Infinitely many solutions for a Schrödinger equation with sign-changing potential and nonlinear term

Long-Jiang Gu*, Huan-Song Zhou†

1 School of Mathematics and Physics, China University of Geosciences Wuhan 430074, China
2 Center for Mathematical Sciences and Department of Mathematics, Wuhan University of Technology Wuhan 430070, China

Abstract: We propose a new variational approach to finding multiple critical points for strongly indefinite problems without assuming the weak upper semicontinuity on the variational functionals. By this approach, we obtain the existence of infinitely many geometrically distinct solutions for a stationary periodic Schrödinger equation, in which the linear part is strongly indefinite and the nonlinear term is allowed to change sign in general ways.

Keywords: Variational method, Strongly indefinite, Elliptic equation, Multiple solutions, Periodic Schrödinger equation, Sign-changing nonlinearity.

2000 Mathematics Subject Classification. 35J20, 35J61, 58E05.

1 Introduction

In this paper, we are concerned with the following well-studied stationary periodic Schrödinger equation

\[
\begin{align*}
-\Delta u + V(x)u &= f(x, u), \quad x = (x_1, x_2, ..., x_N) \in \mathbb{R}^N, \\
u &\in H^1(\mathbb{R}^N), N \geq 2,
\end{align*}
\]

(1.1)

where the potential \( V \) and the nonlinear term \( f \) are sign-changing functions satisfying

(V) \( V \in C(\mathbb{R}^N, \mathbb{R}) \) is 1-periodic in \( x_1, ..., x_N \) and 0 lies in a gap of the spectrum of \(-\Delta + V\), i.e.,

\[
sup\{\sigma(-\Delta + V) \cap (-\infty, 0)\} < 0 < \inf\{\sigma(-\Delta + V) \cap (0, +\infty)\}.
\]

(f₁) \( f \in C(\mathbb{R}^N \times \mathbb{R}) \) is 1-periodic in \( x_1, ..., x_N \), and there exist constants \( C > 0 \) and \( p \in (2, 2^*) \) with \( 2^* = \frac{2N}{N-2} \) if \( N \geq 3 \), and \( 2^* = +\infty \) if \( N = 2 \), such that

\[
|f(x, t)| \leq C(1 + |t|^{p-1}), \text{ for any } (x, t) \in \mathbb{R}^N \times \mathbb{R}.
\]

*Supported by NFSC under Grant No. 12101577 and the Zhejiang Provincial Natural Science Foundation under Grant No. LQ21A010014, Email: gulongjiang0@163.com
†Supported by NFSC under Grants No. 11931012, 11871387, Email: hszhou@whut.edu.cn (Corresponding author)
\( f_2 \) \( \lim_{t \to 0} f(x,t)/t = 0 \) uniformly in \( x \in \mathbb{R}^N \).

The associated variational functional of equation (1.1) is defined by

\[
\varphi(u) := \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 + V(x) u^2 \, dx - \int_{\mathbb{R}^N} F(x,u) \, dx, \quad u \in H^1(\mathbb{R}^N),
\]

where \( F(x,u) := \int_0^u f(x,s) \, ds \). \( \varphi(u) \) is well defined in \( H^1(\mathbb{R}^N) \) and is of class \( C^1 \) under the conditions (V), (f1) and (f2). Moreover, for each \( u \in H^1(\mathbb{R}^N) \),

\[
\varphi'(u) = \int_{\mathbb{R}^N} \nabla u \cdot \nabla \phi \, dx + \int_{\mathbb{R}^N} V(x) u \phi \, dx - \int_{\mathbb{R}^N} f(x,u) \phi \, dx, \quad \text{for any } \phi \in H^1(\mathbb{R}^N),
\]

and \( \varphi'(u) \) is weakly sequentially continuous (see [38, Theorem A.2]).

We say \( u \in H^1(\mathbb{R}^N) \setminus \{0\} \) is a nontrivial weak solution of (1.1) if \( \varphi'(u) \phi = 0 \) for any \( \phi \in H^1(\mathbb{R}^N) \). Two solutions of (1.1), \( u_1 \) and \( u_2 \), are called geometrically distinct if \( u_1 \) and \( u_2 \) are distinct under \( \mathbb{Z}^N \)-translation, i.e., \( u_1(x) \neq u_2(x+a) \) for any \( a \in \mathbb{Z}^N \).

Under condition (V), the continuity and periodicity of \( V \) ensure that the Schrödinger operator \( L := -\Delta + V(x) \) has only continuous spectrum in \( L^2(\mathbb{R}^N) \) (see [34, Theorem XIII.100]). Since \( 0 \) lies in a gap of the spectrum of \( L \), the Hilbert space \( X := \mathcal{D}(|L|^{\frac{1}{2}}) \) can be decomposed into \( X = X^- \oplus X^+ \) such that the quadratic form:

\[
u \rightarrow \int_{\mathbb{R}^N} |\nabla u|^2 + V(x) u^2 \, dx,
\]

is positive definite on \( X^+ \) and negative definite on \( X^- \), both \( X^+ \) and \( X^- \) are infinite-dimensional. Such kind of functional is usually called strongly indefinite, and the associated variational problem is called strongly indefinite problem.

The inner product on \( X = \mathcal{D}(|L|^{\frac{1}{2}}) \) is defined by

\[
\langle u,v \rangle := \langle |L|^{\frac{1}{2}} u, |L|^{\frac{1}{2}} v \rangle_{L^2}, \quad \text{for any } u,v \in X,
\]

where \( \langle \cdot, \cdot \rangle_{L^2} \) is the usual inner product in \( L^2(\mathbb{R}^N) \). The associated norm for \( X \) is

\[
\|u\| := \langle |L|^{\frac{1}{2}} u, |L|^{\frac{1}{2}} u \rangle_{L^2}^{\frac{1}{2}}.
\]

By a similar discussion to the appendix in [15], it follows from \( V(x) \in L^\infty(\mathbb{R}^N) \) that \( X := \mathcal{D}(|L|^{\frac{1}{2}}) = H^1(\mathbb{R}^N) \) and the norms \( \| \cdot \| \) and \( \cdot \|_{H^1} \) are equivalent. Moreover, \( X^+ \) and \( X^- \) are also orthogonal with respect to \( \langle \cdot, \cdot \rangle \).

Let

\[
P : X \rightarrow X^- \quad \text{and} \quad Q : X \rightarrow X^+
\]

be the orthogonal projections, then (1.2) and (1.3) can be rewritten as

\[
\varphi(u) := \frac{1}{2} (-\|Pu\|^2 + \|Qu\|^2) - \int_{\mathbb{R}^N} F(x,u) \, dx,
\]

and

\[
\varphi'(u) \phi = \langle Qu, \phi \rangle - \langle Pu, \phi \rangle - \int_{\mathbb{R}^N} f(x,u) \phi \, dx.
\]
In order to study the strongly indefinite variational problems as above, various variational methods have been developed in the last two decades\([2, 3, 4, 22, 27, 28, 37]\). In paper \([22]\), Kryszewski and Szulkin introduced the so-called \(\tau\)-topology in a Hilbert space (see section 2 for details) and then established a generalized linking theorem for \(\tau\)-upper semi-continuous functionals. By this linking theorem, they proved the existence of solutions for equation \((1.1)\) under \((V), (f_1), (f_2)\) and the (AR) condition:

\[
0 < \gamma F(x, t) \leq tf(x, t) \quad \text{for some } \gamma > 2 \quad \text{and for all } x \in \mathbb{R}^N, t \in \mathbb{R}\setminus\{0\}.
\]

In papers \([4, 5]\), Bartsch and Ding established some critical point theories in general Banach space (see also \([18]\)). Thereafter, based on the \(\tau\)-upper semi-continuity assumption, a series of critical point theorems were developed (see \([6, 24, 26, 35, 36]\)) and used to study the existence of solutions for equation \((1.1)\) with some special nonlinearities, such as, paper \([19]\) for super or asymptotically linear nonlinearities, papers \([1, 32]\) for nonlocal nonlinearities, and papers \([11, 12, 36]\) for critical nonlinearities, etc. However, all of these mentioned papers require that the variational functional satisfies \(\tau\)-upper semi-continuity, which then implies that the nonlinear term in \((1.1)\) must have some kind of positivity, e.g., \(F(x, u) \geq 0\).

Recently, the authors in \([7, 13]\) developed some critical point theorems without assuming the \(\tau\)-upper semi-continuity and equations with certain types of sign-changing nonlinear terms were studied (see also \([14, 25]\)). More precisely, in \([13]\) the authors assumed further in equation \((1.1)\):

\[(f_3)\] For some \(q \in (2, 2^*)\), let

\[
\kappa := \max \left\{ \sup_{u \in X \setminus \{0\}} \frac{\|Pu\|_{L^q}}{\|u\|_{L^q}}, \sup_{u \in X \setminus \{0\}} \frac{\|Qu\|_{L^q}}{\|u\|_{L^q}} \right\} \quad \text{and} \quad \mu_0 := \inf_{u \in X \setminus \{0\}} \frac{\|u\|^2}{\|u\|^2_{L^2}}.
\]

There exist positive constants \(\rho, D_1, D_2\) such that

\[
F(x, t) \geq 0, \quad \tilde{F}(x, t) := \frac{1}{2} uf(x, t) - F(x, t) \geq D_1|t|^q, \quad \text{and} \quad |f(x, t)| \leq D_2|t|^{q-1},
\]

for all \(x \in \mathbb{R}^N\) and \(|t| \geq \rho\). In additional,

\[
2\kappa D_2 \left( \rho^{q-2} + \frac{1}{D_1} \sup_{|t| \leq \rho, x \in \mathbb{R}^N} \left| \frac{\tilde{F}(x, t)}{t^2} \right| \right) + \sup_{|t| \leq \rho, x \in \mathbb{R}^N} \left| \frac{f(x, t)}{t} \right| < \mu_0.
\]

Then, they obtained in \([13]\) a nontrivial solution for equation \((1.1)\) under conditions \((V)\) and \((f_1)-(f_3)\).

**Remark 1.1**

i): Condition \((f_3)\) is mainly used to ensure the boundedness of \((PS)\)-sequence, or \((C)\)-sequence. ii): There are many functions satisfying \((f_1)-(f_3)\), for example, if \(2 < r < q < 2^*\), then

\[
f(x, t) = |t|^{q-2}t - \lambda|t|^{r-2}t, \quad (1.8)
\]
satisfies conditions (f_1)-(f_3) if \( \lambda > 0 \) is sufficiently small (see [13, Remark 1.4]). In fact, for \( f \) given by (1.3), noting

\[
\tilde{F}(x,t) = \frac{q - 2}{2q} |t|^q - \lambda \frac{r - 2}{2r} |t|^r = \frac{q - 2}{4q} |t|^q + \left( \frac{q - 2}{4q} - \lambda \frac{r - 2}{2r} \right) |t|^r,
\]

we take \( D_1 = \frac{q - 2}{4q} \) and \( \rho_1 := \left( \frac{2q(r - 2)}{r(q - 2)} \right)^{\frac{1}{r-q}} \lambda^{\frac{1}{r-q}} \), a direct computation shows that \( \tilde{F}(x,t) = D_1 |t|^q \) if \( t = \rho_1 \) and then \( \tilde{F}(x,t) \geq D_1 |t|^q \) if \( t \geq \rho_1 \). By \( |f(x,t)| \leq |t|^{q-1} + \lambda |t|^{r-1} \), it is easy to see \( |f(x,t)| \leq D_2 |t|^{q-1} \) if \( t \geq \rho_1 \), where \( D_2 = 1 + \frac{r(q - 2)}{2q(r - 2)} \). On the other hand we have \( F(x,t) \geq 0 \) if \( t \geq \rho_2 := \left( \frac{q}{2} \right)^{\frac{1}{r-q}} \lambda^{\frac{1}{r-q}} \). Clearly, by taking \( \rho = \max\{\rho_1, \rho_2\} \) and due to \( \rho \) can be arbitrarily small if \( \lambda > 0 \) is small enough, condition (f_3) is satisfied if \( \lambda > 0 \) is small enough.

The sign-changing nonlinearity in (1.1) plays an important role in nonlinear optics with material mixture of focusing and defocusing (see [8, 23, 29, 30, 31]). For the positive definite case with \( \tau \) levels, i.e., for \( \varphi \) and they first proved two kinds of deformation lemmas for general level sets under different energy levels, i.e., for \( \varphi^{d+\epsilon} \) if \( d \geq \beta + 1 \) and for \( \varphi^{d+\epsilon} |_{\mathcal{N}} \) if \( d < \beta + 1 \), respectively, where \( \mathcal{N} \) is a symmetric \( \tau \)-open set with genus \( \gamma(\mathcal{N}) = 1 \) (see [22, Lemma 4.6]). Then, they proved the existence of infinitely many geometrically distinct solutions of (1.1) under the (AR) condition, the authors defined a special energy level

\[
\beta = \max_{u \in \mathcal{K} \setminus \{0\}} \varphi(u),
\]

where \( \mathcal{K} := \{ u \in H^1(\mathbb{R}^N) : \varphi'(u) = 0 \} \) is assumed to be finite set under the action of \( Z^N \)-translation, and they first proved two kinds of deformation lemmas for general level sets under different energy levels, i.e., for \( \varphi^{d+\epsilon} \) if \( d \geq \beta + 1 \) and for \( \varphi^{d+\epsilon} |_{\mathcal{N}} \) if \( d < \beta + 1 \), respectively, where \( \mathcal{N} \) is a symmetric \( \tau \)-open set with genus \( \gamma(\mathcal{N}) = 1 \) (see [22, Lemma 4.6]). Then, they proved the existence of infinitely many geometrically distinct solutions of (1.1) by an indirect argument originated by [10, 17]. However, because the deformation lemmas in [22] strictly rely on the \( \tau \)-upper semi-continuity, their method seems invalid when \( F \) changes sign. To overcome this difficulty, in this paper we introduce a new energy level

\[
\zeta := \sup_{\|Qu\| \leq \delta + 1} \varphi(u),
\]

where \( \delta := \sup\{\|Qu\| : u \in \mathcal{K} \} \), which is crucial to the proof of deformation lemmas respectively for two symmetric bounded \( \tau \)-compact sets, i.e., for \( M \) if \( \sup_{u \in M} \varphi(u) > \zeta + 1 \), and for \( M \setminus \mathcal{N} \) (by a descending flow on \( \varphi^{\zeta+2} \)) if \( \sup_{u \in M \setminus \mathcal{N}} \varphi(u) < \zeta + 2 \) (see Lemmas 3.2 and 3.4 in Section 3). To prove these deformation lemmas, some new ideas must be used in constructing pseudo-gradient vector fields to avoid using the \( \tau \)-upper semi-continuity. Furthermore, we also need to use a new strategy in proving the split lemma of (PS)-sequence (see Lemma 2.4) for the lack of (AR) condition.

Before giving our main result of the paper, we introduce two further conditions on \( f \) as follows:
(f_4) \( f(x, -t) = -f(x, t) \) for all \( x \in \mathbb{R}^N \) and \( t \in \mathbb{R} \).

(f_5) There exist \( \bar{c} > 0 \) and \( \varepsilon_0 > 0 \) such that
\[
|f(x, t + s) - f(x, t)| \leq \bar{c}|s|(1 + |t|^{p-1})
\]
for all \( x \in \mathbb{R}^N \) and \( t, s \in \mathbb{R} \) with \( |s| \leq \varepsilon_0 \).

Clearly, \( f \) given by (1.8) satisfies all the conditions (f_1)-(f_5) if \( \lambda > 0 \) is sufficiently small.

Finally, we state our main result as follows:

**Theorem 1.1** If the conditions (V) and (f_1)-(f_5) are satisfied, then equation (1.1) has infinitely many geometrically distinct solutions.

Our theorem seems to be the first result on multiple solutions for a periodic Schrödinger equation with both the potential and nonlinear term changing sign. In this case, we lose both the (AR) condition and the \( \tau \)-upper semi-continuity for the variational functional. Moreover, the Fountain Theorem obtained in [20] does not work under the conditions of our Theorem 1.1 because the (PS)-condition (which is needed in [20]) cannot be satisfied when \( \varphi \) is invariant under \( \mathbb{Z}^N \)-translation.

This paper is organized as follows: In section 2, some notations and several useful lemmas are introduced. In section 3, two important deformation lemmas are established. In section 4, Theorem 1.1 is proved by the argument of contradiction based on a mini-max value defined by pseudo-indexes and the two deformation lemmas obtained in section 3.

## 2 Preliminaries

We begin this section by giving some notations and definitions. Let
\[
\mathcal{K} := \{ u \in H^1(\mathbb{R}^N) : \varphi'(u) = 0 \}
\]
be the set of weak solutions of (1.1) and let
\[
\mathcal{F} := \mathcal{K}/\mathbb{Z}^N
\]
be the set of arbitrarily chosen representatives of \( \mathcal{K} \) under the action of \( \mathbb{Z}^N \)-translation. By [13, Theorem 1.5], we know that \( \mathcal{F}\{0\} \neq \emptyset \) if conditions (V) and (f_1)-(f_3) hold.

**Definition 2.1** Let \( \varphi \in C^1(X, \mathbb{R}) \) and \( c \in \mathbb{R} \). A sequence \( \{u_n\} \subset X \) is called a \( (C)^c \)-sequence if
\[
\sup_n \varphi(u_n) \leq c \text{ and } (1 + \|u_n\|)\|\varphi'(u_n)\|_{X'} \to 0 \text{ as } n \to \infty.
\]

**Definition 2.2** Let \( \varphi \in C^1(X, \mathbb{R}) \) and \( c \in \mathbb{R} \). A sequence \( \{u_n\} \subset X \) is called a \( (PS)^c \)-sequence if
\[
\sup_n \varphi(u_n) \leq c \text{ and } \|\varphi'(u_n)\|_{X'} \to 0 \text{ as } n \to \infty.
\]
Clearly, a \((C)^c\)-sequence is a \((PS)^c\)-sequence, but only a bounded \((PS)^c\)-sequence is also a \((C)^c\)-sequence. As in [22, section 2], we have the following definition related to the \(\tau\)-topology.

**Definition 2.3** Let \(\{e_j\}_{j \geq 1}\) be an orthonormal basis of \(X^-\), the \(\tau\)-topology on \(X = X^- \oplus X^+\) is the topology associated with the following norm

\[
\|u\|_\tau := \max\{\sum_{j=1}^{\infty} \frac{1}{2^j}|\langle Pu, e_j \rangle|, \|Q u\|\}, \quad u \in X,
\]

where \(P\) and \(Q\) are given in (L3).

By [22, Remark 2.1(iii)], we know that if \(\{u_n\} \subset X\) is a bounded sequence, then

\[
u_n \overset{n}{\to} u \text{ in } \tau\text{-topology } \iff Pu_n \overset{n}{\rightharpoonup} Pu \text{ weakly in } X^- \quad \text{and} \quad Qu_n \overset{n}{\to} Qu \text{ strongly in } X^+.
\]

(2.3)

Following the paper [35], we give now the definitions on admissible homotopy/map.

**Definition 2.4** Let \(A \subset X\) be a closed subset. For \(T > 0\), a map \(g : [0, T] \times A \to X\) is an admissible homotopy if

- \(g\) is \(\tau\)-continuous in the sense that, for \(\{t_m\} \subset [0, T], \{u_m\} \subset A,\)

\[g(t_m, u_m) \overset{m}{\to} g(t, u) \text{ in } \tau\text{-topology}\]

whenever \(t_m \overset{m}{\to} t\) and \(u_m \overset{m}{\to} u\) in \(\tau\)-topology.

- For any \((t, u) \in [0, T] \times A\), there exists a neighborhood \(W_{(t,u)}\) of \((t, u)\) in the \(|\cdot| \times \tau\)-topology such that

\[
\{v - g(s, v) : (s, v) \in W_{(t,u)} \cap ([0, T] \times A)\}
\]

is contained in a finite-dimensional subspace of \(X\).

In particular, for a map being independent of \(t\), e.g., \(g(t, u) \equiv g(u)\), we have the following definition.

**Definition 2.5** Let \(A \subset X\) be a closed subset. A map \(h : A \to X\) is an admissible map if

- \(h\) is \(\tau\)-continuous and, for any \(u \in A\), there is a \(\tau\)-neighborhood \(W_u\) of \(u\) such that

\[
\{v - h(v) : v \in W_u \cap A\}
\]

is contained in a finite-dimensional subspace of \(X\).

For any integer \(k \geq 1\), let

\[
S_r := \{u \in X : \|u\| = r\} \quad \text{and} \quad X_k := \bigoplus_{j=1}^{k} \mathbb{R} f_j \oplus X^-,
\]

where \(\{f_j\}_{j \geq 1}\) is an orthonormal basis of \(X^+\). Then, we have

**Lemma 2.1** Under the conditions \((\text{V})\) and \((\text{f}_1)\)-(\text{f}_3), let \(\varphi\) be the functional defined by (L3), then

- \((i)\) There exists \(r > 0\) such that

\[
b := \inf_{u \in X^+ \cap S_r} \varphi(u) > 0.
\]
(ii) For $r$ being obtained in (i) and any integer $k \geq 1$, there exists $R_k > r$ such that

$$
\sup_{u \in X_k, \|u\| = R_k} \varphi(u) < a := \inf_{\|u\| \leq r} \varphi(u).
$$

(iii) For any $\delta < +\infty$, there holds

$$
\sup_{\|Qu\| \leq \delta} \varphi(u) < +\infty.
$$

Moreover,

$$
\limsup_{\|Qu\| \to 0} \varphi(u) \leq 0.
$$

Proof. (i) and (iii) directly follow from the steps 1 and 3 in the proof of [13, Lemma 3.1], respectively. Similar to the step 2 of the proof of [13, Lemma 3.1], we know that $\varphi(u) \to -\infty$ as $\|u\| \to -\infty$ and $u \in X_k$, hence part (ii) is also proved. □

Clearly, $a := \inf_{\|u\| \leq r} \varphi(u) \leq b := \inf_{u \in X^+ \cap S_r} \varphi(u)$. By Lemma 2.1(i) and (iii), there exist $r > 0$ and $\delta_0 > 0$ such that

$$
\sup_{\|Qu\| \leq \delta_0} \varphi(u) < b := \inf_{u \in X^+ \cap S_r} \varphi(u).
$$

By (3.18) of [13], we know that, if (V) and (f_1)-(f_3) hold, there exist constants $c_1 > 0$ and $c_2 > 0$, which depend only on $\mu_0$ in (f_3), such that

$$
\varphi(u) \leq c_1\|Qu\|^2 - c_2\|Pu\|^2.
$$

Hence, (2.3) and (2.5) imply that

**Lemma 2.2** Under conditions (V) and (f_1)-(f_3), let $\{u_n\} \subset \varphi_c := \{u \in X : \varphi(u) \geq c\}$ be any sequence with $\|u_n - u_0\|_\tau \to 0$, then

$$
Pu_n \rightharpoonup Pu_0 \text{ weakly in } X^- \text{ and } Qu_n \rightharpoonup Qu_0 \text{ strongly in } X^+.
$$

In [13, Lemma 3.2], it is proved that any $(C)^c$-sequence of $\varphi$, see (1.2), is bounded in $X$ under conditions (V) and (f_1)-(f_3). By an argument almost the same as [13, Lemma 3.2], we can easily prove that

**Lemma 2.3** Let (V) and (f_1)-(f_3) be satisfied. If $\{u_n\} \subset X$ is a $(PS)^c$-sequence of $\varphi$, then

$$
\sup_n \|u_n\| \leq C,
$$

for some $C > 0$ (independent of $n$). □

So, if $\{u_n\}$ is a $(PS)^c$-sequence of $\varphi$, by Lemma 2.3 we may define

$$
M_c := \limsup_{n \to \infty} \|u_n\|,
$$

and $M_c \leq C < +\infty$. Moreover, we have the following compactness result.
Lemma 2.4 Under conditions (V) and (f_1)-(f_5), let \( \{u_m\} \subset X \) be a (PS)\(^c\)-sequence for (1.2), \( \mathcal{K} \) and \( \mathcal{F} \) are defined by (2.1) and (2.2) respectively. If \( \alpha := \inf_{u \in \mathcal{K} \setminus \{0\}} \|u\| > 0 \), then either

(i) \( \limsup_{m \to \infty} \|u_m\| = 0 \), or,

(ii) there exist a positive integer \( l \leq \frac{M^2}{\alpha} \), points \( \bar{u}_1, \ldots, \bar{u}_l \in \mathcal{F} \setminus \{0\} \) \( (\bar{u}_i, i = 1, \ldots, l, \) are not necessarily distinct), a subsequence of \( \{u_m\} \) (still denoted by \( \{u_m\} \)) and sequences \( \{g^i_m\} \subset \mathbb{Z}^N \), \( (i = 1, \ldots, l) \) such that

\[
\lim_{m \to \infty} \|u_m - \sum_{i=1}^{l} (g^i_m + \bar{u}_i)\| = 0,
\]

where \([\cdot]\) denotes the integer part of a real number and \((g * u)(x) := u(x + g)\) for \( g \in \mathbb{Z}^N \).

Proof. For \( u \in X \) we will denote \( u^+ := Qu \) and \( u^- := Pu \) for convenience. By contradiction, if (i) is not satisfied, by Lemma 2.3, \( \{u_m\} \) is also a \((C)^c\)-sequence, then similar to the proof of Theorem 1.5 in [13] (see also Lemma 1.7 in [22]) we know that there exist a weak convergent subsequence of \( \{u_m\} \) (still denoted by \( \{u_m\} \)), a sequence \( \{a_m\} \subset \mathbb{R}^N \) and two constants \( r, \eta > 0 \) such that

\[
\|u_m\|_{L^2(B(a_m, r))} > \eta,
\]

for all \( m \in \mathbb{N} \), where \( B(a_m, r) = \{x \in \mathbb{R}^N : \|x - a_m\| \leq r\} \). We may choose \( \{g_m\} \subset \mathbb{Z}^N \) and set \( v_m := (g_m * u_m) \) such that

\[
\|v_m\|_{L^2(B(0, r + \frac{2\sqrt{N}}{\alpha}))} > \eta, \text{ for all } m \in \mathbb{N}.
\]

It is easy to see that \( \|v_m\| = \|u_m\| \), so, \( \{v_m\} \) is also bounded in \( X \) and there is a subsequence (still denoted by \( \{v_m\} \)) converges to some \( v \in X \) both weakly in \( X \) and strongly in \( L^s_{\text{loc}}(\mathbb{R}^N) \) for \( s \in [2, 2^*) \). Therefore

\[
v \in \mathcal{K} \setminus \{0\}.
\]

Let \( w_m := v_m - v \), we claim that

\[
\lim_{m \to \infty} \varphi'(w_m) = 0,
\]

and

\[
\limsup_{m \to \infty} \|w_m\|^2 \leq M^2 - \|v\|^2. \tag{2.7}
\]

We prove (2.6) first. Let \( \langle \cdot, \cdot \rangle \) be the inner product defined by (1.4), it follows from (1.7) that, for any \( h \in X \) with \( \|h\| = 1 \),

\[
\varphi'(w_m) h = \langle (v_m^+ - v_m^-), h \rangle - \langle (v^+ - v^-), h \rangle - \int_{\mathbb{R}^N} f(x, w_m) h(x) dx. \tag{2.8}
\]

Since \( \varphi'(v) = 0 \) and \( \varphi'(v_m) \rightharpoonup 0 \) in \( X' \), these imply that

\[
\langle (v^+ - v^-), h \rangle = \int_{\mathbb{R}^N} f(x, v) h(x) dx, \tag{2.9}
\]
and
\[ \langle (v^+_m - v^-_m), h \rangle = \int_{\mathbb{R}^N} f(x, v_m)h(x)dx + o(1), \quad \text{as} \ m \to \infty. \] (2.10)

Then, (2.8) together with (2.9) and (2.10) gives that
\[ \varphi'(w_m)h = \int_{\mathbb{R}^N} (f(x, v_m) - f(x, w_m) - f(x, v))h(x)dx + o(1), \quad \text{as} \ m \to \infty. \]

So, we only need to show that
\[ \int_{\mathbb{R}^N} (f(x, v_m) - f(x, w_m) - f(x, v))h(x)dx \to 0, \quad \text{for} \ |h| = 1 \ \text{uniformly.} \]

Since \( v \) is a solution of (1.1), we have \(-\Delta v + c(x)v = 0\), where \( c(x) = V(x) - \frac{f(x,v)}{v} \). It follows from (f1) and (f2) that there exists a \( \delta > 0 \) such that
\[ |f(x, u)| \leq \delta|u| + c_3|u|^{p-1}, \] (2.11)
thus \( c(x) \in L^1_{loc}(\mathbb{R}^N) \) for some \( t > \frac{N}{2} \). By Theorem 4.1 of [21], we have \( v(x) \to 0 \) as \( |x| \to \infty. \)

Then, for any \( \varepsilon > 0 \) and \( R > 0 \), \( B_R = \{ x \in \mathbb{R}^N : |x| < R \} \) with \( R \) large enough, by (f5) and [2.11] there holds
\[ \int_{\mathbb{R}^N \setminus B_R} |(f(x, v_m) - f(x, w_m) - f(x, v))h(x)|dx 
\leq \int_{\mathbb{R}^N \setminus B_R} \bar{c}|v|(1 + |w_m|^{p-1})h + C\int_{\mathbb{R}^N \setminus B_R} |v||h| 
\leq C|v|^2\|h\|^2 + C|v||w_m||^{p-1}\|h\| \leq \frac{\varepsilon}{2}. \] (2.12)

Moreover, since both \( w_m \stackrel{m}{\to} 0 \) in \( L^p(B_R) \) and \( v_m \stackrel{m}{\to} v \) in \( L^p(B_R) \), it follows from Theorem A.2 of [38] that
\[ \int_{B_R} |(f(x, v_m) - f(x, w_m) - f(x, v))h(x)|dx 
\leq \int_{B_R} |(f(x, v_m) - f(x, v))||h(x)|dx + \int_{B_R} |f(x, w_m)||h(x)| \leq \frac{\varepsilon}{2}, \] (2.13)
for \( m \) large enough. Since \( \varepsilon \) is arbitrary, (2.12) and (2.13) show that
\[ \int_{\mathbb{R}^N} |(f(x, v_m) - f(x, w_m) - f(x, v))h(x)|dx \to 0, \] (2.14)
uniformly for \( |h| = 1 \). Hence (2.6) is proved.

Next we prove (2.7). First from \( \varphi'(v) = 0 \), \( \varphi'(v_m) \stackrel{m}{\to} 0 \), \( \varphi'(w_m) \stackrel{m}{\to} 0 \) in \( X' \) and the boundedness of \( \|v_m\| \) and \( \|w_m\| \) we have
\[ \|v\|^2 = \int_{\mathbb{R}^N} f(x, v)(v^+ - v^-), \]
\[ \|v_m\|^2 = \int_{\mathbb{R}^N} f(x, v_m)(v^+_m - v^-_m) + o(1), \]
\[ \|w_m\|^2 = \int_{\mathbb{R}^N} f(x, w_m)(w_m^+ - w_m^-) + o(1). \]

So,
\[
\|w_m\|^2 + \|v\|^2 - \|v_m\|^2
= \int_{\mathbb{R}^N} f(x, w_m)(v_m^+ - v_m^-) - \int_{\mathbb{R}^N} f(x, v_m)(v^+ - v^-)
+ \int_{\mathbb{R}^N} f(x, v)(v^+ - v^-) - \int_{\mathbb{R}^N} f(x, v_m)(v_m^+ - v_m^-) + o(1). \tag{2.15}
\]

Since \( v_m \overset{m}{\rightharpoonup} v \) weakly in \( X \), we have \( w_m \overset{m}{\rightharpoonup} 0 \) weakly in \( X \) and thus
\[
\langle w_m, v^+ \rangle \to 0 \quad \text{as} \quad m \to \infty. \tag{2.16}
\]

By \( \langle \varphi'(w_m), v^\pm \rangle \overset{m}{\to} 0 \), we have
\[
\langle w_m, v^+ \rangle = \int_{\mathbb{R}^N} f(x, w_m)v^+ + o(1),
\]
and
\[
-\langle w_m, v^- \rangle = \int_{\mathbb{R}^N} f(x, w_m)v^- + o(1).
\]

These together with (2.16) imply that
\[
\lim_{m \to \infty} \int_{\mathbb{R}^N} f(x, w_m)v^\pm = 0. \tag{2.17}
\]

By the boundedness of \( \|w_m\| \), we have \( \varphi'(v)w_m^\pm = 0 \), so,
\[
\langle v, w_m^+ \rangle = \int_{\mathbb{R}^N} f(x, v)v^+_m \quad \text{and} \quad -\langle v, w_m^- \rangle = \int_{\mathbb{R}^N} f(x, v)v^-_m. \tag{2.18}
\]

Since \( w_m \overset{m}{\rightharpoonup} 0 \) weakly in \( X \), then \( w^\pm \overset{m}{\rightharpoonup} 0 \) weakly in \( X^\pm \), and (2.18) shows that
\[
\int_{\mathbb{R}^N} f(x, v)w_m^+ = \int_{\mathbb{R}^N} f(x, v)(v_m^+ - v^+) \to 0 \quad \text{as} \quad m \to \infty,
\]
and
\[
\int_{\mathbb{R}^N} f(x, v)w_m^- = \int_{\mathbb{R}^N} f(x, v)(v_m^- - v^-) \to 0 \quad \text{as} \quad m \to \infty.
\]

So,
\[
\int_{\mathbb{R}^N} f(x, v)(v_m^+ - v_m^-) = \int_{\mathbb{R}^N} f(x, v)(v^+ - v^-) + o(1). \tag{2.19}
\]

Then, it follows from (2.15), (2.17) and (2.19) that
\[
\|w_m\|^2 + \|v\|^2 - \|v_m\|^2
= \int_{\mathbb{R}^N} \left( f(x, w_m) + f(x, v) - f(x, v_m) \right)(v_m^+ - v_m^-) + o(1)
\]
By the boundedness of \( \|v_m\| \) and using (2.14) again, we have
\[
\lim_{m \to \infty} \{\|w_m\|^2 + \|v\|^2 - \|v_m\|^2\} = 0,
\]
which leads to (2.7) by using the definition of \( M_c \).

Since \( \|v\| \leq M_c \) (by the weak lower semicontinuity of norms), there are two possibilities which may occur:
If $\|v\| = M_c$, then (2.7) implies that $w_m \to 0$ strongly in $X$, that is (ii) holds with $l = 1$ and $\bar{u}_1 = v$.

If $\|v\| < M_c$, we may go back to the beginning of our proof by simply replacing $\{u_m\}$ and $M_c^2$ by $\{w_m\}$ and $M_c^2 - \|v\|^2$ respectively. Then we can set $\bar{u}_2 \in K$ with $\alpha^2 \leq \|\bar{u}_2\|^2 \leq M_c^2 - \alpha^2$. Repeat this procedure at most $\left\lfloor \frac{M_c^2}{\alpha^2} \right\rfloor$ times, we obtain the conclusion.

Let $l \in \mathbb{N}$ and $A \subset X$ be a finite set, i.e., $A$ contains finite number of elements, define

$$[A, l] := \left\{ \sum_{i=1}^{j} (g_i * v_i) : 1 \leq j \leq l, g_i \in \mathbb{Z}^N, v_i \in A \right\}.$$ 

Then, it follows from [17], see also [16, 22], that

**Lemma 2.5** ([17], Proposition 2.57) For any $l \in \mathbb{N}$, if $A \subset X$ is a finite set, then

$$\inf\{\|v - v'\| : v, v' \in [A, l], v \neq v'\} > 0.$$

We mention that the sets $[A, l]$ defined above are closed related to the so-called $(PS)$-attractors or $(C)$-attractors (see [4]). By Lemmas 2.4 for $F$ defined by (2.2) and $\varphi$ defined by (1.6), we have the following lemma.

**Lemma 2.6** Under conditions (V) and $(f_1)$-$(f_5)$, if $F$ is a finite set and $\{u_m\} \subset X$ is a $(PS)^c$-sequence of $\varphi$, then there exists $l_c \in \mathbb{N}$, which depends only on $c$, such that

$$0 \leq \|u_m - [F, l_c]\|_\tau \leq \|u_m - [F, l_c]\| \to 0 \quad \text{as} \quad m \to \infty.$$

Let

$$B_{X^+}(z, r) := B(z, r) \cap X^+, \quad B(z, r) := \{u \in X : \|u - z\| < r\},$$

then it follows from Lemmas 2.4, 2.6 that

**Lemma 2.7** If conditions (V) and $(f_1)$-$(f_5)$ hold, $F$ is a finite set, then, for any $c \in \mathbb{R}$ and $s \in (0, \frac{r}{4})$ with

$$\mu = \inf\{\|v - v'\| : v, v' \in [QF, l_c], v \neq v'\},$$

where $l_c \in \mathbb{N}$ is given in Lemma 2.4 and depends only on $c$, there exists $\sigma > 0$ such that

$$\|\varphi'(u)\|_{X'} > \sigma \quad \text{for any} \quad u \in \varphi^c \setminus \left( \bigcup_{z \in [QF, l_c]} X^- \oplus B_{X^+}(z, s) \right), \quad \varphi^c := \{u \in X : \varphi(u) \leq c\}.$$
Proof. By contradiction, if such $\sigma$ does not exist, then there exists a sequence

$$\{u_m\} \subset \varphi^c \setminus \left( \bigcup_{z \in [QF, l_c]} X^- \oplus B_{X^+}(z, s) \right)$$

for some $s \in (0, \frac{\mu}{4})$, (2.20)

such that $\varphi'(u_m) \stackrel{m}{\rightharpoonup} 0$ in $X'$. Then, Lemma 2.6 implies that

$$\|u_m - [F, l_c]\| \to 0 \quad \text{as} \quad m \to \infty.$$

So,

$$\|Qu_m - Q[F, l_c]\| \to 0 \quad \text{as} \quad m \to \infty.$$  

On the other hand, by [22, Remark 1.1 (iv)], $Q$ and $\ast$ are commutative, thus

$$Q[F, l_c] = [QF, l_c].$$

So,

$$\|Qu_m - [QF, l_c]\| \to 0 \quad \text{as} \quad m \to \infty,$$

which contradicts (2.20). The proof is complete.

3 Two deformation lemmas

In this section, two deformation lemmas will be proved and we stress that the $\tau$-upper semi-continuity of $\varphi$ is not needed. Assume that (1.1) has only finite geometrically distinct solutions, i.e., $F$ is finite, then

$$\delta := \sup \{\|Qu\| : u \in K\} < +\infty.$$  

(3.1)

By Lemma 2.1 (iii), we have

$$\zeta := \sup_{\|Qu\| \leq \delta + 1} \varphi(u) < +\infty.$$  

(3.2)

Let

$$\Sigma := \{A \subset X : A \text{ is closed and } A = -A\},$$

and

$$\hat{\Sigma} := \{A \in \Sigma : A \text{ is bounded and } \tau\text{-compact}\}.$$  

(3.3)

Since $X$ is a Hilbert space, let $\nabla \varphi$ be given by the formula

$$\langle \nabla \varphi(u), v \rangle = \varphi'(u)v, \quad \text{for all } v \in X,$$

then we have
Lemma 3.1 Under conditions (V), (f1)-(f5), let \( F \) be a finite set and let \( M \in \tilde{\Sigma} \). Denote

\[
\bar{\beta} := \sup_{u \in M} \varphi(u) \text{ and } \beta := \inf_{u \in M} \varphi(u).
\]

If \( \bar{\beta} > \zeta + 1 \), \( \zeta \) is defined in (3.2), then there exists a vector field:

\[
\chi_1(u) : \varphi^\beta_{\beta-3} \to X \text{ with } \varphi^\beta_{\beta-3} := \{ u \in X : \beta - 3 \leq \varphi(u) \leq \bar{\beta} \}
\]

such that

(i) \( \chi_1(u) \) is locally Lipschitz continuous and \( \tau \)-locally Lipschitz \( \tau \)-continuous on \( \varphi^\beta_{\beta-3} \).

(ii) \( \chi_1(u) \) is odd with \( -3 \leq \langle \nabla \varphi(u), \chi_1(u) \rangle \leq 0 \), for any \( u \in \varphi^\beta_{\beta-3} \).

(iii) \( \langle \nabla \varphi(u), \chi_1(u) \rangle \leq -1 \) for any \( u \in E_1 := \varphi^\beta_{\beta-3}\{ u \in X : \|Qu\| < \delta + 1 \} \), for \( \delta \) defined by (3.1).

(iv) There exists \( \sigma_1 > 0 \) such that

\[
\|\chi_1(u)\| < \sigma_1 \text{ for any } u \in \varphi^\beta_{\beta-3} \setminus \left( \bigcup_{z \in [Q,F,t^\beta]} X^- \oplus B_X(z,\frac{\mu}{4}) \right)
\]

where \( \mu \) and \( l^\beta \) are given by Lemma 2.7.

(v) Each \( u \in \varphi^\beta_{\beta-3} \) has a \( \tau \)-open neighborhood, \( U_u \subset X \), of \( u \) such that \( \chi_1(U_u \cap \varphi^\beta_{\beta-3}) \) is contained in a finite-dimensional subspace of \( X \).

Proof. For any \( u \in E_1 \), let

\[
\omega(u) = \frac{2\nabla \varphi(u)}{\|\nabla \varphi(u)\|^2},
\]

then, there exists a \( \tau \)-neighborhood of \( u \), \( V_u \subset X \), satisfying

\[
\|v - u\|_\tau < \min\{\frac{\mu}{8}, \frac{1}{3}\}, \text{ for any } v \in V_u,
\]

such that

\[
1 < \langle \nabla \varphi(v), \omega(u) \rangle < 3, \text{ for any } v \in V_u \cap \varphi^\beta_{\beta-3}\{ u \in X : \|Qu\| < \delta + \frac{1}{2} \}.
\]

Indeed, suppose that such \( V_u \) does not exist, then there exists a sequence \( \{v_n\} \subset \varphi^\beta_{\beta-3} := \{ u \in X : \varphi(u) \geq \beta - 3 \} \) with \( v_n \overset{\tau}{\to} u \) and \( \lim_{n \to \infty} \langle \nabla \varphi(v_n), \omega(u) \rangle \leq 1 \) or \( \geq 3 \). By Lemma 2.2, \( v_n \overset{\lambda}{\to} \) weakly in \( X \), which leads to a contradiction since \( \nabla \varphi \) is weakly continuous. So,

\[
N_1 = \{ V_u : u \in E_1 \} \cup \{ u \in X : \|Qu\| < \delta + 1 \}
\]

forms a \( \tau \)-open covering of \( \varphi^\beta_{\beta-3} \).

Since \( N_1 \) is metric and paracompact, that is, there exists a locally finite \( \tau \)-open covering \( M_1 = \{ M_i : i \in \Lambda \} \) of \( \varphi^\beta_{\beta-3} \) which is finer than \( N_1 \), where \( \Lambda \) is an index set. If \( M_i \subset V_{u_i} \) for some \( u_i \in E_1 \), we take \( \omega_i = \omega(u_i) \) and if \( M_i \subset \{ u \in X : \|Qu\| < \delta + 1 \} \), we take \( \omega_i = 0 \). Let \( \lambda_i : i \in \Lambda \)
be a $\tau$-Lipschitz continuous partition of unity subordinated to $M_1$, then we define the following vector field in $N_1$

$$\xi_1(u) = \sum_{i \in \Lambda} \lambda_i(u) \omega_i, \quad u \in N_1.$$ 

Since the $\tau$-open covering $M_1$ of $N_1$ is locally finite, each $u \in N_1$ belongs to finite many sets $M_i, i \in \Lambda$, that is, for any $u \in N_1$ the sum $\xi_1(u)$ has only finite terms. It follows that, for any $u \in N_1$, there exist a $\tau$-open neighborhood $U_u \subset M_1$ of $u$ and a constant $L_u > 0$ such that $\xi_1(U_u)$ is contained in a finite-dimensional subspace of $X$ and

$$\|\xi_1(v) - \xi_1(w)\| \leq L_u\|v - w\|_\tau,$$ 

for any $v, w \in U_u$.

Define

$$\tilde{\xi}_1(u) = \frac{\xi_1(u) - \xi_1(-u)}{2},$$

and let $\theta \in C^\infty(\R)$ with $0 \leq \theta \leq 1$ be such that

$$\theta(t) = \begin{cases} 
0, & t \leq \delta + \frac{1}{2}, \\
1, & t \geq \delta + \frac{3}{2}, 
\end{cases}$$

where $\delta$ is given in (3.1). Now, we define the vector field $\chi_1$ as follows

$$\chi_1(u) = \begin{cases} 
-\theta(\|Qu\|)\tilde{\xi}_1(u), & u \in N_1, \\
0, & \|Qu\| \leq \delta + \frac{1}{2}, 
\end{cases}$$

then, (i)-(iii), (v) follow directly from the construction of $\chi_1(u)$. By Lemma 2.7 there exists $\sigma > 0$ such that

$$\|\nabla \varphi(u)\| > \sigma, \quad \text{for any } u \in \varphi_\beta \setminus \left( \bigcup_{z \in [Q,F]_{\| \cdot \|}} X^- \oplus B_{X^+}(z, \frac{\mu}{8}) \right),$$

then, $\|\omega(u)\| < \frac{2}{\delta} := \sigma_1$, for any $u \in \varphi_{\beta-3} \setminus \left( \bigcup_{z \in [Q,F]_{\| \cdot \|}} X^- \oplus B_{X^+}(z, \frac{\mu}{8}) \right)$. This together with (3.4) gives (iv). \qed

**Definition 3.1** For each $A \in \tilde{\Sigma}$, define $\mathcal{H}(A)$ to be the class of maps $h : A \rightarrow X$ satisfying

- $h$ is a homeomorphism of $A$ onto $h(A)$ in the original topology, i.e., the topology induced by $\| \cdot \|$, of $X$. 

• $h$ is an odd admissible map which maps bounded set into bounded set;

• $\varphi(h(u)) \leq \varphi(u)$, for any $u \in A$.

**Remark 3.1** For any $A \in \bar{\Sigma}$, $\mathcal{H}(A)$ is nonempty since it always contains the identity. Furthermore, $\mathcal{H}(A)$ is closed under composition, i.e., for any $A \in \bar{\Sigma}$ and $h \in \mathcal{H}(A)$, $g \in \mathcal{H}(h(A))$, there holds $g \circ h \in \mathcal{H}(A)$, since $h(A)$ is also bounded and $\tau$-compact for any $h \in \mathcal{H}(A)$.

Now, we give our first deformation lemma.

**Lemma 3.2 (Deformation Lemma 1)** Under conditions (V), $(f_1)$-$(f_5)$, let $F$ be a finite set and let $M \in \bar{\Sigma}$. If $\beta := \sup_{u \in M} \varphi(u) > \zeta + 1$, $\zeta$ is defined by (3.2), then there exists a map $h \in \mathcal{H}(M)$ such that $h(M) \in \bar{\Sigma}$ and $h(M) \subset \varphi^{\beta - 1}$.

**Proof.** Let $\chi_1(u) : \varphi^{\beta - 3} \to X$ with $\beta := \inf_{u \in M} \varphi(u)$ be the vector field given in Lemma 3.1, we study the following Cauchy problem

\[
\begin{align*}
\frac{d\eta}{dt} &= \chi_1(\eta) \\
\eta(0, u) &= u \in M.
\end{align*}
\]

(3.5)

By the standard theory of ordinary differential equation in Banach space, we know that the initial problem (3.5) has a unique solution on $[0, T_{max})$. Now, we claim that $T_{max} > 1$ for any $u \in M$. In fact, by Lemma 3.1 (ii), if $T_{max} \leq 1$, for any $t \in [0, T_{max})$ and $u \in M$ we know that

\[
\varphi(\eta(t, u)) = \varphi(u) + \int_0^t \frac{d}{ds} \varphi(\eta(s, u))ds = \varphi(u) + \int_0^t (\nabla \varphi(\eta(s, u)), \chi_1(\eta(s, u)))ds \geq \varphi(u) + \int_0^t -3ds \geq \beta - 3T_{max} \geq \beta - 3,
\]

(3.6)

so, $\eta(t, u) \subset \varphi^{\beta - 3}$ for any $(t, u) \in [0, T_{max}) \times M$ where $\chi_1(\eta(t, u))$ is well defined. If $T_{max} \leq 1$, then there exist a sequence $t_m \nearrow T_{max}$ and $u \in M$ such that $\|\chi_1(\eta(t_m, u))\| \to \infty$ as $m \to \infty$. By (3.3), there exists a sequence $\{v_m\} \subset X$ with $v_m - \eta(t_m, u) \rightarrow 0$, $\eta(t_m, u) \rightarrow \min\{\frac{\beta}{3}, \frac{1}{3}\}$ for all $m \geq 1$ satisfying $\nabla \varphi(v_m) \to 0$ in $X$ and $\theta(||Q\eta(t_m, u)||) \neq 0$. Then by Lemma 2.6 passing to a subsequence if necessary, we know that either

(a) there is $z \in [QF, l_{\beta}]$ such that $v_m \in X^- \oplus \left(B_x(z, \frac{|z|}{3})\right)$,

or,

(b) the sequence $\{v_m\}$ enters infinitely many sets of the form $X^- \oplus \left(B_x(z, \frac{|z|}{3})\right)$ where $z \in [QF, l_{\beta}]$.

However, both the cases are impossible. If (a) holds, then $Qv_m \to z$ strongly in $X$. Since $\|v_m\|$ is bounded by Lemma 2.3 we also have $Pv_m \rightharpoonup y$ weakly in $X$. By the weak continuity of $\varphi'$ we know $(y + z) \in K$ hence $\|v_m - K\| \to 0$ and this contradicts with $\theta(||Q\eta(t_m, u)||) \neq 0$. If (b) holds, we may assume that $\eta(t, u)$ leaves $X^- \oplus \left(B_x(z_1, \frac{|z_1|}{3})\right)$ as $t = t_1$ and enters $X^- \oplus \left(B_x(z_2, \frac{|z_2|}{3})\right)$ as $t = t_2$. Then $\eta(t_1, u) - \eta(t_2, u) \geq \frac{1}{2}\mu$. By Lemma 3.1 (ii), there exists $\sigma_1 > 0$ such that
\[ \|\chi_1(u)\| \leq \sigma_1 \text{ for any } u \in \varphi^{3}\big(\bigcup_{z \in [Q,F,t]} X^- \oplus B_{X^+}(z,\frac{\mu}{4})\big). \]

So,
\[ \frac{1}{2} \mu \leq \|\eta_1(t_1,u) - \eta_1(t_2,u)\| \leq \int_{t_1}^{t_2} \|\chi_1(\eta_1(s,u))\|ds \leq \sigma_1(t_2 - t_1), \]
which is a contradiction since \(|t_1 - t_2|\) can be arbitrarily small as \(t_1\) and \(t_2\) arbitrarily close to \(T_{max}\).

Next, we claim that \(h(\cdot) = \eta_1(1,\cdot)\). It is clear that for any \(t \in [0,1]\), \(\eta_1(t,\cdot)\) is an odd homeomorphism from \(M\) to \(\eta_1(t,M)\). Furthermore, by almost the same way as the proof of [38 Lemma 6.8] we can get that \(\eta_1\) is \(\tau\)-continuous and for any \((t,u) \in [0,1] \times M\) there exist a neighborhood \(W_{(t,u)}\) of \((t,u)\) in the \(|\cdot|\times\tau\)-topology such that
\[ \{v - \eta_1(s,v)|(s,v) \in W_{(t,u)} \cap ([0,1] \times M)\} \]
is contained in a finite-dimensional subspace of \(X\).

Then, we claim that \(\eta_1(1,M) \subset \varphi^{3-1}, \tilde{\beta}\) is given by Lemma 3.2. Suppose there exists \(u \in M\) such that \(\varphi(\eta_1(1,u)) > \tilde{\beta} - 1\), then \(\varphi(\eta_1(t,u)) > \tilde{\beta} - 1\) for any \(t \in [0,1]\), since (by Lemma 3.1 (ii)) \(\varphi(\eta_1(t,u))\) is nonincreasing in \(t\). So, \(\eta_1(t,u) \in E_1\) with \(E_1\) given in Lemma 3.1 (iii), for any \((t,u) \in [0,1] \times M\), then
\[ \varphi(\eta_1(1,u)) = \varphi(\eta_1(0,u)) + \int_0^1 (\nabla \varphi(\eta_1(s,u)), \chi_1(\eta_1(s,u)))ds \]
\[ < \varphi(\eta_1(0,u)) + \int_0^1 -1ds \leq \tilde{\beta} - 1, \]
which lead to contradiction.

Finally, in order to prove \(\eta_1(1,\cdot) \in \mathcal{H}(M)\), we only need to show that \(\eta_1(1,M)\) is \(\tau\)-compact and bounded. Indeed, the \(\tau\)-compactness of \(\eta_1(1,M)\) follows directly since \(\eta_1(1,\cdot)\) is \(\tau\)-continuous. Thus, \(Q(\eta_1(1,M))\) is bounded in \(X^+\). On the other hand, by [3.6] we know \(\varphi(\eta_1(1,u)) \geq \beta - 3\) for any \(u \in M\). Together with [2.5] we have \(\eta_1(1,M)\) is bounded. So, let \(h(\cdot) = \eta_1(1,\cdot)\), the proof is complete. \[ \square \]

Next we define a new class of maps related to \(\mathcal{H}(A)\) for convenience.

**Definition 3.2** For \(A \in \tilde{\Sigma}\) (\(\tilde{\Sigma}\) is defined by (3.3)), define \(\tilde{\mathcal{H}}(A)\) to be the set of all maps \(h : A \rightarrow X\) satisfying:

- \(h\) is a homeomorphism of \(A\) onto \(h(A)\) in the original topology, i.e., the topology induced by \(|\cdot|\), of \(X\);
- \(h\) is an odd admissible map which maps bounded set into bounded set;
- \(\varphi(h(u)) \leq \varphi(u)\) for any \(u \in A \cap \varphi_{a-1}\) and \(\varphi(h(u)) \leq a - 1\) for any \(u \in A \cap \varphi^{a-1}\), where \(a\) is given by Lemma 2.1(ii).
Clearly $\mathcal{H}(A) \subset \tilde{\mathcal{H}}(A)$ for any $A \in \tilde{\Sigma}$, so, for $h$ given in Lemma \[3.2\] we have $h \in \tilde{\mathcal{H}}(M)$.

**Lemma 3.3** Under conditions (V), (f\(_1\))-(f\(_5\)), let $\mathcal{F}$ be a finite set and let $M \in \Sigma$ be bounded in $X$, then for any $R > \rho := \sup\{\|u\| : u \in M\}$, there exists a vector field $\chi_2(u) : \varphi^{\zeta+2} \to X$ ($\zeta$ is defined by \[3.2\]) with the following properties:

(i) $\chi_2(u)$ is locally Lipschitz continuous and $\tau$-locally Lipschitz $\tau$-continuous on $\varphi^{\zeta+2}$.

(ii) $\chi_2(u)$ is odd and $\langle \nabla \varphi(u), \chi_2(u) \rangle \leq 0$, for any $u \in \varphi^{\zeta+2}_a$, where $a$ is given by Lemma \[2.1\](ii).

(iii) $\langle \nabla \varphi(u), \chi_2(u) \rangle < -1$, for any

$$u \in \varphi^{\zeta+2}_a \cap \{u \in X : \|u\| \geq \delta_0, \|Qu - [Q\mathcal{F}, \iota_{\zeta+2}]\{0\}\| \geq \frac{\mu}{8}\} \cap \bar{B}_R,$$

where $\bar{B}_R = \{u \in X : \|u\| \leq R\}$, $\delta_0$ is given by \[2.4\], $\iota_{\zeta+2}$ and $\mu$ are given in Lemma \[2.7\].

(iv) There exists $\sigma_2 > 0$ such that $\|\chi_2(u)\| < \sigma_2$ for any $u \in \varphi^{\zeta+2}$.

(v) Each $u \in \varphi^{\zeta+2}$ has a $\tau$-open neighborhood of $u$, $U_u \subset X$, such that $\chi_2(U_u \cap \varphi^{\zeta+2})$ is contained in a finite-dimensional subspace of $X$.

**Proof.** Let

$$N_0 = X^{-} \oplus \bigcup_{z \in [Q\mathcal{F}, \iota_{\zeta+2}]\{0\}} (B_{X^+}(z, \frac{\mu}{16})), $$

and

$$E_2 = \varphi^{\zeta+2} \cap \{u \in X : \|u\| \geq \delta_0\} \cap \{X \backslash N_0\},$$

where $\iota_{\zeta+2}$ and $\mu$ are given by Lemma \[2.7\]. Clearly, there exists $\sigma > 0$ such that

$$\|\nabla \varphi(u)\| > \sigma \text{ for any } u \in E_2. \tag{3.7}$$

Define

$$\omega(u) = \frac{2\nabla \varphi(u)}{\|\nabla \varphi(u)\|^2}, \text{ for } u \in E_2 \cap \bar{B}_R.$$ 

Similar to Lemma \[3.1\] by the weak continuity of $\nabla \varphi$ and \[2.5\], for any $u \in E_2 \cap \bar{B}_R$, there exists a $\tau$-neighborhood of $u$, $V_u \subset X$, such that

$$\langle \nabla \varphi(v), \omega(u) \rangle > 1, \text{ for any } v \in V_u \cap \varphi^{\zeta+2}_a \cap E_2 \cap \bar{B}_R. \tag{3.8}$$

Clearly, (by the convexity of $\bar{B}_R$) $\bar{B}_R$ is $\tau$-closed, so, $X \backslash \bar{B}_R$ is $\tau$-open, then

$$N_2 = \{V_u : u \in E_2 \cap \bar{B}_R\} \cup (X \backslash \bar{B}_R)$$

forms a $\tau$-open covering of $E_2$. Since $N_2$ is metric and paracompact, that is, there exists a locally finite $\tau$-open covering $\mathcal{M}_2 = \{M_i : i \in \Lambda\}$ of $E_2$ finer than $N_2$. Using a similar partition of unity
argument as Lemma 3.1, we can construct a pseudogradient vector field on $\varphi^{\zeta+2}$. To be specific, take $\omega_i = \omega(u_i)$ if $M_i \subset V_{u_i}$ for some $u_i \in E$ and take $\omega_i = 0$ if $M_i \subset X \setminus B_R$. Let

$$\xi_2(u) = \sum_{i \in \Lambda} \lambda_i(u) \omega_i, \ u \in N_2,$$

where $\{\lambda_i : i \in \Lambda\}$ be a $\tau$-Lipschitz continuous partition of unity subordinated to $\{M_i\}$. Since the $\tau$-open covering $\{M_i\}$ is locally finite, each $u \in N_2$ belongs to finite many sets $M_i$. Therefore, for every $u \in N_2$, the sum $\xi_2(u)$ is only a finite sum. It follows that, for any $u \in N_2$, there exists a $\tau$-open neighborhood of $u$, $U_u \subset N_2$, such that $\xi_2(U_u)$ is contained in a finite-dimensional subspace of $X$. By the fact that all norms for a finite-dimensional vector space are equivalent, we know that there exists $L_u > 0$ such that

$$\|\xi_2(v) - \xi_2(w)\| \leq L_u \|v - w\|_\tau, \text{ for any } v, w \in U_u.$$ 

Then it is easy to see that $\xi_2(u)$ is locally Lipschitz continuous and $\tau$-locally Lipschitz $\tau$-continuous. By (3.7) and (3.8), we also have

$$\langle \nabla \varphi(u), \xi_2(u) \rangle > 1 \text{ and } \|\xi_2(u)\| < \frac{2}{\sigma} := \sigma_2, \text{ for any } u \in E_2 \cap B_R \cap \varphi_{a-1},$$

and

$$\langle \nabla \varphi(u), \xi_2(u) \rangle \geq 0, \text{ for any } u \in E_2 \cap \varphi_{a-1}.$$ 

Define

$$\tilde{\xi}_2(u) = \frac{\xi_2(u) - \xi_2(-u)}{2},$$

and take the following two Lipschitz continuous and $\tau$-Lipschitz $\tau$-continuous cut-off functions:

$$\vartheta(u) = \begin{cases} 1, & \text{if } \|u\|_\tau \geq \delta_0, \\ 0, & \text{if } \|u\|_\tau < \frac{2\delta_0}{3}, \end{cases}$$

and

$$\psi(u) = \begin{cases} 1, & \text{if } \|Qu - z\| \geq \frac{\mu}{10}, \text{ for any } z \in [QF, l_{\zeta+2}] \setminus \{0\}, \\ 0, & \text{if } \|Qu - z\| \leq \frac{\mu}{10}, \text{ for any } z \in [QF, l_{\zeta+2}] \setminus \{0\}. \end{cases}$$

Define the vector field $\chi_2 : \varphi^{\zeta+2} \to X$ by

$$\chi_2(u) = \begin{cases} -\vartheta(u)\psi(u)\tilde{\xi}_2(u), & \text{for } u \in N_2, \\ 0, & \text{for } \|Qu\| \leq \frac{2\delta_0}{3} \text{ or } \|Qu - [QF, l_{\zeta+2}] \setminus \{0\}\| \leq \frac{\mu}{10}. \end{cases}$$

Then, it is easy to see that $\chi_2$ is well defined on $\varphi^{\zeta+2}$ and satisfies the properties (i)-(v). □

Before giving our second deformation lemma, we introduce a new class of admissible maps related to $\tilde{H}(A)$. 
Definition 3.3 Let $H \subset X$ be a fixed set, define $\tilde{H}_H$ is the set of all maps $h : H \to X$ satisfying

- $h$ is an admissible odd map and $h$ is homeomorphism in the original topology of $X$;
- if $A \subset H$ is bounded, then $h(A)$ is also bounded;
- $\varphi(h(u)) \leq \varphi(u)$ for any $u \in H \cap \varphi_{a-1}$ and $\varphi(h(u)) \leq a - 1$ for any $u \in H \cap \varphi_{a-1}$, where $a$ is given in Lemma 2.1(ii).

Clearly, for any $h \in \tilde{H}_H$ and $A \in \tilde{\Sigma}$ with $A \subset H$, we have $h|_A \in \tilde{H}(A)$.

For $A \in \Sigma$ we recall the Krasnoselskii genus $\gamma(A)$ of $A$ (see section 7 of [33]):

$$\gamma(A) := \min \{k \in \mathbb{N} : \exists \text{ odd continuous map } \phi : A \to \mathbb{R}^k \setminus \{0\} \}$$

$$\gamma(\emptyset) := 0.$$ \hspace{1cm} (3.9)

Then we have the following second deformation lemma.

Lemma 3.4 (Deformation Lemma 2) Under conditions $(V), (f_1)$-$f_5$, let $F$ be a finite set and let $M \in \tilde{\Sigma}$. If

$$b \leq \tilde{b} := \sup_{u \in M} \varphi(u) < \zeta + 2,$$

and

$$0 < \epsilon < \min \{b - \sup_{u \in M} \varphi(u), \mu, 1\} \text{ with } \sigma_2 \text{ given in Lemma 3.3(iv)},$$

then there exist a symmetric $\tau$-open set $N$ with $\bar{N}$ $\tau$-closed and $\gamma(\bar{N}) = 1$ and $h \in \tilde{H}_{\varphi^{\zeta+2}}$, such that $h(M \setminus N) \subset \varphi^{\beta - \epsilon}$.

Proof. Define

$$N := \bigcup_{z \in [QF, l_{\zeta+2}] \setminus \{0\}} (X^- \oplus B_{X^+}(z, \frac{\mu}{4})) = X^- \oplus \bigcup_{z \in [QF, l_{\zeta+2}] \setminus \{0\}} (B_{X^+}(z, \frac{\mu}{4})), $$

where $l_{\zeta+2}$ and $\mu$ are given in Lemma 2.7. Note $[QF, l_{\zeta+2}]$ is countable and for any $z \in [QF, l_{\zeta+2}]$ $(X^- \oplus B_{X^+}(z, \frac{\mu}{4}))$ is contractible, then

$$\gamma(\bar{N}) = 1.$$ 

Let $\chi_2(u) : \varphi^{\zeta+2} \to X$ be the vector field given in Lemma 3.3, we consider the following Cauchy problem

$$\begin{cases}
\frac{d\eta_2}{dt} = \chi_2(\eta_2) \\
\eta_2(0, u) = u \in \varphi^{\zeta+2}.
\end{cases}$$

By the standard theory of ordinary differential equation in Banach space, we know that the initial problem has a unique solution on $[0, \infty)$. Clearly, $\eta_2(t, u)$ is odd in $u$, furthermore, the similar
argument as the proof of Lemma 6.8 of [38] yields that \( \eta_2 \) is an admissible homotopy. By Lemma 3.3 (ii), we have
\[
\frac{d}{dt} \varphi(\eta_2(t, u)) = \langle \nabla \varphi(\eta_2(t, u)), \chi_2(\eta_2(t, u)) \rangle \leq 0, \quad \text{for } \eta_2(t, u) \in \varphi^{c+2}_{a-1},
\]
where \( a \) is given in Lemma 2.1 (ii).

For any \( \epsilon \) given by (3.10) and let
\[
R = \sigma_2 \epsilon + \rho + 1, \quad \rho := \sup \{ \| u \| : u \in M \},
\]
we claim that \( \{ \eta_2(t, u) : 0 \leq t \leq \epsilon, u \in M \} \subset B_R \). Indeed, since
\[
\eta_2(t, u) = u + \int_0^t \chi_2(\eta_2(s, u))ds,
\]
by Lemma 3.3 (iv) we know that, for any \( t \in [0, \epsilon] \) and \( u \in M \),
\[
\| \eta_2(t, u) \| \leq \| u \| + \int_0^t \| \chi_2(\eta_2(s, u)) \| ds 
\leq \| u \| + \int_0^t \sigma_2 ds \leq \rho + \sigma_2 \epsilon < R. \tag{3.11}
\]
Then by (3.10), we have
\[
\sup_{\| u \| \leq \delta_0} \varphi(u) < b - \epsilon,
\]
that is,
\[
\{ u \in X : \| u \| \leq \delta_0 \} \subset \varphi^{b-\epsilon}.
\]
Moreover, by Lemma 3.3 (iv), for \( t \in [0, \infty) \),
\[
\| \eta_2(t, u) - u \| \leq \int_0^t \| \chi_2(\eta_2(s, u)) \| ds \leq \int_0^t \sigma_2 ds \leq \sigma_2 t.
\]
Let \( t = \epsilon \), this gives
\[
\| \eta_2(\epsilon, u) - u \| \leq \sigma_2 \epsilon < \frac{\mu}{8},
\]
here \( \epsilon < \frac{\mu}{8\sigma_2} \) is required. Then, for any \( u \in M \backslash N \), we have
\[
\eta_2(\epsilon, u) \in \varphi^{c+2} \cap \left( X^- \oplus \bigcup_{z \in [Q, \delta_{c+2}] \backslash \{ 0 \}} (B_{X^+}(z, \frac{\mu}{8})) \right).
\]
Next, we claim that \( \eta_2(\epsilon, M \backslash N) \subset \varphi^{\beta-\epsilon} \). If there exists \( u \in M \backslash N \) such that \( \eta_2(\epsilon, u) \geq \beta - \epsilon \),
then, by Lemma 3.3 (iii) we have
\[ \varphi(\eta_2(\epsilon, u)) = \varphi(\eta_2(0, u)) + \int_0^\epsilon \langle \nabla \varphi(\eta_2(s, u)), \chi_2(\eta_2(s, u)) \rangle ds \]
\[ < \varphi(u) + \int_0^\epsilon -1ds \leq \bar{\beta} - \epsilon, \]

which is a contradiction.

Finally, by Lemma 3.3 (ii), it is easy to see \( \varphi(\eta_2(\epsilon, u)) \leq \varphi(u) \) if \( u \in \varphi^{\epsilon+2}_{a-1} \) and \( \eta_2(\epsilon, \varphi^{a-1}) \subset \varphi^{a-1} \). The boundedness of \( M \) follows directly from (3.11). Further more, if \( M \) is \( \tau \)-compact, then \( M_\downarrow \mathcal{N}, M \cap \mathcal{N} \) and \( \eta_2(\epsilon, M_\downarrow \mathcal{N}) \) are also \( \tau \)-compact since \( \eta_2(\epsilon, \cdot) \) is \( \tau \)-continuous. Let \( h(\cdot) = \eta_2(\epsilon, \cdot) \), the proof is complete. \( \square \)

4 Proof of Theorem 1.1

This section is devoted to prove our Theorem 1.1. For this purpose, we introduce two kinds of pseudo-\( Z_2 \) indexes.

For \( A \in \tilde{\Sigma} \) (defined by (3.3)), we define \( \gamma^*(A) \) as

\[ \gamma^*(A) := \min_{h \in \tilde{\mathcal{H}}(A)} \gamma(h(A) \cap S_r \cap X^+), \quad A \in \tilde{\Sigma}, \]

where \( \gamma \) is the genus (see (3.9)), \( \tilde{\mathcal{H}}(A) \) is defined by Definition 3.2 and \( r \) is obtained in (2.4). The following lemma gives some properties of \( \gamma^* \), its proof can be found in [22], so we omit here.

**Lemma 4.1** Let \( A, B \in \tilde{\Sigma} \).

(i) If \( \gamma^*(A) \neq 0 \), then \( A \neq \emptyset \).

(ii) If \( A \subset B \), then \( \gamma^*(A) \leq \gamma^*(B) \).

(iii) If \( h \in \tilde{\mathcal{H}}(A) \), then \( \gamma^*(h(A)) \geq \gamma^*(A) \).

**Remark 4.1** There are sets of arbitrarily large pseudoindex, \( \gamma^* \), in \( \tilde{\Sigma} \). For any \( k \in \mathbb{Z}^+ \), let \( X_k := (\bigoplus_{j=1}^k \mathbb{R} f_j) \oplus X^- \), where \( \{f_j\}_{j \geq 1} \) is an orthonormal basis of \( X^+ \). Then, by Lemma 2.1(ii), there exists \( R_k > r \) such that

\[ \sup_{u \in X_k, \|u\| = R_k} \varphi(u) < \inf_{\|u\| \leq r} \varphi(u). \]

Take

\[ A := \{ u \in X_k : \|u\| \leq R_k \}. \]

Clearly, \( A \) is bounded and \( \tau \)-compact. Then, noting Lemma 2.1(ii) and arguing exactly as Lemma 4.5 of [22] (see also Proposition 7 of [35]), we know that
\[ \gamma^*(A) \geq k. \]

Furthermore, we need another pseudo-$Z_2$ index defined on $\tilde{\Sigma}_H$ with

\[ \tilde{\Sigma}_H := \{ A \in \tilde{\Sigma} : A \subset H \}, \]

where $H \subset X$ is a fixed set. For any $A \in \tilde{\Sigma}_H$, we define

\[ \gamma^*_H(A) := \min_{h \in \tilde{\mathcal{H}}_H} \gamma(h(A) \cap S_r \cap X^+), \]

where $\tilde{\mathcal{H}}_H$ is defined by Definition 3.3. Here are some basic properties of $\gamma^*_H$:

**Lemma 4.2** Let $A, B \in \tilde{\Sigma}_H$, then

(i) If $\gamma^*_H(A) \neq 0$ then $A \neq \emptyset$.

(ii) $\gamma^*_H(A) \geq \gamma^*(A)$.

(iii) If $A \subset B$ then $\gamma^*_H(A) \leq \gamma^*_H(B)$.

(iv) If $h \in \tilde{\mathcal{H}}_H$ and $h(A) \subset H$ then $\gamma^*_H(h(A)) \geq \gamma^*_H(A)$.

(v) $\gamma^*_H(A \cup B) \leq \gamma^*_H(A) + \gamma(B)$.

**Proof.** (i), (iii) and (iv) follow directly from the properties of genus, here we only give a simple proof of (ii) and (v). Firstly, if $h \in \tilde{\mathcal{H}}_H$, then $h|_A \in \tilde{\mathcal{H}}(A)$, so $\gamma^*_H(A) \geq \gamma^*(A)$, thus (ii) is proved. Then for any $h \in \tilde{\mathcal{H}}_H$, by the subadditivity of genus we have

\[ \gamma^*_H(A \cup B) \leq \gamma(h(A \cup B) \cap S_r \cap X^+) \leq \gamma(h(A) \cap S_r \cap X^+) + \gamma(h|_B(B)). \]

Since $h|_B : B \to h|_B(B)$ is a homeomorphism, we know that $\gamma(h|_B(B)) = \gamma(B)$, and

\[ \gamma^*_H(A \cup B) \leq \gamma(h(A) \cap S_r \cap X^+) + \gamma(B), \]

the proof of (v) is complete. \qed

Before giving the proof of Theorem 1.1, we define a mini-max value through $\gamma^*$:

\[ c_k := \inf_{\gamma^*(A) \geq k} \sup_{u \in A} \varphi(u), \quad A \in \tilde{\Sigma}, \]

then $c_k$ is well-defined for each $k \geq 1$. Moreover,

\[ b \leq c_k \leq c_{k+1}, \quad \text{for any integer } k \geq 1, \]

where $b$ is defined by (2.4). Now we are ready to prove Theorem 1.1.

**Proof of Theorem 1.1** By contradiction, if $\mathcal{F}$ (defined by (2.2)) is finite, i.e., (1.1) has only finite many geometrically distinct solutions, there are two possibilities for $c_k$:
Case (I) There is an integer $k \geq 1$ such that $c_k > \zeta + 1$ ($\zeta$ is given in (3.2)), or,

Case (II) $b \leq c_k \leq \zeta + 1$ for all $k \geq 1$.

In what follows, we show that both cases (I) and (II) are impossible by using the first and second deformation lemmas. If case (I) holds, that is, there exists an integer $k \geq 1$ such that

$$c_k > \zeta + 1,$$

then, there exists $M \in \tilde{\Sigma}$ with $\gamma^*(M) \geq k$ and

$$\zeta + 1 < \sup_{u \in M} \varphi(u) < c_k + \frac{1}{2}.$$  

Hence, by the first deformation Lemma 3.2, there exists $h \in \mathcal{H}(M)$, thus $h \in \tilde{\mathcal{H}}(M)$, such that

$$\sup_{u \in M} \varphi(h(u)) < c_k - \frac{1}{2}.$$  

However, by Lemma 4.1(iii) we have $\gamma^*(h(M)) \geq \gamma^*(M) \geq k$, so,

$$c_k \leq \sup_{u \in h(M)} \varphi(u) < c_k - \frac{1}{2},$$

which is a contradiction, so, case (I) is false.

If case (II) occurs, then

$$b \leq \tilde{c} := \lim_{k \to \infty} c_k \leq \zeta + 1. \quad (4.1)$$

Fix $H = \varphi^{\zeta+2}$ with $\zeta$ given by (3.2) and define

$$\tilde{c}_k := \inf_{\gamma^*_H(A) \geq k} \sup_{u \in A} \varphi(u), \text{ for } A \in \tilde{\Sigma}_H.$$  

By Lemma 4.2 (ii) we have

$$\tilde{c}_k \leq c_k,$$

so, $\tilde{c}_k$ is also bounded from above, then for any integer $k \geq 1$ we have

$$b \leq \tilde{c}_k \leq \tilde{c}_{k+1} \leq \tilde{c} := \lim_{k \to \infty} \tilde{c}_k \leq \tilde{c} \leq \zeta + 1,$$

where $\tilde{c}$ is defined by (4.1).

Let $\epsilon \in (0, 1)$ be given in Lemma 3.4 then there exist $k_1 \in \mathbb{N}$ satisfying

$$\tilde{c} - \frac{\epsilon}{4} < \tilde{c}_{k_1} \leq \tilde{c}, \quad (4.2)$$

and a set $M_1 \in \tilde{\Sigma}_H$ with $\gamma^*_H(M_1) \geq k_1$ such that
\[
\tilde{c} - \frac{\epsilon}{4} < \tilde{c}_{k_1} \leq \sup_{u \in M_1} \varphi(u) < \tilde{c} + \frac{\epsilon}{4} < \zeta + 2.
\]

On the other hand, for any \( M \in \tilde{\Sigma}_H \), let \( \mathcal{N} \) be a \( \tau \)-open set with \( \gamma(\mathcal{N}) = 1 \), by Lemma 4.2(iii)(v), we have

\[
\gamma^*_H(M) = \gamma^*_H((M \setminus \mathcal{N}) \cup (M \cap \mathcal{N})) \leq \gamma^*_H(M \setminus \mathcal{N}) + \gamma(M \cap \mathcal{N})
\]

\[
\leq \gamma^*_H(M \setminus \mathcal{N}) + \gamma(\mathcal{N}) = \gamma^*_H(M \setminus \mathcal{N}) + 1,
\]

i.e.,

\[
\gamma^*_H(M \setminus \mathcal{N}) \geq \gamma^*_H(M) - 1. \tag{4.3}
\]

Now, we are going to deduce a contradiction whenever \( \gamma^*_H(M_1) \) is infinite or finite.

If \( \gamma^*_H(M_1) = +\infty \), then \( \gamma^*_H(M_1 \setminus \mathcal{N}) = +\infty \). By Definition 3.3 with \( H = \varphi^{\zeta+2} \) and the second deformation Lemma 3.4 there exists \( h \in \mathcal{H}_H \) such that

\[
\sup_{u \in M_1 \setminus \mathcal{N}} \varphi(h(u)) < \tilde{c} + \frac{\epsilon}{4} - \epsilon = \tilde{c} - \frac{3\epsilon}{4}.
\]

Lemma 4.2(iv) implies that \( \gamma^*_H(h(M_1 \setminus \mathcal{N})) \geq \gamma^*_H(M_1 \setminus \mathcal{N}) = +\infty \), hence

\[
\tilde{c}_{k_1} \leq \sup_{u \in h(M_1 \setminus \mathcal{N})} \varphi(u) < \tilde{c} - \frac{3\epsilon}{4},
\]

which contradicts with (4.2).

If \( \gamma^*_H(M_1) = \Gamma < +\infty \) with \( \Gamma \geq k_1 \), we have

\[
\tilde{c} - \frac{\epsilon}{4} < \tilde{c}_{k_1} \leq \tilde{c}_{\Gamma+1} \leq \tilde{c},
\]

then there exists an \( M_2 \in \tilde{\Sigma}_H \) with \( \gamma^*_H(M_2) \geq \Gamma + 1 \) such that

\[
\tilde{c} - \frac{\epsilon}{4} < \tilde{c}_{\Gamma+1} \leq \sup_{u \in M_2} \varphi(u) < \tilde{c} + \frac{\epsilon}{4}.
\]

By Lemma 3.4 there exists \( h \in \mathcal{H}_H \) such that

\[
\sup_{u \in M_2 \setminus \mathcal{N}} \varphi(h(u)) < \tilde{c} + \frac{\epsilon}{4} - \epsilon = \tilde{c} - \frac{3\epsilon}{4}.
\]

On the other hand, by (4.3) we have

\[
\gamma^*_H(h(M_2 \setminus \mathcal{N})) \geq \gamma^*_H(M_2 \setminus \mathcal{N}) \geq \gamma^*_H(M_2) - 1 \geq \Gamma \geq k_1.
\]

So,
\[
\bar{c}_k \leq \sup_{u \in h(M_2 \setminus \Lambda)} \varphi(u) < \bar{c} - \frac{3\kappa}{4},
\]
which also contradicts with \([1,2]\). The proof is complete. □

References

[1] N. Ackermann, On a periodic Schrödinger equation with nonlocal superlinear part, Math. Z. 248 (2004), 423-443.

[2] S. Alama, Y. Y. Li, Existence of solutions for semilinear elliptic equations with indefinite linear part, J. Differential Equations 96(1992), 89-115.

[3] S. Alama, Y. Y. Li, On ”Multibump” bound states for certain semilinear elliptic equations, Indiana Univ. Math. J. 41(1992), 983-1026.

[4] T. Bartsch, Y.H. Ding, Deformation theorems on non-metrizable vector spaces and applications to critical point theory, Math. Nachr. 279(2006), 1267-1288.

[5] T. Bartsch, Y.H. Ding, On a nonlinear Schrödinger equation with periodic potential, Math. Ann. 313 (1999), 15-37.

[6] C. J. Batkam, F. Colin, Generalized fountain theorem and applications to strongly indefinite semilinear problems, J. Math. Anal. Appl. 405(2013), 438-452.

[7] F. Bernini, B. Bieganowski, Generalized linking-type theorem with applications to strongly indefinite problems with sign-changing nonlinearities, Calc. Var. Partial Differential Equations 61(2022), 182.

[8] B. Bieganowski, Schrödinger-type equations with sign-changing nonlinearities: a survey, preprint. arXiv:1810.01754.

[9] B. Bieganowski, Solutions of the fractional Schrödinger equation with a sign-changing nonlinearity, J. Math. Anal. Appl. 450(2017), 461-479.

[10] B. Bieganowski, J. Mederski, Nonlinear Schrödinger equations with sum of periodic and vanishing potentials and sign-changing nonlinearities, Commun. Pure Appl. Anal. 17(2018), 143-161.

[11] J. Chabrowski, A. Szulkin, On a semilinear Schrödinger equation with critical Sobolev exponent, Proc. Amer. Math. Soc. 130 (2002), 85-93.

[12] S. Chen, X. Tang, On the planar Schrödinger equation with indefinite linear part and critical growth nonlinearity, Calc. Var. Partial Differential Equations 60(2021), 95.
[13] S. Chen, C. Wang, An infinite-dimensional linking theorem without upper semi-continuous assumption and its applications, J. Math. Anal. Appl. 420(2014), 1552-1567.

[14] S. Chen, L. Xiao, Existence of a nontrivial solution for a strongly indefinite periodic Choquard system, Calc. Var. Partial Differential Equations 54(2015), 599-614.

[15] D. G. Costa, H. Tehrani, Existence and multiplicity results for a class of Schrödinger equations with indefinite nonlinearities, Adv. Differential Equations 8(2003), 1319-1340.

[16] V. Coti Zelati, P. H. Rabinowitz, Homoclinic orbits for second order Hamiltonian systems possessing superquadratic potentials, J. Amer. Math. Soc. 4(1991), 693-727.

[17] V. Coti Zelati, P. H. Rabinowitz, Homoclinic type solutions for a semilinear elliptic PDE on $\mathbb{R}^n$, Comm. Pure Appl. Math. 45(1992), 1217-1269.

[18] Y. H. Ding, Variational Methods for Strongly Indefinite Problems, World Scientific, 2007.

[19] Y. H. Ding, C. Lee, Multiple solutions of Schrödinger equations with indefinite linear part and super or asymptotically linear terms, J. Differential Equations 222(2006), 137-163.

[20] L. J. Gu, H. S. Zhou, An improved fountain theorem and its application, Adv. Nonlinear Stud. 17(2017), 727-738.

[21] Q. Han, F. Lin, Elliptic Partial Differential Equations, Amer. Math. Soc., New York, 2011.

[22] W. Kryszewski, A. Szulkin, Generalized linking theorem with an application to semilinear Schrödinger equation, Adv. Differential Equations 3(1998), 441-472.

[23] P. Kuchment, The mathematics of photonic crystals, Mathem. Modeling in Opt. Sci., 207-272, SIAM, 2001.

[24] G. Li, A. Szulkin, An asymptotically periodic Schrödinger equation with indefinite linear part, Commun. Contemp. Math. 4 (2002), 763-776.

[25] F. Liu, J. Yang, Nontrivial solutions of Schrödinger equations with indefinite nonlinearities, J. Math. Anal. Appl. 334 (2007), 627-645.

[26] S. Liu, Z. Shen, Generalized saddle point theorem and asymptotically linear problems with periodic potential, Nonlinear Anal: Theory, Methods & Applications, 86(2013), 52-57.

[27] J. Mederski, Ground states of a system of nonlinear Schrödinger equations with periodic potentials, Comm. Partial Differential Equations 41(2016), 1426-1440.

[28] J. Mederski, Solutions to a nonlinear Schrödinger equations with periodic potentials and zero on the boundary of the spectrum, Topol. Methods Nonlinear Anal. 46(2015), 755-771.
[29] D. L. Mills, Nonlinear Optics: Basic Concepts, Springer, 2012.

[30] W. Nie, Optical nonlinearity: phenomena, applications, and materials, Adv. Mater. 5(1993), 520-545.

[31] A. Pankov, Periodic nonlinear Schrödinger equation with application to photonic crystals, Milan J. Math. 73(2005), 259-287.

[32] D. Qin, V. Rădulescu, X. Tang, Ground states and geometrically distinct solutions for periodic Choquard-Pekar equations, J. Differential Equations 275(2021), 652-683.

[33] P. H. Rabinowitz, Minimax Methods in Critical Point Theory with Applications to Differential Equations, No. 65, Amer. Math. Soc., 1986.

[34] M. Reed, B. Simon, Methods of Modern Mathematical Physics, Vol. IV, Academic Press, New York, 1975.

[35] H. J. Ruppen, A generalized min-max theorem for functionals of strongly indefinite sign, Calc. Var. Partial Differential Equations 50(2014), 231-255.

[36] M. Schechter, W. M. Zou, Weak linking theorems and Schrödinger equations with critical Sobolev exponent, ESAIM Control Optim. Calc. Var. 9(2003), 601-619.

[37] A. Szulkin, T. Weth, Ground state solutions for some indefinite variational problems, J. Funct. Anal. 257(2009), 3802-3822.

[38] M. Willem, Minimax Theorems, Birkhauser, Boston, 1996.