FALTINGS' DELTA-INVARIANT OF A HYPERELLIPTIC RIEMANN SURFACE

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Abstract. In this note we give a closed formula for Faltings’ delta-invariant of a hyperelliptic Riemann surface.

1. Introduction

In [7], the delta-invariant is introduced for Riemann surfaces, and it is suggested there to search for explicit formulas for this invariant. In this note we give such an explicit formula in the case of a hyperelliptic Riemann surface of arbitrary genus. We note that [7] treats the case of elliptic curves, and that the case of Riemann surfaces of genus 2 has been considered before in [3].

In order to state our result, let us recall some notation and earlier results. Let \( X \) be a compact and connected Riemann surface of genus \( g > 0 \). Let \( G \) be the Arakelov-Green function of \( X \) and let \( \mu \) be the fundamental \((1,1)\)-form on \( X \) as defined in [1], [7]. Let \( S(X) \) be the invariant defined by

\[
\log S(X) := -\int_X \log \|\vartheta\| (gP - Q) \cdot \mu(P).
\]

Here \( \|\vartheta\| \) is the function on \( \text{Pic}_{g-1}(X) \) defined as on [7], p. 401, and \( Q \) can be any point on \( X \). The integral is well-defined and is independent of the choice of the point \( Q \). In our paper [10] we gave an explicit formula for the Arakelov-Green function of \( X \).

Theorem 1.1. Let \( W \) be the classical divisor of Weierstrass points on \( X \). For \( P, Q \) points on \( X \), with \( P \) not a Weierstrass point, we have

\[
G(P, Q)^g = S(X)^{1/g^2} \cdot \prod_{W \in W} \|\vartheta\| (gP - W)^{1/g^3}.
\]

Here the product runs over the Weierstrass points of \( X \), counted with their weights. The formula is also valid if \( P \) is a Weierstrass point, provided that we take the leading coefficients of a power series expansion about \( P \) in both numerator and denominator.

In the same paper, we also gave an explicit formula for the delta-invariant of \( X \). The delta-invariant is a fundamental invariant of \( X \), expressing the proportionality between
two natural metrics on the determinant of the Hodge bundle. For $P$ on $X$, not a Weierstrass point, and $z$ a local coordinate about $P$, we put

$$∥F_z∥(P) := \lim_{Q \to P} \|\vartheta\|(gP - Q)_{|z(P) - z(Q)|^g}.$$ 

Further we let $W_z(\omega)(P)$ be the Wronskian at $P$ in $z$ of an orthonormal basis $\{\omega_1, \ldots, \omega_g\}$ of the differentials $\mathcal{H}^0(X, \Omega^1_X)$ provided with the standard hermitian inner product $(\omega, \eta) \mapsto \frac{1}{2} \int_X \omega \wedge \overline{\eta}$. We define an invariant $T(X)$ of $X$ by

$$T(X) := ∥F_z∥(P)^{-1} \cdot \prod_{W \in \mathcal{W}} \|\vartheta\|(gP - W)^{(g-1)/g^3} \cdot |W_z(\omega)(P)|^2,$$

where again the product runs over the Weierstrass points of $X$, counted with their weights.

It can be checked that this does not depend on the choice of $P$, nor on the choice of local coordinate $z$ about $P$. A more intrinsic definition is possible, see [10], but the above formula will be convenient for us. We remark that $T(X)$ does not involve an integral over $X$, contrary to the invariant $S(X)$.

**Theorem 1.2.** For Faltings’ delta-invariant $\delta(X)$ of $X$, the formula

$$\exp(\delta(X)/4) = S(X)^{-1} \cdot T(X)$$

holds.

In the present paper we make the invariant $T(X)$ explicit in the case that $X$ is a hyperelliptic Riemann surface of genus $g \geq 2$. We relate it to a non-zero invariant $∥\varphi_g∥(X)$ of $X$, the Petersson norm of the modular discriminant associated to $X$, which we introduce in Section 2. As we will see, for hyperelliptic Riemann surfaces this is a very natural invariant to consider. Unfortunately, it is not so clear how to extend its definition to the general Riemann surface of genus $g$.

**Definition 1.3.** We define by $G'$ the modified Arakelov-Green function

$$G'(P, Q) := S(X)^{-1/s^3} \cdot G(P, Q)$$

on $X \times X$.

We prove the following theorem dealing with $G'$ and $T(X)$. Recall that the Weierstrass points of $X$ are just the ramification points of a hyperelliptic map $X \to \mathbb{P}^1$.

**Theorem 1.4.** Let $W$ be a Weierstrass point of $X$. Let $n := (\frac{2g}{g+1})$. Consider the product $\prod_{W' \neq W} G'(W, W')$ running over all Weierstrass points $W'$ different from $W$, ignoring their weights. Then $\prod_{W' \neq W} G'(W, W')$ is independent of the choice of $W$ and the formula

$$\prod_{W' \neq W} G'(W, W')(g-1)^2 = 2(g-1)^2 \cdot \pi^{2g+2} \cdot T(X)^{\frac{g+1}{g}} \cdot ∥\varphi_g∥(X)^{\frac{1}{n}}$$

holds.
The next theorem will be derived in a forthcoming article [11]. The result looks similar to the formula in Theorem 1.4, but the proof is very different.

**Theorem 1.5.** Let \( m := \frac{(2g+2)}{g} \). Then we have

\[
\prod_{(W,W')} G'(W,W')^{n(g-1)} = \pi^{-2(g+2)m} \cdot T(X)^{-(g+2)m} \cdot \|\varphi_g\|(X)^{-\frac{1}{2}(g+1)} ,
\]

the product running over all ordered pairs of distinct Weierstrass points of \( X \).

Combining the above two theorems yields a simple closed formula for the invariant \( T(X) \) in terms of \( \|\varphi_g\|(X) \).

**Theorem 1.6.** Let \( \|\Delta_g\|(X) \) be the modified discriminant \( \|\Delta_g\|(X) := 2^{-4(g+1)n} \cdot \|\varphi_g\|(X) \). Then the formula

\[
T(X) = (2\pi)^{-2g} \cdot \|\Delta_g\|(X)^{-\frac{2g-1}{2g}}
\]

holds.

Combining this with Theorem 1.2 we obtain the following corollary.

**Corollary 1.7.** For Faltings’ delta-invariant \( \delta(X) \) of \( X \), the formula

\[
\exp(\delta(X)/4) = (2\pi)^{-2g} \cdot S(X)^{-g(g-1)/g^2} \cdot \|\Delta_g\|(X)^{-\frac{2g-1}{2g}}
\]

holds.

The significance of this result is that it makes the efficient calculation of the delta-invariant possible for hyperelliptic Riemann surfaces. We have given a demonstration of this in our paper [11].

We remark that in the case \( g = 2 \), an explicit formula for the delta-invariant has been given already by Bost [3]. Apart from the Petersson norm of the modular discriminant, his formula involves an invariant \( \|H\|(X) \). This invariant has properties similar to our \( S(X) \).

The idea of the proof of Theorem 1.5 is quite straightforward: we start with the definition of the invariant \( T(X) \) and the formula for \( G \) in Theorem 1.1 and observe what happens if we let \( P \) approach the Weierstrass point \( W \) on \( X \). Thus, we have to perform a local study around \( W \) of the function \( \prod_{W'} \|\varphi\|(gP-W') \) and of the functions \( \|F_z\|(P) \) and \( W_z(\omega)(P) \) for a suitable local coordinate \( z \). In Section 3 we find a suitable local coordinate on an embedding of \( X \) into its jacobian. In Section 6 we collect the local information that we need in order to complete the proof in Section 7. Some preliminary work on this local information is carried out in the Sections 4 and 5. These two sections form the technical heart of the paper.
2. Modular discriminant

In this section we introduce the modular discriminant \( \varphi_g \) and its Petersson norm \( \| \varphi_g \| \).

The modular discriminant generalises the usual discriminant function \( \Delta \) for elliptic curves.

Let \( g \geq 2 \) be an integer and let \( \mathcal{H}_g \) be the Siegel upper half-space of symmetric complex \( g \times g \)-matrices with positive definite imaginary part. For \( z \in \mathbb{C}^g \) (viewed as a column vector), a matrix \( \tau \in \mathcal{H}_g \) and \( \eta, \eta' \in \frac{1}{2} \mathbb{Z}^g \) we have the theta function with characteristic \( \eta = [\eta'] \) given by

\[
\vartheta[\eta](z; \tau) := \sum_{n \in \mathbb{Z}^g} \exp(\pi i (n + \eta') \tau (n + \eta') + 2\pi i (n + \eta') (z + \eta'')).
\]

For any subset \( S \) of \( \{1, 2, \ldots, 2g + 1\} \) we define a theta characteristic \( \eta_S \) as in [14], Chapter IIIa: let

\[
\eta_{2k} = \left( \begin{array}{c} \tau(0, \ldots, 0, \frac{1}{2}, 0, \ldots, 0) \\ \tau(0, \ldots, 0, \frac{1}{2}, 0, \ldots, 0) \end{array} \right), \quad 1 \leq k \leq g + 1,
\]

\[
\eta_{2k-1} = \left( \begin{array}{c} \tau(0, \ldots, 0, 0, 0, \ldots, 0) \\ \tau(0, \ldots, 0, 0, 0, \ldots, 0) \end{array} \right), \quad 1 \leq k \leq g,
\]

where the non-zero entry in the top row occurs in the \( k \)-th position. Then we put \( \eta_S := \sum_{k \in S} \eta_k \) where the sum is taken modulo 1.

**Definition 2.1.** (Cf. [14], Section 3.) Let \( T \) be the set of subsets of \( \{1, 2, \ldots, 2g + 1\} \) of cardinality \( g + 1 \). Write \( U := \{1, 3, \ldots, 2g + 1\} \) and let \( \circ \) denote the symmetric difference. The modular discriminant \( \varphi_g \) is defined to be the function

\[
\varphi_g(\tau) := \prod_{\tau \in \mathcal{T}} \vartheta[\eta \circ U](0; \tau)^8
\]

on \( \mathcal{H}_g \). The function \( \varphi_g \) is a modular form on \( \Gamma_g(2) := \{ \gamma \in \text{Sp}(2g, \mathbb{Z}) | \gamma \equiv I_{2g} \mod 2 \} \) of weight \( 4r \) where \( r := \frac{(2g + 1)}{2} \).

Consider an equation \( y^2 = f(x) \) where \( f \in \mathbb{C}[X] \) is a monic and separable polynomial of degree \( 2g + 1 \). Write \( f(x) = \prod_{k=1}^{2g+1} (x - a_k) \) and denote by \( D := \prod_{k<l} (a_k - a_l)^2 \) the discriminant of \( f \). Let \( X \) be the hyperelliptic Riemann surface of genus \( g \) defined by \( y^2 = f(x) \). Then \( X \) carries a basis of holomorphic differentials \( \mu_k := x^{k-1} dx/2y \) where \( k = 1, \ldots, g \). Further, in [14], Chapter IIIa, §5 it is shown how, given an ordering of the roots of \( f \), one can construct a canonical symplectic basis of the homology of \( X \). Throughout this paper, we will always work with such a canonical basis of homology, i.e., a certain ordering of the roots of a hyperelliptic equation will always be taken for granted.

Let \( (\mu | \mu') \) be the period matrix of the differentials \( \mu_k \) with respect to a chosen canonical basis of homology. Then \( \mu \) is invertible, and we put \( \tau := \mu^{-1} \mu' \).

**Proposition 2.2.** We have the formula

\[
D^n = \pi^{4gr} (\det \mu)^{-4r} \varphi_g(\tau)
\]
relating the discriminant $D$ of the polynomial $f$ to the value $\varphi_g(\tau)$ of the modular discriminant.

**Proof.** See [13], Proposition 3.2. □

**Definition 2.3.** Let $X$ be a hyperelliptic Riemann surface of genus $g \geq 2$ and let $\tau$ be a period matrix for $X$ formed on a canonical symplectic basis, given by an ordering of the roots of an equation $y^2 = f(x)$ for $X$. Then we write $\|\varphi_g\|(\tau)$ for the Petersson norm $(\det \text{Im} \tau)^{2r} \cdot |\varphi_g(\tau)|$ of $\varphi_g(\tau)$. This does not depend on the choice of $\tau$ and hence it defines an invariant $\|\varphi_g\|(X)$ of $X$.

It follows from Proposition 2.2 that the invariant $\|\varphi_g\|(X)$ is non-zero.

### 3. Local coordinate

For our local computations on our hyperelliptic Riemann surface we need a convenient local coordinate. We find one by embedding the Riemann surface into its jacobian and by taking one of the euclidean coordinates.

Let $X$ be a hyperelliptic Riemann surface of genus $g \geq 2$, let $y^2 = f(x)$ with $f$ monic of degree $2g + 1$ be an equation for $X$, let $\mu_k$ be the differential given by $\mu_k = x^{k-1}dx/2y$ for $k = 1, \ldots, g$, and let $(\mu|\mu')$ be their period matrix formed on a canonical basis of homology. Let $L$ be the lattice in $\mathbb{C}^g$ generated by the columns of $(\mu|\mu')$. We have an embedding $\iota : X \hookrightarrow \mathbb{C}^g/L$ given by integration $P \mapsto \int_P^\infty (\mu_1, \ldots, \mu_g)$. We want to express the coordinates $z_1, \ldots, z_g$, restricted to $\iota(X)$, in terms of a local coordinate about $0 = \iota(\infty)$. This is established by the following lemma. In general, we denote by $O(w_1, \ldots, w_s; d)$ a Laurent series in the variables $w_1, \ldots, w_s$ all of whose terms have total degree at least $d$. We owe the argument to [12].

**Lemma 3.1.** The coordinate $z_g$ is a local coordinate about 0 on $\iota(X)$, and we have

$$z_k = \frac{1}{2(g-k)+1} z_g^{2(g-k)+1} + O(z_g; 2(g - k) + 2)$$

on $\iota(X)$ for $k = 1, \ldots, g$.

**Proof.** We can choose a local coordinate $t$ about $\infty$ on $X$ such that $x = t^{-2}$ and $y = -t^{-(2g+1)} + O(t; -2g)$. For $P \in X$ in a small enough neighbourhood of $\infty$ on $X$ and for a suitable integration path on $X$ we then have

$$z_k(P) = \int_P^{\infty} \frac{x^{k-1}dx}{2y} = \int_0^{\iota(P)} t^{-2(k-1)} \cdot (-2t^{-3}dt -2t^{-(2g+1)} + O(t; -2g))$$

$$= \int_0^{\iota(P)} \left( t^{2(g-k)} + O(t; 2(g - k) + 1) \right) dt$$

$$= \frac{1}{2(g-k)+1} t(P)^{2(g-k)+1} + O(t(P); 2(g - k) + 2).$$
By taking $k = g$ we find $z_g = t + O(t; 2)$ and for $k = 1, \ldots, g - 1$ then

$$z_k = \frac{1}{2(g - k) + 1} z_g^{2(g-k)+1} + O(z_g; 2(g - k) + 2),$$

which is what we wanted. □

4. Schur polynomials

In this section we assemble some facts on Schur polynomials. We will need these facts at various places in the next sections. Fix a positive integer $g$. Consider the ring of symmetric polynomials with integer coefficients in the variables $x_1, \ldots, x_g$. Let $e_r$ be the elementary symmetric functions defined by means of the generating function $E(t) = \sum_{r \geq 0} e_r t^r = \prod_{k=1}^g (1 + x_k t)$.

**Definition 4.1.** Let $d$ be a positive integer and let $\pi = \{\pi_1, \ldots, \pi_h\}$ with $\pi_1 \geq \ldots \geq \pi_h$ be a partition of $d$. The Schur polynomial associated to $\pi$ is the polynomial

$$S_\pi := \det(e_{\pi'_l - k + 1})_{1 \leq k, l \leq h},$$

where $h$ is the length of the partition $\pi$, and where $\pi'$ is the conjugate partition of $\pi$ given by $\pi'_k = \#\{l : \pi_l \geq k\}$, i.e., the partition obtained by switching the associated Young diagram around its diagonal. The polynomial $S_\pi$ is symmetric and has total degree $d$. We denote by $S_g$ the Schur polynomial in $g$ variables associated to the partition $\pi = \{g, g - 1, \ldots, 2, 1\}$. Thus, the formula

$$S_g = \det(e_{g-2k+l+1})_{1 \leq k, l \leq g}$$

holds, and the polynomial $S_g$ has total degree $g(g + 1)/2$.

Let $p_r$ be the elementary Newton functions (power sums) given by the generating function $P(t) = \sum_{r \geq 1} p_r t^{r-1} = \sum_{k \geq 1} x_k/(1 - x_k t)$. The following proposition is then a special case of Theorem 4.1 of [5].

**Proposition 4.2.** The Schur polynomial $S_g$ can be expressed as a polynomial in the $g$ functions $p_1, p_3, \ldots, p_{2g-1}$ only. This polynomial is unique.

**Definition 4.3.** We define $s_g$ to be the unique polynomial in $g$ variables given by Proposition 4.2.

The next proposition is a special case of Theorem 6.2 of [5].

**Proposition 4.4.** Let $s(x_1, \ldots, x_g) \in \mathbb{C}[x_1, \ldots, x_g]$ be a polynomial in $g$ variables such that for any set of $g$ complex numbers $w_1, \ldots, w_g$, the polynomial $s(z_1 - w, z_2 - w^3, \ldots, z_g - w^{2g-1})$ in $w$ either has exactly $g$ roots $w_1, \ldots, w_g$, or vanishes identically, if we give $z$ the value $z = (p_1(w_1, \ldots, w_g), p_3(w_1, \ldots, w_g), \ldots, p_{2g-1}(w_1, \ldots, w_g))$. Then $s$ is equal to the polynomial $s_g$ up to a constant factor.
Definition 4.5. We define $\sigma_g$ to be the polynomial in $g$ variables given by the equation
\[ \sigma_g(z_1, \ldots, z_g) = s_g(z_g, 3z_{g-1}, \ldots, (2g-1)z_1). \]

The following proposition is then the result of a simple calculation.

Proposition 4.6. Up to a sign, the homogeneous part of least total degree of $\sigma_g$ is equal to the Hankel determinant
\[
H(z) = \det \begin{pmatrix}
z_1 & z_2 & \cdots & z_{(g+1)/2} \\
z_2 & z_3 & \cdots & z_{(g+3)/2} \\
\vdots & \vdots & \ddots & \vdots \\
z_{(g+1)/2} & z_{(g+3)/2} & \cdots & z_g
\end{pmatrix}
\]
if $g$ is odd, or
\[
H(z) = \det \begin{pmatrix}
z_1 & z_2 & \cdots & z_{g/2} \\
z_2 & z_3 & \cdots & z_{(g+2)/2} \\
\vdots & \vdots & \ddots & \vdots \\
z_{g/2} & z_{(g+2)/2} & \cdots & z_{g-1}
\end{pmatrix}
\]
if $g$ is even.

We conclude with some more general facts. These can all be found for example in Appendix A to [8].

Proposition 4.7. Let $\pi = \{\pi_1, \ldots, \pi_h\}$ with $\pi_1 \geq \ldots \geq \pi_h$ be a partition. Then the formula
\[
S_\pi(1, \ldots, 1) = \prod_{k<l} \frac{\pi_k - \pi_l + l - k}{l - k}
\]
holds. In particular, $S_g(1, \ldots, 1) = 2^{g(g-1)/2}.$

Definition 4.8. Let $i = (i_1, \ldots, i_d)$ be a $d$-tuple of non-negative integers. The $i$-th generalised Newton function $p^{(i)}$ is defined to be the polynomial
\[ p^{(i)} := p_1^{i_1} \cdot p_2^{i_2} \cdots p_d^{i_d}, \]
where the $p_r$ are the elementary Newton functions.

Proposition 4.9. The set of generalised Newton functions $p^{(i)}$, where $i$ runs through the $d$-tuples $i = (i_1, \ldots, i_d)$ of non-negative integers with $\sum \alpha_i = d$, forms a basis of the $\mathbb{Q}$-vector space of symmetric polynomials with rational coefficients of total degree $d$.

Proposition 4.10. For a partition $\pi$ of $d$ and a $d$-tuple $i = (i_1, \ldots, i_d)$, denote by $\omega_\pi(i)$ the coefficient of the monomial $x_1^{i_1} \cdots x_d^{i_d}$ in $p^{(i)}$. Then the polynomial $S_\pi$ can be expanded on the basis $\{p^{(i)}\}$ of generalised Newton functions of total degree $d$ as $S_\pi = \sum_i \frac{1}{z(i)} \cdot \omega_\pi(i) \cdot p^{(i)}$. Here $z(i) = i_1!1^{i_1} \cdot i_2!2^{i_2} \cdots i_d!d^{i_d}.$
5. Sigma function

We consider again hyperelliptic Riemann surfaces of genus $g \geq 2$, defined by equations $y^2 = f(x)$ with $f$ monic and separable of degree $2g+1$. We write $f(x) = x^{2g+1} + \lambda_1 x^{2g} + \cdots + \lambda_{2g} x + \lambda_{2g+1}$ and denote by $\lambda$ the vector of coefficients $(\lambda_1, \ldots, \lambda_{2g+1})$. In this section we study the sigma function $\sigma(z; \lambda)$ with argument $z \in \mathbb{C}^g$ and parameter $\lambda$. This is a modified theta function, studied extensively in the nineteenth century. Klein observed that the sigma function serves very well to study the function theory of hyperelliptic Riemann surfaces. For us it will be a convenient technical tool for obtaining the local expansions that we need. We will give the definition of the sigma function, as well as its power series expansion in $z, \lambda$. For more details we refer to the *Enzyklopädie der mathematischen Wissenschaften*, Band II, Teil 2, Kapitel 7.XII. A modern reference is [4], where one also finds applications of the sigma function in the theory of the Korteweg-de Vries differential equation.

As before, let $\mu_k$ be the holomorphic differential given by $\mu_k = x^{k-1} dx / 2y$ for $k = 1, \ldots, g$, and let $(\mu | \mu')$ be their period matrix formed on a canonical basis of homology. Let $L$ be the lattice in $\mathbb{C}^g$ generated by the columns of $(\mu | \mu')$. By the theorem of Abel-Jacobi we have a bijective map $\text{Pic}_{g-1}(X) \sim \rightarrow \mathbb{C}^g / L$ given by $\sum k m_k P_k \mapsto \sum k m_k \int_{\infty}^{P_k} (\mu_1, \ldots, \mu_g)$. Denote by $\Theta$ the image of the theta divisor of classes of effective divisors of degree $g - 1$, and let $q : \mathbb{C}^g \rightarrow \mathbb{C}^g / L$ be the projection map. Let $\tau = \mu^{-1} \mu'$. By a fundamental theorem of Riemann, there exists a unique theta-characteristic $\delta$ such that $\vartheta[\delta](z; \tau)$ vanishes to order one precisely along $q^{-1}(\Theta)$.

**Definition 5.1.** Let $\nu$ be the matrix of $A$-periods of the differentials of the second kind $\nu_k := \frac{1}{4g} \sum_{l=k}^{2g-k} (l+1-k) \lambda_{l+k+1} x^l dx$ for $k = 1, \ldots, g$. These differentials have a second order pole at $\infty$ and no other poles. The sigma function is then the function

$$
\sigma(z; \lambda) := \exp(-\frac{1}{2} z \nu \mu^{-1} z) \cdot \vartheta[\delta](\mu^{-1} z; \tau).
$$

Using some of the facts on Schur polynomials from the previous section, we can give the power series expansion of $\sigma(z; \lambda)$. The result is probably well-known to specialists, although we couldn’t find an explicit reference in the literature. For the formulation and the proof we were inspired by [12], as well as by a private communication with the author. For the special case $g = 2$, a somewhat stronger version of the result has been obtained by Grant, see [9], Theorem 2.11.

**Proposition 5.2.** The power series expansion of $\sigma(z; \lambda)$ about $z = 0$ is of the form

$$
\sigma(z; \lambda) = \gamma \cdot \sigma_g(z) + O(\lambda),
$$

where $\sigma_g$ is the polynomial given by Definition 4.2 and where the symbol $O(\lambda)$ denotes a power series in $z, \lambda$ in which each term contains a $\lambda_k$ raised to a positive integral power.
The constant $\gamma$ satisfies the formula

$$\gamma^{8n} = \pi^{4g(r-n)}(\det \mu)^{-4(r-n)}\varphi_g(r).$$

If we assign the variable $z_k$ a weight $2(g-k)+1$, and the variable $\lambda_k$ a weight $-2k$, then the power series expansion in $z, \lambda$ of $\sigma(z; \lambda)$ is homogeneous of weight $g(g+1)/2$.

**Proof.** First of all, the homogeneity of the power series expansion in $z, \lambda$ with respect to the assigned weights follows from an explicit formula for $\sigma(z; \lambda)$ given in [6]. This homogeneity is also mentioned there, cf. the concluding remarks after Corollary 1. Write $\sigma(z; \lambda) = \sigma_0(z) + O(\lambda)$ where $O(\lambda)$ denotes a power series in $z, \lambda$ in which each term contains a $\lambda_k$ raised to a positive integral power. Because of the homogeneity, the series $\sigma_0(z)$ is necessarily a polynomial in the variables $z_1, \ldots, z_g$. By the Riemann vanishing theorem, there is a dense open subset $U \subset \mathbb{C}^{2g+1}$ such that for any $\lambda \in U$, the function $\sigma(z; \lambda)$ satisfies the following property: for any set of $g$ points $P_1, \ldots, P_g$ on the hyperelliptic Riemann surface $X = X_\lambda$ corresponding to $\lambda$, the function $\sigma(z - \int_0^P (\mu_1, \ldots, \mu_g); \lambda)$ in $P$ on $X$ either has exactly $g$ roots $P_1, \ldots, P_g$, or vanishes identically, when we give the argument $z$ the value $z = \sum_k \int_0^P (\mu_1, \ldots, \mu_g)$. In the limit $\lambda \to 0$ we find then, as in the proof of Lemma 3.1 that for any set of $g$ complex numbers $w_1, \ldots, w_g$ the polynomial

$$\sigma_0 \left( \frac{1}{2g-1} (z_g - w^{2g-1}), \frac{1}{2g-3} (z_g - 1 - w^{2g-3}), \ldots, \frac{1}{3} (z_2 - w^3), z_1 - w \right)$$

in $w$ either has exactly $g$ roots $w_1, \ldots, w_g$, or vanishes identically, for the value $z = (p_1(w_1, \ldots, w_g), p_3(w_1, \ldots, w_g), \ldots, p_{2g-1}(w_1, \ldots, w_g))$. By Proposition 4.2 the polynomial $\sigma_0$ is equal to the polynomial $\sigma_g$ up to a constant factor $\gamma$. As to this constant $\gamma$, we find in [2], Section IX a calculation of a constant $\gamma'$ such that $\sigma(z; \lambda) = \gamma' \cdot H(z) + O(z; [(g+3)/2])$, where $H(z)$ is the Hankel determinant from Proposition 4.6 and where now we consider the power series expansion only with respect to the variables $z_1, \ldots, z_g$ and with respect to their usual weight $\deg(z_k) = 1$. By Proposition 4.6 this $\gamma'$ is equal to our constant $\gamma$, up to a sign. We just quote the result of Baker’s computation:

$$\gamma^4 = \vartheta(0; \tau)^4 \cdot \prod_{k,l \in U} (a_k - a_l)^2 / (\ell_1 \ell_3 \cdots \ell_{2g+1}),$$

where $\ell_r := -i \prod_{k \in U \setminus \{r\}} (a_k - a_r) / \prod_{k \in U} (a_k - a_r)$.

Thomae’s formula (cf. [12], Chapter IIIa, §8) says that

$$\vartheta(0; \tau) = (\det \mu)^4 \pi^{-4g} \prod_{k,l \in U} (a_k - a_l)^2 \prod_{k,l \in U} (a_k - a_l)^2.$$

Combining, we obtain $\gamma^8 = D \cdot \pi^{-4g} \cdot (\det \mu)^4$. The formula for $\gamma$ that we gave then follows from Proposition 2.2. \qed
Example 5.3. By way of illustration, we have computed $\sigma_g$ for small $g$:

| $g$ | $\sigma_g$ |
|-----|-------------|
| 1   | $z_1$       |
| 2   | $-z_1 + \frac{1}{3}z_3^3$ |
| 3   | $z_1z_3 - z_2^2 - \frac{1}{3}z_3^3 \pm \frac{1}{15}z_2z_4^2$ |
| 4   | $z_1z_3 - z_2^2 - \frac{1}{3}z_3^3 \pm \frac{1}{15}z_2z_4^2 + \frac{1}{105}z_3z_4^3 \pm \frac{1}{4725}z_5^4$ |

Remark 5.4. As can be seen from Proposition 4.6, the homogeneous part of least total degree (with respect to the usual weight $\deg(z_k) = 1$) of $\sigma_g(z)$ has degree $\lfloor \frac{g+1}{2} \rfloor$. Hence, by a fundamental theorem of Riemann, the theta-characteristic $\delta$ gives rise to a linear system of dimension $\lfloor \frac{g-1}{2} \rfloor$ on $X$.

6. Leading Coefficients

In this section we calculate the leading coefficients of the power series expansions in $z_g$ of the holomorphic functions $\vartheta[\delta](g\mu^{-1}z; \tau)|_{i(X)}$ and $W_{z_g}(\mu)$, the Wronskian in $z_g$ of the basis $\{\mu_1, \ldots, \mu_g\}$.

Proposition 6.1. The leading coefficient of the power series expansion of $\sigma(gz; \lambda)|_{i(X)}$, and hence of $\vartheta[\delta](g\mu^{-1}z; \tau)|_{i(X)}$, is equal to $\gamma \cdot 2^{g(g-1)/2}$.

Proof. By Lemma 5.1 and Proposition 5.2 the power series expansion of $\sigma(gz; \lambda)|_{i(X)}$ has the form

$$\sigma(gz; \lambda)|_{i(X)} = \gamma \cdot \sigma_g \left( g \cdot \frac{g}{2g-1}z_g^{2g-1}, g \cdot \frac{g}{2g-3}z_g^{2g-3}, \ldots, \frac{g}{3g}z_g^3, gz_g \right) + O(z_g; g(g+1)/2 + 1).$$

Hence we need to calculate $\sigma_g \left( g \cdot \frac{g}{2g-1}, g \cdot \frac{g}{2g-3}, \ldots, \frac{g}{3g}, g \right)$. By Definition 4.3 this is equal to $s_g(g, g, \ldots, g)$. But by Proposition 4.2 and Definition 4.3 we have $s_g(g, g, \ldots, g) = S_g(1, 1, \ldots, 1)$, and by Proposition 4.7 we have $S_g(1, 1, \ldots, 1) = 2^{g(g-1)/2}$. The proposition follows. □

Proposition 6.2. The leading coefficient of the power series expansion of the Wronskian $W_{z_g}(\mu)$ is equal to $\pm 2^{g(g-1)/2}$. 
Proof. Expanding the Wronskian yields
\[ W_{z_g}(\mu) = \det \left( \frac{1}{(k-1)!} \frac{d^k z_g}{dz_g^k} \right)_{1 \leq k, l \leq g} = \]
\[ = \begin{pmatrix}
    z_g^{2g-2} & z_g^{2g-4} & \cdots & z_g^2 & 1 \\
    (2g - 2)z_g^{2g-3} & (2g - 4)z_g^{2g-5} & \cdots & 2z_g & 0 \\
    \vdots & \vdots & \ddots & \vdots & \vdots \\
    (2g-2)^g & (2g-4)^g & \cdots & 0 & 0
\end{pmatrix} + O(z_g^g ; (g-1)/2 + 1). \]

Let \( A \) be the matrix of binomial coefficients \( A := \left( \binom{2g-2k}{g-1} \right)_{1 \leq k, l \leq g-1} \). From the expansion it follows that the required leading coefficient is equal to \( \det A \). We will compute this number. First of all note that
\[ \det A = \frac{(2g-2)!/(2g-4)! \cdots 2!}{(g-1)!(g-2)! \cdots 1!} \det \left( \frac{1}{(g-2k)!} \right)_{1 \leq k, l \leq g-1}, \]
where we define \( 1/n! := 0 \) for \( n < 0 \). Now let \( d = g(g-1)/2 \) and consider the ring of symmetric polynomials with integer coefficients in \( g - 1 \) variables. It is well known that for the elementary symmetric functions \( e_r \) we have an expansion
\[ e_r = \frac{1}{r!} \det \begin{pmatrix}
    p_1 & 1 & 0 & \cdots & 0 \\
    p_2 & p_1 & 2 & \cdots & 0 \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    p_{r-1} & p_{r-2} & p_{r-3} & \cdots & r-1 \\
    p_r & p_{r-1} & p_{r-2} & \cdots & p_1
\end{pmatrix}, \]
with \( p_r \) the elementary Newton functions. From Definition 4.1 and this expansion it follows that \( \det(1/(g-2k+l)! \) is the coefficient of \( p_l^d \) in the expansion of \( S_{g-1} \) with respect to the basis of generalised Newton functions. By Proposition 4.1 this coefficient is equal to \( \omega_{g-1}(d)/d! \), where \( \omega_{g-1}(d) \) is the coefficient of \( x_1^{q-1} x_2^{q-2} \cdots x_{g-1} x_g \) in \( p_l^d \). It immediately follows that \( \det(1/(g-2k+l)! = 1/(g-1)!(g-2)! \cdots 1! \). Combining one finds \( \det A = 2^g(g-1)/2 \). \( \square \)

7. Proof of Theorem 1.4

Now we are ready to prove Theorem 1.4. Let \( X \) be a hyperelliptic Riemann surface of genus \( g \geq 2 \), and let \( W \) be one of its Weierstrass points.

Proof of Theorem 1.4. Fix a hyperelliptic equation \( y^2 = f(x) \) for \( X \) with \( f \) monic and separable of degree \( 2g+1 \) that puts \( W \) at infinity. Choose a canonical basis of the homology of \( X \), and form the period matrix \( (\mu | \mu') \) of the differentials \( x^{k-1} dx/2y \) for \( k = 1, \ldots, g \) on this basis. Let \( L \) be the lattice in \( \mathbb{C}^g \) generated by the columns of \( (\mu | \mu') \), and embed \( X \) into \( \mathbb{C}^g/L \) with base point \( W \) as in Section 3. We have the standard euclidean coordinates \( z_1, \ldots, z_g \) on \( \mathbb{C}^g/L \) and according to Lemma 3.1 we have that \( z_g \) is a local coordinate about
would be interesting to have an a priori of the Weierstrass point $W$ on $X$. The weight $w$ of $W$ is given by $w = g(g - 1)/2$. Consider then the following quantities:

\[
A(W') := \lim_{Q \to W} \frac{||\vartheta|| (gQ - W')}{|z_g|^g} \quad \text{for Weierstrass points } W' \neq W; \\
A(W) := \lim_{Q \to W} \frac{||\vartheta|| (gQ - W)}{|z_g|^{w+g}} = \lim_{Q \to W} \frac{||F_{\vartheta}|| (Q)}{|z_g|^w}; \\
B(W) := \lim_{Q \to W} \frac{|W_{\vartheta}(\omega)(Q)|}{|z_g|^w},
\]

where $W_{\omega}(\omega)$ is the Wronskian in $z_g$ of an orthonormal basis $\{\omega_1, \ldots, \omega_g\}$ of $H^0(X, \Omega^1_X)$. We have by Theorem 1.1

\[
G'(W, W')^g = \frac{A(W')}{\prod_{W''} A(W'')^{w/g^2}} \quad \text{for Weierstrass points } W' \neq W,
\]

hence

\[
\prod_{W' \neq W} G'(W, W')^g = \frac{1}{A(W)} \cdot \left( \prod_{W'} A(W') \right)^{\frac{g+1}{2g^2}}.
\]

Further we have by the definition of $T(X)$, letting $P$ approach $W$,

\[
T(X) = A(W)^{-(g+1)} \cdot \left( \prod_{W'} A(W') \right)^{\frac{w(g-1)}{g^2}} \cdot B(W)^2.
\]

Eliminating the factor $\prod_{W'} A(W')$ yields

\[
\prod_{W' \neq W} G'(W, W')^{(g-1)^2} = A(W)^4 \cdot B(W)^{-\frac{2g+2}{g}} \cdot T(X)^{\frac{g+1}{g}}.
\]

Now we use the results obtained in Section 6. Let $\tau = \mu^{-1} \mu'$. A simple calculation gives that $A(W)$ is $(\det \Im \tau)^1/4$ times the absolute value of the leading coefficient of the power series expansion of $\vartheta[\delta](g\mu^{-1} \tau; \omega)[\Delta]$ in $z_g$. Hence by Propositions 5.2 and 6.1 we have

\[
A(W) = 2^{g(g-1)/2} \cdot \pi^{\frac{g}{2} - \frac{n}{2}} \cdot (\det \Im \tau)^{1/4} \cdot |\vartheta(\tau)|^{1/4}.
\]

Next let $\| \cdot \|$ be the metric on $\Lambda^0 H^0(X, \Omega^1_X)$ derived from the hermitian inner product $(\omega, \eta) \mapsto \frac{1}{2} \int_X \omega \wedge \overline{\eta}$ on $H^0(X, \Omega^1_X)$. We then have $||\mu_1 \wedge \ldots \wedge \mu_g||^2 = (\det \Im \tau) \cdot |\det \mu|^2$ by Riemann’s second bilinear relations. This gives $|W_{x_0}(\omega)| = |W_{x_0}(\mu)| \cdot (\det \Im \tau)^{-1/2} \cdot |\det \mu|^{-1}$. From Proposition 6.2 we derive then

\[
B(W) = 2^{g(g-1)/2} \cdot (\det \Im \tau)^{-1/2} \cdot |\det \mu|^{-1}.
\]

Plugging in our results for $A(W)$ and $B(W)$ finally gives the theorem. \Box

Remark 7.1. The fact that the product from Theorem 1.4 is independent of the choice of the Weierstrass point $W$ follows a fortiori from the computations in the above proof. It would be interesting to have an a priori reason for this independence.
Remark 7.2. We have not been able to find in general a formula for $G'(W, W')$ with $W, W'$ just two Weierstrass points. In the case $g = 2$ it can be shown that
\[ G'(W, W')^2 = 2^{1/4} \cdot \| \varphi_2 \|(X)^{-3/64} \cdot \prod_{W'' 
eq W, W'} \| \vartheta \|(W - W' + W''). \]
This formula should be compared with the explicit formula for $G(W, W')$ given in [3], Proposition 4. We guess that in general we have
\[ G'(W, W')^g = A(X) \cdot \prod_{(W_1, \ldots, W_{g-1}) \neq (W', \ldots, W_{g-1})} \| \vartheta \|(W - W' + W_1 + \cdots + W_{g-1}), \]
with $A(X)$ some invariant of $X$. Such a result is consistent with Theorems 1.4 and 1.5.

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