Sketching for Kronecker Product Regression and P-splines

Huaian Diao
hadiao@nenu.edu.cn
Northeast Normal University

Zhao Song
zhaos@utexas.edu
Harvard U & UT-Austin

Abstract

TENSORSKETCH is an oblivious linear sketch introduced in (Pagh, 2013) and later used in (Pham and Pagh, 2013) in the context of SVMs for polynomial kernels. It was shown in (Avron et al., 2014) that TENSORSKETCH provides a subspace embedding, and therefore can be used for canonical correlation analysis, low rank approximation, and principal component regression for the polynomial kernel. We take TENSORSKETCH outside of the context of polynomials kernels, and show its utility in applications in which the underlying design matrix is a Kronecker product of smaller matrices. This allows us to solve Kronecker product regression and non-negative Kronecker product regression, as well as regularized spline regression. Our main technical result is then in extending TENSORSKETCH to other norms. That is, TENSORSKETCH only provides input sparsity time for Kronecker product regression with respect to the 2-norm. We show how to solve Kronecker product regression with respect to the 1-norm in time sublinear in the time required for computing the Kronecker product, as well as for more general p-norms.

1 INTRODUCTION

In the overconstrained least squares regression problem, we are given an $n \times d$ matrix $A$ called the “design matrix”, $n \gg d$, and an $n \times 1$ vector $b$, and the goal is to find an $x$ which minimizes $\|Ax - b\|_2$. There are many variants to this problem, such as regularized versions of the problem in which one seeks an $x$ so as to minimize $\|Ax - b\|_2 + \|Lx\|_2^2$ for a matrix $L$, or regression problems which seek to minimize more robust loss functions, such as $\ell_1$-regression $\|Ax - b\|_1$.

In the era of big data, large scale matrix computations have attracted considerable interest. To obtain reasonable computational and time complexities for large scale matrix computations, a number of randomized approximation algorithms have been developed. For example, in (Woodruff, 2014), it was shown how to output a vector $x' \in \mathbb{R}^d$ for which $\|Ax' - b\|_2 \leq (1 + \epsilon) \min_{x \in \mathbb{R}^d} \|Ax - b\|_2$ in $\text{nnz}(A) + \text{poly}(d/\epsilon)$ time, where $\text{nnz}(A)$ denotes the number of non-zero entries of matrix $A$. We refer the reader to the recent surveys (Kannan and Vempala, 2009; Mahoney, 2011; Woodruff, 2014) for a detailed treatment of this topic.

In this work we focus on regression problems for which the design matrix is a Kronecker product matrix, that is, it has the form $A \otimes B$ for $A \in \mathbb{R}^{n_1 \times d_1}$ and $B \in \mathbb{R}^{n_2 \times d_2}$. Also, $b \in \mathbb{R}^n$, and one seeks to solve the problem $\min_{x \in \mathbb{R}^{d_1 \times d_2}} \| (A \otimes B)x - b \|_2$ in the standard setting, which can also be generalized to regularized and robust variants. One can also ask the question when the design matrix is a Kronecker product of more than two matrices.

Kronecker product matrices have many applications in statistics, linear system theory, signal processing, photogrammetry, multivariate data fitting, etc.; see (Golub and Van Loan, 2013; Van Loan and Pitsianis, 1993; Van Loan, 1992). The linear least squares problem involving the Kronecker product arises in many applications, such as structured linear total least norm on blind deconvolution problems (Oh and Yun, 2005), constrained linear least squares problems with a Kronecker product structure, the bivariate problem of surface fitting and multidimensional density smoothing (Eilers and Marx, 2006).

One way to solve Kronecker product regression is to form the matrix $C = A \otimes B$ explicitly, where $A \in \mathbb{R}^{n_1 \times d_1}, B \in \mathbb{R}^{n_2 \times d_2}$, and then apply a randomized algorithm to $C$. However, this takes at least $\text{nnz}(A) \cdot \text{nnz}(B)$ time and space. It is natural
to ask if it is possible to solve the Kronecker product regression problem in time faster than computing $A \otimes B$. This is in fact the case, as Faussett and Fulton (Faussett and Fulton, 1994) show that one can solve Kronecker product regression in $O(n_1 d_1^2 + n_2 d_2^2)$ time; indeed, the solution vector $x = \text{vec}((B^\top)^\top D^{-1} A^\top)$, where $D = \text{vec}(b)$ and $E$ for a matrix $E$ denotes the operation of stacking the columns of $E$ into a single long vector. While such a computation does not involve computing $A \otimes B$, it is more expensive than what one would like.

A natural question is if one can approximately solve Kronecker product regression in $\text{nnz}(A) + \text{nnz}(B) + \text{poly}(d/e)$ time, which, up to the poly$(d/e)$ term, would match the description size of the input. Another natural question is Kronecker product regression with regularization. Such regression problems arise frequently in the context of splines (Eilers and Marx, 2006). Finally, what about Kronecker product regression with other, more robust, norms such as the $\ell_1$-norm?

1.1 Our Contributions

We first observe that the random linear map TENSORSKETCH, introduced in the context of problems for the polynomial kernel by Pagh (2013), Pham and Pagh (2013), is exactly suited for this task. Namely, in (Avron et al., 2014) it was shown that for a $d$-dimensional subspace $C$ of $\mathbb{R}^n$, represented as an $n \times d$ matrix, there is a distribution on linear maps $S$ with $O(d^2/\epsilon^2)$ rows such that with constant probability, simultaneously for all $x \in \mathbb{R}^d$, $\|SCx\|_2 = (1 + \epsilon)\|Cx\|_2$. That is, $S$ provides an Oblivious Subspace Embedding (OSE) for the column span of $C$. Further, it is known that if $b$ is an $n$-dimensional vector, then one has that $\|S[C,b]x\|_2 = (1 + \epsilon)\|C[b]x\|_2$ for all $x \in \mathbb{R}^{d+1}$. Consequently, to solve the regression problem $\min_{x \in \mathbb{R}^d} \|Cx - b\|_2$, one can instead solve the much smaller problem $\min_{x \in \mathbb{R}^d} \|SCx - Sb\|_2$, and the solution $x'$ to the latter problem will satisfy $\|Cx' - b\|_2 \leq (1 + \epsilon) \min_{x \in \mathbb{R}^d} \|Cx - b\|_2$.

Importantly, if $n = n_1 \cdot n_2$ and there is a basis for the column span of $C$ of the form $A_1 \otimes A_2$, $A_2 \otimes A_3$, . . . , $A_d \otimes A_{d+1}$, then given $A_1, \ldots, A_d$, it holds that $SC$ can be computed in $\text{nnz}(A)$ time, where $A$ is the $n_1 \times d$ matrix whose $i$-th column is $A_i$. Further, given a vector $b$ with $\text{nnz}(b)$ non-zero entries, one can compute $Sb$ in $\text{nnz}(b)$ time. Thus, one obtains a $(1 + \epsilon)$-approximate solution to the regression $\min_{x \in \mathbb{R}^d} \|Cx - b\|_2$ in $\text{nnz}(A) + \text{nnz}(b) + \text{poly}(d/e)$ time.

While not immediately useful for our problem, we show that via simple modifications, the claim about TENSORSKETCH above can be generalized to the case when there is a basis for the column span of $C$ of the form $A_i \otimes B_j$ for arbitrary vectors $A_1, \ldots, A_d, B_1, \ldots, B_d \in \mathbb{R}^{n_1}$ and vectors $B_1, \ldots, B_d \in \mathbb{R}^{n_2}$. That is, we observe that in this case $SC$ can be computed in $\text{nnz}(A) + \text{nnz}(B)$ time, where $A$ is the $n_1 \times d_1$ matrix whose $i$-th column is $A_i$, and $B$ is the $n_2 \times d_2$ matrix whose $i$-th column is $B_i$. In this case $C = A \otimes B$, which is exactly the case of Kronecker product regression. Using the above connection to regression, we obtain an algorithm for Kronecker product regression in $\text{nnz}(A) + \text{nnz}(B) + \text{poly}(d_1d_2/e)$ time. Using the fact that $\|SCx - Sb\|_2 = (1 + \epsilon)\|Cx - b\|_2$ for all $x$, we have in particular that this holds for all non-negative $x$, and so also obtain the same reduction in problem size for non-negative Kronecker product regression, which occurs often in image and text data; see e.g., (Chen and Plemmons, 2010).

The above observation allows us to extend many existing variants of least squares regression to the case when $C$ is a Kronecker product of two matrices. For example, the results in (Avron et al., 2016) for the ridge regression problem $\min_{x \in \mathbb{R}^d} \|Ax - b\|_2^2 + \lambda \|Lx\|_2^2$, where $L$ is an arbitrary matrix. Such problems occur in the context of spline regression (Eilers and Marx, 1996, 2006; Eilers et al., 2015). The number of rows of TENSORSKETCH depends on a generalized notion of statistical dimension depending on the generalized singular values $\gamma_i$ of $[A; L]$, and may be much smaller than $d_1$ or $d_2$. (Avron et al., 2016), only results for $L$ equal to the identity were obtained.

Finally, our main technical result is to extend our results to least absolute deviation Kronecker product regression $\min_{x \in \mathbb{R}^d} \|Cx - b\|_1$, which, since it involves the 1-norm, is often considered more robust than least squares regression (Rousseeuw and Leroy, 2005) and has been widely used in applications such as computer vision (Zheng et al., 2012). We in fact extend this to general $p$-norms but focus the discussion on $p = 1$. We give the first algorithms for this problem that are faster than computing $C = A \otimes B$ explicitly, which would take at least $\text{nnz}(A) + \text{nnz}(B) \geq n_1n_2$ time. Namely, and for simplicity focusing on the case when $n_1 = n_2$, for which the goal is to do better than $n_1^2$ time, we show how to output an $x \in \mathbb{R}^{d_1 \times d_2}$ for which $\|Cx' - b\|_1 \leq (1 + \epsilon) \min_{x \in \mathbb{R}^{d_1 \times d_2}} \|Cx - b\|_1$ in time $n^{3/2} \text{poly}(d_1d_2/e)$. While this is more expensive than...
solving Kronecker product least squares, the 1-norm may lead to significantly more robust solutions. From a technical perspective, TensorSketch when applied to a vector actually destroys the 1-norm of a vector, preserving only its 2-norm, so new ideas are needed. We show how to use multiple TensorSketch matrices to implicitly obtain very crude estimates to the so-called $\ell_1$-leverage scores of $C$, which can be interpreted as probabilities of sampling rows of $C$ in order to obtain an $\ell_1$-subspace embedding of the column span of $C$ (see, e.g., (Woodruff, 2014)). However, since $C$ has $n_1 n_2$ rows, we cannot even afford to write down the $\ell_1$-leverage scores of $C$. We show how to implicitly represent such leverage scores and to sample from them without ever explicitly writing them down. Balancing the phases of our algorithm leads to our overall time.

1.2 Notation

We consider Kronecker Product of $q$ 2-d matrices $A_1 \otimes A_2 \otimes ... \otimes A_q$, where each $A_i \in \mathbb{R}^{n_i \times d_i}$. We denote $A = A_1 \otimes A_2 \otimes ... \otimes A_q, n = \prod_{i=1}^{q} n_i$, and $d = \prod_{i=1}^{q} d_i$. We denote $[n]$ as the set $\{1, 2, 3, \ldots, n\}$. The $l_p$ norm for a vector $x \in \mathbb{R}^d$ is defined as $\|x\|_p = (\sum_{i=1}^{d} |x_i|^p)^{1/p}$, where $x_i$ stands for the $i$th entry of the vector $x$. For any matrix $M$, we use $M_{i,*}$ to represent the $i$th row of $M$ and $M_{*,j}$ as the $j$th column of $M$.

We define a Well-Conditioned Basis and Statistical Dimension as follow (similar definitions can be found in Clarkson (2005); Dasgupta et al. (2009); Sohler and Woodruff (2011); Meng and Mahoney (2013); Song et al. (2017a,b) and Avron et al. (2016)):

**Definition 1.1** (Well-Conditioned Basis). Let $A$ be an $n \times m$ matrix with rank $d$, let $p \in [1, \infty)$, and let $\| \cdot \|_p$ be the dual norm of $\| \cdot \|_p$, i.e., $1/p + 1/q = 1$. Then an $n \times d$ matrix $U$ is an $(\alpha, \beta, p)$-well-conditioned basis for the column space of $A$, if the columns of $U$ span the column space of $A$ and (1) $\|U\|_p \leq \alpha$, and (2) for all $z \in \mathbb{R}^d$, $\|z\|_q \leq \beta \|Uz\|_p$.

Consider the classic Ridge Regression: $\min_x \|Ax - b\|^2_2 + \lambda \|x\|^2_2$. The Statistical Dimension is defined as:

**Definition 1.2** (Statistical Dimension). Let $A$ be an $n \times m$ matrix with rank $d$ and singular values $\sigma_i, i \in [d]$. For $\lambda \geq 0$, the statistical dimension $sd_\lambda(A)$ is defined as the quantity $sd_\lambda(A) = \sum_{i \in [d]} 1/(1 + \lambda/\sigma_i^2)$.

2 BACKGROUND: TensorSketch

We briefly introduce TensorSketch (Pagh, 2013; Avron et al., 2014) and how to apply TensorSketch to the Kronecker product of multiple matrices efficiently without explicitly computing the tensor product.\(^1\)

We want to find a oblivious subspace embedding $S$ such that for any $x \in \mathbb{R}^n$, we have $\|SAx\|_2 = (1 \pm \epsilon)\|Ax\|_2$, where the notation $a = (1 \pm \epsilon)b$ stands for $(1 - \epsilon)b \leq a \leq (1 + \epsilon)b$ for any $a, b \in \mathbb{R}$. Consider the $(i_1, i_2, ..., i_q)$th column of $A$ ($i_j \in [d_j]$): $A_{i_1 \cdots i_q} \otimes A_{2 \cdots i_q} \cdots \otimes A_{q - i_q}$. Assume the sketching target dimension is $m$. TensorSketch is defined using $q$ $3$-wise independent hash functions $h_i : [n_i] \rightarrow [m]$, and $q$ $4$-wise independent sign functions $s_i : [n_i] \rightarrow \{-1, 1\}, \forall i \in [q]$. Define hash function $H : [n_1] \times [n_2] \times \cdots \times [n_q] \rightarrow [m]$ as $H(i_1, i_2, ..., i_q) = ((\sum_{k=1}^{q} h_k(i_k)) \mod m)$, and sign function $S : [n_1] \times [n_2] \times \cdots \times [n_q] \rightarrow \{-1, 1\}$ as $S(i_1, i_2, ..., i_q) = \prod_{i=1}^{q} s_k(i_k)$. Applying TensorSketch to the Kronecker product of vectors $A_{i_1 \cdots i_q}$ is equivalent to applying COUNTSketch (Charikar et al., 2004) defined with $H$ and $S$ to the vector $(A_{i_1 \cdots i_q})$. To apply TensorSketch to $A$, we just need to apply COUNTSketch defined with $H$ and $S$ to the columns of $A$ one by one.

Applying TensorSketch to the Kronecker product of $(A_{i_1 \cdots i_q})$ naively would require at least $O(n)$ time. Pagh (2013) shows that one can apply TensorSketch to the Kronecker product of $(A_{i_1 \cdots i_q})$ without explicitly computing the Kronecker product of these vectors using the Fast Fourier Transformation. Particularly, Pham and Pagh (2013) show that one only needs $O(\sum_{i=1}^{q} \text{nnz}(A_{i_1 \cdots i_q}) + qm \log(m))$ time to compute $S(A_{i_1 \cdots i_q} \otimes \cdots \otimes A_{q - i_q})$ where $A_{i_1 \cdots i_q}$ stands for the $j$th column of $A$. As $A$ has $d$ columns, computing $SA$ takes $O(d(\sum_{i=1}^{q} \text{nnz}(A_i)) + dqm \log(m))$ time, which is much smaller than $O(\prod_{i=1}^{q} n_i)$, which is the least amount of time one needs for explicitly computing $A$. In the rest of the paper, we assume that we compute $SA$ using the above efficient procedure without explicitly computing $A_{i_1 \cdots i_q}$.

3 TENSOR PRODUCT LEAST SQUARES REGRESSION

Consider the tensor product least squares regression problem $\min_x \|Ax - b\|^2_2$ where $A \in \mathbb{R}^{n \times d}$ and $b \in \mathbb{R}^n$. Let $S \in \mathbb{R}^{m \times n}$ be the matrix form of TensorSketch of Section 2. We propose a TensorSketch-type Algorithm 1 for the tensor product regression problem. The following theorem shows that the solution obtained from Alg. 1 is a good approximation of the optimal solution of the original tensor product regression.

Let us define OPT to be the optimal cost of the optimization problem, e.g., $\text{OPT} = \min_x \|Ax - b\|^2_2$. The following theorem shows that Alg. 1 computes a good approximate solution.

**Theorem 3.1.** (Tensor regression) Suppose $\bar{x}$ is the output of Algorithm 1 with TensorSketch $S \in \ldots$
Algorithm 1 Tensor product regression

1: procedure TREGRESSION(A, b, ϵ, δ)
2: \( m \leftarrow (d_1 d_2 \cdots d_q + 1)^2(2 + 3^q)/(\epsilon^2 \delta) \)
3: Choose \( S \) to be an \( m \times (n_1 n_2 \cdots n_q) \) TENSORSKETCH matrix
4: Compute \( S(A_1 \otimes A_2 \otimes \cdots \otimes A_q) \) and \( Sb \)
5: \( \hat{x} \leftarrow \min_x \|S(A_1 \otimes A_2 \otimes \cdots \otimes A_q)x - Sb\|_2 \)
6: return \( \hat{x} \)
7: end procedure

\( \mathbb{R}^{m \times n} \), where \( m = 8(d_1 d_2 \cdots d_q + 1)^2(2 + 3^q)/(\epsilon^2 \delta) \). Then the following approximation \( \|(A_1 \otimes A_2 \otimes \cdots \otimes A_q)x - b\|_2 \leq (1 + \epsilon) \text{OPT} \), holds with probability at least \( 1 - \delta \).

The proof of Theorem 3.1 can be found in Appendix C.1. Theorem 3.1 shows that we can achieve an \( \epsilon \)-close solution by solving a much smaller regression problem with a number of samples of order \( O(\text{poly}(d/\epsilon)) \), which is independent of the large dimension \( n \). Using the technique we introduced in Sec. 2, we can also compute \( SA \) without explicitly computing the tensor product.

We can extend Theorem 3.1 to the nonnegative tensor product regression problem \( \min_{x \geq 0} \|S(A_1 \otimes A_2 \otimes \cdots \otimes A_q)x - Sb\|_2 \). Suppose \( x \) is the optimal solution. Similarly, let \( S \in \mathbb{R}^{m \times (n_1 n_2 \cdots n_q)} \) be the matrix form of TENSORSKETCH of Section 2. If \( \hat{x} \) is the optimal solution to \( \min_{x \geq 0} \|S(A_1 \otimes A_2 \otimes \cdots \otimes A_q)x - Sb\|_2 \), we have the following:

**Corollary 3.2.** (Sketch for tensor nonnegative regression) Suppose \( \hat{x} = \min_{x \geq 0} \|SAx - Sb\|_2 \) with TENSORSKETCH \( S \in \mathbb{R}^{m \times n} \), where \( m = 8(d_1 d_2 \cdots d_q + 1)^2(2 + 3^q)/(\epsilon^2 \delta) \). Then the following approximation \( \|(A_1 \otimes A_2 \otimes \cdots \otimes A_q)x - b\|_2 \leq (1 + \epsilon) \text{OPT} \), holds with probability at least \( 1 - \delta \), where \( \text{OPT} = \min_{x \geq 0} \|(A_1 \otimes A_2 \otimes \cdots \otimes A_q)x - b\|_2 \).

The proof of Corollary 3.2 can be found in Appendix C.1.

4 **P-SPLINES**

B-splines are local basis functions, consisting of low degree (e.g., quadratic, cubic) polynomial segments. The positions where the segments join are called the knots. B-splines have local support and are thus suitable for smoothing and interpolating data with complex patterns. Unfortunately, control over smoothness is limited: one can only change the number and positions of the knots. If there are no reasons to assume that smoothness is non-uniform, the knots will be equally spaced and the only tuning parameter is their (discrete) number. In contrast P-spline (Eilers and Marx, 1996) equally spaces B-splines, discards the derivative completely, and controls smoothness by regularizing the sum of squares of differences of coefficients. Specifically Eilers and Marx (Eilers and Marx, 1996) proposed the P-spline recipe: (1) use a (quadratic or cubic) B-spline basis with a large number of knots, say 10-50; (2) introduce a penalty on (second or third order) differences of the B-spline coefficients; (3) minimize the resulting penalized likelihood function; (4) tune smoothness with the weight of the penalty, using cross-validation or AIC to determine the optimal weight.

We give a brief overview of B-Splines and P-Splines below. Let \( b \) and \( u \), each vectors of length \( n \), represent the observed and explanatory variables, respectively. Once a set of knots is chosen, the B-spline basis \( A \) follows from \( u \). If there are \( d \) basis functions then \( A \) is \( n \times d \). In the case of normally distributed observations the model is \( b = Ax + \epsilon \), with independent errors \( \epsilon \). In the case of B-spline regression the sum of squares of residuals \( \|b - Ax\|_2^2 \) is minimized and the normal equations \( A^T A \hat{x} = A^T b \) are obtained; the explicit solution \( \hat{x} = (A^T A)^{-1} A^T b \). The P-spline approach minimizes the penalized least-squares function

\[
\|b - Ax\|_2^2 + \lambda \|Lx\|_2^2, \tag{1}
\]

where \( L \in \mathbb{R}^{p \times n} \) is a matrix that forms differences of order \( \ell \), i.e., \( Lx = \Delta^\ell x \). Examples of this matrix, for \( \ell = 1 \) and \( \ell = 2 \) are :

\[
L_1 = \begin{bmatrix}
-1 & 1 & 0 & 0 \\
0 & -1 & 1 & 0 \\
0 & 0 & -1 & 1
\end{bmatrix}, \quad
L_2 = \begin{bmatrix}
1 & -2 & 1 & 0 \\
0 & 1 & -2 & 1 \\
0 & 0 & 1 & -2
\end{bmatrix}.
\]

The parameter \( \lambda \) determines the influence of the penalty. If \( \lambda \) is zero, we are back to B-spline regression; increasing \( \lambda \) makes \( \hat{x} \), and hence \( \hat{b} = A\hat{x} \), smoother.

Let \( x^* \) denote argmin\( x \in \mathbb{R}^n \|Ax - b\|_2^2 + \lambda \|Lx\|_2^2 \), and OPT denote \( \|Ax^* - b\|_2^2 + \lambda \|Lx^*\|_2^2 \). In general \( x^* = (A^T A + \lambda L^T L)^{-1} A^T b = A^T (A A^T + \lambda L L^T)^{-1} b \), but \( x^* \) can be found in \( O(mn \|A\| \min(n, d)) \) time using an iterative method (e.g., LSQR). Our first goal in this section is to design faster algorithms that find an approximate \( \hat{x} \) in the following sense:

\[
\|A\hat{x} - b\|_2^2 + \lambda \|L\hat{x}\|_2^2 \leq (1 + \epsilon) \text{OPT}. \tag{2}
\]

4.1 Sketching for P-Spline

We first introduce a new definition of Statistical Dimension that extends the statistical dimension defined for Ridge Regression (i.e., \( L \) is an identity matrix in Eq. 1) (Avron et al., 2016) to P-Spline regression.

The problem (1) can also be analyzed by generalized singular value decomposition (GSVD)
and defined as $sd$ matrix: Let $P$ be the con-

Theorem 4.3. There is a con-

Theorem 4.3. There is a con-

Lemma 4.1. Let $x^* \in \mathbb{R}^d$, $A \in \mathbb{R}^{n \times d}$ and $b \in \mathbb{R}^n$ as above. Let $U_1 \in \mathbb{R}^{n \times d}$ denote the first $n$ rows of an orthogonal basis for $[A \sqrt{n}] \in \mathbb{R}^{(n+p) \times d}$. Let sketching matrix $S \in \mathbb{R}^{m \times n}$ have a distribution such that with constant probability

$$\left( I \right) \|U_1^T S^T U_1 - U_1^T U_1\|_2 \leq 1/4,$$

and

$$\left( II \right) \|U_1^T (S^T - I)(b - Ax^*)\|_2 \leq \sqrt{\text{OPT}}/2.$$}

Let $\bar{x}$ denote $\arg \min_{x \in \mathbb{R}^d} \|S(Ax - b)\|_2^2 + \lambda \|Lx\|_2^2$. Then with probability at least $9/10$,

$$\|A\bar{x} - b\|_2^2 + \lambda \|L\bar{x}\|_2^2 \leq (1 + \epsilon) \text{OPT}.$$}

Define the statistical dimension for P-Splines as follows:

**Definition 4.2 (Statistical Dimension for P-Splines).** For $S$-Spline in Eq. (1), the statistical dimension is defined as $sd_{\lambda}(A, L) = \sum_1^\gamma 1/\left(1 + \lambda/\gamma_i^2\right) + d - p$.

The following theorem shows that there is a sparse subspace embedding matrix $S \in \mathbb{R}^{m \times n}$ (e.g., COUNTSKETCH), with $m \geq K \left(sd_{\lambda}(A, L)/\epsilon + sd_{\lambda}(A, L)^2\right)$, that satisfies Property (I) and (II) of Lemma 4.1, and hence achieves an $\epsilon$-approximation solution to problem 1:

**Theorem 4.3.** (P-Spline regression) There is a constant $K > 0$ such that for $m \geq K(\epsilon^{-1} \cdot sd_{\lambda}(A, L) + d)$ and $S \in \mathbb{R}^{m \times n}$ a sparse embedding matrix (e.g., COUNTSKETCH) with $SA$ computable in $O(nnz(A))$ time, Property (I) and (II) of Lemma 4.1 apply, and with constant probability the corresponding $\bar{x} = \arg \min_{x \in \mathbb{R}^d} \|S(Ax - b)\|_2^2 + \lambda \|Lx\|_2^2$ is an $\epsilon$-approximate solution to $\min_{x \in \mathbb{R}^d} \|b - Ax\|_2^2 + \lambda \|Lx\|_2^2$.

Note $sd_{\lambda}(A, L)$ is upper bounded by $d$. The above theorem shows that the statistical dimension allows us to design smaller sketch matrices whose size only depends on $O(\text{poly}(sd_{\lambda}(A, L)/\epsilon))$ instead of $O(\text{poly}(d/\epsilon))$, without sacrificing the approximation accuracy.

### 4.2 Tensor Sketching for Multi-Dimensional P-Spline

Tensor products allow a natural extension of one-dimensional P-spline smoothing to multi-dimensional P-Spline. We focus on 2-dimensional P-Spline but our results can be generalized to the multi-dimensional setting. Assume that in addition to $u$ we have a second explanatory variable $v$. We have data triples $(u_i, v_j, f(u_i, v_j))$ for $i = 1, \ldots, n_1$ and $j = 1, \ldots, n_2$. We seek a smooth surface $f(u, v)$ which gives a good approximation to the response $b$. Let $A_1, n \times d_1$, be a B-spline basis along $u$, and $A_2, n \times d_2$, be a B-spline basis along $v$. We form the tensor product basis as $A_1 \otimes A_2$. When $n_1$ and $n_2$ are large, we do not compute $A_1 \otimes A_2$. We apply Tensorsketch here to avoid explicitly forming $A_1 \otimes A_2$ to speed up computation.

Let $X = [x_{kl}]$ be a $d_1 \times d_2$ matrix of coefficients. Then, for given $X$, the value fit at $(u, v)$ is $f(u, v) = \sum_k \sum_l A_{2,k,l}(u)x_{kl}$ and so $X$ may be chosen using least squares by minimizing $\sum_{ij} [b_{(i-1)n_2 + j} - f(u_i, v_j)]^2 = \sum_{ij} [b_{(i-1)n_2 + j} - \sum_k A_{2,k,v_i}(u_i)x_{kl}]^2$. Using Krylov product, the above minimization can be written in the form $\min_{\bar{x}} \|b - Ax\|_2$, where $A_1 \otimes A_2 \in \mathbb{R}^{n_1n_2 \times d_1d_2}$ and $x = \text{vec}(X)$. Again the P-spline approach minimizes the penalized least-squares function $\|b - (A_1 \otimes A_2 x)^2\|_2 + \lambda \|Lx\|_2^2$. Consider the tensor p-spline regression problem $\min_{x} \|\bar{x}(A_1 \otimes A_2 \otimes \cdots \otimes A_q)x - b\|_2^2 + \lambda \|Lx\|_2^2$. Let $S \in \mathbb{R}^{m \times n}$ be the matrix form of Tensorsketch of Section 2.

### Algorithm 2 P-Spline Tensor product regression

1: **procedure** PTRegression($A, b, K, L, \epsilon, \delta$)
2:  
3:  
4:  
5:  
6:  
7:  **end procedure**
Algorithm 2 summarizes the procedure for efficiently solving multi-dimensional P-Spline.

Let $A = A_1 \otimes A_2 \otimes \cdots \otimes A_q$. Replacing the matrix $A$ in Theorem 4.3 with $A$, we have the following corollary for multi-dimensional P-Spline:

**Corollary 4.4 (P-Spline tensor regression).** Suppose $\lambda \leq \sigma_t^2 / \epsilon$. There is a constant $K > 0$ such that for $m \geq K (e^{-1} \sigma_t(A) L + \sigma_t(A) L^2)$ and $S \in \mathbb{R}^{m \times n}$ a TensorSketch matrix with $SA$ computable in $O(\text{nnz}(A))$ time, Property (I) and Property (II) of Lemma 4.1 apply, and with constant probability the corresponding $\tilde{x} = \text{argmin}_{x \in \mathbb{R}^n} \|A x - b\|_2 + \lambda \|L x\|_2^2$ is an $\epsilon$-approximate solution to $\text{min}_{x \in \mathbb{R}^n} \|b - A x\|_2^2 + \lambda \|L x\|_2^2$.

5 TENSOR PRODUCT ABSOLUTE DEVIATION REGRESSION

We extend our previous results for the $\ell_2$ norm (i.e., least squares regression) to general $\ell_q$ norms, with a focus on $p = 1$ (i.e., absolute deviation regression). Specifically we consider $\text{min}_x \|A x - b\|_p$, where $A = A_1 \otimes A_2 \otimes \cdots \otimes A_q$. We will show in this section that with probability at least $2/3$, we can quickly find an $\tilde{x}$ for which $\|A \tilde{x} - b\|_p \leq (1 + \epsilon) \text{min}_x \|A x - b\|_p$.

As in Clarkson and Woodruff (2013), for each $i$, in $O(\text{nnz}(A_i) \log(d_i)) + O(r_i^q)$ time we can replace the input matrix $A_i \in \mathbb{R}^{n_i \times d_i}$ with a new matrix with the same column space of $A_i$ and full column rank. We therefore assume $A$ has full rank in what follows.

Suppose $S$ is the TensorSketch matrix defined in Section 2. Let $w_i \in \mathbb{N}$ and assume $w_i | n_i$. Split $A_i$ into $n_i / w_i$ matrices $A_i^{(i)}$, $\ldots$, $A_i^{(n_i / w_i)}$, each $w_i \times d_i$, so that $A_i^{(j)}$ is the submatrix of $A_i$ indexed by the $j$-th block of $w_i$ rows. Note $A$ can be written as:

$$
\begin{bmatrix}
A_1^{(1)} \otimes A_2^{(1)} \otimes \cdots \otimes A_q^{(1)} \\
A_1^{(2)} \otimes A_2^{(2)} \otimes \cdots \otimes A_q^{(2)} \\
\vdots \\
A_{n_1 / w_1}^{(1)} \otimes A_{n_2 / w_2}^{(2)} \otimes \cdots \otimes A_{n_q / w_q}^{(q-1)} \otimes A_q^{(q)}
\end{bmatrix}
$$

For each $A_1^{(i)} \otimes A_2^{(2)} \otimes \cdots \otimes A_q^{(q)}$, we can use the TensorSketch matrix $S_{i_1 i_2 \cdots i_q} \in \mathbb{R}^{m \times \prod_{i=1}^q w_i}$, where we set $m \geq 100 \prod_{i=1}^q d_i^2 (2 + 3^q) / \epsilon^2$, such that with probability at least .99, $\|S_{i_1 i_2 \cdots i_q} A_1^{(i)} \otimes A_2^{(2)} \otimes \cdots \otimes A_q^{(q)} x\|_2 = (1 + \epsilon) \|A_1^{(i)} \otimes A_2^{(2)} \otimes \cdots \otimes A_q^{(q)} x\|_2$ simultaneously for all $x \in \mathbb{R}^{d_1 \cdots d_q}$ as $S_{i_1 i_2 \cdots i_q}$ is an oblivious subspace embedding (Lemma B.3) . Now we use Algorithm 4 from (Liang et al., 2014) to boost the success probability by computing $t = O(\log(1/\delta))$ independent TensorSketch products $S_{i_1 i_2 \cdots i_q} A_1^{(i_1)} \otimes A_2^{(i_2)} \otimes \cdots \otimes A_q^{(i_q)}$, $j = [t]$, each with only constant success probability, and then running a cross validation procedure similar to that in Algorithm 4 of Liang et al. (2014), to find one which succeeds with probability at least $1 - \delta$.

**Lemma 5.1 (Liang et al., 2014).** For $\delta, \epsilon \in (0, 1)$, let $t = O(\log(1/\delta))$ and $m \geq 100 \prod_{i=1}^q d_i^2 (2 + 3^q) / \epsilon^2$.

Running algorithm Algorithm 4 in (Liang et al., 2014) with parameters $t, m$, we can obtain a TENSORSKETCH $S \in \mathbb{R}^{m \times \prod_{i=1}^q w_i}$ such that with probability at least $1 - \delta$, $\|S_{i_1} A_1^{(1)} \otimes A_2^{(2)} \otimes \cdots \otimes A_q^{(q)} x\|_2 = (1 + \epsilon) \|A_1^{(1)} \otimes A_2^{(2)} \otimes \cdots \otimes A_q^{(q)} x\|_2$ for all $x \in \mathbb{R}^{d}$.

After computing a TENSORSKETCH $S_{i_1 i_2 \cdots i_q} \in \mathbb{R}^{m \times \prod_{i=1}^q w_i}$ using lemma 5.1 for each row-block $A_1^{(1)} \otimes \cdots \otimes A_q^{(q)}$, we compose a single TENSORSKETCH for $A$ as $S = \text{diag}(S_{1, \ldots, 1}, S_{1, i_2 \cdots i_q}, \ldots, S_{(n_1 / w_1), \ldots, (n_q / w_q)}) \in \mathbb{R}^{m \times \prod_{i=1}^q n_i / w_i \times \prod_{i=1}^q n_i}$, which is defined as:

$$
\begin{bmatrix}
S_{1, \ldots, 1} \\
\vdots \\
S_{1, i_2 \cdots i_q} \\
\vdots \\
S_{(n_1 / w_1), \ldots, (n_q / w_q)}
\end{bmatrix}
$$

where each block on the diagonal is from Lemma 5.1. Note that $A$ has in total $\prod_{i=1}^q n_i / w_i$ many blocks. Using Lemma 5.1 with a union bound over all blocks of $A$, we have the following theorem which shows $S$ is an oblivious subspace embedding for $A$ in $\ell_2$ norm:

**Theorem 5.2 ($\ell_2$ OSE for tensor matrices).** Given $\delta, \epsilon \in (0, 1)$, let $S \in \mathbb{R}^{m \times \prod_{i=1}^q n_i / w_i \times \prod_{i=1}^q n_i}$ denote the matrix that has $\prod_{i=1}^q n_i / w_i$ diagonal block matrices where each diagonal block $S_{i_1 i_2 \cdots i_q} \in \mathbb{R}^{m \times \prod_{i=1}^q w_i}$ is from Lemma 5.1. With probability at least $1 - \prod_{i=1}^q (n_i / w_i) \delta$, $\|S A x\|_2 = (1 + \epsilon) \|A x\|_2$, $\forall x \in \mathbb{R}^{d}$.

It is known that for any matrix $A \in \mathbb{R}^{n \times r}$, we can compute a change of basis $U \in \mathbb{R}^{r \times r}$ such that $AU$ is an $(\alpha, \beta, p)$ well-conditioned basis of $A$ (see Definition 1.1), in time polynomial with respect to $n, r$ (Dasgupta et al., 2009). Specifically, Theorem 5 in (Dasgupta et al., 2009) shows that we can compute a change of basis $U$ for which $AU$ is a well-conditioned basis of $A$ in time $O(nr^2 \log(n))$ time. However we cannot afford to directly use the results from (Dasgupta et al., 2009) to compute a well-conditioned basis for $A$, which requires time at least $\Omega(n)$. Instead we compute a well-conditioned basis for $A$ through a sketch $SA$ where $S$ is the tensorSketch from Theorem 5.2.

Specifically we define the following procedure:

1. Compute $SA$; 2. Compute a $d \times d$ change of basis matrix $U$
so that $SAU$ is an $(\alpha, \beta, p)$-well-conditioned basis of the column space of $S; A$; 3) Output $AU/(d\gamma_p)$, where
\[ \gamma_p = \sqrt{2^{1/p-1/2}} \text{ for } p \leq 2, \text{ and } \gamma_p = \sqrt{2^{w_1/2-1/p}} \text{ for } p > 2, \]
where $t = \prod_{i=1}^p n_i$ and $w = \prod_{i=1}^p w_i$.
The following Lemma 5.3 (the proof can be found in the Appendix) is the analogue of that in Clarkson et al. (2013) proved for the Fast Johnson Lindenstrauss Transform.

**Lemma 5.3.** For any $p \geq 1$, Condition$(A)$ computes $AU/(d\gamma_p)$ which is an $(\alpha, \beta, \sqrt{3d}(tw)_{1/p-1/2}, p)$-well-conditioned basis of $A$, with probability at least $1 - \prod_{i=1}^p (n_i/w_i)\delta$.

Lemma 5.3 indicates that Condition$(A)$ computes a $(\alpha, \beta, \text{poly}(\max(d, \log n)), p)$-well-conditioned basis. A well-conditioned basis can be used to solve $\ell_p$ regression problems, via sampling a subset of rows of the well-conditioned basis $AU$ with probabilities proportional to the $p$-th power of the $\ell_p$ norm of the rows (Woodruff, 2014). However the first issue for sampling is that we cannot afford to compute $AU$ as this requires $O(mnz(A)d)$ time. To fix this, we apply a Gaussian sketch matrix $G \in \mathbb{R}^{d \times O(\log(n))}$ with i.i.d normal random variables (Drineas et al., 2011) on the right hand side of $AU$. Note that $AUG$ can be computed efficiently by first computing $UG$ and then $A(UG)$ in time $O(d^2 \log(n) + nzz(A) \log(n))$. The second issue is that even with $AUG$, computing the $\ell_p$ norm of each row of $AUG$ takes $O(n)$ time, but we want sublinear time. This leads us to the following sampling technique.

The high level idea is that since $AUG$ only has $O(\log(n))$ columns, we can afford to sample columns of $AUG$ with probability proportional to the $p$-th power of the $\ell_p$ norms of the columns, if we can efficiently estimate the $\ell_p$ norms of the columns (note that naively computing the $\ell_p$ norm of a column also takes $O(n)$ time). Let us denote the first of column of $UG$ as $\mathcal{E} \in \mathbb{R}^{d \times O(\log(n))}$. We focus on how to efficiently estimate the $\ell_p$ norm of the first column of $AUG$, which is $\mathcal{E}$, and all the left columns can be estimated in the same way. We first reshape the vector $A$ into a 2-d matrix

$$ M = A_1 \otimes \ldots \otimes A_q \mathcal{E}(A_{q_1+1} \otimes \ldots \otimes A_q)^T, $$

(3)

where $M \in \mathbb{R}^{(n_1n_2\ldots n_q)(n_{q_1+1}\ldots n_q)}$, $E$ is obtained from reshaping $e$ into a $(d_1d_2\ldots d_{q_1}) \times (d_{q_1+1}\ldots d_q)$ matrix and $q_1 \in [1, q]$ is chosen such that $(n_1n_2\ldots n_q) \approx (n_{q_1+1}\ldots n_q)$. Namely we reshape the column $A$ into a (nearly) square matrix. Focusing now on $p = 1$, note that $\|AE\|_1 = \sum_{i=1}^{n_{q_1+1}} \cdots n_q \|M_{i,\mathcal{E}}\|_1$. Hence to estimate $\|AE\|_1$, we only need to estimate the $\ell_1$ norm of the columns of $M$.

Let us apply a sketch matrix $R \in \mathbb{R}^{O(\log(n)) \times \prod_{i=1}^q n_i}$, whose entries are sampled i.i.d. from the Cauchy distribution, to the left hand side of $M$. For the $i$-th column of $M$, let us define random variables $z_i^l = R_{i,\mathcal{E}}^TM_{i,\mathcal{E}}$, for $l \in [O(\log(n))]$, where $R_{i,\mathcal{E}}$ is the $l$-th row of $R$.

**Algorithm 3** $\ell_1$ tensor product regression

1. **procedure** LITRREGRESSION($A, b, \epsilon, \delta$)
2. Construct a TENSORSKETCH $S \in \mathbb{R}^{(m\prod_{i=1}^q n_i) \times n}$.
3. Run Condition$(A)$ using $S$ to compute $U/(d\gamma_p)$.
4. Generate a Gaussian matrix $G \in \mathbb{R}^{d \times O(\log(n))}\text{ and a Cauchy sketch matrix } R \in \mathbb{R}^{\log(n) \times n}$.
5. **for** each column $e$ in $U/\mathcal{G}$ **do**
6. Reshape $\mathcal{E} \in S$ to $M$ (Eq. 3).
7. Compute $\lambda_i$ and $\lambda_e = \sum_i \lambda_i$ (Eq 4 and 5).
8. **end for**
9. **for** $i \in [\sqrt{\prod_{i=1}^q w_i \text{poly}(d)}]$ **do**
10. Sample a column $(AUG)_{i,\mathcal{E}}$ with probability proportional to $\lambda_{\mathcal{E}}$. \text{ (Eq. 3 and 5)}
11. Reshape $(AUG)_{i,\mathcal{E}}$ to $M$ with $M_{k,j}$ having probability proportional to $\lambda_j$.
12. Sample an entry $kk_j$ with probability proportional to $\lambda_{kk_j}$.
13. Convert $(k, j)$ back to the corresponding row index in $A$, denoted as $r_j$.
14. **end for**
15. $x \leftarrow \min_{\text{for } r_1, r_2, \ldots, r_N \text{th rows of } A} \frac{\|D(A_1 \otimes A_2 \otimes \cdots \otimes A_q) x - Db\|_1}{\sqrt{\prod_{i=1}^q w_i \text{poly}(d)}}$.
16. **return** $x$.
17. **end procedure**

Note that since $AUG$ only has $O(\log(n))$ columns, we can afford to compute the $\ell_1$ norm of all the columns using the above procedure. Let us denote the $\ell_1$ norms of the columns of $AUG$ by $\lambda_1, \lambda_2, \ldots, \lambda_{\log(n)}$. We can sample a column $(AUG)_{i,\mathcal{E}}$ with probability proportional to $\lambda_i$ (Line 10 in Alg. 3). Once we sample a column $(AUG)_{i,\mathcal{E}}$, we need to sample an entry $j \in [n]$
with probability proportional to the absolute value of the entry $|\langle A\mathbf{U}G\rangle_{i,j}|$. As we cannot afford to compute $|\langle A\mathbf{U}G\rangle_{i,j}|$ for all $j \in [n]$, we use the reshaped 2-d matrix $M$ of $\langle A\mathbf{U}G\rangle_{i,j}$. Note that sampling an entry in $M$ with probability proportional to the absolute values of entries of $M$ is equivalent to sampling an entry $j \in [n]$ from $\langle A\mathbf{U}G\rangle_{i,j}$ with probability proportional to the absolute value of the entries of $\langle A\mathbf{U}G\rangle_{i,j}$. We first sample a column $M_{i,j}$ from all the columns of $M$ with probability proportional to $\lambda_j$ for $j \in \prod_{i=1}^{q_1+1} n_i$. We then sample an entry $M_{k,j}$ with probability proportional to $|M_{k,j}|$ for $k \in \prod_{i=1}^{p}\prod_{i=1}^{q_i} n_i$. Noting that $k \in \prod_{i=1}^{p}\prod_{i=1}^{q_i} n_i$ and $j \in \prod_{i=1}^{q_i} n_i$, the pair $(k,j)$ uniquely determines a corresponding row index $r$ in $A$, for some $r \in \prod_{i=1}^{q_i} n_i$. Hence we successfully sample a row from $\mathbf{A}\mathbf{U}\mathbf{G}$ without ever computing the $\ell_p$ norm of the rows. The above sampling procedure is summarized in Line 10 to Line 13 in Alg. 3. We use the above procedure to sample $\prod_{i=1}^{q_i} n_i \log(d)$ rows of $A$. Let $D$ be a diagonal matrix that selects the corresponding sampled rows from $\mathbf{A}$. We can now solve a smaller ADL problem as $\min_{\hat{\mathbf{x}}} \|\mathbf{D} \hat{\mathbf{x}} - \mathbf{D} \mathbf{b}\|_1$. Note that our analysis focuses on the $\ell_1$ norm. We can extend the analysis to general $\ell_p$ norms by using a sketching matrix $R \in \mathbb{R}^{O((\log n) \times \prod_{i=1}^{q_i} n_i)}$ with entries sampled i.i.d. from a $p$-stable distribution for $p \in [1,2]$.

We now present our main theorem and defer the proofs to the appendix:

**Theorem 5.4.** (Main result) Given $\epsilon \in (0,1)$, $\mathbf{A} \in \mathbb{R}^{n \times d}$ and $\mathbf{b} \in \mathbb{R}^n$, Alg. 3 computes $\hat{\mathbf{x}}$ such that

$$\|\mathbf{A}\hat{\mathbf{x}} - \mathbf{b}\|_1 \leq (1 + \epsilon) \min_{\mathbf{x} \in \mathbb{R}^n} \|\mathbf{A}\mathbf{x} - \mathbf{b}\|_1$$

For the special case when $q = 2$, $n_1 = n_2$, the algorithm’s running time is $O(n_1^{3/2} \log(d))$, which is faster than $O(n_1^2)$—the time needed for forming $\mathbf{A}_1 \otimes \mathbf{A}_2$. Note that we can run Alg. 3 $O(\log(1/\delta))$ times independently and pick the best solution among these independent runs to boost the success probability to be $1 - \delta$, for $\delta \in [0,1)$.

### 6 NUMERICAL EXPERIMENTS

We generate matrices $\mathbf{A}_1$, $\mathbf{A}_2$ and $\mathbf{b}$ with all entries sampled i.i.d. from a normal distribution. The baseline we compared to is directly solving regression without sketching. We let $T_1$ be the time for directly solving the regression problem, and $T_2$ be the time of our algorithm. The time ratio is $r_t = T_2/T_1$. The relative residual percentage is defined by $r_c = \frac{100 \|\mathbf{A}_1 \otimes \mathbf{A}_2 \bar{x} - \mathbf{b}\|_2}{\|\mathbf{A}_1 \otimes \mathbf{A}_2 x^* - \mathbf{b}\|_2}$, where $\bar{x}$ is the output of our algorithms and $x^*$ is the optimal solution.

| $m$ | $r_c$ | $r_t$ | $r_c$ | $m$ | $r_c$ | $r_t$ |
|-----|--------|--------|--------|-----|--------|--------|
| $\ell_2$ | 8000 | 0.11 | 0.11 | $\ell_2$ | 12000 | 1.89% | 0.06 |
| 12000 | 1.24% | 0.18 | 12000 | 1.33% | 0.11 |
| 16000 | 1.01% | 0.25 | 16000 | 0.992% | 0.18 |

Throughout the simulations, we use a moderate input matrix size in order to accommodate the brute force algorithm and to compare to the exact solution.

**Example 6.1** ($\ell_2$ Regression). We create a design matrix with moderate size by fixing $n_1 = n_2 = 300$ and $d_1 = d_2 = 15$. Thus $\mathbf{A} \in \mathbb{R}^{8000 \times 450}$. We do 60 rounds to compute the mean values of $r_c$ and $r_t$, which is reported in Table 1 (Left).

From Table 1 (left), we can see that when the number $m$ of sampled rows in Algorithm 1 increases from 8000 to 12000, the mean values of $r_c$ decrease while the mean values of $r_t$ increase. In general we can see that we can achieve around 1% relative error while being 5 times faster than the direct method.

**Example 6.2** ($\ell_1$ Regression). We set $n_1 = n_2 = 300$, $d_1 = d_2 = 15$. For $\min_{\mathbf{x}} \|\mathbf{A}\mathbf{x} - \mathbf{b}\|_1$, we solve it by a Linear Programming solver in Gurobi (Gurobi Optimization, 2016). We tested different numbers of sampled rows $m$ (the number of rows in $D$ in Alg. 3). The results are summarized in Table 1 (Right).

As we can see from Table 1 (right), our method is around 10 times faster than directly solving the problem, with relative error only around 1%.

**Example 6.3** (P-Spline Regression). For P-spline regression, $L_3$ is fixed. We use 30 knots and cubic B-splines. The data $u = (u_l) \in \mathbb{R}^{n_1}$, $v = (v_l) \in \mathbb{R}^{n_2}$ and $\mathbf{b} \in \mathbb{R}^{n_1 \times n_2}$ are generated i.i.d. from the normal distribution. We compute the B-spline basis matrices $\mathbf{A}_1 \in \mathbb{R}^{n_1 \times d_1}$ and $\mathbf{A}_2 \in \mathbb{R}^{n_2 \times d_2}$ separately, if there are $d_1$ and $d_2$ basis functions for the B-spline. We set $n_1 = n_2 = 10^4$, $d_1 = d_2 = 529$. The original and sketched P-spline regression problem are both solved by REGULARIZATION TOOLS (Hansen, 1994) via computing their GSVDs. We test different choices of $\lambda$. The results are shown in Table 2.

From Table 2, sampling only 20% of the rows can give around 0.05% relative error, while the computation time is half of the time of directly solving the p-spline.

### 7 CONCLUSION

We propose algorithms for efficiently solving tensor product least squares regression, regularized P-splines, as well as tensor product least absolute deviation regression, using sketching techniques. Our main contributions are: (1) we apply Tensorsketch to least
Table 2: Example 6.3: the mean values of $r_e$ and $r_t$ with respect to different sampling parameters $m$ and regularization parameters $\lambda$.

| $\lambda$ | $m$  | $r_e$  | $r_t$  |
|-----------|------|--------|--------|
| 1         | 2000 | 4.43e-2% | 0.52   |
|          | 4000 | 2.99e-2% | 0.70   |
|          | 6000 | 7.92e-2% | 0.91   |
| 0.1      | 2000 | 6.98e-2% | 0.47   |
|          | 4000 | 4.07e-2% | 0.72   |
|          | 6000 | 5.41e-2% | 0.94   |
| 0.01     | 2000 | 3.78e-2% | 0.46   |
|          | 4000 | 1.07e-1% | 0.71   |
|          | 6000 | 2.97e-2% | 0.95   |

square regression problems, (2) we propose new statistical dimension measures for P-splines, extending the previous statistical dimension defined only for classic Ridge regression, and (3) we extend Tensorsketch to $\ell_p$ norms and propose an algorithm that can solve tensor product $\ell_1$ regression in time sublinear in the time for explicitly computing the tensor product. Simulation results support our theorems and demonstrate that our algorithms are much faster than brute-force algorithms and can achieve approximate solutions that are close to optimal.

ACKNOWLEDGEMENT

Huaian Diao is supported in part by the Fundamental Research Funds for the Central Universities under the grant 2412017FZ007. Wen Sun is supported in part by Office of Naval Research contract N000141512365.

References

H. Avron, H. Nguyen, and D. Woodruff. Subspace embeddings for the polynomial kernel. In Advances in Neural Information Processing Systems (NIPS), pages 2258–2266, 2014.

H. Avron, K. L. Clarkson, and D. P. Woodruff. Sharper bounds for regression and low-rank approximation with regularization. CoRR, abs/1611.03225, 2016.

L. Carter and M. N. Wegman. Universal classes of hash functions. J. Comput. Syst. Sci., 18(2):143–154, 1979.

M. Charikar, K. Chen, and M. Farach-Colton. Finding frequent items in data streams. Theor. Comput. Sci., 312(1):3–15, 2004.

D. Chen and R. J. Plemmons. Nonnegativity constraints in numerical analysis. In The birth of numerical analysis, pages 109–139. World Sci. Publ., Hackensack, NJ, 2010.

K. Clarkson, P. Drineas, M. Magdon-Ismail, M. Mahoney, X. Meng, and D. P. Woodruff. The fast Cauchy transform and faster robust linear regression. In SODA, 2013.

K. L. Clarkson. Subgradient and sampling algorithms for $\ell_1$ regression. In Proceedings of the sixteenth annual ACM-SIAM symposium on Discrete algorithms (SODA), pages 257–266, 2005.

K. L. Clarkson and D. P. Woodruff. Low rank approximation and regression in input sparsity time. In Symposium on Theory of Computing Conference, STOC’13, Palo Alto, CA, USA, June 1-4, 2013, pages 81–90. https://arxiv.org/pdf/1207.6365, 2013.

A. Dasgupta, P. Drineas, B. Harb, R. Kumar, and M. W. Mahoney. Sampling algorithms and coresets for $\ell_p$ regression. SIAM J. Comput., 38(5):2060–2078, 2009.

P. Drineas, M. Magdon-Ismail, M. W. Mahoney, and D. P. Woodruff. Fast approximation of matrix coherence and statistical leverage. CoRR, abs/1109.3843, 2011.

P. H. Eilers and B. D. Marx. Multidimensional density smoothing with p-splines. In Proceedings of the 21st international workshop on statistical modelling, 2006.

P. H. C. Eilers and B. D. Marx. Flexible smoothing with B-splines and penalties. Statist. Sci., 11(2):89–121, 1996.

P. H. C. Eilers, B. D. Marx, and M. Durbán. Twenty years of P-splines. SORT, 39(2):149–186, 2015. ISSN 1696-2281.

D. W. Fausett and C. T. Fulton. Large least squares problems involving Kronecker products. SIAM J. Matrix Anal. Appl., 15(1):219–227, 1994.

G. H. Golub and C. F. Van Loan. Matrix Computations. Johns Hopkins Studies in the Mathematical Sciences. Johns Hopkins University Press, Baltimore, MD, 2013.

I. Gurobi Optimization. Gurobi optimizer reference manual, 2016. URL http://www.gurobi.com.

P. C. Hansen. Regularization tools: A matlab package for analysis and solution of discrete ill-posed problems. Numerical Algorithms, 6(1):1–35, 1994.

P. Indyk. Stable distributions, pseudorandom generators, embeddings, and data stream computation. J. ACM, 53(3):307–323, 2006.

R. Kannan and S. Vempala. Spectral algorithms. Foundations and Trends in Theoretical Computer Science, 4(3-4):157–288, 2009.
Sketching for Kronecker Product Regression and P-splines

Y. Liang, M.-F. F. Balcan, V. Kanchanapally, and D. Woodruff. Improved distributed principal component analysis. In Advances in Neural Information Processing Systems, pages 3113–3121, 2014.

M. W. Mahoney. Randomized algorithms for matrices and data. Foundations and Trends in Machine Learning, 3(2):123–224, 2011.

X. Meng and M. W. Mahoney. Low-distortion subspace embeddings in input-sparsity time and applications to robust linear regression. In Proceedings of the forty-fifth annual ACM symposium on Theory of computing, pages 91–100. ACM, https://arxiv.org/pdf/1210.3135, 2013.

J. Nelson and H. L. Nguyen. Osnap: Faster numerical linear algebra algorithms via sparser subspace embeddings. In 2013 IEEE 54th Annual Symposium on Foundations of Computer Science (FOCS), pages 117–126. IEEE, https://arxiv.org/pdf/1211.1002, 2013.

S. Oh, S. Kwon and J. Yun. A method for structured linear total least norm on blind deconvolution problem. Applied Mathematics and Computing, 19:151–164, 2005.

R. Pagh. Compressed matrix multiplication. ACM Trans. Comput. Theory, 5(3):9:1–9:17, 2013.

M. Patrascu and M. Thorup. The power of simple tabulation hashing. J. ACM, 59(3):14, 2012.

N. Pham and R. Pagh. Fast and scalable polynomial kernels via explicit feature maps. In Proceedings of the 19th ACM SIGKDD international conference on Knowledge discovery and data mining(KDD), pages 239–247. ACM, 2013.

P. J. Rousseuw and A. M. Leroy. Robust regression and outlier detection, volume 589. John wiley & sons, 2005.

C. Sohler and D. P. Woodruff. Subspace embeddings for the $\ell_1$-norm with applications. In Proceedings of the forty-third annual ACM symposium on Theory of computing (STOC), pages 755–764. ACM, 2011.

Z. Song, D. P. Woodruff, and P. Zhong. Low rank approximation with entrywise $\ell_1$-norm error. In Proceedings of the 49th Annual Symposium on the Theory of Computing (STOC). ACM, https://arxiv.org/pdf/1611.00898, 2017a.

Z. Song, D. P. Woodruff, and P. Zhong. Relative error tensor low rank approximation. arXiv preprint arXiv:1704.08246, 2017b.

C. Van Loan. Computational frameworks for the fast Fourier transform, volume 10 of Frontiers in Applied Mathematics. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1992.
A Background: CountSketch and TensorSketch

We now describe the TensorSketch transform. Let \( m \) be the target dimension. When applied to \( n \)-dimensional vectors, the transform is specified by a 2-wise independent hash function \( h : [n] \to [m] \) and a 2-wise independent sign function \( s : [n] \to \{-1, +1\} \). When applied to \( v \), the value at coordinate \( i \) of the output, \( i = 1, 2, \ldots, m \) is \( \sum_{j \in h^{-1}(i)} s(j)v_j \). Note that CountSketch can be represented as an \( m \times n \) matrix in which the \( j \)-th column contains a single non-zero entry \( s(j) \) in the \( h(j) \)-th row.

We now describe the TensorSketch transform. Suppose we are given points \( v_1, \ldots, v_q \in \mathbb{R}^n \), where \( q = \prod_{i=1}^q |v_i| \). The transform is specified using \( q \) 3-wise independent hash functions \( h_i : [n] \to [m] \), and \( q \) 4-wise independent sign functions \( s_i : [n] \to \{-1, +1\} \), where \( i = 1, \ldots, q \). TensorSketch applied to \( v_1, \ldots, v_q \) is then CountSketch applied to \( \phi(v_1, \ldots, v_q) \) with hash function \( H : [n_2 \cdots n_q] \to [m] \) and sign function \( S : [n_1 n_2 \cdots n_q] \to \{-1, +1\} \) defined as follows:

\[
H(i_1, \ldots, i_q) = h_1(i_1) + h_2(i_2) + \cdots + h_q(i_q) \mod m,
\]

and

\[
S(i_1, \ldots, i_q) = s_1(i_1) \cdot s_2(i_2) \cdots s_q(i_q),
\]

where \( i_j \in [n_j] \). It is well-known that if \( H \) is constructed this way, then it is 3-wise independent. Carter and Wegman (1979); Patrascu and Thorup (2012). Unlike the work of Pham and Pagh Pham and Pagh (2013), which only used that \( H \) was 2-wise independent, our analysis needs this stronger property of \( H \).

The TensorSketch transform can be applied to \( v_1, \ldots, v_q \) without computing \( \phi(v_1, \ldots, v_q) \) as follows. Let \( v_j = (v_{jm}) \in \mathbb{R}^n \). First, compute the polynomials

\[
p_{\ell}(x) = \sum_{i=0}^{B-1} x^i \sum_{j_k : h_k(j_k) = i} v_{jm} \cdot s_{\ell}(j_k),
\]

for \( \ell = 1, 2, \ldots, q \). A calculation Pagh (2013) shows

\[
\prod_{\ell=1}^q p_{\ell}(x) \mod (x^m - 1) = \sum_{i=0}^{B-1} x^i \sum_{(j_1, \ldots, j_q) : H(j_1, \ldots, j_q) = i} v_{j_1} \cdots v_{j_q} S(j_1, \ldots, j_q),
\]

that is, the coefficients of the product of the \( q \) polynomials \( (x^m - 1) \) form the value of TensorSketch\((v_1, \ldots, v_q)\). Pagh observed that this product of polynomials can be computed in \( O(qm \log m) \) time using the Fast Fourier Transform. As it takes \( O(q \max(nz(v_1))) \) time to form the \( q \) polynomials, the overall time to compute TensorSketch\((v)\) is \( O(q \max(nz(v_1)) + m \log m) \).

B TensorSketch is an Oblivious Subspace Embedding (OSE)

Let \( S \) be the \( m \times (n_1 n_2 \cdots n_q) \) matrix such that TensorSketch \((v_1, \ldots, v_q)\) is \( S \cdot \phi(v_1, \ldots, v_q) \) for a randomly selected TensorSketch. Notice that \( S \) is a random matrix. In the rest of the paper, we refer to such a matrix as a TensorSketch matrix with an appropriate number of rows, i.e., the number of hash buckets. We will show that \( S \) is an oblivious subspace embedding for subspaces in \( \mathbb{R}^{n_1 n_2 \cdots n_q} \) for appropriate values of \( m \). Notice that \( S \) has exactly one non-zero entry per column. The index of the non-zero in the column \((i_1, \ldots, i_q)\) is \( H(i_1, \ldots, i_q) = \sum_{j=1}^q h_j(i_j) \mod m \). Let \( \delta_{a,b} \) be the indicator random variable of whether \( S_{a,b} \) is non-zero. The sign of the non-zero entry in column \((i_1, \ldots, i_q)\) is \( S(i_1, \ldots, i_q) = \prod_{j=1}^q s_j(i_j) \). We show that the embedding matrix \( S \) of TensorSketch can be used to approximate matrix product and is an oblivious subspace embedding (OSE).

Theorem B.1. Let \( S \) be the \( m \times (n_1 n_2 \cdots n_q) \) matrix such that

\[
\text{TensorSketch}(v_1, \ldots, v_q)
\]

is \( S \cdot \phi(v_1, \ldots, v_q) \) for a randomly selected TensorSketch. The matrix \( S \) satisfies the following two properties.
1. (Approximate Matrix Product:) Let $A$ and $B$ be matrices with $n_1n_2 \cdots n_q$ rows. For $m \geq (2 + 3\gamma)/(\epsilon^2\delta)$, we have

$$\Pr_S[\|A^T S^T S B - A^T B\|_F^2 \leq \epsilon^2\|A\|_F^2\|B\|_F^2] \geq 1 - \delta.$$ 

2. (Subspace Embedding:) Consider a fixed $k$-dimensional subspace $V$. If $m \geq k^2(2 + 3\gamma)/(\epsilon^2\delta)$, then with probability at least $1 - \delta$, $\|Sx\| = (1 \pm \epsilon)\|x\|$ simultaneously for all $x \in V$.

We establish the theorem via two lemmas as in Avron et al. (2016). The first lemma proves the approximate matrix product property via a careful second moment analysis.

**Lemma B.2** (Approximate matrix product). Let $A$ and $B$ be matrices with $n_1n_2 \cdots n_q$ rows. For $m \geq (2 + 3\gamma)/(\epsilon^2\delta)$, we have

$$\Pr_S[\|A^T S^T S B - A^T B\|_F^2 \leq \epsilon^2\|A\|_F^2\|B\|_F^2] \geq 1 - \delta.$$ 

**Proof.** The proof follows that in Avron et al. (2016). Let $C = A^T S^T S B$. We have

$$C_{u,w} = \sum_{t=1}^m \sum_{i,j \in [n_1n_2 \cdots n_q]} S(i)S(j)\delta_{i,u}\delta_{t,j}A_{i,u}B_{j,w} = \sum_{t=1}^m \sum_{i \neq j \in [n_1n_2 \cdots n_q]} S(i)S(j)\delta_{t,i}\delta_{t,j}A_{i,u}B_{j,w} + (A^T B)_{u,w}.$$ 

Thus, $E[C_{u,w}] = (A^T B)_{u,w}$.

Next, we analyze $E[((C - A^T B)_{u,w})^2]$. We have

$$(C - A^T B)_{u,w} = \sum_{t_1, t_2 = 1}^m \sum_{i_1 \neq j_1, i_2 \neq j_2 \in [n_1n_2 \cdots n_q]} S(i_1)S(i_2)S(j_1)S(j_2)\delta_{t_1,i_1}\delta_{t_1,j_1}\delta_{i_2,i_2}\delta_{i_2,j_2}A_{i_1,u}A_{i_2,u}B_{j_1,w}B_{j_2,w}.$$ 

For a term in the summation on the right hand side to have a non-zero expectation, it must be the case that $E[S(i_1)S(i_2)S(j_1)S(j_2)] \neq 0$. Note that $S(i_1)S(i_2)S(j_1)S(j_2)$ is a product of random signs (possibly with multiplicities) where random signs in different coordinates in $\{1, \ldots, q\}$ are independent and they are 4-wise independent within each coordinate. Thus, $E[S(i_1)S(i_2)S(j_1)S(j_2)]$ is either 1 or 0. For the expectation to be 1, all random signs must appear with even multiplicities. In other words, in each of the $q$ coordinates, the 4 coordinates of $i_1, i_2, j_1, j_2$ must be the same number appearing 4 times or 2 distinct numbers, each appearing twice. All the subsequent claims in the proof regarding $i_1, i_2, j_1, j_2$ agreeing on some coordinates follow from this property.

Let $S_1$ be the set of coordinates where $i_1$ and $i_2$ agree. Note that $j_1$ and $j_2$ must also agree in all coordinates in $S_1$ by the above argument. Let $S_2 \subset [q] \setminus S_1$ be the coordinates among the remaining where $i_1$ and $j_1$ agree. Finally, let $S_3 = [q] \setminus (S_1 \cup S_2)$. All coordinates in $S_3$ of $i_1$ and $j_2$ must agree. Similarly as before, note that $i_2$ and $j_2$ agree on all coordinates in $S_2$ and $i_2$ and $j_1$ agree on all coordinates in $S_3$. We can rewrite $i_1 = (a, b, c), i_2 = (a, e, f), j_1 = (g, b, f), j_2 = (g, e, c)$ where $a = (a_\ell), g = (g_\ell)$ with $\ell \in S_1, b = (b_\ell), c = (e_\ell) with \ell \in S_2 and e = (f_\ell)$ with $\ell \in S_3$.

First we show that the contribution of the terms where $i_1 = i_2$ or $i_1 = j_2$ is bounded by $\frac{2\|A_u\|_F^2\|B_{u'}\|_F^2}{m}$, where $A_u$ is the $u$th column of $A$ and $B_{u'}$ is the $u'$th column of $B$. Indeed, consider the case $i_1 = i_2$. As observed before, we must have $j_1 = j_2$ to get a non-zero contribution. Note that if $t_1 \neq t_2$, we always have $\delta_{t_1,i_1}\delta_{t_2,i_2} = 0$ as $H(i_1)$ cannot be equal to both $t_1$ and $t_2$. Thus, for fixed $i_1 = i_2, j_1 = j_2$,

$$E\left[\sum_{t_1, t_2 = 1}^m S(i_1)S(i_2)S(j_1)S(j_2)\delta_{t_1,i_1}\delta_{t_1,j_1}\delta_{t_2,i_2}\delta_{t_2,j_2}A_{i_1,u}A_{i_2,u}B_{j_1,w}B_{j_2,w}\right]$$

$$= E\left[\sum_{t_1 = 1}^m \delta_{t_1,i_1}\delta_{t_1,j_1}A_{i_1,u}^2B_{j_1,w}\right]$$

$$= \frac{A_{i_1,u}^2B_{j_1,w}}{m}.$$
Thus, we have \( \|A_u\|_F^2 \leq 3\|B_u\|_F^2 \). The case \( i_1 = j_2 \) is analogous.

Next we compute the contribution of the terms where \( i_1 \neq i_2, j_1, j_2 \) i.e., there are at least 3 distinct numbers among \( i_1, i_2, j_1, j_2 \). Notice that \( \mathbb{E}[\delta_{i_1,i}, \delta_{i_1,j}, \delta_{i_2,i}, \delta_{i_2,j}] \leq \frac{1}{m} \) because the \( \delta \)'s are 3-wise independent. For fixed \( i_1, j_1, i_2, j_2 \), there are \( m^2 \) choices of \( i_1, i_2 \) so the total contribution to the expectation from terms with the same \( i_1, j_1, i_2, j_2 \) is bounded by \( m^2 \cdot \frac{1}{m} \|A_{i_1,u}A_{i_2,u}B_{j_1,u'}B_{j_2,u'}\| = \frac{1}{m^2} |A_{i_1,u}A_{i_2,u}B_{j_1,u'}B_{j_2,u'}| \).

Therefore,

\[
\mathbb{E}[(C - A^T B)_{u,u'}^2] 
\leq \frac{2}{m} \|A_u\|_F^2 \|B_u\|_F^2 + \frac{1}{m} \sum_{S_1, S_2, S_3} \sum_{a,b,c} |A(a,b,c)_{u}B(g,b,f)_{u'}A(a,e,f)_{u}B(g,e,c)_{u'}|
\leq \frac{2}{m} \|A_u\|_F^2 \|B_u\|_F^2 + \frac{3^q}{m} \sum_{a,b,c} A^2_{(a,b,c),u} \left( \sum_{a,b,c} B^2_{(g,b,f),u'}A^2_{(a,e,f),u}B^2_{(g,e,c),u'} \right)^{1/2}
\leq \frac{2}{m} \|A_u\|_F^2 \|B_u\|_F^2 + \frac{3^q}{m} \sum_{a,b,c} A^2_{(a,b,c),u} \left( \sum_{b} B^2_{(g,b,f),u'} \right)^{1/2} \left( \sum_{a,c} A^2_{(a,e,f),u}B^2_{(g,e,c),u'} \right)^{1/2}
\leq \frac{2}{m} \|A_u\|_F^2 \|B_u\|_F^2 + \frac{3^q}{m} \|A_u\| \cdot \left( \sum_{a,c} A^2_{(a,e,f),u} \right)^{1/2} \left( \sum_{g,c} B^2_{(g,e,c),u'} \right)^{1/2}
\leq \frac{2}{m} \|A_u\|_F^2 \|B_u\|_F^2 + \frac{3^q}{m} \|A_u\| \cdot \left( \sum_{a,e,f} A^2_{(a,e,f),u} \right)^{1/2} \left( \sum_{g,e,c} B^2_{(g,e,c),u'} \right)^{1/2}
= \frac{(2 + 3^q)\|A_u\|_F^2 \|B_u\|_F^2}{m},
\]

where the second inequality follows from the fact that there are at most \( 3^q \) partitions of \( [q] \) into 3 sets. The other inequalities are from Cauchy-Schwarz.

Combining the above bounds, we have \( \mathbb{E}[(C - A^T B)_{u,u'}^2] \leq \frac{(2 + 3^q)\|A_u\|_F^2 \|B_u\|_F^2}{m} \). For \( m \geq (2 + 3^q)/(\epsilon^2 \delta) \), by the Markov inequality, \( \|A^TS^T SB - A^T B\|_F^2 \leq \epsilon^2 \|A\|_F^2 \|B\|_F^2 \) with probability \( 1 - \delta \).

The second lemma proves that the subspace embedding property follows from the approximate matrix product property.

**Lemma B.3** (Oblivious subspace embeddings). Consider a fixed \( k \)-dimensional subspace \( V \subset \mathbb{R}^{n_1 n_2 \cdots n_q} \). If \( m \geq k^2 (2 + 3^q)/(\epsilon^2 \delta) \), then with probability at least \( 1 - \delta \), \( \|Sx\|_2 = (1 \pm \epsilon)\|x\|_2 \) simultaneously for all \( x \in V \).

**Proof.** Let \( B \) be a \( (n_1 n_2 \cdots n_q) \times k \) matrix whose columns form an orthonormal basis of \( V \). Thus, we have \( B^T B = I_k \) and \( \|B\|_F^2 = k \). The condition that \( \|Sx\|_2 = (1 \pm \epsilon)\|x\|_2 \) simultaneously for all \( x \in V \) is equivalent to the condition that the singular values of \( SB \) are bounded by \( 1 \pm \epsilon \). By Lemma B.2, for \( m \geq (2 + 3^q)/(\epsilon^2 \delta) \), with probability at least \( 1 - \delta \), we have

\[ \|B^T S^T SB - B^T B\|_F^2 \leq \epsilon (k^2)^2 = \epsilon^2 \]

Thus, we have \( \|B^T S^T SB - I_k\|_2 \leq \|B^T S^T SB - B^T B\|_F \leq \epsilon \). In other words, the squared singular values of \( SB \) are bounded by \( 1 \pm \epsilon \), implying that the singular values of \( SB \) are also bounded by \( 1 \pm \epsilon \). Note that \( \|A\|_2 \) for a matrix \( A \) denotes its operator norm.
C Missing Proofs

C.1 Proofs for Tensor Product Least Square Regression

**Theorem 3.1.** (Tensor regression) Suppose $\tilde{x}$ is the output of Algorithm 1 with Tensorsketch $S \in \mathbb{R}^{m \times n}$, where $m = 8(d_1 d_2 \cdots d_q + 1)^2(2 + 3^q)/(\epsilon^2 \delta)$. Then the following approximation $\|(A_1 \otimes A_2 \otimes \cdots \otimes A_q)\tilde{x} - b\|_2 \leq (1 + \epsilon) \text{OPT}$, holds with probability at least $1 - \delta$.

**Proof.** It is easy to see that

$$\|(A_1 \otimes A_2 \otimes \cdots \otimes A_q)x - b\|_2 = \left\|[(A_1 \otimes A_2 \otimes \cdots \otimes A_q) \ 1]egin{bmatrix} x \\ 0 \end{bmatrix}\right\|_2,$$

and identifying

$$y = [(A_1 \otimes A_2 \otimes \cdots \otimes A_q) \ 1]\begin{bmatrix} x \\ 0 \end{bmatrix} \in \mathbb{R}^{n_1 n_2 \cdots n_q}$$

and $y$ is a vector of a subspace $V \subset \mathbb{R}^{n_1 n_2 \cdots n_q}$ with dimension at most $d_1 d_2 \cdots d_q + 1$, we can use Lemma B.3 to conclude that

$$\Pr[\|Sy\|_2 - \|y\|_2 \leq \epsilon \|y\|_2 \geq 1 - \delta]$$

when $m = (d_1 d_2 \cdots d_q + 1)^2(2 + 3^q)/(\epsilon^2 \delta)$.

Thus we have

$$\|(A_1 \otimes A_2 \otimes \cdots \otimes A_q)\tilde{x} - b\|_2 \leq \frac{1}{1 - \epsilon} \|S(A_1 \otimes A_2 \otimes \cdots \otimes A_q)\tilde{x} - Sb\|_2$$

and

$$\|S(A_1 \otimes A_2 \otimes \cdots \otimes A_q)x - Sb\|_2 \leq (1 + \epsilon)\|(A_1 \otimes A_2 \otimes \cdots \otimes A_q)x - b\|_2$$

hold with probability at least $1 - \delta$. Then using a union bound, we have

$$\|(A_1 \otimes A_2 \otimes \cdots \otimes A_q)\tilde{x} - b\|_2 \leq \frac{1}{1 - \epsilon} \|S(A_1 \otimes A_2 \otimes \cdots \otimes A_q)\tilde{x} - Sb\|_2$$

and

$$\|S(A_1 \otimes A_2 \otimes \cdots \otimes A_q)x - Sb\|_2 \leq (1 + \epsilon)\|(A_1 \otimes A_2 \otimes \cdots \otimes A_q)x - b\|_2$$

holds with probability at least $1 - 2\delta$. \hfill $\square$

**Corollary 3.2.** (Sketch for tensor nonnegative regression) Suppose $\tilde{x} = \min_{x \geq 0} \|S Ax - Sb\|_2$ with Tensorsketch $S \in \mathbb{R}^{m \times n}$, where $m = 8(d_1 d_2 \cdots d_q + 1)^2(2 + 3^q)/(\epsilon^2 \delta)$. Then the following approximation $\|(A_1 \otimes A_2 \otimes \cdots \otimes A_q)\tilde{x} - b\|_2 \leq (1 + \epsilon) \text{OPT}$ holds with probability at least $1 - \delta$, where $\text{OPT} = \min_{x \geq 0} \|(A_1 \otimes A_2 \otimes \cdots \otimes A_q)x - b\|_2$.

**Proof.** The proof of Theorem 3.2 is similar to the proof of theorem 3.1. Denote $\tilde{x} = \min_{x \geq 0} \|S Ax - Sb\|_2$ and $x^* = \min_{x \geq 0} \|Ax - b\|_2$. Using Lemma B.3, we have:

$$\|Ax - b\|_2 \leq \frac{1}{1 - \epsilon} \|SAx - Sb\|_2,$$

with probability at least $1 - \delta$, and

$$\|SAx^* - Sb\|_2 \leq (1 + \epsilon)\|Ax^* - b\|_2,$$
with probability at least $1 - \delta$. Hence applying a union bound we have:

$$\|Ax - b\|_2 \leq \frac{1}{1 - \epsilon}\|SAx - Sb\|_2$$

$$\leq \frac{1}{1 - \epsilon}\|SAx^* - Sb\|_2$$

$$\leq \frac{1 + \epsilon}{1 - \epsilon}\|Ax^* - b\|_2,$$

with probability at least $1 - 2\delta$.

\[\Box\]

### C.2 Proofs for P-Splines

**Lemma 4.1.** Let $x^* \in \mathbb{R}^d$, $A \in \mathbb{R}^{n \times d}$ and $b \in \mathbb{R}^n$ as above. Let $U_1 \in \mathbb{R}^{n \times d}$ denote the first $n$ rows of an orthogonal basis for $\frac{A}{\sqrt{\lambda L}} \in \mathbb{R}^{(n+p) \times d}$. Let sketching matrix $S \in \mathbb{R}^{m \times n}$ have a distribution such that with constant probability

$$\text{(I) } \|U_1^T S^T SU_1 - U_1^T U_1\|_2 \leq 1/4,$$

and

$$\text{(II) } \|U_1^T (S^T S - I)(b - Ax^*)\|_2 \leq \sqrt{\epsilon \text{OPT}} / 2.$$ 

Let $\bar{x}$ denote $\text{argmin}_{x \in \mathbb{R}^d}\|S(Ax - b)\|_2^2 + \lambda\|Lx\|_2^2$. Then with probability at least 9/10,

$$\|Ax - b\|_2^2 + \lambda\|L\bar{x}\|_2^2 \leq (1 + \epsilon)\text{OPT}.$$ 

**Proof.** Let $\hat{A} \in \mathbb{R}^{(n+d) \times d}$ have orthonormal columns with $\text{range}(\hat{A}) = \text{range}(\frac{A}{\sqrt{\lambda L}})$. (An explicit expression for one such $\hat{A}$ is given below.) Let $\hat{b} \equiv [\hat{b}_d \ b_*]$. We have

$$\min_{y \in \mathbb{R}^d}\|\hat{A}y - \hat{b}\|_2$$

equivalent to $\|b - Ax\|_2^2 + \lambda\|Lx\|_2^2$, in the sense that for any $\hat{A}y \in \text{range}(\hat{A})$, there is $x \in \mathbb{R}^d$ with $\hat{A}y = [\frac{A}{\sqrt{\lambda L}}]x$, so that $\|\hat{A}y - \hat{b}\|_2^2 = \|\frac{A}{\sqrt{\lambda L}} \ x - \hat{b}\|_2^2 = \|b - Ax\|_2^2 + \lambda\|Lx\|_2^2$. Let $y^* = \text{argmin}_{y \in \mathbb{R}^d}\|\hat{A}y - \hat{b}\|_2$, so that $\hat{A}y^* = [\frac{Ax^*}{\sqrt{\lambda L}x^*}]$. Let $\hat{A} = [U_1 \ U_2]$, where $U_1 \in \mathbb{R}^{n \times d}$ and $U_2 \in \mathbb{R}^{d \times d}$, so that $U_1$ is as in the lemma statement.

We define $\hat{S}$ to be $\begin{bmatrix} S & 0_{m \times d} \\ 0_{d \times n} & I_d \end{bmatrix}$ and $\hat{S}$ satisfies Property (I) and (II) of Lemma 4.1.

Using $\|U_1^T S^T SU_1 - U_1^T U_1\|_2 \leq 1/4$, with constant probability

$$\|\hat{A}^T \hat{S}^T \hat{S} \hat{A} - I_d\|_2 = \|U_1^T S^T SU_1 + U_2^T U_2 - I_d\|_2 = \|U_1^T S^T SU_1 - U_1^T U_1\|_2 \leq 1/4.$$ 

Using the normal equations for Eq. (10), we have

$$0 = \hat{A}^T (\hat{b} - \hat{A}y^*) = U_1^T (b - Ax^*) - \sqrt{\lambda} U_2^T x^*,$$

and so

$$\hat{A}^T \hat{S} (\hat{b} - \hat{A}y^*) = U_1^T S^T S(b - Ax^*) - \sqrt{\lambda} U_2^T x^* = U_1^T S^T S(b - Ax^*) - U_1^T (b - Ax^*).$$

Using Property (II) of Lemma 4.1, with constant probability

$$\|\hat{A}^T \hat{S} (\hat{b} - \hat{A}y^*)\|_2$$

$$= \|U_1^T S^T S(b - Ax^*) - U_1^T (b - Ax^*)\|_2$$

$$\leq \sqrt{\epsilon \text{OPT}} / 2$$

$$= \sqrt{\epsilon / 2}\|\hat{b} - \hat{A}y^*\|_2.$$ 

(12)
It follows by a standard result from (11) and (12) that the solution \( \hat{y} \equiv \arg\min_{y \in \mathbb{R}^d} \| \hat{S}(\hat{A}y - \hat{b}) \|_2 \) has \( \| \hat{A}\hat{y} - \hat{b} \|_2 \leq (1 + \epsilon) \min_{y \in \mathbb{R}^d} \| \hat{A}y - \hat{b} \|_2 \), and therefore that \( \hat{x} \) satisfies the claim of the theorem.

For convenience we give the proof of the standard result: (11) implies that \( \hat{A}^T \hat{S}^T \hat{S} \hat{A} \) has smallest singular value at least 3/4. The normal equations for the unsketched and sketched problems are

\[
\hat{A}^T (\hat{b} - \hat{A}y^*) = 0 = \hat{A}^T \hat{S}^T \hat{S} (\hat{b} - \hat{A}\hat{y}).
\]

The normal equations for the unsketched case imply \( \| \hat{A}\hat{y} - \hat{b} \|_2 \leq \| \hat{A}(\hat{y} - y^*) \|_2 + \| \hat{b} - \hat{A}y^* \|_2 \), so it is enough to show that \( \| \hat{A}(\hat{y} - y^*) \|_2 \leq \epsilon \text{OPT} \). We have

\[
(3/4) \| \hat{y} - y^* \|_2 \leq \| \hat{A}^T \hat{S}^T \hat{S} \hat{A}(\hat{y} - y^*) \|_2
\]

\[
= \| \hat{A}^T \hat{S}^T \hat{S} \hat{A}(\hat{y} - y^*) - \hat{A}^T \hat{S}^T \hat{S} (\hat{b} - \hat{A}y) \|_2
\]

\[
\leq \sqrt{\epsilon \text{OPT}} / 2
\]

so that \( \| \hat{y} - y^* \|_2 \leq (4/3) \epsilon \text{OPT} / 2 \leq \epsilon \text{OPT} \). The lemma follows.

The following lemma computes the statistical dimension \( s\text{d}_\lambda(A, L) \) that will be used for computing the number of rows of sketching matrix \( S \).

**Lemma C.1.** For \( U_1 \) as in Lemma 4.1, \( \|U_1\|_F^2 = s\text{d}_\lambda(A, L) = \sum_{i=1}^p 1/(1 + \lambda/\gamma_i^2) + d - p \), where \( A \) has singular values \( \sigma_i \). Also \( \|U_1\|_2 = \max\{1/\sqrt{1 + \lambda/\gamma_1^2}, 1\} \).

**Proof.** Suppose we have the GSVD of \((A, L)\). Let

\[
D \equiv \begin{bmatrix} \Sigma \Omega^T \Omega & 0_{p \times (n-p)} \\ 0_{(n-p) \times p} & I_{d-p} \end{bmatrix}^{-1/2}.
\]

Then

\[
\hat{A} = U \begin{bmatrix} \Sigma & 0_{p \times (n-p)} \\ 0_{(n-p) \times p} & I_{d-p} \end{bmatrix} D \]

has \( \hat{A}^T \hat{A} = I_d \), and for given \( x \), there is \( y = D^{-1} R Q^T x \) with \( \hat{A}y = \begin{bmatrix} \hat{A} \end{bmatrix} \hat{A} \). We have \( \|U_1\|_F^2 = \|U \begin{bmatrix} \Sigma & 0_{p \times (n-p)} \\ 0_{(n-p) \times p} & I_{d-p} \end{bmatrix} D \|_F^2 = \sum_{i=1}^p 1/(1 + \lambda/\gamma_i^2) + d - p \) as claimed.

**Theorem 4.3.** (P-Spline regression) There is a constant \( K > 0 \) such that for \( m \geq K(\epsilon^{-1} s\text{d}_\lambda(A, L) + s\text{d}_\lambda(A, L)^2) \) and \( S \in \mathbb{R}^{m \times n} \) a sparse embedding matrix (e.g., COUNTSKETCH) with \( SA \) computable in \( O(\text{nnz}(A)) \) time, Property (I) and (II) of Lemma 4.1 apply, and with constant probability the corresponding \( \hat{x} = \arg\min_{x \in \mathbb{R}^e} \| S(Ax - b) \|_2 + \lambda \| Lx \|_2^2 \) is an \( \epsilon \)-approximate solution to \( \min_{x \in \mathbb{R}^e} \| b - Ax \|_2^2 + \lambda \| Lx \|_2^2 \).

**Proof.** Recall that \( s\text{d}_\lambda(A, L) = \|U_1\|_F^2 \). Sparse embedding distributions satisfy the bound for approximate matrix multiplication

\[
\| W^T S H - W^T H \|_F \leq C\| W \|_F \| H \|_F / \sqrt{m},
\]

for a constant \( C \) (Clarkson and Woodruff, 2013; Meng and Mahoney, 2013; Nelson and Nguyên, 2013); this is also true of OSE matrices. We set \( W = H = U_1 \) and use \( \| X \|_2 \leq \| X \|_F \) for all \( X \) and \( m \geq K\|U_1\|_F^2 \) to obtain Property (I) of Lemma 4.1, and set \( W = U_1, H = b - Ax^* \) and use \( m \geq K\|U_1\|_F^2 / \epsilon \) to obtain Property (II) of Lemma 4.1. (Here the bound is slightly stronger than Property (II), holding for \( \lambda = 0 \).) With Property (I) and Property (II), the claim for \( \hat{x} \) from a sparse embedding follows using Lemma 4.1.
C.3 Proofs for Tensor Product $\ell_1$ Regression

Lemma 5.3. For any $p \geq 1$. Condition$(\mathcal{A})$ computes $\mathcal{A}U/(d\gamma_p)$ which is an $(\alpha, \beta\sqrt{3d}(tw)^{1/p-1/2}, p)$-well-conditioned basis of $\mathcal{A}$, with probability at least $1 - \prod_{i=1}^{q}(n_i/w_i)\delta$.

Proof. This lemma is similar to arguments in Clarkson et al. (2013), we simply adjust notation and parameters for completeness. Applying Theorem 5.2, we have that with probability at least $1 - \prod_{i=1}^{q}(n_i/w_i)\delta$, for all $x \in \mathbb{R}^r$, if we consider $y = \mathcal{A}x$ and write $y^T = [z_1^T, z_2^T, \ldots, z_{\prod_{i=1}^{q}n_i/w_i}]^T$, then for all $i \in [\prod_{i=1}^{q}n_i/w_i]$,

$$\sqrt{\frac{2}{\beta}}\|z_i\|_2 \leq \|S_i z_i\|_2 \leq \sqrt{\frac{2}{\beta}}\|z_i\|_2,$$

where $S_i \in \mathbb{R}^{m_i \times \prod_{i=1}^{q}w_i}$. In the following, suppose $m_i = t$. By relating the 2-norm and the $p$-norm, for $1 \leq p \leq 2$, we have

$$\|S_i z_i\|_p \leq \frac{1}{\gamma_{1/p}}\|S_i z_i\|_2 \leq \frac{1}{\gamma_{1/p}}\|S_i z_i\|_2 \leq t^{1/p-1/2} \sqrt{\frac{2}{\beta}}\|z_i\|_2 \leq t^{1/p-1/2} \sqrt{\frac{2}{\beta}}\|z_i\|_2,$$

and similarly,

$$\|S_i z_i\|_p \geq \|S_i z_i\|_2 \geq \frac{1}{\gamma_{1/p}}\|S_i z_i\|_2 \geq \frac{1}{\gamma_{1/p}}\|S_i z_i\|_2 \geq \frac{1}{\gamma_{1/p}}w^{1/2-1/p}\|z_i\|_p, w = \prod_{j=1}^{q}w_j.$$

If $p > 2$, then

$$\|S_i z_i\|_p \leq \|S_i z_i\|_2 \leq \frac{1}{\gamma_{1/p}}\|S_i z_i\|_2 \leq \frac{1}{\gamma_{1/p}}w^{1/2-1/p}\|z_i\|_p,$$

and similarly,

$$\|S_i z_i\|_p \geq \frac{1}{\gamma_{1/p}}w^{1/2-1/p}\|z_i\|_2 \geq \frac{1}{\gamma_{1/p}}w^{1/2-1/p}\|z_i\|_2 \geq \frac{1}{\gamma_{1/p}}t^{1/p-1/2}\|S_i z_i\|_2.$$

Since $\|Ax\|^p_p = \|y\|^p_p = \sum_i \|z_i\|^p_p$ and $\|SAx\|^p_p = \sum_i \|S_i z_i\|^p_p$, for $p \in [1, 2]$ we have with probability $1 - \prod_{i=1}^{q}(n_i/w_i)\delta$

$$\sqrt{\frac{2}{\beta}}w^{1/2-1/p}\|Ax\|_p \leq \|SAx\|_p \leq \sqrt{\frac{2}{\beta}}t^{1/p-1/2}\|Ax\|_p,$$

and for $p \in [2, \infty)$ with probability $1 - \prod_{i=1}^{q}(n_i/w_i)\delta$

$$\sqrt{\frac{2}{\beta}}t^{1/p-1/2}\|Ax\|_p \leq \|SAx\|_p \leq \sqrt{\frac{2}{\beta}}w^{1/2-1/p}\|Ax\|_p.$$

In either case,

$$\|Ax\|_p \leq \beta_p \|SAx\|_p \leq \sqrt{3}(tw)^{1/p-1/2}\|Ax\|_p. \quad (13)$$

We have, from the definition of an $(\alpha, \beta, p)$-well-conditioned basis, that

$$\|SAU\|_p \leq \alpha \quad (14)$$

and for all $x \in \mathbb{R}^d$,

$$\|x\|_q \leq \beta \|SAU x\|_p. \quad (15)$$

Combining (13) and (14), we have that with probability at least $1 - \prod_{i=1}^{q}(n_i/w_i)\delta$,

$$\|AU/(r\gamma_p)\|_p \leq \sum_i \|AU_i/r\gamma_p\|_p \leq \sum_i \|SAU_i/r\|_p \leq \alpha.$$

Combining (13) and (15), we have that with probability at least $1 - \prod_{i=1}^{q}(n_i/w_i)\delta$, for all $x \in \mathbb{R}^r$,

$$\|x\|_q \leq \beta \|SAU x\|_p \leq \beta \sqrt{3}(tw)^{1/p-1/2}\|AU 1/r\gamma_p x\|_p.$$

Hence $AU/(r\gamma_p)$ is an $(\alpha, \beta\sqrt{3}(tw)^{1/p-1/2}, p)$-well-conditioned basis.

$\Box$
Theorem 5.4. (Main result) Given \( \epsilon \in (0, 1) \), \( A \in \mathbb{R}^{n \times d} \) and \( b \in \mathbb{R}^n \), Alg. 3 computes \( \hat{x} \) such that with probability at least 1/2, \( \|Ax - b\| \leq (1 + \epsilon) \min_{x \in \mathbb{R}^d} \|Ax - b\|_1 \). For the special case when \( q = 2, n_1 = n_2 \), the algorithm’s running time is \( O(n_1^{3/2} \text{poly}(\prod_{i=1}^2 d_i / \epsilon)) \).

Proof. For notational simplicity, let us denote \( n_{[q]} \triangleq \prod_{i=1}^q n_i, n_{[q]\backslash [q_1]} \triangleq \prod_{i=1}^q n_i, d_{[q]} \triangleq \prod_{i=1}^q d_i, \) and \( d_{[q]\backslash [q_1]} \triangleq \prod_{i=q_1+1}^q d_i \). For any row-block \( A_{i_1} \otimes \cdots \otimes A_{i_q} \), computing \( S_{i_1\ldots i_q}(A_{i_1} \otimes \cdots \otimes A_{i_q}^{(q-1)}) \) takes \( O(d(\sum_{k=1}^q \text{nnz}(A_{i_k}^{(k)})) + dqm \log(m)) \) (see Sec 2). Hence for \( SA \), it takes:

\[
\left( d \sum_{k=1}^q \text{nnz}(A_k) \prod_{i \in [q] \backslash \{k\}} n_i/w_i \right) \left( dqm \log(m) \prod_{i=1}^q n_i/w_i \right)
\]

where \( S \in \mathbb{R}^{(m \prod_{i=1}^q (n_i/w_i)) \times \prod_{i=1}^q w_i} \) and \( m \geq 100 \prod_{i=1}^q d_i^2 (2 + 3^q) / \epsilon^2 = O(\text{poly}(d/\epsilon)) \). We need to compute an orthogonal factorization \( SA = QR_A \) in \( O(qmd^2) \) and then compute \( U = R_A^{-1} \) in \( O(d^3) \) time. Hence the total running time of Algorithm Condition(\( A \)) is \( O(qmd^2 + d^3) \). Thus the total running time of computing \( SA \) and Condition(\( A \)) is

\[
O \left( \left( \sum_{k=1}^q \text{nnz}(A_k) \prod_{i \in [q] \backslash \{k\}} n_i/w_i \right) \left( \prod_{i=1}^q n_i/w_i \right) \right) \text{poly}(d/\epsilon) + qmd^2 + d^3
\]

We will compute \( UG \) in \( O(d^2 \log n) \) time. We compute \( \tilde{E} = E(A_{q_1+1} \otimes \cdots \otimes A_q)^T \) in \( O(dn_{[q] \backslash [q_1]} n_{[q]}) \) time.

Then we can compute \( R(A_1 \otimes \cdots \otimes A_q) \tilde{E}_j \) in \( O(n_{[q]} d_{[q]} \log n + d_{[q]} n_{[q] \backslash [q_1]} \log n) \) time.

Since computation of the median \( \lambda_i \) takes \( O(\log n) \) time, computing all \( \lambda_i \) and then \( \lambda_e \) takes \( O(n_{[q] \backslash [q_1]} \log n) \) time.

As \( A_{UG} \) has \( O(\log n) \) columns, we need to compute \( \lambda_e \) for each \( A_{UG} \) using the above procedure and hence it takes in total \( O(d(n_{[q]} + n_{[q] \backslash [q_1]})) \log^2 n \) time.

Sampling a column of \( A_{UG} \) using \( \lambda_e \) takes \( O(\log n) \) time, sampling an entry in \( M \) takes in total \( O(n_{[q]} + n_{[q] \backslash [q_1]}) \) time.

Since we need \( \sqrt{\prod_{k=1}^q w_k} \text{poly}(r) \) samples to select rows, the running time is \( d(n_{[q]} + n_{[q] \backslash [q_1]}) \log^2 n \cdot \sqrt{\prod_{k=1}^q w_k} \text{poly}(r) \).

Now for simplicity, we set \( q = 2, n_i = n_0 \) for \( i \in [2] \). Note that it is optimal to choose \( w_i = w \) for \( i \in [2] \). Substituting \( q = 2, n_i = n_0 \) and \( w_i = w \), we that the total running time of Alg. 3:

\[
O \left( dw^{-1} n_0 (\text{nnz}(A_1) + \text{nnz}(A_2)) + w^{-2} n_0^2 \text{poly}(d/\epsilon) + w n_0 \text{poly}(d) \log(n) \right).
\]

For dense \( A_1 \) and \( A_2, \text{nnz}(A_1) + \text{nnz}(A_2) = O(n_0) \) time, and so ignoring poly and log terms that do not depend on \( n_0 \), the total running time can be simplified to:

\[
O(w^{-1} n_0^2 + wn_0).
\]

Setting \( w = \sqrt{n_0} \), we can minimize the above running time to \( O(n_0^{3/2}) \), which is faster than the \( n_0^{5/2} \) time for solving the problem by forming \( A_1 \otimes A_2 \). \qed