A probabilistic formula for gradients of solutions of hypoelliptic Dirichlet problems

Giuseppe Da Prato · Luciano Tubaro

Abstract

We prove a new probabilistic formula for the gradient of the Dirichlet semigroup associated with a class of hypoelliptic operators in a bounded subset of $\mathbb{R}^d$.

Keywords

Hypoelliptic operators · Strong Feller property · Dirichlet problem · Cameron–Martin formula · Cylindrical Wiener processes

Mathematics Subject Classification

35J15 · 60G53 · 60H99 · 60J65

1 Introduction and setting of the problem

We are here concerned with the following Cauchy–Dirichlet problem in $H = \mathbb{R}^d$:

$$
\begin{align*}
\begin{cases}
D_t u(t, x) &= \frac{1}{2} \text{Tr}[C D^2 u(t, x)] + \langle Ax, Du(t, x) \rangle, \\
& \quad (t, x) \in [0, T] \times \partial \mathcal{O}, \\
& \quad u(t, x) = 0, \quad t \in [0, T], \quad x \in \partial \mathcal{O}, \\
u(0, x) &= \varphi(x), \quad x \in \mathcal{O},
\end{cases}
\end{align*}
$$

(1)

where $A$ and $C$ are $d \times d$ matrices, $C$ being symmetric and semi-definite positive, $\mathcal{O}$ is a regular subset of $\mathbb{R}^d$ with boundary $\partial \mathcal{O}$.

---

Giuseppe Da Prato is partially supported by GNAMPA from INDAM.

✉ Luciano Tubaro
luciano.tubaro@unitn.it

Giuseppe Da Prato
daprato@sns.it

1 Scuola Normale Superiore, Piazza dei Cavalieri, 7, 56126 Pisa, Italy
2 University of Trento, Via Sommarive, 14, 38123 Povo Trento, Italy
Formally, a solution of problem (1) is given by the stopped semigroup $u(t, x) := R_T^O \varphi(x)$, $T \geq 0$, defined by

$$R_T^O \varphi(x) = \mathbb{E}[\varphi(X(T, x)) \mathbb{1}_{T \leq \tau}], \quad T \geq 0, \quad \varphi \in B_b(O),$$

where

$$X(t, x) = e^{tA}x + W_A(t), \quad x \in H, \quad t \geq 0.$$ 

Here $W_A$ is the stochastic convolution

$$W_A(t) := \int_0^t e^{(t-s)A} \sqrt{C} dW(s), \quad t \geq 0,$$

and $W(t), t \geq 0$, is an $H$-valued standard Wiener process defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Our basic assumption on $A$ and $C$ is the following:

**Hypothesis 1.1** The matrix $Q_t := \int_0^t e^{sA} Ce^{sA^*} ds$ is non-singular for all $t > 0$.

Hypothesis 1.1 arises in the null controllability for the deterministic system $D_t \xi = A \xi + \sqrt{C} u$, where $u$ is a control, see e.g. [12]; as is well known it implies that the operator

$$K \varphi := \frac{1}{2} \text{Tr}[CD_x^2 \varphi] + \langle Ax D_x \varphi \rangle$$

is hypoelliptic. When $O = \mathbb{R}^d$ the solution of problem (1) is $C^\infty$, as is well known. The situation is different when $O$ is a bounded subset of $\mathbb{R}^d$. We shall assume in what follows that $O_r = \{ g < r \}$, $r > 0$, where

**Hypothesis 1.2** (i) $g : H \to \mathbb{R}$ is a convex function of class $C^1$ such that $g(0) = 0$, $g(x) > 0$ and $g'(x) \neq 0$ for all $x \neq 0$. For any $r > 0$ we set $\overline{O}_r = \{ g < r \}$, $\overline{\partial}O_r = \{ g^{-1}(r) \}$. Moreover, $\overline{O}_r$ is bounded.

(ii) There exist $a, b > 0$ such that $|g(x)| + |g'(x)|_H \leq a + e^{b|x|_H}$ for all $x \in H$.

The goal of this paper is to prove an explicit probabilistic formula for the gradient of $R_t \varphi$, $t > 0$, when $\varphi$ is only bounded and Borel. The main tool is the construction of a suitable translation for which the Cameron–Martin formula applies, see Sect. 2.

We notice that for all $t > 0$, $C^\infty$ regularity of $u(t, x)$ for more general hypoelliptic equations on $\overline{O}_r$ was proved by Cattiaux [4, 5], using Malliavin calculus; however under condition det $C > 0$.

To our knowledge, the existence of the gradient of the solution $u(t, x)$ for $t > 0$ under Hypotheses 1.1 and 1.2 is not known.

We believe that our method could be generalised to Kolmogorov operators of the form

$$K_1 \varphi = \frac{1}{2} \text{Tr}[CD_x^2 \varphi] + \langle Ax + b(x), D_x \varphi \rangle.$$
A probabilistic formula for gradients of solutions

where \( b : H \to H \) is a suitable nonlinear mapping. This will be the object of future work.

2 A Cameron–Martin formula

First we start from an obvious consequence of (2),

\[
R^\psi_T \varphi(x) = \int_{\{g(e^{TAx} + h(s)) \leq r \text{ for all } s \in [0, T]\}} \varphi(h(T) + e^{TAx}) N_{QA_T}(dh), \varphi \in B_b(O, r),
\]

where \( N_{QA_T} \) is the law of \( W_A(\cdot) \) in \( X := L^2(0, T; H) \) or in \( E := C([0, T]; H) \), see Lemma 3.1.

Let \( x \in H \); we cannot eliminate \( x \) in identity (3) making the translation \( h \mapsto h - e^{A}x \) and using the Cameron–Martin formula, because the measures \( N_{e^{A}x, Q} \) and \( N_{Q} \) are singular. For this reason we construct another translation \( h \mapsto h - a(x, \cdot) \) such that \( a(x, \cdot) \) belongs to \( \mathbb{Q}_T(X) \) for all \( x \in H \) (and a fortiori to \( \mathbb{Q}_T^{1/2}(X) \), the Cameron–Martin space of \( N_{QA_T} \)) and such that

\[
a(x, T) = e^{TA}x \quad \text{for all } x \in H
\]

(see Proposition 3.4). Then the measures \( N_{a(x, \cdot), Q_T} \) and \( N_{Q_T} \) are equivalent, so that by the Cameron–Martin Theorem we have

\[
\frac{dN_{a(x, \cdot), Q_T}}{dN_{Q_T}}(h) = \exp\left\{-\frac{1}{2} \left| \mathbb{Q}_T^{-1/2}a(x, \cdot) \right|^2_X + W_{\mathbb{Q}_T^{1/2}a(x, \cdot)}(h) \right\}, \quad x \in H, \ h \in X,
\]

where \( \mathbb{Q}_T^{-1/2} \) is the pseudo-inverse of \( \mathbb{Q}_T^{1/2} \), see e.g. [8, Theorem 2.23]. Now we take advantage of the special form of \( a(x, \cdot) \) to simplify identity (3). We write

\[
\left| \mathbb{Q}_T^{-1/2}a(x, \cdot) \right|^2_X = \langle \mathbb{Q}_T^{-1}a(x, \cdot), a(x, \cdot) \rangle_X =: F(x),
\]

and

\[
W_{\mathbb{Q}_T^{-1/2}a(x, \cdot)}(h) = \left\{ \mathbb{Q}_T^{-1/2}a(x, \cdot), \mathbb{Q}_T^{-1/2}h \right\}_X = \left\{ \mathbb{Q}_T^{-1}a(x, \cdot), h \right\}_X =: G(x, h).
\]

Note that both \( F \) and \( G \) are regular. Now (4) becomes

\[
\frac{dN_{a(x, \cdot), Q_T}}{dN_{Q_T}}(h) = \exp\left\{-\frac{1}{2} F(x) + G(x, h) \right\}, \quad x \in H, \ h \in X,
\]
so that (3) can be written as

\[ R_T^0 \varphi(x) = \int_{[\Gamma(h + d(x, \cdot)) \leq r]} \varphi(h(T)) \exp \left\{ -\frac{1}{2} F(x) + G(x, h) \right\} N_{\mathbb{Q}_T} dh, \quad \varphi \in B(\mathcal{O}_r), \]

where

\[ \Gamma(h + d(x, \cdot)) = \sup_{t \in [0, T]} g(h(t) + d(x, t)), \quad d(x, t) = e^{tA}x - a(x, t), \quad t \in [0, T]. \]

In the integral (6) the variable \( x \) no longer appears under the argument of \( \varphi \). Since the mapping \( x \mapsto \Gamma(h + d(x, \cdot)) \) is continuous, the semigroup \( R_T^0, T > 0, \) is strong Feller that is \( \varphi \in B_b(H) \Rightarrow R_T^0 \varphi \in C_b(H) \) for all \( T > 0, \) see Proposition 3.5.

It is more difficult to show that \( R_T^0 \varphi \) is differentiable for all \( T > 0. \) As expected, this will produce a surface integral which, unfortunately, does not fulfill the classical assumptions of the theory of Airault–Malliavin [1], see also [7]. To overcome this difficulty, we introduce in Sect. 3 an approximating family of operators \( R_{T,n}^0, T > 0, \) for all decompositions \( \{t_j = \frac{jT}{2^n}, j = 0, 1, \ldots, 2^n\} \) of \( [0, T], \) namely by approximating any function \( h \) from \( E \) by step functions. Then we arrive to an identity for \( R_{T,n}^0 \varphi(x) \) (see (17)) that can be easily differentiated with respect to \( x, \) see identity (18). It remains to let \( n \to \infty; \) this is not easy, however, due to the factor

\[ \langle \mathbb{Q}_T^{-1/2}(d_x(x, \cdot)y), \mathbb{Q}_T^{-1/2}h \rangle_{H^{2n}} \]

which appears in identity (18) because \( d_x(x, \cdot)y \) does not belong to the Cameron–Martin space of \( N_{\mathbb{Q}_T}. \) Therefore, some additional work is required, based on the Ehrhard inequality for the Gaussian measure \( N_{\mathbb{Q}_T} \) and Helly’s selection principle, see Sect. 5. After some manipulations, we arrive at the representation formula (31) which is the main result of the paper. Our procedure was partially inspired by the paper of Linde [11], which was dealing, however, with a completely different situation.

We end this section with some notation. For any \( T > 0 \) we consider the law of \( X(\cdot, x) \) both in the Banach space \( E = C([0, T]; H) \) and in the Hilbert space \( X = L^2(0, T; H) \) (in the second case it is concentrated on \( E \) which is a Borel subset of \( X \)). We shall denote by \( | \cdot |_X \) (resp. \( | \cdot |_E \) ) the norm of \( X \) (resp. of \( E \)). The scalar product from two elements \( x, y \in H \) (resp. \( X \) ) will be denoted either by \( \langle x, y \rangle_H \) (resp. \( \langle x, y \rangle_X \) ) or by \( x \cdot y. \) If \( \varphi \in C^1_b(E) \) and \( \eta \in E \) we denote by \( D \varphi(h) \cdot \eta \) the derivative of \( \varphi \) at \( h \) in the direction \( \eta. \)

In what follows several integrals with respect to \( dN_{\mathbb{Q}} \) will be considered, according to the convenience, both in \( X \) and in \( E. \)
3 Strong Feller property

We first recall some properties of the Gaussian measure $N_{Q_T}$. The following lemma is well known, see e.g. [8, Theorem 5.2].

Lemma 3.1 The law of $W_A(\cdot)$ is Gaussian $N_{Q_T}$ both in $E$ and in $X$, where $Q_T$ is given by

\[(Q_T h)(t) = \int_0^T K(t, s) h(s) \, ds, \quad t \in [0, T], \ h \in X,
\]

and

\[K(t, s) = \begin{cases}
\int_0^s e^{(t-r)A} C e^{(s-r)A^*} dr & \text{if } 0 \leq s \leq t \leq T, \\
\int_t^s e^{(t-r)A} C e^{(s-r)A^*} dr & \text{if } 0 \leq t \leq s \leq T.
\end{cases}
\]

We note that there exists an orthonormal basis $(e_j)$ on $X$ and a sequence $(\lambda_j)$ of nonnegative numbers such that

\[Q_T e_j = \lambda_j e_j, \quad j \in \mathbb{N},\]

and an integer $k_0 \geq 0$ such that

\[\lambda_1 = \lambda_2 = \cdots = \lambda_{k_0} = 0, \quad \lambda_j > 0 \quad \text{for all } j > k_0.
\]

If $k_0 = 0$ then $Q_T$ is non-degenerate.

We shall denote by $L_T$ the linear operator from $X$ into itself defined by

\[L_T h(t) = \int_0^t e^{(t-s)A} \sqrt{C} h(s) \, ds, \quad h \in X, \ t \in [0, T].
\]

Its adjoint $L_T^*$ is given by

\[L_T^* g(t) = \int_t^T \sqrt{C} e^{(s-t)A^*} g(s) \, ds, \quad g \in X, \ t \in [0, T].
\]

It is easily checked that $Q_T = L_T L_T^*$. Moreover, the Cameron–Martin space of $N_{Q_T}$ is given by

\[Q_T^{1/2}(X) = L_T(X),
\]

both in $E$ and in $X$, see [8, Corollary B5].
Remark 3.2 If $\det C = 0$ one easily checks that the Gaussian measure $Q_T$ is degenerate and

$$\text{Ker } Q_T = \{ h \in X : L^*_T h = 0 \}.$$  

We shall denote by $Q_T^{-1}$ (resp. $Q_T^{-1/2}$) the pseudo-inverse of $Q_T$ (resp. the pseudo-inverse of $Q_T^{1/2}$).\footnote{Let $S : X \to Y$ be a linear, bounded and compact operator; the pseudo-inverse $S^{-1}$ of $S$ is defined as follows. For any $y \in S(X)$ we denote by $S^{-1} y$ the element of minimal norm from the convex set $\{ x \in X : S(x) = y \}$.} Clearly, the domain of $Q_T^{-1}$ is equal to $Q_T(X)$ and $h \in Q_T^{-1}(X)$ if and only if the following series is convergent in $X$:

$$Q_T^{-1} h = \sum_{j=k_0+1}^{\infty} \lambda_j^{-1} \langle h, e_j \rangle_X e_j.$$  

Similar assertion holds for $Q_T^{-1/2}$.

Now, to introduce the required translation, we first need a lemma.

Lemma 3.3 Let $U := \int_0^T e^{rA} C e^{rA^*} dr$. Then $\det U > 0$.

Proof We have

$$U \geq T/2 \int_{T/2}^T e^{rA} C e^{rA^*} dr = \frac{T}{2} \int_0^{T/2} e^{(T/2+z)A} C e^{(T/2+z)A^*} dz = \frac{T}{2} e^{AT/2} Q_T/2 e^{A^*T/2}.$$  

It follows that $\det U \geq T/2 e^{T \text{Tr} A} \det Q_T/2 > 0$, as claimed. \hfill $\Box$

Proposition 3.4 For all $x \in H$ set

$$u(x, t) := e^{(T-t)A^*} U^{-1} e^{TA} x, \quad t \in [0, T],$$  

and define $a(x, \cdot) := Q_T u(x, \cdot)$. Then $a(x, T) = e^{TA} x$. Moreover, there are $c_T, c_{1, T} > 0$ such that

$$|u(t, x)|_H \leq c_T |x|_H \quad \text{for all } t \in [0, T], \quad x \in H,$$  

and

$$|a(x, t)|_H \leq c_{1, T} |x|_H \quad \text{for all } t \in [0, T], \quad x \in H.$$  

Proof Write

$$a(x, T) = \int_0^T K(T, s) u(x, s) ds.$$
A probabilistic formula for gradients of solutions... 373

\[
\begin{align*}
&= \int_0^T \left( \int_0^s e^{(T-r)A} C e^{(s-r)A^*} dr \right) e^{(T-s)A^*} U^{-1} e^{T Ax} ds \\
&= \int_0^T \left( \int_0^s e^{(T-r)A} C e^{(s-r)A^*} dr \right) U^{-1} e^{T Ax} ds \\
&= \int_0^T (T-r) e^{(T-r)A} C e^{(T-r)A^*} dr U^{-1} e^{T Ax} = e^{T Ax},
\end{align*}
\]

as required. Finally,

\[
|u(x, t)|_H \leq \sup_{s \in [0, T]} \|e^{sA}\|_{\mathcal{L}(H)}^2 \|U^{-1}\|_{\mathcal{L}(H)} |x|_H, \quad t \in [0, T],
\]

so that (9) and (10) follow easily. \(\square\)

Now we prove the first new result of the paper.

**Proposition 3.5** Under Hypotheses 1.1 and 1.2 the semigroup \(R_T^{\alpha r}\), \(T > 0\), is strong Feller.

**Proof** Let \(\varphi \in B(\mathbb{O}_r)\) and \(x_0, x \in \mathbb{O}_r\). Then by (6) we obtain

\[
| R_T^{\alpha r} \varphi(x) - R_T^{\alpha r} \varphi(x_0) |
\leq \|\varphi\|_X \int X \left| \exp \left\{ -\frac{1}{2} F(x) + G(x, h) \right\} \right| N_{\mathbb{Q}_T} (dh) \\
+ \|\varphi\|_X \int X \left\{ 1_{\{ h + d(x_0, \cdot) \leq r \}} \setminus 1_{\{ h + d(x, \cdot) \leq r \}} \right\} \exp \left\{ -\frac{1}{2} F(x_0) + G(x_0, h) \right\} N_{\mathbb{Q}_T} (dh)
\]

\(=: A_1 + A_2.\)

Taking into account (9) we have

\[
F(x) = (\mathbb{Q}_T^{-1} a(x, \cdot), a(x, \cdot))_X = \langle u(x, \cdot), \mathbb{Q}_T u(x, \cdot) \rangle_H \leq \|\mathbb{Q}_T\|_{\mathcal{L}(H)} c_T^2 |x|_H^2
\]

and

\[
|G(x, h)| = |(\mathbb{Q}_T^{-1} a(x, \cdot), h)_X| \leq c_T T |x|_H |h|_X.
\]

Therefore

\[
\exp \left\{ -\frac{1}{2} F(x) + G(x, h) \right\} - \exp \left\{ -\frac{1}{2} F(x_0) + G(x_0, h) \right\}
= \int_0^1 \exp \left\{ -\frac{1}{2} F((1 - \alpha)x_0 + \alpha x) + G((1 - \alpha)x_0 + \alpha x, h) \right\} (x - x_0) \, d\alpha
\leq \int_0^1 \exp \left\{ G((1 - \xi)x_0 + \xi x, h) \right\} |x - x_0| \, d\alpha \leq \exp \left\{ c_T T |x|_H |h|_X \right\} |x - x_0|.
\]
It follows that

\[ A_1 \leq \|\varphi\|_{\infty} \int_X \exp \{c_T T|x|_H |h|_X \} dN_{Q_T} |x - x_0|. \]

Since the integral above is finite, we have \( \lim_{x \to x_0} A_1 = 0 \). Concerning \( A_2 \), we have \( \lim_{x \to x_0} A_2 = 0 \) by the continuity of \( d(x, \cdot) \) and the Dominated Convergence Theorem.

\[ \square \]

4 Approximating family of operators

We define an approximating family of operators \( R^O_{T, n} \varphi \) on \( B_b(\overline{O}) \) setting for all \( n \in \mathbb{N} \),

\[ R^O_{T, n} \varphi(x) = \int_{\{\Gamma_n(h + d(x, \cdot)) \leq r\}} \varphi(h(T)) \exp \left\{ -\frac{1}{2} F(x) + G^n(x, h) \right\} N_{Q_T} (dh), \]

where \( F(x), x \in \overline{O} \), is defined by (5), \( d(x, t) \) by (7) and \( \Gamma_n \) by

\[ \Gamma_n(h + d(x, \cdot)) = \sup \left\{ g(h(t_j) + d(x, t_j)), \ t_j = \frac{jT}{2^n}, \ j = 0, 1, \ldots, 2^n \right\} \]

for all \( h \in E, n \in \mathbb{N} \), and

\[ G^n(x, h) = \sum_{j=1}^{2^n} (u(x, t_j) \cdot h(t_j)) (t_j - t_{j-1}), \ x \in \overline{O}, \ h \in E. \]

Lemma 4.1 (i) It holds

\[ \left| \Gamma_n(h + d(x, \cdot)) - \Gamma_n(h_1 + d(x, \cdot)) \right| \leq a + be^{|h|_E + |h_1|_E}, \ h_1, h_2 \in E. \]

(ii) Moreover \( h \mapsto \Gamma_n(h + d(x, \cdot)) \) belongs to \( W^{1,2}(E, N_{Q_T}) \) and

\[ \left| \Gamma'_n(h + d(x, \cdot)) \cdot d(x, \cdot) \right| \leq (a + be^{2|h|_E})|d(x, \cdot)|. \]

Proof (i) follows from Hypothesis 1.2 (ii) and (ii) is a well-known consequence of the local Lipschitzianity of \( \Gamma_n \).

\[ \square \]

Proposition 4.2 Under Hypotheses 1.1, 1.2 for all \( \varphi \in B(\overline{O}) \),

\[ \lim_{n \to \infty} R^O_{T, n} \varphi(x) = R^O_{T} \varphi(x) \quad \text{for all} \ x \in \overline{O}. \]
Proof Let \( \varphi \in B_b(\overline{O}_r) \). Then

\[
\left| R_{T,n}^O \varphi(x) - R_T^O \varphi(x) \right| \\
\leq \| \varphi \|_{\infty} \int \exp \left\{ -\frac{1}{2} F(x) \right\} \left| \exp \{ G^n(x, h) \} - \exp \{ G(x, h) \} \right| N_{Q_T}(dh) \\
+ \| \varphi \|_{\infty} \int \exp \left\{ -\frac{1}{2} F(x) + G(x, h) \right\} N_{Q_T}(dh).
\]

Taking into account (9), we have

\[
|G^n(x, h)| \leq \sum_{j=1}^{2^n} |u(x, t_j)|_H |h(t_j)|_H (t_j - t_{j-1}) \leq c_T |x|_{C(\overline{O}_r)} |h|_E,
\]

\( x \in \overline{O}_r, \ h \in E \). Now, set

\[
B_n := \left\{ h : g(h(t_j)) + d(x, t_j) \leq r, \ t_j = \frac{jT}{2^n}, \ j = 0, 1, \ldots, 2^n \right\},
\]

\[
B = \left\{ h : g(h(t)) + d(x, t) \leq r, \ t \in [0, T] \right\}.
\]

Then \( B \subset B_n \) and \( \bigcap_{n \in \mathbb{N}} B_n = B \), so that \( N_{Q}(B_n) \downarrow N_{Q}(B) \) as \( n \to \infty \). Moreover,

\[
\lim_{n \to \infty} G^n(x, h) = G(x, h) \quad \text{for all} \quad h \in E, \ \ x \in \overline{O}_r,
\]

and by (8) there is \( c_T > 0 \) such that

\[
\exp \{ G^n(x, h) \} \leq e^{c_T|h|_E} \quad \text{for all} \quad h \in E, \ \ x \in \overline{O}_r.
\]

(13)

The conclusion follows from the Dominated Convergence Theorem. \( \square \)

It is useful to write an expression of \( R_{T,n}^O \varphi \) as a finite dimensional integral. To this purpose we consider the linear mapping

\[
E = C([0, T]; H) \to H^{2^n}, \quad h \mapsto (h(t_1), h(t_2), \ldots, h(t_{2^n})),
\]

\( t_j = \frac{jT}{2^n}, \ j = 0, 1, \ldots, 2^n \), whose law is obviously Gaussian, say \( N_{Q_T,n} \). Then for any \( n \in \mathbb{N} \) and any \( \varphi : H^{2^n} \to \mathbb{R} \) bounded and Borel we have

\[
\int_E \varphi(h(t_1), h(t_2), \ldots, h(t_{2^n})) N_{Q_T}(dh) = \mathbb{E} \left[ \varphi(\overline{W}_A(t_1), \overline{W}_A(t_2), \ldots, \overline{W}_A(t_{2^n})) \right]
\]

\[
= \int_{H^{2^n}} \varphi(\xi_1, \ldots, \xi_{2^n}) N_{Q_T,A}(d\xi_1 \cdots d\xi_{2^n}).
\]

\( \square \) Springer
Now we can write the approximating operator as an integral over $H^{2n}$, namely

$$R_{T,n}^{O_T} \varphi(x) = \int_{\{\Gamma_n(\xi + d(x, \cdot)) \leq r\}} \varphi(\xi_{2n}) \exp\left\{ -\frac{1}{2} F(x) + G^n(x, \xi) \right\} N_{Q_{T,n}}(d\xi),$$

$\xi \in H^{2n},$ where

$$\Gamma_n(\xi + d(x, \cdot)) = \sup \left\{ g(\xi_j + d(x, t_j)), \ t_j = \frac{jT}{2^n}, j = 0, 1, \ldots, 2^n \right\}$$

and

$$G^n(x, \xi) = \sum_{j=1}^{2^n} (u(x, t_j) \cdot \xi_j)(t_j - t_{j-1}), \ x \in \overline{O_T}, \ \xi \in H^{2n}.$$

\textbf{Proposition 4.3} $Q_{T,n}$ has a bounded inverse for all $n \in \mathbb{N}$, so the Cameron–Martin space of $N_{Q_{T,n}}$ is the whole $H^{2n}$.

\textbf{Proof} Let $n \in \mathbb{N}$, then by (15) we have

$$\mathbb{E}\left[e^{i\lambda \sum_{h=1}^{2^n} \langle \xi_h, W_A(t_h) \rangle_H}\right] = e^{-\frac{1}{2} \lambda^2 \langle Q_{T,n} \xi, \xi \rangle H^{2n}}, \ \xi = (\xi_1, \ldots, \xi_{2^n}).$$

We claim that if $Q_{T,n} \xi = 0$ then $\xi = 0$. In fact, if $Q_{T,n} \xi = 0$, we have

$$\mathbb{E}\left[e^{i\lambda \sum_{h=1}^{2^n} \langle \xi_h, W_A(t_h) \rangle_H}\right] = 1$$

and so,

$$\sum_{h=1}^{2^n} \langle \xi_h, W_A(t_h) \rangle_H = 0 \ \text{N}_{Q_{n,T}}\text{-a.s.}$$

Now, setting $L(t) = \int_0^t e^{-sA} dW(s)$, and $\rho_i = e^{-t_i A^*} \xi_i$, we have

$$\langle \rho_1 + \rho_2 + \cdots + \rho_{2^n}, L(t_1) \rangle + \langle \rho_2 + \cdots + \rho_{2^n}, L(t_2) - L(t_1) \rangle + \cdots + \langle \rho_{2^n}, L(t_{2^n}) - L(t_{2^n-1}) \rangle = 0.$$ 

Multiplying both sides by $\langle \rho_1 + \rho_2 + \cdots + \rho_{2^n}, L(t_1) \rangle$ and taking expectation, yields

$$\rho_1 + \rho_2 + \cdots + \rho_{2^n} = 0,$$

since $\mathbb{E}(v, L(t_1))^2 = \langle Q_{T,v} v, v \rangle$ and $Q_{T,v}$ is non-singular by Hypothesis 1.1.

Similarly we obtain $\rho_k + \rho_{k+1} + \cdots + \rho_{2^n} = 0$ for $k = 2, 3, \ldots, 2^n$, which finally implies $\xi = 0$. \qed

\textcopyright Springer
5 Differentiating the approximating family of operators

First note that by (12) we have

$$\mathcal{R}_n^h(x, h) \cdot (d_x(x, \cdot) y) = \sum_{j=1}^{2^n} (u(x, t_j) \cdot d_x(x, t_j) y) (t_j - t_{j-1}),$$

$x \in \overline{\Omega}_r, h \in E$.

**Lemma 5.1** For all $x \in \overline{\Omega}_r, y \in H, n \in \mathbb{N}, \varphi \in B_b(\overline{\Omega}_r)$ we have

$$D_x R_{T,n}^\varphi(x) \cdot y = M_1(n, x, y) + M_2(n, x, y),$$

where

$$M_1(n, x, y) = \int_{\Gamma_n(h + d(x, \cdot))} \varphi(h(T)) \exp \left\{ -\frac{1}{2} F(x) + G^n(x, h) \right\} \times \left( -\frac{1}{2} F_x(x) y + G^n_x(x, h) y - G^n_h(x, h) \cdot (d_x(x, \cdot) y) \right) N_{\mathcal{Q}_T}(dh)$$

and

$$M_2(n, x, y) = \int_{\Gamma_n(h + d(x, \cdot))} \varphi(h(T)) \exp \left\{ -\frac{1}{2} F(x) + G^n(x, h) \right\} \times \langle Q^{-1/2}_{T,n}(d_x(x, \cdot) y), Q^{-1/2}_{T,n} h \rangle_{H^{2^n}} N_{\mathcal{Q}_T}(dh),$$

with

$$\langle Q^{-1/2}_{T,n}(d_x(x, \cdot) y), Q^{-1/2}_{T,n} h \rangle_{H^{2^n}} = \sum_{i,j=1}^{2^n} (Q^{-1}_{T,n})_{i,j} (d_x(x, t_i) y) \cdot h(t_j),$$

where

$$(Q^{-1}_{T,n})_{i,j} = \langle Q^{-1}_{T,n} \psi_i, \psi_j \rangle_{H^{2^n}}, \quad i, j = 1, \ldots, 2^n,$$

and $(\psi_j)$ is the standard orthogonal basis of $H^{2^n}$.

**Proof** We first write identity (11) as

$$R_{T,n}^\varphi(x) = \int_{\Gamma_n(\xi + d(x, \cdot))} \varphi(\xi^{2^n}) \exp \left\{ -\frac{1}{2} F(x) + G^n(x, \xi) \right\} N_{\mathcal{Q}_{T,n}}(d\xi).$$
Then we drop the dependence on $x$ under the domain of integration by making the translation $\xi \mapsto \xi - d(x, \cdot)$ and recalling that $d(x, T) = 0$; we write

$$R_{T,n}^{O_r} \varphi(x) = \int_{\{\Gamma_n(\xi) \leq r\}} \varphi(\xi_{2^n}) \exp \left\{ -\frac{1}{2} F(x) + G^n(x, \xi - d(x, \cdot)) \right\} N_{d(x, \cdot), Q_{T,n}}(d\xi).$$

So, using again the Cameron–Martin Theorem (this is possible thanks to Proposition 4.3), we have

$$R_{T,n}^{O_r} \varphi(x) = \int_{\{\Gamma_n(\xi) \leq r\}} \varphi(\xi_{2^n}) \exp \left\{ -\frac{1}{2} F(x) + G^n(x, \xi - d(x, \cdot)) \right\} \chi_n(x, \xi) N_{Q_{T,n}}(d\xi),$$

(17)

where

$$\chi_n(x, \xi) = \exp \left\{ -\frac{1}{2} \left| Q_{T,n}^{-1/2} d(x, \cdot) \right|_{H^{2^n}} + \left| Q_{T,n}^{-1/2} (\xi - d(x, \cdot)) \right|_{H^{2^n}} \right\}.$$

We now can differentiate $R_{T,n}^{O_r} \varphi(x)$ in any given direction $y \in H$. Taking into account that for any $x, \ y \in H$ we have

$$D_{\xi} \chi_n(x, \xi) \cdot y = \left( Q_{T,n}^{-1/2}(d_{\xi}(x, \cdot) \cdot), Q_{T,n}^{-1/2}(\xi - d(x, \cdot)) \right)_{H^{2^n}} \chi_n(x, \xi),$$

we find

$$D_{\xi} R_{T,n}^{O_r} \varphi(x) \cdot y = \int_{\{\Gamma_n(\xi) \leq r\}} \varphi(\xi_{2^n}) \exp \left\{ -\frac{1}{2} F(x) + G^n(x, \xi - d(x, \cdot)) \right\}$$

$$\times \left[ -\frac{1}{2} F_{x}(x) y + G^n_{x}(x, \xi - d(x, \cdot)) y$$

$$- \left( G^n_{\xi}(x, \xi - d(x, \cdot)), d_{\xi}(x, \cdot) \cdot y \right)_{H^{2^n}}$$

$$+ \left( Q_{T,n}^{-1/2}(d_{\xi}(x, \cdot) \cdot), Q_{T,n}^{-1/2}(\xi - d(x, \cdot)) \right)_{H^{2^n}}$$

$$\times \chi_n(x, \xi) N_{Q_{T,n}}(d\xi).$$

(18)

Here $F_{x}$ and $G_{x}$ denote the derivatives with respect to $x$ of $F$ and $G$ respectively, whereas $G_{\xi}$ is the derivative with respect to $\xi$. Now making the opposite translation $\xi_j \mapsto \xi_j + d(x, t_j), \ j = 0, 1, \ldots, 2^n$, we obtain
A probabilistic formula for gradients of solutions...

\[ D_x R_{T,n}^{\Omega} \varphi(x) \cdot y = \int_{\{ \Gamma_n(x+ \tilde{d}(x, \cdot)) \leq r \}} \varphi( \xi_2 ) \exp \left\{ -\frac{1}{2} F(x) + G^n(x, \xi) \right\} \times \left( -\frac{1}{2} F_x(x) y + G^n_x(x, \xi)y \right. \]
\[ \left. - \langle G^n_\xi(x, \xi), d_x(x, \cdot) y \rangle_{H^n} \right) N_{Q_{T,n}}(d\xi) \]
\[ + \int_{\{ \Gamma_n(x+ \tilde{d}(x, \cdot)) \leq r \}} \varphi( \xi_2 ) \exp \left\{ -\frac{1}{2} F(x) + G^n(h, \xi) \right\} \times \left( \langle Q_{T,n}^{-1/2}(d_x(x, \cdot) y), Q_{T,n}^{-1/2}(\xi) \rangle_{H^n} \right) N_{Q_{T,n}}(d\xi). \]

Finally, we arrive at the conclusion making the change of variables (14).

In Sect. 4 we shall easily prove the existence of the limit of \( M_1(n, x, y) \) as \( n \to \infty \). However, a problem arises, as stated in the introduction, for the term \( M_2(n, x, y) \) due to the factor

\[ \langle Q_{T,n}^{-1/2}(d_x(x, \cdot) y), Q_{T,n}^{-1/2}h \rangle_{H^n}, \]

because \( d_x(x, \cdot) y \) does not belong to \( Q_{T,n}^{1/2}(X) \). So, in the next Lemma 5.6 we look for a different expression of \( M_2(n, x, y) \) that does not contain this term. To this purpose, we recall the definition and some properties of the Sobolev space \( W^{1,p}(E, N_{Q_T}) \). We shall need a result which is a straightforward generalisation of [6, Proposition 6.1.5].

**Lemma 5.2** For any \( \varphi \in C^1_b(E) \) there exists a sequence \( (\varphi_n) \in C^1_b(X) \) such that

(i) \( \lim_{n \to \infty} \varphi_n(h) = \varphi(h) \) for all \( h \in E \).

(ii) \( \lim_{n \to \infty} \langle D\varphi_n(h), \eta \rangle_X = D\varphi(h) \cdot \eta \) for all \( h, \eta \in E \).

**Proof** For any \( \varphi \in C^1_b(E) \) set

\[ \varphi_n : H \to E, \quad x \mapsto \varphi_n(x)(t) = \frac{n}{2} \int_{t-n}^{t+\frac{1}{n}} \hat{\varphi}(s) ds, \quad t \in [0, T], \]

where \( \hat{\varphi}(s) \) is the extension by oddness of \( \varphi(s) \), for \( s \in (-T, 0) \) and \( s \in (T, 2T) \). Then it is easy to check that \( (\varphi_n) \) fulfills (i) and (ii).

The following result is similar to [3, Proposition 4.2].

**Proposition 5.3** For all \( \varphi \in C^1_b(E) \) and any \( \eta \in Q^{1/2}(X) \subset E \) the following integration by parts formula holds:

\[ \int_E D\varphi(h) \cdot \eta N_{Q_T}(dh) = \int_E \varphi(h) \langle Q^{-1/2}h, Q^{-1/2}\eta \rangle_H N_{Q_T}(dh). \]
Proof Let \( \varphi_n \in C^1_b(X) \) be a sequence as in Lemma 5.2, then we have
\[
\int_H (D\varphi_n(h), \eta) x N_{Q_T} (dh) = \int_H \varphi_n(x) \langle Q^{-1/2} h, Q^{-1/2} \eta \rangle_H dN_{Q_T} (dh).
\]
The conclusion follows letting \( n \to \infty \). \( \square \)

**Corollary 5.4** For all \( \varphi, \psi \in C^1_b(E) \) and any \( \eta \in Q^{1/2}(X) \subset E \) the following integration by parts formula holds:
\[
\int_E D\varphi \cdot \eta \psi dN_{Q_T} = - \int_E D\psi \cdot \eta \varphi dN_{Q_T} + \int_E \varphi \psi \langle Q^{-1/2} x, Q^{-1/2} \eta \rangle x dN_{Q_T}.
\]

**Remark 5.5** By (20) it follows, by standard arguments, that the gradient operator \( D \) is closable in \( L^p(E, N_{Q}) \) for any \( p \geq 1 \); we shall still denote by \( D \) its closure and by \( W^{1,p}(E, N_{Q}) \) its domain. Finally, it is well known that every Lipschitz continuous function \( \varphi: E \to \mathbb{R} \) belongs to \( W^{1,p}(E, N_{Q}) \). See Lemma 4.1.

Now we are ready to prove the announced lemma.

**Lemma 5.6** Assume Hypotheses 1.1 and 1.2. Let \( M_2(n, x, y) \) be given by (16). Then for all \( \varphi \in B_b(\overline{\mathbb{R}}) \), \( x \in \overline{\mathbb{R}} \), \( y \in H \), \( n \in \mathbb{N} \), the following identity holds:
\[
M_2(n, x, y) = \int_{\{\Gamma_n(h+d(x, \cdot)) \leq r\}} \varphi(h(T)) D_h \exp \left\{ -\frac{1}{2} F(x) + G^n(x, h) \right\} \times (d_x(x, \cdot) y) N_{Q_T}(dh) \]
\[
+ \lim_{\epsilon \to 0} \frac{1}{2\epsilon} \int_{\{r-\epsilon \leq \Gamma_n(h+d(x, \cdot)) \leq r+\epsilon\}} \varphi(h(T)) (\Gamma'_n(h + d(x, \cdot))) \times (d_x(x, \cdot) y) N_{Q_T}(dh)
\]
\[
= M_{2,1}(n, x, y) + M_{2,2}(n, x, y).
\]

**Proof** By (19) we have
\[
M_2(n, x, y) = \int_{\{\Gamma_n(\xi+d(x, \cdot)) \leq r\}} \varphi(\xi_2^n) \exp \left\{ -\frac{1}{2} F(x) + G^n(h, \xi) \right\} \times \left\{ Q_{-1/2}^T (d_x(x, \cdot) y), Q_{-1/2}^T (\xi) \right\}_{H^{2n}} N_{Q_{T,n}}(d\xi).
\]

Let us first assume in addition that \( \varphi \in C^1(\overline{\mathbb{R}}) \). Then we argue as in [3, Proposition 4.5] defining a mapping \( \theta_\epsilon: \mathbb{R} \to \mathbb{R} \),
\[
\theta_\epsilon(s) = \begin{cases} 0, & \text{if } s \leq r - \epsilon, \\ \frac{1}{\epsilon}(s - r + \epsilon), & \text{if } r - \epsilon \leq s \leq r + \epsilon, \\ 1, & \text{if } s \geq r + \epsilon. \end{cases}
\]


Then we approximate $M_2(n, x, y)$ by setting

$$M_2^\epsilon(n, x, y) = \int_{H^2_n} \theta_\epsilon (\Gamma_n (\xi + d(x, \cdot))) \phi(\xi_2^n) \exp \left\{-\frac{1}{2} F(x) + G^n(x, \xi) \right\}$$

$$\times \left\{ \langle Q_{T, n}^{-1/2} (d_x (x, \cdot), y) , Q_{T, n}^{-1/2} \xi \rangle_{H^2_n} N_{Q_{T, n}} (d\xi) \right\},$$

so that $\lim_{\epsilon \to 0} M_2^\epsilon(n, x, y)$ exists and is given by

$$\lim_{\epsilon \to 0} M_2^\epsilon(n, x, y) = \int_{\left\{ \Gamma_n (\xi + d(x, \cdot)) \leq r \right\}} \phi(\xi_2^n) \exp \left\{-\frac{1}{2} F(x) + G^n(x, \xi) \right\}$$

$$\times \left\{ \langle Q_{T, n}^{-1/2} (d_x (x, \cdot), y) , Q_{T, n}^{-1/2} \xi \rangle_{H^2_n} N_{Q_{T, n}} (d\xi) \right\}.$$

Now, by a classical integration by parts formula, see e.g. [2], we have

$$M_2^\epsilon(n, x, y) = \int_{H^2_n} \theta_\epsilon (\Gamma_n (\xi + d(x, \cdot))) (D_h \phi(\xi_2^n) \cdot d_x (x, T)y)$$

$$\times \exp \left\{-\frac{1}{2} F(x) + G^n(x, \xi) \right\} N_{Q_{T, n}} (d\xi)$$

$$+ \int_{H^2_n} \theta_\epsilon (\Gamma_n (\xi + d(x, \cdot))) \phi(\xi_2^n)$$

$$\times \left\{ D_h \exp \left\{-\frac{1}{2} F(x) + G^n(x, \xi) \right\} \cdot d_x (x, \cdot) y \right\} N_{Q_{T, n}} (d\xi)$$

$$+ \int_{H^2_n} (D_h \theta_\epsilon (\Gamma_n (\xi + d(x, \cdot))) \cdot (d_x (x, \cdot) y) \phi(\xi_2^n)$$

$$\times \exp \left\{-\frac{1}{2} F(x) + G^n(x, \xi) \right\} N_{Q_{T, n}} (d\xi). \quad (22)$$

Taking into account that the first integral vanishes, because $d_x (x, T)y = 0$, and that

$$\left\{ D\theta_\epsilon (\Gamma_n a(\xi + d(x, \cdot))) , (d(x, \cdot)) \right\}_{H^2_n}$$

$$= \theta_\epsilon' (\Gamma_n (\xi + d(x, \cdot))) \langle \Gamma_n' (\xi + d(x, \cdot)) , d(x, \cdot) y \rangle_{H^2_n}$$

$$= \frac{1}{2\epsilon} \left\{ \Gamma_n' (\xi + d(x, \cdot)) , d_x (x, \cdot) y \right\}_{H^2_n} \mathbb{1}_{[r-\epsilon, r+\epsilon]},$$

we deduce by (22), letting $\epsilon \to 0$, that

$$M_2(n, x, y)$$

$$= \int_{\left\{ \Gamma_n (\xi + d(x, \cdot)) \leq r \right\}} \phi(\xi_2^n) \left\{ D_h \exp \left\{-\frac{1}{2} F(x) + G^n(x, \xi) \right\} \cdot d_x (x, \cdot) y \right\} N_{Q_{T, n}} (d\xi)$$

$$+ \lim_{\epsilon \to 0} \frac{1}{2\epsilon} \int_{[r-\epsilon \leq \Gamma_n (\xi + d(x, \cdot)) \leq r+\epsilon]} \left\{ \Gamma_n' (\xi + d(x, \cdot)) , d(x, \cdot) y \right\}_{H^2_n} \phi(\xi_2^n)$$

$$\times \exp \left\{-\frac{1}{2} F(x) + G^n(x, \xi) \right\} N_{Q_{T, n}} (d\xi).$$

 Springer
Then the conclusion of the lemma follows (by the change of variables (14)) when \( \varphi \in C^1(\overline{O_r}) \). The case \( \varphi \in C(\overline{O_r}) \) can be handled by a uniform approximation of \( \varphi \) by \( C^1(\overline{O_r}) \) functions. Finally, if \( \varphi \in B_r(\overline{O_r}) \) we conclude using the strong Feller property of the semigroup, see Proposition 3.5. \( \square \)

We still need to compute the limit in identity (21). This will require the Ehrhard inequality.

### 5.1 Applying the Ehrhard inequality

Define

\[
\Lambda_x(s) := N_{\mathbb{Q}^T}(\Gamma(h + d(x, \cdot)) \leq s),
\]

\[
\Lambda_{n,x}(s) := N_{\mathbb{Q}^T}(\Gamma_n(\xi + d(x, \cdot)) \leq s) \quad \text{for all} \quad s > 0, \quad x \in \overline{O_r}, \quad n \in \mathbb{N}.
\]

Since by Hypothesis 1.2(i), \( g \) is convex, the mapping \( \Gamma(\cdot + d(x, \cdot)) \) (resp. \( \Gamma_n(\cdot + d(x, \cdot)) \)) is convex as well. By applying the Ehrhard inequality (see e.g. [2, Theorem 4.4.1]) we see that for any \( x \in \overline{O_r} \) the real function

\[
[0, +\infty) \rightarrow \mathbb{R}, \quad s \mapsto S_x(s) := \Phi^{-1}(\Lambda_x(s))
\]

(resp. \( [0, +\infty) \rightarrow \mathbb{R}, \quad s \mapsto S_{n,x}(s) := \Phi^{-1}(\Lambda_{n,x}(s)) \)),

where

\[
\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{1}{2} u^2} du, \quad z \in \mathbb{R},
\]

is concave. Note that \( \Phi^{-1} \) maps \((0, 1)\) into \((-\infty, +\infty)\). As a consequence, \( \Lambda_x(\cdot) \) (resp. \( \Lambda_{n,x}(\cdot) \)) is differentiable at any \( s > 0 \) up to a discrete set \( N_\ell \) where there exist the left and the right derivatives; we shall denote by \( D^+_\ell \Lambda_x(s) \) (resp. \( D^+_\ell \Lambda_{n,x}(s) \)) the right derivative at any discontinuity point, and also (with the same symbol) the derivative at the other points.

It follows that \( N_{\mathbb{Q}^T} \circ (\Gamma(h + d(x, \cdot)))^{-1} \) (resp. \( N_{\mathbb{Q}^T} \circ (\Gamma_n(h + d(x, \cdot)))^{-1} \)) is absolutely continuous with respect to the Lebesgue measure \( \ell \) in \( \mathbb{R} \), so that

\[
\frac{dN_{\mathbb{Q}^T} \circ (\Gamma(h + d(x, \cdot)))^{-1}}{d\ell}(s) = D^+_\ell \Lambda_x(s),
\]

(resp. \( \frac{dN_{\mathbb{Q}^T} \circ (\Gamma_n(h + d(x, \cdot)))^{-1}}{d\ell}(s) = D^+_\ell \Lambda_{n,x}(s) \)), \( s > 0 \).

Note that, for any \( x \in H \), \( \Lambda_{n,x}(s) \) is increasing in \( s \) and decreasing in \( n \). Moreover, \( \Lambda_{n,x}(0) = 0 \) and \( \Lambda_{n,x}(s) \uparrow 1 \) as \( s \rightarrow \infty \). Also, it results

\[
D^+ S_x(s) = \sqrt{2\pi} e^{\frac{1}{2} s^2} D^+ \Lambda_x(s).
\]
Now we are going to estimate $D^+_r \Lambda_{n,x}(s)$ independently of $n$, $x$ and $s \in [r/2, 3r/2]$. Then we shall show that $D^+_r \Lambda_{n,x} \to D^+_r \Lambda_x$ as $n \to \infty$.

**Lemma 5.7** There exists $K_r > 0$ independent of $x, n, s$ such that
\[
D^+_r \Lambda_{n,x}(s) \leq K_r \quad \text{for all } x \in \overline{O}_r,\ n \in \mathbb{N},\ s \in [r/2, 3r/2].
\] (25)

Moreover, it results
\[
\lim_{n \to \infty} D^+_r \Lambda_{n,x} = D^+_r \Lambda_x \quad \text{for all } x \in \overline{O}_r.
\] (26)

**Proof** We proceed in three steps.

**Step 1.** There is $l_1 > 0$ such that
\[
0 < l_1 \leq \Lambda_{n,x}(s) \quad \text{for all } x \in \overline{O}_r,\ n \geq 2,\ s \in [r/2, 3r/2].
\] (27)

It is enough to show (27) for $\Lambda_{2,x}(s)$, because $\Lambda_{2,x}(s) \geq \Lambda_{n,x}(s)$ for $n \geq 2$.

Since the convex set $\{ \xi \in H^{2s}_1; \varnothing_2(\xi + d(x, \cdot)) < s \}$ is open and non-empty and the measure $N_{Q_{T,2}}$ is non-degenerate by Proposition 4.3, it follows that there is $l_1(x)$ such that
\[
0 < l_1(x) \leq \Lambda_{n,x}(s) \quad \text{for all } x \in \overline{O}_r,\ n \geq 2,\ s \in [r/2, 3r/2].
\]

Now Step 1 follows because $\overline{O}_r$ is compact.

**Step 2.** There is $l_2 < 1$ such that
\[
\Lambda_{n,x}(s) < l_2 < 1 \quad \text{for all } x \in \overline{O}_r,\ n \in \mathbb{N},\ s \in [r/2, 3r/2].
\]

In fact, thanks to Hypothesis 1.2 (ii), there exists $M > 0$ such that
\[
\Lambda_{n,x}(s) \leq M \quad \text{for all } x \in \overline{O}_r,\ n \in \mathbb{N},\ s \in [r/2, 3r/2].
\]

**Step 3.** Conclusion.

Note first that
\[
S_{n,x}(s) \downarrow S_x(s) \quad \text{as } n \to \infty, \ x \in \overline{O}_r.
\]

The sequence $(S_{n,x}(\cdot))$ is obviously increasing and also concave by the Ehrhard inequality. Therefore, all elements of $(S'_{n,x}(\cdot))$ are positive and decreasing; so, they are BV in the interval $[r/2, 3r/2]$.

We claim that the sequence $(S'_{n,x}(r))$ is equi-bounded in $[r/2, 3r/2]$ in BV norm. To show the claim, it is enough to realize that $(S'_{n,x}(r))$ is equi-bounded at $r_1$ (because it is decreasing). In fact, since $S_{n,x}$ is concave we have if $0 < \epsilon \leq r/2$,
\[
S'_{n,x}(r_1) \leq \frac{1}{\epsilon} (S_{n,x}(r_1 + \epsilon) - S_{n,x}(r_1)) \leq \frac{2}{\epsilon} S_{n,x}(r_2) = \frac{2}{\epsilon} \Phi^{-1}(r_2).
\] (28)
Therefore we can apply Helly’s selection principle, see e.g. [10, Theorem 5, p. 372] to the sequence \((S'_{n,x}(\cdot))\) and conclude that there exists a subsequence of \((S'_{n,x}(\cdot))\) still denoted by \((S'_{n,x}(\cdot))\) that converges in all points of \([r/2, 3r/2]\) to a function \(f(x, \cdot)\).

We claim that \(f(x, s)\) is the derivative of \(S_x(s)\) in \([r/2, 3r/2]\). This follows by an elementary argument. Write

\[
S_{n,x}(s) = \int_{r_1}^{s} S'_{n,x}(v) \, dv, \quad s \in [r/2, 3r/2],
\]

and recall that the real-valued function \(S_{n,x}(\cdot)\) is absolutely continuous thanks to [2, Corollary 4.4.2]. By the Dominated Convergence Theorem it follows that for \(k \to \infty\) we have

\[
S_x(s) = \int_{r_1}^{s} S'_{x}(v) \, dv, \quad s \in [r/2, 3r/2],
\]

which implies

\[
S'_x(s) = f(x, s), \quad s \in [r/2, 3r/2],
\]

as required.

Therefore there is a subsequence of \((S'_{n,x})\) which converges to \(S'_x\) and consequently all the sequence \((S'_{n,x})\) will converge to \(S'_x\). Thus \(\Lambda_x(r)\) has the right derivative for \(r \in [r/2, 3r/2], x \in \overline{O_r}\), and (26) follows.

Finally, taking into account (28) and (24), it results

\[
D^+_r \Lambda_x(r) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} S^2_x(r)} S'_x(r) \leq \frac{1}{\pi \epsilon} \Phi^{-1}(r_2).
\]

So, (25) follows. \qed

The next lemma is devoted to the computation of \(\lim_{\epsilon \to 0} M_{2,2}(n, x, y)\), defined by (21).

**Lemma 5.8** Let \(n \in \mathbb{N}, r > 0, x \in \overline{O_r}, y \in H\). Then

\[
M_{2,2}(n, x, y) = \mathbb{E}_{\mathbb{Q}_T} \left[ \varphi(h(T)) (\Gamma'_n(h + d(x, \cdot))) \cdot (d_x(x, \cdot) y) \mid \Gamma_n(h + d(x, \cdot)) = r \right] D^+_r \Lambda_x(r).
\]

**Proof** Let us recall that by (21) we have

\[
M_{2,2}(n, x, y) = \lim_{\epsilon \to 0} \frac{1}{2\epsilon} \int_{[r-\epsilon \leq \Gamma_n(h+d(x,\cdot)) \leq r+\epsilon]} \varphi(h(T)) (\Gamma'_n(h + d(x, \cdot))) \cdot (d_x(x, \cdot) y) N_{\mathbb{Q}_T}(dh).
\]

\(\square\) Springer
Taking into account (23) it follows that

\[
M_{2,2}(n, x, y) = \lim_{\epsilon \to 0} \frac{1}{2\epsilon} \int_{r-\epsilon}^{r+\epsilon} \mathbb{E}_{N_{Q_T}} \left[ \varphi(h(T)) (\Gamma'_n(h + d(x, \cdot))) \cdot (d_x(x, \cdot) y) \mid \Gamma_n(h + d(x, \cdot)) = s \right] D_{n,T}^+ A_x(s) \, ds.
\]

Note that the existence of a regular distribution of

\[
\mathbb{E}_{N_{Q_T}} \left[ \varphi(h(T)) (\Gamma'_n(h + d(x, \cdot))) \cdot (d_x(x, \cdot) y) \mid \Gamma_n(h + d(x, \cdot)) = s \right]
\]

is granted because $E$ is separable, see e.g. [9, 10.2.2]. It follows that

\[
M_{2,2}(n, x, y) = \mathbb{E}_{N_{Q_T}} \left[ \varphi(h(T)) (\Gamma'_n(h + d(x, \cdot))) \cdot (d_x(x, \cdot) y) \mid \Gamma_n(h + d(x, \cdot)) = r \right] D_{n,T}^+ A_x(r),
\]

by virtue of the Dominated Convergence Theorem. \qed

We are ready now to show the following result.

**Proposition 5.9** Assume Hypotheses 1.1 and 1.2 and let $n \in \mathbb{N}$. Then we have

\[
D_{x,T} R_{T,n}^\varphi(x) \cdot y = \int \varphi(h(T)) \exp \left\{-\frac{1}{2} F(x) + G^n(x, h) \right\} \left(\frac{1}{2} F_x(x) y + G^n_x(x, h) y \right) N_{Q_T}(dh)
\aligned
\left(\Gamma_n(h + d(x, \cdot)) \leq r \right)
\endaligned
\]

\[
+ \mathbb{E}_{N_{Q_T}} \left[ \varphi(h(T)) (\Gamma'_n(h + d(x, \cdot))) \cdot (d_x(x, \cdot) y) \mid \Gamma_n(h + d(x, \cdot)) = r \right] D_{n,T}^+ A_x(r).
\]

Moreover, there is $c_{2,T}(r) > 0$ such that the following estimate holds:

\[
\left| D_{x,T} R_{T,n}^\varphi(x) \right| 
\leq \|\varphi\|_\infty c_{2,T}(r) + \|\varphi\|_\infty \exp \left\{-\frac{1}{2} F(x) \right\}
\times \int \exp \left\{c_T \sqrt{h/E} \left(\frac{1}{2} \|F_x(x)\|_{L(H)} + T c_{2,T}(r) \|U^{-1} \|_{L(H)} \|h\|_E \right) \right\} N_{Q_T}(dh).
\]

**Proof** From Lemmas 5.1, 5.6 and 5.8 we obtain

\[
D_{x,T} R_{T,n}^\varphi(x) \cdot y = M_1(n, x, y) + M_{2,1}(n, x, y) + M_{2,2}(n, x, y)
\]

 Springer
\[ D_x R_{T,n}^{\varphi(x) \cdot y} \]
\[ = \int_{\{ \Gamma_n(h + d(x, \cdot)) \leq r \}} \varphi(h(T)) \exp \left\{ -\frac{1}{2} F(x) + G^n(x, h) \right\} \]
\[ \times \left( -\frac{1}{2} F(x) y + G^n(x, h) y - (G^n_h(x, h) \cdot (d_x(x, \cdot) y)) \right) N_{\mathbb{Q}_T} (dh) \]
\[ + \int_{\{ \Gamma_n(h + d(x, \cdot)) \leq r \}} \varphi(h(T)) \left( D_h \exp \left\{ -\frac{1}{2} F(x) + G^n(x, h) \right\} \cdot (d_x(x, \cdot) y) \right) N_{\mathbb{Q}_T} (dh) \]
\[ + \mathbb{E} \left[ \varphi(h(T)) (\Gamma^n_n(h + d(x, \cdot)) \cdot (d_x(x, \cdot) y) \mid \Gamma_n(h + d(x, \cdot)) = r \right] D_T^+ \Lambda_{n,x}(r). \]

Since
\[ D_h \exp \{ G^n(x, h) \} \cdot (d_x(x, \cdot) y) = \exp \{ G^n(x, h) \} G^n_h(d_x(x, \cdot) y) \cdot (d_x(x, \cdot) y), \]

letting \( n \to \infty \) we obtain, after some simplifications, identity (29). Finally, we prove (30). First by (13) we have
\[ \exp \{ G^n(x, h) \} \leq \exp \{ c_T |h|_E \}. \]

Moreover by (12) it follows that
\[ G^n_h(x, h) = \sum_{j=1}^{2^n} u_x(x, t_j) \cdot h(t_j) (t_j - t_{j-1}), \quad x \in \Theta_r, \quad h \in E, \]

and therefore we have
\[ |G^n_h(x, h)| \leq T \| u_x(x, \cdot) \|_{L(H)} \| h \|_E \leq T c_T^2 \| U^{-1} \|_{L(H)} \| h \|_E, \quad x \in \Theta_r, \quad h \in E. \]

Now, by Hypothesis 1.2 (ii) there exists \( c_{1_T} > 0 \) such that
\[ |\Gamma^n_n(h + d(x, \cdot)) \cdot (d_x(x, \cdot) y)| \leq c_{1,T} |h|_E. \]

Finally, taking into account (25) and Lemma 4.1, we have
\[ |\mathbb{E}_{\mathbb{N}_T} [\varphi(h(T)) (\Gamma^n_n(h + d(x, \cdot)) \cdot (d_x(x, \cdot) y) \mid \Gamma_n(h + d(x, \cdot)) = r] D_T^+ \Lambda_{n,x}(r) | \]
\[ \leq \| \varphi \| \infty c_{1,T} |D_T^+ \Lambda_x(r)| \leq \| \varphi \| \infty c_{1,T} K r. \]

The result is proved. \( \square \)
6 Main results

Now we take \( \varphi \in B_b(H) \), \( T > 0 \) and prove a representation formula for \( D_x R_T^{O_r} \varphi(x) \).

**Theorem 6.1** Assume Hypotheses 1.1 and 1.2. Then there exists the gradient of \( R_T^{O_r} \varphi \) for all \( \varphi \in B_b(\overline{O_r}) \) and it results

\[
D_x R_T^{O_r} \varphi(x) \cdot y = \int_{\{r^{\Gamma(h+d(x,\cdot))} \leq r\}} \varphi(h(T)) \exp\left[ -\frac{1}{2} F(x) + G(x, h) \right] \left(-\frac{1}{2} F_x(x) y + G_x(x, h) y \right) N_{Q_T}(dh) + \mathbb{E}_{N_{Q_T}} \left[ \varphi(h(T)) (\Gamma'(h + d(x, \cdot)) \cdot (d_x(x, \cdot) y) | \Gamma(h + d(x, \cdot)) = r \right] D_r^+ \Lambda_x(r).
\]

**Proof** We recall that by Proposition 5.9 we have

\[
D_x R_T^{O_r,n} \varphi(x) \cdot y = I(n, x, y) + J(n, x, y),
\]

where

\[
I(n, x, y) = \int_{\{r^{\Gamma_n(h+d(x,\cdot))} \leq r\}} \varphi(h(T)) \exp\left[ -\frac{1}{2} F(x) + G^n(x, h) \right] \left(-\frac{1}{2} F_x(x) y + G^n_x(x, h) y \right) N_{Q_T}(dh)
\]

and

\[
J(n, x, y) = \mathbb{E}_{N_{Q_T}} \left[ \varphi(h(T)) \Gamma_n(h + d(x, \cdot)) \cdot (d_x(x, \cdot) y) | \Gamma_n(h + d(x, \cdot)) = r \right] D_r^+ \Lambda_n,x(r).
\]

**Step 1.** Convergence of \( I(n, x, y) \) as \( n \to \infty \).

For all \( x \in \overline{O_r} \) and all \( y \in H \) we have

\[
\lim_{n \to \infty} I(n, x, y) = \int_{\{r^{\Gamma(h+d(x,\cdot))} \leq r\}} \varphi(h(T)) \exp\left[ -\frac{1}{2} F(x) + G(x, h) \right] \left(-\frac{1}{2} F_x(x) y + G_x(x, h) y \right) N_{Q_T}(dh).
\]

This follows by the Dominated Convergence Theorem arguing as in the proof of Proposition 4.2.

**Step 2.** Convergence of \( J(n, x, y) \) as \( n \to \infty \).

Let \( r > 0 \), \( x \in \overline{O_r} \), \( y \in H \). Then, we have

\[
\lim_{n \to \infty} J(n, x, y) = - \mathbb{E} \left[ \varphi(h(T)) (\Gamma'(h + d(x, \cdot)) \cdot (d_x(x, \cdot) y) | \Gamma(h + d(x, \cdot)) = r \right] D_r \Lambda_x(r).
\]

First we notice that \( \Gamma_n(h + d(x, \cdot)) \) converges uniformly to \( \Gamma(h + d(x, \cdot)) \) for any \( x \). Moreover, since the function \( h \mapsto \sup_{t_j} h(t_j) \) is Lipschitz continuous in \( E \) and \( g \)
fulfills Hypothesis 1.2 (ii), it follows that $\Gamma_n(h + d(x, \cdot))$ belongs to a bounded subset of $W^{1,2}(E, N_{Q_T})$ by Lemma 4.1. So, a subsequence of $(\Gamma'_n(h + d(x, \cdot)))$ (which we still denote by $(\Gamma'_n(h + d(x, \cdot)))$ converges to $\Gamma'(h + d(x, \cdot))$ in $L^1(E, N_{Q_T})$.

Now we start from (32) which we write as

$$J(n, x, y) = E[\Psi_n(h) | \Gamma_n(h + d(x, \cdot)) = r] D_r \Lambda_{n,x}(r),$$

where

$$\Psi_n(h) = -\varphi(h(T))(\Gamma'_n(h + d(x, \cdot))) \cdot (d_x(x, \cdot) y).$$

Note that $D_r \Lambda_{n,x}(r) \to D_r \Lambda_x(r)$ as $n \to \infty$ by Lemma 5.7. By Hypothesis 1.2 (ii) we have

$$|\Psi_n(h)| \leq \|\varphi\|_{\infty}(a + e^{b|x|})$$

for all $h \in E$,

so that, there exists $M > 0$ such that $|\Psi_n(h)|_{L^1(E, N_Q)} \leq M$ for all $n \in \mathbb{N}$. Also $\Psi_n(h) \to \Psi(h)$ for all $h \in E$ by Lemma 5.6 (iii). Therefore $\Psi_n \to \Psi$ in $L^1(E, N_Q)$ by the Dominated Convergence Theorem.

Now we can show that

$$\lim_{n \to \infty} E[\Psi_n | \Gamma_n(h + d(x, \cdot)) = r] = E[\Psi | \Gamma(h + d(x, \cdot)) = r].$$

To this aim write

$$|E[\Psi_n | \Gamma_n(h + d(x, \cdot)) = r] - E[\Psi | \Gamma(h + d(x, \cdot)) = r]|$$

$$\leq |E[\Psi_n - \Psi | \Gamma_n(h + d(x, \cdot)) = r]|$$

$$+ |E[\Psi | \Gamma_n(h + d(x, \cdot)) = r] - E[\Psi | \Gamma(h + d(x, \cdot)) = r]|$$

$$:= J_1(n) + J_2(n).$$

Since $\Psi_n \to \Psi$ in $L^1(E, N_Q)$ we have

$$|J_1(n)| \to 0 \quad \text{in} \quad L^1(E, N_Q) \quad \text{as} \quad n \to \infty. \quad (33)$$

Concerning $J_2(n)$, note that

$$\lim_{n \to \infty} E[\Psi | \Gamma_n(h + d(x, \cdot)) = r] = E[\Psi | \Gamma(h + d(x, \cdot)) = r] \quad (34)$$

because $\Gamma_n$ is decreasing to $\Gamma$, see e.g. [9, 10.1.7]. Now Step 2 follows from (33) and (34).

**Step 3.** Existence of $D_x R_T^{Q_T} \varphi$ for all $\varphi \in C_b(\overline{O_T}).$

Let us recall that by Proposition 4.2 and Steps 1, 2 we know that
(i) there exists the limit
\[ \lim_{n \to \infty} R_{T,n}^{O_r} \varphi(x) = R_T^{O_r} \varphi(x) \quad \text{for all } x \in \overline{O_r}, \]

(ii) there exists the limit
\[ \lim_{n \to \infty} D_x R_{T,n}^{O_r} \varphi(x) \cdot y = \Xi(x) \cdot y \quad \text{for all } x \in \overline{O_r}, \quad y \in H, \]

(iii) there exists \( M \|\varphi\|_\infty > 0 \) such that
\[ |R_T^{O_r} \varphi(x)| + |D_x R_T^{O_r} \varphi(x)| \leq M \|\varphi\|_\infty \quad \text{for all } x \in \overline{O_r}. \]

Let now \( x, x_0 \in \overline{O_r} \). Since
\[ R_{T,n}^{O_r} \varphi(x) - R_{T,n}^{O_r} \varphi(x_0) = \int_0^1 (D_x R_{T,n}^{O_r} \varphi)(\alpha x + (1 - \alpha)x_0) \cdot (x - x_0) d\alpha, \]
letting \( n \to \infty \) we obtain, by the Dominated Convergence Theorem,
\[ R_T^{O_r} \varphi(x) - R_T^{O_r} \varphi(x_0) = \int_0^1 \Xi(\alpha x + (1 - \alpha)x_0) \cdot (x - x_0) d\alpha. \]
This implies that \( R_T^{O_r} \varphi(x) \) is differentiable at \( x \) in the direction \( y \) and
\[ D R_T^{O_r} \varphi(x) \cdot y = \Sigma(x) \cdot y. \]

**Step 4.** \( \varphi \in B_b(\overline{O_r}) \).
Since \( R_T^{O_r} \) is strong Feller (Proposition 3.5), we have \( R_T^{O_r} / 2 \in C_b(H) \), so, the conclusion follows starting from \( T/2 \).

**Example 6.2** We consider problem (1) when \( d = 2 \) and
\[ A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \]

Then we have
\[ e^{tA} = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}, \quad e^{tA^*} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \]
and it is easy to see that Hypothesis 1.1 is fulfilled. Let \( U \) be as in Lemma 3.3,
\[ U = \int_0^T r e^{rA} C e^{rA^*} dr = \int_0^T \begin{pmatrix} r & r^2 \\ r^2 & r^3 \end{pmatrix} dr = \frac{1}{12} \begin{pmatrix} 6T^2 & 4T^3 \\ 4T^3 & 3T^4 \end{pmatrix} \]
so that \( \det U > 0 \) and

\[
U^{-1} = \frac{6}{T^2} \begin{pmatrix}
3 & -4/T \\
-4/T & 6/T^2
\end{pmatrix}.
\]

Moreover, by Proposition 3.4 we have

\[
u(x, s) = \frac{6}{T^4} \begin{pmatrix}
T^2 - 2Ts & 2(T - 3s)
\end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad x \in H, \ s \in [0, T].
\]

So,

\[
|u(x, s)|_H \leq T^{-4} C_1(T) |x|_H
\]

and

\[
|K(t, s)|_H \leq C_2(T) \quad \text{for all } t, s \in [0, T],
\]

where \( C_1(T), C_2(T) \) are continuous in \((0, +\infty)\). Moreover

\[
|a(t, x)|_H = |Q_T u(t, x)|_H \leq T^{-3} C_1(T) |x|_H.
\]

Finally, \( G(x, h) = \langle u(t, x), h \rangle_X \leq |h|_X |u(t, x)|. \)

Concerning Hypothesis 1.2, assume that \( g(x) = |x|^2 \). Then

\[
\Lambda(x, r) = \int \mathbb{Q}_T \mathbb{Q}_T^2 (dh).
\]

Note that by (25) we know that \( D_r \Lambda(x, r) \) is uniformly bounded in \( x \). Therefore we can apply Theorem 6.1.

Acknowledgements We thank P. Cattiaux for useful remarks about our paper and E. Priola who pointed out to us the paper by P. Cattiaux.

Open Access This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article’s Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article’s Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit http://creativecommons.org/licenses/by/4.0/.

References

1. Airault, H., Malliavin, P.: Intégration géométrique sur l’espace de Wiener. Bull. Sci. Math. 112(1), 3–52 (1988)
2. Bogachev, V.I.: Gaussian Measures. Mathematical Surveys and Monographs, vol. 62. American Mathematical Society, Providence (1998)
3. Bonaccorsi, S., Da Prato, G., Tubaro, L.: Construction of a surface integral under local Malliavin assumptions, and related integration by parts formulas. J. Evol. Equ. 18(2), 871–897 (2018)
4. Cattiaux, P.: Calcul stochastique et opérateurs dégénérés du second ordre. I. Bull. Sci. Math. 114(4), 421–462 (1990)
5. Cattiaux, P.: Calcul stochastique et opérateurs dégénérés du second ordre. II. Bull. Sci. Math. 115(1), 81–122 (1991)
6. Cerrai, S.: Second Order PDE’s in Finite and Infinite Dimension. Lecture Notes in Mathematics, vol. 1762. Springer, Berlin (2001)
7. Da Prato, G., Lunardi, A., Tubaro, L.: Surface measures in infinite dimension. Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl. 25(3), 309–330 (2014)
8. Da Prato, G., Zabczyk, J.: Stochastic Equations in Infinite Dimensions. 2nd edn. Encyclopedia of Mathematics and its Applications, vol. 152. Cambridge University Press, Cambridge (2014)
9. Dudley, R.M.: Real Analysis and Probability. Cambridge Studies in Advanced Mathematics, vol. 74. Cambridge University Press, Cambridge (2002)
10. Kolmogorov, A.N., Fomin, S.V.: Introductory Real Analysis. Prentice-Hall, Englewood Cliffs (1970)
11. Linde, W.: Gaussian measure of translated balls in Banach spaces. Theory Probab. Appl. 34(2), 307–317 (1986)
12. Zabczyk, J.: Mathematical Control Theory. Foundations and Applications. Birkhäuser, Boston, Systems and Control (1992)

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.