Ghost Inflation

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Abstract

We propose a new scenario for early cosmology, where an inflationary de Sitter phase is obtained with a ghost condensate. The transition to radiation dominance is triggered by the ghost itself, without any slow-roll potential. Density perturbations are generated by fluctuations around the ghost condensate and can be reliably computed in the effective field theory. The fluctuations are scale invariant as a consequence of the de Sitter symmetries, however, the size of the perturbations are parametrically different from conventional slow-roll inflation, and the inflation happens at far lower energy scales. The model makes definite predictions that distinguish it from standard inflation, and can be sharply excluded or confirmed by experiments in the near future. The tilt in the scalar spectrum is predicted to vanish ($n_s = 1$), and the gravity wave signal is negligible. The non-Gaussianities in the spectrum are predicted to be observable: the 3-point function is determined up to an overall $\mathcal{O}(1)$ constant, and its magnitude is much bigger than in conventional inflation, with an equivalent $f_{\text{NL}} \simeq 100$, not far from the present WMAP bounds.
1 Introduction

Inflation is an attractive paradigm for early cosmology. An early de Sitter phase of the universe elegantly solves the horizon and flatness problems. Further imagining that this phase is driven by a slowly varying scalar field with a flat potential gives a beautiful mechanism for generating the observed density perturbations, which have their origin in the quantum fluctuations of this light field during inflation. This minimal picture of slow-roll inflation predicts the fluctuations to be nearly scale invariant and Gaussian, in excellent agreement with observation.

While this qualitative picture is very attractive, it is frustratingly difficult to make much sharper predictions that could definitively exclude or confirm it in future experiments. There are nevertheless some generic expectations for what should be seen. For instance, in most models, the tilt in the scalar spectrum is non-vanishing, and is typically of order 

\[ n_s - 1 \sim (1/N_e) \sim 2 \times 10^{-2}, \]

where \( N_e \) is the number of e-folds to the end of inflation when typical cosmological scales leave the horizon. The current bounds on \( n_s - 1 \) are getting close to this size, while the Planck satellite will improve the sensitivity to \( \sim 3 \cdot 10^{-3} \).

Slow-roll inflation leads to the expectation that Planck should find \( n_s \neq 1 \), but this is not a hard prediction, and if Planck does not find a deviation from \( n_s = 1 \) we cannot definitely exclude it. If a tilt is found, it would be interesting to look for finer structures in the power spectrum, such as bumps or wiggles of various kinds. However, one can always engineer bumps and wiggles in the inflaton potential to reproduce these features in the power spectrum, so there is nothing here either that can exclude or confirm the standard picture with certainty. The observation of a gravitational wave signal in the CMBR would be extremely exciting and it would indicate that the energy density driving the accelerated expansion is very high, of order \( M^4_{\text{GUT}} \). Minimal slow roll inflation predicts a relationship between the tilt of gravity wave spectrum \( n_g - 1 \) and the scalar one \( n_s - 1 \), and if this is found it would be a confirmation of the standard picture. However, we would have to be lucky for the gravity wave spectrum to be observable in near future experiments, to say nothing of the tilt in the gravity wave spectrum, and if the inflationary prediction is found not to hold, it would by no means exclude the theory; deviations from this relationship can come about by integrating out new physics between the Planck and Hubble scales during inflation [1, 2].

The minimal picture of slow-roll inflation does make one very sharp prediction, however: the spectrum is predicted to be Gaussian to a very high degree. For single field inflation models, the 3-point function was recently calculated to leading order in the slow-roll parameters by Maldacena [3]. The dimensionless size of the effect was found to be parametrically at least of the same order as \( \delta \rho / \rho \) itself, (and in fact further suppressed by slow-roll parameters). The prediction is very firm, since the leading non-Gaussianities are generated by the interaction of the inflaton with gravity, and are therefore determined by the symmetries of the theory. However, the magnitude is about a factor of \( \sim 100 \) smaller than what might be observable in the future experiments. Because of this firm prediction,
however, non-Gaussianities are the best and most exciting place to look for deviations from the standard picture of slow-roll inflation. Indeed there are by now a number of different ways of generating density perturbations from sources other than the fluctuations of the slowly rolling inflaton, and in these models it is possible to get larger (and observable) non-Gaussianities [4, 5]. Any observation of non-Gaussianities necessarily implies the presence of additional light fields and a non-standard mechanism for generating density perturbations.

Given the difficulty of conclusively verifying the standard inflationary paradigm, it is worth looking for alternatives, especially if they are more predictive in some way. Given the ease with which a de Sitter phase of expansion obliterates the horizon and flatness problems, it would be nice to keep this, but is there an alternative to the standard slow-roll picture?

In this paper we will describe such a model. Recently, a new possibility of having de Sitter phases in the universe in a way differing from a cosmological constant has been proposed [7], which also modifies gravity in the infrared in a non-trivial and consistent way. It can be thought of as arising from a derivatively coupled ghost scalar field \( \phi \) which “condenses” in a background where it has a non-zero velocity

\[
\langle \dot{\phi} \rangle = M^2 \rightarrow \langle \phi \rangle = M^2 t .
\] (1.1)

The novelty here is that, unlike other scalar fields, the velocity \( \dot{\phi} \) does not redshift to zero as the universe expands, it stays constant, and indeed the energy momentum tensor is identical to that of a cosmological constant. However, the ghost condensate is not a cosmological constant, it is a physical fluid with a physical fluctuations \( \pi \) defined as

\[
\phi = M^2 t + \pi .
\] (1.2)

The ghost condensate then gives an alternative way of realizing de Sitter phases in the universe. The symmetries of the theory allow us to construct a systematic and reliable effective Lagrangian for \( \pi \) and gravity at energies lower than the ghost cut-off \( M \). Neglecting the interactions with gravity, the effective Lagrangian for \( \pi \) (around flat space) has the form

\[
S = \int d^4x \left( \frac{1}{2} \dot{\pi}^2 - \frac{\alpha^2}{2M^2} (\nabla^2 \pi)^2 - \frac{\beta}{2M^2} \dot{\pi} (\nabla \pi)^2 \right) + \cdots
\] (1.3)

where \( \alpha \) and \( \beta \) are order one coefficients. Note that this Lagrangian is non-Lorentz invariant, as should be expected, since the background \( \dot{\phi} = M^2 \) breaks Lorentz invariance spontaneously (the \( \pi \) field can be thought as the Goldstone boson for this symmetry breaking). The low-energy dispersion relation for \( \pi \) is of the form

\[
\omega^2 = \alpha^2 \frac{k^4}{M^2} .
\] (1.4)

It is straightforward to couple this healthy sector to gravity, which leads to a variety of interesting modifications of gravity in the IR, including antigravity and oscillatory modulation of the Newtonian potential at late times and large distances [7].
Given that this theory provides an alternative way of having a de Sitter phase, it is natural to try to use the field \( \phi \) as the inflaton. Since we have a different value for the Hubble scale during inflation than the present, the shift symmetry on \( \phi \) must be broken. However, it is technically natural to assume that there is a potential \( V(\phi) \) which has the form \( e.g. \, V(\phi) = V_0 \) for \( \phi < 0 \), \( V(\phi) = 0 \) for \( \phi > 0 \). The shift symmetry is still good for \( \phi < 0 \) and \( \phi > 0 \), so that this shape is radiatively stable. Of course this is an exaggeration of a smooth shape for the potential, and we could have variations on each side over scales smaller than the cutoff \( M \) of the ghost theory. The theory inflates for \( \phi < 0 \), and evolves to radiation dominance for \( \phi > 0 \). Note that we need not assume that \( V_0 \) itself is of order \( M^4 \).

For instance, as in hybrid inflation, \( \phi \) could couple to matter with shift-symmetry breaking couplings around \( \phi = 0 \); these couplings can trigger a phase transition at \( \phi = 0 \) and \( V_0 \) can be the energy of the false vacuum in the matter sector. We will not concern ourselves with describing any specific model for the transition or with justifying the different energy scales, \( M \) and \( V_0 \), of the model; it suffices to say that the set-up is completely technically natural, \( i.e. \) radiatively stable. From this minimal set-up, however, we can make definite and experimentally relevant predictions for the density perturbations, which is our main focus in this paper.

Let us begin with a qualitative picture and estimates for the size and level of non-Gaussianity in the perturbations, and later derive all the results more formally. There are two important differences here from ordinary inflation. First, there is no slow-roll. Even in the approximation where the potential for \( \phi < 0 \) is \textit{exactly flat}, \( \dot{\phi} = M^2 \) is non-zero, and \( \phi \) is continuously heading towards \( \phi = 0 \) where inflation ends, continuing with about the same velocity afterwards. As we will see in a moment, density perturbations are generated here from the fluctuations of the \( \pi \) field, and are again scale invariant as in inflation. However, since there is no need for any slope in the potential, there is no reason to expect any tilt in the scalar spectrum.

The second important difference with standard inflation concerns the size of the fluctuations in \( \phi \) (or equivalently \( \pi \)). Since the effective Lagrangian for \( \pi \) is non-relativistic, and in particular there are no \( k^2 \) spatial kinetic terms, the fluctuations of \( \pi \) are less suppressed than in a relativistic theory. In a relativistic theory, the fact that scalar fields have scaling dimension 1 tells us that the size of the fluctuations of a scalar field inside a region of size \( R \) is given by \( \sim 1/R \), similarly at a frequency \( E \) it is given by \( E \). In usual inflation, the inflaton fluctuations freeze when they have a typical energy \( E \sim H \), so that their typical size is \( \delta \phi \sim H \). We can determine the size of the fluctuations in our case by a simple scaling argument familiar from power-counting for non-relativistic effective theories \( \mathbb{S} \). Suppose we scale energies by a factor of \( s \), \( E \to sE \), or alternatively \( t \to s^{-1} t \). Clearly, because of the \( \omega^2 \propto k^4 \) dispersion relation, we have to scale \( k \) differently, \( k \to s^{1/2} k \) or \( x \to s^{-1/2} x \). We then determine the scaling dimension of \( \pi \) by requiring the quadratic action to be invariant, and we find that \( \pi \) has scaling dimension 1/4

\[
\pi \to s^{1/4} \pi .
\]  

(1.5)
Now, $\pi$ has mass dimension 1, so the fluctuations at frequencies of order the cutoff $M$ is $\delta \pi_M \sim M$. But the fact that $\pi$ has scaling dimension $1/4$ tells us that the fluctuation at a lower energy $E$ is $\delta \pi_E \sim (EM^3)^{1/4}$. In particular, the size of ghost fluctuations that freeze, as usual, by Hubble friction when its frequency in of order $E \sim H$, is

$$\delta \pi_H \sim (HM^3)^{1/4}.$$ (1.6)

Of course, for consistency of the effective theory, we must have $H \ll M$; note that as expected these fluctuations are much larger than $H$ in this limit.

Let us now estimate the size of the density perturbations. As usual, $\phi$ fluctuates and is stretched out until the Hubble damping becomes important at frequency $E \sim H$; note that this does not correspond to $k \sim H$ but rather, from the dispersion relation, $k \sim \sqrt{HM}$. The fluctuation $\delta \pi_H$ causes inflation to end at slightly different times in different places, and so as usual we have the estimate

$$\frac{\delta \rho}{\rho} \sim H\delta t = \frac{H\delta \pi_H}{\dot{\phi}}.$$ (1.7)

Now both of the differences with the usual story come into play. First, $\dot{\phi} = M^2$, having nothing to do with slow-roll parameters. Second, $\delta \pi_H$ is much larger. We then find

$$\frac{\delta \rho}{\rho} \sim \left(\frac{H}{M}\right)^{5/4},$$ (1.8)

which can be compared with the usual inflationary case

$$\frac{\delta \rho}{\rho} \sim \frac{H}{M_{\text{Pl}}\sqrt{\epsilon}},$$ (1.9)

where $\epsilon$ is the usual slow-roll parameter: $\epsilon \equiv M_{\text{Pl}}^2/2 \cdot (V'/V)^2$. Since $\phi$ continues to have the same velocity today, leading to the IR modifications of gravity described in [7], $M$ is limited to be at most 10 MeV or so: we must have $H \sim 1$ keV at largest for $\delta \rho/\rho \sim 10^{-5}$. We therefore generate the correct magnitude density perturbations at far lower energy scales than usual inflation. Indeed, the maximum energy scale is of order $V_0 \sim H^2 M_{\text{Pl}}^2 \sim (1000 \text{ TeV})^4$. This implies that the gravity wave signal is completely negligible.

We can then reproduce the spectrum of perturbations, with the sharp predictions that (a) $|n_s - 1| = 0$ and (b) there are no gravitational waves. The observation of any of these would decisively rule out our model.

Things get more interesting when we look at the non-Gaussianities in the model. Because all the scales are so much smaller than the Planck scale, there is no effect coming from gravitational interactions. The dominant effect comes from the trilinear interaction in the $\pi$ effective Lagrangian

$$\frac{\beta}{M^2} \dot{\pi}(\nabla \pi)^2.$$ (1.10)
This leads to a non-zero three-point function for the density perturbations. To get an idea of the dimensionless size of this effect, we need to find the dimensionless size of this coupling at an energy of order $H$. We can do this easily since we know the scaling dimension of $t, x$ and $\pi$: we find that the coefficient of this operator has scaling dimension $1/4$; it is an irrelevant operator, but just barely! The dimensionless size of this interaction at the cutoff $M$ is $\sim 1$; scaling it down to an energy of order $H$ we get an estimate for the non-gaussianity of the perturbations

$$NG \sim \left( \frac{H}{M} \right)^{1/4} \sim \left( \frac{\delta \rho}{\rho} \right)^{1/5}.$$  \hfill (1.11)

These are much larger than in standard inflation, where $NG \sim \epsilon \cdot (\delta \rho/\rho)!$ In fact, this estimate suggests that the non-gaussianity may be too large, $\sim 10^{-1}$, but in fact the complete analysis we do below shows that it is naturally just at the level of the present WMAP constraints. The estimate clearly shows the physical origin of the effect, as being due to the unusual fluctuations and scaling dimensions of interactions in the $\pi$ effective theory. Furthermore, this operator is the dominant (least irrelevant) operator at low energies. As such, the dependence of the three-point function on momenta can be unambiguously predicted up to small corrections, as we will see in detail below.

Before turning to the detailed analysis of our scenario, we point out that the idea of an accelerated expansion of the universe driven by a non-minimal kinetic term was already discussed under the name of k-inflation [9]. K-inflation is based on some assumed form for the higher derivative terms, which can only be justified by the knowledge of the fundamental theory. On the contrary our model is based on a consistent effective field theory around the ghost condensate, so that all the predictions are not UV sensitive.

2 Generation of density perturbations

Quantum fluctuations of the $\pi$ field in de Sitter space become, as usual, classical fluctuations at large scales, when they freeze and become constant. In ref. [7] it was shown that the gravitational potential in longitudinal gauge, $\Phi = \Psi$, decays to zero outside the horizon. This means that $\pi$ fluctuations do not gravitate at superhorizon scales in the pure de Sitter background. This is quite intuitive: in the limit in which the shift symmetry $\pi \rightarrow \pi + \text{const.}$ is exact, different parts of the Universe with a different value of $\pi$ are equivalent, so that the metric is not perturbed.

This is rather different from what happens in standard inflation, where the surfaces of constant inflaton are also of constant energy density and the gravitational potential, therefore, does not decay to zero outside the horizon. Our model is similar to the usual slow-roll inflation in the limit in which the potential becomes completely flat, keeping the produced density perturbations constant: $\epsilon \to 0$ with $H/(M_\text{Pl} \sqrt{\epsilon}) = \text{const.}$ In this limit the comoving surfaces of constant inflaton have the only meaning of setting the time to the end of inflation, but there is no metric perturbation before reheating: $\Phi \to 0$ as $\epsilon \to 0$. 5
The calculation of density perturbations will therefore proceed as usual: we can calculate the gauge invariant quantity $\zeta$ \[^{10}\] , proportional to the spatial curvature of comoving surfaces through the position dependent time shift

$$\zeta = - \frac{H}{\dot{\phi}} \pi .$$

(2.1)

This quantity will remain constant outside the horizon, independently of the details of the reheating process and it will seed all the perturbations we observe today. For a simple check of the conservation of $\zeta$ in the limit of instantaneous reheating, see the Appendix.

The situation is in some sense simpler than in conventional slow-roll inflation. In that case we are forced to switch to the $\zeta$ variable at horizon crossing, because we know that $\zeta$ is constant outside the horizon, while the inflaton perturbation will keep on evolving because of the non-zero potential. The evolution of the inflaton perturbation outside the horizon is such that it cancels the time evolution of $H/\dot{\phi}$ in (2.1) to give a constant $\zeta$. In our case both $\zeta$ and $\pi$ are constant outside the horizon before the end of inflation as $H$ and $\dot{\phi}$ are time-independent. So we can also easily follow the surface of constant $\pi$ until reheating.

Let us now calculate the spectrum of density perturbations. In presence of the ghost condensate gravity is modified in the IR and this modification is characterized by a typical time scale $\Gamma^{-1}$, with $\Gamma \sim M^3/M_{Pl}^2$, and a typical length scale $m^{-1}$, with $m \sim M^2/M_{Pl}$ \[^{7}\]. The two scales are different because the condensate obviously breaks Lorentz invariance. To avoid large modification of gravity nowadays we have to require $\Gamma < H_0$, where $H_0$ is the present Hubble constant \[^{7}\]. Obviously this implies that gravity is not modified during inflation, i.e.

$$\Gamma \ll m \ll H .$$

(2.2)

This relation is equivalent to the decoupling limit $M_{Pl} \to \infty$, keeping $H$ fixed. Therefore we can study the $\pi$ Lagrangian alone, without considering the backreaction on the metric through the Einstein equations. At linear level we get

$$\ddot{\pi}_k + 3H \dot{\pi}_k + \frac{\alpha^2}{M^2} \left( \frac{k^4}{a^4} \right) \pi_k = 0 ,$$

(2.3)

where $k$ is the comoving wavevector and $a$ the scale factor. This equation describes a free field in de Sitter space, with the modified dispersion relation $\omega^2 \propto k^4$.

To calculate the spectrum of $\pi$ fluctuations, we quantize the field as usual

$$\pi_k(t) = w_k(t) \hat{a}_k + w_k^*(t) \hat{a}_k^\dagger ,$$

(2.4)

where $w_k(t)$ satisfies (2.3). A qualitative study of eq. (2.3) tells us that the wavefunction goes to a constant when the expansion rate $H$ exceeds the oscillation frequency $\alpha M^{-1}(k/a)^2$: due to the modified dispersion relation the $\pi$ field freezes when its physical wavelength is of order $(\alpha^{-1}HM)^{-1/2}$ and not $H^{-1}$ as usual.
We can easily estimate the asymptotic amplitude of $w_k$, which gives the spectrum of the $\pi$ perturbations. Well before freezing eq. (2.3) describes a harmonic oscillator with slowly varying frequency and with a friction term. Both the variation of frequency and the friction reduce the oscillator energy. Using the equipartition of kinetic and potential energy, it is easy to obtain the time variation of the total energy $E$ of the oscillator

$$\dot{E}(t) = -5HE(t),$$

which implies that the quantity $E(t) \cdot a^5$ is constant before freezing. We can rewrite this using the amplitude of oscillation $\delta \pi(t)$ and the frequency $\omega(t)$ as

$$\delta \pi(t)^2 \omega(t)^2 a^5 = \text{const}.$$ (2.6)

Now we can get the asymptotic amplitude using this conservation law between one time well before freezing (where we set $a = 1$) and the freezing point $\omega(t) \sim H$. At the first point we use the usual flat-space normalization of the wavefunction $\delta \pi \sim \omega^{-1/2}$ and we get the final result

$$\delta \pi_H(k) \sim \frac{(HM^3 \alpha^{-3})^{1/4}}{k^{3/2}},$$ (2.7)

compatible with what we got in the introduction using the scaling dimension of the operators. The spectrum of $\pi$ fluctuations is scale invariant. This is not a surprise; it is just the consequence of time translation invariance in the de Sitter background; every mode freezes at a fixed amplitude when it reaches a certain physical size and this is enough to give a scale invariant spectrum, independently of the dispersion relation of the $\pi$ field and the fact that its modes freeze at a scale smaller than the de Sitter horizon.

The qualitative behavior above can be checked by the explicit solution of eq. (2.3). To simplify the equation we can go to conformal time $d\eta = dt/a$ (we take $\eta = -(aH)^{-1}$ during inflation) and write the equation for the variable $u_k \equiv w_k \cdot a$. We get

$$u_k'' + \left( \frac{\alpha^2 H^2 k^4}{M^2} \cdot \eta^2 - \frac{2}{\eta^2} \right) u_k = 0,$$ (2.8)

where the derivatives are now taken with respect to $\eta$. The solution of eq. (2.8) with the correct flat space limit for very short wavelength (i.e. $\eta \to -\infty$) is given by

$$u_k(\eta) = \sqrt{\frac{\pi}{8}} \sqrt{-\eta} H^{(1)}_{3/4} \left( \frac{Hk^2 \alpha}{2M} \eta^2 \right),$$ (2.9)

where $H$ is the Hankel function, linear combination of Bessel functions: $H^{(1)}_{\nu}(x) = J_{\nu}(x) + iY_{\nu}(x)$. The spectrum of $\pi$ can be calculated from the asymptotic behavior of (2.9) in the limit $\eta \to 0$. The Bessel function $J_{-3/4}$ is dominant and we get

$$u_k(\eta \to 0) \simeq \frac{i \sqrt{2\pi}}{\Gamma(1/4)} \left( \frac{M}{\alpha H} \right)^{3/4} k^{-3/2} \cdot \eta^{-1}.$$ (2.10)
The \( \pi \) spectrum is therefore given by
\[
P_{\pi} = a^{-2} \frac{k^{3}}{2 \pi^{2}} |u_{k}(\eta \to 0)|^{2} = \frac{(HM^{3} \alpha^{-3})^{1/2}}{\pi \Gamma^{2}(1/4)}.
\] (2.11)

From this we get the primordial curvature spectrum through (2.1)
\[
P_{R}^{1/2} = \frac{1}{\sqrt{\pi \Gamma(1/4)}} \frac{(H^{5}M^{3} \alpha^{-3})^{1/4}}{\phi} = \frac{1}{\sqrt{\pi \Gamma(1/4)}} \left( \frac{H}{M} \right)^{5/4} \alpha^{-3/4}.
\] (2.12)

The COBE normalization fixes \( P_{R}^{1/2} \simeq 4.8 \cdot 10^{-5} \).

### 3 Non-gaussianities

One of the sharpest predictions of inflation, at least if the inflaton itself is responsible for the generation of density perturbations, is that these density fluctuations should have a probability distribution very close to Gaussian. There are two reasons behind this property. First of all, the inflaton field must have a very shallow potential which allows a slow-roll phase; this means that the inflaton self-interactions are very small so that it behaves like a free field. Non-linearities coming from General Relativity, even if suppressed by the Planck scale, are in fact bigger than those coming from the inflaton potential, but anyway too small to be experimentally testable: the fluctuations are Gaussian up to a level of \( 10^{-6} \).

One can imagine that additional self-interactions for the inflaton are coming from higher dimension operators with derivatives: these operators are compatible with slow-roll if they respect a shift symmetry on the inflaton \( \phi \to \phi + \text{const} \). The contribution of these operators cannot be very big anyway: the lowest dimension operator, \( (\nabla \phi)^{4} \), has scaling dimension 2, so that its contribution is suppressed by \( (H/\Lambda)^{2} \), where \( \Lambda \) is the cut-off scale. As \( \Lambda \) cannot be too small if we want the classical motion of the inflaton to be under control (i.e. \( \dot{\phi} \ll \Lambda^{2} \)), we get a contribution which cannot be much bigger than \( 10^{-5} \).

In our model we do not have any potential for \( \phi \) during inflation and the only non-linearities come from higher dimension operators. As discussed in the introduction, these interactions are actually much more relevant than in standard inflation, because of the non-relativistic scaling of the operators. The most relevant (or better least irrelevant) operator is
\[
\frac{\beta}{M^{2}} \nabla^{2} \pi \nabla^{2} \pi.
\] (3.1)

It has scaling dimension \([\text{energy}]^{1/4}\), all the others have dimension bigger or equal to \(1/2\) so that they will be less important in calculating the non-gaussianities of the density perturbations. The very small dimension of this operator tells us that this will be much more important than higher dimension operators in standard inflation, which have at least dimension 2. Again we implicitly assumed a decoupling limit, neglecting the backreaction of \( \pi \) on the metric: in this way we neglect all the non-linearities intrinsic to General
Relativity, which are suppressed by $M_{Pl}$ and therefore subleading with respect to the $\pi$ self-interactions.

Let us estimate the size of the non-gaussianities comparing the non-linear correction $\dot{\pi}(\nabla \pi)^2$ with the linear terms in eq. (2.3), close to the freezing point $\omega \simeq H$. The corrections will be given by the ratio

$$
\frac{\beta M^{-2}(HP_\pi^{1/2})(MH^{-1}P_\pi)}{H^2 P_\pi} \simeq \frac{\beta}{\sqrt{\pi} \Gamma(1/4)} \left( \frac{H}{M} \right)^{1/4} \alpha^{-7/4} \simeq 3 \cdot 10^{-2} \cdot \beta \cdot \alpha^{-8/5},
$$

where in the last equality we used the COBE normalization eq. (2.12). Parametrically we obtain the same estimate we got in the introduction using the scaling dimension of the operator. Taking $\alpha \sim \beta \sim 1$ the level of non-gaussianity is quite substantial and close to what is allowed by experiments.

Despite the order one uncertainty we expect that the non-gaussian contributions are experimentally detectable. The angular distribution of the 3-point function clearly depends on the wavefunction of the $\pi$ field and on the specific trilinear interaction and could in principle be used as a smoking gun of the model. We therefore turn to the explicit calculation of the 3-point function given by the interaction $\hat{\pi}(\nabla \pi)^2$.

Before doing the explicit calculation we must clarify the relation between $\pi$ and the gauge invariant quantity $\zeta$. At the end we are obviously interested in the 3-point function of $\zeta$, as this quantity remains constant outside the horizon (also at non-linear order [11]) and it is directly related to the fluctuations we observe today. In doing the calculations it is much easier to calculate the 3-point function of the $\pi$ field and only at the end turn to the $\zeta$ variable. Physically we calculate the 3-point function of the $\pi$ field after its fluctuations are frozen outside the horizon then, before the end of inflation, we switch to comoving coordinates where $\pi$ is constant on surfaces of fixed time, as $\pi$ fixes when inflation ends at each point. The position dependent time shift between the two sets of coordinates is given by (2.1). It is very important to note that this relation is valid also at non-linear order in $\pi$ as $\dot{\phi}$ and $H$ are constant so that they do not depend on $\pi$. This is different from what happens in slow-roll inflation where the relation between $\zeta$ and the inflaton fluctuations is non-linear [3]. In our case there are no additional non-linearities going from the variable $\pi$ to $\zeta$: the two 3-point functions are just proportional through the relation (2.1).

In the calculation of the 3-point function we start from the interaction Lagrangian

$$
L_{int} = -\beta \frac{e^{Ht}}{2M^2} \left[ \ddot{\pi}(\nabla \pi)^2 \right].
$$

We are interested in the value of the $\langle \pi \pi \pi \rangle$ correlator at a time $t$ when the fluctuations are frozen outside the horizon, so we have to evolve it starting from an initial time deep inside the horizon

$$
\langle \pi_{k_1}(t) \pi_{k_2}(t) \pi_{k_3}(t) \rangle = -i \int_{t_0}^{t} dt' \left\langle \left[ \pi_{k_1}(t) \pi_{k_2}(t) \pi_{k_3}(t), \int d^3 x \, H_{int}(t') \right] \right\rangle.
$$
The expression can be evaluated using the decomposition (2.4) of the $\pi$ field, giving

$$
\langle \pi_k \pi_{k_2} \pi_{k_3} \rangle = \frac{i \beta}{M^2} (2\pi)^3 \delta^3 \left( \sum_i \vec{k}_i \right) w_{k_1}(0) w_{k_2}(0) w_{k_3}(0)
$$

$$
\cdot \int_{-\infty}^{0} d\eta \frac{1}{H\eta} w_{k_1}^*(\eta) w_{k_2}^*(\eta) w_{k_3}^*(\eta)(\vec{k}_1 \cdot \vec{k}_2) + \text{symm.} + c.c.
$$

(3.5)

The sum over the two analogous expressions with the derivative acting on the other wavefunctions is indicated with '+symm’. The integral must evaluated with the prescription that the oscillating functions become exponentially decreasing, i.e. below the negative real axis: this fixes the vacuum of the interacting theory at $\eta = -\infty$. Using the explicit expression for the wavefunctions (2.9) we get to the final expression

$$
\langle \pi_k \pi_{k_2} \pi_{k_3} \rangle = -2 \sqrt{\frac{2\pi^3}{3}} \frac{H^5 \beta}{M^2} \left( \frac{M}{\alpha H} \right)^4 (2\pi)^3 \delta^3 \left( \sum_i \vec{k}_i \right)
$$

$$
\cdot \int_{-\infty}^{0} d\eta \eta^{-1} F^*(\eta) F^* \left( \frac{k_2}{k_1} \eta \right) F^* \left( \frac{k_3}{k_1} \eta \right) k_3(\vec{k}_1 \cdot \vec{k}_2) + \text{symm.} + c.c.
$$

(3.6)

where

$$
F(x) = \sqrt{\frac{\pi}{8}} (-x)^{3/2} H^{(1)}_{3/4}(x^2/2).
$$

(3.7)

We can finally translate the result into the $\zeta$ variable, using eq. (2.1)

$$
\langle \zeta_{k_1} \zeta_{k_2} \zeta_{k_3} \rangle = \frac{2 \sqrt{2\pi^{3/2}}}{\Gamma(1/4)^3} \left( \frac{H}{\alpha M} \right)^4 (2\pi)^3 \delta^3 \left( \sum_i \vec{k}_i \right)
$$

$$
\cdot \int_{-\infty}^{0} d\eta \eta^{-1} F^*(\eta) F^* \left( \frac{k_2}{k_1} \eta \right) F^* \left( \frac{k_3}{k_1} \eta \right) k_3(\vec{k}_1 \cdot \vec{k}_2) + \text{symm.}
$$

(3.8)

As there are no poles one can rotate the contour of integration along the direction $\alpha (-1 - i)$ so that it converges exponentially.

In the analysis of the data (see e.g. [13]) it is usually assumed that the non-gaussianities come from a field redefinition

$$
\zeta(x) = \zeta_g(x) - 3 f_{NL} (\zeta_g^2(x) - \langle \zeta_g^2 \rangle),
$$

(3.9)

where $\zeta_g$ is gaussian. This pattern of non-gaussianity, which is local in real space, is characteristic of models in which the non-linearities develop outside the horizon. This happens for all the models in which the fluctuations of an additional light field, different from the inflaton, contribute to the curvature perturbations we observe. In this case nonlinearities come from the evolution of this field outside the horizon and from the conversion mechanism which transforms the fluctuations of this field into density perturbations. Both these sources of non-linearity give a non-gaussianity of the form (3.9) because they occur outside the horizon. Examples of this general scenario are the curvaton models [5], models
with fluctuations in the reheating efficiency \cite{6} and multi-field inflationary models \cite{4}. In the data analyses (3.9) is taken as a simple ansatz and limits are therefore imposed on the scalar variable $f_{NL}$. The angular dependence of the 3-point function in momentum space implied by (3.9) is given by

$$\langle \zeta_{k_1} \zeta_{k_2} \zeta_{k_3} \rangle = (2\pi)^3 \delta^3 \left( \sum_i k_i \right) (2\pi)^4 \left( -\frac{3}{5} f_{NL} P_R^2 \right) \frac{4 \sum_i k_i^3}{\prod_i 2 k_i^3}.$$  

In our case the angular distribution is much more complicated than in the previous expression so the comparison is not straightforward.

We can nevertheless compare the two distributions (3.8) and (3.10) for an equilateral configuration and define in this way an “effective” $f_{NL}$ for $k_1 = k_2 = k_3$. Evaluating numerically the integral in (3.8) for $k_1 = k_2 = k_3$ and using COBE normalization $P_R^{1/2} \simeq 4.8 \times 10^{-5}$ and eq. (2.12) we get to

$$f_{NL}^{\text{equil}} \simeq 85 \cdot \beta \cdot \alpha^{-8/5}.$$  

The result is rather suppressed, taking $\alpha \sim \beta \sim 1$ with respect to the estimate (3.12), which would give $f_{NL} \simeq 10^3$. There are two reasons for this suppression: first of all $\dot{\eta} \rightarrow 0$ near freezing and this suppresses the integral in (3.8). Second the wavefunction is slightly suppressed with respect to its natural size for $\eta \rightarrow 0$ and this gives an additional small suppression. The result must be considered just as an estimate of the effect because order one factors are not under control. On the other hand the angular dependence in (3.8) is well defined and different from all the models generating observable non-gaussianities through an additional light field; the angular pattern represents a potential smoking gun of the model.

The present limit on the non-gaussianity parameter from the WMAP collaboration \cite{13}

$$-58 < f_{NL} < 138 \text{ at } 95\% \text{ C.L.}$$  

is quite close to the estimate (3.11).

We expect that the limit would change appreciably using for the analysis the angular pattern (3.8). The reason is that our pattern is qualitatively quite distinct from the one parametrized by $f_{NL}$. The behavior in the limit in which one of the wavevector goes to zero is, for example, rather different from what happens in (3.10). In this limit one of the fluctuations has a very long wavelength, it exits the horizon and freezes much before the other two and acts as a sort of background. Let us take $k_3$ very small and assume that a spatial derivative acts on this background in the interaction Lagrangian (3.3). The 2-point function $\langle \pi_1 \pi_2 \rangle$ depends on the position on the background wave and it is proportional to $\partial_i \pi_3$ at linear order. This variation of the 2-point function along the $\pi_3$ wave is averaged to zero in calculating the 3-point function $\langle \pi_k \pi_k \pi_{k_3} \rangle$, because the spatial average $\langle \pi_3 \partial_i \pi_3 \rangle$ vanishes. So we are forced to go to second order and we therefore expect the integral in (3.8) to go as $k_3^2$, making the full 3-point function proportional to $k_3^{-1}$, after we multiply
for the $\pi_3$ spectrum. Note that taking the time derivative in (3.3) to act on the background gives a subleading contribution as the wavefunction goes to a constant as $k_3^{-3}$. On the other hand the “local” behavior (3.10) goes as $k_3^{-3}$ in the limit we have considered. In our model the 3-point function is generated by a derivative interaction which favors the correlation of modes freezing roughly at the same time, while the correlation is suppressed for modes of very different wavelength. The same happens in standard inflation when we study the non-gaussianity generated by higher derivative interactions [12]. The implication of very different angular patterns for the experimental limits on non-gaussianities deserves further studies [14].

Before closing this section we want to comment on an assumption we have implicitly made so far. We have imposed on the effective Lagrangian for $\pi$, eq. (1.3), a discrete symmetry $\pi \rightarrow -\pi, t \rightarrow -t$. Without this assumption we should consider an additional trilinear operator with the same scaling dimension as $\dot{\pi}(\nabla\pi)^2$:

$$\frac{\gamma}{M^3}(\nabla\pi)^2\nabla^2\pi,$$

where $\gamma$ is an additional order one coefficient. It is straightforward to repeat the analysis in this case; the qualitative behavior of the 3-point function remains the same.

The $\pi \rightarrow -\pi, t \rightarrow -t$ symmetry is equivalent to $\phi \rightarrow -\phi$ for the ghost action. In absence of this symmetry we can write CPT violating operators in the effective Lagrangian [7]. This leads us to a possible interesting effect in the CMBR which is independent of how density perturbations are generated: the rotation of the polarization vector of photons traveling from the surface of last scattering to us. The origin of this effect is that the ghost background $\langle \dot{\phi} \rangle \neq 0$ breaks the CPT symmetry [7] besides the Lorentz one, so that a term $\epsilon^{\mu\nu\rho}F_{\mu\nu}A_{\rho}$ for the photon field is allowed. This term gives a different dispersion relation for the two circular polarizations and it thus rotates the axis of a linearly polarized photon. Some limits on the size of this operator come from the observation of distant quasars and radio galaxies [15]: the rotation must be smaller than roughly one degree in one Hubble time. Better limits should come in the future from the CMBR experiments observing polarization [16]. Temperature fluctuations on the last scattering surface always give a certain degree of polarization to the radiation, with a typical “gradient” pattern called E-mode. The rotation of the polarization transforms E-modes into “curl” or B-modes giving a correlation between temperature fluctuations and B-modes which would otherwise be zero. While the present sensitivity is comparable to the limits from distant quasars and radio galaxies, planned experiments could reach a sensitivity of $10^{-3}$ degrees in an Hubble time [16]. We stress that the size of the CPT violating operator cannot be predicted as it depends on the coupling of the ghost sector to ordinary matter.

4 Tilting the potential

So far we have assumed that the inflationary potential for $\phi$ is exactly flat, i.e. that the shift symmetry for $\phi$ is not broken except at $\phi = 0$. As we have emphasized, unlike slow-roll
inflation, no tilt in the potential is needed either to end inflation or to generate acceptable density perturbations. However, given that the shift symmetry on $\phi$ must be broken in any case, there may well be a tilt in the potential for $\phi < 0$, although it is technically natural for this tilt to be as small as we like. It is thus interesting to see how our previous results are modified in the presence of a non-zero slope. We expect, in the limit of very big tilt, to recover the predictions of standard slow-roll inflation, where the classical motion of $\phi$ is dominated by the potential term. As a first approximation, we will assume that the potential has a constant slope $V' = \text{const}$. The equation of motion for the homogeneous perturbations $\pi$ then becomes

$$\ddot{\pi} + 3H\dot{\pi} + V' = 0$$ (4.1)

and we quickly reach the solution for which

$$\dot{\pi} = -\frac{V'}{3H}.$$ (4.2)

We see that, because of the potential, we have changed the velocity of the field $\phi$; for $V' < 0$, we have increased the velocity\(^1\). Note that, in order to stay within the regime of validity of the effective theory, the velocity of $\pi$ should still be small compared with $M^2$; this motivates us to define the small parameter

$$\delta^2 \equiv -\frac{V'}{3HM^2}.$$ (4.3)

Now, with this non-zero value of $\dot{\pi}$ in the background, the dispersion relation for $\pi$ is modified by the cubic interaction

$$\left< \dot{\pi} \right> \left( \nabla \pi \right)^2 M^2,$$ (4.4)

so we have (we take $\alpha = \beta = 1$)

$$\omega^2 = \frac{k^4}{M^2} + \delta^2 k^2.$$ (4.5)

As long as the $k^4$ term dominates down to frequencies $\omega \sim H$, the $\delta$ term can be justifiably neglected; this happens as long as

$$\delta \ll \left(H/M\right)^{1/2}.$$ (4.6)

For larger $\delta$ the new $k^2$ term dominates before the modes freeze out, at a cross-over frequency $\omega_{\text{cross}}$ determined by

$$\omega_{\text{cross}} \simeq \delta^2 M.$$ (4.7)

We can estimate the size of the perturbations in this case easily, as it is now just like the usual case for standard inflation, with the only difference that the spatial momenta are rescaled as $k \rightarrow \delta \cdot k$. Thus,

$$\delta \pi_H \sim \frac{H}{\delta^{3/2}}.$$ (4.8)

\(^1\)We will not study the other possibility, $V' > 0$, which leads to a wrong sign spatial kinetic term for $\pi$.
which reduces to the usual result of standard inflation \( \delta \pi_H \sim H \) for \( \delta \to 1 \), as we expected. For the density perturbations we get

\[
\frac{\delta \rho}{\rho} \sim \frac{H \delta \pi_H}{\dot{\phi}} \sim \frac{H^2}{M^2 \delta^{3/2}}. \tag{4.9}
\]

So, ranging over all values of \( \delta \), \( \delta \rho/\rho \) goes from \((H/M)^5/4 \) (see eq. (4.8)) to \((H/M)^2 \).

Let us now estimate the corrections to the spectral index. It is straightforward to check that corrections coming from the variation of \( H \) are completely negligible: the variation of the potential in one Hubble time is of order \( \Delta V \sim \delta^2 M^4 \), which is very small with respect to the total vacuum energy \( V_0 \). Bigger corrections can come from the variation of \( \dot{\phi} \) in eq. (4.9). The effect is of order \( V''/H^2 \) and limits on \( V'' \) come again from the validity of the effective field theory description

\[
\delta^2 M^2 H \simeq |V'| \gtrsim |V''| \cdot \Delta \phi = |V''| \cdot (M^2/H)N_e \Rightarrow |V''| \lesssim \delta^2 H^2/N_e, \tag{4.10}
\]

where \( N_e \) is the typical number of e-folds. So that deviations from the flat spectrum can be as big as \( \delta^2/N_e \): 

\[
|n_s - 1| \lesssim \delta^2/N_e. \tag{4.11}
\]

The biggest contribution to the tilt comes however from the variation of \( \delta \). Assuming a variation of \( \delta \) during observable inflation of order of \( \delta \) itself, we get a contribution to the tilt of order 

\[
|n_s - 1| \lesssim 1/N_e. \tag{4.12}
\]

Therefore, in the regime \( \delta \gtrsim (H/M)^{1/2} \), a rather big deviation from scale invariance is possible.

We can finally also estimate the dimensionless size of the non-Gaussian effects, which still come from the same cubic interaction \( \delta \) as before. For the frequencies \( \omega_{\text{cross}} < \omega < M \) where the \( k^4 \) quadratic spatial kinetic term dominates, this operator still has scaling dimension \( 1/4 \), but for \( H < \omega < \omega_{\text{cross}} \), the \( k^2 \) quadratic term dominates and the operator has its familiar scaling dimension of \( 2 \). Therefore, the dimensionless size of the non-Gaussianity in this case is

\[
\text{NG} \sim \left( \frac{\omega_{\text{cross}}}{M} \right)^{1/4} \times \left( \frac{H}{\omega_{\text{cross}}} \right)^2 \sim \frac{H^2}{M^2 \delta^{7/2}} \sim \frac{\delta \rho}{\rho} \times \frac{1}{\delta^2}, \tag{4.13}
\]

so we see that the non-Gaussianities are always parametrically enhanced relative to \( \delta \rho/\rho \).

This is to be contrasted with slow-roll inflation, where the leading non-Gaussianity is in fact parametrically suppressed relative to \( \delta \rho/\rho \) by slow-roll factors. The same result eq. (4.13) can be obtained directly evaluating the importance of the non-linear interaction \( \dot{\pi}(\nabla \pi)^2 \) with respect to free field terms at horizon crossing.

We note that the result for \( \delta \to 1 \) is what we get in conventional slow-roll inflation, with the addition of higher derivative terms suppressed by a cut-off scale \( M \simeq \dot{\phi}^{1/2} \).
Table 1: Order of magnitude predictions for the tilt in the scalar spectrum and the level of non-gaussianity. First column: ghost inflation limit with flat potential. Second: ghost inflation with non-flat potential. Third: threshold region, higher derivative terms are as important as the tilt in the potential. Fourth: usual slow-roll inflation, higher derivative terms are irrelevant.

| $|n_s - 1|$ | $\delta < (H/M)^{1/2}$ | $(H/M)^{1/2} < \delta < 1$ | $\delta \sim 1$ | Standard slow-roll |
|------|-----------------|----------------------|----------------|------------------|
| NG   | $10^{-2}$       | $10^{-5}\delta^{-2}$ | $10^{-5}$     | $10^{-5}\epsilon$ |

This limit can in fact be approached starting from a conventional slow-roll model with the addition of higher-dimension operators compatible with the shift symmetry of the inflaton. These additional operators will be suppressed by a typical scale $M$ (note that now we do not have $\dot{\phi} \sim M^2$, because the motion of the inflaton is not dominated by these higher derivative terms). The slow-roll picture makes sense for $M^2 \gg \dot{\phi}$, otherwise all the operators become relevant for the classical motion of the inflaton. In the limit $M^2 \sim \dot{\phi} \simeq V'/H$ we lose control of the theory, but we qualitatively approach our ghost model for $\delta \sim 1$; higher dimension operators become as important as the tilt in the potential to describe the classical motion of the inflaton.

In general if we consider higher derivative interactions suppressed by a typical scale $M$ and a potential tilt $V'$, we can look at our model and conventional slow-roll inflation as the two extreme limits $M^2 \simeq \dot{\phi} \gg V'/H$ (the potential tilt is irrelevant) and $M^2 \gg \dot{\phi} \simeq V'/H$ (higher derivative terms are irrelevant). In both the limits we have a trustworthy effective field theory description and they merge for $V'/H \sim M^2$, i.e. $\delta \sim 1$, when the potential tilt and the higher derivative terms give a comparable contribution to the inflaton motion. In this regime both descriptions break down. In the table we summarize the predictions for the tilt in the scalar spectrum and the level of non-gaussianity going from the ghost inflation limit to the usual slow-roll case.

5 Conclusions

We have presented a new way of generating an inflationary de Sitter phase using ghost condensation. The spectrum of density perturbations is sharply predicted to have $n_s = 1$ up to unobservable corrections; gravitational waves are completely unobservable. The non-Gaussianities are expected to be large enough to be observed. Unfortunately because there are two relevant parameters in the $\pi$ effective Lagrangian we cannot completely fix the magnitude of the 3-point function from the observed density perturbations; however, for all parameters of $O(1)$, the size of the effect is at the level of the current WMAP sensitivity. Aside from the overall amplitude, the 3-point correlator $\langle \delta \rho_p(k_1) \delta \rho_p(k_2) \delta \rho_p(k_3) \rangle$ as a function of the spatial momenta $k_1, k_2, k_3$ is completely determined.
We conclude discussing some open issues. As we stressed, all the predictions of our scenario are independent of the details of the reheating process, analogously to standard slow-roll inflation. But interesting processes might occur at the end of ghost inflation. In the region where the shift symmetry is broken we expect that a potential for $\phi$ is generated at some level. As discussed in section 4 this potential will kick $\dot{\phi}$ out of its stable value $\dot{\phi} = M^2$ and $\dot{\phi}$ will be driven back to its original value by the expansion of the universe once the shift symmetry is restored. The stress energy tensor of this homogeneous perturbation $\dot{\pi}$ redshifts like matter $[7]$ ($\rho \propto a^{-3}$) so that it would be interesting to study the evolution of this fluid to see if it could be a viable candidate for dark matter.

Another interesting possibility is that inflation ends with a phase transition, after which the ghost is not condensed anymore: Lorentz symmetry is restored. This scenario allows to raise the scale $M$, because we are not limited anymore by the present modification of gravity. Obviously the phase transition leads us outside the regime of validity of the effective field theory around the ghost condensate, but it is likely that all our results about density perturbations still apply.

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Appendix: conservation of $\zeta$

The use of the formula (2.1) for the calculation of the gauge-invariant quantity $\zeta$ is justified in general by the conservation of $\zeta$ during the reheating process. In this appendix we shall explicitly check the conservation. For simplicity we take the limit in which the time scale associated with reheating is much shorter than the Hubble time scale and assume that spatial components of the stress-energy tensor remain non-singular. In this case we can use the Israel junction condition $[17]$ without a singular matter source to obtain the matching condition for perturbations even without any knowledge about the details of microscopic processes.

Starting with general scalar-type perturbations in the longitudinal gauge,

$$ ds^2 = -[1 + 2\Phi(t, x)] dt^2 + [1 - 2\Phi(t, x)] a^2(t) \delta_{ij} dx^i dx^j, \quad \phi = \phi^{(0)}(t) + \pi(t, x), \quad (A.1) $$

we can perform an infinitesimal gauge transformation

$$ \delta g_{\mu\nu} \rightarrow \delta g_{\mu\nu} - \xi_{\mu}\nu - \xi_{\nu;\mu}, \quad \pi \rightarrow \pi - \xi^\mu \partial_\mu \phi^{(0)}, \quad (A.2) $$

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where
\[
\begin{align*}
\xi_0(t, x) &= -\frac{\pi(t_0, x)}{\phi(0)(t_0)} - \int_{t_0}^{t} \Phi(t', x) dt', \\
\xi_i(t, x) &= -\frac{a^2(t)}{a^2(t')} \int_{t_0}^{t} \frac{\partial_i \xi_0(t', x)}{a^2(t')} dt',
\end{align*}
\]  
(A.3)
so that the metric and the scalar field become
\[
\begin{align*}
ds^2 &= -dt^2 + g_{ij} dx^i dx^j, \\
g_{ij} &= (1 - 2\Psi + 2H\xi_0)a^2 \delta_{ij} - \partial_i \xi_j - \partial_j \xi_i, \\
\phi &= \phi(0) + \pi(t, x) - \pi(t_0, x) - \dot{\phi}(0)(t) \int_{t_0}^{t} \Phi(t', x) dt'.
\end{align*}
\]  
(A.4)
Here, \( H = \dot{a}/a \) and \( t_0 \) is the value of \( t \) when the homogeneous background \( \phi(0)(t) \) reaches the critical value \( \phi_0 \) at which the phase transition is triggered. The coordinate system after the gauge transformation is nothing but the Gaussian normal coordinate system based on the reheating hypersurface \( \phi = \phi_0 \). In the Gaussian normal coordinate system, the Israel junction condition [17] at the reheating hypersurface is
\[
[q_{ij}] = [\dot{q}_{ij}] = 0, \quad [X] \equiv \lim_{t \to t_0^+} X(t) - \lim_{t \to t_0^-} X(t).
\]  
(A.5)
Thus, we obtain
\[
[a] = 0; \quad [H] = 0
\]  
(A.6)
for the unperturbed geometry, and
\[
\begin{align*}
\left[ \frac{\pi}{\phi(0)} \right] &= 0; \quad [\Psi] = 0; \quad \left[ \dot{\Psi} + H\Phi + H \frac{\pi}{\phi(0)} \right] = 0
\end{align*}
\]  
(A.7)
for the linear perturbation.

Since in a pure de Sitter background the gravitational potential at large scales decays [7], it becomes vanishingly small after a long period of de Sitter expansion. Hence, we set \( \Phi = \Psi = 0 \) for \( t < t_0 \). On the other hand, the universe after reheating is dominated by radiation. In the radiation dominated epoch, it is known that the Fourier components \( \Phi_k \) and \( \Psi_k \) of \( \Phi \) and \( \Psi \), respectively, at superhorizon scales behave as
\[
\Phi_k = \Psi_k \simeq A_k + B_k \left( \frac{a(t)}{a(t_0)} \right)^{-3} \quad (t > t_0),
\]  
(A.8)
where \( A_k \) and \( B_k \) are constants. The coefficients \( A_k \) and \( B_k \) are determined by the Israel junction condition as
\[
A_k = -B_k = 2 \frac{\pi_k}{3 \dot{H} \frac{\pi_k}{\phi(0)}} \bigg|_{t=t_0},
\]  
(A.9)
where $\pi_k(t)$ is the Fourier component of $\pi$. Therefore, we obtain the following behavior of $\Phi_k$ and $\Psi_k$ after reheating.

$$
\Phi_k = \Psi_k \simeq A_k \left[ 1 - \left( \frac{a(t_0)}{a(t)} \right)^3 \right] \rightarrow \frac{2}{3} H \pi_k \frac{\dot{\phi}(0)}{\phi'(0)} \bigg|_{t=t_0}.
$$

(A.10)

Finally, with the help of the well-known relation $\zeta = -\frac{3}{2} \Phi$ in radiation-dominated epoch, this shows that

$$
\zeta \to -\frac{H}{2} \pi \frac{\dot{\phi}(0)}{\phi'(0)} \bigg|_{t=t_0},
$$

(A.11)

and that we can safely use the formula (2.1). In the main part of this paper we shall omit the superscript (0) for $\dot{\phi}(0)$.

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