Derivatives of Eisenstein series of weight 2 and intersections of modular correspondences

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Abstract
We give a formula for certain values and derivatives of Siegel series and use them to compute Fourier coefficients of derivatives of the Siegel Eisenstein series of weight $\frac{g}{2}$ and genus $g$. When $g = 4$, the Fourier coefficient is approximated by a certain Fourier coefficient of the central derivative of the Siegel Eisenstein series of weight 2 and genus 3, which is related to the intersection of 3 arithmetic modular correspondences. Applications include a relation between weighted averages of representation numbers of symmetric matrices.

Keywords Eisenstein series · Arithmetic intersection numbers · Modular correspondence

Mathematics Subject Classification 11F30 · 11F32

1 Introduction

1.1 Motivation: On the modular correspondences

Let $j = j' = j(\tau)$ be the elliptic modular $j$-function on the upper half plane. For $m \geq 1$ let $\varphi_{m} \in \mathbb{Z}[j, j']$ be the classical modular polynomial defined by

\[
\varphi_{m}(j, j') = \prod_{\Gamma \subset \Gamma_{0}(m)} \prod_{(c, d) \in \Gamma} j\left(\frac{c \tau + d}{c \tau + d}\right),
\]

where $\Gamma_{0}(m)$ is the group of matrices in $SL(2, \mathbb{Z})$ such that $a \equiv c \equiv 0 \pmod{m}$ and $b \equiv d \equiv 1 \pmod{m}$.

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\[ \varphi_m(j(\tau), j(\tau')) = \prod_{A \in \mathcal{M}} (j(\tau) - j(A\tau')). \]

Put \( S = \text{Spec } \mathbb{Z}[j, j'] \) and \( S_{\mathbb{C}} = \text{Spec } \mathbb{C}[j, j'] \). Let \( T_m \) and \( T_{m,\mathbb{C}} \) be the arithmetic and geometric divisors defined by \( \varphi_m = 0 \). We can view \( S \) as an arithmetic threefold \( S = \mathcal{M} \times_{\text{Spec } \mathbb{Z}} \mathcal{M} \), where \( \mathcal{M} \) is the moduli stack of elliptic curves over \( \mathbb{Z} \), and \( T_m \) as the moduli stack of isogenies of elliptic curves of degree \( m \). In the 19th century Hurwitz has computed the intersection

\[(T_{m_1,\mathbb{C}} \cdot T_{m_2,\mathbb{C}}) := \dim_{\mathbb{C}} \mathbb{C}[j, j']/(\varphi_{m_1}, \varphi_{m_2})\]

of complex curves. Gross and Keating [3] discovered that \((T_{m_1,\mathbb{C}} \cdot T_{m_2,\mathbb{C}})\) is related to the Fourier coefficients of the Siegel Eisenstein series of weight 2 for \( Sp_2(\mathbb{Z}) \). Moreover, they gave an explicit expression for the intersection

\[(T_{m_1} \cdot T_{m_2} \cdot T_{m_3}) := \log \# \mathbb{Z}[j, j']/(\varphi_{m_1}, \varphi_{m_2}, \varphi_{m_3})\]

of 3 arithmetic modular correspondences. It is already mentioned in the introduction of [3] that computations of Kudla or Zagier strongly suggest that deg \( \mathcal{Z}(B) \) equals the \( B \)-th Fourier coefficient of the derivative of the Siegel Eisenstein series of weight 2 for \( Sp_3(\mathbb{Z}) \), up to multiplication by a constant which is independent of \( B \). A complete proof of this identity has been given in [20] (cf. [13]).

The purpose of this paper is to compute the Fourier coefficients of the derivative of the Siegel Eisenstein series of weight 2 for \( Sp_4(\mathbb{Z}) \). One may expect that these coefficients are related to the intersection of 4 modular correspondences. Naively, the fiber product \( \mathcal{T}_{m_1} \times_S \mathcal{T}_{m_2} \times_S \mathcal{T}_{m_1} \times_S \mathcal{T}_{m_3} \) comes up to our mind. However, it may be hard to relate it with the summation of local contribution as in (1.1) of [20] because four modular polynomials never make up a regular sequence in \( \mathbb{Z}[j, j'] \). In the face of this situation, each local contribution itself is an interesting object, and our result (see Theorem 1.2) suggests each local contribution should be described with three elements in \( W(\mathbb{F}_p)[[j, j']] \) which make up a regular sequence, where \( W(\mathbb{F}_p) \) denotes the Witt ring of \( \mathbb{F}_p \). What remains is how we properly subsume the local contributions into the (arithmetic) intersection of four cycles defined by four modular polynomials. This will be handled in a future work or left for the interested readers.

In the intervening years Kudla and others have gone a long way towards proving such relations in much greater generality. In [10], he introduced a certain family of Eisenstein series of genus \( g \) and weight \( \frac{g+1}{2} \). They have an odd functional equation and hence have a natural zero at their center of symmetry. The central derivatives of such series, which he refers to as incoherent Eisenstein series, have a connection with arithmetic algebraic geometry of cycles on integral models of Shimura varieties attached to orthogonal groups of signature \((2, g - 1)\), at least when \( g \leq 4 \). We refer the reader to [16] for \( g = 1 \), to [10, 14, 17] for \( g = 2 \), to [13, 20, 28] for \( g = 3 \), to [15] for \( g = 4 \), and to [18] for an arbitrary positive integer \( g \).

### 1.2 The Fourier coefficients of derivative of Eisenstein series

In this paper we compute the Fourier coefficients of derivatives of incoherent Eisenstein series of genus \( g \) and weight \( \frac{g}{2} \). In this introductory section we will consider classical Eisenstein series of level 1. Let \( g \) be a positive integer that is divisible by 4. Let
be the Siegel Eisenstein series of genus $g$, where $\{C, D\}$ runs over a complete set of representatives of the equivalence classes of coprime symmetric pairs of degree $g$, and $Z$ is a complex symmetric matrix of degree $g$ with positive definite imaginary part $Y$. This series converges absolutely for $\Re s > \frac{g+1}{2}$, admits a meromorphic continuation to the whole $s$-plane and satisfies a functional equation by the general theory of Langlands. It is worth noting that $s = 0$ is the central point on the real axis for this functional equation.

If $\frac{g}{4}$ is even, then $E_g(Z, s)$ is holomorphic at $s = -\frac{1}{2}$ and the $T$-th Fourier coefficient of $E_g(Z, -\frac{1}{2})$ is equal to

$$2 \left( \sum_i \frac{1}{N(L_i, L_i)} \right)^{-1} \sum_i \frac{N(L_i, T)}{N(L_i, L_i)}$$

by the Siegel formula (see [12, 27, 31]), where $\{L_i\}$ is the set of isometry classes of positive definite even unimodular lattices of rank $g$. Here $N(L, L')$ denotes the number of isometries $L' \rightarrow L$ for two quadratic spaces $L, L'$ over $\mathbb{Z}$. In particular, the nondegenerate Fourier coefficients are supported on a single rational equivalence class.

On the other hand, if $\frac{g}{4}$ is odd, then $E_g(Z, s)$ has a zero at $s = -\frac{1}{2}$ by Corollary 5.5 of [31] and Lemma 2.1. Our main object of study in this paper is the derivative

$$\frac{\partial}{\partial s} E_g(Z, s) \bigg|_{s=-1/2} = \sum_{T>0} C_g(T) e^{2\pi \sqrt{-1} \text{tr}(TZ)} + \sum_{\text{other } T} C_g(T, Y) e^{2\pi \sqrt{-1} \text{tr}(TZ)}.$$

Fix a positive definite symmetric half-integral $n \times n$ matrix $T$ and a rational prime $p$. Let $\mathbb{Q}^p$ be a subring of $\mathbb{Q}$, consisting of the numbers of the form $\frac{a}{n}$ with $n \in \mathbb{N}$ and $a \in \mathbb{Z}$. We define the additive character $\varepsilon_p$ of $\mathbb{Q}_p$ by setting $\varepsilon_p(x) = e^{-2\pi \sqrt{-1} x}$ with $y \in \mathbb{Q}^p$ such that $x - y \in \mathbb{Z}_p$. The Siegel series attached to $T$ and $p$ is defined by

$$b_p(T, s) = \sum_{\nu[z] \in \text{Sym}_n(\mathbb{Q}_p)/\text{Sym}_n(\mathbb{Z}_p)} \varepsilon_p(-\text{tr}(Tz)) \nu[z]^{-s},$$

where $\nu[z]$ is the product of denominators of elementary divisors of $z$. Put $D_T = (\det T)$ and we denote the primitive Dirichlet character corresponding to $\mathbb{Q}(\sqrt{D_T})$ by $\chi_T$ and its conductor by $b_T$. Put $\tilde{\varepsilon}_T = \chi_T(p)$. Let $e^T_p = \text{ord}_p D_T$ or $e^T_{p^*} = \text{ord}_p D_T - \text{ord}_p b_T$ according as $n$ is odd or even. There exists a polynomial $F_p^T(X) \in \mathbb{Z}[X]$ such that

$$b_p(T, s) = \gamma_p^T(p^{-s}) F_p^T(p^{-s}),$$

where

$$\gamma_p^T(X) = (1 - X) \prod_{j=1}^{[n/2]} (1 - p^{2j}X^2) \times \left\{ \begin{array}{ll} 1 \frac{1}{1 - \gamma_p^{1/2} p^{1/2} X} & \text{if } n \text{ is odd,} \\ 1 & \text{if } n \text{ is even.} \end{array} \right.$$
Assume that $\frac{g}{4}$ is odd. Let $T$ be a positive definite symmetric half-integral matrix of size $g$.

(1) If $\chi_T = 1$, then $C_g(T) = 0$ unless $\text{Diff}(T)$ is a singleton.

(2) If $\chi_T = 1$ and $\text{Diff}(T) = \{p\}$, then

$$C_g(T) = -\frac{2(g+2)/2}{\zeta(1 - \frac{g}{2})} \prod_{i=1}^{(g-2)/2} \zeta(1 - 2i) \frac{\partial F_p^T}{\partial X}(p^{-g/2}) \prod_{p \not\in \mathcal{D}_T} e^{-g/2} F_p^T(e^{-g/2}).$$

(3) If $\chi_T \neq 1$, then

$$C_g(T) = -\frac{2(g+2)/2}{\zeta(1 - \frac{g}{2})} \prod_{i=1}^{(g-2)/2} \zeta(1 - 2i) \prod_{p \not\in \mathcal{D}_T} e^{-g/2} F_p^T(e^{-g/2}).$$

**Remark 1.1** If $\chi_T \neq 1$, then $L(1, \chi_T) = \frac{\sqrt{b^T}}{\log e} h$ by Dirichlet’s class number formula, where $h$ is the class number of the real quadratic field $\mathbb{Q}(\sqrt{\det T})$ and $e = \frac{t+u\sqrt{b^T}}{2} (t > 0, u > 0)$ is the solution to the Pell equation $t^2 - b^T u^2 = 4$ for which $u$ is smallest.

The following theorem is a special case of Theorem 4.1 and allows us to compute $\frac{\partial F_p^T}{\partial X}(\xi_p^T p^{-g/2})$. For simplicity we here assume $p$ to be odd.

**Theorem 1.1** Let $p$ be an odd rational prime and $T = \text{diag}[t_1, \ldots, t_g]$ with $0 \leq \text{ord}_p t_1 \leq \cdots \leq \text{ord}_p t_g$. Put $T' = \text{diag}[t_1, \ldots, t_{g-1}]$. Suppose that $g$ is even and $p \nmid b^T$. Then

$$F_p^T(\xi_p^T p^{-g/2}) = p^{g/2} F_p^T(\xi_p^T p^{-g/2}).$$

If $\eta_p' = -1$, then

$$\frac{\xi_p^T}{p^{g/2}} \frac{\partial F_p^T}{\partial X}(\xi_p^T p^{2-g/2}) = \frac{F_p^T(\xi_p^T p^{2-g/2})}{p - 1} - \frac{p^{g/2}}{p^{g/2}} \frac{\xi_p^T}{p^{g/2}} \frac{\partial F_p^T}{\partial X}(\xi_p^T p^{-g/2}).$$

Our key ingredient is the explicit formula for $F_p^T(X)$, given by Ikeda and Katsurada in [6], which expresses the polynomial $F_p^T$ in terms of the (naive) extended Gross–Keating datum $H$ of $T$ over $\mathbb{Z}_p$. The polynomial $F_p^T = F_p^{H'}$ is defined in terms of a subset $H' \subseteq H$ for any $p$ in a uniform way. Actually, if $g = 4$, then the values $\frac{\partial F_p^{H'}(p^{-2})}{\partial X}$ and $F_p^{H'}(p^{-1})$ depend only on $(a_1, a_2, a_3)$ if we write $(a_1, a_2, a_3, a_4)$ for the Gross–Keating invariant of $T$ over $\mathbb{Z}_p$.

### 1.3 Applications

#### 1.3.1 On the average of the representation numbers

Theorem 1.1 combined with the Siegel formula will identify (1.1) with four times the average of the representation numbers of a symmetric matrix of size $g - 1$ (see Conjecture 5.1 and Proposition 5.2). The following result is a special case of Proposition 5.2.
If \( T \) is a positive definite symmetric half-integral matrix of size 4 which satisfies \( \chi_T' = 1 \) and \( n_{\ell'} = 1 \) for \( \ell' \neq p \), then there exists a positive definite symmetric half-integral matrix \( T' \) of size 3 such that

\[
\sum_{(E',E)} N(\text{Hom}(E', E), T) \frac{N(\text{Hom}(E', E), T')}{\#\text{Aut}(E)\#\text{Aut}(E')} = 2 \sum_{(E',E)} N(\text{Hom}(E', E), T') \frac{N(\text{Hom}(E', E), T)}{\#\text{Aut}(E)\#\text{Aut}(E')},
\]

where \((E, E')\) extends over all pairs of isomorphism classes of supersingular elliptic curves over \( \overline{\mathbb{F}}_p \).

### 1.3.2 On the Fourier coefficients and the modular correspondences

The factor \( \frac{\partial F_p^B}{\partial X} \left( \frac{1}{p^2} \right) \) appears in Fourier coefficients of central derivatives of incoherent Eisenstein series of genus \( g - 1 \) and weight \( g \frac{1}{2} \), which have close connection with arithmetical geometry on Shimura varieties. We will be mostly interested in the case \( g = 4 \). When \( T_{m_1}, T_{m_2} \) and \( T_{m_3} \) intersect properly, the formula of Gross and Keating in [3] can be stated as follows:

\[
(T_{m_1} \cdot T_{m_2} \cdot T_{m_3}) = \sum_B \deg \mathcal{A}(B),
\]

where \( B \) extends over all positive definite symmetric half-integral matrices with diagonal entries \( m_1, m_2, m_3 \). Here \( \deg \mathcal{A}(B) = 0 \) unless \( \text{Diff}(B) \) consists of a single rational prime \( p \), in which case

\[
\deg \mathcal{A}(B) = - \frac{(\log p)}{2p^2} \frac{\partial F_p^B}{\partial X} \left( \frac{1}{p^2} \right) \sum_{(E',E)} N(\text{Hom}(E', E), B) \frac{N(\text{Hom}(E', E), T)}{\#\text{Aut}(E)\#\text{Aut}(E')}.
\]

The degree \( \deg \mathcal{A}(B) \) equals the \( B \)-th Fourier coefficient of the derivative of the Siegel Eisenstein series of weight 2 and genus 3 up to a negative constant (cf. Theorem 2.2 of [20]). We combine (1.2), Theorem 5.1 and Corollary 5.1 to obtain the following formula:

**Theorem 1.2** If \( T \) is a positive definite symmetric half-integral matrix of size 4, \( \chi_T = 1 \) and \( \text{Diff}(T) \) consists of a single prime number \( p \), then there exists a positive definite symmetric half-integral matrix \( T' \) of size 3 such that

\[
\frac{C_4(T)}{-2^8 \cdot 3^2} = \deg \mathcal{A}(T') + \frac{F_p^T (p^{-1})}{2 \sqrt{p^{t_p} (p - 1)}} \log p \sum_{(E',E)} \frac{N(\text{Hom}(E', E), T')}{\#\text{Aut}(E)\#\text{Aut}(E')},
\]

where \((E, E')\) extends over all pairs of isomorphism classes of supersingular elliptic curves over \( \overline{\mathbb{F}}_p \).

Since \( \text{Hom}(E', E) \) is a quaternary quadratic space, if \( S \) has rank greater than 4, then \( N(\text{Hom}(E', E'), S) = 0 \). Therefore when \( g \geq 5 \), the nature of Fourier coefficients of the derivative of Eisenstein series of weight 2 and genus \( g \) should be much different. The case \( g = 4 \) should be a boundary case. We will explicitly compute \( F_p^T (p^{-1}) \) in Lemma 5.2 and show that
Moreover, Corollary 5.2 says that for a fixed prime number \( p \)
\[
\lim_{\text{ord}_v(\det T) \to \infty} \frac{C_3(T)}{-2^8 \cdot 3^2 \cdot \deg \mathcal{A}(T')} = 1.
\]

### 1.4 Organizations

We now explain the lay-out of this paper. Section 2 extends the notion of incoherent Eisenstein series to the case where the point at which the Eisenstein series is evaluated lies within the left half-plane. We calculate the Fourier coefficients of those Eisenstein series and their derivatives. In Sect. 3 we derive a general formula for Fourier coefficients of derivatives of incoherent Eisenstein series. Section 4 is devoted to a local study of the Siegel series. We give the inductive expression for the special value of the derivative of the Siegel series. Section 5 is devoted to proving Theorem 5.1.

### 2 Notations

For a finite set \( A \), we denote by \( \# A \) the number of elements in \( A \). For a ring \( R \) we denote by \( M_{ij}(R) \) the set of \( i \times j \)-matrices with entries in \( R \) and write \( M_m(R) \) in place of \( M_{m,m}(R) \). The group of all invertible elements of \( M_m(R) \) and the set of symmetric matrices of size \( m \) with entries in \( R \) are denoted by \( \text{GL}_m(R) \) and \( \text{Sym}_m(R) \), respectively. Let \( E_m(R) \) be the set of elements \( (a_{ij}) \in \text{Sym}_m(R) \) such that \( a_{ij} \in 2R \) for every \( i \). For matrices \( B \in \text{Sym}_m(R) \) and \( G \in M_{m,n}(R) \) we use the abbreviation \( B[G] = GBG \), where \( G \) is the transpose of \( G \). If \( A_1, \ldots, A_r \) are square matrices, then \( \text{diag}[A_1, \ldots, A_r] \) denotes the matrix with \( A_1, \ldots, A_r \) in the diagonal blocks and 0 in all other blocks. Let \( 1_m \) be the identity matrix of degree \( m \). Put

\[
S_{pg}(R) = \left\{ \begin{array}{c} G \in \text{GL}_{2g}(R) \\ G \left( \begin{array}{cc} 0 & 1_g \\ -1_g & 0 \end{array} \right) 'G = \left( \begin{array}{cc} 0 & 1_g \\ -1_g & 0 \end{array} \right) \end{array} \right\},
\]

\[
M_g(R) = \left\{ m(A) = \begin{array}{c} A \\ 0 \\ A^{-1} \end{array} \left| A \in \text{GL}_g(R) \right. \right\},
\]

\[
N_g(R) = \left\{ n(B) = \begin{array}{c} 1_g \\ B \\ 0 \end{array} \left| B \in \text{Sym}_g(R) \right. \right\}.
\]

Let \( \mathbb{Z} \) be the set of integers and \( \mu_n \) the group of \( n \)-th roots of unity. If \( x \) is a real number, then we put \( \lfloor x \rfloor = \max \{ m \in \mathbb{Z} \mid m \leq x \} \).

### 3 Eisenstein series

Let \( k \) be a totally real number field with integer ring \( \mathcal{O} \). The set of real places of \( k \) is denoted by \( \mathcal{S}_\infty \). The completion of \( k \) at a place \( v \) is denoted by \( k_v \). Let \( (\cdot, \cdot)_k : k_v^* \times k_v^* \to \mu_2 \) denote the Hilbert symbol. We let \( \mathfrak{p} \) denote a finite place of \( k \) above a prime number \( p \) and do not use the letters \( p, q, \mathfrak{p}, q \) for a real place. Let \( q_p = \# \mathcal{O}/\mathfrak{p} \) be the order of the residue field. We
define the character \( \mathbf{e}_p \) of \( k_p \) by \( \mathbf{e}_p(x) = e(-y) \) with \( y \in \mathbb{Q}^{(p)} \) such that \( \text{Tr}_{k_p/\mathbb{Q}}(x) - y \in \mathbb{Z}_p \) if \( p \) is the rational prime divisible by \( p \). Put \( \mathbf{e}(z) = e^{2\pi i z^2} \) for \( z \in \mathbb{C} \) and \( \mathbf{e}_\infty(z) = \prod_{v \in \mathbb{Q}_\infty} \mathbf{e}(z_v) \) for \( z \in \prod_{v \in \mathbb{Q}_{\infty}} \mathbb{C} \).

Once and for all we fix a positive integer \( g \geq 2 \). Let \((V, (\ , \ ))\) be a quadratic space of dimension \( m \) over \( k_v \). Whenever we speak of a quadratic space, we always assume that \((\ , \ )\) is nondegenerate, i.e., \((u, V) = 0\) implies that \( u = 0 \). Put

\[
s_0 = \frac{1}{2}(m - g - 1).
\]

Given \( u = (u_1, \ldots, u_g) \in V^g \), we write \((u, u)\) for the \( g \times g \) symmetric matrix with \((i, j)\) entry equal to \((u_i, u_j)\). We write \( \det V \) for the element in \( k_v^\times /k_v^{\times 2} \) represented by the determinant of the matrix representation of the bilinear form \((\ , \ )\) with respect to any basis for \( V \) over \( k_v \). We define the character \( \chi^V : k_v^\times \to \mu_2 \) by

\[
\chi^V(t) = (t, (-1)^{m(m-1)/2} \det V)_{k_v}. \tag{2.1}
\]

We normalize our Hasse invariant \( \eta^V \) so that it depends only on the isomorphism class of an anisotropic kernel of \( V \) (cf. \([2, 26]\)).

**Definition 2.1** We associate to the quadratic space \( V \) over \( k_p \) of dimension \( m \) an invariant \( \eta^V \in \mu_2 \) according to the type of \( V \) as follows:

- If \( m \) is odd, then an anisotropic kernel of \( V \) has dimension \( 2 - \eta^V \).
- If \( m \) is even and \( \chi^V \neq 1 \) and if we choose an element \( c \in k_v^\times \) such that \( \chi^V(c) = \eta^V \), then \( V \) is the orthogonal sum of a split form of dimension \( m - 2 \) with the norm form scaled by the factor \( c \) on the quadratic extension of \( k_p \) corresponding to \( \chi^V \).
- If \( m \) is even and \( \chi^V = 1 \), then \( V \) is split or the orthogonal sum of the norm form on the quaternion algebra over \( k_p \) with a split form of dimension \( m - 4 \) according as \( \eta^V = 1 \) or \(-1\).

We denote the set of positive definite symmetric matrices over \( \mathbb{R} \) of rank \( g \) by \( \text{Sym}_g(\mathbb{R})^+ \). Let

\[
\mathfrak{H}_g = \{ X + \sqrt{-1}Y \in \text{Sym}_g(\mathbb{C}) \mid Y \in \text{Sym}_g(\mathbb{R})^+ \}
\]

be the Siegel upper half-space of genus \( g \). The real symplectic group \( \text{Sp}_g(\mathbb{R}) \) acts transitively on \( \mathfrak{H}_g \) by \( GZ = (AZ+B)(CZ+D)^{-1} \) for \( Z \in \mathfrak{H}_g \) and \( G = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{Sp}_g(\mathbb{R}) \). We define the maximal compact subgroups by

\[
K_p = \text{Sp}_g(\mathfrak{o}_p), \quad K_v = \{ G \in \text{Sp}_g(k_v) \mid G(\sqrt{-1}I_g) = \sqrt{-1}I_g \}
\]

for \( v \in \mathbb{Q}_\infty \). We have the Iwasawa decomposition

\[
\text{Sp}_g(k_v) = M_g(k_v)N_g(k_v)K_v.
\]

Denote the two-fold metaplectic cover of \( \text{Sp}_g(k_v) \) by \( \text{Mp}_v \). There is a canonical splitting \( N_g(k_v) \to \text{Mp}_v \). When \( p \) does not divide \( 2 \), we have a canonical splitting \( K_p \to \text{Mp}_p \). We
still use $\mathcal{N}_g(k_v)$ and $K_p$ to denote the images of these splittings. Let $\tilde{K}_v$ denote the pull-back of $K_v$ in $Mp_v$. Define the map $|a(\cdot)| : Mp_v \rightarrow \mathbb{R}_+^\times$ by writing $\tilde{G} = \mathbf{n}(b)\tilde{m} \tilde{k} \in Mp_v$ with $b \in \text{Sym}_g(k_v)$, $a \in GL_g(k_v)$, $\tilde{m} = (m(a), \zeta)$ and $\tilde{k} \in \tilde{K}_v$ and setting $|a(\tilde{G})| = |\det a|$. We refer to Section 1.1 of [31] for additional explanation.

Let $V$ be a quadratic space over $k_v$ and $\omega_v$ the Weil representation of $Mp_v$ with respect to $\mathfrak{e}_v$ on the space $\mathcal{S}(V^\vee)$ of the Schwartz functions on $V^\vee$. We associate to $\varphi \in \mathcal{S}(V^\vee)$ the function on $Mp_v \times \mathbb{C}$ by

$$f^{(s)}_\varphi(\tilde{G}) = (\omega_v(\tilde{G})\varphi(0)|a(\tilde{G})|^{s-s_0}.$$ 

The real metaplectic group acts on the half-space $\mathfrak{S}_g$ through $Sp_g(\mathbb{R})$. There is a unique factor of automorphy $J_v : Mp_v \times \mathfrak{S}_g \rightarrow \mathbb{C}^\times$ whose square descends to the automorphy factor on $Sp(k_v) \times \mathfrak{S}_g$ given by $J_v(G_v, Z_v)^2 = \det(C_vZ_v + D_v)$ for $G_v = \begin{pmatrix} * & * \\ C_v & D_v \end{pmatrix} \in Sp(k_v)$. We define an automorphy factor $J : \prod_{v \in \mathfrak{f}v} (Mp_v \times \mathfrak{S}_g) \rightarrow \mathbb{C}^\times$ by $J(\tilde{G}, Z) = \prod_{v} J_v(\tilde{G}_v, Z_v)$.

Let $\mathbb{A}$ be the adele ring of $k$ and $\mathbb{A}_f$ the finite part of the adele ring. We arbitrarily fix a quadratic character $\chi$ of $\mathbb{A}_f^\times/k^\times$ such that $\chi^\vee = \chi_v$ for all $v$, such that $C_v$ is positive definite for $v \in \mathfrak{S}_\infty$ and such that $\eta^\vee_{\mathfrak{p}} = 1$ for almost all $\mathfrak{p}$. We say that $\mathcal{C}$ is coheren $\mathcal{C}_v$ is coheren $\mathcal{C}_v$ is coheren $\mathcal{C}_v$ is coheren if it is the set of localization of a global quadratic space. Otherwise we call $\mathcal{C}$ incoherent.

One can derive the following from the theorem of Minkowski-Hasse (see Theorem 4.4 of [25]).

**Lemma 2.1** Put $d = [k : \mathbb{Q}]$. When $m$ is odd, $\mathcal{C}$ is coheren if and only if

$$(-1)^{d(m^2-1)/8} \prod_{\mathfrak{p}} \eta_{\mathfrak{p}}^{\mathcal{C}_\mathfrak{p}} = 1.$$ 

When $m$ is even, $\mathcal{C}$ is coheren if and only if $(-1)^{d(m-2)/8} \prod_{\mathfrak{p}} \eta_{\mathfrak{p}}^{\mathcal{C}_\mathfrak{p}} = 1$.

Recall from the beginning of Sect. 2 that $v$ stands for an arbitrary place of $k$ and $\mathfrak{p}$ stands for a finite place of $k$. There is a unique splitting $Sp_g(k) \subseteq Mp_g$ by which we regard $Sp_g(k)$ as the subgroup of the two-fold metaplectic cover $Mp_g$ of $Sp_g(\mathbb{A})$. Let $P_g = M_g \mathcal{N}_g$ be the Siegel parabolic subgroup of $Sp_g$. Given any pure tensor $\varphi = \otimes_{\mathfrak{p}} \varphi_{\mathfrak{p}} \in \mathfrak{S}_g^* \mathcal{S}(C^\vee_{\mathfrak{p}})$, we consider the function

$$f^{(s)}_{\varphi_{\mathfrak{p}}}(\tilde{G}) = \prod_{\mathfrak{p}} f^{(s)}_{\varphi_{\mathfrak{p}}}(\tilde{G}_{\mathfrak{p}}), \quad f^{(s)}_{\varphi_{\mathfrak{p}}}(\tilde{G}_{\mathfrak{p}}) = (\omega_{\mathfrak{p}}(\tilde{G}_{\mathfrak{p}})\varphi_{\mathfrak{p}}(0)|a(\tilde{G}_{\mathfrak{p}})|^{s-s_0}$$

on $Mp_g \times \mathbb{C}$ and the Eisenstein series on $\prod_{v \in \mathfrak{f}v} \mathfrak{S}_g$

$$E(Z, f^{(s)}_{\varphi}) = (\det Y)^{(s-s_0)/2} \sum_{\gamma \in P_g(k) \setminus Sp_g(k)} |\gamma(\gamma, Z)|^{(s-s_0)/2} \gamma^{-s} f^{(s)}_{\varphi}(\gamma),$$
where $Y$ is the imaginary part of $Z$. The series is absolutely convergent for $\Re s > \frac{g+1}{2}$. It admits a meromorphic continuation to the whole plane and its Laurent coefficients define automorphic forms. Moreover, it is holomorphic at $s = s_0$, and if $C$ is coherent, then the Siegel–Weil formula holds by [12].

From now on we require that $m \leq g + 1$. Let $V$ be a totally positive definite quadratic space of dimension $m$ over $k$. We normalize the invariant measure $dh$ on $O(V, k) \setminus O(V, \mathbb{A})$ to have total volume 1 and define the integral

$$I(Z, \varphi) = \int_{O(V, k) \setminus O(V, \mathbb{A})} \Theta(Z, h; \varphi) \, dh$$

of the theta function

$$\Theta(Z, h; \varphi) = \sum_{u \in V(k)^r} \varphi(h^{-1}u)e_{\infty}(\text{tr}((u, u)Z)).$$

Under coherent situation, the Siegel–Weil formula can now be stated as follows:

$$E(Z, f^{(s)}_\varphi)|_{s=s_0} = 2I(Z, \varphi). \quad (2.2)$$

The reader who is interested in this identity can consult Theorem 2.2(i) of [31]. On the other hand, if $C$ is incoherent, then the series $E(Z, f^{(s)}_\varphi)$ has a zero at $s = s_0$ by Corollary 5.5 of [31].

Under incoherent situation, consider the Fourier expansions

$$E(Z, f^{(s)}_\varphi) = \sum_{T \in \text{Sym}_m(k)} A(T, Y, \varphi, s)e_{\infty}(\text{tr}(TZ)),$$

$$\frac{\partial}{\partial s} E(Z, f^{(s)}_\varphi)|_{s=s_0} = \sum_{T \in \text{Sym}_m(k)} C(T, Y, \varphi)e_{\infty}(\text{tr}(TZ)),$$

where

$$Z = X + \sqrt{-1}Y, \quad C(T, Y, \varphi) = \frac{\partial}{\partial s} A(T, Y, \varphi, s)|_{s=s_0}.$$

Put $\text{Sym}^{\text{ad}}_g = \text{Sym}_g(k) \cap \text{GL}_g(k)$. When $T \in \text{Sym}^{\text{ad}}_g$, by Lemma 2.4 of [12] the Fourier coefficient has an explicit expression as an infinite product

$$A(T, Y, \varphi, s) = a(T, Y, s) \prod_p W_T(f^{(s)}_{\varphi_p}) \quad (2.3)$$

for $\Re s \gg 0$, where

$$W_T(f^{(s)}_{\varphi_p}) = \int_{\text{Sym}_g(k_p)} f^{(s)}_{\varphi_p}(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} n(z_p))e_p(\sqrt{-1}\text{tr}(Tz_p)) \, dz_p$$

and $a(T, Y, s)e_{\infty}(\sqrt{-1}\text{tr}(TY))$ is a product of the confluent hypergeometric functions investigated in [21]. Given $T \in \text{Sym}^{\text{ad}}_g$, we define the quadratic form on $V' = k^g$ by $u \mapsto T' u$ and define the Hecke character $\chi^T = \prod_v \chi^T_v$ and the Hasse invariants $\eta^T_v$, where $\chi^T_v$ is
defined in (2.1). Let Diff\((T, C)\) denote the set of places \(v\) of \(k\) such that \(T\) is not represented by \(C_v\). Let Sym\(_g^+\) denote the set of totally positive definite symmetric \(g \times g\) matrices over \(k\).

**Lemma 2.2** Let \(\varphi_p \in S(C_p^g)\) and \(T \in \text{Sym}_{g}^+\).

1. \(a(T, Y, s)\) and \(W_T\left(f_{\varphi_p}^{(s)}\right)\) are entire functions in \(s\).
2. \(\lim_{s \to s_0} W_T\left(f_{\varphi_p}^{(s)}\right) = 0\) unless \(T\) is represented by \(C_p\).
3. If \(m = g\), \(T \in \text{Sym}_{g}^+\), \(\chi^T = \chi\) and \(C\) is incoherent, then Diff\((T, C)\) is a finite set of odd cardinality. Here \(\chi\) is the quadratic character associated to \(C\) as explained in Definition 2.2.

**Proof** The first part is well-known (see [7, 21]). Lemma on p. 73 of [19] implies (2). By assumption \(\text{Diff}(T, C) = \{p \mid \eta^p = -\eta_p^T\}\). Since \(C\) is incoherent, Lemma 2.1 implies \(\prod_p \eta_p^c = -\prod_p \eta_p^T\), which proves (3). \(\square\)

Let \(T \in \text{Sym}_{g}^+\). Then \(a(T, Y, s_0)\) is independent of \(Y\), and so by Lemma 2.2(2), (3) and the Leibniz rule, \(C(T, Y, \varphi)\) is also independent of \(Y\). Put

\[
c_m(T) = a(T, Y, s_0), \quad C(T, \varphi) = C(T, Y, \varphi), \quad D_T = N_{k/\mathbb{Q}}(\det(2T)).
\]

Let \(d_k\) denote the absolute value of the discriminant of \(k\). Note that

\[
c_k(T) = c_g \cdot D_T^{-1/2}, \quad c_g = d_k^{-g(g+1)/4}\left(e\left(\frac{g^2}{8}\right)\frac{2^g \pi^{g^2/2}}{\Gamma_g\left(\frac{g}{2}\right)}\right)^d \tag{2.4}
\]

by applying (1.21K), (3.15) and (4.34K) of [21] with \(\alpha = \frac{g}{2}\) and \(\beta = 0\), where

\[
\Gamma_g(s) = \pi^{g(g-1)/4}\prod_{i=0}^{g-1} \Gamma\left(s - \frac{i}{2}\right).
\]

**Proposition 2.1** Let \(m = g\) and \(T \in \text{Sym}_{g}^+\). Suppose that \(C\) is incoherent. If \(\chi^T = \chi\), then \(C(T, \varphi) = 0\) unless \(\text{Diff}(T, C)\) is a singleton. Moreover, if \(\text{Diff}(T, C) = \{p\}\), then

\[
C(T, \varphi) = c_g D_T^{-1/2} \lim_{s \to -1/2} \beta_T^T(s) \cdot \beta_p^T(s) \frac{\partial W_T\left(f_{\varphi_p}^{(s)}\right)}{\partial s} \prod_{l \neq p} \beta_l^T(s) W_T\left(f_{\varphi_l}^{(s)}\right),
\]

where
\[ \beta^T(s) = \prod_{j=1}^{[(g+1)/2]} \zeta(2s + 2j - 1) \times \begin{cases} 1 \quad & \text{if } 2 \nmid g, \\ L(s + \frac{g+1}{2}, \chi)^{-1} \quad & \text{if } 2 | g, \end{cases} \]

\[ \beta^T_q(s) = \frac{\prod_{j=1}^{[(g+1)/2]} \zeta_q(2s + 2j - 1)}{L(s + \frac{1}{2}, \chi_q \chi_q)} \times \begin{cases} 1 \quad & \text{if } 2 \nmid g, \\ L(s + \frac{g+1}{2}, \chi_q) \quad & \text{if } 2 | g. \end{cases} \]

Note that

\[ \beta^T(s) = \prod_q \beta^T_q(s)^{-1}. \]

**Proof** For given \( \varphi \) and \( T \), let \( \mathbb{S} \) be a finite set of rational primes of \( k \) which contains \( \text{Diff}(T, C) \) and such that if \( q \not\in \mathbb{S} \), then \( q \) does not divide \( 2 \), \( \chi_q \) is unramified, \( e_q \) is of order \( 0 \), \( T \in \text{GL}_q(\mathfrak{o}_q) \), and the restriction of \( f_{\varphi_q}^{(s)} \) to \( K_q \) is \( 1 \). If \( q \in \mathbb{S} \), then

\[ W_T(f_{\varphi_q}^{(s)}) = \beta^T_q(s)^{-1} \]

(2.5)

by [22, Proposition 14.9] and [24, Section A1] (cf. Proposition 3.1 and (4.1)). By (2.3) the \( T \)-th Fourier coefficient of \( E(Z, f_{\varphi}^{(s)}) \) is given by

\[ A(T, Y, \varphi, s) = \beta^T(s)a(T, Y, s) \prod_{q\in \mathbb{S}} \beta^T_q(s)W_T(f_{\varphi_q}^{(s)}). \]  

(2.6)

Notice that the product \( \beta^T_q(s)W_T(f_{\varphi_q}^{(s)}) \) is holomorphic at \( s = -\frac{1}{2} \). Indeed, if \( \chi_q^T = \chi_q \), then \( \beta^T_q(s) \) is holomorphic at \( s = -\frac{1}{2} \) while if \( \chi_q^T \neq \chi_q \), then \( \beta^T_q(s) \) has a simple pole at \( s = -\frac{1}{2} \), but \( W_T(f_{\varphi_q}^{(s)}) \) has a zero at \( s = -\frac{1}{2} \) by Lemma 2.2(2).

Assume that \( \chi^T = \chi \). Then \( \beta^T(s) \) is holomorphic and has no zero at \( s = -\frac{1}{2} \). If \( q \in \text{Diff}(T, C) \), then \( \beta^T_q(s)W_T(f_{\varphi_q}^{(s)}) \) has a zero at \( s = -\frac{1}{2} \) by Lemma 2.2(2), which combined with (2.6) proves the first statement. We obtain the desired formula by differentiating (2.6) at \( s = -\frac{1}{2} \). \( \square \)

**Corollary 2.1** If \( m = g, C \) is incoherent and \( T \in \text{Sym}^+_g \) with \( \chi^T \neq \chi \), then

\[ C(T, \varphi) = c_s D_T^{-1/2} \lim_{s \to -1/2} \frac{\partial \beta^T(s)}{\partial s}(s) \prod_p \beta^T_p(s)W_T(f_{\varphi_p}^{(s)}). \]

**Proof** Since \( \chi_v^T = \text{sgn}^{g(g-1)/2} \) for every \( v \in \mathbb{S}_\infty \), the factor \( \frac{L(s + \frac{1}{2}, \chi^T \chi)}{\zeta(2g+1)} \) of \( \beta^T(s) \) has a zero at \( s = -\frac{1}{2} \) unless \( \chi = \chi^T \). Therefore \( \beta^T(s) \) has a zero at \( s = -\frac{1}{2} \) if \( \chi \neq \chi^T \). We can deduce Corollary 2.1 from (2.6) and the fact that \( W_T(f_{\varphi_q}^{(s)}) = \beta^T_q(s)^{-1} \) for \( q \not\in \mathbb{S} \). \( \square \)

### 4 Fourier coefficients of derivatives of Eisenstein series

For \( t \in \kappa \) there is an 8th root of unity \( \gamma_v(t) \) such that for all Schwartz functions \( \varphi \) on \( k_v \),
\[
\int_{k_v} \phi(x_v)e_v(tx_v^2) \, dx_v = \gamma_v(t)|2t|^{-1/2} \int_{k_v} \mathcal{F}\phi(x_v)e_v\left(-\frac{x_v^2}{4t}\right) \, dx_v,
\]

where \(dx_v\) is the self-dual Haar measure on \(k_v\) with respect to the Fourier transform

\[
\mathcal{F}\phi(y) = \int_{k_v} \phi(x_v)e_v(x_v,y) \, dx_v.
\]

Put

\[
\gamma(C_v) = \varepsilon_v(C_v)\gamma_v\left(\frac{1}{2}\right)^{m-1} \left(\frac{1}{2} \det C_v\right).
\]

Let \(L_p\) be an integral lattice of \(C_p\), i.e., a finitely generated \(\mathcal{O}_p\)-submodule of \(C_p\) which spans \(C_p\) over \(k_p\) and such that \((u,u) \in \mathcal{O}_p\) for every \(u \in L_p\). Note that \(2(u,w) \in \mathcal{O}_p\) for all \(u,w \in L_p\).

Let

\[
L_p^* = \{ u \in C_p \mid 2(u,w) \in \mathcal{O}_p \text{ for every } w \in L_p \}
\]

be its dual lattice. Let \(\text{ch}(L_p^g) \in S(C_p^g)\) be the characteristic function of \(L_p^g\). We write \(S_p\) for the matrix for the quadratic form on \(C_p\) with respect to a fixed basis of \(L_p\). For nondegenerate symmetric matrices \(T \in \frac{1}{2} E_p(C_p)\) and \(S \in \frac{1}{2} E_m(\mathcal{O}_p)\) the local density of representing \(T\) by \(S\) is defined by

\[
\alpha_p(S,T) = \lim_{i \to 0} \phi_p^{ig((g+1)/2m)} A_i(S,T),
\]

where

\[
A_i(S,T) = \#\{ X \in M_{m,g}(\mathcal{O}/\mathfrak{p}^i) \mid S[X] - T \in \mathfrak{p}^i \text{Sym}_g(\mathcal{O}_p) \}.
\]

Let \(b_{\kappa_p}\) denote the norm of the different of \(k_p\) over \(\mathbb{Q}_p\).

**Proposition 3.1** (cf. [10]) For every non-negative integer \(r\) we have

\[
\lim_{s \to r+s_0} W_T(f(s)_{\text{ch}(L_p^g)}) = \frac{\alpha_p\left(S_p \left(\frac{1}{2}, \mathbf{1}_r\right), T\right)}{\gamma(C_p)^g b_{\kappa_p}^{-g/2} [L_p : L_p]^{g/2}}.
\]

Here, \(s_0 = \frac{1}{2}(m-g-1)\) is associated to \(C_p\).

**Proof** Since

\[
\lim_{s \to r+s_0} W_T\left(f(s)_{\text{ch}(L_p^g)}\right) = W_T\left(f(s_0)_{\text{ch}(L_p^g \oplus M_{2g}(\mathcal{O}_p))}\right),
\]

this result can be deduced from the proof of [32, Lemma 8.3(2)]. \(\square\)

Let \(\mathcal{V}\) be a totally positive definite quadratic space of dimension \(g\) over \(k\). Fix an integral lattice \(L\) in \(\mathcal{V}\). Put
\[ L_p = L \otimes_\mathfrak{p} \mathfrak{p}, \quad \text{ch}(L^\mathfrak{p}) = \otimes_\mathfrak{p} \text{ch}(L^\mathfrak{p}). \]

For \( h \in O(V, \mathbb{A}) \) we write \( hL \) for the lattice defined by \((hL)_p = h_p L_p\). Put
\[ K_L = \{ h \in \text{SO}(V, \mathbb{A}) \mid hL = L \}, \quad \text{SO}(L) = \{ h \in \text{SO}(V, k) \mid hL = L \}. \]

**Definition 3.1** We mean by the genus (resp. class) of \( L \) the set of all lattices of the form \( hL \) with \( h \in O(V, \mathbb{A}) \) (resp. \( h \in O(V, k) \)). The proper class of \( L \) consists of all lattices of the form \( hL \) with \( h \in SO(V, k) \).

We write \( \Xi'(L) \) and \( \Xi(L) \) for the sets of classes and proper classes in the genus of \( L \), respectively. Define the mass of the genus of \( L \) by
\[ m'(L) = \sum_{\mathcal{L} \in \Xi'(L)} \frac{1}{\#O(\mathcal{L})}, \quad m(L) = \sum_{\mathcal{L} \in \Xi(L)} \frac{1}{\#\text{SO}(\mathcal{L})}. \]

**Remark 3.1** For each finite prime \( \mathfrak{p} \) there is \( h \in O(V, k_p) \) with \( \det h = -1 \) such that \( hL_\mathfrak{p} = L_\mathfrak{p} \). The genus of \( L \) therefore consists of lattices \( hL \) with \( h \in SO(V, \mathbb{A}) \). We identify \( \Xi(L) \) with double cosets for \( SO(V, k) \backslash SO(V, \mathbb{A})/K_L \) via the map \( h \mapsto hL \).

Lemma 5.6(1) of [23] says that
\[ m(L) = 2m'(L). \quad (3.1) \]

We consider the following sums of representation numbers of \( T \in \text{Sym}_s(k) \):
\[ R'(L, T) = \sum_{\mathcal{L} \in \Xi'(L)} \frac{N(\mathcal{L}, T)}{\#O(\mathcal{L})}, \quad R(L, T) = \sum_{\mathcal{L} \in \Xi(L)} \frac{N(\mathcal{L}, T)}{\#\text{SO}(\mathcal{L})}. \]

where \( N(L, T) = \# \{ u \in L^\mathfrak{p} \mid (u, u) = T \} \).

**Proposition 3.2** Notation being as above, we have
\[ 2 \frac{R(L, T)}{m(L)} = c_T D_T^{-1/2} \lim_{s \to -1/2} \beta_T(s) \prod_{\mathfrak{p}} \beta'_T(\mathfrak{p})(s) W_T \left( \rho_{\text{ch}(L^\mathfrak{p})} \right). \]

**Proof** This equality is nothing but the Siegel formula (cf. [27, Satz 2 on p. 555]). Nevertheless we reproduce its proof here because of its importance for us. Since both sides are zero unless \( V^T \simeq V \) by Lemma 2.2(2), we may identify \( V^T \) with \( V \). As is well-known, there exists \( h \in O(V^T, k) \) such that \( hL_\mathfrak{p} = L_\mathfrak{p} \) and \( \det h = -1 \). Since \( SO(V^T, \mathbb{A}) \backslash O(V^T, \mathbb{A}) = \mu_2(\mathbb{A}) \), we have
\[ I(Z, \text{ch}(L^\mathfrak{p})) = \frac{1}{2} \int_{SO(V^T, k) \backslash SO(V^T, \mathbb{A})} \Theta(Z, h; \text{ch}(L^\mathfrak{p})) \, dh. \]

Choose a finite set of double coset representatives \( \{ h_i \in SO(V^T, \mathbb{A}) \} \) so that
\[ SO(V^T, \mathbb{A}) = \bigsqcup_i SO(V^T, k)h_i K_L. \]
Then
\[
I(Z, \text{ch}(L^g)) = \frac{1}{2} \text{vol}(K_L) \sum_i \frac{\Theta(Z, h_i; \text{ch}(L^g))}{\# \text{SO}(h_iL)}.
\]

Since \(\{h_iL\}\) is a complete set of representatives for \(\Xi(L)\) and since the left coset \(\text{SO}(V^T, k) \setminus \text{SO}(V^T, A)\) can be identified with a disjoint union \(\bigsqcup_i \text{SO}(h_iL) \setminus h_iK_L\) in view of Remark 3.1, we have
\[
2 = \int_{\text{SO}(V^T, k) \setminus \text{SO}(V^T, A)} \text{d}h = \text{vol}(K_L) \sum_i \frac{1}{\# \text{SO}(h_iL)} = \text{vol}(K_L) \text{m}(L).
\]
The \(T\)-th Fourier coefficient of \(I(Z, \text{ch}(L^g))\) is equal to \(\frac{R(L, T)}{\text{m}(L)}\). The Siegel–Weil formula (2.2) and (2.4), (2.6) prove the declared identity. □

An examination of the proof of Proposition 3.2 confirms that
\[
\frac{R(L, T)}{\text{m}(L)} = \frac{R'(L, T)}{\text{m}'(L)}.
\]
We can prove the following result by combining Propositions 2.1 and 3.2.

**Proposition 3.3** We assume that \(\text{Diff}(T, C) = \{\mathfrak{p}\}\), notation and assumption being as in Proposition 2.1. Take an integral lattice \(L\) in \(V^T\) such that
\[
\lim_{s \to -1/2} W_T\left(f_{\text{ch}(L^g)}^{(s)}\right) \neq 0.
\]
If \(\varphi_4 = \text{ch}(L^g)\) for every prime ideal \(\mathfrak{p}\) distinct from \(\mathfrak{p}\), then
\[
C(T, \varphi) = 2 \frac{R(L, T)}{\text{m}(L)} \lim_{s \to -1/2} W_T\left(f_{\text{ch}(L^g)}^{(s)}\right)^{-1} \frac{\partial W_T\left(f_{\varphi_4}^{(s)}\right)}{\partial s}.
\]

## 5 Siegel series

In this section we drop the subscript \(\mathfrak{p}\). Thus \(k\) is a nonarchimedean local field of characteristic zero with integer ring \(\mathfrak{o}\). We denote the maximal ideal of \(\mathfrak{o}\) by \(\mathfrak{p}\) and the order of the residue field \(\mathfrak{o}/\mathfrak{p}\) by \(q\). Fix a prime element \(\varpi\) of \(\mathfrak{o}\). We define the additive order \(\text{ord} : k^\times \to \mathbb{Z}\) by \(\text{ord}(\varpi^i \mathfrak{o}^\times) = i\).

Let \(T \in \frac{1}{2} \mathcal{E}_g(\mathfrak{o})\) with \(\det T \neq 0\). Denote the conductor of \(T^\mathfrak{p}\) by \(\mathfrak{b}^T\). Put
\[ D_T = (-4)^{[g/2]} \det T, \]

\[
\begin{align*}
\xi^T &= \begin{cases} 
1 & \text{if } D_T \in k^{\times 2}, \\
-1 & \text{if } D_T \notin k^{\times 2} \text{ and } b_T = \mathfrak{o}, \\
0 & \text{if } D_T \notin k^{\times 2} \text{ and } b_T \neq \mathfrak{o}.
\end{cases}
\end{align*}
\]

Note that \( \chi^T(\mathfrak{o}) = \xi^T \) if \( b_T = \mathfrak{o} \).

The Siegel series associated to \( T \) is defined by

\[
b(T, s) = \sum_{z \in \text{Sym}_g(k)} \psi(-\text{tr}(Tz))\nu[z]^{-s},
\]

where \( \nu[z] = [z\mathfrak{o}^g + \mathfrak{o}^g : \mathfrak{o}^g] \) and \( \psi \) is an arbitrarily fixed additive character on \( k \) which is trivial on \( \mathfrak{o} \) but nontrivial on \( \mathfrak{p}^{-1} \). By Proposition 14.9 of [22] there exists a polynomial \( A_T(X) \in \mathbb{Z}[X] \) such that \( A_T(q^{-s}) = b(T, s) \). Moreover, this polynomial \( A_T(X) \) is divisible by the following polynomial

\[
\gamma^T(X) = (1 - X) \prod_{j=1}^{[g/2]} (1 - q^{2j}X^2) \times \begin{cases} 
1 & \text{if } g \text{ is odd}, \\
1 \quad & \text{if } g \text{ is even}.
\end{cases}
\]

Put

\[
A_T(X) = \gamma^T(X)E_T(X), \quad \widetilde{F}_T(X) = X^{-c_T/2}F_T(q^{-(g+1)/2}X).
\]

It is proved in [4, 8] that if \( g \) is even, then \( \widetilde{F}_T \in \mathbb{Q}[\sqrt{q}][X, X^{-1}] \) satisfies the functional equation

\[
\widetilde{F}_T(X) = \widetilde{F}_T(X^{-1}).
\]

Let \( C \) be a \( g \)-dimensional quadratic space over \( k \). Recall that \( S \) is the matrix for the quadratic form on \( C \) with respect to a fixed basis of \( L \), where \( L \) is an integral lattice of \( C \) as explained at the beginning of Sect. 3. If \( g \) is even, \( \xi = \chi^C \) is unramified, \( \text{det}(2S) \in \mathfrak{o}^\times \) and \( \eta^T = 1 \), then Lemma 14.8 combined with Proposition 14.3 of [22] gives

\[
\alpha \left( S \perp \frac{1}{2} \begin{pmatrix} \mathbf{1} & \mathbf{1} \end{pmatrix}, T \right) = A_T(\chi(\mathfrak{o})q^{-(g+2)/2}). \tag{4.1}
\]

For the rest of this paper we require \( g \) to be even.

**Proposition 4.1** If \( g \) is even, \( \chi \) is unramified, \( \chi^T = \chi, \eta^T = -1, \eta^C = 1 \) and \( L \) is a self-dual lattice of \( C \), then

\[
\frac{\partial}{\partial s} W_T(\text{ch}(L^s)) \big|_{s=1/2} = -\frac{\sqrt{b_k^g} \log q}{\gamma(C)^g} \frac{e_T}{\sqrt{q}} (\frac{e_T}{\sqrt{q}}) \frac{\partial F_T}{\partial X} \left( \frac{\xi^T}{\sqrt{q}} \right).
\]

**Proof** The formula is analogous to Proposition A.6 of [10], which deals with the case \( \dim C = g + 1 \). Recall that we use the additive character \( \mathbf{e}_p \circ \text{Tr}_{k_p}^k \) on \( k \), where \( p \) is the residual characteristic of \( k \). Let \( \delta_k \) be the different of \( k \) over \( \mathbb{Q}_p \). The factor
\[ b_k^{g/2} = [L : \delta_k L]^{-g/2}[\mathcal{E}_g(O) : \delta_k \mathcal{E}_g(O)] \]

intervenes the formula. Though the proof is similar, we will produce it here. Let \( \varphi = \text{ch}(L^g) \). Since \( \chi(\varphi) = \xi \) and \( s_0 = -\frac{1}{2} \) by assumption, we combine Proposition 3.1 and (4.1) with Lemmas A.2-A.3 of [10] to see that

\[
W_T(f^{(s)}_\varphi) = \gamma(C)^{-g} \sqrt{b_k} A_T \left( \xi^T q^{-(g+1+2s)/2} \right) = \gamma(C)^{-g} \sqrt{b_k} \gamma^T \left( \xi^T q^{-(g+1+2s)/2} \right) F_T \left( \xi^T q^{-(g+1+2s)/2} \right).
\]

By assumption \( \lim_{s \to -1/2} W_T(f^{(s)}_\varphi) = 0 \) in view of Lemma 2.2(2). Since \( \chi^T = \chi \), we have \( \gamma^T \left( \xi^T q^{-(g+2)/2} \right) \neq 0 \) and hence we see that \( F_T \left( \xi^T q^{-(g+2)/2} \right) = 0 \). We can obtain the stated identity by differentiating this equality at \( s = -\frac{1}{2} \).

\[ \square \]

**Definition 4.1** Let \( T = (t_{ij}) \in \left[ \xi \mathcal{E}_g(O) \right] \cap \text{GL}_g(k) \). We denote by \( S(T) \) the set of all nondecreasing sequences \( (a_1, \ldots, a_g) \) of nonnegative integers such that \( \text{ord}(t_{ii}) \geq a_i \) and \( \text{ord}(2t_{ij}) \geq \frac{a_i + a_j}{2} \) for \( 1 \leq i, j \leq g \). The Gross–Keating invariant \( \text{GK}(T) \) of \( T \) is the greatest element of \( \bigcup_{U \in \text{GL}_g(O)} S(T[U]) \) with respect to the lexicographic order.

Here, the lexicographic order is defined as follows: \( (y_1, \ldots, y_g) \) is greater than \( (z_1, \ldots, z_g) \) if there is an integer \( 1 \leq j \leq g \) such that \( y_i = z_i \) for \( i < j \) and \( y_j > z_j \). If \( q \) is odd, then it is easy to compute \( \mathbf{EGK}(T) \); the formula with odd \( q \) will be given in the paragraph just before Theorem 4.1.

Ikeda and Katsurada [6] define a set \( \text{EGK}(T) \) of invariants of \( T \) attached to \( \text{GK}(T) \), which they call the extended Gross–Keating datum of \( T \). They associated to an extended Gross–Keating datum \( H \) a polynomial

\[ \mathcal{F}(H; Y, X) \in \mathbb{Z}[Y, Y^{-1}, X^{1/2}, X^{-1/2}] \]

and show that

\[ \mathcal{F}(\text{EGK}(T); \sqrt{q}, X) = \mathcal{F}(X). \]

When \( g \) is even and \( b^T = \mathbf{a} \), one can associate to \( \text{EGK}(T) \) truncated extended Gross–Keating datum \( \text{EGK}(T)' \) of length \( g - 1 \) by Proposition 4.4 of [6]. By Definitions 4.2-4.4 of [6]

\[
\mathcal{F}(\text{EGK}(T); Y, X) = Y^{e'/2} X^{-(e' + 2)/2} \frac{1 - \xi^{T} Y^{-1} X}{X^{-1} - X} \mathcal{F}(\text{EGK}(T)'; Y, Y X) + Y^{e'/2} X^{(e' + 2)/2} \frac{1 - \xi^{T} Y^{-1} X^{-1}}{X - X^{-1}} \mathcal{F}(\text{EGK}(T)'; Y, Y X^{-1}),
\]

where \( \text{GK}(T) = (a_1, \ldots, a_g) \), \( e = 2 \left[ \frac{a_1 + \cdots + a_g}{2} \right] \) and \( e' = a_1 + \cdots + a_{g-1} \). It is worth noting that since \( b^T = \mathbf{a} \), we have \( e = a_1 + \cdots + a_g = e' \). We put

\[ F^H(X) = (q^{(g+1)/2} X^{e'/2} \mathcal{F}(H; \sqrt{q}, q^{(g+1)/2} X)). \]

Then
\[ F^{EGK(T)}(X) = (q^{(g+1)/2}X)^{\epsilon^T/2} \tilde{F}^{T}(q^{(g+1)/2}X) = F^{T}(X). \] (4.2)

If \( q \) is odd, then \( T \) is equivalent to a diagonal matrix \( \text{diag}[t_1, \ldots, t_g] \) with \( \text{ord} t_1 \leq \ldots \leq \text{ord} t_g \) and the (naive) extended Gross–Keating datum \( EGK(T) = (a_1, \ldots, a_g, \epsilon_1, \ldots, \epsilon_g) \) is given by

\[ a_i = \text{ord} t_i, \quad T^{(i)} = \text{diag}[t_1, \ldots, t_i], \quad \epsilon_i = \begin{cases} \eta^{T(i)} & \text{if } i \text{ is odd}, \\ \xi^{T(i)} & \text{if } i \text{ is even} \end{cases} \]

and \( EGK(T)^{\prime} = (a_1, \ldots, a_{g-1}; \epsilon_1, \ldots, \epsilon_{g-1}) \).

**Theorem 4.1** Assume that \( g \) is even and that \( b^T = 0 \). Then

\[ F^{H}(\xi^T q^{-g/2}) = q^{\epsilon^T/2} F^{H}(\xi^T q^{-g/2}), \]

where we put \( H = EGK(T) \) and \( H^{\prime} = EGK(T)^{\prime} \). If \( \eta^{T} = -1 \), then

\[ \frac{\xi^T}{\sqrt{q}} \frac{\partial F^{H}}{\partial X} \left( \frac{\xi^T}{\sqrt{q}} \right) = \frac{F^{H}(\xi^T q^{(2-g)/2})}{q-1} - \sqrt{q} \frac{\xi^T}{\sqrt{q}} \frac{\partial F^{H'}}{\partial X} \left( \frac{\xi^T}{\sqrt{q}} \right). \]

**Proof** Substituting \( Y = \sqrt{q} \) into \( \tilde{F}(H;Y, X) \), we get

\[
\tilde{F}(H;\sqrt{q}, X) = X^{-(e+2)/2} \frac{1 - \xi^T q^{-1/2}X}{X^1 - X} (\sqrt{q}X)^{\epsilon’/2} \tilde{F}(H';\sqrt{q}, \sqrt{q}X) \\
+ X^{(e+2)/2} \frac{1 - \xi^T q^{-1/2}X^{-1}}{X - X^{-1}} (\sqrt{q}X^{-1})^{\epsilon’/2} \tilde{F}(H';\sqrt{q}, \sqrt{q}X^{-1}) \\
= X^{-(e+2)/2} \frac{1 - \xi^T q^{-1/2}X}{X^1 - X} F^{H'}(q^{(1-g)/2}X) \\
+ X^{(e+2)/2} \frac{1 - \xi^T q^{-1/2}X^{-1}}{X - X^{-1}} F^{H'}(q^{(1-g)/2}X^{-1}).
\]

By letting \( X = \xi^T \sqrt{q} \), we get

\[ (\xi^T \sqrt{q})^{\epsilon’/2} F^{H}(\xi^T q^{-g/2}) = \tilde{F}(H;\sqrt{q}, \xi^T \sqrt{q}) = (\xi^T \sqrt{q})^{\epsilon’/2} F^{H'}(\xi^T q^{-g/2}). \]

In the proof of Proposition 4.1 we have seen by (4.2) that if \( \eta^{T} = -1 \), then

\[ F^{H}(\xi^T q^{-g/2}) = F^{T}(\xi^T q^{-g/2}) = 0 \]

and hence \( F^{H'}(\xi^T q^{-g/2}) = 0 \). We can prove the stated identity by differentiating the equality above at \( X = \xi^T \sqrt{q} \). \( \square \)

We will use the following result in the next section.

**Lemma 4.1** If \( T \) is a split symmetric half-integral matrix of size 4 over \( \mathbb{Z}_p \), namely, \( \chi^T = 1 \) and \( \eta^T = 1 \), then there exists a nondegenerate isotropic symmetric half-integral matrix \( B \) of size 3 over \( \mathbb{Z}_p \) such that \( F^{B}(X) = F^{EGK(T)}(X) \).
Proof If \( p = 2 \), then the existence of \( B \) with \( \text{EGK}(B) = \text{EGK}(T)' \) follows from Proposition 6.4 of [5]. If \( p \) is odd, then \( T \) is equivalent to a diagonal matrix \( \text{diag}[t_1, t_2, t_3, t_4] \) with \( \text{ord} t_1 \leq \text{ord} t_2 \leq \text{ord} t_3 \leq \text{ord} t_4 \). Then we may choose \( B \) as \( \text{diag}[t_1, t_2, t_3] \) by using the argument explained in the paragraph just before Theorem 4.1 so that \( \text{EGK}(B) = \text{EGK}(T)' \). Now we have \( F^B(X) = F^{\text{EGK}(B)}(X) = F^{\text{EGK}(T)'}(X) \) by (4.2). \( \square \)

6 Proofs of the main results

We discuss the classical Eisenstein series of Siegel. For this it is simplest to work over \( k = \mathbb{Q} \). For the moment we let \( g \) be a multiple of 4. Consider the series

\[
E_g(Z, s) = \sum_{\{C,D\}} \det(CZ + D)^{-s/2} |\det(CZ + D)|^{-(2s+1)/2} (\det Y)^{(2s+1)/4}.
\]

Here the sum extends over all symmetric coprime pairs modulo \( \text{GL}_g(\mathbb{Z}) \). Let \( C_p = \mathcal{H}(\mathbb{Q}_p)^{s/2} \) be the split quadratic space of dimension \( g \) over \( \mathbb{Q}_p \). Define \( \varphi = \otimes_p \varphi_p \) by taking \( \varphi_p = \text{ch}(M_{g,g}(\mathbb{Z}_p)) \in \mathcal{S}(C_p^+) \). It is known that

\[
E_g(Z, s) = E(Z, f^{(i)})
\]

(see §IV.2 of [11]). We say that the series is incoherent if the collection \( C \) of local quadratic spaces defining the Eisenstein series is incoherent. The series is incoherent if and only if \( \frac{g}{4} \) is odd due to Lemma 2.1.

Fix a positive definite symmetric half-integral matrix \( T \) of size \( g \). Recall that \( \chi_T \) stands for the primitive Dirichlet character corresponding to \( \chi^T \). Since the series \( E_g(Z, s) \) has level 1, the character \( \chi \) is trivial and \( \beta_p^{(s)} = \gamma^{(p^{-g+1+2s}/2)} \). We see that

\[
\beta_p^{(s)} W_T(f^{(i)}) = \gamma(\mathcal{C})^{-s} F_p^T \left( q^{-(g+1+2s)/2} \right) \quad (5.1)
\]

as in the proof of Proposition 4.1. Recall that \( F_p^T(X) = 1 \) if \( p \) and \( D_T \) are coprime. By (2.6) the \( T \)-th Fourier coefficient of \( E_g(Z, s) \) is given by

\[
A(T, Y, s) = \frac{a(T, Y, s) L(s + \frac{1}{2}, \chi_T)}{\zeta(s + \frac{g+1}{2}) \prod_{i=1}^{g/2} \zeta(2s + 2i - 1) \prod_{p \mid D_T} F_p^T(p^{-(2s+g+1)/2})}.
\]

The \( T \)-th Fourier coefficient of \( \frac{\partial}{\partial s} E_g(Z, s) \bigg|_{s=-1/2} \) is given by

\[
C_g(T) = \frac{\partial}{\partial s} A(T, Y, s) \bigg|_{s=-1/2}.
\]

Recall that \( \text{Diff}(T) = \{ p \mid \eta_p^T = -1 \} \).

Proposition 5.1 Assume that \( \frac{g}{4} \) is odd. Let \( T \in \frac{1}{2} \mathcal{E}_g(Z) \cap \text{Sym}^+_g \).

1. If \( \chi_T = 1 \), then \( C_g(T) = 0 \) unless \( \text{Diff}(T) \) is a singleton.
2. If \( \chi_T = 1 \) and \( \text{Diff}(T) = \{ p \} \), then
\[ C_{g}(T) = -\frac{2^{(g+2)/2} p^{-(g+2)/2} \log p}{\zeta(1 - \frac{g}{2}) \prod_{i=1}^{(g-2)/2} \zeta(1 - 2i)} \ \frac{\partial F^{T}_{\rho}}{\partial X} (p^{-\frac{g}{2}}) \ \prod_{p \neq \ell \mid D_{T}} \ell^{-\frac{g}{2} \ell^{T} F^{T}_{\rho}(\ell^{-\frac{g}{2}})}. \]

(3) If \( \chi_{T} \neq 1 \), then
\[ C_{g}(T) = -\frac{2^{(g+2)/2} L(1, \chi_{T})}{\zeta(1 - \frac{g}{2}) \prod_{i=1}^{(g-2)/2} \zeta(1 - 2i)} \ \prod_{p \mid D_{T}} p^{-\frac{g}{2} \ell^{T} F^{T}_{\rho}(p^{-\frac{g}{2}})}. \]

**Proof** We have already proved (1) in Proposition 2.1. Taking
\[ \zeta(2i) = (-1)^{i} \frac{(2\pi)^{2i}}{2(2i - 1)!} \zeta(1 - 2i) \]
into account, we have
\[ \zeta \left( g \right) \prod_{i=1}^{(g-2)/2} \zeta(2i) = \frac{(2\pi)^{2i} \zeta(1 - \frac{g}{2})}{2^{g/2}(\frac{g}{2} - 1)!} \prod_{i=1}^{(g-2)/2} \frac{\zeta(1 - 2i)}{(2i - 1)!} \]
Recall that \( a(T, Y, -\frac{1}{2}) = \frac{2^{2\pi X^{2}/2}}{\Gamma(\frac{1}{2})^{2}} \) by (2.4). Since
\[ \Gamma \left( g \right) = \frac{(2\pi)^{g/4}}{2^{(g^{2}-2g)/4}} \prod_{i=1}^{(g-2)/2} (2i)!, \ \zeta(0) = -\frac{1}{2}, \ \zeta'(0, \chi_{T}) = \frac{\sqrt{b^{T}}}{2} L(1, \chi_{T}), \]
we get (2) and (3) from Proposition 2.1, Corollary 2.1 and (5.1). Here \( L(0, \chi_{T}) = 0 \) since the positivity of \( D_{T} \) yields \( \chi_{T}(-1) = 1. \)

Now we let \( g = 4 \). By a quaternion algebra over a field \( k \) we mean a central simple algebra over \( k \) of dimension 4. Let \( \mathbb{H}_{p} \) denote the definite quaternion algebra over \( k = \mathbb{Q} \) that ramifies only at a prime number \( p \). The reduced norm \( \text{Nrd} \) on \( \mathbb{H}_{p} \) defines a positive definite quadratic space \( \mathcal{V}_{p} \). Fix a maximal order \( \mathcal{O}_{p} \) of \( \mathbb{H}_{p} \). Let \( \varphi_{\ell} \in \text{SL}(C) \) be the characteristic function of \( M_{2}(\mathbb{Z}_{p})^{-} \) and \( \varphi'_{p} \in \text{SL}(\mathcal{V}_{p}^{4}(\mathbb{Q}_{p})) \) the characteristic function of \( \mathcal{O}_{p}^{4} \otimes \mathbb{Z}_{p} \). We regard \( \varphi' = \varphi'_{p} \otimes (\otimes_{\ell \neq p} \varphi_{\ell}) \) as the characteristic function of \( \mathcal{O}_{p}^{4} \otimes \hat{\mathbb{Z}} \). We write \( S_{p} \) for the matrix representation of \( \mathcal{V}_{p} \) with respect to a \( \mathbb{Z} \)-basis of \( \mathcal{O}_{p} \). Put
\[ S_{0} = \text{diag} \left[ \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right), \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right) \right]. \]

**Lemma 5.1** Let \( T \in \text{Sym}_{4}(\mathbb{Q}_{p}) \).

1. If \( T \notin \frac{1}{2} \mathcal{E}_{4}(\mathbb{Z}_{p}) \), then \( W_{T} \left( \varphi'_{p} \right) \) is identically zero.
2. If \( T \in \frac{1}{2} \mathcal{E}_{4}(\mathbb{Z}_{p}) \) with \( \det T \neq 0 \), \( \chi^{T} = 1 \) and \( \eta_{p}^{T} = -1 \), then
\[
\lim_{s \to -1/2} \frac{W_S(p^{s(s)})}{W_T(p^{s(s)})} \frac{\frac{\partial}{\partial s} W_T(p^{s(s)})}{p W_S(p^{s(s)})} = \left( p^{2-\frac{\partial F^H}{\partial X}(p-2)} - \frac{p^{2-\frac{\partial F^H}{\partial X}(p-2)}}{p-1} \right) \log p,
\]

where we put \( H' = EGK_{\rho}(T)'. \)

**Proof** The first part is trivial. Since
\[
\alpha_p(S, T) = p^{(e_p^p-2)/2} \alpha_p(S, S)
\]
by Hilfssatz 17 of [27], it follows from Proposition 3.1 that
\[
\lim_{s \to -1/2} \frac{W_S(p^{s(s)})}{W_T(p^{s(s)})} = p^{-(e_p^p-2)/2}.
\]

On the other hand, since \( \xi_p^{S_0} = 1 \) and \( F_p^{S_0}(X) = 1 \), we have
\[
W_{S_0}(p^{s(s)}) = \gamma(H(Q_p)^2) - 4 \gamma_{S_0}(p^{-(5+2)/2})
\]
by Proposition 3.1 and (4.1), where \( H(Q_p)^2 \) is the split quaternary quadratic space over \( Q_p \).

It is a special case of Proposition 4.1 that
\[
\frac{\partial}{\partial s} W_T(p^{s(s)}) \bigg|_{s=-1/2} = -\frac{\log p}{\gamma(C_p)^2 p^{2-2} \gamma^{T}(p-2)} \frac{\partial F^T}{\partial X}(p-2).
\]

Since \( \gamma_{S_0}(X) = \gamma^{T}(X) \) and \( \gamma(C_p)^2 = \gamma(V_p)^2 \) by definition, we get
\[
\lim_{s \to -1/2} \frac{\frac{\partial}{\partial s} W_T(p^{s(s)})}{W_{S_0}(p^{s(s)})} = -p^{-2} \frac{\partial F^T}{\partial X}(p-2) \log p.
\]

Theorem 4.1 now gives
\[
\lim_{s \to -1/2} \frac{\frac{\partial}{\partial s} W_T(p^{s(s)})}{W_{S_0}(p^{s(s)})} = \left( p^{(e_p^p-4)/2} \frac{\partial F^H}{\partial X}(p-2) - \frac{F^H(p-1)}{p-1} \right) \log p.
\]

These complete our proof. \( \square \)

Let \( \overline{F}_p \) be an algebraic closure of a finite field \( F_p \) with \( p \) elements. For two supersingular elliptic curves \( E, E' \) over \( \overline{F}_p \), we consider the free \( \mathbb{Z} \)-module \( \text{Hom}(E', E) \) of homomorphisms \( E' \to E \) over \( \overline{F}_p \) together with the quadratic form given by the degree. As \( E \) and \( E' \) are supersingular, \( \text{Hom}(E', E) \) has rank 4 as a \( \mathbb{Z} \)-module. For two quadratic spaces over \( \mathbb{Z} \) we write \( N(L, L') \) for the number of isometries \( L' \to L \).

We are now ready to prove our main result.
Theorem 5.1 If $T \in \frac{1}{2} E_4(\mathbb{Z})$ is positive definite, $\chi_T = 1$ and $\text{Diff}(T)$ consists of a single prime $p$, then

$$C_4(T) = 2^6 \cdot 3^2 \left( p^{-2} \frac{\partial F''_p}{\partial X}(p^{-2}) - \frac{F''_p(p^{-1})}{\sqrt{p}^e (p - 1)} \right) \log p \sum_{(E', E)} N(Hom(E', E), T) \# \text{Aut}(E') \# \text{Aut}(E''),$$

where we put $H' = \text{EGK}_p(T)'$ and where $(E', E)$ extends over all pairs of isomorphism classes of supersingular elliptic curves over $\widetilde{\mathbb{F}}_p$.

Proof Proposition 3.3 and (3.2) applied to $L = O_p$ gives

$$C_4(T) = R'(O_p, T) c \lim_{s \to -1/2} W_{S_p}(f_{\varphi_p}^{(s)}) \frac{\partial}{\partial s} W_T(f_{\varphi_p}^{(s)}) \frac{p W_{S_0}(f_{\varphi_p}^{(s)})}{W_T(f_{\varphi_p}^{(s)})},$$

where

$$c = \frac{2p}{m'(O_p)} \lim_{s \to -1/2} \frac{W_{S_0}(f_{\varphi_p}^{(s)})}{W_{S_p}(f_{\varphi_p}^{(s)})}.$$ 

If $T = S_p$, then we claim that $R'(O_p, S_p) = 1$. To prove this, it suffices to show that $N(\mathcal{L}, S_p) = 0$ if $\mathcal{L}$ is not isometric to $O_p$ and $N(\mathcal{L}, S_p) = \# O(O_p)$, where $\mathcal{L} \in \Xi'(O_p)$. If $N(\mathcal{L}, S_p) \neq 0$, then there is an injection $f : O_p \to \mathcal{L}$ as a lattice preserving the associated quadratic forms. Thus we only need to show that $f$ is surjective. If it is not surjective, then $\mathcal{L}$ and $O_p$ have different discriminant, which is a contradiction to the assumption that $\mathcal{L}$ and $O_p$ are in the same genus.

Note that $R(O_p, T) = 2R'(O_p, T)$ by (3.1) and (3.2), and

$$R(O_p, T) = \sum_{\mathcal{L} \in \Xi(O_p)} \frac{N(\mathcal{L}, T)}{\# \text{SO}(\mathcal{L})} = \sum_{(E', E)} \frac{N(Hom(E', E), T)}{\# \text{Aut}(E) \# \text{Aut}(E')}$$

by Proposition 4.1 of [29]. Our statement follows from Lemma 5.1(2) and the fact that $c = 2^7 \cdot 3^2$. Applying the Siegel formula stated in Proposition 3.2, (2.5) and (3.2) to $T = S_p$, we get the formula

$$\frac{2}{m'(O_p)} = c_4 D_{S_p}^{-1/2} \lim_{s \to -1/2} p^T(s) \cdot p^T(s) W_{S_p}(f_{\varphi_p}^{(s)})$$

It follows that

$$c = pc_4 D_{S_p}^{-1/2} \lim_{s \to -1/2} \prod_{\ell} W_{S_0}(f_{\varphi_{\ell}}^{(s)})$$

$$= c_4 \lim_{s \to -1/2} \prod_{\ell} \gamma_{S_p}(\ell^{-2s+2}) = \frac{c_4}{\gamma(2)^2} \lim_{s \to -1/2} \gamma(2s+1) = 2^7 \cdot 3^2$$

as claimed. \qed
We temporarily let \( g \) be an arbitrary multiple of 4 in the following conjecture:

**Conjecture 5.1** Let \( \mathcal{V} \) be a totally positive definite quadratic space over a totally real number field \( k \) of dimension \( g \). Fix a maximal integral lattice \( L \) of \( \mathcal{V} \). Let \( T \in \frac{1}{2} \mathcal{E}_g(\mathcal{O}) \) be totally positive definite. If \( g \) is even and \( \chi^T = 1 \), then there is a totally positive definite matrix \( T' \in \frac{1}{2} \mathcal{E}_{g-1}(\mathcal{O}) \) such that

\[
R(L, T) = 2R(L, T').
\]

**Proposition 5.2** If \( k = \mathbb{Q} \) and \( g = 4 \), then Conjecture 5.1 is true.

**Proof** Since \( R(L, T) = 0 \) unless \( \chi^T = 1 \) and \( \text{Diff}(T) = \text{Diff}(\mathcal{V}) \), we assume that

\[
\chi^T = 1, \quad \text{Diff}(T) = \text{Diff}(\mathcal{V}).
\]

**Lemma 4.1** gives \( T'_p \in \frac{1}{2} \mathcal{E}_3(\mathbb{Z}_p) \) such that \( F_{p}^{T'} = F_{p}^{\text{EGK}_p(T') \chi^T} \) for every rational prime \( p \). In addition, the proof of Lemma 4.1 yields that \( T' \) is unimodular for almost all primes \( p \). Thus we can find a positive rational number \( 0 < \delta \in \mathbb{Q}^\times \) such that \( \delta^{-1} \det T'_p \in \mathbb{Z}_p^\times \) for every \( p \not\in \text{Diff}(\mathcal{V}) \). For \( p \in \text{Diff}(\mathcal{V}) \) we fix an arbitrary anisotropic ternary quadratic form \( T'_p \) over \( \mathbb{Z}_p \). Recall that \( \alpha_p(S_p, T') \) is independent of the choice of \( T'_p \).

Since \( F_{p}^{\text{Diff}_p} = F_{p}^{T'} \) for \( u \in \mathbb{Z}_p^\times \), there is no harm in assuming that \( \delta = \det T'_p \). Since \( \eta_p^{T'} = 1 \) for \( p \not\in \text{Diff}(\mathcal{V}) \), the Minkowski-Hasse theorem (cf. Lemma 2.1) gives \( z \in \text{Sym}_3(\mathbb{Q}) \) which is positive definite and such that \( z \in T'_p[\text{GL}_3(\mathbb{Q}_p)] \) for every \( p \). Take \( A \in \text{GL}_3(\mathbb{A}_F) \) so that \( z = T'_p[A] \) for every \( p \). We can take \( D \in \text{GL}_3(\mathbb{Q}) \) in such a way that \( AD^{-1} \in \text{GL}_3(\mathbb{Z}_p) \) for every \( p \). Put \( T' = z[D^{-1}] \). Then \( T' \in T'_p[\text{GL}_3(\mathbb{Z}_p)] \) for every \( p \). In particular, \( T' \in \frac{1}{2} \mathcal{E}_3(\mathbb{Z}) \).

In view of (3.2) it suffices to show that

\[
\frac{R'(L, T)}{m'(L)} = 2 \frac{R'(L, T')}{{m'}(L)}.
\]

Since the product \( \prod_p \zeta_p(s) \) is absolutely convergent for \( \Re s > 1 \), we can use \( \zeta_p(2s+1) \) instead of \( \beta_p^-(s) \) as a convergence factor, and Proposition 3.1 gives

\[
2 \frac{R(L, T)}{m(L)} = c_4 D_T^{-1/2} \lim_{s \to -1/2} \zeta(s + \frac{1}{2}) \prod_p \frac{\zeta_p(2s + 1)}{\zeta_p(s + \frac{1}{2})} W_T(f^{(s)}_{\text{ch}(L'_s)}).
\]

Since \( \frac{R'(L, T)}{m'(L)} = \frac{R(L, T)}{m(L)} \) by (3.2),

\[
\frac{R'(L, T)}{m'(L)} = 2^{-1} c_4 D_T^{-1/2} \prod_p \lim_{s \to -1/2} \frac{W_T(f^{(s)}_{\text{ch}(L'_s)})}{2} = 2^4 \pi^4 D_T^{-1/2} \prod_{p \in \text{Diff}(\mathcal{V})} \frac{\alpha_p(S_p, T)}{2[L^*_p : L_p]^2} \prod_{q \not\in \text{Diff}(\mathcal{V})} (1 - q^{-2})^2 F_q(T')(q^{-2})
\]

by (2.4) and Proposition 3.1.

Recall that the nonarchimedean densities for \( p \in \text{Diff}(\mathcal{V}) \) are given by

\[
\alpha_p(S_p, T') = 2(p + 1)(1 + p^{-1}), \quad \alpha_p(S_p, T) = 4p^{3/2}(p + 1)^2.
\]
by [30, Theorem 1.1] and Proposition 5.7 of [1]. Since \([L^* : L] = \prod_{p \in \text{Diff}(\mathcal{V})} p^2\) by assumption and since \(F_q(q^{-2}) = q^{\ell/2}F_q(q^{-2})\) by Theorem 4.1, we obtain

\[
\frac{R'(L, T)}{m'(L)} = d_{\infty}(L, T') 2^3 \prod_{p \in \text{Diff}(\mathcal{V})} \alpha_p(S_p, T') \prod_{q \notin \text{Diff}(\mathcal{V})} (1 - q^{-2})^2 F_q'(q^{-2}),
\]

where

\[
d_{\infty}(L, T') = \frac{\prod_{i=2}^{4} p^{\ell/2}}{[L^* : L]^{3/2}}
\]

denotes the archimedean density. The final expression equals \(2 \frac{R'(L, T')}{m'(L)}\) by the Siegel formula for \(L\) and \(T'\) (cf. [27, Satz 2 on p. 555]).

\[\square\]

**Corollary 5.1** If \(T\) is a positive definite symmetric half-integral matrix of size 4 which satisfies \(\chi^T = 1\) and \(n_{\ell}^T = 1\) for \(\ell \neq p\), then there exists a positive definite symmetric half-integral matrix \(T'\) of size 3 such that

\[
\sum_{(E', E)} \frac{N(\text{Hom}(E', E), T)}{\#\text{Aut}(E)\#\text{Aut}(E')} = 2 \sum_{(E', E)} \frac{N(\text{Hom}(E', E), T')}{\#\text{Aut}(E)\#\text{Aut}(E')},
\]

where \((E, E')\) extends over all pairs of isomorphism classes of supersingular elliptic curves over \(\mathbb{F}_p\).

**Proof** Proposition 4.1 of [29] gives

\[
R(C_p, T') = \sum_{L \in \Xi(C_p)} \frac{N(L, T')}{\#\text{SO}(L)} = \sum_{(E', E)} \frac{N(\text{Hom}(E', E), T')}{\#\text{Aut}(E)\#\text{Aut}(E')}.\]

We can derive Corollary 5.1 from (5.2) and Proposition 5.2.

\[\square\]

Let \(T \in \frac{1}{2} \mathcal{E}_4(\mathbb{Z}_p)\) be an anisotropic symmetric matrix with (naive) extended Gross-Keating invariant \((a_1, a_2, a_3, a_4, \epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4)\). Note that \(\epsilon_1 = \epsilon_4 = 1\) by definition. One can easily see that \(\epsilon_2 \neq 1\) and \(\epsilon_3 = -1\). Proposition 5.3 of [1] gives a partition \(\{1, 2, 3, 4\} = \{i, j\} \cup \{k, l\}\) such that

\[a_i \equiv a_j \not\equiv a_k \equiv a_l \pmod{2}.
\]

**Lemma 5.2**

1. If \(a_1 \not\equiv a_2 \pmod{2}\), then

\[
F'_{p^{(a_1+1)}} = \frac{p^{a_1+1} - 1}{(p - 1)(p^3 - 1)} \left( p^{(a_1+3(a_2+1))/2} - \frac{p^{a_1+1}}{p + 1} \right) - \frac{p^{(a_1+a_2+a_3+1)/2}}{p - 1} \left( (a_1 + 1)p^{(a_1+a_2+a_3+1)/2} - \frac{p^{a_1+1} - 1}{p - 1} \right).
\]
If $a_1 \equiv a_2 \pmod{2}$, then
\[
F_p^{T'}(p^{-1}) = \frac{p^{a_1+1}}{(p-1)(p^3-1)} \left( p^{(a_1+3a_2)/2} - \frac{p^{a_1+1} + 1}{p+1} \right)
- \frac{p^{(a_1+a_2+2a_1+2)/2}}{p-1} \left( (a_1 + 1)p^{(a_1+a_2)/2} - \frac{p^{a_1+1} - 1}{p-1} \right)
+ p^{(a_1+3a_2)/2} \frac{p^{a_1+1} - 1}{p^2-1} (p^{a_1-a_2+1} + 1).
\]

**Proof** We write the naive extended Gross-Keating invariant of $T$ as
\[
EGK_p(T) = (a_1, a_2, a_3, a_4; 1, \varepsilon_2, \varepsilon_3, 1).
\]
Let $\sigma$ be either 1 or 2 according as $a_1 - a_2$ is odd or even. Section 8 of [6] expresses $F_p^{EGK_p(T)}(X)$ in terms of $EGK_p(T)' = (a_1, a_2, a_3; 1, \varepsilon_2, \varepsilon_3)$:
\[
F_p^{EGK_p(T)'}(p^{-2}X) = \sum_{i=0}^{a_1} \sum_{j=0}^{(a_1+a_2-\sigma)/2-i} p^{i+j} X^{i+j}
+ \varepsilon_3 \sum_{i=0}^{a_1} \sum_{j=0}^{(a_1+a_2-\sigma)/2-i} p^{(a_1+a_2-\sigma)/2-j} X^{a_2+i+j+2j}
+ \varepsilon_2^2 p^{(a_1+a_2+2)/2} \sum_{i=0}^{a_1} \sum_{j=0}^{a_2-a_2+2\sigma-4} \varepsilon_2^j X^{a_2-\sigma+2i+j}.
\]
We now specialize the formula to $X = p$ and $\varepsilon_3 = -1$. Then
\[
F_p^{T'}(p^{-1}) = \frac{p^{a_1+1} - 1}{(p-1)(p^3-1)} \left( p^{(a_1+3(a_2-\sigma+2))/2} - \frac{p^{a_1+1} + 1}{p+1} \right)
- \frac{p^{(a_1+a_2+2a_1+\sigma)/2}}{p-1} \left( (a_1 + 1)p^{(a_1+a_2-\sigma+2)/2} - \frac{p^{a_1+1} - 1}{p-1} \right)
+ \varepsilon_2^2 p^{(a_1+3(a_2-\sigma+2))/2} \frac{p^{a_1+1} - 1}{p^2-1} (1 - (\varepsilon_2^2 p^{a_1-a_2+2\sigma-3})/((p-1)(1-\varepsilon_2^2 p))).
\]
Since $\varepsilon_2 = 0$ or $-1$ according as $a_1 - a_2$ is odd or even by Proposition 2.2 of [5] and Proposition 4.8 of [1], we obtain the stated formulas. \qed

The degree $\deg \mathcal{Z}(B)$ is defined in (1.2) for positive definite symmetric half-integral $3 \times 3$ matrices $B$ such that $\text{Diff}(B)$ is a singleton.

**Corollary 5.2** Let $T$ be a positive definite symmetric half-integral $4 \times 4$ matrix such that $\chi_T = 1$ and $\text{Diff}(T) = \{p\}$. Let $\sigma$ be either 1 or 2 according as $a_1 - a_2$ is odd or even. If $\deg \mathcal{Z}(T') \neq 0$, then
\[
\left| \frac{C_4(T)}{-2^8 \cdot 3^2 \cdot \deg \mathcal{Z}(T')} \right| = 1 < \frac{4}{p^{\sigma/p}} \left( p^{-(a_1-3\sigma)/2} + \frac{4p^{-(a_1-a_2)/2}}{a_1+1} \right),
\]
where \( \text{GK}_p(T) = (a_1, a_2, a_3, a_4) \). In particular,
\[
\left| \frac{C_4(T)}{-2^8 \cdot 3^2 \cdot \deg \mathcal{G}(T)} - 1 \right| < \frac{20}{p \sqrt{p}}, \quad \lim_{e_p \to -\infty} \frac{C_4(T)}{-2^9 \cdot 3^2 \cdot \deg \mathcal{G}(T')} = 1.
\]

**Proof** By (2.12) and (2.13) of [30]
\[
-p^2 \frac{\partial F^H}{\partial X}(p^{-2}) \geq (a_1 + 1)p^{(a_1 + a_2)/2} \left( \frac{a_3 - a_2 + 2\sigma}{\sqrt{p}} + \epsilon^2 a_3 - a_2 + 1 \right)
\]
\[
\geq (a_1 + 1)p^{(a_1 + a_2 - (2 - \sigma))/2}.
\]

Recall that if \( \sigma = 1 \), then \( a_1 < a_2 \leq a_3 \leq a_4 \) while if \( \sigma = 2 \), then \( a_1 \leq a_2 < a_3 \leq a_4 \). An examination of the proof of Lemma 5.2 confirms that
\[
\left| \frac{F^H(p^{-1})}{p^2(p-1)} \right| \leq \frac{a_1 + 1}{(p-1)^2}p^{(a_1 + a_2 - a_3 + 2)/2} + \frac{p^{(a_1 + a_2 - a_3 + 3)/2 + 4}}{(p-1)^2(p^3 - 1)}
\]
\[
+ \epsilon^2 \frac{p^{2a_1 + 2 - (a_3 + a_4)/2}}{(p-1)^2(p + 1)} + \epsilon^2 \frac{p^{a_1 + a_2 + 1 - (a_3 + a_4)/2}}{(p-1)^2(p + 1)}
\]
\[
< 4p^{(a_1 + a_2)/2 - 1} \{(a_1 + 1)p^{-a_2/2} + 2p^{-(a_3 - a_1 + 3)/2} + 2\epsilon^2 p^{-(a_1 - a_1 + 1)/2})\}.
\]

Now our proof is completed by Theorem 1.2. \( \square \)

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