On the supnorm form of Leray’s problem for the incompressible Navier-Stokes equations

LINEIA SCHÜTZ, JANAÍNA P. ZINGANO AND PAULO R. ZINGANO

Departamento de Matemática Pura e Aplicada
Universidade Federal do Rio Grande do Sul
Porto Alegre, RS 91509, Brazil

Abstract

We show that $t^{3/4} \| u(\cdot,t) \|_{L^\infty(\mathbb{R}^3)} \rightarrow 0$ as $t \rightarrow \infty$ for all Leray-Hopf’s global weak solutions $u(\cdot,t)$ of the incompressible Navier-Stokes equations in $\mathbb{R}^3$. It is also shown that $t \| u(\cdot,t) - e^{\Delta t} u_0 \|_{L^\infty(\mathbb{R}^3)} \rightarrow 0$ as $t \rightarrow \infty$, where $e^{\Delta t}$ is the heat semigroup, as well as other fundamental new results.

In spite of the complexity of the questions, our approach is elementary and is based on standard tools like conventional Fourier and energy methods.

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1. Introduction

In this work, we derive some new fundamental large time asymptotic properties of (globally defined) Leray-Hopf’s weak solutions [7, 14] of the incompressible Navier-Stokes equations in three-dimensional space,

$$
\begin{align*}
\dot{u} + u \cdot \nabla u + \nabla p &= \Delta u, \\
\nabla \cdot u(\cdot,t) &= 0,
\end{align*}
$$

where $L^2(\mathbb{R}^3)$ denotes the space of functions $u = (u_1, u_2, u_3) \in L^2(\mathbb{R}^3)$ with $\nabla \cdot u = 0$ in distributional sense. In his seminal 1934 paper, Leray [14] showed the existence of (possibly infinitely many) global weak solutions $u(\cdot,t) \in L^2(\mathbb{R}^3)$ which are weakly continuous in $L^2(\mathbb{R}^3)$ and satisfy $u(\cdot,t) \in L^\infty([0,\infty[; L^2(\mathbb{R}^3)) \cap L^2([0,\infty[, H^1(\mathbb{R}^3)),

$$

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with \( \| u(\cdot, t) - u_0 \|_{L^2(\mathbb{R}^3)} \to 0 \) as \( t \to 0 \) and such that the energy inequality

\[
\| u(\cdot, t) \|_{L^2(\mathbb{R}^3)}^2 + 2 \int_0^t \| Du(\cdot, s) \|_{L^2(\mathbb{R}^3)}^2 \, ds \leq \| u_0 \|_{L^2(\mathbb{R}^3)}^2
\]  

(1.2)

holds for all \( t \geq 0 \). Moreover, Leray [14] also showed in his construction that there always exists some \( t_* \gg 1 \) (depending on the solution \( u \)) such that one actually has \( u \in C^\infty(\mathbb{R}^3 \times [t_*, \infty[) \), and, for each \( m \geq 1 \):

\[
u(\cdot, t) \in C^\infty(\mathbb{R}^3 \times [t_*, T], H^m(\mathbb{R}^3)),
\]

(1.3)

for each \( t_* < T < \infty \), that is, \( u(\cdot, t) \in L^\infty_{2\alpha}(\mathbb{R}^3 \times [t_*, \infty[, H^m(\mathbb{R}^3)) \). While the uniqueness of Leray’s solutions remains a fundamental open question to this day, it has been shown by Kato [11] and Masuda [16] (and later by other authors also, see e.g. [10, 24]) that all Leray’s solutions, whether uniquely defined by their initial values or not, must satisfy the important asymptotic property

\[
\lim_{t \to \infty} \| u(\cdot, t) \|_{L^2(\mathbb{R}^3)} = 0
\]

(1.4)

a question left open in [14]. It will prove convenient for our present purposes that we also provide here a new derivation of (1.4) along the lines of the method introduced by Kreiss, Hagstrom, Lorenz and one of the authors in [12, 13] to give a straightforward derivation of the fundamental Schonbek-Wiegner decay estimates [21, 24] for solutions (and their derivatives) of the Navier-Stokes equations in dimension \( n \leq 3 \), under stronger assumptions on the initial data. (See also [18].) It will then be seen that, with some extra steps, one can similarly obtain the new supnorm result

\[
\lim_{t \to \infty} t^{3/4} \| u(\cdot, t) \|_{L^\infty(\mathbb{R}^3)} = 0
\]

(1.5)

which, again, is valid for all Leray-Hopf’s solutions of (1.1), assuming \( u_0 \in L^2(\mathbb{R}^3) \) only. Thus, by interpolation, we have, for any such solution,

\[
\lim_{t \to \infty} t^{\frac{3}{q} - \frac{3}{2q}} \| u(\cdot, t) \|_{L^q(\mathbb{R}^3)} = 0, \quad 2 \leq q \leq \infty,
\]

(1.6)

uniformly in \( q \). The properties (1.4) – (1.6) are well known and easy to obtain (see

\footnote{For the definition of the vector norms involved in (1.2) and other similar expressions throughout the text, see (1.16), (1.17) below.}

\footnote{For a detailed account of this method (mostly due to T. Hagstrom and J. Lorenz), see [17, 19].}
e.g. [3], Theorem 3.3, p. 95) for solutions \( v(\cdot, t) \in L^\infty([t_0, \infty[, L^2(\mathbb{R}^3)) \) of the associated linear heat flow problems

\[
\begin{align*}
v_t &= \Delta v, \quad t > t_0, \\
v(\cdot, t_0) &= u(\cdot, t_0),
\end{align*}
\tag{1.7}
\]

given \( t_0 \geq 0 \) (arbitrary). The solution of (1.7) is given by

\[
v(\cdot, t) = e^{\Delta(t-t_0)}u(\cdot, t_0),
\]

where \( e^{\Delta t} \) denotes the heat semigroup. It is therefore natural to think that the Leray-Hopf’s solutions of (1.1) be closely related to the corresponding heat flows defined in (1.7). In fact, Kato [11] obtained

\[
\lim_{t \to \infty} t^{1/4} \| u(\cdot, t) - v(\cdot, t) \|_{L^2(\mathbb{R}^3)} = 0
\]

for each \( \epsilon > 0 \), and a bit later Wiegner [24] got, using a very involved argument, the sharper result \(^\#\)

\[
\lim_{t \to \infty} t^{1/4} \| u(\cdot, t) - v(\cdot, t) \|_{L^2(\mathbb{R}^3)} = 0
\]

(see [24], Theorem (c), p. 305). Again, a simple proof of (1.8) in the spirit of [12, 13] is provided here (see Section 3), after (1.4) has been obtained. This is useful to pave our way for the corresponding supnorm result obtained in Section 4, viz.,

\[
\lim_{t \to \infty} t \| u(\cdot, t) - v(\cdot, t) \|_{L^\infty(\mathbb{R}^3)} = 0.
\]

\[
\tag{1.9}
\]

By interpolation, it follows from (1.8), (1.9) that

\[
\lim_{t \to \infty} t^{1-\frac{3}{2q}} \| u(\cdot, t) - v(\cdot, t) \|_{L^q(\mathbb{R}^3)} = 0, \quad 2 \leq q \leq \infty,
\]

\[
\tag{1.10}
\]

uniformly in \( q \). It is worth noticing that these results improve the previous estimates

\[
\begin{align*}
\limsup_{t \to \infty} t^{3/4} \| u(\cdot, t) \|_{L^3(\mathbb{R}^3)} < \infty,
\tag{1.11a}
\\
\limsup_{t \to \infty} t^{1-\frac{3}{2q}} \| u(\cdot, t) - v(\cdot, t) \|_{L^q(\mathbb{R}^3)} < \infty
\tag{1.11b}
\end{align*}
\]

obtained by Beirão da Veiga and Wiegner in [3, 25] for finite \( q > 2 \).

\(^\#\) In addition, Wiegner obtains (1.4), (1.8) in the presence of external forces \( f(\cdot, t) \), under suitable assumptions on \( f(\cdot, t) \). Also, he considers the case of arbitrary space dimension \( n \geq 2 \), which is a complicating factor in the analysis. While we can certainly extend our approach to include external forces \( f \) in (1.1), under appropriate assumptions on \( f \) which are slightly different from Wiegner’s, or similarly extend the analysis down to \( n = 2 \), our method is (as that of [12, 13]) limited to \( n \leq 3 \).
Here is a brief overview of what is next. After some important mathematical preliminaries on the Leray-Hopf’s solutions to the Navier-Stokes system (1.1) have been reviewed in Section 2 for later use, along with two new fundamental results given by Theorems 2.2 and 2.3, we turn our attention to the basic $L^2$ estimates (1.4) and (1.8), which are rederived in Section 3 along the lines of [12, 13]. This shows the way to obtain the more difficult estimates (1.5) and (1.9), which is the goal of Section 4. In these two sections, the key point is to first observe that

$$\lim_{t \to \infty} t^{1/2} \| D_u(\cdot, t) \|_{L^2(\mathbb{R}^3)} = 0,$$

(1.12)

from which the desired estimates can be more easily obtained. Although we restrict our attention here to dimension $n = 3$, it will be clearly seen that the method can also be used in the case $n = 2$, which is actually easier since (1.12) turns out to be trivial in this case. Put together, the results for $n = 2, 3$ can be summarized as follows. One has, for each $2 \leq q \leq \infty$ (and $n = 2, 3$):

$$\lim_{t \to \infty} t^{n/2 - n/q} \| u(\cdot, t) \|_{L^q(\mathbb{R}^n)} = 0,$$

(1.13a)

$$\lim_{t \to \infty} t^{n/2 - n/q} \| u(\cdot, t) - v(\cdot, t) \|_{L^q(\mathbb{R}^n)} = 0,$$

(1.13b)

uniformly in $q \in [2, \infty]$, where $v(\cdot, t) = e^{\Delta(t-t_0)}u(\cdot, t_0)$, $t_0 \geq 0$ arbitrary, see (1.7), under the sole assumption that $u_0 \in L^2_\sigma(\mathbb{R}^n)$. A proof (or disproof) of this general property in higher dimensions is apparently still missing in the literature. For $n \leq 3$, everything needed to obtain (1.13) was already known by 1934 after the publication of [14], as the next sections show — and yet it has taken full fifty years before even the easier part of (1.13) could have finally been established! We hope that this shows the power of the ideas presented here, as well as of the approach introduced in [12, 13]. In fact, a deeper combination of these ideas has now led to the complete solution of the full Leray’s problem in dimension $n \leq 3$ [3]: one has, for every $s \geq 0$, and any $0 \leq t_0 \leq t_1 < t$,

$$\lim_{t \to \infty} t^{s/2} \| u(\cdot, t) \|_{H^s(\mathbb{R}^n)} = 0,$$

(1.14a)

$$\lim_{t \to \infty} t^{n/2 + s/2 - 1/2} \| u(\cdot, t) - e^{\Delta(t-t_0)}u(\cdot, t_0) \|_{H^s(\mathbb{R}^n)} = 0,$$

(1.14b)

$$\| e^{\Delta(t-t_0)}u(\cdot, t_0) - e^{\Delta(t-t_1)}u(\cdot, t_1) \|_{H^s(\mathbb{R}^n)} \leq K(n, s) (t_1 - t_0)^{\frac{n}{2}} (t - t_1)^{-\left(\frac{n}{4} + \frac{s}{2}\right)}$$

(1.14c)

for arbitrary Leray solutions in $\mathbb{R}^n$, under the unique assumption of square-integrable,
divergence-free initial data. (For the more involved analysis giving (1.14), see [8]. In the simpler case of dimension $n = 2$, (1.14a) was shown in [2] by a different method. Some related high-order estimates have also been obtained in [12, 18, 21], but under stronger assumptions on the initial data.) Here, $\dot{H}^s(\mathbb{R}^n)$ denotes the homogeneous Sobolev space of all functions $v = (v_1, \ldots, v_n) \in L^2(\mathbb{R}^n)$ such that $|\cdot|^s |\hat{v}(\cdot)| \in L^2(\mathbb{R}^n)$, where $\hat{v}(\cdot)$ stands for the Fourier transform of $v(\cdot)$, with norm $\|v\|_{\dot{H}^s(\mathbb{R}^n)}$ defined by

$$\|v\|_{\dot{H}^s(\mathbb{R}^n)} = \left\{ \int_{\mathbb{R}^n} |\xi|^2 s |\hat{v}(\xi)|^2 d\xi \right\}^{1/2}. \quad (1.15)$$

Thus, one has $t^{m/2} \|D^m u(\cdot, t)\|_{L^2(\mathbb{R}^n)} \to 0$, $t^{3/2 + m/2 - 1/2} \|D^m\{u(\cdot, t) - e^{\Delta t} u_0\}\|_{L^2(\mathbb{R}^n)} \to 0$ (as $t \to \infty$) for every $m \in \mathbb{N}$, and so forth, which are important extensions of (1.12). In (1.15) and throughout the text, $|\cdot|_1$ denotes the Euclidean norm in $\mathbb{R}^n$.

More on notation: boldface letters are used for vector quantities, as in $u(x, t) = (u_1(x, t), u_2(x, t), u_3(x, t))$. Also, $\nabla p \equiv \nabla p(\cdot, t)$ denotes the spatial gradient of $p(\cdot, t)$, $D_j = \partial/\partial x_j$, and $\nabla \cdot u = D_1 u_1 + D_2 u_2 + D_3 u_3$ is the (spatial) divergence of $u(\cdot, t)$.

$\|\cdot\|_{L^q(\mathbb{R}^3)}$, $1 \leq q \leq \infty$, denote the standard norms of the Lebesgue spaces $L^q(\mathbb{R}^3)$, with

$$\| u(\cdot, t) \|_{L^q(\mathbb{R}^3)} = \left\{ \sum_{i = 1}^3 \int_{\mathbb{R}^3} |u_i(x, t)|^q dx \right\}^{1/q} \quad (1.16a)$$

$$\| D u(\cdot, t) \|_{L^q(\mathbb{R}^3)} = \left\{ \sum_{i, j = 1}^3 \int_{\mathbb{R}^3} |D_j u_i(x, t)|^q dx \right\}^{1/q} \quad (1.16b)$$

$$\| D^2 u(\cdot, t) \|_{L^q(\mathbb{R}^3)} = \left\{ \sum_{i, j, \ell = 1}^3 \int_{\mathbb{R}^3} |D_j D_\ell u_i(x, t)|^q dx \right\}^{1/q} \quad (1.16c)$$

if $1 \leq q < \infty$, and $\| u(\cdot, t) \|_{L^\infty(\mathbb{R}^3)} = \max \left\{ \| u_i(\cdot, t) \|_{L^\infty(\mathbb{R}^3)} : 1 \leq i \leq 3 \right\}$ if $q = \infty$. We will also find it convenient in many places to use the following alternative definition for the supnorm of $u(\cdot, t)$:

$$\| u(\cdot, t) \|_\infty = \text{ess sup} \left\{ |u(x, t)|_2 : x \in \mathbb{R}^3 \right\}. \quad (1.17)$$

All other notation, when not standard, will be explained as it appears in the text.

For readers interested mainly in the new results obtained in the present work, one could at this point go directly to Theorems 2.2 and 2.3 in Section 2, and Theorems 4.1 and 4.2 in Section 4, with a quick pass at (2.22) and the discussions in Section 3 and the Appendix, particularly Theorem A.1. The few remaining results may be also worth browsing, as some are not so widely known as they surely deserve to be.
§ 2. Some mathematical preliminaries

In this section, we collect some basic results that will play an important role later in our derivation of (1.4), (1.5), (1.8) and (1.9), and we also introduce two fundamental new results (Theorems 2.2 and 2.3 below). For the construction of Leray-Hopf’s solutions \( u(\cdot, t) \) to the Navier-Stokes equations (1.3), see e.g. [7, 14]. These solutions were originally obtained in [14] by introducing an ingenious regularization procedure which, for convenience, is briefly reviewed next. Taking (any) \( G \in C_0^\infty(\mathbb{R}^n) \) nonnegative with \( \int_{\mathbb{R}^3} G(x) \, dx = 1 \) and setting \( \tilde{u}_{0,\delta}(\cdot) \in C^\infty(\mathbb{R}^3) \) by convolving \( u_0(\cdot) \) with \( G_\delta(x) = \delta^{-n} G(x/\delta), \delta > 0 \), if we define \( u_\delta, p_\delta \in C^\infty(\mathbb{R}^3 \times [0, \infty[) \) as the (unique, globally defined) classical \( L^2 \) solutions of the associated equations

\[
\begin{align*}
\frac{\partial}{\partial t} u_\delta + \tilde{u}_0(\cdot) \cdot \nabla u_\delta + \nabla p_\delta &= \Delta u_\delta, & \nabla \cdot u_\delta(\cdot, t) &= 0, \\
u_\delta(\cdot, 0) &= G_\delta * u_0 \in \bigcap_{m = 1}^{\infty} H^m(\mathbb{R}^3),
\end{align*}
\]  

(2.1a)

(2.1b)

where \( \tilde{u}_0(\cdot) := G_\delta * u_0(\cdot, t) \), it was shown by Leray that, for some sequence \( \delta' \to 0 \), one has the weak convergence property

\[
u_\delta(\cdot, t) \rightharpoonup \nu(\cdot, t) \tag{2.2}
\]

that is, \( u_\delta(\cdot, t) \to u(\cdot, t) \) weakly in \( L^2(\mathbb{R}^3) \), for every \( t \geq 0 \) (see [14], p. 237), with \( u(\cdot, t) \in L^\infty([0, \infty[, L^2_\sigma(\mathbb{R}^3)) \cap L^2([0, \infty[, H^1(\mathbb{R}^3)) \cap C_0^\infty([0, \infty[, L^2(\mathbb{R}^3)) \) continuous in \( L^2 \) at \( t = 0 \) and solving the Navier-Stokes equations (1.1a) in distributional sense. Moreover, the energy inequality (1.2) is satisfied for all \( t \geq 0 \), so that, in particular,

\[
\int_0^{\infty} \| Du(\cdot, t) \|_{L^2(\mathbb{R}^3)}^2 \, dt \leq \frac{1}{2} \| u_0 \|_{L^2(\mathbb{R}^3)}^2.
\]  

(2.3)

Another important property shown in [14] is that \( u \in C^\infty([t_*, \infty[) \) for some \( t_* \gg 1 \), with \( D_m u(\cdot, t) \in L^\infty_{loc}([t_*, \infty[, L^2(\mathbb{R}^3)) \) for each \( m \geq 1 \). This fact (together with Theorems 2.2 and 2.3 below) will greatly simplify our analysis in Sections 3 and 4. Other results needed later have mostly to do with the Helmholtz-Weyl projection of \( - u(\cdot, t) \cdot \nabla u(\cdot, t) \) into \( L^2_\sigma(\mathbb{R}^3) \), that is, the divergence-free field \( Q(\cdot, t) \in L^2(\mathbb{R}^3) \) given by

\[
Q(\cdot, t) := - u(\cdot, t) \cdot \nabla u(\cdot, t) - \nabla p(\cdot, t), \quad \text{a.e. } t > 0.
\]  

(2.4)

For convenience, they are discussed in more detail in the remainder of this section.
Theorem 2.1. For almost every $s > 0$, one has

$$\| e^{\Delta(t-s)}Q(\cdot, s) \|_{L^2(\mathbb{R}^3)} \leq K(t-s)^{-3/4} \| u(\cdot, s) \|_{L^2(\mathbb{R}^3)} \| Du(\cdot, s) \|_{L^2(\mathbb{R}^3)}$$

(2.5a)

and

$$\| e^{\Delta(t-s)}Q(\cdot, s) \|_{\infty} \leq K(t-s)^{-3/4} \| u(\cdot, s) \|_{\infty} \| Du(\cdot, s) \|_{L^2(\mathbb{R}^3)}$$

(2.5b)

for all $t > s$, where $K = (8\pi)^{-3/4}$.

**Proof:** The following argument is adapted from [12]. Considering (2.5a) first, let $\mathbb{F}[f] \equiv \hat{f}$ denote the Fourier transform of a given function $f \in L^1(\mathbb{R}^3)$, viz.,

$$\mathbb{F}[f](k) \equiv \hat{f}(k) := (2\pi)^{-3/2} \int_{\mathbb{R}^3} e^{-ik \cdot x} f(x) \, dx, \quad k \in \mathbb{R}^3$$

(2.6)

(where $i^2 = -1$). Given $\nu(\cdot, s) = (\nu_1(\cdot, s), \nu_2(\cdot, s), \nu_3(\cdot, s)) \in L^1(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$ arbitrary, we get, using Parseval’s identity,

$$\| e^{\Delta(t-s)}\nu(\cdot, s) \|_{L^2(\mathbb{R}^3)}^2 = \| \mathbb{F}[e^{\Delta(t-s)}\nu(\cdot, s)] \|_{L^2(\mathbb{R}^3)}^2$$

$$= \int_{\mathbb{R}^3} e^{-2|k|^2(t-s) \hat{\nu}(k, s)} \, dk$$

$$\leq \| \hat{\nu}(\cdot, s) \|_{\infty}^2 \int_{\mathbb{R}^3} e^{-2|k|^2(t-s)} \, dk$$

$$= \left(\frac{\pi}{2}\right)^{3/2} (t-s)^{-3/2} \| \hat{\nu}(\cdot, s) \|_{\infty}^2,$$

that is,

$$\| e^{\Delta(t-s)}\nu(\cdot, s) \|_{L^2(\mathbb{R}^3)} \leq \left(\frac{\pi}{2}\right)^{3/4} (t-s)^{-3/4} \| \hat{\nu}(\cdot, s) \|_{\infty},$$

(2.7)

where $\cdot \|_2$ denotes the Euclidean norm in $\mathbb{R}^3$ and $\| \hat{\nu}(\cdot, s) \|_{\infty} = \sup \{ \| \hat{\nu}(k, s) \|_2 : k \in \mathbb{R}^3 \}$. As will be shown next, (2.5a) follows from a direct application of (2.7) to $\nu(\cdot, s) = Q(\cdot, s)$.

We need only be able to estimate $\| \hat{Q}(\cdot, s) \|_{\infty}$: because $\mathbb{F}[\nabla P(\cdot, s)](k) = \hat{n} \hat{p}(k, s) k$ and $\sum_{j=1}^3 k_j \hat{Q}_j(k, s) = 0$ (since $\nabla \cdot Q(\cdot, s) = 0$), the vectors $\mathbb{F}[\nabla P(\cdot, s)](k)$ and $\hat{Q}(k, s)$ are orthogonal in $\mathbb{C}^3$, for every $k \in \mathbb{R}^3$. Recalling from (2.4) that $\hat{Q}(k, s) + \mathbb{F}[\nabla P(\cdot, s)](k) = -\mathbb{F}[\nu(\cdot, s) \cdot \nabla \nu(\cdot, s)](k)$, this gives

$$| \hat{Q}(k, s) |_2 \leq | \mathbb{F}[\nu(\cdot, s) \cdot \nabla \nu(\cdot, s)](k) |_2$$

(2.8)

for all $k \in \mathbb{R}^3$, so that we get

$$\| \hat{Q}(\cdot, s) \|_{\infty} \leq \| \mathbb{F}[\nu \cdot \nabla \nu](\cdot, s) \|_{\infty},$$

(2.9)
Now, we have, for each $1 \leq i \leq 3$,\[
|\mathcal{F}[\mathbf{u}(\cdot, s) \cdot \nabla u_i(\cdot, s)](k)| \leq \sum_{j=1}^{3} |\mathcal{F}[u_j(\cdot, s) D_j u_i(\cdot, s)](k)|
\]
\[
\leq (2\pi)^{-3/2} \sum_{j=1}^{3} \|u_j(\cdot, s) D_j u_i(\cdot, s)\|_{L^1(\mathbb{R}^3)}
\]
\[
\leq (2\pi)^{-3/2} \|\mathbf{u}(\cdot, s)\|_{L^2(\mathbb{R}^3)} \|\nabla u_i(\cdot, s)\|_{L^2(\mathbb{R}^3)},
\]
by the Cauchy-Schwarz inequality. (Here, as before, $D_j = \partial / \partial x_j$.) This gives
\[
\|\mathcal{F}[\mathbf{u} \cdot \nabla \mathbf{u}](\cdot, s)\|_{L^\infty} \leq (2\pi)^{-3/2} \|\mathbf{u}(\cdot, s)\|_{L^2(\mathbb{R}^3)} \|D\mathbf{u}(\cdot, s)\|_{L^2(\mathbb{R}^3)}.
\tag{2.10}
\]
From (2.7), (2.9) and (2.10), one gets (2.5a), which shows the first part of Theorem 2.1.

The proof of (2.5b) follows in a similar way, using (2.8) and the elementary estimate $\|e^{\Delta \tau} u\|_{L^\infty(\mathbb{R}^3)} \leq K \tau^{-3/4} \|u\|_{L^2(\mathbb{R}^3)}$ for the heat semigroup, where $\tau > 0$ is arbitrary, and $K = (8\pi)^{-3/4}$. This gives, for any $s > 0$ with both $\|\mathbf{u}(\cdot, s)\|_{\mathbb{R}^3}$ and $\|D\mathbf{u}(\cdot, s)\|_{L^2(\mathbb{R}^3)}$ finite,
\[
\|e^{\Delta(t-s)} \mathcal{Q}(\cdot, s)\|_{L^\infty} \leq K (t-s)^{-3/4} \|\mathcal{Q}(\cdot, s)\|_{L^2(\mathbb{R}^3)}
\]
\[
\leq K (t-s)^{-3/4} \|\mathbf{u}(\cdot, s) \cdot \nabla \mathbf{u}(\cdot, s)\|_{L^2(\mathbb{R}^3)} \quad \text{[by (2.8)]}
\]
\[
\leq K (t-s)^{-3/4} \|\mathbf{u}(\cdot, s)\|_{L^\infty} \|D\mathbf{u}(\cdot, s)\|_{L^2(\mathbb{R}^3)}
\]
for all $t > s$, using Parseval’s identity (twice) and the norm definitions (1.16), (1.17).

Let us notice that, applying the argument above to solutions of the regularized Navier-Stokes equations (2.1), we obtain, in a completely similar way,
\[
\|e^{\Delta(t-s)} \mathcal{Q}_\delta(\cdot, s)\|_{L^2(\mathbb{R}^3)} \leq K (t-s)^{-3/4} \|\mathbf{u}_\delta(\cdot, s)\|_{L^2(\mathbb{R}^3)} \|D\mathbf{u}_\delta(\cdot, s)\|_{L^2(\mathbb{R}^3)} \tag{2.11a}
\]
and
\[
\|e^{\Delta(t-s)} \mathcal{Q}_\delta(\cdot, s)\|_{\infty} \leq K (t-s)^{-3/4} \|\mathbf{u}_\delta(\cdot, s)\|_{L^\infty} \|D\mathbf{u}_\delta(\cdot, s)\|_{L^2(\mathbb{R}^3)} \tag{2.11b}
\]
for all $t > s > 0$, where $K = (8\pi)^{-3/4}$, as before, and
\[
\mathcal{Q}_\delta(\cdot, s) = -\bar{u}_\delta(\cdot, s) \cdot \nabla \mathbf{u}_\delta(\cdot, s) - \nabla p_\delta(\cdot, s).
\tag{2.12}
\]
The estimate (2.11a) is particularly useful, since the regularized solutions $\mathbf{u}_\delta(\cdot, t)$
Given in (2.1) satisfy the energy inequality
\[
\| u_\delta(\cdot,t) \|_{L^2(\mathbb{R}^3)}^2 + 2 \int_0^t \| D u_\delta(\cdot,s) \|_{L^2(\mathbb{R}^3)}^2 \, ds \leq \| u_0 \|_{L^2(\mathbb{R}^3)}^2
\] (2.13)
for all \( t > 0 \) (and \( \delta > 0 \) arbitrary), from which \( \| u_\delta(\cdot,t) \|_{L^2(\mathbb{R}^3)} \), \( \int_0^t \| D u_\delta(\cdot,s) \|_{L^2(\mathbb{R}^3)}^2 \, ds \) can be bounded independently of \( \delta > 0 \). This will be used in Theorems 2.2 and 2.3 below to show that the particular value of \( t_0 \geq 0 \) chosen in defining the heat flow approximations (1.7) is not relevant in regard to the properties (1.8) – (1.10).

**Theorem 2.2.** Let \( u(\cdot,t), t > 0, \) be any particular Leray-Hopf’s solution to (1.1). Given any pair of initial values \( \bar{t}_0 > t_0 \geq 0, \) one has
\[
\| \bar{v}(\cdot,t) - \tilde{v}(\cdot,t) \|_{L^2(\mathbb{R}^3)} \leq \frac{K}{\sqrt{2}} \| u_0 \|_{L^2(\mathbb{R}^3)}^2 (\bar{t}_0 - t_0)^{1/2} (t - \bar{t}_0)^{-3/4}
\] (2.14)
for all \( t > \bar{t}_0, \) where \( v(\cdot,t) = e^{\Delta(t-t_0)} u(\cdot,t_0), \) \( \bar{v}(\cdot,t) = e^{\Delta(t-\bar{t}_0)} u(\cdot,\bar{t}_0) \) are the corresponding heat flows associated with \( t_0, \bar{t}_0, \) respectively, and \( K = (8\pi)^{-3/4}. \)

**Proof:** We start by writing \( v(\cdot,t) \) as
\[
v(\cdot,t) = e^{\Delta(t-t_0)} [u(\cdot,t_0) - u_\delta(\cdot,t_0)] + e^{\Delta(t-t_0)} u_\delta(\cdot,t_0), \quad t > t_0,
\]
with \( u_\delta(\cdot,t) \) given in (2.1), \( \delta > 0. \) Because
\[
u_\delta(\cdot,t_0) = e^{\Delta t_0} \tilde{u}_{0,\delta} + \int_0^{t_0} e^{\Delta(t_0-s)} Q_\delta(\cdot,s) \, ds,
\]
where \( \tilde{u}_{0,\delta} = G_\delta * u_0, Q_\delta(\cdot,s) = -\tilde{u}_{0,\delta}(\cdot,s) \cdot \nabla u_\delta(\cdot,s) - \nabla p_\delta(\cdot,s), \) cf. (2.1b) and (2.12) above, we get
\[
v(\cdot,t) = e^{\Delta(t-t_0)} [u(\cdot,t_0) - u_\delta(\cdot,t_0)] + e^{\Delta t} \tilde{u}_{0,\delta} + \int_0^{t_0} e^{\Delta(t-s)} Q_\delta(\cdot,s) \, ds,
\]
for \( t > t_0. \) Similarly, we have, for \( t > \bar{t}_0: \)
\[
\bar{v}(\cdot,t) = e^{\Delta(t-\bar{t}_0)} [u(\cdot,\bar{t}_0) - u_\delta(\cdot,\bar{t}_0)] + e^{\Delta t} \tilde{u}_{0,\delta} + \int_0^{\bar{t}_0} e^{\Delta(t-s)} Q_\delta(\cdot,s) \, ds.
\]
Hence, we obtain, for the difference \( \bar{v}(\cdot,t) - \tilde{v}(\cdot,t), \) at any \( t > \bar{t}_0, \) the identity
\[
\bar{v}(\cdot,t) - \tilde{v}(\cdot,t) = e^{\Delta(t-\bar{t}_0)} [u(\cdot,\bar{t}_0) - u_\delta(\cdot,\bar{t}_0)] - e^{\Delta(t-t_0)} [u(\cdot,t_0) - u_\delta(\cdot,t_0)] \\
+ \int_{t_0}^{\bar{t}_0} e^{\Delta(t-s)} Q_\delta(\cdot,s) \, ds.
\] (2.15)
Therefore, given any $K \subset \mathbb{R}^3$ compact, we get, for each $t > \tilde{t}_0$, $\delta > 0$:

$$
\| \tilde{v}(\cdot, t) - v(\cdot, t) \|_{L^2(K)} \leq J_{\delta}(t) + \int_{t_0}^{\tilde{t}_0} \| e^{\Delta(t-s)} Q_{\delta}(\cdot, s) \|_{L^2(K)} \, ds
$$

\begin{align*}
\leq J_{\delta}(t) + K \int_{t_0}^{\tilde{t}_0} (t-s)^{-3/4} \| u_\delta(\cdot, s) \|_{L^2(\mathbb{R}^3)} \| D_u(\cdot, s) \|_{L^2(\mathbb{R}^3)} \, ds \\
\leq J_{\delta}(t) + \frac{K}{\sqrt{2}} (\tilde{t}_0 - t_0)^{1/2} \| u_0 \|_{L^2(\mathbb{R}^3)}^2 (t - \tilde{t}_0)^{-3/4}
\end{align*}

by (2.11a), (2.13), where $K = (8\pi)^{-3/4}$ and

$J_{\delta}(t) = \| e^{\Delta(t-\tilde{t}_0)} [u(\cdot, \tilde{t}_0) - u_\delta(\cdot, \tilde{t}_0)] \|_{L^2(K)} + \| e^{\Delta(t-t_0)} [u(\cdot, t_0) - u_\delta(\cdot, t_0)] \|_{L^2(K)}$.

Taking $\delta = \delta' \to 0$ according to (2.2), we get $J_{\delta}(t) \to 0$, since, by Lebesgue’s Dominated Convergence Theorem and (2.2), we have, for any $\sigma, \tau > 0$:

$$
\| e^{\Delta(\tau)} [u(\cdot, \sigma) - u_{\delta'}(\cdot, \sigma)] \|_{L^2(K)} \to 0 \quad \text{as} \quad \delta' \to 0,
$$

recalling that $K$ has finite measure. Hence, we obtain

$$
\| \tilde{v}(\cdot, t) - v(\cdot, t) \|_{L^2(K)} \leq \frac{K}{\sqrt{2}} (\tilde{t}_0 - t_0)^{1/2} \| u_0 \|_{L^2(\mathbb{R}^3)}^2 (t - \tilde{t}_0)^{-3/4}
$$

for each $t > \tilde{t}_0$, and for any compact set $K \subset \mathbb{R}^3$. This is clearly equivalent to (2.14).

Theorem 2.2 greatly simplifies the derivation of the asymptotic property (1.8). For similar reasons, our proof of (1.9) requires the supnorm version of (2.14) above, which is given in the next result.

**Theorem 2.3.** Let $u(\cdot, t), t > 0$, be any particular Leray-Hopf’s solution to (1.1). Given any pair of initial values $\tilde{t}_0 > t_0 \geq 0$, one has

$$
\| v(\cdot, t) - \tilde{v}(\cdot, t) \|_{L^\infty(\mathbb{R}^3)} \leq \frac{\Gamma}{\sqrt{2}} \| u_0 \|_{L^2(\mathbb{R}^3)}^2 (\tilde{t}_0 - t_0)^{1/2} (t - \tilde{t}_0)^{-3/2} \quad (2.16)
$$

for all $t > \tilde{t}_0$, where $v(\cdot, t) = e^{\Delta(t-t_0)} u(\cdot, t_0)$, $\tilde{v}(\cdot, t) = e^{\Delta(t-\tilde{t}_0)} u(\cdot, \tilde{t}_0)$ are the corresponding heat flows associated with $t_0$, $\tilde{t}_0$, respectively, and $\Gamma = (4\pi)^{-3/2}$.

**Proof:** Taking $K \subset \mathbb{R}^3$ compact and $2 < q < \infty$ arbitrary, we get, for each $t > \tilde{t}_0$, $\delta > 0$, recalling (2.12), (2.15):

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for each \( \sigma, \tau > a \) which is easily derived with the Fourier transform. By (2.17) see e.g. [23], Proposition 2.4, p. 13, or [22], Theorem 4.5.1, p. 52; and

\[ \| \tilde{\varphi}(\cdot, t) - \varphi(\cdot, t) \|_{L^q(K)} \leq J_{\delta, q}(t) + \int_{t_0}^{\tilde{t}_0} \| e^{\Delta(t-s)} Q_\delta^2(\cdot, s) \|_{L^q(\mathbb{R}^3)} ds \]

\[ \leq J_{\delta, q}(t) + \int_{t_0}^{\tilde{t}_0} \left[ \frac{4\pi}{t-s} \right]^{-\frac{3}{2}(1-\frac{q}{2})} \| e^{\Delta(t-s)} Q_\delta^2(\cdot, s) \|_{L^2(\mathbb{R}^3)} ds \]

\[ \leq J_{\delta, q}(t) + \gamma_q \int_{t_0}^{\tilde{t}_0} \left( t-s \right)^{-\frac{3}{2}(1-\frac{q}{2})} \| u_\delta(\cdot, s) \|_{L^2(\mathbb{R}^3)} \| Du_\delta(\cdot, s) \|_{L^2(\mathbb{R}^3)} \]

\[ \leq J_{\delta, q}(t) + \frac{\gamma_q}{\sqrt{2}} \left( \tilde{t}_0 - t_0 \right)^{\frac{1}{2}} \| u_0 \|_{L^2(\mathbb{R}^3)}^2 \left( t-\tilde{t}_0 \right)^{-\frac{3}{2}(1-\frac{q}{2})} \]

by (2.11a), (2.13), where \( \gamma_q = (4\pi)^{-\frac{3}{2}(1-\frac{q}{2})} \) and

\[ J_{\delta, q}(t) = \| e^{\Delta(t-\tilde{t}_0)} [ u(\cdot, \tilde{t}_0) - u_\delta(\cdot, \tilde{t}_0) ] \|_{L^q(\mathbb{R}^3)} + \| e^{\Delta(t-t_0)} [ u(\cdot, t_0) - u_\delta(\cdot, t_0) ] \|_{L^q(\mathbb{R}^3)} \]

Taking \( \delta = \delta' \rightarrow 0 \) according to (2.2), we get \( J_{\delta, q}(t) \rightarrow 0 \), since, by Lebesgue’s Dominated Convergence Theorem and (2.2), we have \( \| e^{\Delta t} [ u(\cdot, \sigma) - u_\delta(\cdot, \sigma) ] \|_{L^q(\mathbb{R}^3)} \rightarrow 0 \) as \( \delta' \rightarrow 0 \), for each \( \sigma, \tau > 0 \). Hence, letting \( \delta = \delta' \rightarrow 0 \), we obtain

\[ \| \tilde{\varphi}(\cdot, t) - \varphi(\cdot, t) \|_{L^q(K)} \leq \frac{\gamma_q}{\sqrt{2}} \left( \tilde{t}_0 - t_0 \right)^{\frac{1}{2}} \| u_0 \|_{L^2(\mathbb{R}^3)}^2 \left( t-\tilde{t}_0 \right)^{-\frac{3}{2}(1-\frac{q}{2})} \]

for each \( t > \tilde{t}_0, q > 2 \). This gives, letting \( q \rightarrow \infty \),

\[ \| \tilde{\varphi}(\cdot, t) - \varphi(\cdot, t) \|_{L^\infty(K)} \leq \frac{\Gamma}{\sqrt{2}} \frac{1}{\tilde{t}_0 - t_0} \| u_0 \|_{L^2(\mathbb{R}^3)}^2 \left( t-\tilde{t}_0 \right)^{-\frac{3}{2}} \]

for each \( t > \tilde{t}_0 \), with \( K \subset \mathbb{R}^3 \) compact arbitrary. This estimate clearly implies (2.16). \( \square \)

For the next fundamental result reviewed in this section, given in Theorem 2.4, we will need the following elementary Sobolev-Nirenberg-Gagliardo (SNG) inequalities for arbitrary \( u \in H^2(\mathbb{R}^3) \):

\[ \| u \|_{\infty} \leq K_0 \| u \|_{L^2(\mathbb{R}^3)}^{1/4} \| D^2 u \|_{L^2(\mathbb{R}^3)}^{3/4}, \quad K_0 < 0.678, \quad (2.17a) \]

see e.g. [23], Proposition 2.4, p. 13, or [22], Theorem 4.5.1, p. 52; and

\[ \| Du \|_{L^2(\mathbb{R}^3)} \leq K_1 \| u \|_{L^2(\mathbb{R}^3)}^{1/2} \| D^2 u \|_{L^2(\mathbb{R}^3)}^{1/2}, \quad K_1 = 1, \quad (2.17b) \]

which is easily derived with the Fourier transform. By (2.17a), (2.17b), we then have

\[ \| u \|_{\infty} \| Du \|_{L^2(\mathbb{R}^3)}^{1/2} \leq K_2 \| u \|_{L^2(\mathbb{R}^3)}^{1/2} \| D^2 u \|_{L^2(\mathbb{R}^3)}^{1/2}, \quad K_2 = K_0 K_1^{1/2} < 1. \quad (2.18) \]
\textbf{Theorem 2.4.} Let $u(\cdot, t)$, $t > 0$, be any particular Leray-Hopf’s solution to (1.1). Then, there exists $t_* \gg 1$ ($t_*$ depending on the solution $u$) sufficiently large that $\| Du(\cdot, t) \|_{L^2(\mathbb{R}^3)}$ is a smooth, monotonically decreasing function of $t$ on $[t_*, \infty[$.

\textbf{Proof:} The following argument is adapted from [12], Lemma 2.2. Let $t_0 \geq t_*$ (to be chosen shortly), with $t_* \gg 1$ given in (1.3). Let $t > t_0$. Applying $D_t = \partial / \partial x_i$ to the first equation in (1.1), taking the inner product with $D_t u(\cdot, t)$ and integrating on $\mathbb{R}^3 \times [t_0, t]$, we get, summing over $1 \leq \ell \leq 3$,

$$
\| Du(\cdot, t) \|_{L^2(\mathbb{R}^3)}^2 + 2 \int_{t_0}^t \| D^2 u(\cdot, s) \|_{L^2(\mathbb{R}^3)}^2 \, ds = \leq \| Du(\cdot, t_0) \|_{L^2(\mathbb{R}^3)}^2 + 2 \int_{t_0}^t \| D^2 u(\cdot, s) \|_{L^2(\mathbb{R}^3)} \| Du(\cdot, s) \|_{L^2(\mathbb{R}^3)} \| D^2 u(\cdot, s) \|_{L^2(\mathbb{R}^3)} \, ds
$$

by (2.18), using (1.16) and (1.17). In particular, we have

$$
\| Du(\cdot, t) \|_{L^2(\mathbb{R}^3)}^2 + 2 \int_{t_0}^t \| D^2 u(\cdot, s) \|_{L^2(\mathbb{R}^3)}^2 \, ds \leq \leq \| Du(\cdot, t_0) \|_{L^2(\mathbb{R}^3)}^2 + 2 \int_{t_0}^t [\| u_0 \|_{L^2(\mathbb{R}^3)} \| Du(\cdot, s) \|_{L^2(\mathbb{R}^3)}]^{1/2} \| D^2 u(\cdot, s) \|_{L^2(\mathbb{R}^3)}^2 \, ds
$$

for all $t \geq t_0$. We then choose $t_0 \geq t_*$ such that, by (1.2): $\| u_0 \|_{L^2(\mathbb{R}^3)} \| Du(\cdot, t_0) \|_{L^2(\mathbb{R}^3)} < 1$. In fact, with this choice, it follows from (2.19) that

$$
\| u_0 \|_{L^2(\mathbb{R}^3)} \| Du(\cdot, s) \|_{L^2(\mathbb{R}^3)} < 1 \quad \forall \ s \geq t_0.
$$

[Proof of (2.20): if false, there would be $t_1 > t_0$ such that $\| u_0 \|_{L^2(\mathbb{R}^3)} \| Du(\cdot, s) \|_{L^2(\mathbb{R}^3)} < 1$ for all $t_0 \leq s < t_1$, while $\| u_0 \|_{L^2(\mathbb{R}^3)} \| Du(\cdot, t_1) \|_{L^2(\mathbb{R}^3)} = 1$. Taking $t = t_1$ in (2.19) above, this would give $\| Du(\cdot, t_1) \|_{L^2(\mathbb{R}^3)} \leq \| Du(\cdot, t_0) \|_{L^2(\mathbb{R}^3)}$, so that $\| u_0 \|_{L^2(\mathbb{R}^3)} \| Du(\cdot, t_1) \|_{L^2(\mathbb{R}^3)} \leq \| u_0 \|_{L^2(\mathbb{R}^3)} \| Du(\cdot, t_0) \|_{L^2(\mathbb{R}^3)} < 1$. This contradiction shows (2.20), as claimed. QED (2.20)]

From (2.19) and (2.20), it then follows that

$$
\| Du(\cdot, t) \|_{L^2(\mathbb{R}^3)}^2 + 2 \int_{t_0}^t \| D^2 u(\cdot, s) \|_{L^2(\mathbb{R}^3)}^2 \, ds \leq \| Du(\cdot, t_2) \|_{L^2(\mathbb{R}^3)}^2
$$

for all $t \geq t_2 \geq t_0$, where $\gamma := 1 - \| u(\cdot, t_0) \|_{L^2(\mathbb{R}^3)}^{1/2} \| Du(\cdot, t_0) \|_{L^2(\mathbb{R}^3)}^{1/2}$ is some positive constant. This shows the result, where $t_* = t_0$ with $t_0 \geq t_*$ as chosen in (2.20) above. \(\square\)
Remark 2.1. As shown in [12], one can readily obtain from the proof of Theorem 2.4 that one has $t_{ss} < 0.212 \cdot \| u_0 \|^4_{L^2(\mathbb{R}^3)}$ always. A more elaborated argument developed here in the Appendix produces the much sharper estimate

$$t_{ss} < 0.000753026 \cdot \| u_0 \|^4_{L^2(\mathbb{R}^3)},$$

(2.22)
giving a practical upper bound on how much one should wait in numerical experiments before we may witness any loss of regularity on the part of $u(\cdot, t)$, if this ever happens — in any case, we have $u \in C^\infty(\mathbb{R}^3 \times [t_{ss}, \infty))$. Other estimates for $t_{ss}$ have appeared in the literature, see e.g. [7, 14, 15, 20]; (2.22) is the sharpest of its kind. Whether one can really have $u \not\in C^\infty(\mathbb{R}^3 \times (0, \infty))$ for some Leray-Hopf’s solutions is not really known and remains one of the famous fundamental open questions regarding the Leray-Hopf’s solutions of the Navier-Stokes equations [5, 6, 9].

Our final basic result to be collected in this section for convenience of the reader is the following fundamental property, which is a direct consequence of (1.2), (2.3) and the monotonicity of $\| Du(\cdot, t) \|_{L^2(\mathbb{R}^3)}$ for large $t$, as given in Theorem 2.4 above.

Theorem 2.5. Let $u(\cdot, t)$, $t > 0$, be any particular Leray-Hopf’s solution to (1.1). Then

$$\lim_{t \to \infty} t^{1/2} \| Du(\cdot, t) \|_{L^2(\mathbb{R}^3)} = 0.$$  

(2.23)

Proof: The following argument is taken from [12], Lemma 2.1. If (2.23) were false, there would then exist an increasing sequence $t_\ell \nearrow \infty$ (with $t_\ell \geq t_{ss}$ and $t_\ell \geq 2t_{\ell - 1}$ for all $\ell$, say) and some fixed $\eta > 0$ such that

$$t_\ell \| Du(\cdot, t_\ell) \|^2_{L^2(\mathbb{R}^3)} \geq \eta \quad \forall \ell.$$ 

In particular, this would give

$$\int_{t_{\ell - 1}}^{t_\ell} \| Du(\cdot, s) \|^2_{L^2(\mathbb{R}^3)} \ dt \geq (t_\ell - t_{\ell - 1}) \| Du(\cdot, t_\ell) \|^2_{L^2(\mathbb{R}^3)} \geq \frac{1}{2} t_\ell \| Du(\cdot, t_\ell) \|^2_{L^2(\mathbb{R}^3)} \geq \frac{1}{2} \eta$$

for all $\ell$, in contradiction with (1.2), (2.3). This concludes the proof of (2.23), as claimed. $\square$
2. Proof of the $L^2$ results (1.4) and (1.8)

Now that the basic properties of Leray-Hopf’s solutions given above have been established, it becomes much easier to obtain estimates like (1.4), (1.5), (1.8) or (1.9). In this section, we consider (1.4) and (1.8). Let then $u(\cdot, t) \in L^\infty([0, \infty[, L^2_2(\mathbb{R}^3)) \cap L^2([0, \infty[, H^1(\mathbb{R}^3))$ be any such solution to the initial value problem (1.1a), (1.1b), and let $t_* \gg 1$ be large enough that (1.3) holds. Taking $t_0 \geq t_*$ (arbitrary), we thus have the representation

$$u(\cdot, t) = e^{\Delta(t-t_0)}u(\cdot, t_0) + \int_{t_0}^{t} e^{\Delta(t-s)}Q(\cdot, s) \, ds, \quad t \geq t_0,$$

by Duhamel’s principle, where $Q(\cdot, s)$ is defined in (2.4).

**Theorem 3.1 (Leray’s $L^2$ conjecture).** One has

$$\lim_{t \to \infty} \| u(\cdot, t) \|_{L^2(\mathbb{R}^3)} = 0.$$  \hspace{0.5cm} (3.2)

**Proof:** Given $\epsilon > 0$, let $t_0 \geq t_*$ be chosen large enough so that, by Theorem 2.5,

$$t^{1/2} \| Du(\cdot, t) \|_{L^2(\mathbb{R}^3)} \leq \epsilon \quad \forall \ t \geq t_0.$$  \hspace{0.5cm} (3.3)

From the representation (3.1) for $u(\cdot, t)$, this gives

$$\| u(\cdot, t) \|_{L^2(\mathbb{R}^3)} \leq \| e^{\Delta(t-t_0)}u(\cdot, t_0) \|_{L^2(\mathbb{R}^3)} + \int_{t_0}^{t} \| e^{\Delta(t-s)}Q(\cdot, s) \|_{L^2(\mathbb{R}^3)} \, ds$$

$$\leq \| e^{\Delta(t-t_0)}u(\cdot, t_0) \|_{L^2(\mathbb{R}^3)} + K \int_{t_0}^{t} (t-s)^{-3/4} \| u(\cdot, s) \|_{L^2(\mathbb{R}^3)} \| Du(\cdot, s) \|_{L^2(\mathbb{R}^3)} \, ds$$

$$\leq \| e^{\Delta(t-t_0)}u(\cdot, t_0) \|_{L^2(\mathbb{R}^3)} + K \| u_0 \|_{L^2(\mathbb{R}^3)} \int_{t_0}^{t} (t-s)^{-3/4} \| Du(\cdot, s) \|_{L^2(\mathbb{R}^3)} \, ds$$

$$\leq \| e^{\Delta(t-t_0)}u(\cdot, t_0) \|_{L^2(\mathbb{R}^3)} + K \| u_0 \|_{L^2(\mathbb{R}^3)} \epsilon \int_{t_0}^{t} (t-s)^{-3/4} s^{-1/2} \, ds$$  \hspace{0.5cm} [by (3.3)]

for all $t \geq t_0$, where $K = (8\pi)^{-3/4}$, and where we have used (1.2), (2.5a). Observing that

$$\int_{t_0}^{t} (t-s)^{-3/4} s^{-1/2} \, ds \leq 6 \sqrt{2} \quad \forall \ t \geq t_0 + 1,$$

we then have, for all $t \geq t_0 + 1$:

$$\| u(\cdot, t) \|_{L^2(\mathbb{R}^3)} \leq \| e^{\Delta(t-t_0)}u(\cdot, t_0) \|_{L^2(\mathbb{R}^3)} + \| u_0 \|_{L^2(\mathbb{R}^3)} \epsilon.$$
Theorem 3.2. Given any $t_0 \geq 0$, one has
\[
\lim_{t \to \infty} t^{1/4} \| u(\cdot, t) - e^{\Delta(t-t_0)} u(\cdot, t_0) \|_{L^2(\mathbb{R}^3)} = 0. \tag{3.4}
\]

Proof: By Theorem 2.2, it is sufficient to show (3.4) in the case $t_0 \geq t_\epsilon$, where (3.1) holds. Given $\epsilon > 0$, let $t_\epsilon > t_0$ be large enough that we have, using Theorem 2.5 again,

\[
t^{1/2} \| Du(\cdot, t) \|_{L^2(\mathbb{R}^3)} \leq \epsilon \quad \forall t \geq t_\epsilon. \tag{3.5}
\]

By (3.1) and (1.2), (2.5a), we then get
\[
t^{1/4} \| u(\cdot, t) - e^{\Delta(t-t_0)} u(\cdot, t_0) \|_{L^2(\mathbb{R}^3)} \leq t^{1/4} \int_{t_0}^{t} \| e^{\Delta(t-s)} Q(\cdot, s) \|_{L^2(\mathbb{R}^3)} ds
\leq I(t, t_\epsilon) + K t^{1/4} \int_{t_\epsilon}^{t} (t-s)^{-3/4} \| u(\cdot, s) \|_{L^2(\mathbb{R}^3)} \| Du(\cdot, s) \|_{L^2(\mathbb{R}^3)} ds
\leq I(t, t_\epsilon) + K \| u_0 \|_{L^2(\mathbb{R}^3)} \epsilon t^{1/4} \int_{t_\epsilon}^{t} (t-s)^{-3/4} s^{-1/2} ds \quad [\text{by (3.5)}]
\leq I(t, t_\epsilon) + 0.636 \| u_0 \|_{L^2(\mathbb{R}^3)} \epsilon t^{1/4} (t-t_\epsilon)^{-1/4}
\]
for all $t > t_\epsilon$, where $K = (8\pi)^{-3/4}$ and
\[
I(t, t_\epsilon) = K t^{1/4} \int_{t_0}^{t_\epsilon} (t-s)^{-3/4} \| u(\cdot, s) \|_{L^2(\mathbb{R}^3)} \| Du(\cdot, s) \|_{L^2(\mathbb{R}^3)} ds
\leq K t^{1/4} (t-t_\epsilon)^{-3/4} \int_{t_0}^{t_\epsilon} \| u(\cdot, s) \|_{L^2(\mathbb{R}^3)} \| Du(\cdot, s) \|_{L^2(\mathbb{R}^3)} ds.
\]

Therefore, we obtain
\[
t^{1/4} \| u(\cdot, t) - e^{\Delta(t-t_0)} u(\cdot, t_0) \|_{L^2(\mathbb{R}^3)} \leq (1 + \| u_0 \|_{L^2(\mathbb{R}^3)}) \epsilon
\]
for all $t \gg 1$. This gives (3.4), and our $L^2$ discussion is now complete, as claimed. □
4. Proof of the supnorm results (1.5) and (1.9)

In this section, we follow a similar path to obtain the more delicate supnorm estimates (1.5) and (1.9). Let then \( u(\cdot, t) \in L^\infty([0, \infty[, L^2_0(\mathbb{R}^3)) \cap L^2([0, \infty[, H^1(\mathbb{R}^3)) \) be any given Leray-Hopf’s solution to the Cauchy problem (1.1). Again, we take advantage of the strong regularity properties of \( u(\cdot, t) \) for \( t \geq t_* \gg 1 \) (see (1.3) above), using the representation (3.1) and the fundamental results (2.5b), (2.23) and (3.2) already obtained.

**Theorem 4.1.** One has

\[
\lim_{t \to \infty} t^{3/4} \| u(\cdot, t) \|_{L^\infty(\mathbb{R}^3)} = 0. \tag{4.1}
\]

**Proof:** Given \( 0 < \epsilon \leq 1/2 \), let \( t_0 \geq t_* \) be large enough that, by (2.23) and (3.2) above, we have

\[
t^{1/2} \| Du(\cdot, t) \|_{L^2(\mathbb{R}^3)} \leq \epsilon \quad \forall \ t \geq t_0 \tag{4.2a}
\]
and

\[
\| u(\cdot, t) \|_{L^2(\mathbb{R}^3)} \leq \epsilon \quad \forall \ t \geq t_0. \tag{4.2b}
\]

From the representation (3.1) for \( u(\cdot, t) \), we obtain, by (2.5b) and (4.2a),

\[
\| u(\cdot, t) \|_\infty \leq \| e^{\Delta(t-t_0)} u(\cdot, t_0) \|_\infty + \int_{t_0}^t \| e^{\Delta(t-s)} Q(\cdot, s) \|_\infty ds
\]
\[
\leq \| e^{\Delta(t-t_0)} u(\cdot, t_0) \|_\infty + K \int_{t_0}^t (t-s)^{-3/4} \| u(\cdot, s) \|_\infty \| Du(\cdot, s) \|_{L^2(\mathbb{R}^3)} ds
\]
\[
\leq K (t-t_0)^{-3/4} \| u(\cdot, t_0) \|_{L^2(\mathbb{R}^3)} + K \epsilon \int_{t_0}^t (t-s)^{-3/4} s^{-1/2} \| u(\cdot, s) \|_\infty ds \tag{4.3}
\]
for all \( t > t_0 \), where \( K = (8\pi)^{-3/4} \). That is,

\[
\| u(\cdot, t) \|_\infty \leq K (t-t_0)^{-3/4} \| u(\cdot, t_0) \|_{L^2(\mathbb{R}^3)} + K \epsilon \int_{t_0}^t (t-s)^{-3/4} s^{-1/2} \| u(\cdot, s) \|_\infty ds \tag{4.3}
\]
for all \( t > t_0 \). We claim that this gives

\[
t^{3/4} \| u(\cdot, t) \|_\infty < \epsilon \quad \forall \ t \geq 2(t_0+1), \tag{4.4}
\]
which implies (4.1). In what follows, we will prove (4.4) above. Given any \( \tilde{t} \geq 2(t_0+1) \) (fixed, but otherwise arbitrary), let \( \tilde{t}_1 := \tilde{t}/2 \). Setting

\[
U(t) := (t-\tilde{t}_1)^{3/4} \| u(\cdot, t) \|_\infty, \quad t \geq \tilde{t}_1, \tag{4.5}
\]
we obtain, applying (4.3) [with \( t_0 = \hat{t}_1 \) there]:

\[
U(t) \leq K \| \mathbf{u}(\cdot, \hat{t}_1) \|_{L^2(\mathbb{R}^3)} + K \epsilon (t - \hat{t}_1)^{3/4} \int_{\hat{t}_1}^{t} (t-s)^{-3/4} s^{-1/2} \| \mathbf{u}(\cdot, s) \|_{\infty} ds
\]

\[
= K \| \mathbf{u}(\cdot, \hat{t}_1) \|_{L^2(\mathbb{R}^3)} + K \epsilon (t - \hat{t}_1)^{3/4} \int_{\hat{t}_1}^{t} (t-s)^{-3/4} s^{-1/2} (s - \hat{t}_1)^{-3/4} U(s) ds
\]

\[
\leq K \| \mathbf{u}(\cdot, \hat{t}_1) \|_{L^2(\mathbb{R}^3)} + K \epsilon \hat{t}_1^{-1/2} (t - \hat{t}_1)^{3/4} \int_{\hat{t}_1}^{t} (t-s)^{-3/4} (s - \hat{t}_1)^{-3/4} U(s) ds
\]

for \( t \geq \hat{t}_1 \), so that, by (4.2b), we have

\[
U(t) \leq K \epsilon + K \epsilon \hat{t}_1^{-1/2} (t - \hat{t}_1)^{3/4} \int_{\hat{t}_1}^{t} (t-s)^{-3/4} (s - \hat{t}_1)^{-3/4} U(s) ds, \quad t \geq \hat{t}_1,
\]

where \( K = (8\pi)^{-3/4} \). Thus, setting \( \hat{\mathbf{U}} := \max \{ U(t) : \hat{t}_1 \leq t \leq \hat{t} \} \), we get, for \( \hat{t}_1 \leq t \leq \hat{t} \):

\[
U(t) \leq K \epsilon + K \epsilon \hat{t}_1^{-1/2} (t - \hat{t}_1)^{3/4} \hat{\mathbf{U}} \int_{\hat{t}_1}^{t} (t-s)^{-3/4} (s - \hat{t}_1)^{-3/4} ds
\]

\[
\leq K \epsilon + 8 \sqrt{2} K \epsilon \hat{t}_1^{-1/2} (t - \hat{t}_1)^{1/4} \hat{\mathbf{U}}
\]

\[
\leq K \epsilon + 8 \sqrt{2} K \epsilon \hat{t}_1^{-1/4} \hat{\mathbf{U}}.
\]

Recalling that \( \hat{t}_1 \geq 1, \epsilon \leq 1/2 \), we then get \( U(t) \leq K \epsilon + 8 \sqrt{2} K \epsilon \hat{\mathbf{U}} \leq K \epsilon + 0.504 \hat{\mathbf{U}} \), for each \( \hat{t}_1 \leq t \leq \hat{t} \). This gives \( \hat{\mathbf{U}} \leq K \epsilon + 0.504 \hat{\mathbf{U}} \), that is, \( \hat{\mathbf{U}} < 0.180 \epsilon \). In particular,

\[
\left( \frac{\hat{t}}{2} \right)^{3/4} \| \mathbf{u}(\cdot, \hat{t}) \|_{\infty} = U(\hat{t}) \leq \hat{\mathbf{U}} < 0.180 \epsilon,
\]

so that \( \hat{t}^{3/4} \| \mathbf{u}(\cdot, \hat{t}) \|_{\infty} < \epsilon \), where \( \hat{t} \geq 2 (t_0 + 1) \) is arbitrary. This shows (4.4), as claimed, and the proof of (4.1) is now complete. \( \square \)

In a similar way, (1.9) can be obtained, as shown next.

**Theorem 4.2.** Given any \( t_0 \geq 0 \), one has

\[
\lim_{t \to \infty} t \| \mathbf{u}(\cdot, t) - e^{\Delta(t-t_0)} \mathbf{u}(\cdot, t_0) \|_{L^\infty(\mathbb{R}^3)} = 0. \tag{4.6}
\]

**Proof:** By Theorem 2.2, it is sufficient to show (4.6) in the case \( t_0 \geq t_s \), where (3.1) holds. Given \( 0 < \epsilon \leq 1 \), let \( t_\epsilon > t_0 \) be large enough that we have, from Theorems 2.5 and 4.1,

\[
t^{1/2} \| D\mathbf{u}(\cdot, t) \|_{L^2(\mathbb{R}^3)} \leq \epsilon \quad \forall \ t \geq t_\epsilon, \tag{4.7a}
\]

and

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\( t^{3/4} \| u(\cdot, t) \|_\infty \leq \epsilon \quad \forall \ t \geq t_c. \) (4.7b)

From (3.1), we have

\[
\begin{align*}
t \| u(\cdot, t) - e^{\Delta (t-t_0)} u(\cdot, t_0) \|_\infty & \leq t \int_{t_0}^t \| e^{\Delta (t-s)} Q(\cdot, s) \|_\infty ds \\
& \leq J_1(t) + J_2(t) + J_3(t) \quad (4.8)
\end{align*}
\]

for all \( t > t_c \), where

\[
J_1(t) = t \int_{t_0}^t \| e^{\Delta (t-s)} Q(\cdot, s) \|_\infty ds
\]

\[
\leq (4\pi)^{-3/4} t \int_{t_0}^t (t-s)^{-3/4} \| e^{\frac{1}{2} \Delta (t-s)} Q(\cdot, s) \|_{L^2(\mathbb{R}^3)} ds
\]

\[
\leq (4\pi)^{-3/2} t \int_{t_0}^t (t-s)^{-3/2} \| u(\cdot, s) \|_{L^2(\mathbb{R}^3)} \| Du(\cdot, s) \|_{L^2(\mathbb{R}^3)} ds
\]

\[
\leq (4\pi)^{-3/2} t (t-t_c)^{-3/2} \int_{t_0}^t \| u(\cdot, s) \|_{L^2(\mathbb{R}^3)} \| Du(\cdot, s) \|_{L^2(\mathbb{R}^3)} ds, \quad (4.9a)
\]

by (2.5a), and

\[
J_2(t) = t \int_{t_c}^{\mu(t)} \| e^{\Delta (t-s)} Q(\cdot, s) \|_\infty ds
\]

\[
\leq (4\pi)^{-3/4} t \int_{t_c}^{\mu(t)} (t-s)^{-3/4} \| e^{\frac{1}{2} \Delta (t-s)} Q(\cdot, s) \|_{L^2(\mathbb{R}^3)} ds
\]

\[
\leq (4\pi)^{-3/2} t \int_{t_c}^{\mu(t)} (t-s)^{-3/2} \| u(\cdot, s) \|_{L^2(\mathbb{R}^3)} \| Du(\cdot, s) \|_{L^2(\mathbb{R}^3)} ds
\]

\[
\leq (2\pi)^{-3/2} t (t-t_c)^{-3/2} e^{2 \int_{t_c}^{\mu(t)} s^{-1/2} ds} < 0.090 \epsilon t (t-t_c)^{-1}, \quad (4.9b)
\]

by (2.5a) and (4.7), where \( \mu(t) = (t+t_c)/2 \),

\[
J_3(t) = t \int_{\mu(t)}^t \| e^{\Delta (t-s)} Q(\cdot, s) \|_\infty ds
\]

\[
\leq (8\pi)^{-3/4} t \int_{\mu(t)}^t (t-s)^{-3/4} \| u(\cdot, s) \|_{\infty} \| Du(\cdot, s) \|_{L^2(\mathbb{R}^3)} ds
\]

\[
\leq (8\pi)^{-3/4} t e^{2 \int_{\mu(t)}^t (t-s)^{-3/4} s^{-5/4} ds} < 0.713 \epsilon t (t-t_c)^{-1}, \quad (4.9c)
\]

by (2.5b) and (4.7). Therefore, from (4.8) and (4.9) above, we obtain

\[
t \| u(\cdot, t) - e^{\Delta (t-t_0)} u(\cdot, t_0) \|_\infty < \epsilon \quad \forall \ t \gg 1. \quad (4.10)
\]

This shows (4.6), since \( \epsilon \in (0,1] \) is arbitrary. \( \square \)
**APPENDIX**

Here we show how to obtain the estimate (2.22) given in Section 2. The starting point is the following inequality,

\[
\int \left\{ \sum_{i,j,\ell=1}^{3} |D_{\ell} u_i| |D_{\ell} u_j| |D_{\ell} u_i| \right\} dx \leq K_3^3 \| D u \|^{3/2}_{L^2(\mathbb{R}^3)} \| D^2 u \|^{3/2}_{L^2(\mathbb{R}^3)},
\]

(A.1)

where \( K_3 < 0.581 \) (see Theorem 2.1) is the constant in the Gagliardo-Nirenberg inequality \( \| u \|_{L^1(\mathbb{R}^3)} \leq K_3 \| u \|^{1/2}_{L^2(\mathbb{R}^3)} \| Du \|^{1/2}_{L^2(\mathbb{R}^3)} \). In fact, by repeated application of the Cauchy-Schwarz inequality, we get

\[
\int \left\{ \sum_{i,j,\ell=1}^{3} |D_{\ell} u_i| |D_{\ell} u_j| |D_{\ell} u_i| \right\} dx \leq \| v \|^{3/2}_{L^1(\mathbb{R}^3)} \| D u \|^{3/2}_{L^2(\mathbb{R}^3)} \| D^2 u \|^{3/2}_{L^2(\mathbb{R}^3)}
\]

This gives

\[
\int \left\{ \sum_{i,j,\ell=1}^{3} |D_{\ell} u_i| |D_{\ell} u_j| |D_{\ell} u_i| \right\} dx \leq K_3^3 \| D u \|^{3/2}_{L^2(\mathbb{R}^3)} \| D^2 u \|^{3/2}_{L^2(\mathbb{R}^3)}
\]

by (1.16), as claimed. Now, consider \( \hat{t} > 0 \) satisfying

\[
\hat{t} > \frac{1}{2} K_3^{12} \| u_0 \|^4_{L^2(\mathbb{R}^3)}:
\]

(A.2)

Because (by (1.2)) \( \int_0^\hat{t} \| D u(\cdot, t) \|_{L^2(\mathbb{R}^3)}^2 \leq \frac{1}{2} \| u_0 \|_{L^2(\mathbb{R}^3)}^2 \) there exists \( t' \in (0, \hat{t}) \) so that

\[
\| D u(\cdot, t') \|_{L^2(\mathbb{R}^3)} \leq \| u_0 \|_{L^2(\mathbb{R}^3)} \cdot \frac{1}{\sqrt{2t'}}
\]

(A.3)

Hence, by (A.2), we have \( K_3^3 \| u(\cdot, s) \|^{1/2}_{L^2(\mathbb{R}^3)} \| D u(\cdot, s) \|^{1/2}_{L^2(\mathbb{R}^3)} < 1 \) for all \( s \geq t' \) close to \( t' \). From (1.1), we then get

\[
\| D u(\cdot, t) \|_{L^2(\mathbb{R}^3)}^2 + 2 \int_{t'}^t \| D^2 u(\cdot, s) \|_{L^2(\mathbb{R}^3)}^2 ds
\]

\[
\leq \| D u(\cdot, t') \|_{L^2(\mathbb{R}^3)}^2 + 2 \sum_{i,j,\ell} \int_{t'}^t |D_{\ell} u_i(x, s)| |D_{\ell} u_j(x, s)| |D_{\ell} u_i(x, s)| dx ds
\]

\[
\leq \| D u(\cdot, t') \|_{L^2(\mathbb{R}^3)}^2 + 2 \int_{t'}^t K_3^3 \| D u(\cdot, s) \|^{3/2}_{L^2(\mathbb{R}^3)} \| D^2 u(\cdot, s) \|^{3/2}_{L^2(\mathbb{R}^3)} ds
\]

\[
\leq \| D u(\cdot, t') \|_{L^2(\mathbb{R}^3)}^2 + 2 \int_{t'}^t K_3^3 \| u(\cdot, s) \|^{1/2}_{L^2(\mathbb{R}^3)} \| D u(\cdot, s) \|^{1/2}_{L^2(\mathbb{R}^3)} \| D^2 u(\cdot, s) \|^2_{L^2(\mathbb{R}^3)} ds
\]

(A.4)

for all \( t \geq t' \) close to \( t' \). As in the proof of Theorem 2.4, this gives
\[ K_3^3 \| u(\cdot, t) \|_{L^2(\mathbb{R}^3)}^{1/2} \| Du(\cdot, t) \|_{L^2(\mathbb{R}^3)}^{1/2} < 1, \quad \forall \ t \geq t' \]  
\[ (A.5) \]

and, in particular, as in (A.4) above, we have

\[
\| Du(\cdot, t) \|_{L^2(\mathbb{R}^3)}^2 + 2 \int_{t_0}^t \| D^2 u(\cdot, s) \|_{L^2(\mathbb{R}^3)}^2 \, ds \\
\leq \| Du(\cdot, t_0) \|_{L^2(\mathbb{R}^3)}^2 + 2 \int_{t_0}^t \left[ K_3^3 \| u(\cdot, s) \|_{L^2(\mathbb{R}^3)}^{1/2} \| Du(\cdot, s) \|_{L^2(\mathbb{R}^3)}^{1/2} \right] \| D^2 u(\cdot, s) \|_{L^2(\mathbb{R}^3)}^2 \, ds
\]

for any \( t > t_0 \geq t' \), that is, \( \| Du(\cdot, t) \|_{L^2(\mathbb{R}^3)} \) is monotonically decreasing in \( [t', \infty) \supseteq [\hat{t}, \infty) \), so that, by Leray’s theory, \( u \) must be \( C^\infty \) for \( t > t' \). Recalling (A.2), this completes the proof of (2.22), since \( 1/2 \cdot K_3^{12} < 0.000753026 \). Summarizing it all, we have shown:

**Theorem A.1.** Let \( u_0 \in L^2(\mathbb{R}^3) \), and let \( u(\cdot, t) \) be any Leray-Hopf’s solution of (1.1). Then there exists \( 0 \leq t_{ss} < 0.000753026 \) \( \| u_0 \|_{L^2(\mathbb{R}^3)}^4 \) such that \( u \in C^\infty(\mathbb{R}^3 \times [t_{ss}, \infty)) \) and \( \| Du(\cdot, t) \|_{L^2(\mathbb{R}^3)} \) is finite and monotonically decreasing everywhere in \( [t_{ss}, \infty) \).

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LINEIA SCHÜTZ
Departamento de Matemática Pura e Aplicada
Universidade Federal do Rio Grande do Sul
Porto Alegre, RS 91509-900, Brazil
E-mail: lineia.schutz@ufrgs.br,
       lineiaschutz@yahoo.com.br

JANAÍNA PIRES ZINGANO
Departamento de Matemática Pura e Aplicada
Universidade Federal do Rio Grande do Sul
Porto Alegre, RS 91509-900, Brazil
E-mail: janaina.zingano@ufrgs.br,
       jzingano@gmail.com

PAULO RICARDO DE AVILA ZINGANO
Departamento de Matemática Pura e Aplicada
Universidade Federal do Rio Grande do Sul
Porto Alegre, RS 91509-900, Brazil
E-mail: paulo.zingano@ufrgs.br,
       zingano@gmail.com