Normalizing inclusive rare $B$ decays

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Abstract

The inclusive semileptonic branching ratio is often employed to normalize other inclusive $B$ decays. Using recent determinations of the non-perturbative parameters of the Operator Product Expansion we compute the normalization factor for the branching ratio of $B \to X_s \gamma$ and find a few percent enhancement with respect to previous determinations.
1 Introduction

The partial width of the inclusive $B$ decays to light quarks depends on the fifth power of the $b$ quark mass. In order to reduce the uncertainty that follows from this strong sensitivity and to avoid the large radiative corrections that are sometimes related to the heavy quark mass, the Branching Ratio (BR) of rare decays is usually expressed in terms of the CKM favored semileptonic BR, $\text{BR}_{\text{c\ell\nu}} \equiv \text{BR}[B \rightarrow X_c\ell\nu]$, a quantity that is presently known at the $1\%$ level $[1, 2]$. This choice introduces a marked dependence on the charm quark mass in the calculation of rare decays, but it is very convenient in $b \rightarrow s$ transitions, whose CKM factor $|V_{ts}^*V_{tb}|$ is essentially determined by $|V_{cb}|$ measured in inclusive semileptonic $B$ decays. Moreover, in the case of $B \rightarrow X_s\gamma$ the charm mass dependence due to the normalization to $\text{BR}_{\text{c\ell\nu}}$ is partially compensated by that of the perturbative QCD corrections. In recent years the $B$ factories have performed increasingly detailed studies of semileptonic $B$ decays, providing us with improved determinations of the $b$ and $c$ quark masses and of the Operator Product Expansion (OPE) parameters. In parallel, both the measurements and the theoretical calculations of inclusive rare decays have improved significantly. In this Letter we reconsider the normalization of rare decays and try to assess its uncertainty, taking into account the latest developments. We will concentrate on the radiative inclusive decay of the $B$ meson, $B \rightarrow X_s\gamma$: the Next-to-Next-to-Leading Order (NNLO) calculation of its BR in the Standard Model is quite advanced $[3]$ and its experimental error will soon approach $5\%$. Many of our considerations apply to $B \rightarrow X_c\ell^+\ell^-$ $[4]$ and $B \rightarrow X_u\ell\nu$ as well.

As mentioned already, the unitarity of the CKM matrix implies $|V_{ts}^*V_{tb}|^2 = [1 + \lambda^2(2\rho - 1) + O(\lambda^4)]|V_{cb}|^2 = (0.963 \pm 0.003)|V_{cb}|^2$ $[5]$, $i.e.$ the CKM prefactor of $B \rightarrow X_s\gamma$ is essentially given by $|V_{cb}|$, whose best determination follows from $\text{BR}_{\text{c\ell\nu}}$ and has a $2\%$ error $[1]$. In recent analyses $[3, 6, 7]$ the radiative BR is therefore expressed by

$$\text{BR}_\gamma(E_0) \equiv \text{BR}[B \rightarrow X_s\gamma]_{E_0 > E_0} = \frac{\text{BR}_{\text{c\ell\nu}}}{C} \left( \frac{\Gamma[B \rightarrow X_s\gamma]_{E_0 > E_0}}{|V_{cb}/V_{ub}|^2 \Gamma[B \rightarrow X_u\ell\nu]} \right),$$

where the phase space ratio $C$ is defined by

$$C = \left| \frac{V_{ub}}{V_{tb}} \right|^2 \frac{\Gamma[B \rightarrow X_u\ell\nu]}{\Gamma[B \rightarrow X_u\ell\nu]},$$

In Eq. $[1]$ the radiative width is normalized to the charmless semileptonic decay in order to split the charm mass dependence of the perturbative matrix elements of $b \rightarrow X_s\gamma$ — an $O(\alpha_s)$ two-loop effect — from that due to the normalization — a tree-level effect, contained in $C$ $[6]$. This choice makes the calculation more transparent. One has

$$\left( \frac{\Gamma[B \rightarrow X_s\gamma]_{E_0 > E_0}}{|V_{cb}/V_{ub}|^2 \Gamma[B \rightarrow X_u\ell\nu]} \right) = \left| \frac{V_{ts}^*V_{tb}}{V_{cb}} \right|^2 \frac{6\alpha_{\text{em}}}{\pi} [1 + \delta_{NP}] P(E_0),$$

where $\alpha_{\text{em}}$ is the fine structure constant. The perturbative term $P(E_0)$ is estimated at NNLO in QCD $[3, 7]$ and includes electroweak effects $[8]$, while $\delta_{NP}$ contains the non-perturbative power corrections $[9]$. Fortunately, the $m_c$ dependence of $P(E_0)$, despite being a loop effect, compensates about half of that of $C$. 

1
2 Calculation of C

Like all inclusive widths, the ratio $C$ can be calculated using the OPE and expressed as a double expansion in $\alpha_s$ and inverse powers of the $b$ quark mass, currently known through $O(\alpha_s^2)$ and $O(\Lambda_{QCD}^3/m_b^3)$. $C$ depends sensitively on the $b$ and $c$ quark masses, as well as on the matrix elements of the dimension 5 and 6 operators. This is where the recent experimental studies of the inclusive moments of $B \to X_c e \bar{\nu}$ and $B \to X_s \gamma$ enter in a crucial way. Indeed, the moments of various kinematic distributions provide information on the non-perturbative parameters of the OPE. Global fits to the moments describe successfully a variety of moments and allow for a $40-50\text{MeV}$ determination of $m_c$ and $m_b$, $\sim 10-20\%$ determination of the $1/m_b^2$ and $1/m_c^2$ matrix elements, and a $\sim 2\%$ determination of $|V_{cb}|$ \cite{2,10}. There are different ways to take into account the available information, relying on different assumptions and schemes. We work in the kinetic scheme \cite{11}, where a ‘hard’ cutoff $\mu$ separates perturbative and non-perturbative effects respecting heavy quark relations, and non-perturbative parameters are well-defined and perturbatively stable.

Our starting point are the NNLO expressions for the charmed and charmless total semileptonic widths

\[ \Gamma[B \to X_c e \bar{\nu}] = \frac{G_F^2 m_b^5}{192\pi^3} |V_{cb}|^2 g(r) \left[ 1 + \frac{\alpha_s}{\pi} p_c^{(1)}(r, \mu) + \frac{\alpha_s^2}{\pi^2} p_c^{(2)}(r, \mu) \right. \]

\[ \left. - \frac{\mu_b^2}{2m_b^2} + \left( \frac{1}{2} - \frac{2(1-r)^4}{g(r)} \right) \frac{\mu_b^2 - \rho_{LS}^2 - \rho_D^2}{m_b^2} \right. \]

\[ \left. + \left( \frac{8}{3} \ln r - \frac{10r^4}{3} + \frac{32r^3}{3} - 8r^2 - \frac{32r}{3} + \frac{34}{3} \right) \frac{\rho_D^2}{g(r) m_b^3} \right], \] (4)

\[ \Gamma[B \to X_u e \bar{\nu}] = \frac{G_F^2 m_b^5}{192\pi^3} |V_{ub}|^2 \left[ 1 + \frac{\alpha_s}{\pi} p_u^{(1)}(\mu) + \frac{\alpha_s^2}{\pi^2} p_u^{(2)}(r, \mu) - \frac{\mu_b^2}{2m_b^2} - \frac{3\mu_G^2}{2m_b^2} \right. \]

\[ \left. + \left( \frac{77}{6} + 8\ln \frac{\mu_{WA}}{m_b^2} \right) \frac{\rho_D^2}{m_b^3} + \frac{3\rho_{LS}^2}{2m_b^3} + \frac{32\pi^2}{m_b^3} B_{WA}(\mu_{WA}) \right], \] (5)

where $\alpha_s \equiv \alpha_s^{(n_f=5)} (m_b)$, $r = (m_c/m_b)^2$, $g(r) = 1 - 8r + 8r^3 - r^4 - 12r^2 \ln r$, and all the masses and OPE parameters are defined in the kinetic scheme at finite $m_b$ with $\mu \sim 1\text{GeV}$. The non-perturbative corrections have been computed in \cite{12} and are expressed in terms of the parameters $\mu_b^2, \mu_G^2, \rho_D^2, \rho_{LS}^2$. The matrix element of the Weak Annihilation (WA) operator $B_{WA} \equiv \langle B|O_{WA}^{\pi}|B \rangle$ is poorly known. It is here renormalized in the \(\overline{\text{MS}}\) scheme at the scale $\mu_{WA}$, see \cite{13,14}. We recall that $B_{WA}$ vanishes in the factorization approximation, and that WA is phenomenologically important only to the extent factorization is actually violated.

There is however an $O(1)$ mixing between WA and Darwin operators, and at lowest order in perturbation theory one has $B_{WA}(\mu') = B_{WA}(\mu) - \rho_D^2/2\pi^2 \ln \mu'/\mu$. As factorization may hold only for a certain value $\mu_{WA} = \mu_f$ for which $B_{WA}(\mu_f) = 0$, a change of the scale $\mu_f$ provides a rough measure of the (minimal) violation of factorization induced perturbatively. We neglect intrinsic charm contributions \cite{15}. WA uncertainties make a precise prediction of $C$ problematic at present. Fortunately, they cancel out in Eq.(1) since the radiative BR cannot depend on the non-perturbative features of the charmless semileptonic decay.
The perturbative corrections at $\mu = 0$ (on-shell scheme) are given by

\[
p_c^{(1)}(r, 0) = -\frac{2h(r)}{3g(r)}, \quad p_a^{(1)}(0) = \frac{25}{6} - \frac{2}{3}\pi^2,
\]

\[
p_c^{(2)}(r, 0) = (-3.381 + 7.15\sqrt{r} - 5.18 r)\beta_0^{(4)} + (4.07 - 7.8\sqrt{r}),
\]

\[
p_u^{(2)}(r, 0) = -3.22 \beta_0^{(4)} + 5.54 + (1.73 \ln \sqrt{r} - 2.17)\sqrt{r},
\]

where $h(r)$ is given in App. C of [6], and $\beta_0^{(n_f)} = 11 - \frac{2}{3}n_f$. The $O(\alpha_s^2)$ perturbative corrections are known exactly in both cases [17, 18]. Their $m_c$ dependence is given here in terms of simple interpolation formulas, valid for $m_c/m_b$ between 0.2 and 0.3. While the $O(\alpha_s^2\beta_0)$ part of $p_c^{(2)}$ has been known for some time [19], the remaining (non-BLM) term is a very recent result [18]. The $\mu$-dependence of $p_{a,c}^{(1,2)}(r, \mu)$ can be found exploiting the $\mu$-independence of the widths at each perturbative order and the known $\mu$-dependence of masses and OPE parameters [20].

\[
m_q^{pole} \equiv m_q(0) = m_q(\mu) + \left[\Lambda(\mu)\right]_{pert} + \frac{[\mu_2^2(\mu)]_{pert}}{2m_q(\mu)},
\]

\[
\mu_2^2(\mu) = \mu_2^2(\mu) - [\mu_2^2(\mu)]_{pert}, \quad \rho_D^3(0) = \rho_D^3(\mu) - [\rho_D^3(\mu)]_{pert},
\]

where

\[
[\Lambda(\mu)]_{pert} = \frac{4}{3}C_F\frac{\alpha_s}{\pi}\mu \left(1 + \frac{\alpha_s}{\pi} \left[\frac{\beta_0^{(3)}}{2} + \frac{8}{3} \ln \frac{m_b}{2\mu} + \frac{8}{3}\right] - C_A \left(\frac{\pi^2}{6} - \frac{13}{12}\right)\right),
\]

\[
[\mu_2^2(\mu)]_{pert} = \frac{2}{3}\mu \left[\Lambda(\mu)\right]_{pert} - \frac{C_F\alpha_s^2\beta_0^{(3)}}{\pi^2} \frac{\mu_2^2}{4}, \quad [\rho_D^3(\mu)]_{pert} = \mu^2 \left[\Lambda(\mu)\right]_{pert} - \frac{C_F\alpha_s^2\beta_0^{(3)}}{\pi^2} \frac{2\mu^3}{9},
\]

In the above formulas $C_F = \frac{4}{3}$, $C_A = N_c = 3$, and we have used $n_f = 3$ in $\beta_0$, corresponding to three light massless quarks: indeed, the $m_c$ dependence of the charm loops on gluon lines is not known, and one can approximately decouple them. This is also consistent with the calculation of the moments in the kinetic scheme [21]. A fully consistent $O(\alpha_s^2)$ implementation of the kinetic scheme would require the $O(\alpha_s)$ corrections to the Wilson coefficients of the higher dimensional operators in [44], which is not yet available except for the trivial $\mu_2^2$ term. As for higher order power corrections, we recall that $1/m_b^4$ corrections have been estimated to be tiny in the charmed decay rate [22].

The numerical value of $C$ depends on those of the OPE parameters and of $\alpha_s$. We take as default values the results of the global fit [2], namely

\[
m_b = 4.597 \text{ GeV}, \quad m_c = 1.163 \text{ GeV}, \quad \mu_2^2 = 0.436 \text{ GeV}^2, \quad \mu_G^2 = 0.267 \text{ GeV}^2
\]
\[
\rho_D^3 = 0.213 \text{ GeV}^3, \quad \rho_L^3 = -0.178 \text{ GeV}^3;
\]

where all values are in the kinetic scheme with $\mu = 1\text{ GeV}$. We employ $\alpha_s(m_b) = 0.219$ and set $\mu_{WA} = m_b/2$, obtaining

\[
C = 0.546 - 2.0 B_{WA}(m_b/2)
\]

\[
= 0.625 - 0.028\alpha_s - 0.022\alpha_s^2 - 0.001\mu_G^2 - 0.025\rho_D^3 - 0.001\rho_L^3 - 2.0 B_{WA}(m_b/2)
\]
In the second line we have listed the individual contributions, that are accidentally all negative. Using only BLM corrections and setting \( n_f \) consistently equal to 3, one gets

\[ C = 0.543 - 2.0 B_{WA}(m_b/2), \]

which shows that the \( O(\alpha_s^2) \) corrections are dominated by \( O(\alpha_s^2 \beta_0) \) running coupling effects. The \( O(\alpha_s^2 \beta_0) \) corrections can be absorbed in a rescaling of \( \alpha_s \) in the NLO contribution, using \( \alpha_s(\mu_{BLM}) \). It is clear however that in that case the BLM scale is very low, just above 1 GeV. Following [14] we use \( B_{WA}(m_b/2) \approx 0 \) as central value and vary it between 0 and +0.012 GeV\(^3\), since the positivity of the \( q^2 \) spectrum of \( B \to X_c \ell \nu \) at \( \mu_{WA} = m_b/2 \) suggests a positive WA contribution. Taking into account the correlations among the input parameters [2], and varying \( \mu \) between 0.7 and 1.3 GeV to estimate the residual perturbative error, we find

\[
C = 0.546 \pm 0.017(par) \pm 0.016(pert) \pm 0.0000(WA) = 0.546^{+0.003}_{-0.033}. \tag{7}
\]

This value can be compared with \( C = 0.580 + 1.8 B_{WA}(m_b) \pm 0.016, \) obtained in a global fit to semileptonic and radiative moments [10] where the \( b \) mass is in the 1\( S \) scheme [23]. Using the value of \( \rho_D^3(\mu = 0) \) obtained in [10], we can rewrite it as \( C = 0.582 + 1.8 B_{WA}(m_b/2) \), that differs from Eq. (6) by 6.5\%, exceeding the stated uncertainties. We observe that the fit in [10] is based on older data than that of [2]. The two fits also differ in the perturbative scheme, in several assumptions, and in the estimate of theory errors, as detailed in Refs. [10] and [21,2], resp. Ref. [10], for instance, does not extract the charm mass directly but eliminates it through the heavy meson mass relation

\[
m_b - m_c = \bar{M}_B - \bar{M}_D + \frac{\mu^2}{2} \left( \frac{1}{m_c} - \frac{1}{m_b} \right) + \frac{\rho_D^3 - \bar{\rho}_D^3}{4} \left( \frac{1}{m_c^2} - \frac{1}{m_b^2} \right) + O(1/m_Q^3), \tag{8}
\]

where all quark masses and OPE parameters have the same normalization point \( \mu \). This relation involves an expansion in \( 1/m_c \), rather than \( 1/m_b \), and some non-local operators (the term \( \bar{\rho}_D^3 \)) that do not enter the expressions for the semileptonic widths. In [10] the normalization scale is set to zero and the fit is effectively equivalent to a direct fit to the pole mass difference \( m_b - m_c \). The default procedure in [10] employs other meson mass relations to fix \( \mu^2 \) and \( \rho_D^3 \), treats differently \( \bar{\Lambda} \) terms, includes the moments of \( B \to X_c \gamma \) accounting for neither distribution function effects nor an additional theory error, and estimates theory errors in a different way.

Despite their differences, the two fits give compatible values of \( |V_{cb}| \). But the ratio \( C \), well approximated by \( 1.2 - 2.2 m_c/m_b \) in the physical region, is more sensitive to the exact value of the charm and bottom masses than \( |V_{cb}| \), due to the correlation between \( m_b \) and \( m_c \) in the results of the fits. As illustrated in Fig. 1, the semileptonic rate and moments (in particular, those of the lepton energy distribution) stringently constrain a certain combination of \( m_{c,b} \), given by \( m_b - 0.65 m_c \), that roughly corresponds to constant values of the semileptonic total width for fixed \( |V_{cb}| \). In the fit of [2] the latter gets a 0.3\% relative parametric error, while \( C \approx 1.2 - 2.2 m_c/m_b \) has a 3.3\% error. The charmed inclusive width, roughly proportional to \( m_b^5 (1.2 - 2.2 m_c/m_b) \), has instead a 1.2\% relative uncertainty. As shown in Fig. 1, while the fitted value of \( m_b^5 (1.2 - 2.2 m_c/m_b) \), and consequently of \( |V_{cb}| \), is insensitive to small changes in the data and to the inclusion of radiative moments, the situation is quite different for \( m_c \).
1σ contours in the $m_c, m_b$ plane in the kinetic scheme at $\mu = 1$ GeV from: a) the global fit [2] (small red ellipse in the center); b) same fit without the radiative moments (larger upper right ellipse); c) the Belle fit [24] (larger lower left ellipse); d) the PDG [25] (large, light green central ellipse); e) the sum rules determinations [26] (smaller, light red, center bottom ellipse), after translating them in the kinetic scheme. The solid red and dashed blue lines correspond to constant values of the semileptonic width and of $C$, respectively. $C$ decreases moving to the right of the plot.

and $C$. As indicated by the 1$S$ fit to only Belle data performed by the Belle collaboration [24], the treatment of theoretical errors may similarly have a larger impact on the quark masses than on $|V_{cb}|$. Different determinations of the $c$ and $b$ masses tend to prefer values close to the center of the plot: we show in Fig. 1 the 1σ regions in the $(m_c, m_b)$ plane selected by the PDG [25] and by the $\sigma(e^+e^- \rightarrow \text{hadrons})$ sum rules according to [26], taking into account the non-negligible error introduced by the scheme translation (40 and 50 MeV for $m_b$ and $m_c$, respectively, estimated using the residual scale dependence).

It is instructive to compare results based on the same set of data, like the recent Belle fits [24] that follow the methods of [21] and [10]. Using their kinetic scheme results, in the same way that has led to Eq. (6), we obtain $C = 0.574$, although with a parametric estimate.

\footnote{The Belle data are also included in the global fit of [2].}
error twice larger than above. From the central values of the 1S scheme fit of \[24\] we find \(m_b\text{pole} - m_c\text{pole} = 3.393\text{ GeV}\) that can be employed to eliminate \(m_c\) in Eq. (4), using the 1S scheme for \(m_b\). This is analogous to what was done in Appendix C of \[6\], but includes \(1/m_b^2\) effects that were neglected in that paper. Assuming \(B_{WA}(m_b/2) = 0\), the result is \(C = 0.563^\pm\). In other words, using the same data set the results in the two schemes are closer and the kinetic scheme gives a higher value. The discrepancy between Eq. (6) and \[10\] is likely to be due to different input data, although at some level differences in the fit methods and in the treatment of theory errors may also play a role. The situation is likely to improve with better experimental data and better control of theory errors (an upgrade of the fitting methods is under way \[27\]), but in the meantime the error attached to \(C\) should be treated with caution.

3 Impact on radiative decays

We now move to the calculation of the radiative BR. The power correction \(\delta_{NP}\) to the total inclusive radiative decay normalized to \(B \to X_u \ell \nu\) \[9\] — see Eq. (3) — is

\[
\delta_{NP} = - \left( \frac{\mu_G^2}{27 m_c^2} + \frac{\rho_D^3 - \frac{13}{4} \rho_{LS}^3}{27 m_c^2 m_b} \right) \frac{C^{(0)}(\mu_b)}{C^{(0)\text{eff}}(\mu_b)} - \left( \frac{44}{3} + 8 \ln \frac{\mu_{WA}^2}{m_b^2} \right) \frac{\rho_D^3}{m_b^2} - \frac{32 \pi^2}{m_b^2} B_{WA}(\mu_{WA}),
\]

where \(\mu_b\) is the low-energy scale, that we set equal to 2.5 GeV like in \[7\], \(C^{(0)\text{eff}}(\mu_b) \approx -0.37\) is the effective Wilson coefficients of the operator \(Q_7\) calculated at leading logarithmic order, and \(C^{(0)}(\mu_b) \approx 1.20\) is the appropriate combination of Wilson coefficients of \(Q_{1,2}\) (see \[6\], \[7\]). An additional \(O(\alpha_s \mu^2_e/m_b^2)\) contribution, dependent on the photon energy cut, has been calculated in \[28\], but it is small for \(E_0 \leq 1.6\text{ GeV}\) and we neglect it. In the notation of \[7\], \(\delta_{NP} = N(E_0)/P^{(0)}(\mu_b)\). Since the NNLO calculation of \(P(E_0)\) \[3\] \[7\] has not been performed in the kinetic scheme, all the parameters in Eq. (9) must be understood at \(\mu = 0\), but we convert them to \(\mu = 1\text{ GeV}\) expanding to second order in \(\alpha_s\). We find \(\delta_{NP} = 0.033 - 3.2 B_{WA}(m_b/2)\). Since \(\delta_{NP}\) is correlated with \(C\), a quantity that summarizes all power corrections and the normalization in Eq. (1) is

\[
F \equiv \frac{1 + \delta_{NP}}{C} = 1.600 + 0.075 \alpha_s + 0.061 \alpha_s^2 + 0.047 \rho_G^2 + 0.057 \rho_D^3 + 0.019 \rho_{LS}^3
\]

\[= 1.859 \pm 0.054(\text{par}) \pm 0.045(\text{pert}),
\]

which is independent of both \(B_{WA}\) and \(\mu_{WA}\). As long as the perturbative corrections to Eq. (9) are not available, the numerical value of the charm mass to be employed is quite arbitrary. We have employed our default input value, but using \(m_c \approx 1.5\text{ GeV}\), closer to the pole mass, would decrease \(F\) by 1%. We have accordingly increased the perturbative error in Eq. (10).

We list in the Appendix approximate formulae that allow for an easy determination of \(C\) and \(F\) with different inputs. In the presence of new physics \(C^{(0)\text{eff}}(\mu_b) = C^{(0)\text{eff}}_{7,\text{SM}}(\mu_b) + C^{(0)\text{eff}}_{7,\text{new}}(\mu_b)\)

\(^2\)Employing the same method with the results of \[10\], we get \(C = 0.585\).
and this is expected to be the only change in $F$, leading to

$$F = F_{SM} - (0.06 \pm 0.03) \varepsilon_{\text{new}}$$  \hspace{1cm} (11)$$

with $\varepsilon_{\text{new}} = C_{7, \text{new}}^{(0)\text{eff}}(\mu_b)/C_{7}^{(0)\text{eff}}(\mu_b)$ and $F_{SM}$ given in Eq. (10).

The BR with a photon energy cut at the standard value $E_0 = 1.6$ GeV can be computed from Eq. (1) using the approximate relation

$$P(1.6 \text{ GeV}) \simeq 0.1247 - 0.0572 \left[m_c(m_c) - 1.224 \text{ GeV}\right] + 0.0084 (m_{b}^{1S} - 4.68 \text{ GeV})$$  \hspace{1cm} (12)$$

that can be extracted from [7]. Since this result is obtained with $m_b$ in the $1S$ scheme and $m_c$ in the $\overline{\text{MS}}$ scheme, we perform an explicit change of scheme to the kinetic one, differentiating the NLO contribution to $P(E_0)$ wrt $m_{c,b}$. Inserting Eqs. (10,12) into Eqs. (3) and (1) and using BR$_{cl\nu} = 0.1064$ from [2], one gets $BR_\gamma = 3.28 \times 10^{-4}$ which we choose as our central value. Alternatively, we can convert our kinetic scheme mass inputs to the $1S$ and $\overline{\text{MS}}$ schemes and employ the results in Eq. (12), facing however an extra theoretical error due to the conversion. As both masses enter $P(E_0)$ at $O(\alpha_s)$, one can use one-loop formulas with $\alpha_s(2.5 \text{ GeV}) \approx 0.27$, namely use $m_{b}^{1S} = 4.69 \text{ GeV}$ and $m_c(m_c) = 1.23 \text{ GeV}$ in Eq. (12). This leads to $BR_\gamma = 3.30 \times 10^{-4}$. If the conversion of the kinetic masses to the $1S$ and $\overline{\text{MS}}$ schemes is made via two-loop expressions, $BR_\gamma$ can be as low as $3.21 \times 10^{-4}$, depending on the scale chosen for the $m_c$ conversion. The spread of these values is an estimate of the perturbative higher orders consistent with that given in [3]. All renormalization scales involved in the calculation of $P(E_0)$ are kept to the default value of [7]. The parametric error due to the OPE parameters and BR$_{cl\nu}$ given by the fit follows from the correlation matrix in [2] and amounts to only $1.8\%$, to which we add in quadrature the uncertainty from the CKM factor, $\alpha_s$, $m_t$, and the theory error in Eq. (10), obtaining $3.7\%$. Summarizing, we have

$$BR_\gamma(1.6 \text{ GeV}) = 3.28 \times 10^{-4} \left[1 \pm 0.037 \pm 0.03 \pm 0.03 \pm 0.05\right],$$  \hspace{1cm} (13)$$

where the four errors are due to i) the normalization and parametric uncertainties; ii) the perturbative uncertainty in $P(1.6 \text{ GeV})$; iii) the $m_c$ interpolation of [7]; iv) unknown non-perturbative contributions beyond the OPE. Apart from i), we have employed the same errors as in [3][7] and our central value is about $4\%$ higher than there. The main reason for the shift is the new determination of $C$. Refs. [3][7] employed the most precise value of $C$ available at that time [10]. Our total $3.7\%$ normalization and parametric error is a little larger than the corresponding uncertainty given in [3][7], but the total error is still dominated by the $\pm 5\%$ non-perturbative uncertainty.

The slow convergence of the perturbative series in Eqs. (6,10) is slightly disturbing. It is partly accidental, as both numerator and denominator have $\sim 2\%$ second order perturbative corrections, but with different sign. On the other hand, the kinetic mass definition is not particularly appropriate for a quark with mass $m \sim \mu$. An alternative is to use a hybrid scheme where $m_b$ and the non-perturbative matrix elements are defined in the kinetic scheme, while the charm mass is defined in the $\overline{\text{MS}}$ scheme. This yields a better apparent convergence in both $C$ and $\Gamma[B \rightarrow X_c e\bar{\nu}]$ if one employs $m_c(m_c)$, namely the $\overline{\text{MS}}$ scale of the charm.
mass is set equal to the mass itself. As for the numerical value of $m_c(m_c)$, to simplify the
comparison with Eqs. (6-10) we choose $m_c(m_c) = 1.267$ GeV that is obtained from our input
in (6) through the two-loop perturbative relation with $\alpha_s(m_b)$. The results are
\begin{align}
C &= 0.542 - 1.9 B_{ws}(m_b/2) \\
    &= 0.574 - 0.005\alpha_s - 0.001\alpha_s^2 - 0.004\mu_c^2 - 0.022\rho_b - 0.001\rho_{LS} - 1.9 B_{ws}(m_b/2) \\
F &= 1.886 \\
    &= 1.741 + 0.019\alpha_s + 0.003\alpha_s^2 + 0.047\mu_c^2 + 0.059\rho_b + 0.018\rho_{LS}
\end{align}
(14)
which agree with but converge better than Eqs. (6-10). Since direct fits to $m_c(m_c)$ and
kinetic scheme OPE parameters will soon become available [27], we list in the Appendix
approximate formulae for this option too. The $\mu$-dependence in Eqs. (14) is less than 0.4%.
One can also calculate $C$ and $F$ using the $1S$ definition of the bottom mass and the
$\overline{MS}$ scheme for $m_c$: using again inputs that correspond to those employed in Eqs.(6,10), i.e.
$m_b^{1S} = 4.74$ GeV and $m_c(m_c) = 1.267$ GeV, we find $C = 0.545$ and $F = 1.870$. Our results
are quite stable for changes of scheme.

Finally, let us consider alternatives to the normalization of Eq.(1). Since much of the $m_c$
sensitivity is related to the normalization to $BR_{c\ell\nu}$, one could hope to reduce it by avoiding
$BR_{c\ell\nu}$. This amounts to reexpressing the first ratio in Eq.(1) in terms of the semileptonic
charmless width:

$$
\frac{BR_{c\ell\nu}}{C} = \frac{V_{cb}}{V_{ub}} \left| \frac{\Gamma[B \rightarrow X_u e\bar{\nu}]}{\tau_B} \right|
$$
(15)

where $\tau_B = 1.585(7)$ ps is the lifetime of the $B^0/B^+\pm$ admixture and $\Gamma[B \rightarrow X_u e\bar{\nu}]$ is given
in Eq.(5). As $|V_{ub}|$ drops out, the input parameters necessary to compute the rhs of Eq.(15)
are $|V_{cb}|$, $m_b$, and the OPE parameters. All these parameters, as well as $m_c$ and $BR_{c\ell\nu}$,
can be extracted from a global fit as in [2]. Clearly in that case the radiative BR will be the
same and, because of the various correlations, will have the same error if computed
using the rhs or the lhs of Eq.(15). The rhs is very sensitive to $m_b$ due to the $m_b^5$ factor in
Eq.(5) but insensitive to $m_c$ (which however still enters $P(E_0)$). Moreover, the theoretical
uncertainty in the extraction of $|V_{cb}|$ from $BR_{c\ell\nu}$ affects only the rhs of (15) .\footnote{See eq. (7) in the second paper of Ref. [20] for an estimate.} Neglecting all
the correlations between the OPE parameters and $BR_{c\ell\nu}$, and using the rhs of Eq.(15), we
get a $\sim 5\%$ parametric error instead of $1.8\%$. Even the very small errors on $m_{c,b}$ found in
[26] would lead to a $3.3\%$ uncertainty in this case. This demonstrates the advantage of the
normalization to the semileptonic BR and of a global fit to the OPE parameters, as already
stressed in [7]. In the future, independent and precise determinations of $m_b$ and $m_c$
could be used as additional constraints in the fit.

4 Summary

We have calculated the normalization factor for radiative inclusive $B$ decays in the kinetic
scheme and discussed its uncertainty and dependence on the input parameters, taking into

account the correlations that arise from the fit to the moments of semileptonic and radiative distributions. Using the latest global fit in the kinetic scheme [2] we obtain $C = 0.546^{+0.023}_{-0.033}$. We estimate that adopting our normalization in the first NNLO analysis of [3, 7] would lead to a 4% higher $BR_{\gamma}$ for $E_{\gamma} > 1.6$ GeV, $(3.28 \pm 0.25) \times 10^{-4}$. Since the normalization factor is more sensitive to the value of the $c$ and $b$ quark masses than $|V_{cb}|$, its determination is presently more volatile, but progress will come from improved measurements of the moments, a better understanding of the theoretical uncertainties involved in the fits, and complementary constraints on $m_{c,b}$. First steps to reduce the theory error in the normalization would be to employ the $\overline{\text{MS}}$ charm mass and to compute higher order perturbative corrections, starting with the implementation of [18].

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**Appendix**

The factors $C$ and $F$ defined in Eqs. (2,10) are well approximated by the following formulae, whose coefficients are given in Table 1.

$$C = g(r) \left[ c_1 + c_b \delta_b + c_c \delta_c + c_G \mu_G^2 + c_D \rho_D^3 + c_{LS} \rho_{LS}^3 + c_{\alpha_s} \delta_{\alpha_s} - \frac{32\pi^2}{m_b^3} B(m_b/2) \right]$$

$$F = [g(r)]^{-1} \left[ c_1 + c_b \delta_b + c_c \delta_c + c_D \rho_D^3 + \left( c_G \mu_G^2 + c_{LS} \rho_{LS}^3 \right) /m_c^2 + c_{\alpha_s} \delta_{\alpha_s} \right]$$

Here all parameters are in the kinetic scheme with $\mu = 1$ GeV, except for the charm mass that is either in the kinetic scheme or in the $\overline{\text{MS}}$ scheme, $m_c(m_c)$. Moreover, $\delta_b = m_b - 4.6$ GeV, $\delta_c = m_c - 1.15$ GeV, and $\delta_{\alpha_s} = \alpha_s - 0.22$. Notice that in the range $0.22 \leq \sqrt{r} \leq 0.29$, $g(r) \approx 1.1928 - 2.2443 m_c/m_b$ within 0.03%. The approximate formulae have a precision better than 0.3% in the ranges $4.5 < m_b < 4.7$ GeV, $1 < m_c < 1.3$ GeV, $0.2 < \alpha_s(m_b) < 0.24$. 

| $m_c$ scheme | $c_1$     | $c_b$    | $c_c$   | $c_G$   | $c_D$   | $c_{LS}$ | $c_{\alpha_s}$ |
|-------------|----------|----------|---------|---------|---------|-----------|----------------|
| C           | 0.9185   | 0.035    | -0.001  | -0.021  | -0.186  | 0.005     | -0.53         |
| F           | 1.085    | -0.045   | 0.007   | 0.148   | 0.169   | -0.091    | 0.60          |
| C           | 1.001    | 0.029    | -0.102  | -0.021  | -0.186  | 0.005     | 0.032         |
| F           | 1.001    | -0.035   | 0.112   | 0.148   | 0.169   | -0.091    | -0.01         |

Table 1: Coefficients of the approximate formulae for $C$ and $F$ for different $m_c$ schemes.
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