Simple anomaly-free \(U(1)\) extensions of the Standard Model

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Introduction

Particle physics have problems that can be solved using gauged \(U(1)\) extensions of the Standard Model. Some examples are dark matter \([1, 10]\), neutrino masses \([11, 13]\), strong CP problem \([13, 15]\) and the anomalous magnetic moment of the muon \([16, 18]\). However, in order to be well behaved at high energies a gauge symmetry must be free of anomalies \([19–22]\).

In this paper we will construct some simple anomaly-free \(U(1)\) extensions of the Standard Model. Mainly due to the large number of different extensions and the availability of free parameters, they may be useful for other model builders interested in their use to solve problems of particle physics.

Anomaly-free Abelian extensions

Let \([z_{ij}]\) denote the charges of the Standard Model fermions under a \(U(1)\) gauge symmetry, where \(i \in \{1, 2, 3\}\) refers to the three families and \(j \in \{1, 2, 3, 4, 5\}\) refers to the fermion multiplets. Following the conventions of previous work \([23, 24]\), we will assume that the fermions are taken as lefthanded fields and that the charges \([z_{ij}]\) are given for a normalization of the \(U(1)\) gauge coupling where all charges are integers with no common divisor and the biggest charge, in absolute value, is positive.

With this notation and convention the hypercharges \([y_{ij}]\) of the Standard Model fermions, refered from here on with the letter \(y\), are \(y_1 = 6, y_2 = -3, y_3 = -4, y_4 = 2\) and \(y_5 = 1\) for all \(i \in \{1, 2, 3\}\). We can also present the hypercharges as a triple of row vectors \(y_j = [6, -4, -3, 2, 1]\) or more explicitly in matrix form:

\[
[y_{ij}] = \begin{bmatrix}
y_j \\
y_j \\
y_j
\end{bmatrix} = \begin{bmatrix} E & U & L & D & Q \end{bmatrix} \begin{bmatrix}
6 & -4 & -3 & 2 & 1 \\
6 & -4 & -3 & 2 & 1 \\
6 & -4 & -3 & 2 & 1
\end{bmatrix} \quad \text{1st}
\end{bmatrix} \begin{bmatrix}
6 & -4 & -3 & 2 & 1 \\
6 & -4 & -3 & 2 & 1 \\
6 & -4 & -3 & 2 & 1
\end{bmatrix} \quad \text{2nd}
\end{bmatrix} \begin{bmatrix}
6 & -4 & -3 & 2 & 1 \\
6 & -4 & -3 & 2 & 1 \\
6 & -4 & -3 & 2 & 1
\end{bmatrix} \quad \text{3rd}
\]

A \(U(1)\) extension of the Standard Model is anomaly-free if the charges \([z_{ij}]\) associated with it satisfy the following system of diophantine equations:

\[
0 = \sum_{i,j} g_j z_{ij}^3,
\]

\[
0 = \sum_{i,j} g_j z_{ij} y_{ij}^2,
\]

\[
0 = \sum_{i,j} g_j z_{ij} y_{ij},
\]

\[
0 = \sum_{i,j} \delta_j^\text{grav} g_j z_{ij},
\]

\[
0 = \sum_{i,j} \delta_j^\text{su(2)} g_j z_{ij},
\]

\[
0 = \sum_{i,j} \delta_j^\text{su(3)} g_j z_{ij},
\]

where \(g_j\) is the number of Weyl fermions in the \(j\) multiplet, and \(\delta_j^s\) is equal 1 if multiplet \(j\) is charged under \(s\) and 0 otherwise, with \(s \in \{\text{grav, su(2), su(3)}\}\). More generally, given \(m \in \mathbb{N}\), a \(U(1)_1 \times \cdots \times U(1)_m\) extension of the Standard Model is anomaly-free if the charges \([z_{ij}]\) associated with \(U(1)_\ell\) satisfy \((2)-(7)\) and

\[
0 = \sum_{i,j} g_j z_{ij}^{\ell \ell'} z_{ij}^{\ell''}
\]

for all \(\ell, \ell', \ell'' \in \{1, ..., m\}\).

Recently, an atlas with solutions for the anomaly equations \((2)-(7)\) was constructed \([23]\). Then a general solution was found for the first two equations \([23]\), and for this solution it was given a geometric interpretation \([20]\). While the present work was being written, the full system of equations was solved using this geometric method \([27]\). Other aspects of this system of equations including \(\delta_j\) where also explored \([24, 28]\).

Some simple anomaly-free Abelian extensions

In the Standard Model, hypercharge is anomaly-free for each family, which means that \([y_{ij}]\) satisfy

\[
0 = \sum_j g_j y_{ij}^3,
\]

\[
0 = \sum_j \delta_j y_{ij},
\]
for each $i \in \{1, 2, 3\}$ and $s \in \{\text{grav}, su(2), su(3)\}$. Therefore, we can take a different multiple of it for each family of fermions to obtain a simple anomaly-free $U(1)_s$ extensions of the Standard Model. To be precise, the $U(1)_s$ extension associated with the charges $[z_{ij}] = [k_i y_{ij}]$ since

$$ [k_i y_{ij}] = \begin{bmatrix} k_1 y_{ij} \\ k_2 y_{ij} \\ k_3 y_{ij} \end{bmatrix}, \quad (11) $$

for arbitrary $k_1, k_2, k_3 \in \mathbb{Z}$, is anomaly-free.

It’s easy to check this statement for each anomaly equation (2)-(7). For example, equation (4) is satisfied by $[z_{ij}] = [k_i y_{ij}]$ since

$$ \sum_{i,j} g_j z_{ij}^2 y_{ij} = \sum_i k_i^2 \sum_j g_j y_{ij}^2 = 0, \quad (12) $$

where we used equation (9). In a similar way we can show that the other anomaly equations (2)-(7) are satisfied.

More generally, for any $m \in \mathbb{N}$ the charges (11) can be used to construct an anomaly-free $U(1)_s$ extension of the Standard Model with the $U(1)_s$ charges given by $[k_i y_{ij}]$ with $k_1, k_2$ and $k_3$ arbitrary integers for each $\ell \in \{1, 2, ..., m\}$. It’s also easy to check this statement for $[z_{ij}^\ell] = [k_i^\ell z_{ij}], [z_{ij}^{\ell'}] = [k_i^{\ell'} z_{ij}]$ and $[z_{ij}^{\ell''}] = [k_i^{\ell''} z_{ij}]$ equation (8) gives

$$ \sum_{i,j} g_j k_i^\ell y_{ij} k_i^{\ell'} y_{ij} k_i^{\ell''} y_{ij} = \sum_i k_i^{\ell'} k_i^{\ell''} k_i^{\ell''} \sum_j g_j y_{ij}^2 = 0 \quad (13) $$

for all $\ell, \ell', \ell'' \in \{1, 2, ..., m\}$.

The charges (11) explicitly shows that the space of flavor-dependent anomaly-free $U(1)_s$ extensions of the Standard Model is infinite, since (11) is actually an infinite class of different charges which are determined by the three arbitrary integers $k_1, k_2$ and $k_3$. The availability of these free parameters may be useful because they can be properly chosen to address particular problems of particle physics.

If we further extend the Standard Model with right-handed neutrinos, referred by $j = 6$, the collection of charges $[b_j] = [3, -1, -3, -1, 1, 3]$ corresponding to $U(1)_{B-L}$ is another solutions to the system of anomaly equations for each family. In this case, the combinations

$$ \begin{bmatrix} k_1 y_{ij} + z_N \delta_{ij} \\ k_2 y_{ij} - z_N \delta_{ij} \\ k_3 y_{ij} \end{bmatrix}, \begin{bmatrix} k_1 y_{ij} + z_N \delta_{ij} \\ k_2 y_{ij} - z_N \delta_{ij} \\ k_3 y_{ij} \end{bmatrix}, \begin{bmatrix} k_1 y_{ij} \\ k_2 y_{ij} \\ k_3 y_{ij} \end{bmatrix}, \begin{bmatrix} k_1 b_j \\ k_2 b_j \\ k_3 b_j \end{bmatrix}, \begin{bmatrix} k_1 b_j \\ k_2 b_j \\ k_3 b_j \end{bmatrix}, $$

(14)

with $y_6 \equiv 0$, $z_N \in \mathbb{Z}$ and $\delta_{ij}$ the usual Kronecker delta, are also anomaly-free $U(1)_s$ extensions of the Standard Model. Note that we added opposite charges $z_N$ and $-z_N$ for the pair of right-handed neutrinos in the first and second combinations of (14). This addition does not spoil anomaly cancellations since they form a vector-like pair. These extensions are simple in the sense that anomalies are cancelled for each family.\[10\]

**Generating new anomaly-free $U(1)_s$ extensions**

Given a collection of charges $[z_{ij}]$ and $\sigma = (\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5) \in S_5^3$ where each $\sigma_j \in S_3$ is a permutation of $\{1, 2, 3\}$, we define the family-permutation of $[z_{ij}]$ by $\sigma$ as the new collection

$$ \sigma [z_{ij}] \equiv [z_{ij}] = [z_{\sigma_j(i),j}] \quad (15) $$

In words, the family-permutation of $[z_{ij}]$ permutes the charges of each multiplet among the families in an independent way, that is, $\sigma_j$ may be different from $\sigma_k$ if $j \neq k$. With this definition, the family universality of the hypercharge can be stated as

$$ [y_{ij}] = \sigma [y_{ij}] \quad (16) $$

for all $\sigma \in S_5^3$.

Concerning anomaly-free $U(1)_s$ extensions of the Standard Model we have the following proposition: by permuting the charges of fermions of same type but different families we can generate a new anomaly-free $U(1)_s$ extension from a given one. Formally, if $[z_{ij}]$ satisfy the anomaly-equations (2)-(7), then $\sigma [z_{ij}]$ also satisfy the anomaly equations for all $\sigma \in S_5^3$.

It is straightforward to verify this proposition. For example, the charges $[z_{ij}]$ satisfy the anomaly equation (8) then the charges $[z_{ij}'] = \sigma [z_{ij}]$ satisfy the same equation since

$$ \sum_{i,j} g_j z_{ij}' y_{ij}^2 = \sum_{i,j} g_j z_{\sigma_j(i),j} y_{\sigma_j(i),j}^2 = 0, \quad (17) $$

where we used the fact that hypercharge is family universal. In a similar way we can show that the other anomaly equations (2)-(7) are satisfied by $\sigma [z_{ij}]$.

Note that when we are interested in family-universal $U(1)_s$ extensions, this proposition is of no interest since $\sigma [z_{ij}] = [z_{ij}]$ for all $\sigma \in S_5^3$. However, if we have a non-family-universal anomaly-free $U(1)_s$ extension, then in general $\sigma [z_{ij}] \neq [z_{ij}]$, hence we are able to obtain different anomaly-free extensions by doing family-permutations. To be precise, for each anomaly-free $U(1)_s$ extension there is a total of $|S_5^3| = 6^5 = 7776$ other anomaly-free extensions. If we further add one right-handed neutrino for each family then we will have $|S_5^3| = 46656$ instead. In particular, by applying family-permutation in (14) we can generate $4 \times |S_5^3| = 186624$ different classes of anomaly-free $U(1)_s$ extensions of the Standard Model, the first two with 4 integer parameters that can be freely chosen and the last two with 3.
Conclusions

In this paper we constructed some simple anomaly-free $U(1)$ extensions of the Standard Model. The availability of free parameters in (14) and the possibility of generating new extensions by making family-permutations (15) may help other model builders interested in their use to solve problems of particle physics.

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