigma_2(g(t)) - \frac{1}{2} Ke^{2u(t)} + \mathcal{I}_g(t)(u(t)) = 0, \tag{5.6}
\]
where \( \mathcal{I}_g \) is given by (2.7). Taking the derivative of (5.6) with respect to \( t \) and using that \( u(0) = 0 \) and \( \mathcal{I}_g \) is quadratic in \( u \) we obtain that \( u'(0) \) satisfies
\[
\begin{align*}
\mathcal{I}_g(u'(0)) &= \frac{1}{2} \Lambda_g(h) \quad \text{in } M \\
u'(0) &= 0 \quad \text{on } \partial M.
\end{align*} \tag{5.7}
\]
Hence, by equation (5.4) and using integration by parts we obtain
\[
n \int_M u'(0)dv_g = - \int_M u'(0)\mathcal{I}_g(f)dv_g = - \int_M f\mathcal{I}_g(u'(0))dv_g = - \frac{1}{2} \int_M f\Lambda_g(h)dv_g = - \frac{1}{2} \int_M \langle h, \Lambda^*_g(f) \rangle dv_g, \tag{5.8}
\]
where in the third equality we have used (5.7). Since \( g \) is a critical point of \( V \) in \( \mathcal{M}_\gamma^K \), by (5.2), we have \( \int_M (2nu'(0) + \text{tr}_g h) dv_g = 0 \). Using this and (5.8) we obtain
\[
\int_M \langle h, \Lambda^*_g(f) - g \rangle dv_g = \int_M \langle h, \Lambda^*_g(f) \rangle - \text{tr}_g h \rangle dv_g = 0.
\]
Since \( h \) is any 2-tensor, we conclude that \( \Lambda^*_g(f) = g \).

Now, suppose that \( f \) satisfies the equation (5.3). Let \( h \) be a smooth symmetric 2-tensor in the tangent space of \( g \) in \( \mathcal{M}_\gamma^K \). This implies that \( h|_{\partial M} \equiv 0 \) and \( \Lambda_g(h) = 0 \). Therefore, using integration by parts, we obtain
\[
0 = \int_M f\Lambda_g(h)dv_g = \int_M \langle h, \Lambda^*_g(f) \rangle dv_g \\
- \frac{1}{2(n-2)} \int_{\partial M} (\langle \nabla_\nu (fRic_g) - \delta (fRic_g)(\nu) g, h \rangle + 2\langle h(\nu), \delta (fRic_g) \rangle) d\sigma_g \\
- \frac{n}{2(n-1)} \int_M (\nabla_\nu (fR_g) tr_g h - \langle h(\nu), \nabla (fR_g) \rangle) dv_g \tag{5.9}
\]

since \( h|_{\partial M} \equiv 0 \) and \( f \equiv 0 \) on \( \partial M \) implies that
\[
\langle \nabla_\nu (fRic_g) - \delta (fRic_g)(\nu) g, h \rangle + 2\langle h(\nu), \delta (fRic_g) \rangle = \nabla_\nu (fR_g) tr_g h - \langle h(\nu), \nabla (fR_g) \rangle = 0,
\]
on the boundary. Hence \( g \) is a critical point of the volume functional in \( \mathcal{M}_\gamma^K \).

Theorem 5.2 gives conditions for a constant \( \sigma_2 \)-curvature metric \( g \) to be a critical point of the volume functional in \( \mathcal{M}_\gamma^K \), which is equivalent to the existence of a function \( f \) satisfying \( \Lambda^*_g(f) = g \) in \( M \) with \( f = 0 \) on \( \partial M \). \qed
It would be interesting to know the first variation of the volume functional in $\mathcal{M}_c^K$ if $\Lambda^*_g(f) = g$ in $M$, but $f$ is not necessarily null on $\partial M$. We could not find the expression in the general case, but restrict to a conformal class we have the following.

In 2009, Chen \cite{Chen} introduced the $H_k$-curvature of the boundary of a Riemannian manifold $(M^n, g)$, which for $k = 2$ one has

$$H_2 = \frac{R\overline{g}}{2(n-2)} - \frac{n-1}{6}H^3 - \frac{1}{n-2} \langle A_0, A_\overline{g} \rangle + \frac{1}{2(n-2)} H|A_0|^2,$$

where $A_0 = A - H\overline{g}, \overline{g}$ is the induced metric on the boundary $\partial M$, $A$ is the second fundamental form, $H = \frac{1}{n-1} tr\overline{H}A$ is the mean curvature of the boundary and $A_\overline{g}$ is the Schouten tensor of the boundary $(\partial M, \overline{g})$. The definition for general $k$ can be found in $\cite{Chen1, Chen2}$.

By (1.5) we obtain

$$\frac{\partial}{\partial t} \bigg|_{t=0} \sigma_2(e^{2u}g) = -2T_{g_0}(u) \quad (5.10)$$

and it is well known that

$$\frac{\partial}{\partial t} \bigg|_{t=0} H_2(e^{2u}g) = -3uH_2(g) + T_1(\eta, \nabla_g u) - div\overline{H}(H(g)\nabla u), \quad (5.11)$$

where $T_1$ is defined in (1.6). See $\cite{Chen1, Chen2, Li}$ for details.

**Proposition 5.3.** Let $g \in \mathcal{M}_c^K$ be a smooth metric. Let $f$ be a smooth function on $M$ such that $\Lambda_g^*(f) = g$ on $M$. Consider $\{e^{2u}g\}$ a smooth path of conformal metrics in $\mathcal{M}_c^K$. Then

$$\frac{d}{dt} \bigg|_{t=0} V(e^{2u}g) = \int_{\partial M} fH_2(0)d\sigma_g,$$

where $H_2(t)$ is the $H_2$-curvature of $\partial M$ in $(M, e^{2u}g)$ with respect to the unit outward pointing normal vector.

**Proof.** Note that, by (5.9) we get

$$\int_M f\Lambda_g(h)dv_g = \int_M tr\overline{g}hv_g - \int_{\partial M} fT_1(\eta, \nabla_g u)h d\sigma_g$$

By (5.11), if $u \equiv 0$ in $\partial M$, we get $\frac{\partial}{\partial t} \bigg|_{t=0} H_2(e^{2u}g) = T_1(\eta, \nabla_g u)$.

Thus, by (5.2) we conclude our result. \hfill $\square$

### 6. Critical Metrics in Space Forms

Before presenting rigidity results in space forms, we give some examples of functions satisfying equation (5.3), such functions will be called potential functions.

**Definition 6.1.** Given a connected compact manifold $M$ with smooth connected boundary $\partial M$, we say a metric $g$ on $M$ is a $\sigma_2$-**critical metric** if there exists a potential function $f$ such that

$$\begin{cases} 
\Lambda^*_g(f) = g & \text{in } M \\
f = 0 & \text{on } \partial M. 
\end{cases} \quad (6.1)$$

We will assume that $f^{-1}(0) = \partial M$, which implies that $f$ does not change sign. In the case that $g$ satisfies $\Lambda^*_g(f) = \kappa g$ in $M$, where $\kappa$ is a real constant, we also can assume that $f > 0$ in $M\setminus \partial M$, since for the case $f < 0$, we only need to replace $(f, \kappa)$ by $(-f, -\kappa)$. If $\kappa = 0$ such metrics are called $\sigma_2$-**singular metric** (see $\cite{Li}$).
First we observe that if $g$ is an Einstein metric we can rewrite $\Lambda_{g}^{*}(f)$ as

$$\Lambda_{g}^{*}(f) = \frac{R_{g}}{4n(n-1)} \left( \nabla^{2} f - (\Delta_{g} f) g - \frac{R_{g}}{n} f g \right). \quad (6.2)$$

Now, we shall discuss the volume functional on domains in space forms. First we show two examples of $\sigma_{2}$-critical metrics.

**Example 6.2.** Let $\Omega$ be a geodesic ball in the round sphere $\mathbb{S}^{n}$ with center $p$ and radius $R < \pi/2$. Consider the function $f : \Omega \to \mathbb{R}$ given by

$$f = \frac{4}{n-1} \left( \frac{\cos r}{\cos R} - 1 \right).$$

It is not difficult to see that $f$ satisfies (5.3).

**Example 6.3.** Consider the hyperboloid model for hyperbolic space $\mathbb{H}^{n} = \{x \in \mathbb{R}^{n,1}; t > 0, (x, x)_{L} = -1\}$, where $\mathbb{R}^{n,1}$ is the Minkowski space with the standard flat metric. Let $p = (1, 0, \cdots, 0) \in \mathbb{H}^{n}$, and $\Omega$ be a geodesic ball in the hyperboloid model for hyperbolic space $\mathbb{H}^{n}$ with center $p$ and radius $R$. It is not difficult to see that the function

$$f(t, x_{1}, \cdots, x_{n}) = \frac{4}{n-1} \left( \frac{\cosh r}{\cosh R} - 1 \right),$$

satisfies (5.3), where $r = \cosh^{-1} t$ is the geodesic distance from $(t, x_{1}, \cdots, x_{n})$ to $p$.

Next we show that the only domains in the space forms $\mathbb{H}^{n}$ or $\mathbb{S}^{n}$, on which the canonical metrics are critical points, are geodesic balls.

**Theorem 6.4.** Let $(\Omega, g)$ be a bounded connected domain with smooth boundary $\partial \Omega$ with a fixed boundary metric $\gamma = g|_{\partial \Omega}$ in a simply connected space form in $\mathbb{H}^{n}$ or $\mathbb{S}^{n}$. In the case of $\Omega \subset \mathbb{S}^{n}$, we also assume that $V(\Omega) < \frac{1}{2}V(\mathbb{S}^{n})$. Suppose that $g$ is a critical point of the volume functional $V$ in $M^{K}_{\gamma}$, where $K = \frac{n(n-1)}{8}$. Then, the corresponding space form metric is a critical metric on $\Omega$ if and only if $\Omega$ is a geodesic ball.

**Proof.** We observe that if $\Omega \subset \mathbb{H}^{n}$ or $\Omega \subset \mathbb{S}^{n}$, then $R_{g} = -n(n-1)$ or $R_{g} = n(n-1)$, respectively. In the special case that the volume of $\Omega \subset \mathbb{S}^{n}$ is less than the volume of a hemisphere, $\Omega$ must be strictly contained in the upper hemisphere and the first eigenvalue of $\mathcal{T}_{g}$ is positive by the Faber-Krahn inequality [38]. Under our assumption $g$ is a critical point of the volume functional $V|_{\mathcal{M}^{K}_{\gamma}}$ if and only if there exists a smooth function $f$ on $M$ such that $f = 0$ on $\partial \Omega$ and

$$\begin{cases} R_{g} \nabla^{2} f + \frac{R_{g}^{2}}{n(n-1)} f g = -4ng & \text{in } \Omega \\ f = 0 & \text{on } \partial \Omega. \end{cases} \quad (6.3)$$

The theorem follows by a slightly modification of the proof of Theorem 6 in [32] we obtain the desired result. \hfill \square

The following is an immediate consequence of the theorem.

**Corollary 6.5.** Let $\Omega$ be a connected domain with compact closure in $\mathbb{H}^{n}$ or $\mathbb{S}^{n}$ and with a smooth (possibly disconnected) boundary $\partial \Omega$. Let $g$ be the standard metric in $\Omega$. If $\Omega \subset \mathbb{S}^{n}$, we also assume that $V(\Omega) < \frac{1}{2}V(\mathbb{S}^{n})$. Then, $\Omega$ is a geodesic ball if and only if

$$\int_{\partial \Omega} H_{g}^{2}(0) d\sigma_{g} = 0,$$

for any smooth variation $\{e^{2tu} g\}$ of $g$ in $\mathcal{M}^{K}_{\gamma}$.
Proof. Since Ω is a connected domain with compact closure in \( \mathbb{H}^n \) or \( S^n \), the constant function \( f = -4/(n-1) \) satisfies \( \Lambda^*_g(f) = g \). Then Proposition 5.3 implies

\[
\frac{d}{dt} \bigg|_{t=0} V(e^{2tu}g) = -\frac{4}{n-1} \int_{\partial \Omega} H'_2(0) d\sigma_g.
\]

By Theorem 6.4, \( g \) is a critical point of the volume functional \( V \) in \( \mathcal{M}^K_\gamma \), and so

\[
\int_{\partial \Omega} H'_2(0) d\sigma_g = 0,
\]

for any smooth conformal variation \( \{e^{2tu}g\} \) of \( g \) in \( \mathcal{M}^K_\gamma \). This completes the proof. \( \square \)

In particular, motivated by results of [33] we have

**Proposition 6.6.** Suppose \( (\Omega^n, g) \) is a connected, compact, Einstein manifold with a smooth boundary \( \partial \Omega \). If there is a function \( f \) on \( \Omega \) such that \( f = 0 \) on \( \partial \Omega \) and \( \Lambda^*_g(f) = g \) in \( \Omega \), then \((\Omega, g)\) is isometric to a geodesic ball in a simply connected space form \( \mathbb{H}^n \) or \( S^n \).

**Proof.** We observe that if \( g \) is an Einstein metric then by equation (6.2), we obtain that

\[
\Lambda^*_g(f) = \frac{R_g}{4n(n-1)} L^*_g(f),
\]

where \( L^*_g \) is the \( L^2 \)-formal adjoint of linearization of scalar curvature (see [22]). Note that \( R_g = n(n-1)k \), where \( k = 1 \) or \( k = -1 \). Thus, \( \Lambda^*_g(f) = k/4 L^*_g(f) \). Now, using the same idea that in Lemma 2.1, Lemma 2.2 and Theorem 2.1 in [33] we conclude this result. \( \square \)

**Remark 6.7.** Given a positive constant \( k \) consider a geodesic ball \( \Omega_{-k} \) in the hyperbolic space \( \mathbb{H}^n_{-k} \) with sectional curvature \( -k \). Consider the product \( M = \Omega_{-k} \times S^n_k \) with the product Riemannian metric \( g \). Here \( S^n_k \) is the round sphere with sectional curvature \( k \). It is well known that the canonical metric in \( S^n_k \) is a critical point for the volume functional restricted to the space of Riemannian metric with constant scalar curvature (see [19], for instance). But, using (2.5) and Theorem 5.2 we find that the canonical metric in \( S^n_k \) is a critical point for the volume functional restricted to the space of Riemannian metrics with constant \( \sigma_2 \)-curvature. By the Example 6.3 and Theorem 5.2 we have that the canonical metric in \( \Omega_{-k} \) is a critical point of the volume functional in \( \mathcal{M}^{c-k}_\gamma \).

Note that \((M, g)\) is a smooth compact manifold with nonempty boundary, locally conformally flat and non Einstein manifold, scalar curvature identically zero and \( \sigma_2(g) = -n(n-1)^2 k^2 =: c < 0 \). Since the volume of \( M \) is the product of the volume of \( \Omega_{-k} \) and \( S^n_k \), we conclude that \( g \) is a critical metric of the volume functional restricted to \( \mathcal{M}^{c-k}_\gamma \), where \( \gamma \) is the fixed metric in the boundary \( \partial M \).

### 7. Second variational formula for the volume functional

In this section we will use the second derivative of the \( \sigma_2 \)-curvature given by Proposition 3.4 to find the second variation of the volume functional at a critical metric in \( \mathcal{M}^K_\gamma \). Then we find a direction where this variation is strictly negative.

**Theorem 7.1.** Let \((M, g)\) be a connected compact Riemannian manifold of dimension \( n \) with a nonempty smooth boundary, such that the first Dirichlet eigenvalue of \( -\mathcal{L}_g \) is positive. Suppose \( g \) has constant sectional curvature \( \kappa \) and that there is a smooth function \( f \) on \( M \) satisfying

\[
\begin{cases}
\Lambda^*_g(f) = g & \text{in } M \\
f = 0 & \text{on } \partial M.
\end{cases}
\]

(7.1)
Let $\gamma = g_{T\partial M}$ and let $K$ be the constant that equals the $\sigma_2$-curvature of $g$. Suppose $\{g(t)\}$ is a smooth path of metrics in $\mathcal{M}^n_\gamma$ with $g(0) = g$. Let $V(t)$ be the volume of $(M, g(t))$, then

$$V''(0) = \int_M \left( \frac{1}{4} (tr_g h)^2 + f \left( \frac{1}{8n(n-2)^2} |\Delta h + \nabla^2 tr_g h + 2\delta^* \delta(h) + 2\kappa ((tr_g h) g - h)|^2 + \frac{\kappa}{16} |\nabla h|^2 \right) \right) dv_g,$$

where $h = g'(0)$ and $h' = g''(0)$ and all covariant derivatives are with respect to $g$.

**Proof.** We first note that $V'(0) = 0$ by (7.1) and Theorem 5.2. Note that

$$V''(0) = \int_M \left( \frac{1}{4} (tr_g h)^2 - \frac{1}{2} |h|^2 + \frac{1}{2} tr_g h' \right) dv_g,$$

where $h = g'(0)$ and $h' = g''(0)$. Our aim is to express the last integral in terms of $h$. Applying the fact that $g(t)$ has constant $\sigma_2$-curvature $K$ and Corollary 3.5, we have

$$0 = \sigma_0''(0) = c_n^{-1} I + \Lambda_g(h'),$$

where $c_n = 2(n-2)^2$ and $I$ involves only $h$ and its derivatives. Integrating by parts, we have

$$\int_M f \Lambda_g(h') = \int_M \langle h', \Lambda_g^*(f) \rangle - c_n^{-1} \int_{\partial M} \left( \langle \nabla \nu(f Ric_g) - \delta(f Ric_g)(\nu) g, h' \rangle + 2 \langle h'(\nu), \delta(f Ric_g) \rangle \right)$$

$$- \frac{n}{2(n-1)} \left( \nabla \nu(f R_g) tr_g h' - \langle h'(\nu), \nabla(f R_g) \rangle \right).$$

By (7.1) we have $\langle h', \Lambda_g^*(f) \rangle = tr_g h'$. Using that $h|_{\partial M} \equiv 0$ and $f \equiv 0$ on $\partial M$, as in (5.9), and (7.3) we obtain that

$$\int_M tr_g h' = \int_M f \Lambda_g(h') = -c_n^{-1} \int_M f I dv_g,$$

(7.4)

Since $g$ has constant sectional curvature $\kappa$, then $R_{ijkl} = \kappa (g_{il} g_{jk} - g_{ik} g_{jl})$, $Ric_g = (n-1) \kappa g$, $R_g = n(n-1) \kappa$, $\sigma_2(g) = \frac{n(n-1)}{8} k^2$, $\Lambda_g^*(f) = \frac{n}{4} (\nabla^2 f - (\Delta f) g - (n-1) \kappa f g)$ and $tr_g \Lambda_g^*(f) = -\frac{n-1}{4} \kappa (\Delta_g f + n \kappa f)$. By (7.1) we obtain

$$\Delta_g f = -\frac{4n}{(n-1) \kappa} - n \kappa f \quad \text{and} \quad \nabla^2 f = \left( -\frac{4}{(n-1) \kappa} - \kappa f \right) g.$$

Integrating by parts and (7.5) implies that

$$\int_M f \Delta |h|^2 dv_g = - \int_M |h|^2 \left( n \kappa f + \frac{4n}{(n-1) \kappa} \right) dv_g - \int_{\partial M} |h|^2 \partial_\nu f.$$

Also, in an analogous way as in the proof of [32, Theorem 9], we have

$$\int_M f g^{ij} g^{pq} g^{sl} \nabla_p h_{sji} \nabla_l h_{qij} = \int_M g^{ij} g^{pq} g^{sl} \left( \nabla_p (f h_{sji} \nabla_l h_{qij}) - \nabla_p f h_{sji} \nabla_l h_{qij} - f h_{sji} \nabla_p \nabla_l h_{qij} \right)$$

$$= \int_M g^{ij} g^{pq} g^{sl} \left( -\nabla_l (\nabla_p f h_{sji}) + \nabla_p (\nabla_l f h_{sji}) + \nabla_p f \nabla_{l,sji} h_{qij} - f h_{sji} \nabla_l \nabla_p h_{qij} + g^{mt} f h_{sji} R_{plq} h_{mi} + g^{mt} f h_{sji} R_{pll} h_{qmi} \right).$$
where we used integration by parts and that \( f \equiv 0 \) on \( \partial M \) to infer that the integral of the first term in the r.h.s of the first line is equal to zero. Also, we used in the second equality the Ricci identity. Using integration by parts, (3.4) and (7.5) we get

\[
\int_M f g^{ij} g^{pq} g^s l p_h s_j \nabla_l h q_i = - \int_M g^{ij} g^{pq} \nabla_p f h_v j h_q i - \int_M \left( \kappa f + \frac{4}{(n-1)\kappa} \right) |h|^2 \\
+ \int_M (f(\nabla h, \nabla \delta h) - (n-1)\kappa f|h|^2 + \kappa f(tr_g h)^2 - \kappa f|h|^2 \\
+ g^{ij} g^{pq} g^s l p(f \nabla l h s_j h q_i - f \nabla p \nabla l h s_j h q_i - f \nabla l h s_j \nabla p h q_i))
\]

\[
= \int_M f(\kappa(tr_g h)^2 + 2(h, \delta^* \delta h) - |\delta h|^2) - \int_M (n+1)\kappa f + \frac{4}{(n-1)\kappa} |h|^2 - \int_{\partial M} \nabla_v \nu |h|^2.
\]

Thus

\[
\int_M f(\Delta |h|^2) - g^{ij} g^{pq} g^s l p_h s_j \nabla_l h q_i = \int_M \left( \kappa f - \frac{4}{\kappa} \right) |h|^2 + \int_M f(-\kappa(tr_g h)^2 - 2(h, \delta^* \delta h) + |\delta h|^2).
\]

Therefore, by Corollary 3.5, (7.2), (7.4) and (7.6) we obtain the result. \( \Box \)

**Corollary 7.2.** Under the same assumption of Theorem 7.1, if \( tr_g h = 0 \) and \( \text{div}_g h = 0 \), then

\[
V''(0) = \int_M f \left( \frac{1}{8(n-2)^2} |\Delta_g h - 2\kappa h|^2 + \frac{\kappa^2}{8} |h|^2 + \frac{\kappa}{16} |\nabla h|^2 \right).
\]

Now Theorem D is a consequence of the following result.

**Theorem 7.3.** Let \((\Omega, g)\) be a geodesic ball with compact closure in \(S^n_+ \) and \(\mathbb{H}^n\). Let \(\gamma = g|_{\partial \Omega}\) and \(K\) be the constant equal the \(\sigma_2\)-curvature of \(g\). Suppose \(\{g(t)\}\) is a smooth path of metrics in \(\mathcal{M}_\gamma\) with \(g(0) = g\) and \(g'(0) = h\). Let \(V(t)\) denote the volume of \((\Omega, g(t))\).

a) If \(\Omega \subset S^n_+\), then \(V''(0) > 0\) for any \(h\) satisfying \(\text{div}_g h = 0\) and \(\text{tr}_g h = 0\).

b) If \(\Omega \subset \mathbb{H}^n\), then for any point \(p \in \Omega\) there exists a geodesic ball with radius \(\delta\) centered at \(p\), denoted by \(B(p, \delta)\), depending on \(p\) and \(\Omega\) such that \(V''(0) > 0\) for any \(h\) which has compact support in \(B(p, \delta) \subset (\Omega, g)\) and satisfying \(\text{div}_g h = 0\) and \(\text{tr}_g h = 0\).

**Proof.** Suppose that \(\Omega\) is a geodesic ball of \(S^n_+\), then

\[
V''(0) = \int_M f \left( \frac{1}{8(n-2)^2} |\Delta_g h - 2\kappa h|^2 + \frac{1}{8} |h|^2 + \frac{1}{16} |\nabla h|^2 \right).
\]

Since \(f > 0\) in \(\Omega\), \(V''(0) > 0\).

Next suppose \(\Omega\) is a geodesic ball of \(\mathbb{H}^n\). Then

\[
V''(0) = \int_M f \left( \frac{1}{8(n-2)^2} |\Delta_g h + 2\kappa h|^2 + \frac{1}{8} |h|^2 - \frac{1}{16} |\nabla h|^2 \right).
\]

Given \(p \in \Omega\), we can find \(\delta > 0\) such that \(B(p, \delta) \subset \Omega\) satisfies

\[
2 \min_{B(p, \delta)} f \geq f(p) \quad \text{and} \quad \frac{1}{2} \max_{B(p, \delta)} f \leq f(p).
\]

Then for a compactly supported tensor \(h\) in \(B(p, \delta)\), we have
\[
V''(0) \geq f(p) \int_{B(p, \delta)} \left( \frac{1}{16(n-2)^2} |\Delta g h + 2h|^2 + \frac{1}{16} |h|^2 - \frac{1}{8} |\nabla g h|^2 \right) \\
= f(p) \int_{B(p, \delta)} \left( \frac{1}{16(n-2)^2} \left( |\Delta g h|^2 + 4 \langle h, \Delta g h \rangle + 4|h|^2 \right) + \frac{1}{16} |h|^2 + \frac{1}{8} \langle h, \Delta g h \rangle \right).
\]
Thus,
\[
V''(0) \geq \frac{f(p)}{16(n-2)^2} \left( \lambda_1(B(p, \delta)^2 - (4 + 2(n-2)^2)\lambda_1(B(p, \delta)) + 4 + (n-2)^2 \right) \left( \int_{B(p, \delta)} |h|^2 \right),
\]
where \( \lambda_1(B(p, \delta)) \) is the first eigenvalue of the rough Laplacian on \( B(p, \delta) \).

We conclude that \( V''(0) > 0 \) as \( \lim_{\delta \to 0} \lambda_1(B(p, \delta)) = +\infty \). Hence the result follows for a sufficiently smaller \( \delta \).

**Proof of Theorem D.** As in the proof of Theorem 5.2, we can find a family of metrics \( \tilde{g} = e^{2v(t)}(g_0 + \theta h) \) with constant \( \sigma_2 \)-curvature equal to \( K \), and if \( v = \frac{\partial u}{\partial t} |_{t=0} \) then we have
\[
\begin{cases}
T_{g_0}(v) = \frac{1}{2} \sigma''(0) & \text{in } \Omega \\
v = 0 & \text{on } \partial \Omega.
\end{cases}
\]
If \( h \) satisfies \( h|_{T\partial \Omega} = 0, \text{div}_g h = 0 \) and \( \text{tr}_g h = 0 \), then \( \sigma''(0) = 0 \). Since in a space form the first Dirichlet eigenvalue of \( T_{g}(v) \) is positive, the result follows from Theorem 7.3 and the existence of trace free and divergence free symmetric 2-tensors with prescribed compact support on space forms (see Appendix [32] or [18]).

In the setting of critical metrics of the volume functional with constant scalar curvature, P. Miao and L.F. Tam [32] proved the existence of deformations along which the volume of the standard metric is a strict local maximum. For that, they use a limit argument where the limit metric has zero scalar curvature (in fact in the Euclidean space). This fact was essential to show further the nonexistence of a global volume minimizer in
\[
\mathcal{M}_0^\gamma = \{ g \in \mathcal{M} \ | \ R(g) = 0 \ \text{and} \ g|_{T\partial \mathcal{M}} = \gamma \}.
\]
We observe that using the same technique an analogue of this fact in our context is not possible since \( \sigma_2 \)-critical metrics do not have models in the Euclidean space (see Section 6).

**Conflict of interest statement**

The authors declare that there is no conflict of interest regarding the publication of this article.

**References**

[1] Aronszajn, N. Sur l’unicité du prolongement des solutions des équations aux dérivées partielles elliptiques du second ordre. *C. R. Acad. Sci. Paris* 242 (1956), 723–725.

[2] Baltazar, H., and Ribeiro Jr, E. Critical metrics of the volume functional on manifolds with boundary. *Proceedings of the American Mathematical Society* 145, 8 (2017), 3513–3523.

[3] Barros, A., Diógenes, R., and Ribeiro, E. Bach-flat critical metrics of the volume functional on 4-dimensional manifolds with boundary. *The Journal of Geometric Analysis* 25, 4 (2015), 2698–2715.

[4] Barros, A., Diógenes, R., and Ribeiro, Jr., E. Bach-flat critical metrics of the volume functional on 4-dimensional manifolds with boundary. *J. Geom. Anal.* 25, 4 (2015), 2698–2715.

[5] Battista, R., Diógenes, R., Ranieri, M., and Ribeiro, E. Critical metrics of the volume functional on compact three-manifolds with smooth boundary. *The Journal of Geometric Analysis* 27, 2 (2017), 1530–1547.

[6] Besse, A. L. *Einstein manifolds*. Springer Science & Business Media, 2007.
ON THE $\sigma_2$-CURVATURE AND VOLUME OF COMPACT MANIFOLDS

[7] Branson, T. P., and Gover, A. R. Variational status of a class of fully nonlinear curvature prescription problems. *Calc. Var. Partial Differential Equations* 32, 2 (2008), 253–262.
[8] Brendle, S., and Viaclovsky, J. A. A variational characterization for $\sigma_{n/2}$. *Calc. Var. Partial Differential Equations* 20, 4 (2004), 399–402.
[9] Caffarelli, L., Nirenberg, L., and Spruck, J. The Dirichlet problem for nonlinear second-order elliptic equations. III. Functions of the eigenvalues of the Hessian. *Acta Math.* 155, 3-4 (1985), 261–301.
[10] Case, J. S., Lin, Y.-J., and Yuan, W. Conformally variational Riemannian invariants. *Trans. Amer. Math. Soc.* 371, 11 (2019), 8217–8254.
[11] Case, J. S., Lin, Y.-J., and Yuan, W. Some constructions of formally self-adjoint conformally covariant polydifferential operators. *arXiv:2002.05874* (Feb 2020).
[12] Case, J. S., and Wang, Y. Boundary operators associated to the $\sigma_k$-curvature. *Adv. Math.* 337 (2018), 83–106.
[13] Catino, G., Mastrolia, P., and Monticelli, D. D. A variational characterization of flat spaces in dimension three. *Pacific J. Math.* 282, 2 (2016), 285–292.
[14] Chang, S.-Y. A. *Non-linear elliptic equations in conformal geometry*. Zurich Lectures in Advanced Mathematics. European Mathematical Society (EMS), Zürich, 2004.
[15] Chen, S.-Y. S. Conformal deformation on manifolds with boundary. *Geom. Funct. Anal.* 19, 4 (2009), 1029–1064.
[16] Chow, B., Lu, P., and Ni, L. *Hamilton’s Ricci flow*, vol. 77 of Graduate Studies in Mathematics. American Mathematical Society, Providence, RI; Science Press Beijing, New York, 2006.
[17] Cordes, H. O. Über die eindeutige Bestimmtheit der Lösungen elliptischer Differentialgleichungen durch Angangs- und Randwertaufgaben. *Nachr. Akad. Wiss. Göttingen. Math.-Phys. Kl. IIa.* 1956 (1956), 239–258.
[18] Corvino, J. On the existence and stability of the Penrose compactification. *Ann. Henri Poincaré* 8, 3 (2007), 597–620.
[19] Corvino, J., Eichmair, M., and Miao, P. Deformation of scalar curvature and volume. *Math. Ann.* 357, 2 (2013), 551–584.
[20] Eastham, D. G. The manifold of riemannian metrics, in: Global analysis, berkeley, calif., 1968. In *Proc. Sympos. Pure Math.* (1970), vol. 15, pp. 11–40.
[21] Fang, Y., He, Y., and Zhong, J. Volume comparison theorem with respect to sigma-2 curvature. *arXiv:2111.09532* (2021).
[22] Fischer, A. E., and Marsden, J. E. Deformations of the scalar curvature. *Duke Math. J.* 42, 3 (1975), 519–547.
[23] Gårding, L. An inequality for hyperbolic polynomials. *J. Math. Mech.* 8 (1959), 957–965.
[24] Gursky, M. J., and Viaclovsky, J. A. A new variational characterization of three-dimensional space forms. *Invent. Math.* 145, 2 (2001), 251–278.
[25] Gursky, M. J., and Viaclovsky, J. A. A fully nonlinear equation on four-manifolds with positive scalar curvature. *J. Differential Geom.* 63, 1 (2003), 131–154.
[26] Hu, Z., and Li, H. A new variational characterization of $n$-dimensional space forms. *Trans. Amer. Math. Soc.* 356, 8 (2004), 3005–3023.
[27] Kobayashi, N. Rigidity and stability of Einstein metrics—the case of compact symmetric spaces. *Osaka Math. J.* 17, 1 (1980), 51–73.
[28] Kröncke, K. *Stability of Einstein Manifolds*. PhD thesis, Universität Potsdam, http://opus.kobv.de/ubp/volltexte/2014/6963/, 2014.
[29] Lang, S. *Fundamentals of differential geometry*, vol. 191 of Graduate Texts in Mathematics. Springer-Verlag, New York, 1999.
[30] Lin, Y.-J., and Yuan, W. Deformations of Q-curvature I. *Calc. Var. Partial Differential Equations* 55, 4 (2016), Art. 101, 29.
[31] Lin, Y.-J., and Yuan, W. Deformations of Q-curvature II. *arXiv:2102.05871* (2021).
[32] Miao, P., and Tam, L.-F. On the volume functional of compact manifolds with boundary with constant scalar curvature. *Calc. Var. Partial Differential Equations* 36, 2 (2009), 141–171.
[33] Miao, P., and Tam, L.-F. Einstein and conformally flat critical metrics of the volume functional. *Transactions of the American Mathematical Society* 363, 6 (2011), 2907–2937.
[34] Sattinger, D. H. *Topics in stability and bifurcation theory*. Lecture Notes in Mathematics, Vol. 309. Springer-Verlag, Berlin-New York, 1973.
[35] Schoen, R., and Yau, S.-T. *Lectures on differential geometry*. Conference Proceedings and Lecture Notes in Geometry and Topology, I. International Press, Cambridge, MA, 1994.
[36] Silva Santos, A. Solutions to the singular $\sigma_2$-Yamabe problem with isolated singularities. *Indiana Univ. Math. J.* 66, 3 (2017), 741–790.
[37] **Silva Santos, A., and Andrade, M.** Deformation of the $\sigma_2$-curvature. *Ann. Global Anal. Geom.* 54, 1 (2018), 71–85.

[38] **Sperner, E.** Zur symmetrisierung von funktionen auf sphären. *Mathematische Zeitschrift* 134, 4 (1973), 317–327.

[39] **Viaclovsky, J. A.** Conformal geometry, contact geometry, and the calculus of variations. *Duke Math. J.* 101, 2 (2000), 283–316.

[40] **Viaclovsky, J. A.** Some fully nonlinear equations in conformal geometry. In *Differential equations and mathematical physics (Birmingham, AL, 1999)*, vol. 16 of *AMS/IP Stud. Adv. Math.* Amer. Math. Soc., Providence, RI, 2000, pp. 425–433.

[41] **Viaclovsky, J. A.** Critical metrics for Riemannian curvature functionals. In *Geometric analysis*, vol. 22 of *IAS/Park City Math. Ser.* Amer. Math. Soc., Providence, RI, 2016, pp. 197–274.

[42] **Yuan, W.** Volume comparison with respect to scalar curvature. *arXiv: 1609.08849* (2016).

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