SOME CONGRUENCES FOR TRINOMIAL COEFFICIENTS

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Abstract. We prove several congruences for trinomial coefficients.

1. Introduction

In [3], Pan and Sun proved the following congruence on the sums of binomial coefficients:
\[ \sum_{k=0}^{p-1} \binom{2k}{k+d} \equiv \left( \frac{p-d}{3} \right) \pmod{p}, \] (1.1)
where \( p > 3 \) is a prime, \( 0 \leq d \leq p - 1 \) and \( (\cdot) \) is the Legendre symbol. They also proved that for prime \( p > 3 \) and integer \( 0 \leq d \leq p - 1 \),
\[ \sum_{k=1}^{p-1} \frac{1}{k} \binom{2k}{k+d} \equiv \begin{cases} d^{-1}(-1 + 2(-1)^d + 3 | p-d]) \pmod{p}, & \text{if } 1 \leq d \leq p, \\ 0 \pmod{p}, & \text{if } d = 0, \end{cases} \] (1.2)
where \([A] = 1 \) or \(0\) according to whether the assertion \( A \) holds. Subsequently, Sun and Tauraso [6] extended (1.1) and showed that
\[ \sum_{k=0}^{p-1} \binom{2k}{k+d} \equiv \left( \frac{p-d}{3} \right) + 2pS_d \pmod{p^2}, \] (1.3)
where
\[ S_d = \sum_{0 < k < d} \frac{(-1)^{k-1}}{k} \binom{d-k}{3}. \]

On the other hand, trinomial coefficients \( \binom{n}{k}_2 \) are given by
\[ (1 + x + x^{-1})^n = \sum_{k=-n}^{n} \binom{n}{k}_2 x^k. \]

As G. E. Andrews mentioned [1], trinomial coefficients had been investigated by Euler. And Andrews and R. J. Baxter [2] found the \(q\)-analogues of trinomial coefficients play an important rule in the hard hexagon model. However, there is a similar congruence for trinomial coefficients:
\[ \sum_{k=0}^{n-1} \binom{k}{d}_2 \equiv \begin{cases} (-1)^{\frac{d-1}{2}} \pmod{p}, & \text{if } d \text{ is even}, \\ 0 \pmod{p}, & \text{if } d \text{ is odd}, \end{cases} \] (1.4)
where \( p > 3 \) is prime and \( 0 \leq d < p \). In fact, we shall prove the following stronger result.
Theorem 1.1. Let $p > 3$ be a prime and let $d$ be an integer with $0 \leq d \leq p - 1$. If $d$ is odd, then
\[
\frac{1}{p} \sum_{k=0}^{p-1} \binom{k}{d} \equiv \frac{(-1)^{d+1}}{2} \left( \sum_{k=1}^{(d-1)/2} (-1)^k \frac{1}{k} - 3 \sum_{1 \leq k \leq (d-1)/2} \frac{(-1)^k}{k} \right) \pmod{p}. \tag{1.5}
\]
And if $d$ is even, then
\[
\begin{aligned}
\frac{1}{p} &\left( (-1)^{d/2} \sum_{k=0}^{p-1} \binom{k}{d} \right) - (-1)^{d-1} \\
&\equiv -3 \frac{(-2/p) S_{(p-(d/2))/2} - 2 \sum_{0 \leq j < d/2} \frac{(-1)^j}{2j+1} + 3 \sum_{0 \leq j < d/2 \atop 3 \not| 2j-d} \frac{(-1)^j}{2j+1}}{p} \pmod{p}, \tag{1.6}
\end{aligned}
\]
where the recurrence sequence $\{S_n\}$ is defined as
\[
S_0 = 0, \quad S_1 = 1, \quad S_{n+1} = 4S_n - S_{n-1} \text{ for } n \geq 1.
\]
We also have a congruence for the alternating sums of trinomial coefficients.

Theorem 1.2. Let $p > 3$ be a prime and let $d$ be an integer with $0 \leq d \leq p - 1$. We have
\[
\sum_{k=0}^{p-1} (-1)^k \binom{k}{d} \equiv (-1)^d - 1 + (-1)^d \frac{1}{p} \sum_{k=d+1}^{(p+d-1)/2} \frac{1}{k} \left( \frac{2k-d}{k+1} \right) \pmod{p^2}. \tag{1.7}
\]
Unfortunately, in general, it is not easy to compute
\[
\sum_{k=d+1}^{(p+d-1)/2} \frac{1}{k} \left( \frac{2k-d}{k+1} \right) \pmod{p}.
\]
However, when $d = 0, 1, 2$, we may get

Theorem 1.3. For prime $p > 3$ we have
\[
\begin{aligned}
\sum_{k=0}^{p-1} (-1)^k \binom{k}{0} &\equiv \frac{3\binom{p}{2} - 1}{2} \pmod{p^2}, \tag{1.8} \\
\sum_{k=0}^{p-1} (-1)^k \binom{k}{1} &\equiv 1 - \frac{3\binom{p}{2} - 1}{2} \pmod{p^2} \tag{1.9}
\end{aligned}
\]
and
\[
\sum_{k=0}^{p-1} (-1)^k \binom{k}{2} \equiv -2 + 3p \left( \frac{p}{3} \right) \pmod{p^2}. \tag{1.10}
\]

Quiet recently, Z.-W. Sun told us that with help of the software Mathematica, he found a similar congruence for the sum
\[
\sum_{k=0}^{p-1} \frac{T_k}{3^k}.
\]
Here we shall give a proof of Sun’s congruence.

**Theorem 1.4.** Let $p > 3$ be a prime. Then

$$
\sum_{k=0}^{p^2-1} T_k \equiv \begin{cases} 
p \pmod{p^2} & \text{if } p \equiv 1 \pmod{3}, \\
0 \pmod{p^2} & \text{if } p \equiv 2 \pmod{3}.
\end{cases}
$$

The proofs of the above theorems will be given in the next sections.

2. Proof of Theorem 1.1

**Lemma 2.1.** Suppose that $n > d \geq 0$. Then

$$
\sum_{k=0}^{n-1} \binom{k}{d}_2 = \sum_{k=1}^{n} \binom{n}{k} \binom{k-1}{k+d-1/2}.
$$

**Proof.** Let $[x^d]P(x)$ denote the coefficient of $x^d$ in the expansion of the polynomial $P(x)$. Since

$$
\sum_{k=0}^{n-1} (1 + x + x^{-1})^k = \frac{(1 + x + x^{-1})^n - 1}{x + x^{-1}} = \frac{1}{x^{n-1}} \cdot \frac{(1 + x + x^2)^n - x^n}{1 + x^2},
$$

we have

$$
\sum_{k=0}^{n-1} \binom{k}{d}_2 = \sum_{k=0}^{n-1} [x^d](1 + x + x^{-1})^k = [x^d] \frac{1}{x^{n-1}} \cdot \frac{(1 + x + x^2)^n - x^n}{1 + x^2}
$$

$$
= [x^{n+d-1}] \frac{(1 + x + x^2)^n - x^n}{1 + x^2} = [x^{n+d-1}] \sum_{k=1}^{n} \binom{n}{k} (1 + x^2)^{k-1} x^{n-k}
$$

$$
= \sum_{k=1}^{n} \binom{n}{k} [x^{k+d-1}](1 + x^2)^{k-1} = \sum_{k=1}^{n} \binom{n}{k} \binom{k-1}{k+d-1/2}.
$$

□

Noting that

$$
\binom{p}{k} \equiv 0 \pmod{p} \quad \text{and} \quad \binom{p-1}{k} \equiv (-1)^k \pmod{p}
$$

for $0 < k < p$, (1.4) immediately follows from (2.1).

**Lemma 2.2.** Suppose that $p > 3$ is prime and $0 \leq m < (p-1)/2$. Then

$$
\sum_{j=1}^{(p-1)/2} \frac{1}{j} \binom{2j}{j + m} \equiv \frac{1}{m} (1 - 3[3 \mid p + m]) \pmod{p}.
$$

**Proof.** Let

$$
a_m = \sum_{j=1}^{(p-1)/2} \frac{1}{j} \binom{2j}{j + m}.
$$
Define the polynomials \( \{v_n(x)\}_{n \in \mathbb{N}} \) by
\[
v_0(x) = 2, \quad v_1(x) = x, \quad \text{and} \quad v_{n+1}(x) = xv_n(x) - v_{n-1}(x), \quad n = 1, 2, \ldots.
\]

Applying Theorem 2.1 in [5], we have
\[
m \sum_{j=1}^{n+1} \frac{1}{j} \left( \frac{2j}{j + m} \right) - v_m(-1) = - \sum_{j=0}^{n+1} \binom{p+1}{j} v_{\frac{2j}{2} + m - j}(-1) - 2 \left( \frac{p}{2} + m \right)
\equiv - v_{\frac{2j}{2} + m}(-1) - (p + 1) v_{\frac{2j}{2} + m}(-1)
\equiv - v_{\frac{2j}{2} + m}(-1) - v_{\frac{2j}{2} + m}(-1) = v_{\frac{2j}{2} + m + 1}(-1) \pmod{p}.
\]

Since \( v_n(-1) = 3[3 \mid n] - 1 \) for all \( n \in \mathbb{N} \), we have
\[
a_m = \frac{1}{m} \left( v_m(-1) + v_{\frac{2j}{2} + m}(-1) \right)
\equiv \frac{1}{m} (3[3 \mid m] + 3[3 \mid \frac{p+3}{2} + m] - 2)
\equiv \frac{1}{m} (3[3 \mid m] + 3[3 \mid p - m] - 2)
\equiv \frac{1}{m} (1 - 3[3 \mid p + m]) \pmod{p}.
\]

\[\square\]

Lemma 2.3. Suppose that \( p > 3 \) is prime and \( 0 \leq m < (p - 1)/2 \). Then
\[
\sum_{j=1}^{(p-3)/2} \frac{(-1)^j}{2j + 1} \equiv \left( -\frac{1}{p} \right) \frac{2p-1}{2} \pmod{p}.
\]

Proof. Clearly,
\[
\sum_{j=1}^{(p-3)/2} \frac{(-1)^j}{2j + 1} \equiv \sum_{j=1}^{(p-3)/2} \frac{(-1)^j}{2j + 1} \left( \frac{p - 1}{2i} \right) = \frac{1}{p} \sum_{j=1}^{(p-3)/2} (-1)^j \left( \frac{p}{2j + 1} \right) \pmod{p}.
\]

And
\[
\sum_{j=1}^{(p-3)/2} (-1)^j \left( \frac{p}{2j + 1} \right) = (-1)^{\frac{p-1}{2}} 2^{\frac{p-1}{2}} - (-1)^{\frac{p-1}{2}} = \left( -\frac{2}{p} \right) \left( 2^{\frac{p-1}{2}} - \left( \frac{2}{p} \right) \right).
\]

Finally,
\[
2^{p-1} - 1 = \left( 2^{\frac{p-1}{2}} - \left( \frac{2}{p} \right) \right) \left( 2^{\frac{p-1}{2}} + \left( \frac{2}{p} \right) \right)
= \left( 2^{\frac{p-1}{2}} - \left( \frac{2}{p} \right) \right)^2 + 2 \left( \frac{2}{p} \right) \left( 2^{\frac{p-1}{2}} - \left( \frac{2}{p} \right) \right) \equiv 2 \left( \frac{2}{p} \right) \left( 2^{\frac{p-1}{2}} - \left( \frac{2}{p} \right) \right) \pmod{p^2}.
\]

\[\square\]
Lemma 2.4. Suppose that \( p > 3 \) is prime. Then

\[
\sum_{0 \leq j \leq (p-3)/2} \frac{(-1)^j}{2j + 1} \equiv \left(\frac{-1}{p}\right) \frac{2^{p-1} - 1}{3p} - \left(\frac{-2}{p}\right) \frac{S_{(p+1)/2}}{p} \pmod{p}.
\]

Proof. This is an immediate consequence of [4, Corollary 3.3]. \( \square \)

Now we are ready to prove (1.5) and (1.6).

Proof of (1.5). Suppose that \( d = 2m + 1 \). Let

\[
S_m = \frac{1}{p} \sum_{k=0}^{p-1} \binom{k}{2m+1}_2.
\]

According to Lemma 2.1,

\[
S_m = \frac{1}{p} \sum_{k=1}^{p} \binom{p}{k} \binom{k-1}{k+2m} = \sum_{j=1}^{(p-1)/2} \frac{1}{2j} \binom{p-1}{2j-1} \binom{2j-1}{j+m} \equiv -\frac{1}{2} \sum_{j=1}^{(p-1)/2} \frac{1}{j} \binom{2j-1}{j+m} \pmod{p}.
\]

By (1.2),

\[
\sum_{j=1}^{p-1} \frac{1}{j} \binom{2j}{j} \equiv 0 \pmod{p}.
\]

As

\[
\binom{2j}{j} \equiv 0 \pmod{p}
\]

for \( (p - 1)/2 < j < p \), we obtain that

\[
S_0 \equiv -\frac{1}{2} \sum_{j=1}^{(p-1)/2} \frac{1}{j} \binom{2j-1}{j} = -\frac{1}{4} \sum_{j=1}^{(p-1)/2} \frac{1}{j} \binom{2j}{j} \equiv -\frac{1}{4} \sum_{j=1}^{p-1} \frac{1}{j} \binom{2j}{j} \equiv 0 \pmod{p}.
\]
For $1 \leq m \leq (p - 3)/2$,
\[
S_m = -\frac{1}{2} \sum_{j=1}^{(p-1)/2} \frac{1}{j} \left( \binom{2j}{j+m} - \binom{2j-1}{j+m-1} \right)
= -\frac{1}{2} \sum_{j=1}^{(p-1)/2} \frac{1}{j} \left( \binom{2j}{j+m} \right) + \frac{1}{2} \sum_{j=1}^{(p-1)/2} \frac{1}{j} \left( \binom{2j-1}{j+m-1} \right)
\equiv -\frac{1}{2} \sum_{j=1}^{(p-1)/2} \frac{1}{j} \left( \binom{2j}{j+m} \right) - S_{m-1} \pmod{p}.
\]

Then
\[
S_m \equiv -S_{m-1} - \frac{1}{2} \sum_{j=1}^{(p-1)/2} \frac{1}{j} \left( \binom{2j}{j+m} \right) \pmod{p},
\]
and therefore
\[
S_m \equiv (-1)^m S_0 + \frac{(-1)^{m+1}}{2} \sum_{k=1}^{m} (-1)^k \sum_{j=1}^{(p-1)/2} \frac{1}{j} \left( \binom{2j}{j+k} \right)
\equiv \frac{(-1)^{m+1}}{2} \sum_{k=1}^{m} (-1)^k \sum_{j=1}^{(p-1)/2} \frac{1}{j} \left( \binom{2j}{j+k} \right)
\equiv \sum_{k=1}^{m} \frac{(-1)^k}{k} (1 - 3[3 \mid p + k]) = \sum_{k=1}^{m} \frac{(-1)^k}{k} - 3 \sum_{k=1}^{m} \frac{(-1)^k}{3k+p} \pmod{p}.
\]

This concludes the proof. \qed

Proof of (1.6). Suppose that $d = 2m$. Note that
\[
\binom{k + 1}{2m + 1} = \binom{k}{2m} + \binom{k}{2m + 1} + \binom{k}{2m + 2}.
\]

Hence
\[
\sum_{k=0}^{p-1} \binom{k}{2m} = \sum_{k=0}^{p-1} \binom{k}{2m + 1} - \sum_{k=0}^{p-1} \binom{k}{2m + 2} = \binom{p}{2m + 2}.
\]

Let
\[
S_m = \sum_{k=0}^{p-1} \binom{k}{2m}.
\]

Then for every $j \geq 1$,
\[
S_m = (-1)^j S_{m+j} + \sum_{i=0}^{j-1} (-1)^i \binom{p}{2m + 2i + 1}.
\]
In particular, when \( j = (p - 1)/2 - m \), we have

\[
S_m = (-1)^{(p-1)/2 - m} + \sum_{i=0}^{(p-3)/2 - m} (-1)^i \left( \frac{p}{2m + 2i + 1} \right).
\]

Now

\[
\frac{1}{p} \left( \frac{p}{2m + 2i + 1} \right)_2 = \frac{1}{p} \sum_{k=0}^{p-1} (-1)^k \left( \frac{p}{k} \right) \left( \frac{2(p - k)}{p - k - (2m + 2i + 1)} \right)
\]

\[
= \frac{2}{p - 2m - 2i - 1} \left( \frac{2p - 1}{p - 2m - 2i - 2} \right) + \sum_{k=1}^{p-1} (-1)^k \left( \frac{p - 1}{k - 1} \right) \left( \frac{2(p - k)}{p - k - 2m - 2i - 1} \right)
\]

\[
= \frac{2}{2m + 2i + 1} + \sum_{k=1}^{p-1} \frac{1}{p - k} \left( \frac{2(p - k)}{p - k - 2m - 2i - 1} \right) \pmod{p}.
\]

Therefore

\[
\frac{S_m - (-1)^{(p-1)/2 - m}}{p} \equiv \sum_{i=0}^{(p-3)/2 - m} (-1)^i \left( \frac{2}{2m + 2i + 1} \right) + \sum_{k=1}^{p-1} \frac{1}{k} \left( \frac{2k}{k - 2m - 2i - 1} \right) \pmod{p}
\]

\[
\equiv \sum_{i=0}^{(p-3)/2 - m} (-1)^i \left( \frac{2}{2m + 2i + 1} \right) + \sum_{k=1}^{p-1} \frac{1}{k} \left( \frac{2k}{k + 2m + 2i + 1} \right) \pmod{p}
\]

\[
\equiv \sum_{i=0}^{(p-3)/2 - m} (-1)^i \left( \frac{2}{2m + 2i + 1} \right) + \frac{3}{2m + 2i + 1} \left( [3 \nmid p - (2m + 2i + 1)] - 1 \right) \pmod{p}
\]

\[
= 2 \sum_{i=0}^{(p-3)/2 - m} (-1)^i \frac{1}{2m + 2i + 1} + 3 \sum_{0 \leq i \leq (p-3)/2 - m, 3 \nmid p - (2m + 2i + 1)} (-1)^i \frac{1}{2m + 2i + 1} \pmod{p}.
\]

By Lemma 2.3,

\[
\sum_{i=0}^{(p-3)/2 - m} (-1)^{m+1} \frac{1}{2m + 2i + 1} = \left( \frac{-1}{p} \right) \frac{2^{p-1} - 1}{2p} - \sum_{j=0}^{m-1} \left( \frac{-1}{2j + 1} \right) \pmod{p}.
\]

And by Lemma 2.4,

\[
\sum_{0 \leq i \leq (p-3)/2 - m, 3 \nmid p - (2m + 2i + 1)} (-1)^{m+1} \frac{1}{2m + 2i + 1}
\]

\[
\equiv \left( \frac{-1}{p} \right) \frac{2^{p-1} - 1}{3p} - \left( \frac{-2}{p} \right) \frac{S_{(p-3)/2}}{p} - \sum_{0 \leq i \leq m-1, 3 \nmid p + 2j + 1} \left( \frac{-1}{2j + 1} \right) \pmod{p}.
\]

We are done. \(\square\)
3. Proofs of Theorems 1.2 - 1.4

Proof of Theorem 1.2. Note that
\[
\sum_{k=0}^{p-1} (-1)^k \binom{k}{p-1}^2 = (-1)^{p-1}(p-1) = 1
\]
and
\[
\sum_{k=0}^{p-1} (-1)^k \binom{k}{p-2}^2 = (-1)^{p-2}(p-2) + (-1)^{p-1}(p-1) = -1 + p - 1 = p - 2.
\]
So we may assume that \(d < p - 2\). Since
\[
\sum_{k=0}^{p-1} (-1)^k (1 + x + x^{-1})^k = \frac{1 + (1 + x + x^{-1})^p}{1 + (1 + x + x^{-1})} = \frac{x^p + (1 + x + x^2)^p}{x^{p-1}(1+x)^2},
\]
we have
\[
\sum_{k=0}^{p-1} (-1)^k \binom{k}{d}^2 = \sum_{k=0}^{p-1} (-1)^k [x^d](1 + x + x^{-1})^k = [x^d] \sum_{k=0}^{p-1} (-1)^k (1 + x + x^{-1})^k
\]
\[
= [x^{p+d-1}] \frac{x^p + (1 + x + x^2)^p}{(1+x)^2}
\]
\[
= [x^{d-1}] \frac{1}{(1+x)^2} + [x^{p+d-1}] \sum_{k=0}^{p} \binom{p}{k} x^{2k}(1+x)^{p-k-2}
\]
\[
= (-1)^{d-1}d + \sum_{k=d+1}^{\lfloor(p+d-1)/2\rfloor} \binom{p}{k} \binom{p-k-2}{p+d-1-2k}
\]
\[
= (-1)^{d-1}d + p \sum_{k=d+1}^{\lfloor(p+d-1)/2\rfloor} \frac{1}{k} \binom{p-1}{k-1} \binom{p-k-2}{k-d-1}.
\]
Observe that
\[
\binom{p-1}{k-1} \equiv (-1)^{k-1} \pmod{p}
\]
and
\[
\binom{p-k-2}{k-d-1} = \frac{(p-k-2)(p-k-3) \cdots (p-2k+d)}{(k-d-1)!}
\]
\[
\equiv (-1)^{k-d-1} \frac{(k+2)(k+3) \cdots (2k-d)}{(k-d-1)!}
\]
\[
= (-1)^{k-d-1} \binom{2k-d}{k-d-1} \equiv (-1)^{k-d-1} \binom{2k-d}{k+1} \pmod{p}.
\]
So (1.7) is valid. \(\square\)
Proof of Theorem 1.3. Let $C_n = \binom{2n}{n+1}$ be the Catalan number. Applying (1.7) with $d = 0$, $d = 1$ and $d = 2$ respectively, we get

$$\sum_{k=0}^{p-1} (-1)^k \binom{k}{2} = \sum_{k=1}^{(p-1)/2} \frac{1}{k} \binom{2k}{k+1} = p \sum_{k=1}^{(p-1)/2} C_k \pmod{p^2},$$

and

$$\sum_{k=0}^{p-1} (-1)^k \binom{k}{1} = 1 - p \sum_{k=1}^{(p-1)/2} \frac{1}{k} \binom{2k-1}{k+1} = 1 - \sum_{k=2}^{(p-1)/2} \frac{k-1}{k} C_k \pmod{p^2}.$$

In [3], Pan and Sun have proved that

$$\sum_{k=0}^{p-1} C_k \equiv \frac{3(p^3) - 1}{2} \pmod{p}$$

and

$$\sum_{k=1}^{p-1} \frac{C_k}{k} \equiv \frac{3}{2} \left(1 - \binom{p}{3}\right) \pmod{p}.$$

Clearly,

$$C_0 = C_1 = 1, \quad C_2 = 2, \quad C_{p-1} = \frac{1}{p} \binom{2p-2}{p-1} \equiv -1 \pmod{p}$$

and

$$C_k \equiv 0 \pmod{p} \quad \text{for} \quad (p-1)/2 < k < p-1.$$

Therefore

$$\sum_{k=1}^{(p-1)/2} C_k \equiv \sum_{k=0}^{p-1} C_k \equiv \frac{3(p^3) - 1}{2} \pmod{p}$$

and

$$\sum_{k=1}^{(p-1)/2} \frac{C_k}{k} \equiv \frac{1 - 3(p^3)}{2} \pmod{p}.$$
and
\[
\sum_{k=2}^{(p-1)/2} C_k - \sum_{k=3}^{(p+1)/2} \frac{C_k}{k} \equiv \frac{3\left(\frac{p}{2}\right) - 1}{2} - \frac{1 - 3\left(\frac{p}{2}\right)}{2} + 2 = 3\left(\frac{p}{3}\right) \pmod{p}.
\]

This yields (1.8), (1.9) and (1.10). We are done. □

**Proof of Theorem 1.4.** Clearly
\[
\sum_{k=0}^{p-1} \left(\frac{(1 + x + x^{-1})}{3}\right)^k = \frac{(1 + x + x^{-1})^p/3^p - 1}{(1 + x + x^{-1})/3 - 1}.
\]

Then
\[
\sum_{k=0}^{p-1} \frac{T_k}{3^k} = [x^0] \sum_{k=0}^{p-1} \left(\frac{(1 + x + x^{-1})}{3}\right)^k
\]
\[
= \frac{1}{3^{p-1}} \left[ x^{p-1} \right] \frac{(1 - x^3)^p - (3x(1 - x))^p}{(1 - x)^{p+2}}
\]
\[
= \frac{1}{3^{p-1}} \left[ x^{p-1} \right] \frac{(1 - x^3)^p}{(1 - x)^{p+2}}
\]
\[
= \frac{1}{3^{p-1}} \sum_{0 \leq k < p/3} \left(\frac{p}{k}\right)(-1)^k \binom{2p - 3k}{p - 1 - 3k} (-1)^{p-1-3k}
\]
\[
= \frac{1}{3^{p-1}} \sum_{0 \leq k < p/3} \left(\frac{p}{k}\right)(-1)^{p-1-3k} \binom{2p - 3k}{p - 1 - 3k}
\]
\[
= \frac{1}{3^{p-1}} \left(\frac{2p}{p - 1}\right) + \frac{1}{3^{p-1}} \sum_{0 \leq k < p/3} \left(\frac{p}{k}\right)(-1)^k \binom{2p - 3k}{p - 1 - 3k}
\]
\[
= \frac{2p}{3^{p-1}(p - 1)} \left(\frac{2p - 1}{p - 2}\right) + \frac{p}{3^{p-1}} \sum_{0 \leq k < p/3} \left(\frac{-1}{k}\right) \binom{p - 1}{k} \binom{2p - 3k}{p - 1 - 3k}
\]

It is known that \(\binom{2p-1}{p-2} \equiv (-1)^{p-2} = -1 \pmod{p}\) and
\[
\frac{2p - 3k}{p - 1 - 3k} = \frac{p + (p - 3k)}{p - 1 - 3k} \equiv \frac{p - 3k}{p - 1 - 3k} = p - 3k \equiv -3k \pmod{p}.
\]
Therefore

\[
\sum_{k=0}^{p-1} T_k 3^k \equiv 2p + 3p \left\lfloor \frac{p - 1}{3} \right\rfloor \mod p^2
\]

\[
= \begin{cases} 
  p \ (\text{mod } p^2) & \text{if } p \equiv 1 \ (\text{mod } 3), \\
  0 \ (\text{mod } p^2) & \text{if } p \equiv 2 \ (\text{mod } 3).
\end{cases}
\]

We are done. \qed

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