Essential spectrum of a class of Riemannian manifolds

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Abstract  In this paper we consider a family of Riemannian manifolds, not necessarily complete, with curvature conditions in a neighborhood of a ray. Under these conditions we obtain that the essential spectrum of the Laplace operator contains an interval. The results presented in this paper allow to determine the spectrum of the Laplace operator on unlimited regions of space forms, such as horoball in hyperbolic space and cones in Euclidean space.

Keywords  Laplace operator, essential spectrum, Riemannian manifold.

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1 Introduction

Let $M$ be a simply connected Riemannian manifold. The Laplace operator $\Delta : C_0^\infty (M) \to C_0^\infty (M)$, defined as $\Delta u = \text{div}(\text{grad} \ u)$, is a second order elliptic operator and it has a unique extension $\Delta$ to an unbounded self-adjoint operator on $L^2(M)$. Since $-\Delta$ is positive and symmetric, its spectrum is the set of $\lambda \geq 0$ such that $\Delta + \lambda I$ does not have bounded inverse. Sometimes we say spectrum of $M$ rather than spectrum of $-\Delta$. One defines the essential spectrum $\sigma_{ess}(-\Delta)$ to be those $\lambda$ in the spectrum which are either accumulation points of the spectrum or eigenvalues of infinite multiplicity.

It is well-known that if $M$ is an $n$-dimensional simply connected complete manifold with constant curvature $-c \leq 0$, then its essential spectrum coincides with the spectrum, being such the interval $[(n-1)^2c/4, \infty)$. Moreover, the Decomposition Principle [5] says that the essential spectrum is invariant under compact perturbations of the metric on $M$ and is thus a function of the geometry of the ends. Therefore, it becomes natural the search of geometric conditions, of the ends of the surface, that will determine the essential spectrum of the Laplacian. In 1981, Harold Donnelly in [3] studied the essential spectrum of manifolds which curvature approaches a constant $-c \leq 0$ at infinity. It was shown that the essential spectrum is $[(n-1)^2c/4, \infty)$ if either (i) $M$ is simply connected and negatively curved or (ii) $M$ is a surface with finitely generated fundamental group and an additional decay condition is satisfied for $K + c \to 0$, where $K$ is the Gaussian curvature. In 1992, Escobar and Freire [6] proved that the spectrum of the Laplacian is $[0, \infty)$, using that the sectional curvature is non-negative and the manifold satisfies some additional conditions. In [2], Detang Zhou proved that those additional conditions could be removed. In 1994, Li [9] proved $\sigma_{ess}(-\Delta) = [0, \infty)$ if $M$ has nonnegative Ricci curvatures and a pole. Chen
and Zhiqin Lu [10] proved the same result when the radial sectional curvature is non-negative. Among other results, in [4], Donnelly proved that the essential spectrum is \([0, \infty)\) for manifold with non-negative Ricci curvature and Euclidean volume growth. In 1997, Kumura [7, Theorem 1.2] presented the following result: if \(r\) is the distance function from a pole, then \(\sigma_{\text{ess}}(-\Delta) = [c^2/4, \infty)\) provided
\[
\lim_{n \to \infty} \sup_{r \geq n} |\Delta r - c| = 0.
\]
(1)
Kumura also shows that this result recovers almost all the previous ones mentioned above.

In 1997, J. Wang in [8] proved that, if the Ricci curvature of a manifold \(M\) satisfies 

\[\text{Ric}(M) \geq -\delta/r^2,\]

where \(r\) is the distance to a fixed point, and \(\delta\) is a positive number depending only on the dimension, then the \(L^p\) essential spectrum of \(M\) is \([0, \infty)\) for any \(p \in [1, +\infty]\). In 2011, Zhiqin Lu and Detang Zhou in [11] proved that the \(L^p\) essential spectrum of the Laplacian is \([0, +\infty)\) on a noncompact complete Riemannian manifold when 

\[\lim_{x \to \infty} \text{Ric}_M(x) = 0.\]

The last result we want to recall is a theorem of Donnelly and Li. Fix a point \(p \in M\) and write \(K(r) = \sup\{K(x, \pi)|d(p, x) \geq r\}\) where \(K(x, \pi)\) is the sectional curvature of a two plane \(\pi\) in \(T_x M\). Then the following theorem was proved in [5].

**Theorem 1 (H. Donnelly and P. Li)** Let \(M\) be a complete Riemannian manifold and suppose \(\bar{K}(r) \to -\infty\) as \(r \to \infty\). Then the essential spectrum of \(\Delta\) is empty provided one of the following two side conditions is satisfied: (i) \(M\) is simply connected and negatively curved. (ii) \(M\) is two dimensional and the fundamental group of \(M\) is finitely generated.

In this paper we consider a family of Riemannian manifolds, not necessarily complete, with curvature conditions like (1), but not uniform fashion over \(M\). We require that it holds just in a neighborhood of a ray. With these conditions we obtain that the essential spectrum of the Laplacian contains an interval. This way, we are able to construct an example of a two dimensional negatively curved Riemannian manifold satisfying
\[
\lim_{r \to \infty} K(r, \theta)(\frac{\partial}{\partial r}, \frac{\partial}{\partial \theta}) = -\infty
\]
for all \(\theta \neq 0\) and such that the essential spectrum of this manifold contains the interval \([1/4, \infty)\), see the appendix. This example indicates that, in order for the essential spectrum to be empty, some type of geometric global conditions like Theorem 1 should be necessary. The following Theorem provides some geometric conditions just on a neighborhood of a ray in order to guarantee that the essential spectrum is non-trivial.

**Theorem 2** Let \(M\) be an \(n\)-dimensional Riemannian manifold. Suppose that, in geodesic spherical coordinates, its metric can be written as
\[
g_M = dr^2 + \psi^2(rw)g_{S^{n-1}}
\]
on \(C_a(N) = \{rw : \text{dist}_{S^{n-1}}(w, N) < c_2 e^{-ar}\}\), where \(g_{S^{n-1}} = ds^2 + \sin^2 s g_{S^{n-2}}\) and \(s\) is the distance function in \(S^{n-1}\) to \(N \in S^{n-1}\). Furthermore, the function \(\psi\) satisfies
i) \[ \lim_{w \to N} \frac{\psi(rw)}{\psi(w)} = c > 0 \text{ uniform on the set } C_a(N), \text{ and } c > a \]

ii) \[ \left| \frac{\psi_s(rw)}{\psi(rw)} \right| \leq c_1 \text{ for all } rw \in C_a(N) \]

Then \( [(n-1)^2 c^2 / 4, \infty) \subset \sigma_{ess}(-\Delta) \).

We observe that changing the metric, outside the set \( C_a(N) \), or changing the topology of \( M \), the interval \( [(n-1)^2 c^2 / 4, \infty) \) remains contained in the essential spectrum of \( M \). In fact, we prove that \( [(n-1)^2 c^2 / 4, \infty) \) is contained in the spectrum of \( C_a(N) \). As an application of this theorem we can show that horoball and hyperbolic space has the same spectrum. For this purpose it is sufficient to show that horoball \( \{ rw; \sin^2 \theta \leq (1 + \cos s)^2 \} \) contains a set of the form \( \{ rw; \text{dist}_{S^{n-1}} (w, N) = e^{-r/2} \} \).

In the case where \( c = 0 \), the following theorem implies that the essential spectrum of a cone \( \{ rw; \text{dist}_{S^{n-1}} (w, N) < c_2 \} \) in \( \mathbb{R}^n \) is the interval \( [0, \infty) \).

**Theorem 3** Let \( M \) be a \( n \)-dimensional Riemannian manifold. Suppose that, in geodesic spherical coordinates, its metric can be written as

\[ g_M = dr^2 + \psi^2(rw)g_{S^{n-1}} \]

on the set \( C_0(N) = \{ rw; \text{dist}_{S^{n-1}} (w, N) < c_2 \} \), where \( g_{S^{n-1}} = ds^2 + \sin^2 s g_{S^{n-2}} \) and \( s \) is the distance function in \( S^{n-1} \) to \( N \in S^{n-1} \). Furthermore, assume that function \( \psi \) satisfies

i) \[ \lim_{r \to \infty} \frac{\psi(rw)}{\psi(rw)} = 0 \text{ to each } w \text{ such that } \text{dist}_{S^{n-1}} (w, N) < c_2 \]

ii) \[ \lim_{r \to \infty} \psi(rw) = +\infty \text{ to each } w \text{ such that } \text{dist}_{S^{n-1}} (w, N) < c_2 \]

iii) \[ \left| \frac{\psi_s(rw)}{\psi(rw)} \right| \leq c_1 \frac{1}{r^\gamma} \text{ for all } rw \in C_0(N) \text{ for some } \gamma > 1. \]

Then \( \sigma_{ess}(-\Delta) = [0, \infty) \).

In both theorems, the function \( \psi(rw) \) on the metric satisfies the following conditions:

\( \psi(0) = 0 \) and \( \psi(rw) > 0 \).

For more detail, see reference [10]. The average curvature of the geodesic sphere of \( M \) is given by

\[ \frac{\psi_r(rw)}{\psi(rw)} = \frac{1}{n-1} \Delta r. \]
2 Spectral Theory

A linear operator on a Hilbert space \( \mathcal{H} \) is a pair consisting of a dense linear subspace \( \text{Dom}(A) \) of \( \mathcal{H} \) together with a linear map \( A : \text{Dom}(A) \to \mathcal{H} \). The adjoint operator \( A^* \) is determined by the condition that \( \langle Au, v \rangle = \langle u, A^*v \rangle \) for all \( u \in \text{Dom}(A) \) and \( v \in \text{Dom}(A^*) \).

The domain of \( A^* \) is defined to be the set of all \( v \) for which there exists \( w \in \mathcal{H} \) such that \( \langle Au, v \rangle = \langle u, w \rangle \), for all \( u \in \text{Dom}(A) \). We say that \( A \) is self-adjoint if \( A = A^* \). The spectrum of a linear operator \( A \), \( \sigma(A) \), is defined as follows. We say that a complex number \( z \) does not lie in \( \sigma(A) \) if the operator \((z - A)\) maps \( \text{Dom}(A) \) one-one onto \( \mathcal{H} \), and the inverse \((z - A)^{-1}\) is bounded. The spectrum of any self-adjoint operator is real and non-empty. A complex number is said to be an eigenvalue of such an operator \( A \) if there exists a non-zero \( u \in \text{Dom}(A) \) such that \( Au = \lambda u \). It is entirely possible that no point of the spectrum of \( A \) is an eigenvalue. The discrete spectrum \( \sigma_d(A) \) is defined as the set of all eigenvalues \( \lambda \) of finite multiplicity which are isolated point of the spectrum. The essential spectrum is the set \( \sigma_{\text{ess}}(A) = \sigma(A)/\sigma_d(A) \). A characterization of the essential spectrum is given in following lemma which is a consequence of the spectral theorem [1, Lemma 8.4.1, p.167].

Lemma 4 Let \( A \) be a self-adjoint operator acting on the Hilbert space \( \mathcal{H} \) and let \( \lambda \in \mathbb{R} \). The following are equivalent:

i) \( \lambda \in \sigma_{\text{ess}}(A) \)

ii) For all \( \varepsilon > 0 \) there exists a subspace \( L_\varepsilon \subset \text{Dom}(A) \) with \( \dim(L_\varepsilon) = \infty \) and such that \( \|Au - \lambda u\| \leq \varepsilon \|u\| \) for all \( u \in L_\varepsilon \).

3 Proof of Theorem

First we study the behavior of the function \( \psi \) on the set \( C_a(N) \). Let us prove that for any \( \eta > 0 \) there are \( r_\eta > 0 \) such that

\[
C_1 e^{(c-\eta)r} \leq \psi(rw) \leq C_2 e^{(c+\eta)r} \tag{2}
\]

and

\[
\frac{1}{2} \leq \frac{\psi(rw)}{\psi(rN)} \leq \frac{3}{2} \tag{3}
\]

for all \( r \geq r_\eta \) and \( rw \in C_a(N) \), where \( C_1 \) and \( C_2 \) are positive constants. In fact, by the limit in the item i) of the Theorem for any \( \eta > 0 \), there is \( r_0 \) such that

\[
c - \eta \leq \frac{\psi(rw)}{\psi(rw)} \leq c + \eta
\]

for all \( r \geq r_0 \) and \( rw \in C_a(N) \). Integrating the inequality above from \( r_0 \) to \( r \), we obtain

\[
e^{(c-\eta)(r-r_0)} \leq \frac{\psi(rw)}{\psi(r_0w)} \leq e^{(c+\eta)(r-r_0)} \tag{4}
\]
for all \( r \geq r_0 \) and \( rw \in C_\alpha(N) \). By continuity and positivity of the function \( w \mapsto \psi(r\eta w) \),
\[
\inf_{w \in S^{n-1}} \psi(r_0 w) > 0
\]
and by (4)
\[
0 < C_1 e^{(c-\eta)r} \leq \psi(rw) \leq C_2 e^{(c+\eta)r},
\]
for all \( r \geq r_0 \) and \( rw \in C_\alpha(N) \), where
\[
C_1 = \inf_{w \in S^{n-1}} \psi(r_0 w)e^{-(c-\eta)r_0} \quad \text{and} \quad C_2 = \sup_{w \in S^{n-1}} \psi(r_0 w)e^{-(c+\eta)r_0}.
\]

To prove (3), consider \( \alpha : [0, s] \to S^{n-1} \) the geodesic such that \( \alpha(0) = N \), \( \alpha(s) = w \) and \( \alpha'(t) = \partial/\partial s \). If \( \lambda(t) = \psi(r\alpha(t)) \), by Mean Value Theorem, there is \( t_0 \in (0, s) \) such that
\[
\lambda(s) - \lambda(0) = \lambda'(t_0) s = s g_M(r\alpha(t_0)) (\nabla \psi, r\alpha'(t_0))
\]
\[
\psi(rw) - \psi(rN) = s g_M(r\alpha(t_0)) \left( \psi_r \frac{\partial}{\partial r} + \psi_s \frac{\partial}{\partial s} + \frac{1}{\psi^2} \nabla_{S^2} \psi, r \frac{\partial}{\partial s} \right)
\]
\[
\psi(rw) - \psi(rN) = rs \psi_s(r\alpha(t_0))
\]
By the inequality in the item ii) of the Theorem (2),
\[
|\psi(rw) - \psi(rN)| \leq c_1 rs \psi(r\alpha(t_0))
\]
\[
\left| \frac{\psi(rw)}{\psi(rN)} - 1 \right| \leq \frac{c_1 rs \psi(r\alpha(t_0))}{\psi(rN)}
\]
Since \( \text{dist}_{S^{n-1}}(\alpha(t_0), N) < s = \text{dist}_{S^{n-1}}(w, N) \leq c_1 e^{-ar} \) and using (2)
\[
\left| \frac{\psi(rw)}{\psi(rN)} - 1 \right| \leq C \frac{re^{(c+\eta)r}e^{-ar}}{e^{(c-\eta)r}} = C \frac{r}{e^{(a-2\eta)r}} \to 0
\]
when \( r \to +\infty \), if \( 0 < \eta < a/2 \). Then there is \( r_\eta \geq r_0 \) which we obtain
\[
\frac{1}{2} \leq \frac{\psi(rw)}{\psi(rN)} \leq \frac{3}{2}
\]
for all \( r \geq r_\eta \) and \( rw \in C_\alpha(N) \).

In order to prove the Theorem 2 we will construct, for any \( \lambda > (n-1)^2 \epsilon^2/4 \) and \( \epsilon > 0 \), a sequence of functions \( (u_k) \subset C^\infty_0(M) \) with disjoint supports \( \text{supp} u_j \cap \text{supp} u_k = 0, \) for all \( j \neq k \), such that
\[
||\Delta u_k + \lambda u_k||_{L^2} \leq \epsilon ||u_k||_{L^2}, \quad k = 1, 2, \ldots
\]
(5)

This implies that \( [(n-1)^2 \epsilon^2/4, \infty) \subset \sigma_{ess}(-\Delta) \) by the Lemma 3 and the fact that \( \sigma_{ess}(-\Delta) \) is a closed set. Indeed each function \( u_k \) will have support on \( C_\alpha(N) \) and
\[
u_k(rw) = f(r)g(s)
\]
(6)

where \( s = \text{dist}_{S^{n-1}}(w, N) \). The function \( f \) is defined by
\[
f(r) = f(r, k, p) = F(r)h(r, k, p)
\]
(7)
where

\[ F(r) = v(r)^{-1/2} \cos(\beta r), \]  
\( \beta = \sqrt{\lambda - (n - 1)^2 c^2 / 4}, \)

\[ v(r) = \int_0^r \psi_{n-1}(\tau N) d\tau, \]  
\[ h(r) = h(r, k, p) = H \left( 2(r - r_{k+2p})/(r_{k+4p} - r_k) \right) \]

the is a scaled cut-off function centered at \( r_{k+2p} \), where \( r_k = (2k + 1)\pi / (2\beta) \) is the zero of the function \( \cos(\beta r) \), and \( H \in C^\infty_0(\mathbb{R}) \) is a cut-off function satisfying the conditions

\[
\begin{cases}
H \equiv 1 \text{ on } [-1/2, 1/2] \\
H \equiv 0 \text{ on } \mathbb{R} \setminus [-1, 1] \\
0 \leq H \leq 1 \text{ on } \mathbb{R}.
\end{cases}
\]

The function \( g \) is defined by

\[ g(s) = g(s, k, p) = H(s/\delta_{k,p}) \cos(\pi s/\delta_{k,p}), \]  
\( \delta_{k,p} = c_2 e^{-ar_{k+4p}} \).

Now we will prove that there are \( k_0 > 0 \) and \( p_0 > 0 \) such that the functions \( u_k = f(r, k, p)g(s, k, p) \) defined in (6) satisfies the inequality (5) for all \( k \geq k_0 \) and \( p = p(k) \geq p_0 \).

The function \( v(r) \) defined in (9) satisfies

\[ v'(r) = \psi^{n-1}(r N) \quad \text{and} \quad v''(r) = (n - 1)\psi^{n-2}(r N)\psi_r(r N). \]

By (2), \( \psi(r N) \geq M_1 e^{(c-\eta)r} \) for all \( r \geq r_\eta \) and \( c > \eta > 0 \). Then

\[ \lim_{r \to \infty} v'(r) = \lim_{r \to \infty} v(r) = +\infty \]

and

\[ \lim_{r \to \infty} \frac{v'(r)}{v(r)} = \lim_{r \to \infty} \frac{v''(r)}{v'(r)} = (n - 1) \lim_{r \to \infty} \frac{\psi_r(r N)}{\psi(r N)} = (n - 1)c. \]  

(12)

So there is a \( r_v \geq r_\eta \) such that

\[ \frac{(n - 1)c}{2} \leq \frac{v'(r)}{v(r)} \leq \frac{3(n - 1)c}{2} ; \quad \forall \ r \geq r_v. \]  

(13)

The function \( F(r) \) defined in (8) satisfies

\[ \Delta F + \lambda F = A(r)F + B(rw)F' \]  
(14)

where

\[ A(r) = -\frac{1}{2} \frac{v''}{v'} \cdot \frac{v'}{v} + \frac{1}{4} \left( \frac{v'}{v} \right)^2 + \frac{(n - 1)^2 c^2}{4} \]

and

\[ B(rw) = (n - 1) \frac{\psi_r(rw)}{\psi} - \frac{v'}{v}. \]
Note that, by (12) and i) of the Theorem 2
\[ \lim_{r \to \infty} A(r) = 0 \quad \text{and} \quad \lim_{r \to \infty} B(rw) = 0 \] (15)
uniform on \( C_a(N) \). By (14), the function \( f \) defined in (7) satisfies
\[ \Delta f + \lambda f = A(r)F h + B(r, w)F' h + 2F' h' + F \Delta h. \] (16)

We have the following estimates for the function \( h \) defined in (10)
\[ |h'| \leq \frac{\beta}{4\pi p} \sup |H'| \chi_h \quad \text{and} \quad |h''| \leq \frac{\beta^2}{16\pi^2 p^2} \sup |H''| \chi_h. \] (17)

The Laplacian of the function \( g = g(s) \), defined in (11), is
\[ \Delta g = \frac{(n-3)\psi_s}{\psi^3} g' + \frac{(n-2) \cot(s)}{\psi^2} g' + \frac{1}{\psi^2} g''. \] (18)

We observed that
\[ |g'| \leq \frac{C}{\delta_{k,p}} \chi_B(\delta_{k,p}) \] (19)
and
\[ |g''| \leq \frac{C}{\delta^2_{k,p}} \chi_B(\delta_{k,p}) \] (20)
where \( C \) is independent of \( k \) and \( p \), and \( \chi_B(\delta_{k,p}) : S^{n-1} \to \mathbb{R} \) is the characteristic function of the set \( B(\delta_{k,p}) = \{ w \in S^{n-1} ; \text{dist}_{S^{n-1}} (w, N) \leq \delta_{k,p} \} \).

Finally, the function \( u = u_k(rw) = f(r, k, p) g(s, k, p) \) satisfies
\[ \Delta u = (\Delta f) g + f(\Delta g) \]

Hence by (16) and (18) we conclude
\[ \Delta u + \lambda u = \]
\[ AFgh + BF'gh + 2F'gh' + Fg\Delta h + \frac{(n-3)\psi_s}{\psi^3} g' f + \frac{(n-2) \cot(s)}{\psi^2} g' f + \frac{1}{\psi^2} f g'' \]

By (15), given \( \delta > 0 \), there is \( r_0 > r_v \) such that
\[ |A(r)| \leq \delta \quad \text{and} \quad |B(rw)| \leq \delta \]
for all \( r \geq r_0 \) and \( rw \in C_a(N) \).

By (17) there exists \( r_h \geq r_0 \) such that
\[ \| F' g h' \|_2 \leq \delta \| \chi_h F' g \|_2 \quad \text{and} \quad \| F g \Delta h \|_2 \leq \delta \| \chi_h F g \|_2 \]
for all \( r \geq r_h \), followed by (21)
\[ \| \Delta u + \lambda u \|_2 \leq \]
\[ \delta \left( \| \chi_h F g \|_2 + \| \chi_h F' g \|_2 \right) + C \left\| \psi_s f g' \right\|_2 + C \left\| \psi^2 f g' \right\|_2 + \left\| \frac{1}{\psi^2} f g'' \right\|_2 \]
for all \( r \geq r_h \) and \( rw \in C_a(N) \). We will need to use the technical lemma:
Lemma 5 For the functions $F$, $f$, $g$ and $u$ defined previously, we have the following inequalities

(a) $\|\chi Fg\|_2 \leq C \|u\|_2$
(b) $\|\chi F^\prime g\|_2 \leq C \|u\|_2$
(c) $\|\psi \chi h Fg\|_2 \leq C \|\chi u\|_2$
(d) $\|\psi^2 \chi f^\prime g\|_2 \leq C \|\chi u\|_2$
(e) $\|\psi^2 \chi f'' g\|_2 \leq C \|\chi u\|_2$

where $C_{k,p} = \{rw; r_k \leq r \leq r_{k+4p}, w \in B(\delta_{k,p})\}$ and $C$ is a positive constant independent of $k$ and $p$.

Proof of Lemma: Observe that

$$\|u\|_2^2 = \int_{B(\delta_{k,p})} g^2(s) \int_{r_k}^{r_{k+4p}} \cos^2(\beta r) h^2(r) \frac{\psi^{n-1}(rw)}{v(r)} dr dw \geq \int_{B(\delta_{k,p})} g^2(s) \int_{r_k}^{r_{k+3p}} \cos^2(\beta r) \frac{\psi^{n-1}(rw)}{\psi^{n-1}(rN)} \cdot \frac{v'(r)}{v(r)} dr dw$$

and

$$\|\chi Fg\|_2^2 = \int_{B(\delta_{k,p})} g^2(s) \int_{r_k}^{r_{k+4p}} \cos^2(\beta r) \frac{\psi^{n-1}(rw)}{v(r)} dr dw.$$

We observe of the estimates (13) and (3) that

$$\|u\|_2^2 \geq \frac{(n-1)c}{2} \left( \frac{1}{2} \right)^{n-1} \int_{B(\delta_{k,p})} g^2(s) \int_{r_k}^{r_{k+3p}} \cos^2(\beta r) dr dw \quad (23)$$

and

$$\|\chi Fg\|_2^2 \leq \frac{3(n-1)c}{2} \left( \frac{3}{2} \right)^{n-1} \int_{B(\delta_{k,p})} g^2(s) \int_{r_k}^{r_{k+4p}} \cos^2(\beta r) dr dw.$$

Moreover,

$$\int_{r_k}^{r_{k+4p}} \cos^2(\beta r) dr = 2 \int_{r_k}^{r_{k+3p}} \cos^2(\beta r) dr.$$

The last two inequalities imply

$$\|\chi Fg\|_2 \leq \sqrt{2} \frac{3^{n/2}}{2} \|u\|_2. \quad (24)$$

For the second inequality, using integration by parts

$$\int_{r_k}^{r_{k+4p}} F'(r)^2 \psi^{n-1}(rw) dr = - \int_{r_k}^{r_{k+4p}} F(r) \Delta F(r) \psi^{n-1}(rw) dr.$$
through of the equality (14)
\[ \| \chi h F' g \|_2^2 = \int_{B(\delta_{k,p})} g^2(s) \int_{r_k}^{r_{k+4p}} F' \left[ \lambda F - A(r) F - B(r, w) F' \right] \psi^{n-1}(rw) \, dr \, dw \]
of the limits in (15) it follows
\[ \| \chi h F' g \|_2^2 \leq (\lambda + 1) \int_{B(\delta_{k,p})} g^2(s) \int_{r_k}^{r_{k+4p}} F^2 \psi^{n-1}(rw) \, dr \, dw \]
\[ + \int_{B(\delta_{k,p})} g^2(s) \int_{r_k}^{r_{k+4p}} |F| |F'| \psi^{n-1}(rw) \, dr \, dw \leq (\lambda + \frac{3}{2}) \| \chi h F g \|_2^2 + \frac{1}{2} \| \chi h F' g \|_2^2 \]
and of the inequality (24) we obtain
\[ \| \chi h F' g \|_2 \leq (2\lambda + 3) \| \chi h F g \|_2 \]

Now the third inequality
\[ \left\| \frac{\psi h f'}{\psi^3 f} \right\|_2^2 = \int_{B(\delta_{k,p})} \frac{|g'|^2}{g^2} \int_{r_k}^{r_{k+4p}} \frac{\psi^2}{\psi^6} f^2(r) \psi^{n-1} \, dr \, dw \]
By virtue of the estimate (19) and the hypothesis ii) of the Theorem (2)
\[ \left\| \frac{\psi h f'}{\psi^3 f} \right\|_2^2 \leq \frac{C}{\delta_{k,p}^2} \left[ \inf_{C_{k,p}} \left| \psi \right| \right]^{-4} \int_{B(\delta_{k,p})} \int_{r_k}^{r_{k+4p}} f^2(r) \psi^{n-1} \, dr \, dw \]

By definition, \( f(r) = v^{-1/2} \cos(\beta r) h(r) \) and \( v'(r) = \psi^{n-1}(rN) \), then
\[ \int_{B(\delta_{k,p})} \int_{r_k}^{r_{k+4p}} f^2(r) \psi^{n-1} \, dr \, dw = \int_{B(\delta_{k,p})} \int_{r_k}^{r_{k+4p}} \cos^2(\beta r) h^2(r) \frac{\psi^{n-1}}{v} \, dr \, dw \]
\[ \leq \int_{B(\delta_{k,p})} \int_{r_k}^{r_{k+4p}} \cos^2(\beta r) \frac{\psi^{n-1}(rw) v'}{\psi^{n-1}(rN) v} \, dr \, dw. \]

By the estimates (13) and (3)
\[ \int_{B(\delta_{k,p})} \int_{r_k}^{r_{k+4p}} f^2(r) \psi^{n-1} \, dr \, dw \leq C \int_{B(\delta_{k,p})} \int_{r_k}^{r_{k+4p}} \cos^2(\beta r) \, dr \, dw \]
where
\[ \int_{B(\delta_{k,p})} \, dw = \int_{\mathbb{S}^{n-2}} \int_0^{\delta_{k,p}} \sin^{n-2} s \, ds \, d\xi \]
and \( d\xi \) is the canonical measure of \( \mathbb{S}^{n-2} \).
There exists \( s_0 > 0 \) such that
\[ \frac{1}{2} \leq \frac{\sin s}{s} \leq \frac{3}{2} \]

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for all $0 < s < s_0$. If $0 < \delta_{k,p} < s_0$ we have

$$\int_{B(\delta_{k,p})} dw \leq C \int_{\mathbb{S}^{n-2}} \int_0^{\delta_{k,p}} s^{n-2} ds \, d\xi = C \delta_{k,p}^{n-1}$$

and also

$$\int_0^{\delta_{k,p}/2} \cos^2 \left( \frac{\pi s}{\delta_{k,p}} \right) \sin^{-2} s \, ds \geq C \int_0^{\delta_{k,p}/2} \cos^2 \left( \frac{\pi s}{\delta_{k,p}} \right) s^{n-2} \, ds$$

where

$$\int_0^{\delta_{k,p}/2} \cos^2 \left( \frac{\pi s}{\delta_{k,p}} \right) s^{n-2} \, ds = \frac{\delta_{k,p}^{n-1}}{\pi^{n-1}} \int_0^{\pi/2} \cos^2 \beta s \, s^{n-2} \, ds = C \delta_{k,p}^{n-1}.$$

Then we conclude

$$\int_{B(\delta_{k,p})} dw \leq C \int_{\mathbb{S}^{n-2}} \int_0^{\delta_{k,p}/2} \cos^2 \left( \frac{\pi s}{\delta_{k,p}} \right) \sin^{-2} s \, ds \, d\xi$$

by definition of the function $G_{\delta_{k,p}}$ follow that

$$\int_{B(\delta_{k,p})} dw \leq C \int_{\mathbb{S}^{n-2}} \int_0^{\delta_{k,p}/2} C_{\delta_{k,p}}^2(s) \cos^2 \left( \frac{\pi s}{\delta_{k,p}} \right) \sin^{-2} s \, ds \, d\xi$$

$$\leq C \int_{\mathbb{S}^{n-2}} \int_0^{\delta_{k,p}/2} C_{\delta_{k,p}}^2(s) \cos^2 \left( \frac{\pi s}{\delta_{k,p}} \right) \sin^{-2} s \, ds \, d\xi = C \int_{B(\delta_{k,p})} g^2(s) \, dw.$$

Using the last inequality in the the estimate (26) we have

$$\int_{B(\delta_{k,p})} \int_{r_k}^{r_{k+4p}} f^2(r) \psi^{n-1} \, dr \, dw \leq C \int_{B(\delta_{k,p})} g^2(s) \int_{r_k}^{r_{k+4p}} \cos^2(\beta r) \, dr \, dw \quad (27)$$

using (23), we obten

$$\int_{B(\delta_{k,p})} \int_{r_k}^{r_{k+4p}} f^2(r) \psi^{n-1} \, dr \, dw \leq C \|u\|_2^2 \quad (28)$$

by (25) verified

$$\left\| \frac{\psi_s}{\psi_{\psi^3}} \left( g \right) \right\|_2^2 \leq C \delta_{k,p}^2 \left[ \inf_{C_{k,p}} \frac{\psi}{|\psi|} \right]^{-4} \|u\|_2^2 \quad (29)$$

Now we will show the fourth inequality, using a similar procedure like the last inequality

$$\left\| \frac{\cot s}{\psi^2} f g \right\|_2^2 \leq C \delta_{k,p}^2 \left[ \inf_{C_{k,p}} \frac{\psi}{|\psi|} \right]^{-4} \int_{B(\delta_{k,p})} \int_{r_k}^{r_{k+4p}} \cot^2 s \, \cos^2(\beta r) \, dr \, dw.$$

Since

$$\int_{B(\delta_{k,p})} \cot^2 s \, dw = \int_{\mathbb{S}^{n-2}} \sqrt{\det \xi} \int_0^{\delta_{k,p}} \sin^{-4} s \, ds \, d\xi \leq \leq C \int_{\mathbb{S}^{n-2}} \sqrt{\det \xi} \int_0^{\delta_{k,p}} s^{-4} \, ds \, d\xi = C \delta_{k,p}^{n-3} \int_{\mathbb{S}^{n-2}} \sqrt{\det \xi} \, d\xi$$

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and

\[ \int_{B(\delta_{k,p})} g^2(s) \, dw = \int_{\mathbb{R}^{n-2}} \sqrt{\det \xi} \int_0^{\delta_{k,p}} C_{\delta_{k,p}}^2(s) \cos^2 \left( \frac{\pi s}{\delta_{k,p}} \right) \sin^{n-2} s \, ds \, d\xi \]

\[ \geq \int_{\mathbb{R}^{n-2}} \sqrt{\det \xi} \int_0^{\delta_{k,p}/2} \cos^2 \left( \frac{\pi s}{\delta_{k,p}} \right) \sin^{n-2} s \, ds \, d\xi \]

\[ \geq C \int_{\mathbb{R}^{n-2}} \sqrt{\det \xi} \int_0^{\delta_{k,p}/2} \cos^2 \left( \frac{\pi s}{\delta_{k,p}} \right) s^{n-2} \, ds \, d\xi \]

notice the following

\[ \int_0^{\delta_{k,p}/2} \cos^2 \left( \frac{\pi s}{\delta_{k,p}} \right) s^{n-2} \, ds = \frac{\delta_{k,p}^{n-1}}{n-1} \int_0^{\pi/2} \cos^2 s s^{n-2} \, ds = C \delta_{k,p}^{n-1}. \]

thus

\[ \int_{B(\delta_{k,p})} g^2(s) \, dw \geq C \delta_{k,p}^{n-1} \int_{\mathbb{R}^{n-2}} \sqrt{\det \xi} \, d\xi. \]

However

\[ \int_{B(\delta_{k,p})} \cot^2 s \, ds \, dw \leq C \delta_{k,p}^{-2} \int_{B(\delta_{k,p})} g^2(s) \, dw \]

in this way we ensure

\[ \left\| \frac{\cot(s)}{\psi^2} f g' \right\|_2^2 \leq \frac{C}{\delta_{k,p}} \left[ \inf_{C_{k,p}} |\psi| \right]^{-4} \int_{B(\delta_{k,p})} \int_{r_k}^{r_{k+4p}} g^2(s) \cos^2(\beta r) \, dr \, dw \]

of inequality (23) we deduce

\[ \left\| \frac{\cot(s)}{\psi^2} f g' \right\|_2^2 \leq \frac{C}{\delta_{k,p}} \left[ \inf_{C_{k,p}} |\psi| \right]^{-4} \|u\|_2^2. \quad (30) \]

To finish, using (20) and the same reasoning above

\[ \left\| \frac{1}{\psi^2} f g'' \right\|_2^2 \leq \frac{C}{\delta_{k,p}} \left[ \inf_{C_{k,p}} |\psi| \right]^{-4} \int_{B(\delta_{k,p})} \int_{r_k}^{r_{k+4p}} f^2(r) \psi^{n-1} \, dr \, dw. \]

By (28), we conclude

\[ \left\| \frac{1}{\psi^2} f g'' \right\|_2^2 \leq \frac{C}{\delta_{k,p}} \left[ \inf_{C_{k,p}} |\psi| \right]^{-4} \|u\|_2^2. \quad (31) \]

Continuing the proof of the theorem, consider \( p = \lfloor k/m \rfloor \) (The party integer of \( k/m \), where \( m \in \mathbb{N} \)), so by definition of \( r_{k+4p} \) we obtain \( r_{k+4p} \leq (1 + 4/m)r_k + M \), jointly with the definition \( \delta_{k,p} \) and with the items c), d) and e) above the lemma 5

\[ \left\| \frac{(n - 1) \psi}{\psi^3} f g' \right\|_2^2 \leq C e^2 \left[ \frac{\psi}{e^{4(\eta - \eta)r_k}} \right]^2 \|u\|_2^2 \]

\[ \left\| \frac{(n - 2) \cot(s)}{\psi^2} f g' \right\|_2^2 \leq C e^4 \left[ \frac{\psi}{e^{4(\eta - \eta)r_k}} \right]^2 \|u\|_2^2 \]

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Given $\epsilon > 0$, there $\eta > 0$ and $m \in \mathbb{N}$ where $c - \eta > a(1 + 4/m)$ such that
\[
\| \frac{1}{\psi^2} f g'' \|_2^2 \leq C \frac{e^{4a(1+4/m)r_k}}{e^{4(c-\eta)r_k}} \| u \|_2^2
\]
and jointly with the lemma 5 we deduce of (22) that
\[
\| \Delta u + \lambda u \|_2 \leq \epsilon \| u \|_2	ag{32}
\]
Consider the subspace spanned
\[
G = \left\{ [u(k_0,p_0,\cdot), u(k_0 + 4p_0,p_0,\cdot), u(k_0 + 8p_0,p_0,\cdot), \cdots] \right\}
\]
where $\text{supp}(u(k_0 + 2^i p_0,p_0,\cdot)) \cap \text{supp}(u(k_0 + 2^j p_0,p_0,\cdot)) = \emptyset$, for $i \neq j$, so
\[
\| \Delta u + \lambda u \|_2 < \epsilon \| u \|_2
\]
for all $u \in G$. By the lemma 4, $\lambda \in \sigma_{\text{ess}}(-\Delta)$, which concluded
\[
[ (n-1)^2 \frac{c^2}{4}, \infty ) \subseteq \sigma_{\text{ess}}(-\Delta)	ag{33}
\]
4 Proof of Theorem 3

Let us prove that there is $r_0 > 0$ such that
\[
\frac{1}{2} \leq \frac{\psi(rw)}{\psi(rN)} \leq \frac{3}{2}	ag{34}
\]
for all $r \geq r_0$ and $rw \in C_0(N)$. Consider $\mu : [0,s] \to \mathbb{S}^{n-1}$ the geodesic such that $\mu(0) = N, \mu(s) = w$ and $\mu'(t) = \partial / \partial s$. If $\nu(t) = \psi(r \mu(t))$, by Mean Value Theorem, there is $t_0 \in (0,s)$ such that
\[
\nu(s) - \nu(0) = \nu'(t_0) s = s g_M(r \mu(t_0)) \left( \text{grad} \psi, r \mu'(t_0) \right)
\]
\[
\psi(rw) - \psi(rN) = s g_M(r \mu(t_0)) \left( \psi_r \frac{\partial}{\partial r} + \frac{\psi_s}{\psi^2} \frac{\partial}{\partial s} + \frac{1}{\psi^2} \text{grad}_{\mathbb{S}^{n-2}} \psi, r \frac{\partial}{\partial s} \right)
\]
\[
\psi(rw) - \psi(rN) = rs \psi_s(r \mu(t_0))	ag{35}
\]
Of the hypothesis iii)
\[
\frac{\psi_\tau(r,\tau,\xi)}{\psi(r,\tau,\xi)} \leq \frac{c_1}{r^\gamma}
\]
in $C_0(N)$, integrating with respect to $\tau$ from 0 to $s$, we obtain
\[
\psi(r, s, \xi) \leq \psi(r, 0, \xi) e^{c_1 s / r^\gamma} = \psi(rN) e^{c_1 s / r^\gamma}
\]
thus of iii) and (35)
\[ |\psi(rw) - \psi(rN)| \leq \frac{c_1 c_2 \epsilon^{c_1 c_2/r^7} \psi(rN)}{r^2} \]

since \( \gamma > 1 \), there \( r_0 \) such that

\[ \frac{1}{2} \leq \frac{\psi(rw)}{\psi(rN)} \leq \frac{3}{2} \] (36)

for all \( r \geq r_0 \) and \( rw \in C_0(N) \).

For \( \lambda > 0 \), take \( \alpha > 0 \) such that \( \lambda > \frac{(n-1)^2 \alpha^2}{4} \). Consider the metric

\[ g_{M^\alpha} = dr^2 + \psi^\alpha(rw)^2 g_{S^{n-1}} \]

in \( C_0(N) \), where \( \psi^\alpha(rw) = e^{\alpha r} \psi(rw) \). By hypothesis i), the function \( \psi^\alpha(rw) \) satisfies

\[ \lim_{r \to \infty} \frac{\psi^\alpha(rw)}{\psi^\alpha(rw)} = \lim_{r \to \infty} \left( \frac{\psi^\alpha(rw)}{\psi(rw)} + \alpha \right) = \alpha \]

to each \( w \) such that \( \text{dist}_{S^{n-1}}(w, N) \leq c_2 \).

Given \( \epsilon > 0 \), construct a sequence \( (n_k^\alpha) \) in a manner analogous to (5) that satisfies (5), where \( f_k^\alpha \) is the same of (7) with \( v_\alpha(r) = \int_0^r \psi^\alpha(\tau N)^{n-1} d\tau, r_k = (2k + 1)\pi/2\sqrt{\lambda} \) and \( g(s) = H(s/c_2) \cos(\pi s/c_2) \), knowing that \( g \) satisfies (18) and

\[ \text{supp } g = B(c_2) = \{ w \in S^{n-1} ; \text{dist}_{S^{n-1}}(w, N) \leq c_2 \} \]

Similarly to (21)

\[ \Delta_\alpha u^\alpha + \lambda u^\alpha = A_\alpha(r) F^\alpha g h + B_\alpha(rw)(F^\alpha)' g h + 2(F^\alpha)' g h' + F^\alpha g \Delta h + \]

\[ + \frac{(n-3)\psi^\alpha}{(\psi^\alpha)^3} f^\alpha g' + \frac{(n-2) \cot(s)}{(\psi^\alpha)^2} f^\alpha g' + \frac{1}{(\psi^\alpha)^2} f^\alpha g'' \] (37)

where

\[ A_\alpha(r) = -\frac{1}{2} \frac{\psi''}{\psi^2} + \frac{1}{4} \left( \frac{v'_{\alpha}}{v_{\alpha}} \right)^2 + \frac{(n-1)^2 \alpha^2}{4} \]

and

\[ B_\alpha(rw) = (n-1) \frac{\psi^\alpha}{\psi^\alpha(rw)} - \frac{v'_{\alpha}}{v_{\alpha}} \]

analogous to (12)

\[ \lim_{r \to \infty} \frac{v'_{\alpha}(r)}{v_{\alpha}(r)} = \lim_{r \to \infty} \frac{v''_{\alpha}(r)}{v'_{\alpha}(r)} = (n-1)\alpha \] (38)

then

\[ \lim_{r \to \infty} A_\alpha(r) = 0 \quad \text{and} \quad \lim_{r \to \infty} B_\alpha(rw) = 0 \] (39)

to each \( w \) such that \( \text{dist}_{S^{n-1}}(w, N) \leq c_2 \).

Then, similarly the (22), given \( \delta > 0 \) there \( r_0 \) such that

\[ \| \Delta_\alpha u^\alpha + \lambda u^\alpha \|_{L^2(M^\alpha)} \leq \delta \| \chi_k F^\alpha g \|_{L^2(M^\alpha)} + \delta \| \chi_k (F^\alpha)' g \|_{L^2(M^\alpha)} + \]

\[ + C \left( \frac{\psi^\alpha}{(\psi^\alpha)^3} f^\alpha g' \right)_{L^2(M^\alpha)} + C \left( \frac{\cot(s)}{(\psi^\alpha)^2} f^\alpha g' \right)_{L^2(M^\alpha)} + \left( \frac{1}{(\psi^\alpha)^2} f^\alpha g'' \right)_{L^2(M^\alpha)} \] (40)

for all \( r \geq r_0 \) and \( rw \in C_0(N) \).
Lemma 6 For the functions $F^\alpha$, $f^\alpha$, $g$ and $u^\alpha$ defined previously, we have the following inequalities

(a) $\| \chi_h F^\alpha g \|_2 \leq C \| u^\alpha \|_2$

(b) $\| \chi_h (F')^\alpha g \|_2 \leq C \| u^\alpha \|_2$

(c) $\left\| \frac{\psi^\alpha}{(\psi^\alpha)^3} f^\alpha g' \right\|_{L^2(M^\alpha)}^2 \leq C r_k \inf_{C_{k,p}} \| \psi^\alpha \|^{-2} \| u^\alpha \|_2$

(d) $\left\| \cot(s) \frac{f^\alpha g'}{(\psi^\alpha)^2} \right\|_{L^2(M^\alpha)} \leq C \left( \inf_{C_{k,p}} | \psi^\alpha | \right)^{-2} \| u^\alpha \|_2$

(e) $\left\| \frac{1}{(\psi^\alpha)^2} f^\alpha g'' \right\|_{L^2(M^\alpha)} \leq C \left( \inf_{C_{k,p}} | \psi^\alpha | \right)^{-2} \| u^\alpha \|_2$

where $C_{k,p} = \{ rw; r_k \leq r \leq r_{k+4p}, w \in B(c_2) \}$ and $C$ is a positive constant independent of $k$ and $p$.

Proof of Lemma: The items [(a)] and [(b)] is proved similarly to the items [(a)] and [(b)] of lemma 5. We prove item [(c)], the other follows similarly.

Now the third inequality

$$\left\| \frac{\psi^\alpha}{(\psi^\alpha)^3} f^\alpha g' \right\|_{L^2(M^\alpha)}^2 = \int_{B(c_2)} |g'|^2 \int_{r_k}^{r_{k+4p}} \frac{(\psi^\alpha)^2}{(\psi^\alpha)^6} f^\alpha(r)^2 (\psi^\alpha)^{n-1} dr dw$$

By hypothesis iii) and $\inf_{C_{k,p}} | \psi^\alpha | \leq | \psi^\alpha |$

$$\left\| \frac{\psi^\alpha}{(\psi^\alpha)^3} f^\alpha g' \right\|_{L^2(M^\alpha)}^2 \leq C \left( \inf_{C_{k,p}} | \psi^\alpha | \right)^{-4} \int_{B(c_2)} |g'|^2 \int_{r_k}^{r_{k+4p}} f^\alpha(r)^2 (\psi^\alpha)^{n-1} dr dw$$

since that $f^\alpha(r) = v^{-1/2}_\alpha \cos(\beta r) h(r)$ and $f^\alpha(r) = v^\alpha(r) N^{-1}$, then

$$\int_{B(c_2)} |g'|^2 \int_{r_k}^{r_{k+4p}} f^\alpha(r)^2 (\psi^\alpha)^{n-1} dr dw = \int_{B(c_2)} |g'|^2 \int_{r_k}^{r_{k+4p}} \cos(\beta r) h^2(r) \frac{(\psi^\alpha)^{n-1}}{v^\alpha} dr dw$$

$$\leq \int_{B(c_2)} |g'|^2 \int_{r_k}^{r_{k+4p}} \cos(\beta r) \frac{\psi^\alpha(rw)^n}{\psi^\alpha(rN)^n} \cdot \frac{v^\alpha}{v^\alpha} dr dw.$$

by the estimates (36), (38) and definition of $\psi^\alpha$

$$\left\| \frac{\psi^\alpha}{(\psi^\alpha)^3} f^\alpha g' \right\|_{L^2(M^\alpha)}^2 \leq C \left( \inf_{C_{k,p}} | \psi^\alpha | \right)^{-4} \int_{B(c_2)} |g'|^2 \int_{r_k}^{r_{k+4p}} \cos(\beta r) dr dw \quad (41)$$

also have

$$\| u^\alpha \|_{L^2(M^\alpha)}^2 \geq C \alpha \int_{B(c_2)} g^2(s) \int_{r_k}^{r_{k+3p}} \cos(\beta r) dr dw \quad (42)$$
but
\[ \int_{B(c_2)} |g'(s)|^2 \, dw \leq C \int_{B(c_2)} g(s)^2 \, dw \quad \text{and} \quad \int_{r_k}^{r_{k+4p}} \cos^2(\beta r) \, dr = 2 \int_{r_k}^{r_{k+3p}} \cos^2(\beta r) \, dr. \]
jointly with (41) and (42)
\[ \|\psi^\alpha\|_{L^2(M^\alpha)}^2 \leq \frac{C}{r_k^2} \left( \inf_{C_{\alpha,p}} |\psi^\alpha| \right)^{-4} \|u^\alpha\|_{L^2(M^\alpha)}^2 \tag{43} \]
Continuing the proof of the theorem, in inequality (40) we use the lemma 6, by hypotheses ii), given \( \epsilon > 0 \) there \( r_0 \) such that
\[ \|\Delta u^\alpha + \lambda u^\alpha\|_{L^2(M^\alpha)} \leq \left( \frac{\epsilon}{M} \right) \|u^\alpha\|_{L^2(M^\alpha)} \]
for all \( r \geq r_0 \) and \( rw \in C_0(N) \). Let us now define \( u_k = \mu(r, \alpha)^{-1} u_k^\alpha \in C_0^\infty(M) \) where \( \mu(r, \alpha) = e^{-(n-1)\alpha r^2}/2 \), that notice \( \|u_k\|_{L^2(M)} = \|u_k^\alpha\|_{L^2(M^\alpha)} \). However
\[ \Delta u_k^\alpha = \left[ \Delta u_k + \left( \frac{1}{e^{2\alpha r}} - 1 \right) f_k \Delta g - (n-1)^2 \left( \frac{\alpha^2}{4} + \frac{\alpha \psi_r}{\psi} \right) u_k \right] \mu(r) \]
where \( f_k = \mu(r, \alpha)^{-1} f_k^\alpha \), and then
\[ \left\| \Delta u_k + \left( \frac{1}{e^{2\alpha r}} - 1 \right) f_k \Delta g - (n-1)^2 \left( \frac{\alpha^2}{4} + \frac{\alpha \psi_r}{\psi} \right) u_k + \lambda u_k \right\|_{L^2(M^\alpha)} \leq \left( \frac{\epsilon}{M} \right) \|u^\alpha\|_{L^2(M^\alpha)} \]
in other words
\[ \left\| \Delta u_k + \left( \frac{1}{e^{2\alpha r}} - 1 \right) f_k \Delta g - (n-1)^2 \left( \frac{\alpha^2}{4} + \frac{\alpha \psi_r}{\psi} \right) u_k + \lambda u_k \right\|_{L^2(M)} \leq \left( \frac{\epsilon}{M} \right) \|u\|_{L^2(M)} \]
using triangular inequality
\[ \|\Delta u_k + \lambda u_k\|_{L^2(M)} \leq \left( \frac{\epsilon}{M} \right) \|u\|_{L^2(M)} + C \|f_k \Delta g\|_{L^2(M)} + C \left\| \left( \frac{\alpha^2}{4} + \frac{\alpha \psi_r}{\psi} \right) u_k \right\|_{L^2(M)} \]
Note that
\[ \|f_k \Delta g\|_{L^2(M)} = \|f_k^\alpha \Delta g\|_{L^2(M^\alpha)} = \]
\[ \int_{B(c_2)} \int_{r_k}^{r_{k+4p}} f^\alpha(r)^2 \Delta g^2(\psi^\alpha)^{n-1} \, dr \, dw = \int_{B(c_2)} \int_{r_k}^{r_{k+4p}} \cos^2(\beta r) h^2(r) \Delta g^2 \left( \frac{\psi^\alpha}{\psi^{\alpha}} \right)^{n-1} \, dr \, dw \]
\[ \leq \int_{B(c_2)} \int_{r_k}^{r_{k+4p}} \cos^2(\beta r) \Delta g^2 \frac{\psi^\alpha(rw)^{n-1}}{\psi^\alpha(rN)^{n-1}} \cdot \frac{\psi^\alpha}{\psi^{\alpha}} \, dr \, dw. \]
by the estimates \[ \text{[36], [35]} \] and definition of \( \psi^\alpha \)
\[ \|f_k \Delta g\|_{L^2(M)} \leq C \alpha \int_{B(c_2)} \int_{r_k}^{r_{k+4p}} \cos^2(\beta r) \Delta g^2 \, dr \, dw \]
for (18), hypothesis iii), \( r \geq r_0 \) and \( \text{supp } g = \text{supp } g' = \text{supp } g'' = B(c_2) \), it follows that
\[
\int_{B(c_2)} \Delta g^2 \, dw \leq \int_{B(c_2)} \left[ \frac{(n-3)\psi}{\psi^3} |g'| + \frac{(n-2)|\cot(s)|}{\psi^2} |g'| + \frac{1}{\psi^2} |g''| \right]^2 \, dw \leq \frac{C}{(\inf_{C_{k,p}} |\psi|)^4} \int_{B(c_2)} g^2 \, dw
\]
thus
\[
\| f_k \Delta g \|_{L^2(M)} \leq \frac{C}{(\inf_{C_{k,p}} |\psi|)^4} \int_{B(c_2)} g(s)^2 \int_{r_k}^{r_{k+4p}} \cos^2(\beta r) \, dr \, dw
\]
through inequality (42)
\[
\| f_k \Delta g \|_{L^2(M)} \leq \frac{C}{(\inf_{C_{k,p}} |\psi|)^4} \| u_k \|_{L^2(M)}
\]
of the hypothesis ii), we have
\[
\| f_k \Delta g \|_{L^2(M)} \leq C\epsilon \| u_k \|_{L^2(M)}
\]
doing \( \alpha, |\psi_r/\psi| < \epsilon \), we conclude
\[
\| \Delta u_k + \lambda u_k \|_{L^2(M)} \leq \epsilon \| u_k \|_{L^2(M)}, \quad k = 1, 2, \ldots
\]
by the lemma that guarantee
\[
\sigma_{\text{ess}}(-\Delta) = [0, \infty)
\]

5 Appendix

Consider \( \mathbb{R}^2 = \{(r \cos \theta, r \sin \theta) : r \geq 0, \theta \in [0, 2\pi]\} \) with metric given by \( g = dr^2 + \psi^2(r, \theta)g_{\mathbb{S}^1} \), since \( \psi(r, \theta) = re^{r^2g(\theta)+r} \) where \( g(\theta) = \sin^2(\theta/2) \). Calculating the derivatives of \( \psi \)
\[
\psi_r = \frac{\psi}{r} + (2rg(\theta) + 1)\psi \quad \text{and} \quad \psi_{rr} = \frac{\psi_r}{r} - \frac{\psi}{r^2} + 2g(\theta)\psi + (2rg(\theta) + 1)\psi_r
\]
then we deduce
\[
\left| \frac{\psi_r}{\psi} - 1 \right| = \frac{1}{r} + 2r(\theta/2)^2 \sin^2(\theta/2) (\theta/2)^2
\]
and
\[
K(r, \theta) = -\frac{\psi_{rr}}{\psi}(r, \theta) = -6g(\theta) - \frac{2}{r} - (2rg(\theta) + 1)^2
\]
In the set \( C(\alpha) = \{(r \cos \theta, r \sin \theta) : |\theta| \leq e^{-ar}\} \) we have that
\[
\left| \frac{\psi_r}{\psi} - 1 \right| \leq C e^{-2ar} \to 0
\]
when $r \to +\infty$, however

$$K(r, 0) = -\frac{\psi'}{\psi}(r, 0) \to -1$$

and

$$K(r, \theta) \to -\infty \; ; \; \theta \neq 0$$

when $r \to +\infty$. Clearly

$$\left|\frac{\psi\theta}{\psi}(r, \theta)\right| = r^3 g'(\theta) = r^3 (\theta/2)^2 \frac{\sin(\theta/2)}{\theta/2} \cos(\theta/2) \leq C r^3 e^{-ar} \leq b$$

in $C(a)$, therefore by Theorem [3] the essential spectrum de $\mathbb{R}^2$ with such metric contains the interval $[1/4, +\infty)$.

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