The Seiberg-Witten Equations and the Length Spectrum of Hyperbolic Three-Manifolds

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Abstract. We exhibit the first examples of hyperbolic three-manifolds for which the Seiberg-Witten equations do not admit any irreducible solution. Our approach relies on hyperbolic geometry in an essential way; it combines an explicit upper bound for the first eigenvalue on coexact 1-forms $\lambda^*_1$ on rational homology spheres which admit irreducible solutions together with a version of the Selberg trace formula relating the spectrum of the Laplacian on coexact 1-forms with the volume and complex length spectrum of a hyperbolic three-manifold. Using these relationships, we also provide precise numerical bounds on $\lambda^*_1$ for several hyperbolic rational homology spheres.

In the last three decades, both hyperbolic geometry and Floer homology have played a central role in the study of the geometry and topology of three-dimensional manifolds (see for example [1], [16], [29], [31], [43]). Despite this, and even though both subjects have by now reached their maturity, their mutual interaction (if any) remains extremely mysterious. For example, while the computation of the Floer homology for the Seifert fibered case is very well-understood in explicit, geometric terms [13], [38], the Floer homology of hyperbolic manifolds has eluded similar descriptions. Because Mostow rigidity implies that the geometric invariants of a hyperbolic three-manifold are indeed topological invariants, the following is a very natural yet outstanding problem one encounters.

Question. For a hyperbolic three-manifold $Y$, is there any relationship between the topological invariants arising from the hyperbolic geometry of $Y$ (e.g. the volume, injectivity radius, lengths of geodesics, etc.) and the invariants arising from Floer homology?

In the present paper we discuss, for a hyperbolic-three manifold $Y$ with $b_1(Y) = 0$, a relationship between the existence of irreducible solutions to the Seiberg-Witten equations on $Y$ and the hyperbolic geometry of $Y$. As a testing ground, we explore this relationship for the first 50 manifolds in the Hodgson-Weeks census, which is a (conjecturally complete) list of hyperbolic three-manifolds with volume $< 6.5$ and injectivity radius $> 0.15$ [22]. Our main application is the following.

Theorem 1. Let $Y$ be one of the hyperbolic three-manifolds from the Hodgson-Weeks census listed in Table [7]. Then the Seiberg-Witten equations on $Y$ (for the hyperbolic metric) do not admit any irreducible solution.

The only previously known examples of Riemannian rational homology three-spheres with no irreducible solutions were provided by manifolds with positive scalar curvature, and the

\footnote{This result (and the following Theorem 2) takes as input the computations of the length spectrum provided by the \texttt{length_spectrum}() method of SnapPy version 2.6.1 [8]. These are very accurate (especially for the small manifolds we are dealing with in the paper), but are not yet certified using interval arithmetic in the current version. There is promising work towards this end [14] using the certified hyperbolic structure produced in [21].}
Hantzsche-Wendt manifold (the only rational homology three-sphere with a flat metric), \[30\]. In this sense, the manifolds in Table 1 are also the first examples of hyperbolic three-manifolds for which the set of solutions to the Seiberg-Witten equations is determined explicitly.

| Census label | Volume   | Injectivity radius |
|--------------|----------|--------------------|
| 0            | 0.94270  | 0.29230            |
| 2            | 1.01494  | 0.41572            |
| 3            | 1.26371  | 0.28753            |
| 8            | 1.42361  | 0.17618            |
| 12           | 1.54356  | 0.16768            |
| 13           | 1.54356  | 0.28903            |
| 14           | 1.58316  | 0.27889            |
| 15           | 1.58316  | 0.38874            |
| 16           | 1.58864  | 0.26727            |
| 22           | 1.83193  | 0.26532            |
| 25           | 1.83193  | 0.26531            |
| 28           | 1.88541  | 0.29230            |
| 29           | 1.88541  | 0.19853            |
| 30           | 1.88541  | 0.19853            |
| 31           | 1.88541  | 0.29230            |
| 32           | 1.88591  | 0.20593            |
| 33           | 1.91084  | 0.22107            |
| 39           | 1.96274  | 0.21576            |
| 40           | 1.96274  | 0.28904            |
| 42           | 2.02395  | 0.17922            |
| 44           | 2.02988  | 0.43127            |
| 46           | 2.02988  | 0.27177            |
| 49           | 2.02988  | 0.21564            |

Table 1. The hyperbolic manifolds of Theorem 1

As a direct consequence of Theorem 1, the manifolds in Table 1 are \(L\)-spaces, i.e. their reduced Floer homology groups \(HM\) vanishes. This had been previously shown by Dunfield \[11\] in the setting of Heegaard Floer homology; the latter is known to be isomorphic to monopole Floer homology (see \[28\], \[9\] and subsequent papers). In fact, he has determined exactly which spaces in the Hodgson-Weeks census are \(L\)-spaces; in this regard, Table 1 comprises 23 of the 28 \(L\)-spaces with label less than 49.

Remark 1. As a matter of nomenclature, we will refer to rational homology spheres admitting a metric with no irreducible solutions as minimal \(L\)-spaces.

As mentioned above, the proof of Theorem 1 exploits in an essential way the fact that the underlying manifold is equipped with a hyperbolic metric. While we do not know a direct way to relate the latter to the Seiberg-Witten equations, we use as stepping stone the spectral geometry of the Hodge Laplacian acting on coexact 1-forms. Recall that on a Riemannian 3-manifold with \(b_1(Y) = 0\) the Hodge decomposition implies the direct sum decomposition of 1-forms

\[
\Omega^1(Y) = d\Omega^0(Y) \oplus d^*\Omega^2(Y)
\]
into exact and coexact ones; and the Hodge Laplacian $\Delta = (d + d^*)^2$ preserve such a decomposition. We denote the spectrum of $\Delta$ acting on coexact 1-forms $d^*\Omega^2(Y)$ by

$$0 < \lambda_1^* \leq \lambda_2^* \leq \lambda_3^* \leq \ldots .$$

In the present paper, we will be mostly interested in the first eigenvalue $\lambda_1^*$. While not much is known in general about this quantity, it has recently attracted attention due to its relationship with a deep conjecture of Bergeron and Venkatesh [5] regarding the growth of torsion in the cohomology of arithmetic hyperbolic three-manifolds under towers of coverings. In our setting, its appearance as a stepping stone is synthesized in the following diagram.

Before discussing such relationship, let us point out another of its applications.

**Theorem 2.** For the hyperbolic three-manifolds from the Hodgson-Weeks census listed in Table 2 we can provide precise numerical bounds for the value of $\lambda_1^*$. 

**Remark 2.** It is interesting to compare these computations with the case of the first eigenvalue of functions (or, equivalently, on closed 1-forms). While there are some numerical results in the latter case (especially in the astrophysics literature, see [24], [10]), these are based on heuristic computations. As we will see, a key input in the proof of Theorem 2 is given by the computations of the topological invariants arising from Seiberg-Witten theory.

| Census label | $\lambda_1^*$ lower bound | $\lambda_1^*$ upper bound | Volume | Injectivity radius |
|--------------|---------------------------|---------------------------|--------|-------------------|
| 1            | 0.33749                   | 0.33983                   | 0.98136... | 0.28904...       |
| 4            | 0.61613                   | 0.64594                   | 1.28448... | 0.24015...       |
| 6            | 0.58541                   | 0.60133                   | 1.41406... | 0.39706...       |
| 7            | 0.27882                   | 0.28224                   | 1.41406... | 0.18244...       |
| 9            | 0.43598                   | 0.97651                   | 1.44070... | 0.18076...       |
| 19           | 0.68344                   | 0.82304                   | 1.75712... | 0.35268...       |
| 23           | 0.50310                   | 0.51433                   | 1.83193... | 0.24060...       |
| 24           | 0.31571                   | 0.32022                   | 1.83193... | 0.26531...       |
| 34           | 0.00131                   | 0.00537                   | 1.91221... | 0.24958...       |
| 45           | 0.60516                   | 0.76929                   | 2.02988... | 0.27176...       |
| 47           | 0.37043                   | 0.38036                   | 2.02988... | 0.21563...       |
| 48           | 0.28543                   | 0.29030                   | 2.02988... | 0.27176...       |

Table 2. Bounds for $\lambda_1^*(Y)$ for the hyperbolic manifolds $Y$ of Theorem 2.
The estimate in the diagram above uses the following refinement of the main theorem of [34] as one key input. See also §1 for a more detailed discussion.

**Theorem 3.** Let $Y$ be a hyperbolic rational homology three-sphere. If the Seiberg-Witten equations admit an irreducible solution, then $\lambda_1^2 \leq 2$.

On the other hand, the relationship between hyperbolic geometry and spectral geometry is provided by a specialization of the celebrated Selberg trace formula, which provides, for a Lie group $G$ and a lattice $\Gamma$ in it, a link between geometry and spectral theory (which can be thought as a non-abelian generalization of the classical Poisson summation formula). For simplicity, consider first the case of a compact surface $X$ equipped with a hyperbolic metric, which corresponds to $G = \text{PSL}(2; \mathbb{R})$ and $\Gamma = \pi_1(X)$. In this case, it was proved by Selberg (see [20]) that once we label the eigenvalues

$$0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \ldots$$

of the Laplacian $\Delta$ acting on functions on $X$ as $\lambda_j = r_j^2 + 1/4$ with $r_j \in \mathbb{R}^{\geq 0} \cup [0, 1/2]\sqrt{-1}$, the following identity holds for $g \in C_c^{\mathbb{R}}(\mathbb{R})^{\text{even}}$:

$$\sum_{j=0}^{\infty} \hat{g}(r_j) = \frac{\text{vol}(X)}{2\pi} \int_0^{\infty} \hat{g}(r) \tanh(r) dr + \frac{1}{2} \sum_{\gamma \neq 1} \frac{\ell(\gamma_0)}{e^{\ell(\gamma_0)} - e^{-\ell(\gamma_0)}} g(\ell(\gamma)).$$

The sum on the right hand side runs over all non-trivial closed geodesics $\gamma$ on $X$. These correspond to non-trivial conjugacy classes in $\pi_1(X)$. Here $\ell(\gamma)$ denotes the length of $\gamma$, and $\gamma_0$ is the primitive geodesic of which $\gamma$ is a multiple of.

The Selberg trace formula is a very powerful tool as it allows to extract seemingly inaccessible information regarding spectral geometry of $X$ via the understanding of the lengths of its geodesics; and the latter quantities are directly computable from the traces of the elements $\pi_1(X) \subset \text{PSL}(2; \mathbb{R})$.

In the present paper, we will derive a specialization of the general Selberg trace formula that relates, for a closed oriented hyperbolic three-manifold $Y$, the following quantities:

- on the spectral side, the square roots of the eigenvalues of the Laplacian on coexact 1-forms $t_j = \sqrt{\lambda_j^2}$;
- on the geometric side, the volume $\text{vol}(Y)$ and the complex lengths $\mathbb{C}\ell(\gamma)$ of the closed geodesics $\gamma$ of $Y$.

Recall here that for a closed geodesic $\gamma$ in a hyperbolic three-manifold there is a notion of holonomy $\text{hol}(\gamma)$, namely how an orthonormal framing for the normal bundle of $\gamma$ is rotated under parallel transport along $\gamma$. The complex length of $\gamma$ is then given by

$$\mathbb{C}\ell(\gamma) := \ell(\gamma) + i \text{hol}(\gamma) \in \mathbb{R} + i (\mathbb{R}/2\pi\mathbb{Z}).$$

As in the case of surfaces, these are directly computable in term of the traces of elements $\pi_1(Y) \subset \text{PSL}(2; \mathbb{C})$. The formula is then the following.

**Theorem 4 (Explicit Selberg trace formula for coclosed 1-forms on closed hyperbolic 3-manifolds).** Let $Y$ be a closed oriented hyperbolic three-manifold, and let $H$ be an even, smooth, compactly supported, $\mathbb{R}$-valued function on $\mathbb{R}$. Then the following identity holds.

$$-\frac{1}{2} \hat{H}(0) + \frac{1}{2} \sum_{j=0}^{\infty} \hat{H}(t_j) = \frac{\text{vol}(Y)}{2\pi} (H(0) - H''(0)) + \sum_{\gamma \neq 1} \ell(\gamma_0) \frac{\cos(\text{hol}(\gamma))}{|1 - e^{\mathbb{C}\ell(\gamma)}| \cdot |1 - e^{-\mathbb{C}\ell(\gamma)}|} H(\ell(\gamma)).$$
Figure 1. Surgery diagrams for the Weeks and Meyerhoff manifolds, which are respectively the manifolds labeled 0 and 1 in the Hodgson-Weeks census. The link on the left is the Whitehead link, while the knot on the right is the figure eight knot.

The smoothness of $H$ above can be relaxed to allow much slower decay of $\hat{H}$ (see Theorem 9 for a more precise description of the decay condition).

Taking as input computations of volume and length spectrum of $Y$ provided by SnapPy [8], this formula can be used to show that for a given value $t \in \mathbb{R}^{>0}$, $t^2$ is not an eigenvalue of the Laplacian on coexact 1-forms on $Y$. The specific procedure we use, inspired by the work of Booker and Strombergsson related to Selberg’s 1/4-conjecture [6], is discussed thoroughly in §3.1. Granted this, let us discuss the logic behind the proof of our main results:

- For the spaces in Theorem 1, we will use the Selberg trace formula to show that $t^2$ is not a coclosed, 1-form eigenvalue for any $t^2 \in [0, 2]$. Combined with Theorem 3, this implies that there are no irreducible solutions to the Seiberg-Witten equations.
- For the spaces in Theorem 2, it is known that their reduced Floer homology $HM$ is non-vanishing. This implies that for an arbitrarily small regular perturbation, the Seiberg-Witten equations admit irreducible solutions, so that $\lambda^*_1 \in (0, 2]$. On the other hand, using Theorem 4 one can give in these a precise constraint on which elements in $[0, \sqrt{2}]$ can possibly be eigenvalues, which combined with the existence result from Seiberg-Witten theory implies the precise bounds in Table 2.

We conclude this introduction by discussing the two simplest examples in which our main results apply, see Figure 1. The manifolds in the picture represent the ones labeled 0 and 1 in the Hodgson-Weeks census. Both of these manifolds play a special role in hyperbolic geometry; the manifold on the left, the Weeks manifold, is known to have the smallest volume 0.94... among closed, orientable hyperbolic three-manifolds [17], while the one on the right, the Meyerhoff manifold, which has volume 0.98... was believed to have smallest volume for a long time. Using the surgery diagrams in Figure 1 one can determine their Floer homology $HM$ and show, in particular, that Weeks is an $L$-space while Meyerhoff is not. Such a drastic difference is not reflected in basic quantities that are studied in hyperbolic geometry, as for example these manifolds have very similar volume and injectivity radius. On the other hand, these manifolds are drastically different from the point of view of the spectral geometry of coexact 1-forms, as for Weeks $\lambda^*_1 > 9$ while for Meyerhoff $\lambda^*_1 \sim 0.33$. We will provide a qualitative discussion of this drastic difference, in these and in more general examples, in §5.
Plan of the paper. In §1 we provide some background material on monopole Floer homology, and discuss its relation with spectral geometry and in particular Theorem 3. In §2 we prove the version of the Selberg trace formula relevant to our problem stated in Theorem 4. In §3 we discuss the computational technique of Booker and Strombergsson, and in §4 the outputs are presented. Finally, in §5 we discuss the limitations of our method and some natural problems that arise.

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1. The Seiberg-Witten equations and monopole Floer homology

In this section we review the basic setup of Seiberg-Witten theory on a (closed, oriented, connected) three-manifold \(Y\). We refer the reader to [33] for a more thorough introduction and to [30] for the quintessential reference.

1.1. The geometric setup. Consider on \(Y\) a Riemannian metric and a spin\(c\) structure \(s\). For our purposes, the best way to think about the latter is a rank two hermitian bundle \(S \to Y\) together with a bundle map

\[ \rho : TY \to \text{Hom}(S, S), \]

called Clifford multiplication, satisfying \(\rho(v)^2 = -|v|^2 1_S\). In coordinates, this means that for any oriented frame \(e_1, e_2, e_3\) at a point \(y\), we can find a basis of \(S_y\) so that \(\rho(e_i)\) is the Pauli matrix \(\sigma_i\):

\[ \sigma_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}. \]

We can then consider the configuration space \(\mathcal{C}(Y, s)\) consisting of pairs \((B, \Psi)\) where:

- \(\Psi\) is a spinor, i.e. a section of \(\Gamma(S)\).
- \(B\) is a spin\(c\) connection on \(S\), i.e. a unitary connection for which \(\rho\) is parallel, or, equivalently

\[ \nabla_B(\rho(X)\Psi) = \rho(\nabla X)\Psi + \rho(X)\nabla_B\Psi \]

for any vector field \(X\) and spinor \(\Psi\). Here \(\nabla\) is the Levi-Civita connection and \(\nabla_B\) the covariant derivative associated to \(B\).

The condition (1) implies that the \(SO(3)\)-part of a spin\(c\) connection \(B\) is determined by the Levi-Civita connection; as a consequence, \(B\) is determined by the connection \(B^t\) induced on the determinant line bundle \(\det(S)\). In particular, the space of spin\(c\) connections is an affine space over \(\Omega^1(Y; i\mathbb{R})\).

The space of configurations \(\mathcal{C}(Y, s)\) is acted on by the group of automorphisms of the spin\(c\) structure, i.e. the gauge group \(\mathcal{G}(Y, s) = \text{Maps}(Y, S^1)\), via

\[ u \cdot (B, \Psi) = (u^*B, u \cdot \Psi), \]

where \(u^*B = B - u^{-1}du\) is the pullback connection.
The stabilizer under the gauge group of the configuration \((B, \Psi)\) is trivial when \(\Psi\) is not identically zero. On the other hand the stabilizer of a configuration of the form \((B, 0)\) is given by the constant gauge transformations, so it is identified with \(S^1\). We call the configurations of the first kind irreducible, while the configurations of the second kind reducible.

For a fixed base connection \(B_0\), the Chern-Simons-Dirac functional

\[
\mathcal{L} : \mathcal{C}(Y, s) \to \mathbb{R}
\]

is defined to be

\[
\mathcal{L}(B, \Psi) = -\frac{1}{8} \int_Y (B_t - B_0^t) \wedge (F_{B^t} + F_{B_0^t}) + \frac{1}{2} \int_Y \langle D_B \Psi, \Psi \rangle d\text{vol}.
\]

Here \(F_{B^t}\) denotes the curvature of the connection \(B^t\) (hence a imaginary valued 2-form), and \(D_B\) is the Dirac operator associated to the connection \(B\), i.e. the composition

\[
\Gamma(S) \xrightarrow{\nabla_B} \Gamma(T^* X \otimes S) \xrightarrow{\rho} \Gamma(S).
\]

While the functional is invariant only under the action of the connected component of the gauge group, it descends to a well defined functional

\[
\mathcal{L} : \mathcal{C}(Y, s)/G(Y, s) \to \mathbb{R}/(2\pi^2\mathbb{Z})
\]

on the moduli space of configurations. The critical points of the Chern-Simons-Dirac functional are given by the solutions \((B, \Psi)\) of the system

\[
\begin{align*}
\frac{1}{2} \rho(F_{B^t}) - (\Psi \Psi^*)_0 &= 0 \\
D_B \Psi &= 0
\end{align*}
\]

to which we refer to as the Seiberg-Witten equations on \(Y\).

1.2. Monopole Floer homology and its applications. One can apply the ideas of Morse homology to the Chern-Simons-Dirac functional \(\mathcal{L}\) on the moduli space of configurations in order to define homological invariants of three-manifolds which are topological (i.e. independent of the initial choice of the metric). The final result is a package of invariants called monopole Floer homology \([30]\). There are several complications to be handled, most notably the need to introduce a suitable space of regular perturbations to the equations in order to achieve transversality, and the \(S^1\)-symmetry of the functional. In the setup of \([30]\), the latter is dealt with suitably blowing up the configurations space, and leads to the construction of \(S^1\)-equivariant Floer homology group.

The simplest invariant arising from this construction is the reduced monopole Floer homology group \(HM(Y)\), which plays a central role when studying gluing formulas for the 4-dimensional Seiberg-Witten invariants (see the classical reference \([37]\) for the latter). It has also recently gained attention as it contains significant information regarding three-dimensional geometric structures. An object of central study in three-dimensional geometry are coorientable taut foliations, i.e. coorientable 2-dimensional foliations \(\mathcal{F}\) equipped with a closed 2-form \(\omega\) which is positive on the leaves of \(\mathcal{F}\). While criteria for the existence of such foliations has been provided by Gabai for manifolds with \(b_1(Y) > 0\) \([15]\), a general characterization in the case of rational homology spheres is missing. In this sense, the following Floer theoretic obstruction from \([32]\) plays a central role.
Theorem 5 (Theorem 2.1 of [32]). Suppose $Y$ has $b_1(Y) = 0$. If it admits a coorientable taut foliation, then $HM(Y) \neq 0$.

This highlights the class of $L$-spaces, i.e. three-manifolds $b_1(Y) = 0$ and $HM(Y) = 0$. This notion corresponds to the analogous notion of $L$-space in Heegaard Floer homology (i.e. spaces for which $HF_{red}(Y) = 0$) via the isomorphism between the theories (see [28], [9], and subsequent papers). In fact, it was conjectured by Ozsváth and Szabó that the converse of Theorem 5 holds. Furthermore, the concepts of $L$-spaces and taut foliations are also conjecturally related to the existence of left-invariant orders on the fundamental group of $Y$ [4]. Such conjectures have been proved in the case of graph manifolds [19], and have been verified in some families of hyperbolic three-manifolds [11].

Even though the definition of the invariant $HM(Y)$ involves the solution of certain nonlinear PDEs, its computation can be carried over in several cases (including those in Figure 1) using topological techniques, most notably surgery exact triangle [32]. It can also be computed in a (practically infeasible) purely combinatorial fashion [42].

1.3. Relation with spectral geometry. We will focus from now on the case of a rational homology sphere $Y$. If $Y$ admits a metric for which (suitable small perturbations of) the Seiberg-Witten equations do not admit irreducible solutions, then $HM(Y) = 0$. On the other hand, very little is known in general known about the set of solutions to the Seiberg-Witten equations itself other than manifolds which have positive scalar curvature or are flat (see [30]). In this case, one can show that the equations do not admit irreducible solutions for suitable small perturbations by means of a Bochner type argument involving the Weitzenböck formula. The case of Seifert manifolds can be understood if one studies a different set of equations where the Levi-Civita connection is replaced by a non standard reducible one [38]. As a refinement of argument in the first case, we will now discuss the following.

Theorem 6. Let $Y$ be a rational homology sphere equipped with a Riemannian metric $g$. Let $\lambda_1^*$ be the least eigenvalue of the Laplacian on coexact 1-forms, and $\tilde{s}(p)$ the sum of the two least eigenvalues of the Ricci curvature at $p$. If $\lambda_1^* > -\inf_{p \in Y} \tilde{s}(p)/2$ then on $(Y, g)$ the Seiberg-Witten equations have no irreducible solutions.

Theorem 6 is a slight refinement of the main result of [34], for which the stronger assumption $\lambda_1^* > -\inf_{p \in Y} \tilde{s}(p)$ is required. As for a hyperbolic metric $\tilde{s} = -4$ everywhere, Theorem 5 follows.

While there are qualitative results on the behavior of $\lambda_1^*$ for hyperbolic three-manifolds [25], [36], the goal of this paper is to find examples of hyperbolic three-manifolds for which the explicit bound $\lambda_1^* > 2$ holds. In fact, the slight improvement on the main theorem from [34] provided by the inequality in Theorem 6 will be crucial for drawing conclusions in many of the examples of Theorem 1.

The main theorem of [34] uses, at one important step, the inequality

$$|\nabla \xi|^2 \leq |\Psi|^2 |\nabla_B \Psi|^2$$

for $\xi = \rho^{-1}(\Psi \Psi^*)_0$;

this holds for any configuration $(B, \Psi)$, not necessarily solving the Seiberg-Witten equations. The key observation behind the improvement in Theorem 6 is the following refinement for a configuration $(B, \Psi)$ which does solve the Seiberg-Witten equations.
Proposition 1. Let \((B, \Psi)\) be a solution to the Seiberg-Witten equations, and \(\xi = \rho^{-1}(\Psi\Psi^*)_0\). Then the pointwise identity
\[
|\nabla \xi|^2 + |d\xi|^2 = |\Psi|^2 |\nabla_B \Psi|^2
\]
holds.

Before proving this, let us discuss another nice identity.

Lemma 1. Given a solution \((B, \Psi)\) to the Seiberg-Witten equations, we have
\[
* d\xi = i \Im \langle \Psi, \nabla_B \Psi \rangle.
\]

Proof. Fix an oriented orthonormal frame \(e_1, e_2, e_3\) synchronous at \(p\), and consider a basis of the spinor bundle \(S\) for which \(\rho\) is represented by the Pauli matrices. We will write in this basis \(\Psi = (\alpha, \beta)\) and \(\nabla_B \Psi = (\nabla \alpha, \nabla \beta)\) with
\[
\nabla \alpha = \sum \alpha_i e^i, \quad \nabla \beta = \sum \beta_i e^i.
\]
The Dirac equation \(D_B \Psi = 0\) is then equivalent to the system
\[
i\alpha_1 - \beta_2 + i\beta_3 = 0
\]
\[
-i\beta_1 + \alpha_2 + i\alpha_3 = 0.
\]
We have by definition
\[
(\Psi \Psi^*)_0 = \left[ \frac{1}{2}(|\alpha|^2 - |\beta|^2), \frac{1}{2}(|\beta|^2 - |\alpha|^2) \right].
\]
so that
\[
(3) \quad \xi = -i \left( \frac{1}{2}(|\alpha|^2 - |\beta|^2)e^1 - \Im(\bar{\alpha}\beta)e^2 + Re(\bar{\alpha}\beta)e^3 \right).
\]
We compute the \(-ie^1 \wedge e^2\) component of \(d\xi\) (recalling that \(\Re(iz) = -\Im(z)\)):
\[
= -\Re(\alpha_2 \bar{\alpha} - \beta_2 \bar{\beta}) - \Im(\bar{\alpha}_1 \beta + \bar{\alpha}_1 \beta) 
\]
\[
= -\Re(\alpha_2 \bar{\alpha} - \beta_2 \bar{\beta}) + \Re(i\bar{\alpha}_1 \beta + \bar{\alpha}_1 \beta) 
\]
\[
= -\Re(\alpha_2 \bar{\alpha} - \beta_2 \bar{\beta}) + \Re(-i\alpha_1 \bar{\beta} + \alpha_1 \bar{\beta}) 
\]
\[
= + \Re(\bar{\alpha}(-\alpha_2 + i\beta_1)) + \Re(\bar{\beta}(\beta_2 - i\alpha_1)) 
\]
\[
= + \Re(\alpha \bar{i} \alpha_3) + \Re(\beta \bar{i} \beta_3) 
\]
\[
= - \Im(\alpha_3 \bar{\alpha} + \beta_3 \bar{\beta}).
\]
where we used the Dirac equation. Hence the \(ie^3\) component of \(* d\xi\) is \(\Im(\alpha_3 \bar{\alpha} + \beta_3 \bar{\beta})\), which is the imaginary part of \(\langle \Psi, (\nabla_B)_{e_3} \Psi \rangle\). The computation for the remaining two components is analogous. \(\square\)

Proof of Proposition 4. In the notation of the lemma above, we have
\[
(4) \quad |\Psi|^2 |\nabla_B \Psi|^2 = (|\alpha|^2 + |\beta|^2) \cdot (|\nabla \alpha|^2 + |\nabla \beta|^2).
\]
The computation in the previous lemma shows that
\[
|d\xi|^2 = |\Im(\bar{\alpha}\nabla \alpha + \bar{\beta}\nabla \beta)|^2 = |\Im(\bar{\alpha}\nabla \alpha - \bar{\beta}\nabla \beta)|^2
\]
Recalling that \(\Re(\beta \nabla \beta) = \Re(\bar{\beta} \nabla \beta)\), we have from equation (3) that
\[
|\nabla \xi|^2 = |\Re(\bar{\alpha}\nabla \alpha - \bar{\beta}\nabla \beta)|^2 + |\Im(\bar{\alpha}\nabla \beta + (\nabla \beta)\beta)|^2 + |\Re(\bar{\alpha}\nabla \beta + (\nabla \alpha)\beta)|^2
\]
Therefore
\[ |\nabla \xi|^2 + |d\xi|^2 = |\alpha \nabla \alpha - \beta \nabla \beta|^2 + |\alpha \nabla \beta + (\nabla \alpha)\beta|^2 \]
Expanding, the mixed terms cancel out and we are left with (4). \[\square\]

Finally, we can discuss how the refinement of Theorem 6 works in light of this estimate.

Proof of Theorem 6 Let us quickly review the proof of the slightly weaker inequality of [34] (we refer the reader to the paper for more details). We assume for simplicity of notation that the metric is hyperbolic, so that the Ricci curvature is constantly \(-2\); in particular, we will prove the statement of Theorem 6. Given a solution \((B, \Psi)\) to the Seiberg-Witten equations, the Weitzenböck formula implies the identity
\[ \Delta |\Psi|^2 = 2\langle \Psi, \nabla_B^* \nabla_B \Psi \rangle - 2|\nabla_B \Psi|^2 = -|\Psi|^4 + 3|\Psi|^2 - 2|\nabla_B \Psi|^2. \]
Multiplying this by \(|\Psi|^2\), and integrating over the manifold, we obtain by Green’s identity
\[ (5) \int |\Psi|^6 - |\Psi|^4 + 2|\Psi|^2|\nabla_B \Psi|^2 = -\int |\Psi|^2 \Delta |\Psi|^2 = -\int |d\Psi|^2|^2 \leq 0. \]
The Bochner formula states that on 1-forms \((d + d^* )^2 = \nabla^* \nabla + \text{Ric}\), so that for our form \(\xi = \rho^{-1}(\Psi \Psi^*)_0\), which is coclosed, we have
\[ \|\nabla \xi\|_{L^2}^2 = \|d\xi\|_{L^2}^2 + 2\|\xi\|_{L^2}^2 \geq (2 + \lambda_1^*)\|\xi\|_{L^2}^2 \]
where we used the variational definition of \(\lambda_1^*\) in the last inequality. Hence, the weak inequality (2) implies
\[ \int |\Psi|^2|\nabla_B \xi|^2 \geq \|\nabla \xi\|_{L^2}^2 \geq (2 + \lambda_1^*)\|\xi\|_{L^2}^2 = \frac{1}{4}(2 + \lambda_1^*)\|\Psi\|_{L^4}^4, \]
where we used the pointwise identity \(|\xi|^2 = \frac{1}{4}|\Psi|^4\). Combining this with (5) we get
\[ \int |\Psi|^6 + \frac{1}{2}(\lambda_1^* - 4)|\Psi|^4 \leq 0. \]
so that if \(\lambda_1^* > 4\), \(\Psi\) is identically zero, i.e. the Seiberg-Witten equations have no irreducible solutions.

Let us now show how to refine the inequality. Using the identity in Proposition 1 we obtain
\[ \int |\Psi|^2|\nabla_B \xi|^2 = \|d\xi\|_{L^2}^2 + \|\nabla \xi\|_{L^2}^2 = 2\|d\xi\|_{L^2}^2 + 2\|\xi\|^2 \geq (2 + 2\lambda_1^*)\|\xi\|_{L^2}^2 = \frac{1}{2}(\lambda_1^* + 1)\|\Psi\|_{L^4}^4. \]
Combining this with (5), we see that the inequality
\[ \int |\Psi|^6 + (\lambda_1^* - 2)|\Psi|^4 \leq 0. \]
holds, so that if \(\lambda_1^* > 2\), \(\Psi\) is identically zero. \[\square\]

Let us point out that as a consequence of our discussion, if \(Y\) is a hyperbolic rational homology sphere with \(\lambda_1^* > 2\), then it is an \(L\)-space. The converse of this is not true. For example, consider \(K\) to be the \((-2,3,7)\)-pretzel knot. This is a hyperbolic knot, and it is well known that it admits a lens space (hence \(L\)-space) surgery [12]. In particular, for \(n\) large enough the manifold \(S^3_n(K)\) obtained by \(n\)-surgery is an \(L\)-space [32], and is also hyperbolic by a celebrated result of Thurston. Furthermore, for this family of hyperbolic three-manifolds the
diameter goes to infinity while the volume stays bounded above. Then a result of McGowan implies that \( \lambda_1(Y_n) \) converges to zero, see also §5.

2. The trace formula

Fix a smooth compactly supported test function \( H \) on \( \mathbb{R} \). Our goal in §2 is to explain how, by specializing the trace formula appropriately, the geometric side can be re-expressed as sampling lengths of closed geodesics on \( M = \Gamma \backslash \mathbb{H}^3 \) via \( H \) and the spectral side can be re-expressed as sampling the Fourier transform \( \hat{H} \) at (square roots of) eigenvalues of the Laplacian acting on coexact 1-forms on \( M \). For our intended applications, expressing the trace formula completely explicitly in these terms is crucial.

2.1. The trace formula via representation theory. Let \( G \) be a Lie group and let \( \Gamma \subset G \) be a discrete, cocompact subgroup. Subsequent sections will focus on \( G = \text{PGL}_2(\mathbb{C}) \), the orientation-preserving isometry group of \( \mathbb{H}^3 = \text{PGL}_2(\mathbb{C})/\text{PU}_2 \), but the present discussion holds in full generality.

Let \( dg \) be a Haar measure on \( G \). As a representation of \( G \), the Hilbert space \( L^2(\Gamma \backslash G) \) decomposes as

\[
L^2(\Gamma \backslash G) = \bigoplus_{\pi \in \hat{G}} m_{\Gamma}(\pi) \cdot \pi,
\]

where the sum runs over \( \hat{G} \), all irreducible representation of \( G \). Above,

\[
m_{\Gamma}(\pi) := \dim \text{Hom}_G(\pi, L^2(\Gamma \backslash G)) \in \mathbb{Z}_{\geq 0}
\]

is the multiplicity with which \( \pi \) occurs in the decomposition of \( L^2(\Gamma \backslash G) \). Because \( \Gamma \) is cocompact, the above sum is discrete: \( m_{\Gamma}(\pi) \) is non-zero for only countably many \( \pi \).

**Theorem 7** (Selberg trace formula). For every smooth compactly supported function \( f \) on \( G \), there is an equality

\[
\sum_{\pi} m_{\Gamma}(\pi) \cdot \text{trace}(\pi(fdg)) = \sum_{[\gamma]} \text{vol}(\Gamma_\gamma \backslash G_\gamma, dg_\gamma) \cdot O_\gamma \left( \frac{f}{dg_\gamma} \right).
\]

The right side of \((6)\) is called the geometric side of the trace formula. The left side of \((6)\) is called the spectral side of the trace formula.

Explanation of notation from the above statement of the trace formula is in order:

- For a function \( f \) on \( G \) and a representation \( \pi \) of \( G \), we define

\[
\pi(fdg) := \int_G f(g)\pi(g)dg.
\]

If \( f \) is smooth, compactly supported and \( \pi \) is unitary, then \( \pi(fdg) \) is a compact operator, trace class in fact. On the spectral side of the trace formula, \( \text{trace}(\pi(fdg)) \) denotes its trace.

- The geometric side of the trace formula is summed over all conjugacy classes in \( \Gamma \).

- \( G_\gamma \) and \( \Gamma_\gamma \) respectively denote the centralizer of \( \gamma \) in \( G \) and in \( \Gamma \).

- \( dg_\gamma \) denotes a choice of Haar measure on \( G_\gamma \).
- $O_\gamma \left( f dg, \frac{dg}{dg_\gamma} \right)$ denotes the orbital integral of $f dg$ over the conjugacy class of $\gamma$:

$$O_\gamma \left( f dg, \frac{dg}{dg_\gamma} \right) := \int_{G_\gamma \backslash G} f(g^{-1} \gamma g) \frac{dg}{dg_\gamma}.$$  

Both the orbital integral and the volume of the centralizer $\text{vol}(\Gamma \backslash G_\gamma, dg_\gamma)$ depend on the choice of $dg_\gamma$, but their product does not.

**Proof.** This was proven by Selberg. He computes the trace of the convolution operator $R(f dg)$, where $R$ denotes the regular representation of $G$ on $L^2(\Gamma \backslash G)$ in two different ways. Equality in the trace formula reflects the fact that the trace of this linear operator can be expressed both as the “sum of its eigenvalues” (the spectral side) and as the “sum of the diagonal matrix entries” (the geometric side). □

### 2.2. Notation for $G = \text{PGL}_2(\mathbb{C})$.

Let $B$ denote the upper triangular matrices of $G = \text{PGL}_2(\mathbb{C})$. Let $K = \text{PU}_2(\mathbb{C})$ denote a maximal compact subgroup (the stabilizer of $eK$ in the action of $G$ on $X = G/K$). Let $A$ be the subgroup of diagonal matrices with real entries, both entries having the same sign. Let $U$ be the maximal compact subgroup of $T$ (diagonal unitary matrices). Let $N$ be the subgroup of unipotent upper triangular matrices. Let $W$ be the Weyl group of $T$.

For $S \subset G$ consisting of semisimple elements, let $S_{\text{reg}}$ denote those elements of $s$ which are regular, i.e. for which the centralizer of $S$ is a maximal torus.

#### 2.2.1. Haar measures.

For the rest of §2, we will make the following choice of Haar measures:

- $dk$ denotes the volume 1 Haar measure on $K$.
- $da = du$, where $A = \left\{ \begin{pmatrix} e^u & 0 \\ 0 & 1 \end{pmatrix} \right\}$ and $du$ is standard Lebesgue measure on $\mathbb{R}$.
- $dt = \frac{1}{2\pi} d\theta du$, where $T = \left\{ \begin{pmatrix} e^{u+i\theta} & 0 \\ 0 & 1 \end{pmatrix} \right\}$ and $d\theta$ and $du$ are standard Lebesgue measure on $\mathbb{R}/\mathbb{Z}$ and $\mathbb{R}$.
- $dn$ is the standard Euclidean measure $dxdy$ on $N = \left\{ \begin{pmatrix} 1 & x + iy \\ 0 & 1 \end{pmatrix} \right\}$.
- $dg$ is the Haar measure $da \ dn \ dk$.
- Let $\delta(b) := |\text{det}(\text{Ad}(b))|$, the modulus character of $B$.

### 2.3. The geometric side of the trace formula in geometric terms.

Because $M = \Gamma \backslash \mathbb{H}^3$ is compact, every $1 \neq \gamma \in \Gamma$ is regular hyperbolic: $h^{-1} \gamma h = t_\gamma \in T_{\text{reg}}$. This choice of $h$ is unique up to right multiplication by $N(T)$. Let $dg_\gamma$ on the centralizer $G_\gamma$

$$dg_\gamma = (\text{conjugation by } h)_* dt$$

Because the Haar measure $dt$ is invariant under $N(T)$, the above specification of $dg_\gamma$ is well-defined. In particular,

$$O_\gamma \left( f \frac{dg}{dg_\gamma} \right) = O_{t_\gamma} \left( f \frac{dg}{dt} \right).$$
2.3.1. Regular orbital integrals. Let \( t \in T_{\text{reg}} \) be regular. Let \( f \) be a smooth, compactly supported function on \( G \). The following computation of the orbital integrals of \( f \) is classical (due to Harish-Chandra):

\[
O_t \left( \frac{df}{dt} \right) = \int_N \int_K f(k^{-1}n^{-1}tnk)dnk
\]

The Jacobian of the change of variables \( t^{-1}n^{-1}tn \leftrightarrow n \) is the constant \( \delta(t)^{-1/2}|D(t^{-1})|^{1/2} \), where \( D(t) \) is the Weyl discriminant \( D(t) := \det(1 - \text{Ad}(t)g) \). Thus,

\[
O_t \left( \frac{df}{dt} \right) = |D(t^{-1})|^{-1/2} \delta(t)^{1/2} \int_N \int_K f(k^{-1}tnk)dnk
\]

where

\[
Sf(t) := \delta(t)^{1/2} \int_N \int_K f(t^{-1}tnk)dnk
\]

is the Satake-Harish-Chandra transform of \( f \).

Remark 3. A priori defined only on \( T_{\text{reg}} \), the function \( Sf(t) \) evidently extends to a smooth, compactly supported, \( W \)-invariant function on \( T \).

2.3.2. The Weyl discriminant. Suppose \( t = \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} \in \text{PGL}_2(\mathbb{C}) \). We readily calculate that

\[
D(t) = (1 - z)^2(1 - z^{-1})^2.
\]

2.3.3. Volumes of centralizers of regular elements. Let \( 1 \neq \gamma \in \Gamma \). The centralizer \( \Gamma_{\gamma} \) equals \( \langle \gamma_0 \rangle \), where \( \gamma_0 \in \Gamma \) is primitive. Therefore,

\[
\text{vol}(\Gamma_\gamma \backslash G_\gamma, dg_\gamma) = \text{vol}(\langle \gamma_0 \rangle \backslash G_\gamma, dg_\gamma) = \text{vol}(\langle \gamma_0 \rangle \backslash T, dt).
\]

Suppose

\[
t_{\gamma_0} = \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} \in \text{PGL}_2(\mathbb{C}).
\]

With respect to our chosen Haar measures,

\[
\text{vol}(\langle t_{\gamma_0} \rangle \backslash T, dt) = |z| = \ell(\gamma_0),
\]

where \( \ell(\gamma_0) \) denotes the translation length of \( \gamma_0 \), or equivalently the hyperbolic length of the closed geodesic in \( M = \Gamma \backslash \mathbb{H}^3 \) corresponding to (the conjugacy class of) \( \gamma_0 \).
2.3.4. The identity contribution to the trace formula. With respect to our chosen Haar measure \( dg = da \, dn \, dk \),
\[
\text{vol}(\Gamma \backslash G, dg) = \text{vol}(M).
\]
Thus, the contribution of the identity term to the trace formula equals
\[
\text{vol}(M) \cdot f(1).
\]

The next proposition expresses \( f(1) \) in terms of the Satake-Harish-Chandra transform \( S f(t) \).

**Proposition 2.** There is an equality
\[
f(1) = -c \left( \frac{d^2}{du^2} + \frac{d^2}{d\theta^2} \right) S f|_{t=1}
\]
for some constant \( c > 0 \).

**Proof.** We refer the reader to [27, XI, Lemma 11.1]. \( \square \)

**Remark 4.** Proposition 2 is equivalent to the statement that the Plancherel density on \( \{ \text{tempered } \pi_{it,n} : n \in \mathbb{Z}, t \in \mathbb{R} \} \) is proportional to \( (t^2 + m^2)dt \) on the \( (n = m) \)-component of the (tempered) unitary dual; here \( dt \) is standard Lebesgue measure on \( \mathbb{R} \). See §2.4.1 for notation and for the parametrization of the unitary dual of \( \text{PGL}_2(\mathbb{C}) \).

2.4. The spectral side of the trace formula in geometric terms.

2.4.1. Parametrizing irreducible, unitary representations of \( \text{PGL}_2(\mathbb{C}) \). Below is the complete list of irreducible, unitary representations of \( \text{PGL}_2(\mathbb{C}) \) [27, II §4]:
- The trivial representation \( 1 \).
- Let \( n \in \mathbb{Z}, s \in \mathbb{C} \). Let \( \chi : B \to \mathbb{C}^\times \) denote the character
\[
\chi_{s,n} : \begin{pmatrix} a & * \\ 0 & d \end{pmatrix} \mapsto |a/d|^s \cdot \left( \frac{a/d}{|a/d|} \right)^n =: \chi_s \begin{pmatrix} a & * \\ 0 & d \end{pmatrix} \cdot \chi_n \begin{pmatrix} a & * \\ 0 & d \end{pmatrix}.
\]

Denote by \( \pi_{s,n} \) the unitarily induced representation \( \pi_{s,n} := \text{Ind}^G_B \chi_{s,n} \). Recall that unitarily normalized induction means the ordinary induction of \( \chi \cdot \delta^{1/2} \), where \( \delta \) denotes the modulus character \( \begin{pmatrix} a & * \\ 0 & d \end{pmatrix} \mapsto |a/d|^2 \). All irreducible unitary representations of \( G \), besides the trivial representation, are of the form \( \pi_{s,n} \). However, the condition that \( \pi_{s,n} \) is unitary severely restricts the possible \( s, n \).
- If \( \chi \) is unitary, then the unitary induction \( \text{Ind}^G_B \chi \) admits a natural unitary structure. In particular, \( \pi_{s,n} \) admits a natural \( G \)-invariant inner product if \( s \in i\mathbb{R} \). The representations \( \pi_{s,n}, s \in i\mathbb{R} \), are called unitary principal series representations. They are all irreducible [27, Proposition 2.6].
- For \( s \in [-1, 1] \setminus \{0\} \), the representation \( \pi_{s,0} \) are all irreducible and admit a strange \( G \)-invariant inner product. These representations are known as complementary series representations.

Furthermore, the only coincidences among the representations \( \pi_{s,n} \) are
\[
\pi_{s,n} \cong \pi_{s,-n}.
\]
See Corollary 1 for an explanation of these coincidences.
2.4.2. Calculating $\text{trace}(\pi_{s,n}(fdg))$.

**Proposition 3.** With respect to the Haar measure $dg = da \, dn \, dk$,

$$\text{trace}(\pi_{s,n}(fdg)) = \hat{S}f(\chi_{s,n}^{-1}),$$

where $\hat{\cdot}$ denotes the Fourier transform

$$\hat{F}(\chi) = \int_T F(t)\chi^{-1}(t)\,dt.$$

**Proof.** We refer the reader to [27, X, §3].  \(\square\)

2.4.3. Contribution of the trivial representation to the spectral side.

**Proposition 4.** For the Haar measures $dt, \, dg = da \, dn \, dk$,

$$\text{trace}(1(fdg)) = \frac{1}{|W|} \int_T D(t^{-1})^{1/2} \cdot Sf(t)\,dt.$$

**Proof.** This is a direct consequence of the Weyl integration formula:

$$\text{trace}(1(fdg)) = \int_G f(g)\,dg$$

$$= \frac{1}{|W|} \left| D(t^{-1}) \right| \int_G f(g^{-1}tg)\,dg\,dt$$

$$= \frac{1}{|W|} \int_T \left| D(t^{-1}) \right|^{1/2} \cdot \left( \left| D(t^{-1}) \right|^{1/2} O_t \left( f\left( \frac{dg}{dt} \right) \right) \right) \,dt$$

$$= \frac{1}{|W|} \int_T \left| D(t^{-1}) \right|^{1/2} \cdot Sf(t)\,dt.$$  \(\square\)

2.5. Geometric formulation of the trace formula (preliminary form). Combining Proposition 3 and Proposition 4, the spectral side of the trace formula for $fdg$ equals

$$\text{(7) \ spectral side}(fdg) := \sum_{s,n} m_{\Gamma}(\pi_{s,n}) \cdot \hat{S}f(\chi_{s,n}^{-1}) + \frac{1}{|W|} \int_T \left| D(t^{-1}) \right|^{1/2} \cdot Sf(t)\,dt.$$

Combining the calculation of regular orbital integrals from §2.3.1 with Proposition 2, the geometric side of the trace formula for $fdg$ equals

$$\text{(8) \ geometric side}(fdg) := -c \left( \frac{d^2}{du^2} + \frac{d^2}{dv^2} \right) Sf|_{t=1} + \sum_{[\gamma] \neq 1} |D(t_\gamma^{-1})|^{-1/2} \cdot Sf(t_\gamma),$$

where the sum runs over all non-trivial conjugacy classes in $\Gamma$. The constant $c$ is the same as in the statement of Proposition 2.

Having expressed all terms of the trace formula, applied to $fdg$, in terms of $Sf$, it is essential to understand the image of the Satake-Harish-Chandra transform. This was answered by Bouaziz [7] for all real semisimple groups $G$. We state Bouaziz’s theorem only in the special case $G = \text{PGL}_2(\mathbb{C})$.

**Theorem 8 (Bouaziz).** Every smooth, compactly supported, $W$-invariant function on $T$ is of the form $Sf$ for some smooth, compactly supported function $f$ on $G$.  

Proof. See [7].

Corollary 1 (Coincidences among the \( \pi_{s,n} \)). The representations \( \pi_{s,n} \) and \( \pi_{s',n'} \) are isomorphic iff \( (s', n') = (s, n) \) or \( (s, -n) \).

Proof. The representations \( \pi_{s,n} \) and \( \pi_{s',n'} \) are isomorphic iff they have equal traces, i.e. \( \text{trace}(\pi_{s,n}(fdg)) = \text{trace}(\pi_{s',n'}(fdg)) \) for all smooth, compactly supported functions \( f \) on \( G \). Equivalently,

\[
Sf(\chi_{s,n}^{-1}) = \text{trace}(\pi_{s,n}(fdg)) = \text{trace}(\pi_{s',n'}(fdg)) \text{ by assumption}
\]

for all smooth compactly supported \( f \). By Theorem 8, the latter is equivalent to

\[
\hat{H}(\chi_{s,n}^{-1}) = \hat{H}(\chi_{s',n'}^{-1})
\]

for all even, compactly supported \( H \) on \( T \). This is only possible if \( (s', n') = (s, n) \) or \( (s, -n) \), hence the conclusion. □

Corollary 2 (Preliminary geometric trace formula). Let \( H \) be any smooth, compactly supported, \( W \)-invariant function on \( T \). There is an equality

\[
\sum_{s,n} m_{\Gamma}(\pi_{s,n}) \cdot \hat{F}(\chi_{s,n}^{-1}) + \frac{1}{|W|} \int_T |D(t^{-1})|^{1/2} \cdot F(t) dt
\]

\[
= -c \cdot \text{vol}(M) \cdot \left( \frac{d^2}{d\theta^2} + \frac{d^2}{d\theta^2} \right) F|_{t=1} + \sum_{|\gamma| \neq 1} \epsilon(\gamma_0) \cdot |D(t_\gamma^{-1})|^{-1/2} \cdot F(t_\gamma),
\]

where \( c \) is the constant from Proposition 2.

Proof. This follows by Bouaziz’s characterization of the image of the Satake transform because spectral side(\( fdg \)) = geometric side(\( fdg \)) for all smooth compactly supported test functions \( f \) on \( G \). □

2.6. A trace formula for eigenvalues of coexact 1-forms on \( \Gamma \backslash \mathbb{H}^3 \). Specializing the geometric trace formula of Corollary 2 appropriately, we’ll arrive at a trace formula for the (square roots of) eigenvalues of the Laplacian acting on coexact 1-forms on \( M = \Gamma \backslash \mathbb{H}^3 \).

To do so, we need to express Laplace eigenforms and their eigenvalues on \( M \) in representation theoretic terms.

2.6.1. Differential forms and representation theory. Let \( \mathbb{H}^3 = G/K \). Let \( \theta : g \rightarrow g \) be the Cartan involution fixing \( K \); for \( K = PU_2 \subset G = \text{PGL}_2(\mathbb{C}) \), the corresponding Cartan involution equals \( \theta(Z) = -\overline{Z} \).

Let \( p \) be the -1 eigenspace of \( \theta \). We can naturally identify \( p = T_{eK}(\mathbb{H}^3) \). The Killing form \( B \) induces a positive definite inner product on \( p \):

\[
\langle X, Y \rangle_0 = -B(X, Y).
\]

The inner product \( \langle \cdot, \cdot \rangle_0 \) is \( PU_2 \)-invariant and thus propagates to an invariant metric on all of \( \mathbb{H}^3 \). We call this metric \( g_{\text{Killing}} \). Note that \( \frac{1}{4} g_{\text{Killing}} \) equals the standard curvature -1 metric on \( \mathbb{H}^3 \).
Lemma 2 (Matsushima). For $p = 0, 1, 2, 3$, there is a natural identification
\[ \Omega^p(\Gamma \backslash \mathbb{H}^3) = \text{Hom}_K(\wedge^p \mathfrak{p}, C^\infty(\Gamma \backslash G)) \]
and a Hilbert space decomposition
\[ L^2\Omega^p(\Gamma \backslash \mathbb{H}^3) = \bigoplus_{\pi \in \hat{G}} m_\Gamma(\pi) \cdot \text{Hom}_K(\wedge^p \mathfrak{p}, \pi). \]

In fact, as the next Lemma shows, the decomposition from Matsushima’s Lemma 2 is a refinement of the spectral decomposition of the Laplacian on $p$-forms.

Proposition 5 (Kuga’s Lemma, [2], Lemme 1.1.1). Let $\Gamma \subset G$ be discrete, cocompact. For every irreducible representation $\pi \subset L^2(\Gamma \backslash G)$, every $\wedge^q \pi$-isotypic vector of $\pi$ corresponds to a $q$-form Laplacian eigenvector on $\Gamma \backslash X$ of eigenvalue $\lambda = -C(\pi)$, where $C(\pi)$ denotes the Casimir eigenvalue of $\pi$. Furthermore, every $q$-form Laplacian eigenvector on $\Gamma \backslash X$ of eigenvalue $\lambda$ arises in this way.

Owing to Lemma 2, to understand the (coexact) 1-form spectrum on $M = \Gamma \backslash \mathbb{H}^3$, we need to understand which $\pi_{s,n}$ admit a $\wedge^1 \pi$-isotypic vector. Owing to Lemma and Proposition 5, we need to calculate the Casimir eigenvalue of $\pi_{s,n}$.

2.6.2. $K$-isotypic vectors in $\pi_{s,n}$. We may think of vectors in $\pi_{s,n}$ as functions $j$ on $G$ satisfying $j(bg) = \delta^{1/2}(b) \chi_{s,n}(b) j(g)$. Recall our notation: $B \cap K = U = \left\{ \begin{pmatrix} e^{i\theta} & 0 \\ 0 & 1 \end{pmatrix} \right\}$. Let
\[ \chi_n : U \to \mathbb{C}^\times \]
\[ \begin{pmatrix} e^{i\theta} \\ 0 \end{pmatrix} \to e^{in\theta}, \]
so $\chi_n = \text{Res}_{BG}^L \chi_{s,n}$. There is a natural isomorphism
\[ \text{Res}^K_G \pi_{s,n} \to \text{Ind}^K_U \chi_n \]
\[ j \to j \text{ viewed as a function on } K. \]

The inverse isomorphism:
\[ \text{Ind}^K_U \chi_n \to \text{Res}^K_G \pi_{s,n} \]
\[ j \to \tilde{j}(bk) := \delta^{1/2}(b) \chi_{s,n}(b) j(k). \]

By the Peter-Weyl Theorem,
\[ \text{Ind}^K_U \chi_n = \text{Hom}_U (\chi_{-n}, L^2(K)) \]
\[ = \text{Hom}_U \left( \chi_{-n}, \bigoplus_{\rho} V_\rho \otimes V_\rho^\vee \right) \]
\[ = \bigoplus_{\rho} \text{Hom}_U (\chi_{-n}, V_\rho) \otimes V_\rho^\vee, \]
where $\rho$ runs over all irreducible representations of $K$. Thus, we identify
\[ \text{Hom}_K(\wedge^p \mathfrak{p}, \pi_{s,n}) = \text{Hom}_U(\chi_{-n}, \wedge^p \mathfrak{p}^\vee) \text{ for } p = 0, 1, 2, 3. \]

In particular,
- \( \pi_{s,0} \subset L^2(\Gamma \backslash G) \) gives rise to Laplace eigenforms on \( M = \Gamma \backslash H^3 \) of the same eigenvalue in degrees 0,1,2,3. These equal \( f, df, *f, *df \) for the Laplace eigen 0-form \( f \) corresponding to the \( \wedge^0 p \)-isotypic vector of \( \pi_{s,0} \) by Kuga’s Lemma.
- \( \pi_{s,\pm 1} \subset L^2(\Gamma \backslash G) \) gives rise to Laplace eigenforms of the same eigenvalue in degrees 1,2. These equal \( \omega, d\omega \) for the Laplace eigen 2-form \( \omega \) on \( M \) corresponding to the \( \wedge^2 p \)-isotypic vector of \( \pi_{s,1} \) by Kuga’s Lemma.
- \( \pi_{s,n} \subset L^2(\Gamma \backslash G), n \neq 0, \pm 1, \) does not contribute to the 1-form spectrum of \( M \).

Test functions \( H \) in the geometric trace formula (Corollary 2) isolating the representations \( \pi_{s,\pm 1} \) will thus isolate the coexact 1-form spectrum on \( M \).

2.6.3. The Casimir eigenvalue of \( \pi_{s,n} \). Recall that the Casimir operator \( C \in \text{Center}(U(\mathfrak{g})) \) is defined by

\[
C := \sum_i X_i X_i^\vee,
\]

where \( X_i \) runs over any basis of \( \mathfrak{g} \) and \( X_i^\vee \) runs over its dual basis with respect to the Killing form

\[
B(X, Y) := \text{trace}(\text{ad}(X) \circ \text{ad}(Y)_R).
\]

We write subscript \( \mathbb{R} \) to emphasize that we must view \( \text{ad}(X), \text{ad}(Y) \) as \( \mathbb{R} \) linear transformations of the complex vector space \( \mathfrak{g} \). A direct check then shows

\[
B(X, Y) = 2\mathbb{R} \text{trace}(\text{ad}(X)_\mathbb{C} \circ \text{ad}(Y)_\mathbb{C})
\]

where the subscript \( \mathbb{C} \) emphasizes that, in the second formula for \( B(X, Y) \), we must view \( \text{ad}(X) \) and \( \text{ad}(Y) \) as \( \mathbb{C} \)-linear transformations of the \( \mathbb{C} \)-vector space \( \mathfrak{g} \).

An \( \mathbb{R} \)-basis for \( \mathfrak{g} \) is given by

\[
H = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad iH, iE, iF.
\]

Calculating from the definition, we can write

\[
(9) \quad C = \frac{1}{4} H \cdot H - \frac{1}{4} H - \frac{1}{4} (iH) \cdot (iH) - \frac{1}{4} H + E \cdot \text{(stuff)} + (iE) \cdot \text{(stuff)},
\]

where stuff \( \in \mathfrak{g} \subset U(\mathfrak{g}) \).

The representation \( \pi_{s,n} \) is irreducible. Therefore, \( C \) acts on \( \pi_{s,n} \) by a scalar. Let \( f \in \pi_{s,n}^\infty \), i.e. a smooth function on \( G \) transforming according to the rule \( f(bg) = \delta(b)^{1/2} \chi_{s,n}(b)f(g) \) for all \( b \in B \). It suffices to evaluate \( Cf(1) \).

Note that for all \( T \in U(\mathfrak{g}) \),

\[
(ET \cdot f)(1) = E(Tf)(1) = \left. \frac{d}{dt} \right|_{t=0} (Tf)(e^{tE}) = \left. \frac{d}{dt} \right|_{t=0} (Tf)(1) = 0.
\]


Similarly, \((iE)T \cdot f(1) = 0\). By [9],

\[
(Cf)(1) = \left(\left(\frac{1}{4}H \cdot H - \frac{1}{2}H - \frac{1}{4}(iH) \cdot (iH)\right) \cdot f\right)(1)
\]

\[
= \frac{1}{4} \frac{\partial^2}{\partial u \partial t} |_{(0,0)} f(e^{iu}e^{jtH}) - \frac{1}{2} \frac{\partial}{\partial t} |_{0} f(e^{jtH}) - \frac{1}{4} \frac{\partial^2}{\partial u \partial t} |_{(0,0)} f(e^{jtH}e^{iuH})
\]

\[
= \frac{1}{4} [(s + 1)^2 - 2(s + 1) + n^2]f(1)
\]

\[
= \frac{1}{4} (s^2 + n^2 - 1)f(1).
\]

Therefore

\[
C(\pi_{s,n}) = \frac{1}{4} (s^2 + n^2 - 1).
\]

In particular, by Kuga’s Lemma Proposition[5] the \(\lambda_1\) isotypic vector in \(\pi_{s,\pm 1}\) corresponds to a coexact Laplace eigen 1-form on \(\Gamma \backslash \mathbb{H}^3\) of Laplace eigenvalue \(-\frac{1}{4}s^2\), when \(\mathbb{H}^3\) is endowed with the metric \(g_{\text{Killing}}\). In the standard curvature \(-1\) metric on \(\mathbb{H}^3\), this eigenvalue becomes \(-s^2\).

2.6.4. Specializing the geometric trace formula to isolate coexact 1-forms. Specialize the test function \(F\) from Corollary[2] equal

\[
F\left(\begin{array}{cc}
e^u & i\theta \\
0 & 1
\end{array}\right) = H(u) \cos \theta
\]

for \(H\) an even, compactly supported, \(\mathbb{R}\)-valued function on \(\mathbb{R}\). For this test function \(F\). We unravel every term in the geometric trace formula from Corollary[2] as explicitly as possible:

- Simplifying \(\hat{F}\):

\[
\hat{F}(\chi_{s,n}^{-1}) = \hat{H}(-s) \cos \theta(-n) = \begin{cases} \frac{1}{2} \hat{H}(-s) & \text{if } n = \pm 1 \\ 0 & \text{otherwise.} \end{cases}
\]

- Per the discussion from §2.4.1, the representations \(\pi_{s,\pm 1}\) are unitary iff \(s = it\) for some \(t \in \mathbb{R}\). By Corollary[1] every such representation is isomorphic to \(\pi_{it,\pm 1}\) for a unique \(t \in \mathbb{R}\).

- By Lemma[5] the calculations from §2.6.2 and the relationship between Laplacian and Casimir eigenvalues calculated in §2.6.3 it follows that

\[
m_\Gamma(\pi_{it,1}) + n_\Gamma(\pi_{-it,1}) = m_\Gamma(t^2),
\]

the dimension of the \(t^2\) Laplace eigenspace acting on coexact 1-forms on \(\Gamma \backslash \mathbb{H}^3\).

- Because \(H\) is \(\mathbb{R}\)-valued, \(\hat{F}(\chi_{it,1}) = \hat{F}(\chi_{-it,1})\). Combined with the previous point:

\[
m_\Gamma(\pi_{it,1})\hat{F}(\chi_{it,n}^{-1}) + m_\Gamma(\pi_{-it,1})\hat{F}(\chi_{-it,n}^{-1}) = \frac{1}{2} m_\Gamma(t^2) \hat{H}(t).
\]
- Let $z = e^{u+iθ}$. The contribution of the trivial representation to the geometric trace formula for the test function $F$ equals

$$
\frac{1}{|W|} \int_{T} |D(t^{-1})|^{1/2} F(t) dt
= \frac{1}{2} \cdot \frac{1}{2\pi} \int_{T} |1 - z| \cdot |1 - z^{-1}| \cdot H(u) \cdot \cos θ du dθ \quad \text{because } |W| = 2
= \frac{1}{2} \cdot \frac{1}{2\pi} \int_{T} |z| \cdot |1 - z^{-1}|^2 \cdot H(u) \cdot \cos θ du dθ
= \frac{1}{2} \cdot \frac{1}{2\pi} \int_{T} e^u \cdot (1 - e^{-u-iθ}) \cdot (1 - e^{-u+iθ}) \cdot H(u) \cdot \cos θ du dθ
= \frac{1}{2} \cdot \frac{1}{2\pi} \int_{T} (e^u + e^{-u} - 2 \cos θ) \cdot \cos θ \cdot H(u) du dθ
= -\frac{1}{2} \int H(u) du
= -\frac{1}{2} \hat{H}(0).
$$

- Suppose $t_γ = \left( z = e^{u+iθ} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right) \in T$. The regular terms on the geometric side of the geometric trace formula for the test function $F$:

$$
= \ell(γ) \cdot |D(t_γ^{-1})|^{-1/2} \cdot F(t_γ)
= \ell(γ) \cdot (|1 - z| \cdot |1 - z^{-1}|)^{-1} \cdot H(u) \cos θ
= \ell(γ) \cdot \left( |1 - e^{Cℓ(γ)}| \cdot |1 - e^{-Cℓ(γ)}| \right)^{-1} \cdot H(ℓ(γ)) \cdot \cos(\text{hol}(γ)).
$$

Recall that $ℓ(γ), \text{hol}(γ)$, and $Cℓ(γ)$ respectively denote the length, holonomy, and complex length of the conjugacy class $[γ]$. In particular $Cℓ(γ) := ℓ(γ) + i\text{hol}(γ)$.

- The identity contribution to the trace formula for the test function $F$:

$$
= -c \cdot \text{vol}(M) \cdot \left( \frac{d^2}{du^2} + \frac{d^2}{dθ^2} \right) F|_{t=1}
= -c \cdot \text{vol}(M) \cdot \left( \frac{d^2}{du^2} + \frac{d^2}{dθ^2} \right) (H(u) \cos θ)|_{(u,θ)=(0,0)}
= c \cdot \text{vol}(M) \cdot (H(0) - H''(0)).
$$

Combining everything:

**Corollary 3** (Preliminary geometric trace formula for coexact 1-forms). Let $H$ be any smooth, compactly supported, even, $\mathbb{R}$-valued function on $\mathbb{R}$. There is an equality

$$
= \sum_{λ=\text{coexact 1-form eigenvalue}} \frac{1}{2} m_{Γ}(λ) \cdot \hat{H}(\sqrt{λ}) - \frac{1}{2} \hat{H}(0)
= c \cdot \text{vol}(M) \cdot (H(0) - H''(0)) + \sum_{[γ] \neq 1} \ell(γ) \cdot \left( |1 - e^{Cℓ(γ)}| \cdot |1 - e^{-Cℓ(γ)}| \right)^{-1} \cdot H(ℓ(γ)) \cdot \cos(\text{hol}(γ)).
$$

In the above formula, $c$ is the constant from Proposition 2.
2.6.5. **Evaluating the missing constant $c$ in the geometric trace formula.** We use Weyl’s law to evaluate the missing constant $c$ in the geometric trace formula from Corollary 3.

Fix a smooth, compactly supported, real valued even test function $H$ with $H(0) \neq 0$. Let $H_\nu = H \cdot (e^{i\nu u} + e^{-i\nu u})$. Integrating spectral side of the trace formula from Corollary 3 for the test function $H_\nu$ over $\nu \in [-X, X]$ yields

$$2N_{d*,1}(X) \cdot 2\pi \cdot \frac{1}{2} H(0) + \text{lower order},$$

where $N_{d*,1}(X)$ denotes the number of coexact 1-form eigenvalues on $M = \Gamma \setminus \mathbb{H}^3$ satisfying $\sqrt{\lambda} \leq X$. By Weyl’s law [3, Corollary 2.43],

$$\int_{\nu \in [-X, X]} \text{spectral side for } H_\nu \, d\nu \sim 2N_{d*,1}(X) \cdot 2\pi \cdot \frac{1}{2} H(0) \sim 2 \cdot 2\pi \cdot \frac{1}{2} \cdot \frac{2 \cdot \text{vol}(M)}{(4\pi)^{3/2} \Gamma(3/2 + 1)} X^3.$$

On the other hand,

$$\int_{\nu \in [-X, X]} \text{geometric side for } H_\nu \, d\nu \sim c \cdot \text{vol}(M) \cdot 2 \cdot H(0) \cdot \frac{2X^3}{3}.$$

Equating the above two asymptotic expansions yields

$$c = \frac{1}{2\pi}.$$

2.6.6. **Geometric trace formula for coexact 1-forms: final form.** Having finally evaluated the missing constant $c$ in §2.6.5 we may now state a complete form of the trace formula.

**Corollary 4** (Geometric trace formula for coexact 1-forms: final form). Let $H$ be any smooth, compactly supported, even, $\mathbb{R}$-valued function on $\mathbb{R}$. There is an equality

$$\sum_{\lambda = \text{coexact 1-form eigenvalue}} \frac{1}{2m_H(\lambda)} \cdot \hat{H} \left( \sqrt{\lambda} \right) - \frac{1}{2} \hat{H}(0) = \frac{\text{vol}(M)}{2\pi} \cdot (H(0) - H''(0)) + \sum_{[\gamma] \neq 1} \ell(\gamma_0) \cdot \left( |1 - e^{\ell(\gamma)}| \cdot |1 - e^{-\ell(\gamma)}| \right)^{-1} \cdot H(\ell(\gamma)) \cdot \cos(\text{hol}(\gamma)).$$

**Proof.** This follows immediately from Corollary 3 together with our determinant $c = \frac{1}{2\pi}$ from §2.6.5.

By a limiting argument, we can prove that the trace formula is applicable for a much broader class of test functions $H$. We use this broader class of test functions extensively in §3.1.

**Theorem 9.** Let $\delta > 5/2$. Let $H$ be an even, compactly supported, $\mathbb{R}$-valued test function satisfying $\int_{\mathbb{R}} \left( \left| \hat{H}(t) \right|^2 + \left| \hat{H}'(t) \right|^2 \right) \left( \sqrt{1 + t^2} \right)^{2\delta} < \infty$. Then the trace formula from Corollary 4 is valid for $H$.

**Proof.** See §A.
Suppose we have available a trace formula which expresses a spectral sum

\[ \sum_j \hat{H}(t_j) \]

in explicitly computable terms for every nice test function \( H \). For example,

- The Selberg trace formula for the 0-form spectrum of hyperbolic surfaces has the above form. In that context, \( t_j = \sqrt{\lambda_j - \frac{1}{4}} \) for the eigenvalues of the Laplacian on the hyperbolic surface \( \Gamma \backslash \mathbb{H}^2 \), and the trace formula expresses the spectral sum (10) in terms of \( H \) sampled at lengths of closed geodesics.
- The trace formula from Corollary 4 and Theorem 9 for the coexact 1-form spectrum of hyperbolic 3-manifolds has the above form. In that context, \( t_j = \sqrt{\lambda^*_j} \) for the eigenvalues \( \lambda^*_j \) of the Laplacian acting on coexact 1-forms on the hyperbolic 3-manifold \( \Gamma \backslash \mathbb{H}^3 \), and the trace formula expresses the spectral sum (10) in terms of \( H \) sampled at the lengths of closed geodesics (weighted by their holonomy).

The following simple Lemma underlies the most effective method we know for proving that \( t \neq t_j \) for any \( j \):

**Lemma 3.** Let \( H \) be a nice test function for which the trace formula computing (10) applies. Suppose that \( \hat{H} \geq 0 \) and that

\[ \hat{H}(t) > \sum_j \hat{H}(t_j). \]

Then for every \( j, t \neq t_j \).

**Proof.** If \( t = t_j \), then \( \hat{H}(t) \) is but one summand in the full spectral sum. Because \( \hat{H} \geq 0 \), it must be less than the full spectral sum. \( \square \)

Call a test function \( H \) admissible if \( \hat{H} \geq 0 \) and if the trace formula computing (10) is valid for the test function \( H \). Define

\[ I_{R,t} := \inf_{\hat{H} \text{ admissible}, \hat{H}(t) = 1, \text{supp}(H) \subset [-R,R]} \sum_j \hat{H}(t_j). \]

If \( I_{R,t} < 1 \), then \( t \neq t_j \) for every \( j \); a test function which nearly realizes the infimal value \( I_{R,t} \) is a witness to that fact that \( t \) is not among the \( t_j \) by Lemma 3.

### 3.1. Excluding eigenvalues: the method of Booker and Strombergsson.

While the proofs of the two main results of the paper, Theorems 1 and 2, both involve proving restrictions on the value of \( \sqrt{\lambda^*_1(M)} \), their nature is rather different.

- In Theorem 2 the entire interest lies in finding a narrow window in \([0, \sqrt{2}]\) in which \( \sqrt{\lambda^*_1(M)} \) certifiably lies.
- In Theorem 1 we need only show that \( \sqrt{\lambda^*_1(M)} \notin [0, \sqrt{2}] \). To demonstrate the latter, there is no specific need to find narrow windows in \((\sqrt{2}, \infty)\) in which \( \lambda^*_1(M) \) certifiably lies. However, localizing the value of \( \sqrt{\lambda^*_1(M)} \) gives independently interesting information about \( M \).
Both problems can be attacked with the method of Booker and Strombergsson [6]. But for completeness, we next describe a cruder approach yielding examples for Theorem 1, i.e. $M$ for which $\sqrt{\lambda_1}(M) > \sqrt{2}$.

To find examples for Theorem 1, it is natural to apply the trace formula to admissible test functions $H_0$ for which $\widehat{H}_0$ looks like $1_{[-\sqrt{2},\sqrt{2}]}$ (or any $H_0$ for which $\widehat{H}_0$ is large on $[-\sqrt{2},\sqrt{2}]$ and decays quickly to 0); such $H_0$ might allow us to use Lemma 3 effectively. Regarding the evaluation of $\sum_{j} \widehat{H}_0(t_j)$ via the trace formula, recall that the prime geodesic theorem for closed, hyperbolic 3-manifold $M$ states:

$$\# \{\text{primitive closed geodesics on } M \text{ of length } \leq R \} \sim \frac{e^{2R}}{2R}.$$ 

To evaluate the test function $H_0$ on that many (complex) lengths and sum them is exponentially difficult in $R$. For this reason, it is only possible, in practice, to evaluate the spectral side of the trace formula, via the geometric trace formula from Corollary 4 and Theorem 9 for admissible function $H_0$ supported on $[-R_0, R_0]$ for some relatively small $R_0$. See §4 for discussion of practical choices for $R_0$. Of course, by the uncertainty principle, restricting the support of $H_0$ makes it difficult to localize $\widehat{H}_0$.

We applied the above approach with $H_0(x) = \beta * \beta(5x/2)$ where $\beta(x) = e^{-1/(1-x^2)}$ is a cutoff function. The function $H_0$ is supported in $[-5, 5]$, and we accordingly sampled the geodesics in that range. Using the above, we were able to show that Theorem 1 holds for 19 of the 23 manifolds in Table 1, in particular if $\sum_j \widehat{H}_0(t_j) < 0.01$, then $\lambda_1^* > 2$. Furthermore, because smallness of $\sum_j \widehat{H}_0(t_j)$ correlates strongly with largeness of $\sqrt{\lambda_1^*}$, the size of the latter spectral sum provides heuristic information about the distribution of $\sqrt{\lambda_1^*}$ in our sample of census manifolds; we refer the reader to §5 and in particular Figure 5 for a more detailed discussion.

We emphasize, however, that $I_{R_0,t}$ provides more specific and interesting information about the location of the $t_j$:

- $t = t_j$ implies that $I_{R_0,t} \geq 1$
- The pointwise limit of $I_{R,t}$ is the characteristic function of the $t_j$, i.e. 

$$\lim_{R \to \infty} I_{R,t} = \begin{cases} m_\Gamma(t_j^*) & \text{if } t = t_j \\ 0 & \text{otherwise.} \end{cases}$$ 

In particular, one might hope that if “$M$ is small relative to $R = R_0$,” e.g. if $\text{inj}(M)$ is significantly less than $\frac{1}{2}R$, the function $t \mapsto I_{R_0,t}$ approximates the characteristic function of $\{t_1, t_2, \ldots\}$ (with multiplicities). Furthermore, $t \mapsto I_{R_0,t}$ potentially does better at excluding eigenvalues, via Lemma 3 than any fixed admissible function $H_0$ supported on $[-R_0, R_0]$ because

$$I_{R_0,t} \leq \frac{\sum \widehat{H}_0(t_j)}{\widehat{H}_0(t)}.$$ 

We do not know how to compute the function $t \mapsto I_{R_0,t}$ for any $R_0$ on any hyperbolic 3-manifold $M$. However, the method of Booker and Strombergsson [6] finds an upper bound $J_{R_0,t} \geq I_{R_0,t}$ which is explicitly computable via the trace formula. They applied their method to exclude eigenvalues on (congruence arithmetic) hyperbolic surfaces less than $\frac{1}{2}$, but their
method is equally applicable whenever a trace formula is available in the sense of (10). Their method runs as follows:

- Let $h_0, \ldots, h_n$ be even, $\mathbb{R}$-valued functions on $\mathbb{R}$ supported in $[-R_0, R_0]$ for which $S := \{ h \ast h : h = \sum_i x_i h_i \}$ consists entirely of admissible functions for the trace formula (10). Define

$$J_{R_0,t} := \inf_{H = h \ast h \in S, \mathcal{H}(t) = 1} \sum_j \hat{H}(t_j)$$

$$= \inf_{\sum x_i \hat{h}_i(t) = 1} \sum_j \left( \sum_{i=0}^n x_i \hat{h}_i(t) \right)^2$$

$$= \inf_{\sum x_i \hat{h}_i(t) = 1} \sum_{a,b=0}^n x_a x_b \sum_j \hat{h}_a(t_j) \hat{h}_b(t_j)$$

$$= \inf_{\langle ct, x \rangle = 1} \langle Ax, x \rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes the standard dot product on $\mathbb{R}^n$, $A$ is the matrix with entries

$$A_{a,b} = \begin{cases} 2 \sum_j \hat{h}_a \ast \hat{h}_b(t_j) & \text{if } a \neq b \\ \sum_j \hat{h}_a \ast \hat{h}_a(t_j) & \text{if } a = b \end{cases},$$

and $c_t = \left( \begin{array}{c} \hat{h}_1(t) \\ \vdots \\ \hat{h}_n(t) \end{array} \right)$.

- Clearly, $I_{R_0,t} \leq J_{R_0,t}$ because $I$ is the infimum of the same quantity over a larger space of functions.

- $J_{R_0,t}$ is explicitly computable. It is the minimum of a (positive definite) quadratic form on $\mathbb{R}^n$ subject to a linear constraint. We calculate by Lagrange multipliers:

$$J_{R_0,t} = \frac{1}{\langle A^{-1} c_t, c_t \rangle}.$$

The matrix $A$ is explicitly computable using the trace formula (10).

4. Computations

4.1. Main computation. We discuss the proof of Theorem 1, our main result. We restricted our investigation to manifolds $M$ from the Hodgson-Weeks census with labels from 0, . . . , 49. The Hodgson-Weeks census consists of 11,031 closed, orientable hyperbolic 3-manifolds which is a good approximation to the (finite) set of hyperbolic three-manifolds with volume at most 6.5 and injectivity radius at least 0.15. Volume increases with the census label, and the first manifold in the list (the Weeks manifold) is known to be the compact hyperbolic three-manifold with the least volume. Census manifolds include many of the least complex closed, hyperbolic 3-manifolds, and those census manifolds with labels 0, . . . , 49 are among the least complex of them. For reasons we will explain in §5, the smallest coexact 1-form eigenvalue $\lambda_1^*(M)$ tends to be small for complex $M$. So, we limited our search for $\lambda_1^*(M) > 2$.
to the simplest $M$ we could find. Our computations make essential use of the 3-manifold software SnapPy developed by Culler, Dunfield, and Goesner.

We applied the method of §3.1 to the trace formula from Corollary 4 (more precisely: the slightly broader version from Theorem 9). More specifically, we used the same shape of test functions as in [6] with the following parameters (notation from §3.1):

- $h := \left( \frac{1}{2\pi} \chi_{[-\delta,\delta]} \right)^2$.
- $h_k(x) := \frac{1}{2} (h(x - k\delta) + h(x + k\delta))$. For these choices,
  \[ \sum x_k h_k(t) = \left( \frac{\sin(\delta t)}{\delta t} \right)^2 \sum x_k \cos(k\delta t). \]

- $\delta$ satisfying $\delta \cdot (2n + 4) \leq R$.
- $n = 19$.

It is straightforward to check that the functions $h_a \ast h_b$ satisfy the smoothness hypothesis of Theorem 9 and hence are admissible for the trace formula therein. The test function $h_{x_0,\ldots,x_n} = (\sum x_k h_k)^2$ is supported on $[-(2n + 4)\delta, (2n + 4)\delta]$. Hence, the constraint $\delta \cdot (2n + 4) \leq R$ guarantees that every $h_{x_0,\ldots,x_n}$ is supported on $[-R, R]$.

For the closed, hyperbolic 3-manifold $M$, the only way we know to compute the matrix $A$ from §3.1 is to compute $\text{vol}(M)$ and the full complex length spectrum of $M$ up to real length $R$ and sample these complex lengths via the test functions $h_a \ast h_b$, $0 \leq a, b \leq n + 1$, per the geometric side of the trace formula from Theorem 9 to recover the spectral side. Conveniently, SnapPy has built-in functions in the main class Manifold to compute the volume and the complex length spectrum up to a specified real length cutoff!

4.1.1. Choosing $R$. Heuristically, we expect

\[ H_1 := \hat{h} \left( = \sum x_k h_k \right) \text{ minimizing } \sum \widetilde{h} \ast \hat{h}(t_n) \text{ subject to } \hat{h}(t) = 1 \]
\[ \approx \inf \text{ admissible } H, H(t) = 1, \supp(H) \subseteq [-R, R] \sum \tilde{H}(t_n) \]
\[ =: J_{R,t}. \]

Evidently, $J_{R,t}$ decreases with $R$. So in principle, one would obtain the most useful information by taking $R$ as large as possible. However, enumerating the complex length spectrum up to real length $R$ is prohibitively computationally intensive even for moderately large $R$. Indeed, it is known that the number of primitive geodesics of length at most $R$ is approximately $e^{2R}/2R$ [11]; in practice, the time needed to compute the spectrum seems to be around $Ce^{6R}$ (see Table 3). For practical purposes, $R = 6.5$ seemed to be a reasonable cutoff. For most of the manifolds we tested, this computation took between 20 and 30 minutes (even though in some special cases, including those of Table 3, it took much longer), and we expect the computation for $R = 7$ to typically take about a couple of hours. Of course, this time constraint limits the applicability of our method.

4.1.2. Choosing $n$. For the particular test functions $h_a \ast h_b$ that we use, note that

\[ (h_a \ast h_b)(x) = \frac{1}{4} \sum_{\lambda, \mu \in \{\pm 1\}} (h \ast h)(x + (\lambda a + \mu b)\delta). \]
Table 3. The time (in seconds) needed to compute the spectrum at cutoff $R$ for the manifolds Census 0 and 1 (on an 3.1 GHz Intel Core i5).

| Cutoff | Census 0 | Census 1 |
|--------|----------|----------|
| 4.0    | 0.07     | 0.06     |
| 4.5    | 0.45     | 0.32     |
| 5.0    | 4.66     | 3.40     |
| 5.5    | 86.79    | 62.32    |
| 6.0    | 1290.23  | 1127.64  |

To compute $A$, we calculated the geometric side of the trace formula for the $\sim$ constant $\cdot n$ different test functions $h_n h_k(x + k\delta)$, $k = -2n, \ldots, 2n$. Computing the matrix $A^{-1}$ then requires inverting the $(n + 1) \times (n + 1)$-matrix $A$.

To balance information gained with computational complexity, the specific choice $n = 19$ suited our purposes.

4.2. Results of main computation. We computed function $J_{R_0,t}$, described in general terms in §4.1 relative to the functions $h_n, h_k$, described in §4.1, and the parameters $(R_0, n, \delta) = (6.5, 19, \frac{6.5}{219 + 1})$ explained in §4.1.1, 4.1.2. Recall that if $t^2$ is an eigenvalue of the Laplacian acting on coexact 1-forms on $M$, then $J_{R_0,t}(M) \geq 1$.

4.2.1. Examples of hyperbolic, minimal $L$-spaces. In Table 4, we record

$$\text{PossibleSmallSpectrum}(M) := \{ t \in [0, 4] : J_{R_0,t}(M) \geq 1 \}$$

for several small census manifolds.

For all $M$ listed in Table 4, PossibleSmallSpectrum($M$) is disjoint from $[0, \sqrt{2}]$. If $\sqrt{\lambda_1^R(M)}$ lies in $[0, \sqrt{2}]$ at all, then it must lie in PossibleSmallSpectrum($M$). Because PossibleSmallSpectrum($M$) is disjoint from $[0, \sqrt{2}]$ for all $M$ tabulated above, it follows that $\sqrt{\lambda_1^R(M)} > \sqrt{2}$ for all $M$ from Table 4.

Remark 5. For every entry in the above table, it is in principle possible that $\sqrt{\lambda_1^R} \notin \text{PossibleSmallSpectrum}(M)$ which would mean that $\sqrt{\lambda_1^R} > 4$. To convince ourselves this is not so:

- We applied the trace formula to test functions of the shape $H_a = \left( \frac{d^2}{dx^2} + a^2 \right) e^{-x^2/2}$ for various $0 < a < 4$. The Fourier transform $\hat{H}$ is a constant multiple of $(-t^2 + a^2) e^{-t^2/2}$ and hence is positive if $|t| \leq a$ and negative otherwise. In particular, if $\sum \hat{H}_a(t_n) > 0$, then $\sqrt{\lambda_1^R} \leq a$.

For various $a = a(M)$, chosen near troughs of the graph of $J_{R_0,\cdot}(M)$, the approximate value of $\sum_n \hat{H}_a(t_n)$, as computed via the trace formula from Theorem 9 truncated at $R_0 = 6.5$, was “quite positive.” One could estimate the size of the tail (beyond our cutoff $R_0 = 6.5$) to rigorously prove positivity, but we will not attempt to do so here.

4.2.2. Narrow $\lambda_1^R$-intervals for non-$L$-spaces. In Table 5, we record the same information as in Table 4, for some small census manifolds previously proven to be non-$L$-spaces. In particular, Dunfield has determined exactly which manifolds in the Hodgson-Weeks census are $L$-spaces in the setting of Heegaard Floer homology; in his approach, many of the spaces in the census are shown to be $L$-spaces via surgery exact triangles, using the fact that they are obtained by
Dehn filling on cusped manifolds which admit lens space fillings. More in general, all spaces in the census arise as branched double covers of links in $S^3$, hence their Floer homology can be computed using softwares developed in the setting of bordered Heegaard Floer homology [45]. Via the isomorphism proved in [28], [9], and the subsequent papers, this also provides a list of which manifolds in the Hodgson-Weeks census are $L$-spaces. The Seiberg-Witten equations admit irreducible solutions, Theorem 3 implies that $\sqrt{\lambda^+(M)} \leq \sqrt{2}$ for every non-$L$-space $M$. Thus,

$$\sqrt{\lambda^+(M)} \in [0, \sqrt{2}] \cap \text{PossibleSmallSpectrum}(M)$$

for every non-$L$-space $M$. 

| Census label | Volume         | Injectivity radius | Possible Small Spectrum(M) |
|--------------|---------------|-------------------|---------------------------|
| 0            | 0.94270 ...  | 0.29230 ...       | [2.962, 3.124]            |
| 2            | 1.01494 ...  | 0.41572 ...       | [3.086, 3.302] ∪ [3.977, 4] |
| 3            | 1.26371 ...  | 0.28753 ...       | [2.145, 2.222] ∪ [3.617, 4] |
| 8            | 1.42361 ...  | 0.17618 ...       | [2.031, 2.263] ∪ [3.324, 4] |
| 12           | 1.54356 ...  | 0.16768 ...       | [1.658, 1.686] ∪ [2.478, 2.778] ∪ [3.720, 4] |
| 13           | 1.54356 ...  | 0.28903 ...       | [1.520, 1.672] ∪ [2.108, 2.213] ∪ [3.140, 4] |
| 14           | 1.58316 ...  | 0.27889 ...       | [2.018, 4]               |
| 15           | 1.58316 ...  | 0.38874 ...       | [2.396, 2.595] ∪ [3.248, 4] |
| 16           | 1.58864 ...  | 0.26727 ...       | [1.809, 1.847] ∪ [2.519, 3.013] ∪ [3.221, 4] |
| 22           | 1.83193 ...  | 0.26532 ...       | [1.680, 1.721] ∪ [2.48, 4] |
| 25           | 1.83193 ...  | 0.26531 ...       | [2.323, 2.597] ∪ [3.283, 4] |
| 28           | 1.88541 ...  | 0.29230 ...       | [1.659, 1.689] ∪ [2.543, 4] |
| 29           | 1.88541 ...  | 0.19853 ...       | [1.540, 1.934] ∪ [2.247, 3.554] ∪ [3.951, 4] |
| 30           | 1.88541 ...  | 0.19853 ...       | [1.541, 1.704] ∪ [2.156, 4] |
| 31           | 1.88541 ...  | 0.29230 ...       | [2.172, 3.015] ∪ [3.864, 4] |
| 32           | 1.88591 ...  | 0.20593 ...       | [1.740, 1.794] ∪ [2.491, 4] |
| 33           | 1.91048 ...  | 0.22107 ...       | [1.710, 1.799] ∪ [2.214, 2.731] ∪ [3.012, 4] |
| 39           | 1.96273 ...  | 0.21576 ...       | [2.108, 2.780] ∪ [3.061, 4] |
| 40           | 1.96274 ...  | 0.28904 ...       | [1.842, 1.855] ∪ [2.829, 3.365] ∪ [3.634, 4] |
| 42           | 2.02395 ...  | 0.17922 ...       | [1.779, 4]               |
| 44           | 2.02988 ...  | 0.43127 ...       | [2.717, 4]               |
| 46           | 2.02988 ...  | 0.27177 ...       | [1.992, 4]               |
| 49           | 2.02988 ...  | 0.21564 ...       | [1.681, 1.894] ∪ [2.681, 4] |

Table 4. The hyperbolic manifolds of Theorem 1.
In particular, per Table 5,

\[
\sqrt{\lambda_1^*(\text{Census}_1)} \in [0.580, 0.583]
\]

\[
\sqrt{\lambda_1^*(\text{Census}_4)} \in [0.784, 0.804]
\]

\[
\sqrt{\lambda_1^*(\text{Census}_6)} \in [0.765, 0.776]
\]

\[
\sqrt{\lambda_1^*(\text{Census}_7)} \in [0.528, 0.532]
\]

\]

and so on.

4.3. Pictures bounding PossibleSmallSpectrum(M). Recall that \( J_{R_0,t}(M) \) is designed to approximate

\[
\lim_{R \to \infty} I_{R,t} = \begin{cases} 
    m t \left( \frac{t^2}{J} \right) & \text{if } t = t_j \\
    0 & \text{otherwise.}
\end{cases}
\]

(see §3.1 for further discussion). This bears out in pictures. We include pictures of the graphs of \( t \mapsto J_{R_0,t}(\text{Census}_i) \) for \( i = 0, 1, 2 \).

In all three pictures, we expect the first peak of the graph to occur near

\[
(\sqrt{\lambda_1^*}, \text{multiplicity of the } \lambda_1^*\text{-eigenspace for coexact 1-forms}).
\]

Indeed, the fact that the vertical coordinate just barely exceeds a positive integer is a non-trivial check on our computations. To compute the intervals from PossibleSmallSpectrum(M), we solved for \( J_t(M) = 1 \) (up to tolerance \( 10^{-6} \)) via bisection.

When the graph of \( J_t(M) \) is peaked just barely above vertical coordinate \( m \) for some integer \( m > 1 \), the eigenvalue windows are likely much narrower than we claim. For example, note that

\[
\{ t \in [0, 4] : J_t(\text{Census}_0) \geq 3 \} \subset [3.036, 3.040].
\]
Figure 2. The graph of $t \mapsto J_{R_0,t}(\text{Census}_0)$ for $t \in [0, 4]$.

So if the $\lambda_1^+$-eigenspace for Census 0 really is 3-dimensional, as Figure 2 suggests, then $\sqrt{\lambda_1^+}(\text{Census}_0) \in [3.036, 3.040]$. Likewise, if the $\lambda_1^+$-eigenspace for Census 2 really is 4-dimensional as Figure 4 suggests, then $\sqrt{\lambda_1^+}(\text{Census}_2) \subset [3.177, 3.183]$.

Remark 6. The trace formula is unable to distinguish between two parameters $t_n, t_{n+1}$ which are very close versus equal on the nose. We do not know, in general or even in the particular examples of Census 0 and Census 2, how to compute the multiplicity of an eigenvalue having multiplicity greater than 1. On the other hand, as Census 2 admits an orientation reversing isometry $[23]$, the multiplicities of the eigenvalues are all even; in particular, Figure 4 suggests that the first eigenvalue has multiplicity 4.

5. Limitations and further directions

Even though our results can be seen a first step toward understanding the relation between Floer homology and hyperbolic geometry in dimension three, our approach has some significant limitations; we now discuss these and also some natural questions and problems these lead to.
While our test was successful when studying small manifolds in the census, it can be seen that with few exceptions (e.g. Census 52, 58 and 97) all the manifolds with label bigger than 50 have $\lambda^* \leq 2$. This should be compared with the computations of Dunfield [11], which imply that a big proportion of manifolds in the census are $L$-space. This observation leads the obvious question of whether there are infinitely many manifolds with $\lambda^*_1 > 2$, or the following more general question:

**Question 1.** Fix $\epsilon > 0$. Does the set

$$S_\epsilon = \{ \text{closed, hyperbolic } M : H^*(M, \mathbb{Q}) = H^*(S^3, \mathbb{Q}) \text{ and } \lambda^*_1 > \epsilon \}$$

have any discernable structure? In particular, is $S_\epsilon$ always a finite set?

While we do not have a completely satisfactory answer to the above question, there are some clear restrictions on the local geometry of the elements in $S_\epsilon$. The discussion which follows is inspired by the work of McGowan [36] (which in fact provides more refined estimates regarding the number of small eigenvalues, provided upper bounds on the volume).

Recall that a hyperbolic tube $T$ with complex length $\ell e^{i\theta_0}$ is obtained by quotienting the cylinder

$$\{(r, t, \theta) | 0 \leq r \leq R, 0 \leq T \leq \ell, \theta \in S^1\}$$
Figure 4. The graph of $t \mapsto J_{R_0,t}(\text{Census}_2)$ for $t \in [0, 4]$.

equipped with the hyperbolic metric
\[ dr^2 + \cosh^2 r dt^2 + \sinh^2 r d\vartheta^2. \]
via the identification
\[(r, 0, \vartheta) \sim (r, \ell, \vartheta + \vartheta_0). \]
We refer to $R$ as the radius of the tube. The subset $r = 0$ is a geodesic called the core geodesic. Consider now on a tube $T$ of radius $R$ a 1-form of type $\alpha = f(r) dt$. A form of this kind is always coclosed. Furthermore, we have
\[ d\alpha = f'(r) dr \wedge dt. \]
Now, $|dr \wedge dt| = 1/\cosh(r)$. Choosing $f$ to be a standard pyramid shaped function on $[0, R]$, we see that the Rayleigh quotient of $\alpha$ is approximatively
\[ \frac{\int_T |d\alpha|^2}{\int_T |\alpha|^2} \approx \frac{\int_0^R |f'|^2 dr}{\int_0^R |f|^2 dr}, \]
which converges to zero for $R$ going to infinity. Hence, give $\epsilon > 0$, there is a universal upper bound of the diameter of a tube $T \subset Y$ for a hyperbolic rational homology sphere with $\lambda_1^* > \epsilon$. Using this, we have the following.
Proposition 6. Let \( Y \) be a hyperbolic 3-manifold. There exists \( R, \delta > 0 \) satisfying
- if \( Y \) contains an embedded ball of radius \( R \geq R_0 \), then \( \lambda_1^*(Y) \leq \epsilon \).
- if \( \text{inj}(Y) < \delta \), then \( \lambda_1^*(Y) \leq \epsilon \).

In particular, \( S_\epsilon \) is contained in the set of all \( M \) for which the local injectivity radius function has range contained in \([\delta, R]\).

Proof. For the lower bound, we invoke [17, Theorem 3.2] which says: if there is an embedded geodesic \( \gamma \) of real length \( \ell \), then \( \gamma \) is the core of an embedded tube with radius \( r(\ell) \) with \( r(\ell) \to \infty \) as \( \ell \to 0 \). Because \( \lambda_1^* > \epsilon \) imposes a universal upper bound on the diameter of an embedded tube, the latter implies that \( \ell \) is bounded below. Thus, the injectivity radius, which equals half the length of the shortest closed geodesic ([35], Proposition 4.3.2), must be bounded below too.

To see that there is an upper bound on the local injectivity radius, parametrize the hyperbolic ball of radius \( R \) as \((0, R) \times S^2\) equipped with the metric \( dr^2 + \sinh^2(r)g_{S^2} \), where \( g_{S^2} \) is the metric on the unit sphere in \( \mathbb{R}^3 \). Consider then for a fixed non-zero closed 1-form \( \beta \) on \( S^2 \) the forms of the type \( g_{S^2} \beta \). This are always coclosed, and a computation analogous to (11) shows that its Rayleigh quotient only depends on the Rayleigh quotient of \( g \). In particular, when \( R \) goes to infinity, this can be made to go to zero. \( \square \)

Corollary 5. For every \( \epsilon, V > 0 \) there exists only finitely many hyperbolic three-manifolds with \( \lambda_1^* > \epsilon \) and \( \text{vol} < V \).

Proof. This follows directly from the previous proposition combined with the fact that there are only finitely many manifolds with volume bounded above and injectivity radius bounded below [18]. \( \square \)

One is then lead to ask where do the limitations of our approach stem from. Aside from the applicability of the Booker method to provide effective computations of \( \lambda_1^* \), the main problem is that the bound we are using, i.e. \( \lambda_1^* \leq 2 \) when the Seiberg-Witten equations admit irreducible solutions, is rather crude. In particular:

- it does not use the hyperbolic metric in an essential way. In fact, Theorem [6] shows that \( \lambda_1^*(M) \leq 2 \) provided \( M \) is a Riemannian 3-manifold for which the Seiberg-Witten equations on \( M \) admit irreducible solutions and that \( s(M) = -4 \);
- more importantly, in the proof of Theorem [6] we use the estimate \( ||d\xi||_{L_2}^2 \geq \lambda_1^*||\xi||_{L_2}^2 \). While this holds for any coclosed 1-form \( \xi \) on \( Y \), one could expect that a sharper estimate holds when \( \xi \) arises from a solution to the Seiberg-Witten equations.

For example, we just saw that the smallness of \( \lambda_1^* \) for manifolds with large embedded balls or short geodesics is caused by 1-forms of a special kind; it would be interesting to understand if forms of small Rayleigh quotient on a tube or a ball can arise from the solutions to the Seiberg-Witten equations. More generally, we have the following.

Question 2. Suppose \( M \) is a closed, hyperbolic rational homology sphere. Can one improve upon the upper bound \( \lambda_1^* \leq 2 \), which holds for all Riemannian 3-manifolds \( M \) satisfying \( s(M) = -4 \), using explicit and computable geometric data arising from the hyperbolic geometry of \( M \) (e.g. the injectivity radius)?

In fact, even though our methods are conclusive only in some examples, there seems to be an intriguing correlation between the size of \( \lambda_1^* \) and the property of being \( L \)-spaces (see Figure [5]). A better understanding of this experimental observation could lead to interesting
We have plotted, among the first 100 manifolds in the Hodgson-Weeks census, the $L$-spaces in blue and the non $L$-spaces in red. The $y$-axis records the volume, while the $x$-axis records the value of the spectral sum $\sum H(t_j)$ obtained by using $H_0(x) = \beta * \beta(5x/2)$ where $\beta(x) = e^{-1/(1-t^2)}$ is a cutoff function (see the discussion of the naive attempt in §3.1). The function $H_0$ is supported in $[-5,5]$, and we accordingly need as input the length spectrum with cutoff $R = 5$. Heuristically, the graph should be interpreted as follows: a low value of the spectral sum suggests a big value for $\lambda_{1}^{*}$; in particular, the manifolds with spectral sum $< 0.01$ have $t_1 > \sqrt{2}$.

geometric characterizations of hyperbolic $L$-spaces in terms of explicit quantities of interest in hyperbolic geometry.

We conclude with a final question.

**Question 3.** Is there an $L$-space $Y$ which is not a minimal $L$-space? In other words, is there an $L$-space $Y$, such that for each choice of metric, the Seiberg-Witten equations admit irreducible solutions?

By contrast, the construction of [14] shows that there is always a metric for which the equations admit irreducible solutions.
Appendix A. Limiting argument: Proof of Theorem 9

Proof. Let \( \delta > \frac{5}{2} \). Express \( \delta = \alpha + \beta \) with \( \alpha \geq 1 \) and \( \beta > \frac{3}{2} \). Let \( \langle t \rangle := (1 + t^2)^{\frac{1}{2}} \). Then

\[
\left| \sum_n \hat{H}(t_n) \right| = \left| \sum \hat{H}(t_n) \langle t_n \rangle^{\alpha + \beta} \langle t_n \rangle^{-\alpha} \langle t_n \rangle^{-\beta} \right|
\leq \sqrt{\sum \left| \hat{H}(t_n) \langle t_n \rangle^{\alpha + \beta} \right|^2 \langle t_n \rangle^{-2\alpha} \cdot \sqrt{\sum \langle t_n \rangle^{-2\beta}} \quad \text{by Cauchy-Schwartz.}
\]

By the local Weyl law,

\[ (12) \quad \# \{ t_n \in [N, N + 1] \} \leq DN^2 \]

for some \( D > 0 \). Since \( 2\beta > 3 \), it follows that the second summand above is convergent. Say it equals \( C \). We continue:

\[
\begin{aligned}
   &= \sqrt{C} \cdot \sqrt{\sum \left| \hat{H}(t_n) \langle t_n \rangle^{\alpha + \beta} \right|^2 \langle t_n \rangle^{-2\alpha}} \\
   &\leq \sqrt{C} \cdot \sqrt{\sum_{N=0}^{\infty} \sup_{t \in [N, N+1]} \left| \hat{H}(t) \langle t \rangle^{\alpha + \beta} \right|^2 \cdot (\langle N \rangle^{-2\alpha} \cdot \# \{ t_n \in [N, N + 1] \})} \\
   &\leq \sqrt{C} \sqrt{D} \cdot \sqrt{\sum_{N=0}^{\infty} \sup_{t \in [N, N+1]} \left| \hat{H}(t) \langle t \rangle^{\alpha + \beta} \right|^2} \quad \text{by (12) because } \alpha \geq 1.\end{aligned}
\]

There is also the Sobolev inequality

\[ (13) \quad \sup_{t \in [a,b]} |G(t)|^2 \leq E \cdot \left( ||G||^2_{L^2[a,b]} + ||G'||^2_{L^2[a,b]} \right) \]

for all smooth functions \( G \) on \([a, b]\) and some constant \( E \) uniform in \( b - a \). Applying this to \( G = \hat{H}(t) \langle t \rangle^{\alpha + \beta} \), we have

\[
\begin{aligned}
   ||G'||^2_{L^2[N, N+1]} &\leq \left| \frac{d}{dt} \hat{H}(t) \langle t \rangle^{\alpha + \beta} \right|^2_{L^2[N, N+1]} + \left| \hat{H}(t) \frac{d}{dt} \langle t \rangle^{\alpha + \beta} \right|^2_{L^2[N, N+1]} \\
   &\leq \left| \frac{d}{dt} \hat{H}(t) \langle t \rangle^{\alpha + \beta} \right|^2_{L^2[N, N+1]} + F \left| \hat{H}(t) \langle t \rangle^{\alpha + \beta} \right|^2_{L^2[N, N+1]},
\end{aligned}
\]

for some constant \( F \geq 1 \) independent of \( N \). We continue
Thus, the linear functional

\[ H \mapsto \sum \hat{H}(t_n), \]

corresponding to the regular spectral contribution to the trace formula for the test function \( H \), is continuous in the (weighted) Sobolev space \( W \) defined by the norm \( ||H|| := \sqrt{\sum_{N=0}^{\infty} ||\hat{H}(t)\alpha^+\beta||^2_{L^2[N,N+1]} + \| \frac{d}{dt} \hat{H}(t)\alpha^+\beta||^2_{L^2[N,N+1]} } \). It is also readily checked (using again the Sobolev inequalities) that the linear functionals

\[
H \mapsto H(0) - H''(0)
\]
\[
H \mapsto \hat{H}(0)
\]
\[
H \mapsto \sum_{1 \neq \gamma, \ell(\gamma) \leq R} c_\gamma H(\ell(\gamma)),
\]

corresponding respectively to the identity contribution, and the trivial representation contribution, and the regular geometric contribution to the trace formula for the test function \( H \), are continuous on \( W \).

If \( H \) is supported on \([ -R, R ]\), then for every \( \epsilon > 0 \), there is a sequence \( H_m \) of smooth functions supported on \([ -(R + \epsilon), R + \epsilon ]\) converging to \( H \) in the \( W \)-topology. Take such a sequence \( H_m \) with \( \epsilon \) chosen so that there are no closed geodesics of length in \((R, R + \epsilon)\). It follows that the trace formula is valid for \( H \) by taking the limit of both sides of the trace formula applied to \( H_m \).

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2The sequence of functions we take: approximate \( H \) by \( K * H \) for some smooth approximation to \( \delta_0 \), which we can take to be supported on \([ -\epsilon, \epsilon ]\). All of these functions are supported on \([ -(R + \epsilon), R + \epsilon ]\), and they converge to \( H \) as our approximation to \( \delta_0 \) improves.
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