We discuss a relation between bicomplexes and integrable models, and consider corresponding noncommutative (Moyal) deformations. As an example, a noncommutative version of a Toda field theory is presented.

1 Introduction

Soliton equations and integrable models are known to possess a vanishing curvature formulation depending on a parameter, say \( \lambda \) (cf [1], for example). This geometric formulation of integrable models is easily extended [2, 3] to generalized geometries, in particular in the sense of noncommutative geometry where, on a basic level, the algebra of differential forms (over the algebra of smooth functions) on a manifold is generalized to a differential calculus over an associative (and not necessarily commutative) algebra \( A \).

A bicomplex associated with an integrable model is a special case of a zero curvature formulation. More precisely, let \( \mathcal{M} = \bigoplus_{r \geq 0} \mathcal{M}^r \) be an \( \mathbb{N}_0 \)-graded linear space (over \( \mathbb{R} \) or \( \mathbb{C} \)) and \( d, \delta : \mathcal{M}^r \to \mathcal{M}^{r+1} \) two linear maps satisfying

\[
d^2 = 0, \quad \delta^2 = 0, \quad d \delta + \delta d = 0 \tag{1.1}
\]

(typically as a consequence of certain field equations). Then \((\mathcal{M}, d, \delta)\) is called a bicomplex. Special examples are bi-differential calculi [3]. However, we do not need \( d \) and \( \delta \) to be graded derivations (into some bimodule), i.e., they do not have to satisfy the Leibniz rule.

Given a bicomplex, there is an iterative construction of “generalized conserved densities” in the sense of \( \delta \)-closed elements of the bicomplex (see section 2). In some examples they reproduce directly the conserved quantities of an integrable model. In other examples, the

\[\text{In terms of } d_\lambda = \delta - \lambda d \text{ with a constant } \lambda, \text{ the three bicomplex equations are combined into the single zero curvature condition } d^2_\lambda = 0 \text{ (for all } \lambda).\]
relation is less direct. Anyway, the existence of such a chain of $\delta$-closed elements is clearly a distinguished feature of the model with which the bicomplex is associated.

Noncommutative examples are in particular obtained by starting with a classical integrable model, deforming an associated bicomplex by replacing the ordinary product of functions with the Moyal $\ast$-product and thus arriving at a noncommutative model. As an example, a noncommutative extension of a Toda field theory is considered in section 3. Field theory on noncommutative spaces has gained more and more interest during the last years. A major impulse came from the discovery that a noncommutative gauge field theory arises in a certain limit of string, D-brane and M theory (see [5] and the references cited there). We also refer to [6] for some work on Moyal deformations of integrable models.

2 The bicomplex linear equation

Let us assume that, for some $s \in \mathbb{N}$, there is a (nonvanishing) $\chi^{(0)} \in \mathcal{M}^{s-1}$ with $dJ^{(0)} = 0$ where $J^{(0)} = \delta \chi^{(0)}$. Defining $J^{(1)} = d\chi^{(0)}$, we have $\delta J^{(1)} = -d\delta \chi^{(0)} = 0$ using (1.1). If the $\delta$-closed element $J^{(1)}$ is $\delta$-exact, this implies $J^{(1)} = \delta \chi^{(1)}$ with some $\chi^{(1)} \in \mathcal{M}^{s-1}$. Next we define $J^{(2)} = d\chi^{(1)}$. Then $\delta J^{(2)} = -d\delta \chi^{(1)} = -dJ^{(1)} = -d^2 \chi^{(0)} = 0$. If the $\delta$-closed element $J^{(2)}$ is $\delta$-exact, then $J^{(2)} = \delta \chi^{(2)}$ with some $\chi^{(2)} \in \mathcal{M}^{s-1}$. This can be iterated further and leads to a possibly infinite chain (see the figure below) of elements $J^{(m)}$ of $\mathcal{M}^s$ and $\chi^{(m)} \in \mathcal{M}^{s-1}$ satisfying

$$J^{(m+1)} = d\chi^{(m)} = \delta \chi^{(m+1)} .$$

(2.1)

More precisely, the above iteration continues from the $m$th to the $(m+1)$th level as long as $\delta J^{(m)} = 0$ implies $J^{(m)} = \delta \chi^{(m)}$ with an element $\chi^{(m)} \in \mathcal{M}^{s-1}$. Of course, there is no obstruction to the iteration if $H^s_\delta(\mathcal{M})$ is trivial, i.e., when all $\delta$-closed elements of $\mathcal{M}^s$ are $\delta$-exact. But in general the latter condition is too strong, though in several examples it can indeed be easily verified [3]. Introducing

$$\chi = \sum_{m \geq 0} \lambda^m \chi^{(m)}$$

(2.2)

with a parameter $\lambda$, the essential ingredients of the above iteration procedure are summarized in the linear equation associated with the bicomplex:

$$\delta(\chi - \chi^{(0)}) = \lambda d \chi .$$

(2.3)
Given a bicomplex, we may start with the linear equation (2.3). Let us assume that it admits a (non-trivial) solution \( \chi \) as a (formal) power series in \( \lambda \). The linear equation then leads to \( \delta \chi^{(m)} = d\chi^{(m-1)} \). As a consequence, the \( J^{(m+1)} = d\chi^{(m)} \) are \( \delta \)-exact. Therefore, even if the cohomology \( H^s_\delta(M) \) is not trivial, the solvability of the linear equation ensures that the \( \delta \)-closed \( J^{(m)} \) appearing in the iteration are \( \delta \)-exact.

In all the examples which we presented in [3, 4], the bicomplex space can be chosen as \( M^0 \otimes \Lambda \) where \( \Lambda = \bigoplus_{r=0}^n \Lambda^r \) is the exterior algebra of an \( n \)-dimensional vector space with a basis \( \xi^r, r = 1, \ldots, n \), of \( \Lambda^1 \). It is then sufficient to define the bicomplex maps \( d \) and \( \delta \) on \( M^0 \) since via

\[
\begin{align*}
\sum_{i_1 \ldots, i_r = 1}^n \phi_{i_1 \ldots i_r} \xi^{i_1} \cdots \xi^{i_r} \\
&= \sum_{i_1 \ldots, i_r = 1}^n (d\phi_{i_1 \ldots i_r}) \xi^{i_1} \cdots \xi^{i_r} 
\end{align*}
\]  

(2.4)

(and correspondingly for \( \delta \)) they extend as linear maps to the whole of \( M \).

3 Noncommutative deformation of a Toda model

The \( \ast \)-product on the space \( F \) of smooth functions of two coordinates \( x \) and \( t \) is given by

\[
f \ast h = m \circ e^{\theta P/2} (f \otimes h) = fh + \frac{\theta}{2} \{ f, h \} + O(\theta^2)
\]  

(3.1)

where \( \theta \) is a parameter, \( m(f \otimes h) = fh \) and \( P = \partial_t \otimes \partial_x - \partial_x \otimes \partial_t \). Furthermore, \( \{ , \} \) is the Poisson bracket, i.e., \( \{ f, h \} = (\partial_t f) \partial_x h - (\partial_x f) \partial_t h \). For the calculations below it is helpful to notice that partial derivatives are derivations of the algebra \( (F, \ast) \).

A bicomplex associated with an integrable model can be deformed by replacing the ordinary product of functions with the noncommutative \( \ast \)-product. This then induces a deformation of the integrable model with very special properties since the iterative construction of generalized conservation laws still works. As a specific example, we construct a noncommutative extension of the Toda field theory on an open finite one-dimensional lattice. Other examples can be obtained in the same way.

Let us start from the trivial bicomplex which is determined by

\[
\begin{align*}
\delta \phi &= (\partial_t - \partial_x) \phi \xi^1 + (S - I) \phi \xi^2, \\
d\phi &= -S^T \phi \xi^1 + (\partial_t + \partial_x) \phi \xi^2
\end{align*}
\]  

(3.2)

where \( \phi \) is a vector with \( n \) components (which are functions) and \( S^T \) the transpose of

\[
S = \sum_{i=1}^{n-1} E_{i,i+1}, \\
(E_{i,j})^k_l = \delta^k_i \delta_{j,l}.
\]  

(3.3)

Let \( G \) be an \( n \times n \) matrix of functions which is invertible in the sense \( G^{-1} \ast G = I \) where \( I \) is the \( n \times n \) unit matrix. Now we introduce a “dressing” for \( d \):

\[
\begin{align*}
D\phi &= G^{-1} \ast d(G \ast \phi) = -(L \ast \phi) \xi^1 + (\partial_t + \partial_x + M \ast) \phi \xi^2
\end{align*}
\]  

(3.4)

where

\[
L = G^{-1} \ast S^T \ast G, \quad M = G^{-1} \ast (G_t + G_x).
\]  

(3.5)
Note that $D^2 \phi = G^{-1} \ast d^2(G \ast \phi) = 0$. The only nontrivial bicomplex equation is $\delta D + D \delta = 0$ which reads

$$M_t - M_x = L \ast S - S \ast L.$$  (3.6)

Hence, if this equation holds, then $(\mathcal{F}^n \otimes \Lambda, D, \delta)$ is a bicomplex. Let us now choose

$$G = \sum_{i=1}^{n} G_i E_{ii}$$  (3.7)

with functions $G_i$ for which the invertibility assumption requires $G_i^{-1} \ast G_i = 1$. Then

$$L = \sum_{i=1}^{n-1} G_{i+1}^{-1} \ast G_i E_{i+1,i}, \quad M = \sum_{i=1}^{n} M_i E_{ii}, \quad M_i = G_i^{-1} \ast (\partial_t + \partial_x)G_i.$$  (3.8)

Writing

$$G_i = e^{q_i}(1 + \theta \tilde{q}_i) + \mathcal{O}(\theta^2)$$  (3.9)

we have $G_i^{-1} = e^{-q_i}(1 - \theta \tilde{q}_i) + \mathcal{O}(\theta^2)$ and it follows from (3.6) that the functions $q_i$ have to solve the Toda field theory equations

$$(\partial_t^2 - \partial_x^2)q_i = e^{q_{i-1} - q_i} - e^{q_i - q_{i+1}} \quad i = 2, \ldots, n - 1$$

$$(\partial_t^2 - \partial_x^2)q_1 = -e^{q_1 - q_2}, \quad (\partial_t^2 - \partial_x^2)q_n = e^{q_{n-1} - q_n}.$$  (3.10)

Furthermore, the functions $\tilde{q}_i$ are subject to the following linear equations,

$$(\partial_t^2 - \partial_x^2)\tilde{q}_1 = \{\partial_t q_1, \partial_x q_1\} - e^{q_1 - q_2}(\tilde{q}_1 - \tilde{q}_2)$$

$$(\partial_t^2 - \partial_x^2)\tilde{q}_i = \{\partial_t q_i, \partial_x q_i\} + e^{q_{i-1} - q_i}(\tilde{q}_{i-1} - \tilde{q}_i) - e^{q_i - q_{i+1}}(\tilde{q}_i - \tilde{q}_{i+1})$$

$$(\partial_t^2 - \partial_x^2)\tilde{q}_n = \{\partial_t q_n, \partial_x q_n\} + e^{q_{n-1} - q_n}(\tilde{q}_{n-1} - \tilde{q}_n).$$  (3.11)

A 1-form $J = P \xi^1 + R \xi^2$ is $\delta$-closed iff $(\partial_t - \partial_x)R = (S - I)P$. For $J = \lambda D \chi$ (cf (2.1)) we have $P = -\lambda L \ast \chi$ and $R = \lambda (\partial_t + \partial_x + M \ast \chi)$ and thus

$$\partial_t[\lambda (\partial_t + M \ast \chi)] = \partial_x[\lambda (\partial_t + M \ast \chi)] + P_{i+1} - P_i$$  (3.12)

$(i = 1, \ldots, n)$ where we have to set $P_{n+1} = 0$. Using $P_1 = 0$, we find that

$$Q = \lambda \int dx \sum_{i=1}^{n} (\partial_t \chi_i + G_i^{-1} \ast (\partial_t + \partial_x)G_i) \ast \chi_i$$  (3.13)

is conserved, i.e., $dQ/dt = 0$, provided that the expressions $\partial_x \chi_i + M \ast \chi_i$ vanish at $x = \pm \infty$. In order to further evaluate this expression, we have to explore the linear system associated with the bicomplex. Choosing $\chi^{(0)} = \sum_{i=1}^{n} e_i$, where $e_i$ is the vector with components $(e_i)_j = \delta_{ij}$, we have $\delta \chi^{(0)} = -e_n \xi^2$ and $D \delta \chi^{(0)} = 0$. Now we find

$$Q^{(1)} = \int dx \sum_{i=1}^{n} M_i = \int dx \sum_{i=1}^{n} G_i^{-1} \ast (\partial_t + \partial_x)G_i$$

$$= \int dx \sum_{i=1}^{n} \partial_t q_i + \theta \int dx \sum_{i=1}^{n} \left( (\partial_t + \partial_x)\tilde{q}_i + \frac{1}{2} \{\partial_t + \partial_x)q_i, q_i\} \right) + \mathcal{O}(\theta^2)$$  (3.14)
where we assumed that the $q_i$ vanish at $x = \pm \infty$. The linear system $\delta(\chi - \chi^{(0)}) = \lambda D\chi$ reads

\[
(\partial_t - \partial_x)\chi_1 = 0, \quad (\partial_t - \partial_x)\chi_i = -\lambda G_i^{-1} * G_{i-1} * \chi_{i-1} \quad i = 2, \ldots, n
\]  

(3.15)

and

\[
\begin{align*}
\chi_{i+1} - \chi_i &= \lambda (\partial_t + \partial_x + M_i)\chi_i \quad i = 1, \ldots, n-1 \\
\chi_n &= 1 - \lambda (\partial_t + \partial_x + M_n)\chi_n.
\end{align*}
\]  

(3.16)

(3.17)

Using $\chi_i^{(0)} = 1$, (3.16) yields $\chi_i^{(1)} = M_i$ and thus $\chi_i^{(1)} = -\sum_{k=i}^{n} M_k$. After some manipulations and using (3.10), we obtain

\[
Q^{(2)} = -\int dx \sum_{i=1}^{n} \sum_{k=i}^{n} \left( \partial_t M_k + M_i * M_k \right)
\]

\[
= -\int dx \left( \sum_{i=1}^{n-1} e^{q_i - q_{i+1}} + \frac{1}{2} \left( \sum_{i=1}^{n} (\partial_t + \partial_x)q_i \right)^2 - \frac{1}{2} \sum_{i=1}^{n} [(\partial_t + \partial_x)q_i]^2 \right) + O(\theta)
\]  

(3.18)

where to first order in $\theta$ already a rather complicated expression emerges. At 0th order in $\theta$ one recovers the known conserved charges of the Toda theory.

Infinite-dimensional integrable models possess an infinite set of conserved currents. In contrast to previous approaches to deformations of integrable models (see [6], for example), our approach guarantees, via deformation of the bicomplex associated with an integrable model, that this infinite tower of conservation laws survives the deformation.

References

[1] Faddeev L D and Takhtajan L A, *Hamiltonian Methods in the Theory of Solitons* (Berlin: Springer, 1987).

[2] Dimakis A and Müller-Hoissen F, Soliton equations and the zero curvature condition in noncommutative geometry *J. Phys. A: Math. Gen.* 29 (1996) 7279-7286; Integrable discretizations of chiral models via deformation of the differential calculus *J. Phys. A: Math. Gen.* 29 (1996) 5007-5018; Noncommutative geometry and integrable models *Lett. Math. Phys.* 39 (1997) 69-79; Noncommutative geometry and a class of completely integrable models *Czech. J. Phys.* 48 (1998) 1319-1324.

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2 A “constant of integration” can be added on the rhs. But this would simply lead to an additional term proportional to $Q^{(1)}$ in (3.18).

3 Deforming a Hamiltonian system which is (Liouville) integrable so that the conserved charges are in involution with respect to a symplectic structure, the question arises whether there is a corresponding deformation of the symplectic and Hamiltonian structure such that the deformation preserves the involution property.
[3] Dimakis A and Müller-Hoissen F, Bi-differential calculi and integrable models. *J. Phys. A* **33** (2000) 957-974.

[4] Dimakis A and Müller-Hoissen F, Bi-differential calculus and the KdV equation, *math-ph/9908016*, to appear in *Rep. Math. Phys.*; Bicomplexes and finite Toda lattices, *solv-int/9911006*, to appear in *Quantum Theory and Symmetries*, eds. H.-D. Doebner, V.K. Dobrev, J.-D. Hennig and W. Lücke (Singapore: World Scientific); Bicomplexes and integrable models, in preparation.

[5] Seiberg N and Witten E, String theory and noncommutative geometry, *hep-th/9908142*, *JHEP* **09** (2000) 032.

[6] Strachan I A B, A geometry for multidimensional integrable systems. *J. Geom. Phys.* **21** (1997) 255-278; García-Compeán H, Plebański J F and Przanowski M, Geometry associated with self-dual Yang-Mills and the chiral model approaches to self-dual gravity, *Acta Phys. Polon. B* **29** (1998) 549-571; Takasaki K, Anti-self-dual Yang-Mills equations on noncommutative spacetime, *hep-th/0005194*.