CONSTRUCTION OF A GIBBS MEASURE ASSOCIATED TO
THE PERIODIC BENJAMIN-ONO EQUATION

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Abstract. We define a finite Borel measure of Gibbs type, supported by the Sobolev spaces of negative indexes on the circle. The measure can be seen as a limit of finite dimensional measures. These finite dimensional measures are invariant by the ODE’s which correspond to the projection of the Benjamin-Ono equation, posed on the circle, on the first \(N\), \(N \geq 1\) modes in the trigonometric bases.

1. Introduction, preliminaries and statement of the main result

Let us denote by \(S^1\) the circle identified with \(\mathbb{R}/(2\pi\mathbb{Z})\). For \(u \in \mathcal{D}'(S^1)\) a distribution on \(S^1\), we define its Fourier coefficients as \(\hat{u}(n) \equiv (2\pi)^{-1}u(\exp(-inx)), n \in \mathbb{Z}\). Then, we have the Fourier expansion of \(u\) (cf. [12]),

\[
\hat{u}(n) \equiv \langle n \rangle \equiv \left(1 + n^2\right)^{-1/2},
\]

where \(\langle n \rangle \equiv (1 + n^2)^{1/2}\). For \(s = 0\), \(H^s(S^1) = L^2(S^1)\) and for \(s \geq 0\), the space \(H^s(S^1)\) contains integrable functions on the circle, while for \(s < 0\) the elements

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of $H^s(S^1)$ are not induced by integrable functions via the canonical identification. Denote by $H^s_0(S^1)$ the subset of $H^s(S^1)$ defined as

$$H^s_0(S^1) \equiv \{ u \in H^s(S^1) : \hat{u}(0) = 0, \; \hat{u}(n) = \hat{u}(-n), \; \forall n \in \mathbb{Z}^* \}.$$ 

Notice that the elements of $H^s_0(S^1)$ are real valued distributions. We have that $H^s_0(S^1)$ endowed with the scalar product (1.1) is a real Hilbert space. For $s \geq 0$, the space $H^s_0(S^1)$ contains the real valued functions of $H^s(S^1)$ with zero mean value.

Consider the Cauchy problem for the Benjamin-Ono equation, posed on $S^1$,

$$(\partial_t + H\partial_x^2)u + \partial_x(u^2) = 0, \quad u|_{t=0} = u_0 \in H^s_0(S^1)$$

for some $s \in \mathbb{R}$. In (1.2), $H : H^s_0(S^1) \to H^s_0(S^1)$ denotes the Hilbert transform defined for $w \in H^s_0(S^1)$ as

$$w \mapsto -i \sum_{n \in \mathbb{Z}^*} \text{sign}(n) \hat{w}(n)e^{inx},$$

i.e

$$\hat{H}w(n) \equiv -i \text{sign}(n) \hat{w}(n), \quad n \in \mathbb{Z}^*, \quad \hat{H}w(0) \equiv 0.$$ 

Considering solutions of (1.2) in the space $H^s_0(S^1)$ seems reasonable since by a formal integration of the equation the mean value of $u$ is preserved. If $s < 0$ the expression $u^2$ is a priori not defined and the interpretation of the nonlinear term in (1.2) requires to be done carefully. For $s \geq 0$, it follows from the work of Molinet [11] that (1.2) has a well-defined global in time dynamics in the phase space $H^s_0(S^1)$.

Recall that the Benjamin-Ono equation is an asymptotic model derived from the Euler equation for the propagation of internal long waves (see [2]).

The goal of this paper is to construct a weighted Wiener measure, of Gibbs type associated to (1.2). This construction is in the spirit of the work by Lebowitz-Rose-Speer [8] for the nonlinear Schrödinger equation. As we will see, in the context of (1.2) the construction requires more involved probabilistic arguments compared to [8].

We fix for the remaining part of this paper a positive number $\sigma$. The Gibbs type measure we construct will be a finite Borel measure on $H^{-\sigma}_0(S^1)$. For an integer $N \geq 1$, we consider the finite dimensional sub-space of $H^{-\sigma}_0(S^1)$ defined as follows

$$E_N \equiv \{ u \in H^{-\sigma}_0(S^1) : \hat{u}(n) = 0, \; |n| > N \}.$$
Notice that the elements of $E_N$ are real valued $C^\infty(S^1)$ functions and we may identify $E_N$ with $\mathbb{R}^{2N}$ by specifying a bases of $E_N$. A canonical bases of $E_N$ is formed by $\cos(nx), \sin(nx), 1 \leq n \leq N$. One can also equip $E_N$ with a canonical measure induced by the mapping from $\mathbb{R}^{2N}$ to $E_N$ defined as follows

\begin{align}
(a_1, \cdots, a_N, b_1, \cdots, b_N) & \mapsto \sum_{n=1}^{N} (a_n \cos(nx) + b_n \sin(nx)). \tag{1.3}
\end{align}

Let us denote by $S_N$ the Dirichlet projector defined for $u \in \mathcal{D}'(S^1)$ as

\begin{align}
S_N(u) & \equiv \sum_{|n| \leq N} \hat{u}(n)e^{inx}.
\end{align}

Notice that if $u \in H^{-\sigma}(S^1)$ then $S_N(u) \in E_N$. Let us consider the following ordinary differential equation with phase space $E_N$

\begin{align}
(\partial_t + H \partial_x^2)u_N + S_N(\partial_x(u_N^2)) = 0, \quad u_N|_{t=0} = u_0 \in E_N. \tag{1.4}
\end{align}

Let us decompose $u_N(t, x)$ in the canonical bases as

\begin{align}
u_N(t, x) = \sum_{n=1}^{N} \left(a_n(t) \cos(nx) + b_n(t) \sin(nx)\right), \quad a_n(t), b_n(t) \in \mathbb{R}.
\end{align}

Then, if we set

\begin{align}
c_n(t) & \equiv \frac{1}{2}(a_n(t) - ib_n(t))
\end{align}

we can write

\begin{align}
u_N(t, x) = \sum_{0 < |n| \leq N} c_n(t)e^{inx}, \quad c_n(t) = \overline{c_{-n}(t)}.
\end{align}

Thus (1.4) is an ODE for the coefficients $c_n(t)$, $0 < |n| \leq N$. More precisely for $0 < |n| \leq N$,

\begin{align}
\dot{c}_n(t) & = -i \text{sign}(n) n^2 c_n(t) - in \sum_{0 < |n_1| \leq N, 0 < |n_2| \leq N \atop n = n_1 + n_2} c_{n_1}(t)c_{n_2}(t), \quad c_n(0) = \hat{u}_0(n). \tag{1.5}
\end{align}

Observe that the equation for $n$ is the complex conjugate of the equation for $-n$ and thus (1.5) is a system of $N$ ordinary differential equation for

\begin{align}
c(t) & \equiv (c_1(t), \cdots, c_N(t)) \in \mathbb{C}^N
\end{align}

which can be written in the form $\dot{c} = P(c)$ with $P$ a polynomial of $c, \tilde{c}$ of degree 2 (equivalently one may write an ODE of similar type for $(a_n, b_n)$). Thus we can apply the Cauchy-Lipschitz theorem for ODE’s to (1.5) and deduce that for every real valued $u_0 \in E_N$ there exists a unique local in time solutions of (1.5) on a small
time interval. Moreover, either the solution is global in time or there exists \( T \neq 0 \) such that
\[
\lim_{t \to T} \max_{0 < |n| \leq N} |c_n(t)| = \infty.
\]

Since integrations by parts give
\[
\int_{S^1} (H \partial_x^2 (u_N)) u_N = - \int_{S^1} (H \partial_x (u_N)) (\partial_x (u_N)) = 0
\]
and
\[
\int_{S^1} S_N (\partial_x (u_N^2)) u_N = \int_{S^1} \partial_x (u_N^2) u_N = \frac{2}{3} \int_{S^1} \partial_x (u_N^3) = 0,
\]
by multiplying (1.4) by \( u_N \), we obtain that
\[
\partial_t \left( \int_{S^1} u_N^2 (t, x) dx \right) = 0.
\]

Thus the local solutions of (1.4) satisfy
\[
\sum_{0 < |n| \leq N} |c_n(t)|^2 = \frac{1}{2\pi} \| u_N(t, \cdot) \|^2_{L^2(S^1)} = \frac{1}{2\pi} \| u_0 \|^2_{L^2(S^1)}.
\]

Therefore (1.6) is excluded and thus we obtain that for every \( u_0 \in E_N \) the ODE (1.4) has a unique global in time solution.

The problem (1.4) is a Hamiltonian ODE resulting from the Hamiltonian \( F \) defined by
\[
F(u_N) \equiv - \frac{1}{2} \int_{S^1} (|D_x|^{\frac{1}{2}} (u_N))^2 - \frac{1}{3} \int_{S^1} u_N^3,
\]
where for \( u \in H^s(S^1) \) the operator \( |D_x|^{\frac{1}{2}} \) is defined as Fourier multiplier by
\[
\hat{|D_x|^{\frac{1}{2}} u}(n) \equiv |n|^{\frac{1}{2}} \hat{u}(n), \quad n \in \mathbb{Z}.
\]

Notice that \( H^0_0(S^1) \) is invariant under the action of \( |D_x|^{\frac{1}{2}} \). In addition, we have that
\[
|D_x|^{\frac{1}{2}} \circ |D_x|^{\frac{1}{2}} = H \partial_x
\]
and for real valued \( u, v \in C^\infty(S^1) \),
\[
\int_{S^1} (|D_x|^{\frac{1}{2}} u)(x) v(x) dx = \int_{S^1} u(x) (|D_x|^{\frac{1}{2}} v)(x) dx.
\]

We can write (1.4) as
\[
\partial_t u_N = \frac{d}{dx} \nabla F(u_N)
\]
where $\nabla$ is the $L^2$ gradient on $E_N$. Therefore the Hamiltonian $F$ is also conserved by the flow of (1.4). Let us give a direct proof of this fact. We can write (1.4) as

$$\partial_t u_N + \partial_x (H \partial_x u_N + S_N(u_N^2)) = 0.$$  

Multiplying the last equation by $H \partial_x u_N + S_N(u_N^2)$ and integrating over $S^1$, we get

$$\int_{S^1} (\partial_t u_N)(H \partial_x u_N + S_N(u_N^2)) = 0$$

and thus

$$\frac{1}{2} \partial_t \left( \int_{S^1} |D_x| (u_N) |^2 \right) + \int_{S^1} (\partial_t u_N) S_N(u_N^2) = 0.$$  

On the other hand, using that $\partial_t u_N \in E_N$, we get

$$\int_{S^1} (\partial_t u_N) S_N(u_N^2) = \int_{S^1} (\partial_t u_N)(u_N^2) = \frac{1}{3} \partial_t (\int_{S^1} u_N^3).$$

Therefore $\partial_t (F(u_N(t, \cdot))) = 0$ which implies the Hamiltonian conservation for the solutions of (1.4).

Let us now observe that (1.5) can be written in the coordinates $a^N = (a_1, \cdots, a_N)$, $b^N = (b_1, \cdots, b_N)$ as

$$\partial_t a^N = -J_N \frac{\partial F}{\partial b^N}, \quad \partial_t b^N = J_N \frac{\partial F}{\partial a^N},$$

where $J_N = -\frac{1}{\pi} \text{diag}(1, 2, \cdots, N)$ and

$$F = F(a^N, b^N) = F\left( \sum_{n=1}^{N} (a_n \cos(nx) + b_n \sin(nx)) \right).$$

Indeed, the projection of (1.3) on the mode $\cos(nx)$ is

$$\pi \dot{a}_n + \pi n^2 b_n + \int_{S^1} \partial_x (S_N(u_N^2)) \cos(nx) dx = 0$$

and we may write

$$\int_{S^1} \partial_x (S_N(u_N^2)) \cos(nx) dx = n \int_{S^1} S_N(u_N^2(x)) \sin(nx) dx$$

$$= n \int_{S^1} u_N^2(x) \sin(nx) dx$$

$$= n \frac{\partial}{\partial b_n} \left( \frac{1}{3} \int_{S^1} u_N^3 \right).$$
On the other hand
\[ \int_{S^1} (|D_x|^{\frac{1}{2}} u_N)^2(x) \, dx = \pi \sum_{n=1}^{N} n(a_n^2 + b_n^2) \]
and thus
\[ \pi n^2 b_n = n \frac{\partial}{\partial b_n} \left( \frac{1}{2} \int_{S^1} (|D_x|^{\frac{1}{2}} u_N(x))^2 \, dx \right) \].
Therefore the projection of (1.8) on the mode \( \cos(nx) \) can be written as
\[ \pi \dot{a}_n = n \frac{\partial}{\partial b_n} \left( F \left( \sum_{n=1}^{N} (a_n \cos(nx) + b_n \sin(nx)) \right) \right) \]
Similarly
\[ \pi \dot{b}_n = -n \frac{\partial}{\partial a_n} \left( F \left( \sum_{n=1}^{N} (a_n \cos(nx) + b_n \sin(nx)) \right) \right) \]
and thus, the equation (1.8) may indeed be written in the form (1.9). Since (1.10)
\[ \frac{\partial}{\partial a_N} \left( -J_N \frac{\partial F}{\partial b_N} \right) + \frac{\partial}{\partial b_N} \left( J_N \frac{\partial F}{\partial a_N} \right) = 0 \]
the Liouville theorem for divergence free vector fields (cf. e.g. [14]) applies to (1.9), and thus to (1.4) too. More precisely, if we denote by \( \Phi_N(t) : E_N \to E_N, t \in \mathbb{R} \) the flow of (1.4) then it follows from the Liouville theorem that the Lebesgue measure \( \lambda_N \) on \( E_N \) is invariant by the flow of (1.4). Namely, for every measurable set \( A \subset E_N \) and every \( t \in \mathbb{R} \) one has \( \lambda_N(A) = \lambda_N(\Phi_N(A)) \). Since \( F \) is a conserved quantity for (1.4), we also have that for every \( \beta \in \mathbb{R} \) the Gibbs measure \( \exp(\beta F(u_N)) d\lambda_N(u_N) \) is also invariant by the flow of (1.4). Moreover since the \( L^2 \) norm of \( u_N \) is also a conserved quantity, we have that for every real constant \( c_N \) and every measurable function \( \chi_N : \mathbb{R} \to \mathbb{R} \) the measure
\[ c_N \chi_N \left( \|u_N\|_{L^2(S^1)} \right) \exp(\beta F(u_N)) d\lambda_N(u_N) \]
is also conserved by the flow of (1.4). We are going to show that for a suitable choice of \( c_N \) and \( \chi_N \) the measures (1.11), extended to \( H_0^{-\sigma}(S^1) \), tend to a limit measure which is a finite Borel measure on \( H_0^{-\sigma}(S^1) \), absolutely continuous with respect to a Wiener measure on \( H_0^{-\sigma}(S^1) \) induced by a Gaussian process.

Recall that we identify the Lebesgue measure on \( E_N \) as the image measure under the map (1.3) from \( \mathbb{R}^{2N} \) to \( E_N \). Let us next consider the measure \( d\tilde{\theta}_N \) defined as
\[ d\tilde{\theta}_N \equiv e^{-\pi \sum_{n=1}^{N} n(a_n^2 + b_n^2)} \prod_{n=1}^{N} da_n db_n \].
Notice that \( \tilde{\theta}_N(\mathbb{R}^{2N}) = (N!)^{-1} \). We then consider the probability measure

\[
d\theta_N \equiv N! \, d\tilde{\theta}_N.
\]

We still denote by \( \theta_N \) the measure on \( E_N \) induced from \( \tilde{\theta}_N \) by the mapping \( 1.3 \).

Let us fix a family \( h_n, l_n \in \mathcal{N}(0, 1) \), \( n = 1, 2, \ldots \) of independent identically distributed standard real valued Gaussian variables on a probability space \( (\Omega, \mathcal{A}, p) \). Let us observe that the measure \( \theta_N \) is the distribution of the \( E_N \) valued random variable defined as

\[
\varphi_N(\omega, x) = \sum_{n=1}^{N} \left( \hat{h}_n(\omega) \cos(nx) + \hat{l}_n(\omega) \sin(nx) \right),
\]

where \( \hat{h}_n, \hat{l}_n \in \mathcal{N}(0, 1/\sqrt{2\pi n}) \) are independent identically distributed real Gaussian random variables on \( (\Omega, \mathcal{A}, p) \). Thus we may assume that \( \hat{h}_n(\omega) = (2\pi n)^{-\frac{1}{2}} h_n(\omega) \) and \( \hat{l}_n(\omega) = (2\pi n)^{-\frac{1}{2}} l_n(\omega) \), where \( h_n, l_n \in \mathcal{N}(0, 1) \) are the fixed standard real valued Gaussians. Therefore, if we set

\[
g_n(\omega) \equiv \frac{1}{\sqrt{2}}(h_n(\omega) - il_n(\omega))
\]

then \( (g_n(\omega))_{n=1}^N \) is a sequence of standard independent identically distributed complex Gaussians and

\[
\varphi_N(\omega, x) = \sum_{0 < |n| \leq N} \frac{g_n(\omega)}{2\sqrt{\pi|n|}} e^{inx}, \quad g_n(\omega) = \overline{g_{-n}(\omega)}.
\]

Let us denote by \( L^2(\Omega; H_0^{-\sigma}(S^1)) \) the Banach space of \( H_0^{-\sigma}(S^1) \) valued functions on \( \Omega \) (the integration of such functions being understood in the sense of Bochner integrals). Clearly \( \varphi_N \) is a Cauchy sequence in \( L^2(\Omega; H_0^{-\sigma}(S^1)) \) and hence we can define

\[
\varphi(\omega, x) = \sum_{n \neq 0} \frac{g_n(\omega)}{2\sqrt{\pi|n|}} e^{inx}, \quad g_n(\omega) = \overline{g_{-n}(\omega)}.
\]

as an element of \( L^2(\Omega; H_0^{-\sigma}(S^1)) \). In particular \( \varphi(\omega, \cdot) \in H_0^{-\sigma}(S^1) \) almost surely and the map \( \omega \mapsto \varphi(\omega, \cdot) \) is measurable from \( (\Omega, \mathcal{A}) \) to \( (H_0^{-\sigma}(S^1), \mathcal{B}) \), where \( \mathcal{B} \) denotes the Borel sigma algebra of \( H_0^{-\sigma}(S^1) \). Thus \( \varphi(\omega, x) \) defines a measure \( \theta \) on \( (H_0^{-\sigma}(S^1), \mathcal{B}) \) as follows: if \( A \in \mathcal{B} \) then \( \theta(A) \equiv p(\omega : \varphi(\omega, \cdot) \in A) \). Let \( \chi_R : \mathbb{R} \rightarrow [0, 1] \) be a continuous function with compact support such that \( \chi_R(x) = 1 \) for \( |x| \leq R \). Define the measure \( d\mu_N \) on \( E_N \) as

\[
d\mu_N(u_N) = \chi_R \left( \frac{1}{2} \int_{S^1} u_N^2 \, dx \right)^{-\frac{3}{2}} \int_{S^1} u_N(x)^3 \, dx \, d\theta_N(u_N),
\]
where
\[ \alpha_N \equiv \sum_{n=1}^{N} \frac{1}{n} = \mathbb{E} \left( \| \varphi_N(\omega, \cdot) \|_{L^2(S^1)}^2 \right). \]

Notice that \( \alpha_N \) diverges as \( \log(N) \) for \( N \gg 1 \). Observe that in the coordinates \( a_n, b_n \) given by (1.3) the measure \( \mu_N \) reads
\[ N! \chi_R \left( \| u_N \|_{L^2(S^1)}^2 - \alpha_N \right) e^{2F(u_N) \prod_{n=1}^{N} da_n db_n}, \]
with
\[ u_N = \sum_{n=1}^{N} \left( a_n \cos(nx) + b_n \sin(nx) \right). \]

From the above discussion (see (1.10)) the measure \( \prod_{n=1}^{N} da_n db_n \) is invariant and since \( F \) and the \( L^2 \) norm are conserved under the flow of (1.4), we obtain that \( d\mu_N \) is invariant under the flow of (1.4).

Observe that if \( A \in \mathcal{B} \) is a Borel set of \( H^{-\sigma}(S^1) \) then \( A \cap E_N \) is a Borel set of \( E_N \) (indeed, this is clear for cylindrical sets \( A \) and then can be extends to all \( A \in \mathcal{B} \) using that \( \mathcal{B} \) is the minimal sigma algebra containing all cylindrical sets). We then define the measure \( \rho_N \) which is the natural extension of \( \mu_N \) to \( (H^{-\sigma}(S^1), \mathcal{B}) \). More precisely for every \( A \in \mathcal{B} \) which is a Borel set of \( H^{-\sigma}(S^1) \), we set
\[ \rho_N(A) \equiv \mu_N( A \cap E_N ). \]

We now can state the main result of this paper.

**Theorem 1.** The sequence
\[
\chi_R \left( \| S_N(u) \|_{L^2(S^1)}^2 - \alpha_N \right) e^{-\frac{2}{3} \int_{S^1} (S_N u)(x)^3 dx}
\]
converges in measure, as \( N \to \infty \), with respect to the measure \( \theta \). Denote by \( G(u) \) the limit of (1.13) as \( N \to \infty \). Then for every \( p \in [1, \infty[ \), \( G(u) \in L^p(d\theta(u)) \) and if we set \( d\rho(u) \equiv G(u) d\theta(u) \) then the sequence \( d\rho_N \) converges weakly to \( d\rho \) as \( N \) tends to infinity. More precisely for every continuous bounded function \( h : H^{-\sigma}(S^1) \to \mathbb{R} \) one has
\[
\int_{H^{-\sigma}(S^1)} h(u) d\rho(u) = \lim_{N \to \infty} \int_{H^{-\sigma}(S^1)} h(u) d\rho_N(u).
\]

Our approach to establish Theorem 1 is inspired by the considerations in [4]. The main point in the proof of Theorem 1 is that thanks to the mean value conservation for (1.4) the resonant part of \( \int_{S^1} u_N^3 \) disappears and thus we can get the needed integrability by using some known estimates of the second and third order Wiener
chaos. Observe that in a similar analysis in the context of the 2D NLS [4], the resonant part of the Hamiltonian should be subtracted which leads to a change of the power nonlinearity to a nonlocal one (the Wick ordering).

In order to prove that the measure $\rho$, constructed in Theorem 1 is indeed an invariant measure for the Benjamin-Ono equation a significant PDE problem should be resolved. It would be necessary to establish a well-defined dynamics of (1.2) for a typical element on the statistical ensemble. More precisely, one needs to solve almost surely in $\omega$ the Cauchy problem of (1.2) with data (1.12). Unfortunately, one can prove that the $L^2(S^1)$ of (1.12) is a.s. infinity and thus the $L^2$ well-posedness result of Molinet does not apply for this data. However, the expression (1.12) merely misses to belong to $L^2$ (it belongs a.s. to all $H^s(S^1), s < 0$). Recall that a somehow similar situation occurred in [4] and therefore it is not excluded to construct the flow of (1.2) with data (1.12) a.s. in $\omega$. Observe that local existence would suffice since one may exploit the measure invariance of $\mu_N$ under the flow of (1.4) to get a.s. global solutions (see [3]). In the final section of this paper we give several estimates confirming that one may expect to construct the flow of (1.2) a.s. for data of type (1.12).

Let us observe that one can use the ideas of this paper to perform similar constructions with the higher order conservation lows of the Benjamin-Ono equation in combination with Molinet’s well-posedness analysis. We believe that this provides invariant measures for the Benjamin-Ono equation living on regular spaces. One however still needs to use the Tao’s gauge transform for the truncated ODE in order to get uniform continuity properties of the flow map. We plan to pursue these issues elsewhere.

The remaining part of this paper is organized as follows. In the next section, we prove several elementary inequalities. In Section 3, we recall the hypercontractivity properties of the Ornstein-Uhlenbeck semi-group. In Section 4 we prove Theorem 1. In the last section we prove several PDE estimates related to the random series $\varphi(\omega, x)$ which indicate that one may conjecture that the flow of the Benjamin-Ono equation may be defined for a typical element of the statistical ensemble.

2. Elementary calculus inequalities

In this section, we collect several calculus inequalities, useful for the sequel. Similar inequalities were used systematically by many authors in the context of well-posedness for dispersive equations starting from the work of Kenig-Ponce-Vega [7].
Lemma 2.1. For every $\varepsilon > 0$ there exists $C_\varepsilon \in \mathbb{R}$ such that for every $n \in \mathbb{Z}$,
\[
\sum_{n_1 \in \mathbb{Z} \setminus \{0,n\}} \frac{1}{|n_1||n - n_1|} \leq \frac{C_\varepsilon}{(1 + |n|)^{1-\varepsilon}}.
\]

Proof. From the triangle inequality, $|n| \leq |n_1| + |n - n_1|$. Therefore, either $|n| \leq 2|n_1|$ or $|n| \leq 2|n - n_1|$. Thus it suffices to show that for every $\varepsilon$ there exists $C_\varepsilon \in \mathbb{R}$ such that uniformly in $n$,
\[
\sum_{n_1 \in \mathbb{Z} \setminus \{0,n\}} \frac{1}{|n_1||n - n_1|} \leq C_\varepsilon,
\]
\[
\sum_{n_1 \in \mathbb{Z} \setminus \{0,n\}} \frac{1}{|n_1||n - n_1|} \leq C_\varepsilon.
\]

By a change of the summation $n - n_1 \to m$ we observe that the two inequalities we have to establish are equivalent. Let us prove the second one. We consider two cases.

Case 1. Consider the summation over $n_1$ such that $|n - n_1| \geq \frac{1}{2}|n_1|$. Denote by $I$ the contribution of this region to the summation. Then
\[
I \leq \sum_{n_1 \neq 0} \frac{2^\varepsilon}{|n_1|^{1+\varepsilon}} = C_\varepsilon < \infty.
\]

Case 2. Consider the summation over $n_1$ such that $|n - n_1| \leq \frac{1}{2}|n_1|$. Denote by $II$ the contribution of this region to the sum. The restriction $|n - n_1| \leq \frac{1}{2}|n_1|$ implies that $\frac{3}{2}|n| \leq |n_1| \leq 2|n|$. Thus
\[
II \leq C \sum_{\frac{3}{2}|n| \leq |n_1| \leq 2|n| \atop n_1 \neq 0} \frac{1}{|n_1|} \leq C \log 3.
\]

This completes the proof of Lemma 2.1.

Lemma 2.2. Let us fix $\varepsilon \in [0, 1/4]$. Then there exists $C_\varepsilon > 0$ such that for every $\alpha \in \mathbb{Z}$,
\[
\sum_{n \in \mathbb{Z} \setminus \{0,\alpha\}} \frac{1}{|n|^{|\alpha|\varepsilon}||n - \alpha|^{\frac{1}{2} - \varepsilon}} \leq \frac{C_\varepsilon}{(1 + |\alpha|)^{\frac{1}{2} - \varepsilon}}.
\]

Proof. We can suppose that $\alpha \neq 0$. If $|n - \alpha| \geq \frac{|\alpha|}{2}$ then the contribution of these values of $n$ is bounded by
\[
\left(\frac{2}{|\alpha|}\right)^{\frac{1}{2} - \varepsilon} \sum_{n \neq 0} \frac{1}{|n|^{|\alpha|\varepsilon}||n - \alpha|^{\frac{1}{2} - \varepsilon}} \leq \frac{C_\varepsilon}{|\alpha|^{\frac{1}{2} - \varepsilon}}.
\]
Let us next bound the contribution of those $n$ satisfying $|n - \alpha| \leq \frac{|\alpha|}{2}$. In this case $\frac{|\alpha|}{2} \leq |n| \leq \frac{3|\alpha|}{2}$ and the contribution of those $n$ to the sum is bounded by

$$\frac{C_\epsilon}{|\alpha|^{\frac{1}{2}-\epsilon}} \sum_{\frac{|\alpha|}{2} \leq |n| \leq \frac{3|\alpha|}{2}} \frac{1}{|n||n - \alpha|^{\frac{1}{2}-\epsilon}} \leq \frac{C_\epsilon}{|\alpha|^{\frac{1}{2}-\epsilon}} \sum_{\frac{|\alpha|}{2} \leq |n| \leq \frac{3|\alpha|}{2}} \frac{1}{|n|} \leq \frac{C_\epsilon}{|\alpha|^{\frac{1}{2}-\epsilon}}$$

exactly as in the proof of Lemma 2.1. This completes the proof of Lemma 2.2. □

3. Hypercontractivity properties of the Ornstein-Uhlenbeck semi-group

In this section, we review some $L^p - L^q$ estimates for the heat flow associated to the Hartree-Fock operator $\Delta - x \cdot \nabla$ (see Proposition 3.1 below). We then obtain corollaries, known as bounds on the Wiener chaos, useful for the proof of Theorem 1. For details and background concerning the discussion of this section (in particular for the proof of Proposition 3.1), we refer to [1, 9] and the references therein.

For $d \geq 1$ an integer, we consider the Hilbert space $H \equiv L^2(\mathbb{R}^d, \exp(-|x|^2/2)dx)$ of functions on the euclidean space, square integrable with respect to the Gaussian measure. Then the operator

$$L \equiv \Delta - x \cdot \nabla = \sum_{j=1}^{d} \left( \frac{\partial^2}{\partial x_j^2} - x_j \frac{\partial}{\partial x_j} \right)$$

(3.1)

can be defined as the self adjoint realisation on $L^2(\mathbb{R}^d, \exp(-|x|^2/2)dx)$ of $\Delta - x \cdot \nabla$ with domain

$$D \equiv \left\{ u : u(x) = e^{\mid x \mid^2/4}v(x), \quad v \in D_1 \right\},$$

where

$$D_1 \equiv \left\{ v \in L^2(\mathbb{R}^d) : x^\alpha \partial^\beta v(x) \in L^2(\mathbb{R}^d), \quad \forall (\alpha, \beta) \in \mathbb{N}^{2d}, |\alpha| + |\beta| \leq 2 \right\}.$$  

Indeed, one can directly check that

$$e^{-|x|^2/4} L e^{\mid x \mid^2/4} = \Delta - \left( \frac{|x|^2}{4} - \frac{d}{2} \right).$$

(3.2)

Of course, $x^2/4$ should be seen as $1/2 \int x^2 ydy$. It is well known that $\Delta - |x|^2$ with domain $C^\infty(\mathbb{R}^d)$ is essentially self adjoint on $L^2(\mathbb{R}^d)$. Moreover $D_1$ defined above is the domain of the self adjoint extension. In addition

$$\text{spec}(\Delta - |x|^2) = \left\{ -\sum_{j=1}^{d} (2k_j + 1), \quad k_j \in \{0, 1, 2 \cdots \}, \quad j = 1, \cdots, d \right\}.$$
We now observe that if \( u \) solve \( (\Delta - |x|^2)u = \lambda u \) then \( v(x) \equiv u(x/\sqrt{2}) \) solves
\[
\left( \Delta - \left( \frac{|x|^2}{4} - \frac{d}{2} \right) \right) v = \lambda + \frac{d}{2} v.
\]
Thus, we deduce that \( \Delta - \left( \frac{|x|^2}{4} - \frac{d}{2} \right) \), with domain \( D_1 \), is self-adjoint on \( L^2(\mathbb{R}^d) \) and its spectrum is formed by the integers \( \leq 0 \). Therefore, using (3.2), we obtain that \( L \) has a self-adjoint realisation on \( L^2(\mathbb{R}^d, \exp(-|x|^2/2) dx) \) with domain \( D \). The operator \( L \) is negative with respect to the \( L^2(\mathbb{R}^d, \exp(-|x|^2/2) dx) \) scalar product.

Then the solutions of the linear PDE
\[
(3.3) \quad \partial_t u = Lu, \quad u|_{t=0} = u_0(x) \in H, \quad x \in \mathbb{R}^d, \quad t \in \mathbb{R}^+
\]
are given by the functional calculus of self-adjoint operators by the semi-group \( S(t) = \exp(tL) \), i.e. the solution of (3.3) is given by \( u(t) = S(t)u_0 \). Of course one may also define \( S(t) \) via the Hille-Yosida theorem. It turns out that \( S(t) \) satisfies an amazing “smoothing” property in the scale of \( L^p(\mathbb{R}^d, d\mu_d) \), \( p \geq 2 \), where
\[
d\mu_d(x) = (2\pi)^{-d/2} \exp(-|x|^2/2) dx
\]
(a probability measure on \( \mathbb{R}^d \)). More precisely, a solution starting from \( L^2(\mathbb{R}^d, d\mu_d) \) initial data belongs to any \( L^p(\mathbb{R}^d, d\mu_d) \), \( p > 2 \) (a space smaller than \( H \)) for sufficiently long times. Here is the precise statement.

**Proposition 3.1.** Let us fix \( p \geq 2 \). Then for every \( u_0 \in H \), every \( t \) satisfying \( t \geq \frac{1}{2} \log(p-1) \),
\[
(3.4) \quad \|S(t)u_0\|_{L^p(\mathbb{R}^d, d\mu_d)} \leq \|u_0\|_{L^2(\mathbb{R}^d, d\mu_d)}.
\]

**Remark 3.2.** The exponent 2 in the right hand-side of (3.4) may be substituted by other values \( q < p \) and then the restriction on \( t \) is \( t \geq (1/2) \log((p-1)/(q-1)) \). There is a close correspondence between (3.4) and logarithmic Sobolev inequalities for the Gaussian measure. In addition, hypercontractivity estimates of the spirit of (3.4) are known for many other heat flows.

Thanks to (3.2) the spectrum of \( L \) is formed by the integers \( \leq 0 \) and the eigenfunctions of \( L \) may be described in terms of the Hermite polynomials. The Hermite polynomial \( h_k(x) \), \( k = 0, 1, 2, \ldots \) can be defined via a generating function as
\[
\exp \left( -\lambda x - \frac{\lambda^2}{2} \right) = \sum_{k=0}^\infty \frac{\lambda^k}{\sqrt{k!}} h_k(x).
\]
Notice that \( h_0(x) = 1, h_1(x) = -x, h_2(x) = \frac{1}{\sqrt{2}}(x^2 - 1) \). In what follows, we will only need these three facts about the Hermite polynomials. A bases of eigenfunctions of \( L \) on \( H \) is given by
\[
h_k(x) = h_{k_1}(x_1)h_{k_2}(x_2)\cdots h_{k_d}(x_d),
\]
where $k = (k_1, k_2, \ldots, k_d) \in \mathbb{N}^d$ and $x = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d$. The eigenfunction $h_k$ corresponds to the eigenvalue
\[ \lambda_k = -(k_1 + \cdots + k_d). \]

The following statement will be used in the proof of Theorem 1.

**Proposition 3.3.** Set
\[ \Sigma_d \equiv \left\{ (n_1, n_2, n_3) \in \{1, \cdots, d\}^3 : n_1 \neq n_2, n_1 \neq n_3, n_2 \neq n_3 \right\}. \]

Then
\[ \|H(x)\|_{L^p(\mathbb{R}^d, d\mu_d)} \leq (p - 1)^{\frac{3}{2}} \|H(x)\|_{L^2(\mathbb{R}^d, d\mu_d)}, \]
where
\[ H(x) = \sum_{(n_1, n_2, n_3) \in \Sigma_d} c(n_1, n_2, n_3) x_{n_1} x_{n_2} x_{n_3}, \quad c(n_1, n_2, n_3) \in \mathbb{R}. \]

Proof. The function $H$ is an eigenfunction of $L$ corresponding to an eigenvalue $-3$. Therefore $S(t)H = e^{-3t}H$. Thus Proposition 3.1 yields the bound
\[ \|H\|_{L^p(\mathbb{R}^d, d\mu_d)} \leq \exp(3t)\|H\|_{L^2(\mathbb{R}^d, d\mu_d)}, \]
provided $t \geq \frac{1}{2} \log(p - 1)$. By taking $t = \frac{1}{2} \log(p - 1)$ in the above bound, we complete the proof of Proposition 3.3. \qed

Let us state another bound related to third order Wiener chaos.

**Proposition 3.4.** Set
\[ \Sigma_d \equiv \left\{ (n_1, n_2) \in \{1, \cdots, d\}^2 : n_1 \neq n_2 \right\}. \]

Then
\[ \|H(x)\|_{L^p(\mathbb{R}^d, d\mu_d)} \leq (p - 1)^{\frac{3}{2}} \|H(x)\|_{L^2(\mathbb{R}^d, d\mu_d)}, \]
where
\[ H(x) = \sum_{(n_1, n_2) \in \Sigma_d} c(n_1, n_2) x_{n_1} (x_{n_2}^2 - 1), \quad c(n_1, n_2) \in \mathbb{R}. \]

Proof. Again the function $H$ is an eigenfunction of $L$ corresponding to an eigenvalue $-3$. Therefore we can complete the proof as we did in the proof of Proposition 3.3. \qed

We will also make use of the following inequality.
Proposition 3.5. We have the bound
\[ \|H(x)\|_{L^p(\mathbb{R}^d, d\mu_d)} \leq (p - 1)\|H(x)\|_{L^2(\mathbb{R}^d, d\mu_d)}, \]
where
\[ H(x) = \sum_{n=1}^d c(n)(x_n^2 - 1), \quad c(n) \in \mathbb{R}. \]

Proof. The function \( H \) is an eigenfunction of \( L \) corresponding to an eigenvalue \(-2\). Therefore \( S(t)H = e^{-2t}H \). Thus Proposition 3.1 yields the bound
\[ \|H\|_{L^p(\mathbb{R}^d, d\mu_d)} \leq \exp(2t)\|H\|_{L^2(\mathbb{R}^d, d\mu_d)}, \]
provided \( t \geq \frac{1}{2} \log(p - 1) \). As in the proof of Proposition 3.3 by taking \( t = \frac{1}{2} \log(p - 1) \) in the above bound, we complete the proof. \( \square \)

4. Proof of Theorem 1

In order to deal with the low frequencies we will need the following distributional inequality.

Proposition 4.1. For every \( C_1 > 0 \) and \( C_2 > 0 \), \( \varepsilon > 0 \), \( \alpha > 0 \) there exist \( C > 0 \), \( c > 0 \) such that for every integer \( N \geq 1 \), every \( \lambda \geq 2 \) satisfying \( N \leq \lambda^\alpha \) one has
\[ \theta \left( u \in H_0^{-\sigma}(S^1) : \|S_Nu\|_{L^\infty(S^1)} \geq C_1\lambda, \|S_Nu\|_{L^2(S^1)}^2 \leq C_2 \log \lambda \right) \leq \frac{C}{\exp(c\lambda^{2-\varepsilon})}. \]

Proof. We will need the following Khinchin type inequality.

Lemma 4.2. Let \((l_n(\omega))_{n \in \mathbb{N}}\) be a sequence of independent identically distributed standard real Gaussian random variables. Then for every \( \lambda > 0 \), every sequence \((c_n) \in l^2(\mathbb{N})\) of real numbers,
\[ p \left( \omega : \left| \sum_{n=0}^\infty c_nl_n(\omega) \right| > \lambda \right) \leq 2e^{-2\sum c_n^2}. \]

Proof. The assertion of this lemma follows from the estimates on first order Wiener chaos considered in the previous section. It is also a consequence of the observation that \( \sum c_nl_n \) is a Gaussian in \( \mathcal{N}(0, \sigma^2) \) with \( \sigma^2 = \sum c_n^2 \). We include however here a proof of (4.1) which has the advantage to work for more general systems of independent zero mean value random variables instead of \((l_n(\omega))_{n \in \mathbb{N}}\) (such as Bernoulli variables).
For $t > 0$ to be determined later, using the independence, we obtain that

$$
\int_{\Omega} e^{t \sum_{n \geq 0} c_n l_n(\omega)} d\mu(\omega) = \prod_{n \geq 0} \int_{\Omega} e^{t c_n l_n(\omega)} d\mu(\omega)
$$

$$
= \prod_{n \geq 0} \int_{-\infty}^{\infty} e^{tc_n x} e^{-x^2/2} \frac{dx}{\sqrt{2\pi}}
$$

$$
= \prod_{n \geq 0} e^{(tc_n)^2/2} = e^{(t^2/2) \sum_n c_n^2}.
$$

Using the above calculation, we infer that

$$
e^{(t^2/2) \sum_n c_n^2} \geq e^{t \lambda} \mu(\omega : \sum_{n \geq 1} c_n l_n(\omega) > \lambda)
$$
or equivalently,

$$
\mu(\omega : \sum_{n \geq 1} c_n l_n(\omega) > \lambda) \leq e^{(t^2/2) \sum_n c_n^2} e^{-t \lambda}.
$$

Using that for $a > 0$ the minimum of $f(t) = at^2 - bt$ is $-b^2/4a$, we obtain that

$$
\mu(\omega : \sum_{n \geq 1} c_n l_n(\omega) > \lambda) \leq e^{-\lambda^2/(2 \sum_n c_n^2)}.
$$

In the same way (replacing $c_n$ by $-c_n$), we can show that

$$
\mu(\omega : \sum_{n \geq 1} c_n l_n(\omega) < -\lambda) \leq e^{-\lambda^2/(2 \sum_n c_n^2)}
$$

which completes the proof of Lemma 4.2.

Let us now give the proof of Proposition 4.1. Set

$$A_\lambda \equiv \{ u \in H_{-\sigma}^1(S^1) : \| S_N u \|_{L^\infty(S^1)} \geq C_1 \lambda, \| S_N u \|_{L^2(S^1)}^2 \leq C_2 \log \lambda \}.
$$

Observe that for $\alpha < 2$ the Sobolev embedding applied to $S_N u$ suffices to conclude that for $\lambda \gg 1$ the set $A_\lambda$ is empty. Hence the result is not trivial for $\alpha \geq 2$ (which will be the case in our application of Proposition 4.1). For $\alpha \geq 2$ the Sobolev embedding applied to $S_N u$ does not give a lower bound for $\| S_N u \|_{L^2}$ which is the main source of difficulty. Let us fix $\beta > 2\alpha$. Define the points $x_j \in S^1$, $j = 0, \cdots, [\Lambda^{\beta}]$, where $\Lambda \gg 1$ is to be fixed later by $x_j \equiv (2\pi j)/(\Lambda^{\beta})$. The number $\Lambda$ may depend on $C_1, C_2, \varepsilon, \alpha$ but should be independent of $\lambda$ and $N$. 

□
Notice that \( \text{dist}(x_j, x_{j+1}) \leq 2\pi/(\Lambda \lambda^\beta) \), where \( x_{[\Lambda \lambda^\beta]+1} \equiv x_0 \) and \( \text{dist} \) denotes the distance on \( S^1 \) (i.e. mod \( 2\pi \)). Next, we define the sets \( A_{\lambda,j} \) by

\[
A_{\lambda,j} \equiv \{ u \in H_0^{-\sigma}(S^1) : |S_N u(x_j)| \geq \frac{1}{2} C_1 \Lambda, \|S_N u\|^2_{L^2(S^1)} \leq C_2 \log \Lambda \}.
\]

We claim that for \( \Lambda \gg 1 \),

\[
(4.2) \quad A_\lambda \subset \bigcup_{j=0}^{[\Lambda \lambda^\beta]} A_{\lambda,j}.
\]

Let us prove \((4.2)\). Fix \( u \in A_\lambda \). Let \( x^* \in S^1 \) be such that

\[
|S_N u(x^*)| = \max_{x \in S^1} |S_N u(x)|.
\]

Thus \( |S_N u(x^*)| \geq C_1 \lambda \). Then there exists \( j_0 \in \{0, \cdots, [\Lambda \lambda^\beta]\} \) such that

\[
|x^* - x_{j_0}| \leq \frac{2\pi}{\Lambda \lambda^\beta}.
\]

Then we can write

\[
|S_N u(x^*) - S_N u(x_{j_0})| = \left| \int_{x_{j_0}}^{x^*} (S_N u)'(tx^* + (1-t)x_{j_0}) dt \right| \\
\leq |x^* - x_{j_0}| \frac{2}{\sqrt{\Lambda \lambda^\beta}} \| (S_N u)' \|_{L^2(S^1)} \\
\leq \frac{\sqrt{2\pi}}{\sqrt{\Lambda \lambda^\beta}} N \| S_N u \|_{L^2(S^1)} \\
\leq \frac{\sqrt{2\pi}}{\sqrt{\Lambda \lambda^\beta}} \lambda^\alpha \sqrt{C_2 \log \Lambda}.
\]

Let us choose \( \Lambda \gg 1 \) such that for every \( \lambda \geq 2 \),

\[
\frac{\sqrt{2\pi}}{\sqrt{\Lambda \lambda^\beta}} \lambda^\alpha \sqrt{C_2 \log \Lambda} \leq \frac{1}{2} C_1 \lambda.
\]

Then by the triangle inequality

\[
|S_N u(x_{j_0})| \geq |S_N u(x^*)| - |S_N u(x^*) - S_N u(x_{j_0})| \geq C_1 \lambda - \frac{1}{2} C_1 \lambda = \frac{1}{2} C_1 \lambda.
\]
Hence $u \in A_{\lambda,j_0}$ which proves (4.2). Let us next evaluate $\theta(A_{\lambda,j})$. For that purpose we will make appeal to Lemma 4.2. Observe that

$$\theta(A_{\lambda,j}) = p(\omega : \left| \sum_{n=1}^{N} (2\pi n)^{-\frac{1}{2}} (\cos(nx_j)h_n(\omega) + \sin(nx_j)l_n(\omega)) \right| \geq \frac{1}{2} C_1 \lambda,$$

$$\sum_{n=1}^{N} n^{-1}(h_n^2(\omega) + l_n^2(\omega)) \leq 2C_2 \log \lambda.$$ 

Therefore, by ignoring the $L^2$ restriction and using Lemma 4.2, we obtain that

$$\theta(A_{\lambda,j}) \leq 2e^{-\frac{(C_1 \lambda)^2}{8\kappa}},$$

where

$$\kappa = \sum_{n=1}^{N} \left( \frac{\cos^2(nx_j)}{2\pi n} + \frac{\sin^2(nx_j)}{2\pi n} \right) = \frac{1}{2\pi} \sum_{n=1}^{N} \frac{1}{n}.$$ 

Thus using that $N \leq \lambda^\alpha$, we infer that $\kappa \leq C \log \lambda$, where $C$ is independent of $N$ and $\lambda$. Therefore there exists $c > 0$, depending only on $C_1, C_2, \alpha, \varepsilon$, such that

$$\theta(A_{\lambda,j}) \leq 2e^{-c\lambda^{2-\varepsilon}}.$$

Combining (4.2) and (4.3) implies that

$$\theta(A_{\lambda,j}) \leq 2e^{-c\lambda^{2-\varepsilon}}.$$ 

Thus using that $N \leq \lambda^\alpha$, we infer that $\kappa \leq C \log \lambda$, where $C$ is independent of $N$ and $\lambda$. Therefore there exists $c > 0$, depending only on $C_1, C_2, \alpha, \varepsilon$, such that

(4.3) \hspace{1cm} \theta(A_{\lambda,j}) \leq 2e^{-c\lambda^{2-\varepsilon}}.

Combining (4.2) and (4.3) implies that

$$\theta(A_{\lambda}) \leq \sum_{j=0}^{[\Lambda \lambda^3]} \theta(A_{\lambda,j}) \leq 2(\Lambda \lambda^3 + 1)e^{-c\lambda^{2-\varepsilon}} \leq Ce^{-c\lambda^{2-\varepsilon}},$$

where $C, c > 0$ are independent of $\lambda$ and $N$. This completes the proof of Proposition 4.1. \hfill \Box

Let us define the functions $f_N : H_0^{-\sigma}(S^1) \to \mathbb{R}$ by

$$f_N(u) \equiv \int_{S^1} ((S_N u)(x))^3 \, dx.$$ 

Then we have the following statement.

**Lemma 4.3.** The sequence $(f_N)_{N \geq 1}$ is a Cauchy sequence in $L^2(H_0^{-\sigma}(S^1), \mathcal{B}, d\theta)$. More precisely, for every $\alpha < 1/2$ there exists $C > 0$ such that for every $M > N \geq 1$,

(4.4) \hspace{1cm} \left\| f_M(u) - f_N(u) \right\|_{L^2(H_0^{-\sigma}(S^1), \mathcal{B}, d\theta)} \leq CN^{-\alpha}.

Moreover, for every $M > N \geq 1$, every $p \geq 2$,

(4.5) \hspace{1cm} \left\| f_M(u) - f_N(u) \right\|_{L^p(H_0^{-\sigma}(S^1), \mathcal{B}, d\theta)} \leq Cp^{\frac{3}{2}} N^{-\alpha}.$
Denote by $f(u) \in L^2(H_0^{-\sigma}(S^1), \mathcal{B}, d\theta)$ the limit of $(f_N)_{N \geq 1}$. Let us notice that the result of Lemma 4.3 is displaying some important cancellations since using (for instance) the Fernique integrability theorem one may show that $\int_{S^1} |u|^3 = \infty$, $\theta$ a.s.

Proof of Lemma 4.3. Write

$$\|f_N\|^2_{L^2(H_0^{-\sigma}(S^1), \mathcal{B}, d\theta)} = \int_{H_0^{-\sigma}(S^1)} \int_{S^1} ((S_N u)(x))^3 dx \, d\theta(u)$$

$$= \int_{\Omega} \int_{S^1} ((S_N \varphi(\omega, x))^3 dx \, dp(\omega),$$

where $\varphi(\omega, x)$ is defined by (1.12). For $N \geq 2$, we set

$$\Sigma(N) \equiv ((n_1, n_2, n_3) \in \mathbb{Z}^3 : n_1 + n_2 + n_3 = 0, 0 < |n_1|, |n_2|, |n_3| \leq N).$$

Then

$$\int_{S^1} (S_N \varphi(\omega, x))^3 dx = \frac{1}{8\pi^2} \sum_{(n_1, n_2, n_3) \in \Sigma(N)} \frac{g_{n_1}(\omega) g_{n_2}(\omega) g_{n_3}(\omega)}{\sqrt{|n_1|} \sqrt{|n_2|} \sqrt{|n_3|}}.$$

Next we define $\Sigma_1(N)$ as follows

$$\Sigma_1(N) \equiv ((n_1, n_2, n_3) \in \Sigma(N) : n_1 \neq \pm n_2, n_1 \neq \pm n_3, n_2 \neq \pm n_3).$$

Observe that triples of the form $(n, -n, 0)$ can not belong to $\Sigma(N)$ and therefore, we may write

$$\int_{S^1} (S_N \varphi(\omega, x))^3 = F_1(N, \omega) + F_2(N, \omega),$$

$$F_1(N, \omega) \equiv \frac{3}{8\pi^2} \sum_{0 < |n| \leq N/2} \frac{g_n^2(\omega) g_{2n}(\omega)}{|n|^{3/2}}$$

is the contribution of the terms $(n, n, -2n)$, $(n, -2n, n)$ and $(-2n, n, n)$ and

$$F_2(N, \omega) \equiv \frac{1}{8\pi^2} \sum_{(n_1, n_2, n_3) \in \Sigma_1(N)} \frac{g_{n_1}(\omega) g_{n_2}(\omega) g_{n_3}(\omega)}{\sqrt{|n_1|} \sqrt{|n_2|} \sqrt{|n_3|}}.$$

is the contribution of the remaining terms. Since

$$\|f_M - f_N\|^2_{L^2(H_0^{-\sigma}(S^1), \mathcal{B}, d\omega)}$$

$$= \int_{\Omega} \left| \int_{S^1} ((S_M \varphi(\omega, x))^3 dx - \int_{S^1} ((S_N \varphi(\omega, x))^3 dx \right|^2 dp(\omega),$$
it suffices to show that \((F_j(N, \cdot))_{N \geq 1}, j = 1, 2\) are Cauchy sequences in \(L^2(\Omega)\) satisfying bounds of type \((4.4), (4.5)\). Using the Hölder inequality in the \(\Omega\) integration, we may write

\[
\|F_1(M, \omega) - F_1(N, \omega)\|_{L^2(\Omega)} \leq C \sum_{N/2 < |n| \leq M/2} \frac{\|g_n\|_{L^2(\Omega)}^2 \|g_{2n}\|_{L^2(\Omega)}}{|n|^{3/2}} \\
\leq C \sum_{N/2 < |n| \leq M/2} \frac{1}{|n|^{3/2}} \\
\leq C \frac{\alpha}{N^\alpha} \text{ (recall that } \alpha < 1/2). \]

Thus \((F_1(N, \cdot))_{N \geq 1}\) is a Cauchy sequence in \(L^2(\Omega)\) with the needed quantitative bound. Let us next analyse \(F_2(N, \omega)\). For that purpose, in contrast with \(F_1(N, \omega)\), an orthogonality argument will be needed. For \(M > N \geq 1\), we set

\[
\Lambda(N, M) \equiv \{(n_1, n_2, n_3) \in \mathbb{Z}^3 : n_1 + n_2 + n_3 = 0, n_1 \neq \pm n_2, n_1 \neq \pm n_3, n_2 \neq \pm n_3 \quad 0 < |n_1|, |n_2|, |n_3| \leq M, \max(|n_1|, |n_2|, |n_3|) > N). \]

Therefore, we can write

\[
F_2(M, \omega) - F_2(N, \omega) = \frac{1}{8\pi^2} \sum_{(n_1, n_2, n_3) \in \Lambda(N, M)} \frac{g_{n_1}(\omega)g_{n_2}(\omega)g_{n_3}(\omega)}{\sqrt{|n_1|} \sqrt{|n_2|} \sqrt{|n_3|}}. \]

Observe that if \((n_1, n_2, n_3)\) and \((m_1, m_2, m_3)\) are two triples from \(\Lambda(N, M)\) such that \(\{n_1, n_2, n_3\} \neq \{m_1, m_2, m_3\}\) then

\[
(4.6) \quad \int_\Omega g_{n_1}(\omega)g_{n_2}(\omega)g_{n_3}(\omega)g_{m_1}(\omega)g_{m_2}(\omega)g_{m_3}(\omega)dp(\omega) = 0. \]

Indeed, using the independence, if \(n_{j_1} = -m_{j_2}\) for some \(j_1, j_2 \in \{1, 2, 3\}\) then the integral \((4.6)\) is zero since \(\int_\Omega g_{n_{j_1}}^2(\omega)dp(\omega) = 0\) and \(\pm n_{j_1}\) can not belong to the remaining indexes. In all other cases there is one of the indexes \((n_1, n_2, n_3, m_1, m_2, m_3)\) which is repeated only once and its opposite does not belong to \(\{n_1, n_2, n_3, m_1, m_2, m_3\}\). Therefore, we can write

\[
\|F_2(M, \omega) - F_2(N, \omega)\|_{L^2(\Omega)}^2 \leq C \sum_{(n_1, n_2, n_3) \in \Lambda(N, M)} \frac{1}{|n_1|} \frac{1}{|n_2|} \frac{1}{|n_3|} \\
\leq C \sum_{n_1 \in \mathbb{Z}} \sum_{|n_2| \geq N} \frac{1}{(1 + |n_1|)(1 + |n_2|)(1 + |n_1 + n_2|)}. \]
Using Lemma [2.1] we infer that
\[
\sum_{n_1 \in \mathbb{Z}} \frac{1}{(1 + |n_1|)(1 + |n_1 + n_2|)} \leq \frac{C_\varepsilon}{(1 + |n_2|)^{1-\varepsilon}} + \frac{2}{(1 + |n_2|)} < \frac{C_\varepsilon + 2}{(1 + |n_2|)^{1-\varepsilon}}.
\]
Therefore
\[
\|F_2(M, \omega) - F_2(N, \omega)\|_{L^2(\Omega)}^2 \leq C \sum_{|n_2| \geq N} \frac{1}{(1 + |n_2|)^{2-\varepsilon}} \leq \frac{C_\alpha}{N^{2\alpha}},
\]
provided $1 - 2\alpha > \varepsilon > 0$. Therefore $(F_2(\omega, N))_{N \geq 1}$ is a Cauchy sequence in $L^2(\Omega)$ with the needed quantitative bound. This completes the proof of (4.4). Let us now turn to the proof of (4.5). Write via the triangle inequality,
\[
\|f_M - f_N\|_{L^p(H_0^{-\alpha}(S^1), B,d\theta)}^p = \int_\Omega \left| \int_{S^1} ((S_M \varphi(\omega, x)) \right)^3 dx - \int_{S^1} ((S_N \varphi(\omega, x)) \right)^3 dx \right| dp(\omega)
\leq \left( \|F_1(M, \omega) - F_1(N, \omega)\|_{L^p(\Omega)} + \|F_2(M, \omega) - F_2(N, \omega)\|_{L^p(\Omega)} \right)^p.
\]
Write
\[
F_1(M, \omega) - F_1(N, \omega) = \frac{3}{4\pi^2} \sum_{N/2 < n \leq M/2} \frac{\text{Re}(g_n^2(\omega) g_{2n}(\omega))}{|n|^{\frac{3}{2}}}.
\]
Recall that $g_n(\omega) = \frac{1}{\sqrt{2}}(h_n(\omega) - il_n(\omega))$ and thus one may directly check that
\[
\text{Re}(g_n^2(\omega) g_{2n}(\omega)) = \frac{1}{2\sqrt{2}} \left( (h_n^2(\omega) - 1)h_{2n}(\omega) - (l_n^2(\omega) - 1)h_{2n}(\omega) + 2h_n(\omega)l_n(\omega)l_{2n}(\omega) \right).
\]
Hence we are in the scope of applicability of Propositions 3.3 and 3.4. Consider $\mathbb{R}^{2M}$ parametrized by $(x_1, \ldots, x_M, y_1, \ldots, y_M)$, where $(x_1, \ldots, x_M)$ correspond to the $h_n(\omega)$, $n = 1, \ldots, M$ and where $(y_1, \ldots, y_M)$ correspond to the $l_n(\omega)$, $n = 1, \ldots, M$. Then we will apply Proposition 3.4 (with $d = 2M$) to the function
\[
H_1(x_1, \ldots, x_M, y_1, \ldots, y_M) \equiv \frac{3}{8\sqrt{2\pi^2}} \sum_{N/2 < n \leq M/2} |n|^{-\frac{3}{2}} ((x_n^2 - 1)x_{2n} - (y_n^2 - 1)y_{2n})
\]
and Proposition 3.3 to the function
\[
H_2(x_1, \ldots, x_M, y_1, \ldots, y_M) \equiv \frac{3}{4\sqrt{2\pi^2}} \sum_{N/2 < n \leq M/2} |n|^{-\frac{3}{2}}x_n y_n y_{2n}.
\]
Indeed,
\[ F_1(M, \omega) - F_1(N, \omega) = H_1(h_1(\omega), \cdots, h_M(\omega), l_1(\omega), \cdots, l_M(\omega)) + H_2(h_1(\omega), \cdots, h_M(\omega), l_1(\omega), \cdots, l_M(\omega)). \]

Using the independence, we may write that for \( j = 1, 2, \)
\[
\|H_j(h_1(\omega), \cdots, h_M(\omega), l_1(\omega), \cdots, l_M(\omega))\|_{L^p(\Omega)} = \\
\|H_j(x_1, \cdots, x_M, y_1, \cdots, y_M)\|_{L^p(\mathbb{R}^{2M}(2\pi)^{-M} \exp(-\frac{1}{2}\sum_{n=1}^{M}(x_n^2 + y_n^2)))} dx_1 \cdots dx_M.
\]

Therefore, using Proposition 3.3 and Proposition 3.4 by splitting \( F_1(M, \omega) - F_1(N, \omega) \) into two parts, we obtain that
\[
\|F_1(M, \omega) - F_1(N, \omega)\|_{L^p(\Omega)} \leq C p^{3/2} \|F_1(M, \omega) - F_1(N, \omega)\|_{L^2(\Omega)} \leq C_\alpha p^{3/2} N^{-\alpha},
\]
where \( C_\alpha \) is independent of \( p, M \) and \( N \). Similarly, by developing the product
\[
g_{n_1}(\omega)g_{n_2}(\omega)g_{n_3}(\omega)
\]
for \((n_1, n_2, n_3) \in \Lambda(N, M)\) we observe that the difference \( F_2(M, \omega) - F_2(N, \omega) \) fits in the scope of applicability of Proposition 3.3. We obtain that
\[
\|F_2(M, \omega) - F_2(N, \omega)\|_{L^p(\Omega)} \leq C p^{3/2} \|F_2(M, \omega) - F_2(N, \omega)\|_{L^2(\Omega)} \leq C_\alpha p^{3/2} N^{-\alpha}.
\]
Thus (4.5) is established. This completes the proof of Lemma 4.3.

We have the following standard corollary of Lemma 4.3.

**Corollary 4.4.** Under the assumption of Lemma 4.3, the sequence \((f_N)_{N \geq 1}\) converges in measure to \( f \). More precisely, for every \( \varepsilon > 0, \)
\[
\lim_{N \to \infty} \theta(u \in H_0^{-\alpha}(S^1) : |f(u) - f_N(u)| > \varepsilon) = 0.
\]

**Proof.** This is a consequence of the Tchebishev inequality. \( \square \)

The next lemma is a general feature.

**Lemma 4.5.** Let \( F \) be a real valued measurable function on \( H_0^{-\alpha}(S^1) \). Suppose that there exist \( \alpha > 0, N > 0, k \in \mathbb{N}^* \) and \( C > 0 \) such that for every \( p \geq 2 \) one has
\[
\|F\|_{L^p(d\theta)} \leq CN^{-\alpha} p^{k/2}.
\]
Then there exists \( \delta > 0 \) and \( C_1 > 0 \) depending on \( C \) and \( k \) but independent of \( N \) and \( \alpha \) such that
\[
\int_{H_0^{-\alpha}(S^1)} e^{\delta N \frac{p}{2}} |F(u)|^2 d\theta(u) \leq C_1.
\]
As a consequence for $\lambda > 0$,

$$\theta(u \in H_0^{-\sigma}(S^1) : |F(u)| > \lambda) \leq C_1 e^{\frac{-\delta N^2 \sigma}{k^2} \lambda^2}.$$  \hfill (4.9)

**Proof.** If one is only interested to get (4.9) then it suffices to use the Tchebishev inequality in the context of (4.7) with a suitable $p$ (depending of $\lambda$). Let us now give the proof of the claimed statement (4.8). Write

$$e^{\delta N^2 \sigma |F(u)|^2} = \sum_{n=0}^{k-1} \frac{\delta^n N^{2\alpha n} |F(u)|^{\frac{2n}{k}}}{n!} + \sum_{n=k}^{\infty} \frac{\delta^n N^{2\alpha n} |F(u)|^{\frac{2n}{k}}}{n!}.$$  

If $k \geq 2$, using the Hölder inequality and (4.7), we get for $n = 1, \ldots, k - 1$,

$$\int_{H_0^{-\sigma}(S^1)} |F(u)|^{\frac{2n}{k}} d\theta(u) \leq \|F\|_{L^{\frac{2n}{k}}(d\theta)}^{\frac{2n}{k}} \leq \left[ CN^{-\alpha} (2n)^{\frac{k}{2}} \right]^{\frac{2n}{k}} = C^{\frac{2n}{k}} N^{-\frac{2\alpha n}{k}} (2n)^n.$$  

The Stirling formula provides the existence of a positive constant $\tilde{C}$ such that for every integer $n \geq 1$,

$$\frac{n^n}{n!} \leq \tilde{C} e^{\sqrt{n}}.$$

Therefore, by using (4.7), we obtain that for $n \geq k$,

$$\int_{H_0^{-\sigma}(S^1)} \frac{\delta^n N^{2\alpha n} |F(u)|^{\frac{2n}{k}}}{n!} d\theta(u) \leq \frac{\delta^n N^{2\alpha n} |F(u)|^{\frac{2n}{k}}}{n!} \left[ CN^{-\alpha} (2n)^{\frac{k}{2}} \right]^{\frac{2n}{k}} = \frac{n^n}{n!} \left( \frac{\tilde{C}}{k} e^{\sqrt{n}} \right)^n \leq \frac{\tilde{C}}{\sqrt{n}} \left( \frac{2}{k} C^{\frac{2n}{k}} e^{\sqrt{n}} \right)^n.$$

Summarizing the preceding gives that for $k \geq 2$,

$$\int_{H_0^{-\sigma}(S^1)} e^{\delta N^2 \sigma |F(u)|^2} d\theta(u) \leq 1 + \sum_{n=0}^{k-1} \frac{C^{\frac{2n}{k}} (2n)^n}{n!} \delta^n + \tilde{C} \sum_{n=k}^{\infty} \left( \frac{2}{k} C^{\frac{2n}{k}} e^{\sqrt{n}} \right)^n \leq C_1,$$

provided that $\delta > 0$ is such that

$$\delta < \frac{k}{2C^{\frac{2n}{k}} e^{\sqrt{n}}}.$$  

For $k = 1$, the same bound holds by replacing the term

$$\sum_{n=0}^{k-1} \frac{C^{\frac{2n}{k}} (2n)^n}{n!} \delta^n$$

in the above inequality by zero. This completes the proof of Lemma 4.5. \hfill $\square$

Lemma 4.5 implies the following distributional inequality for $(f_N)_{N \geq 1}$. \hfill $\Box$
Lemma 4.6. For every $\alpha < 1/2$ there exists $C > 0$ and $\delta > 0$ such that for every $M > N \geq 1$, every $\lambda > 0$

$$\theta(u \in H^{-\sigma}_0(S^1) : |f_M(u) - f_N(u)| > \lambda) \leq Ce^{-\delta(N^\alpha \lambda)^{2/3}}.$$ 

Proof. It suffices to combine Lemma 4.3 and Lemma 4.5. □

We next study the limit of $\|S_N(u)\|_{L^2(S^1)}^2 - \alpha_N$ as $N \to \infty$. Let us define the functions $g_N : H^{-\sigma}_0(S^1) \to \mathbb{R}$ by

$$g_N(u) \equiv \|S_N(u)\|_{L^2(S^1)}^2 - \alpha_N.$$ 

We have the following statement.

Lemma 4.7. The sequence $(g_N)_{N \geq 1}$ is a Cauchy sequence in $L^2(H^{-\sigma}_0(S^1), B, d\theta)$. More precisely, there exists $C > 0$ such that for every $M > N \geq 1$,

$$(4.11) \quad \left\|g_M(u) - g_N(u)\right\|_{L^2(H^{-\sigma}_0(S^1), B, d\theta)} \leq CN^{-\frac{1}{2}}.$$ 

Moreover, if we denote by $g(u)$ the limit of $g_N(u)$ in $L^2(H^{-\sigma}_0(S^1), B, d\theta)$ then $g_N(u)$ converges to $g(u)$ in measure:

$$\forall \varepsilon > 0, \ \lim_{N \to \infty} \theta(u \in H^{-\sigma}_0(S^1) : |g(u) - g_N(u)| > \varepsilon) = 0.$$ 

Proof. Write

$$\|g_M - g_N\|_{L^2(H^{-\sigma}_0(S^1), B, d\theta)}^2 = \frac{1}{4} \int_{\Omega} \left| \sum_{N < |n| \leq M} \frac{|g_n(u)|^2 - 1}{|n|} \right|^2 dp(\omega).$$

Thanks to the independence and the normalization of $(g_n(\omega))$ we obtain that for $n_1 \neq n_2$ one has

$$\int_{\Omega} (|g_{n_1}(\omega)|^2 - 1)(|g_{n_2}(\omega)|^2 - 1) dp(\omega) = 0.$$ 

Therefore

$$\|g_M - g_N\|_{L^2(H^{-\sigma}_0(S^1), B, d\theta)}^2 = \sum_{N < |n| \leq M} \frac{c}{|n|^2} \leq \frac{C}{N}.$$ 

This proves 4.11. The convergence of $(g_N(u))$ in measure follows from the Chebyshev inequality. This completes the proof of Lemma 4.7. □

We now prove a distributional inequality for $(g_N)_{N \geq 1}$. 

Lemma 4.8. There exist $C > 0$ and $\delta > 0$ such that for every $M > N \geq 1$, every $\lambda > 0$

$$\theta(u \in H_0^{-\sigma}(S^1) : |g_M(u) - g_N(u)| > \lambda) \leq Ce^{-\delta N^{\frac{1}{2}} \lambda}.$$  

Proof. We have

$$\|g_M - g_N\|_{L^p(H_0^{-\sigma}(S^1), \mathcal{B}, d\theta)}^p = \left(\frac{1}{2}\right)^p \int_\Omega \left| \sum_{N<n\leq M} \frac{|g_n(\omega)|^2 - 1}{n} \right|^p dp(\omega).$$

Recall that $g_n(\omega) = \frac{1}{\sqrt{2}}(h_n(\omega) - il_n(\omega))$ and thus

$$|g_n(\omega)|^2 - 1 = \frac{1}{2}(h_n^2(\omega) - 1) + \frac{1}{2}(l_n^2(\omega) - 1).$$

Therefore, using Proposition 3.5 and (4.11), we obtain that

$$\|g_M - g_N\|_{L^p(H_0^{-\sigma}(S^1), \mathcal{B}, d\theta)} \leq CpN^{\frac{1}{2}}.$$  

A use of Lemma 4.5 completes the proof of Lemma 4.8.  

Combining Lemma 4.3 and Lemma 4.7, we may define the function

$$G : H_0^{-\sigma}(S^1) \rightarrow \mathbb{R}$$

by

$$G(u) \equiv \chi_R(g(u))e^{-\frac{2}{3} f(u)}.$$  

We then have that $G(u)$ is the limit in measure, as $N \rightarrow \infty$, of

$$\chi_R\left(\left\|S_N(u)\right\|_{L^2(S^1)}^2 - \alpha_N\right)e^{-\frac{2}{3} \int_{S^1} (S_Nu(x))^3 dx}.$$  

Indeed since $\chi_R(x)$ and $e^{-\frac{2}{3} x}$ are continuous real functions, we have that $\chi_R(g_N(u))$ and $e^{-\frac{2}{3} f_N(u)}$ converge in the $\theta$ measure to $\chi_R(g(u))$ and $e^{-\frac{2}{3} f(u)}$ respectively. Then we use that the convergence in measure is stable with respect to the product operation to conclude that indeed (4.12) converges to $G(u)$ in measure. Thus the function $G$ is measurable from $(H_0^{-\sigma}(S^1), \mathcal{B}, d\theta)$ to $\mathbb{R}$. We are going to show that in fact $G \in L^p(H_0^{-\sigma}(S^1), \mathcal{B}, d\theta)$ for all finite $p \geq 1$. The main point is the following statement.

**Proposition 4.9.** Let $1 \leq p < \infty$. Then there exists $C > 0$ such that for every $N \geq 1$,

$$\left\|\chi_R\left(\left\|S_Nu\right\|_{L^2(S^1)}^2 - \alpha_N\right)e^{-\frac{2}{3} \int_{S^1} (S_Nu(x))^3 dx}\right\|_{L^p(d\theta(u))} \leq C.$$
Proof. Our goal is to evaluate the function \( \theta(A_\lambda) \), where
\[
A_\lambda \equiv \left( u \in H_0^{-\sigma}(S^1) : \chi_R(\|S_N u\|_{L^2(S^1)}^2 - \alpha_N) e^{-\frac{2}{\theta} \int_{S^1} (S_N u)^3 dx} > \lambda \right)
\]
for \( \lambda \geq 200 \). More precisely, we need to show the convergence and the uniform with respect to \( N \) boundedness of the integral \( \int_0^\infty \lambda^{p-1} \theta(A_\lambda) d\lambda \). Set
\[
(4.13) \quad N_0 \equiv (\log \lambda)^2.
\]
Suppose first that \( N_0 \geq N \). Using the Hölder inequality, we get for \( u \in A_\lambda \),
\[
\left| \int_{S^1} (S_N u)(x)^3 dx \right| \leq C \|S_N u\|_{L^2(S^1)}^2 \|S_N u\|_{L^\infty(S^1)} \leq C \alpha_N \|S_N u\|_{L^\infty(S^1)}
\]
\[
\leq C \log(N) \|S_N u\|_{L^\infty(S^1)} \leq C(\log \log \lambda) \|S_N u\|_{L^\infty(S^1)}.
\]
Hence for every \( \delta > 0 \) there exist \( C \) and \( c \), independent of \( N \), such that
\[
\theta(A_\lambda) \leq \theta \left( u \in H_0^{-\sigma}(S^1) : \|S_N u\|_{L^\infty(S^1)} \geq c(\log \lambda)^{1-\delta} \right) \|S_N u\|_{L^\infty(S^1)} \leq C \log \log \lambda).
\]
Thus, using Proposition 4.1 (with \( (\log \lambda)^{1-\delta} \) instead of \( \lambda \)), we infer that for every \( \varepsilon > 0 \) there exist \( C > 0 \), \( c > 0 \) such that \( \theta(A_\lambda) \leq C \exp(-c(\log \lambda)^{2-\varepsilon}) \leq C L \lambda^{-L} \)
which yields the needed uniform integrability property.

We can therefore suppose in the sequel of the proof that \( N > N_0 \), where \( N_0 \) is defined by (4.13). Consider the set
\[
B_{\lambda, \kappa} \equiv \left( u \in H_0^{-\sigma}(S^1) : |g_N(u) - g_{N_0}(u)| > \kappa \right),
\]
where \( g_N \) is defined by (4.10) and \( \kappa \) is a large constant. Lemma 4.8 yields
\[
\theta(B_{\lambda, \kappa}) \leq C e^{-\delta \kappa(\log \lambda)} = C \lambda^{-\delta \kappa}.
\]
Therefore if \( \kappa \gg 1 \) then \( \mu(B_{\lambda, \kappa}) \leq C \lambda^{-p-10} \). Hence it suffices to evaluate \( \theta(A_\lambda \setminus B_{\lambda, \kappa}) \).
Let us observe that for \( u \in A_\lambda \setminus B_{\lambda, \kappa} \) one has
\[
\|S_{N_0} u\|_{L^2(S^1)}^2 = (\|S_N u\|_{L^2(S^1)}^2 - \alpha_N) - (g_N(u) - g_{N_0}(u)) + \alpha_{N_0}
\]
\[
\leq C + \kappa + C \log(N_0) \leq C \log \log \lambda.
\]
Therefore \( A_\lambda \setminus B_{\lambda, \kappa} \subset C_\lambda \) where
\[
C_\lambda \equiv \left( u \in H_0^{-\sigma}(S^1) : \left| \int_{S^1} (S_N u)(x)^3 dx \right| \geq \frac{3}{2} \log \lambda, \|S_{N_0} u\|_{L^2(S^1)}^2 \leq C \log \log \lambda \right).
\]
We next observe that \( C_\lambda \subset D_\lambda \cup E_\lambda \), where
\[
D_\lambda \equiv \left( u \in H_0^{-\sigma}(S^1) : \left| \int_{S^1} (S_N u)(x)^3 dx \right| \geq \frac{1}{2} \log \lambda, \|S_{N_0} u\|_{L^2(S^1)}^2 \leq C \log \log \lambda \right).
\]
and
\[ E_{\lambda} \equiv \left\{ u \in H_{0}^{-\sigma}(S^1) : \left| \int_{S^1} (S_{N_{0}}u)(x)^{3}dx - \int_{S^1} (S_{N_{0}}u)(x)^{3}dx \right| \geq \frac{1}{2} \log \lambda \right\}. \]

Using Lemma 4.6 we obtain that for every \( \alpha < 1/2 \) there exists \( C > 0 \) and \( \delta > 0 \) such that \( \theta(E_{\lambda}) \leq C e^{-\delta(N_{0} \log \lambda)^{2/3}} \leq C_{L} \lambda^{-L} \) by taking \( \alpha \) close enough to \( 1/2 \) (recall that \( N_{0} = (\log \lambda)^{2} \)). Hence it only remains to evaluate \( \theta(D_{\lambda}) \). Using the Hölder inequality, we obtain that for every \( \delta > 0 \) one has
\[
\left| \int_{S^1} (S_{N_{0}}u)(x)^{3}dx \right| \leq \|S_{N_{0}}u\|_{L^\infty(S^1)} \|S_{N_{0}}u\|_{L^2(S^1)}^{2} \leq C \log \lambda \|S_{N_{0}}u\|_{L^\infty(S^1)}.
\]

Therefore for every \( \delta > 0 \) there exists \( C \) and \( c \) such that
\[
\theta(D_{\lambda}) \leq \theta \left( u \in H_{0}^{-\sigma}(S^1) : \|S_{N_{0}}u\|_{L^\infty(S^1)} \geq c(\log \lambda)^{1-\delta}, \|S_{N_{0}}u\|_{L^2(S^1)} \leq C \log \lambda \right).
\]

Using once again Proposition 4.1 we infer that for every \( \varepsilon > 0 \) there exist \( C > 0 \), \( c > 0 \) such that \( \theta(D_{\lambda}) \leq C \exp(-c(\log \lambda)^{2-\varepsilon}) \leq C_{L} \lambda^{-L} \). Hence we conclude that
\[
\theta(A_{\kappa} \setminus B_{\lambda,\varepsilon}) \leq \theta(C_{\lambda}) \leq \theta(D_{\lambda}) + \theta(E_{\lambda}) \leq C_{L} \lambda^{-L}.
\]

This completes the proof of Proposition 4.9. \( \square \)

Let us now consider the sequence of measurable functions from \((H_{0}^{-\sigma}(S^1), B)\) to \( \mathbb{R} \) defined as
\[
G_{N}(u) \equiv \chi_{R}(\|S_{N_{0}}u\|_{L^2(S^1)}^{2} - \alpha_{N}) e^{-\frac{3}{4} \int_{S^1} (S_{N_{0}}u)(x)^{3}dx}.
\]

Since \( G_{N} \) converges to \( G \) in measure, we obtain that there exists a subsequence \( N_{k} \) such that
\[
G(u) = \lim_{k \to \infty} G_{N_{k}}(u), \quad \text{\( \theta \) a.s.}
\]

Proposition 4.10 implies that there exists a constant \( C \) such that
\[
\|G_{N_{k}}(u)\|_{L^{p}(d\theta(u))} \leq C, \quad \forall k \in \mathbb{N}.
\]

Hence Fatou’s lemma implies that \( G(u) \in L^{p}(d\theta(u)) \) and moreover
\[
\int_{H_{0}^{-\sigma}(S^1)} |G(u)|^{p}d\theta(u) \leq \liminf_{k \to \infty} \int_{H_{0}^{-\sigma}(S^1)} |G_{N_{k}}(u)|^{p}d\theta(u).
\]

Let now \( h \) be a bounded continuous function from \( H_{0}^{-\sigma}(S^1) \) to \( \mathbb{R} \). Our goal is to show that
\[
\lim_{N \to \infty} \int_{H_{0}^{-\sigma}(S^1)} G_{N}(u)h(u)d\theta(u) = \int_{H_{0}^{-\sigma}(S^1)} G(u)h(u)d\theta(u).
\]

Let us fix \( \varepsilon > 0 \). Consider the set
\[
A_{N,\varepsilon} \equiv \left\{ u \in H_{0}^{-\sigma}(S^1) : |G_{N}(u) - G(u)| \leq \varepsilon \right\}.
\]
Denote by \( A_{N,\varepsilon}^c \) the complementary set in \( H_0^{-\sigma}(S^1) \) of \( A_{N,\varepsilon} \). Then, using that \( h \) is bounded, Proposition 4.9 and the Cauchy-Schwarz inequality, we infer that
\[
\left| \int_{A_{N,\varepsilon}^c} (G_N(u) - G(u))h(u)d\theta(u) \right| \leq C\|G_N - G\|_{L^2(d\theta)}[\theta(A_{N,\varepsilon}^c)]^\frac{1}{2} \leq C[\theta(A_{N,\varepsilon}^c)]^\frac{1}{2},
\]
where \( C \) is independent of \( N \) and \( \varepsilon \). On the other hand
\[
\left| \int_{A_{N,\varepsilon}} (G_N(u) - G(u))h(u)d\theta(u) \right| \leq C\varepsilon
\]
and thus we have (4.14) since the convergence in measure of \( G_N \) to \( G \) implies that for a fixed \( \varepsilon \),
\[
\lim_{N \to \infty} \theta(A_{N,\varepsilon}^c) = 0.
\]
This completes the proof of Theorem 1. \[\square\]

Let us observe that for \( R \gg 1 \) the measure \( d\rho(u) \) is not trivial. Indeed, by the estimates on second order Wiener chaos (see Lemma 4.8) we infer that
\[
\theta\left( u \in H_0^{-\sigma}(S^1) : \|S_Nu\|^2_{L^2(S^1)} - \alpha_N > R \right) \leq Ce^{-\delta R},
\]
where \( C > 0 \) and \( \delta > 0 \) are independent of \( R \). Since for \( R \geq 3, \)
\[
\left( u \in H_0^{-\sigma}(S^1) : |g(u)| > R \right) \subset \left( u \in H_0^{-\sigma}(S^1) : |g_N(u)| > R - 2 \right)
\]
\[
\cup \left( u \in H_0^{-\sigma}(S^1) : |g(u) - g_N(u)| > 1 \right)
\]
using the convergence in measure of \( g_N(u) \) to \( g(u) \), we obtain that for \( R \gg 1 \) the set \( (u \in H_0^{-\sigma}(S^1) : |g(u)| \leq R) \) is of positive \( \theta \) measure and thus \( d\rho(u) \) is a nontrivial measure since its (non-negative) density is not vanishing on a set of positive \( \theta \) measure. The result of Theorem 1 implies some additional properties of the convergence of \( \rho_N \) to \( \rho \). For instance we have the following statement.

**Proposition 4.10.** Let \( U \) be an open set of \( H^{-\sigma}(S^1) \). Then
\[
\liminf_{N \to \infty} \rho_N(U) \geq \rho(U).
\]
Let \( V \) be a closed set of \( H^{-\sigma}(S^1) \). Then
\[
\rho(V) \geq \limsup_{N \to \infty} \rho_N(V).
\]
**Proof.** Applying Theorem 1 to \( h = 1 \), we obtain that
\[
\int_{H_0^{-\sigma}(S^1)} G(u)d\theta(u) = \lim_{N \to \infty} \int_{H_0^{-\sigma}(S^1)} G_N(u)d\theta(u).
\]
We set
\[ \beta_N \equiv \int_{H_0^{-\sigma}(S^1)} G_N(u)d\theta(u), \quad \beta \equiv \int_{H_0^{-\sigma}(S^1)} G(u)d\theta(u). \]

If $\beta = 0$ the assertion is trivial. We can therefore suppose that $\beta \neq 0$ and that there exists $N_0$ such that $\beta_N \neq 0$, $\forall N \geq N_0$. Next, we define the probability measures on $(H_0^{-\sigma}(S^1), \mathcal{B})$ as
\[ d\tilde{\rho}_N \equiv \beta_N^{-1}d\rho_N, N \geq N_0, \quad d\tilde{\rho} \equiv \beta^{-1}d\rho. \]

Since $\lim_{N \to \infty} \beta_N = \beta$ (see (4.17), Theorem 1 implies that for every continuous bounded function $h$ from $H_0^{-\sigma}(S^1)$ to $\mathbb{R}$, we have
\[ \int_{H_0^{-\sigma}(S^1)} h(u)d\tilde{\rho}(u) = \lim_{N \to \infty} \int_{H_0^{-\sigma}(S^1)} h(u)d\tilde{\rho}_N(u). \]

But it is known (see e.g. [10, 9]) that the above convergence is in fact equivalent with the fact that for every open set of $H^{-\sigma}(S^1)$ one has
\[ (4.18) \quad \liminf_{N \to \infty} \tilde{\rho}_N(U) \leq \tilde{\rho}(U). \]

Using (4.17), we infer that (4.15) holds. Finally, one obtains (4.16) by passing to complementary sets in (4.18). This completes the proof of Proposition 4.10. □

5. The random distribution $\varphi(\omega, x)$ and the Benjamin-Ono equation

5.1. Behavior of the map $u \mapsto u^2$ on the statistical ensemble. If one is interested to construct solutions of the Benjamin-Ono equation with almost all data $\varphi(\omega, x)$ given by (1.12), in view of the structure of the nonlinearity, it is natural to ask about regularity properties of $\varphi^2(\omega, x)$. Since $\|\varphi(\omega, \cdot)\|_{L^2} = \infty$ a.s. it is natural to project $\varphi^2(\omega, x)$ on the non zero modes. If we denote by $\Pi$ the projector on the non zero modes, we have the following statement.

**Lemma 5.1.** For every $s < 0$ there exists a constant $C$ such that for every $N$,
\[ \mathbb{E}\left(\|\Pi(\varphi_N^2(\omega, x))\|^2_{H^{-s}(S^1)}\right) \leq C. \]

**Remark 5.2.** The nontrivial point is that $C$ is independent of $N$.

**Proof.** Write
\[ \Pi(\varphi_N^2(\omega, x)) = \sum_{0 < |n_1|, |n_2| \leq N, \ n_1 + n_2 \neq 0} \frac{g_{n_1}(\omega)}{2\sqrt{\pi|n_1|}} \frac{g_{n_2}(\omega)}{2\sqrt{\pi|n_2|}} e^{i(n_1 + n_2)x}. \]
Therefore
\[
\| \Pi(\varphi^2_N(\omega, \cdot)) \|^2_{H^s(S^1)} = \frac{1}{2} \sum_{n \neq 0} \langle n \rangle^{2s} \sum_{0 < |n_1| \leq N \atop n_1 + n_2 = n} \frac{g_{n_1}(\omega) g_{n_2}(\omega)}{\sqrt{|n_1| |n_2|}}.
\]

Denote by \( G(\omega) \) the right hand-side of the above equality. Then using the independence of \( g_n(\omega) \) one verifies that
\[
E(G) \leq C \sum_{n \neq 0} |n|^{2s} \sum_{0 < |n_1| \leq N \atop n_1 + n_2 = n} \frac{1}{|n_1| |n_2|} \leq C \sum_{n \neq 0} \sum_{n_1 \in \mathbb{Z} \backslash \{0, n\}} |n|^{2s} |n_1|^{-1} |n - n_1|^{-1}.
\]

Therefore, using Lemma 2.1, we get
\[
E(G) \leq C \varepsilon \sum_{n \neq 0} |n|^{2s} (1 + |n|)^{-1 + \varepsilon} < \infty
\]
provided \( 2s + \varepsilon < 0 \), i.e. \( 0 < \varepsilon < -2s \). This completes the proof of Lemma 5.1. \( \Box \)

5.2. Tao’s gauge transform on the statistical ensemble. In [13], Tao introduces a gauge transform which turns out to be a crucial tool in the low regularity well-posedness of the Benjamin-Ono equation (see [11, 5, 6]). We now study the action of this transform on the functions on the statistical ensemble (1.12). Recall that Tao’s gauge transform is defined by
\[
\Phi(u) = P_+ \left( e^{-i\partial_x^{-1} u} \right) \equiv \Phi(u),
\]
where \( P_+ \) denotes the projector on the positive frequencies. For \( u \in L^2(S^1) \) with zero mean value the gauge transform \( \Phi(u) \) is easily seen to belong to \( L^2(S^1) \). For \( u \in H^s_0(S^1) \), \(-1/2 < s < 0\) one may give sense of the product \( e^{-i\partial_x^{-1} u} \) in \( L^1(S^1) \), since \( \partial_x^{-1} u \in H^{1-s}(S^1) \) a.s. and \( 1 - s > 1/2 \) implies that \( e^{-i\partial_x^{-1} u} \in H^{1+s}(S^1) \) a.s. It is however not a priori clear that for \( u \in H^s_0(S^1) \), \(-1/2 < s < 0\), the transform \( \Phi(u) \) is also in \( H^s_0(S^1) \). This turns out to be the case for \( u = \varphi(\omega, x) \) a.s. in \( \omega \) as shows the next lemma.

**Lemma 5.3.** Let us fix \( s < 0 \). Then \( \Phi(\varphi(\omega, \cdot)) \in H^s_0(S^1) \) a.s.

**Proof.** Write
\[
e^{-i\partial_x^{-1} u} = \sum_{k=1}^{\infty} \frac{(-i\partial_x^{-1} u)^k}{k!}.
\]
Recall (see e.g. [10]) that there exists an a.s. finite real valued random variable \( H(\omega) \) such that for every \( n = 1, 2, \ldots \),
\[
|g_n(\omega)| \leq (\log(1 + n))^{1/2} H(\omega).
\]
It suffices therefore to show that
\[ (5.2) \quad \left\| P_+ \left( (\partial_x^{-1} \varphi(\omega, \cdot))^k \varphi(\omega, \cdot) \right) \right\|_{H^s(S^1)} \leq C(H(\omega))^{k+1}. \]

The proof of (5.2) is based on a repetitive use of Lemma 2.2. The square of the left hand-side of (5.2) can be bounded by
\[ C \sum_{n>0} n^{2s} \left| \sum_{n \neq n_1 + \cdots + n_{k+1}} \frac{g_{n_1}(\omega)}{|n_1|^{\frac{1}{2}}} \frac{g_{n_2}(\omega)}{|n_2|^{\frac{1}{2}}} \cdots \frac{g_{n_{k+1}}(\omega)}{|n_{k+1}|^{\frac{1}{2}}} \right|^2. \]

Next using \( k + 1 \) times (5.1) we bound the last expression as follows
\[ C\varepsilon(H(\omega))^{2(k+1)} \sum_{n>0} n^{2s} \left| \sum_{n \neq n_2 + \cdots + n_k} \frac{1}{|n_2|^{\frac{1}{2}-\varepsilon}} \cdots \frac{1}{|n_{k+1}|^{\frac{1}{2}-\varepsilon}} \right|^2, \]

where \( \varepsilon \in ]0, \frac{1}{4}[. \) Using Lemma 2.2 we bound the above expression by
\[ C\varepsilon(H(\omega))^{2(k+1)} \sum_{n>0} n^{2s} \left| \frac{1}{|n|^{\frac{1}{2}-\varepsilon}} \cdots \frac{1}{|n_{k+1}|^{\frac{1}{2}-\varepsilon}} \right|^2. \]

Finally using \( k - 1 \) more times Lemma 2.2 we eliminate consequently \( n_k, n_{k-1} \) etc. up to \( n_2 \) and thus we bound the last expression by
\[ C\varepsilon(H(\omega))^{2(k+1)} \sum_{n>0} n^{2s} \left| \frac{1}{|n|^{\frac{1}{2}-\varepsilon}} \right|^2 \leq C\varepsilon(H(\omega))^{2(k+1)}, \]

provided \( 0 < \varepsilon < \min(\frac{1}{4}, -s) \). This proves (5.2) and Lemma 5.3 is therefore established. \( \square \)

5.3. Bounds on the second Picard iteration associated to the Benjamin-Ono equation with data \( \varphi(\omega, x) \). If we set \( \sigma(n) \equiv -n|n| \), we then have that for \( w \in H^s_0(S^1) \), \( s \in \mathbb{R} \) the solution of the linearized around the zero solution Benjamin-Ono equation
\[ (\partial_t + H \partial_x^2)u = 0, \quad u|_{t=0} = w \]

is given by
\[ u(t, x) \equiv \exp(-tH \partial_x^2)(w) = \sum_{n \neq 0} e^{it\sigma(n)} e^{inx} \hat{w}(n). \]
If we are interested to solve the Benjamin-Ono equation with initial data \( \varphi(\omega, x) \) then it is useful to study the problem

\[
(\partial_t + H \partial_x^2)u + \partial_x \left( \exp(-tH\partial_x^2)(\varphi(\omega, \cdot)) \right)^2, \quad u|_{t=0} = 0.
\]

If we replace in (5.3) \( \varphi(\omega, x) \) by an \( H^s_0(S^1) \), \( s \geq 0 \) function then it follows from the work by Molinet [11] that the solution of (5.3) is in \( H^s_0(S^1) \). We are now going to show that in the context of (5.3) the solution is a.s. in all \( H^s_0(S^1), s < 0 \).

**Proposition 5.4.** For every \( s < 0 \), the solution \( u \) of (5.3) is a.s. in \( H^s_0(S^1) \).

**Proof.** We have that

\[
\partial_x \left( \exp(-tH\partial_x^2)(\varphi(\omega, \cdot)) \right)^2 = \sum_{n_1 \neq 0, n_2 \neq 0} i(n_1 + n_2) \frac{g_{n_1}(\omega)}{2\sqrt{\pi|n_1|}} \frac{g_{n_2}(\omega)}{2\sqrt{\pi|n_2|}} e^{it(\sigma(n_1)+\sigma(n_2))} e^{i(n_1+n_2)x}.
\]

On the other hand by the Duhamel principle the solution of (5.3) is given by

\[
u(t, x, \omega) = -\int_0^t \exp(-(t-\tau)H\partial_x^2) \left( \partial_x \left( \exp(-\tau H\partial_x^2)(\varphi(\omega, \cdot)) \right)^2 \right) d\tau.
\]

Therefore there exists a numerical constant \( c \) such that

\[
u(t, x, \omega) = c \sum_{n \neq 0} n e^{it\sigma(n)} \left( \sum_{n_1 \neq 0, n} \frac{e^{it(\sigma(n_1)+\sigma(n-n_1)-\sigma(n))} - 1}{\sigma(n_1) + \sigma(n-n_1) - \sigma(n)} \frac{g_{n_1}(\omega)}{\sqrt{|n_1|}} \frac{g_{n-n_1}(\omega)}{\sqrt{|n-n_1|}} \right) e^{inx}.
\]

Using a direct case by case analysis implies that for \( n_1 \neq 0, n, n \neq 0 \),

\[|\sigma(n_1) + \sigma(n-n_1) - \sigma(n)| \geq |n|.
\]

Therefore using the independence of \( g_{n}(\omega) \) and Lemma 2.1 we obtain for \( s < 0 \)

\[\|u(t, \cdot, \cdot)\|_{L^2(\Omega; H^s_0(S^1))}^2 \leq C \sum_{n \neq 0} \sum_{n_1 \neq 0, n} \frac{|n|^{2s}}{|n_1(n-n_1)|} \leq \sum_{n \neq 0} C_\varepsilon |n|^{2s} < \infty,
\]

provided \( \varepsilon > 0 \) being such that \( 2s + \varepsilon < 0 \). This completes the proof of Proposition 5.4. \( \square \)

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