INVERSE SOURCE PROBLEM FOR A ONE-DIMENSIONAL 
TIME-FRACTIONAL DIFFUSION EQUATION AND UNIQUE 
CONTINUATION FOR WEAK SOLUTIONS

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Abstract. In this paper, we obtain the sharp uniqueness for an inverse x-source problem for a one-dimensional time-fractional diffusion equation with a zeroth-order term by the minimum possible lateral Cauchy data. The key ingredient is the unique continuation which holds for weak solutions.

1. Introduction and main results

As a representative of various nonlocal models, time-fractional diffusion equations have attracted consistent attention of multidisciplinary researchers owing to their capability in describing anomalous diffusion (e.g., [27]). In the last decades, fundamental theory has been developed rapidly for time-fractional diffusion equations, represented by the fundamental solution and the well-posedness results established e.g. in [4,11]. Remarkably, it reveals in [19,24] that time-fractional diffusion equations inherit the time analyticity and the maximum principle from their integer counterparts. On the contrary, they differ considerably in view of the long-time asymptotic behavior (see [24]).

In contrast to the above results, some issues remain open in the uniqueness of the lateral Cauchy problem and the unique continuation for time-fractional diffusion equations. As is known, these two properties essentially characterize parabolic equations in the sense of the infinite propagation speed of local information of homogeneous equations (see [22]). However, for time-fractional diffusion equations, most literature only obtained weak unique continuation because additional assumptions were required on the boundary or at the initial time (see [3,8,10,14,15,24,26]).

In this article, we are concerned with an inverse source problem for a one-dimensional time-fractional diffusion equation with a potential term, which is described tentatively:

\[(1.1) \quad d_t^\alpha y(x,t) - y_{xx}(x,t) + p(x)y = \rho(t)f(x) \quad \text{in } (0,1) \times (0,T).\]

Here the Caputo derivative \(d_t^\alpha\) is defined by

\[d_t^\alpha w(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} \frac{dw}{d\tau}(\tau) \, d\tau \quad \text{for } w \in C^1[0,T] \text{ or } W^{1,1}(0,T),\]

where \(0 < \alpha < 1\) and \(\Gamma(\cdot)\) is the gamma function.

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Formulation (1.1) is quite conventional, but the definition of $d_\alpha^t$ requires the differentiability of $y$ in $t$. On the other hand, if for $y$, we require the $H^2(0, 1)$-regularity in $x$ and $L^2(0, T)$-regularity in $t$, then $d_\alpha^t y$ should be in $L^2$-space in $x$ and $t$, which is interpreted to be weaker regularity than the one-time differentiability in time. Moreover, for $\rho f \in L^2((0, 1) \times (0, T))$, it is required that $d_\alpha^t y \in L^2((0, 1) \times (0, T))$. Thus, for such a class of solutions, we are suggested to exploit the class of functions $w = w(t)$ such that $d_\alpha^t w \in L^2(0, T)$, and the class $W^{1,1}(0, T)$ apparently seems narrow for this requirement. Therefore, we start to reformulate the time-fractional derivative in adequate Sobolev spaces. We emphasize that such formulated time-fractional derivatives allow us to work within the regularity specified by (1.5) below.

First we set

$$0C^1[0, T] := \{ w \in C^1[0, T]; w(0) = 0 \}.$$ 

We consider the Caputo derivative $d_\alpha^t w(t)$ ($0 < \alpha < 1$) as an operator from $0C^1[0, T]$ to $L^2(0, T)$. In other words, by setting $\mathcal{D}(d_\alpha^t) = 0C^1[0, T]$, we define the domain $\mathcal{D}(d_\alpha^t)$. The operator $d_\alpha^t$ with domain $0C^1[0, T]$ is not a closed operator, but it admits a unique minimum closed extension, which is denoted by $\partial_\alpha^t$ (e.g., Kubica, Ryszewska and Yamamoto [11]). We can characterize the domain $\mathcal{D}(\partial_\alpha^t)$ as follows. We recall the Sobolev-Slobodecki space $H^\alpha(0, T)$ with the norm $\| \cdot \|_{H^\alpha(0, T)}$ is defined by

$$\|w\|_{H^\alpha(0, T)} := \left( \|w\|_*^2_{L^2(0, T)} + \int_0^T \int_0^T \frac{|w(t) - w(\tau)|^2}{|t - \tau|^{1+2\alpha}} \, dt \, d\tau \right)^{\frac{1}{2}}$$

(see e.g. Adams [1]). We define the Riemann-Liouville integral operator $J^\beta$ for $\beta > 0$ as

$$J^\beta w(t) := \frac{1}{\Gamma(\beta)} \int_0^t (t - \tau)^{\beta-1} w(\tau) \, d\tau, \quad w \in L^2_{\text{loc}}(0, +\infty).$$

Setting

$$H^\alpha(0, T) := \overline{0C^1[0, T]}^{H^\alpha(0, T)},$$

we see that

$$H^\alpha(0, T) = \begin{cases} 
H^\alpha(0, T), & 0 < \alpha < \frac{1}{2}, \\
\{ w \in H^{\frac{1}{2}}(0, T); \int_0^T \frac{|w(t)|^2}{t} \, dt < \infty \}, & \alpha = \frac{1}{2}, \\
\{ w \in H^\alpha(0, T); w(0) = 0 \}, & \frac{1}{2} < \alpha < 1
\end{cases}$$

and

$$\|w\|_{H^\alpha(0, T)} = \begin{cases} 
\|w\|_{H^\alpha(0, T)}, & \alpha \neq \frac{1}{2}, \\
\left( \|w\|_{H^{\frac{1}{2}}(0, T)}^2 + \int_0^T \frac{|w(t)|^2}{t} \, dt \right)^{\frac{1}{2}}, & \alpha = \frac{1}{2}
\end{cases}$$

Then it is known that (see e.g. [6,11])

$$H^\alpha(0, T) = J^\alpha L^2(0, T), \quad 0 < \alpha < 1.$$
The minimum closed extension $\partial_t^\alpha$ of the operator $d_t^\alpha$ with the domain $\mathcal{D}(d_t^\alpha) = \partial C^1[0, T]$ satisfies

$$\partial_t^\alpha = (J^\alpha)^{-1}, \quad \mathcal{D}(\partial_t^\alpha) = H_\alpha(0, T)$$

and there exists a constant depending only on $\alpha$ such that

$$C^{-1}\|w\|_{H_\alpha(0, T)} \leq \|\partial_t^\alpha w\|_{L^2(0, T)} \leq C\|w\|_{H_\alpha(0, T)} \quad \text{for all } w \in H_\alpha(0, T)$$

(see Gorenflo, Luchko and Yamamoto [3], Kubica, Ryszewska and Yamamoto [11], Yamamoto [27]).

Throughout this article, instead of the time-fractional diffusion equation (1.1), we consider

$$(1.2) \quad \partial_t^\alpha y(x, t) - y_{xx}(x, t) + p(x)y(x, t) = \rho(t)f(x) \quad \text{in } (0, 1) \times (0, T).$$

Here $\rho$ and $f$ stand for the temporal and spatial components of the source term, respectively.

In the sequel, we set $F(x, t) = \rho(t)f(x)$ or $F \equiv 0$, and

$$a \in L^2(0, 1), \quad F \in L^2(0, T; L^2(0, 1)), \quad p \in L^\infty(0, 1).$$

In general, we define a solution to a time-fractional diffusion equation with initial value $a \in L^2(0, 1)$ as follows:

$$(1.3) \quad \partial_t^\alpha(u - a)(x, t) - u_{xx}(x, t) + p(x)u(x, t) = F(x, t) \quad \text{in } L^2(0, T; H^{-1}(0, 1))$$

and

$$(1.4) \quad u - a \in H_\alpha(0, T; H^{-1}(0, 1)), \quad u \in L^2(0, T; H^1(0, 1)),$$

where $H^{-1}(0, 1) = (H^1_0(0, 1))'$ is the dual space of $H^1_0(0, 1)$. Here we remark that $u \in L^2(0, T; H^1(0, 1))$ implies $u_{xx} \in L^2(0, T; H^{-1}(0, 1))$. Indeed, for almost all $t \in (0, T)$, since $u_x(\cdot, t) \in L^2(0, 1)$, we see

$$H^{-1}(0, 1) \langle u_{xx}(\cdot, t), \phi \rangle_{H^1_0} = -\langle u_x, \phi \rangle_{L^2(0, 1)} \quad \text{for all } \phi \in H^1_0(0, 1),$$

and

$$|H^{-1}(0, 1) \langle u_{xx}(\cdot, t), \phi \rangle_{H^1_0}| \leq \|u_x(\cdot, t)\|_{L^2(0, 1)} \|\phi\|_{H^1_0(0, 1)}.$$  

Therefore, $u_{xx}(\cdot, t)$, whose derivative is taken in the sense of distribution, can define a bounded linear functional on $H^1_0(0, 1)$, that is, $u_{xx}(\cdot, t) \in H^{-1}(0, 1)$.

This class defined by (1.4) is compatible with the function space for the initial-boundary value problem. For example, attaching (1.3) with the boundary condition $u(0, t) = u(T, t) = 0$ for $0 < t < T$, if $F \in L^2(0, T; H^{-1}(0, 1))$, then we can prove the unique existence of solution to the initial boundary value problem within the above class (e.g., [11]).

Since $H_\alpha(0, T)$ is the closure of the set of $C^1[0, T]$ of $C^1$-functions vanishing at $t = 0$ by the norm of $H^\alpha(0, T)$, we can interpret the first regularity condition in (1.4) as generalized initial condition. In particular, for $\frac{1}{2} < \alpha < 1$, in view of the Sobolev embedding $H_\alpha(0, T) \subset H^\alpha(0, T) \subset C([0, T])$, if $u - a \in H_\alpha(0, T; H^{-1}(0, 1))$, then $u - a \in C([0, T]; H^{-1}(0, 1))$, and so $u$ satisfies the initial condition $u(\cdot, 0) = a$ in $H^{-1}(0, 1)$.

Throughout this article, we assume that a solution $y$ to (1.2) satisfies

$$(1.5) \quad y \in H_\alpha(0, T; H^1(0, 1)).$$

We recall that we consider the zero initial condition in the sense of $y \in H_\alpha(0, T; H^1(0, 1))$.
In addition to the regularity (1.4), for any non-empty open interval $I$ such that $I \subset (0, 1)$, we can prove

$$J^\alpha u \in L^2(0, T; H^2(I)).$$

In particular, the trace theorem yields

$$(J^\alpha u)_x(x_0, \cdot) = (J^\alpha u_x)(x_0, \cdot) \in L^2(0, T)$$

for arbitrary $x_0 \in (0, 1)$. For completeness, we provide the proof of (1.6) in Appendix A.

For (1.2), our target of this article is the uniqueness for the following inverse source problem:

**Problem.** Fix constants $T > 0$ and $x_0 \in (0, 1)$. Let $y \in H_\alpha(0, T; H^1(0, 1))$ satisfy (1.2) with $p \in L^\infty(0, 1)$. Can we uniquely determine $f \in L^2(0, 1)$ by data $y(x_0, \cdot)$ and $(J^\alpha y)_x(x_0, \cdot)$ in $(0, T)$, provided that $\rho$ is given suitably?

By (1.6) and the trace theorem, we note that the data $(J^\alpha y)_x(x_0, \cdot)$ can make sense as a function in $L^2(0, T)$. For a sufficiently smooth initial value $\alpha$, one can observe $y(x_0, t)$ and $y_x(x_0, t)$ in real applications, which means the data of the concentration and its rate of change at a single point. For $a \in L^2(0, 1)$, $y_x(x_0, t)$ does not make sense in $L^2(0, T)$, and more practical observation data can be taken in $I \times (0, T)$ with small open interval $I$ including $x_0$.

In the above problem, by the term $\partial_y^\alpha$, we emphasize that we can consider the zero initial value. On the other hand, boundary values are unknown in the above problem, and we just treat any function $y \in H_\alpha(0, T; H^1(0, 1))$ satisfying (1.2). Therefore, the above problem requires the unique determination of $f(x)$ without data on the lateral boundary $\{0, 1\} \times (0, T)$. Our first main result is concerned with such sharp uniqueness:

**Theorem 1.1.** Fix constants $T > 0$ and $x_0 \in (0, 1)$ arbitrarily. We assume that $y \in H_\alpha(0, T; H^1(0, 1))$ satisfies (1.2) with $p \in L^\infty(0, 1)$, $f \in L^2(0, 1)$ and $\rho \in H^1(0, T)$ satisfying $\rho(0) \neq 0$. Then $J^\alpha y(x_0, \cdot) = (J^\alpha y)_x(x_0, \cdot) = 0$ in $(0, T)$ implies $f \equiv 0$ in $(0, 1)$.

Since $y(x_0, \cdot) \in L^2(0, T)$, we see that $y(x_0, \cdot) = 0$ in $(0, T)$ if and only if $J^\alpha y(x_0, \cdot) = 0$ in $(0, T)$.

With arbitrarily chosen point $x_0 \in (0, 1)$, only two $t$-dependent functions $y(x_0, \cdot)$ and $(J^\alpha y)_x(x_0, \cdot)$ are available for determining $f$. In particular, we do not need the boundary values, which are required in most literature. This turns out to be novel compared with all existing results on inverse problems for time-fractional diffusion equations (see [17] and the references therein).

On the other hand, let us consider

$$y(x, t) = \frac{1}{4\pi^2} (E_{\alpha,1}(-4\pi^2 t^\alpha) - 1) \sin 2\pi x, \quad 0 < x < 1, \quad t > 0,$$

where $E_{\alpha,\beta}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}$ ($\beta > 0$) is the Mittag-Leffler function defined for $z \in \mathbb{C}$. It is known that $E_{\alpha,1}(z)$ is an entire function in $z \in \mathbb{C}$ and $\partial_t^\alpha (E_{\alpha,1}(-4\pi^2 t^\alpha) - 1) = -4\pi^2 E_{\alpha,1}(-4\pi^2 t^\alpha)$ for $t > 0$ (e.g., [20]). Then we can directly verify that

$$\partial_t^\alpha y(x, t) - y_{xx}(x, t) = \rho_0(t)f_0(x), \quad 0 < x < 1, \quad t > 0,$$
where \( \rho_0(t) = -1 \) and \( f_0(x) = \sin 2\pi x \), and that \( y(\frac{1}{t}, t) = 0 \) for \( t > 0 \) in spite of \( f_0 \not= 0 \). This example indicates that only data \( y(x_0, \cdot) \) does not yield the uniqueness even though we have the zero boundary data \( y(0, t) = y(1, t) = 0 \) for \( t > 0 \). In this sense, in Theorem 1.1 data \( y(x_0, \cdot) \) and \( y_x(x_0, \cdot) \) can be considered as the minimum possible for the uniqueness.

In order to prove Theorem 1.1 we need the uniqueness for the lateral Cauchy problem with non-zero initial value \( a \):

\[
\partial_t^\alpha (u(x, t) - a(x)) - u_{xx}(x, t) + p(x)u(x, t) = 0 \quad \text{in} \quad (0, 1) \times (0, T).
\]

Thus we can state our second main result:

**Theorem 1.2.** Fix constants \( T > 0 \) and \( x_0 \in (0, 1) \), \( p \in L^\infty(0, 1) \) arbitrarily. Let \( u \in L^2(0, T; H^1(0, 1)) \) satisfy \( u - a \in H_\alpha(0, T; H^{-1}(0, 1)) \) and \( (1.7) \) with some \( a \in L^2(0, 1) \). Then \( J^\alpha u(x_0, \cdot) = 0 \) in \( (0, T) \) implies \( u = 0 \) in \( (0, 1) \times (0, T) \) and \( a = 0 \) in \( (0, 1) \).

Here by (1.6), we note that \( J^\alpha u \in L^2(0, T; H^2(I)) \) with open interval \( I \) such that \( \bar{I} \subset (0, 1) \) and so again \( J^\alpha u(x_0, \cdot) \in L^2(0, T) \).

In Theorem 1.2 we notice that not only boundary values but also an initial value \( a \) are unknown, and the theorem concludes \( a = 0 \) in \( (0, 1) \) as well as \( u = 0 \) in \( (0, 1) \times (0, T) \). The same result holds for one-dimensional parabolic equations, which is an immediate corollary of the well-known uniqueness of the lateral Cauchy problem (see e.g. Isakov [9]) by treating \((0, x_0) \times (0, T)\) and \((x_0, 1) \times (0, T)\) separately.

The observation point \( x_0 \) in Theorems 1.1, 1.2 is restricted to the open interval \((0, 1)\) because the proof of Theorem 1.2 relies on the interior regularity theory in Gilbarg and Trudinger [5]. The generalization to allowing a boundary point \( x_0 \in \{0, 1\} \) seems not trivial and we will not discuss this issue in this article.

The following corollary is an immediate consequence of Theorem 1.2.

**Corollary 1.3** (classical unique continuation). Choose a constant \( T > 0 \) and a nonempty open interval \( I \subset (0, 1) \) arbitrarily. Let \( u \in L^2(0, T; H^1(0, 1)) \) satisfy \( u - a \in H_\alpha(0, T; H^{-1}(0, 1)) \) and (1.7) with \( p \in L^\infty(0, 1) \). Then \( u = 0 \) in \( I \times (0, T) \) implies \( u \equiv 0 \) in \( (0, 1) \times (0, T) \).

In the case of \( p \equiv 0 \), the uniqueness as in Theorem 1.2 and Corollary 1.3 was proved in Li and Yamamoto [13]. The proof in [13] is not applicable directly to our case \( p \not= 0 \), and moreover [13] requires the higher regularity for the solution \( u \).

As important contribution to the unique continuation for time-fractional partial differential equations, we refer to Lin and Nakamura [16], which proves the uniqueness for more general time-fractional partial differential equations with order \( \alpha \in (0, 1) \cup (1, 2) \) in a bounded domain \( \Omega \subset \mathbb{R}^d \). In [16], the unique continuation was proved under assumptions that the coefficients of the equations are in \( C^\infty \)-class and solutions \( u \) are strong solutions, that is, satisfy

\[
u \in L^2(0, T; H^2(\Omega)) \cap H^\alpha(0, T; L^2(\Omega)),
\]

while we establish the unique continuation within the weaker regularity (1.4). Moreover, for \( 0 < \alpha < \frac{1}{2} \), the space \( H^\alpha(0, T; L^2(\Omega)) \) is not in \( C([0, T]; L^2(\Omega)) \) and so the initial condition requires special cares even if initial values are known. In particular, the regularity
$H^\alpha(0, T; L^2(\Omega))$ itself does not justify any initial conditions for $\alpha < \frac{1}{2}$. As for the formulation of initial condition, we remark that in [16], the fractional derivative $\partial_\alpha^t$ is defined through some extension in $t$ of $u$ from $0 < t < T$ to $-\infty < t < \infty$, and their formulation is not the same as ours. Thus we should understand that our result and the one in [16] are independent, as long as we are limited to the one-dimensional case.

The remainder of this article is organized as follows. In Section 2, we collect some necessary ingredients for proving the main results. Then Sections 3 and 4 are devoted to the proofs of Theorems 1.2 and 1.1, respectively. Concluding remarks will be provided in Section 5, and Appendix A is devoted to the verifications of some technical details.

2. Preliminaries

First we fix some frequently used notations. We set $\mathbb{R}_+ := (0, +\infty)$ and denote the Laplace transform of $w \in L_{1\text{loc}}(\mathbb{R}_+)$ by

$$\hat{w}(s) := \int_{\mathbb{R}_+} w(t) e^{-st} \, dt, \quad s > s_0,$$

provided that the integral converges for some constant $s_0 > 0$. Henceforth we write $\varphi'(x) = \frac{d\varphi}{dx}(x)$, etc. if there is no fear of confusion.

Let $\delta > 0$ be arbitrarily fixed. For $z \in \mathbb{C}$, by $\varphi(x, z)$ we denote the solution to the following initial value problem for a second order ordinary differential equation

$$\begin{cases} -\varphi'' + p(x)\varphi = z^2\varphi, \quad x > \delta, \\ \varphi(\delta) = 1, \quad \varphi'(\delta) = 0. \end{cases} (2.1)$$

It is known (e.g., [12, Theorem 1.1]) that the solution $\varphi$ is analytic with respect to the parameter $z \in \mathbb{C}$. Moreover, we also have the following asymptotic formulae for $\varphi(x, z)$ as $|z| \to \infty$.

**Lemma 2.1.** The solution $\varphi(x, z)$ to (2.1) admits the asymptotic estimate

$$\varphi(x, z) = O(e^{\text{Im}z|x|}) \quad (|z| \to \infty),$$

and more precisely

$$\varphi(x, z) = \cos(z(x-\delta)) + O\left(\frac{e^{\text{Im}z|x|}}{|z|}\right) \quad \text{as} \quad |z| \to \infty. (2.3)$$

Moreover, if $\text{arg} \ z = \pm \frac{\pi}{2}$, then there exists $\eta_0 > 0$ such that for $|z| > \eta_0$, there holds

$$|\varphi'(x, z)| \leq C|z|e^{\text{Im}z}, \quad x > \delta. (2.4)$$

**Proof.** The first two asymptotic estimates for $\varphi(x, z)$ can be found in [12, Chapter 1, Lemma 2.1]. Then it remains to show the inequality (2.4). Indeed, for $z = i\eta$ with $\eta \in \mathbb{R}$, we integrate both sides of the governing equation in (2.1) from $\delta$ to $x$ and employ the initial condition in (2.1) to deduce

$$\varphi'(x, z) = \int_{\delta}^{x} (p(\xi) + \eta^2)\varphi(\xi, z) \, d\xi.$$
Substituting the asymptotic estimate \( \varphi(x, z) \) for \( \varphi(x, z) \) into the above equation yields
\[
|\varphi'(x, z)| \leq C (\|p\|_{L^\infty(0,1)} + \eta^2) \int_0^x e^{\|\eta\| \xi} \, d\xi = \frac{C(\|p\|_{L^\infty(0,1)} + \eta^2)}{|\eta|} e^{\|\eta\|} \leq C|\eta| e^{\|\eta\|}
\]
for \( |\eta| > \eta_0 \), which completes the proof. \( \square \)

Next, we recall two useful results from the complex analysis.

**Lemma 2.2** (generalized Liouville’s theorem). Assume that \( f \) is an entire function and there exist constants \( N \in \mathbb{N} \) and \( R > 0 \) such that \( |f(z)| \leq C|z|^N \) for \( |z| > R \). Then \( f \) is a polynomial of order at most \( N \).

**Lemma 2.3** (Phragmèn-Lindelöf principle). Fix constants \( \theta_2 > \theta_1 \) and let \( F \) be a holomorphic function in a sector \( S := \{z \in \mathbb{C}; \theta_1 < \arg z < \theta_2\} \). Assume that \( F \) is continuous on the closure of \( S \) and \( |F| \leq 1 \) on the boundary of \( S \). If there exist constants \( \gamma \in (0, \frac{\pi}{\theta_2 - \theta_1}) \) and \( C > 0 \) such that
\[
|F(z)| \leq C \exp(C|z|^\gamma), \quad \forall z \in S,
\]
then \( |F| \leq 1 \) in \( S \).

Lemma 2.2 can be proved by using the series theory for analytic functions, e.g., Rudin [21, Theorem 10.22]. The proof of Lemma 2.3 can be found in Stein and Shakarchi [23].

### 3. Proof of Theorem 1.2

Since \( u(x_0, t) = (J^\alpha u)_x(x_0, t) = 0 \) for \( 0 < t < T \), we see
\[
J^{m\alpha} u(x_0, t) = J^{m\alpha} u_x(x_0, t) = 0, \quad 0 < t < T,
\]
where \( (m - 1)\alpha > \frac{5}{2} \). By \( 0 < x_0 < 1 \), we can choose \( \delta \in (0, x_0) \) arbitrarily. Then it suffices to prove \( u(x, t) = 0 \) for \( \delta < x < x_0 \) and \( 0 < t < T \). Indeed, since \( \delta > 0 \) can be arbitrarily small, we see that \( u(x, t) = 0 \) for \( 0 < x < x_0 \) and \( 0 < t < T \). The proof for \( x_0 < x < 1 \) is similar.

To this end, we divide the proof into 5 steps.

**Step 1.** Since it was assumed \( (m - 1)\alpha > \frac{5}{2} \), it follows from the Sobolev embedding that
\[
H^{(m-1)\alpha}(0,T) \subset C^2[0,T].
\]

We can prove
\[
\partial_t^\alpha (J^{m\alpha} (u - a)) = (J^{m\alpha} u)_{xx} - p J^{m\alpha} u.
\]
The above equation must be understood in the distribution sense. The proof of (3.3) is postponed to Appendix A.

Interpreting \( a \) as a constant function in \( t \), we can verify \( a \in L^2(0,T; L^2(0,1)) \) directly. Hence, since \( J^{m\alpha} u, J^{m\alpha} a \in H_\alpha(0,T; L^2(0,1)) \) by \( u \in L^2(0,T; H^1(0,1)) \subset L^2(0,T; L^2(0,1)) \), we see that
\[
\partial_t^\alpha (J^{m\alpha} (u - a)) = \partial_t^\alpha (J^{m\alpha} u - J^{m\alpha} a) = \partial_t^\alpha J^{m\alpha} u - \partial_t^\alpha J^{m\alpha} a = \partial_t^\alpha J^{m\alpha} u - J^{(m-1)\alpha} a.
\]

Therefore, setting
\[
v := J^{m\alpha} u \in J^{m\alpha} L^2(0,T; H^1(0,1)),
\]
by (3.3) we obtain

\begin{equation}
\begin{cases}
\partial_t^\alpha v = v_{xx} - p v + J^{(m-1)\alpha} a & \text{in } (0, 1) \times (0, T), \\
v \in H_{\alpha}(0, T; L^2(0, 1)).
\end{cases}
\label{eq:3.5}
\end{equation}

In addition to \( H_{\alpha}(0, T) \), we need spaces \( H_{\ell+\sigma}(0, T) \) with \( \ell \in \mathbb{N} \) and \( \sigma \in (0, 1) \), which is defined by

\[ H_{\ell+\sigma}(0, T) := \left\{ w \in H^{\ell+\sigma}(0, T); w(0) = \frac{dw}{dt}(0) = \cdots = \frac{d^{\ell-1}w}{dt^{\ell-1}}(0) = 0 \right\}. \]

Then we can readily verify that \( H_{\ell+\sigma}(0, T) = J^{\ell+\sigma}L^2(0, T) \) by \( H_{\sigma}(0, T) = J^\sigma L^2(0, T) \) for \( 0 < \sigma < 1 \).

By (3.4), we see

\[ v \in H_{m\alpha}(0, T; H^1(0, 1)), \quad \partial_t^\alpha v \in H_{(m-1)\alpha}(0, T; H^1(0, 1)). \]

Hence, (3.5) yields

\begin{equation}
v_{xx} - p(x)v = \partial_t^\alpha v - J^{(m-1)\alpha} a \in H_{(m-1)\alpha}(0, T; L^2(0, 1)).
\label{eq:3.6}
\end{equation}

With (3.6), noting \((\delta, x_0) \subset (0, 1)\) and using \( v \in H_{(m-1)\alpha}(0, T; H^1(0, 1)) \), we apply the interior regularity for an elliptic operator \( \frac{d^2}{dx^2} - p(x) \) (e.g., Gilbarg and Trudinger [5, Theorem 8.8]), so that

\begin{equation}v \in H_{(m-1)\alpha}(0, T; H^2(\delta, x_0)).\label{eq:3.7}\end{equation}

Applying the trace theorem to (3.7), by (3.2) we obtain

\[ g := v_x(\delta, \cdot) \in H_{(m-1)\alpha}(0, T) \subset C^2[0, T]. \]

Consequently, we obtain \( g \in H_{(m-1)\alpha}(0, T) = \overline{C^1[0, T]}^{H_{(m-1)\alpha}(0, T)} \subset C^2[0, T] \) and \( g(0) = 0 \).

Similarly, in terms of (3.7), we can see \( v(\cdot, 0) = 0 \) in \((\delta, x_0)\). Therefore, (3.2) and \( H^2(\delta, x_0) \subset C^1[\delta, x_0] \) yield

\[ \begin{cases} g \in C^1[0, T] \cap C^2[0, T], & v(\cdot, 0) = 0 \text{ in } (\delta, x_0), \\
v \in C^2([0, T]; H^2(\delta, x_0)) \subset C^1([0, T]; C^1[\delta, x_0]). \end{cases} \]

Moreover, by (3.1) and (3.7), we have \( v(x_0, t) = v_x(x_0, t) = 0 \) for \( 0 < t < T \).

Now, in place of \( u \), we consider the solution \( v \) to

\begin{equation} \begin{cases} \partial_t^\alpha v = v_{xx} - p v + J^{(m-1)\alpha} a & \text{in } (\delta, x_0) \times (0, T), \\
v_x(\delta, t) = g(t) \in C^1[0, T] \cap C^2[0, T], \quad v(x_0, t) = 0, \quad 0 < t < T, \\
v \in C^1([0, T]; C^1[\delta, x_0]) \end{cases} \label{eq:3.8} \end{equation}

satisfying

\begin{equation} v(x_0, t) = 0, \quad 0 < t < T. \label{eq:3.9} \end{equation}

We construct the following extension \( G \in C^2[0, +\infty) \) of \( g \in C^2[0, T] \). We can find \( G_0 \in C^2[0, +\infty) \) such that \( G_0|_{(0, T)} = g \) and \( \|G_0\|_{C^2[0, T+1]} \leq C\|g\|_{C^2[0, T]} \). Let \( \chi \in C^\infty[0, +\infty) \)
satisfy $\chi(t) = \begin{cases} 1, & t \leq T, \\ 0, & t \geq T + 1. \end{cases}$ We set

$$G(t) = \chi(t)G_0(t), \quad t > 0.$$ Then there holds

$$G \in C^2[0, +\infty), \quad G|_{(0,T)} = g, \quad G|_{(T+1, +\infty)} = 0,$$

$$\|G\|_{C^2[0,T]} \leq C\|g\|_{C^2[0,T]}.$$ Now we mainly consider an initial-boundary value problem:

$$\begin{aligned}
\delta \partial_\tau^2 V &= V_{xx} - pV + J^{(m-1)}V_{xx} \alpha, & \delta < x < x_0, \quad t > 0, \\
V_\delta(\delta, t) &= G(t), \quad V_\delta(x_0, t) = 0, \quad t > 0.
\end{aligned}$$

Then by (3.8), the uniqueness of solution to the initial-boundary value problem yields $V(x,t) = v(x,t)$ for $\delta < x < x_0$ and $0 < t < T$. Taking into consideration $v \in C([\delta, x_0] \times [0,T])$, by (3.9) we derive

$$V(x_0, t) = v(x_0, t) = 0, \quad 0 < t < T.$$\vspace{0.5cm}

**Step 2.** In this step, we estimate $V$. Henceforth, $\| \cdot \|$ and $(\cdot, \cdot)$ denote the norm and the scalar product in $L^2(\delta, x_0)$ respectively if not specified otherwise.

Together with the existence of a solution $V$ to (3.11), we will estimate $\|V_{xx}(\cdot, t)\|$ and $\|\partial_\tau^2 V(\cdot, t)\|$. Let $\{\lambda_n, \varphi_n\}_{n \in \mathbb{N}}$ be the eigensystem of $A_\delta v = -v_{xx} + p(x)v$ with the domain $\mathcal{D}(A_\delta) = \{\eta \in H^2(\delta, x_0); \eta_\delta(\delta) = \eta_{\delta}(x_0) = 0\}$. Here we note that there exists some $n_0 \in \mathbb{N}$ such that $\lambda_1 < \cdots < \lambda_{n_0-1} < 0 \leq \lambda_{n_0} < \lambda_{n_0+1} < \cdots \to \infty$. We define two operators with the domain $L^2(\delta, x_0)$ and the range in itself by

$$S(t)a := \sum_{n=1}^{\infty} (a, \varphi_n) E_{\alpha,1}(-\lambda_n t^\alpha) \varphi_n(x),$$

$$K(t)a := t^{\alpha-1} \sum_{n=1}^{\infty} E_{\alpha,\alpha}(-\lambda_n t^\alpha) (a, \varphi_n) \varphi_n(x),$$

We have the following properties concerning $S(t)$ and $K(t)$.

**Lemma 3.1.** Let $S(t)$ and $K(t)$ be defined in (3.13). Then for $a \in L^2(\delta, x_0)$ and $t > 0$, there hold

$$\|S(t)a\| \leq Ce^{Ct}\|a\|, \quad \lim_{t \to 0^+} \|S(t)a - a\| = 0,$$

$$A_\delta K(t)a = -S'(t)a.$$

Next, we set

$$W(x,t) := V(x,t) + \frac{(x - x_0)^2}{2(x_0 - \delta)} G(t),$$

$$F(x,t) := \frac{(x - x_0)^2}{2(x_0 - \delta)} (\partial_\tau^2 G(t) + p(x)G(t)) - \frac{1}{x_0 - \delta} G(t) + J^{(m-1)}a$$
boundary value problem for
if and only if

(3.17)

for arbitrary $t_0 > 0$.

Lemma 3.2. Let $W$ satisfy (3.10). Then $W$ admits the representation

(3.18)

for $t > 0$, where $K(t)$ was defined in (3.13).

For consistency, we postpone the proofs of Lemmata 3.1 and 3.2 to Appendix A.

Henceforth, $C > 0$ and $C_k > 0$ denote generic constants independent of $t$ but may depend
on $g$ and $p$.

Then we have

$$
\frac{\partial f}{\partial t}(x, t) = \frac{(x-x_0)^2}{2(x_0-\delta)} \left( \frac{d}{dt} \partial^\alpha_t G(t) + p(x) \frac{dG}{dt}(t) \right) - \frac{1}{x_0-\delta} \frac{dG}{dt}(t),
$$

where

$$
\frac{d}{dt} \partial^\alpha_t G(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \frac{G(t-s)}{s^\alpha} ds
\quad = \frac{G'(0)}{\Gamma(1-\alpha)} t^{-\alpha} + \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{G''(t-s)}{s^\alpha} ds
$$

by $G \in C^2[0, +\infty)$. Hence, in terms of (3.10), we deduce

$$
\left| \frac{\partial f}{\partial t}(x, t) \right| \leq \frac{C|G'(0)|}{t^\alpha} + C \int_0^t \frac{|G''(t-s)|}{s^\alpha} ds + C|G'(t)|
\quad \leq C\|G\|_{C^2[0, +\infty)}(1 + t^{-\alpha} + t^{1-\alpha})
\quad \leq C\|g\|_{C^2[0, T]}(1 + t^{-\alpha} + t^{1-\alpha}), \quad \forall x \in (\delta, x_0), \quad \forall t > 0.
$$

Hence, by (3.14), (3.15) and integration by parts, we have

\[
A_\delta W_1(t) = \int_0^t A_\delta K(t-\tau) f(\tau) d\tau = \int_0^t A_\delta K'(\tau) f(t-\tau) d\tau
\quad = -\int_0^t S'(\tau) f(t-\tau) d\tau = \left[ S(\tau) f(t-\tau) \right]_{\tau=t}^{\tau=0} - \int_0^t S(\tau) f'(t-\tau) d\tau
\quad = f(t) - S(t) f(0) - \int_0^t S(\tau) f'(t-\tau) d\tau.
\]
Using (3.18), we obtain
\[ \| A_3 W_1(t) \| \leq C e^{C_1 t} \| f \|_{C^1([0,T];L^2(\delta,x_0))} + \int_0^t C e^{C_2 \tau} \| g \|_{C^2([0,T])} (1 + (t - \tau)^{-\alpha} + (t - \tau)^{1-\alpha}) \, d\tau, \]
indicating
\[ (3.19) \quad \| A_3 W_1(t) \| \leq C(1 + t + t^{1-\alpha} + t^{2-\alpha}) e^{C_1 t} \leq C e^{C_2 t}, \quad t > 0. \]
On the other hand, since \( J^{(m-1)\alpha} a = \frac{1}{\Gamma((m-1)\alpha+1)} t^{(m-1)\alpha} a, \) we have
\[ \| A_3 W_2(t) \| = \left\| \int_0^t S'(t - \tau) J^{(m-1)\alpha} a \, d\tau \right\| \]
\[ = \left\| \int_0^t S'(t - \tau) \frac{1}{\Gamma((m-1)\alpha+1)} s^{(m-1)\alpha} a \, d\tau \right\|. \]
Then, by noting \((m-1)\alpha > 0,\) the integration by parts yields
\[ \int_0^t S'(t - \tau) \tau^{(m-1)\alpha} a \, d\tau = \left[ S(t - \tau) \tau^{(m-1)\alpha} a \right]_{\tau=0}^{\tau=t} + (m-1)\alpha \int_0^t S(t - \tau) \tau^{(m-1)\alpha-1} a \, d\tau \]
\[ = -t^{(m-1)\alpha} + (m-1)\alpha \int_0^t S(t - \tau) \tau^{(m-1)\alpha-1} a \, d\tau \]
and so (3.14) yields
\[ \left\| \int_0^t S'(t - \tau) \tau^{(m-1)\alpha} a \, d\tau \right\| \leq C t^{(m-1)\alpha} + C \int_0^t \tau^{(m-1)\alpha-1} e^{C(t - \tau)} \, d\tau \]
\[ \leq C t^{(m-1)\alpha} + C e^{Ct} \int_{\mathbb{R}_+} \tau^{(m-1)\alpha-1} e^{-C \tau} \, d\tau \]
\[ \leq C t^{(m-1)\alpha} + C e^{Ct} \frac{\Gamma((m-1)\alpha)}{C(m-1)\alpha} \leq C_2 e^{C_2 t}, \quad t > 0. \]
Consequently, (3.17) and (3.19) imply
\[ \| W_{xx}(\cdot,t) \| \leq C \| A_3 W(\cdot,t) \| \leq C_2 e^{C_2 t}, \quad t > 0, \]
that is,
\[ \| V_{xx}(\cdot,t) \| + \| V(\cdot,t) \| \leq C_2 e^{C_2 t}, \quad t > 0. \]
Moreover, by the Sobolev embedding \( H^2(\delta,x_0) \subset C[\delta,x_0], \) we have \( \| V(\cdot,t) \|_{C[\delta,x_0]} \leq C e^{C_2 t} \) for \( t > 0. \) Using the first equation in (3.11), we can estimate \( \partial^\alpha_t V \) as
\[ \| \partial^\alpha_t V(\cdot,t) \| \leq C_2 e^{C_2 t}, \quad t > 0. \]
Therefore,
\[ \| \partial^\alpha_t V(\cdot,t) \|_{L^1(\delta,x_0)} + \| V_{xx}(\cdot,t) \|_{L^1(\delta,x_0)} + \| V(\cdot,t) \|_{C[\delta,x_0]} \leq C_3 e^{C_3 t}, \quad t > 0. \]
Hence, there exists some constant \( s_0 > 0 \) such that for any \( s > s_0 \), we have
\[ \partial_t^s V(x, t) e^{-st}, \quad V_{xx}(x, t) e^{-st}, \quad V(x, t) e^{-st} \in L^1((\delta, x_0) \times \mathbb{R}_+) \]
and
\[ \|V(\cdot, t)\|_{C[\delta, x_0]} \leq C_3 e^{C_3 t}, \quad t > 0. \]
Hence, Fubini’s theorem yields that
\[ \|\partial_t^s V(x, t) e^{-st}\|, \quad \|V_{xx}(x, t) e^{-st}\|, \quad \|V(x, t) e^{-st}\| \]
are integrable in \( t \in \mathbb{R}_+ \) for arbitrarily chosen \( s > s_0 \) and almost all \( x \in (\delta, x_0) \). Thus
\[ \int_0^\infty |\partial_t^s V(x, t)| e^{-st} dt, \quad \int_0^\infty |V_{xx}(x, t)| e^{-st} dt, \quad \int_0^\infty |V(x, t)| e^{-st} dt \]
exist for almost all \( x \in (0, x_0) \) and \( s > s_0 \). This is the same for \( V_{x}(\delta, t) \) and \( V_{x}(x_0, t) \).

Kubica, Ryzsewska and Yamamoto [11, Theorem 2.7] implies \( \partial_t^s \hat{V}(x, s) = s^2 \hat{V}(x, s) \) for \( s > s_0 \). Therefore, we obtain
\[ \left\{ \begin{array}{ll}
\hat{V}(x, s) = \hat{V}_{xx}(x, s) - p(x)\hat{V}(x, s) + s^{-(m-1)\alpha-1}a, & x \in (\delta, x_0), \\
\hat{V}_{x}(\delta, s) = \hat{G}(s), & \hat{V}_{x}(x_0, s) = 0
\end{array} \right. \]
for \( s > s_0 \).

**Step 3.** Recalling the function \( \varphi(x, z) \) defined by (2.1), we set
\[ (3.22) \quad F(z) := \int_\delta^{x_0} a(x)\varphi(x, z) dx, \quad z \in \mathbb{C}. \]
We notice that \( F(z) \) is an entire function on \( \mathbb{C} \) since \( \varphi \) is analytic with respect to \( z \in \mathbb{C} \).

**Lemma 3.3.** The function \( F(z) \) defined in (3.22) satisfies
\[ (3.23) \quad s^{-(m-1)\alpha-1} F(z) = (s^2 + z^2) \int_\delta^{x_0} \hat{V}(x, s)\varphi(x, z) dx + \hat{G}(s) + \hat{V}(x_0, s)\varphi'(x_0, z) \]
for \( s > s_0 \) and \( z \in \mathbb{C} \).

**Proof.** By (3.21), we have
\[ \int_\delta^{x_0} \hat{V}_{xx}(x, s)\varphi(x, z) dx = \left[ \hat{V}_x(x, s)\varphi(x, z) \right]_{x=\delta}^{x=x_0} - \left[ \hat{V}(x, s)\varphi'(x, z) \right]_{x=\delta}^{x=x_0} + \int_\delta^{x_0} \hat{V}(x, s)\varphi_x(x, z) dx \]
\[ = -\hat{G}(s) - \hat{V}(x_0, s)\varphi'(x_0, z) + \int_\delta^{x_0} p(x)\varphi(x, z)\hat{V}(x, s) dx \]
\[ - z^2 \int_\delta^{x_0} \hat{V}(x, s)\varphi(x, z) dx. \]
Lemma 3.4. There exists a sufficiently large integer $N$ such that

$$F(z) = s^{(m-1)\alpha+1} \left( \int_{x_0}^{x_0} (s^\alpha + z^2) \tilde{V}(x,s) \varphi(x,z) \, dx + \tilde{G}(s) + \tilde{V}(x_0,s) \varphi'(x_0,z) \right).$$

Thus the proof of Lemma 3.3 is complete. \qed

The choice $z = \pm i s^{\frac{\alpha}{2}}$ yields $s^\alpha + z^2 = 0$ in (3.23) and thus

$$s^{-(m-1)\alpha-1} F(z) = \varphi'(x_0,z) \tilde{V}(x_0,s) + \tilde{G}(s), \quad z = \pm i s^{\frac{\alpha}{2}}, \ s > s_0.$$

Step 4. Based on (3.24), we can derive an estimate for $F(z)$ defined in (3.22) as follows.

Lemma 3.4. There exists a sufficiently large integer $N$ such that

$$|F(z)| \leq C |z|^N, \quad z = \pm i s^{\frac{\alpha}{2}}, \ s > s_0,$$

where the constant $C > 0$ is independent of all $s > s_0$ and $N$.

Proof. For $z = \pm i s^{\frac{\alpha}{2}}$ for $s > s_0$, we have $|\varphi'(x_0,z)| \leq C |z| e^{\delta |z|}$ by (2.4) in Lemma 2.1. Substituting this into (3.24) implies

$$|F(z)| \leq C s^{(m-1)\alpha+1} |z| e^{\delta |z|} \left| \int_{\mathbb{R}^+} V(x_0,t) e^{-st} \, dt \right| + s^{(m-1)\alpha+1} \left| \int_{\mathbb{R}^+} G(t) e^{-st} \, dt \right|$$

for $z = \pm i s^{\frac{\alpha}{2}}$. From (3.12) and (3.20), it follows that

$$\left| \int_{\mathbb{R}^+} V(x_0,t) e^{-st} \, dt \right| \leq \int_T^{+\infty} |V(x_0,t)| e^{-st} \, dt \leq \int_T^{+\infty} C_3 e^{(C_3-s)t} \, dt = \frac{C_3 e^{(C_3-s)t}}{s-C_3}, \quad s > C_3.$$

On the other hand, by (3.10), we see that

$$\left| \int_{\mathbb{R}^+} G(t) e^{-st} \, dt \right| \leq \int_0^{T+1} |G(t)| e^{-st} \, dt \leq \|G\|_{L^1(0,T+1)}, \quad s > 0.$$

Collecting the above estimates, we reach

$$|F(z)| \leq C_4 s^{(m-1)\alpha+1} \frac{\exp(s^{\frac{\alpha}{2}}) e^{-(s-C_3)T}}{s-C_5} + C_4 s^{(m-1)\alpha+1}, \quad s > C_5,$$

where $C_5 > 0$ is some constant.

Since $\alpha \in (0,1)$, we can dominate $|F(z)| \leq C_6 s^{(m-\frac{1}{2})\alpha+1}$ for $s > C_5$. By the relation $z = \pm i s^{\frac{\alpha}{2}}$, we can further conclude

$$|F(z)| \leq C_7 |z|^{2m-1+\frac{2\alpha}{\alpha}} \leq C_8 |z|^N, \quad z = \pm i s^{\frac{\alpha}{2}}, \ s > C_5$$

with an integer $N \geq 2m - 1 + \frac{2}{\alpha}$, which is the desired estimate. \qed

Next, we show two further properties concerning $F(z)$ below.

Lemma 3.5. The function $F(z)$ defined in (3.22) is a polynomial of order $N$ at most, where the integer $N$ is given in (3.25).
Proof. Introducing $F_N(z) := \frac{F(z)}{(z+1)^N}$, we see that $F_N(z)$ is holomorphic for Re $z > -1$. Furthermore, (3.25) implies that there exists a sufficiently large constant $M > 0$ such that

$$|F_N(z)| \leq \frac{C_8 |z|^N}{|z+1|^N} \leq C_9, \quad \text{arg } z = \pm \frac{\pi}{2}, \ |z| \geq M.$$ 

Here and henceforth the constant $C_k$ can depend also on the constant $M > 0$. Meanwhile, for $\arg z = \pm \frac{\pi}{2}$, we have $|z+1| \geq 1$ and the continuity of the function $\varphi(x, z)$ yields

$$|F_N(z)| = \frac{1}{|z+1|^N} \left| \int_{x_0}^{x_0} a(x) \varphi(x, z) \, dx \right| \leq \int_{\delta}^{x_0} |a(x)\varphi(x, z)| \, dx \leq C_{10}$$

if $|z| \leq M$ and $\arg z = \pm \frac{\pi}{2}$.

Combining the above estimates for $F_N$ yields

$$|F_N(z)| \leq C_{11} \quad \text{for all } z \in \mathbb{C} \text{ satisfying } \arg z = \pm \frac{\pi}{2}.$$

On the other hand, it follows from the asymptotic estimate (2.2) of $\varphi(x, z)$ in Lemma 2.2 that $|\varphi(x, z)| \leq C_{12}$ for any $z \in \mathbb{R}$. Hence, we obtain

$$|F(z)| \leq C_{13} \int_{\delta}^{x_0} |a(x)| \, dx \leq C, \quad z \in \mathbb{R}$$

and finally $|F_N(z)| \leq C$ for $z \in \mathbb{R}$. Again by (2.2), we see

$$|F_N(z)| \leq C_{14} |F(z)| \leq C_{14} \left| \int_{\delta}^{x_0} a(x) \varphi(x, z) \, dx \right| \leq C_{15} \int_{\delta}^{x_0} e^{\text{Im} z |x|} |a(x)| \, dx \leq C_{15} e^{\text{Im} z |x_0|}$$

for Re $z > M$. In the case of $0 < \text{Re } z \leq M$, we can conclude from the continuity of the function $\varphi(x, z)$ that

$$|F_N(z)| \leq C_{14} \|a\|_{L^\infty(0,1)} \|\varphi\|_{L^\infty((0,1) \times \{|z| \leq M\}).}$$

Combining the above estimates, we finally get

$$|F_N(z)| \leq C_{15} e^{\text{Im} z |x_0|}, \quad \text{Re } z > 0.$$ 

Choosing $(\theta_1, \theta_2) = (0, \frac{\pi}{2})$ and $(\theta_1, \theta_2) = (-\frac{\pi}{2}, 0)$, we apply Lemma 2.3 to obtain $|F_N(z)| \leq C$ for all Re $z > 0$. Hence,

$$|F(z)| \leq |z+1|^N, \quad \text{Re } z \geq 0.$$ 

Similarly, by considering $\frac{F(z)}{(z-1)^N}$, we can derive $|F(z)| \leq C|z-1|^N$ for Re $z < 0$. Consequently, according to Lemma 2.2, $F$ must be a polynomial satisfying $\text{deg } F \leq N$. \qed

Lemma 3.6. The function $F(z)$ defined in (3.22) vanishes identically in $\mathbb{C}$. 


Proof. Owing to Lemma 3.3, we can assume that $F(z) = \sum_{j=0}^{N} a_j z^j$ with some $a_j \in \mathbb{C}$. For $z > 0$, let us consider $\lim_{z \to +\infty} F(z)$. From the asymptotic behavior (2.3) of $\varphi(x, z)$ in Lemma 3.1, we obtain

$$\lim_{z \to +\infty} F(z) = \lim_{z \to +\infty} \left( \int_{\delta}^{x_0} a(x) \cos(z(x - \delta)) \, dx + \int_{\delta}^{x_0} a(x) O(|z|^{-1}) \, dx \right)$$

$$= \lim_{z \to +\infty} \int_{\delta}^{x_0} a(x) \cos(z(x - \delta)) \, dx.$$

In view of the Riemann-Lebesgue lemma, we have

$$\lim_{z \to +\infty} \int_{\delta}^{x_0} a(x) \cos(z(x - \delta)) \, dx = 0,$$

that is,

$$\lim_{z \to +\infty} F(z) = \lim_{z \to +\infty} \sum_{j=0}^{N} a_j z^j = 0.$$

If $a_N \neq 0$, then

$$\lim_{z \to +\infty} F(z) = \lim_{z \to +\infty} a_n z^N \left( 1 + \frac{a_{N-1}}{a_N} \frac{1}{z} + \cdots + \frac{a_0}{a_N z^N} \right) = \infty,$$

which is impossible. Therefore, we conclude $a_N = 0$. Repeating the same argument, we see $a_j = 0$ for $j = 0, 1, \ldots, N$, which completes the proof.

\[\square\]

Step 5. Now we are ready to finish the proof of Theorem 1.2. By (3.22) and Lemma 3.6, we see that

$$\int_{\delta}^{x_0} a(x) \varphi(x, z) \, dx = 0, \quad \forall z \in \mathbb{C}.$$

In order to show $a \equiv 0$ in $(\delta, x_0)$, we invoke the Neumann eigensystem $\{(\mu_n, \psi_n)\}$ of the operator $-\frac{d^2}{dx^2} + p(x) + p_0$ in $(\delta, x_0)$, that is,

$$\begin{cases}
-\psi''_n + (p(x) + p_0) \psi_n = \mu_n \psi_n & \text{in } (\delta, x_0), \\
\psi'_n(\delta) = \psi'_n(x_0) = 0.
\end{cases}$$

Here $p_0 > 0$ is a constant sufficiently large so that $\mu_n > 0$ for $n \in \mathbb{N}$. Normalizing $\psi_n$ by $\psi_n(\delta) = 1$, we immediately see that $\varphi \left( \sqrt{\mu_n - p_0} \right) = \psi_n$. Since $\{\psi_n\}$ forms a complete orthogonal basis in $L^2(\delta, x_0)$, it suffices to take $z = \sqrt{\mu_n - p_0}$ in (3.26) to conclude $a \equiv 0$ in $(\delta, x_0)$.

It remains to show $G \equiv 0$. Now the equation (3.24) becomes

$$\int_{\mathbb{R}^+} G(t) e^{-st} \, dt + \varphi'(x_0, z) \int_{\mathbb{R}^+} V(x_0, t) e^{-st} \, dt = 0, \quad z = \pm i \frac{s}{2}.$$

Using (3.10), (3.12), (3.24) and Lemma 3.6 for $z = \pm i \frac{s}{2}$ we deduce

$$\int_{0}^{T+1} G(t) e^{-st} \, dt = \int_{\mathbb{R}^+} G(t) e^{-st} \, dt = -\varphi'(x_0, z) \int_{\mathbb{R}^+} V(x_0, t) e^{-st} \, dt.$$
\[
\int_0^T g(t) e^{-st} dt = -\varphi'(x_0, z) \int_0^{+\infty} V(x_0, t) e^{-st} dt.
\]

Therefore, (2.4) in Lemma 2.1 implies
\[
\left| \int_0^T g(t) e^{-st} dt \right| = \left| \int_0^{T+1} G(t) e^{-st} dt \right| = \left| \int_0^{T+1} G(t) e^{-st} dt - \int_T^{T+1} G(t) e^{-st} dt \right|
\]
\[
\leq \left| \int_0^{T+1} G(t) e^{-st} dt \right| + \int_T^{T+1} |G(t)| e^{-st} dt
\]
\[
\leq |\varphi'(x_0, z)| \int_T^{+\infty} V(x_0, t) e^{-st} dt + \|G\|_{L[0,T+1]} \int_T^{T+1} e^{-st} dt
\]
\[
\leq \|G\|_{L[0,T+1]} e^{-sT} \frac{1-e^{-s}}{s} + C s^{\frac{\alpha}{T}} \exp(C s^{\frac{\alpha}{T}}) \int_T^{+\infty} V(x_0, t) e^{-st} dt, \quad s > \eta_0.
\]

Moreover, by (3.20), we obtain
\[
\left| \int_0^T g(t) e^{-st} dt \right| \leq C s^{-1} e^{-sT} + C s^{\frac{\alpha}{T}} \exp(C s^{\frac{\alpha}{T}}) \frac{e^{-\left(s-C_3\right)T}}{s-C_3}
\]
for \(s > C_3' := \max\{C_3, \eta_0\}\). For any \(\varepsilon > 0\), we see that
\[
\left| \int_0^{T-\varepsilon} g(t) e^{-st} dt \right| \leq C s^{-1} e^{-sT} + C s^{\frac{\alpha}{T}} \exp(C s^{\frac{\alpha}{T}}) \frac{e^{-\left(s-C_3\right)T}}{s-C_3} + \int_{T-\varepsilon}^T |g(t)| e^{-st} dt
\]
\[
\leq C s^{-1} e^{-sT} + C s^{\frac{\alpha}{T}} \exp(C s^{\frac{\alpha}{T}} - \varepsilon s) \frac{e^{C_3'T}}{s-C_3} e^{-s(T-\varepsilon)}
\]
\[
+ \frac{C|e^{-s(T-\varepsilon)} - s^{-sT}|}{s}, \quad s > C_3'.
\]
Choosing \(s > 0\) large enough, we arrive at
\[
\left| \int_0^{T-\varepsilon} g(t) e^{s(T-\varepsilon-t)} dt \right| \leq C, \quad s > C_3'.
\]
Introducing \(\tilde{g}(t) := g(T - \varepsilon - t)\), we immediately have
\[
\left| \int_0^{T-\varepsilon} \tilde{g}(t) e^{st} dt \right| \leq C, \quad s > C_3'.
\]
For \(0 < s \leq C_3'\), we estimate
\[
\left| \int_0^{T-\varepsilon} \tilde{g}(t) e^{st} dt \right| \leq e^{C_3'(T-\varepsilon)} \int_0^{T-\varepsilon} |\tilde{g}(t)| \|g\|_{L^1(0,T-\varepsilon)} dt
\]
Therefore, by defining \(\tilde{G}(z) := \int_0^{T-\varepsilon} \tilde{g}(t) e^{zt} dt\), we have
\[
|\tilde{G}(z)| \leq C_{16} := \max\left\{C, e^{C_3'(T-\varepsilon)} \|g\|_{L^1(0,T-\varepsilon)}\right\}, \quad \arg z = 0.
\]
Meanwhile, for \( \arg z = \frac{\pi}{2} \), it is readily seen that
\[
\left| \tilde{G}(z) \right| \leq \int_0^{T-\varepsilon} |\tilde{g}(t)| \, dt = \|g\|_{L^1(0,T-\varepsilon)}.
\]
For \( 0 < \arg z < \frac{\pi}{2} \), we estimate
\[
\left| \tilde{G}(z) \right| \leq \int_0^{T-\varepsilon} |\tilde{g}(t)| e^{\varepsilon|t|} \, dt \leq \|g\|_{L^1(0,T-\varepsilon)} e^{(T-\varepsilon)|z|}.
\]
Then we can apply Lemma 2.3 with \( \theta_1 = 0 \), \( \theta_2 = \frac{\pi}{2} \) and \( \gamma = 1 \) to obtain
\[
\left| \tilde{G}(z) \right| \leq \max \left\{ C_{16}, \|g\|_{L^1(0,T-\varepsilon)} \right\}, \quad 0 \leq \arg z \leq \frac{\pi}{2}.
\]
Similarly, \( \tilde{G}(z) \) is also bounded for \( -\frac{\pi}{2} \leq \arg z \leq 0 \). On the other hand, it follows immediately from the definition of \( \tilde{G}(z) \) that it is bounded for Re \( z < 0 \). Now that \( \tilde{G}(z) \) is bounded and holomorphic in the whole complex plane, Liouville’s theorem guarantees that \( \tilde{G}(z) \) is a constant. Finally, since \( \lim_{z \to -\infty} \tilde{G}(z) = 0 \), we conclude \( \tilde{G}(z) \equiv 0 \) in \( \mathbb{C} \) and thus \( g \equiv 0 \) in \( (0,T-\varepsilon) \). Since \( \varepsilon > 0 \) can be arbitrarily chosen, we obtain \( g \equiv 0 \) in \( (0,T) \) and eventually \( G \equiv 0 \) in \( \mathbb{R}_+ \).

Finally, by the uniqueness of the solution to the initial-boundary value problem (3.11), we obtain \( V \equiv 0 \) in \( (\delta,x_0) \times (0,T) \). Hence, by (3.12) we conclude that \( v \equiv 0 \) in \( (\delta,x_0) \times (0,T) \). This completes the proof of Theorem 1.2.

4. Proof of Theorem 1.1

In order to prove Theorem 1.1, we show several useful lemmata. Let \( \mu \) be the \((1-\alpha)\)-th Riemann-Liouville derivative of \( \rho \), i.e.,
\[
\frac{d}{dt} (D_t^{1-\alpha} \rho)(t) = \frac{1}{\Gamma(\alpha)} \frac{d}{dt} \int_0^t (t-\tau)^{1-\alpha} \rho(\tau) \, d\tau.
\]

Lemma 4.1. Let \( h \in H_\alpha(0,T) \) and \( \rho \in H^1(0,T) \) satisfy \( \rho(0) \neq 0 \). Then the integral equation
\[
(4.1) \quad h(t) = \int_0^t \mu(t-\tau)w(\tau) \, d\tau
\]
admits a unique solution \( w \in L^2(0,T) \).

Proof. Performing \( J^{1-\alpha} \) on both sides of (1.14) and by direct calculation, we deduce
\[
J^{1-\alpha} h(t) = \int_0^t \rho(t-\tau)w(\tau) \, d\tau.
\]
By \( h \in H_\alpha(0,T) \) and \( \partial_t^\alpha = (J^\alpha)^{-1} \), we see
\[
\frac{d}{dt} (J^{1-\alpha} h) = J^{-1} J^{1-\alpha} h = J^{-1} J^1 \partial_t^\alpha h = \partial_t^\alpha h.
\]
Since $\rho \in H^1(0,T) \subset C[0,T]$ by the Sobolev embedding, we differentiate the above equation to derive

\[\partial_t^{\alpha} h(t) = \rho(0)w(t) + \int_0^t \rho'(t-\tau)w(\tau) \, d\tau.\]

Defining an operator $K : L^2(0,T) \rightarrow L^2(0,T)$ by

\[(Kw)(t) = \int_0^t \rho'(t-\tau)w(\tau) \, d\tau, \quad 0 < t < T,\]

we see that (4.2) can be rephrased as

\[\partial_t^{\alpha} h(t) = \rho(0)w(t) + Kw(t).\]

By $\rho' \in L^2(0,T)$, it is not difficult to verify that $K$ is an integral operator of the Hilbert-Schmidt type. Then it follows immediately from [28, Chapter X.2] that $K$ is a compact operator. Moreover, in view of Grönwall’s inequality, we can show that $\rho(0)w + Kw = 0$ implies $w = 0$. Therefore, it follows from the Fredholm alternative that (4.2) admits a unique solution $w \in L^2(0,T)$. \hfill \Box

**Lemma 4.2** (Duhamel’s principle). Let $f \in L^2(0,1)$, $\rho \in H^1(0,T)$, $\rho(0) \neq 0$ and $y \in H^\alpha(0,T;H^1(0,1))$ satisfy (1.2). If

\[y(\cdot,t) = \int_0^t \mu(t-\tau)u(\cdot,\tau) \, d\tau,\]

then $u \in L^2(0,T;H^1(0,1))$ satisfies $u - f \in H^\alpha(0,T;H^{-1}(0,1))$ and a homogeneous equation with the initial value $f$:

\[\partial_t^{\alpha} (u - f) - u_{xx} + p(x)u = 0 \quad \text{in} \ (0,1) \times (0,T_*) .\]

Here $T_* > 0$ is some constant.

**Remark 4.3.** Concerning Duhamel’s principle for time-fractional partial differential equations in different function spaces, we refer e.g. to [10,18] and the survey [25]. In comparison with existing literature, we do not attach the governing equations with boundary conditions in Lemma 4.2. Therefore, here we only focus on the representation (4.3) instead of the uniqueness issue.

**Proof of Lemma 4.2.** Let $u$ satisfy (4.3) with the regularity assumed in Lemma 4.2. Denoting the right-hand side of (4.3) by $\tilde{y}$, i.e.,

\[\tilde{y}(\cdot,t) = \int_0^t \mu(t-\tau)u(\cdot,\tau) \, d\tau,\]

we can show that $\tilde{y}$ belongs to $H^\alpha(0,T;H^1(0,1))$. Next, we know that (4.4) is equivalent to

\[u = J^\alpha(u_{xx} - p(x)u) + f \quad \text{in} \ L^2(0,T;H^{-1}(0,1)) .\]

Similarly, (4.2) is equivalent to

\[y = J^\alpha(y_{xx} - p(x)y) + (J^\alpha \rho) f = J^\alpha(y_{xx} - p(x)y) + (J^1 \mu) f \quad \text{in} \ L^2(0,T;H^{-1}(0,1)) ,\]
where we used \( \mu = D_t^{1-\alpha} \rho \). We follow the argument used in the proof of [10, Theorem 2.6] to obtain

\[
J^\alpha(\tilde{\gamma}_{xx} - p(x)\tilde{y}) + (J^1 \mu)f = J^\alpha \int_0^t \mu(t - \tau)(u_{xx} - p(x)u)(\tau) d\tau + \left( \int_0^t \mu(\tau) d\tau \right) f
\]

\[
= \int_0^t \mu(t - \tau)\{J^\alpha(u_{xx} - p(x)u)(\tau) + f\} d\tau.
\]

By the Titchmarsh convolution theorem, it turns out that if \( \tilde{\gamma} \) satisfies (4.7), then \( u \) satisfies (4.6), where \( T > 0 \) is replaced by some \( T_* > 0 \). This completes the proof.

Now we are well prepared to complete the proof of Theorem 1.1. Assume that \( y \in H_\alpha(0, T; H^1(0, 1)) \) satisfies (1.2) along with lateral Cauchy data \( J^\alpha y(x_0, t) = (J^\alpha y)_x(x_0, t) = 0 \). Employing Lemma 4.2, we obtain

\[
0 = J^\alpha y(x_0, t) = \int_0^t \mu(t - \tau)J^\alpha u(x_0, \tau) d\tau,
\]

\[
0 = (J^\alpha y)_x(x_0, t) = \int_0^t \mu(t - \tau)J^\alpha u_x(x_0, \tau) d\tau,
\]

where \( u \) satisfies (4.4). According to Lemma 4.1 we immediately conclude \( J^\alpha u(x_0, \cdot) = J^\alpha u_x(x_0, \cdot) = 0 \) in \( (0, T_*) \). Finally, a direct application of Theorem 1.2 indicates \( u \equiv 0 \) in \( (0, 1) \times (0, T_*) \), which indicates \( f \equiv 0 \) in \( (0, 1) \) automatically as the hidden initial value in (1.2).

5. Concluding remarks

In this paper, we obtained novel sharp uniqueness for an inverse \( x \)-source problem for a one-dimensional time-fractional diffusion equation with a potential. With the aid of Duhamel’s principle, the key ingredient reveals to be the uniqueness for the lateral Cauchy problem for the corresponding homogeneous equation. Taking Laplace transform, we changed the original lateral Cauchy problem to an integral equation involving the initial and boundary values of the solution. Then we managed to prove the uniqueness by employing the Phragmén-Lindelöf principle and a generalized Liouville’s theorem (Phragmén-Lindelöf-Liouville argument for short). As a byproduct, we also established a classical unique continuation property.

Let us mention that the Phragmén-Lindelöf-Liouville argument used in the proof heavily relies on the dimension in space. It is interesting to investigate the lateral Cauchy problem for time-fractional diffusion equations in higher spatial dimensions.

Appendix A. Proofs of (1.6), (3.3) and Lemmata 3.1, 3.2

Proof of (1.6). The proof is similar to the one of (3.7). By (1.3) and \( \partial_t^\alpha(u - a) = J^{-\alpha}(u - a) \) for \( u - a \in H_\alpha(0, T; H^{-1}(0, 1)) \), using also \( p \in L^\infty(0, 1) \), we see

\[
u - a - J^\alpha u_{xx}(x, t) + J^\alpha pu = J^\alpha F,
\]

that is,

\[
(J^\alpha u)_{xx}(x, t) = u - a + J^\alpha pu - J^\alpha F \in L^2(0, T; L^2(0, 1)).
\]
Then, since $J^αu \in L^2(0, T; H^1(0, 1))$ by the second regularity condition in (1.4), the interior regularity for an elliptic operator $\frac{d^2}{dx^2}$ (e.g., [28] Theorem 8.8) yields (1.6), and the proof of (1.6) is complete.

**Proof of 3.3.** We set $H^{-2}(0,1) := (H^2_0(0,1))^\prime$, where $H^2_0(0,1) := \{ v \in H^2(0,1) | v(0) = v_x(0) = v(1) = v_x(1) = 0 \}$. First for $\gamma > 0$, we prove

$$J^\gamma \left( h^{-2}(0,1) \langle w(\cdot,t), \psi \rangle_{H^2_0(0,1)} \right) = h^{-2}(0,1) \langle J^\gamma w(\cdot,t), \psi \rangle_{H^2_0(0,1)}$$

for $w \in L^2(0,T; H^{-2}(0,1))$, $\psi \in H^2_0(0,1)$ and almost all $t \in (0,T)$. We note that

$$J^\gamma (w(\cdot,t), \psi)_{L^2(0,1)} = (J^\gamma w(\cdot,t), \psi)_{L^2(0,1)}$$

is directly seen by Fubini’s theorem, but (A.1) with $H^{-2}(0,1) \langle w(\cdot,t), \psi \rangle_{H^2_0(0,1)}$ is not trivial.

We can verify (A.1) as follows. For $w \in L^2(0,T; H^{-2}(0,1))$, by the definition of Bochner’s integral (e.g., Yosida [28]), we can choose a sequence $w_n(x,t) := \sum_{j=1}^{m(n)} a_j^n(x) \chi_{E_j}(t)$ $(n \in \mathbb{N})$ of simple functions such that $w_n \longrightarrow w$ in $L^2(0,T; H^{-2}(0,1))$. Here $m(n) \in \mathbb{N}$, $a_j^n \in H^{-2}(0,1)$, $E_j \subset (0,T)$ are measurable sets, and $\chi_{E}$ indicates the characteristic function of $E$. Then

$$h^{-2}(0,1) \langle w_n(\cdot,t), \psi \rangle_{H^2_0(0,1)} \longrightarrow h^{-2}(0,1) \langle w(\cdot,t), \psi \rangle_{H^2_0(0,1)}$$

in $L^2(0,T)$ as $n \rightarrow \infty$ for all $\psi \in H^2_0(0,1)$ and almost all $t \in (0,T)$. Moreover,

$$J^\gamma \left( h^{-2}(0,1) \langle w_n(\cdot,t), \psi \rangle_{H^2_0(0,1)} \right) = J^\gamma \left( \sum_{j=1}^{m(n)} h^{-2}(0,1) \langle a_j^n, \psi \rangle_{H^2_0(0,1)} \chi_{E_j}(t) \right)$$

for all $n \in \mathbb{N}$, $\psi \in H^2_0(0,1)$ and almost all $t \in (0,T)$. Using that $J^\gamma : L^2(0,T) \longrightarrow L^2(0,T)$ is continuous and letting $n \rightarrow \infty$, we reach (A.1).

Next, we notice that $u \in L^2(0,T; H^1(0,1))$ is a weak solution to (1.7) and satisfies $u - a \in H^2_0(0,1)$, so that we have the weak form:

$$h^{-2}(0,1) \langle \partial_t^\alpha (u - a)(\cdot,t), \psi \rangle_{H^2_0(0,1)} + (u_x(\cdot,t), \psi_x)_{L^2(0,1)} + (pu(\cdot,t), \psi)_{L^2(0,1)} = 0$$

for all $\psi \in H^2_0(0,1)$ and almost all $t \in (0,T)$. Moreover, by integration by parts, we have

$$h^{-2}(0,1) \langle \partial_t^\alpha (u - a)(\cdot,t), \psi \rangle_{H^2_0(0,1)} - (u(\cdot,t), \psi_{xx})_{L^2(0,1)} + (pu(\cdot,t), \psi)_{L^2(0,1)} = 0$$

for all $\psi \in H^2_0(0,1)$ and almost all $t \in (0,T)$. We operate $J^\alpha$ to both sides to have

$$0 = J^\alpha \left( h^{-2}(0,1) \langle \partial_t^\alpha (u - a)(\cdot,t), \psi \rangle_{H^2_0(0,1)} \right) - (J^\alpha u(\cdot,t), \psi_{xx})_{L^2(0,1)}$$

for all $\psi \in H^2_0(0,1)$ and almost all $t \in (0,T)$. Here we used $J^\alpha \langle u(\cdot,t), \psi \rangle_{L^2(0,1)} = (J^\alpha u(\cdot,t), \psi)_{L^2(0,1)}$ etc., which is immediately seen.

Now we apply (A.1) with $\gamma = m\alpha$ to obtain

$$0 = h^{-2}(0,1) \langle J^\alpha \partial_t^\alpha (u - a)(\cdot,t), \psi \rangle_{H^2_0(0,1)} - (J^\alpha u(\cdot,t), \psi_{xx})_{L^2(0,1)}$$

$$+ (p J^\alpha u(\cdot,t), \psi)_{L^2(0,1)}$$
for all $\psi \in \mathcal{H}_0^2(0,1)$ and almost all $t \in (0,T)$. By $u - a \in H_0(0,T;H^{-1}(0,1)) \subset H_0(0,T;H^{-2}(0,1))$, we see $\partial_t^\alpha (u-a) \in L^2(0,T; H^{-2}(0,1))$, so that $J^{^\alpha} \partial_t^\alpha (u-a) = \partial_t^\alpha J^{^\alpha} (u-a)$. Therefore, since $\psi \in \mathcal{H}_0^2(0,1)$ is arbitrary, we reach

$$\partial_t^\alpha J^{^\alpha} (u-a)(\cdot,t) - (J^{^\alpha} u)_{xx}(\cdot,t) + p J^{^\alpha} u(\cdot,t) = 0 \quad \text{in } H^{-2}(0,1)$$

for almost all $t \in (0,T)$. Thus the proof of \[3.3\] is complete. \hfill $\square$

**Proof of Lemma 3.1.** First by direct calculation, it is not difficult to arrive at the following estimate

$$\|S(t)a\|^2 \leq \sum_{n=1}^{\infty} |(a, \varphi_n)|^2 |E_{\alpha,1}(-\lambda_n t^\alpha)|^2.$$  

Decomposing the above summation into two parts according to the signs of $\lambda_n$, we further derive

$$\|S(t)a\|^2 \leq C \sum_{n=1}^{n_0-1} |(a, \varphi_n)|^2 \exp(|\lambda_n|^{1/\alpha} t^\alpha) + C \sum_{n=n_0}^{\infty} |(a, \varphi_n)|^2 \frac{|(a, \varphi_n)|^2}{(1 + |\lambda_n|^\alpha)^2},$$

where we applied the asymptotic estimates for the Mittag-Leffler functions (see Podlubny [20, Theorem 1.5] for the case of $n \leq n_0$ and [20, Theorem 1.6] for that of $n > n_0$). Moreover, from the fact that $|\lambda_1| \geq |\lambda_n| \ (1 \leq n \leq n_0)$, it follows that

$$\|S(t)a\| \leq C \exp(|\lambda_1|^{1/\alpha} t^\alpha) \sum_{n=1}^{\infty} |(a, \varphi_n)|^2 = C \exp(|\lambda_1|^{1/\alpha} t^\alpha) \|a\|.$$

Next, we check the convergence in \[3.14\]. For this, it suffices to evaluate the following series

$$\|S(t)a - a\|^2 = \sum_{n=1}^{\infty} |(a, \varphi_n)|^2 |E_{\alpha,1}(-\lambda_n t^\alpha) - 1|^2.$$  

From the above calculation, it follows that $|E_{\alpha,1}(-\lambda_n t^\alpha)| \leq C \exp(|\lambda_1|^{1/\alpha} t^\alpha)$. Together with the continuity of the Mittag-Leffler function and the dominated convergence theorem, we see that

$$\lim_{t \to 0} \|S(t)a - a\|^2 = \sum_{n=1}^{\infty} |(a, \varphi_n)|^2 \lim_{t \to 0} |E_{\alpha,1}(-\lambda_n t^\alpha) - 1|^2 = 0.$$  

For \[3.15\], we employ the termwise differentiability of the series in $S(t)a$ and the formula

$$E_{\alpha,1}(-\lambda_n t^\alpha) = -\lambda_n t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n t^\alpha)$$

to calculate the derivative $S'(t)a$ as

$$S'(t)a = -\sum_{n=1}^{\infty} t^{\alpha-1} \lambda_n (a, \varphi_n) E_{\alpha,1}(-\lambda_n t^\alpha)\varphi_n.$$  

Then the definition of $K(t)$ implies immediately $A(t)K(t)a = -S'(t)a$. This completes the proof of Lemma 3.1 \hfill $\square$
Proof of Lemma 3.2. Notice that the eigenfunctions \( \{ \varphi_n \} \) form an orthonormal basis of \( L^2(\delta, x_0) \). Then by the Fourier expansion argument e.g. used in the proof of Sakamoto and Yamamoto [24, Theorem 2.1], we similarly obtain that the solution \( W \) to the problem (3.16) admits the following series representation

\[
W(x, t) = \sum_{n=1}^{\infty} \left( \int_0^t \tau^{\alpha-1} E_{\alpha, \alpha}(-\lambda_n \tau^\alpha)(F(\cdot, t - \tau), \varphi_n) \, d\tau \right) \varphi_n(x).
\]

Now it remains to check that the above defined Fourier series converges in the sense of \( L^2(0, t_0; L^2(\delta, x_0)) \). Indeed, introducing

\[
W_N(x, t) := \sum_{n=1}^{N} \left( \int_0^t \tau^{\alpha-1} E_{\alpha, \alpha}(-\lambda_n \tau^\alpha)(F(\cdot, t - \tau), \varphi_n) \, d\tau \right) \varphi_n(x)
\]

for \( N \in \mathbb{N} \), it is readily seen that

\[
\| W_N(\cdot, t) \|^2 \leq \sum_{n=1}^{N} \left( \int_0^t \tau^{\alpha-1} E_{\alpha, \alpha}(-\lambda_n \tau^\alpha)(F(\cdot, t - \tau), \varphi_n) \, d\tau \right)^2.
\]

From Young’s convolution inequality and using \( E_{\alpha, \alpha}(-\lambda_n t^\alpha) \geq 0 \) (see e.g. [24, Lemma 3.3]), it follows that

\[
\| W_N \|^2_{L^2(0, t_0; L^2(\delta, x_0))} \leq \sum_{n=1}^{N} \int_0^{t_0} \left( \int_0^t \tau^{\alpha-1} E_{\alpha, \alpha}(-\lambda_n \tau^\alpha)(F(\cdot, t - \tau), \varphi_n) \, d\tau \right)^2 \, dt
\]

\[
\leq \sum_{n=1}^{N} \left( \int_0^{t_0} \tau^{\alpha-1} E_{\alpha, \alpha}(-\lambda_n t^\alpha) \, dt \right)^2 \int_0^{t_0} |(F(\cdot, t), \varphi_n)|^2 \, dt.
\]

Similarly to the proof of Lemma 3.1, we can see that

\[
|E_{\alpha, \alpha}(-\lambda_n t^\alpha)| \leq C \exp(|\lambda_1|^\frac{1}{\alpha} t), \quad \forall n \in \mathbb{N},
\]

which implies that

\[
\| W_N \|^2_{L^2(0, t_0; L^2(\delta, x_0))} \leq C \sum_{n=1}^{N} \int_0^{t_0} |(F(\cdot, \tau), \varphi_n)|^2 \, dt \leq C \| F \|^2_{L^2(0, t_0; L^2(\delta, x_0))}.
\]

This implies the uniform boundedness of \( \{ W_N \} \) in \( L^2(0, t_0; L^2(\delta, x_0)) \) and by passing \( N \to \infty \), we obtain \( W \in L^2(0, t_0; L^2(\delta, x_0)) \). The proof of Lemma 3.2 is completed. \( \square \)

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REFERENCES

[1] R.A. Adams, Sobolev Spaces, Academic Press, New York, 1975.
[2] E.E. Adams, L.W. Gelhar, Field study of dispersion in an heterogeneous aquifer 2. Spatial moments analysis, Water Resour. Res. 28 (1992) 3293–307.
[3] J. Cheng, C.-L. Lin, G. Nakamura, Unique continuation property for the anomalous diffusion and its application, J. Differential Equations 254 (2013) 3715–3728.
[4] S.D. Eidelman, A.N. Kochubei, Cauchy problem for fractional diffusion equations, J. Differential Equations 199 (2004) 211–255.
[5] D. Gilbarg, N.S. Trudinger, Elliptic Partial Differential Equations of Second Order, Springer-Verlag, Berlin, 2001.
[6] R. Gorenflo, Y. Luchko, M. Yamamoto, Time-fractional diffusion equation in the fractional Sobolev spaces, Fract. Calc. Appl. Anal. 18 (2015) 799–820.
[7] Y. Hatano, N. Hatano, Dispersive transport of ions in column experiments: an explanation of long-tailed profiles, Water Resour. Res. 34 (1998) 1027–1033.
[8] X. Huang, Z. Li, M. Yamamoto, Carleman estimates for the time-fractional advection-diffusion equations and applications, Inverse Problems 35 (2019) 045003.
[9] V. Isakov, Inverse Problems for Partial Differential Equations (Second Edition), Springer, Berlin, 2006.
[10] D. Jiang, Z. Li, Y. Liu, M. Yamamoto, Weak unique continuation property and a related inverse source problem for time-fractional diffusion-advection equations, Inverse Problems 33 (2017) 055013.
[11] A. Kubica, K. Ryszewska, M. Yamamoto, Time-Fractional Differential Equations: A Theoretical Introduction, Springer, Singapore, 2020.
[12] B.M. Levitan, I.S. Sargsian, Introduction to Spectral Theory: Selfadjoint Ordinary Differential Operators, AMS, Providence, 1975.
[13] Z. Li, M. Yamamoto, Unique continuation principle for the one-dimensional time fractional diffusion equation, Fract. Calc. Appl. Anal. 22 (2019) 664–657.
[14] C.-L. Lin, G. Nakamura, Unique continuation property for anomalous slow diffusion equation, Comm. Partial Differential Equations 41 (2016) 749–758.
[15] C.-L. Lin, G. Nakamura, Unique continuation property for multi-terms time fractional diffusion equations, Math. Ann. 373 (2019) 929–952.
[16] C.-L. Lin, G. Nakamura, Classical unique continuation property for multi-term time-fractional evolution equations, Math. Ann. (accepted).
[17] Y. Liu, Z. Li, M. Yamamoto, Inverse problems of determining sources of the fractional partial differential equations, in: Handbook of Fractional Calculus with Applications. Volume 2: Fractional Differential Equations, De Gruyter, Berlin, 2019, pp. 431–442.
[18] Y. Liu, W. Rundell, M. Yamamoto, Strong maximum principle for fractional diffusion equations and an application to an inverse source problem, Fract. Calc. Appl. Anal. 19 (2016) 888–906.
[19] Y. Luchko, M. Yamamoto, Maximum principle for the time-fractional PDEs, in: Handbook of Fractional Calculus with Applications. Volume 2: Fractional Differential Equations, De Gruyter, Berlin, 2019, pp. 299–326.
[20] I. Podlubny, Fractional Differential Equations, Academic Press, San Diego, 1999.
[21] W. Rudin, Real and Complex Analysis, McGraw-Hill, Osborne, 1974.
[22] J.C. Saut, B. Scheurer, Unique continuation for some evolution equations, J. Differential Equations 66 (1987) 118–139.
[23] E.M. Stein, R. Shakarchi, Complex Analysis, Princeton University Press, Princeton, 2003.
[24] K. Sakamoto, M. Yamamoto, Initial value/boundary value problems for fractional diffusion-wave equations and applications to some inverse problems, J. Math. Anal. Appl. 382 (2011) 426–447.
[25] S. Umarov, Fractional Duhamel principle, in: Handbook of Fractional Calculus with Applications. Volume 2: Fractional Differential Equations, De Gruyter, Berlin, 2019, pp. 383–410.
[26] X. Xu, J. Cheng, M. Yamamoto, Carleman estimate for a fractional diffusion equation with half order and application, Appl. Anal. 90 (2011) 1355–1371.
[27] M. Yamamoto, Fractional calculus and time-fractional differential equations: revisit and construction of a theory, Math., 10 (2022) 698.
[28] K. Yosida, Functional Analysis (6th edition), Springer, Berlin, 1980.

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