THREE BALLS THEOREM FOR EIGENFUNCTIONS OF DIRAC OPERATOR IN CLIFFORD ANALYSIS

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Abstract. In this paper we establish the three balls theorem for functions \( u \) satisfying \( Du = \lambda u \) in Clifford analysis, where \( D \) is the Dirac operator. As an application, we generalize Hadamard’s three circles theorem to monogenic function in \( \mathbb{R}^{n+1} \).

Key words: Monogenic Functions, Three Circles Theorem, Frequency Function, Monotonicity

1. Introduction

The famous Hadamard three circles theorem for holomorphic functions is stated as follows:

**Theorem 1.1** (Hadamard,1896). Let \( f \) be a holomorphic function on the annulus \( \{ z \in \mathbb{C} : r_1 \leq |z| \leq r_3 \} \) with \( 0 < r_1 < r_2 < r_3 < \infty \). Denote by \( M(r) \) the maximum of \( |f(z)| \) on the circle \( |z| = r \). Then

\[
\{ M(r_2) \}^{\log \frac{r_3}{r_1}} \leq \{ M(r_1) \}^{\log \frac{r_3}{r_2}} \{ M(r_3) \}^{\log \frac{r_2}{r_1}}.
\]

The theorem was originally announced by J. Hadamard in [9]. Its standard proof could be found in e.g. [6, 14]. It could be shown that the three circles theorem also holds for harmonic functions, as well as subharmonic functions, in \( n \)-dimensional Euclidean spaces (see e.g. [14]). The majority of proofs of Hadamard’s three circles theorem is usually based on the commutativity in the algebra of holomorphic functions.

In recent years, Hadamard’s three circles theorem has been generalized to solutions of various partial differential equations by the frequency function method, which was first introduced by F. Almgren in [2]. The frequency function method leads to an \( L^2 \)-version of Hadamard’s three circles (or balls) theorem. It has been shown that the frequency function method and Hadamard’s three circles theorem have powerful applications in the study of unique continuation for elliptic equations (see e.g. [7]). Hence, Hadamard’s three circles theorem has many interesting applications in partial differential equations and differential geometry (see e.g. [7] [8] [17] [16] [11] [5] [12] and the references therein).

In this paper we consider a generalization of three circles theorem about Dirac operator in Clifford analysis. Let \( \{ e_1, ..., e_n \} \) be an orthonormal basis of \( \mathbb{R}^n \) associated with the rule

\[
e_i e_j + e_j e_i = -2 \delta_{ij}, \quad 1 \leq i, j \leq n,
\]

where \( \delta_{ij} \) is the Kronecker symbol. Denote by \( \mathbb{R}^{(n)} \) the Clifford algebra generated by \( \{ e_1, ..., e_n \} \) over \( \mathbb{R} \), whose elements are of the form \( x = \sum_A x_A e_A \), where \( A = \{ 1 \leq j_1 < j_2 < \cdots < j_l \leq n \} \) runs over all ordered subsets of \( \{ 1, ..., n \} \), \( x_A \in \mathbb{R} \) and \( x_0 = x_0 \) with

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the identity element $e_0 = e_0 = 1$ (see §2 for a brief introduction to Clifford algebra). The Dirac operator is defined as

$$D = \partial_0 + \sum_{j=1}^n \partial_j e_j = \frac{\partial}{\partial x_0} + \sum_A \frac{\partial}{\partial x_j} e_j.$$ 

Suppose that $u(x) = \sum_A u_A(x)e_A$, and $\lambda \in \mathbb{R}$. Then, $u$ is said to be an eigenfunction of $D$ if

$$Du = \lambda u,$$ 

(1.1)

when $\lambda \neq 0$. When $\lambda = 0$, (1.1) means that $u$ is left-monogenic, which generalizes the concept of holomorphic function to $\mathbb{R}^{n+1}$. Therefore, it is natural and significant to study whether there holds a three circles theorem for monogenic functions taking values in $\mathbb{R}^n$. Moreover, we will establish the three balls theorem for (1.1) with general $\lambda$ in this paper. Due to the noncommutativity of Clifford algebra, the standard argument of proving Hadamard’s three circles theorem is invalid in our case (even for $\lambda = 0$). In [13] Abul-Ez, Constales, Morais and Zayed proved a three balls theorem for the so-called special monogenic functions, where the used technique only works for their special case. The three lines theorem in Clifford analysis is given by Peetre and Sjölin in [13]. We also note that in the complex case the three circles theorem can be deduced by the three lines theorem, where the property of complex exponential function plays a role (see [15] pages 386-387). However, this argument is also invalid in the Clifford algebra setting. Therefore, in this paper we will adapt the frequency function method to obtain the $L^2$-version of three balls theorem for $u$ satisfying (1.1). Consequently, we can deduce the $L^\infty$-version of three balls theorem by the Moser iteration method, which is a powerful tool in the theory of elliptic partial differential equations (see e.g. [10]). In particular, the used frequency function was proposed by Zhu in [17] (see also [12]), which is different from the one given in [7].

Our main results are stated as follows.

**Theorem 1.2.** Suppose $u = \sum_A u_A e_A$ satisfies $Du = \lambda u$, and $\alpha \geq 2$. Then, there exist $0 < r_1 < r_2 < 2r_2 < r_3 < \infty$ such that when $\lambda \neq 0$

$$\int_{B_{r_2}} |u(x)|^2 dx \leq C_3 \left( \int_{B_{r_1}} |u(x)|^2 dx \right)^{\frac{C_1}{r_1^2 + C_2}} \left( \int_{B_{r_3}} |u(x)|^2 dx \right)^{\frac{C_2}{r_1^2 + C_2}},$$

(1.2)

where $C_1 = \frac{1}{\log \frac{2r_2}{r_1}}$, $C_2 = \frac{1}{\log \frac{2r_3}{r_2}}$ and

$$C_3 = \frac{2aC_1 + 2bC_2}{3.62(r_1^2 + C_2)} \left( \frac{2(a^2 + b^2 + 5a + 5b + 3)}{(a+2)(b+2)} \right) \left( \frac{2(a^2 + b^2 - 1)}{a+b+2} \right)$$

with $a = \frac{2|\lambda|^2 + |\lambda|}{3}$, $b = \frac{5(a+1)|\lambda| - \frac{2(a+1)(a+2)}{3}}{2(a+1)(a+2)} - \frac{5(a+1)}{18} + \frac{2|\lambda| + 1}{3|\lambda|}$;

when $\lambda = 0$

$$\int_{B_{r_2}} |u(x)|^2 dx \leq C_4 \left( \int_{B_{r_1}} |u(x)|^2 dx \right)^{\frac{C_1}{r_1^2 + C_2}} \left( \int_{B_{r_3}} |u(x)|^2 dx \right)^{\frac{C_2}{r_1^2 + C_2}},$$

(1.3)

where $C_1, C_2$ are given as above, and $C_4 = \frac{2aC_1 + 2bC_2}{3.62(r_1^2 + C_2)}$.
For the case $\lambda = 0$, (1.3) is the $L^2$-version of the three balls inequality for monogenic functions. Then, by the subharmonic inequality we can deduce the following result.

**Corollary 1.1.** Suppose $u = \sum_{A} u_{A}e_{A}$ satisfies $Du = 0$, and $\alpha \geq 2$. Then, there exist $0 < r_{1} < r_{2} < 2r_{2} < r_{3} < \infty$ such that

$$
\|u\|_{L^\infty(B_{r_2})} \leq 3^{\frac{2\alpha}{3\alpha}} C_{3}'(r_{3} - 2r_{2})^{-\frac{2\alpha}{3\alpha}} r_{3}^{\frac{2\alpha}{3\alpha}} \|u\|_{L^\infty(B_{r_1})},
$$

where $C_{1}' = \frac{1}{\log \frac{2(r_{3}+r_{2})}{r_{3}-r_{2}}}$, $C_{2}' = \frac{1}{\log \frac{2(r_{3}+r_{2})}{r_{3}-r_{2}}}$, and $C_{3}' = 3^{\frac{2\alpha}{3\alpha}} C_{1}' C_{2}' C_{3}'$.

**Remark 1.1.** Corollary 1.1 is a generalization of Hadamard’s three circles theorem in the setting of Clifford algebra. On one hand, Corollary 1.1 holds for general monogenic functions, which is more general than that proposed in [3]. On the other hand, since the constant on the right-hand side of (1.4) is bigger than one, in this sense our result is weaker than the classical Hadamard’s three circles theorem. It would be interesting whether there holds a complete generalization of the classical Hadamard’s three circles theorem for general monogenic functions.

It seems that the approach in this paper could not lead to this complete generalization. Nevertheless, the frequency function method can be used to obtain a $L^\infty$-version of the three balls theorem for $\lambda \neq 0$, which is stated as follows.

**Corollary 1.2.** Suppose $u = \sum_{A} u_{A}e_{A}$ satisfies $Du = \lambda u$ with $\lambda \neq 0$, and $\alpha \geq 2$. Then, there exist $0 < r_{1} < r_{2} < 2r_{2} < r_{3} < 1$ such that

$$
\|u\|_{L^\infty(B_{r_2})} \leq M'C_{3}'(r_{3} - 2r_{2})^{-\frac{2\alpha}{3\alpha}} r_{3}^{\frac{2\alpha}{3\alpha}} \|u\|_{L^\infty(B_{r_1})},
$$

where $M'$ is a positive constant, $C_{1}' = \frac{1}{\log \frac{2(r_{3}+r_{2})}{r_{3}-r_{2}}}$, $C_{2}' = \frac{1}{\log \frac{2(r_{3}+r_{2})}{r_{3}-r_{2}}}$, and $C_{3}' = 3^{\frac{2\alpha}{3\alpha}} C_{1}' C_{2}' C_{3}'$.

The paper is organized as follows. In §2 some basic notations and properties of Clifford algebra, and the frequency function are introduced. In §3 the monotonicity of frequency function $N(r)$ is proved. In §4 the main results are proved.

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## 2. Preliminaries

First we introduce some basic notations and properties of Clifford algebras. Let $e_{1}, \ldots, e_{n}$ be basic elements satisfying

$$
e_{j}e_{k} + e_{k}e_{j} = -2\delta_{jk}, \quad j, k = 1, \ldots, n,
$$

where $\delta_{jk}$ is the Kronecker delta function. Let $\mathbb{R}^{n} = \{x = x_{1}e_{1} + \cdots + x_{n}e_{n}; x_{j} \in \mathbb{R}, 1 \leq j \leq n\}$ be identical with the usual Euclidean space $\mathbb{R}^{n}$, and $\mathbb{R}^{n+1} = \{x = x_{0} + \underline{x}; x_{0} \in \mathbb{R}, \underline{x} \in \mathbb{R}^{n}\}$. 

The real Clifford algebra \( \mathbb{R}^{(n)} \) generated by \( \{e_1, ..., e_n\} \) is the associative algebra generated by \( \{e_1, ..., e_n\} \) over the real field \( \mathbb{R} \). The elements of \( \mathbb{R}^{(n)} \) are of the form \( x = \sum_A x_A e_A \), where \( A = \{1 \leq j_1 < j_2 < \cdots < j_l \leq n\} \) runs over all ordered subsets of \( \{1, ..., n\} \), \( x_A \in \mathbb{R} \) with \( x_0 = x_0 \), and \( e_A = e_{j_1} e_{j_2} \cdots e_{j_l} \) with the identity element \( e_0 = e_0 = 1 \). \( \text{Sc}(x) := x_0 \) and \( \text{NSc}(x) := x - \text{Sc}(x) \) are respectively called the scalar part and the non-scalar part of \( x \). We denote the conjugate of \( x \in \mathbb{R}^{(n)} \) by \( \overline{x} = \sum_A x_A \overline{e}_A \), where \( \overline{e}_0 = e_0 \) and \( \overline{e}_j = -e_j \) for \( j \neq 0 \). The norm of \( x \in \mathbb{R}^{(n)} \) is defined as \( |x| := (\text{Sc} x)^{\frac{1}{2}} = (\sum_A |x_A|^2)^{\frac{1}{2}} \). \( x = x_0 + \overline{x} \in \mathbb{R}^{n+1} \) is called a para-vector, and the conjugate of a para-vector \( x \) is \( \overline{x} = x_0 - \overline{x} \). If \( x \) is a para-vector then \( x^{-1} = \overline{x}/|x|^2 \). For more information about Clifford algebras, we refer to [4].

The Dirac operator is defined as
\[
D = \partial_0 + \overline{\partial} = \partial_0 + \sum_{j=1}^n \partial_j e_j = \frac{\partial}{\partial x_0} + \sum_{j=1}^n \frac{\partial}{\partial x_j} e_j.
\]

In Clifford analysis, the Dirac operator plays an important role since it gives rise to the development of monogenic function theory (see e.g. [4]).

In the following we introduce the frequency function used in this paper, which is proposed in [17]. Let \( u = \sum_A u_A e_A = u_0 + \text{NSc}(u) \) satisfy
\[
Du = \lambda u,
\]
where \( \lambda \in \mathbb{R} \). Note that
\[
Du = (\partial_0 + \overline{\partial})(u_0 + \text{NSc}(u)) = \partial_0 u_0 + \overline{\partial} u_0 + \overline{\partial} \text{NSc}(u) + \partial \text{NSc}(u) = \lambda u = \lambda u_0 + \lambda \text{NSc}(u),
\]
which implies that \(-\partial_0 u_0 - \partial \text{NSc}(u) = \partial_0 u_0 + \partial_0 \text{NSc}(u) - \lambda u_0 - \lambda \text{NSc}(u) \). Next we consider
\[
\Delta u = DDu = \lambda (\partial_0 + \overline{\partial})(u_0 + \text{NSc}(u)) = \lambda (\partial_0 u_0 + \partial_0 \text{NSc}(u) - \overline{\partial} u_0 - \overline{\partial} \text{NSc}(u)) = \lambda (2\partial_0 u_0 + 2\partial_0 \text{NSc}(u) - \lambda u_0 - \lambda \text{NSc}(u)),
\]
which is equivalent to
\[
\Delta u_A = \lambda (2\partial_0 u_A - \lambda u_A), \quad \text{for all possible index } A.
\]
For \( \alpha \geq 2 \), define
\[
H(r) = \int_{B_r} |u|^2 (r^2 - |x|^2)^{\alpha} dx = \sum_A \int_{B_r} u_A^2 (r^2 - |x|^2)^{\alpha} dx = \sum_A \int u_A^2 (r^2 - |x|^2)^{\alpha},
\]
where \( B_r \) is the ball centered at the origin with radius \( r \). For the sake of simplicity, we let \( n_1 = n + 1 \), and write \( \int_{B_r} \) as \( \int \) when there is no confusion. By taking the derivative for \( H(r) \) with respect to \( r \), we have
\[
H'(r) = \frac{2\alpha + n_1}{r} H(r) + \frac{1}{r(\alpha + 1)} I(r),
\]
where

\[ I(r) = \int |\nabla u|^2(r^2 - |x|^2)^{\alpha+1} dx + \sum_A \int u_A \Delta u_A (r^2 - |x|^2)^{\alpha+1} dx. \tag{2.8} \]

The frequency function \( N(r) \) is defined as

\[ N(r) = \frac{I(r)}{H(r)}. \]

3. Monotonicity of frequency function

In the following lemma we prove a monotonic result for \( N(r) \), which plays a crucial role in proving the main results.

**Lemma 3.1.** Suppose \( u = \sum_A u_A e_A \) satisfies \( Du = \lambda u \). Then, for \( \lambda \neq 0 \), \( e^{6|\lambda| r}(N(r) + \rho(r)) \) is nondecreasing, where \( p(r) = ar^2 + br + c \) with \( a = \frac{2|\lambda|^2 + |\lambda|}{3}, b = \frac{5(\alpha + 1)|\lambda| - \frac{2|\lambda| + 1}{9} \alpha}{18} \) and \( c = \frac{2(\alpha + 1)(1 + \alpha)}{3} - \frac{5(\alpha + 1)}{18} + \frac{2|\lambda| + 15\alpha}{4|\lambda|} \). For \( \lambda = 0 \), we have that \( N(r) \) is nondecreasing.

**Proof.** Firstly, by taking the derivative for \( I(r) \) in (2.8) with respect to \( r \), we have

\[ I'(r) = 2(\alpha + 1) \int r |\nabla u|^2(r^2 - |x|^2)^{\alpha} dx + 2(\alpha + 1) \sum_A \int r u_A \Delta u_A (r^2 - |x|^2)^{\alpha} dx \]

\[ = \frac{2(\alpha + 1)}{r} \int |\nabla u|^2(r^2 - |x|^2)^{\alpha+1} - \frac{1}{r} \int |\nabla u|^2(x, \nabla(r^2 - |x|^2)^{\alpha+1}) dx \]

\[ + 2(\alpha + 1) \sum_A \int r u_A \Delta u_A (r^2 - |x|^2)^{\alpha} dx. \tag{3.9} \]

Using integration by parts, we have

\[ I'(r) = \frac{2(\alpha + 1) + n_1}{r} \int |\nabla u|^2(r^2 - |x|^2)^{\alpha+1} + \frac{1}{r} \int \langle \nabla |\nabla u|^2, x \rangle (r^2 - |x|^2)^{\alpha+1} dx \]

\[ + 2(\alpha + 1) \sum_A \int r u_A \Delta u_A (r^2 - |x|^2)^{\alpha} dx. \]

Note that

\[ \int \langle \nabla |\nabla u_A|^2, x \rangle (r^2 - |x|^2)^{\alpha+1} dx \]

\[ = 2 \sum_{i,j} \int x_i \partial_j u_A \partial_i \partial_j u_A (r^2 - |x|^2)^{\alpha+1} dx \]

\[ = -2 \int |\nabla u_A|^2(r^2 - |x|^2)^{\alpha+1} dx - 2 \int \Delta u_A \langle \nabla u_A, x \rangle (r^2 - |x|^2)^{\alpha+1} \]

\[ + 4(\alpha + 1) \int \langle \nabla u_A, x \rangle (r^2 - |x|^2)^{\alpha} dx. \]

Consequently,

\[ I'(r) = \frac{2\alpha + n_1}{r} \int |\nabla u|^2(r^2 - |x|^2)^{\alpha+1} - \frac{2}{r} \sum_A \int \Delta u_A \langle \nabla u_A, x \rangle (r^2 - |x|^2)^{\alpha+1} \]

\[ + \frac{4(\alpha + 1)}{r} \sum_A \int \langle \nabla u_A, x \rangle (r^2 - |x|^2)^{\alpha} dx + 2(\alpha + 1) \sum_A \int r u_A \Delta u_A (r^2 - |x|^2)^{\alpha} dx. \]
Using (2.8), we can rewrite \( I'(r) \) as

\[
I'(r) = \frac{2\alpha + n_1}{r} I(r) + \frac{4(\alpha + 1)}{r} \sum_A \int (\nabla u_A, x)^2 (r^2 - |x|^2)^\alpha \, dx
\]

\[
- \frac{2}{r} \sum_A \int \triangle u_A (\nabla u_A, x)(r^2 - |x|^2)^{\alpha+1} \, dx - \frac{n_1 - 2}{r} \sum_A \int u_A \triangle u_A (r^2 - |x|^2)^{\alpha+1} \, dx
\]

\[
+ \frac{2(\alpha + 1)}{r} \sum_A \int |x|^2 u_A \triangle u_A (r^2 - |x|^2)^\alpha \, dx.
\]

We also note that

\[
- \frac{2}{r} \int \triangle u_A (\nabla u_A, x)(r^2 - |x|^2)^{\alpha+1} \, dx
\]

\[
= - \frac{2\lambda}{r} \int (2\partial_0 u_A - \lambda u_A) (\nabla u_A, x)(r^2 - |x|^2)^{\alpha+1} \, dx
\]

\[
= - \frac{4\lambda}{r} \int \partial_0 u_A (\nabla u_A, x)(r^2 - |x|^2)^{\alpha+1} \, dx + \frac{2\lambda^2}{r} \int u_A (\nabla u_A, x)(r^2 - |x|^2)^{\alpha+1} \, dx
\]

\[
\geq - \frac{2 |\lambda|}{r} \int |\partial_0 u_A|^2 |x|(r^2 - |x|^2)^{\alpha+1} \, dx + \frac{2 |\lambda|}{r} \int |\nabla u_A|^2 |x|(r^2 - |x|^2)^{\alpha+1} \, dx
\]

\[
- \frac{|\lambda|^2}{r} \int u_A^2 |x|(r^2 - |x|^2)^{\alpha+1} \, dx - \frac{|\lambda|^2}{r} \int |\nabla u_A|^2 |x|(r^2 - |x|^2)^{\alpha+1} \, dx
\]

\[
\geq - 4 |\lambda| \int |\nabla u_A|^2 (r^2 - |x|^2)^{\alpha+1} \, dx - |\lambda|^2 \int u_A^2 (r^2 - |x|^2)^{\alpha+1} \, dx
\]

\[
- |\lambda|^2 \int u_A^2 (r^2 - |x|^2)^{\alpha+1} \, dx.
\]

and

\[
- \frac{n_1 - 2}{r} \int u_A \triangle u_A (r^2 - |x|^2)^{\alpha+1} \, dx
\]

\[
= - \frac{(n_1 - 2)\lambda}{r} \int u_A (2\partial_0 u_A - \lambda u_A)(r^2 - |x|^2)^{\alpha+1} \, dx
\]

\[
= - \frac{2(n_1 - 2)\lambda}{r} \int u_A \partial_0 u_A (r^2 - |x|^2)^{\alpha+1} \, dx + \frac{(n_1 - 2)\lambda^2}{r} \int u_A^2 (r^2 - |x|^2)^{\alpha+1} \, dx
\]

\[
= - \frac{2(n_1 - 2)(\alpha + 1)\lambda}{r} \int u_A^2 x_0 (r^2 - |x|^2)^{\alpha+1} \, dx + \frac{(n_1 - 2)\lambda^2}{r} \int u_A^2 (r^2 - |x|^2)^{\alpha+1} \, dx
\]

\[
\geq -2(n_1 - 2)(\alpha + 1) |\lambda| \int u_A^2 (r^2 - |x|^2)^{\alpha} \, dx,
\]
and

\[
\frac{2(\alpha + 1)}{r} \int |x|^2 u_A \Delta u_A (r^2 - |x|^2)^{\alpha} dx
\]

\[
= \frac{2(\alpha + 1)}{r} \int |x|^2 u_A (2\partial_0 u_A - \lambda u_A) (r^2 - |x|^2)^{\alpha} dx
\]

\[
= \frac{2(\alpha + 1)}{r} \int |x|^2 \partial_0 u_A (r^2 - |x|^2)^{\alpha} dx - \frac{2(\alpha + 1)}{r} \int |x|^2 u_A^2 (r^2 - |x|^2)^{\alpha} dx
\]

\[
= - \frac{4(\alpha + 1)}{r} \int x_0 u_A^2 (r^2 - |x|^2)^{\alpha} dx + \frac{4\alpha(\alpha + 1)}{r} \int |x|^2 x_0 u_A^2 (r^2 - |x|^2)^{\alpha - 1} dx
\]

\[
- \frac{2(\alpha + 1)}{r} \int |x|^2 u_A^2 (r^2 - |x|^2)^{\alpha} dx.
\]

By simplifying the coefficients, we obtain

\[
\frac{2(\alpha + 1)}{r} \int |x|^2 u_A \Delta u_A (r^2 - |x|^2)^{\alpha} dx
\]

\[
\geq - 4(\alpha + 1)|\lambda| \int u_A^2 (r^2 - |x|^2)^{\alpha} dx - \frac{2\alpha(\alpha + 1)|\lambda|}{r} \int |x|^2 |x_0|^2 u_A^2 (r^2 - |x|^2)^{\alpha - 1} dx
\]

\[
- \frac{2\alpha(\alpha + 1)|\lambda|}{r} \int |x|^2 |x_0|^2 u_A^2 (r^2 - |x|^2)^{\alpha - 1} dx - \frac{2(\alpha + 1)|\lambda|^2}{r} \int |x|^2 u_A^2 (r^2 - |x|^2)^{\alpha} dx
\]

\[
\geq - 4(\alpha + 1)|\lambda| \int u_A^2 (r^2 - |x|^2)^{\alpha} dx - 4\alpha(\alpha + 1)|\lambda| r^2 \int u_A^2 (r^2 - |x|^2)^{\alpha - 1} dx
\]

\[
- 2(\alpha + 1)|\lambda|^2 r \int u_A^2 (r^2 - |x|^2)^{\alpha} dx.
\]

(3.12)

Combining (3.10), (3.11) and (3.12), we have

\[
- \frac{2}{r} \sum_A \int \Delta u_A \nabla u_A (r^2 - |x|^2)^{\alpha + 1} - \frac{n_1 - 2}{r} \sum_A \int u_A \Delta u_A (r^2 - |x|^2)^{\alpha + 1} dx
\]

\[
+ \frac{2(\alpha + 1)}{r} \sum_A \int |x|^2 u_A \Delta u_A (r^2 - |x|^2)^{\alpha} dx
\]

\[
\geq - 4|\lambda| \sum_A \int |\nabla u_A|^2 (r^2 - |x|^2)^{\alpha + 1} dx - |\lambda|^2 \sum_A \int u_A^2 (r^2 - |x|^2)^{\alpha + 1} dx
\]

\[
- |\lambda|^2 \sum_A \int u_A^2 (r^2 - |x|^2)^{\alpha + 1} dx - 2(n_1 - 2)(\alpha + 1)|\lambda| \sum_A \int u_A^2 (r^2 - |x|^2)^{\alpha} dx
\]

\[
- 4(\alpha + 1)|\lambda| \sum_A \int u_A^2 (r^2 - |x|^2)^{\alpha} dx - 4\alpha(\alpha + 1)|\lambda| r^2 \sum_A \int u_A^2 (r^2 - |x|^2)^{\alpha - 1} dx
\]

\[
- 2(\alpha + 1)|\lambda|^2 r \int u_A^2 (r^2 - |x|^2)^{\alpha} dx.
\]
\[\begin{align*}
&= -4|\lambda| \sum_A \int |\nabla u_A|^2 (r^2 - |x|^2)^{\alpha+1} dx - 4|\lambda| \sum_A \int u_A \Delta u_A (r^2 - |x|^2)^{\alpha+1} dx \\
&\quad + 4|\lambda| \sum_A \int u_A \Delta u_A (r^2 - |x|^2)^{\alpha+1} dx - |\lambda|^2 \sum_A \int u_A^2 (r^2 - |x|^2)^{\alpha+1} dx \\
&\quad - |\lambda|^2 \sum_A \int u_A^2 (r^2 - |x|^2)^{\alpha+1} dx - 2(n_1 - 2)(\alpha + 1)|\lambda| \sum_A \int u_A^2 (r^2 - |x|^2)^{\alpha} dx \\
&\quad - 4(\alpha + 1)|\lambda| \sum_A \int u_A^2 (r^2 - |x|^2)^{\alpha-1} dx - 4\alpha(\alpha + 1)|\lambda| r^2 \sum_A \int u_A^2 (r^2 - |x|^2)^{\alpha} dx \\
&\quad - 2(\alpha + 1)|\lambda|^2 r \int u_A^2 (r^2 - |x|^2)^{\alpha} dx,
\end{align*}\]

which gives that

\[\begin{align*}
\text{(3.13)}
= & -4|\lambda|I(r) + 8(\alpha + 1)|\lambda| \lambda \sum_A \int u_A^2 x_0 (r^2 - |x|^2)^{\alpha} dx - 4|\lambda|^3 \sum_A \int u_A^2 (r^2 - |x|^2)^{\alpha+1} \\
&\quad - |\lambda|^2 \sum_A \int u_A^2 (r^2 - |x|^2)^{\alpha+1} dx - |\lambda|^2 \sum_A \int u_A^2 (r^2 - |x|^2)^{\alpha+1} dx \\
&\quad - 2(n_1 - 2)(\alpha + 1)|\lambda| \sum_A \int u_A^2 (r^2 - |x|^2)^{\alpha} dx - 4(\alpha + 1)|\lambda| \sum_A \int u_A^2 (r^2 - |x|^2)^{\alpha} dx \\
&\quad - 4\alpha(\alpha + 1)|\lambda| r^2 \sum_A \int u_A^2 (r^2 - |x|^2)^{\alpha-1} dx - 2(\alpha + 1)|\lambda|^2 r \int u_A^2 (r^2 - |x|^2)^{\alpha} dx \\
&\geq -4|\lambda|I(r) - 8(\alpha + 1)|\lambda|^2 r H(r) - 4|\lambda|^3 r^2 H(r) - 2|\lambda|^2 r^2 H(r) - 2(n_1 - 2)(\alpha + 1)|\lambda| H(r) \\
&\quad - 4(\alpha + 1)|\lambda| H(r) - 2(\alpha + 1)|\lambda| r H'(r) - 2(\alpha + 1)|\lambda|^2 r H(r) \\
&= -4|\lambda|I(r) - (10(\alpha + 1)|\lambda|^2 r + 4|\lambda|^3 r^2 + 2|\lambda|^2 r^2 + 2n_1(\alpha + 1)|\lambda|) H(r) \\
&\quad - 2(\alpha + 1)|\lambda| r \left( \frac{2\alpha + n_1}{r} H(r) + \frac{1}{r(\alpha + 1)} I(r) \right) \\
&= -6|\lambda|I(r) - (10(\alpha + 1)|\lambda|^2 r + 4|\lambda|^3 r^2 + 2|\lambda|^2 r^2 + 2n_1(\alpha + 1)|\lambda| + 2(\alpha + 1)(2\alpha + n_1)|\lambda|) H(r).
\end{align*}\]

Therefore, we have

\[\begin{align*}
I'(r)
&\geq \frac{2\alpha + n_1}{r} I(r) + \frac{4(\alpha + 1)}{r} \sum_A \int \langle \nabla u_A, x \rangle^2 (r^2 - |x|^2)^{\alpha} dx - 6\lambda I(r) \\
&\quad - (10(\alpha + 1)|\lambda|^2 r + 4|\lambda|^3 r^2 + 2|\lambda|^2 r^2 + 2n_1(\alpha + 1)|\lambda| + 2(\alpha + 1)(2\alpha + n_1)|\lambda|) H(r).
\end{align*}\]
Consequently,

\[ N'(r) = \frac{1}{H^2} (I'(r)H(r) - I(r)H'(r)) \]

\[ \geq \frac{1}{H^2} \frac{4(\alpha + 1)}{r} \left( \int H \sum_A \langle \nabla u_A, x \rangle^2 (r^2 - |x|^2) \, dx - \frac{1}{4(\alpha + 1)^2} I^2 \right) \]

\[ - 6|\lambda|N(r) - (10(\alpha + 1)|\lambda|^2 r + 4|\lambda|^3 r^2 + 2|\lambda|^2 r^2 + 4(\alpha + 1)(\alpha + n_1)|\lambda|). \]

Let \( K = \sum_A K_A = \sum_A \int \langle \nabla u_A, x \rangle^2 (r^2 - |x|^2) \, dx \). By Cauchy-Schwarz's inequality, we have

\[ \frac{1}{4(\alpha + 1)^2} I^2 = \left( \int \langle x, \nabla u_A \rangle u_A (r^2 - |x|^2) \right)^2 \]

\[ \leq \int \langle x, \nabla u_A \rangle^2 (r^2 - |x|^2) \, dx \int u_A^2 (r^2 - |x|^2) \, dx \]

\[ = K_A H_A. \]

Consequently,

\[ \frac{1}{4(\alpha + 1)^2} I^2 = \frac{1}{4(\alpha + 1)^2} \left( \sum_A I_A^2 + 2 \sum_{|A|<|B|} I_A I_B \right) \]

\[ \leq \frac{1}{4(\alpha + 1)^2} \left( \sum_A I_A^2 + 2 \sum_{|A|<|B|} |I_A||I_B| \right) \]

\[ \leq \sum_A K_A H_A + 2 \sum_{|A|<|B|} \sqrt{K_A H_A K_B H_B} \]

\[ \leq \sum_A K_A H_A + \sum_{|A|<|B|} (K_A H_B + K_B H_A) \]

\[ = (\sum_A K_A)(\sum_B H_B) \]

\[ = KH. \]

Therefore,

\[ N'(r) \geq -6|\lambda|N(r) - (10(\alpha + 1)|\lambda|^2 r + 4|\lambda|^3 r^2 + 2|\lambda|^2 r^2 + 4(\alpha + 1)(\alpha + n_1)|\lambda|), \]

which implies that \( e^{6|\lambda| r} (N(r) + p(r)) \) is nondecreasing for \( \lambda \neq 0 \), where \( p(r) = ar^2 + br + c \) with \( a = \frac{2|\lambda|^2 + |\lambda|}{3} \), \( b = \frac{5(\alpha + 1)}{9}|\lambda| - \frac{2|\lambda| + 1}{9} \) and \( c = \frac{2(\alpha + 1)(\alpha + n_1)}{3} - \frac{5(\alpha + 1)}{18} + \frac{2|\lambda| + 1}{54|\lambda|} \).

When \( \lambda = 0 \), it becomes \( Du = 0 \), which means that \( u \) is monogenic. The estimate of \( I'(r) \) is given as follows, i.e.,

\[ I'(r) \geq \frac{2\alpha + n_1}{r} I(r) + \frac{4(\alpha + 1)}{r} \sum_A \int \langle \nabla u_A, x \rangle^2 (r^2 - |x|^2) \, dx, \]
and consequently,
\[
N'(r) = \frac{1}{H^2}(I'(r)H(r) - I(r)H'(r)) \\
\geq \frac{1}{H^2}\frac{4(\alpha + 1)}{r} \left( H \sum_A \left( \langle \nabla u_A, x \rangle^2 (r^2 - |x|^2)^\alpha dx - \frac{1}{4(\alpha + 1)^2} I^2 \right) \right) \\
\geq 0.
\]
Thus \(N(r)\) is nondecreasing. \(\square\)

4. Three Balls Inequality

In this section we will prove the main results.

**Proof of Theorem 1.2.** Define
\[
h(r) = \int_{B_r} |u(x)|^2 dx,
\]
where \(B_r\) is the ball centering at the origin with radius \(r\). There hold
\[
H(r) \leq r^{2\alpha} h(r)
\]
and
\[
H(2r) \geq 3^{\alpha} r^{2\alpha} h(r),
\]
where \((4.14)\) follows from the fact
\[
h(r) = \int_{B_r} |u|^2 dx \leq \int_{B_r} |u|^2 (1 + \frac{r^2 - |x|^2}{4r^2 - r^2})^\alpha dx \leq \int_{B_{2r}} |u|^2 (1 + \frac{r^2 - |x|^2}{4r^2 - r^2})^\alpha dx = \frac{H(2r)}{3^{\alpha} r^{2\alpha}}.
\]
By \((2.17)\), we have that
\[
\frac{H'(r)}{H(r)} = \frac{2\alpha + n_1}{r} + \frac{N(r)}{r(\alpha + 1)}.
\]
Integrating the above equality from \(r_1\) to \(2r_2\), we have
\[
\log \frac{H(2r_2)}{H(r_1)} = \int_{r_1}^{2r_2} \frac{2\alpha + n_1}{r} dr + \int_{r_1}^{2r_2} \frac{N(r)}{r(\alpha + 1)} dr \\
= (2\alpha + n_1) \log \frac{2r_2}{r_1} + \int_{r_1}^{2r_2} \frac{e^{6\lambda_r} (N(r) + p(r))}{r(\alpha + 1)e^{6\lambda_r}} dr - \int_{r_1}^{2r_2} \frac{a_r^2 + br + c}{r(\alpha + 1)} dr.
\]
Thus
\[
\log \frac{H(2r_2)}{H(r_1)} \leq (2\alpha + n_1) \log \frac{2r_2}{r_1} + \frac{e^{12\lambda_r} (N(2r_2) + p(2r_2))}{(\alpha + 1)e^{6\lambda_r_1}} \log \frac{2r_2}{r_1} - \frac{c}{\alpha + 1} \log \frac{2r_2}{r_1} \\
- \frac{1}{\alpha + 1} \left( \frac{a}{2}(2r_2)^2 - \frac{a}{2}(r_2^2) \right) + (2r_2 - r_1)\right).
\]
Similarly, integrating from \(2r_2\) to \(r_3\), we have
\[
\log \frac{H(r_3)}{H(2r_2)} = \int_{2r_2}^{r_3} \frac{2\alpha + n_1}{r} dr + \int_{2r_2}^{r_3} \frac{N(r)}{r(\alpha + 1)} dr \\
= (2\alpha + n_1) \log \frac{r_3}{2r_2} + \int_{2r_2}^{r_3} \frac{e^{6\lambda_r} (N(r) + p(r))}{r(\alpha + 1)e^{6\lambda_r}} dr - \int_{2r_2}^{r_3} \frac{a_r^2 + br + c}{r(\alpha + 1)} dr.
\]
Then, 
\[
\log \frac{H(r_3)}{H(2r_2)} \geq (2\alpha + n_1) \log \frac{r_3}{2r_2} + \frac{e^{12\lambda r_2}(N(2r_2) + p(2r_2))}{(\alpha + 1)e^{6\lambda r_3}} \log \frac{r_3}{2r_2} - \frac{c}{\alpha + 1} \log \frac{r_3}{2r_2} 
\]
and consequently,
\[
\log \frac{H(2r_2)}{H(r_1)} + \frac{c}{\alpha + 1} - (2\alpha + n_1) + \frac{1}{\log \frac{r_3}{2r_2}(\alpha + 1)} \left( \frac{a}{2} (r_2^3 - (2r_2)^2) + b(r_2 - 2r_1) \right)
\]
Hence,
\[
\leq \frac{\log H(2r_2)}{\log \frac{r_3}{2r_2}} + \frac{c}{\alpha + 1} - (2\alpha + n_1) + \frac{1}{\log \frac{r_3}{2r_2}(\alpha + 1)} \left( \frac{a}{2} (r_3^2 - (2r_2)^2) + b(r_3 - 2r_2) \right)
\]
and consequently,
\[
\log \left( \frac{H(2r_2)}{H(r_1)} \right) \leq \log \left( \frac{H(r_3)}{H(2r_2)} \right) + \log \frac{r_3}{2r_2} + \frac{1}{\log \frac{r_3}{2r_2}(\alpha + 1)} \left( \frac{a}{2} (r_3^2 - (2r_2)^2) + b(r_3 - 2r_2) \right)
\]
which is equivalent to
\[
(H(2r_2))^{\log \frac{r_3}{2r_2}} + \log \frac{r_3}{2r_2} + \frac{1}{\log \frac{r_3}{2r_2}(\alpha + 1)} \left( \frac{a}{2} (r_3^2 - (2r_2)^2) + b(r_3 - 2r_2) \right)
\]
Combining the above inequality with (1.14) and (1.15), we have that
\[
3^\alpha(C_1 + C_2) \frac{2\alpha C_1 + C_2}{r_2} (h(r_2))^C_1 + C_2 \leq e^{\frac{\Phi((r_3^2 - (2r_2)^2) + h(r_3 - 2r_2))}{(\alpha + 1)\log \frac{r_3}{2r_2}}} (H(r_1))^{\frac{c_1}{C_1 + C_2}} (H(r_3))^{\frac{c_2}{C_1 + C_2}}.
\]
This implies that
\[
\int_{B_{r_2}} |u|^2 dx \leq C_3 \left( \int_{B_{r_1}} |u|^2 dx \right)^{\frac{c_1}{C_1 + C_2}} \left( \int_{B_{r_3}} |u|^2 dx \right)^{\frac{c_2}{C_1 + C_2}},
\]
where \( C_1 = \frac{1}{\log \frac{r_3}{2r_2}} \), \( C_2 = \frac{1}{\log \frac{r_3}{2r_2}} \) and
\[
C_3 = \left( \frac{2\alpha C_1 + C_2}{r_1} + \frac{2\alpha C_2}{r_2} \right) \left( \frac{3^\alpha C_1 + C_2}{r_2} \right) e^{\frac{\Phi((r_3^2 - (2r_2)^2) + h(r_3 - 2r_2))}{(\alpha + 1)\log \frac{r_3}{2r_2}}} (H(r_1))^{\frac{c_1}{C_1 + C_2}} (H(r_3))^{\frac{c_2}{C_1 + C_2}}.
\]
In particular, when \( \lambda = 0 \), we have that
\[
H(2r_2) \log \frac{r_3}{2r_2} + \log \frac{r_3}{2r_2} \leq H(r_1) \log \frac{r_3}{2r_2} + \log \frac{r_3}{2r_2}.
\]
Consequently, the three balls inequality becomes
\[
\int_{B_{r_2}} |u|^2 dx \leq C_4 \left( \int_{B_{r_1}} |u|^2 dx \right)^{\frac{c_1}{C_1 + C_2}} \left( \int_{B_{r_3}} |u|^2 dx \right)^{\frac{c_2}{C_1 + C_2}},
\]
where \( C_1 = \frac{1}{\log \frac{2r_2}{r_1}} \), \( C_2 = \frac{1}{\log \frac{2r_2}{r_1}} \) and \( C_4 = \frac{2a_1 c_1 c_2 2a_1 c_1 c_2}{3^{a_1 + 1}} \). \( \square \)

**Proof of Corollary 1.1.** When \( \lambda = 0 \), we can easily deduce the three balls inequality in the \( L^\infty \)-norm from Theorem 1.2. In fact, by the subharmonicity of \( |u|^2 \) we have that, for \( x \in B_{r_1} \),

\[
|u(x)|^2 \leq \frac{\Gamma\left(\frac{n_2}{2} + 1\right)}{\pi^{n_2}} \int_{B_{r_1}(x)} |u(y)|^2 dy \leq \frac{\Gamma\left(\frac{n_3}{2} + 1\right)}{\pi^{n_3}} \int_{B_{r_3}} |u(y)|^2 dy,
\]

where \( \Gamma(\cdot) \) is the Gamma function. This implies that

\[
\|u\|_{L^\infty(B_{r_0})} \leq \left(\frac{\Gamma\left(\frac{n_2}{2} + 1\right)}{\pi^{n_2}}\right)^{\frac{1}{2}} r_0^{-\frac{n_2}{2}} \|u\|_{L^2(B_{r_0})},
\]

or equivalently,

\[
\|u\|_{L^\infty(B_{r_0})} \leq \left(\frac{\Gamma\left(\frac{n_3}{2} + 1\right)}{\pi^{n_3}}\right)^{\frac{1}{2}} (\delta - \rho)^{-\frac{n_3}{2}} \|u\|_{L^2(B_{r_3})}.
\]

Since \( r_3 > 2r_2 > r_2 > r_1 \), we have

\[
\|u\|_{L^\infty(B_{r_3})} \leq \left(\frac{\Gamma\left(\frac{n_2}{2} + 1\right)}{\pi^{n_2}}\right)^{\frac{1}{2}} (r_3 + r_2)^{-\frac{n_2}{2}} \|u\|_{L^2(B_{r_3 + r_2})}.
\]

Note that \( r_3 > \frac{2(r_3 + r_2)}{3} > r_2 > r_1 \). Then, we have

\[
\|u\|_{L^2(B_{r_3 + r_2})} \leq C'_4 \|u\|_{L^2(B_{r_1})} \|u\|_{L^2(B_{r_3})},
\]

where \( C'_4 = \frac{1}{\log \frac{2(r_3 + r_2)}{3r_1}} \). \( C'_2 = \frac{1}{\log \frac{2r_2}{r_1}} \) and \( C'_4 = \frac{3^{a_1 c_1 c_2 2a_1 c_1 c_2}}{4^a (r_3 + r_2)^{2a_1 + 1}} \).

Hence,

\[
\|u\|_{L^\infty(B_{r_3})} \leq 3^{\frac{n_2}{2}} \left(\frac{\Gamma\left(\frac{n_2}{2} + 1\right)}{\pi^{n_2}}\right)^{\frac{1}{2}} C'_4 (r_3 - 2r_2)^{-\frac{n_2}{2}} \left(\frac{C'_1 + c_2}{C'_1 + c_2}ight)^{\frac{1}{2}} \|u\|_{L^2(B_{r_1})} \|u\|_{L^2(B_{r_3})}.
\]

To prove Corollary 1.2 we need the following lemma.

**Lemma 4.1.** Suppose that \( Du = \lambda u \) with \( \lambda \neq 0 \). Then, for \( 0 < r < R < 1 \), there holds

\[
\|u\|_{L^\infty(B_r)} \leq \tilde{M}(R - r)^{-\frac{n_2}{2}} \|u\|_{L^2(B_R)},
\]

where \( \tilde{M} \) is a positive constant.
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Proof. Firstly, for all possible index $A$, we note that

$$\triangle u_A = 2\lambda \delta u_A - \lambda^2 u_A.$$ 

By using the standard Moser iteration method for such $u_A$ (see e.g. [10]), there exists a positive constant $\widetilde{M}$ such that

$$\|u_A\|_{L^\infty(B_r)} \leq \widetilde{M}(R-r)^{-\frac{\nu}{2}}\|u_A\|_{L^2(B_r)}.$$ 

Therefore,

$$\|u\|_{L^\infty(B_r)} \leq \widetilde{M}(R-r)^{-\frac{\nu}{2}}\|u\|_{L^2(B_r)}.$$ 

□

Proof of Corollary [1.2]. By Lemma 4.1 we have that, for $0 < r_1 < r_2 < 2r_2 < r_3 < 1$,

$$\|u\|_{L^\infty(B_{r_2})} \leq \widetilde{M} \left(\frac{r_3 + r_2}{3} - r_2\right)^{-\frac{\nu}{2}}\|u\|_{L^2(B_{r_3})},$$

where $\widetilde{M}$ is a positive constant given in Lemma 4.1. Note that $1 > r_3 > \frac{2(r_1+r_2)}{3} > \frac{r_1+r_2}{3} > r_2 > r_1$. Then, applying Theorem 1.2 we have that

$$\|u\|_{L^2(B_{r_3})} \leq C_3^{\prime}\|u\|_{L^2(B_{r_1})} \|u\|_{L^2(B_{r_2})},$$

where $C_1^{\prime} = \frac{1}{\log \frac{2(r_1+r_2)}{r_2}}$, $C_2^{\prime} = \frac{1}{\log \frac{2(r_2+2r_2)}{r_2}}$, and

$$C_3^{\prime} = \frac{2n_1}{4((r_3+r_2))^{2n_1}} e^{\frac{C_1^{\prime} n_1}{(n_1+c_2)} - \frac{C_2^{\prime} n_1}{(n_1+c_2)}}.$$ 

Therefore,

$$\|u\|_{L^\infty(B_{r_3})} \leq 3^{\frac{n_1}{2}} \widetilde{M} C_3^{\prime}(r_3 - 2r_2)^{-\frac{n_1}{2}} \|u\|_{L^2(B_{r_1})} \|u\|_{L^2(B_{r_2})} \leq 3^{\frac{n_1}{2}} \widetilde{M} C_3^{\prime}(r_3 - 2r_2)^{-\frac{n_1}{2}} \|u\|_{L^\infty(B_{r_1})} \|u\|_{L^\infty(B_{r_2})} \leq M^{\prime} C_3^{\prime}(r_3 - 2r_2)^{-\frac{n_1}{2}} \|u\|_{L^\infty(B_{r_1})} \|u\|_{L^\infty(B_{r_2})}.$$ 

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