A NOTE ON WEIGHTED HOMOGENEOUS SICIAK-ZAHARYUTA EXTREMAL FUNCTIONS

BARBARA DRINOVEC DRNOVŠEK AND RAGNAR SIGURDSSON

Abstract. We prove that for any given upper semicontinuous function \( \varphi \) on an open subset \( E \) of \( \mathbb{C}^n \setminus \{0\} \), such that the complex cone generated by \( E \) minus the origin is connected, the homogeneous Siciak-Zaharyuta function with the weight \( \varphi \) on \( E \), can be represented as an envelope of a disc functional.

Introduction. Let \( \mathcal{L} \) denote the Lelong class on \( \mathbb{C}^n \) and \( \mathcal{L}^h \) the subclass of functions \( u \) which are logarithmically homogeneous. Let \( \varphi : E \to \overline{\mathbb{R}} \) be a function on a subset \( E \) of \( \mathbb{C}^n \) taking values in the extended real line \( \overline{\mathbb{R}} \). The Siciak-Zaharyuta extremal function \( V_{E,\varphi} \) with weight \( \varphi \) is defined by

\[
V_{E,\varphi} = \sup\{ u \in \mathcal{L} ; u|_E \leq \varphi \}.
\]

The homogeneous Siciak-Zaharyuta extremal function \( V_{E,\varphi}^h \) with weight \( \varphi \) is defined similarly with \( \mathcal{L}^h \) in the role of \( \mathcal{L} \). In the special case when \( \varphi = 0 \) we only write \( V_E \) (and \( V_E^h \)) and we call this function the (homogeneous) Siciak-Zaharyuta extremal function for the set \( E \). The function \( V_E \) (\( V_E^h \)) is also called the (homogeneous) pluricomplex Green function for \( E \) with pole at infinity.

Theorem 1. Let \( \varphi : E \to \mathbb{R} \cup \{-\infty\} \) be an upper semicontinuous function on an open subset \( E \) of \( \mathbb{C}^n \setminus \{0\} \). Assume that there exists a function in \( \mathcal{L}^h \) dominated by \( \varphi \) on \( E \). Then the largest logarithmically homogeneous function \( \mathcal{C}E \to \mathbb{R} \cup \{-\infty\} \) dominated by \( \varphi \) on \( E \) is upper semicontinuous on \( \mathbb{C}^*E \) and it is of the form \( \log \varrho_{E,\varphi} \), where

\[
\varrho_{E,\varphi}(z) = \inf\{|\lambda|e^{\varphi(z/\lambda)} ; \lambda \in \mathbb{C}^*, z/\lambda \in E\}, \quad z \in \mathbb{C}^*E.
\]

If \( \mathbb{C}^*E \) is connected, then for every \( z \in \mathbb{C}^n \)

\[
V_{E,\varphi}^h(z) = \inf\left\{ \int_T \log \varrho_{E,\varphi}(f_1,\ldots,f_n) \, d\sigma ; f \in \mathcal{O}(\overline{D},\mathbb{P}^n), f = [f_0 : \cdots : f_n], f(T) \subset \mathbb{C}^*E, f_0(0) = 1, (f_1(0),\ldots,f_n(0)) = z \right\}.
\]

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If $CE = \mathbb{C}^n$, then for every $z \in \mathbb{C}^n$

\begin{equation}
V_{E,\varphi}^n(z) = \inf \left\{ \int_T \log q_{E,\varphi} \circ f \, d\sigma ; f \in O(\mathbb{D}, \mathbb{C}^n), f(0) = z \right\}.
\end{equation}

A disc envelope formula is a formula where the values of a function $F$ defined on a complex space $X$ with values on the extended real line $\mathbb{R}$ are given as $F(z) = \inf \{ H(f) ; f \in B(z) \}$, where $H$ is disc functional, i.e., a function defined on some subset $\mathcal{A}$ of $O(\mathbb{D}, X)$, the set of analytic discs in $X$, with values on $\mathbb{R}$, $B$ is a subclass of $\mathcal{A}$, and $B(z)$ consists of all of $f \in B$ with center $z = f(0)$.

The formula (2) is an example of a disc envelope formula, where $\mathcal{A}$ consists of all closed analytic discs with value in the projective space, i.e., elements $f$ in $O(\mathbb{D}, \mathbb{P}^n)$ which map the unit circle $T$ into $\mathbb{C}^*E$, $H(f)$ is the integral, and $B$ is the subset of $\mathcal{A}$ consisting of discs with $f_0(0) = 1$. We identify a point $[1 : z] \in \mathbb{P}^n$ with the point $z \in \mathbb{C}^n$.

For general information on the Siciak-Zaharyuta extremal function see Siciak [8, 9, 10, 11, 12] and Zaharyuta [13]. The first disc envelope formula for $V_E$ was proved by Lempert in the case when $E$ is an open convex subset of $\mathbb{C}^n$ with real analytic boundary. (The proof is given in Momm [5, Appendix].) Lárusson and Sigurdsson [2] proved disc envelope formulas for $V_E$ for open connected subsets $E$ of $\mathbb{C}^n$. Magnusson and Sigurdsson [4] generalized this result and obtained a disc formula for $V_{E,\varphi}$ in the case when $\varphi$ is an upper semicontinuous function on an open connected subset $E$ of $\mathbb{C}^n$. Drnovšek and Forstnerič [1] proved disc envelope formulas for $V_E$ for open subsets $E$ of an irreducible and locally irreducible algebraic subvariety of $\mathbb{C}^n$. Recently, Magnusson [3] established disc envelope formulas for the global extremal function in projective space.

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Notation. Let $\mathbb{D}$ denote the unit disc in $\mathbb{C}$, $T$ the unit circle, and $\sigma$ the arc length measure on $T$ normalized to 1. An analytic disc is a holomorphic map $f : \mathbb{D} \to X$, where $X$ is some complex space. We let $O(\mathbb{D}, X)$ denote the set of all analytic discs. We say that the disc is closed if it extends as a holomorphic map to some neighbourhood of the closed unit disc $\overline{\mathbb{D}}$ with values in $X$ and we let $O(\overline{\mathbb{D}}, X)$ denote the set of all closed analytic discs in $X$. The point $z = f(0) \in X$ is called the center of $f$.

For a subset $X$ of $\mathbb{C}^n$ we let $\mathcal{USC}(X)$ denote the set of all upper semicontinuous functions on $X$, and for open subset $U$ of $\mathbb{C}^n$ we denote by $\mathcal{PSH}(U)$ the set of all plurisubharmonic functions on $U$. The Lelong class $\mathcal{L}$ consists of all $u \in \mathcal{PSH}(\mathbb{C}^n)$ such that $u - \log^+ |\cdot|$ is bounded above and $\mathcal{L}^h$ is the
subclass of all logarithmically homogeneous functions, i.e., functions satisfying \( u(\lambda z) = u(z) + \log |\lambda| \) for \( \lambda \in \mathbb{C}^* \) and \( z \in \mathbb{C}^n \). Observe that every such function takes the value \(-\infty\) at the origin. For every subset \( E \) of \( \mathbb{C}^n \) we set \( \mathcal{C}E = \{ \lambda z ; \lambda \in \mathbb{C}, z \in E \} \), \( \mathcal{C}^*E = \{ \lambda z ; \lambda \in \mathbb{C}^*, z \in E \} \) and we call \( \mathcal{C}E \) the complex cone generated by \( E \). Note that complex cones are suitable sets for the domains of definition of logarithmically homogeneous functions.

Let \( \mathbb{P}^n \) denote the \( n \)-dimensional projective space, \( \pi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}^n \) the natural projection, \( (z_0: \cdots : z_n) \mapsto [z_0 : \cdots : z_n] \), and identify \( \mathbb{C}^n \) with the subset of all \( [z_0 : \cdots : z_n] \) with \( z_0 \neq 0 \) and, in particular, the point \( z \in \mathbb{C}^n \) with \( [1 : z] \in \mathbb{P}^n \). The hyperplane at infinity is \( H_\infty = \pi(\mathbb{C}^n \setminus \{0\}) \), where \( Z_0 \) is the hyperplane in \( \mathbb{C}^{n+1} \) defined by the equation \( z_0 = 0 \). Then \( \mathbb{P}^n = \mathbb{C}^n \cup H_\infty \).

**Review of a few results.** Assume that \( \psi : X \to \mathbb{R} \cup \{-\infty\} \) is a measurable function on a subset \( X \) of \( \mathbb{C}^n \), such that there is \( u \in \mathcal{L} \) satisfying \( u|X \leq \psi \). It is an easy observation that a function \( u \in \mathcal{PSH}(\mathbb{C}^n) \) is in \( \mathcal{L} \) if and only if the function

\[
(z_0, \ldots, z_n) \mapsto u(z_1/z_0, \ldots, z_n/z_0) + \log |z_0|
\]

extends as a plurisubharmonic function from \( \mathbb{C}^{n+1} \setminus Z_0 \) to \( \mathbb{C}^{n+1} \setminus \{0\} \). Let \( v \) denote this extension. Take \( f = [f_0 : \cdots : f_n] \in \mathcal{O}(\mathbb{B}, \mathbb{P}^n) \) with \( f_0(0) = 1 \), \((f_1(0), \ldots, f_n(0)) = z \), satisfying \( f(T) \subset X \), and define \( \tilde{f} = (f_0, \ldots, f_n) \in \mathcal{O}(\mathbb{B}, \mathbb{C}^{n+1} \setminus \{0\}) \). Then the subaverage property of \( v \circ \tilde{f} \) and the Riesz representation formula applied to \( \log |f_0| \) give (see [4, p. 243])

\[
u(z) = u(f_1(0), \ldots, f_n(0)) + \log |f_0(0)| = v \circ \tilde{f}(0)
\]

\[
\leq \int_\mathbb{T} u(f_1/f_0, \ldots, f_n/f_0) \, d\sigma + \int_\mathbb{T} \log |f_0| \, d\sigma
\]

\[
\leq \int_\mathbb{T} \psi(f_1/f_0, \ldots, f_n/f_0) \, d\sigma - \sum_{a \in f^{-1}(H_\infty)} m_{f_0}(a) \log |a|.
\]

For an open connected \( X \subset \mathbb{C}^n \) and \( \psi \in \mathcal{USC}(X) \), Magnússon and Sigurdsson [4, Theorem 2] proved that for every \( z \in \mathbb{C}^n \)

\[
V_{X,\psi}(z) = \inf \left\{ -\sum_{a \in f^{-1}(H_\infty)} m_{f_0}(a) \log |a| + \int_\mathbb{T} \psi(f_1/f_0, \ldots, f_n/f_0) \, d\sigma \right\}
\]

\[
f \in \mathcal{O}(\mathbb{B}, \mathbb{P}^n), \ f(T) \subset X, \ f_0(0) = 1, \ (f_1(0), \ldots, f_n(0)) = z \}
\]

Our main result, Theorem [1] will follow from this formula and the following

**Proposition 2.** Let \( \varphi : E \to \mathbb{R} \cup \{-\infty\} \) be a function on a subset \( E \subset \mathbb{C}^n \setminus \{0\} \) such that there exists \( u \in \mathcal{L}^h \) satisfying \( u|E \leq \varphi \). Let \( \hat{\varphi} : \mathcal{C}E \to \mathbb{R} \cup \{-\infty\} \) be the supremum of all logarithmically homogeneous functions on \( \mathcal{C}E \) dominated by \( \varphi \) on \( E \). Then the following hold:
(i) \( \tilde{\phi} \) is logarithmically homogeneous on \( CE \) and for every \( z \in C^*E \)

\[ \phi(z) = \inf \{ \varphi(\lambda z) - \log |\lambda| ; \lambda \in C^* \text{ and } \lambda z \in E \}, \]

(ii) \( V_{E,\varphi}^h = V_{E,\tilde{\phi}}^h = V_{C^*E,\tilde{\phi}}^h \).

If, in addition to the above, \( C^*E \) is nonpluripolar and \( \varphi \in USC(E) \) then

(iii) \( \tilde{\phi} \in USC(C^*E) \) and \( V_{E,\tilde{\phi}}^h = V_{C^*E,\tilde{\phi}}^h \),

(iv) if \( CE = C^n \), then \( \tilde{\phi} \in USC(C^n) \) and

\[ V_{E,\tilde{\phi}}^h = \sup \{ u \in PSH(C^n) ; u \leq \tilde{\phi} \}. \]

Proof. (i) It is easy to see that the supremum of any family of logarithmically homogeneous functions defined on a complex cone is a logarithmically homogeneous function provided the family is bounded from above at any point of the cone. Take \( z \in C^*E \) and choose \( \lambda \in C^* \) such that \( \lambda z \in E \). For any logarithmically homogeneous function \( u \) on \( CE \) dominated by \( \varphi \) on \( E \) we have

\[ u(z) = u(\lambda z) - \log |\lambda| \leq \varphi(\lambda z) - \log |\lambda| \]

which implies that the family is bounded from above at \( z \). Since all logarithmically homogeneous functions take the value \(-\infty\) at the origin the family is bounded from above at any point of the cone.

Let \( \psi \) denote the function on \( C^*E \) whose value at \( z \) is given by the right hand side of the equation (5). For a logarithmically homogeneous function \( u \) on \( CE \), dominated by \( \varphi \) on \( E \), we have \( u(z) \leq \varphi(\lambda z) - \log |\lambda| \) for any \( \lambda \in C^* \) such that \( \lambda z \in E \) by (i). Taking infimum over all \( \lambda \in C^* \) with \( \lambda z \in E \) shows that \( u \leq \psi \) on \( C^*E \). Hence \( \tilde{\phi} \leq \psi \) on \( C^*E \). To prove the converse inequality note that

\[ (i) \quad \psi(\mu z) = \inf \{ \varphi(\lambda \mu z) - \log |\lambda| ; \lambda \in C^* \text{ and } \lambda \mu z \in E \} \]
\[ = \inf \{ \varphi(\lambda \mu z) - \log |\lambda \mu| ; \lambda \in C^* \text{ and } \lambda \mu z \in E \} + \log |\mu| \]
\[ = \psi(z) + \log |\mu| \]

for any \( z \in C^*E \) and \( \mu \in C^* \) thus the map \( \psi \) is logarithmically homogeneous. Since \( \psi \leq \varphi \) on \( E \) we get \( \psi \leq \tilde{\phi} \).

(ii) Since \( \varphi \geq \tilde{\phi} \) on \( E \) and \( E \subset C^*E \) we have \( V_{E,\varphi}^h \geq V_{E,\tilde{\phi}}^h \geq V_{C^*E,\tilde{\phi}}^h \). For proving the two equalities we take \( u \in L^h \) with \( u|E \leq \varphi \). By (i) we obtain \( u \leq \tilde{\phi} \) on \( C^*E \) which implies \( V_{C^*E,\tilde{\phi}}^h \geq V_{E,\varphi}^h \).

(iii) To prove that \( \tilde{\phi} \) is upper semicontinuous take \( z_0 \in C^*E \) and \( c > \tilde{\phi}(z_0) \).
We need to show that \( c > \tilde{\phi}(z) \) for all \( z \) in some neighbourhood \( U \) of \( z_0 \). We choose \( \lambda_0 \in C^* \) such that \( \lambda_0 z_0 \in E \) and such that \( c > \varphi(\lambda_0 z_0) - \log |\lambda_0| \).
Since \( \varphi \in USC(E) \) there exists an open neighbourhood \( U \) of \( z_0 \) such that \( \lambda_0 z \in E \) and \( c > \varphi(\lambda_0 z) - \log |\lambda_0| \) for all \( z \in U \). By (i) we have \( c > \tilde{\phi}(z) \) for all \( z \in U \).

Since \( L^h \subset L \) we have \( V_{C^*E,\tilde{\phi}}^h \leq V_{C^*E,\tilde{\phi}}^h \). For proving the opposite inequality we take \( u \in L \) such that \( u \leq \tilde{\phi} \) on \( C^*E \). Then \( u(\lambda z) - \log |\lambda| \leq \tilde{\phi}(\lambda z) - \log |\lambda| = \tilde{\phi}(z) \) for all \( z \in C^*E \) and \( \lambda \in C^* \). Let \( v \) be the upper
semicontinuous regularization of the function \(\sup\{u(\lambda \cdot) - \log |\lambda|; \lambda \in \mathbb{C}^*\}\) on \(\mathbb{C}^n\). We have \(u \leq v \leq \tilde{\varphi}\) on \(\mathbb{C}^*E\) and since \(\mathbb{C}^*E\) is nonpluripolar and \(\tilde{\varphi}\) is locally bounded above on \(\mathbb{C}^*E\), we have \(v \in \mathcal{L}\). A similar calculation as in (7) shows that \(v\) is logarithmically homogeneous, which proves the opposite inequality.

(iv) The fact that \(\tilde{\varphi}\) is upper semicontinuous at 0 easily follows from the fact that \(\tilde{\varphi}\) is bounded from above on the unit sphere and that it is logarithmically homogeneous. By (iii) we get \(V_{E,\tilde{\varphi}}^h = V_{\mathbb{C}^*E,\tilde{\varphi}}\) and it is easy to see that in the case \(\mathbb{C}E = \mathbb{C}^n\) the latter equals \(V_{\mathbb{C}^n,\tilde{\varphi}}\).

Let \(P_{\tilde{\varphi}}\) denote the function whose value at \(z\) is given by the right hand side of the equation. Since \(\mathcal{L} \subset \mathcal{PSH}(\mathbb{C}^n)\) it follows \(V_{\mathbb{C}^n,\tilde{\varphi}} \leq P_{\tilde{\varphi}}\). To prove the opposite inequality, it is enough to show that \(P_{\tilde{\varphi}} \in \mathcal{L}\). Since \(\tilde{\varphi} \in \mathcal{USC}(\mathbb{C}^n)\) it follows that \(P_{\tilde{\varphi}}\) is the largest plurisubharmonic function on \(\mathbb{C}^n\) dominated by \(\tilde{\varphi}\). By upper semicontinuity the map \(\tilde{\varphi}\) is bounded from above on the unit sphere in \(\mathbb{C}^n\) by some constant \(M \in \mathbb{R}\). Since \(\tilde{\varphi}\) is logarithmically homogeneous we get

\[
P_{\tilde{\varphi}}(lz) \leq \tilde{\varphi}(lz) \leq \log |\lambda| + M = \log |lz| + M
\]

for any \(z \in \mathbb{C}^n\), \(|z| = 1\), and \(\lambda \in \mathbb{C}^*\). It follows that \(P_{\tilde{\varphi}} \in \mathcal{L}\). \(\square\)

**Proof of Theorem 1.** By Proposition 2 the largest logarithmically homogeneous function \(\tilde{\varphi}: \mathbb{C}E \to \mathbb{R} \cup \{-\infty\}\) dominated by \(\varphi\) on \(E\) is upper semicontinuous on \(\mathbb{C}^*E\) and \(q_{E,\varphi} = e^{\tilde{\varphi}(z)} = \inf\{|\lambda|e^{\varphi(z)}; \lambda \in \mathbb{C}^*, z/\lambda \in E\}\) which proves (1).

If we take \(X = \mathbb{C}^*E\) and \(\psi = \tilde{\varphi}\) in (1), then logarithmic homogeneity of \(\tilde{\varphi}\) on \(\mathbb{C}^*E\) implies that

\[
\int_\mathcal{T} \tilde{\varphi}(f_1/f_0, \ldots, f_n/f_0) \, d\sigma = \int_\mathcal{T} \tilde{\varphi}(f_1, \ldots, f_n) \, d\sigma - \int_\mathcal{T} \log |f_0| \, d\sigma.
\]

If \(f_0(0) = 1\), then the Riesz representation formula gives

\[
\sum_{a \in f^{-1}(H_\infty)} m_{f_0}(a) \log |a| + \int_\mathcal{T} \log |f_0| \, d\sigma = 0,
\]

which implies that the right hand side of (1) reduces to

\[
V_{\mathbb{C}^*E,\tilde{\varphi}}(z) = \inf \left\{ \int_\mathcal{T} \tilde{\varphi}(f_1, \ldots, f_n) \, d\sigma; f \in \mathcal{O}(\overline{\mathbb{B}}, \mathbb{P}^n), \right. \\
\left. f(\mathcal{T}) \subset \mathbb{C}^*E, f_0(0) = 1, (f_1(0), \ldots, f_n(0)) = z \right\},
\]

thus (2) follows from Proposition 2 (iii).

If \(\mathbb{C}E = \mathbb{C}^n\) then Proposition 2 (iv) and Poletsky theorem [6, 7] imply

\[
V_{E,\varphi}^h = \sup\{u \in \mathcal{PSH}(\mathbb{C}^n); u \leq \tilde{\varphi}\} = \inf \left\{ \int_\mathcal{T} \log q_{E,\varphi} \circ f \, d\sigma; f \in \mathcal{O}(\overline{\mathbb{B}}, \mathbb{C}^n), f(0) = z \right\}
\]

which proves (3). \(\square\)
**Observation.** In the special case $\varphi = 0$ we write $\rho_E$ for $\rho_{E,\varphi}$. The function $\rho_E$ is absolutely homogeneous of degree 1, i.e., $\rho_E(z\zeta) = |z|\rho_E(\zeta)$. Thus, if $E$ is a balanced domain, i.e., $\overline{E} = E$, then $\rho_E$ is its Minkowski function.

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Faculty of Mathematics and Physics, University of Ljubljana, Institute of Mathematics, Physics and Mechanics, Jadranska 19, 1000 Ljubljana, Slovenia

E-mail address: barbara.drinovec@fmf.uni-lj.si

Department of Mathematics, School of Engineering and Natural Sciences, University of Iceland, IS-107 Reykjavík, Iceland

E-mail address: ragnar@hi.is