Open strings, Born–Infeld action and the heat kernel

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Abstract

In the derivation of the Born-Infeld action for the case with a nontrivial boundary of the string world sheet the appearance of a new term changes the conformal anomaly. This may have many consequences, especially also in the study of generalized interacting brane systems.

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1 Introduction

An essential ingredient for the proper formulation of systems consisting of strings and (D-) branes is the Born-Infeld (BI) action. The pioneering works \([1–4]\) derive it from the condition that the beta-function for the string vanishes, when the string is embedded in an external gauge field. From the extensive literature on the application of this action in brane theory and its vast number of consequences \([5]\) we only refer to some recent work \([6]\) which is also based upon older results \([7,8]\).

In our present note we address some delicate point in the derivation of the BI action which is related to the existence of a nontrivial boundary. The use of the heat kernel technique appears to be a necessity in this context. However, in order to obtain a well-posed spectral problem it turns out that also a new rule for Euclidean continuation has to be introduced. We also rely heavily on the seminal paper of O. Alvarez \([9]\) who already a long time ago strongly emphasized the consistent (gauge-invariant) treatment of the boundary. In our present case these techniques are extended by the introduction of an (abelian) gauge field.

2 The sigma model

To simplify the discussion we represent the string by a sigma model action with Euclidean metric both on the world sheet and in space-time

\[
S = \frac{1}{2\pi\alpha'} \left[ \frac{1}{2} \int_{\mathcal{M}} \sqrt{h} \, h^{ab} \partial_a X_\mu \partial_b X^\mu + \int_{\partial\mathcal{M}} d\tau A_\mu \partial_\tau X^\mu \right]
\]  

(1)

The string tension \(2\pi\alpha'\) which is usually written as a factor in front of the boundary term has been absorbed in \(A_\mu\) for convenience. We suppose that the metric \(h_{ab}\) is flat, but the boundary \(\partial\mathcal{M}\) can be of an arbitrary shape and may contain several connected components. We ignore non-trivial background fields in the closed string sector \((G_{\mu\nu} = \delta_{\mu\nu})\) and do not include a dilaton interaction and the \(B\)-field. The generalization is straightforward for the case when these fields are taken into account. Let \(N^a\) be an inward pointing unit normal to the boundary. We choose the coordinate system in such way that on \(\partial\mathcal{M}\) the vectors \(N_a dz^a, d\tau\) form an orthonormal pair.

It should be emphasized that the boundary term in (1) is real in contrast to the Euclidean actions of \([1,3]\). There the relative factor of \(i\) originated from the continuation of the volume element \(\sqrt{-h} \rightarrow i\sqrt{h}\) in the first term. We suggest to also rotate \(A_\mu\) to \(iA_\mu\) during continuation to the Euclidean space so that the factor \(i\) is compensated. Actually, it does not matter which particular continuation is chosen as long as one gets a meaningful theory in Euclidean space so that a proper continuation back to the physical Minkowski space is possible. As will be demonstrated below, our rule of continuation has the crucial advantage to provide a well-posed spectral problem in Euclidean space, as opposed to the
usual approach [1, 3] when the factor $i$ enters the boundary conditions for real fields and thus makes the boundary value problem ill-defined. The fact that parity odd fields can get an imaginary factor in the Euclidean space has been observed long ago [10] in the context of chiral theories. It is easy to see that the field $A_\mu$ is of odd parity from the world volume point of view. Indeed, the last term in (1) can be rewritten as \[ \int_M \partial_a (\varepsilon^{ab} A_\mu \partial_b X^\mu). \] Here $A_\mu$ couples to the parity-odd quantity $\varepsilon^{ab}$. Whenever it is possible to compare our results to those of previous related papers [1, 3] where this mathematical subtlety was just ignored, they are compatible after the replacement $A \to iA$.

Let us expand the action (1) around an arbitrary background $\bar{X}$, $X = \bar{X} + \xi$. To calculate the one-loop (in the field theory sense) effective action we need only the part which is quadratic in $\xi$:

\begin{equation}
S_2 = \frac{1}{2\pi \alpha'} \left[ \frac{1}{2} \int_M d^2z \sqrt{h} \varepsilon^{ab} \partial_a \xi^\mu \partial_b \xi^\mu + \frac{1}{2} \int_{\partial M} \left( F_{\nu\mu} \xi^\nu \xi^\mu + \dot{\bar{X}}^\mu \partial_\nu F_{\rho\mu} \xi^\nu \xi^\rho \right) \right] \tag{2}
\end{equation}

Clearly the one-loop effective action is

\begin{equation}
W = \frac{1}{2} \log \det (-\Delta_{\delta_{\mu\nu}}), \tag{3}
\end{equation}

with the scalar Laplacian $\Delta$ and with the boundary condition

\begin{equation}
(-\partial_N \delta_{\mu\nu} + F_{\mu\nu} \partial_\tau + \dot{\bar{X}}^\rho \partial_\rho (\partial_\mu F_{\nu\rho}))\xi^\mu|_{\partial M} = 0 \tag{4}
\end{equation}

where $\partial_N = N^a \partial_a$ is the derivative with respect to the inward pointing unit normal vector $N$. It is useful to rewrite (1) by adding and subtracting a term with $\mu \leftrightarrow \nu$ as

\begin{equation}
B\xi|_{\partial M} = \left( \partial_N + \frac{1}{2} (\partial_\tau \Gamma + \Gamma \partial_\tau) + S \right) \xi|_{\partial M} = 0 \tag{5}
\end{equation}

where

\begin{equation}
\Gamma_{\mu\nu} = -F_{\mu\nu}, \quad S_{\mu\nu} = -\frac{1}{2} \dot{\bar{X}}^\rho \partial_\rho F_{\mu\nu} + \partial_\mu F_{\nu\rho}. \tag{6}
\end{equation}

Thus the boundary condition (5) also ensures the hermiticity of the Laplace operator. Indeed,

\begin{equation}
\int_M (\xi'_{\mu} \Delta \xi^\mu - \xi^\mu \Delta \xi'_{\mu}) = \int_{\partial M} (-\xi_{\mu} \partial_N \xi^\mu + \xi^\mu \partial_N \xi_{\mu}) = 0 \tag{7}
\end{equation}

if $\xi$ and $\xi'$ satisfy (3), when the hermiticity of $\frac{1}{2} (\partial_\tau \Gamma + \Gamma \partial_\tau) + S$ is used. The boundary condition (5) contains tangential derivatives and belongs to the Gilkey-Smith class\footnote{Sometimes the boundary conditions with tangential derivatives are called “mixed” [11]. In the theory of the heat equation asymptotics the name “mixed” is reserved for a completely different type of the boundary conditions [12].}.
3 The heat kernel expansion

In this section we collect some basic information on the heat kernel expansion for Gilkey–Smith boundary conditions. The spectral geometry of such boundary conditions was first studied in [13, 14]. The increased interest for that heat kernel expansion with these boundary conditions in recent years was motivated primarily by one-loop calculations in quantum cosmology [15–19].

Consider an operator of the Laplace type $D = -\nabla^a \nabla_a + E$ acting in a smooth vector bundle over a smooth Riemannian manifold $\mathcal{M}$ of dimension $m$. $\nabla$ denotes a covariant derivative with respect to a certain connection on $\mathcal{M}$. $E$ is an endomorphism (a matrix-valued function). Let the boundary operator be

$$B = \nabla_N + \frac{1}{2}(\Gamma^i \nabla_i + \nabla_i \Gamma^i) + S,$$

where $\Gamma^i$ and $S$ are some matrices depending on the coordinates on the boundary, $i = 1, \ldots, m - 1$. The covariant derivative $\nabla$ contains both the standard Levi-Civita connection of the boundary and the restriction of the bundle connection to the boundary. The operator (8) defines the boundary condition $B\phi|_{\partial \mathcal{M}} = 0$. The Laplace operator $D$ is symmetric if the $\Gamma^i$’s are antihermitian and $S$ is hermitian. In our case the matrices (8) satisfy this requirement.

If $f$ is a smooth function on $\mathcal{M}$, there is an asymptotic series as $t \to +0$ of the form

$$\text{Tr}(f \exp(-tD)) = \sum_{n \geq 0} t^{-m/2 + n} a_n(f, D, B),$$

where $n = 0, 1, 1, \ldots$. The functional trace in the space of square integrable functions is denoted by Tr. The heat kernel (1) is well defined only if the boundary value problem is strongly elliptic. A criterion for strong ellipticity for the boundary conditions (8) has been proved recently in [19]. Roughly speaking, it requires the absolute values of the eigenvalues of $\Gamma^i$ to be smaller than 1. In the present context the points where the strong ellipticity is lost correspond to critical values of the electric field, which were discussed for Minkowski signature in [8].

The bulk terms in the heat kernel expansion do not depend on $\Gamma$. The leading boundary contributions to the heat kernel can be taken directly from refs. [14, 17]

$$a_{1/2}(f, P, B) = \frac{1}{(4\pi)^{(m-1)/2}} \int_{\partial \mathcal{M}} \text{tr}(\gamma f),$$

$$a_1(f, P, B) = \frac{1}{(4\pi)^{m/2}} \int_{\partial \mathcal{M}} \text{tr} \left(f(b_0 k + b_2 S + \sigma_1 k_{ij} \Gamma^i \Gamma^j) + b_1 \nabla_N f \right),$$

where $k_{ij}$ is the extrinsic curvature of the boundary. Here tr is the ordinary
matrix trace. The functions \( b_0, b_1, b_2, \gamma, \sigma_1 \) are

\[
\begin{align*}
\gamma &= \frac{1}{4} \left[ \frac{2}{\sqrt{1+\Gamma^2}} - 1 \right], \\
b_1 &= \frac{1}{\sqrt{1+\Gamma^2}} \text{Artanh}(\sqrt{-\Gamma^2}) - \frac{1}{2}, \\
b_2 &= \frac{2}{1+\Gamma^2}, \\
b_0 + \sigma_1 \Gamma^2 &= \frac{1}{3},
\end{align*}
\]

(11)

In two dimensions the functions \( b_0 \) and \( \sigma_1 \) enter the heat kernel expansion only in the combination \( b_0 + \sigma_1 \Gamma^2 \) due to the identity \( k_{ij} \Gamma^i \Gamma^j = k \Gamma^2 \), valid on the one-dimensional boundary.

In this paper we are not discussing the contribution of the zero modes to the path integral. Therefore, strictly speaking the equations (10) and (11) are valid only up to some “global” effects.

### 4 Beta function and the conformal anomaly

Next we make use of the \( \zeta \)-function regularization [20]. The zeta function of an elliptic operator \( D \) is defined as

\[
\zeta_D(s) = \text{Tr}(D^{-s}).
\]

(12)

In term of the zeta function (12) the effective action (3) reads

\[
W = -\frac{1}{2s} \zeta_D(0) - \frac{1}{2} \zeta_D'(0),
\]

(13)

where the prime denotes differentiation with respect to \( s \).

By using the well known relation between the zeta function and the heat kernel coefficients \( \zeta_D(0) = a_1(1, D, B) \) the divergent part of the effective action at \( s \to 0 \) may be written as

\[
W_{\text{div}} = -\frac{1}{2s} \frac{1}{4\pi} \int_{\partial M} d\tau \left[ -\dot{X}^\rho (\partial_\nu F_{\mu\rho} + \partial_\mu F_{\nu\rho})(1 + F^2)^{-1} + \frac{1}{3} k \delta^\nu_\nu \right].
\]

(14)

The first term on the right hand side of (14) can be represented as \((1/2\pi) \int_{\partial M} d\tau G_\mu \dot{X}^\mu \). From this we can read off the beta function (in the notations of [3])

\[
\beta^A_\mu \propto (\partial_\rho F_{\nu\mu})(1 + F^2)^{-1}.
\]

(15)

After collecting the factors of \( i \) which originate from our prescription for Euclidean continuation it gives the same equation of motion as in [3] and, therefore,
reproduces the variation of the BI action \([1]\) for the \(A_\mu\). It should be stressed that
to derive this result we neither supposed a special geometry of the world sheet,
nor had to neglect higher derivatives of \(F_{\mu\nu}\).

The second term under the integral in (14), which is proportional to the
dimension of the target space \(\delta_{\nu\nu}\), does not depend on \(F\)
and is always present in
the theory of open strings \([9]\). Since we had assumed that the scalar curvature
of the two-manifold \(\mathcal{M}\) is zero, this term can be expressed in terms of the Euler
characteristic of \(\mathcal{M}\):
\[
2\pi \chi(\mathcal{M}) = \int_{\partial \mathcal{M}} d\tau k.
\]

We next turn to the conformal anomaly. An infinitesimal conformal transforma-
tion \(\delta h_{ab} = (\delta k)h_{ab}\) with a local parameter \(\delta k\) produces the trace of the
(effective) energy momentum tensor
\[
\delta W_{\text{ren}} = \frac{1}{2} \int_{\mathcal{M}} d^2 z \sqrt{h} \delta h^{ab} T_{ab} = -\frac{1}{2} \int_{\mathcal{M}} d^2 z \sqrt{h} \delta k(x) T^a_a(x),
\]
where the \(W_{\text{ren}}\) is the second (finite) term in (13). It is quite important that both
the Laplace operator and the boundary operator transform covariantly under the
metric rescaling
\[
\Delta \to (1 - k + \ldots) \Delta,
\]
\[
B \to (1 - \frac{k^2}{2} + \ldots) B.
\]
The second property (18) follows from our assumption that \(N_a dz^a, d\tau\) are two
orthonormal vectors – which we used to derive the equation (4). It guarantees
that the functional space defined by (4) is invariant under the conformal trans-
formations.

With the definition of a generalized \(\zeta\)-function
\[
\zeta(s|\delta k, D) = \text{Tr}(\delta k D^{-s})
\]
the variation in (16) can be identified with
\[
\delta W_{\text{ren}} = -\frac{1}{2} \zeta(0|\delta k, D),
\]
where we used \(\delta \zeta_D(s) = s \text{Tr}(D^{-s} \delta k)\). Combining (21) and (16) we obtain
\[
\zeta(0|\delta k, D) = \int d^2 z \sqrt{h} \delta k(z) T^a_a(x).
\]
By a Mellin transformation one can show that \(\zeta(0|\delta k, D) = a_1(\delta k, D, B)\). The
(smeared) conformal anomaly reads:
\[
\int_{\mathcal{M}} \sqrt{h} d^2 z f(z) T^a_a(z) = \frac{1}{4\pi} \int_{\partial \mathcal{M}} d\tau \left[ f(\tau) \left( \frac{1}{3} k \delta^\nu_{\nu} - 2 \dot{X}^\rho (\partial_{\nu} F_{\rho\mu})(1 + F^2)^{-1}_{\nu\mu} \right) \right. \\
+ \left. (\nabla_N f) \left( (\sqrt{-F^2})_{\mu\nu}^{-1/2} \text{Artanh}(\sqrt{-F^2})_{\nu\mu} - \frac{1}{2} \delta_{\mu\nu} \right) \right].
\]
In the limit $F \to 0$ the conformal anomaly (22) coincides with the standard expression [4]. The second term on the first line appears quite naturally and is a manifestation of the BI action. The second line contains a somewhat unusual contribution. To the best of our knowledge it has not been obtained before. To perform a full-scale analysis of this term one should include the dilaton field in the “bare” action (1). We postpone this to a future more detailed publication. In any case, the last term in (22) suggests a very interesting interplay between the gauge sector and the conformal sector of string theory.

5 Conclusions

By a careful interpretation of the transition to Euclidean space we are able to reformulate the problem of the string in the presence of a nontrivial boundary where the string is coupled to an abelian background field. As a consequence of the effect that our operator, appearing in the boundary condition, is hermitian we arrive at a well-posed elliptic problem in the sense of the heat kernel technique where known formulas from that field can be applied directly. Our central result is the one for the conformal anomaly at the boundary, Eq. (22). Beside the standard term leading to the BI action for the gauge field from the vanishing of the beta-function and a term proportional to the extrinsic curvature at the boundary (which was known for a long time [4]) we find a new contribution which depends on the gauge field $F_{\mu\nu}$. It should be stressed that our whole argument (in contrast to previous ones for the BI action) contains no restriction on the vanishing of derivatives for $F_{\mu\nu}$. Also no special geometry of the world volume need be assumed. This means that our calculations are valid for an arbitrary number of string loops. Of course, higher loop corrections (in the sense of quantum field theory) will contain higher derivatives of $F_{\mu\nu}$ [21].

We see a rather wide range of applications of our present result, the most obvious extension being the presence of other fields [22, 23] which, however, should not present any new technical difficulties. Also the indication for a further contribution to the anomaly at the intersection of branes possibly could shed new light upon the problems encountered for interacting strings and branes [24]. There could be even a relation to the very recent work on the noncommutative geometry approach for strings and branes [25].

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Note added

After this paper has been completed and posted on the net we were informed by Professor Osborn that our expression for the conformal anomaly (22) (that is our main result) is contained in the Appendix of his paper \[26\]. We find it however quite striking that such important result is not widely recognised in modern literature on strings and branes. Therefore, we decided to leave our preprint on the net and add this short note.

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