BLOCK CODES ON POMSET METRIC

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ABSTRACT. Given a regular multiset $M$ on $[n] = \{1, 2, \ldots, n\}$, a partial order $R$ on $M$, and a label map $\pi : [n] \to \mathbb{N}$ defined by $\pi(i) = k_i$ with $\sum_{i=1}^{n} \pi(i) = N$, we define a pomset block metric $d(P_m, \pi)$ on the direct sum $\mathbb{Z}_{k_1}^n \oplus \mathbb{Z}_{k_2}^n \oplus \cdots \oplus \mathbb{Z}_{k_n}^n$ of $\mathbb{Z}_m^N$ based on the pomset $P = (M, R)$. The pomset block metric extends the classical pomset metric introduced by I. G. Sudha and R. S. Selvaraj and generalizes the poset block metric introduced by M. M. S. Alves et al over $\mathbb{Z}_m$. The space $(\mathbb{Z}_m^N, d(P_m, \pi))$ is called the pomset block space and we determine the complete weight distribution of it. Further, $I$-perfect pomset block codes for ideals with partial and full counts are described. Then, for block codes with chain pomset, the packing radius and Singleton bound are established. The relation between MDS codes and $I$-perfect codes for any ideal $I$ is investigated. Moreover, the duality theorem for an MDS pomset block code is established when all the blocks have the same size.

1. Introduction

The major problem of coding theory is to find the largest minimum distance $d$ of any $k$-dimensional linear code of length $n$ for any integer $n > k \geq 1$. Numerous researchers have worked on the structure of posets and defined various spaces such as poset space [4] including crown space [1] and hierarchical poset space [12], RT space [17] including NRT space [14] and so on. They investigated the properties such as packing radius, Singleton bound, maximum distance separability, weight distribution, and perfectness of codes with those spaces. Over the past two decades, the study of block codes has sparked several significant developments in the communication field, such as experimental design, high-dimensional numerical integration, and cryptography. In 2006, $\pi$-block codes of length $N$ over $\mathbb{F}_q$ were introduced by K. Feng et al. [8], which is another generalization of codes with Hamming metric. Further, it is extended to $(P, \pi)$-block codes by M. M. S. Alves et al. [2], by introducing a partial order relation on the block positions $[n]$ of an $N$-tuple in $\mathbb{F}_q^N$ consisting of $n$-blocks in each $N$-tuple, thereby providing an extension to the poset codes [4] as well. Thus, it kept the researchers to concentrate on the exploration of block codes with various metrics, allowing one to study the class of posets (such as hierarchical posets, NRT posets, etc.) that admit certain properties of the block codes (2, 5, 6, 7, 8, 11, 14, 16). This has emerged as a useful research topic in the digital communication world.

A space equipped with pomset metrics is a recently introduced one by I. G. Sudha, R. S. Selvaraj [18], which is a generalization of Lee space [13], in particular, and poset space [4], in general, over $\mathbb{Z}_m$. Further, the MacWilliam type identities

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were determined \[19\], maximum distance separability, and perfectness of codes \[20\] were studied on codes with pomset metric. A pomset is a partially ordered multiset, and for more information, one can see \[3, 9, 10\] and \[18\]. Of late, L. Panek \[15\] introduced the notion of weighted coordinates pomset metric, a general form of the pomset metric without using the multiset structure. But, as the notion of ideals \(I\) with partial count and those with full count in pomsets affect the study of \(I\)-perfect codes and hence MDS codes, it is felt by the authors to pursue the exploration of block codes by retaining the flavour of multiset set up.

In poset space or poset block spaces, the \(I\)-balls are linear, which is not the case with pomset spaces. There are two kinds of ideals in pomsets \[20\]: ideals with full count and ideals with partial count. So, we extend the concept of \(I\)-balls (\(I\) is an ideal in the pomset) to the pomset block space \(Z_m^N\) to study the \(I\)-perfect pomset codes. We will see that if \(I\) is an ideal with a full count then the \(I\)-ball is linear, whereas this is not the case for ideals with a partial count. Thus, there is a need to investigate the \(I\)-perfect pomset block codes for both ideals with partial count and full count (see Theorem \[3, 10\] and \[3, 18\]). However, these \(I\)-perfect codes are more comfortable to study than the \(r\)-perfect codes because the \(I\)-balls are submodules of \(Z_m^N\).

This paper introduces the pomset block metric for codes of length \(N\) over the ring \(Z_m\). Section 2 provides the preliminaries and basic results on multisets, relations on multisets, and pomset. In Section 3, we define the \((Pm, \pi)\)-metric (or pomset block metric) on \(Z_m^N\). Further, we define \(r\)-balls, \(I\)-balls and explore their structures. We determine the complete weight distribution of \((Pm, \pi)\)-space. An \(I\)-perfect pomset block code for an ideal \(I\) with full count is found. But, an \(I\)-perfect pomset block code \(C\) for an ideal \(I\) with a partial count is constructed by imposing certain conditions on the subset \(C\) of \(Z_m^N\). Section 4 completely deals with block codes with chain pomsets. There we obtain the packing radius and Singleton bound for pomset block codes. Then the relationship of MDS poset block codes with MDS pomset block codes is found. Moreover, the relationship of MDS \((Pm, \pi)\)-codes with \(I\)-perfect \((Pm, \pi)\)-codes is studied and the duality theorem for such \((Pm, \pi)\)-codes when all blocks have the same length is established.

2. Preliminaries

A multiset \(M\) (in short, mset) is a collection of elements that may contain duplicates. These elements are drawn from a set \(A\) and the multiset \(M\) is described through a count function \(C_M : A \rightarrow \mathbb{N}\) such that \(C_M(a) = c\) for \(a \in A\). For example, if \(A = \{a_1, a_2, \ldots, a_n\}\) then \(M = \{c_1/a_1, c_2/a_2, \ldots, c_n/a_n\}\) is a multiset drawn from the set \(A\) where \(c_i/a_i \in M\) represents an element \(a_i \in A\) appearing \(c_i\) times in \(M\). The cardinality of a multiset \(M\) is \(|M| = \sum_{a \in A} C_M(a)\).

For \(a \in \mathbb{N}\), if \(c_i = h\) for every \(i\) then \(M\) is called a regular mset with height \(h\). Given a multiset \(M\), we call the set \(M^* \triangleq \{i \in A : C_M(i) > 0\}\) to be the root set of \(M\). Let \(M_1\) and \(M_2\) be two mssets drawn from the set \(A\). If \(C_{M_1}(a) \leq C_{M_2}(a)\) for all \(a \in A\), then \(M_1\) is called as submset (in short, submset) of \(M_2\) \((M_1 \subseteq M_2)\) and in addition, if \(C_{M_1}(a) < C_{M_2}(a)\) for an \(a \in M_2\) then \(M_1\) is said to be a proper submset of \(M_2\) \((M_1 \subset M_2)\). \(M_1 = M_2\) iff \(M_1 \subseteq M_2\) and \(M_2 \subseteq M_1\). Union and intersection of multisets \(M_1\) and \(M_2\) are defined as: \(M_1 \cup M_2 \triangleq \{C_{M_1 \cup M_2}(a) / a : C_{M_1 \cup M_2}(a) = \max\{C_{M_1}(a), C_{M_2}(a)\}\}\) for all \(a \in A\).
and $M_1 \cap M_2 \triangleq \{C_{M_1 \cap M_2}(a) : C_{M_1 \cap M_2}(a) = \min\{C_{M_1}(a), C_{M_2}(a)\} \}$ for all $a \in A$, respectively.

The mset space $[A]^r$ is the set of all multisets $M$ drawn from $A$ such that $C_M(a) \leq r$ for every $a \in A$. Let $M_1, M_2 \in [A]^r$, then their mset sum $M_1 \oplus M_2 \triangleq \{C_{M_1 \oplus M_2}(a) : C_{M_1 \oplus M_2}(a) = \min\{C_{M_1}(a) + C_{M_2}(a), r\} \}$ for all $a \in A$. The mset difference of $M_2$ from $M_1$ is $M_1 \ominus M_2 \triangleq \{C_{M_1 \ominus M_2}(a) : C_{M_1 \ominus M_2}(a) = \max\{C_{M_1}(a) - C_{M_2}(a), 0\} \}$ for all $a \in A$. Cartesian product: $M_1 \times M_2 = \{mn/(m/a, n/b) : m/a \in M_1 \text{ and } n/b \in M_2\}$. The notation $k/(m/a, n/b)$ means that $a$ is appearing $m$ times in $M_1$, $b$ is appearing $n$ times in $M_2$ and the pair $(a,b)$ is appearing $k$ times in $M_1 \times M_2$ where $1 \leq k \leq mn$. The subset $R$ of $M \times M$ is said to be an mset relation on $M$ if for every $(m/a, n/b) \in R$ one has $C_R(m/a, n/b) = C_M(a) \cdot C_M(b)$. For $(m/a, n/b) \in R$, the count of $a$ and $b$ in the ordered pair $(m/a, n/b)$ is denoted as $C_M(a) = m$ and $C_M(b) = n$ respectively.

A mset relation $R \subseteq M \times M$ is said to be partially ordered mset relation (in short, pomset relation) if it satisfies the following: (1) for every $p/a \in M$, $p/aRq/b$ (reflexive), (2) $p/aRq/b$ and $q/bRp/a$ imply $p = q, a = b$ (anti-symmetric), and (3) $p/aRq/b$ and $q/bRr/c$ imply $p/aRr/c$ (transitive). The pair $(M, R)$ is called a partially ordered multiset (or pomset) and is denoted by $P$. An element $q/b \in M$ is said to be a maximal element of $P$ if there is no element $t/c \in M$ such that $q/bRt/c$. An element $r/a \in M$ is said to be a minimal element of $P$ if there is no element $t/c \in M$ such that $t/cRr/a$. $P$ is said to be a chain if every distinct pair of elements from $M$ is comparable in $P$. $P$ is said to be an antichain if no distinct pair of elements from $M$ is comparable in $P$.

A subset $I$ of $M$ is called an ideal of $P$ if $p/a \in I$ and $q/b R p/a (b \neq a)$ imply $q/b \in I$. An ideal generated by an element $p/a \in M$ is defined as $\langle p/a \rangle = \{p/a\} \cup \{q/b \in M : q/bRp/a\}$. An ideal generated by a subset $I$ of $M$ is defined as $\langle I \rangle = \bigcup_{p/a \in I} \langle p/a \rangle$. An ideal $I$ is said to be of full count if $C_I(i) = C_M(i)$ for every $i \in I$; otherwise, it is said to be an ideal with a partial count. Throughout the paper, $\mathcal{J}(P)$ denotes the set of all ideals in $P$ and $\mathcal{J}(P)^r$ denotes the set of all ideals in $P$ with cardinality $r$.

**Example 2.1.** Let $M = \{h/i_j : j \in [5]\}$ be a regular mset of height $h$ and let $R = \{h^2/(i_j, i_j, h), h^2/(h, i_k) : j \in [5]\}$ be a pomset relation defined on $M$. Then $P = (M, R)$ is a pomset. Some of the ideals of $P = (M, R)$ are: (a) $I_1 = \{(h_1, i_1)\} = h/i_1, h_1/i_1$ for any $l \in \{2, 3, 4, 5\}$ and $h_1 \leq h$; (b) $I = \{h/i_1, h/i_2, h/i_3\}$; (c) $J = \{h/i_1, h/i_2, h/i_3, h/i_4, h/i_5\}$. If $h = h$ then $I$, $I$ and $J$ are ideals with full counts. If $h_1 < h$ then these are ideals of $P$ with partial counts. But, if we take $R' = \{h^2/(i_j, i_j, h), h^2/(h/i_1, h/i_k) : j, k \in \{1, 2, 3, 4, 5\}, k \neq 1 \text{ and } h_2 < h^2\}$ then $R'$ is not an mset relation because the counts of the ordered pairs $(h/i_1, h/i_k)$ are $h_2 < h^2$. Hence, $P = (M, R')$ is not a pomset.

**Proposition 2.1.** Let $P = (M, R)$ be a pomset where $M \in [A]^r$. If $I, J \in \mathcal{J}(P)$, then the following holds:

(i) $(I \cup J) = (I) \cup (J)$ and $(I \cap J) = (I) \cap (J)$.

(ii) $(I \oplus J) \subseteq (I) \oplus (J)$ if $M$ is a regular mset with height $r$.

**Proposition 2.2.** Let $P = (M, R)$ be a pomset and $0 \leq m \leq n \leq |M|$. Then for each $I \in \mathcal{J}^n(P)$ there exists a $J \in \mathcal{J}^m(P)$ such that $J \subseteq I$. Moreover, for each $J \in \mathcal{J}^m(P)$ there exists an $I \in \mathcal{J}^n(P)$ such that $I \supseteq J$. 
Definition 2.2 (Dual Pomset). For a pomset $P = (M, R)$, its dual pomset $\tilde{P} = (\tilde{M}, \tilde{R})$ is defined with the same underlying mset $M$ of height $r$ such that $p/a R q/b$ in $\tilde{P}$ if and only if $q/b R p/a$ in $P$. As a result, the order ideals of $\tilde{P}$ are precisely the complements of the order ideals of $P$, that is, $\mathcal{F}(\tilde{P}) = \{I^c : I \in \mathcal{F}(P)\}$, where the compliment of $I$ with respect to $M$ is $I^c \triangleq \{C_M(i) : C_M(i) = r - C_M(i)\}$ for all $i \in M^+$. 

Example 2.3. For the pomset $P$ given in Example 2.1 the dual is $\tilde{P} = (\tilde{M}, \tilde{R})$ with the multiset relation $\tilde{R} = (h^2/(h/i_j, h/i_j), h^2/(h/i_k, h/i_k), j, k \in \{1, 2, 3, 4, 5\})$. The maximal elements of the pomset $P$ viz., $h/i_2, h/i_3, h/i_4$, and $h/i_5$ are the minimal elements of its dual $\tilde{P}$.

3. Pomset Block Metric Space ($(Pm, \pi)$-space)

For the regular multiset $M = \{|\frac{m}{2}| : i \in [n]\}$ of height $\frac{m}{2}$ drawn from $[n]$ with a partial order $R$, the pair $\bar{P} = (M, R)$ is a pomset. For any positive integer $k$ and $x = (x_1, x_2, \ldots, x_k) \in Z_m^k$, the support of $x$ with respect to Lee weight is defined to be the multiset $\text{supp}_L(x) = \{c/i : x_i \neq 0, c = w_L(x_i)\}$ where $w_L(x_i) = \min\{x_i, m - x_i\}$ is the Lee weight of $x_i \in Z_m$. Let $\pi$ be a label map from $[n]$ to $\mathbb{N}$ defined by $\pi(i) = k_i$ with $\sum_{i=1}^n \pi(i) = N$ and consider the space $Z_m^N$ as the direct sum of modules $Z_m, \ldots, Z_m$; that is, $Z_m^N = Z_m \oplus Z_m \oplus \ldots \oplus Z_m$. Every $N$-tuple $v$ in $Z_m^N$ is expressed uniquely as $v = v_1 \oplus v_2 \oplus \ldots \oplus v_n$ where $v_i = (v_{i_1}, v_{i_2}, \ldots, v_{i_k}) \in Z_m^{k_i}$. For $v_i \in Z_m^{k_i}$, if $\text{Max}_c \text{supp}_L(v_i) \triangleq \max_{v_j \in \text{supp}_L(v_i)} \{|w_L(v_{i,j})|\}$ denote the maximum among the Lee weights of the components of $v_i$, we define the pomset block weight of $v \in Z_m^N$ as $\text{supp}(Pm, \pi)(v) = \{c_i/i : v_i \neq 0 \text{ and } c_i = \text{Max}_c \text{supp}_L(v_i)\}$ a subset of $M$.

Subsequently, the pomset block weight or $(Pm, \pi)$-weight of $v$ is $w(Pm, \pi)(v) \triangleq |\text{supp}(Pm, \pi)(v)|$ and the pomset block distance or $(Pm, \pi)$-distance between $u, v \in Z_m^N$ is $d(Pm, \pi)(u, v) \triangleq w(Pm, \pi)(u - v)$. Now, the $(Pm, \pi)$-distance is indeed a metric on $Z_m^N$ as given in:

Theorem 3.1. The pomset block distance $d(Pm, \pi)(\ldots)$ is a metric on $Z_m^N$.

Proof. Let $u, v, w \in Z_m^N$. As $|\text{supp}(Pm, \pi)(u - v)| \geq 0$ so is $d(Pm, \pi)(u, v) \geq 0$. Clearly, $d(Pm, \pi)(u, v) = 0$ iff $u = v$. As $w_L(-v_i) = w_L(v_i)$ for each $i$ and $\ell$, so is $\text{supp}_L(-v) = \text{supp}_L(v)$ and hence $w(Pm, \pi)(-v) = w(Pm, \pi)(v)$. Thus, $d(Pm, \pi)(u, v) = d(Pm, \pi)(v, u)$. Since $\text{Max}_c \text{supp}_L(u_i + v_i) \leq \text{Max}_c \text{supp}_L(u_i) + \text{Max}_c \text{supp}_L(v_i)$, then $\text{supp}(Pm, \pi)(u + v) \subseteq \text{supp}(Pm, \pi)(u) \oplus \text{supp}(Pm, \pi)(v)$. It follows from the Proposition 2.4.2 that, $(\text{supp}(Pm, \pi)(u + v)) \subseteq (\text{supp}(Pm, \pi)(u) \oplus \text{supp}(Pm, \pi)(v))$. Thus, $w(Pm, \pi)(u + v) = |\text{supp}(Pm, \pi)(u + v)| \leq w(Pm, \pi)(u) + w(Pm, \pi)(v)$. Hence, $d(Pm, \pi)(u, v) \leq d(Pm, \pi)(u, w) + d(Pm, \pi)(w, v)$. 

The metric $d(Pm, \pi)(\ldots)$ on $Z_m^N$ is called as the pomset block metric or $(Pm, \pi)$-metric. The pair $(Z_m^N, d(Pm, \pi))$ is said to be a pomset block space or $(Pm, \pi)$-space.

If $k_i = 1$ for every $i \in [n]$ then $\text{Max}_c \text{supp}_L(v_i) = w_L(v_i)$ and $\text{supp}(Pm, \pi)(v) = \{k/i : v_i \neq 0 \text{ and } k = w_L(v_i)\}$. Here, the pomset block weight becomes pomset weight and the pomset block space becomes the pomset space [15].
Example 3.2. Consider $\mathbb{Z}_3^8 = \mathbb{Z}_3^2 \oplus \mathbb{Z}_3^4 \oplus \mathbb{Z}_3^4 \oplus \mathbb{Z}_3^3 \oplus \mathbb{Z}_3^3$ with $N = 18$ and $n = 6$. Let $x$ be a partial order relation on $[6] = \{1, 2, 3, 4, 5, 6\}$ such that $3/1 \leq 3/2 \leq 3/4$; $3/5 \leq 3/6$. Then the set $A = \{x = x_1 \oplus x_2 \oplus 0 \oplus 0101 \oplus x_5 \oplus 20 : x_1 \in \mathbb{Z}_3^2, \ x_2 \in \mathbb{Z}_3^4, \ x_5 \in \mathbb{Z}_3^3\}$ gives a set of vectors in $\mathbb{Z}_7^8$ in which $(P_m, \pi)$-weight of any $x \in A$ is $|\langle \text{supp}_{P_m, \pi}(x) \rangle| = |\{1/4, 2/6\}| = 3 + 3 + 1 + 3 + 2 = 12.$

Any subset $C$ of a $(P_m, \pi)$-space is said to be a poset block code or a $(P_m, \pi)$-code. The minimum distance of $C$ is $d_{(P_m, \pi)}(C) \triangleq \min \{d_{(P_m, \pi)}(c_1, c_2) : c_1, c_2 \in C\}$. If $C$ is linear, $d_{(P_m, \pi)}(C) = \min \{w_{(P_m, \pi)}(c) : 0 \neq c \in C\}$. As $w_{(P_m, \pi)}(v) \leq n[\frac{m}{2}]$ for any $v \in \mathbb{Z}_m^N$, the minimum distance of any $(P_m, \pi)$-code $C$ is bounded above by $n[\frac{m}{2}]$.

3.1. Balls and $I$-perfect codes in $(P_m, \pi)$-space. The poset block ball or $(P_m, \pi)$-ball with center $u \in \mathbb{Z}_m^N$ and radius $r$ (or r-ball centered at $u$) is defined as $B_{(P_m, \pi)}(u, r) = \{v \in \mathbb{Z}_m^N : d_{(P_m, \pi)}(u, v) \leq r\}$ whereas an r-sphere centered at $u$ is defined as $S_{(P_m, \pi)}(u, r) = \{v \in \mathbb{Z}_m^N : d_{(P_m, \pi)}(u, v) = r\}$. If $u = 0$, they will also be denoted as $B_r$ and $S_r$, respectively.

Given a poset $P = (M, R)$ on a regular multiset $M$ of height $\left\lceil \frac{m}{2} \right\rceil$ drawn from $[n]$, we define a poset $P = ([n], \preceq_P)$ induced by $P$ to be such that $i \preceq_P j$ in $P$ iff $p/i \mathcal{R} j$ in $P$.

It is easy to see that if $I$ is an ideal in poset $P$ then $I^*$ is an ideal in the poset $P$ induced by $P$. Now, for a $u \in \mathbb{Z}_m^N$, its $\pi$-support is $\text{supp}_\pi(u) = \{i \in [n] : u_i \neq 0\}$ and for an ideal $I^*$ in $P$, the $I^*$-ball centered at $u$ with respect to $(P, \pi)$-metric is $B_{I^*}(u) \triangleq \{v \in \mathbb{Z}_m^N : \text{supp}_\pi(u - v) \subseteq I^*\}$. The r-ball centered at $u$ with respect to $(P, \pi)$-metric is $B_{(P, \pi)}(u, r) \triangleq \{v \in \mathbb{Z}_m^N : d_{(P, \pi)}(u, v) \leq r\}$. Let $I^*(P)$ denotes the set of all ideals in $P$ of cardinality $r$. Let $d_{(P, \pi)}(C)$ be the minimum distance of a code $C$ of length $N$ over $\mathbb{Z}_m$ with respect to $(P, \pi)$-metric.

Since $|\langle \text{supp}_\pi(v) \rangle| \leq w_{(P, \pi)}(v) \leq \left\lceil \frac{|v|}{2} \right\rceil |\langle \text{supp}_\pi(v) \rangle| \ \forall \ v \in \mathbb{Z}_m^N$, we have $d_{(P, \pi)}(u, v) \leq d_{(P, \pi)}(u, v) \leq \frac{|v|}{2} |d_{(P, \pi)}(u, v)|$ for all $u, v \in \mathbb{Z}_m^N$. Then, $d_{(P, \pi)}(C) \leq \frac{|v|}{2} |d_{(P, \pi)}(C)|$. Thus, we have the following:

**Proposition 3.1.** Let $d_{(P_m, \pi)}(C)$ and $d_{(P, \pi)}(C)$ be the minimum distances of a code $C$ of length $N$ over $\mathbb{Z}_m$, with respect to $(P_m, \pi)$-metric and $(P, \pi)$-metric respectively. Then $\left[\frac{d_{(P, \pi)}(C) - 1}{2}\right] \leq d_{(P, \pi)}(C) - 1$.

Let $r = t[\frac{m}{2}] + s$, $1 \leq s \leq \frac{|v|}{2}$ and $1 \leq t \leq n - 1$. If $v \in B_{(P, \pi)}(u, t)$ then $\text{supp}_{(P, \pi)}(u - v) \subseteq I$ for an $I \in I^*(P)$. Then, $w_{(P, \pi)}(u - v) \leq t[\frac{m}{2}]$. Thus, we have $\mathbf{Theorem 3.3.}$ For $x \in \mathbb{Z}_m^N$ and $1 \leq t \leq n - 1$, $B_{(P, \pi)}(x, t) \subseteq B_{(P, \pi)}(x, t[\frac{m}{2}])$.

**Definition 3.4** ($r$-perfect $(P_m, \pi)$-code). A $(P_m, \pi)$-code $C$ of length $N$ over $\mathbb{Z}_m$ is said to be $r$-perfect if the $r$-balls centered at the codewords of $C$ are pairwise disjoint and their union covers the entire space $\mathbb{Z}_m^N$.

For a subset $I$ in $P$, the I-ball centered at $u$ is $B_I(u) \triangleq \{v \in \mathbb{Z}_m^N : \text{supp}_{P_m, \pi}(u - v) \subseteq I\}$ and the I-sphere centered at $u$ is $S_I(u) \triangleq \{v \in \mathbb{Z}_m^N : \text{supp}_{P_m, \pi}(u - v) = I\}$. For $v \in B_I(u)$, it is not necessary that $\langle \text{supp}_{P_m, \pi}(u - v) \rangle \subseteq I$. If the subset $I$ is an ideal in $P$ then $\langle \text{supp}_{P_m, \pi}(u - v) \rangle \subseteq I$ will always hold. Hence, for an ideal $I$ in $P$, the I-ball centered at $u$ can also be defined as $B_I(u) \triangleq \{v \in \mathbb{Z}_m^N : \langle \text{supp}_{P_m, \pi}(u - v) \rangle \subseteq I\}$.
and \( d_{(P_m, \pi)}(u, v) \leq |I| \) for each \( v \in B_I(u) \). Moreover, the \( I \)-sphere centered at \( u \) is \( S_I(u) \triangleq \{ v \in \mathbb{Z}_m^N : \text{supp}_{(P_m, \pi)}(u - v) = I \} \). In short, the \( B_I \) and \( S_I \) denote the \( I \)-ball and \( I \)-sphere centered at \( 0 \in \mathbb{Z}_m^N \), respectively. Let \( \mathcal{F}_I^*(\mathbb{P}) \) denote the collection of all ideals with cardinality \( i \) having \( j \) maximal elements. Then 

\[
\bigcup_{j=1}^{\min\{i,n\}} \mathcal{F}_j^*(\mathbb{P}) = \mathcal{F}_I^*(\mathbb{P}).
\]

Let \( \mathcal{F}(I) = \{ c_{i_1}/i_1, c_{i_2}/i_2, \ldots, c_{i_j}/i_j \} \) be the submset of maximal elements in the ideal \( I \).

**Proposition 3.2.** Let \( I \) be a submset (or ideal) in \( \mathbb{P} \) and \( v \in \mathbb{Z}_m^N \). Then \( S_I(v) \) is the translate (or coset) of \( S_I \) containing \( v \), that is, \( S_I(v) = v + S_I \). Moreover, \( B_I(v) = v + B_I \).

**Proof:** If \( u \in S_I(v) \) then \( \text{supp}_{(P_m, \pi)}(u - v) = I \) so that \( u - v \in S_I \) and thus, \( u = v + (u - v) \in v + S_I \). Let \( y \in S_I \). Then \( \text{supp}_{(P_m, \pi)}(v - (v + y)) = \text{supp}_{(P_m, \pi)}(-y) = \text{supp}_{(P_m, \pi)}(y) = I \). Thus, \( v + y \in S_I(v) \) for all \( y \in S_I \). Hence, \( S_I(v) = v + S_I \). Then, \( B_I(v) = v + B_I \) follows.

**Remark 3.5.** In a similar way as above, for \( v \in \mathbb{Z}_m^N \), an \( r \)-sphere and an \( r \)-ball centered at \( v \) can also be expressed as \( S_{(P_m, \pi)}(v, r) = v + S_r \) and \( B_{(P_m, \pi)}(v, r) = v + B_r \) respectively.

**Proposition 3.3.** Two vectors \( u \) and \( v \) belong to the same \( I \)-ball for some \( I \in \mathcal{F}^r(\mathbb{P}) \) if and only if \( d_{(P_m, \pi)}(u, v) \leq r \).

**Proposition 3.4.** Every \( r \)-ball is the union of all \( I \)-balls where \( I \in \mathcal{F}^r(\mathbb{P}) \), that is, \( B_{(P_m, \pi)}(v, r) = \bigcup_{I \in \mathcal{F}^r(\mathbb{P})} B_I(v) \).

Given an ideal \( I \), for each \( i_t \in \text{Max}(I)^* \) with \( C_I(i_t) = C_{i_t} \), we set

\[
N_{i_t} = \begin{cases} 
2 & \text{if } c_{i_t} < \frac{m}{2} \\
1 & \text{if } c_{i_t} = \frac{m}{2}.
\end{cases}
\]

**Theorem 3.6.** Let \( I \) be an ideal in \( \mathbb{P} \). Then the cardinality of \( I \)-sphere centered at \( 0 \in \mathbb{Z}_m^N \) is

\[
|S_I| = \prod_{i_t \in \text{Max}(I)^*} \delta(c_{i_t}) \prod_{I \notin \text{Max}(I)} m^{k_{i_t}}
\]

where \( c_{i_t} = C_I(i_t) \) and \( \delta(c_{i_t}) = \begin{cases} (2c_{i_t} + 1)^{k_{i_t}} - (2c_{i_t} - 1)^{k_{i_t}} & \text{if } c_{i_t} \neq \frac{m}{2} \\
m^{k_{i_t}} - (m - 1)^{k_{i_t}} & \text{if } c_{i_t} = \frac{m}{2}.
\end{cases} \)

**Proof:** Let \( \text{Max}(I) = \{ c_{i_1}/i_1, c_{i_2}/i_2, \ldots, c_{i_j}/i_j \} \). Let \( S_I \) be the \( I \)-sphere centered at \( 0 \). Choose \( y = y_1 \oplus y_2 \oplus \ldots \oplus y_n \in \mathbb{Z}_m^N \) such that if \( i_t \in \text{Max}(I)^* \) then \( c_{i_t} \neq 0 \) and \( \text{Max}_c\text{supp}_L y_{i_t} = c_{i_t} \) and if \( i_t \notin \text{Max}(I)^* \) then \( y_{i_t} = 0 \). Thus, \( y \in S_I \).

Case(1) If \( i_t \in \text{Max}(I)^* \) is with full count \( \lfloor \frac{m}{2} \rfloor \) in the ideal \( I \), the total number of choices for such \( y_{i_t} = (y_{i_{t1}}, y_{i_{t2}}, \ldots, y_{i_{tk_{i_t}}}) \) in \( \mathbb{Z}_m^{k_{i_t}} \) with \( \text{Max}_c\text{supp}_L y_{i_t} = \lfloor \frac{m}{2} \rfloor \) is

\[
= \binom{k_{i_t}}{1}N_{i_t}(m - N_{i_t})^{k_{i_t} - 1} + \binom{k_{i_t}}{2}N_{i_t}^2(m - N_{i_t})^{k_{i_t} - 2} + \ldots + \binom{k_{i_t}}{k_{i_t}}N_{i_t}^{k_{i_t}}
\]

\[
= m^{k_{i_t}} - (m - N_{i_t})^{k_{i_t}}.
\]
Case(2) If \( i_t \in Max(I)^* \) is with partial count say \( c_{i_t} < \lceil \frac{m}{2} \rceil \) in the ideal \( I \), the total number of choices for such \( y_{i_t} = (y_{i_{t_1}}, y_{i_{t_2}}, \ldots, y_{i_{t_{k_{i_t}}}}) \) in \( \mathbb{Z}_m^{k_{i_t}} \) with \( \text{Max}_c \text{supp}_I y_{i_t} = c_{i_t} < \lceil \frac{m}{2} \rceil \) is

\[
\left( \frac{k_{i_t}}{1} \right) (2(2c_{i_t} - 1))^{k_{i_t}-2} + \left( \frac{k_{i_t}}{2} \right) 2^2(2c_{i_t} - 1)^{k_{i_t}-2} + \cdots + \left( \frac{k_{i_t}}{k_{i_t}} \right) 2^{k_{i_t}}
\]

\[= (2c_{i_t} + 1)^{k_{i_t}} - (2c_{i_t} - 1)^{k_{i_t}}\]

Hence the result. \( \square \)

Since \( S_I(u) = u + S_I \), \( |S_r| = \sum_{j=1}^{min\{i,n\}} \sum_{I \in \mathcal{I}'(P)} |S_I| \), the following result follows from Theorem 3.6.

**Corollary 3.7.** Let \( I \) be an ideal in \( P \). Then the cardinality of \( I \)-sphere centered at \( u \in \mathbb{Z}_m^N \) is

\[|S_{(P_m,\pi)}(u,r)| = \sum_{j=1}^{min\{i,n\}} \sum_{I \in \mathcal{I}'(P)} \prod_{i_t \in Max(I)^*} \delta(c_{i_t}) \prod_{t \in I \setminus Max(I)} m^{k_{i_t}}.\]

where \( c_{i_t} = C_I(i_t) \) and \( \delta(c_{i_t}) = \begin{cases} (2c_{i_t} + 1)^{k_{i_t}} - (2c_{i_t} - 1)^{k_{i_t}} & \text{if } c_{i_t} \neq \lceil \frac{m}{2} \rceil \\ m^{k_{i_t}} - (m - 1)^{k_{i_t}} & \text{if } c_{i_t} = \lceil \frac{m}{2} \rceil. \end{cases} \)

From Proposition 3.3 we have

\[B_{(P_m,\pi)}(u,r) = \bigcup_{I \in \mathcal{I}'(P)} B_I(u) = \bigcup_{r=1}^{n} \bigcup_{I \in \mathcal{I}'(P)} S_I(u).\]

Since cardinality of an \( r \)-ball does not depend on its centre, \( |B_{(P_m,\pi)}(0,r)| = 1 + \sum_{j=1}^{r} |S_{(P_m,\pi)}(0,j)| = 1 + \sum_{i=1}^{\min\{i,n\}} \sum_{j=1}^{\min\{i,n\}} \sum_{I \in \mathcal{I}'(P)} |S_I| \). Thus, we have the following:

**Corollary 3.8.** The cardinality of \( (P_m,\pi) \)-ball centered at \( u \in \mathbb{Z}_m^N \) with radius \( r \) is

\[|B_{(P_m,\pi)}(u,r)| = 1 + \sum_{i=1}^{r} \sum_{j=1}^{\min\{i,n\}} \sum_{I \in \mathcal{I}'(P)} \prod_{i_t \in Max(I)^*} \delta(c_{i_t}) \prod_{t \in I \setminus Max(I)^*} m^{k_{i_t}}.\]

where \( c_{i_t} = C_I(i_t) \) and \( \delta(c_{i_t}) = \begin{cases} (2c_{i_t} + 1)^{k_{i_t}} - (2c_{i_t} - 1)^{k_{i_t}} & \text{if } c_{i_t} \neq \lceil \frac{m}{2} \rceil \\ m^{k_{i_t}} - (m - 1)^{k_{i_t}} & \text{if } c_{i_t} = \lceil \frac{m}{2} \rceil. \end{cases} \)

Now, the \( (P_m,\pi) \)-weight distribution of \( \mathbb{Z}_m^N \) is obtained as follows: For each \( 1 \leq r \leq n \lceil \frac{m}{2} \rceil \), let \( A_r = \{ u = u_1 \oplus u_2 \oplus \ldots \oplus u_n \in \mathbb{Z}_m^N : w_{(P_m,\pi)}(u) = r \} \) and \( A_0 = \{ 0 \} \). Then, \( u \in A_r \) if \( u \in S_r \). Thus, we have:

**Proposition 3.5.** For any \( 1 \leq r \leq n \lceil \frac{m}{2} \rceil \), the number of \( N \)-tuples \( x \in \mathbb{Z}_m^N \) having \( w_{(P_m,\pi)}(x) = r \) is \( |A_r| = |S_r| \).
3.2. Ideals with full count. In [20], it was shown that unlike the results in poset space [11], I-balls are no more linear subspaces of \( \mathbb{Z}_m^N \) in the pomset space when the ideal \( I \) is with a partial count. Thus, there is a need to investigate the properties of I-balls with respect to pomset block space as well. The following result is a block version to the result in [20] ref. Proposition 3] and its proof follows the similar pattern.

**Proposition 3.6.** Let \( \mathbb{P} \) be a pomset on a regular multiset \( M \) and \( \tilde{\mathbb{P}} \) be the dual pomset of \( \mathbb{P} \). If \( I \) is an ideal with full count in \( \mathbb{P} \), then

(a) \( B_I \) is a submodule of \( \mathbb{Z}_m^N \) and \( |B_I| = m^{\sum_i k_i} \).

(b) For \( u \in \mathbb{Z}_m^N \), \( B_I(u) \) is the coset of \( B_I \) containing \( u \), ie. \( B_I(u) = u + B_I \).

(c) For \( u, v \in \mathbb{Z}_m^N \), the two I-balls \( B_I(u) \) and \( B_I(v) \) are either identical or disjoint. Moreover, \( B_I(u) = B_I(v) \) if and only if \( \text{supp}(P_m, \pi)(u - v) \subseteq I \).

(d) \( \tilde{B}_{I^*}^N(0) = B_{I^*}(0) \).

Hence, \( \mathbb{Z}_m^N \), \( d(P_m, \pi) \)-space can be partitioned into I-balls for an ideal \( I \) with full count. If \( t \) be the number of distinct I-balls in the partition of \( \mathbb{Z}_m^N \), then \( t = m^{N - \sum_i k_i} \). If \( k_i = k \) for each \( i \) in \( [n] \), then \( t = m^{(n-|I|)}k \).

**Definition 3.9.** A \((P_m, \pi)\)-code \( \mathbb{C} \subseteq \mathbb{Z}_m^N \) is said to be an I-perfect code if the I-balls centered at the codewords of \( \mathbb{C} \) are pairwise disjoint and their union is \( \mathbb{Z}_m^N \).

**Theorem 3.10.** If \( I \) is an ideal with full count in \( \mathbb{P} \) then there exists an I-perfect pomset block code.

**Proof.** By Proposition 3.6, the space \( \mathbb{Z}_m^N \) can be partitioned into I-balls. The number of I-balls is \( m^{N - \sum_i k_i} \). Then the set of \( N \)-tuples formed by picking exactly one \( N \)-tuple from each I-ball forms an I-perfect \((P_m, \pi)\)-code \( \mathbb{C} \) of length \( N \) over \( \mathbb{Z}_m \) and \( |\mathbb{C}| = m^{N - \sum_i k_i} \). Thus, \( d(P_m, \pi) \)(\( \mathbb{C} \)) > |\( I \)|.

**Lemma 3.11.** Let \( I \) be an ideal with full count. If \( \mathbb{C} \) is an I-perfect pomset block code of cardinality \( m^s \) and \( k_i = k \) for all \( i \) then \( |I^*| = \frac{kn}{k} \). Moreover if \( k = s \) then \( |I^*| = n - 1 \).

**Proof.** Since \( \mathbb{C} \) is an I-perfect pomset block code, \( |\mathbb{C}| |B_I| = m^N \) and \( |B_I| = m^{k|I^*|} \) imply that \( |I^*| = \frac{N}{k} \).

**Theorem 3.12.** Let \( \mathbb{C} \subseteq \mathbb{Z}_m^N \) be a linear pomset block code of cardinality \( m^s \) and \( I \) be an ideal with full count in \( \mathbb{P} \). Then, \( \mathbb{C} \) is I-perfect if and only if \( \mathbb{C}^\perp \) is \( I^\perp \)-perfect where \( I^\perp \) is an ideal in \( \tilde{\mathbb{P}} \).

**Proof.** Suppose that \( \mathbb{C} \) is I perfect. Let \( B_{I^\perp} \) be an I-ball centered at 0 in \( \mathbb{C} \), then \( \sum_i k_i = N - s \). Since \( I \) is an ideal with full count in \( \mathbb{P} \), then \( I^\perp \) is an ideal with a full count in \( \tilde{\mathbb{P}} \). So, \( \sum_i k_i = s \). Let \( B_{I^\perp} \) be an I^\perp-ball centered at 0 in \( \mathbb{C}^\perp \). Then \( |B_{I^\perp}^*| = m^s \). We have to prove that \( \mathbb{C}^\perp \) is \( I^\perp \)-perfect. It is enough to show that \( |B_{I^\perp} \cap \mathbb{C}^\perp| = 1 \). Suppose that \( u \in B_{I^\perp} \cap \mathbb{C}^\perp \). From Proposition 3.6(u) \( u \in B_{I^\perp} \) and thus \( u \cdot x = 0 \forall x \in B_I \). Since \( \mathbb{C} \) is I-perfect each \( y \in \mathbb{Z}_m^N \) occurs exactly in one I-ball, say \( B_I \) for some \( c \in \mathbb{C} \), so that \( y = c + x \) for some \( x \in B_I \). Thus, \( u \cdot y = 0 \) for every \( y \in \mathbb{Z}_m^N \) which means \( u = 0 \). Hence, \( |B_{I^\perp} \cap \mathbb{C}^\perp| = 1 \). The converse follows in a similar fashion. □
3.3. Ideals with a partial count. Now, for an ideal \( I \) with partial count, \( I \)-balls may not hold all the properties arrived at so far. Consider an ideal \( I \) with partial count in the pomset \( \mathbb{P} \). There exist an \( i \in I^* \) such that \( 0 < C_I(i) < \left\lceil \frac{m}{2} \right\rceil \). Let \( A_p = \{ i_1, i_2, \ldots, i_{p_i} \} \) be the set of those elements in \( I^* \) that have partial count in \( I \) and \( A_f = \{ j_1, j_2, \ldots, j_{f_i} \} \) be the set of those elements in \( I^* \) that have full count in \( I \). Thus, \( I^* = A_p \cup A_f \) with \( |A_p| = I_p \) and \( |A_f| = I_f \). Let \( c_f \) be the count of \( \ell \) in \( I \). Clearly, \( 1 \leq c_s \leq \left\lceil \frac{m}{2} \right\rceil - 1 \) when \( 1 \leq s \leq I_p \) and \( c_{js} = \left\lceil \frac{m}{2} \right\rceil \) for all \( 1 \leq k \leq I_f \). Then, the \( I \)-ball \( B_I \) consists of \( N \)-tuples of the form \( v_1 \oplus v_2 \oplus \ldots \oplus v_n \in \mathbb{Z}_m^N \) where \( v_i = (v_{i_1}, v_{i_2}, \ldots, v_{i_k}) \in \mathbb{Z}_m^k \) such that \( v_{is} \in \{0, \pm1, \pm2, \ldots, \pm c_{is}\} \) if \( i \in A_p \), \( v_{is} \in \mathbb{Z}_m \) if \( i \in A_f \) and \( v_{is} = 0 \) if \( i \notin I^* \). Clearly, \( B_I \) is not a group with respect to addition of \( N \)-tuples and hence it is not a submodule of \( \mathbb{Z}_m^N \). Moreover, the following is hold concerning the cardinality of \( B_I \):

**Proposition 3.7.** If \( I \) is an ideal with partial count in the pomset \( \mathbb{P} \) then the cardinality of \( I \)-ball is \( |B_I| = m^{k_I} \prod_{i \in A_p} (1 + 2c_i)^{k_i} \).

**Remark 3.13.** Let \( k_i = k \forall i \) and \( I \) be an ideal with partial count in the pomset \( \mathbb{P} \). Then the cardinality of \( I \)-ball is \( |B_I| = m^{k_I} \prod_{i \in A_p} (1 + 2c_i)^k \).

Since the \( I \)-balls centered at any two vectors are either disjoint or identical for an ideal \( I \) with a full count, cosets of \( B_I \) can partition the space \( \mathbb{Z}_m^N \) into \( I \)-balls and one can determine \( I \)-perfect pomset block codes. But, in the case of an ideal \( I \) with partial count, the \( I \)-balls need not be disjoint from one another. Translates of \( B_I \) for such an ideal \( I \) need not be disjoint always. We will identify if there exists any representative vectors \( u, v \in \mathbb{Z}_m^N \) such that the translates \( u + B_I \) and \( v + B_I \) are disjoint, in the line of \([20]\).

3.4. Partition of \( \mathbb{Z}_m^k \). Some results from \([20]\) are restated here which will help to find the partition of \( \mathbb{Z}_m^k \) for any positive integer \( k \). Let \( t \) be an integer such that \( 0 \leq t \leq \left\lceil \frac{m}{2} \right\rceil - 1 \). Let \( S = \{ a \in \mathbb{Z}_m : w_L(a) \leq t \} \) and \( S' = \{ a \in \mathbb{Z}_m : w_L(a) > t \} \). Thus, \( S = \{0, \pm1, \pm2, \ldots, \pm t\} \), \( |S| = 2t + 1 \) (\( 3 \leq 2t + 1 \leq m - 1 \)), \( S' = \{t + 1, t + 2, \ldots, m - t - 1\} \), \( S \cap S' = \{\} \) and \( |S'| = m - 2t - 1 \). If \( a \in S \setminus \{0\} \) then \( a + S = -a + S \). Moreover, \( 2t + 1 | m - 2t - 1 \) if and only if \( 2t + 1 | m \); that is, \( |S| = |S'| \) if and only if \( |S| = |m| \).

**Proposition 3.8** ([20]). \( S \) and \( a + S \) are neither disjoint nor identical for any \( a \in S \setminus \{0\} \).

\( S \) can be written as disjoint union of two subsets \( S_1 = \{0, 1, \ldots, t\} \), and \( S_2 = \{m - t, m - t + 1, \ldots, m - 1\} \). If \( 2t + 1 | m - 2t - 1 \) and \( \ell = \frac{m - 2t - 1}{2t + 1} \), then \( m = (\ell + 1)(2t + 1) \). Clearly, \( b \in S' \) iff \( t < b < m - t \). Hence, \( j(2t + 1) + a \in S' \) for all \( a \in S \) whenever \( 1 \leq j \leq t \).\n
**Lemma 3.14** ([20]). Let \( 2t + 1 | m - 2t - 1 \) and \( \ell = \frac{m - 2t - 1}{2t + 1} \). Then the following hold:

(i) For each \( i = 1, 2, \ldots, \ell \), we have \( a + i(2t + 1)(\mod m) \notin S \) for any \( a \in S \).

(ii) For \( i \neq j \), the translates \( i(2t + 1) + S \) and \( j(2t + 1) + S \) are disjoint where \( 1 \leq i, j \leq \ell \). Moreover, \( \bigcup_{0 \leq i \leq \ell} i(2t + 1) + S = \mathbb{Z}_m \).

(iii) \( 2t + 1 \) is a submodule of \( \mathbb{Z}_m \).
Hence, whenever $2t + 1 \mid m - 2t - 1$, there exist translates of $S$ that partition $\mathbb{Z}_m$. If $2t + 1 \nmid m - 2t - 1$ then the translates of $S$ will not form a partition of $\mathbb{Z}_m$. For further details, one can refer to [20].

Considering the findings above, we now define $S^k = \{(s_1,s_2,\ldots,s_k) \in \mathbb{Z}_m^k : s_j \in S\}$ and $S^k' = \{(s'_1,s'_2,\ldots,s'_k) \in \mathbb{Z}_m^k : s'_j \in S'\}$. As S is not a subgroup of $\mathbb{Z}_m$, $S^k$ is not a subgroup of $\mathbb{Z}_m^k$. Clearly, $|S^k| = (2t+1)^k$ and $|S^k'| = (m-2t-1)^k$.

**Proposition 3.9.** $S^k$ and $x + S^k$ are neither disjoint nor identical for any $x \in S^k \setminus \{0\}$.

**Proof.** Let $x = (x_1,x_2,\ldots,x_k) \in S^k \setminus \{0\}$, where $x_i \in S$. Let

$$y = (y_1,y_2,\ldots,y_k) \text{ where } y_i = \begin{cases} -(x_i - 1) + t & \text{if } 1 \leq x_i \leq t \\ -(x_i + 1) - t & \text{if } -t \leq x_i < -1. \end{cases}$$

Clearly, $y_i \in S$ and $x_i + y_i \in S'$ for each $i$. Thus, $y \in S^k$ and $x + y \in x + S^k$. Moreover, $x + y \in S^k$ (as $x_i + y_i \in S'$) which means $x + y \not\in S^k$. Hence, $S^k$ and $x + S^k$ are not identical. Also, $x \in S^k$ and $x \in x + S^k$. Thus, $S^k$ and $x + S^k$ are neither disjoint nor identical for any $x \in S^k \setminus \{0\}$. $\square$

We have $j(2t + 1) + a \in S'$ for all $0 \not\equiv a \in S, 1 \leq j \leq \ell$ whenever $2t + 1 \mid m - 2t - 1$ and $\ell = \frac{m-2t-1}{2t+1}$. Now, we define $T = \{j(2t + 1) : 0 \leq j \leq \ell\}$ and $T^k = \{v = (v_1,v_2,\ldots,v_k) \in \mathbb{Z}_m^k : v_i \in T\}$. Clearly, $|T^k| = (\ell + 1)^k = (\frac{m}{2t+1})^k$.

**Lemma 3.15.** Let $2t + 1 \mid m - 2t - 1$ and $\ell = \frac{m-2t-1}{2t+1}$. Then the following hold:

(i) For each $v \in T^k \setminus \{0\}$, $S^k$ and $v + S^k$ are disjoint.
(ii) For $v \neq v' \in T^k \setminus \{0\}$, the translates $v + S^k$ and $v' + S^k$ are disjoint. Moreover, $\bigcup_{v \in T^k} v + S^k = \mathbb{Z}_m^k$.
(iii) $\{v = (v_1,v_2,\ldots,v_k)\}$ is a submodule of $\mathbb{Z}_m^k$, where $v_j \in \langle 2t + 1 \rangle$ for all $1 \leq j \leq k$.

Hence, if $2t + 1 \mid m - 2t - 1$ then the translates of $S^k$ form a partition of $\mathbb{Z}_m^k$.

**Proof.**

(i) Let $v \in T^k \setminus \{0\}$ and $x \in S^k$. Suppose that $x_j$ and $v_j$ be the $j^{th}$ coordinate of $x$ and $v$ respectively. Since $v_j \in T$, from Lemma 3.14(i), $v_j + x_j \in S'$ for any $x_j$ in $S$. Thus, $v + x \not\in S^k$ for any $x \in S^k$. Hence, $S^k$ and $v + S^k$ are disjoint.

(ii) Suppose that the translates $v + S^k$ and $v' + S^k$ are not disjoint for some $v \not\equiv v' \in T^k \setminus \{0\}$. Now, $v$ and $v'$ will differ in at least one position say $i_0$ where $v_{i_0} = j(2t + 1) = v'_{i_0} = p(2t + 1)$ for some $j \neq p$. Then, there exist a $u \in (v + S^k) \cap (v' + S^k)$, so that $u = v + x$ and $u = v' + y$ for some $x,y \in S^k$. $(v_1,v_2,\ldots,v_k) + (x_1,x_2,\ldots,x_k) = (v'_1,v'_2,\ldots,v'_k) + (y_1,y_2,\ldots,y_k)$ implies that $v_{i_0} + x_{i_0} = v'_{i_0} + y_{i_0}$ for all $1 \leq i \leq k$. Then, $v_{i_0} + x_{i_0} \in v'_{i_0} + S$ and $v'_{i_0} + y_{i_0} \in v_{i_0} + S$. Thus, $j(2t + 1) + S$ and $p(2t + 1) + S$ are not disjoint which contradicts the Lemma 3.14(ii).

(iii) Let $H = \{v = (v_1,v_2,\ldots,v_k)\}$ where $v_j \in \langle 2t + 1 \rangle$ for all $1 \leq j \leq k$. From Lemma 3.14(iii), we have $(2t + 1)$ is a submodule of $\mathbb{Z}_m$. Consider two vectors $u,v \in H$ where $u + w = (u_1,u_2,\ldots,u_k) + (w_1,w_2,\ldots,w_k) = (u_1 + w_1,u_2 + w_2,\ldots,u_k + w_k)$. Since $u_j,w_j \in \langle 2t + 1 \rangle$, $u_j + w_j \in \langle 2t + 1 \rangle$ for all $j$ and thus, $u + w \in H$. For $a \in \mathbb{Z}_m$, $au \in H$. Hence, $H$ is a submodule of $\mathbb{Z}_m^k$. $\square$
Remark 3.16. If $2t + 1 \mid m - 2t - 1$ then the translates of $S^k$ will not form a partition of $\mathbb{Z}_m^k$.

Now, we will find the $I$-perfect pomset block code $C$ of length $N$ for an ideal $I$ with a partial count. Refer the notations $A_p$ and $A_I$ in Section 3.3 with regard to the ideal $I$. We denote the partial count of each $i \in A_p$ in $I$ as $t_i$. Clearly, $1 \leq t_i \leq \lfloor \frac{m}{2i} \rfloor - 1$ for $1 \leq s \leq I_p$. Suppose that $2t_i + 1 \mid m - 2t_i - 1$ for an $i \in \{i_1, i_2, \ldots, i_p\}$ and $\ell_i = \frac{m - 2t_i - 1}{2t_i + 1}$. The foregoing results connected with $S^k$ and $T^k$ for a specified $t$, can now be used for each $t_i$.

Let $D_i = \{0, \ldots, 0, v_i, 0, \ldots, 0\} \in \mathbb{Z}_m^N : v_i \in T^k_i\}$. Let $v = (0, \ldots, 0, v_i, 0, \ldots, 0)$ and $v' = (0, \ldots, 0, v'_i, 0, \ldots, 0)$ be the two distinct $N$-tuples in $D_i$. Then $B_I(v) = v + B_I$ and $B_I(v') = v' + B_I$ where

$$B_I = \left\{ x_1 \oplus x_2 \oplus \ldots \oplus x_n \in \mathbb{Z}_m^N : x_i = \begin{cases} x_i \in \mathbb{Z}_m^k, & \text{for } i \in I^* - A_p \\ x_i \in S^k_i, & \text{for } i \in A_p \\ 0, & \text{for } i \in [n] - I^* \end{cases} \right\}$$

Since $v_i \neq v'_i$, $v_i + S^k_i$ and $v'_i + S^k_i$ are disjoint by Lemma 3.15(ii). Thus, the $I$-balls centered at distinct $N$-tuples $v$ and $v'$ in $D_i$ are disjoint.

This allows one to construct a set of $N$-tuples such that the $I$-balls centered at vectors in this set are pairwise disjoint. For this, the ideal $I$ with a partial count must be such that $2t_i + 1 \mid m - 2t_i - 1$ for each $i \in A_p = \{i_1, i_2, \ldots, i_p\}$.

Now, define $D_{A_p} = D_{i_1} \oplus D_{i_2} \oplus \ldots \oplus D_{i_p}$ and $D_{[n] - I^*} = \{v_1 \oplus v_2 \oplus \ldots \oplus v_n \in \mathbb{Z}_m^N : v_i = 0 \text{ when } i \in I^*\}$. Note that, for a $v \in D_{A_p}$, $v_i = 0 \forall i \notin A_p$ and for a $v \in D_{[n] - I^*}$, $v_i = 0 \forall i \notin [n] - I^*$.

As $I$-balls centered at the vectors in $D_i$ are disjoint, the $I$-balls centered at the vectors in $D_{A_p}$ are also disjoint.

Theorem 3.17. Let $I$ be an ideal with partial count and $t_j$ be the partial count of $j$ in $I$. If $2t_j + 1 \mid m - 2t_j - 1$ for each $j \in A_p = \{i_1, i_2, \ldots, i_p\}$, then the $I$-balls centered at the $N$-tuples of $D_{A_p}$ =

$$\left\{ v = v_1 \oplus v_2 \oplus \ldots \oplus v_n \in \mathbb{Z}_m^N : v_i \in \mathbb{Z}_m^k \text{ and } v_i = \begin{cases} 0, & \text{for } i \notin A_p \\ v_i \in T^k_i, & \text{for } i \in A_p \end{cases} \right\}$$

are disjoint. Moreover, $|D_{A_p}| = \prod_{j \in A_p} (1 + \ell_j)^{k_j}$ where $\ell_j = \frac{m - 2t_j - 1}{2t_j + 1}$.

Now, we can construct the $I$-perfect pomset block code by considering the direct sum of $D_{A_p}$ and $D_{[n] - I^*}$.

Theorem 3.18. Let $I$ be an ideal with partial count and $t_j$ be the partial count of $j$ in $I$. If $2t_j + 1 \mid m - 2t_j - 1$ for each $j \in A_p = \{i_1, i_2, \ldots, i_p\}$, then $D =

$$\left\{ v = v_1 \oplus v_2 \oplus \ldots \oplus v_n \in \mathbb{Z}_m^N : v_i \in \mathbb{Z}_m^k \text{ and } v_i = \begin{cases} 0, & \text{for } i \notin I^* - A_p \\ v_i \in T^k_i, & \text{for } i \in A_p \\ v_i \in \mathbb{Z}_m^k, & \text{for } i \in [n] - I^* \end{cases} \right\}$$

is an $I$-perfect $(P_m, \pi)$-code of length $N$ over $\mathbb{Z}_m$. Moreover, $|D| = \sum_{k_i \in [n] - I^*} \prod_{j \in A_p} (1 + \ell_j)^{k_j}$ where $\ell_j = \frac{m - 2t_j - 1}{2t_j + 1}$.
4. MDS and I-Perfect Codes with Chain Pomset

Throughout this section, $P = (M, R)$ is considered to be a chain. Then $|\mathcal{J}^r(P)| = 1$ for each $r \leq n(\frac{m}{2})$. For the ideal $I \in \mathcal{J}^r(P)$, $B_I(x) = B(P_{m,\pi})(x, r)$ for any $x \in \mathbb{Z}_m^N$. Moreover, the poset $P = ([n], \preceq)$ induced by the pomset $P$ is also a chain. Each ideal in $P$ or in $P$ has a unique maximal element. Let $c_{i_t}/i_t$ be the maximal element of $\langle \text{supp}(P_{m,\pi}(x)) \rangle$. Then,

$$w(P_{m,\pi})(x) = c_{i_t} + (|\{c_{i_t}/i_t\}^*| - 1)|\frac{m}{2}|.$$ 

In this section, the Singleton bound and the packing radius for pomset block codes are obtained. Furthermore, we look into the connection between MDS pomset block codes and I-Perfect (r-perfect) pomset block codes.

**Lemma 4.1.** Let $P$ be a chain pomset. Then a code $C \subseteq \mathbb{Z}_m^N$ is an r-perfect $(P_m, \pi)$-code if and only if $C$ is an I-perfect $(P_m, \pi)$-code for the ideal $I \in \mathcal{J}^r(P)$.

Let $r = t(\frac{m}{2}) + s$ where $s \in \{1, 2, \ldots, \frac{m}{2}\}$ and $t \geq 0$. Let $B(P_{m,\pi})(x, r)$ and $B(P,\pi)(x, r)$ denote the $r$-balls centered at $x$ having radius $r$ with respect to the $(P_m, \pi)$-metric and $(P, \pi)$-metric respectively. If $y \in B(P,\pi)(x, r)$, then $w(P)(x - y) = (\text{supp}(x - y)) \leq t + 1$. Hence, $y \in B(P,\pi)(x, t + 1)$. Thus, we have

**Theorem 4.2.** Let $x \in \mathbb{Z}_m^N$ and $w$ be a weight on $\mathbb{Z}_m$. If $r = t(\frac{m}{2}) + s$, $1 \leq s \leq \frac{m}{2}$ and $t \geq 0$, then $B(P_m,\pi)(x, r) \subseteq B(P,\pi)(x, t + 1)$. Furthermore, $B(P_m,\pi)(x, r) = B(P,\pi)(x, t)$ iff $r = t(\frac{m}{2})$.

**Definition 4.3.** The packing radius of a code $C$ with respect to any metric $d$ is the greatest integer $r$ such that the $r$-balls centered at any two distinct codewords are disjoint.

Let $R(P_{m,\pi})(C)$ and $R(P,\pi)(C)$ denote the packing radius of the code $C \subseteq \mathbb{Z}_m^N$ with respect to the $(P_m, \pi)$-metric and $(P, \pi)$-metric respectively. Let $R(P,\pi)(C) = t$ and $R_s = (t - 1)(\frac{m}{2}) + s$ where $1 \leq s \leq \frac{m}{2}$. Then, from Theorem 4.2, $B(P_m,\pi)(x, R_s) \subseteq B(P,\pi)(x, t)$ for each positive integer $s \leq \frac{m}{2}$. Hence, $R(P_{m,\pi})(C) \geq R_s$ for every $0 < s \leq \frac{m}{2}$. If $s = \frac{m}{2}$, then from Theorem 4.2, $R(P,\pi)(C) = R(P_m,\pi)(C) = t(\frac{m}{2})$. Thus, packing radius of any $(P_m, \pi)$-block code $C$ is $R(P_{m,\pi})(C) = \lfloor \frac{m}{2} \rfloor R(P,\pi)(C)$. As seen in [14] (Theorem 5), $R(P,\pi)(C) = d(P,\pi)(C) - 1$ when $P$ is a chain. Thus, packing radius of any $(P_m, \pi)$-block code $C$ is $R(P_{m,\pi})(C) = \lfloor \frac{m}{2} \rfloor (d(P,\pi)(C) - 1)$.

**Corollary 4.4** (Packing radius). Packing radius of a $(P_m, \pi)$-block code $C \subseteq \mathbb{Z}_m^N$ is $R(P_{m,\pi})(C) = \lfloor \frac{m}{2} \rfloor (d(P,\pi)(C) - 1)$.

When $P$ is a chain, the Singleton bound [7] for a $(P, \pi)$-code $C$ is $\sum_{j \in J} k_j \leq N - \lfloor \log_m |C| \rfloor$ where $J$ is an ideal in $P$ with $|J| = d(P,\pi)(C) - 1$. Since the poset $P$ is induced by the pomset $P$, there exist an ideal $I \in P$ with $|I| \leq d(P_{m,\pi})(C) - 1$ and $|I^*| = \lfloor \frac{d(P_{m,\pi})(C) - 1}{\frac{m}{2}} \rfloor$. From Proposition 3.1, we have $|I^*| \leq |J|$ and $I^* \subseteq J$ as $P$ is a chain. Thus, $\sum_{i \in I^*} k_i \leq \sum_{j \in J} k_j$. Hence, we have:

**Theorem 4.5** (Singleton bound). Let $C$ be a chain pomset block code of length $N$ over $\mathbb{Z}_m$ with minimum distance $d(P_{m,\pi})(C)$. Then max $\sum_{i \in I^*} k_i \leq N - \lfloor \log_m |C| \rfloor$.
Theorem 4.10. Let $C$ be a pomset block code of length $N$ over $\mathbb{Z}_m$. Then the following holds:

(i) if $k_i = k$ for all $i \in [n]$ then $\left\lfloor \frac{d_{(P_m, \pi)}(C) - 1}{k} \right\rfloor \leq n - \left\lfloor \log_m |C| \right\rfloor$.

(ii) if $k_1 \geq k_2 \geq \ldots \geq k_n$ then $rk_n \leq N - \left\lfloor \log_m |C| \right\rfloor$ and $rk_n \leq \sum_{i=1}^{n} k_i \leq rk_1$.

(iii) if $k_i = 1$ for all $i \in [n]$, then the Singleton bound for a pomset block code becomes that for a pomset code; that is, $\left\lfloor \frac{d_{(P_m, \pi)}(C) - 1}{k} \right\rfloor \leq n - \left\lfloor \log_m |C| \right\rfloor$.

Definition 4.7. A pomset block code $C \subseteq \mathbb{Z}_m^N$ of length $N$ over $\mathbb{Z}_m$ is said to be maximum distance separable (MDS) if it attains its Singleton bound.

Theorem 4.8. Let $k_i = k \forall i \in [n]$ and $C$ be a $(P_m, \pi)$-code of length $N$ over $\mathbb{Z}_m$ with minimum distance $d_{(P_m, \pi)}(C)$. If $C$ is MDS then $\left\lfloor \frac{m}{n-k} \right\rfloor \left( n - \left\lfloor \log_m |C| \right\rfloor \right) + 1 \leq d_{(P_m, \pi)}(C) \leq \left\lfloor \frac{m}{n-k} \right\rfloor \left( n - \left\lfloor \log_m |C| \right\rfloor \right) + 1$.

Proof. Since $C$ is MDS and $k_i = k \forall i \in [n]$, we have $\left\lfloor \frac{d_{(P_m, \pi)}(C) - 1}{k} \right\rfloor \leq n - \left\lfloor \log_m |C| \right\rfloor$. Thus, $n - \left\lfloor \log_m |C| \right\rfloor \leq d_{(P_m, \pi)}(C) \leq n - \left\lfloor \log_m |C| \right\rfloor + 1$. Hence, $\left\lfloor \frac{m}{n-k} \right\rfloor \left( n - \left\lfloor \log_m |C| \right\rfloor \right) + 1 \leq d_{(P_m, \pi)}(C) \leq \left\lfloor \frac{m}{n-k} \right\rfloor \left( n - \left\lfloor \log_m |C| \right\rfloor \right) + 1$. Therefore, if $k_i = k \forall i \in [n]$ and $C$ is a $(P_m, \pi)$-code of length $N$ over $\mathbb{Z}_m$ with minimum distance $d_{(P_m, \pi)}(C)$, then $C$ cannot be an MDS pomset code whenever $1 \leq d_{(P_m, \pi)}(C) \leq \left\lfloor \frac{m}{n-k} \right\rfloor \left( n - \left\lfloor \log_m |C| \right\rfloor \right)$ or $d_{(P_m, \pi)}(C) > \left\lfloor \frac{m}{n-k} \right\rfloor \left( n - \left\lfloor \log_m |C| \right\rfloor \right) + 1$.

Now we will examine the maximum distance separability of codes with respect to $(P_m, \pi)$-metric and $(P, \pi)$-metric.

Theorem 4.9. If $C$ is an MDS code with respect to $(P_m, \pi)$-metric then $C$ is an MDS code with respect to $(P, \pi)$-metric.

Proof. Suppose that $C$ is MDS with respect to $(P_m, \pi)$-metric. Then $\sum_{i \in I} k_i = N - \left\lfloor \log_m |C| \right\rfloor$, where $|I^*| = \left\lfloor \frac{d_{(P_m, \pi)}(C) - 1}{k} \right\rfloor$. Since $|I^*| \leq d_{(P, \pi)}(C) - 1$ by Proposition 3.1, we have $\sum_{i \in I^*} k_i \leq \sum_{i \in J} k_i$ for an ideal $J$ in $P$ with $|J| = d_{(P, \pi)}(C) - 1$ as $I^* \subseteq J$. Thus, $N - \left\lfloor \log_m |C| \right\rfloor \leq \sum_{i \in J} k_i$. Hence $C$ is MDS with respect to $(P, \pi)$-metric.

Theorem 4.10. Let $k_i = k$ for each $i \in [n]$ and $C$ be a $(P_m, \pi)$-code of length $N$ over $\mathbb{Z}_m$ with cardinality $m^s$ for some $s > 0$. If $C$ is MDS then $C$ is $I$-perfect for all $I \in \mathcal{F}(\mathbb{P})^{n-2}(P)$.

Proof. Let $C$ be an MDS $(P_m, \pi)$-code. As $P$ is a chain, $|\mathcal{F}(\mathbb{P})^{n-2}(P)| = 1$ and so $I \in \mathcal{F}(\mathbb{P})^{n-2}(P)$ must be an ideal with full count with $I^* = n - \frac{k}{\omega}$. As $\mathbb{Z}_m^N$ can be partitioned into $I$-balls, let $\ell$ be the number of $I$-balls in this partition. Then $\ell|B_I| = m^s$ which gives $\ell = |C|$ as $|B_I| = m^{|I^*|}$. Since $C$ is MDS, $\left\lfloor \frac{d_{(P_m, \pi)}(C) - 1}{k} \right\rfloor = n - \frac{k}{\omega}$ by Corollary 4.6 (1) and thus $d_{(P_m, \pi)}(C) > |I|$. Therefore, any two $I$-balls...
centered at distinct codewords of $C$ must be disjoint and $|C||B_I| = m^N$. Hence $C$ is $I$-perfect. 

**Theorem 4.11.** Let $k_i = k$ for each $i \in [n]$ and $C \subseteq \mathbb{Z}_m^n$ be a $(P_m, \pi)$-code of length $N$ over $\mathbb{Z}_m$. If $C$ is $I$-perfect for the ideal $I \in \mathfrak{A}(\mathbb{P}; n - \lceil \log_m |C| \rceil)(\mathbb{P})$ then $C$ is MDS.

**Proof.** If $C$ is $I$-perfect for the ideal $I \in \mathfrak{A}(\mathbb{P}; n - \lceil \log_m |C| \rceil)(\mathbb{P})$, then $\frac{m}{k}|n - \lceil \log_m |C| \rceil|$, is an integer and $d(P_m, \pi)(C) > \lceil \frac{m}{k}|n - \lceil \log_m |C| \rceil| \rceil + 1$. Thus, $\lceil \frac{d(P_m, \pi)(C) - 1}{k} \rceil \geq n - \lceil \log_m |C| \rceil$. Hence $C$ is MDS. 

**Theorem 4.12.** Let $k_i = k$ for each $i \in [n]$. Then every $I$-perfect $(P_m, \pi)$-code over $\mathbb{Z}_m$ is MDS.

**Proof.** Let $C$ be an $I$-perfect $(P_m, \pi)$-code. Then $d(P_m, \pi)(C) > |I|$. If $I$ is an ideal with full count then $|B_I| = m^{|I|}$, $|C||B_I| = m^N$ and thus, we have $\sum_{i \in I} k_i = N - \lceil \log_m |C| \rceil$. Hence $C$ is an MDS block code. Now, if $I$ is an ideal with partial count then $|I| = |I'| + t$ for some ideal $I'$ with full count and $0 < t \leq \lceil \frac{m}{k} \rceil - 1$. Since $C$ is an $I$-perfect code, $|C||B_I| = m^N$, which means $|C|(2t + 1)|B_I| = m^N$. Taking log to the base $m$ both sides we get, $\sum_{j \in I'} k_j = nk - \log_m |C| - k \log_m (2t + 1)$ so that 

$$|I'| = n - \frac{\log_m |C|}{k} - \log_m (2t + 1).$$

Since $0 < t \leq \lceil \frac{m}{k} \rceil - 1$, we get $0 < \log_m (2t + 1) < 1$. Since $\log_m |C| + k \log_m (2t + 1)$ is an integer, then 

$$\frac{\log_m |C|}{k} + \log_m (2t + 1) = \frac{\log_m |C|}{k} + \log_m (2t + 1) \leq \frac{\log_m |C|}{k} + \log_m (2t + 1) + 1 = \frac{\log_m |C|}{k} + 1 = \frac{\log_m |C|}{k}.$$ 

As $d(P_m, \pi)(C) > |I| = |I'| + t$, we have $\frac{d(P_m, \pi)(C) - 1}{k} > n - \frac{\log_m |C|}{k} - \log_m (2t + 1) + 1 \geq n - \frac{\log_m |C|}{k} - \log_m (2t + 1) \geq n - \frac{\log_m |C|}{k}$. Thus, $\lceil \frac{d(P_m, \pi)(C) - 1}{k} \rceil \geq n - \lceil \frac{\log_m |C|}{k} \rceil$. Hence, $C$ is MDS.

For a chain poset, the following is the duality theorem of an MDS $(P_m, \pi)$-code when all the blocks are of same length.

**Theorem 4.13 (Duality Theorem).** Let $\hat{\mathbb{P}}$ be the dual poset of the chain $\mathbb{P}$ on $M$ and $\pi$ be a labeling of $[n]$ with $k_1 = k_2 = \ldots = k_n = k$. Let $C$ be an $(P_m, \pi)$-code of length $N$ over $\mathbb{Z}_m$ with cardinality $m^s$ for some $s > 0$. Then the following statements are equivalent:

1. $C$ is an MDS $\mathbb{P}$-code.
2. $C$ is an $I$-perfect $\mathbb{P}$-code for all $I \in \mathfrak{S}(\hat{\mathbb{P}}; |n - \hat{\pi}|)(\hat{\mathbb{P}})$.
3. $C^\perp$ is an $I^\perp$-perfect $\hat{\mathbb{P}}$-code for all $I^\perp \in \mathfrak{S}(\hat{\mathbb{P}}; |\hat{\pi}|)(\hat{\mathbb{P}})$.
4. $C^\perp$ is an MDS $\hat{\mathbb{P}}$-code of cardinality $m^{N-s}$.

**Proof.** The proof is straightforward from Theorem 4.10 and Theorem 4.11. 

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