Abstract

A colouring of a graph $G = (V, E)$ is a mapping $c: V \rightarrow \{1, 2, \ldots\}$ such that $c(u) \neq c(v)$ for every two adjacent vertices $u$ and $v$ of $G$. The List $k$-Colouring problem is to decide whether a graph $G = (V, E)$ with a list $L(u) \subseteq \{1, \ldots, k\}$ for each $u \in V$ has a colouring $c$ such that $c(u) \in L(u)$ for every $u \in V$. Let $P_t$ be the path on $t$ vertices and let $K_{1,s}$ be the graph obtained from the $(s+1)$-vertex star $K_{1,s}$ by subdividing each of its edges exactly once.

Recently, Chudnovsky, Spirkl and Zhong (DM 2020) proved that List $3$-Colouring is polynomial-time solvable for $(K_{1,s}, P_t)$-free graphs for every $t \geq 1$ and $s \geq 1$. We generalize their result to List $k$-Colouring for every $k \geq 1$. Our result also generalizes the known result that for every $k \geq 1$ and $s \geq 0$, List $k$-Colouring is polynomial-time solvable for $(sP_1 + P_5)$-free graphs, which was proven for $s = 0$ by Hoàng, Kamiński, Lozin, Sawada, and Shu (Algorithmica 2010) and for every $s \geq 1$ by Couturier, Golovach, Kratsch and Paulusma (Algorithmica 2015).

We show our result by proving boundedness of an underlying width parameter. Namely, we show that for every $k \geq 1$, $s \geq 1$, $t \geq 1$, the class of $(K_k, K_{1,s}, P_t)$-free graphs has bounded mim-width and that a corresponding branch decomposition is "quickly computable" for these graphs.

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Ae and the other in \( \overline{Ae} \). A matching \( F \subseteq E(G) \) of \( G \) is induced if there is no edge in \( G \) between vertices of different edges of \( F \). We let cut\( \text{cutm}_{G}(Ae, \overline{Ae}) \) be the size of a maximum induced matching in \( G[Ae, \overline{Ae}] \). The mim-width \( \text{mimw}_{G}(T, \delta) \) of \( (T, \delta) \) is the maximum value of \( \text{cutm}_{G}(Ae, \overline{Ae}) \) over all edges \( e \in E(T) \). The mim-width \( \text{mimw}(G) \) of \( G \) is the minimum value of \( \text{mimw}_{G}(T, \delta) \) over all branch decompositions \( (T, \delta) \) for \( G \). See Figure 1 for an example.

![Figure 1](image-url) An example of a graph \( G \) with a branch decomposition \( (T, \delta) \). The partition \( (Ae, \overline{Ae}) \) of \( V(G) \) in the rightmost figure witnesses that \( \text{mimw}_{G}(T, \delta) \geq 1 \). It can be easily seen that \( \text{mimw}_{G}(T, \delta) \leq 1 \) and so \( \text{mimw}(G) = 1 \).

Vatshelle [37] proved that every class of bounded clique-width, or equivalently, bounded boolean-width, module-width, NLC-width or rank-width, has bounded mim-width, and that the converse is not true. That is, he proved that there exist graph classes of bounded mim-width that have unbounded clique-width. This means that proving that a problem is polynomial-time solvable for graph classes of bounded mim-width yields more tractable graph classes than doing this for clique-width. Hence, mim-width has greater modeling power than clique-width.

However, the trade-off is that fewer problems admit such an algorithm, as we explain below by means of a relevant example, namely the classical colouring problem. Moreover, computing mim-width is NP-hard [36] and it is not possible to approximate in polynomial time the mim-width of a graph within a constant factor unless \( \text{NP} = \text{ZPP} \) [36]. It remains a challenging open problem to develop a polynomial-time algorithm for computing a branch decomposition with mim-width \( f(k) \) for a graph with mim-width \( k \). However, the latter has been shown possible for special graph classes \( G \). In such a case, we say that the mim-width of \( G \) is quickly computable. We can then develop a polynomial-time algorithm for the problem of interest via dynamic programming over the computed branch decomposition. We refer to [1, 2, 3, 5, 6, 7, 13, 22, 23, 24, 25] for a wide range of examples of graph classes and problems for which such dynamic programming algorithms have been obtained.

As mentioned, in this paper we focus on Graph Colouring, a central problem in Discrete Mathematics, Theoretical Computer Science and beyond. A colouring of a graph \( G = (V, E) \) is a mapping \( c : V \to \{1, 2, \ldots\} \) that gives each vertex \( u \in V \) a colour \( c(u) \) in such a way that, for every two adjacent vertices \( u \) and \( v \), we have that \( c(u) \neq c(v) \). If for every \( u \in V \) we have \( c(u) \in \{1, \ldots, k\} \), then we say that \( c \) is a \( k \)-colouring of \( G \). The Colouring problem is to decide whether a given graph \( G \) has a \( k \)-colouring for some given integer \( k \geq 1 \). If \( k \) is fixed, that is, not part of the input, we call this the \( k \)-COLOURING problem. A classical
result of Lovász [30] states that \(k\)-\textsc{Colouring} is \(\text{NP}\)-complete even if \(k = 3\). The \textsc{Colouring} problem is an example of a problem that distinguishes between classes of bounded mim-width and bounded clique-width: it is polynomial-time solvable for every graph class of bounded clique-width [27] but \(\text{NP}\)-complete for circular-arc graphs [14], a class of graphs of mim-width at most 2 and for which mim-width is quickly computable [1]. When we fix \(k\), we no longer have this distinction, as \(k\)-\textsc{Colouring}, for every fixed integer \(k \geq 1\), is polynomial-time solvable for a graph class whose mim-width is bounded and quickly computable [7].

We consider the following generalization of \(k\)-\textsc{Colouring}. For an integer \(k \geq 1\), a \(k\)-list assignment of a graph \(G = (V,E)\) is a function \(L\) that assigns each vertex \(u \in V\) a list \(L(u) \subseteq \{1,2,\ldots,k\}\) of admissible colours for \(u\). A colouring \(c\) of \(G\) respects \(L\) if \(c(u) \in L(u)\) for every \(u \in V\). For a fixed integer \(k \geq 1\), the \(k\)-\textsc{Colouring} problem is to decide whether a given graph \(G\) with a \(k\)-list assignment \(L\) admits a colouring that respects \(L\). Note that for \(k_1 \leq k_2\), \(k_1\)-\textsc{Colouring} is a special case of \(k_2\)-\textsc{Colouring} and that by setting \(L(u) = \{1,\ldots,k\}\) for every \(u \in V\), we obtain the \(k\)-\textsc{Colouring} problem.

Given an instance \((G,L)\) of \(k\)-\textsc{Colouring}, one can construct an equivalent instance \(G'\) of \(k\)-\textsc{Colouring} by adding a clique on new vertices \(u_1,\ldots,u_k\) to \(G\) and adding an edge between \(u_i\) and \(v \in V(G)\) if and only if \(i \notin L(u)\) (see, for example, [31]). Kwon [29] observed that \(\text{mimw}(G') \leq \text{mimw}(G) + k\) and thus, as \(k\)-\textsc{Colouring} is polynomial-time solvable for graph classes whose mim-width is bounded and quickly computable [7], for every fixed integer \(k \geq 1\), this leads to the following:

\begin{quote}
\textbf{Theorem 1} ([29]). For every \(k \geq 1\), \(k\)-\textsc{Colouring} is polynomial-time solvable for a graph class whose mim-width is bounded and quickly computable.
\end{quote}

In this paper we show that a number of known polynomial-time results for \(k\)-\textsc{Colouring} on special graph classes can be obtained, and strengthened, by applying Theorem 1.

The classes that we consider belong to the framework of hereditary graph classes. A graph class is \textit{hereditary} if it is closed under vertex deletion. It is well known and not difficult to see that hereditary graph classes are exactly those classes characterized by a (unique) set \(\mathcal{F}\) of minimal forbidden induced subgraphs. If \(|\mathcal{F}| = 1\) or \(|\mathcal{F}| = 2\), we say that the hereditary graph class is \textit{monogenic} or \textit{bigenic}, respectively. In a recent study [5], boundedness or unboundedness of mim-width has been determined for all monogenic classes and a large number of bigenic classes. These results imply that a monogenic graph class has bounded mim-width if and only if it has bounded clique-width [5] but that this equivalence does not always hold for bigenic graph classes. As we focus on hereditary graph classes, our work can be seen as a continuation of the research in [5].

\textbf{Related Work}

We first need to introduce some more terminology. A graph \(G\) is \textit{\(H\)-free}, for some graph \(H\), if it contains no \textit{induced} subgraph isomorphic to \(H\), that is, we cannot modify \(G\) into \(H\) by a sequence of vertex deletions. For a set of graphs \(\{H_1,\ldots,H_p\}\), a graph is \((H_1,\ldots,H_p)\)-free if it is \(H_i\)-free for every \(i \in \{1,\ldots,p\}\). We denote the disjoint union of two graphs \(G_1\) and \(G_2\) by \(G_1 + G_2 = (V(G_1) \cup V(G_2), E(G_1) \cup E(G_2))\). We let \(P_r\) and \(K_r\) denote the path and complete graph on \(r\) vertices, respectively.

The complexity of \textsc{Colouring} for \(H\)-free graphs has been settled for every graph \(H\) [28], but there are still infinitely many open cases for \(k\)-\textsc{Colouring} restricted to \(H\)-free graphs when \(H\) is a \textit{linear forest}, that is, a disjoint union of paths. We refer to [15] for a survey and to [8, 10, 26] for updated summaries. In particular, Hoàng et al. [20] proved that for every
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integer $k \geq 1$, $k$-COLOURING is polynomial-time solvable for $P_5$-free graphs. Their proof is in fact a proof for LIST $k$-COLOURING. The result of [20] was generalized by Couturier et al. [11] as follows:

▶ **Theorem 2 ([11]).** For every $k \geq 1$ and $s \geq 0$, LIST $k$-COLOURING is polynomial-time solvable for $(sP_1 + P_5)$-free graphs.

For $r \geq 1$ and $s \geq 1$, we let $K_{r,s}$ denote the complete bipartite graph with partition classes of size $r$ and $s$. The graph $K_{1,s}$ is also known as the $(s + 1)$-vertex star. The 1-subdivision of a graph $G$ is the graph obtained from $G$ by subdividing each edge of $G$ exactly once. We denote the 1-subdivision of a star $K_{1,s}$ by $K_{1,s}^1$, in particular $K_{1,2}^1 = P_5$. Very recently, Chudnovsky, Spirkl and Zhong proved the following result:

▶ **Theorem 3 ([10]).** For every $s \geq 1$ and $t \geq 1$, LIST 3-COLOURING is polynomial-time solvable for $(K_{1,s}^1, P_t)$-free graphs.

For every $s \geq 1$ and $t \geq 2s + 5$, the class of $(K_{1,s+2}^1, P_t)$-free graphs contains the class of $(sP_1 + P_5)$-free graphs. Hence, Theorem 3 generalizes Theorem 2 in the case $k = 3$. As $K_{1,s}$ is an induced subgraph of $K_{1,s}^1$, Theorem 3 also generalizes the following result in the case $r = 1$:

▶ **Theorem 4 ([17]).** For every $k \geq 1$, $r \geq 1$, $s \geq 1$ and $t \geq 1$, LIST $k$-COLOURING is polynomial-time solvable for $(K_{r,s}^1, P_t)$-free graphs.

**Our Results**

We prove the following result:

▶ **Theorem 5.** For every $r \geq 1$, $s \geq 1$ and $t \geq 1$, the mim-width of the class of $(K_r, K_{1,s}^1, P_t)$-free graphs is bounded and quickly computable.

We may assume without loss of generality that an instance of LIST $k$-COLOURING is $K_{k+1}$-free, for otherwise it is a no-instance. Hence, combining Theorem 5 with Theorem 1 enables us to generalize both Theorems 2 and 3:

▶ **Corollary 6.** For every $k \geq 1$, $s \geq 1$ and $t \geq 1$, LIST $k$-COLOURING is polynomial-time solvable for $(K_{1,s}^1, P_t)$-free graphs.

Corollary 6 is tight in the following sense. Let $L_{1,s}$ denote the subgraph obtained from $K_{1,s}^1$ by subdividing one edge exactly once; in particular $L_{1,2} = P_6$. Then, as LIST 4-COLOURING is NP-complete for $P_6$-free graphs [16], we cannot generalize Corollary 6 to $(L_{1,s}, P_t)$-free graphs for $k \geq 4$, $s \geq 2$ and $t \geq 6$. Moreover, the mim-width of $(K_1, P_6)$-free graphs is unbounded [5] and so we cannot extend Theorem 5 to $(K_r, L_{1,s}, P_t)$-free graphs, for $r \geq 4$, $s \geq 2$ and $t \geq 6$, either.

Theorem 5 has other applications as well. Firstly, as mentioned earlier, there are many problems known to be XP parameterized by mim-width, so Theorem 5 implies that these problems are polynomial-time solvable for this graph class; in particular, this is the case for the broad class of problems known as Locally Checkable Vertex Subset and Vertex Partitioning problems. For a graph $G$, let $\omega(G)$ denote the size of a maximum clique in $G$. Chudnovsky et al. [9] gave for the class of $(K_{1,3}^1, P_6)$-free graphs an $n^{O(\omega(G)^3)}$-time algorithm for MAX PARTIAL H-COLOURING, a problem equivalent to INDEPENDENT SET if $H = P_3$ and to ODD CYCLE TRANSVERSAL if $H = P_2$. In other words, MAX PARTIAL H-COLOURING is polynomial-time solvable for $(K_{1,3}^1, P_6)$-free graphs with bounded clique number. Moreover,
they observed that Max Partial $H$-Colouring is polynomial-time solvable for graph classes whose mim-width is bounded and quickly computable. Hence, Theorem 5 generalizes their result for Max Partial $H$-Colouring to $(K^1_{2s},P_t)$-free graphs with bounded clique number, for any $s \geq 1$ and $t \geq 1$. However, the running time of the corresponding algorithm is worse than $n^{O((\omega(G))^3)}$ (see [9] for details).

It remains to prove Theorem 5, which we do in the next section. In Section 3 we give some directions for future work.

## 2 The Proof of Theorem 5

We first state two lemmas. The first lemma shows that given a partition of the vertex set of a graph $G$, we can bound the mim-width of $G$ in terms of the mim-width of the graphs induced by each part and the mim-width between any two of the parts.

**Lemma 7.** Let $G$ be a graph, and let $(X_1,\ldots,X_p)$ be a partition of $V(G)$ such that $\text{cutmim}_G(X_i, X_j) \leq c$ for all distinct $i, j \in \{1,\ldots,p\}$, and $p \geq 2$. Then

$$\text{mimw}(G) \leq \max \left\{ c \left[ \left( \frac{p}{2} \right)^2 \right], \max_{i \in \{1,\ldots,p\}} \{ \text{mimw}(G[X_i]) + c(p-1) \} \right\}.$$  

Moreover, if $(T_i, \delta_i)$ is a branch decomposition for $G[X_i]$ for each $i$, then we can construct, in $O(p)$ time, a branch decomposition $(T, \delta)$ for $G$ with

$$\text{mimw}(T, \delta) \leq \max \left\{ c \left[ \left( \frac{p}{2} \right)^2 \right], \max_{i \in \{1,\ldots,p\}} \{ \text{mimw}(T_i, \delta_i) + c(p-1) \} \right\}.$$  

**Proof.** We construct a branch decomposition $(T, \delta)$ for $G$ with the desired mim-width as follows. Let $T_0$ be an arbitrary subcubic tree having $p$ leaves $\ell_1,\ldots,\ell_p$. Fix for each $i \in \{1,\ldots,p\}$ a branch decomposition $(T_i, \delta_i)$ for $G[X_i]$. For each $i \in \{1,\ldots,p\}$, we choose an arbitrary leaf vertex $v_i$ of $T_i$, we identify $v_i$ with $\ell_i$ calling the resulting vertex $\ell_i$, and we create a new pendant edge incident to $\ell_i$, where the new leaf vertex adjacent to $\ell_i$ is called $v_i$. Then $T$ is a subcubic tree whose set of leaves is the disjoint union of the leaves of $T_i$ for each $i \in \{1,\ldots,p\}$. See Figure 2, for example. For a leaf $v$ of $T$, we set $\delta(v) = \delta_i(v)$, where $v$ is a leaf of $T_i$. Now $(T, \delta)$ is a branch decomposition for $G$, and clearly this branch decomposition can be constructed in $O(p)$ time. It remains to prove the upper bound for mimw$(T, \delta)$.

Consider $e \in E(T)$ and the partition $(A_e, \overline{A_e})$ of $V(G)$. If $e \in E(T_0)$, then $A_e = \bigcup_{j \in J} X_j$ for some $J \subseteq \{1,\ldots,p\}$. If $e \in E(T_i)$ for some $i \in \{1,\ldots,p\}$, then either $A_e$ or $\overline{A_e}$ is properly contained in $X_i$. The only other possibility is that $e$ is one of the newly created pendant edges, in which case either $A_e$ or $\overline{A_e}$ has size 1.

First suppose $e \in E(T_0)$, so $A_e = \bigcup_{j \in J} X_j$ for some $J \subseteq \{1,\ldots,p\}$. We claim that $\text{cutmim}_G(A_e, \overline{A_e}) \leq c \left( \left( \frac{\delta}{2} \right)^2 \right)$. Let $M$ be a maximum-sized induced matching in $G[A_e, \overline{A_e}]$. Let $K = \{1,\ldots,p\} \setminus J$. For each $j \in J$ and $k \in K$, there are at most $c$ edges of $M$ with one end in $X_j$ and the other end in $X_k$, since $\text{cutmim}_G(X_j, X_k) \leq c$. Thus $\text{cutmim}_G(A_e, \overline{A_e}) \leq c|J||K|$, where $|J| + |K| = p$. As $c|J||K| \leq c \left( \left( \frac{\delta}{2} \right) \right) \left( \left( \frac{\delta}{2} \right) \right) = c \left( \left( \frac{\delta}{2} \right)^2 \right)$, the claim follows.

Now suppose $e \in E(T_i)$ for some $i \in \{1,\ldots,p\}$, so, without loss of generality, $A_e$ is properly contained in $X_i$. We claim that $\text{cutmim}_G(A_e, \overline{A_e}) \leq \text{mimw}(G[X_i]) + c(p-1)$. Consider a maximum-sized induced matching $M$ in $G[A_e, \overline{A_e}]$. As $A_e \subseteq X_i$, all the edges of $M$ have one end in $X_i$. For each $j \in \{1,\ldots,p\}$ with $j \neq i$, there are at most $c$ edges of $M$ with one end in $X_j$, since $\text{cutmim}_G(X_i, X_j) \leq c$. Since there are at most $\text{mimw}(G[X_i])$ edges of $M$ with both ends in $X_i$, we deduce that $\text{cutmim}_G(A_e, \overline{A_e}) \leq \text{mimw}(G[X_i]) + c(p-1)$, as claimed. The lemma follows.
A clique in a graph is a set of pairwise adjacent vertices. An independent set is a set of pairwise non-adjacent vertices. A dominating set is a set $D$ of vertices such that every vertex not in $D$ is adjacent to at least one vertex in $D$. Ramsey’s Theorem states that for all positive integers $k$ and $\ell$, there exists an integer $R(k, \ell)$ such that every graph on at least $R(k, \ell)$ vertices contains a clique of size $k$ or an independent set of size $\ell$. A well-known, rough bound for $R(k, \ell)$ is $R(k, \ell) \leq \lceil \frac{k + \ell - 2}{k - 1} \rceil (k + \ell - 2)^{k - 1}$.

For $r \geq 1$ and $s, t \geq 1$, let $M(r, s, t) = (1 + R(r + 1, R(r + 1, s)))^{t-2}$. The next lemma has been proven by Chudnovsky, Spirkl and Zhong [10] for the case where $r = 3$. The proof of the lemma is analogous to the proof in [10] for the case where $r = 3$: replace each occurrence of “4” in the proofs of Lemmas 13 and 15 in [10] by “$r + 1$”.

**Lemma 8 (cf. [10]).** For every $r \geq 1$, $s \geq 1$ and $t \geq 1$, a connected $(K_{r+1}, K_1^1, s, P_t)$-free graph contains a dominating set of size at most $M(r, s, t)$.

We are now ready to prove Theorem 5. We in fact prove the following theorem, Theorem 9, which gives an explicit bound on the mim-width; Theorem 5 then follows from this.

**Theorem 9.** Let $r \geq 1$, $s \geq 1$ and $t \geq 1$, and let $G$ be a $(K_r, K_1^1, s, P_t)$-free graph. Then $\text{mimw}(G) \leq g(r, s, t)$ where $g(r, s, t) = 2(r + s - 1)^2(r + 1)^{t(t+1)}$, and a branch decomposition $(T, \delta)$ of $G$ with $\text{mimw}(T, \delta) \leq g(r, s, t)$ can be found in polynomial time.

**Proof.** We may assume without loss of generality that $G$ is connected. We use induction on $r$. If $r \leq 2$, then $G$ is $K_2$-free, so $\text{mimw}(T, \delta) = 0$ for any branch decomposition $(T, \delta)$ of $G$, whereas $g(r, s, t)$ is positive for all $s, t \geq 1$; so the theorem holds trivially in this case.

Suppose that $r \geq 3$. By Lemma 8, we find that $G$ has a dominating set $D$ of size at most $M(r - 1, s, t)$. Moreover, we can find $D$ in polynomial time by brute force (or we can apply the $O(tn^2)$-time algorithm of [10]). We let $p = |D|$, so $p \leq M(r - 1, s, t)$.
Let \( f(r, s, t) = (r + s - 1)^2 (t + 1)^2 \). We will show that there is a branch decomposition \((T', \delta')\) of \( G - D\) with mimw\((T', \delta')\) \(\leq f(r, s, t)\). The theorem will then follow: to see this, observe that if \((T', \delta')\) is such a branch decomposition, then we can readily extend \((T', \delta')\) to a branch decomposition \((T, \delta)\) for \( G\) with mim-width at most \( f(r, s, t) + p \leq f(r, s, t) + \max(r - 1, s, t) \leq g(r, s, t)\). Namely, we can obtain \( T \) in polynomial time from \( T'\) and an arbitrary subcubic tree \( T''\) with \( p + 2\) leaves by identifying a leaf of \( T'\) with a leaf of \( T''\). So it remains to prove that mimw\((G - D)\) \(\leq f(r, s, t)\), and that we can find a branch decomposition witnessing this bound, in polynomial time.

Let \( V = V(G) \). We partition \( V \) as follows. We first fix an arbitrary ordering \( d_1, \ldots, d_p \) on the vertices of \( D\). Let \( X_1 \) be the set of vertices in \( V \setminus D\) adjacent to \( d_1\). For \( i \in \{2, \ldots, p\}\), let \( X_i = \{d \in V \setminus D\} \) be such a branch decomposition, then we can readily extend \( T\) in polynomial time from \( T'\) and an arbitrary subcubic tree \( T''\) with \( p + 2\) leaves by identifying a leaf of \( T'\) with a leaf of \( T''\). So it remains to prove that mimw\((G - D)\) \(\leq f(r, s, t)\), and that we can find a branch decomposition witnessing this bound, in polynomial time.

Consider two sets \( X_i, X_j \) with \( i < j\). We claim that cutmim\(_G\)(\(X_i, X_j\)) \(\leq c = R(r - 1, R(r - 1, s))\). Towards a contradiction, suppose that cutmim\(_G\)(\(X_i, X_j\)) \(\geq c\). Then, by construction, there exist two sets \( A = \{a_1, a_2, \ldots, a_c\} \subseteq X_i\) and \( B = \{b_1, b_2, \ldots, b_c\} \subseteq X_j\), each of size \( c\), such that \(\{a_1 b_1, \ldots, a_c b_c\}\) is a set of \( c\) edges with the property that \( G\) does not contain any edges \(a_i b_j\) for \( i \neq j\) (note that edges \(a_i a_j\) and \(b_i b_j\) may exist in \( G\)).

As \( G[X_i]\) is \( K_{r-1}\)-free, and \( |A| = c = R(r - 1, R(r - 1, s))\), Ramsey’s Theorem tells us that \( G[A]\) contains an independent set \( A'\) of size \( c' = R(r - 1, s)\). Assume without loss of generality that \( A' = \{a_1, a_2, \ldots, a_c\}\). Let \( B' = \{b_1, b_2, \ldots, b_c\}\). As \( G[X_j]\) is \( K_{r-1}\)-free, \( G[B']\) contains an independent set \( B''\) of size \( s\). Assume without loss of generality that \( B'' = \{b_1, \ldots, b_s\}\). By construction, \( d_i\) is adjacent to every vertex of \( \{a_1, \ldots, a_s\} \subseteq X_i\) and non-adjacent to every vertex of \( \{b_1, \ldots, b_s\} \subseteq X_j\). Hence, \(\{a_1, a_2, b_1, \ldots, b_s, d_i\}\) induces a \( K_{1,s}^1\) in \( G\), a contradiction. We conclude that cutmim\(_G\)(\(X_i, X_j\)) \(\leq c\).

Now, by Lemma 7, we have

\[
\text{mimw}(G - D) \leq \max \{c \left[ \frac{(p - 1)}{r} \right] \left[ \frac{(p - 1)}{s} \right], \max_{i \in \{1, \ldots, p\}} \{\text{mimw}(G[X_j])\} + c(p - 1)\} \leq \max \{cp^2, f(r - 1, s, t) + M(r - 2, s, t) + cp\}.
\]

Recall that \( R(k, \ell) \leq (k + \ell - 2)^k - 1\). We observe that \( R(k, R(k, \ell)) \leq (k + \ell - 2)^k - 1\). Hence, \( c = R(r - 1, R(r - 1, s)) \leq (r + s - 3)^{(r - 1) (r - 2)}\) and \( p \leq M(r - 1, s, t) = (1 + R(r, R(r, s)))^{t - 2} \leq (1 + (r + s - 1)^{r (r + 1) + 1})^{t - 2} \). Thus

\[
\begin{align*}
cp^2 & \leq (r + s - 3)^{(r - 1) (r - 2)} \left((r + s - 1)^{r (r + 1) + 1}\right)^{2(t - 2)} \\
& \leq (r + s - 1)^{(r + 1)^2 (2(r + 1)^2 (t - 2) - (r + s - 1)^{(2(r + 1)^2 t - 1)} \leq f(r, s, t),
\end{align*}
\]
$f(r-1, s, t) + M(r-2, s, t) + cp$

$\leq (r + s - 2)2^{r(t+1)} + ((r + s - 2)(r-1)r+1)t-2 + (r + s - 3)(r-1)(r-2) \left( (r + s - 1)^2(r+1)+1 \right)^t-2$

$\leq (r + s - 1)2^{r(t+1)} \left( (r + s - 1)^2 r+1 + 1 + (r + s - 1)(r+1)(t-1) \right)$

$\leq (r + s - 1)2^{r(t+1)} \left( (r + s - 1)^2 r+1 \right)$

$= (r + s - 1)2^{r(t+1)+1}$

$\leq f(r, s, t)$.

So mimw$(G - D) \leq f(r, s, t)$ and the theorem follows by induction.

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**3 Conclusions**

We proved in Corollary 6 that for every $k \geq 1$, $s \geq 1$ and $t \geq 1$, List $k$-Colouring is polynomial-time solvable for $(K_{1^k}, P_t)$-free graphs by showing that the mim-width of these graphs is bounded and quickly computable. Huang [21] proved that 4-Colouring is NP-complete for $P_7$-free graphs and that 5-Colouring is NP-complete for $P_6$-free graphs. It is also known that List 4-Colouring is NP-complete for $P_6$-free graphs [16]. However, the List 3-Colouring problem is polynomial-time solvable for $P_7$-free graphs [4] and the computational complexities of 3-Colouring and List 3-Colouring are open for $P_t$-free graphs if $t \geq 8$. In particular, we do not know any integer $t$ such that 3-Colouring or List 3-Colouring are NP-complete for $P_t$-free graphs. Recently, Pilipczuk, Pilipczuk and Rzążewski [35] gave, for every $t \geq 3$, a quasi-polynomial-time algorithm for 3-Colouring on the class of $\{C_{t+1}, C_{t+2}, \ldots\}$-free graphs; note that this class contains, for $t \geq 2$, the class of $P_t$-free graphs as a subclass. Hence, an extension of Corollary 6, which will require more research into the structure of $P_t$-free graphs, might still be possible for $k = 3$. We leave this for future work.

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