Optimal Control for Acrobot with Two-link Manipulators

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Abstract. This paper presents a unified treatment of the optimal control of under-actuated two-link manipulators. Firstly, a nonlinear invertible transfer is introduced to simplify design of controller, then several optimal control laws are established by maximum principle. Controller for swing-up area is based on performance index, while controller for attractive area is based on LQR (linear quadric regulator). It is proved that optimal control is not singular, invariant sets are obtained by theoretical analysis. The strategy for Acrobat is to use optimal control law to pump manipulators into linearizable area, then switch control law to LQR, which drives system to up-straight position. Local and global stability were analyzed in detail under control strategy using LaSalle’s invariance principle and WCLF (non-smooth control Lyapunov function). Simulation and comparison with previous results show that the proposed approach here is valid and of advantages.

Keywords: Acrobat, optimal control, WCLF, LaSalle’s invariance principle, performance index.

1. Introduction
Acrobot is an underactuated two-link robot that moves in a vertical plane [1]. This kind of robot has great advantages in terms of weight, cost, and energy consumption due to the reduction of the driving device at the elbow joint. At the same time, the reduction of the driving device also causes the dynamic model of the robot to be constrained by second order nonholonomic conditions, so it is very difficult to control it [2].

In the past three decades, to achieve the stability control goal of Acrobot at the vertical upward equilibrium point, scholars have conducted in-depth research and proposed a variety of control methods. These methods be summarized into two steps. Firstly, the appropriate control law is designed to drive the system to the invariant set of the system. The motion area of the system is called the rocking area. Then linearize the system in the invariant set and adopt the optimal linear control law for the linearized model. The moving region from the invariant set to the target equilibrium point is called the equilibrium region [3-5]. The control change between the two areas belongs to the switching control. The control method of the rocking area includes partial feedback linearization method, zero dynamic system method and Lyapunov method[1][6-7]. The control problem of the equilibrium region has been solved by mature linear system theory, including pole placement method[8], quadratic performance index optimal control method[1], and various artificial intelligence algorithms[6]. The algorithm based on performance indexes in the shake-up area, such as the control method based on the shortest time MTC performance index, first appeared in the literature[9-10]. In [11], the MTC problem is extended to allow the performance index to take into account the control torque. The text [12] proposes a reversible transformation different from the text [1] [10], which simplifies the controller
design of the shake-up area. Other representative design methods are described in the literature [12-17].

From the current published literature, the performance indexes of optimal control are basically obtained under the premise that the control is limited, and the design result is embodied in the form of bang-bang control. Control limitation is not general in actual engineering, but bang-bang control jumps due to control quantity switching, so the actual control effect needs to be verified by experiments rather than by simulation experiments. In addition, the literature [3] [18] discusses the local and global stability problems, and the reference [19] discusses the problems of nonlinear dynamics analysis and global stabilization, but few literatures discuss the stability of the system under the optimal control law. The normal equation based on the minimum value principle is only the necessary condition for the optimal performance index. Therefore, it is necessary to explore the stability problem under the control law. Based on a Lyapunov function, the stability problems of three different performance indexes under optimal control are discussed in this paper, including the control law under two conditions of control limitation and control freedom. A reversible nonlinear transformation was introduced to reduce the complexity of the controller design. Validity and superiority of two simulation example verification methods at the end of the paper.

2. Acrobot's Control Problem

Acrobot's model structure is shown in Figure 1, where \( m_i \) represents the mass of the \( i \)-th rod, and \( l_i \) represents the length of the \( i \)-th rod, and \( l_{ic} \) represent the distance from the \( i \) joint to the center of mass of \( i \), and \( I_i \) represents the inertia of the \( i \)-th rod relative to the center of mass, \( q_1 \) represents the angle of the first rod relative to the vertical y-axis, \( q_2 \) indicates the angle of the second rod relative to the first rod, \( \tau_2 \) is the control force acting on the second link, \( g \) is the acceleration of gravity.

![Figure 1. Model structure of an Acrobot.](image)

Let \( \mathbf{q} = [q_1, q_2]^T \), then the kinetic equation of Acrobot satisfies the following conditions:

\[
M(q)\ddot{q} + H(q, \dot{q}) + G(q) = \tau
\]

In the above formula

\[
H(q, \dot{q}) = \begin{bmatrix} -\theta_1 (q_1^2 + 2\dot{q}_1\dot{q}_2) \sin q_2, \theta_1 \dot{q}_1^2 \sin q_2 \end{bmatrix}^T, \quad G(q) = \begin{bmatrix} -\theta_1 \sin q_1 - \theta_2 \sin q_{12}, -\theta_1 \sin q_{12} \end{bmatrix}^T, \quad \tau = [0, \tau_2]^T,
\]

\[
M(q) = \begin{bmatrix} \theta_1 + \theta_2 + 2\theta_1 \cos q_1 + \theta_1 \cos q_2, \theta_2 + \theta_1 \cos q_2 \end{bmatrix}^T
\]

where \( q_{12} = q_1 + q_2, \quad \theta_j (j = 1, \ldots, 5) \) represents the structural parameters of Acrobot,

\[
\theta_1 = m_1 l_{g1}^2 + m_2 l_1^2 + I_1, \quad \theta_2 = m_2 l_{g2}^2 + I_2, \quad \theta_3 = m_1 l_{g1} l_{g2}, \quad \theta_4 = m_1 gl_{g1}, \quad \theta_5 = m_2 gl_{g2}.
\]

Take \( \mathbf{x} = [x_1, x_2, x_3, x_4] = [q_1, q_2, \dot{q}_1, \dot{q}_2] \), then formula (1) can be rewrite as follows [9]:

\[
\dot{x}(t) = f(x) + b(x)\tau_2 = F(x, \tau_2)
\]
In the above formula $f(x) = \begin{bmatrix} x_3 & x_4 & f_3(x) & f_4(x) \end{bmatrix}^T$, $b(x) = \begin{bmatrix} 0 & 0 & b_1(x) & b_2(x) \end{bmatrix}^T$, $f_3(x), f_4(x), b_1(x), b_2(x)$ is a nonlinear function, they are as follows:

$$\begin{pmatrix} f_3(x) \\ f_4(x) \end{pmatrix} = M^{-1}(q) \begin{pmatrix} -H(q, \dot{q}) - G(q) \\ b(x) \end{pmatrix} \quad (4)$$

$$b_1(x) = -(\theta_2 + \theta_1 \cos(x_2)) / \gamma(x) \quad (5)$$

$$b_2(x) = (\theta_1 + \theta_2 + 2\theta_1 \cos(x_2)) / \gamma(x) \quad (6)$$

where $\gamma(x) = (\theta_1 \theta_2 - \theta_3 \cos^2(x_2))$.

Equation (6) shows that the control coefficient matrix is a nonlinear coefficient moment associated only with $x_2$.

The energy of the system at any position is expressed as follows:

$$E(q) = E(q, \dot{q}) = T(q, \dot{q}) + U(q) \quad (7)$$

where $T(q, \dot{q}) = \frac{1}{2} \dot{q}^T M(q) \dot{q}, \ U(q) = \sum_{j=1}^{2} U_j(q) = \sum_{j=1}^{2} m_j g \gamma_j (q)$.

Acrobot's rocking control target is from the vertical downward stable equilibrium point $x_a = [\pi, 0, 0, 0]'$ to the vertical upward unstable equilibrium point $x_d = [0, 0, 0, 0]'$.

### 3. System Transformation and Optimal Control Law

#### 3.1. System Equivalent Transformation

The control coefficient matrix contains a nonlinear function, which brings difficulties to the design of the control law. To this end, a full rank transform $z = Tx$ is introduced:

$$T = \begin{bmatrix} I_{2 \times 2} & 0 \\ 0 & D(x_2) \end{bmatrix} \quad (8)$$

In the above formula

$$D(x_2) = \begin{bmatrix} 1 & -b_1(x_2) \\ b_2(x_2) & 1 \\ 0 & b_2(x_2) \end{bmatrix} \quad (9)$$

Then the system (3) can be converted as follows:

$$\dot{z}(t) = f(z) + br_2 \quad (10)$$

In the above formula

$$f(z) = \begin{bmatrix} D^{-1}z_3 \\ f_3(z) \\ f_4(z) \end{bmatrix}, b = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T \quad (11)$$

$$\begin{pmatrix} f_3(z) \\ f_4(z) \end{pmatrix} = \alpha(z_2) \begin{bmatrix} pd_1(z_2) & 0 \\ 0 & pd_2(z_2) \end{bmatrix} D^{-1} \begin{bmatrix} Z_3 \\ Z_4 \end{bmatrix} + D(z_2) \begin{bmatrix} f_3(x) \\ f_4(x) \end{bmatrix}_{x=\tau^t z} \quad (12)$$

where

$$\alpha(z) = \frac{z_4}{b_2(z_2)}, \quad pd_1(z) = \frac{\partial b_1(z_2)}{\partial z_2} b_2(z_2) - \frac{\partial b_2(z_2)}{\partial z_2} b_1(z_2), \quad pd_2(z) = -\frac{\partial b_2(z_2)}{\partial z_2}.$$

Equation (8) is a full rank transform, and $\|T\|$ is bounded, so $x \to 0 \Leftrightarrow z \to 0$. 

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3.2. Three Typical Performance Indexes and Optimal Control Laws

This article discusses the control questions of the second type of performance indexes as follows:

\[ I : J = \theta(t_f) + S(t_f) \]  
\[ \text{(13)} \]

\[ II : J = \theta(t_f) + \frac{1}{2} \int_0^{t_f} R \tau_2^2 \]  
\[ \text{(14)} \]

\[ III : J = \int_0^{t_f} \frac{1}{2} (z^T Q z + R \tau_2^2) dt \]  
\[ \text{(15)} \]

where \( \theta(t_f) = t_f \), and \( S(t_f) = E(z) - E_d + \varepsilon \) is the terminal condition, \( R, \varepsilon > 0 \) is a suitable positive number, \( Q \) is a semi-definite matrix.

Remark 1: The terminal condition is generally taken as a certain invariant set of the system.

Type I is the MTC index, Type II considers the input control energy consumption, and type III considers the steady change of state (attitude).

3.2.1 Optimal control law. A: Optimal control law for the first performance index.

Hamilton function: \( H(z, \lambda, \tau_2) = \lambda^T (f(z) + b \tau_2) \), by the optimal control principle, estate of \( \lambda \) and the regular equation are \[ \lambda(t_f) = \frac{\partial H}{\partial \lambda} = f(z) + b \tau_2 \]
\[ \text{(16)} \]

The corresponding cross-section conditions and initial conditions are as follows:

\[ \lambda(t_f) = \frac{\partial \theta}{\partial z} + \left( \frac{\partial S}{\partial z(t_f)} \right)^T \nu(t_f), H(t_f) = \frac{\partial \theta}{\partial t_f}, \frac{\partial S}{\partial t_f}, z(0) = z_0 \]
\[ \text{(17)} \]

where \( \lambda, \nu \) is a Lagrange multiplier.

In practical engineering, the motor provides a control torque that is always limited, so it is assumed that control is limited without loss of generality. Give the definition as follows.

Definition 1: If \( u \) can switch from \( u = \pm M \) to \( u = \mp M \) a limited number of times, then \( u \) is called bang-bang control. If the switching function is only a finite number of times, then at the zero point of the switching function, \( u \) is almost always \( \pm M \rightarrow \mp M \). \[ \text{[22][23]} \]

Theorem 1: The MTC control law of the shortest time problem must be non-singular bang-bang control.

Demonstration: \( H(z, \lambda, \tau_2) = \lambda^T (f(z) + b \tau_2) = \lambda^T f(z) + \lambda_4 \tau_2 \), if you want
\[ \min_{t_2} H(z, \lambda, \tau_2) = \lambda^T f(z) + \lambda_4 \tau_2 \] to be established, it is obvious that the control torque is as follows:

\[ \tau_2 = \begin{cases} 
-M & \lambda_4 > 0 \\
\text{indeterminacy} & \lambda_4 = 0 \\
+M & \lambda_4 < 0 
\end{cases} \]
\[ \text{(18)} \]

Prove that the non-singular process is below. If the switching function is always 0 for a certain \( (t_k, t_{k+1}) \), the control amount is uncertain. This situation is called a singular problem. At this time, \( \dot{\lambda} = 0, \lambda_4 = 0 \), when \( \forall t \in (t_k, t_{k+1}) \), the formula below can be obtained by equation (16).
The solution of Equation (19) is $\lambda_1, \lambda_2, \lambda_3$. By the uniqueness of the solution, the solution is linearly independent, so if the formula (20) is true, the following formula must be established.

$$\frac{\partial f_1}{\partial z_4} \lambda_1 + \frac{\partial f_2}{\partial z_4} \lambda_2 + \frac{\partial f_3}{\partial z_4} \lambda_3 = 0$$

(20)

Because of $f_1(z) = z_1 + b_1(z_2)z_2 \Rightarrow \frac{\partial f_1(z)}{\partial z_4} = b_1(z_2) \neq 0$, $\lambda_4(t)$ has a finite number of zeros, the optimal control must not be singular. Proof is done.

B: Optimal control law for class II performance indexes

For the second type of performance indexes, our conclusions are as follows:

Theorem 2: If the control is unconstrained, the control law of the minimization performance index (15) is:

$$\tau_2 = -\frac{\lambda_4}{R}$$

(22)

Proof:

$$J = \theta(t_f) + \frac{1}{2} \int_0^{t_f} R \tau_2^2 dt$$

(23)

$$H(z, \lambda, \tau_2) = \frac{1}{2} R \tau_2^2 + \lambda^T (f(z) + b \tau_2) = \lambda^T f(z) + \frac{1}{2} R \tau_2^2 + \lambda_4 \tau_2$$

$$= \lambda^T f(z) - \frac{1}{2R} \lambda_4^2 + \frac{R}{2} (\tau_2 + \frac{\lambda_4}{R})^2$$

(24)

Obviously, the control torque of $\min_{\tau_2} H(z, \lambda, \tau_2)$ takes the format of (22). Proof is done.

Remark 4: Equation (22) is continuous, so the motion stability of bang-bang control law is better than the comparison.

Deduction 1: For the case where there is a constraint $|\tau_2| \leq M$, the minimization performance index (15) control law takes the following form.

$$\tau_2 = \begin{cases} 
+M & \frac{\lambda_4}{R} \leq -M \\
\frac{\lambda_4}{R} & -M \leq \frac{\lambda_4}{R} \leq +M \\
-M & \frac{\lambda_4}{R} \geq +M 
\end{cases}$$

(25)

This conclusion is easy to draw from Theorem 2. Proof process is ignored.

C: The corresponding Hamiltonian function of the optimal control law of the Class III performance index is as follows:
Using the same method as Theorem 2 and Inference 1, it is easy to prove the following conclusions.

Theorem 3: If the control is unconstrained, the control law of the minimum class performance index (15) is taken as the form (18), in which

\[ H(z, \lambda, \tau_z) = \frac{1}{2}(z^T Qz + R \tau_z^2) + \lambda^T (f(z) + b \tau_z) \]  

(26)

\[ \dot{z} = \frac{\partial H}{\partial \lambda} = f(z) + b \tau_z \]

(27)

\[ \dot{\lambda} = -\frac{\partial H}{\partial z} = -Qz - \frac{\partial f^T(z)}{\partial z} \lambda \]

Deduction 2: Let \( |\tau_z| \leq M \), the optimization performance index (15) is controlled in the form of equation (22), where the equation of motion and the normal equation satisfy the equation (27).

4. Design Balance Zone Controller

As mentioned earlier, the control law of the rocking area only guarantees that the Acrobot moves to the invariant set but does not ensure that the system reaches the desired balance point, so another controller is needed to achieve the goal. If the system is operated to a region that meets the linearization conditions, linearize the Acrobot's dynamic equation at equilibrium point \( z_d \), which can be achieved using linear system theory. First, the linearization is as follows:

\[ \dot{z} = Az + B \tau_z \]  

(28)

Take the performance index as

\[ J = \frac{1}{2} \int \left( z^T Qz + R \tau_z^2 \right) dt \]  

(29)

then the controller is \( \tau_z = -R^{-1} B^T Pz \), where \( P \) is the solution of the matrix equation.

\[ PA + A^T P - PBR^{-1} B^T P + Q = 0 \]  

(30)

5. Stability Analysis

5.1. Stability Analysis of the Rocking Area

To comparison with the results of the relevant literature, the following analysis is described in terms of the original system without transformation.

The following result can be obtained by the equation (7) combined with the system motion equation (3).

\[ E(x) = \frac{x_1^2}{2} (\theta_1 + \theta_2 + \theta_3) + x_3 x_4 (\theta_1 + \theta_3 \cos x_3) + \frac{\theta_2 x_2^2}{2} + \theta g \cos x_3 + \theta g \cos(x_1 + x_3) \]  

(31)

Let \( E_s = E(x) - E_d \), where \( E_d \) is the potential energy of the system at the apex. The Lyapunov function is as follows:

\[ \begin{aligned}
V(x) &= \frac{1}{2} E_s^2 + \delta \\
E_s &= (\theta_1 + \theta_3) g
\end{aligned} \]  

(32)

where \( \delta > 0 \) is a suitable normal number, the effect of which is shown in the following global stability analysis. The derivative of equation (31) along time of system (3) is as follows:

\[ \dot{V} = E_s \dot{E}_s = E_s x_3 \tau_2 \]  

(33)

Theorem 4: Existence of bang-bang control rate \( \tau_2 \) makes \( \dot{V} \leq 0 \).

Proof: Take \( \tau_2 = -M \text{sign}(E_s x_3) \) and get the conclusion immediately.
Note: M is an arbitrary positive number, so any size bang-bang control can make $V \leq 0$. Discussed below is a set of states (Invariant set) that make $V=0$.

Definition 2 \[\text{WCLF}]:\] There is a functional $V(x): \mathbb{R}^4 \rightarrow \mathbb{R}$ that is continuous and differentiable. Except for $x=0$, it is positive definite. If there is a control law $\tau_2$, there is always the following formula.

$$L_{F(x,\tau_2)} = \frac{\partial V}{\partial x} F(x,\tau_2) \leq 0$$ \hspace{1cm} (34)

Then $V$ is a generalized control Lyapunov function (WCLF) in which the system acts at $\tau_2$.

According to Theorem 4, the optimal control law for the three performance indexes defined in this paper is WCLF.

Definition 3: If $x(0) \in \Omega$, $\forall t \geq 0$ and the motion state of equation (3) $x(t) \in \Omega$, then $\Omega$ is invariant set.

Lemma 1 \[\text{Lasalle invariant set principle}]: Let $\Omega$ be an invariant set of system (3), $\Psi = \{x(t) \in \Omega | V(x) = 0\}$, $\Phi$ is the largest invariant set (LIS) in $\Psi$, then all solutions $x(t)$ converge from $\Psi$ to $\Phi$.

Set $\Psi$ is the solution of system (3) at $V = 0$, so $x_4 = 0$. By the $E_x = 0$ in equation (33), so $E_x = \text{constant}$. Note that the equation of motion of system (3) is $\dot{x}_2 = x_4 = 0$, so $x_2 = \text{constant}$.

We point out that $x_4 = 0, E_x = \text{constant}(\neq 0)$ is not sustainable.

$[t_k, t_{k+1}]$ is an arbitrary time period, let $\tau_2 = M$, and the following formula is obtained from equation (7).

$$E_x = \frac{x_1^2}{2} (\theta_1 + \theta_2 + \theta_3) + \theta_4 g \cos x_1 + \theta_5 g \cos(x_1 + x_2) = \text{constant}$$ \hspace{1cm} (35)

Since $x_4 = 0, x_2 = \text{constant}$, the above formula (3) can be reduced to the following formula:

$$A_1 x_3^2 + B_1 \cos x_1 + C_1 \sin(x_1) = W_1$$ \hspace{1cm} (36)

where $A_1, B_1, C_1, W_1$ is the associated constant.

Equation (4) is substituted into the relevant parameters, when $x_i = 0$, the following formula can be obtained.

$$f_4 = \frac{(\theta_1 + \theta_2 + 2\theta_1 \cos(x_2))\theta_2 \sin(x_2)^2 - (\theta_2 + \theta_2 \cos x_2)\sin(x_1 + x_2) + (\theta_2 \theta g + \theta \theta g \cos x_1)\sin(x_1)}{-\theta_2 \theta + \theta_3 \cos^2(x_2)}$$ \hspace{1cm} (37)

Then $\dot{x}_4 = 0$ of the system (3) is obtained as follows:

$$M \text{ or } M = \tau_2 = \frac{f_4}{b_2}$$ \hspace{1cm} (38)

Because $x_2 = \text{constant}$, simplification of the above formula can get the following formula:

$$A_2 x_3^2 + B_2 \cos x_1 + C_2 \sin(x_1) = W_2$$ \hspace{1cm} (39)

where $A_2, B_2, C_2, W_2$ is the associated constant.

Eliminate $x_3 = 0$ from equations (33) and (36) to get the formula below.

$$B_1 \cos x_1 + C_1 \sin(x_1) = W_3$$ \hspace{1cm} (40)
where $B, C_1, W_3$ is the associated constant. It can be concluded that $x_1$ must be a constant, so that $x_3 = 0$.

The following shows that $x_3 = 0, x_4 = 0$ is impossible. The following formula is obtained by the system (3).

$$\begin{align*}
  f_3 + b_1 t_2 &= 0 \\
  f_4 + b_2 t_2 &= 0
\end{align*}$$

In the above formula:

$$f_3 = \frac{\theta_2 \theta_4 g \sin(x_1) - \theta_1 \theta_2 \cos x_2 \sin(x_1 + x_3)}{\theta_2 \theta_1 + \theta_3^2 \cos^2(x_2)}$$

$$f_4 = \frac{-\left(\theta_1 \theta_3 + \theta_2 \theta_3 \cos x_3\right) \sin(x_1 + x_2) + \theta_2 \theta_4 g + \theta_2 \theta_4 g \cos x_3 \sin(x_1)}{\theta_2 \theta_1 + \theta_3^2 \cos^2(x_2)}$$

Divide the two equations of (40) and sort out the following formula:

$$-\frac{m_{22}(x) g_1(x) + m_{12}(x) g_2(x)}{m_{21}(x) g_1(x) - m_{11}(x) g_2(x)} = -\frac{m_{12}(x)}{m_{11}(x)}$$

Therefore, $|M| = 0$, which contradicts that $M$ is a positive definite matrix, so $E_s = 0$.

Consider the following invariant sets:

$$0 \neq h(x) = \begin{cases}
  x_4 = 0 \\
  x_2 = \text{constant} \\
  E_x = 0 \\
  f_4(x) = 0
\end{cases}$$

Because $h(x)$ is a non-zero constant vector, $\dot{h} = 0$.

$$0 = \dot{h}(t) = \frac{\partial h}{\partial x} \dot{x}$$

$$\frac{\partial h}{\partial x} = \begin{bmatrix}
  0 & 1 & 0 & 0 \\
  0 & 0 & 0 & 1 \\
  \frac{\partial f_4}{\partial x_1} & \frac{\partial f_4}{\partial x_2} & \frac{\partial f_4}{\partial x_3} & \frac{\partial f_4}{\partial x_4}
\end{bmatrix}$$

Calculate the determinant of equation (47) as follows:

$$\left| \frac{\partial h}{\partial x} \right| = \frac{\partial f_4}{\partial x_1} \frac{\partial E}{\partial x_2} - \frac{\partial f_4}{\partial x_2} \frac{\partial E}{\partial x_1}$$

In the above formula:

$$\frac{\partial f_4}{\partial x_3} = \frac{1}{\gamma(x_2)} \left[ -(c_2 + c_3 \cos x_3) 2 x_4 c_3 \sin x_2 - 2 x_5 c_3 \sin x_2 (c_1 + c_2 + 2 c_3 \cos x_2) \right]$$
\[
\frac{\partial E_x}{\partial x_i} = -c_4 g \sin x_1 - c_5 g \sin(x_i + x_2) \quad (50)
\]

\[
\frac{\partial f_4}{\partial x_i} = \frac{1}{\gamma(x_2)} [-c_4 g \cos x_1 (c_2 + c_3 \cos x_2) + c_5 g \cos(x_i + x_2) (c_1 + c_3 \cos x_2)] \quad (51)
\]

\[
\frac{\partial E_x}{\partial x_3} = x_1 (c_1 + c_2 + 2c_3 \cos x_3) + x_4 (c_2 + c_3 \cos x_2) \quad (52)
\]

Only when \( x_3 = 0, x_4 = 0 \), the following formula is obtained.

\[
\left. \frac{\partial h}{\partial x} \right| = 0 \quad (53)
\]

So (47) has a non-zero solution \( \dot{x} \neq 0 \).

Bring \( x_3, x_4 \) into (7) to get the following formula:

\[
E_x = \theta_4 \cos x_1 + \theta_4 x_1 + x_2 - (\theta_4 + \theta_4) g = 0 \quad (54)
\]

Summarize the above analysis and draw the following conclusions:

Theorem 5: Under the action of the system (3) bang-bang control law, the maximum invariant set corresponding to Lyapunov of equation (31) is as follows:

\[
(x_1, x_4) = (0, 0) \quad (55)
\]

\[
\theta_4 \cos x_1 + \theta_4 \cos(x_1 + x_4) - (\theta_4 + \theta_4) g = 0 \quad (56)
\]

The stability of the system under the control law (18) is discussed below.

Theorem 6: Suppose there is a set \( \Lambda \) of control laws \( \Lambda : r \left| \dot{V}(r) \right| \leq 0 \), so \( \frac{\lambda}{R} \in \Lambda \).

Proof: Through theorem 2, we can know that the bang-bang control law is

\[
\tau = -M \text{sign}(\lambda_4), \dot{V} \leq 0 \quad (33) \text{to get } \dot{V} = -M \dot{e}_x \dot{x}_4 \text{sign}(\lambda_4) \leq 0.
\]

if \( \lambda_4 > 0, \dot{V} = -M \dot{e}_x \dot{x}_4 \leq 0 \Rightarrow -\frac{M \dot{e}_x}{R} \dot{x}_4 \lambda_4 \leq 0; \)

if \( \lambda_4 < 0, \dot{V} = M \dot{e}_x \dot{x}_4 \leq 0 \Rightarrow -M \dot{e}_x \dot{x}_4 \lambda_4 \leq 0; \)

Proof is done.

Note: The bang-bang control law is a discontinuous jump control. The control law (19) is continuous. Theorem 6 indicates that the continuous control law also ensures that the system tends to be invariant in the pumping zone.

5.2. Global Stability Analysis

The control law of the shake-up zone needs to switch to the control law of the balance zone. The stability analysis between the two is based on different Lyapunov functions, so the global stability needs to be verified.

Let \( V'(x) \) be the Lyapunov function of the panning region satisfying equation (31),

\[
V'(x) = \frac{1}{2} x^T P x, \quad \text{where } P \text{ satisfies the equation (39). Define } J(x) = V_1(x) + V_2(x).
\]

Appropriate choice of \( \delta \) makes \( J(x(t_i)) \geq J(x(t_2)), \forall t_1 < t_2 \), where \( t_0 \) is the switching moment. According to the theorem 2 in Literature [3], the switching control is stable.
6. Simulation Results and Analysis

To avoid setting \( S(t_f) \) in advance and the difficulty of solution \( v \), We divide the motion space \( \Sigma \) into two parts:

\[
\Sigma = S + A
\]

(57)

where \( S : x_1^2 + x_2^2 \geq \varepsilon_1^2 \cap x_3^2 + x_4^2 \geq \varepsilon_2^2 \), \( A = \Sigma - S \) is the movement space of the balance zone.

\[
\tau_2 = \begin{cases} 
\tau_2^{(1)} & \text{if } x \in S \\
\tau_2^{(2)} & \text{else} 
\end{cases}
\]

(58)

The algorithm of control is to find the initial value condition \( \lambda(0) \) to stabilize the system in the vertical equilibrium position. The algorithm can be solved using ode45 provided by MATLAB. The simulation model is established using the function ode45 of the numerical solution of the ordinary differential equation in MATLAB. The model parameters of Acrobot are given in Table 1.

Table 1. Acrobot model parameters.

|                  | \( m_1(\text{kg}) \) | \( I_1(\text{kg} \cdot \text{m}^2) \) | \( l_1(\text{m}) \) | \( l_2(\text{m}) \) | \( m_2(\text{kg}) \) | \( I_2(\text{kg} \cdot \text{m}^2) \) | \( l_3(\text{m}) \) | \( l_4(\text{m}) \) |
|------------------|----------------------|------------------|------------------|------------------|------------------|------------------|------------------|------------------|
| Acrobot          | 1.00                 | 8.33 \times 10^{-2} | 1.00             | 0.50             | 1.00             | 0.33             | 2.00             | 1.00             |

6.1. Simulation Results of the Optimal Control Law of the First Type of Performance Index under the Control of the Pendulum

\[ M = 100, \quad \lambda(0) = [-5.82 4.186 -5.72 -7.6]' \], State coefficient matrix \( Q=I \), Control law coefficient matrix \( R=0.5 \), \( K= [-236.49 -94.81 -102.11 16.85] \), \( \varepsilon_1 = \varepsilon_2 = 0.2 \), the simulation results are shown in Figure 2 below.

Figure 2. Simulation results with the balancing region being 0.2.

Figure 3 below shows the process of bang-bang control before switching. It can be seen from the figure that the number of switching of the optimal control law is 1.

Figure 3. Switch of bang-bang control law.
It can be seen from Figure 2 that the system switches the control law at 0.35s. The number of switches controlled by bang-bang is 1 before switching, and system is stable near 5.5s. Bang-Bang control can make the acrobot swing into the balance zone faster than the literature [3] and other similar documents, but the acceleration of the acrobot is difficult to control to the minimum when entering the balance zone, resulting in a longer balance time of the linear quadratic controller.

6.2. The Optimal Control Law of the Second Type of Performance Index is the Simulation Result under the Pendulum Control

\[ \dot{z}(0) = [-2.976 6.617 1.705 0.994]' \]

State coefficient matrix \( Q = I \), Control law coefficient matrix \( R = 0.5 \), Linear quadratic controller feedback gain matrix \( K = [-236.49 -94.81 -102.11 16.85] \), other parameters are the same as before. The simulation results are shown in Figure 4.

![Simulation results with the balancing region being 0.2.](image)

The simulation results show that the system switches the control law at 2.93s. It is stable near 4.5s. Since the optimal control law of the Class II performance index is continuous, the control torque is flexible, which is better than the bang-bang control law. Compared with the first type of performance index, the control law energy is considered due to the performance index, so the required torque is also small.

7. Conclusion

In summary, this paper studies the controller design of two-link Acrobot based on the optimal performance index, optimal control law under three performance indexes is discussed in detail. Optimal control law includes two types, one is Bang-Bang, and another is continuous. The law proposed here are all proved asymptotic stable. The strategy of departed area is effective by simulation. It is a subject that needs further research in the future to extend the method in this paper to four-link and above robotic arms.

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