Non shifted calculus of variations on time scales with \( \nabla \)-differentiable \( \sigma \)

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Abstract

In calculus of variations on general time scales, an integral Euler-Lagrange equation is usually derived in order to characterize the critical points of non shifted Lagrangian functionals, see e.g. [R.A.C. Ferreira and co-authors, Optimality conditions for the calculus of variations with higher-order delta derivatives, Appl. Math. Lett., 2011].

In this paper, we prove that the \( \nabla \)-differentiability of the forward jump operator \( \sigma \) is a sharp assumption in order to obtain an Euler-Lagrange equation of differential form. Furthermore, this differential form allows us to prove a Noether-type theorem providing an explicit constant of motion for differential Euler-Lagrange equations admitting a symmetry.

Keywords: Time scale; calculus of variations; Euler-Lagrange equations; Noether’s theorem.

AMS Classification: 34N05; 39A12; 39A13; 39A10.

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1 Introduction

The time scale theory was introduced by S. Hilger in his PhD thesis [19] in 1988 in order to unify discrete and continuous analysis. The general idea is to extend classical theories on an arbitrary non empty closed subset \( \mathbb{T} \) of \( \mathbb{R} \). Such a closed subset \( \mathbb{T} \) is called a time scale. In this paper, it is supposed bounded with \( a = \min \mathbb{T} \) and \( b = \max \mathbb{T} \). Hence, the time scale theory establishes the validity of some results both in the continuous case \( \mathbb{T} = [a, b] \) and in the purely discrete case \( \mathbb{T} = \{a = t_0 < t_1 < \ldots < t_N = b\} \). Moreover, it also treats more general models involving both continuous and discrete time elements, see e.g. [15] [29] for dynamical population whose

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generations do not overlap. Another example of application is to consider \( \mathbb{T} = \{a\} \cup \{a + \lambda^N\} \) with \( 0 < \lambda < 1 \) allowing the time scale theory to cover the quantum calculus [29].

Since S. Hilger defined the \( \Delta \)- and \( \nabla \)-derivatives on time scales, many authors have extended to time scales various results from the continuous or discrete standard calculus theory. We refer to the surveys [1,2,10,11] of M. Bohner et al. In the continuous case \( \mathbb{T} = [a, b] \), let us mention that the operators \( \Delta \) and \( \nabla \) coincide with the usual derivative operator \( d/dt \). In the discrete case \( \text{card}(\mathbb{T}) < \infty \), \( \Delta \) is the usual forward Euler approximation of \( d/dt \), i.e. \( \Delta u(t_k) = (u(t_{k+1}) - u(t_k))/(t_{k+1} - t_k) \) and similarly, \( \nabla \) is the usual backward Euler approximation of \( d/dt \), i.e. \( \nabla u(t_k) = (u(t_k) - u(t_{k-1}))/\langle t_k - t_{k-1} \rangle \).

**Context in shifted calculus of variations.** The pioneering work on calculus of variations on general time scales is due to M. Bohner in [8]. In particular, he obtains a necessary condition for local optimizers of Lagrangian functionals of type

\[
\mathcal{L}(u) = \int_a^b L(u(\tau), u(\Delta(\tau)), \tau) \Delta \tau, \tag{1}
\]

where \( u^\sigma = u \circ \sigma \) (\( \sigma \) is the forward shift operator explicitly defined in Section [11]), \( u^\Delta \) is the \( \Delta \)-differential of \( u \) and \( \int \Delta \tau \) is the Cauchy \( \Delta \)-integral defined in [10] p.26.

Precisely, M. Bohner characterizes the critical points of \( \mathcal{L} \) as the solutions of the following \( \Delta \circ \Delta \)-differential Euler-Lagrange equation, see [8] Theorem 4.2:

\[
\left[ \frac{\partial L}{\partial u^\sigma(u^\sigma, u^\Delta, \cdot)} \right]^\Delta (t) = \frac{\partial L}{\partial x}(u^\sigma(t), u^\Delta(t), t). \tag{EL_{\Delta\circ\Delta}^{\Delta\circ\Delta}}
\]

Here, the notation \( \Delta \circ \Delta \) refers to the composition of \( \Delta \) with itself in the left-hand term of \( \text{EL}_{\Delta\circ\Delta}^{\Delta\circ\Delta} \).

As it is mentioned in [8], this work recovers the usual continuous case \( \mathbb{T} = [a, b] \) (where \( \sigma \) is the identity) where the critical points of Lagrangian functionals of type

\[
\mathcal{L}(u) = \int_a^b L(u(\tau), du/d\tau, \tau) d\tau \tag{2}
\]

are characterized by the solutions of the well known continuous Euler-Lagrange equation (see e.g. [4] p.12) given by

\[
\frac{d}{dt} \left[ \frac{\partial L}{\partial u}(u, du/dt, \cdot) \right] (t) = \frac{\partial L}{\partial x}(u(t), du/dt(t), t). \tag{3}
\]

Moreover, the work of M. Bohner in [8] also recovers the following discrete case \( \text{card}(\mathbb{T}) < \infty \) where the critical points of discrete Lagrangian functionals of type

\[
\mathcal{L}(u) = \sum_{k=0}^{N-1} (t_{k+1} - t_k)L(u(t_{k+1}), \Delta u(t_k), t_k) \tag{4}
\]

are characterized by the solutions of the well known discrete Euler-Lagrange equation (see e.g. [3]) given by

\[
\Delta \left[ \frac{\partial L}{\partial u}(u^\sigma, \Delta u, \cdot) \right] (t_k) = \frac{\partial L}{\partial x}(u(t_{k+1}), \Delta u(t_k), t_k). \tag{5}
\]

In what follows, we will speak about \( \mathcal{L} \) (defined in [11]) as a shifted Lagrangian functional in reference to the presence of \( u^\sigma \) (instead of \( u \)) in its definition. Note that this characteristic has no consequences on the continuous case but let us mention the presence of \( u(t_{k+1}) \) (instead of \( u(t_k) \)) in the discrete case. We will see that the difference is important at the discrete level and \textit{a fortiori} at the time scale one too. In particular, this shifted framework does not cover the variational integrators studied in [18,26]. We refer to the next paragraph for more details.

Since the publication of [8], the shifted calculus of variations is widely investigated in several directions: with double integral [9], with higher-order \( \Delta \)-derivatives [14], with non fixed boundary
conditions and transversality conditions [20], with double integral mixing \(\Delta\)- and \(\nabla\)-derivatives [25], with higher-order \(\nabla\)-derivatives [27], etc. We also refer to [21] [22] for shifted optimal control problems. Let us mention that shifted variational problems are particularly suitable (in comparison with the non shifted ones) because of the emergence of a shift in the integration by parts formula on time scales (see [10] Theorem 1.77 p.28) given by:

\[
\int_a^b u(\tau) v^{\Delta}(\tau) \Delta \tau = u(b) v(b) - u(a) v(a) - \int_a^b u^{\Delta}(\tau) v^\sigma(\tau) \Delta \tau.
\] (6)

Context in non shifted calculus of variations. As mentioned in the previous paragraph, the shifted calculus of variations on general time scales developed in [3] does not cover an important class of problems. Let us mention that authors of [12] [13] only obtain an integral form of Euler-Lagrange equation in contrary to the classical continuous and discrete cases where a differential form is provided. Moreover, note that a differential form is also given in the shifted calculus of variations on general time scales (see [10, Theorem 1.77 p.28]) given by:

\[
\int_a^b u(\tau) v^{\Delta}(\tau) \Delta \tau = u(b) v(b) - u(a) v(a) - \int_a^b u^{\Delta}(\tau) v^\sigma(\tau) \Delta \tau.
\] (6)

Recall that the above discrete Euler-Lagrange equation (8) corresponds to the variational integrator obtained and well studied in [18] [26]. In particular, it is an efficient numerical scheme for the optimal control problems. Nevertheless, a serious obstruction is due to the presence of \(\sigma\) in the upper bound of the \(\Delta\)-integral in (EL\(_{\text{int}}\)). In fact, we will exhibit a counter-example (see Example 1) showing that we cannot \(\nabla\)-differentiate the integral Euler-Lagrange equation (EL\(_{\text{int}}\)) on general time scales. It leads us to consider a subclass of time scales, see [12, 13]. In these papers, the critical points of (non shifted) Lagrangian functionals

\[
\mathcal{L}(u) = \sum_{k=0}^{N-1} (t_{k+1} - t_k)L(u(t_k), \Delta u(t_k), t_k)
\] (7)

are characterized by the solutions of the well known discrete Euler-Lagrange equation (see e.g. [18]) given by

\[
\nabla \left[ \frac{\partial L}{\partial v}(u, \Delta u, \cdot) \right](t_k) = \frac{t_{k+1} - t_k}{t_k - t_{k-1}} \frac{\partial L}{\partial x}(u(t_k), \Delta u(t_k), t_k).
\] (8)

Note that authors of [12] [13] only obtain an integral form of Euler-Lagrange equation in contrary to the classical continuous and discrete cases where a differential form is provided. Moreover, note that a differential form is also given in the shifted framework of [3]. Hence, it exists a gap in the literature on (non shifted) calculus of variations on time scales and we will see that it cannot be trivially filled in a general framework.

The objective of this paper is to derive, in the non shifted case, a differential Euler-Lagrange equation of type

\[
\left[ \frac{\partial L}{\partial v}(u, \Delta u, \cdot) \right]^{\nabla}(t) = \omega(t) \frac{\partial L}{\partial x}(u(t), \Delta u(t), t)
\] (10)

allowing in particular to recover the non shifted discrete case given by (6).

In this way, a natural idea is to apply the \(\nabla\)-derivative on (EL\(_{\text{int}}\)). Nevertheless, a serious obstruction is due to the presence of \(\sigma\) in the upper bound of the \(\Delta\)-integral in (EL\(_{\text{int}}\)). In fact, we will exhibit a counter-example (see Example 1) showing that we cannot \(\nabla\)-differentiate the integral Euler-Lagrange equation (EL\(_{\text{int}}\)) on general time scales. It leads us to consider a subclass of time scales...
scales. Precisely, we prove that the $\nabla$-differentiability of $\sigma$ is a sharp assumption on the time scale in order to derive a $\nabla \circ \Delta$-differential Euler-Lagrange equation of type (10) as a characterization of the critical points of the (non shifted) Lagrangian functional $L$ defined in (9). Moreover, in such a case, an interesting phenomena is the direct emergence of this assumption in (10) since we prove that $\omega = \sigma \nabla$.

In this paper, we study the consequences of the $\nabla$-differentiability of $\sigma$ on the structure of $\mathbb{T}$. We will see how this assumption allows us to apply $\nabla$ on (EL$_{\Delta \text{Diff}}$) in order to obtain (10) with $\omega = \sigma \nabla$. Note that the $\nabla$-differentiability of $\sigma$ is not a loss of generality since it is satisfied in the continuous case $\mathbb{T} = [a, b]$ (with $\sigma \nabla = 1$) and in the discrete case card($\mathbb{T}$) $< \infty$ (with $\sigma \nabla(t_k) = (t_{k+1} - t_k)/(t_k - t_{k-1})$). As a consequence, our main result both recovers the usual continuous case (3) and the non shifted discrete case (8).

**Derivation of Noether-type results.** In shifted calculus of variations, we refer to the paper [7] studying the existence of constant of motion for $\Delta \circ \Delta$-differential Euler-Lagrange equations (EL$_{\Delta \Delta \text{Diff}}$). We refer to [28] for a similar study with $\nabla$-derivatives. The common strategy is to generalize the celebrated Noether’s theorem [24, 31] to time scales. Precisely, under invariance assumption on the Lagrangian $L$, authors prove that a conservation law can be obtained.

In the non shifted calculus of variations, note that the non differential form of (EL$_{\Delta \text{Int}}$) is an obstruction in order to develop the same strategy. A direct application of our main result is then to provide a Noether-type theorem based on the differential form (10). This will be done in Section 4.

Finally, it has to be noted that our whole study is made in terms of Lagrangian functionals involving $\Delta$-integral and $\Delta$-derivative. However, all results can be analogously derived for Lagrangian functionals involving $\nabla$-integral and $\nabla$-derivative. This will be done in Section 5.

**Organization of the paper.** We first give basic recalls on time scale calculus in Section 2.1 and on non shifted calculus of variations on general time scales in Section 2.2. Finally, Section 2.3 is devoted to our main result (Theorem 1). A study of time scales with continuous $\sigma$ is provided in Section 3.1 and with $\nabla$-differentiable $\sigma$ in Section 3.2. The results obtained in these two previous sections allow us to prove our main result. Finally, we prove a Noether-type theorem in Section 4.

We conclude this paper with the analogous results for non shifted calculus of variations defined in terms of $\nabla$-integral and $\nabla$-derivative, see Section 5.

## 2 Non shifted calculus of variations on time scales with $\nabla$-differentiable $\sigma$

In this paper, $\mathbb{T}$ denotes a bounded time scale with $a = \min(\mathbb{T})$, $b = \max(\mathbb{T})$ and card($\mathbb{T}$) $\geq 3$. In Section 2.1 we give basic recalls about time scale calculus.

In Section 2.2, we recall the non shifted calculus of variations on general time scales originally developed in [13], see also [12, Section 9]. In particular, the $\Delta$-integral Euler-Lagrange equation (EL$_{\Delta \text{Int}}$) is given as a necessary condition for local optimizers of non shifted Lagrangian functionals, see Proposition 3.

In Section 2.3, under the assumption of $\nabla$-differentiability of $\sigma$, our main result provides a $\nabla \circ \Delta$-differential Euler-Lagrange equation of type (10) as a necessary condition for local optimizers of non shifted Lagrangian functionals, see Equation (EL$_{\Delta \Delta \text{Diff}}$) in Theorem 1. We also prove that this assumption is sharp, see Example 1.

### 2.1 Basic recalls on time scale calculus

We refer to the surveys [1, 2, 10, 11] for more details on time scale calculus. The backward and forward jump operators $\rho, \sigma : \mathbb{T} \rightarrow \mathbb{T}$ are respectively defined by:

$$\forall t \in \mathbb{T}, \rho(t) = \sup\{s \in \mathbb{T}, s < t\} \text{ and } \sigma(t) = \inf\{s \in \mathbb{T}, s > t\},$$

(11)
where we put sup\(\emptyset = a\) and inf\(\emptyset = b\). A point \(t \in \mathbb{T}\) is said to be left-dense (resp. left-scattered, right-dense and right-scattered) if \(\rho(t) = t\) (resp. \(\rho(t) < t, \sigma(t) = t\) and \(\sigma(t) > t\)). Let LD (resp. LS, RD and RS) denote the set of all left-dense (resp. left-scattered, right-dense and right-scattered) points of \(\mathbb{T}\). The graininess (resp. backward graininess) function \(\mu : \mathbb{T} \rightarrow \mathbb{R}^*\) (resp. \(\nu : \mathbb{T} \rightarrow \mathbb{R}^*\)) is defined by \(\mu(t) = \sigma(t) - t\) (resp. \(\nu(t) = t - \rho(t)\)) for any \(t \in \mathbb{T}\).

We set \(\mathbb{T}^c = \mathbb{T}\{\rho(b), b\}, \mathbb{T}_n = \mathbb{T}\{a, \sigma(a)\}\) and \(\mathbb{T}^c = \mathbb{T}^c \cap \mathbb{T}_n\). Note that \(\mathbb{T}^c_n \neq \emptyset\) since \(\text{card}(\mathbb{T}) \geq 3\). Let us recall the usual definitions of \(\Delta\)- and \(\nabla\)-differentiability. A function \(u : \mathbb{T} \rightarrow \mathbb{R}^n\), where \(n \in \mathbb{N}^*\), is said to be \(\Delta\)-differentiable at \(t \in \mathbb{T}^c\) (resp. \(\nabla\)-differentiable at \(t \in \mathbb{T}_n\)) if the following limit exists in \(\mathbb{R}^n\):

\[
\lim_{s \to t \atop s \neq \sigma(t)} \frac{u(\sigma(t)) - u(s)}{\sigma(t) - s} \quad \text{(resp.} \quad \lim_{s \to t \atop s \neq \rho(t)} \frac{u(s) - u(\rho(t))}{s - \rho(t)} \text{)}.
\]

In such a case, this limit is denoted by \(u^\Delta(t)\) (resp. \(u^\nabla(t)\)). Let us recall the following results on \(\Delta\)-differentiability, see [10, Theorem 1.16 p.5] and [10, Corollary 1.68 p.25]. The analogous results for \(\nabla\)-differentiability are also valid.

**Proposition 1.** Let \(u : \mathbb{T} \rightarrow \mathbb{R}^n\) and \(t \in \mathbb{T}^c\). The following properties hold:

1. if \(u\) is \(\Delta\)-differentiable at \(t\), then \(u\) is continuous at \(t\).

2. if \(t \in \text{RS}\) and if \(u\) is continuous at \(t\), then \(u\) is \(\Delta\)-differentiable at \(t\) with:

\[
u^{\Delta}(t) = \frac{u(\sigma(t)) - u(t)}{\mu(t)}.
\]

3. if \(t \in \text{RD}\), then \(u\) is \(\Delta\)-differentiable at \(t\) if and only if the following limit exists in \(\mathbb{R}^n\):

\[
\lim_{s \to t \atop s \neq \rho(t)} \frac{u(t) - u(s)}{t - s}.
\]

In such a case, this limit is equal to \(u^\Delta(t)\).

**Proposition 2.** Let \(u : \mathbb{T} \rightarrow \mathbb{R}^n\). Then, \(u\) is \(\Delta\)-differentiable on \(\mathbb{T}^c\) with \(u^\Delta = 0\) if and only if there exists \(c \in \mathbb{R}^n\) such that \(u(t) = c\) for every \(t \in \mathbb{T}\).

From Proposition 1 and for every \(t \in \text{RS}\), note that a function \(u\) is \(\Delta\)-differentiable at \(t\) if and only if \(u\) is continuous at \(t\). Still from Proposition 1, note that every \(\Delta\)-differentiable function on \(\mathbb{T}^c\) is continuous on \(\mathbb{T}\).

Recall that a function \(u\) is said to be rd-continuous on \(\mathbb{T}\) if it is continuous at every \(t \in \text{RD}\) and if it admits a left-sided limit at every \(t \in \text{LD}\), see [10, Definition 1.58 p.22]. We respectively denote by \(C^0_{\text{rd}}(\mathbb{T})\) and \(C^1_{\text{rd}}(\mathbb{T})\) the functional spaces of rd-continuous functions on \(\mathbb{T}\) and of \(\Delta\)-differentiable functions on \(\mathbb{T}^c\) with rd-continuous \(\Delta\)-derivative. Recall the following results, see [10, Theorem 1.60 p.22]:

- \(\sigma\) is rd-continuous.
- if \(u \in C^0_{\text{rd}}(\mathbb{T})\), the composition \(u^\sigma = u \circ \sigma\) is rd-continuous.
- if \(u \in C^0_{\text{rd}}(\mathbb{T})\), the composition \(f \circ u\) with any continuous function \(f\) is rd-continuous.

Let us denote by \(\int \Delta t\) the Cauchy \(\Delta\)-integral defined in [10, p.26]. For every \(u \in C^0_{\text{rd}}(\mathbb{T}^c)\), recall that the function \(U\), defined by \(U(t) = \int_t^a u(\tau)\Delta t\) for every \(t \in \mathbb{T}\), is the unique \(\Delta\)-antiderivative of \(u\) (in the sense that \(U^\Delta = u\) on \(\mathbb{T}^c\)) vanishing at \(t = a\), see [10, Theorem 1.74 p.27]. In particular, we have \(U \in C^1_{\text{rd}}(\mathbb{T})\).
2.2 Recalls on non shifted calculus of variations on general time scales

In this section, we recall the non shifted calculus of variations on general time scales originally developed in [13], see also [12, Section 9]. Let \( L \) be a Lagrangian i.e. a continuous map of class \( C^1 \) in its two first variables:

\[
L : \mathbb{R}^n \times \mathbb{R}^n \times T_\kappa \rightarrow \mathbb{R}
\]

\[
(x, v, t) \mapsto L(x, v, t)
\]

(15)

and let \( \mathcal{L} \) be the following (non shifted) Lagrangian functional:

\[
\mathcal{L} : C^{1,\Delta}_{rd}(T) \rightarrow \mathbb{R}
\]

\[
u \mapsto \int_a^b L(u(\tau), u^\Delta(\tau), \tau) \Delta \tau.
\]

(16)

In this section, our aim is to give a necessary condition for local optimizers of \( L \) (with or without boundary conditions on \( t = a \) and \( t = b \)). In this way, we introduce the following notions and notations:

- \( C^{1,\Delta}_{rd,0}(T) = \{ w \in C^{1,\Delta}_{rd}(T), w(a) = w(b) = 0 \} \) is called the set of variations of \( \mathcal{L} \).
- \( u \in C^{1,\Delta}_{rd}(T) \) is said to be a critical point of \( \mathcal{L} \) if \( DL(u)(w) = 0 \) for every \( w \in C^{1,\Delta}_{rd,0}(T) \). Let us precise that \( DL(u)(w) \) denotes the Gâteaux-differential of \( \mathcal{L} \) at \( u \) in direction \( w \).

In particular, if \( u \) is a local optimizer of \( \mathcal{L} \), then \( u \) is a critical point of \( \mathcal{L} \). Finally, let us recall the following characterization of the critical points of \( \mathcal{L} \), see [12, Theorem 11] or [13, Corollary 1].

**Proposition 3.** Let \( u \in C^{1,\Delta}_{rd}(T) \). Then, \( u \) is a critical point of \( \mathcal{L} \) if and only if there exists \( c \in \mathbb{R}^n \) such that:

\[
\frac{\partial L}{\partial v}(u(t), u^\Delta(t), t) = \int_a^t \sigma(t) \frac{\partial L}{\partial x}(u(\tau), u^\Delta(\tau), \tau) \Delta \tau + c,
\]

(EL\(_{int}\))

for every \( t \in T_\kappa \).

Hence, Proposition 3 provides a necessary condition for local optimizers of \( \mathcal{L} \). Precisely, if \( u \) is a local optimizer of \( \mathcal{L} \), then there exists \( c \in \mathbb{R}^n \) such that \( u \) satisfies the \( \Delta \)-integral Euler-Lagrange equation (EL\(_{int}\)). We refer to Example 1 in Section 2.3 for an example of application of Proposition 3.

2.3 Main result

In this paper, we aim to rewrite the \( \Delta \)-integral Euler-Lagrange equation (EL\(_{int}\)) as a \( \nabla \circ \Delta \)-differential one of type \( \text{EL}_{\text{diff}}^{\Delta} \). Precisely, we prove the following result under the assumption of \( \nabla \)-differentiability of \( \sigma \).

**Theorem 1** (Main result). Let us assume that \( \sigma \) is \( \nabla \)-differentiable on \( T_\kappa \) and let \( u \in C^{1,\Delta}_{rd}(T) \). Then, \( u \) is a critical point of \( \mathcal{L} \) if and only if \( u \) is solution of the following \( \nabla \circ \Delta \)-differential Euler-Lagrange equation:

\[
\left[ \frac{\partial L}{\partial v}(u, u^\Delta, \cdot), \right]^{\nabla}(t) = \sigma^{\nabla}(t) \frac{\partial L}{\partial x}(u(t), u^\Delta(t), t),
\]

(EL\(_{\text{diff}}^{\Delta} \))

for every \( t \in T_\kappa \).

**Proof.** We refer to Corollary 1 in Section 3.2.

Note that this result both recovers the usual continuous and discrete Euler-Lagrange equations given by (3) and (8) in Introduction. Indeed, as it is mentioned in Example 3 in Section 3.2 the following properties are satisfied:
• if $\mathbb{T} = [a, b]$, $\sigma$ is $\nabla$-differentiable on $\mathbb{T}_K$ with $\sigma^{\nabla} = 1$.

• if $\text{card}(\mathbb{T}) < \infty$, $\sigma$ is $\nabla$-differentiable on $\mathbb{T}_K$ with $\sigma^{\nabla} = \mu/\nu$.

Let us prove, via the following simple example, that the assumption of $\nabla$-differentiability of $\sigma$ is sharp for the validity of Theorem 1.

Example 1. Let us consider $n = 1$, $L(x, v, t) = x + v^2/2$ and $u \in C^{1,\Delta}_{\text{rd}}(\mathbb{T})$ defined by $u(t) = \int_{t_0}^{t} \sigma(\tau) \Delta \sigma$ for every $t \in \mathbb{T}$. Since $u$ satisfies $\{\text{EL}_{\text{int}}\}$, we conclude that $u$ is a critical point of $\mathcal{L}$, see Proposition 2. However, note that $\partial L/\partial v(u, u^\Delta, \cdot) = u^\Delta = \sigma$ and consequently, $\{\text{EL}_{\text{diff}}\}$ is not satisfied if $\sigma$ is not $\nabla$-differentiable.

We refer to Example 2 for time scales with continuous and non continuous $\sigma$. We respectively refer to Examples 3 and 4 for time scales with $\nabla$-differentiable and non $\nabla$-differentiable $\sigma$.

3 Time scales with $\nabla$-differentiable $\sigma$

In this section, we first study the consequences on $\mathbb{T}$ of the continuity of $\sigma$ in Section 3.1. Then, we study the consequences on $\mathbb{T}$ of its $\nabla$-differentiability in Section 3.2. In particular, the results obtained in these sections allow us prove our main result (Theorem 1).

3.1 Continuity of $\sigma$

Let us prove the following characterizations of the continuity of $\sigma$.

Proposition 4. Let $t \in \mathbb{T}_K$. The following properties are equivalent:

1. $\sigma$ is continuous at $t$;

2. $\sigma \circ \rho(t) = t$;

3. $t \notin \mathbb{R} \cap \mathbb{L} \cap \mathbb{D}$.

Proof. Let us prove that 1. implies 2.. By contradiction, let us assume that $\sigma \circ \rho(t) \neq t$. Necessarily, we have $t \in \mathbb{R} \cap \mathbb{L} \cap \mathbb{D}$. As a consequence, $t \neq a$ since $t \in \mathbb{T}_K$. Then, let $(s_k) \subset \mathbb{T}$ be a sequence such that $s_k < t$ for any $k \in \mathbb{N}$ and $s_k \to t$. Thus, we have $\sigma(s_k) < t < \sigma(t)$ for any $k \in \mathbb{N}$ and consequently, $(\sigma(s_k))$ does not tend to $\sigma(t)$. This is a contradiction with the continuity of $\sigma$ at $t$.

Let us prove that 2. implies 3.. If $t \in \mathbb{R} \cap \mathbb{L} \cap \mathbb{D}$, then $\sigma \circ \rho(t) = \sigma(t) \neq t$.

Let us prove that 3. implies 1.. By contradiction, let us assume that $\sigma$ is not continuous at $t$. As a consequence, there exists $\varepsilon > 0$ and a monotone sequence $(s_k)$ such that $s_k \to t$ and $|\sigma(t) - \sigma(s_k)| \geq \varepsilon$ for every $k \in \mathbb{N}$. Firstly, let us assume that $(s_k)$ is decreasing. Then, we have $t < s_k < s_{k-1}$ and then $t \leq \sigma(t) \leq \sigma(s_k) \leq s_{k-1}$ for any $k \in \mathbb{N}^*$. It is a contradiction since $s_{k-1} \to t$. Secondly, let us assume that $(s_k)$ is increasing. As a consequence, $t \in \mathbb{L} \cap \mathbb{D}$ and then $t \in \mathbb{R} \cap \mathbb{D}$ (see 3.). Finally, we have $s_k \to t$ and then, $s_{k-1} < \sigma(s_k) \leq t = \sigma(t)$ for any $k \in \mathbb{N}^*$. It is a contradiction since $s_{k-1} \to t$. In both cases, we have obtained a contradiction.

Note that $\sigma$ is continuous at $a$. Indeed, if $a \in \mathbb{R}$, then $a \in \mathbb{T}_K$, $a \notin \mathbb{R} \cap \mathbb{L} \cap \mathbb{D}$ and then, Proposition 4 concludes in this case. If $a \in \mathbb{R}$, then $a$ is isolated and thus, $\sigma$ is continuous at $a$.

Consequently, $\sigma$ is continuous at $t \in \mathbb{T}$ if and only if $t = a$ or (non exclusive) $t \notin \mathbb{R} \cap \mathbb{L} \cap \mathbb{D}$. Hence, $\sigma$ is continuous at $t \in \mathbb{T}$ if and only if $t = a$ or $t \in \mathbb{R} \cap \mathbb{D}$.

Finally, we conclude that $\sigma$ is continuous on $\mathbb{T}$ if and only if $\mathbb{R} \cap \mathbb{L} \cap \mathbb{D}$ is right-scattered and left-dense. A similar remark is already done in [10] Example 1.55. Let us see some examples and counter-examples.

Example 2. 1. If $\mathbb{T} = [a, b]$, $\sigma$ is continuous on $\mathbb{T}$.
2. If \( \text{card}(\mathbb{T}) < \infty \), \( \sigma \) is continuous on \( \mathbb{T} \).

3. If \( \mathbb{T} = \{0, 1\} \cup [2, 3] \), \( \sigma \) is continuous on \( \mathbb{T} \).

4. If \( \mathbb{T} = [-1, 0] \cup \{1/k \mid k \in \mathbb{N}^* \} \), \( \sigma \) is continuous on \( \mathbb{T} \).

5. If \( \mathbb{T} = [0, 1] \cup [2, 3] \), \( \sigma \) is not continuous at \( 1 \in \mathbb{R} \cap \mathbb{L} \setminus \{0\} \).

6. If \( \mathbb{T} \) is the usual Cantor set (see [10, Example 1.47 p.18]), \( \sigma \) is not continuous at \( 1/3 \in \mathbb{R} \cap \mathbb{L} \setminus \{0\} \).

Let us give some remarks.

**Remark 1.** If \( \sigma \) is continuous on \( \mathbb{T} \), then every \( t \in \mathbb{R} \) is isolated. Consequently, in such a case, every function is directly continuous and \( \Delta \)-differentiable at every \( t \in \mathbb{R} \), see Proposition 4.

**Remark 2.** If \( \sigma \) is continuous on \( \mathbb{T} \), note that every rd-continuous function is directly continuous on \( \mathbb{T} \). Indeed, if \( \sigma \) is continuous on \( \mathbb{T} \), every rd-continuous function is continuous at every \( t \in \mathbb{R} \) (by definition) and is continuous at every \( t \in \mathbb{L} \) (see Remark 7). Finally, in such a case, \( C^0_{\text{rd}}(\mathbb{T}) \) coincides with the set of continuous functions. Similarly, \( C^1_{\text{rd}}(\mathbb{T}) \) coincides with the set of \( \Delta \)-differentiable functions on \( \mathbb{T} \) with continuous \( \Delta \)-derivative.

**Remark 3.** Let us give a short discussion on the notion of regular time scale originally introduced in [10, Definition 9]. We refer to [10, 30] for other applications of this notion. Recall that \( \mathbb{T} \) is said to be regular if for every \( t \in \mathbb{T} \), \( \sigma \circ \rho(t) = \rho \circ \sigma(t) = t \). In particular, for every regular time scale, \( a \) is necessarily right-dense and \( b \) is necessarily left-dense, see [10, Proposition 10]. Hence, the discrete bounded time scales are not regular. The regularity of a time scale is then a relative restrictive assumption. Consequently, we suggest the introduction of the following weakened notion: a time scale is said to be quasi-regular if \( \sigma \) and \( \rho \) are continuous on \( \mathbb{T} \). Hence, a time scale is quasi-regular if and only if \( \sigma \circ \rho(t) = t \) for every \( t \in \mathbb{T} \) and \( \rho \circ \sigma(t) = t \) for every \( t \in \mathbb{T} \). Such a weakened notion allows to cover the discrete bounded time scales.
3.2 $\nabla$-differentiability of $\sigma$

From Proposition 3 we derive the following result.

**Proposition 5.** The following properties are satisfied:

1. if $\sigma$ is continuous at $t \in \mathbb{LS}$, then $\sigma$ is directly $\nabla$-differentiable at $t$ with $\sigma^\nabla(t) = \mu(t)/\nu(t)$.

2. if $\sigma$ is continuous on $\mathbb{T}$, then $\sigma$ is $\nabla$-differentiable on $\mathbb{T}_K$ if and only if for every $t \in \text{LD} \cap \mathbb{T}_K$, the following limit exist in $\mathbb{R}$:

$$\lim_{s \to t, s \neq t} \frac{\sigma(s) - \sigma(t)}{s - t}. \quad (17)$$

In such a case, this limit is equal to $\sigma^\nabla(t)$.

**Proof.** Let us prove the first point. From Proposition 3 and since $\sigma$ is continuous at $t \in \mathbb{LS} \subset \mathbb{T}_K$, $\sigma$ is directly $\nabla$-differentiable at $t$ with:

$$\sigma^\nabla(t) = \frac{\sigma(t) - \sigma(\rho(t))}{\nu(t)} = \frac{\sigma(t) - t}{\nu(t)} = \frac{\mu(t)}{\nu(t)}. \quad (18)$$

since $\sigma \circ \rho(t) = t$ from Proposition 3.

Let us prove the second point. Since $\sigma$ is continuous on $\mathbb{T}$, $\sigma$ is directly $\nabla$-differentiable at every $t \in \mathbb{LS}$ from the first point. Consequently, $\sigma$ is $\nabla$-differentiable on $\mathbb{T}_K$ if and only if $\sigma$ is $\nabla$-differentiable at every $t \in \text{LD} \cap \mathbb{T}_K$ i.e. if and only if for every $t \in \text{LD} \cap \mathbb{T}_K$, the following limit exists in $\mathbb{R}$:

$$\lim_{s \to t, s \neq t} \frac{\sigma(s) - \sigma(t)}{s - t}. \quad (19)$$

To conclude, it is sufficient to note that the continuity of $\sigma$ implies $\text{LD} \cap \mathbb{T}_K \subset \text{RD}$, see Proposition 3. The proof is complete.

Let us give some examples of time scale with $\nabla$-differentiable $\sigma$.

**Example 3.**

1. If $\mathbb{T} = [a, b]$, $\sigma$ is $\nabla$-differentiable on $\mathbb{T}_K$ with $\sigma^\nabla = 1$.

2. If $\text{card}(\mathbb{T}) < \infty$, $\sigma$ is $\nabla$-differentiable on $\mathbb{T}_K$ with $\sigma^\nabla = \mu/\nu$.

3. If $\mathbb{T} = \{0\} \cup \{z_k, k \in \mathbb{N}\}$ where $(z_k)$ is a decreasing positive sequence tending to 0 and if $\lim_{k \to \infty} z_{k-1}/z_k$ exists (denoted by $\ell$), then $\sigma$ is $\nabla$-differentiable on $\mathbb{T}_K$. In particular, we have $\sigma^\nabla(0) = \ell$. Indeed, let $(s_k) \subset \mathbb{T}$ be a positive sequence tending to 0. Then, for every $k \in \mathbb{N}$, there exists $p_k \in \mathbb{N}$ such that $s_k = z_{p_k}$. Since $s_k \to 0$, we have $p_k \to +\infty$. Finally, we obtain:

$$\lim_{k \to \infty} \frac{\sigma(s_k) - 0}{s_k - 0} = \lim_{k \to \infty} \frac{z_{p_k-1}}{z_{p_k}} = \ell. \quad (20)$$

4. Application: if $\mathbb{T} = \{0\} \cup \{1/r^k, k \in \mathbb{N}\}$ with $r > 1$, then $\sigma$ is $\nabla$-differentiable on $\mathbb{T}_K$. In particular, we have $\sigma^\nabla(0) = r$.

5. Similarly to 3., we can prove that if $\mathbb{T} = \{0\} \cup \{z_k, k \in \mathbb{N}\}$ where $(z_k)$ is an increasing negative sequence tending to 0 and if $\lim_{k \to \infty} z_{k+1}/z_k$ exists (denoted by $\ell$), then $\sigma$ is $\nabla$-differentiable on $\mathbb{T}_K$. In particular, we have $\sigma^\nabla(0) = \ell$. 

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9
Theorem 2. Let the following properties:

6. Application: if \( T = \{0\} \cup \{-1/r^k, \ k \in \mathbb{N}\} \), then \( \sigma \) is \( \nabla \)-differentiable on \( T_k \). In particular, we have \( \sigma^\nabla(0) = 1/r \).

7. Similarly to 3., we can prove that if \( T = [-1,0] \cup \{z_k, \ k \in \mathbb{N}\} \) where \( (z_k) \) is a decreasing positive sequence tending to 0 and if \( \lim_{k \to \infty} z_{k-1}/z_k = 1 \), then \( \sigma \) is \( \nabla \)-differentiable on \( T_k \). In particular, we have \( \sigma^\nabla(0) = 1 \).

8. Application: if \( T = [-1,0] \cup \{1/k^2, \ k \in \mathbb{N}^+\} \), then \( \sigma \) is \( \nabla \)-differentiable on \( T_k \). In particular, we have \( \sigma^\nabla(0) = 1 \).

9. Similarly to 3., we can prove that if \( T = \{0\} \cup \{z_k^-, k \in \mathbb{N}\} \cup \{z_k^+, k \in \mathbb{N}\} \) where \( (z_k^-) \) (resp. \( (z_k^+) \)) is an increasing negative (resp. decreasing positive) sequence tending to 0 and if \( \lim_{k \to \infty} z_{k-1}/z_k = \lim_{k \to \infty} z_{k-1}^+/z_k^+ = \ell \), then \( \sigma \) is \( \nabla \)-differentiable on \( T_k \). In particular, we have \( \sigma^\nabla(0) = \ell \). Note that, in such a case, we can only have \( \ell = 1 \) since \( z_{k+1}^-/z_k^- < 1 < z_{k-1}^+/z_k^+ \) for every \( k \in \mathbb{N} \).

10. Application: if \( T = \{0\} \cup \{-1/k, k \in \mathbb{N}\} \cup \{1/k, k \in \mathbb{N}\} \) with \( z_k^- = -1/k \) and \( z_k^+ = 1/k^2 \) for every \( k \in \mathbb{N} \), then \( \sigma \) is \( \nabla \)-differentiable on \( T_k \). In particular, we have \( \sigma^\nabla(0) = 1 \).

Let us give some examples of time scale with continuous but non \( \nabla \)-differentiable \( \sigma \).

**Example 4.**
1. If \( T = \{0\} \cup \{1/k!, k \in \mathbb{N}\} \), then \( \sigma \) is not \( \nabla \)-differentiable in 0 since \( k!/(k-1)! = k \) tends to \( +\infty \).
2. If \( T = [-1,0] \cup \{1/2^k, k \in \mathbb{N}\} \), then \( \sigma \) is not \( \nabla \)-differentiable in 0 since \( 2^k/2^{k-1} = 2 \) does not tend to 1.
3. If \( T = \{0\} \cup \{\pm 1/2^k, k \in \mathbb{N}\} \), then \( \sigma \) is not \( \nabla \)-differentiable in 0 since \( 2^k/2^{k-1} = 2, 2^k/2^{k+1} = 1/2 \) and \( 2 \neq 1/2 \).

Examples 2, 3 and 4 allow to get a better understanding of the restrictions imposed on a time scale by the \( \nabla \)-differentiability of \( \sigma \). Indeed, we conclude that such a time scale has to satisfied the following properties:

- Due to the continuity of \( \sigma \), no point (except \( a \)) can be right-scattered and left-dense.
- Due to the \( \nabla \)-differentiability of \( \sigma \), the density in a dense point is not "too weak", in contrary to 1. in Example 4. Secondly, in a left- and right-dense point different to \( a \) and \( b \), the left and the right densities have to be "homogeneous" with limit equal to 1, as in 7., 9. of Example 3. and in contrary to 2., 3. of Example 4.

Finally, the most important result of this section is the following one.

**Theorem 2.** Let \( u : T \to \mathbb{R}^n \) and let \( t \in T_k^a \). If the two following properties are satisfied:

- \( \sigma \) is \( \nabla \)-differentiable at \( t \);
- \( u \) is \( \Delta \)-differentiable at \( t \);
then, \( u^\sigma \) is \( \nabla \)-differentiable at \( t \) with \((u^\sigma)^\nabla(t) = \sigma^\nabla(t)u^\Delta(t)\).

**Proof.** Since \( \sigma \) is continuous at \( t \), recall that \( \sigma \circ \rho(t) = t \) from Proposition 4. We distinguish two cases: \( t \in \text{LS} \) and \( t \in \text{RD} \).

- Firstly, let us consider that \( t \in \text{LS} \). Since \( \sigma \) is continuous at \( t \), we have \( \sigma^\nabla(t) = \mu(t)/\nu(t) \), see Proposition 5. If moreover \( t \in \text{RS} \), then \( u^\Delta(t) = (u(\sigma(t)) - u(t))/\mu(t) \) and since \( t \) is isolated, \( u^\sigma \) is \( \nabla \)-differentiable at \( t \) with:

\[
(u^\sigma)^\nabla(t) = \frac{u^\sigma(t) - u^\sigma(\rho(t))}{\nu(t)} = \frac{u(\sigma(t)) - u(t)}{\nu(t)} = \frac{\mu(t)}{\nu(t)}u^\Delta(t) = \sigma^\nabla(t)u^\Delta(t). \tag{21}
\]

In the contrary case \( t \in \text{RD} \), since \( \sigma \) is continuous at \( t \) and since \( u \) is continuous at \( t = \sigma(t) \), we deduce that \( u^\sigma \) is continuous at \( t \in \text{LS} \). Then, from Proposition 4, \( u^\sigma \) is \( \nabla \)-differentiable at \( t \) with:

\[
(u^\sigma)^\nabla(t) = \frac{u^\sigma(t) - u^\sigma(\rho(t))}{\nu(t)} = \frac{u(\sigma(t)) - u(t)}{\nu(t)} = 0, \tag{22}
\]

since \( \sigma(t) = t \). However, in this case, we have \( \sigma^\nabla(t) = \mu(t)/\nu(t) = 0 \). Consequently, we also retrieve \((u^\sigma)^\nabla(t) = \sigma^\nabla(t)u^\Delta(t)\) in this case.

- Secondly, let us consider that \( t \in \text{RD} \). Since \( \sigma \) is continuous at \( t \) and since \( u \) is \( \Delta \)-differentiable at \( t \), we deduce that \( u^\sigma \) is \( \nabla \)-differentiable at \( t \) with:

\[
\lim_{s \to t, s \neq t} \frac{u^\sigma(s) - u^\sigma(t)}{s - t} = \lim_{s \to t} \frac{\sigma(s) - t}{s - t} \frac{u(\sigma(s)) - u(t)}{\sigma(s) - t} = \sigma^\nabla(t)u^\Delta(t). \tag{23}
\]

In the previous limit, since \( \sigma \) is continuous at \( t \in \text{LD} \cap \text{RD} \), we have used that \( s \to t, s \neq t \) implies that \( \sigma(s) \to \sigma(t) = t \), \( \sigma(s) \neq t \).

The proof is complete. \( \square \)

From Theorem 2, the following corollary is directly derived.

**Corollary 1.** Let \( u : \mathbb{T} \to \mathbb{R}^n \). If the following properties are satisfied:

- \( \sigma \) is \( \nabla \)-differentiable on \( \mathbb{T}_\kappa \);
- \( u \) is \( \Delta \)-differentiable on \( \mathbb{T}^\kappa \);

then, \( u^\sigma \) is \( \nabla \)-differentiable at every \( t \in \mathbb{T}^\kappa \) with \((u^\sigma)^\nabla(t) = \sigma^\nabla(t)u^\Delta(t)\).  

From Propositions 2, 3 and Corollary 1, our main result (Theorem 1) is proved. We conclude this section by introducing the following Leibniz formula useful in order to derive the Noether-type Theorem 3 in Section 4.

**Proposition 6** (Leibniz formula). Let \( u, v : \mathbb{T} \to \mathbb{R}^n \) and \( t \in \mathbb{T}^\kappa \). If the following properties are satisfied:

- \( \sigma \) is \( \nabla \)-differentiable at \( t \);
- \( u \) is \( \Delta \)-differentiable at \( t \);
- \( v \) is \( \nabla \)-differentiable at \( t \);

then, \( u^\sigma \cdot v \) is \( \nabla \)-differentiable at \( t \) and the following Leibniz formula holds:

\[
(u^\sigma \cdot v)^\nabla(t) = u(t) \cdot v^\nabla(t) + \sigma^\nabla(t)u^\Delta(t) \cdot v(t). \tag{24}
\]

**Proof.** Since \( \sigma \) is continuous at \( t \in \mathbb{T}_\kappa \), we have \( \sigma \circ \rho(t) = t \) from Proposition 4. From Theorem 2 \( u^\sigma \) is \( \nabla \)-differentiable at \( t \) with \((u^\sigma)^\nabla(t) = \sigma^\nabla(t)u^\Delta(t) \). Finally, from the usual Leibniz formula on time scale (see [10] Theorem 1.20 p.8), we have \( u^\sigma \cdot v \) is \( \nabla \)-differentiable at \( t \) with:

\[
(u^\sigma \cdot v)^\nabla(t) = u^\sigma(\rho(t)) \cdot v^\nabla(t) + (u^\sigma)^\nabla(t) \cdot v(t) = u(t) \cdot v^\nabla(t) + \sigma^\nabla(t)u^\Delta(t) \cdot v(t). \tag{25}
\]

The proof is complete. \( \square \)
4 Application to a Noether-type theorem

We first review the definition of a one-parameter family of infinitesimal transformations of $\mathbb{R}^n$.

**Definition 1.** Let $\eta > 0$. A map $\Phi$ is said to be a one-parameter family of infinitesimal transformations of $\mathbb{R}^n$ if $\Phi$ is a map of class $C^2$:

$$\Phi : [-\eta, \eta] \times \mathbb{R}^n \rightarrow \mathbb{R}^n \quad (\theta, x) \rightarrow \Phi(\theta, x),$$

such that $\Phi(0, \cdot) = \text{Id}_{\mathbb{R}^n}$.

The action of a one-parameter family of infinitesimal transformations of $\mathbb{R}^n$ on a Lagrangian allows us to introduce the notion of symmetry for a $\nabla \circ \Delta$-differential Euler-Lagrange equation $\text{EL}^{\text{diff}}$.

**Definition 2.** Let $\Phi$ be a one-parameter family of infinitesimal transformations of $\mathbb{R}^n$. A Lagrangian $L$ is said to be invariant under the action of $\Phi$ if for every solution $u$ of $\text{EL}^{\text{diff}}$ and every $t \in T_\kappa$, the map

$$\theta \mapsto L(\Phi(\theta, u(t)), \Phi(\theta, u)\Delta(t), t)$$

is constant. In such a case, $\Phi$ is said to be a symmetry of the $\nabla \circ \Delta$-differential Euler-Lagrange equation $\text{EL}^{\text{diff}}$ associated.

The most classical examples of invariance of a Lagrangian under the action of a one-parameter family of infinitesimal transformations of $\mathbb{R}^n$ are given by quadratic Lagrangians and rotations:

**Example 5.** Let us consider $n = 2$, $L(x, v, t) = \|x\|^2 + \|y\|^2$, $\eta = \pi > 0$ and $\Phi$ defined by:

$$\Phi : [-\pi, \pi] \times \mathbb{R}^2 \rightarrow \mathbb{R}^2 \quad (\theta, x_1, x_2) \rightarrow \left( \begin{array}{c} \cos(\theta) - \sin(\theta) \\ \sin(\theta) \cos(\theta) \end{array} \right) \times \left( \begin{array}{c} x_1 \\ x_2 \end{array} \right).$$

Then, for every $u \in C^1_{\text{rd}}(T)$, every $(\theta, t) \in [-\pi, \pi] \times T_\kappa$, we have $\Phi(\theta, u)\Delta(t) = \Phi(\theta, u^\Delta(t))$. Consequently, for every $u \in C^1_{\text{rd}}(T)$ and every $T_\kappa$, one can easily prove that the map

$$\theta \mapsto L(\Phi(\theta, u(t)), \Phi(\theta, u)\Delta(t), t)$$

is independent of $\theta$.

Finally, on time scales with $\nabla$-differentiable $\sigma$, we prove the following Noether-type theorem providing a constant of motion for $\nabla \circ \Delta$-differential Euler-Lagrange equations $\text{EL}^{\text{diff}}$, admitting a symmetry.

**Theorem 3** (Noether). Let us assume that $\sigma$ is $\nabla$-differentiable on $T_\kappa$ and let $\Phi$ be a one-parameter family of infinitesimal transformations of $\mathbb{R}^n$. If $L$ is invariant under the action of $\Phi$, then for every solution $u$ of $\text{EL}^{\text{diff}}$, there exists $c \in \mathbb{R}$ such that:

$$\frac{\partial L}{\partial v}(u(t), u^\Delta(t), t) \cdot \frac{\partial \Phi}{\partial \theta}(0, u^\sigma(t)) = c,$$

for every $t \in T_\kappa$.

**Proof.** Let $u$ be a solution of $\text{EL}^{\text{diff}}$. Let us differentiate the map given by Equation (27) with respect to $\theta$ and let us invert the operators $\Delta$ and $\partial/\partial \theta$ from the $C^2$-regularity of $\Phi$. We obtain for every $\theta \in [-\eta, \eta]$ and every $t \in T_\kappa$:

$$\frac{\partial L}{\partial x}(\Phi(\theta, u(t)), \Phi(\theta, u)\Delta(t), t) \cdot \frac{\partial \Phi}{\partial \theta}(\theta, u(t)) + \frac{\partial L}{\partial v}(\Phi(\theta, u(t)), \Phi(\theta, u)\Delta(t), t) \cdot \frac{\partial \Phi}{\partial \theta}(\theta, u)\Delta(t) = 0.$$
Taking $\theta = 0$, it holds for every $t \in T^\kappa_\kappa$:

$$\frac{\partial L}{\partial x}(u(t), u^\Delta(t), t) \cdot \frac{\partial \Phi}{\partial \theta}(0, u(t)) + \frac{\partial L}{\partial v}(u(t), u^\Delta(t), t) \cdot \frac{\partial \Phi}{\partial \theta}(0, u^\Delta(t)) = 0.$$  \hfill (32)

Finally, multiplying this last equality by $\sigma \nabla(t)$ and using that $u$ is solution of $[\text{EL}_\text{diff}^\Delta \nabla]$ on $T^\kappa_\kappa$, we obtain:

$$\left[ \frac{\partial L}{\partial v}(u, u^\Delta, \cdot) \right] \nabla(t) \cdot \frac{\partial \Phi}{\partial \theta}(0, u(t)) + \sigma \nabla(t)(u(t), u^\Delta(t), t) \cdot \frac{\partial \Phi}{\partial \theta}(0, u^\Delta(t)) = 0,$$  \hfill (33)

for every $t \in T^\kappa_\kappa$. Finally, from the Leibniz formula obtained in Proposition 6 it holds:

$$\left[ \frac{\partial L}{\partial v}(u, u^\Delta, \cdot) \frac{\partial \Phi}{\partial \theta}(0, u) \sigma \right] \nabla(t) = 0,$$  \hfill (34)

for every $t \in T^\kappa_\kappa$. From Proposition 2 the proof is complete.  \hfill \Box

Note that this theorem both recovers the usual Noether’s theorems given in the continuous case \cite[p.88]{4} and in the (non shifted) discrete case \cite[Theorem 6.4]{18}.

## 5 The $\nabla$-analogous results

We conclude this paper with the following remark. The whole study made in this paper can be analogously derived for non shifted calculus of variations with Lagrangian functionals written with a $\nabla$-integral dependent on a $\nabla$-derivative.

Precisely, let us assume that $\rho$ is $\Delta$-differentiable. Then, the following $\Delta \circ \nabla$-differential Euler-Lagrange equation on $T^\kappa_\kappa$:

$$\left[ \frac{\partial L}{\partial v}(u, u^\nabla, \cdot) \right] \Delta(t) = \rho^\Delta(t) \frac{\partial L}{\partial x}(u(t), u^\nabla(t), t)$$  \hfill (EL$^\Delta$)$\Delta \nabla$

characterizes the critical points of the following (non shifted) Lagrangian functional:

$$\mathcal{L} : C^1_{\text{id}}(T) \rightarrow \mathbb{R},$$

$$u \mapsto \int_a^b L(u(\tau), u^\nabla(\tau), \tau) \nabla \tau.$$  \hfill (35)

In particular, a necessary condition for local optimizers of $\mathcal{L}$ is to be a solution of $[\text{EL}_\text{diff}^\Delta \nabla]$.

Moreover, let us assume that $L$ is invariant under the action of a one-parameter family $\Phi$ of infinitesimal transformations of $\mathbb{R}^n$ in the sense that for every solution $u$ of $[\text{EL}_\text{diff}^\Delta \nabla]$ and every $t \in T^\kappa_\kappa$, the map

$$\theta \mapsto L(\Phi(\theta, u(t)), \Phi(\theta, u^\nabla(t), t))$$  \hfill (36)

is constant. Then, for every solution $u$ of $[\text{EL}_\text{diff}^\Delta \nabla]$, there exists $c \in \mathbb{R}$ such that:

$$\frac{\partial L}{\partial v}(u(t), u^\nabla(t), t) \cdot \frac{\partial \Phi}{\partial \theta}(0, u^\nabla(t)) = c,$$  \hfill (37)

for every $t \in T^\kappa_\kappa$.

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