GRavitational Field Equations and Theory of Dark Matter and Dark Energy

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Abstract. The main objective of this article is to derive a new set of gravitational field equations and to establish a new unified theory for dark energy and dark matter. The new gravitational field equations with scalar potential \( \varphi \) are derived using the Einstein-Hilbert functional, and the scalar potential \( \varphi \) is a natural outcome of the divergence-free constraint of the variational elements. Gravitation is now described by the Riemannian metric \( g_{ij} \), the scalar potential \( \varphi \) and their interactions, unified by the new gravitational field equations. Associated with the scalar potential \( \varphi \) is the scalar potential energy density \( \frac{c^4}{8\pi G} \phi = \frac{c^4}{8\pi G} g^{ij} D_i D_j \varphi \), which represents a new type of energy caused by the non-uniform distribution of matter in the universe. The negative part of this potential energy density produces attraction, and the positive part produces repelling force. This potential energy density is conserved with mean zero: \( \int_M \Phi dM = 0 \). The sum of this new potential energy density \( \frac{c^4}{8\pi G} \phi \) and the coupling energy between the energy-momentum tensor \( T_{ij} \) and the scalar potential field \( \varphi \) gives rise to a new unified theory for dark matter and dark energy: The negative part of this sum represents the dark matter, which produces attraction, and the positive part represents the dark energy, which drives the acceleration of expanding galaxies. In addition, the scalar curvature of space-time obeys \( R = \frac{8\pi G c^4}{c^4} T + \Phi \). Furthermore, the new field equations resolve a few difficulties encountered by the classical Einstein field equations.

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1. Introduction and Summary

The main aim of this article is an attempt to derive a new theory for dark matter and dark energy, and to derive a new set of gravitational field equations.

The primary motivation of this study is the great mystery of the dark matter and dark energy. The natural starting point for this study is to fundamentally examine the Einstein field equations, given as follows:

\[ R_{ij} - \frac{1}{2} g_{ij} R = \frac{8\pi G}{c^4} T_{ij}, \]

where \( R_{ij} \) is the Ricci curvature tensor, \( R \) is the scalar curvature, \( g_{ij} \) is the Riemannian metric of the space-time, and \( T_{ij} \) is the energy-momentum tensor of matter; see among many others [1]. The Einstein equations can also be derived using the Principle of Lagrangian Dynamics to the Einstein-Hilbert functional:

\[ F(g_{ij}) = \int_M R\sqrt{-g}dx, \]

whose Euler-Lagrangian is exactly \( R_{ij} - \frac{1}{2} g_{ij} R \), which is the left hand side of the Einstein field equations (1.1). It is postulated that this Euler-Lagrangian is balanced by the symmetric energy-momentum tensor of matter, \( T_{ij} \), leading to the Einstein field equations (1.1). The Bianchi identity implies that the left hand side of the Einstein equations is divergence-free, and it is then postulated and widely accepted that the energy-momentum tensor of matter \( T_{ij} \) is divergence-free as well.

However, there are a number of difficulties for the Einstein field equations:

First, the Einstein field equations failed to explain the dark matter and dark energy, and the equations are inconsistent with the accelerating expansion of the galaxies. In spite of many attempts to modify the Einstein gravitational field equation to derive a consistent theory for the dark energy, the mystery remains.
Second, we can prove that there is no solution for the Einstein field equations for the spherically symmetric case with cosmic microwave background (CMB). One needs clearly to resolve this inconsistency caused by the non-existence of solutions.

Third, from the Einstein equations (1.1), it is clear that

\[ R = \frac{4\pi G}{c^4} T, \]

where \( T = g^{ij} T_{ij} \) is the energy-momentum density. A direct consequence of this formula is that the discontinuities of \( T \) give rise to the same discontinuities of the curvature and the discontinuities of space-time. This is certainly an inconsistency which needs to be resolved.

Fourth, it has been observed that the universe is highly non-homogeneous as indicated by e.g. the "Great Walls", filaments and voids. However, the Einstein equations do not appear to offer a good explanation of this inhomogeneity.

These observations strongly suggest that further fundamental level examinations of the Einstein equations are inevitably necessary. It is clear that any modification of the Einstein field equations should obey three basic principles:

- the principle of equivalence,
- the principle of general relativity, and
- the principle of Lagrangian dynamics.

The first two principles tell us that the spatial and temporal world is a 4-dimensional Riemannian manifold \( (M, g_{ij}) \), where the metric \( \{g_{ij}\} \) represents gravitational potential, and the third principle determines that the Riemannian metric \( \{g_{ij}\} \) is an extremum point of the Lagrangian action. There is no doubt that the most natural Lagrangian in this case is the Einstein-Hilbert functional as explained in many classical texts of general relativity.

The key observation for our study is a well-known fact that the Riemannian metric \( g_{ij} \) is divergence-free. This suggests two important postulates for deriving a new set of gravitational field equations:

- The energy-momentum tensor \( T_{ij} \) of matter need not to be divergence-free due to the presence of dark energy and dark matter; and
- The field equations obey the Euler-Lagrange equation of the Einstein-Hilbert functional under the natural divergence-free constraint, with divergence defined at the extremum Riemannian metric \( g \):

\[ D^k_g X_{kl} = 0, \quad X_{kl} = X_{lk}. \]

Here \( D^k_g \) is the contra-variant derivative with respect to the extremum point \( g \), and \( X_{ij} \) are the variational elements. Namely, for any \( X = \{X_{ij}\} \) with \( D^k_g X_{ij} = 0 \),

\[ \lim_{\lambda \to 0} \frac{1}{\lambda} [F(g_{ij} + \lambda X_{ij}) - F(g_{ij})] = (\delta F(g_{ij}), X) = 0. \]

For this purpose, an important part of this article is to drive an orthogonal decomposition theorem of tensors on Riemannian manifolds, which we shall explain further in the last part of this Introduction.

Under these two postulates, using the orthogonal decomposition theorem of tensors, we derive the following new set of gravitational field equations with scalar
potential:

\[(1.4) \quad R_{ij} - \frac{1}{2} g_{ij} R = - \frac{8 \pi G}{c^4} T_{ij} - D_i D_j \varphi,\]

where the scalar function \( \varphi : M \to \mathbb{R} \) is called the scalar potential.

The corresponding conservations of mass, energy and momentum are then replaced by

\[(1.5) \quad \text{div} \left( D_i D_j \varphi + \frac{8 \pi G}{c^4} T_{ij} \right) = 0,\]

and the energy-momentum density \( T = g^{ij} T_{ij} \) and the scalar potential energy density \( \frac{c^4}{8 \pi G} \Phi = \frac{c^4}{8 \pi G} g^{ij} D_i D_j \varphi \) satisfy

\[(1.6) \quad R = \frac{8 \pi G}{c^4} T + \Phi,\]
\[(1.7) \quad \int_M \Phi \sqrt{-g} dx = 0.\]

The scalar potential energy density \( \frac{c^4}{8 \pi G} \Phi \) has a number of important physical properties:

1. Gravitation is now described by the Riemannian metric \( g_{ij} \), the scalar potential \( \varphi \) and their interactions, unified by the new gravitational field equations \((1.4)\).

2. This scalar potential energy density \( \frac{c^4}{8 \pi G} \Phi \) represents a new type of energy/force caused by the non-uniform distribution of matter in the universe. This scalar potential energy density varies as the galaxies move and matter of the universe redistributes. Like gravity, it affects every part of the universe as a field.

3. This scalar potential energy density \( \frac{c^4}{8 \pi G} \Phi \) consists of both positive and negative energies. The negative part of this potential energy density produces attraction, and the positive part produces repelling force. The conservation law \((1.7)\) amounts to saying that the total scalar potential energy density is conserved.

4. The sum of this new potential energy density \( \frac{c^4}{8 \pi G} \Phi \) and the coupling energy between the energy-momentum tensor \( T_{ij} \) and the scalar potential field \( \varphi \), as described e.g. by the second term in the right-hand side of \((1.9)\), gives rise to a new unified theory for dark matter and dark energy: The negative part of \( \varepsilon \) represents the dark matter, which produces attraction, and the positive part represents the dark energy, which drives the acceleration of expanding galaxies.

5. The scalar curvature of space-time obeys \((1.6)\). Consequently, when there is no normal matter present (with \( T = 0 \)), the curvature \( R \) of space-time is balanced by \( R = \Phi \). Therefore, there is no real vacuum in the universe.

6. The universe with uniform distributed matter leads to identically zero scalar potential energy, and is unstable. It is this instability that leads to the existence of the dark matter and dark energy, and consequently the high non-homogeneity of the universe.

Hereafter, we further explore a few direct consequences of the above new gravitational field equations.
First, the new field equations are consistent with the spherically symmetric case with cosmic microwave background (CMB). Namely, the existence of solutions in this case can be proved.

Second, our new theory suggests that the curvature $R$ is always balanced by $\Phi$ in the entire space-time by (1.6), and the space-time is no longer flat. Namely, the entire space-time is also curved and is filled with dark energy and dark matter. In particular, the discontinuities of $R$ induced by the discontinuities of the energy-momentum density $T$, dictated by the Einstein field equations, are no longer present thanks to the balance of $\Phi$.

Third, this scalar potential energy density should be viewed as the main cause for the non-homogeneous distribution of the matter/galaxies in the universe, as the dark matter (negative scalar potential energy) attracts and dark energy (positive scalar potential energy) repels different galaxies; see (1.9) below.

Fourth, to further explain the dark matter and dark energy phenomena, we consider a central matter field with total mass $M$ and radius $r_0$ and spherical symmetry. With spherical coordinates, the corresponding Riemannian metric must be of the following form:

$$(1.8) \quad ds^2 = -e^{2u}dt^2 + e^{2v}dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2),$$

where $u = u(r)$ and $v = v(r)$ are functions of the radial distance. With the new field equations, the force exerted on an object with mass $m$ is given by

$$(1.9) \quad F = mMG \left[ -\frac{1}{r^2} - \frac{1}{\delta} \left( 2 + \frac{\delta}{r} \right) \varphi' + \frac{Rr}{\delta} \right], \quad R = \Phi \quad \text{for} \quad r > r_0,$$

where $\delta = 2GM/c^2$, $R$ is the scalar curvature, and $\varphi$ is the scalar potential. The first term is the classical Newton gravitation, the second term is the coupling interaction between matter and the scalar potential $\varphi$, and the third term is the interaction generated by the scalar potential energy density $\frac{c^4}{8\pi G} \Phi$ as indicated in (1.6) ($R = \Phi$ for $r > r_0$). In this formula, the negative and positive values of each term represent respectively the attracting and repelling forces. It is then clear that the combined effect of the second and third terms in the above formula represent the dark matter, dark energy and their interactions with normal matter.

Also, importantly, this formula is a direct representation of the Einstein’s equivalence principle. Namely, the curvature of space-time induces interaction forces between matter.

In addition, one can derive a more detailed version of the above formula:

$$(1.10) \quad F = mMG \left[ -\frac{1}{r^2} + \frac{\delta}{r} \left( 2 + \frac{\delta}{r} \right) \varepsilon r^2 + \frac{Rr}{\delta} + \frac{1}{\delta} \left( 2 + \frac{\delta}{r} \right) r^2 \int_r^{r_0} Rdr \right],$$

where $\varepsilon > 0$. The conservation law (1.7) of $\Phi$ suggests that $R$ behaves as $r^{-2}$ for $r$ sufficiently large. Consequentaly the second term in the right hand side of (1.10) must dominate and be positive, indicating the existence of dark energy.

In fact, the above formula can be further simplified to derive the following approximate formula:

$$(1.11) \quad F = mMG \left[ -\frac{1}{r^2} - \frac{k_0}{r} + k_1 r \right],$$

$$(1.12) \quad k_0 = 4 \times 10^{-18} \text{km}^{-1}, \quad k_1 = 10^{-57} \text{km}^{-3}.$$
Again, in (1.11), the first term represents the Newton gravitation, the attracting second term stands for dark matter and the repelling third term is the dark energy.

The mathematical part of this article is devoted to a rigorous derivation of the new gravitational field equations.

First, as mentioned earlier, the field equations obey the Euler-Lagrange equation of the Einstein-Hilbert functional under the natural divergence-free constraint, with divergence defined at the extremum Riemannian metric $g$:

$$\delta F(g_{ij}), X = 0 \quad \forall \ X = \{X_{ij}\} \text{ with } D^i g_{ij} = 0, \ X_{ij} = X_{ji}.$$  

As the variational elements $X$ are divergence-free, (1.13) does not imply $\delta F(g_{ij}) = 0$, which is the classical Einstein equations. In fact, (1.13) amounts to saying that $\delta F(g_{ij})$ is orthogonal to all divergence-free tensor fields $X$.

Hence we need to decompose general tensor fields on Riemannian manifolds into divergence-free and gradient parts. For this purpose, an orthogonal decomposition theorem is derived in Theorem 3.1. In particular, given an $(r,s)$ tensor field $u \in L^2(T^r_s M)$, we have

$$(1.14) \quad u = \nabla \psi + v, \ \text{div } v = 0, \ \psi \in H^1(T^r_s M).$$

The gradient part is acting on an $(r,s-1)$ tensor field $\psi$.

Second, restricting to a $(0,2)$ symmetric tensor field $u$, the gradient part in the above decomposition is given by

$$\nabla \psi \quad \text{with } \psi = \{\psi_i\} \in H^1(T^0_1 M).$$

Then using symmetry, we show in Theorem 3.2 that this $(0,1)$ tensor $\psi$ can be uniquely determined, up to addition to constants, by the gradient of a scalar field $\varphi$:

$$\psi = \nabla \varphi, \ \varphi \in H^2(M),$$

and consequently we obtain the following decomposition for general symmetric $(0,2)$ tensor fields:

$$(1.15) \quad u_{ij} = v_{ij} + D_i D_j \varphi, \quad D^i v_{ij} = 0, \ \varphi \in H^2(M).$$

Finally, for the symmetric and divergence free $(0,2)$ field $\delta F(g_{ij})$, which is the Euler-Lagrangian of the Einstein-Hilbert functional in (1.13) and is orthogonal to all divergence-free fields, there is a scalar field $\varphi \in H^2(M)$ such that

$$\delta F(g_{ij}) = D_i D_j \varphi,$$

which, by adding the energy-momentum tensor $T_{ij}$, leads to the new gravitational field equations (1.4).

We remark here that the orthogonal decompositions (1.14) and (1.15) are reminiscent of the orthogonal decomposition of vectors fields into gradient and divergence parts, which are crucial for studying incompressible fluid flows; see among many others [7, 8].

This article is divided into two parts. The physically inclined readers can go directly to the physics part after reading this Introduction.
Part 1. Mathematics

2. Preliminaries

2.1. Sobolev spaces of tensor fields. Let \((M, g_{ij})\) be an \(n\)-dimensional Riemannian manifold with metric \((g_{ij})\), and \(E = T^r_m M\) be an \((r, s)\)-tensor bundle on \(M\). A mapping \(u : M \rightarrow E\) is called a section of the tensor-bundle \(E\) or a tensor field. In a local coordinate system \(x\), a tensor field \(u\) can be expressed component-wise as follows:

\[
u = \left\{ u^{i_1 \cdots i_r}_{j_1 \cdots j_r}(x) \mid 1 \leq i_1, \cdots, i_s, j_1, \cdots, j_r \leq n \right\},
\]

where \(u^{i_1 \cdots i_r}_{j_1 \cdots j_r}(x)\) are functions of \(x \in U\). The section \(u\) is called \(C^r\)-tensor field or \(C^r\)-section if its components are \(C^r\)-functions.

For any real number \(1 \leq p < \infty\), let \(L^p(E)\) be the space of all \(L^p\)-integrable sections of \(E\):

\[
L^p(E) = \left\{ u : M \rightarrow E \mid \int_M |u|^p dx < \infty \right\},
\]

equipped with the norm

\[
||u||_{L^p} = \left[ \int_M |u|^p dx \right]^{1/p} = \left[ \int_M \sum |u^{i_1 \cdots i_r}_{j_1 \cdots j_r}|^p dx \right]^{1/p}.
\]

For \(p = 2\), \(L^2(E)\) is a Hilbert space equipped with the inner product

\[
(u, v) = \int_M g_{j_1 k_1} \cdots g_{j_r k_r} g^{i_1 l_1} \cdots g^{i_s l_s} u^{i_1 \cdots i_r}_{j_1 \cdots j_r} v^{k_1 \cdots k_r}_{l_1 \cdots l_r} \sqrt{-g} dx,
\]

where \((g_{ij})\) is Riemannian metric, \((g^{ij}) = (g_{ij})^{-1}\), \(g = \det(g_{ij})\), and \(\sqrt{-g} dx\) is the volume element.

For any positive integer \(k\) and any real number \(1 \leq p < \infty\), we can also define the Sobolev spaces \(W^{k,p}(E)\) to be the subspace of \(L^p(E)\) such that all covariant derivatives of \(u\) up to order \(k\) are in \(L^p(E)\). The norm of \(W^{k,p}(E)\) is always denoted by \(\| \cdot \|_{W^{k,p}}\).

As \(p = 2\), the spaces \(W^{k,p}(E)\) are Hilbert spaces, and are usually denoted by

\[
H^k(E) = W^{k,2}(E) \quad \text{for} \ k \geq 0,
\]

equipped with inner product \((\cdot, \cdot)_{H^k}\) and norm \(\| \cdot \|_{H^k}\).

2.2. Gradient and divergent operators. Let \(u : M \rightarrow E\) be an \((r, s)\)-tensor field, with the local expression

\[
u = \left\{ u^{i_1 \cdots i_r}_{j_1 \cdots j_r} \right\}.
\]

Then the gradient of \(u\) is defined as

\[
\nabla u = \left\{ D_k u^{i_1 \cdots i_r}_{j_1 \cdots j_r} \right\},
\]

where \(D = (D_1, \cdots, D_n)\) is the covariant derivative. It is clear that the gradient \(\nabla u\) defined by (2.4) is an \((r, s + 1)\)-tensor field:

\[
\nabla u : M \rightarrow T^{r+1}_s M.
\]

We define \(\nabla^* u\) as

\[
\nabla^* u = \{ g^{kl} D_l u \} : M \rightarrow T^{r+1}_s M \quad \text{for} \ u \ \text{as in} \ (2.3).
\]
For an \((r+1, s)\)-tensor field \(u = \{u^{i_1, \ldots, i_r}_{i_{s+1}, \ldots, i_s}\}\), the divergence of \(u\) is defined by
\[
\text{div} \ u = \{D_{i_1} u^{i_1, \ldots, i_r}_{i_{s+1}, \ldots, i_s}\}.
\]
Therefore, the divergence \(\text{div} \ u\) defined by (2.6) is an \((r, s)\)-tensor field. Likewise, for an \((r, s+1)\)-tensor field
\[
u = \{u^{i_1, \ldots, i_r}_{i_{s+1}, \ldots, i_s}\},
\]
the following operator is also called the divergence of \(\nu\),
\[
\text{div} \ \nu = \{D_{i_1} u^{i_1, \ldots, i_r}_{i_{s+1}, \ldots, i_s}\},
\]
where \(D_{i_1} = g^{lk} D_k\), which is an \((r, s)\)-tensor field.

For the gradient operators (2.4)-(2.5) and the divergent operators (2.6)-(2.7), it is well known that the following integral formulas hold true; see among others [2].

**Theorem 2.1.** Let \((M, g_{ij})\) be a closed Riemannian manifold. If \(u\) is an \((r-1, s)\)-tensor and \(v\) is an \((r, s)\)-tensor, then we have
\[
(\nabla^* u, v) = - (u, \text{div} \ v),
\]
where \(\nabla^* u\) is as in (2.7) and \(\text{div} \ v\) is as in (2.6), the inner product \((\cdot, \cdot)\) is as defined by (2.1). If \(u\) is an \((r, s-1)\)-tensor and \(v\) is an \((r, s)\)-tensor, then
\[
(\nabla u, v) = - (u, \text{div} \ v),
\]
where \(\nabla u\) is as in (2.4) and \(\text{div} \ v\) is as in (2.7).

**Remark 2.1.** If \(M\) is a manifold with boundary \(\partial M \neq \emptyset\), and \(u|_{\partial M} = 0\) or \(v|_{\partial M} = 0\), then the formulas (2.8) and (2.9) still hold true.

**2.3. Acute-angle principle.** Let \(H\) be a Hilbert space equipped with inner product \((\cdot, \cdot)\) and norm \(\| \cdot \|\), and \(G : H \to H\) be a mapping. We say that \(G\) is weakly continuous if for any sequence \(\{u_n\} \subset H\) weakly converging to \(u_0\), i.e.
\[
u_n \rightharpoonup u_0 \text{ in } H,
\]
we have
\[
\lim_{n \to \infty} (G \nu_n, v) = (G \nu_0, v) \quad \forall v \in H.
\]
If the operator \(G\) is linear and bounded, then \(G\) is weakly continuous. The following theorem is called the acute-angle principle [5].

**Theorem 2.2.** If a mapping \(G : H \to H\) is weakly continuous, and satisfies
\[
(Gu, u) \geq \alpha \|u\|^2 - \beta,
\]
for some constants \(\alpha, \beta > 0\), then for any \(f \in H\) there is a \(u_0 \in H\) such that
\[
(Gu_0 - f, v) = 0 \quad \forall v \in H.
\]

**3. Orthogonal Decomposition for Tensor Fields**

**3.1. Main theorems.** The aim of this section is to derive an orthogonal decomposition for \((r, s)\)-tensor fields with \(r + s \geq 1\) into divergence-free and gradient parts. This decomposition plays a crucial role for the theory of gravitational field, dark matter and dark energy developed later in this article.

Let \(M\) be a closed Riemannian manifold, and \(v \in L^2(T^*_r M)\) \((r + s \geq 1)\). We say that \(v\) is divergence-free, i.e., \(\text{div} \ v = 0\), if for all \(\nabla \psi \in L^2(T^*_r M)\),
\[
(v, \nabla \psi) = 0.
\]
Here \( \psi \in H^1(T_s^{-1}M) \) or \( H^1(T_{s-1}^r M) \), and \( \langle \cdot, \cdot \rangle \) is the \( L^2 \)-inner product defined by \([2.1]\).

We remark that if \( v \in H^1(T_s^r M) \) satisfies \([3.1]\), then \( v \) is weakly differentiable, and \( \operatorname{div} v = 0 \) in the \( L^2 \)-sense. If \( v \in L^2(T_s^r M) \) is not differential, then \([3.1]\) means that \( v \) is divergence-free in the distribution sense.

**Theorem 3.1** (Orthogonal Decomposition Theorem). Let \( M \) be a closed Riemannian manifold, and \( u \in L^2(T_s^r M) \) with \( r + s \geq 1 \). The following assertions hold true:

1. The tensor field \( u \) has the following orthogonal decomposition:
   \[
   u = \nabla \varphi + v,
   \]
   where \( \varphi \in H^1(T_s^{-1}M) \) or \( \varphi \in H^1(T_{s-1}^r M) \), and \( \operatorname{div} v = 0 \).
2. If \( M \) is compact, then \( u \) can be orthogonally decomposed into
   \[
   u = \nabla \varphi + v + h,
   \]
   where \( \varphi \) and \( v \) are as in \([3.2]\), and \( h \) is a harmonic field, i.e.
   \[
   \text{div} v = 0, \quad \text{div} h = 0, \quad \nabla h = 0.
   \]

   In particular the subspace of all harmonic tensor fields in \( L^2(T_s^r M) \) is of finite dimensional:
   \[
   H(T_s^r M) = \{ h \in L^2(T_s^r M) | \nabla h = 0, \text{ div } h = 0 \}, \quad \dim H < \infty.
   \]

**Remark 3.1.** The above orthogonal decomposition theorem implies that \( L^2(E) \) \((E = T_s^r M)\) can be decomposed into
   \[
   L^2(E) = G(E) \oplus L^2_D(E) \quad \text{for } \partial M = \emptyset,
   \]
   \[
   L^2(E) = G(E) \oplus H(E) \oplus L^2_N(E) \quad \text{for } M \text{ compact.}
   \]

Here \( H \) is as in \([3.5]\), and
   \[
   G(E) = \{ v \in L^2(E) | \nabla \varphi, \varphi \in H^1(T_{s-1}^r M) \},
   \]
   \[
   L^2_D(E) = \{ v \in L^2(E) | \text{ div } v = 0 \},
   \]
   \[
   L^2_N(E) = \{ v \in L^2(E) | \nabla v \neq 0 \}.
   \]

They are orthogonal to each other:
   \[
   L^2_D(E) \perp G(E), \quad L^2_N(E) \perp H(E), \quad G(E) \perp H(E).
   \]

**Remark 3.2.** The dimension of the harmonic space \( H(E) \) is related with the bundle structure of \( E = T_s^r M \). It is conjectured that
   \[
   \dim H = k \quad \text{the degree of freedom of } E.
   \]

Namely \( k \) is the integer that \( E \) can be decomposed into the Whitney sum of a \( k \)-dimensional trivial bundle \( E^k = M \times \mathbb{R}^k \) and a nontrivial bundle \( E_1 \), i.e.
   \[
   E = E_1 \oplus E^k.
   \]

**Proof of Theorem 3.1.** We proceed in several steps as follows.

**Step 1 Proof of Assertion (1).** Let \( u \in L^2(E), E = T_s^r M \) \((r + s \geq 1)\). Consider the equation
   \[
   \Delta \varphi = \operatorname{div} u \quad \text{in } M,
   \]
   where \( \Delta \) is the Laplace operator defined by
   \[
   \Delta = \operatorname{div} \nabla.
   \]
Without loss of generality, we only consider the case where \( \text{div} \, u \in T^*_{\mathbb{R}} M \).
It is clear that if the equation (3.6) has a solution \( \varphi \in H^1(\tilde{E}) \), then by (3.7), the following vector field must be divergence-free
\[
(3.8) \quad v = u - \nabla \varphi \in L^2(E).
\]
Moreover, by (3.1) we have
\[
(3.9) \quad (\nabla \varphi - u, \nabla \psi) = 0 \quad \forall \psi \in H^1(\tilde{E}).
\]
Namely \( v \) and \( \nabla \varphi \) are orthogonal. Therefore, the orthogonal decomposition \( u = v + \nabla \varphi \) follows from (3.8) and (3.9).

It suffices then to prove that (3.6) has a weak solution \( \varphi \in H^1(\tilde{E}) \):
\[
(3.10) \quad (\nabla \varphi - u, \nabla \psi) = 0 \quad \forall \psi \in H^1(\tilde{E}).
\]
To this end, let
\[
H = H^1(\tilde{E}) \setminus \tilde{H},
\]
\[
\tilde{H} = \{ \psi \in H^1(\tilde{E}) | \nabla \psi = 0 \}.
\]
Then we define a linear operator \( G : H \to H \) by
\[
(3.11) \quad (G\varphi, \psi) = (\nabla \varphi, \nabla \psi) \quad \forall \psi \in H.
\]
It is clear that the linear operator \( G : H \to H \) is bounded, weakly continuous, and
\[
(3.12) \quad (G\varphi, \varphi) = (\nabla \varphi, \nabla \varphi) = ||\varphi||^2.
\]
Based on Theorem 2.2 for any \( f \in H \), the equation
\[
\Delta \varphi = f \quad \text{in} \ M
\]
has a weak solution \( \varphi \in H \). Hence for \( f = \text{div} \, u \) the equation (3.6) has a solution, and Assertion (1) is proved. In fact the solution of (3.6) is unique. We remark that by the Poincaré inequality, for the space \( H = H^1(\tilde{E}) \setminus \{ \psi | \nabla \psi = 0 \} \), (3.12) is an equivalent norm of \( H \). In addition, by Theorem 2.1 the weak formulation (3.10) for (3.6) is well-defined.

**Step 2 Proof of Assertion (2).** Based on Assertion (1), we have
\[
H^k(E) = H^k_{\mathbb{D}} \oplus G^k,
\]
\[
L^2(E) = L^2_{\mathbb{D}} \oplus G,
\]
where
\[
H^k_{\mathbb{D}} = \{ u \in H^k(E) | \text{div} \, u = 0 \},
\]
\[
G^k = \{ u \in H^k(E) | u = \nabla \psi \}.
\]
Define an operator \( \hat{\Delta} : H^2_{\mathbb{D}}(E) \to L^2_{\mathbb{D}}(E) \) by
\[
(3.13) \quad \hat{\Delta} u = P \Delta u,
\]
where \( P : L^2(E) \to L^2_{\mathbb{D}}(E) \) is the canonical orthogonal projection.
We know that the Laplace operator \( \Delta \) can be expressed as
\[
(3.14) \quad \Delta = \text{div} \nabla = D^k D_k = g^{kl} \frac{\partial^2}{\partial x^k \partial x^l} + B,
\]
where $B$ is the lower order derivative linear operator. Since $M$ is compact, the Sobolev embeddings $H^2(E) \hookrightarrow H^1(E) \hookrightarrow L^2(E)$ are compact, which implies that the lower order derivative operator

$$B : H^2(M, \mathbb{R}^N) \rightarrow L^2(M, \mathbb{R}^N)$$

is compact, where the integer $N$ is the dimension of the tensor bundle $E$. According to the elliptic operator theory, the elliptic operator in (3.14)

$$A = g^{kl} \frac{\partial^2}{\partial x^k \partial x^l} : H^2(M, \mathbb{R}^N) \rightarrow L^2(M, \mathbb{R}^N)$$

is a linear homeomorphism. Therefore the operator in (3.14) is a linear completely continuous field

$$\Delta : H^2(E) \rightarrow L^2(E),$$

which implies that the operator of (3.13) is also a linear completely continuous field:

$$\tilde{\Delta} = P \Delta : H_D^2(E) \rightarrow L_D^2(E).$$

By the spectral theorem of completely continuous fields [6, 8], the space

$$\tilde{H} = \{ u \in H_D^2(E) | \tilde{\Delta} u = 0 \}$$

is finite dimensional, and is the eigenspace of the eigenvalue $\lambda = 0$. By Theorem 2.1, for $u \in \tilde{H}$

$$\int_M (\tilde{\Delta} u, u) \sqrt{-g} dx = \int_M (\Delta u, u) \sqrt{-g} dx \quad \text{(by div } u = 0)$$

$$= - \int_M (\nabla u, \nabla u) \sqrt{-g} dx$$

$$= 0 \quad \text{(by } \tilde{\Delta} u = 0).$$

It follows that

$$u \in \tilde{H} \iff \nabla u = 0 \Rightarrow H = \tilde{H},$$

where $H$ is the harmonic space as in (3.5). Thus we have

$$L_D^2(E) = H \oplus L_2^N(E),$$

$$L_2^N(E) = \{ u \in L_D^2(E) | \nabla u \neq 0 \}.$$

The proof of Theorem 3.1 is complete. \hfill \Box

3.2. **Uniqueness of the orthogonal decomposition.** In Theorem 3.1 a tensor field $u \in L^2(T^r_s M)$ with $r + s \geq 1$ can be orthogonally decomposed into

$$u = \nabla \varphi + v \quad \text{for } \partial M = \emptyset,$$

$$u = \nabla \varphi + v + h \quad \text{for } M \text{ compact.} \quad (3.15)$$

Now we address the uniqueness problem of the decomposition (3.15). In fact, if $u$ is a vector field or a co-vector field, i.e.

$$u \in L^2(TM) \text{ or } u \in L^2(T^* M),$$

then the decomposition of (3.15) is unique.

We can see that if $u \in L^2(T^r_s M)$ with $r + s \geq 2$, then there are different types of the decompositions of (3.15). For example, for $u \in L^2(T^3_2 M)$, the local expression of $u$ is given by

$$u = \{ u_{ij}(x) \}. \quad (3.15)$$
In this case, \( u \) has two types of decompositions:

\[
\begin{align*}
(3.16) & \quad u_{ij} = D_i \varphi_j + v_{ij}, \quad D^i v_{ij} = 0, \\
(3.17) & \quad u_{ij} = D_j \psi_i + w_{ij}, \quad D^i w_{ij} = 0.
\end{align*}
\]

It is easy to see that if \( u_{ij} \neq u_{ji} \) then both (3.16) and (3.17) are two different decompositions of \( u_{ij} \). Namely

\[
\{v_{ij}\} \neq \{w_{ij}\}, \quad (\varphi_1, \ldots, \varphi_n) \neq (\psi_1, \ldots, \psi_n).
\]

If \( u_{ij} = u_{ji} \) is symmetric, \( u \) can be orthogonally decomposed into the following two forms:

\[
\begin{align*}
(3.18) & \quad u_{ij} = v_{ij} + D_i \varphi_j, \quad D^i v_{ij} = 0, \\
(3.19) & \quad u_{ij} = w_{ij} + D_j \psi_i, \quad D^i w_{ij} = 0.
\end{align*}
\]

By \( u_{ij} = u_{ji} \) we have \( D_k u_{kj} = D_k u_{jk} \). Hence, (3.18) and (3.19) are the same, and \( \varphi = \psi \). Therefore, the symmetric tensors \( u_{ij} \) can be written as

\[
\begin{align*}
(3.20) & \quad u_{ij} = v_{ij} + D_i \varphi_j, \quad D^i v_{ij} = 0, \\
(3.21) & \quad u_{ij} = w_{ij} + D_j \varphi_i, \quad D^i w_{ij} = 0.
\end{align*}
\]

From (3.20)-(3.21) we can deduce the following theorem.

**Theorem 3.2.** Let \( u \in L^2(T^0_2 M) \) be symmetric, i.e. \( u_{ij} = u_{ji} \), and the first Betti number \( \beta_1(M) = 0 \) for \( M \). Then the following assertions hold true:

1. \( u \) has a unique orthogonal decomposition if and only if there is a scalar function \( \varphi \in H^2(M) \) such that \( u \) can be expressed as \( (3.22) \)

\[
\begin{align*}
(3.22) & \quad u_{ij} = v_{ij} + D_i \varphi_j, \\
v_{ij} = v_{ji}, \quad D^i v_{ij} = 0.
\end{align*}
\]

2. \( u \) can be orthogonally decomposed in the form of (3.22) if and only if \( u_{ij} \) satisfy

\[
(3.23) \quad \frac{\partial}{\partial x^j}(D^k u_{ki}) - \frac{\partial}{\partial x^i}(D^k u_{kj}) = \frac{\partial}{\partial x^j} \left( R^k_j \frac{\partial \varphi}{\partial x^k} \right) - \frac{\partial}{\partial x^i} \left( R^k_i \frac{\partial \varphi}{\partial x^k} \right),
\]

where \( R^k_j = g^{ki} R_{ij} \) and \( R_{ij} \) are the Ricci curvature tensors.

3. If \( v_{ij} \) in (3.20) is symmetric: \( v_{ij} = v_{ji} \), then \( u \) can be expressed by (3.22).

**Proof.** We only need to prove Assertions (2) and (3).

We first prove Assertion (2). It follows from (3.20) that

\[
(3.24) \quad \frac{\partial}{\partial x^j}(D^k u_{ki}) - \frac{\partial}{\partial x^i}(D^k u_{kj}) = \frac{\partial \Delta \varphi_i}{\partial x^j} - \frac{\partial \Delta \varphi_j}{\partial x^i},
\]

where \( \Delta = D^k D_k \). By the Weitzenböck formula [4],

\[
(3.25) \quad \Delta \varphi_i = -(\delta d + d \delta) \varphi_i - R^k_i \varphi_k,
\]
and \((\delta d + d\delta)\) is the Laplace-Beltrami operator. We know that for \(\omega = \varphi_i dx^i\),

\[
d\omega = 0 \quad \iff \quad \varphi_i = \frac{\partial \varphi}{\partial x^i},
\]

\[
d\delta \omega = \nabla(\tilde{\Delta} \varphi) \quad \iff \quad \varphi_i = \frac{\partial \varphi}{\partial x^i},
\]

where \(\nabla\) is the gradient operator, and

\[
\tilde{\Delta} \varphi = -\frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^i} \left( \sqrt{-g} g^{ij} \frac{\partial \varphi}{\partial x^j} \right).
\]

Namely

\[(\delta d + d\delta) \varphi_i = \frac{\partial}{\partial x^i} \tilde{\Delta} \varphi \iff \varphi_i = \frac{\partial \varphi}{\partial x^i}.
\]

Hence, we infer from (3.25) that

\[(3.26) \quad \Delta \varphi_i = -\frac{\partial}{\partial x^i} \tilde{\Delta} \varphi - R^k_i \frac{\partial \varphi}{\partial x^k} \iff \varphi_i = \frac{\partial \varphi}{\partial x^i}.
\]

Thus, by (3.24) and (3.26), we obtain that

\[
\frac{\partial}{\partial x^j}(D^k u_{ki}) - \frac{\partial}{\partial x^i}(D^k u_{kj}) = \frac{\partial}{\partial x^i}(R^k_i \frac{\partial \varphi}{\partial x^k}) - \frac{\partial}{\partial x^j}(R^k_i \frac{\partial \varphi}{\partial x^k})
\]

holds true if and only if the tensor \(\psi = (\varphi_1, \cdots, \varphi_n)\) in (3.20) is a gradient \(\psi = \nabla \varphi\). Assertion (2) is proven.

Now we verify Assertion (3). Since \(v_{ij}\) in (3.20) is symmetric, we have

\[(3.27) \quad D_i \varphi_j = D_j \varphi_i.
\]

Note that

\[(3.28) \quad D_i \varphi_j = \frac{\partial \varphi_j}{\partial x^i} - \Gamma^k_{ij} \varphi_k,
\]

where \(\Gamma^k_{ij}\) is the Levi-Civita connection, and \(\Gamma^k_{ij} = \Gamma^k_{ji}\). We infer then from (3.27) that

\[(3.29) \quad \frac{\partial \varphi_j}{\partial x^i} = \frac{\partial \varphi_i}{\partial x^j}.
\]

By assumption, the 1-dimensional homology of \(M\) is zero,

\[
H_1(M) = 0,
\]

and it follows from the de Rham theorem and (3.29) that

\[
\varphi_k = \frac{\partial \varphi}{\partial x^k},
\]

for some scalar function \(\varphi\). Thus Assertion (3) follows and the proof is complete. \(\square\)

**Remark 3.3.** The conclusions of Theorem 3.2 are also valid for second-order contra-variant symmetric tensors \(u = \{u^{ij}\}\), and the decomposition is given as follows:

\[
u^{ij} = v^{ij} + g^{ik} g^{jl} D_k D_l \varphi,
\]

\[
D_i v^{ij} = 0, \quad v^{ij} = v^{ji}, \quad \varphi \in H^2(M).
\]
4. Variational Principle for Functionals of Riemannian Metric

4.1. General theory. Hereafter we always assume that $M$ is a closed manifold. A Riemannian metric $G$ on $M$ is a mapping

\[(4.1)\quad G : M \to T_0^*M = T^*M \otimes T^*M,\]

which is symmetric and nondegenerate, i.e., in a local coordinate $(v, x)$, $G$ can be expressed as

\[(4.2)\quad G = \{g_{ij}(x)\} \quad \text{with} \quad g_{ij} = g_{ji},\]

and the matrix $(g_{ij})$ is invertible on $M$:

\[(4.3)\quad (g_{ij}) = (g_{ij})^{-1}.\]

If we regard a Riemannian metric $G = \{g_{ij}\}$ as a tensor field on manifold $M$, then the set of all metrics $G = \{g_{ij}\}$ on $M$ constitutes a topological space, called the space of Riemannian metrics on $M$. We denote

\[(4.4)\quad G^{-1} = \{g^{ij}\} : M \to T_0^*M = TM \otimes TM.\]

The space for Riemannian metrics on $M$ is denied by $W^{m,2}(M, g)$ as in (4.2),

\[(4.5)\quad W^{m,2}(M, g) = \left\{ G \mid G \in W^{m,2}(T_0^*M), G^{-1} \in W^{m,2}(T_0^*M), \right.\]

$G$ is the Riemannian metric on $M$ as in (4.2),

which is a metric space, but not a Banach space. However, it is a subspace of the direct sum of two Sobolev spaces $W^{m,2}(T_0^*M)$ and $W^{m,2}(T_0^2M)$:

\[W^{m,2}(M, g) \subset W^{m,2}(T_0^*M) \oplus W^{m,2}(T_0^2M).\]

A functional defined on $W^{m,2}(M, g)$:

\[(4.6)\quad F : W^{m,2}(M, g) \to \mathbb{R}\]

is called the functional of Riemannian metric. Usually, the functional (4.6) can be expressed as

\[(4.7)\quad F(g_{ij}) = \int_M f(x, g_{ij}, \cdots, \partial^m g_{ij}) \sqrt{-g} dx.\]

Since $(g^{ij})$ is the inverse of $(g_{ij})$, we have

\[(4.8)\quad g_{ij} = \frac{1}{g} \times \text{the cofactor of } g^{ij}.\]

Therefore, $F(g_{ij})$ in (4.7) also depends on $g^{ij}$, i.e. putting (4.8) in (4.7) we get

\[(4.9)\quad F(g^{ij}) = \int_M f(x, g^{ij}, \cdots, D^m g_{ij}) \sqrt{-g} dx.\]

We note that although $W^{m,2}(M, g)$ is not a linear space, for a given element $g_{ij} \in W^{m,2}(M, g)$ and any symmetric tensor $X_{ij}$ and $X^{ij}$, there is a number $\lambda_0 > 0$ such that

\[(4.10)\quad g_{ij} + \lambda X_{ij} \in W^{m,2}(M, g) \quad \forall 0 \leq |\lambda| < \lambda_0,\]

\[g^{ij} + \lambda X^{ij} \in W^{m,2}(M, g) \quad \forall 0 \leq |\lambda| < \lambda_0.\]
Due to (4.10), we can define the derivative operators of the functional $F$, which are also called the Euler-Lagrange operators of $F$, as follows

\begin{align}
\delta_* F : W^{m,2}(M,g) &\to W^{-m,2}(T^2_0 M), \\
\delta^* F : W^{m,2}(M,g) &\to W^{-m,2}(T^0_2 M),
\end{align}

where $W^{-m,2}(E)$ is the dual space of $W^{m,2}(E)$, and $\delta_* F, \delta^* F$ are given by

\begin{align}
(\delta_* F(g_{ij}), X) &= \frac{d}{d\lambda} F(g_{ij} + \lambda X_{ij})|_{\lambda=0}, \\
(\delta^* F(g^{ij}), X) &= \frac{d}{d\lambda} F(g^{ij} + \lambda X^{ij})|_{\lambda=0}.
\end{align}

For any given metric $g_{ij} \in W^{m,2}(M,g)$, the value of $\delta_* F$ and $\delta^* F$ at $g_{ij}$ are second-order contra-variant and covariant tensor fields respectively, i.e.

$$
\delta_* F(g_{ij}) : M \to TM \otimes TM,
$$

$$
\delta^* F(g_{ij}) : M \to T^* M \otimes T^* M.
$$

Moreover, the equations

\begin{align}
\delta_* F(g_{ij}) &= 0, \\
\delta^* F(g_{ij}) &= 0,
\end{align}

are called the Euler-Lagrange equations of $F$, and the solutions of (4.15) and (4.16) are called the extremum points or critical points of $F$.

**Theorem 4.1.** Let $F$ be the functionals defined by (4.6) and (4.9). Then the following assertions hold true:

1. For any $g_{ij} \in W^{m,2}(M,g), \delta_* F(g_{ij})$ and $\delta^* F(g_{ij})$ are symmetric tensor fields.
2. If $\{g^{ij}\} \in W^{m,2}(M,g)$ is the extremum point of $F$, then $\{g^{ij}\}$ is also an extremum point of $F$, i.e. $\{g^{ij}\}$ satisfies (4.15) and (4.16) if and only if $\{g^{ij}\}$ satisfies (4.15) and (4.16).
3. $\delta_* F$ and $\delta^* F$ have the following relation

$$
(\delta^* F(g_{ij}))^{kl} = -g^{kr} g^{ls} (\delta_* F(g_{ij}))_{rs},
$$

where $(\delta^* F)^{kl}$ and $(\delta_* F)_{kl}$ are the components of $\delta^* F$ and $\delta_* F$ respectively.

**Proof.** We only need to verify Assertion (3). Noting that

$$
g_{ik} g^{kj} = \delta_i^j,
$$

we have the variational relation

$$
\delta(g_{ik} g^{kj}) = g_{ik} \delta g^{kj} + g^{kj} \delta g_{ik} = 0.
$$

It implies that

\begin{align}
\delta g^{kl} &= -g^{ki} g^{lj} \delta g_{ij}, \\
\lambda X_{ij} &= \delta g_{ij}, \quad \lambda X^{ij} = \delta g^{ij}, \quad \lambda \neq 0 \text{ small}.
\end{align}

In addition, in (4.13) and (4.14),

$$
(\delta_* F)_{kl} = \lambda X_{ij}, \quad (\delta^* F)^{kl} = \lambda X^{ij}.
$$

Therefore, by (4.17) we get

$$
((\delta_* F)_{kl}, \delta g^{kl}) = -((\delta_* F)_{kl}, g^{ki} g^{lj} \delta g_{ij}) = (g^{ki} g^{lj} (\delta_* F)_{kl}, \delta g_{ij}) = ((\delta^* F)^{ij}, \delta g_{ij}).
$$
Hence
\[(δ^* F)^{ij} = -g^{ki} g^{lj} (δ^* F)_{kl}.\]
Thus Assertion (3) follows and the proof is complete. □

4.2. Scalar potential theorem for constraint variations. We know that the critical points of the functional \( F \) in (4.6) are the solution (4.18)
\[δF(g_{ij}) = 0,\]
in the following sense
(4.19)
\[(δF(g_{ij}), X) = \frac{d}{dλ} F(g^{ij} + λX^{ij})|_{λ=0} = \int_M (δF(g_{ij}))_{kl} X^{kl} \sqrt{-g} dx = 0 \ ∀ X^{kl} = X^{lk} in L^2(E),\]
where \( E = TM ⊗ TM \). Hence, the critical points of functionals of Riemannian metrics are not solutions of (4.18) in the usual sense.

It is easy to see that \( L^2(TM ⊗ TM) \) can be orthogonally decomposed into the direct sum of the symmetric and contra-symmetric spaces, i.e.

\[L^2(E) = L^2_s(E) ⊕ L^2_c(E),\]
(4.20)
\[L^2_s(E) = \{ u ∈ L^2(E) \mid u_{ij} = u_{ji} \},\]
\[L^2_c(E) = \{ v ∈ L^2(E) \mid v_{ij} = −v_{ji} \}.\]

Since \( δF \) is symmetric, by (4.20) the extremum points \( \{g_{ij}\} \) of \( F \) satisfy more general equality

(4.21)
\[(δF(g_{ij}), X) = 0 \ ∀ X = \{ X_{ij} \} ∈ L^2(E).\]

Thus, we can say that the extremum points of functionals of the Riemannian metrics are solutions of (4.18) in the usual sense of (4.21), or are zero points of the variational operators
\[δF : W^{m,2}(M, g) → W^{-m,2}(E).\]

Now we consider the variations of \( F \) under the divergence-free constraint. In this case, the Euler-Lagrangian equations with symmetric divergence-free constraints are equivalent to the Euler-Lagrangian equations with general divergence-free constraints. Hence we have the following definition.

**Definition 4.1.** Let \( F : W^{m,2}(M, g) → ℝ \) be a functional of Riemannian metric. A metric tensor \( \{g_{ij}\} ∈ W^{m,2}(M, g) \) is called an extremum point of \( F \) with divergence-free constraint, if \( \{g_{ij}\} \) satisfies
(4.22)
\[(δF(g_{ij}), X) = 0 \ ∀ X = \{ X_{ij} \} ⊂ L^2_D(E),\]
where \( L^2_D(E) \) is the space of all divergence-free tensors:
\[L^2_D(E) = \{ X ∈ L^2(E) \mid \text{div } X = 0 \}.\]

It is clear that an extremum point satisfying (4.22) is not a solution of (4.18). Instead, we have the scalar potential theorem for the extremum points of divergence free constraint (4.22), which is based on the orthogonal decomposition theorems. This result is also crucial for the gravitational field equations and the theory of dark matter and dark energy developed later.
Theorem 4.2 (Scalar Potential Theorem). Assume that the first Betti number of $M$ is zero, i.e. $\beta_1(M) = 0$. Let $F$ be a functional of the Riemannian metric. Then there is a $\varphi \in H^2(M)$ such that the extremum points \( \{g_{ij}\} \) of $F$ with divergence-free constraint satisfy

\[(\delta F(g_{ij}))_{kl} = D_k D_l \varphi.\]

Proof. Let \( \{g_{ij}\} \) be an extremum point of $F$ under the constraint \( (4.22) \). Namely, $\delta F(g_{ij})$ satisfies

\[\int_M (\delta F(g_{ij}))_{kl} X^{kl} \sqrt{-g} dx = 0 \quad \forall X = \{X_{kl}\} \text{ with } D_k X^{kl} = 0.\]

By Theorem 3.1, $\delta^* F(g_{ij})$ can be orthogonally decomposed as

\[(\delta F(g_{ij}))_{kl} = v_{kl} + D_k \psi_l, \quad D^k v_{kl} = 0.\]

By Theorem 2.1, for any $D_k X^{kl} = 0$,

\[(D_k \psi_l, X^{kl}) = \int_M D_k \psi_l X^{kl} \sqrt{-g} dx = -\int_M \psi_l D_k X^{kl} \sqrt{-g} dx = 0.\]

Therefore it follows from (4.24)-(4.26) that

\[\int_M v_{kl} X^{kl} \sqrt{-g} dx = 0 \quad \forall D_k X^{kl} = 0.\]

Let $X^{kl} = g^{ki} g^{lj} v_{ij}$. Since $D_k g_{ij} = D_k g^{ij} = 0$,

we have

\[D_k X^{kl} = D_k (g^{ki} g^{lj} v_{ij}) = g^{lj} (g^{ik} D_k v_{ij}) = g^{lj} D^i v_{ij} = 0,\]

thanks to $D^i v_{ij} = 0$. Inserting $X^{kl} = g^{ki} g^{lj} v_{ij}$ into (4.27) leads to

\[||v||^2_2 = \int_M g^{ki} g^{lj} v_{kl} v_{ij} \sqrt{-g} dx = 0,\]

which implies that $v = 0$. Thus, (4.25) becomes

\[(\delta F(g_{ij}))_{kl} = D_k \psi_l.\]

By Theorem 4.1, $\delta F$ is symmetric. Hence we have

\[D_k \psi_l = D_l \psi_k.\]

It follows from (4.28) that

\[\frac{\partial \psi_l}{\partial x^k} = \frac{\partial \psi_k}{\partial x^l}.\]

By assumption, the first Betti number of $M$ is zero, i.e. the 1-dimensional homology of $M$ is zero: $H_1(M) = 0$. It follows from the de Rham theorem that if

\[d(\psi_k dx^k) = \left( \frac{\partial \psi_k}{\partial x^l} - \frac{\partial \psi_l}{\partial x^k} \right) dx^l \wedge dx^k = 0,\]

then there exists a scalar function $\varphi$ such that

\[d\varphi = \frac{\partial \varphi}{\partial x^k} dx^k = \psi_k dx^k.\]

Thus, we infer from (4.29) that

\[\psi_l = \frac{\partial \varphi}{\partial x^l} \text{ for some } \varphi \in H^2(M).\]
Therefore we get (4.23) from (4.28). The theorem is proved. □

If the first Betti number $\beta_1(M) \neq 0$, then there are $N = \beta_1(M)$ number of 1-forms:

$$\omega_j = \psi_j^k dx^k \in H^1_\partial(M) \quad \text{for} \quad 1 \leq j \leq N,$$

which constitute a basis of the 1-dimensional de Rham homology $H^1_\partial(M)$. We know that the components of $\omega_j$ are co-vector fields:

$$\psi^j = (\psi^j_1, \cdots, \psi^j_n) \in H^1(T^*M) \quad \text{for} \quad 1 \leq j \leq N,$$

which possess the following properties:

$$\frac{\partial \psi^j_k}{\partial x_l} = \frac{\partial \psi^j_l}{\partial x_k} \quad \text{for} \quad 1 \leq j \leq N,$$

or equivalently,

$$D_l \psi^j_k = D_k \psi^j_l \quad \text{for} \quad 1 \leq j \leq N.$$

Namely, $\nabla \psi^j \in L^2(T^*M \otimes T^*M)$ are symmetric second-order contra-variant tensors. Hence Theorem 4.2 can be extended to the non-vanishing first Betti number case as follows.

**Theorem 4.3.** Let the first Betti number $\beta_1(M) \neq 0$ for $M$. Then for the functional $F$ of Riemannian metrics, the extremum points $\{g_{ij}\}$ of $F$ with the constraint (4.22) satisfy the equations

$$\delta F(g_{ij})_{kl} = D_k D_l \varphi + \sum_{j=1}^N \alpha_j D_k \psi^j_l,$$

where $N = \beta_1(M)$, $\alpha_j$ are constants, $\varphi \in H^2(M)$, and the tensors $\psi^j = (\psi^j_1, \cdots, \psi^j_n) \in H^1(T^*M)$ are as given by (4.31).

The proof of Theorem 4.3 is similar to Theorem 4.2 and is omitted here.

**Remark 4.1.** By the Hodge decomposition theory, the 1-forms $\omega_j$ in (4.30) are harmonic:

$$d\omega_j = 0, \delta \omega_j = 0 \quad \text{for} \quad 1 \leq j \leq N,$$

which implies that the tensors $\psi^j$ in (4.32) satisfy

$$(\delta d + d\delta)\psi^j = 0 \quad \text{for} \quad 1 \leq j \leq N.$$

According to the Weitzenböck formula (3.25), we obtain from (4.33) that

$$D^k D_k \psi^j_l = -R^k_l \psi^j_k \quad \text{for} \quad 1 \leq j \leq N,$$

for $\psi^j = (\psi^j_1, \cdots, \psi^j_n)$ in (4.32).

**Remark 4.2.** Theorem 4.2 is derived for deriving new gravitational field equation in the next section for explaining the phenomena of dark matter and dark energy. The condition that $\beta_1(M) = 0$ means that any loops in the manifold $M$ can shrink to a point. Obviously, our universe can be considered as a 4-dimensional manifold satisfying this condition.
Part 2. Physics

5. Gravitational Field Equations

5.1. Einstein-Hilbert functional. The general theory of relativity is based on three basic principles: the principle of equivalence, the principle of general relativity, and the principle of Lagrangian dynamics. The first two principles tell us that the spatial and temporal world is a 4-dimensional Riemannian manifold \((M, g_{ij})\), where the metric \(\{g_{ij}\}\) represents gravitational potential, and the third principle determines that the Riemannian metric \(\{g_{ij}\}\) is an extremum point of the Lagrangian action, which is the Einstein-Hilbert functional.

Let \((M, g_{ij})\) be an \(n\)-dimensional Riemannian manifold. The Einstein-Hilbert functional (5.1)

\[
F : W^{2,2}(M, g) \rightarrow \mathbb{R}
\]

is defined by

\[
F(g_{ij}) = \int_M \left( R + \frac{8 \pi G}{c^4} g^{ij} S_{ij} \right) \sqrt{-g} dx,
\]

where \(W^{2,2}(M, g)\) is defined by (4.5), \(R = g^{kl} R_{kl}\) and \(R_{kl}\) are the scalar and the Ricci curvatures, \(S_{ij}\) is the stress tensor, \(G\) is the gravitational constant, and \(c\) is the speed of light.

The Euler-Lagrangian of the Einstein-Hilbert functional \(F\) is given by

\[
\delta F(g_{ij}) = R_{ij} - \frac{1}{2} g_{ij} R + \frac{8 \pi G}{c^4} T_{ij},
\]

where \(T_{ij}\) is the energy-momentum tensor given by

\[
T_{ij} = S_{ij} - \frac{1}{2} g_{ij} S + g^{kl} \frac{\partial S_{kl}}{\partial g^{ij}}, \quad S = g^{kl} S_{kl},
\]

and the Ricci curvature tensor \(R_{ij}\) is given by

\[
R_{ij} = \frac{1}{2} g^{kl} \left( \frac{\partial^2 g_{kl}}{\partial x^i \partial x^j} + \frac{\partial^2 g_{ij}}{\partial x^k \partial x^l} - \frac{\partial^2 g_{il}}{\partial x^j \partial x^k} - \frac{\partial^2 g_{kj}}{\partial x^i \partial x^l} \right)
+ g^{kl} g_{rs} \left( \Gamma^r_{kl} \Gamma^s_{ij} - \Gamma^r_{il} \Gamma^s_{kj} \right),
\]

\[
\Gamma^k_{ij} = \frac{1}{2} g^{kl} \left( \frac{\partial g_{il}}{\partial x^j} + \frac{\partial g_{jl}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^l} \right).
\]

By (5.3)-(5.5), the Euler-Lagrangian \(\delta F(g_{ij})\) of the Einstein-Hilbert functional is a second order differential operator on \(\{g_{ij}\}\), and \(\delta F(g_{ij})\) is symmetric.

5.2. Einstein field equations. The General Theory of Relativity consists of two main conclusions:

1) The space-time of our world is a 4-dimensional Riemannian manifold \((M^4, g_{ij})\), and the metric \(\{g_{ij}\}\) represents gravitational potential.

2) The metric \(\{g_{ij}\}\) is the extremum point of the Einstein-Hilbert functional \((5.2)\). In other words, gravitational field theory obeys the principle of Lagrange dynamics.

\(^1\) The matter tensor is included here as well.
The principle of Lagrange dynamics is a universal principle, stated as:

**Principle of Lagrange Dynamics.** For any physical system, there are a set of state functions
\[ u = (u_1, \ldots, u_N), \]
which describe the state of this system, and there exists a functional \( L \) of \( u \), called the Lagrange action:

\[
L(u) = \int_0^T \int_{\Omega} \mathfrak{L}(u, Du, \ldots, D^m u) dx dt, \tag{5.6}
\]
such that the state \( u \) is an extremum point of \( L \). Usually the function \( \mathfrak{L} \) in (5.6) is called the Lagrangian density.

Based on this principle, the gravitational field equations are the Euler-Lagrange equations of the Einstein-Hilbert functional:

\[
\delta F(g_{ij}) = 0, \tag{5.7}
\]
which are the classical Einstein field equations:

\[
R_{ij} - \frac{1}{2} g_{ij} R = -\frac{8\pi G}{c^4} T_{ij}. \tag{5.8}
\]

By the Bianchi identities, the left hand side of (5.8) is divergence-free, i.e.

\[
D^i (R_{ij} - \frac{1}{2} g_{ij} R) = 0. \tag{5.9}
\]

Therefore it is required in the general theory of relativity that the energy-momentum tensor \( \{T_{ij}\} \) in (5.8) satisfies the following energy-momentum conservation law:

\[
D^i T_{ij} = 0 \quad \text{for } 1 \leq j \leq n. \tag{5.10}
\]

**5.3. New gravitational field equations.** Motivated by the mystery of dark energy and dark matter and the other difficulties encountered by the Einstein field equations as mentioned in Introduction, we introduce in this section a new set of field equations, still obeying the three basic principles of the General Theory of Relativity.

Our key observation is a well-known fact that the Riemannian metric \( g_{ij} \) is divergence-free. This suggests us two important postulates for deriving the new gravitational field equations:

- The energy-momentum tensor of matter need not to be divergence-free due to the presence of dark energy and dark matter; and
- The field equation obeys the Euler-Lagrange equation of the Einstein-Hilbert functional under the natural divergence-free constraint.

Under these two postulates, by the Scalar Potential Theorem, Theorem 4.2 if the Riemannian metric \( \{g_{ij}\} \) is an extremum point of the Einstein-Hilbert functional (5.2) with the divergence-free constraint (4.22), then the gravitational field equations are taken in the following form:

\[
R_{ij} - \frac{1}{2} g_{ij} R = -\frac{8\pi G}{c^4} T_{ij} - D_i D_j \varphi, \tag{5.11}
\]

where \( \varphi \in H^2(M) \) is called the scalar potential. We infer from (5.9) that the conservation laws for (5.11) are as follows

\[
\text{div} (D_i D_j \varphi + \frac{8\pi G}{c^4} T_{ij}) = 0. \tag{5.12}
\]
Using the contraction with $g^{ij}$ in (5.11), we have

$$R = \frac{8\pi G}{c^4} T + \Phi,$$

where

$$T = g^{ij} T_{ij}, \quad \Phi = g^{ij} D_i D_j \varphi,$$

represent respectively the energy-momentum density and the scalar potential density. Physically this scalar potential density $\Phi$ represents potential energy caused by the non-uniform distribution of matter in the universe. One important property of this scalar potential is

$$\int_M \Phi \sqrt{-g} dx = 0,$$

which is due to the integration by parts formula in Theorem 2.1. This formula demonstrates clearly that the negative part of this quantity $\Phi$ represents the dark matter, which produces attraction, and the positive part represents the dark energy, which drives the acceleration of expanding galaxies. We shall address this important issue in the next section.

5.4. Field equations for closed universe. The topological structure of closed universe is given by

$$M = S^1 \times S^3,$$

where $S^1$ is the time circle and $S^3$ is the 3-dimensional sphere representing the space. We note that the radius $R$ of $S^3$ depends on time $t \in S^1$,

$$R = R(t), \quad t \in S^1,$$

and the minimum time $t_0$,

$$t_0 = \min_t R(t)$$

is the initial time of the Big Bang.

For a closed universe as (5.15), by Theorem 4.2, the gravitational field equations are in the form

$$R_{ij} - \frac{1}{2} g_{ij} R = -\frac{8\pi G}{c^4} T_{ij} - D_i D_j \varphi + \alpha D_i \psi_j,$$

(5.16)

$$\Delta \psi_j + g^{ik} R_{ij} \psi_k = 0,$$

$$D_i \psi_j = D_j \psi_i,$$

where $\Delta = D^k D_k, \varphi$ the scalar potential, $\psi = (\psi_0, \psi_1, \psi_2, \psi_3)$ the vector potential, and $\alpha$ is a constant. The conservation laws of (5.16) are as follows

$$\Delta \psi_j = \frac{1}{\alpha} \Delta \left( \frac{\partial \varphi}{\partial x^j} \right) + \frac{8\pi G}{\alpha c^4} D^k T_{kj}.$$

6. Interaction in a Central Gravitational Field

6.1. Schwarzschild solution. We know that the metric of a central gravitational field is in a diagonal form [1]:

$$ds^2 = g_{00} c^2 dt^2 + g_{11} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2),$$

(6.1)

and physically $g_{00}$ is given by

$$g_{00} = -\left(1 + \frac{2}{c^2} \psi\right),$$

(6.2)
where $\psi$ is the Newton gravitational potential; see among others [1].

If the central matter field has total mass $M$ and radius $r_0$, then for $r > r_0$, the metric (6.1) is the well known Schwarzschild solution for the Einstein field equations (5.8), and is given by

$$ds^2 = -\left(1 - \frac{2MG}{c^2 r}\right)c^2dt^2 + \frac{dr^2}{\left(1 - \frac{2MG}{c^2 r}\right)} + r^2(d\theta^2 + \sin^2 \theta d\varphi^2).$$

We derive from (6.2) and (6.3) the classical Newton gravitational potential

$$\psi = -\frac{MG}{r}.$$

6.2. New gravitational interaction model. We now consider the metric determined by the new field equations (5.11), from which we derive a gravitational potential formula replacing (6.4).

Equations (5.11) can be equivalently expressed as

$$R_{ij} = -\frac{8\pi G}{c^4}(T_{ij} - \frac{1}{2}g_{ij}T) - (D_i D_j \varphi - \frac{1}{2}g_{ij}\Phi),$$

where

$$T = g^{kl}T_{kl}, \quad \Phi = g^{kl}D_k D_l \varphi.$$

For the central matter field with total mass $M$ and radius $r_0$, by the Schwarzschild assumption, for $r > r_0$, there exists no matter, i.e.

$$T_{ij} = 0.$$

Therefore the conservation laws of (6.5) are

$$\Delta \left(\frac{\partial \varphi}{\partial x^k}\right) = 0 \quad \text{for } k = 0, 1, 2, 3.$$

The tensors $g_{ij}$ in (6.1) are written as

$$g_{00} = -e^u, \quad g_{11} = e^v, \quad g_{22} = r^2, \quad g_{33} = r^2 \sin^2 \theta,$n$\quad u = u(r), \quad v = v(r).$$

Noting that the central field is spherically symmetric, we assume that

$$\varphi = \varphi(r) \quad \text{is independent of } t, \theta, \varphi.$$

$$r \gg \frac{2MG}{c^2}.$$

For the metric (6.7), the non-zero components of the Levi-Civita connection are as follows

$$\Gamma^1_{00} = \frac{1}{2}e^uu' = v', \quad \Gamma^1_{11} = \frac{1}{2}v, \quad \Gamma^1_{22} = -re^{-v},$$

$$\Gamma^1_{33} = -re^{-v} \sin^2 \theta, \quad \Gamma^0_{10} = \frac{1}{2}u', \quad \Gamma^2_{12} = \frac{1}{r},$$

$$\Gamma^2_{33} = -\sin \theta \cos \theta, \quad \Gamma^3_{13} = \frac{1}{r}, \quad \Gamma^3_{23} = \frac{\cos \theta}{\sin \theta}.$$

Hence the Ricci tensor

$$R_{ij} = \frac{\partial \Gamma^k_{ik}}{\partial x^j} - \frac{\partial \Gamma^k_{ij}}{\partial x^k} + \Gamma^k_{ij} \Gamma^r_{kr} - \Gamma^k_{ij} \Gamma^r_{kr}.$$
are given by
\[
R_{00} = -e^{-u-v} \left[ \frac{u''}{2} + \frac{u'}{r} + \frac{u'}{4} (u' - v') \right],
\]
\[
R_{11} = \frac{u''}{2} - \frac{v'}{r} + \frac{u'}{4} (u' - v'),
\]
\[
R_{22} = e^{-v} \left[ 1 - e^v + \frac{r}{2} (u' - v') \right],
\]
\[
R_{33} = R_{22} \sin^2 \theta,
\]
\[
R_{ij} = 0 \quad \forall i \neq j.
\]
(6.12)

Furthermore, we infer from (6.7), (6.9) and (6.11) that
\[
D_i D_j \varphi - \frac{1}{2} g_{ij} \Phi = 0, \quad \forall i \neq j,
\]
\[
D_0 D_0 \varphi - \frac{1}{2} g_{00} \Phi = \frac{1}{2} e^{-u-v} \left[ \varphi'' - \frac{1}{2} (u' + v' - \frac{4}{r}) \varphi' \right],
\]
\[
D_1 D_1 \varphi - \frac{1}{2} g_{11} \Phi = \frac{1}{2} \left[ \varphi'' - \frac{1}{2} (u' + u' + \frac{4}{r}) \varphi' \right],
\]
\[
D_2 D_2 \varphi - \frac{1}{2} g_{22} \Phi = -\frac{\sigma^2}{2} e^{-v} \left[ \varphi'' + \frac{1}{2} (u' - v') \varphi' \right],
\]
\[
D_3 D_3 \varphi - \frac{1}{2} g_{33} \Phi = \sin^2 \theta \left( D_2 D_2 \varphi - \frac{1}{2} g_{22} \Phi \right).
\]
(6.13)

Thus, by (6.12) and (6.13), the equations (6.5) are as follows
\[
u'' + \frac{2u'}{r} + \frac{u'}{2} (u' - v') = \varphi'' - \frac{1}{2} (u' + v' - \frac{4}{r}) \varphi',
\]
\[
u'' - \frac{2u'}{r} + \frac{u'}{2} (u' - v') = -\varphi'' + \frac{1}{2} (u' + v' + \frac{4}{r}) \varphi',
\]
\[
u' - v' + \frac{2}{r} (1 - e^v) = r (\varphi'' + \frac{1}{2} (u' - v') \varphi').
\]
(6.14) (6.15) (6.16)

6.3. Consistency. We need to consider the existence and uniqueness of solutions of the equations (6.14)-(6.16). First, in the vacuum case, the classical Einstein equations are in the form
\[
R_{kk} = 0 \quad \text{for } k = 1, 2, 3,
\]
(6.17)

two of which are independent. The system contains two unknown functions, and therefore for a given initial value (as \( u' \) is basic in (6.17)),
\[
u'(r_0) = \sigma_1, \quad v(r_0) = \sigma_2, \quad r_0 > 0,
\]
(6.18)

the problem (6.17) with (6.18) has a unique solution, which is the Schwarzschild solution
\[
u' = \frac{\varepsilon_2}{r^2} \left( 1 - \frac{\varepsilon_1}{r} \right)^{-1}, \quad v = -\ln \left( 1 - \frac{\varepsilon_1}{r} \right),
\]
where
\[
\varepsilon_1 = r_0 (1 - e^{-\sigma_2}), \quad \varepsilon_2 = r_0^2 \left( 1 - \frac{\varepsilon_1}{r_0} \right) \sigma_1.
\]
Now if we consider the influence of the cosmic microwave background (CMB) for the central fields, then we should add a constant energy density in equations (6.17):

\[(T_{ij}) = \begin{pmatrix} -g_{00}\rho & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} .\]

Namely,

\[(6.20) \quad R_{00} = \frac{4\pi G}{c^4} g_{00}\rho, \quad R_{11} = 0, \quad R_{22} = 0,\]

where \(\rho\) is the density of the microwave background, whose value is \(\rho = 4 \times 10^{-31}\) kg/m\(^3\). Then it is readily to see that the problem (6.20) with (6.18) has no solution. In fact, the divergence-free equation \(D^i T_{ij} = 0\) yields that

\[\Gamma_{10} T_{00} = \frac{1}{2} u' \rho = 0,\]

which implies that \(u' = 0\). Hence \(R_{00} = 0\), a contradiction to (6.20). Furthermore, if we regard \(\rho\) as an unknown function, then the equations (6.20) still have no solutions.

On the other hand, the new gravitational field equations (6.14)-(6.16) are solvable for the microwave background as the number of unknowns are the same as the number of independent equations.

Equations (6.14)-(6.16) have the following equivalent form:

\[u'' = \left(\frac{1}{r} + \frac{u'}{2}\right) (v' - u') + \frac{\varphi'}{r},\]

\[\varphi'' = -\frac{1}{r^2} (e^\varphi - 1) + \frac{1}{2} \varphi' v' + \frac{1}{r} u',\]

\[v' = -\frac{1}{r^2} (e^\varphi - 1) - \frac{r}{2} \varphi' u',\]

equipped with the following initial values:

\[(6.22) \quad u'(r_0) = \alpha_1, \quad v(r_0) = \alpha_2, \quad \varphi'(r_0) = \alpha_3, \quad r_0 > 0.\]

It is classical that (6.21) with (6.22) possesses a unique local solution. In fact, we can prove that the solution exists for all \(r > r_0\).

6.4. Gravitational interaction. We now derive the gravitational interaction formula from the basic model (6.14)-(6.16).

First, we infer from (6.14)-(6.16) that

\[u' + v' = \frac{r \varphi''}{1 + \frac{2}{\tau} \varphi'},\]

\[u' - v' = \frac{1}{1 - \frac{2}{\tau} \varphi'} \left[\frac{2}{r} (e^\varphi - 1) + r \varphi''\right].\]

Consequently,

\[u' = \frac{1}{1 - \frac{2}{\tau} \varphi'} \frac{1}{r} (e^\varphi - 1) + \frac{r \varphi''}{1 - (\frac{2}{\tau} \varphi')^2}.\]
By (6.2) and (6.7), we have
\[ \psi = \frac{c^2}{2} (e^u - 1). \]
As the interaction force \( F \) is given by \( F = -m \nabla \psi \),
it follows from (6.23) and (6.25) that
\[ F = mc^2 \left[ -\frac{1}{1 - \frac{\delta}{2}} r (e^v - 1) - \frac{r \varphi''}{1 - \left( \frac{\delta}{2} \varphi' \right)^2} \right], \]
(6.26)
\[ \varphi'' = -e^v R + \frac{1}{2} (u' - v' + \frac{4}{r}) \varphi'. \]
(6.27)
Of course, the following energy balance and conservation law hold true as well:
\[ R = \frac{8\pi G}{c^4} T + \Phi, \]
\[ \int_0^\infty e^{u+v} r^2 \Phi dr = 0. \]
where \( R \) is the scalar curvature and \( \Phi = g^{kl} D_k D_l \varphi \). Equation (6.27) is derived by solving \( \varphi'' \) using (6.28).

6.5. Simplified formulas. We now consider the region: \( r_0 < r < r_1 \). Physically, we have
\[ |\varphi'|, |\varphi''| \ll 1. \]
Hence \( u \) and \( v \) in (6.26) can be replaced by the Schwarzschild solution:
\[ u_0 = \ln \left( 1 - \frac{\delta}{r} \right), \quad v_0 = -\ln \left( 1 - \frac{\delta}{r} \right), \quad \delta = \frac{2GM}{c^2}. \]
As \( \delta/r \) is small for \( r \) large, by (6.29), the formula (6.28) can be expressed as
\[ F = \frac{mc^2}{2} \left[ -\frac{\delta}{r^2} - \varphi'' \right]. \]
(6.31)
This is the interactive force in a central symmetric field. The first term in the parenthesis is the Newton gravity term, and the added second term \(-r \varphi''\) is the scalar potential energy density, representing the dark matter and dark energy.

In addition, replacing \( u \) and \( v \) in (6.27) by the Schwarzschild solution (6.30), we derive the following approximate formula:
\[ \varphi'' = \left( \frac{2}{r} + \frac{\delta}{r^2} \right) \varphi' - R. \]
(6.32)
Consequently we infer from (6.31) that
\[ F = mMG \left[ -\frac{1}{r^2} - \frac{1}{\delta} \left( 2 + \frac{\delta}{r} \right) \varphi' + \frac{Rr}{\delta} \right], \quad R = \Phi \quad \text{for} \ r > r_0. \]
(6.33)
The first term is the classical Newton gravitation, the second term is the coupling interaction between matter and the scalar potential \( \varphi \), and the third term is the interaction generated by the scalar potential energy density \( \Phi \). In this formula, the negative and positive values of each term represent respectively the attracting and repelling forces.
Integrating (6.32) yields (omitting $e^{-\delta/r}$)

\[
\phi' = -\varepsilon_2 r^2 - r^2 \int r^{-2} R dr,
\]

where $\varepsilon_2$ is a free parameter. Hence the interaction force $F$ is approximated by

\[
F = mMG \left[ -\frac{1}{r^2} + \left( 2 + \frac{\delta}{r} \right) \varepsilon r^2 + \frac{1}{\delta} \left( 2 + \frac{\delta}{r} \right) r^2 \int r^{-2} R dr \right],
\]

where $\varepsilon = \varepsilon_2 \delta^{-1}$, $R = \Phi$ for $r > r_0$, and $\delta = 2MG/c^2$. We note that based on (6.28), for $r > r_0$, $R$ is balanced by $\Phi$, and the conservation of $\Phi$ suggests that $R$ behaves like $r^{-2}$ as $r$ sufficiently large. Hence for $r$ large, the second term in the right hand side of (6.34) must be dominate and positive, indicating the existence of dark energy.

We note that the scalar curvature is infinite at $r = 0$: $R(0) = \infty$. Also $R$ contains two free parameters determined by $u'$ and $v$ respectively. Hence if we take a first order approximation as

\[
R = -\varepsilon_1 + \frac{\varepsilon_0}{r} \quad \text{for } r_0 < r < r_1 = 10^{21} \text{km},
\]

where $\varepsilon_1$ and $\varepsilon_0$ are free yet to be determined parameters. Then we deduce from (6.34) and (6.36) that

\[
\phi' = -\varepsilon \delta r^2 - \varepsilon_1 r + \frac{\varepsilon_0}{2}.
\]

Therefore,

\[
F = mMG \left[ -\frac{1}{r^2} - \frac{\varepsilon_0}{2r} + \varepsilon_1 + \left( \varepsilon \delta + \varepsilon_1 \delta^{-1} \right) r + 2\varepsilon r^2 \right].
\]

Physically it is natural to choose

$\varepsilon_0 > 0$, $\varepsilon_1 > 0$, $\varepsilon > 0$.

Also, $\varepsilon_1$ and $2\varepsilon_2 \delta^{-1} r^2$ are much smaller than $(\varepsilon \delta + \varepsilon_1 \delta^{-1}) r$ for $r \leq r_1$. Hence

\[
F = mMG \left[ -\frac{1}{r^2} - \frac{k_0}{r} + k_1 r \right].
\]

where $k_0$ and $k_1$ can be estimated using the Rubin law of rotating galaxy and the acceleration of the expanding galaxies:

\[
k_0 = 4 \times 10^{-18} \text{km}^{-1}, \quad k_1 = 10^{-57} \text{km}^{-3}.
\]

We emphasize here that the formula (6.38) is only a simple approximation for illustrating some features of both dark matter and dark energy.

7. Theory of Dark Matter and Dark Energy

7.1. Dark matter and dark energy. Dark matter and dark energy are two of the most remarkable discoveries in astronomy in recent years, and they are introduced to explain the acceleration of the expanding galaxies. In spite of many attempts and theories, the mystery remains. As mentioned earlier, this article is an attempt to develop a unified theory for the dark matter and dark energy.

A strong support to the existence of dark matter is the Rubin law for galactic rotational velocity, which amounts to saying that most stars in spiral galaxies orbit
at roughly the same speed. Namely, the orbital velocity $v(r)$ of the stars located at radius $r$ from the center of galaxies is almost a constant:

$$v(r) = \text{constant for a given galaxy.}$$

Typical galactic rotation curves [3] are illustrated by Figure 7.1(a), where the vertical axis represents the velocity (km/s), and the horizontal axis is the distance from the galaxy center (extending to the galaxy radius).

![Figure 7.1](image)

**Figure 7.1.** (a) Typical galactic rotation curve by Rubin, and (b) theoretic curve by the Newton gravitation law.

However, observational evidence shows discrepancies between the mass of large astronomical objects determined from their gravitational effects, and the mass calculated from the visible matter they contain, and Figure 7.1(b) gives a calculated curve. The missing mass suggests the presence of dark matter in the universe.

In astronomy and cosmology, dark energy is a hypothetical form of energy, which spherically symmetrically permeates all of space and tends to accelerate the expansion of the galaxies.

The High-Z Supernova Search Team in 1998 and the Supernova Cosmology Project in 1998 published their observations which reveal that the expansion of the galaxies is accelerating. In 2011, a survey of more than $2 \times 10^5$ galaxies from Austrian astronomers confirmed the fact. Thus, the existence of dark energy is accepted by most astrophysicist.

### 7.2. Nature of dark matter and dark energy.

With the new gravitational field equation with the scalar potential energy, and we are now in position to derive the nature of the dark matter and dark energy. More precisely, using the revised Newton formula derived from the new field equations:

$$F = mMG \left( -\frac{1}{r^2} - \frac{k_0}{r} + k_1 r \right),$$

we determine an approximation of the constants $k_0, k_1$, based on the Rubin law and the acceleration of expanding galaxies.

First, let $M_r$ be the total mass in the ball with radius $r$ of the galaxy, and $V$ be the constant galactic rotation velocity. By the force equilibrium, we infer from (7.2) that

$$\frac{V^2}{r} = M_r G \left( \frac{1}{r^2} + \frac{k_0}{r} - k_1 r \right),$$
which implies that

\begin{equation}
M_r = \frac{V^2 r}{G \left(1 + k_0r - k_1r^3\right)}.
\end{equation}

This matches the observed mass distribution formula of the galaxy, which can explain the Rubin law (7.3).

Second, if we use the classical Newton formula

\[ F = -\frac{mMG}{r^2}, \]

to calculate the galactic rotational velocity \( v_r \), then we have

\begin{equation}
\frac{v_r^2}{r} = \frac{M_rG}{r^2}.
\end{equation}

Inserting (7.4) into (7.5) implies

\begin{equation}
v_r = \frac{V}{\sqrt{1 + k_0r - k_1r^3}}.
\end{equation}

As \( 1 \gg k_0 \gg k_1 \), (7.6) can approximatively written as

\[ v_r = V \left(1 - \frac{1}{2}k_0r + \frac{1}{4}k_0^2r^2\right), \]

which is consistent with the theoretic rotational curve as illustrated by Figure 7.1(b). It implies that the distribution formula (7.4) can be used as a test for the revised gravitational field equations.

Third, we now determine the constants \( k_0 \) and \( k_1 \) in (7.2). According to astronomical data, the average mass \( M_{r_1} \) and radius \( r_1 \) of galaxies is about

\begin{equation}
M_{r_1} = 10^{11} M_\odot \simeq 2 \times 10^{41} \text{kg},
\end{equation}

\begin{equation}
r_1 = 10^4 \sim 10^5 \text{pc} \simeq 10^{18} \text{km},
\end{equation}

where \( M_\odot \) is the mass of the Sun.

Taking \( V = 300 \text{km/s} \), then we have

\begin{equation}
\frac{V^2}{G} = 8 \times 10^{23} \text{kg/km}.
\end{equation}

Based on physical considerations,

\begin{equation}
k_0 \gg k_1 r_1 \quad (r_1 \text{ as in (7.7)})
\end{equation}

By (7.7) + (7.9), we deduce from (7.4) that

\begin{equation}
k_0 = \frac{V^2}{G M_{r_1}} \left(\frac{1}{r_1} - \frac{1}{r_1^3}\right) = 4 \times 10^{-18} \text{km}^{-2}.
\end{equation}

Now we consider the constant \( k_1 \). Due to the accelerating expansion of galaxies, the interaction force between two clusters of galaxies is repelling, i.e. for (7.2),

\[ F \geq 0, \quad r \geq \bar{r}, \]

where \( \bar{r} \) is the average distance between two galactic clusters. It is estimated that

\[ \bar{r} = 10^8 \text{pc} \simeq 10^{20} \sim 10^{21} \text{km}. \]

We take

\begin{equation}
\bar{r} = \frac{1}{\sqrt{2}} \times 10^{20} \text{km}
\end{equation}
as the distance at which $F = 0$. Namely,

$$k_1\bar{r} - \frac{k_0}{\bar{r}} - \frac{1}{\bar{r}^2} = 0.$$

Hence we derive from (7.10) and (7.11) that

$$k_1 = k_0\bar{r}^{-2} = 10^{-57}\text{km}^{-3}.$$

Thus, the constants $k_0$ and $k_1$ are estimated by

(7.12) \hspace{1cm} k_0 = 4 \times 10^{-18}\text{km}^{-2}, \hspace{1cm} k_1 = 10^{-57}\text{km}^{-3}.

In summary, for the formula (7.2) with (7.12), if the matter distribution $M_r$ is in the form

(7.13) \hspace{1cm} M_r = \frac{V^2}{G} \frac{r}{1 + k_0\bar{r}},

then the Rubin law holds true. In particular, the mass $\bar{M}$ generated by the revised gravitation is

$$\bar{M} = M_T - M_{r_1} = \frac{V^2}{G} r_1 - \frac{V^2}{G} \frac{r_1}{1 + k_0 r_1}, \hspace{1cm} r_1 \text{ as in (7.7)},$$

where $M_T = V^2 r_1 / G$ is the theoretic value of total mass. Hence

$$\frac{\bar{M}}{M_T} = \frac{k_0 r_1}{1 + k_0 r_1} = \frac{4}{5}.$$

Namely, the revised gravitational mass $\bar{M}$ is four times of the visible matter $M_{r_1} = M_T - \bar{M}$. Thus, it gives an alternative explanation for the dark matter.

In addition, the formula (7.2) with (7.12) also shows that for a central field with mass $M$, an object at $r > \bar{r}$ ($\bar{r}$ as in (7.11)) will be exerted a repelling force, resulting the acceleration of expanding galaxies at $r > \bar{r}$.

Thus the new gravitational formula (7.2) provides a unified explanation of dark matter and dark energy.

7.3. Effects of non-homogeneity. In this section, we prove that if the matter is homogeneously distributed in the universe, then the scalar potential $\varphi$ is a constant, and consequently the scalar potential energy density is identically zero: $\Phi \equiv 0$.

It is known that the metric for an isotropic and homogeneous universe is given by

(7.14) \hspace{1cm} ds^2 = -c^2 dt^2 + a^2 \left[ \frac{dr^2}{1 - kr^2} + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2) \right],

which is called the Friedmann-Lemaître-Robertson-Walker (FLRW) metric, where the scale factor $a = a(t)$ represents the cosmological radius, and $k$ takes one of the three numbers: $-1, 0, 1$.

For the FLRW metric (7.14), the energy-momentum tensor $\{T_{ij}\}$ is given by

(7.15) \hspace{1cm} T_{ij} = \text{diag}(\rho c^2, g_{11}p, g_{22}p, g_{33}p),

where $\rho$ is the mass density, $p$ is pressure, and

$$g_{11} = \frac{a^2}{1 - kr^2}, \hspace{1cm} g_{22} = a^2 r^2, \hspace{1cm} g_{33} = a^2 r^2 \sin^2 \theta.$$
By (7.14) and (7.15), the Einstein field equations (1.1) are reduced to two equations:

\( a'' = -\frac{4\pi G}{3} \left( \rho + \frac{3p}{c^2} \right) a, \)  

\( (7.16) \)

\( aa'' + 2(a')^2 + 2kc^2 = 4\pi G \left( \rho - \frac{p}{c^2} \right) a^2, \)  

\( (7.17) \)

and the conservation law \( \text{div} \ T_{ij} = 0 \) gives

\( (7.18) \)

Then only two of the above three equations (7.16)-(7.18) are independent, and are called the Friedman equations.

On the other hand, for the metric (7.14) with (7.15), the new gravitational field equations (1.4) with scalar potential are reduced to

\( a'' = -\frac{4\pi G}{3} \left( \rho + \frac{3p}{c^2} \right) a + \frac{1}{6} \varphi'' a - \frac{1}{2} a' \varphi', \)  

\( (7.19) \)

\( \frac{a''}{a} + 2 \left( \frac{a'}{a} \right)^2 + \frac{2kc^2}{a^2} = 4\pi G \left( \rho - \frac{p}{c^2} \right) - \frac{\varphi''}{2} - \frac{\varphi' a'}{2 a}, \)  

\( (7.20) \)

and the conservation equation

\( (7.21) \)

Again two of the above three equations (7.19)-(7.21) are independent. It follows from (7.19) and (7.20) that

\( (a')^2 = \frac{8\pi G}{3} a^2 \rho - \frac{1}{3} \varphi'' a^2 - kc^2. \)  

\( (7.22) \)

In the following we shall prove that

\( \nabla \varphi = 0, \)  

\( (7.23) \)

Namely, \( \varphi = \text{constant}. \) In fact, let \( \varphi'' \) and \( \rho \) have the form:

\( \rho = \frac{\theta}{a^3}, \quad \varphi'' = \frac{\psi}{a^3}. \)  

\( (7.24) \)

Inserting (7.24) in (7.19), (7.21) and (7.22), we arrive at

\( a'' = \frac{4\pi G \theta}{3 a^2} + \frac{1}{6} \psi + \frac{4\pi G p}{c^2} a \theta - \frac{1}{2} a' \varphi', \)  

\( (7.25) \)

\( (a')^2 = \frac{8\pi G \theta}{3 a} + \frac{1}{3} \psi = -kc^2, \)  

\( (7.26) \)

\( \psi' = 8\pi G \theta' + 24\pi G a^2 \rho' c^2. \)  

\( (7.27) \)

Multiplying both sides of (7.25) by \( a' \) we obtain

\( \frac{1}{2} \frac{d}{dt} \left[ (a')^2 - \frac{8\pi G \theta}{3 a} + \frac{1}{3} \psi \right] + \frac{4\pi G \theta'}{3 a} - \frac{1}{6} \psi' = -\frac{4\pi G}{c^2} paa' - \frac{1}{2} (a')^2 \varphi'. \)  

\( (7.28) \)

It follows then from (7.26)-(7.28) that

\( \frac{1}{2} (a')^2 \varphi' = 0, \)  

which implies that (7.23) holds true.
The conclusion (7.23) indicates that if the universe is in the homogeneous state, then the scalar potential energy density \( \frac{c^4}{8\pi G}\Phi \) is identically zero: \( \Phi = 0 \). This fact again demonstrates that \( \varphi \) characterizes the non-uniform distribution of matter in the universe.

8. Conclusions

We have discovered new gravitational field equations (1.4) with scalar potential under the postulate that the energy momentum tensor \( T_{ij} \) needs not to be divergence-free due to the presence of dark energy and dark matter:

\[
R_{ij} - \frac{1}{2} g_{ij} R = -\frac{8\pi G}{c^4} T_{ij} - D_i D_j \varphi,
\]

With the new field equations, we have obtained the following physical conclusions:

First, gravitation is now described by the Riemannian metric \( g_{ij} \), the scalar potential \( \varphi \) and their interactions, unified by the new gravitational field equations (1.4).

Second, associated with the scalar potential \( \varphi \) is the scalar potential energy density \( \frac{c^4}{8\pi G}\Phi \), which represents a new type of energy/force caused by the non-uniform distribution of matter in the universe. This scalar potential energy density varies as the galaxies move and matter of the universe redistributes. Like gravity, it affects every part of the universe as a field.

This scalar potential energy density \( \frac{c^4}{8\pi G}\Phi \) consists of both positive and negative energies. The negative part of this potential energy density produces attraction, and the positive part produces repelling force. Also, this scalar energy density is conserved with mean zero:

\[
\int_M \Phi dM = 0.
\]

Third, using the new field equations, for a spherically symmetric central field with mass \( M \) and radius \( r_0 \), the force exerted on an object of mass \( m \) at distance \( r \) is given by (see (6.33)):

\[
F = mMG \left[ -\frac{1}{r^2} - \frac{1}{\delta} \left( 2 + \frac{\delta}{r} \right) \varphi' + \frac{R r}{\delta} \right], \quad R = \Phi \quad \text{for} \ r > r_0.
\]

where \( \delta = 2MG/c^2 \).

Fourth, the sum \( \varepsilon = \varepsilon_1 + \varepsilon_2 \) of this new potential energy density

\[
\varepsilon_1 = \frac{c^4}{8\pi G}\Phi
\]

and the coupling energy between the energy-momentum tensor \( T_{ij} \) and the scalar potential field \( \varphi \)

\[
\varepsilon_2 = -\frac{c^4}{8\pi G} \left( \frac{2}{r} + \frac{2MG}{c^2 r^2} \right) \frac{d\varphi}{dr},
\]

gives rise to a new unified theory for dark matter and dark energy: The negative part of \( \varepsilon \) represents the dark matter, which produces attraction, and the positive part represents the dark energy, which drives the acceleration of expanding galaxies.
Fifth, the scalar curvature $R$ of space-time obeys:

$$R = \frac{8\pi G}{c^4} T + \Phi.$$ 

Consequently, when there is no normal matter present (with $T = 0$), the curvature $R$ of space-time is balanced by $R = \Phi$. Therefore, there is no real vacuum in the universe.

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