Research Article

On a Conjecture about the Saturation Number of Corona Product of Graphs

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Let $G = (V_G, E_G)$ be a simple and connected graph. A set $M \subseteq E_G$ is called a matching if no two edges of $M$ have a common endpoint. A matching $M$ is maximal if it cannot be extended to a larger matching in $G$. The smallest size of a maximal matching is called the saturation number of $G$. In this paper, we confirm a conjecture of Alikhani and Soltani about the saturation number of corona product of graphs. We also present the exact value of $s(G \ast H)$ where $H$ is a randomly matchable graph.

1. Introduction

All graphs considered in this paper are connected and simple; that is, they do not have loops and multiple edges [1–3]. For notation and graph theory terminology, we in general follow [11, 12, 15].

Let $G = (V_G, E_G)$ be a graph. A collection of edges $M_G \subseteq E_G$ is called a matching of $G$ if no two edges of $M_G$ are adjacent. The vertices incident to the edges of a matching $M_G$ are said to be saturated by $M_G$ (or $M_G$-saturated); the others are said to be unsaturated (or $M_G$-unsaturated). A matching whose edges meet all vertices of $G$ is called a perfect matching of $G$. If there does not exist a matching $M_G$ in $G$ such that $|M_G| < |M_G'|$, then $M_G$ is called a maximum matching of $G$. A matching $M_G$ is maximal if it cannot be extended to a larger matching in $G$. The cardinality of any maximum matching, $\nu(G)$, and the cardinality of any smallest maximal matching in $G$, $s(G)$, are called the matching number and the saturation number of $G$, respectively.

If any maximal matching in $G$ is also perfect (i.e., if $s(G) = |V_G|/2$), then $G$ is called randomly matchable.

Smallest maximal matchings have a wide range of applications in real-world problems. For example, application of smallest maximal matchings related to a telephone switching network was presented in [4]. Finding a smallest maximal matching is NP-hard even for special family of graphs (such as planar graphs), see [4–6]. Also, one can find some bounds for this invariant in [7–10]. See [10, 12, 13] for more details on this topic. See [11–13] Recently, Alikhani and Soltani presented the following conjecture about the saturation number of corona product of graphs.

Conjecture. [14] Let $G$ and $H$ be two graphs and $|V_G| = n$. Then,

$$ns(H) \leq s(G \ast H) \leq ns(H) + \nu(G) + l,$$

where $\nu(G)$ is the size of a maximum matching $M_G$ of the graph $G$ and $l$ is the number of $M$-unsaturated vertices of $G$.

In this paper, we confirm this conjecture. We also present some more efficient results on the saturation number of corona product of graphs.

For two graphs, $G = (V_G, E_G)$ and $H = (V_H, E_H)$. The corona product of $G$ and $H$, denoted by $G \ast H$, is obtained from one copy of $G$ and $|V_G|$ copies of $H$ by joining each vertex of the $i$th copy of $H$, $i \in [1, \ldots, |V_G|]$, to the $i$th vertex of $G$, cf. [15]. In the following, for $g \in V_G$, $H_g$ shows the copy of $H$ in $G \ast H$ corresponding to $g$.

2. Main Results

The first result of this section is the proof of the conjecture mentioned in the previous section.
**Theorem 1.** Let $G$ and $H$ be two graphs and $|V_G| = n$. Then,

\[ ns(H) \leq s(G \circ H) \leq ns(H) + v(G) + l, \tag{2} \]

where $v(G)$ is the size of a maximum matching $M_G$ of the graph $G$ and $l$ is the number of $M$-unsaturated vertices of $G$.

**Proof.** First, we prove the upper bound. Let $M_G$ be a maximum matching of $G$, and $M_H$ a maximal matching in $H$ that $|M_H| = s(H)$. Also, suppose that vertices $g_1, \ldots, g_l$ are $M$-unsaturated vertices of $G$. There are two cases for $H$.

**Case 1.** $H$ is a randomly matchable. Suppose that $M_G'$ is a maximal matching in $G$ that $|M_G'| = s(G)$. Set

\[ M = M_G' \cup \left( \bigcup_{i=1}^{n} M_{H_i} \right), \tag{3} \]

where $M_{H_i}$ is the $i$th copy of $M_H$, $i \in \{1, \ldots, n\}$, in $H_i$. Clearly, $M$ is a maximal matching in $G \circ H$. Thus,

\[ s(G \circ H) \leq |M| = |M_G'| \cup \left( \bigcup_{i=1}^{n} M_{H_i} \right) = s(G) + ns(H) < ns(H) + v(G) + l. \tag{4} \]

**Case 2.** $H$ is not a randomly matchable. Thus, $M_H$ is not a perfect matching. Suppose that $h_j$ is a $M_H$-unsaturated of $H$. Set

\[ M = M_G' \cup \left( \bigcup_{i=1}^{n} M_{H_i} \right) \cup \left( \bigcup_{i=1}^{l} \{ h_i, g_i \} \right), \tag{5} \]

where $h_i$ is the copy of $h_j$ in $H_i$ corresponding to $g_i$. Easily one can check that $M$ is a maximal matching in $G \circ H$. Therefore,

\[ s(G \circ H) \leq |M| = |M_G'| \cup \left( \bigcup_{i=1}^{n} M_{H_i} \right) \cup \left( \bigcup_{i=1}^{l} \{ h_i, g_i \} \right) = ns(H) + v(G) + l. \tag{6} \]

Now, we prove the lower bound. Let $M$ be a maximal matching of $G \circ H$. We consider two below cases for $M$.

**Case 1.** $M$ does not have any edges $e$ so that $e$ has one end in $G$ and one end in a copy of $H$. Hence, $|M \cap E_{H_i}| \geq s(H)$, and consequently, $ns(H) \leq s(G \circ H)$.

**Case 2.** Suppose \{ $g_1h_1', g_2h_2', \ldots, g_kh_k'$ \} be all edges of $M$ such that $g_i \in V_G$ and $h_j' \in V_{H_i}$. Then, for each $i \in \{1, \ldots, k\}$, we have $|M \cap E_{H_i}| = s(H)$. Also, for each $i \in \{1, \ldots, k\}$, we have $s(H) - 1 \leq |M \cap E_{H_i}| \leq s(H)$. On the other hand,

\[ \{ g_1h_1', g_2h_2', \ldots, g_kh_k' \} \cup \left( \bigcup_{i=1}^{n} \left( M \cap E_{H_i} \right) \right) \leq M. \tag{7} \]

Therefore, $ns(H) \leq s(G \circ H)$.

The next theorem gives the exact value of $s(G \circ H)$ for some family of graphs.

**Theorem 2.** Let $G$ be a graph of order $n$. If $H$ is a randomly matchable graph, then

\[ s(G \circ H) = ns(H). \tag{8} \]

**Proof.** By Theorem 1, we have $s(G \circ H) \geq ns(H)$. Then, it is sufficient to prove that $s(G \circ H) \leq ns(H)$. Suppose $h_1'$ is the copy of $h_j$ in $H_i$ corresponding to $g_j$. Let $M_H$ be a maximal matching of $H$, and $M_{H_i}$ is the $i$th copy of $M_H$, $i \in \{1, \ldots, n\}$, in $H_i$. Assume that $V_G = \{ g_1, \ldots, g_n \}$ and $h_1', h_2', \ldots, h_l'$. Set

\[ M = \{ g_1h_1', g_2h_2', \ldots, g_nh_n' \} \cup \left( \bigcup_{i=1}^{n} \left( M_{H_i} \setminus \{ h_i' \} \right) \right). \tag{9} \]

(For more illustration, see Figure 1 which is $C_3 \circ C_4$. Suppose $C_4 := h_1, h_2, h_3, h_4$. Consider the maximal matching $M_{C_4} = \{ h_1h_3, h_2h_4 \}$. Since $C_4$ is a randomly matchable graph, then $M = \{ g_1h_1', g_2h_2', g_3h_3', g_4h_4' \} \cup \{ h_1h_1', h_2h_2', h_3h_3', h_4h_4' \}$). According to this fact that $H$ is a randomly matchable graph, then $M$ is a maximal matching in $G \circ H$. Thus,

\[ s(G \circ H) \leq |M| = \left| \{ g_1h_1', g_2h_2', \ldots, g_nh_n' \} \cup \left( \bigcup_{i=1}^{n} \left( M_{H_i} \setminus \{ h_i' \} \right) \right) \right|. \tag{10} \]

On the other hand, $\left| \{ g_1h_1', g_2h_2', \ldots, g_nh_n' \} \cup \left( \bigcup_{i=1}^{n} \left( M_{H_i} \setminus \{ h_i' \} \right) \right) \right| = n(s(H) - 1) + n = ns(H)$. Therefore, $s(G \circ H) \leq ns(H)$. 

**Data Availability**

No data were used to support this study.

**Conflicts of Interest**

The authors declare that they have no conflicts of interest.
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