Resonant Mode Coupling in δ Scuti Stars

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Abstract

Delta Scuti (δ Sct) variables are intermediate-mass stars that lie at the intersection of the main sequence and the instability strip on the Hertzsprung–Russell diagram. Various lines of evidence indicate that nonlinear mode interactions shape their oscillation spectra, including the particularly compelling detection of resonantly interacting mode triplets in the δ Sct star KIC 8054146. Motivated by these observations, we use the theory of three-mode coupling to study the strength and prevalence of nonlinear mode interactions in 14 δ Sct models that span the instability strip. For each model, we calculate the frequency detunings and nonlinear coupling strengths of $\sim 10^4$ unique combinations of mode triplets. We find that all the models contain at least $\sim 100$ well-coupled triplets whose detunings and coupling strengths are consistent with the triplets identified in KIC 8054146. Our results suggest that resonant mode interactions can be significant in δ Sct stars and may explain why many exhibit rapid changes in amplitude and oscillation period.

1. Introduction

The δ Sct stars are pulsating variables that are on or slightly past the main sequence and have masses between 1.5 and 2.5$M_\odot$, effective temperatures between 6400 and 8600 K, and stellar types between A and F (for reviews, see Breger 2000; Goupil et al. 2005; Catelan & Smith 2015; Guzik 2021; Aerts 2021). Their oscillations are driven by the opacity ($\kappa$-) mechanism in the second partial-ionization zone of helium and consist of low-order $p$- and $g$-modes with oscillation periods ranging from 15 minutes to 8 hr (frequencies from a few to 100 cycles per day). The photometric amplitudes are often a few millimagnitudes, but can exceed a tenth of a magnitude in high-resolution data. A recent cataclysm in KIC 8054146. Our results find that the fundamental and first-overtone radial modes of the HADS star KIC 5950759 exhibits a linear period change $P/\Delta P \approx 10^{-6}$ yr−1 over a timescale of several years, at least two orders of magnitude higher than predicted by evolutionary models. Both Breger & Pamyatnykh (1998) and Bowman et al. (2021) suggest (see also Blake et al. 2003) that the rapid period changes could be the result of nonlinear mode interactions. These interactions can enable a fast transfer of energy among the modes and induce rapid amplitude-dependent variations in their oscillation periods.

Bowman et al. (2016) find further evidence of nonlinear mode interactions in their ensemble study of Kepler δ Sct stars. Of the 983 stars they analyze, 603 exhibit at least one pulsation mode that varies significantly in amplitude over four years. While some of these amplitude variations are due to the star being in a binary, they conclude that some must be due to processes intrinsic to the star and that nonlinear mode interactions are a likely culprit. In a detailed analysis of KIC 5892969, Barceló Forteza et al. (2015) likewise report finding

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amplitude variations that they argue can be attributed to nonlinear mode interactions. Analogous to the Balona (2021) study of TESS δ Sct stars cited above, Bowman et al. (2016) find no obvious correlation between the appearance of amplitude variations and stellar parameters such as the surface gravity log g or Teff.

Somewhat separately, Bowman et al. (2016) note that in many δ Sct stars, the visible pulsation-mode energy is not conserved over the four-year Kepler observations (see also Bowman & Kurtz 2014) and that low-frequency peaks can sometimes be associated with combination frequencies of high-frequency peaks. They propose that this may be due to a transfer of energy via nonlinear mode interactions from visible modes into nonvisible modes (angular degree l ≥ 3) or into the low-frequency modes.

Arguably the most definitive observational evidence of nonlinear mode interactions comes from the analysis of the Kepler δ Sct star KIC 8054146 by Breger & Montgomery (2014). From the oscillation spectrum, they identify several mode triplets with properties consistent with the theory of nonlinear three-mode coupling in which two parent modes driven by the k- mechanism nonlinearly excite a daughter mode. In particular, for each triplet, they find that (i) the frequencies of the modes combine such that the magnitude of their detuning Δabc ≡ ωa ± ωb ± ωc is very small (≤ 10−5 times the individual frequencies), (ii) the O−C phase shift of the daughter mode c varies in time as φc(t) = φa(t) ± φb(t), and (iii) the daughter amplitude varies in proportion to the product of the parent amplitudes, i.e., Ac(t) = μAa(t)Ab(t), where the constant μ is a measure of the nonlinear coupling strength.

There are a number of theoretical studies that consider nonlinear mode interactions in pulsating main-sequence stars (e.g., Moskalik 1985; Dziembowski & Krolickowska 1985; Dziembowski et al. 1988; Moskalik & Buchler 1990; Buchler et al. 1997; Nowakowski 2005). However, a comprehensive analysis investigating their impact on the oscillation modes in δ Sct stars has not been carried out. In this paper, we take some initial steps toward achieving this goal. In Section 2 we present the formalism of three-mode coupling and derive the relation between the coupling strength μ and other stellar and mode parameters. In Section 3 we describe our calculational methods including how we search for strongly coupled modes in our δ Sct stellar models. In Section 4 we present the results of our analysis and compare them with the measurements of mode coupling in KIC 8054146. We summarize in Section 5 and discuss how future theoretical work can help further quantify the impact of nonlinear mode coupling in δ Sct stars.

2. Resonant Three-mode Coupling

The position x of a fluid element in an unperturbed star is related to its perturbed position x′ at time t via x′ = x + ξ(x, t), where ξ(x, t) is the Lagrangian displacement vector. Since we are interested in computing weakly nonlinear mode interactions, we account for the equation of motion for ξ(x, t) to second order in perturbation theory,

$$\rho \ddot{\xi} = f_1[\xi] + f_2[\xi, \xi],$$

where f1 and f2 are the linear and second-order forces, respectively (see Schenk et al. 2001; Weinberg et al. 2012). We solve for ξ(x, t) by expanding the phase-space vector of the displacement in terms of its linear eigenmodes,

$$\xi(x, t) = \sum_a q_a(t) \left[ \xi_a(x) - i\omega_a \xi_a(x) \right],$$

which allows us to recast the equation of motion as a set of nonlinearly coupled equations for the dimensionless amplitudes qα(t). Here ξα(x) and ωα are the eigenfunctions and eigenfrequencies of a linear eigenmode, and the sums are over all quantum numbers a (angular degree l, azimuthal number mα, and radial order nα) and both frequency signs (±ωα). Reality conditions impose a relation between the amplitudes of modes with positive and negative frequencies (Schenk et al. 2001).

For simplicity, we neglect stellar rotation in our analysis1, and therefore, the eigenfunctions are degenerate in mα. We normalize the eigenfunctions such that

$$2\omega_a^2 \int d^3x \rho |\xi_a|^2 = E_a,$$

where ρ is the density, and Eα = GM2/R is the characteristic energy of a star of mass M and radius R. The energy of a mode is then Ea = |qa|2Eα.

Consider a system of three coupled modes, which we label by subscripts a, b, and c. If we plug the expansion given by Equation (2) into Equation (1), use the orthogonality of the eigenfunctions, and add a linear damping term, we obtain an amplitude equation for mode a of

$$\dot{q}_a + (i\omega_a + \gamma_a)q_a = i\omega_ab^- \kappa_{abc} q^b q^c,$$

and similarly for modes b and c. Here asterisks denote complex conjugation, γa is the linear damping rate (if γa > 0) or growth rate of the mode (γa < 0; in general γa > 0 unless otherwise specified), and $\kappa_{abc}$ is the three-mode coupling coefficient. The latter is dimensionless and symmetric in the three indices and is found by computing

$$\kappa_{abc} = \frac{1}{E_\alpha} \int d^3x \xi_a \cdot f_2[\xi_b, \xi_c].$$

We describe how we calculate γa and $\kappa_{abc}$ in Section 3.

We are interested in a three-mode system in which two of the modes are directly excited by the k- mechanism and the third mode is excited on account of its nonlinear coupling to the other two modes (we assume it is stable to the k- mechanism). We refer to the former two modes as the parents and label them with subscripts a and b, and we refer to the latter mode as the daughter and label it with subscript c. When the magnitude of the triplet detuning

$$\Delta_{abc} = \omega_a ± \omega_b ± \omega_c$$

is small, the parents can potentially drive the daughter to large amplitude. This type of nonlinear inhomogeneous driving was called direct resonance by Dziembowski (1982). Another type

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1 Although some δ Sct stars rotate rapidly (including KIC 8054146), their spin frequencies are typically much lower than the frequency of the modes, with the exception, perhaps, of the low-frequency g-modes. For example, KIC 8054146 has a measured rotational velocity sin i = 300 ± 20 km s−1, and Breger et al. (2012) estimate that its spin frequency is approximately three cycles day−1 (corresponding to 70% of its Keplerian breakup frequency). Since this is at least 10 times lower than its p-mode frequencies, we do not expect rotation to drastically modify their eigenfunctions. However, the same is not true of the g-modes, and the impact of rotation on nonlinear mode coupling should be investigated in future work.
of three-mode coupling often considered in the literature is the parametric instability, which involves one parent resonantly exciting two daughters. Although the parametric instability might also impact the observed power spectra of δ Sct stars, we focus here on direct resonance. This is because Breger & Montgomery (2014) find strong evidence for direct resonance in their observations of KIC 8054146, and Bowman et al. (2016) argue that it can account for the amplitude modulations of many δ Sct stars observed by Kepler.

We now calculate the amplitude of a daughter mode \( q_d(t) \) excited by direct resonance with two parents. Since the parent energies will be nearly constant on the timescale of the oscillation periods,\(^2\) we can write \( q_d(t) = A_d e^{-i(\omega_d t + \delta_d)} \), and similarly for mode \( b \), where the amplitude \( A_d \) is \( \sqrt{E_d/E_a} \) and the phase lag \( \delta_d \) are real constants. We ignore the feedback of the daughter on the parents, which should be a good approximation as long as the parent energies \( E_a \) and \( E_b \) are higher than the energy of the daughter. The daughter amplitude equation is then just that of a driven harmonic oscillator with forcing frequency \( \omega_d + \omega_b \) and driving force \( i\omega_d \kappa_{abc} A_d A_b e^{i(\delta_d + \delta_b)} \). Writing \( q_d = A_d e^{i(\omega_d t + \delta_d + \delta_b)} \), the daughter amplitude equation then gives

\[
(i\Delta_{abc} + \gamma_c)A_d e^{i\delta_d} = i\omega_k \kappa_{abc} A_d A_b e^{i(\delta_d + \delta_b)}. \tag{7}
\]

If we multiply each side of this equation by its complex conjugate, we obtain

\[
A_c = \mu A_d A_b, \tag{8}
\]

where the nonlinear coupling strength

\[
\mu = \frac{|\omega_d \kappa_{abc}|}{\sqrt{\Delta_{abc}^2 + \gamma_c^2}}. \tag{9}
\]

If instead we separate Equation (7) into its real and imaginary parts and take their ratio, we obtain

\[
\tan(\delta_d + \delta_b) = \frac{\Delta_{abc} \sin \delta_c - \gamma_c \cos \delta_c}{\Delta_{abc} \cos \delta_c + \gamma_c \sin \delta_c}. \tag{10}
\]

We thus see that if \( |\Delta_{abc}| \gg \gamma_c \) then \( \delta_d \approx \delta_b + \delta_b \), whereas if \( \gamma_c \gg |\Delta_{abc}| \) then \( \delta_d \approx \delta_c + \delta_b - \pi/2 \). This is the standard result that for a driven damped harmonic oscillator, the driving and response are in phase if the detuning is large (compared to the damping) and \( \pi/2 \) out of phase if the detuning is small.

Breger & Montgomery (2014) present a similar relation for \( \mu \) (their Equation (5)), although it looks slightly different. This is partly because they adopt a different eigenfunction normalization. More importantly, their expression neglects detuning and effectively assumes \( |\Delta_{abc}| \ll \gamma_c \). However, we will see that this is not generally true: for the triplets with the largest \( \mu \), \( \Delta_{abc} \) rather than \( \gamma_c \) often limits the magnitude of \( \mu \).

### 3. Calculational Methods

In Section 3.1 we describe our δ Sct models and how we solve for their linear eigenmodes, and in Section 3.2 we describe how we calculate their linear damping rates. In

\(^2\)The observed amplitude modulations are on timescales of \( \geq 100 \) days (Breger & Montgomery 2014; Bowman et al. 2016), and thus we can treat the parent amplitudes as nearly constant on the timescale of the oscillation periods (\( \lesssim \) hours). Their amplitudes are determined by a combination of linear driving by the \( \kappa \)-mechanism and nonlinear damping processes (including presumably nonlinear mode coupling) that are not well understood.


3.2. Linear-mode Damping

The two principle sources of linear dissipation acting on the oscillation modes are radiative and turbulent damping. The former is due to dissipation of the mode-induced temperature fluctuations by radiative diffusion. The latter is due to dissipation of the mode-induced fluid displacements by turbulent eddies within the convective core (the thin surface convection zone contributes very little to the damping).

We calculate the radiative damping rate of a mode $\gamma^{(\text{rad})}$ from the GYRE solution of the nonadiabatic oscillation equations. For all other parts of our calculations (eigenfrequencies, displacements, etc.), we use the solution of the adiabatic oscillation equations.\(^3\) In order to assign $\gamma^{(\text{rad})}$ to an adiabatic eigenmode, we take its values as a function of the nonadiabatic eigenfrequencies and use interpolation to assign it to the adiabatic eigenfrequencies (the two frequencies differ only slightly).

We find the turbulent damping rate by computing

\[
\gamma^{(\text{turb})} = \frac{\omega_n^2}{E_*} \int dr \, \rho \nu_{\text{turb}} F(r),
\]

\(^3\) This is primarily because the expression we use to compute $\nu_{\text{turb}}$ assumes adiabatic eigenmodes. Since the damping rates are all much lower than the mode frequencies we consider, the modes are adiabatic to a good approximation.
where \(F(r)\) depends on the eigenfunction displacement and is given by the expression found in Higgins & Kopal (1968; see also Lai 1994). The turbulent effective viscosity \(v_{\text{turb}}\) depends on the ratio of the convective turnover frequency (provided by MESA) to the mode frequency and is reduced when this ratio is low. To calculate \(v_{\text{turb}}\), we use the power-law expression given in Duguid et al. (2020) from a fit to their numerical simulations.

In Figure 4 we show \(\gamma_a^{(\text{rad})}\) and \(\gamma_a^{(\text{turb})}\) as a function of mode frequency for all the \(l=2\) modes of the 2.0\(M_\odot\), 7696K model (the other models yield similar results). We omit from the plot the linearly unstable modes whose radiative damping rates are negative (we discuss these modes in Section 4). We see that the radiative damping dominates at essentially all frequencies, often by a factor of \(~100\), and therefore, the total damping rate

\[
\gamma_a^{(\text{tot})} = \gamma_a^{(\text{rad})} + \gamma_a^{(\text{turb})} \approx \gamma_a^{(\text{rad})}.
\]

The \(p\)-mode damping rates are several orders of magnitude higher than the \(g\)-mode damping rates because the \(p\)-modes have much lower mode inertias (see Aerts et al. 2010). Moreover, because the efficiency of radiative dissipation increases with decreasing wavelength, we see that \(\gamma_a^{(\text{rad})}\) increases with increasing \(p\)-mode frequency and decreasing \(g\)-mode frequency.

### 3.3. Nonlinear-coupling Coefficient \(\kappa_{abc}\)

We calculate \(\kappa_{abc}\) using Equations (A55) through (A62) in Weinberg et al. (2012). The modes couple only if they satisfy the angular selection rules \(|\ell_b - \ell_c| \leq \ell_a \leq \ell_b + \ell_c\) with \(\ell_a + \ell_b + \ell_c\) even and \(m_a + m_b + m_c = 0\). The angular selection rules help restrict the number of triplets to consider in our search for high \(\mu\) values. In Section 4 we show that the triplets with the highest \(\mu\) have \(\kappa_{abc} \sim 1-100\).

As representative examples of the \(\kappa_{abc}\) calculation, in Figure 2 we show the cumulative integral \(\kappa_{abc}(<r) = \int_0^r (d\kappa_{abc}/dr) \, dr\) for a triplet of three \(p\)-modes and in Figure 3 for a triplet of three \(g\)-modes. For the triplet shown in Figure 2, \(\kappa_{abc} \approx 60\), and most of the contribution to \(\kappa_{abc}\) (i.e., most of the nonlinear coupling) occurs in the outer layers of the star. This triplet has fractional detuning \(\Delta_{abc}/\omega_c \approx 1.3 \times 10^{-4}\), \(\gamma_c = 9.9 \times 10^{-8} \, \text{s}^{-1}\), and \(\mu = 3.7 \times 10^5\). For the triplet shown in Figure 3, \(\kappa_{abc} \approx 2.3\) and most of the contribution occurs in the deep interior just outside the convective-radiative interface. This is because that is where the modes of the triplet have their largest displacement. This triplet has \(\Delta_{abc}/\omega_c \approx 3.0 \times 10^{-7}\), \(\gamma_c = 1.0 \times 10^{-12} \, \text{s}^{-1}\), and \(\mu = 7.6 \times 10^6\).

### 3.4. Search for Triplets with Large Coupling Strength \(\mu\)

We are interested in finding the triplets with the highest values of the nonlinear coupling strength \(\mu\). From Equation (9), we see that this requires finding triplets with large \(\kappa_{abc}\) and small \(\Delta_{abc}/\omega_c\) and \(\gamma_c/\omega_c\). For each of our \(\delta\) Sct models, we search for the triplets with the largest \(\mu\) by scanning over all combinations of mode triplets in our sample of \(\approx 150\) modes. Since computing \(\kappa_{abc}\) is the most expensive part of the \(\mu\) calculation, we restrict our search to triplets with detuning \(|\Delta_{abc}| < 0.15\sqrt{GM/R^3}\). In practice, this does not affect our results because the largest \(\mu\) all have much smaller detunings than this. As we show in Section 3.4.1, the magnitude of the smallest detunings can be understood as resulting from the repeated drawing of three random numbers from a uniform distribution. Note that when calculating \(\Delta_{abc}\), we must account for all combinations of mode frequency signs (see Equation (6)) because the phase-space mode expansion includes both positive and negative frequencies for each \(\omega_a\). This means that the daughters are not necessarily the highest-frequency mode of the triplet; indeed, for many of our largest \(\mu\), they are the intermediate-frequency mode (Breger et al. 2012 also find this among the triplets they identify in KIC 8054146).

#### 3.4.1. Minimum Detuning \(\Delta_{abc}\)

In Section 4 we show that all of our \(\delta\) Sct models have minimum fractional detunings \(\Delta_{abc}/\omega_c \approx 10^{-3}\). We can roughly understand what sets this minimum as follows. For each model, the mode frequencies lie between some minimum and maximum, \(\omega_{\text{min}} \leq \omega_a \leq \omega_{\text{max}}\), corresponding to the highest-order \(g\)- and \(p\)-modes. To a reasonable approximation, we can treat the \(\approx 150\) calculated modes from each model as though their frequencies were uniformly distributed between these two values. We therefore assume that \(\epsilon \equiv (\omega_a - \omega_{\text{min}})/(\omega_{\text{max}} - \omega_{\text{min}})\) is uniformly distributed between 0 and 1, and write \(\Delta_{abc} = \omega_a + \omega_b - \omega_c\), as \(\epsilon = a + b - c\), where \(\epsilon \equiv (\Delta_{abc} - \omega_{\text{min}})/(\omega_{\text{max}} - \omega_{\text{min}})\) is similar to the fractional detuning \(\Delta_{abc}/\omega_c\). For a random draw of \(\{a, b, c\}\), the

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4 Although a fully self-consistent calculation would account for turbulence in the oscillation equations because radiative damping generally dominates, GYRE’s neglect of the latter is unlikely to be a significant shortcoming.

5 Since the modes are relatively low order, they do not satisfy the asymptotic relations appropriate for high-order modes, in which the period (\(g\)-modes) or frequency (\(p\)-modes) spacings are nearly constant for fixed \(l\).
probability, we therefore need \( \sim 1/\epsilon_0 \) draws of \( \{a, b, c\} \) before we obtain an \( |e| < \epsilon_0 \). For \( 0 < l \leq 3 \), there are eight combinations of \( \{l_a, l_b, l_c\} \) that satisfy the selection rules (see Section 3.3) and because \( -20 < n < 20 \), there are about \( \left( \frac{40}{3} \right) \approx 10^4 \) combinations of \( \{n_a, n_b, n_c\} \). Our search therefore has \( \approx 10^5 \) unique combinations of \( \{a, b, c, \} \), and we can expect a minimum \( |e| \approx 10^{-5} \), consistent with our calculated minimum \( \Delta_{abc}/\omega_c \).

Breger & Montgomery (2014) find that the triplets in KIC 8054146 have fractional detunings \( \lesssim 10^{-6} \), somewhat smaller than our calculated minimum. This could be because we do not account for rotation, which lifts the degeneracy in \( m_a \). It therefore increases the total number of unique triplet combinations and further reduces the minimum \( \Delta_{abc}/\omega_c \).

### 4. Results

In Section 4.1 we show results for a representative \( \delta \) Sct model with \( M = 2.0M_\odot \), \( T_{\text{eff}} = 7696 \text{ K} \), and \( \log g = 3.9 \). We choose this model because it lies near the middle of the instability strip (see Figure 1) and because KIC 8054146 has similar \( T_{\text{eff}} \) and \( \log g \) (Breger et al. 2012; Breger & Montgomery 2014). In Section 4.2 we show results for our other \( \delta \) Sct models, which we find are generally similar to those of our representative model. In Section 4.3 we compare our results to the triplets identified in KIC 8054146.

#### 4.1. Representative \( \delta \) Sct Model

In Figure 5 we show the triplets with the largest coupling strengths \( \mu > 10^3 \) from our representative model (\( M = 2.0M_\odot \), \( T_{\text{eff}} = 7696 \text{ K} \), \( \log g = 3.9 \)). We see that there are many triplets with \( \mu > 10^3 \) and even a few with \( \mu > 10^5 \). The four panels, from left to right, show how \( \mu \) depends on the coupling coefficient \( \kappa_{abc} \), the fractional detuning \( \Delta_{abc}/\omega_c \), the daughter damping rate \( \gamma_c \), and \( \Delta_{abc}/\gamma_c \). Although the coupling strength can be as large as \( \mu \approx 10^6 \) even for coupling coefficients as small as \( \kappa_{abc} \approx 1 \), the largest \( \mu \) tend to have \( \kappa_{abc} > 10 \). These are generally the triplets containing three \( p \)-modes (gray points).

Nonetheless, other types of triplets involving different combinations of \( p- \) and \( g \)-modes (colored points) can still have \( \mu \gtrsim 10^3 \).

The second panel shows that the largest \( \mu \) are more likely to have small detunings, with \( \Delta_{abc}/\omega_c \lesssim 10^{-5} \) (in Section 3.4.1 we explain why the minimum \( \Delta_{abc}/\omega_c \approx 10^{-7} \)). By contrast, the third panel shows that the largest \( \mu \) are not especially sensitive to \( \gamma_c \); if anything, they are more likely to have larger \( \gamma_c \) (those in which the daughter is a \( p \)-mode; see Figure 4). This is because for the majority of strongly coupled triplets, \( \Delta_{abc} \lesssim \gamma_c \), as can be seen in the fourth panel. Thus, by Equation (9), \( \mu \) tends to be limited by the magnitude of \( \Delta_{abc} \) rather than \( \gamma_c \).

In Figure 6 we show (in red) the amplitude of the daughter modes \( A_c = |q_c| \) as a function of their frequency for the representative \( \delta \) Sct model. This figure is like a power spectrum, albeit a fairly artificial one. Specifically, to calculate \( A_c \), we use Equation (8) assuming the parents (in blue) are all at amplitude \( A_p = A_b = 10^{-6} \). While this choice of parent amplitude is essentially arbitrary, the reason we choose \( A_p = A_b = 10^{-6} \) is because then the best coupled triplets (\( \mu \approx 10^5 \)) have \( A_c \approx 10^{-7} \) to \( 10^{-6} \). Thus, our choice ensures that the daughter amplitudes are comparable to their parents’ amplitudes, which is similar to the daughter modes observed in KIC 8054146 (Breger & Montgomery 2014). Interestingly, we see in Figure 6 that while most of the high-amplitude daughters are \( p \)-modes (frequencies greater than about 15 cycles day\(^{-1} \); see the vertical black line), the \( g \)-mode daughters can also have significant amplitudes. This feature is also observed in KIC 8054146.

A notable difference between our artificial power spectrum (Figure 6) and the observed power spectrum of KIC 8054146 (see Figure 1 in Breger & Montgomery 2014) is that ours has a much higher density (per unit frequency) of parent and daughter modes with large amplitudes. Although we do not know for certain, we suspect that this is because in our treatment, we assume that every \( p- \) and \( g \)-mode parent is unstable to the \( \kappa \)-mechanism and linearly driven to \( A_p = A_g = 10^{-6} \). In reality, as the power spectrum of KIC 8054146 indicates, only a subset of these modes will be linearly unstable and driven to sufficiently large amplitudes to be detectable\(^7 \) (and to likewise drive daughters to detectable amplitudes).

\(^7\) Note, however, that Breger & Montgomery (2014) only analyze the 20 dominant modes and their harmonics. The full spectrum presented in Breger et al. (2012) contains many more modes.
Another notable difference between Figure 6 and the observed power spectrum is that ours is based on mode amplitudes, whereas the observed spectrum is based on surface flux perturbations. Although the procedure for deriving the latter from the former is known (see, e.g., Dziembowski 1977), the results can be sensitive to how the regions near the photosphere are treated, especially in stellar models like ours with thin surface convection zones (Pfahl et al. 2008). Investigating the relation between the intrinsic amplitudes of nonlinear coupled modes and their observed flux variations and how it relates to the measured values of $\mu$ is an important next step in the analysis and is left to future work.

In order to determine which modes are unstable to the $\kappa$-mechanism, we could turn to the GYRE solution of the nonadiabatic oscillation equations and find which modes have $\gamma_a^{(\text{rad})} < 0$. Goldstein & Townsend (2020) use this approach to find the unstable modes in models of $\beta$ Cephei stars whose oscillations are driven by the iron-bump $\kappa$-mechanism. However, in practice, we find that only low-order modes with $n \approx 2$ have $\gamma_a^{(\text{rad})} < 0$. Moreover, when we include turbulent dissipation $\gamma_a^{(\text{turb})}$, which GYRE does not account for, many of these modes have total damping $\gamma_a^{(\text{tot})} > 0$. Compared to the observed spectra of $\delta$ Sct stars, GYRE seems to find too few unstable modes, and the range of frequencies is too low ($f \lesssim 40$ cycles day$^{-1}$). The origin of this discrepancy is unclear, but it might be related to the larger problem of the unknown mode-selection process noted in the introduction. This important caveat aside, we find two triplets whose parents are both unstable according to GYRE ($\gamma_a^{(\text{tot})} < 0$) and whose $\mu$ is large enough ($\mu \simeq 10^4$) to drive a daughter to significant amplitude (again assuming $A_g = A_g = 10^{-6}$). These two triplets are listed in Table 1 in the Appendix (see the lines starting with asterisks among the triplets of the representative model).

### 4.2. Our Other $\delta$ Sct Models

In Figure 7 we show the largest $\mu$ triplets in rank order for each of our 14 $\delta$ Sct models. We find that all the models save one have more than 100 triplets with $\mu > 10^4$ and a few triplets with $\mu > 10^5$. Triplet with strong nonlinear coupling are thus a common feature of our $\delta$ Sct models.

In Table 1 in the Appendix, we provide more detailed information about the triplets with the three largest $\mu$ for each model. Most of these triplets consist of three $p$-modes, although some consist of three $g$-modes or a mix of $p$- and $g$-modes. With only a few exceptions, the daughter is either the lowest-frequency or intermediate-frequency mode of the triplet. While high-frequency daughters tend not to be among the highest-$\mu$ triplets (those with $\mu \sim 10^4$–$10^5$), Figure 8 shows that the triplets with $\mu \sim 10^5$ often do contain high-frequency high-order $\rho$-mode daughters ($f \gtrsim 60$ cycles day$^{-1}$, $n \gtrsim 10$). It also shows that they sometimes contain low-frequency $g$-mode
daughters ($f \lesssim 15$ cycles day$^{-1}$, $n < 0$), even though their $\kappa_{abc}$ is somewhat smaller (see Figure 5).

As with the representative model, we also list in Table 1 the two largest $\mu$ triplets whose parents are both unstable, i.e., $\gamma_{\mu}^{(100)} < 0$ according to GYRE (not all models have these triplets). In general, they consist of low-order modes and have $\mu \gtrsim 10^3$.

4.3. Comparison to the δ Sct Star KIC 8054146

The stellar parameters of our representative δ Sct model ($M = 2.0M_\odot$, $T_{\text{eff}} = 7696$ K, and $\log g = 3.9$) are similar to those of the Kepler δ Sct star KIC 8054146 studied by Breger & Montgomery (2014). They found several resonant triplets in their analysis of the star’s power spectrum, and focus on three of them in particular. Their frequencies ($f_a$, $f_b$, $f_c$) in units of cycles per day are (25.9509, 60.4346, 34.4836), (25.9509, 63.3680, 37.4170), and (25.9509, 66.2988, 40.3479). All three share a common mode ($f_c = 25.9509$ cycles day$^{-1}$), and to the precision provided in the paper, they have detunings $|f_a - f_b + f_c| < 10^3$ cycles day$^{-1}$, corresponding to fractional detunings $|\Delta_{abc}| / f_c < 3 \times 10^{-6}$ (given the four-year data set, the frequencies cannot be measured to better than $\sim 10^{-4}$ cycles day$^{-1}$, and only an upper bound can be placed on the detunings). They find that the amplitudes vary over the four-year Kepler observations according to Equation (8). This allows them to identify the parent and daughter modes in each triplet because they are observed to vary in concert as $A_i(t) = \mu A_i(t)A_i(t)$. Consistent with the mode-coupling interpretation, they also find that their phase shifts vary as $\phi_i(t) = \phi_i(t) + \phi_0(t)$. The daughter is the intermediate-frequency mode in all three triplets (mode $f_c$ in the list above), and from its amplitude variation, they measure a coupling strength in the range 7000 $\lesssim \mu \lesssim 60,000$.

This range of measured $\mu$ is consistent with the values we find in our calculations (not counting the few triplets we find with $\mu \gtrsim 10^5$). Moreover, as can be seen from Table 1 in the Appendix, our representative δ Sct model contains triplets whose other properties are similar to those found in KIC 8054146, especially the triplet with the third-largest $\mu$ in the list. Specifically, their frequencies and detunings are similar, and because $|\Delta_{abc}| > \gamma_c$, we expect their phase shift, like the phase lag, to satisfy $\phi_i \simeq \phi_0 + \phi_b$ (see Section 2). Some of the triplets in KIC 8054146 also contain a mix of $p$- and $g$-modes, like ours. However, given that we find hundreds of triplets with $\mu \gtrsim 10^3$, this similarity between the largest $\mu$ triplets listed in Table 1 and those observed in KIC 8054146 could just be a coincidence. As we discuss below, a more definitive comparison requires a time-dependent network calculation that also models the parent driving.

5. Summary and Conclusions

Motivated by the observational evidence of nonlinear mode interactions in δ Sct stars, especially in KIC 8054146, we carried out a theoretical investigation of the prevalence and strength of resonant three-mode coupling in 14 models of δ Sct stars that span the instability strip. For each model, we found all the eigenmodes with angular degree $0 \leq l \leq 3$ and radial order $-20 \leq n \leq 20$, corresponding to $g$- and $p$-modes with frequencies in the range $1 \lesssim f \lesssim 100$ cycles day$^{-1}$. We computed the linear damping rate $\gamma$ of each mode due to radiative and turbulent dissipation and found that the former typically dominates. We then searched for all mode triplets $(a, b, c$ that satisfy the three-mode angular selection rules and computed their detuning $\Delta_{abc} = \omega_a \pm \omega_b \pm \omega_c$ and coupling coefficient $\kappa_{abc}$. Last, we rank-ordered the triplets according to their nonlinear coupling strength $\mu = |\omega_c \kappa_{abc}| / \sqrt{|\Delta_{abc}|^2 + \gamma_c^2}$.

According to the theory of nonlinear three-mode coupling, two parent modes $a$ and $b$ driven by the $\kappa$-mechanism to amplitudes $A_a$ and $A_b$ will excite a daughter mode $c$ to an amplitude $A_c = \mu A_a A_b$. In all of our models, we found at least 10 triplets with $\mu \gtrsim 10^3$, and in many models, there were $> 100$ such triplets. These $\mu$ values are broadly consistent with those directly measured by Breger & Montgomery (2014) in their analysis of KIC 8054146. We found that the triplets with large $\mu$ consist of various combinations of $p$- and $g$-modes (e.g., three $p$-modes, three $g$-modes, two $p$-mode parents, and a $g$-mode daughter), which is also true of the triplets found in KIC 8054146.

Our results suggest that resonant three-mode interactions can be significant in δ Sct stars, and the ubiquity of large $\mu$ across all of our models may explain why amplitude variations (Bowman et al. 2016) and large $P/P$ (Breger & Pamyatnykh 1998; Blake et al. 2003; Bowman et al. 2021) are commonly observed. However, our analysis did not model the parent driving and therefore did not solve for the overall mode amplitudes, nor for their relation to the surface flux perturbations. Thus, we cannot yet define the extent to which mode coupling impacts the observed oscillation spectra. Moreover, we only focused on direct nonlinear forcing, in which two parent modes excite a daughter mode. Another form of three-mode coupling that might impact the oscillation spectra is the parametric instability, in which a parent mode excites two daughter modes (see, e.g., Dziembowski 1982). Unlike direct forcing, the parametric instability is only triggered if a parent is above a threshold amplitude. Dziembowski & Krolikowska (1985) showed that it is a potentially important mechanism for limiting oscillation amplitudes and transferring energy across the modes (see also Dziembowski et al. 1988). A further complication is that the daughters can themselves excite granddaughters, and the granddaughters can excite great-granddaughters, and so on, either through direct forcing or through the parametric instability, or through both mechanisms.

In order to make further progress on this problem, a time-dependent mode network calculation is needed. This would entail solving a large set of nonlinearly coupled amplitude equations that account for the parent driving, multiple generations of $p$- and $g$-modes, and both forms of three-mode coupling. Weinberg et al. (2021; see also Weinberg & Arras 2019) carried out a similar calculation in their study of solar-like oscillations in red giants, although there the parents were driven by stochastic turbulent motions, and not by the $\kappa$-mechanism. They found that depending on the evolutionary state of the red giant, the secondary modes (daughters, granddaughters, etc.) could significantly suppress the parent amplitudes and efficiently transfer the parents’ energy among thousands of higher-order higher-degree modes. The structure of a red giant, and therefore of the mode coupling, is of course very different from that of a δ Sct star. Most notably, the nonlinear mode interactions in a red giant are concentrated in the stellar core and involve mixed modes. In a δ Sct star, by contrast, we found that many of the triplets consist of three $p$-modes whose nonlinear couplings peak near the stellar surface.
These surface interactions might impact the oscillation spectra more directly and leave a detectable time-dependent imprint on not just the mode amplitudes, but also on their frequencies and line widths.

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Software: MESA (Paxton et al. 2011, 2013, 2015, 2018, 2019; Jermnyn et al. 2023, http://mesa.sourceforge.net), GYRE (Townsend & Teitler 2013; Townsend et al. 2018, https://gyre.readthedocs.io/en/stable/).

Appendix

Table of Triplets with Large Nonlinear Coupling Strength

In Table 1, we list the properties of the triplets with the three largest values of the coupling strength $\mu$ for each of our 14 $\delta$ Scuti models. We also list two triplets that contain linearly unstable parent modes ($\kappa_{\ell a}^{\text{ind}} > 0$) according to GYRE’s solutions of the nonadiabatic oscillation equations.

| $l_a$ | $l_b$ | $l_c$ | $n_a$ | $n_b$ | $n_c$ | $f_a$ | $f_b$ | $f_c$ | $\log \gamma_a$ | $\log \gamma_b$ | $\log \Delta_{abc}$ | $\kappa_{abc}$ | $\log \mu$ |
|------|------|------|------|------|------|------|------|------|------------|------------|----------------|-------------|----------|
| 3  | 3 | 0 | 19 | 10 | 7 | 61 | 38 | 23 | −1.9 | −2.0 | −3.5 | −4.6 | −46.3 | 5.2 |
| 0 | 1 | 1 | 17 | 13 | −1 | 52 | 41 | 10 | −1.6 | −1.5 | −6.1 | −4.7 | 2.1 | 5.0 |
| 0 | 3 | 3 | 8 | 15 | 5 | 26 | 50 | 24 | −2.7 | −1.9 | −3.0 | −4.5 | 77.8 | 4.9 |
| * 3 | 2 | 1 | 2 | 4 | 12 | 19 | 20 | 39 | −5.0 | −5.3 | −2.2 | −2.2 | 49.7 | 3.7 |
| * 1 | 2 | 1 | 4 | 4 | 12 | 18 | 20 | 39 | −5.2 | −5.3 | −2.2 | −3.1 | 34.5 | 3.7 |

| $l_a$ | $l_b$ | $l_c$ | $n_a$ | $n_b$ | $n_c$ | $f_a$ | $f_b$ | $f_c$ | $\log \gamma_a$ | $\log \gamma_b$ | $\log \Delta_{abc}$ | $\kappa_{abc}$ | $\log \mu$ |
|------|------|------|------|------|------|------|------|------|------------|------------|----------------|-------------|----------|
| * 0 | 1 | 1 | 6 | 5 | −14 | 23 | 21 | 2 | −3.7 | −4.3 | −6.0 | −2.1 | 0.8 | 2.0 |

This table contains the properties of the triplets with the largest values of the coupling strength $\mu$ from each $\delta$ Scuti model.
Table 1 (Continued)

| $l_a$ | $l_b$ | $l_c$ | $n_a$ | $n_b$ | $n_c$ | $f_a$ | $f_b$ | $f_c$ | $\log \gamma_a$ | $\log \gamma_b$ | $\log \gamma_c$ | $\log \Delta_{abc}$ | $\kappa_{abc}$ | $\log \mu$ |
|-------|-------|-------|-------|-------|-------|-------|-------|-------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| 3     | 3     | 2     | 4     | 4     | 13    | 32    | 32    | 64    | $-5.3$          | $-5.3$          | $-2.2$          | $-2.4$          | 38.9            | 3.7             |

$M = 2.0M_\odot$, $T_{\text{eff}} = 8328$ K, $\log g = 4.10$

| 1     | 1     | 2     | $-13$ | $-18$ | $-13$ | 2     | 1     | 3     | $-8.0$          | $-7.0$          | $-5.8$          | 2.4             | 6.1             |
| 1     | 2     | 3     | 15    | 5     | 7     | 82    | 36    | 47    | $-2.0$          | $-4.8$          | $-3.5$          | $-3.0$          | 70.4            | 4.9             |
| 1     | 3     | 2     | 15    | 8     | 4     | 82    | 52    | 31    | $-1.8$          | $-3.0$          | $-4.6$          | $-3.2$          | 46.7            | 4.8             |

$M = 1.85M_\odot$, $T_{\text{eff}} = 7350$ K, $\log g = 3.96$

| 0     | 3     | 3     | 7     | 15    | 6     | 33    | 71    | 38    | $-3.4$          | $-1.9$          | $-3.9$          | $-4.3$          | 132.7           | 5.9             |
| 2     | 0     | 2     | 17    | 11    | 4     | 77    | 49    | 29    | $-1.8$          | $-2.2$          | $-2.0$          | $-3.5$          | 73.1            | 5.7             |
| 3     | 3     | 0     | 6     | 15    | 7     | 38    | 71    | 33    | $-3.8$          | $-1.8$          | $-3.4$          | $-3.6$          | 121.2           | 5.5             |

$M = 1.85M_\odot$, $T_{\text{eff}} = 7765$ K, $\log g = 4.01$

| 0     | 2     | 2     | $-1$  | 4     | 15    | 14    | 29    | $-6.6$ | $-7.1$          | $-4.0$          | $-3.4$          | 1.3             | 3.5             |
| 1     | 2     | 3     | 1     | 5     | 15    | 19    | 34    | $-6.5$ | $-6.2$          | $-3.8$          | $-3.5$          | 1.1             | 3.5             |

$M = 1.85M_\odot$, $T_{\text{eff}} = 7967$ K, $\log g = 4.13$

| 2     | 1     | 1     | 8     | 17    | 7     | 54    | 100   | 47    | $-3.0$          | $-1.9$          | $-3.8$          | $-4.3$          | 123.4           | 5.9             |
| 3     | 0     | 3     | 18    | 11    | 6     | 111   | 65    | 46    | $-2.0$          | $-2.4$          | $-4.0$          | $-3.9$          | 80.6            | 5.7             |
| 3     | 0     | 3     | 19    | 12    | 6     | 116   | 70    | 46    | $-2.0$          | $-2.2$          | $-4.0$          | $-4.3$          | 35.7            | 5.5             |

$M = 1.7M_\odot$, $T_{\text{eff}} = 7492$ K, $\log g = 4.14$

| 3     | 2     | 2     | 17    | 11    | 4     | 109   | 75    | 35    | $-1.7$          | $-2.1$          | $-5.0$          | $-4.4$          | 30.3            | 5.9             |
| 1     | 2     | 1     | 17    | 10    | 5     | 104   | 67    | 38    | $-1.8$          | $-2.3$          | $-4.3$          | $-3.9$          | 70.0            | 5.8             |
| 1     | 1     | 2     | 16    | 8     | 6     | 99    | 53    | 45    | $-1.9$          | $-3.0$          | $-3.6$          | $-4.1$          | 122.3           | 5.7             |

$M = 1.7M_\odot$, $T_{\text{eff}} = 7750$ K, $\log g = 4.23$

| 2     | 1     | 2     | $-1$  | $-2$  | 27    | 16    | 11    | $-7.2$ | $-7.5$          | $-9.0$          | $-4.1$          | $-0.4$          | 3.7             |
| 2     | 2     | 2     | $-2$  | 1     | 5     | 15    | 25    | 40    | $-8.9$          | $-6.8$          | $-4.2$          | $-4.1$          | 0.4             | 3.6             |

$M = 1.7M_\odot$, $T_{\text{eff}} = 7829$ K, $\log g = 4.26$

| 3     | 0     | 3     | 18    | 11    | 6     | 140   | 82    | 58    | $-1.8$          | $-2.5$          | $-3.8$          | $-4.3$          | 129.5           | 5.9             |
| 2     | 1     | 3     | 18    | 6     | 123   | 65    | 58    | $-1.9$ | $-3.2$          | $-3.8$          | $-4.3$          | 118.1           | 5.8             |
| 3     | 0     | 3     | 19    | 11    | 7     | 147   | 91    | 56    | $-1.8$          | $-2.3$          | $-3.9$          | $-4.0$          | 87.8            | 5.8             |

$M = 1.7M_\odot$, $T_{\text{eff}} = 7829$ K, $\log g = 4.26$

Note. The subscripts $a$ and $b$ label the parents, and subscripts $c$ labels the daughter. The columns are the angular degree $l$, the radial order $n$, the mode frequency $f$ (in cycles per day), the total damping rate (in units of $\log \gamma = \log10(\gamma/\omega_\text{a})$), the frequency detuning (in units of $\log \Delta = \log10(\Delta/\omega_\text{a})$), the coupling coefficient $\kappa_{abc}$ and $\log \mu$. For each model, the first three rows show the largest $\mu$ triplets. The following two rows (indicated by asterisks) show the largest $\mu$ triplets whose parents both have negative damping rates (we then give $\log \gamma = \log10(-\gamma/\omega_\text{a})$; note that not all models have such triplets). Two of the triplets containing three $g$-modes repeat with different parent-daughter combinations, and we only list the triplet with the largest $\mu$.

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