THE CANONICAL CLASS OF $\overline{M}_{0,n}(\mathbb{P}^r, d)$ AND ENUMERATIVE GEOMETRY

RAHUL PANDHARIPANDE

0. Summary

Let $\mathbb{C}$ be the field of complex numbers. Let the Severi variety

$$S(0, d) \subset \mathbb{P} \left( \mathbb{H}^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d)) \right)$$

be the quasi-projective locus of irreducible, nodal rational curves. Let $\overline{S}(0, d)$ denote the closure of $S(0, d)$. Let $p_1, \ldots, p_{3d-2}$ be $3d-2$ general points in $\mathbb{P}^2$. Consider the subvariety $C_d \subset \overline{S}(0, d)$ corresponding to curves passing through all the points $p_1, \ldots, p_{3d-2}$. $C_d$ is a complete curve in $\mathbb{P} \left( \mathbb{H}^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d)) \right)$. Let $N_d$ be the degree of $C_d$. $N_d$ is determined by the recursive relation ([K-M], [R-T]):

$$N_1 = 1$$

$$\forall d > 1, \quad N_d = \sum_{i+j=d, i,j>0} N_i N_j \left( i^2 j^2 \left( \frac{3d-4}{3i-2} - \frac{i^3 j}{3i-1} \right) \right).$$

For $d \geq 3$, $C_d$ is singular. The arithmetic genus $g_d$ of $C_d$ is determined by:

$$g_1 = 0,$$

$$g_2 = 0,$$

$$2g_d - 2 = \frac{6d^2 + 5d - 15}{2d} N_d + \frac{1}{4d} \sum_{i=1}^{d-1} N_i N_{d-i} \left( 15i^2 (d-i)^2 - 8d(i(d-i) - 4d) \right) \left( \frac{3d-2}{3i-1} \right).$$

The last formula holds for $d \geq 3$. The geometric genus $\tilde{g}_d$ of the normalization $\tilde{C}_d$ is determined by ($d \geq 1$):

$$2\tilde{g}_d - 2 = -\frac{3d^2 - 3d + 4}{2d^2} N_d + \frac{1}{4d^2} \sum_{i=1}^{d-1} N_i N_{d-i}(id-i^2) \left( (9d+4)i(d-i) - 6d^2 \right) \left( \frac{3d-2}{3i-1} \right).$$

These genus formulas are established by adjunction and intersection on Kontsevich’s space of stable maps $\overline{M}_{0,n}(\mathbb{P}^r, d)$.

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1. The Canonical Class of $\overline{M}_{0,n}(\mathbb{P}^r, d)$

Let $\overline{M}_{0,n}(\mathbb{P}^r, d)$ be the coarse moduli space of degree $d$, Kontsevich stable maps from $n$-pointed, genus 0 curves to $\mathbb{P}^r$. Foundational treatments of $\overline{M}_{0,n}(\mathbb{P}^r, d)$ can be found in [Al], [P1], [K], and [K-M]. Only the case $r \geq 2$ will be considered here. Let $\mathcal{L}_p$ denote the line bundle obtained on $\overline{M}_{0,n}(\mathbb{P}^r, d)$ by the $p^{th}$ evaluation map ($1 \leq p \leq n$). Let $\Delta$ be the set of boundary divisors. Let $\mathcal{H}$ denote the divisor of maps meeting a fixed codimension 2 linear space of $\mathbb{P}^r$. $\mathcal{H} = \emptyset$ if $d = 0$. In [P2], it is shown the classes

$$\{\mathcal{L}_p\} \cup \Delta \cup \{\mathcal{H}\}$$

span $Pic(\overline{M}_{0,n}(\mathbb{P}^r, d)) \otimes \mathbb{Q}$.

The canonical class of the stack $\overline{M}_{0,n}(\mathbb{P}^r, d)$ has the following coarse moduli interpretation. $\overline{M}_{0,n}(\mathbb{P}^r, d)$ is an irreducible variety with finite quotient singularities. When $r \geq 2$, the automorphism-free locus $\overline{M}_{0,n}^*(\mathbb{P}^r, d) \subset \overline{M}_{0,n}(\mathbb{P}^r, d)$ is a nonsingular, fine moduli space with codimension 2 complement except when ([P2])

$$[0, n, r, d] = [0, 0, 2, 2].$$

For $r \geq 2$ and $[0, n, r, d] \neq [0, 0, 2, 2]$, the first Chern class of the cotangent bundle to the moduli space $\overline{M}_{0,n}(\mathbb{P}^r, d)$ yields the canonical class in $Pic(\overline{M}_{0,n}(\mathbb{P}^r, d)) \otimes \mathbb{Q}$.

Let $P = \{1, 2, \ldots, n\}$ be the set of markings ($P$ may be the empty set). The boundary components are in bijective correspondence with data of weighted partitions $(A \cup B, d_A, d_B)$ where:

(i.) $A \cup B$ is a partition of $P$.
(ii.) $d_A + d_B = d$, $d_A \geq 0$, $d_B \geq 0$.
(iii.) If $d_A = 0$ (resp. $d_B = 0$), then $|A| \geq 2$ (resp. $|B| \geq 2$).
Define $D_{i,j}$ to be the reduced sum of boundary components with $d_A = i$ and $|A| = j$. Note $0 \leq i \leq d$ and $0 \leq j \leq n$. The divisors $D_{i,n}$, $D_{i,\infty}$, $D_{i,-\infty}$, $D_{i,\lambda}$ are equal to 0 by stability. Also, $D_{i,|} = D_{(-),\lambda}$.

Consider first the case $d = 0$. $\overline{M}_{0,n}(\mathbb{P}^r, 0) \cong \overline{M}_{0,n} \times \mathbb{P}^r$. It suffices to determine the canonical class of $\overline{M}_{0,n}$.

**Proposition 1.** The canonical class $K_{\overline{M}}$ of $\overline{M}_{0,n}$ is determined in $\text{Pic}(\overline{M}_{0,n}) \otimes \mathbb{Q}$ by:

\[
K_{\overline{M}} = \sum_{i=0}^{\frac{d}{2}} \left( ((\lambda - |) - \infty) - \infty \right) D_{i,|}.
\]  

(1)

The canonical class has a different form in case $d > 0$, $n = 0$, $r \geq 2$.

**Proposition 2.** The canonical class $K_{\overline{M}}$ of $\overline{M}_{0,0}(\mathbb{P}^r, d)$ ($d > 0, r \geq 2$) is determined in $\text{Pic}(\overline{M}_{0,0}(\mathbb{P}^r, d)) \otimes \mathbb{Q}$ by:

\[
K_{\overline{M}} = -\left( [\infty + \infty] \right) H + \sum_{i=\infty}^{\frac{d}{2}} \left( \left( [\infty + \infty] \right) \left( [\infty - \infty] \right) - \infty \right) D_{i,j}.
\]  

(2)

Finally, when $d > 0$, $n > 0$, $r \geq 2$, the form of the canonical class is the following:

**Proposition 3.** The canonical class of $K_{\overline{M}}$ of $\overline{M}_{0,n}(\mathbb{P}^r, d)$ ($d > 0$, $n > 0$, $r \geq 2$) is determined in $\text{Pic}(\overline{M}_{0,n}(\mathbb{P}^r, d)) \otimes \mathbb{Q}$ by:

\[
K_{\overline{M}} = -\left( [\infty + \infty] \right) H - \sum_{\lambda=\infty}^{\frac{d}{2}} \left( \left( [\infty + \infty] \right) \left( [\infty - \infty] \right) - \infty \right) L \sqrt{\mathcal{C}}
\]  

\[+ \sum_{i=0}^{\frac{d}{2}} \sum_{j=0}^{n} \left( \left( (r + 1)(d - i)d + 2d^2j - 4dij + 2ni^2 \right) - 2 \right) D_{i,|}.
\]  

Equation (1) can be derived from the explicit construction of $\overline{M}_{0,n}$ described in [F-M]. Equations (1-3) will be established here via intersections with curves.
2. Computing The Canonical Class

2.1. Curves in $\overline{M}_{0,n}(\mathbb{P}^r, d)$. By Proposition (2) of [P2], the canonical projection

$$\text{Pic}(\overline{M}_{0,n}(\mathbb{P}^r, d)) \otimes \mathbb{Q} \rightarrow \mathbb{N} \cong (\overline{M}_{0,n}(\mathbb{P}^r, d)) \otimes \mathbb{Q}$$

is an isomorphism. Hence, the canonical class of $\overline{M}_{0,n}(\mathbb{P}^r, d)$ can be established via intersections with curves. Curves can easily be found in $\overline{M}_{0,n}(\mathbb{P}^r, d)$. The notation of [P2] is recalled here.

Let $C$ be a nonsingular, projective curve. Let $\pi : S = \mathbb{P}^1 \times C \rightarrow C$. Select $n$ sections $s_1, \ldots, s_n$ of $\pi$. A point $x \in S$ is an intersection point if two or more sections contain $x$. Let $\mathcal{N}$ be a line bundle on $S$ of type $(d, k)$. Let $z_l \in H^0(S, \mathcal{N})$ $(0 \leq l \leq r)$ determine a rational map $\mu : S \rightarrow \mathbb{P}^r$ with simple base points. A point $y \in S$ is a simple base point of degree $1 \leq e \leq d$ if the blow-up of $S$ at $y$ resolves $\mu$ locally at $y$ and the resulting map is of degree $e$ on the exceptional divisor $E_y$.

The set of special points of $S$ is the union of the intersection points and the simple base points. Three conditions are required:

1. There is at most one special point in each fiber of $\pi$.
2. The sections through each intersection point $x$ have distinct tangent directions at $x$.
3. (i.) $d = 0$. No $n - 1$ sections pass through a point $x \in S$.
   (ii.) $d > 0$. If at least $n - 1$ sections pass through a point $x \in S$, then $x$ is not a simple base point of degree $d$.

Condition (3.ii) implies there are no simple base points of degree $d$ if $n = 0$ or 1. Let $\overline{S}$ be the blow-up of $S$ at the special points. It is easily seen $\overline{\mu} : \overline{S} \rightarrow \mathbb{P}^r$ is Kontsevich stable family of $n$-pointed, genus 0 curves over $C$. Condition (2) ensures the strict transforms of the sections are disjoint. Condition (3) implies Kontsevich stability. There is a canonical morphism $\lambda : C \rightarrow \overline{M}_{0,n}(\mathbb{P}^r, d)$. Condition (1) implies $C$ intersects the boundary components transversally.

2.2. $\overline{M}_{0,n}$. Curves in $\overline{M}_{0,n}$ are obtained by the above construction (omitting the map $\mu$). Let $s_1, \ldots, s_n$ be $n$ sections of $\pi : S = \mathbb{P}^1 \times C \rightarrow C$ satisfying (1), (2), (3.i). For $1 \leq \alpha \leq n$, let $s_{\alpha}$ be of type $(1, \sigma_{\alpha})$ on $S = \mathbb{P}^1 \times C$. Let $\overline{\pi} : \overline{S} \rightarrow C$ be the blow-up of $S$ as above. Let $s_1, \ldots, s_n$ also denote the transformed sections of $\overline{\pi}$. Let $Q$ denote the
points of $C$ lying under the special points of $S$. There is a canonical sequence
\[ 0 \to R^1\pi_*\left(\omega^*_\pi\left(-\sum_1^n s_\alpha\right)\right) \to \lambda^*\left(T_{\overline{M}_{0,n}}\right) \to \bigoplus_{q \in \mathbb{Q}} \mathbb{C}_{\pi} \to \mathcal{V}. \]

(See, for example, [K].) Hence $C \cdot K_{\overline{M}} = -\lceil | \lceil - \sum_1^n f_\alpha \rceil \rceil \rceil - C \cdot \sum_{|\in \mathbb{Q}} D_{\pi}.$

The degree of $R^1\pi_*\left(\omega^*_\pi\left(-\sum_1^n s_\alpha\right)\right)$ is determined by the Grothendieck-Riemann-Roch formula. Let $x_j$ for $2 \leq j \leq n - 2$ be the number of intersection points of $S$ which lie on exactly $j$ sections. If $j \neq n/2$, $C \cdot D_{\pi} = \sum_1^n x_j$. If $j = n/2$, $C \cdot D_{\pi} = \sum_1^n x_j$. G-R-R yields:
\[\deg\left(R^1\pi_*\left(\omega^*_\pi\left(-\sum_1^n s_\alpha\right)\right)\right) = \sum_1^n 2\sigma_\alpha + \sum_2^{n-2} (1-j)x_j.\]

By the transverse intersection condition, the following relation must hold:
\[\sum_1^n \sigma_\alpha = \frac{1}{n-1} \sum_2^{n-2} \left(\frac{j^2 - j}{2}\right)x_j.\]

Combining equations yields:
\[C \cdot K_{\overline{M}} = \sum_{\in \mathbb{Q}} \left(1 - \in - \frac{|\in - |}{- - \infty}\right) \sum_1^n x_j = \sum_2^{n-2} \left(\frac{j(n-j)}{n-1} - 2\right)x_j.\]

Hence both sides of equation (I) have the same intersection numbers with $C$. Let $D$ be any nonsingular curve in $\overline{M}_{0,n}$ which intersects the boundary transversely. The universal family over $D$ can be blown-down to a projective bundle $\pi : T \to D$. The above calculation covers the case where $T = \mathbb{P}^1 \times D$. The general case (in which $T$ is any $\mathbb{P}^1$-bundle) is identical. Since $A^1(\overline{M}_{0,n})$ is spanned by curves meeting the boundary transversely, Proposition (I) is immediate.

2.3. $\overline{M}_{0,0}(\mathbb{P}^r, d)$. The case $d > 0$, $n = 0$, $r \geq 2$ is now considered. Let $\pi : S \to C$, $\overline{\pi} : \overline{S} \to \mathbb{P}$ be a family of stable maps as above. There is canonical exact sequence
\[0 \to \pi_*\left(\omega^*_\pi\right) \to \pi_*\overline{\pi}^*\left(T_{\overline{\pi}}\right) \to \lambda^*\left(T_{\overline{M}_{0,0}(\mathbb{P}^r, d)}\right) \to \bigoplus_{p \in \mathbb{Q}} \mathbb{C}_{\pi} \to \mathcal{V}.\]
Hence $C \cdot K_{\overline{M}} = + \left\lfloor \frac{n}{2} \right\rfloor \left( \pi_*(\omega_{\overline{M}}^\natural) - \left\lfloor \frac{n}{2} \right\rfloor \left( \pi_*(\eta^\natural (T_{\overline{P}^r})) \right) \right) - \sum_{i=0}^{n} d_{i,j}. \] Let $x_i$ for $1 \leq i \leq d - 1$ be the number of simple base points of $\mu : S - \rightarrow \mathbb{P}^r$ of degree exactly $i$. If $i \neq d/2$, $C \cdot D_{i,j} = \delta_i + \delta_{i-1}. \] If $i = d/2$, $C \cdot D_{i,j} = \delta_i$. Via G-R-R,

$$\deg(\pi_*(\omega_{\overline{M}}^\natural)) = - \sum_{i=1}^{d-1} x_i,$$

$$\deg(\pi_*(\eta^\natural (T_{\overline{P}^r}))) = (r + 1)(d + 1)k - \frac{\sum_{i=1}^{d-1} (r + 1)(i^2 + i)}{2} x_i.$$

Combining equations yields:

$$C \cdot K_{\overline{M}} = -(\nabla + \infty)([\nabla + \infty]) + \sum_{i=1}^{\infty} - (\nabla + \infty)(\{ \epsilon + \}) - \epsilon \delta_i.$$

Finally $C \cdot \mathcal{H}$ must be computed:

$$C \cdot \mathcal{H} = \epsilon [\nabla - \sum_{i=1}^{\infty} \epsilon \delta_i].$$

These equations (plus algebra) verify Proposition (2). As before, the complete proof requires the above calculation for any $\mathbb{P}^1$-bundle $\pi : S \rightarrow C$. Again, the generalization to this case is trivial.

2.4. $\overline{M}_{0,n}(\mathbb{P}^r, d)$. Finally, the case $d > 0$, $n > 0$, $r \geq 2$ is considered. Let $\pi : \overline{S} \rightarrow C$, $\overline{\mu} : \overline{S} \rightarrow \mathbb{P}$ be a family of stable maps as above. Let $s_1, \ldots, s_n$ be $n$ sections of $\pi : S = \mathbb{P}^1 \times \mathbb{C} \rightarrow \mathbb{C}$ satisfying (1), (2), (3). For $1 \leq \alpha \leq n$, let $s_\alpha$ be of type $(1, \sigma_\alpha)$ on $S = \mathbb{P}^1 \times C$. There is a canonical exact sequence

$$0 \rightarrow \pi_*(\omega_{\overline{M}}) \rightarrow \pi_* (\omega_{\overline{S}} \oplus \pi_*(\eta^\natural (T_{\overline{P}^r}))) \rightarrow \lambda^*(T_{\overline{M}_{0,n}(\mathbb{P}^r, d)}) \rightarrow \bigoplus_{p \in Q} \mathbb{C} \rightarrow 0.$$

Hence $C \cdot K_{\overline{M}} = + \left\lfloor \frac{n}{2} \right\rfloor \left( \pi_*(\omega_{\overline{M}}^\natural) - (\omega_{\overline{M}}^\cdot \sum_{\alpha} f_\alpha) - \left\lfloor \frac{n}{2} \right\rfloor \left( \pi_*(\eta^\natural (T_{\overline{P}^r})) \right) \right) - \sum_{j=0}^{d-1} \sum_{i=1}^{n} d_{i,j}. \] Let $z_{i,j}$ for $0 \leq i \leq d$ and $0 \leq j \leq n$ be the number of special points of $S$ that are simple base points of degree exactly $i$ and that lie on exactly $j$ sections. If $i \neq d/2$ or $j \neq n/2$, then $C \cdot D_{i,j} = \delta_i + \delta_{i-1}. \] If $i = d/2$ and $j = n/2$, then $C \cdot D_{i,j} = \delta_i. \] Via G-R-R,

$$\deg(\pi_*(\omega_{\overline{M}}^\natural)) = - \sum_{i=0}^{d} \sum_{j=0}^{n} z_{i,j},$$

$$\deg(\pi_*(\eta^\natural (T_{\overline{P}^r}))) = (r + 1)(d + 1)k - \frac{\sum_{i=0}^{d} \sum_{j=0}^{n} (r + 1)(i^2 + i)}{2} z_{i,j}.$$
A simple calculation yields:

$$\omega^* = \sum_1^n \alpha \cdot \sum_{j=0}^d \sum_{i=0}^n j z_{i,j}.$$ 

The two additional intersection numbers are:

$$C \cdot H = \langle \| - \sum_{j=t} \sum_{i=t} \epsilon_{i,j} \rangle,$$

$$C \cdot \sum_1^n \mathcal{L} = \langle \| + \sum_{i=\infty} \sum_{j=t} \epsilon_{i,j} \rangle.$$

Now algebra yields the desired equality of intersections that establishes Proposition (3). Again the calculation must be done in case $\pi : S \to C$ is a $\mathbb{P}^1$ bundle.

3. The genus of $C_d, \tilde{C}_d$

3.1. Singularities. Let $C_d \subset \mathbb{S}(0,d)$ be the dimension 1 subvariety corresponding to curves passing through $3d - 2$ general points $p_1, \ldots, p_{3d-2}$ in $\mathbb{P}^2$. Let $\tilde{C}_d \subset \overline{M}_{0,0}(\mathbb{P}^2, d)$ be the dimension 1 subvariety corresponding to maps passing through $p_1, \ldots, p_{3d-2}$. The singularities of $C_d, \tilde{C}_d$ will be analyzed.

Let $[\mu] \in \overline{M}_{0,0}(\mathbb{P}^2, d)$ correspond to an automorphism-free map with domain $\mathbb{P}^1$. There is a normal sequence on $\mathbb{P}^1$:

$$0 \to T_{\mathbb{P}^1} \xrightarrow{d\mu} \mu^*(T_{\mathbb{P}^2}) \to \text{Norm} \to 0.$$ 

The tangent space to $\overline{M}_{0,0}(\mathbb{P}^2, d)$ is $H^0(\mathbb{P}^1, \text{Norm})$. If $\mu$ is an immersion, $\text{Norm} \cong \mathcal{O}_{\mathbb{P}^1}(3d - 2)$. If $\mu$ is not an immersion $\text{Norm}$ will have torsion. A 1-cuspidal rational plane curve is an irreducible rational plane curve with nodal singularities except for exactly 1 cusp. If $\mu$ corresponds to a 1-cuspidal rational curve, then there is a sequence:

$$0 \to \mathbb{C}, \to \mathbb{N} \xrightarrow{p} \mathcal{O}_{\mathbb{P}^1}([\theta - \theta]) \to \mathbb{C}$$

where $p$ is the point of $\mathbb{P}^1$ lying over the cusp. Since $3d - 2$ distinct points of $\mathbb{P}^1$ always impose independent conditions on $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(3d - 2))$ and $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(3d - 3))$, Lemma (1) has been established:

Lemma 1. Let $[\mu] \in \tilde{C}_d$ be a point corresponding to an irreducible, nodal or 1-cuspidal rational curve with all singularities distinct from $p_1, \ldots, p_{3d-2}$. $\tilde{C}_d$ is nonsingular at $[\mu]$. 

7
The corresponding analysis for $C_d$ is more involved.

**Lemma 2.** Let $x \in C_d$ be a point corresponding to an irreducible, nodal rational curve with nodes distinct from $p_1, \ldots, p_{3d-2}$. $C_d$ is nonsingular at $x$.

**Proof.** Let $X \subset \mathbb{P}^2$ be the plane curve corresponding to $x \in C_d$. $S(0, d)$ is nonsingular at $x$ with tangent space $H^0(\tilde{X}, \mathcal{O}_{\mathbb{P}^2}(d) - N)$ where $N$ is the divisor of points of $\tilde{X}$ lying over the nodes of $X$. The additional points correspond to $3d - 2$ distinct points of $\tilde{X}$. Since $3d - 2$ distinct points on $\mathbb{P}^1$ impose $3d - 2$ independent linear conditions on sections of $\mathcal{O}_{\mathbb{P}^2}(d) - N \cong \mathcal{O}_{\mathbb{P}^1}(3d - 2)$, it follows $C_d$ is nonsingular at $x$. \qed

Actually, $\overline{M}_{0,0}(\mathbb{P}^2, d)$ and $S(0, d)$ are isomorphic on the irreducible, nodal locus. Hence Lemma (2) is unnecessary.

**Lemma 3.** Let $x \in C_d$ be a point corresponding to a 1-cuspidal rational plane curve with all singularities distinct from $p_1, \ldots, p_{3d-2}$. $C_d$ is cuspidal at $x$.

**Proof.** The versal deformation space of the cusp $Z_0^2 + Z_1^3$ is 2 dimensional:

$$Z_0^2 + Z_1^3 + aZ_1 + b.$$

The locus in the versal deformation space corresponding to equigeneric deformations is determined by the cuspidal curve $4a^3 + 27b^2 = 0$.

Let $X$ be the plane curve corresponding to $x$. Let $q \in X$ be the cusp. Let $\tilde{X}$ the normalization of $X$. Let $p_1, \ldots, p_{3d-2}$, and the 2 dimensional subscheme supported on $q$ and pointing in the direction of the tangent cone of $X$ all together impose independent conditions on the linear system of degree $d$ plane curves. First, this independence will be established.

Let $A$ be the subspace of $H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d))$ passing through the nodes, points, and the subscheme of length 2. As before, let $N$ denote the divisor of $\tilde{X}$ lying above the nodes. There is a natural left exact sequence obtained by pulling back sections to $\tilde{X}$:

$$0 \to \mathbb{C} \to A \to H^0(\tilde{X}, \mathcal{O}_{\mathbb{P}^2}(d) - N - \langle \mathcal{O}_{\mathbb{P}^2}(-1) \rangle_{\tilde{X}} - \langle \mathcal{O}_{\mathbb{P}^2}(-2) \rangle_{\tilde{X}}).$$
By counting conditions,
\[ \dim(A) \geq \frac{(d + 1)(d + 2)}{2} - \frac{(d - 1)(d - 2)}{2} + 1 - 3d + 2 - 2 = 1 \]
with equality if only if the conditions are independent. Since
\[ \deg(\mathcal{O}_{\mathbb{P}^2}(d-N-p_1-\ldots-p_{3d-2}-3p)) = d^2-(d-1)(d-2)+2-3d+2-3 = -1, \]
\[ \dim(A) = 1 \] and the conditions are independent.

By the independence result above, the deformations of \( X \) parameterized by the linear system of plane curves through the nodes and the points \( p_1, \ldots, p_{3d-2} \) surjects on the 2 dimensional versal deformation space of the cusp. The locus of equigeneric deformations of \( X \) through the points \( p_1, \ldots, p_{3d-2} \) is étale locally equivalent to the cusp in the versal deformation space of the cusp. \( \square \)

**Lemma 4.** Let \( [\mu] \in \hat{C}_d \) (resp. \( x \in C_d \)) be a point corresponding to an irreducible, nodal, rational curve with a node at \( p_1 \) and nodes distinct from \( p_2, \ldots, p_{3d-2} \). \( \hat{C}_d \) is nodal at \( [\mu] \) (resp. \( C_d \) is nodal at \( x \)).

**Proof.** If suffices to prove the result for \( \hat{C}_d \). The divisor \( D_1 \subset M_{0,0}(\mathbb{P}^2, d) \) corresponding to curves passing through the point \( p_1 \) has two nonsingular branches with a normal crossings intersection at \( [\mu] \). Let \( r, s \in \mathbb{P}^1 \) lie over \( p_1 \in \mathbb{P}^2 \). The two nonsingular branches have the following tangent spaces at \( X \):
\[ H^0(\mathbb{P}^1, \text{Norm}(-r), H^0(\mathbb{P}^1, \text{Norm}(-s)). \]
The remaining \( 3d - 3 \) points impose independent conditions on each of these tangent spaces. Étale locally at \( [\mu] \), \( \hat{C}_d \) is the intersection of the union of linear spaces of dimensions \( 3d - 2 \) meeting along a subspace of dimension \( 3d - 3 \) with general linear space of codimension \( 3d - 3 \). Hence, \( \hat{C}_d \) is nodal at \( [\mu] \). \( \square \)

**Lemma 5.** Let \( x \in C_d \) be a point corresponding to the union of two irreducible, nodal, rational curves with degrees \( i \) and \( d - i \) meeting transversely with nodes (including component intersection points) distinct from \( p_1, \ldots, p_{3d-2} \). Also assume the components of degrees \( i, d - i \) contain \( 3i - 1, 3(d - i) - 1 \) points respectively. \( C_d \) has the singularity type of the coordinate axes at the origin in \( \mathbb{C}^2 \).

**Proof.** The nodes (including the intersections of the two components of \( X \)) and the points \( p_1, \ldots, p_{3d-2} \) necessarily impose \( (d + 1)(d + 2)/2 \)}
independent conditions on $H^0(P^2, \mathcal{O}_{P^2}(d))$. This independence can be established as follows. Let $\tilde{X}$ be the normalization of $X$ (note $\tilde{X}$ is the disjoint union of two $P^1$'s). Let $A \subset H^0(P^2, \mathcal{O}_{P^2}(d))$ be the linear series passing through all the nodes of $X$. There is an exact sequence of vector spaces 

$$0 \to C \to A \to H^0(\tilde{X}, \mathcal{O}_{P^2}(d - N)) \to \mathcal{N}.$$

As before, $N$ is the divisor preimage of the nodes of $X$. Certainly only a 1 dimensional subspace of $A$ corresponding to the equation of $X$ vanishes on $\tilde{X}$. Surjectivity of the above sequence follows by a dimension count:

$$\dim(A) \geq \frac{(d + 1)(d + 2)}{2} - \frac{(d - 1)(d - 2)}{2} - 1 = 3d - 1,$$

$$h^0(\tilde{X}, \mathcal{O}_{P^2}(d - N)) = d^2 - (d - 1)(d - 2) - 2 + 2 = 3d - 2.$$

The points $p_1, \ldots, p_{3d - 2}$ are distinct on $\tilde{X}$ and impose independent conditions on $H^0(\tilde{X}, \mathcal{O}_{P^2}(d - N))$ by the assumption of their distribution (and the fact $\tilde{X}$ is a disjoint union of $P^1$'s).

At $x \in \mathcal{S}(0, d)$, the closed Severi variety has $id - i^2$ nonsingular branches (one for each intersection point). Let $q \in P^2$ be an intersection point of the two components of $X$. The tangent space $T(q)$ to the branch of $\mathcal{S}(0, d)$ corresponding to $q$ is simply the linear subspace $T(q) \subset H^0(P^2, \mathcal{O}_{P^2}(d))$ of polynomials that vanish at all the nodes of $X$ besides $q$. Let $V \subset H^0(P^2, \mathcal{O}_{P^2}(d))$ be the linear subspace of polynomials that vanish at the non-intersection nodes of $X$ and the points $p_1, \ldots, p_{3d - 2}$. $\mathcal{C}_d$ at $x$ is étale locally equivalent to the intersection 

$$V \cap (T(q_1) \cup T(q_2) \cup \cdots \cup T(q_{id - i^2})).$$

Note $V \cong \mathcal{O}^{3 - i^2}$. Since the nodes of $X$ and the points $p_1, \ldots, p_{3d - 2}$ impose independent conditions on $H^0(P^2, \mathcal{O}_{P^2}(d))$, the Lemma is proven.

The last case to be consider is the analogue of Lemma (5) for $\hat{C}_d$: when $[\mu] \in \hat{C}_d$ corresponds to a map with reducible domain and image satisfying the conditions of (5). This case can be handled directly. However, it is easier to observe that at such $[\mu]$, $\overline{M}_{0,0}(P^2, d)$ is locally isomorphic to the nonsingular branch in the proof of Lemma (5) determined by the attaching point of the two components. The singularity analysis then shows $[\mu] \in \hat{C}_d$ is a nonsingular point.
For general points \( p_1, \ldots, p_{3d-2} \), every point \( x \in C_d, [\mu] \in \hat{C}_d \) corresponds to exactly one of the three cases covered by Lemmas (1-5). Hence the singularities of \( C_d, \hat{C}_d \) are established.

3.2. The Arithmetic Genus. Consider the moduli space \( \overline{M}_{0,0}(\mathbb{P}^2, d) \) for \( d \geq 3 \) (to avoid \([0, 0, r, d] = [0, 0, 2, 2]\)). For general points \( p_1, \ldots, p_{3d-2} \), the intersection cycle

\[
\hat{C}_d = H_\infty \cap \mathcal{H}_\varepsilon \cap \cdots \cap \mathcal{H}_{3\varepsilon - \varepsilon}
\]

is a curve in \( \overline{M}_{0,0}(\mathbb{P}^2, d) \). \( H_i \) is the divisor of maps passing through the point \( p_i \). By the analysis in section (3.1), \( \hat{C}_d \) is nonsingular except for nodes. The nodes occur precisely at the points \([\mu] \in \hat{C}_d \) corresponding to a nodal curve with a node at some \( p_i \). Since, for general points, \( \hat{C}_d \subset \overline{M}_{0,0}(\mathbb{P}^2, d) \), the arithmetic genus \( \hat{g}_d \) of \( \hat{C}_d \) can be determined by the formula for the canonical class and adjunction (\( d \geq 3 \)):

\[
2\hat{g}_d - 2 = (\mathcal{K}_{\overline{M}} + (\exists [-\varepsilon])H) \cdot \mathcal{H}^{3\varepsilon - \varepsilon}.
\]

A computation of these intersection numbers in terms of the numbers \( N_d \) yields for all \( d \geq 3 \):

\[
2\hat{g}_d - 2 = \frac{(2d - 3)(3d + 1)}{2d}N_d + \frac{1}{4d} \sum_{i=1}^{d-1} N_i N_{d-i} \left(3i^2(d-i)^2 - 2di(d-i)\right) \left(\frac{3d - 2}{3i - 1}\right).
\]

The natural map \( \hat{C}_d \to C_d \) is a partial desingularization. The arithmetic genus of \( \hat{C}_d \) differs from the arithmetic genus of \( \hat{C}_d \) only by the contribution of the singularities of Lemma (3) and (5). Consider first the cusps in \( C_d \) determined by Lemma (3). The number of these cusps is exactly the number of 1-cuspidal, degree \( d \), rational curves through \( 3d - 2 \) points in the plane. In [P2] it is shown there are

\[
\frac{3d - 3}{d}N_d + \frac{1}{2d} \sum_{i=1}^{d-1} N_i N_{d-i} \left(3i^2(d-i)^2 - 2di(d-i)\right) \left(\frac{3d - 2}{3i - 1}\right)
\]

1-cuspidal, degree \( d \), rational curves through \( 3d - 2 \) points. Each cusp contributes 1 to the arithmetic genus of \( C_d \). The singularities of Lemma (5) contribute

\[
\frac{1}{2} \sum_{i=1}^{d-1} N_i N_{d-i} \left(i(d-i) - 1\right) \left(\frac{3d - 2}{3i - 1}\right)
\]

to the arithmetic genus of \( C_d \). The formula for the arithmetic genus of \( C_d \) can be deduced by adding these contributions to the formula for \( \hat{g} \):
\[2g_d - 2 = \frac{6d^2 + 5d - 15}{2d}N_d + \frac{1}{4d} \sum_{i=1}^{d-1} N_i N_{d-i} \left(15i^2(d-i)^2 - 8di(d-i) - 4d\right) \left(\frac{3d - 2}{3i - 1}\right).\]

3.3. **The Geometric Genus.** The geometric genus, \(g(\tilde{\mathcal{C}}_d)\) is simple to compute. By Bertini’s Theorem, the curve \(\tilde{\mathcal{C}}_d\) determined in \(\overline{M}_{0,3d-2}(\mathbb{P}^2, d)\) by \(3d - 2\) general points is nonsingular and contained in the automorphism-free locus \(\overline{M}_{0,3d-2}(\mathbb{P}^2, d)\). There is a sequence of natural maps exhibiting \(\tilde{\mathcal{C}}_d\) as the normalization of both \(\hat{\mathcal{C}}_d\) and \(C_d\):

\[\tilde{\mathcal{C}}_d \to \hat{\mathcal{C}}_d \to C_d.\]

The genus of \(\tilde{\mathcal{C}}_d\) can be determined by the formula for the canonical class and adjunction:

\[2\tilde{g}_d - 2 = \left(\frac{K_{\overline{M}} + \sum_{\epsilon=0}^{\exists-\epsilon} \mathcal{L}_{\epsilon}}{\infty} \cdot \frac{\mathcal{L}_{\infty}}{\infty} \cdot \frac{\mathcal{L}_{e|\epsilon}}{\infty} \right)\]

\[= K_{\overline{M}} \cdot \frac{\mathcal{L}_{\infty}}{\infty} \cdot \frac{\mathcal{L}_{e|\epsilon}}{\infty}.\]

A computation of these intersection numbers in terms of the numbers \(N_d\) yields for all \(d \geq 1\):

\[2\tilde{g}_d - 2 = \frac{-3d^2 - 3d + 4}{2d^2} N_d + \frac{1}{4d^2} \sum_{i=1}^{d-1} N_i N_{d-i} (id-i^2) \left((9d+4)i(d-i) - 6d^2\right) \left(\frac{3d - 2}{3i - 1}\right).\]

3.4. **The difference** \(\hat{g}_d - \tilde{g}_d\). Let \(d \geq 3\). The natural map \(\hat{\mathcal{C}}_d \to \tilde{\mathcal{C}}_d\) is a desingularization. \(\tilde{\mathcal{C}}_d\) has only nodal singularity. The difference, \(\hat{g}_d - \tilde{g}_d\), equals the number of nodes of \(\tilde{\mathcal{C}}_d\). Let \(M_d\) be the number of irreducible, nodal, rational degree \(d\) plane curves with a node at a fixed point and passing through \(3d - 3\) general point in \(\mathbb{P}^2\). From the description of the nodes of \(\tilde{\mathcal{C}}_d\), it follows:

\[\hat{g}_d - \tilde{g}_d = (3d - 2) M_d.\]

Values for low degree \(d\) are tabulated below:
The formula for $M_d$ (for $d \geq 3$) is:

$$M_d = \frac{d^2 - 1}{d^2} N_d - \frac{1}{4d^2} \sum_{i=1}^{d-1} N_i N_{d-i} (i^2 - i)(6d + 4)(d - i - 2d^2) \left( \frac{3d - 2}{3i - 1} \right).$$

An alternative method of computing $g_3$ is the following. $S(0,3)$ is simply the degree 12 discriminant hypersurface in the linear system of plane cubics. Therefore, $C_3$ is a degree 12 plane curve of arithmetic genus $11 \cdot 10/2 = 55$. In fact, $C_3$ has 24 cusp, 28 nodes, and geometric genus 3.

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Department of Mathematics, University of Chicago

rahl@math.uchicago.edu