Boundary algebraic Bethe Ansatz for a nineteen vertex model with $U_q[\text{osp}(2|2)^{(2)}]$ symmetry

R S Vieira and A Lima-Santos

Departamento de Física, Universidade Federal de São Carlos, caixa-postal 676, CEP. 13565-905, São Carlos, SP, Brasil
E-mail: rsvieira@df.ufscar.br and dals@df.ufscar.br

Received 28 July 2017
Accepted for publication 6 August 2017
Published 27 September 2017

Online at stacks.iop.org/JSTAT/2017/093107
https://doi.org/10.1088/1742-5468/aa85ca

Abstract. The boundary algebraic Bethe Ansatz for a supersymmetric nineteen vertex-model constructed from a three-dimensional representation of the twisted quantum affine Lie superalgebra $U_q[\text{osp}(2|2)^{(2)}] \simeq U_q[\text{C}(2)^{(2)}]$ is presented. The eigenvalues and eigenvectors of Sklyanin’s transfer matrix, with diagonal reflection $K$-matrices, are calculated and the corresponding Bethe Ansatz equations are obtained. A numerical analysis for a small-length chain is also presented.

Keywords: quantum integrability (Bethe Ansatz), integrable spin chains and vertex models, solvable lattice models
1. The model

Over the last decades, great interest has been aroused in the study of supersymmetric integrable systems. In fact, supersymmetry is now present in several fields of contemporary mathematics and physics, ranging from condensed matter physics to superstring theory. We can cite, for instance, the graded generalizations of Hubbard and $t-J$ models [1–3], which play an important role in condensed matter physics, and also the search for solutions of the graded Yang–Baxter equation (YBE) [4–9], which gave origin to important algebraic constructions as the supersymmetric Hopf algebras and quantum groups [10].

More recently, the integrability of supersymmetric models also proved to be important in superstring theory, more specifically in the AdS/CFT correspondence [11–14].

The most powerful and beautiful method to analyze these integrable quantum systems probably is the algebraic Bethe Ansatz (ABA) [15–17]. This technique allows one to diagonalize the transfer matrix of a given integrable quantum system in an analytical way. The ABA was originally applied to systems with periodic boundary conditions but after the work of Sklyanin [18], integrable models with non-periodic boundaries could also be handled. Further generalizations [19, 20] showed that the ABA can be applied to several classes of integrable models, described by different Lie algebras and superalgebras, with both periodic as non-periodic boundary conditions. In the case of the periodic ABA, the fundamental object is a $R$-matrix, solution of the YBE, while in the case of the boundary ABA, other fundamental ingredient is necessary: the $K$-matrices (or reflection matrices), which are solutions of the boundary Yang–Baxter equations (a.k.a. reflection equations) [18–20].
The aba was successfully applied to nineteen vertex models. In fact, after Tarasov [21] used this technique to solve the Izergin–Korepin model [22] with periodic boundary conditions, the Zamolodchikov–Fateev vertex-model [23] was also solved by Lima–Santos [24]. The boundary aba for these vertex models were performed in [25, 26] together with the supersymmetric sl(2|1) and osp(2|1) vertex models. Several other important results were obtained for nineteen vertex-models—see, for instance, [27] and references therein.

In this work we shall study another graded three-state nineteen-vertex model with reflecting boundary conditions. We derive the boundary aba for a supersymmetric nineteen-vertex model that was presented by Yang and Zhang in [28]. The $R$-matrix associated with this model is constructed from a three-dimensional free boson representation $V$ of the twisted quantum affine Lie superalgebra $U_q[osp(2|2)^{(2)}] \simeq U_q[C(2)^{(2)}]$ and the periodic aba for this model was presented in [29]. We would like to emphasize that vertex-models described by Lie superalgebras—and, in particular, by twisted Lie superalgebras—are usually the most complex ones, which is due, of course, to the high complexity of such Lie superalgebras [30–35]. In fact, even the reflection $K$-matrices of those models were not yet completely classified, although a great advance in this direction has been obtained in the last years [36–41]. In particular, the reflection $K$-matrices of the Yang–Zhang model were derived recently by us in [27].

Since we shall deal here with a supersymmetric system, it will be helpful to remember first the basics of the graded Lie algebras [42]. Let $W = V \oplus U$ be a $\mathbb{Z}_2$-graded vector space where $V$ and $U$ denote its even and odd parts, respectively. In a $\mathbb{Z}_2$-graded vector space we associate a gradation $p(i)$ to each element $e_i$ of a given basis of $V$. In the present case, we shall consider only a three-dimensional representation of the twisted quantum affine Lie superalgebra $U_q[osp(2|2)^{(2)}]$ with a basis $E = \{\epsilon_1, \epsilon_2, \epsilon_3\}$ and the grading $p(1) = 0$, $p(2) = 1$ and $p(3) = 0$. Multiplication rules in the graded vector space $W$ differ from the ordinary ones by the appearance of additional signs. For example, the graded tensor product of two homogeneous even elements $A \in \text{End}(V)$ and $B \in \text{End}(V)$ turns out to be defined by the formula,

$$A \otimes^g B = \sum_{i,j,k,l=1}^{d} (-1)^{p(i)p(k)+p(j)p(k)} A_{ij} B_{kl} (\epsilon_{ij} \otimes \epsilon_{kl}) ,$$

where $d$ (in the present case, $d = 3$) is the dimension of the vector space $V$ and $\epsilon_{ij}$ are the Weyl matrices ($\epsilon_{ij}$ is a matrix in which all elements are null, except that element on the $[i,j]$ position, which equals 1). In the same fashion, the graded permutation operator $P^g$ is defined by

$$P^g = \sum_{i,j=1}^{d} (-1)^{p(i)p(j)} (\epsilon_{ij} \otimes \epsilon_{ji}) ,$$

and the graded transposition $A^{tg}$ of a matrix $A \in \text{End}(V)$ as well as its inverse graded transposition, $A^{t^g}$, are defined, respectively, by

$$A^{tg} = \sum_{i,j=1}^{d} (-1)^{p(i)p(j)+p(i)} A_{ji} \epsilon_{ij} , \quad A^{t^g} = \sum_{i,j=1}^{d} (-1)^{p(i)p(j)+p(j)} A_{ji} \epsilon_{ij} .$$

https://doi.org/10.1088/1742-5468/aa85ca 3
so that $A^{\tau g \tau g} = A^{\tau g \tau g} = A$. Finally, the graded trace of a matrix $A \in \text{End}(V)$ is given by

$$\text{tr}^g(A) = \sum_{i=1}^{d} (-1)^{\mu(i)} A_{ii} e_{ii}. \quad (4)$$

In the graded case, both the periodic YBE [4–9],

$$R_{12}(x) R_{13}(xy) R_{23}(y) = R_{23}(y) R_{13}(xy) R_{12}(x), \quad (5)$$

as well as the boundary YBE [18–20],

$$R_{12}(x/y) K_1^-(x) R_{21}(xy) K_2^-(y) = K_2^-(y) R_{12}(xy) K_1^-(x) R_{21}(x/y), \quad (6)$$

can be written in the same way as in the non-graded case: it is only necessary to employ graded operations instead of the usual operations [20].

The $R$-matrix, solution of the graded YBE (5), associated with the Yang–Zhang vertex-model [28] can be written, up to a normalizing factor and employing a different notation, as follows:

$$R(x) = \begin{pmatrix}
    r_1(x) & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & r_2(x) & 0 & r_5(x) & 0 & 0 & 0 & 0 \\
    0 & 0 & r_3(x) & 0 & r_6(x) & 0 & r_7(x) & 0 \\
    0 & s_5(x) & 0 & r_2(x) & 0 & 0 & 0 & 0 \\
    0 & 0 & s_6(x) & 0 & r_4(x) & 0 & r_6(x) & 0 \\
    0 & 0 & 0 & 0 & r_2(x) & 0 & r_5(x) & 0 \\
    0 & 0 & s_7(x) & 0 & s_6(x) & 0 & r_3(x) & 0 \\
    0 & 0 & 0 & 0 & s_5(x) & 0 & r_2(x) & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 & r_1(x)
\end{pmatrix},$$

where the amplitudes $r_i(x)$ and $s_i(x)$ are given respectively by

$$r_1(x) = q^2 x - 1, \quad (8)$$

$$r_2(x) = q(x - 1), \quad (9)$$

$$r_3(x) = q(q + x)(x - 1)/(qx + 1), \quad (10)$$

$$r_4(x) = q(x - 1) - (q + 1)(q^2 - 1)x/(qx + 1), \quad (11)$$

$$r_5(x) = q^2 - 1, \quad (12)$$

$$r_6(x) = -q^{1/2}(q^2 - 1)(x - 1)/(qx + 1), \quad (13)$$

$$r_7(x) = (q - 1)(q + 1)^2/(qx + 1), \quad (14)$$

$$s_5(x) = (q^2 - 1)x = x r_5(x), \quad (15)$$

$$s_6(x) = -q^{1/2}(q^2 - 1)x(x - 1)/(qx + 1) = x r_6(x), \quad (16)$$

https://doi.org/10.1088/1742-5468/aa85ca
Boundary algebraic Bethe Ansatz for a nineteen vertex model with $\mathcal{U}_q[\text{osp}(2|2)]$ symmetry

\[ s_\gamma(x) = (q - 1)(q + 1)^2 x^2 / (qx + 1) = x^2 r_\gamma(x). \]  

(17)

This $R$-matrix has the following properties or symmetries [27]:

- Regularity: $R_{12}(1) = f(1)^{1/2} P_{12}^g$,

(18)

- Unitarity: $R_{12}(x) = f(x) R_{21}^{-1}(x^{-1})$,

(19)

- Super-PT: $R_{12}(x) = R_{21}^{t_2 t_1^3}(x)$,

(20)

- Crossing: $R_{12}(x) = g(x) \left[ V_1 R_{12}^{y_1} (y^{-1} x^{-1}) V_1^{-1} \right]$,  

(21)

where

\[ f(x) = r_1(x) r_1 \left( \frac{1}{x} \right) = (q^2 x - 1) \left( \frac{q}{x} - 1 \right), \quad g(x) = -\frac{q x (x - 1)}{q x + 1}. \]  

(22)

Here, $t_1^\tau$ and $t_2^\tau$ mean graded partial transpositions in the first and second vector spaces, respectively; $\tau_1^\tau$ and $\tau_2^\tau$ the corresponding inverse operations. Besides, $\eta = -q$ is the crossing parameter while

\[ M = V^{t_\tau} V = \text{diag} (1/q, 1, q) \]  

(23)

is the crossing matrix.

Solutions of the boundary YBE (6) for this vertex model were presented recently in [27]. The most general regular diagonal reflection $K$-matrix,

\[ K^{-}(x) = \text{diag} \left( k_{1,1}^{-}(x), k_{2,2}^{-}(x), k_{3,3}^{-}(x) \right), \]  

(24)

of the Yang–Zhang vertex model—which is of interest in the present work—has the following entries [27]:

\[ k_{1,1}^{-}(x) = 1 + \frac{1}{2} \left( x^2 - 1 \right) \beta_{1,1}, \]  

(25)

\[ k_{2,2}^{-}(x) = x \left[ \frac{(\beta_{2,2} - \beta_{1,1}) x + (\beta_{1,1} - \beta_{2,2} + 2)}{(\beta_{1,1} - \beta_{2,2} + 2) x + (\beta_{2,2} - \beta_{1,1})} \right] \left[ 1 + \frac{1}{2} \left( x^2 - 1 \right) \beta_{1,1} \right], \]  

(26)

\[ k_{3,3}^{-}(x) = x^2 \left[ \frac{(\beta_{1,1} - \beta_{2,2} + 2) - q x (\beta_{2,2} - \beta_{1,1})}{(\beta_{2,2} - \beta_{1,1}) x + q (\beta_{2,2} - \beta_{1,1})} \right] \times \left[ \frac{(\beta_{2,2} - \beta_{1,1} - 2) - (\beta_{2,2} - \beta_{1,1}) x}{(\beta_{2,2} - \beta_{1,1} - 2) x + q (\beta_{2,2} - \beta_{1,1})} \right] \left[ 1 + \frac{1}{2} \left( x^2 - 1 \right) \beta_{1,1} \right], \]  

(27)

where $\beta_{1,1}$ and $\beta_{2,2}$ are the boundary free-parameters of the solutions. Notice that the properties (18)–(21), enjoyed by the $R$-matrix (7), ensure the existence of the dual reflection equation,

\[ R_{12}(x/y) K_{1}^{\dagger}(x) M_{1}^{-1} R_{21} \left( \frac{\eta}{x y} \right) M_{1} K_{2}^{\dagger}(y) = M_{1} K_{2}^{\dagger}(y) R_{12} \left( \frac{\eta}{x y} \right) M_{1}^{-1} K_{1}^{\dagger}(x) R_{12}(x/y), \]  

(28)

https://doi.org/10.1088/1742-5468/aa85ca
as described by Bracken et al. [20]. Besides, the dual reflection matrices $K^+(x)$ which are solutions of the dual reflection equation (28) can determined by the following isomorphism [20, 27]

$$K^+(x) = K^-(q^{-1}x^{-1})M,$$  \hspace{1cm} (29)

with a new set of boundary free-parameters (say, with $\alpha_{i,j}$ replacing $\beta_{i,j}$). It is to be noticed that special values for the boundary free-parameters lead to particular reflection matrices, for instance, the quantum group invariant solutions $K^-(x) = I$ and $K^+(x) = M$.

2. The boundary algebraic Bethe Ansatz

The aba for quantum integrable systems containing diagonal boundaries was developed by Sklyanin [18] for integrable systems described by symmetric $R$-matrices. Menisezcu and Nepomechie extended Sklyanin’s formalism to get account of non-periodic $R$-matrices [19] and the graded case was developed further by Bazhanov and Shadrikov [9] and also Bracken et al [20].

The fundamental ingredient of the boundary aba is the Sklyanin transfer matrix,

$$t(x) = \text{tr}_a\left[K_a^+(x)U(x)\right], \quad \text{where,} \quad U(x) = T(x)K_a^-(x)T^{-1}(1/x),$$  \hspace{1cm} (30)

is the Sklyanin monodromy and

$$T(x) = R_{aN}(x) \cdots R_{a1}(x), \quad T^{-1}(1/x) = R_{1a}(x) \cdots R_{Na}(x)/f(x)^N.$$  \hspace{1cm} (31)

The operators $R_{aq}(x)$ appearing in the expressions above act in $\text{End} (V_a \otimes V_q)$, where $V_a$ denotes the auxiliary space and $V_q$, for $1 \leq q \leq N$, are the quantum spaces associated with a lattice of $N$ sites. Notice moreover that the expression for $T^{-1}(1/x)$ follows from the unitarity property (19) enjoyed by the $R$-matrix (7).

The main feature of the boundary aba is that the transfer matrix (30) commutes with itself for any values of the spectral parameters $x$ and $y$, that is,

$$[t(x), t(y)] = 0, \quad \forall \{x, y\} \in \mathbb{C},$$  \hspace{1cm} (32)

which can be proved through the (boundary) Yang–Baxter algebra plus the symmetries enjoyed by the $R$-matrix of the respective integrable model [18–20]. This means that $t(x)$ can be thought as the generator of infinitely many conserved quantities in involution, which justifies the name of integrable to systems that can be solved by the boundary aba. In particular, the Hamiltonian is a such conserved charge, and it is given by,

$$H = \sum_{i=1}^{N-1} H_{i,i+1} + \frac{1}{2} \left. \frac{dK_a^-(x)}{dx} \right|_{x=1} + \frac{\text{tr}_a^q[K_a^+(1)H_{N,a}]}{\text{tr}_a^q[K_a^+(1)]},$$  \hspace{1cm} (33)

where

$$H_{i,i+1} = \left. \frac{d}{dx} \left[P_{i,i+1}R_{i,i+1}(x)\right] \right|_{x=1}, \quad 1 \leq i \leq N - 1.$$  \hspace{1cm} (34)
We remark, however, that the free boson realization of the $U_q[\text{osp}(2|2)^{(2)}]$ quantum Lie superalgebra considered by Yang–Zhang does not have a classical limit as $q \to 1$ [28]. This particularity prevent us to study the Gaudin magnets through the off-shell ABA for this model. Nevertheless, other realizations as, for instance, that one presented in [33], could provide other vertex models with $U_q[\text{osp}(2|2)^{(2)}]$ symmetry that do have a classical limit.

2.1. The reference state

In order to find the eigenvalues and eigenvectors of the transfer matrix through the ABA method, it is necessary to know at least one of its eigenvectors that is simple enough so that the corresponding eigenvalue can be directly computed [18]. This simple eigenvector is called reference state.

To find the reference state is useful to rewrite the transfer matrix in the Lax representation, that is, as a three-by-three operator valued matrix, say,

$$U(x) = \begin{pmatrix} A_1(x) & B_1(x) & B_2(x) \\ C_1(x) & A_2(x) & B_3(x) \\ C_2(x) & C_3(x) & A_3(x) \end{pmatrix}. \quad (35)$$

In this representation, the diagonal (graded) Sklyanin’s transfer matrix (30), becomes,

$$t(x) = k_{1,1}^{-1}(x)A_1(x) - k_{2,2}^{+}(x)A_2(x) + k_{3,3}^{+}(x)A_3(x), \quad (36)$$

and we can easily verify that the following state

$$\Psi_0 = (1, 0, 0)^t$$

is an eigenvector of the transfer matrix and, hence, it is the reference state we were looking for. In fact, the action of the monodromy elements over $\Psi_0$ can be evaluated following [25, 26], and it reads,

$$A_i(x)\Psi_0 = \alpha_i(x)\Psi_0, \quad B_i(x)\Psi_0 \neq \xi\Psi_0, \quad C_i(x)\Psi_0 = 0\Psi_0, \quad 1 \leq i \leq 3. \quad (38)$$

where $\xi$ can be any complex number and

$$\alpha_1(x) = k_{1,1}(x)r_1^{2N}(x)/f^N(x), \quad (39)$$

$$\alpha_2(x) = F_1(x)\alpha_1(x) + [k_{2,2}(x) - F_1(x)k_{1,1}(x)]r_2^{2N}(x)/f^N(x), \quad (40)$$

$$\alpha_3(x) = [F_2(x) - F_1(x)F_3(x)]\alpha_1(x) + F_3(x)\alpha_2(x)$$

$$+ [k_{3,3}(x) - F_3(x)k_{2,2}(x) - F_4(x)k_{1,1}(x)]r_3^{2N}(x)/f^N(x), \quad (41)$$

where,

$$F_1(x) = s_5(x^2)/r_1(x^2), \quad (42)$$

$$F_2(x) = s_7(x^2)/r_1(x^2), \quad (43)$$

$$F_3(x) = -\frac{r_1(x^2)s_5(x^2) - r_5(x^2)s_7(x^2)}{r_1(x^2)r_4(x^2) + r_5(x^2)s_5(x^2)}, \quad (44)$$

https://doi.org/10.1088/1742-5468/aa85ca
Boundary algebraic Bethe Ansatz for a nineteen vertex model with $U_q[osp(2|2)^{(2)}]$ symmetry

$$F_4(x) = \frac{r_4(x^2)s_7(x^2) + s_5(x^2)s_5(x^2)}{r_1(x^2)r_4(x^2) + r_5(x^2)s_5(x^2)}$$

(45)

Therefore, the action of the transfer matrix on $\Psi_0$ reads:

$$t(x)\Psi_0 = k_{1,1}^+(x)\alpha_1(x) - k_{2,2}^+(x)\alpha_2(x) + k_{3,3}^+(x)\alpha_3(x).$$

(46)

It will be more convenient, however, to introduce a new set of diagonal operators, namely,

$$D_1(x) = A_1(x),$$

(47)

$$D_2(x) = A_2(x) - F_1(x)A_1(x),$$

(48)

$$D_3(x) = A_3(x) - F_1(x)A_2(x) - [F_2(x) - F_1(x)F_3(x)]A_1(x),$$

(49)

so that their action on the reference state $\Psi_0$ simplifies to

$$D_1(x)\Psi_0 = \delta_1(x)\Psi_0, \quad D_2(x)\Psi_0 = \delta_2(x)\Psi_0, \quad D_3(x)\Psi_0 = \delta_3(x)\Psi_0,$$

(50)

with

$$\delta_1(x) = k_{1,1}^-(x)r_1^{2N}(x)/f^N(x),$$

(51)

$$\delta_2(x) = [k_{2,2}^-(x) - F_1(x)k_{1,1}^-(x)]r_2^{2N}(x)/f^N(x),$$

(52)

$$\delta_3(x) = [k_{3,3}^-(x) - F_3(x)k_{2,2}^-(x) - F_4(x)k_{1,1}^-(x)]r_3^{2N}(x)/f^N(x).$$

(53)

Similarly, the transfer matrix can be rewritten as

$$t(x) = \Omega_{1,1}(x)D_1(x) + \Omega_{2,2}(x)D_2(x) + \Omega_{3,3}(x)D_3(x),$$

(54)

with

$$\Omega_{1,1}(x) = k_{1,1}^+(x) - F_1(x)k_{2,2}^+(x) + F_2(x)k_{3,3}^+(x),$$

(55)

$$\Omega_{2,2}(x) = -k_{2,2}^+(x) + F_3(x)k_{3,3}^+(x),$$

(56)

$$\Omega_{3,3}(x) = k_{3,3}^+(x),$$

(57)

so that their action on the reference state reads now,

$$t(x)\Psi_0 = \Omega_{1,1}^+(x)\delta_1(x) + \Omega_{2,2}^+(x)\delta_2(x) + \Omega_{3,3}^+(x)\delta_3(x).$$

(58)

### 2.2. The 1-particle state

Once the action of the transfer matrix over the reference state is determined, we can turn our attention to the excited states. In the framework of the boundary $\text{ABA}$, these excited states are constructed from the action of the creators operators over the reference state $\Psi_0$. Actually, only the operators $B_1$ and $B_2$ are needed (the excited states...
are independent of operator $B_3$, see [21, 25, 26]). Furthermore, we can verify that the action of $B_2$ on $\Psi_0$ is proportional to a double action of $B_1$ on $\Psi_0$ (in the sense that $B_2$ rises the magnon number of the system twice when compared to $B_1$). From this follows that the first excited state, named here the 1-particle state, should be defined as,

$$\Psi_1(x_1) = B_1(x_1) \Psi_0.$$  \hfill (59)

The new variable $x_1$—called rapidity—must be determined in order to $\Psi_1(x_1)$ be indeed an eigenvector of the transfer matrix; we shall see below that this requirement is provided by the Bethe Ansatz equation of the 1-particle state.

From (58) and (59) we can write down the action of $t(x)$ over $\Psi_1(x_1)$:

$$t(x)\Psi_1(x_1) = \left[ \Omega_{1,1}(x)D_1(x)B_1(x_1) + \Omega_{2,2}(x)D_2(x)B_1(x_1) + \Omega_{3,3}(x)D_3(x)B_1(x_1) \right] \Psi_0. \hfill (60)$$

Notice that we shall need to know the commutation relations between the diagonal operators $D_1(x)$, $D_2(x)$ and $D_3(x)$ with $B_1(x_1)$ in order to evaluate the action of $t(x)$ on $\Psi_1(x_1)$. These commutation relations are provided by the fundamental relation of the boundary $ABA$—equation (A.1)—and they are presented in the appendix. Making use of the commutation relations (A.2)–(A.4) and simplifying the results, we can realize that the action of $t(x)$ on $\Psi_1(x_1)$ can be written as follows:

$$t(x)\Psi_1(x_1) = \tau_1(x|x_1) \Psi_1(x_1) + \beta_1(x_1) B_1(x) \Psi_0 + \beta_3(x_1) B_3(x) \Psi_0, \hfill (61)$$

where,

$$\begin{align*}
\tau_1(x|x_1) &= a_1(x, x_1)\Omega_{1,1}(x)\delta_1(x) + a_2(x, x_1)\Omega_{2,2}(x)\delta_2(x) + a_3(x, x_1)\Omega_{3,3}(x)\delta_3(x), \\
\beta_1(x_1) &= \Omega_{1,1}(x) \left[ a_2(x_1)\delta_1(x_1) + a_3(x_1)\delta_2(x_1) \right] \\
&\quad + \Omega_{2,2}(x) \left[ a_2(x_1)\delta_1(x_1) + a_3(x_1)\delta_2(x_1) \right] \\
&\quad + \Omega_{3,3}(x) \left[ a_3(x_1)\delta_1(x_1) + a_3(x_1)\delta_2(x_1) \right], \\
\beta_3(x_1) &= \Omega_{2,2}(x) \left[ a_2(x_1)\delta_1(x_1) + a_3(x_1)\delta_2(x_1) \right] \\
&\quad + \Omega_{3,3}(x) \left[ a_2(x_1)\delta_1(x_1) + a_3(x_1)\delta_2(x_1) \right].
\end{align*} \hfill (62-64)$$

Now, if $\Psi_1(x_1)$ is an eigenstate of $t(x)$, then we must have $\beta_1^2(x_1) = \beta_1^3(x_1) = 0$. This provides the Bethe Ansatz equation of the 1-particle state that implicitly fixes the rapidity $x_1$:

$$\begin{align*}
\delta_2(x_1) &= \Omega_{1,1}(x)a_2(x, x_1) + \Omega_{2,2}(x)a_2^3(x, x_1) + \Omega_{3,3}(x)a_3^2(x, x_1) \\
&= \frac{1}{\Omega_{2,2}(x)a_3^2(x, x_1) + \Omega_{3,3}(x)a_3^3(x, x_1)},
\end{align*} \hfill (65)$$

After simplify we can verify that both $\beta_1^2(x_1)$ as $\beta_1^3(x_1)$ vanish and also that the right-hand-side of (65) does not actually depend on the spectral parameter $x$, as it should. The conclusion in that $\Psi_1(x_1)$ is an eigenvector of the transfer matrix with eigenvalue $\tau_1(x|x_1)$ given by (62).
2.3. The 2-particle state

In the construction of the next excited state—the 2-particle state—both the operators $B_1$ and $B_2$ should be taken into account. This is necessary because both $B_2 \Psi_0$ as $B_1 B_1 \Psi_0$ are in the same sector (i.e. both states have the same magnon number). Therefore, the most general 2-particle state should be constructed through a linear combination of these operators and we can verify *a posteriori* that the adequate linear combination is as follows:

$$
\Psi_2 (x_1, x_2) = B_1 (x_1) B_1 (x_2) \Psi_0 + \lambda (x_1, x_2) B_2 (x_1) \Psi_0 = B_1 (x_1) \Psi_1 (x_2) + B_2 (x_1) \Psi_0.
$$

(66)

The coefficient $\lambda (x_1, x_2)$ of this linear combination can be fixed by the condition that

$$
\Psi_2 (x_2, x_1) = B_1 (x_2) B_1 (x_1) \Psi_0 + \lambda (x_2, x_1) B_2 (x_2) \Psi_0 = \omega (x_1, x_2) \Psi_2 (x_1, x_2),
$$

(67)

for some phase function $\omega (x_1, x_2)$ [21, 25, 26]. Making use of the commutation relation (A.11) between $B_1 (x_1)$ and $B_1 (x_2)$, we get that

$$
\Psi_2 (x_2, x_1) = b^1_1 (x_2, x_1) B_1 (x_1) B_1 (x_2) \Psi_0 + [b^1_2 (x_2, x_1) \delta_1 (x_2) + b^3_1 (x_2, x_1) \delta_2 (x_2)] B_2 (x_1) \Psi_0
$$

\[ + [b^1_4 (x_2, x_1) \delta_1 (x_1) + b^4_5 (x_2, x_1) \delta_2 (x_1) + \lambda (x_2, x_1) B_2 (x_2)] \Psi_0 \]

(68)

from which follows that

$$
\omega (x_1, x_2) = b^1_1 (x_2, x_1), \quad \lambda (x_1, x_2) = -b^1_4 (x_1, x_2) \delta_1 (x_2) - b^5_4 (x_1, x_2) \delta_2 (x_2),
$$

(69)

where we made use of the following properties:

$$
\omega (x_1, x_2) \omega (x_2, x_1) = 1, \quad \frac{b^1_2 (x_2, x_1)}{b^1_1 (x_2, x_1)} = \frac{b^3_1 (x_2, x_1)}{b^4_5 (x_2, x_1)} = -\omega (x_1, x_2).
$$

(70)

Once $\lambda (x_1, x_2)$ is determined, we can find the action of the transfer matrix on the 2-particle state. To this end, will be necessary to use several other commutation relations besides the previous ones, namely, the commutation relations provided by (A.5)–(A.7), (A.20) and (A.22). Although this computation maybe somewhat extensive, we can verify that the action of $t(x)$ over $\Psi_2 (x_1, x_2)$ can be written as,

$$
t(x) \Psi_2 (x_1, x_2) = \tau_2 (x) \Psi_2 (x_1, x_2) + \beta^1_1 (x_1, x_2) B_1 (x) \Psi_1 (x_2)
$$

\[ + \beta^1_2 (x_1, x_2) B_1 (x) \Psi_1 (x_1) + \beta^3_1 (x_1, x_2) B_3 (x) \Psi_1 (x_2)
$$

\[ + \beta^3_2 (x_1, x_2) B_3 (x) \Psi_1 (x_1) + \beta^2_{12} (x_1, x_2) B_2 (x) \Psi_0,
$$

(71)

where,

$$
\tau_2 (x | x_1, x_2) = \sum_{j=1}^3 \Omega_{j,j} (x) \delta_j (x) a^j_1 (x, x_1) a^j_2 (x, x_2),
$$

(72)

and

$$
\beta^1_1 (x_1, x_2) = \delta_1 (x_1) a^1_1 (x_1, x_2) \sum_{j=1}^3 \Omega_{j,j} (x) a^j_1 (x, x_1)
$$

\[ + \delta_2 (x_1) a^2_1 (x_1, x_2) \sum_{j=1}^3 \Omega_{j,j} (x) a^j_2 (x, x_1),
$$

(73)

https://doi.org/10.1088/1742-5468/aa85ca
Boundary algebraic Bethe Ansatz for a nineteen vertex model with $U_q[osp(2|2)(2)]$ symmetry

\[ \beta_2^1 (x_1, x_2) = \omega (x_2, x_1) \beta_1^1 (x_2, x_1), \]  

(74)

\[ \beta_4^3 (x_1, x_2) = \delta_1 (x_1) a_1^1 (x_1, x_2) \sum_{j=2}^{3} \Omega_{j,j}(x) a_4^j (x, x_1) \]

\[ + \delta_2 (x_1) a_1^2 (x_1, x_2) \sum_{j=2}^{3} \Omega_{j,j}(x) a_5^j (x, x_1), \]  

(75)

\[ \beta_2^3 (x_1, x_2) = \omega (x_2, x_1) \beta_3^3 (x_2, x_1), \]  

(76)

\[ \beta_{12}^2 (x_1, x_2) = \sum_{\{i,j\}=1}^{2} \delta_i(x_1) \delta_j(x_2) H_{ij}(x_1, x_2), \]  

(77)

with

\[ H_{11} (x_1, x_2) = b_1^1 (x_1, x) \sum_{i=1}^{3} \Omega_{i,i}(x) a_1^i (x_1, x_1) a_2^i (x, x_2) - b_1^1 (x_1, x_2) \sum_{i=1}^{3} \Omega_{i,i}(x) a_3^{3+i} (x, x_1) \]

\[ + b_3^0 (x_1, x) \sum_{i=2}^{3} \Omega_{i,i}(x) a_1^i (x, x_1) a_4^i (x, x_2) \]

\[ + \left[ c_6^0 (x_1, x_2) + c_6^1 (x_1, x_2) \right] \sum_{i=1}^{3} \Omega_{i,i}(x) a_6^i (x, x_1) \]

\[ + \left[ c_6^3 (x_1, x_2) + c_6^{10} (x_1, x_2) \right] \sum_{i=1}^{3} \Omega_{i,i}(x) a_7^i (x, x_1), \]  

(78)

\[ H_{12} (x_1, x_2) = b_1^1 (x_1, x) \sum_{i=1}^{3} \Omega_{i,i}(x) a_1^i (x_1, x_1) a_3^i (x, x_2) - b_1^1 (x_1, x_2) \sum_{i=1}^{3} \Omega_{i,i}(x) a_2^{3+i} (x, x_1) \]

\[ + b_3^0 (x_1, x) \sum_{i=2}^{3} \Omega_{i,i}(x) a_1^i (x, x_1) a_5^i (x, x_2) + c_9^1 (x_1, x_2) \sum_{i=1}^{3} \Omega_{i,i}(x) a_6^i (x, x_1) \]

\[ + c_{10}^3 (x_1, x_2) \sum_{i=1}^{3} \Omega_{i,i}(x) a_7^i (x, x_1), \]  

(79)

\[ H_{21} (x_1, x_2) = b_1^1 (x_1, x) \sum_{i=1}^{3} \Omega_{i,i}(x) a_1^i (x_1, x_1) a_2^i (x, x_2) - b_1^1 (x_1, x_2) \sum_{i=1}^{3} \Omega_{i,i}(x) a_3^{3+i} (x, x_1) \]

\[ + b_3^0 (x_1, x) \sum_{i=2}^{3} \Omega_{i,i}(x) a_1^i (x, x_1) a_4^i (x, x_2) \]

\[ + \left[ c_7^1 (x_1, x_2) + c_{10}^1 (x_1, x_2) \right] \sum_{i=1}^{3} \Omega_{i,i}(x) a_6^i (x, x_1) \]

\[ + \left[ c_7^3 (x_1, x_2) + c_{11}^3 (x_1, x_2) \right] \sum_{i=1}^{3} \Omega_{i,i}(x) a_7^i (x, x_1), \]  

(80)

https://doi.org/10.1088/1742-5468/aa85ca
and

\[ H_{22}(x_1, x_2) = b^3_3(x_1, x) \sum_{i=1}^{3} \Omega_{i,i}(x) a^i_1(x, x_1) a^i_3(x, x_2) - b^3_5(x_1, x_2) \sum_{i=1}^{3} \Omega_{i,i}(x) a^i_3(x, x_1) \]
\[ + b^5_1(x_1, x) \sum_{i=2}^{3} \Omega_{i,i}(x) a^i_1(x, x_1) a^i_5(x, x_2) + c^5_{11}(x_1, x_2) \sum_{i=1}^{3} \Omega_{i,i}(x) a^i_6(x, x_1) \]
\[ + c^5_{12}(x_1, x_2) \sum_{i=1}^{3} \Omega_{i,i}(x) a^i_7(x, x_1). \]  

(81)

Now, in order to \( \Psi_2(x_1, x_2) \) given at (66) be an eigenstate of the transfer matrix (58), all terms on (71) but the first one must vanish. This is indeed true, provided that the BAE of the 2-particle state,

\[ \frac{\delta_2(x_1)}{\delta_1(x_1)} = - \frac{a^1_1(x_1, x_2)}{a^1_1(x_1, x_2)} \left( \Omega_{2,2}(x) a^2_1(x, x_1) + \Omega_{3,3}(x) a^3_1(x, x_1) \right) \],

(82)

\[ \frac{\delta_2(x_2)}{\delta_1(x_2)} = - \frac{a^1_1(x_2, x_1)}{a^1_1(x_2, x_1)} \left( \Omega_{2,2}(x) a^2_1(x, x_2) + \Omega_{3,3}(x) a^3_1(x, x_2) \right) \],

(83)

are satisfied. Moreover, we can realize again that all dependence of the BAE on the spectral parameter \( x \) is only apparent, as they should. Therefore, provided that the BAE (82) and (83) are satisfied, \( \Psi_2(x_1, x_2) \) as given by (66) will be an eigenvector of the transfer matrix (58) with eigenvalue \( \tau_2(x|x_1, x_2) \) given by (72).

2.4. The general \( n \)-particle state

From the previous cases we can figure out what should be the appropriated \( n \)-particle state of the transfer matrix (58). It follows that \( \Psi_n(x_1, \ldots, x_n) \) can be defined through a recurrence relation of the form,

\[ \Psi_n(x_1, \ldots, x_n) = B_1(x_1) \Psi_{n-1}(x^\times_1) + \sum_{i=2}^{n} \lambda_i(x_1, \ldots, x_n) B_2(x_1) \Psi_{n-2}(x^\times_i, x^\times_i), \]

(84)

for \( n > 2 \), where \( \Psi_0 \) and \( \Psi_1(x_1) \) are given, respectively, by (37) and (59). We have also introduced the notation,

\[ \Psi_{n-1}(x^\times_i) = \prod_{k=1, k \neq i}^{n} B_1(x_k), \quad \Psi_{n-2}(x^\times_i, x^\times_j) = \prod_{k=1, k \neq i, j}^{n} B_1(x_k). \]

(85)

The functions \( \lambda_i(x_1, \ldots, x_n) \) appearing in (84) can be determined imposing the following exchange conditions,

\[ \Psi_n(x_1, \ldots, x_i, x_{i-1}, \ldots, x_n) = \omega(x_{i-1}, x_i) \Psi_n(x_1, \ldots, x_{i-1}, x_i, \ldots, x_n), \quad 2 \leq i \leq n, \]

(86)
and using the commutations relations between the creator operators in order to put them in a well-ordered form (see appendix). This lead us to the expressions,

\[ \omega(x_{i-1}, x_i) = b_1^i(x_{i+1}, x_i), \quad 2 \leq i \leq n, \]  

and

\[
\lambda_k(x_1, \ldots, x_n) = - \prod_{j=2}^{k-1} \omega(x_k, x_j) \left\{ b_1^1(x_1, x_k) \delta_1(x_k) \prod_{i=2, i \neq k}^n a_1^i(x_k, x_i) \right. \\
+ b_5^1(x_1, x_k) \delta_2(x_k) \prod_{i=2, i \neq k}^n a_1^2(x_k, x_i) \left\}, \quad 2 \leq k \leq n. \tag{88} \]

The action of \( t(x) \) on \( \Psi_n(x_1, \ldots, x_n) \) can be computed using many others commutation relations presented in the appendix. It follows that this action can be written as,

\[
t(x) \Psi_n(x_1, \ldots, x_n) = \tau_n(x|x_1, \ldots, x_n) \Psi_n(x_1, \ldots, x_n) \\
+ \sum_{i=1}^n \beta_1^i(x_1, \ldots, x_n) B_1(x) \Psi_{n-1}(x_i^x) \\
+ \sum_{i=1}^n \beta_3^i(x_1, \ldots, x_n) B_3(x) \Psi_{n-1}(x_i^x) \\
+ \sum_{\{i,j\}=1, i<j} \beta_{i,j}^2(x_1, \ldots, x_n) B_2(x) \Psi_{n-2}(x_i^x, x_j^x), \tag{89} \]

where,

\[
\tau_n(x|x_1, \ldots, x_n) = \sum_{j=1}^3 \Omega_{j,j}(x) \delta_j(x) \prod_{i=1}^n a_1^j(x, x_i), \tag{90} \]

\[
\beta_1^i(x_1, \ldots, x_n) = \prod_{l=1}^{k-1} \omega(x_k, x_l) \delta_1(x_k) \sum_{j=1}^3 \Omega_{j,j}(x) a_2^j(x, x_k) \prod_{i=1, i \neq k}^n a_1^i(x_k, x_i) \\
+ \prod_{l=1}^{k-1} \omega(x_k, x_l) \delta_2(x_2) \sum_{j=1}^3 \Omega_{j,j}(x) a_3^j(x, x_i) \prod_{i=1, i \neq k}^n a_1^2(x_k, x_i), \tag{91} \]

\[
\beta_3^i(x_1, \ldots, x_n) = \prod_{l=1}^{k-1} \omega(x_k, x_l) \delta_1(x_k) \sum_{j=2}^3 \Omega_{j,j}(x) a_4^j(x, x_k) \prod_{i=1, i \neq k}^n a_1^i(x_k, x_i) \\
+ \prod_{l=1}^{k-1} \omega(x_k, x_l) \delta_2(x_2) \sum_{j=2}^3 \Omega_{j,j}(x) a_5^j(x, x_i) \prod_{i=1, i \neq k}^n a_2^2(x_k, x_i), \tag{92} \]

https://doi.org/10.1088/1742-5468/aa85ca

J. Stat. Mech. (2017) 093017
Boundary algebraic Bethe Ansatz for a nineteen vertex model with $U_q[osp(2|2)]$ symmetry

and

$$
\beta_{i,j}^2 (x_1, \ldots, x_n) = \prod_{k=1}^{i-1} \omega (x_k, x_i) \prod_{k=1}^{j-1} \omega (x_k, x_j) \sum_{\{p,q\}=1}^2 \delta_p(x_i) \delta_q(x_j) H_{pq} (x_i, x_j) \\
\quad \times \prod_{k=1, k \neq (i,j)}^n a_i^n(x_i, x_k) \prod_{l=1, l \neq (i,j)}^n a_j^n(x_i, x_l).
$$

(93)

The requirement that $\Psi_n(x_1, \ldots, x_n)$ be an eigenstate of the transfer matrix means that all terms in (89) that are not proportional to $\Psi_n(x_1, \ldots, x_n)$ itself must vanish. This lead us to the BAE of the general $n$-particle state:

$$
\frac{\delta_2(x_k)}{\delta_1(x_k)} = -\left[ \frac{\Omega_{2,2}(x) a_2^2(x, x_k) + \Omega_{3,3}(x) a_3^2(x, x_k)}{\Omega_{2,2}(x) a_2^2(x, x_k) + \Omega_{3,3}(x) a_3^2(x, x_k)} \right] \prod_{j=1, j \neq k}^n \frac{a_i^1(x_k, x_j)}{a_i^2(x_k, x_j)}, \quad 1 \leq k \leq n.
$$

(94)

3. Explicit results

Making use of the amplitudes of the $R$-matrix (7), the expressions for the elements of the diagonal reflection $K$-matrix (24) and the coefficients of the commutation relations presented in the appendix, we can explicitly write down the results of the boundary BAE for the Yang–Zhang model. It follows that the $n$-state eigenvector of the transfer matrix is given by (84) where,

$$
\omega (x, y) = \frac{(q^2 x - y) (q y + x)}{(q^2 y - x) (q x + y)}.
$$

(95)

and

$$
\lambda_k(x_1, \ldots, x_n) = -\prod_{j=2}^{k-1} \frac{(q^2 x_k - x_j) (q x_j + x_k)}{(q^2 x_j - x_k) (q x_k + x_j)} \\
\quad \times \left\{ \frac{\sqrt{q} (q^2 - 1) (x_k^2 - 1)}{(q x_k + x_1) (q^2 x_k^2 - 1)} \delta_1(x_k) \prod_{i=2, i \neq k}^n \frac{(x_i x_k - 1) (q^2 x_i - x_k)}{(x_i - x_k) (q^2 x_i x_k - 1)} \\
\quad - \frac{1 - q^2}{\sqrt{q} (q x_1 x_k + 1)} \delta_2(x_k) \prod_{i=2, i \neq k}^n \frac{(q^2 - 1) (x_i^2 - 1) x_i}{(x_i - x_k) (q^2 x_i^2 - 1)} \right\}, \quad 2 \leq k \leq n.
$$

(96)

The eigenvalues of the Sklyanin transfer matrix (58) are given by

https://doi.org/10.1088/1742-5468/aa85ca
\[
\tau_n (x_1, \ldots, x_n) = \frac{(q^3 x_1^2 + 1) \left[ 1 - \frac{1}{2} (x_1 - 1) (\beta_{1,1} - \beta_{2,2}) \right]}{q^4 x_1^2 (q x_1^2 + 1)} \delta_1 (x)
\]
\[
\times \left[ \beta_{1,1} (q^2 x_1^2 - 1) - 2q^2 x_1^2 \right] \prod_{i=1}^{n} \left\{ \frac{(x x_i - 1) (x - q^2 x_i)}{(x - x_i) (q^2 x_i x - 1)} \right\} \delta_2 (x)
\]
\[
\times \left[ \frac{(q x + 1) (\beta_{1,1} - \beta_{2,2}) + 2q x}{(q x + 1) (\beta_{1,1} - \beta_{2,2}) + 2} \right] \frac{\delta_3 (x)}{q (x + x) (q x x_i + 1)},
\]
where
\[
\delta_1 (x) = \left[ 1 + \frac{1}{2} (x^2 - 1) \beta_{1,1} \right] \left( \frac{q^2 x_1^2 - 1}{q^2 x_1 - 1} \right),
\]
\[
\delta_2 (x) = -\frac{x (x^2 - 1) \left[ 1 + \frac{1}{2} (x^2 - 1) \beta_{1,1} \right]}{(q x^2 - 1)} \frac{(q^2 x - 1) (\beta_{1,1} - \beta_{2,2}) - 2}{(x - 1) (\beta_{1,1} - \beta_{2,2}) + 2x} \frac{[q (x - 1)]^{2N}}{[(q^2 - x) (q^2 x - 1) / x]^N},
\]
\[
\delta_3 (x) = -\frac{x^2 (q^2 x + 1) \left[ 1 + \frac{1}{2} (x^2 - 1) \beta_{1,1} \right]}{q (q x^2 + 1)} \frac{(q^2 x - 1) (\beta_{1,1} - \beta_{2,2}) - 2}{(x - 1) (\beta_{1,1} - \beta_{2,2}) + 2x} \frac{[q (x - 1) (q + x) / (q x + 1)]^{2N}}{[(q^2 - x) (q^2 x - 1) / x]^N}.
\]
Finally, the BAE are:
\[
\left[ \frac{q (x_k - 1)}{q^2 x_k - 1} \right]^{2N} = \left[ \frac{(x_k - 1) (\beta_{1,1} - \beta_{2,2}) + 2x_k}{(1 - q^2 x_k) (\beta_{1,1} - \beta_{2,2}) + 2} \right] \frac{q (q x_k + 1) (\beta_{1,1} - \beta_{2,2}) + 2q x_k}{(q x_k + 1) (\beta_{1,1} - \beta_{2,2}) + 2q x_k} \times \prod_{j=1, j \neq k}^{n} \left\{ \frac{q (q x_j + x_k) (q x_j x_k + 1)}{(x_j + q x_k) (q^2 x_j x_k + 1)} \right\}, \quad 1 \leq k \leq n.
\]
Boundary algebraic Bethe Ansatz for a nineteen vertex model with $U_q[osp(2|2)(2)]$ symmetry

4. Numerical analysis

For a small-length chain, the results above can be checked numerically. As an example, let us consider a chain with three sites, that is, let us assume that $N = 3$. In this case, both the monodromy as the transfer matrix defined at (30) can be explicitly constructed without difficult. The Sklyanin monodromy becomes an operator with values in $\text{End}(V_a \otimes V_1 \otimes V_2 \otimes V_3)$ and, therefore, consists in an 81-by-81 matrix. Taking the (graded) trace of the monodromy with respect to the auxiliary space $V_a$, we get the transfer matrix, which turns out to be an operator acting only on the quantum spaces $V_1$, $V_2$ and $V_3$. Hence, the transfer matrix consists in a 27-by-27 matrix. The eigenvalues of the transfer matrix can be computed numerically as we give numerical values for the parameters $x$, $q$, $\beta_{1,1}$, $\beta_{1,2}$ and $\beta_{2,2}$. In this example, we shall consider the (randomly generated) values, $x = 1.2970895172$, $q = 0.3438435138$, $\beta_{1,1} = 0.6057011678$ and $\beta_{2,2} = 0.5210113587$. We remark however that only 17 of the 27 eigenvalues of the transfer matrix are actually distinct, which is due to the symmetry of the system regarding inversion of the spins.

In the framework of the ABA, on the other hand, we usually divide the spectrum of the transfer matrix into sectors, according to the magnon number $M$ associated with the possible chain configurations. (We say that a spin pointing to up ($\uparrow$), to center ($\circ$), or to down ($\downarrow$) has a magnon number equal to 0, 1 or 2, respectively, and that the total magnon number of the chain is given by the sum of the magnon numbers associated with all its sites.) In this way, the reference state corresponds to the sector $M = 0$, which physically corresponds to a configuration in which all spins point up, while the $n$-particle states correspond to the configurations in which $M = n$, that is, they are physically formed by any combination of $k$ spins pointing down and $l$ spins pointing to the center, in such a way that $2k + l = n$. Therefore, for a chain of length $N = 3$, we have in total 7 sectors, corresponding to the values of $M$ ranging from 0 to 6. The number $C$ of spin configurations in each sector $n$ and the respective spin configurations for this chain is presented in table 1.

| $n$ | Spin configurations for $N = 3$ | $C$ |
|-----|---------------------------------|-----|
| 0   | ↑↑↑                            | 1   |
| 1   | ↑↑○                            | 3   |
| 2   | ↑↓↓                            | 6   |
| 3   | ○↓↓                            | 7   |
| 4   | ○○○                            | 6   |
| 5   | ○○●                            | 3   |
| 6   | ↓↓↓                            | 1   |

In order to compute the eigenvalues of the transfer matrix in the framework of the ABA, we need to solve the BAE, since the eigenvalues given by (90) depend implicitly on the rapidities (i.e. on the solutions of the BAE). Here we remark as well that for $N = 3$ is not necessary to go up to $n = 6$, as we could expect from table 1. The solutions for
Table 2. The distinct eigenvalues of the transfer matrix for $N = 3$, the corresponding sectors they belong and the respective Bethe roots, solutions of the Bethe Ansatz equations.

| # | Eigenvalue          | n | $x_1$              | $x_2$                              | $x_3$                             |
|---|---------------------|---|--------------------|------------------------------------|-----------------------------------|
| 1 | 15.55187750         | 3 | -0.49331775        | -0.64061596 - 2.8368673i           | 2.419486 - 1.613782i              |
| 2 | 12.62320656         | 2 | -0.77355394        | 2.67270048 + 1.14668192i           |                                   |
| 3 | 12.38842171         | 3 | -0.80973407        | 2.68478139 + 1.11810384i           | 146.118915                        |
| 4 | 11.07294406 + 1.40073055i | 2 | -2.97255121 + 3.69294020i | 1.84033474 - 1.74912001i          |                                   |
| 5 | 11.07294406 - 1.40073055i | 2 | -1.11874174 + 1.38986550i | 2.41472578 - 2.29504181i          |                                   |
| 6 | 10.75678367 + 1.54722986i | 3 | -3.04722568 + 3.91529532i | 2.47592193 + 2.32282182i          | 151.915113 + 2.332858i           |
| 7 | 10.75678367 - 1.54722986i | 3 | -1.04708263 + 1.34536728i | 1.81697233 + 1.70461876i          | 151.915113 - 2.332858i           |
| 8 | 9.60816570          | 1 | 2.47143937 + 1.53303449i |                                   |                                   |
| 9 | 9.48135517          | 3 | -0.01661866        | 2.46824901 + 1.53816584i          | 210.240415                        |
| 10| 9.38955066          | 2 | 0.0429863          | 2.46547555 + 1.54260741i          |                                   |
| 11| 6.35026824          | 1 | 0.42382585 + 2.87725198i |                                   |                                   |
| 12| 6.25970589          | 3 | -0.01656047        | 0.40109637 + 2.88050846i          | 210.299378                        |
| 13| 6.19420896          | 2 | 0.04283620         | 0.38270488 + 2.88300959i          |                                   |
| 14| 4.06297970          | 0 |                 |                                   |                                   |
| 15| 4.00955707          | 2 | -0.01431615        | 208.23451827                      |                                   |
| 16| 3.99751827          | 3 | -0.01423742 - 0.00092488i | 0.00649761 - 0.01190777i          | 208.2906560 + 0.10370i            |
| 17| 3.98300662          | 1 | 0.04083511         |                                   |                                   |
$n = \{4,5,6\}$ provide the same eigenvalues as that obtained from the cases $n = \{2,1,0\}$, respectively, which is due to the above mentioned symmetry of the system regarding the inversion of the spins. This is very welcome, since the BAE are very difficult to solve, even numerically. In fact, the BAE are highly ill-conditioned: their roots are very close to each other, which requires a high accuracy in the computations; there are singular solutions (in the present case, when some root equals one of the values $0, \pm1, \pm1/q^2$) and also there are other non-physical solutions (e.g., when two or more roots have the same value) that must be discarded. Different solutions may also lead to the same eigenvalue, for example those solutions differing only by a permutation of the Bethe roots, or, sometimes, roots differing only by a complex conjugation. For more details about the complexity BAE, see [43].

Notwithstanding the complexity of solving the BAE, we were able to verify the correctness of the boundary ABA in the present case. The solutions of the BAE (disregarding nonphysical and equivalent solutions) are presented together with the corresponding eigenvalues in the table 2. Although we do not write down the eigenvectors, we remark that they can be obtained from the expressions for the Bethe roots and the eigenvalues through (84).

5. Conclusion

In this work we derived the boundary ABA for the supersymmetric nineteen vertex model constructed from a three-dimensional free boson representation $V$ of the twisted quantum affine Lie superalgebra $U_q(\text{osp}(2|2)^{(2)}) \simeq U_q[C(2)^{(2)}]$. The $R$-matrix of this model was introduced by Yang and Zhang in [28] and the corresponding reflection $K$-matrices were derived by us recently in [27]. The eigenvalues and eigenvectors of Sklyanin’s transfer matrix with diagonal reflection $K$-matrices were determined, as well as the corresponding Bethe Ansatz equations. Explicit results and a numerical analysis were also presented.

Acknowledgments

This work was supported in part by Brazilian Research Council (CNPq), grant #310625/2013-0 and FAPESP, grant #2011/18729-1.

Appendix. The fundamental commutation relations

To perform the boundary ABA, we need to know how the diagonal operators $D_1$, $D_2$ and $D_3$ pass through the creator operators $B_1$, $B_2$ and $B_3$ (as an intermediate step, we shall also need known how the $C$’s operators pass through the $B$’s). These exchange rules are provided by the commutation relations that can be derived from the so-called fundamental relation of the boundary ABA:

$$R_{12}(x/y)U_1(x)R_{21}(xy)U_2(y) = U_2(y)R_{12}(xy)U_1(x)R_{21}(x/y),$$

(A.1)
In fact, writing $U(x)$ in the Lax representation as (35), and using the relations (42)–(45), the following commutation relations can be obtained (for details about how these expressions are obtained, please see [25, 26]):

\[
D_1(x)B_1(y) = a_1^1(x,y)B_1(y)D_1(x) + a_2^1(x,y)B_1(x)D_1(y) + a_3^1(x,y)B_1(x)D_2(y) + a_0^1(x,y)B_2(x)C_1(y) + a_2^1(x,y)B_2(x)C_3(y) + a_3^1(x,y)B_2(y)C_1(x) ,
\]

(A.2)

\[
D_2(x)B_1(y) = a_1^2(x,y)B_1(y)D_2(x) + a_2^2(x,y)B_1(x)D_1(y) + a_3^2(x,y)B_1(x)D_2(y) + a_4^2(x,y)B_2(x)D_1(y) + a_2^2(x,y)B_2(x)D_2(y) + a_3^2(x,y)B_2(y)D_1(x) + a_2^2(x,y)B_2(y)D_3(x) ,
\]

(A.3)

\[
D_3(x)B_1(y) = a_1^3(x,y)B_1(y)D_3(x) + a_2^3(x,y)B_1(x)D_1(y) + a_3^3(x,y)B_1(x)D_2(y) + a_4^3(x,y)B_3(x)D_1(y) + a_3^3(x,y)B_3(x)D_2(y) + a_3^3(x,y)B_2(x)D_1(y) + a_3^3(x,y)B_2(y)D_3(x) ,
\]

(A.4)

\[
D_1(x)B_2(y) = a_1^4(x,y)B_2(y)D_1(x) + a_2^4(x,y)B_2(x)D_1(y) + a_3^4(x,y)B_2(x)D_2(y) + a_4^4(x,y)B_4(x)D_1(y) + a_4^4(x,y)B_1(y)B_1(x)B_3(y) + a_5^4(x,y)B_1(y)B_2(y)B_3(y) ,
\]

(A.5)

\[
D_2(x)B_2(y) = a_1^5(x,y)B_2(y)D_2(x) + a_2^5(x,y)B_2(x)D_1(y) + a_3^5(x,y)B_2(x)D_2(y) + a_4^5(x,y)B_4(x)D_1(y) + a_3^5(x,y)B_3(x)D_1(y) + a_3^5(x,y)B_2(y)D_1(y) + a_3^5(x,y)B_2(y)D_3(y) ,
\]

(A.6)

\[
D_3(x)B_2(y) = a_1^6(x,y)B_2(y)D_3(x) + a_2^6(x,y)B_2(x)D_1(y) + a_3^6(x,y)B_2(x)D_2(y) + a_4^6(x,y)B_4(x)D_1(y) + a_5^6(x,y)B_3(x)D_1(y) + a_6^6(x,y)B_2(x)D_1(y) + a_5^6(x,y)B_2(y)B_3(y) ,
\]

(A.7)

\[
D_1(x)B_3(y) = a_1^7(x,y)B_3(y)D_1(x) + a_2^7(x,y)B_1(y)B_1(x)D_1(y) + a_3^7(x,y)B_1(x)D_2(y) + a_4^7(x,y)B_3(y)D_1(y) + a_5^7(x,y)B_2(y)B_3(y)D_1(y) + a_6^7(x,y)B_2(y)B_3(y)D_2(y) + a_7^7(x,y)B_2(x)C_1(y) + a_7^7(x,y)B_2(x)C_3(y) + a_8^7(x,y)B_2(y)B_3(x) + a_9^7(x,y)B_2(y)B_3(x) ,
\]

(A.8)

\[
D_2(x)B_3(y) = a_1^8(x,y)B_3(y)D_2(x) + a_2^8(x,y)B_1(y)B_1(x)D_2(y) + a_3^8(x,y)B_1(x)D_3(y) + a_4^8(x,y)B_3(y)D_1(y) + a_5^8(x,y)B_3(x)D_1(y) + a_6^8(x,y)B_2(y)B_3(y)D_1(y) + a_7^8(x,y)B_2(x)C_1(y) + a_8^8(x,y)B_2(x)C_3(y) + a_9^8(x,y)B_2(y)B_3(x) + a_{10}^8(x,y)B_2(y)B_3(x) ,
\]

(A.9)

\[
D_3(x)B_3(y) = a_1^9(x,y)B_3(y)D_3(x) + a_2^9(x,y)B_1(y)B_1(x)D_3(y) + a_3^9(x,y)B_1(x)B_3(y)D_1(y) + a_4^9(x,y)B_3(x)D_1(y) + a_5^9(x,y)B_2(y)B_3(y)D_1(y) + a_6^9(x,y)B_2(x)C_1(y) + a_7^9(x,y)B_2(x)C_3(y) + a_8^9(x,y)B_2(y)B_3(x) + a_9^9(x,y)B_2(y)B_3(x) + a_{10}^9(x,y)B_2(y)B_3(x) ,
\]

(A.10)
The commutation relations among the creator operators $B_1$, $B_2$ and $B_3$ themselves are:

\begin{align}
B_1(x)B_1(y) &= b_1^1(x, y)B_1(y)B_1(x) + b_2^1(x, y)B_2(y)D_1(x) + b_3^1(x, y)B_2(y)D_2(x) \\
&\quad + b_1^2(x, y)B_2(x)D_1(y) + b_2^3(x, y)B_2(x)D_2(y), \quad (A.11) \\
B_2(x)B_1(y) &= b_1^2(x, y)B_1(y)B_2(x) + b_2^2(x, y)B_2(y)B_1(x) + b_3^2(x, y)B_2(y)B_3(x), \quad (A.12) \\
B_3(x)B_1(y) &= b_1^3(x, y)B_1(y)B_3(x) + b_2^3(x, y)B_2(y)B_1(x) + b_3^3(x, y)B_2(y)B_2(D_1(x) \\
&\quad + b_3^4(x, y)B_2(y)D_2(x) + b_2^5(x, y)B_2(y)B_3(x) + b_3^6(x, y)B_2(y)D_1(y) \\
&\quad + b_3^7(x, y)B_2(x)D_2(y), \quad (A.13) \\
B_1(x)B_2(y) &= b_1^4(x, y)B_2(y)B_1(x) + b_2^4(x, y)B_2(y)B_2(x) + b_3^4(x, y)B_2(y)B_1(x) \\
&\quad + b_3^5(x, y)B_2(x)B_2(y), \quad (A.14) \\
B_2(x)B_2(y) &= B_2(y)B_2(x), \quad (A.15) \\
B_3(x)B_2(y) &= b_3^6(x, y)B_2(y)B_3(x) + b_3^7(x, y)B_2(y)B_3(x) + b_3^8(x, y)B_2(y)B_3(x) \\
&\quad + b_3^9(x, y)B_2(x)B_2(y), \quad (A.16) \\
B_1(x)B_3(y) &= b_1^8(x, y)B_3(y)B_1(x) + b_2^8(x, y)B_2(y)B_1(x) + b_3^8(x, y)B_2(y)D_1(x) \\
&\quad + b_3^9(x, y)B_2(x)D_2(y) + b_3^8(x, y)B_2(y)B_3(x) + b_3^9(x, y)B_2(y)B_3(x) \\
&\quad + b_3^8(x, y)B_2(y)B_3(x) + b_3^9(x, y)B_2(y)B_3(x), \quad (A.17) \\
B_2(x)B_3(y) &= b_2^9(x, y)B_3(y)B_2(x) + b_3^9(x, y)B_2(y)B_2(x) + b_3^9(x, y)B_2(y)B_1(x) \\
&\quad + b_3^9(x, y)B_2(x)B_3(x), \quad (A.18) \\
B_3(x)B_3(y) &= b_3^9(x, y)B_3(y)B_3(x) + b_3^9(x, y)B_2(y)B_3(x) + b_3^9(x, y)B_2(y)B_3(x) \\
&\quad + b_3^9(x, y)B_2(x)B_2(y) + b_3^9(x, y)B_2(y)B_3(x) + b_3^9(x, y)B_2(y)B_3(x) \\
&\quad + b_3^9(x, y)B_2(y)B_3(x) + b_3^9(x, y)B_2(y)B_3(x), \quad (A.19)
\end{align}

Notice that we have chosen an appropriated partial order for the creator operators, namely, that $B_i(x_k) < B_j(x_l)$ if, and only if, $x_k < x_l$. This partial order is important for the implementation of the boundary ABa, since it is necessary to use these commutation relations until all operators be well ordered — i.e. so that all diagonal operators $D_i$, and all annihilator operators $C_i$, be at right of the creator operators $B_j$ and, further, that among the creator operators themselves, we have always $B_i < B_j$.

Finally, the commutation relations between the $C$'s and the $B$'s are:

\begin{align}
C_1(x)B_1(y) &= c_1^1(x, y)B_1(y)C_1(x) + c_2^1(x, y)B_1(y)C_3(x) + c_3^1(x, y)B_1(y)C_3(x) \\
&\quad + c_4^1(x, y)B_2(x)C_3(y) + c_5^1(x, y)B_2(x)C_2(x) + c_6^1(x, y)B_1(y)D_1(x) \\
&\quad + c_7^1(x, y)D_2(x)D_3(y) + c_8^1(x, y)D_2(x)D_3(y) + c_9^1(x, y)D_2(x)D_3(y) \\
&\quad + c_{10}^1(x, y)D_2(x)D_1(y) + c_{11}^1(x, y)D_2(x)D_2(y), \quad (A.20)
\end{align}
Boundary algebraic Bethe Ansatz for a nineteen vertex model with \( U_q[osp(2|2)^{(2)}] \) symmetry

\[
C_2(x) B_1(y) = c_1^2(x, y) B_3(y) C_2(x) + c_2^2(x, y) C_1(y) D_1(x) + c_3^2(x, y) C_1(x) D_1(y)
+ c_4^2(x, y) C_1(y) D_2(x) + c_5^2(x, y) C_1(x) D_2(y) + c_6^2(x, y) C_1(y) D_3(x)
+ c_7^2(x, y) C_3(x) D_1(y) + c_8^2(x, y) C_3(x) D_2(y) + c_9^2(x, y) C_1(x) D_1(y)
+ c_{10}^2(x, y) C_3(x) D_1(y) + c_{11}^2(x, y) C_3(y) D_1(x) + c_{12}^2(x, y) C_1(x) D_2(y)
+ c_{13}^2(x, y) C_3(x) D_2(y) + c_{14}^2(x, y) C_3(y) D_2(x) + c_{15}^2(x, y) C_3(y) D_3(x), \tag{A.21}
\]

\[
C_3(x) B_1(y) = c_1^3(x, y) B_1(y) C_3(x) + c_2^3(x, y) B_1(y) C_1(x) + c_3^3(x, y) B_1(x) C_3(y)
+ c_4^3(x, y) B_2(y) C_2(x) + c_5^3(x, y) B_3(x) C_3(y) + c_6^3(x, y) D_1(y) D_1(x)
+ c_7^3(x, y) D_1(y) D_2(x) + c_8^3(x, y) D_1(y) D_3(x) + c_9^3(x, y) D_1(x) D_1(y)
+ c_{10}^3(x, y) D_1(x) D_2(y) + c_{11}^3(x, y) D_2(x) D_1(y) + c_{12}^3(x, y) D_2(x) D_2(y)
+ c_{13}^3(x, y) D_3(x) D_1(y) + c_{14}^3(x, y) D_3(x) D_2(y), \tag{A.22}
\]

\[
C_1(x) B_2(y) = c_1^4(x, y) B_2(y) C_1(x) + c_2^4(x, y) B_2(y) C_3(x) + c_3^4(x, y) B_2(x) C_1(y)
+ c_4^4(x, y) B_2(x) C_3(y) + c_5^4(x, y) B_1(y) D_1(x) + c_6^4(x, y) B_1(y) D_2(x)
+ c_7^4(x, y) B_3(y) D_1(x) + c_8^4(x, y) B_3(y) D_2(x) + c_9^4(x, y) B_3(x) D_1(y)
+ c_{10}^4(x, y) B_1(x) D_2(y) + c_{11}^4(x, y) B_1(x) D_3(y) + c_{12}^4(x, y) B_3(x) D_1(y)
+ c_{13}^4(x, y) B_3(x) D_2(y) + c_{14}^4(x, y) B_3(y) D_3(y), \tag{A.23}
\]

\[
C_2(x) B_2(y) = c_1^5(x, y) B_2(y) C_2(x) + c_2^5(x, y) B_2(y) C_2(y) + c_3^5(x, y) B_1(y) C_1(x)
+ c_4^5(x, y) B_1(x) C_1(y) + c_5^5(x, y) B_1(x) C_3(y) + c_6^5(x, y) B_1(x) C_3(y)
+ c_7^5(x, y) B_3(y) C_1(x) + c_8^5(x, y) B_3(y) C_3(y) + c_9^5(x, y) B_3(x) C_3(y)
+ c_{10}^5(x, y) D_1(x) D_2(y) + c_{11}^5(x, y) D_1(x) D_1(y) + c_{12}^5(x, y) D_1(x) D_1(y)
+ c_{13}^5(x, y) D_2(x) D_1(y) + c_{14}^5(x, y) D_3(x) D_1(y) + c_{15}^5(x, y) D_1(x) D_3(y)
+ c_{16}^5(x, y) D_1(x) D_2(y) + c_{17}^5(x, y) D_2(x) D_1(y) + c_{18}^5(x, y) D_2(x) D_2(y)
+ c_{19}^5(x, y) D_2(x) D_3(y) + c_{20}^5(x, y) D_2(x) D_3(y) + c_{21}^5(x, y) D_1(x) D_2(y)
+ c_{22}^5(x, y) D_3(x) D_1(y) + c_{23}^5(x, y) D_2(x) D_3(y) + c_{24}^5(x, y) D_3(x) D_3(y)
+ c_{25}^5(x, y) D_3(x) D_3(y), \tag{A.24}
\]

\[
C_3(x) B_2(y) = c_1^6(x, y) B_2(y) C_3(x) + c_2^6(x, y) B_2(y) C_3(y) + c_3^6(x, y) B_1(y) D_1(x)
+ c_4^6(x, y) B_1(x) D_1(y) + c_5^6(x, y) B_1(y) D_2(x) + c_6^6(x, y) B_1(x) D_2(y)
+ c_7^6(x, y) B_1(y) D_3(x) + c_8^6(x, y) B_1(x) D_3(y) + c_9^6(x, y) B_2(y) C_1(x)
+ c_{10}^6(x, y) B_2(x) C_1(y) + c_{11}^6(x, y) B_3(y) D_1(x) + c_{12}^6(x, y) B_3(x) D_1(y)
+ c_{13}^6(x, y) B_3(y) D_2(x) + c_{14}^6(x, y) B_3(x) D_2(y) + c_{15}^6(x, y) B_3(y) D_3(x)
+ c_{16}^6(x, y) B_3(x) D_3(y), \tag{A.25}
\]

https://doi.org/10.1088/1742-5468/aa85ca
\[ C_1(x)B_3(y) = c_1^7(x, y) B_3(y) C_1(x) + c_3^7(x, y) B_3(x) C_1(y) + c_3^5(x, y) B_1(y) C_1(x) \\
+ c_1^7(x, y) B_1(y) C_3(x) + c_3^7(x, y) B_1(x) C_3(y) + c_3^7(x, y) B_3(y) C_2(x) \\
+ c_1^7(x, y) B_3(x) C_3(y) + c_3^8(x, y) D_1(x) D_1(y) + c_1^9(x, y) D_1(x) D_1(y) \\
+ c_{10}^7(x, y) D_2(x) D_1(y) + c_{11}^7(x, y) D_1(x) D_2(y) + c_{12}^7(x, y) D_1(x) D_3(y) \\
+ c_{13}^7(x, y) D_1(x) D_2(y) + c_{14}^7(x, y) D_2(x) D_1(y) + c_{15}^7(x, y) D_2(x) D_2(y) \\
+ c_{16}^7(x, y) D_2(x) D_3(y) + c_{17}^7(x, y) D_1(x) D_3(y) \]  
(A.26)

\[ C_2(x)B_3(y) = c_1^8(x, y) B_3(y) C_2(x) + c_3^8(x, y) B_3(x) C_2(y) + c_3^8(x, y) C_1(y) D_1(x) \\
+ c_1^8(x, y) C_1(x) D_1(y) + c_3^8(x, y) C_1(y) D_2(x) + c_3^6(x, y) C_1(x) D_2(y) \\
+ c_1^8(x, y) C_1(y) D_3(x) + c_3^8(x, y) C_1(x) D_3(y) + c_3^8(x, y) C_3(y) D_1(x) \\
+ c_{10}^8(x, y) C_3(x) D_1(y) + c_{11}^8(x, y) C_3(y) D_2(x) + c_{12}^8(x, y) C_3(x) D_2(y) \\
+ c_{13}^8(x, y) C_3(x) D_3(y) + c_{14}^8(x, y) C_3(x) D_1(y) + c_{15}^8(x, y) C_1(y) D_1(x) \\
+ c_{16}^8(x, y) C_3(x) D_1(y) + c_{17}^8(x, y) C_3(y) D_1(x) + c_{18}^8(x, y) C_1(x) D_2(y) \\
+ c_{19}^8(x, y) C_1(y) C_3(y) D_1(x) + c_{20}^8(x, y) C_3(x) D_2(y) + c_{21}^8(x, y) C_3(x) D_2(x) \\
+ c_{22}^8(x, y) C_3(x) D_3(y) + c_{23}^8(x, y) C_3(y) D_3(y) \]  
(A.27)

\[ C_3(x)B_3(y) = c_1^9(x, y) B_3(y) C_3(x) + c_2^9(x, y) B_3(x) C_3(y) + c_3^9(x, y) B_1(y) C_3(x) \\
+ c_4^9(x, y) B_1(x) C_1(y) + c_3^9(x, y) B_1(y) C_3(x) + c_6^9(x, y) B_1(x) C_3(y) \\
+ c_1^9(x, y) B_3(y) C_2(x) + c_3^9(x, y) B_2(x) C_2(y) + c_3^9(x, y) B_3(y) C_1(x) \\
+ c_{10}^9(x, y) D_1(x) D_1(y) + c_{11}^9(x, y) D_1(x) D_1(y) + c_{12}^9(x, y) D_2(x) D_1(y) \\
+ c_{13}^9(x, y) D_1(x) D_2(y) + c_{14}^9(x, y) D_3(x) D_1(y) + c_{15}^9(x, y) D_1(x) D_3(y) \\
+ c_{16}^9(x, y) D_1(x) D_2(y) + c_{17}^9(x, y) D_2(x) D_1(y) + c_{18}^9(x, y) D_2(x) D_2(y) \\
+ c_{19}^9(x, y) D_2(x) D_3(y) + c_{20}^9(x, y) D_2(x) D_3(y) + c_{21}^9(x, y) D_1(x) D_3(y) \\
+ c_{22}^9(x, y) D_3(x) D_1(y) + c_{23}^9(x, y) D_2(x) D_3(y) + c_{24}^9(x, y) D_3(x) D_2(y) \\
+ c_{25}^9(x, y) D_3(x) D_3(y) \]  
(A.28)

The fundamental relation (A.1) also provides the commutation relations between the operators $C_i$ and $D_j$, which can be used to eliminate some terms in some commutation relations. However, these supplementary commutation relations are not necessary in the implementation of the boundary $ABA$ and they will not be presented (the reader can consult references [25, 26] for this purpose). Next we write down the coefficients of the commutation relations presented above. We also restrict ourselves to the coefficients that appear explicitly in the text only; the others expressions (which are usually very cumbersome and, as a matter of fact, not necessary) can also be found following the lines of [25, 26].
\[ a_1(x, y) = \frac{r_1(y/x)r_2(xy)}{r_1(xy)r_2(y/x)}, \]  

\[ a_2(x, y) = -F_1(y) \frac{r_5(xy)}{r_1(xy)} - \frac{r_2(xy)s_5(y/x)}{r_1(xy)r_2(y/x)}, \]  

\[ a_3(x, y) = -\frac{r_5(xy)}{r_1(xy)}, \]  

\[ a_6(x, y) = \frac{r_6(xy)s_5(y/x)}{r_1(xy)r_2(y/x)}, \]  

\[ a_7(x, y) = -\frac{r_7(xy)}{r_1(xy)}, \]  

\[ a_2(x, y) = \left[ F_1(x) + \frac{s_5(x/y)s_5(xy)}{r_2(xy)r_2(xy)} \right] \left[ F_1(y) \frac{r_5(xy)}{r_1(xy)} + \frac{r_2(xy)s_5(x/y)}{r_1(xy)r_2(y/x)} \right] + F_1(y) \frac{r_4(xy)s_5(xy)}{r_2(xy)r_2(xy)} - \frac{r_1(xy)r_4(xy)s_5(xy)}{r_1(xy)r_2(xy)^2} + \frac{r_6(xy)}{r_2(xy)} \frac{r_1(xy)s_5(xy)s_6(xy)}{r_1(xy)r_2(xy)r_3(xy)} - \frac{s_5(x/y)s_6(xy)}{r_2(xy)r_3(xy)}, \]  

\[ a_3(x, y) = \frac{r_5(xy)}{r_1(xy)} F_1(x) + \frac{s_5(x/y)s_5(xy) + r_1(xy)r_4(xy)}{r_1(xy)r_2(xy)r_2(xy)}, \]  

\[ a_4(x, y) = \frac{r_6(xy)}{r_2(xy)} F_1(y) - \frac{r_4(xy)r_6(xy)}{r_2(xy)r_3(xy)}, \]  

\[ a_5(x, y) = \frac{r_6(xy)}{r_2(xy)}, \]  

\[ a_6(x, y) = -\frac{F_1(x)r_6(xy)s_5(y/x)}{r_1(xy)r_2(y/x)} - \frac{r_6(xy)}{r_2(xy)} \left[ \frac{s_5(x/y) + r_1(xy)r_6(xy)s_5(xy)s_6(xy)}{r_1(xy)r_2(xy)r_2(xy)r_3(xy)} \right] + \frac{r_6(xy)s_5(xy)}{r_1(xy)r_2(xy)r_2(xy)} \left[ \frac{r_1(xy)r_4(xy)}{r_2(y/x)} - \frac{s_5(x/y)s_5(y/x)}{r_2(y/x)} \right], \]  

\[ a_7(x, y) = F_1(x) + \frac{s_5(x/y)[s_7(xy)s_5(xy) - r_1(xy)r_5(xy)]}{r_1(xy)r_2(xy)r_2(xy)}, \]
\[ a_1^3 (x, y) = \frac{r_2(x/y)(r_6(x/y)s_6(x/y) + r_2(x/y)^2)}{r_2(x/y)r_3(x/y)r_3(x/y)}, \]  
\[ a_2^3 (x, y) = \left[ \frac{F_1(y)r_5(x/y)}{r_1(x/y)} + \frac{r_2(x/y)s_5(x/y)}{r_1(x/y)r_2(x/y)} \right] s_6(x/y)s_6(x/y)s_5(x/y)s_5(x/y) F_3(x) + F_2(x) \]  
\[ + \frac{s_7(x/y)s_7(x/y)}{r_3(x/y)r_3(x/y)} \left( F_1(x) - \frac{s_5(x/y)s_5(x/y)}{r_2(x/y)r_2(x/y)} \right) \left\{ s_6(x/y)s_6(x/y)r_6(x/y) - r_2(x/y)r_3(x/y)r_3(x/y) \right\} \]  
\[ a_3^3 (x, y) = \frac{r_4(x/y)s_6(x/y)s_6(x/y)s_6(x/y)}{r_2(x/y)r_2(x/y)r_3(x/y)r_3(x/y)} - \frac{s_6(x/y)s_7(x/y)}{r_3(x/y)r_3(x/y)} \]  
\[ a_4^3 (x, y) = \frac{r_3(x/y)s_6(x/y)(r_6(x/y)s_6(x/y) + r_2(x/y)^2)}{r_2(x/y)r_3(x/y)r_3(x/y)} F_1(y) \]  
\[ + \frac{[r_3(x/y)r_6(x/y) - r_3(x/y)r_6(x/y)F_1(y)]}{r_2(x/y)r_3(x/y)} F_3(x) \]  
\[ - \frac{s_6(x/y)[r_6(x/y)s_6(x/y) + r_2(x/y)^2]}{r_2(x/y)r_3(x/y)^2}, \]  
\[ a_5^3 (x, y) = \frac{s_6(x/y)[r_6(x/y)s_6(x/y) + r_2(x/y)^2]}{r_2(x/y)r_3(x/y)r_3(x/y)} - \frac{r_6(x/y)}{r_2(x/y)} F_3(x), \]  

\url{https://doi.org/10.1088/1742-5468/aa85ca}
\[
\begin{align*}
a^3_6(x, y) &= \left[ \frac{s_6(xy)s_6(x/y)}{r_3(x/y)r_3(xy)} - F_3(x) \right] \\
&\times \left\{ \frac{r_1(x/y)r_4(x/y)s_5(xy)}{r_2(x/y)r_2(xy)s_5(y/x)} - F_1(x) \right\} \frac{s_5(y/x)r_6(xy)}{r_1(xy)r_2(y/x)} \\
&- \left[ \frac{s_5(x/y)}{r_3(x/y)} + \frac{r_1(x/y)r_6(xy)s_5(xy)s_6(x/y)}{r_1(xy)r_2(xy)r_3(x/y)} \right] \left[ \frac{r_6(x/y)}{r_2(x/y)} - \frac{r_2(x/y)s_6(xy)}{r_3(x/y)r_3(xy)} \right] \\
&- \left[ \frac{s_5(y/x)s_5(xy)}{r_2(x/y)r_2(xy)} + \frac{r_1(x/y)s_7(xy)}{r_3(x/y)r_3(xy)} \right] \frac{r_6(xy)s_5(xy)}{r_1(xy)r_2(x/y)} + \frac{r_6(xy)s_5(y/x)}{r_1(xy)r_2(y/x)} F_2(x) \\
&+ \frac{[s_6(xy)s_6(xy)F_1(x) - s_7(x/y)s_7(xy)]r_6(xy)s_5(y/x)}{r_1(xy)r_2(y/x)r_3(xy)r_3(xy)} \\
\end{align*}
\]

\[
\begin{align*}
a^3_7(x, y) &= \left[ \frac{s_6(xy)s_6(x/y)}{r_3(x/y)r_3(xy)} - F_3(x) \right] - \frac{r_7(xy)s_6(x/y)s_6(xy)}{r_1(xy)r_3(x/y)r_3(xy)} F_1(x) \\
&+ \frac{r_7(xy)s_7(x/y)}{r_1(xy)} F_2(x) + \left[ \frac{s_7(xy)}{r_1(xy)} - \frac{r_1(xy)s_7(x/y)}{r_3(x/y)r_3(xy)} \right] \frac{r_7(xy)s_7(x/y)}{r_3(x/y)r_3(xy)}, \\
b^1_1(x, y) &= \frac{s_6(xy)[r_6(xy)s_6(xy) + r_2(xy)^2]}{r_2(xy)r_3(xy)r_3(xy)} - \frac{r_6(xy)}{r_2(xy)} F_3(x), \\
b^1_2(x, y) &= \frac{r_1(xy)[r_3(x/y)r_6(xy)F_1(x) - r_3(xy)r_6(y/x)]}{r_2(xy)[r_3(x/y)r_4(y/x) - r_6(y/x)s_6(x/y)]}, \\
b^1_3(x, y) &= \frac{r_1(xy)r_3(y/x)r_6(xy)}{r_2(xy)[r_3(y/x)r_4(y/x) - r_6(y/x)s_6(x/y)]}, \\
b^1_4(x, y) &= \frac{r_6(xy)}{r_2(xy)} F_1(y) \\
&+ \frac{r_3(xy)[r_6(x/y)s_7(y/x) - r_3(xy)s_6(y/x)]}{r_2(xy)[r_3(x/y)r_4(y/x) - r_6(y/x)s_6(x/y)]}, \\
b^1_5(x, y) &= \frac{r_6(xy)}{r_2(xy)}, \\
c^1_6(x, y) &= \frac{r_2(xy)r_5(x/y)}{r_1(xy)r_2(x/y)} F_1(x) + \frac{s_5(xy)}{r_1(xy)}, \\
c^1_7(x, y) &= \frac{r_2(xy)r_5(x/y)}{r_1(xy)r_2(x/y)}.
\end{align*}
\]
Boundary algebraic Bethe Ansatz for a nineteen vertex model with $\mathcal{U}_q[\text{osp}(2|2)^{(2)}]$ symmetry

\begin{align}
\textbf{c}_0^1(x, y) &= -\frac{r_2(xy)s_5(x/y)}{r_1(xy)r_2(x/y)} F_1(y) - \frac{r_5(x/y)}{r_1(x/y)} F_1(x) F_1(y), \\
\textbf{c}_1^1(x, y) &= -\frac{r_5(x/y)}{r_1(x/y)} F_1(x) - \frac{r_2(xy)s_5(x/y)}{r_1(xy)r_2(x/y)}, \\
\textbf{c}_{10}^1(x, y) &= -\frac{r_5(x/y)}{r_1(x/y)} F_1(y), \\
\textbf{c}_{11}^1(x, y) &= -\frac{r_5(x/y)}{r_1(x/y)}, \\
\textbf{c}_0^3(x, y) &= \frac{s_5(x/y)s_6(x/y)}{r_2(xy)r_3(x/y)} \left[ \frac{r_2(xy)s_5(x/y)}{r_1(xy)r_2(x/y)} F_1(x) + \frac{s_5(x/y)}{r_1(x/y)} \right] \\
&\quad + \frac{r_4(xy)s_6(x/y)}{r_2(xy)r_3(x/y)} F_1(x) - \frac{r_3(xy)r_6(x/y)}{r_2(xy)r_3(x/y)} F_2(x) - \frac{s_6(x/y)s_7(x/y)}{r_2(xy)r_3(x/y)}, \\
\textbf{c}_3^1(x, y) &= -\frac{r_3(xy)r_6(x/y)}{r_2(xy)r_3(x/y)} F_3(x) + \frac{r_5(x/y)s_5(x/y)s_6(x/y)}{r_1(xy)r_2(x/y)r_3(x/y)} + \frac{r_4(xy)s_6(x/y)}{r_2(xy)r_3(x/y)}, \\
\textbf{c}_8^1(x, y) &= -\frac{r_3(xy)r_6(x/y)}{r_2(xy)r_3(x/y)}, \\
\textbf{c}_0^2(x, y) &= -\frac{s_5(x/y)s_6(x/y)}{r_2(xy)r_3(x/y)} \left[ \frac{r_2(xy)s_5(x/y)}{r_1(xy)r_2(x/y)} F_1(x) + \frac{s_5(x/y)}{r_1(x/y)} F_1(x) F_1(y) \right] \\
&\quad + \frac{s_6(x/y)s_7(x/y)}{r_2(xy)r_3(x/y)} F_1(y) + \frac{r_6(x/y)}{r_2(xy)} F_2(x) F_1(y) - \frac{r_4(xy)s_6(x/y)}{r_2(xy)r_3(x/y)} F_1(x) F_1(y), \\
\textbf{c}_{10}^3(x, y) &= -\frac{s_5(x/y)s_6(x/y)}{r_2(xy)r_3(x/y)} \left[ \frac{r_5(x/y)}{r_1(xy)} F_1(x) + \frac{r_2(xy)s_5(x/y)}{r_1(xy)r_2(x/y)} \right] \\
&\quad - \frac{r_4(xy)s_6(x/y)}{r_2(xy)r_3(x/y)} F_1(x) + \frac{r_6(x/y)}{r_2(xy)} F_2(x) + \frac{s_6(x/y)s_7(x/y)}{r_2(xy)r_3(x/y)}, \\
\textbf{c}_{11}^3(x, y) &= -\frac{r_4(xy)s_6(x/y)}{r_2(xy)r_3(x/y)} F_1(y) - \frac{r_5(xy)s_5(x/y)s_6(x/y)}{r_1(xy)r_2(xy)r_3(xy)} F_1(y) + \frac{r_6(x/y)}{r_2(xy)} F_1(y) F_3(x), \\
\textbf{c}_{12}^3(x, y) &= \frac{r_6(x/y)}{r_2(xy)} F_3(x) - \frac{r_4(xy)s_6(x/y)}{r_2(xy)r_3(x/y)} - \frac{r_5(xy)s_5(x/y)s_6(x/y)}{r_1(xy)r_2(xy)r_3(x/y)}. 
\end{align}

ORCID iDs

R S Vieira e https://orcid.org/0000-0002-8343-7106

https://doi.org/10.1088/1742-5468/aa85ca
Boundary algebraic Bethe Ansatz for a nineteen vertex model with $U_q[osp(2|2)^{(2m)}]$ symmetry

References

[1] Esler F H L, Korepin V E and Schoutens K 1992 Phys. Rev. Lett. 68 2960
[2] Esler F H L and Korepin V E 1992 Phys. Rev. B 46 9147
[3] Martins M J and Ramos P B 1998 Nucl. Phys. B 522 413–70
[4] McGuire J B 1964 J. Math. Phys. 5 622–36
[5] Yang C N 1967 Phys. Rev. Lett. 19 1312
[6] Yang C N 1968 Phys. Rev. 168 1920
[7] Baxter R J 1972 Ann. Phys. 70 193–228
[8] Baxter R J 1978 Phil. Trans. R. Soc. A 289 315–46
[9] Bazhanov V V and Shadrikov A G 1987 Theor. Math. Phys. 73 1302–12
[10] Drinfel’d V G 1988 J. Sov. Math. 41 898–915
[11] Maldacena J 1999 Int. J. Theor. Phys. 38 1113–33
[12] Minahan J A and Zarembo K 2003 J. High Energy Phys. JHEP03(2003)013
[13] Beisert N and Staudacher M 2003 Nucl. Phys. B 670 439–63
[14] Bena I, Polchinski J and Roiban R 2004 Phys. Rev. D 69 046002
[15] Takhtadzhian L A and Faddeev L D 1979 Russ. Math. Surv. 34 11–68
[16] Sklyanin E K 1982 J. Sov. Math. 19 1546–96
[17] Korepin V E, Bogoliubov N M and Izergin A G 1997 Quantum Inverse Scattering Method and Correlation Functions vol 3 (Cambridge: Cambridge University Press)
[18] Sklyanin E K 1988 J. Phys. A: Math. Gen. 21 2375
[19] Mezincescu L and Nepomechie R I 1991 J. Phys. A: Math. Gen. 24 L17
[20] Bracken A J, Ge X Y, Zhang Y Z and Zhou H Q 1998 Nucl. Phys. B 516 588–602
[21] Tarasov V O 1988 Theor. Math. Phys. 76 793–803
[22] Izergin A G and Korepin V E 1981 Commun. Math. Phys. 79 303–16
[23] Zamolodchikov A B and Fateev V A 1980 A model factorized S-matrix and an integrable spin-1 Heisenberg chain Yadernaya Fizika 32 581–590 (in Russian)
Zamolodchikov A B and Fateev V A 1980 A model factorized S-matrix and an integrable spin-1 Heisenberg chain Sov. J. Nucl. Phys. 32 298–303
[24] Lima-Santos A 1999 J. Phys. A: Math. Gen. 32 1819
[25] Kurak V and Lima-Santos A 2004 Nucl. Phys. B 699 595–631
[26] Kurak V and Lima-Santos A 2005 J. Phys. A: Math. Gen. 38 2359
[27] Vieira R S and Lima Santos A 2017 Phys. Lett. A 381 3015–3020
[28] Yang W L and Zhang Y Z 1999 Phys. Lett. A 261 252–8
[29] Yang W L and Zhou H Q 1999 Phys. Lett. A 261 252–8
[30] Frappat L, Sciarrino A and Sorba P 1989 Commun. Math. Phys. 121 457–500
[31] Frappat L, Sciarrino A and Sorba P 2000 Dictionary on Lie Algebras and Superalgebras vol 10 (London: Academic)
[32] Khoroshkin S, Lukierski J and Tolstoy V 2001 Commun. Math. Phys. 220 537–60
[33] MacKay N and Zhao L 2001 J. Phys. A: Math. Gen. 34 6313
[34] Ransingh B 2013 Int. J. Pure Appl. Math. 84 539–47
[35] Xu Y and Zhang R B 2016 arXiv:1607.01142
[36] Lima-Santos A 2009 J. Stat. Mech. P04005
[37] Lima-Santos A 2009 J. Stat. Mech. P07045
[38] Lima-Santos A 2009 J. Stat. Mech. P08006
[39] Lima-Santos A and Galleas W 2010 Nucl. Phys. B 833 271–97
[40] Vieira R S and Lima-Santos A 2013 J. Stat. Mech. P02011
[41] Vieira R S and Lima-Santos A 2017 Reflection matrices with $U_q[osp(2|2)^{(2m)}]$ symmetry J. Phys. A: Math. Theor. 50
[42] Kac V G 1977 Commun. Math. Phys. 53 31–64
[43] Vieira R S and Lima-Santos A 2015 Phys. Lett. A 379 2150–3

https://doi.org/10.1088/1742-5468/aa85ca