BIPARTITE MINORS

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Abstract. We introduce a notion of bipartite minors and prove a bipartite analog of Wagner’s theorem: a bipartite graph is planar if and only if it does not contain $K_{3,3}$ as a bipartite minor. Similarly, we provide a forbidden minor characterization for outerplanar graphs and forests. We then establish a recursive characterization of bipartite $(2,2)$-Laman graphs — a certain family of graphs that contains all maximal bipartite planar graphs.

1. Introduction

Wagner’s celebrated theorem [5], [2, Theorem 4.4.6] provides a characterization of planar graphs in terms of minors: a graph $G$ is planar if and only if it contains neither $K_5$ nor $K_{3,3}$ as a minor. Unfortunately, a minor of a bipartite graph is not always bipartite as contracting edges destroys 2-colorability. Here, we introduce a notion of a bipartite minor: an operation that applies to bipartite graphs and outputs bipartite graphs. We then prove a bipartite analog of Wagner’s theorem: a bipartite graph is planar if and only if it does not contain $K_{3,3}$ as a bipartite minor. Similarly, we provide a forbidden bipartite minor characterization for bipartite outerplanar graphs and forests.

All the graphs considered in this note are simple graphs. A graph with vertex set $V$ and edge set $E$ is denoted by $G = (V, E)$. We denote the edge connecting vertices $i$ and $j$ by $ij$. A graph is bipartite if there exists a bipartition (or bicoloring in red and blue) of the vertex set $V$ of $G$, $V = A \sqcup B$, in such a way that no two vertices from the same part are connected by an edge. When discussing bipartite graphs, we fix such a bipartition and write $G = (A \sqcup B, E)$; we refer to $A$ and $B$ as parts or sides of $G$.

As bipartite planar graphs with $n \geq 3$ vertices have at most $2n - 4$ edges, and as all their subgraphs are also bipartite and planar, and hence satisfy the same restriction on the number of edges, it is natural to consider the family of maximal bipartite graphs possessing this property. Specifically, we say that a bipartite graph $G = (A \sqcup B, E)$ with $|A| \geq 2$ and $|B| \geq 2$ is $(2,2)$-Laman if (i) $G$ has exactly $2(|A| + |B|) - 4$ edges, and (ii) every subgraph $H$ of $G$ with at least 3 vertices has at most $2|V(H)| - 4$ edges. Note that the family of $(2,2)$-Laman graphs is strictly larger than that of maximal bipartite planar graphs: indeed, taking $n \geq 2$ copies of $K_{3,3}$ minus an edge, and gluing all these
copies together along the two vertices of the missing edge, produces a graph on $4n + 2$
n vertices with $8n$ edges; this graph is $(2, 2)$-Laman, but it is not planar.

Our second main result is a recursive characterization of $(2, 2)$-Laman graphs. We
remark that the name $(2, 2)$-Laman is motivated by Laman’s theorem [4] from rigidity
theory of graphs, and its relation to a recent theory of rigidity for bipartite graphs can
be found in [3]. As such, this paper is a part of a project to understand notions of mi-
nors and graph-rigidity for bipartite graphs as well as to understand higher-dimensional
generalizations.

The rest of this note is organized as follows: in Section 2 we define bipartite mi-
nors and prove the bipartite analog of Wagner’s theorem and analogous theorems for
bipartite outerplanar graphs and forests (deferring treatment of some of the cases to the
Appendix). Then in Section 3 we discuss $(2, 2)$-Laman graphs.

2. Wagner’s theorem for bipartite graphs

We start by defining a couple of basic operations on (bipartite) graphs. If $G = (V, E)$
is a graph and $v$ is a vertex of $G$, then $G - v$ denotes the induced subgraph of $G$ on
the vertex set $V - \{v\}$. If $G = (V = A \uplus B, E)$ is a bipartite graph and $u, v$ are two vertices
from the same part, then the contraction of $u$ with $v$ is a graph $G'$ on the vertex set
$V - \{u\}$ obtained from $G$ by identifying $u$ with $v$ and deleting the extra copy from each
double edge that was created. Observe that $G'$ is also bipartite.

Recall that if $G$ is a graph and $C$ is a cycle of $G$, then $C$ is non-separating if the removal
of the vertices of $C$ from $G$ does not increase the number of connected components. A
cycle $C$ is induced (or chordless) if each two nonadjacent vertices of $C$ are not connected
by an edge in $G$. Induced non-separating cycles are known in the literature as peripheral
cycles.

We now come to the main definition of this section.

**Definition 2.1.** Let $G$ be bipartite graph. We say that a graph $H$ is a bipartite minor of
$G$, denoted $H <_b G$, if there is a sequence of graphs $G = G_0, G_1, \ldots, G_t = H$ where for
each $i$, $G_{i+1}$ is obtained from $G_i$ by either deletion (of a vertex or an edge) or admissible
contraction. A contraction of a vertex $u$ with a vertex $v$ in $G_i$ is called admissible if $u$
and $v$ have a common neighbor in $G_i$, and at least one of these common neighbors, say
$w$, is such that the path $(u, w, v)$ is a part of a peripheral cycle in $G_i$.

For instance, applying an admissible contraction to an 8-cycle results in a 6-cycle plus
an edge attached to this cycle at one vertex. Note that since each admissible contraction
identifies two vertices that have a common neighbor, these two vertices are from the same
part of the graph. Thus all bipartite minors are bipartite graphs. The importance of the
notion of bipartite minors is explained by the following result that can be considered as
a bipartite analog of Wagner’s theorem.

**Theorem 2.2.** A bipartite graph $G$ is planar if and only if $G$ does not contain $K_{3,3}$ as
a bipartite minor.
Proof. First assume that \( G \) is planar. We may assume that \( G \) is connected. To verify that \( G \) does not contain \( K_{3,3} \) as a bipartite minor, it suffices to show that deletions and admissible contractions preserve planarity. This is clear for deletions. For admissible contractions, consider an embedding of \( G \) in a 2-sphere \( S^2 \), and let \( C \) be a peripheral cycle of \( G \) that contains a path \((u,v)\). By the Jordan-Schönflies theorem, the complement of the image of \( C \) in \( S^2 \) consists of two components, each homeomorphic to an open 2-ball. As \( C \) is peripheral, one of these components contains no vertices/edges of \( G \), and hence is a face of the embedding of \( G \). Contracting \( u \) with \( v \) “inside this face” produces an embedding of the resulting graph in \( S^2 \).

Assume now that \( G \) is not planar. We must show that \( G \) contains \( K_{3,3} \) as a bipartite minor. By Kuratowski’s theorem [2, Theorem 4.4.6], \( G \) contains a subgraph \( H \) that is a subdivision of either \( K_5 \) or \( K_{3,3} \). Hence, it only remains to show that \( K_{3,3} \) is a bipartite minor of \( H \). We first treat the case where an edge \( e \) of the original \( K_5 \) (or \( K_{3,3} \)) is subdivided at least twice. Let \( C \) be a peripheral cycle of the original \( K_5 \) (or \( K_{3,3} \)) that contains \( e \) (there exists such \( C \) — a 3-cycle for \( K_5 \) and a 4-cycle for \( K_{3,3} \)), let \( C' \) be the subdivision of \( C \) in \( H \), and let \((a, b, a')\) be a path of length two in \( H \) that is contained in \( e \). Then \((a, b, a')\) is a part of a peripheral cycle \( C' \) in \( H \), and hence contracting \( a' \) with \( a \) is an admissible contraction in \( H \). Performing this contraction and then deleting \( b \), we obtain a new bipartite subdivision \( H' \) of \( K_5 \) (or \( K_{3,3} \)) that subdivides \( e \) with fewer interior vertices than \( H \), but agrees with \( H \) on all other edges of \( K_5 \) (\( K_{3,3} \), respectively). Thus, we can assume that each edge of the original \( K_5 \) (or \( K_{3,3} \)) is subdivided at most once. By symmetry, this reduces the problem of finding \( K_{3,3} \) as a bipartite minor of \( H \) to finding \( K_{3,3} \) as a bipartite minor of the nine bipartite graphs described below. These cases are treated in the Appendix.

There are three bipartite graphs which are subdivisions of \( K_5 \) to consider, denoted by \( G(i) \) for \( i = 5, 4, 2 \), where \( G(i) \) is the graph obtained from \( K_5 \) by coloring \( i \) of its vertices red, the other \( 5 - i \) blue, then subdividing each monochromatic edge once, and coloring the subdivision vertex red/blue so that its color is opposite to that of the vertices of the original monochromatic edge. For example, \( G(5) \) is the barycentric subdivision of \( K_5 \), endowed with a 2-coloring. Note that no edge of \( K_5 \) that connects two vertices of opposite colors is subdivided.

There are six bipartite graphs which are subdivisions of \( K_{3,3} \) to consider, denoted by \( G(i,j) \) and defined as follows. Let \( X \) and \( Y \) be the two sides of \( K_{3,3} \). Then \( G(i,j) \) is the graph obtained from \( K_{3,3} \) by first (i) coloring red exactly \( i \) vertices from \( X \) and \( j \) vertices from \( Y \), and coloring blue the other \( 3 - i - j \) vertices; then (ii) subdividing each monochromatic edge once, and coloring the subdivision vertex red/blue as before, so that a proper 2-coloring is obtained. Note that as before, no edge of \( K_{3,3} \) that connects two vertices of opposite colors is subdivided. Up to symmetry, the six graphs we need to consider are \( G(3,3), G(3,2), G(3,1), G(3,0), G(2,2), G(2,1) \). (Observe that \( G(3,0) = K_{3,3} \), and so this case is trivial.) \( \square \)

Remark 2.3. It is worth noting that the barycentric subdivision of \( K_5 \) (which is a bipartite graph) does not contain subgraphs homeomorphic to \( K_{3,3} \).
A graph is outerplanar if it can be embedded in the plane in such a way that all of
the vertices lie on the outer boundary. Equivalently, a graph \( G \) is outerplanar if adding
a new vertex to \( G \) and connecting it to all vertices of \( G \) results in a planar graph; we
denote this graph by \( \hat{G} \). Outerplanar graphs are characterized by not having as a minor
\( K_4 \) and \( K_{2,3} \). Here is a bipartite analog of this result for bipartite minors; the proof is
similar to the proof of Theorem 2.2.

**Theorem 2.4.** A bipartite graph \( G \) is outerplanar if and only if \( G \) does not contain \( K_{2,3} \)
as a bipartite minor.

**Proof.** First assume that \( G \) is outerplanar. To verify that \( G \) does not contain \( K_{2,3} \) as a
bipartite minor, it suffices to show that deletions and admissible contractions preserve
outerplanarity. This is clear for deletions. To deal with admissible contractions, consider
the graph \( \hat{G} \) defined right before the statement of the theorem. Then \( \hat{G} \) is planar,
although not bipartite, and if \( G' \) is obtained from \( G \) by an admissible contraction of \( a' \)
with \( a \), then \( \hat{G}' = (\hat{G})' \). (Note that a peripheral cycle of \( G \) is also a peripheral cycle of
\( \hat{G} \), so the same contraction is admissible in \( \hat{G} \).) As \( (\hat{G})' \) is planar (the same argument
as in the proof of Theorem 2.2 applies), we infer that \( G' \) is outerplanar.

Next assume that \( G \) is not outerplanar. By a result of Chartrand and Harary [1],
\( G \) contains a subdivision of either \( K_4 \) or \( K_{2,3} \). Let \( H \) be such a subgraph of \( G \). As
in the proof of Theorem 2.2, we may assume that each edge of the original \( K_4 \) (\( K_{2,3} \),
respectively) is subdivided at most once. Thus, it suffices to show that \( K_{2,3} \) is a bipartite
minor of each of the following nine bipartite graphs. Given the coloring below, we can
find \( K_{2,3} \) with three red vertices and two blue ones as a bipartite minor.

For subdivisions of \( K_4 \), we need to consider \( H = H_{(i)} \) for \( i = 4, 3, 2 \), where in \( H_{(i)} \)
every \( i \) of the original vertices of \( K_4 \) are red (the other \( 4 - i \) vertices are blue). For sub-
divisions of \( K_{2,3} \), let \( X \) and \( Y \) be the two sides of \( K_{2,3} \), with \( |X| = 2 \) and \( |Y| = 3 \), and for
the bipartite subdivisions \( H_{(i,j)} \) as above, with exactly \( i \) red vertices from \( X \) and \( j \) red ver-
tices from \( Y \), we need to consider \( H \) being one of \( H_{(2,3)}, H_{(1,3)}, H_{(0,3)}, H_{(2,2)}, H_{(1,2)}, H_{(2,1)} \).
We leave the verification of these nine cases to the readers. \( \square \)

Similarly, the following results hold (we omit their easy proofs). We define an apex
planar graph (respectively, apex outerplanar graph) to be any graph \( G \) that has a vertex
\( v \) such that \( G - v \) is planar (respectively, outerplanar).

**Theorem 2.5.** A bipartite graph \( G \) is a forest if and only if \( G \) does not contain \( K_{2,2} \) as
a bipartite minor.

**Theorem 2.6.** Deletions and admissible contractions preserve apex planarity; in other
words, the family of bipartite apex planar graphs is closed under bipartite minors. Sim-
ilarly, the family of bipartite apex outerplanar graphs is closed under bipartite minors.

However, there are examples showing that neither linkless embeddability nor embed-
dability into closed surfaces other than spheres is preserved under bipartite minors.

A natural question at this point to consider is then
3. \((2, 2)\)-Laman graphs

We now turn our discussion to \((2, 2)\)-Laman graphs. Note that if \(G = (V = A \uplus B, E)\) is \((2, 2)\)-Laman, then every vertex of \(G\) has degree at least two: indeed, if \(v\) were a vertex of degree one, then \(G - v\) would have \(2(|A| + |B| - 1) - 3\) edges instead of at most \(2(|A| + |B| - 1) - 4\) edges allowed by the definition of \((2, 2)\)-Laman graphs. Moreover, if \(G\) is \((2, 2)\)-Laman and \(v\) is a vertex of degree two, then either \(G\) is \(K_{2, 2}\) or \(G - v\) is also \((2, 2)\)-Laman. Finally, since \(G\) has fewer than \(2|V|\) edges, there is a vertex of \(G\) that has degree at most three. Hence, we can assume that \(G\) is a graph with minimal degree three. The following theorem can thus be considered as a recursive characterization of \((2, 2)\)-Laman graphs.

**Theorem 3.1.** Let \(G = (V, E)\) be a bipartite \((2, 2)\)-Laman graph with minimal degree three. Then every vertex \(v\) of degree three has two neighbors \(x, y\) with the property that there exists a vertex \(p\) that is adjacent to \(y\) and not adjacent to \(x\), and such that the graph \(G' = (G - v) \cup xp\) is \((2, 2)\)-Laman.

**Proof.** If \(X \subseteq V\), we write \(E(X) = |E(G[X])|\) — the cardinality of the edge set of the subgraph of \(G\) induced by \(X\). A subset \(X\) of \(V\) is critical if \(|X| \geq 3\) and \(E(X) = 2|X| - 4\); equivalently, if \(|X| \geq 3\) and \(G[X]\) is \((2, 2)\)-Laman or \(K_{2, 1}\).

Let \((A, B)\) be a bipartition of \(V\). In what follows vertices called \(a_i\) belong to \(A\), and vertices called \(b_i\) belong to \(B\). Suppose that \(a_0 \in A\) is a vertex of degree 3 and let \(b_1, b_2, b_3\) be the neighbors of \(a_0\). We prove the theorem in several steps, which we number below by \((i)^*\).

\((0^*)\) Every subset \(X \subseteq V\) with \(|X| \geq 2\) such that \(G[X]\) is not an edge satisfies \(E(X) \leq 2|X| - 4\).

**Proof.** This is immediate from the definition of \((2, 2)\)-Laman graphs.

\((1^*)\) At least two neighbors of \(a_0\) have non-neighbors in \(A\).

**Proof.** The subgraph induced on \(A \cup \{b_1, b_2, b_3\}\) has \(|A| + 3\) vertices and hence at most \(2|A| + 2\) edges. Thus if \(b_1, b_2\) are adjacent to all of \(A\) then \(b_3\) has degree at most two, a contradiction.

Let \(b_1, b_2\) be as guaranteed in \((1^*)\). Let \(Z \subseteq V\) be maximal with \(2|Z| - 4\) edges, containing \(b_1, b_2\) and not containing \(a_0\) (possibly \(Z = \{b_1, b_2\}\)).

\((2^*)\) The element \(b_3\) is not in \(Z\) and has at most one neighbor in \(Z\). In particular, \(b_3\) has a neighbor in \(A \setminus (Z \cup \{a_0\})\), and so the latter set is nonempty.

**Proof.** If \(b_3 \in Z\) then \(Z \cup \{a_0\}\) violates Laman condition (ii), a contradiction, and so \(b_3 \notin Z\). Now \(G[Z \cup \{a_0\}]\) is \((2, 2)\)-Laman, hence the Laman condition (ii) for \(Z \cup \{a_0, b_3\}\) shows that \(b_3\) has at most one neighbor in \(Z\). As \(\deg(b_3) \geq 3\) the rest of \((2^*)\) follows.
Denote by $M(b_i)$ the set of non-neighbors of $b_i$ in $A \setminus (Z \cup \{a_0\})$, and w.l.o.g. assume $|M(b_1)| \geq |M(b_2)|$. Note that the maximality of $Z$ implies that

(3*) Every vertex in $A \setminus (Z \cup \{a_0\})$ has at most one neighbor in $Z$.

In particular, we will use the following:

(4*) No element in $A \setminus (Z \cup \{a_0\})$ is a neighbor of both $b_1$ and $b_2$.

(5*) The set $A_1$ of neighbors of either $b_2$ or $b_3$ in $M(b_1)$ is nonempty.

Proof. Either there exists $p \in M(b_1) \setminus M(b_2)$ (that is, $p$ is a neighbor of $b_2$ but not of $b_1$), in which case we are done, or $M(b_1) = M(b_2)$. In the latter case $M(b_1) = A \setminus (Z \cup \{a_0\})$ by (4*). Hence by (2*), there is a vertex $p \in M(b_1)$ that is a neighbor of $b_2$. The statement follows.

We need to prove that there is $p \in A_1$ such that no critical set contains $p$ and $b_1$ and not $a_0$. Assume the contrary. Then for every $p \in A_1$ there is some critical set $X_p$ containing $p$ and $b_1$, and not containing $a_0$. We will reach a contradiction to (5*). We may assume that the sets $X_p$ are maximal with these properties.

(6*) If $p \in A_1$ then $b_2 \notin X_p$.

Proof. Assume by contradiction $b_2 \in X_p$. By maximality of $Z$ and as $p \notin Z$, it is enough to show that $X_p \cup Z$ is critical. Indeed,

$$E(Z \cup X_p) \geq E(Z) + E(X_p) - E(Z \cap X_p) = 2(|Z| + |X_p|) - 8 - E(Z \cap X_p) \geq 2(|Z| + |X_p|) - 8 - (2|Z \cap X_p| - 4) \geq 2|Z \cup X_p| - 4,$$

where the middle inequality is by (0*), as $\{b_1, b_2\} \subseteq Z \cap X_p$ are two vertices on the same side. □

(7*) For every $p \in A_1$, $b_1$ has at least two neighbors in $X_p$.

Proof. Since $X_p$ is critical containing two non-adjacent vertices from opposite sides (namely $p$ and $b_1$), we deduce that $|X_p| \geq 4$. Then $E(X_p \setminus b_1) \leq 2(|X_p| - 1) - 4$ by Laman condition (ii), and so $b_1$ has at least two neighbors in $X_p$. □

Let $X_0 \subseteq V$ (in our notations $0 \notin V$) consist of $b_1$ and its neighbors different from $a_0$. For notational convenience, identify $A_1$ with $\{1, 2, \ldots, s\}$, and let $Y_s = X_0 \cup X_1 \cup \cdots \cup X_s$. Let $k = |X_0| - 1$, namely the degree of $b_1$ in $G[V \setminus a_0]$ (hence $k \geq 2$).

(8*) $E(Y_s) \geq 2|Y_s| - 4 - (k - 2)$.

Proof. Let $Y_i = \cup_{0 \leq j \leq i} X_j$. We show by induction on $i$ that $E(Y_i) \geq 2|Y_i| - 4 - (k - 2)$. The case $i = 0$ trivially holds with equality. For $i \geq 1$, $|X_0 \cap X_i| \geq 3$ by (7*), so applying Laman condition (ii) to $Y_{i-1} \cap X_i$ we see that

$$E(Y_i) \geq E(Y_{i-1}) + E(X_i) - E(Y_{i-1} \cap X_i) \geq (2|Y_{i-1}| - 4 - (k - 2)) + (2|X_i| - 4) - (2|Y_{i-1} \cap X_i| - 4) = 2|Y_{i-1} \cup X_i| - 4 - (k - 2),$$

as desired. □
Next we will show that $Z \cup Y_s$ is critical.

(9*) $Z \cap Y_s \subseteq X_0$, and thus $Z \cap Y_s = Z \cap X_0$.

Proof. Assume the contrary. Then there is $i$ such that $Z \cap X_i$ has a non-neighbor of $b_i$. Hence by (0*), $E(Z \cap X_i) \leq 2|Z \cap X_i| - 4$. Thus, our usual yoga shows $E(Z \cup X_i) \geq 2|Z \cup X_i| - 4$. By maximality of $X_i$, we conclude that $Z \subseteq X_i$, and so $b_2 \in X_i$. This contradicts (6*). \[\square\]

(10*) $Z \cup Y_s$ is critical.

Proof. Denote by $d_i$ the number of neighbors of $b_i$ in $A \setminus (Z \cup \{a_0\})$. By (9*), $E(Z \cap Y_s) = k - d_1$ and $|Z \cap Y_s| = k - d_1 + 1$. There are $d_2$ edges from $b_2 \in Z$ into $A_1 \subseteq Y_s \setminus Z$ by (4*). Therefore,

$$E(Z \cup Y_s) \geq E(Z) + E(Y_s) - E(Z \cap Y_s) + d_2 \geq$$

$$(2|Z| - 4) + (2|Y_s| - 4 - (k - 2)) - (k - d_1) + d_2 =$$

$$2|Z \cup Y_s| - 4 + (d_2 - d_1),$$

and by $|M(b_1)| \geq |M(b_2)|$ we have $d_2 - d_1 \geq 0$. Thus $Z \cup Y_s$ is critical. \[\square\]

By maximality of $Z$, (10*) implies $Y_s \subseteq Z$. Thus $A_1 \subseteq Z$. Hence by the definition of $A_1$, $A_1$ must be empty. This contradicts (5*) and completes the proof of the theorem. \[\square\]

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4. Appendix

For each of the nine graphs described in the proof of Theorem 2.2, we describe the sequences of deletions and admissible contractions yielding $K_{3,3}$ as a bipartite minor. For each contraction we indicate a peripheral cycle showing that the contraction is admissible, called a witness cycle.

We start with the subdivisions of $K_5$. Denote the vertices of $K_5$ by $v_i$ where $i \in [5]$ (here $[n] = [1, n] = \{1, 2, \cdots, n\}$), and the subdivision vertex of the edge $v_i v_j$ in $G_{(i)}$ by $v_{ij} = v_{ji}$. When referring to vertices after performing contractions on $G_{(i)}$, we use any representative from the vertices of $G_{(i)}$; this should cause no confusion.

Case 1: $G_{(5)}$. Here all $v_i$ where $i \in [5]$ are red.

1. Contract $v_{15}$ with $v_{13}$. Witness cycle: $(v_{15}v_{13}v_{35}v_5v_5)$.
2. Contract $v_{25}$ with $v_{23}$. Witness cycle: $(v_{25}v_{23}v_{35}v_5v_5)$.
3. Contract $v_{45}$ with $v_{14}$. Witness cycle: $(v_{15}v_{44}v_{14}v_{15}v_5)$. 
Call the resulting graph $H$. The induced subgraph of $H$ on the 3 red vertices $v_1, v_3, v_5$ and the 3 blue vertices $v_{15}, v_{25}, v_{45}$ is $K_{3,3}$.

**Case 2:** $G_{(4)}$. Here all $v_i$ where $i \in [4]$ are red, $v_5$ is blue.

1. Contract $v_{34}$ with $v_{23}$. Witness cycle: $(v_{34}v_3v_{23}, v_2v_{24}v_4)$.
2. Contract $v_{12}$ with $v_{14}$. Witness cycle: $(v_{12}v_1v_{14}, v_4v_{24}v_2)$.
3. Contract $v_{12}$ with $v_{13}$. Witness cycle: $(v_{12}v_1v_{13}, v_3v_{32}v_2)$.

Call the resulting graph $H$. The induced subgraph of $H$ on the 3 red vertices $v_2, v_3, v_4$ and the 3 blue vertices $v_{34}, v_{12}, v_1$ is $K_{3,3}$.

**Case 3:** $G_{(2)}$. Here all $v_i$ where $i \in [2]$ are red, the other 3 vertices of $G_{(2)}$ are blue.

1. Contract $v_{34}$ with $v_{35}$. Witness cycle: $(v_{34}v_3v_{35}, v_5v_{45}v_4)$.

Call the resulting graph $H$. The induced subgraph of $H$ on the 3 red vertices $v_1, v_2, v_{34}$ and the 3 blue vertices $v_3, v_4, v_5$ is $K_{3,3}$.

We now turn to the six subdivisions of $K_{3,3}$. Let the sides of $K_{3,3}$ be $X = \{v_1, v_2, v_3\}$ and $Y = \{v_4, v_5, v_6\}$. We use the same notation $v_{ij}$ as in the case of subdivisions of $K_5$.

**Case 1:** $G_{(3,3)}$. Here all $v_i$ where $i \in [6]$ are red.

1. Contract $v_{15}$ with $v_{35}$. Witness cycle: $(v_{15}v_5v_{35}, v_3v_{34}v_4v_{14}v_1)$.
2. Contract $v_{14}$ with $v_{24}$. Witness cycle: $(v_{14}v_4v_{24}, v_2v_{26}v_6v_{16}v_1)$.
3. Contract $v_{26}$ with $v_{36}$. Witness cycle: $(v_{26}v_6v_{36}, v_3v_{34}v_4v_{24}v_2)$.
4. Contract $v_{1}$ with $v_{6}$. Witness cycle: $(v_1v_{16}v_6, v_{36}v_2v_{24})$.
5. Contract $v_{2}$ with $v_{5}$. Witness cycle: $(v_2v_{25}v_5, v_{35}v_3v_{34}v_4v_{24})$.
6. Delete $v_{16}$, then delete $v_{25}$.
7. Contract $v_{3}$ with $v_{4}$. Witness cycle: $(v_3v_{34}v_4, v_{24}v_1v_{36})$.

Call the resulting graph $H$. The induced subgraph of $H$ on the 3 red vertices $v_1, v_2, v_3$ and the 3 blue vertices $v_{15}, v_{14}, v_{26}$ is $K_{3,3}$.

**Case 2:** $G_{(3,2)}$. Here all $v_i$ where $i \in [5]$ are red and $v_6$ is blue.

1. Contract $v_{15}$ with $v_{35}$. Witness cycle: $(v_{15}v_5v_{35}, v_3v_{34}v_4v_{14}v_1)$.
2. Contract $v_{14}$ with $v_{24}$. Witness cycle: $(v_{14}v_4v_{24}, v_2v_{25}v_5v_{15}v_1)$.
3. Contract $v_{3}$ with $v_{4}$. Witness cycle: $(v_3v_{34}v_4, v_{24}v_2v_{25}v_5v_{15})$.
4. Contract $v_{2}$ with $v_{5}$. Witness cycle: $(v_2v_{25}v_5, v_{15}v_1v_6)$.

Call the resulting graph $H$. The induced subgraph of $H$ on the 3 red vertices $v_1, v_2, v_3$ and the 3 blue vertices $v_{15}, v_{14}, v_6$ is $K_{3,3}$.

**Case 3:** $G_{(3,1)}$. Here all $v_i$ where $i \in [4]$ are red and $v_5, v_6$ are blue.

1. Contract $v_{24}$ with $v_{34}$. Witness cycle: $(v_{24}v_4v_{34}, v_3v_5v_2)$.
2. Contract $v_{1}$ with $v_{4}$. Witness cycle: $(v_1v_{14}v_4, v_{34}v_3v_5)$.

Call the resulting graph $H$. The induced subgraph of $H$ on the 3 red vertices $v_1, v_2, v_3$ and the 3 blue vertices $v_{24}, v_5, v_6$ is $K_{3,3}$. 

Case 4: $G_{(2,2)}$. Here all $v_i$ where $i \in [2, 5]$ are red and $v_1, v_6$ are blue.

1. Contract $v_{24}$ with $v_{34}$. Witness cycle: $(v_{34}v_4v_{24}, v_2v_{25}v_5v_{53}v_3)$.
2. Contract $v_{25}$ with $v_{35}$. Witness cycle: $(v_{25}v_5v_{35}, v_3v_{34}v_2)$.
3. Contract $v_4$ with $v_5$. Witness cycle: $(v_4v_1v_5, v_{25}v_3v_{34})$.
4. Contract $v_1$ with $v_6$. Witness cycle: $(v_1v_{16}v_6, v_2v_{25}v_4)$.

Call the resulting graph $H$. The induced subgraph of $H$ on the 3 red vertices $v_2, v_3, v_4$ and the 3 blue vertices $v_{24}, v_{25}, v_6$ is $K_{3,3}$.

Case 5: $G_{(2,1)}$. Here all $v_i$ where $i \in [2, 4]$ are red and $v_1, v_5, v_6$ are blue.

1. Contract $v_{16}$ with $v_{15}$. Witness cycle: $(v_{16}v_1v_{15}, v_5v_3v_6)$.
2. Contract $v_{24}$ with $v_{34}$. Witness cycle: $(v_{24}v_4v_{34}, v_3v_6v_2)$.
3. Contract $v_1$ with $v_{34}$. Witness cycle: $(v_1v_4v_{34}, v_2v_5v_{15})$.

Call the resulting graph $H$. The induced subgraph of $H$ on the 3 red vertices $v_{16}, v_2, v_3$ and the 3 blue vertices $v_1, v_5, v_6$ is $K_{3,3}$.

Case 6: $G_{(3,0)}$. In this case $G_{(3,0)} = H = K_{3,3}$. This completes the proof of Theorem 2.2. □