Unitarity of the Aharonov-Bohm Scattering Amplitudes

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Abstract

We discuss the unitarity relation of the Aharonov-Bohm scattering amplitude with the hope that it distinguishes between the differing treatments which employ different incident waves. We find that the original Aharonov-Bohm scattering amplitude satisfies the unitarity relation under the regularization prescription whose theoretical foundation does not appear to be understood. On the other hand, the amplitude obtained by Ruijsenaars who uses plane wave as incident wave also satisfies the unitarity relation but in an unusual way.

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I. INTRODUCTION AND SUMMARY

In their pioneering work Aharonov and Bohm [1] examined the scattering of nonrelativistic charged particles off a magnetic flux of infinitesimal radius. We shall call the process as the AB scattering in this paper. At the very least the problem serves as an idealized system which exhibits the Aharonov-Bohm effect.

While more than one-third of the century has been passed since their paper there seem to still exist some disagreements among the literatures on the treatment and the interpretation of the AB scattering. Ruijsenaars [2] and others [3] advocates the viewpoint that one has to take plane wave as incident wave, as opposed to the original treatment by Aharonov and Bohm. These authors’ treatment entails the S-matrix which contains a delta-function peaked in the forward direction.

The treatment of the AB scattering with incident plane wave was critically examined by Hagen [4]. He pointed out that taking the asymptotic limit \( r \rightarrow \infty \) and summing over angular momenta do not commute with each other owing to the violent infrared (= high angular momentum) behavior of the scattering amplitude. Thus, the usual definition of the phase shift, which involves the procedure of taking asymptotic limit in each partial wave, does not work. The observation casts serious doubt on the treatment with incident plane wave, but it does not appear to be the last ward to settle the controversy.

In this paper we examine the unitarity of the the S-matrix of the AB scattering in hoping that it may discriminate differing treatments in the literatures. There is a “naive” argument that the delta-function in the forward direction is physically meaningless because it cannot be directly observed. This is not correct because the forward scattering amplitude is related with the total cross section by the optical theorem. This was the original motivation which leads us to the study of unitarity relation of the AB scattering amplitude.

In fact, the problem is slightly more complicated. As some readers might have noticed the original AB scattering amplitude diverges in the forward direction, rendering the detection of the delta-function contribution difficult. However, it is also true that the unitarity relation
in non-forward direction contains the information of forward scattering amplitude in its right-hand-side (RHS). (See below.) Therefore, it appears that the setting of the problem itself seems to be meaningful and our investigation has started along this line of thought.

In carrying out this investigation we have encountered the new feature of the problem that we never expected before actually engaging the work. Our conclusions at hand are as follows:

(1) With the choice of plane wave as incident wave the unitarity relation of the AB scattering amplitude holds but in a contrived way as will be explained in Sec. V.

(2) The scattering amplitude obtained by Aharonov and Bohm satisfies the unitarity relation if one employs the suitable regularization prescription which is consistent with positivity of RHS of the unitarity relation near the forward direction. Unfortunately, we fail in our attempts at deriving it in a physically reasonable way and thereby placing the regularization prescription on a firm theoretical ground, as we will describe in Secs. III and IV.

Thus, we have not made our original goal of distinguishing between two different choices of incident wave. Instead, we learn some lessons, and in particular uncover the necessity of an “ic-prescription” which, to our knowledge, does not seem to be noticed before.

In Sec. II the basic formulas of the AB scattering problem are briefly reviewed to define our notations. We follow the notation of the original paper by Aharonov and Bohm [1]. The expressions of scattering amplitudes by Aharonov and Bohm and by Ruijsenaars are also recollected. In Sec. III the unitarity relation of Aharonov-Bohm’s scattering amplitude is examined. The necessity of a phenomenological regularization prescription is noted. In Sec. IV some unsuccessful trials for deriving the regularization prescription by exploiting hard-core potential, or finite-radius magnetic flux are described. In Sec. V the unitarity relation of Ruijsenaars’ scattering amplitude is discussed. Sec. VI summarizes our investigations. In Appendix the unitarity relation of the AB scattering amplitude with modified incident wave is derived by utilizing the method of Landau and Lifshitz.
II. THE AHARONOV-BOHM SCATTERING AMPLITUDES

We follow the original notation of Aharonov and Bohm and denote the flux parameter as \( \alpha = -e\Phi/2\pi \) where \( \Phi \) is the magnetic flux. The Schrödinger equation takes the form in polar coordinate as

\[
\left[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \left( \frac{\partial}{\partial \theta} + i\alpha \right)^2 + k^2 \right] \psi = 0, \tag{1}
\]

where we treat the problem in a two-dimensional setting by ignoring the separated z-direction and \( k \) is the wave number. The regular solution of the Schrödinger equation which vanishes at the origin (i.e., the location of the magnetic flux) can be written as

\[
\psi_{AB}(r, \theta) = \sum_{m=-\infty}^{\infty} (-i)^{|m+\alpha|} J_{|m+\alpha|}(kr)e^{im\theta} \tag{2}
\]

There is a variety of ways of extracting the asymptotic form of the wave function at spatial infinity; the original method of Aharonov and Bohm [1], the method of contour deformation by Berry and coworkers [5], and the Takabayasi method [6] which utilizes an integral representation of the Bessel function. All these methods agree with each other and result in the expression

\[
\psi_{AB}(r, \theta) \xrightarrow{r \to \infty} e^{-ikr \cos \theta + \alpha \theta} + \frac{e^{ikr}}{\sqrt{r}} f_{AB}(\theta) \tag{3}
\]

\[
f_{AB}(\theta) = -\frac{1}{\sqrt{2\pi k}} e^{\frac{i\pi}{4}} (-1)^{-[\alpha]} e^{-i([\alpha]+\frac{1}{2})\theta} \sin \frac{\pi \alpha}{\cos \frac{\theta}{2}} \tag{4}
\]

where \([\alpha]\) denotes the largest integer which is less than or equal to \( \alpha \). Notice that we take the convention that the incident wave moves to the negative \( x \) direction and the forward scattering corresponds to \( \theta = \pm \pi \).

On the other hand, the asymptotic form obtained by Ruijsenaars [2] takes the form

\[
\psi_{AB}(r, \theta) \xrightarrow{r \to \infty} e^{-ikr \cos \theta} + \frac{e^{ikr}}{\sqrt{r}} f_{R}(\theta) \tag{5}
\]

\[
f_{R}(\theta) = -\sqrt{\frac{2\pi}{k}} e^{-\frac{i\pi}{4}} \left[ (1 - \cos \pi \alpha) \delta(\theta - \pi) \\
+ \frac{i}{\pi} (-1)^{[\alpha]} \sin \pi \alpha e^{-i[\alpha]\theta} \frac{1}{e^{i\theta} + 1} \right] \tag{6}
\]
where $P$ denotes the principal value prescription.

An important distinction between the Aharonov-Bohm and the Ruijsenaars scattering amplitudes is that the former is not defined at the forward direction whereas the latter is. It is implicit [7] in the original paper by Aharonov and Bohm [1] and was emphasized in [3] that the scattering amplitude (3) is defined except for the narrow cone $|\theta| > \pi - O[(kr)^{-1/2}]$. While it is defined in the forward direction, the square modulus of the Ruijsenaars scattering amplitude does not appear to be well defined. A possible way of obtaining finite forward scattering amplitude by modifying the boundary condition at the origin has recently been put forward by Giacconi et al. [8]. It is also argued by Stelitano [9] that the time-dependent formulation is necessary for consistent treatment of the AB scattering at around the forward direction.

III. UNITARITY OF THE AHARANOV-BOHM SCATTERING AMPLITUDE; PHENOMENOLOGICAL APPROACH

Let us define the scattering amplitude (3) apart from the forward direction, or more precisely, up to an infinitesimal value of $\pi - \theta$, by defining it at $r \to \infty$. We then discuss the unitarity relation of the Aharonov-Bohm scattering amplitude.

In Appendix we follow the method described by Landau and Lifshitz [10] to derive the unitarity relation of the scattering amplitude corresponding to the choice of incident wave as in (3). It reads

$$e^{i(\alpha - \frac{1}{2})\pi} f(\theta) - e^{-i(\alpha - \frac{1}{2})\pi} f^*(-\theta) = i\sqrt{\frac{k}{2\pi}} \int_{-\pi}^{\pi} d\theta' f^*(\theta') f(\theta' + \theta + \pi) \quad (7)$$

We have to remark that we take the viewpoint in this paper that the scattering amplitude in (3)

Using (3) the left-hand-side (LHS) of the unitarity relation can be expressed as

$$\text{LHS} = -i \sqrt{\frac{2}{\pi k}} (-1)^{[\alpha]} e^{-i(\alpha + \frac{1}{2})}\frac{\sin^2 \pi \alpha}{\cos \frac{\theta}{2}} \quad (8)$$
To compute RHS the expression of the scattering amplitude (4) is not enough; we need to specify certain “i\(\epsilon\) prescription” to dictate how to make detour around the singularities. Notice that the Ruijsenaars amplitude does have such prescription, taking the principal value, as indicated in (6).

Lacking any known “i\(\epsilon\)-prescriptions” we try to identify it via the Berry et al.’s method for deriving the asymptotic form of the wave function. They use the integral representation of the Bessel function [12]

\[ J_\nu(z) = \frac{1}{2\pi} \int_C dt \exp[i(\nu t - z \sin t)] \] (9)

where \(C\) is the contour starting from \(-\pi + i\infty\) and goes down to \(-\pi\) and traverse to \(+\pi\) on the real axis, and then goes up to \(+\pi + i\infty\). If we insert (9) into (2) the summation over \(m\) converges if we add small positive imaginary part on the contour along the real axis. One obtains

\[ \psi_{AB} = \frac{1}{2\pi} \int_C dt e^{-ikr \sin t} \left[ \frac{\exp\{-i[(t - \frac{\pi}{2})(\alpha - [\alpha] - 1) + ([\alpha] + 1)\theta]\}}{1 - \exp[i(t - \frac{\pi}{2} - \theta)]} + \frac{\exp\{i[(t - \frac{\pi}{2})(\alpha - [\alpha]) - [\alpha]\theta]\}}{1 - \exp[i(t - \frac{\pi}{2} + \theta)]} \right] \] (10)

Note that we differ in sign from [5] in defining the flux parameter \(\alpha\).

Then, we deform the contour \(C\) into \(C'\) which passes through \(-\frac{\pi}{2} + i\epsilon\) and moves down into the lower \(t\)-plane and again goes up to the top of the upper \(t\)-plane by passing through \(+\frac{\pi}{2} + i\epsilon\), as described in [4]. Through the process of the deformation we pick up a pole at somewhere on the real axis \(-\frac{\pi}{2} < t < \frac{\pi}{2}\). The pole term comes from the first (second) term in (10) provided that \(-\pi < \theta < 0 (0 < \theta < \pi)\). As shown by Berry et al. the pole term gives rise to the incident wave of Aharonov and Bohm as in (3). The remaining contribution comes from the saddle point at \(t = \pm \frac{\pi}{2}\). The saddle point at \(t = -\frac{\pi}{2}\) produces the scattering wave (4). The saddle point at \(t = \frac{\pi}{2}\), which is potentially dangerous because it would produce incoming wave, makes no contribution owing to the vanishing residue. We believe it natural to keep small positive imaginary part in computing the saddle-point contribution at \(t = -\frac{\pi}{2}\).
Namely, we do saddle-point integration at \( t = -\frac{\pi}{2} + i\epsilon \). It lead to the regularized form of the AB scattering amplitude

\[
 f_{AB}(\theta) = -\frac{1}{\sqrt{2\pi k}} e^{-\frac{\pi}{2}} (-1)^{[\alpha]} e^{-i[\alpha] \theta} \\
 \times \left[ \frac{1}{1 + e^{i(\theta - i\epsilon)}} e^{i\alpha \pi} - \frac{1}{1 + e^{i(\theta + i\epsilon)}} e^{-i\alpha \pi} \right] 
\]

(11)

Having specified the regularization prescription we are ready to compute RHS of the unitarity relation. Alas we have a trouble; RHS vanishes.

Since the simplest regularization prescription fails we look for a phenomenologically successful regularization prescription. It turns out that the solution is given by the \( \theta - i\epsilon \) prescription. That is, we replace \( e^{i(\theta + i\epsilon)} \) in the second term in the square bracket in (11) into \( e^{i(\theta - i\epsilon)} \). (Note that the first term already meets the requirement.) Under the regularization procedure just specified one can easily compute RHS of the unitarity relation. Changing the integration variable into \( z = e^{i\theta} \) it can be expressed as

\[
 \text{RHS} = \frac{-4}{\sqrt{(2\pi)^3 k}} (-1)^{\alpha} e^{-i[\alpha] \theta} \sin^2 \pi \alpha \oint dz \frac{e^{-i\theta}}{(z + e^{i\epsilon})(z - e^{-i\theta} e^{-i\epsilon})} 
\]

where the integration contour is along the circle \( |z| = 1 \). The \( i\epsilon \)-prescription dictates to pick up the pole at \( z = e^{-i\theta} \) and the resulting expression of RHS coincides with LHS in (8).

Thus, we have shown that the Aharonov-Bohm scattering amplitude satisfies the unitarity relation provided that the phenomenological \( \theta - i\epsilon \) prescription is employed.

### IV. LOOKING FOR REGULARIZATION PRESCRIPTION

It is natural to expect that the \( \theta - i\epsilon \) prescription can naturally be derived from certain physical regularization procedures which are able to regulates the divergence of forward scattering amplitude. The most natural possibility is to introduce a small but finite radius of the magnetic flux. Unfortunately, the solution becomes complicated. Therefore, we postpone the investigation of this case to the end of this section and start with the simpler problem of setting up the hard core potential of radius \( R \), keeping the width of the magnetic
The exact solution of this problem is again given by Berry et al. [5]. The wave function takes the form

\[ \psi(r) = \psi_{AB}(r) - \psi_R(r) \]  

\[ \psi_R(r) = \sum_{m=-\infty}^{+\infty} b_m (-i)^{|m+\alpha|} e^{im\theta} H^{(1)}_{|m+\alpha|}(kr) \]

where

\[ b_m = \frac{J_{|m+\alpha|}(kR)}{H^{(1)}_{|m+\alpha|}(kR)}. \]

Here, \( H^{(1)}(z) \) is the Hankel function of the first kind.

We use the integral representation of the Hankel function [12]

\[ H^{(1)}_{\nu}(z) = \frac{1}{\pi} \int_{C_1} dt \exp[i(\nu t - z \sin t)]. \]

The contour \( C_1 \) runs from \( t = x_+ + i\infty \) to \( t = x_- - i\infty \) by passing through \( t = -\frac{\pi}{2} \), where \(-\pi < x_+ < -\frac{\pi}{2} \) and \(-\frac{\pi}{2} < x_- < 0 \). We compute \( \psi_R \) by using the saddle-point approximation at \( t = -\frac{\pi}{2} \) to evaluate the nonzero \( R \) correction to the AB scattering wave. For this purpose it suffices to keep the leading order in \( kR \) in (14). Then, the integrand does not develop pole singularities unlike the case of evaluating \( \psi_{AB} \). The dominant contribution comes from the saddle point at \( t = -\frac{\pi}{2} \). Using the small \( z \) behavior of the Bessel functions

\[ \frac{J_{\nu}(z)}{H^{(1)}_{\nu}(z)} \rightarrow \frac{i\pi}{\Gamma(\nu)\Gamma(1+\nu)} \left( \frac{z}{2} \right)^{2\nu} \]

we obtain

\[ \psi_R(r) \rightarrow i e^{ikr} \left( \frac{2\pi}{ikR} \right)^{\frac{1}{2}} \left[ \frac{e^{-i|\alpha|\theta} e^{-i\pi(\alpha-|\alpha|)}}{\Gamma(\alpha - |\alpha|)\Gamma(\alpha - |\alpha| + 1)} \left( \frac{kR}{2} \right)^{2(\alpha-|\alpha|)} \right. \]

\[ \left. - \frac{e^{-i(|\alpha|+1)\theta} e^{i\pi(\alpha-|\alpha|)}}{\Gamma(|\alpha| - \alpha + 1)\Gamma(|\alpha| - \alpha + 2)} \left( \frac{kR}{2} \right)^{2(|\alpha|-\alpha+1)} \right] \]

On the other hand the order \( \epsilon \) correction expected from the \( \theta - i\epsilon \) prescription takes the form

\[ \psi(r) \rightarrow \psi_{AB}(r) - \left( \frac{2\pi}{ikr} \right)^{\frac{1}{2}} \frac{e^{ikr}}{8\pi \cos^2(\theta/2)} \left[ e^{i(|\alpha|+1)\theta} e^{i\pi(\alpha-|\alpha|)} \delta e^{-i\alpha|\theta} e^{-i\pi(\alpha-|\alpha|)} \right] \]
where we have regulated the first and the second terms of (10) by replacing \( \theta \) by \( \theta - i \epsilon \) and \( \theta - i \delta (\epsilon > 0, \delta > 0) \), respectively.

In spite of their similarity, the correction terms in (18) and (19) differ by two important respects; The two terms differ in sign in (18) whereas those in (19) have the same relative sign. Also there exists an extra over-all \( i \) in (18) relative to the correction terms in (19). Thus, we conclude that the finite width regularization does not give rise to the requested \( \theta - i \epsilon \) prescription.

One can repeat the similar calculation for the case of finite radius magnetic flux. In this case the vector potential may be taken as

\[
A_\theta(r) = \begin{cases} 
\frac{\Phi_r}{2\pi R^2} & (r < R) \\
\frac{\Phi}{2\pi r} & (r > R)
\end{cases}
\]  

(20)

The solution of the Schrödinger equation in the outer region is given by (14). The one in the inner region is given by the Whittaker function

\[
\psi_{\text{inner}}(r) = \frac{1}{\sqrt{\alpha}} \left( \frac{R}{r} \right)^{\lambda + \frac{|m|}{2}} e^{i m \theta} M_{\lambda - \frac{m}{2}, \frac{|m|}{2}} \left[ \alpha \left( \frac{r}{R} \right)^2 \right]
\]

(21)

where \( \lambda = \frac{(kR)^2}{4\alpha} \). One can determine \( a_m \) and \( b_m \) by matching the wave functions and their derivatives at \( r = R \). We obtain

\[
b_m = \frac{J_{|m|}(kR) \left\{ 2\alpha M'_{\lambda - \frac{m}{2}, \frac{|m|}{2}}(\alpha) - M_{\lambda - \frac{m}{2}, \frac{|m|}{2}}(\alpha) \right\} - kR J'_{|m|}(kR) M_{\lambda - \frac{m}{2}, \frac{|m|}{2}}}{H_{|m|}(kR) \left\{ 2\alpha M'_{\lambda - \frac{m}{2}, \frac{|m|}{2}}(\alpha) - M_{\lambda - \frac{m}{2}, \frac{|m|}{2}}(\alpha) \right\} - kR H'_{|m|}(kR) M_{\lambda - \frac{m}{2}, \frac{|m|}{2}}}
\]

(22)

and a similar expression for \( a_m \).

We can go through the same analysis as before and we end up the same result as in (18) but with the first and the second terms in (18) being multiplied by

\[
\left( |\alpha| + |\alpha| + 1 \right) M_{\kappa+1,|\kappa|}(\alpha) - M_{\kappa,|\kappa|}(\alpha) \\
\left( |\alpha| - |\alpha| + 1 \right) M_{\kappa-1,|\kappa|}(\alpha) - M_{\kappa,|\kappa|}(\alpha)
\]

(23)

and

\[
\left( |\alpha| + 1 - |\alpha| \right) M_{-\frac{1}{2},|\kappa+\frac{1}{2}|}(\alpha) - M_{\kappa+\frac{1}{2},|\kappa+\frac{1}{2}|}(\alpha) \\
\left( |\alpha| + 1 + |\alpha| + 2 \right) M_{\kappa+\frac{3}{2},|\kappa+\frac{3}{2}|}(\alpha) - M_{\kappa+\frac{1}{2},|\kappa+\frac{1}{2}|}(\alpha)
\]

(24)

respectively, where \( \kappa \equiv \frac{|\alpha|}{2} \). Because of the surviving “\( i \)-problem” the finite width magnetic flux does not give the required \( \theta - i \epsilon \) prescription.
V. UNITARITY OF THE RUIJSENAARS AMPLITUDE

The unitarity relation of the Ruijsenaars amplitude is even more subtle. Having employed the plane wave as incident wave the unitarity relation is different from (7); one can repeat the same procedure as before and the result is obtained by setting $\alpha = 0$ in (7). Then, we have a curious result. LHS vanishes except for the forward direction,

$$\text{LHS} = 2i\sqrt{\frac{2\pi}{k}}(1 - \cos \pi \alpha)\delta(\theta - \pi).$$

(25)

One can easily show that RHS gives rise to the same expression as (25) thanks to the principal value prescription. Therefore, the unitarity relation holds in the sense that both LHS and RHS give the identical delta-function contribution in the forward direction, and vanish elsewhere. We do not know any other examples of scattering problem whose scattering amplitude possesses such curious property.

VI. CONCLUSION

We summarize our investigation in this paper.

(1) The scattering amplitude obtained by Aharonov and Bohm satisfies the unitarity relation under the phenomenological $\theta - i\epsilon$ prescription. Unfortunately, we neither succeeded to systematically derive the prescription nor pinned down its physical meaning.

(2) The Ruijsenaars scattering amplitude obeys unitarity relation in a contrived way that both LHS and RHS vanish anywhere at $\theta \neq \pi$.

(3) To our understanding of the problem the unitarity relation of the AB scattering amplitude does not appear to select out the unique treatment of the incident wave that one has to employ.

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Appendix

We derive the unitarity relation of the Aharonov-Bohm scattering amplitude by following the method of Landau and Lifshitz [10]. We consider the asymptotic form of the wave function

$$\psi = e^{-ikr \cos (\theta'-\theta)} - i\alpha (\theta' - \theta) + e^{ikr} \sqrt{rf} (\theta' - \theta), \quad (26)$$

which describes the scattering of the incident wave coming from the direction of angle $\theta$ to the direction of angle $\theta'$. The angles $\theta$ and $\theta'$ are measured from the $x$-axis. The basic strategy of Landau and Lifshitz is to consider the superpositions of the wave function (26) with arbitrary weight functions $F(\theta)$, which also describe certain scattering processes, and to demand the conservation of the fluxes of the incoming and the outgoing waves.

To this goal we decompose the incident wave into the incoming and the outgoing waves. Using the representation of the plane wave by the sum of the Bessel functions

$$e^{ikr \cos \theta} = \sum_{m=-\infty}^{\infty} i^m e^{im\theta} J_m(kr), \quad (27)$$

and noting the asymptotic form of the Bessel function

$$J_\nu(z) \to \frac{2}{\sqrt{\pi z}} \cos \left( z - (2\nu + 1)\frac{\pi}{4} \right), \quad (28)$$

one can show that
\[ e^{-ikr \cos(\theta' - \theta) - i\alpha(\theta' - \theta)} \rightarrow e^{-i\alpha(\theta' - \theta)} \left( \frac{2\pi}{kr} \right)^{\frac{1}{2}} \left[ e^{i(kr - \frac{\pi}{4})} \delta(\theta' - \theta - \pi) + e^{-i(kr - \frac{\pi}{4})} \delta(\theta' - \theta) \right] \] \quad (29)

We then obtain

\[ \int F(\theta) \psi d\theta \rightarrow_{r \to \infty} \left( \frac{2\pi}{kr} \right)^{\frac{1}{2}} \left[ e^{-i(kr - \frac{\pi}{4})} F(\theta') + e^{i(kr - \frac{\pi}{4} - \alpha\pi)} \hat{S} F(\theta' - \pi) \right] \] \quad (30)

where \( \hat{S} \) is the S-matrix;

\[ \hat{S} = 1 + e^{i\frac{\pi}{4} \sqrt{k} \hat{f}}, \] \quad (31)

and the operator \( \hat{f} \) acts as

\[ \hat{f} F(\theta' - \pi) = \frac{e^{i\alpha\pi}}{\sqrt{2\pi}} \int_{\theta' - \pi}^{\theta' + \pi} F(\theta) f(\theta' - \theta) d\theta \] \quad (32)

The unitarity of the S matrix, \( \hat{S}^\dagger \hat{S} = 1 \), which follows from the conservation of the probability current, leads to

\[ e^{-i\pi \frac{\pi}{4} \hat{f}} - e^{i\pi \frac{\pi}{4} \hat{f}^\dagger} = i\sqrt{k} \hat{f} \hat{f}^\dagger, \] \quad (33)

By operating the both sides of this equation to \( F(\theta' - \pi) \) we obtain

\[ e^{i(\alpha - \frac{\pi}{4})} \hat{f}(\theta' - \theta) - e^{-i(\alpha - \frac{\pi}{4})} \hat{f}^\dagger(\theta - \theta') \]

\[ = i\sqrt{k} \left\{ \int_{\theta' - \pi}^{\theta' + \pi} f^*(\theta'' - \theta') f(\theta'' - \theta + \pi) d\theta'' + \int_{\theta - \pi}^{\theta + \pi} f^*(\theta'' - \theta') f(\theta'' - \theta - \pi) d\theta'' \right\} \] \quad (34)

We set \( \theta = 0 \) and replace \( \theta' \) by \( \theta \) to derive the unitarity relation (7)
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