Normalizers of parabolic subgroups of Coxeter groups

DANIEL ALLCOCK

We improve a bound of Borcherds on the virtual cohomological dimension of the nonreflection part of the normalizer of a parabolic subgroup of a Coxeter group. Our bound is in terms of the types of the components of the corresponding Coxeter subdiagram rather than the number of nodes. A consequence is an extension of Brink’s result that the nonreflection part of a reflection centralizer is free. Namely, the nonreflection part of the normalizer of parabolic subgroup of type $D_5$ or $A_m$ odd is either free or has a free subgroup of index 2.

20F55

Suppose $\Pi$ is a Coxeter diagram, $J$ is a subdiagram and $W_J \subseteq W_\Pi$ is the corresponding inclusion of Coxeter groups. The normalizer $N_{W_\Pi}(W_J)$ has been described in detail by Borcherds [2] and Brink and Howlett [4]. Such normalizers have significant applications to working out the automorphism groups of Lorentzian lattices and K3 surfaces; see [2] and its references. $N_{W_\Pi}(W_J)$ falls into 3 pieces: $W_J$ itself, another Coxeter group $W_\Omega$ and a group $\Gamma_\Omega$ of diagram automorphisms of $W_\Omega$. The last two groups are called the “reflection” and “nonreflection” parts of the normalizer. Borcherds bounded the virtual cohomological dimension of $\Gamma_\Omega$ by $|J|$. Our Theorems 1, 3 and 4 give stronger bounds, in terms of the types of the components of $J$ rather than the number of nodes. There are choices involved in the definition of $W_\Omega$ and $\Gamma_\Omega$, and our bound in Theorem 3 applies regardless of how these choices are made (Theorem 1 is a special case). Theorem 4 improves this bound when $W_\Omega$ is “maximal”. In this case, when $J = D_5$ or $A_m$ odd, $\Gamma_\Omega$ turns out to either be free or have an index 2 subgroup that is free. This extends Brink’s result [3] that $\Gamma_\Omega$ is free when $J = A_1$.

The author is grateful to the Clay Mathematics Institute, the Japan Society for the Promotion of Science, and Kyoto University for their support and hospitality.

We follow the notation of Borcherds [2] and refer to Humphreys [5] for general information about Coxeter groups. Suppose $(W_\Pi, \Pi)$ is a Coxeter system, which is to say that $W_\Pi$ is a Coxeter group and $\Pi$ is a standard set of generators. The Coxeter diagram is the graph whose nodes are $\Pi$, with an edge between $s_i, s_j \in \Pi$ labeled by the order $m_{ij}$ of $s_is_j$, when $m_{ij} > 2$. $W_\Pi$ acts isometrically on a real inner product space $V_\Pi$ with basis (the simple roots) $\Pi$ and inner products defined in terms of the $m_{ij}$. The (open) Tits cone $K$ is an open convex subset of $V_\Pi^*$ on which
\(W_\Pi\) acts properly discontinuously with fundamental chamber \(C_\Pi\). (Our \(C_\Pi\) and \(K\) are “missing” the faces corresponding to infinite parabolic subgroups of \(W_\Pi\).) The standard generators act on \(V_\Pi^*\) by reflections across the hyperplanes containing the facets of \(C_\Pi\), and they also act on \(V_\Pi\) by reflections. For a root \(\alpha\) (ie, a \(W_\Pi\)–image of a simple root) we write \(\alpha^\perp\) for \(\alpha\)’s mirror, meaning the fixed-point set in \(K\) of the reflection associated to \(\alpha\).

Now let \(J \subseteq \Pi\) be a spherical subdiagram, ie, one corresponding to a finite subgroup of \(W_\Pi\), and let \(W_{\text{min}}\) be the group generated by the reflections in \(W_\Pi\) that act trivially on \(V_J \subseteq V_\Pi\). This is the “reflection” part of \(N_{W_\Pi}(W_J)\), or rather the strictest possible interpretation of this idea. It corresponds to Borcherds’ \(W_\Omega\) in the case that the groups he calls \(\Gamma_\Pi\) and \(\Gamma_J\) are trivial; see the discussion after Lemma 2. Let \(J^\perp := \bigcap_{\alpha \in J} \alpha^\perp\), pick a component \(C_{\text{min}}^\alpha\) of the complement of \(W_{\text{min}}\)’s mirrors in \(J^\perp\), and define \(C_{\text{min}}\) as its closure (in \(J^\perp\)). By definition, \(W_{\text{min}}\) is a Coxeter group, and the general theory of these groups shows that \(C_{\text{min}}\) is a chamber for it. The “nonreflection” part of \(N_{W_\Pi}(W_J)\) means the subgroup \(\Gamma_{\text{min}}\) of \(W_{\text{min}}\) preserving \(J\) (regarded as a set of roots) and sending \(C_{\text{min}}\) to itself. The reason for the first condition is to discard the trivial part of \(N_{W_\Pi}(W_J)\), namely \(W_J\) itself. That is, \(W_{\text{min}}; \Gamma_{\text{min}}\) is a complement to \(W_J\) in \(N_{W_\Pi}(W_J)\). We write \(\Gamma_{\text{min}}^\vee\) for the subgroup of \(\Gamma_{\text{min}}\) acting trivially on \(J\) (equivalently, on \(V_J\)). The reason for passing to this (finite-index) subgroup is that \(\Gamma_{\text{min}}\) often contains torsion and therefore has infinite cohomological dimension for boring reasons.

**Theorem 1** \(\Gamma_{\text{min}}^\vee\) acts freely on a contractible cell complex of dimension at most

\[
(1) \quad \#A_1 + \#D_{m>4} + \#E_6 + \#I_2(5) + 2(\#A_{m>1} + \#D_4),
\]

where \(\#X_m\) means the number of components of \(J\) isomorphic to a given Coxeter diagram \(X_m\). In particular, the cohomological dimension of \(\Gamma_{\text{min}}^\vee\) is at most (1).

Borcherds’ result [2, Theorem 4.1] has \(|J|\) in place of (1), but treats a more general group \(\Gamma_\Omega\), of which \(\Gamma_{\text{min}}\) is a special case. The more general case follows from this one, in Theorem 3 below.

**Proof** First we prove for \(x \in C_{\text{min}}^\alpha\) that its stabilizer \(\Gamma_{\text{min},x}^\vee\) is trivial. The \(W_\Pi\)–stabilizer of \(x\) is some \(W_{\Pi}；\text{conjugate} W_x\) of a spherical parabolic subgroup of \(W_\Pi\). So \(W_x\) acts on \(V_\Pi\) as a finite Coxeter group. It is well-known that any vector stabilizer in such an action is generated by reflections, so the subgroup \(W_x; J\) fixing \(J\) pointwise is generated by reflections. Observe that any reflection in \(W_x; J\) lies in \(W_{\text{min}}\). Since \(x\) lies in the interior \(C_{\text{min}}^\alpha\) of \(C_{\text{min}}\), it is fixed by no reflection in \(W_{\text{min}}\), so there can be no
reflection in $W_x J$, so $W_x J = 1$. It is easy to see that $W_x J$ contains $\Gamma_{\text{min}, x}^\vee$, so we have proven that $\Gamma_{\text{min}}^\vee$ acts freely on $C_{\text{min}}^o$.

The component $C_{\text{min}}^o$ is contractible because it is convex, and it obviously admits an equivariant deformation-retraction to its dual complex. So it suffices to show that the dual complex has dimension at most (1). Suppose $\phi \subseteq J^\perp$ is a face of a chamber of $W_\Pi$, with codimension in $J^\perp$ larger than (1); we must show $\phi \cap C_{\text{min}}^o = \emptyset$.

For some $w \in W_\Pi$, $w \phi$ is a face of $C_\Pi$ whose corresponding set of simple roots $I' \subseteq \Pi$ contains $J' := w(J) \cong J$. By the codimension hypothesis on $\phi$, $|I'| - |J'|$ is more than (1). Applying the lemma below to $J'$ and $I'$, we see that $W_{I'}$ contains a reflection $r$ fixing $J'$ pointwise. Since $r \in W_{I'}$, its mirror contains $w \phi$. So $w^{-1} r w$ is a reflection fixing $J$ pointwise (so it lies in $W_{\text{min}}$), whose mirror contains $\phi$. Since $C_{\text{min}}^o$ is a component of the complement of the mirrors of $W_{\text{min}}$, it is disjoint from $\phi$, as desired. \hfill \Box

**Lemma 2** If $J$ lies in a spherical Coxeter diagram $I \subseteq \Pi$ whose cardinality exceeds that of $J$ by more than (1), then $W_I$ contains a reflection fixing $J$ pointwise.

**Remark** Equality in (1) holds when $I$ extends the $A_m$, $D_m$, $E_6$ and $I_2(5)$ components of $J$ by $A_1 \rightarrow A_2$, $A_{m>1} \rightarrow D_{m+2}$, $D_4 \rightarrow E_6$, $D_{m>4} \rightarrow D_{m+1}$, $E_6 \rightarrow E_7$ and $I_2(5) \rightarrow H_3$. One can check in these cases that the conclusion of the lemma fails.

**Proof** We may suppose $I = \Pi$, by discarding the rest of $\Pi$. Working one component at a time, it suffices to prove the lemma under the additional hypothesis that $\Pi$ is connected. We now consider the various possibilities for $\Pi$, and suppose $W_\Pi$ contains no reflections fixing $V_J$ pointwise. That is, we assume $W_{\text{min}} = 1$. In each case we will show that $|\Pi| - |J|$ is at most (1).

The $\Pi = A_n$ case is a model for the rest. Suppose the component of $J$ nearest one end of $\Pi$ has type $A_m$ and does not contain that end. Then it must be adjacent to that end (since $W_{\text{min}} = 1$), so together with the end it forms an $A_{m+1}$. We conjugate by the long word in $W(A_{m+1})$, which exchanges the two $A_m$ diagrams in $A_{m+1}$ and fixes the roots in the other components of $J$. The result is that we may suppose without loss that $J$ contains that end of $\Pi$. Repeating the argument to move the other components of $J$ toward that end, we may suppose that there is exactly one node of $\Pi$ between any two consecutive components of $J$. And the other end of $\Pi$ is either in $J$ or adjacent to it. It is now clear that $|\Pi| - |J|$ is the number of components of $J$, or one less than this. Since every component of $J$ has type $A$, $|\Pi| - |J|$ is at most (1). This finishes the proof in the $\Pi = A_n$ case.

If $\Pi = B_n = C_n$ then we begin by shifting any type $A$ components of $J$ as far as possible from the double bond. If $J$ has no $B_m$ then $J$ contains one end of the double bond.
bond, and we get $|\Pi| - |J|$ equal to the number of components of $J$, all of which have type $A$. If $J$ has a $B_m$ then the node after it (if there is one) must be adjacent to some type $A$ component of $J$. This is because $W(B_{m+1})$ contains a reflection acting trivially on $V_{B_m}$. This is easy to see in the model of $W(B_{m+1})$ as the isometry group of $\mathbb{Z}^{m+1}$. It follows that $|\Pi| - |J|$ is the number of components of $J$ of type $A$.

In the $\Pi = D_{n>3}$ case, one can use the shifting trick to reduce to one of the cases

\[ (2) \]

where the filled nodes are those in $J$ and the dashes indicate a chain of nodes with no two consecutive unfilled nodes. (Except for the dashes on the left in the last 3 diagrams, which indicate chains of filled nodes.) In every case we get

$$|\Pi| - |J| \leq \#A_1 + \#D_{m \geq 4} + 2 \#A_{m > 1}.$$ 

The most interesting case is $A_{n-2} \to D_n$, at the top left.

We will treat the case $\Pi = E_8$ and leave the similar $E_6$ and $E_7$ cases to the reader. If $J$ has a $D_4$, $D_5$ or $E_6$ component, then it must also have a type $A$ component, and then $|\Pi| - |J| \leq 2 \#D_4 + \#D_5 + \#A_{m \geq 1}$, as desired. $J$ cannot be $D_6$ or $E_7$, because then $W_{\text{min}}$ would contain the reflection in the lowest root of $E_8$, which extends $E_8$ to the affine diagram $\tilde{E}_8$. So we may suppose $J$’s components have type $A$. In order for $|\Pi| - |J|$ to exceed (1), we must have $J = A_{m \leq 5}, A_3 A_1, A_2 A_1$ or $A_{2m \leq 3}$. But none of these cases can occur, because in each of them we may shift $J$’s components around so that some node of $\Pi$ is not joined to $J$.

The remaining cases are $\Pi = F_4, H_3, H_4$ and $I_2$, the last case including $G_2 = I_2(6)$. The facts required to treat these cases are that if $J = B_2$ or $B_3$ in $\Pi = F_4$ then $W_{\text{min}}$ contains a reflection, and similarly in the $J = H_3 \subseteq H_4 = \Pi$ case. The first fact is visible inside a $B_3$ or $B_4$ root system inside $F_4$. To see the second, observe that the root stabilizer in $H_4$ contains Coxeter groups of types $A_2$ and $I_2(5)$, visible in the centralizers of the two end reflections of $H_4$ (which are conjugate). So the root stabilizer can only be $W(H_3)$, which is to say that the $H_3$ root system is orthogonal to a root.

The greater generality obtained by Borcherds is the following. Let $\Gamma_{\Pi}$ be a group of diagram automorphisms of $\Pi$, acting on $V_{\Pi}$ and $K$ in the obvious way. The goal is to understand $N_{W_{\Pi} \Gamma_{\Pi}}(W_J)$. Again we discard the boring part of this normalizer by passing to the subgroup $W_J'$ preserving the set of roots $J \subseteq \Pi$. Let $W_\Omega$ be any

*Algebraic & Geometric Topology, Volume 12 (2012)*
subgroup of $W'_J$ which contains $W_{\text{min}}$ and is generated by elements which act on $J^\perp$ by reflections. We define $C^{\circ}_\Omega$, $C_\Omega$ and $\Gamma_\Omega$ as for $C^{\circ}_{\text{min}}$, $C_{\text{min}}$ and $\Gamma_{\text{min}}$, and define $\Gamma^{\vee}_\Omega$ as the subgroup of $\Gamma_\Omega \cap W_\Pi$ acting trivially on $J$. (Borcherds left $\Gamma^{\vee}_\Omega$ unnamed and defined $W_\Omega$ in terms of auxiliary groups $R \trianglelefteq \Gamma_\Pi \trianglelefteq \text{Aut} J$; his $W_\Omega$ has the properties assumed here.) The inclusion $W_{\text{min}} \subseteq W_\Omega$ is the source of the subscript “min”, but note that $C_{\text{min}}$ and $\Gamma_{\text{min}}$ are larger than $C_\Omega$ and $\Gamma_\Omega$. We can now recover Borcherds’ result [2, Theorem 4.1] with our (1) in place of $|J|$.  

**Theorem 3** Theorem 1 holds with $\Gamma^{\vee}_{\text{min}}$ replaced by $\Gamma^{\vee}_\Omega$.  

**Proof** The freeness of the action follows from the same argument. (This is why $\Gamma^{\vee}_\Omega$ is defined as a subgroup of $\Gamma_\Omega \cap W_\Pi$ rather than just $\Gamma_\Omega$.) The essential point for the rest of the proof is that $W_\Omega$ contains $W_{\text{min}}$, so the decomposition of $J^\perp$ into chambers of $W_\Omega$ refines that of $W_{\text{min}}$. This shows $C^{\circ}_\Omega \subseteq C^{\circ}_{\text{min}}$. So the dual complex of $C^{\circ}_\Omega$ has dimension at most that of $C^{\circ}_{\text{min}}$, and we can apply Theorem 1. \[ \square \]

The point of considering $W_\Omega$ rather than $W_{\text{min}}$ is that it is larger and so $\Gamma_\Omega$ will be smaller than $\Gamma_{\text{min}}$. This is good since the nonreflection part is more mysterious than the reflection part. So it is natural to define $W_{\text{max}}$ by setting $\Gamma_\Pi = 1$ and taking $W_\Omega$ as large as possible, ie, $W_{\text{max}}$ is the subgroup of $W'_J$ generated by the transformations which act on $J^\perp$ by reflections.

This is the largest possible “universal” $W_\Omega$, although a larger $W_\Omega$ is possible if $\Pi$ admits suitable diagram automorphisms. For example, $\Gamma_\Pi$ might contain elements acting on $C_\Pi$ by reflections. I don’t know other examples, although probably there are some.

We define $C^{\circ}_{\text{max}}$, $C_{\text{max}}$, $\Gamma_{\text{max}}$ and $\Gamma^{\vee}_{\text{max}}$ as above. The next theorem follows from Lemma 5 in exactly the same way that Theorem 1 follows from Lemma 2.

**Theorem 4** The dimension of the dual complex of $C^{\circ}_{\text{max}}$, hence the cohomological dimension of $\Gamma^{\vee}_{\text{max}}$, is bounded above by

\[
\#D_5 + \#A_{m \text{ odd}} + 2 \#A_{m \text{ even}}.
\]

**Remarks** (i) If $J$ has no $A_m$ or $D_5$ component then $\Gamma^{\vee}_{\text{max}} = 1$ and $\Gamma_{\text{max}}$ is finite. This is Borcherds’ [2, Example 5.6].

(ii) If $J = D_5$ or $A_{m \text{ odd}}$ then $\Gamma^{\vee}_{\text{max}} \subseteq N_{W_\Pi}(W_J)$ is free. Also, since $|\text{Aut } J| \leq 2$, $\Gamma^{\vee}_{\text{max}}$ has index 1 or 2 in $\Gamma_{\text{max}}$. Therefore the nonreflection part $\Gamma_{\text{max}}$ of $N_{W_\Pi}(W_J)$ has a free subgroup of index 1 or 2.
(iii) If \( J = A_1 \) then \( \Gamma_{\text{min}} = \Gamma_{\text{max}} = \Gamma_{\text{max}}^\vee \) has cohomological dimension \( \leq 1 \). This recovers Brink’s result [3] that \( \Gamma_{\text{min}} \) is free.

(iv) If \( J = A_{m \text{ even}} \) then the conclusion \( \dim(\text{dual of } C_{\text{min}}^\circ) \leq 2 \) suggests that \( \Gamma_{\text{max}} \) is often comprehensible, like the \( J = A_6 \) example of [2, Example 5.4].

**Lemma 5** If \( J \) lies in a spherical Coxeter diagram \( I \subseteq \Pi \), whose cardinality exceeds that of \( J \) by more than (3), then \( W_I \) contains an element preserving the set \( J \) of roots and acting on \( J^\perp \) by a reflection.

**Proof** This is essentially the same as for Lemma 2, using the following additional ingredients. For example, when \( I = D_n \) one can use them to show that the fifth, seventh, eighth and tenth cases of (2) are impossible, while the first can only occur when \( n \) is even.

First, if \( J = E_6 \subseteq E_7 = I \) then \( W_I \) contains the negation of \( V_I \), which we follow by the long word in \( W_I \) to send \(-J\) back to \( J \). The composition is the claimed element of \( W_I \). The same argument applies if \( J = I_2(5) \subseteq H_3 = I \).

Second, if \( J = A_{m \text{ odd}} \subseteq D_{m+2} = I \) as in the first diagram of (2), then consider the long word in \( W_I \). It negates \( J \) and exchanges and negates the two simple roots in \( I - J \). Following this by the long word in \( W_I \) yields the claimed element of \( W_I \). (When \( m \) is even, the long word in \( W_I \) negates the simple roots in \( I - J \) without exchanging them, so it doesn’t act on \( J^\perp \) by a reflection.)

Third, if \( J = D_{m \geq 3} \subseteq D_{m+1} = I \) then consider the model of \( W_I \) as the group generated by permutations and evenly many negations of \( m + 1 \) coordinates, with \( W_I \) the corresponding subgroup for the first \( m \) coordinates. Letting \( \sigma \) be the negation of the last two coordinates, and following it by the element of \( W_I \) sending \( \sigma(J) \) back to \( J \), gives the claimed element of \( W_I \). \( \square \)

There is a nice geometrical interpretation of the freeness of \( \Gamma_{\text{min}} \) in the case \( J = A_1 \), developed further in [1]. Namely, the natural map \( C_{\text{min}}^\circ \to C_{\text{min}}^\circ / \Gamma_{\text{min}} \subseteq K / W_\Pi = C_\Pi \) is the universal cover of its image. The image is got by discarding all the codimension 2 faces of \( C_\Pi \) corresponding to even bonds in \( \Pi \), discarding all codimension 3 faces, and taking the component corresponding to \( J \). This identifies \( \Gamma_{\text{min}} \) with the fundamental group of \( J \)'s component of the “odd” subgraph of \( \Pi \) in a natural manner.

One can extend this picture to the case \( J \neq A_1 \), but complications arise. First, one must take \( W_\Omega \) to be normal in \( W_\Pi ; \Gamma_\Pi \). Second, while \( C_\Omega^\circ \to C_\Omega^\circ / \Gamma_\Omega \) is a covering space, the image \( C_\Omega^\circ / \Gamma_\Omega \) of \( C_\Omega^\circ \) in \( C_\Pi \) is the quotient of \( C_\Omega^\circ / \Gamma_\Omega^\vee \) by the finite group \( \Gamma_\Omega / \Gamma_\Omega^\vee \). Usually, \( C_\Omega^\circ \to C_\Omega^\circ / \Gamma_\Omega \) is only an orbifold cover since \( \Gamma_\Omega \) often has torsion. The
top-dimensional strata of \( C^\circ_\Omega / \Gamma_\Omega \) correspond to the “associates” of the inclusion \( J \to \Pi \) in the sense of [2; 4]. Suppose \( J' \subseteq \Pi \) is (the image of) an associate and \( I' \) is a spherical diagram containing it. Then the face of \( C^\circ_\Pi \) corresponding to \( I' \), minus lower-dimensional faces, lies in \( C^\circ_\Omega / \Gamma_\Omega \) just if \( W_{I'} \) contains no element preserving \( J' \), acting on it in a manner constrained by the choice of \( W_\Omega \), and acting on \( J'^\perp \) by a reflection. From this perspective, Lemmas 2 and 5 amount to working out two cases of Borcherds’ notion of “\( R \)-reflectivity”. The orbifold structure on \( C^\circ_\Omega / \Gamma_\Omega \) is essentially the same information as Borcherds’ classifying category for \( \Gamma_\Omega \).

References

[1] **D Allcock**, *Reflection centralizers in Coxeter groups*, in preparation

[2] **R E Borcherds**, *Coxeter groups, Lorentzian lattices, and K3 surfaces*, Internat. Math. Res. Notices (1998) 1011–1031 MR1654763

[3] **B Brink**, *On centralizers of reflections in Coxeter groups*, Bull. London Math. Soc. 28 (1996) 465–470 MR1396145

[4] **B Brink, R B Howlett**, *Normalizers of parabolic subgroups in Coxeter groups*, Invent. Math. 136 (1999) 323–351 MR1688445

[5] **J E Humphreys**, *Reflection groups and Coxeter groups*, Cambridge Studies in Adv. Math. 29, Cambridge Univ. Press (1990) MR1066460

Department of Mathematics, University of Texas at Austin
1 University Station C1200, Austin TX 78712, USA

allcock@math.utexas.edu

http://www.math.utexas.edu/~allcock

Received: 13 September 2011
