Hypothesis Testing for Functional Linear Models via Bootstrapping

Yinan Lin∗ and Zhenhua Lin†

Department of Statistics and Data Science, National University of Singapore

Abstract

Hypothesis testing for the slope function in functional linear regression is of both practical and theoretical interest. We develop a novel test for the nullity of the slope function, where testing the slope function is transformed into testing a high-dimensional vector based on functional principal component analysis. This transformation fully circumvents ill-posedness in functional linear regression, thereby enhancing numeric stability. The proposed method leverages the technique of bootstrapping max statistics and exploits the inherent variance decay property of functional data, improving the empirical power of tests especially when the sample size is limited or the signal is relatively weak. We establish validity and consistency of our proposed test when the functional principal components are derived from data. Moreover, we show that the test maintains its asymptotic validity and consistency, even when including all empirical functional principal components in our test statistics. This sharply contrasts with the task of estimating the slope function, which requires a delicate choice of the number (at most in the order of \(\sqrt{n}\)) of functional principal components to ensure estimation consistency. This distinction highlights an interesting difference between estimation and statistical inference regarding the slope function in functional linear regression. To the best of our knowledge, the proposed test is the first of its kind to utilize all empirical functional principal components.

Keywords: uniform bootstrap approximation; slope function; uniform Gaussian approximation; ill-posedness; max statistic.

∗stayina@nus.edu.sg
†linz@nus.edu.sg
1 Introduction

Functional data are nowadays common in practice and have been extensively studied in the past decades. For a comprehensive treatment on the subject of functional data analysis, we recommend the monographs Ramsay and Silverman (2005) and Kokoszka and Reimherr (2017) for an introduction, Ferraty and Vieu (2006) for nonparametric functional data analysis, Hsing and Eubank (2015) from a theoretical perspective, and Horváth and Kokoszka (2012) and Zhang (2013) with a focus on statistical inference.

Functional linear models that pair a response variable with a predictor variable in a linear way, where at least one of the variables is a function, play an important role in functional data analysis. A functional linear model (FLM), in its general form that accommodates both functional responses and/or functional predictors, can be mathematically represented by

\[ Y - \mathbb{E}Y = \beta(X - \mathbb{E}X) + Z, \] (1)

where \( Y, Z \in \mathcal{Y}, X \in \mathcal{X} \), with \((\mathcal{X}, \langle \cdot, \cdot \rangle_1)\) and \((\mathcal{Y}, \langle \cdot, \cdot \rangle_2)\) being two separable Hilbert spaces respectively endowed with the inner products \( \langle \cdot, \cdot \rangle_1 \) and \( \langle \cdot, \cdot \rangle_2 \), and \( \beta \), called the slope operator, is an unknown Hilbert–Schmidt operator between \( \mathcal{X} \) and \( \mathcal{Y} \). The variable \( Z \), representing a random error, is assumed to be centered, of finite variance, and independent of \( X \). The model (1) includes the following popular models as special cases: the scalar-on-function model with a scalar response and a functional predictor, the function-on-function model with both functional response and predictor, the function-on-vector model (also known as the varying coefficient model in Shen and Faraway (2004)) with a functional response and multiple scalar predictors, the model with mixed-type predictors (Cao et al., 2020), and the partial functional linear model (Shin, 2009); see Section S10 of Lin and Lin (2024) for more details. These models have been investigated, for example, among many others, by Cardot et al. (1999, 2003b); Yao et al. (2005); Hall and Horowitz (2007); James et al. (2009); Yuan and Cai (2010); Zhou et al. (2013); Lin et al. (2017); Shen and Faraway (2004); Zhang (2011); Zhu et al. (2012); Cao et al. (2020); Wang et al. (2020); Shin (2009); Kong et al. (2016), with a focus on estimation of the slope operator in one of these models.

It is also of practical importance to check whether the predictor \( X \) has influence on the response \( Y \) in the postulated model (1), which corresponds to whether the slope operator is null and can be cast into the following hypothesis testing problem

\[ H_0 : \beta = 0 \quad \text{v.s.} \quad H_a : \beta \neq 0. \] (2)

This problem has been investigated in the literature, with more attention given to the scalar-on-function model. For example, among many others, Hilgert et al. (2013) proposed Fisher-type parametric tests with random projection to empirical functional principal components by using
multiple testing techniques, Lei (2014) introduced an exponential scan test by utilizing the estimator for $\beta$ proposed in Hall and Horowitz (2007) that is based on functional principal component analysis, Qu and Wang (2017) developed generalized likelihood ratio test using smoothing splines, and Xue and Yao (2021), exploiting the techniques developed for post-regularization inferences, constructed a test for the case that there are an ultrahigh number of functional predictors. For the function-on-function model, Kokoszka et al. (2008) proposed a weighted $L_2$ test statistic based on functional principal component analysis, and Lai et al. (2021) developed a goodness-of-fit test based on generalized distance covariance. For the function-on-vector model, Shen and Faraway (2004); Zhang (2011); Smaga (2019) proposed functional F-tests while Zhu et al. (2012) considered a wild bootstrap method.

In this paper, we develop a novel approach to testing (2) under the model (1), with the following distinct features. First, by exploiting principal component analysis of $X$ and $Y$, we propose a suitable transformation that transfers the test on the slope operator to a test on a high-dimensional vector of entries $\nu_{jk} := \mathbb{E}(\langle X - \mathbb{E}X, \phi_j \rangle_1 \langle Y - \mathbb{E}Y, \psi_k \rangle_2)$, where $\phi_j$ and $\psi_k$ are population principal components of $X$ and $Y$, respectively; see Section 2 for details. While there exist methods (e.g., Kokoszka et al., 2008; Hilgert et al., 2013; Lei, 2014; Su et al., 2017) that transfer testing (2) into testing vectors, these vectors consist of the coefficients of $\beta$ with respect to $\phi_j$ and $\psi_k$. These coefficients however involve $\lambda_j^{-1}$ of the eigenvalues $\lambda_j$ that decay to zero in the setting of functional data, thereby facing the issue of ill-posedness. In contrast, the novelty of our transformation lies in eliminating $\lambda_j^{-1}$ and thus fully circumventing the ill-posed problem. In particular, it allows for incorporating all empirical principal components into our test statistic. Consequently, unlike existing counterparts, our test does not require an intricate choice of the number of empirical principal components, thereby enhancing numeric stability and potentially increasing the test’s power.

Second, observing that the entries $\nu_{jk}$ are population means of the random variables $\langle X - \mathbb{E}X, \phi_j \rangle_1 \langle Y - \mathbb{E}Y, \psi_k \rangle_2$, we propose to construct a max-type test statistic and bootstrap it via exploiting the inherent variance decay patterns of these random variables, achieving higher power especially when the sample size is limited and/or the signal is weak. While the strategy of bootstrapping a max-type statistic has been explored for testing the mean function (e.g., Lopes et al., 2020; Lin et al., 2022), it has not been studied in the substantially more challenging setting of functional linear regression. For example, in contrast to the works of Lopes et al. (2020); Lin et al. (2022), which utilize a fixed and known projection basis, our test distinctively employs empirical principal components as a projection basis to construct the max-type test statistic. The integration of principal components and their empirical versions is crucial in our context, as the aforementioned transformation for fully circumventing ill-posedness relies explicitly on the principal components of $X$, while these principal components are unknown and practically estimated from
data. Estimating these principal components introduces nontrivial variability into the bootstrap procedure and substantially complicates the theoretical investigation; see Section 3 for details.

In addition to the above methodological contributions, our theoretical studies not only establish validity and consistency of the proposed test, but also uncover a practical and theoretical distinction between estimation and statistical inference about the slope operator $\beta$ when principal components are employed. For estimating the slope function, a delicate choice of the number $p$ of principal components is required to achieve consistent estimation (e.g., Hall and Horowitz, 2007); typically, $p \ll \sqrt{n}$ for data of $n$ observations. In sharp contrast, we show that, both numerically and theoretically, our test is valid and consistent even when all the $n$ principal components are included (i.e., $p = n$); note that at most $n$ principal components can be derived from $n$ observations. This not only eliminates the intricate tuning step for selecting the number of principal components, but also potentially improves numeric power of the test, particularly when the signal of the slope operator is tied to some high-order principal components. This finding highlights a critical distinction between estimation and inference about the slope operator/function in functional data analysis. To the best of our knowledge, this is the first test utilizing all $p = n$ principal components. In contrast, at most $p \lesssim \sqrt{n}$ principal components are allowed by the previous studies in the general case (e.g., Cardot et al., 2003a; Müller and Stadtmüller, 2005; Hilgert et al., 2013; Lei, 2014; Su et al., 2017; Choi and Reimherr, 2018; Kokoszka et al., 2008; Shin, 2009).

In our theoretical investigation, to accommodate the situation that empirical principal components are adopted for conducting the aforementioned transformation, we establish validity and consistency of the proposed test uniformly for a family of test statistics induced by a class of orthonormal bases; we show that the empirical principal components fall into this class of bases with probability tending to one. We achieve this by deriving uniform Gaussian and bootstrap approximations of distributions of the corresponding family of max statistics. Consequently, our theoretical analyses are materially different from, and encounter considerably more challenges than, the analyses in Lopes et al. (2020) and Lin et al. (2022), which focus on only a single max statistic. For example, as discussed in Section S1 of Lin and Lin (2024), our analyses involve random elements in an infinite-dimensional Hilbert space, and thus the framework of Lopes et al. (2020) for finite-dimensional Euclidean spaces does not directly apply. As another example, the orthonormal bases within the class, aside from the principal components $\phi_j$ and $\psi_k$, may destroy the variance decay patterns induced by the principal components, which presents a significant challenge to the techniques of Lopes et al. (2020); Lin et al. (2022). Overcoming this challenge requires us to nontrivially and more effectively exploiting the basic property that $X$ and $Y$ have finite total variance, i.e., $\mathbb{E}\langle X, X \rangle_1 < \infty$ and $\mathbb{E}\langle Y, Y \rangle_2 < \infty$. These, along with other distinctions presented in our proofs, are materially different from the analysis in Lopes et al. (2020) and Lin et al. (2022).

The rest of the paper is organized as follows. We describe the proposed test in Section 2
and analyze its theoretical properties in Section 3. We then proceed to showcase its numerical performance via simulation studies in Section 4 and illustrate its applications in Section 5. We conclude the article with a remark in Section 6. All proofs are provided in Lin and Lin (2024).

2 Methodology

Without loss of generality, we assume $X$ and $Y$ in (1) are centered, i.e., $\mathbb{E}X = 0$ and $\mathbb{E}Y = 0$. Such an assumption, adopted also in Cai et al. (2006), is practically satisfied by replacing $X_i$ with $X_i - \bar{X}$ and replacing $Y_i$ with $Y_i - \bar{Y}$, where $\bar{X} = \frac{1}{n-1} \sum_{i=1}^{n} X_i$ and $\bar{Y} = \frac{1}{n-1} \sum_{i=1}^{n} Y_i$. This simplifies the model (1) to

$$Y = \beta(X) + Z. \quad (3)$$

We assume $\mathbb{E}\|X\|_1^2 < \infty$ and $\mathbb{E}\|Y\|_2^2 < \infty$ where $\| \cdot \|_1$ and $\| \cdot \|_2$ are norms induced respectively by $\langle \cdot, \cdot \rangle_1$ and $\langle \cdot, \cdot \rangle_2$, so that the covariances of $X$ and $Y$ exist. Our goal is to test (2) based on the independently and identically distributed (i.i.d.) realizations $(X_1, Y_1), \ldots, (X_n, Y_n)$. In addition, we assume that $X_i$ and $Y_i$ are fully observed when they are functions. This assumption is pragmatically satisfied when $X_i$ and $Y_i$ are observed in a dense grid of their defining domains, as the observations in the grid can be interpolated to form an accurate approximation to $X_i$ and $Y_i$. Thanks to modern technologies, such densely observed functional data are nowadays common in many fields, such as medicine and healthcare (Zhu et al., 2012; Chang and McKeague, 2022), meteorology (Burdejova et al., 2017; Shang, 2017) and finance (Müller et al., 2011; Tang and Shi, 2021). The case that $X_i$ and $Y_i$ are only observed in a sparse grid is much more challenging and is left for future research.

For $x \in \mathcal{X}$ and $y \in \mathcal{Y}$, the tensor product operator $(x \otimes y) : \mathcal{X} \rightarrow \mathcal{Y}$ is defined by

$$(x \otimes y)z = \langle x, z \rangle_1 y$$

for all $z \in \mathcal{X}$. The tensor product $x \otimes z$ for $x, z \in \mathcal{X}$ is defined analogously. For example, if $\mathcal{X} = \mathbb{R}^q$, then $x \otimes z = zx^\top$ for $x, z \in \mathcal{X}$, and if $\mathcal{X} = L^2(\mathcal{T})$, $f \otimes g$ is represented by the function $(f \otimes g)(s, t) = f(s)g(t)$, for $f, g \in L^2(\mathcal{T})$. With the above notation, the covariance operator of a random element $X$ in the Hilbert space $\mathcal{X}$ is given by $C_X = \mathbb{E}(X \otimes X)$. For example, if $\mathcal{X} = \mathbb{R}^q$ then $C_X = \mathbb{E}(XX^\top)$ and if $\mathcal{X} = L^2(\mathcal{T})$ then $(C_X f)(t) = \int_{\mathcal{T}} \mathbb{E}\{X(s)X(t)\} f(s)ds$ for $f \in L^2(\mathcal{T})$ and all $t \in \mathcal{T}$.

By Mercer’s theorem, the operator $C_X$ admits the decomposition

$$C_X = \sum_{j_1=1}^{d_X} \lambda_{j_1} \phi_{j_1} \otimes \phi_{j_1}, \quad (4)$$

where $\lambda_1 > \lambda_2 > \cdots > 0$ are eigenvalues, $\phi_1, \phi_2, \ldots$ are the corresponding eigenelements that are orthonormal, and $d_X$ is the dimension of $\mathcal{X}$; for example, $d_X = q$ if $\mathcal{X} = \mathbb{R}^q$ and $d_X = \infty$ if
\( \mathcal{X} = L^2(T) \). Similarly, the operator \( C_Y \) is decomposed by

\[
C_Y = \sum_{j_2=1}^{d_Y} \rho_{j_2} \psi_{j_2} \otimes \psi_{j_2},
\]

with eigenvalues \( \rho_1 > \rho_2 > \cdots > 0 \) and the corresponding eigenelements \( \psi_1, \psi_2, \ldots \). Without loss of generality, we assume \( \phi_1, \phi_2, \ldots \) form a complete orthonormal system (CONS) of \( \mathcal{X} \) and \( \psi_1, \psi_2, \ldots \) form a CONS of \( \mathcal{Y} \); otherwise, we can simply redefine \( \mathcal{X} \) and \( \mathcal{Y} \) to be the closed subspaces spanned by the respective eigenelements corresponding to the nonzero eigenvalues, since \( \beta \) in (3) can only be identified within the space of Hilbert–Schmidt operators between these subspaces.

Let \( \mathfrak{B}_{HS}(\mathcal{X}, \mathcal{Y}) \) be the set of Hilbert–Schmidt operators from \( \mathcal{X} \) to \( \mathcal{Y} \) (see Definition 4.4.2 in Hsing and Eubank, 2015), and note that \( \beta \in \mathfrak{B}_{HS}(\mathcal{X}, \mathcal{Y}) \). Since \( \phi_1, \phi_2, \ldots \) and \( \psi_1, \psi_2, \ldots \) are CONS, \( \beta \) can be represented as

\[
\beta = \sum_{j_1=1}^{d_X} \sum_{j_2=1}^{d_Y} b_{j_1,j_2} \phi_{j_1} \otimes \psi_{j_2},
\]

where each \( b_{j_1,j_2} \in \mathbb{R} \) is the generalized Fourier coefficient. Consequently, the null hypothesis in (2) is equivalent to \( b_{j_1,j_2} = 0 \) for all \( j_1 \) and \( j_2 \). It turns out that the coefficients are linked to the cross-covariance operator \( \mathbb{E}(X \otimes Y) \). Specifically, with \( \nu_{j_1,j_2} = \langle \mathbb{E}(X \otimes Y), \phi_{j_1} \otimes \psi_{j_2} \rangle \), we have the following proposition that connects \( b_{j_1,j_2} \) and \( \nu_{j_1,j_2} \); special cases of this connection have been exploited for example by Cai et al. (2006); Hall and Horowitz (2007); Kokoszka et al. (2008).

**Proposition 2.1.** \( \nu_{j_1,j_2} = \mathbb{E}(\langle X, \phi_{j_1} \rangle_1 \langle Y, \psi_{j_2} \rangle_2) \) and \( b_{j_1,j_2} = \lambda_{j_1}^{-1} \nu_{j_1,j_2} \).

Because \( \lambda_{j_1} \to 0 \) as \( j_1 \to \infty \), estimating the coefficients \( b_{j_1,j_2} \) becomes an ill-posed problem (Hall and Horowitz, 2007) and hence a direct test on the coefficients is difficult. To overcome the challenge of ill-posedness, we go one step further to observe that, \( b_{j_1,j_2} = 0 \) is equivalent to \( \nu_{j_1,j_2} = 0 \) for all \( j_1 \) and \( j_2 \), as the eigenvalues \( \lambda_{j_1} \) are nonzero according to remark right after (5). Therefore, a test on \( b_{j_1,j_2} \) can be further transformed into a test on \( \nu_{j_1,j_2} \), and this consequently eliminates the difficulty of estimating the reciprocals of the eigenvalues and the associated complications in deriving the asymptotic distribution of the test statistic. This further transformation, previously not exploited in the literature, is elegantly simple and effective for fully circumventing ill-posedness of functional linear regression in the context of testing nullity of the slope operator. Moreover, testing \( \nu_{j_1,j_2} = 0 \) is much more manageable as each \( \nu_{j_1,j_2} \) is the mean of some random variable according Proposition 2.1.

Following from the above discussions, we consider testing \( \nu_{j_1,j_2} = 0 \) for \( j_1 = 1, \ldots, p_1 \) when \( d_X = \infty \), and analogously, for \( j_2 = 1, \ldots, p_2 \) when \( d_Y = \infty \). Here, \( p_1 \) and \( p_2 \) are integers that may grow with the sample size; when \( d_X < \infty \) or \( d_Y < \infty \), one may opt for \( p_1 = d_X \) or \( p_2 = d_Y \), respectively. Given \( p_1 \) and \( p_2 \), we focus on the truncated vector \( \nu = (\nu_{j_1,j_2} : (j_1,j_2) \in \mathcal{P}) \) comprising
where \( q \) acts as a surrogate for the \( p \)-dimensional centered Gaussian distribution with the covariance matrix \( \hat{\Sigma} \). Then, the bootstrap counterparts of \( M \) and \( L \) are given by

\[
M^* = \max_{1 \leq j \leq p} \frac{S_{n,j}^*}{\hat{\sigma}_j^2} \quad \text{and} \quad L^* = \min_{1 \leq j \leq p} \frac{S_{n,j}^*}{\hat{\sigma}_j^2},
\]

respectively. Intuitively, the distribution of \( M^* \) provides an approximation to the distribution of \( M \) when the sample size is sufficiently large, while the distribution of \( M \) acts as a surrogate for the distribution of \( T_U \) under \( H_0 \); we justify this intuition in Theorems 3.5 and 3.6. Therefore, we reject the null hypothesis if \( T_U > q_{M^*}(1 - \varrho/2) \) or \( T_L < q_{L^*}(\varrho/2) \), where \( q_{M^*}(\cdot) \) and \( q_{L^*}(\cdot) \) are quantile functions of \( M^* \) and \( L^* \) respectively. In particular, both quantile functions can be practically approximated via resampling from the distribution \( N_p(0, \hat{\Sigma}) \).
Specifically, for a sufficiently large integer $B$, e.g., $B = 1000$, for each $b = 1, \ldots, B$, we independently draw $S_{n,b} \sim N_p(0, \hat{\Sigma})$ and compute $M_{*,b}$ and $L_{*,b}$. The quantiles $q_{M^*}(1 - \varrho/2)$ and $q_{L^*}(\varrho/2)$ are then respectively estimated by the empirical $1 - \varrho/2$ quantile $\hat{q}_{M}(1 - \varrho/2)$ of $M_{*,1}, \ldots, M_{*,B}$ and the $\varrho/2$ quantile $\hat{q}_{L}(\varrho/2)$ of $L_{*,1}, \ldots, L_{*,B}$.

In practice, the eigenelements $\phi_{j_1}$ and $\psi_{j_2}$ are unknown. To test (7), we need to estimate $\phi_{j_1}$ and $\psi_{j_2}$ from data. Specifically, $\phi_{j_1}$ is estimated by the eigenelement corresponding to the $j_1$th eigenvalue of the sample covariance operator $\hat{C}_X = n^{-1} \sum_{i=1}^{n} X_i \otimes X_i$, and similarly, $\psi_{j_2}$ is estimated by the eigenelement corresponding to the $j_2$th eigenvalue of $\hat{C}_Y = n^{-1} \sum_{i=1}^{n} Y_i \otimes Y_i$. The tuning parameter $\tau \in [0, 1)$ controls the degree of partial standardization in (8) and is the key to exploiting the decay variance. The work of Lin et al. (2022) provides a strategy to select a value of $\tau$ that maximizes the empirical power of the test, where the projection basis is fixed and known. In our numeric studies presented in Section 4, we found that the same selection strategy empirically works well even when the projection bases are estimated from data in our case.

As aforementioned in the introduction, in the previous studies of functional linear regression, it is crucial to determine a delicate value for $p_1$ and $p_2$ (typically $\lesssim \sqrt{n}$) to ensure estimation consistency or test validity. In contrast, our theoretical results in the next section show that our test remains asymptotically valid and consistent even with the choice of $p_1 = p_2 = n$; note that at most $n$ empirical principal components can be derived from $n$ observations. This choice of $p_1$ and $p_2$ is also validated by our numeric studies in Section 4. Consequently, unlike the existing counterparts, our test eliminates the nontrivial requirement of selecting the number of principal components, leading to enhanced numeric simplicity and stability.

3 Theory

We begin with introducing some notations. The symbol $\ell^2$ denotes the set of sequences that are square summable. For a matrix $A$, we write $\|A\|_F = \left(\sum_{i,j} A_{ij}^2\right)^{1/2}$ for its Frobenius norm and $\|A\|_\infty = \max_{i,j} |A_{ij}|$ for its max norm, where $A_{ij}$ is the element of $A$ at position $(i, j)$. For a random variable $\xi$ and a real number $\theta \in (0, 2]$, the $\psi_\theta$-Orlicz (quasi-)norm is defined as $\|\xi\|_{\psi_\theta} = \inf\{t > 0 : \mathbb{E}[\exp(|\xi|/t^\theta)] \leq 2\}$, where the cases of $\theta = 1$ and $\theta = 2$ correspond to the sub-exponential and sub-Gaussian random variables, respectively. We also use $\mathcal{L}(\xi)$ to denote the distribution of $\xi$ and define the Kolmogorov distance between random variables $\xi$ and $\eta$ by $d_K(\mathcal{L}(\xi), \mathcal{L}(\eta)) = \sup_{t \in \mathbb{R}} |\mathbb{P}(\xi \leq t) - \mathbb{P}(\eta \leq t)|$. For two sequences $\{a_n\}$ and $\{b_n\}$ with non-negative elements, $a_n = o(b_n)$ or $a_n \ll b_n$ means $a_n/b_n \to 0$ as $n \to \infty$, and $a_n = O(b_n)$ means $a_n \leq cb_n$ for some constant $c > 0$ and all sufficiently large $n$. Moreover, we write $a_n \lesssim b_n$ if $a_n = O(b_n)$, write $a_n \gtrsim b_n$ if $b_n = O(a_n)$, and write $a_n \asymp b_n$ if $a_n \lesssim b_n$ and $a_n \gtrsim b_n$. Also, define $a_n \lor b_n = \max\{a_n, b_n\}$ and $a_n \land b_n = \min\{a_n, b_n\}$. 

8
Consider the untruncated vector \( V_i^\infty = (\xi_{ij}, \zeta_{ij}, j_1, j_2 \geq 1) \) with \( \xi_{ij_1} = \langle X_i, \phi_{j_1} \rangle \) and \( \zeta_{ij_2} = \langle Y_i, \psi_{j_2} \rangle \). Our first assumption concerns the tail behavior of \( V_i^\infty \).

**Assumption 3.1 (Tail behavior).** The random vector \( V_i^\infty \) satisfies the following two tail conditions:

\[
\mathbb{P}(\|V_i^\infty - \mathbb{E}V_i^\infty\|_2 \geq t) \leq 2e^{-\frac{t^2}{K}} \tag{9}
\]

\[
\mathbb{E} \langle V_i^\infty - \mathbb{E}V_i^\infty, x \rangle \|_{\psi_1} \leq K \left( \mathbb{E}[\|V_i^\infty - \mathbb{E}V_i^\infty\|^2] \right)^{1/2} \tag{10}
\]

for some \( K > 0 \), all \( t \geq 0 \), and all \( x \in \ell^2 \) with \( \|x\|_2 = 1 \).

Condition (9), extending Jin et al. (2019) to sub-exponentiality and potentially infinite-dimensional vectors, ensures that the \( \ell_2 \)-norm of the centered \( V_i^\infty \) has a sub-exponential tail. For example, it holds when both \( X \) and \( Y \) have sub-Gaussian norms, i.e., when \( \|X\|_1 \) and \( \|Y\|_2 \) are sub-Gaussian. Condition (10), e.g., satisfied by normal random elements and vectors, extends its benchmark counterparts in Lopes et al. (2020); Antonini (1997); Vershynin (2018); Giessing and Fan (2020); Cai et al. (2022) to the infinite-dimensional setting.

To state the next assumption, let \( R(d_1, d_2) \in \mathbb{R}^{d_2 \times d_1} \) denote the correlation matrix of random variables \( \{\langle X, \phi_{j_1} \rangle, \langle Y, \psi_{j_2} \rangle : 1 \leq j_1 \leq d_1, 1 \leq j_2 \leq d_2 \} \). That is, \( R(d_1, d_2) \) corresponds to the cross-products of the leading \( d_1 \) principal component scores of \( X \) and the leading \( d_2 \) principal component scores of \( Y \).

**Assumption 3.2 (Structural assumptions).**

\( i \) \( \mathbb{E}[\langle X_1, \phi_{j_1} \rangle^2] \leq C\lambda_j^2 \) for some \( C > 1 \) and all \( j_1 \geq 1 \). The eigenvalues \( \lambda_{j_1} \) for \( j_1 \geq 1 \) and \( \rho_{j_2} \) for \( j_2 \geq 1 \) are positive, and there are constants \( \alpha_1 > 2 \) and \( \alpha_2 > 1 \), not depending on \( n \), such that

\[
\lambda_{j_1} \approx j_1^{-\alpha_1} \quad \text{and} \quad \rho_{j_2} \approx j_2^{-\alpha_2}. \tag{11}
\]

Moreover,

\[
\max_{j_1 \geq 1} |b_{j_1, j_2}| \lesssim \rho_{j_2}, \quad \text{for all} \ j_2 \geq 1. \tag{12}
\]

\( ii \) Let \( \tilde{\alpha} = \max\{\alpha_1/2, \alpha_2\} \) and \( \alpha = \min\{\alpha_1/2, \alpha_2\} \). For any constant \( \delta \in (0, 1/2) \) and an arbitrarily small number \( \delta_0 > 0 \), define \( \ell_n = \left[ n^{\frac{\delta}{\alpha_1(1-\alpha)}} \right] \) and \( m_n = \ell_n^{\frac{2\alpha}{\alpha_1}} - \delta_0 \) and the class

\[
\mathcal{R}(\ell_n, m_n) = \left\{ R^o \in \mathbb{R}^{\ell_n \times \ell_n} : R^o \text{ is a sub-matrix of } R([m_n^{1/2}], [m_n^{1/2}]) \right\}.
\]

Assume

\[
\sup_{R^o \in \mathcal{R}(\ell_n, m_n)} \|R^o\|_F^2 \lesssim \ell_n^{2-\delta}. \]

The condition on \( \mathbb{E}[\langle X_1, \phi_{j_1} \rangle^2] \) was previously adopted in Cai and Hall (2006); Hall and Horowitz (2007); Lei (2014). The condition (11) imposes a smoothness requirement on the covariance operators.
of $X$ and $Y$, which is often needed in analyzing convergence rates involving functional data; for example, similar requirements are adopted in Meister (2011); Cai et al. (2018). Such condition is connected to the so-called Sacks-Ylvisaker condition (Yuan et al., 2010). For example, when the covariance $C_X$ satisfies the Sacks-Ylvisaker condition of order $s > 0$, the corresponding $j$th eigenvalue is of the order $j^{−2(s+1)} \ll j^{-2}$. For a scalar-on-function model, $p_2 = d_Y = 1$, and thus the requirement for $\rho_{j_2}$ in (11) and the condition on the generalized Fourier coefficients $b_{j_1j_2}$ of $\beta$ in (12) are automatically satisfied. In addition, (12) is considerably weaker than the requirement in Cai et al. (2006); Hall and Horowitz (2007) for the scalar-on-function regression model. Condition (ii) in the above assumption is in analogy to Condition (3.2) of Lopes and Yao (2022). As mentioned in Lopes and Yao (2022), since $\sup_{R^o \in \mathcal{R}(\ell_n,m_n)} \|R^o\|_F^2 \leq \ell_n^2$ and $\delta$ could be taken arbitrarily small, the condition (ii), which cannot be substantially weakened in general, appears not restrictive. In addition, this condition applies only to the small set of the variables that involve the $m_n$ leading eigen-bases, while the other variables can have an unrestricted correlation structure.

Our last assumption imposes some conditions on the growth rate of $p$ relative to $n$, where we recall that $p = p_1p_2$.

**Assumption 3.3.** We require $p \lesssim n^{\alpha_0}$ for a (arbitrarily large) fixed number $\alpha_0 > 0$.

Under this assumption, $p_1$ and $p_2$, the numbers of potential scalar predictors or principal component scores, are allowed to grow with the sample size at a polynomial rate of $n$. Hence, in the scenario where $d_X = \infty$ and/or $d_Y = \infty$, the assumption accommodates the situation where $p_1 = n$ and/or $p_2 = n$, which corresponds to the maximal number of empirical eigenfunctions that can be derived from $n$ observations. This is markedly distinct from estimation problems that involve empirical eigenfunctions (e.g., Hall and Horowitz, 2007), in which at most $O(\sqrt{n})$ leading eigenfunctions can be utilized in order to guarantee consistent estimation of $\beta$ and the eigenfunctions (e.g., Wahl, 2022). In contrast, for testing the slope function, $p_1 = n$ and/or $p_2 = n$ are allowed, as demonstrated by the proposed test procedure. This suggests that, for statistical inference about the slope function in functional data analysis, we may include all $n$ empirical eigenfunctions, without requiring consistent estimation of all eigenfunctions being involved.

As mentioned previously, the eigenelements $\phi = \{\phi_{j_1}\}_{j_1=1}^{p_1}$ and $\psi = \{\psi_{j_2}\}_{j_2=1}^{p_2}$ are often unknown, and practitioners may use alternative orthonormal elements $\tilde{\phi} = \{\tilde{\phi}_{j_1}\}_{j_1=1}^{p_1}$ and $\tilde{\psi} = \{\tilde{\psi}_{j_2}\}_{j_2=1}^{p_2}$, such as the empirical eigenelements $\hat{\phi} = \{\hat{\phi}_{j_1}\}_{j_1=1}^{p_1}$ and $\hat{\psi} = \{\hat{\psi}_{j_2}\}_{j_2=1}^{p_2}$, which may differ from $\phi$ and $\psi$. In this case, all quantities depending on $\phi$ and $\psi$, such as $M$ and $S_n$, will be computed by using $\tilde{\phi}$ and $\tilde{\psi}$. We write, for example, $M(\tilde{\phi}, \tilde{\psi})$ and $S_n(\tilde{\phi}, \tilde{\psi})$, to indicate the dependence on $\tilde{\phi}$ and $\tilde{\psi}$, and note that $M = M(\phi, \psi)$ and $S_n = S_n(\phi, \psi)$. 

10
For two integers \(d_1, d_2 \geq 1\), for two orthonormal sequences \(\{\tilde{\phi}_{j_1}\}_{j_1=1}^{d_1}\) and \(\{\tilde{\psi}_{j_2}\}_{j_2=1}^{d_2}\), define

\[
U_X^{d_1}(\tilde{\phi}) = \begin{pmatrix}
\langle \phi_1, \tilde{\phi}_1 \rangle & \cdots & \langle \phi_1, \tilde{\phi}_{d_1} \rangle \\
\vdots & \ddots & \vdots \\
\langle \phi_{d_1}, \tilde{\phi}_1 \rangle & \cdots & \langle \phi_{d_1}, \tilde{\phi}_{d_1} \rangle
\end{pmatrix}
\quad \text{and} \quad
U_Y^{d_2}(\tilde{\psi}) = \begin{pmatrix}
\langle \psi_1, \tilde{\psi}_1 \rangle & \cdots & \langle \psi_1, \tilde{\psi}_{d_2} \rangle \\
\vdots & \ddots & \vdots \\
\langle \psi_{d_2}, \tilde{\psi}_1 \rangle & \cdots & \langle \psi_{d_2}, \tilde{\psi}_{d_2} \rangle
\end{pmatrix}.
\]

Let \(W_d(\tilde{\phi}, \tilde{\psi}) = U_X^{d_1}(\tilde{\phi}) \otimes U_Y^{d_2}(\tilde{\psi})\), where \(\otimes\) denotes the Kronecker product of two matrices, be the transformation matrix consisting of the leading eigenelements. With the truncation numbers \(p_1\) and \(p_2\) introduced in Section 2, consider a class \(\mathcal{F}_p\) of \((\tilde{\phi}, \tilde{\psi})\) with 
\[\tilde{\phi} = \{\tilde{\phi}_{j_1}\}_{j_1=1}^{p_1}\] and \(\tilde{\psi} = \{\tilde{\psi}_{j_2}\}_{j_2=1}^{p_2}\), such that 1) \(\tilde{\phi}\) and \(\tilde{\psi}\) are respectively orthonormal sequences, and 2)

\[
\max \left\{ \|U_X^{d_1}(\tilde{\phi}) - I_{d_1}\|_\infty, \|U_Y^{d_2}(\tilde{\psi}) - I_{d_2}\|_\infty, \|W_d(\tilde{\phi}, \tilde{\psi}) - I_d\|_\infty \right\} \leq c a_n \tag{13}
\]

for a sufficiently large constant \(c > 0\), where \(I_r\) represents the \(r \times r\) identity matrix for any integer \(r > 0, d = d_1 d_2\), and we set

\[
a_n = k_n^{-2\alpha} \quad \text{with} \quad k_n = \ell_n \frac{\log \frac{n}{d} + \log \log n}{\log \frac{n}{d}},
\]

\[
d_1 = \min\{[h_n^{1/2}], p_1\} \quad \text{and} \quad d_2 = \min\{[h_n^{1/2}], p_2\} \quad \text{with} \quad h_n = n^{-\frac{1}{2(\alpha + 1)}}
\]

throughout this paper. Note that \(d \lesssim h_n\), with the above choices.

Th condition \((13)\), for example, is satisfied by the leading empirical eigenbases with probability tending to one, according to the following proposition.

**Proposition 3.4.** Let \(\{\tilde{\phi}_{j_1}\}_{j_1=1}^{d_1}\) and \(\{\tilde{\psi}_{j_2}\}_{j_2=1}^{d_2}\) be the leading empirical eigenvalues of \(\hat{C}_X\) and \(\hat{C}_Y\) defined in Section 2, respectively. If \(X\) and \(Y\) are sub-Gaussian random elements in \(\mathcal{X}\) and \(\mathcal{Y}\) satisfying Assumption 3.2, then for \(1 \ll t \ll \max\{d_1^{-2(\alpha_1 + 1)} n, d_2^{-2(\alpha_2 + 1)} n\}\), with probability at least \(1 - e^{-t + 2 \log(2d)}\), we have

\[
\max \left\{ \|U_X^{d_1}(\hat{\phi}) - I_{d_1}\|_\infty, \|U_Y^{d_2}(\hat{\psi}) - I_{d_2}\|_\infty, \|W_d(\hat{\phi}, \hat{\psi}) - I_d\|_\infty \right\} \lesssim (d_1^{\alpha_1 + 1} \lor d_2^{\alpha_2 + 1}) \sqrt{t/n}.
\]

Consequently, for any \(q > 0\) such that \(t \asymp n^q (k_n d^{2(\alpha + 1)}) \gg \log d\) with \(d = \max\{d_1, d_2\}\), we have

\[
\max \left\{ \|U_X^{d_1}(\hat{\phi}) - I_{d_1}\|_\infty, \|U_Y^{d_2}(\hat{\psi}) - I_{d_2}\|_\infty, \|W_d(\hat{\phi}, \hat{\psi}) - I_d\|_\infty \right\} \lesssim k_n^{-q}
\]

with probability tending to one. In particular, it holds for \(q = 2\alpha\).

**Remark 1.** In the case \(d_X = d_Y = \infty\) so that in practice we set \(p_1 = p_2 = n\), the requirement \((13)\) in \(\mathcal{F}_p\), with the choice of \(d_1\) and \(d_2\), is imposed only on the leading \([h_n^{1/2}] \ll p_1\) eigen-bases of \(C_X\) and the leading \([h_n^{1/2}] \ll p_2\) eigen-bases of \(C_Y\). In particular, there are no conditions on the remaining eigen-bases. The condition \((13)\) would enforce that the variances of the coordinates of \(V_i(\hat{\phi}, \hat{\psi})\) partially exhibit a decay pattern similar to that of the variances of \(V_i\); see Proposition
S2.7 in Lin and Lin (2024) for details. This is one of the key properties we exploit for establishing the validity and consistency of the proposed test even when we take \( p_1 = p_2 = n \) that is set in our numeric implementation. Another key property we heavily exploit is that \( X \) and \( Y \) have finite total variances, i.e., \( \mathbb{E}\|X - \mathbb{E}X\|^2 < \infty \) and \( \mathbb{E}\|Y - \mathbb{E}Y\|^2 < \infty \) or equivalently \( \sum_{j_1=1}^{12} \lambda_{j_1} < \infty \) and \( \sum_{j_2=1}^{12} \rho_{j_2} < \infty \).

The class \( \mathcal{F}_p \) gives rise to a class of test statistics \( T_U(\hat{\phi}, \hat{\psi}) \) and \( T_L(\hat{\phi}, \hat{\psi}) \). Below we analyze the uniform asymptotic power and size over this class of test statistics; the asymptotic properties of the proposed test by using \( T_U = T_U(\phi, \psi) \) and \( T_L = T_L(\phi, \psi) \), as well as their empirical versions \( T_U(\hat{\phi}, \hat{\psi}) \) and \( T_L(\hat{\phi}, \hat{\psi}) \), then follow as direct consequences, since the class \( \mathcal{F}_p \) contains \((\phi, \psi)\) and further \((\hat{\phi}, \hat{\psi})\) with probability tending to one according to Proposition 3.4. To this end, we first establish three approximation results related to the test statistics, namely, the Gaussian approximation, the bootstrap approximation and the approximation with empirical variances, uniformly over the class \( \mathcal{F}_p \). Compared with their non-uniform counterparts in Lopes et al. (2020); Lin et al. (2022); Lopes and Yao (2022), these general uniform approximations, maybe of independent interest, require considerably different and more challenging proofs. For example, as mentioned in Remark 1, we heavily exploit the property that \( X \) and \( Y \) have finite total variances in our proofs, which is materially different from the aforementioned previous works. Below we consider only the max statistic while note that similar results hold for the min statistic.

We start with defining the Gaussian counterpart of \( M(\hat{\phi}, \hat{\psi}) \) by

\[
\hat{M}(\hat{\phi}, \hat{\psi}) = \max_{1 \leq j \leq p} \frac{\hat{S}_{n,j}(\hat{\phi}, \hat{\psi})}{\sigma_j^T(\hat{\phi}, \hat{\psi})},
\]

where \( \hat{S}_n(\hat{\phi}, \hat{\psi}) \sim N_p(0, \Sigma(\hat{\phi}, \hat{\psi})) \). The following result shows that the distribution of \( M(\hat{\phi}, \hat{\psi}) \) converges to the distribution \( \hat{M}(\hat{\phi}, \hat{\psi}) \) at a near \( 1/\sqrt{n} \) rate uniformly over \( \mathcal{F}_p \).

**Theorem 3.5** (Uniform Gaussian approximation). For any sufficiently small number \( \delta \in (0, 1/2) \), if Assumptions 3.1–3.3 hold, then

\[
\sup_{(\hat{\phi}, \hat{\psi}) \in \mathcal{F}_p} d_K\left( \mathcal{L}(M(\hat{\phi}, \hat{\psi})), \mathcal{L}(\hat{M}(\hat{\phi}, \hat{\psi})) \right) \lesssim n^{-1/2+\delta}.
\]

In the proposed test, a bootstrap strategy is used to estimate the distribution of \( \hat{M}(\hat{\phi}, \hat{\psi}) \), which is justified by the following result.

**Theorem 3.6** (Uniform bootstrap approximation). For any sufficiently small number \( \delta \in (0, 1/4) \), if Assumptions 3.1–3.3 hold, then there is a constant \( c > 0 \), not depending on \( n \), such that the event

\[
\sup_{(\hat{\phi}, \hat{\psi}) \in \mathcal{F}_p} d_K\left( \mathcal{L}(\hat{M}(\hat{\phi}, \hat{\psi})), \mathcal{L}(M^*(\hat{\phi}, \hat{\psi})|D) \right) \leq cn^{-1/4+\delta}
\]

occurs with probability at least \( 1-cn^{-1} \), where \( \mathcal{L}(M^*(\hat{\phi}, \hat{\psi})|D) \) represents the distribution of \( M^*(\hat{\phi}, \hat{\psi}) \) conditional on the observed data \( D = (X_i, Y_i)_{i=1}^n \).
Remark 2. The rate of near $n^{-1/4}$ in the bootstrap approximation is primarily due to the requirement of uniform convergence over $\mathcal{F}_p$ of growing complexity. Specifically, for this uniform convergence, we need to establish a uniform Gaussian-to-Gaussian comparison result between the Gaussian and bootstrap counterparts of the max statistic over the $k_n$ leading components; details can be found in Lemma S3.2 of Lin and Lin (2024). Our strategy is to transform these leading components induced by ($\tilde{\phi}, \tilde{\psi}$) into the components induced by the population eigenelements ($\phi, \psi$). This transformation involves an infinite-dimensional operator, destroying some key finite-dimensional structures possessed in the analysis of Lopes et al. (2020), such as Proposition C.1 therein. Instead, we apply a Gaussian comparison result for general covariance structures, such as Lemma S5.6 in Lin and Lin (2024), leading to the rate $n^{-1/4+\delta}$.

In reality, the variances $\sigma_j^2$ are estimated by $\hat{\sigma}_j^2$, and the max statistic is pragmatically computed by

$$\hat{M}(\tilde{\phi}, \tilde{\psi}) = \max_{1 \leq j \leq p} \frac{S_{n,j}(\tilde{\phi}, \tilde{\psi})}{\hat{\sigma}_j^2(\tilde{\phi}, \tilde{\psi})}.$$ 

Below we show that the distribution of this practical max statistic converges to the distribution of the original max statistic uniformly over the class $\mathcal{F}_p$.

**Theorem 3.7.** For any sufficiently small number $\delta \in (0, 1/2)$, if Assumptions 3.1–3.3 hold, then

$$\sup_{(\tilde{\phi}, \tilde{\psi}) \in \mathcal{F}_p} d_K\left(\mathcal{L}(\hat{M}(\tilde{\phi}, \tilde{\psi})), \mathcal{L}(M(\tilde{\phi}, \tilde{\psi}))\right) \lesssim n^{-1/2+\delta}.$$ 

With the triangle inequality, Theorems 3.5–3.7 together imply that, with probability at least $1 - cn^{-1}$,

$$\sup_{(\tilde{\phi}, \tilde{\psi}) \in \mathcal{F}_p} d_K\left(\mathcal{L}(\hat{M}(\tilde{\phi}, \tilde{\psi})), \mathcal{L}(M^*(\tilde{\phi}, \tilde{\psi})|\mathcal{D})\right) \leq cn^{-1/4+\delta}$$

for some constant $c > 0$ not depending on $n$. This eventually leads to Theorem 3.8 and Theorem 3.9 for uniform validity and consistency of the proposed test respectively.

**Theorem 3.8.** For any sufficiently small number $\delta \in (0, 1/4)$, if Assumptions 3.1–3.3 hold, then for any $\varrho \in (0, 1)$, we have

$$\sup_{(\tilde{\phi}, \tilde{\psi}) \in \mathcal{F}_p} \text{SIZE}(\varrho, \tilde{\phi}, \tilde{\psi}) \leq \varrho + O(n^{-1/4+\delta}),$$

where $\text{SIZE}(\varrho, \tilde{\phi}, \tilde{\psi})$ is the probability of rejecting the null hypothesis by using the bases ($\tilde{\phi}, \tilde{\psi}$) at the significance level $\varrho$ when the null hypothesis in (2) is true.

**Theorem 3.9.** Suppose Assumptions 3.1–3.3 hold. Then,
(1) for any fixed \( \varrho \in (0, 1) \), one has
\[
\sup_{(\tilde{\phi}, \tilde{\psi}) \in \mathcal{F}_p} |q_{M^*}(\tilde{\phi}, \tilde{\psi})(\varrho)| \leq c \sqrt{\log n}
\]
with probability at least \( 1 - cn^{-1} \), where \( c > 0 \) is a constant not depending on \( n \), and

(2) for some constant \( c > 0 \) not depending on \( n \), one has
\[
P \left( \sup_{(\tilde{\phi}, \tilde{\psi}) \in \mathcal{F}_p} \frac{\max_{1 \leq j \leq p} \sigma_j^2(\tilde{\phi}, \tilde{\psi})}{\sigma_{\max}^2(\tilde{\phi}, \tilde{\psi})} < 2 \right) \geq 1 - cn^{-1},
\]
where \( \sigma_{\max}(\tilde{\phi}, \tilde{\psi}) = \max\{\sigma_j(\tilde{\phi}, \tilde{\psi}) : 1 \leq j \leq p\} \).

Consequently, if
\[
\nu_0 = \max_{1 \leq j_1, j_2 \leq \lfloor h_n^1 \rfloor} |E(\langle X, \phi_{j_1} \rangle_1 \langle Y, \psi_{j_2} \rangle_2)| \geq c_0 \max\{\sigma^\tau_{\max}(\phi, \psi)n^{-1/2}\log(n), a_n\}
\]
for a sufficiently large constant \( c_0 > 0 \), where \( h_n = n^{1/(2(\alpha+1))} \), then with \( p_1 \geq \lfloor h_n^{1/2} \rfloor \) and \( p_2 \geq \lfloor h_n^{1/2} \rfloor \), the null hypothesis in (2) is rejected uniformly over \( \mathcal{F}_p \) with probability tending to one, that is,
\[
P \left( \forall (\tilde{\phi}, \tilde{\psi}) \in \mathcal{F}_p : T_U(\tilde{\phi}, \tilde{\psi}) > q_{M^*}(\tilde{\phi}, \tilde{\psi})(1 - \varrho/2) \text{ or } T_L(\tilde{\phi}, \tilde{\psi}) < q_{M^*}(\tilde{\phi}, \tilde{\psi})(\varrho/2) \right) \to 1,
\]
as \( n \to \infty \).

The above theorem, where the condition on \( p_1 \) and \( p_2 \) is clearly satisfied by \( p_1 = p_2 = n \), implies that the proposed test exhibits local power of the order \( a_n \vee (n^{-1/2}\log n) \). It also implies that, for any fixed alternative, the power of the proposed test converges to one as \( n \to \infty \).

4 Simulation Studies

To illustrate the numerical performance of the proposed method, we consider three families of models. For each family, we consider various settings; see below for details. In all settings, \( Y \) is computed from (1) with \( EY = 1 \).

For each setting, we consider different sample sizes, namely, \( n = 50 \) and \( n = 200 \), to investigate the impact of \( n \) on the power of a test. For the proposed test, we set \( p_1 = n \) when \( d_X = \infty \) and \( p_2 = n \) when \( d_Y = \infty \), i.e., we do not need to tune the parameters \( p_1 \) and \( p_2 \). The tuning parameter \( \tau \) is selected by the method described in Lin et al. (2022). Finally, we independently perform \( R = 1000 \) replications for each setting, based on which we compute the empirical size as the proportion of rejections among the \( R \) replications when the null hypothesis is true and compute the empirical power as the proportion of rejections when the alternative hypothesis is true. In all settings, the significance level is \( \varrho = 0.05 \).
Figure 1: Empirical size \((r = 0)\) and power \((r > 0)\) of the proposed method (red-solid), the exponential scan method (blue-dashed) and the Fisher-type method (black-dotted) for the scalar-on-function family.

**Scalar-on-function.** The functional predictor \(X\) is a centered Gaussian process with the following Matérn covariance function

\[
C(s, t) = \sigma^2 \frac{2^{1-\nu}}{\Gamma(\nu)} \left( \sqrt{2\nu \frac{|s-t|}{\rho}} \right)^\nu K_\nu \left( \sqrt{2\nu \frac{|s-t|}{\rho}} \right),
\]

where \(\Gamma\) is the gamma function and \(K_\nu\) is the modified Bessel function of the second kind. Here, we fix \(\nu = 1\), \(\rho = 1\) and \(\sigma = 1\). The noise \(Z\) is sampled from the Laplacian distribution with zero mean and unit variance, so that the distribution of \(Y\) is non-Gaussian.

We consider the slope operator as \(\beta(t) = rg(t)\) for \(r \in \mathbb{R}\) with the following distinct functions \(g(t)\):

- (Sparsest) \(g(t) = 1\);
- (Sparse) \(g(t) = \sum_{j=1}^3 \frac{11}{4} (j + 2)^{-1} \phi_j(t)\);
- (Dense) \(g(t) = \sum_{j=1}^{100} \frac{12}{4} (j + 2)^{-1} \phi_j(t)\) for \(K = 100\);
- (Densest) \(g(t) = \frac{6}{4} t^2 e^t\).

The parameter \(r = 0, 0.1, \ldots, 1\) controls the strength of the signal. The case of \(r = 0\) corresponds to the null hypothesis, while the case of \(r > 0\) corresponds to the alternative hypothesis and the power of a test is expected to increase as \(r\) increases. In the sparse setting, \(g(t)\) is formed by only a
Figure 2: Empirical size ($r = 0$) and power ($r > 0$) of the proposed method (red-solid) and the chi-squared test (blue-dashed) for the function-on-function family.

few principal components, while in the dense and densest settings, $g(t)$ contains considerably more components and thus represents challenging settings.

We compare the proposed method with the exponential scan method (Lei, 2014) and a Fisher-type method (Hilgert et al., 2013), where for the proposed method the bases $\tilde{\phi}$ and $\tilde{\psi}$ are pragmatically taken to be the empirical eigenelements of the sample covariance operators $\hat{C}_X$ and $\hat{C}_Y$, respectively. From the results shown in Figure 1, we see that the proposed method controls the empirical type-I error well and has empirical power increasing with the sample size and approaching one as the signal, quantified by $r$, becomes stronger. Moreover, the proposed test outperforms the other two methods by a large margin.

Function-on-function. The functional predictor $X$ is sampled as in the scalar-on-function case, while the noise process $Z$ is represented by

$$Z(t) = \sum_{j=1}^{k} \eta^{(j)}(t) \phi_j(t)$$

for $k = 50$, where $\phi_1(t) \equiv 1$, $\phi_{2j}(t) = \sqrt{2} \cos(2j\pi t)$ and $\phi_{2j+1}(t) = \sqrt{2} \sin(2j\pi t)$, and for each $j = 1, \ldots, k$, $\eta^{(j)}$ is a random variable following the centered Laplacian distribution $\text{Laplace}(0, \sqrt{\lambda_j/2})$. Consequently, the process $Y$ is non-Gaussian.

For the slope operator, we consider $\beta(s, t) = rg(s, t)$ with $g(s, t) \in \mathbb{R}$ being one of the following:

- (Sparsest) $g(s, t) = \frac{5}{7}$;
- (Sparse) $g(s, t) = \sum_{k=1}^{3} \sum_{j=1}^{3} \frac{10}{\pi (j+2)^{1/2}(k+2)^{1/2}} \phi_j(s) \phi_k(t)$;
Figure 3: Empirical size ($r = 0$) and power ($r > 0$) of the proposed method (red-solid) and the F-test (blue-dashed) for the function-on-vector family.

- (Dense) $g(s, t) = \sum_{k=1}^{K} \sum_{j=1}^{K} \phi_{j}(s)\phi_{k}(t)\frac{9}{4}(j+2)^{1/4}(k+2)^{1/2}$ with $K = 100$;

- (Densest) $g(s, t) = \frac{10}{4}(st)^{2}\sqrt{e^{(s+t)/2}}$.

The dense case represents a challenging setting as $\beta$ contains a large number of relatively weaker spectral signals, in contrast with the sparse case in which the spectral signals of $\beta$ are stronger.

We compare the proposed method with the chi-squared test (Kokoszka et al., 2008). The chi-squared test is also based on functional principal component analysis, but unlike our method, it requires a delicate choice of the number of principal components, as the choice has a visible influence on the performance of the test. In our simulations, we take the leading $K = 4$ principal components as in Kokoszka et al. (2008); we also tried various values for $K$ and found that overall $K = 4$ yields the best results for the chi-squared test. According to the results shown in Figure 2, the proposed method consistently outperforms the chi-squared test, and particularly has much larger power when the sample size is small or the signal is sufficiently sparse.

**Function-on-vector.** The vector predictor $X \in \mathbb{R}^{q}$ with $q = 5$ follows the centered multivariate Laplacian distribution with the covariance matrix $\Sigma = ADA^{\top}$ for $D = \text{diag}(\lambda_{1}, \ldots, \lambda_{q})$, where $\lambda_{j} = j^{-1.5}$ and $A$ is an orthogonal matrix that is randomly generated and then remains fixed throughout the studies. The noise $Z$ is sampled as in the function-on-function case.

For the slope operator, we set $\beta(t) = rg(t)$ with $g(t) \in \mathbb{R}^{q}$, where for each $j = 1, \ldots, q$, the $j$th component $g^{j}(t)$ of $g(t)$ is one of the following:

- (Sparsest) $g^{j}(t) = \frac{11}{10}$.
\begin{itemize}
  \item (Sparse) \( g^j(t) = \sum_{k=1}^{3} \sum_{j=1}^{3} \frac{11}{4} \frac{\phi_{j/2}(t) \phi_k(t)}{(j+2)^{k+2}}; \)
  \item (Dense) \( g^j(t) = \sum_{k=1}^{K} \sum_{j=1}^{K} \frac{6}{4} \frac{\phi_{j/2}(t) \phi_k(t)}{(j+2)^{k+2}} \) with \( K = 100; \)
  \item (Densest) \( g^j(t) = \frac{11}{4} \frac{(i-1)}{(q-1)^2} \sqrt{\frac{1}{\pi}} \sqrt{t} \sqrt{e^{t/4}}. \)
\end{itemize}

Similar to the function-on-function family, the dense case represents a more challenging setting for the proposed method. We compare the proposed method with the F-test developed by Zhang (2011). From the results shown in Figure 3, we observe that the proposed method is more powerful when the signal is relatively weak while the F-test has slightly higher power when the signal is strong.

## 5 Data Application

We apply the proposed method to study physical activities using data collected from wearable devices and available in the National Health and Nutrition Examination Survey (NHANES) 2005-2006. Over seven consecutive days, each participant wore a wearable device that for each minute recorded the average physical activity intensity level (ranging from 0 to 32767) in that minute. As the wearable devices were not waterproof, participants were advised to remove the devices when they swam or bathed. The devices were also removed when the participants were sleeping. For each subject, an activity trajectory, denoted by \( A(t) \) for \( t \in [0, 7] \), was collected. In our study, trajectories with missing values or unreliable readings are excluded.

To eliminate the effect of circadian rhythms that vary among participants, instead of the raw activity trajectories, we follow the practice in Chang and McKeague (2022); Lin et al. (2022) to consider the activity profile \( Y(s) = \text{Leb}(\{t \in [0, 7] : A(t) \geq s\}) \) for \( s = 1, \ldots, 32767 \), where Leb denotes the Lebesgue measure on \( \mathbb{R} \). The zero intensity values are also excluded since they may represent no activities like sleeping or intense activities like swimming. After these pre-processing steps, for the \( i \)-th subject, we obtain an activity profile \( Y_i(s) \) which is regarded as a densely observed function. Our goal is to study the effect of age on the activity profile. As children and adults, as well as males and females, have different activity patterns, we conduct the study on each group separately by using the proposed test, where the tuning parameter \( \tau \) is selected by using the method of Lin et al. (2022).

First, we consider children with age from 6 to 17, including 6 and 17, and focus on the intensity spectrum \([1, 1000]\) as children are found to have more moderate activities (WHO, 2020). As shown in Table 1, the age seems no impact on the activity profile for female children, but has significant impact for male children. By inspecting the mean activity profile curves in Figure 4, we see the visible differences for different age groups among male children, in contrast with the visually
Table 1: The p-values for testing the effect of age on the activity pattern, with sample size in the parentheses.

|                     | Male     | Female   |
|---------------------|----------|----------|
| p-value (age 6-17)  | 0.0046   | 0.1778   |
|                     | (962)    | (952)    |
| p-value (age 18-35) | 0.1336   | 0.0282   |
|                     | (623)    | (823)    |

Figure 4: Mean activity profile curves among male children (left) and female children (right) for different age groups, namely, age 6-8 (red-solid), age 9-10 (blue-dashed), age 11-12 (black-dotted), 13-14 (purple-dash-dotted) and age 15-17 (green-dashed).

indistinguishable differences among female children. In particular, our test results and Figure 4 together suggest that on average young male children tend to be significantly more active than elder male children.

Now we consider the young adults with age from 18 to 35, and focus on the intensity spectrum [1, 3000]. As shown in Table 1, there is significant difference of mean activity profiles among female young adults, while the difference is not significant among the males. This also agrees with the mean activity profiles shown in Figure 5, where we observe significant difference in the mean activity profiles among different age groups of females, especially on the intensity spectrum [300, 1500], while the difference is less pronounced among males.

6 Concluding Remarks

In this paper, we developed a novel approach for testing nullity of the slope function in functional linear regression. This method fully circumvents the challenge of ill-posedness without requiring an intricate choice of the number of principal components, thereby enhancing numeric stability
Figure 5: Mean activity profile curves among young female (top-left) and male (top-right) adults with their zoom-in regions (bottom) on the intensity spectrum [300, 1500] in different age groups, namely, age 18-21 (red-solid), age 22-25 (blue-dashed), age 26-29 (black-dotted), age 30-33 (purple-dash-dotted) and age 33-35 (green-dashed).

and potentially improving the test’s power. We also uncovered an interesting distinction between estimating and inferring the slope function. Specifically, for estimation to be consistent, it can incorporate no more than $\sqrt{n}$ empirical principal components. However, for the inference purpose, the method allows the use of all empirical principal components while maintaining its asymptotic validity and consistency. To the best of our knowledge, our test is the first of its kind to utilize all $n$ empirical principal components.

While we focused on fully or densely observed functional data in this work, in real-world applications, there are applications with sparsely observed data, such as longitudinal/panel data in medicine and econometrics. Adapting our framework to accommodate such sparse data presents a challenging yet worthwhile future research direction. Moreover, investigating the theoretical
transition between densely and sparsely observed data is intriguing. This exploration could yield valuable insights for practical data applications.

**Acknowledgement**

This research is partially supported by NUS startup grant A-0004816-01-00 and MOE AcRF Tier 1 grant A-0008522-00-00.

**Supplementary Material**

Supplementary material contains proofs for the theoretical results of this paper, some auxiliary results used in the proofs and a concentration inequality for empirical eigenelements. (PDF)

**References**

Antonini, R. G. (1997), “Subgaussian random variables in Hilbert spaces,” *Rendiconti del Seminario Matematico della Università di Padova*, 98, 89–99.

Burdejova, P., Härdle, W., Kokoszka, P., and Xiong, Q. (2017), “Change point and trend analyses of annual expectile curves of tropical storms,” *Econometrics and statistics*, 1, 101–117.

Cai, T. T. and Hall, P. (2006), “Prediction in functional linear regression,” *The Annals of Statistics*, 34, 2159–2179.

Cai, T. T., Hall, P., et al. (2006), “Prediction in functional linear regression,” *The Annals of Statistics*, 34, 2159–2179.

Cai, T. T., Zhang, A. R., and Zhou, Y. (2022), “Sparse group lasso: Optimal sample complexity, convergence rate, and statistical inference,” *IEEE transactions on information theory*, 68, 5975–6002.

Cai, T. T., Zhang, L., and Zhou, H. H. (2018), “Adaptive functional linear regression via functional principal component analysis and block thresholding,” *Statistica Sinica*, 28, 2455–2468.

Cao, G., Wang, S., and Wang, L. (2020), “Estimation and inference for functional linear regression models with partially varying regression coefficients,” *Stat*, 9, e286.

Cardot, H., Ferraty, F., Mas, A., and Sarda, P. (2003a), “Testing hypotheses in the functional linear model,” *Scandinavian Journal of Statistics*, 30, 241–255.
Cardot, H., Ferraty, F., and Sarda, P. (1999), “Functional linear model,” *Statistics & Probability Letters*, 45, 11–22.

— (2003b), “Spline estimators for the functional linear model,” *Statistica Sinica*, 13, 571–591.

Chang, H.-w. and McKeague, I. W. (2022), “Empirical Likelihood-Based Inference for Functional Means with Application to Wearable Device Data,” *Journal of the Royal Statistical Society Series B: Statistical Methodology*, 84, 1947–1968.

Choi, H. and Reimherr, M. (2018), “A geometric approach to confidence regions and bands for functional parameters,” *Journal of the Royal Statistical Society: Series B (Statistical Methodology)*, 80, 239–260.

Ferraty, F. and Vieu, P. (2006), *Nonparametric Functional Data Analysis: Theory and Practice*, New York: Springer-Verlag.

Giessing, A. and Fan, J. (2020), “Bootstrapping $\ell_p$-Statistics in High Dimensions,” .

Hall, P. and Horowitz, J. L. (2007), “Methodology and convergence rates for functional linear regression,” *The Annals of Statistics*, 35, 70–91.

Hilgert, N., Mas, A., and Verzelen, N. (2013), “Minimax adaptive tests for the functional linear model,” *The Annals of Statistics*, 41, 838–869.

Horváth, L. and Kokoszka, P. (2012), *Inference for functional data with applications*, Springer Series in Statistics, Springer.

Hsing, T. and Eubank, R. (2015), *Theoretical Foundations of Functional Data Analysis, with an Introduction to Linear Operators*, Wiley.

James, G. M., Wang, J., and Zhu, J. (2009), “Functional linear regression that’s interpretable,” *The Annals of Statistics*, 37, 2083–2108.

Jin, C., Netrapalli, P., Ge, R., Kakade, S. M., and Jordan, M. I. (2019), “A short note on concentration inequalities for random vectors with subgaussian norm,” *arXiv preprint arXiv:1902.03736*.

Kokoszka, P., Maslova, I., Sojka, J., and Zhu, L. (2008), “Testing for lack of dependence in the functional linear model,” *Canadian Journal of Statistics*, 36, 207–222.

Kokoszka, P. and Reimherr, M. (2017), *Introduction to Functional Data Analysis*, Chapman and Hall/CRC.
Kong, D., Xue, K., Yao, F., and Zhang, H. H. (2016), “Partially functional linear regression in high dimensions,” *Biometrika*, 103, 147–159.

Lai, T., Zhang, Z., and Wang, Y. (2021), “Testing independence and goodness-of-fit jointly for functional linear models,” *Journal of the Korean Statistical Society*, 50, 380–402.

Lei, J. (2014), “Adaptive Global Testing for Functional Linear Models,” *Journal of the American Statistical Association*, 109, 624–634.

Lin, Y. and Lin, Z. (2024), “Supplementary Material for “Hypothesis Testing for Functional Linear Models via Bootstrapping”.”

Lin, Z., Cao, J., Wang, L., and Wang, H. (2017), “Locally Sparse Estimator for Functional Linear Regression Models,” *Journal of Computational and Graphical Statistics*, 26, 306–318.

Lin, Z., Lopes, M. E., and Müller, H.-G. (2022), “High-dimensional MANOVA via Bootstrapping and its Application to Functional Data and Sparse Count Data,” *Journal of the American Statistical Association*, to appear.

Lopes, M. E., Lin, Z., and Müller, H.-G. (2020), “Bootstrapping max statistics in high dimensions: Near-parametric rates under weak variance decay and application to functional data analysis,” *The Annals of Statistics*, 48, 1214–1229.

Lopes, M. E. and Yao, J. (2022), “A sharp lower-tail bound for Gaussian maxima with application to bootstrap methods in high dimensions,” *Electronic Journal of Statistics*, 16, 58–83.

Meister, A. (2011), “ASYMPTOTIC EQUIVALENCE OF FUNCTIONAL LINEAR REGRESSION AND A WHITE NOISE INVERSE PROBLEM,” *The Annals of Statistics*, 39, 1471–1495.

Müller, H.-G., Sen, R., and Stadtmüller, U. (2011), “Functional data analysis for volatility,” *Journal of Econometrics*, 165, 233–245.

Müller, H.-G. and Stadtmüller, U. (2005), “Generalized functional linear models,” *Ann. Statist.*, 33, 774–805.

Qu, S. and Wang, X. (2017), “Optimal Global Test for Functional Regression,” *arxiv*.

Ramsay, J. O. and Silverman, B. W. (2005), *Functional Data Analysis*, Springer Series in Statistics, New York: Springer, 2nd ed.

Shang, H. L. (2017), “Functional time series forecasting with dynamic updating: An application to intraday particulate matter concentration,” *Econometrics and statistics*, 1, 184–200.
Shen, Q. and Faraway, J. (2004), “An F test for linear models with functional responses,” *Statistica Sinica*, 14, 1239–1257.

Shin, H. (2009), “Partial functional linear regression,” *Journal of Statistical Planning and Inference*, 139, 3405–3418.

Smaga, L. (2019), “General linear hypothesis testing in functional response model,” *Communications in Statistics-Theory and Methods*, 1–16.

Su, Y.-R., Di, C.-Z., and Hsu, L. (2017), “Hypothesis testing in functional linear models,” *Biometrics*, 73, 551–561.

Tang, C. and Shi, Y. (2021), “Forecasting High-Dimensional Financial Functional Time Series: An Application to Constituent Stocks in Dow Jones Index,” *Journal of Risk and Financial Management*, 14, 343.

Vershynin, R. (2018), *High-dimensional probability: An introduction with applications in data science*, vol. 47, Cambridge university press.

Wahl, M. (2022), “Lower bounds for invariant statistical models with applications to principal component analysis,” in *Annales de l’Institut Henri Poincare (B) Probabilites et statistiques*, Institut Henri Poincaré, vol. 58, pp. 1565–1589.

Wang, D., Zhao, Z., Yu, Y., and Willett, R. (2020), “Functional Linear Regression with Mixed Predictors,” *arXiv preprint arXiv:2012.00460*.

WHO (2020), “WHO Physical activity Fact Sheet,” https://www.who.int/news-room/fact-sheets/detail/physical-activity.

Xue, K. and Yao, F. (2021), “Hypothesis testing in large-scale functional linear regression,” *Statistica Sinica*, 31, 1101–1123.

Yao, F., Müller, H.-G., and Wang, J.-L. (2005), “Functional linear regression analysis for longitudinal data,” *The Annals of Statistics*, 33, 2873–2903.

Yuan, M. and Cai, T. T. (2010), “A reproducing kernel Hilbert space approach to functional linear regression,” *The Annals of Statistics*, 38, 3412–3444.

Yuan, M., Cai, T. T., et al. (2010), “A reproducing kernel Hilbert space approach to functional linear regression,” *The Annals of Statistics*, 38, 3412–3444.

Zhang, J.-T. (2011), “Statistical inferences for linear models with functional responses,” *Statistica Sinica*, 21, 1431–1451.
— (2013), *Analysis of variance for functional data*, London: Chapman & Hall.

Zhou, J., Wang, N.-Y., and Wang, N. (2013), “Functional Linear Model with Zero-value Coefficient Function at Sub-regions,” *Statistica Sinica*, 23, 25–50.

Zhu, H., Li, R., and Kong, L. (2012), “Multivariate varying coefficient model for functional responses,” *The Annals of Statistics*, 40, 2634–2666.
Supplementary Material for “Hypothesis Testing for Functional Linear Models via Bootstrapping”

Yinan Lin and Zhenhua Lin

Department of Statistics and Data Science, National University of Singapore

**Organization.** In Section S1, we present mean models related to the test sequence, preparing for the proofs of Theorems 3.5–3.7. These proofs are detailed in Sections S2 and S3. Section S4 is devoted to the proof for Theorem 3.9. We have compiled some auxiliary results, instrumental in these proofs, in Section S5. An example of Assumption 3.1 is provided in Section S6. The proof of Proposition 2.1 is given in Section S7. Additionally, Section S8 presents the proof of Proposition 3.4, which is based on a concentration inequality established in Section S9. In Section S10 we provide a list of special cases of the model (1).

**Notation and convention.** Recall the eigen-decomposition of the covariance operators as follows:

\[ C_X = \sum_{j=1}^{\infty} \lambda_j \phi_j \otimes \phi_j, \quad C_Y = \sum_{j=1}^{\infty} \rho_j \psi_j \otimes \psi_j, \quad (S1) \]

and consider the Karhunen-Loève expansions:

\[ X_i = \sum_{j=1}^{\infty} (X_i, \phi_j)\phi_j = \sum_{j=1}^{\infty} \sqrt{\lambda_j} \eta_{ij}^X \phi_j \in X, \quad Y_i = \sum_{j=1}^{\infty} (Y_i, \psi_j)\psi_j = \sum_{j=1}^{\infty} \sqrt{\rho_j} \eta_{ij}^Y \psi_j \in Y, \]

where \( \eta_{ij}^X \)'s (and similarly \( \eta_{ij}^Y \)'s) are zero-mean, unit variance random variables that are uncorrelated for each \( i,j \). The variance of the random variable \( (X_i, \phi_{j1})_1 \) is \( \lambda_{j1} \) for each \( i,j_1 \), and similarly, the variance of the random variable \( (Y_i, \psi_{j2})_2 \) is \( \rho_{j2} \) for each \( i,j_2 \). In addition, let \( \kappa_j = \mathbb{E}[\langle Z_i, \psi_j \rangle^2] \).

We now summarize some general notation. The notation \( \langle \cdot, \cdot \rangle \) is used for an inner product in a specified Hilbert space, clear from the context. Let \( \mathbb{N} \) denote the set of natural numbers, excluding zero. Denote by \( \ell^2 \) the space of square summable sequences consisting of elements of the form \( (x_1, x_2, \ldots) \), where \( x_j \in \mathbb{R} \) and \( \sum_{j=1}^{\infty} x_j^2 < \infty \). As usual, we equip \( \ell^2 \) with the inner product \( \langle x, y \rangle = \sum_{j=1}^{\infty} x_j y_j \) for any \( x, y \in \ell^2 \). Let \( V^\infty \in \ell^2 \) represent a random sequence taking values in \( \ell^2 \), defined via a probability measure on \( \ell^2 \). For \( S = \mathbb{R}^q \) with \( q \in \mathbb{N} \) or \( S = \ell^2 \), the vector \( e_j \in S \) denotes the standard basis vector with one at the \( j \)th coordinate and zero elsewhere. For a random variable \( X \in \mathbb{R} \), denote \( \|X\|_{L_p} = (\mathbb{E}[X^q])^{1/q} \) for \( q \in \mathbb{N} \). The constants \( c, C, c_1, C_1, \ldots \) may vary with the context, but do not depend on \( p, n \) or \( (\hat{\phi}, \hat{\psi}) \in \mathcal{F}_p \). In addition, an “absolute constant” refers to a constant independent of \( p, n \) and \( (\hat{\phi}, \hat{\psi}) \in \mathcal{F}_p \). For an operator \( U \), let \( U^\dagger \) denote its adjoint, and

1
For a sequence \( \{c_j\} \), \( c_{(j)} \) denotes its decreasingly ordered entries. The notation \( (j) \) refers to the index of \( c_{(j)} \) in the original sequence \( \{c_j\} \). In cases of multiple sequences, symbols like \( (j)_1 \) and \( (j)_2 \) are used for indices in different sequences. For instance, \( \{\lambda_{(j)_1}\} \) and \( \{\rho_{(j)_2}\} \) represent the decreasingly ordered sequences of \( \{\lambda_{(j)}\}_{j=1}^{\infty} \) and \( \{\rho_{(j)}\}_{j=1}^{\infty} \) respectively.

Unless otherwise stated, quantities depending on \((\hat{\phi}, \hat{\psi})\) are marked with a tilde, while those without it depend on \((\phi, \psi)\). For example, \( \hat{V}_i = V_i(\hat{\phi}, \hat{\psi}) \) and \( V_i = V_i(\phi, \psi) \). We also use the symbol with a hat like \( \hat{\sigma} = \hat{\sigma}(\hat{\phi}, \hat{\psi}) \) to indicate an empirical quantity related to \((\hat{\phi}, \hat{\psi})\). Our primary focus is on the case where \( d_X = d_Y = \infty \); the cases that \( d_X \) or \( d_Y \) is finite can be analyzed in an almost identical (and simpler) fashion.

For \( j_1, j_2 \in \mathbb{N} \), denote \( \xi_{i j_1} = \langle X_i, \phi_{j_1} \rangle_1 \) and \( \xi_{i j_2} = \langle Y_i, \psi_{j_2} \rangle_2 \). When the context is clear, the subscripts of the inner products are sometimes omitted. For two positive integers \( k_1 \) and \( k_2 \), let \( \mathcal{P}(k_1, k_2) = \{ (j_1, j_2) \in \mathbb{N}^2 : 1 \leq j_1 \leq k_1, 1 \leq j_2 \leq k_2 \} \). When \( k_1 = p_1 \) and \( k_2 = p_2 \), \( \mathcal{P} = \mathcal{P}(p_1, p_2) \) is used for brevity. We introduce a bijection \( \Phi : \mathcal{P} \rightarrow \mathcal{O} \) with \( \mathcal{O} = \{1, \ldots, p\} \) for \( p = p_1 p_2 \), such that \( \Phi((j_1, j_2)) = j \) for each \((j_1, j_2) \in \mathcal{P} \) and \( j \in \mathcal{O} \), and \( \nu = \{ \nu_j : j = \Phi((j_1, j_2)) \} \in \mathbb{R}^p \) is the mean vector rearranged via the map \( \Phi \) with elements \( \nu_j = E[\xi_{ij_1} \xi_{ij_2}] \). The specific choice of \( \Phi \) and the arrangement of \( \nu_j \) do not play a role in our proof, but the introduction of \( \Phi \) can facilitate the discussion in some contexts. Given \( \mathcal{P} \) and \( \Phi \), for \( j \in \{1, \ldots, p\} \), the \( j \)-th element of \( V_i = (\xi_{ij_1} \xi_{ij_2} : (j_1, j_2) \in \mathcal{P}) \) corresponds to the basis functions \( \phi_{j_1} \) and \( \psi_{j_2} \) for some \((j_1, j_2) \in \mathcal{P} \) such that \( \Phi((j_1, j_2)) = j \). Similarly, for the infinite sequence \( V_i^\infty = (\xi_{ij_1} \xi_{ij_2} : (j_1, j_2) \in \mathbb{N}^2) \), the indexing \((j_1, j_2) \in \mathbb{N}^2 \) indicates the position of \( \xi_{ij_1} \xi_{ij_2} \) in \( V_i^\infty \). This indexing convention applies to \( \hat{V}_i \), \( \hat{V}_i^\infty \), and other related sequences, including their expected values. Finally, it is straightforward to extend \( \Phi \) on \( \mathcal{P}(\infty, \infty) = \{ (j_1, j_2) \in \mathbb{N}^2 : j_1 \geq 1, j_2 \geq 1 \} \).

### S1 Mean Models Related to the Test

Recall that, for each \( i = 1, \ldots, n \), we define \( V_i^\infty(\phi, \psi) := V_i^\infty = (\xi_{ij_1} \xi_{ij_2} : (j_1, j_2) \in \mathbb{N}^2) \in \ell^2 \). Here, \( \xi_{ij_1} = \langle X_i, \phi_{j_1} \rangle_1 \) and \( \xi_{ij_2} = \langle Y_i, \psi_{j_2} \rangle_2 \), forming the infinite test sequence with respect to the eigenbasis of the covariance operators. Additionally, \( V_i(\phi, \psi) := V_i = (\xi_{ij_1} \xi_{ij_2} : (j_1, j_2) \in \mathcal{P}) \in \mathbb{R}^p \) with \( p = p_1 p_2 \) is the truncated counterpart of \( V_i^\infty \). Let \( \Pi_p^\infty \) be the projection operator from \( \ell^2 \) to \( \mathbb{R}^p \), such that \( V_i = \Pi_p^\infty V_i^\infty \). In other words, \( \Pi_p^\infty \) acts on \( V_i^\infty \) to yield the \( p \) coordinates comprising the \( p \)-dimensional vector \( V_i \).

Before presenting our proofs, we first present two related mean models for \( V_i^\infty \) and \( V_i \). Specifically, we model

\[
V_i^\infty = \nu^\infty + \epsilon_i^\infty, \quad i = 1, \ldots, n, \tag{S2}
\]

where \( \nu^\infty = E[V_i^\infty] \) is the expectation of \( V_i^\infty \), \( \epsilon_i^\infty = V_i^\infty - \nu^\infty \). Let \( \Sigma^\infty = E[(V_i^\infty - \nu^\infty) \odot (V_i^\infty - \nu^\infty)] \) denote the covariance operator of \( V_i^\infty \), and it is also the covariance operator of \( \epsilon_i^\infty \). As a covariance operator, \( \Sigma^\infty \) is nonnegative-definite and trace-class (Theorem 7.2.5, Hsing and Eubank, 2015). Based on \( \Sigma^\infty \) and the
relation $V_i = \Pi_p^\infty V_i^\infty$, we derive a mean model for $V_i$:

$$V_i = \nu + \Pi_p^\infty \epsilon_i^\infty, \quad i = 1, \ldots, n, \quad (S3)$$

where $\nu = \mathbb{E} V_i = \Pi_p^\infty \nu^\infty$, and the covariance matrix $\Sigma$ of $V_i$ satisfies the relation that

$$\Sigma = \mathbb{E}[(V_1 - \nu)(V_1 - \nu)^\top] = \Pi_p^\infty \Sigma^\infty (\Pi_p^\infty)^\dagger.$$

When the basis $(\tilde{\phi}, \tilde{\psi})$ is used, we obtain the test sequence $\tilde{V}_i^\infty$ and the corresponding truncated counterpart $\tilde{V}_i$, which satisfies the relation $\tilde{V}_i = \tilde{\Pi}_p^\infty \tilde{V}_i^\infty$ for a projection operator, from $\ell^2$ to $\mathbb{R}^p, \tilde{\Pi}_p^\infty$. Similar to (S2) and (S3), we have representations for $\tilde{V}_i^\infty$ and $\tilde{V}_i$ as follow:

$$\tilde{V}_i^\infty = \tilde{\nu}^\infty + \tilde{\epsilon}_i^\infty, \quad i = 1, \ldots, n, \quad (S4)$$

and

$$\tilde{V}_i = \tilde{\nu} + \tilde{\Pi}_p^\infty \tilde{\epsilon}_i, \quad i = 1, \ldots, n, \quad (S5)$$

where the corresponding quantities are similarly defined as in (S2) and (S3). Based on Proposition S5.4, we have $\tilde{\Sigma}^\infty = \tilde{U}^\dagger \Sigma \tilde{U}$.

S2   Proofs of Theorem 3.5 and Theorem 3.7

In all the proofs, for a given pair of bases $(\tilde{\phi}, \tilde{\psi}) \in \mathcal{F}_p$, we may notationally omit the dependence on $(\tilde{\phi}, \tilde{\psi})$ for related quantities for brevity.

S2.1 Proof of Theorem 3.5

Proof of Theorem 3.5. Consider the inequality

$$\sup_{(\tilde{\phi}, \tilde{\psi}) \in \mathcal{F}_p} d_K(\mathcal{L}(M_p(\tilde{\phi}, \tilde{\psi})), \mathcal{L}(\tilde{M}_p(\tilde{\phi}, \tilde{\psi}))) \leq \sup_{(\tilde{\phi}, \tilde{\psi}) \in \mathcal{F}_p} \tilde{I}_n + \sup_{(\tilde{\phi}, \tilde{\psi}) \in \mathcal{F}_p} \tilde{\Pi}_n + \sup_{(\tilde{\phi}, \tilde{\psi}) \in \mathcal{F}_p} \tilde{\Pi}_n, \quad (S6)$$

where we define

$$\tilde{I}_n = d_K\left(\mathcal{L}(M_p(\tilde{\phi}, \tilde{\psi})), \mathcal{L}(M_{k_n}(\tilde{\phi}, \tilde{\psi}))\right), \quad (S7)$$

$$\tilde{\Pi}_n = d_K\left(\mathcal{L}(M_{k_n}(\tilde{\phi}, \tilde{\psi})), \mathcal{L}(\tilde{M}_p(\tilde{\phi}, \tilde{\psi}))\right), \quad (S8)$$

$$\tilde{\Pi}_n = d_K\left(\mathcal{L}(\tilde{M}_{k_n}(\tilde{\phi}, \tilde{\psi})), \mathcal{L}(\tilde{M}_p(\tilde{\phi}, \tilde{\psi}))\right). \quad (S9)$$
Below, we show that the term $\tilde{I}_n$ is at most $Cn^{-\frac{1}{2}+\delta}$ in Proposition S2.1 for an absolute constant $C > 0$. Later, we establish corresponding results for $\tilde{I}_n$ and $\tilde{II}_n$ in Proposition S2.2. These results together complete the proof of Theorem 3.5.

\textbf{Proposition S2.1.} Fix any sufficiently small number $\delta \in (0, 1/2)$, and suppose the conditions of Theorem 3.5 hold. Then, for each $(\tilde{\phi}, \tilde{\psi}) \in \mathcal{F}_p$,

$$\tilde{I}_n \leq Cn^{-\frac{1}{2}+\delta}.$$  \hfill (S10)

for a constant $C > 0$ not depending on $p$, $n$ or $(\tilde{\phi}, \tilde{\psi})$.

Proof. For the truncated test sequence $\tilde{V}_i \in \mathbb{R}^p$, an alternative representation to the mean model (S5) can be formulated as:

$$\tilde{V}_i = \tilde{\nu} + \tilde{\Sigma}^{1/2}\epsilon_i,$$

where $\epsilon_i = \tilde{\Sigma}^{-1/2}\epsilon_i \in \mathbb{R}^p$, $\epsilon_i = \tilde{V}_i - \tilde{\nu} \in \mathbb{R}^p$ and $\Sigma = \mathbb{E}[^t\epsilon_i\epsilon_i] \in \mathbb{R}^{p \times p}$.

Under this representation, $\epsilon_i$ is sub-exponential by Proposition S2.6. Consequently, the desired result can be derived as demonstrated in the proof of Proposition B.1 in Lopes et al. (2020).

\textbf{Proposition S2.2.} Fix any sufficiently small number $\delta \in (0, 1/2)$, and suppose the conditions of Theorem 3.5 hold. Then, for each $(\tilde{\phi}, \tilde{\psi}) \in \mathcal{F}_p$,

$$\tilde{I}_n \leq Cn^{-\frac{1}{2}+\delta} \quad \text{and} \quad \tilde{II}_n \leq Cn^{-\frac{1}{2}+\delta}.$$  \hfill (S11)

for a constant $C > 0$ not depending on $p$, $n$ or $(\tilde{\phi}, \tilde{\psi})$.

Proof. We only prove the bound for $\tilde{I}_n$, since the same argument applies to $\tilde{II}_n$. We can check that for any fixed real number $t$,

$$\left| \mathbb{P}\left( \max_{1 \leq j \leq p} \tilde{S}_{n,j}/\tilde{\sigma}_j^2 \leq t \right) - \mathbb{P}\left( \max_{j \in \mathcal{J}_n} \tilde{S}_{n,j}/\tilde{\sigma}_j^2 \leq t \right) \right| = \mathbb{P}(A(t) \cap B(t)),$$

where we define the events

$$A(t) = \left\{ \max_{j \in \mathcal{J}_n} \tilde{S}_{n,j}/\tilde{\sigma}_j^2 \leq t \right\} \quad \text{and} \quad B(t) = \left\{ \max_{j \in \mathcal{J}_n^c} \tilde{S}_{n,j}/\tilde{\sigma}_j^2 > t \right\},$$  \hfill (S12)

where for any $d \in \{1, \ldots, p\}$, $\mathcal{J}_d$ denotes the set of indices corresponding to the $d$ largest values among the standard deviations $\tilde{\sigma}_1, \ldots, \tilde{\sigma}_p$ of the elements in $\tilde{V}_1$, and $\mathcal{J}_d^c$ denotes the complement of $\mathcal{J}_d$ in $\{1, \ldots, p\}$. Also, for any pair of real numbers $t_{1,n}$ and $t_{2,n}$ satisfying $t_{1,n} \leq t_{2,n}$, it is straightforward to check that the following inclusion holds for all $t \in \mathbb{R}$,

$$A(t) \cap B(t) \subset A(t_{2,n}) \cup B(t_{1,n}).$$  \hfill (S13)
Applying a union bound, and then taking the supremum over $t \in \mathbb{R}$, we obtain

$$\tilde{I}_n \leq \mathbb{P}(A(t_{2,n})) + \mathbb{P}(B(t_{1,n})).$$

The remainder of the proof consists in selecting $t_{1,n}$ and $t_{2,n}$ so that $t_{1,n} \leq t_{2,n}$ and that the probabilities $\mathbb{P}(A(t_{2,n}))$ and $\mathbb{P}(B(t_{1,n}))$ are sufficiently small. Below, Lemma S2.3 shows that if $t_{1,n}$ and $t_{2,n}$ are chosen as

$$t_{1,n} = c_1 \cdot \ell_n^{-\frac{1}{2}(1-\tau)} \cdot \log(n) \quad (S14)$$

$$t_{2,n} = c_2 \cdot \ell_n^{-\frac{1}{2}(1-\tau)} \cdot \sqrt{\log(d_n)}, \quad (S15)$$

for certain absolute constants $c_1, c_2 > 0$ and $d_n$ defined in the proof of Lemma S2.3, then $\mathbb{P}(A(t_{2,n}))$ and $\mathbb{P}(B(t_{1,n}))$ are at most of the order $n^{-\frac{1}{2}+\delta}$. Furthermore, the inequality $t_{1,n} \leq t_{2,n}$ holds for all large $n$, due to the definition of $\ell_n$ and both definitions of $k_n$. 

\textbf{Lemma S2.3.} Fix any sufficiently small number $\delta \in (0,1/2)$, and suppose the conditions of Theorem 3.5 hold. Then, there exists a constant $C > 0$, not depending on $p$, $n$ or $(\tilde{\phi}, \tilde{\psi})$, that can be selected according to the definitions of $t_{1,n}$ (S14) and $t_{2,n}$ (S15), such that

$$\mathbb{P}(A(t_{2,n})) \leq C n^{-\frac{1}{2}+\delta}, \quad (a)$$

and

$$\mathbb{P}(B(t_{1,n})) \leq C n^{-1}, \quad (b)$$

\textbf{Proof.} Define the parameter

$$\omega = 2\delta^2/\iota,$$

where $\iota = 4 \vee (3\alpha(1-\tau))$. Clearly, $\omega \in (0, \delta)$ for any choice of $\delta \in (0,1/2)$. Also, define the integer

$$d_n = \lceil (\frac{\omega^2}{4} r(\tilde{R}(\ell_n))) \vee 1 \rceil,$$

where $r(A)$ is the stable rank of a non-zero positive semidefinite matrix $A$ defined as $r(A) = [\text{tr}(A)]^2/\|A\|_F^2$. Note that since $r(\tilde{R}(\ell_n)) \leq \ell_n$, and thus the inequalities $d_n \leq \ell_n \leq k_n \leq p$ hold for all $n$. Also, using Proposition S2.7, for a sufficiently small $\delta$, we have the following lower bound on $d_n$,

$$d_n \geq c r(\tilde{R}(\ell_n)) = \frac{\iota^2}{[\tilde{R}(\ell_n)]_F^2} \geq c \ell_n^{\delta} \geq c n^{\delta^2/\iota}. \quad (S16)$$
Part (a). Due to Proposition S2.1 and the fact that \( \tilde{J}_{\ell n} \subset \tilde{J}_{k n} \), we have

\[
\mathbb{P}(A(t_{2,n})) \leq \mathbb{P}\left( \max_{j \in \tilde{J}_{k n}} \tilde{S}_{n,j}/\sigma_j \leq t_{2,n} \right) + \Pi_n
\]

\[
\leq \mathbb{P}\left( \max_{j \in \tilde{J}_{k n}} \tilde{S}_{n,j}/\sigma_j \leq t_{2,n} \right) + c n^{-\frac{1}{2} + \delta}.
\]

(S17)

To bound the probability in the last line, consider some generic random variables \( \{Z_j\} \) and positive scalars \( \{a_j\} \) indexed by \( j \in \tilde{J}_{\ell n} \), as well as a constant \( b \) such that \( \max_{j \in \tilde{J}_{\ell n}} a_j \leq b \). Then,

\[
\mathbb{P}\left( \max_{j \in \tilde{J}_{\ell n}} Z_j \leq t_{2,n} \right) \leq \mathbb{P}\left( \max_{j \in \tilde{J}_{\ell n}} a_j Z_j \leq b t_{2,n} \right).
\]

(S18)

Based on Proposition S2.7, there is a positive constant \( c_0 \), not depending on \( p, n \) or \( (\tilde{\phi}, \tilde{\psi}) \), such that the inequality \( \tilde{\sigma}_j^{-(1-\tau)} \leq \tilde{c}_n^{-(1-\tau)/2} / c_0 \) holds for all \( j \in \tilde{J}_{\ell n} \). Accordingly, we will use (S18) with the choices \( a_j = \tilde{\sigma}_j^{-(1-\tau)}, b = \tilde{c}_n^{-(1-\tau)/2} / c_0 \), and \( Z_j = \tilde{S}_{n,j}/\tilde{\sigma}_j \). Also, we may choose \( c_2 \) in the definition of \( t_{2,n} \) to be \( c_2 = \omega \log_2(2(1-\omega)) \), which implies \( b t_{2,n} = \omega \sqrt{2(1-\omega) \log(d_n)} \). Under these choices, the inequality (S18) becomes

\[
\mathbb{P}\left( \max_{j \in \tilde{J}_{\ell n}} \tilde{S}_{n,j}/\tilde{\sigma}_j \leq t_{2,n} \right) \leq \mathbb{P}\left( \max_{j \in \tilde{J}_{\ell n}} \tilde{S}_{n,j}/\tilde{\sigma}_j \leq \omega \sqrt{2(1-\omega) \log(d_n)} \right).
\]

Now, we apply Theorem 2.3 in Lopes and Yao (2022) to the right-hand side, with \( (\ell_n, d_n, \omega, \omega) \) playing the roles of \( (N, k, \epsilon, \delta) \) therein. Hence,

\[
\mathbb{P}\left( \max_{j \in \tilde{J}_{\ell n}} \tilde{S}_{n,j}/\tilde{\sigma}_j \leq t_{2,n} \right) \leq c d_n^{-(1-\omega)/2} \left( \log(d_n) \right)^{1-\omega/2} \omega \sqrt{2(1-\omega) \log(d_n)}.
\]

Furthermore, using the lower bound on \( d_n \) from (S16), there is an absolute constant \( c \), such that

\[
\mathbb{P}\left( \max_{j \in \tilde{J}_{\ell n}} \tilde{S}_{n,j}/\tilde{\sigma}_j \leq t_{2,n} \right) \leq c b \left( \omega \log_2(2(1-\omega)) \right)^{1-\omega/2} \omega \sqrt{2(1-\omega) \log(d_n)} \leq n^{-\frac{1}{2} + \delta},
\]

which, together with (S17), implies (a).

Part (b). To establish (b), define \( q = \max\left\{ \frac{2}{(1-\tau)\bar{\omega}} \log(n), 3 \right\} \). For any \( t > 0 \), we have the tail bound

\[
\mathbb{P}(B(t_{1,n})) \leq t^{-q} \left\| \max_{j \in \tilde{J}_{k n}} \tilde{S}_{n,j}/\sigma_j \right\|_{L^q}^q.
\]

(S19)

Due to Lemma S2.9, \( \left\| \tilde{S}_{n,j}/\sigma_j \right\|_q \leq cd_j \) for all \( j = 1, \ldots, p \), and thus, from Proposition S2.7, we have

\[
\left\| \max_{j \in \tilde{J}_{k n}} \tilde{S}_{n,j}/\sigma_j \right\|_{L^q}^q \leq \sum_{j \in \tilde{J}_{k n}} \left\| \tilde{S}_{n,j}/\sigma_j \right\|_{L^q}^q \leq (cq)^q \sum_{j \in \tilde{J}_{k n}} \sigma_j^{q(1-\tau)}.
\]
\[ \left\langle (cq)^q \sum_{j=k_n+1}^p j^{-q(1-\tau)/2} \right\rangle \leq \frac{(cq)^q}{q^{(1-\tau)/2-1}} k_n^{-q(1-\tau)/2+1}. \]

One can check that \( q \asymp \log(n) \) and \( \frac{cq}{q^{(1-\tau)/2-1}} n^{1/q} \leq 1 \). Hence, if we take \( t = e \max_{j \in \mathcal{J}_n} \left\| \tilde{S}_{n,j}/\tilde{\sigma}_j \right\|_{L^q} \) in (S19), then there is a choice of \( c_1 \) for which \( t_{1,n} \) satisfies \( t \leq t_{1,n} \), and furthermore,

\[ \mathbb{P}(B(t_{1,n})) \leq \mathbb{P}(B(t)) \leq e^{-q} \leq n^{-1}. \]

\[ \square \]

### S2.2 Proof of Theorem 3.7

**Proof.** We reuse the proof of Theorem 3.4 in Lin et al. (2022), where the key ingredients have been analyzed in the proofs of Theorems 3.5 and 3.6. Specifically, in that paper, it is mentioned that Assumption 4 therein can be replaced with the condition \( n^{-1/2} \log^3(p) \ll 1 \) which is satisfied here due to our Assumption 3.3. By examining their proof, we find that it is sufficient to prove Lemma E.7 and Lemma E.8 in Lin et al. (2022) for the sequence \( \{\tilde{V}_i\}_{i=1}^n \) satisfying the assumptions in Lopes et al. (2020). We fulfill these requirements by Lemmas S2.4 and S2.5 below.

For the following two lemmas, we define

\[ \hat{M}_d = \max_{j \in \mathcal{J}_d} \tilde{S}_{n,j}/\tilde{\sigma}_j, \quad (S20) \]

where for any \( d \in \{1, \ldots, p\} \), recall \( \mathcal{J}_d \) denotes a set of indices corresponding to the \( d \) largest values among the standard deviations \( \tilde{\sigma}_1, \ldots, \tilde{\sigma}_p \) of the elements in \( \tilde{V}_1 \), i.e. \( \{\tilde{\sigma}_1, \ldots, \tilde{\sigma}_p\} = \{\tilde{\sigma}_j : j \in \mathcal{J}_d\} \).

**Lemma S2.4.** Under the conditions of Theorem 3.7, for some constant \( c > 0 \), not depending on \( p \) or \( n \), we have

\[ \sup_{(\hat{\phi}, \hat{\psi}) \in \mathcal{F}_p} \mathbb{P} \left( |\hat{M}_{k_n} - \hat{M}_{k_n}| > r_n \right) \leq cn^{-1}, \]

where \( \hat{M}_{k_n} \) is defined according to (S20) and \( r_n = cn^{-1/2} \log^{5/2}(n) \).

**Proof.** Note that Lemma E.4 and Lemma E.3 in Lin et al. (2022), which are the key ingredients for proving Lemma E.7 in Lin et al. (2022), can be replaced by Lemma D.6 and Lemma D.4 in Lopes et al. (2020), which have been analogously established in Lemmas S3.3 and S2.9 respectively under our settings. \( \square \)

The next lemma is in analogy to Lemma E.8 in Lin et al. (2022) and can be established also by a similar argument that leads to Lemma E.8 in Lin et al. (2022) for each given \( (\hat{\phi}, \hat{\psi}) \in \mathcal{F}_p \). We thus omit the proof.
Lemma S2.5. Under Assumptions 3.1 and 3.3, for any number $D \geq 2$, there are positive constants $c$ and $c_D$, not depending on $p$ or $n$, such that the event

$$
\sup_{(\hat{\phi},\hat{\psi}) \in \mathcal{F}_p} \mathbb{P}\left( \max_{1 \leq j \leq p} \left| \gamma_j \right| - 1 \leq c_D (\log^2(n) + \log^3(p)) n^{-\frac{1}{2}} \right) \geq 1 - cn^{-D}
$$

S2.3 Structure-related Propositions and Technical Lemmas

For each pair $(\hat{\phi},\hat{\psi}) \in \mathcal{F}_p$, the random element $\hat{\epsilon}_1^\infty$ from (S4) exhibits a sub-exponential behavior in $\ell^2$.

**Proposition S2.6.** Under Assumption 3.1, for each pair $(\hat{\phi},\hat{\psi}) \in \mathcal{F}_p$, there exists an absolute constant $K_0 > 0$, such that

$$
\sup_{x^\infty \in S_\infty} \|\langle \hat{\epsilon}_1^\infty, x^\infty \rangle\|_{\psi_1} \leq K_0 \left( \mathbb{E}\left( \langle \hat{\epsilon}_1^\infty, x^\infty \rangle^2 \right) \right)^{1/2},
$$

where $S_\infty$ denotes the unit sphere in $\ell^2$. Moreover, for $\tilde{\epsilon}_1 = \tilde{\Sigma}^{-1/2} \hat{\epsilon}_1$ with $\tilde{\epsilon}_1 = \tilde{\Pi}_p^\infty \hat{\epsilon}_1^\infty$, $\tilde{\Sigma} = \mathbb{E}[\tilde{\epsilon}_1 \tilde{\epsilon}_1^\top]$ and the projection operator $\tilde{\Pi}_p^\infty$ defined in Section S1, it satisfies

$$
\sup_{x \in S_p} \|\langle \tilde{\epsilon}_1^\infty, x^\infty \rangle\|_{\psi_1} \leq K_0,
$$

where $S_p$ denotes the unit sphere in $\mathbb{R}^p$.

**Proof.** Recall that $\epsilon_1^\infty = V_1^\infty - \nu^\infty$, where $\nu^\infty = \mathbb{E}\left[ V_1^\infty \right]$, as defined. For any $x^\infty \in S_\infty$, according to Assumption 3.1, there exists an absolute constant $K_0 > 0$, such that

$$
\|\langle \hat{\epsilon}_1^\infty, x^\infty \rangle\|_{\psi_1} \leq K_0 \left( \mathbb{E}\langle \hat{\epsilon}_1^\infty, x^\infty \rangle^2 \right)^{1/2}.
$$

From Proposition S5.4, we deduce that there is a unitary operator $\hat{U}$ on $\ell^2 \times \ell^2$ such that $V_1^\infty = \hat{U}^\dagger V_1^\infty$ and $\hat{\nu}^\infty = \hat{U}^\dagger \nu^\infty$. Consequently,

$$
\tilde{\epsilon}_1^\infty = \tilde{V}_1^\infty - \hat{\nu}^\infty = \hat{U}^\dagger (V_1^\infty - \nu^\infty) = \hat{U}^\dagger \epsilon_1^\infty,
$$

owing to the linearity of $\hat{U}^\dagger$. Therefore, for any $x^\infty \in S_\infty$ and noting $\|\hat{U} x^\infty\|_2 = 1$,

$$
\|\langle \epsilon_1^\infty, x^\infty \rangle\|_{\psi_1} = \|\langle \hat{U}^\dagger \epsilon_1^\infty, x^\infty \rangle\|_{\psi_1} = \|\langle \tilde{\epsilon}_1^\infty, \hat{U} x^\infty \rangle\|_{\psi_1} \\
\leq K_0 \left( \mathbb{E}\langle \tilde{\epsilon}_1^\infty, \hat{U} x^\infty \rangle^2 \right)^{1/2} = K_0 \left( \mathbb{E}\langle \epsilon_1^\infty, x^\infty \rangle^2 \right)^{1/2}

= K_0 \left( \mathbb{E}\langle \tilde{\epsilon}_1^\infty, x^\infty \rangle^2 \right)^{1/2}.
$$

Considering a unit vector $x \in S_p$, we observe that $(\tilde{\Pi}_p^\infty)^\dagger x$ belongs to $\ell^2$ and has unit length. Consequently,

$$
\|\langle \tilde{\epsilon}_1, x \rangle\|_{\psi_1} = \|\langle \epsilon_1^\infty, (\tilde{\Pi}_p^\infty)^\dagger x \rangle\|_{\psi_1} \leq K_0 \left( \mathbb{E}\langle \epsilon_1^\infty, (\tilde{\Pi}_p^\infty)^\dagger x \rangle^2 \right)^{1/2} = K_0 \left( \mathbb{E}\langle \tilde{\epsilon}_1, x \rangle^2 \right)^{1/2}.
$$
Lastly, for \( \tilde{\varepsilon}_1 = \tilde{\Sigma}^{-\frac{1}{2}} \varepsilon_1 \) where \( \tilde{\Sigma} = \mathbb{E}[\varepsilon_1 \varepsilon_1^\top] \), it holds that

\[
\sup_{x \in S_p} \| \langle \varepsilon_1, x \rangle \|_{\psi_1} = \sup_{x \in S_p} \| \langle \tilde{\Sigma}^{-\frac{1}{2}} \varepsilon_1, x \rangle \|_{\psi_1} \\
\leq \sup_{x \in S_p} K_0 \left( \mathbb{E}(\varepsilon_1, \tilde{\Sigma}^{-\frac{1}{2}} x)^2 \right)^{\frac{1}{2}} = \sup_{x \in S_p} K_0 \left( \mathbb{E}[x_p^\top \tilde{\Sigma}^{-\frac{1}{2}} \varepsilon_1 \tilde{\Sigma}^{-\frac{1}{2}} x] \right)^{\frac{1}{2}} \\
= \sup_{x \in S_p} K_0 \| x \|_2 = K_0.
\]

This confirms that \( \tilde{\varepsilon}_1 \) is a sub-exponential random vector. \( \square \)

We demonstrate below that \( \tilde{\Sigma} \) shares with \( \Sigma \) the same variance decay and correlation structures. In particular, \( \tilde{\Sigma} \) satisfies the variance decay condition (Condition (i) of Assumption 2.2) in Lopes et al. (2020) and the correlation structure condition (Equation (3.2)) in Lopes and Yao (2022).

**Proposition S2.7.** Let the variance sequence \( \{\tilde{\sigma}_j^2\}_{j=1}^p \) and the correlation matrix \( \tilde{R} \) of \( \tilde{V}_1 \) be induced by \( (\tilde{\varphi}, \tilde{\psi}) \in \mathcal{F}_p \). Recall \( k_n = \ell_n(\min(1,1)) \), where \( \ell_n \) is defined in Assumption 3.2(ii). Suppose Assumptions 3.1-3.3 hold with a sufficiently small \( \delta \in (0,1/2) \).

(i) There are positive constants \( c_0, c > 0 \), not depending on \( p, n \) or \( (\tilde{\varphi}, \tilde{\psi}) \), such that

\[
\tilde{\sigma}_j \geq c_0 j^{-\bar{\alpha}/2} \quad \text{for all } j = 1, \ldots, k_n
\]

and

\[
\tilde{\sigma}_j \leq c j^{-1/2} \quad \text{for all } j = 1, \ldots, p.
\]

(ii) There is a constant \( c_R > 0 \), not depending on \( p, n \) or \( (\tilde{\varphi}, \tilde{\psi}) \), such that

\[
\sup_{(\tilde{\varphi}, \tilde{\psi}) \in \mathcal{F}_p} \| \tilde{R}(\ell_n) \|_F^2 \leq c_R \ell_n^{2-\delta}.
\]

**Proof.** Part (i). For the case of \( j = 1, \ldots, k_n \), consider the index set \( \mathcal{P}(k_{n,1}, k_{n,2}) = \{(j_1, j_2) : j_1 = 1, \ldots, k_{n,1}, j_2 = 1, \ldots, k_{n,2}\} \) where \( k_{n,1} = k_{n,2} = [k_n^{1/2}] \). Given \( \mathcal{P}(k_{n,1}, k_{n,2}) \), Lemma S5.3 ensures that, for any \( j = k_{n,1}(j_2 - 1) + j_1 \) with \( j_1 = 1, \ldots, k_{n,1}, j_2 = 1, \ldots, k_{n,2}, \) \( |\tilde{\sigma}_j^2 - \sigma_j^2| \leq c_1 a_n \) for \( a_n = k_n^{-2\bar{\alpha}} \) and sufficiently small \( \delta \). Hence, \( \tilde{\sigma}_j^2 \geq \sigma_j^2 - c_1 a_n \). Similar to (S27), we also have \( \sigma_j^2 \geq c_0 j^{-\alpha} \) in this setting, which further implies \( \tilde{\sigma}_j^2 \geq c_0 j^{-\alpha} - c_1 a_n \). As \( j \leq k_n \), for an appropriate constant \( c_0 > 0 \), it is seen that \( \tilde{\sigma}_j \geq c_0 j^{-\bar{\alpha}/2} \) for each \( j = 1, \ldots, k_n \). Consequently,

\[
\tilde{\sigma}_j \geq c_0 j^{-\bar{\alpha}/2}.
\]

For \( j = k_n + 1, \ldots, p \), the relation \( \tilde{\Sigma}^\infty = \tilde{U}^\top \Sigma^\infty \tilde{U} \) from Section S1 implies

\[
\sum_{j=1}^{\infty} \tilde{\sigma}_j^2 = \| \tilde{\Sigma}^\infty \|_{TR} = \sum_{j=1}^{\infty} \langle \Sigma^\infty g_j, g_j \rangle = \sum_{j=1}^{\infty} \langle \Sigma^\infty \tilde{U} g_j, \tilde{U} g_j \rangle
\]

9
\[
\sum_{j=1}^{\infty} \langle \Sigma f_j, f_j \rangle = \| \Sigma \|_{TR} = \sum_{j=1}^{\infty} \sigma_j^2 < \infty,
\] (S22)

where \( \{g_j, j \geq 1\} \) is a CONS of \( \ell^2 \) and \( \{f_j, j \geq 1\} \) is another CONS determined by \( \{g_j, j \geq 1\} \). Let \( S := \sum_{j=1}^{\infty} \sigma_j^2 < \infty \), and note that \( S \) is not depending on \( p, n \) or \( (\tilde{\phi}, \tilde{\psi}) \). Applying Lemma S5.5, we establish

\[
\tilde{\sigma}(j) \leq cj^{-1/2},
\]

where \( c = \sqrt{S} \) is not depending on \( p, n \) or \( (\tilde{\phi}, \tilde{\psi}) \).

Part (ii). We first claim that the indices of the \( \ell_n \) largest elements among \( \{\tilde{\sigma}_j\}_{j=1}^{p} \) is a subset of the indices of \( \{\sigma_j\}_{j=1}^{m_n} \) for \( j = \Phi(j_1, j_2) \) and \( (j_1, j_2) \in P(m_n, 1, m_n, 2) \), where \( P(m_n, 1, m_n, 2) = \{(j_1, j_2) : j_1 = 1, \ldots, m_n, 1; j_2 = 1, \ldots, m_n, 2\} \), \( m_n = [m_n^2] \), \( m_n \) is defined in Assumption 3.2, and \( \Phi \) is defined at the beginning of this supplementary material. This claim stems from Lemma S5.3. Specifically, by Remark S2 and \( m_n \ll h_n \) for sufficiently small \( \delta \), we have \( \tilde{\sigma}_j^2 \geq \sigma_j^2 - c_1 a_n \) with \( a_n = k_n^{2\alpha} \) for each \( j = \Phi(j_1, j_2) \) and \( (j_1, j_2) \in P(m_n, 1, m_n, 2) \). Hence,

\[
\sum_{j=m_n+1}^{m_n} \tilde{\sigma}_j^2 \leq \sum_{j=m_n+1}^{m_n} \sigma_j^2 - c_1 m_n a_n.
\]

Based on this inequality and the relation (S22), one has

\[
\sum_{j=m_n+1}^{\infty} \tilde{\sigma}_j^2 \leq \sum_{j=m_n+1}^{\infty} \sigma_j^2 + c_1 m_n a_n.
\]

By Lemma S5.1, \( \sigma_j^2 \geq \lambda_j \rho_{j2} \) for each \( j = \Phi(j_1, j_2) \) with \( (j_1, j_2) \in P(\infty, \infty) \), where \( P(\infty, \infty) \) is introduced at the beginning of this supplementary material. Therefore, by Assumption 3.2,

\[
\sum_{j=m_n+1}^{\infty} \sigma_j^2 \leq \sum_{j=m_n+1}^{\infty} \lambda_j \rho_{j2} + \sum_{j=m_n+1}^{\infty} \lambda_j \rho_{j2} \leq \sum_{j=m_n}^{\infty} \lambda_j + \sum_{j=m_n}^{\infty} \rho_{j2} \\
\leq \sum_{j=m_n}^{\infty} \lambda_j^{a_1} + \sum_{j=m_n}^{\infty} \rho_{j2}^{a_2} \leq m_n^{-(\alpha_1-1)} + m_n^{-(\alpha_2-1)} \leq m_n^{-(\bar{\alpha}-1)/2},
\] (S23)

where \( \bar{\alpha} = \min\{\alpha_1/2, \alpha_2\} \). Recalling \( m_n = \ell_n^{\frac{2\alpha}{\alpha} + \delta_0} \), and noting \( m_n \ll k_n^{\bar{\alpha}} \) for \( k_n = \ell_n^{\frac{(4\alpha_1+2\alpha_2)}{\bar{\alpha}}} \), we have

\[
\sup_{j=m_n+1, \ldots, p} \tilde{\sigma}_j^2 \leq \sum_{j=m_n+1}^{\infty} \tilde{\sigma}_j^2 \leq c_2 m_n^{\bar{\alpha}/2} + c_1 m_n a_n \ll \ell_n^{\bar{\alpha}} + k_n^{\bar{\alpha}} \ll \ell_n^{\bar{\alpha}},
\]

which implies \( \sup_{j=m_n+1, \ldots, p} \tilde{\sigma}_j \ll \ell_n^{\bar{\alpha}/2} \ll \sigma_{(\ell_n)} \). That is, the \( \ell_n \) largest elements among \( \{\tilde{\sigma}_j\}_{j=1}^{p} \) cannot appear in \( \{\tilde{\sigma}_j\}_{j=m_n+1}^{p} \) when \( n \) is sufficiently large, which proves the claim.

In addition, for each \( i, j = 1, \ldots, m_n \), \( \sigma_i^2 \geq k_n^{-\alpha} = \sqrt{a_n} \) and \( \sigma_j^2 \geq k_n^{-\bar{\alpha}} = \sqrt{a_n} \) according to Lemma S2.8, and
hence Lemma S5.3 further ensures that
\[ |\tilde{R}_{ij}| = \frac{|\tilde{\Sigma}_{ij}|}{\sqrt{\tilde{\Sigma}_{ii}\tilde{\Sigma}_{jj}}} \leq \frac{|\Sigma_{ij}| + a_n}{\sqrt{\Sigma_{ii} - a_n\Sigma_{jj} - a_n}} \leq \frac{|\Sigma_{ij}| + a_n}{\sqrt{\Sigma_{ii}\Sigma_{jj}}} + \sqrt{a_n} = |R_{ij}| + \sqrt{a_n}. \]

Based on the above observation, we conclude that
\[ \sup_{(\tilde{\phi}, \tilde{\psi}) \in \mathcal{F}_p} \|\tilde{R}(\ell_n)\|_{\mathcal{F}}^2 \leq 2 \sup_{R \in \mathcal{R}(\ell_n, m_n)} \|R^o\|_{\mathcal{F}}^2 + 2\ell_n^2 k_n^{-2\delta} \leq \ell_n^{2-\delta} + \ell_n \times \ell_n^{2-\delta}, \]
where the second inequality holds for \( k_n = \ell_n^{\frac{(4\alpha_2+2)\alpha}{(1+\delta)\alpha}} \), by noting \( \tilde{\alpha} > 1 \).

Let \( \sigma(j) \) be the decreasingly ordered elements in the sequence \( \sigma_1, \ldots, \sigma_p \), which represents the standard deviations of \( V_1 \), as induced by the eigenfunction pair \((\phi, \psi)\). The following lemma shows that the sequence \( \{\sigma_j\} \) follows a decay pattern.

**Lemma S2.8.** Given any sufficiently small number \( \delta \in (0, 1/2) \), such that Assumption 3.2 hold, there is an absolute constant \( c' \geq 0 \) such that
\[ \sigma(j) \geq c'j^{-\frac{\delta}{2}} \quad \text{for } j = 1, \ldots, k_n \] (S24)
with \( k_n = \ell_n^{\frac{(4\alpha_2+2)\alpha}{(1+\delta)\alpha}} \). Moreover, there is an absolute constant \( c > 0 \) such that
\[ \sigma(j) \leq cj^{-\frac{1}{2}} \] (S25)
for \( j = 1, \ldots, p \).

**Proof.** From Lemma S5.1, the elements of \( \Sigma \) satisfy the following relation:
\[ \Sigma_{(r_1, r_2)(j_1, j_2)} \begin{cases} \asymp \lambda_{r_1} \rho_{r_2} & j_1 = r_1, j_2 = r_2, \\ \lesssim \sqrt{\lambda_{r_1} \lambda_{r_2}} \rho_{r_1} \rho_{r_2} & \text{o.w.} \end{cases} \]
for each pair \((r_1, r_2), (j_1, j_2) \in \mathcal{P} \), with \( \mathcal{P} \) defined at the beginning of this supplementary material. Assuming for simplicity \( p_1, p_2 \geq k_n^{1/2} \) (other cases can be similarly discussed), we define
\[ k_{n, 1} = k_{n, 2} = \lfloor k_n^{1/2} \rfloor. \] (S26)
Consider \( \mathcal{P}(k_{n, 1}, k_{n, 2}) = \{(j_1, j_2) : j_1 = 1, \ldots, k_{n, 1}, j_2 = 1, \ldots, k_{n, 2}\} \). Given the form of \( \Sigma_{(r_1, r_2)(j_1, j_2)} \), rearrange the elements in \( V_i \) such that the \( j \)th coordinate in \( V_i \) is \( \xi_{i j_1} \xi_{j_2} \) with \( j = p_1(j_2 - 1) + j_1 \). The order of the variance at coordinate \( j \) of \( V_i \) is then
\[ \sigma_j^2 = \Sigma_{jj} \asymp \lambda_{j_1} \rho_{j_2}. \]
Recalling $\lambda_j \sim j_1^{-\alpha_1}$ and $\rho_{j_2} \sim j_2^{-\alpha_2}$ by Assumption 3.2, it follows that for each $(j_1, j_2) \in P(kn, 1, kn, 2)$, there exist absolute constants $c_0, c'_0 \in (0, 1)$, such that

$$\sigma_j \geq \sqrt{c'_0}\lambda_{j_1}\rho_{j_2} \geq \sqrt{c_0j_1^{-\alpha_1}j_2^{-\alpha_2}} \geq \sqrt{c_0(p_1(j_2 - 1) + j_1)^{-\alpha}} = \sqrt{c_0j^{-\frac{\theta}{2}}}. \quad (S27)$$

Therefore, $\sigma_{(j)} \geq \sigma_1 \geq \sqrt{c_0} \cdot 1^{-\alpha/2}$ is evident. That is, (S24) holds for $j = 1$. For $j \in \{2, \ldots, kn\}$, we consider two scenarios:

- If $\sigma_{(j)} \geq \sigma_j$, then $\sigma_{(j)} \geq \sigma_1 \geq \sqrt{c_0}j^{-\frac{\theta}{2}}$ is clear;

- If $\sigma_{(j)} < \sigma_j$, then $\sigma_1$ is the among the $j - 1$ largest elements. This implies at least one of $\sigma_1, \ldots, \sigma_{j-1}$ is not in $\{\sigma_{(1)}, \ldots, \sigma_{(j-1)}\}$. Thus, $\sigma_{(j)} \geq \sigma_l \geq \sqrt{c_0}l^{-\frac{\theta}{2}} \geq \sqrt{c_0}j^{-\frac{\theta}{2}}$ for some $1 \leq l \leq j - 1$.

Consequently, for $j = 1, \ldots, kn$, we always have

$$\sigma_{(j)} \geq \sqrt{c_0}j^{-\frac{\theta}{2}}.$$

This proves the first statement of the proposition.

For the second statement, given Assumption 3.2(i), with $S := \sum_{j=1}^{\infty} \sigma_j^2 < \infty$, Lemma S5.5 ensures $\sigma_{(j)}^2 \leq Sj^{-1}$ for every $j$. Hence, by setting $c = \sqrt{S}$, we deduce

$$\sigma_{(j)} \leq cj^{-1/2},$$

for $j = 1, \ldots, p.$ \hfill \Box

**Lemma S2.9.** Suppose the conditions of Theorem 3.5 hold, and let $q = \max\{\frac{2}{(1-\tau)^2}, \log(n), 3\}$. Then, there is an absolute constant $c > 0$ not depending on $p$, $n$ or $({\tilde{\phi}}, {\tilde{\psi}})$, such that, for any $j \in \{1, \ldots, p\}$, we have

$$\sup_{({\tilde{\phi}}, {\tilde{\psi}}) \in \mathcal{F}_p} \left\| \frac{1}{\sigma_j} \tilde{S}_{n,j} \right\|_{L^q} \leq cq. \quad (S28)$$

**Proof.** For each $({\tilde{\phi}}, {\tilde{\psi}}) \in \mathcal{F}_p$, given that $q > 2$, Lemma S2.10 yields

$$\left\| \frac{1}{\sigma_j} \tilde{S}_{n,j} \right\|_{L^q} \leq c_1q \cdot \max\left\{ \left\| \frac{1}{\sigma_j} \tilde{S}_{n,j} \right\|_{L^2}, n^{-1/2+1/q} \left\| \frac{1}{\sigma_j} (\tilde{V}_{1,j} - \tilde{\nu}_j) \right\|_{L^q} \right\}, \quad (S29)$$

for an absolute constant $c_1 > 0$. It is evident that,

$$\left\| \frac{1}{\sigma_j} \tilde{S}_{n,j} \right\|_{L^2}^2 = \text{var}\left( \frac{1}{\sigma_j}, \tilde{S}_{n,j} \right) = 1.$$

Furthermore, considering the standard basis vector $e_j$ in $\ell^2$ with $\|e_j\|_2 = 1$, we have $\|\tilde{V}_{1,j} - \tilde{\nu}_j\|_{\psi_1} = \|(\tilde{e}_j^\infty, e_j)\|_{\psi_1} \leq$
$K \tilde{\sigma}_j$ by Proposition S2.6. This implies that \( \frac{1}{\sigma_j} (\tilde{V}_{1,j} - \tilde{\nu}_j) \) is a sub-exponential random variable, then we have

\[
\left\| \frac{1}{\sigma_j} (\tilde{V}_{1,j} - \tilde{\nu}_j) \right\|_{L^q} \leq K' q,
\]

for an absolute constant $K' > 0$. Applying this result to bound (S29) leads to

\[
\left\| \frac{1}{\sigma_j} \tilde{S}_{n,j} \right\|_{L^q} \leq c q \cdot \max \left\{ 1, n^{-1/2+1/q} q \right\},
\]

where $c > 0$ is an absolute constant. Finally, the proposed choice of $q$ ensures that the right-hand side of the last equation is of the order $q$, which completes the proof. \( \square \)

The following inequalities are due to Johnson et al. (1985).

**Lemma S2.10** (Rosenthal’s inequality with best constants). Fix $r \geq 1$ and put $\log(r) \coloneqq \max\{\log(r), 1\}$. Let $\xi_1, \ldots, \xi_m$ be independent random variables satisfying $E[|\xi_j|^r] < \infty$ for all $1 \leq j \leq m$. Then, there is an absolute constant $c > 0$ such that the following two statements are true.

(i). If $\xi_1, \ldots, \xi_m$ are non-negative random variables, then

\[
\left\| \sum_{j=1}^{m} \xi_j \right\|_{L^r} \leq c \cdot \frac{r}{\log(r)} \cdot \max \left\{ \left\| \sum_{j=1}^{m} \xi_j \right\|_{L^r}, \left( \sum_{j=1}^{m} \left\| \xi_j \right\|_{L^r} \right)^{1/r} \right\}. \tag{S30}
\]

(ii). If $r > 2$, and the random variables $\xi_1, \ldots, \xi_m$ all have mean 0, then

\[
\left\| \sum_{j=1}^{m} \xi_j \right\|_{L^r} \leq c \cdot \frac{r}{\log(r)} \cdot \max \left\{ \left\| \sum_{j=1}^{m} \xi_j \right\|_{L^2}, \left( \sum_{j=1}^{m} \left\| \xi_j \right\|_{L^2} \right)^{1/r} \right\}. \tag{S31}
\]

### S3 Proof of Theorem 3.6

In all the proofs, for a given pair of bases $(\tilde{\phi}, \tilde{\psi}) \in \mathcal{F}_p$, we may notationally omit the dependence of $(\tilde{\phi}, \tilde{\psi})$ of related quantities for brevity.

**Proof.** We first have the following decomposition:

\[
\sup_{(\tilde{\phi}, \tilde{\psi}) \in \mathcal{F}_p} d_K \left( \mathcal{L}(\tilde{M}_p(\tilde{\phi}, \tilde{\psi})), \mathcal{L}(\tilde{M}_p' (\tilde{\phi}, \tilde{\psi})|D) \right) \leq \sup_{(\tilde{\phi}, \tilde{\psi}) \in \mathcal{F}_p} \tilde{I}_n' + \sup_{(\tilde{\phi}, \tilde{\psi}) \in \mathcal{F}_p} \tilde{II}_n'(D) + \sup_{(\tilde{\phi}, \tilde{\psi}) \in \mathcal{F}_p} \tilde{III}_n'(D),
\]

where

\[
\tilde{I}_n' = d_K \left( \mathcal{L}(\tilde{M}_p(\tilde{\phi}, \tilde{\psi})), \mathcal{L}(\tilde{M}_{kn}(\tilde{\phi}, \tilde{\psi})) \right),
\]

\[
\tilde{II}_n'(D) = d_K \left( \mathcal{L}(\tilde{M}_{kn}(\tilde{\phi}, \tilde{\psi})), \mathcal{L}(\tilde{M}_{kn}' (\tilde{\phi}, \tilde{\psi})\big| D) \right),
\]

\[
\tilde{III}_n'(D) = d_K \left( \mathcal{L}(\tilde{M}_{kn}' (\tilde{\phi}, \tilde{\psi})\big| D), \mathcal{L}(\tilde{M}_p' (\tilde{\phi}, \tilde{\psi})\big| D) \right).
\]
where $k_n = \ell_n^{\frac{\log(1+\frac{\log n}{n})}{\log 2}}$.

In Section S2.1, we established that the term $\sup_{(\tilde{\phi}, \tilde{\psi})\in \mathcal{F}_p} \tilde{I}_n$ is bounded by the rate $n^{-1/2+\delta}$. Therefore, it suffices to demonstrate that both $\sup_{(\tilde{\phi}, \tilde{\psi})\in \mathcal{F}_p} \tilde{I}_n'(D)$ and $\sup_{(\tilde{\phi}, \tilde{\psi})\in \mathcal{F}_p} \tilde{I}_n'(D)$ are similarly bounded with probability at least $1-cn^{-1}$ for some constant $c > 0$.

Define, for any $t \in \mathbb{R}$,

$$A'(t) = \left\{ \max_{j \in \hat{J}_n} \tilde{S}_{n,j}/\tilde{\sigma}_j^2 \leq t \right\} \quad \text{and} \quad B'(t) = \left\{ \max_{j \in \hat{J}_n} \tilde{S}_{n,j}/\tilde{\sigma}_j^2 > t \right\}.$$  

For $\sup_{(\tilde{\phi}, \tilde{\psi})\in \mathcal{F}_p} \tilde{I}_n'(D)$, given any number $t_{1,n}' \leq t_{2,n}'$, we observe

$$\sup_{(\tilde{\phi}, \tilde{\psi})\in \mathcal{F}_p} \tilde{I}_n'(D) \leq \sup_{(\tilde{\phi}, \tilde{\psi})\in \mathcal{F}_p} \mathbb{P}(A'(t_{2,n}')) + \sup_{(\tilde{\phi}, \tilde{\psi})\in \mathcal{F}_p} \mathbb{P}(B'(t_{1,n}')),$$

where

$$t_{1,n}' = c_1' \cdot k_n^{\frac{1}{\min\{1, \frac{\log n}{n}\}}} \cdot \log^{3/2}(n) \quad \text{(S32)}$$

$$t_{2,n}' = c_2' \cdot \ell_n^{\frac{\delta}{2}} \cdot (1-\delta) \cdot \sqrt{\log(d_n)} \quad \text{(S33)}$$

for some absolute constants $c_1, c_2 > 0$, and $d_n$ is defined in the proof of Lemma S2.3. Note that for such choices, $t_{1,n}' \leq t_{2,n}'$ holds for all sufficiently large $n$ with $k_n = \ell_n^{\frac{\log(1+\frac{\log n}{n})}{\log 2}}$.

For $\mathbb{P}(A'(t_{2,n}'))$,

$$\sup_{(\tilde{\phi}, \tilde{\psi})\in \mathcal{F}_p} \mathbb{P}(A'(t_{2,n}')) \leq \sup_{(\tilde{\phi}, \tilde{\psi})\in \mathcal{F}_p} \mathbb{P}\left( \max_{j \in \hat{J}_n} \tilde{S}_{n,j}/\tilde{\sigma}_j^2 \leq t_{2,n}' \right) + \sup_{(\tilde{\phi}, \tilde{\psi})\in \mathcal{F}_p} \tilde{I}_n'(D).$$

The first term is at the rate $n^{-1/2+\delta}$ by following arguments similar to those used to establish (S17) in Lemma S2.3, and the second term will be addressed later. In addition, defining $\bar{W}^* = \max_{j \in \hat{J}_n} \tilde{S}_{n,j}/\tilde{\sigma}_j^2$, for $q \geq \log(n)$ and $b_n = t_{1,n}'/e$, for a suitably chosen $c_1'$, we claim

$$\sup_{(\tilde{\phi}, \tilde{\psi})\in \mathcal{F}_p} \left( \mathbb{E}[|\bar{W}^*|^q]D \right)^{\frac{1}{q}} \leq b_n \quad \text{(S34)}$$

with probability tending to one, and further

$$\sup_{(\tilde{\phi}, \tilde{\psi})\in \mathcal{F}_p} \mathbb{P}(B'(t_{1,n}')) = \sup_{(\tilde{\phi}, \tilde{\psi})\in \mathcal{F}_p} \mathbb{P}(\bar{W}^* > t_{1,n}') = \sup_{(\tilde{\phi}, \tilde{\psi})\in \mathcal{F}_p} \mathbb{P}(\bar{W}^* > eb_nD) \leq e^{-q} \leq n^{-1}.$$
Thus, if $\sup_{(\tilde{\phi},\tilde{\psi}) \in F_p} \tilde{\Pi}_n'(D) \leq n^{-1/2+\delta}$ and (S34) holds with probability approaching one, then

$$\sup_{(\tilde{\phi},\tilde{\psi}) \in F_p} \tilde{\Pi}_n'(D) \leq n^{-1/2+\delta} + n^{-1/2+\delta} + n^{-1} \approx n^{-1/2+\delta}$$

also holds with probability approaching one. The inequality (S34) can be established by using arguments similar to those in Lemma C.1 in Lopes et al. (2020) with the help of Lemma S3.1.

It remains to establish $\sup_{(\tilde{\phi},\tilde{\psi}) \in F_p} \tilde{\Pi}_n'(D) \leq n^{-1/2+\delta}$. Applying the triangle inequality, we obtain

$$\sup_{(\tilde{\phi},\tilde{\psi}) \in F_p} \tilde{\Pi}_n'(D) \leq \sup_{(\tilde{\phi},\tilde{\psi}) \in F_p} d_K\left(\mathcal{L}(\tilde{M}_{k_n}),\mathcal{L}(\tilde{M}_{n}|D)\right) + \sup_{(\tilde{\phi},\tilde{\psi}) \in F_p} d_K\left(\mathcal{L}(\tilde{M}_{n}|D),\mathcal{L}(\tilde{M}_{n}|D)\right), \quad (S35)$$

where $\tilde{M}_{k_n} = \max_{j \in \mathcal{J}_{k_n}} \tilde{S}_{n,j}/\tilde{\sigma}_j$. The first term in (S35) is addressed in Lemma S3.2. To tackle the second term in (S35), we further break it down into

$$\sup_{(\tilde{\phi},\tilde{\psi}) \in F_p} d_K\left(\mathcal{L}(\tilde{M}_{n},|D)\right) \leq \sup_{(\tilde{\phi},\tilde{\psi}) \in F_p} \sup_k \mathbb{P}\left(|\tilde{M}_n^* - t| \leq r_n|D\right) + \sup_{(\tilde{\phi},\tilde{\psi}) \in F_p} \mathbb{P}\left(|\tilde{M}_n^* - \tilde{M}_n^*| > r_n|D\right),$$

where $r_n = cn^{-1/2} \log^{5/2}(n)$. The second term in the above inequality without the sup quantifier was previously addressed in Lopes et al. (2020) using Lemma D.8 from that paper. Here, Equation (S38) from our Lemma S3.1 and Lemma S3.3 ensure that the corresponding uniform version remains valid. For the first term, the key is to analyze

$$\inf_{(\tilde{\phi},\tilde{\psi}) \in F_p} \min_{j \in \mathcal{J}_{k_n}} \tilde{\sigma}_j^{1-\gamma},$$

which is handled in Lemma S3.4. With this, the anti-concentration inequality for a Gaussian vector (Nazarov’s inequality) can be applied to bound $\sup_{(\tilde{\phi},\tilde{\psi}) \in F_p} \sup_{t \in \mathbb{R}} \mathbb{P}\left(|\tilde{M}_n^* - t| \leq r_n|D\right)$. Incorporating these bounds, with probability approaching one, we arrive at

$$\sup_{(\tilde{\phi},\tilde{\psi}) \in F_p} d_K\left(\mathcal{L}(\tilde{M}_{n},|D)\right) \leq n^{-1/2+\delta}. \quad (S36)$$

Thus, by combining Lemma S3.2 and (S36), we conclude that

$$\sup_{(\tilde{\phi},\tilde{\psi}) \in F_p} \tilde{\Pi}_n'(D) \leq n^{-1/4+\delta}$$

also holds with probability approaching one, thereby completing the proof.

\[
\text{Lemma S3.1. Assume the conditions of Theorem 3.6 hold for a sufficiently small constant } \delta \in (0,1/2). \text{ Let}
\]
\[ q = \max \left\{ \frac{2(4\alpha_0+2)}{(1-\tau)}, \log(n), 4 \right\} \text{ and } s = q(1 - \tau), \] and consider the random variables \( \hat{s} \) and \( \hat{t} \) defined by

\[
\hat{s} = \left( \sum_{j \in \mathcal{J}_n} \hat{\delta}_j^s \right)^{1/2} \quad \text{and} \quad \hat{t} = \left( \sum_{j \in \mathcal{J}_n} \hat{\delta}_j^t \right)^{1/2}
\]

where \( k_n = \ell_n^{(4\alpha_0+2)/s} \). Then, there is a constant \( c > 0 \), not depending on \( p, n \) or \((\hat{\phi}, \hat{\psi})\), such that

\[
P \left( \sup_{(\hat{\phi}, \hat{\psi}) \in \mathcal{F}_p} \hat{s} \geq \frac{cq}{(q(1 - \tau)/(4\alpha_0 + 2) - 1)^{1/2}} k_n^{-1/(4\alpha_0+2)+1/s} \right) \leq e^{-q}, \quad (S37)
\]

and

\[
P \left( \sup_{(\hat{\phi}, \hat{\psi}) \in \mathcal{F}_p} \hat{t} \geq c\sqrt{q} \right) \leq e^{-q}. \quad (S38)
\]

**Proof.** By Markov's inequality,

\[
P \left( \sup_{(\hat{\phi}, \hat{\psi}) \in \mathcal{F}_p} \hat{s} \geq e \left\| \sup_{(\hat{\phi}, \hat{\psi}) \in \mathcal{F}_p} \hat{s} \right\|_q \right) \leq e^{-q}.
\]

It suffices to bound \( \left\| \sup_{(\hat{\phi}, \hat{\psi}) \in \mathcal{F}_p} \hat{s} \right\|_q \) (and similarly for \( \sup_{(\hat{\phi}, \hat{\psi}) \in \mathcal{F}_p} \hat{t} \)). Through calculation, we obtain

\[
\left\| \sup_{(\hat{\phi}, \hat{\psi}) \in \mathcal{F}_p} \hat{s} \right\|_q = \left( \mathbb{E} \left( \sup_{(\hat{\phi}, \hat{\psi}) \in \mathcal{F}_p} \left( \sum_{j \in \mathcal{J}_n} \hat{\delta}_j^s \right)^{1/s} q^{1/q} \right) \right)^{1/q} = \left( \mathbb{E} \left( \sup_{(\hat{\phi}, \hat{\psi}) \in \mathcal{F}_p, j \in \mathcal{J}_n} \hat{\delta}_j^s \right)^{1/s} q^{1/q} \right)^{1/q}.
\]

Note \((j)\) may differ for various \((\hat{\phi}, \hat{\psi})\). We aim to find a bound for

\[
I_j := \left( \mathbb{E} \left( \sup_{(\hat{\phi}, \hat{\psi}) \in \mathcal{F}_p} \hat{\delta}_j^2 \right)^{q/2} \right)^{1/q}, \quad (S40)
\]

which would be further used in \((S39)\) and leads to

\[
\left\| \sup_{(\hat{\phi}, \hat{\psi}) \in \mathcal{F}_p} \hat{s} \right\|_q \leq \left( \sum_{j = k_n+1}^p I_j^s \right)^{1/s}. \quad (S41)
\]
Following Proposition S5.4, $\tilde{V}_i^\infty = \tilde{U}^*V_i^\infty$ for a unitary operator $\tilde{U}$ on $\ell^2 \times \ell^2$. For each $j$,

$$
\hat{\sigma}^2_j = \frac{1}{n} \sum_{i=1}^{n} (\tilde{V}_{ij} - \bar{V}_j)^2 = \frac{1}{n} \sum_{i=1}^{n} \tilde{V}_{ij}^2 - \bar{V}_j^2
$$

$$
\leq \frac{1}{n} \sum_{i=1}^{n} \tilde{V}_{ij}^2 = \frac{1}{n} \sum_{i=1}^{n} (\tilde{u}_j, V_i^\infty)^2 = \hat{\sigma}^2_j + \frac{1}{n} \sum_{i=1}^{n} ((\tilde{u}_j, V_i^\infty)^2 - \hat{\sigma}^2_j),
$$

(S42)

where $\hat{\sigma}^2_j = \mathbb{E}[(\tilde{u}_j, V_i^\infty)^2]$ and $\tilde{u}_j \in \ell^2$ is a deterministic sequence of unit length. For the first term on the right-hand side of (S42), Proposition S2.7 implies that

$$
\hat{\sigma}^2_j \leq cj^{-1},
$$

(S43)

where $c > 0$ is an absolute constant. We now focus on the second term in (S42).

Setting $Q = q/2$ and noting that by definition, $Q > 2$, we introduce i.i.d. Rademacher random variables $w_i, i = 1, \ldots, n$, independent of $V_i^\infty, i = 1, \ldots, n$. Consider the envelope function $F(x) = \|x\|_2$ with $x \in \ell^2$ for $\sup_{(\tilde{\phi}, \tilde{\psi}) \in \mathcal{F}_p} (\tilde{u}_j, x)$, and let $M = \max_{1 \leq i \leq n} F(V_i^\infty)$. Utilizing these notations and symmetrization inequalities, we derive

$$
\mathbb{E} \left( \sup_{(\tilde{\phi}, \tilde{\psi}) \in \mathcal{F}_p} n^{-1} \sum_{i=1}^{n} (\tilde{u}_j, V_i^\infty)^2 - \hat{\sigma}^2_j \right)^Q
$$

$$
\leq \mathbb{E} \left( 2 \sup_{(\tilde{\phi}, \tilde{\psi}) \in \mathcal{F}_p} n^{-1} \sum_{i=1}^{n} w_i (\tilde{u}_j, V_i^\infty)^2 \right)^Q
$$

$$
= \frac{2^Q}{n^Q} \mathbb{E} \left( \sup_{(\tilde{\phi}, \tilde{\psi}) \in \mathcal{F}_p} n^{-1} \sum_{i=1}^{n} w_i (\tilde{u}_j, V_i^\infty)^2 \right)^Q
$$

$$
\leq \frac{2^{Q+2}}{n^{Q}} \mathbb{E} \left( \sup_{(\tilde{\phi}, \tilde{\psi}) \in \mathcal{F}_p} n \sum_{i=1}^{n} w_i (\tilde{u}_j, V_i^\infty) \right)^M^Q \left( M \right)^{Q/2}
$$

$$
\leq \frac{2^{Q+2}}{n^{Q}} \mathbb{E} \left( \sup_{(\tilde{\phi}, \tilde{\psi}) \in \mathcal{F}_p} n \sum_{i=1}^{n} w_i (\tilde{u}_j, V_i^\infty) \right)^{2Q} \left( M \right)^{Q/2}
$$

$$
= \frac{2^{Q+2}}{n^{Q}} \left( M \right)^{Q/2} \left( \mathbb{E} \left( \sup_{(\tilde{\phi}, \tilde{\psi}) \in \mathcal{F}_p} n \sum_{i=1}^{n} w_i (\tilde{u}_j, V_i^\infty) \right)^{2Q} \right)^{1/2}
$$

$$
\leq \frac{2^{Q+2}}{n^{Q}} \left( M \right)^{Q/2} \left( \mathbb{E} \left( \sup_{(\tilde{\phi}, \tilde{\psi}) \in \mathcal{F}_p} n \sum_{i=1}^{n} w_i (\tilde{u}_j, V_i^\infty) \right)^{2Q} \right)^{1/2}
$$

$$
\leq C \frac{2^{Q+2}}{n^{Q}} \left( \frac{2KQ}{\log(2Q)} \right)^{Q} \left( \mathbb{E} \left( \sup_{(\tilde{\phi}, \tilde{\psi}) \in \mathcal{F}_p} n \sum_{i=1}^{n} w_i (\tilde{u}_j, V_i^\infty) \right)^{2Q} \right)^{1/2}
$$

(S44)

for some $K > 0$, where the first line is due to Lemma 2.3.1 in van der Vaart and Wellner (1996), the third line is from Theorem 4.12 in Ledoux and Talagrand (1991), the fourth line is owing to the Cauchy-Schwartz inequality, and the last line is a result of Theorem 6.20 in Ledoux and Talagrand (1991). Combining this
bound and (S43) with (S40), we find that there exists an absolute constant $c_1 > 0$,

$$I_2^2 \leq C j^{-1} + \frac{c_1}{n \log Q} \|M\|_{2Q} \left( \mathbb{E} \left( \sup_{(\tilde{\phi}, \tilde{\psi}) \in \mathcal{F}_p} \left| \sum_{i=1}^{n} w_i (\tilde{u}_j, V_i^\infty) \right| \right) + \|M\|_{2Q} \right).$$

(S45)

Given that every $V_i^\infty, i = 1, \ldots, n$, is norm-subexponential (Equation (9)) by Assumption 3.1, we have $\|M\|_{2Q} \leq c_2 n^{1/(2Q)} 2Q$.

For the other term in (S45), first note that

$$\mathbb{E} \left( \sup_{(\tilde{\phi}, \tilde{\psi}) \in \mathcal{F}_p} \left| \sum_{i=1}^{n} w_i (\tilde{u}_j, V_i^\infty) \right| \right)
\leq 2 \mathbb{E} \left( \sup_{(\tilde{\phi}, \tilde{\psi}) \in \mathcal{F}_p} \left| \sum_{i=1}^{n} (\tilde{u}_j, V_i^\infty) - \mathbb{E}(\tilde{u}_j, V_i^\infty) \right| \right) + c_2 \sqrt{n},$$

(S46)

where the second line is from Lemma 2.3.6 in van der Vaart and Wellner (1996), and the third line holds due to the Khintchine’s inequality (Exercise 2.6.6, Vershynin, 2018) and the fact that $\mathbb{E}(\tilde{u}_j, V_i^\infty) \leq \mathbb{E} |V_i^\infty|_2 \leq C_1 < \infty$ for an absolute constant $C_1 > 0$ and any $(\tilde{\phi}, \tilde{\psi})$. To further bound the right-hand side in (S46), we consider the following decomposition:

$$\mathbb{E} \left( \sup_{(\tilde{\phi}, \tilde{\psi}) \in \mathcal{F}_p} \left| \sum_{i=1}^{n} (\tilde{u}_j, V_i^\infty) - \mathbb{E}(\tilde{u}_j, V_i^\infty) \right| \right)
\leq \mathbb{E} \left( \sup_{b \in B_m} \sum_{i=1}^{n} (b, V_i^m) - \mathbb{E}(b, V_i^m) \right) + \mathbb{E} \left( \sup_{(\tilde{\phi}, \tilde{\psi}) \in \mathcal{F}_p} \left| \sum_{i=1}^{n} \tilde{u}_{il} \left( \sum_{i=l+1}^{n} (V_i^\infty - \mathbb{E} V_i^\infty) \right) \right| \right)
\leq \mathbb{E} \left( \sup_{b \in B_m} \sum_{i=1}^{n} (b, V_i^m) - \mathbb{E}(b, V_i^m) \right) + \mathbb{E} \left[ \sup_{(\tilde{\phi}, \tilde{\psi}) \in \mathcal{F}_p} \left( \sum_{i=l+1}^{n} \tilde{u}_{il}^2 \left( \sum_{i=l+1}^{n} (V_i^\infty - \mathbb{E} V_i^\infty) \right) \right)^{1/2} \right]
\leq \mathbb{E} \left( \sup_{b \in B_m} \sum_{i=1}^{n} (b, V_i^m) - \mathbb{E}(b, V_i^m) \right) + \mathbb{E} \left[ \sum_{i=l+1}^{n} \mathbb{E} (V_i^\infty - \mathbb{E} V_i^\infty)^2 \right]^{1/2}
\leq \mathbb{E} \left( \sup_{b \in B_m} \sum_{i=1}^{n} (b, V_i^m) - \mathbb{E}(b, V_i^m) \right) + \sqrt{n},$$

where, $m = \log n$, $V_i^m = \Pi_m V_i^\infty \in \mathbb{R}^m$, and $B_m = \{ b \in \mathbb{R}^m : |b|_2 \leq 1 \}$ is the unit $\ell_2$-ball in $\mathbb{R}^m$. In the above, the second and third inequalities are due to the Cauchy-Schwartz inequality and the Jensen’s inequality respectively, and $r_n = \sum_{i=l+1}^{n} \sigma_i^2 = o(1)$ since the sequence $\{ \sigma_i^2 : j \geq 1 \}$ is summable (see also the proof of Lemma S2.8). Moreover, by the Dudley’s inequality for sub-exponential processes (see, for example, Problem
12.10 in Tropp (2022)), there is an absolute constant $C_1 > 0$, such that
\[
\mathbb{E} \left( \sup_{b \in B_m} \left| \sum_{i=1}^{n} (\langle b, V_i^m \rangle - \mathbb{E}\langle b, V_i^m \rangle) \right| \right) \leq C_1 n^{1/2}.m.
\]
This leads to
\[
\mathbb{E} \left( \sup_{(\tilde{\phi}, \tilde{\psi}) \in \mathcal{F}_p} \left| \sum_{i=1}^{n} (\langle \tilde{u}_j, V_i^\infty \rangle - \mathbb{E}\langle \tilde{u}_j, V_i^\infty \rangle) \right| \right) \leq C n^{1/2} \log n
\]
for an absolute constant $C > 0$.

By Combining the above result with (S45) and (S46), under Assumption 3.3, we deduce that for sufficiently large $n$,
\[
I_j \leq c_j^{-1/2} + c_2 q (n^{-\frac{1}{4} + \frac{1}{s}})^{1/2} \log n \leq c_0 q p^{-1/(4\alpha_0 + 2)} \leq c_0 q j^{-1/(4\alpha_0 + 2)},
\]
holds for $j = k_n + 1, \ldots, p$. Applying this result to (S41), we can then infer that
\[
\left\| \sup_{(\tilde{\phi}, \tilde{\psi}) \in \mathcal{F}_p} \tilde{\phi} \right\|_q \leq c_0 \frac{q}{(q(1-\tau)/(4\alpha_0 + 2) - 1)^{1/s}} n^{-1/(4\alpha_0 + 2) + 1/s},
\]
where we use the fact $s/(4\alpha_0 + 2) = q(1-\tau)/(4\alpha_0 + 2) > 1$ according to the definition of $q$. Similarly, we can also establish (S38) for $\sup_{(\tilde{\phi}, \tilde{\psi}) \in \mathcal{F}_p} \tilde{\phi}$.

\[
\begin{align*}
\text{Lemma S3.2.} \quad &\text{Under the conditions of Theorem 3.6, for the first term in (S35), it can be established that} \\
&\sup_{(\tilde{\phi}, \tilde{\psi}) \in \mathcal{F}_p} d_K \left( \mathcal{L}(\tilde{M}_{k_n}), \mathcal{L}(\tilde{M}_{k_n}^* \mathcal{D}) \right) \leq cn^{-1/4+\delta},
\end{align*}
\]
with probability at least $1 - cn^{-1}$.

\[
\text{Proof.} \quad \text{Considering } \tilde{M}_{k_n}, \text{ it is the coordinate-wise maximum of a Gaussian vector drawn from } N_{k_n}(0, \mathfrak{S}), \text{where } \mathfrak{S} = \tilde{D}_{k_n}^{-1/2} \tilde{\Pi}_{k_n}^{-1} \tilde{\Sigma}_{\infty}(\tilde{\Pi}_{k_n}^{-1})^t \tilde{D}_{k_n}^{-1/2}. \text{Here, } \tilde{D}_{k_n} \text{ is the diagonal matrix } \tilde{D}_{k_n} = \text{diag}(\tilde{\sigma}_{(1)}, \ldots, \tilde{\sigma}_{(k_n)}) \text{ with decreasingly elements } \tilde{\sigma}_{(1)} \geq \cdots \geq \tilde{\sigma}_{(k_n)} \text{ of } \{\tilde{\sigma}_j, 1 \leq j \leq k_n\}, \text{ and } \tilde{\Pi}_{k_n}^\infty = \tilde{\Pi}_{k_n} \tilde{\Pi}_{k_n}^\infty \text{ is a projection operator from } \ell^2 \text{ to } \mathbb{R}^{k_n}. \text{Meanwhile, } \tilde{M}_{k_n}^* \text{ is the coordinate-wise maximum of a Gaussian vector drawn from } N_{k_n}(0, \breve{\mathfrak{S}}), \text{where } \breve{\mathfrak{S}} = \tilde{D}_{k_n}^{-1/2} \tilde{\Pi}_{k_n} \tilde{W}_{n}(\tilde{\Pi}_{k_n}^{-1})^t \tilde{D}_{k_n}^{-1/2} \text{ and } \tilde{W}_{n} \text{ is defined by:}
\end{align*}
\]
\[
W_{n} = \frac{1}{n} \sum_{i=1}^{n} (\tilde{e}_i^\infty - \tilde{\bar{e}}^\infty) \otimes (\tilde{e}_i^\infty - \tilde{\bar{e}}^\infty),
\]
with $\tilde{\bar{e}}^\infty = n^{-1} \sum_{i=1}^{n} \tilde{e}_i^\infty$.

To bound the desired term, we will apply Lemma S5.6. To this end, observe that, for any $(\tilde{\phi}, \tilde{\psi}) \in \mathcal{F}_p$ and each $j = 1, \ldots, k_n$,
\[
\mathfrak{S}_{jj} = \tilde{\sigma}_{(j)}^2/\tilde{\sigma}_{(j)}^2 = \tilde{\sigma}_{(j)}^{2(1-\tau)} \geq c_0 j^{-\alpha(1-\tau)} \geq c_0 k_n^{-\alpha(1-\tau)},
\]
(S50)
where the second inequality is derived from Proposition S2.7. Following Lemma S5.6,

\[
\sup_{(\tilde{\phi}, \tilde{\psi}) \in F_p} d_k \left( \mathcal{L}(\tilde{M}_{kn}), \mathcal{L}(\tilde{M}^*_{kn}) |D \right) \leq \frac{C_1}{\min_{1 \leq j \leq k_n} |\mathcal{G}_{j1} - \mathcal{G}_{jj}|} \Delta^{1/2} \log(k_n),
\]  
(S51)

with \( C_1 \) being an absolute constant and \( \Delta = \max_{1 \leq j \leq k_n} |\mathcal{G}_{j1} - \mathcal{G}_{jj}| \). Additionally, leveraging Proposition S2.7 again, we arrive at

\[
\Delta \leq \| \mathcal{G} - \tilde{\mathcal{G}} \|_2 \leq c_0 k_n \|	ilde{\Pi}^\infty_k \tilde{\Sigma}^\infty_k (\tilde{\Pi}^\infty_k)^\dagger - \tilde{\Pi}^\infty_k \tilde{W}_n (\tilde{\Pi}^\infty_k)^\dagger \|_2.
\]

Combining the above inequality and (S50) with (S51), we obtain

\[
\sup_{(\tilde{\phi}, \tilde{\psi}) \in F_p} d_k \left( \mathcal{L}(\tilde{M}_{kn}), \mathcal{L}(\tilde{M}^*_{kn}) |D \right) \leq C_2 k_n^{\alpha(1-\tau/2)} \log(k_n) \sup_{(\tilde{\phi}, \tilde{\psi}) \in F_p} \| \tilde{\Pi}^\infty_k \tilde{\Sigma}^\infty_k (\tilde{\Pi}^\infty_k)^\dagger - \tilde{\Pi}^\infty_k \tilde{W}_n (\tilde{\Pi}^\infty_k)^\dagger \|_2^{1/2}. 
\]  
(S52)

To bound the spectral norm term, for any pair \((\tilde{\phi}, \tilde{\psi}) \in F_p\), consider the unit sphere \( \mathcal{S}_{kn} \) in \( \mathbb{R}^{k_n} \). The spectral norm term can be expressed as:

\[
\| \tilde{\Pi}^\infty_k \tilde{\Sigma}^\infty_k (\tilde{\Pi}^\infty_k)^\dagger - \tilde{\Pi}^\infty_k \tilde{W}_n (\tilde{\Pi}^\infty_k)^\dagger \|_2 = \max_{v \in \mathcal{S}_{kn}} |\tilde{v}^\top \left( \tilde{\Pi}^\infty_k \tilde{\Sigma}^\infty_k (\tilde{\Pi}^\infty_k)^\dagger - \tilde{\Pi}^\infty_k \tilde{W}_n (\tilde{\Pi}^\infty_k)^\dagger \right) v|.
\]

For a given \( \tilde{v} \) in \( \mathcal{S}_{kn} \),

\[
\tilde{v}^\top \tilde{\Pi}^\infty_k \tilde{W}_n (\tilde{\Pi}^\infty_k)^\dagger \tilde{v} = \frac{1}{n} \sum_{i=1}^{n} (\tilde{\epsilon}^\infty_i - \tilde{\epsilon}^\infty, (\tilde{\Pi}^\infty_k)^\dagger \tilde{v})^2 = \frac{1}{n} \sum_{i=1}^{n} (\epsilon^\infty_i - \epsilon^\infty, \tilde{U}^\dagger (\tilde{\Pi}^\infty_k)^\dagger \tilde{v})^2 = \frac{1}{n} \sum_{i=1}^{n} (\epsilon^\infty_i - \epsilon^\infty, v)^2,
\]

where the second identity is due to Proposition S5.4 for a unitary operator \( \tilde{U} \) on \( \ell^2 \times \ell^2 \), and \( v = \tilde{U}^\dagger (\tilde{\Pi}^\infty_k)^\dagger \tilde{v} \in \ell^2 \) with \( \|v\|_2 = 1 \). Putting \( \xi_{v,i} = (\epsilon^\infty_i, v) \), which is sub-exponential due to Assumption 3.1, we then consider the following representation:

\[
\tilde{v}^\top \left( \tilde{\Pi}^\infty_k \tilde{W}_n (\tilde{\Pi}^\infty_k)^\dagger - \tilde{\Pi}^\infty_k \tilde{\Sigma}^\infty_k (\tilde{\Pi}^\infty_k)^\dagger \right) \tilde{v} = \left( \frac{1}{n} \sum_{i=1}^{n} \xi^2_{v,i} \right) - \left( \frac{1}{n} \sum_{i=1}^{n} \xi_{v,i} \right)^2,
\]

with \( \mathbb{E} \xi_{v,i} = 0 \). Thus, we get

\[
\sup_{(\tilde{\phi}, \tilde{\psi}) \in F_p} \| \tilde{\Pi}^\infty_k \tilde{\Sigma}^\infty_k (\tilde{\Pi}^\infty_k)^\dagger - \tilde{\Pi}^\infty_k \tilde{W}_n (\tilde{\Pi}^\infty_k)^\dagger \|_2 = \sup_{v \in \mathcal{S}_{\infty}} \left| \frac{1}{n} \sum_{i=1}^{n} \xi^2_{v,i} \right| - \left( \frac{1}{n} \sum_{i=1}^{n} \xi_{v,i} \right)^2 
\]

\[
\leq \sup_{v \in \mathcal{S}_{\infty}} \left| \frac{1}{n} \sum_{i=1}^{n} \left( \xi^2_{v,i} - \mathbb{E} \xi^2_{v,i} \right) \right| + \left( \sup_{v \in \mathcal{S}_{\infty}} \left| \frac{1}{n} \sum_{i=1}^{n} \xi_{v,i} \right| \right)^2, 
\]  
(S53)

where \( \mathcal{S}_{\infty} \) denotes the unit sphere in \( \ell^2 \).

Utilizing symmetrization arguments similar to those used in analyzing the second term in (S42), specifically the derivation for (S44) and the upper bound for the second term in (S45), we can assert that, there exists
an absolute constant $c_1, c_2 > 0$ such that

$$
P\left( \sup_{v \in S^\infty} \left| \frac{1}{n} \sum_{i=1}^{n} (\xi_{v,i}^2 - E\xi_{v,i}^2) \right| > \frac{c_1 \log^2(n)}{\sqrt{n}} \right) \leq \frac{c_1}{n},$$

and

$$
P\left( \left( \sup_{v \in S^\infty} \left| \frac{1}{n} \sum_{i=1}^{n} \xi_{v,i} \right| \right)^2 > \left( \frac{c_2 \log(n)^2}{n} \right) \right) = P\left( \sup_{v \in S^\infty} \left| \frac{1}{n} \sum_{i=1}^{n} \xi_{v,i} \right| > \frac{c_2 \log(n)}{\sqrt{n}} \right) \leq \frac{c_2}{n}.
$$

Consequently, combining the above two probability inequalities with (S53), we conclude that

$$
\sup_{(\tilde{\phi}, \tilde{\psi}) \in F_p} \| \tilde{\Pi}_{k_n}^\infty \tilde{S}_n^\infty (\tilde{\Pi}_{k_n}^\infty)^\dagger - \tilde{\Pi}_{k_n}^\infty \tilde{W}_n (\tilde{\Pi}_{k_n}^\infty)^\dagger \|_2 \leq \frac{c \log^2(n)}{\sqrt{n}},
$$

holds with probability at least $1 - cn^{-1}$. Combining this result with (S52), and observing that the term $k_n^{\tilde{a}(1-\tilde{a}/2)} \log(k_n) \ll n^{\delta}$ for a constant $c > 0$, with $\delta$ redefined to $\delta/c$, we finally obtain that

$$
\sup_{(\tilde{\phi}, \tilde{\psi}) \in F_p} d_K (L(\tilde{M}_{k_n}), L(\tilde{M}_{k_n}^*|D)) \leq cn^{-\frac{1}{4} + \delta},
$$

holds with probability at least $1 - cn^{-1}$.

---

**Lemma S3.3.** Suppose the conditions of Theorem 3.6 hold, and fix any $j \in \tilde{J}_{k_n}$, where $\tilde{J}_{k_n}$ denotes the set of indices corresponding to the $k_n$ largest values in $\{\tilde{\sigma}_1, \ldots, \tilde{\sigma}_p\}$ and $k_n = \log_{\min(1, \gamma)} n / n^{\alpha/2}$. Then, for any number $D \geq 1$, there are positive absolute constants $c$ and $c_1(D)$, not depending on $p$, $n$ or $(\tilde{\phi}, \tilde{\psi})$, such that the event

$$
\sup_{(\tilde{\phi}, \tilde{\psi}) \in F_p} \left| \frac{\tilde{\sigma}_j^2}{\sigma_j} - 1 \right| \leq c_1(D) \frac{\log^2(n) k_n^{\alpha/2}}{\sqrt{n}}, \quad (\tilde{\phi}, \tilde{\psi}) \in F_p (S54)
$$

holds with probability at least $1 - cn^{-D}$.

**Proof.** It is equivalent to showing that, for any $j \in \tilde{J}_{k_n}$, with probability at least $1 - cn^{-D}$, we have

$$
|\tilde{\sigma}_j^2 - \sigma_j^2| \leq c_1(D) \frac{\log^2(n) k_n^{\alpha/2}}{\sqrt{n}}, \quad \forall (\tilde{\phi}, \tilde{\psi}) \in F_p.
$$

Noting that, by Proposition S2.7, $\sigma_j^2 \geq c_0 k_n^{-\alpha/2}$ for any $j \in \tilde{J}_{k_n}$, it suffices to show

$$
P \left( \exists (\tilde{\phi}, \tilde{\psi}) \in F_p, \ |\tilde{\sigma}_j^2 - \sigma_j^2| \geq c_1(D) \frac{\log^2(n) k_n^{\alpha/2}}{\sqrt{n}} \right)
$$

$$
\leq P \left( \exists (\tilde{\phi}, \tilde{\psi}) \in F_p, \ |\tilde{\sigma}_j^2 - \sigma_j^2| \geq c_2(D) \frac{\log^2(n) k_n^{\alpha/2-\alpha/2}}{\sqrt{n}} \right)
$$

$$
= P \left( \sup_{(\tilde{\phi}, \tilde{\psi}) \in F_p} \ |\tilde{\sigma}_j^2 - \sigma_j^2| \geq c_2(D) \frac{\log^2(n)}{\sqrt{n}} \right) \leq cn^{-D}.
$$
As in (S42), for any \( j \in \tilde{J}_{kn} \),
\[
\hat{\sigma}_j^2 = \frac{1}{n} \sum_{i=1}^{n} (\tilde{V}_{ij} - \bar{\tilde{V}}_j)^2 = \frac{1}{n} \sum_{i=1}^{n} \tilde{V}_{ij}^2 - \bar{\tilde{V}}_j^2 = \hat{\sigma}_j^2 + \frac{1}{n} \sum_{i=1}^{n} ((\tilde{u}_j, V_i^\infty)^2 - \hat{\sigma}_j^2) - \left( \frac{1}{n} \sum_{i=1}^{n} (\tilde{u}_j, V_i^\infty)^2 \right),
\]
where \( \hat{\sigma}_j^2 = \mathbb{E}[(\tilde{u}_j, V_i^\infty)^2] \) and \( \tilde{u}_j \in \ell^2 \) is a deterministic sequence of unit length. Therefore,
\[
\sup_{\tilde{\sigma} \in F_p} \left| \hat{\sigma}_j^2 - \tilde{\sigma}_j^2 \right| \leq \sup_{\tilde{\sigma} \in F_p} \left| \frac{1}{n} \sum_{i=1}^{n} ((\tilde{u}_j, V_i^\infty)^2 - \tilde{\sigma}_j^2) \right| + \sup_{\tilde{\sigma} \in F_p} \left| \frac{1}{n} \sum_{i=1}^{n} (\tilde{u}_j, V_i^\infty)^2 \right|^2. \tag{S55}
\]
For the first term on the right-hand side of (S55), Markov’s inequality with \( q = \max\{D \log(n), 3\} \) leads to
\[
P \left( \sup_{\tilde{\sigma} \in F_p} \left| \frac{1}{n} \sum_{i=1}^{n} ((\tilde{u}_j, V_i^\infty)^2 - \tilde{\sigma}_j^2) \right| \geq e \| \sup_{\tilde{\sigma} \in F_p} \left| \frac{1}{n} \sum_{i=1}^{n} (\tilde{u}_j, V_i^\infty)^2 \right| \|_{L_q} \right) \leq e^{-q} \leq n^{-D}.
\]
We now turn to derive an upper bound for
\[
\left\| \sup_{\tilde{\sigma} \in F_p} \left| \frac{1}{n} \sum_{i=1}^{n} ((\tilde{u}_j, V_i^\infty)^2 - \tilde{\sigma}_j^2) \right| \right\|_{L_q} = \mathbb{E} \left( \sup_{\tilde{\sigma} \in F_p} \left| \frac{1}{n} \sum_{i=1}^{n} (\tilde{u}_j, V_i^\infty)^2 - \tilde{\sigma}_j^2 \right| \right)^{q/2}.
\]
This term has been similarly analyzed in the proof of Lemma S3.1. Specifically, by similar arguments for (S44) and the upper bound for the second term in (S45), it can be derived that
\[
\left\| \sup_{\tilde{\sigma} \in F_p} \left| \frac{1}{n} \sum_{i=1}^{n} ((\tilde{u}_j, V_i^\infty)^2 - \tilde{\sigma}_j^2) \right| \right\|_{L_q} \leq c_3(D) \frac{\log^2(n)}{\sqrt{n}},
\]
for some constant \( c_3(D) > 0 \) depending on \( D \). Hence,
\[
P \left( \sup_{\tilde{\sigma} \in F_p} \left| \frac{1}{n} \sum_{i=1}^{n} ((\tilde{u}_j, V_i^\infty)^2 - \tilde{\sigma}_j^2) \right| \geq c_3(D) \frac{\log^2(n)}{\sqrt{n}} \right) \leq n^{-D}. \tag{S56}
\]
A similar argument can be applied on the second term on the right-hand side of (S55), and gives that
\[
P \left( \sup_{\tilde{\sigma} \in F_p} \left| \frac{1}{n} \sum_{i=1}^{n} (\tilde{u}_j, V_i^\infty)^2 \right| > \frac{c_4(D) \log^2(n)}{n} \right) = \mathbb{P} \left( \sup_{\tilde{\sigma} \in F_p} \left| \frac{1}{n} \sum_{i=1}^{n} (\tilde{u}_j, V_i^\infty)^2 \right| > \frac{c_4(D) \log(n)}{\sqrt{n}} \right) \leq n^{-D}, \tag{S57}
\]
for some constant \( c_4(D) > 0 \) depending on \( D \). Combining (S56) and (S57) with (S55) completes the proof.

**Lemma S3.4.** Suppose the conditions of Theorem 3.6 hold and define the parameter \( D = \tilde{\alpha}(1 - \tau)/2 \). Then, for \( k_n = \ell_n^{(4nq + 2)/(2q - 1)} \), there is a constant \( c > 0 \), not depending on \( p \), \( n \) or \( (\tilde{\phi}, \tilde{\psi}) \), such that the event
\[
\inf_{(\tilde{\phi}, \tilde{\psi}) \in \mathcal{F}_p} \min_{j \in \tilde{J}_{kn}} \hat{\sigma}_j^{1 - \tau} \geq c k_n^{-D}
\]
holds with probability at least \( 1 - cn^{-1} \).
Lemma S3.3 ensures that for each \( j \in \tilde{J}_{kn} \), the event

\[
\sup_{(\tilde{\phi}, \tilde{\psi}) \in F_p} \left| \frac{\hat{\sigma}_j^2}{\sigma_j^2} - 1 \right| \leq \frac{c_1 \log^2(n) k_n^{3/2}}{\sqrt{n}} = \begin{cases} 
\max_{(j_1, j_2) \in P(\lfloor h_1/2 \rfloor, \lfloor h_1/2 \rfloor)} \nu_j \geq c_0 \max \{ \sigma_{\max} n^{-1/2} \log(n), a_n \} 
\end{cases}
\]

holds with probability at least \( 1 - c_1 n^{-2} \) for an absolute constant \( c_1 > 0 \). Combining this result with Proposition S2.7 implies that, for each \( j \in \tilde{J}_{kn} \), the event

\[
\hat{\sigma}_j^2 \geq \hat{\sigma}_j^2 \left( 1 - \frac{c_1 \log^2(n) k_n^{3/2}}{\sqrt{n}} \right) \geq c_2 k_n^{-a}, \quad \forall (\tilde{\phi}, \tilde{\psi}) \in F_p
\]

is of probability at least \( 1 - c_1 n^{-2} \), for an absolute constant \( c_2 > 0 \). Therefore, for each \( j \in \tilde{J}_{kn} \), \( \inf_{(\tilde{\phi}, \tilde{\psi}) \in F_p} \hat{\sigma}_j^{1-\tau} \geq c_2 k_n^{-D} \) holds with probability at least \( 1 - c_1 n^{-2} \). Finally, applying a union bound, and considering \( k_n \ll n \), we conclude that

\[
\inf_{(\tilde{\phi}, \tilde{\psi}) \in F_p} \min_{j \in \tilde{J}_{kn}} \hat{\sigma}_j^{1-\tau} = \min_{1, \ldots, k_n} \inf_{(\tilde{\phi}, \tilde{\psi}) \in F_p} \hat{\sigma}_j^{1-\tau} \geq c k_n^{-D},
\]

holds with probability at least \( 1 - cn^{-1} \).

**S4 Proof of Theorem 3.9**

Proof. Part 2 is handled in Lemma S4.1, while part 1 can be derived as in the proof of Theorem 3.3 in Lin et al. (2022) by replacing the condition event therein with

\[
E = \{ \sup_{(\tilde{\phi}, \tilde{\psi}) \in F_p} d_{k,i} (L(\tilde{\phi}, \tilde{\psi}), L(M^*(\tilde{\phi}, \tilde{\psi}))) \leq n^{-1/4+\delta} \},
\]

which has shown to hold with probability at least \( 1 - cn^{-1} \) according to Theorems 3.5 and 3.6.

Now we prove the uniform consistency of the proposed test over \( F_p \). Consider the power function

\[
P\left( \forall (\tilde{\phi}, \tilde{\psi}) \in F_p : \{ \tilde{T}_U > q_M(1 - g/2) \} \cup \{ \tilde{T}_L < q_L(g/2) \} \right) | A_0, \quad (*)
\]

where

\[
A_0 = \left\{ \max_{(i_1, i_2) \in P(\lfloor h_1/2 \rfloor, \lfloor h_1/2 \rfloor)} |\nu_j| \geq c_0 \max \{ \sigma_{\max} n^{-1/2} \log(n), a_n \} \right\}
\]

for a sufficiently large constant \( c_0 \). Note that \( A_0 \) can be divided into two parts.

- **Positive part**: Consider the case

\[
A_0^+ = \left\{ \max_{(i_1, i_2) \in P(\lfloor h_1/2 \rfloor, \lfloor h_1/2 \rfloor)} |\nu_j| \geq c_0 \max \{ \sigma_{\max} n^{-1/2} \log(n), a_n \} \right\}
\]

Let \( j_* \) be the index at which the maximum is achieved in \( A_0^+ \). Define

\[
A_1 = \left\{ \sup_{(\hat{\phi}, \hat{\psi}) \in F_p} |q_{\hat{M}^*}(1 - \varrho/2)| \leq c_0 \log^{1/2} n \right\},
\]

\[
A_2 = \left\{ \sup_{(\hat{\phi}, \hat{\psi}) \in F_p} \max_{1 \leq j \leq \rho} \frac{\hat{\sigma}_j^2}{\sigma_{\max}^2} < 2 \right\},
\]

\[
A_3 = \left\{ \max_{1 \leq j \leq \rho} \sup_{(\hat{\phi}, \hat{\psi}) \in F_p} |\bar{V}_j - \bar{v}_j| \leq c_2 n^{-1/2} \log(n) \right\},
\]

\[
A = A_1 \cap A_2 \cap A_3,
\]

for some constants \( c_1, c_2 > 0 \). Note that, according to Lemmas S4.1 and S4.2 and the discussion in the first paragraph, we have

\[
\mathbb{P}(A) = 1 - \mathbb{P}(A_1^c \cup A_2^c \cup A_3^c) \geq 1 - \mathbb{P}(A_1^c) - \mathbb{P}(A_2^c) - \mathbb{P}(A_3^c) \geq 1 - c'/n,
\]

for some absolute constant \( c' > 0 \). Moreover, under event \( A_2 \), it appears that, for each \( j = 1, \ldots, p \),

\[
\sup_{(\hat{\phi}, \hat{\psi}) \in F_p} \hat{\sigma}_j^2 \leq C_1 \sup_{(\hat{\phi}, \hat{\psi}) \in F_p} \hat{\sigma}_{\max}^2,
\]

(S58)

for some constant \( C_1 > 0 \). In the proof of Part (ii) in Proposition S2.7, we have shown that the index of the largest element among \( \{ \hat{\sigma}_j \}_{j=1}^p \) is in the indices of \( \{ \sigma_j \}_{j=1}^{m_1} \) for \( j = \Phi(j_1, j_2) \) and \( (j_1, j_2) \in \mathcal{P}(m_{n,1}, m_{n,2}) \), where \( \mathcal{P}(m_{n,1}, m_{n,2}) = \{(j_1, j_2) : j_1 = 1, \ldots, m_{n,1}, j_2 = 1, \ldots, m_{n,2}\} \), \( m_{n,1} = m_{n,2} = [m/n] \), \( m_n \) is defined in Assumption 3.2, and \( \Phi \) is defined at the beginning of this supplementary material. The Equation (S23) also indicates that the largest element among \( \{ \sigma_j \}_{j=1}^p \) is in \( \{ \sigma_j \}_{j=1}^{m_1} \), as \( \sup_{m_1 \leq j \leq \rho} \hat{\sigma}_j^2 \leq \sum_{j=m_1}^{\infty} \sigma_j^2 \ll 1 \leq \max_{1 \leq j \leq \rho} \sigma_j^2 \). Therefore,

\[
\hat{\sigma}_{\max}^2 = \max_{1 \leq j \leq \rho} \hat{\sigma}_j^2 = \max_{1 \leq j \leq m_n} \hat{\sigma}_j^2 \leq C_2 \max_{1 \leq j \leq m_n} (\sigma_j^2 + a_n) = C_2 \max_{1 \leq j \leq \rho} (\sigma_j^2 + a_n) \leq \sigma_{\max}^2,
\]

for some constant \( C_2 > 0 \) and the inequality is from Lemma S5.3. Combining the above result with (S58), we find that, under event \( A_2 \)

\[
\sup_{(\hat{\phi}, \hat{\psi}) \in F_p} \hat{\sigma}_j^2 \leq C_3 \sigma_{\max}^2,
\]

(S59)

for some constant \( C_3 > 0 \).

Then, the power function (*) can be bounded from below by

\[
(*) = \mathbb{P} \left( \bigcap_{(\hat{\phi}, \hat{\psi}) \in F_p} \left( \left\{ \hat{T}_U > q_{\hat{M}^*}(1 - \varrho/2) \right\} \cup \left\{ \hat{T}_L < q_{L}(\varrho/2) \right\} \right) \right) \left| A_0^+ \right|
\]

24
\[ \geq P \left( \bigcap_{(\tilde{\phi}, \tilde{\psi}) \in \mathcal{F}_p} \left\{ \tilde{T}_U > q_M, (1 - \theta/2) \right\} \right| A_0^+, A \right) P(A \mid A_0^+) \]

\[ = P \left( \bigcap_{(\tilde{\phi}, \tilde{\psi}) \in \mathcal{F}_p} \left\{ \max_{1 \leq j \leq p} \frac{\tilde{V}_j}{\sqrt{n}} > q_M, (1 - \theta/2) \right\} \right| A_0^+, A \right) P(A) \]

\[ \geq P \left( \inf_{(\tilde{\phi}, \tilde{\psi}) \in \mathcal{F}_p} \max_{1 \leq j \leq p} \frac{\tilde{V}_j}{\sqrt{n}} > \sup_{(\tilde{\phi}, \tilde{\psi}) \in \mathcal{F}_p} q_M, (1 - \theta/2) \right| A_0^+, A \right) P(A) \]

\[ \geq P \left( \inf_{(\tilde{\phi}, \tilde{\psi}) \in \mathcal{F}_p} \max_{1 \leq j \leq p} \frac{\tilde{V}_j}{\sqrt{n}} > c_1 \log^{1/2} n \right| A_0^+, A \right) P(A) \quad \text{(due to } A_1) \]

\[ \geq P \left( \max_{(\tilde{\phi}, \tilde{\psi}) \in \mathcal{F}_p} \inf_{1 \leq j \leq p} \frac{\tilde{V}_j - \tilde{\nu}_j + \bar{\nu}_j}{\sqrt{n}} > c_1 \log^{1/2} n \right| A_0^+, A \right) P(A) \quad \text{(due to } A_3) \]

\[ \geq P \left( \inf_{(\tilde{\phi}, \tilde{\psi}) \in \mathcal{F}_p} \nu_j - c_3 a_n - c_2 \log(n)/\sqrt{n} > c_1 \log^{1/2} n \right| A_0^+, A \right) P(A) \quad \text{(due to Lemma S5.3)} \]

\[ \geq P \left( \forall (\tilde{\phi}, \tilde{\psi}) \in \mathcal{F}_p : \nu_j - c_3 a_n - c_2 \log(n)/\sqrt{n} > c_1 (\hat{\sigma}_j / \sqrt{n}) \log^{1/2} n \right| A_0^+, A \right) P(A) \quad \text{(due to } A_2 \text{ and } (S59)) \]

\[ \geq P \left( \nu_j \geq c_n \right| A_0^+, A \right) \left( 1 - \frac{\theta}{n} \right) \to 1, \]

where \( c_n = c_0 \cdot \max\{ \sigma_{\max}^{-1/2} \log(n), a_n \} \), for a sufficiently large \( c_0 \).

- The case of \( \nu_1 \leq -c \max\{ \sigma_{\max}^{-1/2} \log(n), a_n \} \): This is similar to the above one, and corresponds to the other part of the critical region, i.e., \( \{ \tilde{T}_L < q_L, (\theta/2) \} \).

\[ \square \]

**Lemma S4.1.** Suppose Assumptions 3.1-3.3 hold. Then, for some constant \( c > 0 \) not depending on \( p, n \) or \((\tilde{\phi}, \tilde{\psi})\), one has

\[ P \left( \sup_{(\tilde{\phi}, \tilde{\psi}) \in \mathcal{F}_p} \frac{\max_{1 \leq j \leq p} \hat{\sigma}_j^2}{\sigma_{\max}^2} < 2 \right) \geq 1 - cn^{-1}, \quad (S60) \]

where \( \hat{\sigma}_{\max} = \max\{ \hat{\sigma}_j : 1 \leq j \leq p \} \).

**Proof.** First, we note that

\[ P \left( \sup_{(\tilde{\phi}, \tilde{\psi}) \in \mathcal{F}_p} \frac{\max_{1 \leq j \leq p} \hat{\sigma}_j^2}{\sigma_{\max}^2} \geq 2 \right) \leq P \left( \forall (\tilde{\phi}, \tilde{\psi}) \in \mathcal{F}_p : \max_{1 \leq j \leq p} \hat{\sigma}_j^2 \geq 2\hat{\sigma}_{\max} \right). \]
Define

\[ A^\circ = \left\{ \forall (\tilde{\phi}, \tilde{\psi}) \in \mathcal{F}_p : \max_{j \in \overset{\circ}{J}_{kn}} \hat{\sigma}^2_j \geq 2\sigma^2_{\text{max}} \right\}, \]

\[ B^\circ = \left\{ \forall (\tilde{\phi}, \tilde{\psi}) \in \mathcal{F}_p : \max_{j \not\in \overset{\circ}{J}_{kn}} \hat{\sigma}^2_j \geq 2\sigma^2_{\text{max}} \right\}, \]

where \( \overset{\circ}{J}_{kn} \) is the complement (possibly the empty set) of \( \tilde{J}_{kn} \) in \( \{1, \ldots, p\} \), and \( k_n = \ell_n^{\frac{(4m_0 + 2)\alpha}{2\bar{\alpha}}} \). With \( t^\circ = 2 \max_{1 \leq j \leq p} \inf_{(\tilde{\phi}, \tilde{\psi}) \in \mathcal{F}_p} \hat{\sigma}^2_j \), to prove (S60), it is sufficient to show

\[ \mathbb{P}(A^\circ) \lesssim n^{-1}, \quad (S61) \]

and when \( \overset{\circ}{J}_{kn} \neq \emptyset \) that

\[ \mathbb{P}(B^\circ) \lesssim n^{-1}. \quad (S62) \]

Regarding (S61), Lemma S3.3 indicates that for each \( j \in \overset{\circ}{J}_{kn} \), with probability at least \( 1 - cn^{-3} \),

\[ \sup_{(\tilde{\phi}, \tilde{\psi}) \in \mathcal{F}_p} \left( \frac{\hat{\sigma}^2_j}{\sigma^2_{\text{max}}^2} - 1 \right) \leq c \log^2(n) k_n^{\bar{\alpha}/2} / \sqrt{n}, \]

which implies

\[ \hat{\sigma}^2_j \leq \sigma^2_j \left( 1 + c \log^2(n) k_n^{\bar{\alpha}/2} / \sqrt{n} \right) \leq 2\sigma^2_{\text{max}} \]

for sufficiently large \( n \) and for all \( (\tilde{\phi}, \tilde{\psi}) \in \mathcal{F}_p \). Therefore,

\[ \mathbb{P}(A^\circ) \leq \sum_{j \in \overset{\circ}{J}_{kn}} \mathbb{P}\left( \forall (\tilde{\phi}, \tilde{\psi}) \in \mathcal{F}_p : \hat{\sigma}^2_j \geq 2\sigma^2_{\text{max}} \right) \leq cn^{-2} \leq n^{-1}. \quad (S63) \]

To prove (S62), consider the random variable

\[ W = \max_{j \in \overset{\circ}{J}_{kn}} \sup_{(\tilde{\phi}, \tilde{\psi}) \in \mathcal{F}_p} \hat{\sigma}^2_j. \]

By a similar argument to the one in the proof of Lemma S3.1 for establishing the upper bound on (S39), it can be shown that \( \|W\|_{L_q} \ll 1 \) for \( q = \max\{16, \log(n)\} \). Concurrently, by the similar argument of Part (i) in Proposition S2.7, we can infer that \( t^\circ \gtrsim 1 \). Therefore, applying Markov’s inequality yields

\[ \mathbb{P}(B^\circ) \leq \mathbb{P}\left( W \geq \sup_{(\tilde{\phi}, \tilde{\psi}) \in \mathcal{F}_p} 2\sigma^2_{\text{max}} \right) \leq \mathbb{P}(W \geq e\|W\|_{L_q}) \leq e^{-q} \leq n^{-1}, \]

as desired. \[ \square \]

**Lemma S4.2.** Under Assumptions 3.2 and 3.3, for any \( j = 1, \ldots, p \), there is an absolute constant \( c > 0 \),
such that the event
\[ \sup_{(\tilde{\phi}, \tilde{\psi}) \in \mathcal{F}_n} |\tilde{V}_j - \tilde{\nu}_j| \leq c\sqrt{n} \log n, \]
holds with probability at least \( 1 - cn^{-1} \).

**Proof.** Recall \( \tilde{V}_j = n^{-1} \sum_{i=1}^{n} \tilde{V}_{ij} \) and \( \tilde{\nu}_j = \mathbb{E}\tilde{V}_{ij} \). Given Proposition S5.4, we can express
\[ |\tilde{V}_j - \tilde{\nu}_j| = \left| n^{-1} \sum_{i=1}^{n} (\tilde{u}_j, V_{ij}) - \mathbb{E}(\tilde{u}_j, V_{ij}) \right|, \]
where \( \tilde{u}_j \in \ell^2 \) is a deterministic sequence of unit length. The desired result then follows by Markov’s inequality in conjunction with the symmetrization techniques similar to those used in the proof of Lemma S3.1. \( \square \)

### S5 Auxiliary Results

For the infinite sequence \( V_i^\infty = (\xi_{ij}, \zeta_{ij}^j, j_1, j_2 \geq 1) \) with \( \xi_{ij} = (X_i, \phi_j)_1 \) and \( \zeta_{ij}^j = (Y_i, \psi_j)_2 \), we say the element \( \xi_{ij}, \zeta_{ij}^j \) is at the position \( (j_1, j_2) \). Recall that \( \kappa_{j_2} = \mathbb{E}(Z, \psi_{j_2})^2 \).

**Lemma S5.1.** Suppose that Assumption 3.2 holds. For \( (r_1, r_2), (j_1, j_2) \in \mathbb{N}^2 \), the covariance between the positions \( (r_1, r_2) \) and \( (j_1, j_2) \) of \( V_i^\infty (\phi, \psi) \) is given by
\[
\Sigma_{(r_1, r_2), (j_1, j_2)}^\infty \begin{cases} \lambda_{r_1} \rho_{r_2} & j_1 = r_1, j_2 = r_2, \\ \sqrt{\lambda_{r_1} \lambda_{j_1} \rho_{r_2} \rho_{j_2}} & \text{o.w.} \end{cases}
\]

**Remark S1.** Recall that \( \Sigma \) is the covariance matrix of \( V_i^\infty (\phi, \psi) \) as defined in Section 2. By definition, for each \( r, j \in \{1, \ldots, p\} \), the element in \( \Sigma \) at position \( (r, j) \) is \( \Sigma_{rj} = \Sigma_{(r_1, r_2), (j_1, j_2)}^\infty \) for \( (r_1, r_2), (j_1, j_2) \in \mathcal{P} \) with \( \Phi(r_1, r_2) = r \) and \( \Phi(j_1, j_2) = j \), where \( \mathcal{P} \) and \( \Phi \) are introduced at the beginning of this supplementary material.

**Proof.** By definition, for \( (r_1, r_2), (j_1, j_2) \in \mathbb{N}^2 \), we have
\[
\Sigma_{(r_1, r_2), (j_1, j_2)}^\infty = \mathbb{E}[(V_i^\infty_{(r_1, r_2)} - \nu_{(r_1, r_2)}^\infty)(V_i^\infty_{(j_1, j_2)} - \nu_{(j_1, j_2)}^\infty)] + \mathbb{E}[V_i^\infty_{(r_1, r_2)} V_i^\infty_{(j_1, j_2)}] - \nu_{(r_1, r_2)}^\infty \nu_{(j_1, j_2)}^\infty.
\]

Below we calculate \( \nu_{(r_1, r_2)}^\infty \nu_{(j_1, j_2)}^\infty \) and \( \mathbb{E}[V_i^\infty_{(r_1, r_2)} V_i^\infty_{(j_1, j_2)}] \) separately.

First, for \( \nu^\infty \), we have
\[
\nu_{(j_1, j_2)}^\infty = \mathbb{E}[(X_1, \phi_{j_1})(Y_1, \psi_{j_2})] = \mathbb{E}[(X_1, \phi_{j_1})(\beta(X_1) + Z_1, \psi_{j_2})] \\
= \mathbb{E}[(X_1, \phi_{j_1})(\beta(X_1), \psi_{j_2})] + \mathbb{E}[(X_1, \phi_{j_1})(Z_1, \psi_{j_2})] \\
= \sum_{k=1}^{\infty} b_{k j_2} \mathbb{E}[(X_1, \phi_{j_1})(X_1, \phi_{k})] = b_{j_1 j_2} \lambda_{j_1}.
\]
Therefore,
\[
\nu_{(r_1, r_2)}^{(j_1, j_2)} = b_{r_1 r_2} b_{j_1 j_2} \lambda_{r_1} \lambda_{j_1}, \quad (r_1, r_2), (j_1, j_2) \in \mathbb{N}^2.
\] (S64)

Next, we turn to \(\mathbb{E}[V_{1(r_1, r_2)}^{\infty} V_{1(j_1, j_2)}^{\infty}]\). For any \((r_1, r_2), (j_1, j_2) \in \mathbb{N}^2\),
\[
V_{1(r_1, r_2)}^{\infty} V_{1(j_1, j_2)}^{\infty} = \langle X_1, \phi_{r_1} \rangle \langle Y_1, \psi_{r_2} \rangle \langle X_1, \phi_{j_1} \rangle \langle Y_1, \psi_{j_2} \rangle
\]
\[
= \langle X_1, \phi_{r_1} \rangle (\beta(X_1) + Z_1, \psi_{r_2}) \langle X_1, \phi_{j_1} \rangle (\beta(X_1) + Z_1, \psi_{j_2})
\]
\[
= \langle X_1, \phi_{r_1} \rangle (\beta(X_1) + (Z_1, \psi_{r_2})) \langle X_1, \phi_{j_1} \rangle (\beta(X_1) + (Z_1, \psi_{j_2})).
\]

Given the independence of \(X_1\) and \(Z_1\), we further deduce that
\[
\mathbb{E}[V_{1(r_1, r_2)}^{\infty} V_{1(j_1, j_2)}^{\infty}] = \mathbb{E}[\langle X_1, \phi_{r_1} \rangle \langle X_1, \phi_{j_1} \rangle (\beta(X_1), \psi_{r_2}) (\beta(X_1), \psi_{j_2})]
\]
\[
+ \mathbb{E}[\langle X_1, \phi_{r_1} \rangle \langle X_1, \phi_{j_1} \rangle \mathbb{E}[\langle Z_1, \psi_{r_2} \rangle (Z_1, \psi_{j_2})]] (S65)
\]

- When \(j_1 = r_1\) and \(j_2 = r_2\), (S65) reduces to
\[
\mathbb{E}[V_{1(r_1, r_2)}^{\infty}] = \mathbb{E}[\langle X_1, \phi_{r_1} \rangle^2 \langle \beta(X_1), \psi_{r_2} \rangle^2 + \mathbb{E}[\langle X_1, \phi_{r_1} \rangle^2 \mathbb{E}[\langle Z_1, \psi_{r_2} \rangle^2]], \quad (S66)
\]

and (S64) reduces to
\[
(\nu_{(r_1, r_2)}^{\infty})^2 = b_{r_1 r_2}^2 X_{r_1}^2.
\]

Based on the above two expressions, we derive a lower bound for \(\Sigma_{(r_1, r_2)}^{\infty}\) for any \(r_1, r_2 \in \mathbb{N}\) as
\[
\Sigma_{(r_1, r_2)}^{\infty} = \mathbb{E}[\langle V_{1(r_1, r_2)}^{\infty} \rangle^2] - (\nu_{(r_1, r_2)}^{\infty})^2 \geq \mathbb{E}[\langle X_1, \phi_{r_1} \rangle^2 \mathbb{E}[\langle Z_1, \psi_{r_2} \rangle^2] - (\nu_{(r_1, r_2)}^{\infty})^2
\]
\[
= \lambda_{r_1} \rho_{r_2} - b_{r_1 r_2}^2 \lambda_{r_1}^2 \geq \lambda_{r_1} \rho_{r_2} - \rho_{r_2}^2 \lambda_{r_1}^2 \geq \lambda_{r_1} \rho_{r_2},
\]

where the the second inequality is due to Lemma S5.2 and (12) in Assumption 3.2(i). On the other hand, we have the following upper bound,
\[
\Sigma_{(r_1, r_2)}^{\infty} = \mathbb{E}[\langle V_{1(r_1, r_2)}^{\infty} \rangle^2] - (\nu_{(r_1, r_2)}^{\infty})^2
\]
\[
= \left| \sum_{k=1}^{b_{r_1 r_2}} b_{k r_2}^2 \mathbb{E}[\langle X_1, \phi_{r_1} \rangle^2 \langle X_1, \phi_k \rangle^2] + \sum_{k \neq 1} b_{k r_2} b_{k r_2} \mathbb{E}[\langle X_1, \phi_{r_1} \rangle^2 \langle X_1, \phi_k \rangle \langle X_1, \phi_k \rangle] \right|
\]
\[
+ \mathbb{E}[\langle X_1, \phi_{r_1} \rangle^2 \mathbb{E}[\langle Z_1, \psi_{r_2} \rangle^2] - b_{r_1 r_2}^2 \lambda_{r_1}^2]
\]
\[
\leq b_{r_1 r_2}^2 \mathbb{E}[\langle X_1, \phi_{r_1} \rangle^4] - \lambda_{r_1}^2 + \sum_{k \neq r_1} b_{k r_2}^2 \mathbb{E}[\langle X_1, \phi_{r_1} \rangle^2 \langle X_1, \phi_k \rangle^2]
\]
\[
+ \sum_{k \neq 1} b_{k r_2} \|b_{k r_2}\| \mathbb{E}[\langle X_1, \phi_{r_1} \rangle^2 \langle X_1, \phi_k \rangle \langle X_1, \phi_k \rangle] + \mathbb{E}[\langle X_1, \phi_{r_1} \rangle^2 \mathbb{E}[\langle Z_1, \psi_{r_2} \rangle^2]]
\]
\[ \leq b_{r_1, r_2}^2 \left( \mathbb{E}[(X_1, \phi_{r_1})^4] - \lambda_{r_1}^2 \right) + \sum_{k \neq \nu_1} b_{r_1, r_2}^2 \left( \mathbb{E}[(X_1, \phi_{r_1})^4] \right)^{1/2} \left( \mathbb{E}[(X_1, \phi_{\nu_1})^4] \right)^{1/2} \]
\[ + \sum_{k \neq \nu_1} |b_{r_1, r_2}^2 b_{k, r_2}^2 \mathbb{E}[(X_1, \phi_{r_1})^4] \mathbb{E}[(X_1, \phi_{\nu_1})^4] \mathbb{E}[(X_1, \phi_{k})^4] \mathbb{E}[(X_1, \phi_{\nu_1})^4] \mathbb{E}[(X_1, \phi_{k})^4] \]}
\[ \mathbb{E}[(Z_1, \psi_{r_2})^2] \]
\[ \leq b_{r_1, r_2}^2 \lambda_{r_1}^2 + \sum_{k \neq \nu_1} b_{r_1, r_2}^2 \lambda_{r_1} \lambda_{k} + \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} |b_{l, r_2}^2 |b_{k, r_2}^2 |\lambda_{r_1} \sqrt{\lambda_{k}} + \lambda_{r_1} \rho_{r_2} \]
\[ \leq \lambda_{r_1}^2 \rho_{r_2}^2 + \lambda_{r_1} \rho_{r_2}^2 + \lambda_{r_1} \rho_{r_2}^2 + \lambda_{r_1} \rho_{r_2} \]
\[ \leq \lambda_{r_1} \rho_{r_2} \]

where the third inequality holds when \( \mathbb{E}[(\psi_{r_1}^X)^4] \leq C \) for some \( C > 1 \), and the fourth inequality holds since we assume \( \lambda_{k} \leq k^{-\alpha_1} \) for some constant \( \alpha_1 > 2 \).

- When \( j_1 = r_1 \) and \( j_2 \neq r_2 \), \( S65 \) reduces to

\[ \mathbb{E}[V_{r_1, r_2}^{\infty} V_{j_1, j_2}^{\infty}] = \mathbb{E}[(X_1, \phi_{r_1})^2 (\beta(X_1), \psi_{r_2}) (\beta(X_1), \psi_{j_2})] + \mathbb{E}[(X_1, \phi_{r_1})^2] \mathbb{E}[(Z_1, \psi_{r_2}) (Z_1, \psi_{j_2})], \] 
\( S67 \)

and \( S64 \) reduces to

\[ \nu_{r_1, r_2}^\infty \nu_{j_1, j_2}^\infty = b_{r_1, r_2} b_{r_1, j_2} \lambda_{r_1}^2. \]

For the second term on the right-hand side of \( S67 \), observing

\[ 0 = \mathbb{E}(Y, \psi_{r_2}) (Y, \psi_{j_2}) = \sum_{l=1}^{\infty} b_{l, r_2} b_{l, j_2} \lambda_{l} + \mathbb{E}[(Z, \psi_{r_2}) (Z, \psi_{j_2})], \]

we find that, for any \( r_2, j_2 \in \mathbb{N} \), it holds that

\[ |\mathbb{E}[(Z, \psi_{r_2}) (Z, \psi_{j_2})]| \leq \sum_{l=1}^{\infty} |b_{l, r_2}^2 |b_{l, j_2}^2 |\lambda_{l} \leq \rho_{r_2} \rho_{j_2}. \]

Combining this with \( S67 \), then we can deduce an upper bound for \( |\Sigma_{r_1, r_2}^{\infty}(j_1, j_2)| \) as follows:

\[ |\Sigma_{r_1, r_2}^{\infty}(j_1, j_2)| \leq \left| \mathbb{E}[(X_1, \phi_{r_1})^2 (\beta(X_1), \psi_{r_2}) (\beta(X_1), \psi_{j_2})] + \nu_{r_1, r_2}^\infty \nu_{j_1, j_2}^\infty \right| + \lambda_{r_1} \rho_{r_2} \rho_{j_2} \]
\[ \leq \mathbb{E}[(X_1, \phi_{r_1})^2 (\beta(X_1), \psi_{r_2}) (\beta(X_1), \psi_{j_2})] + \lambda_{r_1}^2 \rho_{r_2} \rho_{j_2} + \lambda_{r_1} \rho_{r_2} \rho_{j_2} \]
\[ = \left| \mathbb{E}[(X_1, \phi_{r_1})^2 (\sum_{k=1}^{\infty} b_{k, r_2} (X_1, \phi_{k})) (\sum_{l=1}^{\infty} b_{l, j_2} (X_1, \phi_{l}))] \right| + \lambda_{r_1}^2 \rho_{r_2} \rho_{j_2} + \lambda_{r_1} \rho_{r_2} \rho_{j_2} \]
\[ \leq \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} |b_{k, r_2}^2 |b_{l, j_2}^2 |(\mathbb{E}[(X_1, \phi_{r_1})^4] (X_1, \phi_{k}) (X_1, \phi_{j_2})] + \lambda_{r_1}^2 \rho_{r_2} \rho_{j_2} + \lambda_{r_1} \rho_{r_2} \rho_{j_2} \]
\[ \leq \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} |b_{k, r_2}^2 |b_{l, j_2}^2 |(\mathbb{E}[(X_1, \phi_{r_1})^4] \mathbb{E}[(X_1, \phi_{k})^4] \mathbb{E}[(X_1, \phi_{j_2})^4] \mathbb{E}[(X_1, \phi_{k})^4])^{1/4} \]
\[ + \lambda_{r_1}^2 \rho_{r_2} \rho_{j_2} + \lambda_{r_1} \rho_{r_2} \rho_{j_2} \]
where the sixth inequality due to $\lambda_k \asymp k^{-\alpha_1}$ for $\alpha_1 > 2$.

- When $j_1 \neq r_1$, (S65) reduces to

$$
\mathbb{E}[V_{\ell_1,r_2}^\infty V_{\ell_1,j_2}^\infty] = \mathbb{E}[\langle X_1, \phi_{\ell_1}\rangle \langle X_1, \phi_{j_2}\rangle \langle \beta(X_1), \psi_{r_2}\rangle \langle \beta(X_1), \psi_{j_2}\rangle].
$$

(S68)

By a similar argument as the one for the case when $j_1 = r_1$ and $j_2 \neq r_2$, one can show that

$$
|\Sigma_{(r_1,r_2)(j_1,j_2)}^\infty| \lesssim \sqrt{\lambda_{r_1} \lambda_{j_1} \rho_{r_2} \rho_{j_2}}.
$$

In summary, we have

$$
\Sigma_{(r_1,r_2)(j_1,j_2)}^\infty \begin{cases}
\asymp \lambda_{r_1} \rho_{r_2} & j_1 = r_1, j_2 = r_2, \\
\lesssim \sqrt{\lambda_{r_1} \lambda_{j_1} \rho_{r_2} \rho_{j_2}} & \text{ow.}
\end{cases}
$$

\[\square\]

**Lemma S5.2.** Suppose the triple $(X,Y,Z)$ follows the model (3). Recall that for $j_2 \geq 1$, $\kappa_{j_2} = \mathbb{E}[Z, \psi_{j_2}]$ with $\psi_{j_2}$ from (S1). Then, under Assumption 3.2(i), $\{\kappa_{j_2}\}_{j_2=1}^\infty$ shares the same decay pattern as $\{\rho_{j_2}\}_{j_2=1}^\infty$. That is, $\kappa_{j_2} \asymp \rho_{j_2} \asymp j_2^{-\alpha_2}$ for $j_2 \geq 1$.

**Proof.** For $j_2 \geq 1$, we observe

$$
\rho_{j_2} = \mathbb{E}[\langle Y_1, \psi_{j_2}\rangle^2] = \mathbb{E}[\langle \beta(X_1) + Z_1, \psi_{j_2}\rangle^2]
$$

$$
= \mathbb{E}[\langle \beta(X_1), \psi_{j_2}\rangle^2] + \mathbb{E}[\langle Z_1, \psi_{j_2}\rangle^2] = \mathbb{E}[\langle \sum_{k=1}^\infty b_{kj_2} \sqrt{\lambda_k} \eta_k^X \rangle^2] + \kappa_{j_2}
$$

$$
= \mathbb{E}[\sum_{k=1}^\infty b_{kj_2}^2 \lambda_k (\eta_k^X)^2] + \kappa_{j_2} = \sum_{k=1}^\infty b_{kj_2}^2 \lambda_k + \kappa_{j_2},
$$

where the fifth identity is from the Cauchy product and the uncorrelatedness among $\eta_k^X$’s. Therefore, with the condition (12) in Assumption 3.2, we have $\kappa_{j_2} \asymp \rho_{j_2}$ for $j_2 \geq 1$. This shows that, $\{\rho_{j_2}\}_{j_2=1}^\infty$ and $\{\kappa_{j_2}\}_{j_2=1}^\infty$ share the same decay pattern. \[\square\]
respectively. Let \( \tilde{\nu}_J = \mathbb{E}[\tilde{V}_{1,J}] \) and \( \nu_J = \mathbb{E}[V_{1,J}] \). Also let \( \tilde{\Sigma}_J \) and \( \Sigma_J \) be respectively the sub-matrices of \( \tilde{\Sigma} \) and \( \Sigma \) whose columns and rows are in the index set \( J \). The following lemma provides insight into the relationship between the mean vectors \( \nu_J \) and \( \tilde{\nu}_J \) and the relationship between the covariance matrices \( \tilde{\Sigma}_J \) and \( \Sigma_J \).

**Lemma S5.3.** Under Assumptions 3.2 and 3.3, we have

\[
\sup_{(\tilde{\phi}, \tilde{\psi}) \in \mathcal{F}_p} \| \tilde{\Sigma}_J - \Sigma_J \|_\infty \leq a_n + b_n
\]

and

\[
\sup_{(\tilde{\phi}, \tilde{\psi}) \in \mathcal{F}_p} \| \tilde{\nu}_J - \nu_J \|_\infty \leq a_n + b_n,
\]

with \( a_n = k_n^{-2\alpha} \) and \( b_n = h_n^{-(\alpha-1)/2} \). Moreover, if \( d_X < \infty \) and \( d_Y < \infty \), then \( \sup_{(\tilde{\phi}, \tilde{\psi}) \in \mathcal{F}_p} \| \tilde{\Sigma}_J - \Sigma_J \|_\infty \leq a_n \) and \( \sup_{(\tilde{\phi}, \tilde{\psi}) \in \mathcal{F}_p} \| \tilde{\nu}_J - \nu_J \|_\infty \leq a_n \).

**Remark S2.** With the choice of \( h_n \), and for a sufficiently small but fixed \( \delta > 0 \) in the definition of \( \ell_n \) in Assumption 3.2, it follows that \( a_n \gg b_n \), which further leads to \( \sup_{(\tilde{\phi}, \tilde{\psi}) \in \mathcal{F}_p} \| \tilde{\Sigma}_J - \Sigma_J \|_\infty \leq a_n \) and \( \sup_{(\tilde{\phi}, \tilde{\psi}) \in \mathcal{F}_p} \| \tilde{\nu}_J - \nu_J \|_\infty \leq a_n \).

**Proof.** Below we only provide a proof for the bound on \( \| \tilde{\Sigma}_J - \Sigma_J \|_\infty \); the bound on \( \| \tilde{\nu}_J - \nu_J \|_\infty \) can be established in a similar fashion. We suppress \( (\tilde{\phi}, \tilde{\psi}) \) from \( W_\text{p}(\tilde{\phi}, \tilde{\psi}) \) to further simplify notation. In addition, the constants hidden in "\( \xi \)" below are independent of \( (\tilde{\phi}, \tilde{\psi}) \), \( n \) and \( p \).

Define

\[
W_{(r_1, r_2)(j_1, j_2)}^\infty = \langle \tilde{\phi}_{r_1}, \phi_{j_1} \rangle \langle \tilde{\psi}_{r_2}, \psi_{j_2} \rangle,
\]

for \((r_1, r_2), (j_1, j_2) \in \mathbb{N}^2\); see also the remark on the general notations about \((r_1, r_2)\) and \((j_1, j_2)\) at the beginning of this supplementary material. Similar to the proof of Lemma S2.8, we focus on the case \( p_1, p_2 \geq h_n^{1/2} \), and introduce the index sets \( \mathcal{P}_{n,1} = \{1, \ldots, n_{h,1}\} \) and \( \mathcal{P}_{n,2} = \{1, \ldots, n_{h,2}\} \). Clearly, \( \mathcal{P}_n = \mathcal{P}_{n,1} \times \mathcal{P}_{n,2} \) for any \( r, j \in \mathcal{J} \), define \( W_{rj}^\infty = W_{(r_1, r_2)(j_1, j_2)}^\infty \) for \((r_1, r_2), (j_1, j_2) \in \mathcal{P}_n \) with \( r = \Phi(r_1, r_2) \) and \( j = \Phi(j_1, j_2) \). Note that, for any \( r, j \in \mathcal{J} \), \( W_{rj} = W_{rj}^\infty \) and \( |W_{rj}^\infty| \leq 1 \). Finally, by the definition of \( \mathcal{F}_p \), max\(|W_{rr} - 1|, |W_{rj}|\) \( \leq a_n \) for any \( r, j \in \mathcal{J} \) with \( r \neq j \), and max\(\{|\tilde{\phi}_{r_1}, \phi_{j_1}|, |\tilde{\psi}_{r_2}, \psi_{j_2}|\}\) \( \leq a_n \) for any \((r_1, r_2), (j_1, j_2) \in \mathcal{P}_n \) with \( r_1 \neq j_1 \) and \( r_2 \neq j_2 \).

For any \((r_1, r_2), (j_1, j_2) \in \mathcal{P}_n \) with \( r = \Phi(r_1, r_2) \) and \( j = \Phi(j_1, j_2) \),

\[
\tilde{\Sigma}_{rj} - \Sigma_{rj} = \mathbb{E}[\tilde{V}_{1,r} \tilde{V}_{1,j}] - \mathbb{E}[\tilde{V}_{1,r}] \mathbb{E}[\tilde{V}_{1,j}] - \Sigma_{rj}
\]

\[
= \mathbb{E}[(X_1, \tilde{\phi}_{r_1})(Y_1, \tilde{\psi}_{r_2})|X_1, \tilde{\phi}_{j_1}, Y_1, \tilde{\psi}_{j_2}] - \mathbb{E}[(X_1, \tilde{\phi}_{r_1})(Y_1, \tilde{\psi}_{r_2})|X_1, \tilde{\phi}_{j_1}, Y_1, \tilde{\psi}_{j_2}] - \Sigma_{rj}
\]

\[
= \sum_{m_1=1}^{\infty} \sum_{m_2=1}^{\infty} \sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} \langle \tilde{\phi}_{r_1}, \phi_{m_1} \rangle \langle \tilde{\psi}_{r_2}, \psi_{m_2} \rangle \langle \tilde{\phi}_{j_1}, \phi_{n_1} \rangle \langle \tilde{\psi}_{j_2}, \psi_{n_2} \rangle \Sigma_{(m_1, m_2)(n_1, n_2)} - \Sigma_{rj}
\]

\[
\therefore \tilde{\Sigma}_{rj} - \Sigma_{rj} = \sum_{m_1=1}^{\infty} \sum_{m_2=1}^{\infty} \sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} W_{(r_1, r_2)(m_1, m_2)} W_{(j_1, j_2)(n_1, n_2)} \Sigma_{(m_1, m_2)(n_1, n_2)} - \Sigma_{rj}.
\]
\[= A + A_0 + A_1 + A_2 + A_3 + A_4,\]

where \(A\) and \(A_k, k = 0, \ldots, 4\) are defined as follows. First, we define \(A = W_{rr} W_{jj} \Sigma_{rj} - \Sigma_{rj}\). Then, we observe that, in the summation, the range of \(m_k, l_k, k = 1, 2\) can be divided into two parts according to the sets \(P_{n,1}\) and \(P_{n,2}\). For \(m_1\) and \(l_1\), the range is \(P_{n,1} \cup P_{n,1}'\), where \(P_{n,1}' = \{h_{n,1} + 1, \ldots, \infty\}\). For \(m_2\) and \(l_2\), the range is \(P_{n,2} \cup P_{n,2}'\), where \(P_{n,2}' = \{h_{n,2} + 1, \ldots, \infty\}\). We call \(P_{n,1}'\) and \(P_{n,1}''\) the infinite summation parts. With these notation, we define

\[
A_0 = \sum_{m_1, l_1 \in P_{n,1}} \sum_{m_2, l_2 \in P_{n,2}} W_{rm} W_{jl} \Sigma_{ml} - W_{rr} W_{jj} \Sigma_{rj}
\]

\[
A_1 = \sum_{m_1 \in P_{n,1}} \sum_{l_1 \in P_{n,1}} \sum_{m_2, l_2 \in P_{n,2}} W_{(r_1, r_2)}^{(1)}(m_1, m_2) W_{(j_1, j_2)}^{(1)}(l_1, l_2) \Sigma_{(m_1, m_2)}^{(1)}(l_1, l_2)
\]

\[
+ \sum_{m_2 \in P_{n,2}} \sum_{l_2 \in P_{n,2}} W_{(r_1, r_2)}^{(1)}(m_1, m_2) W_{(j_1, j_2)}^{(1)}(l_1, l_2) \Sigma_{(m_1, m_2)}^{(1)}(l_1, l_2)
\]

\[
+ \sum_{l_1 \in P_{n,1}} \sum_{m_1 \in P_{n,1}} \sum_{m_2, l_2 \in P_{n,2}} W_{(r_1, r_2)}^{(1)}(m_1, m_2) W_{(j_1, j_2)}^{(1)}(l_1, l_2) \Sigma_{(m_1, m_2)}^{(1)}(l_1, l_2)
\]

\[
A_4 = \sum_{m_1, l_1 \in P_{n,1}} \sum_{m_2, l_2 \in P_{n,2}} W_{(r_1, r_2)}^{(1)}(m_1, m_2) W_{(j_1, j_2)}^{(1)}(l_1, l_2) \Sigma_{(m_1, m_2)}^{(1)}(l_1, l_2)
\]

where \(\Sigma_{(m_1, m_2)}^{(1)}(l_1, l_2)\) are defined in Lemma S5.1. Here, the subscript \(k\) in \(A_k, k = 0, \ldots, 4\) indicates the number of infinite summation parts in each summation term. For instance, \(A_1\) contains four terms and each term includes one infinite summation part; \(A_4\) has one term and this term consists of four infinite summation parts. The terms \(A_2, A_3\) are defined in a similar way. Below we bound these terms.

We first verify that the summation of all elements \(|\Sigma_{(m_1, m_2)}^{(1)}(l_1, l_2)|\) is finite via the following calculation:

\[
\sum_{m_1, l_1 \in P_{n,1}} \sum_{m_2, l_2 \in P_{n,2}} |\Sigma_{(m_1, m_2)}^{(1)}(l_1, l_2)| = \sum_{m_1 = 1}^{\infty} \sum_{m_2 = 1}^{\infty} |\Sigma_{(m_1, m_2)}^{(1)}(m_1, m_2)| + \sum_{m_1, l_1 \in P_{n,1}} \sum_{m_2, l_2 \in P_{n,2}} |\Sigma_{(m_1, m_2)}^{(1)}(l_1, l_2)|
\]

\[
\leq \sum_{m_1 = 1}^{\infty} \sum_{m_2 = 1}^{\infty} \lambda m_1 \rho m_3 + \sum_{m_1, l_1 \in P_{n,1}} \sum_{m_2, l_2 \in P_{n,2}} \sqrt{\lambda m_1} \lambda l_1 \rho m_1 \rho m_2
\]

\[
\leq \sum_{m_1 = 1}^{\infty} \sum_{m_2 = 1}^{\infty} \lambda m_1 \rho m_2 + \left( \sum_{m_1 = 1}^{\infty} \sum_{m_2 = 1}^{\infty} \sqrt{\lambda m_1} \rho m_2 \right)^2
\]

\[
\times \left( \sum_{m_1 = 1}^{\infty} m_1^{-\alpha_1} \right) \left( \sum_{m_2 = 1}^{\infty} m_2^{-\alpha_2} \right) + \left( \sum_{m_1 = 1}^{\infty} m_1^{-\alpha_1/2} \right)^2 \left( \sum_{m_2 = 1}^{\infty} m_2^{-\alpha_2/2} \right)^2
\]

\[= 32\]
\[ < \infty.\]

For \( A \), since \( 1 - ca_n \leq W_{rr} \leq 1 + ca_n \), we have \( A \leq (1 + ca_n)^2 \Sigma r_j - \Sigma r_j \leq ca_n \) and \( A \geq (1 - ca_n)^2 \Sigma r_j - \Sigma r_j \geq -ca_n \). This shows \( |A| \leq ca_n \).

For \( A_0 \), we either have \( m \neq r \) or \( l \neq j \) for each term in the summation. Therefore, the order of \( |W_{rm}| \) or \( |W_{jl}| \) is bounded by \( a_n \). Consequently,

\[
|A_0| \leq ca_n \sum_{m_k, l_k = 1}^{\infty} \left| \Sigma_{(m_1, m_2)(l_1, l_2)} \right| \leq ca_n.
\]

For \( A_4 \), with Assumption 3.2(i) and Lemma S5.2,

\[
|A_4| \leq \sum_{m_1 h_{n,1} + 1}^{\infty} \sum_{m_2 h_{n,2} + 1}^{\infty} \sum_{l_1 h_{n,1} + 1}^{\infty} \sum_{l_2 h_{n,2} + 1}^{\infty} \left| \Sigma_{(m_1, m_2)(l_1, l_2)} \right|
\]

\[
\leq \sum_{m_1 h_{n,1} + 1}^{\infty} \sum_{m_2 h_{n,2} + 1}^{\infty} \lambda_{m_1} \rho_{m_2} \sum_{l_1 h_{n,1} + 1}^{\infty} \sum_{l_2 h_{n,2} + 1}^{\infty} \lambda_{l_1} \rho_{l_2}
\]

\[
\leq \left( \sum_{m_1 h_{n,1} + 1}^{\infty} \frac{m_1^{-\alpha_1}}{m_2 h_{n,2} + 1} \right) \left( \sum_{m_2 h_{n,2} + 1}^{\infty} \frac{m_2^{-\alpha_2}}{m_1 h_{n,1} + 1} \right) + \left( \sum_{m_1 h_{n,1} + 1}^{\infty} \frac{m_1^{-\alpha_1/2}}{m_2 h_{n,2} + 1} \right) \left( \sum_{m_2 h_{n,2} + 1}^{\infty} \frac{m_2^{-\alpha_2}}{m_1 h_{n,1} + 1} \right)^2
\]

\[
= \frac{h_{n,1}^{1-\alpha_1} h_{n,2}^{1-\alpha_2}}{\alpha_1 - 1} + \frac{h_{n,1}^{1-\alpha_1/2} h_{n,2}^{1-\alpha_2}}{(\alpha_1/2 - 1/2) - 1}^2
\]

\[
\leq h_n^{1-\alpha_1}.
\]

For \( A_1 \), we analyze one of its terms; the other terms can be bounded in a similar fashion. Observe

\[
\sum_{m_1 \in \mathcal{P}_{n,1}, l_1 \in \mathcal{P}_{n,1}} \sum_{m_2 \in \mathcal{P}_{n,2}, l_2 \in \mathcal{P}_{n,2}} \left| W_{(r_1, r_2)}(m_1, m_2) W_{(j_1, j_2)}(l_1, l_2) \Sigma_{(m_1, m_2)(l_1, l_2)} \right|
\]

\[
= \sum_{m_1 h_{n,1} + 1}^{\infty} \sum_{m_2 h_{n,2} + 1}^{\infty} \left| \left( \tilde{\phi}_{r_1}, \phi_{m_1} \right) \left( \tilde{\psi}_{r_2}, \psi_{m_2} \right) \left( \tilde{\phi}_{j_1}, \phi_{l_1} \right) \left( \tilde{\psi}_{j_2}, \psi_{l_2} \right) \Sigma_{(m_1, m_2)(l_1, l_2)} \right|
\]

\[
= \sum_{m_1 \in \mathcal{P}_{n,1}, l_1 \in \mathcal{P}_{n,1}, m_2, l_2 \in \mathcal{P}_{n,2}} \left| \left( \tilde{\phi}_{r_1}, \phi_{m_1} \right) \left( \tilde{\psi}_{r_2}, \psi_{m_2} \right) \left( \tilde{\phi}_{j_1}, \phi_{l_1} \right) \left( \tilde{\psi}_{j_2}, \psi_{l_2} \right) \Sigma_{(m_1, m_2)(l_1, l_2)} \right|
\]

\[
+ \sum_{m_1 \in \mathcal{P}_{n,1}} \left| \left( \tilde{\phi}_{r_1}, \phi_{m_1} \right) \left( \tilde{\psi}_{r_2}, \psi_{m_2} \right) \left( \tilde{\phi}_{j_1}, \phi_{l_1} \right) \left( \tilde{\psi}_{j_2}, \psi_{l_2} \right) \Sigma_{(m_1, r_2)(j_1, j_2)} \right|
\]

\[
\leq ca_n \sum_{m_1 h_{n,1} + 1}^{\infty} \sum_{m_2 h_{n,2} + 1}^{\infty} \left| \Sigma_{(m_1, m_2)(l_1, l_2)} \right| + \sum_{m_1 h_{n,1} + 1}^{\infty} \left| \Sigma_{(m_1, r_2)(j_1, j_2)} \right|
\]

\[
\leq a_n + h_n^{1-\alpha_1}.
\]
where the first inequality is due to the facts that $|W_{jl}| \lesssim a_n$ for any $j, l \in J$ with $l \neq j$ and $|\langle \tilde{\psi}_{r_2}, \psi_{m_2} \rangle| \lesssim a_n$ for any $m_2 \in \mathcal{Y}$. Similarly, one can show $|A_2| + |A_3| \lesssim c(a_n + b_n)$. Combining the above bounds for $A$ and $A_1, \ldots, A_4$, we conclude that $\|\hat{\Sigma}_{\mathcal{J}} - \Sigma_{\mathcal{J}}\|_\infty \lesssim c(a_n + b_n)$.

If $d_X < \infty$ and $d_Y < \infty$, there are no infinite sum parts in the above summations of $A_1, \ldots, A_4$, and thus all of these terms can be upper bounded by the order of $a_n$. \hfill \Box

The following proposition states the relation between the corresponding test sequences induced by different pairs of bases of $\mathcal{X}$ and $\mathcal{Y}$. As discussed in Section 2, these spaces are characterized by dimensions $d_X$ and $d_Y$, which can either be finite or infinite. For the sake of brevity and clarity in our notations, our discussion will primarily focus on the scenario where both $d_X$ and $d_Y$ are infinite. However, that the analysis for cases involving finite dimensions for either $\mathcal{X}$ or $\mathcal{Y}$ would follow from a similar argument.

**Proposition S5.4.** For orthonormal bases $\{\tilde{\phi}_j\}_{j=1}^\infty$ and $\{\tilde{\psi}_j\}_{j=1}^\infty$ of $\mathcal{X}$ and $\mathcal{Y}$ respectively, and the corresponding sequence $\tilde{V}_i = (\tilde{\xi}_{ij}, \tilde{\zeta}_{ij} : (j_1, j_2) \in \mathbb{N}^2)$, where $\tilde{\xi}_{ij} = \langle X_i, \tilde{\phi}_j \rangle$, $\tilde{\zeta}_{ij} = \langle Y_i, \tilde{\psi}_j \rangle$, there is a unitary operator $\tilde{U}$ (possibly depending on $(\tilde{\phi}, \tilde{\psi})$) on $\ell^2 \times \ell^2$ such that

$$\tilde{V}_i = \tilde{U}V_i,$$

where $V_i$ is defined before Assumption 3.1 and $\tilde{U}^\dagger$ is the adjoint of $\tilde{U}$.

**Proof.** For $\mathcal{X}$, given the CONS $\{\phi_j\}_{j=1}^\infty$ and $\{\tilde{\phi}_j\}_{j=1}^\infty$, we define mappings $T_X, \tilde{T}_X : \mathcal{X} \to \ell^2$ as

$$T_X : x \mapsto (\langle x, \phi_j \rangle)_{j=1}^\infty,$$

$$\tilde{T}_X : x \mapsto (\langle x, \tilde{\phi}_j \rangle)_{j=1}^\infty.$$

These mappings are both well-defined and bijective.

Let $F_{X,i}^\infty = ((X_i, \phi_j))_{j=1}^\infty$ and $\tilde{F}_{X,i}^\infty = ((X_i, \tilde{\phi}_j))_{j=1}^\infty$ represent the Fourier coefficients of $X_i$ with respect to $\{\phi_j\}_{j=1}^\infty$ and $\{\tilde{\phi}_j\}_{j=1}^\infty$, respectively. Given that $T_X^\dagger F_{X,i}^\infty = X = \tilde{T}_X^\dagger \tilde{F}_{X,i}^\infty$, we deduce

$$\tilde{F}_{X,i}^\infty = U_X^\dagger F_{X,i}^\infty,$$

where $U_X^\dagger$ is the adjoint operator of the operator $U_X = T_X \tilde{T}_X^\dagger$, the composition of $T_X$ and $\tilde{T}_X^\dagger$. Given this construction, we claim $U_X$ is a unitary operator from $\ell^2$ to $\ell^2$. That is, $U_X$ is surjective and satisfies $\langle U_X g, U_X h \rangle = \langle g, h \rangle$ for all $g, h \in \ell^2$. Firstly, for any $\tilde{g} = (\tilde{g}_j)_{j=1}^\infty \in \ell^2$, let $G = \tilde{T}_X \tilde{g}$ and observe $G = \sum_{j=1}^\infty \tilde{g}_j \tilde{\phi}_j \in \mathcal{X}$. Thus, $g = T_X G = U_X \tilde{g}$ consist of the Fourier coefficients of $G$ with respect to $\{\phi_j\}_{j=1}^\infty$, which implies $g \in \ell^2$. This indicates $U_X$ is surjective. Secondly, for any $g, h \in \ell^2$,

$$U_X g = \left( \sum_{j=1}^\infty g_j (\tilde{\phi}_j, \phi_k) k \geq 1 \right) \in \ell^2 \quad \text{and} \quad U_X h = \left( \sum_{j=1}^\infty h_j (\tilde{\phi}_j, \phi_k) k \geq 1 \right) \in \ell^2.$$
Setting \( e_j = ((\tilde{\phi}_j, \phi_k), k \geq 1) \in \ell^2 \), therefore,

\[
\langle U_X g, U_X h \rangle = \langle \sum_{j=1}^{\infty} g_j e_j, \sum_{l=1}^{\infty} h_l e_l \rangle = \sum_{j=1}^{\infty} \sum_{l=1}^{\infty} g_j h_l \langle e_j, e_l \rangle
\]

confirming the unitarity of \( U_X \). A similar process applies to \( Y \), with the similar definitions and notations, leading to

\[
\tilde{F}_{Y_i} = U_{Y_i} F_{Y_i}^\infty,
\]

where \( U_{Y_i} = T_{Y_i} T_{Y_i}^\dagger \) is a unitary operator.

We then define \( \tilde{U} = U_X \otimes U_Y \), whose definition follows from the tensor product of linear maps, as an operator on \( \ell^2 \times \ell^2 \). It can be verified that \( \tilde{U} \) is unitary, with \( \tilde{U}^\dagger = U_{Y_i}^\dagger \otimes U_{Y_i}^\dagger \) serving as its adjoint operator. Therefore, we establish that

\[
\tilde{V}_i = \tilde{F}_{X_i} \otimes \tilde{F}_{Y_i} = U_{X_i} F_{X,i}^\infty \otimes U_{Y_i} F_{Y,i}^\infty = (U_{X_i} \otimes U_{Y_i}) (F_{X,i}^\infty \otimes F_{Y,i}^\infty) = \tilde{U} V_i^\infty.
\]

\[\Box\]

Lemma S5.5. For an infinite sequence \( \{a_j : j \geq 1\} \) with non-negative entries satisfies \( S = \sum_{j=1}^{\infty} a_j < \infty \), for all \( j \geq 1 \), we have

\[a_{(j)} \leq S j^{-1},\]

where \( a_{(j)} \) denotes the \( j \)th largest element in the sequence \( \{a_j : j \geq 1\} \).

Proof. For each \( j \geq 1 \),

\[j \cdot a_{(j)} \leq j \cdot \frac{1}{j} \sum_{k=1}^{j} a_{(k)} \leq \sum_{k=1}^{\infty} a_{(k)} = \sum_{k=1}^{\infty} a_k = S.\]

\[\Box\]

Lemma S5.6. If \( Z_1 \) and \( Z_2 \) are centered Gaussian random vectors in \( \mathbb{R}^p \) with covariance matrices \( \Sigma_1 \) and \( \Sigma_2 \), respectively, and \( \Sigma_2 \) is such that \( \Sigma_{jj}^2 \geq b \) for all \( j = 1, \ldots, p \) for some absolute constant \( b > 0 \), then

\[
\sup_{y \in \mathbb{R}^p} |\mathbb{P}(Z_1 \leq y) - \mathbb{P}(Z_2 \leq y)| \leq \frac{C}{b}(\Delta \log^2 p)^{1/2},
\]

where \( C > 0 \) is an absolute constant, \( \Delta = \max_{1 \leq i, k \leq p} |\Sigma_{ij}^1 - \Sigma_{jk}^2| \), and the symbol \( \leq \) for two vectors refers to coordinately smaller relation.

Remark S3. This lemma corresponds to Proposition 2.1 in Chernozhukov et al. (2022). In the original version, the upper bound is expressed as \( C'(\Delta \log^2 p)^{1/2} \), with \( C' > 0 \) being a constant that depends only on
the minimum variance level \( b \). By examining their proofs, it becomes evident that the explicit form of this constant should be \( C' = C/b \), as presented in this lemma. This specific dependency arises from applying a Gaussian anti-concentration inequality, wherein the explicit relationship between the constant in the upper bound and \( b \) can be found in Chernozhukov et al. (2017). Hence, the proof of this lemma is omitted.

S6 An Example for Assumption 3.1 under FLM

In Section 3, we have mentioned that Condition (9) in Assumption 3.1 holds when both \( \|X\|_1 \) and \( \|Y\|_2 \) are sub-Gaussian. Below, we provide a more concrete example for Condition (9). For clarity, we consider a scalar-on-function model and simplify the notation by dropping the subscript from \( V_{11}^\infty \). In this context, \( Y \) and \( Z \) are real-valued scalar variables, while \( X \) and \( \beta \) belong to the space of square-integrable functions, \( L^2(T) \), on a compact interval \( T \subset \mathbb{R} \). By the Karhunen-Loève expansion \( X = \sum_{j=1}^{\infty} \sqrt{\lambda_j} \eta_j X \varphi_j \), where the pairs \((\lambda_j, \varphi_j)\) consisted of the eigenvalues and eigenfunctions of the covariance operator \( C_X = \mathbb{E}[X \otimes X] \), and \( \eta_j \)'s are independent random variables with zero mean and unit variance.

Lemma S6.1. Under Assumption 3.2(i), Condition (9) in Assumption 3.1 can be satisfied if \( \eta_j \)'s and \( Z \) are all sub-Gaussian.

Proof. Since \( \{\varphi_j : j \geq 1\} \) forms a CONS of \( L^2(T) \), we express the slope function \( \beta \) as \( \beta = \sum_{k=1}^{\infty} b_k \varphi_k \).

Therefore, each coordinate \( V_{11}^\infty \) in \( V^\infty \) can be written as

\[
V_{1j}^\infty = \left( \sum_{k=1}^{\infty} b_k \sqrt{\lambda_k} \eta_k X + Z \right) \sqrt{\lambda_j} \eta_j X := A_0 \cdot A_j,
\]

where \( A_0 = \sum_{k=1}^{\infty} b_k \sqrt{\lambda_k} \eta_k X + Z \) and \( A_j = \sqrt{\lambda_j} \eta_j X \).

From our assumption, we have

\[
\|A_0\|_{\psi_2} \leq C_1 \sum_{k=1}^{\infty} b_k \sqrt{\lambda_k}, \quad \|A_j\|_{\psi_2} \leq C_2 \sqrt{\lambda_j}.
\]

Hence, by the basic properties of sub-exponential random variables, for each \( j \in \mathbb{N} \),

\[
\|V_{1j}^\infty - \mathbb{E}V_{1j}^\infty\|_{\psi_1} \leq \|A_0\|_{\psi_2} \|A_j\|_{\psi_2} \leq C_3 \sqrt{\lambda_j} \sum_{k=1}^{\infty} b_k \sqrt{\lambda_k}.
\]

Next, applying Lemma A.1 in Götze et al. (2021),

\[
\|(V_{1j}^\infty - \mathbb{E}V_{1j}^\infty)^2\|_{\psi_{1/2}} \leq \|V_{1j}^\infty - \mathbb{E}V_{1j}^\infty\|^2_{\psi_1} \leq C_4^2 \lambda_j \left( \sum_{k=1}^{\infty} b_k \sqrt{\lambda_k} \right)^2 \leq C_4^2 \lambda_j \left( \sum_{k=1}^{\infty} b_k^2 \right) \left( \sum_{k=1}^{\infty} \lambda_k \right) \leq C_4 \lambda_j,
\]

where the last inequality is due to Assumption 3.2(i). Based on this fact, by Lemma A.2 in Götze et al.
(2021), one obtains
\[ \left\| V^\infty - E[V^\infty] \right\|_{L^q}^2 \leq \sum_{j=1}^{\infty} \left\| (V^\infty_j - E[V^\infty_j])^2 \right\|_{L^q} \leq C_5 \sum_{j=1}^{\infty} \left\| (V^\infty_j - E[V^\infty_j])^2 \right\|_{\psi_{1/2}}^2 \leq C_6 \sum_{j=1}^{\infty} \lambda_j q^2 \leq C_0 q^2, \]

for all \( q \geq 1 \). According to the equivalent definitions (see, for example, Theorem 1 in Vladimirova et al. (2020)) of an \( \alpha \)-sub-exponential (or sub-Weibull) random variable, the above result implies \( \| V^\infty - E[V^\infty] \|_2^2 \) is an \( \alpha \)-sub-exponential random variable with \( \alpha = 1/2 \).

Therefore, for all \( t \geq 0 \),
\[ \mathbb{P}( \| V^\infty - E[V^\infty] \|_2^2 \geq t ) = \mathbb{P}( \| V^\infty - E[V^\infty] \|_2^2 \geq t^2 ) \leq 2 \exp \left( -t^2/2K \right) = 2 \exp \left( -t/K \right), \]

for an absolute constant \( K > 0 \). \( \square \)

S7 Proof of Proposition 2.1

Proof. By expansion (6), the FLM (1) in Section 1 can be written as
\[ Y = \sum_{j_1=1}^{\infty} \sum_{j_2=1}^{\infty} b_{j_1,j_2} \langle X, \phi_{j_1} \rangle_1 \psi_{j_2} + Z. \]

Accordingly,
\[ \nu = E[X \otimes Y] = E \left[ X \otimes \left( \sum_{j_1=1}^{\infty} \sum_{j_2=1}^{\infty} b_{j_1,j_2} \langle X, \phi_{j_1} \rangle_1 \psi_{j_2} \right) \right] = \sum_{j_1=1}^{\infty} \sum_{j_2=1}^{\infty} b_{j_1,j_2} \mathbb{E}[ \langle X, \phi_{j_1} \rangle_1 X] \otimes \psi_{j_2} = \sum_{j_1=1}^{\infty} \sum_{j_2=1}^{\infty} b_{j_1,j_2} (C \phi_{j_1}) \otimes \psi_{j_2} = \sum_{j_1=1}^{\infty} \sum_{j_2=1}^{\infty} b_{j_1,j_2} \lambda_{j_1} \phi_{j_1} \otimes \psi_{j_2}. \]

Given the above, we find that
\[ \nu_{j_1,j_2} = \langle \nu, \phi_{j_1} \otimes \psi_{j_2} \rangle = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} b_{mn} \lambda_m \langle \phi_m \otimes \psi_n, \phi_{j_1} \otimes \psi_{j_2} \rangle = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} b_{mn} \lambda_m \langle \phi_m, \phi_{j_1} \rangle_1 \langle \psi_n, \psi_{j_2} \rangle_2 = \lambda_j b_{j_1,j_2}, \]

which shows that \( b_{j_1,j_2} = \lambda_{j_1}^{-1} \nu_{j_1,j_2} \). By Fubini’s Theorem,
\[ \nu_{j_1,j_2} = \langle \nu, \phi_{j_1} \otimes \psi_{j_2} \rangle = \langle E[X \otimes Y], \phi_{j_1} \otimes \psi_{j_2} \rangle = E[\langle X \otimes Y, \phi_{j_1} \otimes \psi_{j_2} \rangle] = E[(X, \phi_{j_1})_1(Y, \psi_{j_2})_2]. \]
S8  Proof of Proposition 3.4

Proof. Recall the Karhunen–Loève expansion of the covariance operator \( C_X = \mathbb{E}[X \otimes X] = \sum_{j=1}^d \lambda_j \phi_j \otimes \phi_j \).

Define \( P_r = \phi_r \otimes \phi_r \) and its empirical version \( \hat{P}_r = \hat{\phi}_r \otimes \hat{\phi}_r \). Let \( g_{1,r} := \lambda_r - \lambda_{r-1} \) for \( r \geq 1 \), and \( \bar{g}_{1,r} := \min\{g_{1,r-1}, g_{1,r}\} \) for \( r \geq 2 \) with \( \bar{g}_{1,1} = g_{1,1} \). Since we assume that \( \lambda_j \asymp j^{-\alpha_1} \) for some \( \alpha_1 > 1 \), the effective rank

\[
r(C_X) = \frac{\text{tr}(C_X)}{\|C_X\|_2}
\]

of the covariance operator \( C_X \) is always bounded, where \( \text{tr}(C_X) \) and \( \|C_X\|_2 \) are respectively the trace and the operator norm of the operator \( C_X \). Moreover, \( g_{1,r} \asymp r^{-(\alpha_1+1)} \geq d_1^{-(\alpha_1+1)} \) when \( r \leq d_1 \), and \( \|C_X\|_2 \) is bounded.

From Proposition S9.4, for any \( u, v \in X \), the event \( \mathcal{E}_X(u, v) \)

\[
|\langle (\hat{P}_r - P_r) u, v \rangle| \leq \frac{c}{\bar{g}_{1,r}} \sqrt{\frac{t}{n}} |u||v|
\]

occurs with probability at least \( 1 - e^{-t} \). Consider the following two cases.

- When \( j = r \). Take \( u = v = \phi_r \) and we have

\[
|\langle (\hat{P}_r - P_r) u, v \rangle| = |\langle (\hat{P}_r - P_r) \phi_r, \phi_r \rangle| = |\langle \hat{P}_r \phi_r, \phi_r \rangle - 1| = 1 - \langle \hat{\phi}_r, \phi_r \rangle^2.
\]

Then, under the event \( \mathcal{E}_X(\phi_r, \phi_r) \), \( |\langle \hat{\phi}_r, \phi_r \rangle| \geq 1 - c \sqrt{t/(n\bar{g}_{1,r}^2)} = 1 - o(1) \) under the assumption \( t \ll n\bar{g}_{1,r}^2 \) and further

\[
|1 - \langle \hat{\phi}_r, \phi_r \rangle| = \frac{1 - \langle \hat{\phi}_r, \phi_r \rangle^2}{1 + \langle \hat{\phi}_r, \phi_r \rangle} \leq \frac{c}{\bar{g}_{1,r}} \sqrt{\frac{t}{n}}
\]

for all sufficiently large \( n \).

- When \( j \neq r \). Take \( u = \phi_j \) and \( v = \phi_r \) and we have

\[
|\langle (\hat{P}_r - P_r) u, v \rangle| = |\langle (\hat{P}_r - P_r) \phi_j, \phi_r \rangle| = |\langle \hat{P}_r \phi_j, \phi_r \rangle| = |\langle \hat{\phi}_j, \phi_r \rangle| |\langle \hat{\phi}_r, \phi_r \rangle|.
\]

Under the event \( \mathcal{E}_X(\phi_r, \phi_r) \cap \mathcal{E}_X(\phi_j, \phi_r) \), we have

\[
|\langle \hat{\phi}_j, \phi_r \rangle| = \frac{|\langle (\hat{P}_r - P_r) u, v \rangle|}{|\langle \hat{\phi}_r, \phi_r \rangle|} = \frac{|\langle (\hat{P}_r - P_r) u, v \rangle|}{1 + o(1)} \leq \frac{c}{\bar{g}_{r,r}} \sqrt{\frac{t}{n}}.
\]
Similarly, for the transformation matrix $U^d_Y(\hat{\psi})$ defined in Section 3, we have

$$\langle \psi_r, \hat{\psi}_j \rangle \leq c \frac{g_{2, r}}{\bar{g}_{1, r}} \sqrt{\frac{t}{n}} \quad j \neq r,$$

$$\langle \psi_r, \hat{\psi}_r \rangle - 1 \leq c \frac{g_{2, r}}{\bar{g}_{1, r}} \sqrt{\frac{t}{n}} \quad j = r,$$

under the event $\mathcal{E}_Y(\psi_j, \psi_r) \cap \mathcal{E}_Y(\psi_r, \psi_r)$, where $\mathcal{E}_Y(u, v)$ is defined in analogy to $\mathcal{E}_X(u, v)$ and $\bar{g}_{2, r} \geq d_2^{\alpha \varphi + 1}$

is defined in analogy to $\bar{g}_{1, r}$ by replacing $\{\lambda_j\}_{j=1}^{d_1}$ with $\{\rho_j\}_{j=1}^{d_2}$.

Below, we focus on $W := W_d(\hat{\phi}, \hat{\psi}) = U^d_X(\hat{\phi}) \circ U^d_Y(\hat{\psi})$, while the results for $U^d_X(\hat{\phi})$ and $U^d_Y(\hat{\psi})$ can be similarly analyzed. For $W$, we have $W_{r j} = \langle \phi_{r_1}, \hat{\phi}_{j_1} \rangle \langle \psi_{r_2}, \hat{\psi}_{j_2} \rangle$, where $r = \Phi(r_1, r_2)$ and $j = \Phi(j_1, j_2)$ with $\Phi$ defined at the beginning of this supplementary material. Let $a_{n, 1} = \frac{1}{\bar{g}_{1, d_1}} \sqrt{\frac{t}{n}}$ and $a_{n, 2} = \frac{1}{\bar{g}_{2, d_2}} \sqrt{\frac{t}{n}}$

with suitable $t$ such that $\frac{t}{n} = o(1)$. Below we divide the proof into three cases under the event $\mathcal{W}(r, j) := \mathcal{E}_X(\phi_{r_1}, \phi_{r_1}) \cap \mathcal{E}_X(\phi_{j_1}, \phi_{j_1}) \cap \mathcal{E}_Y(\psi_{r_2}, \psi_{r_2}) \cap \mathcal{E}_Y(\psi_{r_2}, \psi_{j_2})$.

- $j_1 = r_1$ and $j_2 = r_2$, which implies $j = r$. Under the event $\mathcal{W}(r, j)$, since $|\langle \phi_{r_1}, \hat{\phi}_{r_1} \rangle - 1| \leq c a_{n, 1}$ and $|\langle \psi_{r_2}, \hat{\psi}_{r_2} \rangle - 1| \leq c a_{n, 2}$, we have $(1 - c a_{n, 1})(1 - c a_{n, 2}) \leq \langle \phi_{r_1}, \hat{\phi}_{r_1} \rangle \langle \psi_{r_2}, \hat{\psi}_{r_2} \rangle \leq (1 + c a_{n, 1})(1 + c a_{n, 2})$, which implies that

$$c[a_{n, 1} a_{n, 2} - (a_{n, 1} + a_{n, 2})] \leq W_{i i} - 1 \leq c[a_{n, 1} a_{n, 2} + (a_{n, 1} + a_{n, 2})],$$

and further

$$|W_{i i} - 1| \leq c \max\{a_{n, 1}, a_{n, 2}\} = c \max\left\{\frac{1}{g_{1, d_1}}, \frac{1}{g_{2, d_2}}\right\} \sqrt{\frac{t}{n}}.$$

- $j_1 \neq r_1$ and $j_2 \neq r_2$. In this case, under the event $\mathcal{W}(r_1, r_2, j_1, j_2)$, we have

$$|W_{r j}| \leq c \frac{t}{g_{1, d_1} g_{2, d_2} n} \leq c \max\left\{\frac{1}{g_{1, d_1}}, \frac{1}{g_{2, d_2}}\right\} \sqrt{\frac{t}{n}}.$$

- $j_1 = r_1$ and $j_2 \neq r_2$; the case of $j_1 \neq r_1$ and $j_2 = r_2$ is analyzed in a similar fashion. Under the event $\mathcal{W}(r, j)$, we have

$$c(1 - a_{n, 1}) a_{n, 2} \leq W_{r j} \leq c(1 + a_{n, 1}) a_{n, 2},$$

which further implies that

$$|W_{r j}| \leq c a_{n, 2} \leq \max\left\{\frac{1}{g_{1, d_1}}, \frac{1}{g_{2, d_2}}\right\} \sqrt{\frac{t}{n}},$$

and

$$|W_{i j}| \leq c a_{n, 1} \leq \max\left\{\frac{1}{g_{1, d_1}}, \frac{1}{g_{2, d_2}}\right\} \sqrt{\frac{t}{n}}.$$

Let $a_{n, t} = \max\{d_1^{\alpha \varphi + 1}, d_2^{\alpha \varphi + 1}\} \sqrt{t/n} > 0$ with a suitable $t$ such that $a_{n, t} = o(1)$. Note that $\mathbb{P}(\mathcal{W}(r, j)^c) \leq \mathbb{P}(\mathcal{E}_X(\phi_{r_1}, \phi_{j_1})^c) + \mathbb{P}(\mathcal{E}_X(\phi_{r_1}, \phi_{r_1})^c) + \mathbb{P}(\mathcal{E}_Y(\psi_{r_2}, \psi_{j_2})^c) + \mathbb{P}(\mathcal{E}_Y(\psi_{r_2}, \psi_{j_2})^c) \leq 4e^{-t}$. Then we conclude that, for
a suitable constant $C > 0$, 

$$P(\|W - I_d\|_\infty \geq C_{a,n,t}) = P\left(\bigcup_{r,j} \{|W_{rj} - (I_d)_{rj}| \geq C_{a,n,t}\}\right)$$

$$\leq \sum_{r,j} P(|W_{rj} - (I_d)_{rj}| \geq C_{a,n,t})$$

$$\leq \sum_{r,j} P\left(|W_{rj} - (I_d)_{rj}| \geq C \max \left\{\frac{1}{g_{1,d}}, \frac{1}{g_{2,d}}\right\} \sqrt{\frac{t}{n}}\right)$$

$$\leq \sum_{r,j} P(W(r,j)^c)$$

$$\leq \sum_{r,j} 4e^{-t} \leq e^{-t + 2\log(2d)},$$

which is equivalent to $\|W - I_d\|_\infty \leq C_{a,n,t}$ with probability at least $1 - e^{-t + 2\log(2d)}$. 

\[\square\]

**S9 A Concentration Inequality for $\langle \phi_r, \hat{\phi}_j \rangle$**

The proof of Proposition 3.4 depends on Proposition S9.4 to be stated below, which in turn depends on the following lemmas from Koltchinskii et al. (2016). In this section, the notation $|C|_2$ is denoted as the operator norm for an operator $C$.

**Lemma S9.1.** Let $X, X_1, \ldots, X_n$ be i.i.d. centered sub-Gaussian random elements in a Hilbert space with covariance $C_X = E(X \otimes X)$. For all $q \geq 1$,

$$\left(E|\hat{C}_X - C_X|_2^q\right)^{1/q} \leq \|C_X\|_2 \max\left\{\sqrt{\frac{r(C_X)}{n}}, \frac{r(C_X)}{n}\right\}.$$ 

**Lemma S9.2.** Let $X, X_1, \ldots, X_n$ be i.i.d. centered sub-Gaussian random vectors in a Hilbert space with covariance $C_X = E(X \otimes X)$. There exist a constant $C_1 > 0$ such that for all $t \geq 1$ with probability at least $1 - e^{-t}$,

$$|\hat{C}_X - C_X|_2 \leq C_1|C_X|_2 \left[\sqrt{\frac{r(C_X)}{n}} \vee \frac{r(C_X)}{n} \vee \sqrt{\frac{t}{n}} \vee \frac{t}{n}\right].$$

Let $E = \hat{C}_X - C_X$ and $\bar{g}_r = \bar{g}_{1,r}$ as in the proof of Proposition 3.4. Denote

$$O_r = \sum\frac{1}{\lambda_r - \lambda_s}P_s,$$

with $P_s = \phi_s \otimes \phi_s$. 

...
Lemma S9.3. It holds that $$\| \hat{P}_r - P_r \|_2 \leq 4 \frac{\| E \|_2}{\bar{g}_r}.$$ Moreover, $$\hat{P}_r - P_r = L_r(E) + S_r(E),$$ where $$L_r(E) = O_r E P_r + P_r E O_r$$ and $$\| S_r(E) \|_2 \leq 14 \left( \frac{\| E \|_2}{\bar{g}_r} \right)^2.$$

Remark S4. Although Lemma S9.1 and Lemma S9.2 in Koltchinskii et al. (2016) are stated for Gaussian elements, Remark 1 therein mentions that such results also hold for centered sub-Gaussian elements. Also, the Lemma S9.3 is without a Gaussian assumption.

Now we are ready to state and prove the proposition.

Proposition S9.4. Let $$X, X_1, \ldots, X_n$$ be i.i.d. centered sub-Gaussian random elements in a separable Hilbert space $$\mathcal{X}$$ with covariance $$C_X = \mathbb{E}(X \otimes X).$$ If Assumptions 3.1–3.2 hold, then for a constant $$c > 0,$$ for all $$r > 0$$ and $$1 \ll t \ll n,$$ with probability at least $$1 - e^{-t},$$ we have

$$\left\langle (\hat{P}_r - P_r) u, v \right\rangle \leq \frac{c}{\bar{g}_r} \sqrt{\frac{t}{n}} \| u \| \| v \|,$$

for any fixed $$u, v \in \mathcal{X}.$$

Proof. It is seen that

$$\left\langle (\hat{P}_r - P_r) u, v \right\rangle = \left\langle (\hat{P}_r - \mathbb{E}\hat{P}_r) u, v \right\rangle + \left\langle (\mathbb{E}\hat{P}_r - P_r) u, v \right\rangle$$

$$\leq \left\langle (\hat{P}_r - \mathbb{E}\hat{P}_r) u, v \right\rangle + \left\| \mathbb{E}\hat{P}_r - P_r \right\|_2 \| u \| \| v \|,$$

where $$\| u \|$$ denotes the norm of $$u$$ in $$\mathcal{X}$$. As in Corollary 1 and Theorem 4 of Koltchinskii et al. (2016), it remains to bound $$\left\langle (\hat{P}_r - \mathbb{E}\hat{P}_r) u, v \right\rangle$$ and $$\left\| \mathbb{E}\hat{P}_r - P_r \right\|_2.$$ It is sufficient to consider the case with sufficiently large $$n$$ and bounded $$r(C_X).$$

From Lemma S9.3, $$\hat{P}_r - P_r = L_r + S_r,$$ where $$L_r = O_r E P_r + P_r E O_r$$ and $$E = \hat{C}_X - C_X.$$ In addition, $$\mathbb{E}L_r = 0$$ due to $$\mathbb{E}E = 0.$$ Then we have

$$\left\| \mathbb{E}\hat{P}_r - P_r \right\|_2 = \left\| \mathbb{E}(\hat{P}_r - P_r) \right\|_2 = \left\| \mathbb{E}(L_r + S_r) \right\|_2$$

$$= \left\| \mathbb{E}S_r \right\|_2 \leq \left\| S_r \right\|_2$$

$$\leq c \frac{\| E \|^2_2}{\bar{g}_r^2} \leq c \frac{\| C_X \|^2_2 r(C_X)}{\bar{g}_r^2 n}$$

for a constant not depending on $$r$$ or $$n,$$ where the last line is based on Lemma S9.1.

Now we analyze the term $$\left\langle (\hat{P}_r - \mathbb{E}\hat{P}_r) u, v \right\rangle.$$ As in Lemma S9.3, let $$R_r = (\hat{P}_r - P_r) - \mathbb{E}(\hat{P}_r - P_r) = \hat{P}_r - \mathbb{E}\hat{P}_r - L_r.$$ Then,

$$\left\langle (\hat{P}_r - \mathbb{E}\hat{P}_r) u, v \right\rangle \leq \left\langle R_r u, v \right\rangle + \left\langle L_r u, v \right\rangle.$$

According to a discussion after Theorem 3 of Koltchinskii et al. (2016), $$\left\langle L_r u, v \right\rangle$$ can be written as a sum
of i.i.d. sub-exponential random variables with the assumption that \( X_i \) are sub-Gaussian random elements. By using a Bernstein inequality we then have

\[
|\langle L_r, u, v \rangle| \leq D_1 \frac{\|C_X\|_2}{g_r} \sqrt{\frac{t \|u\| \|v\|}{n}} \tag{S69}
\]

with probability at least \( 1 - e^{-t} \), for some constant \( D_1 > 0 \).

To bound the term \(|\langle R_r, u, v \rangle|\), without loss of generality, assume \( \|u\| \leq 1 \) and \( \|v\| \leq 1 \). First, we observe that

\[
R_r = S_r - \mathbb{E}S_r,
\]

and

\[
|\langle S_r, u, v \rangle| \leq \|S_r\|_2 \leq \|\hat{P}_r - P_r\|_2 + \|L_r\|_2 \leq \|\hat{P}_r - P_r\|_2 + \sqrt{2} \frac{\|E\|_2}{g_r} \leq 6 \frac{\|E\|_2}{g_r},
\]

where the third inequality is based on the fact \( L_r \leq \sqrt{2g_r^{-1}} \|E\|_2 \) from Koltchinskii et al. (2016). Then, we use Lemma S9.2 to derive that, with probability at least \( 1 - e^{-t} \),

\[
|\langle S_r, u, v \rangle| \leq c \frac{\|C_X\|_2}{g_r} \left( \sqrt{\frac{r(C_X)}{n}} \sqrt{\frac{r(C_X)}{n}} \sqrt{\frac{t}{n} \vee \frac{t}{n}} \right)
\]

for a constant not depending on \((u, v)\). By taking \( t = \log 2 \), we see that \( 1 - e^{-t} = 1/2 \) and consequently observe that

\[
\text{Med} \leq c \frac{\|C_X\|_2}{g_r} \left( \sqrt{\frac{r(C_X)}{n}} \sqrt{\frac{r(C_X)}{n}} \sqrt{\frac{1}{n} \vee \frac{1}{n}} \right),
\]

where Med is the median of \(|\langle S_r, u, v \rangle|\). This implies that, all \( t \geq 1 \), with probability at least \( 1 - e^{-t} \),

\[
|\langle S_r, u, v \rangle - \text{Med}| \leq c \frac{\|C_X\|_2}{g_r} \left( \sqrt{\frac{r(C_X)}{n}} \sqrt{\frac{r(C_X)}{n}} \sqrt{\frac{t}{n} \vee \frac{t}{n}} \right).
\]

Integrating the tail of this exponential bound yields that, for some \( D_2 > 0 \),

\[
|\mathbb{E}\langle S_r, u, v \rangle - \text{Med}| \leq \mathbb{E}|\langle S_r, u, v \rangle - \text{Med}| \leq D_2 \frac{\|C_X\|_2}{g_r} \left( \sqrt{\frac{r(C_X)}{n}} \sqrt{\frac{r(C_X)}{n}} \sqrt{\frac{1}{n} \vee \frac{1}{n}} \right),
\]

which in turn implies that one can replace Med by the expectation \( \mathbb{E}\langle S_r, u, v \rangle \) in the concentration bound to conclude that, with some \( D_3 > 0 \) and with probability at least \( 1 - 2e^{-t} \),

\[
|\langle R_r, u, v \rangle| = |\langle S_r, u, v \rangle - \mathbb{E}\langle S_r, u, v \rangle| \leq D_3 \frac{\|C_X\|_2}{g_r} \left( \sqrt{\frac{r(C_X)}{n}} \sqrt{\frac{r(C_X)}{n}} \sqrt{\frac{t}{n} \vee \frac{t}{n}} \right), \tag{S70}
\]

where by adjusting the constant \( D_3 \) one can replace \( 1 - 2e^{-t} \) by \( 1 - e^{-t} \).
Combining (S69) and (S70) yields, with probability at least $1 - e^{-t}$,
\[
|\langle (\hat{P}_r - E P_r) u, v \rangle| \leq D_1 \|C_X\|_2 \frac{t}{g_r} \sqrt{\frac{t}{n}} \|u\| \|v\| + D_2 \frac{\|r(C_X)^\vee\|}{g_r} \left( \sqrt{\frac{r(C_X)}{n}} \vee \sqrt{\frac{t}{n}} \vee \frac{t}{n} \right) \|u\| \|v\|
\]
\[
\leq c \frac{1}{g_r} \sqrt{\frac{t}{n}} \|u\| \|v\|,
\]
and further
\[
|\langle (\hat{P}_r - P_r) u, v \rangle| \leq c \frac{1}{g_r} \sqrt{\frac{t}{n}} \|u\| \|v\| + c \frac{1}{g^2 n} \|u\| \|v\| \leq c \frac{1}{g_r} \sqrt{\frac{t}{n}} \|u\| \|v\|.
\]
where we relies on $1 \ll t \ll n$, and the boundedness of $r(C_X)$ and $\|C_X\|_2$.

Remark S5. It can be shown that if $X$ and $Z$ in (1) are sub-Gaussian random elements in $\mathcal{X}$ and $\mathcal{Y}$, respectively, then $Y$ from (1) is also a sub-Gaussian random variable in $\mathcal{Y}$. Proposition S9.4 then holds also for $Y$, which is needed in the proof of Proposition 3.4.

S10 Special Cases of Model (1)

- The scalar-on-function model: Taking $\mathcal{Y} = \mathbb{R}$, $\mathcal{X} = L^2(T) = \{ f : T \to \mathbb{R} \mid \int_T |f(t)|^2 dt < \infty \}$ for an interval $T \subset \mathbb{R}$, endowed with their canonical inner products respectively, and $\beta(x) = \int x(t) \hat{\beta}(t) dt$ for some function $\hat{\beta} : T \to \mathbb{R}$, the model (1) becomes
  \[
  Y - EY = \int_T \{X(t) - E X(t)\} \hat{\beta}(t) dt + Z.
  \]

- The function-on-function model: Taking $\mathcal{Y} = L^2(T_0)$, $\mathcal{X} = L^2(T_1)$ for some intervals $T_0, T_1 \subset \mathbb{R}$, endowed with their respective canonical inner products, and $\beta(x) = \int x(s) \hat{\beta}(s, \cdot) ds$ for some function $\hat{\beta} : T_1 \times T_0 \to \mathbb{R}$, the model (1) becomes
  \[
  Y(t) - EY(t) = \int_{T_1} \{X(s) - E X(s)\} \hat{\beta}(s, t) ds + Z(t).
  \]

- The function-on-vector model, also known as the varying coefficient model in Shen and Faraway (2004): Taking $\mathcal{Y} = L^2(T)$ for some interval $T \subset \mathbb{R}$, $\mathcal{X} = \mathbb{R}^q$ for some positive integer $q$, endowed with their respective canonical inner products, and $\beta(x) = x^T \hat{\beta}(\cdot)$ for some function $\hat{\beta} : T \to \mathbb{R}^q$, the model (1) becomes
  \[
  Y(t) - EY(t) = \sum_{j=1}^q (X_j - E X_j) \hat{\beta}_j(t) + Z(t).
  \]

- The model with mixed-type predictors (Cao et al., 2020): Take $\mathcal{Y} = L^2(T_0)$ for some interval $T_0 \subset \mathbb{R}$ endowed with its canonical inner product, and take $\mathcal{X} = L^2(T_1) \oplus \cdots \oplus L^2(T_d) \oplus \mathbb{R}^q$ endowed with the inner product of the direct sum of Hilbert spaces $\langle \cdot, \cdot \rangle_{DS}$ (e.g., see Section I.6 of Conway, 2007), for some
positive integers \(d\) and \(q\), along with some intervals \(T_1, \ldots, T_d \subseteq \mathbb{R}\). Let \(\beta(x) = \sum_{k=1}^{d} \int g_k(s) \tilde{\gamma}_k(s, \cdot) \, ds + v^T \tilde{\eta}(\cdot)\) for \(x = (g_1, \ldots, g_d, v) \in \mathcal{X}\) with \(\tilde{\eta}_k: T_0 \to \mathbb{R}^q\) and real-valued functions \(\tilde{\gamma}_k: T_k \times T_0 \to \mathbb{R}\). The model (1), with \(X = (G_1, \ldots, G_d, V)\), then becomes

\[
Y(t) - \mathbb{E}Y(t) = \sum_{k=1}^{d} \int_{T_k} \{G_k(s) - \mathbb{E}G_k(s)\} \tilde{\gamma}_k(s, t) \, ds + (V - \mathbb{E}V)^T \tilde{\eta}(t) + Z(t).
\]

- The partial functional linear model (Shin, 2009): Take \(\mathcal{Y} = \mathbb{R}\) and \(\mathcal{X} = L^2(T_1) \oplus \cdots \oplus L^2(T_d) \oplus \mathbb{R}^q\) endowed with the inner product of the direct sum of Hilbert spaces \((\cdot, \cdot)_{DS}\) for some positive integers \(d\) and \(q\), along with some intervals \(T_1, \ldots, T_d \subseteq \mathbb{R}\). Let \(\beta(x) = \sum_{k=1}^{d} \int g_k(t) \tilde{\gamma}_k(t) \, dt + v^T \tilde{\eta}\) for \(x = (g_1, \ldots, g_d, v) \in \mathcal{X}\), with \(\tilde{\eta} \in \mathbb{R}^q\) and real-valued functions \(\tilde{\gamma}_k: T_k \to \mathbb{R}\). The model (1), with \(X = (G_1, \ldots, G_d, V)\), then becomes

\[
Y - \mathbb{E} = \sum_{k=1}^{d} \int_{T_k} \{G_k(s) - \mathbb{E}G_k(s)\} \tilde{\gamma}_k(s) \, ds + (V - \mathbb{E}V)^T \tilde{\eta} + Z.
\]

References

Cao, G., Wang, S., and Wang, L. (2020), “Estimation and inference for functional linear regression models with partially varying regression coefficients,” Stat, 9, e286.

Chernozhukov, V., Chetverikov, D., and Kato, K. (2017), “Detailed proof of Nazarov’s inequality,” arXiv preprint arXiv:1711.10696.

Chernozhukov, V., Chetverikov, D., Kato, K., and Koike, Y. (2022), “Improved Central Limit Theorem and bootstrap approximations in high dimensions,” The Annals of Statistics, 50, 2562–2586.

Conway, J. B. (2007), A Course in Functional Analysis, vol. 96, New York, NY: Springer New York, 2nd ed.

Götze, F., Sambale, H., and Simulis, A. (2021), “Concentration inequalities for polynomials in \(\alpha\)-subexponential random variables,” Electronic Journal of Probability, 26, 1 – 22.

Hsing, T. and Eubank, R. (2015), Theoretical Foundations of Functional Data Analysis, with an Introduction to Linear Operators, Wiley.

Johnson, W. B., Schechtman, G., and Zinn, J. (1985), “Best constants in moment inequalities for linear combinations of independent and exchangeable random variables,” The Annals of Probability, 234–253.

Koltchinskii, V., Lounici, K., et al. (2016), “Asymptotics and concentration bounds for bilinear forms of spectral projectors of sample covariance,” in Annales de l’Institut Henri Poincaré, Probabilités et Statistiques, Institut Henri Poincaré, vol. 52, pp. 1976–2013.

Ledoux, M. and Talagrand, M. (1991), Probability in Banach Spaces: isoperimetry and processes, vol. 23, Springer Science & Business Media.

Lin, Z., Lopes, M. E., and Müller, H.-G. (2022), “High-dimensional MANOVA via Bootstrapping and its Application to Functional Data and Sparse Count Data,” Journal of the American Statistical Association,
Lopes, M. E., Lin, Z., and Müller, H.-G. (2020), “Bootstrapping max statistics in high dimensions: Near-parametric rates under weak variance decay and application to functional data analysis,” *The Annals of Statistics*, 48, 1214–1229.

Lopes, M. E. and Yao, J. (2022), “A sharp lower-tail bound for Gaussian maxima with application to bootstrap methods in high dimensions,” *Electronic Journal of Statistics*, 16, 58–83.

Shen, Q. and Faraway, J. (2004), “An F test for linear models with functional responses,” *Statistica Sinica*, 14, 1239–1257.

Shin, H. (2009), “Partial functional linear regression,” *Journal of Statistical Planning and Inference*, 139, 3405–3418.

Tropp, J. A. (2022), “ACM 217: Probability in High Dimensions,”

van der Vaart, A. and Wellner, J. (1996), *Weak Convergence and Empirical Processes: With Applications to Statistics*, New York: Springer.

Vershynin, R. (2018), *High-dimensional probability: An introduction with applications in data science*, vol. 47, Cambridge university press.

Vladimirova, M., Girard, S., Nguyen, H., and Arbel, J. (2020), “Sub-Weibull distributions: Generalizing sub-Gaussian and sub-Exponential properties to heavier tailed distributions,” *Stat*, 9, e318.