Ice melting in a turbulent stratified shear flow

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In order to improve our understanding of ice melting in oceans, here we report an idealized numerical study of phase changes between a solid and an adjacent turbulent, stratified shear flow. We use the phase-field method to dynamically couple the Navier-Stokes equations for the fluid with the heat equation for the solid. We investigate the evolution of an initially-flat solid boundary overlying a pressure-driven turbulent flow. We assume a linear equation of state and change the sign of the thermal expansion coefficient, such that the background density stratification is either stable, neutral or unstable. We find that channels aligned with the direction of the mean flow are generated spontaneously by phase changes at the fluid-solid interface. Streamwise vortices in the fluid, the interface topography and the temperature field in the solid influence each other and adjust until a statistical steady state is obtained. The crest-to-trough amplitude of the channels are larger than about 10\(\delta\nu\) in all cases, with \(\delta\nu\) the viscous length scale, but are much larger and more persistent for an unstable stratification than for a neutral or stable stratification. This happens because a stable stratification makes cool melt water buoyant such that it shields the channel from further melting, whereas an unstable stratification makes cool melt water sink, allowing further melting by rising hot plumes. The statistics of flow velocities and melt rates are investigated, and we find that channels and keels emerging in our simulations do not significantly change the mean drag coefficient.

1. Introduction

Ice-ocean interactions play an important role in the climate system. In West Antarctica, ocean changes have led to increased melting and thinning of the ice shelves, floating glacial ice that buttresses the Antarctic Ice Sheet (Pritchard \textit{et al.} 2012; Rignot \textit{et al.} 2013), and this may result in large sea-level rise by 2100 (Kennicutt \textit{et al.} 2019). Earth’s global climate drives the large-scale conditions for the melting of icebergs, ice shelves and marine-terminating glaciers. However, ice melting is ultimately controlled by the efficiency of turbulence in drawing warm water from the fluid interior toward the ice-ocean boundary, a local process which remains poorly constrained. The study of ice-ocean interactions is generally difficult but perhaps most so at scales smaller than a few tens of meters, owing to the difficulty in observing small length scales and fast time scales, and simultaneously resolving the slow evolution of the ice-ocean boundary and fast turbulent features, which dominate the dynamical variability at small scales. Ocean currents have velocities in the range of centimeters to one meter per second, resulting in rapid eddy turnover times of tenths of seconds (Davis & Nicholls 2019), whereas melt rates are relatively slow, in the

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range of a few centimeters per day for horizontal faces (Dutrieux et al. 2014) to a few meters per day for vertical faces (Sutherland et al. 2019).

Heat and salt fluxes across the ice-ocean boundary layer drive melting. A drag coefficient $C_D$, heat transfer coefficient $\Gamma_T$ and salt transfer coefficient $\Gamma_S$ between the fluid interior and the boundary are typically required in order to estimate the melt rate of ice in regional and global climate models unable to represent turbulence explicitly due to limited grid resolution. The melt rate $\dot{m}$ is often inferred from the so-called three-equation model (Holland & Jenkins 1999, Vreugdenhil & Taylor 2019)

$$\rho_i L \dot{m} = c_w \rho_w C_D^{1/2} U \Gamma_T (T - T_b), \quad (1.1a)$$

$$\rho_i S_b \dot{m} = c_w \rho_w C_D^{1/2} U \Gamma_S (S - T_b), \quad (1.1b)$$

$$T_b = \lambda_1 S_b + \lambda_2 + \lambda_3 P_b, \quad (1.1c)$$

where $\rho_i$ is the ice density, $L$ is the latent heat of fusion per unit mass, $c_w$ is the heat capacity of water, $\rho_w$ is the water density, $T_b$, $S_b$ and $P_b$ are the temperature, salinity and pressure at the boundary, $\lambda_i$ ($i = 1, 2, 3$) are thermodynamic constants and $T$, $S$, and $U$ are the resolved values of temperature, salinity and flow speed at the outer edge of the boundary layer. Equations (1.1a)–(1.1b) form parameterized versions of the true Stefan and salt-conservation conditions relating the interface velocity to heat and salt fluxes at the boundary, and equation (1.1c) is the linearized liquidus condition for salt water (Woods 1992).

An increasing number of studies have now attempted to model ice-bounded ocean turbulence accurately, i.e. either using direct numerical simulations (Gayen et al. 2016, Keitzl et al. 2016a, Mondal et al. 2019) or large-eddy simulations (Vreugdenhil & Taylor 2019), allowing the investigation of ice-ocean interactions at small scales and the evaluation of the accuracy of the three-equation model (1.1). For simplicity, all such simulations have so far assumed a fixed and hydraulically smooth ice-ocean interface. As a result, the variability of heat and salt fluxes with basal roughness (McPhee 2008) and basal topography, i.e. topographical features at scales much larger than the submillimetric viscous sublayer thickness but smaller than the ten-meter-scale Ekman layer thickness, remain largely unknown. Relatedly, the physical mechanisms leading to the generation and saturation of topographical features such as meter-scale scallops at the ice-ocean interface remain largely unknown, except for a recent theoretical study that provides a prediction for the generation of two-dimensional ice scallops as a function of the far-field velocity via a linear instability mechanism of the coupled water-ice system with parameterized flow nonlinearities (Claudin et al. 2017).

Importantly, the ice-ocean boundary has many topographical features at both large (e.g. Stanton et al. 2013, Dutrieux et al. 2014, Gourmelen et al. 2017) and small (meter) scales. For instance, three-dimensional topographical features have been observed on the underside of icebergs (Hobson et al. 2011) and sea ice (Lucieer et al. 2016) at meter scales, on the underside of ice shelves (Nicholls et al. 2006) and sea ice (Wadhams et al. 2006) at tens-of-meters scales, and in laboratory experiments (Bushuk et al. 2019) at tens-of-centimeters scales. Thus, it is expected that the ice-ocean interface exhibits topographical variability at scales that can impact the efficiency of momentum, heat and salt fluxes across the boundary layer.

There are several lines of evidence to suggest that equations (1.1) do not fully describe ice melting in seawater. Bushuk et al. (2019) reported a doubling of the drag coefficient for flows over tens-of-centimeters long scallops, compared to no scallops, in laboratory experiments. McConnochie & Kerr (2017) found that equation (1.1) is unable to predict melt rates of a vertical ice face for flow velocities less than 4 cm/s. Sutherland et al. (2019) showed that measured melt rates at a marine-terminating glacier in Alaska were up to 100 times larger than modeled melt
rates using equation (1.1). Kimura et al. (2015) showed that (1.1) significantly over-predicts melting in the presence of quiescent, double-diffusive conditions. The effect of topography and roughness on drag, heat and salt fluxes do not account for all of the melt rate discrepancies reported in the literature, but the study of Bushuk et al. (2019), the observed discrepancies of McConnochie & Kerr (2017) Sutherland et al (2019) and the sensitivity of large-scale model outputs to basal friction parameters (Gwyther et al. 2015) reinforce the consensus that high-resolution investigations of melting in the presence of ice topography and roughness are required in order to constrain melting parameterizations and overcome the associated large uncertainties in projected sea-level rise (Dinniman et al. 2016).

Here we report direct numerical simulation results on the evolution—due to melting and freezing—of a layer of ice adjacent to a turbulent and stratified shear flow. Our primary goal is to demonstrate the capability of the model in tracking the spontaneous evolution of the ice-water interface, and to investigate the influence of stratification on the topography obtained. Thus, for simplicity we consider a pure water flow, i.e. we neglect salt effects, and we assume that the ice and water have the same thermodynamical properties, i.e. reference density and thermal diffusivity. Our model solves for the evolution of the fluid and the solid simultaneously using the phase-field method. The phase-field method is a one-domain two-phase fixed-grid method that was originally developed by the metallurgy community (Wang et al. 1993; Karma & Rappel 1998; Beckermann et al. 1999). Recent studies have demonstrated its applicability for fluid dynamics and environmental flow problems (Favier et al. 2019; Purseed et al. 2020), which we extend here to the specific case of ice–ocean interactions. Other methods that simultaneously solve for the evolution of a fluid phase and a solid phase include the enthalpy method (Ulvrovˇa et al. 2012), the level set method (Gibou et al. 2007), the lattice-Boltzmann method (Rabbanipour Esfahani et al. 2018) and two-domain moving-boundary methods (Ulvrovˇa et al. 2012). The main advantage of the phase-field method over these other methods is that it can be implemented relatively easily in any fluid solver.

The main result of our paper is that topographical features spontaneously emerge at the ice-water interface due to uneven melting of the solid boundary. We investigate the effect of background density stratification and we demonstrate that the topography is dominated by keels and channels that are aligned with the direction of the mean flow in all cases. The paper is organized as follows. In §2 we describe the phase-field method, the governing equations and the numerical method. In §3 we present and discuss the direct numerical simulation results obtained for three different background stratifications. In §4 we conclude. Finally, in appendices A-D we provide additional details about the method and derivations.

2. Model

2.1. Phase-field method

We investigate the generation of topography at the interface between an immovable solid and a pressure-driven fluid flow (also known as Poiseuille or channel flow), due to uneven melting or solidification (figure 1). The fluid is in a channel of depth $2H$, length $L_x = 4\pi H$ and width $L_y = 2\pi H$, and the overlying solid has initial thickness $H/2$. We define a Cartesian coordinate system $(x, y, z)$ centered on the channel mid-plane with $z$-axis vertically upward, i.e. opposite to gravity, and use superscripts $(f)$ and $(s)$ to denote variables in the fluid and the solid, respectively. The fluid velocity $\mathbf{u}^{(f)}$ and pressure $p^{(f)}$ evolve according to the Navier-Stokes equations under the Boussinesq approximation. The fluid density is related to temperature through the linear equation of state $\rho^{(f)} = \rho_0 (1 - \alpha T^{(f)})$, i.e. we neglect the density maximum of water around 4°C (Léard et al. 2020), with $\rho_0$ the reference density and $\alpha$ the thermal expansion coefficient.
The temperature in the fluid, $T^{(f)}$, and solid, $T^{(s)}$, evolve according to the heat equation. For a freshwater flow, the interface temperature must be at the temperature of melting, $T_m$, and the movement of the interface is governed by the Stefan condition, i.e.

$$T^{(f)} = T^{(s)} = T_m,$$

(2.1a)

$$\mathcal{L} v_n = \frac{c_p}{k} \left( q_n^{(f)} - q_n^{(s)} \right),$$

(2.1b)

with $v_n$ the interface velocity normal to the interface and directed toward the solid phase, and $q_n$ the heat flux normal to the interface and directed toward the solid phase; $\mathcal{L}$ is the latent heat of fusion per unit mass, $c_p$ is the specific heat capacity at constant pressure of the fluid and $k$ is the thermal conductivity.

Here we use a volume-penalization method [Angot et al., 1999], which is a type of immersed boundary method, combined with the phase-field method, in order to solve for phase-change processes and the evolution of the variables in the fluid and the solid simultaneously. Specifically, we solve the Navier-Stokes equations and the heat equation combined with an equation for the fluid fraction $\phi$, i.e.

$$\frac{\partial \mathbf{u}}{\partial t} + \phi (\mathbf{u} \cdot \nabla) \mathbf{u} = \nu \nabla^2 \mathbf{u} - \frac{1}{\rho_0} \nabla p + \alpha g \hat{z} + \frac{\Pi}{\rho_0} \frac{x}{\tau} (1 - \phi) \mathbf{u},$$

(2.2a)

$$\frac{\partial T}{\partial t} + \phi (\mathbf{u} \cdot \nabla) T = \kappa \nabla^2 T - \frac{\mathcal{L}}{c_p} \frac{\partial \phi}{\partial t},$$

(2.2b)

$$\frac{\partial \phi}{\partial t} = a \nabla^2 \phi + b \phi (1 - \phi) [2\phi - 1 + c(T - T_m)],$$

(2.2c)

$$\nabla \cdot \mathbf{u} = 0,$$

(2.2d)

with $\mathbf{u} = (u, v, w)$, $p$, and $T$ defined in both the fluid and solid, i.e. assuming that the fluid and solid phases form a single domain, such that we drop the superscripts $(f)$ and $(s)$. In equations (2.2), $\nu$ is the constant kinematic viscosity, $g$ the gravity acceleration, $\Pi$ is the imposed pressure-gradient force and $\kappa$ is the constant thermal diffusivity; $\tau$, $a$, $b$ and $c$ are parameters related to volume penalization and the phase-field method, which we define later, and $\hat{z}$ and $\hat{x}$ are the unit vectors of the $z$ and $x$ axis, respectively.

The fluid fraction, $\phi$, also known as the phase-field variable or order parameter, satisfies a forced diffusion equation (2.2c) with parameters tuned such that $\phi$ transitions continuously from 1 in the fluid to 0 in the solid across a diffuse interface whose thickness is smaller than all physical length scales in the problem (cf. appendix A). $\phi$ is introduced in the momentum, heat and continuity equations, (2.2a), (2.2b) and (2.2c) respectively, in order to modulate locally the importance of each physical processes based on the component’s phase. For instance, the last term on the right-hand-side of equation (2.2a) is a linear (penalization) damping term, which is active in the solid but inactive in the fluid. The second term on the right-hand-side of the heat equation (2.2b) is a heat sink or source that represents the consumption or release of latent heat associated with melting or solidification. The third and fourth terms on the right-hand-side of equation (2.2a) represent the buoyancy force and the imposed pressure-gradient force, respectively. In the limit of infinitesimally-small diffuse interface thickness of the phase field, it has been shown that the dynamics of the fluid-solid interface governed by equations (2.2) converges to the exact Stefan conditions (2.1), and that the fluid velocity converges to 0 at the fluid-solid interface, thus mimicking a no-slip boundary. Here we multiply by $\phi$ the advective terms in equations (2.2a)–(2.2b), such that they are zero in the solid phase. Previous studies have used both damped and undamped advective terms and we discuss the impact of our choice on the results in appendix B.
2.2. Dimensionless equations

Equations (2.2) can be non-dimensionalized in order to identify the set of independent parameters controlling the coupled fluid-solid problem. Let us define dimensionless variables, denoted by tildes, such as

\[(x, y, z) = (H\tilde{x}, H\tilde{y}, H\tilde{z}), \quad t = \tau_\kappa \tilde{t}, \quad u = u_t\tilde{u}, \quad T = T_m + \Delta T\tilde{T}, \quad p = p_0 u_t^2 \tilde{p}, \quad \phi = \tilde{\phi}, \quad (2.3)\]

where \(H\) is the channel half-depth, \(\tau_\kappa = H^2/\kappa\) is the thermal diffusive time scale, \(u_t = \kappa/H\) is a thermal-diffusion velocity scale and \(\Delta T = T_b - T_m\) is the temperature scale. Substituting variables (2.3) into equations (2.2), and dropping tildes, we obtain the dimensionless equations

\[
\begin{align*}
\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} &= Pr \nabla^2 \mathbf{u} - \nabla p + PrRa T \mathbf{a} + 2Pr^2Re \mathbf{k} - \frac{1}{\Gamma} \mathbf{u}, \\
\frac{\partial T}{\partial t} + (\mathbf{u} \cdot \nabla) T &= \nabla^2 T - St \frac{\partial \phi}{\partial t}, \\
\frac{\partial \phi}{\partial t} &= A\nabla^2 \phi + B(1 - \phi)(2\phi - 1 + CT), \\
\nabla \cdot \mathbf{u} &= 0, \quad (2.4a, b, c, d)
\end{align*}
\]

where the Prandtl number, \(Pr = v/\kappa\), the centreline Reynolds number, \(Re = \Pi H^3/(2\rho_0 v^2)\), the Rayleigh number, \(Ra = \alpha g \Delta T H^3/(\kappa v)\), and the Stefan number, \(St = S/\epsilon_{\kappa} \Delta T\) are the physical dimensionless parameters of the problem; and \(\Gamma = \tau v H^2/\kappa, A = a/\kappa, B = b/(\kappa/H^2)\) and \(C\) are additional non-physical parameters that are prescribed based on numerical constraints of the volume-penalization and the phase-field methods (cf. appendix A). The problem is fully specified once \(Pr, Re, Ra\) and \(St\) are known and the boundary conditions are prescribed. Here we enforce a no-slip, fixed-temperature condition at the top of the ice, i.e. \(\mathbf{u} = 0\) and \(T = T_0 < 0\) at \(z = 1.5\). Note that the melting temperature is \(T = 0\) in dimensionless units. For numerical expediency we consider a half-channel flow, such that we impose free-slip, fixed-temperature conditions on the bottom boundary \(z = 0\), i.e. \(\partial_z u = \partial_z v = w = 0\) and \(T = 1\) at \(z = 0\). The initial interface position is \(z = 1\) and we note \((l_x, l_y, l_z) = (4\pi, 2\pi, 1.5)\) the domain lengths in dimensionless space. The initial condition in the fluid is a half-channel laminar Poiseuille flow superimposed with divergence-free white noise for the velocity fluctuations. Note that here we will generally discuss our results in terms of the steady-state friction or shear Reynolds number, \(Re_s\), and the friction Richardson number, \(Ri_s\),

\[
Re_s = \sqrt{2Re} = \sqrt{\frac{\Pi H^3}{\rho_0 v^2}}, \quad Ri_s = \frac{-Ra}{PrRe} = -\frac{2\rho_0 \alpha \Delta T g}{\Pi}, \quad (2.5)
\]

since they are more commonly used than \(Re\) and \(Ra\) in turbulent channel flow studies (García-Villalba & del Alamo 2011, Zonta & Soldati 2018).

We investigate the effect of background density stratification on the generation of topography at the fluid-solid interface by considering three distinct values of \(Ri_s\), i.e. \(Ri_s = 40, Ri_s = 0\) and \(Ri_s = -40\), for which the stratification is stable, neutral and unstable, respectively. Physically, this is obtained for a thermal expansion coefficient \(\alpha < 0\), \(\alpha = 0\) and \(\alpha > 0\), leading to a density difference between the bottom boundary and the fluid-solid interface \(\Delta \rho > 0\), \(\Delta \rho = 0\) and \(\Delta \rho < 0\), respectively. For simplicity and numerical expediency the other parameters are fixed such that the flow is turbulent, i.e. \(Re_s = 150\) \((Re = 11250), Pr = 1\) and \(St = 1\). For reference, the Rayleigh number for the unstable stratification case \((Ri_s = -40)\) is \(Ra = 4.5 \times 10^5\), which is above the instability onset for Rayleigh-Bénard convection rolls in the streamwise direction \((Ra \approx 1101)\) of thermally stratified plane Poiseuille flow (Chandrasekhar 1961, Gage & Reid 1968). We solve equations (2.4) using the open-source pseudo-spectral direct numerical simulation (DNS).
Figure 1. (a) Simulation snapshot showing the temperature field in the fluid (red colormap) and the solid (blue colormap), and the velocity vectors at select locations (arrows), for $Ri_*=0$ and at statistical steady state. (b) Zoom-in on a region of (a). (c) Variation of the phase field from $\phi = 1$ in the fluid to $\phi = 0$ in the solid along the vertical solid line drawn in (b). The non-dimensional lengths in $x$, $y$ and $z$ directions are $l_x = 4\pi$, $l_y = 2\pi$ and $l_z = 1.5$, respectively.

| stratification | $Ri_*$ | $Re_b^I$ | $Re_b^{III}$ | $Nu_b^I$ | $q'$ | $Nu_b^{III}$ | $\xi_{III}$ | $\xi_-^I$, $\xi_+^I$ | $10^3C_D^I$ | $10^3C_D^{III}$ |
|----------------|-------|----------|--------------|----------|-----|-------------|-------------|---------------------|------------|--------------|
| stable         | 40    | 2720     | 2630         | 3.02     | 2.83| 3.09        | 1.042       | (2.4,2.2)          | 5.9        | 6.7          |
| neutral        | 0     | 2290     | 2240         | 4.58     | 4.48| 4.75        | 1.025       | (5.2,4.1)         | 8.6        | 9.2          |
| unstable       | -40   | 1970     | 1910         | 6.74     | 6.69| 7.75        | 1.037       | (30,15)           | 11.3       | 12.7         |

Table 1. Simulation parameters and selected output variables. Note that $Re_*=11250$, $Pr = 1$ and $St = 1$ in all simulations; also $Ri_* = -40$ corresponds to $Ra = -4.5 \times 10^5$. The output bulk Reynolds numbers $Re_b^I$ and $Re_b^{III}$ and Nusselt numbers $Nu_b^I$ and $Nu_b^{III}$ are volume-averaged and time-averaged over 50 friction time units before the end of stages I and III, respectively. $q'$ is the conductive heat flux through the ice imposed as an initial condition at the beginning of stage II. $\xi_{III}$, $\xi_-$ and $\xi_+$ are the mean interface position, the maximum amplitude of the keels and the maximum amplitude of the channels averaged over 50 friction time units at the end of stage III. $C_D^I$ and $C_D^{III}$ are the drag coefficients averaged over 50 friction time units at the end of stage I and stage III, respectively.

code DEDALUS (Burns et al. 2019). We use 256 Fourier modes in $x$ and $y$ directions and a compound Chebyshev basis with a total of 288 modes in the $z$ direction unless stated otherwise (cf. details in appendix C). We use a two-step implicit-explicit Runge-Kutta scheme for time integration. The CFL condition is typically set to 0.2 in the transient initial stage and 0.4 later on. At statistical steady state, the time step is typically $10^3$ to $10^4$ times smaller than the friction time scale $1/(Re_*Pr)$, which is equal (in terms of dimensional variables) to $H$ divided by the steady-state friction velocity. We run each simulation for about 4 diffusive time scales, or 600 friction time scales, which takes roughly 2 million time steps, such that the total cost of the project is on the order of 1 million CPU hours. Figure 1(a) shows a snapshot of the temperature field in the fluid (red colormap) and the solid (blue colormap), as well as the velocity vectors (arrows) at select locations for $Ri_* = 0$. Figure 1(b) shows that the variations of the phase field along the thick solid line drawn in figure 1(b); the transition from $\phi = 1$ in the fluid to $\phi = 0$ in the solid occurs over a very thin diffuse interface of thickness $\approx 0.007H$. Simulation parameters and output variables are provided in table 1.
2.3. Variables of interest

We define the friction velocity, the bulk velocity and the Nusselt number as

$$ u_s = \sqrt{-\frac{\langle \tau_d + \tau_v + \tau_w \rangle}{\langle \phi \rangle}} , \quad u_b = \langle u \rangle \langle \phi \rangle , \quad Nu = \langle q \rangle , \quad (2.6) $$

respectively (cf. details in appendix B), where overbar denotes the horizontal average, and $\langle \cdot \rangle \equiv \int d\gamma / \gamma$ denotes the volume average, i.e. such that $\langle \phi \rangle$ is the mean fluid fraction. In equation (2.6), $\tau_d$, $\tau_v$ and $\tau_w$ are the linear damping, viscous and Reynolds shear stresses, and $q = wT - \partial_z T$ is the heat flux. Note that, at statistical steady state, $\langle \tau_d + \tau_v + \tau_w \rangle$ is approximately a linear function of $z$ and $q$ is approximately depth invariant, in agreement with channel flow simulations of a pure fluid (cf. appendix B for details on stresses and depth-independent variables using the phase-field method). We denote by $\xi(z,y)$ the fluid-solid interface position, where

$$ \bar{\xi}(x,y) = \int_0^{l_z} \phi dz , \quad (2.7) $$

such that $\bar{\xi} = \langle \phi \rangle$ (note that one could alternatively define $\xi$ as $\phi(z = \bar{\xi}) = 0.5$ or $T(z = \bar{\xi}) = 0$). The melt rate is then $\bar{m} = \partial_t \bar{\xi}$ and we define the drag coefficient of the fluid-solid boundary as $C_D = 2(u_s/u_b)^2$ (García-Villalba & del Alamo 2011). Temporal fluctuations of the variables of interest will be mainly reported in terms of the friction time $t_\ast$, which is defined as $t_\ast = Re Pr t$. On several occasions we will show depth variations of variables in terms of the distance from the interface, $\chi = \bar{\xi} - z$.

3. Results and discussion

The key findings of our work are that (i) turbulent flows induce uneven melting, which can lead to the spontaneous generation of topographical features at the fluid-solid interface, and that (ii) stratification affects interface topography. Thus, after a discussion of the evolution of global flow variables (3.1), we directly present the results of the topographical features generated at the fluid-solid boundary (3.2). We then investigate the interplay between the turbulent flow, the topography, and the melting (3.3, 3.4), and finally discuss the evolution of the mean interface position and the statistics of melting (3.5).

3.1. Simulation stages and global flow variables

We show in figure 2 the friction velocity $u_s$ (left axis), the bulk velocity $u_b$ (left axis), and the Nusselt number $Nu$ (right axis) for stable (figure 2(a)), neutral (figure 2(b)), and unstable stratification (figure 2(c)). Each one of our three simulations can be broken down into three stages, which are highlighted by different colors in figure 2. First, grey/blue colors highlight results obtained during the spin-up phase (stage I; $t \leq t_\ast^{(c)}$), i.e. obtained while running a simple fluid-only channel flow simulation in $0 \leq z \leq 1$ without/buoyancy effects (we turn on buoyancy at $t = t_\ast^{(b)}$; cf. appendix C for more details). Second, orange colors highlight results obtained when we add the solid phase and volume penalization is turned on, i.e. obtained while solving for the variables in both the fluid and solid, but with $\phi = 0.5 \{ 1 - \tanh [2(z - \bar{\xi})/\delta] \}$ prescribed (stage II; $t_\ast^{(c)} < t \leq t_\ast^{(II)}$), where $\delta$ is the thickness of the diffuse interface (see appendix A). Finally, green colors highlight the results of the fully-coupled fluid-solid problem with melting, i.e. obtained while solving equations (2.4) with all variables freely evolving (stage III; $t > t_\ast^{(II)}$). The temperature in the solid is initialized at the beginning of stage II as

$$ T = -q^s(z - 1) , \quad 1 \leq z \leq 1.5 , \quad (3.1) $$
where $q^s$ is the initial conductive heat flux through the solid, yielding the fixed-temperature condition $T = T_i = -q^s/2$ at the top of the solid. The difference between the heat flux in the fluid and the conductive heat flux in the solid in stage II controls whether the solid melts or the fluid solidifies once phase changes are turned on in stage III. Here, we set $q^s$ to be slightly smaller than the heat flux in the fluid at the end of stage I, which we denote $Nu^I$, such that the solid melts slowly at the beginning of stage III in all three simulations (see further discussion in \((3.5)\)).

The bulk Reynolds and Nusselt numbers at the end of stages I and III are defined as

$$
Re_b^I = \int_{t^I}^{t^I_c} \frac{u_b dt_s}{Pr \Delta s}, \quad Re_b^{III} = \int_{t^III}^{t^III_c} \frac{u_b dt_s}{Pr \Delta s}, \quad Nu^I = \int_{t^I}^{t^I_c} \frac{Nu dt_s}{\Delta s}, \quad Nu^{III} = \int_{t^III}^{t^III_c} \frac{Nu dt_s}{\Delta s},
$$

with $\Delta s = 50$ and are reported with $q^s$ in table \([1]\).

Buoyancy effects are turned off for $t_s \leq t_b^I$ (gray colors), such that the results of figure \([2]\) are exactly the same for all three simulations until $t_s = t_b^I$. Upon turning on buoyancy, i.e. for $t_s > t_b^I$ (blue colors), the Nusselt number and bulk velocity deviate from the neutral case (figure \([2]\)b)), but with opposite behaviors: $Nu$ decreases while $u_b$ increases with stabilizing buoyancy effects (figure \([2]\)a)), and $Nu$ increases while $u_b$ decreases with destabilizing buoyancy effects (figure \([2]\)c)). The friction velocity, on the other hand, remains close to $u^*_s$ throughout stages I and III (cf. reported values in table \([1]\)).

The effect of background stratification on bulk velocity and heat fluxes are well known from channel-flow studies, and the important point is that the heat flux is the variable that changes the most with buoyancy effects; specifically, $Nu^I = 3.02, 4.58$ and $7.75$ for $Ri_s = 40, 0$ and $-40$ respectively (cf. table \([1]\)). It is worth noting that while $Nu$ remains the same between stage Ic and stage II (in a time-average sense; orange colors), $u_s$ and $u_b$ show some variations as a result of turning on volume penalization and adding a solid phase. The large dip of $u_s$ at $t_s \approx t_b^I$ is merely the result of a sudden deceleration of the mean flow close to the interface, due to the addition of linear damping, but $u_s$ quickly returns to its statistically steady state value of $u_s \approx 150$. The drop of the bulk velocity is similarly due to the added linear damping. However, unlike the dip in $u_s$, the drop in $u_b$ persists at all times, implying that volume penalization results in anomalous drag on the mean flow. Here, the relative drop of bulk velocity is in the order of $5\%$ and the profiles of temperature and velocity close to the fluid-solid interface in stage II reproduce closely those in stage Ic (see appendix \([A]\)). Therefore, we consider the discrepancy to be small enough not to warrant a computationally costly increase in resolution or further tuning of the phase-field parameters.

When melting is turned on, i.e. for $t_s > t_b^I$ (green colors), global flow variables show different behavior depending on $Ri_s$. For the stable case, $u_s, u_b, Nu$ exhibit moderately-large fluctuations (as in previous stages), but do not exhibit any time-mean deviation (figure \([2]\)a)). For the neutral case, we find a small increase in $u_s, u_b, Nu$ (figure \([2]\)b)). For the unstable case, we find that $u_s$ and $Nu$ increase substantially, while $u_b$ stays relatively constant. The analysis presented in the next sections explain this behavior. Eventually, all simulations reach a statistical steady state.

We show in figure \([3]\) the temporal evolution of another global variable, namely, the drag coefficient, $C_D = 2(u_s/u_b)^2$, which is of significant interest in inferring melt rates from resolved variables in coarse models (cf. equation \([1.1]\)). The drag coefficient decreases/increases significantly at $t_s = t_b^I$, i.e. when the stratification becomes stable/unstable, in agreement with previous studies (García-Villalba & del Álamo 2011). In stage II, $C_D$ increases because $u_b$ decreases moderately upon turning on volume penalization (cf. figure \([2]\)). In stage III, $C_D$ has similar values as in stages I and II (cf. reported values in table \([1]\)), showing that is not modified by the topographical features obtained in DNS, perhaps because they are aligned with the main flow direction (see section \([5.2]\)).
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Figure 2. Friction velocity $u_*$ and bulk velocity $u_b$ divided by 20 (left axes) and Nusselt number $Nu$ (right axes) as functions of time $t_*$ for (a) stable ($Ri_*= 40$), (b) neutral ($Ri_*= 0$) and (c) unstable ($Ri_*= -40$) density stratification. Successive simulation stages are shown by different colors and separated by vertical dashed lines and may be described as follows: low-resolution spin up, neutral channel flow, stratified channel flow, simulation with volume penalization and simulation with melting. There is an intermediate stage $t_* \in [t_{ib}^b, t_{ic}^b]$ for (b) during which $Ri_*= 20$. See the text and appendix C for more details.
3.2. Spontaneous generation of channels and keels

The mean interface position does not vary significantly in our simulations, due to our choice of initial and boundary conditions for the solid (see §3.5), but uneven melting by the turbulent flow can still generate large-amplitude topography, which we discuss in this section.

We show snapshots of the two-dimensional fluid-solid interface $\xi$ at the end of stage III in figures 4(a-c), which correspond to stable, neutral and unstable stratification, respectively. In all three cases, the topography is dominated by channels (troughs in the ice; red colors), and keels (excursions of ice into the fluid; blue colors), aligned with the streamwise direction. We show in figures 4(d-f) the Hövomoller diagrams of the channels and keels by plotting $\xi' = \bar{\xi} - \tilde{\xi}$ in $(t^*, y)$ plane for all of stage III, where $\bar{\xi}$ is the $x$-averaged interface deformation, i.e. $\bar{\xi} = \int_0^l x \xi dx/l_x$. The channels and keels have similar amplitude throughout all of stage III, suggesting that they are the characteristic topographical features of our simulations at all times (see Movies 1-3 in Supplementary Material). The amplitude of the biggest channels, $\xi_+$ (maximum of $\xi'$), and the amplitude of the deepest keels, $\xi_-$ (minus the minimum of $\xi'$), increase with decreasing $Ri^*$ (i.e. from (d) to (f)). The crest-to-trough amplitude is roughly 5, 10 and 45 times the viscous length scale $\delta_v = 1/Re_s$ for stable, neutral and unstable stratification, respectively (note that $\delta_v$ is roughly equal to the diffuse interface thickness; cf. appendix A). Thus, the crest-to-trough amplitude is of the same order as the viscous sublayer thickness for stable and neutral stratification, but extends beyond the buffer layer and into the log layer for the case of unstable stratification (figures 4(c),(f)).

Figures 4(d-f) show that the viability of channels and keels increases with decreasing stratification: channels and keels are short lived with stable stratification but long lived with unstable stratification. For stable stratification (figures 4(d)), the separation of scales between the topography lifetime (about 10 friction time units) and the diffusion time scale across the solid layer (about 100 friction time units), suggests that the interface evolution is purely driven by the flow dynamics. For neutral stratification, figure 4(e)) shows that channels and keels can drift, merge, split, decay and spontaneously appear over time scales of tens to hundreds of friction time units, highlighting a possible interplay between interface evolution and the fixed-temperature condition at the top solid boundary. For unstable stratification (figures 4(f)), the channels and keels become time invariant and their amplitudes saturate because of the top solid boundary condition, which plays a key role in the interface evolution.

**Figure 3.** Drag coefficient $C_D$ as a function of time $t^*$ for unstable, neutral and stable stratification. The colors highlight different simulation stages as in figure 2.
3.3. Coupled dynamics of the fluid and solid phases

The emergence of channels and keels can be the result of either (i) a passive response of the interface to uneven melting patterns driven by the turbulent flow, or (ii) a fully-coupled interplay between fluid turbulence, interface topography, and temperature in the solid. Here, we investigate the relevance of regimes (i) and (ii) to each of our simulations by presenting both the flow dynamics and the temperature in the solid. Variables averaged in the $x$ direction are denoted by a tilde ($\tilde{}$), while variables averaged in the $x$ direction minus the horizontal mean are denoted by a prime ($'$).
We first show in figure 5 a snapshot of the heat flux $\tilde{q}$ in the fluid and of the conductive heat flux $-\partial_z \tilde{T}$ in the solid toward the end of stage III (note that we subtract $Nu^{III}$ in order to highlight fluctuations and that $-\partial_z \tilde{T}$ provides a more accurate measure than $\tilde{q}$ of the heat flux in the solid; cf. appendix B). In all three simulations, the spanwise fluctuations of the heat flux anomaly are one order of magnitude (or more) larger in the fluid where convection is active than in the solid where there is no movement. This confirms that convection in the fluid is the primary driver in shaping interface topography on short time scales, i.e. shorter than the diffusion time scale, which is on the order of 100 friction time units. Fluctuations in the heat flux anomaly increase in the solid phase as well as in the fluid phase with decreasing $Ri_s$ (i.e. from top to bottom row): enhancement of the heat flux fluctuations are the result of buoyancy effects in the fluid and of larger interface deformation in the solid. All panels show that $-\partial_z \tilde{T} - Nu^{III}$ in the solid is
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**Figure 6.** (a) Heat flux anomaly $q'$ in the fluid at $z = 0.5$ as a function of time and spanwise coordinate for the simulation with unstable stratification ($Ri_s = -40$). The two vertical dashed lines highlight the times at which we turn on volume penalization and melting. (b) Spanwise spectrum of the heat flux $\tilde{q}$ in the fluid at $z = 0.5$ (solid lines) and at a distance $\chi = 0.04$ (6 wall units) from the interface (dashed lines). The spectra are averaged over 100 friction time units and we use black and red colors to denote results in stage I and III, respectively. The blue dotted line shows the time-averaged spectrum of $\tilde{\xi}$.

Generally positive (green) over a channel and negative (pink) over a keel. In the case of stable and neutral stratification, $-\partial_z P - Nu^{III}$ decays from large fluctuation values near the interface to almost 0 near the top boundary, suggesting no significant influence of the top boundary on the temperature field in the solid (i.e. increasing the ice thickness while adjusting the top temperature to conserve the heat flux would not change the results). On the other hand, in the case of unstable stratification (figure 5(c)), $\tilde{q}$ fluctuations remain large near the top boundary, suggesting that there is a backreaction from the fixed-temperature top-boundary condition, $T = -q_s/2$ at $z = 1.5$, on the interface evolution. The backreaction from the top boundary for $Ri_s = -40$ is obtained because the position of the channels and keels becomes rapidly stationary (contrary to the cases $Ri_s = 0$ and $Ri_s = 40$), such that the temperature field in the solid has time to adjust diffusively and balance the growth of the channels and keels (cf. appendix D for more details on the temperature field in the solid).

The emergence of channels and keels is consistent with the well-documented presence and importance of near-wall streamwise streaks and vortices in stratified shear flows (Pirozzoli et al. 2017; Zonta & Soldati 2018), but their amplitude varies with $Ri_s$. In the case of stable stratification (figure 5(a)), buoyancy effects inhibit the generation of large topographical features (discussed in 3.4), such that channels and keels have small amplitudes and do not feed back onto the flow. In the case of neutral stratification, buoyancy is turned off, such that the solid boundary deforms more and can affect the flow dynamics. Figure 5(a) shows that the heat flux in the fluid close to the boundary is usually larger where there are channels (e.g. $y \approx 1.5, 2.5$) than where there are keels (e.g. $y = 0.9$), suggesting a topographic influence on the flow. For the case of unstable stratification (figure 5(c)), two streamwise rolls aligned with the direction of the flow and filling the entire depth dominate the fluid dynamics. These flow features are reminiscent of Rayleigh-Bénard convection rolls as observed in channel flow simulations with unstable stratification (Pirozzoli et al. 2017), which here appear locked within the interface deformation pattern (figure 5(c)).

The interplay between the solid boundary and the flow dynamics for the case of unstable stratification is further highlighted in figure 6. Figure 6(a) is the Hövsmoller diagram of the heat flux $q'$ in the middle of the fluid ($z = 0.5$) through stage I, stage II and stage III (shown by blue, orange and green colors in figure 2(c)). Two mid-depth streamwise rolls whose positions are locked are evident from $t_s \approx 370$ onward, which is about the same time as when the two
channels and keels become large in figure 4(f). Similar rolls can be inferred for $t_s < 370$ but are weaker and meander. Figure 4(b) shows the spanwise spectrum of $\tilde{q}$ (left axis) in the middle of the fluid (solid lines) and near the solid boundary (dashed lines) at the end of stages I (red colors) and III (black colors). In all cases, the spectrum peaks at wavenumber $L_y k_y / 2\pi = 2$, which is close to the critical wavenumber $L_y k_c / 2\pi = 3.11$ of convection instability ($\lambda_c = 2.016$), suggesting that Rayleigh-Bénard convection is already active in stage I. However, with melting turned on, the peak is significantly amplified, especially for the spectrum near the boundary (dashed lines), which also shows amplification of higher harmonics, consistent with the spectrum of the interface itself (dotted blue line; right axis). These result suggest that Rayleigh-Bénard rolls are energized more than any other fluid features once melting is turned on, because they best couple with the interface topography evolution as a result of melting or freezing. Note that the spectra of the heat flux near the boundary and of the interface have a sawtooth-like pattern due to the non-sinusoidal shape of the interface and numerical confinement in the spanwise direction.

We next show in figure 7(a-f) vertical profiles of the mean streamwise velocity $U$ and the root-mean-square (rms) vertical velocity, $w_{rms}$, where here mean and rms are defined using a horizontal and temporal average. The results are shown for all three stages and for stable (two left-most columns), neutral (two middle columns) and unstable stratification (two right-most
columns). For stable and neutral conditions, there is a strong overlap of all curves, suggesting that the topography doesn’t influence the mean profiles, while for unstable stratification, there is a small deviation of the stage III profiles (solid lines). It may be noted that the vertical profiles are not symmetric with respect to $z = 0.5$. This is because our velocity boundary conditions are different on the top (no-slip) and bottom (free-slip) boundaries.

We investigate in figure [7]g-l whether the mean profiles vary in the spanwise direction in such a way that they correlate with the $x$-averaged interface deformation. To do so we consider the mean profiles under the largest keel (downward triangles), i.e. at $y = y_k$ where $y_k$ is the minimum in $y$ of $\tilde{\xi}$ at each time step, separately from the mean profiles under the largest channel (upward triangle), i.e. at $y = y_c$ where $y_c$ is the minimum in $y$ of $\tilde{\xi}$ at each time step, where mean now denotes streamwise and temporal averaging. For stable stratification, there is no distinction between the profiles under keels and channels. However, for neutral and unstable stratification, the two profiles depart in such a way that the streamwise velocity is larger under channels than under keels, and the vertical velocity rms (defined using a temporal and $(x,y)$ average for each $\chi$) is larger under keels than under channels. These results indicate a noticeable influence of the topography on the flow. For the case of neutral stratification, the separation of the profiles is maximum for $\chi < 0.3$ and then vanishes, suggesting a local influence of the topography on the flow dynamics, while for unstable stratification the effect of the topography is felt throughout the entire depth due to coupling with the Rayleigh-Bénard rolls. It may be noted that the profiles of $w_{\text{rms}}$ under the largest keels and channels are both larger than the plane-average profile shown by the dotted line in figure [7]l. This is expected because Rayleigh-Bénard convection promotes both localized intense upwellings and intense downwellings under channels and keels. In fact, away from the main channel and keel the profiles decrease rapidly, as can be seen from the blue and green lines.

In order to gain further insight into the statistics of the flow interacting with the melting boundary, we show in figure [8] the probability density functions (pdfs) of the streamwise velocity (left panels), the vertical velocity (middle panels) and the temperature gradient (right panels), for stable (top row), neutral (middle row) and unstable stratification (bottom row). For the velocities, the pdfs are shown both in the middle of the fluid, at $z = 0.5$, and near the boundary, at $z = \xi - 0.04$ (i.e. 6 wall units into the fluid). We find little difference between the different stage I (dashed lines) and stage III (solid lines) for the streamwise and vertical velocities, suggesting limited influence of the topography on the overall flow morphology, although the streamwise velocity in figure [8]g (left panel) has a negative tail with higher probabilities in stage III than in stage I. The temperature gradient at the interface (figures [8]c,f,i) does vary noticeably between stage I and III; however, this difference is due to the phase-field method rather than to a fundamental change in flow morphology since the pdfs in stage I and II (not shown) show significant overlap.

While melting has little effect on the pdfs, there are several notable asymmetries in the pdfs. Most importantly, the temperature gradient has a rapidly-decaying positive tail but a slowly-decaying negative tail, an asymmetry that may have profound implications for melting. The pdfs of the streamwise velocity near the boundary are also asymmetric, featuring a slowly-decaying positive tail but a rapidly-decaying negative tail. We have further separated the pdfs of the velocities in figure [8] based on the sign of the local temperature anomaly, compared to the plane- and time-averaged value. Blue curves denote pdfs obtained for negative temperature anomaly, i.e. representative of water influenced by the cold top boundary, while red curves denote pdfs obtained for positive temperature anomaly, i.e. representative of water influenced by the warm bottom boundary. The cold-temperature pdfs are shifted to the left of the warm-temperature pdfs for the streamwise velocity (left column), which suggests that negative stream-
Figure 8. Probability density functions (pdfs) for stable (top row), neutral (middle row) and unstable stratification conditions (bottom row) of the streamwise velocity fluctuations (a,d,g; at $z = 0.5$ and $z = \xi - 0.04$), vertical velocity fluctuations (b,e,h; at $z = 0.5$ and $z = \xi - 0.04$) and temperature gradient fluctuations (c,f,i; at $z = \xi$). Results are shown for (---) stage I and (Ł) stage III. Blue and red colors in the panels for the streamwise and vertical velocity fluctuations show the pdfs conditioned on negative and positive local temperature fluctuations (compared to the horizontal mean). Double overlines denote temporal and spatial averaging.

Wise velocity is more often associated with cold water coming from the top boundary. Also, for the vertical velocity at $z = \xi - 0.04$ (right panels of the middle column), the positive tail of the warm-temperature pdfs (red) is larger than the negative tail of the cold-temperature pdfs (blue), suggesting more extreme warm upwelling events than cold downwelling events just outside of the viscous sublayer. These two results demonstrate that the near-wall flow dynamics have multiple asymmetries, which can be responsible for the asymmetry in the temperature gradient at the boundary.

3.4. Reversing the stratification

In the case of stable stratification (positive $Ri_*$), melting produces cool melt water at the freezing temperature ($T = 0$), which is more buoyant than the remaining fluid with temperature $T > 0$. Thus, channels and keels are limited to small amplitudes in figure 5(a), because cool melt water buoyantly accumulates wherever channels emerge, such that it inhibits further melting. In the case of unstable stratification (negative $Ri_*$) the opposite is true, i.e. dense melt water is
evacuated from troughs in the ice. We observe that horizontal divergence of melt water from the channels’ centreline promotes stronger upwelling of warm and buoyant bottom fluid, which drives further melting. This feedback carves large-amplitude channels in the solid, eventually saturating due to thermal adjustment in the solid. Figure 5(c) clearly shows the upwelling of warm fluid under the channels’ centreline as well as the downwelling of cool melt water along the channel flanks and down from the tip of the keels.

The hypothesis of melt water pooling in channels and inhibiting their growth cannot be verified within the stable stratification results because of its small interface deformation. Therefore, we have run a fourth simulation starting from the final time of the simulation with unstable stratification (and large interface deformation) but with an increasing Richardson number such that the fluid eventually becomes stably stratified, using intermediate steps so that the flow doesn’t relax to a laminar state. Specifically, we substituted $Ri_s = -40$ with $Ri_s(t_s) = -40[1 - f(t_s,5)] + 20[2 + f(t_s,12) + f(t_s,19)]$, (3.3)

with $f(t_s, \tau_s) = \tanh(t_s - t_0^\text{III} - \tau_s)$, such that $Ri_s \approx 40$ for $t_s > t_0^\text{III} + 19$. We show the results of this run in figure 9 where blue/red colors highlight x-averaged temperature values in the solid/fluid phase, while arrows denote x-averaged velocity vectors ($\tilde{v}, \tilde{w}$) in the $(y,z)$ plane. Figure 9(a) shows the results at time $t_s = t_0^\text{III} + 1$ ($t_0^\text{III} = 526$), i.e. when $Ri_s \approx -40$. The stratification is unstable such that the flow features strong upwelling of warm water below the channels and strong downwelling of cool water along and under the keels, akin to Rayleigh-Bénard rolls locked into the deformed-interface pattern. A pair of counter-rotating streamwise rolls is clearly visible below each one of the two channels. Figures 9(b-c) show the results at times $t_s = t_0^\text{III} + 13$ and $t_s = t_0^\text{III} + 21$, i.e. when $Ri_s \approx 0.7$ and $Ri_s \approx 39$, respectively. At these times, the stratification is stable and the flows of melt water, starting from the keels, converge toward the channels’ centreline. The Rayleigh-Bénard cells are disrupted, such that the heat flux through the fluid goes down. Freezing occurs everywhere and the solid front advances into the fluid on average. However, freezing is faster in the channels because of the convergence of buoyant cold melt water, which leads to rapid refreezing of the initial troughs.

3.5. Melting

In this section we discuss the evolution of the mean interface position $\bar{\xi}$ with time and the statistics of melting $\bar{m} = \partial_t \bar{\xi}$ at statistical steady state. We first show the evolution of $\bar{\xi}$ in DNS as a function of time $t_s$ in figure 10(a) for all three simulations (solid lines). As indicated in section 2, we have initialized the temperature field in the ice in stage II such that the conductive heat flux through the solid is slightly less than the heat flux going through the fluid, i.e. $q^f < \bar{Nu} \bar{f}$ (cf. table 1). Thus, $\bar{\xi}$ increases with time initially. The conductive heat flux in the ice also increases as $\bar{\xi}$ increases (in a plane-averaged sense), eventually balancing the heat flux coming from the fluid such that $\bar{\xi}$ saturates. Under the assumption of small interface deformations, it is possible to predict the evolution of the mean interface position over time using a reduced model. As a first approximation, we consider that the topography has no effect on the temperature in the solid, i.e. we assume that the heat flux through the solid is simply equal to the temperature difference between the interface and the top boundary divided by the mean ice thickness $h$ (cf. details in appendix D which assumes quasi steady state). Then, the evolution equation for $\bar{\xi}$ becomes

$$\frac{d\bar{\xi}}{dt} = q^f - \frac{h_0 q^f}{h}$$

(3.4)

where $h_0 = 1/2$ is the initial ice thickness, $q^f$ is the heat flux in the fluid, and we recall $\bar{\xi} = \bar{\xi}_0 = 1$ at $t = t^\text{II}$. For simplicity we take $q^f$ to be a constant diagnosed from the simulations. The results of equation (3.4) for $q^f = \bar{Nu} \bar{f}$, i.e. obtained when setting $q^f$ to the average heat flux before
melting is turned on, are shown by the dotted lines in figure 10(a). The overlap between the reduced model and DNS results at early times is good for unstable stratification (as expected) but is poor for stable and neutral stratification. The disagreement with \( q_f = Nu_L \) at early times arises because the temperature in the solid is slightly above 0 because of volume penalization, such that there is some artificially large melting at the beginning of stage III (cf. appendix A). At later time, the DNS results and the model results shown by the dotted lines diverge because the heat flux \( Nu \) increases rapidly once melting is turned on, as can be seen in figure 10(b). For unstable stratification, the agreement with \( q_f = Nu_L \) is relatively good until \( t_s \approx t_{II}^* + 50 \) (cf. red dotted line), i.e. before the Rayleigh-Bénard rolls are energized, as discussed in §3.3 and seen in figure 6(a).

In order to account for the increase in heat flux through the fluid enabled by melting and
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The dotted (resp. dashed) lines show the position of the interface predicted from the reduced model (3.4) with $\text{Nu}^I$ (resp. $\text{Nu}^{III}$) as heat flux in the fluid. The dash-dot line shows the result of the higher-order model (D1), (D10) for unstable stratification. (b) Time history of the heat flux relative to $\text{Nu}^I$, i.e. $(\text{Nu}^{III} - \text{Nu}^I)/\text{Nu}^I$ (solid lines). We use a rolling mean over a $\Delta_\ast = 20$ friction time units window in order to remove some of the most rapid large-amplitude oscillations. The horizontal dashed lines show $(\text{Nu}^{III} - \text{Nu}^I)/\text{Nu}^I$.

The generation of topography we show with dashed lines in figure 10(a) the result of equation (3.4) with $q^f = \text{Nu}^{III}$, which is the heat flux at statistical steady state with melting turned on. For stable and neutral stratification, there is a good agreement between the model results and the mean interface position at late times. For unstable stratification, however, equation (3.4) with $q^f = \text{Nu}^{III}$ overestimates the final value of $\xi - 1$ by a factor two (approximately), suggesting that topography plays a non-negligible role on the heat flux in the solid. We show in figure 10(a) a prediction of $\xi$ for unstable stratification obtained using a more accurate higher-order model (red dash-dot line), which takes into account interface deformation (cf. appendix D). The higher-order prediction overlaps well with the DNS results at late times, demonstrating that melting and the generation of topography changes the heat flux through both the fluid and the solid. Specifically, the topography makes the solid more efficient at evacuating heat, i.e. the steady state ice layer thickness is larger with topographical features than without for the same $q^f$.

We finally show in figure 11 the pdfs of interface deformation and melt rate. The pdf of interface deformation (figure 11(a)) for stable stratification is almost Gaussian and so symmetric with respect to the mean. For neutral stratification, an asymmetry develops, with the median shifting to the right, i.e. toward small positive values, but with negative values being more extreme. The same asymmetry is amplified for unstable stratification, with a narrow peak appearing to the right of the mean and a shift of the negative tail toward more extreme values. The asymmetry in the pdfs of $\xi$ is expected since channels are typically flatter and more widespread than keels, which are more pointed, as can be seen in figure 4. The pdfs for the melt rate $\dot{m}$ are shown in figure 11(b). The temporal and spatial average $\bar{m}$, which is subtracted from the pdf, is close to 0 in all cases, since $\bar{\xi}$ has reached a statistically steady state at late times such that there is no mean melting, as can be seen in figure 10(a). The pdfs of melt rate have their median shifted to the left of the mean, i.e. toward negative values, and an increased probability of the positive tail compared to the negative tail. In other words, the interface is more commonly freezing slowly ($\dot{m} < 0$), but occasionally melts rapidly ($\dot{m} > 0$). While the turbulent flow can drive rapid melting independently from what happens in the solid, freezing necessarily involves slow diffusive processes in the solid; thus, it is not completely unexpected that there is an asymmetry in the melting/freezing statistics. Nevertheless, we expect that some of the melting/freezing
asymmetry is also related to the near-wall dynamics, which feature coherent structures such as streamwise streaks and vortices that may be associated with the asymmetric velocity and temperature gradient pdfs shown in section 3.3.

4. Concluding remarks

We have shown that channels and keels spontaneously emerge as the dominant topographical features of an ice boundary adjacent to a turbulent stratified shear flow with $Re_\ast = 150$, $Pr = 1$ and $St = 1$. We have investigated the effect of the background density stratification on topography generation and found that the amplitude of the channels and keels increases with decreasing stratification. For unstable stratification ($Ri_\ast = -40$), the channels and keels couple strongly with Rayleigh-Bénard rolls, which are energised and locked within the interface deformation pattern. For neutral stratification, a similar correlation is obtained between the flow dynamics and the interface deformation pattern, although it is weaker, limited to the near-wall region and that there is no locking mechanism, i.e. the topography drifts. For neutral ($Ri_\ast = 0$) and stable stratification ($Ri_\ast = 40$) the saturation of the channels and keels is due to the fluid dynamics; for stable stratification, buoyancy effects clearly inhibit channel formation. For unstable stratification, the saturation is due to the top fixed-temperature condition. With an imposed heat flux condition, the entire solid melts rapidly and entirely for unstable stratification (not shown), suggesting that the choice of boundary conditions at the top of the solid can be critical. Note that the unbounded growth of the fluid layer for unstable stratification is due to the positive feedback that melting has on the effective $Ra$ number of the convective fluid. As the solid melts, the effective $Ra$ number increases, leading to further melting, which is stopped only if diffusion in the solid can eventually balance the increasing heat flux in the fluid.

The analysis of the melt rate statistics indicates that there is an asymmetry in melting and freezing, which may be related to the different melting/freezing dynamics (freezing relying primarily on slow diffusive processes in the solid) but also asymmetries in the flow statistics. Specifically, melting is highly localized and intense while freezing is widespread but weak. While beyond the scope of this study, it would be useful to identify whether coherent features of the near-wall turbulent flow, such as streamwise streaks and vortices, correlate preferentially with either melting or freezing events.

The drag coefficient changes significantly depending on the stratification but is only weakly affected by the generation of topographical features, which is not unexpected in our case since
channels and keels are smooth in the direction of the flow. This prompts the question of whether we should expect topographical features variable in the streamwise direction in other parameter regimes, for which the drag coefficient may change significantly. Scallops are the most ubiquitous three-dimensional features of the ice-ocean boundary, which are known to double the drag coefficient in some conditions based on the most recent laboratory experiments (Bushuk et al. 2019). Previous experimental and theoretical works have found that the friction Reynolds number based on the scallop wavelength $\lambda$ must satisfy $Re^*_s = Re_s \lambda/H \geq O(1000 - 10000)$ (Blumberg & Curl 1974; Thomas 1979; Claudin et al. 2017) for scallops to emerge. Here, $Re_s = 150$ and $L_x/H = 4\pi$ such that the expected scallop wavelength is $\lambda \sim O(10H - 100H)$, which may fit into our domain. However, the coupling between the flow dynamics and topography leading to scallops and the prediction $Re^*_s \geq O(1000 - 10000)$ may need to be revised for our case, since we assume a small latent heat (Stefan number) in order to increase melt rates for numerical expediency. Importantly, Bushuk et al. (2019) reported that scallops emerged in less than 12 hours—a threshold based on practical limits—only for large velocities, i.e. larger than 0.6 m/s. Thus, it is also possible that scallops could be obtained with our current setup but that they would take so long to form that we did not observe them. In conclusion, three-dimensional scallops may be within reach for direct numerical simulations but would require significant high-performance computing time and ideally a dedicated fluid-solid simulation code.

For water, it would be useful in future studies to consider higher Prandtl and Stefan numbers, such as $Pr \approx 7$ and $St \approx 50$. Higher $Pr$ results in thinner thermal boundary layers, which could impact the near-wall dynamics and e.g. the asymmetry between melting and freezing. Higher $St$ results in slower melt rates, which could significantly change how interface patterns couple with transient flow features. In the case of unstable stratification, we might still expect that Rayleigh-Bénard rolls couple with the interface deformation pattern for high $St$, since they are stationary flow features. For neutral stratification, however, the interface evolves over time scales similar to those of the flow dynamics for $St = 1$ (figure 4(e)), such that increasing $St$ might significantly decrease the sensitivity of the interface topography to fluid anomalies.

Many additional studies are now required to improve our understanding of ice-ocean interactions at small scales. Regarding the effect of topography on melting, an exploration is required of the assumptions and theoretical predictions of Claudin et al. (2017) on the formation and saturation of scallops. It is also necessary to investigate the effect of stratification on such predictions, since most studies of scallops have assumed neutral stratification, even when dealing with an unstable stratification configuration (Toppaladoddi & Wettlaufer 2019). Finally, it would be useful to include salt effects and to push simulations to higher $Re_s$, $Pr$ and $St$, even though such simulations would be computationally more demanding than those reported in this manuscript. It would also be interesting to investigate potential analogies between ice patterns due to melting and the formation of sand ripples and dunes, which have and continue to be extensively studied (e.g. Charru et al. 2013; Courrech du Pont et al. 2014).

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Figure 12. Kinetic energy in the solid divided by kinetic energy in the fluid as a function of time $t_*$, Red, gold and blue colors denote results obtained for unstable, neutral and stable stratification, respectively.

Declaration of Interests

The authors report no conflict of interest.

Appendix A. Phase-field method

The phase-field method transforms the discontinuous two-phase two-domain problem into a continuous two-phase one-domain problem, which can be solved numerically using a pre-existing fluid code. In order to reproduce the original problem correctly, the resolution and parameters $A$, $B$, $C$ and $\Gamma$ of the phase-field equations (2.4) must satisfy several constraints (described in detail in a paper in preparation), which can be verified a posteriori by diagnosing the flow properties in the solid and fluid phases. Here, for our choice of resolution, we have $A = 6/(5St)$, $B = (16/\delta^2) \times 6/(5St)$ and $C = 1$, with $\delta$ chosen such that it is equal to 2 times the local grid size at $z = 1$ (initial interface position); $\Gamma = (\delta/2.648228)^2$ and we require step size always smaller than $\Gamma/2$.

We first assess the effect of the phase-field method and choice of parameters on the flow variables by showing in figure 12 the ratio of the kinetic energy averaged over the solid volume, $KE_s$, divided by the kinetic energy averaged over the fluid volume. Figure 12 shows that $KE_s/KE_f < 10^{-4}$, i.e. velocities in the fluid penetrate only very weakly into the solid.

Let us denote $\delta$ the thickness of the diffuse phase-field interface over which $\phi$ transitions from 1 in the fluid to 0 in the solid (see figure 1(c)). A constraint is that $\delta$, which is an artificial length scale, must be smaller than any physical length scale in the problem, while at the same time being larger than the grid size since it must be resolved numerically. We demonstrate in figure 13 that $\delta$ is smaller than the viscous sublayer thickness, which is the smallest and most relevant physical length scale close to the solid-fluid interface. Figures 13a-c show the wall-normalized velocity $U^+$ (left axes) and wall-normalized temperature $T^+$ (right axes) as functions of wall units $z^+$ ($z^+ \geq 0$ denote positions in the fluid while $z^+ < 0$ denote positions in the solid) in stages I and II for stable, neutral and unstable stratification, respectively. In the viscous and thermal sublayers, which extend from $z^+ = 0$ to $z^+ \approx 5$, we expect a linear scaling for both $U^+$ and $T^+$ with $z^+$, shown by the solid dashed lines. This linear scaling is perfectly satisfied by the DNS results in stage I (blue circles and blue crosses) as well as the DNS results in stage II (orange circles and crosses), except for $|z^+| < \delta$ (shown by the vertical solid lines), i.e. within the diffuse interface, which is expected since this is where the dynamics is artificially controlled by the phase-field equation. It may be noted that $T^+$ (orange crosses) is anomalously large for $z^+ < \delta$, suggesting reliance on the phase-field equation.
and in fact deviates from the true solution (blue crosses) slightly outside the diffuse interface. This discrepancy is due to the fact that the heat flux in the fluid is larger than the heat flux in the solid in stage II. The interface being fixed in stage II, the heat imbalance results in the heating of the solid, such that $T^+ = 0$ occurs at $z^+ < 0$ away from the fixed interface position $z^+ = 0$ (note that we use a symmetric logarithmic scale with a linear threshold at $|z^+| = 0.1$). By shifting the temperature profile to the right such that $T^+ = 0$ is aligned with $z^+ = 0$ (red pluses), we recover a perfect linear scaling for the temperature both within the thermal sublayer and the solid.

Far from the top boundary, i.e. e.g. for $z^+ \approx 100$, $U^+$ shows a steeper scaling with $z^+$ for stable stratification than for neutral or unstable stratification. This is a consequence of buoyancy effects, which have already been reported (García-Villalba & del Álamo 2011).

Appendix B. Depth-independent variables

At statistical steady state, stratified pressure-driven flows between solid boundaries have linearly-varying shear stress $\tau = \tau_v + \tau_w$, with $\tau_v = \partial_z u$ the viscous stress and $\tau_w = -\overline{w'u}$ the Reynolds stress, and depth-independent heat flux $\overline{q} = \overline{w'T}$, where overbar denotes horizontal and time averaging. These conservation equations for the vertical fluxes of momentum and heat are at the origin of the definitions of the friction velocity and Nusselt number, which typically read $u_* = \sqrt{-(\tau_v + \tau_w)|z=1}$ and $Nu = \int_0^1 \overline{q}dz$ (assuming $z = 1$ is the top of the fluid), respectively. With the phase-field method, these conservation equations are modified and some of the modifications are reflected in the definitions of $u_*$ and $Nu$ in equation (2.6). In particular, $u_*$ in equation (2.6) includes the linear damping term $\tau_d = -\int_0^1 \hat{\phi}u/\Gamma dz/l_z$ that comes from the last term on the right-hand-side of equation (2.4a). The true conservation of vertical momentum and heat fluxes based on governing equations (2.4) read

$$\frac{\partial}{\partial z}(\tau + \overline{\tau}) = -2Pr^2Re, \quad \frac{\partial}{\partial z}(q + \overline{q}) = 0.$$
where \(\tilde{\tau}\) and \(\tilde{q}\) are the anomalous stress and heat flux due to damping of the advective terms in equations (2.4) and read

\[
\tilde{\tau} = \int_0^{l_z} -(1 - \phi)(u \cdot \nabla)udz/l_z, \quad \tilde{q} = \int_0^{l_z} -(1 - \phi)(u \cdot \nabla)Tdz/l_z.
\]  

(2.2)

The existence of anomalous stress and heat fluxes means that the friction velocity and Nusselt numbers as defined in equation (2.4) are based on a total stress and heat flux, which are not rigorously depth invariant.

We show in figures 14(a-c) the Reynolds stress \(\tau_{uv}\), the viscous stress \(\tau_v\), the linear damping stress \(\tau_d\) and the Reynolds stress plus the anomalous stress \(\tau_v + \tilde{\tau}\) for stable, neutral and unstable stratification, respectively. Importantly, \(\tau_v\) and \(\tau_v + \tilde{\tau}\) overlap well, showing that the anomalous stress is negligible. The results of figures 14(d-f) further confirms that the anomalous stress is negligible in all simulations: the (approximate) total stress (solid lines) decreases linearly with \(z\) in all stages and overlap well with \(\tau_v + \tau_w + \tau_d + \tilde{\tau}\), i.e. the total stress that includes the anomalous stress. We show in figures 14(g-i) \(\bar{q}\) (solid lines) and \(\bar{q} + \tilde{q}\) (thin dashed lines). For stable and neutral stratification, \(\bar{q}\) and \(\bar{q} + \tilde{q}\) are constants with depth and overlap perfectly, suggesting that the anomalous heat flux is negligible. For unstable stratification, we obtain similar results for stages I and II. For stage III, however, \(\bar{q}\) is not perfectly constant, deviates from \(\bar{q} + \tilde{q}\) and peaks at \(z \approx 1.15\), which is roughly the height of the channels. The relative discrepancy between \(\bar{q}\) and \(\bar{q} + \tilde{q}\) is on the order of 5% and is a result of the damping of the advective terms in the momentum and heat equations (2.4). We expect that this discrepancy would decrease with increased resolution. Previous studies have alternatively considered advective terms with the same damped form as here, with a divergence damped form, i.e. \((u \cdot \nabla)(\phi u)\), or without any damping. There is no proof that any of these methods is more efficient than the other two. However, we would recommend using either one of the latter two methods, i.e. not the method used in this paper, in order to simplify the analysis of depth-invariant variables.

**Appendix C. Additional details on the simulation stages**

In this section we give additional details on the simulation stages and sub-stages. In stage I we solve equations (2.4a)–(2.4b) and (2.4d) with \(\phi \equiv 1\) and a no-slip fixed-temperature top boundary condition, i.e. \(u = u\) and \(T = 0\) at \(z = 1\). We use a straightforward half channel flow configuration, i.e. without a solid domain, with 64 Chebyshev modes in the vertical. In stage II we add a solid layer of thickness 0.5 on top of the fluid domain and we use a compound Chebyshev basis stitched at \(z = 1.2\) with 256 (resp. 32) Chebyshev modes in the lower (resp. upper) region. The compound Chebyshev basis allows a high vertical resolution near the interface’s initial position. We solve equations (2.4a)–(2.4b) and (2.4d) with \(\phi\) prescribed, i.e. not varying in time (cf. main text). In stage III we solve equations (2.4) with all variables freely evolving and we use the same spectral resolution as in stage II.

Stage I of our simulations can be further broken down into three sub-stages. In stage Ia \((t_a < t_{a}^{*}\); cf. light gray in figure 2) we run a low-resolution (128 Fourier modes in \(x\) and \(y\) directions and 32 Chebyshev modes in \(z\) direction) spin-up simulation of an initially-laminar flow superposed with three-dimensional velocity perturbations and no buoyancy effects, i.e. \(R_i = 0\). In stage Ib we increase the resolution (256 Fourier modes in \(x\) and \(y\) directions and 64 Chebyshev modes in \(z\) direction) but keep \(R_i = 0\) \((t_a^{lb} \leq t_a < t_a^{lc}\); cf. dark gray in figure 2). In stage Ic we turn on buoyancy effects \((t_a^{lc} \leq t_a < t_a^{lc}\); cf. blue in figure 2), i.e. we substitute \(R_i = 0\) with

\[
R_i(t) = R_i(t_a^{lc}) \tanh \left( t_a - t_a^{lc} \right)
\]  

(C 1)
Figure 14. (a-c) Reynolds stress $\tau_w$ (dotted lines), viscous stress $\tau_v$ (dashed lines), linear damping stress $\tau_d$ (solid lines) and Reynolds stress plus anomalous stress $\tau_w + \tilde{\tau}$ (for stages II and III; black dashed lines) averaged in time over 20 friction time units and horizontal planes as functions of depth $z$ for stable, neutral and unstable stratification, respectively (see the text for more details). Blue, orange and green colors denote results obtained in stage I, II and III, respectively (same in (d-f) and (g-i)). (d-f) Total stress, i.e. $\tau_w + \tau_v + \tau_d$ (solid lines), and total stress plus the anomalous stress, i.e. $\tau_w + \tau_v + \tau_d + \tilde{\tau}$ (thin dashed lines), averaged in time over 20 friction time units and horizontal planes as functions of depth $z$ for stable, neutral and unstable stratification, respectively. Note that the full stress is shifted to the right by 10000 (20000) between stage II and stage I and between stage III and stage II for the case of stable (unstable) stratification for clarity as all curves overlap otherwise. (g-i) Heat flux $q$ (solid lines) and heat flux plus anomalous heat flux $q + \tilde{q}$ (thin dashed lines) averaged in time over 20 friction time units and horizontal planes as functions of depth $z$ for stable, neutral and unstable stratification, respectively. Note that the heat flux is shifted to the right by 2.5 (5) between stage II and stage I and between stage III and stage II for the case of stable (unstable) stratification for clarity.
with \( t_f^{lb} \) the time at the end of stage \( \text{Ib} \), such that \( Ri_* \) transitions smoothly (over the time scale of one friction time unit) from 0 to the target value listed in table 1. Note that in the case of stabilizing buoyancy effects, we found that the turbulent flow relaxes to a laminar state when using \( (C) \). In order to avoid this we used an intermediate stage with a more moderate target \( Ri_* = 20 \) (sub-stage \( \text{Ici} \)) before transitioning to \( Ri_* = 40 \), using a similar equation as \( (C) \).

### Appendix D. Higher-order interface evolution model

In this section we derive a reduced model for the evolution of the mean interface position \( \bar{\xi} \), or ice thickness \( h = h_0 - (\bar{\xi} - \xi_0) \), with \( h_0 = 1/2 \) and \( \xi_0 = 1 \) the initial ice thickness and interface position, which takes into account interface deformation. Under steady-state assumption (instantaneous temperature diffusion), the evolution of \( \bar{\xi} \) is controlled by the difference between the input heat flux from the fluid, \( q^f \), and the mean heat flux at the ice top, \( q_{top} \), i.e.

\[
\frac{d \bar{\xi}}{dt} = q^f - q_{top}.
\]

At leading order, we assume that the interface is flat such that \( q_{top} = h_0 q^s / h \), which yields equation (3.4). At higher order, we take into account interface deformation, which can change \( q_{top} \), using regular perturbation (Favier et al. 2019). Specifically, we seek a solution of the \( x \)-averaged steady-state heat equation, i.e. (dropping tilde for \( x \)-averaged variables)

\[
\nabla^2 T = 0, \quad \xi(y) \leq z \leq L_z, \quad (D2)
\]

\[
T = -h_0 q^s, \quad z = L_z, \quad (D3)
\]

\[
T = 0, \quad z = \xi(y), \quad (D4)
\]

using a perturbation series of the form

\[
T(y,z) = T^{(0)}(z) + T^{(1)}(y,z) + T^{(2)}(y,z) + ..., \quad (D5)
\]

with \( \partial_z T^{(0)} = 0 \) and \( T^{(i)} \sim O(\varepsilon^i) \), where \( \varepsilon \ll 1 \) is the dimensionless amplitude of the interface deformation. For simplicity, here we approximate the interface deformation as

\[
\xi(y) = \bar{\xi} + \varepsilon \cos ky. \quad (D6)
\]

The leading-order solution is

\[
T^{(0)} = -\left( \frac{z - \bar{\xi}}{L_z - \bar{\xi}} \right) h_0 q^s, \quad (D7)
\]

the first-order solution is

\[
T^{(1)} = \left( \frac{\varepsilon h_0 q^s}{L_z - \bar{\xi}} \right) \cos ky \frac{\sinh k (L_z - z)}{\sinh k (L_z - \bar{\xi})}, \quad (D8)
\]

and the second-order solution is

\[
T^{(2)} = \frac{\varepsilon^2 h_0 q^s}{2 \left( L_z - \bar{\xi} \right) \tanh k \left( L_z - \bar{\xi} \right)} \left[ \frac{L_z - z}{L_z - \bar{\xi}} \right. + \cos 2kx \frac{\sinh 2k (L_z - z)}{\sinh 2k (L_z - \bar{\xi})} \left. \right], \quad (D9)
\]

such that the second-order accurate formula for the mean heat flux at the top of the ice reads

\[
q_{top} = -\partial_z T(z = L_z) \approx \frac{h_0 q^f}{h} \left[ 1 + \frac{\varepsilon^2 k}{2h \tanh kh} \right], \quad (D10)
\]
which differs from the leading-order heat flux only at second order. The prediction for the evolution of the mean interface position for unstable stratification with $q_f = Nu^{HI}$, which is shown by the red dashed lines in figure [10]a, is based on equation (D1) with $q_{top}$ given by (D10) and with $\varepsilon = 0.137$ and $k = 2$ (as obtained from best-fit of the true interface topography at steady state for unstable stratification). We note that the quasi-steady state assumption may affect the prediction of the transient evolution of the mean interface position adversely but has no effect on the final value, which is our primary interest.

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