DYNAMICS OF A DENGUE FEVER TRANSMISSION MODEL WITH CROWDING EFFECT IN HUMAN POPULATION AND SPATIAL VARIATION

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Abstract. Dengue fever is a virus-caused disease in the world. Since the high infection rate of dengue fever and high death rate of its severe form dengue hemorrhagic fever, the control of the spread of the disease is an important issue in the public health. In an effort to understand the dynamics of the spread of the disease, Esteva and Vargas [2] proposed a SIR v.s. SI epidemiological model without crowding effect and spatial heterogeneity. They found a threshold parameter $R_0$, if $R_0 < 1$, then the disease will die out; if $R_0 > 1$, then the disease will always exist.

To investigate how the spatial heterogeneity and crowding effect influence the dynamics of the spread of the disease, we modify the autonomous system provided in [2] to obtain a reaction-diffusion system. We first define the basic reproduction number in an abstract way and then employ the comparison theorem and the theory of uniform persistence to study the global dynamics of the modified system. Basically, we show that the basic reproduction number is a threshold parameter that predicts whether the disease will die out or persist. Further, we demonstrate the basic reproduction number in an explicit way and construct suitable Lyapunov functionals to determine the global stability for the special case where coefficients are all constant.

1. Introduction. Dengue fever is an arbovirus disease in the tropical regions of the world, and temporal or sporadic in the subtropical and temperate regions. The symptoms of dengue fever include fever, headache, muscle and joint pains. More seriously, it will occur blood plasma leakage or the dengue shock syndrome and potential to death. Dengue disease is transmitted to humans by the bite of Aedes mosquitoes. Four serotypes (I ∼ IV) have been identified. Infection by any single type of virus usually gives lifelong immunity to that type, but only short-term immunity to the other serotypes ([25]). The mosquitoes never recover from the
infection and their infective period ends with their death ([3]). Since the high infection rate of dengue fever and high death rate of its severe form dengue hemorrhagic fever, the control of the spread of the disease is always an important issue in the public health. It is known that the relatively rate of vertical transmission in the main vector of dengue (A. aegypti.) is low ([10, 18]). In an effort to understand the dynamics of the spread of the disease, Esteva and Vargas [2] proposed a SIR v.s. SI epidemiological model. Basically, they studied the mechanisms that allows the invasion and persistence of a serotype of dengue in a region. Their mathematical model for the dynamics of dengue disease contains only one type of virus and ignore the disease-related death rate.

In the following, we shall briefly review the model proposed in [2]. Let \( S_H, I_H, \) and \( R_H \) denote the number of the susceptible, infectious and immune class in the human population; \( S_V, I_V \) denote the number of the susceptible, infectious class in the mosquito population. Thus, \( N_H := S_H + I_H + R_H \) and \( N_V := S_V + I_V \) represent the population sizes of human and mosquitoes, respectively. The constants \( \mu_H, \mu_V, \) and \( \gamma_H \) represent the birth, death and recover rate of human species; \( A \) and \( \mu_V \) denote the recruitment and the per capita mortality rate of mosquitoes, respectively. For each species, flow from the susceptible class into the infectious class depends on the biting rate of the mosquitoes, the transmission probabilities together with disease-related death rate. More precisely, \( \beta_H \) is the transmission probability from infectious mosquitoes to susceptible humans; \( \beta_V \) is the transmission probability from infectious humans to susceptible mosquitoes. Thus a human receives \( b \frac{N_V}{N_H + m} \) bites per unit of time, and a mosquito takes \( b \frac{N_H}{N_H + m} \) human blood meals per unit of time. The force of infection for human population is given by

\[
\beta_H b \frac{N_V}{N_H} \frac{N_H}{N_H + m} I_V S_H = \frac{\beta_H b}{N_H + m} S_H I_V,
\]

while the force of infection for vector population is given by

\[
\beta_V b \frac{N_H}{N_H + m} I_H S_V = \frac{\beta_V b}{N_H + m} S_V I_H,
\]

where \( \beta_H \) is the transmission probability from infectious mosquitoes to susceptible humans; \( \beta_V \) is the transmission probability from infectious humans to susceptible mosquitoes. Then we get the following system which is closely related to the one in [2]:

\[
\begin{align*}
S_H'(t) &= \mu_H N_H - \frac{\beta_H b}{N_H + m} S_H I_V - \mu_d S_H, \\
I_H'(t) &= \frac{\beta_H b}{N_H + m} S_H I_V - (\mu_d + \gamma_H) I_H, \\
R_H'(t) &= \gamma_H I_H - \mu_d R_H, \\
S_V'(t) &= A - \frac{\beta_V b}{N_H + m} S_V I_H - \mu_V S_V, \\
I_V'(t) &= \frac{\beta_V b}{N_H + m} S_V I_H - \mu_V I_V, \\
S_H(0), I_H(0), R_H(0), S_V(0), I_V(0) &\geq 0.
\end{align*}
\]

We note that system (1) coincides with the one in [2] if we assume \( \mu_b = \mu_d \). Esteva and Vargas [2] employed the results of the theory of competitive systems to determine the global dynamics of (1) under the assumption \( \mu_b = \mu_d \). More precisely, they found a threshold parameter \( R_0 \), if \( R_0 < 1 \), then the disease-free equilibrium
is globally stable, or equivalently, the disease will die out; if \( R_0 > 1 \), then the only endemic equilibrium is globally stable, which means that the disease will always exist.

In this paper, we shall modify the standard model (1) to incorporate the crowding effect and species movements in spatially heterogeneous environments. Let \( \Omega \) be a spatial habitat with smooth boundary \( \partial \Omega \). We consider a closed environment in the sense that the fluxes for each of these subpopulations are zero. Corresponding to this, we shall propose the Neumann boundary conditions to the equations on the boundary. Finally, the crowding effect terms (see, e.g., [8]) in the susceptible class, the infectious class and the immune class in the human population are respectively described by

\[
c(x)S_H N_H, \ c(x)I_H N_H \text{ and } c(x)R_H N_H.
\]

With all these assumptions, the disease dynamics can be described by the following system of differential equations:

\[
\begin{align}
\frac{\partial S_H}{\partial t} &= d_H \Delta S_H + \mu_0 N_H - c(x)S_H N_H - \frac{\beta_H(x)b(x)}{N_H+m(x)} S_H I_V - \mu_d S_H, \\
\frac{\partial I_H}{\partial t} &= d_H \Delta I_H + \frac{\beta_H(x)b(x)}{N_H+m(x)} S_H I_V - c(x)I_H N_H - (\mu_d + \gamma_H) I_H, \\
\frac{\partial R_H}{\partial t} &= d_H \Delta R_H + \gamma_H I_H - c(x)R_H N_H - \mu_d R_H, \quad x \in \Omega, \quad t > 0, \\
\frac{\partial S_V}{\partial t} &= d_V \Delta S_V + A(x) - \frac{\beta_V(x)b(x)}{N_H+m(x)} S_V I_D - \mu_V S_V, \\
\frac{\partial I_V}{\partial t} &= d_V \Delta I_V + \frac{\beta_V(x)b(x)}{N_H+m(x)} S_V I_D - \mu_V I_V, \\
\frac{\partial R_V}{\partial t} &= \frac{\partial R_H}{\partial t} = \frac{\partial S_V}{\partial t} = \frac{\partial I_V}{\partial t} = 0, \quad x \in \partial \Omega, \quad t > 0.
\end{align}
\]

Here, the spatial dependent functions \( A(x), b(x), c(x), m(x), \beta_H(x), \beta_V(x) \) are assumed to be positive; \( \Delta \) is the usual Laplacian operator; \( d_H > 0, \ d_V > 0 \) denote the diffusion coefficients for human and mosquitoes, respectively. Notice that the system (2) reduces to (1) if all the coefficients functions are constants, and \( c = d_H = d_V = 0 \).

The organization of this paper is as follows. In section 2, we first study the model (2) in a spatially variable habitat. By the theory of monotone dynamical systems and uniformly persistent, we determine a threshold number that predicts the disease persistence or extinction. In section 3, we consider the model (2) where all the coefficients are habitat independent (i.e. positive constants). We are able to construct an appropriate Lyapunov functional to discuss the global attractiveness of the steady-state solutions. Finally, a brief discussion is given in section 4.

2. The heterogeneous model. This section is devoted to the study of the dynamics of the system (2). Before demonstrating the limiting system for (2), we first consider the following scalar reaction-diffusion equation

\[
\begin{align}
\frac{\partial w}{\partial t} &= d \Delta w + g(x) - D(x)w, \quad x \in \Omega, \quad t > 0, \\
\frac{\partial w}{\partial \nu} &= 0, \quad x \in \partial \Omega, \quad t > 0,
\end{align}
\]

where \( d > 0; \ D(x) \) and \( g(x) \) are continuous and positive functions on \( \overline{\Omega} \). Then we have the following results.

Lemma 2.1. [13, Lemma 1] The system (3) admits a unique positive steady state \( w^*(x) \) which is globally asymptotically stable in \( C(\Omega, \mathbb{R}) \). Moreover, if \( g(x) \equiv g, \ D(x) \equiv D, \forall x \in \Omega, \) then \( w^*(x) = \frac{g}{D} \).
Since $\overline{N}_H = \overline{S}_H + \overline{T}_H + \overline{R}_H$ and $\overline{N}_V = \overline{S}_V + \overline{T}_V$, it follows from (2) that $\overline{N}_H$ and $\overline{N}_V$ satisfy the following equations respectively:

$$\begin{align*}
\frac{\partial \overline{N}_H}{\partial t} &= d_H \Delta \overline{N}_H + (\mu_b - \mu_d) \overline{N}_H - c(x) \overline{N}_H^2, \ x \in \Omega, \ t > 0, \\
\frac{\partial \overline{N}_H}{\partial \nu} &= 0, \ x \in \partial \Omega, \ t > 0.
\end{align*}$$

and

$$\begin{align*}
\frac{\partial \overline{N}_V}{\partial t} &= d_V \Delta \overline{N}_V + A(x) - \mu_V \overline{N}_V(t), \ x \in \Omega, \ t > 0, \\
\frac{\partial \overline{N}_V}{\partial \nu} &= 0, \ x \in \partial \Omega, \ t > 0.
\end{align*}$$

System (4) is a logistic equation and it is well-known that the reaction-diffusion equation (4) admits a unique positive steady state $K(x)$ such that (see, e.g., [16, page 506] and [28, Theorem 3.1.5 and the proof of Theorem 3.1.6]):

$$\lim_{t \to \infty} \overline{N}_H(x,t) = K(x) \text{ uniformly in } x \in \bar{\Omega},$$

for all solutions with nonnegative and nonzero initial data provided that $\mu_b > \mu_d$.

From (5) and Lemma 2.1, it follows that there exists a unique continuous function $\sigma(x)$ which is positive on $\bar{\Omega}$ such that

$$\lim_{t \to \infty} \overline{N}_V(x,t) = \sigma(x) \text{ uniformly in } x \in \bar{\Omega}.$$  

We assume that $\mu_b > \mu_d$ and $(u_1, u_2, u_3) := (\overline{S}_H, \overline{T}_H, \overline{T}_V)$, then one concludes that the limiting system for (2) takes the form:

$$\begin{align*}
\frac{\partial u_1}{\partial t} &= d_H \Delta u_1 + \mu_b K(x) - \frac{\beta(x)b(x)}{K(x) + m(x)} u_1 u_3 - D_1(x) u_1, \\
\frac{\partial u_2}{\partial t} &= d_H \Delta u_2 + \frac{\beta(x)b(x)}{K(x) + m(x)} u_1 u_3 - D_2(x) u_2, \ x \in \Omega, \ t > 0, \\
\frac{\partial u_3}{\partial t} &= d_V \Delta u_3 + \frac{\beta(x)b(x)}{K(x) + m(x)} (\sigma(x) - u_3) u_2 - \mu_V u_3, \ x \in \Omega, \ t > 0,
\end{align*}$$

where

$$D_1(x) := c(x)K(x) + \mu_d \text{ and } D_2(x) := c(x)K(x) + \mu_d + \gamma_H.$$  

Let $X := C(\Omega, \mathbb{R}^3)$ be the Banach space with the supremum norm $\| \cdot \|_X$. Define $X^+ := C(\Omega, \mathbb{R}^3_+)$, then $(X, X^+)$ is a strongly ordered space. Let $X^+_q := \{ \phi = (\phi_1, \phi_2, \phi_3) \in X^+ : 0 \leq \phi_3(x) \leq \sigma(x), \ \forall \ x \in \Omega \}$.

Suppose that $T_1(t), T_2(t), T_3(t) : C(\bar{\Omega}, \mathbb{R}) \to C(\bar{\Omega}, \mathbb{R})$ are the $C_0$ semigroups associated with $d_H \Delta - D_1(\cdot), d_H \Delta - D_2(\cdot)$ and $d_V \Delta - \mu_V$ subject to the Neumann boundary condition, respectively. It then follows that for any $\phi \in C(\bar{\Omega}, \mathbb{R}), t \geq 0$,

$$\begin{align*}
(T_1(t)\phi)(x) &= \int_{\Omega} \Gamma_1(t, x, y)\phi(y)dy, \\
(T_2(t)\phi)(x) &= \int_{\Omega} \Gamma_2(t, x, y)\phi(y)dy, \\
(T_3(t)\phi)(x) &= e^{-\mu_V t} \int_{\Omega} \Gamma_3(t, x, y)\phi(y)dy,
\end{align*}$$

where $\Gamma_1$, $\Gamma_2$ and $\Gamma_3$ are the Green functions associated with $d_H \Delta - D_1(\cdot), d_H \Delta - D_2(\cdot)$ and $d_V \Delta$ subject to the Neumann boundary conditions, respectively.

From [20, Section 7.1 and Corollary 7.2.3], it follows that $T_i(t) : C(\bar{\Omega}, \mathbb{R}) \to C(\bar{\Omega}, \mathbb{R})$ is compact and strongly positive, $\forall \ t \geq 0$ and $i = 1, 2, 3$. Furthermore, $T(t) := (T_1(t), T_2(t), T_3(t)) : X \to X$, $t \geq 0$, is a $C_0$ semigroup (see, e.g., [17]).
Let \( A_i : D(A_i) \to C(\bar{\Omega}, \mathbb{R}) \) be the generator of \( T_i(t), \ i = 1, 2, 3 \). Then \( T(t) : X \to X \) is a \( C_t \) semigroup generated by the operator \( A := (A_1, A_2, A_3) \) defined on \( D(A) := D(A_1) \times D(A_2) \times D(A_3) \). Define \( F = (F_1, F_2, F_3) : X^+ \to X \) by

\[
F_1(\phi)(x) = \mu_h K(x) - \frac{\beta H(x) b(x)}{K(x) + m(x)} \phi_1 \phi_3,
\]

\[
F_2(\phi)(x) = \frac{\beta H(x) b(x)}{K(x) + m(x)} \phi_1 \phi_3,
\]

\[
F_3(\phi)(x) = \frac{\beta V(x) b(x)}{K(x) + m(x)} (\sigma(x) - \phi_3) \phi_2,
\]

\( \forall \ x \in \Omega \) and \( \phi = (\phi_1, \phi_2, \phi_3) \in X^+_\sigma \). Then (8) can be rewritten as the following abstract differential equation

\[
\begin{align*}
\frac{du}{dt} &= Au + F(u), \ t > 0, \\
u(\cdot, 0) &= \phi \in X^+_\sigma,
\end{align*}
\]

or it can be rewritten as the following integral equation

\[
u(t) = T(t) \phi + \int_0^t T(t-s) F(u(\cdot, s)) ds.
\]

**Lemma 2.2.** *For every initial value function \( \phi \in X^+_\sigma \), system (8) has a unique mild solution \( u(x, t, \phi) \) on \( (0, \tau_0) \) with \( u(\cdot, 0, \phi) = \phi \), where \( \tau_0 \leq \infty \). Furthermore, \( u(\cdot, t, \phi) \in X^+_\sigma \), \( \forall t \in (0, \tau_0) \) and \( u(x, t, \phi) \) is a classical solution of (8), \( \forall t > 0 \).*

**Proof.** By [14, Corollary 4] or [20, Theorem 7.3.1], it suffices to show that

\[
\lim_{h \to 0^+} \text{dist}(\phi + h F(\phi), X^+_\sigma) = 0, \ \forall \phi \in X^+_\sigma.
\]  

Let \( \tilde{K}_1 := \max_{x \in \Omega} \frac{\beta H(x) b(x)}{K(x) + m(x)} \) and \( \tilde{K}_2 := \max_{x \in \Omega} \frac{\beta V(x) b(x)}{K(x) + m(x)} \). For any \( \phi \in X^+_\sigma \) and \( h \geq 0 \), we have

\[
\phi + h F(\phi) = \begin{bmatrix}
\phi_1 + h \mu_h K(\cdot) - \frac{\beta H(x) b(x)}{K(x) + m(x)} \phi_1 \phi_3 \\
\phi_2 + h \frac{\beta H(x) b(x)}{K(x) + m(x)} \phi_1 \phi_3 \\
\phi_3 + h \frac{\beta V(x) b(x)}{K(x) + m(x)} (\sigma(\cdot) - \phi_3) \phi_2
\end{bmatrix}
\]

\[
\geq \begin{bmatrix}
\phi_1 [1 - h \tilde{K}_1 \phi_3] \\
\phi_2 \\
\phi_3 [1 - h \tilde{K}_2 \phi_2]
\end{bmatrix},
\]

and

\[
\sigma - \left\{ \phi_3 + h \frac{\beta V b}{K(x) + m(x)} (\sigma - \phi_3) \phi_2 \right\}
\]

\[
\geq \sigma - \left\{ \phi_3 + h \tilde{K}_2 (\sigma - \phi_3) \phi_2 \right\}
\]

\[
= \left( \sigma - \phi_3 \right) \left[ 1 - h \tilde{K}_2 \phi_2 \right].
\]

The above inequalities imply that (13) holds and thus the lemma is proved.

We are in a position to show that solutions of system (8) exist globally on \( [0, \infty) \) and converge to a compact attractor in \( X^+_\sigma \).
Lemma 2.3. For every initial value functions \( \phi \in \mathbb{X}_\sigma^+ \), system (8) has a unique solution \( u(x,t,\phi) \) on \([0, \infty)\) with \( u(\cdot,0,\phi) = \phi \) and the semiflow \( \Phi(t) : \mathbb{X}_\sigma^+ \to \mathbb{X}_\sigma^+ \) generated by (8) is defined by

\[
\Phi(t)\phi = u(\cdot, t, \phi), \quad t \geq 0.
\]

Furthermore, the semiflow \( \Phi(t) : \mathbb{X}_\sigma^+ \to \mathbb{X}_\sigma^+ \) has a global compact attractor in \( \mathbb{X}_\sigma^+ \), \( \forall \ t \geq 0 \).

Proof. By Lemma 2.2, it follows that \( u_3(\cdot,t,\phi) \leq \sigma, \forall \ t \in [0, \tau_\phi) \). Let \( U = u_1 + u_2 \). Then it follows from the first two equations of (8) that \( U \) is dominated by the following system

\[
\begin{align*}
\frac{\partial U}{\partial t} &= d_H \Delta U_1 + \mu_b \bar{K} - \bar{D} U, \quad x \in \Omega, \ t > 0, \\
\frac{\partial \bar{U}}{\partial \nu} &= 0, \quad x \in \partial \Omega, \ t > 0,
\end{align*}
\]

(14)

where \( \bar{K} := \max_{x \in \Omega} K(x) \) and \( \bar{D} := \min_{x \in \Omega} \{D_1(x), D_2(x)\} \). By Lemma 2.1, it follows that \( \bar{K} \) is globally attractive in \( C(\bar{\Omega}, \mathbb{R}) \) for (14). By the standard parabolic comparison theorem (see, e.g., [20, Theorem 7.3.4]), it follows that \( U(\cdot,t,\phi) \) is bounded on \([0, \tau_\phi) \). Then it follows from the positiveness of \( u_1 \) and \( u_2 \) that \( u_1(\cdot,t,\phi) \) is bounded on \([0, \tau_\phi) \), \( \forall \ i = 1, 2 \). It then follows that

\[
u(\cdot,t,\phi) = (u_1(\cdot,t,\phi), u_2(\cdot,t,\phi), u_3(\cdot,t,\phi))
\]

is bounded on \([0, \tau_\phi) \), and hence \( \tau_\phi = \infty \) for each \( \phi \in \mathbb{X}_\sigma^+ \). Therefore, system (8) defines a semiflow \( \Phi : \mathbb{X}_\sigma^+ \to \mathbb{X}_\sigma^+ \) by

\[
(\Phi(t)\phi)(x) = u(x,t,\phi), \quad t \geq 0, \quad x \in \bar{\Omega}.
\]

In fact, for any \( \phi \in \mathbb{X}_\sigma^+ \), we have some \( t_1 := t_1(\phi) \) such that \( U(\cdot,t,\phi) \leq 2 \frac{\mu_b \bar{K}}{D_1}, \forall \ t > t_1 \). It then follows from the positiveness of \( u_1 \) and \( u_2 \) that \( u_i(\cdot,t,\phi) \leq 2 \frac{\mu_b \bar{K}}{D_1}, \forall \ t > t_1, \ i = 1, 2 \). Then the semiflow \( \Phi(t) : \mathbb{X}_\sigma^+ \to \mathbb{X}_\sigma^+ \) is point dissipative. Obviously, \( \Phi(t) : \mathbb{X}_\sigma^+ \to \mathbb{X}_\sigma^+ \) is compact, \( \forall \ t > 0 \). By [4, Theorem 3.4.8], it follows that \( \Phi(t) : \mathbb{X}_\sigma^+ \to \mathbb{X}_\sigma^+ \), \( t \geq 0 \), has a global compact attractor. \( \square \)

The following results will play an important role in establishing the persistence of (8).

Lemma 2.4. Suppose \( u(x,t,\phi) \) is the solution of system (8) with \( u(\cdot,0,\phi) = \phi \in \mathbb{X}_\sigma^+ \).

(i) If there exists some \( t_0 \geq 0 \) such that \( u_i(\cdot,t_0,\phi) \neq 0 \), for some \( i \in \{2, 3\} \), then

\[
u(\cdot,t,\phi) > 0, \quad \forall \ x \in \bar{\Omega}, \ t > t_0;
\]

(ii) For any \( \phi \in \mathbb{X}_\sigma^+ \), we always have

\[
u(\cdot,t,\phi) > 0, \quad \forall \ x \in \bar{\Omega}, \ t > 0 \quad \text{and}
\]

\[
\lim_{t \to \infty} \inf u_1(\cdot,t,\phi) \geq \frac{\mu_b \bar{K}}{\sigma \bar{K}_1 + D_1} \quad \text{uniformly for} \ x \in \bar{\Omega},
\]

where \( \bar{K} := \min_{x \in \Omega} K(x) \), \( \sigma := \max_{x \in \Omega} \sigma(x) \), \( \bar{K}_1 := \max_{x \in \Omega} \frac{\beta_H \bar{b}(x)}{K(x) + m(x)} \) and

\[
D_1 := \max_{x \in \Omega} D_1(x).
\]

Proof. By Lemma 2.2, it follows that \( 0 \leq u_3(x,t,\phi) \leq \sigma(x), \forall \ x \in \bar{\Omega}, \ t \geq 0 \). Thus, it is easy to see that \( u_2 \) and \( u_3 \) satisfy the following inequalities, respectively:

\[
\begin{align*}
\frac{\partial u_2}{\partial t} \geq d_H \Delta u_2 - D_2(x) u_2, \quad x \in \Omega, \ t > 0, \\
\frac{\partial u_3}{\partial \nu} = 0, \quad x \in \partial \Omega, \ t > 0,
\end{align*}
\]
and
\[
\begin{cases}
\frac{\partial u_k}{\partial t} \geq d_H \Delta u_k + \mu_h \hat{K} - [\sigma \hat{K}_1 + \hat{D}_1]u_1, \quad x \in \Omega, \quad t > 0, \\
\frac{\partial u_k}{\partial t} = 0, \quad x \in \partial \Omega, \quad t > 0.
\end{cases}
\]

By similar arguments to those in [7, Lemma 2.1] and [26, Proposition 3.1], it follows from the strong maximum principle (see, e. g., [19, p. 172, Theorem 4]) and the Hopf boundary lemma (see, e. g., [19, p. 170, Theorem 3]) that part (i) is valid.

From the first equation of (8), it is obvious that \( u_1 \) satisfies
\[
\begin{cases}
\frac{\partial u_k}{\partial t} \geq d_H \Delta u_k + \mu_h \hat{K} - [\sigma \hat{K}_1 + \hat{D}_1]u_1, \quad x \in \Omega, \quad t > 0, \\
\frac{\partial u_k}{\partial t} = 0, \quad x \in \partial \Omega, \quad t > 0.
\end{cases}
\]

Let \( v_1(x, t, \phi) \) be the solution of
\[
\begin{cases}
\frac{\partial v_1}{\partial t} = d_H \Delta v_1 + \mu_h \hat{K} - [\sigma \hat{K}_1 + \hat{D}_1]v_1, \quad x \in \Omega, \quad t > 0, \\
\frac{\partial v_1}{\partial t} = 0, \quad x \in \partial \Omega, \quad t > 0, \\
v_1(x, 0, \phi) = \phi_1.
\end{cases}
\]

By the standard parabolic comparison theorem (see, e. g., [20, Theorem 7.3.4]), it follows that \( u_1(x, t, \phi) \geq v_1(x, t, \phi) > 0 \), \( \forall \ t > 0, \quad x \in \Omega \). Furthermore, Lemma 2.1 implies that
\[
\lim_{t \to \infty} u_1(x, t, \phi) \geq \frac{\mu_h \hat{K}}{\sigma \hat{K}_1 + \hat{D}_1},
\]
uniformly for \( x \in \bar{\Omega} \). Thus the proof of Part (ii) is complete.

In order to find the disease-free equilibrium (infection-free steady state), we let the densities of the diseased compartments (\( u_2 \) and \( u_3 \)) be zero, we get the following equation for the density of susceptible human,
\[
\begin{cases}
\frac{\partial u_k}{\partial t} = d_H \Delta u_k + \mu_h \hat{K}(x) - D_1(x)w, \quad x \in \Omega, \quad t > 0, \\
\frac{\partial u_k}{\partial t} = 0, \quad x \in \partial \Omega, \quad t > 0.
\end{cases}
\]

By Lemma 2.1, it is easy to see that system (17) has a positive steady state \( u^*_1(x) \), which is globally asymptotically stable in \( C(\bar{\Omega}, \mathbb{R}) \). Linearizing system (8) at the disease-free equilibrium \( (u^*_1(x), 0, 0) \), we get the following cooperative system for the infectious human and vector population, respectively:
\[
\begin{align*}
\frac{\partial w_2}{\partial t} &= d_H \Delta w_2 - D_2(x)w_2 + \frac{\beta_H(x)h(x)}{K(x)+m(x)}u^*_1(x)w_3, \quad x \in \Omega, \quad t > 0, \\
\frac{\partial w_3}{\partial t} &= d_V \Delta w_3 - \mu_V w_3 + \sigma(x) \frac{\psi(x)h(x)}{K(x)+m(x)}w_2, \quad x \in \Omega, \quad t > 0, \\
\frac{\partial w_2}{\partial \nu} &= \frac{\partial w_3}{\partial \nu} = 0, \quad x \in \partial \Omega, \quad t > 0.
\end{align*}
\]

We first consider the following generalized version of system (18):
\[
\begin{align*}
\frac{\partial w_2}{\partial t} &= d_H \Delta w_2 - D_2(x)w_2 + \frac{\beta_H(x)h(x)}{K(x)+m(x)}h(x)w_3, \quad x \in \Omega, \quad t > 0, \\
\frac{\partial w_3}{\partial t} &= d_V \Delta w_3 - \mu_V w_3 + (\sigma(x) - \rho) \frac{\psi(x)h(x)}{K(x)+m(x)}w_2, \quad x \in \Omega, \quad t > 0, \\
\frac{\partial w_2}{\partial \nu} &= \frac{\partial w_3}{\partial \nu} = 0, \quad x \in \partial \Omega, \quad t > 0,
\end{align*}
\]
where \( h(x) > 0 \) and \( 0 \leq \rho < \sigma(x) \), \( \forall \ x \in \bar{\Omega} \). Note that if one choose \( h = u^*_1 \) and \( \rho = 0 \) in (19) then we get system (18).

Substituting \( \psi_1(x, t) = e^{\lambda \psi_1(x)} \), \( i = 2, 3 \), into (19) and we get the following eigenvalue problem associated with (19):
\[
\begin{align*}
\lambda \psi_2 &= d_H \Delta \psi_2 - D_2(x)\psi_2 + \frac{\beta_H(x)h(x)}{K(x)+m(x)}h(x)\psi_3, \quad x \in \Omega, \\
\lambda \psi_3 &= d_V \Delta \psi_3 - \mu_V \psi_3 + (\sigma(x) - \rho) \frac{\psi(x)h(x)}{K(x)+m(x)}\psi_2, \quad x \in \Omega, \\
\frac{\partial \psi_2}{\partial \nu} &= \frac{\partial \psi_3}{\partial \nu} = 0, \quad x \in \partial \Omega,
\end{align*}
\]

Equation for the density of susceptible human,
By a similar argument as in [20, Theorem 7.6.1], we have the following results:

**Lemma 2.5.** For $h(x) > 0$, $\forall x \in \bar{\Omega}$ and $0 < \rho < \sigma$, the eigenvalue problem (20) has a principal eigenvalue, denoted by $\lambda^*(h, \rho)$ which is associated with a strongly positive eigenfunction.

The basic reproductive number, which is defined as the average number of secondary infections generated by a single infected individual introduced into a completely susceptible population, is one of the important quantities in epidemiology. For models described by ordinary differential equations (finite dimensions), [1, 24] provide a standard procedure for defining and computing the basic reproductive number by using the next generation matrix.

In the following, we shall adopt the same ideas as in [13, 27] to define the basic reproduction ratio for the reaction-diffusion system (8). Let

$$S(t)\varphi := (T_2(t)\varphi_2, T_3(t)\varphi_3), \; \forall \varphi := (\varphi_2, \varphi_3) \in C(\bar{\Omega}, \mathbb{R}^2),$$

where $T_2(t)$ and $T_3(t)$ are defined in (10) and (11) respectively. It then follows that $S(t)$ is a positive $C_0$-semigroup on $C(\bar{\Omega}, \mathbb{R}^2)$. We further define a positive linear operator $C$ on $C(\bar{\Omega}, \mathbb{R}^2)$ by

$$C(\varphi)(x) := (C_2(\varphi)(x), C_3(\varphi)(x)), \; \forall \varphi \in C(\bar{\Omega}, \mathbb{R}^2), \; x \in \bar{\Omega},$$

where

$$C_2(\varphi)(x) := \frac{\beta_H(x)b(x)u_1^*(x)}{K(x) + m(x)} \varphi_3(x)$$

and $C_3(\varphi)(x) := \frac{\sigma(x)\beta_V(x)b(x)}{K(x) + m(x)} \varphi_2(x)$.

In order to define the basic reproduction ratio for system (8), we assume that both human and vector individuals are near the disease-free equilibrium $(u_1^*(x), 0, 0)$, and introduce infectious human and vector individuals at time $t = 0$, where the distribution of initial infectious human and vector individuals is described by $\varphi := (\varphi_2, \varphi_3) \in C(\bar{\Omega}, \mathbb{R}^2)$. Thus, it is easy to see that $S(t)\varphi$ represents the distribution of infective human and vector individuals at time $t \geq 0$.

Consequently, at time $t \geq 0$, the distribution of new infective human is

$$\frac{\beta_H(x)b(x)u_1^*(x)}{K(x) + m(x)}T_3(t)\varphi_3.$$ 

Thus, the distribution of total new infective human is:

$$\int_0^\infty \frac{\beta_H(x)b(x)u_1^*(x)}{K(x) + m(x)}T_3(t)\varphi_3 dt = \frac{\beta_H(x)b(x)u_1^*(x)}{K(x) + m(x)} \int_0^\infty T_3(t)\varphi_3 dt = C_2 \left( \int_0^\infty S(t)\varphi dt \right).$$

Similarly, the distribution of total new infective vector is:

$$\int_0^\infty \frac{\sigma(x)\beta_V(x)b(x)}{K(x) + m(x)}T_2(t)\varphi_2 dt = \frac{\sigma(x)\beta_V(x)b(x)}{K(x) + m(x)} \int_0^\infty T_2(t)\varphi_2 dt = C_3 \left( \int_0^\infty S(t)\varphi dt \right).$$

It then follows that

$$L(\varphi) := C \left( \int_0^\infty S(t)\varphi dt \right)$$

represents the distribution of the total infective population generated by initial infectious human and vector individuals $\varphi := (\varphi_2, \varphi_3)$, and hence, $L$ is the next infection operator. We define the spectral radius of $L$ as the basic reproduction ratio for system (8), that is,

$$R_0 := r(L).$$

By the general results in [23] and the same arguments as in [27, Lemma 2.2], we have the following observation.

**Lemma 2.6.** $R_0 − 1$ and $\lambda^*(u_1^*, 0)$ have the same sign.
Now we are ready to prove the main result of this section, which indicates that \( R_0 \) is a threshold index for disease persistence.

**Theorem 2.7.** Suppose \( u(x, t, \phi) \) is the solution of system (8) with \( u(\cdot, 0, \phi) = \phi \in X_+^\dagger \). Then the following statements hold.

(i) If \( R_0 < 1 \), then the disease free equilibrium \((u_1^*, 0, 0)\) is globally attractive in \( X_+^\dagger \);

(ii) If \( R_0 > 1 \), then system (8) admits at least one positive steady state \( \bar{u}(x) \) and there exists an \( \eta > 0 \) such that for any \( \phi \in X_+^\dagger \) with \( \phi_i \neq 0 \) for \( i = 2, 3 \), we have

\[
\liminf_{t \to \infty} u_i(x, t) \geq \eta, \quad \forall \ i = 1, 2, 3,
\]

uniformly for all \( x \in \Omega \).

**Proof.** We first assume that \( R_0 < 1 \). It then follows from Lemma 2.6 that \( \lambda^*(u_1^*, 0) < 0 \). By the continuity, there is a \( \rho_0 > 0 \) such that \( \lambda^*(u_1^* + \rho_0, 0) < 0 \). From the first equation of (8), it follows that

\[
\begin{cases}
\frac{\partial u_1}{\partial t} \leq d_H \Delta u_1 + \mu_u K(x) - D_1(x)u_1, \quad x \in \Omega, \ t > 0, \\
\frac{\partial u_1}{\partial t} = 0, \quad x \in \partial \Omega, \ t > 0.
\end{cases}
\tag{24}
\]

From (17), (24) and the comparison principle, it follows that there is a \( t_0 := t_0(\phi) \) such that

\[
u_1(x, t, \phi) \leq u_1^*(x) + \rho_0, \quad \forall \ t \geq t_0, \quad x \in \Omega.
\]

Thus,

\[
\begin{cases}
\frac{\partial u_2}{\partial t} \leq d_H \Delta u_2 + \frac{\beta_H(x,b(x))}{K(x) + \rho(x)} (u_1^*(x) + \rho_0)u_3 - D_2(x)u_2, \ x \in \Omega, \ t \geq t_0, \\
\frac{\partial u_2}{\partial t} = 0, \quad x \in \partial \Omega, \ t \geq t_0,
\end{cases}
\tag{25}
\]

By Lemma 2.5, there is a strongly positive eigenfunction \( \hat{\psi} := (\hat{\psi}_2, \hat{\psi}_3) \) corresponding to \( \lambda^*(u_1^* + \rho_0, 0) \). For any given \( \phi \in X_+^\dagger \), there exists some \( \alpha > 0 \) such that \( (u_2(x, t_0, \phi), u_3(x, t_0, \phi)) \leq \alpha \hat{\psi}(x), \ \forall \ x \in \Omega \). Note that the following linear system

\[
\begin{cases}
\frac{\partial w_2}{\partial t} = d_H \Delta w_2 + \frac{dH(x)b(x)}{K(x)+\rho(x)} (u_1^*(x) + \rho_0)w_3 - D_2(x)w_2, \ x \in \Omega, \ t \geq t_0, \\
\frac{\partial w_2}{\partial t} = 0, \quad x \in \partial \Omega, \ t \geq t_0,
\end{cases}
\tag{26}
\]

admits a solution \( \alpha e^{\lambda^*(u_1^* + \rho_0, 0)(t-t_0)} \hat{\psi}(x), \ \forall \ t \geq t_0 \). The comparison principle implies that

\[
u_2(x, t, \phi), u_3(x, t, \phi)) \leq \alpha e^{\lambda^*(u_1^* + \rho_0, 0)(t-t_0)} \hat{\psi}(x), \ \forall \ t \geq t_0,
\]

and it then follows that \( \lim_{t \to \infty} (u_2(x, t, \phi), u_3(x, t, \phi)) = 0 \) uniformly for \( x \in \Omega \). Then, the equation for \( u_1 \) is asymptotic to the reaction-diffusion equation (17) and then we get \( \lim_{t \to \infty} u_1(x, t, \phi) = u_1^*(x) \) uniformly for \( x \in \Omega \) by the theory for asymptotically autonomous semiflows (see, e.g., [22, Corollary 4.3]). Thus Part (i) is proved.

We are in a position to deal with the case where \( R_0 > 1 \). Obviously, Lemma 2.6 implies \( \lambda^*(u_1^*, 0) > 0 \). Let

\[
\mathcal{W}_0 = \{ \phi \in X_+^\dagger : \phi_2(\cdot) \neq 0 \text{ and } \phi_3(\cdot) \neq 0 \},
\]

and \[ \partial \mathbb{W}_0 = \mathbb{X}^+_0 \setminus \mathbb{W}_0 = \{ \phi \in \mathbb{X}^+_0 : \phi_2(\cdot) \equiv 0 \text{ or } \phi_3(\cdot) \equiv 0 \}. \]

By Lemma 2.4, it follows that for any \( \phi \in \mathbb{W}_0 \), we have \( u_i(x, t, \phi) > 0 \), \( \forall x \in \Omega \), \( t > 0 \), \( i = 2, 3 \). In other words, \( \Phi(t)\mathbb{W}_0 \subseteq \mathbb{W}_0 \), \( \forall t \geq 0 \). Let \[ M_\partial := \{ \phi \in \partial \mathbb{W}_0 : \Phi(t)\phi \in \partial \mathbb{W}_0, \forall t \geq 0 \}, \]

and \( \omega(\phi) \) be the omega limit set of the orbit \( O^+(\phi) := \{ \Phi(t)\phi : t \geq 0 \} \).

Claim: \( \omega(\psi) = \{(u_1^*, 0, 0)\}, \forall \psi \in M_\partial \).

Since \( \psi \in M_\partial \), we have \( \Phi(t)\psi \in M_\partial \), \( \forall t \geq 0 \). Thus \( u_2(\cdot, t, \psi) \equiv 0 \) or \( u_3(\cdot, t, \psi) \equiv 0 \), \( \forall t \geq 0 \). In case where \( u_3(\cdot, t, \psi) \equiv 0 \), \( \forall t \geq 0 \). Then \( u_1 \) satisfies the reaction-diffusion equation (17), \( \forall t \geq 0 \) and hence we get \( \lim_{t \to \infty} u_1(x, t, \psi) = u_1^*(x) \) uniformly for \( x \in \Omega \). On the other hand, it is easy to see that \( \lim_{t \to \infty} u_2(x, t, \psi) = 0 \) uniformly for \( x \in \Omega \) from the equation of \( u_2 \) in (8). In case where \( u_3(\cdot, t_0, \psi) \not\equiv 0 \), for some \( t_0 \geq 0 \). Then Lemma 2.4 implies that \( u_3(x, t, \psi) > 0 \), \( \forall x \in \Omega \), \( \forall t > t_0 \). Hence, \( u_2(\cdot, t, \psi) \equiv 0 \), \( \forall t > t_0 \). In view of the \( u_3 \) equation in (8), it is easy to see that \( \lim_{t \to \infty} u_3(x, t, \psi) = 0 \) uniformly for \( x \in \Omega \). Again, the equation for \( u_1 \) is asymptotic to the reaction-diffusion equation (17) and the theory for asymptotically autonomous semiflows (see, e.g., [22, Corollary 4.3]) implies that \( \lim_{t \to \infty} u_1(x, t, \psi) = u_1^*(x) \) uniformly for \( x \in \Omega \). Hence, \( \omega(\psi) = \{(u_1^*, 0, 0)\}, \forall \psi \in M_\partial \).

Since \( \lambda^*(u_1^*, 0) > 0 \), there exists a sufficiently small positive number \( \delta_0 \) such that \( \lambda^*(u_1^* - \delta_0, \delta_0) > 0 \).

Claim: \( (u_1^*, 0, 0) \) is a uniform weak repeller for \( \mathbb{W}_0 \) in the sense that

\[ \limsup_{t \to \infty} \| \Phi(t)\phi - (u_1^*, 0, 0) \| \geq \delta_0, \forall \phi \in \mathbb{W}_0. \]

Suppose, by contradiction, there exists \( \phi_0 \in \mathbb{W}_0 \) such that

\[ \limsup_{t \to \infty} \| \Phi(t)\phi_0 - (u_1^*, 0, 0) \| < \delta_0. \]

Then, there exists \( t_1 > 0 \) such that \( u_1(x, t, \phi_0) > u_1^*(x) - \delta_0 \) and \( u_3(x, t, \phi_0) < \delta_0 \), \( \forall t \geq t_1 \), \( x \in \Omega \). Thus \( u(x, t, \phi_0) \) satisfies

\[ \begin{cases} \frac{\partial u_2}{\partial t} \geq d_1 u_2 + \frac{\beta(x)(x) (x)}{\gamma(x) + m(x)} (u_1(x) - \delta_0) u_3 - D_2(x) u_2, & x \in \Omega, \ t \geq t_1, \\ \frac{\partial u_3}{\partial t} \geq d_2 u_3 + \frac{\beta(x)(x) (x)}{\gamma(x) + m(x)} (\sigma(x) - \delta_0) u_2 - \mu \nu u_3, & x \in \Omega, \ t \geq t_1, \\ \frac{\partial \phi}{\partial t} = \frac{\partial \psi}{\partial t}, & 0, \ x \in \partial \Omega, \ t \geq t_1. \end{cases} \]  

By Lemma 2.5, we can let \( \tilde{\psi} := (\tilde{\psi}_2, \tilde{\psi}_3) \) be the strongly positive eigenfunction corresponding to \( \lambda^*(u_1^* - \delta_0, \delta_0) \). Since \( u_i(x, t, \phi_0) > 0 \), \( \forall x \in \Omega, \ t > 0 \), \( i = 2, 3 \), there exists \( \epsilon_0 > 0 \) such that \( (u_2(x, t_1, \phi_0), u_3(x, t_1, \phi_0)) \geq \epsilon_0 \tilde{\psi} \). Note that \( \epsilon_0 e^{\lambda^*(u_1^* - \delta_0, \delta_0)(t-t_1)} \tilde{\psi} \), \( t \geq t_1 \), is a solution of the following linear system:

\[ \begin{cases} \frac{\partial u_2}{\partial t} = d_1 u_2 + \frac{\beta(x)(x) (x)}{\gamma(x) + m(x)} (u_1(x) - \delta_0) u_3 - D_2(x) u_2, & x \in \Omega, \ t \geq t_1, \\ \frac{\partial u_3}{\partial t} = d_2 u_3 + \frac{\beta(x)(x) (x)}{\gamma(x) + m(x)} (\sigma(x) - \delta_0) u_2 - \mu \nu u_3, & x \in \Omega, \ t \geq t_1, \\ \frac{\partial \phi}{\partial t} = \frac{\partial \psi}{\partial t} = 0, & x \in \partial \Omega, \ t \geq t_1. \end{cases} \]

The comparison principle implies that

\[ (u_2(x, t, \phi_0), u_3(x, t, \phi_0)) \geq \epsilon_0 e^{\lambda^*(u_1^* - \delta_0, \delta_0)(t-t_1)} \tilde{\psi}, \forall t \geq t_1, \ x \in \Omega. \]

Since \( \lambda^*(u_1^* - \delta_0, \delta_0) > 0 \), it follows that \( u(x, t, \phi_0) \) is unbounded. This contradiction proves the claim. Define a continuous function \( p : \mathbb{X}^+_\nu \to [0, \infty) \) by

\[ p(\phi) := \min_{x \in \Omega} \{ \min \{ \phi_2(x), \phi_3(x) \} : x \in \Omega \}, \forall \phi \in \mathbb{X}^+_\nu. \]
By Lemma 2.4, it follows that $p^{-1}(0, \infty) \subseteq \mathbb{W}_0$ and $p$ has the property that if $p(\phi) > 0$ or $\phi \in \mathbb{W}_0$ with $p(\phi) = 0$, then $p(\Phi(t)\phi) > 0, \forall \ t > 0$. That is, $p$ is a generalized distance function for the semiflow $\Phi(t) : \mathbb{X}^+_t \to \mathbb{X}^+_t$ (see, e.g., [21]). From the above claims, it follows that any forward orbit of $\Phi(t)$ in $M_0$ converges to $(u^*_1, 0, 0)$ which is isolated in $\mathbb{X}^+_t$ and $W^s(u^*_1, 0, 0) \cap \mathbb{W}_0 = \emptyset$, where $W^s(u^*_1, 0, 0)$ is the stable set of $(u^*_1, 0, 0)$ (see [21]). It is obvious that there is no cycle in $M_0$ from $(u^*_1, 0, 0)$ to $(u^*_1, 0, 0)$. By [21, Theorem 3], it follows that there exists an $\tilde{\eta} > 0$ such that

$$\min_{\psi \in \omega(\phi)} p(\psi) > \tilde{\eta}, \ \forall \ \phi \in \mathbb{W}_0.$$ 

Hence, $\liminf_{t \to \infty} u_i(\cdot, t, \phi) \geq \tilde{\eta}, \ \forall \ \phi \in \mathbb{W}_0, \ i = 2, 3$. From Lemma 2.4, there exists an $\eta$ satisfying $0 < \eta \leq \tilde{\eta}$ such that

$$\liminf_{t \to \infty} u_i(\cdot, t, \phi) \geq \eta, \ \forall \ \phi \in \mathbb{W}_0, \ i = 1, 2, 3.$$ 

Hence, the uniform persistence stated in the conclusion (ii) are valid. By [15, Theorem 3.7 and Remark 3.10], it follows that $\Phi(t) : \mathbb{W}_0 \to \mathbb{W}_0$ has a global attractor $A_0$. It then follows from [15, Theorem 4.7] that $\Phi(t)$ has an equilibrium $\tilde{u}(\cdot) \in \mathbb{W}_0$. Further, Lemma 2.4 implies that $\tilde{u}(\cdot)$ is a positive steady state of (8). The proof is complete. \hfill \Box

**Remark 1.** By similar arguments to those in [26, Section 3.2] or [9, Section 5], we can further lift the dynamics of the limiting system (8) to the full system (2) by the concept of “chain transitive sets” (see, e.g., [28]).

3. **The homogeneous model.** In this section, we consider the reaction-diffusion system (2) in the case where all the coefficients are positive constants and one can obtain the following limiting system by using the same arguments in the previous section:

$$\begin{cases}
\frac{\partial u_1}{\partial t} = d_{11}\Delta u_1 + \mu_b K - \frac{\beta_{b1} u_1 u_3}{K + m} - \mu_b u_1, \\
\frac{\partial u_2}{\partial t} = d_{22}\Delta u_2 + \frac{\beta_{b1} u_1 u_3}{K + m} - (\mu_b + \gamma_H) u_2, \ x \in \Omega, \ t > 0, \\
\frac{\partial u_3}{\partial t} = d_{33}\Delta u_3 + \frac{\beta_{b2} u_1 u_3}{K + m} - (\mu_a + \mu) u_3, \\
\frac{\partial u_4}{\partial t} = \frac{\partial u_5}{\partial t} = \frac{\partial u_6}{\partial t} = 0, \ x \in \partial\Omega, \ t > 0,
\end{cases}$$

(29)

where $K = \frac{\mu_b - \mu_a}{c}$ (when $\mu_b > \mu_a$) and $\sigma = \frac{A}{\mu}$. By Lemma 2.1, it is easy to see that $(K, 0, 0)$ is the disease-free steady-state solution of the system (29). From (9), it follows that

$$D_1 = \mu_b \quad \text{and} \quad D_2 = \mu_b + \gamma_H.$$ 

By similar arguments to those in [27, Theorem 2.1], we can show that the basic reproduction ratio $R_0$ equals the spectral radius of the following $2 \times 2$ matrix:

$$J = \begin{bmatrix}
\beta_{b1} b & 0 \\
\beta_{b2} b & \frac{\mu}{\mu_a} \cdot \frac{1}{\mu_b + \gamma_H} \cdot \frac{1}{m} \cdot \frac{K}{\mu} \cdot \frac{1}{\mu} \\
\frac{\beta_{b1} b}{K + m} \cdot \frac{A}{\mu} & \frac{\mu}{\mu_a} \cdot \frac{1}{\mu_b + \gamma_H} \cdot \frac{1}{m} \cdot \frac{K}{\mu} \cdot \frac{1}{\mu}
\end{bmatrix},$$

and hence, we have the following formula for $R_0$:

**Lemma 3.1.** For the system (29), the basic reproduction ratio is given by

$$R_0 = \sqrt{\frac{\beta_{b1} b}{K + m} \cdot \frac{A}{\mu b} \cdot \frac{1}{\mu_b + \gamma_H} \cdot \frac{1}{m} \cdot \frac{K}{\mu} \cdot \frac{1}{\mu}}.$$ 

(30)
We nondimensionalize the system (29) with the following relations:
\[ s = \frac{u_1}{K}, \quad u = \frac{u_2}{K}, \quad v = \frac{\mu v}{A}u_3. \]

Then the system (29) becomes
\[
\begin{aligned}
s_t &= d_1 \Delta s + \mu (1 - s) - \alpha sv \equiv d_1 \Delta s + f_1(s, u, v), \quad x \in \Omega, \ t > 0, \\
u_t &= d_1 \Delta u + \alpha sv - \gamma u \equiv d_1 \Delta u + f_2(s, u, v), \quad x \in \Omega, \ t > 0, \\
\frac{\partial s}{\partial \nu} = \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, \ x \in \partial \Omega, \ t > 0, \\
s(x, 0) &= s^0(x), \ u(x, 0) = u^0(x), \ v(x, 0) = v^0(x) \geq 0, \neq 0, \ x \in \Omega.
\end{aligned}
\]

where \( d_1 = \frac{d \mu}{K}, \ d_2 = \frac{\mu v}{A}dV \) and
\[
\begin{aligned}
\mu &= \mu_b, \quad \gamma = \mu_b + \gamma_H, \quad \theta = \mu v, \quad \alpha = \frac{A}{\mu v} \cdot \frac{b \beta_H}{K + m}, \quad \delta = \frac{bK \beta v}{K + m}.
\end{aligned}
\]

Here, we assume that the admissible initial data \( s^0(x), u^0(x) \) and \( v^0(x) \) are in the set
\[
\mathcal{Y} := \{(s^0(x), u^0(x), v^0(x)) \in C(\Omega, \mathbb{R}^3_+) : 0 \leq v^0(x) \leq 1, \ \forall \ x \in \bar{\Omega}\}.
\]

By similar arguments to those in the previous section and the standard theory for parabolic equations, the unique solution \((s(x, t), u(x, t), v(x, t))\) of (31) exists and is positive on \( \mathcal{Y} \).

It is easy to see that \( E_0 := (1, 0, 0) \) is the disease-free steady state solution of the system (31). Further, \( E_* := (s^*, u^*, v^*) = \left( \frac{\delta \mu + \gamma \theta}{\delta (\mu + \gamma)}, \frac{\mu (\alpha \delta - \gamma \theta)}{\gamma \delta (\mu + \gamma)}, \frac{\mu (\alpha \delta - \gamma \theta)}{\alpha (\delta \mu + \gamma \theta)} \right) \) is the unique constant positive steady state solution of (31) provided that \( \alpha \delta > \gamma \theta \), where
\[
\alpha s^* v^* = \gamma u^* \quad \text{and} \quad \delta (1 - v^*) u^* = \theta v^*.
\]

In the following, we shall adopt a technique of Lyapunov functional (see, e.g., [6, 11, 12]) to study the global attractiveness of the positive steady state \((s^*, u^*, v^*)\). Before we state our results, we first note that \( \mathbf{R}_0 = \sqrt{\frac{\alpha \delta}{\gamma \theta}} \) by using the relations (32).

**Theorem 3.2.** Let \( \mathbf{R}_0 = \sqrt{\frac{\alpha \delta}{\gamma \theta}} \). Then the following statements hold

(i) If \( \mathbf{R}_0 > 1 \), then \( E_* \) is globally asymptotically stable in the interior of \( \mathcal{Y} \);

(ii) If \( \mathbf{R}_0 < 1 \), then \( E_0 \) is globally asymptotically stable in \( \mathcal{Y} \).

**Proof.** For Part (i), we define
\[
V(s, u, v) = s - s^* \ln s + u - u^* \ln u - \frac{\gamma}{\delta} (v^* \ln v + (1 - v^*) \ln (1 - v)),
\]
and
\[
W(t) = \int_{\Omega} V(s(x, t), u(x, t), v(x, t)) dx,
\]
where \((s(x,t), u(x,t), v(x,t))\) is an arbitrary positive solution of (31). A direct computations shows

\[
V_s(s, u, v)f_1(s, u, v) + V_u(s, u, v)f_2(s, u, v) + V_v(s, u, v)f_3(s, u, v)
= (\mu(1 - s) - \alpha s v) \left(1 - \frac{s^*}{s}\right) + \alpha s v - \gamma u - \frac{u^*}{u} (\alpha s v - \gamma u)
- \frac{\gamma}{\delta} \left(\frac{v^*}{v} - \frac{1 - v^*}{1 - v}\right) (\delta(1 - v) u - \theta v)
= 2\gamma u^* + \alpha s^* v + \frac{\gamma}{\delta} \frac{v^* - v}{1 - v} - \frac{\mu}{s} (s - s^*)^2 - \gamma u^* \frac{s^*}{s} - \gamma u^* \frac{v^*}{s} - \gamma u^* \frac{u^*}{s} - \gamma v^* \frac{u^*}{s} + \frac{2}{\delta} \left(\frac{\gamma u^*}{1 - s^*} - \gamma u^* \frac{s^*}{s} \frac{v^*}{u^*}\right)
\leq 2\gamma u^* + \theta \frac{v^* - v}{1 - v} - \gamma u^* - \frac{\mu}{s} (s - s^*)^2.
\]

Since

\[
\alpha s^* = \gamma \frac{u^*}{v^*}, \quad \theta = u^* \frac{1 - v^*}{v^*},
\]
we have

\[
V_s(s, u, v)f_1(s, u, v) + V_u(s, u, v)f_2(s, u, v) + V_v(s, u, v)f_3(s, u, v)
\leq \gamma u^* \frac{v^*}{v^*} + \gamma u^* \frac{1 - v^*}{v^*} \frac{v^* - v}{1 - v} - \gamma u^* - \frac{\mu}{s} (s - s^*)^2
= - \frac{\gamma}{1 - v^*} \left(\frac{v^* - v}{v^*}\right)^2 - \frac{\mu}{s} (s - s^*)^2 \leq 0.
\]

Hence, it follows that

\[
W(t) = \int_\Omega [V_s(s, u, v) s_t + V_u(s, u, v) u_t + V_v(s, u, v) v_t] dx
= \int_\Omega \left(1 - \frac{s^*}{s}\right) (d_1 \Delta s) + \left(1 - \frac{u^*}{u}\right) (d_1 \Delta u) - \frac{\gamma}{\delta} \left(\frac{v^*}{v} - \frac{1 - v^*}{1 - v}\right) (d_2 \Delta v) \right] dx
+ \int_\Omega [V_s(s, u, v)f_1(s, u, v) + V_u(s, u, v)f_2(s, u, v) + V_v(s, u, v)f_3(s, u, v)] dx
= - d_1 \frac{s^*}{s^2} |\nabla s|^2 + d_1 \frac{u^*}{u^2} |\nabla u|^2 \right) dx - d_2 \frac{\gamma}{\delta} \left(\frac{v^*}{v^2} + \frac{1 - v^*}{(1 - v^2)^2}\right) |\nabla v|^2 dx
+ \int_\Omega [V_s(s, u, v)f_1(s, u, v) + V_u(s, u, v)f_2(s, u, v) + V_v(s, u, v)f_3(s, u, v)] dx
\leq 0.
\]

Therefore, \(W\) is a Lyapunov functional for the system (31), namely, for any \(t > 0\), \(W(t) \leq 0\) along trajectories. Let \(C := \{(s, u, v) \in \mathbb{Y} : \dot{W} = 0\}\). Notice that

\[
\dot{W} = 0 \iff (s, u, v) = (s^*, u^*, v^*).
\]

Since the forward orbit of (31) is compact and \(\lim_{t \to \infty} (s(\cdot, t), u(\cdot, t), v(\cdot, t)) \to C\) by LaSalle Invariant Principle (see, e.g., [5, Theorem 4.3.4]). Thus, \((s^*, u^*, v^*)\) is globally asymptotically stable for (31). The proof of Part (i) is complete.
We next point out that the result in Part (ii) is a special case of that in Theorem 2.7 (i). For the completeness, we provide the following functional
\[ W(t) = \int_{\Omega} \mathcal{V}(s(x,t), u(x,t), v(x,t)) dx, \]
where \( \mathcal{V}(s, u, v) := u + \frac{\gamma}{\delta} v \) and \( (s(x,t), u(x,t), v(x,t)) \) is an arbitrary positive solution of (31). By computations,
\[
\dot{W}(t) = \int_{\Omega} \left[ \mathcal{V}_s(s, u, v)s_t + \mathcal{V}_u(s, u, v)u_t + \mathcal{V}_v(s, u, v)v_t \right] dx \\
= \int_{\Omega} \left[ 0 + d_1 \Delta u + \frac{\gamma}{\delta} d_2 \Delta v \right] dx + \int_{\Omega} \left[ -\gamma vu + \alpha(s - 1)v + (\alpha \delta - \gamma \theta) \frac{v^2}{\delta} \right] dx \\
= \int_{\Omega} \left[ -\gamma vu + \alpha(s - 1)v + (\alpha \delta - \gamma \theta) \frac{v^2}{\delta} \right] dx \leq 0.
\]
Therefore, \( W \) is a Lyapunov functional for the system (31), namely, for any \( t > 0, W(t) \leq 0 \) along trajectories. Let \( \mathcal{C} := \{(s, u, v) \in \mathcal{Y} : \dot{W} = 0 \} \). Notice that
\[
\dot{W} = 0 \iff v = 0 \text{ or } \gamma u - \alpha s + \theta \frac{\gamma}{\delta} = 0 \iff v = 0 \text{ or } u = 0 \text{ or } s = 1.
\]
Since the forward orbit of (31) is compact and \( \lim_{t \to \infty} (s(\cdot, t), u(\cdot, t), v(\cdot, t)) \to \mathcal{C} \) by LaSalle Invariant Principle (see, e. g., [5, Theorem 4.3.4]). Thus, \((1, 0, 0)\) is globally asymptotically stable for (31). We complete the proof.

4. Discussion. In this paper, we studied the qualitative behavior of solutions of a reaction-diffusion system (see (2)), which is used to describe the dynamics of the spread of dengue fever. If the coefficients are spatial dependent, we employed the comparison theorems and the property of eigenvalue problems to establish criterium for the uniform persistence property of system (2) (see Theorem 2.7 and Remark 1). If the coefficients are all constants (see (29)), then we took the advantages of the method of Lyapunov functionals to obtain the global dynamics of (29) (see Theorem 3.2). Notice that our findings could be viewed as a generalization to those obtained in Esteva and Vargas [2].

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