Planck-scale effects for Chandrasekhar model and TOV equations

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In a recent paper by Camacho a class of Planck-scale effects that is of interest from a quantum-gravity perspective was studied within the framework of the Chandrasekhar model of white dwarfs, also hoping to use observations of white dwarfs to constrain (or search for) the Planck-scale effects. We generalize Camacho’s results by considering a broader class of Planck-scale effects, and taking into account general-relativistic corrections to the Chandrasekhar model. The Planck-scale effects do turn out to be remarkably amplified to macroscopic scales, but only in a regime of ultra-high densities where the Chandrasekhar model no longer provides a meaningful physical description of stars. While our results appear to rule out the possibility of observations of white dwarfs that would provide meaningful experimental insight on the relevant Planck-scale effects, we argue that the analysis does provide some elements of intuition that could be valuable in quantum-gravity research, particularly in as much as they can contribute to a shift of focus from ultra-small length scales to ultra-high densities.

I. INTRODUCTION

Over the last decade a few experimental/observational contexts have been identified in which remarkably some ultrasmall effects introduced at the Planck scale, of the type that can be of interest for the study of the quantum-gravity problem, could be tested (see, e.g., Refs. [1, 2, 3, 4, 5, 6, 7, 8]). This had a significant impact on the perspective adopted by many of those studying the quantum-gravity problem, especially in light of the fact that up to the mid 1990s it was instead assumed as a given that the smallness of the characteristic distance scale of quantum gravity, the “Planck length” $L_P \sim 10^{-35} m$, would render these effects forever inaccessible. While this is certainly exciting, it should be stressed that the very few examples of tests of Planck-scale physics that have been identified so far provide us only with opportunities to explore the Planck-scale regime in a rather limited way. In particular, most of these tests involve (more or less directly) the laws of propagation of free particles in vacuum, and therefore provide only a rather narrow window on the quantum-gravity realm.

The recent paper in Ref. [9], by Camacho, could be rather relevant for these concerns, since it focuses on a context that has very limited overlap with the properties of free particles in vacuum: Camacho considers the implications of possible Planck-scale modifications of the energy-momentum “dispersion” relation for the analysis of the Chandrasekhar model [10] of white dwarfs. And the results of the preliminary analysis reported by Camacho in Ref. [9] were rather exciting, since they provided evidence of a rather striking amplification of the Planck-scale effects in the limit in which the mass of the white dwarf approaches the Chandrasekhar limit.

Due to the preliminary nature of Camacho’s investigation it remained unclear whether this amplification could be turned into a resource from a phenomenology perspective, and also the nature (conceptual origin) of the mechanism producing such a large amplification was not explored. But clearly the encouragement those preliminary results provided for the prospect of testing Planck-scale effects in macroscopic systems, should provide sufficient motivation for further investigation.

A first follow-up study was reported in Ref. [11], providing a first few more in-depth indications, showing in particular that the amplification of Planck-scale effects reported by Camacho is unlikely to provide the basis for a valuable phenomenological programme, at least not in the context of the Chandrasekhar model and of its Newtonian description of gravitational effects.

We are here mostly interested in characterizing this amplification mechanism from a conceptual perspective, possibly in ways that could inspire the search of other options for an associated phenomenological programme (the Chandrasekhar-model option, as observed in Ref. [11] and further deduced here, is indeed not viable). And from a broader perspective we are also interested in this amplification mechanism in relation to its implications for the intuition that guides quantum-gravity research: this intuition nearly exclusively refers, as the regime where quantum-gravity effects are nonnegligible, to the regime of ultra-short (Planckian) length scales, but we were hoping (and indeed we found, as here reported) that a possible alternative/complementary perspective might arise from investigation of Camacho’s amplification mechanism. From this perspective besides further analysis of the Newtonian case (which we also give here) it is valuable to contemplate a possible role for general-relativistic effects, and this aspect, not considered in Refs. [2, 3, 11], is indeed here preliminarily investigated.
We are also interested in these issues because of their relevance for another aspect of the recent quantum-gravity literature, which concerns the possibility of “Planck-scale deformations” of spacetime symmetries \[12, 13, 14, 15, 16, 17, 18\]. Both Ref. \[9\] and Ref. \[11\] introduced Planck-scale modifications of the dispersion relation in such a way that the underlying spacetime symmetries would be broken. Some of the new aspects we here introduce in the analysis are suitable for the description of modified dispersion relations in a deformed-symmetry scenario. And in this respect we report preliminary evidence that this type of analysis can be used to discriminate between the two possibilities.

II. CHANDRASEKHAR MODEL WITH MODIFIED DISPERSION RELATION

It is convenient for us to start by summarizing briefly the argument proposed by Camacho in Ref. \[9\], so that the perspective from which we are planing to contribute will be clearer. The analysis of Ref. \[9\] adopts a class of candidate modifications of the dispersion relation, parametrized by a parameter \(\eta\) in terms of the Planck scale \(E_p\) (we use units such as \(c=\hbar=1\))

\[
E^2 - m^2 = p^2 + \eta p^2 \left( \frac{E}{E_p} \right),
\]

which has been much studied in the literature (see, e.g., Refs. \[1, 12, 13, 19, 20, 21, 22\]). And Ref. \[9\] observes that such a modification of the dispersion relation would primarily affect the analysis of the Chandrasekhar model for white dwarfs by producing a correspondingly modified expression for the zero point energy of the system:

\[
E_0 = \frac{2V}{\hbar^3} \int_0^{p_F} dp \, 4\pi p^2 \sqrt{p^2 + m_e^2} \left[ 1 + \frac{\eta}{2} \frac{p^2}{p^2 + m_e^2} \left( \frac{\sqrt{p^2 + m_e^2}}{E_p} \right) \right],
\]

From Eq. (2) one finds

\[
E_0 = \frac{m_e^4 V}{\pi^2} \left( f(x_F) + \frac{\eta}{2} \left( \frac{m_e}{E_p} \right) g(x_F) \right)
\]

in terms of the dimensionless variable \(x_F \equiv \frac{p_F}{m_e}\) and with

\[
f(x_F) = \frac{x_F}{4} (x_F^2 + 1)^{3/2} - \frac{x_F}{8} \sqrt{x_F^2 + 1} - \frac{1}{8} \ln \left( x_F + \sqrt{x_F^2 + 1} \right),
\]

\[
g(x_F) = \int_0^{x_F} x^4 \, dx = \frac{1}{5} x_F^5.
\]

The result was then used by Camacho \[9\] to study the implications of the quantum-gravity effects for the pressure due to Pauli repulsion in the ultrarelativistic \((x_F >> 1)\) limit:

\[
P_0 = -\frac{\partial E_0}{\partial V} \simeq \frac{m_e^4}{12\pi^2} \left( x_F^4 - x_F^2 + \frac{4}{5} \frac{m_e^2 x_F^5}{E_p} \right).
\]

This is rather significant, at least conceptually, since within the Chandrasekhar model \[10\], the stability of white dwarfs is due to the equilibrium between gravitational pressure and Pauli repulsion. Therefore the result of Eq. \(6\) should balance the gravitational pressure, which in the Chandrasekhar model is described using Newtonian gravity:

\[
P_0 = \frac{G}{4\pi} \left( \frac{8m_p}{9\pi} \right)^2 m_e^4 \frac{M^2}{R^4}.
\]

\(^1\) Camacho considered the more general case of a quantum-gravity correction of the form \(\eta p^2 (E/E_p)^\alpha\) with the additional parameter \(\alpha\) giving the power of the leading-order quantum-gravity correction. For simplicity we here focus on the most optimistic scenario in which \(\alpha = 1\). Since we are investigating whether there are any chances of significant effects it makes sense to focus on the case in which the effects would be strongest.
where we introduced the notation $m_e$ for electron mass, $m_p$ for the proton mass, and
\[ \bar{M} = \frac{9\pi}{8m_p} M, \quad \bar{R} = m_e R. \] (8)
Camacho's quantum-gravity modified pressure-balance equation therefore takes the form
\[ K \frac{\bar{M}^{4/3}}{R^{4}} - K \frac{\bar{M}^{2/3}}{R^{2}} + 4 \eta \frac{m_e}{E_p} K \frac{\bar{M}^{5/3}}{R^{5}} = K' \bar{M}^{2} / \bar{R}^{4}, \] (9)
in which $K = (\frac{m^4}{12\pi^2})$ and $K' = \frac{G}{4\pi} (\frac{8m_p}{9\pi})^2 m_e^4$.

Solving Eq. (9) for the radius one finds
\[ \bar{R} = \bar{M}^{\frac{2}{3}} \sqrt{1 - \frac{K'}{K} \bar{M}^{\frac{2}{3}} - \frac{4}{5} \frac{m_e}{E_p} \frac{1}{2(1 - \frac{K'}{K} \bar{M}^{\frac{2}{3}})}}, \] (10)
which is most intelligibly characterized in terms of a correction to the Chandrasekhar result:
\[ R = R_{Chan} \left( 1 - \frac{2}{5} \frac{m_e}{E_p} \frac{1}{(1 - \frac{M}{M_0})^{1/2}} \right), \] (11)
where
\[ R_{Chan} = \frac{3}{2} \frac{\pi^{\frac{1}{2}}}{m_e} \left( \frac{M}{m_p} \right)^{\frac{1}{2}} \sqrt{1 - \left( \frac{M}{M_0} \right)^{\frac{2}{3}}}, \] (12)
is the Chandrasekhar radius and $M_0 = \left( \frac{9\pi}{8m_p} \right)^2 \left( \frac{1}{m_e} \right)^{\frac{2}{3}}$ is called Chandrasekhar limit and represents the maximum stable mass for a white dwarf in the Chandrasekhar model.

The remarkable aspect of Camacho's result (11) is the huge amplification of the quantum-gravity effect for $M \rightarrow M_0$: the quantum-gravity correction has the inevitable factor with Planck-scale suppression $\frac{m_e}{E_p}$ (and this is of order $\sim 10^{-22}$) but in this instance it also involves a factor $(1 - (\frac{M}{M_0})^{\frac{2}{3}})$ which formally diverges as $M \rightarrow M_0$. For some value of $M$ that is very close but finitely smaller than $M_0$ the correction is of $O(1)$. This result is of clear significance from a conceptual perspective, and we shall provide further elements to underline this significance here. From the conceptual perspective it is however interesting to investigate whether it is possible that this unexpected feature is a peculiarity of scenarios with broken Lorentz symmetry, or could be also a feature of scenarios with deformation of Lorentz symmetry in the sense of “doubly-special relativity” \cite{12, 13}. Both in Camacho’s original paper \cite{9} and in a follow up study \cite{11} there was no evidence that this feature might be prevented in symmetry-deformation scenarios, but we shall here show that this is a possibility.

Most importantly, one would hope that this unexpected and striking feature could be exploited for experimental searches of quantum-gravity effects, in the spirit of the recent literature on “Quantum-Gravity Phenomenology” \cite{1, 2, 3, 4, 5, 6, 7, 8}. From this perspective Camacho \cite{9} focused his investigations on two white dwarfs, specifically G156-64 and Wolf 485 A, arguing that a comparison of the properties of these two white dwarfs could provide valuable information on the correction term in his formula (here reported in Eq. (11)). One of our objectives is a more careful analysis of the possibility of using data on white dwarfs like G156-64 and Wolf 485 A as opportunities to place meaningful constraints on the model. And our first observation on this point is that these white dwarfs cannot be described in the ultrarelativistic regime, on which Camacho focused, since for them the condition $x_F >> 1$ is not satisfied. This will lead us to generalize Camacho’s analysis to include proper handling of the nonrelativistic regime, which (besides the specifics of white dwarfs like G156-64 and Wolf 485 A) could also be of rather general interest.

We also plan to consider a somewhat broader class of features both for the Planck-scale effects and in the characterization of the stars. The only study that so far has extended Camacho’s original observation was reported in Ref. \cite{11} and focused mainly on improving the picture advocated by Camacho in the direction of obtaining a more robustly quantitative analysis, including in particular the effects of a non-homogeneous mass-density distribution within the star, but still focusing exclusively on Planck-scale effects in the dispersion relation and on a Newtonian description of gravitational effects.

We will offer also a preliminary investigation of general-relativistic corrections (Section VI), and, perhaps most importantly, we combine Planck-scale effects for the dispersion relation with Planck-scale effects in the law of composition of momenta. This type of combination of Planck-scale effects is relevant for establishing whether the mechanism under study is characteristic of quantum-gravity pictures with a breakdown of spacetime symmetries or could also be found in pictures with “deformation” of symmetries.
III. DEFORMED INTEGRATION MEASURE

Both Ref. [9] and Ref. [11] considered modifications of the Chandrasekhar model implied by quantum-gravity scenarios with Planck-scale modifications of the dispersion relation. And in both papers it was assumed that the results obtained would be applicable to any proposal with modified dispersion relation, independently of other structures present in the model, and in particular that they would be applicable equally well to scenarios with breakdown of Lorentz symmetry and scenarios with deformed Lorentz symmetry.

The proposal [12, 13] of deformed Lorentz symmetry, in the DSR sense ("doubly-special relativity"), has generated rather significant interest and a relatively large literature (see, e.g., Refs. [12, 13, 14, 15, 16, 17, 18]). Already with the first results [12, 13] of this research programme it was established that modified dispersion relation of the type (1) can be introduced in a deformed-symmetry scenario but only if accompanied by a modification of the law of composition of momentum. The studies reported in Ref. [9] and Ref. [11] focus on the fact that there is a clear explicit role for the dispersion relation in the Chandrasekhar analysis, while there is no explicit role for the law of composition of momentum. However, we want to here stress that, as shown already in Refs. [23, 24], the law of composition of momenta affects the rules of integration over energy-momentum space, and these are crucially relevant for the Chandrasekhar analysis.

The considerations reported in Refs. [23, 24], while showing robustly that integration over energy-momentum is affected, are insufficient to fully specify the new rules of integration. Still the arguments reported in Refs. [23, 24] imply that the net effect should be describable in terms of a deformed measure of integration of the type

\[ d^4p \rightarrow e^{\theta E/E_F}d^4p, \]

(13)

with \( \theta \) a parameter that will need to be adjusted on the basis of future better insight on the mechanism, but should most likely be an integer multiple of \( \eta \), with one of the values \( \eta, -\eta, -3\eta \), according to Ref. [23], or the value \( 3\eta \), according to Ref. [24].

In our Planck-scale modification of the Chandrasekhar model we shall take into account both of modifications of the dispersion relation and of modifications of the integration measure.

In closing this section let us give a simple illustrative derivation of deformed integration measure that follows from a modified law of composition of momenta. For this illustrative example we focus on the following modified law of composition of momenta

\[ p+q = \{ E(p) + E(q), \rho + e^{-\eta E(p)/E_F} \}, \]

(14)

which is in use in the literature on the \( \kappa \)-Minkowski noncommutative spacetime [23, 25, 26, 27]. We look for the implications of this modified law of composition of momenta for the function \( F'(p) \) which is the integrand of \( F(p) \): \( F = \int F' dp \). And let us observe that the law of composition of momenta [14] suggests that, for each spatial component of momentum,

\[ F'(p) = \frac{F(p+\Delta p) - F(p)}{\Delta p} = \frac{F(p + \Delta p e^{-\eta E/p}) - F(p)}{\Delta p} \approx \frac{\partial F(p)}{\partial p} e^{-\eta E/p} \cdot \]

(15)

This in turn suggests that in one spatial dimension one should have

\[ F = \int F' dp = \int F' dp = \int \frac{\partial F(p)}{\partial p} e^{-\eta E/p} dp, \]

(16)

amounting effectively to a change of integration measure with respect to the standard case: \( dp \rightarrow e^{-\eta E/E_F} dp \). In the case here of interest, with 3 spatial and 1 time dimension, one ends up with

\[ d^4p \rightarrow e^{-3\eta E/E_F}d^4p, \]

(17)

which is indeed compatible with [13] for \( \theta = -3\eta \).

IV. CHANDRASEKHAR MODEL WITH MODIFIED DISPERSION RELATION AND MODIFIED MOMENTUM COMPOSITION

In light of the observations reported in the previous section we are interested in an analysis of somewhat broader scope than the ones reported in Refs. [9, 11], contemplating the implications for the Chandrasekhar model of both a
modification of type \( \text{II} \) of the dispersion relation and of a modification of the law of composition of momenta such that (on the basis of the observations reported in the preceding section, and approximating \( e^{\theta E/E_F} \approx (1 + \theta E/E_p) \))

\[
\left(1 + \theta \frac{E}{E_p}\right) d^4p. \tag{18}
\]

This deformation of the integration measure, which results from a corresponding deformation of the law of composition of momenta, affects the Chandrasekhar description of white dwarfs already at the level of the relationship between the Fermi momentum \( p_F \), the number \( N \) of fermions in the system and the volume \( V \) of the star:

\[
N = \frac{2V}{(2\pi)^4} \int_0^{p_F} p^2 \left(1 + \theta \frac{E}{E_p}\right) dp. \tag{19}
\]

In particular, in the ultrarelativistic limit this leads to

\[
p_F = \left(\frac{3N}{4V} (2\pi)^2\right)^{\frac{1}{3}} \left(1 - \frac{1}{4} \frac{\theta E}{E_p} \left(3 \frac{N}{4V}(2\pi)^2\right)^{\frac{1}{3}}\right), \tag{20}
\]

Here the Planck-scale correction is such that for positive \( \theta \) it decreases (with respect to the case without Planck-scale corrections) the value of the Fermi momentum at a given density, or equivalently one can describe the effect as going in the direction of allowing, for given value of the Fermi momentum, to place more particles in a given volume. The opposite holds for negative \( \theta \), which is a case with increased value of the Fermi momentum at a given density, or lower density at given Fermi momentum.

For the determination of zero-point energy both the modification of the dispersion relation and the modification of the integration measure play a role:

\[
E_0 = \frac{2V}{h^3} \int_0^{p_F} dp \left(1 + \theta \frac{\sqrt{p^2 + m_f^2}}{E_p}\right) 4\pi p^2 \sqrt{p^2 + m_e^2} \left(1 + \frac{\eta}{2} \frac{p^2}{E_p} \left(\frac{\sqrt{p^2 + m_e^2}}{E_p}\right)\right). \tag{21}
\]

The Planck-scale features we are introducing in the Chandrasekhar model are all encoded in the Fermi momentum and in the zero-point energy. We are going to first consider the ultrarelativistic behaviour, and then, in the next section, offer some observations on the nonrelativistic limit.

Concerning the equations of state one straightforwardly finds\(^2\):

\[
\epsilon(x_F) = \frac{E_0}{V} = \frac{3}{2} K \left(x_F (1 + 2x_F^2) \sqrt{1 + x_F^2} - \ln(x_F + \sqrt{1 + x_F^2}) + \frac{1}{3} m_f (\theta + \frac{\eta}{2} x_F^2 + x_F^3)\right), \tag{22}
\]

\[
P_0(x_F) = -\frac{\partial E_0}{\partial V} = \frac{1}{2} K \left(x_F (2x_F^2 - 3) \sqrt{1 + x_F^2} + \ln(x_F + \sqrt{1 + x_F^2}) + \frac{16}{5} m_f (\theta + \frac{\eta}{2} x_F^3)\right), \tag{23}
\]

where \( K = \frac{m^2}{4x_F^2} \), \( \epsilon \) is the energy density, \( P_0 \) is the pressure of the fermionic gas, and, also taking into account \( \frac{\dot{M}}{R} \),

\[
x_F = \frac{p_F}{m_e} = \frac{\dot{M}^4}{R} \left(1 - \theta \frac{m_e}{4E_p} \frac{\dot{M}^4}{R}\right). \tag{24}
\]

These observations allow us to derive a form of the pressure-balance equation which takes into account both the modification of the dispersion relation and the modification of the law of composition of momenta

\[
K \frac{\dot{M}^{4/3}}{R^{2/3}} \left(1 - \theta \frac{m_e}{E_p} \frac{\dot{M}^4}{R}\right) - K \frac{\dot{M}^{2/3}}{R^{2/3}} \left(1 + \theta \frac{m_e}{2E_p} \frac{\dot{M}^4}{R}\right) +
\]

\[
+ \frac{\eta}{2} \frac{m_e}{E_p} K \frac{\dot{M}^{4/3}}{R^{2/3}} = K' \frac{\dot{M}^2}{R^2}, \tag{25}
\]

\(^2\) Some of the results in this section are given in terms of the mass \( m_f \) of some fermions. Within the context of the Chandrasekhar model, on which we focus in this section, \( m_f \) should be taken as the electron mass, but in Section VI we shall refer to these formulas again, and within the perspective of Section VI \( m_f \) could still be the mass of electrons, but could also be the mass of neutrons.
where on the left-hand side we used Eq. (23), while the right-hand side describes the pressure due to the Newtonian potential, as done in Eq. (7).

From Eq. (25) one straightforwardly obtains (also see analogous derivation in Section II) a relationship between radius and mass of the white dwarf, which is most insightfully described in terms of a correction to the analogous relationship that holds in the standard Chandrasekhar model:

$$R = R_{\text{Chan}} \left(1 - \left(\frac{13\theta + 4\eta}{10}\right) \frac{m_e}{E_p} \frac{1}{\left(1 - \left(\frac{M}{m_0}\right)^2\right)^\frac{1}{2}}\right).$$

(26)

This result, which characterizes our variant of the Chandrasekhar model in the ultrarelativistic regime, exactly reproduces Camacho’s formula (11) upon taking $(13/4\theta + \eta) \rightarrow \eta$. The introduction of a deformed measure of integration, required in frameworks with a modified law of composition of momenta, has not produced stronger or different effects, but it is noteworthy that the effects due to the deformed measure of integration, with parameter $\theta$, are exactly of the same order of magnitude of the ones induced by the modification of the dispersion relation. This implies that in particular frameworks with only a modified law of composition of momenta (but no modification of the dispersion relation) would still predict exactly the same features. And on the other hand in frameworks with both a modified dispersion relation and a modified law of composition of momenta one cannot a priori exclude a cancellation of the main Planck-scale effect (the cancellation would occur if $\theta = -4/13\eta$).

In closing this section we observe that by casting the result in the form (26) (which we here adopted for allowing quick comparison to the results of Ref. [9]) some of the implications might be overlooked. The form of Eq. (26) underlines that the effects become significant when the mass of the white dwarf gets very close to the Chandrasekhar mass, and this is a characterization that provides little intuition for the type of features that should be sought in order to find an amplification of Planck-scale effects in contexts that are different from the one of the study of white dwarfs. We notice however that Eq. (26) can be rewritten equivalently as follows

$$R = R_{\text{Chan}} \left(1 - \frac{3}{2} \left(\frac{13\theta + 4\eta}{10}\right) \frac{\pi^2}{m_p m_e^2 E_p} \rho_{\text{Chan}}\right),$$

(27)

where $\rho_{\text{Chan}} = M/(4\pi R_{\text{Chan}}^3/3)$, and this provides the intuition that the effects are amplified in presence of ultra-high densities ($\rho_{\text{Chan}} \sim 2m_p m_e^2 E_p/(3\pi^2)$). There is a certain “quantum-gravity folklore” assuming that strong effects could only arise for systems of Planck-length size, but this result (in spite of its “mere academic” nature) provides encouragement for a new intuition according to which one could look for systems of extremely high density (perhaps very-unusually high density) but not necessarily Planck-length size.

V. NONRELATIVISTIC LIMIT

The Planck-scale effects that can be introduced in the ultrarelativistic limit of the Chandrasekhar model are conceptually very significant, because they provide a rather remarkable example of amplification of Planck-scale effects, and particularly because this could encountered in the description of a macroscopic system (in admittedly “ideal” conditions of ultra-high densities). However, while it is indeed intriguing conceptually, in the specific context of the Chandrasekhar model these results should at present be viewed as of merely academic interest, since the effects of the Planck-scale corrections become significant only for very small white-dwarf radius, and for real stars of such small radius and high densities the Chandrasekhar model is completely inapplicable, particularly as a result of the fact that it ignores the implications of weak interactions (which are definitely not negligible in those regimes [28]).

As mentioned, as a possible opportunity of realistic application of Planck-scale effects for the Chandrasekhar model Camacho [2] considered two white dwarfs, G156-64 and Wolf 485 A, but, as we already stressed above, the ultrarelativistic limit on which Camacho focused is not relevant for these white dwarfs since $x_F \approx 0.68$ for G156-64 and $x_F \approx 0.50$ for Wolf 485 A. And actually the majority of white dwarf stars are in the nonrelativistic $x_F \lesssim 1$ regime (the biggest $x_F$ is the one of Syrius B which is $x_F \approx 1.08$).

These observations motivate us to further extend the analysis proposed by Camacho (to which we already added the feature of a possible modification of the law of composition of momenta) by considering also the nonrelativistic regime. For this we must first establish the relationship between $x_F$, $M$ and $R$ in the nonrelativistic limit. From Eq. (19), assuming indeed $x_F \ll 1$, one finds

$$p_F = \left(\frac{3}{4} \frac{N}{V} (2\pi)^2\right)^\frac{1}{2} \left(1 - \frac{1}{5} \frac{\theta}{E_p} \frac{(\frac{3}{4} \frac{N}{V} (2\pi)^2)^{\frac{1}{2}}}{2m}\right),$$

(28)
which leads to
\[
x_F = \frac{M^+}{R} \left( 1 - \frac{1}{10} \frac{m_e \bar{M}^2}{E_p R^2} \right).
\] (29)

This must be taken into account in the analysis of the zero-point energy, which is given by
\[
P_0 = -\frac{\partial E_0}{\partial V} = \left( \frac{m_e^4}{15\pi^2} \right) \left( x_F^5 + (\eta + \theta) \frac{m_e}{E_p} x_F^5 \right),
\] (30)
as easily found assuming \( x_F \ll 1 \) in Eq. (23).

We therefore find, combining this result (30) with Eq. (17), that the Planck-scale-modified pressure-balance equation in the nonrelativistic limit takes the form
\[
\frac{4}{5} \frac{\bar{M}^2}{R^3} \left( 1 + (\theta + \eta) \frac{m_e}{E_p} \right) = K' \frac{\bar{M}^2}{R^4},
\] (31)
which can be equivalently described as the following relationship between mass and radius of the star:
\[
\bar{R} = \frac{4}{5} \frac{K}{K'} \frac{\bar{M}^2}{R^3} \left( 1 + (\theta + \eta) \frac{m_e}{E_p} \right).
\] (32)

This result excludes the possibility of any meaningful phenomenological studies of these effects in the nonrelativistic, \( x_F \ll 1 \), regime, since we are finding that the correction to the mass-radius formula is purely governed by the ratio \( m_e/E_p (\sim 10^{-22}) \).

VI. DEFORMED TOLMAN-OPPENHEIMER-VOLKOFF EQUATIONS

Our main conclusion is that the picture proposed by Camacho is probably not relevant for the phenomenology of white dwarfs (which appeared to be Camacho’s original motivation), but might carry a rather significant message from the conceptual perspective, since it shows, within a sort of “toy model”, that there is some chance of finding macroscopic manifestations of Planck-scale effects. It appears likely that future searches of more realistic descriptions of macroscopic Planck-scale effects would have to take into account general relativistic effects, rather than be confined (like Camacho’s and Chandrasekhar’s analyses) to the Newtonian limit. As a first contribution in this direction in this section we offer a preliminary characterization of how the class of Planck-scale effects we are considering would affect the Tolman-Oppenheimer-Volkoff (TOV) equations \[31, 32\], which provide a way for introducing general-relativistic effects in the Chandrasekhar picture.

The starting point for our considerations are the TOV equations\(^3\):
\[
\begin{align*}
\frac{dm}{dr} &= 4\pi \epsilon(P) r^2, \\
\frac{dP}{dr} &= -\frac{P + \epsilon(P)}{r(r-2m)} (4\pi r^3 + m(r)).
\end{align*}
\] (33)

Here \( r \) is the distance from the center of the star, and the equations simply give the general-relativistic description of the radial pressure gradient, under the assumption of spherically symmetric mass distribution (so that \( M = \int_0^R m(r)dr = \int_0^R 4\pi r^2 \epsilon(r)dr \)).

These TOV equations are solvable upon providing an equation of state, giving \( \epsilon(P) \). we shall of course be interested in the outcome of these TOV equations for the case in which the equation of state is obtained from \[29\] , so that it is an equation of state that takes into account our Planck-scale effects.

On the basis of the observations reported in the previous sections we expect the Planck-scale effects to be significant only in the ultrarelativistic regime, and we therefore focus on solving\(^4\) the TOV equations for \( x_F \gg 1 \). For this reason we will solve TOV equations in the ultrarelativistic regime, in which it is convenient to make use of the variable
\[
t = 4 \ln \left( x_F + \sqrt{1 + x_F^2} \right),
\] (34)

\(^3\) In this section we also set \( G = 1 \)

\(^4\) For \( x_F \gg 1 \) solving the TOV equations is of little interest from a strict astrophysics perspective because of stability concerns \[24\]. From our perspective the analysis of the regime \( x_F \gg 1 \) of the TOV equations is still meaningful, since we are here only interested in estimating the possible magnitude of the effects.
also taking into account that, as stressed in Section IV, our Planck-scale effects of modification of the law of composition of momenta also introduce Planck-scale corrections, described in Eq. (24), for the formula that gives $x_F$ (whereas $x_F$ would not be affected by any Planck-scale effects in scenarios that exclusively modify the dispersion relation).

In the ultrarelativistic regime ($x_F \gg 1$ and, as a result, also $t \gg 1$) the TOV equations, if one indeed assumes that the Planck-scale-corrected $\epsilon(P)$ is obtained from (22)-(23), take the form\(^5\)

\[
\begin{align*}
\frac{dm}{dr} & = \frac{1}{2} r^{-2} e^t + \frac{1}{2} r^{-2}(2\theta + \eta) \frac{m_f}{E_p} e^{\frac{1}{2} t} \\
\frac{dt}{dr} & = -\frac{4}{(1-r^2(\theta+\eta))} \left( e^t + m(r) \right) - \frac{r^3}{(1-r^2(\theta+\eta))} \left( 2\theta + \eta \right) \frac{m_f}{E_p} e^{\frac{1}{2} t}
\end{align*}
\]

From these combined equations one can straightforwardly obtain the explicit form of the $r$-dependence of $m$ and of (the exponential of) $t$ at leading order in $m_f/E_p$:

\[
\begin{align*}
e^t(r) & = \frac{3}{7} \frac{1}{r^2} + \frac{m_f}{E_p} \left( \frac{3}{7} \frac{1}{r^2} \right)^{\frac{1}{2}} \left( \frac{884}{441} \theta + \frac{271}{196} \eta \right) r^{-\frac{3}{2}} \\
m(r) & = \frac{3}{14} r + \frac{m_f}{E_p} \left( \frac{3}{7} \frac{1}{r^2} \right)^{\frac{1}{2}} \left( \frac{2335}{882} \theta + \frac{335}{196} \eta \right) r^{\frac{3}{2}}
\end{align*}
\]

These results can in turn be used to establish the relation between mass and energy density, which is given by

\[
m(\epsilon) \simeq \left( \frac{3}{14} \right)^{\frac{1}{2}} \epsilon_0^{\frac{1}{2}} \left( 1 + \frac{m_f}{E_p} f(\theta, \eta) \epsilon_0^{\frac{1}{2}} \right)
\]

where $\epsilon_0$ is the central density of the star and $f(\theta, \eta) = \frac{1}{7} \left( \frac{2777}{441} \theta + \frac{1611}{996} \eta \right) + 2^{\frac{4}{7}} \left( \frac{2}{7} \theta + \eta \right)$.

It is noteworthy that our result (38) shows that the Planck-scale effects can be significant for ultra-high densities ($\epsilon_0 \sim (E_p/m_f)^4$), without necessarily requiring the abstraction of a star of Planck-length size. We are therefore encountering once again the characteristics already discussed for our analysis of the modified Chandrasekhar model.

\section*{VII. CLOSING REMARKS}

The fact, established over the last decade, that there are some (however rare) windows on the Planck-scale realm represents a, previously unexpected, significant opportunity for quantum-gravity research. Now that we have the first few examples of phenomenological analyses establishing access to effects introduced genuinely at the Planck scale we have in some sense the luxury of starting to worry about the fact that these first few examples only probe certain specific aspects of the Planck-scale realm, and look for ways to achieve a broader ability to investigate this realm. The study we here reported, taking off from related previous analyses reported in Refs. \cite{9, 11}, starts to set the stage for new ways to probe the Planck-scale realm in the context of macroscopic systems in astrophysics.

We unfortunately found that it is unlikely that, as originally hoped by Camacho \cite{9}, near-term observations of white dwarfs might play a role in the phenomenology of certain Planck-scale effects. Yet the result of our analysis provides encouragement for the idea that there is indeed a new regime of phenomena were tiny Planck-scale effects could be observably large, and it is the regime of extremely high densities. The densities that appear to be required for observably large Planck-scale effects are ultra-high, and it might require a dedicated multi-stage research programme to identify the most promising applications that might allow to unveil such effects. And such a programme will need to face the challenges posed by the still only partial ordinary-physics understanding of some of the most interesting macroscopic systems in astrophysics, particularly when gravitational collapse (and associated high densities) is involved. But the payoff that could expected appears to be well worth the effort, since such a novel window on the Planck-scale realm could have particularly significant impact on our ability to investigate the quantum-gravity problem.

Also for what concerns the more academic/conceptual side of the issues here discussed these studies should motivate further investigation, particularly for what concerns the discrimination between scenarios with broken spacetime symmetries and scenarios with deformed spacetime symmetries in the DSR sense. As discussed in Section III through

\footnote{For compactness, in this section we work in units such that $K = 2/(3\pi)$, which are not unusual in studies of TOV equations. For a neutron gas this means that the unit of length has been fixed to be $l = 1.36 \times 10^4 \text{m}$, while the unit of mass is $\mu = 1.83 \times 10^{33} \text{Kg}$.}
refinements of the analysis here reported it should be possible to gain further insight on the type of observables that can be used to discriminate between these two alternative scenarios for the fate of spacetime symmetries at the Planck scale.

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