ARRANGEMENTS OF HOMOTHETS OF A CONVEX BODY II

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Abstract. A family of homothets of an o-symmetric convex body $K$ in $d$-dimensional Euclidean space is called a Minkowski arrangement if no homothet contains the center of any other homothet in its interior. We show that any pairwise intersecting Minkowski arrangement of a $d$-dimensional convex body has at most $2 \cdot 3^d$ members. This improves a result of Polyanskii (Discrete Mathematics 340 (2017), 1950–1956).

Using similar ideas, we also give a proof the following result of Polyanskii: Let $K_1, \ldots, K_n$ be a sequence of homothets of the o-symmetric convex body $K$, such that for any $i < j$, the center of $K_j$ lies on the boundary of $K_i$. Then $n = O(3^d)$.

1. Introduction

We use the notation $[n] = \{1, 2, \ldots, n\}$. A convex body $K$ in the $d$-dimensional Euclidean space $\mathbb{R}^d$ is a compact convex set with non-empty interior, and is o-symmetric if $K = -K$. A (positive) homothet of $K$ is a set of the form $\lambda K + v := \{\lambda k + v : k \in K\}$, where $\lambda > 0$ is the homothety ratio, and $v \in \mathbb{R}^d$ is a translation vector. If $K$ is o-symmetric, we also call $v$ the center of the homothet $\lambda K + v$. An arrangement of homothets of $K$ is a collection $\{\lambda_i K + v_i : i \in [n]\}$. A Minkowski arrangement of an o-symmetric convex body $K$ is a family $\{v_i + \lambda_i K\}$ of homothets of $K$ such that none of the homothets contains the center of any other homothet in its interior. This notion was introduced by L. Fejes Tóth [3] in the context of Minkowski’s fundamental theorem on the minimal determinant of a packing lattice for a symmetric convex body, and was further studied by him in [4, 5], by Böröczky and Szabó in [2], and in connection with the Besicovitch covering theorem by Füredi and Loeb [6]. Recently, Minkowski arrangements have been used to study a problem arising in the design of wireless networks [10]. In [9] it was shown that the largest cardinality of a pairwise intersecting Minkowski arrangement of homothets of an o-symmetric convex body in $\mathbb{R}^d$
is $O(3^d d \log d)$. This was improved to $3^{d+1}$ by Polyanskii [11]. We make the following slight improvement.

**Theorem 1.1.** For any $a$-symmetric convex body $K$ in $\mathbb{R}^d$, a pairwise intersecting Minkowski arrangement has at most $2 \cdot 3^d$ members.

Note that the $d$-cube has $3^d$ pairwise intersecting translates that form a Minkowski arrangement. The proof uses ideas from [8] and [7].

In [9], bounds on pairwise intersecting Minkowski arrangements were used to give an upper bound of $O(6^d d^2 \log d)$ on the length of a sequence of homothets $v_i + \lambda_i K$ of an $a$-symmetric convex body $K$ such that $v_j \in \text{bd}(v_i + \lambda_i K)$ whenever $j > i$. This bound was improved to $O(3^d d)$ by Polyanskii [11]. We use some similar ideas to the proof of Theorem 1.1 to give a short proof of this result of Polyanskii.

**Theorem 1.2** (Polyanskii [11]). Let $K$ be an $a$-symmetric convex body, and $v_1, v_2, \ldots, v_n \in \mathbb{R}^d$. Let $\lambda_1, \lambda_2, \ldots, \lambda_{n-1} > 0$, and assume that for any $1 \leq i < j \leq n$ we have $v_j \in \text{bd}(v_i + \lambda_i K)$. Then $n = O(3^d d)$.

Clearly, when $K$ is the cube, $n = 2^d$ is attained. It would be interesting to find better bounds for the maximum size of a family satisfying the conditions of Theorem 1.2.

The interest in this result is that it gives the upper bound $kO(3^d d)$ to the cardinality of a set in a $d$-dimensional normed space in which only $k$ non-zero distances occur between pairs of points. This is currently the best known upper bound if $k = \Omega(3^d d)$ (see [12] for a survey of this problem).

2. Proof of Theorem 1.1

**Theorem 2.1.** Let $d \geq 1$. Suppose that there exists an $a$-symmetric convex body $K$ in $\mathbb{R}^d$ which has a pairwise intersecting Minkowski arrangement of $n$ homothets. Then there exists a set $\{x_1, \ldots, x_n\}$ of $n$ points in $\mathbb{R}^{d+1}$ such that $o \notin \text{conv}\{x_1, \ldots, x_n\}$, and for any distinct $i, j \in [n], i < j$, there exists a non-zero linear functional $f_{ij} : \mathbb{R}^{d+1} \rightarrow \mathbb{R}$ with

$$|f_{ij}(x_k)| \leq |f_{ij}(x_i) - f_{ij}(x_j)| \quad \text{for all } k \in [n].$$

(2.1)

We remark that the converse of the above theorem does not hold. We describe a simple counterexample for $d = 1$. On the one hand, clearly, a pairwise intersecting Minkowski arrangement of intervals in $\mathbb{R}$ has at most two members. On the other hand, there is a set of 5 points on the plane satisfying the conclusion of Theorem 2.1. Indeed, let $\{x_1, \ldots, x_5\}$ be the vertex set of a regular pentagon, with $o$ just outside the pentagon, close to the midpoint of an edge. It is easy to see that for any pair $x_i, x_j$ of vertices, there is a line through $o$ such that the projections $\pi(x_k)$ of the vertices onto the line are all within distance $|\pi(x_i) - \pi(x_j)|$ of $o$.

The above remark is to be contrasted with the equivalence in the following result, which generalizes part of Theorem 1.4 of [7].
Theorem 2.2. Given $\lambda \geq 1$, and $D \in \mathbb{Z}, D \geq 1$. Then the following statements are equivalent.

(i) There exists a set $\{x_1, \ldots, x_n\}$ of $n$ points in $\mathbb{R}^D$, such that $\alpha \notin \text{conv}\{x_1, \ldots, x_n\}$, and for any distinct $i, j \in [n], i < j$ there exists a non-zero linear functional $f_{ij}: \mathbb{R}^D \rightarrow \mathbb{R}$ with

\[ |f_{ij}(x_k)| \leq \frac{\lambda}{2} |f_{ij}(x_i) - f_{ij}(x_j)| \quad \text{for all } k \in [n]. \]

(ii) There is an $o$-symmetric convex set $L$ in $\mathbb{R}^D$ that has $n$ non-overlapping translates $L + t_1, \ldots, L + t_n$, each intersecting $(\lambda - 1)L$, with $o \notin \text{conv}\{t_1, \ldots, t_n\}$.

It follows that $K$ supports $\lambda \in \rho(2.1)$. Let $i, j$ distinct

\[ \text{By the Minkowski property,} \]

\[ \text{Theorem 2.3. Let } K \text{ be an } o \text{-symmetric convex set in } \mathbb{R}^D \text{ with } D \geq 2, \] and let $\alpha K + t_1, \ldots, \alpha K + t_n$ be $n$ non-overlapping translates of $\alpha K$ with $\alpha > 0$ such that each translate intersects $K$, and $o \notin \text{int}(\text{conv}\{t_1, \ldots, t_n\})$. Then

\[ n \leq \frac{(1 + 2\alpha)^{D-1}(1 + 3\alpha)}{2\alpha^D}. \]

This theorem is a slight modification of Theorem 1.5 of [7]. There the translates of $\alpha K$ touch $K$, whereas here they may overlap with $K$. Theorem 2.3 is sharp for $\alpha = 1$. Indeed, let $K$ be the cube $[-1,1]^D$, and consider the $2 \cdot 3^{D-1}$ translation vectors $\{t \in \{-2,0,2\}^D : t^{(1)} \geq t^{(2)}\}$.

Combining Theorems 2.1, 2.2 and 2.3 (with $\lambda = 2$, $K = (\lambda - 1)L = L$, $\alpha = \frac{1}{\lambda - 1} = 1$), we immediately obtain Theorem 1.1.

3. Proof of Theorem 2.1

Let the Minkowski arrangement by $\{v_i + \lambda_i K : i \in [n]\}$, where $\lambda_i > 0$ and $v_i \in \mathbb{R}^d$ for each $i \in [n]$. Let $x_i = (\lambda_i^{-1} v_i, \lambda_i^{-1}) \in \mathbb{R}^d \times \mathbb{R}$, $i \in [n]$. Fix distinct $i, j \in \{1, \ldots, n\}$. We will find a linear $f: \mathbb{R}^d \times \mathbb{R} \rightarrow \mathbb{R}$ that satisfies (2.1). Let $\varphi: \mathbb{R}^d \rightarrow \mathbb{R}$ be a linear functional such that $\varphi(x) \leq \|x\|_K$ for all $x \in \mathbb{R}^d$ and $\varphi(v_j - v_i) = \|v_j - v_i\|_K$. (Thus, $\varphi^{-1}(1)$ is a hyperplane that supports $K$ at $\|v_j - v_i\|_K^{-1}(v_j - v_i)$.)

Since any two homothets $v_k + \lambda_k K$ and $v_l + \lambda_l K$ intersect, any two of the compact intervals $\varphi(v_k + \lambda_k K)$ and $\varphi(v_l + \lambda_l K)$ intersect in $\mathbb{R}$. By Helly’s Theorem in $\mathbb{R}$, there exists $\alpha \in \bigcap_{i=1}^n \varphi(v_i + \lambda_i K)$. Since $\varphi(v_i + \lambda_i K) = [\varphi(v_i) - \lambda_i, \varphi(v_i) + \lambda_i]$ and $\varphi(v_j + \lambda_j K) = [\varphi(v_j) - \lambda_j, \varphi(v_j) + \lambda_j]$, we have

\[ \varphi(v_j) - \lambda_j \leq \alpha \leq \varphi(v_i) + \lambda_i. \]

By the Minkowski property,

\[ \varphi(v_j - v_i) = \|v_j - v_i\|_K \geq \max\{\lambda_i, \lambda_j\}. \]

It follows that

\[ \varphi(v_i) \leq \alpha \leq \varphi(v_j). \]
We set \( f = (\varphi, -\alpha) \in (\mathbb{R}^d \times \mathbb{R})^* \), that is, define \( f(x) = \varphi(v) - \alpha \mu \), where \( x = (v, \mu) \in \mathbb{R}^d \times \mathbb{R} \). We show that \( f(x_j - x_i) \geq 1 \), and \( |f(x_k)| \leq 1 \) for all \( k \in \{1, \ldots, n\} \). This will show that (2.1) is satisfied, which will finish the proof.

\[
\begin{align*}
\varphi(v_j) - \alpha &= \varphi(\lambda_j^{-1}v_j - \lambda_i^{-1}v_i) - \alpha(\lambda_j^{-1} - \lambda_i^{-1}) \\
&= \frac{\varphi(v_j) - \alpha}{\lambda_j} + \frac{\alpha - \varphi(v_i)}{\lambda_i} \\
&\geq \frac{\varphi(v_j) - \alpha + \alpha - \varphi(v_i)}{\max\{\lambda_i, \lambda_j\}} \\
&= \frac{\|v_j - v_i\|_K}{\max\{\lambda_i, \lambda_j\}} \geq 1.
\end{align*}
\]

Since \( \alpha \in \varphi(v_k + \lambda_kK) \), there exists \( x \in K \) such that \( \varphi(v_k + \lambda_kx) = \alpha \). Therefore,

\[
|f(x_k)| = |\varphi(\lambda_k^{-1}v_k) - \alpha \lambda_k^{-1}| = |\varphi(x)| \leq \|x\|_K \leq 1.
\]

\[\square\]

4. Proof of Theorem 1.2

The following proof is very similar to the proof of Theorem 2.1. Without loss of generality, \( \min \lambda_i = 1 \). Denote the unit ball of \( \|\cdot\|_K \) by \( K \). Let \( x_i = (\lambda_i^{-1}v_i, \lambda_i^{-1}) \in \mathbb{R}^d \times \mathbb{R} \), \( i = 1, \ldots, n - 1 \). Let \( N \geq 1 \), to be fixed later. For each \( m = 0, \ldots, N \), let

\[
X_m = \{x_i : i \in [n - 1], |N \log_2 \lambda_j| \equiv m \text{ (mod } N + 1)\}.
\]

Then \( X_0, \ldots, X_N \) partition \( \{x_1, \ldots, x_{n-1}\} \) into \( N + 1 \) parts. Fix \( x_i, x_j \in X_m \) such that \( 1 \leq i < j < n \). We will find a linear \( f : \mathbb{R}^d \times \mathbb{R} \rightarrow \mathbb{R} \) such that (2.2) is satisfied for all \( x_k \in X_m \) and \( \lambda = 2 - 2^{1/N} \). Let \( \varphi : \mathbb{R}^d \rightarrow \mathbb{R} \) be a linear functional such that \( \varphi(v) \leq \|v\|_K \) for all \( v \in \mathbb{R}^d \) and

\[
\varphi(v_j - v_i) = \|v_j - v_i\|_K = \lambda_j.
\]

(Thus, \( \varphi^{-1}(1) \) is a hyperplane that supports \( K \) at \( \|v_j - v_i\|_K^{-1} (v_j - v_i) \).)

Since any two homothets \( v_k + \lambda_kK \) and \( v_l + \lambda_lK \) intersect in their interiors, any two of the open intervals \( \varphi(v_k + \lambda_k \text{ int } K) \) and \( \varphi(v_l + \lambda_l \text{ int } K) \) intersect in \( \mathbb{R} \). By Helly’s Theorem in \( \mathbb{R} \), there exists \( \alpha \in \bigcap_{l=1}^n \varphi(v_l + \lambda_l \text{ int } K) \). Since \( \varphi(v_i + \lambda_i \text{ int } K ) = (\varphi(v_i) - \lambda_i, \varphi(v_i) + \lambda_i) \) and \( \varphi(v_j + \lambda_j \text{ int } K ) = (\varphi(v_j) - \lambda_j, \varphi(v_j) + \lambda_j) \), we have

\[
\varphi(v_j) - \lambda_j < \alpha < \varphi(v_i) + \lambda_i.
\]

By (4.1), we can rewrite this as

\[
-\lambda_i < \varphi(v_i) - \alpha < \lambda_j - \lambda_i.
\]

We set \( f = (\varphi, -\alpha) \in (\mathbb{R}^d \times \mathbb{R})^* \), that is, for \( x = (v, \mu) \in \mathbb{R}^d \times \mathbb{R} \), we let \( f(x) = \varphi(v) - \alpha \mu \). It remains to show that \( f(x_j - x_i) > 2 - 2^{1/N} \), and \( |f(x_k)| \leq 1 \) for all \( k \in \{0, \ldots, n\} \), since this will show that (2.2) is satisfied.
with $\lambda = 2 - 2^{1/N}$. By applying Theorems 2.2 and 2.3 with $\lambda = 2/(2 - 2^{1/N}) = 2 + \frac{\log 4}{N} + O(N^{-2})$, $K = (\lambda - 1)L$ and $\alpha = 1/(\lambda - 1) = 2^{1-1/N} - 1$, we obtain $|X_m| \leq (1 + \lambda/2)(1 + \lambda)^d$, and it follows that

$$n - 1 \leq (N + 1)(1 + \lambda/2)(1 + \lambda)^d.$$  

If we choose $N = d$, we obtain $\lambda = 2 + \frac{\log 4}{d} + O(d^{-2})$ and $n = 3^dO(d)$, which would finish the proof.

By definition of $X_m$,

$$\lfloor N \log_2 \lambda_j \rfloor - \lfloor N \log_2 \lambda_i \rfloor = kN$$  

for some $k \in \mathbb{Z}$.

If $k \geq 1$, then $N \log_2 \lambda_j - N \log_2 \lambda_i > N$, hence $\lambda_j/\lambda_i > 2$. However, we also have

$$\lambda_i = \|v_i - v_j\| \geq \|v_j - v_n\| - \|v_n - v_i\| = \lambda_j - \lambda_i,$$

a contradiction. Therefore, $k \leq 0$, that is, $\lfloor N \log_2 \lambda_j \rfloor - \lfloor N \log_2 \lambda_i \rfloor \leq 0$. This gives $N \log_2 \lambda_j - N \log_2 \lambda_i < 1$ and

$$(4.3) \quad \frac{\lambda_j}{\lambda_i} < 2^{1/N}.$$  

It follows that

$$f(x_j - x_i) = \varphi(\lambda_j^{-1}v_j - \lambda_i^{-1}v_i) - \alpha(\lambda_j^{-1} - \lambda_i^{-1})$$  

$$= \frac{\varphi(v_j) - \alpha}{\lambda_j} + \frac{\alpha - \varphi(v_i)}{\lambda_i}$$  

$$= \frac{\varphi(v_i) + \lambda_i - \alpha}{\lambda_j} + \frac{\alpha - \varphi(v_i)}{\lambda_i}$$  

$$> 2^{-1/N}(\varphi(v_i) + \lambda_i - \alpha) + \alpha - \varphi(v_i)$$  

$$> \frac{2^{-1/N}(\varphi(v_i) + \lambda_i - \alpha) + \alpha - \varphi(v_i)}{\lambda_i}$$  

$$= 2^{-1/N} + \frac{(1 - 2^{-1/N})(\alpha - \varphi(v_i))}{\lambda_i}$$  

$$> \frac{(1 - 2^{-1/N})(\lambda_i - \lambda_j)}{\lambda_i}$$  

$$= 1 - (1 - 2^{-1/N})\frac{\lambda_j}{\lambda_i}$$  

$$> 1 - (1 - 2^{-1/N})2^{1/N}$$  

$$= 2 - 2^{1/N}.$$  

Since $\alpha \in \varphi(v_k + \lambda_kK)$, there exists $x \in K$ such that $\varphi(v_k + \lambda_kx) = \alpha$. Therefore,

$$|f(x_k)| = |\varphi(\lambda_k^{-1}v_k) - \alpha\lambda_k^{-1}| = |\varphi(x)| \leq \|x\|_K \leq 1.$$  

$\square$
5. Proof of Theorem 2.2

Assume that (i) holds. Let $C := \cap_{i \neq j} S_{ij}$ be the intersection of the $\alpha$-symmetric slabs $S_{ij} := \{p \in \mathbb{R}^D : |f_{ij}(p)| \leq \frac{1}{2} |f_{ij}(x_i) - f_{ij}(x_j)|\}$. By assumption, $C \supseteq \{x_1, \ldots, x_n\}$. For each $i \in [n]$, let $C_i := \frac{\lambda x_i + C}{\lambda + 1}$ be the homothetic copy of $C$ with center of homothety $x_i$, and of ratio $\frac{1}{\lambda + 1}$. It is an easy exercise that the $C_i$s are non-overlapping. Moreover, by the symmetry of $C$, we have $\frac{\lambda - 1}{\lambda + 1} x_i \subseteq C_i \cap \frac{\lambda - 1}{\lambda + 1} C$. Thus, for $L := \frac{1}{\lambda + 1} C$, and $t_i := \frac{\lambda}{\lambda + 1} x_i$, (ii) holds as promised.

Next, assume that (ii) holds. Fix $i, j \in [n], i \neq j$. Since $L + t_i$ and $L + t_j$ are non-overlapping, there is a linear functional $f$ such that the two real intervals $s_i := f(L + t_i)$ and $s_j := f(L + t_j)$ do not overlap. These two intervals are of equal length, which we denote by $w$. Thus, we have

$$w \leq |f(t_i) - f(t_j)|. \quad (5.1)$$

On the other hand, $s_k := f(L + t_k)$ is also a real interval of length $w$ for any $k \in [n]$; and $s_0 := f((\lambda - 1)L)$ is a 0-symmetric real interval of length $(\lambda - 1)w$, which intersects each $s_k$. Thus, for the center $f(t_k)$ of $s_k$, we have $|f(t_k)| \leq \frac{(\lambda - 1)w}{2} + \frac{w}{2} = \frac{\lambda w}{2}$. Now, (5.1) yields $|f(t_k)| \leq \frac{1}{2} |f(t_i) - f(t_j)|$. Thus, we may set $f_{ij} := f$. This argument is valid for any $i$ and $j$, thus, with $x_i := t_i$, we obtain (i).

6. Proof of Theorem 2.3

The proof is an almost verbatim copy of the proof of Theorem 1.5 of [7]. There are two points of difference, which we will note.

We recall Lemma 3.1. of [7], which is a slightly more general version of the Lemma of [1].

Lemma 6.1. Let $f$ be a function on $[0, 1]$ with the properties $f(0) \geq 0$, $f$ is positive and monotone increasing on $(0, 1)$, and $f(x) = (g(x))^k$ for some concave function $g$ and $k > 0$. Then

$$F(y) := \frac{1}{f(y)} \int_0^y f(x) \, dx$$

is strictly increasing on $(0, 1]$.

Proof of Theorem 2.3. Clearly, we may assume that $K$ is bounded, otherwise, by a projection, we can reduce the dimension. Let $\alpha K + t_1, \alpha K + t_2, \ldots, \alpha K + t_n$ be pairwise non-overlapping translates of $\alpha K$ that intersect $K$. By the assumptions of the theorem, there is a non-zero vector $v \in \mathbb{R}^D$ such that $a_i := \langle t_i, v \rangle \geq 0$ for $i \in [n]$. Set $h(x) := \{p \in \mathbb{R}^D : \langle p, v \rangle = x\}$. Without loss of generality, we may assume that $h(-1)$ and $h(1)$ are supporting hyperplanes of $K$.

Clearly, $\alpha K + t_i$ is between $h(-\alpha)$ and $h(1 + 2\alpha)$, and it is contained in $(1 + 2\alpha)K$, for $i \in [n]$. 

\begin{equation}
\int_{-\alpha}^{1+2\alpha} V_{D-1} \left( \left( \bigcup_{i=1}^{n} \alpha K + t_i \right) \cap h(x) \right) \, dx = n\alpha^D V_D(K).
\end{equation}

\begin{equation}
\int_{0}^{1+2\alpha} V_{D-1} \left( \left( \bigcup_{i=1}^{n} \alpha K + t_i \right) \cap h(x) \right) \, dx \\
\leq \int_{0}^{1+2\alpha} V_{D-1} \left( ((1+2\alpha)K \cap h(x)) \right) \, dx = \frac{(1+2\alpha)^d}{2} V_D(K).
\end{equation}

We note that this was the first point of difference from the proof in [7]: here, we do not subtract the contribution of $K$ in the total volume on the right hand side of the inequality.

Set $f(x) := V_{D-1} (\alpha K \cap h(x-\alpha))$, and observe that the conditions of Lemma 6.1 are satisfied by $f$ (with $k = D - 1$, by the Brunn–Minkowski inequality). We may assume that $a_1, \ldots, a_m \leq \alpha < a_{m+1}, \ldots, a_n$. By Lemma 6.1,

\begin{align*}
\int_{-\alpha}^{\alpha} V_{D-1} \left( \left( \bigcup_{i=1}^{n} \alpha K + t_i \right) \cap h(x) \right) \, dx &= \sum_{i=1}^{m} \int_{0}^{\alpha-a_i} f(x) \, dx \\
&\leq \sum_{i=1}^{m} \int_{0}^{\alpha} f(x) \, dx \frac{f(\alpha-a_i)}{f(\alpha)} = \frac{\alpha^d V_D(K)}{2f(\alpha)} \sum_{i=1}^{m} V_{D-1} ((\alpha K + t_i) \cap h(0)) \\
&= \frac{\alpha^d V_D(K)}{2f(\alpha)} V_{D-1} \left( \left( \bigcup_{i=1}^{m} \alpha K + t_i \right) \cap h(0) \right) \\
&\leq \frac{\alpha^d V_D(K)}{2f(\alpha)} \left[ V_{D-1} \left( ((1+2\alpha)K \cap h(0)) \right) \right] = \frac{\alpha(1+2\alpha)^{D-1}}{2} V_D(K).
\end{align*}

We note that this was the second point of difference from the proof in [7]: again, the contribution of $K$ to the volume is not subtracted.

This inequality, combined with (6.1) and (6.2), yields (2.3). □

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