LATTICE STRUCTURE ON BOUNDED HOMOMORPHISMS BETWEEN TOPOLOGICAL LATTICE RINGS

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Abstract. Suppose $X$ is a locally solid lattice ring. It is known that there are three classes of bounded group homomorphisms on $X$ whose topological structures make them again topological rings. In this note, we consider lattice structure on them; more precisely, we show that, under some mild assumptions, they are locally solid lattice rings.

1. Introduction and Preliminaries

It is a natural direction to consider lattice structures on known algebraic concepts like groups or vector spaces. Moreover, topological aspects of them and essence of their connections with order structures are interesting in their own rights as well as in applications in other disciplines.

The concept of a lattice group ($\ell$-group, for short) was firstly investigated in [2, 4]. In addition, topological $\ell$-groups as an extension of topological Riesz spaces are appeared in [9, 10], at first. Although, Riesz spaces are widely investigated in many directions for decades, lattice groups are rarely considered in the literatures; only recently, a comprehensive reference announced regarding basic properties of topological $\ell$-groups (see [3] for more details).

Nevertheless, the notion of a lattice ring ($\ell$-ring) is even considered less than $\ell$-groups in the contexts. To our best knowledge, it is initially investigated in [3, 5]. The situation got stricter while adding topological notion to them; the earliest special literature is [12].

Note that since topological $\ell$-groups are a generalization of topological Riesz spaces which contain many known and applicable objects like Banach lattices and examples therein, they are investigated in more details at least in the contexts so that topological $\ell$-rings seem to be largely unexplored with respect to topological $\ell$-groups. On the other hand, topological rings arise almost in many directions of topological fields; for example, the completion of a topological field is always a topological ring. Moreover, the set of all real continuous functions on a Hausdorff topological space, the set of all matrices defined on a field, are examples of rings which are widely useful in the literatures. So, it is of independent interest to discover different directions of rings such as topological and order notions; topological and order aspects are considered in several contexts, separately (see [1, 6, 11, 12], for example) but using both order and topological ones have been investigated not so much.

In [5], Mirzavaziri and the author considered three non-equivalent classes of bounded group homomorphisms on a topological ring and endowed them with appropriate topologies which make them again topological rings. Now, suppose $X$ is a locally solid $\ell$-ring. In this note, our attempt is to consider lattice structures on these classes of bounded homomorphisms. In fact, we show that under some mild hypotheses, they configure locally solid $\ell$-rings.

For recent progress on topological $\ell$-groups as well as basic expositions on these notions, see [6, 13]. Finally, for undefined terminology, general theme about $\ell$-rings and the related subjects, we refer the reader to [5, 13].

Let us first, recall some required notions and terminology. Suppose $X$ is a topological ring. A set $B \subseteq X$ is called bounded if for each zero neighborhood $W \subseteq X$, there is a zero neighborhood $V \subseteq X$ such that $VB \subseteq W$ and $BV \subseteq W$. 

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A topological lattice ring (\(\ell\)-ring) is a topological ring which is simultaneously an \(\ell\)-group such that the multiplication and order structure are compatible via the inequality \(|x \cdot y| \leq |x| \cdot |y|\); for more details, we refer the reader to [6].

A Birkhoff and Pierce ring (\(f\)-ring) is a lattice ordered ring with this property: \(a \land b = 0\) and \(c \geq 0\) imply that \(ca \land b = ac \land b = 0\). For ample facts regarding this subject, see [5]. It was initially presented by Birkhoff and Pierce in [3] to illustrate some understandable examples in lattice ring theory and apparently, it turned out to have many interesting and fruitful tools among the category of lattice rings.

Now, we recall some definition we need in the sequel (see [8] for further notifications about these facts). It should be mentioned here that in [8], the authors used the notion \(B(X,Y)\) for rings of all bounded group homomorphisms between topological rings; in this note, we replace it with \(\text{Hom}(X,Y)\) in compatible with [7] for homomorphisms as well as to show their nature as a homomorphism not an operator.

**Definition 1.** Let \(X\) and \(Y\) be two topological rings. A group homomorphism \(T : X \to Y\) is said to be

1. **\(nr\)-bounded** if there exists a zero neighborhood \(U \subseteq X\) such that \(T(U)\) is bounded in \(Y\);
2. **\(br\)-bounded** if for every bounded set \(B \subset X\), \(T(B)\) is bounded in \(Y\).

The set of all \(nr\)-bounded (\(br\)-bounded) homomorphisms from a topological ring \(X\) to a topological ring \(Y\) is denoted by \(\text{Hom}_{nr}(X,Y)\) (\(\text{Hom}_{br}(X,Y)\)). We write \(\text{Hom}(X)\) instead of \(\text{Hom}(X,X)\).

Now, assume \(X\) is a topological ring. The class of all \(nr\)-bounded group homomorphisms on \(X\) equipped with the topology of uniform convergence on some zero neighborhood is denoted by \(\text{Hom}_{nr}(X)\). Observe that a net \((S_\alpha)\) of \(nr\)-bounded homomorphisms converges uniformly on a neighborhood \(U\) to a homomorphism \(S\) if for each neighborhood \(V\) there exists an \(\alpha_0\) such that for each \(\alpha \geq \alpha_0\), \((S_\alpha - S)(U) \subset V\).

The class of all \(br\)-bounded group homomorphisms on \(X\) endowed with the topology of uniform convergence on bounded sets is denoted by \(\text{Hom}_{br}(X)\). Note that a net \((S_\alpha)\) of \(br\)-bounded homomorphisms uniformly converges to a homomorphism \(S\) on a bounded set \(B \subset X\) if for each zero neighborhood \(V\) there is an \(\alpha_0\) with \((S_\alpha - S)(B) \subset V\) for each \(\alpha \geq \alpha_0\).

The class of all continuous group homomorphisms on \(X\) equipped with the topology of \(cr\)-convergence is denoted by \(\text{Hom}_{cr}(X)\). A net \((S_\alpha)\) of continuous homomorphisms \(cr\)-converges to a homomorphism \(S\) if for each zero neighborhood \(W\), there is a neighborhood \(U\) such that for every zero neighborhood \(V\) there exists an \(\alpha_0\) with \((S_\alpha - S)(U) \subset VW\) for each \(\alpha \geq \alpha_0\).

Note that \(\text{Hom}_{nr}(X)\), \(\text{Hom}_{br}(X)\), and \(\text{Hom}_{cr}(X)\) form subrings of the ring of all group homomorphisms on \(X\), in which, the multiplication is given by function composition.

### 2. Main Results

**Remark 1.** Suppose \(G\) is a locally solid \(\ell\)-group. We have seen in [13] that \(\text{Hom}^b_{cr}(G)\), the group of all order bounded \(nb\)-bounded homomorphisms, \(\text{Hom}^c_{cr}(G)\), the group of all order bounded continuous homomorphisms and \(\text{Hom}^b_{br}(G)\), the group of all order bounded \(bb\)-bounded homomorphisms, are locally solid \(\ell\)-groups. It can be verified easily that they are in fact locally solid \(\ell\)-rings where the multiplication is determined via function composition.

**Remark 2.** Compatible with homomorphisms on a topological \(\ell\)-group, not every order bounded group homomorphism between topological \(\ell\)-rings is bounded and vise versa.

Suppose \(X = \mathbb{R}^\mathbb{N}\), the ring of all sequences with product topology, coordinate-wise ordering and pointwise multiplication. Consider the identity group homomorphism \(I\) on \(X\). It is indeed order bounded but not \(nr\)-bounded (see [8, Example 2.1]). Moreover, if we replace pointwise multiplication in \(\mathbb{R}^\mathbb{N}\) with zero one, then the identity group homomorphism is still order bounded but neither \(nr\)-nor \(br\)-bounded. Suppose \(X = \ell_\infty\) with the usual norm topology and \(Y\) is \(\ell_\infty\).
with the product topology inherited from $\mathbb{R}^2$; both of them, with coordinate-wise ordering and pointwise multiplication are topological $\ell$-rings. Then the identity group homomorphism from $Y$ into $X$ is order bounded but not continuous, certainly.

We recall that topology $\tau$ on a topological $\ell$-ring $(X, \tau)$ is Fatou if $X$ has a base of zero neighborhoods which are order closed.

**Lemma 1.** Suppose $X$ is a Dedekind complete locally solid $f$-ring with Fatou topology and $\text{Hom}_{\text{nr}}^b(X)$ is the ring of all order bounded $\text{nr}$-bounded group homomorphisms. Then $\text{Hom}_{\text{nr}}^b(X)$ is an $\ell$-ring.

**Proof.** It suffices to prove that for a homomorphism $T \in \text{Hom}_{\text{nr}}^b(X)$, $T^+ \in \text{Hom}_{\text{nr}}^b(X)$. By [13] Theorem 1, for each positive $x \in X$, we have

$$T^+(x) = \sup\{T(u) : 0 \leq u \leq x\}.$$ 

Choose a zero neighborhood $U \subseteq X$ such that $T(U)$ is bounded. So, for arbitrary neighborhood $W$, there is a zero neighborhood $V$ with $VT(U) \subseteq W$. Therefore, for each $x \in U_+$ and for each $y \in V_+$, $yT(x) \in W$, so that using [5] Theorem 3.15, solidity of zero neighborhoods $U, V$, and order closedness of $W$, yields that $T^+(U)$ is also bounded.

**Remark 3.** It is proved in [9] Theorem 4.1 that locally solidness of a topological $\ell$-group $G$ is equivalent to uniform continuity of the most lattice operations of $G$. Now, consider topological $\ell$-ring $X$. Zero neighborhoods of $X$ and locally solidness of them lay on group and lattice structures of $X$ so that we can have a similar statement for locally solidness of zero neighborhoods in $X$. Moreover, note that by [6] Remark 2, uniform continuity of the modulus is not an equivalent condition for locally solidness of a topological $\ell$-group. This is an vital difference between Riesz spaces and $\ell$-groups.

**Theorem 1.** Suppose $X$ is a Dedekind complete locally solid $f$-ring with Fatou topology. Then $\text{Hom}_{\text{nr}}^b(X)$ is locally solid with respect to the uniform convergence topology on some zero neighborhood.

**Proof.** Let $T \in \text{Hom}_{\text{nr}}^b(X)$ and $x \in X_+$. By [13] Theorem 1, we have

$$T^+(x) = \sup\{T(u) : 0 \leq u \leq x\}.$$ 

Now, suppose $(T_\alpha)$ is a net of order bounded $\text{nr}$-bounded group homomorphisms that converges uniformly on some zero neighborhood $U \subseteq X$ to the homomorphism $T$ in $\text{Hom}_{\text{nr}}^b(X)$. Choose arbitrary neighborhood $W \subseteq X$. Fix $x \in U_+$. Recall that for two subsets $A, B$ in an $\ell$-ring, we have $\sup(A) - \sup(B) \leq \sup(A - B)$. Thus,

$$\sup\{T_\alpha(u) : 0 \leq u \leq x\} - \sup\{T(u) : 0 \leq u \leq x\} \leq \sup\{(T_\alpha - T)(u) : 0 \leq u \leq x\}.$$ 

There exists an $\alpha_0$ such that $(T_\alpha - T)(U) \subseteq W$ for each $\alpha \geq \alpha_0$. Therefore, using the order closedness of neighborhood $W$ and solidity of neighborhood $U$, we have

$$T_\alpha^+(x) - T^+(x) \leq (T_\alpha - T)^+(x) \in W.$$ 

Now, using Remark 3 yields the desired result.

The following lemma may be known; to our best knowledge, we could not find any proof for it; we present a proof for the sake of completeness.

**Lemma 2.** Suppose $X$ is a locally solid $f$-ring. Then, the solid hull of a bounded set is also bounded.
Proof. Suppose $B \subseteq X$ is bounded. Then, by usual definition of a solid hull, we have

$$\text{Sol}(B) = \{ x \in X, \exists y \in B : |x| \leq |y| \}. $$

Let $W$ be an arbitrary zero neighborhood of $X$. There exists a zero neighborhood $V$ with $VB \subseteq W$. For each $x \in \text{Sol}(B)$, there is $y \in B$ such that $|x| \leq |y|$ so that for each $z \in V$, the inequality $|zx| = |z||x| \leq |z||y| = |zy|$ in connection with solidity of zero neighborhood $W$, imply that $V\text{Sol}(B) \subseteq W$, as we wanted. \hfill \Box

**Lemma 3.** Suppose $X$ is a Dedekind complete locally solid $f$-ring with Fatou topology and $\text{Hom}^b_{\alpha}(X)$ is the ring of all order bounded $br$-bounded group homomorphisms. Then $\text{Hom}^b_{\alpha}(X)$ is an $\ell$-ring.

Proof. It suffices to prove that for a homomorphism $T \in \text{Hom}^b_{\alpha}(X)$, $T^+ \in \text{Hom}^b_{\alpha}(X)$. By [13, Theorem 1], we have

$$T^+(x) = \sup \{ T(u) : 0 \leq u \leq x \}. $$

Fix a bounded set $B \subseteq X$. Without loss of generality, we may assume that $B$ is also bounded; otherwise consider the solid hull of $B$ which is by Lemma [2] bounded. So, for arbitrary neighborhood $W$, there is a zero neighborhood $V$ with $VT(B) \subseteq W$. Therefore, for each $x \in B_+$ and for each $y \in V_+$, $yT(x) \in W$, so that using [5, Theorem 3.15], solidity of zero neighborhood $V$ and bounded set $B$, and order closedness of $W$, we see that $T^+(B)$ is also bounded. \hfill \Box

**Theorem 2.** Suppose $X$ is a Dedekind complete locally solid $f$-ring with Fatou topology. Then $\text{Hom}^b_{\alpha}(X)$ is locally solid with respect to the uniform convergence topology on bounded sets.

Proof. Let $T \in \text{Hom}^b_{\alpha}(X)$ and $x \in X_+$. By [13, Theorem 1], we have

$$T^+(x) = \sup \{ T(u) : 0 \leq u \leq x \}. $$

Now, suppose $(T_\alpha)$ is a net of order bounded $br$-bounded group homomorphisms that converges uniformly on bounded sets to the homomorphism $T$ in $\text{Hom}^b_{\alpha}(X)$. Choose arbitrary neighborhood $W \subseteq X$. Fix $x \in B_+$. By Lemma [2] $B$ can be considered solid. Recall that for two subsets $A, C$ in an $\ell$-ring, we have $\sup(A) - \sup(C) \leq \sup(A - C)$. Thus,

$$\sup \{ T_\alpha(u) : 0 \leq u \leq x \} - \sup \{ T(u) : 0 \leq u \leq x \} \leq \sup \{ (T_\alpha - T)(u) : 0 \leq u \leq x \}. $$

There exists an $\alpha_0$ such that $(T_\alpha - T)(B) \subseteq W$ for each $\alpha \geq \alpha_0$. Therefore, using the order closedness of neighborhood $W$ and solidity of bounded set $B$, we have

$$T_\alpha^+(x) - T^+(x) \leq (T_\alpha - T)^+(x) \in W. $$

\hfill \Box

**Lemma 4.** Suppose $X$ is a Dedekind complete locally solid $f$-ring with Fatou topology and $\text{Hom}^b_{cr}(X)$ is the ring of all order bounded continuous group homomorphisms. Then $\text{Hom}^b_{cr}(X)$ is an $\ell$-ring.

Proof. It suffices to prove that for a homomorphism $T \in \text{Hom}^b_{cr}(X)$, $T^+ \in \text{Hom}^b_{cr}(X)$. By [13, Theorem 1], for any $x \in X_+$, we have

$$T^+(x) = \sup \{ T(u) : 0 \leq u \leq x \}. $$

Choose arbitrary zero neighborhood $W$. There exists a zero neighborhood $U$ with $T(U) \subseteq W$. Therefore, for each $x \in U_+$, $T(x) \in W$, so that $T^+(x) \in W$ using solidity of $U$ and order closedness of $W$. Thus, we see that $T^+(U) \subseteq W$. \hfill \Box

**Theorem 3.** Suppose $X$ is a Dedekind complete locally solid $f$-ring with Fatou topology. Then $\text{Hom}^b_{\alpha}(X)$ is locally solid with respect to the $cr$-convergence topology.
Proof. Let $T \in \text{Hom}^b_{cr}(X)$ and $x \in X_+$. By [13 Theorem 1], we have

$$T^+(x) = \sup\{T(u) : 0 \leq u \leq x\}.$$ 

Now, suppose $(T_\alpha)$ is a net of order bounded continuous group homomorphisms that $cr$-converges to the homomorphism $T$ in $\text{Hom}^b_{cr}(X)$. Choose arbitrary neighborhood $W \subseteq X$. There is a zero neighborhood $U \subseteq X$ such that for each zero neighborhood $V \subseteq X$ there exists an $\alpha_0$ with $(T_\alpha - T)(U) \subseteq VW$ for each $\alpha \geq \alpha_0$. Recall that for two subsets $A, B$ in an $\ell$-ring, we have $\sup(A) - \sup(B) \leq \sup(A - B)$. Thus,

$$\sup\{T_\alpha(u) : 0 \leq u \leq x\} - \sup\{T(u) : 0 \leq u \leq x\} \leq \sup\{(T_\alpha - T)(u) : 0 \leq u \leq x\}.$$ 

Therefore, using the order closedness of neighborhoods $V, W$ and solidness of zero neighborhood $U$, we have

$$T_\alpha^+(x) - T^+(x) \leq (T_\alpha - T)^+(x) \in VW.$$ 

This would complete the proof. 

□

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