Weakly coupled oscillators play an important role in understanding collective behaviour of large populations. They are often used to model the dynamics of a variety of systems that arise in nature, even though they are quite different. Synchronization is one of the interesting phenomena observed in these systems where the interacting oscillators under the influence of coupling would have a common frequency. Particularly, these systems show an extremely complex clustering behavior as a function of the coupling strength. In spite of their differences, the above mentioned systems can be described using simple models of coupled phase equations such as the Kuramoto model. This paper analyzes the behaviour of individual oscillators in the vicinity of the critical coupling where all the oscillators evolve in synchrony with each other.

I. INTRODUCTION

Systems of coupled oscillators can describe problems in physics, chemistry, biology, neuroscience and other disciplines. They have been widely used to model several phenomena such as: Josephson junction arrays, multimode lasers, vortex dynamics in fluids, biological information processes, neurodynamics. These systems have been observed to synchronize themselves to a common frequency, when the coupling strength between these oscillators is increased. The synchronization features of many of the above mentioned systems, in spite of the diversity of the dynamics, might be described using simple models of weakly coupled phase oscillators such as the Kuramoto model. Finite range interactions are more realistic for the description of many physical systems, although finite range coupled systems are difficult to analyze and to solve analytically. However, in order to figure out the collective phenomena when finite range interactions are considered, it is of fundamental importance to study and to understand the nearest neighbour interactions, which is the simplest form of the local interactions. In this context, a simplified version of the Kuramoto model with nearest neighbour coupling in a ring topology, which we shall refer to as locally coupled Kuramoto model (LCKM), represents a good candidate to describe the dynamics of coupled systems with local interactions such as Josephson junctions, coupled lasers, neurons, chains with disorders, multi-cellular systems in biology and in communication systems. For instance, it has been shown that the equations of the resistively shunted junction which describe a ladder array of overdamped, critical-current disordered Josephson junctions that are current biased along the rungs of the ladder can be expressed by a LCKM. In nearest neighbours coupled Rössler oscillators the phase synchronization can be described by the LCKM. Therefore, LCKM can provide a way to understand phase synchronization in coupled systems, for example, in locally coupled lasers, where local interactions are dominant. Coupled phase oscillators described by LCKM can also be used to model the occurrence of travelling waves in neurons. In communication systems, unidirectionally coupled Kuramoto model can be used to describe an antenna array. Such unidirectionally coupled Kuramoto models can be considered as a special case of the LCKM and it often mimics the same behaviour.

One of the important features of the local model is that the properties of individual oscillators can be easily analyzed to study the collective dynamics while one has to rely on the average quantities, in a mean field approxi-
We give a summary of the results and conclusions. In particular, we point out the mechanism that leads to sive bursts at the transition to complete synchronization. Finally, in Sec. III we present a brief overview on the dynamics of local Kuramoto. This is done by analyzing the behaviour of each individual oscillator at the onset of synchronization. For this purpose we consider the equations governing the phase differences at the onset of synchronization. In particular, we identify that the cosine of only one among the phase differences becomes zero. Based on this property we derive the expression for the critical coupling strength for complete synchronization in the local Kuramoto model with different boundary conditions.

In this paper we address the mechanism that leads to a complete synchronization in the Kuramoto model with local coupling. This is done by analyzing the behaviour of each individual oscillator at the onset of synchronization. For this purpose we consider the equations governing the phase differences at the onset of synchronization. In particular, we identify that the cosine of only one among the phase differences becomes zero. Based on this property we derive the expression for the time interval between bursting behaviour of the instantaneous frequencies of each individual oscillator in the vicinity of critical coupling strength. Our analysis shows that the transition to complete synchronization occurs due to a saddle node bifurcation in agreement with the earlier studies. Further we deduce the expressions for the phases and frequencies of the individual oscillators at the onset of complete synchronization.

This paper is organized as follows. In Sec. II we present a brief overview on the dynamics of local Kuramoto model. Then we analyze the behaviour of the phase differences and the time interval between successive bursts at the transition to complete synchronization. In particular, we point out the mechanism that lead to complete phase locking at the critical coupling strength. Based on this we deduce the forms of phases and frequencies at the onset of synchronization. Finally, in Sec. III we give a summary of the results and conclusions.

II. BEHAVIOUR OF PHASES AND FREQUENCIES AT THE ONSET OF SYNCHRONIZATION

Even when there has been an extensive exploration of the dynamics of the Kuramoto model (global coupling among all oscillators), the local model of nearest neighbour interactions, which can be considered as a diffusive version of the Kuramoto model, has been receiving attention only recently. The LCKM is expressed as 23,24,25,26,27,28,29.

$$\hat{\theta}_i = \omega_i + \frac{K}{3} [\sin(\theta_{i+1} - \theta_i) + \sin(\theta_{i-1} - \theta_i)],$$

(1)

here $\omega_i$ are the natural frequencies, $K$ is the coupling strength, $\theta_i$ is the instantaneous phase, $\theta_i$ is the instantaneous frequency and $i = 1, 2, \ldots, N$. Many interesting features of the LCKM remain unknown, especially an analytic solution, which would be of great importance in understanding the mechanism that leads to synchronization. In order to find such an analytic solution, one should study carefully the temporal evolution of frequency and phase of each individual oscillator in the neighbourhood of the critical coupling for complete synchronization.

If we consider the oscillators in a ring, with periodic boundary conditions $\theta_{i+1} = \theta_i$, the nonidentical oscillators cluster in time averaged frequency, until they completely synchronize to a common value of the average frequency $\omega_0 = \frac{1}{N} \sum_{i=1}^{N} \omega_i$, at a critical coupling $K_{c}$. At $K \geq K_{c}$ the phases and frequencies are time independent and all the oscillators remain synchronized. In Fig. I we show the synchronization tree for a periodic system with $N = 15$ oscillators, where the elements which compose each one of the major clusters that merge into one at $K_{c}$, are indicated in each branch.
two oscillators which have phase difference to get a solution of the above eq. (2) show that for only $|K|$ value of critical coupling strength $K_c$, transition to complete synchronization occurs. However the determination of which two oscillators among $N$ oscillators that have $|\sin \phi_i^*| = 1$, remains difficult. From the study of the temporal evolution of phases and frequencies of each individual oscillator, it has also been found numerically that, at the onset of synchronization $K \lesssim K_c$, the values of $\theta_i(t)$ and $\phi_i(t)$ remain equal to $\omega_0$ and zero, respectively, for a certain time interval $T$. During this time $T$ a stable phase-locked solution exists, then they burst$^{18,24,25}$, and this stable phase-locked solution is lost. In between bursts, the phases remain fixed and then they have an abrupt change (phase-slip behaviour) by an amount which depends on the initial values of the frequencies $\omega_i^{24,25}$, corresponding to the burst in the frequencies, while the quantities $\sum_{i=1}^{N} \phi_i = 0$ and $\sum_{i=1}^{N} \phi_i^* = 0$ are always preserved by the topology. Integrated with the above information, it has been shown by numerical investigation that the time interval $T$ blows up as $K$ becomes close to $K_c$ and $T \to \infty$ at $K_c$. All these information leads to conclude that there is a saddle-node bifurcation at $K_c$ and the synchronization-desynchronization transition at the critical coupling can be interpreted using this knowledge.

In this work, we perform numerical investigations of the temporal evolution of the phases and frequencies for the individual oscillators in order to arrive to specific conditions which will lead to criteria to obtain an analytic solution. A detailed study of all quantities $\sin \phi_i^*$ at $K_c$ for several values of $N$ and for different sets of $\omega_i$, shows that there is only one value of phase difference between two neighbouring oscillators $\phi_i^* = \theta_{i+1}^* - \theta_i^*$ for which $|\sin \phi_i^*| = 1$, while for all other values, $i \neq l$ $|\sin \phi_i^*| \neq 1$. In Fig. 2 we show $\phi_i^*$ for a case of $N = 15$ as time progresses at the critical coupling $K_c$, with the same initial frequencies of Fig. 1. We see that the value of $|\sin \phi_i^*| = 1$, is for $l = 13$ and that this quantity $|\sin \phi_i^*| = 1$ holds for only one value of phase difference $\phi_l = \pi/2$ where these two oscillators $l+1$ and $l$ belong to different clusters, and these two nearest neighbours oscillators are always at the borders between the major clusters that merge at $K_c$, which can be seen from Fig. 1. We find the same result for different initial frequencies $\omega_i$ and for different values of $N$. In addition, the sign of $\sin \phi_i^*$ is negative for $\omega_i > \omega_{i+1}$ and positive for the reverse.

The knowledge of the burst and phase slip (in the vicinity of $K_c$) of the quantities $\phi_i(t)$ and $\phi_i(t)$, respectively, as well as the finding of $|\sin \phi_i^*| = 1$ (at $K_c$), will allow us to rewrite equation (2), for the index $l$ as:

$$\phi_l = B (A - 2 \sin \phi_l),$$

where $A = \frac{3(\omega_l + \omega_{l+1})}{K} + \sin \phi_{l-1} + \sin \phi_{l+1}$ and $B = \frac{\omega_l}{K}$. Eq. (3) takes the form of a phase synchronization of two coupled limit-cycles$^{20}$. At $K_c$, $\phi_l = \pm \pi/2$ and, $\phi_{l-1}^*$ and $\phi_{l+1}^*$ are constants and time independent and $A = 2$. A detailed numerical study shows also that, at the onset of synchronization, $A \approx 2$ and the values of $\phi_l$, $\phi_{l-1}$ and $\phi_{l+1}$ remain equal to their values at $K_c$, for a time interval $T$. The values of $K_c$ and $A$ for different number of oscillators $N$ from numerical simulations are tabulated in Table. I. It is clear that in all the cases, $A \approx 2$ when $K$ approaches $K_c$. The relation $A \approx 2$ is found to be valid for different choices of initial frequencies $\omega_i$ for each $N$ in the vicinity of $K_c$. Further, it should be noted that when the time interval $T \to \infty$ one can find that $A = 2$. The time interval $T$ can be found analytically, according to eq. (3), to be

$$T \approx \frac{3 \pi \sqrt{2}}{K \sqrt{A}} \frac{1}{A - 2}.$$  

In Fig. 3 we clearly see that $T$ blows up as $A$ becomes close to 2 (where $K$ goes to $K_c$), for the case of $N = \text{Table I: Calculated values of } K_c \text{ and } A \text{ for different values of } N.$

| $N$ | $K_c$ | $A$ | $N$ | $K_c$ | $A$ |
|-----|-------|-----|-----|-------|-----|
| 3   | 0.85041227 | 1.9994 | 20 | 4.95830014 | 2.0002 |
| 5   | 3.17082713 | 2.0001 | 25 | 3.64106038 | 1.9989 |
| 10  | 3.54701035 | 1.9996 | 50 | 9.45720049 | 1.9993 |
| 15  | 3.87023866 | 2.0000 | 100 | 12.7232087 | 1.9985 |

| $N$ | $K_c$ | $A$ |
|-----|-------|-----|
| 1.9994 | 2.0002 |
| 1.9989 | 1.9993 |
| 1.9985 | 1.9993 |
15. We find that $T$ blows up as $(A - 2)^{-0.5}$ which is a numerical proof that a saddle-node bifurcation occurs at $K_c$. Assuming that $\sin \phi_{l-1}$ and $\sin \phi_{l+1}$ remain constant for a time interval $T$, in the vicinity of $K_c$, and equal to their values at $K_c$ (which has been verified numerically), we find that $\frac{dK}{dA} \approx K_c$. Table II shows this fact where the error is small and decreases as $K$ approaches $K_c$. Therefore, eq. (4) takes the form:

\[
\dot{\phi}_l \approx \frac{L}{K_c} \left( \frac{\alpha^2 \sec^2 \left( \frac{i}{2} \alpha L \phi \right)}{K_c} + \left\{ \frac{1}{K_c} \left[ \alpha \tan \left( \frac{i}{2} \alpha L \phi \right) \pm K \right] \right\}^2, \tag{7b}
\]

where $\alpha = \sqrt{K_c^2 - K^2}$ and $L = \frac{2}{\alpha}$. Eqs. (7a) and (7b) show that, at $K_c$, the values $\sin \phi_l = \pm 1$ which lead to $\dot{\phi}_l = 0$. Also, it can be seen that in the vicinity of $K_c$, $\sin \phi_l = \pm 1$ and $\dot{\phi}_l = 0$ for a period $T$. The ($+$) sign in Eqs. (7) is corresponding to the case $\omega_{l+1} > \omega_l$ and the ($-$) sign for the reverse.

In order to understand the mechanism of full synchronization which occurs at $K_c$, we use the fact that $\sin \phi_l = \pm 1$ and each $\theta^* = \omega_0$, where these quantities remain unchanged for $T$ in the vicinity of $K_c$. Hence, from system (1), we are able to get the following relations:

\[
\sin \phi_{l+1}^* = \frac{3}{K_c} \sum_{m=1}^{N-l} (\omega_0 - \omega_{l+m}) \pm \sin \phi_l^*, \quad (8a)
\]

\[
\sin \phi_{l-n}^* = -\frac{3}{K_c} \sum_{n=1}^{l-1} (\omega_0 - \omega_{l-n-1}) \pm \sin \phi_l^*. \quad (8b)
\]

Using this fact, we write the following equations, in addition to eq. (7):

\[
\phi_{l-n} \approx \sin^{-1} (a_n \pm \sin \phi_l), \quad (9a)
\]

\[
\dot{\phi}_{l-n} \approx \frac{\cos \phi_l \phi_i}{\sqrt{1 - (a_n \pm \sin \phi_l)^2}}, \quad (9b)
\]

\[
\phi_{l+m} \approx \sin^{-1} (a_m \pm \sin \phi_l), \quad (9c)
\]

\[
\dot{\phi}_{l+m} \approx \frac{\cos \phi_l \phi_i}{\sqrt{1 - (a_m \pm \sin \phi_l)^2}}, \quad (9d)
\]

where $a_n = \sum_{i=1}^{n} (\omega_0 - \omega_{l+1})$ with $n = 1, 2, 3, \ldots, l-1$ and $a_m = \sum_{j=1}^{m} (\omega_0 - \omega_{l+m})$ with $m = 1, 2, 3, \ldots, N-l$. It is clearly seen that according to the above equation, each $\phi_l$ can be expressed in terms of $\phi_l$ and consequently each $\dot{\phi}_l$ can be expressed in terms of $\phi_l$ and $\dot{\phi}_l$. Therefore, all values of $\phi_l$ will be shifted from each other by some constant which is determined by the location of the indexes $l - n$ and $l + m$ relative to oscillators with indexes $l$ and $l + 1$. This is shown in Fig. (2), where $\sin \phi_l$ values are shifted from each others at $K_c$. Therefore, at $K_c$, what occurs to $\phi_l$ and $\dot{\phi}_l$ due to saddle-node bifurcation diffuses through the ring via interaction between neighboring oscillators. This means that, at the vicinity of $K_c$, the value of $\phi_l$ has an abrupt change after being constant for a time $T$, caused by a burst behaviour of $\phi_l$ after being zero for the same time interval $T$. The abrupt change of $\phi_l$ produces a sudden change in the values of $\phi_l$ of their neighbours, while the bursting behaviour of $\phi_l$ in turn yields bursts in $\phi_l$ ($i \neq l$). In order to demonstrate this fact, we plot the temporal evolution of $\phi_{l-1}$ and $\dot{\phi}_l$, in the vicinity of $K_c$, according to numerical simulation of eq. (1) in Fig. 4a while we plot both quantities according to numerical calculation by Zheng et. al. 24, 25.
 FIG. 4: (Color online) Time evolution of $\dot{\phi}_{13}$ and $\dot{\phi}_6$ according to (a) system (1) and (b) equations (7b) and (9b), at $K = 3.870226709$, for 15 oscillators with the same initial conditions of Fig. 1

III. SUMMARY AND CONCLUSIONS

In summary, we have analyzed the conditions on the phase differences for the onset of complete synchronization at the critical coupling strength in a Kuramoto-like model with nearest neighbour coupling. Such condition, which is $|\sin \phi^*_l| = 1$ (or $\cos \phi^*_l = 0$), allows us to solve the equations of the phase differences analytically. Also, we found that full synchronization occurs always when the quantity $A = 2$ at $K_c$. Due to the diffusive nature of the LCKM, complete synchronization of all oscillators to a common value can be interpreted and understood once we have an analytic forms for $\phi_l$ and $\dot{\phi}_l$. However, it is still difficult to determine analytically the number of oscillators in each cluster which merge into one at $K_c$. Therefore, one cannot allocate straightforwardly the two nearest neighbour oscillators which would have $|\sin \phi^*_l| = 1$. On the other hand, a detailed numerical study on the temporal evolution of phases, phase-differences and frequencies of oscillators at the borders of the clusters that merge into larger one at onset of complete synchronization helps us to determine the neighbouring oscillators which have $\sin \phi^*_l = \pm 1$. Such analysis can also be used to understand the partial synchronization that leads to the formation of small clusters for coupling strengths below the critical coupling strength $K_c$. Of course analysis of the simplest case of locally coupled phase oscillators can help to understand models with local interactions where amplitudes and phases are included. In such cases, a detailed study of the time evolution of amplitudes and phases can reveal a better understanding of the mechanism of synchronization. The present analysis can also be applicable to models in higher dimensions such as that for dislocations in solids which includes local nearest neighbour interactions. Furthermore, the present approach can be extended to understand the underlying mechanism in the case of locally coupled Kuramoto models with time delay (or phase delay) introduced between the coupled oscillators. In addition, the mechanism of synchronization in LCKM for open and fixed boundaries can be studied in a similar manner to the present work as well as for the case of unidirectional LCKM. We also want to mention that the scaling law given by eq. (5) has been found experimentally in a transition to phase synchronization in $CO_2$ lasers and in electronic circuits. On the other hand one can not make a direct comparison between the mechanism of synchronization discussed here in LCKM and the scaling law that has been found in experiments since the physical systems are not necessarily the same.

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