Semicontinuity of Singularity Invariants
in Families of Formal Power Series

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Abstract. The problem we are considering came up in connection with the classification of singularities in positive characteristic. Then it is important that certain invariants like the determinacy can be bounded simultaneously in families of formal power series parametrized by some algebraic variety. In contrast to the case of analytic or algebraic families, where such a bound is well known, the problem is rather subtle, since the modules defining the invariants are quasi-finite but not finite over the base space. In fact, in general the fibre dimension is not semicontinuous and the quasi-finite locus is not open. However, if we pass to the completed fibers in a family of modules we can prove that their fiber dimension is semicontinuous under some mild conditions. We prove this in a rather general framework by introducing and using the completed and the Henselian tensor product, the proof being more involved as one might think. Finally we apply this to the Milnor number and the Tjurina number in families of hypersurfaces and complete intersections and to the determinacy in a family of ideals.

Mathematics Subject Classification (2000). 13B35, 13B40, 14A15, 14B05, 14B07.
Keywords. Formal power series, completed tensor product, Henselian tensor product, semicontinuity, Milnor number, Tjurina number, determinacy.

Introduction

In connection with the classification of singularities defined by formal power series over a field a fundamental invariant is the modality of the singularity (with respect to some equivalence relation like right or contact equivalence). To determine the modality one has to investigate adjacent singularities that appear in nearby fibers. This cannot be done by considering families over complete local rings but one has to consider families of power series parametrized by some algebraic variety in the neighbourhood of a given point. To determine potential adjacencies, an important tool is the semicontinuity of certain singularity invariants like, for example, the
Milnor or the Tjurina number. Another basic question is if the determinacy of an ideal can be bounded by a semicontinuous invariant. In the complex analytic situation the answer to these questions is well known and positive, but for formal power series the problem is much more subtle as one might think at the first glance. This is mainly due to the fact that ideals or modules that define the invariants are quasi-finite but not finite over the base space.

The modality example shows that the questions treated in this paper are rather natural and appear in important applications. Moreover, the semicontinuity in general is a very basic property with numerous applications in many other contexts. Therefore we decided to choose a rather general framework with families of modules presented by matrices of power series and parametrized by the spectrum of some Noetherian ring. It is not difficult to see that the fiber dimension is in general not semicontinuous and that the quasi-finite locus is in general not open (in contrast the case of ring maps of finite type, where the quasi-finite locus is open by Zariski’s Main Theorem), see Examples 19 and 20. It turns out that the situation is much more satisfactory if we consider not the fibers but the completed fibers and we prove the desired semicontinuity for the completed fiber dimension under some conditions on the family. To guarantee that the completed fibre families behave well under base change we introduce the notion of a (partial) completed tensor product and study its properties in sections 1.1 and 1.2.

Unfortunately, we cannot prove the semicontinuity of the completed fibre dimension in full generality. We prove it if either the base ring has dimension one (in section 1.3) or if the presentation matrix has polynomials or algebraic power series as entries (in section 1.5). Together, these cases cover most applications. To treat the latter case, we use Henselian rings and the Henselian tensor product, for which we give a short account in section 1.4. It would be interesting to know, if the result holds for presentation matrices with arbitrary power series as entries or if there are counterexamples.

In section 2 we apply our results to singularity invariants. We discuss and compare first the notions of regularity and smoothness (over a field) and show that both notions coincide for the completed fibres (Lemma 52). Under the restrictions mentioned above, we prove the semicontinuity of the Milnor number and Tjurina number for hypersurfaces (section 2.2) and the Tjurina number for complete intersections (section 2.4) as well as an upper bound for the determinacy of an ideal (section 2.3). Since the base ring may be the integers, our results are of some interest for computational purposes. For example, if a power series has integer coefficients then the Milnor number over the rationals is bounded by the Milnor number modulo just one prime number if this is finite (see Corollary 55).

We assume all rings to be associative, commutative and with unit. Throughout the paper \( \mathbb{k} \) denotes an arbitrary field, \( A \) a ring, \( R = A[[x]] \), \( x = (x_1, \ldots, x_n) \), the formal power series ring over \( A \) and \( M \) an \( R \)-module. For our main results we will assume that \( A \) is Noetherian and that \( M \) is finitely generated as \( R \)-module.
1. Quasi-finite modules and semicontinuity

1.1. The completed tensor product

Let $A$ be a ring $R = A[[x]]$ and $M$ an $R$-module. For any prime ideal $p$ of $A$ let $k(p) = A_p/pA_p$ be the residue field of $p$. $k(p) = \text{Quot}(A/p)$ is the quotient field of $A/p$ and hence $k(p) = A/p$ if $p$ is a maximal ideal. We consider $M$ via the canonical map $A \rightarrow R$ as $A$-module and set

$$M(p) := \frac{M \otimes_A k(p) = (M \otimes_A A_p) \otimes_{A_p} k(p) = M \otimes_A k(p)}{M \otimes_A k(p)},$$

which is called the fiber of $M$ over $p$. $M(p)$ is a vector space over $k(p)$ and its dimension is denoted by $d_p(M) := \dim_{k(p)} M(p)$.

$M$ is called quasi-finite over $p$ if $d_p(M) < \infty$. We are interested in the behavior of $d_p(M)$ if $p$ varies in Spec $A$, in particular in finding conditions under which $d_p(M)$ is semicontinuous on Spec $A$.

We say that a function $d : \text{Spec } A \rightarrow \mathbb{R}$, $p \mapsto d_p$, is (upper) semicontinuous at $p$ if $p$ has an open neighbourhood $U \subset \text{Spec } A$ such that $d_q \leq d_p$ for all $q \in U$.

$d$ is semicontinuous on Spec $A$ if it is semicontinuous at every $p \in \text{Spec } A$.

For finitely presented $A$-modules $M$ the semicontinuity of $p \mapsto d_p(M)$ is true and well known (cf. Lemma 1). However, in many applications $M$ is not finitely generated over $A$ but finite over some $A$-algebra $R$. Such a situation appears naturally in algebraic geometry, when one considers families of schemes or of coherent sheaves over Spec $A$. But then it is assumed that the ring $R$ is either of (essentially) finite type over $A$ (in algebraic geometry) or an analytic $A$-algebra (in complex analytic geometry). When we study families of singularities defined by formal power series (cf. Section 2), we have to consider $R = A[[x]]$, which is not of finite type over $A$. As far as we know, this situation has not been systematically studied and it leads to some perhaps unexpected results. For example, $d_p(M)$ is in general not semicontinuous on Spec $A$ (cf. Examples 19, 20).

It turns out that the situation is much more satisfactory if we pass from the usual fibres to the completed fibres, that is, we consider the completed fiber dimension

$$\hat{d}_p(M) := \dim_{k(p)} M(p)^\wedge,$$

where $M(p)^\wedge$ is the $\langle x \rangle$-adic completion of the $R(p)$-module $M(p)$. To guarantee that the completed fibres behave well when $p$ varies in Spec $A$, we introduce the notion of a completed tensor product below.

For a finitely presented $A$-module $M$ the semicontinuity of $p \mapsto d_p(M)$ is well known (in this case $\hat{d}_p(M) = d_p(M)$ by Proposition 3):

**Lemma 1.** If $M$ is a finitely presented $A$-module then $d_p(M)$ is semicontinuous on Spec $A$. Moreover, if $M$ is $A$-flat, then $d_p(M)$ is locally constant on Spec $A$. 
Proof. Fix \( \mathfrak{p} \in \text{Spec } A \). We may assume that \( d_p(M) \) is finite. Consider a presentation
\[
A^p \xrightarrow{P} A^q \to M \to 0,
\]
of \( M \) with matrix \( P = (p_{ij}) \), \( p_{ij} \in A \).
Applying \( \otimes_A k(\mathfrak{p}) \) to this sequence we get the exact sequence of vector spaces
\[
k(\mathfrak{p})^q \xrightarrow{P} k(\mathfrak{p})^q \to M(\mathfrak{p}) \to 0,
\]
with entries of \( P_{\mathfrak{p}} \) being the images of \( p_{ij} \) in \( k(\mathfrak{p}) \). Then \( d_p(M) = q - \text{rank}(P_{\mathfrak{p}}) \) and since \( \text{rank}(P_{\mathfrak{p}}) \leq \text{rank}(P_q) \) for all \( q \) in some neighbourhood \( U \) of \( \mathfrak{p} \), the claim follows.

If \( M \) is flat, then \( M_{\mathfrak{p}} \) is free over the local ring \( A_{\mathfrak{p}} \) for a given \( \mathfrak{p} \in \text{Spec } A \).
By [Mat86], Theorem 4.10 (ii) (and its proof) there exists an \( f / \mathfrak{p} \) such that \( M_f \) is a free \( A_f \)-module of some rank \( r \) and hence \( d_q(M) = r \) for \( q \) in the open neighbourhood \( D(f) \) of \( \mathfrak{p} \).
\[\square\]

We introduce now the completed tensor product. Let us denote by
\[
\langle x \rangle := \langle x_1, ..., x_n \rangle_R
\]
the ideal in \( R \) generated by \( x_1, ..., x_n \). More generally, if \( S \) is an \( R \)-algebra, then \( \langle x \rangle_S \) denotes the ideal in \( S \) generated by (the images of) \( x_1, ..., x_n \).

For an \( R \)-module \( N \) denote by
\[
N^\wedge := \lim_{\leftarrow} N/\langle x \rangle^m N
\]
the \( \langle x \rangle \)-adic completion of \( N \). If \( N \) is also an \( S \)-module for some \( R \)-algebra \( S \), then \( \langle x \rangle^m N = (\langle x \rangle_S)^m N \), and hence the \( \langle x \rangle \)-adic completion and the \( \langle x \rangle_S \)-adic completion of \( N \) coincide.

**Definition 2.** Let \( A \) be a ring, \( R = A[[x]] \), \( B \) an \( A \)-algebra and \( M \) an \( R \)-module.
We define the completed tensor product of \( R \) and \( B \) over \( A \) as the ring
\[
R \hat{\otimes}_A B := \lim_{\leftarrow} \left( (R/\langle x \rangle^m) \otimes_A B \right)
\]
and the completed tensor product of \( M \) and \( B \) over \( A \) as the module
\[
M \hat{\otimes}_A B := \lim_{\leftarrow} \left( (M/\langle x \rangle^m M) \otimes_A B \right).
\]
If \( N \) is an \( A \)-module, we define the \( R \)-module
\[
M \hat{\otimes}_A N := \lim_{\leftarrow} \left( (M/\langle x \rangle^m M) \otimes_A N \right)
\]
and call it the completed tensor product of \( M \) and \( N \) over \( A \).

One reason why we consider the completed tensor product is that it provides the right base change property in the category of rings of the form \( A[[x]] \) by the following Proposition 3.1.

**Proposition 3.** The completed tensor product has the following properties (assumptions as in Definition 2).
1. \( R \otimes_A B = (R \otimes_A B)^\wedge = B[[x]] \).

2. \( M \otimes_A N = (M \otimes_A N)^\wedge \).

3. If \( M \) is finitely presented over \( R \) and \( N \) finitely presented over \( B \), then
   \[ M \otimes_A N \cong (M \otimes_A N) \otimes_{R \otimes_A B} (R \otimes_A B). \]

4. The canonical map \( M \otimes_A N \to M \otimes_A N \) is injective if \( A \) is Noetherian, \( M \)
   finite over \( R \) and \( N \) finite over \( A \).

5. If \( M \) is a finite \( A \)-module then \( \langle x \rangle^m \subset \text{Ann}_R(M) \) for some \( m \) and
   \( M \otimes_A N = M \otimes_A N \) for every \( A \)-module \( N \).

Proof. 1. We have \( \lim \limits_{\leftarrow} \langle [x]/(x)^m \otimes_A B \rangle = \lim \limits_{\leftarrow} \langle [x]/(x)^m \otimes_A B \rangle = \lim \limits_{\leftarrow} B[x]/(x)^m = B[[x]] \),
   showing the second equality. The first follows since \((R/(x)^m) \otimes_A B = (R \otimes_A B)/(x)^m(R \otimes_A B) \).

2. Since \( (M/(x)^m M) \otimes_A N = (M \otimes_A N)/(x)^m(M \otimes_A N) \) the equality follows
   and that \( (M \otimes_A N)^\wedge \) is the \( \langle x \rangle \) as well as the \( \langle x \rangle_{R \otimes_A B} \) \-adic completion of \( M \otimes_A N \).

3. If \( M \) resp. \( N \) are finitely presented over \( R \) resp. \( B \), then \( M \otimes_A N \) is finitely
   presented over \( R \otimes_A B \). Hence we can apply (the proof of) [AM69, Proposition
   10.13] and use 1. to show the isomorphism.

4. If \( A \) is Noetherian then \( R \) is Noetherian. If \( M \) is finitely generated over \( R \)
   and \( N \) finitely generated over \( A \) then \( M \otimes_A N \) is finitely generated over \( R \). The
   injectivity follows from 2. and [AM69, Theorem 10.17 and Corollary 10.19], since
   \( \langle x \rangle \) is contained in the Jacobson radical of \( R \) by Lemma 16.

5. Let us see that the assumption implies that \( \langle x \rangle^m \) is contained in the annihilator of \( M \) for some \( m \):
   Otherwise (we may assume \( M \neq 0 \)) we get a decreasing sequence of non-zero submodules
   \( \langle x \rangle^{k} M \subset \langle x \rangle^{k+1} M \). We may assume that \( \langle x \rangle^{k+1} M \neq 0 \)
   for all \( k \). The sequence \( x^{k+1} M \) is strictly decreasing by Nakayama’s lemma since \( M \)
   is finitely generated over \( R \) and since \( \langle x \rangle \) is contained in the Jacobson radical of \( R \)
   (Lemma 16). Hence we get infinitely many different elements \( x^{m} \in M \), which
   cannot be generated over \( A \) by finitely many elements from \( M \). This contradicts
   the assumption and hence \( \langle x \rangle^{m} M = 0 \) for some \( m \). Therefore \( M \otimes_A N = M \otimes_A N \)
   by definition of the completed tensor product. \( \Box \)

Corollary 4. Let \( A \) be a ring, \( R = A[[x]] \) and \( B \) an \( A \)-algebra. The completed tensor
product is right-exact on the category of finitely presented modules. That is,
\[
M' \to M \to M'' \to 0, \text{ resp.} \\
N' \to N \to N'' \to 0
\]
be exact sequences of finitely presented \( R \)-modules resp. \( B \)-modules. Then the sequences of finitely presented \( R \otimes_A B \)-modules
\[
M' \otimes_A N \to M \otimes_A N \to M'' \otimes_A N \to 0, \text{ resp.} \\
M \otimes_A N' \to M \otimes_A N \to M \otimes_A N'' \to 0
\]
are exact.
Definition 2 generalizes the completed tensor product of formal $A$ the analytic tensor product for analytic algebras (cf. [GR71, Chapter III.5]). Thus, a property of the tensor product in the category of formal $A$-algebras, analogous to the analytic tensor product for analytic algebras (cf. [GR71, Chapter III.5]). Thus, Definition 2 generalizes the completed tensor product of formal $A$-algebras.

Corollary 5. (i) $R \hat{\otimes}_A A = R$.
(ii) If $S$ is multiplicatively closed in $A$ then $A[[x]] \hat{\otimes}_A (S^{-1}A) = (S^{-1}A)[[x]]$.
(iii) For any $R$-module $M$ we have $M \hat{\otimes}_A A = M^\wedge$.
(iv) If $M$ is finitely presented over $R$ and $N$ finitely presented over $A$, then $M \hat{\otimes}_A N = M \otimes_A N$.

Proof. (i) and (ii) follow immediately from Proposition 3.1. (iv) follows from (i) and Proposition 3.3 with $B = A$ and (iii) is a special case of (iv).

Applying Corollary 4 and Proposition 3.1 we get

Corollary 6. If $A[[x]]^p \to A[[x]]^q \to M \to 0$ is an $A[[x]]$-presentation of $M$ and $B$ an $A$-algebra, then

$M \hat{\otimes}_A B = \text{coker} \ (B[[x]]^p \to B[[x]]^q)$.

Remark 7. Let $A$ be Noetherian and $B$ an $A$-algebra. If $\langle x \rangle$ is contained in the Jacobson radical of $R \otimes A B$, then $R \hat{\otimes}_A B$ is faithfully flat over $R \otimes A B$, by Proposition 3.1 and [Mat86, Theorem 8.14]. Note however that although $\langle x \rangle$ is contained in the Jacobson radical of $R$ by Lemma 16, it need not be in the Jacobson radical of $R \otimes A B$ (cf. Example 14).

Example 8. Let $\langle f_1, \ldots, f_k \rangle \subset R = A[[x]]$ be an ideal and $M = A[[x]]/\langle f_1, \ldots, f_k \rangle$. If $p$ is a prime ideal in $A$ then $R \otimes_A A_p = A_p[[x]]$ and form Corollary 6 we get $M \hat{\otimes}_A A_p = A_p[[x]]/\langle f_1, \ldots, f_k \rangle$. If $k(p)$ is the residue field of $A$ at $p$ then $M \hat{\otimes}_A k(p) = k(p)[[x]]/\langle f_1, \ldots, f_k \rangle$, something what one expects as fiber of $M$ over $p$. While $A_p[[x]]$ and $k(p)[[x]]$ are nice local rings, the subrings $R \otimes_A A_p \subset A_p[[x]]$ and $R \otimes_A k(p) \subset k(p)[[x]]$ are in general not local if $p$ is not a maximal ideal (see Example 14).

Remark 9. Proposition 3.1 with $B = A[[y]]$, $y = (y_1, \ldots, y_m)$, implies

$A[[x]] \hat{\otimes}_A A[[y]] = A[[x, y]]$.

Now let $A$ be Noetherian. If $I$ resp. $J$ are ideals in $A[[x]]$ resp. $A[[y]]$, we get from Corollary 4

$A[[x]]/I \hat{\otimes}_A A[[y]]/J = A[[x, y]]/(I, J)A[[x, y]]$.

We call an $A$-algebra a formal $A$-algebra if it is isomorphic to an $A$-algebra $A[[x]]/I$. For two formal $A$-algebras $B = A[[x]]/I$ and $C = A[[y]]/J$ the completed tensor product can be defined as $B \hat{\otimes}_A C = A[[x, y]]/(I, J)A[[x, y]]$. It has the usual universal property of the tensor product in the category of formal $A$-algebras, analogous to the analytic tensor product for analytic algebras (cf. [GR71, Chapter III.5]). Thus, Definition 2 generalizes the completed tensor product of formal $A$-algebras.
1.2. Fiber and completed fiber

Let again \( A \) be a ring and \( M = A[[x]] \)-module. We introduce the completed fiber \( \hat{M}(p) \) and the completed fiber dimension \( \hat{d}_p M \) of \( M \) for \( p \in \text{Spec} A \) and compare it with the usual fiber \( M(p) \) and the usual fiber dimension \( d_p M \).

At the end of this section we give examples, showing that semicontinuity of \( d_p(M) \) does not hold in general on \( \text{Spec} A \), even if \( A = \mathbb{C}[t] \) or \( A = \mathbb{Z} \) (Examples 19 and 20). However, we show in the next sections 1.3 and 1.5 that, under some conditions, semicontinuity holds for the completed fiber dimension \( \hat{d}_p(M) \).

**Notation 10.** We have canonical maps

\[
A \xrightarrow{i} R \xrightarrow{\pi} R/\langle x \rangle \xrightarrow{j} A
\]

with \( i \circ \pi \circ j = \text{id} \) and for an ideal \( I \subset R \) we set \( \overline{I} := \pi(I) \). On the level of schemes we have the maps \( \text{Spec} A \xrightarrow{i} V(\langle x \rangle) \xrightarrow{\pi} \text{Spec} R \xrightarrow{j} \text{Spec} A \), with \( i^*(p) = \langle p, x \rangle \), \( j^*(\langle p, x \rangle) = \langle p, x \rangle \cap A = p \) for \( p \in \text{Spec} A \). We denote by \( n_p := \langle p, x \rangle = \langle p, x_1, \ldots, x_n \rangle_R \) the ideal in \( R \) generated by \( p \in \text{Spec} A \) and \( x_1, \ldots, x_n \). The family \( j^* : \text{Spec} R \to \text{Spec} A \) has the trivial section \( \sigma = (i \circ \pi)^* : \text{Spec} A \to \text{Spec} R, p \mapsto n_p \), and the composition \( h := \pi \circ j : A \cong R/\langle x \rangle \) induces an isomorphism

\[
h^* : V(\langle x \rangle) \xrightarrow{\cong} \text{Spec} A,
\]

the restriction of \( j^* \) to \( V(\langle x \rangle) \). We call \( R_p := R \otimes_A A_p \) the stalk of \( R \) over \( p \). \( R_p \) is not a local ring, its local ring at \( n_p \) is \( (R_p)_{n_p} = R_{n_p} \) with residue field \( k(n_p) = k(p) \) (by Lemma 12).

If \( M \) is an \( R \)-module, we call \( M_p = M \otimes_A A_p \) the stalk of \( M \) over \( p \) and we are interested in the behavior of \( M \) along the section \( \sigma \). However, we are not interested in the \( R(p) \)-modules \( M(p) \) since \( R(p) \) is not a power series ring (and does not behave nicely). We are interested in the completed stalk \( \hat{M}_p \) and in the completed fibers \( \hat{M}(p) \), which we introduce now.

**Definition 11.** Let \( A \) be a ring, \( R = A[[x]] \), \( M \) an \( R \)-module and \( p \in \text{Spec} A \).

1. We set \( \hat{R}_p := R \hat{\otimes}_A A_p \), a local ring isomorphic to \( A_p[[x]] \) (by Proposition 3), and call the \( \hat{R}_p \)-module

\[
\hat{M}_p := \hat{M} \hat{\otimes}_A A_p
\]

the completed stalk of \( M \) over \( p \).

2. The ring \( \hat{R}(p) := R \hat{\otimes}_A k(p) \) is called the completed fiber of \( R \) over \( p \). It is a local ring isomorphic to \( k(p)[[x]] \) (Proposition 3). The \( \hat{R}(p) \)-module

\[
\hat{M}(p) := \hat{M} \hat{\otimes}_A k(p) = \hat{M}_p \otimes_{A_p} k(p)
\]

is called the completed fiber of \( M \) over \( p \).
3. \( \hat{M}(p) \) is a \( k(p) \)-vector space and we call its dimension
\[
\hat{d}_p(M) := \dim_{k(p)} \hat{M}(p)
\]
the completed fiber dimension of \( M \) over \( p \).

4. \( M \) is called quasi-completed-finite over \( p \) if \( \hat{d}_p(M) < \infty \).

The map \( A \to R \) induces a map of local rings \( A_p \to R_{n_p} \) and for an \( R \)-module \( M \) we have the fiber \( M(p) = M \otimes_A k(p) \) of \( M \) w.r.t. \( A \to R \) and the fiber
\[
M_{n_p}(p) = M_{n_p} \otimes_{A_p} k(p) = M_{n_p}/pM_{n_p},
\]
of \( M_{n_p} \) w.r.t. \( A_p \to R_{n_p} \). The fibers are in general different but the completed fibers coincide by the Lemma 15 (if \( M \) is finitely \( R \)-presented).

Let us now compare the fiber \( M(p) \) with its completed fiber \( \hat{M}(p) \).

**Lemma 12.** For any \( R \)-module \( M \) the following holds.

(i) \( \hat{M}_p = (M_p)^\wedge \) and \( \hat{M}(p) = M(p)^\wedge. \)

(ii) \( n_p \) is a prime ideal in \( R \) with \( n_p \cap A = p \) and the residue field of \( n_p \) in \( R \) satisfies \( k(n_p) = k(p) \).

(iii) If \( n \) is any prime ideal in \( R \) containing \( (x) \), then \( n = n_p \) with \( p = n \cap A \in \text{Spec} \, A \).

**Proof.** Statement (i) follows from Proposition 3. The first statement of (ii) follows since \( R/n_p = A/p \) is an integral domain. Since \( R/n_p = A/p \) we have \( k(n_p) = \text{Quot}(R/n_p) = \text{Quot}(A/p) = k(p) \). (iii) is obvious. \( \square \)

**Remark 13.** We have strict flat inclusions \( A_p \subsetneq R_p \subsetneq R_{n_p} \subsetneq A_p[[x]] \) of rings that are Noetherian if \( A \) is Noetherian.

The strictness is easy to see. E.g. \( g_0 + \sum_{|a| \geq 1}(g_a/h_0)x^a \), \( g_0 \notin p \), with arbitrary \( h_0 \in R \setminus n_p \), is a unit in \( A_p[[x]] \) but it is not contained in \( R_{n_p} \), where only finitely many different denominators are allowed. We have \( R_q = S^{-1}R \), with \( S \) the multiplicative set \( A \setminus p \) and \( R_{n_p} = (R_p)_{n_p} \). Since localization preserves flatness ([AM69, Corollary 3.6]) and the Noether property ([AM69, Proposition 7.3]), the inclusions \( A_p \subset R_p \subset R_{n_p} \) are flat and the rings are Noetherian if \( A \) is Noetherian.

The flatness of \( A_p[[x]] \) over \( R_{n_p} \) follows, since the first is the \( (x) \)-adic completion of the second by Lemma 12 (i).

The rings \( R_p \) and \( R_{n_p} \) are “strange” subrings of \( A_p[[x]] \). The ring \( A_p[[x]] \) is of interest in applications (cf. section 2), while the rings \( R_p \) and \( R_{n_p} \) are of minor interest. By the following lemma we have \( (R_p)^\wedge = (R_{n_p})^\wedge = A_p[[x]] \).

**Example 14.** As an example let \( A = k[t] \) and \( R = A[[x]] \) with \( t \) and \( x \) one variable, \( p = (0) \in \text{Spec} \, A \). We have \( A_p = k(p) = k(t) \) and
\[
R_p = k[t][[x]] \otimes_{k[t]} k(t) = \{ g/h \mid g \in k[t][[x]], h \in k(t) \setminus 0 \},
\]
\[
g = g_0 + \sum_{i \geq 1}g_i x^i, \ g_i \in k[t], \text{ a subring strictly contained in } R \hat{\otimes}_A A_p = k(t)[[x]].
\]

• \( (x) \) is contained in the Jacobson radical of \( R \hat{\otimes}_A A_p \) by Lemma 16.
• \langle x \rangle \) is not contained in the Jacobson radical of \( R_p \).

To see this, note that the element \( t - x \) is a unit in \( R \otimes_A A_p \), since \( 1/t - x = 1/t \sum_{i \geq 0} (x/t)^i \), but not in \( R \otimes_A A_p \). Hence the ideal \( \langle t - x \rangle \) is contained in a maximal ideal \( m \subset R \otimes_A A_p \). If \( x \in m \) then \( t \in m \) contradicting that \( t \) is a unit in \( R \otimes_A A_p \).

• The rings \( R_p \) and \( R(p) \) are in general not local.

Since \( R_p/\langle x \rangle = k(t) \), the ideal \( \langle x \rangle \) is another maximal ideal and \( R_p = R(p) \) (\( p = (0) \)) is not local.

**Lemma 15.** Let \( M \) be a finitely presented \( R \)-module and \( p \in \text{Spec} \ A \).

1. We have isomorphisms

\[
M_p \cong M_{n_p} \otimes_A A_p = M_{n_p} \otimes_A A = (M_{n_p})^\wedge.
\]

2. \( \hat{M}(p) \cong (M_{n_p}/pM_{n_p})^\wedge \).

3. If \( M = \text{coker} (A[[x]]^p \xrightarrow{T} A[[x]]^q) \) then

\[
\hat{M}_p = M \otimes_A A_p = \text{coker} (A_p[[x]]^p \xrightarrow{T} A_p[[x]]^q),
\]

\[
\hat{M}(p) = \text{coker} (k(p)[[x]]^p \xrightarrow{T} k(p)[[x]]^q).
\]

Note that \( \hat{R}_p = A_p[[x]] = (R_p)^\wedge \cong (R_{n_p})^\wedge \) and \( \hat{R}(p) = k(p)[[x]] = R(p)^\wedge \cong (R_{n_p}/pR_{n_p})^\wedge \) are local rings but \( R_p \not\cong R_{n_p} \) and \( R(p) \not\cong R_{n_p}/pR_{n_p} \), since \( R_p \) and \( R(p) \) are in general not local.

**Proof.** 1. The natural inclusion \( R_{n_p} = A[[x]]_{n_p} \hookrightarrow A_p[[x]] \) is given as follows. Let \( h/g \in R_{n_p} \) with \( h, g \in R, g \notin n_p \) and write \( g = g_0 - g_1 \) with \( g_0 \in A \) and \( g_1 \in \langle x \rangle R \). Then \( g \notin n_p = (p, x) \) iff \( g_0 \notin p \) and \( g \) is a unit in \( R_{n_p} \) iff its image in \( A_p[[x]] \) is a unit. We get

\[
h/g = \left(1 - g_1/g_0\right) = g_0^{-1}h \sum_{i \geq 0} (g_1/g_0)^i \in A_p[[x]].
\]

Now it is not difficult to see that the induced map \( A[[x]]_{n_p}/\langle x \rangle^m \to A_p[[x]]/\langle x \rangle^m \) is bijective (a finite sum \( \sum_{i=0}^{m-1}(a_\alpha/b_\alpha)x^\alpha, a_\alpha, b_\alpha \in A, b_\alpha \notin p \) in \( A_p[[x]] \) can be written as \( 1/b \sum_{i=0}^{m-1}(a_\alpha/b_\alpha')x^\alpha \) with \( b = \prod b_\alpha \notin n_p, b_\alpha' = b/b_\alpha \in A \), and hence is in \( A[[x]]_{n_p} \)). We get

\[
R_{n_p} \otimes_A A = \lim_{\leftarrow} A[[x]]_{n_p}/\langle x \rangle^m \otimes_A A = \lim_{\leftarrow} A_p[[x]]/\langle x \rangle^m = A_p[[x]]
\]

and also \( R_{n_p} \otimes_A A_p = A_p[[x]] = R_{n_p}^\wedge \). Now apply Corollary 4 to the presentation of \( M \) and deduce the claim for \( M \otimes_A A_p \).

2. \( \hat{M}(p) = M \otimes_A (A_p/pA_p) = (M \otimes_A A_p)/p(M \otimes_A A_p) = M_{n_p}^\wedge /pM_{n_p}^\wedge \) by Corollary 5 (iv) and the first statement of this lemma. Since \( M_{n_p} \) is finitely presented over \( R_{n_p} \) we have \( M_{n_p}^\wedge = M_{n_p} \otimes_{R_{n_p}} R_{n_p}^\wedge \), which implies the result.

3. This follows from Corollary 6. \( \Box \)

Over maximal ideals the fiber and the completed fiber coincide:
Lemma 16. Let $A$ be Noetherian and $M$ a finitely generated $R$-module. For $a \subset A$ a maximal ideal the following holds.

(i) 
$$
\hat{M}(a) = M(a), \quad \hat{d}_a(M) = d_a(M).
$$

(ii) 
$$
R/aR = R(a) = k(a)[[x]] \quad \text{and} \quad aR \text{ is a prime ideal in } R.
$$

(iii) 
$$
n_a \text{ is a maximal ideal of } R \text{ and any maximal ideal of } R \text{ is of the form } n_a \text{ for some } a \in \text{Max } A. \text{ Hence } \langle x \rangle \text{ is contained in the Jacobson radical of } R.
$$

(iv) 
$$
M(a) = M/aM \cong M_{n_a}/aM_{n_a}.
$$

Proof. (i) Since $a$ is maximal, $k(a) = A/a$ is a finite $A$-module. Corollary 5 (iv) implies $M(a) = M \hat{\otimes}_A A/a = M \otimes_A A/a = M(a)$.

(ii) This follows from (i) and the fact that $R/aR = k(a)[[x]]$ is integral.

(iii) Cf. [Mat86, §1, Example 1] and [AM69, Chapter 1, Exercise 5].

(iv) $M/aM = M \otimes_A A/a = M(a) = M(\hat{a}) = M(a) = M(\hat{a}) = \text{coker}(R_{n_a} \hat{\otimes}_A A/a) = R_{n_a} \hat{\otimes}_A A/a = M_{n_a}/aM_{n_a}$, as in the proof of Lemma 15.

We denote by $\text{Supp}_A(M) := \{p \in \text{Spec } A \mid M_p \neq 0\}$ the support of $M$ as $A$-module and by $\text{Supp}_R(M) \subset \text{Spec } R$ its support as $R$-module. Let $\text{Ann}_A(M) \subset A$ resp. $\text{Ann}_R(M) \subset R$ denote the annihilator of $M$ as $A$-module resp. $R$-module.

Remark 17. If $M$ is a finite $R$-module then $\text{Supp}_R(M) = V(\text{Ann}_R(M))$, which is closed in $\text{Spec } R$.

If $A$ is Noetherian then we have also $\text{Supp}_A(M) = V(\text{Ann}_A(M))$ and hence $\text{Supp}_A(M)$ is closed in $\text{Spec } A$ (in general the support of a not finitely generated module is not closed). To see this, let $p \in \text{Spec } A$ and note that $M_p = 0$ iff $\forall m \in M \exists f \in A, f \notin p, fm = 0$. Since $M$ is finitely generated over $R$, $\exists f \in A, f \notin p$, s.t. $f$ kills these generators, and hence $fM = 0$. Therefore $M_p = 0 \iff \exists f \notin p, fM = 0 \iff \text{Ann}_A(M) \notin p \iff p \notin V(\text{Ann}_A(M))$, implying the claim.

We show now that the set $\{p \in \text{Spec } A \mid \hat{d}_p(M) \neq 0\}$ is also closed but it may be stricly contained in $\text{Supp}_A(M)$ as Example 19 4. shows.

Lemma 18. Let $A$ be Noetherian and $M$ a finite $R$-module. Then

$$
V(\langle x \rangle) \cap \text{Supp}_R(M) \xrightarrow{\cong} \{p \in \text{Spec } A \mid \hat{d}_p(M) \neq 0\}
$$

and hence the set $\{p \in \text{Spec } A \mid \hat{d}_p(M) = 0\}$ is open in $\text{Spec } A$.

Proof. Let $A[[x]]^p \xrightarrow{T} A[[x]]^q \rightarrow M \rightarrow 0$ be a presentation of $M$, $T = (t_{ij}), t_{ij} \in A[[x]]$, and $p \in \text{Spec } A$. Then $k(p)[[x]]^p \xrightarrow{T'} k(p)[[x]]^q \rightarrow \hat{M}(p) \rightarrow 0$ is a presentation of $\hat{M}(p)$ with $T' = (t'_{ij}), t'_{ij} \in k(p)[[x]]$, the induced map (Corollary 6).

Now $\hat{M}(p) = 0$ iff $T'$ is surjective, i.e., iff the 0-th Fitting ideal (the ideal of $q$-minors) of $T'$ contains a unit $u' \in k(p)[[x]]$. Write $u'$ as $u' = u_0 + u_1$ with $u_0 \in k(p) \setminus \{0\}, u_1 \in \langle x \rangle k(p)[[x]]$. Since Fitting ideals are compatible with base change, the 0-th Fitting ideal $F_0 \subset A[[x]]$ of $M$ contains an element $u = u_0 + u_1 \in A[[x]]$ with $u_0 \in A, u_1 \in \langle x \rangle A[[x]]$, that maps to $u'$ under $A[[x]] \rightarrow A_p[[x]] \rightarrow A_p/pA_p[[x]]$. 

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Example 20. The following example may be of interest for arithmetic and computational purposes. It goes along similar lines as Example 19. Let \( h \) happen for \( M \).

Example 19. Let \( A = K[t], K \) an algebraically closed field, \( R = A[[x]], \) and \( M = R/(t - x), \) with \( t \) and \( x \) one variable. The following properties illustrate the difference between the fibers and the completed fibers. Let \( a = (t - c), c \in K, \) denote the maximal ideals in \( A.\)

1. \( M \) is not finitely generated over \( A, d_a(M) = \hat{d}_a(M) < \infty \) for \( a \in \text{Max} A \) and \( d_a(M) \) is semicontinuous on \( A: \)

   \[ M \cong K[[t]] \text{ as } A\text{-module via } f(x,t) \mapsto f(t,t), \text{ hence it is not finite over } A. \]

   \[ M(a) \cong K[[t]]/(t - c) \text{ and } d_a(M) = 1 \text{ if } c = 0 \text{ and } 0 \text{ if } c \neq 0. \]

2. \( d_p(M) \) is not semicontinuous on Spec \( A:\)

   The prime ideal \( (0) \) is contained in every neighbourhood of \( a = (t) \) in Spec \( A. \)

   It satisfies \( k((0)) = K(t) \) and we get \( M((0)) \cong K[[t]] \otimes_A K(t) = K((t)) \), the field of formal Laurent series. Since \( \dim_{K(t)} K((t)) = \infty, d_{(0)}(M) = \infty, \)

   while \( d_a(M) \leq 1 \) for \( a \in \text{Spec } A \setminus \langle 0 \rangle. \)

3. \( d_p(M) \) is semicontinuous on Spec \( A:\)

   We have \( M((0)) = K(t)[[x]]/(t - x) \) by Corollary 6. Since \( t \) is a unit in \( K(t), \)

   \[ d_{(0)}(M) = 0. \]

4. \( M(a) = M_a/aM_a = 0 \) does not imply \( M_a = 0:\)

   In fact, we have \( M_{(t-c)} \cong K[[t]]/(t-c) \) (as \( K[t]\)-module), which contains \( K((t)) \) if \( c \neq 0 \) (since \( t \notin (t-c) \)) while \( M((t-c)) = 0 \) for \( c \neq 0 \) by 1. We have
   \[ \{ p \mid d_p(M) \neq 0 \} = \{ (t) \} \subseteq \overset{\neq}{\{ p \mid d_p(M) \neq 0 \}} = \{ (t), (0) \} \subseteq \text{Supp}_A(M) = \text{Spec } A. \]

5. \( M \) is flat over \( A. \) By 1. and 3. we cannot expect any continuity of \( d_p(M) \) or \( d_p(M) \) on \( A \) or on Spec \( A \) for flat \( A\)-modules.

6. The quasi-finite locus of \( A \to A[[x]]/(t-x) \) is not open:

   The quasi-finite locus \( \{ p \in \text{Spec } A \mid d_p(M) < \infty \} \) is Spec \( A \setminus \langle 0 \rangle \) by 1. and 2. Recall that if \( B \) is a ring of finite type over \( A, \) then the quasi-finite locus of \( A \to B \) is open by Zariski’s main Theorem (cf. [Sta19, 10.122]).

7. The quasi-completed-finite locus of \( A \to A[[x]]/(t-x) \) is open:

   Let us call \( \{ p \in \text{Spec } A \mid d_p(M) < \infty \} \) the \textit{quasi-completed-finite locus}. It is Spec \( A \) in our example. If semicontinuity of \( d_p(M) \) holds (Theorem 24 and 41) then the quasi-completed-finite locus is open.

Example 20. The following example may be of interest for arithmetic and computational purposes. It goes along similar lines as Example 19.

Let \( A = \mathbb{Z}, \) \( R = \mathbb{Z}[x], \) and \( M = R/(x-p), p \in \mathbb{Z} \) a prime number. Since \( R = \lim_{\to} \mathbb{Z}[x]/\langle x \rangle^n \) we obtain \( M = \lim_{\to} \mathbb{Z}/p^n = \mathbb{Z}/(p), \) the ring of \( p\)-adic integers.
Now let \( \langle q \rangle \in \text{Max} \mathbb{Z} \).
If \( q \neq p \) then \( q \) is a unit in \( \mathbb{Z}_{(p)} \) hence in \( \hat{\mathbb{Z}}_{(p)} \) and \( M \otimes_{\mathbb{Z}} \mathbb{Z}/q = M/\langle q \rangle M = \hat{\mathbb{Z}}_{(p)}/q \hat{\mathbb{Z}}_{(p)} = 0 \).
If \( q = p \) then \( M \otimes_{\mathbb{Z}} \mathbb{Z}/p = \hat{\mathbb{Z}}_{(p)}/p \hat{\mathbb{Z}}_{(p)} = \mathbb{Z}/p \).
Hence \( d_{\langle q \rangle}(M) = \dim_{\mathbb{Z}/q} M \otimes_{\mathbb{Z}} \mathbb{Z}/q \) is 0 if \( q \neq p \) and 1 if \( q = p \).

On the other hand, looking at the prime ideal \( \langle 0 \rangle \) we get
\[
\hat{M}(\langle 0 \rangle) = M \otimes_{\mathbb{Z}} \mathbb{Q} = \mathbb{Q}[x]/\langle x - p \rangle = 0,
\]
while
\[
M(\langle 0 \rangle) = M \otimes_{\mathbb{Z}} \mathbb{Q} = \hat{\mathbb{Z}}_{(p)} \otimes_{\mathbb{Z}} \mathbb{Q} = \text{Quot}(\hat{\mathbb{Z}}_{(p)})
\]
has dimension \( d_{\langle 0 \rangle}(M) = \dim_{\mathbb{Q}} \text{Quot}(\hat{\mathbb{Z}}_{(p)}) = \infty \).

To see the last equality in the formula for \( M(\langle 0 \rangle) \) one checks that the following diagram has the universal property of the tensor product:
\[
\begin{array}{ccc}
\hat{\mathbb{Z}}_{(p)} & \xrightarrow{i_1} & \text{Quot}(\hat{\mathbb{Z}}_{(p)}) = (\hat{\mathbb{Z}}_{(p)})_p \\
\downarrow{i_2} & & \downarrow{j_1} \\
\mathbb{Z} & \xrightarrow{j_2} & \mathbb{Q}
\end{array}
\]
Here \( i_1, i_2 \) and \( j_2 \) are the canonical inclusions and \( j_1 \) is given as follows: if \( \alpha, \beta \in \mathbb{Z} \), \( p \nmid \beta \), then \( j_1\left(\frac{\alpha}{p^m} \mathbb{Z}/p^m\right) = \frac{1}{p^m} \alpha \mathbb{Z}/p^m \), \( \frac{\alpha}{p^m} \in \hat{\mathbb{Z}}_{(p)} \) since \( p \nmid \beta \). The universality of the diagram is easily seen. If \( T \) is a \( \mathbb{Z} \) - algebra and \( \phi : \hat{\mathbb{Z}}_{(p)} \to T \) and \( \psi : \mathbb{Q} \to \mathbb{Z} \) are \( \mathbb{Z} \) - algebra homomorphisms then the morphism \( \sigma : (\hat{\mathbb{Z}}_{(p)})_p \to T \), given as \( \sigma(\alpha/p^m) = \phi(\alpha)\psi(1/p^m), p \nmid \alpha \) is the unique one, making the obvious diagram commutative.

### 1.3. Semicontinuity over a 1-dimensional ring

In this section \( A \) will be Noetherian and \( M \) a finitely generated \( R \)-module (except we say otherwise). Then \( R = A[x] \) is Noetherian and \( M \) is finitely presented as \( R \)-module. At the moment we can prove the semicontinuity of \( d_q(M) \) on \( \text{Spec} A \) in full generality only under certain assumptions on the irreducible components of \( \text{Supp}_R(M) \). This includes the case \( \dim A = 1 \) where \( \dim A \) denotes the Krull dimension of \( A \). The case of arbitrary Noetherian \( A \) is treated in the next section under the assumption that the presentation matrix of \( M \) is polynomial.

In an important special case semicontinuity holds for arbitrary \( A \):

**Proposition 21.** Let \( A \) be Noetherian and \( M \) a finitely generated \( R \)-module. If \( \text{Supp}_R(M) \subset V(\langle x \rangle) \) then semicontinuity of \( \hat{d}_p(M) \) holds at any \( p \in \text{Spec} A \). In fact, \( \text{Supp}_R(M) \subset V(\langle x \rangle) \) iff \( M \) is finitely generated as \( A \)-module and in this case \( \hat{d}_q(M) = d_q(M) \) for all \( q \in \text{Spec} A \).
Proof. Since \( V(\text{Ann}_R(M)) = \text{Supp}_R(M) \subset V(\langle x \rangle) \), we have \( \langle x \rangle \subset \sqrt{\text{Ann}_R(M)} \) and there exists an \( m \) such that \( \langle x \rangle^m \subset \text{Ann}_R(M) \). We get a surjection

\[
A[[x]]/(x)^m \to R/\text{Ann}_R(M)
\]

and since \( A[[x]]/(x)^m \) is finitely generated over \( A \) this holds for \( R/\text{Ann}_R(M) \) too. Since \( M \) is finitely generated over \( R/\text{Ann}_R(M) \) it is finitely generated over \( A \) and hence finitely presented. By Lemma 1 there there is an open neighborhood \( U \) of \( p \) in \( \text{Spec } A \) such that \( d_q(M) \leq d_p(M) \), \( q \in U \). By Proposition 3 (iii), \( \hat{M}(q) = M(q) \) for all \( q \in \text{Spec } A \), showing the first claim. Moreover, in Proposition 3.5 it was shown that the \( A \)-finiteness of \( M \) implies that \( \langle x \rangle^m \subset \text{Ann}_R(M) \) for some \( m \). \( \square \)

Before we formulate the next result, we introduce some notations to be used throughout this section. Consider a minimal primary decomposition of \( \text{Ann}_R(M) \),

\[
\text{Ann}_R(M) = \bigcap_{i=1}^r Q_i \subset R.
\]

Since \( M \) is finitely generated over \( R \), \( \text{Supp}_R(M) = V(\text{Ann}_R(M)) = \bigcup_{i=1}^r V(Q_i) \) and \( \text{dim } M = \text{dim } \text{Supp}_R(M) \).

Let \( P_1, \ldots, P_s \subset R \) be the minimal associated primes of \( \langle x \rangle \). Since they correspond via \( h : A \cong R/\langle x \rangle \) to the minimal associated primes \( \hat{P}_1, \ldots, \hat{P}_s \) of \( A \), we have \( \text{dim } V(P_j) \leq \text{dim } A \).

**Lemma 22.** For \( p \in \text{Spec } A \) the following holds:

1. Let \( A' \) be the reduction of \( A \), \( R' = A'[[x]] \) and \( M' \) the \( R' \)-module \( M \otimes_R R' \). Then \( \hat{M}'(p) \cong \hat{M}(p) \) and hence \( \hat{d}_p(M') = \hat{d}_p(M) \).
2. Let \( Q \subset R \) be an ideal. Then \( \hat{d}_p(M/QM) \leq \hat{d}_p(M) \).
3. If \( Q_i \subsetneq \mathfrak{n}_p \) for some \( 1 \leq i \leq r \), then \( \hat{d}_q(M) = 0 \) for \( q \) in some open neighborhood of \( p \) in \( \text{Spec } A \).
4. If \( Q_i \subsetneq \mathfrak{n}_p \) and \( \text{dim } V(Q_i) > \text{dim } A_p \) for some \( 1 \leq i \leq r \), then \( \hat{d}_p(M) = \infty \).
5. Let \( U = \text{Spec } B \subset \text{Spec } A \) be an affine open neighborhood of \( p \) and \( M_U = M \otimes_A B \) the restriction of \( M \) to \( U \). Then \( \hat{M}_B(q) = \hat{M}(q) \) for all \( q \in U \).

**Proof.** 1. Since \( p \in \text{Spec } A \) contains the nilpotent elements, \( A'/p' = A/p \), where \( p' \) is the image of \( p \) in \( A' \), and hence the residue field does not change if we pass from \( A \) to \( A' \). By Proposition 3 (i) we have \( \hat{R}'(p') = R' \otimes_{A'} k(p) = k(p)[[x]] = \hat{R}(p) \).

Consider a presentation \( R^{p} \xrightarrow{T} R^{q} \to M \to 0 \) of \( M \). Applying \( \otimes_R \hat{R}' \), we get a presentation of \( M', R'^{p} \xrightarrow{T'} R'^{q} \to M' \to 0 \). Apply \( \otimes_A k(p) \) to the first resp. \( \hat{\otimes}_A k(p) \) to the second exact sequence above. The sequences stay exact by Corollary 4. Since \( (\hat{R}(p))^k = (\hat{R}'(p'))^k \) it follows that the canonical morphism \( M \to M' \) induces an isomorphism \( \hat{M}(p) \cong \hat{M}'(p) \).

2. Since \( (M/QM) \hat{\otimes}_A k(p) = M \hat{\otimes}_A k(p)/Q(M \hat{\otimes}_A k(p)) \) by Corollary 4, the result follows.
3. If $Q_i \not\in \mathfrak{n}_p$, then $\mathfrak{n}_p \not\in V(Q_i) \subset \text{Supp}_R(M)$, i.e., $M_{\mathfrak{n}_p} = 0$ and the result follows from Lemma 18.

4. Set $\hat{R} := R/Q_i$ and $\hat{M} := M/Q_i M$. Then $\dim \hat{R}_{\mathfrak{n}_p} = \dim V(Q_i) > \dim A_p$ by assumption. Considering $\hat{M}$ as $R$-module, we have $\hat{M} \otimes_A k(p) = M(p)^\wedge = (M_{\mathfrak{n}_p}/p M_{\mathfrak{n}_p})^\wedge$ by Lemma 15. Since the $\hat{R}_{\mathfrak{n}_p}$-modules $\hat{M}(p)$ and its $(x)$-adic completion $\hat{M}(p)^\wedge$ have the same Hilbert-Samuel function w.r.t. $\mathfrak{n}_p$, their dimension coincides (c.f. [GP08, Corollary 5.6.6]). Moreover, $\mathfrak{p}\hat{R}_{\mathfrak{n}_p}$ is the annihilator of $M_{\mathfrak{n}_p}/p M_{\mathfrak{n}_p}$ and therefore $\dim \hat{M}(p)^\wedge = \dim \hat{R}_{\mathfrak{n}_p}/\mathfrak{p}\hat{R}_{\mathfrak{n}_p}$.

We apply now [Mat86, Theorem 15.1] to the map of local rings $A_{\mathfrak{p}} \to \hat{R}_{\mathfrak{n}_p}$ and get that $\dim \hat{R}_{\mathfrak{n}_p}/\mathfrak{p}\hat{R}_{\mathfrak{n}_p} \geq \dim \hat{R}_{\mathfrak{n}_p} - \dim A_{\mathfrak{p}} > 0$ and hence $\dim_{k(p)} M(p)^\wedge = \infty$.

Then $d_q(M) = \dim_{k(p)} M(p)^\wedge = \infty$ by 2. of this lemma.

5. We may assume that $B = A_f$ for some $f \not\in \mathfrak{p}$. Since $A_q = (A_f)_q$ for $q \in U = D(f)$, we have $k(q) = A_f \otimes_A k(q)$. Now Proposition 3 implies $M_{A_f}(q) = (M \otimes_A A_f) \otimes_A k(q) = (M \otimes_A A_f) \otimes_A k(q))^\wedge = M(q)^\wedge = M(q)$.

\begin{prop}
Fix $\mathfrak{p} \in \text{Spec} A$. Let $Q_1, \ldots, Q_r$ be the primary components of $\text{Ann}_R(M)$ and $P_1, \ldots, P_s$ the minimal primes of $V(\langle x \rangle)$. After renumeration we have two disjoint sets of primary components of $\text{Ann}_R(M)$ (any one may be empty):

I. $Q_1, \ldots, Q_k$ such that for each $1 \leq i \leq k$ there exists a $j$ with $P_j \subset \sqrt{Q_i}$.

II. $Q_{k+1}, \ldots, Q_r$ such that for each $k+1 \leq i \leq r$ we have $P_j \not\subset \sqrt{Q_i}$ for all $j$.

We set

\begin{align*}
Q_I & := \bigcap_{i=1}^k Q_i, & Q_I & := \bigcap_{i=k+1}^r Q_i, \\
M_I & := M/Q_I M, & M_{II} & := M/Q_{II} M.
\end{align*}

1. If case II does not occur (i.e., $k = r$), then there is an open neighbourhood $U$ of $\mathfrak{p}$ in Spec $A$ such that $d_q(M) \leq d_p(M)$ for all prime ideals $q \in U$.

2. If $\dim V(\langle x \rangle) \cap V(Q_I) \cap V(Q_{II}) \leq 0$ then there is an open neighbourhood $U$ of $\mathfrak{p}$ in Spec $A$ such that $d_q(M) \leq d_p(M)$ for all prime ideals $q \in U$.

\begin{proof}
Note that $V(Q_I) \subset V(\langle x \rangle)$ and $V(Q_{II}) \not\subset V(\langle x \rangle)$ for $k+1 \leq i \leq r$.

1. follows from Proposition 21, since the assumption says that $\text{Supp}(M) = \bigcup_{i=1}^r V(Q_i) \subset \bigcup_{i=1}^r V(P_j) = V(\langle x \rangle)$.

2. We renumerate the $P_j$ such that $P_1, \ldots, P_t$ satisfy $P_j \subset \sqrt{Q_i}$ for some $i = 1, \ldots, k$ and $P_{t+1}, \ldots, P_s$ satisfy $P_j \not\subset \sqrt{Q_i}$ for any $i = 1, \ldots, r$. We set

\[ P_I := \bigcap_{j=1}^t P_j, \text{ satisfying } P_I \subset \sqrt{Q_I}. \]

We have $\text{Supp}_R(M_I) = V(Q_I) \subset V(P_I) \subset V(\langle x \rangle)$. By Proposition 21 there is an open neighborhood $U_I$ of $\mathfrak{p}$ in Spec $A$ such that

\[ d_q(M_I) \leq d_p(M_I), \qquad q \in U_I. \]

\end{proof}
We set
\[ d := \dim R/(\Ann_R(M) + \langle x \rangle) \]
and make induction on \( d \). If \( d = 0 \) then \( \dim \Supp(M) \cap V(\langle x \rangle) = 0 \) and \( \mathfrak{p} \) is an isolated point of \( \{ q \in \Spec A \mid \hat{\delta}_q(M) \neq 0 \} \) by Lemma 18 and the theorem is trivially true in this case. Hence we may assume that \( d \geq 1 \). By Lemma 22 we may assume that

1. \( A \) is reduced (Lemma 22 (1)),
2. \( \overline{Q_i} \subset n_i \) for all \( 1 \leq i \leq r \) (Lemma 22 (3)),
3. \( \dim V(Q_i) \leq d \) for all \( 1 \leq i \leq r \) (Lemma 22 (4.), assuming \( (\beta) \)).

We have \( \dim V(Q_{\overline{I}}) \leq d \) by (\( \gamma \)). Since \( P_j \not\subset \sqrt{Q_j} \) and since \( V(Q_i) \) and \( V(P_j) \) are irreducible we have for each \( k + 1 \leq i \leq r \) and for \( t + 1 \leq j \leq s \)
\[ \dim V(Q_i) \cap V(P_j) < \min\{\dim V(Q_i), \dim V(P_j)\} \]
In particular \( \dim V(\langle x \rangle) \cap V(Q_{\overline{I}}) < d \) and \( \dim R/\Ann_R(M_{\overline{I}}) + \langle x \rangle < d \) since \( Q_{\overline{I}} \subset \Ann_R(M_{\overline{I}}) \).

By the induction hypothesis there exists an open neighbourhood \( U_2 \subset \Spec A \) of \( \mathfrak{p} \) such that
\[ \hat{\delta}_q(M_{\overline{I}}) \leq \hat{\delta}_p(M_{\overline{I}}), \ q \in U_2. \] (2)
By Lemma 22 (2), we have
\[ \hat{\delta}_q(M_{I}), \hat{\delta}_q(M_{\overline{I}}) \leq \hat{\delta}_q(M), \ q \in \Spec A. \] (3)
Note that
\[ M_{I,q} = M_q, \ n_q \in V(Q_I) \setminus V(Q_I \cap V(Q_{\overline{I}})), \]
\[ M_{\overline{I},q} = M_q, \ n_q \in V(Q_{\overline{I}}) \setminus V(Q_I \cap V(Q_{\overline{I}})). \] (4)
Using Notation 10, we set \( V_1 = U_1 \cap V(\overline{Q}_I) \setminus (V(Q_I) \cap V(Q_{\overline{I}})) \) and \( V_2 = U_2 \cap V(Q_I) \setminus (V(Q_I) \cap V(Q_{\overline{I}}) \cap V(Q_I \cap V(Q_{\overline{I}}))). \) From (1), (2), (3) and (4), we deduce
\[ \hat{\delta}_q(M) = \hat{\delta}_q(M_{I}) \leq \hat{\delta}_p(M), \ q \in V_1, \]
\[ \hat{\delta}_q(M) = \hat{\delta}_q(M_{\overline{I}}) \leq \hat{\delta}_p(M), \ q \in V_2. \] (5)
The set \( U := V_1 \cup V_2 \cup (V(Q_I) \cap V(Q_{\overline{I}}) \cap (V_1 \cup V_2)) \) is an open neighbourhood of \( \mathfrak{p} \). In general we cannot compare \( \hat{\delta}_p(M) \) with \( \hat{\delta}_q(M) \) for \( q \in V(\overline{Q}_I) \cap V(Q_{\overline{I}}), \) except if this intersection consists just of \( \mathfrak{p} \), which is the case if \( V(Q_I) \cap V(Q_{\overline{I}}) \cong V(\langle x \rangle) \cap V(Q_I \cap V(Q_{\overline{I}}) \cap V(Q_I) \cap V(Q_{\overline{I}})) \) has dimension 0. Then the result follows from (5). \( \square \)

As a corollary we get the following theorem, which was already proved for maximal ideals and for \( A = k[t] \) in [GPh19].

**Theorem 24.** Let \( A \) be Noetherian, \( M \) a finitely generated \( R \)-module and \( \mathfrak{p} \in \Spec A \). If \( \dim A_{\mathfrak{p}} \leq 1 \) then there is an open neighbourhood \( U \) of \( \mathfrak{p} \) in \( \Spec A \) such that
\[ \hat{\delta}_q(M) \leq \hat{\delta}_p(M) \text{ for all } q \in U. \]
Proof. We may assume that \( \hat{d}_p(M) < \infty \) and, as shown in the proof of Proposition 23, that \( n_p \supset \text{Ann}_R(M) = Q_I \cap Q_{II} \). By Proposition 21, the result holds if case II. does not occur. If it occurs, then \( \dim A_p \leq 1 \) implies \( \dim R/(\text{Ann}_R(M) + (x)) \leq 1 \) and \( \dim V((x)) \cap V(Q_I) \cap V(Q_{II}) = 0 \) as shown in the proof of Proposition 23 and the result follows. \( \square \)

Corollary 25. Let \( A = \mathbb{Z} \) and \( M \) be a finitely generated \( \mathbb{Z}[[x]] \)-module, \( x = (x_1 \cdots x_n) \), given by a presentation

\[
\mathbb{Z}[[x]]^r \to \mathbb{Z}[[x]]^s \to M \to 0.
\]

Denote by

\[
M_p := \hat{M}(p) = \text{coker} \left( \mathbb{F}_p[[x]]^r \xrightarrow{T} \mathbb{F}_p[[x]]^s \right)
\]

if \( p \in \mathbb{Z} \) is a prime number and by

\[
M_0 := \hat{M}(0) = \text{coker} \left( \mathbb{Q}[[x]]^r \xrightarrow{T} \mathbb{Q}[[x]]^s \right)
\]

the induced modules.

1. If \( \dim_{\mathbb{F}_p} M_p < \infty \) then \( \dim_{\mathbb{F}_p} M_p \geq \dim_{\mathbb{Q}} M_0 \). Moreover, for all except finitely many prime numbers \( q \in \mathbb{Z} \), \( \dim_{\mathbb{F}_p} M_p \geq \dim_{\mathbb{Q}_q} M_q \).

2. If \( \dim_{\mathbb{Q}} M_0 < \infty \) then \( \dim_{\mathbb{Q}} M_0 \geq \dim_{\mathbb{F}_p} M_q \) for all except finitely many prime numbers \( q \in \mathbb{Z} \), and hence \( = \) for all except finitely many prime numbers \( q \in \mathbb{Z} \).

The first part of statement 1. follows, since \( (0) \) is in every neighbourhood of \( p \). In particular \( \dim_{\mathbb{Q}} M_0 \) is finite if \( \dim_{\mathbb{F}_p} M_p \) is finite for some prime number \( p \).

1.4. Henselian rings and Henselian tensor product

In this section we recall some basic facts about Henselian rings and introduce similarly to the complete tensor product a Henselian tensor product. For details about Henselian rings see [Sta19] or [KPR75]. The Henselian tensor product is needed in Section 1.5 for polynomially presented modules. We start with some basic facts about étale ring maps.

Definition 26. 1. A ring map \( \phi : A \to B \) is called étale if it is flat, unramified\(^1\) and of finite presentation.

2. \( \phi \) is called standard étale if \( B = (A[T]/F)G, F,G \in A[T] \), the univariate polynomial ring, \( F \) monic and \( F' \) a unit in \( B \).

3. \( \phi \) is called étale at \( q \in \text{Spec}(B) \) if there exist \( g \in B \setminus q \) such that \( A \to B_g \) is étale.

The following proposition lists some basic properties of étale maps. The results can be found in section 10.142 of [Sta19].

Proposition 27. 1. The map \( A \to A_f \) is étale.

\(^1\phi \) is unramified if it is of finite type and the module of Kähler differentials \( \Omega_{B/A} \) vanishes. \( \phi \) is of finite presentation if \( B \cong A[x_1, \ldots, x_n]/(f_1, \ldots, f_m) \) as \( A \)-algebras.
2. A standard étale map is an étale map.
3. The composition of étale maps is étale.
4. A base change of étale maps is étale.
5. An étale map is open.
6. An étale map is quasi-finite.
7. Given \( \phi : A \to B \) and \( g_1, \ldots, g_m \in B \) generating the unit ideal\(^2\) such that \( A \to B_{g_i} \) is étale for all \( i \) then \( \phi : A \to B \) is étale.
8. Let \( \phi : A \to B \) be étale then there exist \( g_1, \ldots, g_m \in B \) generating the unit ideal such that \( A \to B_{g_i} \) is standard étale for all \( i \).
9. Let \( S \subset A \) be a multiplicatively closed subset and assume that \( \phi' : S^{-1}A \to B' \) is étale. Then there exist an étale map \( \phi : A \to B \) such that \( B' = S^{-1}B \) and \( \phi' = S^{-1}\phi \).
10. Let \( \phi' : A/I \to B' \) be étale for some ideal \( I \subset A \) then there exist an étale map \( \phi : A \to B \) such that \( B' = B/IB \) and the obvious diagram commutes.

**Definition 28.** Let \( A \) be a ring and \( I \subset A \) an ideal. \( A \) is called Henselian with respect to \( I \) if the following holds\(^3\): (Univariate Implicit Function Theorem): Let \( F \in A[T] \), the univariate polynomial ring, such that \( F(0) \in I \) and \( F'(0) \) is a unit modulo \( I \) then there exists \( a \in I \) such\(^4\) that \( F(a) = 0 \).

Next we associate to any pair \((A, I)\), \( I \subset A \) an ideal, the Henselization \( A^h_I \), i.e. the ”smallest” Henselian ring with respect to \( I \), such that \( A^h_I \subset \hat{A}_I = \lim_{\leftarrow} (A/I^n) \) the \( I \)-adic completion.

**Definition 29.**
1. Let \( A \) be a ring and \( I \subset A \) an ideal. The ring
   \[
   A^h_I = \lim_{\leftarrow} \left( B \mid A \to B \text{ an étale ring map inducing } A/I \cong B/IB \right)
   \]
   is called the Henselization of \( A \) with respect to \( I \).
2. The Henselization of \( A[x] \), \( A \) any ring, \( x = (x_1, \ldots, x_n) \), with respect to \( I = \langle x \rangle \) is denoted by \( A\langle x \rangle \).

The Henselization has the following properties (cf. section 15.11 and 15.12 of [Sta19]):

**Proposition 30.** Let \( A \) be ring and \( I \subset A \) an ideal.
1. \( A^h_I \) is Henselian with respect to \( I^h = IA^h_I \) and \( A/I^m = A^h_I/(I^h)^m \) for all \( m \).
2. \( A \) is Henselian with respect to \( I \) if and only if \( A = A^h_I \).
3. If \( A \) is Noetherian then the canonical map \( A \to A^h_I \) is flat.
4. If \( A \) is Noetherian then the canonical map \( A^h_I \to \hat{A}_I \) is faithfully flat and \( \hat{A}_I \) is the \( I^h \)-adic completion of \( A^h_I \).

\(^2\)i.e., \( \text{Spec}(A) = \cup D(g_i) \).
\(^3\)Note, that (similarly to the \( I \)-adic completion) the condition implies that \( I \) is contained in the Jacobson radical of \( A \). If we start with an ideal contained in the Jacobson radical then it is enough to consider monic polynomials \( F \) in the definition.
\(^4\)Note that \( a \) is uniquely determined by the condition \( a \in I \), [KPR75].
Remark 31. The definition of the Henselization implies that $A^h_I$ is contained in the algebraic closure of $A$ in $\hat{A}_I$. If $A$ is excellent then $A^h_I$ is the algebraic closure of $A$ in $\hat{A}_I$. This is even true under milder conditions, see [KPR75]. In this situation $C\langle x \rangle$ is called the ring of algebraic power series of $C[[x]]$.

Next we prove a lemma which we need later in the applications.

Lemma 32. Let $A$ be a ring and $p \in \text{Spec}(A)$ a prime ideal. Let $C = A_p$, $I = pC$ and $f_1, \ldots, f_m \in C^h_I$. Then there exist an étale map $A \rightarrow B$ such that
1. $f_1, \ldots, f_m \in B$,
2. there exists a prime ideal $q \in \text{Spec}(B)$ such that $q \cap A = p$.

Proof. By definition we have
$$C^h_I = \lim\limits_{\longrightarrow} (D | C \rightarrow D \text{ étale inducing } C/I = D/ID).$$

We choose $D$ from the inductive system above such that $f_1, \ldots, f_m \in D$. Since $(C, I)$ is a local ring and $C/I = D/ID$, the ideal $ID$ is a maximal ideal in $D$ and we have $ID \cap C = I$. Using Proposition 27 (8) for the multiplikatively closed system $S = A \setminus p$ we find an étale map $A \rightarrow B'$ such that $D = S^{-1}B'$. This implies that $f_1, \ldots, f_m \in B'_g$ for a suitable $g \in S$. Let $B = B'_g$ and $q = ID \cap B$ then $A \rightarrow B$ is étale having the properties 1. and 2. □

Next we define the Henselization of an $A$-module $M$ with respect to an ideal $I \subset A$ similarly to the definition of the Henselization of $A$ with respect to $I$.

Definition 33. Let $A$ be a ring, $I \subset A$ an ideal and $M$ an $A$-module. The module
$$M^h_I = \lim\limits_{\longrightarrow} (M \otimes_A B | A \rightarrow B \text{ étale inducing } A/I = B/IB)$$
is called the Henselization of $M$ with respect to $I$.

Lemma 34. $M^h_I = M \otimes_A A^h_I$.

Proof. The lemma follows since the direct limit commutes with the tensor product (cf. 10.75.2 [Sta19]). □

Definition 35. Let $A$ be a ring, $R = A\langle x \rangle$, $B$ an $A$-algebra and $M$ an $R$-module. We define the henselian tensor product of $R$ and $B$ over $A$ as the ring
$$R \otimes^h_A B := \lim\limits_{\longrightarrow} (C | B[x] \rightarrow C \text{ étale inducing } B = C/\langle x \rangle C) = B(x).$$

We have
$$M \otimes^h_A B := \lim\limits_{\longrightarrow} (M \otimes_A C | B[x] \rightarrow C \text{ étale inducing } B = C/\langle x \rangle C) = M \otimes_A B(x).$$

The Henselian tensor product has similar properties as the complete tensor product. Especially we obtain the following lemma.

\footnote{For the definition of excellence see 15.51 [Sta19].}
Lemma 36. If $A(x)^p \xrightarrow{T} A(x)^q \to M \to 0$ is an $A(x)$-presentation of $M$ then 

$$M \otimes_A^h B = \operatorname{coker}(B(x)^p \xrightarrow{T} B(x)^q).$$

In particular $R \otimes_A^h k(p) = k(p)(x)$ for $p \in \text{Spec } A$.

Definition 37. Let $A$ be a ring, $R = A(x)$ and $M$ an $R$-module. We define for $p \in \text{Spec } A$ the $R \otimes_A^h k(p) = k(p)(x)$-module

$$M_*^h(p) := M \otimes_A^h k(p)$$

and call it the Henselian fiber of $M$ over $p$. Moreover, we set

$$d_\text{b}^h(M) := \dim_{k(p)} M_*^h(p).$$

1.5. Semicontinuity for polynomially presented modules

Let $A$ be Noetherian and $M$ finitely generated as $R = A[[x]]$-module. Then $M$ is finitely $R$-presented and in this section we assume that $M$ has a polynomial presentation matrix. That is, there exists a presentation

$$R^p \xrightarrow{T} R^q \to M \to 0$$

with $T = (t_{ij})$ a $q \times p$ matrix such that $t_{ij} \in A[x]$ or $t_{ij} \in A(x)$, $x = (x_1, \ldots, x_n)$. Under this assumption we shall prove the semicontinuity of $\hat{d}_b(M)$ for $b \in \text{Spec } A$.

Let $R_0 = A(x)$ be the Henselization of $A[x]$ with respect to $(x)$ and $M_0 = \operatorname{coker}(R_0^p \xrightarrow{T} R_0^q)$. Then using the $(x)$-adic completion we obtain $R_0^\wedge = R$ and $M_0^\wedge = M$.

Lemma 38. Let $B \supset A$ be an $A$-algebra of finite type, $b \in \text{Spec } B$ and $a = b \cap A$. Then

$$\hat{d}_a(M) < \infty \Leftrightarrow \hat{d}_b(M \otimes_A B) < \infty \Leftrightarrow d_b^h(M_0 \otimes_A^h B) < \infty$$

and

$$\hat{d}_a(M) = \hat{d}_b(M \otimes_A B) = d_b^h(M_0 \otimes_A^h B).$$

Proof. We have

$$\hat{a}(M) = \dim_{k(a)}(M \otimes_A k(a))$$

$$\hat{b}(M \otimes_A B) = \dim_{k(b)}(M \otimes_A B \otimes_B k(b))$$

$$d_b^h(M \otimes_A^h B) = \dim_{k(b)}(M \otimes_A^h B \otimes_B^h k(b))$$

and

$$M \otimes_A k(a) = \operatorname{coker}(k(a)[[x]]^p \xrightarrow{T} k(a)[[x]]^q)$$

$$M \otimes_A B \otimes_B k(b) = \operatorname{coker}(k(b)[[x]]^p \xrightarrow{T} k(b)[[x]]^q)$$

$$M_0 \otimes_A^h B \otimes_B^h k(b) = \operatorname{coker}(k(b)[x]^p \xrightarrow{T} k(b)[x]^q)$$

with $T = (t_{ij})$ and $\hat{t}_{ij}$ the induced elements in $k(a)[x]$ resp. $k(b)[x]$. 

Lemma 39. Let $B_d M$ that $\hat{\mathcal{B}}_\mathcal{M}$ implies that there is an open neighbourhood $\tilde{\mathcal{B}}(\mathcal{M})$.

Choose $b$ in $B$ that $\hat{\mathcal{B}}_\mathcal{M}$ with respect to its maximal ideal. We set $\hat{\mathcal{B}}_\mathcal{M}(\mathcal{B})$. This gives the first equality in the Lemma. Since $B\langle x \rangle/\langle x \rangle^N = B[\langle x \rangle]/\langle x \rangle^N$ we get the remaining claims.

\[ \text{Corollary 40.} \quad \text{Let} \quad (A, \mathfrak{m}, \mathfrak{k}) \quad \text{be a local Noetherian Henselian ring and} \quad R \quad \text{a local quasi-finite (i.e.} \quad \dim R/\mathfrak{m}R < \infty) \quad A\text{-algebra. Then} \quad R \quad \text{is a finite} \quad A\text{-algebra.} \]

\text{Proof.} This is an immediate consequence of Lemma 10.149.3 (13) of [Sta19] since $R$ is local.

\[ \text{Corollary 40.} \quad \text{Let} \quad (A, \mathfrak{m}, \mathfrak{k}) \quad \text{be a local Noetherian Henselian ring and} \quad R \quad \text{a local quasi-finite} \quad A\text{-algebra. If} \quad M \quad \text{is a finitely generated} \quad R\text{-module with} \quad \dim \mathfrak{m} M/\mathfrak{m}M < \infty, \quad \text{then} \quad M \quad \text{is a finitely generated} \quad A\text{-module.} \]

\text{Proof.} Passing from $R$ to $R/\text{Ann}_R(M)$ we may assume that $\text{Ann}_R(M) = 0$. In this case $\dim \mathfrak{m} M/\mathfrak{m}M < \infty$ implies $\dim \mathfrak{m} R/\mathfrak{m}R < \infty$. Lemma 39 implies that $R$ is a finitely generated $A$-module. Since $M$ is finitely generated over $R$ it follows that $M$ is a finitely generated $A$-module.

\[ \text{Theorem 41.} \quad \text{Let} \quad A \quad \text{be a Noetherian ring,} \quad R = A[\langle x \rangle], \quad x = (x_1, \ldots, x_n), \quad \text{and} \quad M \quad \text{a finitely generated} \quad R\text{-module admitting a presentation} \]

\[ R^n \rightarrow R^d \rightarrow M \rightarrow 0 \]

\text{with polynomial presentation matrix} $T = (t_{ij}), \quad t_{ij} \in A[\langle x \rangle]$. \text{Fix} $\mathfrak{p} \in \text{Spec} A$ with $d_\mathfrak{p}(M) < \infty$. \text{Then there is an open neighbourhood} $U$ of $\mathfrak{p}$ in $\text{Spec} A$ such that $d_\mathfrak{q}(M) \leq d_\mathfrak{p}(M)$ for all $\mathfrak{q} \in U$.

\text{The same result holds if the presentation matrix has entries in} \quad A(\langle x \rangle)$. \]

\text{Proof.} Recall that $R_0 = A(\langle x \rangle)$ is the Henselization of $A[\langle x \rangle]$ with respect to $\langle x \rangle$ and $M_0 = \text{coker}(R_0^n \rightarrow R_0^d)$. Denote by $A^h$ the henselization of the local ring $A_\mathfrak{p}$ with respect to its maximal ideal. We set $R^h := A^h(\langle x \rangle)$ and $M^h := \text{coker}(R^h)^n \rightarrow (R^h)^d) = M_0 \otimes A^h R^h$. Then Lemma 38 implies $d_\mathfrak{p}(M) = d_\mathfrak{p}(M^h)$ and Corollary 40 that $M^h$ is a finitely generated $A^h$-module. Lemma 32 implies that there is an étale neighbourhood $\pi : \text{Spec} B \rightarrow \text{Spec} A$ of $\mathfrak{p}$ such that $M_0 \otimes A^h B = \text{coker}((R_0 \otimes A^h B)^n \rightarrow (R_0 \otimes B^h A^h)^d)$ is a finitely generated $B$-module and $M_0 \otimes A^h B \otimes A^h A^h = M^h$. Choose $b \in \text{Spec} B$ such that $b \cap A = \mathfrak{p}$. This is possible because of Lemma 32. Corollary 40 and Lemma 1 imply that there is an open neighbourhood $U \subset \text{Spec} B$.

\[ \text{Note that} \quad R^h \quad \text{is the Henselization of} \quad A_\mathfrak{p}[\langle x \rangle] \quad \text{with respect to the maximal ideal} \quad \langle \mathfrak{p}, x \rangle. \]
of $b$ such that for $c \in \tilde{U}$ we have $d_c(M_0 \otimes_A^h B) \leq d_b(M_0 \otimes_A^h B)$. Since $\pi$ is étale it is open (Proposition 27), $U := \pi(\tilde{U})$ is an open neighbourhood of $p$ in $\text{Spec } A$ and for any $q \in U \cap \text{Spec } A$ there exists a $c \in \tilde{U} \cap \text{Spec } B$ with $c \cap A = q$. From Lemma 38 we obtain $\hat{d}_q(M) = d_c(M_0 \otimes_A^h B) \leq d_b(M_0 \otimes_A^h B) = \hat{d}_p(M)$. □

**Remark 42.** The important property of local Heselian rings is that a quasi-finite module is already finite (Corollary 40). The same holds for analytic rings and we have a similar semicontinuity result. Let $\mathcal{O} / \mathcal{O}_Z \in \{ \mathcal{O} / \mathcal{O}_C, \mathcal{O} / \mathcal{O}_B \}$ and $A = \mathcal{O} / \mathcal{O}_Z \{ y \} / \mathcal{I}$ with $\mathcal{O} / \mathcal{O}_Z \{ y \} = \mathcal{O} / \mathcal{O}_Z \{ y_1, \ldots, y_s \}$ the convergent power series ring and $R = A\{ x \}, x = (x_1, \ldots, x_n)$.

2. Singularity invariants

2.1. Isolated singularities and flatness

Recall that a local Noetherian ring $(A, \mathfrak{m})$ is said to be regular if $\mathfrak{m}$ can be generated by $\dim A$ elements. A Noetherian ring $A$ is said to be regular if the local ring $A_p$ is regular for all $p \in \text{Spec } A$. For arbitrary Noetherian rings the regular locus $\text{Reg}_A := \{ p \in \text{Spec } A \mid A_p \text{ is regular} \}$ need not be open in $\text{Spec } A$. However, $\text{Reg}_A$ is open if $A$ is a complete Noetherian local ring ([Mat86, Corollary of Theorem 30.10]) and the non-regular locus $\{ p \in \text{Spec } A \mid A_p \text{ is not regular} \}$ is closed.

However, in our situation of families of power series, the notion of formal smoothness is more appropriate than that of regularity. Formal smoothness is a relative notion and refers to a morphism, while regularity is an absolute property of the ring. The notions are related as follows. Let $(A, \mathfrak{m})$ be a local ring containing a field $\mathbb{K}$. If $A$ is formally smooth over $\mathbb{K}$ (w.r.t. the $\mathfrak{m}$-adic topology) then $A$ is regular and the converse holds if the residue field $A/\mathfrak{m}$ is separable over $\mathbb{K}$ (see Remark 44). Hence formal smoothness of $A$ over $\mathbb{K}$ coincides with regularity if $\mathbb{K}$ is a perfect field. The notions do also coincide for arbitrary $\mathbb{K}$ if $A$ is the quotient ring of a formal power series ring over $\mathbb{K}$ by an ideal (cf. Lemma 49).

We recall now basic facts about formal smoothness. For details and proofs see [Mat86] and [Maj10].

**Definition 43.** Let $A$ be a ring, $B$ an $A$-algebra defined by $\phi : A \to B$ and $I$ an ideal in $B$. The $A$-algebra $B$ is called formally smooth with respect to the $I$-adic topology (for short $B$ is $I$-smooth over $A$) if for any $A$-algebra $C$ and any continuous\(^7\) $A$-algebra homomorphism $u : B \to C/N$, $N$ an ideal in $C$ with $N^2 = 0$, there exist $\sigma : B \to C$ such that $\pi \sigma = u$.

\[\begin{array}{ccc}
B & \xrightarrow{u} & C/N \\
\phi \downarrow & \nearrow \sigma & \downarrow \pi \\
A & \xrightarrow{v} & C
\end{array}\]

\(^7\)Here we consider $B$ with the $I$-adic topology and $C/N$ with the discrete topology; $u$ is continuous if $u(I^m) = 0$ for some $m$. 

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If $I = 0$ then $B$ is called a formally smooth $A$-algebra.

**Remark 44.** 1. A formally smooth map of finite presentation is smooth ([Sta19] Proposition 10.137.13).
2. $A[x], x = (x_1, \ldots, x_n)$, is smooth over $A$ ([Sta19] Lemma 10.137.4).
3. $A[[x]]$ is $(x)$-smooth over $A$ ([Mat86] page 215).
4. Let $(A, m)$ be a local ring containing a field $k$.
   (a) $A$ is $m$-smooth over $k$ iff $A$ is geometrically regular, i.e. $A \otimes_k k'$ is a regular ring for every finite extension field $k'$ of $k$ ([Mat86, Theorem 28.7]).
   (b) Assume that $A/m$ is separable over $k$. Then $A$ is $m$-smooth over $k$ iff $A$ is regular ([Mat86] Lemma 1, page 216).

We now generalize example 1 on page 215 of [Mat86].

**Lemma 45.** Let $A$ be a ring, $B$ a $A$-algebra, $I$ an ideal in $B$ and $\hat{B}$ the $I$-adic completion of $B$. $\phi : A \rightarrow B$ is $I$-smooth iff $\hat{\phi} : A \rightarrow \hat{B}$ is $I\hat{B}$-smooth.

**Proof.** Assume that $B$ is $I$-smooth over $A$ and consider the following commutative diagram:

$$
\begin{array}{c}
\hat{B} \\
\downarrow \hat{\phi} \\
A \\
\end{array}
\quad
\begin{array}{c}
\hat{B} \\
\downarrow \hat{\phi} \\
A \\
\end{array}
\begin{array}{c}
\downarrow \sigma \\
\downarrow \pi \\
\end{array}
\begin{array}{c}
C/N \\
\end{array}
$$

with $N^2 = 0$. We have to prove that there exists $\hat{\sigma}$ such that $\pi \hat{\sigma} = \hat{u}$. Since $\hat{u}$ is continuous there exist $m$ such that $\hat{u}(I^m \hat{B}) = 0$. Let $i : B \rightarrow \hat{B}$ be the canonical map such that $\hat{\phi} = i\phi$. The $I$-smoothness of $B$ implies that there exists $\sigma : B \rightarrow C$ such that $\sigma \pi = \hat{u}$. $\hat{u}(I^n \hat{B}) = 0$ implies $\sigma(I^m) \subset N$. Since $N^2 = 0$ we obtain $\sigma(I^{2m}) = 0$. We obtain the following commutative diagram:

$$
\begin{array}{c}
\hat{B} \\
\downarrow i \\
B \\
\end{array}
\quad
\begin{array}{c}
\downarrow i_{2m} \\
\downarrow \sigma_{2m} \\
A \\
\end{array}
\begin{array}{c}
\downarrow \phi \\
\downarrow \sigma_{2m} \\
\downarrow \pi \\
\end{array}
\begin{array}{c}
B/I^{2m} \\
\downarrow \phi \\
A \\
\end{array}
\begin{array}{c}
\downarrow i_{2m} \\
\downarrow \sigma_{2m} \\
\downarrow \pi \\
\end{array}
\begin{array}{c}
C/N \\
\end{array}
$$

Now we define $\hat{\sigma} = \sigma_{2m} i_{2m}$. This proves that $\hat{\phi} : A \rightarrow \hat{B}$ is $I\hat{B}$-smooth.
Now assume that $\hat{\phi} : A \to \hat{B}$ is $I\hat{B}$-smooth. Consider the following commutative diagram:

$$
\begin{array}{ccc}
B & \xrightarrow{u} & C/N \\
\phi \downarrow & & \downarrow \pi \\
A & \xrightarrow{\pi} & C
\end{array}
$$

with $N^2 = 0$. We have to prove that there exists $\sigma$ such that $\pi \sigma = u$. Since $\phi : A \to \hat{B}$ is $I\hat{B}$-smooth there exists $\hat{\sigma} : \hat{B} \to C$ with $\pi \hat{\sigma} = \hat{u}$. Now we define $\sigma = \hat{\sigma}$ and obtain $\pi \sigma = u$.

The following important theorem is due to Grothendieck ([Mat86] Theorem 28.9).

**Theorem 46.** Let $(A, m)$ be a local ring and $(B, n)$ a local $A$-algebra. Let $\hat{B} = B/\mathfrak{m}B$ and $\bar{n} = n/\mathfrak{m}B$. Then $B$ is $n$-smooth over $A$ if $\hat{B}$ is $\bar{n}$-smooth over $A/\mathfrak{m}$ and $B$ is flat over $A$.

**Definition 47.** Let $A$ be a ring and $B$ an $A$-algebra defined by $\phi : A \to B$. We define the smooth locus of $\phi$ by

$$\text{Sm}(\phi) := \{ P \in \text{Spec}(B) | A_{\phi^{-1}(P)} \to B_P \text{ is } P\text{-smooth} \}.$$ 

and the singular locus of $\phi$ by

$$\text{Sing}(\phi) := \text{Spec}(B) \setminus \text{Sm}(\phi).$$

**Remark 48.** Let $A$ be a ring and $B$ a $A$-algebra defined by $\phi : A \to B$. The Theorem 46 of Grothendieck implies that

$$\text{Sm}(\phi) = \{ P \in \text{Spec}(B) | A_Q \to B_P, Q = \phi^{-1}(P), \text{is flat and } B_P/QB_P \text{ is } P\text{-smooth over } k(Q) \}.$$ 

Now let $k$ be a field, $k[[x]]$, $x = (x_1, \ldots, x_n)$, the formal power series ring over $k$ and $I$ an ideal in $(x)k[[x]]$. If $I$ is generated by $f_1, \ldots, f_m$ we denote by $\text{Jac}(I)$ the Jacobian matrix $\left( \partial f_j/\partial x_i \right)$ and by $I_k(\text{Jac}(I))$ the ideal generated by the $k \times k$-minors of $\text{Jac}(I)$ (which is independent of the chosen generators $f_j$). The following lemma gives equivalent conditions for the maximal ideal $(x) \in B = k[[x]]/I$ to be contained in the smooth locus $\text{Sm}(\phi)$ of the map $\phi : k \to B$ (Remark 48).

**Lemma 49.** If $\dim k[[x]]/I = d$ the following are equivalent.

1. $k[[x]]/I$ is $(x)$-smooth over $k$.
2. $k[[x]]/I$ is regular.
3. $I_d(\text{Jac}(I)) = k[[x]]$ (Jacobian criterion).
4. $k[[x]]/I \cong k[[y_1, \ldots, y_d]]$.

**Proof.** The equivalence of 1. and 2. follows from [Mat86, Lemma 1, p. 216], the equivalence of 3. and 4. is the inverse mapping theorem for formal power series.\(^8\)

\(^8\)Given $f_1, \ldots, f_n \in k[[x_1, \ldots, x_n]]$ then $\det \left( \frac{\partial f_j}{\partial x_i} \right)$ is a unit iff $k[[x_1, \ldots, x_n]] = k[[f_1, \ldots, f_n]]$ ([GLS07, Theorem 1.1.18]).
Obviously 4. implies 2. From [Mat86, Theorem 29.7, p. 228 in] we deduce that 2. implies 4. □

**Remark 50.** Part of the lemma can be generalized by extending the proof of Theorem 30.3 in [Mat86] as follows:

Let \( P \) be a prime ideal in \( \mathbb{k}[x] \) containing \( I = \langle f_1, \ldots, f_m \rangle \) and \( m \) the maximal ideal of \( A = \mathbb{k}[x]_P/I\mathbb{k}[x]_P \). Then \( I_d(Jac(I)) = \mathbb{k}[x]_P \) implies that \( A \) is \( m \)-smooth over \( \mathbb{k} \) (or geometric regular by Remark 44).

We use the Jacobian criterion to define the singular locus of ideals in power series rings over a field.

**Definition 51.**

1. If \( B = \mathbb{k}[[x]]/I \) is pure \( d \)-dimensional (i.e. \( \dim B/P = d \) for all minimal primes \( P \in \text{Spec} B \)) we define the singular locus of \( B \) (or of \( I \)) as

\[
\text{Sing}(B) = V(I + I_d(Jac(I))).
\]

2. If \( B \) is not pure dimensional we consider the minimal primes \( P_1, \ldots, P_r \) of \( B \). Then \( B/P_i \) is pure dimensional and we define the singular locus of \( B \) as

\[
\text{Sing}(B) = \bigcup_{i=1}^r \text{Sing}(B/P_i) \cup \bigcup_{i \neq j} V(P_i) \cap V(P_j),
\]

which is a closed subvariety of \( \text{Spec} B \). The points in \( \text{Spec} B \setminus \text{Sing}(B) \) are called non-singular points of \( B \).

3. We say that \( \mathbb{k}[[x]]/I \) (or \( I \)) has an isolated singularity (at 0) if the maximal ideal \( (x) \) is an isolated point of \( \text{Sing}(\mathbb{k}[[x]]/I) \) or if \( (x) \) is a non-singular point.

Note that \( \text{Sing}(B) \) carries a natural scheme structure given by the Fitting ideal \( I + I_d(Jac(I)) \subset \mathbb{k}[[x]] \) if \( B \) is pure \( d \)-dimensional. In general we endow \( \text{Sing}(B) \) with its reduced structure.

Now let us consider families. Let \( A \) be a Noetherian ring, \( F_1, \ldots, F_m \in \langle x \rangle A[[x]], I \subset A[[x]] \) the ideal generated by \( F_1, \ldots, F_m \) and set \( B := A[[x]]/I \).

We describe now the smooth locus of the map \( \phi : A \to B \) along the section \( \sigma : \text{Spec} A \to \text{Spec} B, p \mapsto n_p = (x, p), \) of \( \text{Spec} \phi \).

For \( p \in \text{Spec} A \) denote by \( F_i(p) \) the image of \( F_i \) in \( k(p)[[x]] \). Note that \( F_1(p), \ldots, F_m(p) \) generate the ideal \( \overline{I}(p) \subset k(p)[[x]] \), and that we have (by Lemma 15.3) for the completed fiber of \( \phi \) over \( p \)

\[
\hat{B}(p) = (B_{n_p}/pB_{n_p})^\wedge = k(p)[[x]]/\overline{I}(p).
\]

The maximal ideals of the local rings of the fiber \( B_{n_p}/pB_{n_p} \) and the completed fiber \( \hat{B}(p) \) are generated by \( n_p/p = \langle x \rangle \). Assume that \( \phi : A \to B \) is flat. Then the theorem of Grothendieck says

\[
n_p \in \text{Sm}(\phi) \iff B_{n_p} \text{ is } n_p\text{-smooth over } A_p \iff B_{n_p}/pB_{n_p} \text{ is } \langle x \rangle\text{-smooth over } k(p).
\]
Lemma 52. With the above notations assume that \( \phi : A \to B \) is flat. Denote by

\[
\text{Sing}_\sigma(\phi) := \{ n_p \in \text{Spec } B \mid B_{n_p} \text{ is not } n_p\text{-smooth over } A_p \}
\]

the singular locus of \( \phi \) along the section \( \sigma \). Then

\[
\text{Sing}_\sigma(\phi) = \{ n_p \in \text{Spec } B \mid \hat{B}(p) \text{ is not regular} \}.
\]

Proof. \( B_{n_p}/pB_{n_p} \) is \( \langle x \rangle \)-smooth over \( k(p) \) iff \( (B_{n_p}/pB_{n_p})^* = k(p)[[x]]/I(p) \) is \( \langle x \rangle \)-smooth over \( k(p) \) by Lemma 45. The claim follows from Lemma 49. \( \square \)

Since we assumed \( B \) to be flat over \( A \), we have \( \dim \hat{B}(p) = \dim B_{n_p} - \dim A_p \) (by [Mat86, Theorem 15.1]). If \( \phi \) is of pure relative dimension \( d \) (i.e. \( \hat{B}(p) \) is pure \( d \)-dimensional for all \( p \)) then Lemma 49 implies

\[
\text{Sing}_\sigma(\phi) = \{ n_p \in \text{Spec } B \mid I_d(\text{Jac}(I))(p) \text{ is a proper ideal of } k(p)[[x]] \},
\]

where \( \text{Jac}(I) \) is the Jacobian matrix \( \langle \partial F_j/\partial x_i \rangle \) and \( I_d(\text{Jac}(I)) \subset A[[x]] \) the ideal defined by the \( d \times d \)-minors.

2.2. Milnor number and Tjurina number of hypersurface singularities

Let \( k \) be a field and \( f \in k[[x]], x = (x_1, \ldots, x_n) \) a formal power series. The most important invariants are the Milnor number \( \mu(f) \) and the Tjurina number \( \tau(f) \), defined as

\[
\mu(f) = \dim_k k[[x]]/j(f),
\]

\[
\tau(f) = \dim_k k[[x]]/(f, j(f)),
\]

where \( j(f) = \langle \partial f/\partial x_1, \ldots, \partial f/\partial x_n \rangle \) is the Jacobian ideal of \( f \). We say that \( f \) has an isolated critical point (at 0) resp. an isolated singularity (at 0) if \( \mu(f) < \infty \) resp. \( \tau(f) < \infty \). Note that \( \tau(f) < \infty \) iff \( k[[x]]/(f) \) has an isolated singularity in the sense of Definition 51.

Remark 53. Let \( \text{char}(k) = 0 \). It is proved in [BGM12, Theorem 2] that for \( f \in \langle x \rangle \), \( \mu(f) < \infty \Leftrightarrow \tau(f) < \infty \) but it is easy to see that this is not true in positive characteristic. We have always \( \tau(f) \leq \mu(f) \) and \( \tau(f) = \mu(f) \Leftrightarrow f \in j(f) \). If \( k = \mathbb{C} \) and if \( f \in \langle x \rangle^2 \) has an isolated singularity, this is equivalent to \( f \) being quasi homogeneous by a theorem of K. Saito (see [Sa71]). His proof generalises to any algebraically closed field of characteristic zero (cf. [BGM11, Theorem 2.1]).

We consider now families of singularities. Let \( A \) be a Noetherian ring and \( F \in R = A[[x]] \). Set

\[
j(F) := \langle \partial F/\partial x_1, \ldots, \partial F/\partial x_n \rangle
\]

and for \( p \in \text{Spec } A \) denote by \( F(p) \) the image of \( F \) in \( k(p)[[x]] \). Then the Milnor number

\[
\mu(F(p)) = \dim_{k(p)} k(p)[[x]]/j(F(p))
\]

and the Tjurina number

\[
\tau(F(p)) = \dim_{k(p)} k(p)[[x]]/(F(p), j(F(p)))
\]

are defined, and we deduce now the semicontinuity of \( \mu(F(p)) \) and \( \tau(F(p)) \).
Proposition 54. Let $A$ be Noetherian, $F \in R = A[[x]]$ and $p \in \text{Spec } A$. Assume that $\dim A = 1$ or $F \in A[x]$. Then $\mu(F(p))$ and $\tau(F(p))$ are semicontinuous at $p \in \text{Spec } A$.

Proof. Set $M = R/j(F)$ resp. $M = R/(F, j(F))$, then $\text{Supp}_R(M) = V(j(F))$ resp. $\text{Supp}_R(M) = V((F, j(F)))$. Using Lemma 15 we get $\hat{d}_q(M) = \mu(F(q))$ resp. $\hat{d}_q(M) = \tau(F(q))$ for $q \in \text{Spec } A$. The result follows from Theorem 24 and Theorem 41.

Corollary 55. Let $F \in \mathbb{Z}[x]$, $p \in \mathbb{Z}$ a prime number and denote by $F_p$ the image of $F$ in $\mathbb{F}_p[[x]]$ and by $F_0$ the image of $F$ in $\mathbb{Q}[[x]]$.

If $\mu(F_p)$ is finite, then $\mu(F_p) \geq \mu(F_0)$ and $\mu(F_p) \geq \mu(F_0)$ for all except finitely many prime numbers $q \in \mathbb{Z}$. In particular, if $\mu(F_p)$ is finite for some $p$ then $\mu(F_0)$ is finite.

If $\mu(F_0)$ is finite, then $\mu(F_0) \geq \mu(F_q)$ (and hence “=”) for all except finitely many prime numbers $q \in \mathbb{Z}$.

The same holds for the Tjurina number.

Example 56. We illustrate the corollary by a simple example. Let $F = F_0 = x^p + x^{p+1} + y^q$ with $p, q$ prime numbers. Then $\mu(F_0) = (p-1)(q-1)$, $\mu(F_p) = p(q-1)$ $\geq \mu(F_0)$ while $\mu(F_q) = \infty$. Moreover, for any prime number $r \neq p, q$ we have $\mu(F_r) = \mu(F_0)$.

2.3. Determinacy of ideals

Let $I$ be a proper ideal of $k[[x]]$ and $f_1, \ldots, f_m$ a minimal set of generators of $I$. $I$ is called contact $k$-determined if for every ideal $J$ of $k[[x]]$ that can be generated by $m$ elements $g_1, \ldots, g_m$ with $g_i - f_i \in (x)^{k+1}$ for $i = 1, \ldots, m$, the local $k$-algebras $k[[x]]/I$ and $k[[x]]/J$ are isomorphic. $I$ is called finitely contact determined if $I$ is contact $k$-determined for some $k$. It is easy to see (cf. [GPh19, Proposition 4.3]) that these notions depend only on the ideal and not on the set of generators.

The ideal $I$ or the ring $k[[x]]/I$ is called a complete intersection if $\dim k[[x]]/I = n - m$ and an isolated complete intersection singularity (ICIS) if it has moreover an isolated singularity.

Set $f = (f_1, \ldots, f_m) \in k[[x]]^m$ and denote by $\langle \partial f/\partial x_1, \ldots, \partial f/\partial x_n \rangle$ the submodule of $k[[x]]^m$, generated by the $m$-tuples $\partial f/\partial x_i = (\partial f_1/\partial x_i, \ldots, \partial f_m/\partial x_i)$, $i = 1, \ldots, n$. We define

$$T_I := k[[x]]^m/I_k[[x]]^m + \langle \partial f/\partial x_1, \ldots, \partial f/\partial x_n \rangle.$$

If $I$ is a complete intersection, then $\tau(I) := \dim_k T_I$ is called the Tjurina number of $I$. For a complete intersection $T_I$ is concentrated on the singular locus of $k[[x]]/I$ (Definition 51) and $\tau(I)$ is finite iff $I$ has an isolated singularity. This follows from [GPh19, Lemma 3.1], where it is shown that the ideals $I + I_{n-m}(\text{Jac}(I))$ and $\text{Ann}_{k[[x]]}(T_I)$ have the same radical.

The module $T_I$ is used in the following theorem.
Proposition 59. We show first that being a regular sequence in a flat family of power series in $A$ is infinitely contact determined.

(i) $I$ is finitely generated.

(ii) $\dim_k T_I < \infty$.

(iii) $R/I$ is an isolated complete intersection singularity.

If one of these condition is satisfied then $I$ is contact (2 $\dim_k T_I - \text{ord}(I) + 2$) determined, where $\text{ord}(I) = \max\{k \mid I \subset (x)^{k}\}$. The implications (iii) $\Leftrightarrow$ (ii) $\Rightarrow$ (i) hold for any field $k$, as well as (i) $\Rightarrow$ (ii) for hypersurfaces.

Proposition 58. Let $A$ be Noetherian, $F_1, \ldots, F_m \in \langle x \rangle A[[x]]$. Let $I \subset A[[x]]$ be the ideal generated by $F_1, \ldots, F_m$ and $I(p) \subset k(p)[[x]]$, $p \in \text{Spec} A$, the ideal generated by $F_1(p), \ldots, F_m(p) \in k(p)[[x]]$.

Assume either that $\dim A = 1$ and $F_i \in \langle x \rangle A[[x]]$ or $\dim A \geq 1$ and $F_i \in \langle x \rangle A(x)$, $i = 1, \ldots, m$. Then any $p \in \text{Spec} A$ has an open neighbourhood $U \subset \text{Spec} A$ such that for all $q \in U$, $\dim_{k(p)} T_{I(p)} \geq \dim_{k(q)} T_{I(q)}$.

2.4. Tjurina number of complete intersection singularities

We show first that being a regular sequence in a flat family of power series in $R = A[[x]]$ is an open property.

Proposition 59. Let $A$ be a Noetherian ring, $F_i \in \langle x \rangle R$, $i = 1, \ldots, m$ and $M$ a finitely generated $R$-module. For $p \in \text{Spec} A$ we denote by $F_i(p)$ the image of $F_i$ in $\hat{R}(p) = k(p)[[x]]$ and by $F_{np}$ the image of $F_i$ in $R_{np}(p)$ (cf. Definition 11).

(i) If $p \in \text{Spec} A$ then $F_1(p), \ldots, F_m(p)$ is an $\hat{M}(p)$-sequence iff $F_{1np}, \ldots, F_{mp}$ is an $M_{np}(p)$-sequence.

(ii) Let $F_1, \ldots, F_m$ be an $M$-sequence and let $M/(F_1, \ldots, F_m)M$ be $A$-flat. Then $F_1(p), \ldots, F_m(p)$ is an $\hat{M}(p)$-sequence for all $p \in \text{Spec} A$.

(iii) Let $p \in \text{Spec} A$ and $F_1(p), \ldots, F_m(p)$ an $\hat{M}(p)$-sequence. If $M/(F_1, \ldots, F_m)M$ is flat over $A$, then there exists an open neighbourhood $U$ of $p$ in $\text{Spec} A$ such that $F_1(q), \ldots, F_m(q)$ is a $M(q)$-sequence for all $q \in U$.

Proof. Set $M_0 = M, M_i = M/(F_1, \ldots, F_i)M$ and consider for $i = 1, \ldots, m$ the exact sequence

$$0 \to K_{i-1} \to M_{i-1} \xrightarrow{F_i} M_{i-1} \to M_i \to 0,$$

with $K_{i-1}$ the kernel of $F_i$.

(i) By Lemma 15.2 $\hat{R}(p) = R_{np}(p)^{\wedge}$ and $\hat{M}(p) = M_{i, np}(p)^{\wedge}$ for all $i$ and hence

$$F_i(p) = F_{i, np}^{\wedge} : (M_{i-1, np}(p))^{\wedge} \to (M_{i-1, np}(p))^{\wedge}.$$

Since $M_{i, np}(p)$ is a finite $R_{np}$-module we have (by [AM69, Proposition 10.13]) $\hat{M}(p) = M_{i, np}(p) \otimes_{R_{np}} R_{np}^{\wedge}$. Moreover $R_{np}^{\wedge}$ is faithfully flat over the local ring $R_{np}$.
([Mat86, Theorem 8.14]). Hence $F_{in_p} : M_{i-1,n_p}(p) \rightarrow M_{i-1,n_p}(p)$ is injective iff $F_i(p) : M_{i-1}(p) \rightarrow M_{i-1}(p)$ is injective.

(ii) By assumption $K_{i-1} = 0$ for $i = 1, \ldots, m$ and $M/(F_1, \ldots, F_m)M$ is $A$-flat. By Lemma 16 the Jacobson radical of $R$ contains $(x)$ and $F_1, \ldots, F_m$ is a regular sequence contained in the Jacobson radical. Hence $M_i$ is $A$-flat and $M_{i,n_p}$ is $A_p$-flat for all $i$ (repeated application of [Mat86, Theorem 22.2]). Tensoring $0 \rightarrow M_{i-1,n_p} \rightarrow M_{i-1,n_p} \rightarrow M_{i,n_p} \rightarrow 0$ with $\otimes_{A_p} k(p)$ we get an exact sequence $0 \rightarrow M_{i-1,n_p}(p) \rightarrow M_{i-1,n_p}(p)) \rightarrow M_{i,n_p}(p) \rightarrow 0$ for $i = 1, \ldots, m$ by [Mat86, Theorem 22.3]. Now apply (i).

(iii) Localizing the exact sequence (*) at $n_p$ we get an exact sequence of finite $R_{n_p}$-modules. Taking the $(x)$-adic completion, the sequence stays exact and we see that $(K_{i-1,n_p})^\wedge = \ker (F_{in_p}^\wedge : (M_{i-1,n_p})^\wedge \rightarrow (M_{i-1,n_p})^\wedge)$. By Lemma 15 $K_{i-1}(p) = (M_{i-1,n_p})^\wedge \otimes_{A_p} k(p)$ and $F_i(p) = F_{in_p}^\wedge \otimes_{A_p} k(p)$, and by assumption $F_i(p)$ is injective. We apply now repeatedly [Mat86, Theorem 22.5 ] to $A_p \rightarrow R_{n_p}$ to get that $(K_{i-1,n_p})^\wedge = K_{i-1,n_p} \otimes_{R_{n_p}} R_{n_p}^\wedge = 0$ and that $(M_{i,n_p})^\wedge = M_{i,n_p} \otimes_{R_{n_p}} R_{n_p}^\wedge$ is flat over $A_p$ for all $i$. Since $R_{n_p}^\wedge$ is faithfully flat over $R_{n_p}$ this implies $K_{i-1,n_p} = 0$ and that $M_{i,n_p}$ is flat over $A_p$.

The support of the $R$-module $K_{i-1}$ is closed and hence $(K_{i-1})^\wedge_{n_q} = 0$ for $q$ an open neighbourhood $U$ of $p$ in Spec $A$. Moreover the flatness of $M/(F_1, \ldots, F_m)M$ implies that $M_{n_q}^\wedge/(F_1, \ldots, F_m)M_{n_q}^\wedge$ is $A_q$-flat. Applying [Mat86, Theorem 22.5 ] now to $F_{in_q}^\wedge : (M_{i-1})^\wedge_{n_q} \rightarrow (M_{i-1})^\wedge_{n_q}$ we get that $K_{i-1}(q) \rightarrow K_{i-1}(q)$ is injective and that $F_i(q), \ldots, F_m(q)$ is an $M(q)$-sequence. □

**Proposition 60.** Let $A$ be a Noetherian ring and $I \subseteq (x)A[[x]]$ an ideal generated by $F_1, \ldots, F_m$, such that $A[[x]]/I A[[x]]$ is $A$-flat. For $p \in \text{Spec} A$ denote by $\bar{I}(p) \subset k(p)[[x]]$ the ideal generated by $F_1(p), \ldots, F_m(p)$.

1. If $\bar{I}(p)$ is a complete intersection, then $\bar{I}(q)$ is a complete intersection for $q$ in an open neighbourhood of $p$ in Spec $A$.

2. If $\bar{I}(p)$ is an ICIS and $\dim A = 1$ or $\dim A \geq 1$ and $I \subseteq (x)A(x)$, then $\bar{I}(q)$ is an ICIS with $\tau(\bar{I}(p)) \geq \tau(\bar{I}(q))$ for $q$ in an open neighbourhood of $p$ in Spec $A$.

**Proof.** 1. We may assume that $F_1(p), \ldots, F_m(p)$ is a $k(p)[[x]]$-sequence. By Proposition 59 $F_1(q), \ldots, F_m(q)$ is a $k(q)[[x]]$-sequence, hence $\bar{I}(q)$ is a complete intersection, for $q$ in an open neighbourhood of $p$ in Spec $A$.

2. follows from Proposition 58 since $\dim_{k(q)} T_{\bar{I}(q)}$ for $\bar{I}(q) \subset k(q)[[x]]$ a complete intersection is the Tjurina number of $\bar{I}(q)$. □

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