Polyharmonic Hardy Spaces on the Klein-Dirac Quadric with Application to Polyharmonic Interpolation and Cubature Formulas

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Abstract

In the present paper we introduce a new concept of Hardy type space naturally defined on the Klein-Dirac quadric. We study different properties of the functions belonging to these spaces, in particular boundary value problems. We apply these new spaces to polyharmonic interpolation and to interpolatory cubature formulas.

1 Introduction

In one-dimensional mathematical analysis, Interpolation Theory and Quadrature formulas are intimately related, cf. [35], [11]. This relation causes a similarity between the approaches for estimation of the remainders of Interpolation and Quadrature. One approach for estimation of the error in Interpolation theory is related to Lagrange formula and uses higher derivatives of the interpolated function (cf. [35], chapter 3, Theorem 4, and [11], Theorem 3.1.1). The second approach uses analyticity of the interpolated function and Hermite formula (cf. [35], chapter 3, Theorem 5, and [11], Theorema 3.6.1). In a similar way, already A. Markov has estimated the error of a quadrature formula for differentiable functions in $C^N(I)$ defined on the interval $I$ by means of its $N$–th derivative (cf. [35], chapter 7.1, and Davis [11], p. 344). The second approach estimates the error of a quadrature formula for certain classes of functions $f$ which are analytic on some open set $D$ in $C$ containing the interval $I$, (cf. [12], chapter 4.6, see also [35], chapter 12.2). However, in both Interpolation and Quadrature, the first approach is usually not very practical beyond derivatives of order five, see [50].

Interpolation and Approximation of integrals in the multivariate case is a much more difficult task. In Numerical Analysis, instead of quadrature formula the notion of cubature formula is often used, see [51], [55], [52] and the recent survey [10]. In contrast to the univariate case there is no satisfactory error analysis available in the multivariate case, cf. [55], [5], and part 4 in the last Russian edition of the classical monograph [35]. Let us mention that the area of quadrature domains which has received a lot of interest recently presents an interesting multidimensional alternative and we refer to [14].

The present research continues the study of estimates of polyharmonic interpolation and polyharmonic interpolatory cubature formulas initiated in [30]; in
the present paper we consider interpolation and cubature formulas in the ball in $\mathbb{R}^d$, while in [30] the case of an annular region was considered.

1.1 Gauss-Almansi formula

In [18] polyharmonic interpolation has been considered for functions defined in the ball in $\mathbb{R}^d$. In the same spirit, in [27] and [25] we have introduced a new multivariate cubature formula $C_N(f)$ in the ball depending on a parameter $N \in \mathbb{N}$ which approximates the integral

$$\int_{B_R} f(x) \, d\mu(x)$$

for continuous functions $f : B_R \to \mathbb{C}$ defined on the ball

$$B_R = \{ x \in \mathbb{R}^d : |x| < R \},$$

where $|x|$ denotes the euclidean norm of $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$.

The exact definition of the polyharmonic interpolation formula and of the polyharmonic cubature formula $C_N(f)$ will be explained in Section 6. A major purpose of the present paper is to provide an error analysis for a class of functions on the ball $B_R$ which exhibit a certain type of analytical behavior.

Let us introduce the necessary notions and notations. Let $S^{d-1} := \{ x \in \mathbb{R}^d : |x| = 1 \}$ be the unit sphere endowed with the rotation invariant measure $d\theta$. We shall write $x \in \mathbb{R}^d$ in spherical coordinates $x = r\theta$ with $\theta \in S^{d-1}$. Let $\mathcal{H}_k(\mathbb{R}^d)$ be the set of all harmonic homogeneous complex-valued polynomials of degree $k$. Then $f \in \mathcal{H}_k(\mathbb{R}^d)$ is called a solid harmonic and the restriction of $f$ to $S^{d-1}$ a spherical harmonic of degree $k$ and we set

$$a_k := \dim \mathcal{H}_k(\mathbb{R}^d),$$

see [54], [49], [1], [24] for details. Throughout the paper we shall assume that the set of functions

$$Y_{k,\ell} : \mathbb{R}^d \to \mathbb{C}, \text{ for } \ell = 1, \ldots, a_k,$$

is an orthonormal basis of $\mathcal{H}_k(\mathbb{R}^d)$ with respect to the scalar product

$$\langle f, g \rangle_{S^{d-1}} := \int_{S^{d-1}} f(\theta) \overline{g(\theta)} d\theta.$$ 

Recall that due to the homogeneity of $Y_{k,\ell}(x)$ we have the identity $Y_{k,\ell}(x) = r^k Y_{k,\ell}(\theta)$ for $x = r\theta$.

Our polyharmonic Interpolation and polyharmonic Cubature $C_N(f)$ approximating the integral (11) are based on the Laplace–Fourier series of the continuous function $f : B_R \to \mathbb{C}$, defined by the formal expansion

$$f(r\theta) = \sum_{k=0}^{\infty} \sum_{\ell=1}^{a_k} f_{k,\ell}(r) Y_{k,\ell}(\theta)$$

(5)
where the Laplace–Fourier coefficient $f_{k,\ell}(r)$ is defined by

$$f_{k,\ell}(r) = \int_{S^{d-1}} f(r\theta) Y_{k,\ell}(\theta) \, d\theta$$

(6)

for any positive real number $r$ with $r < R$ and $a_k$ is defined in (3). There is a strong interplay between algebraic and analytic properties of the function $f$ and those of the Laplace-Fourier coefficients $f_{k,\ell}$. For example, if $f(x)$ is a polynomial in the variable $x = (x_1, \ldots, x_d)$ then the Laplace-Fourier coefficient $f_{k,\ell}$ is of the form $f_{k,\ell}(r) = r^k p_{k,\ell}(r^2)$ where $p_{k,\ell}$ is a univariate polynomial, see e.g. in [54] or [51]. Hence, the Laplace-Fourier series (5) of a polynomial $f(x)$ is equal to

$$f(x) = \sum_{k=0}^{\deg f} \sum_{\ell=1}^{a_k} p_{k,\ell}(|x|^2) Y_{k,\ell}(x) = \sum_{k=0}^{\deg f} \sum_{\ell=1}^{a_k} |x|^k p_{k,\ell}(|x|^2) Y_{k,\ell}(\theta)$$

(7)

where $\deg f$ is the total degree of $f$ and $p_{k,\ell}$ is a univariate polynomial of degree $\leq \deg f - k$. This representation is often called the Gauss representation. A similar formula is valid for a much larger class of functions. Let us recall that a function $f : G \to \mathbb{C}$ defined on an open set $G$ in $\mathbb{R}^d$ is called polyharmonic of order $N$ if $f$ is $2N$ times continuously differentiable and

$$\Delta^N u(x) = 0$$

(8)

for all $x \in G$ where $\Delta = \frac{\partial^2}{\partial x_1^2} + \ldots + \frac{\partial^2}{\partial x_d^2}$ is the Laplace operator and $\Delta^N$ the $N$-th iterate of $\Delta$. The theorem of Almansi states that for a polyharmonic function $f$ of order $N$ defined on the ball $B_R = \{x \in \mathbb{R}^d : |x| < R\}$ there exist univariate polynomials $p_{k,\ell}(r)$ of degree $\leq N - 1$ such that

$$f(x) = \sum_{k=0}^{\infty} \sum_{\ell=1}^{a_k} p_{k,\ell}(|x|^2) Y_{k,\ell}(x) = \sum_{k=0}^{\infty} \sum_{\ell=1}^{a_k} |x|^k p_{k,\ell}(|x|^2) Y_{k,\ell}(\theta)$$

(9)

where convergence of the sum is uniform on compact subsets of $B_R$, see e.g. [51], [3], [2].

To end this technical introduction, let us remind some estimates for spherical harmonics which we will need below:

1. For every multiindex $\alpha$ and for every integer $k \geq 1$ holds

$$|D^\alpha_\theta Y_{k,\ell}(\theta)| \leq C k^{|\alpha| + \frac{d-2}{2}}$$

for $\theta \in S^{d-1}$, (10)

see [49], p. 120. Here $D^\alpha_\theta$ is the multi-index notation for the derivative with respect to $\theta \in S^{d-1}$.

2. A function $f(\theta)$ defined on $S^{d-1}$ is real analytic if its Laplace-Fourier expansion $f(\theta) = \sum_{k=0}^{\infty} Y_k(\theta)$ (where we have put $Y_k(\theta) = \sum_{\ell=1}^{a_k} f_{k,\ell} Y_{k,\ell}(\theta)$) satisfies

$$\|Y_k(\theta)\|_{L^2(S^{d-1})} < C e^{-\eta k}$$

for $k \geq 0$, for some constants $C, \eta > 0$; see [51].
1.2 Complexification of the ball in $\mathbb{R}^d$, related to the ball of the Klein-Dirac quadric

We want to study analytical extensions of functions $f$ defined on the ball using the Laplace-Fourier series (9). Our strategy is to require minimal assumptions on the functions $f$; thus instead of the standard approach where one works with functions $f$ which are a priori analytically extendible to a fixed domain $U$ in the complex space $\mathbb{C}^d$ (as in [2]) we shall require only that we can extend the function $x = r\theta \mapsto f(r\theta)$ to an analytic function $z\theta \mapsto f(z\theta)$, so we only complexify the radial variable $r$ to a complex variable $z$. Henceforth we will use the following terminological convention: If the function $f(r\theta)$ possesses an analytic extension with respect to $r$ we call the extended function $f(z\theta)$ "$r$-analytic complexification" or "$r$-analytic continuation". Thus, for the $r$-complexification of the function $f$ one should expect from equation (6) that the Laplace-Fourier coefficient $f_{k,\ell}(r)$ extends to an analytic function of one variable. Hence, we consider the analytically continued functions on domains in the set $\mathbb{C} \times S^{d-1}$.

We obtain the following important proposition.

**Proposition 1** The set of functions

$$B_{k,\ell,N} := \left\{ b_{k,\ell,j}(r,\theta) = r^{k+2j}Y_{k,\ell}(\theta) : k \geq 0, \, \ell = 1, 2, ..., a_k, \, j = 0, 1, ..., N - 1 \right\}$$

is a basis for the multivariate polynomials which are polyharmonic of order $N$.

The proof follows by representation (7).

**Remark 2** We will see later that the basis $B_{k,\ell,N}$ is a natural generalization of the basis $\{r^j\}_{j=0}^{N-1}$ for the polynomials in the one-dimensional case.

The main approach in the present paper is to consider the $r$-complexification of the functions $f$ defined on the ball $B_R \subset \mathbb{R}^d$ in the form $f(z\theta)$. In particular, for polyharmonic functions $f$ we can provide a formal expression for the $r$-complexification, using representation (9), by the following formula:

$$f(z\theta) = \sum_{k=0}^{\infty} \sum_{\ell=1}^{a_k} p_{k,\ell}(z^2)Y_{k,\ell}(\theta) = \sum_{k=0}^{\infty} \sum_{\ell=1}^{a_k} z^k p_{k,\ell}(z^2)Y_{k,\ell}(\theta)$$

The question of convergence will be addressed in the course of the paper. From this formula we make a crucial observation: the complexification $f(z\theta)$ depends only on the values $z\theta$ and not on the coordinates of the pair $(z, \theta)$. Indeed, this is due to the equality $z^2 = (z\theta, z\theta)$ where for the vectors $w, u \in \mathbb{C}^d$ we have the non-Hermitian product by putting $(u, w) := \sum_{j=1}^{d} u_j w_j$. Thus the function $f(z\theta)$ is defined on the space

$$\mathbb{C} \times S^{d-1}/\mathbb{Z}_2 = \left\{ z\theta : z \in \mathbb{C}, \, \theta \in S^{d-1} \right\},$$

where the factor in $\mathbb{Z}_2$ means identification of the points $(z, \theta)$ and $(-z, -\theta)$ in $\mathbb{C} \times S^{d-1}$. But the set $\mathbb{C} \times S^{d-1}/\mathbb{Z}_2 \subset \mathbb{C}^d$ is one of the possible representations of the famous Klein-Dirac quadric.
Definition 3 We define the **Klein-Dirac quadric** by putting

\[ \text{KDQ} := \mathbb{C} \times \mathbb{S}^{d-1}/\mathbb{Z}_2, \]  

(13)

We shall see that the \( r \)-analytic continuation of the solutions of the polyharmonic equation are in fact "the analytic functions" naturally defined on the Klein-Dirac quadric KDQ. The main interest of the present paper is devoted to the Function theory on the complexified ball \( B_R \) in KDQ defined by

\[ \mathcal{B}_R := \{ z \theta : |z| < R, \theta \in \mathbb{S}^{d-1} \} = \mathbb{D}_R \times \mathbb{S}^{d-1}/\mathbb{Z}_2, \]  

(14)

where \( \mathbb{D}_R \) is the open disc of radius \( R \) in \( \mathbb{C} \), i.e. \( \mathbb{D}_R := \{ z \in \mathbb{C} : |z| < R \} \); for \( R = 1 \) as usually one puts \( \mathbb{D}_1 = \mathbb{D} \) and \( B = B_1 \).

Some enlightening comments about the **Klein-Dirac quadric** KDQ defined in (13) are in order.

Remark 4 This quadric has been originally introduced in a special case by Felix Klein in his Erlangen program in 1870, where he put forth his correspondence between the lines in complex projective 3-space and a general quadric in projective 5-space. The physical relevance of the quadric and the relation to the conformal motions of compactified Minkowski space-time had been exploited by Paul Dirac in 1936 [13]. The Klein-Dirac quadric plays an important role also in Twistor theory, where it is related to the complexified compactified Minkowski space [42]. Apparently, the term "Klein-Dirac quadric" for arbitrary dimension \( d \), has been coined by the theoretical physicist I. Todorov, cf. e.g. [39], [40]. In these references important aspects of the Function theory on the ball in KDQ have been considered in the context of Conformal Quantum Field Theory (CFT). In the context of CFT Laurent expansions appear in a natural way as the field functions in the higher dimensional conformal vertex algebras (using a complex variable parametrization of compactified Minkowski space); see in particular formula (4.43) in [41], as well as the references [39], [40].

Our main novelty will be a multivariate generalization of the classical Hardy space \( H^2(\mathbb{D}) \) called **polyharmonic** Hardy space on the ball \( \mathcal{B}_R \) to be introduced in Definition 11. We will denote this space by \( H^2(\mathcal{B}_R) \). For simplicity sake we will restrict ourselves to considering the space \( H^2(B) \).

Running ahead of the events, let us say that the name polyharmonic comes from the fact that \( H^2(\mathcal{B}_R) \) may be obtained as a limit of the complexifications of the polyharmonic functions in the ball (12): we take the closure of all finite sums of the type

\[ u(z, \theta) = \sum_{k=0}^{\infty} \sum_{\ell=1}^{a_k} u_{k,\ell} \left( z^2 \right) z^k Y_{k,\ell}(\theta), \]  

(15)

where \( u_{k,\ell}(\cdot) \) are algebraic polynomials of degree \( \leq N - 1 \); such functions \( u \) satisfy \( \Delta^N u(x) = 0 \). In view of the Polyharmonic Paradigm announced in [24], the space \( H^2(\mathcal{B}_R) \) generalizes the classical Hardy spaces which are obtained as limits of algebraic polynomials, where the degree of a polynomial is replaced by the **degree of polyharmonicity**.
The Hardy space $H^2(B_R)$ will be a Hilbert space and we will provide a Cauchy type kernel, which is the analog and a generalization to the Hua-Aronszajn kernel in the ball (cf. [2], p. 125, Corollary 1.1). Let us note that the last is a multidimensional generalization of the classical Cauchy kernel $\frac{1}{z-a}$ from Complex Analysis (more about Cauchy kernels see in [22], [32]).

**Remark 5** The reader familiar with the Cartan classification of classical domains, may remark that the boundary $\partial B_R$ is very close to the Shilov boundary

$\{ (e^{i\varphi}r\theta) : \varphi \in [0, \pi] , \ 0 < r < R, \ \theta \in S^{d-1} \}$

of the so-called Cartan classical domain $\mathcal{R}_{IV}$ (called also "Lie-ball") equal to

$$\hat{B}_1 := \{ \xi + i\eta \in \mathbb{C}^d : \xi, \eta \in \mathbb{R}^d, \ q(\xi + i\eta) < 1 \}$$

where

$$q(\xi + i\eta) = \sqrt{|\xi|^2 + |\eta|^2 + 2\sqrt{|\xi|^2 |\eta|^2 - \langle \xi, \eta \rangle^2}.$$  

This has been considered from the point of view of several complex variables in the monograph of Hua [21], and for the study of the polyharmonic functions of infinite order in the monographs [2] (see in particular p. 59 and 126) and [3].

The Hardy spaces defined in Definition 11 can be identified with the Hardy space of holomorphic functions on the Lie ball in $\mathbb{C}^d$, cf. [50]; this correspondence will be given a thorough consideration in [28].

**Remark 6** Let us define the annulus in the Klein-Dirac quadric $KDQ$ as the set

$\tilde{A}_{a,b} := \{ z\theta \in KDQ : a < |z| < b, \ \theta \in S^{d-1} \}$.

The Function theory on $\tilde{A}_{a,b}$ would help to relate the present results to previous obtained by us. We have seen in [30] that an interesting, consistent and fruitful Function theory is available only on the set

$$A_{a,b} = \{ (z, \theta) : a < |z| < b, \ \theta \in S^{d-1} \}$$

which is a subset of $\mathbb{C} \times S^{d-1}$. This is due to the fact that the $r-$ analytic continuations of the solutions of the polyharmonic equations in the annulus $A_{a,b} \subset \mathbb{R}^d$ live on the set $A_{a,b}$ but not on the set $\tilde{A}_{a,b}$! The point is that in the case of the annulus $A_{a,b}$ we cannot identify the point $z\theta$ with $(z, \theta)$, in other words, $(z, \theta)$ is not identified with $(-z, -\theta)$.

The paper is organized as follows: in Section 2 we recall background material about the Hardy space $H^2(B_R)$. In Section 3 we introduce the polyharmonic Hardy space $H^2(B_b)$ on the ball $B$ of the Klein-Dirac quadric. We prove that it is a Hilbert space, a maximum principle, and infinite-differentiability of the functions in $H^2(B_b)$. In Section 4.1 we construct a Cauchy type kernel for $H^2(B_b)$.

In Section 4 we prove other main properties of the space ..., which generalize similar properties of the one-dimensional Hardy spaces. In Section 5 we characterize the polyharmonic functions which are extendible to the Hardy space $H^2(B_b)$. In Section 6 we prove some of the main results of the paper, about the error estimate of the polyharmonic interpolation, and about the polyharmonic interpolatory cubature formulas, generalizing the polyharmonic Gauss-Jacobi cubature formulas introduced in [23], [27].
2 Classical Hardy spaces – a reminder

Hardy spaces are a bridge between Harmonic and Complex Analysis. This is based on the fact that the Taylor coefficients of a function \( f(x) \) on the real line \( \mathbb{R} \) are at the same time the coefficients of an orthogonal expansion of the analytic continuation \( f(z) \) with respect to the basis \( \{ z^j \}_{j \geq 0} \) which is orthogonal on the circle. Thus in a certain sense the setting of the Hardy space represents a study of the properties of the real functions having Taylor coefficients with \( \sum |a_j|^2 < \infty \) by the methods of Complex Analysis. Since we are generalizing before all the Hardy space \( H^2 \) we will recall the main results about it. Let us put

\[
M(f; r) := \left\{ \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\varphi})|^2 \, d\varphi \right\}^{1/2}.
\]

Then for every function \( f \) which is analytic in the disc \( \mathbb{D} \) we define the Hardy space norm

\[
\|f\|_{H^2(\mathbb{D})} := \sup_{r < 1} M(f; r).
\]

**Remark 7** For every analytic function \( f \) the function \( M(f; r) \) is an increasing function of \( r \), cf. Theorem 17.6 in [44].

A basic fact is that \( H^2 \) is a Hilbert space and may be identified with the limit values on the circle which coincide with \( L^2(\mathbb{S}^1) \). For every \( g \in L^2(\mathbb{S}^1) \) we have the norm defined by

\[
\|g\|_{L^2(\mathbb{S}^1)}^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |g(e^{i\varphi})|^2 \, d\varphi,
\]

and the Fourier coefficients

\[
\hat{g}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(e^{i\varphi}) e^{-i\pi \varphi} \, d\varphi \quad \text{for } n \in \mathbb{Z}.
\]

Some of the main properties of the classical Hardy spaces \( H^2 \) are summarized in the following theorem (cf. Theorem 17.10 in [44]).

**Theorem 8** 1. An analytic function \( f \) on \( \mathbb{D} \) of the type

\[
f(z) = \sum_{j=0}^{\infty} f_j z^j \quad \text{for } z \in \mathbb{D}
\]

belongs to \( H^2(\mathbb{D}) \) if and only if

\[
\sum_{j=0}^{\infty} |f_j|^2 < \infty; \quad (17)
\]

in that case

\[
\|f\|_{H^2}^2 = \sum_{j=0}^{\infty} |f_j|^2.
\]
2. If \( f \in H^2 (\mathbb{D}) \) then \( f \) has radial limits \( f^* (e^{i\varphi}) \) at almost all points on the circle \( S \) and \( f^* \in L^2 (S) \). The Riesz condition holds, i.e.

\[
\begin{align*}
    f_j^* &= 0 \quad \text{for all } j < 0; \\
    f_j^* &= f_j \quad \text{for all } j \geq 0.
\end{align*}
\]

The \( L^2 \)–approximation holds

\[
\lim_{r \to 1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\varphi}) - f^* (e^{i\varphi})|^2 \, d\varphi = 0.
\]

The integral of Poisson and of Cauchy of \( f^* \) recover \( f \), i.e.

\[
\begin{align*}
    f(z) &= \frac{1}{2\pi} \int_0^{2\pi} P_r (\varphi - t) f^* (e^{it}) \, dt \\
    f(z) &= \frac{1}{2\pi i} \int_{\Gamma} \frac{f^*(\zeta)}{\zeta - z} \, d\zeta
\end{align*}
\]

where \( \Gamma \) is the positively oriented circle \( S \).

3. The mapping \( f \mapsto f^* \) is an isometry of \( H^2 \) on the subspace of \( L^2 (S) \) which consists of those \( g \) for which \( \hat{g}(j) = 0 \) for all \( j < 0 \).

Let us recall the famous theorem of brothers F. and M. Riesz which concludes the absolute continuity of a Borel measure on \( S^1 \) only from the annihilation of half of its Fourier coefficients, [44, Theorem 17.13].

**Theorem 9** Let \( \mu \) be a complex valued Borel measure on the circle \( S \). If

\[
\int_0^{2\pi} e^{im\varphi} d\mu (t) = 0 \quad \text{for } m \geq 1,
\]

then the measure \( \mu \) is absolutely continuous with respect to the Lebesgue measure, i.e. there exists a function \( f^* \in L^1 (S) \) such that \( d\mu (t) = \frac{1}{2\pi} f^* (t) \, dt \) for \( t \in [0, 2\pi] \).

3 The polyharmonic Hardy space \( H^2 (\mathcal{B}) \) on the ball of the Klein-Dirac quadric

At first we observe that the basis functions \( b_{k,\ell,j} (r,\theta) \) defined in [11] have a natural \( r \)–analytic extension

\[
b_{k,\ell,j} (z,\theta) := z^{2j+k} Y_{k,\ell} (\theta).
\]

Let us present some heuristics for explaining our main goal: We want to define the Hardy space \( H^2 (\mathcal{B}_R) \) as a space of functions which are uniform limits of sequences of complexified polynomials \( P (z\theta) \) on compacts of the ball \( \mathcal{B}_R \) in KDKQ. For that reason we need an appropriate inner product. Let us make the **important observation** that there is a natural inner product where the basic functions \( \{20\} \) are orthogonal. Hence, for functions defined on the set \( S^1 \times S^{d-1} \) we introduce the inner product

\[
\langle f, g \rangle := \frac{1}{2\pi} \int_{S^d-1} \int_0^{2\pi} f (e^{i\varphi}, \theta) \overline{g (e^{i\varphi}, \theta)} \, d\varphi d\theta.
\]

8
The crux of our approach is the orthogonality of the basis functions \( b_{k,\ell,j} \) in Definition 10 on the boundary of the Klein-Dirac quadric \( S^1 \times S^{d-1}/\mathbb{Z}_2 \), i.e.,
\[
\langle b_{k,\ell,j}, b_{k',\ell',j'} \rangle = \delta_{k,k'} \delta_{\ell,\ell'} \delta_{j,j'},
\]
where the Kronecker symbol \( \delta \) means \( \delta_{\alpha,\beta} = 1 \) for \( \alpha = \beta \) and 0 for \( \alpha \neq \beta \).

This property is a remarkable generalization of the orthogonality of the basis \( \{ z^j \}_{j \geq 0} \) on the circle \( S^1 \) and traces the analogy to the one-dimensional case.

Further, we provide some arguments about the proper definition of the norm on the prospective Hardy space. The objects of our polyharmonic Hardy space \( H^2(\mathcal{B}) \) will be functions \( f(z,\theta) \) which are representable as infinite sums in the \( L^2 \) sense, and are absolutely and uniformly convergent on compacts with \( |z| < 1 \):
\[
f(z,\theta) = \sum_{k=0}^{\infty} \sum_{\ell=1}^{\infty} \left( \sum_{j=0}^{\infty} f_{k,\ell,j} z^{2j} \right) z^k Y_{k,\ell}(\theta).
\]
Further, we provide some arguments about the proper definition of the norm on \( H^2(\mathcal{B}) \) having the representation
\[
f(z) = \sum_{k=0}^{\infty} a_k z^{k+2j},
\]
respectively the norm on \( H^2(\mathcal{B}) \) is the inherited from \( H^2(\mathbb{D}) \).

Definition 11 We define the polyharmonic Hardy space \( H^2(\mathcal{B}) \) on the unit ball \( \mathcal{B} = B_1 \) of the Klein-Dirac quadric defined in (14), as the space of functions \( f \) given by the Laplace-Fourier series (23) with coefficients \( f_{k,\ell} \in H^2(\mathbb{D}) \) satisfying
\[
\|f\| := \sqrt{\sum_{k=0}^{\infty} \sum_{\ell=1}^{\infty} \|f_{k,\ell}\|_2^2} < \infty.
\]

Remark 12 The reader may note that Definition 11 mimics the definition of the classical Hardy spaces which are obtained as the closure of the polynomials.
hence
\[ \sum_{k=0}^{\infty} \sum_{\ell=1}^{a_k} \| f_{k,\ell} \|_{H^2(\mathbb{D})}^2 = \sum_{k=0}^{\infty} \sum_{\ell=1}^{a_k} \int_0^{2\pi} |f_{k,\ell}(e^{i\varphi})|^2 \, d\varphi = \sum_{k=0}^{\infty} \sum_{\ell=1}^{a_k} \int_0^{2\pi} |f_{k,\ell}^* (z^\ell)|^2 \, d\varphi < \infty. \] (26)

This implies
\[ \lim_{r \to 1^-} 2\pi \int_0^{2\pi} \int_{S^{d-1}} |f(re^{i\varphi}, \theta)|^2 \, d\varphi d\theta = \sum_{k=0}^{\infty} \sum_{\ell=1}^{a_k} \int_0^{2\pi} |f_{k,\ell}(e^{i\varphi})|^2 \, d\varphi < \infty, \]

hence, it follows that \( f \in L^2(\mathbb{D} \times S^{d-1}) \), and due to (23), also \( f \in L^2(B) \).

Note that as in the classical Hardy spaces the inner product (21) is good only for the polynomials but might not be well defined for arbitrary functions having bad boundary behavior. For that reason we have to change the definition of this inner product.

Definition 13 We put
\[ \langle f, g \rangle_{H^2(B)} := \frac{1}{2\pi} \lim_{r \to 1^-} \int_{S^{d-1}} \int_0^{2\pi} f(ze^{i\varphi}, \theta) \overline{g(ze^{i\varphi}, \theta)} \, d\varphi d\theta. \] (27)

The following theorem justifies our arguments above and is an analog to results for the classical Hardy spaces, cf. Theorem 8 (or Theorem 17.10 in [44]).

Theorem 14 1. The space \( H^2(B) \) is complete.
2. It coincides with the space of functions \( f \) having representation
\[ f(z, \theta) = \sum_{k=0}^{\infty} \sum_{\ell=1}^{a_k} \sum_{j=0}^{\infty} f_{k,\ell,j} z^{k+2j} Y_{k,\ell}(\theta) \] (28)
with coefficients satisfying
\[ \sum_{k=0}^{\infty} \sum_{\ell=1}^{a_k} \sum_{j=0}^{\infty} |f_{k,\ell,j}|^2 < \infty. \] (29)
3. The norm of \( f \) is given by
\[ \| f \|_{H^2(B)} = \left\{ \sum_{k=0}^{\infty} \sum_{\ell=1}^{a_k} \sum_{j=0}^{\infty} |f_{k,\ell,j}|^2 \right\}^{1/2}. \]
4. Every element \( f \in H^2(B) \) is the limit of a sequence of polynomials \( P_N \in \mathcal{P} \) which satisfy
\[ \Delta^N P_N(x) = 0 \quad \text{for } x \in B \subset \mathbb{R}^d. \]
We will prove only some of the above statements.

**Proof.** 1. It is obvious by a direct estimation that condition \( f \in H^2(B) \) implies the absolute convergence of the series (28) on every compact \( K \subset \mathbb{D} \times \mathbb{S}^{d-1} \). Hence, every function defined by the series (28) and satisfying (29) also satisfies

\[
\sum_{k=0}^{\infty} \sum_{\ell=1}^{d_k} \sum_{j=0}^{N} |f_{k,\ell,j}|^2 < \infty,
\]

which shows that the space \( H^2(B) \) is a tensor sum of the \( \ell^2 \) spaces \( \left\{ f_{k,\ell,j} \right\}_{j \geq 0} \) taken over all indices \((k, \ell)\). Now, in a standard manner, the completeness of the space \( H^2(B) \) follows from the completeness of the space of sequences \( \ell^2 \).

4. We have to see that every function defined by the series (28) and satisfying (29) is a limit in the norm of \( H^2(B) \) of a sequence of polynomials. Indeed, for every \( \varepsilon > 0 \) we may find integers \( k_1 > 0 \) and \( N > 0 \) such that

\[
\sum_{k=k_1+1}^{\infty} \sum_{\ell=1}^{d_k} \sum_{j=N+1}^{\infty} |f_{k,\ell,j}|^2 < \varepsilon.
\]

Then the polynomial \( P \) defined by

\[
P(x) := \sum_{k=0}^{k_1} \sum_{\ell=1}^{d_k} \sum_{j=0}^{N} f_{k,\ell,j} r^{k+2j} Y_{k,\ell}(\theta)
\]

satisfies with \( z = re^{i\varphi} \) the following:

\[
\frac{1}{2\pi} \int_{\mathbb{S}^{d-1}} \int_0^{2\pi} |f(z\theta) - P(z\theta)|^2 d\varphi d\theta \leq \sum_{k=k_1+1}^{\infty} \sum_{\ell=1}^{d_k} \sum_{j=N+1}^{\infty} |f_{k,\ell,j}|^2 \leq \varepsilon.
\]

This implies

\[
\|f - P\|_{H^2(B)} \leq \varepsilon,
\]

which ends the proof. 

**Remark 15** The essence of Theorem 14 is that, as in the classical Hardy spaces, only the information about \( f \) in the real domain, provided by the Laplace-Fourier coefficients

\[
f(x) = \sum_{k=0}^{\infty} \sum_{\ell=1}^{d_k} f_{k,\ell} (r^2)^{k+2j} Y_{k,\ell}(\theta),
\]

determine when does \( f \) belong to \( H^2(B) \). All we need to know is that they satisfy the convergence condition (29), \( \sum_{k=0}^{\infty} \sum_{\ell=1}^{d_k} \sum_{j=0}^{\infty} |f_{k,\ell,j}|^2 < \infty \).

Theorem 14 suggests the representation

\[
H^2(B) = \bigoplus_{k,\ell} H^2_{k,\ell},
\]
where the tensor sum is understood in the sense of equality (24), and the prime in the symbol \( \bigoplus_{k,\ell} \) means that only sums are taken which are convergent as (29)-(30).

The following result follows from the representation in (25).

**Proposition 16** For every function \( f \in H^{2,k}(\mathcal{D}) \) the following formula of Cauchy type holds:

\[
f(\zeta) = \frac{1}{2\pi i} \int_{\Gamma_1} \frac{z}{z^2 - \zeta^2} \frac{\zeta^k}{z^k} f(z) \, dz;
\]

here \( \zeta \in \mathcal{D} \).

### 3.1 Cauchy type kernel for \( H^2(B) \) and Hua-Aronszajn type formula

The orthogonality of the basis \( \{b_{k,\ell,j}\} \) hints us to construct a Cauchy type kernel and a corresponding formula which reproduces the multivariate polynomials by using their values on the set \( S^1 \times S^{d-1}/\mathbb{Z}_2 \). By exploiting the above orthogonality, we obtain such formula easily, following the general principles of constructing kernels, by putting:

\[
K(z',\theta';z,\theta) := \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} a_k \sum_{\ell=1}^{b_k,\ell,j}(\zeta;\theta') b_k,\ell,j(z;\theta) \tag{31}
\]

\[
= \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sum_{\ell=1}^{b_k,\ell,j}(\zeta;\theta') \zeta^{2j+k}Y_{k,\ell}(\theta') z^{2j+k}Y_{k,\ell}(\theta).
\]

Note that this kernel is **absolutely convergent** for every \( \zeta \) and \( \theta \) with \( |\zeta| < 1 \), \( \theta' \in S^{d-1} \), and \( z = e^{i\phi} \), due to the estimates for the spherical harmonics (10). By the orthogonality property (22), for every polynomial \( P \) we obtain the Cauchy type formula formula

\[
P(\zeta,\theta') = \langle K(\zeta,\theta';\cdot),P(\cdot) \rangle_+ \tag{32}
\]

\[
= \frac{1}{2\pi} \int_{S^1} \int_{0}^{2\pi} K(\zeta,\theta';z,\theta) P(z,\theta) \, d\phi d\theta.
\]

Let us remark that no such formula is available in the real domain. Formula (32) is a strong motivation to consider further the consequences of the inner product (24).

Obviously,

\[
K(\zeta,\theta';z,\theta) = \sum_{j=0}^{\infty} (\zeta z)^j \sum_{k=0}^{\infty} \sum_{\ell=1}^{b_k,\ell,j}(\zeta;\theta') b_k,\ell,j(z;\theta)
\]

\[
= \frac{1}{1 - \zeta^2 z^2} \sum_{k=0}^{\infty} \sum_{\ell=1}^{b_k,\ell,j}(\zeta z)^k Y_{k,\ell}(\theta') Y_{k,\ell}(\theta)
\]

which shows the absolute convergence for \( |\zeta z| < 1 \). Let us recall that the usual Poisson kernel (see [51], chapter 2, Theorem 1.9) is given by

\[
K_P(r,\theta,\theta') := \frac{1 - r^2}{|\theta - r\theta'|^d} = \sum_{k=0}^{d} \sum_{\ell=1}^{b_k,\ell,j}(\theta') Y_{k,\ell}(\theta) Y_{k,\ell}(\theta') \quad \text{for } r < 1
\]
in every dimension $d \geq 2$. This expression is obviously close to the above expression for $K (\zeta, \theta'; z, \theta)$. We will apply the idea for the $r$–complexification to the kernel $K_P (r, \theta, \theta')$ and we will relate it to the Cauchy type kernel $K (\zeta, \theta'; z, \theta)$.

**Proposition 17** For every complex number $w$ with $|w| < 1$ the following equality holds:

$$\sum_{k=0}^{\infty} \sum_{\ell=1}^{d_k} w^k Y_{k, \ell} (\theta) Y_{k, \ell} (\theta') = \sum_{k=0}^{\infty} w^k Z_{\theta}^{(k)} (\theta') = \frac{1 - w^2}{(1 - 2w \langle \theta, \theta' \rangle + w^2)^{\frac{d}{2}}},$$  \hspace{1cm} (33)

where $Z_{\theta}^{(k)} (\theta')$ are the zonal harmonics, see [54]. Let us put $\cos \phi = \langle \theta, \theta' \rangle$.

For $d = 2$ we have

$$K_P (w, \theta, \theta') = \frac{1}{2\pi} + \frac{1}{\pi} \sum_{k=1}^{\infty} w^k \cos k \phi \pi = \frac{1}{2\pi} \frac{1 - w^2}{1 - 2w \cos \phi + w^2}$$  \hspace{1cm} (34)

and for $d > 2$,

$$K_P (w, \theta, \theta') = \sum_{k=0}^{\infty} w^k c_k, d P^k \lambda (\cos \phi)$$  \hspace{1cm} (35)

where

$$c_k, d = \frac{1}{\omega_{d-1}} \frac{2k + d - 2}{d - 2}, \hspace{1cm} \lambda = \frac{d - 2}{2}$$

and $P^k \lambda$ are the Legendre polynomials.

Hence, the $r$–complexification of the Poisson kernel $K_P (r, \theta, \theta')$ is given by

$$K_P (w, \theta, \theta') = \frac{1 - w^2}{(1 - 2w \langle \theta, \theta' \rangle + w^2)^{\frac{d}{2}}},$$  \hspace{1cm} (36)

and

$$K (\zeta, \theta'; z, \theta) = \frac{1}{1 - \zeta^2 z^2 K_P (\zeta z, \theta', \theta')},$$  \hspace{1cm} (37)

where

$$K (\zeta, \theta'; z, \theta) = \frac{1}{1 - 2w \cos \phi + w^2} = \frac{1}{(1 - e^{i\phi} w) (1 - e^{-i\phi} w)}.$$  \hspace{1cm} (38)

**Proof.** The proof uses the following results from [54].

1. Formula (34) follows from formula (2.7) in chapter 4 in [54].
2. Formula (35) follows from Lemma 2.8, Theorem 2.10 and Theorem 2.14 in chapter 4 in [54].

By the estimates for the spherical harmonics in [10] it follows that the left-hand side of (33) is absolutely and uniformly convergent on every compact $K \subset D \times S^{d-1} \times S^{d-1}$. Representation (36) follows from formula

$$1 - 2w \cos \phi + w^2 = (1 - e^{i\phi} w) (1 - e^{-i\phi} w).$$  \hspace{1cm} (38)
The definition of the kernel \( K(\zeta, \theta'; z, \theta) \) shows that for all polynomials \( P \) with real coefficients holds

\[
P(\zeta \theta') = \frac{1}{2\pi \omega_d} \int_{S^{d-1}} \int_0^{2\pi} K(\zeta, \theta'; z, \theta) \overline{P(z, \theta)} d\varphi d\theta \quad \text{for } z = e^{i\varphi}.
\]

Since \( \overline{z} = z^{-1} \), a change in the integration \( \varphi \to -\varphi \) shows that

\[
P(\zeta \theta') = \frac{1}{2\pi \omega_d} \int_{S^{d-1}} \int_0^{2\pi} K(\zeta, \theta'; \frac{1}{z}, \theta) \overline{P(z, \theta)} d\varphi d\theta
\]

\[
= \frac{1}{2\pi \omega_d} \int_{S^{d-1}} \int \frac{1}{z} K(\zeta, \theta'; \frac{1}{z}, \theta) P(z, \theta) dz d\theta
\]

where \( \Gamma_1 \) is the positively oriented circle \( S^1 \). On the other hand we see that

\[
\frac{1}{z} K(\zeta, \theta'; \frac{1}{z}, \theta) = \frac{1}{z} \frac{1}{\left(1 - \frac{2\zeta}{z} \theta \cdot \theta' + \frac{\zeta^2}{z^2}\right)^{\frac{d}{2}}}
\]

\[
= \frac{z^{d-1}}{\left(z^2 - 2\zeta z (\theta \cdot \theta') + \zeta^2\right)^{\frac{d}{2}}}.
\]

The last by definition is up to a factor the Hua-Aronszajn kernel, see [2], p. 126.

The above identities motivate the definition of the well-known Hua-Aronszajn kernel \( H(\zeta, \theta'; z, \theta) \) given by

\[
H(\zeta, \theta'; z, \theta) = \frac{1}{\omega_d} \frac{z^{d-1}}{\left(\zeta^2 - 2\zeta z (\theta \cdot \theta') + z^2\right)^{\frac{d}{2}}}, \quad (39)
\]

where \( \omega_d = \pi^{d/2}/\Gamma(d/2) \) is the surface of the sphere (see [2], p. 122 Theorem 1.1, and p. 126, Remark 1.4, where up to a factor it is called Cauchy kernel for the Cartan classical domain \( \mathcal{D}_IV \)).

From above we see that the following equality holds:

\[
H(\zeta, \theta'; z, \theta) = \frac{1}{\omega_d z} \frac{z}{\left(1 - \zeta^2 / z^2\right)^{\frac{d}{2}}} \sum_{k=0}^{\infty} \sum_{\ell=1}^{\alpha_k} (\zeta/z)^k Y_{k, \ell}(\theta') Y_{k, \ell}(\theta)
\]

\[
= \frac{1}{\omega_d z} \frac{1}{\left(1 - 2\zeta/z (\theta \cdot \theta') + \zeta^2/z^2\right)^{\frac{d}{2}}}. \quad (40)
\]

### 4 Main properties of the Hardy spaces \( H^2(\mathcal{B}) \)

In the next theorem we provide a generalization of the classical boundary value properties of the Hardy spaces \( H^2 \), see e.g. Theorem 15.10, 17.12 and 17.13 in [44].
Theorem 18  Let \( f \in H^2(B) \).

1. **Fatou type theorem:** For \( r \to 1^- \), and for almost all \( \varphi \in [0, 2\pi] \) and \( \theta \in S^{d-1} \), the function \( f(re^{i\varphi}) \) has a radial limit which we denote by \( f^*(e^{i\varphi}, \theta) \), and which satisfies \( f^*(e^{i\varphi}, \theta) \in L^2(S \times S^{d-1}) \). If the expansion of the function \( f \) is given by (28),

\[
f(z, \theta) = \sum_{k=0}^{\infty} \sum_{\ell=1}^{\infty} \sum_{j=0}^{\infty} f_{k,\ell,j} z^{k+2j} Y_{k,\ell} (\theta) \quad \text{for all } |z| < 1, \theta \in S^{d-1},
\]

then \( f^*(e^{i\varphi}, \theta) \) is given by the Laplace-Fourier series

\[
f^*(e^{i\varphi}, \theta) = \sum_{k=0}^{\infty} \sum_{\ell=1}^{\infty} \sum_{j=0}^{\infty} f_{k,\ell,j}^* z^{k+2j} Y_{k,\ell} (\theta) \quad \text{for } z = e^{i\varphi}.
\]  

2. The following limiting relation holds,

\[
\lim_{r \to 1^-} \frac{1}{2\pi} \int_0^{2\pi} \int_{S^{d-1}} |f(re^{i\varphi}) - f^*(e^{i\varphi})|^2 \, d\varphi d\theta = 0.
\]

3. Let the Laplace-Fourier series of the function \( f^*(e^{i\varphi}, \theta) \in L^2(S \times S^{d-1}) \) be given by

\[
f^*(e^{i\varphi}, \theta) = \sum_{k=0}^{\infty} \sum_{\ell=1}^{\infty} \sum_{j=0}^{\infty} f_{k,\ell,j}^* z^{k+2j} Y_{k,\ell} (\theta) \quad \text{for } z = e^{i\varphi}.
\]

Then the Fourier coefficients of the functions \( f_{k,\ell,j}^* \) satisfy the following zero conditions, which we call **Riesz type conditions**:

\[
f_{k,\ell,j}^* = 0 \quad \text{for all } j \neq k, k+2, k+4, \ldots.
\]  

These conditions are equivalent to (the usual form of Riesz conditions)

\[
\int_0^{2\pi} \int_{S^{d-1}} f^*(e^{i\varphi}, \theta) e^{-ij\varphi} Y_{k,\ell} (\theta) \, d\varphi d\theta = 0 \quad \text{for all } j \neq k, k+2, k+4, \ldots.
\]

4. The following **Cauchy-Hua-Aronszajn** formula holds

\[
f(\zeta, \theta') = \frac{1}{2\pi i} \int_{1} \int_{S^{d-1}} H(\zeta, \theta'; z, \theta) f^*(z, \theta) \, dz \, d\theta
\]

\[
= \frac{1}{2\pi \omega_d} \int_0^{2\pi} \int_{S^{d-1}} K(\zeta, \theta'; \frac{1}{z}, \theta) f^*(z, \theta) \, d\varphi d\theta
\]

\[
= \frac{1}{2\pi \omega_d} \int_0^{2\pi} P_r(2\varphi - 2\varphi') \int_{S^{d-1}} K_P \left( \frac{\zeta}{z}, \theta', \theta \right) f^*(z, \theta) \, d\theta d\varphi,
\]

where we use the notations \( z = e^{i\varphi} \) and \( \zeta = re^{i\varphi'} \), the kernel \( K_P \) is the **Cauchy type kernel** (37), and \( P_r(\varphi) \) is the usual two-dimensional Poisson kernel. We call the kernel

\[
P_r(2\varphi - 2\varphi') K_P \left( \frac{\zeta}{z}, \theta', \theta \right)
\]

15
the modified Poisson type kernel. Here the contour $\Gamma_1$ is the positively oriented circle $\mathbb{S}^1$, or a bigger contour which encircles it.

5. **Dirichlet problem** with $L^2$ data: If a function $f^* (e^{i\varphi}, \theta) \in L^2 (\mathbb{S}^1 \times \mathbb{S}^{d-1})$ satisfies the Riesz type conditions (42) then there exists an unique function $f \in H^2 (\mathcal{B})$ which has as a "non-tangential limit" the function $f^* (e^{i\varphi}, \theta)$ in the sense

$$\lim_{r \to 1} \int_{\mathbb{S}^{d-1}} \int_0^{2\pi} \left| f (re^{i\varphi}, \theta) - f^* (e^{i\varphi}, \theta) \right|^2 \, d\varphi d\theta = 0.$$ 

6. Every function $f^* \in L^2 (\mathbb{S}^1 \times \mathbb{S}^{d-1})$ which satisfies the Riesz type conditions (42) belongs to the space $L^2 (\mathbb{S}^1 \times \mathbb{S}^{d-1}/\mathbb{Z}_2)$. The map $f \rightarrow f^*$ is an **isometry** between $H^2 (\mathcal{B})$ and the subspace of $L^2 (\mathbb{S}^1 \times \mathbb{S}^{d-1}/\mathbb{Z}_2)$.

7. The norm is given by

$$\|f\|_{H^2 (\mathcal{B})}^2 = \frac{1}{2\pi} \int_{\mathbb{S}^{d-1}} \int_0^{2\pi} |f^* (e^{i\varphi} \theta')|^2 \, d\varphi d\theta'.$$

**Proof.** 1. We will omit the proof of this non-trivial but standardly proved statement.

2. By Theorem 14 the function $f$ is represented by a series (28) satisfying $\sum_{k=0}^{\infty} \sum_{j=0}^{\infty} |f_{k,j}|^2 < \infty$. Let us consider the function

$$f^* (e^{i\varphi}, \theta) := \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} f_{k,j} z^{k+2j} Y_{k,j} (\theta)$$

for $z = e^{i\varphi}$.

Since the functions $\{e^{i\varphi} Y_{k,j} (\theta) : j \in \mathbb{Z}, \text{ all } (k, \ell)\}$ form a basis of the space $L^2 (\mathbb{S}^1 \times \mathbb{S}^{d-1})$, it follows that $f^* \in L^2 (\mathbb{S}^1 \times \mathbb{S}^{d-1})$. By Parseval's theorem we see that

$$\frac{1}{2\pi} \int_{\mathbb{S}^{d-1}} \int_0^{2\pi} \left| f (re^{i\varphi} \theta) - f^* (e^{i\varphi}, \theta) \right|^2 \, d\varphi d\theta = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} |f_{k,j}|^2 (1 - r^{k+2j})^2.$$

It is easy to see that for $\rho \to 1^-$ the last tends to 0.

The proof of 3.) is evident due to the expansion (11).

4. From formula (10) it follows that for every function $g (e^{i\varphi}, \theta) \in L^2 (\mathbb{S}^1 \times \mathbb{S}^{d-1})$ holds

$$\frac{1}{2\pi i \omega_d} \int_{\Gamma_1} \int_{\mathbb{S}^{d-1}} z^{d-1} f^* (z, \theta) \, dz \, d\theta =$$

$$= \frac{1}{2\pi \omega_d} \int_{0}^{2\pi} \int_{\mathbb{S}^{d-1}} K (\zeta, \zeta'; z, \theta) f^* (\zeta, \theta) \, d\varphi d\theta$$

for $z = e^{i\varphi}$.

From the series representation of $f$ and $f^*$, by the definition of the kernel $K (\zeta, \zeta'; z, \theta)$ it follows

$$f (\zeta \theta') = \frac{1}{2\pi} \int_{0}^{2\pi} \int_{\mathbb{S}^{d-1}} K (\zeta, \zeta'; z, \theta) f^* (\zeta, \theta) \, d\varphi d\theta.$$

Further, since

$$P_r (\varphi - \varphi') = \sum_{j=-\infty}^{\infty} r^{|j|} e^{ij (\varphi' - \varphi)}$$

16
\[ K_P \left( \frac{\zeta}{z}, \theta, \theta' \right) = \sum_{k=0}^{\infty} \sum_{\ell=1}^{a_k} r^k e^{ik(\varphi' - \varphi)} Y_{k,\ell}(\theta) Y_{k,\ell}(\theta') \]

the product \( P_r^2 (2\varphi - 2\varphi') K_P \left( \frac{\zeta}{z}, \theta, \theta' \right) \) has terms which in the integral will vanish due to the Riesz conditions satisfied by \( f^* \), and there will remain only the terms corresponding to \( K (\zeta, \theta'; z, \theta) \), which proves the formula.

5. Every function \( f^* (e^{i\varphi}, \theta) \) which satisfies the Riesz type conditions (42) has a Laplace-Fourier series of the type (41), say

\[ f^* (e^{i\varphi}, \theta) = \sum_{k=0}^{\infty} \sum_{\ell=1}^{a_k} \sum_{j=0}^{\infty} a_{k,\ell,j} z^{k+2j} Y_{k,\ell}(\theta) \]

for \( z = e^{i\varphi} \).

Since \( f^* (e^{i\varphi}, \theta) \in L^2 (S^1 \times S^{d-1}) \), the coefficients satisfy \( \sum_{k=0}^{\infty} \sum_{\ell=1}^{a_k} \sum_{j=0}^{\infty} |a_{k,\ell,j}|^2 < \infty \), hence, we may define the function

\[ f (z\theta) := \sum_{k=0}^{\infty} \sum_{\ell=1}^{a_k} \sum_{j=0}^{\infty} a_{k,\ell,j} z^{k+2j} Y_{k,\ell}(\theta) ; \]

the limiting property follows as in item 2).

Finally, item 6.) follows from the above arguments.

Theorem 18 shows that we may solve Boundary Value Problems in the spaces \( H^2 (B) \) which is an essential advantage over the situation with the holomorphic functions in \( C^d \) and alternative definitions of Hardy space in several dimensions, see [53], [45], [54], [46].

By Theorem 18 the following space:

\[ \left\{ f^* \in L^2 (S \times S^{d-1}) : f^* \text{ satisfies the Riesz type condition (42)} \right\} \]

is isomorphic to the space \( H^2 (B) \). The Cauchy-Hua-Aronszajn formula (44) generalizes the Cauchy formula in \( C \) and the Poisson formula in \( \mathbb{R}^d \) at the same time.

**Remark 19** Let us formulate a conjecture about an analog to brothers’ Riesz theorem: Let the complex valued Borel measure \( \mu (\varphi, \theta) \) be given on \( S^1 \times S^{d-1} \) with \( \varphi \in [0, 2\pi] \) and \( \theta \in S^{d-1} \). Assume that for all indices \( (k, \ell) \) holds

\[ \int_0^{2\pi} \int_{S^{d-1}} \mathcal{P} Y_{k,\ell}(\theta) \, d\mu (\varphi, \theta) = 0 \quad \text{for } j \neq k, k+2, k+4, ... \quad (46) \]

Is the measure \( \mu \) “absolutely continuous”, i.e. does there exist a function \( f^* \) which is in \( L^1 (S^1 \times S^{d-1}) \) such that \( \int_{S^1 \times S^{d-1}} \, d\mu (\varphi, \theta) = f^* (\varphi, \theta) \, d\varphi d\theta \) ? It seems that the answer in this form is negative, but a positive answer needs some additional properties of the measure \( \mu \). A thorough discussion to this question will be considered in [28]. Let us remark that a genuine analog of the brothers Riesz theorem is difficult to achieve for all approaches to Hardy spaces, cf. [7], [19], [43], [53], [54], [46].
More generally, for every mixed derivative \( B_{\alpha, \theta} \) of the ball \( \text{(weak) maximum principle holds:} \)

\[
\text{by (44)) we may define a function } F_{\theta}(\zeta, \theta) \text{ on the interior (for } |\zeta| = r < 1 \text{) of the ball } B \text{ of the Klein-Dirac quadric. Let us put } F_{\theta}(e^{i\varphi}, \theta) = g(e^{i\varphi}, \theta) \text{ for } r = 1. \text{ We conjecture that the function } F_{\theta} \text{ is continuos on the closure } B. \text{ Let us note that in the classical case the Poisson kernel is used to prove similar statement, see chapter 2, Theorem 1.9 in [44]. Here we expect that the modified Poisson kernel (39) will of central importance for the solution.}

4.1 Maximum principle

In the classical case of the Hardy spaces, the maximum principle is intimately related to the Cauchy formula in \( \mathbb{C} \) or to the Poisson formula in \( \mathbb{R}^d \) (see the proof of the completeness of \( H^p \) in Remark 17.8 in [44]). A weak form of maximum principle allows to prove that the elements of \( H^2 \) are uniform limits of polynomials on compact subsets of \( \mathbb{D} \). Here we prove analog to this for the polyharmonic Hardy space \( H^2(B) \). In the next theorem we see that the explicit form for the Cauchy-Hua-Aronszajn kernel is essential for proving a maximum principle.

**Theorem 21** Let \( f \in H^2(B) \). For every \( q \) with \( 0 < q < 1 \) we have the following (weak) maximum principle holds:

\[
|f(\zeta, \theta)| \leq (1 - q)^{-d} \|f\|_{H^2(B)} \quad \text{for all } |\zeta| \leq q, \quad \zeta \in S^{d-1}.
\]

More generally, for every mixed derivative \( D^\alpha_{\zeta, \theta} \) with respect to the variables \( \zeta \) and \( \theta \), we have the maximum principle

\[
|D^\alpha f(\zeta, \theta)| \leq C_1 \times (1 - q)^{-d-|\alpha|} \left[ \binom{d}{\frac{d}{2}} \left( \binom{d}{\frac{d}{2}} + 1 \right) \cdots \left( \binom{d}{\frac{d}{2}} + |\alpha| \right) \right] |\alpha||f|_{H^2(B)}
\]

for all \( |\zeta| \leq q \), and \( \zeta \in S^{d-1} \);

here the constant \( C_1 > 0 \) is independent of \( \alpha \). Respectively, for real \( \zeta = r \theta \) this gives an estimate for \( D^\alpha f(x) \).

**Proof.** Indeed, for \( |\zeta| \leq q \) and \( z = e^{i\varphi} \), if we put \( w = \zeta/z \), by the Hua-Aronszajn formula [44], it follows

\[
|f(\zeta, \theta)| \leq \frac{1}{2\pi} \int_{S^{d-1}} \int_0^{2\pi} \left| K\left(\zeta, \theta^*; \frac{1}{z}, \theta \right) |f^* (z\theta^*)| |dz| d\theta^* \right| \leq \frac{1}{2\pi} \int_{S^{d-1}} \int_0^{2\pi} \left| (1 - 2w\langle \theta, \theta^* \rangle + w^2)^{-\frac{d}{2}} \right|^2 d\varphi d\theta^* \right|^{1/2} \times \frac{1}{2\pi} \int_{S^{d-1}} \int_0^{2\pi} \left| f^* (e^{i\varphi} \theta^*) \right|^2 d\varphi d\theta^* \right|^{1/2}.
\]

Here we apply (35) with \( \cos \psi = \langle \theta, \theta^* \rangle \), which implies the inequality

\[
|(1 - 2w\langle \theta, \theta^* \rangle + w^2)| = |(1 - e^{i\varphi} w) (1 - e^{-i\varphi} w)| \geq (1 - q)^2,
\]

18
hence,
\[
\left| (1 - 2w \langle \theta, \theta' \rangle + w^2)^{\frac{1}{2}} \right| = \left| (1 - e^{i\psi}w) \left( 1 - e^{-i\psi}w \right) \right|^d \\
\geq (1 - q)^d.
\]

By Theorem 18 this implies
\[
|f(\zeta \theta)| \leq (1 - q)^{-d} \|f\|_{H^2(B)},
\]
which ends the proof of the first part of our statement.

In the same way, by differentiating under the sign of the integral in the Hua-Aronszajn formula, we obtain the estimate for the derivatives,
\[
D_\zeta^\alpha \left( \left( 1 - 2\frac{\zeta}{z} \langle \theta, \theta' \rangle + \frac{\zeta^2}{z^2} \right)^{-\frac{1}{2}} \right).
\]

By the Leibniz differentiation formula, the differentiation with respect to \(\zeta\) and \(\theta\) gives the following estimate
\[
\left| D_\zeta^\alpha \left( \left( 1 - 2\frac{\zeta}{z} \langle \theta, \theta' \rangle + \frac{\zeta^2}{z^2} \right)^{-\frac{1}{2}} \right) \right| \leq C \left( \frac{1}{1 - q} \right)^{d+|\alpha|} \left| \left( \frac{d}{2} \right) \left( -\frac{d}{2} - 1 \right) \cdots \left( -\frac{d}{2} - |\alpha| \right) \right| \times |\alpha|,
\]
and this ends the proof.

We have the following immediate corollary about the regularity of the functions in the space \(H^2(B)\).

**Corollary 22** The functions in \(H^2(B)\) belong to \(C^\infty(B)\).

The proof follows from the maximum principle in Theorem 21 since every \(f \in H^2(B)\) and the derivatives of \(f\) are uniform limits of a sequence of polynomials \(P_N(z\theta)\) and the respective derivatives of \(P_N(z\theta)\) on every compact sets \(K \times S^{d-1}\) where the compact \(K \subset \mathbb{D}\).

### 4.2 Real analytic functions and the space \(H^2(B)\)

As is the case with the classical analytic functions the elements of \(H^2(B)\) are real analytic which we prove in the next theorem.

**Theorem 23** Let \(F \in H^2(B)\). Then the function \(f(x) = F(r\theta)\) is real-analytic in the ball \(B \subset \mathbb{R}^d\).

**Proof.** We use the maximum principle: By a change of the variables, and a routine estimation with the Leibnitz rule, we obtain
\[
|D^\alpha f(x)| \leq C \max_{|\beta| = |\alpha|} \left| D^\beta_r \theta f(r \theta) \right|.
\]
The last is estimated by the maximum principle on compacts, by Theorem 21, which gives
\[ |D^3_{r,\theta} f (r\theta)| \leq C \frac{|\beta|!}{M_q^{|\beta|}} \leq C_1 \frac{|\beta|!}{M_q^{|\beta|}} \quad \text{for } r \leq q < 1. \]

The statement of the theorem follows by an equivalent definition of real-analytic function (cf. [2], p. 17).

5 BVPs for the polyharmonic operator \( \Delta^N \) and the spaces \( H^2 (\mathcal{B}) \)

5.1 The one-dimensional case

Let us provide some heuristical observations from the one-dimensional case.

As we already said, the Hardy space setting provides interpretation to real Taylor series \( f (t) = \sum_{j=0}^{\infty} a_j t^j \) for which we have \( \sum_{j=0}^{\infty} |a_j|^2 < \infty \) as the Fourier series \( \sum_{j=0}^{\infty} a_j e^{ij\varphi} \) in the Hardy space \( H^2 (\mathcal{D}) \). Thus the Hardy spaces provide a different way to encode the information of real-domain data. There is an alternative way to encode the information, by means of boundary data, e.g. by means of
\[ \left\{ \frac{d^{2j} f}{dt^{2j}} (-1), \frac{d^{2j} f}{dt^{2j}} (1) \right\}_{j \geq 0}. \]

It is more convenient to consider further differential operators defined on the functions on \( S \).

Let \( F \) be a function defined on \( S \). Indeed, if we have a function \( F \) which is representable as a series
\[ F (\varphi) = \sum_{j=0}^{\infty} a_j e^{ij\varphi} \]
then we obtain
\[ DF (\varphi) = \sum_{j=1}^{\infty} ja_j e^{i(j-1)\varphi}, \]
where we have put
\[ DF (e^{i\varphi}) := \frac{1}{i e^{i\varphi}} F' (e^{i\varphi}). \]

We want to investigate for which sequences \( \{c_j\}_{j=0}^{\infty} \) and \( \{d_j\}_{j=0}^{\infty} \) there exists a function \( F \in H^2 (\mathcal{D}) \) such that
\[ D^{2j} F (-1) = c_j, \quad D^{2j} F (1) = d_j \quad \text{for } j = 0, 1, 2, \ldots. \]

since \( D^{2j} \) is the operator \( \Delta^j \) in the one-dimensional case. The matrix of the
system for the unknown coefficients \(a_j\) is given by
\[
\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & \cdots \\
1 & -1 & 1 & -1 & 1 & 1 & 1 & \cdots \\
2 & -1 & 3 & -2 & 4 & -3 & 5 & \cdots \\
4! & -5! & 6! / 2! & & & & & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots 
\end{pmatrix}.
\]
We see that even in the one-dimensional case the map \(f^* \rightarrow \Delta^j f|_{\mathbb{R}^{d-1}}\) is not trivial; this is in fact the map to the Taylor coefficients
\[
\{D^{2j}F(-1), D^{2j}F(1)\}_{j \geq 0} \iff \{F^{(j)}(0)\}_{j \geq 0} \iff F(z) = \sum_{j=0}^{\infty} \frac{F^{(j)}(0)}{j!} z^j.
\]

### 5.2 Which polyharmonic functions are in \(H^2(\mathcal{B})\)?

The polyharmonic function in the one-dimension case satisfy \(d^N P_{N-1}(t)/dt^N = 0\) and are polynomials. Their complexifications \(P_N(z)\) belong to the Hardy spaces in the disc. On the other hand, not all polyharmonic function (of fixed finite order \(N\)) belong to the polyharmonic Hardy spaces \(H^2(\mathcal{B})\). In the present section we characterize those polyharmonic functions which belong to \(H^2(\mathcal{B})\).

We will characterize the polyharmonic functions by means of their boundary properties. The main point is that the polyharmonic functions in a domain \(D \subset \mathbb{R}^d\) (and more general domains) may be "parametrized" by considering the Dirichlet problem
\[
\Delta^N u = 0 \quad \text{in } D
\]
\[
\Delta^j u = g_j \quad \text{on } \partial D \quad \text{for } j = 0, 1, 2, \ldots, N - 1.
\]
By means of the classical Green formulas we may find \(u(x)\) for \(x \in D\). On the other hand, if the function \(u\) which has a \(r\)-complexification to ball \(B\) of the Klein-Dirac quadric, then by means of Cauchy type formula (44) we may recover \(u(z, \theta)\) using its values \(u(e^{i \varphi}, \theta)\) on the boundary \(\partial B = \mathbb{S} \times \mathbb{S}^{d-1}/\mathbb{Z}_2\). Thus we see the boundary values \(u(e^{i \varphi}, \theta)\) provide an alternative parametrization for the polyharmonic functions. It is essential, as in the one-dimensional case, to provide a relation between the boundary data \(\{g_j\}_{j=0}^{N-1}\) and \(u(e^{i \varphi}, \theta)\).

The following theorem establishes the link between the polyharmonic BVPs and the boundary values of the elements in \(H^2(\mathcal{B})\).

**Theorem 24** Let \(u\) be a polyharmonic function of order \(N \geq 1\) in the ball \(B \subset \mathbb{R}^d\), i.e. \(\Delta^N u(x) = 0\) in \(B\). Then the \(r\)-complexification of \(u\) satisfies \(u \in H^2(\mathcal{B})\) if and only if
\[
\Delta^j u|_{\partial B} \in H^{-j}(\mathbb{S}^{d-1}), \quad \text{for } j = 0, 1, \ldots, N - 1,
\]
where \(H^s(\mathbb{S}^{d-1})\) denotes the Sobolev space of exponent \(s\).

**Proof.** First recall some properties of the polyharmonic operator \(\Delta^N\). The following operator is defined by
\[
L_{(k)} f := \frac{1}{r^{d+k-1}} \frac{d}{dr} \left[ r^{d+2k-1} \frac{d}{dr} \left[ \frac{1}{r^k} f \right] \right],
\]
see p. 152 in [24]. For $f(r) = r^{k+2s}$, $k, s \geq 0$, we obtain

\[ L_{(k)} f(r) = \frac{1}{r^{d+k+1}} \frac{d}{dr} \left[ r^{d+2k-1} \frac{1}{r^d} \right] f \left( \frac{r}{r_k+2s} \right) = \frac{1}{r^{d+k-1}} \frac{d}{dr} \left[ r^{d+2k-1} \frac{d}{dr} \left( r^{2s} \right) \right] \]

\[ = 4s \left( \frac{d}{2} + k + s - 1 \right) r^{k+2s-2}. \]

Hence, for every $j \leq s$ holds

\[ L_{(k)} \left[ r^{k+2s} \right] = 4^j s (s-1) \cdots (s-j+1) \times \left( \frac{d}{2} + k + s - 1 \right) \cdots \left( \frac{d}{2} + k + s - j \right) r^{k+2s-2j}. \]

Let us put

\[ \gamma_{j,s}^k := L_{(k)} \left[ r^{k+2s} \right]_{r=1}. \]

Obviously,

\[ \gamma_{j,s}^k = 0 \quad \text{for } s \leq j - 1. \]

Let the function $u(x)$ have the expansion

\[ u(x) := \sum_{k=0}^{\infty} \sum_{\ell=1}^{N-1} \sum_{j=0}^{m} a_{k,\ell,j} r^{k+2j} Y_{k,\ell}(\theta), \]

and for the boundary data we have the Laplace-Fourier series expansion

\[ g_m(\theta) := \sum_{k=0}^{\infty} \sum_{\ell=1}^{N-1} g_{k,\ell}^m Y_{k,\ell}(\theta). \]

Hence, by formula in [24], p. 165 for the Laplace operator, we have

\[ g_m(\theta) = \Delta^m u(x)_{|r=1} = \sum_{k=0}^{\infty} \sum_{\ell=1}^{N-1} \sum_{j=0}^{m} a_{k,\ell,j} L_{(k)} \left[ r^{k+2j} \right]_{|r=1} Y_{k,\ell}(\theta) \]

\[ = \sum_{k=0}^{\infty} \sum_{\ell=1}^{N-1} \sum_{j=0}^{m} a_{k,\ell,j} \gamma_{m,j}^k Y_{k,\ell}(\theta). \]

By comparing the coefficients we obtain the system of equations

\[ g_{k,\ell}^m = \sum_{j=m}^{N-1} u_{k,\ell,j} \gamma_{m,j}^k \quad \text{for } m = 0, 1, \ldots, N-1. \]

Hence, the map $u \leftrightarrow \{g_j\}_{j=0}^{N-1}$ is given by the infinitely many matrices for $k = 0, 1, 2, \ldots,$

\[ U_k = \begin{pmatrix}
1 & 1 & 1 & \cdots & 1 \\
0 & \gamma_{1,1}^k & \gamma_{1,2}^k & \cdots & \gamma_{1,N-1}^k \\
0 & 0 & \gamma_{2,2}^k & \cdots & \gamma_{2,N-1}^k \\
& \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & \cdots & \gamma_{N-1,N-1}^k
\end{pmatrix}. \]
The result follows from the Lemma 25 below, and by application of the Cauchy-Bunyakowski-Schwarz inequality to the norm
\[
\|u\|_{H^2(B)}^2 = \sum_{j=0}^{N-1} \sum_{k=0}^{\infty} \sum_{\ell=1}^{a_k} |u_{k,\ell,j}|^2
\]
with equations (50).

We have the following technical lemma.

**Lemma 25** The solution of the system (50) is given by
\[
u_{k,\ell,j} = \sum_{p=j}^{N-1} \frac{D_{k,p}}{D_k} \times \frac{g_{k,\ell}^p}{\left(\frac{d}{2} + k\right)^j}
\]
where \(D_k = \det U_k\), and \(D_{k,p}^{p,j}\) are the appropriate minors of \(U_k\) given by the Cramer’s rule. These coefficients satisfy
\[
\frac{D_{k,p}^{p,j}}{D_k} \to C_{p,j} \quad \text{for } k \to \infty,
\]
for appropriate constants \(C_{p,j}\).

**Proof.** In the system (50) we divide the \(j\)-th row by \(\left(\frac{d}{2} + k\right)^{j-1}\). On the right-hand side we obtain the vector \(\left(g_{k,\ell}^p\right)_{j=0}^{N-1}\), and the coefficients of the system of the left-hand side (50) become equal to
\[
\frac{\gamma_{j,s}^k}{\left(\frac{d}{2} + k\right)^j} \quad \text{for } j \leq s \leq N - 1.
\]
From formula (48) for the constants \(\gamma_{j,s}^k\) we see that
\[
\frac{\gamma_{j,s}^k}{\left(\frac{d}{2} + k\right)^j} \to F_{j,s} \quad \text{for } k \to \infty,
\]
for some appropriate constants \(F_{j,s}\) which are non-zero. This ends the proof since the minors \(D_{k,p}^{p,j}\) and \(D_k\) are computed through the values \(\gamma_{j,s}^k\).

We see that Theorem 24 provides us with a large class of functions defined on the ball \(B \subset \mathbb{R}^d\) which are extendable to the ball \(B\) of the Klein-Dirac quadric. It is also possible to consider functions which are in a certain sense polyharmonic of infinite order, as those studied by Aronszajn, Lelong, Avanissian, and others, cf. the references in [2], [3], [29].

6 Error estimate of Polyharmonic Interpolation and Cubature formulas

The topic of estimation of quadrature formulas for analytic functions is a widely studied one. Beyond the classical monographs [35], [12], we provide further and
more recent publications, as [5], [16], [17], [23], [31], [37]. No references may be found though for the multivariate case, for cubature formulas, even in the fundamental monographs as [51], [55], [52]: see also the recent survey [10].

Our main framework of Interpolation and Cubature was defined in [27], [18], [25]. It has further brought to life the multivariate complexification and the polyharmonic Hardy spaces.

We will consider first polyharmonic interpolation, and as second, polyharmonic Cubature formulas for approximating integrals of the type

\[ \int_B f(x) d\mu(x) \]

over the unit ball \( B \subset \mathbb{R}^d \), where \( \mu(x) \) is a signed measure of special type. The main purpose of the present section is to find error estimates for the Interpolation and Cubature in the case of functions \( f \in H^2(B) \).

6.1 One-dimensional case

First, following the classical scheme outlined in [12], p. 303−306, (see also chapter 12 in [35]), we will remind how one finds the error for the classical Quadrature formulas.

Let the points \( t_0, t_1, \ldots, t_N \) belong to the interval \([a,b]\). We define

\[ \omega_N(z) = (z - t_0)(z - t_1) \cdots (z - t_N). \]

Let the function \( f \) be analytic in a simply connected (open) domain \( D \subset \mathbb{C} \) containing the interval \([a,b]\) with boundary \( \partial D = \Gamma \). Then the interpolation polynomial \( P_N(t) \) satisfying \( P_N(t_j) = f(t_j) \) for \( j = 0, 1, \ldots, N \) is given by

\[ P_N(z) = \sum_{j=0}^{N} f(t_j) \frac{\omega(z)}{\omega'(t_j)(z - t_j)} = f(z) - \frac{1}{2\pi i} \int_{\Gamma} \frac{\omega(z)f(t)}{\omega(t)(t - z)} dt, \]

where \( \Gamma \) is considered as a contour oriented counterclockwise. Hence, the remainder is

\[ f(z) - P_N(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{\omega(z)f(t)}{\omega(t)(t - z)} dt. \]

Now, if \( \mu \) is a non-negative Stieltjes measure, say \( d\mu(t) = w(t) dt \), the quadrature formula

\[ \int_{a}^{b} f(t) d\mu(t) \approx \sum_{j=0}^{N} \lambda_j f(t_j) \]

is called interpolatory quadrature formulas of degree \( N \) if it satisfies the following equality

\[ \sum_{j=0}^{N} \lambda_j Q(t_j) = \int_{a}^{b} P_N(t) d\mu(t) \]
for every polynomial $Q_N$ of degree $\leq N$. This implies that
\[
\lambda_j = \int_a^b \frac{\omega(t)}{(t - t_j) \omega'(t_j)} d\mu(t);
\]
cf. [12], p. 303, or Krylov, [35], chapter 12. Hence, for the error of such formula we obtain
\[
E(f) := \int_a^b (f(z) - P_N(z)) d\mu(z) = \frac{1}{2\pi i} \int_a^b \left( \int_\Gamma \frac{\omega(t) f(t)}{\omega(t)(t - z)} dt \right) d\mu(z).
\]
This may be directly estimated by
\[
|E(f)| \leq \frac{L_\Gamma}{2\pi} \max_{t \in \Gamma} |f(t)| \times \frac{1}{d^{N+1}} \frac{(b - a) D^{N+1}}{\delta_{\Gamma}} \int_a^b |d\mu(z)|
\]
where $d := \min_j (\text{dist}(t_j, \Gamma))$, $D := \max_j (\text{dist}(t_j, \{a, b\}))$, $\delta_{\Gamma} := \min \text{dist} ([a, b], \Gamma)$, and $L_\Gamma$ is the length of the contour $\Gamma$.

### 6.2 Multivariate Interpolation

Now we consider the multivariate case.

Let us assume that the number $b$ satisfies
\[
0 < b < 1.
\]
Choose the domain $D = B_1$ and a function $f \in H^2(B)$, assuming that $f$ has the expansion
\[
f(z, \theta) = \sum_{k=0}^{\infty} \sum_{\ell=1}^{a_k} f_{k,\ell} (z^2) z^k Y_{k,\ell} (\theta) \quad \text{for } |z| < 1, \text{ and } \theta \in S^{d-1}.
\]
Let $N \geq 0$ be a fixed integer. We will consider polyharmonic interpolation which has been studied in [18]. Let the points $\{r_{k,\ell,j}\}_{j=0}^{N}$ belong to the interval $[0, b]$. We consider the following series:
\[
P_N(z, \theta) = \sum_{k=0}^{\infty} \sum_{\ell=1}^{a_k} p_{k,\ell} (z^2) z^k Y_{k,\ell} (\theta),
\]
where and $p_{k,\ell}$ are polynomials of degree \leq $N$ satisfying the interpolation conditions
\[
p_{k,\ell} (r_{k,\ell,j}^2) = f_{k,\ell} (r_{k,\ell,j}^2) \quad \text{for } j = 0, 1, \ldots, N.
\]
We prove below that the series (51) is convergent.

For the remainder of this interpolation we have
\[
f(z, \theta) - P_N(z, \theta) = \sum_{k=0}^{\infty} \sum_{\ell=1}^{a_k} \left[ f_{k,\ell} (z^2) - p_{k,\ell} (z^2) \right] z^k Y_{k,\ell} (\theta).
\]
Now define as above the functions
\[
\omega_{k,\ell} (z) = (z - r_{k,\ell,0}^2) (z - r_{k,\ell,1}^2) \cdots (z - r_{k,\ell,N}^2),
\]
where and $r_{k,\ell,j}$ are the points of the domain $D = B_1$.
and consider the oriented contour $\Gamma(t) = e^{it}$ for $t \in [0, 2\pi]$. For all $z$ with $|z| \leq b$ and $\theta \in S^{d-1}$, and obtain the estimate

$$|f(z, \theta) - P_N(z, \theta)| \leq \sum_{k=0}^{\infty} \sum_{\ell=1}^{a_k} \left| \int_{\Gamma} \frac{\omega_{k, \ell}(\tau) f_{k, \ell}(\tau^2) 2\tau}{\omega_{k, \ell}(\tau^2)(\tau^2 - z^2)} d\tau \right| z^k Y_{k, \ell}(\theta) \leq C \frac{2^{N+2}}{(1 - b)^{N+2}} \sum_{k=0}^{\infty} \sum_{\ell=1}^{a_k} \int_{\Gamma} |f_{k, \ell}(\tau^2)| |d\tau| \times b^k Y_{k, \ell}(\theta) \leq C \frac{2^{N+2}}{(1 - b)^{N+2}} \|f\|_{H^2(B_b)} b^k k^{\frac{d+2}{d-2}}.$$ 

Since $f \in H^2(B)$ the last inequality shows, after application of Cauchy-Bunyakowski-Schwarz inequality, that the series $\sum_{k=0}^{\infty} \sum_{\ell=1}^{a_k} \left[ f_{k, \ell}(z^2) - p_{k, \ell}(z^2) \right] z^k Y_{k, \ell}(\theta)$ is absolutely and uniformly convergent. Hence, the series (51) representing the function $P_N(z, \theta)$ is also such.

Above we have outlined the most important arguments for proving the following:

**Theorem 26** Let $f \in H^2(B)$. Let the points $\{r_{k, \ell,j}\}_{j=0}^{N}$ belong to the interval $[0, b]$, where $b < 1$. Then the function $P_N(z, \theta)$ defined by the series (57) is polyharmonic of order $N+1$ in the ball $B_b \subset \mathbb{R}^d$ and belongs to the polyharmonic Hardy space $H^2(B_b)$, while the following inequality holds:

$$\|P_N\|_{H^2(B_b)} \leq C_{N,b} \|f\|_{H^2(B)}.$$

### 6.3 Multivariate Polyharmonic Cubature

The class of **pseudo-positive** measures used for our cubature formula $C_N(f)$ is now defined in the following way: a signed measure $\mu$ with support in $B_R \subset \mathbb{R}^d$ is **pseudo-positive with respect to the orthonormal basis** $Y_{k, \ell}, \ell = 1, \ldots, a_k$, $k \in \mathbb{N}_0$ if the inequality

$$\int_{\mathbb{R}^d} h(|x|) Y_{k, \ell}(x) d\mu(x) \geq 0$$

holds for every non-negative continuous function $h : [a, b] \to [0, \infty)$ and for all $k \in \mathbb{N}_0, \ell = 1, 2, \ldots, a_k$. Let us note that every signed measure $d\mu$ with bounded variation may be represented (non-uniquely) as a difference of two pseudo-positive measures. We refer to [27] for instructive examples of pseudo-positive measures.
Let the pseudo-positive (signed) measure $d\mu$ be given in the ball $B_b \subset \mathbb{R}^d$.
For all indices $(k,\ell)$ the component measures are defined by

$$d\mu_{k,\ell} (r) := \int_{S^{d-1}} Y_{k,\ell} (\theta) \, d\mu (r \theta) \geq 0 \quad \text{for all } r \in [0, b] ; \quad (54)$$

here the integral is symbolical with respect to the variables $\theta$. Rigorously, the component measure $d\mu_{k,\ell} (r)$ is defined for the functions $g(r)$ on the interval $[0, b]$ by means of the equality

$$\int_0^b g(r) \, d\mu_{k,\ell} (r) := \int_{B_b} g(r) \, Y_{k,\ell} (\theta) \, d\mu (x) ;$$

cf. [25], [27].

In [25], [27], we have considered a special type of Cubature formula, the so-called polyharmonic Gauss-Jacobi Cubature formula. Here however we will consider more generally, interpolatory polyharmonic Cubature formulas and will prove their convergence and error estimate for them. The case of the annulus has been considered by us in [30].

Let us fix $(k,\ell)$. We assume that there exist points $t_{k,\ell,j}$, $j = 0, 1, \ldots, N$, belonging to the interval $[0, b]$, and numbers $\{\lambda_{k,\ell,j}\}_{j=1}^N$, such that the following interpolatory quadrature formula holds:

$$\int_0^b Q(r) \, d\mu_{k,\ell} (r) = \sum_{j=0}^N \lambda_{k,\ell,j} Q (t_{k,\ell,j}) \quad \text{for every } Q \in V_{k,N} ; \quad (55)$$

here the set $V_{k,N}$ is given by

$$V_{k,N} = \{r^{k+2j}\}_{j=0}^N .$$

We define the polyharmonic interpolatory Cubature formula by

$$C_N (f) := \sum_{k=0}^{\infty} \sum_{\ell=1}^{a_k} \sum_{j=0}^N \lambda_{k,\ell,j} f_{k,\ell} (t_{k,\ell,j}) . \quad (56)$$

For the interpolation polyharmonic function $P_N$ defined in [31] we obtain equality

$$\int_{B} P_N (x) \, d\mu (x) = C_N (P_N) .$$

Hence, the remainder of the polyharmonic Cubature formula is given by

$$E (f) = \int_{B} \left( f (r, \theta) - P_N (r, \theta) \right) \, d\mu (z, \theta)$$

$$= \sum_{k=0}^{\infty} \sum_{\ell=1}^{a_k} \int_0^1 \left[ f_{k,\ell} (r^2) - p_{k,\ell} (r^2) \right] r^k d\mu_{k,\ell} (r) ,$$
which implies the estimate

\[ |E(f)| \leq \sum_{k=0}^{\infty} \sum_{\ell=1}^{a_k} \left| \int_0^1 \left[ f_{k,\ell}(r^2) - p_{k,\ell}(r^2) \right] r^k d\mu_{k,\ell}(r) \right| \leq \sum_{k=0}^{\infty} \sum_{\ell=1}^{a_k} \left| \int_0^1 \int_{\Gamma} \frac{\omega_{k,\ell}(r^2) f_{k,\ell}(r^2)}{\omega_{k,\ell}(r^2)} d\tau \times r^k d\mu_{k,\ell}(r) \right| \leq \frac{C_N}{(1-b)^{N+2}} \sum_{k=0}^{\infty} \sum_{\ell=1}^{a_k} \|f_{k,\ell}\|_{H^2(B)} \times \int_0^1 r^k d\mu_{k,\ell}(r). \]

This proves the following result:

**Theorem 27** Let \( f \in H^2(B) \). Let the points \( \{r_{k,j}\}_{j=0}^{N} \) belong to the interval \([0,b]\) with \( b < 1 \). Then the polyharmonic cubature formula defined by (56) with remainder \( E(f) = \int_B f(x) d\mu(x) - C_N(f) \) satisfies the following estimate

\[ |E(f)| \leq \frac{C_N}{(1-b)^{N+2}} \sum_{k=0}^{\infty} \sum_{\ell=1}^{a_k} \|f_{k,\ell}\|_{H^2(B)} \times \int_0^1 r^k d\mu_{k,\ell}(r). \]

7 Conclusions

1. Our research may be considered as a contribution to the topic of analytic continuation of solutions to elliptic equations (in particular, harmonic functions), see the discussion and references to the works of V. Avanissian, P. Lelong, C. Kiselman, J. Siciak, M. Jarnicki, T. du Cros, on p. 54–55 in [2], [3], [38], [15], [26], [29]. Our construction of \( r \)-analytic continuation is applicable to domains as annuli, strips and other domains with symmetry in \( \mathbb{R}^d \). The case of the annulus has been considered in [30], while the case of strip and other domains will be considered in [28].

2. The concept of polyharmonic Hardy spaces appears to be a new multivariate concept which differs from the existing approaches in several complex variables, cf. [53], [54], [45], [46], [9], [48], [50].

3. We have seen that the space of \( r \)-analytic functions on the Klein-Dirac quadric provides an useful setting for estimation of the remainders in Interpolation and Cubature. Although the space of such functions is \( 1 - 1 \) mapped to a Hardy space of holomorphic functions of several complex variables on the Lie ball, our approach has a non-trivial counterpart on the annulus which is not obtained from \( C^d \) constructions, cf. [30]. Our approach which is based on \( r \)-analytic continuation of solutions to elliptic equations (in particular, polyharmonic functions) provides non-trivial constructions of Hardy spaces on complexified annulus, strip and other symmetric domains in \( \mathbb{R}^d \), which are not obtained by the standard approach to holomorphic functions in \( C^d \).
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