WEAKENING CONVERGENCE CONDITIONS OF A
POTENTIAL REDUCTION METHOD FOR TENSOR
COMPLEMENTARITY PROBLEMS

XIAOFEI LIU AND YONG WANG*
School of Mathematics, Tianjin University
Tianjin 300350, China

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Abstract. Recently, under the condition that the included tensor in the tensor complementarity problem is a diagonalizable and positive definite tensor, the convergence of a potential reduction method for tensor complementarity problems is verified in a potential reduction method for tensor complementarity problems. Journal of Industrial and Management Optimization, 2019, 15(2): 429–443. In this paper, we improve the convergence of this method in the sense that the condition we used is strictly weaker than the one used in the above reference. Preliminary numerical results indicate the effectiveness of the potential reduction method under the new condition.

1. Introduction. As a generalization of the linear complementarity problem [4], the tensor complementarity problem (TCP for short) has been introduced and investigated in [3, 11, 22], which is a specific class of nonlinear complementarity problems [6]. The TCP has many applications, including n-person non-cooperative games [10], hypergraph clustering [12], a class of traffic equilibrium problems [12], and so on.

Recently, a multitude of researchers have focused on the properties and theories of TCPs. For example, Bai, Huang and Wang [1] showed that the tensor complementarity with a strong $P$ tensor has a unique solution. Che, Qi and Wei [3] established the non-emptiness and compactness of the solution set for the TCP with the underlying tensor being positive definite; furthermore, they showed that the tensor complementarity problem with a diagonalizable and positive definite tensor has a unique solution. Theoretical advances in tensor complementarity problems are summarized in [11].

With the rapid growth in the theory of the TCP, the studies of the numerical algorithm for solving the TCP are becoming more and more important. Luo, Qi and Xiu [18] proposed a method to find the sparsest solution to a TCP with a $Z$ tensor by reformulating the TCP as an equivalent polynomial programming problem. Liu, Li and Vong [17] proposed a modulus-based nonsmooth Newton’s method for
solving TCPs. Xie, Li and Xu [27] established an iterative method for finding the least solution to the TCP. Du and Zhang [5] proposed a mixed integer programming approach to solve the TCP. Han [9] proposed a continuation method for the TCP with a strictly semi-positive tensor. Recently, Guan and Li [7, 16] proposed linearized methods for solving the TCP with an $M$-tensor. Refer to [21] for a survey. Particularly, in the paper [28], the authors proposed a potential reduction method to solve the TCP, and under the condition that the involved tensor in the TCP is diagonalizable and positive definite, the convergence of the method is proved. But the convergence condition in [28] is relatively strict, which limits the use of the potential reduction method.

In this paper, we focus on improving the convergence condition of the potential reduction method proposed by Zhang, Chen and Zhao in [28]. In fact, we prove the convergence of the potential reduction method under a new condition that the involved tensor in the TCP is a strong $P$ tensor. In addition, we also prove that a diagonalizable and positive definite tensor must be a strong $P$ tensor, but the opposite is not true. This result implies that the convergence condition proposed in this paper is weaker than that in [28] and extends the scope of application for the potential reduction method. Finally, the numerical experiments show the effectiveness of the algorithm under the new condition.

The rest of this paper is organized as follows. In the next section, we first recall some basic concepts and results. In Section 3, we use the potential reduction method to solve the TCPs with a strong $P$ tensor, and prove the convergence of this method. Then we discuss the relationship between a diagonalizable and positive definite tensor and a strong $P$ tensor. With the help of two examples and one theorem, we obtain the desired result that diagonalizable and positive definite tensors are a subclass of strong $P$ tensors. In Section 4, numerical experiments are presented to show that the potential reduction method under the new condition is effective. Finally, the conclusions are given in the last section.

2. Preliminaries. In this paper, we use $[n]$ to denote the set $\{1, 2, \ldots, n\}$ for any positive integer $n$. We use $\mathbb{R}$ to denote the set of all real numbers, $\mathbb{R}^n$ the set of all $n$-dimensional vectors, $\mathbb{R}_+^n$ the set of all $n$-dimensional nonnegative vectors, and $\mathbb{R}^n_{++}$ the set of all $n$-dimensional positive vectors. For any $x, y \in \mathbb{R}^n$, we simply write the column vector $(x^\top, y^\top)^\top$ as $(x, y)^\top$, and we also use $\text{diag}(x)$ to denote the diagonal matrix whose the $i$-th diagonal element is $x_i$ for any $i \in [n]$. If not specified, $\| \cdot \|$ denotes the $l_2$-norm of vectors. For any positive integers $m$ ($m \geq 2$) and $n$, an $m$-order $n$-dimensional real tensor $\mathcal{A} = (a_{i_1i_2\ldots i_m})$ is a multiple array, where $a_{i_1i_2\ldots i_m} \in \mathbb{R}$ for any $i_j \in [n]$ and $j \in [m]$. The set of all $m$-order $n$-dimensional real tensors is denoted by $\mathbb{R}^{[m,n]}$.

More than a decade, tensors have been studied extensively and many applications of tensors have been found [20]. Given $\mathcal{A} = (a_{i_1i_2\ldots i_m}) \in \mathbb{R}^{[m,n]}$, $\mathcal{A}x^{m-1}$ is an $n$-dimensional vector, which the $i$-th element is

$$(\mathcal{A}x^{m-1})_i := \sum_{i_1i_2\ldots i_m=1}^n a_{i_1i_2\ldots i_m}x_{i_1}x_{i_2}\cdots x_{i_m}, \quad \forall i \in [n].$$

And $\mathcal{A}x^m$ is defined as

$$\mathcal{A}x^m := x^\top (\mathcal{A}x^{m-1}) = \sum_{i_1i_2\ldots i_m=1}^n a_{i_1i_2\ldots i_m}x_{i_1}x_{i_2}\cdots x_{i_m}.$$
Given $\mathcal{A} \in \mathbb{R}^{[m,n]}$ and $q \in \mathbb{R}^n$, the tensor complementarity problem is to find a point $x \in \mathbb{R}^n$ such that
\[ x \geq 0, \quad F(x) = \mathcal{A}x^{m-1} + q \geq 0, \quad \text{and} \quad x^\top F(x) = 0, \]
which is denoted by TCP($\mathcal{A}$, $q$). Especially, when $m = 2$ (i.e., $\mathcal{A}$ reduces to a matrix), the TCP($\mathcal{A}$, $q$) reduces to the linear complementarity problem; when $F(x)$ becomes a general nonlinear function, the TCP($\mathcal{A}$, $q$) is a nonlinear complementarity problem. So, the TCP can be regarded as an extension of the linear complementarity problem or a special case of the nonlinear complementarity problem.

The following classes of tensors will be used in our discussions.

**Definition 2.1.** Let $\mathcal{A} = (a_{i_1 \cdots i_m}) \in \mathbb{R}^{[m,n]}$. Then $\mathcal{A}$ is said to be

(i) a **positive definite tensor** [19] if $\mathcal{A}x^m > 0$ for all $x \in \mathbb{R}^n$ and $x \neq 0$.

(ii) a **strictly positive definite tensor** [26] if and only if
\[ (x - y)^\top (\mathcal{A}x^{m-1} - \mathcal{A}y^{m-1}) > 0 \]
for any $x, y \in \mathbb{R}^n$ with $x \neq y$.

(iii) a **P tensor** [22] if for each $x \in \mathbb{R}^n \setminus \{0\}$, there exists an index $i \in [n]$ such that $x_i \neq 0$ and $x_i(\mathcal{A}x^{m-1})_i > 0$.

(iv) a **strong P tensor** [1] if for all pairs of distinct vectors $x$ and $y$ in $\mathbb{R}^n$,
\[ \max_{i \in [n]} (x_i - y_i)(\mathcal{A}x^{m-1} - \mathcal{A}y^{m-1})_i > 0. \]

(v) an **R tensor** [23] if the following system is inconsistent
\[
\begin{cases}
0 \neq x \geq 0, & t \geq 0, \\
(\mathcal{A}x^{m-1})_i + t = 0, & \text{if } x_i > 0, \\
(\mathcal{A}x^{m-1})_i + t \geq 0, & \text{if } x_i = 0.
\end{cases}
\]

(vi) an **$R_0$ tensor** [23] if the above system (1) is inconsistent for $t = 0$.

From **Definition 2.1**, it is easy to see that each strictly positive definite tensor must be a positive definite tensor, a strictly positive definite tensor is a strong $P$ tensor, and a strong $P$ tensor is a $P$ tensor. In addition, a positive definite tensor is a $P$ tensor [19], a $P$ tensor is an $R$ tensor [11]. Furthermore, a positive definite tensor is an $R$ tensor, and is also an $R_0$ tensor.

In this paper, we also need the following concepts of functions.

**Definition 2.2.** Let a mapping $F : K \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^n$. Then $F$ is said to be

(i) a **P-function** [6] if for all pairs of distinct vectors $x$ and $y$ in $K$,
\[ \max_{i \in [n]} (x_i - y_i)(F_i(x) - F_i(y)) > 0; \]

(ii) a **$P_0$-function** [6] if for every $x \neq y$, there is an index $i$ such that
\[ x_i \neq y_i \quad \text{and} \quad (x_i - y_i)(F_i(x) - F_i(y)) \geq 0. \]

From **Definition 2.1** (iv) and **Definition 2.2** (i), it is easy to see that for any $q \in \mathbb{R}^n$ and any strong $P$ tensor $\mathcal{A} \in \mathbb{R}^{[m,n]}$, the function $F(x) = \mathcal{A}x^{m-1} + q$ is a $P$ function, and vice versa. Furthermore, every $P$ function is a $P_0$ function, and the Jacobian of every continuous differentiable $P_0$ function at any point is a $P_0$ matrix [25].
Let $\mathcal{A} \in \mathbb{R}^{[m,n]}$ and $B$ be a real square matrix of the size $n \times n$. Then their product on mode-$k$ (denoted by $\mathcal{A} \times_k B$) is a new $m$-order $n$-dimensional real tensor by virtue of the definition in [15], and the $i_1i_2\cdots j_k\cdots i_m$-th element of $\mathcal{A} \times_k B$ is

$$(\mathcal{A} \times_k B)_{i_1i_2\cdots j_k\cdots i_m} = \sum_{i_k=1}^{n} a_{i_1i_2\cdots i_k\cdots i_m} B_{j_ki_k}.$$ 

**Definition 2.3.** [3] Suppose that $\mathcal{A} \in \mathbb{R}^{[m,n]}$ is symmetric, i.e., $a_{\tau(i_1i_2\cdots i_m)} = a_{i_1i_2\cdots i_m}$ for any permutation $\tau(i_1i_2\cdots i_m)$ of $i_1i_2\cdots i_m$. Then $\mathcal{A}$ is said to be diagonalizable if $\mathcal{A}$ can be represented as

$$\mathcal{A} = \mathcal{D} \times_1 B \times_2 B \cdots \times_m B,$$

where $B$ is an invertible $n \times n$ real matrix and $\mathcal{D} \in \mathbb{R}^{[m,n]}$ is a diagonal tensor, i.e., all elements except for $d_{i_1i_2\cdots i_m}$ (for all $i \in [n]$) are zero. Particularly, the diagonal tensor $\mathcal{D} \in \mathbb{R}^{[m,n]}$ is called an identity tensor if its diagonal elements are all ones.

The Jacobian matrix of $\mathcal{A}x^{m-1}$ plays an important role in the potential reduction method. To represent the Jacobian matrix conveniently, we need to partially symmetrize the tensor $\mathcal{A} = (a_{i_1i_2\cdots i_m})$. Specifically, we let $\tilde{\mathcal{A}} := (\tilde{a}_{i_1i_2\cdots i_m})$, which is called to be partial symmetric or semi-symmetric, with

$$\tilde{a}_{i_1i_2\cdots i_m} = \frac{1}{(m-1)!} \sum_{\tau(i_2\cdots i_m)} a_{i_1\tau(i_2\cdots i_m)}.$$ 

Thus, it holds $\tilde{\mathcal{A}}x^{m-1} = \mathcal{A}x^{m-1}$ and the Jacobian matrix of $\mathcal{A}x^{m-1}$ is $(m - 1)\mathcal{A}x^{m-2}$. Furthermore, we have the following conclusion.

**Theorem 2.4.** If $\mathcal{A}$ is a strong $P$ tensor, so is $\tilde{\mathcal{A}}$.

**Proof.** From (iv) of Definition 2.1, we get that for any distinct $x$ and $y$ in $\mathbb{R}^n$,

$$\max_{i \in [n]} (x_i - y_i)(\mathcal{A}x^{m-1} - \mathcal{A}y^{m-1})_i > 0.$$ 

Since $\mathcal{A}x^{m-1} = \mathcal{A}x^{m-1}$ for any $x \in \mathbb{R}^n$, we get

$$\max_{i \in [n]} (x_i - y_i)(\tilde{\mathcal{A}}x^{m-1} - \tilde{\mathcal{A}}y^{m-1})_i > 0.$$ 

Hence, $\tilde{\mathcal{A}}$ is a strong $P$ tensor.

**Theorem 2.5.** [2] Suppose that $G(x) : D \subseteq \mathbb{R}^n \to \mathbb{R}$ is a second order continuous differentiable function on a nonempty convex set $D$. If the Hesse matrix $\nabla^2 G(x)$ is positive definite on $D$, then $G(x)$ is a strictly convex function on $D$.

For a diagonalizable and positive definite tensor, the following conclusion shows the property of the Jacobian matrix $\nabla F(x)$, where $F(x) = \mathcal{A}x^{m-1} + q$ and $x \in \mathbb{R}^n$.

**Theorem 2.6.** [3] Let $\mathcal{A}$ be a diagonalizable and positive definite tensor. Then the Jacobian matrix $\nabla F(x) = (m - 1)\mathcal{A}x^{m-2}$ (here $\mathcal{A}$ is symmetric from Definition 2.3) is positive definite with $x \neq 0$.

According to the above two theorems, we can draw a conclusion that if $\mathcal{A}$ is a diagonalizable and positive definite tensor, $\mathcal{A}x^m$ is a strictly convex function on $\mathbb{R}^n$.

**Theorem 2.7.** [1] Suppose that $\mathcal{A} \in \mathbb{R}^{[m,n]}$ is a strong $P$ tensor. Then the TCP($\mathcal{A}, q$) has a unique solution for any $q \in \mathbb{R}^n$. 


3. Main results. In [28], a potential reduction method was considered to solve the TCP, and the convergence was proved under the condition that the involved tensor is diagonalizable and positive definite. The numerical experiments at the end of the article [28] verified the effectiveness of the method. But, on the other hand, the convergence condition is a little strict, which limits the use for this method to solve the TCP. In this section, we also consider the potential reduction method to solve the TCP. Particularly, under a new condition, the convergence of the algorithm is proved. Furthermore, we also show that the new convergence condition is weaker than that in [28].

3.1. The potential reduction method under a new condition. Recall that a TCP\((\mathcal{A}, q)\) is to find \(x \in \mathbb{R}^n\) such that
\[
x \geq 0, \quad \mathcal{A} x^{m-1} + q \geq 0, \quad \text{and} \quad x^\top (\mathcal{A} x^{m-1} + q) = 0.
\] (2)

Through this section, we suppose that the tensor \(\mathcal{A}\) is a strong \(P\) tensor defined by (iv) of Definition 2.1, so the TCP\((\mathcal{A}, q)\) has a unique solution from Theorem 2.7.

In this subsection, we state the potential reduction method to solve the TCP\((\mathcal{A}, q)\) and prove its convergence. For any \(x, y \in \mathbb{R}^n\), the TCP\((\mathcal{A}, q)\) (2) can be equivalently reformulated as the following system [28]:
\[
x \geq 0, \quad y \geq 0, \quad H(x, y) = \left( \begin{array}{c} x \circ y \\
y - \mathcal{A} x^{m-1} - q \end{array} \right) = 0,
\]
where the symbol \(\circ\) represents the Hadamard product, i.e., \(x \circ y = (x_1 y_1, x_2 y_2, \cdots, x_n y_n)^\top\).

For interior point methods, if each iterative point \((x, y)^\top \in \mathbb{R}_+^n \times \mathbb{R}_+^n\) satisfies \(y = \mathcal{A} x^{m-1} + q\), then the method is called a feasible interior point method. Otherwise, it is called an infeasible interior point method. Since \(y = \mathcal{A} x^{m-1} + q\) is a nonlinear function, it is difficult to ensure that the iterative sequence always satisfies the feasibility [14]. For convenience, we define \(z := (x, y)^\top\) and \(\Delta z := (\Delta x, \Delta y)^\top\). Then, let the set
\[
\Omega_{++} := \{ z \in \mathbb{R}_+^n \times \mathbb{R}_+^n : \| y - \mathcal{A} x^{m-1} - q \|_{\infty} < \epsilon \},
\]
where \(\epsilon > 0\) is a small tolerance and \(\| \cdot \|_{\infty}\) denotes the \(l_{\infty}\)-norm of vectors.

**Remark 1.** Recall that a tensor \(\mathcal{A}\) is called an \(S\) tensor if and only if there exists \(x > 0\) such that \(\mathcal{A} x^{m-1} > 0\) [24, Definition 3.1]. From [8, Theorem 2.1], \(\mathcal{A}\) is an \(S\) tensor if and only if TCP\((\mathcal{A}, q)\) is strictly feasible for any \(q \in \mathbb{R}^n\). It is easy to know from [8, Remark 2.1] that a strong \(P\) tensor is an \(S\) tensor. So, for any \(q \in \mathbb{R}^n\), there exists at least a point \(\tilde{x} > 0\) such that \(\mathcal{A} \tilde{x}^{m-1} + q > 0\) with \(\mathcal{A}\) being a strong \(P\) tensor. Thus, let \(\tilde{y} := \mathcal{A} \tilde{x}^{m-1} + q + \rho e\), where \(\rho \in (0, 1)\) and \(e\) is the vector of all ones, then \((\tilde{x}, \tilde{y})^\top \in \Omega_{++}\). Hence, the set \(\Omega_{++}\) is nonempty with a strong \(P\) tensor \(\mathcal{A}\).

In the iteration process of the algorithm, we would ensure that the iterative sequence belongs to the set \(\Omega_{++}\).

Let \(x \in \mathbb{R}_+^n, y \in \mathbb{R}_+^n, D_x = \text{diag}(x), D_y = \text{diag}(y)\). In order to obtain the Newton iterative direction, we need the following result.
Theorem 3.1. Suppose that \( A \) is a strong P tensor. Then for all \( z \in \Omega_{++} \), the matrix
\[
H'(z) = \begin{bmatrix}
D_y & D_x \\
-(m-1)A x^{m-2} & I
\end{bmatrix}
\]
is nonsingular, where \( I \) is the identity matrix with proper size.

Proof. Since \( A \) is a strong P tensor, \( A x^{m-1} + q \) is a P function, and then a P0 function. Thus, the Jacobian \( (m-1)A x^{m-2} \) is a P0 matrix [25]. Therefore, we can conclude that the matrix \( H'(z) \) is nonsingular from [13, Lemma 4.1]. \( \square \)

From Theorem 3.1, we can construct the following Newton method for \( H(z) + H'(z) \Delta z = 0 \) at a given point \( z \in \Omega_{++} \),
\[
\begin{bmatrix}
D_y & D_x \\
-(m-1)A x^{m-2} & I
\end{bmatrix} \Delta z = \begin{bmatrix}
-D_x y \\
-(m-1)A x^{m-2} + q
\end{bmatrix}.
\]
In order to enhance the performance of the algorithm, similar to [28], we use the inexact Newton equation as follows:
\[
\begin{bmatrix}
D_y & D_x \\
-(m-1)A x^{m-2} & I
\end{bmatrix} \Delta z = \begin{bmatrix}
-D_x y + \beta z^\top y e \\
-(m-1)A x^{m-2} + q
\end{bmatrix},
\]
where \( \beta \in [0,1) \) is a constant, and \( e \) is the vector of all ones.

In order to implement the potential reduction method, we also adopt the same potential function as that in [28]:
\[
\phi(z) := (n + s) \ln \left( x^\top y + \|y - A x^{m-1} - q\|^2 \right) - \sum_{i=1}^n \ln x_i y_i,
\]
where \( s > 0 \) is a parameter.

The following conclusions can be found in the paper [28].

Theorem 3.2. [28] If \( \lim_{k \to +\infty} \phi(z^k) = -\infty \), where \( \{z^k\} \subset \Omega_{++} \) and \( z^k = (x^k, y^k)^\top \), then \( \lim_{k \to +\infty} (x^k)^\top y^k = 0 \) and \( \lim_{k \to +\infty} (y^k - A(x^k)^{m-1} - q) = 0 \).

Theorem 3.3. [28] Let \( z = (x, y)^\top \in \Omega_{++} \) and \( \Delta z = (\Delta x, \Delta y)^\top \) be a solution of (3). Then
\[
\nabla \phi(z)^\top \Delta z \leq - 1 - \beta s < 0
\]
and there exists a scalar \( \bar{\theta} > 0 \), such that for all \( \theta \in (0, \bar{\theta}] \),
\[
z + \theta \Delta z \in \Omega_{++}, \quad \phi(z + \theta \Delta z) - \phi(z) \leq - \alpha \theta (1 - \beta) s < 0.
\]

Theorem 3.4. [28] Let \( \{z^k\} \subset \Omega_{++} \). If \( \lim_{k \to +\infty} z^k = \bar{z} \) for some \( \bar{z} = (\bar{x}, \bar{y})^\top \in \Omega_{++} \), and \( \bar{x}_j \bar{y}_j = 0 \) for some \( j \in [n] \), then \( \bar{z} \) is a solution of the TCP\((A, q)\).

For convenience and completeness, we describe the potential reduction method in the follows.

Algorithm 3.1. (The potential reduction method)

**Step 0** (Initialization) Let \( s > 0, \epsilon > 0, \alpha \in (0, 1), \beta \in [0, 1), \sigma > 0, \) and \( \rho \in (0, 1) \) be given. Choose any \( z^0 := (x^0, y^0)^\top \in \Omega_{++} \) and \( \beta_0 \in [0, \beta] \). Set \( k := 0 \).

**Step 1** (Direction generation) Solve the system of equations (3) to obtain the search direction \( \Delta z^k := (\Delta x^k, \Delta y^k)^\top \).
Step 2 (Stepsize determination) Let \( l_k \) be the smallest nonnegative integer such that the following two conditions hold:

\[
z_k^k + \sigma_\rho^l \Delta z_k \in \Omega_++ \quad \text{and} \quad \phi(z_k^k + \sigma_\rho^l \Delta z_k) - \phi(z_k^k) \leq -\alpha \sigma_\rho^l \left(1 - \beta_k\right) s.
\]

Set \( z_k^{k+1} = z_k^k + \sigma_\rho^l \Delta z_k \).

Step 3 (Termination verification) If \( z_k^{k+1} \) satisfies

\[
f(z_k^{k+1}) = (x_k^{k+1})^\top y_k^{k+1} + (\| y_k^{k+1} - \mathcal{A}(x_k^{k+1})^{m-1} - q \|)^2 < \epsilon,
\]

or

\[
z_k^{k+1} - z_k^k < \epsilon \quad \text{and} \quad x_j^{k+1} y_j^{k+1} = 0 \quad \text{for some} \quad j \in [n],
\]

stop. Then \( z_k^{k+1} \) is the approximate solution of the TCP(\( \mathcal{A}, q \)). Otherwise, pick any \( \beta_{k+1} \in (0, \beta) \) and return to Step 1 with \( k = k + 1 \).

Remark 2. (i) Under the condition that \( \mathcal{A} \) is a strong \( P \) tensor, Step 1 is well defined from Theorem 3.1. From Theorem 3.3, Step 2 is well defined. From Theorems 3.2 and 3.4, the termination rules are reasonable. That is, if \( z_k^{k+1} \) is a solution of TCP(\( \mathcal{A}, q \)), \( z_k^{k+1} \) must satisfy one of the two stopping rules.

(ii) It is obvious that the framework of Algorithm 3.1 is the same one as Algorithm 1 in [28], but it should be noticed that we discuss Algorithm 3.1 under the condition that the involved tensor is a strong \( P \) tensor. While the algorithm considered in [28] is discussed under the condition that the involved tensor is a diagonalizable and positive definite tensor. In next subsection, we will show that these two classes of tensors are different and demonstrate that the diagonalizable and positive definite tensors are a subclass of the strong \( P \) tensors. So, Algorithm 3.1 widens the scope of application of the algorithm in [28].

Next we will prove the convergence of Algorithm 3.1. First, we give the following conclusion.

Theorem 3.5. Let \( \mathcal{A} \in \mathbb{R}^{[m,n]} \) be a strong \( P \) tensor. If Algorithm 3.1 is initialized at \( z_0 = (x^0, y^0)^\top \in \Omega_{++} \), then it generates an iterative sequence \( \{z_k\} = \{(x_k^k, y_k^k)^\top\} \subset \Omega_{++} \). Furthermore, the sequences \( \{(x_k^k)^\top y_k^k\} \) and \( \{\| y_k^k - \mathcal{A}(x_k^k)^{m-1} - q \|^2\} \) are bounded.

Proof. From Remark 1, there exists \( z_0 = (x^0, y^0)^\top \in \Omega_{++} \) for any \( \epsilon > 0 \). Therefore, the iterative sequence \( \{z_k\} = \{(x_k^k, y_k^k)^\top\} \subset \Omega_{++} \) from Step 2 of Algorithm 3.1. For convenience, we omit the superscript of \( z_k = (x_k^k, y_k^k)^\top \) for \( k \geq 1 \) in the next.

From the definition of \( \phi \), we have

\[
\phi(z) = n \ln \left( x^\top y + \| y - \mathcal{A} x^{m-1} - q \|^2 \right)
+ s \ln \left( x^\top y + \| y - \mathcal{A} x^{m-1} - q \|^2 \right) - \sum_{i=1}^n \ln x_i y_i
\geq n \ln x^\top y + s \ln \left( x^\top y + \| y - \mathcal{A} x^{m-1} - q \|^2 \right) - \sum_{i=1}^n \ln x_i y_i.
\]

From Jensen’s inequality, it follows that \( \ln \frac{x^\top y}{n} \geq \frac{1}{n} \sum_{i=1}^n \ln x_i y_i \). Therefore,

\[
n \ln x^\top y - \sum_{i=1}^n \ln x_i y_i = n \left( \ln \frac{x^\top y}{n} - \frac{1}{n} \sum_{i=1}^n \ln x_i y_i \right) + n \ln n \geq n \ln n.
\]
Furthermore, we can get that
\[ \phi(z) - n \ln n \geq s \ln \left( x^\top y + \| y - \mathcal{A} x^{m-1} - q \|_2 \right). \]
Together with \( \{ \phi(z^k) \} \) being decreasing, we obtain that
\[ x^\top y + \| y - \mathcal{A} x^{m-1} - q \|_2^2 \leq \exp \left\{ \frac{\phi(z) - n \ln n}{s} \right\} \leq \exp \left\{ \frac{\phi(z^0) - n \ln n}{s} \right\}, \]
which implies that the sequence \( \{ (x^k)^\top y^k \} + \{ \| y^k - \mathcal{A} x^{m-1} - q \|_2 \} \) is bounded for any \( (x^k, y^k)^\top \in \Omega_+^+ \). Then, both the sequence \( \{ (x^k)^\top y^k \} \) and the sequence \( \{ \| y^k - \mathcal{A} x^{m-1} - q \|_2 \} \) are bounded.

**Theorem 3.6.** Suppose that \( \mathcal{A} \) is a strong \( P \) tensor. Then the iterative sequence \( \{ z^k \} \) generated by Algorithm 3.1 satisfies the following properties:

(i) the iterative sequence \( \{ z^k \} \) is bounded;

(ii) every accumulation point of \( \{ z^k \} \) is a solution of the TCP(\( \mathcal{A}, q \)).

Proof. (i) Suppose that the sequence \( \{ z^k \} = \{ (x^k, y^k)^\top \} \) is unbounded, then \( \{ x^k \} \) is unbounded. Otherwise, assume that \( \{ x^k \} \) is bounded. Since \( \{ (x^k, y^k)^\top \} \in \Omega_+^+ \), it yields that \( \| y^k \|_\infty < \| \mathcal{A}(x^k)^{m-1} + q \|_\infty + \epsilon \) from the triangle inequality of norm. So, we get that the sequence \( \{ y^k \} \) is also bounded, which contradicts the unboundedness of \( \{ z^k \} \). Denote \( u^k := \frac{x^k}{\| x^k \|} \) and \( v^k := \frac{y^k}{\| y^k \|} \), then
\[ \| v^k \|_\infty = \left\| \frac{y^k}{\| y^k \|} \right\|_\infty < \left\| \mathcal{A}(x^k)^{m-1} + q \right\|_\infty + \epsilon \leq \left\| \mathcal{A}(u^k)^{m-1} \right\|_\infty + \frac{\| q \|_\infty + \epsilon}{\| x^k \|^{m-1}}. \]
Therefore, when \( k \) is large sufficiently, the sequences \( \{ u^k \} \) and \( \{ v^k \} \) are all bounded, and then they have convergent subsequences, respectively. Without loss of generality, we denote the convergent subsequences by \( \{ u^k \} \) and \( \{ v^k \} \) and the limit points by \( u^* \) and \( v^* \), respectively. Since \( \| u^k \| = 1 \), it follows that \( u^* \neq 0 \), and then \( (u^k, v^k)^\top \rightarrow (u^*, v^*)^\top \neq 0 \) as \( k \rightarrow \infty \).

Consider the next two expressions
\[ (u^k)^\top v^k = \frac{(x^k)^\top y^k}{\| x^k \|^{m-1}}, \quad (4) \]
\[ \left\| v^k - \mathcal{A}(u^k)^{m-1} - \frac{q}{\| x^k \|^{m-1}} \right\|^2 = \left\| y^k - \mathcal{A}(x^k)^{m-1} - q \right\|^2. \quad (5) \]
From Theorem 3.5, both the sequences \( \{ (x^k)^\top y^k \} \) and \( \{ \| y^k - \mathcal{A}(x^k)^{m-1} - q \|_2 \} \) are bounded. As \( k \rightarrow \infty \), from (4) and (5) we can get that
\[ \lim_{k \rightarrow \infty} (u^k)^\top v^k = 0, \quad (6) \]
and
\[ \lim_{k \rightarrow \infty} \left\| v^k - \mathcal{A}(u^k)^{m-1} - \frac{q}{\| x^k \|^{m-1}} \right\|^2 = \lim_{k \rightarrow \infty} \left\| v^k - \mathcal{A}(u^k)^{m-1} \right\|^2 = 0. \quad (7) \]
(6) and (7) imply that \( (u^*, v^*)^\top \) is a nonzero solution of the TCP(\( \mathcal{A}, 0 \)). On the other hand, since \( \mathcal{A} \) is a strong \( P \) tensor, then an \( R_0 \) tensor, the TCP(\( \mathcal{A}, 0 \)) has the unique solution 0. A contradiction happens. So, the sequence \( \{ z^k \} \) is bounded.

(ii) For the second result of the theorem, the proof is similar to that of [28, Theorem 4.6 (ii)], and here we omit it.
3.2. Relationship between two class of tensors. In subsection 3.1, we prove the convergence of the potential reduction method with the condition that the concerned tensor is a strong $P$ tensor. In order to show that the convergence condition is weaker than that of [28], in which the concerned tensor is a diagonalizable and positive definite tensor, we will discuss the relationship between strong $P$ tensors and diagonalizable and positive definite tensors. To this end, we first discuss the relationship between strong $P$ tensors and positive definite tensors.

First, we adopt the example from [1] for a positive definite tensor to illustrate that a positive definite tensor is not necessary a strong $P$ tensor.

**Example 3.1.** [1] Let $\mathcal{A} = (a_{i_1i_2i_3i_4}) \in \mathbb{R}^{[4, 2]}$, where $a_{1111} = 1, a_{1222} = -1, a_{2111} = 1, a_{2222} = -1, a_{2211} = 1$ and all others $a_{i_1i_2i_3i_4} = 0$.

Obviously, $\mathcal{A} x^3 = \begin{bmatrix} x_1^3 - x_2^3 + x_1 x_2^2, x_2^3 - x_1^3 - x_2 x_1^2 \end{bmatrix}^\top$. Then for any $x \neq 0$,

$$x^\top \mathcal{A} x^3 = x_1^4 - x_1 x_2^3 + x_1^2 x_2^2 + x_2^4 - x_1^3 x_2 + x_1 x_2^3 = (x_1^2 + x_2^2)(x_1^2 - x_1 x_2) > 0.$$

Therefore, $x^\top \mathcal{A} x^3 = \mathcal{A}_S x^4 > 0$ for any $x \neq 0$, which implies $\mathcal{A}$ is a positive definite tensor. However, $\mathcal{A}$ is not a strong $P$ tensor from [1, Example 4.2].

Then, whether a strong $P$ tensor must be a positive definite tensor or not? Next example gives a negative answer.

**Example 3.2.** Let $\mathcal{A} = (a_{i_1i_2i_3i_4}) \in \mathbb{R}^{[4, 2]}$, where $a_{1111} = 1, a_{1222} = -2, a_{2222} = 1$ and all others $a_{i_1i_2i_3i_4} = 0$.

Obviously, $\mathcal{A} x^3 = \begin{bmatrix} x_1^3 - 2 x_2^3, x_2^3 \end{bmatrix}^\top$. Then for any $x \neq 0$, $x^\top \mathcal{A} x^3 = x_1^4 - 2 x_1 x_2^3 + x_2^4$. For any $x \neq y$, we have

$$\begin{aligned}
(x_1 - y_1)(\mathcal{A} x^3)_1 - (\mathcal{A} y^3)_1 &= (x_1 - y_1)(x_1^3 - 2 x_2^3 - y_1^3 + 2 y_2^3), \\
(x_2 - y_2)(\mathcal{A} x^3)_2 - (\mathcal{A} y^3)_2 &= (x_2 - y_2)(x_2^3 - y_2^3).
\end{aligned}$$

So,

(i) if $x_2 \neq y_2$, then $(x_2 - y_2)(x_2^3 - y_2^3) > 0$.
(ii) if $x_2 = y_2$, then $(x_1 - y_1)(x_1^3 - 2 x_2^3 - y_1^3 + 2 y_2^3) = (x_1 - y_1)(x_1^3 - y_1^3) > 0$.

Therefore, by Definition 2.1 (iv) we can conclude that $\mathcal{A}$ is a strong $P$ tensor.

However, $\mathcal{A}$ is not a positive definite tensor. In fact, if we take $x = (0.5, 0.5)^\top$, then $\mathcal{A} x^4 = 0$, which implies that $\mathcal{A}$ is not a positive definite tensor.

Examples 3.1 and 3.2 demonstrate that the class of strong $P$ tensors and the class of positive definite tensors cannot contain each other. Furthermore, it is easy to verify that for the 4-order 2-dimensional identity tensor, it is a strong $P$ tensor and also a positive definite tensor. Therefore, the intersection of the class of strong $P$ tensors and the class of positive definite tensors is nonempty. On the other hand, we can prove that if a positive definite tensor is also diagonalizable, it is also a strong $P$ tensor.

**Theorem 3.7.** If $\mathcal{A}$ is a diagonalizable and positive definite tensor, then it is a strong $P$ tensor.

**Proof.** Since $\mathcal{A}$ is a diagonalizable and positive definite tensor, by Theorem 2.6 we can draw the conclusion that $\mathcal{A} x^{n-2}$ is a positive definite matrix. Denoting $f(x) := \mathcal{A} x^n$ for any $x \in \mathbb{R}^n$, by Theorem 2.5 we know $f(x)$ is a strictly convex
function on \( \mathbb{R}^n \). According to the property of strictly convex function, for \( \forall x, y \in \mathbb{R}^n, x \neq y \), we have
\[
\begin{align*}
    f(y) - f(x) &> \nabla f(x)^\top(y - x), \\
    f(x) - f(y) &> \nabla f(y)^\top(x - y).
\end{align*}
\]
Add the above two inequalities, then we can get
\[
\langle \mathcal{A} x^{m-1} - \mathcal{A} y^{m-1}, x - y \rangle > 0,
\]
which implies that \( \mathcal{A} \) is a strictly positive definite tensor. Thus, \( \mathcal{A} \) is a strong \( P \) tensor.

From Example 3.2, it has been known that a strong \( P \) tensor may not be a positive definite tensor, then not be a diagonalizable and positive definite tensor. Together with Theorem 3.7, we can know that diagonalizable and positive definite tensors are a subclass of strong \( P \) tensors. Considering Example 3.1, Example 3.2 and Theorem 3.7 together, we can get the relationships among three classes of tensors, which is shown in Figure 1.

![Figure 1. The relationships among three classes of tensors.](image)

From Figure 1, it is clear for the relationship between diagonalizable and positive definite tensors and strong \( P \) tensors, which implies that the new convergence condition presented in this paper is wider than that in [28]. Therefore, we weaken the convergence condition of the potential reduction method and extend the range of application of the method.

4. Numerical experiments. In this section, we implement Algorithm 3.1 to solve the TCPs with strong \( P \) tensors. All procedures are written in Matlab code and our numerical experiments will be performed via Matlab R2017a on a 2.20 GHz dell computer (Inter(R) Core(TM) CPU i5-5200U). In these experiments, we set up the parameters as follows:
\[
s = 1, \quad \alpha = 0.6, \quad \bar{\beta} = 0.95, \quad \sigma = 0.8, \quad \rho = 0.4, \quad \epsilon = 10^{-8}.
\]
In the record of the results for all examples, we use “\( z^0 \)” denotes the initial point in Algorithm 3.1, “\( \bar{z} \)” the approximal solution, “Iter” the iterative number and “Time(s)” the running time in seconds.

**Example 4.1.** Consider the TCP(\( \mathcal{A}, q \)), where \( q = (-0.1, 0.1)^\top \) and the tensor \( \mathcal{A} = (a_{ij, k, l}) \in \mathbb{R}^{[4,2]} \) satisfies that \( a_{1,1,1} = 1, a_{1,1,2} = 1, a_{1,2,2} = -3, a_{2,2,2} = 1 \) and all others \( a_{i,j,k,l} = 0 \).
Since \((x_2 - y_2)((\mathcal{A}x^3)_2 - (\mathcal{A}y^3)_2) = (x_2 - y_2)(x_2^3 - y_2^3) > 0\) with \(x_2 \neq y_2\), and \\
\((x_1 - y_1)((\mathcal{A}x^3)_1 - (\mathcal{A}y^3)_1) = (x_1 - y_1)(x_1^3 - y_1^3) + (x_1 - y_1)x_2^2 > 0\) with \(x_2 = y_2\) \\
and \(x_1 \neq y_1\), \(\mathcal{A}\) is a strong \(P\) tensor. However, \(\mathcal{A}\) is not a positive definite since \\
\(\mathcal{A}x^3 = 0\) with \(x = (0.5, 0.5)^\top\).

In order to get an initial point \((x^0, y^0)^\top \in \Omega_{++}\), we first take a feasible point of \\
the TCP(\(\mathcal{A}, q\)). To this end, let \\
\[\mathcal{A}x^3 + q = (x_1^3 - 3x_2^3 + x_1x_2^2 - 0.1, x_2^3 + 0.1)^\top > 0.\]

By simply calculating, it can be verified that for any scalar \(a > \frac{1}{3}\), \(x^0 = (3a, a)^\top > 0\) \\
satisfies the above inequality. So, with \\
\[x^0 = (3a, a)^\top \quad \text{and} \quad y^0 = (27a^3 - 0.1 + 0.8\epsilon, a^3 + 0.1 + 0.8\epsilon)^\top\]
for any \(a > \frac{1}{3}\), \((x^0, y^0)^\top \in \Omega_{++}\) is the desired point.

Since \(\mathcal{A}\) is not semi-symmetric, the Jacobian matrix of 
\(\mathcal{A}x^3 + q = 3\mathcal{A}x^2\).
We take \(\beta_0 = 0.5\) and different initial points with \(a = 0.4, 0.8, 1.2, 1.6, 2.0, 2.4\) in this \\
example. The numerical results are listed in Table 1.

**Table 1. Numerical Results for Example 4.1**

| \((z^0)^\top\) | Iter | Time(s) | \((z^*)^\top\) |
|---------------|------|---------|---------------|
| (1.2, 0.4, 1.6280, 0.1640) | 40 | 0.3584 | (0.4642, 0.0000, 0.0000, 0.1000) |
| (2.4, 0.8, 13.7240, 0.6120) | 46 | 0.3714 | (0.4642, 0.0000, 0.0000, 0.1000) |
| (3.6, 1.2, 46.5560, 1.8280) | 49 | 0.3776 | (0.4642, 0.0000, 0.0000, 0.1000) |
| (4.8, 1.6, 110.4920, 4.1960) | 52 | 0.4612 | (0.4642, 0.0000, 0.0000, 0.1000) |
| (6.0, 2.0, 215.9000, 8.1000) | 54 | 0.3880 | (0.4642, 0.0000, 0.0000, 0.1000) |
| (7.2, 2.4, 373.1480, 13.9240) | 56 | 0.4293 | (0.4642, 0.0000, 0.0000, 0.1000) |

From Table 1, we can see that with different initial points, the iterative number has small difference. And for different initial points, the approximal solutions obtained by Algorithm 3.1 have no difference.

In the following example, we discuss the relationship between the algorithm’s results and the different values of \(\beta_0\).

**Example 4.2.** Consider the TCP(\(\mathcal{A}, q\)), where \(q = (0, 0)^\top\) and \(\mathcal{A} \in \mathbb{R}^{[4,2]}\) satisfies that 
\(a_{1111} = 1, a_{1222} = -3, a_{2222} = 1\) and all others \(a_{i_1i_2i_3i_4} = 0\).

Similar to the proof of Example 3.2, it is easy to verify that \(\mathcal{A}\) is a strong \(P\) tensor but not a positive definite tensor. To let 
\(\mathcal{A}x^3 + q = (x_1^3 - 3x_2^3, x_2^3)^\top\) be positive, it is enough to take 
\(x^0 = (3a, a)^\top\) for any scalar \(a > 0\). So, with 
\[x^0 = (3a, a)^\top \quad \text{and} \quad y^0 = (24a^3 + 0.8\epsilon, a^3 + 0.8\epsilon)^\top\]
for any \(a > 0\), \((x^0, y^0)^\top \in \Omega_{++}\) is the initial point.

Since \(\mathcal{A}\) is semi-symmetric, the Jacobian matrix of 
\(\mathcal{A}x^3 + q = 3\mathcal{A}x^2\).
In this example, \(\beta_0\) is chosen as different numbers and the initial point is taken with \(a = 0.3\). The numerical results are listed in Table 2.

From Table 2, we can see that Algorithm 3.1 can effectively stop in a very short time for any \(\beta_0\), and needs fewer iterative steps with the decreasing of \(\beta_0\). For different \(\beta_0\), the proximal solutions obtained by Algorithm 3.1 have small difference.

In the following example, we consider the TCP with a higher order strong \(P\) tensor.
Example 4.3. Consider the TCP($\mathcal{A}$, $q$), where different values of $q$ is chosen and $\mathcal{A} \in \mathbb{R}^{[6,2]}$ satisfies that $a_{111111} = 1, a_{122222} = -3, a_{222222} = 1$ and all others $a_{i_1i_2i_3i_4i_5i_6} = 0$.

Similar to the proof of Example 3.2, it is easy to verify that $\mathcal{A}$ is a strong $P$ tensor but not a positive definite tensor. It follows $\mathcal{A}x^5 = (x_1^5 - 3x_2^5, x_2^5)^\top$.

In this example, we take 3 different vectors of $q$. To obtained an initial point $(x^0, y^0)^\top \in \Omega_{++},$

- with $q = (-1, 1)^\top$, from $(x_1^5 - 3x_2^5 - 1, x_2^5 + 1)^\top > 0$, it is enough to take $x^0 = (2a_1, a_1)^\top$ and $y^0 = (29a_1^2 - 1 + 0.8\epsilon, a_1^2 + 1 + 0.8\epsilon)^\top$ for any scalar $a_1 > \frac{1}{\sqrt{29}}$.
- with $q = (-6, -2)^\top$, from $(x_1^5 - 3x_2^5 - 6, x_2^5 - 2)^\top > 0$, it is enough to take $x^0 = (2a_2, a_2)^\top$ and $y^0 = (29a_2^2 - 6 + 0.8\epsilon, a_2^2 - 2 + 0.8\epsilon)^\top$ for any scalar $a_2 > \sqrt{2}$.
- with $q = (36, -19)^\top$, from $(x_1^5 - 3x_2^5 + 36, x_2^5 - 19)^\top > 0$, it is enough to take $x^0 = (2a_3, a_3)^\top$ and $y^0 = (29a_3^2 + 36 + 0.8\epsilon, a_3^2 - 19 + 0.8\epsilon)^\top$ for any scalar $a_3 > \sqrt{19}$.

In this example, $\mathcal{A}$ is semi-symmetric. We also set $\beta_0 = 0.5$. For each $q$, we take two different initial points. Specially, we take $a_1 = 1, 2.1, a_2 = 1.4, 4$ and $a_3 = 2, 12$, respectively. The numerical results with different vectors of $q$ are listed in Table 3.

| $q^\top$ | $(z^0)^\top$ | Iter | Time(s) | $(z^*)^\top$ |
|---------|-------------|------|--------|-------------|
| $(-1, 1)$ | $(2, 1, 28, 2)$ | 47   | 0.3207 | $(1, 0, 0, 1)$ |
| $(-1, 1)$ | $(4.2, 2.1, 1183.3893, 41.8410)$ | 67   | 0.4496 | $(1, 0, 0, 1)$ |
| $(-6, -2)$ | $(2.8, 1.4, 149.9690, 3.3782)$ | 52   | 0.3200 | $(1.6438, 1.1487, 0, 0)$ |
| $(-6, -2)$ | $(8, 4, 29690, 1022)$ | 129  | 1.3179 | $(1.6438, 1.1487, 0, 0)$ |
| $(36, -19)$ | $(4, 2, 964, 13)$ | 66   | 0.4528 | $(1.8384, 1.8020, 0, 0)$ |
| $(36, -19)$ | $(24, 12, 7216164, 248813)$ | 700  | 7.4286 | $(1.8384, 1.8020, 0, 0)$ |

From Table 3, it can be seen that for the same $q$ and different initial points, the iterative number and the running time of Algorithm 3.1 have obvious difference, respectively. However, for the same $q$, there is no change for the proximal solution obtained by Algorithm 3.1.
5. **Conclusion.** Motivated by the potential reduction method proposed in the paper [28], we also consider to use the potential reduction method to solve a class of TCPs in this paper. However, because the involved strong $P$ tensors in this class of TCPs may not be such tensors, i.e., diagonalizable and positive definite tensors, required by [28], the convergence of the method cannot be guaranteed by the theory in [28]. The main contribution of this paper is to prove the convergence of the potential reduction method under the new condition that the concerned tensor in the TCP is a strong $P$ tensor. Moreover, we also show that a diagonalizable and positive definite tensor must be a strong $P$ tensor, but not vice versa, which implies that the new convergence condition contains the one discussed in [28]. Thus, through the study in this paper, the convergence condition for the potential reduction method becomes weaker and its scope of application becomes wider. Finally, the numerical experiments show the effectiveness of the method. In the future, it is worth considering how to further weaken the convergence condition of the potential reduction method, how to design other effective algorithm to solve TCPs and how to distinguish a strong $P$ tensor effectively.

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E-mail address: lx0176210126.com
E-mail address: wang_yong@tju.edu.cn