Determining a Riemannian Metric from Minimal Areas

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Abstract

We prove that if \((M, g)\) is a topological 3-ball with a \(C^4\)-smooth Riemannian metric \(g\), and mean-convex boundary \(\partial M\) then knowledge of least areas circumscribed by simple closed curves \(\gamma \subset \partial M\) uniquely determines the metric \(g\), under some additional geometric assumptions. These are that \(g\) is either a) \(C^3\)-close to Euclidean or b) satisfies much weaker geometric conditions which hold when the manifold is to a sufficient degree either thin, or straight.

In fact, the least area data that we require is for a much more restricted class of curves \(\gamma \subset \partial M\). We also prove a corresponding local result: assuming only that \((M, g)\) has strictly mean convex boundary at a point \(p \in \partial M\), we prove that knowledge of the least areas circumscribed by any simple closed curve \(\gamma\) in a neighbourhood \(U \subset \partial M\) of \(p\) uniquely determines the metric near \(p\). Additionally, we sketch the proof of a global result with no thin/straight or curvature condition, but assuming the metric admits minimal foliations “from all directions”.

The proofs rely on finding the metric along a continuous sweep-out of \(M\) by area-minimizing surfaces; they bring together ideas from the 2D-Calderón inverse problem, minimal surface theory, and the careful analysis of a system of pseudo-differential equations.

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1 Introduction

The classical boundary rigidity problem in differential geometry asks whether knowledge of the distance between any two points on the boundary of a Riemannian manifold is sufficient to identify the metric up to isometries that fix the boundary. Manifolds for which this is the case are called boundary rigid. One motivation for the problem comes from seismology, if one seeks to determine the interior structure of the Earth from measurements of travel times of seismic waves. There are counterexamples to boundary rigidity: intuitively, if the manifold has a region of large positive curvature in the interior, length-minimizing geodesics between boundary points need not pass through this region. A way to rule out such counterexamples is to assume the manifold is simple, meaning that any two points can be joined by a unique minimizing geodesic and that the boundary is strictly convex. Michel \cite{17} conjectured that simple manifolds are boundary rigid. Special cases have been proved by Michel \cite{17}, Gromov \cite{10}, Croke \cite{8}, Lassas, Sharafutdinov, and Uhlmann \cite{13}, Stefanov and Uhlmann \cite{21}, and Burago and Ivanov \cite{5,6}. In two dimensions, the conjecture was settled by Pestov and Uhlmann \cite{19}. Moving away from the simplicity assumption, important recent work of Stefanov, Uhlmann and Vasy solved a local version of the rigidity problem in a neighbourhood of any strictly convex point of the boundary, and obtained a corresponding global rigidity result for manifolds that admit a foliation satisfying a certain convexity condition. (see \cite{22} and earlier references given there).

In this paper, we consider the following higher dimensional version of the boundary rigidity problem, where in lieu of lengths of geodesics the data consists of areas of minimal surfaces.

**Question.** Given any simple closed curve $\gamma$ on the boundary of a Riemannian 3-manifold $(M,g)$, suppose the area of any least-area surface $Y_\gamma \subset M$ circumscribed by the curve is known. Does this information determine the metric $g$?

Under certain geometric conditions, we show that the answer is yes. In some cases, we only require the area data for a much smaller subclass of curves on the boundary.

Theories posited by the AdS/CFT correspondence also provide strong physics motivation to consider the problem of using knowledge of the areas of certain submanifolds to determine the metric. Loosely speaking, the AdS/CFT correspondence states that the dynamics of $(n+2)$-dimensional supergravity theories modelled on an Anti-de Sitter (AdS) space are equivalent to the quantum physics modelled by $(n+1)$-dimensional conformal field theories (CFT) on the boundary of the AdS space (see \cite{14}). This equivalence is often referred to as holographic duality. Analogous to the problem of boundary rigidity, one of the main goals of the AdS/CFT correspondence is to use conformal field theory information on the boundary to determine the metric on the AdS spacetime which encodes information about the corresponding gravity dynamics.
Towards this goal, Maldecena [15] has proposed that given a curve on the boundary of the AdS spacetime, the renormalized area of the minimal surface bounding the curve contains information about the expectation value of the Wilson loop associated to the curve. More recently, Ryu and Takayanagi [20] have conjectured that given a region $A$ on the boundary of an $(n+2)$-dimensional AdS spacetime, the entanglement entropy of $A$ is equivalent to the renormalized area of the area-minimizing surface $Y_\gamma$ with boundary $\gamma = \partial A$ (see Figure 2). The AdS/CFT correspondence is often studied in Riemannian signature, where one must consider a Riemannian, asymptotically hyperbolic manifold $(M, g)$, for which one knows information on the boundary and the renormalized area for minimal surfaces bounded by closed loops on the boundary. Hence, one is led to the question: Given a collection of simple closed curves on the boundary-at-infinity of $(M, g)$, does knowledge of the renormalized area of the area-minimizing surfaces bounded by these curves allow us to recover the metric? And even locally: Does knowledge of renormalized areas of loops lying in a given domain $V \subset M$ allow one to reconstruct the bulk metric, and in how large a (bulk) region is the reconstruction possible? Answers in the affirmative may provide new methods to describe the relationship between gravity theories and conformal field theories.

Here we do not consider boundaries at infinity, but rather the finite-boundary problem. We will also be working with foliations of the boundary $\partial M$ by a family of simple closed loops, and we will need the area-minimizers bounded by these loops to yield a foliation of $M$.\footnote{Loosely speaking, a \textit{foliation} of an $n$-manifold $M$ is a continuous, 1-parameter family of $k$-submanifolds, $k < n$, which sweep out $M$.} We use knowledge of the areas of the surfaces\footnote{These surfaces will always be homeomorphic to a disc.} of least area bounded by such curves to find the metric. We present two such results below, as well as a local determination result. All these rely on knowledge of areas for one foliation of our manifold by area-minimizers, and the area data for all area-minimizing perturbations of this foliation by (nearby) area-minimizing foliations. It is not clear what the \textit{minimal} knowledge of areas required to determine the

Figure 1: Simple closed curve $\gamma$ on $\partial M$. 
interior metric is. We suspect that our assumptions on what areas are known are essentially optimal. We propose a conjecture (only partially addressed here) which stipulates that they are sufficient:

**Conjecture** (Boundary rigidity for least area data). Let \((M, g)\) be a Riemannian 3-ball which admits a foliation by properly embedded, area-minimizing surfaces. Suppose that for this foliation and any nearby perturbation, we know the areas of the leaves.

Then this information determines \(g\) up to boundary-fixing isometries.

We prove this conjecture for particular classes of Riemannian manifolds. To describe in detail these classes, we make the following definitions.

**Definition 1.1.** Let \((M, g)\) be a Riemannian 3-manifold. For \(k \in \mathbb{N}\), we say the metric \(g\) is \(\epsilon\)-**\(C^k\)-close** to the Euclidean metric \(g_\mathbb{E}\) on \(\mathbb{R}^3\) if there exists a global coordinate chart \((x^\alpha)\), \(\alpha = 1, 2, 3\), on \(M\) for which we have

\[
\|g_{\alpha\beta}(x) - (g_\mathbb{E})_{\alpha\beta}(x)\|_{C^k(M)} < \epsilon
\]

for all \(\alpha, \beta = 1, 2, 3\).

From this point onwards when we say a metric is \(C^k\)-close to Euclidean we mean that it is \(\epsilon\)-\(C^k\)-close, for some sufficiently small \(\epsilon > 0\).

**Definition 1.2.** We say a Riemannian manifold with boundary \((M, g)\) is \((K, \epsilon_0, \delta_0)\)-**thin** if for some parameters \(K, \epsilon_0, \delta_0 > 0\) the following holds:
1. there exist global coordinates \((y^\alpha), \alpha = 1, 2, 3\) on \((M, g)\) such that the surfaces \(Y(t) := \{y^3 = \text{constant} = t\}\) are properly embedded and area-minimizing discs, \(\{Y(t) : t \in (-1, 1)\} = M\), and the coordinates \((y^1, y^2, y^3)\) are regular Riemannian coordinates in the sense that the Beltrami coefficient \(\mu(y^1, y^2, y^3)\) satisfies \(|\partial^k_{y^i} \mu| \leq 10/(\epsilon_0)^2\) for \(i = 1, 2\) (see (2.4));

2. \(\frac{1}{2} < ||g^{33}||_{L^\infty(M)} < 2\), and for \(i, j \in \{1, 2\}\), and \(k, l \in \{0, 1, 2\}\), with \(0 < k + l \leq 2\), \(||\partial^k_{y^i} \partial^l_{y^j} g^{33}||_{L^\infty(M)} \leq K \epsilon_0^{-(k+l)+1}\);

3. for \(i, j, k \in \{1, 2\}\), and \(\beta, \gamma \in \{0, 1, 2, 3, 4\}\) with \(\beta + \gamma \leq 4\), \(||\partial^\beta_{y^i} \partial^\gamma_{y^j} g^{33}||_{L^\infty(M)} \leq K \epsilon_0^{-(\beta+\gamma)}\);

4. for each \(t\), \(\text{Area}[Y(t)]\) is bounded above by \(4\pi \epsilon_0^2\);

5. The Riemann curvature tensor \(Rm_g\) of the metric \(g\) and the second fundamental forms \(A(t)\) of each \(Y(t)\) satisfy the bounds\(^3\)

\[
||A(t)|| \leq \sqrt{\delta_0} \cdot \epsilon_0^{-1}, ||\nabla A(t)|| \leq \sqrt{\delta_0} \cdot \epsilon_0^{-2}, ||Rm_g|| \leq \delta_0 \cdot \epsilon_0^{-2}, ||\nabla Rm_g|| \leq \delta_0 \cdot \epsilon_0^{-3}
\]

\[\text{(a) The case when } \epsilon_0 << 1. \text{ The thin manifold is allowed to be “bent”}\].

\[\text{(b) The case when } \epsilon_0 > 1. \text{ This fatter manifold is required to be “straight”}\].

Figure 3: Depiction of \((K, \epsilon_0, \delta_0)\)-thin manifolds.

Let us describe how the parameters \(K, \epsilon_0, \delta_0\) correspond to different bounds on the geometry of the minimal foliated \((M, g)\):

The parameter \(\epsilon_0\) in requirement 4 corresponds to a weak notion of “girth” of the manifold. Note that in conjunction with bounds on the geodesic curvature on the boundary \(\gamma(t) = \partial Y(t)\) and the bound in 5, requirement 4 implies bounds on the diameter of each leaf \(Y(t)\). Thus a small \(\epsilon_0\) implies the manifold has thin girth.

Requirement 5 bounds the curvature of the ambient manifold, as well as the extrinsic geometry of the minimal leaves. These bounds can be large when \(\delta_0\) is fixed and \(\epsilon_0\) is taken

\[^3\nabla\) is the connection of the metric \(g\) on \(Y\) and \(\nabla\) is the connection of \(g\).\]
small enough. Requirement 1 (weakly) bounds the intrinsic geometry of the leaves, while the estimates in requirements 2 and 3 bound the “straightness” of the metric $g$. In particular, the functions $\partial_i g^{33}$, $\partial_i g^{3k}$ vanish when the vector fields $\partial_3$ are hypersurface-orthogonal affine geodesic vector fields, i.e. the metric is expressed in Fermi-type coordinates

$$g = (dx^3)^2 + \sum_{i,j=1}^{2} g_{ij}(x^1, x^2, x^3) dx^i dx^j.$$  

Thus, the bounds in 2 and 3 measure the departure from this straight picture; smallness of $K$ should be thought of as a nearly straight minimal foliated manifold.

With Definitions 1.1 and 1.2 in mind, we work with Riemannian manifolds belonging to either of the next two classes:

**Definition 1.3.** Let $(M, g)$ be a $C^4$-smooth, Riemannian manifold which has the properties

1. $M$ homeomorphic to a 3-dimensional ball in $\mathbb{R}^3$,  
2. $(M, g)$ has $C^4$-smooth, mean convex boundary $\partial M$,  
3. there is a foliation of $\partial M$ by simple closed curves, $\{\gamma(t) : t \in (-1, 1)\} = \partial M$, which induces a foliation $\{Y(t) : t \in (-1, 1)\} = M$ by properly embedded, area minimizing surfaces and satisfies a regularity assumption: The geodesic curvatures of the curves $\gamma(t) \subset \partial M$ obey

$$||\kappa|| \leq 2 \text{Area}[Y(t)]^{-1/2}, ||\nabla \kappa|| \leq 2 \text{Area}[Y(t)]^{-1}.$$  

If additionally the metric $g$ is $\epsilon_0 - C^3$-close to Euclidean for some small $\epsilon_0 > 0$, we say $(M, g)$ is of Class 1. If $(M, g)$ is $(K, \epsilon_0, \delta_0)$-thin, for some sufficiently small $\delta_0 > 0$, and $K, \epsilon_0 > 0$ are such that $K \epsilon_0$ is sufficiently small we say $(M, g)$ is of Class 2.

In these settings, we prove:

**Theorem 1.4.** Let $(M, g)$ be a manifold of Class 1 or Class 2 above, and $g|_{\partial M}$ be given. Let $\{\gamma(t) : t \in (-1, 1)\} = \partial M$ and $\{Y(t) : t \in (-1, 1)\} = M$ be as in Definition 1.3. Suppose that for each curve $\gamma(t)$ and any nearby perturbation $\gamma(s, t) \subset \partial M$, we know the area of the properly embedded surface $Y(s, t)$ which solves the least-area problem for $\gamma(s, t)$.

Then the knowledge of these areas uniquely determines the metric $g$ (up to isometries which fix the boundary).

We note that for our first result the curvature is required to be small. For the second result, the curvature can be very large, but this will be compensated by the thinness condition. (The requirement on the geodesic curvature is a technical condition imposed to ensure that a certain extension of our surfaces to infinity can be performed while preserving the bounds we have).

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4Note: as $t \to \pm 1$, the loops $\gamma(t)$ and the surfaces $Y(t)$ collapse to points on $\partial M$.  
5We also note that the area bound on the leaves $Y(t) = \{x^3 = t\}$ together with the geodesic curvature bounds, the bounds on the ambient curvature, and bounds on $||A||_g$ imply diameter bounds on each $Y(t)$, of the form: $\text{diam}(Y(t)) \leq 4 \sqrt{\text{Area}[Y(t)]}$.  
6We do not keep track of the constants but careful tracking of the proof could yield universal bounds.
Our $C^3$-close to Euclidean or $(K, \epsilon_0, \delta_0)$-thin assumption\footnote{We remark, but do not prove, that under such assumptions we expect that the existence of area minimizing foliations for $(M, g)$ can be derived by a perturbation argument.} is technically needed for very different reasons than those in [13 21 5], as we use it to obtain unique solvability of the system of pseudodifferential equations for the metric components which we will describe later in this paper. We impose that $\partial M$ is mean convex to ensure the solvability of the Plateau problem for a given simple curve on the boundary of $M$ (see [16]). However, this is not necessary; it is easy to see that one could have foliations on $C^3$-close to Euclidean or $(K, \epsilon_0, \delta_0)$-thin manifolds without mean convex boundary. Our hypothesis on foliations of the manifold by area-minimizing surfaces has some similarity with that of [22]. However, the proofs are completely different. The existence of foliations of $(M, g)$ by properly embedded, area-minimizing surfaces ensures that there are no unreachable regions trapped between minimal surfaces bounded by the same curve, thus avoiding obstructions to uniqueness analogous to the ones for the boundary distance problem.

As a consequence of Theorem 1.4, we have the local result:

**Theorem 1.5.** Let $(M, g)$ be a $C^4$-smooth, Riemannian manifold with boundary $\partial M$. Assume that $\partial M$ is both $C^4$-smooth and mean convex at $p \in \partial M$. Let $U \subset \partial M$ be a neighbourhood of $p$, and let $\{\gamma(t) : t \in (-1, 1)\}$ be a foliation of $U$ by simple, closed curves which satisfy the estimates in Definition 1.3. Suppose that $g|_U$ is known, and for each $\gamma(t)$ and any nearby perturbation $\gamma(s, t) \subset U$, we know the area of the properly embedded surface $Y(s, t)$ which solves the least-area problem for $\gamma(s, t)$. Then, there exists a neighbourhood $V \subset M$ of $p$ such that, up to isometries which fix the boundary, $g$ is uniquely determined on $V$.

The methods that are developed in this paper can be used to prove further results. We highlight one such further result below and provide the sketch of the proof (for brevity’s sake) at the end of Section 4.

The result below again proves the uniqueness of the metric given knowledge of minimal areas. We do not impose any thinness or small curvature restrictions. Instead we assume that the manifold admits a foliation by strictly mean-convex spheres that shrink down to a point, and minimal foliations from “all directions” and that the areas of all these minimal surfaces and their perturbations are known. More precisely, we define the following class of manifolds.

**Definition 1.6.** A Riemannian manifold $(M, g)$ admits minimal foliations from all directions if

- $(M, g)$ has mean convex boundary $\partial M$,
- there exists a foliation $\{N(r) : r \in [0, 1]\} = M$ by (strictly) mean convex surfaces $N(r) \subset M$ with $N(0) := \partial M$, and $\lim_{r \to 1} N(r) \to q$, where $q \in M$ is a point and the mean curvature of $N(r)$ tends to $+\infty$ as $r \to 1$.
- for every $r \in [0, 1)$ and $p \in N(r)$, there is a foliation $\{\gamma(t, p) : t \in (-1, 1)\} = \partial M$ by simple closed curves, which induces a fill-in $\{Y(t, p) : t \in (-1, 1)\} = M$ by properly embedded, area minimizing surfaces, with the property that $Y(t_0)$ is tangent to $N(r)$ at $p$ for some $t_0 \in (-1, 1)$.
Figure 4: Illustration of the property of “admits minimal foliations from all directions”.

**Theorem 1.7.** Let \((M, g)\) be a \(C^4\)-smooth Riemannian manifold which admits minimal foliations from all directions, and let \(g|\partial M\) be given. Suppose that for all \(p \in M\) and for each \(\gamma(t, p)\) as in Definition 1.6, and any nearby perturbation \(\gamma(s, t, p) \subset \partial M\), we know the area of the properly embedded surface \(Y(s, t, p)\) which solves the least-area problem for \(\gamma(s, t, p)\).

Then the knowledge of these areas uniquely determines the metric \(g\) (up to isometries which fix the boundary).

**Remark 1.8.** We note that our theorems (and extensions of these results that one can obtain by adapting these methods) use foliations of the unknown manifold \((M, g)\) by a family of area-minimizing discs. Not all foliations of \(\partial M\) by closed loops \(\gamma(t)\) yield fill-ins by area minimizing surfaces \(Y(t)\) which foliate \(M\). For example, if \((M, g)\) admits a closed minimal surface in the interior, then the set of area-minimizers \(Y(t)\) that fill-in the loops \(\gamma(t)\) must contain “gaps”.

We note that recently Haslhofer and Ketover \[11\] derived that a positive-mean-curvature foliation as required in Definition 1.2 does exist, under the first assumption, and assuming the non-existence of a closed minimal surface inside \((M, g)\). Now, regarding the third requirement of Definition 1.2, it is interesting to note that the area data function \(\text{Area}[Y(t)]\), seen as a function of \(t\), detects whether the area-minimizing fill-ins \(\{Y(t) : t \in (-1, 1)\}\) yield a foliation or display gaps.

**Proposition 1.9.** Let \((M, g)\) be a Riemannian manifold with mean convex boundary \(\partial M\). Let \(\{\gamma(t) : t \in (-1, 1)\} = \partial M\) be a foliation of \(\partial M\). Let \(A(t)\) denote the area of area-minimizing surfaces \(Y(t)\) that bound \(\gamma(t)\).

Then there is a foliation of \(M\) by fill-ins of area minimizing surfaces that bound \(\gamma(t)\) if and only if \(A(t)\) is a \(C^1\)-smooth function of \(t\).

We provide a proof of Proposition 1.9 in the Appendix.

\[\text{This function is well-defined even when there are multiple area-minimizing surfaces bounding } \gamma(t).\]
1.1 Outline of the main ideas.

We briefly describe below our approach to the proof of Theorem 1.4. The main strategy to reconstructing the metric is to use the background foliation of \((M, g)\) by area-minimizing, topological discs,

\[ M = \{ Y(t) : t \in (-1, 1) \}, \]

to progressively solve for the metric by moving “upwards” along the foliation.

The metric is solved for in a normalized gauge: the metric \(g\) is expressed in a new coordinate system \(\{x_1, x_2, x_3\}\) such that \(x_3\) is constant on each leaf \(Y(t)\) of the foliation, and \(x_1, x_2\) restricted to each leaf \(Y(t)\) are isothermal coordinates for \(g|_{Y(t)}\). The non-uniqueness of isothermal coordinates is fixed by moving to an auxiliary extension \((\tilde{M}, \tilde{g})\) of our manifold and imposing a normalization at infinity.

In such a coordinate system, there are four non-zero independent entries of the metric:

\[
    g = \begin{pmatrix}
        e^{2\phi} & 0 & g_{31} \\
        0 & e^{2\phi} & g_{32} \\
        g_{13} & g_{23} & g_{33}
    \end{pmatrix}.
\]

The proof then proceeds in reconstructing the components of \(g^{-1}\).

Our main strategy is to not use the area data directly, but rather the second variation of areas, which is also known (since it corresponds to the second variation of a known functional). We show in Section 2 (Proposition 1.10) that the second variation of area yields knowledge of the Dirichlet-to-Neumann map for the stability operator

\[
    \mathcal{J} := \Delta_{g|_{Y}} + (\text{Ric}_g(\vec{n}, \vec{n}) + ||A||_g^2)
\]

associated to each of the minimal surfaces \(Y(t)\). In fact we can learn the Dirichlet-to-Neumann map for the associated Schrödinger operator

\[
    \Delta_{g|_{E}} + e^{2\phi} \left( \text{Ric}_g(\vec{n}, \vec{n}) + ||A||_g^2 \right),
\]

in isothermal coordinates. The existence of the foliation \(\{Y(t) : t \in (-1, 1)\}\) implies that the equation

\[
    \Delta_{g|_{E}} \psi + e^{2\phi} \left( \text{Ric}_g(\vec{n}, \vec{n}) + ||A||_g^2 \right) \psi = 0,
\]

has positive solutions. Thus, the operator (1.2) has strictly negative eigenvalues (see for instance [3]), and its potential is of the form covered in [18] (see also [12]). The main result in [18] implies that such a potential can be determined from the Dirichlet-to-Neumann map. We therefore also have knowledge of all solutions of (1.2) for given boundary data. That is, we first prove

**Proposition 1.10.** Let \((M, g)\) be a \(C^4\)-smooth, Riemannian manifold which is homeomorphic to a 3-dimensional ball in \(\mathbb{R}^3\), and has mean convex boundary \(\partial M\). Let \(g\) be given on \(\partial M\). Let \(\gamma\) be a given simple closed curve on \(\partial M\), and set \(Y_\gamma \subset M\) to be a surface of least area.
bounded by \( \gamma \). Suppose that the stability operator on \( Y_\gamma \) is non-degenerate, and that for \( \gamma \) and any nearby perturbation \( \gamma(s) \), the area of the least-area surface \( Y_{\gamma(s)} \) enclosed by \( \gamma(s) \) is known.

Equip a neighbourhood of \( Y_\gamma \) with coordinates \((x^\alpha)\) such that on \( Y_\gamma \), \( x^3 = 0 \) and \((x^1, x^2)\) are isothermal coordinates. Then,

1. the first and second variations of the area of \( Y_\gamma \) determine the Dirichlet-to-Neumann map associated to the boundary value problem

\[
\Delta g_E \psi + e^{2\phi} \left( \text{Ric}_g(\vec{n}, \vec{n}) + ||A||_g^2 \right) \psi = 0,
\]

on \( Y_\gamma \), where \( e^{2\phi} g_E = e^{2\phi} [(dx^1)^2 + (dx^2)^2] \) is the metric on \( Y_\gamma \) in the coordinates \((x^1, x^2)\).

2. Knowledge of the first and second variations of the area of \( Y_\gamma \) determines any solution \( \psi(x) \) to the above boundary value problem, in the above isothermal coordinates.

This then yields information on the (first variation of) the position of nearby minimal surfaces \( Y(s,t) \) relative to any of our \( Y(t) \) without having found any information yet on the metric. This first variation of position is expressed in the above isothermal coordinates, thus at this point we make full use of the invariance of solutions of the two equations (1.1) and (1.2) above under conformal changes of the underlying 2-dimensional metric.

We note that in the chosen isothermal coordinates, the component \( \sqrt{g^{33}} \) is the lapse function associated to the foliation:

**Definition 1.11.** Let \( \Omega \subset \mathbb{R}^2 \) be a domain with boundary, and \( f(\cdot, t) : \Omega \times [0, 1] \hookrightarrow M \) a foliation of \( M \) by the surfaces \( Y(t) := f(\Omega, t) \). Set \( \vec{n}_t \) to be a unit normal to \( Y(t) \). Then, the normal component of the variational vector

\[
g \left( f_\ast \left( \frac{\partial}{\partial t}, \vec{n}_t \right) \right) =: \psi,
\]

is called the lapse function of the foliation \( f \).

As the lapse function is a Jacobi field on any such area-minimizing surface, by Proposition 1.10 knowledge of the solutions to equation (1.2) directly yields the component \( g^{33} \) of the inverse metric, everywhere on \( M \). To find further components, we will apply the above strategy not only to our background foliation, but also to a suitable family of first variations of our background foliation. For each point \( p \in Y(t) \subset M \) we can identify two 1-parameter families of foliations \( Y(s,t) \) for which the first variation of \( p \) vanishes at \( p \), but the first variation of tangent space is either in the direction \( \partial_1 \) or \( \partial_2 \). This is done in Subsection 4.1.2 below. These variations lead to a nonlocal system of equations involving \( g^{31}, g^{32}, \) and \( \phi \) (in fact for the differences \( \delta g^{31}, \delta g^{32}, \delta \phi \) of these quantities for two putative metrics with the same area data). We then solve for \( \delta g^{31}, \delta g^{32} \) in terms of the last quantity \( \delta \phi \). This involves the inversion of a system of pseudodifferential equations. The assumption of close to Euclidean or thin/straight is used at this point in a most essential way.

Invoking the knowledge of \( g^{33} \) obtained above for each of the new foliations \( Y(s,t) \) at \( p \) and linearizing in \( s \) yields new equations on \( g^{31}, g^{32} \) at the chosen point \( p \). These equations
involve the (still unknown) conformal factor $\phi$, but also the first variations of the isothermal coordinates $(x^1, x^2)$.

This results in a non-local system of the equations. Thus so far for each $p \in M$ we have derived three equations on the four unknown components of the metric $g$. The required fourth equation utilizes the fact that each $Y(t)$ is minimal. This results in an evolution equation (in $t = x^3$) on the components $\phi, g^{13}, g^{23}$. We prove the uniqueness for the resulting system of equations in Section 4. We note that it is at this point that the existence of a global foliation (without “gaps”) is used in the most essential way.

**Outline of the paper:** In Section 2 we explicitly describe how we asymptotically extend $(M, g_1)$ and $(M, g_2)$ and construct the coordinates systems on $(M, g_1)$ and $(M, g_2)$ we work with. We also prove Proposition [1,10]. In Section 3, we collect arguments for determining the lapse function in our chosen coordinates and the first variation of the lapse, as well as the pseudodifferential equation governing the evolution from leaf to leaf of the metric components of $g_1$ and $g_2$ which are purely tangent to the leaves. In Section 4, we give the proofs of all theorems.

**Notation:** We use the Einstein summation notation throughout this paper. Thus an instance of an index in an up and down position indicates a sum over the index; e.g. $T^i T_j := \Sigma_{i=j} T^i T_j$. Denote by $\nabla$ the Levi-Civita connection associated to $g$, and $\text{Rm}_g$ the Riemann curvature tensor of $(M, g)$. We’ll use the following sign convention for the curvature tensor:

$$\text{Rm}_g(U, V)W := \nabla_V \nabla_U W - \nabla_U \nabla_V W + \nabla_{[U, V]} W.$$ 

The Ricci curvature of $M$ is $\text{Ric}_g(U, V) := \text{tr}_g \text{Rm}_g(U, V, V)$. If $u : M \to \mathbb{R}$, we write $\nabla u := \text{grad}(u)$.

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## 2 Asymptotically Flat Extension and Conformal Maps

In this section, we define asymptotically flat extensions of the 3-manifolds we work with. For technical reasons, we also define an extended foliation $\hat{Y}(t)$ of $M$ and a coordinate system $(x^a)$ adapted to the foliation $\hat{Y}(t)$ such that $x^3$ is constant on the leaves $Y(t)$ and $(x^1, x^2)$ are isothermal for the metric restricted to the leaves $x^3 = \text{constant}$. We then prove that knowledge of the area of any area-minimizing surface near $\hat{Y}(t)$ determines the Dirichlet-to-Neumann map for the stability operator on $\hat{Y}(t)$ in our preferred coordinates $(x^a)$, and from this information, we also determine the image of $\hat{Y}(t)$ under our chosen isothermal coordinate map.

From this point onwards all estimates we write out will involve a constant $C > 0$. Unless stated otherwise, the constant $C$ will depend on the parameters $K, \delta_0$ in the setting of Class 2, or will be a uniformly fixed parameter in the setting of Class 1.

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9So only now we use the minimal surface equation directly, not its first variation.
2.1 Extension to an Asymptotically Flat Manifold

In this section, we let \((M, g)\) be a \(C^4\)-smooth, 3-manifold that satisfies the assumptions of Theorem 1.4. In particular, we assume that \((M, g)\) admits foliations by properly embedded, area minimizing surfaces. In this setting, we use such a foliation to equip \((M, g)\) with a preferred coordinate system. The preferred coordinate system we construct will be used in several of the proofs of this paper to simplify computations and derive relevant equations for the components of the metric.

To construct the desired coordinates, let \(\Omega \subset \mathbb{R}^2\) be a bounded \(C^{2, \alpha}\) domain for some \(\alpha > 0\), and let \(\gamma : \partial \Omega \times (-1, 1) \rightarrow \partial M\) be a given foliation of the boundary by embedded, closed curves. Let \(f(\cdot, t) : \Omega \rightarrow M\) solve Plateau’s problem for \(\gamma(\cdot, t)\), for each \(t \in (-1, 1)\). In particular, \(f(\cdot, \cdot)\) defines a foliation of \(M\) by properly embedded, codimension 1, area-minimizing surfaces such that \(f(\partial \Omega \times \{t\}) = \gamma(\cdot, t)\) for each \(t \in (-1, 1)\). We denote the leaves of the foliation \(f\) in \(M\) by \(Y(t) := f(\Omega \times \{t\})\).

Our first choice of coordinate is the parameter identifying each minimal surface \(Y(t)\): label the coordinate \(x^3 = t\). Now, to obtain two other coordinate functions, we will choose conformal coordinates \((x^1, x^2)\) on each leaf \(Y(t)\) of the foliation. Then \((x^1, x^2, x^3)\) will be a global coordinate system on \((M, g)\). However, there are many choices for conformal coordinates \((x^1, x^2)\) on a 2-dimensional surface \(Y(t)\). To remove this ambiguity, we extend \((M, g)\) to an asymptotically flat manifold and impose decay conditions at infinity on the conformal coordinates on the extension of \(Y(t)\) which renders these coordinates unique.

To this end, let \((z^1, z^2, t) : M \rightarrow \Omega \times (-1, 1)\) be the regular coordinates on \(M\) stipulated by our Theorems for manifold Classes 1 and 2, respectively. Considering a fixed extension operator for metrics (and using our assumed bounds on curvature and second fundamental forms), we may smoothly extend the metric \(g|_{\partial M}\) to a tubular neighbourhood \(N\) of \(\partial M\), preserving (up to a multiplicative factor) the bounds on curvature and on the second fundamental form of \(Y(t)\) assumed in our Theorems.

Let \(g_E\) denote the Euclidean metric on \(M := \mathbb{R}^2 \times (-1, 1)\), and let \(\chi : M \rightarrow \mathbb{R}\) be a smooth cutoff function such that \(\chi|_M = 1\) and \(\chi = 0\) outside \(M \cup N\). We then extend \((M, g)\) to an asymptotically flat manifold \((M, g)\) with metric \(g\) defined as

\[
g := \chi g + (1 - \chi) g_E.
\]

Again via an extension operator, we obtain a smooth extension \(f : \mathbb{R}^2 \times (-1, 1) \rightarrow M\) of the foliation \(f\). The smooth (but not necessarily minimal with respect to \(g\)) extension of \(Y(t)\) to \(M\) is then \(f(\mathbb{R}^2, t) := Y(t) \supseteq \mathbb{R}^2\).

A well known result of Ahlfors [1] then gives the unique existence of isothermal coordinates on \(Y(t)\) which are normalized at infinity. That is, for some \(p > 2\), there exists a conformal map

\[
\Phi(t) : \mathbb{R}^2 \rightarrow \mathbb{R}^2, \quad z := z^1 + iz^2 \mapsto x^1 + ix^2
\]

in \(L^p(\mathbb{R}^2)\) satisfying

\[
\bar{\partial}\Phi(t) = \mu(t) \partial\Phi(t) \quad \text{and} \quad \Phi(z, t) - z = L^p(\mathbb{R}^2),
\]

where for some \(\mu(t) > 0\) and \(L^p(\mathbb{R}^2)\) is the space of functions on \(\mathbb{R}^2\) whose \(p\)-th power is integrable over \(\mathbb{R}^2\).

\[
\frac{\partial\Phi(t)}{\partial z} = \mu(t) \frac{\partial\Phi(t)}{\partial x}
\]

where for some \(\mu(t) > 0\).
with dilation $\mu(t) := \frac{g_{11} - g_{22} + 2g_{12}}{g_{11} + g_{22} + 2\sqrt{|g|}}$. In these coordinates, $\Phi(t)$ pushes forward $g_t := g|_{y(t)}^\nu$ to a metric conformal to the Euclidean metric on $\mathbb{R}^2$:

$$g_t = e^{2\phi(x,t)}[(dx^1)^2 + (dx^2)^2]$$

for some conformal factor $\phi(\cdot, t) : \mathbb{R}^2 \rightarrow \mathbb{R}$. We call the images $(x^1, x^2)$ isothermal coordinates on $\Omega$, and denote the conformal image of $\Omega$ as $\hat{\Omega}(t) := \Phi(\hat{\Omega}, t)$.

Since $g$ is $C^4$-smooth by construction, $\mu_t$ is $C^4$-smooth in $t$. As shown in [2], $\Phi(z, t)$ is a $C^4$-smooth map in $t$. Therefore,

$$\Phi(z(\cdot), t(\cdot)) : M \rightarrow \hat{\Omega}(t) \times (-1, 1)$$

$$\Phi(z(p), t(p)) = (x^1, x^2, x^3)$$

defines a global coordinate chart on $(M, g)$ in this chart, the metric takes the form

$$g = g_{3\alpha}dx^3dx^\alpha + g_t,$$

where $g_t$ is conformally flat for each $t$, and additionally $g_{3k} = 0$ for $k = 1, 2$ outside a compact set containing $M$. In particular, the metric $g$ restricted to $M$ is written as

$$g = g_{3\alpha}dx^3dx^\alpha + e^{2\phi(t)}[(dx^1)^2 + (dx^2)^2].$$

Next we prove that for such a coordinate system $\Phi(z, t) = (x^1, x^2, x^3)$ on $(M, g)$ as described above, knowledge of the areas of properly embedded area minimizing surfaces in $(M, g)$ determines the conformal map $\Phi(\cdot, t)$ on the complement $\mathbb{R}^2 \setminus \Omega$, for every $t \in (-1, 1)$. To prove such a statement, we first show that the knowledge of a Dirichlet-to-Neumann map for a non-degenerate Schrödinger operator on $\Omega$ determines the conformal map $\Phi$ satisfying (2.4) and (2.5) on the open set $Z := \mathbb{R}^2 \setminus \Omega$ (see [4], [23], for similar results). Then, we prove that the knowledge of the area of any properly embedded area minimizing surface in $(M, g)$ determines the Dirichlet-to-Neumann map associated to the stability operator on $Y(t)$ and its perturbations.

In the proofs below, we will construct solutions to the Dirichlet problem for the Schrödinger operator. Precise asymptotic statements will be given in terms of a weighted $L^2$ space. The particular weighted space we require has norm

$$||f||_{L^2_\delta(Z)} = \left( \int_Z |f(w)|^2(1 + |w|^2)^{-\delta} dw \right)^{\frac{1}{2}}.$$

The following propositions also recall the construction of isothermal coordinates on a domain in $\mathbb{R}^2$ as well as existence and uniqueness for an exterior problem which will be crucial to the proofs of the theorems in this paper.

**Proposition 2.1.** Let $\Omega \subset \mathbb{R}^2$ be a bounded, $C^2$-smooth domain. Let $g$ be a $C^2$-smooth Riemannian metric on $\Omega$, and $g$ to be a $C^2$-smooth extension of $g$ to $\mathbb{R}^2$ with

$$g = g_\mathbb{E} \text{ outside a large compact set containing } \Omega,$$

$$g = g \text{ on } \Omega,$$

where $g_\mathbb{E}$ is the Euclidean metric on $\mathbb{R}^2$. Write $Z := \mathbb{R}^2 \setminus \Omega$ and $\nu$ for the outward pointing unit normal vector field to $\partial \Omega$. Let $(z^1, z^2)$ be coordinates on $\mathbb{R}^2$. 

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1. For some $p > 2$, there exists a unique conformal map $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ satisfying
\[
\Phi(z) - z \in L^p(\mathbb{R}^2) \quad (2.6)
\]
\[
\Phi^*(g) = 2^{\phi(x)}[(dx_1)^2 + (dx_2)^2]. \quad (2.7)
\]

2. Let $V \in L^\infty(\Omega)$ and suppose $0$ is not a Dirichlet eigenvalue of $\Delta_g + V$. Let $\Lambda : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$ be the Dirichlet-to-Neumann map associated to $\Delta_g \psi + V$.

Then there exists and $R_1 > 0$ such that for any $\xi \in \mathbb{C}$ with $|\xi| > R_1$ and $1 - \frac{1}{p} < \delta < 1$, there exists a unique solution $\psi(\cdot, \xi)$ to the exterior problem
\[
e^{-iz\xi} \psi(\cdot, \xi) - 1 \in L^2_{-\delta}(Z), \quad (2.8)
\]
\[
\Delta_g \psi(\cdot, \xi) = 0 \text{ on } Z, \quad (2.9)
\]
\[
g(\nabla \psi(\cdot, \xi), \nu) = \Lambda(\psi(\cdot, \xi)) \text{ on } \partial\Omega. \quad (2.10)
\]

Moreover,
\[
||e^{-i\Phi(z)\xi} \psi(\cdot, \xi) - 1||_{L^2_{-\delta}(Z)} < C|\xi|, \quad (2.11)
\]
for some constant $C > 0$ and all $\xi \in \mathbb{C}$ with $|\xi| > R_1$.

Proof. 1. This statement is proved in [1].

2. We will prove existence and uniqueness to (2.8), (2.9), (2.10) by transforming the problem into a Euclidean one via the map $\Phi$.

First, note that for $z = z^1 + iz^2 \in \partial\Omega$, in the coordinates $\Phi(z) = x^1 + ix^2 =: x \in \partial\Phi(\Omega)$ the Dirichlet-to-Neumann map $\Lambda$ is given by the bilinear form
\[
(\chi, \Lambda(\psi)) = \int_{\partial\Omega} \chi(z) g(\nabla \psi(z), \nu(z)) \, dS
\]
\[
= \int_{\partial\Phi(\Omega)} \tilde{\chi}(x) g_E(\nabla \tilde{\psi}(x), e^{-\phi(x)} \tilde{\nu}(x)) e^{\phi(x)} \, d\tilde{S}
\]
\[
= \int_{\partial\Phi(\Omega)} \tilde{\chi}(x) g_E(\tilde{\psi}(x), \tilde{\nu}(x)) \, d\tilde{S}
\]
\[= (\tilde{\chi}, \tilde{\Lambda}((\tilde{\psi})).
\]
Here $\chi, \psi \in H^{1/2}(\partial\Omega)$, $\tilde{\chi}(x) := \chi \circ \Phi^{-1}(x^1 + ix^2)$, $\tilde{\psi}(x) := \psi \circ \Phi^{-1}(x^1 + ix^2)$, and $\tilde{\nu}$ is the outward pointing unit normal vector field to $\partial\Phi(\Omega)$, with respect to the metric $g_E$.

The boundary value problem (2.9), (2.10) expressed in the conformal coordinates given by $\Phi$ becomes
\[
\Delta_{g_E} \tilde{\psi}(\cdot, \xi) = 0 \text{ on } \Phi(Z), \quad (2.12)
\]
\[
g_E(\nabla \tilde{\psi}(\cdot, \xi), \tilde{\nu}) = \tilde{\Lambda}(\tilde{\psi}(\cdot, \xi)) \text{ on } \Phi(\partial\Omega). \quad (2.13)
\]
We claim condition (2.8) is equivalent to
\[ e^{-ix\bar{\psi}(x, \xi)} - 1 \in L^2_{-\delta}(\Phi(Z)). \] (2.14)

To show this assertion, we need the following lemma:

**Lemma 2.2.** For \( f \in L^2_{-\delta}(\Phi(Z)) \), \( ||f||_{L^2_{-\delta}(\Phi(Z))} \leq C ||f \circ \Phi||_{L^2_{-\delta}(Z)} \), for some constant \( C > 0 \).

**Proof.** By a simple change of variable,
\[
||f||^2_{L^2_{-\delta}(\Phi(Z))} = \int_{\Phi(Z)} |f(x)|^2 (1 + |x|^2)^{-\delta} \, dx
= \int_Z |f(\Phi(z))|^2 (1 + |\Phi(z)|^2)^{-\delta}|\Phi'(z)|^2 \, dz.
\]

Choose \( R > 0 \) large so that \( \Omega \subset B_R(0) \). Consider \( z \in Z \) satisfying \( |z| > 2R \). We have \( \Phi(z) - z \in L^p(\mathbb{R}) \) and \( |\Phi(z) - z|^p \) is subharmonic. Then, the Mean Value Property for \( |\Phi(z) - z|^p \) over the ball \( B_z(|z| - R) \) gives
\[
|\Phi(z) - z|^p \leq \left( \frac{1}{\text{Vol}(B_z(|z| - R))} \right) \int_{B_z(|z| - R)} |\Phi(w) - w|^p \, dw
\leq C \frac{||\Phi(w) - w||_{L^p(Z)}^p}{(|z| - R)^2}
\leq C \frac{||\Phi(w) - w||_{L^p(Z)}^p}{|z|^2}. \tag{2.15}
\]

On the other hand, for \( z \in Z \) and \( |z| > 3R \),
\[
|\Phi'(z) - 1| \leq \frac{1}{2\pi} \int_{|w-z|=R} \frac{|\Phi(w) - w|}{|(w-z)^2|} \, |dw|
\leq C \int_{|w-z|=R} \frac{|\Phi(v) - v||_{L^p(Z)}^2}{|w|^2} \, \frac{1}{|w-z|^2} \, |dw|
\leq C \frac{1}{|R|^\frac{2}{p}} \tag{2.17}
\leq C \frac{1}{|z|^\frac{2}{p}}. \tag{2.18}
\]

Therefore
\[
||f||^2_{L^2_{-\delta}(\Phi(Z \cap \{ |z| > 3R \}))} = \int_{Z \cap \{ |z| > 3R \}} |f(\Phi(z))|^2 (1 + |\Phi(z)|^2)^{-\delta}|\Phi'(z)|^2 \, dz
\leq C \int_{Z \cap \{ |z| > 3R \}} |f(\Phi(z))|^2 (1 + |z|^2)^{-\delta} \, dz.
\]
Correspondingly, for \( \{ z \in Z : |z| < 3R \} \),
\[
\int_{\Phi(Z \cap \{|z| < 3R\})} |f(x)|^2 (1 + |x|^2)^{-\delta} \, dx \leq C \int_{Z \cap \{|z| < 3R\}} |f(\Phi(z))|^2 (1 + |z|^2)^{-\delta} \, dz
\]
since \( \Phi \) is a \( C^1 \)-smooth diffeomorphism on \( B_R(0) \).

So indeed,
\[
\|f\|_{L^2_\delta(\Phi(Z))} \leq C\|f \circ \Phi\|_{L^2_\delta(Z)}.
\]

Using the previous lemma,
\[
\|e^{-iz\xi \tilde{\psi}(x, \xi)} - 1\|_{L^2_\delta(\Phi(Z))} \leq C\|e^{-i\Phi(z)\xi} \tilde{\psi}(z, \xi) - 1\|_{L^2_\delta(Z)}
\]
\[
\leq C\|e^{-i\Phi(z)\xi} \tilde{\psi}(z, \xi) - 1\|_{L^\infty(Z)}\|e^{-iz\xi \tilde{\psi}(z, \xi)} - 1\|_{L^2_\delta(Z)}
\]
\[
+ C\|e^{-i\Phi(z)\xi} \tilde{\psi}(z, \xi) - 1\|_{L^2_\delta(Z)} + C\|e^{-iz\xi \tilde{\psi}(z, \xi)} - 1\|_{L^2_\delta(Z)}.
\]

We assume \( \|e^{-iz\xi \tilde{\psi}(z, \xi)} - 1\|_{L^2_\delta(Z)} < \infty \); to prove \( \|e^{-iz\xi \tilde{\psi}(z, \xi)} - 1\|_{L^2_\delta(\Phi(Z))} < \infty \), it remains to show \( \|e^{-i\Phi(z)\xi} \tilde{\psi}(z, \xi) - 1\|_{L^2_\delta(Z)} < \infty \) and \( \|e^{-i\Phi(z)\xi} \tilde{\psi}(z, \xi) - 1\|_{L^\infty(Z)} < \infty \).

Since \( \Phi(z) - z \to 0 \) as \( |z| \to \infty \), \( \Phi(z) \in L^\infty(Z) \). Thus \( \|e^{-i\Phi(z)\xi} \tilde{\psi}(z, \xi) - 1\|_{L^\infty(Z)} \) is bounded. Now we show a bound for \( \|e^{-i\Phi(z)\xi} \tilde{\psi}(z, \xi) - 1\|_{L^2_\delta(Z)} \). To do this, we use the estimate (2.15) from Lemma 2.2 to obtain for \( |z| > 2R \)
\[
\left| e^{-i\Phi(z)\xi} - 1 \right| \leq C(|\xi|)|\Phi(z) - z|
\]
\[
\leq C(|\xi|)\|\Phi(w) - w\|_{L^p(Z)} |z|^{-\frac{\delta}{p}}.
\]

Therefore,
\[
\|e^{-i\Phi(z)\xi} - 1\|_{L^2_\delta(Z)} = \int_Z \left| e^{-i\Phi(z)\xi} - 1 \right|^2 (1 + |z|^2)^{-\delta} \, dz
\]
\[
= \int_{Z \cap \{|z| > 3R\}} \left| e^{-i\Phi(z)\xi} - 1 \right|^2 (1 + |z|^2)^{-\delta} \, dz
\]
\[
+ \int_{Z \cap \{|z| < 3R\}} \left| e^{-i\Phi(z)\xi} - 1 \right|^2 (1 + |z|^2)^{-\delta} \, dz
\]
\[
\leq C(|\xi|)\int_{Z \cap \{|z| > 3R\}} \frac{\|\Phi(w) - w\|^2_{L^p(Z)}}{|z|^{\frac{\delta}{p}}(1 + |z|^2)^{-\delta}} \, dz \, |z| \, dz + C(|\xi|),
\]
which is finite since we take \( 1 > \delta > 1 - \frac{1}{p} \). Hence
\[
e^{-iz\xi \tilde{\psi}(z, \xi)} - 1 \in L^2_{\delta}(\Phi(Z)).
\]

By a similar argument as above, if (2.14) holds, then (2.8) holds.
Now we construct solutions to (2.14), (2.12), and (2.13). Consider \( \tilde{\psi}(\cdot, \xi) : \mathbb{R}^2 \to \mathbb{R} \) defined by the integral equation

\[
\tilde{\psi}(x, \xi) = e^{iz(x^1 + ix^2)\xi} - G_{\xi} * (\tilde{V} \psi_1(\cdot, \xi)),
\]

(2.20)

where

\[ G_{\xi}(w) = \frac{e^{iz(w^1 + iw^2)}}{4\pi^2} \int_{\mathbb{R}^2} \frac{e^{iw\cdot \zeta}}{|\zeta|^2 + 2\xi(\zeta^1 + i\zeta^2)} \, d\zeta \]

is Faddeev’s Green function (see [9], [18], [24]), and \( \tilde{\psi} \) solutions satisfy (2.23), (2.24) which additionally satisfy (2.11) holds for

\[ \partial_{\nu} \tilde{\psi} = \Phi(\Omega) \text{ in } \Phi(\Omega) \text{ and is equal on } \mathbb{R}^2 \setminus \Omega. \]

In addition, from the estimate (2.21) for \( \tilde{\psi}(x, \xi) \) with \( e^{-iz(x^1 + ix^2)\xi} \tilde{\psi}(x, \xi) - 1 \in L^2_{-\delta}(\mathbb{R}^2) \), for \( |\xi| \) sufficiently large. Moreover, for large \( |\xi| \), these solutions satisfy

\[ ||e^{-iz(x^1 + ix^2)\xi} \tilde{\psi}(x, \xi) - 1||_{L^2_{-\delta}(\mathbb{R}^2)} \leq \frac{C}{|\xi|}. \]

(2.21)

Consider the pullback \( \psi(z, \xi) := \tilde{\psi}(\Phi_k(z), \xi) \). The functions \( \psi(z, \xi) \) then satisfy the exterior problem (2.8), (2.9), (2.10). In addition, from the estimate (2.21) for \( \tilde{\psi}(x, \xi) \), the estimate (2.11) holds for \( \psi(z, \xi) \). This proves existence. Uniqueness for \( \psi \) follows from the uniqueness of \( \tilde{\psi} \).

Proposition 2.3. Let \((z_1, z_2)\) be coordinates on \( \mathbb{R}^2 \), and \( \Omega \subset \mathbb{R}^2 \) be a bounded domain with Lipschitz boundary \( \partial \Omega \). Set \( g_1, g_2 \) to be two \( C^2 \)-smooth Riemannian metrics on \( \Omega \).

For \( k = 1, 2 \), let \( g_k, \Phi_k, V_k \in L^\infty(\Omega) \) be as in Proposition 2.1. Define by \( \Lambda_k, k = 1, 2 \), the Dirichlet-to-Neumann maps associated to \( \Delta_{g_k} \psi_k + V_k \psi_k \) in \( \Omega \).

Then, if \( \Lambda_1 = \Lambda_2 \), the conformal maps \( \Phi_1(z) = z^1 + iz^2, \Phi_2(z) = y^1 + iy^2 \) are equal on the exterior set \( Z := (\mathbb{R}^2 \setminus \Omega) \cup \partial \Omega \). In particular, the Dirichlet-to-Neumann maps determine the domain \( \Phi_1(\Omega) = \Phi_2(\Omega) \).

Proof. Extend \( V_k, k = 1, 2 \) to all of \( \mathbb{R}^2 \) such that \( V_k = 0 \) outside a compact set containing \( \Omega \), and \( V_1 = V_2 \) is known on \( \mathbb{R}^2 \setminus \Omega \).

Let \( \xi \in \mathbb{C} \setminus \{0\} \) and \( \delta \) satisfy the conditions of Proposition 2.1 for \( g_1 \) and \( g_2 \). For \( k = 1, 2 \), consider the exterior problems

\[
\psi_k(\cdot, \xi) \in L^2_{\text{loc}}(Z) \text{ and } e^{-iz\xi} \psi_k(\cdot, \xi) - 1 \in L^2_{-\delta}(Z),
\]

(2.22)

\[
\Delta_{g_k} \psi_k(\cdot, \xi) + V_k \psi_k(\cdot, \xi) = 0 \text{ on } Z
\]

(2.23)

\[
g_k(\nabla \psi_k(\cdot, \xi), \nu) = \Lambda_k \psi_k(\cdot, \xi) \text{ on } \partial \Omega,
\]

(2.24)

where \( \nu \) is the outward pointing unit normal vector field to \( \partial \Omega \).

By Proposition 2.1, there exists for \( k = 1, 2 \) unique families of solutions \( \psi_k(\cdot, \xi) \) to (2.22), (2.23), (2.24) which additionally satisfy

\[ ||e^{-i\Phi_k(z)} \psi_k(\cdot, \xi) - 1||_{L^2_{-\delta}(Z)} \leq \frac{C}{|\xi|}. \]

(2.25)
Since we imposed $V_1 = V_2$ on $Z$, and from the assumption that $\Lambda_1 = \Lambda_2$, $\psi(z, \xi)$ solves the same problem as $\psi_2(z, \xi)$ on $Z$. Proposition 2.1 gives uniqueness of the solutions $\psi_k(z, \xi)$ to the exterior problems (2.22), (2.23), (2.24); thus $\psi_1(z, \xi) = \psi_2(z, \xi)$ on $Z$. we now show that this together with (2.23) implies $\Phi_1 = \Phi_2$ on $Z$.

Write $\psi(z, \xi) := \psi_1(z, \xi) = \psi_2(z, \xi)$. From the estimates on $\psi(z, \xi)$, we have

$$\frac{2C}{|\xi|} \geq \left| e^{-i\Phi_2(z)\xi} \psi(z, \xi) - e^{-i\Phi_1(z)\xi} \psi(z, \xi) \right|_{L^2_\delta(Z)}$$

$$\geq \left| e^{i(\Phi_1 - \Phi_2)(z)\xi} - 1 \right| e^{-i\Phi_1(z)\xi} \psi(z, \xi) \right|_{L^2_\delta(Z)}.$$

(2.26)

(2.27)

Using the above estimate and proof by contradiction, we show $\Phi_1(z) = \Phi_2(z)$ for $z \in Z$. Suppose $|\Phi_1(z_0) - \Phi_2(z_0)| > 0$ for some $z_0 \in Z$. Without loss of generality, we may assume that $\text{Re}(\Phi_1(z_0) - \Phi_2(z_0)) > 0$. By the continuity of $\Phi_j$, $j = 1, 2$, there exists $\epsilon > 0$ and $c > 0$ such that

$$\text{Re}(\Phi_1(z) - \Phi_2(z)) > c$$

for $z \in B_{z_0}(\epsilon)$.

Consider $\xi = 0 + i\xi^2$, for $\xi^2 < 0$. We find

$$\left| e^{i(\Phi_1 - \Phi_2)(z)\xi} - 1 \right| \geq \left| e^{i(\Phi_1 - \Phi_2)(z)\xi} - 1 \right| - 1$$

$$\geq \left| e^{-\text{Re}(\Phi_1(z_0) - \Phi_2(z_0))\xi^2} - 1 \right|$$

$$\geq \left| e^{-c\xi^2} - 1 \right|,$$

for all $z \in B_{z_0}(\epsilon)$.

Taking $\xi^2 \to -\infty$, we have $\left| e^{i(\Phi_1 - \Phi_2)(z)\xi} - 1 \right| \to \infty$. This violates (2.27), since the right hand side goes to zero as $|\xi| \to \infty$.

Therefore, $|\Phi_1(z) - \Phi_2(z)| = 0$ on $Z$. This completes the proof.

We note some further useful bounds on the conformal factors $\phi_t$ for future use, which stem from the above estimates on $\Phi'$, along with the Gauss equation and standard elliptic estimates applied to the equation $\Delta g \phi = -\frac{1}{2} R(g_t)$ (where $R(g_t)$ is the Gauss curvature of the metric $g_t|_{Y_t}$):

**Lemma 2.4.** The conformal factors $\phi_t$ have small $C^2$ norm in the close to Euclidean setting. In the thin/straight setting, the conformal factors $\phi_t$ satisfy the bounds:

$$|\phi_t| \leq C\delta_0^2, \quad |\partial \phi_t| \leq C\delta_0 e^{-1}, \quad |\partial^2 \phi_t| \leq C\delta_0 e^{-2},$$

for some universal constant $C > 0$; here $\partial$ is the coordinate derivative.

**Sketch of Proof.** The argument is based on standard elliptic estimates. It is convenient to consider the re-scaling of the metric $g_t$ by a factor $(\text{Area}[Y(t)])^{-1}$. Denote the resulting metric by $\tilde{g}_t$. We also rescale the underlying isothermal coordinates by $(\text{Area}[Y(t)])^{-1/2}$, and
denote the conformal factor for $\tilde{g}_t$ over the new coordinate system by $e^{2\tilde{\phi}_t}$. The function $\tilde{\phi}_t$ is just the push-forward of $\phi_t$ under the dilation map (with dilation factor $(\text{Area}[Y(t)])^{-1/2})$.

In the equation

$$\Delta_{\tilde{g}_t} \tilde{\phi}_t = -\frac{1}{2} R(\tilde{g}_t)$$

The right hand side now has a $W^{1,p}$ norm bounded by $C\delta_0$, for any $p > 2$. The $C^2$ norm of the metric $\tilde{g}_t$ is uniformly bounded, via the bounds on the Beltrami coefficient and the assumed curvature bounds; thus we derive:

$$||\tilde{\phi}_t||_{W^{4,p}(\mathbb{R}^2)} \leq C\delta_0 \Rightarrow ||\tilde{\phi}_t||_{C^2} \leq C\delta_0.$$  

The above estimates imply the claimed bounds in the original metric $g_t$.

**Remark 2.5.** We also make note of a consequence of the above bounds, which will be useful further down:

Given the formula

$$(\Phi')^T g_t \Phi' =: \Phi_*(g_t) = e^{2\phi_t}[(dx^1)^2 + (dx^2)^2]$$

the $C^2$ bounds that we were assuming for the metric $g$ over $M$ in the original coordinate system continue to hold when $g_t$ is expressed in the new isothermal coordinate system $(x^1, x^2)$, up to increasing the constant in those bounds by a fixed amount.

### 2.2 Least Area Data Implies Dirichlet-to-Neumann Data

Once again, in this section we assume that $(M, g)$ is a $C^4$-smooth, Riemannian manifold which is homeomorphic to a 3-dimensional ball in $\mathbb{R}^3$. Further, we suppose that $(M, g)$ has $C^4$-smooth, mean convex boundary $\partial M$.

Recall that for a properly embedded minimal surface $\Omega \hookrightarrow M$, we defined the stability operator as $\Delta_{g_Y} + (\text{Ric}_g(n, n) + ||A||_g^2)(\text{see } (1.1))$. Then,

**Definition 2.6.** The **Dirichlet-to-Neumann map** associated to the stability operator is the map

$$\Lambda_{g_Y} : H^{1/2}(\partial Y_\gamma) \rightarrow H^{-1/2}(\partial Y_\gamma)$$

$$\Lambda_{g_Y} (\psi) := g_{Y_\gamma} (\nabla \psi, \nu)|_{\partial Y_\gamma},$$

where $\psi$ solves $[1.1]$, $\nu$ is the outward pointing normal vector field to the boundary $\partial Y_\gamma$ and $g_{Y_\gamma}$ is the metric induced on $Y_\gamma$ by $g$.

We restate Proposition 2.3 in a form which connects the Dirichlet-to-Neumann map of the stability operator with knowledge of the isothermal coordinates (normalized at infinity) outside an open set.

**Corollary 2.7.** Let $g_1, g_2$ be two $C^4$-smooth metrics on $M$, and $\Omega \subset \mathbb{R}^2$ be a bounded domain. Suppose $g_1 = g_2$ on $\partial M$. Let $\Omega \hookrightarrow Y_k \subset M$ for $k = 1, 2$ be properly embedded, area-minimizing surfaces in $(M, g_k)$ with $\partial Y_1 = \partial Y_2$. 

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As defined earlier, let \( g_k \) be the extensions of \( g_k \) to an asymptotically flat 3-manifold \( M \supset \Omega \). Set \( \Phi_k \) to be the unique conformal maps inducing isothermal coordinates on the extensions of \( Y_k \).

Consider the Dirichlet-to-Neumann map \( \Lambda_{g_k} \) associated to the stability operator
\[
\Delta_{g_k|_{Y_k}} + \text{Ric}_{g_k}(\vec{n}_k, \vec{n}_k) + ||A_{Y_k}||^2_{g_k},
\]
k = 1, 2. If \( \Lambda_{g_1} = \Lambda_{g_2} \), then \( \Phi_1 = \Phi_2 \) on \((\mathbb{R}^2 \setminus \Omega) \cup \partial \Omega\).

Key to all subsequent proofs in this paper, we now show that our minimal area data determines the Dirichlet-to-Neumann map associated to the stability operator on an area minimizing surface with boundary.

**Proposition 2.8.** Let \( g|_{\partial M} \) be given. Suppose that

1. \( M \) admits a properly embedded, area-minimizing foliation \( Y(t) \)
2. the area of each \( Y(t) \) and any nearby perturbation of \( Y(t) \) by area-minimizing surfaces is known.

Then, the first variations of the area of \( Y(t) \) determine the angle at which \( Y(t) \) cuts the boundary of \( M \).

**Proof.** Write \( \gamma(t) = \partial Y(t) \subset \partial M \). Let \( \gamma(s, t) : (-\eta, \eta) \times (-1, 1) \to \partial M \) be a one parameter variation of \( \gamma(t) \) by simple closed curves, chosen so that \( \frac{\partial}{\partial s}|_{s=0} \gamma(s, t) \) is a vector field tangent to the boundary. Denote by \( Y(s, t) \) the area minimizing surface circumscribed by \( \gamma(s, t) \), and \( A(s, t) \) the area of \( Y(s, t) \). Write \( X_0 := \frac{\partial}{\partial s}|_{s=0} Y(s, t) \). By standard computations the first variation in the area of \( Y(t) \) is
\[
\frac{\partial}{\partial s} A(s, t) \bigg|_{s=0} = \int_{Y(t)} g(X_0, H) \, d\text{Vol}_{g_t} + \int_{\partial Y(t)} g(X_0, \nu) \, dS,
\]
where \( g_t \) is the metric restricted to \( Y(t) \) and \( \nu \) is the unit outward pointing normal vector to \( \partial Y(t) \) and tangent to \( Y(t) \). Since we have assumed knowledge of the area of any minimal perturbation of \( Y(t) \), we know \( \frac{\partial}{\partial s} A(s, t) \bigg|_{s=0} \); further, the fact that \( Y(t) \) is minimal implies
\[
\frac{\partial}{\partial s} A(s, t) \bigg|_{s=0} = \int_{\partial Y(t)} g(X_0, \nu) \, dS.
\]
Since \( X_0 \) is known, and is allowed to be any vector field tangent to the boundary, we have determined the angle at which \( Y(t) \) cuts the boundary of \( M \).

Note: the above argument holds for arbitrary dimension. 

**Proposition 2.9.** Let \( g|_{\partial M} \) be given.

Let \( \gamma(t) \), \( t \in (-1, 1) \) be a 1-parameter family of simple closed curves which foliate \( \partial M \). For each \( t \), let \( Y(t) \) be the area-minimizing surface which is bounded by \( \gamma(t) \); we assume that \( \{Y(t) : t \in (-1, 1)\} \) defines a foliation of \( M \). Suppose that for each \( t \), the area of \( Y(t) \) and any nearby perturbation of \( Y(t) \) by area-minimizing surfaces is known.

Then, this data determines the Dirichlet-to-Neumann map associated to the stability operator on \( Y(t) \).
Proof. It is convenient to consider variations of $Y(t)$ that are normal to it at the boundary curve $\gamma$. From such variations, we discover information about the Dirichlet-to-Neumann map associated to the stability operator on each $Y(t)$. Such variations need not arise as variations of $\gamma(t)$ on the boundary of $M$; hence we smoothly extend $M$ and work with this extension. Let $N$ to be a tubular neighbourhood of $\partial M$. Let $\tilde{M} := N \cup M$ and extend $g$ to a $C^4(\tilde{M})$-smooth metric $\tilde{g}$ on $M$. We further impose that $N$ was chosen so that the Riemannian manifold $(\tilde{M}, \tilde{g})$ has mean convex boundary $\partial \tilde{M}$.

Next, we construct an auxiliary family of unique, area-minimizing surfaces in $\tilde{M}$ which we will vary normally to obtain information about a 1-parameter family of Dirichlet-to-Neumann maps which, loosely speaking, are close in some sense to the Dirichlet-to-Neumann map we seek to identify on $Y(t)$. Towards this end, for each fixed $t$, we select a 1-parameter family simple closed curves in $\tilde{M}\setminus M$ which are $C^4(\tilde{M})$-close to $\gamma(t)$; here $\epsilon \in [0,1]$ and $\gamma(t,0) := \gamma(t)$.

We have the following two facts: since we have assumed that $\gamma(t)$ bounds a unique, area-minimizing surface, so too for every $t$ and small enough $\epsilon > 0$ the curves $\gamma(t,\epsilon)$ bound a unique, area-minimizing surface. Thus, given some bounded domain $\Omega \subset \mathbb{R}^2$, there exists embeddings

$$\tilde{f}_{t,\epsilon} : \Omega \to \tilde{M},$$

$$\tilde{f}_{t,\epsilon}\big|_{\partial\Omega} = \gamma(t,\epsilon),$$

such that $\tilde{f}_{t,\epsilon}(\Omega) =: \tilde{Y}(t,\epsilon)$ is the unique surface which solves the least area problem for $\gamma(t,\epsilon)$.

Now we describe normal variations of $\gamma(t,\epsilon)$; for every $s \in [0,1]$, define $\tilde{\gamma}(s,t,\epsilon)$ to be a simple closed curve satisfying $\frac{d}{ds}\big|_{s=0} \tilde{\gamma}(s,t,\epsilon)$ is parallel to the unit normal vector field $\tilde{n}_{t,\epsilon}$ on the surface $\tilde{Y}(t,\epsilon)$. Here we write $\tilde{\gamma}(0,t,\epsilon) = \tilde{\gamma}(t,\epsilon)$. Once more, the variations $\tilde{\gamma}(s,t,\epsilon)$ are $C^4(\tilde{M})$-close to $\gamma(t)$, and since $\gamma(t)$ bounds a unique, area-minimizing surface, for each $s$, the curves $\tilde{\gamma}(s,t,\epsilon)$ bound a unique, area-minimizing surface.
In particular, there exist embeddings \( \bar{f}_{s,t,\epsilon} : \Omega \to \bar{M} \) satisfying

\[
\bar{f}_{s,t,\epsilon} \big|_{\partial \Omega} = \bar{\gamma}(t, \epsilon),
\]

\[
\bar{f}_{s,t,\epsilon}(\Omega) = \bar{Y}(s, t, \epsilon)
\]
solves the least area problem for \( \bar{\gamma}(s, t, \epsilon) \), and

\[
\frac{d}{ds} \bar{f}_{s,t,\epsilon} \bigg|_{s=0} = \psi_{t,\epsilon} \bar{n}_{t,\epsilon}.
\]

Moreover, the \( C^4(\bar{M}) \)-smooth function \( \psi_{t,\epsilon} : \bar{Y}(t, \epsilon) \to \mathbb{R} \) solves the boundary value problem

\[
\Delta_{\bar{g}_{t,\epsilon}} \psi_{t,\epsilon} + \left( \text{Ric}_{\bar{g}}(\bar{n}_{t,\epsilon}, \bar{n}_{t,\epsilon}) + ||A_{t,\epsilon}||^2_\bar{g} \right) \psi_{t,\epsilon} = 0, \quad \text{on } \bar{Y}(t, \epsilon) \tag{2.28}
\]

\[
\psi_{t,\epsilon} \big|_{\partial \bar{Y}(t, \epsilon)} = \psi^\circ_{t,\epsilon},
\]

for prescribed boundary data \( \psi^\circ_{t,\epsilon} := g(V, \bar{n}_{t,\epsilon}) \), \( V := \frac{d}{ds} \bar{\gamma}_{s,t,\epsilon} \big|_{s=0} \). Here \( \bar{g}_{t,\epsilon} \) is the metric \( \bar{g} \) restricted to \( \bar{Y}(t, \epsilon) \) and \( A_{t,\epsilon} \) is the second fundamental form of \( \bar{Y}_{t,\epsilon} \).

We know the metric on \( \bar{M} \setminus M \cup \partial M \). Therefore, by the following lemma (Lemma 2.10), we determine the intersection of \( \bar{Y}(s, t, \epsilon) \cap \partial M \). From this, the area of \( \bar{Y}(s, t, \epsilon) \), denoted by \( \text{Area}(s, t, \epsilon) \), is found. An easy computation shows that for each \( (s, t, \epsilon) \), the second variation in area is the number given by

\[
\frac{\partial^2}{\partial s^2} \text{Area}(s, t, \epsilon) \bigg|_{s=0} = \int_{\partial \bar{Y}(t, \epsilon)} \psi_{t,\epsilon} \bar{g}(\nabla \psi_{t,\epsilon}, \nu_{t,\epsilon}) + \bar{g}(\nabla V, \nu_{t,\epsilon}) \, dS
\]

\[
\quad - \int_{\bar{Y}(t, \epsilon)} \psi_{t,\epsilon} \Delta_{\bar{g}_{t,\epsilon}} \psi_{t,\epsilon} + \psi^\circ_{t,\epsilon} \left( \text{Ric}_{\bar{g}}(\bar{n}, \bar{n}) + ||A||^2_\bar{g} \right) \, d\text{Vol}_{\bar{g}_{Y}(t)}
\]

\[
= \int_{\partial \bar{Y}(t, \epsilon)} \psi_{t,\epsilon} \bar{g}(\nabla \psi_{t,\epsilon}, \nu_{t,\epsilon}) \, dS + \int_{\partial \bar{Y}(t, \epsilon)} \bar{g}(\nabla V, \nu_{t,\epsilon}) \, dS,
\]

where \( \nu_{t,\epsilon} \) is the outward pointing normal to \( \partial \bar{Y}(t, \epsilon) \) and tangent to \( \bar{Y}(t, \epsilon) \).

The quantity \( \int_{\partial \bar{Y}(t, \epsilon)} \bar{g}(\nabla V, \nu_{t,\epsilon}) \, dS \) appearing in the above second variation of area is known, since the metric \( \bar{g}_{t,\epsilon} \) is given on \( \bar{M} \setminus M \), and both \( \nu_{t,\epsilon} \) and \( V \) are known on \( \bar{\gamma}(t, \epsilon) \). Therefore,

\[
\int_{\partial \bar{Y}(t, \epsilon)} \psi_{t,\epsilon} \bar{g}(\nabla \psi_{t,\epsilon}, \nu_{t,\epsilon}) \, dS = \frac{\partial^2}{\partial s^2} \text{Area}(s, t, \epsilon) \bigg|_{s=0} - \int_{\partial \bar{Y}(t, \epsilon)} \bar{g}(\nabla V, \nu_{t,\epsilon}) \, dS
\]

\[
= \text{known quantity.} \tag{2.30}
\]

Then, by polarizing, our area data has determined the functional

\[
L(\phi^\circ, \psi^\circ) := \int_{\partial \bar{Y}(t, \epsilon)} \phi^\circ \bar{g}(\nabla \psi, \nu_{t,\epsilon}) \, dS
\]

for any functions \( \phi^\circ, \psi^\circ : \partial \bar{Y}(t, \epsilon) \to \mathbb{R} \in C^2(\bar{Y}(t, \epsilon)) \).

In particular, we have learned the Dirichlet-to-Neumann map

\[
A_{t,\epsilon}(\psi^\circ) := \bar{g}(\nabla \psi, \nu_{t,\epsilon}) |_{\partial \bar{Y}(t, \epsilon)}
\]

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associated to equation (2.29). We remark that the operators

\[ \mathcal{J}_{t, \epsilon} := \Delta \bar{g}_{t, \epsilon} + \text{Ric}_{\bar{g}}(\bar{n}_{t, \epsilon}, \bar{n}_{t, \epsilon}) + ||A_{t, \epsilon}||^2_\bar{g} \]

are non-degenerate for \( \epsilon > 0 \) small enough, since the eigenvalues of \( \mathcal{J}_{t, 0} \) depend continuously on the strictly negative eigenvalues of \( J_{t, 0} \). Hence the Dirichlet-to-Neumann map \( \Lambda_{t, \epsilon} \) is well-defined for each \( t, \epsilon \).

Now since the surfaces \( \bar{Y}(t, \epsilon) \) are \( C^4(\bar{M}) \)-close to \( \bar{Y}(t) \), as \( \epsilon \to 0 \) the component functions of the metrics \( \bar{g}_{t, \epsilon} \) tend to those of \( \bar{g}_{t, 0} \) in the \( C^4(\bar{M}) \)-norm. Also, for each \( t \), the potentials \( \left( \text{Ric}_{\bar{g}}(\bar{n}_{t, \epsilon}, \bar{n}_{t, \epsilon}) + ||A_{t, \epsilon}||^2_\bar{g} \right) \) converge to \( \left( \text{Ric}_{\bar{g}}(\bar{n}_{t, 0}, \bar{n}_{t, 0}) + ||A_{t, 0}||^2_\bar{g} \right) \) in \( C^1(\bar{M}) \) as \( \epsilon \to 0 \).

Finally, since each \( \psi_{t, \epsilon} \) depends continuously on \( g_{t, \epsilon} \) and \( \left( \text{Ric}_{\bar{g}}(\bar{n}_{t, \epsilon}, \bar{n}_{t, \epsilon}) + ||A_{t, \epsilon}||^2_\bar{g} \right) \), the functions \( \psi_{t, \epsilon} \) converge to \( \psi_{t, 0} \) in \( C^4(\bar{M}) \). Take the limit as \( \epsilon \to 0 \) of (2.30). Since \( Y(t) := Y(t, 0) \), on the original leaf \( Y(t) \) we learn

\[
\int_{\partial \bar{Y}(t)} \psi_{t, 0} \bar{g}(\nabla \psi_t, \nu_t) \, dS = \frac{\partial^2}{\partial s^2} \text{Area}(s, t, 0) \bigg|_{s=0} + \text{known quantity},
\]

and thus determine the Dirichlet-to-Neumann map

\[ \Lambda_t(\psi^t) := \bar{g}(\nabla \psi, \nu_t)|_{\partial \bar{Y}(t)} \]

associated to the stability operator on \( Y(t) \).

Now we prove our assumption about the boundaries and areas of \( \bar{Y}(t, \epsilon) \).

**Lemma 2.10.** Suppose that \((M, g)\) admits a foliation by properly embedded, area-minimizing surfaces. Let \((\bar{M}, \bar{g})\) be a smooth extension of \((M, g)\) such that \(g|_M = g\), \(\bar{g}\) is known on \((\bar{M} \setminus M) \cup \partial M\), and \(\partial \bar{M}\) is mean convex.

Let \(\gamma(t)\) be a given 1-parameter family of simple closed curves which foliate \(\partial M\), and let \(Y(t)\) be unique, area-minimizing leaves of the foliation induced on \(M\) by solving the least-area problem for \(\gamma(t)\). Suppose that for each \(t\), the area of \(Y(t)\) and any nearby perturbation of \(Y(t)\) by area-minimizing surfaces is known.

For each fixed \(t\), choose \(\tilde{\gamma}(t, \epsilon), \epsilon \in [0, 1]\) to be a family of simple closed curves which lie in \(\bar{M} \setminus (M \cup \partial M)\), and satisfy \(\tilde{\gamma}(t, 0) = \gamma(t)\) and \(\tilde{\gamma}(t, \epsilon)\) is \(C^4(\bar{M})\)-close to \(\gamma(t)\). Define \(Y(t)\) and \(\bar{Y}(t, \epsilon)\) to be the surface of least area which bound \(\gamma(t)\) and \(\tilde{\gamma}(t, \epsilon)\), respectively.

Then,

\[ a. \text{ We know the closed curve given by the intersection } c(t, \epsilon) := \bar{Y}(t, \epsilon) \cap \partial M. \]

\[ b. \text{ We know the area of } \bar{Y}(t, \epsilon), \text{ with respect to } \bar{g}. \]

**Proof.** a. Let \(c(t, \epsilon)\) the curve given by \(\bar{Y}(t, \epsilon) \cap \partial M\). Consider the set \(\Sigma\) of all simple closed curves on \(\partial M\) which are \(C^4(\bar{M})\)-close to \(c(t, \epsilon)\). For any curve \(\sigma \in \Sigma\), denote by \(Y_\sigma \subset M\) the surface which minimizes the area enclosed by \(\sigma\).

For \(\sigma \in \Sigma\), let \(A_\sigma\) be the area-minimizing annulus which lies between \(\sigma\) and \(\tilde{\gamma}(t, \epsilon)\). The metric \(\bar{g}\) is known on \((\bar{M}(r) \setminus M) \cup \partial M\), so for any such annulus \(A_\sigma\), we can determine the inward pointing (with respect to \(A_\sigma\)) unit normal vector field \(\bar{\nu}_\sigma\) tangent to \(A_\sigma\) and normal to the curve \(\sigma\).
Now, given any $\sigma \in \Sigma$ the first variations in the area of $Y_{\sigma}$ determine the angle at which $Y_{\sigma}$ cut the boundary of $M$ (see Proposition \[2.8\]). Thus, we may determine the outward pointing (with respect to $Y_{\sigma}$) unit normal vector fields $\nu_{\sigma}$ which are tangent to $Y_{\sigma}$, and normal to the curve $\sigma$.

Consider the annulus $A_c(t, \epsilon)$. Notice that the vectors $\nu_{c(t, \epsilon)}$ and $\tilde{\nu}_{c(t, \epsilon)}$ are collinear, since the surface $\bar{Y}(t, \epsilon)$ is smooth at the curve $c(t, \epsilon)$. We claim that $c(t, \epsilon)$ is the only curve in $\Sigma$ with this property.

To show the uniqueness of $c(t, \epsilon)$, suppose that $\sigma^2 \in \Sigma$ is a curve such that $A_{\sigma^2}$ is the minimal annulus for which $\nu_{\sigma^2}$ and $\tilde{\nu}_{\sigma^2}$ are collinear on $\partial M$. Note for any $p \in \sigma^2$, the tangent space $T_pA_{\sigma^2}$ coincides with the tangent space $T_pY_{\sigma^2}$ since they are both spanned by $\nu_{\sigma^2}$ and any vector tangent to $\sigma^2$. Hence, $A_{\sigma^2} \cup Y_{\sigma^2}$ is a $C^1(M)$ surface which minimizes area bounded by $\gamma(t, \epsilon)$. This fact follows by general minimal surface theory, but we include a brief proof. We claim that $A_{\sigma^2} \cup Y_{\sigma^2}$ is in fact a smooth minimal surface and further $A_{\sigma^2} \cup Y_{\sigma^2} \equiv \bar{Y}(t, \epsilon)$.

To prove that $A_{\sigma^2} \cup Y_{\sigma^2}$ is smooth, we express it as a graph of a function $z$ and show that the derivatives of $z$ exist and are continuous. To this end, let $T_{\sigma^2} \subset M$ be the surface obtained by following geodesics $c_\rho(\theta)$ with $\theta \in \sigma^2$ and initial direction $\frac{\partial}{\partial \theta}c_\rho(0) = \nu_{\sigma^2}(\theta)$; that is $T_{\sigma^2} := \{ p \in M : p = c_\rho(\theta), \text{for some } \rho \in [-1, 1], \theta \in \sigma^2 \}$.

Express $T_{\sigma^2}$ in the natural coordinate system $(\rho, \theta)$. View $A_{\sigma^2}$ as a graph of a function $z = z(\rho, \theta)$ over $T_{\sigma^2}$. Again, we repeat that since $A_{\sigma^2} \cup Y_{\sigma^2}$ is smooth away from $\rho = 0$, to show $A_{\sigma^2} \cup Y_{\sigma^2}$ is smooth, it suffices to show that the second order derivatives of $z(\rho, \theta)$ at $\rho = 0$ are continuous. This follows from standard elliptic regularity; the surfaces $A_{\sigma^2}$ and $Y_{\sigma^2}$ are minimal, hence $z = z(\rho, \theta)$ solves the minimal surface equation

$$\text{div}_g \left( \frac{\nabla z}{||\nabla z||_g} \right) = 0.$$  

Let $(\rho, \theta, z)$ form a local coordinate system near $T_{\sigma^2}$. Since $A_{\sigma^2}$ agrees with $T_{\sigma^2}$ on $\sigma^2$ to first order, $z(0, \theta) = 0$ and $\partial_\rho z(0, \theta) = \partial_\theta z(0, \theta) = 0$. The minimal surface equation for $A_{\sigma^2}$ written in our chosen coordinates is

$$0 = \text{div}_g \left( \frac{\nabla z}{||\nabla z||_g} \right) = \frac{||\nabla z||_g \text{div}_g(\nabla z) - g(\nabla z, g(\nabla \nabla z, \nabla z))}{||\nabla z||_g^2}$$

Substituting $\rho = 0$ into the above equation and using $z(0, \theta) = 0$, $\partial_\rho z(0, \theta) = \partial_\theta z(0, \theta) = 0$, we determine $\nabla_N \nabla_N z$, where $N := \frac{\nabla z}{||\nabla z||_g}$ is the unit normal to $A_{\sigma^2}$ at $\rho = 0$. Likewise $Y_{\sigma^2}$ agrees with $T_{\sigma^2}$ on $\sigma^2$ to first order and is also a minimal surface, so by similar analysis we determine $\nabla_N \nabla_N z$, where $N := \frac{\nabla z}{||\nabla z||_g}$ is the unit normal to $Y_{\sigma^2}$ at $\rho = 0$. In particular, at $\rho = 0$ we have $\nabla_N \nabla_N z = \nabla_N \nabla_N z$.

So $z = z(\rho, \theta, \epsilon)$ is $C^2(\bar{M}(r))$ on $A_{\sigma^2} \cup Y_{\sigma^2}$. Since $A_{\sigma^2}$ agrees with $Y_{\sigma^2}$ at $\sigma^2$ up to second order, we have by elliptic regularity that $A_{\sigma^2} \cup Y_{\sigma^2}$ is smooth.

Now we show that $A_{\sigma^2} \cup Y_{\sigma^2}$ is unique. Since we have $\bar{Y}(t, \epsilon)$ is a unique area minimizer for each $t$, the perturbed minimal surfaces $\bar{Y}(t, \epsilon)$ are unique for $\epsilon$ small enough. Now, by construction $A_{\sigma^2} \cup Y_{\sigma^2}$ is $C^1$-close to $\bar{Y}(t, \epsilon)$, and hence the surface $A_{\sigma^2} \cup Y_{\sigma^2}$ is unique.

In particular, the uniqueness of both $A_{\sigma^2} \cup Y_{\sigma^2}$ and $\bar{Y}(t, \epsilon)$ implies that $A_{\sigma^2} \cup Y_{\sigma^2} \equiv \bar{Y}(t, \epsilon)$. Therefore, $\sigma^2 \equiv c(t, \epsilon)$.  

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b. From part a, for any $t$ we may determine the curves $\gamma(t, \epsilon)$ cut by the intersection $\tilde{Y}(t, \epsilon) \cap \partial M$. In particular, we can find the area of the annulus $\tilde{Y}(t, \epsilon) \setminus (\tilde{Y}(t, \epsilon) \cap M)$. We have

$$\text{Area}(\tilde{Y}(t, \epsilon)) = \text{Area}(\tilde{Y}(t, \epsilon) \setminus (\tilde{Y}(t, \epsilon) \cap M)) + \text{Area}(\tilde{Y}(t, \epsilon) \cap M).$$

Since the metric $\tilde{g}$ is known on $\tilde{Y}(t, \epsilon) \setminus (\tilde{Y}(t, \epsilon) \cap M)$, we may compute this area. Since we assumed the knowledge of any minimal surface $M$, the area of $\tilde{Y}(t, \epsilon) \cap M$ is known. Therefore, $\text{Area}(\tilde{Y}(t, \epsilon))$ is known.

Proposition 1.10. Let $(M, g)$ be a $C^4$-smooth, Riemannian manifold which is homeomorphic to a 3-dimensional ball in $\mathbb{R}^3$, and has mean convex boundary $\partial M$. Let $g$ be given on $\partial M$. Let $\gamma$ be a given simple closed curve on $\partial M$, and set $Y_\gamma \subset M$ to be a surface of least area bounded by $\gamma$. Suppose that the stability operator on $Y_\gamma$ is non-degenerate, and that for $\gamma$ and any nearby perturbation $\gamma(s)$, the area of the least-area surface $Y_{\gamma(s)}$ enclosed by $\gamma(s)$ is known.

Equip a neighbourhood of $Y_\gamma$ with coordinates $(x^\alpha)$ such that on $Y_\gamma$, $x^3 = 0$ and $(x^1, x^2)$ are isothermal coordinates. Then,

1. the first and second variations of the area of $Y_\gamma$ determine the Dirichlet-to-Neumann map associated to the boundary value problem

$$\nabla g_0 \psi + e^{2\phi} (\text{Ric}_g(\vec{n}, \vec{n}) + ||A||_g^2) \psi = 0,$$

$$\psi|_{\partial Y_\gamma} = g\left(\frac{d}{ds} \gamma(s) \bigg|_{s=0}, \vec{n}\right),$$

on $Y_\gamma$, where $e^{2\phi}g_\mathbb{E} = e^{2\phi}[(dx^1)^2 + (dx^2)^2]$ is the metric on $Y_\gamma$ in the coordinates $(x^1, x^2)$.

2. Knowledge of the first and second variations of the area of $Y_\gamma$ determines any solution $\psi(x)$ to the above boundary value problem with given boundary data $\psi|_{\partial Y_\gamma}$, in the above isothermal coordinates.

Proof. Without loss of generality, set $t = 0$. Let $g_0$ be the metric $g$ restricted to $Y(0)$. From Lemma 2.9, the minimal area data enables us to find the Dirichlet-to-Neumann map $\Lambda_{g_0}(\psi) := g(\nabla \psi, \nu)|_{\partial \Omega}$ associated to the boundary value problem for the stability operator

$$\Delta_{g_0} \psi + \left(\text{Ric}_g(\vec{n}, \vec{n}) + ||A||_g^2\right) \psi = 0$$

$$\psi|_{\partial \Omega} = \psi_0$$

on $\hat{\Omega}$, \hspace{1cm} (2.31)

$$\text{on } \partial \hat{\Omega}. \hspace{1cm} (2.32)$$

In our chosen coordinate system, the metric $g_0$ pulled back to $\hat{\Omega}$ takes the form $g_0 = e^{2\phi}g_\mathbb{E}$. In these coordinates, the problem \hspace{1cm} (2.31) is transformed to

$$\Delta_{g_0} \psi + e^{2\phi} \left(\text{Ric}_g(\vec{n}, \vec{n}) + ||A||_g^2\right) \psi = 0$$

$$\psi|_{\partial \Omega} = \psi_0$$

on $\hat{\Omega}$, \hspace{1cm} (2.33)

$$\text{on } \partial \hat{\Omega}, \hspace{1cm} (2.34)$$
and the solutions \( \psi(x) \) of (2.31) are the same as the solutions \( \psi(x) \) of (2.33).

Using the isothermal coordinates \((x^1, x^2)\),

\[
\int_{\partial \tilde{\Omega}} \psi \Lambda_{g_0}(\psi) \, dS := \int_{\partial \tilde{\Omega}} \psi \nu(\psi) \, dS \\
= \int_{\partial \tilde{\Omega}} \psi e^{-\phi} \nu(\psi) e^\phi \, d\tilde{S} \\
= \int_{\partial \tilde{\Omega}} \psi \Lambda_{g_0}(\psi) \, d\tilde{S}
\]

where \( \nu = e^{-\phi} \nu \) is the unit outward pointing normal with respect to the Euclidean metric \( g_E \), and \( \nabla \) denotes the gradient of \( \psi := \psi(x^1, x^2) \) with respect to the metric \( g_E \).

By polarizing, the knowledge of the area of any minimal surface in \( M \) has determined the Dirichlet-to-Neumann map \( \Lambda_{g_E} \) associated to the Schrödinger equation in (2.33), with respect to the Euclidean metric.

Employing the result in [12] for linear Schrödinger equations, the Dirichlet-to-Neumann map \( \Lambda_{g_E} \) determines the potential

\[
e^{2\phi} \left( \text{Ric}_g(\tilde{n}, \tilde{n}) + ||A||^2_g \right)
\]

on \( \tilde{\Omega} \). Now that we know this potential in coordinates \((x^1, x^2)\), all solutions \( \psi(x^1, x^2) \) to the Dirichlet problem (2.33) are known.

\[\square\]

3 Equations for the Components of the Inverse Metric

**Proposition 3.1.** Let \((M, g)\) be a \( C^4 \)-smooth, Riemannian manifold. Let \( \gamma(t) \) be a foliation of \( \partial M \) by simple closed curves. Suppose that \((M, g)\) admits a non-degenerate foliation by properly embedded, area-minimizing surfaces \( Y(t) \) with \( \partial Y(t) = \gamma(t) \). Further, suppose that for \( \gamma(t) \) and any nearby perturbation \( \gamma(s, t) \subset \partial M \), the area of the least-area surface \( Y(s, t) \) enclosed by \( \gamma(s, t) \) is known.

As in section 2.1, extend \((M, g)\) to an asymptotically flat manifold \((M, g)\) and extend each \( Y(t) \) smoothly to \( Y(t) \) in \( M \). Equip \( M \) with coordinates \((x^\alpha), \alpha = 1, 2, 3\) such that \( x^3 = t \) and for each \( t \) fixed \( x^1, x^2 : Y(t) \to \mathbb{R} \) are the unique conformal coordinates given by Proposition 2.1.

In these coordinates, \( g^{33} := ||\nabla x^3||_g^2 \), may be recovered on \( M \) from the area data.

**Proof.** Recall \( \nabla x^3 := \text{grad}(x^3) \). Set \( \tilde{n} := \frac{\nabla x^3}{||\nabla x^3||_g} \) to be a unit normal vector field on \( Y(t) \) for \( t \in (-1, 1) \), and write \( g_t \) for the restriction of the metric \( g \) to the surface \( Y(t) \).

For each fixed \( t \), we may view the nearby leaves of the foliation \( Y(t + \delta t) \) as a variation of \( Y(t) \) by area-minimizing surfaces. From this viewpoint, the variation is captured by the vector field

\[
\frac{\partial}{\partial x^3} := \partial_3.
\]
The associated lapse function is
\[
g(\partial_3, \vec{n}) = g\left(\partial_3, \frac{\nabla x^3}{||\nabla x^3||_g}\right) = ||\nabla x^3||_g.
\]

Recall \(x^k : Y(t) \to \mathbb{R}, k = 1, 2,\) are conformal on \(Y(t)\). Since \(||\nabla x^3||_g : Y(t) \to \mathbb{R}\) is a nontrivial solution of the Jacobi equation
\[
\Delta_{g_t} \omega + (\text{Ric}_g(\vec{n}, \vec{n}) + ||A||^2_g) \omega = 0 \tag{3.35}
\]
on \(Y(t)\) (see appendix A.3), the stability operator \(\Delta_{g_t} + (\text{Ric}_g(\vec{n}, \vec{n}) + ||A||^2_g)\) is non-degenerate for each \(t\).

Therefore, written in the coordinates \((x^\alpha)\), for \(x^3 = t\) fixed, the function \(||\nabla x^3||_g\) solves
\[
\Delta_{g_3} \psi + e^{2\phi(t)} (\text{Ric}_g(\vec{n}, \vec{n}) + ||A||^2_g) \psi = 0 \tag{3.36}
\]
on \(Y(t)\), where the metric on \(Y(t)\) is expressed as \(g_t = e^{2\phi(t)}[(dx^1)^2 + (dx^2)^2] =: e^{2\phi(t)}g_E\).

Now, we know \(g|_{\partial M}\) and the curves \(\partial Y(t) = (x^3)^{-1}(t) \cap \partial M\). Thus, the function \(||\nabla x^3||_g\) on \(\partial Y(t)\) is known. By Proposition \[\ref{proposition:existence-lapse} \] we determine the lapse function \(||\nabla x^3||_g\) on \(Y(t)\), in the conformal coordinate system given by \((x^1, x^2, x^3 = t)\). Since we now know \(||\nabla x^3||_g\) on \(Y(t)\) for any \(t \in (-1, 1)\), we have determined \(||\nabla x^3||_g\) on \(M\).

We have
\[
g^{33} := g(dx^3, dx^3) = ||\nabla x^3||_g^2,
\]
in the chosen coordinates \((x^\alpha), \alpha = 1, 2, 3\). Hence, in these coordinates, the metric component \(g^{33}\) is known on \(M\).

\[\qed\]

**Lemma 3.2.** Let \((M, g)\) be a \(C^4\)-smooth, Riemannian manifold. Let \(\gamma(t)\) be a foliation of \(\partial M\) by simple closed curves. Suppose that \((M, g)\) admits a foliation by properly embedded, area-minimizing surfaces \(Y(t)\) with \(\partial Y(t) = \gamma(t)\). Further, suppose that for \(\gamma(t)\) and any nearby perturbation \(\gamma(s, t) \subset \partial M\), the area of the least-area surface \(Y(s, t)\) enclosed by \(\gamma(s, t)\) is known.

Extend \(M\) and each \(Y(t)\) to asymptotically flat manifolds \(\tilde{M}\) and \(\tilde{Y}(t)\) as defined in section \[\ref{section:asymptotically-flat} \]. Further, set \((x^1, x^2) = \Phi(\cdot, t) : Y(t) \to \mathbb{R}^2\) to be unique isothermal coordinates on \(Y(t)\) given by Proposition \[\ref{proposition:isothermal-coordinates}\]. Write \(\tilde{\Omega}(t) := \Phi(Y(t))\). Set \(x^3 = t\).

Consider a point \(p \in Y(t)\). Let \(h : [0, S] \times \tilde{\Omega}(t) \to \tilde{M}, h(s, x^1, x^2, t) = Y(s, t) = \gamma(t)\) be a variation of \(Y(t) \subset \tilde{M}\) by properly embedded, area-minimizing surfaces which has the property that the component of \(h_* \left( \frac{\partial}{\partial t}|_{s=0} \right)\) projected onto the normal vector field to \(Y(t)\), denoted by \(\psi_p = \psi_p(x^1, x^2)\), vanishes at \(p\). Set \((x^1_s, x^2_s) = \Phi(\cdot, s, t) : Y(s, t) \to \mathbb{R}^2\) to be the unique isothermal coordinates on the extended, new foliation \(Y(s, t)\).

Then, the linearization of \(||\nabla x^3||_g\) at the point \(p\) is
\[
\frac{d}{ds} ||\nabla x^3||_g(p) \bigg|_{s=0} = g^{3\alpha}(p) \partial_\alpha \psi_p(x^1(p), x^2(p)) + \partial_\alpha ||\nabla x^3||_g(p) x^\alpha(p), \tag{3.37}
\]

\[\footnote{It is not always the case that such a variation exits. We do not prove the existence here, but later in Section 4.}\]
where \( \dot{x}^\alpha := \frac{d}{ds} x^k_{s=0} \) is the first variation in the coordinate functions \( x^\alpha_s \) at \( p \), for \( \alpha = 1, 2, 3 \).

Moreover, the quantity \( \frac{d}{ds} \|\nabla x^3_s\|_g(p) \big|_{s=0} \) is known in the coordinates \((x^\alpha), \alpha = 1, 2, 3\).

**Figure 6:** Depiction of the leaves \( Y(s,t) \).

**Proof.** Let \( \vec{n} := \frac{\nabla x^3_s}{\|\nabla x^3_s\|_g} \) denote the unit normal vector field to \( Y(t) \).

Via Taylor expansion, the new coordinate functions \((x^\alpha_s), \alpha = 1, 2, 3 \) on \( Y(s,t) \) in terms of the “original” coordinate functions \((x^\alpha) \) on \( Y(t) \) are expressed as

\[
x^\alpha_s = x^\alpha + s \dot{x}^\alpha + \mathcal{O}(s^2).
\]

Then, linearizing \( \|\nabla x^3_s\|_g^2(p) \) about \( s = 0 \),

\[
\frac{d}{ds} \bigg|_{s=0} \|\nabla x^3_s\|_g^2(p) = \frac{d}{ds} \bigg|_{s=0} \left[ \left( \|\nabla x^3_s\|_g^2 + 2sg(\nabla x^3_s, \nabla \dot{x}^3_s) \right) \circ (p) + \mathcal{O}(s^2) \right]
\]

\[
= 2g(\nabla x^3_s, \nabla \dot{x}^3_s)(p) + \partial_\alpha \|\nabla x^3_s\|_g^2(p) \dot{x}^\alpha(p)
\]

\[
= 2g^{3\alpha}(p) \partial_\alpha \|\nabla x^3_s\|_g^2(p) \dot{x}^\alpha(p)
\]

\[
= 2\|\nabla x^3_s\|_g^2(p) g^{3\alpha}(p) \partial_\alpha \psi_p(x^1(p), x^2(p)) + \partial_\alpha \|\nabla x^3_s\|_g^2(p) \dot{x}^\alpha(p),
\]

at the chosen point \( p \).

Now, by Proposition 3.1, the function \( \|\nabla x^3_s\|_g^2(p) \) is known on \( M \) for \( s \geq 0 \). Hence \( \frac{d}{ds} \bigg|_{s=0} \|\nabla x^3_s\|_g^2(p) \) is known.

Since the left hand side equation (3.37) is known, and the function \( \psi_p \) as defined above is known from Proposition 1.10, we would like to use this equation to solve for \( g^{3k}, k = 1, 2 \). However, solving equation (3.37) is complicated due to the presence of the terms containing \( \dot{x}^k, k = 1, 2 \). It will thus serve our purposes to find an expression for \( \dot{x}^k \) in terms of \( g^{13}, g^{23} \) and \( \phi \). The calculations for such an expression are carried out below.

**Lemma 3.3.** Let \((M,g), \bar{\Omega}(t), Y(t), Y(s,t), p \in Y(t)\), and \( \psi_p : \bar{\Omega}(t) \to \mathbb{R} \) be as defined in Lemma 3.2. For \( \alpha = 1, 2, 3 \), let \( x^\alpha : Y(t) \to \mathbb{R} \) and \( x^\alpha_s : Y(s,t) \to \mathbb{R} \) be the coordinate systems...
on $Y(t)$ and $Y(s,t)$, as defined in Lemma 3.2 and write $\dot{x}^k : Y(t) \to \mathbb{R}$, $k = 1, 2$ for the first order change in the isothermal coordinates $x^k_s$.

Then on $Y(t)$, for given variations of $\partial Y(t)$ the functions $\dot{x}^k$, $k = 1, 2$ are determined via a Poisson equation

$$\Delta_{g_s} \dot{x}^k = \mathcal{F}^k(g_1^{13}, g_2^{23}, \phi, \psi_p, d\psi_p, p),$$

where $\mathcal{F}^k$ is given explicitly below in (3.43) and (3.44), $\phi = \phi(x^1, x^2, t)$ is the conformal factor on $Y(t)$, and $\mathcal{F}^k$ is a second order differential operator acting on $g_1^{13}$, $g_2^{23}$, and $\phi$.

**Proof.** Without loss of generality, fix $t = 0$ and consider $Y(0)$. Write $g_0$ for the metric induced by $g$ on $Y(0)$, and $g_{s,0} := g|_{Y(s,0)}$ for the metric induced on the leaves $Y(s,0)$. Recall from Lemma 3.2 we express the foliation $Y(s,0)$ as embeddings $h : [0, S] \times \tilde{\Omega}(0) \to M$ into the extension $\tilde{M}$ of $M$; that is, $h(s, x^1, x^2, 0) = Y(s,0)$.

The equation (3.38) which expresses $x^k_s$ in terms of $x^k$ is

$$x^k_s = x^k + s \dot{x}^k + O(s^2).$$

Now, to compute how $\dot{x}^k$ depends on the components of the metric $g$, we linearize $x^k_s$ in $s$.

The conformal coordinates $(x^k_s)$ on the leaves $Y(s,0)$ are harmonic functions, and thus satisfy

$$\Delta_{g_{s,0}} x^1_s = 0 = \Delta_{g_{s,0}} x^2_s.$$
Linearizing about \( s = 0 \) and noting \( \partial_j x^k_s = \delta^k_j \), we derive

\[
0 = \frac{d}{ds}[\Delta_{g_{s,0}} x^k_s]
\]

\[
= \left[ \frac{dg_{ij}^s}{ds} \partial_i \partial_j x^k_s - \frac{d}{ds}[g_{ij}^s \Gamma^k_{ij}(g_{s,0})] + \Delta_{g_{s,0}} \frac{d}{ds} x^k_s + O(s) \right]_{s=0}
\]

\[
= 0 - (g_0)^{ij} \Gamma^k_{ij}(g_0) - g_{0,0}^{ij} \Gamma^k_{ij}(g_0) + \Delta_{g_{0,0}} \frac{d}{ds} x^k_s |_{s=0}
\]

\[
= -(g_0)^{ij} \Gamma^k_{ij}(g_0) - \frac{1}{2} g_{0,0}^{ij} k^l \nabla_j (g_0)_{il} + \nabla_i (g_0)_{jl} - \nabla_l (g_0)_{ij} + \Delta_{g_{0,0}} x^k
\]

\[
= -\hat{g}^{ij} \Gamma^k_{ij}(g_0) - g_{0,0}^{ij} \nabla_j (g_0)_{kl} + \frac{1}{2} g_{0,0}^{kl} \nabla_l (g_0)_{ij} + \Delta_{g_{0,0}} x^k,
\]

on \( Y(0) \), for \( k = 1, 2 \).

To perform further analysis, we require an expression for the linearization \( \hat{g}_0 \) of the induced metric on the leaves \( Y(s, 0) \), as well as the Christoffel symbols associated to the metric \( g_{0,0} := g_0 \) on \( Y(0) \).

In all computations which follow, let \( i, j, k, l, m, p \) sum over \( 1, 2 \) and \( \alpha, \beta, \gamma \) sum over \( 1, 2, 3 \).  In the coordinates \((x^\alpha)\), \( g_0 = e^{2\phi} g_\xi \).  Hence, the Christoffel symbols of \( g_0 \) are

\[
\Gamma^k_{ij}(g_0) = \Gamma^k_{ij}(e^{2\phi} g_\xi)
\]

\[
= \Gamma^k_{ij}(g_\xi) + g_\xi^{k,ij} \partial_i \phi + g_\xi^{k,ij} \partial_j \phi - (g_\xi)_{ij} g_\xi^{kl} \partial_l \phi
\]

\[
= g_\xi^{k,ij} \partial_i \phi + g_\xi^{k,ij} \partial_j \phi - (g_\xi)_{ij} g_\xi^{kl} \partial_l \phi.
\]

For ease of computation of the linearization \( \hat{g}_0 \), we employ Gaussian coordinates adapted to \( Y(0) \): for \( i = 1, 2 \), define the coordinate vector fields \( X_i := h(\cdot, s, 0)_s (\frac{\partial}{\partial x^i}) \) and \( X_s := h(\cdot, s, 0)_s (\frac{\partial}{\partial s}) \). Then in these coordinates, the components of the metric \( g_{s,0} \) induced on the leaves \( Y(s, 0) \) are given by \((g_{s,0})_{ij} := g(X_i, X_j) \).

Now Taylor expand \( g_{s,0} \) in terms of \( s \): \( g_{s,0} = g_0 + s\hat{g}_0 + O(s^2) \). Then,

\[
(\hat{g}_0)_{ij} := \frac{d}{ds}(g_{s,0})_{ij} \bigg|_{s=0} = \frac{d}{ds}g(X_i, X_j) \bigg|_{s=0}
\]

\[
= [g(\nabla_{X_s} X_i, X_j) + g(X_i, \nabla_{X_s} X_j)]_{s=0}
\]

\[
= g(\nabla_{X_i} (\psi_p \vec{n}), X_j) + g(X_i, \nabla_{X_j} (\psi_p \vec{n}))
\]

\[
= -2\psi_p A_{ij},
\]

where \( A_{ij} \) are the components of the second fundamental form of \( Y(0) \). Thus, the first order change in \( g_{s,0} \) is given by the coordinate free expression

\[
\hat{g}_0 = -2\psi_p A.
\]

Recall \( g_{0,0} = g_0 \), and substitute \( \hat{g}_0 = -2\psi_p A \) into equation (3.39); the resulting PDE describes the first variation in \( s \) of the coordinates \((x^i)\):

\[
\Delta_{g_0} x^k = -2\psi_p A^{ij} \Gamma^k_{ij}(g_0) - 2g_{0}^{ij} \nabla_j (g_0)_{kl} \psi_p A_{il} + g_{0}^{kl} \nabla_l (g_0)_{ij} \psi_p A_{kl}
\]

\[
= -2\psi_p A^{ij} \Gamma^k_{ij}(g_0) - 2g_{0}^{ij} \nabla_j (\psi_p A^k_i).
\]

(3.40)

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Note the term $g_0^{kl}\nabla_l(g_0^{ij} \psi_p A_{ij})$ is zero since the surface $Y(0)$ is minimal.

We now expand each term in equation (3.40) in terms of the components of $g$ and $g^{-1}$ which we aim to uniquely determine. To this end, a quick calculation gives that in the coordinates $(x^\alpha)$, the normal vector field to the leaves $Y(t)$ is

$$\vec{n} := \frac{\nabla x^3}{||\nabla x^3||_g} = \frac{1}{||\nabla x^3||_g} g^{\alpha \beta} \partial_\beta x^3 \partial_\alpha = \frac{1}{||\nabla x^3||_g} g^{\alpha \beta} \partial_\alpha.$$

Hence the components of the second fundamental form are

$$A_{ij} = -\frac{1}{2||\nabla x^3||_g} g^{3\alpha}(\partial_\alpha g_{ij} + \partial_j g_{i\alpha} - \partial_i g_{ij}).$$

Raising an index and noting $-g_{ij} \partial_i g^{3\alpha} = \partial_i g_{ij} g^{3\alpha}$ then gives

$$A^k_i = -\frac{e^{-2\phi}(g_\phi)^{jk}}{2||\nabla x^3||_g} (g_{ij} \partial_j g^{3\alpha} + g_{io} \partial_j g^{3\alpha} + g^{3\alpha} \partial_\alpha g_{ij}).$$

(3.41)

So we calculate a factor of the first term in (3.40) to be

$$-2||\nabla x^3||_g g_0^{im} A^j_m \Gamma^k_{ij}(g_0) = e^{-2\phi} \cdot (g_\phi)^{im} \cdot e^{-2\phi}(g_\phi)^{jl} \cdot (g_{io} \partial_j g^{3\alpha} + g_{mo} \partial_j g^{3\alpha} + g^{3\alpha} \partial_\alpha g_{jl}).$$

$$= 2e^{-4\phi} \left\{ g_\phi^{km} g_\phi^{jl} g_{3m} \partial_m g^{33} \partial_j \phi + g_\phi^{im} h^{k} g_\phi^{jl} g_{3m} \partial_m g^{33} \partial_j \phi \right. $$

$$\left. - g^{k}_\phi g_\phi^{jl} g_{3m} \partial_m g^{33} e^{2\phi} \partial_j \phi + g_\phi^{km} e^{2\phi} \partial_m g^{3j} \partial_j \phi \right. $$

$$+ g^{km} e^{2\phi} \partial_m g^{3k} \partial_i \phi - g^{km} \partial_m g^{3m} e^{4\phi} \partial_j \phi + 0 \right\}.$$

For the second term in (3.40), using (3.41), observe the partial coordinate derivatives of the components of the second fundamental form are

$$2\partial_j(A^k_i) = -2\partial_j \phi A^k_i - \frac{1}{||\nabla x^3||_g} \partial_j ||\nabla x^3||_g A^k + \frac{e^{-2\phi}(g_\phi)^{km}}{||\nabla x^3||_g} \left\{ \partial_j h^{km} \partial_i g^{3\alpha} + g_{om} \partial_j g^{3\alpha} \partial_\alpha \phi \right. $$

$$\left. + \partial_j h^{km} \partial_i g^{3\alpha} + g_{om} \partial_j g^{3\alpha} + 2e^{2\phi}(g_\phi)^{im} \partial_j g^{3\alpha} \partial_\alpha \phi \right. $$

$$\left. + 2e^{2\phi}(g_\phi)^{im} g^{3\alpha} \partial_j \partial_\alpha \phi - 4e^{2\phi}(g_\phi)^{im} g^{3\alpha} \partial_\alpha \phi \partial_j \phi \right\}.$$

Substituting the expressions for $\partial_j(A^k_i)$ and $g^{im} A^j_m \Gamma^k_{ij}(g_0)$ above into equation (3.40), the equation for the first order change in conformal coordinates is given below (indices run over...
values $\alpha \in \{1, 2, 3\}$ and $i, j, k, l, m \in \{1, 2\}$.

$$\Delta_{g_0} \dot{x}^k = -2\psi_p A^{ij}_{\alpha} \Gamma^k_{ij}(g_0) - 2g_0^{ij} \nabla_j(\psi_p A^k)$$

$$= -2\psi_p g^{im} A^m_{ij} \Gamma^k_{ij}(g_0) - 2g_0^{ij} \nabla_j(\psi_p) A^k_i - 2g_0^{ij} \psi_p \partial_j A^k_i - \Gamma^m_{ij}(g_0) A^k_m + \Gamma^k_{mj}(g_0) A^m_i$$

$$= -2g_0^{ij} \nabla_j(\psi_p) A^k_i - 2\psi_p g_0^{ij} \partial_j A^k_i - 4\psi_p g^{im} A^k_m \Gamma^k_{ij}(g_0)$$

$$= -g_0^{ij} \nabla_j(\psi_p) e^{-2\phi/(g_E)} \frac{\nabla^k g_{ij}}{||\nabla x^3||_g} (g_{ij} \partial_i \partial_j g^{3\alpha} + g_{ij} \partial_i g^{3\alpha} + g^{3\alpha} \partial_i g_{ij})$$

$$+ g_0^{ij} \psi_p \partial_j \phi e^{-2\phi/(g_E)} \frac{1}{||\nabla x^3||_g} \left( \partial_j g_0 \partial_i g^{3\alpha} + g_0 \partial_i g^{3\alpha} + g^{3\alpha} \partial_i g_{ij} \right)$$

$$+ g_0^{ij} \psi_p \partial_j \phi e^{-2\phi/(g_E)} \frac{\nabla^k g_{ij}}{||\nabla x^3||_g} \left\{ \partial_j g_0 \partial_i g^{3\alpha} + g_0 \partial_i g^{3\alpha} + \partial_j g_0 \partial_i g_{ij} + g_0 \partial_i \partial_j g_{ij} \right\}$$

$$- g_0^{ij} \psi_p \partial_j \phi e^{-2\phi/(g_E)} \frac{\nabla^k g_{ij}}{||\nabla x^3||_g} \left\{ \partial_j g_0 \partial_i g^{3\alpha} + g_0 \partial_i g^{3\alpha} + \partial_j g_0 \partial_i g_{ij} + g_0 \partial_i \partial_j g_{ij} \right\}$$

We may express this complicated PDE schematically as

$$\Delta_{g_0} \dot{x}^k = e^{-2\phi} \psi_p A^{ijk}_{\alpha} \partial_l \partial_j g^{3\alpha} - e^{-2\phi} \psi_p B^{ijk}_{\alpha} \partial_l \partial_i \phi = e^{-2\phi} \psi_p C^{ijk}_{\alpha} \partial_l \partial_j g^{3\alpha}$$

$$+ e^{-2\phi} \psi_p D^{ijk}_{\alpha} \partial_l g^{3\alpha} \partial_j g^{3\alpha} + e^{-2\phi} \psi_p E^{ijk}_{\alpha} \partial_l \partial_i \phi + e^{-2\phi} \psi_p F^{ijk}_{\alpha} \partial_l \partial_j g^{3\alpha}$$

$$\Delta_{g_0} g_0 = e^{-2\phi} \Delta_{g_0} g_0;$$

so on $Y(0)$ we have the equation

$$\Delta_{g_0} x^k = F^k(g_0, \phi, \psi_p, p)$$

where the differential operator $F^k(g_0, \phi, \psi_p, p)$ is defined in (3.43).

Equation (3.44) together with equations of the form of (3.37) will allow us to solve for the metric components, $g^{2k}$, $k = 1, 2$, in terms of the conformal factor, $\phi$. Thus we require one more equation to ultimately determine all components of the metric. Such an equation will be provided by a transport-type equation for the conformal factor $\phi = \phi(x^1, x^2, t)$ on the leaf $Y(t)$, which we derive in the next proposition.
Proposition 3.4. Let $(M, g)$ be a $C^4$-smooth, compact, 3-dimensional Riemannian manifold and let $\gamma(t)$ be a foliation of $\partial M$ by simple closed curves. Suppose that $(M, g)$ admits a foliation by properly embedded, area-minimizing surfaces $Y(t)$ with $\partial Y(t) = \gamma(t)$. Further, suppose that for $\gamma(t)$ and any nearby perturbation $\gamma(s, t) \subset \partial M$, the area of the least-area surface $Y(s, t)$ enclosed by $\gamma(s, t)$ is known.

Extend $M$ and each $Y(t)$ to asymptotically flat manifolds $M$ and $Y(t)$ as defined in section 2.1 Further, set $(x^1, x^2) = \Phi(\cdot, t) : Y(t) \rightarrow \mathbb{R}^2$ to be unique isothermal coordinates on $Y(t)$ given by Proposition 2.1. Set $x^3 = t$.

Then, the evolution in $x^3$ of the conformal factor $\phi = \phi(x^1, x^2, t)$ is described by the transport-type equation

$$g^{31}\partial_1\phi + g^{32}\partial_2\phi + g^{33}\partial_3\phi + \frac{1}{2}\partial_k g^{3k} - \frac{1}{2}g^{3k}\partial_k \log(g^{33}) = 0,$$

where $g_k := e^{2\phi}g_{x^k}$ is the metric on the leaf $Y(t)$ in the coordinate system $(x^\alpha)$.

**Proof.** Recall the mean curvature of $Y(t)$ is the trace of the second fundamental form: $H(Y(t)) := A^i_i$ (we do not average over the dimension).

As demonstrated in the proof of Lemma 3.3, equation (3.41), the second fundamental form may be written as

$$A^i_i = -\frac{e^{-2\phi}}{2||\nabla x^3||_g}(g_{x^i j} \partial_j g^{3\alpha} + g_{\alpha j} \partial_j g^{3\alpha} + g^{3\alpha} \partial_\alpha g_{ij}),$$

in the coordinates $(x^\alpha)$, $\alpha = 1, 2, 3$.

Therefore the mean curvature of $Y(t)$ is given by

$$H := A^i_i = -\frac{e^{-2\phi}}{||\nabla x^3||_g}(g_{x31} \partial_1 g^{33} + e^{2\phi} \partial_k g^{3k} + g_{x32} \partial_2 g^{33} + 2e^{2\phi} g^{3\alpha} \partial_\alpha \phi),$$

where $k$ sums over 1, 2.

Since $Y(t)$ is minimal for each $t \in \mathbb{R}$, $H(Y(t)) = 0$ provides the differential equation

$$0 = e^{-2\phi}(g_{x31} \partial_1 g^{33} + g_{x32} \partial_2 g^{33}) + (\partial_k g^{3k} + 2g^{3\alpha} \partial_\alpha \phi)$$

which we rewrite as

$$g^{31} \partial_1 \phi + g^{32} \partial_2 \phi + g^{33} \partial_3 \phi + \frac{1}{2}\partial_k g^{3k} - \frac{e^{-2\phi}}{2}(g_{x31} \partial_1 g^{33} + g_{x32} \partial_2 g^{33}) = 0. \tag{3.46}$$

As shown in the appendix, we can express the components $g_{x31}, g_{x32}$ in terms of the components of the inverse metric as

$$g_{31} = -\frac{g^{31}}{g^{33}} e^{2\phi}, \quad g_{32} = -\frac{g^{32}}{g^{33}} e^{2\phi}.$$

Substituting the above into equation (3.46), we obtain

$$g^{31} \partial_1 \phi + g^{32} \partial_2 \phi + g^{33} \partial_3 \phi + \frac{1}{2}\partial_k g^{3k} - \frac{1}{2}g^{31} \partial_1 \log(g^{33}) + g^{32} \partial_2 \log(g^{33}) = 0. \tag{3.45}$$

**Remark.** Notice that if $g^{13}, g^{23},$ and $g^{33}$ were known functions on $Y(t)$, then equation (3.45) reduces to a simple first order, linear differential equation for the conformal factor $\phi$, which can be easily solved.
4 Proof of the Main Theorems

In this section, we prove the main theorems stated in the introduction. We first prove:

**Theorem 1.4.** Let \((M, g)\) be a manifold of Class 1 or Class 2, and \(g_{|\partial M}\) be given. Let \(\{\gamma(t) : t \in (-1,1)\} = \partial M\) and \(\{Y(t) : t \in (-1,1)\} = M\) be as in Definition 1.3. Suppose that for each curve \(\gamma(t)\) and any nearby perturbation \(\gamma(s, t) \subset \partial M\), we know the area of the properly embedded surface \(Y(s, t)\) which solves the least-area problem for \(\gamma(s, t)\).

Then the knowledge of these areas uniquely determines the metric \(g\) (up to isometries which fix the boundary).

4.1 Proof of Theorem 1.4

**Proof of Theorem 1.4.** To show \(g_1\) is isometric to \(g_2\) on \(M\), we construct coordinate systems on \((M, g_1)\) and \((M, g_2)\), and explicitly construct a diffeomorphism \(F : (M, g_1) \to (M, g_2)\) which maps one coordinate system to the other. In this setting, we prove that the components of the inverses of the metrics \(g_1\) and \(F^*(g_2)\) satisfy

\[g_1^{\alpha\beta} - F^*(g_2)^{\alpha\beta} = 0.\]

This equation implies our uniqueness result.

4.1.1 Construction of the diffeomorphism \(F : (M, g_1) \to (M, g_2)\):

As in section 2.1, extend \((M, g_1)\) to an asymptotically flat manifold \((\tilde{M}, g_1) := (\mathbb{R}^2 \times (-1,1), g_1)\). Smoothly extend each leaf \(Y_1(t)\) to an asymptotically flat manifold \(\tilde{Y}_1(t)\) as defined in section 2.1. Further, set \((x^1, x^2) = \Phi_1(\cdot, t) : \tilde{Y}_1(t) \to \mathbb{R}^2\) to be unique isothermal coordinates on \(Y_1(t)\) given by Proposition 2.1. Write \(\tilde{\Omega}_1(t) = \Phi_1(Y_1(t))\). Set \(x^3 = t\).

Let \(Y_2(t), t \in (-1,1)\), be a foliation of \((M, g_2)\) by properly embedded, area minimizing surfaces which is found by solving the least-area problem for \(\gamma(t)\). As in section 2.1, we also extend \((M, g_2)\) to an asymptotically flat manifold \((\tilde{M}, g_2) := (\mathbb{R}^2 \times (-1,1), g_2)\) and smoothly extend each leaf \(Y_2(t)\) to an asymptotically flat manifold \(\tilde{Y}_2(t)\). As above, we set \((y^1, y^2) = \Phi_2(\cdot, t) : \tilde{Y}_2(t) \to \mathbb{R}^2\) to be unique isothermal coordinates on \(Y_2(t)\) given by Proposition 2.1. Write \(\tilde{\Omega}_2(t) := \Phi_1(Y_2(t))\). Set \(y^3 = t\).

Then, define

\[F : (M, g_1) \to (M, g_2)\]

\[F(p) = \Phi_2^{-1} \circ \Phi_1(p).\]

From Proposition 2.3 in section 2, we know \(\tilde{\Omega}_1(t) = \tilde{\Omega}_2(t)\) for all \(t \in (-1,1)\) and \(F = \text{Id}\) on \(\tilde{M} \setminus \partial M\). The restriction of \(F\) to \((M, g_1)\) is then our desired diffeomorphism.

**Notation:** Abusing notation, we write \(g_2\) for the pulled-back metric \(F^*(g_2)\) on \(M\) in all that follows. Note that in the \((x^\alpha)\) coordinates, the metrics \(g_1\) and \(g_2\) take the form

\[g_1 = (g_1)_{3\alpha} dx^3 dx^\alpha + e^{2\phi_1(t)} [(dx^1)^2 + (dx^2)^2]\]

\[g_2 = (g_2)_{3\alpha} dx^3 dx^\alpha + e^{2\phi_2(t)} [(dx^1)^2 + (dx^2)^2].\]
Further, since Proposition 2.3 shows our area data determines $\tilde{\Omega}_1(t) = \tilde{\Omega}_2(t)$ for all $t \in (-1,1)$, by choosing a new family of conformal maps, we may assume that $\tilde{\Omega}_1(t) = \tilde{\Omega}_2(t) = D(r(t))$ for some disc of radius $r(t) > 0$, where $r(t)$ is chosen via the requirement:

$$4\pi r^2(t) = \text{Area}[Y(t)].$$

(We write $r$ instead of $r(t)$ below for simplicity of notation). We note that given the regularity assumptions on the boundaries of $Y(t)$, the conformal factors corresponding to the two new conformal maps satisfy the same bounds as the conformal factor over the domain $\tilde{\Omega}_1(t) = \tilde{\Omega}_2(t)$, in Lemma 2.4 up to a uniformly bounded multiplicative factor.

We make this choice for simplicity in the proofs to follow. Again abusing notation, we still write $(x^1, x^2) = \Phi_1$ and $(y^1, y^2) = \Phi_2$ for the resulting maps to the discs $D(r(t))$.

**Lemma 4.1.** In the coordinate system described above, $g^{\alpha\beta}_1 - g^{\alpha\beta}_2 = 0$ on $M \setminus M$, and $g^{33}_1 = g^{33}_2 = 0$ on $M$.

*Proof.* Since $F = \text{Id}$ on $M \setminus M$ and $g^{\alpha\beta}_1 - g^{\alpha\beta}_2 = 0$ on $M \setminus M$, by the hypothesis on $M$ and the conformal factors corresponding to the two new conformal maps, we may assume that $\tilde{\Omega}_1(t) = \tilde{\Omega}_2(t)$, in Lemma 2.4 up to a uniformly bounded multiplicative factor.

Next, we prove uniqueness for all the remaining metric components by showing the differences $g^{\alpha\beta}_1 - g^{\alpha\beta}_2$ vanish on $M$. First, in the coordinates $(x^\alpha)$, we derive a system of equations for the differences

$$\delta g^{3j} := g^{3j}_1 - g^{3j}_2, \text{ and } \delta \phi := \phi_1 - \phi_2$$

on $M$, $j = 1, 2$. Then, using Lemma 3.2 and Lemma 3.3 we will express $\delta g^{3j}$ as a linear combination of pseudodifferential operators acting on $\delta \phi$ or $\partial_3 \delta \phi$. The requirement that $(M, g_i)$, $i = 1, 2$, are either $C^3$-close to Euclidean or $(K, \epsilon_0, \delta_0)$-thin will play a crucial role here. After this has been achieved, it will suffice to show that $\delta \phi = \phi_1 - \phi_2 = 0$.

We use Proposition 3.4 and the pseudodifferential expressions for $\delta g^{3j}$ to obtain a hyperbolic Cauchy problem for $\delta \phi$. Here too, our assumption that $(M, g_i)$, $i = 1, 2$, are either $C^3$-close to Euclidean or $(K, \epsilon_0, \delta_0)$-thin will be key to obtaining the hyperbolic problem for $\delta \phi$. The desired result $\delta \phi = \phi_1 - \phi_2 = 0$ will then follow from a very standard energy argument.

### 4.1.2 Derivation of a system of equations for the metric components:

First consider the foliation $\{Y_1(t) : t \in (-1,1)\}$ of $(M, g_1)$. Notice that in the coordinates $(x^6)$, the gradient of the function $x^3 : M \to \mathbb{R}$ is

$$\nabla x^3 := g^{\alpha\beta}_1 \partial_\beta x^3 \partial_\alpha = g^{\alpha\beta}_1 \partial_\alpha.$$
Geometrically, the components of the inverse metric $g_1^{3\alpha}$, for $\alpha = 1, 2, 3$, correspond to rescalings of the components of the normal vector field $\vec{n}_1 := \frac{\nabla x^3}{||\nabla x^3||_{g_1}}$ to a leaf $Y_1(t) := \{x^3 = t\}$. Further, we saw above that if we view $Y_1(t + \delta t)$ as a variation of $Y_1(t)$ by area-minimizing surfaces, the lapse function associated to the foliation $\{Y_1(t) : t \in (-1, 1)\}$, is

$$g_1(\partial_3, \vec{n}_1) := ||\nabla x^3||_{g_1} = \sqrt{g_1^{33}}.$$

Below we will consider two chosen variations $\{Y_{i,1}(s,t) : t \in (-1,1), s \in [0,S]\}$, $i = 1, 2$, of the foliation $Y_1(t)$, $t \in (-1,1)$. Knowledge of the areas of the leaves $Y_1(t)$ and $Y_{i,1}(s,t)$ $i = 1, 2$ allows us to recover information about the new lapse function for the new foliation, and we use this to find equations which describe the differences $\delta g^{3k}$.

We construct two variations of $Y_1(t)$ as follows: Consider a point $p \in Y_1(t)$. For $i = 1, 2$, define the maps $h_i : [0,S] \times D(r(t)) \rightarrow M$, $h_i(s,x^1,x^2,t) =: Y_{i,1}(s,t)$ to be a variation of $Y_1(t) \subset M$ by properly embedded, area-minimizing surfaces which has the property that the component of $(h_i)_*\left(\frac{\partial}{\partial s}\big|_{s=0}\right)$ projected onto the normal to $Y(t)$, denoted by

$$\psi_{p,i,1} = g \left((h_i)_*\left(\frac{\partial}{\partial s}\big|_{s=0}\right), \vec{n}_1\right),$$

vanishes at $p$. We further require that $\nabla \psi_{p,i,1}(x(p)) = \frac{\partial}{\partial x^3}$. We write $(x_s^1, x_s^2, s, t) = \Phi_1(\cdot, s, t) : Y_1(s,t) \rightarrow \mathbb{R}^2$ for the unique isothermal coordinates on a smooth, asymptotically flat extension $Y_1(s,t)$ of the new foliation $Y_1(s,t)$ (see Proposition 2.1).

The existence of the desired foliations $Y_{i,1}(s,t) := h_i(s,x_i^1,x_i^2)$, $s \in [0,S]$, $t \in (-1,1)$, is equivalent to the existence of the desired functions $\psi_{p,i,1}$. In particular, the existence of the new foliations $Y_{i,1}(s,t) := h_i(s,x^1,x^2,t)$, $s \in [0,S]$, $t \in (-1,1)$, follows from the (uniform) non-degeneracy of the stability operator on each leaf $Y_i(t)$. In the coordinate system $(x^\alpha)$, $\alpha = 1, 2, 3$, we claim that from Proposition 1.10 for each $t \in (-1,1)$ and for each $p \in Y_1(t) \subset M$, we may construct two distinct, nontrivial solutions $\psi_{p,i,1} \in C^2(\mathbb{R}^2)$, $i = 1, 2$, of the Jacobi equation

$$\Delta_{g_1} \psi_{p,i,1} + e^{2\phi_1} (Ric_{g_1}(\vec{n}, \vec{n}) + ||A||_{g_1}^2) \psi_{p,i,1} = 0$$
on $\mathbb{R}^2$, which additionally satisfy

$$\psi_{p,i,1}(x(p)) = 0,$$

$$\nabla \psi_{p,i,1}(x(p)) = \frac{\partial}{\partial x^3}.$$

Moreover, from the area data for each $Y_1(t)$ and nearby perturbation, Proposition 1.10 implies that these functions $\psi_{p,i,1}$ are known in our chosen coordinates system $(x^\alpha)$, $\alpha = 1, 2, 3$.

In the setting of the first case of Theorem 1.4, we have imposed that all the metric components are $C^3$-close to Euclidean, and we will show this is enough to obtain the functions $\psi_{p,i,1}$. In the setting of the second case of Theorem 1.4, the condition on the size of each of the the radii of $D(r(t))$ will play the crucial role in place of the $C^3$-close assumption, and we will show that we are able to construct the functions $\psi_{p,i,1}$ in this case too. We also seek such solutions $\psi_{p,i,1}$ which have suitable bounds in the following function spaces:
Let \( \Omega \subset \mathbb{R}^2 \). For \( k = 0, 1, 2, \ldots \), define
\[
C^k_x(\Omega) := \left\{ f(x, p) : \Omega \times \Omega \to \mathbb{R} \left| \frac{\partial^{i+j}}{\partial x^i \partial x^j} f \text{ is continuous for all } i + j \leq k \right. \right\}
\]
\[
C^k_p(\Omega) := \left\{ f(x, p) : \Omega \times \Omega \to \mathbb{R} \left| \frac{\partial^{i+j}}{\partial p^i \partial p^j} f \text{ is continuous for all } i + j \leq k \right. \right\}.
\]

**Lemma 4.2.** Let \((M, g), \epsilon_0, \) and \(Y(t)\) satisfy the conditions of the first or second case of Theorem \ref{thm:1.4}. Write \( \bar{n} \) for the unit normal vector field to \( Y(t) \), and \( A \) the second fundamental form of \( Y(t) \). For \( 0 < r(t) \leq \epsilon_0 \), set \((x^1, x^2) \subset D(r(t))\) to be isothermal coordinates on \( Y(t) \), so \( g|_{Y(t)} := e^{2\phi(t)}g_\mathbb{E} \). Suppose \( V := e^{2\phi(t)}(\text{Ric}_g(\bar{n}, \bar{n}) + ||A||^2_g) \in W^{1,q}(D(r(t))) \) with \( q > 2 \).

Then, for any \( p \in D(r(t)) \), and fixed \( i = 1, 2 \), there exists a function \( \psi_{p,i} \in C^3_2(D(r(t))) \) which satisfies
\[
1. (\Delta g_\mathbb{E} + V)\psi_{p,i} = 0 \text{ on } D(r(t)),
\]
\[
2. \psi_{p,i}(p) = 0,
\]
\[
3. \frac{\partial}{\partial r^j} \psi_{p,i}(p) = \delta_{ij}.
\]
Moreover, for \( j, k, l = 1, 2 \), there is a \( C > 0 \) independent of \( p \) (and depending on \( \delta_0 \) in Case 2) such that
\[
4. ||\psi_{p,i} - (x^j - p^j)\delta_{ij}||_{C^3_2(D(r(t)))} \leq Cr^3,
\]
\[
5. ||\partial_{x^j} \psi_{p,i}(x) - \delta_{ij}||_{C^3_2(D(r(t)))} \leq Cr^3,
\]
\[
6. ||\partial_{x^k} \partial_{x^j} \psi_{p,i}(x)||_{C^3_2(D(r(t)))} \leq Cr,
\]
\[
7. ||\partial_{p^j} \psi_{p,i}(x) + \delta_{ij}||_{C^3_2(D(r(t)))} \leq Cr,
\]
\[
8. ||\partial_{p^k} \partial_{p^j} \psi_{p,i}(x)||_{C^3_2(D(r(t)))} \leq Cr,
\]
\[
9. ||\partial_{p^j} \partial_{p^k} \psi_{p,i}(x)||_{C^3_2(D(r(t)))} \leq C,
\]
\[
10. ||\partial_{p^j} \partial_{x^j} \psi_{p,i}(x)||_{C^3_2(D(r(t)))} \leq Cr,
\]
\[
11. ||\partial_{p^j} \partial_{p^j} \partial_{x^j} \psi_{p,i}(x)||_{C^3_2(D(r(t)))} \leq C,
\]
\[
12. ||\partial_{p^j} \partial_{x^k} \partial_{x^j} \psi_{p,i}(x)||_{C^3_2(D(r(t)))} \leq C,
\]
where \( \partial_{p^j} \psi_{p,i} \) denotes differentiation with respect to the \( j \)-th component of \( p \). Lastly, we have
\[
13. \text{the area data for } (M, g) \text{ determines the function } \psi_{p,i}(x) \text{ on } D(r(t)).
\]

**Proof.** Part 1. In the first case of Theorem \ref{thm:1.4} we are assuming that the metric \( g \) is \( C^3 \)-close to Euclidean, hence the norm \( ||V||_\infty \) is small. Then, by setting \( r(t) = 1 \) in the proof of Part 2 below, we obtain the desired claims.

Part 2. Now, let \((M, g), 0 < r \leq \epsilon_0, D(r), \) and \( Y(t) \) be as in the second case of Theorem \ref{thm:1.4}. We seek a function \( \psi_{p,i} : D(r) \to \mathbb{R} \) with the properties 1–13. Without loss of generality, set \( i = 1 \) and write \( \psi_p := \psi_{p,i} \) for simplicity. We also simply write \( Y := Y(t), r := r(t), \) and \( \phi := \phi(t) \) below.
We will construct $\psi_p$ from linear combinations of solutions which are close to either $x^1$, $x^2$, or 1 in $C^2(D(r))$.

To this end, let $\chi^1$ be a function which solves the boundary value problem

$$
(\Delta_{g_{\mathcal{E}}} + V)\chi^1 = 0 \quad \text{on } D(r),
$$

$$
\chi^1 = x^1 \quad \text{on } \partial D(r).
$$

Rescale the coordinates $(x^i)$, $i = 1, 2$, to $(\tilde{x}^i)$, $i = 1, 2$, so that we work over the unit disk $D(1)$: define $f : D(1) \to D(r)$ to be the change of coordinates map $f(\tilde{x}) = r\tilde{x} = x$. Set $\tilde{x}^1 := \frac{1}{r}\chi^1 \circ f$, $\tilde{V} := V \circ f$, and $\tilde{p} = f^{-1}(p)$. Then

$$
f^*(\Delta_{g_{\mathcal{E}}} + V) = r^{-2}\Delta_{g_{\mathcal{E}}} + \tilde{V},
$$

and we seek solutions $\tilde{\chi}^1$ to

$$
(\Delta_{g_{\mathcal{E}}} + r^2\tilde{V})\tilde{\chi}^1 = 0, \quad \text{on } D(1) \tag{4.48}
$$

$$
\tilde{\chi}^1 = \tilde{x}^1 \quad \text{on } \partial D(1). \tag{4.49}
$$

Since $(M, g)$ is $(K, \epsilon_0, \delta_0)$-thin and using the definition of $V$ and the bounds in Lemma 2.4, the potential satisfies the estimate $|||V|||_{W^{1,q}(D(r))} < \frac{\delta_0}{\epsilon_0}$ for some small $\delta_0 > 0$; this combined with the fact $r < \epsilon_0$ gives $|||r^2\tilde{V}|||_{W^{1,q}(D(1))} \leq \delta_0$. Now, as $\delta_0$ is small, the operator $(\Delta_{g_{\mathcal{E}}} + r^2\tilde{V}) : W^{3,q}(D(1)) \to W^{1,q}(D(1))$ is invertible, and the norm of the inverse is bounded by a constant depending on $\delta_0$ for the Class 2 case. Therefore there exists a unique solution $\tilde{\chi}^1$ to (4.48), (4.49).

Now consider

$$
(\Delta_{g_{\mathcal{E}}} + r^2\tilde{V}) [\tilde{\chi}^1 - \tilde{x}^1] = -(r^2\tilde{V})\tilde{x}^1 \quad \text{on } D(1),
$$

$$
\tilde{\chi}^1 - \tilde{x}^1 = 0 \quad \text{on } \partial D(1). \tag{4.50}
$$

Since $r$ is chosen to be small, for $q > 2$ we have via Sobolev embedding

$$
|||\tilde{\chi}^1 - \tilde{x}^1|||_{C^2(D(1))} \leq C|||\tilde{\chi}^1 - \tilde{x}^1|||_{W^{3,q}(D(1))}
$$

$$
\leq C\left[|||(r^2\tilde{V})\tilde{x}^1|||_{W^{1,q}(D(1))}\right]
$$

$$
\leq C\left[r^2|||\tilde{V}\tilde{x}^1|||_{L^q(D(1))} + r^2|||\tilde{\nabla}(\tilde{V}\tilde{x}^1)|||_{L^q(D(1))}\right]
$$

$$
\leq Cr^2 \tag{4.50}
$$

for $C > 0$ depending on $\delta_0$, $q$, and $D(1)$.

Therefore, the function $\tilde{\chi}^1$ satisfies the estimate

$$
|||\tilde{\chi}^1 - \tilde{x}^1|||_{C^2(D(1))} \leq Cr^2. \tag{4.51}
$$

In particular, the estimate (4.51) implies

$$
|||\partial_{\tilde{x}^1} \tilde{\chi}^1 - 1|||_{C^1(D(1))} \leq Cr^2, \tag{4.52}
$$

$$
|||\partial_{\tilde{x}^2} \tilde{\chi}^1 - 0|||_{C^1(D(1))} \leq Cr^2. \tag{4.53}
$$

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Similar to above, we construct a function $\tilde{\chi}^2$, which solves the boundary value problem

\[
(\Delta_{g_0} + r^2 \tilde{V}) \tilde{\chi}^2 = 0 \quad \text{on } D(1), \\
\tilde{\chi}^2 = \tilde{x}^2 \quad \text{on } \partial D(1),
\]

and satisfies

\[
||\tilde{\chi}^2 - \tilde{x}^2||_{C^2(D(1))} \leq Cr^2.
\] (4.54)

Further, let $\omega$ solve

\[
(\Delta_{g_0} + r^2 \tilde{V}) \omega = 0 \quad \text{on } D(1), \\
\omega = 1 \quad \text{on } \partial D(1).
\]

As argued above, the function $\omega$ satisfies

\[
||\omega - 1||_{C^2(D(1))} \leq Cr^2.
\] (4.55)

Now, for $j = 1, 2$, and $\tilde{p} \in D(1)$ fixed, the function

\[
\gamma^j(\tilde{p}, \tilde{x}) := \omega(\tilde{p})\tilde{x}^j(\tilde{x}) - \tilde{\chi}^j(\tilde{p}) \omega(\tilde{x})
\]

solves

\[
(\Delta_{g_0} + r^2 \tilde{V}) \gamma^j = 0 \quad \text{for } D(1), \\
\gamma^j(\tilde{p}, \tilde{x}) = \omega(\tilde{p})\tilde{x}^j(\tilde{x}) - \tilde{\chi}^j(\tilde{p}) \omega(\tilde{x}) \quad \text{on } \partial D(1),
\]

and has the property $\gamma^j(\tilde{p}) = 0$. The function $\gamma^j$ also obeys the following estimates. From (4.51), (4.54), and (4.55),

\[
||\gamma^j(\tilde{p}, \tilde{x}) - (\tilde{x}^j - \tilde{p}^j)||_{C^0(D(1))} = ||\omega(\tilde{p})\tilde{x}^j(\tilde{x}) - \tilde{\chi}^j(\tilde{p}) \omega(\tilde{x}) - (\tilde{x}^j - \tilde{p}^j)||_{C^0(D(1))}
\]

\[
\leq ||\omega(\tilde{p})\tilde{x}^j(\tilde{x}) - \omega(\tilde{p})\tilde{x}^j||_{C^0(D(1))} + ||\tilde{\chi}^j(\tilde{p}) \omega(\tilde{x}) - \tilde{x}^j||_{C^0(D(1))}
\]

\[
+ ||\tilde{x}^j(\tilde{p}) \omega(\tilde{x}) - \tilde{p}^j \omega(\tilde{x})||_{C^0(D(1))} + ||\tilde{p}^j \omega(\tilde{x}) - \tilde{p}^j||_{C^0(D(1))}
\]

\[
\leq Cr^2,
\] (4.56)

where $C > 0$ is a constant independent of $\tilde{p}$. Similarly, we find for $i, j = 1, 2$,

\[
||\partial_{\tilde{x}^i} \gamma^j(\tilde{p}, \tilde{x}) - \delta^j_i||_{C^0(D(1))} \leq Cr^2,
\] (4.57)

\[
||\partial_{\tilde{p}^i} \gamma^j(\tilde{p}, \tilde{x}) + \delta^j_i||_{C^0(D(1))} \leq Cr^2,
\] (4.58)

\[
||\partial_{\tilde{x}^i} \partial_{\tilde{x}^j} \gamma^j(\tilde{p}, \tilde{x})||_{C^0(D(1))} \leq Cr^2,
\] (4.59)

\[
||\partial_{\tilde{p}^i} \partial_{\tilde{x}^j} \gamma^j(\tilde{p}, \tilde{x})||_{C^0(D(1))} \leq Cr^2,
\] (4.60)

where the constant is independent of $\tilde{p}$.
Consider $\gamma^1$ and $\gamma^2$ as above. From (4.51), (4.54), and (4.55),

\[
||\partial_{x^1} \gamma^1(\tilde{p}, \tilde{p}) \partial_{x^2} \gamma^2(\tilde{p}, \tilde{p}) - \partial_{x^2} \gamma^1(\tilde{p}, \tilde{p}) \partial_{x^1} \gamma^2(\tilde{p}, \tilde{p})||_{C^0_2(D(1))} \sim 1 + Cr^2.
\]  
(4.61)

Then, the function

\[
\tilde{\psi}_p(\tilde{x}) := \frac{\partial_{x^2} \gamma^2(\tilde{p}, \tilde{p}) \gamma^1(\tilde{p}, \tilde{x}) - \partial_{x^2} \gamma^1(\tilde{p}, \tilde{p}) \gamma^2(\tilde{p}, \tilde{x})}{\partial_{x^1} \gamma^1(\tilde{p}, \tilde{p}) \partial_{x^2} \gamma^2(\tilde{p}, \tilde{p}) - \partial_{x^2} \gamma^1(\tilde{p}, \tilde{p}) \partial_{x^1} \gamma^2(\tilde{p}, \tilde{p})}
\]

is well defined and satisfies conditions 2-3. By linearity of the operator $(\Delta g_\delta + r^2 \tilde{V})$, $\tilde{\psi}_p$ satisfies condition 1:

\[
(\Delta g_\delta + r^2 \tilde{V}) \tilde{\psi}_p = 0 \quad \text{on } D(1),
\]

\[
\tilde{\psi}_p = f_\delta \quad \text{on } \partial D(1),
\]

where the boundary data is explicitly given by

\[
f_\delta(\tilde{x}) := \frac{\partial_{x^2} \gamma^2(\tilde{p}, \tilde{p}) [\omega(\tilde{p}) \tilde{x}^1 - \tilde{\chi}^1(\tilde{p})] - \partial_{x^2} \gamma^1(\tilde{p}, \tilde{p}) [\omega(\tilde{p}) \tilde{x}^2 - \tilde{\chi}^2(\tilde{p})]}{\partial_{x^1} \gamma^1(\tilde{p}, \tilde{p}) \partial_{x^2} \gamma^2(\tilde{p}, \tilde{p}) - \partial_{x^2} \gamma^1(\tilde{p}, \tilde{p}) \partial_{x^1} \gamma^2(\tilde{p}, \tilde{p})}.
\]

Now, we prove preliminary estimates for $\tilde{\psi}_p$ and its derivatives which will enable us to obtain the desired function $\psi_p$ and estimates claimed for $\psi_p$. First, we show

\[
||\tilde{\psi}_p - (\tilde{x}^1 - \tilde{p}^1)||_{C^0_2(D(1))} \leq C r^2,
\]

for some constant $C > 0$ independent of $\tilde{p}$. Indeed, from the inequalities (4.51)–(4.56) and (4.57)–(4.60),

\[
||\tilde{\psi}_p(\tilde{x}) - (\tilde{x}^1 - \tilde{p}^1)||_{C^0_2(D(1))} = \left|\left| \frac{\partial_{x^2} \gamma^2(\tilde{p}, \tilde{p}) \gamma^1(\tilde{p}, \tilde{x}) + \partial_{x^2} \gamma^1(\tilde{p}, \tilde{p}) \gamma^2(\tilde{p}, \tilde{x})}{\partial_{x^1} \gamma^1(\tilde{p}, \tilde{p}) \partial_{x^2} \gamma^2(\tilde{p}, \tilde{p}) - \partial_{x^2} \gamma^1(\tilde{p}, \tilde{p}) \partial_{x^1} \gamma^2(\tilde{p}, \tilde{p})} - (\tilde{x}^1 - \tilde{p}^1) \right|\right|_{C^0_2(D(1))} \leq C r^2.
\]

We perform similar computations and use the inequalities (4.51)–(4.56) and (4.57)–(4.60) to derive an estimate for $\partial_{x^j} \tilde{\psi}_p(\tilde{x})$, $j = 1, 2$.

\[
||\partial_{x^j} \tilde{\psi}_p(\tilde{x}) - \delta^j_1||_{C^0_2(D(1))} = \left|\left| \frac{\partial_{x^2} \gamma^2(\tilde{p}, \tilde{p}) \partial_{x^j} \gamma^1(\tilde{p}, \tilde{x}) + \partial_{x^2} \gamma^1(\tilde{p}, \tilde{p}) \partial_{x^j} \gamma^2(\tilde{p}, \tilde{x})}{\partial_{x^1} \gamma^1(\tilde{p}, \tilde{p}) \partial_{x^2} \gamma^2(\tilde{p}, \tilde{p}) - \partial_{x^2} \gamma^1(\tilde{p}, \tilde{p}) \partial_{x^1} \gamma^2(\tilde{p}, \tilde{p})} - \delta^j_1 \right|\right|_{C^0_2(D(1))} \leq C r^2.
\]

40
Additionally, for $j, k = 1, 2$, and using (4.57)-(4.60), we obtain the inequality

\[
\| \partial_{\tilde{x}_i} \partial_{\tilde{x}_j} \tilde{\psi}_p(\tilde{x}) \|_{C^0(D(1))} \leq \frac{\| \partial_{\tilde{x}_2} \gamma^2(\tilde{p}, \tilde{p}) \partial_{\tilde{x}_2} \gamma^1(\tilde{p}, \tilde{x}) \|_{C^0(D(1))}}{\| \partial_{\tilde{x}_2} \gamma^1(\tilde{p}, \tilde{p}) \partial_{\tilde{x}_2} \gamma^2(\tilde{p}, \tilde{p}) - \partial_{\tilde{x}_2} \gamma^1(\tilde{p}, \tilde{p}) \partial_{\tilde{x}_2} \gamma^2(\tilde{p}, \tilde{p}) \|} + \frac{\| \partial_{\tilde{x}_2} \gamma^1(\tilde{p}, \tilde{p}) \partial_{\tilde{x}_2} \gamma^2(\tilde{p}, \tilde{p}) \|_{C^0(D(1))}}{\| \partial_{\tilde{x}_2} \gamma^1(\tilde{p}, \tilde{p}) \partial_{\tilde{x}_2} \gamma^2(\tilde{p}, \tilde{p}) - \partial_{\tilde{x}_2} \gamma^1(\tilde{p}, \tilde{p}) \partial_{\tilde{x}_2} \gamma^2(\tilde{p}, \tilde{p}) \|} \leq C r^2.
\]

(4.65)

By an analogous computation we find the claimed estimate

\[
\| \partial_{\tilde{x}_i} \partial_{\tilde{x}_j} \tilde{\psi}_p(\tilde{x}) \|_{C^0(D(1))} \leq C r.
\]

(4.66)

Now we show estimates for the derivatives $\partial_{\tilde{p}_i} \tilde{\psi}_p, \partial_{\tilde{p}_i} \partial_{\tilde{p}_j} \tilde{\psi}_p, \partial_{\tilde{p}_i} \partial_{\tilde{x}_j} \tilde{\psi}_p$, and $\partial_{\tilde{p}_i} \partial_{\tilde{x}_j} \tilde{\psi}_p$, where $i, j, k = 1, 2$.

First we show $\| \partial_{\tilde{p}_i} \tilde{\psi}_p + \delta_i^t \|_{C^1(D(1))}$. By definition,

\[
\partial_{\tilde{p}_i} \tilde{\psi}_p(\tilde{x}) := \frac{\partial_{\tilde{x}_i} \gamma^2(\tilde{p}, \tilde{p}) \partial_{\tilde{x}_i} \gamma^1(\tilde{p}, \tilde{x}) \|_{C^1(D(1))}}{\| \partial_{\tilde{x}_2} \gamma^1(\tilde{p}, \tilde{p}) \partial_{\tilde{x}_2} \gamma^2(\tilde{p}, \tilde{p}) - \partial_{\tilde{x}_2} \gamma^1(\tilde{p}, \tilde{p}) \partial_{\tilde{x}_2} \gamma^2(\tilde{p}, \tilde{p}) \|} + \frac{\| \partial_{\tilde{x}_2} \gamma^1(\tilde{p}, \tilde{p}) \partial_{\tilde{x}_2} \gamma^2(\tilde{p}, \tilde{p}) \|_{C^1(D(1))}}{\| \partial_{\tilde{x}_2} \gamma^1(\tilde{p}, \tilde{p}) \partial_{\tilde{x}_2} \gamma^2(\tilde{p}, \tilde{p}) - \partial_{\tilde{x}_2} \gamma^1(\tilde{p}, \tilde{p}) \partial_{\tilde{x}_2} \gamma^2(\tilde{p}, \tilde{p}) \|} \leq C r^2.
\]

(4.65)
Thus,

$$
||\partial_{p^i} \tilde{\psi}_p(x) + \delta_1^i||_{C^0_b(D(1))} \leq ||\partial_{p^i} \partial_{x^2} \gamma^2(\tilde{p}, \tilde{p}) \gamma^1(\tilde{p}, \tilde{x}) \partial_{x^1} \gamma^1(\tilde{p}, \tilde{p}) \partial_{x^2} \gamma^2(\tilde{p}, \tilde{p})||_{C^0_b(D(1))}
+ ||\partial_{p^i} \partial_{x^2} \gamma^2(\tilde{p}, \tilde{p}) \gamma^1(\tilde{p}, \tilde{x}) \partial_{x^1} \gamma^1(\tilde{p}, \tilde{p}) \partial_{x^2} \gamma^2(\tilde{p}, \tilde{p})||_{C^0_b(D(1))}
+ ||[\partial_{p^i} \gamma^1(\tilde{p}, \tilde{x}) + \delta_1^1 \partial_{x^1} \gamma^1(\tilde{p}, \tilde{p})] \partial_{x^2} \gamma^2(\tilde{p}, \tilde{p})||_{C^0_b(D(1))}
+ ||[\partial_{p^i} \gamma^1(\tilde{p}, \tilde{x}) + \delta_1^1 \partial_{x^1} \gamma^1(\tilde{p}, \tilde{p})] \partial_{x^2} \gamma^2(\tilde{p}, \tilde{p})||_{C^0_b(D(1))}
+ ||\partial_{p^i} \partial_{x^2} \gamma^1(\tilde{p}, \tilde{p}) \gamma^2(\tilde{p}, \tilde{x}) \partial_{x^1} \gamma^1(\tilde{p}, \tilde{p}) \partial_{x^2} \gamma^2(\tilde{p}, \tilde{p})||_{C^0_b(D(1))}
+ ||\partial_{x^2} \gamma^2(\tilde{p}, \tilde{p}) \gamma^1(\tilde{p}, \tilde{x}) \partial_{x^1} \gamma^1(\tilde{p}, \tilde{p}) \partial_{x^2} \gamma^2(\tilde{p}, \tilde{p})||_{C^0_b(D(1))}
+ ||\partial_{x^2} \gamma^1(\tilde{p}, \tilde{p}) \partial_{p^i} \gamma^2(\tilde{p}, \tilde{x}) \partial_{x^1} \gamma^1(\tilde{p}, \tilde{p}) \partial_{x^2} \gamma^2(\tilde{p}, \tilde{p})||_{C^0_b(D(1))}
+ ||\partial_{x^2} \gamma^1(\tilde{p}, \tilde{p}) \partial_{p^i} \gamma^2(\tilde{p}, \tilde{x}) \partial_{x^1} \gamma^1(\tilde{p}, \tilde{p}) \partial_{x^2} \gamma^2(\tilde{p}, \tilde{p})||_{C^0_b(D(1))}
+ ||\partial_{x^2} \gamma^1(\tilde{p}, \tilde{p}) \partial_{p^i} \gamma^2(\tilde{p}, \tilde{x}) \partial_{x^1} \gamma^1(\tilde{p}, \tilde{p}) \partial_{x^2} \gamma^2(\tilde{p}, \tilde{p})||_{C^0_b(D(1))}
+ ||\partial_{x^2} \gamma^2(\tilde{p}, \tilde{p}) \gamma^1(\tilde{p}, \tilde{x}) \partial_{p^i} \partial_{x^1} \gamma^1(\tilde{p}, \tilde{p}) \partial_{x^2} \gamma^2(\tilde{p}, \tilde{p})||_{C^0_b(D(1))}
+ ||\partial_{x^2} \gamma^2(\tilde{p}, \tilde{p}) \gamma^1(\tilde{p}, \tilde{x}) \partial_{p^i} \partial_{x^1} \gamma^1(\tilde{p}, \tilde{p}) \partial_{x^2} \gamma^2(\tilde{p}, \tilde{p})||_{C^0_b(D(1))}
+ ||\partial_{x^2} \gamma^2(\tilde{p}, \tilde{p}) \gamma^1(\tilde{p}, \tilde{x}) \partial_{p^i} \partial_{x^1} \gamma^1(\tilde{p}, \tilde{p}) \partial_{x^2} \gamma^2(\tilde{p}, \tilde{p})||_{C^0_b(D(1))}.
$$

From (4.57)–(4.60), we see

$$
||\partial_{p^i} \tilde{\psi}_p(x) + \delta_1^i||_{C^0_b(D(1))} \leq Cr^2.
$$

(4.67)

By analogous computations, \( \tilde{\psi}_p(x) \) satisfies

$$
||\partial_{p^i} \partial_{p^j} \tilde{\psi}_p(x)||_{C^0_b(D(1))} \leq Cr^2, \quad ||\partial_{p^i} \partial_{x^j} \tilde{\psi}_p(x)||_{C^0_b(D(1))} \leq Cr^2, \quad (4.68)
$$

$$
||\partial_{x^j} \partial_{x^j} \tilde{\psi}_p(x)||_{C^0_b(D(1))} \leq Cr^2, \quad ||\partial_{p^i} \partial_{x^j} \tilde{\psi}_p(x)||_{C^0_b(D(1))} \leq Cr^2, \quad (4.69)
$$

$$
||\partial_{p^i} \partial_{p^j} \partial_{x^j} \tilde{\psi}_p(x)||_{C^0_b(D(1))} \leq Cr^2, \quad ||\partial_{p^i} \partial_{p^j} \partial_{x^j} \tilde{\psi}_p(x)||_{C^0_b(D(1))} \leq Cr^2, \quad (4.70)
$$

for \( i, j, k = 1, 2 \).

Having constructed \( \tilde{\psi}_p \) which satisfies the estimates, we rescale \( \tilde{\psi}_p \) to achieve the desired function \( \psi_p \) on the disk \( D(r) \). Define

$$
\psi_p(x) := r \tilde{\psi}_p \left( \frac{x}{r} \right).
$$

We claim that this is the function which satisfies conditions 1-12.

By construction, \( \psi_p \) satisfies conditions 1, 2, and 3. We now prove the estimates 4–11. Note that the coordinates change as

$$
x = r \tilde{x}, \quad p = r \tilde{p}.
$$

Hence, the derivatives change as

$$
\partial_{x^j} = r \partial_{\tilde{x}^j}, \quad \partial_{p^j} = r \partial_{\tilde{p}^j}.
$$
where \( j = 1, 2 \). In particular we have
\[
\partial_{\tilde{x}j} \tilde{\psi}_p(\tilde{x}) = \partial_{xj} \psi_p(x),
\]
\[
\partial_{\tilde{x}k} \partial_{\tilde{x}l} \tilde{\psi}_p(\tilde{x}) = r \partial_{\tilde{x}k} \partial_{\tilde{x}l} \tilde{\psi}_p(\tilde{x}),
\]
\[
\partial_{\tilde{p}j} \partial_{\tilde{x}k} \tilde{\psi}_p(\tilde{x}) = r^2 \partial_{\tilde{p}j} \partial_{\tilde{x}k} \tilde{\psi}_p(\tilde{x}),
\]
\[
\partial_{\tilde{p}j} \partial_{\tilde{p}k} \partial_{\tilde{x}l} \tilde{\psi}_p(\tilde{x}) = r^2 \partial_{\tilde{p}j} \partial_{\tilde{p}k} \partial_{\tilde{x}l} \tilde{\psi}_p(\tilde{x}),
\]
\[
\partial_{\tilde{p}j} \tilde{\psi}_p(\tilde{x}) = \partial_{\tilde{p}j} \psi_p(x),
\]
\[
\partial_{\tilde{p}j} \partial_{\tilde{p}k} \partial_{\tilde{x}l} \tilde{\psi}_p(\tilde{x}) = r \partial_{\tilde{p}j} \partial_{\tilde{p}k} \partial_{\tilde{x}l} \tilde{\psi}_p(\tilde{x}),
\]
for \( j, k, l = 1, 2 \). By the change of coordinates and the estimates \([4.62] – [4.70]\) for \( \tilde{\psi}_p \), we have \( \psi_p \) obeys the estimates 4–12.

Now for the last condition 13: By Proposition \([1.10]\), the area data for \((M, g)\) determines \( \psi_p \).

Thus, \( \psi_p \) is the desired function which satisfies conditions 1–13.

Now, from the above lemma, for \( i = 1, 2 \), and each fixed \( t \in (-1, 1) \), and \( p \in Y_1(t) \), there exists families of foliations \( Y_{i,1}(s, t) \) given by embeddings
\[
h_{i,1}(., ., t) : [0, S] \times D(r(t)) \to M \subset M
\]
\[
(h_{i,1})_s \left( \frac{\partial}{\partial s} \right) |_{s=0} = \psi_{p,1,i} m_1,
\]
for which \( \psi_{p,i,1} : \mathbb{R}^2 \to \mathbb{R} \) has the properties defined in Lemma \([4.2]\) on the disc \( D(r(t)) \subset \mathbb{R}^2 \);

\[
\Delta_{g_{i1}} \psi_{p,i,1} + e^{2\phi_1} (\text{Ric}_{g_1}(\tilde{m}, \tilde{m}) + ||A||^2_{g_1}) \psi_{p,i,1} = 0
\]
on \( D(r(t)) \), and
\[
\psi_{p,i,1}(f_1(p)) = 0,
\]
\[
\nabla \psi_{p,i,1}(f_1(p)) = \frac{\partial}{\partial x_i}.
\]

The induced variation of the coordinate \( x^3 \) is written in Taylor expanded form as
\[
x^3_i(s) = x^3 + s \tilde{x}_i^3 + O(s^2).
\]
As shown by Proposition \([3.1]\) the knowledge of the areas of \( Y_{i,1}(s, t) := h_{i,1}(D(r(t)), s, t) \) determines the functions
\[
||\nabla x^3_i(s)||^2_{g_1}(p)
\]
on \( \mathbb{R}^2 \), in the coordinates \( (x^\alpha) \), for all \( s \in [0, S] \), \( i = 1, 2 \).

Linearizing in \( s \), by Lemma \([3.2]\) we obtain a nonlinear, non-local, coupled system of equations for \( \tilde{g}_l^{k3} \), \( k = 1, 2 \) of the form
\[
\frac{d}{ds} ||\nabla x^3_i(s)||^2_{g_1}(p) |_{s=0} = g_1^{3\alpha} \partial_\alpha \psi_{p,1,i}(x^1, x^2) + \partial_\alpha ||\nabla x^3||_{g_1(p)} \tilde{x}_i^\alpha,
\]
\[
\frac{d}{ds} ||\nabla x^3_2(s)||^2_{g_1}(p) |_{s=0} = g_1^{3\alpha} \partial_\alpha \psi_{p,2,i}(x^1, x^2) + \partial_\alpha ||\nabla x^3||_{g_1(p)} \tilde{x}_2^\alpha,
\]
\[43\]
where the first order change in conformal coordinates \( \dot{x}_i^\alpha \), \( \alpha = 1, 2, 3 \), depends on \( p, \psi_{p,i,1}, \delta g \) and the first and second derivatives of \( \delta g \), as shown in Lemma 3.3. By Proposition 3.1, our area information determines the functions \( ||\nabla x^3_i(s)||_{g_1}(p) \) and \( ||\nabla x^3_i(s)||_{g_2}(p) \) in the coordinates \((x^\alpha)\).

By the very same argument as above, if we consider instead the foliation \( Y_2(t) \) of \((M, g_2)\), we may obtain a nonlinear system of equations for \( g_2^{k3} \), \( k = 1, 2 \) of the form

\[
\frac{d}{ds} ||\nabla y^3_i(s)||_{g_2}(p) \bigg|_{s=0} = g_2^{3\alpha}\partial_{\alpha}\psi_{p,1,2}(x^1, x^2) + \partial_{\alpha}||\nabla x^3||_g(p)\dot{y}^\alpha + F \circ \Phi_1, \tag{4.73}
\]

\[
\frac{d}{ds} ||\nabla y^3_i(s)||_{g_2}(p) \bigg|_{s=0} = g_2^{3\alpha}\partial_{\alpha}\psi_{p,2,2}(x^1, x^2) + \partial_{\alpha}||\nabla x^3||_g(p)\dot{y}^\alpha + F \circ \Phi_1, \tag{4.74}
\]

where \( \dot{y}^\alpha \) depends on \( p, \psi_{p,i,2}, \delta g \) and the first and second derivatives of \( \delta g \), as shown in Lemma 3.3, and the functions \( \psi_{p,i,2} \in H^2(\mathbb{R}^2), i = 1, 2 \) are solutions of the Jacobi equation

\[ \Delta_{g_2}\psi_{p,i,2} + e^{2\phi_2} \left( [\text{Ric}_{g_2}(\tilde{n}_2, \tilde{n}_2) + ||A||^2_{g_2}] \circ F \circ \Phi_1 \right) \psi_{p,i,2} = 0 \]

on \( \mathbb{R}^2 \), which additionally satisfy the conditions of Lemma 4.2 on \( D(r) \). Again, employing Proposition 3.1, our area information determines the functions \( ||\nabla y^3_i(s)||_{g_2}(p) \) and \( ||\nabla y^3_i(s)||_{g_2}(p) \) in the coordinates \((x^\alpha)\).

**Lemma 4.3.** In the coordinates \((x^\alpha)\), \( \psi_{p,i,1} = \psi_{p,i,2} \) for \( i = 1, 2 \) on \( \mathbb{R}^2 \).

**Proof.** Fix \( x^3 = y^3 = t \), and consider the leaves \( Y_1(t) \) and \( Y_2(t) \). Since the foliations \( Y_1(t) \) and \( Y_2(t) \) agree outside the disc \( D(r(t)) \times \{1\} \equiv D(r(t)) \subset \mathbb{R}^2 \), so do the functions \( \psi_{p,i,1} \) and \( \psi_{p,i,2} \). As shown by Proposition 2.9, our area data determines the Dirichlet-to-Neumann maps associated to the operators

\[
J_1 := \Delta_{g_1} + \text{Ric}_{g_1}(\tilde{n}, \tilde{n}) + ||A||^2_{g_1}
\]

\[
J_2 := \Delta_{g_2} + \text{Ric}_{g_2}(\tilde{n}, \tilde{n}) + ||A||^2_{g_2}
\]

on \( D(r(t)) \), in the coordinates \((x^\alpha)\) and \((y^\alpha)\) respectively. Here \( g_{k,i} \) denotes the metric induced on \( Y_k(t) \) by \( g_{k}, k = 1, 2 \). Via the map \( F \), we can express both the operators in the coordinate system \((x^\alpha)\) as

\[
J_1 := \Delta_{g_k} + e^{2\phi_1} \left( \text{Ric}_{g_k}(\tilde{n}, \tilde{n}) + ||A||^2_{g_k} \right) \circ F \circ \Phi_1.
\]

\[
J_2 := \Delta_{g_k} + e^{2\phi_2} \left( \text{Ric}_{g_k}(\tilde{n}, \tilde{n}) + ||A||^2_{g_k} \right) \circ F \circ \Phi_1.
\]

From Proposition 1.10, the Dirichlet-to-Neumann maps to these operators is also determined as expressed in the coordinates \((x^\alpha)\). By construction the foliation \( Y_1(t) \) agrees with the foliation \( Y_2(t) \) on the boundary of \( M \). By hypothesis, for each fixed \( t \) the area of \( Y_1(t) \) as measured by \( g_1 \) is equal to the area of \( Y_2(t) \) as measured by \( g_2 \). Thus, Proposition 2.9, the Dirichlet-to-Neumann maps associated to the operators \((4.75)\) and \((4.76)\) agree as maps \( H^{\frac{1}{2}}(\partial D(r(t))) \to H^{-\frac{1}{2}}(\partial D(r(t))) \). By the linear result in [12], the potential functions

\[ e^{2\phi_1} \left( \text{Ric}_{g_1}(\tilde{n}, \tilde{n}) + ||A||^2_{g_1} \right) \]

\[ e^{2\phi_2} \left( \text{Ric}_{g_2}(\tilde{n}, \tilde{n}) + ||A||^2_{g_2} \right) \circ F \circ \Phi_1 \]

agree on \( D(r(t)) \), as written in the coordinates \((x^\alpha)\).

So in the coordinates \((x^\alpha)\), \( \psi_{p,i,1} = \psi_{p,i,2} \) on \( D(r(t)) \) for \( i = 1, 2 \).
With the above lemma in hand, we simplify notation a bit and write
\[ \psi_{p,i} := \psi_{p,i,1} = \psi_{p,i,2} \]
for \( i = 1, 2 \).

Now in the above setting, we may obtain pseudodifferential equations which relate the differences of the metric components \( \delta \phi, \delta g^{31}, \) and \( \delta g^{32} \):

**Lemma 4.4.** For each \( p \in M \), the unknown differences \( \delta \phi, \delta g^{31}, \delta g^{32} \) satisfy the following system of equations:

\[
\begin{align*}
0 &= \delta g^{3k}(p) \partial_k \psi_{p,1}(x(p)) + \partial_k \|[x^3]|_{g_0}(p) \delta \phi^k(p) \\
0 &= \delta g^{3k}(p) \partial_k \psi_{p,2}(x(p)) + \partial_k \|[x^3]|_{g_0}(p) \delta \phi^k(p) \\
0 &= g_1^{k3} \partial_k(\delta \phi) + g_1^{33} \partial_3(\delta \phi) + \delta g^{k3}(\partial_3 \delta \phi - \frac{1}{2} \partial_k \log(g_1^{33})) + \frac{1}{2} \partial_k(\delta g^{3k}).
\end{align*}
\]

Furthermore, by construction \( \delta \phi = 0 \), and \( \delta g^{3k} = 0 \) for \( k = 1, 2 \) on \( M \setminus M \cup \partial M \).

**Proof.** These equations follow from the previous lemma and from taking the difference of the equations on the leaves \( Y_1(t) \) and \( Y_2(t) \), \( t \in (-1, 1) \), given in Lemma 3.2 and Proposition 3.4.

4.1.3 **Expressing \( \delta g^{3k} \) via pseudodifferential operators:**

Our next goal is to solve equations (4.77) and (4.78) by expressing \( \delta g^{3k}, k = 1, 2 \), as a linear combination of pseudodifferential operators acting on \( \delta \phi \) and \( \partial_3 \delta \phi \). To do this, we use the fact that \((M, g_i), i = 1, 2\) are either \( C^3\)-close to Euclidean or \((K, \epsilon_0, \delta_0)\)-thin. Furthermore, the assumptions of either \( C^3\)-close to Euclidean or \((K, \epsilon_0, \delta_0)\)-thin allow us to obtain estimates for the pseudodifferential operators acting on \( \delta \phi \) and \( \partial_3 \delta \phi \) which appear in the expressions for \( \delta g^{3k}, k = 1, 2 \). From these estimates, we show equation (4.79) can be written as a hyperbolic pseudodifferential operator acting on \( \delta \phi \). A standard energy argument then gives uniqueness for the conformal factors, which implies uniqueness for the metrics \( g_1 \) and \( g_2 \).

**Proposition 4.5 (\( \delta g^{31} \) and \( \delta g^{32} \) are \( \Psi \)DOs).** At each \( p \in M \), the above differences \( \delta g^{31} \) and \( \delta g^{32} \) can be expressed as

\[
\delta g^{31}(p) =: P_{-1}^1(\delta \phi, p) + Q_{-2}^1(\partial_3 \delta \phi, p) \\
\delta g^{32}(p) =: P_{-1}^2(\delta \phi, p) + Q_{-2}^2(\partial_3 \delta \phi, p).
\]

where \( P_{-1}^k : L^2(D(r(t)))) \rightarrow H^1(D(r(t)))) \), \( Q_{-2}^k : L^2(D(r(t)))) \rightarrow H^2(D(r(t)))) \), \( k = 1, 2, s > 0 \), are respectively order \(-1\) and \(-2\) pseudodifferential operators in the tangential directions \( \partial_k \), \( k = 1, 2 \).

Further, we have the estimates

\[
||P_{-1}^k(\delta \phi)||_{H^1} \leq C\epsilon_0 ||\delta \phi||_{L^2} \\
||Q_{-2}^k(\partial_3 \delta \phi)||_{H^2} \leq C\epsilon_0 ||\partial_3 \delta \phi||_{L^2} \\
||\partial_j Q_{-2}^k(\partial_3 \delta \phi)||_{L^2} \leq C\epsilon_0 ||\partial_3 \delta \phi||_{L^2}
\]

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in the first case of Theorem 1.4 and

\[ \|P_{k_1}^k(\delta \phi)\|_{H^1} \leq CK\|\delta \phi\|_{L^2} \]  
\[ \|Q_{k_2}^k(\partial_3 \delta \phi)\|_{H^2} \leq CK\|\partial_3 \delta \phi\|_{L^2}, \]  
\[ \|\partial_3 Q_{k_2}^k(\partial_2 \delta \phi)\|_{L^2} \leq CK_0\|\partial_3 \delta \phi\|_{L^2} \]

in the second case of Theorem 1.4. In both settings, \( C > 0 \) is a uniform constant independent of \( r(t) \) and \( E_0 \) (but depending on \( \delta_0 \) in Definition 1.2 in the second case of Theorem 1.4), and \( j, k \in \{1, 2\} \).

**Proof.** First we derive the operators \( P_{k_1}, Q_{k_2} \). Let \( w \in M \), and set \( \Delta_{g_2} \equiv \frac{\partial^2}{\partial(w^2)} + \frac{\partial^2}{\partial(w^2)} \). From Lemma 3.3, for each of our choices \( \psi \) where the differential operator \( \delta \) on the the disc \( \alpha, \beta \) \( g \) the metric \( \text{components of } F \text{ in the conformal coordinates (} x^1, x^2 \text{) on } Y_1(t) \) is given schematically as

\[ \Delta_{g_2} \dot{x}^k = \psi_{p,i}A_{\alpha}^{ijk} \partial_i \partial_j g_{1}^{3\alpha} + \psi_{p,i}B_{\alpha}^{mk} \partial_m \partial_\alpha \phi_1 + \psi_{p,i}C_{\alpha}^{mjk} \partial_m \phi_1 \partial_j g_{1}^{3\alpha} \]
\[ + \psi_{p,i}D_{\alpha}^{mjk} \partial_m \partial_j g_{1}^{3\beta} \psi_{p,i}F_{\alpha}^{mka} \partial_m \phi_1 \phi_1 \]
\[ + (\psi_{p,i}H_{\alpha}^{m} + \nabla_j \psi_{p,i}I_{\alpha}^{mjk}) \partial_m \partial_j g_{1}^{3\alpha} + (\psi_{p,i}J_{\alpha}^{1k} + \nabla_m \psi_{p,i}K_{\alpha}^{mka}) \partial_m \partial_\alpha \phi_1, \]
\[ =: F^k(g_{1}^{13}, g_{1}^{23}, \phi_1, \psi_{p,i}), \]  

(4.88)

with an analogous equation for the conformal coordinates \( (y^1, y^2) \) on \( Y_2(t) \), with respect to the metric \( g_2 \). Here the functions \( A_{\alpha}^{mjk}, \ldots, J_{\alpha}^{mka} \) appearing in (4.88) are polynomials in the components of \( g_1 \) and \( g_1^{-1} \) and the function \( e^{\phi_1} \) and \( e^{-\phi_1} \). Here \( \partial_i \equiv \partial_{w^i} \). The indices take values \( \alpha, \beta \in \{1, 2, 3\} \) and \( i, j, k, l, m \in \{1, 2\} \).

The differences in the conformal coordinate functions \( \delta \dot{x}^k(w) := x^k - y^k, k = 1, 2, \) satisfy on the the disc \( D(r) \) \( (r = r(t)) \) an equation of the form

\[ \Delta_{g_2} \delta \dot{x}^k(w) = \delta F^k(w, \delta \phi, \delta g, g_1, g_2, \psi_{p,i}) \]

where the differential operator \( \delta F^k \) is written schematically as

\[ \delta F^k(w) = \psi_{p,i}A_{\alpha}^{mkl} \partial_i \partial_j g_{2}^{3m}(w) + \psi_{p,i}B_{\alpha}^{mk} \partial_m \partial_\alpha \delta \phi(w) + (\psi_{p,i}C_{\alpha}^{mjk} + \partial_j \psi_{p,i}D_{\alpha}^{mka})(w) \partial_m \partial_\alpha \delta \phi(w) \]
\[ + (\psi_{p,i}E_{\alpha}^{1} + \partial_j \psi_{p,i}F_{\alpha}^{1m})(w) \partial_\alpha \delta \phi^{3m}(w) \]
\[ + (\psi_{p,i}G_{\alpha}^{k} + \partial_j \psi_{p,i}H_{\alpha}^{k})(w) \delta \phi^{3m}(w), \]  

(4.89)

for functions \( A_{\alpha}^{mkl}, \ldots, H_{\alpha}^{kl} \), which depend on \( w \in D(r) \) and are polynomials in the unknown metric coefficients \( g_{1}^{13}, g_{1}^{23}, e^{\phi_1}, e^{-\phi_1} \) and \( g_{2}^{13}, g_{2}^{23}, e^{\phi_2}, e^{-\phi_2} \) and their first and second derivatives at \( w \) as follows:

\( \bar{A}_{\alpha}^{mkl}, \bar{B}_{\alpha}^{mka} \) contain no derivatives of the metric coefficients,
\( \bar{C}_{\alpha}^{mka}, \bar{C}_{\alpha}^{mk}, \bar{E}_{\alpha}^{1m}, \bar{E}_{\alpha}^{k} \) contain up to first derivatives, and
\( \bar{D}_{\alpha}, \bar{D}_{\alpha}^{1}, \bar{F}_{\alpha}^{k}, \bar{F}_{\alpha}^{kl} \) contain up to second derivatives.

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Moreover, the functions $\bar{A}_{m}^{jkl}, \ldots, \bar{F}_{2m}^{kl}$, are linear in the above derivatives of the metric components and are bounded in $L^{\infty}(D(r))$ as follows. In the case where $g_{1}$ and $g_{2}$ are $C^{3}$-close to Euclidean,

\begin{align}
||\bar{A}_{m}^{jkl}||_{L^{\infty}}, ||\bar{B}^{j\alpha}||_{L^{\infty}} & \leq C, \\
||\partial \bar{A}_{m}^{jkl}||_{L^{\infty}}, ||\partial \bar{B}^{j\alpha}||_{L^{\infty}} & \leq C\epsilon_{0}, \\
||\bar{C}_{1}^{\alpha}||_{L^{\infty}}, \ldots, ||\bar{E}_{2m}^{jkl}||_{L^{\infty}} & \leq C\epsilon_{0}, \\
||\partial \bar{A}_{m}^{jkl}||_{L^{\infty}}, ||\partial \bar{B}^{j\alpha}||_{L^{\infty}} & \leq C\epsilon_{0}, \\
||\partial \bar{C}_{1}^{\alpha}||_{L^{\infty}}, \ldots, ||\partial \bar{E}_{2m}^{jkl}||_{L^{\infty}} & \leq C\epsilon_{0}, \\
||\bar{D}_{1}||_{L^{\infty}}, \ldots, ||\bar{F}_{2m}^{kl}||_{L^{\infty}} & \leq C\epsilon_{0}
\end{align}

and

\begin{align}
||\partial \bar{D}_{1}||_{L^{\infty}}, \ldots, ||\partial \bar{F}_{2m}^{kl}||_{L^{\infty}} & \leq C
\end{align}

for a uniform constant $C > 0$ independent of $\epsilon_{0}$.

In the case where $g_{1}$ and $g_{2}$ are $(K, \epsilon_{0}, \delta_{0})$-thin, we have the estimates

\begin{align}
||\bar{A}_{m}^{jkl}||_{L^{\infty}}, ||\bar{B}^{j\alpha}||_{L^{\infty}} & \leq C, \\
||\partial \bar{A}_{m}^{jkl}||_{L^{\infty}}, ||\partial \bar{B}^{j\alpha}||_{L^{\infty}} & \leq C\epsilon_{0}^{-1}, \\
||\bar{C}_{1}^{\alpha}||_{L^{\infty}}, \ldots, ||\bar{E}_{2m}^{jkl}||_{L^{\infty}} & \leq C\epsilon_{0}^{-1}, \\
||\partial \bar{A}_{m}^{jkl}||_{L^{\infty}}, ||\partial \bar{B}^{j\alpha}||_{L^{\infty}} & \leq C\epsilon_{0}^{-2}, \\
||\partial \bar{C}_{1}^{\alpha}||_{L^{\infty}}, \ldots, ||\partial \bar{E}_{2m}^{jkl}||_{L^{\infty}} & \leq C\epsilon_{0}^{-2}, \\
||\bar{D}_{1}||_{L^{\infty}}, \ldots, ||\bar{F}_{2m}^{kl}||_{L^{\infty}} & \leq C\epsilon_{0}^{-2}
\end{align}

and

\begin{align}
||\partial \bar{D}_{1}||_{L^{\infty}}, \ldots, ||\partial \bar{F}_{2m}^{kl}||_{L^{\infty}} & \leq C\epsilon_{0}^{-4}
\end{align}
These estimates follow from Lemma 2.4 and the estimates for \(g_1, g_2\), and their derivatives in the definition of either \(C^3\)-close to Euclidean or \((K, \epsilon_0, \delta_0)\)-thin, which continue to hold in the coordinates \((x^\alpha), \alpha \in \{1, 2, 3\}\) (see Remark 2.5).

Now, given (4.88) and since \(\delta x^k_i = 0\) on \(\partial M\), at a point \(p \in M\) we have the expression

\[
\delta x^k_i(p) = \int_{D(r)} G(p, w)\delta F^k(w) \, dw,
\]

where \(G(p, w)\) is the Dirichlet Green’s function on the disc \(D(r)\):

\[
\Delta_{g^0} G(p, w) = \delta(p-w) \quad \text{for } x(p) \in D(r)
\]

\[
G(p, w) = 0 \quad \text{for } x(p) \in \partial D(r).
\]

Since \(\delta F^k\) vanishes on \(\partial D(r)\), using (4.89) and integrating by parts, (4.110) becomes

\[
\delta x^k_i(p) = \int_{D(r)} G(p, w)\delta F^k(w) \, dw
\]

\[
= \int_{D(r)} G(p, w)\delta F^k(w) \, dw + \int_{D(r)} \partial D(p) G(p, w)\delta F^k(w) \, dw
\]

\[
\overset{IBP}{=} K^k_{i,1}(\delta g^{31})(p) + K^k_{i,2}(\delta g^{32})(p) + L^k_{i,1}(\delta \phi)(p) + L^k_{i,2}(\partial \delta \phi)(p),
\]

where

\[
K^k_{i,j}(f)(p) := \int_{D(r)} K^k_{i,j}(p, w) f(w) \, dw,
\]

\[
L^k_{i,j}(f)(p) := \int_{D(r)} L^k_{i,j}(p, w) f(w) \, dw
\]

are integral operators with kernels defined as

\[
K^k_{i,1}(p, w) := \partial_i \partial_j [G(p, w)\psi_{p,i} \bar{A}^{jk}(w)] - \partial_j [G(p, w)\psi_{p,i} \bar{E}^{j1}_i + \partial_i \psi_{p,i} \bar{E}^{jk}_2](w)
\]

\[
+ G(p, w)\psi_{p,i} \bar{F}^{k1}_i + \partial_i \psi_{p,i} \bar{F}^{jk}_2,
\]

\[
(4.111)
\]

\[
K^k_{i,2}(p, w) := \partial_i \partial_j [G(p, w)\psi_{p,i} \bar{A}^{jk2}(w)] - \partial_j [G(p, w)\psi_{p,i} \bar{E}^{j2}_i + \partial_i \psi_{p,i} \bar{E}^{jk}_2](w)
\]

\[
+ G(p, w)\psi_{p,i} \bar{F}^{k2}_i + \partial_i \psi_{p,i} \bar{F}^{jk}_2,
\]

\[
(4.112)
\]

\[
L^k_{i,1}(p, w) := \partial_j \partial_i [G(p, w)\psi_{p,i} \bar{B}^{jk1}(w)] - \partial_i [G(p, w)\psi_{p,i} \bar{C}^{k1}_i + \partial_j \psi_{p,i} \bar{C}^{jk1}_2](w)
\]

\[
+ G(p, w)\psi_{p,i} \bar{D}_i + \partial_j \psi_{p,i} \bar{D}^k_2(w),
\]

\[
(4.113)
\]

\[
L^k_{i,2}(p, w) := \partial_j [G(p, w)\psi_{p,i} \bar{B}^{jk2}(w)] + G(p, w)\psi_{p,i} \bar{C}^{k2}_i + \partial_j \psi_{p,i} \bar{C}^{jk2}_2(w),
\]

\[
(4.114)
\]

which have singularities of order \(-1, -1, -1, -1\) respectively. Above we denote \(\partial_i := \partial_{x^i}\), and the indices take values \(\alpha, \beta \in \{1, 2, 3\}\) and \(i, j, k, l, m \in \{1, 2\}\).

To solve for \(\delta g^{3k}, k = 1, 2\) in each of the settings where the metrics area either \(C^3\)-close to Euclidean or \((K, \epsilon_0, \delta_0)\)-thin, we require that the operators \(K^k_{i,j}\), and \(L^k_{i,1}, i, j = 1, 2\) be
Lemma 4.6. For \( f \in L^2(D(r)) \), the operator \( T \) given by

\[
T(f)(p) := \int_{D(r)} G(p, w) \psi_{p,i}(w) f(w) \, dw
\]

lies in \( H^3(D(r)) \), and \( ||T(f)||_{H^3(D(r))} \leq C ||f||_{L^2(D(r))} \) for some universal constant \( C > 0 \) (depending on \( \delta_0 \) in Case 2). In particular,

**Lemma 4.6.** For \( f \in L^2(D(r)) \), the operator \( T \) given by

\[
T(f)(p) := \int_{D(r)} G(p, w) \psi_{p,i}(w) f(w) \, dw
\]
maps \( L^2(D(r)) \rightarrow H^2(D(r)) \). Moreover, \( T \) satisfies

\[
||T(f)||_{H^1(D(r))} \leq C r^3-j ||f||_{L^2(D(r))}
\]

for \( j \in \{0, 1, 2\} \) and some universal constant \( C > 0 \) independent of \( r \) and \( \epsilon_0 \).

Proof. Notice that by Lemma 4.2 the functions \( \psi_{p,i} \) vanish at the point \( p \), so

\[
\Delta_{ge} T(f)(p) = \psi_{p,i}(p)f(p) + \int 2 \delta^{jk} \partial_{\rho_j} G(p, w) \partial_{\rho_k} \psi_{p,i}(w)f(w) \, dw
\]

\[
+ \int G(p, w) \Delta_{ge} \psi_{p,i}(w) f(w) \, dw
\]

\[
= 2T_1(f)(p) + T_2(f)(p),
\]

where

\[
T_1(f)(p) = \int \delta^{jk} \partial_{\rho_j} G(p, w) \partial_{\rho_k} \psi_{p,i}(w)f(w) \, dw,
\]

\[
T_2(f)(p) = \int G(p, w) \Delta_{ge} \psi_{p,i}(w) f(w) \, dw,
\]

and all the integrals here and below are computed over \( D(r) \). Further, since \( G(p, w) = 0 \) on \( \partial D(r) \), so too \( T(f)(p) = 0 \) on \( \partial D(r) \). It thus will suffice for us to show that the functions \( T_1(f) \) and \( T_2(f) \)

\[
T_1(f)(p) = \int \delta^{jk} \partial_{\rho_j} G(p, w) \partial_{\rho_k} \psi_{p,i}(w)f(w) \, dw,
\]

\[
T_2(f)(p) = \int G(p, w) \Delta_{ge} \psi_{p,i}(w) f(w) \, dw,
\]

are uniformly bounded in \( H^1(D(r)) \) in terms of the \( L^2(D(r)) \) norm of \( f \), and with a constant whose dependence on the various parameters is as above.

We first show \( ||T_2(f)||_{L^2} \leq C r^3 \log r ||f||_{L^2} \) and \( ||T_2(f)||_{H^1} \leq C r^2 \log r ||f||_{L^2} \) for \( f \in L^2(D(r)) \). Observe

\[
||T_2(f)||_{L^2} \leq C \sup_p ||\Delta_{ge} \psi_{p,i}(w)||_{C^0_g} ||\log ||g||_{C^1_g} || ||f||_{L^2}
\]

\[
\leq C r^2 \log r \sup_p ||\Delta_{ge} \psi_{p,i}(w)||_{C^0_g} ||f||_{L^2},
\]

where \( C > 0 \) is a uniform constant. From the estimates for \( \psi_{p,i} \) in Lemma 4.2 \( ||\Delta_{ge} \psi_{p,i}(w)||_{C^0_g} \leq C r \) for some constant \( C \) (which depends on \( \delta_0 \) in the second case of Theorem 1.4). Thus

\[
T_2 : L^2(D(r)) \rightarrow L^2(D(r)), \quad ||T_2(f)||_{L^2} \leq C r^3 \log r ||f||_{L^2}.
\]

Additionally, for \( j = 1, 2 \)

\[
\partial_{p^j} T_2(f)(p) = \int \partial_{p^j} G(p, w) \Delta_{ge} \psi_{p,i}(w)f(w) + G(p, w) \partial_{p^j} \Delta_{ge} \psi_{p,i}(w) f(w) \, dw.
\]
We estimate
\[
\left\| \int \partial_p G(p, w) \Delta_{g}\psi_{p,i}(w) f(w) \, dw \right\|_{L^2} \leq C \sup_p \|\Delta_{g}\psi_{p,i}(w)\|_{C^0_w} \left\| \frac{1}{\|w\|_{g_e}} \right\|_{L^1} \|f(w)\|_{L^2}
\]
and similarly
\[
\left\| \int G(p, w) \partial_{p^j} \Delta_{g}\psi_{p,i}(w) f(w) \, dw \right\|_{L^2} \leq C \sup_p \|\partial_{p^j} \Delta_{g}\psi_{p,i}(w)\|_{C^0_w} \|\log \|w\|_{g_e}\|_{L^1} \|f(w)\|_{L^2}
\]
where we used \(|\|\partial_{p^j} \Delta_{g}\psi_{p,i}(w)\|_{C^0_w} \leq C\) for some constant \(C\) (see Lemma 4.2). In summary, \(T_2 : L^2(D(r)) \to H^1(D(r))\) and \(||T_2(f)||_{H^1} \leq C r^2 (1 + \log r) ||f||_{L^2}||.

As above, we have \(T_1 : L^2(D(r)) \to L^2(D(r))\) and ||\(T_1(f)||_{L^2} \leq C r ||f||_{L^2}||.

Next, we prove \(\partial_{p^l} T_1 : L^2(D(r)) \to L^2(D(r)), l = 1, 2,\) and \(||\partial_{p^l} T_1(f)||_{L^2} \leq C ||f||_{L^2}||. We have
\[
\partial_{p^l} T_1(f)(p) = \int \delta^{j^k} \partial_{p^j} \partial_{p^l} G(p, w) \partial_{p^k} \psi_{p,i}(w) f(w) + \delta^{j^k} \partial_{p^j} G(p, w) \partial_{p^k} \partial_{p^l} \psi_{p,i}(w) f(w) \, dw
\]
\[
=: I_1(f)(p) + I_2(f)(p).
\]
First, in view of Lemma 4.2, the Taylor expansion of \(\partial_{p^l} \psi_{p,i}\) about \(p = w\) yields
\[
\partial_{p^l} \psi_{p,i}(w) = \delta_{ij} + \nabla_p \partial_{p^j} \psi_{p,i}(p) \cdot (w - p) + (w - p) \cdot \nabla_p \nabla_p \partial_{p^j} \psi_{p,i}(p) \cdot (w - p) + O(||w - p||_{g_e}^3),
\]
with \(||\nabla_p \partial_{p^j} \psi_{p,i}(p)||_{g_e} \leq C r\) and \(||\nabla_p \nabla_p \partial_{p^j} \psi_{p,i}(p)||_{g_e} \leq C\) for some constants (which depend on \(\delta_0\) in the Class 2 case of Theorem 1.4). Then,
\[
I_1(f)(p) = - \int \delta^{j^k} \partial_{p^j} \partial_{p^l} G(p, w) f(w) \, dw
\]
\[
+ \int \delta^{j^k} \partial_{p^j} \partial_{p^l} G(p, w) \nabla_{p^k} \nabla_{p^l} \psi_{p,i}(p) [\delta_{k}^a (w - p)^a + \delta_{k}^b (w - p)^b] f(w) \, dw
\]
\[
+ \int \delta^{j^k} \partial_{p^j} \partial_{p^l} G(p, w) \cdot O(||w - p||_{g_e}^2) f(w) \, dw
\]
\[
=: J_1(f)(p) + J_2(f)(p) + J_3(f)(p).
\]
Since \(G(p, w)\) is the Dirichlet Green’s function over \(D(r)\), there is a uniform constant such that
\[
||J_1(f)||_{L^2} \leq C ||f||_{L^2}.
\]
The operators \(J_2, J_3\) and \(I_2\) are weakly singular, and bounded on \(L^2(D(r))\) by estimates similar to those for \(T_2\) above. Hence \(\partial_{p^l} T_1 : L^2(D(r)) \to L^2(D(r))\) and \(||\partial_{p^l} T_1(f)||_{L^2} \leq C ||f||_{L^2}||.

Therefore, we have
\[
||\Delta_{g} T(f)(p)||_{H^1} = ||2 T_1(f)(p) + T_2(p)(f)(p)||_{H^1} \leq C ||f||_{L^2}.
\]
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By standard elliptic regularity, \( T : L^2(D(r)) \to H^3(D(r)) \) with \( \|T(f)\|_{H^3(D(r))} \leq C\|f\|_{L^2(D(r))} \) for some universal constant \( C > 0 \) independent of \( r \).

From the \( L^2(D(r)) \) estimates for \( T_1 \) and \( T_2 \) we have

\[
\|\Delta g_{ij}T(f)(p)\|_{L^2} = \|2T_1(f)(p) + T_2(p)(f)(p)\|_{L^2} \leq Cr[1 + r^2 \log r]\|f\|_{L^2}.
\]

We now use this estimate to prove (4.119).

The inverse \( G \) of the Dirichlet Laplacian on \( D(r) \) satisfies

\[
\|G(f)\|_{H^3(D(r))} \leq Cr^{2-j}\|f\|_{L^2(D(r))}
\]

for \( j \in \{0, 1, 2\} \). Thus,

\[
\|T(f)\|_{H^3(D(r))} = \|G\Delta g_{ij}T(f)\|_{H^3(D(r))} \leq C r^{2-j}\|G\Delta g_{ij}T(f)\|_{L^2(D(r))} \leq C r^{3-j}\|f\|_{L^2(D(r))},
\]

which proves estimate (4.119).

Together with Lemma 4.6 above, the \( L^\infty(D(r)) \) estimates (4.90)-(4.99) or (4.100)-(4.109), for the functions \( \bar{A}_{jkm}, \ldots, F_{pnm}, j, k, l, m \in \{1, 2\} \) and their first and second derivatives allow us to prove the estimates (4.115), (4.117), and (4.118) for the operators \( K_{i,j} \) and \( \mathcal{L}_{i,1}, \mathcal{L}_{i,2} \), \( i, j \in \{0, 1, 2\} \) respectively. For example, a term in the operator \( K_{i,j} \) has a kernel of the form

\[
U(p, w) = \partial_w^j(G(p, w)\psi_p(w))\partial_w^L\bar{A}(w),
\]

with \( j + l \leq 2 \). Consequently, given the above bounds for the operator \( T \) holds, the norm of the operator \( U \) with kernel \( U(p, w) \) from \( L^2(D(r)) \) into \( H^{3-j}(D(r)) \) satisfies

\[
\|U(f)\|_{H^{3-j}} \leq \|\partial_w^j\bar{A}\|_{L^\infty}\|\partial_w^lT\|_{L^2} \|\partial_w^j\bar{A}\|_{H^{3-j}}\|f\|_{L^2}.
\]

Proceeding term by term, Lemma 4.6 and estimates of the form (4.126) prove (4.115), (4.117), and (4.118). To show (4.116), for \( h \in H^2(D(r)) \), we note that the kernels of the terms in \( \partial_w^2K_{i,j}(\bar{h})(p) \) are of the form

\[
\partial_p^2\partial_w^L(G(p, w)\psi_p(w))\partial_w^L\bar{A}(w).
\]

for \( j + l = 2 \). When \( l \neq 0 \), Lemma 4.6 and estimates of the form (4.126) give the desired bound. Thus, to prove (4.116), when the kernel has a term with \( l = 0 \) it suffices to show that for \( i, j \in \{1, 2\} \), the operator

\[
U(h)(p) := \int_{D(r)}\partial_w\partial_w^j(G(p, w)\psi_p(w))\bar{A}(w)h(w)\,dw
\]

maps \( H^2(D(r)) \to H^2(D(r)) \), bounded.

**Lemma 4.7.** We have

\[
\|U(h)\|_{H^2(D(r))} \leq C\|h\|_{H^2(D(r))},
\]

(4.128)
Proof. Integration by parts twice yields

\[
U(h)(p) = \int_{D(r)} G(p,w)\psi_p(w)\partial_{w,i}\partial_{w,j}[\bar{A}(w)h(w)] \, dw + \int_{\partial D(r)} \nu^i\partial_{w,j}[G(p,w)\psi_p(w)]\bar{A}(w)h(w) \, dS
\]

\[
:= U_1(h)(p) + U_2(h)(p).
\]

(4.130)

Above \(i, j \in \{1, 2\}\), and each of the functions \(\nu^i(w)\) are equal to either \(\cos(\theta)\) or \(\sin(\theta)\). Notice that we have only one nontrivial boundary integral \(U_2(h)(p)\) as \(G(p, w) = 0\) for \(w \in \partial D(r)\).

Consider the operator \(U_1(h)\). We have

\[
U_1(h)(p) = T(\partial_{w,i}\partial_{w,j}[\bar{A}(w)h(w)])
\]

(4.132)

where \(T : L^2 \to H^3\) is defined in Lemma 4.6. Therefore, inequality (4.128) for \(U_1\) follows from Lemma 4.6 and the estimates (4.90)–(4.99) or respectively (4.100)–(4.109), for the function \(\bar{A}(w)\) and its derivatives.

Now consider \(U_2(h)(p)\). Since \(G(p, w)\) and its tangential derivative are zero for \(w \in \partial D(r)\), we can rewrite \(U_2(h)\) as

\[
U_2(h) = \int_{\partial D(r)} \nu^i(w)\nu^j(w)\partial G(p,w)\frac{\partial \psi_p(w)}{\partial \nu(w)}\bar{A}(w)h(w) \, dz(\theta) \, dS(w)
\]

Let \(v_{ij}(w)\) be the harmonic extension of \(\nu^i(w)\nu^j(w) = \frac{w^i w^j}{r^2}\) to \(D(r)\). Then, using Green’s identity we have

\[
U_2(h) = v_{ij}(p)\psi_p(p)\bar{A}(p)h(p) - \int_{D(r)} G(p,w)\Delta_{gel}[v_{ij}(w)\psi_p(w)\bar{A}(w)h(w)] \, dw.
\]

The first term vanishes since \(\psi_p(p) = 0\). The integral term is bounded on \(H^2(D(r))\) uniformly in \(r\) by arguments similar to those in Lemma 4.6. Here we use \(|v_{ij}(w)| \leq 1, |\nabla v_{ij}(w)| \leq \frac{1}{r}\), as well as the pointwise estimates on \(\psi_p(w)\) (see Lemma 4.2), \(\bar{A}\) (see (4.90)–(4.99) or (4.100)–(4.109)), and their derivatives up to second order.

\[
\square
\]

Employing this strategy to each term in appearing in the kernels (4.111)–(4.114), in the close to Euclidean case we have:

\[
K_{i,j}^k : L^2(D(r)) \to H^1(D(r)), \quad ||K_{i,j}^k(f)||_{H^1} \leq C\epsilon_0||f||_{L^2}, \quad (4.133)
\]

\[
K_{i,j}^k : H^2(D(r)) \to H^2(D(r)), \quad ||K_{i,j}^k(f)||_{H^2} \leq C\epsilon_0||f||_{H^2}, \quad (4.134)
\]

\[
\mathcal{L}_{i,1}^k : L^2(D(r)) \to H^1(D(r)), \quad ||\mathcal{L}_{i,1}^k(f)||_{H^1} \leq C\epsilon_0||f||_{L^2}, \quad (4.135)
\]

\[
\mathcal{L}_{i,2}^k : L^2(D(r)) \to H^1(D(r)), \quad ||\mathcal{L}_{i,2}^k(f)||_{H^1} \leq C\epsilon_0||f||_{L^2}, \quad (4.136)
\]

\[
\mathcal{L}_{i,2}^k : L^2(D(r)) \to H^2(D(r)), \quad ||\mathcal{L}_{i,2}^k(f)||_{H^2} \leq C\epsilon_0||f||_{L^2}, \quad (4.137)
\]

where \(C > 0\) is a uniform constant independent of \(\epsilon_0\).
In the \((K, \epsilon_0, \delta_0)\)-thin case, from the estimates \((4.100)\)–\((4.109)\) the differentiated terms
\(\partial_{w_i} A_{kl}^m, \ldots, \partial_{w_i} F_{2m}^{kl} \) \(i, j, k, l, m \in \{1, 2\}, \alpha \in \{1, 2, 3\}\) in each of the kernels \((4.111)\)–\((4.114)\) will satisfy \(L^\infty(D(r))\) bounds.

Then by Lemma \ref{lem:bound_per} we bound each of the operators \(K_{i,j}^k\) and \(L_{i,j}^k\) term by term to obtain
\begin{align}
K_{i,j}^k : L^2(D(r)) &\rightarrow L^2(D(r)), \quad ||K_{i,j}^k(f)||_{L^2} \leq C \epsilon_0^{-2}r^3 ||f||_{L^2}, \quad (4.138) \\
K_{i,j}^k : L^2(D(r)) &\rightarrow H^1(D(r)), \quad ||K_{i,j}^k(f)||_{H^1} \leq C ||f||_{L^2}, \quad (4.139) \\
K_{i,j}^k : H^2(D(r)) &\rightarrow H^2(D(r)), \quad ||K_{i,j}^k(f)||_{H^2} \leq C \epsilon_0^{-2}r^3 ||f||_{H^2}, \quad (4.140)
\end{align}
\begin{align}
L_{i,1}^k : L^2(D(r)) &\rightarrow L^2(D(r)), \quad ||L_{i,1}^k(f)||_{L^2} \leq C \epsilon_0^{-2}r^3 ||f||_{L^2}, \quad (4.141) \\
L_{i,1}^k : L^2(D(r)) &\rightarrow H^1(D(r)), \quad ||L_{i,1}^k(f)||_{H^1} \leq C ||f||_{L^2}, \quad (4.142) \\
L_{i,2}^k : L^2(D(r)) &\rightarrow L^2(D(r)), \quad ||L_{i,2}^k(f)||_{L^2} \leq C \epsilon_0^{-1}r^3 ||f||_{L^2}, \quad (4.143) \\
L_{i,2}^k : L^2(D(r)) &\rightarrow H^1(D(r)), \quad ||L_{i,2}^k(f)||_{H^1} \leq C \epsilon_0^{-1}r^2 ||f||_{L^2}, \quad (4.144) \\
L_{i,2}^k : L^2(D(r)) &\rightarrow H^2(D(r)), \quad ||L_{i,2}^k(f)||_{H^2} \leq C ||f||_{L^2}, \quad (4.145)
\end{align}
where the constant \(C > 0\) is independent of \(r, \epsilon_0\) and in the second class depends only on \(\delta_0\) as in Definition \ref{def:deltaclasses}. With the above estimates in hand, we may define the relevant inverse operators to solve for \(\delta g^{31}\) and \(\delta g^{32}\) in terms of \(\delta \phi, \partial_3 \delta \phi\). Indeed, the equations \((4.77), (4.78)\), which describe \(\delta g^{31}\) and \(\delta g^{32}\), can be written in terms of the operators \(K_{i,j}^k\) and \(L_{i,j}^k\), \(i, j, k = 1, 2\), via the system
\begin{align}
[I - K](\delta g^{31}, \delta g^{32}) &= \mathcal{L}(\delta \phi, \partial_3 \delta \phi) \quad (4.146)
\end{align}
where
\begin{align}
I &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \\
\epsilon_k := \partial_k ||\nabla x^3||_{g_1}, \\
K &= \begin{pmatrix} \epsilon_k K_{1,1}^k & \epsilon_k K_{1,2}^k \\ \epsilon_k K_{2,1}^k & \epsilon_k K_{2,2}^k \end{pmatrix}, \\
\mathcal{L} &= \begin{pmatrix} \epsilon_k L_{1,1}^k & \epsilon_k L_{1,2}^k \\ \epsilon_k L_{2,1}^k & \epsilon_k L_{2,2}^k \end{pmatrix},
\end{align}
k \in \{1, 2\}.

To solve this system for \(\delta g^{31}\), \(\delta g^{32}\) in terms of \(\delta \phi, \partial_3 \delta \phi\), we need to invert \(I - K\). It will suffice to show that the operators \(\epsilon_k K_{i,j}^k\) \(i, j, k = 1, 2\), have small norms as operators \(L^2(D(r)) \rightarrow L^2(D(r))\). We show the necessary smallness requirement in each case of Theorem \ref{thm:main_thm}. Further, for \(i, j = 1, 2\) and \(k \in \{1, 2\}\), we will derive \(H^1(D(r))\) estimates for \(\epsilon_k K_{i,j}^k\) and \(\epsilon_k L_{i,1}^k\), as well as \(H^2(D(r))\) estimates for \(\epsilon_k L_{i,2}^k\). The estimates for \(\epsilon_k K_{i,j}^k\) we use to invert the system \((4.146)\), and also together with the estimates for \(\epsilon_k L_{i,2}^k\) we prove \((4.85)\)–\((4.87)\).
Consider $\varepsilon_k K^k_{i,j}$, and let $f \in L^2(D(r))$. Then,

$$
||\varepsilon_k K^k_{i,j}(f)||_{L^2} \leq ||\varepsilon_k||_{L^\infty} ||K^k_{i,j}(f)||_{L^2}
$$

and

$$
||\partial(\varepsilon_k K^k_{i,j})(f)||_{L^2} \leq ||\varepsilon_k||_{L^\infty} ||\partial K^k_{i,j}(f)||_{L^2} + ||\partial\varepsilon_k||_{L^\infty} ||K^k_{i,j}(f)||_{L^2}.
$$

Note

$$
|\varepsilon_k(p)| := |\partial_k||\nabla x^3||_g^1(p)| = |\partial_k\sqrt{g^1_{33}(p)}| \leq C||\nabla g^3_{33}||_{L^\infty}, \quad (4.147)
$$

$$
|\partial\varepsilon_k(p)| \leq C||\nabla g^3_{13}||_{L^\infty}, \quad (4.148)
$$

for some constant universal constant $C$ independent. Hence,

$$
||\varepsilon_k K^k_{i,j}(f)||_{L^2} \leq C||\nabla g^3_{33}||_{L^\infty} ||K^k_{i,j}(f)||_{L^2}, \quad (4.149)
$$

$$
||\partial(\varepsilon_k K^k_{i,j})(f)||_{L^2} \leq C \left(||\nabla g^3_{33}||_{L^\infty} ||\partial K^k_{i,j}(f)||_{L^2} + ||\nabla\nabla g^3_{33}||_{L^\infty} ||K^k_{i,j}(f)||_{L^2}\right), \quad (4.150)
$$

$$
||\partial(\varepsilon_k K^k_{i,j})(f)||_{L^2} \leq C \left(||\nabla g^3_{33}||_{L^\infty} ||\partial K^k_{i,j}(f)||_{L^2} + 2||\nabla\nabla g^3_{33}||_{L^\infty} ||\partial K^k_{i,j}(f)||_{L^2} + ||\nabla\nabla g^3_{13}||_{L^\infty} ||K^k_{i,j}(f)||_{L^2}\right). \quad (4.151)
$$

By analogous computations, we obtain estimates for the operators $\mathcal{L}_{i,j}$, $i, j = 1, 2$ (for some universal constant $C$):

$$
||\varepsilon_k \mathcal{L}^k_{i,j}(f)||_{L^2} \leq C||\nabla g^3_{13}||_{L^\infty} ||\mathcal{L}^k_{i,j}(f)||_{L^2}, \quad (4.152)
$$

$$
||\partial(\varepsilon_k \mathcal{L}^k_{i,j})(f)||_{L^2} \leq C \left(||\nabla g^3_{33}||_{L^\infty} ||\partial \mathcal{L}^k_{i,j}(f)||_{L^2} + ||\nabla\nabla g^3_{13}||_{L^\infty} ||\mathcal{L}^k_{i,j}(f)||_{L^2}\right). \quad (4.153)
$$

$$
||\partial(\varepsilon_k \mathcal{L}^k_{i,j})(f)||_{L^2} \leq C \left(||\nabla g^3_{33}||_{L^\infty} ||\partial \mathcal{L}^k_{i,j}(f)||_{L^2} + 2||\nabla\nabla g^3_{33}||_{L^\infty} ||\partial \mathcal{L}^k_{i,j}(f)||_{L^2} + ||\nabla\nabla g^3_{13}||_{L^\infty} ||\mathcal{L}^k_{i,j}(f)||_{L^2}\right). \quad (4.154)
$$

Now we consider the first case of Theorem 1.4. Recall in this case, we are assuming that the metric $g_1$ and $g_2$ are $C^3$-close to Euclidean. From (4.133) and (4.149)–(4.150), the operators $\varepsilon_k K_{i,j} : L^2 \to L^2$ have norm controlled by $\varepsilon_0$, which is small. Thus $I - K : L^2 \to L^2$ is invertible.

In the second case of Theorem 1.4 we are assuming that the metric $g_1$ and $g_2$ are $(K, \varepsilon_0, \delta_0)$-thin for some $K > 0$ and sufficiently small $\delta_0, \varepsilon_0 > 0$. Using (4.138) and (4.149)–(4.150), together with the bounds for the operators $\varepsilon_k K_{i,j} : L^2 \to L^2$, obey

$$
||\varepsilon_k K_{i,j}(f)||_{L^2} \leq C||\nabla g^3_{33}||_{L^\infty} \varepsilon_0^{-2} r^3 ||(f)||_{L^2} \quad (4.155)
$$

$$
\leq C||\nabla g^3_{13}||_{L^\infty} \varepsilon_0^{-2} r^3 ||(f)||_{L^2}. \quad (4.156)
$$

Since $0 < r < \varepsilon_0$, we derive that

$$
||\nabla g^3_{33}||_{L^\infty} \varepsilon_0^{-2} r^3 \leq K\varepsilon_0,
$$
which is sufficiently small; this implies that $I - K : L^2 \to L^2$ is invertible.

Therefore, in both cases the system (4.146) is solvable in terms of $\delta \phi$ and $\partial_3 \delta \phi$:

$$
\delta g^{31} = P_{-1}^1(\delta \phi) + Q_{-2}^1(\partial_3 \delta \phi),
$$

$$
\delta g^{32} = P_{-1}^2(\delta \phi) + Q_{-2}^2(\partial_3 \delta \phi),
$$

where $P_{-1}^k$, $Q_{-2}^k$, $k = 1, 2$ are respectively order $-1$ and $-2$ pseudodifferential operators in the tangential directions $\partial_k$, $k = 1, 2$, given by the compositions

$$
\left( \frac{P_{-1}^1(\delta \phi)}{P_{-1}^2(\delta \phi)} \right) := (I - K)^{-1} L \left( \frac{\delta \phi}{0} \right),
$$

$$
\left( \frac{Q_{-2}^1(\partial_3 \delta \phi)}{Q_{-2}^2(\partial_3 \delta \phi)} \right) := (I - K)^{-1} L \left( \frac{0}{\partial_3 \delta \phi} \right).
$$

The estimates (4.149)–(4.153) together with (4.133)–(4.137) or respectively (4.138)–(4.145) give the claimed inequalities (4.85) and (4.86). Indeed, under the $\epsilon_0 C^3$-close to Euclidean assumption, from (4.152), (4.133)–(4.137), and the $L^2$ bound on $(I - K)^{-1}$, we have

$$
||P_{-1}^k(\delta \phi)||_{L^2} \leq C \epsilon_0 ||\nabla g_{13}^3||_{L^\infty} ||\delta \phi||_{L^2}
$$

$$
||Q_{-2}^k(\partial_3 \delta \phi)||_{L^2} \leq C \epsilon_0 ||\nabla g_{13}^3||_{L^\infty} ||\partial_3 \delta \phi||_{L^2}.
$$

By (4.152) and (4.153), the operator $\partial P_{-1}^k : L^2 \to L^2$, $k = 1, 2$, satisfies

$$
||\partial P_{-1}^k(\delta \phi)||_{L^2} \leq ||(I - K)^{-1}||_{L^2 \to L^2} \left[ ||\partial(\epsilon_j L_{1,1}^j)(\delta \phi)||_{L^2} + ||\partial(\epsilon_j L_{1,1}^j)(\delta \phi)||_{L^2} \right]
$$

$$
+ ||\partial(I - K)^{-1}||_{L^2 \to L^2} \left[ ||\epsilon_j L_{2,1}^j(\delta \phi)||_{L^2} + ||\epsilon_j L_{2,1}^j(\delta \phi)||_{L^2} \right]
$$

$$
\leq C \epsilon_0 \left( ||\nabla g_{13}^3||_{L^\infty} + ||\nabla g_{13}^3||_{L^\infty} \right) ||\delta \phi||_{L^2}
$$

$$
+ C \epsilon_0 \left( ||\nabla g_{13}^3||_{L^\infty} + ||\nabla g_{13}^3||_{L^\infty} \right) ||\nabla g_{13}^3||_{L^\infty} ||\delta \phi||_{L^2},
$$

since

$$
\partial(I - K)^{-1} = (I - K)^{-1} \partial K(I - K)^{-1};
$$

which is bounded from $L^2 \to L^2$ since $(I - K)^{-1}$ and $\partial K$ are bounded from $L^2 \to L^2$.

Similarly, inequalities (4.152) and (4.153) imply for $\partial Q_{-2}^k : L^2 \to L^2$, $k = 1, 2$,

$$
||\partial Q_{-2}^k(\partial_3 \delta \phi)||_{L^2} \leq ||(I - K)^{-1}||_{L^2 \to L^2} \left[ ||\partial(\epsilon_j L_{1,2}^j)(\partial_3 \delta \phi)||_{L^2} + ||\partial(\epsilon_j L_{1,2}^j)(\partial_3 \delta \phi)||_{L^2} \right]
$$

$$
+ ||\partial(I - K)^{-1}||_{L^2 \to L^2} \left[ ||\epsilon_j L_{2,2}^j(\partial_3 \delta \phi)||_{L^2} + ||\epsilon_j L_{2,2}^j(\partial_3 \delta \phi)||_{L^2} \right]
$$

$$
\leq C \epsilon_0 \left( ||\nabla g_{13}^3||_{L^\infty} + ||\nabla g_{13}^3||_{L^\infty} \right) ||\partial_3 \delta \phi||_{L^2}
$$

$$
+ C \epsilon_0 \left( ||\nabla g_{13}^3||_{L^\infty} + ||\nabla g_{13}^3||_{L^\infty} \right) ||\nabla g_{13}^3||_{L^\infty} ||\partial_3 \delta \phi||_{L^2},
$$

$$
\leq C \epsilon_0^2 ||\delta \phi||_{L^2},
$$

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Further,

\[ ||\partial Q^k_2(\partial_3 \delta \phi)||_{L^2} \leq ||(I - \mathcal{K})^{-1}||_{L^2 \to L^2} \left[ ||\partial \partial (\varepsilon_j \mathcal{L}^j_{1,2}) (\partial_3 \delta \phi)||_{L^2} + ||\partial \partial (\varepsilon_j \mathcal{L}^j_{2,2}) (\partial_3 \delta \phi)||_{L^2} \right] + 2 ||\partial (I - \mathcal{K})^{-1}||_{L^2 \to L^2} \left[ ||\partial (\varepsilon_j \mathcal{L}^j_{1,2}) (\partial_3 \delta \phi)||_{L^2} + ||\partial (\varepsilon_j \mathcal{L}^j_{2,2}) (\partial_3 \delta \phi)||_{L^2} \right] + ||\partial \partial (I - \mathcal{K})^{-1}||_{H^2 \to L^2} \left[ ||\varepsilon_j \mathcal{L}^j_{1,2} (\partial_3 \delta \phi)||_{H^2} + ||\varepsilon_j \mathcal{L}^j_{2,2} (\partial_3 \delta \phi)||_{H^2} \right] \]

(4.158)

is an estimate for \( \partial Q^k_2 \), \( k = 1, 2 \). We claim that when the metrics are of Class 1,

\[ ||\partial Q^k_2(\partial_3 \delta \phi)||_{L^2} \leq C \epsilon_0 ||\partial_3 \delta \phi||_{L^2} \]

for a universal constant \( C \).

To see this, recall \( \mathcal{K} \) is comprised of the operators \( \varepsilon_j \mathcal{K}^k_{i,j} \), which obey estimates (4.149)–(4.151). This together with

\[ \partial \partial (I - \mathcal{K})^{-1} = (I - \mathcal{K})^{-1} \partial \partial \mathcal{K} (I - \mathcal{K})^{-1} - 2 (I - \mathcal{K})^{-1} \partial \mathcal{K} (I - \mathcal{K})^{-1} \partial \mathcal{K} (I - \mathcal{K})^{-1} \]

(4.159)

implies \( \partial \partial (I - \mathcal{K})^{-1} : H^2 \to L^2 \) is bounded. From the argument below (4.157), we know \( (I - \mathcal{K})^{-1}, \partial (I - \mathcal{K})^{-1} : L^2 \to L^2 \) are uniformly bounded. Then (4.149)–(4.154) proves the claim.

This proves the inequalities (4.82), (4.83), and (4.84).

Now we show the inequalities (4.85)–(4.87) for the second case of Theorem 1.4. From (4.152) together with (4.138)–(4.145), plus the bound on \( (I - \mathcal{K})^{-1} \) we have

\[ ||P^k_{-1}(\delta \phi)||_{L^2} \leq C ||(I - \mathcal{K})^{-1}||_{L^2 \to L^2} ||\nabla g^{33}_1||_{L^\infty} \epsilon_0^{-2} r^3 ||\delta \phi||_{L^2} \leq CK \epsilon_0^{-2} r^3 ||\delta \phi||_{L^2} \]

\[ ||Q^k_{-2}(\partial_3 \delta \phi)||_{L^2} \leq C ||(I - \mathcal{K})^{-1}||_{L^2 \to L^2} ||\nabla g^{33}_1||_{L^\infty} \epsilon_0^{-1} r^3 ||\partial_3 \delta \phi||_{L^2} \leq CK \epsilon_0^{-1} r^3 ||\partial_3 \delta \phi||_{L^2}, \]

where \( C \) is a universal constant. By the same computation as above, using (4.149)–(4.150), (4.152)–(4.153), and (4.138)–(4.145), \( \partial P^k_{-1} : L^2 \to L^2, k = 1, 2 \), satisfies

\[ ||\partial P^k_{-1}(\delta \phi)||_{L^2} \leq ||(I - \mathcal{K})^{-1}||_{L^2 \to L^2} ||\partial (\varepsilon_j \mathcal{L}^j_{1,1}) (\delta \phi)||_{L^2} + ||\partial (\varepsilon_j \mathcal{L}^j_{2,1}) (\delta \phi)||_{L^2} + ||\partial (\varepsilon_j \mathcal{L}^j_{3,1}) (\delta \phi)||_{L^2} \]

\[ \leq C \left( ||\nabla \nabla g^{33}_1||_{L^\infty} \epsilon_0^{-2} r^3 + ||\nabla g^{33}_1||_{L^\infty} \right) ||\delta \phi||_{L^2} + C \left( ||\nabla \nabla g^{33}_1||_{L^\infty} \epsilon_0^{-2} r^3 + ||\nabla g^{33}_1||_{L^\infty} \right) ||\nabla g^{33}_1||_{L^\infty} \epsilon_0^{-2} r^3 ||\delta \phi||_{L^2}, \]

where \( C \) is a universal constant and having used \( 0 < r < \epsilon_0 \) and the estimates for \( ||\nabla^k g^{33}_1||_{L^\infty} \) in Definition 1.2. This shows (4.85) in the \( (K, \delta_0, \epsilon_0) \)-thin case of Theorem 1.4.
Now, to prove (4.86), we calculate as before
\[
||\partial Q^{-1}_2(\partial_3 \delta \phi)||_{L^2} \leq ||(I - \mathcal{K})^{-1}||_{L^2 \rightarrow L^2} ||\partial (\varepsilon_j \mathcal{L}_{1,2}^i)(\partial_3 \delta \phi)||_{L^2} + ||\partial (\varepsilon_j \mathcal{L}_{2,2}^i)(\partial_3 \delta \phi)||_{L^2} \\
+ ||\partial (I - \mathcal{K})^{-1}||_{L^2 \rightarrow L^2} ||\varepsilon_j \mathcal{L}_{1,2}^i(\partial_3 \delta \phi)||_{L^2} + ||\varepsilon_j \mathcal{L}_{2,2}^i(\partial_3 \delta \phi)||_{L^2},
\]
where we used estimates (4.149)–(4.150), (4.152)–(4.153), and (4.138)–(4.145).

Additionally, to show (4.87), we use estimates (4.149)–(4.150), (4.152)–(4.153), and (4.138)–(4.145) in (4.158) to obtain
\[
||\partial Q^{-1}_2(\partial_3 \delta \phi)||_{L^2} \leq C \left( ||\varepsilon_j \mathcal{L}_{1,2}^{33}||_{L^\infty} \epsilon_0^{-1} + \epsilon_0 + \epsilon_0 \right) (||\varepsilon_j \mathcal{L}_{1,2}^{33}||_{L^\infty} \epsilon_0^{-1} + \epsilon_0 + \epsilon_0),
\]
where (4.149)–(4.150), (4.152)–(4.153), and (4.138)–(4.145). Summarizing in the second case of Theorem 1.4, from the bounds for ||\nabla^{k_3}||_{L^\infty} in the definition of (K, \delta_0, \epsilon_0)-thin, and since 0 < r < \epsilon_0, we have for some universal constant C
\[
||\partial P^{-1}_k(\delta \phi)||_{L^2} \leq CK||\delta \phi||_{L^2},
\]
\[
||\partial Q^{-1}_2(\partial_3 \delta \phi)||_{L^2} \leq CK\epsilon_0||\partial_3 \delta \phi||_{L^2}
\]
(4.160)
\[
||\partial Q^{-1}_2(\partial_3 \delta \phi)||_{L^2} \leq CK||\partial_3 \delta \phi||_{L^2}
\]
(4.161)
which completes the proof.

\[
\square
\]

4.1.4 Reduction to uniqueness for \( \delta \phi \)

From Proposition 4.5, we readily obtain the following uniqueness result:

**Lemma 4.8.** If \( \delta \phi \equiv 0 \) on \( M \), then
\[
g_{11}^{11} = g_{22}^{11}, \quad g_{11}^{22} = g_{22}^{22}, \quad g_{11}^{31} = g_{22}^{31}, \quad g_{21}^{32} = g_{22}^{32}, \quad g_{12}^{31} = g_{22}^{31}, \quad g_{21}^{21} = g_{22}^{21},
\]
on \( M \).
Proof. From Lemma 4.1 \( \delta g^{33} := g_1^{33} - g_2^{33} = 0 \) on \( M \).

Substituting \( \delta \phi \equiv 0 \) into the pseudodifferential expressions (4.80) and (4.81), gives \( \delta g^{31} = 0 \) and \( \delta g^{32} = 0 \) on \( M \). From simple algebraic equations for the other metric components, \( \delta \phi = \delta g^{31} = \delta g^{32} = \delta g^{33} = 0 \) implies \( g_1^\alpha{}^\beta = g_2^\alpha{}^\beta \), on \( M \) for \( \alpha, \beta = 1, 2, 3 \).

In light of the above Lemma, to conclude \( g_1 = g_2 \) in our chosen coordinates, it only remains to prove \( \delta \phi \equiv 0 \) on \( M \). Below we show that from the pseudodifferential expressions (4.80) and (4.81), the equation (4.79) for \( \delta \phi \) may be expressed as a hyperbolic Cauchy problem for \( \delta \phi \) with initial data \( \delta \phi = 0 \) on \( Y(0) \). Then, using a standard energy estimate we prove \( \delta \phi = 0 \) on \( M \) as desired.

### 4.1.5 A hyperbolic Cauchy problem for \( \delta \phi \):

Substituting the expressions (4.80), (4.81) into equation (4.79) gives us the following evolution equation for \( \delta \phi \) on \( M \):

\[
0 = g_1^{33} \partial_3 \delta \phi + g_1^{31} \partial_1 (\delta \phi) + g_1^{32} \partial_2 (\delta \phi) + \left( \partial_k \phi_2 - \frac{1}{2} \partial_k \log (g_1^{33}) \right) P_{-1}^k (\delta \phi) + \left( \partial_k \phi_2 - \frac{1}{2} \partial_k \log (g_1^{33}) \right) Q_{-2}^k (\partial_3 \delta \phi) + \frac{1}{2} \partial_k P_{-1}^k (\delta \phi) + \frac{1}{2} \partial_k Q_{-2}^k (\partial_3 \delta \phi). \tag{4.162}
\]

Now since \( P_{-1}^k, Q_{-2}^k \) are pseudodifferential operators of order \(-1\) and \(-2\) respectively, \( \partial_k P_{-1}^k (\delta \phi) \), \( \partial_k Q_{-2}^k (\partial_3 \delta \phi) \) are respectively order 0 and \(-1\) pseudodifferential operators in the tangential directions \( \partial_k \). So, equation (4.162) takes the form

\[
(I - Q_0) \partial_3 \delta \phi + Q_1(\delta \phi) = 0, \tag{4.163}
\]

where

\[
Q_1 = \frac{g_1^{31}}{g_1^{33}} \partial_1 (\delta \phi) + \frac{g_1^{32}}{g_1^{33}} \partial_2 + \frac{1}{g_1^{33}} \left( \partial_k \phi_2 - \frac{1}{2} \partial_k \log (g_1^{33}) \right) P_{-1}^k + \frac{1}{2 g_1^{33}} \partial_k P_{-1}^k \tag{4.164}
\]

is an order 1 pseudodifferential operator and

\[
Q_0 = \frac{1}{g_1^{33}} \left( \partial_k \phi_2 - \frac{1}{2} \partial_k \log (g_1^{33}) \right) Q_{-2}^k + \frac{1}{2 g_1^{33}} \partial_k Q_{-2}^k \tag{4.165}
\]

is a pseudodifferential operator of order 0, both of which act only in the tangential directions \( \partial_1 \) and \( \partial_2 \). Now, by Lemma 2.4 and Sobolev embedding, \( \phi_2 \) and \( \partial_0 \phi_2 \) are uniformly bounded in \( L^\infty (D(r)) \). This together with the bounds given in Lemma 4.5 shows that \( Q_1 : L^2 (D(r)) \to L^2 (D(r)) \) is uniformly bounded. We now argue that \( I - Q_0 : L^2 (D(r)) \to L^2 (D(r)) \) is invertible.

From (4.87) in Proposition 4.5,

\[
\| \partial Q_{-2}^k (\partial_3 \delta \phi) \|_{L^2 (D(r))} \leq C K \epsilon_0 \| \partial_3 \delta \phi \|_{L^2 (D(r))},
\]

for \( k = 1, 2 \). Therefore,

\[
\| (I - Q_0) \partial_3 \delta \phi \|_{L^2 (D(r))} \geq \| (I - C K \epsilon_0) \partial_3 \delta \phi \|_{L^2 (D(r))}.
\]

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Since \( \epsilon_0 > 0 \) is small, \( (I - Q_0) \) is invertible.

Inverting \( (I - Q_0) \), we derive a hyperbolic Cauchy problem for \( \delta \phi \) of the form

\[
\begin{align*}
\partial_t \delta \phi + \tilde{Q}_1 (\delta \phi) &= 0 \quad \text{on } M \\
\delta \phi(t) &= 0 \quad \text{on } \partial M,
\end{align*}
\]

where \( \tilde{Q}_1 = (I - Q_0)^{-1} Q_1 (I - Q_0)^{-1} \)

\[
\lim_{t \to 1} ||\delta \phi(t)||_{H^1(D(r))} = 0
\]

where \( \delta \phi \in C^3(\bar{M}) \), the Cauchy problem \((4.166)\) has a unique solution \( \delta \phi \equiv 0 \).

**Proof.** We use a standard energy argument. First we show that \( \tilde{Q}_1 + \tilde{Q}^*_1 : L^2(D(r)) \to L^2(D(r)) \) is bounded. Expanding,

\[
\tilde{Q}_1 + \tilde{Q}^*_1 = (I - Q_0)^{-1} Q_1 + Q^*_1 (I - Q_0)^{-1}
\]

Now, \( Q_1 = \frac{\partial^1}{\partial t^1} \partial_1 + \frac{\partial^2}{\partial t^1} \partial_2 + \text{l.o.t.,} \) thus \( Q_1 + Q^*_1 \) is a pseudodifferential operator of order 0 and is uniformly bounded from \( L^2(D(r)) \to L^2(D(r)) \). Using Proposition \( 4.5 \), \( (I - Q_0)^{-1} \) and \( Q^*_1 \) are uniformly bounded operators from \( L^2(D(r)) \to L^2(D(r)) \), thus \( (I - Q_0)^{-1} Q^*_1 : L^2(D(r)) \to L^2(D(r)) \) is uniformly bounded.

It remains to show that \( Q^*_1 (I - Q_0)^{-1} : L^2(D(r)) \to L^2(D(r)) \) is uniformly bounded.

Since the leading order term in \( Q^*_1 \) is a regular first order differential operator and

\[
\partial_j (I - Q_0)^{-1} = (I - Q_0)^{-1} \partial_j Q^*_0 (I - Q_0)^{-1},
\]

it suffices to bound \( \partial_j Q^*_0 : L^2(D(r)) \to L^2(D(r)) \). By definition,

\[
\partial_j Q^*_0 := \partial_j \left[ \frac{1}{g^1_{33}} \left( \partial_k \phi_2 - \frac{1}{2} \partial_k \log(g^1_{33}) \right) Q^k_{-2} + \frac{1}{2 g^1_{33}} \partial_k Q^k_{-2} \right]^*,
\]

where \( j, k \in \{1, 2\} \). By inspection of \( \partial_j Q^*_0 \) above, it suffices to uniformly bound \( \partial_j Q^k_{-2} \) and \( \partial_k Q^k_{-2} \) from \( L^2(D(r)) \to L^2(D(r)) \). These bounds are achieved by estimate \( 4.86 \) in Proposition \( 4.5 \) for \( Q^k_{-2} : L^2(D(r)) \to H^2(D(r)) \).

Now, let \( (\delta \phi, \delta \phi) := ||\delta \phi||_2^2 := \int_{D(r)} |\delta \phi(x)|^2 \, dx \, dx. \) Recall that \( r = r(t) \). Differentiating with respect to \( t \),

\[
\partial_t ||\delta \phi||_2^2 = \int_{\partial D(r(t))} |\delta \phi(x)|^2 \, dS + (|\tilde{Q}_1 + \tilde{Q}^*_1|)(\delta \phi, \delta \phi)
\]

where we used Cauchy-Schwarz and the fact \( \delta \phi = 0 \) on \( \partial D(r(t)) \) for each \( t \in (-1, 1) \).

By Gronwall’s inequality,

\[
\partial_t ||\delta \phi(t)||_2^2 \leq C ||\delta \phi(-1)||_2^2 = 0,
\]

therefore \( \delta \phi \equiv 0 \) on \( \bigcup_{t \in (-1, 1)} D(r(t)) \).

With the above lemma in hand, we conclude \( g_1 = F^*(g_2) \) as desired.
4.2 Proofs of Theorems 1.5 and 1.7

We now show a purely local result near a point on the boundary.

![Figure 8: Neighbourhood near a point on the manifold \((M,g)\).](image)

**Theorem 1.5.** Let \((M,g)\) be a \(C^4\)-smooth, Riemannian manifold with boundary \(\partial M\). Assume that \(\partial M\) is both \(C^4\)-smooth and mean convex at \(p \in \partial M\). Let \(U \subset \partial M\) be a neighbourhood of \(p\), and let \(\{\gamma(t) : t \in (-1,1)\}\) be a foliation of \(U\) by simple, closed curves which satisfy the estimates in Definition 1.3. Suppose that \(g|_U\) is known, and for each \(\gamma(t)\) and any nearby perturbation \(\gamma(s,t) \subset U\), we know the area of the properly embedded surface \(Y(s,t)\) which solves the least-area problem for \(\gamma(s,t)\). Then, there exists a neighbourhood \(V \subset M\) of \(p\) such that, up to isometries which fix the boundary, \(g\) is uniquely determined on \(V\).

**Proof.** Let \(V \subset M\) be a neighbourhood near \(p \in M\) for which we know the above area information. Further, we may choose \(V\) sufficiently small so that \((V,g|_V)\) is a \((K,\epsilon_0,\delta_0)\)-thin manifold for some parameters \(K,\epsilon_0,\delta_0 > 0\). Since \(\partial M\) is mean convex at \(p\), we may further choose \(V\) so that \(V \cap \partial M\) is mean convex. In this case, the induced foliation \(Y(t) \subset V\) by properly embedded, area-minimizing curves is non-degenerate.

Applying Theorem 1.4 to \((V,g|_V)\), we may recover the metric \(g\) near \(p \in M\).

Finally we sketch the proof of Theorem 1.7 which we restate below.

**Theorem 1.7.** Let \((M,g)\) be a \(C^4\)-smooth Riemannian manifold which admits foliations from all directions, and let \(g|_{\partial M}\) be given. Suppose that for all \(p \in M\) and for each \(\gamma(t,p)\) as in Definition 1.6 and any nearby perturbation \(\gamma(s,t,p) \subset \partial M\), we know the area of the properly embedded surface \(Y(s,t,p)\) which solves the least-area problem for \(\gamma(s,t,p)\).

Then the knowledge of these areas uniquely determines the metric \(g\) (up to isometries which fix the boundary).
Sketch of proof. Consider two metrics \( g_1, g_2 \), as in the assumption of our Theorem. Applying Theorem 1.5 we derive that \( g_1 \) restricted to 
\[
\bigcup_{r \in [0, \epsilon)} N(r),
\]
for some small \( \epsilon \) is isometric to \( g_2 \) restricted on an open set of \( M \) which has nonempty intersection with \( \partial M \).

We will then show that this is true for all \( r_0 < 1 \). This will conclude our argument.

To do this, we can apply an open-closed argument: Let \( R \subset [0, 1) \) be the largest connected set of values for which \( g_1 \) is isometric to a portion of \( g_2 \) over 
\[
\bigcup_{r \in R} N(r).
\]
We wish to show that \( R = [0, 1) \). The continuity of the metrics implies that \( R \) is relatively closed; it remains to show it is open. We assume that \( R = [0, r_0] \) with \( r_0 < 1 \), and we will reach a contradiction.

To derive the contradiction we need to show that for any point \( p \in N(r_0) \) there is a small neighbourhood \( \Omega \) of \( p \) for which \( g_1 | \Omega \) is isometric to a region of \( (M, g_2) \) containing \( p \).

Consider the minimal surface \( Y(t_0, p) \) (for \( g_1 \)) for which \( p \in Y(t_0, p) \) and \( Y(t_0, p) \) is tangent to \( N(r_0) \) at \( p \). We can employ the same strategy as in the previous theorems, considering “tilts” \( Y(s, t, p) \subset M \) of the surfaces \( Y(t, p) \) with \( t - t_0 \) suitably small (see Lemma 3.2). One difference are the bounds on the “tilt” functions \( \psi_{p,i}, i = 1, 2 \) (see Lemma 4.2) can now have very large norms in \( W^{3,p} \), in the absence of a thinness condition. Accordingly \( g^{33}, |Rm|, |A| \) and their derivatives can be large in the relevant norms. However restricting attention to \( t \in [t_0, t_0 + \zeta] \) for some \( \zeta > 0 \) small enough, the corresponding functions \( \delta \phi, \delta g^{33} \) appearing in the analogous system (4.77)–(4.79) are now supported on discs of size \( \eta > 0 \), where \( \eta > 0 \) can be chosen to be arbitrarily small.

This (employing the argument in Lemma 4.5) provides the invertibility of the operators \( I - K \) in equation (4.146) and \( (I - Q_0) \) in equation (4.163) over \( Y(s, t, p) \) for \( t \in [t_0, t_0 + \zeta] \), with \( \zeta > 0 \) and \( |s| \) small enough, yielding that \( g_1, g_2 \) are isometric over
\[
\left( \bigcup_{v \in [t_0, t_0 + \eta]} Y(v, p) \right) \cap \left( \bigcup_{r \in [r_0, 1)} N(r) \right).
\]
This can be done for all \( p \in N(r_0) \) (and the set of these points form a compact set), thus proving that \( g_1, g_2 \) are isometric up to 
\[
\bigcup_{r \in [1, r_0 + \eta_0)} N(r).
\]

\[\text{\footnote{Here “size” is measured by the corresponding area.}}\]
5 Appendix

5.1 Proof of Proposition 1.9

Proof. The “only if” part of the statement is easy and follows by a standard perturbation argument for the minimal surface equation. Let us show that if \( A(t) \) is \( C^1 \) then the area minimizing surfaces \( Y(t) \) are unique for each \( t \in (0, 1) \).

We argue by contradiction and assume that this is not the case. Then, there exists a \( T < 1 \) for which two things hold:

- The family of surfaces \( Y(t), t \in (0, T) \) are the unique area-minimizing fill-ins of \( \gamma(t) \) for \( t < T \), and the surfaces \( Y(t) \) converge to an area-minimizing surface \( Y_-(T) \) which fills in \( \gamma(T) \).
- There exists a sequence \( t_i, t_i > T \) with \( t_i \to T^+ \) so that some area-minimizing surfaces \( Y(t_i) \) filling in \( \gamma(t_i) \) converge to an area-minimizer \( Y_+(T) \) filling in \( \gamma(T) \) with \( Y_+(T) \neq Y_-(T) \).

To see this, we observe that for some small \( \epsilon > 0 \) the mean-convexity of the boundary the fill-ins \( Y(t), t < \epsilon \) are indeed unique, and thus define a foliation. We choose \( T \) to be the maximum value of \( \tau \geq \epsilon \) for which the family of surfaces \( Y(t), t \in (0, \tau) \) are the unique area minimizers, and thus define a \( C^1 \) foliation.

It follows that the left limit of these surfaces \( Y(t), t < \tau \) must be an area-minimizer; we denote this limit by \( Y_-(T) \). The right limit is denoted similarly by \( Y_+(T) \). Now, since \( T \) was chosen to be maximal, there must exist a sequence \( t_i \) as described in the second requirement above.

Therefore, to derive our claim we need to show that

\[
Y_+(T) \cap Y_-(T) = \emptyset.
\]

This will hold if:

\[
Y_+(T) \cap \left( \bigcup_{t < T} Y(t) \right) = \emptyset. \tag{5.167}
\]

Equation (5.167) follows by the maximum principle, using the fact that \( Y(t), t < T \) is a foliation.

Now, let \( \nu_- \) be the inward pointing unit normal vector field along \( \gamma(T) \) that is tangent to \( Y_-(T) \), and \( \nu_+ \) the inward pointing unit normal vector field along \( \gamma(T) \) that is tangent to \( Y_+(T) \). Using the Hopf maximum principle, we know that \( \nu_+ \) is transverse everywhere to \( Y_-(T) \) and points above \( Y_-(T) \).

Let us now derive a contradiction to the assumption that \( A(t) \) is \( C^1 \)-smooth in \( t \). On the one hand, using the first variation of area formula for minimal-surfaces, the left derivative of \( A(t) \) at \( t = T \) must equal:

\[
\int_{\gamma(T)} \nu_- \cdot \dot{\gamma}(T).
\]

Here we note that the vector \( \dot{\gamma}(T) \) is tangent to \( \partial M \), and captures the first variation in the foliation \{\( \gamma(t) : t \in (-1, 1) \)\} at \( t = T \). This follows from the first property of \( \gamma(T) \). From the
second property of \( \gamma(T) \) we find that the lim-sup of the right derivative of \( A(t) \) at \( T \) must be bounded below by
\[
\int_{\gamma(T)} \nu_+ \cdot \dot{\gamma}(T).
\]
Since the integral
\[
\int_{\gamma(T)} (\nu_- - \nu_+) \cdot \dot{\gamma}(T)
\]
is never zero, we have derived that \( A(t) \) cannot be differentiable at \( t = T \) and we have derived our contradiction.

5.2 Algebraic Relationships Between the Components of \( g \) and \( g^{-1} \)

Let \((x^\alpha), \alpha = 1, 2, 3\) be a local coordinate system on a Riemannian manifold \((M, g)\) such that in these coordinates the metric \(g\) takes the form
\[
g = \begin{pmatrix}
e^{2\phi} & 0 & g_{13} \\
0 & e^{2\phi} & g_{23} \\
g_{31} & g_{32} & g_{33}
\end{pmatrix},
\]
where the functions \(g_{13} = g_{31}\) and \(g_{23} = g_{32}\). Then, simple cofactor expansion gives
\[
g^{-1} := \begin{pmatrix}
g^{11} & g^{12} & g^{13} \\
g^{21} & g^{22} & g^{23} \\
g^{31} & g^{32} & g^{33}
\end{pmatrix}
\]
\[
= - \det(g^{-1}) \begin{pmatrix}
e^{2\phi} g_{33} - (g_{32})^2 & g_{31} g_{32} e^{2\phi} & -g_{13} \\
g_{31} g_{32} e^{2\phi} & e^{2\phi} g_{33} - (g_{31})^2 & -g_{23} \\
-g_{31} & -g_{32} & e^{2\phi}
\end{pmatrix}.
\]
Thus we have the following relationships:
\[
\det(g^{-1}) = e^{-2\phi} g^{33},
\]
\[
g_{31} = \frac{g^{31}}{\det(g^{-1})} = -\frac{g^{31}}{g^{33}} e^{2\phi},
\]
\[
g_{32} = \frac{g^{32}}{\det(g^{-1})} = -\frac{g^{32}}{g^{33}} e^{2\phi}.
\]
Now, the determinant of \(g^{-1}\) is
\[
\det(g^{-1}) = \det(g)^{-1}
\]
\[
= \left[e^{2\phi} g_{33} + (g_{31})^2 + (g_{32})^2\right]^{-1}.
\]
So
\[
g^{33} = -\det(g^{-1}) e^{-2\phi}
\]
\[
e^{-2\phi}
\]
\[
= \left[e^{2\phi} g_{33} + (g_{31})^2 + (g_{32})^2\right]^{-1}.
\]
and manipulating the above we obtain an expression for $g_{33}$ in terms of $g^{31}, g^{32}$ and $\phi$:

$$g_{33} = -\frac{1}{g_{33}} - \frac{(g_{31})^2 + (g_{32})^2}{(g_{33})^2} e^{2\phi}.$$  

Therefore, we derive the following expressions for the components of $g^{-1}$ in the $\partial_1, \partial_2$ directions in terms of the functions $g^{33}, g^{31}, g^{32}$ and $\phi$:

$$g^{12} = g^{21} = \frac{g^{31} g^{32}}{g^{33}},$$

$$g^{11} = -e^{-2\phi} - \frac{(g^{13})^2 + 2(g^{23})^2}{g^{33}},$$

$$g^{22} = -e^{-2\phi} - \frac{2(g^{13})^2 + (g^{23})^2}{g^{33}}.$$  

Lastly, we may compute

$$e^{-2\phi} \left[ g_{31} \partial_1 g^{33} + g_{32} \partial_2 g^{33} \right] = e^{-2\phi} \left[ -\frac{g_{31}}{g^{33}} e^{2\phi} \partial_1 g^{33} - \frac{g_{32}}{g^{33}} e^{2\phi} \partial_2 g^{33} \right]$$

$$= -g_{31} \partial_1 \log(g^{33}) - g_{32} \partial_2 \log(g^{33}).$$

**References**

[1] Lars V. Ahlfors, *Lectures on quasiconformal mappings*, University Lecture Series, Vol 38. 2nd Ed. American Mathematical Society, Providence, RI. 2006.

[2] Lars Ahlfors and Lipman Bers, *Riemann’s Mapping Theorem for variable metrics* Annals of Mathematics, Second Series, Annals of Mathematics. 72(2):385-404. 1960.

[3] M. Aizenman and B. Simon, *Brownian motion and Harnack’s inequality for Schrödinger operators*, Comm. Pure Appl. Math.Annals of Mathematics. 35:209-273. 1982.

[4] Kari Astala, Matti Lassas, and Lassi Päivärinta, *The borderlines of the invisibility and visibility for Calderón’s inverse problem*, Analysis and PDE. 9:43–98. 2016.

[5] Dmitri Burago and Sergei Ivanov, *Boundary rigidity and filling volume minimality of metrics close to a flat one*, Ann. of Math. (2). 171(2):1183–1211. 2010.

[6] Dmitri Burago and Sergei Ivanov, *Area minimizers and boundary rigidity of almost hyperbolic metrics*, Duke Math. J. 162(7):1205–1248. 2013.

[7] Tobias H. Colding and William P. Minicozzi, II, *A course in minimal surfaces*, Graduate Studies in Mathematics, Vol. 121, AMS. 2011.

[8] Christopher B. Croke, *Rigidity and the distance between boundary points*, J. Differential Geom. 33:445-464. 1991.

[9] Ludvig D. Faddeev, *Increasing solutions of the Schrödinger equation*, Soviet Physics Doklady. 10:1033-1035. 1966.
10 Mikhael Gromov, *Filling Riemannian manifolds*, J. Differential Geom. 18(1):1–147. 1983

11 Robert Haslhofer, Daniel Ketover, *Minimal two-spheres in three-spheres*, arXiv:1708.06567v2 [math.DG]. 2017.

12 Victor Isakov and Adrian Nachman, *Global uniqueness for a two-dimensional semilinear elliptic inverse problem*, Transactions of the American Mathematical Society, 347(9). 1995.

13 Matti Lassas, Vladimir Sharafutdinov, and Gunther Uhlmann *Semiglobal boundary rigidity for Riemannian metrics*, Mathematische Annalen, Springer. 325(4):767–793:2003.

14 Juan Maldacena, *The large N limit of superconformal field theories and supergravity*, Adv. Theor. Math. Phys. 2(2):231–252. 1998.

15 Juan Maldacena, *Wilson loops in large N field theories*, Phys. Rev. Lett. 80(22): 4859–4862. 1998.

16 William H. Meeks III and Shing Tung Yau, *The classical Plateau problem and the topology of three-dimensional manifolds. The embedding of the solution given by Douglas-Morrey and an analytic proof of Dehn’s lemma*, Topology. 21(4):409–442. 1982.

17 René Michel, *Sur la rigidité imposée par la longueur des géodésiques*, Invent. Math. 65(1):71–83. 1981/82.

18 Adrian Nachman, *Global uniqueness for a two-dimensional inverse boundary value problem*, Ann. of Math. (2), 143(1):71–96. 1996.

19 Leonid Pestov and Gunther Uhlmann, *Two dimensional compact simple Riemannian manifolds are boundary distance rigid*, Ann. of Math. (2). 161(2):1093–1110. 2005

20 Shinsei Ryu and Tadashi Takayanagi, *Holographic derivation of entanglement entropy from the anti-de Sitter space/conformal field theory correspondence*, Phys. Rev. Lett. 96(18):181602, 4. 2006

21 Plamen Stefanov and Gunther Uhlmann, *Boundary rigidity and stability for generic simple metrics*, Journal of the American Mathematical Society. 18 (4):975–1003. 2005.

22 Plamen Stefanov, Gunther Uhlmann, and Andras Vasy, *Local and Global Boundary Rigidity and the Geodesic X-Ray Transform in the Normal Gauge*, arXiv:1702.03638v2 [math.DG]. 2017.

23 Ziqi Sun and Gunther Uhlmann, *Anisotropic inverse problems in two dimensions*, Inverse Problems. 19:1001-1010. 2003.

24 John Sylvester and Gunther Uhlmann, *A global uniqueness theorem for an inverse boundary value problem*, Ann. of Math. (2). 125(1):153–169. 1987.