The way in which different types of dynamics unfold in complex networks is intrinsically related to the propagation of activation along nodes, which is strongly affected by the network connectivity. In this work we investigate to which extent a time-varying signal emanating from a specific node is modified as it diffuses, at the equilibrium regime, along uniformly random (Erdős-Rényi) and scale-free (Barabási-Albert) networks. The degree of preservation is quantified in terms of the Pearson cross-correlation between the original signal and the derivative of the signals appearing at each node along time. Several interesting results are reported. First, the fact that quite distinct signals are typically obtained at different nodes in the considered networks implies mean-field approaches to be completely inadequate. It has also been found that the peak and lag of the similarity time-signatures obtained for each specific node are strongly related to the respective distance between that node and the source node. Such a relationship tends to decrease with the average degree of the networks. Also, in the case of the lag, a less intense relationship is verified for scale-free networks. No relationship was found between the dispersion of the similarity signature and the distance to the source.

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I. INTRODUCTION

One of the most important current issues in complex systems research regards the identification of relationships between the connectivity structure of a given network and the properties of different types of dynamics (e.g. synchronization, Ising and integrate-and-fire) unfolding on that network, giving rise to the so-called structure-dynamics paradigm. The present work focuses on the investigation of the alterations of activation signals as they diffuse, at the equilibrium regime, along networks. More specifically, a given node is selected as the source of the activation, which is subsequently diffused among the neighboring nodes, allowing the signals arising at each node to be compared to the original activation. The choice of such a linear dynamics monitored with respect to pairs of nodes (i.e. the source and the node where the signal is observed) is justified for the following reasons: (i) as diffusion plays an important role in many linear systems, investigations of signal diffusion can provide valuable initial insights about a large number of natural and artificial systems; (ii) because diffusion dynamics is involved in many non-linear dynamics (e.g. reaction-diffusion), such studies can yield preliminary insights about those more sophisticated dynamics; and (iii) the consideration of the activations at individual nodes provides a much more comprehensive characterization of the structure-dynamics relationship than typical mean-field approximations, which are only valid when the activations are very similar amongst nodes.

The conservation along time of the activation injected into the network implies the signals at the nodes to progressively increase. In order to avoid this trend, we consider the first time derivative of the signals at each node, which guarantees null signal average. Thus, the transformation undergone along time as the activation is transported from the source to a given node $i$ can be quantified in terms of the Pearson correlation coefficient between the derivative of the signal arising at that node and the original activation signal.

The Pearson correlation, instead of the traditional correlation or covariance, is adopted here in order to normalize the magnitude of the obtained similarity signals, which becomes comprised between -1 and 1. This also removes the effect of the respective magnitudes of the two signals being compared. The similarity between the observed signal and the original signal injected at the source allows several interpretations. Its maximum absolute value (peak) reflects how intensely the original signal appears within the observed node. At the same time, the time where such a peak appears indicates the relative delay of the observed signal with respect to the source. Observe that a high value of peak magnitude does not necessarily mean that the original signal is accurately reproduced at the observed node, as large dispersions around the peak suggests cluttered versions of the original signal. Therefore, we also quantify such a dispersion in terms of the entropy of the Pearson correlation.

In order to provide a first glimpse about how the network structure influences the diffusion of time signals, we briefly discuss a simple case example. Figure 3(a-b) shows two networks receiving activation from the respective source node (node number 1 in both cases). The activation consists of just a single Kronecker delta of intensity 2 at time step $t = 1$. For simplicity’s sake, we
assume that the signal diffusion is related to self-avoiding random walks, a non-linear dynamics closely related to traditional random walks. Let us focus on the activation arising at node 3. In Figure 3(a), the signal arising at that node will consist simply of the original signal delayed by 1 time-step. However, because of the additional path connecting nodes 1 and 3 in the network in Figure 3(b), the signal resulting at node 3 will be a linear combination (2 terms) of delayed versions of the original signal.

It is clear from the example above that the effect of the network structure on the propagation of the activation signals along the network is strongly related to its topology, in this case to the number of paths of different lengths existing between the source and each other individual node. Therefore, every network characterized by varying path statistics cannot be properly described or analyzed in terms of mean-field approximations of the signals taken along the whole network. Though implying in different dynamics, where portions of the signal are send backwards at each step, and at the equilibrium regime, the linear diffusion (traditional random walks) studied in this work can be expected to be affected in similar ways by the network structure.

The current work investigates the propagation of time-signals through Erdős-Rényi (ER) and Barabási-Albert (BA) complex networks, at the equilibrium regime, as implied by the linear dynamical process of random walks. The signals observed at each node are differentiated along time and compared to the original activation by using the Pearson correlation between those signals. Three related measurements are taken into account: the magnitude of the correlation peak, its position along time (lag), as well as the correlation dispersion (quantified in terms of entropy). These experiments are performed with respect to ER and BA networks with fixed size (200 nodes) and two average degrees, namely 2 and 4. The consideration of these two theoretical complex network models allows us to investigate how different types of connectivity affect the diffusion of the time-varying signals. A number of interesting results are obtained and discussed, including the identification of well-defined relationships between the peak magnitude, lag and dispersion with the respective distance to the source node.

This work starts by presenting the basic concepts related to complex networks representation and characterization. A complex network is a graph exhibiting a particularly complex structure. The current work focuses on weighted, undirected networks. A network of this type can be fully represented in terms of its adjacency matrix $K$, such that each undirected edge between nodes $i$ and $j$, with $i, j \in \{1, 2, \ldots, N\}$, implies $K(i, j) = K(j, i) = 1$, with $K(i, j) = K(j, i) = 0$ being enforced otherwise. Observe that the adjacency matrix $K$ has dimension $N \times N$. The degree of a node $i$, hence $k(i)$, corresponds to the number of edges attached to it. Two nodes are adjacent if they share different extremities of an edge. Two edges are adjacent if they share a single node. A walk is a sequence of adjacent edges or nodes. A path is a walk where nodes are edges cannot be repeated. The length of a path is henceforth understood to mean the number of its constituent edges. The shortest path between two nodes corresponds to the path comprised between those nodes which contains the smallest number of edges. We consider two theoretical models of networks, namely the uniformly random Erdős-Rényi and the scale-free Barabási-Albert structures (e.g. [1, 2, 3]), which are generated as described in [2].

The diffusion of activations in a complex network can be effectively obtained in terms of its respective transition matrix $S$, which can be obtained from the adjacency
forcing signal arriving at node \(i\) with probability in this work are uniformly random, Bernoulli realizations

zeros and then changing each of the zeros to one with

tween two signals. Let work, we resort to the Pearson correlation coefficient be-
the same length can be quantified in several ways. In this
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because we are using discrete time signals
activation implied by traditional random walks, we take
each node as a consequence of the conservation of the
activations along time which will be otherwise observed at

The similarity between any two time-varying signals of
the same length can be quantified in several ways. In this
work, each of the nodes in the network, henceforth called the source, receives a non-zero forcing signal along time.

All activation signals propagated from the source node in this work are uniformly random, Bernoulli realizations with probability \(\alpha\). Thus, a forcing signal \(s(t)\) of duration \(L\) can be generated by starting with a vector of \(L\) zeros and then changing each of the zeros to one with fixed probability \(\alpha\). During the investigations reported in Section III of this work, 8 periods of the random input signal are fed into the system, and the Pearson correlation is calculated only for the penultimate period, when the dynamics is already at the equilibrium regime.

In order to remove the progressive increase of the activations along time which will be otherwise observed at each node as a consequence of the conservation of the activation implied by traditional random walks, we take into account in our analysis the first time derivative of those signals. Because we are using discrete time signals with unit time step, their time derivative can be obtained by subtracting the value of the signal at the current time from the value at the previous time.

The similarity between any two time-varying signals of the same length can be quantified in several ways. In this work, we resort to the Pearson correlation coefficient between two signals. Let \(a(t)\) and \(b(t)\), with \(t = 1, 2, \ldots, L\), be two time-varying signals. First we standardize each of these signals. In the case of signal \(a(t)\), its standardization can be calculated by making, for each \(t\),

\[ A(t) = \frac{a(t) - \langle a \rangle}{\sigma_a}, \quad (4) \]

where \(\langle a \rangle\) is the time-average of \(a(t)\) and \(\sigma_a\) is the standard deviation of the values of \(a(t)\). The Pearson correlation coefficient between the two signals can now be calculated as

\[ P_{a,b}(t) = \frac{1}{L-1} \sum_{i=1}^{L-t+1} A(i)B(i + t - 1) \quad (5) \]

The Pearson correlation coefficient resulting signal, i.e. \(P(t)\), has values which are necessarily between -1 and 1. Values close to 1 indicate that one of the signals is almost completely contained into the other signal at that lag \(t\). Contrariwise, values close to 0 mean that the two signals are quite different for that lag. Values near to -1 indicate that the a similar version of the original signal appears with negative sign. Observe also that the preliminary standardizations of both signals ensure that the correlation becomes invariant to multiplications of any of the signals by a constant, as well as respective additions with a constant. The Pearson correlation coefficients between two time-varying signals of length \(L\) is also a time-varying signal of length \(L\), which in this work is called similarity signature.

In order to characterize the similarity between the original signal and the derivatives of the signals arising at each of the nodes in a given network, we resort to the following three features extracted from the Pearson correlation signal: (i) the maximum absolute value ('peak') of the Pearson correlation, which is proportional to the maximum overlap between the two signals being compared; (ii) the distance from the time where the peak occurs and the origin of the respective period, which corresponds to the delay in which the maximum similarity is obtained; and (iii) the entropy of a whole Pearson correlation period, which quantifies how dispersed the identified signal is. The latter measurement is obtained by understanding the Pearson correlation period as a probability density (it is therefore normalized so that its overall sum is 1) and using the classical definition of entropy. Thus, low entropy indicates that the derivative of the signal at the respective node contains a relatively uncluttered version of the original signal within itself.

### A. A Simple Example of Signal Diffusion

Let us illustrate the above concepts with respect to the simple network shown in Figure 2. This network is composed of four main branches deriving from node 1, each one representing a different type of connectivity: (a) a chain of nodes extending from node 1 to node 4; (b) three alternative paths with different lengths between node 1 and 8; (c) a cycle with 6 edges including node 15; and (d) node 1 connected to a hub (node 18).

Figure 3 shows three periods (each with \(L = 200\) time steps) of a random signal with \(\alpha = 0.05\) (a), injected at node 1 of the network in Figure 2 as well as the time derivatives of the signals appearing at nodes 4, 8, 15 and 18, followed by the respective similarity signatures with
the injected signal, as quantified by the Pearson correlation. It can be observed from all the signal derivatives that they are composed by a linear combination of a basic signal packet corresponding to the derivative of the system response to a single impulse. In other words, the signal derivative at a given node can be obtained by convolving the original signal with the derivative of the system response obtained for that node. Observe also that the latter signal involves alternance of positive and negative values, so that their linear combination can lead to increase or decrease of the resulting signal, depending on the interval between successive input pulses. As a consequence of the small size of the network in Figure 2, the transient dynamics can hardly be discerned and is limited to the first period. Because of the finite duration of the signals, the last period in the Pearson coefficients shows a decay as it approaches the final time steps. Observe also that the Pearson correlation coefficients (indicated as ‘Whole Pearson’ in the figure) for each node always exhibit a peak near the beginning of each period. The magnitude and lag, measured with respect to the beginning of each period, of the peaks resulted similar for nodes 4, 15 and 18 (just below 0.04) and more than doubled (0.085) for node 8. The lags obtained for nodes 4, 8, 15 and 18 were 2, 1, 3, and 2 time steps, respectively. Except for node 18, these lags tended to reflect the distance from the respective nodes to the source of activation. Another important aspect of the similarities, quantified in terms of the Pearson correlation coefficients, concerns the dispersion of activations around the peak, a feature which is strongly affected by the dispersion of the system impulse response. Largest dispersions can be observed for nodes 15 and 18, followed by nodes 4 and 8. The smaller dispersion and higher magnitude of the peak obtained for the latter case are to a substantial extent a consequence of the path of length 2 edges interconnecting node 8 to the source, as well as the fact that more activation is driven from the source to that node through the three paths going from the source to node 8.

Though simple, the previous example allows a number of preliminary insights about how the connectivity structure of a network can affect the diffusion of the signal emanating from the source. The most important effects of the network structure on the signal arising at a given node include: (i) the tendency of the peak lag to reflect the distance from node \( i \) to the source; (ii) the tendency of cycles and hubs to imply in more disperse and longer system impulse response at node \( i \) (implying less sharp peaks); (iii) the potential importance of the number of paths of different lengths between the source and \( i \). In the following section we report a more systematic investigation about how the diffusion time-varying signals is affected by distinct network structures. Because both the magnitude and lag (and potentially the dispersion) of the peaks seem to be strongly related to the distance to the source node, special attention is given to correlating the signals similarity with the respective distances.

FIG. 2: A simple network used to illustrate the diffusion of an activation signal emanating from node 1.

### III. RESULTS AND DISCUSSION

All networks in this article have \( N = 200 \) nodes. Two different average degree values have been considered, namely \( \langle k \rangle = 2 \) and \( \langle k \rangle = 4 \). The random signals were obtained for \( \alpha = 0.1 \) and basic period \( L = 200 \), repeated \( T = 8 \) times, implying the total signal duration to be equal to 1600 time steps.

A total of 1000 realizations were performed for each network configuration, defined by \( N \) and \( \langle k \rangle \). Different random signals were used at each realization, and the source was always placed at node 1. The activation along time at each node was Pearson correlated with the original signal in order to obtain a quantification of the similarity between those two signals. Three properties of the similarity signatures — namely their peak, lag and entropy — were then Pearson correlated with the respective distances to the source node.

Figure 4 refers the correlations between the logarithm of peak magnitude and the logarithm of the distance to the source obtained for ER and BA networks with \( \langle k \rangle = 2 \) and \( \langle k \rangle = 4 \). Each group of relative frequency histograms in this figure corresponds to: (i) Pearson correlation coefficient between the logarithm of the similarity peak magnitude and the logarithm of the distance to the source node (top histogram, \( f(P) \)); (b) slope of the linear regression obtained for the same relationship (middle histogram, \( f(m) \)) and (c) respective intercept (bottom histogram, \( f(c) \)).

Surprisingly, quite similar relative frequency histograms of the Pearson correlations have been obtained for ER and BA with the same average degrees. This result implies that both networks have similar properties.
FIG. 3: The original, uniformly random, signal injected into node 1 of the network in Figure 2 and the derivatives of the signals at nodes 4, 8, 15 and 21, followed by the respective similarities quantified in terms of the Pearson correlation coefficient between two signals.
as far as the presence of the original signal within the observed derivatives is concerned. However, the slope and intercept of the respective linear regressions resulted rather different between ER and BA with distinct average degrees. It is also clear from Figure 4 that the similarity to the activation at the source node exhibits strong negative correlations with the respective distance to the source for both ER and BA networks with $\langle k \rangle = 2$. This is a consequence of the fact that nodes more distant to the source tend to be connected to that node through paths with large dispersion of lengths, hubs and cycles, therefore contributing to a stronger intermixing of delayed versions of the original signal. For $\langle k \rangle = 2$, it is clear from the obtained histograms that the similarity to the original activation at each node can be predicted with good accuracy from the distance between that node and the source. This result establishes a strong relationship between the structure and the dynamics for signal diffusion in networks. The slope of the linear regression between the similarity and distance to the source is substantially larger for the ER model than the respective BA counterparts, indicating that the similarity with the source signal varies more intensely with the distance to the source in the case of the ER structures. Very little variation is observed for BA for both average degrees. Both the Pearson correlation and the slope of the linear regression of the relationship between the similarity and the distance are drastically reduced with the increase in the average node degree. This is because more intense connectivity between nodes tends to create more alternative paths of different lengths between nodes, hubs and cycles, contributing to the intermixing of the several delayed versions of the original signal while defining the signals at each node.

Additional insights about the interplay between the dynamics and structure in signal diffusion through networks can be gathered by considering the lag between the detected signal and the original activation. Figure 5 refers to the correlations between the lag and the distance between each respective node and the source, being organized in the same ways as Figure 4. Strong positive correlations can be identified between the lag and the distance. However, unlike the results obtained for similarity and distance (Fig. 4), markedly distinct Pearson correlation coefficient values were obtained for ER and BA for both average degrees, with smaller correlations being observed in the latter cases.

The relative frequency histograms (not shown in this work) of the Pearson correlations obtained between the entropies of the similarity signatures and the respective distances to the source of activation indicate, in average, that the similarity entropy tends not to correlate with the distance to the source. This is a surprising result because it could be expected that the dispersion would increase with the distance from the source as a consequence of the additional intermixings of the signals obtained for longer distances. The absence of correlations between the similarity entropy and the distances to the source is possibly a consequence of the fact that, in traditional random walks, the signals are allowed to backdiffuse freely through the whole network. However, additional investigations (possibly involving self-avoiding dynamics) are needed in order to better understand such a lack of correlation.

IV. CONCLUDING REMARKS

Great attention has been focused along the last years on the important subject of relating structure and dynamics in complex systems. The current article reported an investigation about the effect of complex network structure on the progressive modifications of a time-varying signal, issued from a specific source node, as it diffuses at the equilibrium regime. The undevoted modifications were quantified in terms of three features (peak magnitude, lag and entropy) derived from the Pearson correlation between the original signal and the derivative of signals observed at specific nodes. Though little studied at the individual level, the linear dynamics of time-varying signal diffusion provides an interesting case of structure-dynamics in complex network and is direct and indirectly related to several important types of natural and artificial linear and non-linear dynamics. In particular, relatively little attention has been focused on the problem of transmission of time-varying signals, which implies the preservation of the sequence of their constituent parts. Another distinguishing feature of the currently reported investigation is that the relationship between the network structure and dynamics has been considered with respect to individual pairs of nodes (source and destination nodes), and not by taking averages through the network. This is particularly important given that quite distinct dynamics have been observed in our experiment while considering different pairs of nodes. In this work, the consideration of individual pairwise relationships has been allowed by taking into account not by taking the average of the observed features through the network, but by obtaining the relative frequency histograms of the Pearson correlations calculated between those features and the distance to the source while considering pairs of nodes involving the source node and all the other nodes in the networks.

A number of interesting results were obtained with respect to a systematic investigation (1000 realizations) taking into account ER and BA structures. The first important result is that the similarity peak is almost always obtained near the beginning of each period, though with varying lags, which indicates that the original signal is to a certain degree contributing to the variation of the signals arising at each node. In addition, a strong negative correlation was observed between the logarithm of the similarity peak magnitude and the logarithm of the respective distances to the source node, with similar values resulting for ER and BA with equivalent average degrees. This result indicates that the effect of the original signal
FIG. 4: The relative frequency histograms describing the correlations between the similarity peak magnitude and the distance to the source node for ER and BA complex networks with $N = 200$ nodes and $\langle k \rangle = 2$ and 4. Each vertical group shows three relative frequency histograms with respect to the Pearson correlation between the peak magnitude and the distance to the source (top), as well as the slope (middle) and intercept (bottom) of the respective linear regression.
FIG. 5: The relative frequency histograms describing the correlations between the similarity lag and the distance to the source node for ER and BA complex networks with $N = 200$ nodes and $\langle k \rangle = 2$ and 4. Each vertical group shows three relative frequency histograms with respect to the Pearson correlation between the peak magnitude and the distance to the source (top), as well as the slope (middle) and intercept (bottom) of the respective linear regression.

on the variations of the signals observed along time at each node tends to decrease strong and steadily with the distance to the source. High positive correlation values were also identified between the peak lag and the distance to the source node. However, different correlation intensities were observed for ER and BA structures in this case, with larger correlations being obtained for the former type of network. So, the lag is more strongly de-
fined by the distance to the source node in the case of ER networks. In both cases (i.e. correlations between peak magnitude and distance to the source and between peak lag and distance to the source), the correlations tended to substantially decrease for networks with larger average degree. The correlation between the similarity entropy and the distance to the source tended to present null average, suggesting no clear relationship between these two features. Additional investigations are required in order to better understand the latter effect.

The above results have immediate implications to many areas, such as the investigation of the quality of communication channels, identification of positions in networks, and digital signal processing. In the former case, the network connectivity between each pair of nodes (generalized connectivity or superedges [4, 5]) can be understood as a communicating channel between those two nodes. The methodology reported in this article, as well as related extensions, can be immediately applied in order to quantify the fidelity and delay of each of such channels.

The possibility to identify the position of nodes in non-geographical networks is directly related to the correlations identified in the present work between the similarity peak magnitude and lag with the distance to the source node. Thus, given a network node and its respective time-response to a given input signal, it is possible to estimate its distance to the source node by considering the slope and intercept of the linear regression between the logarithm of the magnitude or the lag with respect to the distance to the source. In such a case, the source would be acting as a beacon, establishing a one-coordinate axes through the network (the distances from each node to the beacon). It would be particularly interesting to investigate how the incorporation of additional beacons could contribute to a one-to-one identification of all the nodes in terms of their distances to each of several beacons.

The diffusion of time-varying signals in complex networks is a subject which is intrinsically related to digital signal processing and linear digital filters (e.g. [6]). Indeed, the latter are structures composed by delays, adders, and multiplications by constant values. The linear transformation performed on the input signal has traditionally been investigated in terms of the z-transform, a method intrinsically suited to treat time-discrete signals. Thus, it is possible to represent and study the diffusion dynamics in complex networks by using such concepts, i.e. digital filters and z-transforms. At the same time, it would be interesting to apply complex networks concepts, such as shortest path distances between pairs of nodes, to the area of digital filters. The main difference between the work reported in this article and traditional digital filters is that the latter area typically involves structures which are typically much smaller and simpler than complex networks.

The reported work also paves the way to a number of further related investigations. Immediate developments could take into account other types of networks, correlations with measurements other than the distance to the source, as well as investigate finite-size effects by taking into account networks with different numbers of nodes. In particular, given that the diffusion effects on the propagated signals seem to be related to the number of paths of different lengths between the source and destination nodes, it would be interesting to study the relationship between such statistics (e.g. [5]) and the similarity between the signals at the source and destinations. It would also be especially interesting to repeat the presently reported study to non-linear self-avoiding dynamics, so as to find out the specific effects of backward signal propagation allowed by the linear random walks adopted in the current work. Even more sophisticated dynamics such as integrate-and-fire can be investigated from the perspective of the conceptual and methodological framework proposed in the current work. As a more leisurely project, it would be possible to transform the derivative signals into some audio format and hear the sounds arising at different nodes while the network is stimulated by a given signal (such as a piece of music) injected at a specific node.

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