BFT Embedding of Second-Class Systems

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The embedding procedure of Batalin, Fradkin, and Tyutin, which allows to convert a second-class system into first-class, is pushed beyond the formal level. We study nonrelativistic as well as relativistic systems. We explicitly construct, in all cases, the variables of the converted first-class theory in terms of those of the corresponding second-class one. Moreover, we only conclude about the equivalence between these two different kind of theories after comparing their respective spectra of excitations.

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I. INTRODUCTION

Constrained systems possessing only second-class constraints, also known as second-class systems, are to be quantized by abstracting the basic equal-time commutators (anticommutators) from the corresponding Dirac brackets, the constraints thereby translating into strong operator relations. Then, the classical-quantum transition may be afflicted by ambiguities, which recognize as common origin the noncanonical structure of the Dirac brackets. This problem does not arise in connection with first-class systems, i.e., constrained systems whose constraints are all first-class, because in this case one can retain the canonical structure for the equal-time commutators (anticommutators) while imposing the constraints as restrictions on the states. Thus, it would be desirable to be able of converting any second-class system into first-class. This is precisely what the Batalin-Fradkin-Tyutin embedding procedure (BFT) does for us.

However, the gauge invariant quantization procedure described above, which was also proposed by Dirac, is not operational in most cases of physical interest. For instance, for non-Abelian gauge theories no one has yet succeeded in constructing a vacuum state of finite norm being annihilated by the corresponding Gauss law constraint. In fact, the quantization of non-Abelian gauge field theories is to be performed in a fixed gauge, where constraints and gauge conditions form a set of second-class constraints. Again, the equal-time commutators are to be abstracted from the corresponding Dirac bracket, which bring us back to the problem we were trying to avoid.

We believe that the importance of the BFT conversion mechanism rests more on the fact that it provides an efficient tool to generate a set of quantum mechanically equivalent theories. This paper is dedicated to a detailed study of this equivalence.

We start, in Section 2, by proposing a second-class nonrelativistic system (a toy model) whose quantization via Dirac brackets can be fully carried out and is free of ambiguities. In Section 3, the BFT embedding procedure is used to generate the corresponding first-class counterpart. After verifying the existence of a unitary gauge, we implement a canonical transformation which enable us to construct all the phase-space variables of the first-class theory in terms of those of the second-class one. As we shall see, the first-class theory can be formulated in terms of gauge invariant variables only. Also, the number of degrees of freedom of the gauge invariant theory is larger than those of the second-class theory. Hence, it becomes non trivial to establish in what sense these two theories are classically and quantum mechanically equivalent. The outcomes from the Stückelberg embedding mechanism, for the same problem, are discussed in Section 4.

The self-dual (SD) model of Townsend, Pilch and Van Nieuwenhuizen has recently served as a testing ground for applying the BFT embedding procedure in the relativistic case. After its conversion into first-class, the SD model appears to be quantum mechanically equivalent to the Maxwell-Chern-Simons (MCS) theory when formulated in a Coulomb like gauge. In Section 5 we generalize the strategy of Section 3, based on canonical transformations, and build up the phase-space variables of the MCS theory, in any arbitrary gauge, in terms of those of the SD model. Then, the SD and the MCS theories will be shown to be rigorously equivalent irrespective of any gauge election.

The models in the previous sections only contain bosonic variables. In Section 6 we present the BFT embedding of the Proca-Wentzel field minimally coupled to fermions. As known, this is a second-class theory possessing bosonic as well as fermionic second-class constraints. This time is a non-linear transformation which enables us to write the converted first-class theory solely in terms of gauge invariant fields.

The conclusions are contained in Section 7.

II. A TOY MODEL

We devote this section to study the classical and quantum dynamics of the nonrelativistic model whose Lagrangian is

$$L = \frac{1}{2} m \omega \left( q^a \epsilon_{ab} q^b - \omega q^a g_{ab} q^b \right). \quad (2.1)$$

Here, \( m \) is a mass parameter, \( a \) runs from 1 to 2, \( g_{ab} \) is the metric tensor of a two-dimensional Euclidean space and \( \epsilon_{ab} \) is the completely antisymmetric tensor (\( \epsilon_{12} = +1 \)). The first term in the right-hand side of (2.1) is reminiscent of the Chern-Simons structure in three space-time dimensions, while the second is a “mass” term.

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1See, for instance, Refs.
The Lagrange equations of motion deriving from (2.1),
\[ \dot{q}^a + \omega \epsilon^{ab} q_b = 0 \] (2.2)
tell us that the system under analysis is just a two dimensional particle in uniform circular motion, i.e.,
\[ q^a q_a = (q^1)^2 + (q^2)^2 = \text{constant} \] (2.3)
with (constant) angular velocity equal to \( \omega \). Clearly, the energy \( H^{(0)} \) and the angular momentum \( M^{(0)} \),
\[ M^{(0)} = \frac{H^{(0)}}{\omega} = \frac{m \omega}{2} q^a q_a = \text{constant} \] (2.4)
are conserved quantities.

We shall denote by \( p_a \) the momentum canonically conjugate to the coordinate \( q^a \). Then, within the Hamiltonian framework, the system is characterized by the primary constraints
\[ T_a^{(0)} = p_a + \frac{1}{2} m \omega \epsilon_{ab} q^b \approx 0 \] (2.5)
while the canonical Hamiltonian reads
\[ H^{(0)} = \frac{m \omega^2}{2} q^a q_a \] (2.6)
Since the Poisson bracket \( \{.,.\}_P \)
\[ \{T_a^{(0)}, T_b^{(0)}\}_P = m \omega \epsilon_{ab} \] (2.7)
do not vanishes, the persistence in time of the primary constraints does not give rise to secondary constraints and, hence, all the constraints are second-class. The usual counting reveals that only one independent degree of freedom is present in (2.1).

The system can be quantized through the Dirac bracket quantization procedure (DBQP). According to this, all phase space variables are to be promoted to operators obeying an equal-time commutator algebra which is to be abstracted from the corresponding Dirac bracket algebra. For the model under analysis, one easily finds that
\[ [q^a, q^b] = -\frac{i \hbar}{m \omega} \epsilon^{ab} \] (2.8a)
\[ [q^a, p_b] = \frac{i \hbar}{2} \delta^a_b \] (2.8b)
\[ [p_a, p_b] = -\frac{i \hbar}{4} m \omega \epsilon_{ab} \] (2.8c)
As for the quantum mechanical Hamiltonian it can be read off directly from (2.6), in view of the absence of ordering ambiguities in the classical-quantum transition.

Within the algebra (2.3) the constraints (2.5) hold as strong identities. Therefore, we use them to eliminate from the game \( q^2 \) and \( p_2 \). We, also, define
\[ q \equiv q^1 = \frac{2}{m \omega} p_2 \] (2.9a)
\[ p = 2 p_1 = -m \omega q^2 \] (2.9b)
as the variables spanning the reduced phase-space \( (\Gamma^*) \) of the system. As required (2.4), they verify the canonical commutation relation \( [q,p] = i \hbar \). Then, from (2.6) follows that the reduced phase-space Hamiltonian is

\[ \text{Throughout this paper the sign of weak equality (\( \approx \)) is used in the sense of Dirac}. \]

\[ \text{We shall not distinguish between a quantum operator and its classical counterpart}. \]
\[ H^{(0)*} = \frac{1}{2m} p^2 + \frac{m}{2} \omega^2 q^2 , \quad (2.10) \]

which describes an one dimensional harmonic oscillator with mass \( m \) and proper frequency \( \omega \). We shall designate by \( |n \rangle, n = 0, +1, +2, \ldots \), the eigenstates of \( H^{(0)*} \), and by \( E_n \),

\[ E_n = (n + 1/2) \hbar |\omega| , \quad (2.11) \]

the corresponding eigenvalues.

The expression for the quantum mechanical angular momentum operator can not be obtained by promoting \( (2.4) \) to the quantum regime. Indeed, if one insists on \( M^{(0)*} = H^{(0)*}/\omega \), \( M^{(0)*} \) fails to annihilate the ground state \( |n = 0 \rangle \). Because of this, we substitute \( (2.4) \) by

\[ M^{(0)*} = H^{(0)*} \omega - \frac{1}{2} \hbar |\omega| . \quad (2.12) \]

To summarize, \( M^{(0)*} \) and \( H^{(0)*} \) possess common eigenstates and the eigenvalues of \( M^{(0)} \) are \( m_n \hbar \) with

\[ m_n = n \epsilon(\omega) , \quad (2.13) \]

where \( \epsilon \) denotes the sign function.

We shall also need, for future purposes, the functional formulation of the quantum dynamics. The Green functions generating functional \( (W) \) for second-class systems was derived by Senjanovic \[15\]. Since in the present case the determinant \( \text{det} \left[ T^{(0)}_a, T^{(0)}_b \right]_p \) is just a number, the expression for \( W \) reduces to

\[ W = N \int \prod_{a=1}^2 Dq^a \prod_{a=1}^2 Dp_a \left( \prod_{a=1}^2 \delta[p_a + \frac{1}{2} m \omega \epsilon_{ab} q^b] \right) \times \exp \left[ i \int_{-\infty}^{+\infty} dt \left( p_a q^a - \frac{m \omega^2}{2} q^a q_a \right) \right] , \quad (2.14) \]

where \( N \) is a normalization constant. After performing the momentum integrations one obtains

\[ W = N \int \prod_{a=1}^2 Dq^a \exp \left[ i \int_{-\infty}^{+\infty} dt m \omega \left( q^a \epsilon_{ab} \dot{q}^b - \omega q^a g_{ab} q^b \right) \right] , \quad (2.15) \]

which says that the effective Lagrangian coincides with \( (2.7) \). A further integration on \( q^2 \) yields

\[ W = N \int [Dq] \exp \left[ i \int_{-\infty}^{+\infty} dt \left( \frac{m}{2} \dot{q}^2 - \frac{m}{2} \omega^2 q^2 \right) \right] , \quad (2.16) \]

where we have used \( (2.9a) \). This confirms that the reduced system is an one dimensional harmonic oscillator. The analysis of the toy model as a second-class system is by now complete.

III. BFT EMBEDDING OF THE TOY MODEL

We next use the BFT \[2\] procedure to convert the second-class system described in the previous Section into first-class\[1\]. For this purpose, one starts by introducing an additional pair of canonical variables (coordinate \( u^a \) and momentum \( s_a \)) for each second-class constraint. The new constraints and the new Hamiltonian are found, afterwards, through an iterative scheme which, in the present case, ends after a finite number of steps. Presently, the BFT conversion procedure yields

\[ T^{(0)}_a \longrightarrow T^{(0)}_a = T^{(0)}_a + T^{(1)}_a = p_a + \epsilon_{ab} \left( \frac{1}{2} m \omega q^b + \sqrt{m \omega} z^b \right) \approx 0 , \quad (3.1) \]

\(^4\)For a detailed description of the BFT procedure we refer the reader to the original papers in Ref. \[2\].
\[
H^{(0)} \rightarrow H = H^{(0)} + H^{(1)} + H^{(2)} = \frac{m\omega^2}{2} \left( q^a + \frac{1}{\sqrt{m\omega}} z^a \right)^2 ,
\]
where the following definition
\[
z^a = -\frac{1}{2} \dot{u}^a - \epsilon^{ab} s_b ,
\]
has been introduced. We emphasize that the \(z^a\)'s are composite objects whose Poisson bracket algebra,
\[
\{z^a, z^b\}_P = -\epsilon^{ab} ,
\]
derives from the canonical algebra
\[
\{u^a, u^b\}_P = 0 , \quad \{s_a, s_b\}_P = 0 , \quad \{u^a, s_b\}_P = \delta^a_b .
\]
One can easily check that the new constraints and the new Hamiltonian are, as required, strong under involution, i.e.,
\[
\{T_a, T_b\}_P = 0 , \quad \{T_a, H\}_P = 0 .
\]
The converted system is, indeed, first-class and obeys an Abelian involution algebra.

We construct next the unitarizing Hamiltonian \((H_U)\) and the corresponding Green functions generating functional \((W_\chi)\). If we denote by
\[
\Psi \equiv \bar{C}_a \chi^a - \bar{P}_a \lambda^a ,
\]
\[
\Omega \equiv \bar{\pi}_a P^a + T_a C^a ,
\]
the gauge fixing fermion function and the BRST charge, respectively, one has that
\[
H_U = H - \{\Psi, \Omega\}_P .
\]
Here, \(C^a\) and \(\bar{C}_a\) are ghost coordinates and \(\bar{P}_a\) and \(P^a\) their respective canonical conjugate momenta. Furthermore, \(\lambda^a\) is the Lagrange multiplier associated with the constraint \(T_a\) and \(\bar{\pi}_a\) is its canonical conjugate momentum. The gauge conditions \(\chi^a\) are to be chosen such that
\[
det \{\chi^a, T_b\}_P \neq 0 .
\]
The generating functional \(W_\chi\), corresponding to the unitarizing Hamiltonian \((H_U)\), is
\[
W_\chi = \mathcal{N} \int [D\sigma] \exp(iA_U) ,
\]
where the unitarizing action \(A_U\) is given by
\[
A_U = \int \left( p_a \dot{q}^a + s_a \dot{u}^a + \bar{\pi}_a \dot{\lambda}^a + \bar{C}_a \dot{P}^a + \bar{P}_a \dot{C}^a - H_U \right) ,
\]
and the integration measure \([D\sigma]\) involves all the variables appearing in \(A_U\). To complete the characterization of the converted system, we mention that under an infinitesimal supertransformation generated by \(\Omega\), the phase space variables change as follows.
\[
\begin{align*}
\delta q^a & \equiv [q^a, \Omega]_p \varepsilon = C^a \varepsilon, \\
\delta p_a & \equiv [p_a, \Omega]_p \varepsilon = \frac{m\omega}{2} \epsilon_{ab} C^b \varepsilon, \\
\delta u^a & \equiv [u^a, \Omega]_p \varepsilon = \sqrt{m\omega} C^a \varepsilon, \\
\delta s_a & \equiv [s_a, \Omega]_p \varepsilon = -\sqrt{\frac{m\omega}{4}} \epsilon_{ab} C^b \varepsilon.
\end{align*}
\]

where \(\varepsilon\) is an infinitesimal fermionic parameter.

We now focus on Eq. (3.11) and restrict ourselves to consider gauge conditions which do not depend upon \(\lambda^a\) and/or \(\bar{\pi}_a\). Then, the rescaling \(\chi^a \rightarrow \chi^a/\beta\), \(\bar{\pi}_a \rightarrow \bar{\pi}_a/\beta\) and \(\bar{C}_a \rightarrow \beta \bar{C}_a\) allows, at the limit \(\beta \rightarrow 0\), to carry out all the integrals over the ghosts and multiplier variables \([16,17]\), with the result

\[
W_\chi = \mathcal{N} \int \prod_{a=1}^{2} Dq^a [\prod_{a=1}^{2} Dp_a] [\prod_{a=1}^{2} Du^a] [\prod_{a=1}^{2} Ds_a] \det [\chi^a, T_b]_p \\
\times \left( \prod_{a=1}^{2} \delta[T_a] \right) \left( \prod_{a=1}^{2} \delta[\chi^a] \right) \exp \left[ i \int_{-\infty}^{+\infty} dt \left( p_a \dot{q}^a + s_a \dot{u}^a - H \right) \right].
\]

The structure of the Hamiltonian (see Eq. (3.2)) suggests the change of variables \(u^a \rightarrow u'^a = z^a, s_a \rightarrow s'_a = s_a\), whose Jacobian is a nonvanishing real number. Since \(H\) does not depend upon \(s'_a\), the corresponding integration is straightforward and after performing it one obtains

\[
W_\chi = \mathcal{N} \int \prod_{a=1}^{2} Dq^a [\prod_{a=1}^{2} Dp_a] [\prod_{a=1}^{2} Dz^a] \det [\chi^a, T_b]_p \\
\times \left( \prod_{a=1}^{2} \delta[T_a] \right) \left( \prod_{a=1}^{2} \delta[\chi^a] \right) \exp \left[ i \int_{-\infty}^{+\infty} dt \left( p_a \dot{q}^a + \frac{1}{2} z^a \epsilon_{ab} \dot{z}^b - H \right) \right].
\]

Notice that, up to a surface term, \(1/2 \int dt z^a \epsilon_{ab} \dot{z}^b\) can be rewritten in the standard canonical form \(\int dt z^1 \dot{z}^2\). The interpretation of \(\dot{z}^1\) as the canonical conjugate momentum of \(z^2\), suggested by \([3,3]\), is then possible but not mandatory. Finally, we take advantage of the functions \(\delta[T_a]\) to carry out the integrals on \(p_a\), thus arriving at the following final expression for \(W_\chi\)

\[
W_\chi = \mathcal{N} \int \prod_{a=1}^{2} Dq^a [\prod_{a=1}^{2} Dz^a] \det [\chi^a, T_b]_p \left( \prod_{a=1}^{2} \delta[\chi^a] \right) \exp \left( i \int_{-\infty}^{+\infty} dt L_{BFT} \right),
\]

where

\[
L_{BFT} = \frac{m\omega}{2} \left( x^a \epsilon_{ab} \dot{x}^b - \omega x^a g_{ab} x^b \right)
\]

and

\[
x^a \equiv q^a + \frac{1}{\sqrt{m\omega}} s^a.
\]

One can easily verify, from \([3.3], (3.13a), (3.13c)\) and \((3.13d)\), that \(x^a\) is gauge invariant.

Clearly, the subsidiary conditions \(z^a \approx 0\) enables one to recover the original second-class theory (see \((2.14)\)) and, therefore, define the unitary gauge. This is enough to secure that the first-class theory \((3.16)\) is equivalent to the second-class theory \((2.14)\) from which we started.
Most investigations on the BFT embedding procedure end at the level of our expression (3.15). This leave out of consideration a whole of possibilities whose analysis is one of our purpo ses in this work. To exemplify what we mean by this, we perform in (3.14) the canonical transformation

\[ q^a \rightarrow Q^a = \frac{1}{2} q^a + \frac{1}{m \omega} \epsilon^{ab} p_b - \frac{1}{\sqrt{4m \omega}} u^a + \frac{1}{\sqrt{4m \omega}} \epsilon^{ab} s_b , \]  

\[ p_a \rightarrow P_a = \frac{1}{2} p_a - \frac{m \omega}{4} \epsilon^{ab} q^b - \frac{m \omega}{16} \epsilon_{ab} u^b - \frac{m \omega}{4} s_a , \]  

\[ u^a \rightarrow U^a = \frac{1}{2} q^a - \frac{1}{m \omega} \epsilon^{ab} p_b + \frac{1}{\sqrt{4m \omega}} u^a + \frac{1}{\sqrt{4m \omega}} \epsilon^{ab} s_b , \]  

\[ s_a \rightarrow S_a = \frac{1}{2} T_a , \]  

which, after some algebra, enables one to rewrite

\[ W_X = \mathcal{N} \int \left[ \prod_{a=1}^2 DQ^a \right] \left[ \prod_{a=1}^2 DP_a \right] \left[ \prod_{a=1}^2 DU^a \right] \left[ \prod_{a=1}^2 DS_a \right] \det \left[ \epsilon^{ab} \right] \]  

\[ \times \left( \prod_{a=1}^2 \delta[S_a] \right) \left( \prod_{a=1}^2 \delta[\chi^a] \right) \exp \left[ i \int_{-\infty}^{+\infty} dt \left( P_a \dot{Q}^a + S_a \dot{U}^a - K \right) \right] , \]  

where the transformed Hamiltonian \( K \) is given by

\[ K = \frac{1}{2m} P_a g^{ab} P_b + \frac{\omega}{2} Q^a \epsilon^{ab} P_b + \frac{m \omega^2}{8} Q^a g_{ab} Q^b . \]  

Notice that \( Q^a, P_a, \) and \( S_a \) are gauge invariant phase-space variables, as can be seen from (3.13) and (3.19). Moreover, the canonical transformation (3.14) has been chosen in such a way that the constraints are simply given by the equations \( S_a = 0 \). Thus, only the coordinates \( U^a \) are affected by gauge transformations and, furthermore, \( U^a \approx 0 \) are admissible gauge conditions, to be selected from now on. Hence, after carrying out the \( S, U \) and \( P \) integrals in (3.21) one arrives at

\[ W_{U=0} = \mathcal{N} \int \left[ \prod_{a=1}^2 DQ^a \right] \exp \left( i \int_{-\infty}^{+\infty} dt L_{U=0} \right) , \]  

where the effective Lagrangian \( (L_{U=0}) \)

\[ L_{U=0} = \frac{m}{2} \dot{Q}^a g_{ab} \dot{Q}^b - \frac{m \omega}{2} Q^a \epsilon_{ab} \dot{Q}^b , \]  

only contains gauge invariant degrees of freedom.

Observe that the “mass” term in (2.1) has been replaced by a standard kinetic energy term. Then, \( L_{U=0} \) does not describe a constrained system but a regular one possessing truly two independent degrees of freedom. On the other hand, \( L \) involves, as we already pointed out, only one independent degree of freedom. Since the BFT conversion procedure should no alter the physics, \( L_{U=0} \) and \( L \) must be equivalent. We shall next investigate this equivalence in detail.

Let us first look at the classical dynamics arising from \( L_{U=0} \). The Lagrange equations of motion are found to read

\[ \ddot{Q}^a + \omega \epsilon^{ab} \dot{Q}_b = 0 , \]  

implying that

\[ \dot{Q}^a + \omega \epsilon^{ab} Q_b = C^a , \]  

where \( C^a, a = 1,2 \) are constants of motion. The Cartesian form of the trajectory is easily found to be

\[ \left( Q^1 + \frac{C_2}{\omega} \right)^2 + \left( Q^2 - \frac{C_1}{\omega} \right)^2 = \text{constant} , \]
which are just circles centered at $Q^1 = -C^2, Q^2 = C^1$. Also, the energy ($K$) and the angular momentum ($M$),

$$K = \frac{m}{2} \dot{Q}^a \dot{Q}_a ,$$

$$M = \frac{E}{\omega} - \frac{m}{2\omega} C^a C_a ,$$

turn out to be conserved quantities. By comparing these results with the corresponding ones in Section 2, we conclude that the second-class rotator was converted into a first-class system just by turning arbitrary the position of the center of rotation.

We turn next into quantizing $L_{U=0}$. For a regular (unconstrained) system the equal-time commutation algebra must be abstracted from the corresponding Poisson bracket algebra. Then,

$$[Q^a, Q^b] = 0 ,$$

$$[Q^a, P_b] = i\hbar \delta^a_b ,$$

$$[P_a, P_b] = 0 .$$

Due to the absence of ordering ambiguities, the Hamiltonian operator can be read off from (3.21). On the other hand, in terms of phase-space variables the angular momentum operator is found to read

$$M = Q_a e^{ab} P_b .$$

It will prove convenient to introduce destruction ($A_\pm$) and creation ($A_\pm^\dagger$) operators of definite helicity defined as

$$A_\pm = \frac{1}{\sqrt{2}} (A_1 \mp i A_2) ,$$

$$A_\pm^\dagger = \frac{1}{\sqrt{2}} (A_1^\dagger \pm i A_2^\dagger) ,$$

where $A_a$ and $A_a^\dagger$, $a = 1, 2$, are, respectively, standard destruction and creation operators, i.e.,

$$A_a \equiv \left( \frac{m|\omega|}{4\hbar} \right)^\dagger Q_a + i (m\hbar|\omega|)^\dagger P_a ,$$

$$A_a^\dagger \equiv \left( \frac{m|\omega|}{4\hbar} \right)^\dagger Q_a - i (m\hbar|\omega|)^\dagger P_a .$$

The equal-time algebra verified by $A_\pm$ and $A_\pm^\dagger$,

$$[A_r, A_s] = 0 ,$$

$$[A_r^\dagger, A_s^\dagger] = 0 ,$$

$$[A_r, A_s^\dagger] = \delta_{rs} ,$$

where $r = +, -$ and $s = +, -$, follows from (3.29), (3.31) and (3.32).

Now, the Hamiltonian and the angular momentum operators can be cast as

$$K = (N_+ + N_-) \frac{\hbar|\omega|}{2} + \frac{\hbar|\omega|}{2} + (N_+ - N_-) \frac{\hbar\omega}{2} ,$$

$$M = \hbar (N_+ - N_-) ,$$

where

$$N_+ \equiv A_1^\dagger A_+ ,$$

$$N_- \equiv A_- A_1^\dagger .$$
We shall denote by $|n_+,n_->$ the common eigenstates of the hermitean commuting operators $N_+$ and $N_-$. They are labeled by the semidefinite positive integers $n_+, n_-$. Then, the eigenvalue problems for $K$ and $M$ read, respectively,

\begin{align}
K |n_+,n_-> &= E_{n_+,n_-} |n_+,n_-> , \\
M |n_+,n_-> &= m_{n_+,n_-} \hbar |n_+,n_-> ,
\end{align}

where

\begin{equation}
E_{n_+,n_-} = \left\{ \frac{[1 + \epsilon(\omega)]}{2} n_+ + \frac{[1 - \epsilon(\omega)]}{2} n_- + \frac{1}{2} \right\} \hbar |\omega| ,
\end{equation}

and

\begin{equation}
m_{n_+,n_-} = n_+ - n_- .
\end{equation}

The dependence of $E_{n_+,n_-}$ on $n_+$ and $n_-$ is rather peculiar. In fact, for $\omega > 0$ the right hand side of Eq. (3.38) reduces to $(n_+ + 1/2)\hbar |\omega|$, while for $\omega < 0$ it goes into $(n_- + 1/2)\hbar |\omega|$. Hence, in either case, the energy eigenvalue spectrum is that of the second-class system (see Eq. (2.11)). As for the angular momentum, we observe that $m_{n_+,n_-}$ depends simultaneously on $n_+$ and $n_-$, irrespective of the sign of $\omega$. Thus, the energy levels of the first-class system are (infinitely) degenerate, whereas those of the second-class system are not (see Eq. (2.13)). For both systems, the range of eigenvalues of the angular momentum operator is the same. It is in this sense that these systems are quantum mechanically equivalent.

Before closing this Section, we would like to confirm that the degeneracy of the energy levels is related to the arbitrariness in the position of the point around which the classical motion takes place. To this end, we start by noticing that the right hand sides in Eqs. (3.21) and (3.30) can both be diagonalized by means of the canonical transformation

\begin{align}
Q^1 &= \frac{1}{\sqrt{2}} (\eta + \sqrt{2} \rho) , \\
Q^2 &= \frac{\sqrt{2}}{m\omega} (p_\eta - \frac{1}{\sqrt{2}} \rho_\eta) , \\
P_1 &= \frac{1}{\sqrt{2}} (p_\eta + \frac{1}{\sqrt{2}} \rho_\eta) , \\
P_2 &= -\frac{m\omega}{2\sqrt{2}} (\eta - \sqrt{2} \rho) .
\end{align}

The replacement of (3.40) into (3.21) and (3.30) yields, respectively,

\begin{equation}
K = \frac{1}{2m} \rho_\eta^2 + \frac{m}{2} \omega^2 \rho^2 ,
\end{equation}

and

\begin{equation}
M = \frac{1}{\omega} K - \frac{1}{4} \left[ \frac{\rho_\eta^2}{2m} + \frac{m}{2} \left( \frac{\omega}{2} \right)^2 \eta^2 \right] .
\end{equation}

The decoupling of the sector $\eta, p_\eta$ from the dynamics should be noticed. Moreover, from (3.28) and (3.42) follows that the constants of motion $C^a, a = 1, 2$ can be written in terms of $\eta$ and $p_\eta$ as

\begin{align}
C^1 &= \frac{\sqrt{2}}{m} p_\eta , \\
C^2 &= -\frac{\omega}{\sqrt{2}} \eta .
\end{align}

In fact, $\eta$ and $p_\eta$ are, up to proportionality constants, the Noether charges associated with the invariance of $L_{U=0}$ under the global translations $Q^a \rightarrow Q^a + C^a$. As already pictured, what we have at hand is a rotator whose location in the plane $Q^1$ and $Q^2$ is arbitrary.

This new formulation of the converted model is to be quantized, again, by abstracting the equal-time commutators from the corresponding Poisson brackets. The only nonvanishing commutators, then, are
\[ [\rho, p_{\rho}] = i\hbar \quad \text{(3.44a)} \]
\[ [\eta, p_{\eta}] = i\hbar \quad \text{(3.44b)} \]

and, consequently, from (3.41), (3.42) and (3.44) one obtains
\[ [K, M] = [K, \eta] = [K, p_{\eta}] = 0 \quad \text{(3.45)} \]

Thus, we have four conserved observables \((K, M, \eta, p_{\eta})\) but not all of them are mutually commuting, since
\[ [M, \eta] = \frac{2i\hbar}{m\omega} p_{\eta} \neq 0 \quad \text{(3.46a)} \]
\[ [M, p_{\eta}] = -\frac{i\hbar m\omega}{2} \eta \neq 0 \quad \text{(3.46b)} \]

We select \(K\) and \(M\) as the maximal set of commuting observables. Their common eigenstates will be labeled by the corresponding eigenvalues, \(E_n\) and \(m_{n,\bar{n}}\), respectively. According to (3.41) the Hamiltonian \(K\) is that of a harmonic oscillator of mass \(m\) and proper frequency \(|\omega|\). Therefore,
\[ E_n = \left( n + \frac{1}{2} \right) \hbar |\omega| \quad \text{(3.47)} \]

where \(n\) is a positive semidefinite integer. This confirms that the energy eigenvalue spectrum is that of the second-class system (see (2.11)). As for the angular momentum, we observe that the first term in the right-hand side of (3.42) is just \(K\) whereas the second is, up to a sign, proportional to the Hamiltonian operator of a harmonic oscillator of mass \(m\) and proper frequency \(|\omega|/2\). Hence,
\[ m_{n,\bar{n}} = (n - \bar{n}) \epsilon(\omega) \quad \text{(3.48)} \]

where \(n\) and \(\bar{n}\) are positive semidefinite integers. As it must be the case, this is in agreement with (3.39). It is clear that the degeneracy of the energy levels \(E_n\) is due to the presence of \(\bar{n}\) in (3.48). In turn, \(\bar{n}\) originates from the harmonic oscillator in the sector \(\eta\) and \(p_{\eta}\). As we already pointed out, these are the variables that, in the classical limit \((\hbar \to 0)\), determine the location of the center of rotation in the plane \(Q_1, Q_2\).

**IV. STÜCKELBERG EMBEDDING OF THE DISCRETE SYSTEM**

The aim of this Section is to compare the outcomes of the BFT and the Stückelberg embeddings in connection with the nonrelativistic system defined by (2.1). The Stückelberg embedding consists in replacing, in (2.1), \(q^a \to q^a + y^a\), thus obtaining
\[ L_S = \frac{1}{2} m \omega \left[ (q^a + y^a)^2 \epsilon_{ab} (\dot{q}^b + \dot{y}^b) - \omega (q^a + y^a) g_{ab} (\dot{q}^b + \dot{y}^b) \right] \quad \text{(4.1)} \]

We shall designate by \(p_a\) and \(w_a\) the momenta canonically conjugate to \(q^a\) and \(y_a\), respectively. From the defining equations for \(p_a\) and \(w_a\) follows that the extended system is characterized by the primary second-class constraints
\[ \Omega_a = w_a + \frac{1}{2} m \omega \epsilon_{ab} (\dot{q}^b + \dot{y}^b) \approx 0 \quad \text{(4.2)} \]

the primary first-class constraints
\[ \bar{T}_a = p_a - w_a \approx 0 \quad \text{(4.3)} \]

and the canonical Hamiltonian
\[ H_S = \frac{m \omega^2}{2} (q^a + y^a) (q_a + y_a) \quad \text{(4.4)} \]

There are no secondary constraints. From (4.3) follows that the infinitesimal gauge transformations generated by the first-class constraints leave the Lagrangian (4.1) invariant, as must be the case. The second-class constraints can
be eliminated by introducing partial Dirac brackets (\(\Delta\)-brackets) with respect to them \([1,4–6]\). For the nonvanishing \(\Delta\)-brackets one obtains

\[
\begin{align*}
[q^a, p_b]_\Delta &= \delta^a_b, \\
[p_a, p_b]_\Delta &= -\frac{m\omega}{4} \epsilon_{ab}, \\
[p_a, y^b]_\Delta &= \frac{1}{2} \delta^a_b, \\
[p_a, w_b]_\Delta &= -\frac{m\omega}{4} \epsilon_{ab}, \\
[y^a, y^b]_\Delta &= -\frac{1}{2} \delta^a_b, \\
[y^a, w_b]_\Delta &= -\frac{m\omega}{4} \epsilon_{ab}, \\
w_a, w_b]_\Delta &= -\frac{m\omega}{4} \epsilon_{ab}.
\end{align*}
\] (4.5a-g)

Within the \(\Delta\)-bracket algebra the second-class constraints hold as strong identities. We now use this fact to eliminate from the game the variables \(w^a, a = 1, 2\). As seen from (4.2) and (4.3), the first-class constraints in the reduced phase space can be cast as

\[
\bar{T}^*_a = p_a + \frac{1}{2} m \omega \epsilon_{ab} (q^b + y^b) \approx 0.
\] (4.6)

We subject, afterwards, the remaining variables to the transformation

\[
\begin{align*}
q^a &\rightarrow \bar{q}^a = q^a, \\
p_a &\rightarrow \bar{p}_a = p_a - \frac{1}{2} m \omega \epsilon_{ab} y^b, \\
y^a &\rightarrow \bar{z}^a = \sqrt{m \omega} y^a.
\end{align*}
\] (4.7a-c)

The variables \(\bar{q}^a, \bar{p}_a\) are canonical,

\[
[q^a, \bar{p}_b]_\Delta = \delta^a_b,
\] (4.8)

while

\[
[z^a, z^b]_\Delta = -\epsilon^{ab}.
\] (4.9)

All other \(\Delta\)-brackets vanish. In terms of these new variables, the first-class constraints and the Hamiltonian of the reduced phase-space are found to read, respectively,

\[
\bar{T}^*_a = \bar{p}_a + \epsilon_{ab} \left( \frac{1}{2} m \omega \bar{q}^b + \sqrt{m \omega} \bar{z}^b \right) \approx 0,
\] (4.10)

\[
H^*_S = \frac{m \omega^2}{2} \left( \bar{q}^a + \frac{1}{\sqrt{m \omega}} \bar{z}^a \right)^2.
\] (4.11)

From (4.3), (4.4), (4.10) and (4.11) one concludes that the BFT and the Stuckelberg embeddings lead to equivalent results. The only subtle point to be noticed is that the \(z^a\)’s are composite variables, while the \(\bar{z}^a\)’s are basic phase-space variables. Nevertheless, the second-class constraints involving the \(\bar{z}^a\)’s give rise to a Dirac bracket which is numerically equal to the Poisson bracket obeyed by the \(z^a\)’s.

V. BFT EMBEDDING OF THE SELF-DUAL MODEL

On a semiclassical level, the SD model has been shown \([18,19]\) to be equivalent to the Maxwell-Chern-Simons (MCS) theory \([12,13,20]\). That this equivalence holds on the level of the Green functions was proved in Ref. \([21]\). Lately, the second-class constraints of the SD model were successfully converted into first-class by means of the BFT embedding.
procedure. It was then found that the SD model and the MCS theory in a Coulomb like gauge are just different gauge-fixed versions of a parent theory [9–11].

In this Section we go further on and construct explicitly, for any gauge, the phase space variables of the MCS theory in terms of those of the SD model. As we shall see, the strategy developed in Sections 2 and 3 will be of great help for putting the equivalence between the SD and the MCS theories on a more firm basis.

The dynamics of the SD theory is described by the Lagrangian density [8,18]

\[ \mathcal{L}^{SD} = \frac{1}{2\theta} \epsilon^{\mu\nu\rho} \left( \partial_\mu f_\nu \right) f_\rho + \frac{1}{2} f^\mu f_\mu, \]

(5.1)

where \( \theta \) is a parameter with dimensions of mass. We use natural units \((c = \hbar = 1)\) and our metric is \( g_{00} = -g_{11} = -g_{22} = 1 \). The fully antisymmetric tensor \( \epsilon^{\mu\nu\rho} \) is normalized such that \( \epsilon^{012} = 1 \) and we define \( \epsilon^{ij} \equiv \epsilon^{0ij} \). Repeated Greek indices sum from 0 to 2 while repeated Latin indices sum from 1 to 2. Within the Hamiltonian framework, the SD model is characterized by the primary constraints [9–11,21]

\[ T_0^{(0)} = \pi_0 \approx 0, \]

(5.2a)

\[ T_i^{(0)} = \pi_i + \frac{1}{2\theta} \epsilon_{ij} f^j \approx 0, \quad i = 1, 2, \]

(5.2b)

the secondary constraint

\[ T_3^{(0)} = \frac{1}{\theta} \left( f^0 - \frac{1}{\theta} \epsilon_{ij} \partial^i f^j \right) \approx 0, \]

(5.3)

and the canonical Hamiltonian

\[ H_{SD}^{(0)} = \int \mathcal{d}^2x \left( -\frac{1}{2} f^\mu f_\mu + \frac{1}{\theta} \epsilon_{ij} f^0 \partial^i f^j \right). \]

(5.4)

We denote by \( \pi_\mu \) the momentum canonically conjugate to the field variable \( f^\mu \). All constraints are second-class.

The quantization of the SD model as a second-class theory was carried out in detail in Ref. [22]. The Heisenberg equations of motion together with the equal-time commutation relations are solved by

\[ f^\mu(\pm)(x) = \frac{1}{2\pi} \int \frac{d^2k}{\sqrt{2\omega_k \theta}} \exp \left[ \pm i (\omega_k x^0 - \vec{k} \cdot \vec{x}) \right] f^\mu(\pm)(\vec{k}), \]

(5.5)

where \( \omega_k \equiv +\sqrt{|\vec{k}|^2 + \theta^2} \) and

\[ f^\mu(+)\!(\vec{k}) = \varepsilon^\mu(\vec{k}) a^{(+)\!(\vec{k})}, \]

(5.6a)

\[ f^\mu(-)\!(\vec{k}) = \varepsilon^\mu(\vec{k}) a^{(-)\!(\vec{k})}. \]

(5.6b)

Here, \( a^{(\pm)}\!(\vec{k}) \) are creation and annihilation operators and \( \varepsilon^\mu(\vec{k}) \) is the polarization vector. Observe that the system under analysis possesses three coordinates, three momenta and four second-class constraints. Thus, as in the particle case in Section 2, one is left with only one independent degree of freedom. The determination of \( \varepsilon^\mu(\vec{k}) \) led to [22]

\[ \varepsilon^0(\vec{k}) = \frac{1}{|\theta|} \vec{k} \cdot \varepsilon(0), \]

(5.7a)

\[ \varepsilon^j(\vec{k}) = \varepsilon^j(0) + \frac{\varepsilon(0) \cdot \vec{k}}{(\omega_k + |\theta|) |\theta|} k^j, \]

(5.7b)

where

\[ \varepsilon^0(0) = 0, \]

(5.8a)

\[ \varepsilon^j(0) = -i \frac{|\theta|}{\theta} \varepsilon^j(0) \varepsilon^0(0). \]

(5.8b)

As for the spin of the of the SD quanta, it was found to be \( \pm 1 \) depending upon the sign of \( \theta \).
The outcomes of applying the BFT embedding procedure to the SD theory have already been reported in the literature and we shall merely quote here the results [4, 11, 21].

\[
\begin{align*}
\mathcal{T}_0^{(0)} &\rightarrow \mathcal{T}_0 = \pi_0 - \frac{1}{\theta} \phi^0 \approx 0, \quad (5.9a) \\
\mathcal{T}_i^{(0)} &\rightarrow \mathcal{T}_i = \pi_i + \frac{1}{\theta} \epsilon_{ij} \left( \frac{1}{2} f^j + \phi^j \right) - \frac{1}{\theta^2} \epsilon_{ij} \partial_j \phi^0 \approx 0, \quad (5.9b) \\
\mathcal{T}_3^{(0)} &\rightarrow \mathcal{T}_3 = \frac{1}{\theta} \left( f^0 + \phi^3 \right) - \frac{1}{\theta^2} \epsilon_{ij} \partial_i \left( f^0 + \phi^3 \right) \approx 0, \quad (5.9c)
\end{align*}
\]

and

\[
H_{SD}^{(0)} \rightarrow H_{SD} = \int d^2x \left[ \frac{1}{2} (f^i + \phi^i)^2 + \frac{1}{2} (f^0 + \phi^3)^2 - m (f^0 + \phi^3) \mathcal{T}_3 \right]. \quad (5.10)
\]

As demanded [2], a new pair of canonical phase-space variables (coordinates \(u^0, u^i, u^3\) and momenta \(\mathcal{P}_0, \mathcal{P}_i, \mathcal{P}_3\)) for each second-class constraint has been introduced. By definition

\[
\begin{align*}
\phi^0 &\equiv -\frac{1}{2} u^0 - \theta \mathcal{P}_3, \\
\phi^i &\equiv -\frac{1}{2} u^i + \partial^i \mathcal{P}_3 - \theta \epsilon^{ij} \mathcal{P}_j, \\
\phi^3 &\equiv -\frac{1}{2} u^3 + \theta \mathcal{P}_0 + \partial^i \mathcal{P}_i.
\end{align*}
\]

Then, the only nonvanishing Poisson brackets among the \(\phi^i\)'s are, as required [2],

\[
\begin{align*}
[\phi^0(\vec{x}), \phi^3(\vec{y})]_{\mathcal{P}} &= -\theta \delta(\vec{x} - \vec{y}), \\
[\phi^i(\vec{x}), \phi^j(\vec{y})]_{\mathcal{P}} &= -\theta \epsilon^{ij} \delta(\vec{x} - \vec{y}), \\
[\phi^i(\vec{x}), \phi^j(\vec{y})]_{\mathcal{P}} &= -\epsilon^3_{ij} \delta(\vec{x} - \vec{y}).
\end{align*}
\]

Since the extended constraints verify, by construction, an Abelian algebra [2], the Green functions generating functional \(\mathcal{W}_\chi\) is given by

\[
\mathcal{W}_\chi = \mathcal{N} \int \prod_{\mu=0}^{2} \mathcal{D}f^\mu \prod_{\mu=0}^{2} \mathcal{D}\pi_\mu \prod_{a=0}^{3} \mathcal{D}u^a \prod_{a=0}^{3} \mathcal{D}\mathcal{P}_a \det[\chi^a, \mathcal{T}_a] P \times \left( \prod_{\alpha=0}^{3} \delta[\mathcal{T}_a] \right) \left( \prod_{\alpha=0}^{3} \delta[\chi^a] \right) \exp \left[ i \int d^2x \left( \pi_\mu f^\mu + \mathcal{P}_a u^a - \mathcal{H}_{SD} \right) \right], \quad (5.13)
\]

where \(\mathcal{H}_{SD}\) is the Hamiltonian density corresponding to \(H_{SD}\) and \(\chi^a, a = 0, 1, 2, 3\) are the gauge conditions. As we did in the particle case, we first perform the change variables \(u^a \rightarrow u'^a = \phi^a, \mathcal{P}_a \rightarrow \mathcal{P}'_a = \mathcal{P}_a\), whose jacobian is nonsingular and can be lumped into the normalization constant \(\mathcal{N}\). Since \(\mathcal{H}_{SD}\) does not depend upon \(\mathcal{P}'_a\), the corresponding integrals can be carried out at once, yielding,

\[
\mathcal{W}_\chi = \mathcal{N} \int \prod_{\mu=0}^{2} \mathcal{D}f^\mu \prod_{\mu=0}^{2} \mathcal{D}\pi_\mu \prod_{a=0}^{3} \mathcal{D}\phi^a \det[\chi^a, \mathcal{T}_a] P \left( \prod_{\alpha=0}^{3} \delta[\mathcal{T}_a] \right) \left( \prod_{\alpha=0}^{3} \delta[\chi^a] \right) \exp \left[ i \int d^2x \left( \pi_\mu f^\mu + \frac{1}{2\theta} \phi^i \epsilon_{ij} \dot{\phi}^j - \frac{1}{\theta} \phi^3 \dot{\phi}^0 - \frac{1}{\theta^2} \phi^0 \epsilon_{ij} \dot{\phi}^i \dot{\phi}^j - \mathcal{H}_{SD} \right) \right]. \quad (5.14)
\]

We use next the extended primary constraints \(\mathcal{T}_0 = 0, \mathcal{T}_i = 0\) to perform the momentum integrals, thus arriving to

\[
\mathcal{W}_\chi = \mathcal{N} \int \prod_{\mu=0}^{2} \mathcal{D}f^\mu \prod_{a=0}^{3} \mathcal{D}\phi^a \det[\chi^a, \mathcal{T}_a] P \left( \prod_{\alpha=0}^{3} \delta[\chi^a] \right) \exp \left[ i \int d^2x \mathcal{L}_{SD}^{SD}(C^\mu) \right]. \quad (5.15)
\]
This is the analog of (3.16). The role of the nonrelativistic variables $x^a$ (see Eq.(3.18)) is now played by the variables $C^\mu$, defined as
\begin{align}
C^0 &= f^0 + \phi^3, \\
C^i &= f^i + \phi^i.
\end{align}
(5.16a)
(5.16b)

The variables $C^\mu$ are gauge invariant. To see this, we recall that the generator of infinitesimal gauge transformations ($G$) is, by definition \cite{1}, a linear combination of the first-class constraints, i.e.,
\begin{equation}
G = \int d^2 x \varepsilon^a T_a, \tag{5.17}
\end{equation}
where the $\varepsilon^a$ are infinitesimal gauge parameters and $a$ runs from 0 to 3. Then, under infinitesimal gauge transformation the phase space variables can be seen to change as follows
\begin{align}
\delta f^0 &= \varepsilon^0, \\
\delta f^i &= \varepsilon^i, \\
\delta u^0 &= \varepsilon^3, \\
\delta u^i &= \varepsilon^i, \\
\delta \pi_0 &= -\frac{1}{m} \varepsilon^3, \\
\delta \pi_i &= \frac{1}{2m} \epsilon_{ij} \varepsilon^j + \frac{1}{m^2} \epsilon_{ij} \partial^j \varepsilon^3, \\
\delta P_0 &= -\frac{1}{2m} \varepsilon^0 + \frac{1}{m^2} \epsilon_{ij} \partial^j \varepsilon^3, \\
\delta P_i &= -\frac{1}{2m} \epsilon_{ij} \varepsilon^j - \frac{1}{2m^2} \epsilon_{ij} \partial^j \varepsilon^3, \\
\delta P_3 &= \frac{1}{2m} \varepsilon^3. 
\end{align}
(5.18a)
(5.18b)
(5.18c)
(5.18d)
(5.18e)
(5.18f)
(5.18g)
(5.18h)
(5.18i)
(5.18j)

From (5.16) and (5.18) follows that the $C^\mu$'s are, as asserted, gauge invariant fields. Moreover, the subsidiary conditions $\chi^a = \phi^a = 0$ return us back to the original second-class theory and, hence, define the unitary gauge.

We turn next into investigating the existence of alternative formulations for the converted theory. Guided by the particle case, we perform the canonical transformation
\begin{align}
f^0 &\rightarrow A^0 = f^0, \\
\pi_0 &\rightarrow P_0 = \pi_0 + \frac{1}{2\theta} u^0 + \mathcal{P}_3, \\
f^i &\rightarrow A^i = \frac{1}{2} (f^i - u^i) + \theta \epsilon_{ij} (\pi_j + P_j) + \frac{1}{\theta} \partial^j u^0, \\
\pi_i &\rightarrow P_i = \frac{1}{2} (\pi_i - \mathcal{P}_i) - \frac{1}{4\theta} \epsilon_{ij} (f^j + u^j) - \frac{1}{\theta} \epsilon_{ij} \partial^j P_3, \\
u^0 &\rightarrow U^0 = u^0, \\
\mathcal{P}_0 &\rightarrow N_0 = \mathcal{P}_0 + \frac{1}{2\theta} f^0 + \frac{1}{4\theta} \partial^j (\pi_i - \mathcal{P}_i) - \frac{3}{8\theta^2} \epsilon_{ij} \partial^j (f^j + u^j), \\
u^i &\rightarrow U^i = \frac{1}{2} (u^i + f^i) - \theta \epsilon_{ij} (\pi_j - P_j) - 3 \partial^i \mathcal{P}_3, \\
\mathcal{P}_i &\rightarrow N_i = \frac{1}{2} T_i, \\
u^3 &\rightarrow U^3 = u^3 - f^0 + \frac{1}{2} \partial^j (\pi_i + \mathcal{P}_i) + \frac{5}{4\theta} \epsilon_{ij} \partial^j (f^j + u^j), \\
\mathcal{P}_3 &\rightarrow N_3 = \mathcal{P}_3. 
\end{align}
(5.19a)
(5.19b)
(5.19c)
(5.19d)
(5.19e)
(5.19f)
(5.19g)
(5.19h)
(5.19i)
(5.19j)

In terms of the new variables, the functional integral in the right hand side of (5.13) can be cast as
\[ W_\chi = \mathcal{N} \int \prod_{\mu=0}^{2} DA^\mu \prod_{\mu=0}^{2} DP_\mu \prod_{a=0}^{3} DU^a \prod_{a=0}^{3} DN_a \det[\chi^a, T_b]_P \times \left( \prod_{a=0}^{3} \delta(T_a) \right) \left( \prod_{a=0}^{3} \delta[\chi^a] \right) \exp \left[ i \int d^3 x \left( P_\mu \dot{A}^\mu + N_a \dot{U}^a - K \right) \right] , \tag{5.20} \]

where the transformed Hamiltonian density \( K \) is given by
\[
K = \frac{\theta^2}{2} P_i P_i - \frac{\theta}{2} P_i \epsilon^{ij} A^j + \frac{1}{4\theta^2} F_{ij} F^{ij} + \frac{1}{8} A^i A^i \\
- \frac{1}{2\theta} \epsilon_{ij} F^{ij} G - G^2 , \tag{5.21} \]

with
\[
F^{ij} = \partial^i A^j - \partial^j A^i , \tag{5.22} \]

and
\[
G = \partial^k P_k + \frac{1}{2\theta} \epsilon_{kl} \partial^k A^l . \tag{5.23} \]

Also notice that, in terms of the new variables, the constraints translate into
\[
T_0 = P_0 , \tag{5.24a} \\
T_i = 2 N_i , \tag{5.24b} \\
T_3 = N_0 - \frac{1}{2\theta} U^3 - \frac{1}{2\theta} G . \tag{5.24c} \]

We now focus on the right hand side \([5.20]\). Since \( K \) does not depend upon \( U^a \) and \( N_a \), the partial gauge fixing \( \chi^i = U^i = 0 \) does not imply in a physically meaningful restriction and enables us to carry out the \( U^i \) and \( N_i \) integrals. The situation is quite analogous to that encountered in the particle case. Afterwards, the constraint \( T_3 \) is exponentiated by means of the auxiliary variable \( \lambda^3 \) and the integrals on \( N_0 \) and \( U^3 \) are performed. The integrals on \( N_3 \) and \( \lambda^3 \) are also carried out and one arrives at
\[
W_\chi = \mathcal{N} \int \prod_{\mu=0}^{2} DA^\mu \prod_{\mu=0}^{2} DP_\mu \prod_0^{U^0} \det[\chi^{a'}, T_{b'}]_P \times \delta(P_0) \delta[\chi^0] \delta[\chi^3] \exp \left[ i \int d^3 x \left( P_\mu \dot{A}^\mu - U^0 \dot{G} - K \right) \right] , \tag{5.25} \]

where \( a' \) and \( b' \) only take the values 0 and 3. The subsequent integration on \( U^0 \) produces \( \delta[\dot{G}] \) which, up to a field independent determinant, is proportional to \( \delta[G] \). Hence, the final form for \( W_\chi \) is
\[
W_\chi = \mathcal{N} \int \prod_{\mu=0}^{2} DA^\mu \prod_{\mu=0}^{2} DP_\mu \det[\chi^{a'}, T_{b'}]_P \delta[P_0] \delta[\chi^0] \delta[\dot{P}_k] \delta[\epsilon_{ik} \partial^k A^l] \delta[\chi^3] \times \exp \left\{ i \int d^3 x \left[ P_\mu \dot{A}^\mu - \left( \frac{\theta^2}{2} P_i P_i - \frac{\theta}{2} P_i \epsilon^{ij} A^j + \frac{1}{4\theta^2} F_{ij} F^{ij} + \frac{1}{8} A^i A^i \right) \right] \right\} , \tag{5.26} \]

which is just the phase-space path integral describing the MCS theory in any arbitrary canonical gauge \([21]\). Also, from Eqs.\((5.18)\) and \((5.19)\) one can confirm that, under infinitesimal gauge transformations, the field variables \( A^\mu \) and \( P_\mu \) do transform as the MCS variables, i.e.,
\[
\delta A^0 = \varepsilon^0 , \tag{5.27a} \\
\delta P_0 = 0 , \tag{5.27b} \\
\delta A^i = \frac{1}{2\theta} \partial^i \varepsilon^3 , \tag{5.27c} \\
\delta P_i = \frac{1}{4\theta^2} \epsilon_{ik} \partial^k \varepsilon^3 . \tag{5.27d} \]
After performing the momentum integrations, one finds that the effective Lagrangian arising from \[ \text{Eq.}(5.27) \] is the well known MCS Lagrangian density \[ L_{\text{MCS}} \equiv L_{U=0} = -\frac{1}{4\rho^2} F_{\mu \nu} F^{\mu \nu} + \frac{1}{4\rho^4} \epsilon_{\mu \nu \rho \sigma} F^{\mu \nu} A_\sigma . \] (5.28)

As in the particle case, the mass term in \[ \text{Eq.}(5.1) \] was replaced by a “kinetic energy” term. However, unlike in the particle case, \[ L_{\text{MCS}} \] can not be fully written in terms of gauge invariant fields. In fact, \[ L_{\text{MCS}} \] does not describe a regular theory but a first-class one, possessing two first-class constraints \[ [20]. \] The counting of degrees of freedom (three coordinates, three momenta, two first-class constraints and two gauge conditions) reveals that, as in the SD model, only one independent degree of freedom is present in the MCS theory. Since \[ L_{\text{MCS}} \] derives from \[ L_{\text{SD}} \] through the BFT conversion procedure, they must be equivalent. In fact, as demonstrated in Ref. \[ [22], \] the particle content of the SD model is that of the MCS model in the Coulomb gauge, while the polarization vector of the SD quanta (see \[ \text{Eqs.}(5.7) \] and \[ (5.8) \]) is that of the massive MCS quanta in the Landau gauge\[ 1 \]. It is in this sense that \[ L_{\text{MCS}} \] and \[ L_{\text{SD}} \] describe equivalent physics.

We believe to have generalized the proof of equivalence between the SD and the MCS models presented in Refs. \[ [9-11, 21], \] where the election of specific functional forms for \( \chi^0 \) and \( \chi^3 \) was essential to establish this correspondence. Notice, moreover, that our proof of equivalence is not restricted to demonstrate the equality between two functional integrals, but also involves an explicit construction of the phase-space variables of the MCS theory in terms of those of the SD theory. This is precisely the meaning of the canonical transformation in \[ \text{Eq.}(5.14) \].

VI. BFT EMBEDDING OF THE PROCA-WENTZEL THEORY COUPLED TO FERMIONS

We shall present in this Section the BFT embedding of the 3 + 1-dimensional Proca-Wentzel field \( (B^\alpha) \) minimally coupled to fermions \( (\psi, \bar{\psi}) \). This is second-class theory involving bosonic and fermionic degrees of freedom and exhibiting bosonic and fermionic constraints. Our starting point is the Lagrangian density

\[
L_{\text{PW}} = -\frac{1}{4} F_{\alpha \beta}^B F^{B, \alpha \beta} + \frac{\mu^2}{2} B^\alpha B_\alpha + \frac{i}{2} \bar{\psi} \gamma^\alpha \partial_\alpha \psi - \frac{i}{2} (\partial_\alpha \bar{\psi}) \gamma^\alpha \psi - M \bar{\psi} \psi + g \bar{\psi} \gamma^\alpha B_\alpha \psi ,
\]  

\[ (6.1) \]

where \( F_{\alpha \beta}^B = \partial_\alpha B_\beta - \partial_\beta B_\alpha, \mu \) is the vector boson mass, \( M \) is the fermion mass and \( g \) is a coupling constant. Our metric is \( g_{00} = -g_{11} = -g_{22} = -g_{33} = +1 \). This model possesses primary and secondary second-class bosonic constraints. They are, respectively,

\[
b_0^{(0)} = \pi_0^B \approx 0 , \quad (6.2a)
\]

\[
b_1^{(0)} = \frac{1}{m} \left( \partial^i \pi_i^B + m^2 B^0 + g \bar{\psi} \gamma^0 \psi \right) \approx 0 . \quad (6.2b)
\]

The second-class fermionic constraints,

\[
f_a^{(0)} = \pi_{\bar{\psi} a} - \frac{i}{2} \gamma_{ab} \psi_b \approx 0 , \quad (6.3a)
\]

\[
f_a^{(0)} = \pi_{\psi a} - \frac{i}{2} \bar{\psi}_b \gamma_{ba} \approx 0 , \quad (6.3b)
\]

are all primary constraints, while the canonical Hamiltonian reads

\[
H_{\text{PW}}^{(0)} = \int d^3x \left\{ \frac{1}{2} \dot{\pi}_i^B \dot{\pi}_i^B + \frac{1}{4} F_{ij}^B F^{B, ij} + \frac{\mu^2}{2} B^i B^i + \frac{i}{2} \left[ \partial_i \bar{\psi} \right] \gamma^i \psi - \frac{i}{2} \bar{\psi} \gamma^i \left[ \partial_i \bar{\psi} \right] \right.
\]

\[
+ g \bar{\psi} \gamma^i \psi B^i + M \bar{\psi} \psi - B^0 \left( \partial^i \pi_i^B + \frac{\mu^2}{2} B^0 + g \bar{\psi} \gamma^0 \psi \right) \right\} . \quad (6.4)
\]

\[ ^5 \text{Recall that in the Landau gauge the MCS theory exhibits massive and massless (gauge) excitations.} \]

\[ ^6 \text{The BFT embedding of the free Proca-Wentzel field has already been reported in the literature. See Refs. \[ 22 \].} \]
Here, $\pi^\mu \mu, \pi_{\bar{\psi}_a}$ and $\pi_{\bar{\bar{\psi}}_a}$ are the momenta canonically conjugate to $B^\mu, \psi_a$ and $\bar{\psi}_a$, respectively, and $a$ is a spinor index running from 1 to 4. Notice that $\tilde{f}^{(0)} = -f^1 \gamma^0$.

Now, we use again the BFT procedure to convert the system into first-class. For the bosonic sector of the constraints one obtains

$$
\begin{align}
\delta b_0^{(0)} &\to b_0^{(0)} = b_0^{(0)} + \frac{1}{m} \phi^2 - \lambda + m \Phi^1 \approx 0, \\
\delta b_1^{(0)} &\to b_1^{(0)} = b_1^{(0)} + \frac{1}{m} \phi^2 - \lambda + m \Phi^1 \approx 0, \\
&= \frac{1}{m} \left[ \phi^2 - \lambda + m \Phi^1 \right] \approx 0, \\
\end{align}
$$

whereas the fermionic sector of the constraints extends as follows

$$
\begin{align}
\delta f_a^{(0)} &\to f_a^{(0)} + \frac{1}{2} \gamma^0 \left( \psi + \frac{1}{2} \xi \right) \approx 0, \\
\delta \bar{f}_a^{(0)} &\to \bar{f}_a^{(0)} + \frac{1}{2} \gamma^0 \left( \bar{\psi} + \bar{\xi} \right) \approx 0.
\end{align}
$$

The extension of the Hamiltonian is more involved and one ends up with

$$
\begin{align}
H_{PW}^{(0)} &\to H_{PW}^{(0)} + H_{PW}^{(1)} + H_{PW}^{(2)} + H_{PW}^{(3)} \\
&= \int d^3 x \left\{ \frac{1}{2} \pi^B_i \pi_i^B + \frac{1}{4} F_{ij}^B F^{B,ij} + m^2 B^i B^i + \frac{\mu^2}{2} (B^0 - \Phi^0)^2 - \mu (B^0 - \Phi^0) b_1 \\
&+ \frac{i}{2} \left[ \partial_i (\bar{\psi} + \frac{1}{2} \xi) \right] \gamma^i (\psi + \frac{1}{2} \xi) - \frac{i}{2} \left[ \partial_i (\bar{\psi} + \frac{1}{2} \xi) \right] \gamma^i \left[ \partial_i (\psi + \frac{1}{2} \xi) \right] \\
&+ \left[ \bar{\psi} + \frac{1}{2} \bar{\xi} \right] \gamma^i (\psi + \frac{1}{2} \xi) B^i + M (\bar{\psi} + \frac{1}{2} \bar{\xi}) (\psi + \frac{1}{2} \xi) + m \Phi^1 \partial_i B^i - \frac{1}{2} \Phi^1 \nabla^2 \Phi^1 \right\},
\end{align}
$$

where we denoted by $\Phi^0, \Phi^1$ and $\bar{\xi}, \xi$ the BFT bosonic and fermionic embedding variables, respectively. Presently, we omit the detailed construction of these composite objects in terms of canonical variables but only mention that they are required to obey the Poisson bracket relations

$$
\begin{align}
\{ \Phi^A (x^0, \bar{x}), \Phi^B (x^0, \bar{y}) \}_P & = \frac{1}{\mu} \epsilon^{AB} \delta (\bar{x} - \bar{y}), \\
\{ \xi_a (x^0, \bar{x}), \bar{\xi}_b (x^0, \bar{y}) \}_P & = -4i \gamma^0_{ab} \delta (\bar{x} - \bar{y}),
\end{align}
$$

where the superscripts $A$ and $B$ run from 0 to 1. Needless to say, the Poisson bracket between fermionic variables is symmetric. We would like to stress that, as a consequence of the embedding procedure, the number of fermions doubles. One can check that the constraints $b_A, f_A, \bar{f}_a$ and $H$ are strong under involution, thus characterizing an Abelian first-class theory. By definition, the Hermitian generator of infinitesimal gauge transformations is

$$
G_{PW} = \int d^3 x \left( \lambda^A \mu^a + i \bar{\alpha}^a f_a + i \bar{f}_a \alpha^a \right),
$$

where $\lambda^A, \alpha^a$ and $\bar{\alpha}^a = +\bar{\alpha}^a \gamma^0$ are space-time dependent gauge parameters. Then, under infinitesimal gauge transformations the field in the game change as

$$
\begin{align}
\delta B^0 & = \lambda^0, \\
\delta B^i & = -\frac{1}{m} \partial^i \lambda^1, \\
\delta \Phi^0 & = \lambda^0, \\
\delta \Phi^1 & = \lambda^1, \\
\delta \psi_a & = i \alpha^a,
\end{align}
$$

We are indebted to Prof. J Barcelos-Neto for a discussion about this effect.
\[ \delta \bar{\psi}_a = - i \bar{\alpha}^a, \quad (6.10f) \]
\[ \delta \xi_a = - \frac{2ig}{\mu} (\psi_a + \frac{1}{2} \xi_a) \lambda^1 - 2 i \alpha^a, \quad (6.10g) \]
\[ \delta \bar{\xi}_a = + \frac{2ig}{\mu} (\bar{\psi}_a + \frac{1}{2} \bar{\xi}_a) \lambda^1 + 2 i \bar{\alpha}^a. \quad (6.10h) \]

We look next for the generating functional of Green functions \( W_{\chi, \eta, \bar{\eta}} \). Since the theory is Abelian, the ghosts are easily integrated out and one is left with

\[
W_{\chi, \eta, \bar{\eta}} = \mathcal{N} \int [D\Sigma] \det [\chi^A, b_A] \det [\bar{\eta}^a, f_a] \det [\eta^a, \bar{f}_a] \left( \prod_{A=0}^{4} \delta [b_A] \right) \left( \prod_{A=0}^{4} \delta [\chi^A] \right) \left( \prod_{a=1}^{4} \delta [\eta^a] \right) \left( \prod_{a=1}^{4} \delta [\bar{\eta}^a] \right) \exp (iA_{\text{PW}F}), \quad (6.11) \]

where \( \chi^A \) are the gauge fixing functions for the bosonic sector of constraints and \( \eta^a \) and \( \bar{\eta}^a \) are the corresponding ones for the fermionic sector. These subsidiary conditions are to be chosen as to make all the Faddeev-Popov determinants, in Eq. (6.11), nonvanishing and field-independent. The integration measure \([D\Sigma]\) involves all variables appearing in the action \( A_{\text{PW}F} \), which, in turn, reads

\[
A_{\text{PW}F} = \int d^4x \left( \pi_\mu B^\mu + \pi_\psi \bar{\psi} + \bar{\psi} \pi_\bar{\psi} + \frac{i}{4} \bar{\xi} \gamma^0 \xi + \frac{\mu}{2} \Phi^A \epsilon_{AB} \Phi^B - \mathcal{H}_{\text{PW}F} \right), \quad (6.12) \]

with \( \int d^4x \mathcal{H}_{\text{PW}F} = H_{\text{PW}F} \). The integrations over the fermionic momenta can be carried out at once by using \( f_a = 0 \) and \( \bar{f}_a = 0 \). Then, the gauge is partially fixed by choosing \( \chi^0 = B^0 \). This, together with \( b_0 = 0 \), enables one to integrate out the sector \( B^0, \pi_0^B \). Afterwards, the constraint \( \bar{b}_1 = 0 \) is exponentiated by means of the Lagrange multiplier \( \mu B^0 \), which brings back \( B^0 \) into the game. Finally, by restricting the remaining gauge conditions not to depend upon \( \Phi^0 \) and the momenta \( \pi_i^B \), the corresponding integrals can also be carried out. Thus, one arrives at

\[
W_{\chi, \eta, \bar{\eta}} = \mathcal{N} \int \left[ \prod_{\alpha} DB^\alpha \right] [D\theta][D\bar{\psi}][D\psi][D\bar{\xi}][D\xi] \det [\chi^1, b_1] \det [\bar{\eta}^a, f_a] \det [\eta^a, \bar{f}_a] \delta [\chi^1] \left( \prod_{a=1}^{4} \delta [\eta^a] \right) \left( \prod_{a=1}^{4} \delta [\bar{\eta}^a] \right) \exp \left( i \int d^4x L_{\text{BFT}}^{\text{PW}F} \right), \quad (6.13) \]

where

\[
L_{\text{BFT}}^{\text{PW}F} = - \frac{1}{4} F_{\alpha \beta}^B F^{B, \alpha \beta} + \frac{m^2}{2} (B^\alpha - \partial^\alpha \theta) (B_\alpha - \partial_\alpha \theta) \\
+ \frac{i}{2} \bar{\psi} \left( \gamma^0 \bar{\psi} + \frac{1}{2} \bar{\xi} \right) D_\alpha (B) \left( \psi + \frac{1}{2} \xi \right) - \frac{i}{2} D_\alpha (\bar{B}) \left( \bar{\psi} + \frac{1}{2} \bar{\xi} \right) \gamma^0 \left( \psi + \frac{1}{2} \xi \right) \\
- M \left( \bar{\psi} + \frac{1}{2} \bar{\xi} \right) \left( \psi + \frac{1}{2} \xi \right). \quad (6.14) \]

As usual, \( D_\alpha (B) \equiv \partial_\alpha - i g B_\alpha \) and we have replaced \( \Phi^1 \) by \( -\mu \theta \).

From (6.10) follows that \( L_{\text{BFT}}^{\text{PW}F} \) remains invariant under gauge transformations. Clearly, the unitary gauge conditions \( \chi^1 = \theta = 0, \bar{\eta} = \xi = 0 \) and \( \eta = \xi = 0 \) lead us back to the original theory, defined by \( L^{\text{PW}F} \).

We end this work by remarking that, as in the particle case, the theory can be fully phrased in terms of local gauge invariant variables. To see how this come about, we start by introducing the composite fields \( \Psi, \bar{\Psi} \) and \( B^\alpha \) via the non-linear transformation

\[
\Psi \equiv e^{-i g \theta} \left( \psi + \frac{1}{2} \xi \right), \quad (6.15a) \\
\bar{\Psi} \equiv e^{+i g \theta} \left( \bar{\psi} + \frac{1}{2} \bar{\xi} \right), \quad (6.15b) \\
B^\alpha \equiv B^\alpha - \partial^\alpha \theta, \quad (6.15c) 
\]

which, from (6.10), are seen to be effectively gauge invariant. It is now easy to see that \( L_{\text{BFT}}^{\text{PW}F} \) can be casted as

\[ \ldots \]
\[
\mathcal{L}_{BFT}^{PW} = - \frac{1}{4} F_{\alpha\beta}^B F^{B,\alpha\beta} + \frac{\mu^2}{2} B^\alpha B_\alpha \\
+ \frac{i}{2} \bar{\Psi} \gamma^\alpha [D_\alpha (B) \Psi] - \frac{i}{2} \left[ D^*_\alpha (B) \bar{\Psi} \right] \gamma^\alpha \Psi - M \bar{\Psi} \Psi .
\]  

Therefore, the theories described by \( \mathcal{L}_{PW}^{BFT} \) and \( \mathcal{L}_{BFT}^{PW} \) are not only equivalent but identical.

### VII. CONCLUSIONS

The BFT conversion procedure provides a systematics to generate a set of first-class theories equivalent to a given second-class one. However, we learnt that no \textit{a priori} statements can be made about the kind of first-class theories arising through the BFT mechanism.

Indeed, in the particle case we succeeded in finding a canonical transformation where all the first-class constraints \( T_a \) became a subset \( S_a \) of the transformed momenta. Then, only the variables \( U^a \), canonically conjugate to \( S_a \), were affected by the gauge transformations. Moreover, it turned out possible to write the Hamiltonian \( K \) in terms gauge independent variables only. \( Q^a \), and \( P_a \).

The situation for the SD model was qualitatively different. As indicated in Eq.(5.24), only the converted primary constraints \( T_0 \) and \( T_i \) turned, after the canonical transformation, into momenta. The converted secondary constraint \( T_3 \) remains a combination of the phase-space variables \( A^i \) and \( P_i \) (see Eq.(5.24c)). As consequence, the coordinates of the MCS theory are gauge dependent objects.

Any attempt of relating the results summarized above with the existence or not of secondary second-class constraints was destroyed by the Proca-Wentzel theory. There, the presence of secondary second-class constraint was not enough to prevent us of writing the converted first-class theory only in terms of gauge invariant fields.

It has recently appeared in the literature the BFT embeddings of the massive Yang-Mills theory \cite{26} and of the non-Abelian SD model \cite{27}. The generalization for these cases of our technique mounted on canonical transformations is currently under progress.
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