Minkowski space is locally the Noldus limit of Poisson process causets

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Abstract

A Poisson process \( P_\lambda \) on \( \mathbb{R}^d \) with causal structure inherited from the usual Minkowski metric on \( \mathbb{R}^d \) has a normalised discrete causal distance \( D_\lambda(x, y) \) given by the height of the longest causal chain normalised by \( \lambda^{1/d} d \). We prove that \( P_\lambda \) restricted to a compact set \( Q \) converges in probability in the sense of Noldus (2004 Class. Quantum Grav. 21 839–50) to \( Q \) with the Minkowski metric.

Keywords: causal sets, discrete Lorentzian geometry, random geometry, Poisson process

1. Introduction

The notion of a causal set or causet introduced in [5] is a proposed framework for developing a theory of quantum gravity. It posits that spacetime is in some sense a continuous approximation of a set of points with a partial order given by causal precedence. It is ontologically parsimonious, and hence elegant, but being only in its nascency, it still lacks much to be a full theory of quantum gravity. Much work has been done by examining the behaviour of random causets in Minkowski space as an attempt to derive an appropriate notion of convergence [6, 8, 12] and dynamics [11], including the promising transitive percolation model. Some of the most interesting work has arisen from the relation to random partial orders in combinatorics [6].

The appropriate notion of convergence is an important question. Some promising work has been made using the notion of poset limits [6]. For a geometric analyst used to studying convergence of metric spaces, the obvious answer is to generalise the notion of Gromov–Hausdorff convergence to causal-metric spaces. This was done by Noldus in [10], who constructed a distance between compact subsets of causal distance spaces \( (Q_1, d_1) \) \( (Q_2, d_2) \) presently denoted...
which Noldus showed forms a complete metric on the class of compact causal distance spaces [9].

In this paper we study the convergence in the Noldus metric of a Poisson process \( P \) on \( \mathbb{R}^d \) with the Lorentzian metric \((dx)^2 - \sum_{i=2}^{d}(dx)^2\). The process \( P \) inherits a causal structure from \( \mathbb{R}^d \), which allows us to define a discrete causal-metric on \( P \) given by

\[
D_N(x, y) = H(x, y)/(c_d \lambda^d)
\]

for any two points \( x, y \in P \), where \( c_d \) is a number which only depends on the dimension and \( H(x, y) \) is the height of the causal set in the interval \( (x, y) \). We prove the following theorem:

**Theorem 1.** For every dimension \( d \in \mathbb{N} \) there are numbers \( K_d, C_{1,d}, C_{2,d} \) such that for every interval \( \langle Q \rangle = \langle x, y \rangle \subset \mathbb{R}^d \), every \( \varepsilon > 0 \) if a density \( \lambda \) satisfies

\[
K_d \lambda^{-1/2d} \leq \varepsilon \leq 8K_d \log \lambda,
\]

then

\[
\mathbb{P}(d_N((P \cap Q, D_N), (Q, D))) \geq \varepsilon) \leq C_{1,d} h(Q)^{2d+1} \varepsilon^{2(1-d)} \lambda^{1/d} \exp(-C_{2,d} \varepsilon^2 \lambda^{1/d}/\log^3 \lambda \varepsilon^d),
\]

where \( h(Q) \) is the Minkowskian height of \( Q \). In particular \( P \cap Q \to_N Q \) in probability as \( \lambda \to \infty \).

The proof of the theorem is straightforward, and can be seen by bounding the Noldus distance to a uniform lattice, and applying the following corollary of two later theorems which is claimed by Bollobás and Brightwell in [4]:

**Corollary 2.** Let \( H_\lambda \) denote the greatest length of a chain in \( P \cap \langle 0, 1 \rangle \subset \mathbb{R}^d \). For every \( d \in \mathbb{N} \) there are numbers \( c_d \) and \( K_d \) such that for every \( 2 \leq \mu \leq \lambda^{1/2d}/\log \log \lambda \)

\[
\mathbb{P}\left( |H_\lambda - \lambda^{1/d} c_d| \geq \frac{K_d \lambda^{1/(2d)} \log^{3/2} \lambda}{\log \log \lambda} \right) \leq 4\mu^2 \exp(-\mu^2).
\]

**Proof.** This follows directly by combing theorems 4 and 5 noting that \( c_\lambda \) is defined to be \( \mathbb{E}H_\lambda \lambda^{-1/d} \). Theorems 4 and 5 are direct generalisations of the corresponding result of Bollobás and Brightwell, which as they state only requires a little effort (provided here for the benefit of the reader).

The idea of using the height of maximal chains to define a timelike distance, is essentially quantised proper time, i.e. proper time is allowed to take values only in \( \eta \mathbb{N} \), for some scale factor \( \eta \).

While theorem 1 nicely formalises an intuitive notion of convergence of the Poisson process causet to Minkowski space, it is hoped that the real benefit will be in the quantitative estimate. In particular the dependence of convergence on the size of the set \( Q \), the density of the process, and the magnitude of the error are all quantitatively bounded. In analysis many convergence results depend on sequentially applying estimates dependent on each other. In
this case if one wished to study the convergence of a process dependent on an underlying Poisson process causet, e.g. models of waves or spinor fields, the estimate could prove useful.

2. Preliminaries

We consider Minkowski space $\mathbb{R}^d$ with the Lorentzian metric

$$dx^2 = (dx^1)^2 - \sum_{i=2}^{d} (dx^i)^2.$$  

This defines a natural causal distance on $\mathbb{R}^d$ by

$$d(x, y) = \begin{cases} \max \left\{0, (y^1 - x^1)^2 - \sum_{i=2}^{d} (y^i - x^i)^2\right\}, & \text{if } x_1 < y_1, \\ 0, & \text{if } x_1 \geq y_1. \end{cases}$$

**Definition 3.** For our purposes a causal distance $d : X \times X \to \mathbb{R}^+$ satisfies the following three properties

1. For every $x \in X$

$$d(x, x) = 0,$$

2. For every $x, y \in X$ if $d(x, y) > 0$ then $d(y, x) = 0$, and

3. if $d(x, y) > 0$ and $d(y, z) > 0$ then

$$d(x, y) + d(y, z) \leq d(x, z).$$

Although causal distances superficially resemble distances on classical metric spaces, the reverse triangle inequality makes working with them highly nonintuitive.

We can define the timelike height of a set $Q$ to be

$$h(Q) = \sup_{x, y \in Q} d(x, y).$$

Most of our results will depend on the height of the set under investigation. The notion will also be useful in rescaling the sets under consideration to a standard reference set.

We define the causal future of a point $x$ to be

$$I^+(x) = \{ y : d(x, y) > 0 \},$$

and the causal past of $y$ to be

$$I^-(y) = \{ x : d(x, y) > 0 \}.$$

Then for any two points $x, y$ we define the spacetime interval

$$\langle x, y \rangle = I^+(x) \cap I^-(y).$$

In particular for Minkowski space

$$h(\{x, y\}) = d(x, y).$$

Because Minkowski space is preserved under Lorentz transformations, it is useful to introduce the Minkowski diamond, which is the set
This slight abuse of notation is introduced for legibility’s sake, as the latter while more technically correct is less readable and less intuitive than the former.

Most importantly the standard Minkowski diamond is the set $\langle 0, 1 \rangle$. This has $d$-dimensional Lebesgue measure $C_d$, i.e.

$$\left|\langle 0, 1 \rangle\right| = C_d.$$

Consequently, because every non-empty spacetime interval $\langle x, y \rangle$ can be translated and then Lorentz boosted into a Minkowski diamond of the form $\langle 0, d(x, y) \rangle$, it follows that

$$\left|\langle x, y \rangle\right| = C_d d(x, y)^d.$$

Given two causal distance spaces $(X_1, d_1)$ and $(X_2, d_2)$ we say they are $\varepsilon$ close if there are maps $\psi : X_1 \to X_2$ and $\varphi : X_2 \to X_1$ such that

$$\sup_{x, y \in X_1} |d_1(x, y) - d_2(\psi(x), \psi(y))| \leq \varepsilon$$

and

$$\sup_{x, y \in X_2} |d_2(x, y) - d_1(\varphi(x), \varphi(y))| \leq \varepsilon.$$

The Noldus distance is

$$d_N((X_1, d_1), (X_2, d_2)) = \inf \{ \varepsilon : (X_1, d_1) \text{ and } (X_2, d_2) \text{ are $\varepsilon$ close} \}.$$

For brevity’s sake we will omit the causal distance functions when it is clear, i.e.

$$d(X_1, X_2) \text{ will mean } d((X_1, d_1), (X_2, d_2))$$

when $X_1$ and $X_2$ can be obviously (within the context of this article) be assigned causal distances.

Noldus introduced his notion of distance [10], as a generalisation of Gromov’s notion of Gromov–Hausdorff distance for metric spaces. The important properties proved in [10] and [9] are that this is indeed a distance on the class of compact causal distance spaces.

3. Causets and poisson processes

A causal set or causet, is a set $X$ with a partial order $\leq$. In other words it is a partially ordered set or poset. We say that $x < y$ if $x \leq y$ and $x \neq y$. A causet has a natural causal distance given by

$$D(x, y) = \sup \{ N : x < x_1 < \ldots < x_N < y \}.$$ 

Causets were proposed as a potential perspective for quantum gravity in [5]. Although causets are posets, it seems beneficial to refer to them as causets in the context of quantum gravity, to emphasise their relation to causality. They are usually required to be locally finite. We omit that requirement so that Minkowski space can be considered as a causal set.

One idea of the causet programme has been to use Poisson processes as a toy model of discretised spacetime. The primary benefit of a Poisson process, as opposed to deterministic discretisation of spacetime, is that it is Lorentz invariant, i.e. given a Lorentz boost $T$, and a Poisson process with density $\lambda$, $P(TP_0)$ is also a Poisson process with density $\lambda$. 


A Poisson process with density \( \lambda \) is defined by the following properties,

1. The probability of finding \( n \) points in a set \( A \) is given by
   \[
   \mathbb{P}(|A \cap P_n| = n) = e^{-|A|\lambda} \frac{|A|^n}{n!},
   \]
   where \(|\cdot|\) is either the cardinality or Lebesgue measure appropriately.

2. For \( A \) and \( B \) disjoint sets \( A \cap B \), \( P(A) \) and \( P(B) \) are independent random variables.

A Poisson process can be constructed by subdividing a \( \sigma \)-finite measure space \( X, \mu \) into countably many disjoint sets of finite measure \( A_i \) and for each \( A_i \) defining a Poisson random variable \( N_i \) distributed like \( \lambda|A_i| \), and \( x'_1, \ldots, x'_k \) uniform random variables in \( A_i \).

Given a Poisson process with density \( \lambda \), \( P_i \), on a causal distance space \( (X, d) \) equipped with a \( \sigma \)-finite measure \( \mu \), \( P_i \) inherits a causal structure from \( (X, d) \) via \( x \leq y \) if and only if \( x = y \) or \( d(x, y) > 0 \). Poisson processes were used as tools to study random partial orders. The field is vigorous, and active, but the relevant citations for this paper are Bollobás and Brightwell papers [3, 4].

For a finite causet, we can define the height \( H(Q) \) of a subset \( Q \) to be
   \[
   H(Q) = \max \{ N : \exists x_1 < \ldots < x_N \in Q \}.
   \]

For the standard Minkowski diamond, we introduce a random variable \( H \) as \( H = H(P_i \cap (0, 1)) \). An important property that follows immediately from Lorentz invariance and the scaling properties of Poisson processes is
   \[
   H(P_i \cap (x, y)) \sim H_{d/(x,y)}^d.
   \]

### 4. Concentration of measure

The goal of this section is to apply a ‘little effort’ and modify the proof of Bollobás and Brightwell in [4] to apply to Minkowski space as they claim following their theorem 1. The first step is to modify the proof of theorem 3 from the case of the Cartesian order to that of the standard Minkowski diamond. The only modifications are minor modifications to certain quantities.

**Theorem 4.** Let \( H_i = H(P_i \cap (0, 1)) \). For every \( 2 \leq \mu \leq \lambda^{1/2d} \log \lambda \)
   \[
   \mathbb{P} \left( H_i - \mathbb{E}H_i > \mu K_d \lambda^{1/2d} \log \lambda \right) \leq 4 \mu^2 \exp(-\mu^2).
   \]

**Proof.** One difference between the Minkowski diamond and the Cartesian order is the division of our domain of interest into strips. In Minkowski space we just divide our set into strips by the time coordinate \( x_1 \) instead of the sum \( \sum x_i \). The strips are defined as \( X_j = \{(x_1, \ldots, x_d) : (j-1)/m \leq x_1 \leq j/m \} \)

Now following Bollobás and Brightwell we define a new random variable
   \[ H' = \text{the length of the longest chain } C \text{ such that } |C \cap X_j| \leq 2^{d+1} \frac{\log \lambda}{\log \log \lambda}. \]

Note in the case of the Cartesian order there is a value \( 2(d+1) \) as opposed to \( 2^{d+1} \) in the definition of \( H' \). We let \( k = 2^{d+1} \frac{\log \lambda}{\log \log \lambda} \) and \( m = [d \lambda^{1/d}] \). We let
\[ a = \mu k \sqrt{2m}. \]

And apply the following lemma from [4]:

**Lemma (Lemma 4 verbatim from [4]).** Suppose \( Z = Z(U) \) is a random variable, where \( U = (U_1, \ldots, U_n) \) and the \( U_i \) are chosen independently from probability spaces \( \Omega_i \). Suppose also that, whenever \( U \) and \( V \) differ in only one coordinate (i.e. \( U_i = V_i \) for all but index one index), we have \( |Z(U) - Z(V)| \leq k \). Then, for any real \( a \), we have

\[
\mathbb{P}(|Z - \mathbb{E}[Z]| > a) \leq 2 \exp(-a^2/2m^2).
\]

Now for any point in a strip. This follows because the positive light cone from a point intersected with a strip \( X_{ij} \) intersects at most \( 2^d \) cubes of side length \( 1/m \).

From which it follows that

\[
\mathbb{P}\left(|H' - \mathbb{E}(H')| > \mu 2^{d+1}\sqrt{2d} \frac{\lambda^{1/2d} \log \lambda}{\log \log \lambda}\right) \leq 2 \exp(-\mu^2).
\]

As in [4] we introduce a random variable \( T \) which bounds \( H - H' \). For every \( J \in \{0, \ldots, 2^{d/4}\} \), let \( S_j = \{x \in \mathbb{P} : \lambda^{-1/4} j_i \leq x_i \leq \lambda^{-1/4} (j_i + 1)\} \). Let \( T_j = \max \{0, |S_j| - k/2^{d+1}\} \), and let \( T = \sum_j T_j \). As in [4, lemma 6], \( H - H' \leq T \). This is because we can delete at most \( T \) points from \( S \) so that there are no more than \( k/2^{d+1} \) points in each cube. Now let \( x \) and \( y \) denote the lowest and highest points in a layer. Then the chain from \( x \) to \( y \) passes through at most \( 2^d \) cubes, so the chain has length at most \( k \) in each strip.

The proof of lemma 7 in [4] is unchanged for Minkowski space, because it is just a claim on the size of deviation for a Poisson Process over a subset of cubes. Consequently for \( \lambda \) sufficiently large

\[
\mathbb{P}\left(T > \frac{\mu \lambda^{1/2d} \log \lambda}{\log \log \lambda}\right) \leq 2 \mu^2 e^{-\mu^2},
\]

and \( \mathbb{E}T \leq 1 \).

Combining the bound on \( |H' - \mathbb{E}(H')| \) and the bound on \( |H - H'| \) with \( T \), yields the result with \( K_d = 2^{d+1}\sqrt{2d} \).

Now we can prove the convergence of the expected height. Let

\[
c_\lambda := \frac{\mathbb{E}H_\lambda}{\lambda^{1/2d}}.
\]

From [3] we know \( c_\lambda \rightarrow c \).

**Theorem 5.** For \( \lambda \) sufficiently large

\[
c \geq c_\lambda \geq c_\lambda - \frac{K_d \log^{3/2} \lambda}{\lambda^{1/2d} \log \log \lambda}.
\]

**Proof.** As in [4], we examine the sub-intervals of \((0, 2)\), given by \((0, 1)\) and \((1, 2)\). Both are of height 1, and volume \( C_d \). Consequently \( P_3 \cap (0, 2) \) is distributed as \( \delta_2(P_2 \cap (0, 1)) \), where \( \delta_2 \) is dilation of \( \mathbb{R}^d \) by a factor of 2. Because dilation does not change the causal structure, by studying the sub-intervals we can take advantage of the self-similar causal structure. Furthermore \((0, 1)\) and \((1, 2)\) are identically distributed. As a result
Then as before we consider the longest chain $C$, and let $x$ denote its midpoint. Let $h$ be the minimum of $d(0, x)$ and $d(x, 2)$. Consequently $h \leq 1$. Without loss of generality assume that $h = d(0, x)$, hence $|\langle 0, x \rangle| \leq |\langle 0, 1 \rangle|$ and hence $H_h(\langle 0, x \rangle) \leq H_h(\langle 0, 1 \rangle)$. Thus

$$P(x \text{ is a midpoint of a chain whose length is greater than } 2E H_h$$

$$+ 2 \mu \lambda^{1/2d} \log \lambda / \log \log \lambda)$$

$$\leq P(H_h(\langle 0, x \rangle) \geq H_h + \mu K_d \lambda^{1/2d} \log \lambda / \log \log \lambda)$$

$$\cdot P(H_h(\langle x, 2 \rangle) \geq H_h + \mu K_d \lambda^{1/2d} \log \lambda / \log \log \lambda)$$

$$\leq P(H_{2^d}(\langle 0, 1 \rangle) - \mathbb{E} H_{2^d} \geq \mu K_d \lambda^{1/2d} \lambda^{1/2} \log (h^2 \lambda) / \log (h^2 \lambda))$$

$$\leq 4\mu^2 \exp(-\mu^2).$$

Assume that there are less than $2^{d+1}\lambda$ points in $\langle 0, 2 \rangle$. Then

$$P \left( H_{2^d} \leq 2E H_h + 2\mu K_d \lambda^{1/2d} \log \lambda / \log \log \lambda \right) \leq 2^{d+1}\lambda \mu^2 e^{-\mu^2}.$$

Consequently (following [4]) we set $\mu = 2\sqrt{\log \lambda}$ and

$$r_0 = 2\mu K_d \lambda^{1/2d} \log \lambda / \log \log \lambda.$$

For $\lambda$ big enough we have that

$$P(H_{2^d} \geq 2E H_h + r_0) \leq 2^{-d}\lambda^{-1}.$$

Using the bound $H_{2^d} \leq |S|$ for large deviations:

$$\mathbb{E} H_{2^d} \leq 2E H_h + r_0 + 2^{d+1}\lambda \mathbb{P}(H_{2^d} \geq 2E H_h + r_0) + \sum_{m > 2^{d+1}} m \mathbb{P}(|S| = m)$$

$$\leq 2E H_h + 4K_d \lambda^{1/2d} \log^{3/2} \lambda / \log \log \lambda + 2 + 1.$$

From this it follows that

$$c_{2^d} \leq c_\lambda + \frac{3K_d \log^{3/2} \lambda}{\lambda^{1/2d} \log \log \lambda}.$$

And hence

$$c_\lambda \geq \limsup_{\mu \to \infty} c_\mu - \sum_{j=0}^{\infty} \frac{3K_d \log^{3/2} (2^j \lambda)}{(2^{j+1} \lambda)^{1/2d} \log (2^{j+1} \lambda)}$$

$$\geq \limsup_{\mu \to \infty} c_\mu - \frac{12K_d \log^{3/2} \lambda}{\lambda^{1/2d} \log \log \lambda}.$$

5. Convergence in probability

Let $P_\lambda$ be a Poisson process of density $\lambda$ on $\mathbb{R}^d$. Let $Q_\lambda$ denote $P_\lambda \cap Q$ for every subset $Q \subset \mathbb{R}^d$. We define a causal distance on $\mathbb{R}^d$ by
Restricted to $Q$, this satisfies the required properties. Let $h = h(Q) = \varepsilon^2/16\sqrt{d}$. Let $B_r(Q) = \{ x \in \mathbb{R}^d : \exists y \in Q \ | x - y | < \eta \}$, and let $\Lambda_r = (\eta \mathbb{Z})^d \cap B_r(Q)$.

**Lemma 6.** For every $\varepsilon > 0$, the lattice $\Lambda_r$ satisfies

1. For every $x \in Q$ there are points $x_+ = x_0^+ + \varepsilon /4\sqrt{d}$ and $x_- = x_0^- - \varepsilon /4\sqrt{d}$ such that $d(x_+, x_-) \leq 2\varepsilon /4\sqrt{d}$, and $x \in \langle x_+, x_- \rangle$;
2. For every pair of points $x, y \in Q$ we have $d(x, y) \leq d(x, y) + \varepsilon /4$;
3. For every pair of points $x, y \in Q$ we have $d(x, y) + d(x, y) \leq d(x, y)$;
4. For every $x, y \in \Lambda_r$ either $d(x, y) = 0$ or $d(x, y) \geq \varepsilon^2/4\sqrt{d}$.

**Proof**

1. Given a point $x \in Q$ choose the euclidean nearest point in $\Lambda_r$, $x_0 = (x_0^+, x_0^-)$.

   Then $x_+ = x_0^+ + \varepsilon /4\sqrt{d}$, $x_- = x_0^- - \varepsilon /4\sqrt{d}$ are candidates satisfying the condition.

2. Here we consider the maximum of $\max_{n \in \mathbb{Z}} \max_{\xi \in \mathbb{Z}^{d-1}} \sqrt{n_1^2 + n_2^2 + \xi^2}$, where $n_1 \in \mathbb{Z}$ and $\xi \in \mathbb{Z}^{d-1}$, subject to $n_1^2 > \xi^2$. Assume further that $n_1 = n_2 + 4\lceil \sqrt{d} \rceil$. Then the difference is $\sqrt{4n + 16 + \eta} - \sqrt{\eta}$.

   Thus the maximal difference is $\sqrt{4n + 16}$.

   Now if we scale by $\eta$, and note that the maximum for $n + 4$ is $(h(Q) + 4\eta)/\eta$, we get the result.

3. This follows from $x_- \leq x \leq x_+$ and $y_- \leq y \leq y_+$.

4. This is because the lattice the distance will always be given by $\eta \sqrt{2}$ for $z \in \mathbb{N}$. \square

We now bound the Noldus distance of our lattice with the random causal distance, compared to

**Lemma 7.** For every $d$ there are numbers $C_{1,d}$ and $C_{2,d}$ such that for every $\varepsilon > 0$ and $\lambda$ satisfying $K_d \lambda^{1/2d} \leq \varepsilon \leq 8K_d \log \lambda$

$$P \left( \sup_{x,y \in \Lambda_r} |d(x, y) - D_h(x, y)| \geq \varepsilon /2 \right) \leq C_{1,d} h(Q)^{2d+1} \varepsilon^{2(d-1)} \lambda^{1/2d} e^{-C_{2,d} \varepsilon^2 \lambda^{1/2d} / \log^8 \lambda}.$$

**Proof.** We merely coarsely estimate, noting that $d(x, y) \geq \varepsilon^2 /4\sqrt{d}$ and hence $|\langle x, y \rangle| \geq \varepsilon^2 /4\sqrt{d}$, so for any pair of points $x, y \in \Lambda_r$, it follows that
\[ P\left( \frac{H(x, y)}{\lambda^{1/d}} - d(x, y) \right) \geq \varepsilon/4 \leq P\left( \left| \frac{H_{d(x,y)}}{c_d(x, y)\lambda^{1/d}} - 1 \right| \geq \mu c_d \frac{\log^{3/2} \lambda d(x, y)^d}{\lambda^{1/d} \sqrt{d(x, y) \log \log d(x, y)^d}} \right), \]

where \( \mu = \varepsilon^{\lambda^{2d}} \sqrt{d(x, y) \log \log d(x, y)^d}/8K_d \lambda^{3/2} d(x, y)^d. \) Consequently for \( \lambda \) sufficiently large this is bounded by

\[ \mu^2 e^{-\mu^2} \leq e^{2\lambda^{1/d} h(Q)} K_d^{-2} e^{-3(h(Q) + \log^{3/2} \lambda^{2d}).} \]

We coarsely estimate the number of pairs of points as proportional to

\[ h(Q)^d \epsilon \log \epsilon \leq 2^{2d} d h(Q)^d \epsilon^{2d} \] to yield the result. \( \square \)

We are now equipped to

**Prove theorem 1.** We consider the inclusion map \( Q_\lambda \rightarrow Q \) and the map \( Q \rightarrow Q_\lambda \), which takes \( x \) to the Euclidean nearest point in \( Q_\lambda \). We note that

\[ D_\lambda(x_+, y_-) \leq D_\lambda(x, y) \leq D_\lambda(x_-, y_+) \]

and hence

\[ d(x, y) - |d(x, y) - d(x_+, y_-)| - |D_\lambda(x_+, y_-) - d(x_+, y_-)| \]

\[ \leq D_\lambda(x, y) \leq d(x, y) + |d(x_-, y_+) - d(x, y)| + |D_\lambda(x_-, y_+) - D_\lambda(x_-, y_+)|. \]

Thus the event \( \sup_{x,y \in Q_\lambda} |d(x, y) - D_\lambda(x, y)| \geq \varepsilon \) is implied by the event

\[ \sup_{x,y \in Q_\lambda} |d(x_+, y_-) - D_\lambda(x_+, y_-)| \leq \varepsilon/4, \]

because of lemma 6 and hence

\[ P\left( \sup_{x,y \in Q_\lambda} |d(x, y) - D_\lambda(x, y)| \geq \varepsilon \right) \leq C_1 d h(Q)^{2d+1} \varepsilon \log^{3/2} \lambda^{2d}. \] (1)

Note that

\[ |d(x, y) - D_\lambda(\psi(x), \psi(y))| \leq |d(x, y) - d(\psi(x), \psi(y))| + |d(\psi(x), \psi(y) - D_\lambda(\psi(x), \psi(y))| \leq \varepsilon/2 + |d(\psi(x), \psi(y) - D_\lambda(\psi(x), \psi(y))|. \]

Consequently the event \( \sup_{x,y \in Q_\lambda} |d(x, y) - D_\lambda(\psi(x), \psi(y))| \geq \varepsilon \) is implied by

\[ \sup_{x,y \in Q_\lambda} |d(x, y) - D_\lambda(x, y)| \leq \varepsilon/2. \]

The proof then follows by applying (1) with \( \varepsilon \) replaced by \( \varepsilon/2. \) \( \square \)

The proof of theorem 4 should generalise rather naturally to the case of a Poisson process on a Lorentzian manifold. One could use the time coordinate for a stationary spacelike slice, and subdivide that. Similarly the division into cubes could be done under sufficiently regular coordinates, to guarantee the appropriate bounds on the probability of deviation.

Theorem 5 will prove more challenging to generalise. A sketch proof of the qualitative convergence of the mean was given in [1], but to generalise this result the rate of convergence needs to be made uniform over our approximating lattice as in the proof of theorem 5. This will be more challenging, as the self similarity of a Poisson process to itself on smaller subsets was integral in the proof. This will not hold in general for a Lorentzian manifold. Nonetheless by virtue of being Lorentzian, on sufficiently small scales everything would be well approximated by the Minkoswki case. One avenue of attack would thus be to try to cover a timelike curve in a Lorentzian manifold by a series of spacetime intervals. As the size of the intervals gets smaller they will be better approximated by the Minkoswki case, and this
should give a bound on the probability of deviation of a maximal chain in the causet, from the geodesic in the underlying manifold.

The notion of Noldus convergence clearly formalises an intuitive notion of convergence of causets, allowing several questions to be amenable to mathematical analysis, but in order to extrapolate appropriate dynamics for causets one would need to ascribe an appropriate notion of convergence for certain other differential geometric, and operator theoretic concepts, like curvature, spinor fields, connections, the d’Alembertian, etc [2]. Some interesting work in the Riemannian case could pave the way [7]. Therein a Gromov–Hausdorff convergence is combined with a notion of convergence of the Tangent space, to derive convergence of geodesics, curvature, and the Laplace operator. Although the framework is not immediately generalisable, it provides strong hints of an appropriate direction.

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