Towards breaking the Omega–bias degeneracy in density–velocity comparisons

Michał J. Chodorowski
Copernicus Astronomical Center, Bartycka 18, 00–716 Warsaw, Poland

1 INTRODUCTION

Comparisons between the cosmic density and peculiar velocity fields of galaxies are potentially a powerful tool to measure the cosmological parameter $\Omega$. This is so because in the gravitational instability paradigm the density and the velocity fields are tightly related and the relation between them depends on $\Omega$. In the linear regime, i.e., when the density fluctuations are significantly smaller than unity, the density–velocity relation (DVR) is

$$\delta(r) = -f^{-1}(\Omega) \nabla \cdot v(r). \quad (1)$$

Here, $\delta$ is the mass density fluctuation field, $v$ is the peculiar velocity field, $f(\Omega) \approx \Omega^{0.6}$ and I express distances in units of km s$^{-1}$. Galaxies are ‘test particles’ which trace the velocity field induced by the mass distribution. However, there are both theoretical arguments (e.g., Kaiser 1984; Davis et al. 1985; Bardeen et al. 1986; Dekel & Silk 1986; Cen & Ostriker 1992; Kauffmann, Nusser & Steinmetz 1997; Blanton et al. 1998; Dekel & Lahav 1998) and observational evidence (e.g., Davis & Geller 1976; Dressler 1980; Giovanelli, Haynes & Chincarini 1986; Santiago & Strauss 1992; Loveday et al. 1996; Hermit et al. 1996; Guzzo et al. 1997; Giavalisco et al. 1998; Tegmark & Bromley 1998) that they are biased tracers of the mass distribution itself. In the simplest model of linear and local biasing, the galaxy density fluctuation field is

$$\delta_{gal}(r) = -b f(\Omega) \nabla \cdot v(r). \quad (3)$$

Thus, a comparison between the fields in question based on linear theory yields not an estimate of $\Omega$ directly but rather of a quantity $\beta \equiv f(\Omega)/b$. This has been called the ‘Omega–bias degeneracy problem’. To break this degeneracy in the IRAS–POTENT comparison, Dekel et al. (1993) used nonlinear effects. These effects in any case should be accounted for in analyzing the data, since the observed fields are currently smoothed over scales which are mildly nonlinear. Dekel et al. (1993) stressed that the limits on $\Omega$ they derived ‘are valid only in the approximation that the linear biasing model is correct’; simply put, they assumed linear biasing. However, nonlinear corrections to the linear biasing relation (2) are of the same order as the corrections to the linear DVR (1), so in general if one accounts for nonlinear dynamics one should also account for

$$\delta_{gal}(r) = -b f(\Omega) \nabla \cdot v(r). \quad (3)$$

$\star$ This terminology in the context of linear density–velocity comparisons makes little sense to me, since in general we can speak about degeneracy of parameters only if we have a number of constraints on them. This is the case when attempting to estimate $\Omega$ from the multipoles of the redshift-space power spectrum. Here, we have just one constraint. Still, the terminology has become standard so I will use it.
nonlinear biasing as well. Such an analysis has been performed recently by Bernardou et al. (1999; hereafter B99), resulting in a method of disentangling $\Omega$ from (a priori) nonlinear bias.

The method of B99 assumes that both fields are given in real space, while the observed density field, like, e.g., IRAS, is given in redshift space. In density–velocity comparisons, the real-space density field is commonly reconstructed from the redshift-space one using an iterative technique (Yahil et al. 1991, Strauss et al. 1992), though one-step procedures have been also invented (Fisher et al. 1995; Taylor & Valen-
tine 1999). All these inversion methods aim at correcting the positions of galaxies for their peculiar velocities. However, to predict the velocities using an integral version of equation (5) or its mildly nonlinear extension, one has to assume the value of $\beta$. This is a serious drawback of real-space comparisons; they necessarily assume the value of a parameter which is to be subsequently estimated.

One might think that a way out of this problem is to perform the comparison in redshift space. Such an analysis has been done by Nusser & Davis (1994), but using linear theory only. Nusser & Davis (1994) applied the Zel’dovich approximation to model the redshift-space DVR, but subsequently restricted their analysis to linear regime. The reason was, as they pointed out, that the redshift-space velocity field is rotational at second-order, so it cannot be reconstructed from its radial component only. Since in the present paper I aim at disentangling $\Omega$ and bias by using nonlinear effects in the DVR, the redshift-space analysis is also inappropriate.

A true way out of this dilemma is to relate the real-space velocity field directly to the redshift-space density field. Such a relation allows us to avoid at the same time problems with the reconstruction of the three-dimensional velocity field in redshift space and of the density field in real space. One might ask whether this makes any sense, since the fields in question are defined in different spaces. However, the point is that they are not: real-space and redshift-space are useful concepts, but in fact they are merely two different coordinate systems. I will explain this in more detail later on.

In the present paper I derive a second-order relation between the redshift space density and the real space velocity. Next I demonstrate that, as in ‘real-to-real’ comparisons, second-order terms in this relation offer a method for disentangling $\Omega$ and bias. Now the method is self-consistent because it does not make any a priori assumption about the value of $\beta$. I give completely worked out an example of nonlinear but local bias. The case of stochastic bias is left for further study.

The paper is organized as follows: in Section 2 I derive the described DVR up to second-order in perturbation theory. Specifically, in Subsection 2.1 I derive it in the case of unsmoothed fields; in Subsection 2.2 I extend it to the case of smoothed evolved fields, in a form directly applicable to so-called density–density comparisons. Next, I compute the expected scatter in the latter relation (Section 3). In Section 4 I derive a similar relation between the galaxy density field and the velocity field under an assumption of a nonlinear but local bias. In Section 5 I show how such a relation can be used to break the $\Omega$–bias degeneracy in density–density comparisons. Summary and conclusions are in Section 6.

2 REDSHIFT-SPACE DENSITY VERSUS REAL-SPACE VELOCITY

The purpose here is to derive a second-order local relation between the redshift-space density field and the real-space peculiar velocity field. Intrinsic motions of galaxies along the line of sight cause distortion in galaxy redshift surveys. On one hand, this distortion complicates measurements of the statistical properties of the large scale galaxy distribution and also cosmic density–velocity comparisons. On the other hand, it contains valuable information on the structure of the peculiar velocity fields. Since the amplitude of peculiar velocities depends on $\Omega$, measurement of the degree of the distortion can serve as a method of measuring this parameter (see Hamilton 1997; a review on linear redshift distortions). Similarly to density–velocity comparisons, linear studies of redshift distortions yield merely an estimate of $\beta$ and in order to disentangle $\Omega$ from bias, nonlinear effects have been recently investigated (Taylor & Hamilton 1996; Heavens, Matarrese & Verde 1998; Scoccimarro et al. 1998).

On large scales ($\gtrsim$ several $h^{-1}$ Mpc), coherent flows lead to compression of structures along the line of sight. This compression reaches maximum at a turnaround radius, where the structure appears totally collapsed along the line of sight and thereafter begins to invert itself. On small scales ($\ll$ a few $h^{-1}$ Mpc), high velocities of galaxies in clusters stretch out the structure in redshift space, creating the so-called ‘finger-of-God effects’. These effects are commonly interpreted as an evidence for virialisation of clusters. However, this is not necessarily the case. During three-dimensional highly-nonlinear gravitational collapse (but still before shell crossing), a structure attains infall velocities which are typically much bigger compared to its size multiplied by the Hubble constant. Therefore, young unvirialized clusters, still partly in an infall phase, may also appear in redshift space as “fingers-of-God” (Chodorowski 1990).

A perturbative approach which I will adopt here is valid for scales bigger than turnaround. Then the ‘compression effect’ dominates and as we will see below, the redshift-space overdensity is increased compared to the corresponding real-space overdensity.

2.1 Unsmoothed fields

At the beginning my analysis will be similar to that of Heavens, Matarrese & Verde (1998), who studied nonlinear corrections to the redshift-space distortion of the power spectrum. Let $\mathbf{r}$ and $\mathbf{s}$ be the coordinates in real and redshift space, respectively, with the Local Group at the origin. The redshift-space position of a galaxy is related to its real-space position by (Kaiser 1987)

$$s = r \left( 1 + \frac{u(r)}{r} \right).$$

Here, $u(r) \equiv \mathbf{v} \cdot r/r$ and velocities are expressed relative to the Local Group. Let $\rho$ and $\rho_s$ be the density fields in real

† Obviously, the transformation from real space to redshift space does not conserve the volume element and in general may be even non-invertible.
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The velocity evaluated at \( \mathbf{r} \) varies in space, due to selection effects; I will denote the mean densities as \( \phi(r) \) and \( \phi_s(r) \). The density contrast in redshift space is then defined by \( 1 + \delta_r(s) = \rho_r(s)/\phi_s(s) \); similarly for the real-space density contrast.

Numbering correction galaxies implies \( \rho_r(s) d^3s = \rho(\mathbf{r}) d^3\mathbf{r} \). This yields

\[
\left[ 1 + \delta_r(s) \right] \phi_s(s) = \left[ 1 + u(r) \right]^{-2} \left[ 1 + u'(r) \right]^{-1} \times \left[ 1 + \delta(r) \right] \phi(r),
\]

where \( \dot{r} \equiv \partial/\partial r \). I will make the standard distant-observer approximation: that the survey is large enough so that the modes with wavenumbers \( k \lesssim r^{-1} \) are negligible. Then in the above expression we can neglect the term \( u(r)/r \); furthermore, a Taylor expansion of \( \phi(s) \) yields \( \phi(r) \) plus a negligible correction. Second-order expansion of the above equation is then

\[
\delta_r(s) = \delta(r) - u'(r) - \delta(r)u'(r) + [u'(r)]^2.
\]

I will now use the fact that in real space the density field is up to second order a local function of the two velocity scalars: the expansion (the velocity divergence), \( \theta \), and the shear, \( \Sigma \). Specifically (Chodorowski 1997; cf. Gramann 1993, Catelan et al. 1995, Mancinelli & Yahil 1995)

\[
\delta(r) = -f^{-1} \theta(r) + \frac{\Omega}{\Omega_0} f^{-2} \theta^2(r) - \frac{\Omega}{\Omega_0} f^{-3} \Sigma^2(r).
\]

Here,

\[
\Sigma^2 \equiv \Sigma_i \Sigma_{ij},
\]

\[
\Sigma_{ij} \equiv \frac{1}{2} \left( \partial v_i/\partial r_j + \partial v_j/\partial r_i \right) - \frac{1}{2} \delta_{ij} \theta,
\]

\[
\theta \equiv \nabla \cdot \mathbf{v},
\]

and I have neglected the weak \( \Omega \)-dependence. The symbol \( \delta^K \) denotes the Kronecker delta. Using expression (8) in (6) gives up to second order

\[
\delta_r(s) = -f^{-1} \theta(r) - u'(r) + f^{-1} \theta(r) u'(r) - f^{-1} \theta(r) [u'(r)]^2 + \frac{\Omega}{\Omega_0} f^{-2} \left[ \theta^2(r) - \frac{1}{2} \Sigma^2(r) \right].
\]

Thus, the redshift-space density field at a position \( s \) is a function of the derivatives of the real-space velocity field at a corresponding position \( r \). This is in fact a local relation between the redshift-space density and the real-space velocity. Real-space and redshift-space are useful notions, but actually they are merely two different mappings of the same physical space to \( \mathbb{R}^3 \). The coordinate \( r \) and the corresponding coordinate \( s \), given by equation (6), label the same point in physical space. Therefore, any local relation between the redshift-space quantity-one and the real-space quantity-two naturally expresses quantity-one at a position \( s \) in terms of quantity-two at a position \( r \). However, this is inconvenient, because in the process of comparison, for each position \( r \) we should, given the velocity field, seek the corresponding \( s \).

Fortunately, the value of any real-space function \( g \) at \( r \) can be expressed as its value at \( \hat{r} \equiv s \) plus Taylor-expansion corrections, which will depend on the velocity. To second order we have

\[
g(r) = g(s) - u(s) g'(s).
\]

This allows us to cast the RHS side of Equation (11), involving the velocity evaluated at \( r \), to another form, but also requiring only the velocity, evaluated at \( \hat{r} \equiv s \). (I have used a ‘check’ mark to emphasize that \( \hat{r} \) is a real-space point but in general different from \( r \), which is related to \( s \) by eq. (6). The result is the relation between the redshift-space density and the real-space velocity, all evaluated at \( s \):

\[
\delta_r(s) = -f^{-1} \theta - u' + [u(f^{-1} \theta + u')] + \frac{\Omega}{\Omega_0} f^{-2} \left( \theta^2 - \frac{1}{2} \Sigma^2 \right). (13)
\]

What does it exactly mean? It means that if we select a point \( s \) in redshift space, to estimate the redshift-space density contrast at this point we need to evaluate the RHS of the above equation at a real-space point \( \hat{r} = s \). Conversely, if we select first a point \( r \) in real space, evaluating the RHS of the above equation at this point we estimate the redshift-space density contrast at a redshift-space point \( \hat{s} = r \). In other words, we can now treat the two fields as if they were given in the same coordinate system: in fact we are now comparing the density with the velocity at two (slightly) different physical points (since \( \hat{s} \neq s \)), but the Taylor terms in the resulting DVR provide the necessary correction.

I reiterate that equation (13) involves real-space derivatives of the real-space velocity field. Having measured the peculiar velocity of a galaxy, we have a choice: we can assign the velocity either to the galaxy’s distance, or to its redshift. The former procedure leads to construction of the velocity field in real space; the latter to construction of the velocity field in redshift space. Thus, the definition of the redshift-space velocity field, \( \mathbf{vs} \), is \( \mathbf{vs}(s) \equiv \mathbf{v}(r) \). This field is not vorticity-free: \( \nabla \times \mathbf{vs}(s) = \nabla \times \mathbf{v}(r) \neq \nabla \times \mathbf{v}(r) \), the last quantity being the vorticity of the velocity field in real space, which vanishes. On the other hand, the difference between \( \nabla \times \mathbf{vs}(s) \) and \( \nabla \times \mathbf{v}(r) \), and hence the resulting vorticity in redshift space, is already of second-order in velocity. In sum, while the linear velocity field in redshift space is irrotational the nonlinear one is not and therefore cannot be reconstructed from its radial component only.

As stated earlier, in density–velocity comparisons a common procedure is to first reconstruct the real-space density from the redshift-space one. Relation (13) provides a basis to avoid this step. The application of this relation to density–density comparisons is obvious: (a) take the appropriate derivatives of the real-space velocity field at \( r \); (b) compare the result directly to the redshift-space density contrast at \( s = r \). As for velocity–velocity comparisons, relation (13) can be straightforwardly used to reconstruct the real-space velocity field directly from the redshift-space density field. The real-space velocity field is irrotational, thus describable as a gradient of some potential. Relation (13) reduces then to a differential equation (in real space) for the velocity potential, with the source term \( S(r) = \delta_r(s = r) \). This equation can be solved perturbatively: at linear order, by the method of Nusser & Davis (1994) at second order similarly, with the source term modified by the nonlinear terms in velocity, approximated by linear solutions. In the present paper I concentrate on the application of the relation (13) to density–density comparisons; the application to velocity–velocity comparisons will be addressed elsewhere.

The second-order DVR (13) has been derived in a quite simple and natural way. There is, however, a more formal

\[ \text{The linear part of equation (13) coincides with equation (6) of Nusser & Davis (1994) in the distant observer limit.} \]
way of deriving it and, more importantly, generalizing for all orders. For convenience, introduce a scaled velocity field, 
\[ \tilde{v} = -f^{-1}v. \] (14)

Equation (13) then reads
\[ \delta_s|_{s=r} = \left\{ \tilde{\theta} + f \tilde{u}' + f \left[ \tilde{u}(\tilde{\theta} + f \tilde{u}') \right]' + \frac{4}{21} \left( \tilde{\theta}^2 - \frac{3}{2} \Sigma_s^2 \right) \right\}_r. \] (15)

where for any function \( g, \tilde{g} = -f^{-1}g. \) Applying the approach of Scoccimarro et al. (1998), in Appendix A I show that in general
\[ \delta_s|_{s=r} = \sum_{n=0}^{\infty} \frac{1}{n!} f^n \left[ \tilde{u}'(\delta + f \tilde{u}') \right]^{(n)}_r. \] (16)

Here, the superscript \( (n) \) denotes \( \frac{\partial^n}{\partial z^n} \). This equation will yield the desired DVR when we substitute for the real-space density contrast \( \delta(r) \) its local estimate from velocity; in other words, when we make use of the real-space DVR. Equation (13) is easily obtained from (16) when the second-order relation (8) is used and the series above is consequently truncated at second-order \( (n = 1) \) terms. However, extensions of real-space DVRs beyond second order have been studied both analytically and numerically (Nusser et al. 1999; Mancinelli et al. 1994; Chodorowski \& Lokas 1997, hereafter CL97, Chodorowski et al. 1998; B99; Ganon et al. 1999), resulting in a robust analytical description of the DVR in the mildly nonlinear regime. Specifically, B99 and Ganon et al. (1999) found nonlinear formulas for density in terms of the velocity derivatives, which have proven to be very accurate, when tested against N-body simulations. One might thus think that using them in equation (13) would yield a very good description of the mildly nonlinear local relation between the redshift-space density and the real-space velocity.

In fact, this is not so simple and in the present paper I restrict the analysis to second order. The reason is that equations (14) or (15) are not directly applicable to real data, since the observed density and velocity fields are always smoothed over some scale. As in real space, the analog of equation (13) or of its higher-order extension – for smoothed fields will significantly differ from its unsmeothed counterpart (see the next subsection). Therefore, smoothing should be accounted for. This, however, turns out to be a non-trivial task, even in real space (CL97, Chodorowski et al. 1998, B99), and calculations in redshift space are even more difficult. Restricting the analysis to second order will enable me to straightforwardly calculate the effects of smoothing. Furthermore, as I will show it later, a second-order analysis is sufficient for the purpose of the present paper, i.e., oulining a idea of breaking the degeneracy between \( \Omega \) and bias in density–velocity comparisons by nonlinear effects. Therefore, including orders higher than second would make the analysis more complicated than is necessary here.

At the end of this subsection, let us compare contributions of the linear and nonlinear terms in relation (13). To simplify the comparison, consider a spherical top-hat overdensity. Then \( \Sigma^2 = 0, \theta = \text{const}, \tilde{u}' = \frac{4}{7} \theta, \) and hence
\[ \delta_s = - \left( 1 + \frac{4}{7} f \right) f^{-1} \delta + \frac{1}{21} \left( 1 + \frac{4}{7} f + \frac{7}{2} f^2 \right) f^{-2} \theta^2. \] (17)

The real-space counterpart of the above ‘redshift-to-real’ relation can be obtained by setting the \( f(\Omega) \) factors in the parentheses equal to zero, yielding correctly a special case of equation (13):
\[ \delta = -f^{-1} \delta + \frac{4}{7} f^{-2} \theta^2. \] (18)

The amplitude of the second-order term relative to the linear one in this real-space relation is given for \( \theta = 1 \) by \( (4/21)f^{-1} \approx 0.2f^{-1} \). Hence, while for a low-\( \Omega \) Universe real-space nonlinear effects can be significant, for a high-\( \Omega \) \( (\Omega \sim 1) \) Universe they are rather weak (Dekel et al. 1993). However, nonlinear effects in the relation (13) are significantly stronger. Peculiar velocities increase the redshift-space overdensity at both orders, but while the linear term is increased only by 33% for \( \Omega = 1 \), the second-order term is increased by more than a factor of three. Thus, the relative importance of the nonlinear terms in the ‘redshift-to-real’ DVR is significantly greater than of those in the real-space DVR, and so is the sensitivity of the degeneracy-breaking method presented here.

### 2.2 Smoothed fields

As stated earlier, the density and the velocity fields reconstructed from observations are always smoothed over some (large) scale \( R \). At linear order, the DVR for smoothed evolved fields follows immediately from its counterpart for unsmoothed fields. From the linear part of equation (13) and equation (14) we have \( \delta^n = \nabla_r \cdot (\tilde{u} \tilde{\theta}^{(n)}) + f (\tilde{u} \tilde{\theta}^{(n)}), \) because the operations of smoothing \( (\tilde{u}) \) and differentiation commute. However, the operations of smoothing and multiplication do not and the derivation of an analog of equation (13) for smoothed fields including second-order terms is already a non-trivial task. Here I will derive the relation in a form applicable to so-called density–density comparisons, i.e., expressing density in terms of velocity. In such comparisons, a large-scale velocity field is used to reconstruct the associated mass density field, which is subsequently compared to the galaxian density field. To simplify the notation, I will from now on implicitly assume that all fields are smoothed at scale \( R \) and will drop the superscript \( R \) on all variables.

Being a non-local operation, smoothing inevitably introduces some scatter in any local relation between nonlinear variables. As an estimator of the smoothed density from the smoothed velocity we cannot therefore do better than to adopt the conditional mean, the mean redshift-space density contrast \( \delta(\Sigma) = \left\langle \frac{\partial \Sigma}{\partial \Sigma}, \frac{\partial \theta}{\partial \Sigma}, \frac{\partial \delta}{\partial \Sigma}, ... \right\rangle \), because the operations of smoothing \( (\Sigma) \) and differentiation commute. In practice, however, this is not the best thing to do. (Leaving aside extreme complexity of such a calculation!) The differentiation of the noisy peculiar velocity field should be done with great care and even so it still leads to dangerous biases (Dekel et al. 1998). The current incarnation of the POTENT algorithm, as an estimator of the real-space mass density from the velocity field uses a nonlinear formula but involving only first-order velocity

\[ \delta_s = - \left( 1 + \frac{4}{7} f \right) f^{-1} \delta + \frac{1}{21} \left( 1 + \frac{4}{7} f + \frac{7}{2} f^2 \right) f^{-2} \theta^2. \] (17)
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derivatives (Dekel et al. 1998, Sigad et al. 1998). Second-order derivatives of the velocity fields reconstructed from current catalogs of peculiar velocities probably cannot be reliably estimated at all. It would be thus more reasonable to compute the quantity \( \langle \delta_s \rangle_{|\vec{b}} \).

However, a simple method of breaking the Omega–bias degeneracy in density–density comparisons, which I will present later on, relies on a relation between\[^\dagger\]\n
\[ \langle \delta_s \rangle_{|\vec{b}} = f(\vec{u}^{'})_{|\vec{b}} + (\vec{\theta} + \Delta_2)_{|\vec{b}}, \]

where

\[ \Delta_2 \equiv f(\vec{u} + f\vec{u}^{'})^2 + \frac{1}{\theta^2} (\vec{\theta}^2 - \frac{2}{3} \Sigma^2). \]

Define now the variable\[^\ddagger\]

\[ \gamma \equiv \vec{\theta} \pm \Delta_2. \]

CL97 derived a general expression for the conditional mean for two mildly nonlinear smoothed variables, given Gaussian initial conditions, assumed also here. An additional assumption was that the variables are equal at linear order. Expanding \( \gamma \) to second order we have \( \gamma = \vec{\theta} + \theta_2 + \Delta_2 \), where \( \theta_2 \) are \( n \)-th order perturbative contributions to \( \vec{\theta} \); in general \( \theta_2 \) is \( \mathcal{O}(\theta) \). One can see that \( \Delta_2 \), like \( \theta_2 \), is already \( \mathcal{O}(\theta^2) \). Thus, the variables \( \gamma \) and \( \vec{\theta} \) are indeed equal at linear order. To second order for such variables (CL97),

\[ \langle \gamma \rangle_{|\vec{b}} = \vec{\theta} + a_2 (\vec{\theta}^2 - \langle \vec{\theta}^2 \rangle). \]

Here, \( a_2 \equiv (S_{1r} - S_{2r})/6 \) and \( S_{2r}, S_{1r} \) denotes the skewness of the variable \( \gamma \). Using the definition \( \theta_2 \) of \( \gamma \), at leading order we obtain:

\[ a_2 = - \left( \frac{1}{6} + \frac{1}{3} f \right) \left( \frac{1}{6} \right) \theta + a_2 (\vec{\theta}^2 - \langle \vec{\theta}^2 \rangle). \]

By isotropy we have

\[ \langle \vec{u}^{'})_{|\vec{b}} = \frac{1}{\vec{\theta}}. \]

This, together with equations \( \theta_2 \) and \( \gamma \), used in equation \( \theta_2 \) yields

\[ \langle \delta_s \rangle_{|\vec{b}} = \left( 1 + \frac{1}{3} f \right) \vec{\theta} + a_2 (\vec{\theta}^2 - \langle \vec{\theta}^2 \rangle), \]

or, transforming back to the plain velocity divergence,

\[ \langle \delta_s \rangle_{|\vec{b}} = - \left( 1 + \frac{1}{3} f \right) f^{-1} \theta + a_2 f^{-1} (\vec{\theta}^2 - \langle \vec{\theta}^2 \rangle). \]

\(^\dagger\) Chodorowski 1997 derived a related expression for the real-space density, \( \langle \delta_s \rangle_{|\vec{b}} \).

\(^\ddagger\) Obviously, this is always the case when the velocity field is described by one variable.

This is a second-order relation between the redshift-space mass density, at \( \vec{s} = r \), and the real-space velocity divergence, at \( r \). The calculation of the coefficient \( a_2 \) for different smoothing windows is outlined in Appendix \( a_2 \). In analyzes of observational data the fields are commonly smoothed with a Gaussian window. The value of \( a_2 \) for a Gaussian smoothing function is given by equation \( a_2 \).

The coefficient \( a_2 \) is a measure of the degree of nonlinearity in the DVR. Since a method for breaking the Omega–bias degeneracy, which I will present in Section \( a_2 \) relies on the value of \( a_2 \), we should compute it accurately. Smoothing clearly affects the value of \( a_2 \) (see Appendix \( a_2 \)). For simplicity, consider the case of a top-hat smoothing. From equation \( a_2 \) it follows that for \( \Omega = 1 \), the value of \( a_2 \) for unsmoothed fields, \( a_2(\infty) \), overestimates the true value by almost 40% for the spectral index \( n = -1 \) and by more for bigger values of \( n \). Thus, smoothing changes the value of the coefficient \( a_2 \) significantly so it was indeed necessary to account for it. On the other hand, it is worth noting that the true value for \( \Omega = 1 \) and \( n = -1 \) is still more than a factor of two bigger than its real-space counterpart, equation \( a_2 \).

3 SCATTER IN THE RELATION

A local relation between the density and the velocity divergence derived in previous Section has a scatter, present already at linear order. The r.m.s. value of the scatter is given by the square root of the conditional variance. At linear order we have

\[ \langle \langle \delta_s - \langle \delta_s \rangle_{|\vec{b}} \rangle^2 \rangle_{|\vec{b}} = f^2 \left( \langle \vec{u}' - \frac{1}{3} \vec{\theta} \rangle^2 \right)_{|\vec{b}} \]

\[ = \frac{1}{\theta^2} f^2 \vec{\theta}^2 - \frac{2}{3} f^2 \vec{\theta} \langle \vec{\theta}^2 \rangle_{|\vec{b}} + f^2 \langle \vec{u}' \rangle^2_{|\vec{b}}. \]

The quantity \( \langle \vec{u}' \rangle_{|\vec{b}} \) is given by equation \( \theta_2 \). The calculation of the quantity \( \langle \vec{u}'^2 \rangle_{|\vec{b}} \) is presented in Appendix \( a_2 \). The result is given by equation \( \theta_2 \), where \( \sigma_{\theta}^2 \) is the variance of the field \( \theta \). This yields \( \langle \delta_s - \langle \delta_s \rangle_{|\vec{b}} \rangle^2 \rangle_{|\vec{b}} \), or

\[ \langle \langle \delta_s - \langle \delta_s \rangle_{|\vec{b}} \rangle^2 \rangle_{|\vec{b}}^{1/2} = \frac{1}{3} f^2 \sigma_{\theta}^2. \]

The scatter is of the order of the redshift-space correction to the linear term in equation \( \theta_2 \). When compared to the total linear term in equation \( \theta_2 \), the scatter is relatively small: for \( \vec{s} \sim \sigma_{\theta} \), it is \( \sim 20\% \) for \( \Omega = 1 \) and accordingly less for smaller \( \Omega \). This implies that redshift-space density is well (though not as well as real-space density) correlated with real-space velocity-divergence.

4 GALAXIAN DENSITY VERSUS VELOCITY

As already stated in the introduction, it is currently clear that galaxies are biased tracers of the mass distribution. In this section I will derive a relation between the redshift-space galaxy density field and the real-space velocity field under an assumption of a nonlinear but local bias. This is only a toy model for bias because there are good reasons to believe that bias is in fact somewhat stochastic (Dekel & Lahav 1998; Pen 1998; Tegmark & Peebles 1998; Blanton et al. 1998; Tegmark & Bromley 1998). However, a number of important conclusions can be drawn already from this model.
Equation [10] has been derived from conservation of galaxy numbers, so the densities appearing in it are in fact the galaxy densities \( \delta_{\text{s}}^{(s)} \) and \( \delta_{\text{r}}^{(s)} \) (redshift- and real-space galaxy density contrasts respectively). Hence, a more general form of equation (10) is

\[
\delta_{\text{s}}^{(s)}|_{\text{s} = r} = \sum_{n=0}^{\infty} \frac{1}{n!} \left[ \bar{u}^n \left( \bar{\delta}_{\text{s}}^{(s)} + f \bar{u} \right) \right]^{(n)} |_{r},
\]

(29)

the form \([10]\) implicitly assumes no bias between the distribution of galaxies and mass. In a local bias model, the real-space galaxy density contrast is assumed to be in general a nonlinearity, but local function of the mass density contrast (Fry & Gaztalaiga 1993; see also Juszkiewicz et al. 1995),

\[
\delta_{\text{s}}^{(s)}(r) = \mathcal{N}[\delta(r)].
\]

(30)

Given a (nonlinear) local formula for the real-space mass density in terms of the real-space (first-order) velocity derivatives, \( \delta(r) = \mathcal{F}[\partial v_{\text{r}}/\partial r_j(r)] \), equation (23) yields

\[
\delta_{\text{s}}^{(s)}|_{\text{s} = r} = \sum_{n=0}^{\infty} \frac{1}{n!} \left\{ \bar{u}^n \left[ \mathcal{N} \circ \mathcal{F}[\partial v_{\text{r}}/\partial r_j] + f \bar{u} \right] \right\}^{(n)} |_{r}.
\]

(31)

Up to second order, \( \mathcal{F} \) is given by expression [8] and

\[
\mathcal{N}[\delta(r)] = b \delta(r) + \frac{2}{3} b_2 \left( \delta^2 - \langle \delta^2 \rangle \right).
\]

(32)

Here, \( b \) and \( b_2 \) are respectively the linear and nonlinear (second-order) bias parameters. The term \( \delta^2 \) ensures that the mean value of \( \delta^{(s)} \) vanishes, as required. Using this in equation (11) and truncating the series at second-order terms we obtain

\[
\delta_{\text{s}}^{(s)}|_{\text{s} = r} = \left\{ b \delta + f \bar{u} + f \left[ \bar{u}(b \delta + f \bar{u}) \right] \right\} + \frac{4}{n!} \frac{b}{n!} \left( \delta^2 - \frac{2}{3} \hat{\Sigma}^2 \right) + \frac{2}{n!} \bar{u} \left( \delta^2 - \langle \delta^2 \rangle \right).
\]

(33)

This is a second-order local relation between the redshift-space galaxy density and the real-space velocity, derived under the assumption of local bias. This relation is for unsmoothed fields. To account for the effects of smoothing and to enable a model-independent density--velocity comparison we proceed analogously to when constructing the corresponding relation for the redshift-space mass density. Namely, we compute the expectation value of the redshift-space galaxy density given the velocity divergence. The result is

\[
\langle \delta_{\text{s}}^{(s)} \rangle |_{\delta} = \left( b + \frac{1}{3} f \right) \theta + a_2^{(s)} (\theta^2 - \langle \theta^2 \rangle),
\]

(34)

or, transforming back to the plain velocity divergence,

\[
\langle \delta_{\text{s}}^{(s)} \rangle |_{\theta} = -\left( b + \frac{1}{3} f \right) f^{-1} \theta + a_2^{(s)} f^{-2} (\theta^2 - \langle \theta^2 \rangle).
\]

(35)

The calculation of the coefficient \( a_2^{(s)} \) is outlined in Appendix B2. The result for Gaussian-smoothed fields is given by formula (B22). The coefficient \( a_2^{(s)} \) depends on three parameters: \( \Omega \), \( b \) and \( b_2 \).

In section B I argued that in the related mass-density versus velocity-divergence relation the nonlinear corrective term has fairly big amplitude. However, in the above expression for the galaxy density it may even vanish, if in equation (B22) for the coefficient \( a_2^{(s)} \) the nonlinear biasing term of negative sign cancels exactly the terms due to nonlinear dynamics. Negative nonlinear biasing of the IRAS density field has been inferred from a preliminary analysis of its bispectrum (Scoccimarro, private communication). This makes physical sense, since the IRAS galaxies are underrepresented in the cores of rich galaxy clusters, which is effectively a nonlinear antibiasing operation on the mass density field. This may partly explain why in the IRAS–POTENT comparison the measured nonlinear effects are weak (Dekel et al. 1993), although in real-space analyses they are generally weaker anyway.

5 BREAKING THE \( \Omega \)--BIAS DEGENERACY

Second-order equation (33) can be used to estimate \( \Omega \) separately from bias in density--density comparisons. As first pointed out by Dekel et al. (1993), the key point of the idea of breaking the \( \Omega \)--bias degeneracy by nonlinear effects is not to correct for them in the density reconstruction, but rather to use them as additional information. The degeneracy-breaking method presented here is similar to that proposed by B99. However, the method is now self-consistent, because it no longer needs to make any a priori assumptions about the values of estimated parameters. To apply the method of B99, we should first reconstruct the nonlinear real-space density field. In order to do this, we would have to assume simultaneously the values of \( \Omega \), linear bias and nonlinear (quadratic) bias. Here, we relate the real-space velocity field directly to the redshift-space density field and this problem disappears.

First steps of the degeneracy-breaking analysis can be outlined as follows: the output of a POTENT-like reconstruction machinery should be real-space plain velocity divergence \( \theta \), a quantity derived directly from the data. Next, the velocity divergence should be compared to the redshift-space galaxy density, also a model-independent quantity. Since \( \Omega \) and bias factors appear only in the coefficients of the density versus velocity-divergence relation (eq. 33), this comparison is clearly model-independent. Specifically, the redshift-space galaxy density should be plotted against the velocity divergence and a second-order polynomial should be fitted to the (hopefully) apparent correlation. The fitted linear and quadratic coefficients give us two equations for three variables \( \Omega \), \( b \) and \( b_2 \). We need therefore an additional constraint on these variables.

As this constraint we can adopt the large-scale galaxy density skewness, since it is a second order effect, involving \( b \) and \( b_2 \). Galactic skewness can be measured only in redshift space. Therefore, we need a theoretical relation between the redshift-space galaxy density skewness \( S_{3a}^{(g)} \) (which we can measure) and the redshift-space mass density skewness \( S_{3a} \) (which we can compute). Such a relation will depend on the real-space bias factors \( b \) and \( b_2 \). This dependence may be quite complicated, since the operations of bias and redshift-space mapping do not commute.

The issue of non-commutivity of bias and redshift-space mapping was addressed in some detail by Scoccimarro, Couchman & Frieman (1998). Specifically, they compared the redshift-space hierarchical amplitude for the galaxy field, \( Q_{3a}^{(g)} \), to that predicted if bias and the mapping commuted. (The hierarchical amplitude is defined as the ratio of the bispectrum to the relevant sum of products of the power spectra.) In the local bias model, the real-space galaxy hi-
erarchical amplitude, \( Q^{(6)} \), is related to the real-space mass hierarchical amplitude, \( Q \), by \( Q^{(6)} = Q / b + b_2 / b^2 \) (Fry 1994). If the assumption of commutivity were correct, this would imply

\[
Q^{(6)} = Q_s / b + b_2 / b^2 ,
\]

where \( Q_s \) is the redshift-space mass hierarchical amplitude.

Scoccimarro et al. (1998) showed that the actual \( Q^{(6)} \) differs from \( Q_s / b + b_2 / b^2 \). However, the differences are quite small: at most (for some values of angle) 20% for \( \Omega = 1 \) and respectively smaller for smaller \( \Omega \). Moreover, as a function of angle, they fluctuate in sign. The galaxy skewness is essentially an integral over the galaxy hierarchical amplitude. Therefore, the differences between the actual skewness and its approximate value from an analog of equation (35) will largely cancel out. Using equation (36), a simple calculation yields

\[
S_{3s}^{(6)} = S_{3s} / b + b_2 / b^2 .
\]

For the reasons stated above, we can expect this equation to be accurate to \( \sim 10\% \) or less.

It should be emphasized that approximate commutivity applies only to particular combinations of some statistical quantities and not to the quantities alone. In the case of skewness, this combination is the ratio of the third moment to the square of the second. It is straightforward to show that at leading order, the mass second moment is \( \langle \delta^2 \rangle = (1 + \frac{\beta}{2}) \langle \delta^2 \rangle \) and the galaxy second moment is \( \langle \delta^2 \rangle = b^2 (1 + \frac{\beta}{2} \rangle \). However, in the quadratic coefficient they clearly do not. Therefore, \( \beta \) is the redshift-space mass hierarchical amplitude computed directly by Bouchet et al. (1993). Kim & Strauss (1998) pointed out that sparse sampling of the density field was measured directly by Bouchet et al. (1993; see also CL97) is that they assumed linear biasing. However, if we account for nonlinear dynamics, in general we should account for nonlinear biasing as well. There is also another problem which plagues all real-space comparisons, even these performed at the linear level, aiming merely at inferring the estimate of \( \beta \). Namely, in the process of reconstruction of the real-space density from the redshift-space density one has to assume the value of \( \beta \), the parameter which is to be subsequently estimated.

With nonlinear biasing accounted for in the comparison, the resulting system of equations for bias parameters and \( \Omega \) is underconstrained. As recently pointed out by B99, an additional constraint on \( \Omega \) and bias can be inferred from the density field alone (namely, its skewness), so additional types of observations are unnecessary. In other words, there is enough information in the density field and the corresponding velocity field to disentangle \( \Omega \) from bias. As for the problem with the real-space density reconstruction, Nusser & Davis (1994) proposed to solve it by performing the comparison directly in redshift space.

One might thus think that to break the \( \Omega \)-bias degeneracy in density–velocity comparisons by nonlinear effects one should: (a) account for nonlinear biasing, (b) add an additional constraint on it from the density field (e.g., the skewness) and (c) perform the comparison in redshift space. However, here we face a new problem: the redshift-space velocity field is rotational at second order, so in the nonlinear regime it cannot be reconstructed from its radial component only. Since the method of breaking the degeneracy is based on nonlinear terms in the density–velocity relation, the redshift-space analysis is also inappropriate.

A way out of this dilemma is to relate the redshift-space density field directly to the real-space velocity field. Such a relation allows one to avoid simultaneously problems with the reconstruction of the density in real space and of the three-dimensional velocity in redshift space. In the present paper I derived this relation explicitly up to second-order terms. The relation is local, i.e., it expresses density at a given point in terms of the velocity derivatives at the same point. (Strictly speaking, it expresses the redshift-space density at

\[
\left[ c_2 + \frac{1}{6} \left( S_{3s}^{(6)} - b^{-1} S_{3s} \right) \right] \beta^{-2} \left( \theta^2 - \sigma_\theta^2 \right) ,
\]

where \( c_2 \) is given by equation (B25). The parameters \( \Omega \) and \( \beta \) appearing in the above relation are not degenerate: while in the linear coefficient they appear only via their combination \( \beta \), in the quadratic coefficient they clearly do not. Therefore, measuring the linear coefficient and the parabolic correction in the galaxian density versus velocity–divergence relation one can, at least in principle, measure \( \Omega \) and \( \beta \) separately.

6 SUMMARY AND CONCLUSIONS

The main purpose of the present paper is to outline a self-consistent method of breaking the \( \Omega \)-bias degeneracy in density–velocity comparisons by nonlinear effects. A problem which was present in previous similar attempts (Dekel et al. 1993; see also CL97) is that they assumed linear biasing. However, if we account for nonlinear dynamics, in general we should account for nonlinear biasing as well. There is also another problem which plagues all real-space comparisons, even these performed at the linear level, aiming merely at inferring the estimate of \( \beta \). Namely, in the process of reconstruction of the real-space density from the redshift-space density one has to assume the value of \( \beta \), the parameter which is to be subsequently estimated.

** When attempting to reconstruct the nonlinear real-space density field, this problem would get even worse: one would have to assume simultaneously the values of \( \beta \), linear bias and nonlinear (quadratic) bias.
a point $\hat{s} = r$ in terms of the real-space velocity derivatives at $r$.) Therefore, it can be straightforwardly applied to so-called density–velocity comparisons and I did this here. It can be also applied to velocity–velocity comparisons; I will address this point elsewhere.

Next, I rigorously accounted for the effects of smoothing of the evolved density and velocity fields. This was necessary because smoothing changes the degree of nonlinearity in the density–velocity relation considerably.

Finally, I derived a second-order relation between the redshift-space \textit{galaxy} density and the real-space velocity divergence, under an assumption of nonlinear but local bias. This relation (eq. 35) enables one to perform density–velocity comparisons in a model-independent way. When combined with an additional measurement of the skewness of the galaxy density field (eq. 38), it also enables in principle to solve for $\Omega$ and bias separately in density–density comparisons. In short, the idea is to measure the amplitude of the nonlinear corrective term in the relation.

The density versus the velocity-divergence relation has a scatter even at linear order. This scatter is relatively small when compared to the linear term in equation (35). This means that the linear estimate of $\beta$ is affected by the scatter in a rather weak way. In any case, a fairly large number of independent volumes in the comparison will enable to suppress this scatter, allowing one to estimate the linear coefficient and hence $\beta$ accurately.

The idea of breaking the $\Omega$–bias degeneracy presented here is somewhat related to that given by Pen (1998). In order to measure bias and $\Omega$ separately, Pen proposed to use simultaneously redshift distortions of the power spectrum and of the skewness. However, while in his method the estimate of $\Omega$ is based mostly on the redshift distortion of the galaxy skewness (which is quite a small effect), here it is based on the nonlinear effects in the redshift-space density versus real-space velocity relation. These effects are considerably stronger than in the corresponding real-space relation.

Quality of the current velocity data is probably too poor to perform a degeneracy-breaking analysis similar to proposed here. The scatter in the (real-space) density–velocity diagram of the IRAS–POTENT comparison by Sigad et al. (1988) is huge and it results in significant errors in the estimate of $\beta$. If even the linear coefficient is to some extent uncertain we cannot hope to reliably estimate the nonlinear corrections.

The analysis in its present form is still very simplistic. Points that should be addressed in further studies in the first place are: (a) stochastic nature of biasing; (b) higher than second-order effects, which may play a role for scales smaller than $\simeq 12$ h$^{-1}$ Mpc; (c) relaxing the distant-observer assumption, not strictly valid for current density–velocity comparisons. Recently, Taruya & Soda (1998) have derived a mildly nonlinear galaxy density–mass density relation in a stochastic bias model. The calculation of the galaxy density–velocity divergence relation in such a model is mathematically entirely analogous. Higher-orders effects are complex, but tractable; here I computed them for unsmoothed fields. Similarly, calculation of the density–velocity relation without the simplifying assumption of a distant observer, though more complex, is certainly possible.

The purpose of this paper was to perform a first step towards self-consistent breaking the $\Omega$-bias degeneracy in density–velocity comparisons by nonlinear effects. I showed that the effects in the comparisons proposed here are fairly strong and that the fact that they are of the same order as possible nonlinearities in the biasing relation is not a fundamental obstacle. The method of breaking the $\Omega$-bias degeneracy presented here can be worked out in full detail when data of sufficient quality exists. I mean primarily catalogs of less noisy, more densely and homogeneously sampled peculiar velocities.

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APPENDIX A: DERIVATION OF EQUATION (16)

In this appendix I present the derivation of exact relation (16), expressing the redshift-space density at \( \hat{s} = \hat{r} \) in terms of the real-space density and velocity at \( r \). In the distant-observer approximation, the line-of-sight is taken as a fixed direction, which will be denoted here by \( \hat{s} \equiv \hat{r} \). The real-to-redshift mapping (3) reduces then to \( s = r - f\hat{u}(r)\hat{z} \), where \( \hat{u} \) is the \( z \)-component of the scaled velocity field, defined by equation (14). From the conservation of the number of galaxies we have 
\[
1 + \delta_{\hat{s}}(s) = J(r)[1 + \delta(r)],
\]
where \( J(r) \) is the Jacobian of the mapping. In the plane-parallel approximation, \( J(r) = 1 - f\hat{u}'(r) \) (exactly), where \( \hat{r} \equiv \partial \hat{r}/\partial z \). This yields
\[
\delta_{\hat{s}}(s) = \frac{\delta(r) + 1 - J(r)}{J(r)} \equiv \frac{\delta(r) + f\hat{u}'(r)}{1 - f\hat{u}'(r)}
\]
(eq. 3 of Scoccimarro et al. 1998). Fourier transforming the above equation we obtain
\[
\delta_{\hat{s}}(k) \equiv \int \frac{d^3s}{(2\pi)^3} e^{-i k \cdot s} \delta_{\hat{s}}(s) = \int \frac{d^3r}{(2\pi)^3} \int e^{-i k \cdot r} e^{i f k \cdot \hat{u}(r)} \left[ \delta(r) + f\hat{u}'(r) \right]
\]
(eq. 4 of Scoccimarro et al. 1998). Thus we have expressed the Fourier transform of the density contrast in redshift space as a real-space integral involving the real-space density and velocity fields. The inverse Fourier transform of equation (A2) is
\[
\delta_{\hat{s}}(s) = \int d^3k \sum_{n=0}^{\infty} \frac{\left[ f\hat{u}(r) \right]^n}{n!} (ik_s)^n e^{i k \cdot (s-r)} = \sum_{n=0}^{\infty} \frac{(-1)^n f^n \hat{u}^n(r)}{n!} \frac{\partial^n}{\partial z^n} e^{i k \cdot (s-r)},
\]
where I have changed the order of integration. Expanding the first exponent of the integrand we have
\[
e^{i f k \cdot \hat{u}(r)} = \sum_{n=0}^{\infty} \frac{(-1)^n f^n \hat{u}^n(r)}{n!} \frac{\partial^n}{\partial z^n},
\]
\[
\delta_{\hat{s}}(s) = \int d^3r \sum_{n=0}^{\infty} \frac{(-1)^n f^n \hat{u}^n(r)}{n!} \frac{\partial^n}{\partial z^n} \int \frac{d^3k}{(2\pi)^3} e^{i k \cdot (s-r)}.
\]
The integral over \( k \) yields the Dirac delta distribution, \( \delta_{\hat{s}}(s-r) \). Integrating the remaining integral over \( r \) by parts we obtain
\[
\delta_{\hat{s}}(s) = \sum_{n=0}^{\infty} \frac{(-1)^n f^n}{n!} \int \frac{d^3r \hat{u}^n(r)}{\partial z^n} \left[ \delta(r) + f\hat{u}'(r) \right] \partial^n \frac{\partial^n}{\partial z^n} e^{i k \cdot (s-r)},
\]
\[
(\text{A6})
\]
where the superscript \( \cdot e^{i(n)} \) denotes \( \partial^n/\partial z^n \). Thus, if we select a point \( s \) in redshift space, to estimate \( \delta_{\hat{s}}(s) \) we evaluate the expression on the RHS of the above equation at a real-space point \( \hat{r} = s \). Conversely, if we first fix a point \( r \) in real space, evaluating the RHS of the above equation at \( r \) we estimate the redshift-space density contrast at a redshift-space point \( \hat{s} = r \). This yields equation (16).

APPENDIX B: CALCULATION OF THE COEFFICIENT \( \alpha_2 \)

B1 The case of no bias

Our purpose here is to compute the coefficient \( \alpha_2 \), given by equation (23), for smoothed evolved fields. The calculation is much easier when performed in Fourier space. Denote the Fourier transform of the linear real-space density field \( \delta_l \) by \( \epsilon(k) \). According to linear theory \( \delta_l = \delta_1 \), so \( \epsilon(k) \) is also the Fourier transform of the linear scaled velocity divergence. We have
\[
\langle \hat{\theta}_l^2 \rangle = \sigma^2(R),
\]
\[
(\text{B1})
\]
where \( \sigma^2(R) \) is the linear variance of the real-space density field smoothed over a scale \( R \). To compute the quantity \( \langle \hat{\theta}_l^2 \Delta_2 \rangle \), we need the Fourier transform of \( \Delta_2 \) (defined by eq. 20). Following Heavens et al. (1998) I find that
\[
\Delta_2(k) = \int \frac{d^3p d^3q}{(2\pi)^2} \delta_D(k-p-q) T_2(p,q) \epsilon(p) \epsilon(q),
\]
\[
(\text{B2})
\]
where
\[
T_2(p,q) = \frac{2}{7} \left[ 1 - \cos^{2} \vartheta \right] + \frac{f}{2} \left[ \mu_p^2 + \mu_q^2 + \mu_p \mu_q \left( \frac{p}{q} + \frac{q}{p} \right) \right] + f^2 \left[ \frac{\mu_p^2 \mu_q^2}{2} + \left( \mu_p \mu_q \right)^2 \right].
\]
\[
(\text{B3})
\]
In the above, cosine \( \cos \vartheta = \hat{p} \cdot \hat{q} \) and for any vector \( \hat{k} \equiv k/k \). The remaining quantities are \( \mu_p = \hat{p} \cdot \hat{s} \) and \( \mu_q = \hat{q} \cdot \hat{s} \), where \( \hat{s} \) is a unit vector from the observer to the galaxy; in the distant-observer approximation one assumes \( \hat{s} \) is constant within the smoothing radius.

For Gaussian initial fluctuations, straightforward calculation yields
\[ \frac{1}{2} \langle \tilde{\phi}^2 \Delta_2 \rangle = \int \frac{d^3p \, d^3q}{(2\pi)^6} W(p) W(q) W(|p + q|) P(p) P(q) T_2(p, q) \]  
(B4)

(see e.g. Juszkiewicz, Bouchet & Colombi 1993). Here, \( P \) is the real-space linear mass power spectrum and \( W \) is the Fourier transform of the window function. The calculation of the above integral with the kernel \( T_2 \) given by equation (B3) is similar to the calculation of the skewness of the redshift-space density field. Following Hivon et al. (1995) I note that the angular integration of the above expression for a fixed \( \vartheta \) to the calculation of the skewness of the redshift-space density field. Following Hivon et al. (1995) I note that the angular integration of the above expression for a fixed \( \vartheta \) is, up to a multiplicative factor, equivalent to averaging over different orientations of \( s \). We then have \( p = (0, 0, 1), q = (\sin \vartheta, 0, \cos \vartheta) \) and \( s = (\cos \varphi \sin \vartheta', \sin \varphi \sin \vartheta', \cos \vartheta') s \), so

\[ \mu_p = \cos \vartheta', \]  
(B5)

\[ \mu_q = \cos \varphi \sin \vartheta' \sin \vartheta + \cos \vartheta' \cos \vartheta, \]  
(B6)

and the averaging over the polynomials of \( \mu_p \) and \( \mu_q \) given \( \vartheta \) is effectively over the trigonometric functions of \( \varphi' \) and \( \vartheta' \). The result is

\[ \langle T_2(p, q) \rangle_{\mu_p \mu_q} |_{\vartheta} = \frac{2}{7} + \frac{f}{3} + \frac{f^2}{15} + \left( \frac{f}{6} + \frac{f^2}{10} \right) \cos \vartheta \left( \frac{p}{q} + \frac{q}{p} \right) - \left( \frac{2}{7} - \frac{2f^2}{15} \right) \cos^2 \vartheta. \]  
(B7)

Using equations (B1) and (B4) in equation (23) yields

\[ a_2 = \int \frac{d^3p \, d^3q}{(2\pi)^6} W(p) W(q) W(|p + q|) P(p) P(q) \langle T_2(p, q) \rangle, \]  
(B8)

where \( T_2 \) is given by expression (B3). The calculation of the coefficient \( a_2 \) is now entirely analogous to the calculation of the skewness of the real-space density field. This is so because the averaged kernel has a similar functional dependence on \( \vartheta, p \) and \( q \) to that of the second-order solution for the real-space density field (see e.g. Juszkiewicz et al. 1993), with different coefficients. In the case of unsmoothed fields \( W(k) \equiv 1 \) and we obtain

\[ a_2^{(un)} = \frac{4}{21} + \frac{f}{3} + \frac{f^2}{9}. \]  
(B9)

Comparing (B9) with expressions (17) and (26), we see that \( a_2^{(un)} \) is equal to the value predicted by the spherical collapse model.

To study the effects of smoothing I will consider power spectra with a power-law form

\[ P(k) = C k^n, \quad -3 \leq n \leq 1, \]  
(B10)

where \( C \) is a normalization constant. For more realistic spectra, the value of \( a_2 \) can be very well approximated by the result for scale-free spectra with the effective index at a smoothing scale defined as

\[ n_{eff} = -R \frac{d \sigma^2(R)}{dR} - 3 \]  
(B11)

(97). For the case of a spherical top-hat smoothing function, using the results of Bernardeau (1994) I find that

\[ a_2^{(TH)} = a_2^{(un)} - (n + 3) \left( \frac{f}{18} + \frac{f^2}{30} \right). \]  
(B12)

For the most interesting case of Gaussian smoothing, using the results of Lokas et al. (1995) I obtain

\[ a_2^{(G)} = \frac{4}{21} F(5/2) + \frac{f}{6} \left[ F(3/2) - \frac{n}{3} F(5/2) \right] + \frac{f^2}{10} \left[ F(3/2) - \frac{3n + 8}{9} F(5/2) \right]. \]  
(B13)

Here,

\[ F(3/2) \equiv 2F_1 \left( \frac{n + 3}{2}, \frac{n + 3}{2}; \frac{3}{2}, \frac{1}{4} \right), \]  
(B14)

\[ F(5/2) \equiv 2F_1 \left( \frac{n + 3}{2}, \frac{n + 3}{2}; \frac{5}{2}, \frac{1}{4} \right) \]  
(B15)

and \( 2F_1 \) is the hypergeometric function. The hypergeometric function can be expanded in a series of powers of \( n + 3 \) which converges rapidly. Fitting the expansion by a polynomial in \( n + 3 \) in the range \(-3 \leq n \leq 1 \) yields low-order accurate fits. Namely,

\[ F(r) = 1 + d_2^{(r)} (n + 3)^2 + d_3^{(r)} (n + 3)^3, \]  
(B16)

where

\[ d_2^{(3/2)} = 0.03239, \quad d_3^{(3/2)} = 0.009364 \]  
(B17)

and

\[ d_2^{(5/2)} = 0.02183, \quad d_3^{(5/2)} = 0.003531. \]  
(B18)
The real-space velocity divergence. All calculations are similar to those in the previous Subsection. Using equation (33), analogously to equations (19)–(20) we can write

\[ a^\Delta \equiv \frac{4}{21} F(5/2) . \]  

**B2 The case of bias**

Our purpose here is to compute the coefficient \( a^\delta \), entering in the relation (24) between the redshift-space galaxy density and the real-space velocity divergence. All calculations are similar to those in the case of no bias, presented in the previous Subsection. Using equation (24), analogously to equations (19)–(21) we can write

\[ \langle \delta^\delta \rangle |_{\delta} = b \left( 1 + \frac{1}{3} b \right) \delta + \langle \delta^\Delta \rangle |_{\delta} , \]  

where

\[ \Delta^\delta \equiv f \left( \bar{u}(\hat{b} + f \bar{u}) \right) + \frac{4}{21} b \left( \bar{\theta}^2 - \frac{1}{3} \bar{\Sigma}^2 \right) + \frac{1}{2} b_2 \left( \bar{\theta}^2 - \langle \bar{\theta}^2 \rangle \right) . \]  

The coefficient \( a^\Delta \) is given by the following equation (cf. eq. 28):

\[ a^\Delta = \frac{1}{2} \langle \bar{\theta}^2 \rangle^{-2} \left( \bar{\theta}^2 \Delta^\Delta \right) . \]  

The Fourier transform of \( \Delta^\Delta \) is

\[ \Delta^\Delta(k) = \int \frac{d^3k}{(2\pi)^3} \delta_D(k - p - q) T_2(p,q) c(p) c(q) - 4\pi^2 b_2 \sigma^2 \delta_D(k) , \]  

where already an averaged form of the kernel is

\[ (T_2^\delta(p,q))_{\mu\nu\rho} = \frac{2b}{7} + \frac{3b_2}{4} + \frac{bc}{3} + \frac{f_2}{4} - \frac{b}{15} \frac{f_2}{10} \cos^2 \theta \left( \frac{p}{q} + \frac{q}{p} \right) + \left( \frac{2b}{7} - \frac{2f_2}{15} \right) \cos^2 \theta . \]  

In the expression (B24), the last term is the Fourier transform of the term \((b_2/2)\bar{\theta}^2\).

Using equations (B24) and (B26) yields (cf. eq. B18)

\[ a^\delta = \int \frac{d^3k}{(2\pi)^3} \sigma^2 \delta_D(k - p - q) P(p) P(q) (T_2^\delta(p,q)) - b_2^2 / 4 . \]  

For Gaussian-smoothed fields, the result of the integration is

\[ a^\delta \equiv \frac{b_2}{2} + b^2 c_2 . \]  

where

\[ c_2 = \frac{4b}{21} F(5/2) + \frac{\beta}{6} \left( F(3/2) - \frac{n}{3} F(5/2) \right) + \frac{\beta^2}{10} \left( F(3/2) - \frac{3n}{9} F(5/2) \right) , \]  

\[ \beta \equiv f(\Omega)/b \quad \text{and} \quad F(n/2) \quad \text{are defined in equations (B14)–(B15). In the limit of no bias (} b = 1 \text{ and} \ b_2 = 0, \text{ equations (B24)–(B28) reduce to equation (B13), as expected. The result for a top-hat smoothing can be obtained by replacing the factors } F(3/2) \text{ and}\ F(5/2) \text{ by unity. The result for unsmoothed fields then follows from setting the spectral index equal to } -3. \]

### APPENDIX C: GAUSSIAN CONDITIONAL AVERAGES

Our purpose in this Section is to compute the quantity \( \langle \bar{u}'(\hat{\theta}) \rangle \) at leading, i.e. linear, order. We may assume therefore that \( \bar{u}' \) and \( \theta \) are Gaussian variables. For such variables,

\[ \langle \bar{u}' \rangle \langle \hat{\theta} \rangle \]  

Here, \( \sigma_{\bar{u}'}^2 (\sigma_{\hat{\theta}}^2) \) is the variance of the variable \( \bar{u}' (\hat{\theta}) \) and \( \rho \) is the correlation coefficient defined as

\[ \rho = \frac{\sigma_{\bar{u}'} \sigma_{\hat{\theta}}}{\sigma_{\bar{u}'} \sigma_{\hat{\theta}}} . \]  

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\[ r = \frac{\langle \tilde{u}' \tilde{\theta} \rangle}{\sigma_{\tilde{u}'} \sigma_{\tilde{\theta}}} . \]  
\hspace{1cm} \text{(C2)}

Using results of Appendix B we have \( \sigma_{\tilde{u}'}^2 = (1/5) \sigma_{\tilde{\theta}}^2 \) and \( r = \sqrt{5}/3 \). This yields

\[ \langle \tilde{u}' \rangle_{\tilde{\theta}} = \frac{1}{3} \tilde{\theta} , \]  
\hspace{1cm} \text{(C3)}

a result which is obvious by isotropy. The conditional variance is

\[ \langle \tilde{u}^2 \rangle_{\tilde{\theta}} - \langle \tilde{u}' \rangle_{\tilde{\theta}}^2 = (1 - r^2) \sigma_{\tilde{u}'}^2 = \frac{4}{15} \sigma_{\tilde{\theta}}^2 , \]  
\hspace{1cm} \text{(C4)}

hence finally

\[ \langle \tilde{u}^2 \rangle_{\tilde{\theta}} = \frac{1}{3} \tilde{\theta}^2 + \frac{4}{15} \sigma_{\tilde{\theta}}^2 . \]  
\hspace{1cm} \text{(C5)}

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