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Appendix

Alzheimer’s Disease Modelling and Staging through Independent Gaussian Process Analysis of Spatio-Temporal Brain Changes

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A. Lower bound derivation

In this section we detail the derivation of the lower bound:

\[
\log(p(Y, C|\sigma, \lambda)) = \log\left[ \int_A \int_S p(Y|A, S, \sigma)p(C|S', \lambda)p(A)p(S|S', \lambda) dAdSdS' \right]
\]

\[
= \log\left[ \int_A \int_S p(Y|A, S, \sigma)p(C|S', \lambda)p(A)p(S'|S, \lambda)p(S) dAdSdS' \right]
\]

(1)

If we know S this completely determines S', thus we have:

\[
\int p(S'|S, \lambda) dS' = 1
\]

which gives us:

\[
\log(p(Y, C|\sigma, \lambda)) = \log\left[ \int_A \int_S p(Y|A, S, \sigma)p(C|S', \lambda)p(A)p(S) q_1(A)q_2(S) dAdS \right]
\]

\[
= \log[\mathbb{E}_{A \sim q_1, S \sim q_2}\left[ \frac{p(Y|A, S, \sigma)p(C|S', \lambda)p(A)p(S)}{q_1(A)q_2(S)} \right]]
\]

\[
\geq \mathbb{E}_{A \sim q_1, S \sim q_2}\left[ \log\left[ \frac{p(Y|A, S, \sigma)p(C|S', \lambda)p(A)p(S)}{q_1(A)q_2(S)} \right] \right]
\]

(2)

This is obtained thanks to Jensen’s inequality. Finally this leads us to:

\[
\mathbb{E}_{A \sim q_1, S \sim q_2}\left[ \log\left[ \frac{p(Y|A, S, \sigma)p(C|S', \lambda)p(A)p(S)}{q_1(A)q_2(S)} \right] \right] = \mathbb{E}_{A \sim q_1, S \sim q_2}\left[ \log[p(Y|A, S, \sigma)] \right]
\]

\[
+ \mathbb{E}_{S \sim q_2}\left[ \log(P(C|S', \lambda)) \right]
\]

\[
- \mathcal{D}[q_1(A|Y)||p(A)]
\]

\[
- \mathcal{D}[q_2(S|Y)||p(S)]
\]

(3)
In the Method section we introduced the approximation \( q_1(A) = \prod_{n=1}^{N_s} \mathcal{N}(\mu_n, \Sigma(\alpha, \beta)) \).

The covariance matrix is shared by all the spatial processes which gives us the set of spatial parameters:

\[
\psi = \{\mu_n, n \in [1, N_s], \alpha, \beta\}
\]  

(4)

Following [5] we introduce for each GP two vectors, \( \Omega_n \) with a prior \( p(\Omega_n) = \mathcal{N}(0, \frac{1}{\lambda_n} I) \) for each element and \( W_n \) with a prior \( p(W_n) = \mathcal{N}(0, I) \), such that \( S_n(t) = \Phi(t \Omega_n)W_n \). Where \( \Phi \) is chosen to obtain a RBF kernel as explained in [7]. We define the approximated distributions \( q_3(W_n) = \prod_j \mathcal{N}(m_{n,j}, s_{n,j}^2) \) and \( q_4(\Omega_n) = \prod_j \mathcal{N}(\alpha_{n,j}, \beta_{n,j}^2) \) of \( p(W_n) \) and \( p(\Omega_n) \). Using these approximations and following [5], we can derive a lower bound for \( S \) with the same technique than above.

We have the set of temporal parameters:

\[
\theta = \{m_n, s_n, \alpha_n, \beta_n, l_n, n \in [1, N_s]\}
\]  

(5)

Now we can obtain every term of [3]. The Kullback-Leibler of a multivariate Gaussian has a closed-from:

\[
\mathcal{D}[q_1(A|X)||p(A)] = \frac{1}{2} \sum_{n=1}^{N_s} Tr(\Sigma) + \mu_n^T \mu_n - F - \log[\text{det}(\Sigma)]
\]  

(6)

Using the factorized form of \( q_2 \) and the fact that the different Gaussian processes are independent from each other we can write:

\[
\mathcal{D}[q_2(S|X)||p(S)] = \sum_{n=1}^{N_s} \mathcal{D}[q_3(W_n)||p(W_n)] + \mathcal{D}[q_4(\Omega_n)||p(\Omega_n)]
\]  

(7)

Since the approximations \( q_3 \) and \( q_4 \) and their respective priors are normally distributed we have an analytic formula for both Kullback-Leibler divergences.

\[
\mathcal{D}[q_3(W_n)||p(W_n)] = \frac{1}{2} \sum_j s_{n,j}^2 + \mu_{n,j}^2 - 1 - \log(s_{n,j}^2)
\]  

(8)

\[
\mathcal{D}[q_4(\Omega_n)||p(\Omega_n)] = \frac{1}{2} \sum \beta_{n,j}^2 l_n + \alpha_{n,j}^2 l_n - 1 - \log(\beta_{n,j}^2 l_n)
\]  

(9)

As in [10] we employ the reparameterization trick to have an efficient way of sampling the expectations of [3]. Thus we have:

\begin{align*}
- & W_{n,j} = m_{n,j} + s_{n,j} \epsilon_{n,j} \\
- & \Omega_{n,j} = \alpha_{n,j} + \beta_{n,j} \zeta_{n,j} \\
- & A_n = \mu_n + \Sigma_n^{\frac{1}{2}} \kappa_n
\end{align*}

Which gives us:

\begin{align*}
\mathbb{E}_{A \sim q_1, S \sim q_2}[\log(p(Y|A, S, \sigma))] &= \mathbb{E}_{\epsilon, \zeta, \kappa}[\log(p(Y|m, s, \alpha, \beta, \mu, \Sigma, \sigma))] \\
\mathbb{E}_{S \sim q_2}[\log(p(C|S', \lambda))] &= \mathbb{E}_{\epsilon, \zeta}[\log(p(C|m, s, \alpha, \beta, \lambda))]
\end{align*}

(10)

(11)

Where \( \epsilon_{n,j} \sim \mathcal{N}(0, 1) \), \( \zeta_{n,j} \sim \mathcal{N}(0, 1) \) and \( \kappa_n \sim \mathcal{N}(0, I) \).
B. Kronecker factorization

Here we detail how to split the covariance matrix in a Kronecker product of three matrices along each spatial dimensions. We have:

$$\Sigma_{i,j}(\alpha, \beta) = \alpha \exp\left(-\frac{||u_i - u_j||^2}{2\beta}\right)$$  \hspace{1cm} (12)

We can use the separability properties of the exponential to decompose the covariance between two locations $u_i = (x_i, y_i, z_i)$ and $u_j = (x_j, y_j, z_j)$:

$$\Sigma_{i,j}(\alpha, \beta) = \alpha \exp\left(-\frac{(x_i - x_j)^2}{2\beta}\right) \exp\left(-\frac{(y_i - y_j)^2}{2\beta}\right) \exp\left(-\frac{(z_i - z_j)^2}{2\beta}\right)$$  \hspace{1cm} (13)

So $\Sigma$ can be decomposed into the Kronecker product of 1D processes:

$$\Sigma = \Sigma_x \otimes \Sigma_y \otimes \Sigma_z$$  \hspace{1cm} (14)

Allowing us to deal with large-size matrices.
C. Comparison with ICA

We performed a comparison of our algorithm with ICA on a similar example than in [3.1]. However the data was generated in a simplified setting since ICA can’t be applied when the time associated to each image is unknown. To do so we assigned the ground truth parameter $t_p$ beforehand. The goal was to compare the separation performances of both our algorithm and ICA, on data generated with three latent spatio-temporal processes. In Figure 1 we observe that the sources estimated by ICA are more noisy and uncertain than the ones estimated by our method, highlighting the performances of our algorithm in terms of sources separation.
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