On Numbers of Tuples of Nilpotent Matrices over Finite Fields under Simultaneous Conjugation

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Abstract

The problem of classifying tuples of nilpotent matrices over a field under simultaneous conjugation is considered “hopeless”. However, for any given matrix order over a finite field, the number of concerned orbits is always finite. This paper gives a closed formula for the number of absolutely indecomposable orbits using the same methodology as Hua [5]; those orbits are non-splittable over field extensions. As a consequence, those numbers are always polynomials in the cardinality of the base field with integral coefficients. It is conjectured that those coefficients are always non-negative.

1 Introduction

Let $q$ be a prime power and $\mathbb{F}_q$ be the finite field with $q$ elements. For any positive integer $n$, let $\mathcal{M}_n(\mathbb{F}_q)$ be the matrix algebra which consists of all $n \times n$ matrices over $\mathbb{F}_q$, $GL(n, \mathbb{F}_q) \subset \mathcal{M}_n(\mathbb{F}_q)$ be the General Linear Group consisting of all invertible ones and $\mathcal{N}_n(\mathbb{F}_q) \subset \mathcal{M}_n(\mathbb{F}_q)$ be the subset of all nilpotent ones.

Let $g$ be a fixed positive integer and $\mathcal{M}_n(\mathbb{F}_q)^g$ be the set of all $g$-tuples of $n \times n$ matrices over $\mathbb{F}_q$ and $\mathcal{N}_n(\mathbb{F}_q)^g$ be the set of all $g$-tuples of nilpotent ones, i.e.,

$$\mathcal{M}_n(\mathbb{F}_q)^g = \{(M_1, M_2, \ldots, M_g) \mid M_i \in \mathcal{M}_n(\mathbb{F}_q), 1 \leq i \leq g\},$$

$$\mathcal{N}_n(\mathbb{F}_q)^g = \{(M_1, M_2, \ldots, M_g) \mid M_i \in \mathcal{N}_n(\mathbb{F}_q), 1 \leq i \leq g\}.$$

$GL(n, \mathbb{F}_q)$ acts on $\mathcal{M}_n(\mathbb{F}_q)^g$ by simultaneous conjugation, i.e.,

$$GL(n, \mathbb{F}_q) \times \mathcal{M}_n(\mathbb{F}_q)^g \rightarrow \mathcal{M}_n(\mathbb{F}_q)^g$$

$$(T, (M_1, M_2, \ldots, M_g)) \rightarrow (T^{-1}M_1T, T^{-1}M_2T, \ldots, T^{-1}M_gT).$$

It is obvious that $\mathcal{N}_n(\mathbb{F}_q)^g$ is closed under the action of $GL(n, \mathbb{F}_q)$. Every $g$-tuple of matrices $(M_1, M_2, \ldots, M_g) \in \mathcal{M}_n(\mathbb{F}_q)^g$ gives rise to a representation of the free algebra $\mathbb{F}_q(x_1, x_2, \ldots, x_g)$ by the following mapping:

$$\mathbb{F}_q(x_1, x_2, \ldots, x_g) \rightarrow \mathcal{M}_n(\mathbb{F}_q)$$

$$x_i \rightarrow M_i \ (i = 1, 2, \cdots, g).$$
Conversely, any finite dimensional representation of $\mathbb{F}_q\langle x_1, x_2, \ldots, x_g \rangle$ is determined by a $g$-tuple from $M_n(\mathbb{F}_q)^g$. It is obvious that two $g$-tuples are in the same orbit if and only if their corresponding representations are isomorphic.

**Definition 1.1.** An orbit of $M_n(\mathbb{F}_q)^g/GL(n, \mathbb{F}_q)$ is said to be indecomposable if its corresponding representation of the free algebra $\mathbb{F}_q\langle x_1, x_2, \ldots, x_g \rangle$ is indecomposable; it is absolutely indecomposable if its corresponding representation is absolutely indecomposable.

Thus, an orbit $GL(n, \mathbb{F}_q) \cdot (M_1, M_2, \ldots, M_g)$ is absolutely indecomposable if there does not exist an invertible matrix $T$ over the algebraic closure of $\mathbb{F}_q$, such that

$$(T^{-1}M_1T, T^{-1}M_2T, \ldots, T^{-1}M_gT) = \left(\begin{bmatrix} A_1 & 0 \\ 0 & B_1 \end{bmatrix}, \begin{bmatrix} A_2 & 0 \\ 0 & B_2 \end{bmatrix}, \ldots, \begin{bmatrix} A_g & 0 \\ 0 & B_g \end{bmatrix}\right),$$

where $A_i, B_i$ for $1 \leq i \leq g$ are square matrices over $\overline{\mathbb{F}}_q$.

Let $M_g(n, q) (I_g(n, q), A_g(n, q))$ be the number of orbits (indecomposable orbits, absolutely indecomposable orbits respectively) of $g$-tuples of nilpotent $n \times n$ matrices over $\mathbb{F}_q$ under simultaneous conjugation. It will be shown in later sections that $M_g(n, q), I_g(n, q)$ and $A_g(n, q)$ are all polynomials in $q$ with rational coefficients. The case for general $g$-tuples has been studied in Hua [5]. It is widely known that $A_g(n, q)$’s are of significant importance because of their deep connections with Geometric Invariant Theory, Quantum Group Theory and Representation Theory of Kac-Moody Algebras (Kac [6], Ringel [8] and Hausel [4]).

### 2 Minimal building blocks of conjugacy classes of $GL(n, \mathbb{F}_q)$

Let $\mathbb{N}$ be the set of all positive integers, $\mathcal{P}$ be the set of partitions of all positive integers, i.e.,

$$\mathcal{P} = \{ (\lambda_1, \lambda_2, \ldots, \lambda_k) \mid k \in \mathbb{N}, \lambda_i \in \mathbb{N}, \lambda_i \geq \lambda_{i+1} \geq 1, 1 \leq i \leq k \}.$$ 

The unique partition of 0 is (0). Let $\Phi$ the set of monic irreducible polynomials in $\mathbb{F}_q[x]$ with $x$ excluded. A **partition valued function** on $\Phi$ is a function $\delta : \Phi \mapsto \mathcal{P} \cup \{(0)\}$. $\delta$ has **finite support** if $\delta(f) = (0)$ except for finitely many $f$ in $\Phi$.

Let $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_{n-1}x^{n-1} + x^n \in \mathbb{F}_q[x]$ and $c(f)$ be its **companion matrix**, i.e.,

$$c(f) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix}.$$
For any $m \in \mathbb{N} \setminus \{0\}$, let $J_m(f)$ be the Jordan block matrix of order $m$ with $c(f)$ on the main diagonal, i.e.,

$$J_m(f) = \begin{bmatrix} c(f) & I & 0 & \cdots & 0 \\ 0 & c(f) & I & \cdots & 0 \\ & & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & I \\ 0 & 0 & 0 & \cdots & c(f) \end{bmatrix}_{m \times m},$$

where $I$ is the identity matrix of order $\text{deg}(f)$. For $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \in \mathcal{P}$, let $J_{\lambda}(f)$ be the direct sum of $J_{\lambda_i}(f)$, i.e.,

$$J_{\lambda}(f) = J_{\lambda_1}(f) \oplus J_{\lambda_2}(f) \oplus \cdots \oplus J_{\lambda_k}(f),$$

which stands for

$$\begin{bmatrix} J_{\lambda_1}(f) & 0 & \cdots & 0 \\ 0 & J_{\lambda_2}(f) & \cdots & 0 \\ & & & \ddots \\ 0 & 0 & \cdots & J_{\lambda_k}(f) \end{bmatrix}.$$ 

Rational Canonical Form Theorem implies that, for any matrix $M \in \text{GL}(n, \mathbb{F}_q)$, there exists a unique partition valued function $\delta$ on $\Phi$ with finite support such that $\sum_{f \in \Phi} \text{deg}(f) \cdot |\delta(f)| = n$ and $M$ is conjugate to

$$\bigoplus_{f \in \Phi, \delta(f) \neq (0)} J_{\delta(f)}(f).$$

For this reason, $J_{\lambda}(f)$ where $\lambda \in \mathcal{P}$ and $f \in \Phi$ are called minimal building blocks of conjugacy classes of $\text{GL}(n, \mathbb{F}_q)$. Rational Canonical Form for non-invertible matrices does exist as long as $\Phi$ admits $x$ as its member.

### 3 Nilpotent matrices commuting with minimal building blocks

Any partition $\lambda \in \mathcal{P}$ can be written in its “exponential form” $(1^{n_1}2^{n_2}3^{n_3}\cdots)$, which means there are exactly $n_i$ parts in $\lambda$ equal to $i$ for all $i \geq 1$. The weight of $\lambda$, denoted by $|\lambda|$, is $\sum_{i \geq 1} i n_i$, and the length of $\lambda$, denoted by $l(\lambda)$, is $\sum_{i \geq 1} n_i$. Let $\varphi_r(q) = (1 - q)(1 - q^2)\cdots(1 - q^r)$ for $r \in \mathbb{N}$ and $\varphi_0(q) = 1$. Furthermore, define $b_\lambda(q) = \prod_{i \geq 1} \varphi_{n_i}(q)$.

**Definition 3.1.** For any matrix of order $m \times n$, the arm length of index $(i, j)$ is one plus the number of minimal moves from $(i, j)$ to $(1, n)$, where diagonal
moves are not permitted. Thus the arm length distribution is as follows:

\[
\begin{pmatrix}
    n & n-1 & \ldots & 3 & 2 & 1 \\
    n + 1 & n & \ldots & 4 & 3 & 2 \\
    n + 2 & n + 1 & \ldots & 5 & 4 & 3 \\
    \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
    m + n & m + n - 1 & \ldots & m + 2 & m + 1 & m \\
\end{pmatrix}_{m \times n}
\]

The arm rank of a matrix \( M = [a_{ij}] \) of order \( m \times n \), denoted by \( \text{ar}(M) \), is the largest arm length of indexes of non-zero elements of \( M \), i.e.,

\[\text{ar}(M) = \max \{ \text{arm length of } (i, j) \mid a_{ij} \neq 0 \text{ where } 1 \leq i \leq m, 1 \leq j \leq n \}.\]

**Definition 3.2.** A matrix \( M = [a_{ij}] \) of order \( m \times n \) is of type-U if it satisfies the following conditions:

- \( a_{ij} = a_{st} \) if \((i, j)\) and \((s, t)\) have the same arm length,
- the arm rank of \( M \) is at most \( \min(m, n) \).

Thus a type-U matrix has either the following form when \( m \geq n \):

\[
\begin{pmatrix}
    a_1 & a_2 & \ldots & a_{n-1} & a_n \\
    0 & a_1 & \ldots & a_{n-2} & a_{n-1} \\
    \vdots & \vdots & \ddots & \vdots & \vdots \\
    0 & 0 & \ldots & a_1 & a_2 \\
    0 & 0 & \ldots & 0 & a_1 \\
    \vdots & \vdots & \ddots & \vdots & \vdots \\
    0 & 0 & \ldots & 0 & 0 \\
\end{pmatrix}_{m \times n}
\]

or the following form when \( m \leq n \):

\[
\begin{pmatrix}
    0 & \ldots & 0 & a_1 & a_2 & \ldots & a_{m-1} & a_m \\
    0 & \ldots & 0 & 0 & a_1 & \ldots & a_{m-2} & a_{m-1} \\
    \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots \\
    0 & \ldots & 0 & 0 & \ldots & a_1 & a_2 \\
    0 & \ldots & 0 & 0 & \ldots & 0 & a_1 \\
\end{pmatrix}_{m \times n}
\]

**Theorem 3.1** (Turnbull & Aitken [9]). Let \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \) be a partition with \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 1 \) and \( f(x) = x - a_0 \) with \( a_0 \in \mathbb{F}_q \), then any matrix over \( \mathbb{F}_q \) that commutes with \( J_{\lambda}(f) \) can be written as a \( k \times k \) block matrix in the following form:

\[
\begin{pmatrix}
    U_{11} & U_{12} & \ldots & U_{1k} \\
    U_{21} & U_{22} & \ldots & U_{2k} \\
    \vdots & \vdots & \ddots & \vdots \\
    U_{k1} & U_{k2} & \ldots & U_{kk} \\
\end{pmatrix}
\]
where submatrix $U_{ij}$ is a type-U matrix over $F_q$ of order $\lambda_i \times \lambda_j$ for all $(i, j)$ where $1 \leq i, j \leq k$.

As an example, let $\lambda = (3, 2, 2)$ and $f(x) = x - t \in F_q[x]$, $E$ a generic matrix that commutes with $J_\lambda(f)$, then

\[
J_\lambda(f) = \begin{bmatrix}
t & 1 & 0 & 0 & 0 & 0 \\
0 & t & 1 & 0 & 0 & 0 \\
0 & 0 & t & 1 & 0 & 0 \\
0 & 0 & 0 & t & 1 & 0 \\
0 & 0 & 0 & 0 & t & 1 \\
0 & 0 & 0 & 0 & 0 & t
\end{bmatrix},
\]

\[
E = \begin{bmatrix}
a & b & c \\
0 & a & b \\
0 & 0 & a \\
0 & r & s \\
r & 0 & 0 \\
u & v & j
\end{bmatrix}.
\]

**Theorem 3.2** (Fine & Herstein [3]). For any positive integer $n$, the number of nilpotent $n \times n$ matrices over $F_q$ is equal to $q^{n^2 - n}$.

For a partition $\lambda = (\lambda_1, \lambda_2, \lambda_3, \ldots)$, let $\lambda' = (\lambda_1', \lambda_2', \lambda_3', \ldots)$ be its conjugate partition, which means that $\lambda_i'$ is the number of parts in $\lambda$ that are greater than or equal to $i$ for all $i \geq 1$.

**Definition 3.3.** Let $\lambda, \mu$ be two partitions and $\lambda' = (\lambda_1', \lambda_2', \lambda_3', \ldots)$, $\mu' = (\mu_1', \mu_2', \mu_3', \ldots)$ be their conjugate partitions. The “inner product” of $\lambda$ and $\mu$ is defined as follows:

\[
\langle \lambda, \mu \rangle = \sum_{i \geq 1} \lambda_i' \mu_i'.
\]

**Lemma 3.1** (Hua [5]). Let $\lambda = (1^{m_1} 2^{m_2} 3^{m_3} \ldots)$ and $\mu = (1^{n_1} 2^{n_2} 3^{n_3} \ldots)$ be two partitions in their “exponential form”, then there holds:

\[
\langle \lambda, \mu \rangle = \sum_{i \geq 1} \sum_{j \geq 1} \min(i, j)m_i n_j.
\]

**Theorem 3.3.** For any partition $\lambda \in \mathcal{P}$ and $f(x) = x - a_0 \in F_q[x]$, the number of nilpotent matrices over $F_q$ that commute with $J_\lambda(f)$ is $q^{\langle \lambda, \lambda \rangle - \ell(\lambda)}$.

**Proof.** Suppose that $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 1$ and let $\mathcal{E}$ be the set of all matrices over $F_q$ that commute with $J_\lambda(f)$, i.e.,

\[
\mathcal{E} = \{ M \in M_n(F_q) \mid M J_\lambda(f) = J_\lambda(f) M, v = |\lambda| \}.
\]

$\mathcal{E}$ is indeed the endomorphism algebra of the representation of $F_q[x]$ induced by $J_\lambda(f)$. Theorem 3.1 implies that

\[
\mathcal{E} = \left\{ \begin{bmatrix} U_{11} & U_{12} & \ldots & U_{1k} \\
U_{21} & U_{22} & \ldots & U_{2k} \\
\vdots & \vdots & \ddots & \vdots \\
U_{k1} & U_{k2} & \ldots & U_{kk} \end{bmatrix} \mid \begin{array}{c}
U_{ij} \text{ is a type-U matrix over } F_q, \\
U_{ij} \text{ is of order } \lambda_i \times \lambda_j \text{ for } 1 \leq i, j \leq k
\end{array} \right\}.
\]
Let \((1^{n_1}2^{n_2}3^{n_3} \ldots)\) be the “exponential form” for \(\lambda\). Thus there are \(n_i\) parts equal to \(i\). \(E\) has finite dimension over \(F_q\), the dimension that is contributed by the submatrices of order \(i \times j\) for all pairs \((i, j)\) is:

- \(\min(i, j)n_injnj\) if \(i \neq j\),
- \(n_i^2\) if \(i = j\).

Thus the dimension of \(E\) is:

\[
\sum_{i \geq 1} \sum_{j \geq 1, j \neq i} \min(i, j)n_injnj + \sum_{i \geq 1} n_i^2 = \sum_{i \geq 1, j \geq 1} \min(i, j)n_injnj = (\lambda, \lambda).
\]

It follows that the order of \(E\) is:

\[
|E| = q^{(\lambda, \lambda)}.
\]

Let \(D\) be the subspace of \(E\) defined as follows:

\[
D = \left\{ \begin{bmatrix} D_{11} & D_{12} & \ldots & D_{1k} \\
D_{21} & D_{22} & \ldots & D_{2k} \\
\vdots & \vdots & \ddots & \vdots \\
D_{k1} & D_{k2} & \ldots & D_{kk} \end{bmatrix} \in E \mid \begin{array}{l}
D_{ij} \text{ is a matrix of order } \lambda_i \times \lambda_j, \\
\lambda_i = 0 \text{ if } \lambda_i \neq \lambda_j, \\
D_{ij} = aI \text{ for some } a \in F_q \text{ if } \lambda_i = \lambda_j
\end{array} \right\},
\]

and \(N\) be the subspace of \(E\) defined by:

\[
N = \left\{ \begin{bmatrix} N_{11} & N_{12} & \ldots & N_{1k} \\
N_{21} & N_{22} & \ldots & N_{2k} \\
\vdots & \vdots & \ddots & \vdots \\
N_{k1} & N_{k2} & \ldots & N_{kk} \end{bmatrix} \in E \mid \begin{array}{l}
N_{ij} \text{ is a type-U matrix of order } \lambda_i \times \lambda_j, \\
\ar(N_{ij}) = \lambda_i - 1 \text{ if } \lambda_i = \lambda_j
\end{array} \right\}.
\]

It is evident that \(E\) is a direct sum of \(D\) and \(N\) as a vector space. It can be verified that \(N\) is a two-sided ideal of \(E\). Furthermore \(N\) is nilpotent, i.e., every element in \(N\) is nilpotent. Thus every matrix \(E \in E\) can be uniquely written as a sum of a matrix from \(D\) and a matrix from \(N\), i.e.,

\[
E = D + N \text{ for some } D \in D \text{ and } N \in N.
\]

Since \(N\) is a nilpotent two-sided ideal of \(E\), \(E\) is nilpotent if and only if \(D\) is nilpotent.

As an example, if \(\lambda = (3, 2, 2)\), then any matrix in \(E\) can be written as a sum as follows:

\[
\begin{bmatrix}
a & 0 & 0 & 0 & 0 & 0 \\
0 & a & 0 & 0 & 0 & 0 \\
0 & 0 & a & 0 & 0 & 0 \\
0 & 0 & 0 & d & 0 & h \\
0 & 0 & 0 & 0 & d & 0 \\
0 & 0 & 0 & j & 0 & f \\
0 & 0 & 0 & j & 0 & f
\end{bmatrix}
+ \begin{bmatrix}
0 & b & c & l & m & p & q \\
0 & 0 & b & 0 & l & 0 & p \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & r & s & e & 0 & i & 0 \\
0 & 0 & r & 0 & 0 & 0 & 0 \\
0 & u & v & 0 & k & 0 & g \\
0 & 0 & u & 0 & 0 & 0 & 0
\end{bmatrix}.
\]
Every matrix $D \in \mathcal{D}$ can be viewed as a diagonal block matrix. Since there are $n_i$ parts equal to $i$ in $\lambda$, the block corresponding to $i^{n_i}$ is:

$$D_i = \begin{bmatrix} a_{11}I & a_{12}I & \cdots & a_{1n_i}I \\
a_{21}I & a_{22}I & \cdots & a_{2n_i}I \\
\vdots & \vdots & \ddots & \vdots \\
a_{n_i1}I & a_{n_i2}I & \cdots & a_{n_in_i}I \end{bmatrix},$$

where $I$ is the identity matrix of order $i$ and all $a_{ij} \in \mathbb{F}_q$. Thus $D$ is nilpotent if and only if every diagonal block $D_i$ is nilpotent. Since $D_i$ is conjugate to the direct sum of $i$ copies of the following matrix:

$$d_i = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n_i} \\
a_{21} & a_{22} & \cdots & a_{2n_i} \\
\vdots & \vdots & \ddots & \vdots \\
a_{n_i1} & a_{n_i2} & \cdots & a_{n_in_i} \end{bmatrix},$$

$D_i$ is nilpotent if and only if $d_i$ is nilpotent. Since the number of nilpotent matrix of order $n_i$ over $\mathbb{F}_q$ is $q^{n_i^2-n_i}$ by Theorem 3.2, the number of nilpotent matrices in $\mathcal{D}$ is:

$$q^{\sum_{i \geq 1} n_i^2-n_i}.$$

The dimension of $\mathcal{N}$ is:

$$\dim(\mathcal{E}) - \dim(\mathcal{D}) = \langle \lambda, \lambda \rangle - \sum_{i \geq 1} n_i^2.$$

Thus the order of $\mathcal{N}$ is:

$$|\mathcal{N}| = q^{\langle \lambda, \lambda \rangle - \sum_{i \geq 1} n_i^2}.$$

Putting it all together, the number of nilpotent matrices in $\mathcal{E}$ is:

$$q^{\langle \lambda, \lambda \rangle - \sum_{i \geq 1} n_i^2} \cdot q^{\sum_{i \geq 1} n_i^2-n_i} = q^{\langle \lambda, \lambda \rangle - \sum_{i \geq 1} n_i} = q^{\langle \lambda, \lambda \rangle - l(\lambda)}.$$

This finishes the proof. \hfill \Box

**Theorem 3.4.** For any partition $\lambda \in \mathcal{P}$ and any monic irreducible polynomial $f \in \mathbb{F}_q[x]$, the number of nilpotent matrices over $\mathbb{F}_q$ that commute with $J_\lambda(f)$ is $q^{d((\lambda,\lambda)-l(\lambda))}$, where $d$ is the degree of $f$.

**Proof.** Suppose that $d > 1$ as the case for $d = 1$ has been proved in Theorem 3.3. Let $c(f)$ be the companion matrix for $f$ and $\langle c(f) \rangle$ be the subalgebra of $\mathcal{M}_d(\mathbb{F}_q)$ generated by $c(f)$. Since $f$ is the characteristic equation of $c(f)$, $c(f)$ satisfies the polynomial $f$, i.e., $f(c(f)) = 0$. Since $f$ is irreducible, $f$ is the minimal polynomial satisfied by $c(f)$. This implies that $I, c(f), c(f)^2, \ldots, c(f)^{d-1}$ form a basis for $\langle c(f) \rangle$ over $\mathbb{F}_q$; i.e.,

$$\langle c(f) \rangle = \left\{ \sum_{i=0}^{d-1} a_i c(f)^i \mid a_i \in \mathbb{F}_q, 0 \leq i \leq d-1 \right\}.$$
Thus $\langle c(f) \rangle$ is a commutative subalgebra of $\mathcal{M}_d(\mathbb{F}_q)$ and the following map is an isomorphism:

$$\mathbb{F}_q[x]/(f(x)) \mapsto \langle c(f) \rangle \mapsto c(f).$$

Since $f$ is irreducible, $\mathbb{F}_q[x]/(f(x))$ is isomorphic to the finite field $\mathbb{F}_{q^d}$, and hence $\langle c(f) \rangle$ is a finite field with $q^d$ elements.

When $\deg(f) > 1$, Theorem 3.1 still holds as long as all submatrices $U_{ij}$ take values from the finite field $\langle c(f) \rangle$. All arguments in the proof of Theorem 3.3 still work with $\mathbb{F}_q$ being replaced by $\langle c(f) \rangle$. Thus Theorem 3.3 implies the desired results.

### 4 Calculating numbers of absolutely indecomposable orbits

Let $\mathbb{Q}$ be the rational number field, $\mathbb{Q}[[X]]$ be the ring of formal power series in $X$ over $\mathbb{Q}$, $\mathbb{Q}(q)$ be the field of rational functions in $q$ over $\mathbb{Q}$, and $\mathbb{Q}(q)[[X]]$ be the ring of formal power series in $X$ over $\mathbb{Q}(q)$. Let $\phi_n(q)$ be the number of monic irreducible polynomials with degree $n$ in $\mathbb{F}_q[x]$ with $x$ excluded. It is known that for any positive integer $n$,

$$\phi_n(q) = \frac{1}{n} \sum_{d|n} \mu(d)(q^d - 1),$$

where the sum runs over all divisors of $n$ and $\mu$ is the Möbius function. Following Hua [5], let

$$P(X, q) = 1 + \sum_{\lambda \in P} q^{\varphi(\lambda, \lambda) - l(\lambda)} X^{[\lambda]}.$$  

**Theorem 4.1.** The following identity holds in $\mathbb{Q}[[X]]$:

$$1 + \sum_{n=1}^{\infty} M_g(n, q) X^n = \prod_{d=1}^{\infty} (P(X^d, q^d))^{\phi_d(q)}.$$  

**Proof.** The method applied in Theorem 4.3 from Hua [5] still works here. In current context, the Burnside orbit counting formula is applied to $N_g(\mathbb{F}_q)^g/\text{GL}(n, \mathbb{F}_q)$ and the number of points fixed by $J_\lambda(f)$ is equal to $q^{\deg(f)\varphi(\lambda, \lambda) - l(\lambda)}$ by Theorem 3.3. Repeating the arguments there yields the desired result.

**Definition 4.1.** Define rational functions $H_g(n, q)$ for all positive integer $n$ as follows:

$$\log (P(X, q)) = \sum_{n=1}^{\infty} H_g(n, q) X^n,$$

where $\log$ is the formal logarithm, i.e., $\log(1 + x) = \sum_{i \geq 1} (-1)^{i-1} x^i/i.$
Theorem 4.2. The following identity holds for all positive integer $n$:

$$A_g(n, q) = (q - 1) \sum_{d | n} \frac{\mu(d)}{d} H_q \left( \frac{n}{d}, q^d \right).$$

Proof. This is the counterpart of Theorem 4.6 from Hua [5] with slight adjustment on the definition of $H_q(n, q)$, same arguments apply.

Analogues of Theorem 4.6 of Hua [5] have been proved by Bozec, Schiffmann & Vasserot [1] for Lusztig nilpotent varieties and their variants using techniques from Algebraic Geometry. Their definition of nilpotency is stronger than the one used here. In the language of $\lambda$-ring and Adams operator, Theorem 4.2 is equivalent to the following identities in the ring of formal power series $\mathbb{Q}(q)[[X]]$:

$$\sum_{n=1}^{\infty} A_g(n, q) X^n = (q - 1) \log(P(X, q)),$$

$$P(X, q) = \exp \left( \frac{1}{q - 1} \sum_{n=1}^{\infty} A_g(n, q) X^n \right).$$

For the definitions of operator Log and Exp, we refer to the Appendix in Mozgovoy [7].

$H_q(n, q)$’s are rational functions in $q$, so are $A_g(n, q)$’s. As $A_g(n, q)$’s take integer values for all prime powers $q$, $A_g(n, q)$’s must be polynomials in $q$ with rational coefficients. It follows from Lemma 2.9 of Bozec, Schiffmann & Vasserot [1] that $A_g(n, q) \in \mathbb{Z}[q]$. Kac [6] implies that the degree of polynomial $A_g(n, q)$ is at most $(g - 1)n^2$. $I_g(n, q)$ and $M_g(n, q)$ can be calculated by the following identities:

$$I_g(n, q) = \sum_{d | n} \sum_{r | d} \mu \left( \frac{d}{r} \right) A_g \left( \frac{n}{d}, q^r \right),$$

$$1 + \sum_{n=1}^{\infty} M_g(n, q) X^n = \prod_{n=1}^{\infty} (1 - X^n)^{-I_g(n, q)}.$$

The first identity is the counterpart of the first identity of Theorem 4.1 from Hua [5] and the second identity is a consequence of the Krull–Schmidt Theorem from representation theory. It follows that $I_g(n, q)$ and $M_g(n, q)$ are polynomials in $q$ with rational coefficients for all $n \geq 1$.

Theorem 4.3. Let

$$A_g(n, q) = \sum_{s=0}^{(g-1)n^2} a_{n,s} q^s,$$

where $a_{n,s} \in \mathbb{Z}$. Then the following identity holds in $\mathbb{Q}(q)[[X]]$:

$$P(X, q) = \prod_{n=1}^{\infty} \prod_{s=0}^{(g-1)n^2} \prod_{i=0}^{\infty} (1 - q^{s+i} X^n)^{a_{n,s}}.$$
Proof. This is the counterpart of Theorem 4.9 from Hua [5], same arguments apply.

In the context of representations of quivers over finite fields, Kac [6] conjectured that the constant term of the polynomial counting isomorphism classes of absolutely indecomposable representations with a given dimension vector is the same as the root multiplicity of the dimension vector in the corresponding Kac-Moody algebra. This conjecture was proved by Crawley-Boevey and Van den Bergh [2] for indivisible dimension vectors and by Hausel [4] in general, which confirms that Theorem 4.9 from Hua [5] is a $q$-deformation of Weyl-Kac denominator identity. Thus Theorem 4.3 here may also be regarded as a $q$-deformation of Weyl-Kac denominator identity for some generalized Kac-Moody algebra.

When $g = 1$, Jordan Canonical Form Theorem shows that there exists only one indecomposable nilpotent matrix over $\mathbb{F}_q$ of a given order up to conjugation, where the unique conjugacy class is the Jordan matrix with eigenvalue 0:

$$
\begin{bmatrix}
0 & 1 & 0 & \ldots & 0 \\
0 & 0 & 1 & \ldots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & 1 \\
0 & 0 & 0 & \ldots & 0
\end{bmatrix}
$$

This implies that $A_1(n, q) = 1$ for all $n \geq 1$. Thus Theorem 4.3 amounts to the following identity:

$$
1 + \sum_{\lambda \in \mathcal{P}} \frac{q^{-l(\lambda)}}{b_\lambda(q^{-1})} X^{|\lambda|} = \prod_{n=1}^{\infty} \prod_{i=0}^{\infty} (1 - q^i X^n).
$$

Conjecture 4.1. For any $g \geq 2$ and $n \geq 1$, all coefficients of the polynomial $A_g(n, q)$ are non-negative integers.

This conjecture is supported by the following observations generated by a Python program based on Theorem 4.2:

- $A_2(1, q) = 1$,
- $A_2(2, q) = 2q$,
- $A_2(3, q) = q^4 + 3q^2 + 2q$,
- $A_2(4, q) = q^9 + q^7 + q^6 + 4q^3 + 2q^3 + 7q^3 + 4q^2 + 2q$,
- $A_2(5, q) = q^{16} + q^{14} + q^{13} + 2q^{12} + 2q^{11} + 4q^{10} + 4q^9 + 7q^8 + 8q^7 + 13q^6 + 13q^5 + 16q^4 + 14q^3 + 7q^2 + 2q$,
- $A_2(6, q) = q^{25} + q^{23} + q^{22} + 2q^{21} + 2q^{20} + 4q^{19} + 3q^{18} + 7q^{17} + 7q^{16} + 10q^{15} + 11q^{14} + 19q^{13} + 17q^{12} + 28q^{11} + 29q^{10} + 39q^9 + 40q^8 + 53q^7 + 48q^6 + 52q^5 + 40q^4 + 25q^3 + 8q^2 + 2q$.


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