Incompressible limit for the two-dimensional isentropic Euler system with critical initial data

Taoufik Hmidi and Samira Sulaiman
Mathematics Research Institute of Rennes, Université de Rennes 1, Campus de Beaulieu, 35 042 Rennes Cedex, France
(thmidi@univ-rennes1.fr)

(MS received 9 March 2012; accepted 29 August 2013)

We study the low-Mach-number limit for the two-dimensional isentropic Euler system with ill-prepared initial data belonging to the critical Besov space $B^2_{2,1}$. By combining Strichartz estimates with the special structure of the vorticity, we prove that the lifespan of the solutions goes to infinity as the Mach number goes to zero. We also prove strong convergence results of the incompressible parts to the solution of the incompressible Euler system.

1. Introduction

This work is devoted to the study of the compressible isentropic Euler equations described by the following Cauchy problem:

$$\begin{align*}
\rho(\partial_t u + u \cdot \nabla u) + \nabla P &= 0, \\
\partial_t \rho + \text{div}(\rho u) &= 0, \\
(u, \rho)|_{t=0} &= (u_0, \rho_0).
\end{align*}
$$

(1.1)

Here, $u: [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ stands for a velocity field with $T > 0$, and $\rho$ is the density of a gas which is assumed to be everywhere positive ($\rho > 0$). The pressure $P$ satisfies the polytropic $\gamma$ law

$$P(\rho) = \rho^\gamma,$$

with $\gamma > 1$ the adiabatic exponent.

Local well-posedness can be proven by using the symmetrization approach of Kawashima et al. [21], which transforms (1.1) to a symmetric quasilinear hyperbolic system. Upon introducing the sound speed

$$c = 2\frac{\sqrt{\gamma}}{\gamma - 1} \rho^{(\gamma - 1)/2},$$

the system (1.1) may be reduced to

$$\begin{align*}
\partial_t u + u \cdot \nabla u + \gamma c \nabla c &= 0, \\
\partial_t c + u \cdot \nabla c + \gamma c \text{div} u &= 0, \\
(u, c)|_{t=0} &= (u_0, c_0).
\end{align*}
$$

(1.2)
with $\bar{\gamma} \triangleq (\gamma - 1)/2$. Klainerman and Majda proved local well-posedness of (1.2) in the framework of Sobolev spaces $H^s$, $s > \frac{1}{2}d + 1$ (see [16, 17]). It is also well known that we can formally derive the incompressible Euler equations from the compressible system (1.2). This can be done by taking the sound speed close to a constant, taking the velocity very small and looking at large timescales. More precisely, we scale the unknowns as follows:

$$u(t, x) = \bar{\gamma}c_0\varepsilon v_\varepsilon(x\bar{\gamma}c_0t, x), \quad c(t, x) = c_0 + \bar{\gamma}c_0\varepsilon c_\varepsilon(x\bar{\gamma}c_0t, x).$$

This gives us the system

$$\begin{align*}
\partial_t v_\varepsilon + v_\varepsilon \cdot \nabla v_\varepsilon + \bar{\gamma}c_\varepsilon \nabla c_\varepsilon + \frac{1}{\varepsilon} \nabla c_\varepsilon &= 0, \\
\partial_t c_\varepsilon + v_\varepsilon \cdot \nabla c_\varepsilon + \bar{\gamma}c_\varepsilon \text{div} v_\varepsilon + \frac{1}{\varepsilon} \text{div} v_\varepsilon &= 0,
\end{align*}$$

$$\begin{aligned}
(v_\varepsilon, c_\varepsilon)|_{t=0} &= (v_{0,\varepsilon}, c_{0,\varepsilon}).
\end{aligned}$$

(1.3)

The small parameter $\varepsilon$ is called the Mach number and measures the compressibility of the fluid; for more details about the derivation of this last model, we refer the reader to [11, 16, 22] and the references therein.

The rigorous derivation of the incompressible Euler equations is known to be far from complete, and depends on several factors, such as the geometry of the domain where the fluid is assumed to evolve and the state of the initial data. We have assumed that the gas fills the whole space and so there are no boundary effects. We shall therefore limit our discussion to the influence of the initial data.

The first result in this direction goes back to Klainerman and Majda [16, 17], who proved the existence of unique solutions for a short time which may be taken independent of the parameter $\varepsilon$. They furthermore established the strong convergence of solutions towards solutions of the incompressible Euler equations in the well-prepared case, that is, where $(\text{div} v_{0,\varepsilon}, \nabla c_{0,\varepsilon}) = O(\varepsilon)$ as $\varepsilon \to 0$. Recall that the incompressible inviscid flow is described by

$$\begin{align*}
\partial_t v + v \cdot \nabla v + \nabla p &= 0, \\
\text{div} v &= 0, \\
v|_{t=0} &= v_0.
\end{align*}$$

(1.4)

The well-preparedness assumption is crucial in order to get a uniform bound for $(\partial_t v_\varepsilon)$, and to apply in turn the Aubin–Lions compactness lemma.

In the ill-prepared case, where $(v_{0,\varepsilon}, c_{0,\varepsilon})$ is only assumed to be bounded in some Sobolev spaces $H^s$ with $s > \frac{1}{2}d + 1$ and the incompressible parts of $(v_{0,\varepsilon})$ converge to some vector field $v_0$, the situation is more delicate. This is due to the time derivative $\partial_t v_\varepsilon$, which is of size $O(1/\varepsilon)$. To overcome this difficulty, Ukai [27] used the dispersion of highly oscillating acoustic waves in order to show that the compressible parts disappear as the Mach number tends to zero. We point out that this problem was already discussed in several papers and for more general models (see, for example, [1, 3, 8–10, 16–20, 22, 24, 27]).

In contrast to the incompressible Euler system in two space dimensions, the system (1.1) develops singularities in finite time and for some supersonic smooth initial data (see, for example, [23, 26]). As for the lifespan $T_\varepsilon$ of the compressible
Incompressible limit for the 2D isentropic Euler system

solutions and its asymptotic behaviour for small Mach number, some partial results are known in the literature. First, according to [17], we have \( \lim \inf_{\varepsilon \to 0} T_{\varepsilon} \geq T \), with \( T \) the lifespan of the incompressible solution. This result was established only for the well-prepared case and with initial data belonging to the Sobolev space \( H^s \) with \( s > \frac{1}{2}d + 2 \). The proof relies on perturbation theory, which is no longer valid for weaker regularities \( \frac{1}{2}d + 2 > s > \frac{1}{2}d + 1 \). Thus, it follows from the preceding result that, for planar motion, one has \( \lim_{\varepsilon \to 0} T_{\varepsilon} = +\infty \). We point out that this result can be improved by using Strichartz estimates for the subcritical regularity \( H^s \) with \( s > 2 \), leading to the lower bound \( T_{\varepsilon} \geq C \log \log \frac{1}{\varepsilon} \) (see remark 1.2). Second, it seems that we can get better information about the lifespan of the solutions when the initial data have some special structure. In [2], Alinhac proved that, in two space dimensions and for axisymmetric non-vanishing compactly supported initial data, the lifespan of the solution is equivalent to \( \frac{1}{\varepsilon} \). In dimension 3 and for irrotational velocity, Sideris [26] established that the solutions are almost global in time, that is, their lifespans are bounded below by \( e^{C/\varepsilon} \). Finally, we mention the results of [13,25], which deal with global existence under suitable conditions on the initial data: the initial density must be small and have compact support, and the spectrum of \( \nabla u_0 \) must be far away from the negative real numbers.

In this paper we shall investigate the incompressible limit problem in two space dimensions with critical regularities, meaning that the initial data belong to the borderline Besov space \( B^2_{2,1} \). Our result is as follows.

**Theorem 1.1.** Let \( \{ (v_{0,\varepsilon}, c_{0,\varepsilon}) \}_{0<\varepsilon \leq 1} \) be a family of initial data such that

\[
\sum_{q \geq -1} 2^{2q} \sup_{0<\varepsilon \leq 1} \| (\Delta_q v_{0,\varepsilon}, \Delta_q c_{0,\varepsilon}) \|_{L^2} < +\infty.
\]

Then the system (1.3) has a unique solution \( (v_{\varepsilon}, c_{\varepsilon}) \in C([0,T_{\varepsilon}]; B^2_{2,1}) \) with

\[
\lim_{\varepsilon \to 0} T_{\varepsilon} = +\infty.
\]

Moreover, the acoustic parts of the solutions vanish as \( \varepsilon \to 0 \):

\[
\lim_{\varepsilon \to 0} \| (\text{div} v_{\varepsilon}, \nabla c_{\varepsilon}) \|_{L^1_T L^\infty} = 0.
\]

Assume in addition that \( \lim_{\varepsilon \to 0} \| \mathbb{P} v_{0,\varepsilon} - v_0 \|_{L^2} = 0 \) for some \( v_0 \in B^2_{2,1} \). Then the incompressible parts \( (\mathbb{P} v_{\varepsilon})_\varepsilon \) converge to the solution \( v \) of the system (1.4) associated to the initial data \( v_0 \). More precisely, for every \( T > 0 \),

\[
\lim_{\varepsilon \to 0} \| \mathbb{P} v_{\varepsilon} - v \|_{L^T_\varepsilon B^2_{2,1}} = 0.
\]

Here, \( \mathbb{P} = v - \nabla \Delta^{-1} \text{div} v \) denotes the Leray projection over solenoidal vector fields.

Before discussing the important ingredients of the proof, some useful remarks are in order.

**Remark 1.2.** The result of theorem 1.1 will be generalized in theorem 5.1, which gives precise information about the lifespan of solutions. Specifically, we may take

\[
T_{\varepsilon} \geq C_0 \log \log \Psi \left( \log \frac{1}{\varepsilon} \right),
\]

where \( \Psi \) is an increasing function.
Remark 1.6. The Strichartz estimates for $||\div (\varphi_\varepsilon, \nabla c_\varepsilon)||_{L^q_t L^\infty_x}$ that we are led to use are, in some sense, critical, which we may see as follows. It is known that, for the linear operator, Strichartz estimates allow a gain of $1/4$ derivative on the initial data compared with Sobolev embeddings, and thus we only need to require $(\varphi_{0,\varepsilon}, c_{0,\varepsilon}) \in B_{2,1}^{1/4}\}. However, in treating the nonlinear term as a source term, this method requires the solutions to be in the space $B_{2,1}^{11/4}$, which scales above the critical space $B_{2,1}^{2}$. To remedy to this loss of regularity, we first establish Strichartz estimates for $||\div (\varphi_{0,\varepsilon}, c_{0,\varepsilon})||_{L^q_t L^\infty_x}$ with an explicit polynomial rate in $\varepsilon$. Next, we interpolate these Strichartz estimates with energy estimates in order to prove

$$T_\varepsilon \geq C_0 \log \log \frac{1}{\varepsilon}.$$  

For more details, see remark 5.2.

Remark 1.3. If the initial data satisfy $\sup_{0<\varepsilon \leq 1} ||(\varphi_{0,\varepsilon}, c_{0,\varepsilon})||_{H^{s}} < \infty$ for some $s > 2$, then this family satisfies the boundedness assumption of theorem 1.1. To see this, we write

$$\sum_{q \geq -1} 2^{2q} \sup_{0<\varepsilon \leq 1} ||(\Delta_q \varphi_{0,\varepsilon}, \Delta_q c_{0,\varepsilon})||_{L^2} \leq \sum_{q \geq -1} 2^{q(2-s)} \sup_{0<\varepsilon \leq 1} 2^{2q} ||(\Delta_q \varphi_{0,\varepsilon}, \Delta_q c_{0,\varepsilon})||_{L^2} \leq C \sup_{0<\varepsilon \leq 1} ||(\varphi_{0,\varepsilon}, c_{0,\varepsilon})||_{H^{s}}.$$

Remark 1.4. The initial data are assumed to be bounded, but we can weaken this assumption by allowing the compressible parts to blow up sufficiently slowly as $\varepsilon \to 0$. We can implement the ideas developed in [11], but for the sake of a clear presentation we omit this discussion here.

Remark 1.5. In the second part of theorem 1.1, the velocity $\varphi_0$ belongs naturally to the space $L^2$, but according to the assumption

$$\sum_{q \geq -1} 2^{2q} \sup_{0<\varepsilon \leq 1} ||\Delta_q \varphi_{0,\varepsilon}||_{L^2} < \infty,$$

combined with the weak compactness of the Besov space $B_{2,1}^{2}$, we obtain $\varphi_0 \in B_{2,1}^{2}$. Moreover, we get strong convergence to $v_0$ in the space $B_{2,1}^{2}$. Indeed, for $N \in \mathbb{N}^{*}$, we can write

$$||\varphi_0 - v_0||_{B_{2,1}^{2}} \lesssim 2^{2N} ||\varphi_0 - v_0||_{L^2} + \sum_{q \geq N} 2^{2q} \left( ||\Delta_q \varphi_0||_{L^2} + \sup_{0<\varepsilon \leq 1} ||\Delta_q \varphi_{0,\varepsilon}||_{L^2} \right).$$

Letting $\varepsilon$ go to zero gives

$$\limsup_{\varepsilon \to 0} ||\varphi_0 - v_0||_{B_{2,1}^{2}} \lesssim \sum_{q \geq N} 2^{2q} \left( ||\Delta_q \varphi_0||_{L^2} + \sup_{0<\varepsilon \leq 1} ||\Delta_q \varphi_{0,\varepsilon}||_{L^2} \right),$$

and therefore when $N$ goes to $\infty$ we obtain the desired result.

Remark 1.7. The Strichartz estimates for $\sup_{0<\varepsilon \leq 1} ||(\varphi_{0,\varepsilon}, c_{0,\varepsilon})||_{L^q_t L^\infty_x}$ that we are led to use are, in some sense, critical, which we may see as follows. It is known that, for the linear operator, Strichartz estimates allow a gain of $1/4$ derivative on the initial data compared with Sobolev embeddings, and thus we only need to require $(\varphi_{0,\varepsilon}, c_{0,\varepsilon}) \in B_{2,1}^{1/4}$. However, in treating the nonlinear term as a source term, this method requires the solutions to be in the space $B_{2,1}^{11/4}$, which scales above the critical space $B_{2,1}^{2}$. To remedy to this loss of regularity, we first establish Strichartz estimates for $\sup_{0<\varepsilon \leq 1} ||(\varphi_{0,\varepsilon}, c_{0,\varepsilon})||_{L^q_t L^\infty_x}$ with an explicit polynomial rate in $\varepsilon$. Next, we interpolate these Strichartz estimates with energy estimates in order to prove

$$T_\varepsilon \geq C_0 \log \log \frac{1}{\varepsilon}.$$  

For more details, see remark 5.2.
that \( \lim_{\varepsilon \to 0} \| (\text{div} v_\varepsilon, \nabla c_\varepsilon) \|_{L^1_T L^\infty} = 0 \). More precisely, we use the Strichartz estimates for low frequencies, and we use energy estimates for high frequencies. For the energy estimates, there is no explicit decay in frequency compared with the subcritical regularity \( H^s, s > 2 \), and this is the sense in which we mean that the Strichartz estimates are critical.

Let us now briefly describe the main difficulties that arise when we try to work in the critical framework and give the basic ideas used in solving them. There are essentially two principal difficulties: the first difficulty has a compressible nature and concerns the Strichartz estimate for the acoustic parts \( \| (\text{div} v_\varepsilon, \nabla c_\varepsilon) \|_{L^1_T L^\infty} \), which is critical in the sense of remark 1.6. To remedy to this problem, we shall proceed in the spirit of [14]: we construct a non-decreasing profile \( \Psi \) such that \( \lim_{q \to +\infty} \Psi(q) = +\infty \), which satisfies

\[
\sup_{\varepsilon} \sum_q \Psi(q) 2^{2q} \| (\Delta_q v_0, \Delta_q c_0) \|_{L^2} < +\infty.
\]

Then, by performing energy estimates, we prove that the solutions have the same additional decay in the time interval \([0, T_\varepsilon]\). This allows us to conclude that there is no energy transfer to higher frequencies. Consequently, interpolating between this estimate and the Strichartz estimate for the lower degree quantity

\[
\| (\nabla \Delta^{-1} v_\varepsilon, c_\varepsilon) \|_{L^1_T L^\infty},
\]

we get that the acoustic parts vanish as the Mach number goes to zero. We point out that the rate convergence is implicit, but it can be related to the distribution of the energy of the initial data for high frequencies.

The second difficulty has an incompressible nature and concerns the Beale–Kato–Majda criterion [5]. It asserts that for the system (1.4) we can propagate the initial regularity in the full time interval \([0, T]\), provided that we can control the quantity \( \| \omega \|_{L^1_T L^\infty} \), where \( \omega \) denotes the vorticity of the velocity. But this criterion is only known to be valid for the subcritical case \( H^s, s > \frac{1}{2}d + 1 \). For a critical framework, such as the Besov space \( B^{d/2+1}_{2,1} \), this criterion should be replaced by control of the stronger norm \( \| \omega \|_{L^1_T B^{0}_{2,1}} \). In dimension 2, global existence in this critical framework was demonstrated some years ago by Vishik [28], who was able to prove the following persistence result with a linear growth in the Besov space \( B^{0}_{\infty,1} \):

\[
\| \omega(t) \|_{B_{\infty,1}^0} \leq C \| \omega_0 \|_{B_{\infty,1}^0} \left( 1 + \int_0^t \| \nabla v(\tau) \|_{L^\infty} \, d\tau \right).
\]

Vishik’s method relies on a composition result in the Besov space \( B^{0}_{\infty,1} \) with a logarithmic growth. We emphasize that in [15] another proof was given which has the advantage of fitting with general models such as transport diffusion equations. Now, in order to get a lower bound for the lifespan, we need to establish a similar estimate for the compressible case. This is more subtle due to the nonlinearities in the vorticity equation and to the lack of incompressibility. We recall that the vorticity \( \omega_\varepsilon = \partial_1 v_2^\varepsilon - \partial_2 v_1^\varepsilon \) satisfies the compressible transport equation

\[
\partial_t \omega_\varepsilon + v_\varepsilon \cdot \nabla \omega_\varepsilon + \omega_\varepsilon \text{div} v_\varepsilon = 0.
\]
In theorem 4.1 we shall prove the following estimate: for $p \in [1, +\infty[,$
\[
\|\omega_\varepsilon(t)\|_{B_0^0,1} \leq C \|\omega_0,\varepsilon\|_{B_0^0,1} \left(1 + \exp(C\|\nabla v_\varepsilon\|_{L^1_t L^\infty})\right)\|\text{div} v_\varepsilon\|_{L^1_t B_2^{2/p},1}^2 \times \left(1 + \int_0^t \|\nabla v_\varepsilon(\tau)\|_{L^\infty} \, d\tau\right).
\]

We observe that when $\text{div} v_\varepsilon = 0$ we get Vishik's estimate. Our proof is in the spirit of [15] but with significant modifications. In the first step, we use Lagrangian coordinates combined with a filtration procedure to get rid of the compressible part. In the second step we establish a suitable splitting of the vorticity and use a dynamical interpolation method.

For the incompressible limit, we first prove strong convergence in $L^\infty_{\text{loc}}(\mathbb{R}^+; L^2)$ by using Strichartz estimates. This leads, via some standard interpolation inequalities, to strong convergence in Besov spaces $B_{2,1}^s,$ for all $s < 2.$ However, for strong convergence in the space of the initial data $B_{2,1}^2$ and in order to rule out any energy transfer, we establish an additional frequency decay of the energy uniformly with respect to $t$ and $\varepsilon.$ In other words, from the assumption on the initial data, we can find a non-decreasing function $\Psi$ independent of $\varepsilon$ such that $(v_0,\varepsilon, c_0,\varepsilon)$ belongs to $B_{2,\Psi}^2; \text{see definition 2.2.}$ This regularity will be conserved in the full interval $[0, T_\varepsilon[,$ and thus the energy transfer for higher frequencies cannot occur.

The paper is organized as follows: we recall in §2 some basic results about Littlewood–Paley operators and Besov spaces. In §3, we gather some energy estimates for heterogeneous Besov spaces $B_{2,1}^s,$ and we establish some useful Strichartz estimates for the acoustic parts of the fluid. In §4, we establish a logarithmic estimate for a compressible transport model. In §5, we generalize the result of theorem 1.1 and we give the proofs of the main results.

2. Basic tools

In this section, we recall the so-called Littlewood–Paley operators and give some of their elementary properties. We also introduce some function spaces and review some important lemmas that will be used later.

2.1. Littlewood–Paley operators

We denote by $C$ any positive constant that may change from line to line and by $C_0$ a real positive constant depending on the size of the initial data. We shall use the following notation: for any non-negative real numbers $A$ and $B,$ the notation $A \lesssim B$ means that there exists a positive constant $C$ independent of $A$ and $B$ and such that $A \leq CB.$

To define Besov spaces we first introduce the dyadic partition of the unity (see, for example, [7]). There are two non-negative radial functions, $\chi \in \mathcal{D}(\mathbb{R}^2)$ and $\varphi \in \mathcal{D}(\mathbb{R}^2 \setminus \{0\}),$ such that
\[
\chi(\xi) + \sum_{q \neq 0} \varphi(2^{-q} \xi) = 1, \quad \forall \xi \in \mathbb{R}^2,
\]
\[
\sum_{q \in \mathbb{Z}} \varphi(2^{-q} \xi) = 1, \quad \forall \xi \in \mathbb{R}^2 \setminus \{0\},
\]
and

$$|p - q| \geq 2 \implies \text{supp } \varphi(2^{-p} \cdot) \cap \text{supp } \varphi(2^{-q} \cdot) = \emptyset,$$

$$q \geq 1 \implies \text{supp } \chi \cap \text{supp } \varphi(2^{-q} \cdot) = \emptyset.$$

Letting $$u \in S'(\mathbb{R}^2)$$, we define the Littlewood–Paley operators by

$$\Delta_{-1} u = \chi(D) u, \quad \forall q \geq 0, \quad \Delta_q u = \varphi(2^{-q}D) u \quad \text{and} \quad S_q u = \sum_{-1 \leq p < q} \Delta_p u.$$

We can easily check that

$$u = \sum_{q \in \mathbb{Z}} \Delta_q u, \quad \forall u \in S'(\mathbb{R}^2).$$

Moreover, the Littlewood–Paley decomposition satisfies the property of almost orthogonality: for any $$u, v \in S'(\mathbb{R}^2),$$

$$\Delta_p \Delta_q u = 0 \quad \text{if} \quad |p - q| \geq 2,$$

$$\Delta_p (S_{q-1} u \Delta_q v) = 0 \quad \text{if} \quad |p - q| \geq 5.$$

Note that the above operators $$\Delta_q$$ and $$S_q$$ continuously map $$L^p$$ uniformly into itself with respect to $$q$$ and $$p$$. We define the homogeneous operators in the same way:

$$\dot{\Delta}_q u = \varphi(2^{-q}D) v \quad \text{and} \quad \dot{S}_q u = \sum_{j \leq q-1} \dot{\Delta}_j u.$$

Note that these operators are of convolution type. For example, for $$q \in \mathbb{Z},$$ we have

$$\dot{\Delta}_q u = 2^{2q} \hat{h}(2^q \cdot) * u \quad \text{with} \quad h \in \mathcal{S}, \quad \hat{h}(\xi) = \varphi(\xi).$$

Now we recall Bernstein inequalities (see, for example, [7]).

**Lemma 2.1.** There exists a constant $$C > 0$$ such that for all $$q \in \mathbb{N}, k \in \mathbb{N}$$ and for every tempered distribution $$u$$ we have

$$\sup_{|\alpha| = k} \|\partial^\alpha S_q u\|_{L^b} \leq C^k 2^{q(k+2(1/a-1/b))} \|S_q u\|_{L^a} \quad \text{for} \ b \geq a \geq 1,$$

$$C^{-k} 2^{qk} \|\Delta_q u\|_{L^a} \leq \sup_{|\alpha| = k} \|\partial^\alpha \Delta_q u\|_{L^a} \leq C^k 2^{qk} \|\Delta_q u\|_{L^a}.$$

**2.2. Besov spaces**

Now we shall define the Besov spaces by using Littlewood–Paley operators. Let $$(p, r) \in [1, +\infty]^2$$ and $$s \in \mathbb{R}$$. Then the non-homogeneous Besov space $$B^s_{p,r}$$ is the set of tempered distributions $$u$$ such that

$$\|u\|_{B^s_{p,r}} := (2^q \|\Delta_q u\|_{L^p})_{L^r} < +\infty.$$

The homogeneous Besov space $$\dot{B}^s_{p,r}$$ is given by the set of $$u \in S'(\mathbb{R}^d)$$ up to polynomials such that

$$\|u\|_{\dot{B}^s_{p,r}} := (2^q \|\dot{\Delta}_q u\|_{L^p})_{L^r} < +\infty.$$
We remark that the usual Sobolev space $H^s$ coincides with $B^s_{2,2}$ for $s \in \mathbb{R}$ and the Hölder space $C^s$ coincides with $B^s_{\infty,\infty}$ when $s$ is not an integer.

The following embeddings are an easy consequence of Bernstein inequalities:

$$B^s_{p_1,r_1} \hookrightarrow B^{s+2(1/p_2-1/p_1)}_{p_2,r_2}, \quad p_1 \leq p_2, \quad r_1 \leq r_2.$$ 

Letting $T > 0$ and $\rho \geq 1$, we denote by $L^{\rho}_T B^s_{p,r}$ the space of tempered distributions $u$ such that

$$\|u\|_{L^{\rho}_T B^s_{p,r}} := \|(2^{qs}\|\Delta_q u\|_{L^p})^\rho\|_{L^r_T} < +\infty.$$ 

Now we shall introduce the heterogeneous Besov spaces, which are an extension of the classical Besov spaces.

**Definition 2.2.** Let $\Psi : \{-1\} \cup \mathbb{N} \to \mathbb{R}_+^*$ be a given function.

(i) We say that $\Psi$ belongs to the class $\mathcal{U}$ if the following conditions are satisfied:

(a) $\Psi$ is a non-decreasing function;

(b) there exists $C > 0$ such that

$$\sup_{x \in \mathbb{N} \cup \{-1\}} \frac{\Psi(x+1)}{\Psi(x)} \leq C.$$ 

(ii) We define the class $\mathcal{U}_{\infty}$ by the set of functions $\Psi \in \mathcal{U}$ that also satisfy $\lim_{x \to +\infty} \Psi(x) = +\infty$.

(iii) Let $s \in \mathbb{R}$, $(p,r) \in [1, +\infty]^2$ and let $\Psi \in \mathcal{U}$. We define the heterogeneous Besov space $B^s_{p,r,\Psi}$ as follows:

$$u \in B^s_{p,r,\Psi} \iff \|u\|_{B^s_{p,r,\Psi}} = (\Psi(q)2^{qs}\|\Delta_q u\|_{L^p})^\rho \|_{L^r_T} < +\infty.$$ 

**Remark 2.3.**

- From condition (b) of the above definition, we see that the profile $\Psi$ has at most an exponential growth: there exists $\alpha > 0$ such that

$$\Psi(-1) \leq \Psi(q) \leq \Psi(-1)2^{\alpha q}, \quad \forall q \geq -1.$$ 

When the profile $\Psi$ has an exponential growth, i.e. $\Psi(q) = 2^{\alpha q}$ with $\alpha \in \mathbb{R}_+$, the space $B^s_{p,r,\Psi}$ reduces to the classical Besov space $B^{s+\alpha}_{p,r}$.

- Condition (b) seems to be necessary for the definition of $B^s_{p,r,\Psi}$: it allows us to get a coherent definition that is independent of the choice of the dyadic partition.

The following lemma, proved in [14], is important for the proof of theorem 1.1. Roughly speaking, it says that any element of a given Besov space is always more regular than the prescribed regularity.

**Lemma 2.4.** Let $s \in \mathbb{R}$, $p \in [1, +\infty]$, $r \in [1, +\infty]$ and $f \in B^s_{p,r}$. Then there exists a function $\Psi \in \mathcal{U}_{\infty}$ such that $f \in B^s_{p,r,\Psi}$.
This yields the following result.

**Corollary 2.5.** Let $s \in \mathbb{R}$, $p \in [1, +\infty]$, $r \in [1, +\infty]$ and let $(g_\varepsilon)_{0 < \varepsilon \leq 1}$ be a family of smooth functions satisfying

$$
\left( \sum_{q \geq 1} 2^{qsr} \sup_{0 < \varepsilon \leq 1} \| \Delta_q g_\varepsilon \|_{L^r}^r \right)^{1/r} < +\infty.
$$

Then there exists $\Psi \in \mathcal{U}_\infty$ such that $g_\varepsilon \in B^s_{p,r}$ with uniform bounds, that is,

$$
\sup_{0 < \varepsilon \leq 1} \left( \sum_{q \geq -1} \Psi^r(q) 2^{qsr} \sup_{0 < \varepsilon \leq 1} \| \Delta_q g_\varepsilon \|_{L^r}^r \right)^{1/r} < +\infty.
$$

**Proof.** From the assumption, we have

$$
\sum_{q \geq 1} u_q < +\infty
$$

with $u_q := 2^{qsr} \sup_{0 < \varepsilon \leq 1} \| \Delta_q g_\varepsilon \|_{L^r}$. Then, by using lemma 2.4, there exists $\Psi \in \mathcal{U}_\infty$ such that $g_\varepsilon \in B^s_{p,r}$, that is,

$$
\left( \sum_{q \geq -1} \Psi^r(q) 2^{qsr} \sup_{0 < \varepsilon \leq 1} \| \Delta_q g_\varepsilon \|_{L^r}^r \right)^{1/r} < +\infty.
$$

The proof of the corollary is now complete. \qed

For the proof of the next proposition, see, for example, [8,11].

**Lemma 2.6.** Let $u$ be a smooth vector field, not necessarily of zero divergence. Let $f$ be a smooth solution of the transport equation

$$
\partial_t f + u \cdot \nabla f = g, \quad f|_{t=0} = f_0,
$$

such that $f_0 \in B^s_{p,r}(\mathbb{R}^2)$ and $g \in L^1_{\text{loc}}(\mathbb{R}_+; B^s_{p,r})$. Then the following assertions hold.

(i) Let $(p, r) \in [1, \infty]^2$ and $s \in [0, 1]$. Then

$$
\| f(t) \|_{B^s_{p,r}} \leq C e^{CV(t)} \left( \| f_0 \|_{B^s_{p,r}} + \int_0^t e^{-CV(\tau)} \| g(\tau) \|_{B^s_{p,r}} \, d\tau \right),
$$

where

$$
V(t) = \int_0^t \| \nabla u(\tau) \|_{L^\infty} \, d\tau
$$

and $C$ is a constant depending on $s$.

(ii) Let $s \in [1, 0]$, $r \in [1, +\infty]$ and $p \in [2, +\infty]$ with $s + 2/p > 0$. Then

$$
\| f(t) \|_{B^s_{p,r}} \leq C e^{CV_p(t)} \left( \| f_0 \|_{B^s_{p,r}} + \int_0^t e^{-CV_p(\tau)} \| g(\tau) \|_{B^s_{p,r}} \, d\tau \right),
$$

with $V_p(t) = \| \nabla u \|_{L^1 L^\infty} + \| \text{div} u \|_{L^1 B^s_{p,\infty}}$. 

We now prove the following result, which will be needed later.

**Lemma 2.7.** Let $v$ be a vector field such that $v \in B^{1,1}_\infty \cap L^2$ and let $\omega$ be its vorticity. Then we have

$$
\|\nabla v\|_{L^\infty} \lesssim \|v\|_{L^2} + \|\text{div } v\|_{B^{0,1}_\infty} + \|\omega\|_{B^{0,1}_\infty}.
$$

**Proof.** We split the velocity into incompressible and compressible parts: $v = P v + Q v$. Then we have

$$
curl v = \text{curl } P v.
$$

Using the Bernstein inequality, the continuity of $\Delta_q P : L^p \to L^p$, for all $p \in [1, \infty]$ uniformly in $q$ and $\|\Delta_q v\|_{L^\infty} \sim 2^{-q} \|\Delta_q \omega\|_{L^\infty}$,

$$
\|\nabla P v\|_{L^\infty} \leq \|\Delta_{-1} \nabla P v\|_{L^\infty} + \sum_{q \in \mathbb{N}} \|\Delta_q \nabla P v\|_{L^\infty}
\lesssim \|\Delta_{-1} P v\|_{L^2} + \sum_{q \in \mathbb{N}} 2^q \|\Delta_q P v\|_{L^\infty}
\lesssim \|v\|_{L^2} + \sum_{q \in \mathbb{N}} \|\Delta_q \omega\|_{L^\infty}
\lesssim \|v\|_{L^2} + \|\omega\|_{B^{0,1}_\infty}.
$$

(2.1)

On the other hand, using the Bernstein inequality leads to

$$
\|\nabla Q v\|_{L^\infty} = \|\nabla^2 \Delta^{-1} \text{div } v\|_{L^\infty}
\leq \|\Delta_{-1} \nabla^2 \Delta^{-1} \text{div } v\|_{L^\infty} + \sum_{q \geq 0} \|\Delta_q \nabla^2 \Delta^{-1} \text{div } v\|_{L^\infty}
\lesssim \|v\|_{L^2} + \|\text{div } v\|_{B^{0,1}_\infty}.
$$

Combining this estimate with (2.1) gives

$$
\|\nabla v\|_{L^\infty} \lesssim \|v\|_{L^2} + \|\text{div } v\|_{B^{0,1}_\infty} + \|\omega\|_{B^{0,1}_\infty}.
$$

This completes the proof of the lemma.

3. Preliminaries

This section is devoted to some useful estimates for the system (1.3) that will be crucial for the proof of the main results. We shall discuss some energy estimates for the full system (1.3) and give some Strichartz estimates for the acoustic operator.

3.1. Energy estimates

Here we list two energy estimates for (1.3). The first one is very classical and is concerned with the $L^2$-estimate. However, the second one deals with the energy estimates in the heterogeneous Besov space $B^{s,\Psi}_{2,1}$. All the spaces of the initial data are constructed over the Lebesgue space $L^2$ in order to remove the singular terms and get uniform estimates with respect to the Mach number $\varepsilon$. The full proof can be found in [14], but for the convenience of the reader we shall sketch it here.
Proposition 3.1. Let \((v_\varepsilon, c_\varepsilon)\) be a smooth solution of (1.3) and \(\Psi \in \mathcal{U}\) (see definition 2.2).

(i) \(L^2\)-estimate: there exists \(C > 0\) such that, for all \(t \geq 0\),

\[
\|(v_\varepsilon, c_\varepsilon)(t)\|_{L^2} \leq C\|(v_{0,\varepsilon}, c_{0,\varepsilon})\|_{L^2} e^{C\|\text{div} v_\varepsilon\|_{L^1} t}.
\]

(ii) Besov estimates: for \(s > 0\), \(r \in [1, +\infty]\), there exists \(C > 0\) such that

\[
\|(v_\varepsilon, c_\varepsilon)(t)\|_{B_{s,r}^2} \leq C\|(v_{0,\varepsilon}, c_{0,\varepsilon})\|_{B_{s,r}^2} e^{C\varepsilon t}
\]

with

\[
V_\varepsilon(t) := \|\nabla v_\varepsilon\|_{L^1} + \|\nabla c_\varepsilon\|_{L^1}.
\]

Remark 3.2. We emphasize that for \(\Psi \equiv 1\) the space \(B_{s,r}^2\) reduces to the classical Besov space \(B_{s,r}^2\). Therefore, we get the known energy estimate (see, for example, [11])

\[
\|(v_\varepsilon, c_\varepsilon)(t)\|_{B_{s,r}^2} \leq C\|(v_{0,\varepsilon}, c_{0,\varepsilon})\|_{B_{s,r}^2} e^{CV_\varepsilon(t)}.
\] (3.1)

Proof. (i) Taking the \(L^2\) inner product of the first equation of (1.3) with \(v_\varepsilon\) and integrating by parts yields

\[
\frac{1}{2} \frac{d}{dt} \|v_\varepsilon(t)\|_{L^2}^2 - \frac{1}{2} \int_{\mathbb{R}^2} (|v_\varepsilon|^2 + \bar{\gamma}|c_\varepsilon|^2) \text{div} v_\varepsilon \, dx - \frac{1}{\varepsilon} \int_{\mathbb{R}^2} c_\varepsilon \text{div} v_\varepsilon \, dx = 0.
\]

Multiplying the second equation of (1.3) by \(c_\varepsilon\) and integrating by parts gives

\[
\frac{1}{2} \frac{d}{dt} \|c_\varepsilon(t)\|_{L^2}^2 + \left(\bar{\gamma} - \frac{1}{2} \int_{\mathbb{R}^2} |c_\varepsilon|^2 \text{div} v_\varepsilon \, dx\right) + \frac{1}{\varepsilon} \int_{\mathbb{R}^2} c_\varepsilon \text{div} v_\varepsilon \, dx = 0.
\]

Thus, by adding these identities, one finds

\[
\frac{d}{dt} (\|v_\varepsilon(t)\|_{L^2}^2 + \|c_\varepsilon(t)\|_{L^2}^2) = \int_{\mathbb{R}^2} \text{div} v_\varepsilon (|v_\varepsilon|^2 + (1 - \bar{\gamma})|c_\varepsilon|^2) \, dx \leq \|\text{div} v_\varepsilon(t)\|_{L^2} (\|v_\varepsilon(t)\|_{L^2}^2 + \|c_\varepsilon(t)\|_{L^2}^2).
\]

By integrating in time we infer that

\[
\|v_\varepsilon(t)\|_{L^2}^2 + \|c_\varepsilon(t)\|_{L^2}^2 \leq \|v_{0,\varepsilon}\|_{L^2}^2 + \|c_{0,\varepsilon}\|_{L^2}^2 + \int_0^t \|\text{div} v_\varepsilon(\tau)\|_{L^2} (\|v_\varepsilon(\tau)\|_{L^2} + \|c_\varepsilon(\tau)\|_{L^2}) \, d\tau.
\]

According to Gronwall’s inequality,

\[
\|v_\varepsilon(t)\|_{L^2}^2 + \|c_\varepsilon(t)\|_{L^2}^2 \leq (\|v_{0,\varepsilon}\|_{L^2}^2 + \|c_{0,\varepsilon}\|_{L^2}^2) \exp(C\|\text{div} v_\varepsilon\|_{L^1} t).
\]
(ii) First, for \( q \geq -1 \) we set \( f_q := \Delta_q f \). Then, localizing in frequency the equations (1.3) gives
\[
\partial_t v_{\varepsilon,q} + v_{\varepsilon} \cdot \nabla v_{\varepsilon,q} + \gamma_{\varepsilon} \nabla c_{\varepsilon,q} + \frac{1}{\varepsilon} \nabla c_{\varepsilon,q} = -[\Delta_q, v_{\varepsilon} \cdot \nabla] v_{\varepsilon} - [\Delta_q, c_{\varepsilon} \cdot \nabla] c_{\varepsilon} := F_{\varepsilon,q}^1,
\]
\[
\partial_t c_{\varepsilon,q} + v_{\varepsilon} \cdot \nabla c_{\varepsilon,q} + \gamma_{\varepsilon} \text{div} v_{\varepsilon,q} + \frac{1}{\varepsilon} \text{div} v_{\varepsilon,q} = -[\Delta_q, v_{\varepsilon} \cdot \nabla] c_{\varepsilon} - \gamma [\Delta_q, c_{\varepsilon}] \text{div} v_{\varepsilon} := F_{\varepsilon,q}^2,
\]
\[
(v_{\varepsilon,q}, c_{\varepsilon,q})|_{t=0} = (\Delta_q v_{0,\varepsilon}, \Delta_q c_{0,\varepsilon}).
\]
Similarly to the above \( L^2 \) energy estimate, we obtain
\[
\frac{1}{2} \frac{d}{dt} \| (v_{\varepsilon,q}, c_{\varepsilon,q})(t) \|^2_{L^2} = \frac{1}{2} \int_{\mathbb{R}^2} \text{div} v_{\varepsilon,q} (|v_{\varepsilon,q}|^2 + |c_{\varepsilon,q}|^2) \, dx - \gamma \int_{\mathbb{R}^2} c_{\varepsilon}(v_{\varepsilon,q} \cdot \nabla c_{\varepsilon,q} + c_{\varepsilon,q} \text{div} v_{\varepsilon,q}) \, dx
\]
\[
+ \int_{\mathbb{R}^2} (F_{\varepsilon,q}^1 v_{\varepsilon,q} + F_{\varepsilon,q}^2 c_{\varepsilon,q}) \, dx
\]
\[
= \frac{1}{2} \int_{\mathbb{R}^2} \text{div} v_{\varepsilon,q} (|v_{\varepsilon,q}|^2 + |c_{\varepsilon,q}|^2) \, dx + \gamma \int_{\mathbb{R}^2} \nabla c_{\varepsilon} \cdot (c_{\varepsilon,q} v_{\varepsilon,q}) \, dx
\]
\[
+ \int_{\mathbb{R}^2} (F_{\varepsilon,q}^1 v_{\varepsilon,q} + F_{\varepsilon,q}^2 c_{\varepsilon,q}) \, dx
\]
\[
\leq \frac{1}{2} (\| \text{div} v_{\varepsilon}(t) \|_{L^\infty} + \gamma \| \nabla c_{\varepsilon}(t) \|_{L^\infty}) (\| (v_{\varepsilon,q}, c_{\varepsilon,q})(t) \|^2_{L^2} + \| (F_{\varepsilon,q}^1, F_{\varepsilon,q}^2) \|_{L^2} (\| (v_{\varepsilon,q}, c_{\varepsilon,q})(t) \|_{L^2},
\]
where in the last line we have used the Young inequality \(|ab| \leq \frac{1}{2}(a^2 + b^2)\). This implies that
\[
\frac{d}{dt} \| (v_{\varepsilon,q}, c_{\varepsilon,q})(t) \|_{L^2} \lesssim (\| \text{div} v_{\varepsilon}(t) \|_{L^\infty} + \| \nabla c_{\varepsilon}(t) \|_{L^\infty})
\]
\[
\times \| (v_{\varepsilon,q}, c_{\varepsilon,q})(t) \|_{L^2} + \| (F_{\varepsilon,q}^1, F_{\varepsilon,q}^2) \|_{L^2}.
\]
Multiplying this inequality by \( \Psi(q)^{2s/q} \) and taking the \( \ell^r \) norm yields
\[
\frac{d}{dt} \| (v_{\varepsilon}, c_{\varepsilon})(t) \|_{B^{1,s}_{2,r}} \lesssim (\| \text{div} v_{\varepsilon}(t) \|_{L^\infty} + \| \nabla c_{\varepsilon}(t) \|_{L^\infty})
\]
\[
\times \| (v_{\varepsilon}, c_{\varepsilon})(t) \|_{B^{1,s}_{2,r}} + (\Psi(q)^{2s/q} \| (F_{\varepsilon,q}^1, F_{\varepsilon,q}^2) \|_{L^2})_{L^r}.
\]
Now, we use the following commutator estimate, which was proved in [14]. For \( s > 0, r \in [1, +\infty] \) and \( \Psi \in \mathcal{U} \), we then have
\[
(\Psi(q)^{2s/q} \| [\Delta_q, v \cdot \nabla] u \|_{L^2})_{L^r} \lesssim \| \nabla v \|_{L^\infty} \| u \|_{B^{s,s}_{2,r}} + \| \nabla u \|_{L^\infty} \| v \|_{B^{s,s}_{2,r}}.
\]
Consequently,
\[
\frac{d}{dt} \| (v_{\varepsilon}, c_{\varepsilon})(t) \|_{B^{s,s}_{2,r}} \lesssim \| \text{div} v_{\varepsilon}(t) \|_{L^\infty} + \| \nabla v_{\varepsilon}(t) \|_{L^\infty} + \| \nabla c_{\varepsilon}(t) \|_{L^\infty}) (\| (v_{\varepsilon}, c_{\varepsilon})(t) \|_{B^{s,s}_{2,r}}
\]
\[
\lesssim \| \nabla v_{\varepsilon}(t) \|_{L^\infty} + \| \nabla c_{\varepsilon}(t) \|_{L^\infty}) (\| (v_{\varepsilon}, c_{\varepsilon})(t) \|_{B^{s,s}_{2,r}}.
\]
Incompressible limit for the 2D isentropic Euler system

Integrating in time and using Gronwall’s inequality gives
\[ \| (v_{\varepsilon,q}, c_{\varepsilon,q}) (t) \|_{B^2_{2,r}} \leqslant C \| (v_{0,\varepsilon}, c_{0,\varepsilon}) \|_{B^2_{2,r}} e^{CV_{\varepsilon}(t)}. \]

This completes the proof of the proposition. \( \square \)

### 3.2. Strichartz estimates

In this paragraph we intend to establish some classical Strichartz estimates for the acoustic parts. We shall in particular show that the averaging in time of the compressible part of the velocity \( v_{\varepsilon} \) and the sound speed \( c_{\varepsilon} \) will lead to vanishing quantities when \( \varepsilon \) goes to zero. This phenomenon is due to the wave structure governing the acoustic parts and it was used few decades ago by Ukai [27] in order to study the incompressible limit in the framework of the ill-prepared initial data.

First, we rewrite (1.3) in the following form:
\[
\begin{align*}
\partial_t v_{\varepsilon} + \frac{1}{\varepsilon} \nabla c_{\varepsilon} &= -v_{\varepsilon} \cdot \nabla v_{\varepsilon} - \gamma c_{\varepsilon} \nabla c_{\varepsilon} := f_{\varepsilon}, \\
\partial_t c_{\varepsilon} + \frac{1}{\varepsilon} \text{div} v_{\varepsilon} &= -v_{\varepsilon} \cdot \nabla c_{\varepsilon} - \gamma c_{\varepsilon} \text{div} v_{\varepsilon} := g_{\varepsilon}, \\
(v_{\varepsilon}, c_{\varepsilon}) |_{t=0} &= (v_{0,\varepsilon}, c_{0,\varepsilon}).
\end{align*}
\]

Denote by \( Qv := \nabla \Delta - \frac{1}{\varepsilon} \text{div} v \) the compressible part of the velocity \( v \). Then the coupled quantity
\[
\Gamma_{\varepsilon} := Qv_{\varepsilon} - i\nabla|D|^{-1}c_{\varepsilon} \quad \text{with} \quad |D| = (-\Delta)^{1/2}
\]

satisfies the following wave equation:
\[
\left( \partial_t + \frac{i}{\varepsilon} |D| \right) \Gamma_{\varepsilon} = Qf_{\varepsilon} - i\nabla|D|^{-1}g_{\varepsilon}.
\]

Similarly, we can easily check that the quantity
\[
\Upsilon_{\varepsilon} := |D|^{-1} \text{div} v_{\varepsilon} + ic_{\varepsilon}
\]
obeys the wave equation
\[
\left( \partial_t + \frac{i}{\varepsilon} |D| \right) \Upsilon_{\varepsilon} = |D|^{-1} \text{div} f_{\varepsilon} + ig_{\varepsilon}.
\]

Now we are in a position to use the following Strichartz estimates stated in space dimension 2. For the proof see, for example, [4, 11, 12].

**Lemma 3.3.** Let \( \psi \) be a solution of the wave equation
\[
\left( \partial_t + \frac{i}{\varepsilon} |D| \right) \psi = G, \quad \psi |_{t=0} = \psi_0.
\]

Then there exists an absolute constant \( C \) such that for every \( T > 0 \),
\[
\| \psi \|_{L^r_t L^p} \leqslant C e^{1/4 - 1/2p} (\| \psi_0 \|_{B^{1/4 - 3/2p}_{2,1}} + \| G \|_{L^1_t B^{1/4 - 3/2p}_{2,1}})
\]
for all \( p \in [2, +\infty] \) and \( r = 4 + 8/(p - 2) \).
This result enables us to get the following estimate.

**Proposition 3.4.** Let $(v_{0,\varepsilon}, c_{0,\varepsilon})$ be a bounded family in $B^{7/4}_{2,1}$. Then any smooth solution of (1.3) defined in the time interval $[0, T]$ satisfies

$$
\|(Q v_{\varepsilon}, c_{\varepsilon})\|_{L^4_t L^\infty_x} \lesssim C_0 \varepsilon^{1/4}(1 + T)e^{CV_{\varepsilon}(T)},
$$

where $C_0$ depends on the quantity $\sup_{0 < \varepsilon \leq 1} \|(v_{0,\varepsilon}, c_{0,\varepsilon})\|_{B^{7/4}_{2,1}}$ and

$$
V_{\varepsilon}(T) := \|\nabla v_{\varepsilon}\|_{L^4_t L^\infty_x} + \|\nabla c_{\varepsilon}\|_{L^4_t L^\infty_x}.
$$

**Proof.** We apply lemma 3.3 to (3.3) with $p = +\infty$. Then we obtain

$$
\|\Gamma_{\varepsilon}\|_{L^4_t L^\infty_x} \lesssim \varepsilon^{1/4}(\|\Gamma_{\varepsilon}\|_{B^{3/4}_{2,1}} + \|Q f_{\varepsilon} - i\nabla |D|^{-1} g_{\varepsilon}\|_{L^4_t B^{3/4}_{2,1}}).
$$

Using the continuity of the Riesz operators $Q$ and $\nabla |D|^{-1}$ on the homogeneous Besov spaces, combined with the embedding $B^{3/4}_{2,1} \hookrightarrow B^{3/4}_{2,1}$ and Hölder inequality yields

$$
\|\Gamma_{\varepsilon}\|_{L^4_t L^\infty_x} \lesssim \varepsilon^{1/4}(\|(v_{0,\varepsilon}, c_{0,\varepsilon})\|_{B^{3/4}_{2,1}} + \|(f_{\varepsilon}, g_{\varepsilon})\|_{L^4_t B^{3/4}_{2,1}})
\lesssim \varepsilon^{1/4}(\|(v_{0,\varepsilon}, c_{0,\varepsilon})\|_{B^{3/4}_{2,1}} + T\|(f_{\varepsilon}, g_{\varepsilon})\|_{L^\infty_t B^{3/4}_{2,1}}). \quad (3.5)
$$

It remains to estimate $\|(f_{\varepsilon}, g_{\varepsilon})\|_{L^\infty_t B^{3/4}_{2,1}}$. For this purpose we use the following law product, which can be proved, for example, by using Bony’s decomposition:

$$
\|f \cdot \nabla g\|_{B^{3/4}_{2,1}} \lesssim \|f\|_{L^\infty} \|g\|_{B^{3/4}_{2,1}} + \|g\|_{L^\infty} \|f\|_{B^{7/4}_{2,1}}.
$$

Thus, we get

$$
\|(f_{\varepsilon}, g_{\varepsilon})\|_{B^{3/4}_{2,1}} \lesssim \|(v_{\varepsilon}, c_{\varepsilon})\|_L \|(v_{\varepsilon}, c_{\varepsilon})\|_{B^{7/4}_{2,1}}
\lesssim \|(v_{\varepsilon}, c_{\varepsilon})\|_{B^{7/4}_{2,1}},
$$

where in the last line we have used the embedding $B^{7/4}_{2,1} \hookrightarrow L^\infty$. Therefore, (3.1) yields

$$
\|(f_{\varepsilon}, g_{\varepsilon})\|_{L^\infty_t B^{3/4}_{2,1}} \lesssim \|(v_{0,\varepsilon}, c_{0,\varepsilon})\|_{B^{7/4}_{2,1}} e^{CV_{\varepsilon}(T)}.
$$

Plugging this inequality into (3.5), we infer that

$$
\|\Gamma_{\varepsilon}\|_{L^4_t L^\infty_x} \lesssim \varepsilon^{1/4}(\|(v_{0,\varepsilon}, c_{0,\varepsilon})\|_{B^{3/4}_{2,1}} + T\|(v_{0,\varepsilon}, c_{0,\varepsilon})\|_{B^{7/4}_{2,1}} e^{CV_{\varepsilon}(T)})
\lesssim C_0 \varepsilon^{1/4}(1 + T)e^{CV_{\varepsilon}(T)}.
$$

Note that the real part of $\Gamma_{\varepsilon}$ is the compressible part of the velocity $v_{\varepsilon}$. Then

$$
\|Q v_{\varepsilon}\|_{L^4_t L^\infty_x} \lesssim C_0 \varepsilon^{1/4}(1 + T)e^{CV_{\varepsilon}(T)}.
$$

We can prove in a similar way that

$$
\|T_{\varepsilon}\|_{L^4_t L^\infty_x} \lesssim C_0 \varepsilon^{1/4}(1 + T)e^{CV_{\varepsilon}(T)},
$$

and therefore

$$
\|c_{\varepsilon}\|_{L^4_t L^\infty_x} \lesssim C_0 \varepsilon^{1/4}(1 + T)e^{CV_{\varepsilon}(T)}.
$$

The proof of the proposition 3.4 is now complete. \qed
4. Logarithmic estimate

The purpose of this section is to study the linear growth of the norm $B^0_{\infty,1}$ for the following compressible transport model:

\[
\begin{aligned}
\partial_t f + v \cdot \nabla f + f \text{ div } v &= 0, \\
 f|_{t=0} &= f_0.
\end{aligned}
\]  

(4.1)

We point out that the dynamics of the vorticity for system (1.3) is governed by an equation of type (4.1). In the framework of the incompressible vector fields, that is $\text{div } v = 0$, Vishik [28] established the following linear growth for Besov regularity of index zero:

\[
\|f(t)\|_{B^0_{\infty,1}} \leq C \|f_0\|_{B^0_{\infty,1}} \left( 1 + \int_0^t \|\nabla v(\tau)\|_{L^\infty} \, d\tau \right).
\]

A simple proof for this result was given in [15], where Vishik’s result was extended to a transport diffusion model. Our main goal here is to validate the previous linear growth for the compressible model and this is very crucial for the proof of the theorem 1.1. Our result reads as follows.

**Theorem 4.1.** Let $v$ be a smooth vector field and let $f$ be a smooth solution of the transport equation (4.1). Then, for every $1 \leq p < +\infty$, there exists a constant $C$ depending only on $p$ such that

\[
\|f(t)\|_{B^0_{\infty,1}} \leq C \|f_0\|_{B^0_{\infty,1}} \left( 1 + \exp(C \|\nabla v\|_{L^1_t L^\infty_x}) \|\text{div } v\|_{L^p_t L^2_x} \right.
\]

\[
\times \left. \left( 1 + \int_0^t \|\nabla v(\tau)\|_{L^\infty} \, d\tau \right) \right).
\]

**Proof.** First we observe that in the incompressible case the above estimate reduces to Vishik’s estimate. Here, we shall try to use the same approach of [15], but, as we shall see, the lack of incompressibility brings more technical difficulties. Let us denote by $\psi$ the flow associated to the velocity $v$:

\[
\psi(t, x) = x + \int_0^t v(\tau, \psi(\tau, x)) \, d\tau, \quad (x, t) \in \mathbb{R}^2 \times \mathbb{R}_+.
\]

We shall make use of the following estimate for the flow and its inverse $\psi^{-1}$:

\[
\|\nabla (\psi(\tau, \psi^{-1}(t, \cdot)))\|_{L^\infty} \leq \exp \left( \left| \int_\tau^t \|\nabla v(s)\|_{L^\infty} \, ds \right| \right).
\]  

(4.2)

We set $g(t, x) = f(t, \psi(t, x))$. Then it is clear that $g$ satisfies the equation

\[
\partial_t g(t, x) + (\text{div } v)(t, \psi(t, x)) g(t, x) = 0.
\]

Therefore, we obtain

\[
g(t, x) = f_0(x) \exp \left( - \int_0^t (\text{div } v)(\tau, \psi(\tau, x)) \, d\tau \right).
\]
It follows that
\[ f(t, x) = f_0(\psi^{-1}(t, x)) \exp \left( - \int_0^t (\text{div} \ v)(\tau, \psi(\tau, \psi^{-1}(t, x))) \, d\tau \right). \]

Set \( h(x) = e^x - 1 \) and
\[ W(t, x) = - \int_0^t (\text{div} \ v)(\tau, \psi(\tau, \psi^{-1}(t, x))) \, d\tau. \]

Then we can write
\[ f(t, x) = f_0(\psi^{-1}(t, x))(h(W(t, x)) + 1). \]

Thus, the problem reduces to establishing a composition law in \( B^{0,1}_\infty \) space. First we claim that we have the following law product, which can be obtained by using Bony’s decomposition [6]: for \( p < \infty \),
\[ \|uv\|_{B^{0,1}_\infty} \leq C\|u\|_{B^{2/p}_p} \|v\|_{B^{0,1}_\infty}. \]

Therefore,
\[ \|f(t)\|_{B^{0,1}_\infty} \leq C\|f_0(\psi^{-1}(t, \cdot))\|_{B^{0,1}_\infty}(\|h(W(t, \cdot))\|_{B^{2/p}_p} + 1). \] (4.3)

At this stage we need the following result:
\[ \|h(W(t, \cdot))\|_{B^{2/p}_p} \leq C\|W(t, \cdot)\|_{B^{2/p}_p} e^{C\|W(t)\|_{L^\infty}}. \]

This estimate can be proved as follows. By definition we have
\[ \|h(W(t, \cdot))\|_{B^{2/p}_p} \leq \sum_{n \geq 1} \frac{1}{n!} \|W^n(t, \cdot)\|_{B^{2/p}_p}. \]

According to the law product
\[ \|u^2\|_{B^{2/p}_p} \leq C\|u\|_{L^\infty} \|u\|_{B^{2/p}_p}, \quad p < +\infty, \]
we can easily show by induction that, for \( n \geq 1 \),
\[ \|W^n(t, \cdot)\|_{B^{2/p}_p} \leq C^{n-1} \|W(t, \cdot)\|_{L^\infty}^{n-1} \|W(t, \cdot)\|_{B^{2/p}_p}. \]

Therefore,
\[ \|h(W(t, \cdot))\|_{B^{2/p}_p} \leq \|W\|_{B^{2/p}_p} \sum_{n \geq 1} \frac{C^{n-1}}{n!} \|W\|_{L^\infty}^{n-1} \]
\[ \leq \|W\|_{B^{2/p}_p} \sum_{n \geq 0} \frac{C^n \|W\|_{L^\infty}^n}{(n + 1)!} \]
\[ \leq \|W\|_{B^{2/p}_p} \sum_{n \geq 0} \frac{C^n \|W\|_{L^\infty}^n}{n!} \]
\[ \leq \|W\|_{B^{2/p}_p} e^{C\|W\|_{L^\infty}}. \]
This ends the proof of the desired inequality. Thus, it follows that
\[ \|h(W(t, \cdot))\|_{B^2_{p,1}} \lesssim \|W(t, \cdot)\|_{B^2_{p,1}} e^{C\|W(t)\|_{L^\infty}} \]
\[ \lesssim \|W(t, \cdot)\|_{B^2_{p,1}} \exp \left( C \int_0^t \|\text{div} \, v(\tau)\|_{L^\infty} \, d\tau \right). \] (4.4)

To estimate \( W(t) \) we set \( k_\tau(t, x) = \text{div} \, v(\tau, \psi^{-1}(t, x)) \). Then \( k_\tau(t, \psi(t, x)) = \text{div} \, v(\tau, \psi(\tau, x)) \). It follows that
\[ \partial_\tau k_\tau + v \cdot \nabla k_\tau = 0, \]
\[ k_\tau(\tau, x) = \text{div} \, v(\tau, x). \]

It remains to estimate \( \|k_\tau(t)\|_{B^2_{p,1}} \). Towards this aim, we use lemma 2.6(i) with \( p > 2 \)
\[ \|k_\tau(t)\|_{B^2_{p,1}} \leq C \|\text{div} \, v(\tau)\|_{B^2_{p,1}} \exp \left( C \int_0^t \|\nabla v(s)\|_{L^\infty} \, ds \right). \]

Combining the definition of \( W \) with this latter estimate implies
\[ \|W(t)\|_{B^2_{p,1}} \leq \int_0^t \|k_\tau(t')\|_{B^2_{p,1}} \, dt' \]
\[ \leq C \int_0^t \|\text{div} \, v(\tau)\|_{B^2_{p,1}} \exp \left( C \int_\tau^t \|\nabla v(s)\|_{L^\infty} \, ds \right) \, d\tau. \]

Plugging this estimate into (4.4) gives
\[ \|h(W(t, \cdot))\|_{B^2_{p,1}} \leq C e^{C\|\nabla v\|_{L^1 \cdot L^\infty}} \int_0^t \|\text{div} \, v(\tau)\|_{B^2_{p,1}} \, d\tau. \]

By inserting this estimate into (4.3) one gets
\[ \|f(t)\|_{B^0_{\infty,1}} \leq C \|f_0(\psi^{-1}(t, \cdot))\|_{B^0_{\infty,1}} \left( 1 + \exp(C\|\nabla v\|_{L^1 \cdot L^\infty}) \int_0^t \|\text{div} \, v(\tau)\|_{B^2_{p,1}} \, d\tau \right). \] (4.5)

It remains to estimate \( \|f_0(\psi^{-1}(t, \cdot))\|_{B^0_{\infty,1}} \). Set \( w(t, x) = f_0(\psi^{-1}(t, x)) \). Then it satisfies the transport equation
\[ \partial_\tau w + v \cdot \nabla w = 0, \]
\[ w(0, x) = f_0(x). \]

We split the initial data into Fourier modes \( f_0 = \sum_{q \geq -1} \Delta_q f_0 \) and, for each frequency \( q \geq -1 \), we denote by \( \tilde{w}_q \) the unique global solution of the initial-value problem
\[ \partial_\tau \tilde{w}_q + v \cdot \nabla \tilde{w}_q = 0, \]
\[ \tilde{w}_q(0, \cdot) = \Delta_q f_0. \] (4.6)

According to lemma 2.6(ii), we have
\[ \|\tilde{w}_q(t)\|_{L^\infty} \leq C \|\Delta_q f_0\|_{L^\infty} e^{C_v(t)} \left( \forall 0 < s < \min \left( 1, \frac{2}{p} \right) \right), \]
with $V(t) = \|\nabla v\|_{L^1_t L^\infty} + \|\text{div} \, v\|_{L^1_t B^2_{p,1}}$. Combined with the definition of Besov spaces, for $j, q \geq -1$ this yields
\[ \|\Delta_j \tilde{w}_q(t)\|_{L^\infty} \leq C2^{-s|j-q|}\|\Delta_q f_0\|_{L^\infty}e^{CV(t)}. \] (4.7)

By linearity and uniqueness of the solution, it is obvious that
\[ w = \sum_{q=-1}^{\infty} \tilde{w}_q. \]

From the definition of Besov spaces we have
\[ \|w(t)\|_{B^0_{\infty,1}} \leq \sum_{|j-q| \geq N} \|\Delta_j \tilde{w}_q(t)\|_{L^\infty} + \sum_{|j-q| < N} \|\Delta_j \tilde{w}_q(t)\|_{L^\infty}, \] (4.8)

where $N \in \mathbb{N}$ is to be chosen later. To deal with the first sum we use (4.7):
\[ \sum_{|j-q| \geq N} \|\Delta_j \tilde{w}_q(t)\|_{L^\infty} \lesssim 2^{-Ns} \sum_{q \geq 1} \|\Delta_q f_0\|_{L^\infty}e^{CV(t)} \]
\[ \lesssim 2^{-Ns}\|f_0\|_{B^0_{\infty,1}}e^{CV(t)}. \] (4.9)

For the second sum in the right-hand side of (4.8), we use the fact that the operator $\Delta_j$ maps $L^\infty$ into itself uniformly with respect to $j$,
\[ \sum_{|j-q| < N} \|\Delta_j \tilde{w}_q(t)\|_{L^\infty} \lesssim \sum_{|j-q| < N} \|\tilde{w}_q(t)\|_{L^\infty}. \]

Applying the maximum principle to system (4.6) yields
\[ \|\tilde{w}_q(t)\|_{L^\infty} \leq \|\Delta_q f_0\|_{L^\infty}. \]

So, it holds that
\[ \sum_{|j-q| < N} \|\Delta_j \tilde{w}_q(t)\|_{L^\infty} \lesssim N\|f_0\|_{B^0_{\infty,1}}. \]

The outcome is the following:
\[ \|w(t)\|_{B^0_{\infty,1}} \lesssim \|f_0\|_{B^0_{\infty,1}} \left(2^{-Ns}e^{CV(t)} + N\right). \]

Choosing
\[ N = \left[\frac{CV(t)}{s \log 2} + 1\right], \]

we get
\[ \|w(t)\|_{B^0_{\infty,1}} \leq C\|f_0\|_{B^0_{\infty,1}} \left(1 + \int_0^t (\|\nabla v(\tau)\|_{L^\infty} + \|\text{div} \, v(\tau)\|_{B^2_{p,1}}) \, d\tau\right). \]

Therefore,
\[ \|f_0(\psi^{-1}(t, \cdot))\|_{B^0_{\infty,1}} \leq C\|f_0\|_{B^0_{\infty,1}} \left(1 + \int_0^t (\|\nabla v(\tau)\|_{L^\infty} + \|\text{div} \, v(\tau)\|_{B^2_{p,1}}) \, d\tau\right). \]
Inserting this estimate into (4.5) gives
\[
\|f(t)\|_{B_{\infty,1}^0} \leq C\|f_0\|_{B_{\infty,1}^0} (1 + \exp(C\|v\|_{L^1_tL^\infty})\|\text{div } v\|_{L^1_tL^2_{p,1}}^2) \\
\times \left(1 + \int_0^t \|\nabla v(\tau)\|_{L^\infty} \, d\tau\right).
\]
This is the desired result.

5. Proofs of the main results

In this section we firstly extend the result of theorem 1.1 to the framework of the heterogeneous Besov spaces $B_{2,1}^{2,\Psi}$. Secondly, we show how to deduce the proof of theorem 1.1, and ultimately the rest of the paper will be devoted to the discussion of the proof of the general statement given in theorem 5.1.

5.1. General statement

We shall give a generalization of the theorem 1.1.

**Theorem 5.1.** Let $\Psi \in \mathcal{U}_\infty$ and let $\{(v_{0,\varepsilon}, c_{0,\varepsilon})\}_{0<\varepsilon \leq 1}$ be a bounded family in $B_{2,1}^{2,\Psi}$ of initial data, that is,
\[
\sup_{0<\varepsilon \leq 1} \|(v_{0,\varepsilon}, c_{0,\varepsilon})\|_{B_{2,1}^{2,\Psi}} := M_0 < +\infty.
\]
Then the system (1.3) admits a unique solution $(v_{\varepsilon}, c_{\varepsilon}) \in C([0,T_\varepsilon]; B_{2,1}^{2,\Psi})$, with
\[
T_\varepsilon = C_0 \log \log \{\Psi(\log(\varepsilon^{-1}))\}.
\]
Moreover, there exists $\eta > 0$ such that, for small $\varepsilon$ and for all $0 \leq T \leq T_\varepsilon$,
\[
\|\text{div } v_{\varepsilon}, \nabla c_{\varepsilon}\|_{L^1_\varepsilon L^\infty} \leq C_0 \{\Psi(-\log \varepsilon)\}^{-\eta}, \quad \|\omega_{\varepsilon}(t)\|_{L^\infty} \leq C_0, \\
\|\nabla v_{\varepsilon}\|_{L^1_\varepsilon L^\infty} \leq C_0 e^{C_0 T}, \quad \|(v_{\varepsilon}, c_{\varepsilon})(T)\|_{B_{2,1}^{2,\Psi}} \lesssim C_0 \exp(e^{C_0 T}).
\]

The constant $C_0$ may depend on $M_0$ and on the profile $\Psi$.

Assume in addition that the incompressible parts $(Pv_{0,\varepsilon})_{0<\varepsilon \leq 1}$ converge in $L^2$ to some $v_0$. Then the incompressible parts of the solution converge strongly to the global solution of the system (1.4) with initial data $v_0$. More precisely, for all $T > 0$ and for all $\tilde{\Psi} \in \mathcal{U}$ such that
\[
\lim_{q \to +\infty} \frac{\tilde{\Psi}(q)}{\Psi(q)} = 0
\]
we have
\[
\lim_{\varepsilon \to 0} \|Pv_{\varepsilon} - v\|_{L^\infty B_{2,1}^{2,\Psi}} = 0.
\]
Remark 5.2.

(i) We observe that the result for the subcritical regularities is a special case of the above theorem. Indeed, let \( \{(v_{0,\varepsilon}, c_{0,\varepsilon})\}_{0<\varepsilon\leq 1} \) be a bounded family in \( H^s, s > 2 \). It is plain that the following embedding holds: \( H^s \hookrightarrow B^{2\Psi}_{2,1} \), with \( \Psi(q) = 2^{q\alpha} \) and \( 0 < \alpha < s - 2 \). Thus, by applying theorem 5.1 we get that

\[
T_\varepsilon = C_0 \log \log \left\{ \frac{1}{\varepsilon} \right\}.
\]

(ii) If \( \Psi \) has a polynomial growth: \( \Psi(q) = (q + 2)^{\alpha}, \alpha > 0 \), then it is easily seen that \( \Psi \in \mathcal{U}_\infty \) and

\[
T_\varepsilon = C_0 \log \log \log \left\{ \frac{1}{\varepsilon} \right\}.
\]

Before going further into the details of the proof of the theorem 5.1, we shall first show how to deduce the result of the theorem 1.1.

5.2. Proof of theorem 1.1

Since

\[
\sum_{q \geq -1} 2^{2q} \sup_{0<\varepsilon\leq 1} \| (\Delta_q v_{0,\varepsilon}, \Delta_q c_{0,\varepsilon}) \|_{L^2} < +\infty,
\]

we deduce from corollary 2.5 the existence of a profile \( \Psi \in \mathcal{U}_\infty \) such that the family \( (v_{0,\varepsilon}, c_{0,\varepsilon})_{0<\varepsilon\leq 1} \) is uniformly bounded in \( B^{2\Psi}_{2,1} \). Then according to theorem 5.1, there exists a unique solution \( (v_\varepsilon, c_\varepsilon) \) belonging to the space \( \mathcal{C}([0, T_\varepsilon]; B^{2\Psi}_{2,1}) \) with

\[
T_\varepsilon = C_0 \log \log \{ \Psi(\log(\varepsilon^{-1})) \}.
\]

Therefore, it is clear that \( (v_\varepsilon, c_\varepsilon) \in \mathcal{C}([0, T_\varepsilon]; B^{2}_{2,1}) \) and

\[
\lim_{\varepsilon \to 0} T_\varepsilon = +\infty.
\]

For the incompressible limit, we can apply the second part of theorem 5.1 with the profile \( \tilde{\Psi} \) equal to a non-negative constant and thus the space \( B^{2\tilde{\Psi}}_{2,1} \) reduces to the usual Besov space \( B^{2}_{2,1} \).

We now give the complete proof of theorem 5.1, which will be done in several steps. We start first with estimating the lifespan of the solutions and we discuss at the end the incompressible limit problem.

5.3. Lifespan of the solutions

We shall give an a priori bound of \( T_\varepsilon \) and show that the acoustic parts vanish when the Mach number goes to zero.

Proof. Using lemma 2.7, we get

\[
\| \nabla v_\varepsilon(t) \|_{L^\infty} \lesssim \| v_\varepsilon(t) \|_{L^2} + \| \text{div} v_\varepsilon(t) \|_{B^{0}_{2,1}} + \| \omega_\varepsilon(t) \|_{B^{0}_{2,1}}.
\]
Integrating in time and using proposition 3.1, we obtain
\[ \|\nabla v_\varepsilon\|_{L^p_t L^p} \lesssim \|v_\varepsilon\|_{L^1_t L^2} + \|\text{div} v_\varepsilon\|_{L^1_t B^0_{p,1}} + \|\omega_\varepsilon\|_{L^1_t B^0_{\infty,1}} \]
\[ \lesssim C_0 T \exp(C \|\nabla v_\varepsilon\|_{L^1_t L^\infty}) + \|\text{div} v_\varepsilon\|_{L^1_t B^0_{p,1}} + \|\omega_\varepsilon\|_{L^1_t B^0_{\infty,1}} \]
\[ \lesssim C_0 (1 + T) \exp(C \|\nabla v_\varepsilon\|_{L^1_t B^0_{p,1}}) + \|\omega_\varepsilon\|_{L^1_t B^0_{\infty,1}}. \]

Now we recall the following notation:
\[ V_\varepsilon(T) = \|\nabla v_\varepsilon\|_{L^p_t L^p} + \|\nabla c_\varepsilon\|_{L^1_t L^\infty}. \]

Then it follows that
\[ V_\varepsilon(T) \leq C_0 (1 + T) \exp(C \|\text{div} v_\varepsilon, \nabla c_\varepsilon\|_{L^1_t B^0_{p,1}}) + \|\omega_\varepsilon\|_{L^1_t B^0_{\infty,1}}. \quad (5.1) \]

Applying theorem 4.1 gives
\[ \|\omega_\varepsilon(t)\|_{B^0_{p,1}} \]
\[ \lesssim C |\omega_\varepsilon|_{B^0_{\infty,1}} (1 + \exp(C \|\nabla v_\varepsilon\|_{L^1_t L^\infty})) \|\text{div} v_\varepsilon\|_{L^1_t B^2_{p,1}}^2 (1 + \|\nabla v_\varepsilon\|_{L^1_t L^\infty}) \]
\[ \lesssim C_0 (1 + C^CV_\varepsilon(t) \|\nabla v_\varepsilon\|_{L^1_t B^2_{p,1}}^2) (1 + V_\varepsilon(t)). \quad (5.2) \]

To estimate \( \|\text{div} v_\varepsilon\|_{B^2_{p,1}} \), we use the following interpolation inequality: for \( 2 < p < \infty \),
\[ \|\text{div} v_\varepsilon\|_{B^2_{p,1}} \lesssim \|\text{div} v_\varepsilon\|_{B^{1/2}_{2,1}}^{2/p} \|\text{div} v_\varepsilon\|_{B^{1/2}_{\infty,1}}^{1-2/p}. \]

Integrating in time and using Bernstein and Hölder inequalities gives
\[ \|\text{div} v_\varepsilon\|_{L^1_t B^{2/p}_{p,1}} \lesssim T^{2/p} \|\text{div} v_\varepsilon\|_{L^p_t B^{1/2}_{p,1}} \|\text{div} v_\varepsilon\|_{L^p_t B^{1/2}_{\infty,1}} \]
\[ \lesssim T^{2/p} \|\text{div} v_\varepsilon\|_{L^p_t B^{1/2}_{2,1}} \|\text{div} v_\varepsilon\|_{L^1_t B^{1/2}_{\infty,1}}. \]

Choose \( p = 4 \) and using proposition 3.1(ii) yields
\[ \|\text{div} v_\varepsilon\|_{L^1_t B^{1/2}_{4,1}} \lesssim C_0 T e^C V_\varepsilon(T) \|\text{div} v_\varepsilon\|_{L^1_t B^0_{\infty,1}}. \quad (5.3) \]

Putting together (5.2) and (5.3) and integrating in time we find
\[ \|\omega_\varepsilon\|_{L^1_t B^0_{\infty,1}} \lesssim C_0 \int_0^T (1 + t e^C V_\varepsilon(t)) \|\nabla v_\varepsilon\|_{L^1_t B^0_{\infty,1}} (1 + V_\varepsilon(t)) \, dt. \]

Inserting this estimate into (5.1) yields
\[ V_\varepsilon(T) \leq C_0 (1 + T) \exp(C \|\text{div} v_\varepsilon, \nabla c_\varepsilon\|_{L^1_t B^0_{\infty,1}}) \]
\[ + C_0 \int_0^T (1 + t e^C V_\varepsilon(t)) \|\nabla v_\varepsilon\|_{L^1_t B^0_{\infty,1}} (1 + V_\varepsilon(t)) \, dt. \quad (5.4) \]

It remains to estimate \( \|\text{div} v_\varepsilon, \nabla c_\varepsilon\|_{L^1_t B^0_{\infty,1}} \). For this purpose we shall develop an interpolation argument between the Strichartz estimates for lower frequencies and the energy estimates for higher frequencies. More precisely, we assume a value
$N \in \mathbb{N}^*$, which will be judiciously fixed later. Then, using the Bernstein inequality combined with the continuity of the operator $Q$ on the Lebesgue space $L^2$, we find

$$\|\text{div} v \varepsilon\|_{B_{\infty,1}^0} = \|\text{div} Q v \varepsilon\|_{B_{\infty,1}^0} \leq \sum_{-1 \leq q \leq N} \|\Delta_q \text{div} Q v \varepsilon\|_{L^\infty} + \sum_{q > N} \|\Delta_q \text{div} Q v \varepsilon\|_{L^2} \lesssim \sum_{-1 \leq q \leq N} 2^q \|\Delta_q Q v \varepsilon\|_{L^\infty} + \sum_{q > N} 2^q \|\Delta_q Q v \varepsilon\|_{L^2} \lesssim 2^N \|Q v \varepsilon\|_{L^\infty} + \|v \varepsilon\|_{B_{2,1}^2}.$$ 

We have used the fact that the profile $\Psi$ is a non-decreasing function. Now, integrating in time and using propositions 3.4 and 3.1(ii) gives

$$\|\text{div} v \varepsilon\|_{L^1_T B_{\infty,1}^0} \lesssim 2^N T^{3/4} \|Q v \varepsilon\|_{L^\infty_T L^\infty} + \frac{T}{\Psi(N)} \|v \varepsilon\|_{L^\infty_T B_{2,1}^2} \lesssim C_0 (T^{3/4} (1 + T) e^{CV_{v}(T)} \left( \varepsilon^{1/4} 2^N + \frac{1}{\Psi(N)} \right)) \lesssim C_0 (1 + T^{7/4}) e^{CV_{v}(T)} \left( \varepsilon^{1/4} 2^N + \frac{1}{\Psi(N)} \right).$$

By similar computations we get

$$\|\nabla c \varepsilon\|_{L^1_T B_{\infty,1}^0} \lesssim C_0 (1 + T^{7/4}) e^{CV_{v}(T)} \left( \varepsilon^{1/4} 2^N + \frac{1}{\Psi(N)} \right).$$

Therefore,

$$\|(\text{div} v \varepsilon, \nabla c \varepsilon)\|_{L^1_T B_{\infty,1}^0} \lesssim C_0 (1 + T^{7/4}) e^{CV_{v}(T)} \left( \varepsilon^{1/4} 2^N + \frac{1}{\Psi(N)} \right).$$

Now, we choose $N$ such that

$$N \approx \log \left( \frac{1}{\varepsilon^{1/8}} \right).$$

Hence, we find

$$\|(\text{div} v \varepsilon, \nabla c \varepsilon)\|_{L^1_T B_{\infty,1}^0} \lesssim C_0 (1 + T^{7/4}) e^{CV_{v}(T)} \left( \varepsilon^{1/8} + \frac{1}{\Psi \left( \log(1/\varepsilon^{1/8}) \right)} \right).$$

According to the remark 2.3, the profile $\Psi$ has at most an exponential growth: there exists $\alpha > 0$ such that

$$\forall x \geq -1, \quad \Psi(x) \leq C e^{\alpha x}.$$ 

This implies

$$\varepsilon^{1/8} \leq \frac{C^{1/\alpha}}{\Psi^{1/\alpha} \left( \log(1/\varepsilon^{1/8}) \right)}. \quad (5.5)$$

Let $\beta = \min(1, 1/\alpha)$. Then

$$\|(\text{div} v \varepsilon, \nabla c \varepsilon)\|_{L^1_T B_{\infty,1}^0} \lesssim C_0 (1 + T^{7/4}) e^{CV_{v}(T)} \Phi(\varepsilon), \quad (5.6)$$
with
\[ \Phi(\varepsilon) := \frac{1}{\Psi^3(\log(1/\varepsilon))}, \]
where we have used property (b) of the definition 2.2. Plugging (5.6) into (5.4) and using Gronwall’s inequality, we obtain
\[
V_\varepsilon(T) \leq C_0(1 + T) \exp\{C_0(1 + T^{7/4})\Phi(\varepsilon)e^{CV_\varepsilon(T)}\}
+ C_0 \int_0^T (1 + (1 + t^{11/4})e^{CV_\varepsilon(t)\Phi(\varepsilon)}(1 + V_\varepsilon(t)) \, dt
\leq C_0 e^{C_0 T} \exp\{C_0(1 + T^{15/4})\Phi(\varepsilon)e^{CV_\varepsilon(T)}\}. \tag{5.7}
\]
We choose \( T_\varepsilon \) such that
\[ e^{C_0 T_\varepsilon} = \Phi^{-1/2}(\varepsilon). \tag{5.8} \]
Then we claim that for small \( \varepsilon \) and \( 0 \leq t \leq T_\varepsilon \) we have
\[ e^{CV_\varepsilon(t)} \leq \Phi^{-2/3}(\varepsilon). \tag{5.9} \]
Indeed, set
\[ J_\varepsilon := \{ t \in [0, T_\varepsilon]; \ e^{CV_\varepsilon(t)} \leq \Phi^{-2/3}(\varepsilon) \}. \]
Observe that this set is non-empty since \( 0 \in J_\varepsilon \). It is also closed by the continuity of the mapping \( t \mapsto V_\varepsilon(t) \). It remains to prove that \( J_\varepsilon \) is an open subset of \([0, T_\varepsilon]\) and thus we deduce \( J_\varepsilon = [0, T_\varepsilon] \). Let \( t \in J_\varepsilon \). Then using (5.7) we get, for small \( \varepsilon \),
\[ e^{CV_\varepsilon(t)} \leq C_0 \exp\{\exp\{C_0 T_\varepsilon + C_0 T_\varepsilon^{15/4}\Phi^{1/3}(\varepsilon)\}\}. \tag{5.10} \]
From (5.8), we deduce that
\[ C_0 T_\varepsilon = \log \log \Phi^{-1/2}(\varepsilon)
\approx \log \log \Psi\left(\log \left(\frac{1}{\varepsilon}\right)\right). \]
Since
\[ \lim_{\varepsilon \to 0} \Phi^{1/3}(\varepsilon)\{\log \log \Phi^{-1/2}(\varepsilon)\}^{15/4} = 0, \]
for sufficiently small \( \varepsilon \) and for \( t \in [0, T_\varepsilon] \), we get
\[ e^{CV_\varepsilon(t)} \leq 2C_0 \Phi^{-1/2}(\varepsilon)
< \Phi^{-2/3}(\varepsilon). \]
This proves that \( t \) belongs to the interior of \( J_\varepsilon \) and consequently \( J_\varepsilon \) is an open subset of \([0, T_\varepsilon]\). Finally, we get \( J_\varepsilon = [0, T_\varepsilon] \). From the previous estimate, we obtain, for all \( T \in [0, T_\varepsilon] \),
\[ (1 + T^{15/4})e^{CV_\varepsilon(T)\Phi(\varepsilon)} \leq \{\log \log \Phi^{-1/2}(\varepsilon)\}^{15/4}\Phi^{1/3}(\varepsilon) \leq 1. \tag{5.11} \]
Inserting (5.11) into (5.7) for \( 0 \leq T \leq T_\varepsilon \) yields
\[ V_\varepsilon(T) \leq C_0 e^{C_0 T}. \tag{5.12} \]
In particular, we have obtained
\[ \| \nabla v_\varepsilon \|_{L^1_t L^\infty} \leq C_0 e^{C_0 T}. \]

Plugging the estimate (5.11) into (5.6), for small \( \varepsilon \) and for \( T \in [0, T_\varepsilon] \) we find
\[ \|( \text{div } v_\varepsilon, \nabla c_\varepsilon) \|_{L^1_t B^0_{2,1}} \leq C_0 \left( \frac{\log \log \Phi^{-1/2}(\varepsilon)}{\Phi^{-1/2}(\varepsilon)} \right)^{7/4} \Phi^1/4(\varepsilon). \] (5.13)

Using proposition 3.4, (5.9) and (5.5) for small \( \varepsilon \), we obtain
\[ \| Q v_\varepsilon \|_{L^4_t L^\infty} + \| c_\varepsilon \|_{L^4_t L^\infty} \leq C_0 \varepsilon^{1/4}(1 + T) e^{CV_\varepsilon(T)} \leq C_0 \varepsilon^{1/4} \Phi^{-2/3}(\varepsilon) \log \log \Phi^{-1/2}(\varepsilon) \leq C_0 \varepsilon \Phi(\varepsilon). \] (5.14)

Let us now move to the estimate of the vorticity. First, recall that \( \omega_\varepsilon \) satisfies the compressible transport equation
\[ \partial_t \omega_\varepsilon + v_\varepsilon \cdot \nabla \omega_\varepsilon + \omega_\varepsilon \text{ div } v_\varepsilon = 0. \]

Consequently, by using Gronwall’s inequality, we get
\[ \| \omega_\varepsilon(t) \|_{L^\infty} \leq \| \omega_0,\varepsilon \|_{L^\infty} \exp(\| \text{div } v_\varepsilon \|_{L^1_t L^\infty}). \]

It suffices to apply (5.13), leading to the following estimate: for all \( T \in [0, T_\varepsilon] \),
\[ \| \omega_\varepsilon(T) \|_{L^\infty} \leq C_0. \]

Finally, to estimate \( \|(v_\varepsilon, c_\varepsilon)(T)\|_{B^2_{2,1}} \) we use proposition 3.1 combined with (5.12) in order to find
\[ \|(v_\varepsilon, c_\varepsilon)(T)\|_{B^2_{2,1}} \leq C \|(v_0,\varepsilon, c_0,\varepsilon)\|_{B^2_{2,1}} e^{CV_\varepsilon(T)} \leq C_0 \exp\{e^{C_0 T}\}. \] (5.15)

This completes the proof of the first part of theorem 5.1.

5.4. Incompressible limit

In this paragraph we shall sketch the proof of the second part of theorem 5.1 which deals with the incompressible limit.

Proof. We shall proceed in two steps. In the first, we prove that for any fixed \( T > 0 \) the family \( \{P(v_\varepsilon)\}_\varepsilon \) converges strongly in \( L^2_\varepsilon L^2 \), when \( \varepsilon \to 0 \), to the solution \( v \) of the incompressible Euler equations with initial data \( v_0 \). It is worth pointing out that similarly to remark 1.5 we can show that the limit \( v_0 \) also belongs to the same space \( B^2_{2,1} \). In the second step we show how to get the strong convergence in the Besov spaces \( B^2_{2,1} \). We mention that the limit system (1.4) is globally well-posed if \( v_0 \in B^2_{2,1} \). Indeed, combining the embedding \( B^2_{2,1} \hookrightarrow B^2_{2,1} \) with Vishik’s
result [28], we obtain that system (1.4) admits a unique global solution such that \( v \in C(\mathbb{R}_+; B^{3,\Psi}_{2,1}) \), with the following Lipschitz bound:

\[
\|\nabla v(t)\|_{L^\infty} \leq C_0 e^{C_0 t}.
\]

The latter \textit{a priori} estimate together with proposition 3.1, which remains valid for (1.4), give the global persistence of the initial regularity \( B^{3,\Psi}_{2,1} \). More precisely, we get

\[
\|v(t)\|_{B^{3,\Psi}_{2,1}} \leq C_0 \exp(C_0 t).
\]  

(5.16)

Let \( \varepsilon > 0 \) and set

\[
\zeta_\varepsilon := v_\varepsilon - v, \quad w_\varepsilon = P v_\varepsilon - v.
\]

By applying Leray’s projector \( P \) to the first equation of (1.3), we obtain

\[
\partial_t P v_\varepsilon + P (v_\varepsilon \cdot \nabla v_\varepsilon) = 0.
\]

Taking the difference between the above equation and (1.4) yields

\[
\partial_t w_\varepsilon + P (v_\varepsilon \cdot \nabla \zeta_\varepsilon) + P (\zeta_\varepsilon \cdot \nabla v) = 0.
\]

We take the \( L^2 \) inner product of the above equation with \( w_\varepsilon \). Integrating by parts and using the identities

\[
\zeta_\varepsilon = w_\varepsilon + Qv_\varepsilon, \quad Pw_\varepsilon = w_\varepsilon,
\]

we get

\[
\frac{1}{2} \frac{d}{dt} \|w_\varepsilon(t)\|_{L^2}^2 = - \int_{\mathbb{R}^2} (v_\varepsilon \cdot \nabla \zeta_\varepsilon) P w_\varepsilon \, dx - \int_{\mathbb{R}^2} (\zeta_\varepsilon \cdot \nabla v) P w_\varepsilon \, dx
\]

\[
= - \int_{\mathbb{R}^2} (v_\varepsilon \cdot \nabla (w_\varepsilon + Qv_\varepsilon)) w_\varepsilon \, dx - \int_{\mathbb{R}^2} ((w_\varepsilon + Qv_\varepsilon) \cdot \nabla v) w_\varepsilon \, dx
\]

\[
= \frac{1}{2} \int_{\mathbb{R}^2} |w_\varepsilon|^2 \text{div} \, v_\varepsilon \, dx - \int_{\mathbb{R}^2} (w_\varepsilon \cdot \nabla v) w_\varepsilon \, dx
\]

\[
- \int_{\mathbb{R}^2} (v_\varepsilon \cdot \nabla Qv_\varepsilon + Qv_\varepsilon \cdot \nabla v) w_\varepsilon \, dx
\]

\[
\leq \left( \frac{1}{2} \|\text{div} \, v_\varepsilon(t)\|_{L^\infty} + \|\nabla v(t)\|_{L^\infty}\right) \|w_\varepsilon(t)\|_{L^2}^2
\]

\[
+ \left( \|v_\varepsilon(t)\|_{L^2} \|\nabla Qv_\varepsilon(t)\|_{L^\infty} + \|Qv_\varepsilon(t)\|_{L^\infty} \|\nabla v(t)\|_{L^2}\right) \|w_\varepsilon(t)\|_{L^2}.
\]

This implies that

\[
\frac{d}{dt} \|w_\varepsilon(t)\|_{L^2} \leq \left( \frac{1}{2} \|\text{div} \, v_\varepsilon(t)\|_{L^\infty} + \|\nabla v(t)\|_{L^\infty}\right) \|w_\varepsilon(t)\|_{L^2}
\]

\[
+ \|v_\varepsilon(t)\|_{L^2} \|\nabla Qv_\varepsilon(t)\|_{L^\infty} + \|Qv_\varepsilon(t)\|_{L^\infty} \|\nabla v(t)\|_{L^2}.
\]

Integrating in time and using Gronwall’s inequality yield

\[
\|w_\varepsilon(t)\|_{L^2} \leq (\|w_\varepsilon^0\|_{L^2} + F_\varepsilon(t)) \exp\left( \frac{1}{2} \|\text{div} \, v_\varepsilon\|_{L^1 L^\infty} + \|\nabla v\|_{L^1 L^\infty}\right),
\]

(5.17)

where

\[
F_\varepsilon(t) := \|v_\varepsilon\|_{L^\infty L^2} \|\nabla Qv_\varepsilon(t)\|_{L^1 L^\infty} + \|Qv_\varepsilon\|_{L^1 L^\infty} \|\nabla v\|_{L^\infty L^2}
\]

\[
\lesssim \|v_\varepsilon\|_{L^\infty L^2} \|\nabla Qv_\varepsilon\|_{L^1 L^\infty} + \|Qv_\varepsilon\|_{L^1 L^\infty} \|\omega\|_{L^\infty L^2}.
\]
with \( \omega \) the vorticity of \( v \). In the last line we have used the estimate \( \| \nabla v \|_{L^p} \lesssim \| \omega \|_{L^p} \), which is valid for every \( p \in ]1, +\infty[ \). To estimate \( \| \nabla Q \varepsilon \|_{L^\infty} \) we use the Bernstein inequality:

\[
\| \nabla Q \varepsilon \|_{L^\infty} \leq \| \Delta_1 \nabla Q \varepsilon \|_{L^\infty} + \sum_{q \geq 0} \| \Delta_q \nabla Q \varepsilon \|_{L^\infty} + \| \Delta_q \text{div} \varepsilon \|_{L^\infty} + \| \text{div} \varepsilon \|_{B^0_{\infty,1}}.
\]

Integrating in time gives

\[
\| \nabla Q \varepsilon \|_{L^1_t L^\infty} \leq \| \Delta_1 \|_{L^1_t L^\infty} + \| \Delta_q \|_{L^1_t L^\infty} \lesssim \| \Delta_q \|_{L^1_t L^\infty} + \| \text{div} \varepsilon \|_{L^1_t B^0_{\infty,1}}.
\]

By virtue of (5.13) and (5.14) one has, for small \( \varepsilon \) and \( t \in [0, T] \),

\[
\| \nabla Q \varepsilon \|_{L^1_t L^\infty} + \| \Delta_q \|_{L^1_t L^\infty} \lesssim C_0 \Phi^{1/4}(\varepsilon).
\]

On the other hand, from proposition 3.1(i) and (5.13) we get

\[
\| \nabla Q \varepsilon \|_{L^1_t L^\infty} \leq C_0.
\]

To estimate \( \| \omega(t) \|_{L^2} \) we use the fact that the vorticity is transported for the incompressible flow and thus

\[
\| \omega(t) \|_{L^2} = \| \omega_0 \|_{L^2}.
\]

Therefore, for \( t \in [0, T] \) and for small \( \varepsilon \) we obtain

\[
F_\varepsilon(t) \leq C_0 \Phi^{1/4}(\varepsilon).
\]

According to Vishik’s result [28] we have, for all \( t \geq 0 \),

\[
\| \nabla v(t) \|_{L^\infty} \leq C_0 e^{C_0 t}.
\]

Inserting the previous estimates into (5.17), we find that, for all \( t \in [0, T] \) and small values of \( \varepsilon \),

\[
\| \nabla v(t) \|_{L^\infty} \leq C_0 \exp(e^{C_0 t})(\| P v_{0,\varepsilon} - v_0 \|_{L^2} + \Phi^{1/4}(\varepsilon)).
\]

This proves the strong convergence in \( L^\infty_{loc}(\mathbb{R}_+; L^2) \).

Let us now turn to the proof of the strong convergence in the space \( L^\infty T \text{B}_{2,1} \Psi \). We recall that \( \Psi \) is any element of the set \( \mathcal{U} \) (see definition 2.2) and satisfies, in addition, the assumption

\[
\lim_{q \to +\infty} \frac{\hat{\Psi}(q)}{\Psi(q)} = 0.
\]

By using (5.15) and the continuity of the Leray projection on the spaces \( \text{B}_{2,1} \Psi \), we get, for \( t \in [0, T] \) and for small \( \varepsilon \),

\[
\| P v_\varepsilon(t) \|_{\text{B}_{2,1} \Psi} \leq C_0 \exp(e^{C_0 t}).
\]
Now let $N \in \mathbb{N}^*$ be an arbitrary number. Then by (5.18), (5.16) and (5.19) we obtain
\[
\|w_\varepsilon(t)\|_{B^2_2,\tilde{\Psi}^{2,1}}
\leq C_0 \exp(e^{C_0 t})(\|Pv_{0,\varepsilon} - v_0\|_{L^2} + \phi^{1/4}(\varepsilon))\tilde{\Psi}(N)2^{2N} + \rho_N \|((P\varepsilon,v)(t))\|_{B^2_\infty,\tilde{\Psi}^{2,1}}
\leq C_0 \exp(e^{C_0 t})(\|Pv_{0,\varepsilon} - v_0\|_{L^2} + \phi^{1/4}(\varepsilon))\tilde{\Psi}(N)2^{2N} + \rho_N,
\]
with
\[
\rho_N := \max_{q \geq N} \Psi(q).
\]
We remark that $(\rho_N)_N$ is a sequence decreasing to zero. Consequently,
\[
\limsup_{\varepsilon \to 0^+} \|w_\varepsilon\|_{L^\infty_T B^2_2,\tilde{\Psi}^{2,1}} \leq C_0 \exp(e^{C_0 T})\rho_N.
\]
Combining this estimate with $\lim_{N \to +\infty} \rho_N = 0$, we get
\[
\lim_{\varepsilon \to 0^+} \|w_\varepsilon\|_{L^\infty_T B^2_2,\tilde{\Psi}^{2,1}} = 0.
\]
This completes the proof of the strong convergence.

Acknowledgements

We thank the anonymous referee for helpful comments.

References

1. T. Alazard. Incompressible limit of the nonisentropic Euler equations with the solid wall boundary conditions. Adv. Diff. Eqns 10 (2005), 19–44.
2. S. Alinhac. Temps de vie des solutions régulières des équations d’Euler compressible axisymétriques en dimension deux. Invent. Math. 111 (1993), 627–670.
3. K. Asano. On the incompressible limit of the compressible Euler equation. Jpn. J. Appl. Math. 4 (1987), 455–488.
4. H. Bahouri, J. Y. Chemin and R. Danchin. Fourier analysis and nonlinear partial differential equations. Grundlehren der Mathematischen wissenschaften, vol. 343 (Springer, 2011).
5. J. T. Beale, T. Kato and A. Majda. Remarks on the breakdown of smooth solutions for the 3-D Euler equations. Commun. Math. Phys. 94 (1984), 61–66.
6. J.-M. Bony. Calcul symbolique et propagation des singularités pour les équations aux dérivées partielles non linéaires. Annales Scient. Éc. Norm. Sup. 14 (1981), 209–246.
7. J.-Y. Chemin. Perfect incompressible fluids (Oxford University Press, 1998).
8. R. Danchin. Zero Mach number limit in critical spaces for compressible Navier-Stokes equations. Annales Scient. Éc. Norm. Sup. 35 (2002), 27–75.
9. B. Desjardins and E. Grenier. Low Mach number limit of viscous compressible flows in the whole space. Proc. R. Soc. Lond. A 455 (1999), 2271–2279.
10. B. Desjardins, E. Grenier, P.-L. Lions and N. Masmoudi. Incompressible limit for solutions of the isentropic Navier-Stokes equations with Dirichlet boundary conditions. J. Math. Pures Appl. 78 (1999), 461–471.
11 A. Dutrifoy and T. Hmidi. The incompressible limit of solutions of the two-dimensional compressible Euler system with degenerating initial data. *Commun. Pure Appl. Math.* 57 (2004), 1159–1177.

12 J. Ginibre and G. Velo. Generalized Strichartz inequalities for the wave equation. *J. Funct. Analysis* 133 (1995), 50–68.

13 M. Grassin. Global smooth solutions to Euler equations for a perfect gas. *Indiana Univ. Math. J.* 47 (1998), 1397–1432.

14 T. Hmidi. Low Mach number limit for the isentropic Euler system with axisymmetric initial data. *J. Inst. Math. Jussieu* 12 (2013), 335–389.

15 T. Hmidi and S. Keraani. Incompressible viscous flows in borderline Besov spaces. *Arch. Ration. Mech. Analysis* 189 (2008), 283–300.

16 S. Klainerman and A. Majda. Singular limits of quasilinear hyperbolic systems with large parameters and the incompressible limit of compressible fluids. *Commun. Pure Appl. Math.* 34 (1981), 481–524.

17 T. Hmidi. Low Mach number limit for the isentropic Euler system with axisymmetric initial data. *J. Inst. Math. Jussieu* 12 (2013), 335–389.

18 T. Hmidi and S. Keraani. Incompressible viscous flows in borderline Besov spaces. *Arch. Ration. Mech. Analysis* 189 (2008), 283–300.

19 P.-L. Lions and N. Masmoudi. The incompressible limit of the non-isentropic Euler equations. *Arch. Ration. Mech. Analysis* 158 (2001), 61–90.

20 A. Majda. *Compressible fluid flow and systems of conservation laws in several space variables*. Applied Mathematical Sciences, vol. 53 (Springer, 1984).

21 T. Makino, S. Ukai and S. Kawashima. Sur la solution à support compact de l’équation d’Euler compressible. *Jpn. J. Appl. Math.* 3 (1986), 249–257.

22 G. Métivier and S. Schochet. The incompressible limit of the non-isentropic Euler equations. *Arch. Ration. Mech. Analysis* 158 (2001), 61–90.

23 S.-K. Lin. On the incompressible limit of the slightly compressible viscous fluid flows. In *Nonlinear waves*, GAKUTO International Series on Mathematical Sciences and Their Applications, vol. 10, pp. 277–282 (Tokyo: Gakkōtosho, 1997).

24 S. Klainerman and A. Majda. Compressible and incompressible fluids. *Commun. Pure Appl. Math.* 35 (1982), 629–651.

25 T. Hmidi. Low Mach number limit for the isentropic Euler system with axisymmetric initial data. *J. Inst. Math. Jussieu* 12 (2013), 335–389.

26 C.-K. Lin. On the incompressible limit of the slightly compressible viscous fluid flows. In *Nonlinear waves*, GAKUTO International Series on Mathematical Sciences and Their Applications, vol. 10, pp. 277–282 (Tokyo: Gakkōtosho, 1997).

27 P.-L. Lions and N. Masmoudi. Incompressible limit for a viscous compressible fluid. *J. Math. Pures Appl.* 77 (1998), 585–627.

28 S. Ukai. The incompressible limit and the initial layer of the compressible Euler equation. *J. Math. Kyoto Univ.* 26 (1986), 323–331.

29 S. Ukai. The incompressible limit and the initial layer of the compressible Euler equation. *J. Math. Kyoto Univ.* 26 (1986), 323–331.

30 M. Vishik. Hydrodynamics in Besov spaces. *Arch. Ration. Mech. Analysis* 145 (1998), 197–214.

(Received 5 December 2014)