ON THE THIRD LAW OF BLACK HOLE DYNAMICS

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Abstract

The third law of black hole dynamics states that the surface gravity (temperature) of black hole cannot be reduced to zero in finite sequence of physical interactions. We argue that the same is true when surface gravity is replaced by gravitational charge. We demonstrate that the prescribed window for infalling energy and radiation pinches off as extremality ($M^2 = a^2 + Q^2$) is approached.

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Parallel to the laws of thermodynamics, the laws of black hole dynamics (BHD) were formulated comprehensively by Bardeen, Carter and Hawking [1]. The identification of temperature with surface gravity of the hole was clinched by the Hawking radiation [2] which followed from application of quantum theory in general relativity. Subsequently Israel has given the precise formulation and proof of the third law [3]. Although the law has been proved in an elegant way using sophisticated global analysis, a clear and straightforward demonstration of its working has its own merit and usefulness [4]. We shall argue that the law can also be stated by replacing surface gravity by gravitational charge enclosed by the horizon. Gravitational charge of a hole can be defined by the flux of red-shifted proper acceleration across the closed 2-surface defined by the horizon [5,6] and value of the acceleration at the horizon defines the surface gravity. Thus the two quantities are intimately related. By considering variation in gravitational charge and applying the area non-decrease theorem, we would like to exhibit how it cannot be reduced to zero in finite sequence of physical processes. It turns out that as extremality \((M^2 = a^2 + Q^2)\) is approached, the window for allowed range of parameters of infalling energy and radiation pinches off.

The third law of thermodynamics has two essentially equivalent statements; (a) isothermal reversible processes turn isentropic as temperature of a system approaches zero, and (b) temperature cannot be reduced to zero in finite number of physical operations [7]. There is yet another stronger version due to Planck which states that the entropy of any system tends to an absolute constant, which can be taken as zero, as temperature tends to zero.

In the case of a black hole, surface gravity tends to zero either as \(M^2 \rightarrow a^2 + Q^2\) for charged and rotating black hole or \(M \rightarrow \infty\), where \(M, a = J/M,\) and \(Q\) refer respectively to mass, specific angular momentum and electric charge of the hole. In either case area of the hole, which is analogous to entropy, does not go to zero. Hence Planck’s version has no analogue in classical BHD. However it has been well recognised that thermodynamic description will not be tenable for extremal black hole. In particular it has recently been argued on topological and quantum considerations [8,9] that extremal \((M^2 = a^2 + Q^2)\) black hole is qualitatively different from non-extremal black hole. The identification of area with entropy is true only for the latter and is not true for the former. For the extremal case entropy can be deduced separately and it vanishes, thus according well with Planck’s version. The conclusion is that area of the event horizon does not always measure entropy of the hole. It does so only for non-extremal black holes while for extremal case area is though finite and non-zero but entropy turns out to be zero. This is a new proposal based primarily on non-classical considerations.
Let us look at the familiar mass formula for a black hole,
\[ M = \frac{\kappa}{4\pi} A + 2wJ + Q\phi \]  
(1)

where all symbols have the usual meaning. In thermodynamical sense the first term on the right should measure the internal energy (enthalpy) while the remaining two terms indicate the work done on the hole. The internal energy, \( M_I \), can be identified with the effective gravitational charge of the hole [5], which is defined by [6],
\[ M_g = \frac{1}{4\pi} \int g \cdot ds = M - \frac{a^2}{r_+} - \frac{Q^2}{r_+} = (M^2 - a^2 - Q^2)^{1/2} \]  
(2)

where the integration is taken over the closed 2-surface defined by the horizon, and \( g = -\alpha \nabla (\ln \alpha) \), \( \alpha \) is the norm of the timelike corotating vector \( \frac{\partial}{\partial t} + w \frac{\partial}{\partial \varphi} \). This follows from application of the Gauss theorem to red-shifted proper acceleration \( g \), its norm represents surface gravity when evaluated at the horizon. This is in fact the Komar integral [10] for the corotating timelike vector \( \frac{\partial}{\partial t} + w \frac{\partial}{\partial \varphi} \), over the horizon. It is therefore the Komar mass of the hole evaluated at the horizon [6]. Unfortunately this vector is not Killing in general and hence the integral does not yield an invariant mass. However it does give a good measure for \( r \rightarrow r_+ \) and \( r \rightarrow \infty \), because in these limits the vector does tend to be Killing. It will give
\[ M_g = M_I = \frac{\kappa}{4\pi} A = \frac{2M^2}{M} - M \]  
(3)

where
\[ M^2_{ir} = \frac{1}{2} M r_+ = \frac{A}{16\pi}, \quad r_+ = M + (M^2 - a^2 - Q^2)^{1/2}. \]  
(4)

\( M_{ir} \) is called the irreducible mass of the hole [11]. Note that \( M_I = (\kappa/4\pi)A \) is not a defining relation for \( M_g \), which is the measure of flux of \( g \) across the closed 2-surface defined by the horizon. However \( g \) also defines the surface gravity and we have \( M_g = M_I \).

\( M_g \) tends to zero as extremality is reached. That is gravitational charge contained inside the horizon vanishes for an extremal black hole. If entropy of the hole is to depend upon gravitational charge contained in the hole, it should also vanish for an extremal hole as argued in [8,9]. There is however no relation connecting entropy with gravitational charge. All we can say is that area of the horizon is not a measure of entropy of extremal hole.
Here an analogy can be drawn between extremal and non-extremal black holes, and photons and ordinary particles, indicating their characteristic difference. The analogy suggests that gravitational charge $M_g$ corresponds to rest mass, we have $M_g^2 = M^2 - a^2 - Q^2$ analogus $m^2 = E^2 - p^2$. For photon rest mass energy vanishes and its entire energy is kinetic while for extremal hole its gravitational charge vanishes and its entire energy is rotational and/or electromagnetic. The third law simply states that a non-extremal black hole cannot be converted into an extremal one (an ordinary particle cannot be accelerated into a photon). Like photon it has to be born like that. For photon the converse is also true, i.e. a photon cannot be converted into an ordinary particle. This does not appear to be the case for black hole, for nothing prohibits classically to add mass to an extremal hole to make it non-extremal. It has however been argued on the basis of radiation properties of black holes that extremal hole can be in equilibrium with thermal radiation at any temperature and hence it can radiate at any rate independent of temperature. It can thus be thought that extremal black hole always radiates in such a way when matter and radiation fall into it so as to keep itself extremal [8]. This seems to make the analogy with photon perfect.

Another equivalent statement of the third law could be that gravitational charge of the hole cannot be reduced to zero in finite sequence of physical processes. Further as argued by Hawking, et.al [8], that if it happens to be zero, then no finite sequence of physical operations can make it non-zero. The latter statement would however have to be formulated as a separate law. That means black holes are characterised into two distinct classes by vanishing and non-vanishing of their gravitational charge and the two classes are qualitatively different and non-interchangeable exactly in the similar sense as photons are from ordinary particles.

From eqn. (3) let us consider variation in gravitational charge,

$$
\delta M_g = - \left( 1 + \frac{2M^2}{M_g^2} \right) \delta M + 4 \frac{M_g}{M} \delta M_g.
$$

(5)

This will tend to zero as extremality ($M^2 = a^2 + Q^2$) is approached, both terms tend to the same limit and cancel each other. The process tends to be isenthalpic. All particles that tend to decrease $M_g$ are scattered off by the hole as $M^2 \rightarrow a^2 + Q^2$.

For simplicity let us now specialise to a rotating hole, while all our results will hold good even when $Q \neq 0$. If we consider $\delta M_g \leq 0$ and $\delta A \geq 0$, the former implies an upper limit while the latter a lower limit on $\delta M/\delta J$ falling into the
hole, and then we obtain the following window for permissible range,

\[ \frac{a}{2Mr_+} \leq \frac{dM}{dJ} \leq \frac{a}{M^2 + a^2} < \infty. \]  

(6)

Now both lower and upper limits tend to 1/2M as \( M^2 \rightarrow a^2 \), thus completely pinching off the window. This clearly demonstrates that \( M_g = M_I \) can never be reduced to zero and all interactions turn isenthalpic and isentropic (reversible) as extremality is approached.

We shall next consider variation in surface gravity. The analogue of (6) will now be

\[ \frac{a}{2Mr_+} \leq \frac{dM}{dJ} \leq \frac{aM}{-M^3 + 3Ma^2 - (M^2 - a^2)^{3/2}} < \infty \]  

(7)

where we have used

\[ A = 8\pi Mr_+ , \]  

(8)

\[ \delta A = 8\pi \left[ \frac{2Mr_+\delta M - a\delta J}{(M^2 - a^2)^{1/2}} \right] \]  

(9)

and

\[ \kappa = \frac{(M^2 - a^2)^{1/2}}{2Mr_+} = \frac{M_g}{2Mr_+} = \frac{M_g}{4M^2_{ir}} \]  

(10)

\[ \delta \kappa = \frac{\delta M[-M^2 + 3a^2 - \frac{1}{M}(M^2 - a^2)^{3/2}] - a\delta J}{2Mr_+^2 \sqrt{M^2 - a^2}} . \]  

(11)

Here again either side in (7) will tend to the same limit 1/2M, yielding the same conclusion that surface gravity cannot be reduced to zero and all interactions turn isothermal and isentropic (reversible) as extremality is approached.

Thus all interactions with black hole that point towards extremality turn isenthalpic, isentropic, isothermal and reversible as extremality is approached. No finite sequence of physical interactions can reduce the surface gravity and gravitational charge of black hole to zero. A non-extremal hole cannot evolve into an extremal one. The converse of this statement has also been discussed and justified by Hawking, et.al [8] but it cannot be established from these purely classical considerations. The argument crucially rests on radiation properties of the hole which are governed by quantum considerations.
As demonstrative simple examples, let us consider one, evolution of surface gravity of a hole along the isentropic (constant area - analogous to adiabatic process in thermodynamics) path as shown in Fig.1, and second, along the constant angular momentum path as shown in Fig.2. In the former case there is a monotonic evolution and which is reversible, while Fig 2 depicts the evolution from extremality to decreasing $a/M$. In the latter case, $\kappa$ will increase initially as matter is added into a (nearly) extremal hole and attain maximum value at $a^2/M^2 = \sqrt{3}(2 - \sqrt{3}) \approx 0.46 \ (a/M \approx 0.68)$, and will then start decreasing as mass begins to dominate over rotation. Here the process is irreversible and hence it will not trace the same curve in the reverse direction.

There have been several attempts to define quasi-local energy of black hole spacetimes (for example, [12,13]). It is supposed to be a measure of energy contained inside some compact surface. It may be noted that gravitational charge as defined in (2) does not agree with the quasi-local energy. This point has been specifically discussed in [6]. In our discussion, it is gravitational charge which acts as a source for gravitational attraction - the surface gravity. In this respect it measures a physically meaningful property of the hole. We have demonstrated explicitly that it cannot be reduced to zero through any finite sequence of operations.

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Fig. 2