ALGEBRAIC GROUPS WHOSE ORBIT CLOSURES CONTAIN ONLY FINITELY MANY ORBITS

VLADIMIR L. POPOV

Abstract. We explore connected affine algebraic groups $G$, which enjoy the following finiteness property (F): for every algebraic action of $G$, the closure of every $G$-orbit contains only finitely many $G$-orbits. We obtain two main results. First, we classify such groups. Namely, we prove that a connected affine algebraic group $G$ enjoys property (F) if and only if $G$ is either a torus or a product of a torus and a one-dimensional connected unipotent algebraic group. Secondly, we obtain a characterization of such groups in terms of the modality of action in the sense of V. Arnol’d. Namely, we prove that a connected affine algebraic group $G$ enjoys property (F) if and only if for every irreducible algebraic variety $X$ endowed with an algebraic action of $G$, the modality of $X$ is equal to $\dim X - \max_{x \in X} G \cdot x$.

1. Introduction

The phenomenon of finiteness of the sets of orbits in orbit closures of algebraic actions of a connected affine algebraic group $G$, which arises under certain restrictions on $G$ and actions, has been known for a long time and plays an essential role in several mathematical theories. Since every $G$-orbit is dense and open in its closure, it is the phenomenon of finiteness of the sets of $G$-orbits in algebraic $G$-varieties containing dense open $G$-orbit or, saying differently, in equivariant open embeddings of algebraic homogeneous spaces of $G$.

The first example is that of a torus $G$. In this case, every $G$-orbit closure contains only finitely many $G$-orbits. This is a key fact of the theory of toric embeddings [TE73] (see, e.g., also [Ful93]).

Historically, the next example, which generalizes the previous one, is the class of all equivariant open embeddings of a fixed algebraic homogeneous space $G/H$ of a connected complex reductive group $G$: by [Akh85], every such embedding contains only finitely many $G$-orbits if and only if $H$ is a spherical subgroup of $G$. This fact is a key ingredient of the theory of spherical embeddings [LV83] (see, e.g., also [Tim11]). The earliest manifestation of this example is the case of parabolic $H$: then every equivariant embedding of $G/H$ coincides with $G/H$. 
One more example is obtained if $G$ is unipotent: in this case, every quasiaffine $G$-orbit closure coincides with this orbit [Ros61, Thm. 2]. This is an important fact of the algebraic transformation group theory.

In the present paper, we consider the absolute case, i.e., that in which no conditions on actions under consideration are imposed. Namely, we explore connected affine algebraic groups $G$ such that for every algebraic $G$-action, every $G$-orbit closure contains only finitely many $G$-orbits. The example of tori shows that such groups do exist.

First, we solve the classification problem for such groups. The answer we found turned out to be rather unexpected for us. Namely, our first main result is the following theorem (which is a part of Theorem 3 below).

**Theorem 1.** Let $G$ be a connected affine algebraic group. The following properties are equivalent:

(i) every $G$-orbit closure of every algebraic action of $G$ contains only finitely many $G$-orbits;

(ii) $G$ is either a torus or a direct product of a torus and a connected one-dimensional unipotent algebraic group.

Secondly, we find that such groups play a special role in the theory of modality [PV94, 5.2] that goes back to Arnol’d’s works on the theory of singularities [Arn75]. The modality of an algebraic action of $G$ is the maximal number of parameters, on which an algebraic family of $G$-orbits of this action may depend. The actions of modality 0 are precisely that with a finite number of $G$-orbits. In this paper we prove that the groups enjoying the above properties (i), (ii) can be characterized in terms of modality. Namely, our second main result is the following theorem (which is a part of Theorem 3 below).

**Theorem 2.** The conditions (i), (ii) in Theorem 1 hold if and only if for every $G$-action $\alpha$ on an irreducible algebraic variety $X$,

$$\text{the modality of } \alpha \text{ is equal to } \dim X - \max_{x \in X} \dim G \cdot x. \quad (1)$$

Property (1) means that the maximal number of parameters, on which an algebraic family of $G$-orbits in $X$ may depend, is attained on a family, which is open in $X$. If (1) holds, we say that $\alpha$ is a *modality-regular* action.

For a given $G$, Theorems 1, 2 reveal when the phenomenon of modality-regular action holds unconditionally, i.e., for every algebraic action of $G$. In some conditional forms the manifestations of this phenomenon has been discovered earlier.

For instance, it follows from [Vin86, Thms. 2 and 3] that if $G$ is either a Borel subgroup $B$ of a connected complex reductive group $R$ or the unipotent radical $U$ of $B$, then the restriction to $G$ of any action of $R$ on an irreducible algebraic variety is modality-regular.
Theorems 1, 2 imply that apart from $B$ and $U$, there are other closed subgroups $H$ of $R$ such that the restriction to $H$ of any algebraic action of $R$ on an irreducible algebraic variety is modality-regular. Namely, this property also holds if $H$ is either a torus or a product of a torus and a connected one-dimensional unipotent algebraic group. To the best of my knowledge, at this writing (April 2020) a complete classification of subgroups $H$ enjoying this property is not known.

Note that such subgroups $H$ yield examples of non-extendable algebraic actions: in view of Theorems 1, 2, if $H$ is not a torus or a product of a torus and a one-dimensional connected unipotent algebraic group, then there exists an algebraic action of $H$ on an irreducible algebraic variety $X$, which can not be extended up to an algebraic action of $R$ on $X$.

Some other conditional forms of the manifestations of the phenomenon of modality-regular action are found in [Pop17]; for instance, it is proved that if $G$ is reductive, then every visible action of $G$ on an irreducible affine variety is modality-regular.

This paper is organized as follows. Our central result, Theorem 3, is formulated in Section 4 and proved in Section 7. Section 3 contains the materials about modality, which is necessary for stating Theorem 3. In Sections 5 and 6 are collected some auxiliary results on property (F) from the statement of Theorem 3 and on modality, which we use in the proof of this theorem. In Section 2 are collected some conventions, notation, and terminology.

Acknowledgement. I am grateful to J.-P. Serre for remarks.

2. Conventions, notation, and terminology

We fix an algebraically closed field $k$ of arbitrary characteristic. In what follows, we freely use the viewpoint, standard notation and conventions of [Bor91], [Spr98], [PV94], where also the proofs of unreferenced claims and/or the relevant references can be found.

A variety means an algebraic variety over $k$ (so an algebraic group means an algebraic group over $k$). If all irreducible components of a variety $X$ have the same dimension, then $X$ is called equidimensional. Topological terms are related to the Zariski topology.

Below all actions of algebraic groups on varieties are assumed to be algebraic (i.e., regular/morphic). If an algebraic group $G$ acts on a variety $X$, we say that $X$ is a $G$-variety.

The product of $d$ copies of $G_n$ (respectively $G_m$) is denoted by $G_n^d$ (respectively $G_m^d$). If an algebraic group is isomorphic to $G_m^d$ for some $d$, it is called a torus.
3. Modality

Let $H$ be a connected algebraic group. Any irreducible $H$-variety $F$ such that all $H$-orbits in $F$ have the same dimension $d$ is called an (algebraic) family of $H$-orbits depending on

$$\text{mod}(H:F) := \dim F - d$$

(2)

parameters; the integer $\text{mod}(H:F)$ is called the modality of $F$. If $F \rightarrow F\!/:H$ is a rational geometric quotient of this action (i.e., the geometric quotient of an $H$-stable dense open subset of $F$; such a quotient exists by the Rosenlicht theorem [PV94, Thm. 4.4]), then

$$\text{mod}(H:F) = \dim F\!/:H = \text{tr deg}_k(F)^H$$

(3)

and $F\!/:H$ may be informally viewed as the variety parametrizing typical $H$-orbits in $F$.

Given an $H$-variety $Y$, we denote by $\mathcal{F}(Y)$ the set of all locally closed $H$-stable subsets of $Y$, which are the families. The integer

$$\text{mod}(H:Y) := \max_{F \in \mathcal{F}(Y)} \text{mod}(H:F),$$

(4)

is then called the modality of the $H$-variety $Y$.

If $G$ is a (not necessarily connected) algebraic group and $X$ is a $G$-variety, then by definition,

$$\text{mod}(G:X) := \text{mod}(G^0 : X),$$

where $G^0$ is the identity component of $G$.

It readily follows from the definition that if $Z$ is a locally closed $G$-stable subset of $X$, then

$$\text{mod}(G:X) \geq \text{mod}(G:Z).$$

(5)

Recall that, for every integer $s$, the set $\{x \in X \mid \dim G \cdot x \leq s\}$ is closed in $X$. Whence, for every locally closed (not necessarily $G$-stable) subset $Z$ in $X$,

$$Z^{\text{reg}} := \{z \in Z \mid \dim G \cdot z \geq \dim G \cdot x \text{ for every } x \in Z\}$$

(6)

is a nonempty open subset of $Z$.

The definition of modality implies that (4) still holds if $\mathcal{F}(Y)$ is replaced by the set of all maximal (with respect to inclusion) families in $Y$, i.e., by the sheets of $Y$ [PV94, Sect. 6.10]. Recall that there are only finitely many sheets of $Y$. If $Y$ is irreducible, then $Y^{\text{reg}}$ is a sheet, called regular, which is open and dense in $Y$. By (3),

$$\text{mod}(H:Y^{\text{reg}}) = \text{tr deg}_k(Y)^H.$$  

(7)

Similarly, (4) still holds if $\mathcal{F}(Y)$ is replaced by the set of all $H$-stable irreducible locally closed (or closed) subsets of $Y$, and $\text{mod}(H:F)$ by $\text{tr deg}_k(F)^H$. 


The aforesaid shows that \( \text{mod}(G : X) = 0 \) if and only if the set of all \( G \)-orbits in \( X \) is finite.

The existence of regular sheets leads to defining the following distinguished class of algebraic group actions:

**Definition 1.** An irreducible \( G \)-variety \( X \) and the action of \( G \) on \( X \) are called *modality-regular* if \( \text{mod}(G : X) = \text{mod}(G : X^{\text{reg}}) \).

It follows from (2), (6), and Definition 1 that an irreducible \( G \)-variety \( X \) is modality-regular if and only if

\[
\text{mod}(G : X) = \dim X - \max_{x \in X} \dim G \cdot x.
\]

**4. Main result: formulation**

**Theorem 3.** Let \( G \) be a connected affine algebraic group. The following properties are equivalent:

(\( F \)) there are only finitely many \( G \)-orbits in every \( G \)-variety containing an open \( G \)-orbit;

(\( M \)) every irreducible \( G \)-variety is modality-regular;

(\( G \)) \( G \) is either a torus or a direct product of a torus and a connected one-dimensional unipotent algebraic group.

**Remarks.**

1. Recall that every connected one-dimensional unipotent algebraic group is isomorphic to \( \mathbb{G}_a \) (see, e.g., [Spr98, 3.4.9]).

2. Let \( G \) be a connected affine algebraic group. The following properties are equivalent:

(i) \( G \) is a product of a torus and a connected one-dimensional unipotent algebraic group.

(ii) \( G \) is nilpotent and its unipotent radical is one-dimensional.

Therefore, property (\( G \)) is equivalent to the property

(\( G' \)) \( G \) is nilpotent and its unipotent radical is at most one-dimensional.

**5. Auxiliary results: property (\( F \))**

This section contains some auxiliary results about property (\( F \)) from the formulation of Theorem 3 that will be used in its proof. First we explore behaviour of this property under passing to a subgroup and a quotient group. Then we explore it for two-dimensional connected solvable affine algebraic groups, and, in conclusion, for semisimple affine algebraic groups.
5.1. Passing to a subgroup and a quotient group.

**Lemma 1.** Let $G$ be a connected affine algebraic group and let $H$ be its closed subgroup. If $G$ enjoys property (F), then

(a) $H$ enjoys property (F);

(b) $G/H$, for normal $H$, enjoys property (F).

**Proof.** (a) Arguing on the contrary, suppose there exists an irreducible $H$-variety $Y$ with infinitely many $H$-orbits, one of which, say, $O$, is open in $Y$. Since the action canonically lifts to the normalization [Ses63], we may (and shall) assume that $Y$ is normal. Then, by [Sum74, Lem. 8], we have $Y = \bigcup_{i \in I} U_i$, where each $U_i$ is an $H$-stable quasi-projective open subset of $Y$. As $Y$ is irreducible, each $U_i$ contains $O$. Since in the Zariski topology any open covering contains a finite subcovering, there is $i_0 \in I$ such that $U_{i_0}$ contains infinitely many $H$-orbits. Therefore replacing $Y$ by $U_{i_0}$, we may (and shall) assume that $Y$ is quasi-projective. Then, by [Ser58, 3.2] (see also [PV94, Thm. 4.9]), the homogeneous fiber space $X := G \times^H Y$ over $G/H$ with the fiber $Y$ is an algebraic variety. Since for the action of $H$ on $Y$ there are infinitely many orbits one of which is open, the natural action of $G$ on $X$ enjoys these properties as well; see [PV94, Thm. 4.9]. This contradicts the condition that $G$ enjoys property (F), thereby proving (a).

(b) Assume, again arguing on the contrary, that there is an irreducible algebraic $G/H$-variety $X$ with infinitely many $G/H$-orbits, one of which is open in $X$. Since the canonical homomorphism $G \to G/H$ determines an action of $G$ on $X$ whose orbits coincide with $G/H$-orbits, this contradicts the condition that $G$ enjoys property (F), thereby proving (b). \[\square\]

5.2. Two-dimensional connected solvable affine algebraic groups.

Let $S$ be a two-dimensional connected solvable affine algebraic group. Then $S = T \ltimes S_u$, where $T$ is a maximal torus and $S_u$ is the unipotent radical of $S$. There are only the following three possibilities (S1), (S2), and (S3) for $S$:

(S1) $S_u$ is trivial, i.e., $S$ is a torus.

Such $S$ enjoys property (F) (see the first example in Introduction).

(S2) The following equality holds: $\dim T = \dim S_u = 1$.

In this case, $T$ and $S_u$ are isomorphic to respectively to $G_m$ and $G_a$ and there is $d \in \mathbb{Z}$ such that $S$ is isomorphic to the group $S(d) := G_m \ltimes G_a$, in which the group operation is defined by the formula

$$(t_1, u_1)(t_2, u_2) := (t_1 t_2, t_1^d u_1 + u_2), \quad t_i \in G_m, u_i \in G_a.$$  \hspace{1cm} (8)

Indeed, fix an isomorphism $\theta: G_a \to S_u$. For any $t \in T$, the map $S_u \to S_u, u \mapsto tu t^{-1}$, is an automorphism, therefore, there is a character $\chi: T \to G_m$ such that $t\theta(u)t^{-1} = \theta(\chi(t)u)$ for all $u \in G_a, t \in T$; whence the claim.
The group $S(d)$ is commutative if and only if $d = 0$.

**Proposition 1.** Every group $S(d)$ for $d \neq 0$ does not enjoy property (F).

**Proof.** It follows from (8) that

$$S(d) \to \text{GL}_2, \quad (t, u) \mapsto \begin{pmatrix} t^d & 0 \\ u & 1 \end{pmatrix},$$

is a representation of $S(d)$. It determines the following linear action of $S(d)$ on $A^2$: 

$$g \cdot a := (a_1 t^d, a_1 u + a_2) \quad \text{for} \quad g = (t, u) \in S(d), \quad a = (a_1, a_2) \in A^2. \quad (9)$$

From (9) and $d \neq 0$ we immediately infer that the fixed point set of this action is the line $\ell := \{(a_1, a_2) \in A^2 \mid a_1 = 0\}$ whose complement $A^2 \setminus \ell$ is a single orbit. Thus, the $S(d)$-variety $A^2$ contains infinitely many orbits one of which is open. This completes the proof. □

(S3) $T$ is trivial, i.e., $S$ is unipotent.

**Proposition 2.** Every two-dimensional connected unipotent affine algebraic group $S$ does not enjoy property (F).

**Proof.** First, by [Spr98, 6.3.4] there exists a one-dimensional connected closed subgroup of $S$ lying in the center of $S$. Since $\dim S = 2$, this implies that there is an exact sequence of group homomorphisms

$$0 \to \mathbf{G}_a \to S \to \mathbf{G}_a \to 0, \quad (10)$$

such that $\iota(G_a)$ is a central subgroup of $S$. Thus $S$ is a central extension of $G_a$ by $G_a$.

We consider two cases, of which the first is simpler than the second.

The consideration in the first case is based on the following

**Claim.** The group $G_a^d$ does not enjoy property (F) for every $d \geq 2$.

**Proof of Claim.** The action of $G_a^d$ on itself by translations is the action on the affine space $A^d$ defined by the formula

$$u \cdot a := (a_1 + u_1, \ldots, a_d + u_d) \quad \text{for} \quad u = (u_1, \ldots, u_d) \in G_a^d, \quad a = (a_1, \ldots, a_d) \in A^d.$$

We indentify $A^d$ with the affine chart

$$P^d_d := \{(b_0 : \ldots : b_d) \in P^d \mid b_d \neq 0\} \quad (11)$$

of the projective space $P^d$. Then the following formula extends this action to the action of $G_a^d$ on $P^d$:

$$u \cdot b := (b_0 + u_1 b_d : \ldots : b_{d-1} + u_d b_d : b_d) \quad \text{for} \quad u = (u_1, \ldots, u_d) \in G_a^d, \quad b = (b_0 : \ldots : b_d) \in P^d.$$
For the latter action, $P_d^d$ is an open orbit, and the hyperplane $P^d \setminus P_d^d$ is pointwise fixed. For $d \geq 2$, this hyperplane contains infinitely many points, whence the claim. □

The two cases mentioned above are that of char $k = 0$ and char $k \neq 0$. Note that in the first version of this paper [Pop17] only the simpler case of char $k = 0$ has been considered, as at the time of writing we did not have a proof for char $k \neq 0$.

**First case:** char $k = 0$.

The fact that in this case $S$ does not enjoy property (F) immediately follows from the above Claim, because char $k = 0$ implies that $S$ is isomorphic to $G^2_a$. Indeed, using the notation of (10), take an element in $S \setminus \iota(G_a)$. Since char $k = 0$, it lies in a connected closed one-dimensional subgroup $D$ of $S$; see [Bor91, Rem. in II.7.3]. Both $\iota(G_a)$ and $D$ are isomorphic to $G_a$, centralize each other, and $\iota(G_a) \cap D = e$ because, in view of char $k = 0$, $G_a$ is the closure of any its nontrivial cyclic subgroup. This implies that $\iota(G_a) \times D \to S$, $(g, h) \mapsto gh$ is an embedding of algebraic groups, and therefore, an isomorphism because dim $S = 2$.

**Second case:** char $k = p > 0$.

In this case, it is no longer true that every two-dimensional connected unipotent affine algebraic group (even commutative) is isomorphic to $G^2_a$. In [Pop19], we proved that the two-dimensional Witt group $W_2(p)$ does not enjoy property (F). Below we give a proof in the general case, developing the idea used in [Pop19].

Since the underlying variety of $S$ is isomorphic to $A^2$ (see [Gr58, Prop. 2]), we can (and shall) identify $S$ with $A^2$ endowed with a group operation $\ast$, which is described below.

Let $x_0, x_1 \in k[A^2]$ be the standard coordinate functions on $A^2$:

$$x_i(a) = a_i \text{ for every } a = (a_0, a_1) \in A^2.$$  \hspace{1cm} (12)

The classes of equivalent central extensions of $G_a$ by $G_a$ are classified by $H^2(G_a, G_a)$ (see, e.g., [DFPS08, Chap. 2]). In [DG70, II, §3, 4.6], it is shown that $H^2(G_a, G_a)$ is a free $\text{End}(G_a)$-module, having the following family of 2-cocycles as a basis modulo the coboundaries:

$$\sum_{i=1}^{p-1} c_i x_0^i x_1^{p-i}, \text{ where } c_i := (p - 1)!/i!(p - i)!, \hspace{1cm} (13)$$

(informally, the first polynomial in (13) is $((x_0^p + x_1^p) - (x_0 + x_1)^p)/p$.) In view of the existence of an exact sequence (10), this implies that there are polynomials $f, h_1, \ldots, h_d \in k[z]$ (where $z$ is a variable over $k$) and positive integers $m_1, \ldots, m_d$ such that, for all $(a_0, a_1), (b_0, b_1) \in A^2$, we
have
\[(a_0, a_1) \ast (b_0, b_1)\]
\[= \left( a_0 + b_0, a_1 + b_1 + f\left( \sum_{i=1}^{p-1} c_i a_0^i b_0^{p-i} \right) + \sum_{j=1}^{d} h_j(a_0 b_0^{m_j}) \right). \tag{14}\]

We consider \(k[x_0, x_1] = k[S]\) as an \(S\)-module with respect to the action of \(S\) by left translations (see [Bor91, I.1.9]). It then follows from (14), (12) that for every \(s = (s_0, s_1) \in S\), we have
\[s^{-1} \cdot x_0 = x_0 + s_0, \tag{15}\]
\[s^{-1} \cdot x_1 = x_1 + s_1 + f\left( \sum_{i=1}^{p-1} c_i s_0^i x_0^{p-i} \right) + \sum_{j=1}^{d} h_j(s_0 x_0^{m_j}). \tag{16}\]

We now fix an integer \(n\) such that
\[\deg(f\left( \sum_{i=1}^{p-1} c_i x_0^i x_1^{p-i} \right) + \sum_{j=1}^{m} h_j(x_0 x_1^{m_j})) \leq n. \tag{17}\]

It then follows from (15), (16), and (17) that the \(k\)-linear span \(V\) of the sequence of functions
\[1, x_0, x_0^2, \ldots, x_0^n, x_1 \tag{18}\]
is an \((n + 2)\)-dimensional \(S\)-stable linear subspace of \(k[x_0, x_1]\), and if
\[\rho_V : S \rightarrow GL(V)\]
is the linear representation determined by the natural action of \(S\) on \(V\), then there are polynomials \(g_0, g_1, \ldots, g_{n-1} \in k[z]\) such that
\[\deg(g_i) \leq n \quad \text{for all } i, \tag{19}\]
and, for every \(s = (s_0, s_1) \in S\), the matrix of \(\rho_V(s^{-1})\) relative to basis (18) has the form
\[R_{g^{-1}} := \begin{pmatrix}
1 & s_0 & s_0^2 & s_0^3 & s_0^4 & \ldots & s_0^{n-1} & s_1 \\
0 & 1 & \binom{2}{1}s_0 & \binom{3}{2}s_0^2 & \binom{4}{3}s_0^3 & \ldots & \binom{n-1}{n-2}s_0^{n-2} & q_{n-1}(s_0) \\
0 & 0 & 1 & \binom{2}{1}s_0 & \binom{3}{2}s_0^2 & \ldots & \binom{n-2}{n-3}s_0^{n-3} & q_{n-2}(s_0) \\
0 & 0 & 0 & 1 & \binom{3}{2}s_0 & \ldots & \binom{n-3}{n-4}s_0^{n-4} & q_{n-3}(s_0) \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & 1 & \binom{n}{1}s_0 & q_1(s_0) \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & g_0(s_0) \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}. \tag{20}\]

Next, we fix a positive integer \(r\). It follows from (15) and \(\text{char } k = p\) that the \(k\)-linear span \(U\) of the sequence of functions
\[1, x_0^{p^r} \tag{21}\]
is a two-dimensional $S$-stable linear subspace of $k[x_0, x_1]$, and if $\rho_U : S \to \text{GL}(U)$ is the linear representation determined by the natural action of $S$ on $U$, then, for every $s = (s_0, s_1) \in S$, the matrix of $\rho_U(s^{-1})$ relative to basis (21) has the form

$$P_{s^{-1}} := \begin{pmatrix} 1 & s_0^{p^r} \\ 0 & 1 \end{pmatrix}. \quad (22)$$

We now identify $V \oplus U$ with the affine space $\mathbb{A}^{n+4}$ by means of the bijection

$$V \oplus U \to \mathbb{A}^{n+4}, \quad (\sum_{i=0}^n a_i x_0^i + a_{n+1} x_1 + a_{n+2} x_0^{p^r}) \mapsto (a_0, a_1, \ldots, a_{n+3}),$$

and, in turn, identify $\mathbb{A}^{n+4}$ with the standard affine chart

$$\mathbb{P}^{n+4}_{n+4} := \{(b_0 : b_1 : \ldots : b_{n+3}) \mid b_{n+3} \neq 0\}$$

of the projective space $\mathbb{P}^{n+4}$ by means of the bijection

$$\mathbb{A}^{n+4} \to \mathbb{P}^{n+4}_{n+4}, \quad (a_0, a_1, \ldots, a_{n+3}) \mapsto (a_0 : a_1 : \ldots : a_{n+3} : 1).$$

The linear action of $S$ on $\mathbb{A}^{n+4}$ determined by $\rho_V \oplus \rho_U$ extends to the following action of $S$ on $\mathbb{P}^{n+4}$: if $s \in S$ and

$$Q_{s^{-1}} := \begin{pmatrix} R_{s^{-1}} & 0 \\ 0 & P_{s^{-1}} \end{pmatrix}, \quad (23)$$

then

$$s^{-1} \cdot (b_0 : b_1 : \ldots : b_{n+4}) := (b'_0 : b'_1 : \ldots : b'_{n+4}), \quad \text{where} \quad \begin{pmatrix} b'_0 \\ b'_1 \\ \vdots \\ b'_{n+3} \end{pmatrix}^T = Q_{s^{-1}} \begin{pmatrix} b_0 \\ b_1 \\ \vdots \\ b_{n+3} \end{pmatrix}^T, \quad b'_{n+4} = b_{n+4}. \quad (24)$$

It follows from (20), (22), (23), (24) that the line

$$\ell := \{(b_0 : b_1 : \ldots : b_{n+4}) \in \mathbb{P}^{n+4} \mid b_i = 0 \text{ for every } i \neq 0, n + 2\} \quad (25)$$

is pointwise fixed with respect to this action of $S$ on $\mathbb{P}^{n+4}$:

$$\ell \subseteq (\mathbb{P}^{n+4})^S. \quad (26)$$

We shall now show that if we take $r$ such that

$$p^r > n, \quad (27)$$

then there exists an $S$-orbit in $\mathbb{P}^{n+4}$, whose closure contains the line $\ell$; this and (26) will then complete the proof.

Namely, assume that (27) holds and consider the $S$-orbit of the point

$$v := (1 : 1 : \ldots : 1) \in \mathbb{P}^{n+4}. \quad (28)$$

It follows from (20), (22), (23), (24) that this orbit $S \cdot v$ is the image of the morphism

$$\varphi : \mathbb{A}^{2} \to \mathbb{P}^{n+4}, \quad s \mapsto (f_0(s) : f_1(s) : \ldots : f_{n+3}(s) : 1), \quad (29)$$
where
\begin{align*}
f_0 &= 1 + x_0 + x_0^2 + \cdots + x_0^n + x_1, \\
f_1 &= 1 + \binom{n}{1}x_0 + \binom{n}{2}x_0^2 + \cdots + \binom{n}{n-1}x_0^{n-1} + q_{n-1}(x_0), \\
f_2 &= 1 + \binom{n}{1}x_0 + \binom{n}{2}x_0^2 + \cdots + \binom{n}{n-2}x_0^{n-2} + q_{n-2}(x_0), \\
f_3 &= 1 + \binom{n}{1}x_0 + \binom{n}{2}x_0^2 + \cdots + \binom{n}{n-3}x_0^{n-3} + q_{n-3}(x_0), \\
\vdots & \quad \quad \vdots \\
f_{n-1} &= 1 + \binom{n}{1}x_0 + q_1(x_0), \\
f_n &= 1 + q_0(x_0), \\
f_{n+1} &= 1, \\
f_{n+2} &= 1 + x_0^r, \\
f_{n+3} &= 1.
\end{align*}

(29)

Let \( A^1_* := A^1 \setminus \{0\} \). We assign to every \( a \in k \) the morphism
\[
\psi_a: A^1_* \rightarrow A^2, \quad t \mapsto (t^{-1}, at^{−r}).
\]

(30)

Formulas (28), (29), (30) show that \((\varphi \circ \psi_a)(t)\) for every \( t \in A^1_* \) is the point
\[
(f_0(t^{-1}, at^{−r})): f_1(t^{-1}, at^{−r}): \ldots : f_n(t^{-1}, at^{−r}): 1:1 + t^{−r}: 1:1) \in \mathbb{P}^{n+4}.
\]

Since \( t \neq 0 \), this point coincides with
\[
(t^{−r}f_0(t^{-1}, at^{−r})): t^{−r}f_1(t^{-1}, at^{−r}): \ldots : t^{−r}f_n(t^{-1}, at^{−r}): 1+t^{−r}: t^{−r}: t^{−r}).
\]

(31)

Denote by \( k[z]_+ \) the \( k \)-linear span of \( \{z^d \mid d > 0\} \) in \( k[z] \). From (29), (27), and (19) we infer the existence of polynomials \( u_0, u_1, \ldots , u_n \in k[z]_+ \) such that
\[
t^{−r}f_i(t^{-1}, at^{−r}) = \begin{cases} 
a + u_0(t) & \text{for } i = 0, \\
u_i(t) & \text{for } i = 1, \ldots , n.
\end{cases}
\]

(32)

From (31) and (32) we conclude that the morphism
\[
\varphi \circ \psi_a: A^1_* \rightarrow S \cdot v \subseteq \mathbb{P}^{n+4}
\]
uniquely extends up to a morphism
\[
\varepsilon_a: A^1 \rightarrow \mathbb{P}^{p+4}
\]
such that
\[
\varepsilon_a(0) = (a: 0: 0: \ldots : 0: 1: 0: 0).
\]

By (25) we have \( \varepsilon_a(0) \in \ell \). Since \( a \) is an arbitrary element of \( k \), we infer that the closure of \( S \cdot v \) in \( \mathbb{P}^{n+4} \) contains a nonempty subset of \( \ell \); hence it contains the whole \( \ell \) as well. This completes the proof of Proposition 2. \( \Box \)
5.3. Semisimple affine algebraic groups.

**Proposition 3.** Every nontrivial connected semisimple algebraic group $G$ does not enjoy property (F).

**Proof.** Let $\alpha$ be a root of $G$ with respect to a maximal torus and let $G_\alpha$ be the centralizer of the torus $(\ker \alpha)^0$ in $G$. The commutator group $(G_\alpha, G_\alpha)$ is isomorphic to either SL$_2$ or PSL$_2$ (see, e.g., [Spr98, 7.1.2, 8.1.4]). Correspondingly, the Borel subgroups of $(G_\alpha, G_\alpha)$ are isomorphic to either $S(1)$ or $S(2)$ (see Subsection 5.2(S2) above). Hence, by Proposition 1, they do not enjoy property (F). The claim then follows from Lemma 1(a). \qed

6. Auxiliary results: modality

The following lemma helps to practically compute the modality and will be used in the proof of Theorem 3.

**Lemma 2.** Let $G$ be an algebraic group, let $X$ be a $G$-variety, and let $\{C_i\}_{i \in I}$ be a collection of the subsets of $X$ such that

(i) $I$ is finite;

(ii) $\bigcup_{i \in I} C_i = X$;

(iii) the closure $\overline{C_i}$ of $C_i$ in $X$ is irreducible for every $i \in I$;

(iv) every $C_i$ is $G$-stable;

(v) all $G$-orbits in $C_i$ have the same dimension $d_i$ for every $i \in I$.

Then the following hold:

(a) $\mod(G : X) = \max_{i \in I} (\dim \overline{C_i} - d_i)$;

(b) if $X$ is irreducible, then $X = \overline{C_{i_0}}$ for some $i_0$, and $\mod(G : X^{\text{reg}}) = \dim X - d_{i_0}$.

**Proof.** By (iii), we have a family $\overline{C_i}^{\text{reg}}$, and (v) implies $C_i \subseteq \overline{C_i}^{\text{reg}}$. Whence

$$\mod(G : \overline{C_i}^{\text{reg}}) = \dim \overline{C_i} - d_i. \tag{33}$$

From (5) and (33), we infer that $\mod(G : X) \geq \max_{i \in I} (\dim \overline{C_i} - d_i)$.

To prove the opposite inequality let $Z \in \mathcal{F}(X)$ be a family of $s$-dimensional $G$-orbits such that $\mod(G : X) = \dim Z - s$ and let $J := \{i \in I | Z \cap C_i \neq \emptyset\}$. By (ii), we have $Z = \bigcup_{j \in J} (Z \cap \overline{C_j})$. Since $Z$ is irreducible and, by (i), $J$ is finite, there is $j_0 \in J$ such that $Z \subseteq \overline{C_{j_0}}$. As $Z \cap C_{j_0} \neq \emptyset$, we have $s = d_{j_0}$. Therefore, $\mod(G : X) = \dim Z - s \leq \dim \overline{C_{j_0}} - d_{j_0}$. This proves (a).

By (ii), $\bigcup_{i \in I} \overline{C_i} = X$. If $X$ is irreducible, then, in view of (i), this equality implies the existence of $i_0$ such that $X = \overline{C_{i_0}}$. This and (33) prove (b). \qed
Lemma 3. Let $G$ be a connected algebraic group, let $X$ and $Y$ be the irreducible $G$-varieties, and let $\varphi: X \longrightarrow Y$ be a rational $G$-equivariant map.

(i) If $\varphi$ is dominant, then $\text{mod}(G : X^{\text{reg}}) \geq \text{mod}(G : Y^{\text{reg}})$. If, moreover, $\dim X = \dim Y$, then $\text{mod}(G : X^{\text{reg}}) = \text{mod}(G : Y^{\text{reg}})$.

(ii) If $\varphi$ is a surjective morphism, then $\text{mod}(G : X) \geq \text{mod}(G : Y)$.

Proof. The inequality in (i) follows from (7) because $\varphi$ determines a $G$-equivariant field embedding $\varphi^*: k(Y) \hookrightarrow k(X)$.

Assume that $\dim X = \dim Y$. Then, by the fiber dimension theorem, the fibers of $\varphi$ over the points of an open subset of $Y$ are finite. Whence, for every point $x$ of an open subset of $X$, the equality $\dim G \cdot x = \dim G \cdot \varphi(x)$ holds. This implies the following equality:

$$m_X := \max_{x \in X} \dim G \cdot x = m_Y := \max_{y \in Y} \dim G \cdot y. \quad (34)$$

From (34) and (2) we obtain $\text{mod}(G : X^{\text{reg}}) = \dim X - m_X = \dim Y - m_Y = \text{mod}(G : Y^{\text{reg}})$. This proves (i).

To prove (ii), consider a family $F$ in $Y$ such that

$$\text{mod}(G : Y) = \text{mod}(G : F). \quad (35)$$

If $\varphi$ is a surjective morphism, then $\varphi: \varphi^{-1}(F) \rightarrow F$ is a surjective morphism. As $F$ is irreducible, there is an irreducible component $\tilde{F}$ of $\varphi^{-1}(F)$ such that $\varphi: \tilde{F} \rightarrow F$ is a surjective morphism. Since $\varphi$ is $G$-equivariant and $G$ is connected, $\tilde{F}$ is $G$-stable, so the latter morphism is $G$-equivariant. Hence

$$\text{mod}(G : X) \geq \text{mod}(G : \tilde{F}^{\text{reg}}) \geq \text{mod}(G : F^{\text{reg}}) = \text{mod}(G : F) = \text{mod}(G : Y) \quad (36)$$

(in (36), the first inequality follows from (4), and the second from (i); the first equality follows from $F = F^{\text{reg}}$, and the second from (35)). This proves (ii). \[\Box\]

Recall [Ses63] that if $G$ is an algebraic group, $X$ is an irreducible $G$-variety, and $\nu: \tilde{X} \rightarrow X$ is its normalization, then the action of $G$ on $X$ lifts to $\tilde{X}$ so that $\nu$ becomes $G$-equivariant.

Corollary 1. In the above notation,

(i) $\text{mod}(G : \tilde{X}) = \text{mod}(G : X)$;

(ii) $X$ is modality-regular if and only if $\tilde{X}$ is.

Lemma 4. Let $T$ be a torus and let $Y$ be an irreducible $T$-variety. Then

(i) the stabilizer of any point of an open subset of $Y$ is the kernel of $T$-action on $Y$;

(ii) $Y$ is modality-regular.
Proof. First, we may (and shall) assume that $T$ acts on $Y$ faithfully. Next, by Corollary 1, replacing $Y$ by $\tilde{Y}$, we may (and shall) assume that $Y$ is normal. By [Sum74, Cor. 2, p. 8], then $Y$ is covered by $T$-stable affine open subsets. Whence we may (and shall) assume that $Y$ is affine.

(i) As $Y$ is affine, by [PV94, Thm. 1.5], we may (and shall) assume that $Y$ is a closed $T$-stable subset of a finite-dimensional algebraic $T$-module and $Y$ does not lie in a proper $T$-submodule of $V$. The action of $G$ on $V$ is faithful because that on $Y$ is. As $T$ is a torus, $V$ is a direct sum of the $T$-weight subspaces. Let $U$ be the complement in $V$ to the union of these subspaces. The stabilizer of any point of $U$ coincides with the kernel of the action on $V$, hence is trivial. As, by construction, $Y \cap U \neq \emptyset$, this proves (i).

(ii) In view of Definition 1, the proof of [Vin86, Prop. 1] can be viewed as that of (ii). For the sake of completeness, below is a different proof.

By (i), we have $\text{mod}(T : Y^{\text{reg}}) = \dim Y - \dim T$. Let $S$ be a sheet in the $T$-variety $Y$, and let $T_0$ be the kernel of the action of $T$ on $S$. Then $\text{mod}(T : S) = \dim S - (\dim T - \dim T_0)$.

Since $Y$ is affine, the morphism $\pi : Y \rightarrow Y/T_0 = : \text{Spec} k[Y]^{T_0}$ induced by the identity embedding $k[Y]^{T_0} \hookrightarrow k[Y]$ is the categorical quotient for the action of $T_0$ on $Y$. As $T_0$ act on $Y$ faithfully, (i) yields $\dim Y/T_0 \leq \dim Y - \dim T_0$.

Since $S$ is pointwise fixed by $T_0$ and $\pi$ separates closed $T_0$-orbits (see [PV94, Thm. 9.4]), we have $\dim \pi(S) = \dim S$; whence $\dim S \leq \dim Y/T_0 \leq \dim Y - \dim T_0$. Combining this information, we obtain

$$\text{mod}(T : S) = \dim S - \dim T + \dim T_0 \leq \dim Y - \dim T_0 - \dim T + \dim T_0 = \text{mod}(T : Y^{\text{reg}}).$$

This completes the proof. \qed

7. Main result: proof

We shall prove the implications $(M) \Rightarrow (F) \Rightarrow (G) \Rightarrow (M)$.

1. The implication $(M) \Rightarrow (F)$ is clear.

2. We now turn to the proof of the implication $(F) \Rightarrow (G)$.

Assume that $G$ enjoys property $(F)$. Let $R$ be the radical of $G$. Since the group $G/R$ is semisimple, our assumption, Lemma 1, and Proposition 3 entail that $G/R$ is trivial, i.e., $G$ is solvable. Whence $G = T \ltimes U$, where $T$ is a maximal torus and $U$ is the unipotent radical of $G$. We should show that either $U$ is trivial or $U$ is isomorphic to $G_a$ and $G$ is commutative. Arguing on the contrary, we suppose that this is not so.

Then $U$ is a nontrivial unipotent group. Hence there exists a chain $\{e\} = U_1 \subsetneq U_2 \subsetneq \cdots \subsetneq U_d = U$ of closed connected subgroups, normal
in $G$, such that $d \geq 2$, and the successive quotients are one-dimensional; see [Bor91, 10.6].

We claim that $d = 2$. Indeed, if this is not the case, the above chain contains $U_3$. Since $\dim U_3 = 2$, Proposition 2 and Lemma 1 yield contradiction with the fact that $G$ enjoys property (F). Thus $d = 2$; whence $U$ is isomorphic to $G_a$.

Next, the assumption that $G$ is not commutative means that the conjugating action of $T$ on $U$ is nontrivial. As $T$ is generated by its one-dimensional subtori, there is such a subtorus $T'$ not lying in the kernel of this action. Then $T'U$ is a noncommutative closed connected two-dimensional subgroup of $G$; see [Bor91, 2.2]. Hence it is isomorphic to $S(n)$ for some $n \neq 0$; see case (S3) in Section 5. By Proposition 1 and Lemma 1, this is impossible since $G$ enjoys property (F). This contradiction proves the implication (F) $\Rightarrow$ (G).

3. Now we turn to the proof of the last implication (G) $\Rightarrow$ (M).

Assume that (G) holds and $G$ acts on an irreducible variety $X$. We should show that this action is modality-regular. In view of Lemma 4(ii), we should consider only the case, where $G$ is a direct product of two subgroups:

$$G = T \times U, \quad T \text{ is a torus, } U \text{ is isomorphic to } G_a.$$  \hfill (37)

Below, exploring the actions of the subgroups of $G$ on $X$, we always mean the actions obtained by restricting the given action of $G$ on $X$.

We may (and shall) assume that $G$ acts on $X$ faithfully. In view of Corollary 1(ii), replacing $X$ by its normalization, we also may (and shall) assume that $X$ is normal.

Notice that since the elements of $T$ (respectively, $U$) are semisimple (respectively, unipotent), and the $G$-stabilizers of points of $X$, being closed in $G$, contain the Jordan decomposition components of their elements, for these stabilizers we have, in view of (37), the equalities:

$$G_x = T_x \times U_x \quad \text{for every } x \in X.$$  \hfill (38)

As $G$ acts on $X$ faithfully, from (38), (37), and Lemma 4(i) we infer that

$$G_x \text{ is finite for every } x \in X^{\text{reg}}.$$  \hfill (39)

Let $S$ be a sheet of the action of $T$ on $X$. As $T$ and $U$ commute and both are connected, $S$ is $U$-stable and every sheet $C$ of the action of $U$ on $S$ is $T$-stable, hence $G$-stable. Consider the set of all $C$’s, obtained in this way when $S$ runs over all sheets of the action of $T$ on $X$. This set is finite; we fix a numbering of its elements: $C_1, \ldots, C_n$. The construction and the condition $\dim U = 1$ yield the following:

(C1) $X = C_1 \cup \ldots \cup C_n$;
(C2) every $C_i$ is a locally closed irreducible $G$-stable subset of $X$;
(C3) all $T$-orbits in $C_i$ have the same dimension $d_i$ for every $i$;
(C4) for every \(i\), either \(C_i^U = C_i\) or \(\dim U \cdot x = 1\) for all \(x \in C_i\).

The construction implies that \(X^{\text{reg}}\) is one of these subsets; we assume that

\[ X^{\text{reg}} = C_1. \]  

(40)

In view of (C3) and (39), we have

\[ \text{mod}(T : C_i) = \dim C_i - d_i \quad \text{for every } i, \]

\[ d_1 = \dim T. \]  

(41)

By Lemma 4(ii), the action of \(T\) on \(X\) is modality-regular, so (41) yields

\[ \dim X - \dim T \geq \dim C_i - d_i \quad \text{for every } i. \]  

(42)

From (38), (C3), (C4), we deduce that

\[ \text{mod}(G : C_i) = \begin{cases} 
\dim C_i - d_i & \text{if } C_i^U = C_i, \\
\dim C_i - d_i - 1 & \text{if } C_i^U = \emptyset.
\end{cases} \]  

(43)

In view of (37), (39), we have

\[ \text{mod}(G : X^{\text{reg}}) = \dim X - \dim T - 1. \]  

(44)

Arguing on the contrary, we now suppose that the action of \(G\) on \(X\) is not modality-regular, i.e.,

\[ \text{mod}(G : X) > \text{mod}(G : X^{\text{reg}}). \]  

(45)

Then, as a first step, we shall find a certain \(C_{i_0}\) that has some special properties. The next step will be analysing these properties which eventually will lead us to a sought-for contradiction.

Namely, by (45) and Lemma 2, there is \(i_0\) such that

\[ \text{mod}(G : X^{\text{reg}}) < \text{mod}(G : C_{i_0}). \]  

(46)

Combining (42), (43), (44), (46), we obtain

\[ \dim C_{i_0} - d_{i_0} - 1 \leq \dim X - \dim T - 1 \]

\[ \overset{(44)}{=} \text{mod}(G : X^{\text{reg}}) \]

\[ \overset{(46)}{<} \text{mod}(G : C_{i_0}) \]

\[ \overset{(43)}{=} \begin{cases} 
\dim C_{i_0} - d_{i_0} & \text{if } C_{i_0}^U = C_{i_0}, \\
\dim C_{i_0} - d_{i_0} - 1 & \text{if } C_{i_0}^U = \emptyset.
\end{cases} \]  

(47)

In turn, from (47) we infer the following:

\[ C_{i_0}^U = C_{i_0}, \]  

(48)

\[ \dim C_{i_0} - d_{i_0} = \dim X - \dim T. \]  

(49)
Denote by $T_{i_0}$ be the identity component of the kernel of the action of $T$ on $C_{i_0}$ and consider in $G$ the closed subgroup

$$H := T_{i_0} \times U. \tag{50}$$

By (C3) and Lemma 4(i), we have

$$\dim T_{i_0} = \dim T - d_{i_0}, \tag{51}$$

$$\dim H = \dim T - d_{i_0} + 1. \tag{52}$$

From (48) and the definitions of $T_{i_0}$ and $H$ we infer that

$$C_i = C_i^H. \tag{53}$$

By [Sum74, Cor. 2, p. 8], as $X$ is normal, it is covered by the $T_{i_0}$-stable affine open subsets. Whence there is a $T_{i_0}$-stable affine open subset $A$ in $X$ such that

$$A \cap C_{i_0} \text{ is a dense open subset of } C_{i_0}, \tag{54}$$

$$A \cap X_{\text{reg}} \text{ is a dense open subset of } A. \tag{55}$$

Consider the categorical quotient for the affine $T_{i_0}$-variety $A$:

$$\pi: A \to A \!\!/ T_{i_0} =: \text{Spec } k[A]^{T_{i_0}}. \tag{56}$$

By (39), we have $\dim T_{i_0} \cdot x = \dim T_{i_0}$ for every $x \in X_{\text{reg}}$. This, the fiber dimension theorem, the $T_{i_0}$-equivariance of $\pi$, and the equality $\dim A = \dim X$ then yield:

$$\dim A \!\!/ T_{i_0} \leq \dim A - \dim T_{i_0} \overset{\text{(51)}}{=} \dim X - \dim T + d_{C_{i_0}} \overset{\text{(49)}}{=} \dim C_{i_0}. \tag{57}$$

On the other hand, since $k[A]^{T_{i_0}}$ separates disjoint closed $T_{i_0}$-stable subsets of $A$ (see [PV94, Thm. 9.4]), we have

$$\dim C_{i_0} \overset{\text{(53)}}{=} \dim \pi(C_{i_0}) \leq \dim A \!\!/ T_{i_0} \tag{58}$$

From (56), (57) we obtain the equalities

$$\dim C_{i_0} = \dim A \!\!/ T_{i_0} = \dim A - \dim T_{i_0}. \tag{59}$$

In turn, from (58), (54), (55), and the fiber dimension theorem, we deduce the existence of a dense open subset $Q$ of $A \!\!/ T_{i_0}$ that enjoys the following properties:

$$Q \subseteq \pi(A \cap C_{i_0}) \cap \pi(A \cap X_{\text{reg}}), \tag{59}$$

$$\pi^{-1}(q) \text{ is equidimensional of dimension } \dim T_{i_0} \text{ for every } q \in Q. \tag{60}$$

Now take a point $x \in \pi^{-1}(Q) \cap X_{\text{reg}}$. In view of (39), we have

$$\dim T_{i_0} \cdot x = \dim T_{i_0}. \tag{61}$$
As orbits are open in their closures, and $T_{i_0} \cdot x \subseteq \pi^{-1}(\pi(x))$, from (60), (61) we infer that $T_{i_0} \cdot x$ is a dense open subset of an irreducible component of the fiber $\pi^{-1}(\pi(x))$. In view of (59), this fiber contains a point $s \in C_{i_0}$, so we have

$$\pi^{-1}(\pi(x)) = \pi^{-1}(\pi(s)).$$

(62)

As, by (53), the point $s$ is $T_{i_0}$-fixed, it lies in the closure of $T_{i_0} \cdot x$ in $A$ (and a fortiori in $X$); see [PV94, Thm. 4.7]. Thus $T_{i_0} \cdot x$ belongs to the set $\mathcal{S}$ of all $T_{i_0}$-orbits $O$ in $X$ that enjoy the following properties:

(a) $\dim O = \dim T_{i_0}$;
(b) the closure $\overline{O}$ of $O$ in $X$ contains $s$.

We claim that $\mathcal{S}$ is finite. Indeed, if a $T_{i_0}$-orbit $O$ belongs to $\mathcal{S}$, then $\overline{O} \cap A$ is an open neighbourhood of $s$ in $\overline{O}$, therefore $\overline{O} \cap A \neq \emptyset$. Whence $O$ lies in $A$ and contains $s$ in its closure in $A$. This and (62) show that $O$ is a $\dim T_{i_0}$-dimensional $T_{i_0}$-orbit of $\pi^{-1}(\pi(x))$; whence, as above, $O$ is a dense open subset of an irreducible component of $\pi^{-1}(\pi(x))$. The claim now follows from the finiteness of the set of irreducible components of $\pi^{-1}(\pi(x))$.

The finiteness of $\mathcal{S}$ implies that the union of all $T_{i_0}$-orbits from $\mathcal{S}$ is a locally closed subset $Z$ of $X$ whose irreducible components are these orbits. As we proved above, one of these components is $T_{i_0} \cdot x$. Since $U$ commutes with $T_{i_0}$ and, by (48), $s$ is a $U$-fixed point, the subset $Z$ is $U$-stable. The connectedness of $U$ then entails that each irreducible component of this subset is $U$-stable. In particular, $T_{i_0} \cdot x$ is $U$-stable. Whence $T_{i_0} \cdot x$ is $H$-stable and therefore we have

$$H \cdot x = T_{i_0} \cdot x.$$

(63)

In view of (39), (50), (37), we now obtain the sought-for contradiction:

$$\dim T_{i_0} + 1 = \dim H = \dim H \cdot x \overset{(63)}{=} \dim T_{i_0} \cdot x = \dim T_{i_0}.$$

This completes the proof. $\square$

References

[Akh85] D. N. Akhiezer, Actions with a finite number of orbits, Funct. Analysis Appl. 19 (1985), no. 1, 1–4.

[Arn75] V. I. Arnold, Critical points of smooth functions, in: Proceedings of the International Congress of Mathematicians, Vol. 1 (Vancouver, BC, 1974), Canad. Math. Congress, Montreal, Que., 1975, pp. 19–39.

[Bor91] A. Borel, Linear Algebraic Groups, 2nd enlarged ed., Graduate Texts in Mathematics, Vol. 126, Springer-Verlag, 1991.

[DG70] M. Demazure, P. Gabriel, Groupes Algébriques, Masson & Cie, Paris, North-Holland, Amsterdam, 1970.

[DFPS08] A. Di Bartolo, G. Falcone, P. Plaumann, K. Strambach, Algebraic Groups and Lie Groups with Few Factors, Lecture Notes in Mathematics, Vol. 1944, Springer-Verlag, Berlin, 2008.
[Ful93] W. Fulton, Introduction to Toric Varieties, Annals of Mathematics Studies, Vol. 131, Princeton University Press, Princeton, NJ, 1993.

[Gr58] A. Grothendieck, Torsion homologique et sections rationnelles, in: Séminaire Claude Chevalley 3 (1958), exp. no. 5, Secrétariat mathem. Paris, 1958, pp. 1–29.

[TE73] G. Kempf, F. Knudsen, D. Mumford, B. Saint-Donat, Toroidal Embeddings I, Lecture Notes in Mathematics, Vol. 339, Springer-Verlag, Berlin, 1973.

[LV83] D. Luna, Th. Vust, Plongements d’espaces homogènes, Comment. Math. Helv. 58 (1983), no. 2, 186–245.

[Pop17] V. L. Popov, Algebraic groups whose orbit closures contain only finitely many orbits, arXiv:1707.069v1 (2017).

[Pop17] V. L. Popov, Modality of representations, and packets for θ-groups, in: Lie Groups, Geometry, and Representation Theory. A Tribute to the Life and Work of Bertram Kostant, Progress in Mathematics, Vol. 326, Birkhäuser Basel, Basel, 2018, pp. 459–579.

[Pop19] V. L. Popov, Orbit closures of the Witt group actions, Proc. Steklov Inst. Math. 307 (2019), 193–197.

[PV94] V. L. Popov, E. B. Vinberg, Invariant Theory, in: Algebraic Geometry IV, Encyclopaedia of Mathematical Sciences, Vol. 55, Springer-Verlag, Berlin, 1994, pp. 123–284.

[Ros61] M. Rosenlicht, On quotient varieties and the affine embedding of certain homogeneous spaces Trans. Amer. Math. Soc. 101 (1961), no. 2, 211–223.

[Ser55] J.-P. Serre, Faisceaux algébriques cohérents, Ann. Math. 61 (1955), 197–278.

[Ser58] J.-P. Serre, Espaces fibrés algébriques, in: Anneaux de Chow et Applications, Séminaire Claude Chevalley, Vol. 3, Exp. no. 1 (Secrétariat mathématique, Paris, 1958), pp. 1–37.

[Ses63] C. S. Seshadri, Some results on the quotient space by algebraic group of automorphisms, Math. Annalen 149 (1963), 286–301.

[Spr98] T. A. Springer, Linear Algebraic Groups. Second Edition, Progress in Mathematics, Vol. 9, Birkhäuser, Boston, 1998.

[Sum74] H. Sumihiro, Equivariant completion, J. Math. Kyoto Univ. (JMKYAZ) 14–1 (1974), 1–28.

[Tim11] D. A. Timashev, Homogeneous Spaces and Equivariant Embeddings, Encyclopaedia of Mathematical Sciences, Vol. 138, Subseries Invariant Theory and Algebraic Transformation Spaces, Vol. VIII, Springer, Heidelberg, 2011.

[Vin86] E. B. Vinberg, Complexity of action of reductive groups, Funct. Analysis Appl. 20 1986, no. 1, 1–11.

STEKLOV MATHEMATICAL INSTITUTE, RUSSIAN ACADEMY OF SCIENCES, GUBKINA 8, MOSCOW 119991, RUSSIA

E-mail address: popovvl@mi-ras.ru