Operator Approach to Complexity : Excited State

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ABSTRACT

The operator approach to evaluate the complexity in field theory is presented. We directly regard the transformation between the creation operators of referent and target states as the quantum gate. We calculate the geodesic length on the associated transformation group manifold of quantum circuit and reproduce the known value of ground-state complexity. Since that in the operator approach it needs not to use the explicit form of the wave function we can study the complexity in the excited state. Although the transformation matrix is very large we can transform it to the direct products of several $R$ and $GL(2, R)$, and obtain the associated complexity. We calculate the complexity in the first excited state ($n=2$), second excited state ($n=3$) and third excited state ($n=4$). After the proper normalization we find that the square of length of the quantum circuit in nth excited state is $n \cdot 2^{n-1}$ times that in the ground state. We also sketch a possible proof.

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1 Introduction

Complexity plays an important role in understanding how spacetime emerges from field theory degrees of freedom within the AdS/CFT correspondence [1], besides the entanglement entropy [5]. It relates to the tensor network models and involves the dynamics of black hole interiors [6]. In the context of the eternal AdS-Schwarzchild black hole the wormhole which connects the two sides grows linearly with time [7], which is conjectured to dual to the growth of complexity of the dual CFT state [8, 9, 10].

In the context of AdS/CFT two interesting proposals are used to evaluate the holographic complexity. The first is complexity=volume (CV) conjecture [11] which suggests that complexity is dual to the volume of an extremal (codimension one) bulk surface anchored to a certain time slice in the boundary. The second is complexity=action (CA) conjecture [12, 13, 14] which identifies the complexity with the gravitational action evaluated on a particular bulk region, known as the Wheeler-DeWitt (WDW) patch.

Complexity is the number of operations $\{O^I\}$ needed to transform a reference state $|\psi_R\rangle$ to a target state $|\psi_T\rangle$. The operators are called as quantum gates and the more gates we need the more complex the target state was. We can define the affine parameter “s” associated to an
unitary operator $U(s)$ and use a set of function $Y^I(s)$ to characterize the quantum circuit. The unitary operation which connects the reference state and target state is

$$U(s) = P e^{-\int_0^s Y^I(s) \mathcal{O}^I}, \quad |\psi_R\rangle = U(0)|\psi_R\rangle, \quad |\psi_T\rangle = U(1)|\psi_R\rangle$$ (1.1)

The circuit depth $D[U]$ which also called as cost function is defined by

$$D[U] = \int_0^1 ds \sqrt{\sum_I |Y^I(s)|^2}$$ (1.2)

Use it the complexity $C$ is defined by

$$C = \min_{\{Y^I\}} D[U]$$ (1.3)

For the model example which has been studied the gates are the group generators. Thus the minimum in $C$ means that we are to calculate the geodesic in the Riemann space of group manifold.

The calculation along this line was initial in papers [15] and [16] which considered the free scalar field. Later, it was extended to study the free fermions [17, 18], quenched system [19, 20], coherence state [21], and interacting model [22]. The investigate in [15] had shown that once the cost function is chosen to be

$$D[U] = \int_0^1 ds \sum_I |Y^I(s)|^2$$ (1.4)

the field theory calculation could match to gravity method.

The most studies in field theory are considering the Gaussian ground state [15, 17, 18] in the free field or exponential type wavefunction in interacting model [22]. In free field there are two basic coordinate: $x_1, x_2$ while in interacting model there are 5 basic coordinate: $x_1, x_2, x_1 x_2, x_1^2, x_2^2$. The transformations between the coordinate of referent and target state are regarded as the quantum gates. Using the transformation group, which is $GL(2, R)$ in free field and chosen to be $R \times R \times R \times GL(2, R)$ in interacting model, to represent the gate the complexity is the related to the geodesic length on the group manifold of quantum circuit.

In this paper we will present the operator approach to evaluate the complexity in field theory. We regard the transformation between the create operators of referent and target state as the quantum gate and, in the similar way, calculate the geodesic length on the associated group manifold. Since that in the operator approach we need not to use the explicit form of the wave function we can study the complexity in the excited state, in which the SHO wavefunction is not pure exponential form and it seems that the coordinate space approach is hard to work.

The paper is organized as follows. In section 2, we quickly review the method in [15] to see how the transformation between the wavefunctions of the referent state and target state can be

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3While the Bogoliubov transformation method was used in [16], fermion theory [18] and quench mode [19, 20] our scheme is simple and more general.
used to calculate the complexity of the ground state. In section 3, we setup the notation and describe the transformation between the creation operators in the referent state and target state. We find that the transformation can be reduced to $R \times R \times R \times R$ after orthogonal transformation. Then, we calculate circuit depth $D[U]$ and the complexity. The result reproduces the known value of ground-state complexity in [15].

In section 4, we study the complexity in the first excited state ($n=2$) and in section 5, we study the complexity in the second excited state ($n=3$). We see that although the dimension of transformation matrix is very large we can transform it to the direct products of severe $R$ and $GL(2, R)$, and obtain the associated complexity. We find that the square of length of the quantum circuit is $n \cdot 2^{n-1}$ times that in the ground state. In section 6, we present a simple algorithm to calculate the complexity in the $n$th excited state and sketch a possible proof. Final remarks appear in the section 7. In appendix A we discuss the geodesic length after transforming gate matrix to become diagonal form. In appendix B we derive a summation formula which is used to prove the complexity formula.

2 Free Scalar Field and SHO : Wavefunction Approach

We consider the Hamiltonian of a free scalar field in $d$ spacetime dimensions by following the notation in [15]

$$H = \frac{1}{2} \int d^{d-1}x \left[ \pi(x)^2 + \bar{\nabla}\phi(x)^2 + m^2 \phi(x)^2 \right]. \quad (2.1)$$

Placing the theory on a square lattice with lattice spacing $\delta$ the Hamiltonian is described by

$$H = \frac{1}{2} \sum_{\vec{n}} \left\{ \frac{p(\vec{n})^2}{\delta^d-1} + \delta^{-1} \left[ \frac{1}{\delta^2} \sum_i \left( (\phi(\vec{n}) - \phi(\vec{n} - \hat{x}_i))^2 + m^2 \phi(\vec{n})^2 \right) \right] \right\} \quad (2.2)$$

where $\hat{x}_i$ are unit vectors pointing along the spatial directions of the lattice. By redefining $X(\vec{n}) = \delta^{d/2} \phi(\vec{n})$, $P(\vec{n}) = p(\vec{n})/\delta^{d/2}$, $M = 1/\delta$, $\omega = m$ and $\Omega = 1/\delta$ the Hamiltonian becomes

$$H = \sum_{\vec{n}} \left\{ \frac{P(\vec{n})^2}{2M} + \frac{1}{2M} \left[ \omega^2 X(\vec{n})^2 + \Omega^2 \sum_i \left( X(\vec{n}) - X(\vec{n} - \hat{x}_i) \right)^2 \right] \right\} \quad (2.3)$$

where $\omega = m$ and $\Omega = 1/\delta$. The resulting theory is essentially a quantum mechanical problem with an infinite family of coupled (one-dimensional) harmonic oscillators.

Consider a simple case of two coupled harmonic oscillators:

$$H = \frac{1}{2} \left[ \hat{p}_1^2 + \hat{p}_2^2 + \omega^2 (x_1^2 + x_2^2) + \Omega^2 (x_1 - x_2)^2 \right] \quad (2.4)$$

where $x_1, x_2$ label their spatial positions, after setting $M_1 = M_2 = 1$ for simplicity. The Hamiltonian expressed in terms of the normal modes is

$$H = \frac{1}{2} \left( \hat{p}_+^2 + \hat{\omega}_+^2 \hat{x}_+^2 + \hat{p}_-^2 + \hat{\omega}_-^2 \hat{x}_-^2 \right) \quad (2.5)$$
where
\[ \tilde{x}_\pm \equiv \frac{1}{\sqrt{2}}(x_1 \pm x_2), \quad \tilde{p}_\pm \equiv \frac{1}{\sqrt{2}}(p_1 \pm p_2), \quad \tilde{\omega}_+^2 = \omega^2, \quad \tilde{\omega}^- = \omega^2 + 2\Omega^2 \] (2.6)

The normalized ground-state wave function is
\[ \psi_0(\tilde{x}_+, \tilde{x}_-) = \psi_0^+(\tilde{x}_+)\psi_0^-(\tilde{x}_-) = \left(\frac{\tilde{\omega}_+\tilde{\omega}_-}{\sqrt{\pi}}\right)^{1/4} \exp\left[-\frac{1}{2}\left(\tilde{\omega}_+\tilde{x}_+^2 + \tilde{\omega}_-\tilde{x}_-^2\right)\right], \] (2.7)

We can express this wave function in terms of the physical positions of the two masses:
\[ \psi_0(x_1, x_2) = \left(\frac{\omega_1\omega_2 - \beta^2}{\sqrt{\pi}}\right)^{1/4} \exp\left[-\frac{\omega_1}{2}x_1^2 - \frac{\omega_2}{2}x_2^2 - \beta x_1x_2\right], \] (2.8)

where
\[ \omega_1 = \omega_2 = \frac{1}{2}(\tilde{\omega}_+ + \tilde{\omega}_-), \quad \beta \equiv \frac{1}{2}(\tilde{\omega}_+ - \tilde{\omega}_-) \] (2.9)

Above Gaussian wave functions is the target state. The reference state, as that in [15], is chosen to be following factorized Gaussian state in which the two masses are unentangled
\[ \psi_R(x_1, x_2) = \left(\frac{\omega_0}{\sqrt{\pi}}\right)^{1/4} \exp\left[-\frac{\omega_0}{2}\left(x_1^2 + x_2^2\right)\right]\] (2.10)

where \(\omega_0\) is a free parameter which characterizes our reference state.

After choosing the reference and target states we have to find a desired unitary transformation \(U\) which implements \(\psi_T = U\psi_R\). In the wavefunction approach the transformation is considered to relate the the exponential in target state wavefunction (2.8) and in reference state wavefunction (2.10), i.e.
\[ \omega_0\left(x_1^2 + x_2^2\right) \rightarrow \omega_1 x_1^2 + \omega_2 x_2^2 + 2\beta x_1x_2 \] (2.11)

We can use the basic vector \(\psi_R = (\tilde{x}_1, \tilde{x}_2)\), with \(\tilde{x}_1 = \sqrt{\omega_0}x_1, \tilde{x}_2 = \sqrt{\omega_0}x_2\) to express above transformation in a matrix form
\[ \psi_R^T \cdot \psi_R \rightarrow \psi_R^T U^T \cdot U \psi_R = \psi_R^T \cdot M_w \cdot \psi_R, \quad M_w = \begin{pmatrix} \frac{\omega_1}{\omega_0} & \beta \\ \frac{\omega_2}{\omega_0} & \frac{\omega_2}{\omega_0} \end{pmatrix} \] (2.12)

in which \(M_w\) denotes the transformation matrix in wavefunction approach.

We can describe matrix \(U(s)\) as a \(R \times SL(2, R)\) group element
\[ U_{GL(2, R)}(s) = e^y(s) \begin{pmatrix} \cos \tau(s) \cosh \rho(s) - \sin \tau(s) \sinh \rho(s) & -\sin \tau(s) \cosh \rho(s) + \cos \tau(s) \sinh \rho(s) \\ \sin \tau(s) \cosh \rho(s) + \cos \tau(s) \sinh \rho(s) & \cos \tau(s) \cosh \rho(s) + \sin \tau(s) \sinh \rho(s) \end{pmatrix} \] (2.13)

The solution of \(U^T(1)U(1) = M_w\) is
\[ y(1) = \frac{1}{4} \ln \frac{\tilde{\omega}_+\tilde{\omega}_-}{\omega_0^2}, \quad \rho(1) = \frac{1}{4} \ln \frac{\tilde{\omega}_-}{\tilde{\omega}_+} \] (2.14)
which gives the geodesic distance

\[ D^2 = 2(y(1))^2 + \rho(1)^2 = \frac{1}{4} \left( (\ln \frac{\tilde{\omega}_+}{\omega_0})^2 + (\ln \frac{\tilde{\omega}_-}{\omega_0})^2 \right) \]  \quad (2.15)

Above is that described by Jefferson and Myers in [15].

The method relies on the wavefunction and had been extended to study an interacting model [22] in which the two basic coordinates \( x_1, x_2 \) are extended to 5 basic coordinates \( x_1, x_2, x_1 x_2, x_1^2, x_2^2 \). However the states they considered are pure exponential type and thus are constrained on ground state. For the excited states the wavefunctions are not described by the pure exponential, for example the wave function of the first excited state of SHO is \( x e^{-\frac{1}{2} \omega x^2} \), and it is hard to work in the wavefunction approach.

Thus, in the following sections we will turn to the operator approach, which need not the explicit form of wavefunction and can be used to study the complexity in excited states.

## 3 Operator Approach to Complexity in Ground State

In the second quantization the target state is created by \( a_+ a_- \) and the reference state is created by \( a_1^\dagger a_2^\dagger \), i.e.

\[ \psi_T = a_+ a_- |0\rangle, \quad \psi_R = a_1^\dagger a_2^\dagger |0\rangle \]  \quad (3.1)

where

\[ a_+ = \sqrt{\frac{\omega_+}{2}} x_+ + i \frac{1}{\sqrt{2\omega_+}} p_+, \quad a_- = \sqrt{\frac{\omega_-}{2}} x_- + i \frac{1}{\sqrt{2\omega_-}} p_- \]  \quad (3.2)

\[ a_1^\dagger = \sqrt{\frac{\omega_0}{2}} x_1 + i \frac{1}{\sqrt{2\omega_0}} p_1, \quad a_2^\dagger = \sqrt{\frac{\omega_0}{2}} x_2 + i \frac{1}{\sqrt{2\omega_0}} p_2 \]  \quad (3.3)

The relations between the operators are:

\[ a_+^\dagger = \frac{1}{2\sqrt{2}} \left( (\alpha + \alpha^{-1})(a_2^\dagger + a_1^\dagger) + (\alpha - \alpha^{-1})(a_2 + a_1) \right) \]  \quad (3.4)

\[ a_-^\dagger = \frac{1}{2\sqrt{2}} \left( (\gamma + \gamma^{-1})(-a_2^\dagger + a_1^\dagger) + (\gamma - \gamma^{-1})(-a_2 + a_1) \right) \]  \quad (3.5)

where

\[ \alpha = \sqrt{\frac{\tilde{\omega}_+}{\omega_0}}, \quad \gamma = \sqrt{\frac{\tilde{\omega}_-}{\omega_0}} \]  \quad (3.6)

We see that both of operators \( a_+^\dagger \) and \( a_-^\dagger \) depend on the four kinds operators \( a_1^\dagger, a_1, a_2^\dagger, a_2 \). Thus to consider the transformation between target state operators and referent state operators we have to consider 4 by 4 matrix in below

\[ \begin{pmatrix} a_+^\dagger \\ a_-^\dagger \\ a_+ \\ a_- \end{pmatrix} = \tilde{M}_{op}^{(0)} \begin{pmatrix} a_1^\dagger \\ a_2^\dagger \\ a_1 \\ a_2 \end{pmatrix} \]  \quad (3.7)
where
\[
\tilde{M}_{\text{op}}^{(0)}(0) = \frac{1}{2\sqrt{2}} \begin{pmatrix}
\alpha + \alpha^{-1} & \alpha + \alpha^{-1} & \alpha - \alpha^{-1} & \alpha - \alpha^{-1} \\
\gamma + \gamma^{-1} & -(\gamma + \gamma^{-1}) & \gamma - \gamma^{-1} & -(\gamma - \gamma^{-1}) \\
\alpha - \alpha^{-1} & \alpha - \alpha^{-1} & \alpha + \alpha^{-1} & \alpha + \alpha^{-1} \\
\gamma - \gamma^{-1} & -(\gamma - \gamma^{-1}) & \gamma + \gamma^{-1} & -(\gamma + \gamma^{-1})
\end{pmatrix}
\]
(3.8)

\(\tilde{M}_{\text{op}}^{(0)}\) is the transformation matrix of ground state in operator approach.

We put the transformation matrix into block form through SO(4) transformation by a matrix
\[
\begin{pmatrix}
\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\
0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 0 \\
0 & \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}}
\end{pmatrix}
\]
(3.9)

then \(\tilde{M}_{\text{op}}^{(0)} \rightarrow M_{\text{op}}^{(0)}\) where
\[
M_{\text{op}}^{(0)} = \begin{pmatrix}
M_1 & 0 \\
0 & M_2
\end{pmatrix}, \quad M_1 = \frac{1}{\sqrt{2}} \begin{pmatrix}
\alpha & \alpha \\
\gamma & -\gamma
\end{pmatrix}, \quad M_2 = \frac{1}{\sqrt{2}} \begin{pmatrix}
\alpha^{-1} & \alpha^{-1} \\
\gamma^{-1} & -\gamma^{-1}
\end{pmatrix}
\]
(3.10)

and
\[
\begin{pmatrix}
\hat{a}_1^+ + \hat{a}_+ \\
\hat{a}_2^+ + \hat{a}_- \\
\hat{a}_1^+ - \hat{a}_+ \\
\hat{a}_2^+ - \hat{a}_-
\end{pmatrix} = M_{\text{op}}^{(0)} \begin{pmatrix}
\hat{a}_1^+ + a_1 \\
\hat{a}_2^+ + a_2 \\
\hat{a}_1^+ - a_1 \\
\hat{a}_2^+ - a_2
\end{pmatrix}
\]
(3.11)

The gate matrix becomes \(GL(2, R) \times GL(2, R)\). Now we can use the \(GL(2, R)\) matrix representation in (2.13) to parameter the matrix \(M_1\) and \(M_2\). For the matrix \(M_1\) we find that
\[
y(1) = \frac{1}{4} \ln \frac{\tilde{\omega}_+ \tilde{\omega}_-}{\omega_0^2}, \quad \rho(1) = \frac{1}{4} \ln \frac{\tilde{\omega}_+}{\tilde{\omega}_-}
\]
(3.12)

It is interesting to see that the values are exact those in (2.14) despite \(M_w \neq M_1\). Thus the geodesic length is just that in wavefunction approach. For the matrix \(M_2\)
\[
y(1) = -\frac{1}{4} \ln \frac{\tilde{\omega}_+ \tilde{\omega}_-}{\omega_0^2}, \quad \rho(1) = -\frac{1}{4} \ln \frac{\tilde{\omega}_-}{\tilde{\omega}_+}
\]
(3.13)

which gives the same geodesic length as from \(M_1\).

At first sight, the total geodesic from \(GL(2, R) \times GL(2, R)\) length will be double comparing to that in wavefunction approach. In fact, in the operator approach to consider the transformation, for completeness, we had to use four operators \(\hat{a}_1^+, \hat{a}_2^+, a_+, a_-\) (which implies four operators, \(a_1^+, a_2^+, a_1, a_2\) all together, not just two operators \(\hat{a}_1^+, \hat{a}_-\) which create the target state. Thus, to count the proper number of the quantum gate we have to reduce the summation and
\[
D_{\text{ground state}}^2 = \frac{1}{2} (D_{M_1}^2 + D_{M_2}^2) = \frac{1}{4} \left( \left( \ln \frac{\tilde{\omega}_+}{\omega_0} \right)^2 + \left( \ln \frac{\tilde{\omega}_-}{\omega_0} \right)^2 \right)
\]
(3.14)
In this way the operator approach fits to wavefunction approach (2.15). Notice that the SO(4) transformation im transformation matrix is corresponding to the SO(4) transformation of group manifold, like as a rotation, and does not modify the geodesic length.

While above algorithm is reasonable we can slightly improve it for simplicity. After a linear combination we can put the transformation matrix into a block form

\[
\bar{M}^{(0)}_{\text{op}} = \begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix}, \quad \bar{M}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} \alpha & 0 \\ 0 & \gamma \end{pmatrix}, \quad \bar{M}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} \alpha^{-1} & 0 \\ 0 & \gamma^{-1} \end{pmatrix}
\]  

(3.15)

and

\[
\begin{pmatrix} a_+^\dagger + a_+ \\ a_+^\dagger + a_- \\ a_-^\dagger - a_+ \\ a_-^\dagger - a_- \end{pmatrix} = \bar{M}^{(0)}_{\text{op}} \begin{pmatrix} (a_1^\dagger + a_1) + (a_2^\dagger + a_2) \\ (a_1^\dagger + a_1) - (a_2^\dagger + a_2) \\ (a_1^\dagger - a_1) + (a_2^\dagger - a_2) \\ (a_1^\dagger - a_1) - (a_2^\dagger - a_2) \end{pmatrix}
\]  

(3.16)

Now that the gate matrix \(\bar{M}^{(0)}_{\text{op}}\) is a \(R \times R \times R \times R\) matrix. As discussed in appendix A the associated geodesic length \(D_{\text{ground state}}\) becomes

\[
D_{\text{ground state}}^2 = \frac{1}{2} \cdot \left( (\ln \alpha)^2 + (\ln \gamma)^2 + (\ln \alpha^{-1})^2 + (\ln \gamma^{-1})^2 \right) \quad (3.17)
\]

\[
= \frac{1}{4} \left( (\ln \tilde{\omega}_+)^2 + (\ln \tilde{\omega}_-)^2 \right) \quad (3.18)
\]

In this way the operator approach fits to wavefunction approach (2.15).

Notice that we add a normalization factor \(\frac{1}{2}\) in (3.17) to fit the known value. The reason is explained before. The property tells us that in considering the nth excited state (ground state is \(n=1\)) we have to add a factor \(\frac{1}{2n}\) to obtain the correct value of complexity. In following section we follow the argument to add this normalization to the complexity in nth excited state. We calculate the complexity in first excited state (\(n=2\)) and second excited state (\(n=3\)) and, after the proper normalization, find that the square of length of the quantum circuit is \(n \cdot 2^{n-1}\) times that in the ground state.

4 Operator Approach to Complexity in First Excited State

To study the excited state we first define the useful notations for simplicity

\[
A_+ \equiv a_+^\dagger + a_+, \quad A_- \equiv a_-^\dagger + a_-, \quad B_+ \equiv a_+^\dagger - a_+, \quad B_- \equiv a_-^\dagger - a_-
\]  

(4.1)

\[
A_1 \equiv a_1^\dagger + a_1, \quad A_2 \equiv a_2^\dagger + a_2, \quad B_1 \equiv a_1^\dagger - a_1, \quad B_2 \equiv a_2^\dagger - a_2
\]  

(4.2)

which satisfy the commutative relations

\[
[A_+, A_-] = [A_+, B_] = [B_+, B_-] = [B_+, A_-] = 0
\]  

(4.3)

\[
[A_+, B_+] = [A_-, B_-] = -1
\]  

(4.4)
Using the new operators $A_J$ and $B_J$ the matrix transformation relations in (3.16) becomes

\[
\begin{pmatrix}
A_+ \\
A_-
\end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix}
\alpha & \alpha \\
\gamma & -\gamma
\end{pmatrix} \begin{pmatrix}
A_1 \\
A_2
\end{pmatrix}, \quad \begin{pmatrix}
B_+ \\
B_-
\end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix}
\alpha^{-1} & \alpha^{-1} \\
\gamma^{-1} & -\gamma^{-1}
\end{pmatrix} \begin{pmatrix}
B_1 \\
B_2
\end{pmatrix}
\]

i.e.

\[
A_+ = \frac{\alpha}{\sqrt{2}} (A_1 + A_2), \quad A_- = \frac{\gamma}{\sqrt{2}} (A_1 - A_2)
\]

\[
B_+ = \frac{\alpha^{-1}}{\sqrt{2}} (B_1 + B_2), \quad B_- = \frac{\gamma^{-1}}{\sqrt{2}} (B_1 - B_2)
\]

Now there are four basic states : $(A_+, A_-, B_+, B_-)$. Thus the product of them will produce 16 basic states. The associated transformation matrix will become $16 \times 16$ which, at first sight, is too large to be studied. In fact the $16 \times 16$ is formed from four block matrix and can be transformed to be a products of $R$ matrix.

Let us consider the first block. While the ground state complexity is studied by operator $(A_+, A_-)$ the excite state complexity is studied by the product of operator $A_+$ with $A_-$. Denote the target state operator by $(AA)_T$ and reference state operator by $(AA)_R$ in which

\[
(\begin{pmatrix}
A_+ \\
A_-
\end{pmatrix})_T = \begin{pmatrix}
A_+ A_+ \\
A_- A_- \\
A_+ A_- \\
A_- A_+
\end{pmatrix}, \quad (\begin{pmatrix}
A_+ \\
A_-
\end{pmatrix})_R = \begin{pmatrix}
A_1 A_1 \\
A_2 A_2 \\
A_1 A_2 \\
A_2 A_1
\end{pmatrix}
\]

the transformations in (4.6) lead to

\[
(\begin{pmatrix}
A_+ \\
A_-
\end{pmatrix})_T = \tilde{M}(AA) (\begin{pmatrix}
A_+ \\
A_-
\end{pmatrix})_R, \quad \tilde{M}(AA) = \frac{1}{2} \begin{pmatrix}
\alpha^2 & \alpha^2 & \alpha^2 & \alpha^2 \\
\gamma^2 & \gamma^2 & -\gamma^2 & -\gamma^2 \\
\alpha\gamma & -\alpha\gamma & 0 & 0 \\
\alpha\gamma & -\alpha\gamma & 0 & 0
\end{pmatrix}
\]

Through $SO(4)$ transformation the gate matrix becomes $R \times R \times GL(2,R)$, i.e.

\[
\tilde{M}(AA) \rightarrow \tilde{M}(AA) = \frac{1}{2} \begin{pmatrix}
\alpha^2 & \alpha^2 & 0 & 0 \\
\gamma^2 & -\gamma^2 & 0 & 0 \\
0 & 0 & \alpha\gamma & 0 \\
0 & 0 & 0 & \alpha\gamma
\end{pmatrix}
\]

Now, as that in the ground state, after rearrangement, the gate matrix can be transformed to a diagonal form, i.e.

\[
\begin{pmatrix}
A_+ A_+ \\
A_- A_- \\
A_+ A_- \\
A_- A_+
\end{pmatrix} = M_{(AA)} \begin{pmatrix}
A_1 A_1 + A_2 A_2 + A_+ A_- + A_- A_+ \\
A_1 A_1 + A_2 A_2 - A_+ A_- - A_- A_+ \\
A_1 A_1 - A_2 A_2 \\
A_1 A_1 - A_2 A_2
\end{pmatrix}
\]

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where

\[
M_{(AA)} = \frac{1}{2} \begin{pmatrix}
\alpha^2 & 0 & 0 & 0 \\
0 & \gamma^2 & 0 & 0 \\
0 & 0 & \alpha\gamma & 0 \\
0 & 0 & 0 & \alpha\gamma
\end{pmatrix}
\]  

(4.12)

Thus the gate matrix \(M_{(AA)}\) is a \(R \times R \times R \times R\) matrix and can be calculated as before. The geodesic length \(D_{(AA)}\) becomes

\[
D^2_{(AA)} = \frac{1}{2^2} \left( \left( \ln \frac{\tilde{\omega}_+}{\omega_0} \right)^2 + \left( \ln \frac{\tilde{\omega}_-}{\omega_0} \right)^2 + 2 \left( \frac{1}{2} (\ln(\tilde{\omega}_+) + \ln(\tilde{\omega}_-)) \right)^2 \right)
\]  

(4.13)

The next block is that with \(A \rightarrow B\). The geodesic formulas are just those by replacement \(\alpha \rightarrow \alpha^{-1}, \gamma \rightarrow \gamma^{-1}\) and

\[
M_{(BB)} = \frac{1}{2} \begin{pmatrix}
\alpha^{-2} & 0 & 0 & 0 \\
0 & \gamma^{-2} & 0 & 0 \\
0 & 0 & \alpha^{-1}\gamma^{-1} & 0 \\
0 & 0 & 0 & \alpha^{-1}\gamma^{-1}
\end{pmatrix}
\]  

(4.14)

Thus we have the simple relations :

\[D_{(BB)} = D_{(AA)}\]  

(4.15)

The symmetry property had been shown in the ground state case.

Consider the third block in which the state is the product of operator \(A\) with \(B\) :

\[
(AB)_T = \begin{pmatrix}
A_+B_+ \\
A_+B_- \\
A_-B_+ \\
A_-B_-
\end{pmatrix}, \quad \text{and} \quad (AB)_R = \begin{pmatrix}
A_1B_1 \\
A_1B_2 \\
A_2B_1 \\
A_2B_2
\end{pmatrix}
\]  

(4.16)

The associated matrix transformation is

\[
(AB)_T = M_{(AB)} (AB)_R, \quad \tilde{M}_{(AB)} = \frac{1}{2} \begin{pmatrix}
1 & 1 & 1 & 1 \\
\alpha\gamma^{-1} & -\alpha\gamma^{-1} & \alpha\gamma^{-1} & -\alpha\gamma^{-1} \\
\gamma\alpha^{-1} & \gamma\alpha^{-1} & -\gamma\alpha^{-1} & -\gamma\alpha^{-1} \\
1 & -1 & -1 & 1
\end{pmatrix}
\]  

(4.17)

As before, through \(SO(4)\) transformation we have a block matrix

\[
\tilde{M}_{(AB)} \rightarrow \tilde{\tilde{M}}_{(AB)} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & \alpha\gamma^{-1} & -\alpha\gamma^{-1} \\
0 & 0 & \gamma\alpha^{-1} & \gamma\alpha^{-1}
\end{pmatrix}
\]  

(4.18)
which is $R \times R \times GL(2, R)$ matrix. After rearrangement the matrix $\tilde{M}_{(AB)}$ becomes a diagonal matrix $M_{(AB)}$

\[
\tilde{M}_{(AB)} \rightarrow M_{(AB)} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & \alpha \gamma^{-1} & 0 \\
0 & 0 & 0 & \gamma \alpha^{-1}
\end{pmatrix}
\] (4.19)

The associated geodesic length $D_{(AB)}$ can be evaluated as before. The result is

\[
D_{(AB)}^2 = \frac{1}{2} \left( 2 \left( \frac{1}{2} \left( \ln \left( \frac{\tilde{\omega}^+}{\omega_0} \right) - \ln \left( \frac{\tilde{\omega}^-}{\omega_0} \right) \right) \right)^2 \right)
\] (4.20)

The another block is that with $A \leftrightarrow B$. The geodesic formula is just that by replacement $\alpha \leftrightarrow \alpha^{-1}$, $\gamma \leftrightarrow \gamma^{-1}$ and

\[
M_{(BA)} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & \gamma \alpha^{-1} & 0 \\
0 & 0 & 0 & \alpha \gamma^{-1}
\end{pmatrix}
\] (4.21)

Thus we have a simple relation :

\[
D_{(BA)} = D_{(AB)}
\] (4.22)

Totally there are four contributions from four blocks and the total complexity is

\[
D_{\text{first excited state}}^2 = \left( D_{(AA)}^2 + D_{(AB)}^2 + D_{(BA)}^2 + D_{(BB)}^2 \right)
\] (4.23)

\[
= 4 \times \frac{1}{4} \left( \left( \ln \left( \frac{\tilde{\omega}^+}{\omega_0} \right) \right)^2 + \left( \ln \left( \frac{\tilde{\omega}^-}{\omega_0} \right) \right)^2 \right)
\] (4.24)

which is four times that in the ground state (2.15). Notice that we have already added a normalization factor $\frac{1}{2^2}$ in (4.23) inside each terms of $D_{(AA)}, D_{(AB)}, D_{(BA)}, D_{(BB)}$ to obtain the final result.

5 Operator Approach to Complexity in Second Excited State

We now have to use six creation operators to obtain the second excited state, i.e. $a_+^+ a_+^+ a_+^+ a_-^+ a_-^+ a_-^+ |0\rangle$. Use the notation in previous section we see that there are eight blocks to be considered :

\[
[A(AA)]; [A(AB)]; [A(BA)]; [A(BB)]; [B(AA)]; [B(AB)]; [B(BA)]; [B(BB)]
\] (5.1)

where states $(AA)$ and $(AB)$ are defined in left-hand side of (4.11) and (4.16) respectively. While, at first sight it is hard to explicitly calculate them we can use the symmetry property : $A_J \leftrightarrow B_J$, $(\alpha, \gamma) \leftrightarrow (\alpha^{-1}, \gamma^{-1})$ to simplify the calculation. In face we only need to calculate three blocks among eight and others can be quickly obtained from symmetry property. This property had been used in previous section to find the complexity in first excited state.
5.1 Block [A(AA)]

Consider the first block: [A(AA)]. Using left-hand side of (4.11) to represent (AA) and matrix (4.12) to represent the gate matrix of block [A(AA)] then, after transformation it becomes two diagonal matrices:

\[
M_{A(AA)}^{(1)} = \frac{1}{2\sqrt{2}} \cdot \alpha \cdot \begin{pmatrix}
\alpha^2 & 0 & 0 & 0 \\
0 & \gamma^2 & 0 & 0 \\
0 & 0 & \alpha \gamma & 0 \\
0 & 0 & 0 & \alpha \gamma
\end{pmatrix}
\]

\[\text{(5.2)}\]

\[
M_{A(AA)}^{(2)} = \frac{1}{2\sqrt{2}} \cdot \gamma \cdot \begin{pmatrix}
\alpha^2 & 0 & 0 & 0 \\
0 & \gamma^2 & 0 & 0 \\
0 & 0 & \alpha \gamma & 0 \\
0 & 0 & 0 & \alpha \gamma
\end{pmatrix}
\]

\[\text{(5.3)}\]

The geodesic length can be calculated as before. Using the symmetry property: \((\alpha, \gamma) \rightarrow (\alpha^{-1}, \gamma^{-1})\) we get the transformations matrix in block [B(BB)]. Both have the same geodesic length.

5.2 Block [A(AB)]

Consider the second block: [A(AB)]. Now, associative law in matrix implies [A(AB)] = [(AA)B]]. Using left-hand side of (4.11) to represent (AA) and matrix (4.12) to represent the transformation the gate matrix of block [A(AB)] can be transformed to a diagonal matrix:

\[
M_{A(AB)}^{(1)} = \frac{1}{2\sqrt{2}} \cdot \alpha^{-1} \cdot \begin{pmatrix}
\alpha^2 & 0 & 0 & 0 \\
0 & \gamma^2 & 0 & 0 \\
0 & 0 & \alpha \gamma & 0 \\
0 & 0 & 0 & \alpha \gamma
\end{pmatrix}
\]

\[\text{(5.4)}\]

\[
M_{A(AB)}^{(2)} = \frac{1}{2\sqrt{2}} \cdot \gamma^{-1} \cdot \begin{pmatrix}
\alpha^2 & 0 & 0 & 0 \\
0 & \gamma^2 & 0 & 0 \\
0 & 0 & \alpha \gamma & 0 \\
0 & 0 & 0 & \alpha \gamma
\end{pmatrix}
\]

\[\text{(5.5)}\]

The geodesic length can be calculated as before. Using the symmetry property: \((\alpha, \gamma) \leftrightarrow (\alpha^{-1}, \gamma^{-1})\) we get the transformations matrix in blocks [A(BB)], [B(AA)] and [B(BA)]. They all have the same geodesic length.

5.3 Block [B(AB)]

Consider the sixth block: [B(AB)]. Using left-hand side of (4.16) to represent (AA) and matrix (4.19) to represent the transformation the gate matrix of block [B(AB)] can be transformed to
a diagonal matrix:

\[ M^{(1)}_{B(AB)} = \frac{1}{2\sqrt{2}} \cdot \alpha^{-1} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \alpha^{-1} & 0 \\ 0 & 0 & 0 & \gamma \alpha^{-1} \end{pmatrix} \]

\[ (5.6) \]

\[ M^{(2)}_{B(AB)} = \frac{1}{2\sqrt{2}} \cdot \gamma^{-1} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \alpha^{-1} & 0 \\ 0 & 0 & 0 & \gamma \alpha^{-1} \end{pmatrix} \]

The geodesic length can be calculated as before. Using the symmetry property : \((\alpha, \gamma) \leftrightarrow (\alpha^{-1}, \gamma^{-1})\) we get the transformations matrix in Block \([A(BA)]\). Both have the same geodesic length.

5.4 Geodesic length of Second Excited State

After summation we find

\[ D^2_{\text{second excited state}} = \frac{1}{2^3} \cdot \left( 2D^2_{A(AA)} + 4D^2_{A(AB)} + 2D^2_{B(AB)} \right) \]

\[ = 12 \times \frac{1}{4} \left( \left( \ln \frac{\tilde{\omega}_+}{\omega_0} \right)^2 + \left( \ln \frac{\tilde{\omega}_-}{\omega_0} \right)^2 \right) \]

\[ (5.8) \]

\[ (5.9) \]

which is twelve times that in the ground state \((2.15)\). Notice that we have added a normalization factor \(\frac{1}{2^3}\) to obtain the final result.

6 Complexity in nth Excited State

6.1 A Simple Algorithm

From the investigations in previous sections we can find a simple rule to calculate the complexity in the nth excited state. The crucial point is that, while in finding the transformation matrix the basic state of referent state is relevant we can linear combine them to form the diagonal form. Note also that an over all normalization of the referent state does not affect the geodesic length.

Using these observations we can begin with the basic relation in \((4.6)\) and \((4.7)\). We rewrite them in below

\[ A_+ = \frac{\alpha}{\sqrt{2}} (A_1 + A_2) = \alpha X, \quad A_- = \frac{\gamma}{\sqrt{2}} (A_1 - A_2) = \gamma Y \]

\[ (6.1) \]

\[ B_+ = \frac{\alpha^{-1}}{\sqrt{2}} (B_1 + B_2) = \alpha^{-1} Z, \quad B_- = \frac{\gamma^{-1}}{\sqrt{2}} (B_1 - B_2) = \gamma^{-1} W \]

\[ (6.2) \]
Regarding \((A_{\pm}, B_{\pm})\) as four target states while \((X,Y,Z,W)\) as four referent states the corresponding transformation is a \(4 \times 4\) diagonal matrix with element \((\alpha, \gamma, \alpha^{-1}, \gamma^{-1})\), which essentially is the matrix in (3.15). Thus we get the complexity of ground state.

To study the first excited states we can product the four target states and transformation matrix is a diagonal matrix. The matrix elements are the coefficients in below relation

\[
(\alpha X + \gamma Y + \alpha^{-1}W + \gamma^{-1}Z)^2 = \frac{W^2}{\gamma^2} + \frac{2\alpha W X}{\gamma} + 2WY + \frac{2WZ}{\alpha \gamma} + \alpha^2 X^2 + 2\alpha \gamma XY + 2XZ + \gamma^2 Y^2 + \frac{2\gamma YZ}{\alpha} + \frac{Z^2}{\alpha^2}
\] (6.3)

Note that despite the coefficients do not respect the non-commutative property between operator states \((X,Y,W,Z)\) they have shown the correct number, which is the necessary to be used to calculate the complexity\(^2\). Therefore, we can drop the basic fields \(X,Y,Z,W\) and, for example, in the second excited states the transformation matrix elements can be read from the coefficients in below relation

\[
(\alpha + \gamma + \alpha^{-1} + \gamma^{-1})^3 = \alpha^3 + \frac{1}{\alpha^3} + 3\alpha^2 \gamma + \frac{3\alpha^2}{\gamma} + \frac{3\gamma}{\alpha^2} + \frac{3\gamma^2}{\alpha^2} + 3\alpha \gamma^2 + \frac{3\alpha}{\gamma^2} + \frac{3\alpha^2}{\gamma^2} + 9\alpha + \frac{9}{\alpha} + \gamma^3 + \frac{1}{\gamma^3} + 9\gamma + \frac{9}{\gamma}
\] (6.4)

The coefficients are essentially these shown in previous sections. In the same way we can find the matrix element in the third and higher excited state.

Using above diagonal element of transformation matrix we can calculate the geodesic length through a simple rule

\[
\alpha^k \gamma^m \to \left[ \frac{k}{2} \ln(\tilde{\omega}_+) + \frac{m}{2} \ln(\tilde{\omega}_-) \right]^2
\] (6.5)

which have been used in previous sections.

Using above simple algorithm we can, just through simple calculations, quickly find that

\[
D_{n=1}^2 = 1 \times \frac{1}{4} \left( \left( \ln \frac{\tilde{\omega}_+}{\omega_0} \right)^2 + \left( \ln \frac{\tilde{\omega}_-}{\omega_0} \right)^2 \right)
\] (6.6)

\[
D_{n=2}^2 = 4 \times \frac{1}{4} \left( \left( \ln \frac{\tilde{\omega}_+}{\omega_0} \right)^2 + \left( \ln \frac{\tilde{\omega}_-}{\omega_0} \right)^2 \right)
\] (6.7)

\[
D_{n=3}^2 = 12 \times \frac{1}{4} \left( \left( \ln \frac{\tilde{\omega}_+}{\omega_0} \right)^2 + \left( \ln \frac{\tilde{\omega}_-}{\omega_0} \right)^2 \right)
\] (6.8)

\[
D_{n=4}^2 = 32 \times \frac{1}{4} \left( \left( \ln \frac{\tilde{\omega}_+}{\omega_0} \right)^2 + \left( \ln \frac{\tilde{\omega}_-}{\omega_0} \right)^2 \right)
\] (6.9)

\[\Rightarrow D_n^2 = n \cdot 2^{n-1} \times \frac{1}{4} \left( \left( \ln \frac{\tilde{\omega}_+}{\omega_0} \right)^2 + \left( \ln \frac{\tilde{\omega}_-}{\omega_0} \right)^2 \right)
\] (6.10)

\(^2\)For example the terms \(\alpha(XY + YX), 2\alpha XY\) and \(2\alpha YX\) give the same contribution to the geodesic length.
after imposing the proper normalization factor $\frac{1}{2^n}$. We have checked the relations up to $n=4$. Results indicate that the square of length of the quantum circuit in nth excited state is $n \cdot 2^{n-1}$ times that in the ground state, which is shown in below.

6.2 Formula’s Proof

Using previous arguments the transformation matrix of nth excited state is read from the coefficients in the following quadrinomial expansion

\[
(\alpha + \alpha^{-1} + \gamma + \gamma^{-1})^n = \sum_{i,j,k,\ell \geq 0} \alpha^i (\alpha^{-1})^j \gamma^k (\gamma^{-1})^\ell C_{ijkl}^n
\]

\[
= \sum_{i,j,k,\ell \geq 0} \alpha^{i-j} \gamma^{k-\ell} C_{ijkl}^n
\]

(6.11)

where the summation is constrained by $i + j + k + \ell = n$. $C_{ijkl}^n$ is the coefficients in quadrinomial expansion. To calculate the geodesic length from above coefficients in the quadrinomial expansion we can use the simple rule in (6.5). Thus

\[
D_n^2 = \frac{1}{2^n} \sum_{i,j,k,\ell \geq 0} \left[ \frac{i-j}{2} \ln(\omega_+ + \omega_0) + \frac{k-\ell}{2} \ln(\omega_- + \omega_0) \right]^2 C_{ijkl}^n
\]

\[
= \left( \frac{1}{2^n} \sum_{i,j,k,\ell \geq 0} \left( \frac{i-j}{2} \right)^2 C_{ijkl}^n \right) \cdot \left( \ln(\omega_+ + \omega_0) \right)^2 + \left( \frac{1}{2^n} \sum_{i,j,k,\ell \geq 0} \left( \frac{k-\ell}{2} \right)^2 C_{ijkl}^n \right) \cdot \left( \ln(\omega_- + \omega_0) \right)^2
\]

\[
= n \cdot 2^{n-1} \times \frac{1}{4} \left( \left( \ln(\omega_+ + \omega_0) \right)^2 + \left( \ln(\omega_+ - \omega_0) \right)^2 \right)
\]

(6.12)

in which the cross term in second line is odd in $i,j$ and becomes zero automatically. The summation of coefficients in remain two terms are equal and give the above result to fit the formula. We perform the summation in the appendix B.

7 Discussions

In this paper we have presented the operator approach to calculate the complexity in excited state of SHO. We find that the square length of the quantum circuit in nth excited state is $n \cdot 2^{n-1}$ times that in the ground state after a proper normalization. Although we only study the SHO the extending to field theory is like as that in [15] and thus, the corresponding value in nth excited state of free scalar field theory is just $n \cdot 2^{n-1}$ times that in ground state.

Some more studies are needed to clarify the property of the operator approach to calculate the complexity:

1. We only calculate the excited states up to $n=4$. To make sure that the square of length of the quantum circuit in nth excited state is simply $n \cdot 2^{n-1}$ times that in the ground state
we have found an algebra summation theorem to exactly prove it. It is interesting to find the holographic method to see the property.

2. We only study the free boson theory in this paper. The extension to interacting theory and fermion theory is deserved to be studied.

3. Since that the operator approach needs not to use the wavefunction it can be used to study the complexity in the spin system.

These problems are under studied.

A Geodesic Length in Equivalent Gate Matrix

Consider following two gate matrix

\[
M_1 = \begin{pmatrix} \alpha & \alpha \\ \gamma & -\gamma \end{pmatrix} ; \quad M_2 = \begin{pmatrix} \alpha & 0 \\ 0 & \gamma \end{pmatrix}
\]

(A.1)

where parameters \( \alpha \) and \( \gamma \) could be any values. Since that

\[
M_1 \cdot \begin{pmatrix} \Psi_a \\ \Psi_b \end{pmatrix} = M_2 \cdot \begin{pmatrix} \Psi_a + \Psi_b \\ \Psi_a - \Psi_b \end{pmatrix}
\]

(A.2)

we say that matrix \( M_2 \) is the matrix \( M_1 \) after rearrangement (or linear combination), which mentioned and repeatedly used in this paper. See, for example, (3.16) and (4.11).

We can regard both matrix as \( \text{GL}(2,\mathbb{R}) \) and use the function form in (2.13) to calculate the geodesic lengths. Both give the same geodesic length

\[
D_{\text{GL}(2,\mathbb{R})}^2 = 2\left( \frac{1}{2}(\ln \alpha + \ln \gamma) \right)^2 + 2\left( \frac{1}{2}(\ln \alpha - \ln \gamma) \right)^2 = (\ln \alpha)^2 + (\ln \gamma)^2
\]

(A.3)

On the other hand we can regard the second matrix as \( R \times R \) and express it in a general matrix

\[
M_{R \times R} = \begin{pmatrix} e^{y_1} & 0 \\ 0 & y_2 \end{pmatrix}
\]

(A.4)

The associated metric and geodesic length are

\[
d_{R \times R}^2 = \text{tr}(de^{y_1} e^{-y_1}) \text{tr}(de^{y_1} e^{-y_1}) + \text{tr}(de^{y_2} e^{-y_2}) \text{tr}(de^{y_2} e^{-y_2}) = dy_1^2 + dy_2^2
\]

\[
D_{R \times R}^2 = (\ln \alpha)^2 + (\ln \gamma)^2
\]

(A.5)

(A.6)

which is consistent with (A.3). The relation can be applied to any diagonal matrix and is extensively used in this paper.
**B Summation Formula**

Begin with the definition

\[(a + b + c + d)^n = \sum_{i,j,k,\ell \geq 0} a^i b^j c^k d^\ell C_{ijkl}^n \]  

(B.1)

Considering derivative one time and two times with respective \(a\), and derivative respective \(a\) then \(b\), then let \(a = b = c = d = 1\) we get three relations

\[\sum_{i,j,k,\ell \geq 0} i C_{ijkl}^n = n4^{n-1} \]  

(B.2)

\[\sum_{i,j,k,\ell \geq 0} i(i-1) C_{ijkl}^n = n(n-1)4^{n-2} \]  

(B.3)

\[\sum_{i,j,k,\ell \geq 0} i j C_{ijkl}^n = n(n-1)4^{n-2} \]  

(B.4)

Use above relations we find a summation relation

\[\sum_{i,j,k,\ell \geq 0} \left(\frac{i - j}{2}\right)^2 C_{ijkl}^n = \frac{n4^n}{8} \]  

(B.5)

which is used to obtain the general formula (6.12)

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