Abstract

Let $V$ be a finite set of points in the plane. We present a 2-local algorithm that constructs a plane $\frac{4\sqrt{3}}{9}$-spanner of the unit-disk graph $UDG(V)$. This algorithm makes only one round of communication and each point of $V$ broadcasts at most 5 messages. This improves the previously best message-bound of 11 by Araújo and Rodrigues (Fast localized Delaunay triangulation, Lecture Notes in Computer Science, volume 3544, 2004).

1 Introduction

A wireless ad hoc network consists of a finite set $V$ of wireless nodes. Each node $u$ in $V$ is a point in the plane that can communicate directly with all points of $V$ within $u$’s communication range. If this range is one unit for each point, then the network is modeled by the unit-disk graph $UDG(V)$ of $V$. This (undirected) graph has $V$ as its vertex set and any two distinct vertices $u$ and $v$ are connected by an edge if and only if the Euclidean distance $|uv|$ between $u$ and $v$ is at most one unit.

In order for two points that are more than one unit apart to be able to communicate, the points of $V$ use a so-called local algorithm (to be defined below) to construct a subgraph $G$ of $UDG(V)$. This subgraph should have the property that it supports efficient routing of messages, i.e., there should be a simple and efficient protocol that allows any point of $V$ to send a message to any other point of $V$.

In this paper, we present a local algorithm that constructs a subgraph $G$ of $UDG(V)$ that satisfies the following properties:

1. Each point $u$ of $V$ stores a set $E(u)$ of edges that are incident on $u$. The edge set of $G$ is equal to $\bigcup_{u \in V} E(u)$.

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2. The edge sets $E(u)$ with $u \in V$ are consistent: For any two points $u$ and $v$ in $V$, $(u, v)$ is an edge in $E(u)$ if and only if $(u, v)$ is an edge in $E(v)$.

3. The graph $G$ is plane: If we consider each edge $(u, v)$ to be the straight-line segment joining $u$ and $v$, then no two edges of $G$ cross. The graph being plane is useful, because several algorithms are known for routing messages in a plane subgraph of $UDG(V)$; see, e.g., Bose et al. [3] and Karp and Kung [6].

4. The graph $G$ is a $t$-spanner of $UDG(V)$, for some constant $t > 1$: For each edge $(u, v)$ of $UDG(V)$, the graph $G$ contains a path between $u$ and $v$ whose Euclidean length is at most $t|uv|$. Observe that this implies that shortest-path distances in $UDG(V)$ are approximated, within a factor of $t$, by shortest-path distances in $G$. Thus, this property implies that the total distance traveled by a message, when using $G$, is not much larger than the minimum distance that needs to be traveled in $UDG(V)$.

1.1 Local Algorithms

As mentioned above, we model a wireless ad hoc network by the unit-disk graph $UDG(V)$, where $V$ is a finite set of points in the plane. The points of $V$ want to construct a communication graph $G$ (which is a subgraph of $UDG(V)$) using a distributed and local algorithm. In this section, we formalize this notion and introduce the complexity measures that we will use to analyze the efficiency of such algorithms.

The points of $V$ can communicate with each other by broadcasting messages. If a point $u$ of $V$ broadcasts a message, then each point of $V$ within Euclidean distance one from $u$ receives the message. Each point of $V$ can perform computations based on its location and all information received from other points. Informally, an algorithm is called local, if the computation performed at each point $u$ of $V$ is based only on its location and the locations of all points that are within distance $k$ (in $UDG(V)$) from $u$, for some small integer $k \geq 1$. Thus, in a local algorithm, information cannot “travel” over a “large” distance.

To define this notion formally, let $\delta_{UDG}(u, v)$ denote the Euclidean length of a shortest path between the points $u$ and $v$ in the graph $UDG(V)$. For any integer $k \geq 1$, let

$$N_k(u) = \{ v \in V : \delta_{UDG}(u, v) \leq k \}.$$  

Observe that $u \in N_k(u)$.

Let $\mathcal{A}(V)$ be a distributed algorithm that runs on a set $V$ of points in the plane, and let $\mathcal{A}(u; V)$ denote the computation performed by point $u$. As is common in this field, we assume that, at the start of the algorithm, each point $u$ of $V$ knows the locations (i.e., the $x$- and $y$-coordinates) of all points in $N_1(u)$. Thus, the set $N_1(u)$ can be considered to be the input for $u$.

For any point $u$ of $V$, we denote by $T_u(V)$ the trace of the computation performed by $\mathcal{A}(u; V)$. Thus, $T_u(V)$ contains the sequence of all computing and broadcasting operations performed by $\mathcal{A}(u; V)$ when each point $v$ of $V$ runs algorithm $\mathcal{A}(v; V)$.

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1Two edges are said to cross if they are not collinear and there exists a (unique) point that is in the relative interior of both edges.
Definition 1 For an integer \( k \geq 1 \), we say that \( \mathcal{A}(V) \) is a \( k \)-local algorithm, if for each point \( u \) of \( V \),
\[
T_u(V) = T_u(N_k(u)).
\]

In other words, for every point \( u \) of \( V \), the following holds: If we run the entire distributed algorithm \( \mathcal{A} \) with \( V \) replaced by \( N_k(u) \), then the computation performed by \( u \) does not change (even though the computations performed by other points may change).

A \( k \)-local algorithm runs in parallel on all points of \( V \), where each point \( u \) performs an alternating sequence of computation steps and broadcasting steps in a synchronized manner. In a computation step, point \( u \) performs some computation based on the subset of \( N_k(u) \) that is known to \( u \) at that moment. (For example, \( u \) may compute the Delaunay triangulation of this subset; we consider this to be one computation step.) In a broadcasting step, point \( u \) broadcasts a (possibly empty) sequence of messages, which is received by all points in \( N_1(u) \).

In this paper, a message is defined to be the location of a point in the plane (which need not be an element of \( V \)). The efficiency of a local algorithm will be expressed in terms of the following measures:

1. The value of \( k \). The smaller the value of \( k \), the “more local” the algorithm is.
2. The maximum number of messages that are broadcast by any point of \( V \). The goal is to minimize this number.
3. The number of communication rounds, which is defined to be the maximum number of broadcasting steps performed by any point in \( V \). This number measures the (parallel) time for the entire algorithm to complete its computation. Again, the goal is to minimize this number.

1.2 Previous Work

Above, we have defined the notion of a \( t \)-spanner of the unit-disk graph \( UDG(V) \). For a real number \( t > 1 \), a graph \( G \) is called a \( t \)-spanner of the point set \( V \) if for any two elements \( u \) and \( v \) of \( V \), there exists a path in \( G \) between \( u \) and \( v \) whose length is at most \( t|uv| \). The problem of constructing \( t \)-spanners for point sets has been studied intensively in computational geometry; see the book by Narasimhan and Smid [9] for a survey.

Since we are concerned with plane spanners of the unit-disk graph, our algorithm will be based on the Delaunay Triangulation \( DT(V) \) of \( V \); see, e.g., the textbook by de Berg et al. [4]. Recall that \( DT(V) \) is the plane graph with vertex set \( V \) in which any two distinct points \( u \) and \( v \) are connected by an edge if and only if there exists a disk \( D \) such that (i) \( u \) and \( v \) are the only points of \( V \) that are on the boundary of \( D \) and (ii) no point of \( V \) is in the interior of \( D \). Also, three points \( u \), \( v \), and \( w \) determine a triangular face of \( DT(V) \) if and only if the disk having \( u \), \( v \), and \( w \) on its boundary does not contain any point of \( V \) in its interior. Keil and Gutwin [7] have shown that \( DT(V) \) is a \( \frac{4\pi\sqrt{3}}{9} \)-spanner of \( V \). To extend this result to unit-disk graphs, it is natural to consider subgraphs of \( UDel(V) \), which is defined to be the intersection of the Delaunay triangulation and the unit-disk graph of \( V \). It has
been shown by Bose et al. [2] that UDel(V) is a \( \frac{4\pi\sqrt{3}}{9} \)-spanner of UDG(V). Unfortunately, constructing UDel(V) using a \( k \)-local algorithm, for any constant value of \( k \), is not possible: Consider an edge \((u, v)\) in UDel(V) whose empty disk \( D \) is very large. In order for a \( k \)-local algorithm to verify that no point of \( V \) is in the interior of \( D \), information about the points of \( V \) must travel over a large distance to \( u \) or \( v \). Clearly, this is possible only if the value of \( k \) is very large. Because of this, researchers have considered the problem of designing local algorithms that construct a plane subgraph of UDG(V) which is a supergraph of UDel(V). Obviously, by the result of [2], such a graph is also a \( \frac{4\pi\sqrt{3}}{9} \)-spanner of UDG(V).

Gao et al. [5] proposed a 2-local algorithm that constructs a plane subgraph of UDG(V) which is a supergraph of UDel(V). However, the number of messages broadcast by a single point of \( V \) can be as large as \( \Theta(n) \), where \( n \) is the number of elements of \( V \). This result was improved by Li et al. [8]: They presented a 2-local algorithm that constructs such a graph in four communication rounds and in which each point broadcasts at most 49 messages.

Currently, the best result for computing a plane \( t \)-spanner (for some constant \( t \)) of the unit-disk graph UDG(V) is by Araújo et al. [1]. They presented a 2-local algorithm which computes such a spanner in one communication round and in which each point broadcasts at most 11 messages.

1.3 Our Result

In this paper, we improve the upper bound of Araújo et al. [1] on the message complexity for each point of \( V \) from 11 to 5:

**Theorem 1** Let \( V \) be a finite set of points in the plane. There exists a 2-local algorithm that computes a plane and consistent \( \frac{4\pi\sqrt{3}}{9} \)-spanner of the unit-disk graph of \( V \). This algorithm makes one communication round and each point of \( V \) broadcasts at most 5 messages.

The rest of this paper is organized as follows. In Section 2 we present a preliminary 2-local algorithm that computes, in one communication round, a subgraph of UDG(V). In this algorithm, each point of \( V \) broadcasts at most 6 messages. We present a rigorous proof of the fact that the graph computed by this algorithm is a plane and consistent \( \frac{4\pi\sqrt{3}}{9} \)-spanner of UDG(V). In Section 3 we make a simple modification to the algorithm of Section 2 which reduces the message complexity for each point of \( V \) from 6 to 5. We then show that the new algorithm and the algorithm of Section 2 compute the same graph. Thus, this will prove Theorem 1. We conclude in Section 4 with some directions for future work.

Throughout the rest of this paper, we assume that the points in the set \( V \) are in general position (meaning that no three points of \( V \) are collinear and no four points of \( V \) are co-circular). We also assume that the unit-disk graph UDG(V) is connected. We will use the following notation:

- \( D(a, b, c) \) denotes the disk having the three points \( a, b, \) and \( c \) on its boundary.
- \( D(c; r) \) denotes the disk centered at the point \( c \) and having radius \( r \).
• \( \Delta(a, b, c) \) denotes the triangle having the three points \( a, b, \) and \( c \) as its vertices.

• \( \partial D \) denotes the boundary of the disk \( D \).

• \( \text{int}(D) \) denotes the interior of the disk \( D \).

• Let \( v, x, \) and \( y \) be points of \( V \), where \( v \neq y \). Assume there exists a disk \( D \) such that \( N_1(x) \cap \partial D = \{v, y\} \) and \( N_1(x) \cap \text{int}(D) = \emptyset \). We denote such a disk \( D \) by \( \text{Del}_x(v, y) \). Observe that \( \text{Del}_x(v, y) \) is a certificate for the fact that \( (v, y) \) is an edge in the Delaunay triangulation of the point set \( N_1(x) \).

2 A Preliminary Algorithm

In this section, we present a 2-local algorithm that constructs a graph, called the plane localized Delaunay graph \( \text{PLDG}(V) \), whose vertex set is a finite set \( V \) of points in the plane. The algorithm computes \( \text{PLDG}(V) \) in one communication round and each point of \( V \) broadcasts at most 6 messages. We will prove that \( \text{PLDG}(V) \) is a plane and consistent supergraph of \( \text{UDel}(V) \).

In the construction, each point \( v \) of \( V \) runs algorithm \( \text{PLDG}(v) \) in parallel. Let \( N_v = N_1(v) \), i.e., \( N_v = \{u \in V : |uv| \leq 1\} \). Recall that we assume that, at the start of the algorithm, point \( v \) knows the locations of all points in \( N_v \). Algorithm \( \text{PLDG}(v) \) first computes the Delaunay triangulation \( \text{LDT}(v) \) of the set \( N_v \). Then, for each triangular face \( \Delta(u, v, w) \) in \( \text{LDT}(v) \) for which \( \angle uvw > \frac{\pi}{3} \), algorithm \( \text{PLDG}(v) \) broadcasts the location \( v \) together with the center of the disk \( D(u, v, w) \) containing \( u, v, \) and \( w \) on its boundary.

In the final step, algorithm \( \text{PLDG}(v) \) checks the validity of all edges that are incident on \( v \) in \( \text{LDT}(v) \) and removes those edges which cause a crossing. To be more precise, let \( x \) be a point in \( N_v \), and assume that \( v \) receives a center \( c'_i \) from \( x \). Algorithm \( \text{PLDG}(v) \) considers the unit-disk \( D(v; 1) \) centered at \( v \) and the disk \( D(c'_i; |c'_i x|) \) centered at \( c'_i \) that contains \( x \) on its boundary. The algorithm knows that \( \partial D(c'_i; |c'_i x|) \) contains exactly three points which define a triangular face in the Delaunay triangulation \( \text{LDT}(x) \) of \( N_x \). Point \( x \) is one of these three points; let \( p \) and \( q \) be the other two points. Assume that the set \( N_v \) contains exactly two points of \( \{x, p, q\} \), say \( x \) and \( p \). Thus, algorithm \( \text{PLDG}(v) \) knows the points \( x \) and \( p \), but it does not know \( q \). The algorithm computes \( \text{arc}_i \), which is defined to be the (open) portion of \( \partial D(c'_i; |c'_i x|) \) which is not contained in \( D(v; 1) \). Even though the algorithm does not know the exact location of the third point \( q \), it does know that \( q \) is on \( \text{arc}_i \). The algorithm chooses an arbitrary point \( z' \) on \( \text{arc}_i \) such that \( |xz'| \leq 1 \) or \( |pz'| \leq 1 \) and acts as if \( \Delta(x, p, z') \) is a triangular face in \( \text{LDT}(x) \). (Observe that, since \( q \in \text{arc}_i \) and \( |xq| \leq 1 \), the algorithm can choose such a point \( z' \). Also, \( z' \) is not necessarily a point of \( V \).)

The algorithm now considers each edge \( (v, y) \) in \( \text{LDT}(v) \) (where, possibly, \( v = p \), \( y = p \), or \( y = x \)) and uses the triangle \( \Delta(x, p, z') \) to decide whether or not to remove \( (v, y) \): Since \( (v, y) \) is an edge in \( \text{LDT}(v) \), algorithm \( \text{PLDG}(v) \) can compute a disk \( D = \text{Del}_v(v, y) \) such that (i) \( v \) and \( y \) are the only points of \( N_v \) that are on the boundary of \( D \) and (ii) the interior of \( D \) does not contain any point of \( N_v \). If \( \text{arc}_i \) is fully contained in the interior of \( \text{Del}_v(v, y) \), then
Algorithm PLDG($v$)
1. let $N_v = \{ u \in V : |uv| \leq 1 \}$;
2. compute the Delaunay triangulation $LDT(v)$ of $N_v$;
3. let $E(v)$ be the set of all edges in $LDT(v)$ that are incident on $v$;
4. let $\Delta_v$ be the set of all triangular faces $\Delta(u,v,w)$ in $LDT(v)$ for which $\angle uvw > \frac{\pi}{3}$;
5. let $k$ be the number of elements in $\Delta_v$;
6. if $k \geq 1$
7. then let $c_1, \ldots, c_k$ be the centers of the circumcircles of all triangles in $\Delta_v$;
8. broadcast the sequence $(v, c_1, \ldots, c_k)$;
9. for each sequence $(x, c'_1, \ldots, c'_m)$ received
10. do for $i = 1$ to $m$
11. do let $D(c'_i; |c'_i x|)$ be the disk with center $c'_i$ that contains $x$ on its boundary;
12. if $\partial D(c'_i; |c'_i x|)$ contains exactly two points of $N_v$
13. then let $p$ be the point in $(N_v \setminus \{x\}) \cap \partial D(c'_i; |c'_i x|)$;
14. let $arc_i$ be the (open) arc on $\partial D(c'_i; |c'_i x|)$ that is not contained in the unit-disk $D(v; 1)$ centered at $v$;
15. let $z'$ be an arbitrary point on $arc_i$ with $|xz'| \leq 1$ or $|pz'| \leq 1$;
16. for each edge $(v, y)$ in $E(v)$
17. do let $\text{Del}_v(v, y)$ be a disk $D$ such that $N_v \cap \partial D = \{v, y\}$ and $N_v \cap int(D) = \emptyset$;
18. if $arc_i$ is contained in the interior of $\text{Del}_v(v, y)$ and the line segment $vy$ crosses at least one of the line segments $xz'$ and $pz'$
19. then remove $(v, y)$ from $E(v)$

Figure 1: The plane localized Delaunay graph algorithm.

the algorithm knows that $q$ is contained in the interior of $\text{Del}_v(v, y)$ (even though it does not know the exact location of $q$) and, therefore, $\text{Del}_v(v, y)$ is not a certificate that $(v, y)$ is an edge in the Delaunay triangulation of the entire set $V$. The algorithm now checks if the line segment $vy$ crosses any of the two line segments $xz'$ and $pz'$ and, if so, removes the edge $(v, y)$. Observe that if $(v, y)$ is not an edge of the Delaunay triangulation $DT(V)$, the algorithm still keeps it as long as it does not cross any other edge.

The formal algorithm is given in Figure 1. An illustration, with the special cases when $v = p$, $y = p$, or $y = x$, is given in Figure 2.

Running algorithm PLDG($v$) for all points $v$ of $V$ in parallel will be referred to as running algorithm PLDG($V$). We denote by $E(v)$ the edge set that is computed by algorithm PLDG($v$). Observe that each edge in $E(v)$ is incident on the point $v$. Let $E = \cup_{v \in V} E(v)$ and let PLDG($V$) denote the graph with vertex set $V$ and edge set $E$.

In the rest of this section, we will prove a sequence of lemmas which lead to the proof that PLDG($V$) is a plane and consistent supergraph of $UDel(V)$; see Lemmas 5, 7 and 8.
Figure 2: Illustrating algorithm PLDG(v).
We start with a simple, but fundamental lemma:

Lemma 1 Let $S = \{u, v, w, z\}$ be a set of four points in the plane in general position, such that $|uv| \leq 1$, $|wz| \leq 1$, and the line segments $uw$ and $wz$ cross. Then there exists a point $x$ in $S$ such that $|xy| \leq 1$ for all $y$ in $S$.

Proof. Let $c$ be the intersection of the line segments $uv$ and $wz$; see Figure 3. By the triangle inequality, we have $|uw| \leq |uc| + |cw|$ and $|wz| \leq |wc| + |cz|$. Since $|uv| = |uc| + |cv| \leq 1$ and $|wz| = |wc| + |cz| \leq 1$, we have $|uw| + |vz| \leq |uv| + |wz| \leq 2$. Therefore, at least one of $uw$ and $vz$ has length at most 1. Without loss of generality, assume that $|uw| \leq 1$. By a symmetric argument, at least one of $uz$ and $vw$ has length at most 1. Without loss of generality, assume that $|uz| \leq 1$. Then all distances $|uv|$, $|uw|$, and $|uz|$ are at most 1 and, thus, we can take $x$ to be the point $u$.

The following lemma implies that for every edge $(v, y)$ in $E(v)$, the edge $(v, y)$ is in the Delaunay triangulation $\text{LDT}(y)$ of the set $N_y$.

Lemma 2 Let $v$ and $y$ be two distinct points of $V$ and assume that $(v, y)$ is not an edge in $LDT(y)$. Then, after algorithm $\text{PLDG}(V)$ has terminated, $(v, y)$ is not an edge in $E(v)$.

Proof. First assume that $(v, y)$ is not an edge in $LDT(v)$. Then, since $E(v)$ is a subset of the edge set of $LDT(v)$, $(v, y)$ is not an edge in $E(v)$.

From now on, we assume that $(v, y)$ is an edge in $LDT(v)$. Observe that $|vy| \leq 1$. Since $(v, y)$ is not an edge in $LDT(y)$, there exist two points $p$ and $q$ in $V$ such that the triangle $\Delta(y, p, q)$ is a triangular face in $LDT(y)$ and $vy$ crosses $pq$; see Figure 4. Observe that the points $p$, $q$, $v$, and $y$ are pairwise distinct. In the rest of the proof, we will do the following:

1. We first show that algorithm $\text{PLDG}(y)$ broadcasts the center of the circumcircle of $\Delta(y, p, q)$. Since $|vy| \leq 1$, $v$ will receive this center.

2. We then show that, when algorithm $\text{PLDG}(v)$ considers the center of $\Delta(y, p, q)$, it deletes the edge $(v, y)$. As a result, the edge $(v, y)$ is not in $E(v)$.
Let \( c'_i \) be the center of the circumcircle of \( \Delta(y, p, q) \) and consider the corresponding disk \( D(c'_i; |c'_iy|) \), i.e., the disk with center \( c'_i \) that contains \( y, p, \) and \( q \) on its boundary. Recall that \( D(v; 1) \) denotes the unit-disk centered at \( v \) and \( N_v = \{ u \in V : |uv| \leq 1 \} \).

Since \(|vy| \leq 1\) and \( \Delta(y, p, q) \) is a triangular face in \( LDT(y) \), \( v \) is not contained in \( D(c'_i; |c'_iy|) \). Since \( vy \) crosses \( pq \), this implies that any disk \( D \) with \( v \) and \( y \) on its boundary contains at least one of \( p \) and \( q \) (otherwise, \( \partial D \) and \( \partial D(c'_i; |c'_iy|) \) intersect more than twice).

We first claim that \( \partial D(c'_i; |c'_iy|) \) contains exactly two points of \( N_v \). Since \( y \in N_v \), this means that we claim that exactly one of \( p \) and \( q \) is in \( N_v \). We prove this by contradiction. First assume that neither \( p \) nor \( q \) is in \( N_v \). Then both \( p \) and \( q \) are outside \( D(v; 1) \). Let \( D \) be the disk with diameter \( vy \). Since \(|vy| \leq 1\), \( D \) is contained in \( D(v; 1) \). Thus, neither \( p \) nor \( q \) is contained in \( D \), which is a contradiction, because \( D \) contains \( v \) and \( y \) on its boundary. Now assume that both \( p \) and \( q \) are in \( N_v \). Then, since any disk with \( v \) and \( y \) on its boundary contains one of \( p \) and \( q \), it follows that \((v, y)\) is not an edge in \( LDT(y) \), which is again a contradiction.

Thus, we have shown that \( \partial D(c'_i; |c'_iy|) \) contains exactly two points of \( N_v \). We may assume without loss of generality that \( y, p \in N_v \) and \( q \notin N_v \).

Consider the triangle \( \Delta(v, y, q) \). Since \(|yv| \leq 1, |yq| \leq 1, \) and \(|vq| > 1\), we have \( \angle vyq > \frac{\pi}{3} \). Since \( \angle pyq > \angle vyq \), it follows that \( \angle pyq > \frac{\pi}{3} \). Since \( \Delta(y, p, q) \) is a triangular face in \( LDT(y) \), algorithm PLDG(y) broadcasts a sequence in line 8 which contains the center \( c'_i \) of \( D(c'_i; |c'_iy|) \).

As we have mentioned above, since \(|vy| \leq 1\), \( v \) receives the sequence broadcast by PLDG(y). This sequence contains the center \( c'_i \) together with the point \( y \). When algorithm PLDG(v) considers \( c'_i \), it discovers that \( \partial D(c'_i; |c'_iy|) \) contains exactly two points of \( N_v \); as we have seen above, these points are \( y \) and \( p \). Thus, the condition in line 12 is
satisfied. In line 14, algorithm PLDG($v$) computes the open arc $arc_{i}$, which is the part of $\partial D(c'_i;|c'_i|)$ that is not contained in $D(v;1)$. Observe that even though PLDG($v$) does not know the location of the point $q$, the algorithm knows that it is on $arc_{i}$. Let $Del_{v}(v,y)$ be the disk that is computed by PLDG($v$) in line 17. This disk has the properties that $N_v \cap \partial Del_{v}(v,y) = \{v,y\}$ and $N_v \cap \text{int}(Del_{v}(v,y)) = \emptyset$.

We show that $arc_{i}$ is contained in the interior of $Del_{v}(v,y)$; thus, the first condition in line 18 is satisfied. Let $w$ be the intersection between $vy$ and $\partial D(c'_i;|c'_i|)$, let $\hat{w}p\hat{y}$ be the arc on $\partial D(c'_i;|c'_i|)$ with endpoints $w$ and $y$ and which contains $p$, and let $\hat{y}qw$ be the arc on $\partial D(c'_i;|c'_i|)$ with endpoints $y$ and $w$ and which contains $q$. Since $|vy| \leq 1$, $|vq| > 1$, and $q \in \hat{y}qw$, we have $\hat{w}p\hat{y} \subseteq D(v;1)$ (because otherwise, $\partial D(v;1)$ and $\partial D(c'_i;|c'_i|)$ intersect more than twice). It follows that $arc_{i} \subseteq \hat{y}qw$. Since $|vp| \leq 1$, we have $p \notin Del_{v}(v,y)$. Therefore, $\partial Del_{v}(v,y)$ and $\hat{w}p\hat{y}$ intersect twice. Since $\partial Del_{v}(v,y)$ and $\partial D(c'_i;|c'_i|)$ cannot intersect more than twice, it follows that $arc_{i}$ is contained in the interior of $Del_{v}(v,y)$.

Consider the point $z'$ on $arc_{i}$ that is chosen in line 15 of algorithm PLDG($v$). We will show that $vy$ crosses $pz'$; thus, the second condition in line 18 is also satisfied. Assume, by contradiction, that $vy$ does not cross $pz'$. Since the line through $v$ and $y$ separates $p$ from the two points $q$ and $z'$, and since $vy$ crosses $pq$, it follows that $y$ or $v$ is in the triangle $\Delta(p,q,z')$. However, since $y$, $p$, $q$, and $z'$ are on the circle $\partial D(c'_i;|c'_i|)$, $y$ cannot be in $\Delta(p,q,z')$. Also, since $\Delta(p,q,z')$ is contained in $D(c'_i;|c'_i|)$ and since $v \in N_y$, $v$ cannot be in $\Delta(p,q,z')$, because otherwise, $\Delta(y,p,q)$ would not be a triangular face in $LDT(y)$. Thus, we have shown that $vy$ crosses $pz'$.

By inspecting algorithm PLDG($v$), it follows that it removes, in line 19, the edge $(v,y)$ from the edge set $E(v)$. This completes the proof. 

For the following geometric lemma, refer to Figure 5.

**Lemma 3** Let $p$ and $q$ be two points with $|pq| \leq 1$, let $D$ be a disk containing $p$ and $q$ on its boundary, and let $D_{cap}$ be the part of $D$ that is bounded by the line segment $pq$ and the minor arc $\hat{p}q$ on $\partial D$ between $p$ and $q$. Then $|xy| \leq 1$ for all $x$ and $y$ in $D_{cap}$.

**Proof.** Consider the disk $D' = D(\frac{p+q}{2};\frac{|pq|}{2})$ with diameter $pq$. Since $\hat{p}q$ is the minor arc on $\partial D$ between $p$ and $q$, $D_{cap}$ is completely contained in $D'$. Therefore, if $x$ and $y$ are points in $D_{cap}$, then these points are contained in $D'$. Since the diameter of $D'$ is at most 1, it follows that $|xy| \leq 1$.

The next lemma will form the basis for our claim that the graph $PLDG(V)$ is plane.

**Lemma 4** Let $x, q, v$, and $y$ be four pairwise distinct points of $V$. Assume that $|xq| \leq 1$, $|vx| \leq 1$, $|xy| \leq 1$, $|vy| \leq 1$, $xq$ crosses $vy$, $(x,q)$ is an edge in $LDT(x)$, and $(v,y)$ is an edge in $LDT(y)$. Then, after algorithm PLDG($V$) has terminated, $(v,y)$ is not an edge in $E(y)$.

**Proof.** If $(v,y)$ is not an edge in $LDT(v)$, then the claim follows from Lemma 2. In the rest of the proof, we assume that $(v,y)$ is an edge in $LDT(v)$. We have to show that algorithm
PLDG($y$) removes the edge $(v, y)$ from $E(y)$. Thus, we have to show that there exists a point $x'$ in $N_y$ which broadcasts the center of the circumcircle of some triangular face in $LDT(x')$ and, based on this information, PLDG($y$) removes $(v, y)$. We will use the edge $(x, q)$ to prove that such a point $x'$ exists. We assume, without loss of generality, that $vy$ is horizontal and $v$ is to the right of $y$. For each $x' \in V \setminus \{v, y\}$, let $Q_{vy}(x') = \{q' \in V \setminus \{v, y\} : (x', q') \text{ is an edge in } LDT(x') \text{ and } x'q' \text{ crosses } vy\}$.

We define $X_{vy} = \{x' \in V \setminus \{v, y\} : |x'y| \leq 1, |x'v| \leq 1, Q_{vy}(x') \neq \emptyset\}$. Since $q \in Q_{vy}(x)$, we have $Q_{vy}(x) \neq \emptyset$. Since $|xy| \leq 1$ and $|xv| \leq 1$, we have $x \in X_{vy}$ and, therefore, $X_{vy} \neq \emptyset$.

Let $x'$ be the leftmost point in $X_{vy}$. Let $q'$ be the point in $Q_{vy}(x')$ such that the intersection between $x'q'$ and $vy$ is closest to $y$. We assume, without loss of generality, that $x'$ is above the line through $vy$. Since $x'q'$ crosses $vy$, the point $q'$ is below the line through $vy$. Observe that $x'$, $q'$, $v$, and $y$ are pairwise distinct.

By definition, $(x', q')$ is an edge in $LDT(x')$. Let $p'$ be the point of $V$ such that $\Delta(x', p', q')$ is a triangular face in $LDT(x')$ and $p'$ is to the left of the directed line from $q'$ to $x'$; refer to Figure 6. Since $y \in N_{x'}$ and $y$ is to the left of this line, the point $p'$ exists. Observe that $p'$ may be equal to $y$.

The following two facts imply that (i) $p'$ is not below the line through $vy$, and (ii) in the case when $p' \neq y$, $p'q'$ crosses $vy$: First, since $y \in N_{x'}$ and $\Delta(x', p', q')$ is a triangular face in $LDT(x')$, the point $y$ cannot be in $\Delta(x', p', q')$. Second, by our choice of $q'$, the line segments $x'p'$ and $vy$ do not cross.

In the rest of the proof, we will prove the following two claims:
1. Algorithm PLDG\((x')\) broadcasts the center of the circumcircle of \(\Delta(x', p', q')\). Since \(|x'y| \leq 1\), \(y\) will receive this center.

2. When algorithm PLDG\((y)\) considers the center of the circumcircle of \(\Delta(x', p', q')\), it deletes the edge \((v, y)\). As a result, the edge \((v, y)\) is not in \(E(y)\).

Let \(c'_i\) be the center of the circumcircle of \(\Delta(x', p', q')\) and consider the corresponding disk \(D(c'_i; |c'_ix'|)\). Recall that \(D(x'; 1)\) denotes the unit-disk centered at \(x'\).

Since \(|x'y| \leq 1\), \(|x'v| \leq 1\), and \(\Delta(x', p', q')\) is a triangular face in \(LDT(x')\), neither \(v\) nor \(y\) is contained in the interior of \(D(c'_i; |c'_ix'|)\). Moreover, since \(v \notin \{x', p', q'\}\), \(v\) is not contained in \(\partial D(c'_i; |c'_ix'|)\). Finally, in the case when \(y \neq p'\), \(y\) is not contained in \(\partial D(c'_i; |c'_ix'|)\). Since \(vy\) crosses \(x'q'\), it follows that any disk \(D\) with \(v\) and \(y\) on its boundary contains at least one of \(x'\) and \(q'\) (because otherwise, \(\partial D\) and \(\partial D(c'_i; |c'_ix'|)\) intersect more than twice).

We now show that \(|yq'| > 1\). Assume, by contradiction, that \(|yq'| \leq 1\). Since \((v, y)\) is an edge in \(LDT(y)\), there exists a disk \(Del_y(v, y)\) having the property that \(N_y \cap \partial Del_y(v, y) = \{v, y\}\) and \(N_y \cap int(Del_y(v, y)) = \emptyset\). Since both \(x'\) and \(q'\) are in \(N_y\), neither of these two points is contained in \(Del_y(v, y)\), which is a contradiction. Since \((v, y)\) is an edge in \(LDT(v)\), a symmetric argument implies that \(|vq'| > 1\).

Consider the triangle \(\Delta(v, y, q')\). Since \(|vq'| > 1\), \(|yq'| > 1\), and \(|vy| \leq 1\), we have \(\angle vq'y < \frac{\pi}{3}\). Since \(\angle x'q'p' \leq \angle vq'y\) it follows that \(\angle x'q'p' < \frac{\pi}{3}\). Next, consider the triangle \(\Delta(x', p', q')\). Since \(\angle x'p'q' + \angle p'x'q' > \frac{2\pi}{3}\), at least one of \(\angle x'p'q'\) and \(\angle p'x'q'\) is larger than
Below, we will prove that $\angle p'x'q' > \frac{\pi}{3}$. Since $\Delta(x', p', q')$ is a triangular face in $LDT(x')$, this will imply that algorithm PLDG($x'$) broadcasts a sequence in line 8 which contains the center $c_i'$ of $D(c_i'; |c_i'x'|)$.

Assume, by contradiction, that $\angle p'x'q' \leq \frac{\pi}{3}$. Then $\angle x'p'q' > \frac{\pi}{3}$. Since $|x'p'| \leq 1$, $|x'q'| \leq 1$ and $\angle p'x'q' \leq \frac{\pi}{3}$, we have $|p'q'| \leq 1$. We first prove, again by contradiction, that $\Delta(x', p', q')$ is not a triangular face in $LDT(p')$. Thus, we assume that it is a triangular face in $LDT(p')$. Since $vy$ crosses $x'q'$, and $(v, y)$ is an edge in $LDT(y)$, we have $p' \neq y$ (because otherwise, $LDT(p')$ would not be plane). Refering again to Figure \(\square\) let $u_1$ and $u_2$ be the two intersection points between $D(c_i'; |c_i'x'|)$ and $vy$, where $u_1$ is to the left of $u_2$. We have $\angle u_1q'u_2 \leq \angle vy < \frac{\pi}{3}$. Consider the arc $u_1x'u_2$ on $\partial D(c_i'; |c_i'x'|)$ with endpoints $u_1$ and $u_2$ that contains $x'$. This arc is a minor arc on $\partial D(c_i'; |c_i'x'|)$. Since $p' \in u_1x'u_2$ and $p'$ is to the left of the line through $x'q'$, it follows that $p'$ is to the left of $x'$. Then, by our choice of $x'$, we have $p' \notin X_{vy}$. Thus, by the definition of $X_{vy}$, we have (i) $|p'y| > 1$ or (ii) $|p'v| > 1$ or (iii) $Q_{vy}(p') = \emptyset$. Since $q' \in Q_{vy}(p')$, (iii) does not hold. Since $vy$ and $p'q'$ cross, $|vy| \leq 1$, $|p'q'| \leq 1$, $|yq'| > 1$, and $|vq'| > 1$, it follows from Lemma \(\square\) that $|p'y| \leq 1$ and $|p'v| \leq 1$; thus, neither (i) nor (ii) holds, which is a contradiction. We conclude that $\Delta(x', p', q')$ is not a triangular face in $LDT(p')$.

We continue deriving a contradiction to the assumption that $\angle p'x'q' \leq \frac{\pi}{3}$. Since $\Delta(x', p', q')$ is a triangular face in $LDT(x')$ but not in $LDT(p')$, there exists at least one point $w$ of $V$ in the interior of $D(c_i'; |c_i'x'|)$ such that $|x'w| > 1$ and $|p'w| \leq 1$. Let $W$ be the set of all such points $w$, i.e.,

$$W = \{w \in V : w \in int(D(c_i'; |c_i'x'|)), |x'w| > 1, |p'w| \leq 1\}.$$

For each $w \in W$, let $R_w$ be the radius of the circle through $p'$ and $w$ and whose center is on $p'c_i'$. Let $w$ be a point in $W$ for which $R_w$ is minimum. Let $D_w$ be the disk centered on $p'c_i'$ that contains $p'$ and $w$ on its boundary. Observe that $D_w$ is contained in $D(c_i'; |c_i'x'|)$. Also, no point of $W$ is in the interior of $D_w$. It follows that $(p', w)$ is an edge in $LDT(p')$.

We have seen above that $p'$ is to the left of $x'$. It follows that $p' \notin X_{vy}$. Thus, by the definition of $X_{vy}$, we have (i) $|p'y| > 1$, or (ii) $|p'v| > 1$, or (iii) $Q_{vy}(p') = \emptyset$, or (iv) $p' = y$. The arguments above show that neither (i) nor (ii) holds. Assume that (iv) does not hold, i.e., $p' \neq y$. We show that $w \in Q_{vy}(p')$; this will imply that (iii) does not hold. Since $w \in int(D(c_i'; |c_i'x'|))$, we have $w \neq v$ and $w \neq y$. Thus, in order to show that $w \in Q_{vy}(p')$, it suffices to show that $p'w$ crosses $vy$. Consider again the two intersection points $u_1$ and $u_2$ between $D(c_i'; |c_i'x'|)$ and $vy$, where $u_1$ is to the left of $u_2$. As we have seen before, the arc $u_1x'u_2$ on $\partial D(c_i'; |c_i'x'|)$ is a minor arc. Since $|u_1u_2| \leq |vy| \leq 1$, it follows from Lemma \(\square\) that $w$ is below the line through $v$ and $y$ (because otherwise, we would have $|wx'| \leq 1$, contradicting the fact that $w \in W$). Thus, since $p'$ and $w$ are on opposite sides of the line through $v$ and $y$, and since $w \in D(c_i'; |c_i'x'|)$, this shows that $p'w$ crosses $vy$. As mentioned above, this implies that (iii) does not hold. We conclude that (iv) holds, i.e., $p' = y$. In the triangle $\Delta(p', x', q')$, we have $|p'x'| \leq 1$, $|x'q'| \leq 1$, and $|p'q'| = |yq'| > 1$. It follows that $\angle p'x'q' > \frac{\pi}{3}$, which is a contradiction.

Thus, we have obtained a contradiction to the assumption that $\angle p'x'q' \leq \frac{\pi}{3}$. As a
result, we conclude that \( \angle p'x'q' > \frac{\pi}{3} \). As we mentioned before, this implies that algorithm PLDG\((x')\) broadcasts a sequence in line \(8\) which contains the center \(c'_i\) of \(D(c'_i; |c'_ix'|)\) (which is the circumscribing disk of the triangular face \(\Delta(x', p', q')\) in \(LDT(x')\)).

Since \( |yx'| \leq 1 \), \( y \) receives the sequence broadcast by algorithm PLDG\((x')\). This sequence contains the center \(c'_i\) together with the point \(x'\). Recall that \( |yq'| > 1 \). Let \( D \) be the disk whose boundary contains \( y, v \), and the “north pole” of \( D(c'_i; |c'_ix'|) \). Since \( \angle vq'y < \frac{\pi}{3} \), the center of \( D \) is below the line through \( v \) and \( y \). Since \( |vy| \leq 1 \), it then follows from Lemma 3 that \( |yp'| \leq 1 \). Thus, when algorithm PLDG\((y)\) considers \(c'_i\), it discovers that the boundary of the disk \(D(c'_i; |c'_ix'|)\) contains exactly two points of \( N_y \); these are the points \(x'\) and \(p'\). Algorithm PLDG\((y)\) computes the open arc \(arc_i\), which is the part of \(\partial D(c'_i; |c'_ix'|)\) that is not contained in the unit-disk \(D(y; 1)\) centered at \(y\). The algorithm knows that the third point \(q'\) on \(\partial D(c'_i; |c'_ix'|)\) is somewhere on \(arc_i\). Let \(Del_y(v, y)\) be the disk that is computed in line \(17\) of algorithm PLDG\((y)\). This disk has the properties that \(N_y \cap \partial Del_y(v, y) = \{v, y\}\) and \(N_y \cap \text{int}(Del_y(v, y)) = \emptyset\). By the same argument as in the proof of Lemma 2, \(arc_i\) is contained in the interior of \(Del_y(v, y)\). Moreover, \(arc_i\) is below the line through \(v\) and \(y\).

Consider the point \(z'\) on \(arc_i\) that is chosen in line \(15\) of algorithm PLDG\((y)\). We will show that \(vy\) crosses \(x'z'\). Assume, by contradiction, that \(vy\) does not cross \(x'z'\). Since the line through \(v\) and \(y\) separates \(x'\) from the two points \(q'\) and \(z'\), and since \(vy\) crosses \(x'q'\), it follows that \(v\) or \(y\) is in the triangle \(\Delta(x', q', z')\). Thus, \(v\) or \(y\) is in the interior of the disk \(D(c'_i; |c'_ix'|)\). This is a contradiction, because \(|x'v| \leq 1\), \(|x'y| \leq 1\), and \(\Delta(x', p', q')\) is a triangular face in \(LDT(x')\). Thus, we have shown that \(vy\) crosses \(x'z'\).

It now follows from the description of the algorithm that PLDG\((y)\) removes the edge \((v, y)\) from \(E(y)\). This completes the proof of the lemma.

Based on the previous lemmas, we can now prove that PLDG\((V)\) is plane:

**Lemma 5** PLDG\((V)\) is a plane graph.

**Proof.** The proof is by contradiction. Assume that PLDG\((V)\) contains two crossing edges \((v, y)\) and \((x, q)\). By Lemma 4, one of the points in \( \{x, q, v, y\} \) is within distance 1 from the other three points. We may assume without loss of generality that \(|xq| \leq 1\), \(|xv| \leq 1\), and \(|xy| \leq 1\). By Lemma 2, \((v, y)\) is an edge in \(LDT(v)\) and in \(LDT(y)\), and \((x, q)\) is an edge in \(LDT(x)\).

Since all conditions in Lemma 4 are satisfied, \((v, y)\) is not an edge in \(E(y)\). Also, the conditions in Lemma 4 with \(v\) and \(y\) interchanged, are satisfied. Therefore, \((v, y)\) is not an edge in \(E(v)\). Thus, \((v, y)\) is not an edge in PLDG\((V)\), which is a contradiction.

The following lemma summarizes the different scenarios when algorithm PLDG\((v)\) removes an edge \((v, y)\) from the edge set \(E(v)\):

**Lemma 6** Let \(v\) and \(y\) be two distinct points of \(V\) such that \((v, y)\) is an edge in \(LDT(v)\). Assume that algorithm PLDG\((v)\) removes \((v, y)\) from \(E(v)\). Then, there exist three pairwise distinct points \(x\), \(p\), and \(q\) in \(V\) such that
1. $\Delta(x, p, q)$ is a triangular face in $LDT(x)$,
2. $v \neq x$, $|vx| \leq 1$, $|vp| \leq 1$, $|vq| > 1$,
3. neither $v$ nor $y$ is in the interior of the disk $D(x, p, q)$, and
4. (a) if $y \neq x$, $v \neq p$, and $y \neq p$, the line segment $vy$ crosses both the line segments $xq$ and $pq$,
   (b) if $y = x$, the line segment $vy$ crosses the line segment $pq$,
   (c) if $v = p$, the line segment $vy$ crosses the line segment $xq$,
   (d) if $y = p$, the line segment $vy$ crosses the line segment $xq$.

**Proof.** Since algorithm PLDG($v$) removes ($v, y$) from $E(v)$, there exists a point $x$ in $N_v \setminus \{v\}$ which broadcasts the center $c'_i$ of the circumcircle of a triangular face $\Delta(x, p, q)$ in $LDT(x)$, such that the following holds:

1. Consider the disk $D(c'_i; |c'_ix|) = D(x, p, q)$ with center $c'_i$ that contains $x$, $p$, and $q$ on its boundary. Then, according to line 12 of algorithm PLDG($v$), $\partial D(c'_i; |c'_ix|)$ contains exactly two points of $N_v$. Since we assume that no four points of $V$ are cocircular, $x$, $p$, and $q$ are the only points of $V$ that are on $\partial D(c'_i; |c'_ix|)$. Thus, since $x \in N_v$, exactly one of $p$ and $q$ is in $N_v$. We may assume without loss of generality that $p \in N_v$ and $q \notin N_v$.

2. Consider the unit-disk $D(v; 1)$ centered at $v$. Let $arc_i$ be the arc on $\partial D(c'_i; |c'_ix|)$ that is not contained in $D(v; 1)$, let $v'$ be the point on $arc_i$ with $|xz'| \leq 1$ or $|pz'| \leq 1$ that is chosen by algorithm PLDG($v$) in line 15, and let $Del_v(v, y)$ be the disk chosen in line 17. Thus, $v$ and $y$ are the only points of $N_v$ that are on $\partial Del_v(v, y)$ and no point of $N_v$ is in the interior of $Del_v(v, y)$. Then, by line 18 of algorithm PLDG($v$), $arc_i$ is contained in the interior of $Del_v(v, y)$ and $vy$ crosses at least one of $xz'$ and $pz'$.

The first two claims in the lemma hold for the points $x$, $p$, and $q$.

We now prove the third claim. Since $|xv| \leq 1$ and $\Delta(x, p, q)$ is a triangular face in $LDT(x)$, $v$ is not in the interior of the disk $D(x, p, q)$. We prove by contradiction that $y$ is not in the interior of $D(x, p, q)$. Thus, we assume that $y$ is in the interior of this disk. Then, again since $\Delta(x, p, q)$ is a triangular face in $LDT(x)$, we have $|xy| > 1$. Recall that (i) $|xz'| \leq 1$ or $|pz'| \leq 1$ and (ii) $vy$ crosses at least one of $xz'$ and $pz'$. Therefore, we distinguish four cases and derive a contradiction for each of them.

**Case 1:** $|xz'| \leq 1$ and $vy$ crosses $x'z'$.

Since $|vy| \leq 1$ and $|vz'| > 1$, Lemma[1] implies that $|xy| \leq 1$, which is a contradiction.

**Case 2:** $|xz'| \leq 1$ and $vy$ does not cross $x'z'$.

In this case, $vy$ crosses $pz'$. Observe that $x \neq y$, $v \neq p$, and $y \neq p$. Also, $v \notin D(x, p, q)$.

The following observations lead to a contradiction:
Case 3: \(|xz'| > 1\) and \(vy\) crosses \(xz'\).

The following observations lead to a contradiction:

- Since \(|xp| \leq 1\) and \(|xz'| \leq 1\), each point in the triangle \(\Delta(x, p, z')\) has distance at most one to \(x\). Therefore, \(y \notin \Delta(x, p, z')\).

- The line segment \(vy\) crosses \(px\). This follows from the facts that \(vy\) does not cross \(xz'\), \(vy\) crosses \(p'\), \(v \notin D(x, p, q)\), \(y \in \text{int}(D(x, p, q))\), and \(y \notin \Delta(x, p, z')\).

- The line segment \(px\) is disjoint from \(arc_i\): Since \(|vp| \leq 1\) and \(|vx| \leq 1\), each point on \(px\) has distance at most one to \(v\). Thus, \(px \subseteq D(v; 1)\). However, \(arc_i\) and \(D(v; 1)\) are disjoint.

- \(arc_i\) and \(v\) are on the same side of the line through \(p\) and \(x\): Assume this is not the case. Since neither \(p\) nor \(x\) is in \(\text{Del}_v(v, y)\) and since \(arc_i\) is in the interior of \(\text{Del}_v(v, y)\), it follows that \(\partial \text{Del}_v(v, y)\) and \(\partial D(x, p, q)\) intersect more than twice. This is a contradiction.

- Let \(\tilde{px}\) be the arc on \(\partial D(x, p, q)\) between \(p\) and \(x\) that does not contain \(z'\). We claim that \(\tilde{px}\) is a major arc. To prove this, assume that it is a minor arc. The observations above imply that \(y\) is in the region of \(D(x, p, q)\) that is bounded by \(px\) and \(\tilde{px}\). Since \(|xp| \leq 1\), it then follows from Lemma 3 that \(|xy| \leq 1\), which is a contradiction.

- Let \(v'\) be the intersection between \(xx\) and \(\partial D(x, p, q)\). Let \(pv'x\) be the arc on \(\partial D(x, p, q)\) between \(p\) and \(x\) that contains \(v'\). Since \(|vp| \leq 1\), \(|vx| \leq 1\), and \(pv'x\) is a minor arc, we know that \(pv'x\) is contained in \(D(v; 1)\). Since \(q\) is on \(pv'x\), it follows that \(|vq| \leq 1\), which is a contradiction.

Case 4: \(|xz'| > 1\) and \(vy\) does not cross \(xz'\).

In this case, \(vy\) crosses \(pz'\). The following observations lead to a contradiction:
The line segments $vy$ and $px$ do not cross: If they do cross, then the same analysis as in Case 2 leads to a contradiction.

As in Case 3, the line segments $vy$ and $qx$ do not cross.

Since $vy$ crosses $pz'$, but $vy$ neither crosses $xz'$ nor $xp$, and since $y \in \text{int}(D(x, p, q))$ and $v \notin \text{int}(D(x, p, q))$, the point $y$ is in the triangle $\Delta(x, p, z')$.

Since $|xp| \leq 1$ and $|xq| \leq 1$, each point in the triangle $\Delta(x, p, q)$ has distance at most one to $x$. Therefore, since $|xy| > 1$, we have $y \notin \Delta(x, p, q)$. In particular, $q \neq z'$.

As in Case 2, the line segment $px$ is disjoint from $arc_i$. Thus, $q$ and $z'$ are on the same side of the line through $p$ and $x$.

Assume without loss of generality that $px$ is horizontal, $p$ is to the left of $x$, and both $q$ and $z'$ are above the line through $p$ and $x$.

Let $\widehat{pqx}$ be the arc on $\partial D(x, p, q)$ that is above $px$. If this arc is a minor arc, then, since $|px| \leq 1$ and using Lemma 3, we have $|xz'| \leq 1$, which is a contradiction. Thus, $\widehat{pqx}$ is a major arc.

Assume that $y$ is on or below $px$. Since the arc on $\partial D(x, p, q)$ that is below $px$ is a minor arc, it follows from Lemma 3 that $|xy| \leq 1$, which is a contradiction. Thus, $y$ is above $px$.

Assume that $y$ is to the right of $xq$. Since $y$ is contained in $\Delta(x, p, z')$, the point $z'$ is on the arc on $\partial D(x, p, q)$ between $x$ and $q$ that is to the right of $xq$. Recall that $vy$ crosses neither $xz'$ nor $xq$. It follows that $vy$ crosses $qz'$, which is a contradiction, because $q$ and $z'$ are on the same side of the line through $v$ and $y$.

We conclude that $y$ is to the left of $xq$. Since $y$ is contained in $\Delta(x, p, z')$ but not in $\Delta(x, p, q)$, the point $z'$ is on the arc on $\partial D(x, p, q)$ between $p$ and $q$ that is to the left of $xq$.

Assume that $v$ is above the line through $x$ and $y$. Since $vy$ crosses $pz'$, $y$ is contained in the triangle $\Delta(x, p, v)$. However, since $|xv| \leq 1$ and $|xp| \leq 1$, this implies that $|xy| \leq 1$, which is a contradiction. Thus, $v$ is below the line through $x$ and $y$.

Since both $q$ and $z'$ are above the line through $v$ and $y$, $y$ is contained in the triangle $\Delta(x, v, q)$. However, since $|xv| \leq 1$ and $|xq| \leq 1$, this implies that $|xy| \leq 1$, which is a contradiction.

To conclude, in each of the four cases above, we have obtained a contradiction to the assumption that $y$ is in the interior of $D(x, p, q)$. Therefore, we have proved the third claim in the lemma.

It remains to prove the fourth claim in the lemma. First assume that $y \neq x$, $v \neq p$, and $y \neq p$. We first show that $x$ and $p$ are on the same side of the line through $v$ and $y$. 
Assume, by contradiction, that $x$ and $p$ are on opposite sides of this line. Since both $x$ and $p$ are in $N_e$, neither of these two points is contained in $\text{Del}_e(v, y)$. On the other hand, since $\text{arc}_i \subseteq \text{int}(\text{Del}_e(v, y))$ and $z' \in \text{arc}_i$, the point $z'$ is in the interior of $\text{Del}_e(v, y)$. We also know that neither $v$ nor $y$ is contained in $D(c_i^e; |c_i^e|) = D(x, p, q)$. Since $\partial D(x, p, q)$ contains the points $x$, $p$, and $z'$, it follows that the boundaries of $\text{Del}_e(v, y)$ and $D(x, p, q)$ intersect more than twice. This is a contradiction.

Assume, without loss of generality, that $vy$ is horizontal and both $x$ and $p$ are above the line through $v$ and $y$. Then $z'$ is below this line. Since $\text{arc}_i \cap D(v; 1) = \emptyset$, it follows that the entire arc $\text{arc}_i$ is below this line. In particular, $q$ is below the line through $v$ and $y$. Since neither $v$ nor $y$ is contained in $D(x, p, q)$, since $vy$ intersects $\partial D(x, p, q)$ twice, and since $vy$ separates $q$ from $x$ and $p$, it follows that $vy$ crosses both the line segments $xq$ and $pq$.

It remains to prove the special cases in the fourth claim. First assume that $y = x$. Since $vy$ does not cross $xz' = yz'$, we know that $vy$ crosses $pz'$, which implies that $v \neq p$ and $y \neq p$. Since the line through $v$ and $y$ separates $p$ from $q$ and $z'$, it follows that $vy$ crosses $pq$.

Next assume that $v = p$. Since $vy$ does not cross $pz' = vz'$, we know that $vy$ crosses $xz'$, which implies that $y \neq x$. Since $q$ and $z'$ are on the same side of the line through $v$ and $y$, it follows that $vy$ crosses $xq$.

Finally, assume that $y = p$. Since $vy$ does not cross $pz' = yz'$, we know that $vy$ crosses $xz'$. Since the line through $v$ and $y$ separates $x$ from $q$ and $z'$, it follows that $vy$ crosses $xq$. This completes the proof of the lemma.

We can now prove that $\text{PLDG}(V)$ is consistent:

**Lemma 7** The graph $\text{PLDG}(V)$ is consistent: For any two distinct points $v$ and $y$ of $V$, $(v, y)$ is an edge in $E(v)$ if and only if $(v, y)$ is an edge in $E(y)$.

**Proof.** The proof is by contradiction. Assume there is a pair $(v, y)$ which is an edge in $E(y)$ but not in $E(v)$. Then $(v, y)$ is an edge in $\text{LDT}(y)$ and, by Lemma 6, $(v, y)$ is an edge in $\text{LDT}(v)$. Since $(v, y)$ is not an edge in $E(v)$, it has been removed by algorithm $\text{PLDG}(v)$. Thus, by Lemma 6 there exist three pairwise distinct points $x$, $p$, and $q$ in $V$ such that (i) $\Delta(x, p, q)$ is a triangular face in $\text{LDT}(x)$, (ii) $v \neq x$, $|vx| \leq 1$, $|vq| > 1$, and (iii) the line segment $vy$ crosses at least one of the line segments $pq$ and $xq$.

Assume that $vy$ does not cross $xq$. Then $vy$ crosses $pq$ and, by the fourth claim in Lemma 6, $y = x$. Thus, since $(v, y)$ is an edge in $\text{LDT}(y) = \text{LDT}(x)$ and using (i), it follows that $\text{LDT}(x)$ is not plane, which is a contradiction.

Thus, $vy$ crosses $xq$. This implies that the points $x$, $q$, $v$, and $y$ are pairwise distinct. It follows from (i) that $(x, q)$ is an edge in $\text{LDT}(x)$ and $|xq| \leq 1$. Since $|vy| \leq 1$, $|xq| \leq 1$, $|vq| > 1$, and since $vy$ crosses $xq$, it follows from Lemma 6 that $|xy| \leq 1$. Thus, all conditions in Lemma 6 are satisfied. As a result, algorithm $\text{PLDG}(y)$ deletes the edge $(v, y)$ from $E(y)$. This is a contradiction.

Recall that $\text{UDel}(V)$ denotes the intersection of the Delaunay triangulation and the unit-disk graph of $V$. We next show that $\text{PLDG}(V)$ contains $\text{UDel}(V)$. 

18
Lemma 8 The graph UDel(V) is a subgraph of PLDG(V).

Proof. Let (v, y) be an edge of UDel(V). We will show that (v, y) is an edge in E(v). By definition, |vy| ≤ 1 and (v, y) is an edge in the Delaunay triangulation of V. Therefore, (v, y) is also an edge in the Delaunay triangulation LDT(v) of N_v and, thus, (v, y) is added to the edge set E(v) in line 3 of algorithm PLDG(v). We have to show that algorithm PLDG(v) does not remove (v, y) in line 19.

Assume that (v, y) is removed in line 19 of algorithm PLDG(v). By Lemma 6, there exist three pairwise distinct points x, p, and q in V such that (i) neither v nor y is in the interior of the disk D(x, p, q) and (ii) the line segment vy crosses at least one of the line segments xq and pq.

Assume that vy crosses xq. Then, the points v, y, x, and q are pairwise distinct. Observe that p may be equal to v or y. Let D be an arbitrary disk having v and y on its boundary, and assume that neither x nor q is contained in D. Then it follows from (i) and (ii) that the boundaries of D and D(x, p, q) intersect more than twice, which is a contradiction. Thus, D contains at least one of x and q. Since D was arbitrary, this contradicts the fact that (v, y) is an edge in the Delaunay triangulation of V.

By a symmetric argument, the case when vy crosses pq also leads to a contradiction to the fact that (v, y) is an edge in the Delaunay triangulation of V.

In the next lemma, we summarize the results obtained in this section. Recall that a message is defined to be the location of a point in the plane.

Lemma 9 Let V be a finite set of points in the plane. The distributed algorithm PLDG(v), where v ranges over all points in V, is a 2-local algorithm that computes a plane and consistent \( \frac{4\pi\sqrt{3}}{9} \)-spanner PLDG(V) of the unit-disk graph of V. This algorithm makes one communication round and each point of V broadcasts at most 6 messages.

Proof. Let v be a point of V. Lines 1–8 of algorithm PLDG(v) depend only on the points in N_v. Lines 9–19 depend only on information received from nodes x in N_v; this information was computed in lines 1–8 of algorithm PLDG(x) and, thus, depends only on the points in N_x. It follows that algorithm PLDG(v) is 2-local. Algorithm PLDG(v) broadcasts a sequence of messages only once, in line 8. Therefore, there is only one round of communication. In the Delaunay triangulation LDT(v) of N_v, there are at most 5 triangular faces \( \Delta(u, v, w) \) with \( \angle uvw > \frac{\pi}{3} \). Therefore, the sequence that is broadcast in line 8 contains at most 6 points. Thus, PLDG(v) broadcasts at most 6 messages.

By Lemmas 5 and 7, the graph PLDG(V) is plane and consistent. By Lemma 8, PLDG(V) is a supergraph of UDel(V). Since UDel(V) is a \( \frac{4\pi\sqrt{3}}{9} \)-spanner of the unit-disk graph UDG(V) of V (see Bose et al. [2]), the graph PLDG(V) is a \( \frac{4\pi\sqrt{3}}{9} \)-spanner of UDG(V).
3 The Final Algorithm

We have seen that in algorithm PLDG, each point of $V$ broadcasts at most 6 messages. In this section, we improve this upper bound to 5. We obtain this improvement, by making the following modification to the algorithm: The sequence $(v, c_1, \ldots, c_k)$ that is broadcast in line 8 of algorithm PLDG($v$) contains the location of the sender $v$. In our new algorithm, point $v$ sends only the sequence $(c_1, \ldots, c_k)$ of centers. Thus, any point that receives this sequence does not know that the sequence was broadcast by $v$. Assume that $v$ receives a center $c_i'$ from some node $x$ in $N_v$. Since $v$ does not know that $c_i'$ was broadcast by $x$, line 11 in algorithm PLDG($v$) has to be modified. In the new algorithm, $v$ computes a point $x'$ in $N_v \setminus \{v\}$ that is closest to $c_i'$ and uses the disk $D(c_i'; |c_i'x'|)$ to decide whether or not to remove an edge $(v, y)$.

The new algorithm, which we denote by PLDG', is given in Figure 7. We denote by $E'(v)$ the edge set that is computed by algorithm PLDG'(v). Let $E' = \cup_{v \in V} E'(v)$ and let PLDG'(V) denote the graph with vertex set $V$ and edge set $E'$.

Recall that $E(v)$ denotes the edge set that is computed by algorithm PLDG($v$) and PLDG(V) denotes the graph with vertex set $V$ and edge set $\cup_{v \in V} E(v)$. We claim that PLDG(V) = PLDG'(V); thus, the new algorithm PLDG' computes the same graph as algorithm PLDG. In order to prove this claim, it suffices to show that algorithm PLDG($v$) removes an edge $(v, y)$ from $E(v)$ if and only if algorithm PLDG'(v) removes the edge $(v, y)$ from $E'(v)$. We will show this in the following two lemmas.

Lemma 10 Let $v$ be an element of $V$ and let $(v, y)$ be an edge of the Delaunay triangulation $LDT(v)$ of the set $N_v$. If algorithm PLDG($v$) removes $(v, y)$ from $E(v)$, then algorithm PLDG'(v) removes $(v, y)$ from $E'(v)$.

Proof. By Lemma 6 there exist three pairwise distinct points $x$, $p$, and $q$ in $V$ such that

1. $\Delta(x, p, q)$ is a triangular face in $LDT(x)$,
2. $v \neq x$, $|vx| \leq 1$, $|vp| \leq 1$, $|vq| > 1$,
3. neither $v$ nor $y$ is in the interior of the disk $D(x, p, q)$.

In fact, in algorithm PLDG($v$), $v$ receives from $x$ the center $c_i'$ of the disk $D(c_i'; |c_i'x|) = D(x, p, q)$. Since $|vx| \leq 1$, in algorithm PLDG'(v), $v$ receives the center $c_i'$, but does not know that it was broadcast by $x$. Consider the point $x'$ that is computed in line 11 of algorithm PLDG'(v). Thus, $x'$ is a point in $N_v \setminus \{v\}$ that is closest to $c_i'$. Since $x \in N_v \setminus \{v\}$, we have $|c_i'x'| \leq |c_i'x|$. We claim that $|c_i'x'| = |c_i'x|$. To prove this claim, assume, by contradiction, that $|c_i'x'| < |c_i'x|$. Then $x'$ is in the interior of the disk $D(x, p, q)$. Since $\Delta(x, p, q)$ is a triangular face in $LDT(x)$, we have $|xx'| > 1$.

Consider the disk $Del_v(v, y)$ that is computed in line 17 of algorithm PLDG($v$). Recall that $N_v \cap \partial Del_v(v, y) = \{v, y\}$ and $N_v \cap \text{int}(Del_v(v, y)) = \emptyset$. Since both $x$ and $x'$ are in $N_v$, neither of these two points is in the interior of $Del_v(v, y)$. It follows from line 18 of algorithm PLDG($v$) that $q$ is in the interior of $Del_v(v, y)$ (because $q \in \text{arc}_i$).
Algorithm PLDG’(v)
1. let $N_v = \{ u \in V : |uv| \leq 1 \}$;
2. compute the Delaunay triangulation $LDT(v)$ of $N_v$;
3. let $E'(v)$ be the set of all edges in $LDT(v)$ that are incident on $v$;
4. let $\Delta_v$ be the set of all triangular faces $\Delta(u, v, w)$ in $LDT(v)$ for which $\angle uvw > \frac{\pi}{3}$;
5. let $k$ be the number of elements in $\Delta_v$;
6. if $k \geq 1$
7. then let $c_1, \ldots, c_k$ be the centers of the circumcircles of all triangles in $\Delta_v$;
8. broadcast the sequence $(c_1, \ldots, c_k)$;
9. for each sequence $(c'_1, \ldots, c'_m)$ received
10. do for $i = 1$ to $m$
11. do compute a point $x'$ in $N_v \setminus \{ v \}$ that is closest to $c'_i$;
12. let $D(c'_i; |c'_ix'|)$ be the disk with center $c'_i$ that contains $x'$ on its boundary;
13. if $\partial D(c'_i; |c'_ix'|)$ contains exactly two points of $N_v$
14. then let $p'$ be the point in $(N_v \setminus \{ x' \}) \cap \partial D(c'_i; |c'_ix'|)$;
15. let $arc_i = (\partial D(c'_i; |c'_ix'|)) \setminus D(v; 1)$;
16. let $Z = \{ z' \in arc_i : |x'z'| \leq 1 \text{ or } |p'z'| \leq 1 \}$;
17. if $arc_i \neq \emptyset$ and $Z \neq \emptyset$
18. then let $z'$ be an arbitrary element of $Z$;
19. for each edge $(v, y)$ in $E'(v)$
20. do let $Del_v(v, y)$ be a disk $D$ such that $N_v \cap \partial D = \{ v, y \}$ and $N_v \cap \text{int}(D) = \emptyset$;
21. if $arc_i$ is contained in the interior of $Del_v(v, y)$
22. then remove $(v, y)$ from $E'(v)$

Figure 7: The improved plane localized Delaunay graph algorithm.
Assume, without loss of generality that \( vy \) is horizontal, \( v \) is to the right of \( y \), and \( q \) is below the line through \( v \) and \( y \); refer to Figure 8.

Since \(|vq| > 1\) and \(|vy| \leq 1\), we have \( \angle yqv < \pi/2 \). Let \( \hat{y}v \) be the arc on \( \partial Del_v(v, y) \) with endpoints \( y \) and \( v \) that contains the north pole of \( \partial Del_v(v, y) \). Then \( \hat{y}v \) is a minor arc.

Let \( u_1 \) and \( u_2 \) be the intersections between \( \partial Del_v(v, y) \) and \( \partial D(x, p, q) \), where \( u_1 \) is to the left of \( u_2 \). Then both \( u_1 \) and \( u_2 \) are contained in \( \hat{y}v \) and, therefore, by Lemma 3 \( |u_1u_2| \leq 1 \).

Let \( \hat{u}_1\hat{u}_2 \) be the arc on \( \partial D(x, p, q) \) with endpoints \( u_1 \) and \( u_2 \) that contains the north pole of \( \partial D(x, p, q) \). Since \( \angle u_1qu_2 \leq \angle yqv \leq \pi/2 \), \( \hat{u}_1\hat{u}_2 \) is a minor arc.

Recall that \( x \not\in int(Del_v(v, y)) \). Also, if \( x \in \partial Del_v(v, y) \), then \( x = y \). It follows that \( x \) is not below the line through \( u_1 \) and \( u_2 \). By a similar argument, \( x' \) is not below this line. Since both \( x \) and \( x' \) are in \( D(x, p, q) \) and since \( \hat{u}_1\hat{u}_2 \) is a minor arc, it follows from Lemma 3 that \( |xx'| \leq 1 \), which is a contradiction.

Thus, we have shown that \( |c'_x'x'| = |c'_xx| \). Recall that \( p \) is the point that is computed in line 13 of algorithm PLDG(\( v \)). Consider the point \( p' \) that is computed in line 14 of algorithm PLDG'(\( v \)). Since \( D(c'_x'|c'_xx|) = D(c'_x'c'_xx'|) \), we have \( \{x, p\} = \{x', p'\} \). In other words, algorithm PLDG'(\( v \)) knows the points \( x \) and \( p \), but does not know which of them is \( x \) and which of them is \( p \).

Since lines 15–19 of algorithm PLDG(\( v \)) are symmetric in \( x \) and \( p \), and since lines 16–22 of algorithm PLDG'(\( v \)) are symmetric in \( x' \) and \( p' \), it follows that the behaviors of PLDG(\( v \)) and PLDG'(\( v \)) with respect to the edge \( (v, y) \) are identical. Therefore, algorithm PLDG'(\( v \)) removes the edge \( (v, y) \) from \( E'(v) \).

**Lemma 11** Let \( v \) be an element of \( V \) and let \( (v, y) \) be an edge of the Delaunay triangulation \( LDT(v) \) of the set \( N_v \). If algorithm PLDG'(\( v \)) removes \( (v, y) \) from \( E'(v) \), then algorithm...
**Proof.** Since algorithm PLDG′(v) removes (v, y) from E′(v), there exist three pairwise distinct points x, p, and q in V such that

1. \(\Delta(x, p, q)\) is a triangular face in LDT(x),
2. algorithm PLDG′(x) broadcasts the center \(c_i\) of the disk \(D(x, p, q) = D(c_i; |c_i x|)\),
3. \(v \neq x, |vx| \leq 1\),
4. \(v\) receives the center \(c_i\) (but does not know that it was broadcast by \(x\)).

Consider the point \(x'\) that is computed in line 11 of algorithm PLDG′(v). Thus, \(x'\) is a point in \(N_v \setminus \{v\}\) that is closest to \(c_i\). Since \(x \in N_v \setminus \{v\}\), we have \(|c_i x'| \leq |c_i x|\). In the rest of the proof, we will show that \(|c_i x'| = |c_i x|\). As in the proof of Lemma 10, this will imply that algorithm PLDG(v) removes the edge \((v, y)\) from \(E(v)\).

The proof of the claim that \(|c_i x'| = |c_i x|\) is by contradiction. Thus, we assume that \(|c_i x'| < |c_i x|\). Then \(x'\) is in the interior of the disk \(D(x, p, q)\). Since \(\Delta(x, p, q)\) is a triangular face in LDT(x), we have \(|xx'| > 1\). Since \(|vx| \leq 1\), \(v\) is not in the interior of \(D(x, p, q)\).

Consider the disk \(D(c_i; |c_i x'|)\). It follows from line 13 of algorithm PLDG′(v) that the boundary of this disk contains exactly two points of \(N_v\); \(x'\) is one of these two points, let \(p'\) be the other one. Thus, \(p'\) is the point that is computed in line 14 of algorithm PLDG′(v). Since \(y \in N_v \setminus \{v\}\), the point \(y\) is not in the interior of \(D(c_i; |c_i x'|)\).

Consider the disk \(Del_v(v, y)\) that is computed in line 20 of algorithm PLDG′(v). Then \(N_v \cap \partial Del_v(v, y) = \{v, y\}\) and \(N_v \cap int(Del_v(v, y)) = \emptyset\). Since \(|vx'| \leq 1\) and \(|vp'| \leq 1\), neither \(x'\) nor \(p'\) is in the interior of \(Del_v(v, y)\).

Consider the point \(z'\) on \(arc_i = (\partial D(c_i; |c_i x'|)) \setminus D(v; 1)\) that is computed in line 18 of algorithm PLDG′(v). It follows from line 21 that \(z'\) is in the interior of \(Del_v(v, y)\) and \(vy\) crosses at least one of \(x'z'\) and \(p'z'\).

Since lines 11–22 of algorithm PLDG′(v) are symmetric with respect to \(x'\) and \(p'\), we may assume without loss of generality that \(vy\) crosses \(x'z'\). Thus, \(x' \notin \{v, y\}\) and \(x'\) and \(z'\) are on opposite sides of the line through \(v\) and \(y\).

We may assume without loss of generality that \(vy\) is horizontal, \(v\) is to the right of \(y, x'\) is above the line through \(v\) and \(y\), and \(z'\) is below this line.

We claim that \(y\) is in the interior of \(D(x, p, q)\). The proof is by contradiction; thus, we assume that \(y \notin int(D(x, p, q))\). Observe that \(z' \in int(D(x, p, q))\) and recall that \(z' \in int(Del_v(v, y))\). Since \(|vz'| > 1\) and \(|vy| \leq 1\), we have \(\angle yz'v < \pi/2\). Therefore, the upper arc on \(\partial Del_v(v, y)\) with endpoints \(y\) and \(v\) is a minor arc. Let \(u_1\) and \(u_2\) be the two intersection points between \(\partial Del_v(v, y)\) and \(\partial D(x, p, q)\); refer to Figure 9. It follows from Lemma 3 that \(|u_1 u_2| \leq 1\). Since \(\angle u_1 z' u_2 \leq \angle yz'v < \pi/2\), the upper arc on \(\partial D(x, p, q)\) with endpoints \(u_1\) and \(u_2\) is a minor arc. Since both \(x\) and \(x'\) are in \(D(x, p, q)\) and on or above the line through \(u_1\) and \(u_2\), it follows, again from Lemma 3, that \(|xx'| \leq 1\), which is a contradiction.

Thus, we have shown that \(y \in int(D(x, p, q))\). Since \(\Delta(x, p, q)\) is a triangular face in LDT(x), this implies that \(|xy| > 1\).
We next claim that \( x \) is below the line through \( v \) and \( y \). We prove this claim by contradiction. Thus, we assume that \( x \) is above this line. Observe that \( y \) is to the left of the vertical line through \( c'_i \). We have seen above that \( \angle yz'v < \pi/2 \). Therefore, the arc on \( \partial D(c'_i; |c'_i x'|) \) that is not below the line through \( v \) and \( y \) is a minor arc. It follows that \( c'_i \) is below the line through \( v \) and \( y \).

Consider the disk \( D(x, p, q) \). We translate the center \( c'_i \) of this disk horizontally to the right. During the translation, we change the disk so that \( x \) stays on its boundary. We stop the translation as soon as one of \( v \) and \( y \) is on the boundary of the moving disk. Let \( c^* \) be the center of the new disk \( D^* \). Since \( c^* \) is below the line through \( v \) and \( y \), the arc on \( \partial D^* \) that is not below the line through \( v \) and \( y \) is a minor arc. First assume that \( y \) is on the boundary of \( D^* \). Then it follows from Lemma 3 that \( |xy| \leq 1 \), which is a contradiction. Thus, \( y \) is in the interior of \( D^* \) and \( v \) is on the boundary of \( D^* \). Then, the disk \( D(y; |yv|) \) contains the point \( x \) and, therefore, \( |yx| \leq |yv| \leq 1 \), which is also a contradiction.

We conclude that \( x \) is below the line through \( v \) and \( y \). Let \( y' \) be the leftmost intersection between \( yv \) and \( \partial D(c'_i; |c'_i x'|) \), and let \( v' \) be the intersection between \( yv \) and \( \partial D(c'_i; |c'_i x'|) \). We translate the center of the disk \( D(c'_i; |c'_i x'|) \) along the line through \( y' \) and \( c'_i \), such that the center moves away from \( y' \). During the translation, we change the disk so that \( y' \) stays on its boundary. We stop the translation as soon as \( v' \) is on the boundary of the moving disk; refer Figure 10. Let \( D^{**} \) be the resulting disk. Let \( v'' \neq v' \) be the second intersection point between \( \partial D(c'_i; |c'_i x|) \) and \( \partial D^{**} \). Then \( |y'y'| = |y'y''| \). Since \( |yv'| \leq |yv| \leq 1 \) and \( |yx| > 1 \), the point \( x \) is not in the disk \( D(y'; |y'y'|) \). Therefore, the point \( x \) is on the clockwise arc on \( D(c'_i; |c'_i x|) \) from \( v' \) to \( v'' \). Observe that both \( x \) and \( x' \) are contained in \( D^{**} \). It follows that any disk having \( y' \) and \( v' \) on its boundary contains at least one of \( x \) and \( x' \). This, in turn, implies that any disk having \( y \) and \( v \) on its boundary contains at least one of \( x \) and

\[ y = x = u_1, u_2 \]

\[ y = x = u_1, u_2 \]

\[ y = x = u_1, u_2 \]
Figure 10: An illustration of the proof of Lemma 11. The point \( y \) cannot be in the interior of \( D(c'_i; |c'_ix'|) \).

\[ D \left( c'_i; |c'_ix'| \right) \]

In particular, \( \text{Del}_v(v, y) \) contains at least one of \( x \) and \( x' \), which is a contradiction. This completes the proof of the lemma.

By Lemmas 10 and 11, algorithms PLDG\((v)\) and PLDG'\((v)\) compute the same graph. Therefore, the proof of Theorem 1 can be completed as in the proof of Lemma 9 and by observing that the sequence that is broadcast in line 8 of algorithm PLDG'\((v)\) contains at most 5 points.

4 Concluding Remarks

We have presented a 2-local algorithm that constructs the Plane Localized Delaunay Graph PLDG\((V)\) of any finite set \( V \) of points in the plane. This graph is a plane and consistent \( \frac{4\pi\sqrt{3}}{9} \)-spanner of the unit-disk graph UDG\((V)\). Our algorithm makes only one communication round and each point of \( V \) broadcasts at most 5 messages. We leave as an open problem the question of whether a 2-local algorithm exists in which each point broadcasts less than 5 messages.

In general, the maximum degree of any vertex in the graph PLDG\((V)\) can be linear in the size of \( V \). It is still open whether there is a communication-efficient localized algorithm that constructs a bounded-degree plane spanner of the unit-disk graph.
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