Quantum Foundations in the Light of Quantum Information

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Abstract

In this paper, I try to cause some good-natured trouble. The issue at stake is when will we ever stop burdening the taxpayer with conferences and workshops devoted—explicitly or implicitly—to the quantum foundations? The suspicion is expressed that no end will be in sight until a means is found to reduce quantum theory to two or three statements of crisp physical (rather than abstract, axiomatic) significance. In this regard, no tool appears to be better calibrated for a direct assault than quantum information theory. Far from being a strained application of the latest fad to a deep-seated problem, this method holds promise precisely because a large part (but not all) of the structure of quantum theory has always concerned information. It is just that the physics community has somehow forgotten this.

1 Imprimatur

The title of the NATO Advanced Research Workshop that gave birth to this volume was “Decoherence and its Implications in Quantum Computation and Information Transfer.” It was a wonderful meeting—the kind most of us lick our lips for year after year, with little hope of ever tasting. It combined the best of science with the exotic solitude of an island far, far away. One could not help but have a creative thought shaken loose with each afternoon’s gusty wind. Indeed, it was a meeting that will make NATO proud. But, as any attendee can tell you, the most popular pastime—in spite of those windy beaches and dark tans—was an
activity just half-devoted to the conference title. The life of the party was all the talks and conversations on “Decoherence and its Implications in Quantum Foundations.”

In this article, I am going to make an admission at the outset. Despite the industry of work it has spawned for so many conscientious colleagues, and the absolute importance its understanding holds for implementing quantum information technologies, I simply cannot see that decoherence bears in any way on the quantum foundation problem. In saying this, I am well aware of the jeopardy I bring upon myself. For I am but one person—one, in fact, who makes no bones about how limited his knowledge is—while my decoherence colleagues tell me that they are many. The press tells me that they are the “new orthodoxy.” Try as I might, though, I just do not get it. Something about their program does not click in my head.

The root of this problem could be many things, of course—not least of which might be my supreme thickheadedness. There is, however, one thing I know for sure: My want of understanding does not come from a lack of trying to understand. To the extent that I am a relatively reasonable person—and presumably as conscientious as the enthusiasts of the “new orthodoxy”—it seems to me this is a datum that should not go unnoticed. Why might it be that an honest outsider is having so much trouble coming to grips with something the indoctrinated profess to see with great clarity? I cannot stop myself from thinking, “Where there is smoke, there is smoke.”

But let me put that aside: What I wish to get at with this self-sanctioned imprimatur is not a detailed criticism of the decoherence-based quantum-foundations programs. I say all these things instead to provide a particular instantiation (and a little background) for what I deem to be a significantly larger problem in the quantum-foundations efforts to date. My reluctance, or more accurately, my inability, to toe the line for any of the rival quantum political parties—be they the Bohmians [1], the Consistent Historians [2], the Einselectionists [3], the Spontaneous Collapsicans [4], or the Everettistas [5]—springs from a distrust captured vividly by the image of a political convention. The relevant point is, what are their platforms?

Throughout the 2000 presidential campaign in America, the Green Party accused the Democrats and the Republicans of having no difference whatsoever in their platforms. The Democrats and Republicans were appalled. Likewise, to what I suspect will be a jaw-dropping shock to the quantum-party leaders, I declare that I can see little to no difference in any of their beliefs. They all look equally pale to me. For though everyone seems to want a little reality—i.e., something in the theory that they can point to and say, “There, that term is what is real in the universe even when there are no physicists about”—none are willing to dig very deep for it.

What I mean by this deliberately provocative statement is that in spite of the differences in what the various parties are willing to label[1] “real” in quantum theory[2] they nonetheless

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1Or add to the theory, as the case may be.

2Very briefly, a cartoon of the positions might be as follows. For the Bohmians, “reality” is captured by supplementing the state vector with an actual trajectory in coordinate space. For the Everettistas, it is the universal wave function and the universe’s Hamiltonian. (Depending upon the faction, though, these two entities are sometimes supplemented with the terms in various Schmidt decompositions of the universal state vector with respect to a preconceived tensor-product structure.) For the Spontaneous Collapsicans it is again the state vector—though now for the individual system—but Hamiltonian dynamics is supplemented...
all proceed from essentially the same *abstract* starting point. It is nothing other than the standard textbook accounts of the *axioms* of quantum theory.

### The Platform for Most Quantum Foundations Ventures: The Axioms (plain and simple)

1. For every system, there is a complex Hilbert space $\mathcal{H}$.
2. States of the system correspond to projection operators onto $\mathcal{H}$.
3. Those things that are observable *somehow* correspond to the eigenprojectors of Hermitian operators.
4. Isolated systems evolve according to the Schrödinger equation.

“But what nonsense is this,” you must be asking. “Where else could they start!?!?” The main issue is this, and no one has said it more clearly than Carlo Rovelli. Where present-day quantum-foundation studies have stagnated in the stream of history is not so unlike where the physics of length contraction and time dilation stood before Einstein’s 1905 paper on special relativity.

The Lorentz transformations have the name that they do, rather than, say, the Einstein transformations, for a simple historical reason: Lorentz had published some of them as early as 1895. Indeed one could say that most of the empirical predictions of special relativity were in place well before Einstein came onto the scene. But that was of little consolation to the pre-Einsteinian physics community striving so hard to make sense of electromagnetic phenomena and the luminiferous ether. Precisely because the *only* justification for the Lorentz transformations appeared to be their *empirical adequacy*, they remained a mystery to be conquered. More particularly, this was a mystery that heaping further *ad hoc* (mathematical) structure onto could not possibly solve.

What was being begged for in the years between 1895 and 1905 was an understanding of the *origin* of that abstract, mathematical structure—some simple, crisp *physical* statements with respect to which the necessity of the mathematics would be indisputable. Einstein supplied that and became one of the greatest physicists of all time. He reduced the mysterious structure of the Lorentz transformations to two simple statements that could be written in any common language:

with an objective collapse mechanism. For the Consistent Historians “reality” is captured with respect to an initial quantum state and a Hamiltonian by the addition of a set of preferred positive-operator valued measures (POVMs)—they call them consistent sets of histories—along with a truth-value assignment within each of those sets. For the Einselectionists, I leave it as an exercise to the reader.

3To be fair, they do, each in their own way, contribute minor modifications to the *meanings* of a few *words* in the axioms. But that is essentially where the effort stops.

4Though, FitzGerald had considered length contraction as early as 1889.
1) the speed of light in empty space is independent of the speed of its source,
2) physics should appear the same in all inertial reference frames.

The deep significance of this for the quantum problem should stand out and speak overpoweringly to anyone who admits the simplicity of these principles.

Einstein’s move effectively stopped all further debate on the origins of the Lorentz transformations. Outside of the time of the Nazi regime in Germany, I suspect there have been less than a handful of conferences devoted to “interpreting” them. More importantly, with the supreme simplicity of Einstein’s principles, physics became ready for “the next step.” Is it possible to imagine that any mind—even Einstein’s—could have made the leap to general relativity directly from the original, abstract structure of the Lorentz transformations? A structure that was only empirically adequate? I would say no. Indeed, one might question what wonders we will find by pursuing the same strategy of simplification for the quantum foundations.

| Symbolically, where we are: | Where we need to be: |
|-----------------------------|-----------------------|
| $x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}}$ | Speed of light is constant. |
| $t' = \frac{t - vx/c^2}{\sqrt{1 - v^2/c^2}}$ | Physics is the same in all inertial frames. |

The task is not to make sense of the quantum axioms by heaping more structure, more definitions, more science-fiction imagery on top of them, but to throw them away wholesale and start afresh. We should be relentless in asking ourselves: From what deep physical principles might we derive this exquisite mathematical structure? Those principles should be crisp; they should be compelling. They should stir the soul. When I was in junior high school, I sat down with Martin Gardner’s book *Relativity for the Million* and came away with an understanding of the subject that sustains me even today: The concepts were strange to my everyday world, but they were clear enough that I could get a grasp of them knowing little more mathematics than arithmetic. One should expect nothing less for a proper foundation to the quantum. Until we can explain the essence of the theory to a junior high-school or high-school student—*the essence, not the mathematics!*—and have them walk away with a deep, lasting memory, I well believe we will have not understood a thing about quantum foundations.

But I am not fooling myself. I know that anyone with a vested interest in any of the existing quantum interpretations will be quick to point out every hole, every nonnecessity in the sermon I just gave. Indeed, I can feel the upcoming wrath of their email as I write this sentence. I have no retort. Only a calm confidence that if progress is not made in
this direction, 100 years from now the political pollster will still have a niche at the latest
quantum foundations meeting \cite{7}.

So, throw the existing axioms of quantum mechanics away and start afresh! But how
to proceed? I myself see no alternative but to contemplate deep and hard the tasks, the
techniques, and the implications of quantum information theory. The reason is simple, and
I think inescapable. Quantum mechanics has always been about information. It is just that
the physics community has somehow forgotten this.

| Quantum Mechanics: |
|-------------------|
| \textit{The Axioms and Our Imperative!} |

| 1. States correspond to density operators $\rho$ over a Hilbert space $\mathcal{H}$. |
| 2. Measurements correspond to positive operator-valued measures (POVMs) $\{E_b\}$ on $\mathcal{H}$. |
| 3. $\mathcal{H}$ is a complex vector space, not a real vector space, not a quaternionic module. |
| 4. Systems combine according to the tensor product of their separate vector spaces, $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$. |
| 5. Between measurements, states evolve according to trace-preserving completely positive linear maps. |
| 6. By way of measurement, states evolve (up to normalization) via outcome-dependent completely positive linear maps. |
| 7. Probabilities for the outcomes of a measurement obey the Born rule for POVMs $\text{tr}(\rho E_b)$. |

This table is my plea to the community. Our foremost task should be to go to each and
every axiom of quantum theory and give it an information theoretic justification if we can.
Only when we are finished picking off all the terms (or combinations of terms) that can be interpreted as information—subjective information—will we be in a position to make real
progress. The raw distillate that is left behind, miniscule though it may be, will be our first
glimpse of what quantum mechanics is trying to tell us about nature itself.

2 Introduction

This paper is about taking that plea to heart, though it contributes only a small amount
to the labor it asks. Just as in the founding of quantum mechanics, this is not something
that will spring forth from a lone mind in the shelter of a medieval college. It is a task for
a community with diverse but productive points of view. The quantum information com-
munity is nothing if not that. “Philosophy is too important to be left to the philosophers,”
John Archibald Wheeler once said. Likewise, I am apt to say the same for the quantum
foundations.

The structure of the remainder of the paper is as follows. In Section 3 “Why Infor-
mation?” I reiterate the cleanest argument I know of that the quantum state is solely an
expression of information—the information one has about a quantum system. It has no ob-
jective reality in and of itself. The argument is then refined by considering the phenomenon
of quantum teleportation [8].

In Section 4 “Information About What?,” I tackle that very question head-on. The answer is, “nothing more than the potential consequences of our experimental inter-
ventions into nature.” Once freed from the notion that quantum measurement ought to
be about revealing traces of some preexisting property [10] (or beable [11]), one finds no
particular reason to take the standard account of measurement (in terms of complete sets
of orthogonal projection operators) as a basic notion. Indeed quantum information theory,
with its emphasis on the utility of generalized measurements or positive operator-valued
measures (POVMs) [12], suggests one should take those entities as the basic notion instead.
The productivity of this point of view is demonstrated by the beautifully simple Gleason-like
derivation of the quantum probability rule recently found by Paul Busch [14] and, indepen-
dently, by Joseph Renes and collaborators [15]. Contrary to Gleason’s original theorem [16],
this theorem works just as well for two-dimensional Hilbert spaces, and even for Hilbert
spaces over the field of rational numbers.

In Section 5 “Whither Bayes Rule?,” I ask why one should expect the rule for updating
quantum state assignments upon the completion of a measurement to take the form it actu-
ally does. Along the way, I give a simple derivation that one’s information always increases
on average for any quantum mechanical measurement that does not itself discard informa-
tion. (Despite the appearance otherwise, this is not a tautology!) More importantly, the
proof technique used for showing the theorem indicates an extremely strong analogy between
quantum collapse and Bayes’ rule in classical probability theory: Up to an overall unitary
“readjustment” of one’s knowledge (that takes into account details of the measurement in-
teraction as well as one’s initial state of knowledge), quantum collapse is precisely Bayesian
conditionalization. This in turn gives even more impetus for the assumptions behind the
Gleason-like theorem of the previous section.

In Section 6 “Wither Entanglement?,” I ask whether entanglement is all it is cracked-
up to be as far as quantum foundations are concerned. In particular, I give a simple derivation
of the tensor-product rule for combining Hilbert spaces of individual systems into a larger
composite system. To no surprise, once again this comes from Gleason-like considerations for local measurements in the presence of classical communication—the very bread and butter of much of quantum information theory.

In Section 7 “Unknown Quantum States?,” I tackle the conundrum posed by these very words. Despite the phrase’s ubiquitous use in the quantum information literature, what can an unknown state possibly be? A quantum state—from the present point of view, explicitly someone’s information—must always be known by someone, if it exists at all. On the other hand, for many an application in quantum information, it would be quite a contrivance to imagine there is always someone in the background with extra knowledge of the system being measured or manipulated. The solution, at least in the case of quantum state tomography [13], is found through a quantum mechanical version of de Finetti’s classic theorem on “unknown probabilities.” This reports work from Refs. [17] and [18]. Maybe one of the most interesting things about the theorem is that it fails for Hilbert spaces over the field of real numbers, suggesting that perhaps the whole discipline of quantum information might not be well defined in that imaginary world.

Finally, in Section 8 “The Oyster and the Quantum,” I flirt with the most tantalizing question of all: Why the quantum? There are no answers here, but I do not discount that there will be one within 20 years. In this regard no platform seems firmer for the leap than the very existence of quantum cryptography and quantum computing. The world is sensitive to our touch. It has a kind of “Zing!” [5] that makes it fly off in ways that were not imaginable classically. The whole structure of quantum mechanics—it is speculated—may be nothing more than the optimal method of reasoning and processing information in the light of such a fundamental (wonderful) sensitivity.

3 Why Information?

Einstein was the master of clear thought. I have already expressed my reasons for thinking this in the arena of electromagnetic phenomena. Likewise, I would say he possessed the same great penetrating power when it came to analyzing the quantum. For even there, he was immaculately clear and concise in his expression. In particular, he was the first person to say in absolutely unambiguous terms why the quantum state should be viewed as information (or, to say the same thing, as a representation of one’s knowledge).

His argument was simply that a quantum-state assignment for a system can be forced to go one way or the other by interacting with a part of the world that should have no causal connection with the system of interest. The paradigm here is of course the one well

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5Dash, verve, vigor, vim, zip, pep, punch, pizzazz!
known through the Einstein, Podolsky, Rosen paper [19], but simpler versions of the train of thought had a long pre-history with Einstein [20].

The best was in essence this. Take two spatially separated systems $A$ and $B$ prepared in some entangled quantum state $|\psi^{AB}\rangle$. By performing the measurement of one or another of two observables on system $A$ alone, one can *immediately* write down a new state for system $B$. Either the state will be drawn from one set of states $\{|\phi^B_i\rangle\}$ or another $\{|\eta^B_i\rangle\}$, depending upon which observable is measured. The key point is that it does not matter how distant the two systems are from each other, what sort of medium they might be immersed in, or any of the other fine details of the world. Einstein concluded that whatever these things called quantum states be, they cannot be “real states of affairs” for system $B$ alone. For, whatever the real, objective state of affairs at $B$ is, it should not depend upon the measurements one can make on a causally unconnected system $A$.

Thus one must take it seriously that the new state (either a $|\phi^B_i\rangle$ or a $|\eta^B_i\rangle$) represents partial knowledge about system $B$. In making a measurement on $A$, one learns something about $B$, but that is where the story ends. The state change cannot be construed to be something more physical than that. More particularly, the state itself cannot be viewed as more than a reflection of the knowledge gained through the measurement. Expressed in the language of Einstein, the quantum state cannot be a “complete” description of the quantum system.

Here is the way Einstein put it to Michele Besso in a 1952 letter [22]:

> What relation is there between the “state” (“quantum state”) described by a function $\psi$ and a real deterministic situation (that we call the “real state”)? Does the quantum state characterize completely (1) or only incompletely (2) a real state? . . .

I reject (1) because it obliges us to admit that there is a rigid connection between parts of the system separated from each other in space in an arbitrary way (instantaneous action at a distance, which doesn’t diminish when the distance increases). Here is the demonstration: . . .

If one considers the method of the present quantum theory as being in principle definitive, that amounts to renouncing a complete description of real states. One could justify this renunciation if one assumes that there is no law for real states—i.e., that their description would be useless. Otherwise said, that would mean: laws don’t apply to things, but only to what observation teaches us about them. (The laws that relate to the temporal succession of this partial knowledge are however entirely deterministic.)

Now, I can’t accept that. I think that the statistical character of the present theory is simply conditioned by the choice of an incomplete description.

There are two issues in this letter that are worth disentangling. 1) Rejecting the rigid connection of all nature—that is to say, admitting that the very notion of separate systems

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6Generally there need be hardly any relation between the two sets of states: only that when the states are weighted by their probabilities, they mix together to form the initial density operator for system $B$ alone. For a precise statement of this freedom, see Ref. [21].

7The rigid connection of all nature, on the other hand, is exactly what the Bohmians and Everettistas do embrace, even glorify. So, I suspect these words will fall on deaf ears with them. But similarly would they fall on deaf ears with the believer who says that God wills each and every event in the universe and no further explanation is needed. No point of view should be dismissed out of hand: the overriding issue is simply which view will lead to the most progress, which view has the potential to close the debate, which view will give the most new phenomena for the physicist to have fun with?
has any meaning at all—one is led to the conclusion that a quantum state cannot be a complete specification of a system. It must be information, at least in part. This point should be placed in contrast to the other well-known facet of Einstein’s thought: namely, 2) an unwillingness to accept such an “incompleteness” as a necessary trait of the physical world.

It is quite important to recognize that the first issue does not entail the second. Einstein had that firmly in mind, but he wanted more. His reason for going the further step was, I think, well justified at the time [23]:

There exists [...] a simple psychological reason for the fact that this most nearly obvious interpretation is being shunned. For if the statistical quantum theory does not pretend to describe the individual system (and its development in time) completely, it appears unavoidable to look elsewhere for a complete description of the individual system; in doing so it would be clear from the very beginning that the elements of such a description are not contained within the conceptual scheme of the statistical quantum theory. With this one would admit that, in principle, this scheme could not serve as the basis of theoretical physics.

But the world has seen much in the mean time. The last seventeen years have given confirmation after confirmation that the Bell inequality (and several variations of it) are indeed violated by the physical world. The Kochen-Specker no-go theorems have been meticulously clarified to the point where simple textbook pictures can be drawn of them [24]. Incompleteness, it seems, is here to stay: The theory prescribes that no matter how much we know about a quantum system—even when we have maximal information about it—there will always be a statistical residue. There will always be questions that we can ask of a system for which we cannot predict the outcomes. In quantum theory, maximal information is simply not complete information [23]. But neither can it be completed. As Wolfgang Pauli once wrote to Markus Fierz [20], “The well-known ‘incompleteness’ of quantum mechanics (Einstein) is certainly an existent fact somehow-somewhere, but certainly cannot be removed by reverting to classical field physics.” Nor, I would add, will the mystery of that “existent fact” be removed by attempting to give the quantum state anything resembling an ontological status.

The complete disconnectedness of the quantum-state change rule from anything to do with spacetime considerations is telling us something deep: The quantum state is information. Subjective, incomplete information. Put in the right mindset, this is not so intolerable. It is a statement about our world. There is something about the world that keeps us from ever getting more information than can be captured through the formal structure of quantum mechanics. Einstein had wanted us to look further—to find out how the incomplete information could be completed—but perhaps the real question is, “Why can it not be completed?”

Indeed I think this is one of the deepest questions we can ask and still hope to answer. But first things first. The more immediate question for anyone who has come this far—and one that deserves to be answered forthright—is what is this information symbolized by a $|\psi\rangle$ actually about? I have hinted that I would not dare say that it is about some kind of hidden variable (as the Bohmian might) or even about our place within the universal wavefunction (as the Everettista might).
Perhaps the best way to build up to an answer is to be true to the title of this paper: quantum foundations in the light of quantum information. Let us forage the phenomena of quantum information to see if we might first refine Einstein’s argument. One need look no further than to the phenomenon of quantum teleportation. Not only can a quantum-state assignment for a system be forced to go one way or the other by interacting with another part of the world of no causal significance, but, for the cost of two bits, one can make that quantum state assignment anything one wants it to be.

Such an experiment starts out with Alice and Bob sharing a maximally entangled pair of qubits in the state $|\psi_{AB}\rangle = \sqrt{\frac{1}{2}} (|0\rangle|0\rangle + |1\rangle|1\rangle)$. Bob then goes to any place in the universe he wishes. Alice in her laboratory prepares another qubit with any state $|\psi\rangle$ that she ultimately wants to impart onto Bob’s system. She performs a Bell-basis measurement on the two qubits in her possession. In the same vein as Einstein’s thought experiment, Bob’s system immediately takes on the character of one of the states $|\psi\rangle$, $\sigma_x|\psi\rangle$, $\sigma_y|\psi\rangle$, or $\sigma_z|\psi\rangle$. But that is only insofar as Alice is concerned. Since there is no (reasonable) causal connection between Alice and Bob, it must be that these states represent the possibilities for Alice’s new knowledge of Bob’s system.

If now Alice broadcasts the result of her measurement to the world, Bob may complete the teleportation protocol by performing one of the four Pauli rotations ($I$, $\sigma_x$, $\sigma_y$, $\sigma_z$) on his system, conditioning it on the information he receives. The result, as far as Alice is concerned, is that Bob’s system finally resides predictably in the state $|\psi\rangle$.

How can Alice convince herself that such is the case? Well, if Bob is willing to reveal his location, she just need walk to his site and perform the YES-NO measurement: $|\psi\rangle\langle\psi|$ vs. $I - |\psi\rangle\langle\psi|$. The outcome will be a YES with probability one if all has gone well in carrying out the protocol. Thus, for the cost of a measurement on a causally disconnected system and two bits worth of causal action on the system of actual interest—i.e., one of the four Pauli rotations—Alice can sharpen her predictability to complete certainty for any YES-NO observable she wishes.

Roger Penrose argues in his book *The Emperor’s New Mind* that when a system “has” a state $|\psi\rangle$ there ought to be some property in the system (in and of itself) that corresponds to its “$|\psi\rangle$’ness.” For how else could the system be prepared to reveal a YES in the case that Alice actually checks it? Asking this rhetorical question with a sufficient amount of command is enough to make many a would-be informationist weak in knees. But there is a crucial oversight implicit in its confidence, and we have already caught it in action. If Alice fails to reveal her information to anyone else in the world, there is no one else who can predict the qubit’s ultimate revelation with certainty. More importantly, there is nothing in quantum mechanics that gives the qubit the power to stand up and say YES all by itself: If Alice does not take the time to walk over to it and interact with it, there is no revelation. There is only the confidence in Alice’s mind that, should she interact with it, she could predict the consequence of that interaction.

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8 As far as Bob is concerned, nothing whatsoever changes about the system in his possession: It started in the completely mixed state $\rho = \frac{1}{2}I$ and remains that way.

9 As far as Bob is concerned, nothing whatsoever changes about the system in his possession: It started in the completely mixed state $\rho = \frac{1}{2}I$ and remains that way.
4 Information About What?

[S]urely, the existence of [the] world is the primary experimental fact of all, without which there would be no point to physics or any other science; and for which we all receive new evidence every waking minute of our lives. This direct evidence of our senses is vastly more cogent than are any of the deviously indirect experiments that are cited as evidence for the Copenhagen interpretation.

— E. T. Jaynes, 1986

The criticism of the Copenhagen interpretation of the quantum theory rests quite generally on the anxiety that, with this interpretation, the concept of “objective reality” which forms the basis of classical physics might be driven out of physics. . . . [T]his anxiety is groundless . . . At this point we realize the simple fact that natural science is not Nature itself but a part of the relation between Man and Nature, and therefore is dependent on Man.

— Werner Heisenberg, 1955

There are great rewards in being a new parent. Not least of all is the opportunity to have a close-up look at a mind in formation. I have been watching my two year old daughter learn things at a fantastic rate, and though there have been untold numbers of lessons for her, there have also been a sprinkling for me. For instance, I am just starting to see her come to grips with the idea that there is a world independent of her desires. What strikes me is the contrast between this and the concomitant gain in confidence I see grow in her everyday that there are aspects of existence she actually can control. The two go hand in hand. She pushes on the world, and sometimes it gives in a way that she has learned to predict, and sometimes it pushes back in a way she has not foreseen (and may never be able to). If she could manipulate the world to the complete desires of her will, I am quite sure, there would be little difference between wake and dream.

But the main point is that she learns from her forays into the world. In my more cynical moments, I find myself thinking, “How can she think that she’s learned anything at all? She has no theory of measurement. She leaves measurement completely undefined. How can she have any true stake to knowledge?”

Hideo Mabuchi once told me, “The quantum measurement problem refers to a set of people.” And though that is a bit harsh, maybe it also contains a bit of the truth. With the physics community making use of theories that tend to last between 100 and 300 years, we are apt to forget that scientific views of the world are built from the top down, not from the bottom up. The experiment is the basis of all that we know to be firm. But an experiment is an active intervention into the course of nature on the part of the experimenter; it is not contemplation of nature from afar. We set up this or that experiment to see how nature reacts. It is the conjunction of myriads of such interventions and their consequences that we record into our data books.

But I must stress that I am not so positivistic as to think that physics should somehow be grounded on a primitive notion of “sense impression” as the philosophers of the Vienna Circle did. The interventions and their consequences that an experimenter records, have no option but to be thoroughly theory-laden. It is just that, in a sense, they are by necessity at least one theory behind. No one got closer to the salient point than Heisenberg (in a quote he attributed to Einstein many years after the fact):

It is quite wrong to try founding a theory on observable magnitudes alone. In reality the very opposite happens. It is the theory which decides what we can observe. You must appreciate
We tell ourselves that we have learned something new when we can distill from the data a compact description of all that was seen and—even more tellingly—when we can dream up further experiments to corroborate that description. This is the minimal requirement of science. If, however, from such a description we can further distill a model of a free-standing “reality” independent of our interventions, then so much the better. I have no bone to pick with reality. It is the most solid thing we can hope for from a theory. Classical physics is the ultimate example in that regard. It gives us a compact description, but it can give much more if we want it to.

The important thing to realize, however, is that there is no logical necessity that such a worldview always be obtainable. If the world is such that we can never identify a reality—a free-standing reality—then we must be prepared for that too. That is where quantum theory in its most minimal and conceptually simplest dispensation seems to stand. It is a theory whose terms refer predominately to our interface with the world. It is a theory that cannot go the extra step that classical physics did without “writing songs I can’t believe, with words that tear and strain to rhyme.” It is a theory not about observables, not about beables, but about “dingables.” We tap the bell with our gentle touch and listen for its beautiful ring.

So what are the ways we can intervene on the world? What are the ways we can push it and wait for its unpredictable reaction? The usual textbook story is that those things that are measurable correspond to Hermitian operators. Or perhaps to say it in more modern language, to each observable there corresponds a set of orthogonal projection operators \{\Pi_i\} over a complex Hilbert space \(\mathcal{H}_d\) that form a complete resolution of the identity,

\[
\sum_i \Pi_i = I .
\]

The index \(i\) labels the potential outcomes of the measurement (or intervention, to slip back into the language promoted above). When an observer possesses a state of knowledge \(\rho\)—captured most generally by a mixed-state density operator—quantum mechanics dictates that he can expect the various outcomes with a probability

\[
P(i) = \text{tr}(\rho \Pi_i) .
\]

11Woody Allen said it best: “I hate reality, but, you know, where else can you get a good steak dinner?” Spending my childhood in Texas, where beef is the staple of most meals, the reader should realize that it has been no easy journey for me to come to my present view of quantum mechanics!
The best justification for this probability rule comes by way of Andrew Gleason’s amazing 1957 theorem \[16\]. For, it states that the standard rule is the *only* rule that satisfies a very simple kind of noncontextuality for measurement outcomes \[32\]. In particular, if one contemplates measuring two distinct observables \(\{\Pi_i\}\) and \(\{\tilde{\Pi}_i\}\) which happen to share a single projector \(\Pi_k\), then the probability of outcome \(k\) is independent of which observable it is associated with. More formally, the statement is this. Let \(P_d\) be the set of projectors associated with a (real or complex) Hilbert space \(\mathcal{H}_d\) for \(d \geq 3\), and let \(f : P_d \to [0,1]\) be such that

\[
\sum_i f(\Pi_i) = 1
\]

whenever a set of projectors \(\{\Pi_i\}\) forms an observable. The theorem concludes that there exists a density operator \(\rho\) such that

\[
f(\Pi) = \text{tr}(\rho \Pi).
\]

In fact, in a single blow, Gleason’s theorem derives not only the probability rule, but also the state-space structure for quantum mechanical states (i.e., that it corresponds to the convex set of density operators).

In itself this is no small feat, but the thing that makes the theorem an “amazing” theorem is the shear difficulty required to prove it \[33\]. Note that no restrictions have been placed upon the function \(f\) beyond the ones mentioned above. There is no assumption that it need be differentiable, nor that it even need be continuous. All of that, and linearity too, comes from the structure of the observables—i.e., that they are complete sets of orthogonal projectors onto a linear vector space.

Nonetheless, one should ask: Does this theorem really give the physicist a clearer vision of where the probability rule comes from? Astounding feats of mathematics are one thing; insight into physics is another. The two are often at opposite ends of the spectrum. As fortunes turn however, a unifying strand can indeed be drawn by viewing quantum foundations in the light of quantum information.

The place to start is to drop the fixation that the basic set of observables in quantum mechanics are complete sets of orthogonal projectors. In quantum information theory it has been found to be extremely convenient to expand the notion of measurement to also include general positive operator-valued measures (POVMs) \[24\, 34\]. In other words, in place of the usual textbook notion of measurement, \(\text{any}\) set \(\{E_b\}\) of positive-semidefinite operators on \(\mathcal{H}_d\) that forms a resolution of the identity, i.e., that satisfies

\[
\langle \psi | E_b | \psi \rangle \geq 0, \quad \text{for all } |\psi\rangle \in \mathcal{H}_d
\]

and

\[
\sum_b E_b = I,
\]

counts as a measurement. The outcomes of the measurement are identified with the indices \(b\), and the probabilities of the outcomes are computed according to a generalized Born rule,

\[
P(b) = \text{tr}(\rho E_b).
\]
The set \( \{E_b\} \) is called a POVM, and the operators \( E_b \) are called POVM elements. (In the nonstandard language promoted earlier, the set \( \{E_b\} \) signifies an intervention into nature, while the individual \( E_b \) represent the potential consequences of that intervention.) Unlike standard measurements, there is no limitation on the number of values the index \( b \) can take. Moreover, the \( E_b \) may be of any rank, and there is no requirement that they be mutually orthogonal.

The way this expansion of the notion of measurement is usually justified is that any POVM can be represented formally as a standard measurement on an ancillary system that has interacted in the past with the system of actual interest. Indeed, suppose the system and ancilla are initially described by the density operators \( \rho_S \) and \( \rho_A \) respectively. The conjunction of the two systems is then described by the initial quantum state

\[
\rho_{SA} = \rho_S \otimes \rho_A .
\]

An interaction between the systems via some unitary time evolution leads to a new state

\[
\rho_{SA} \rightarrow U \rho_{SA} U^\dagger .
\]

Now, imagine a standard measurement on the ancilla. It is described on the total Hilbert space via a set of orthogonal projection operators \( \{I \otimes \Pi_b\} \). An outcome \( b \) will be found, by the standard Born rule, with probability

\[
P(b) = \text{tr} \left( (\rho_S \otimes \rho_A) U (I \otimes \Pi_b) U^\dagger \right) .
\]

The number of outcomes in this seemingly indirect notion of measurement is limited only by the dimensionality of the ancilla’s Hilbert space—in principle, there can be arbitrarily many.

As advertised, it turns out that the probability formula above can be expressed in terms of operators on the system’s Hilbert space alone: This is the origin of the POVM. If we let \(|s_\alpha\rangle\) and \(|a_c\rangle\) be an orthonormal basis for the system and ancilla respectively, then \(|s_\alpha\rangle|a_c\rangle\) will be a basis for the composite system. Using the cyclic property of the trace in Eq. (10), we get

\[
P(b) = \sum_\alpha \langle s_\alpha| (\rho_S \otimes \rho_A) U (I \otimes \Pi_b) U^\dagger |s_\alpha\rangle |a_c\rangle .
\]

Letting \(\text{tr}_A\) and \(\text{tr}_S\) denote partial traces over the system and ancilla, respectively, it follows that

\[
P(b) = \text{tr}_S \left( \rho_S E_b \right) ,
\]

where

\[
E_b = \text{tr}_A \left( (I \otimes \rho_A) U (I \otimes \Pi_b) U^\dagger \right)
\]

is an operator acting on the Hilbert space of the original system. This proves half of what is needed, but it is also straightforward to go in the reverse direction—i.e., to show that for any POVM \( \{E_b\} \), one can pick an ancilla and find operators \( \rho_\lambda, U, \) and \( \Pi_b \) such that Eq. (13) is true.
Putting this all together, there is a sense in which standard measurements capture everything that can be said about quantum measurement theory [34]. As became clear above, a way to think about this is that by learning something about the ancillary system through a standard measurement, one in turn learns something about the system of real interest. Indirect though it may seem, this can be a powerful technique, sometimes revealing information that could not have been revealed otherwise [35]. A very simple example is where a sender has only a single qubit available for the sending one of three potential messages. She therefore has a need to encode the message in one of three preparations of the system, even though the system is a two-state system. To recover as much information as possible, the receiver might (just intuitively) like to perform a measurement with three distinct outcomes. If, however, he were limited to a standard quantum measurement, he would only be able to obtain two outcomes. This—perhaps surprisingly—generally degrades his opportunities for recovery.

What I would like to bring up is whether this standard way of justifying the POVM is the most productive point of view one can take. Might any of the mysteries of quantum mechanics be alleviated by taking the POVM as a basic notion of measurement? Does the POVM’s utility portend a larger role for it in the foundations of quantum mechanics?

| Standard Measurements | Generalized Measurements |
|-----------------------|--------------------------|
| \{\Pi_i\}            | \{E_b\}                  |
| \langle \psi | \Pi_i | \psi \rangle \geq 0, \forall | \psi \rangle | \langle \psi | E_b | \psi \rangle \geq 0, \forall | \psi \rangle |
| \sum_i \Pi_i = I    | \sum_b E_b = I            |
| \( P(i) = \text{tr}(\rho \Pi_i) \) | \( P(b) = \text{tr}(\rho E_b) \) |
| \Pi_i \Pi_j = \delta_{ij} \Pi_i | ——— |

I try to make the point dramatic in my lectures by exhibiting a transparency of the table above. On the left-hand side there is a list of various properties for the standard notion of a quantum measurement. On the right-hand side, there is an almost identical list of properties for the POVMs. The only difference between the two columns is that the right-hand one is missing the orthonormality condition required of a standard measurement. The question I ask the audience is this: Does the addition of that one extra assumption really make the process of measurement any less mysterious? Indeed, I imagine myself teaching quantum mechanics for the first time and taking a vote with the best audience of all, the students:

“Which set of postulates for quantum measurement would you prefer?” I am quite sure they

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12I am making the safe bet that they will be lucky enough to not yet be conditioned by years of squabbles in quantum foundations.
would respond with a blank stare. But that is the point! It would make no difference to them, and it should make no difference to us. The only issue worth debating is which notion of measurement will allow us to see more deeply into quantum mechanics.

Therefore let us pose the question that Gleason did, but with POVMs. In other words, let us suppose that the sum total of ways an experimenter can intervene on a quantum system corresponds to the full set of POVMs on its Hilbert space $H_d$. It is the task of the theory to give him probabilities for the various consequences of his interventions. Concerning those probabilities, let us (in analogy to Gleason) assume only that whatever the probability for a given consequence $E_c$ is, it does not depend upon whether $E_c$ is associated with the POVM $\{E_b\}$ or, instead, any other one $\{\tilde{E}_b\}$. This means we can assume there exists a function

$$f : \mathcal{E}_d \rightarrow [0,1],$$

where

$$\mathcal{E}_d = \left\{ E : 0 \leq \langle \psi | E | \psi \rangle \leq 1, \ \forall | \psi \rangle \in H_d \right\},$$

such that whenever $\{E_b\}$ forms a POVM,

$$\sum_b f(E_b) = 1.$$  

(In general, we will call any function satisfying $f(E) \geq 0$ and $\sum_b f(E_b) = \text{const.}$ a frame function, in analogy to Gleason’s nonnegative frame functions.)

It will come as no surprise, of course, that a Gleason-like theorem must hold for the function in Eq. (14). Namely, it can be shown that there must exist a density operator $\rho$ for which

$$f(E) = \text{tr}(\rho E).$$  

This was recently shown by Paul Busch and, independently, by Joseph Renes and collaborators. What is surprising however is the utter simplicity of the proof. Let us exhibit the whole thing right here and now.

First, consider the case where $H_d$ and the operators on it are defined only over the field of (complex) rational numbers. It is no problem to see that $f$ is “linear” with respect to positive combinations of operators that never go outside $\mathcal{E}_d$. For consider a three-element POVM $\{E_1, E_2, E_3\}$. By assumption $f(E_1) + f(E_2) + f(E_3) = 1$. However, we can also group the first two elements in this POVM to obtain a new POVM, and must therefore have $f(E_1 + E_2) + f(E_3) = 1$. In other words, the function $f$ must be additive with respect to a fine-graining operation:

$$f(E_1 + E_2) = f(E_1) + f(E_2).$$

Similarly for any two integers $m$ and $n$,

$$f(E) = m f\left(\frac{1}{m}E\right) = n f\left(\frac{1}{n}E\right).$$

Suppose $\frac{n}{m} \leq 1$. Then if we write $E = nG$, this statement becomes:

$$f\left(\frac{n}{m}G\right) = \frac{n}{m}f(G).$$
Thus we immediately have a kind of limited linearity on $\mathcal{E}_d$.

One might imagine using this property to cap off the theorem in the following way. Clearly the full $d^2$-dimensional vector space $\mathcal{O}_d$ of Hermitian operators on $\mathcal{H}_d$ is spanned by the set $\mathcal{E}_d$ since that set contains, among other things, all the projection operators. Thus, we can write any operator $E \in \mathcal{E}_d$ as a linear combination

$$E = \sum_{i=1}^{d^2} \alpha_i E_i$$

(21)

for some fixed operator-basis $\{E_i\}_{i=1}^{d^2}$. “Linearity” of $f$ would then give

$$f(E) = \sum_{i=1}^{d^2} \alpha_i f(E_i).$$

(22)

So, if we define $\rho$ by solving the $d^2$ linear equations

$$\text{tr}(\rho E_i) = f(E_i),$$

(23)

we would have

$$f(E) = \sum_i \alpha_i \text{tr}(\rho E_i) = \text{tr} \left( \rho \sum_i \alpha_i E_i \right) = \text{tr}(\rho E)$$

(24)

and essentially be done. (Positivity and normalization of $f$ would require $\rho$ to be an actual density operator.) But the problem is that in expansion (21) there is no guarantee that the coefficients $\alpha_i$ can be chosen so that $\alpha_i E_i \in \mathcal{E}_d$.

What remains to be shown is that $f$ can be extended uniquely to a function that is truly linear on $\mathcal{O}_d$. This too is rather simple. First, take any positive semi-definite operator $E$. We can always find a positive rational number $g$ such that $E = gG$ and $G \in \mathcal{E}_d$. Therefore, we can simply define $f(E) \equiv g f(G)$. To see that this definition is unique, suppose there are two such operators $G_1$ and $G_2$ (with corresponding numbers $g_1$ and $g_2$) such that $E = g_1 G_1 = g_2 G_2$. Further suppose $g_2 \geq g_1$. Then $G_2 = \frac{g_2}{g_1} G_1$ and, by the homogeneity of the original unextended definition of $f$, we obtain $g_2 f(G_2) = g_1 f(G_1)$. Furthermore this extension retains the additivity of the original function. For suppose that neither $E$ nor $G$, though positive semi-definite, are necessarily in $\mathcal{E}_d$. We can find a positive rational number $c \geq 1$ such that $\frac{1}{c}(E + G)$, $\frac{1}{c} E$, and $\frac{1}{c} G$ are all in $\mathcal{E}_d$. Then, by the rules we have already obtained,

$$f(E + G) = c f \left( \frac{1}{c} (E + G) \right) = c f \left( \frac{1}{c} E \right) + c f \left( \frac{1}{c} G \right) = f(E) + f(G).$$

(25)

Let us now further extend $f$’s domain to the full space $\mathcal{O}_d$. This can be done by noting that any operator $H$ can be written as the difference $H = E - G$ of two positive semi-definite operators. Therefore define $f(H) \equiv f(E) - f(G)$, from which it also follows that $f(-G) = -f(G)$. To see that this definition is unique suppose there are four operators $E_1$, $E_2$, $G_1$, and $G_2$, such that $H = E_1 - G_1 = E_2 - G_2$. It follows that $E_1 + G_2 = E_2 + G_1$. Applying $f$ (as extended in the previous paragraph) to this equation, we obtain $f(E_1) + f(G_2) = f(E_2) + f(G_1)$ so that $f(E_1) - f(G_1) = f(E_2) - f(G_2)$. Finally, with this new extension, full linearity can be checked immediately. This completes the proof as far as the (complex) rational number field is concerned: Because $f$ extends uniquely to a linear functional on $\mathcal{O}_d$, we can indeed go through the steps of Eqs. (21) through (24) without worry.
There are two things that are significant about this much of the proof. First, in contrast to Gleason’s original theorem, there is nothing to bar the same logic from working when $d = 2$. This is quite nice because much of the community has gotten in the habit of thinking that there is nothing particularly “quantum mechanical” about a single qubit. Indeed, because orthogonal projectors on $\mathcal{H}_2$ can be mapped onto antipodes of the Bloch sphere, it is known that the measurement-outcome statistics for any standard measurement can be mocked-up through a noncontextual hidden-variable theory. What this result shows is that that simply is not the case when one considers the full set of POVMs as one’s potential measurements.

The other important thing is that the theorem works for Hilbert spaces over the rational number field: one does not need to invoke the full power of the continuum. This contrasts with the surprising result of Meyer [36] that the standard Gleason theorem fails in such a setting. The present theorem hints at a kind of resiliency to the structure of quantum mechanics that falls through the mesh of the standard Gleason result: The probability rule for POVMs does not actually depend so much upon the detailed workings of the number field.

The final step of the proof, indeed, is to show that nothing goes awry when we go the extra step of reinstating the continuum.

In other words, we need to show that the function $f$ (now defined on the set $\mathcal{E}_d$ complex operators) is a continuous function. This comes about in simple way from $f$’s additivity. Suppose for two positive semi-definite operators $E$ and $G$ that $E \leq G$ (i.e., $G - E$ is positive semi-definite). Then trivially there exists a positive semi-definite operator $H$ such that $E + H = G$ and through which the additivity of $f$ gives $f(E) \leq f(G)$. Let $c$ be an irrational number, and let $a_n$ be an increasing sequence and $b_n$ a decreasing sequence of rational numbers that both converge to $c$. It follows for any positive semi-definite operator $E$, that

$$f(a_n E) \leq f(c E) \leq f(b_n E) ,$$

which implies

$$a_n f(E) \leq f(c E) \leq b_n f(E) .$$

Since $\lim a_n f(E)$ and $\lim b_n f(E)$ are identical, by the “pinching theorem” of elementary calculus, they must equal $f(c E)$. This establishes that we can consistently define

$$f(c E) = cf(E) .$$

Reworking the extensions of $f$ in the last inset (but with this enlarged notion of homogeneity), one completes the proof in a straightforward manner.

Of course we are not getting something from nothing. The reason the present derivation is so easy in contrast to the standard proof is that mathematically the assumption of POVMs as the basic notion of measurement is significantly stronger than the usual assumption. Physically, though, I would say it is just the opposite. Why add extra restrictions to the notion of measurement when they only make the route from basic assumption to practical usage more circuitous than it need be?
Still, no assumption should be left unanalyzed if it stands a chance of bearing fruit. Indeed, one can ask what is so very compelling about the noncontextuality property (of probability assignments) that both Gleason’s original theorem and the present version make use of. Given the picture of measurement as a kind of invasive intervention into the world, one might expect the very opposite. One is left wondering why measurement probabilities do not depend upon the whole context of the measurement interaction. Why is $p(b)$ not of the form $f(b, \{E_c\})$? Is there any precedent for the usual assumption?

5 Whither Bayes’ Rule?

And so you see I have come to doubt
All that I once held as true
I stand alone without beliefs
The only truth I know is you.
— Paul Simon, Kathy’s Song

Quantum states are states of knowledge, not states of nature. That statement is the cornerstone of this paper. Thus, in searching to make sense of the remainder of quantum mechanics, one strategy ought to be to seek guidance from the most developed avenue of “knowledge theory” to date—Bayesian probability theory. Indeed, the very aim of Bayesian theory is to develop reliable methods of reasoning and making decisions in the light of incomplete knowledge. To what extent does that structure mesh with the seemingly independent structure of quantum mechanics? To what extent are there analogies; to what extent distinctions?

This section is about turning a distinction into an analogy. The core of the matter is the manner in which states of knowledge are updated in the two theories. At first sight, they appear to be quite different in character. To see this, let us first explore how quantum mechanical states change when information is gathered.

In older accounts of quantum mechanics, one often encounters the “collapse postulate” as a basic statement of the theory. One hears things like, “Axiom 5: Upon the completion of a measurement of a Hermitian observable $H$, the system is left in an eigenstate of $H$.” In quantum information, however, it has become clear that it is useful to broaden the notion of measurement, and with it, the analysis of how a state can change in the process. The foremost reason for this is that the collapse postulate is simply not true in general: Depending upon the exact nature of the measurement interaction, there may be any of a large set of possibilities for the final state of a system.

The broadest (consistent) notion of state change arose in the theory of “effects and operations”. The statement is this. Suppose one’s initial state for a quantum system is a density operator $\rho$, and a POVM $\{E_b\}$ is measured on that system. Then, according to this formalism, the state after the measurement can be any state $\rho_b$ of the form

$$\rho_b = \frac{1}{\text{tr}(\rho E_b)} \sum_i A_{bi} \rho A_{bi}^\dagger,$$  \hspace{1cm} (29)
where
\[ \sum_i A_{bi}^\dagger A_{bi} = E_b . \] (30)

Note the immense generality of this formula. There is no constraint on the number of indices \( i \) in the \( A_{bi} \) and these operators need not even be Hermitian.

The usual justification for this kind of generality—just as in the case of the commonplace justification for the POVM formalism—comes about by imagining that the measurement arises in an indirect fashion rather than as a direct attack. In other words, the primary system is pictured to interact with an ancilla first, and only then subjected to a “real” measurement on the ancilla alone. The trick is that one posits a kind of projection postulate on the primary system due to this process. This assumption has a much safer feel than the raw projection postulate since, after the interaction, no measurement on the ancilla should cause a physical perturbation to the primary system.

More formally, we can start out by following Eqs. (8) and (9), but in place of Eq. (10) we must make an assumption on how the system’s state changes. For this one invokes a kind of “projection-postulate-at-a-distance.”

\[ \rho_b = \frac{1}{P(b)} \text{tr}_\lambda \left( (I \otimes \Pi_b)U(\rho_s \otimes \rho_\lambda)U^\dagger (I \otimes \Pi_b) \right) . \] (31)

The reason for invoking the partial trace is to make sure that any hint of a state change for the ancilla remains unaddressed.

To see how expression (31) makes connection to Eq. (29), denote the eigenvalues and eigenvectors of \( \rho_\lambda \) by \( \lambda_\alpha \) and \( |a_\alpha\rangle \) respectively. Then \( \rho_s \otimes \rho_\lambda \) can be written as
\[ \rho_s \otimes \rho_\lambda = \sum_\alpha \sqrt{\lambda_\alpha} |a_\alpha\rangle \rho_s \langle a_\alpha| \sqrt{\lambda_\alpha} , \] (32)

and, expanding Eq. (31), we have
\[ \rho_b = \frac{1}{P(b)} \sum_\beta \langle a_\beta|(I \otimes \Pi_b)U^\dagger(\rho_s \otimes \rho_\lambda)U(I \otimes \Pi_b)|a_\beta\rangle \]
\[ = \frac{1}{P(b)} \sum_{\alpha\beta} \left( \sqrt{\lambda_\alpha} \langle a_\beta|(I \otimes \Pi_b)U^\dagger|a_\alpha\rangle \right) \rho_s \left( \langle a_\alpha|U(I \otimes \Pi_b)|a_\beta\rangle \sqrt{\lambda_\alpha} \right) . \] (33)

A representation of the form in Eq. (29) can be made by taking
\[ A_{ba\beta} = \sqrt{\lambda_\alpha} \langle a_\alpha|U(I \otimes \Pi_b)|a_\beta\rangle \] (34)

and lumping the two indices \( \alpha \) and \( \beta \) into the single index \( i \). Indeed, one can easily check that Eq. (30) holds. This completes what we had set out to show. However, just as with the case of the POVM \( \{E_b\} \), one can always find a way to reverse engineer the derivation:

\(^{13}\)David Mermin has also recently emphasized this point in Ref. [41].

\(^{14}\)As an aside, it should be clear from the construction in Eq. (34) that there are many equally good representations of \( \rho_b \). For a precise statement of the latitude of this freedom, see Ref. [42].
Given a set of $A_{bi}$, one can always find a $U$, a $\rho_A$, and set of $\Pi_b$ such that Eq. (31) becomes true.

Of course the old collapse postulate is contained within the extended formalism as a special case: There, one just takes both sets $\{E_b\}$ and $\{A_{bi} = E_b\}$ to be sets of orthogonal projectors. Let us take a moment to think about this special case in isolation. What is distinctive about it is that it captures in the extreme a common folklore associated with the measurement process. For it tends to convey the image that measurement is a kind of gut wrenching violence: In one moment the state is a $\rho = |\psi\rangle\langle\psi|$, while in the very next it is a $\Pi_i = |i\rangle\langle i|$. Moreover, such a wild transition need depend upon no details of $|\psi\rangle$ and $|i\rangle$; in particular the two states may even be almost orthogonal to each other. In density-operator language, there is no sense in which $\Pi_i$ is contained in $\rho$: the two states are simply in distinct places of the operator space.

Contrast this with the description of information gathering that arises in Bayesian probability theory. There, an initial state of knowledge is captured by a probability distribution $p(h)$ for some hypothesis $H$. The way gathering a piece of data $d$ is taken into account in assigning one’s new state of knowledge is through Bayes’ conditionalization rule. That is to say, one expands $p(h)$ in terms of the relevant joint probability distribution and picks off the appropriate term:

$$p(h) = \sum_d p(h,d) = \sum_d p(d)p(h|d)$$

$$\downarrow$$

$$p(h) \xrightarrow{d} p(h|d) ,$$

where $p(h|d)$ satisfies the tautology

$$p(h|d) = \frac{p(h,d)}{p(d)} .$$

How gentle this looks in comparison to quantum collapse! When one gathers new information, one simply refines one’s old knowledge in the most literal of senses. It is not as if the new state is incommensurable with the old. It was always there; it was just initially averaged in with various other potential states of knowledge.

Why does quantum collapse not look more like Bayes’ rule? Is quantum collapse really a more violent kind of change, or might it be an artifact of a problematic representation? By this stage, it should come as no surprise to the reader that dropping the ancilla from our image of generalized measurements will be the first step to progress. Taking the transition from $\rho$ to $\rho_b$ in Eqs. (29) and (30) as the basic statement of what quantum measurement is is a good starting point.

To accentuate a similarity between Eq. (29) and Bayes’ rule, let us first contemplate cases of it where the index $i$ takes on a single value. Then, we can conveniently drop that index and write

$$\rho_b = \frac{1}{P(b)} A_b \rho A_b^\dagger ,$$

(38)
where

\[ E_b = A_b^\dagger A_b . \]  

(39)

In a loose way, one can say that measurements of this sort are the most efficient they can be for a given POVM \( \{ E_b \} \): For, a measurement interaction with an explicit \( i \)-dependence may be viewed as “more truly” a measurement of a finer-grained POVM that just happens to throw away some of the information it gained. Let us make this point more precise.

Notice that Bayes’ rule has the property that one’s uncertainty about a hypothesis can be expected to decrease upon the acquisition of data. This can be made rigorous, for instance, by gauging uncertainty with the Shannon entropy function [43],

\[ S(H) = - \sum_h p(h) \log p(h) . \]  

(40)

This number is bounded between 0 and the logarithm of the number of hypotheses in \( H \), and there are several reasons to think of it as a good measure of uncertainty. Perhaps the most important of these is that it quantifies the number of YES-NO questions one can expect to ask per instance of \( H \), if one’s only means to ascertain the outcome is from a colleague who knows the actual result [44]. Under this quantification, the lower the Shannon entropy, the more predictable a measurement’s outcomes.

Because the function \( f(x) = -x \log x \) is concave on the interval \([0, 1]\), it follows that,

\[
S(H) = - \sum_h \left( \sum_d p(d)p(h|d) \right) \log \left( \sum_d p(d)p(h|d) \right) \\
\geq - \sum_d p(d) \sum_h p(h|d) \log p(h|d) . \\
= \sum_d p(d)S(H|d) \\
\equiv S(H|D) .
\]  

(41)

Indeed we hope to find a similar statement for how the result of efficient quantum measurements decrease uncertainty or impredictability. But, what can be meant by a decrease of uncertainty through quantum measurement? I have argued strenuously that the information gain in a measurement cannot be information about a preexisting reality. The way out of the impasse is simple: The uncertainty that decreases in quantum measurement is the uncertainty one expects for the results of potential future measurements.

There are at least two ways of quantifying this that are worthy of note. The first has to do with the von Neumann entropy of a density operator \( \rho \):

\[ S(\rho) = -\text{tr} \rho \log \rho = - \sum_{k=1}^d \lambda_k \log \lambda_k , \]  

(42)

where the \( \lambda_k \) signify the eigenvalues of \( \rho \). (We use the convention that \( \lambda \log \lambda = 0 \) whenever \( \lambda = 0 \) so that \( S(\rho) \) is always well defined.)

The intuitive meaning of the von Neumann entropy can be found by first thinking about the Shannon entropy. Consider any standard measurement \( \mathcal{P} \) consisting of \( d \) one-dimensional
orthogonal projectors $\Pi_i$. The Shannon entropy for the outcomes of this measurement is given by
\[
H(\mathcal{P}) = -\sum_{i=1}^{d} (\text{tr}\rho\Pi_i) \log (\text{tr}\rho\Pi_i) .
\] (43)

A natural question to ask is: With respect to a given density operator $\rho$, which measurement $\mathcal{P}$ will give the most predictability over its outcomes? As it turns out, the answer is any $\mathcal{P}$ that forms a set of eigenprojectors for $\rho$. When this obtains, the Shannon entropy of the measurement outcomes reduces to simply the von Neumann entropy of the density operator. The von Neumann entropy, then, signifies the amount of unpredictability one achieves by way of a standard measurement in a best case scenario. Indeed, true to one’s intuition, one has the most knowledge by this account when $\rho$ is a pure state—for then $S(\rho) = 0$. Alternatively, one has the least knowledge when $\rho$ is proportional to the identity operator—for then any measurement $\mathcal{P}$ will have outcomes that are all equally likely.

The best case scenario for predictability, however, is a limited case, and not very indicative of the density operator as a whole. Since the density operator contains, in principle, all that can be said about every possible measurement, it seems a shame to throw away the vast part of that information in our considerations.

This leads to a second method for quantifying uncertainty in the quantum setting. For this, we again rely on the Shannon information as our basic notion of unpredictability. The difference is we evaluate it with respect to a “typical” measurement rather than the best possible one. But typical with respect to what? The notion of typical is only defined with respect to a given measure on the set of measurements.

Regardless, there is a fairly canonical answer. There is a unique measure $d\Omega_\Pi$ on the space of one-dimensional projectors that is invariant with respect to all unitary operations. That in turn induces a canonical measure $d\Omega_\mathcal{P}$ on the space of von Neumann measurements $\mathcal{P}$. Using this measure leads to the following quantity
\[
\overline{S}(\rho) = \int H(\Pi) d\Omega_\mathcal{P}
= -d \int (\text{tr}\rho\Pi) \log (\text{tr}\rho\Pi) d\Omega_\Pi ,
\] (44)

which is intimately connected to the so-called quantum “subentropy” of Ref. [47]. This mean entropy can be evaluated explicitly in terms of the eigenvalues of $\rho$ and takes on the expression
\[
\overline{S}(\rho) = \frac{1}{\ln 2} \left( \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{d} \right) + Q(\rho)
\] (45)

where the subentropy $Q(\rho)$ is defined by
\[
Q(\rho) = -\sum_{k=1}^{d} \left( \prod_{i \neq k} \frac{\lambda_k}{\lambda_k - \lambda_i} \right) \lambda_k \log \lambda_k .
\] (46)

In the case where $\rho$ has degenerate eigenvalues, $\lambda_l = \lambda_m$ for $l \neq m$, one need only reset them to $\lambda_l + \epsilon$ and $\lambda_m - \epsilon$ and consider the limit as $\epsilon \to 0$. The limit is convergent and hence
$Q(\rho)$ is finite for all $\rho$. With this, one can also see that for a pure state $\rho$, $Q(\rho)$ vanishes. Furthermore, since $S(\rho)$ is bounded above by $\log d$, we know that

$$0 \leq Q(\rho) \leq \log d - \frac{1}{\ln 2} \left( \frac{1}{2} + \cdots + \frac{1}{d} \right) \leq \frac{1 - \gamma}{\ln 2},$$

(47)

where $\gamma$ is Euler’s constant. This means that for any $\rho$, $Q(\rho)$ never exceeds approximately 0.60995 bits.

The interpretation of this result is the following. Even when one has maximal information about a quantum system—i.e., one has a pure state for it—one can predict almost nothing about the outcome of a typical measurement [23]. In the limit of large $d$, the outcome entropy for a typical measurement is just a little over a half bit away from its maximal value. Having a mixed state for a system, reduces one’s predictability even further, but indeed not by that much: The small deviation is captured by the function in Eq. (46), which becomes a quantification of uncertainty in its own right.

The way to get at a quantum statement of Eq. (41) is to make use of the fact that $S(\rho)$ and $Q(\rho)$ are both concave in the variable $\rho$ [18]. That is, for either function, we have

$$F(t\rho_0 + (1-t)\rho_1) \geq tF(\rho_0) + (1-t)F(\rho_1),$$

(48)

for any density operators $\rho_0$ and $\rho_1$ and any real number $t \in [0, 1]$. However, the result does not arise in the trivial fashion it did for classical case of Eq. (41). This is because generally, as already emphasized,

$$\rho \neq \sum_b P(b)\rho_b$$

(49)

for $\rho_b$ defined as in Eq. (48). One must be slightly more roundabout.

The key is in noticing that

$$\rho = \rho^{1/2}I\rho^{1/2}$$

$$= \sum_b \rho^{1/2}E_b\rho^{1/2}$$

$$= \sum_b P(b)\tilde{\rho}_b$$

(50)

where

$$\tilde{\rho}_b = \frac{1}{P(b)}\rho^{1/2}E_b\rho^{1/2} = \frac{1}{P(b)}\rho^{1/2}A_b^\dagger A_b\rho^{1/2}.$$  

(51)

What is special about this decomposition of $\rho$ is that for each $b$, $\rho_b$ and $\tilde{\rho}_b$ have the same eigenvalues. This follows since $X^\dagger X$ and $XX^\dagger$ have the same eigenvalues, for any operator $X$. In the present case, setting $X = A_b\rho^{1/2}$ does the trick. Using the fact that both $S(\rho)$ and $Q(\rho)$ depend only upon the eigenvalues of $\rho$ we obtain:

$$S(\rho) \geq \sum_b P(b)S(\rho_b)$$

(52)

$$Q(\rho) \geq \sum_b P(b)Q(\rho_b),$$

(53)
as we had been hoping for. Thus, in performing an efficient quantum measurement of a
POVM \( \{ E_b \} \), an observer can expect to be left with less uncertainty than he started with.\footnote{By differing methods, a strengthening of this result in terms of a majorization property can be found in Refs.\cite{48} and \cite{49}.}

In this sense, quantum “collapse” does indeed have some of the flavor of Bayes’ rule. But we can expect more, and the derivation above hints at just the right ingredient: \( \rho_b \) and \( \tilde{\rho}_b \) have the same eigenvalues! To see the impact of this, let us once again explore the content of Eqs.\ (38) and (39). A common way to describe their meaning is to use the operator polar-decomposition theorem \cite{50} to rewrite Eq. (38) in the form

\[
\rho_b = \frac{1}{P(b)} U_b E_b^{1/2} \rho E_b^{1/2} U_b^\dagger ,
\]

where \( U_b \) is a unitary operator. Since—subject only to the constraint of efficiency—the operators \( A_b \) are not determined any further than Eq. (39), \( U_b \) can be \textit{any} unitary operator whatsoever. Thus, a customary way of thinking of the state-change process is to break it up into two conceptual pieces. First there is a “raw collapse”:

\[
\rho \rightarrow \sigma_b = \frac{1}{P(b)} E_b^{1/2} \rho E_b^{1/2} .
\]

Then, subject to the details of the measurement interaction and the particular outcome \( b \), one imagines the measuring device enforcing a further kind of “feedback” on the measured system:

\[
\sigma_b \rightarrow \rho_b = U_b \sigma_b U_b^\dagger .
\]

But this break down of the transition is a purely conceptual game.

Since the \( U_b \) are arbitrary to begin with, we might as well break down the state-change process into the following (nonstandard) conceptual components. First one imagines an observer refining his initial state of knowledge and simply plucking out a term corresponding to the “data” collected:

\[
\rho = \sum_b P(b) \tilde{\rho}_b
\]

\[
\rho \rightarrow \tilde{\rho}_b .
\]

Finally, there may be a further “mental readjustment” of the observer’s knowledge, taking into account details both of the measurement interaction and the observer’s initial state of knowledge. This is enacted via some (formal) unitary operation \( V_b \):

\[
\tilde{\rho}_b \rightarrow \rho_b = V_b \tilde{\rho}_b V_b^\dagger .
\]

Putting the two processes together, one has the same result as the usual picture.

The resemblance between the process in Eq. (58) and the classical Bayes’ rule of Eq. (36) is unmistakable.\footnote{Earlier allusions to a resemblance between quantum collapse and Bayes’ rule can be found in Ref.\cite{51}.} By this way of viewing things, quantum collapse is indeed not such a
violent state of affairs after all. Quantum measurement is nothing more, and nothing less, than a refinement and a readjustment of one’s state of knowledge. More general state changes of the form Eq. (23) come about similarly, but with a further step of coarse-graining (i.e., throwing away information that was in principle accessible).

Let us look at two limiting cases of efficient measurements. In the first, we imagine an observer whose initial state of knowledge \( \rho = |\psi\rangle\langle\psi| \) is a maximal state of knowledge. By this account, no measurement whatsoever can refine that state of knowledge. This follows because, no matter what \( \{E_b\} \) is,

\[
\rho^{1/2}E_b\rho^{1/2} = P(b)|\psi\rangle\langle\psi|.
\] (60)

The only state change that can come about from such a measurement must be purely of the “mental readjustment” sort: We learn nothing new; we just change what we can predict as a consequence of experimental intervention. In particular, when the POVM is an orthogonal set of projectors \( \{\Pi_i = |i\rangle\langle i|\} \) and the state-change mechanism is the von Neumann collapse postulate, this simply corresponds to a readjustment according to the unitary operators

\[
U_i = |i\rangle\langle\psi|.
\] (61)

At the opposite end of things, we can contemplate measurements that have no causal connection at all to the system being measured. This could come about, for instance, by interacting with one side of an entangled pair of systems and using the consequence of that intervention to update one’s knowledge about the other side. In such a case, one can show that the state change is purely of the refinement variety (with no further mental readjustment). For instance, consider a pure state \( |\psi^{AB}\rangle \) whose Schmidt decomposition takes the form

\[
|\psi^{AB}\rangle = \sum_i \sqrt{\lambda_i}|a_i\rangle|b_i\rangle.
\] (62)

An efficient measurement on the \( A \) side of this leads to a state update of the form

\[
|\psi^{AB}\rangle\langle\psi^{AB}| \longrightarrow (A_b \otimes I)|\psi^{AB}\rangle\langle\psi^{AB}|(A_b^\dagger \otimes I).
\] (63)

Tracing out the \( A \) side, then gives

\[
\text{tr}_A\left( A_b \otimes I|\psi^{AB}\rangle\langle\psi^{AB}|A_b^\dagger \otimes I\right) = \sum_{ijk} \sqrt{\lambda_j \lambda_k} \langle a_i|A_b \otimes I|a_j\rangle \langle b_j|a_k\rangle \langle b_k|A_b^\dagger \otimes I|a_i\rangle
\]

\[
= \sum_{ijk} \sqrt{\lambda_j \lambda_k} \langle a_i|A_b^\dagger A_b|a_j\rangle \langle b_j|a_k\rangle |b_k|\langle b_k|
\]

\[
= \sum_{jk} \sqrt{\lambda_j \lambda_k} \langle b_k|U A_b^\dagger A_b U^\dagger U|b_j\rangle |b_k|\langle b_k|
\]

\[
= \sum_{jk} \sqrt{\lambda_j \lambda_k} \langle b_j|U A_b^\dagger A_b U^\dagger U^\dagger |b_k\rangle |b_j| \langle b_k|
\]

\[
= \rho^{1/2}(U A_b^\dagger A_b U^\dagger)^\dagger \rho^{1/2}.
\] (64)
where $\rho$ is the initial quantum state on the $B$ side, $U$ is the unitary operator connecting the $|a_i\rangle$ basis to the $|b_i\rangle$ basis, and $^T$ represents taking a transpose with respect to the $|b_i\rangle$ basis. Since the operators

$$F_b = \left( U A_b^\dagger A_b U^\dagger \right)^T$$

(65)
go together to form a POVM, we indeed have the claimed result.

In summary, the lesson here is that it turns out to be rather easy to think of quantum collapse as a noncommutative variant of Bayes’ rule. In fact it is just in this that one starts to get a feel for (at least) a partial reason for Gleason’s noncontextuality assumption. In the setting of classical Bayes’ conditionalization we have just that: The probability of the transition $p(h) \rightarrow p(h|d)$ is governed solely by the local probability $p(d)$. The transition does not care about how we have partitioned the rest of the potential transitions. That is, it does not care whether $d$ is embedded in a two outcome set $\{d, \neg d\}$ or whether it is embedded in a three outcome set, $\{d, e, \neg d \lor e\}$, etc. Similarly with the quantum case. The probability for a transition from $\rho$ to $\rho_0$ cares not whether our refinement is of the form

$$\rho = P(0)\rho_0 + \sum_{b=1}^{17} P(b)\rho_b$$

or of the form

$$\rho = P(0)\rho_0 + P(18)\rho_{18},$$

(66)
as long as

$$P(18)\rho_{18} = \sum_{b=1}^{17} P(b)\rho_b$$

(67)

What could be a simpler generalization of Bayes’ rule?

Indeed, leaning on that, we can restate the discussion of the “measurement problem” at the beginning of Section 4 in slightly more technical terms. Go back to the classical setting of Eqs. (35) and (37) where an agent has a probability distribution $p(h, d)$ over two sets of hypotheses. Marginalizing over the possibilities for $d$, one obtains the agent’s initial state of knowledge $p(h)$ for the hypothesis $h$. If he gathers an explicit piece of data $d$, he should use Bayes’ rule to update his knowledge about $h$ to $p(h|d)$.

The question is this: Is the transition

$$p(h) \rightarrow p(h|d)$$

(68)
a mystery we should contend with? If someone asked for a physical description of that transition, would we be able to give an explanation? After all, one value for $h$ is true and always remains true: there is no transition in it. One value for $d$ is true and always remains true: there is no transition in it. The only transition is in the knowledge $p(h)$. To put the issue into perspective for the quantum measurement problem, let us ask: Should we not have a detailed theory of how the brain works before we can trust in the validity of Bayes’ rule?

The answer is, “Of course not!” Bayes’ rule, and with it all of probability theory, is an intellectual construct that stands beyond the details of physics. George Boole called probability theory a law of thought \[52\]. Its calculus specifies the optimal way an agent should reason and make decisions when faced with incomplete information. In this way, probability theory is but a generalization of Aristotelian logic\[17\]—a construct very few would

\[17\] In addition to Ref. \[39\], many further materials concerning this point of view can be downloaded from the Probability Theory As Extended Logic web site maintained by G. L. Bretthorst. http://bayes.wustl.edu/.
accept as being tied to the explicit details of the physical world.\footnote{18}

The formal similarities between Bayes’ rule and quantum collapse may be telling us how to finally cut the Gordian knot of the measurement problem. Namely, it may be telling us that it is simply not a problem at all! Indeed, drawing on the analogies between the two theories, one is left with a spark of insight: perhaps the better part of quantum mechanics is simply “law of thought” \footnote{53}. Perhaps the structure of the theory denotes the optimal way to reason and make decisions in light of some fundamental situation, waiting to be ferreted out in a more satisfactory fashion.

This much we know: That “fundamental situation”—whatever it is—must be an ingredient Bayesian probability theory does not have. There must be something to drive a wedge between the two theories. Probability theory alone is too general of a structure. Narrowing it will require input from the world about us.

6 Wither Entanglement?

When two systems, of which we know the states by their respective representatives, enter into temporary physical interaction due to known forces between them, and when after a time of mutual influence the systems separate again, then they can no longer be described in the same way as before, viz. by endowing each of them with a representative of its own. I would not call that one but rather the characteristic trait of quantum mechanics, the one that enforces its entire departure from classical lines of thought. By the interaction the two representatives (or $\psi$-functions) have become entangled.

— Erwin Schrödinger, 1935

Quantum entanglement certainly gets a load of airplay these days. By most accounts it is the main ingredient in quantum information theory and quantum computing \footnote{54}, and it is the main mystery of the quantum foundations \footnote{55}. But what is it? Where does it come from?

The predominant purpose it has served in this paper has been as a kind of background. For it, more than any other ingredient in quantum mechanics, has clinched the issue of “information about what?” in the author’s mind: That information cannot be about a preexisting reality (a hidden variable) unless we are willing to renge on our very reason for rejecting the quantum state’s objective reality in the first place. What I am alluding to here is the conjunction of the Einstein argument reported in Section 3 and the phenomena of the Bell inequality violations by quantum mechanics. Putting those points together gave us that the information symbolized by a $|\psi\rangle$ must be information about the potential consequences of our interventions into the world.

\footnote{18We have, after all, used simple Aristotelian logic in making deductions from all our physical theories to date: from Aristotle’s physics to quantum mechanics to general relativity and even string theory.}
But, now I would like to turn the tables and ask whether the structure of our potential interventions—the POVMs—can tell us something about the origin of entanglement. Could it be that the concept of entanglement is just a minor addition to the much deeper point that measurements correspond to refinements of density operators (i.e., the substance of the two preceding sections)?

The technical translation of this question is, why do we combine systems according to the tensor product rule? There are certainly innumerable ways to combine two Hilbert spaces \( \mathcal{H}_A \) and \( \mathcal{H}_B \) to obtain a third \( \mathcal{H}_{AB} \). We could take the direct sum of the two spaces \( \mathcal{H}_{AB} = \mathcal{H}_A \oplus \mathcal{H}_B \). We could take their Grassmann product \( \mathcal{H}_{AB} = \mathcal{H}_A \wedge \mathcal{H}_B \)[56]. We could take scads of other things. But instead we take their tensor product,

\[
\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B .
\] (69)

Why?

Could it arise from the selfsame considerations we have already made our mainstay—from a noncontextuality property for measurement-outcome probabilities? The answer is yes, and the theorem I am about demonstrate owes much in inspiration to Ref. [57].

Here is the scenario. Suppose we have two quantum systems, and we can make a measurement on each. On the first, we can measure any POVM on the \( d_A \)-dimensional Hilbert space \( \mathcal{H}_A \); on the second, we can measure any POVM on the \( d_B \)-dimensional Hilbert space \( \mathcal{H}_B \). Moreover, suppose we may condition the second measurement on the nature and the outcome of the first, and vice versa. That is to say—walking from \( A \) to \( B \)—we could first measure \( \{E_i\} \) on \( A \), and then, depending on the outcome \( i \), measure \( \{F^i_j\} \) on \( B \). Similarly—walking from \( B \) to \( A \)—we could first measure \( \{F_j\} \) on \( B \), and then, depending on the outcome \( j \), measure \( \{E^j_i\} \) on \( A \). So that we have valid POVMs, we must have

\[
\sum_i E_i = I \quad \text{and} \quad \sum_j F^i_j = I \quad \forall i ,
\] (70)

and

\[
\sum_i E^j_i = I \quad \forall j \quad \text{and} \quad \sum_j F_j = I ,
\] (71)

for these sets of operators. Let us denote by \( S_{ij} \) an ordered pair of operators, either of the form \( (E_i, F^i_j) \) or of the form \( (E^j_i, F_j) \), as appearing above. Let us call a set of such operators \( \{S_{ij}\} \) a locally-measurable POVM tree.

Suppose now that—just as with the POVM-version of Gleason’s theorem in Section 4—the joint probability \( P(i,j) \) for the outcomes of such a measurement should not depend upon which tree \( S_{ij} \) is embedded in: This is essentially the same assumption we made there, but now applied to local measurements on the separate systems. In other words, let us suppose there exists a function

\[
f : \mathcal{E}_{d_A} \times \mathcal{E}_{d_B} \longrightarrow [0, 1]
\] (72)

such that

\[
\sum_{ij} f(S_{ij}) = 1 \quad \text{(73)}
\]

This, one might think, is the very essence of having two systems rather than one—that we can probe them independently.
whenever the $S_{ij}$ satisfy either Eq. (70) or Eq. (71).

Note in particular that Eq. (72) makes no use of the tensor product: The domain of $f$ is the Cartesian product of the two sets $\mathcal{E}_{d_A}$ and $\mathcal{E}_{d_B}$. The notion of a local measurement on the separate systems is enforced by the requirement that the ordered pairs $S_{ij}$ satisfy the side conditions of Eqs. (70) and (71). This, of course, is not the most general kind of local measurement one can imagine—more sophisticated measurements could involve multiple ping-ponging between $A$ and $B$ as in Ref. [58]—but the present restricted class is already sufficient for fixing the probability rule for local measurements.

The theorem is this: If $f$ satisfies Eqs. (72) and (73) for all locally-measurable POVM trees, then there exists a density operator $\tilde{\rho}$ on $\mathcal{H}_A \otimes \mathcal{H}_B$ such that

$$f(E, F) = \text{tr}(\tilde{\rho}(E \otimes F)) .$$

If $\mathcal{H}_A$ and $\mathcal{H}_B$ are defined over the field of complex numbers, then $\tilde{\rho}$ is unique. Uniqueness does not hold, however, if the underlying field is the real numbers.

The proof of this statement is almost a trivial extension of the proof in Section 4. One again starts by showing additivity, but this time in the two variables $E$ and $F$ separately. For instance, for a fixed $E \in E_{d_A}$, define

$$g_E(F) = f(E, F) ,$$

and consider two locally-measurable POVM trees

$$\{(I - E, F_i), (E, G_\alpha)\} \quad \text{and} \quad \{(I - E, F_i), (E, H_\beta)\} ,$$

where $\{F_i\}$, $\{G_\alpha\}$, and $\{H_\beta\}$ are arbitrary POVMs on $\mathcal{H}_B$. Then Eq. (73) requires that

$$\sum_i g_{I-E}(F_i) + \sum_\alpha g_E(G_\alpha) = 1$$

and

$$\sum_i g_{I-E}(F_i) + \sum_\beta g_E(H_\beta) = 1 .$$

From this it follows that,

$$\sum_\alpha g_E(G_\alpha) = \sum_\beta g_E(H_\beta) = \text{const} .$$

That is to say, $g_E(F)$ is a frame function in the sense of Section 4. Consequently, we know that we can use the same methods as there to uniquely extend $g_E(F)$ to a linear functional on the complete set of Hermitian operators on $\mathcal{H}_B$. Similarly, for fixed $F \in \mathcal{E}_{d_B}$, we can define

$$h_F(E) = f(E, F) ,$$

and prove that this function too can be uniquely extended to a linear functional on the Hermitian operators on $\mathcal{H}_A$.

The linear extensions of $g_E(F)$ and $h_F(E)$ can be put together in a simple way to give a full bilinear extension to the function $f(E, F)$. Namely, for any two Hermitian operators
A and $B$ on $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively, let $A = \alpha_1 E_1 - \alpha_2 E_2$ and $B = \beta_1 F_1 - \beta_2 F_2$ be decompositions such that $\alpha_1, \alpha_2, \beta_1, \beta_2 \geq 0$, $E_1, E_2 \in \mathcal{E}_d$, and $F_1, F_2 \in \mathcal{E}_d$. Then define

$$f(A, B) \equiv \alpha_1 g_{E_1}(B) - \alpha_2 g_{E_2}(B).$$

To see that this definition is unique, take any other decomposition $A = \tilde{\alpha}_1 \tilde{E}_1 - \tilde{\alpha}_2 \tilde{E}_2$. Then we have

$$f(A, B) = \tilde{\alpha}_1 g_{\tilde{E}_1}(B) - \tilde{\alpha}_2 g_{\tilde{E}_2}(B) = \tilde{\alpha}_1 f(\tilde{E}_1, B) - \tilde{\alpha}_2 f(\tilde{E}_2, B) = \beta_1 (\tilde{\alpha}_1 f(\tilde{E}_1, F_1) - \tilde{\alpha}_2 f(\tilde{E}_1, F_1)) - \beta_2 (\tilde{\alpha}_1 f(\tilde{E}_2, F_2) - \tilde{\alpha}_2 f(\tilde{E}_2, F_2)) = \beta_1 (\alpha_1 f(E_1, F_1) - \alpha_2 f(E_1, F_1)) - \beta_2 (\alpha_1 f(E_1, F_2) - \alpha_2 f(E_2, F_2)) = \alpha_1 f(E_1, B) - \alpha_2 f(E_2, B) = \alpha_1 g_{E_1}(B) - \alpha_2 g_{E_2}(B),$$

which is as desired.

With bilinearity for the function $f$ established, we have essentially the full story [56, 59]. For, let $\{E_i\}$, $i = 1, \ldots, d_A^2$, be a complete basis for the Hermitian operators on $\mathcal{H}_A$ and let $\{F_j\}$, $j = 1, \ldots, d_B^2$, be a complete basis for the Hermitian operators on $\mathcal{H}_B$. If $E = \sum_i \alpha_i E_i$ and $F = \sum_j \beta_j F_j$, then

$$f(E, F) = \sum_{ij} \alpha_i \beta_j f(E_i, F_j).$$

Define $\tilde{\rho}$ to be a linear operator on $\mathcal{H}_A \otimes \mathcal{H}_B$ satisfying the $(d_A d_B)^2$ linear equations

$$\text{tr}(\tilde{\rho}(E_i \otimes F_j)) = f(E_i, F_j).$$

Such an operator always exists. Consequently we have,

$$f(E, F) = \sum_{ij} \alpha_i \beta_j \text{tr}(\tilde{\rho}(E_i \otimes F_j)) = \text{tr}(\tilde{\rho}(E \otimes F)).$$

Enforcing positivity and normalization for the function $f$ proves the main point of the theorem.

For complex Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$, the uniqueness of $\tilde{\rho}$ comes about because the set $\{E_i \otimes F_j\}$ forms a complete basis for the Hermitian operators on $\mathcal{H}_A \otimes \mathcal{H}_B$. For real Hilbert spaces, however, the analog of the Hermitian operators are the symmetric operators. The dimensionality of the space of symmetric operators on a real Hilbert space $\mathcal{H}_d$ is $\frac{1}{2} d(d + 1)$, rather than the $d^2$ it is for the complex case. This means that in the steps above only

$$\frac{1}{4} d_A d_B (d_A + 1)(d_B + 1)$$

equations will appear in Eq. (84), whereas

$$\frac{1}{2} d_A d_B (d_A d_B + 1)$$

(87)
are needed to uniquely specify a $\rho$. For instance take $d_A = d_B = 2$. Then Eq. (86) gives nine equations, while Eq. (87) requires ten.

Let us emphasize the striking feature of this way of deriving the tensor product rule for combining separate quantum systems: It is built on the very concept of local measurement. There is nothing “spooky” or “nonlocal” about it; there is nothing in it resembling “passion at a distance” [61]. Indeed, one need not even consider probability assignments for the outcomes of measurements of the “nonlocality without entanglement” variety [58] in order to uniquely fix the probability rule. That is—to give an example on $H_3 \otimes H_3$—one need not consider standard measurements like $\{E_b = |\psi_b\rangle\langle\psi_b|\}$, where

$$|\psi_1\rangle = |1\rangle|1\rangle, \quad |\psi_2\rangle = |0\rangle|0 + 1\rangle, \quad |\psi_3\rangle = |0\rangle|0 - 1\rangle, \quad |\psi_4\rangle = |2\rangle|1 + 2\rangle, \quad |\psi_5\rangle = |2\rangle|1 - 2\rangle, \quad |\psi_6\rangle = |1 + 2\rangle|0\rangle, \quad |\psi_7\rangle = |1 - 2\rangle|0\rangle, \quad |\psi_8\rangle = |0 + 1\rangle|2\rangle, \quad |\psi_9\rangle = |0 - 1\rangle|2\rangle$$

with $|0\rangle$, $|1\rangle$, and $|2\rangle$ forming an orthonormal basis on $H_3$, and $|0 + 1\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$, etc. This is a measurement that takes neither the form of Eq. (70) nor (71). It stands out instead in that, even though all its POVM elements are tensor product operators—i.e., they have no quantum entanglement—it still cannot be measured by local means, even with the elaborate ping-ponging strategies mentioned earlier.

Thus, the tensor product rule, and with it quantum entanglement, seems to be more a statement of locality than anything else. It, like the probability rule, is more a product of the structure of the observables—that they are POVMs—combined with noncontextuality. In searching for the secret ingredient to drive a wedge between general Bayesian probability theory and quantum mechanics, it seems that the direction not to look is toward quantum entanglement. Perhaps the trick instead is to dig deeper into the Bayesian toolbox.

7 Unknown Quantum States?

My thesis, paradoxically, and a little provocatively, but nonetheless genuinely, is simply this:

QUANTUM STATES DO NOT EXIST.

The abandonment of superstitious beliefs about the existence of Phlogiston, the Cosmic Ether, Absolute Space and Time, ..., or Fairies and Witches, was an essential step along the road to scientific thinking. The quantum state, too, if regarded as something endowed with some kind of objective existence, is no less a misleading conception, an illusory attempt to exteriorize or materialize the information we possess.

— the ghost of Bruno de Finetti

The hint of a more fruitful direction can be found by trying to make sense of one of the most commonly used phrases in quantum information theory. It is the unknown quantum
Figure 1: What can the term “unknown state” mean if quantum states are taken exclusively to be states of knowledge rather than states of nature? When we say that a system has an unknown state, must we always imagine a further observer whose state of knowledge is symbolized by some $|\psi\rangle$, and it is the identity of the symbol that we are ignorant of?

Figure 1: What can the term “unknown state” mean if quantum states are taken exclusively to be states of knowledge rather than states of nature? When we say that a system has an unknown state, must we always imagine a further observer whose state of knowledge is symbolized by some $|\psi\rangle$, and it is the identity of the symbol that we are ignorant of?

state. There is hardly a paper in quantum information that does not make use of it. Unknown quantum states are teleported [8], protected with quantum error correcting codes [62], and used to check for quantum eavesdropping [63]. The list of uses grows each day. But what can the term possibly mean? In an information-based interpretation of quantum mechanics, it is an overt oxymoron: If quantum states, by their very definition, are states of knowledge and not states of nature, then the state is known by someone—at the very least, by the person who wrote it down.

Thus, if a phenomenon ostensibly invokes the concept of an unknown state in its formulation, that unknown state had better be shorthand for a more basic situation (even if that basic situation still awaits a complete analysis). This means that for any phenomenon using the idea of an unknown quantum state in its description, we should demand that either

1. The owner of the unknown state—a further decision-making agent or observer—be explicitly identified. (In this case, the unknown state is merely a stand-in for the unknown state of knowledge of an essential player who went unrecognized in the original formulation.) Or,

2. If there is clearly no further agent or observer on the scene, then a way must be found to reexpress the phenomenon with the term “unknown state” completely banished from its formulation. (In this case, the end-product of the effort will be a single quantum state used for describing the phenomenon—namely, the state that actually captures the describer’s state of knowledge throughout.)

This Section reports the work of Ref. [17] and [18], where such a project is carried out for the experimental practice of quantum-state tomography [13]. The usual description of
Figure 2: To make sense of quantum tomography, must we go to the extreme of imagining a “man in the box” who has a better description of the systems than we do? How contrived our usage would be if that were so!

tomography is this. A device of some sort, say a nonlinear optical medium driven by a laser, repeatedly prepares many instances of a quantum system, say many temporally distinct modes of the electromagnetic field, in a fixed quantum state $\rho$, pure or mixed [64]. An experimentalist who wishes to characterize the operation of the device or to calibrate it for future use may be able to perform measurements on the systems it prepares even if he cannot get at the device itself. This can be useful if the experimenter has some prior knowledge of the device’s operation that can be translated into a probability distribution over states. Then learning about the state will also be learning about the device. Most importantly, though, this description of tomography assumes that the precise state $\rho$ is unknown. The goal of the experimenter is to perform enough measurements, and enough kinds of measurements (on a large enough sample), to estimate the identity of $\rho$.

This is clearly an example where there is no further player on whom to pin the unknown state as a state of knowledge. Any attempt to find such a missing player would be entirely artificial: Where would the player be placed? On the inside of the device the tomographer is trying to characterize? The only available course is the second strategy above—to banish the idea of the unknown state from the formulation of tomography.

To do this, we once again take our cue from Bayesian probability theory [38, 39, 40]. As emphasized previously, in Bayesian theory probabilities—just like quantum states—are

\[ \begin{align*}
\psi_1 &\quad \psi_2 &\quad \psi_3 &\quad \ldots \\
&\quad &\quad &\quad \\
&\quad &\quad &\quad \\
&\quad &\quad &\quad \\
&\quad &\quad &\quad
\end{align*} \]

20 Placing the player here would be about as respectable as George Berkeley’s famous patch to his philosophical system of idealism. The difficulty is captured engagingly by a limerick of Ronald Knox and its anonymous reply:

There was a young man who said, “God : Must think it exceedingly odd : If he finds that this tree : Continues to be : When there’s no one about in the Quad.” REPLY: “Dear Sir: Your astonishment’s odd. : I am always about in the Quad. : And that’s why the tree : Will continue to be, : Since observed by Yours faithfully, God.”
not objective states of nature, but rather measures of credible belief, reflecting one’s state of knowledge. In light of this, it comes as no surprise that one of the most overarching Bayesian themes is to identify the conditions under which a set of decision-making agents can come to a common belief or probability assignment for a random variable even though their initial beliefs may differ \[40\]. Following that theme is the key to understanding the essence of quantum-state tomography.

Indeed, classical Bayesian theory encounters almost precisely the same problem as our unknown quantum state through the widespread use of the phrase “unknown probability” in its domain. This is an oxymoron every bit as egregious as unknown state.

The procedure analogous to quantum-state tomography in Bayesian theory is the estimation of an unknown probability from the results of repeated trials on “identically prepared systems.” The way to eliminate unknown probabilities from this situation was introduced by Bruno de Finetti in the early 1930s \[67\]. His method was simply to focus on the equivalence of the repeated trials—namely, that what is really of importance is that the systems are indistinguishable as far as probabilistic predictions are concerned. Because of this, any probability assignment \(p(x_1, x_2, \ldots, x_N)\) for multiple trials should be symmetric under permutation of the systems. As innocent as this conceptual shift may sound, de Finetti was able to use it to powerful effect. For, with his representation theorem, he showed that any multi-trial probability assignment that is permutation-symmetric for an arbitrarily large number of trials—de Finetti called such multi-trial probabilities exchangeable—is equivalent to a probability for the “unknown probabilities.”

Let us outline this in a little more detail. In an objectivist description of \(N\) “identically prepared systems,” the individual trials are described by discrete random variables \(x_n \in \{1, 2, \ldots, k\}, n = 1, \ldots, N\), and the probability in the multi-trial hypothesis space is given by an independent identically distributed distribution

\[
p(x_1, x_2, \ldots, x_N) = p_{x_1}p_{x_2} \cdots p_{x_N} = p_1^{n_1}p_2^{n_2} \cdots p_k^{n_k}.
\]  

(89)

The numbers \(p_j\) describe the objective, “true” probability that the result of a single experiment will be \(j\) \((j = 1, \ldots, k)\). The variable \(n_j\), on the other hand, describes the number of times outcome \(j\) is listed in the vector \((x_1, x_2, \ldots, x_N)\). But this description—for the objectivist—only describes the situation from a kind of “God’s eye” point of view. To the experimentalist, the “true” probabilities \(p_1, \ldots, p_k\) will very often be unknown at the outset. Thus, his burden is to estimate the unknown probabilities by a statistical analysis of the experiment’s outcomes.

In the Bayesian approach, however, it does not make sense to talk about estimating a true probability. Instead, a Bayesian assigns a prior probability distribution \(p(x_1, x_2, \ldots, x_N)\) on the multi-trial hypothesis space and uses Bayes’ theorem to update the distribution in the light of his measurement results. The content of de Finetti’s theorem is this. Assuming only that

\[
p(x_{\pi(1)}, x_{\pi(2)}, \ldots, x_{\pi(N)}) = p(x_1, x_2, \ldots, x_N)
\]  

(90)

for any permutation \(\pi\) of the set \(\{1, \ldots, N\}\), and that for any integer \(M > 0\), there is a distribution \(p_{N+M}(x_1, x_2, \ldots, x_{N+M})\) with the same permutation property such that

\[
p(x_1, x_2, \ldots, x_N) = \sum_{x_{N+1}, \ldots, x_{N+M}} p_{N+M}(x_N, x_{N+1}, \ldots, x_{N+M}),
\]  

(91)

35
then \( p(x_1, x_2, \ldots, x_N) \) can be written uniquely in the form
\[
p(x_1, x_2, \ldots, x_N) = \int_{S_k} P(\vec{p}) p_{x_1} p_{x_2} \cdots p_{x_N} \, d\vec{p},
\]
where \( \vec{p} = (p_1, p_2, \ldots, p_k) \), and the integral is taken over the simplex of such distributions
\[
S_k = \left\{ \vec{p} : p_j \geq 0 \text{ for all } j \text{ and } \sum_{j=1}^{k} p_j = 1 \right\}.
\]
Furthermore, the function \( P(\vec{p}) \geq 0 \) is required to be a probability density function on the simplex:
\[
\int_{S_k} P(\vec{p}) \, d\vec{p} = 1.
\]
This representation theorem, the unsatisfactory concept of an unknown probability vanishes from the description in favor of the fundamental idea of assigning an exchangeable probability distribution to multiple trials.

With this representation theorem, the unsatisfactory concept of an unknown probability vanishes from the description in favor of the fundamental idea of assigning an exchangeable probability distribution to multiple trials.

This cue in hand, it is easy to see how to reword the description of quantum-state tomography to meet our goals. What is relevant is simply a judgment on the part of the experimenter—notice the essential subjective character of this “judgment”—that there is no distinction between the systems the device is preparing. In operational terms, this is the judgment that all the systems are and will be the same as far as observational predictions are concerned. At first glance this statement might seem to be contentless, but the important point is this: To make this statement, one need never use the notion of an unknown state—a completely operational description is good enough. Putting it into technical terms, the statement is that if the experimenter judges a collection of \( N \) of the device’s outputs to have an overall quantum state \( \rho^{(N)} \), he will also judge any permutation of those outputs to have the same quantum state \( \rho^{(N)} \). Moreover, he will do this no matter how large the number \( N \) is. This, complemented only by the consistency condition that for any \( N \) the state \( \rho^{(N)} \) be derivable from \( \rho^{(N+1)} \), makes for the complete story.

The words “quantum state” appear in this formulation, just as in the original formulation of tomography, but there is no longer any mention of unknown quantum states. The state \( \rho^{(N)} \) is known by the experimenter (if no one else), for it represents his state of knowledge. More importantly, the experimenter is in a position to make an unambiguous statement about the structure of the whole sequence of states \( \rho^{(N)} \): Each of the states \( \rho^{(N)} \) has a kind of permutation invariance over its factors. The content of the quantum de Finetti representation theorem \([65, 17]\) is that a sequence of states \( \rho^{(N)} \) can have these properties, which are said to make it an exchangeable sequence, if and only if each term in it can also be written in the form
\[
\rho^{(N)} = \int_{D_d} P(\rho) \rho^{\otimes N} \, d\rho,
\]
where
\[
\rho^{\otimes N} = \underbrace{\rho \otimes \rho \otimes \cdots \otimes \rho}_{N\text{-fold tensor product}}
\]
(96)
Here \( P(\rho) \geq 0 \) is a fixed probability distribution over the density operator space \( \mathcal{D}_d \), and
\[
\int_{\mathcal{D}_d} P(\rho) \, d\rho = 1 ,
\]
where \( d\rho \) is a suitable measure.

The interpretive import of this theorem is paramount. For it alone gives a mandate to the term unknown state in the usual description of tomography. It says that the experimenter can act as if his state of knowledge \( \rho^{(N)} \) comes about because he knows there is a “man in the box,” hidden from view, repeatedly preparing the same state \( \rho \). He does not know which such state, and the best he can say about the unknown state is captured in the probability distribution \( P(\rho) \).

The quantum de Finetti theorem furthermore makes a connection to the overarching theme of Bayesianism stressed above. It guarantees for two independent observers—as long as they have a rather minimal agreement in their initial beliefs—that the outcomes of a sufficiently informative set of measurements will force a convergence in their state assignments for the remaining systems \([18]\). This “minimal” agreement is characterized by a judgment on the part of both parties that the sequence of systems is exchangeable, as described above, and a promise that the observers are not absolutely inflexible in their opinions. Quantitatively, the latter means that though \( P(\rho) \) may be arbitrarily close to zero, it can never vanish.

This coming to agreement works because an exchangeable density operator sequence can be updated to reflect information gathered from measurements by another quantum version of Bayes’s rule for updating probabilities \([18]\). Specifically, if measurements on \( K \) systems yield results \( D_K \), then the state of additional systems is constructed as in Eq. (95), but using an updated probability on density operators given by
\[
P(\rho|D_K) = \frac{P(D_K|\rho)P(\rho)}{P(D_K)} .
\]
Here \( P(D_K|\rho) \) is the probability to obtain the measurement results \( D_K \), given the state \( \rho^{\otimes K} \) for the \( K \) measured systems, and
\[
P(D_K) = \int_{\mathcal{D}_d} P(D_K|\rho) \, P(\rho) \, d\rho
\]
is the unconditional probability for the measurement results. For a sufficiently informative set of measurements, as \( K \) becomes large, the updated probability \( P(\rho|D_K) \) becomes highly peaked on a particular state \( \rho_{D_K} \) dictated by the measurement results, regardless of the prior probability \( P(\rho) \), as long as \( P(\rho) \) is nonzero in a neighborhood of \( \rho_{D_K} \). Suppose the two observers have different initial beliefs, encapsulated in different priors \( P_i(\rho) \), \( i = 1, 2 \). The measurement results force them to a common state of knowledge in which any number \( N \) of additional systems are assigned the product state \( \rho_{D_K}^{\otimes N} \), i.e.,
\[
\int P_i(\rho|D_K) \rho^{\otimes N} \, d\rho \rightarrow \rho_{D_K}^{\otimes N},
\]
independent of \( i \), for \( K \) sufficiently large.

This shifts the perspective on the purpose of quantum-state tomography: It is not about uncovering some “unknown state of nature,” but rather about the various observers’ coming
to agreement over future probabilistic predictions. In this connection, it is interesting to note that the quantum de Finetti theorem and the conclusions just drawn from it work only within the framework of complex vector-space quantum mechanics. For quantum mechanics based on real Hilbert spaces, the connection between exchangeable density operators and unknown quantum states does not hold.

A simple counterexample is the following. Consider the $N$-system state

$$\rho^{(N)} = \frac{1}{2} \rho^+ \otimes_N + \frac{1}{2} \rho^- \otimes_N ,$$

where

$$\rho_+ = \frac{1}{2}(I + \sigma_2)$$  and  $$\rho_- = \frac{1}{2}(I - \sigma_2)$$

and $\sigma_1, \sigma_2, \sigma_3$ are the Pauli matrices. In complex-Hilbert-space quantum mechanics, Eq. (101) is clearly a valid density operator: It corresponds to an equally weighted mixture of $N$ spin-up particles and $N$ spin-down particles in the $y$-direction. The state $\rho^{(N)}$ is thus exchangeable, and the decomposition in Eq. (101) is unique according to the quantum de Finetti theorem.

But now consider $\rho^{(N)}$ as an operator in real-Hilbert-space quantum mechanics. Despite its ostensible use of the imaginary number $i$, it remains a valid quantum state. This is because, upon expanding the right-hand side of Eq. (101), all the terms with an odd number of $\sigma_2$’s cancel away. Yet, even though it is an exchangeable density operator, it cannot be written in de Finetti form Eq. (102) using only real symmetric operators. This follows because $i\sigma_2$ cannot be written as a linear combination of $I, \sigma_1, \sigma_3$, while a real-Hilbert-space de Finetti expansion as in Eq. (102) can only contain those three operators. Hence the de Finetti theorem does not hold in real-Hilbert-space quantum mechanics.

In classical probability theory, exchangeability characterizes those situations where the only data relevant for updating a probability distribution are frequency data, i.e., the numbers $n_j$ in Eq. (103). The quantum de Finetti representation shows that the same is true in quantum mechanics: Frequency data (with respect to a sufficiently robust measurement) are sufficient for updating an exchangeable state to the point where nothing more can be learned from sequential measurements. That is, one obtains a convergence of the form Eq. (104), so that ultimately any further measurements on the individual systems will be statistically independent. That there is no quantum de Finetti theorem in real Hilbert space means that there are fundamental differences between real and complex Hilbert spaces with respect to learning from measurement results.

Finally, in summary, let us hang on the point of learning for just a little longer. The quantum de Finetti theorem shows that the essence of quantum-state tomography is not in revealing an “element of reality” but in deriving that various agents (who agree some minimal amount) can come to agreement in their ultimate quantum-state assignments. This is not the same thing as the stronger statement that “reality does not exist.” It is simply that one need not go to the extreme of taking the “unknown quantum state” as being objectively real to make sense of the experimental practice of tomography.

\[21\] Technically, this means any POVM $\{E_b\}$ whose elements span the space of Hermitian operators. See Ref. [17] for details.
One is left with the feeling—an almost salty feeling\footnote{Working under the presumption that no interpretation of quantum mechanics is worth its salt unless it raises as many technical questions as it answers philosophical ones.}—that perhaps this is the whole point to quantum mechanics. That is: Perhaps the missing ingredient for narrowing the structure of Bayesian probability down to the structure of quantum mechanics has been in front of us all along. It finds no better expression than in the taking account of the limitations the physical world poses to our ability to come to agreement.

8 The Oyster and the Quantum

The significance of this development is to give us insight into the logical possibility of a new and wider pattern of thought. This takes into account the observer, including the apparatus used by him, differently from the way it was done in classical physics. In the new pattern of thought we do not assume any longer the detached observer, occurring in the idealizations of this classical type of theory, but an observer who by his indeterminable effects creates a new situation, theoretically described as a new state of the observed system.

Nevertheless, there remains still in the new kind of theory an objective reality, inasmuch as these theories deny any possibility for the observer to influence the results of a measurement, once the experimental arrangement is chosen. Particular qualities of an individual observer do not enter the conceptual framework of the theory.

— Wolfgang Pauli, 1954

A grain of sand falls into the shell of an oyster and the result is a pearl. The oyster’s sensitivity to the touch is the source of one of our most beautiful gems. In the 75 years that have passed since the founding of quantum mechanics, only the last 10 have turned to a view and an attitude that may finally reveal the essence of the theory. The quantum world is sensitive to the touch, and that may well be one of the best things about it. Quantum information science—with its three prongs of quantum information theory, quantum cryptography, and quantum computing—leads the way in telling us how to quantify that sentence. Quantum algorithms can be exponentially faster than classical algorithms. Secret keys can be encoded into physical systems in such a way as to reveal whether information has been gathered about them. The list of triumphs keeps growing.

The key to so much of this has been simply in a change of attitude. This can be seen by going back to almost any older textbook on quantum mechanics: Nine times out of ten, the Heisenberg uncertainty relation is presented in a way that conveys the feeling that we have been short-changed by the physical world.
“Look at classical physics, how nice it is: We can measure a particle’s position and momentum with as much accuracy as we would like. How limiting quantum theory is instead. We are stuck with

$$\Delta x \Delta p \geq \frac{1}{2}\hbar,$$

and there is just nothing we can do about it. The task of physics is just to sober up to this state of affairs and make the best of it.”

How this contrasts with the point of departure of quantum information science! There the task is not to ask what limits quantum mechanics places upon us, but what novel, productive things we can do in the quantum world that we could not have done otherwise. In what ways is the quantum world fantastically better than the classical one?

If one is looking for something “real” in quantum theory, what more direct tack could one take than to look to its technologies? People may argue about the objective reality of the wave function ad infinitum, but few would argue about the existence of quantum cryptography as a solid prediction of the theory. Why not take that or a similar effect as the grounding for what quantum mechanics is trying to tell us about nature?

Let us try to give this imprecise set of thoughts some shape by molding quantum cryptography into the vision built up in the previous sections. For quantum key distribution it is essential to be able to prepare a physical system in one or another quantum state drawn from some fixed nonorthogonal set [63, 68]. These nonorthogonal states are used to encode a potentially secret cryptographic key to be shared between the sender and receiver. The information an eavesdropper seeks is about which quantum state was actually prepared in each individual transmission. What is novel here is that the encoding of the proposed key into nonorthogonal states forces the information-gathering process to induce a disturbance to the overall set of states. That is, the presence of an active eavesdropper transforms the initial pure states into a set of mixed states or, at the very least, into a set of pure states with larger overlaps than before. This action ultimately boils down to a loss of predictability for the sender over the outcomes of the receiver’s measurements and, so, is directly detectable by the receiver (who reveals some of those outcomes for the sender’s inspection). More importantly, there is a direct connection between the statistical information gained by an eavesdropper and the consequent disturbance she must induce to the quantum states in the process. As the information gathered goes up, the necessary disturbance also goes up in a precisely formalizable way [39].

Note the two ingredients that appear in this scenario. First, the information gathering or measurement is grounded with respect to one observer (in this case, the eavesdropper), while the disturbance is grounded with respect to another (here, the sender). In particular, the disturbance is a disturbance to the sender’s previous information—this is measured by her diminished ability to predict the outcomes of certain measurements the legitimate receiver might perform. No hint of any variable intrinsic to the system is made use of in this formulation of the idea of “measurement causing disturbance.”

The second ingredient is that one must consider at least two possible nonorthogonal preparations in order for the formulation to have any meaning. This is because the information gathering is not about some classically-defined observable—i.e., about some unknown hidden variable or reality intrinsic to the system—but is instead about which of the unknown
states the sender actually prepared. The lesson is this: Forget about the unknown preparation, and the random outcome of the quantum measurement is information about nothing. It is simply “quantum noise” with no connection to any preexisting variable.

How crucial is this second ingredient—that is, that there be at least two nonorthogonal states within the set under consideration? We can address its necessity by making a shift in the account above: One might say that the eavesdropper’s goal is not so much to uncover the identity of the unknown quantum state, but to sharpen her predictability over the receiver’s measurement outcomes. In fact, she would like to do this at the same time as disturbing the sender’s predictions as little as possible. Changing the language still further to the terminology of Section 4, the eavesdropper’s actions serve to sharpen her information about the potential consequences of the receiver’s further interventions on the system. (Again, she would like to do this while minimally diminishing the sender’s previous information about those same consequences.) In the cryptographic context, a byproduct of this effort is that the eavesdropper ultimately comes to a more sound prediction of the secret key. From the present point of view, however, the importance of this change of language is that it leads to an almost Bayesian perspective on the information–disturbance problem.

As previously emphasized, within Bayesian probability the most significant theme is to identify the conditions under which a set of decision-making agents can come to a common probability assignment for some random variable in spite of the fact that their initial probabilities differ. One might similarly view the process of quantum eavesdropping. The sender and the eavesdropper start off initially with differing quantum state assignments for a single physical system. In this case it so happens that the sender can make sharper predictions than the eavesdropper about the outcomes of the receiver’s measurements. The eavesdropper, not satisfied with this situation, performs a measurement on the system in an attempt to sharpen those predictions. In particular, there is an attempt to come into something of an agreement with the sender but without revealing the outcomes of her measurements or, indeed, her very presence.

It is at this point that a distinct property of the quantum world makes itself known. The eavesdropper’s attempt to surreptitiously come into alignment with the sender’s predictability is always shunted away from its goal. This shunting of various observer’s predictability is the subtle manner in which the quantum world is sensitive to our experimental interventions.

And maybe this is our crucial hint! The wedge that drives a distinction between Bayesian probability theory in general and quantum mechanics in particular is perhaps nothing more than this “Zing!” of a quantum system that is manifested when an agent interacts with it. It is this wild sensitivity to the touch that keeps our knowledge and beliefs from ever coming into too great of an alignment. The most our knowledge about the potential consequences of our interventions on a system can come into alignment is captured by the mathematical structure of a pure quantum state $|\psi\rangle$. Take all possible information-disturbance curves for a quantum system, tie them into a bundle, and that is the long-awaited property, the input we have been looking for from nature.

Or, at least, that is the speculation. Look at that bundle long and hard and we might just find that it stays together without the help of our tie.
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