Partition of unity with mixed quantum states

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Abstract

The completeness of quantum state space, is usually expressed as \(\sum_{m=0}^{\infty} |m\rangle \langle m| = 1\), where \(\{|m\rangle\}\) is selected set of quantum states (basis). Density matrix \(|m\rangle \langle m|\) describes a pure quantum state. In this paper, by virtue of the summation method within the normally ordered product of operators we propose and show that the completeness relation can also be represented or partitioned in terms of some mixed states, such as binomial states and negative binomial states. Thus the view on the structure of Fock space is widen and the connotation of Fock space is enriched. See the matter in this sight, experimentalists may have interests to prepare the binomial- and negative binomial states.

Keywords: Decomposition of unity; Fock space; mixed state; binomial states; negative binomial states

1 Introduction

In quantum mechanics and quantum optics theory, the fundamental concept of Fock space is frequently used for describing photons’ creation and/or annihilation, and the photon number state is denoted by

\[ |m\rangle = \frac{a^m}{\sqrt{m!}} |0\rangle, \]

which is a discrete pure state. Here \(a^\dagger\) is bosonic creation operator satisfying the commutative relation \( [a, a^\dagger] = 1 \). The \(|m\rangle\)'s form a complete set of Fock space, i.e.,

\[ \sum_{m=0}^{\infty} |m\rangle \langle m| = 1, \] \hspace{1cm} (2)

in other words, the unity operator can be expanded by the discrete pure state \(|m\rangle \langle m|\). The unity operator in Fock space can be also expanded by the continuous pure states

\[ \int \frac{d^2\alpha}{\pi} |\alpha\rangle \langle \alpha| = 1, \] \hspace{1cm} (3)

where \(|\alpha\rangle\) is the coherent states

\[ |\alpha\rangle = e^{a^\dagger \alpha - a^\dagger \ast a} |0\rangle. \] \hspace{1cm} (4)

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By virtue of the integration within normally product of operators and the normally ordering form of the vacuum state

\[ |0\rangle \langle 0| = e^{-a^\dagger a} : \]

we can re-express (3) as

\[ \int \frac{d^2\alpha}{\pi} |\alpha\rangle \langle \alpha| = \int \frac{d^2\alpha}{\pi} : e^{-(a^\dagger - a)(a - a)} : = 1 \] (6)

We have many different types of quantum light field, which are usually described by density operators (pure or mixed states) \[1\]. For instance, density operator \( \rho_c \) of the chaotic field is

\[ \rho_c = \sum_{m=0}^{\infty} \gamma (1 - \gamma)^m |m\rangle \langle m|, \] (7)

An interesting question, which has been overlooked for long, is: can unity operator be expanded in terms of some density operators which describe mixed states? If yes, then what are these mixed states? In this paper, we shall point out that unity operator in Fock space can be expressed in terms of the binomial state (BS) and the negative binomial state (NBS) respectively.

We shall demonstrate this with the method of summation within the normally ordered operator because the creation and annihilation operators are permutable within the normally ordering symbol (denoted by : :) \[2, 4, 5\]. Our new partition of unity in Fock space can deepen people’s understanding about the structure of Fock space and enrich the connotation of Fock space. Once theoretical physicists are certain that some mixed states can constitute Fock space, experimentalists are challenged to bring them about.

## 2 Unity in Fock space partitioned by the binomial states

Based on the binomial distribution \( \binom{n}{l} \sigma^l (1 - \sigma)^{n-l}, \ 0 < \sigma < 1 \), we can construct the density operator of binomial states of light field

\[ \sum_{l=0}^{n} \binom{n}{l} \sigma^l (1 - \sigma)^{n-l} |l\rangle \langle l| \equiv \rho_n (\sigma), \] (8)

where \(|l\rangle = \frac{\sqrt{n!}}{\sqrt{l!}} |0\rangle\) is the number state and \( \text{tr} \rho_n (\sigma) = 1 \) is the result of the binomial theorem. Here we should emphasize that \( \rho_n (\sigma) \) denotes a mixed state, sharply different from the pure state \( \sum_{l=0}^{n} \sqrt{\binom{n}{l} \sigma^l (1 - \sigma)^{n-l}} |l\rangle \equiv |\psi\rangle \) introduced in Ref.\[6\]. Using the generating function of Laguerre polynomials \( L_n (x) \),

\[ (1 - z)^{-1} e^{z - tz^2} = \sum_{n=0}^{\infty} L_n (x) z^n, \] (9)

and noticing that operators \( a^\dagger \) and \( a \) are permutable within the normally ordering symbol : :, we can make a decomposition of unity as the following

\[ 1 = : e^{a^\dagger a} e^{-a^\dagger a} : \]

\[ = \sigma \frac{1}{1 - (1 - \sigma)} : e^{\frac{i\pi}{\sigma} a^\dagger a} e^{-a^\dagger a} : \]

\[ = \sigma \sum_{n=0}^{\infty} (1 - \sigma)^n : L_n \left( \frac{\sigma}{\sigma - 1} a^\dagger a \right) e^{-a^\dagger a} : , \] (10)
where the power series representation of the Laguerre polynomials $L_n(x)$ is

$$L_n(x) = \sum_{l=0}^{n} \binom{n}{l} \frac{(-1)^l}{l!} x^l.$$  \hspace{1cm} (11)

Using the normally ordered form of the vacuum projector in Eq.(5), we can rewrite Eq.(9) as

$$1 = \sigma \sum_{n=0}^{\infty} (1 - \sigma)^n \sum_{l=0}^{n} \binom{n}{l} \frac{(-1)^l}{l!} \left(\frac{\sigma}{\sigma - 1}\right)^l a^l \rho l |0\rangle \langle 0| a^l : (12)$$

Using Eqs.(8), we can write (12) in the compact form

$$1 = \sigma \sum_{n=0}^{\infty} \rho_n (\sigma),$$  \hspace{1cm} (13)

which indicates that the unity can be represented or partitioned in terms of the binomial states. This guides us to theoretically discuss how this mixed state can be obtained experimentally.

In the next section we will show that when a pure number state $|l\rangle \langle l|$ passes through an amplitude dissipation channel, the output state is just the binomial state.

### 3 Preparation of the binomial state

The master equation for amplitude dissipation channel is

$$\frac{d}{dt} \rho(t) = \kappa [2a \rho a^\dagger - a^\dagger a \rho - \rho a^\dagger a],$$  \hspace{1cm} (14)

where $\kappa$ is the damping constant. Its solution is

$$\rho(t) = \sum_{n=0}^{\infty} \frac{T^n}{n!} \exp \left[-\kappa t a^\dagger a\right] a^n \rho_0 a^n \exp \left[-\kappa t a^\dagger a\right], \hspace{1cm} T = 1 - e^{-2\kappa t}.$$  \hspace{1cm} (15)

Thus using Eqs.(8), we can write (12) in the compact form

$$1 = \sigma \sum_{n=0}^{\infty} \rho_n (\sigma),$$  \hspace{1cm} (13)

which indicates that the unity can be represented or partitioned in terms of the binomial states. This guides us to theoretically discuss how this mixed state can be obtained experimentally.

In the next section we will show that when a pure number state $|l\rangle \langle l|$ passes through an amplitude dissipation channel, the output state is just the binomial state.
4 Partition of unity in Fock space by negative binomial states

Next we consider whether the unity of Fock space can be partitioned in terms of negative binomial states. For this purpose, we employ the following sum rearranging formula

\[
\sum_{n=0}^{\infty} \sum_{l=0}^{n} A_{n-l} B_l = \sum_{s=0}^{\infty} \sum_{m=0}^{\infty} A_s B_m, \tag{16}
\]

and rewrite Eq. (12) as

\[
1 = \sigma \sum_{n=0}^{\infty} \rho_n (\sigma) \tag{17}
\]

\[
= \sigma \sum_{n=0}^{\infty} \sum_{l=0}^{n} \binom{n}{n-l} (1-\sigma)^{n-l} \sigma^l |l\rangle \langle l|
\]

\[
= \sum_{m=0}^{\infty} \sum_{s=0}^{\infty} \binom{m+s}{s} (1-\sigma)^s \sigma^{m+1} |m\rangle \langle m|, \tag{18}
\]

where \((\binom{m+s}{s})(1-\sigma)^s \sigma^{m+1}\) is the negative binomial distribution \(8, 9, 10, 11\).

Letting \(1-\sigma = \gamma\) in Eq. (17), we have

\[
1 = \sum_{s=0}^{\infty} \sum_{m=0}^{\infty} \binom{m+s}{m} \gamma^s (1-\gamma)^{m+1} |m\rangle \langle m|, \tag{19}
\]

By defining

\[
\sum_{m=0}^{\infty} \binom{m+s}{m} \gamma^{s+1} (1-\gamma)^m |m\rangle \langle m| \equiv \rho_s (\gamma),
\]

which is named negative binomial state (it is also a mixed state), and using the negative binomial theorem

\[
(1 + x)^{-s-1} = \sum_{m=0}^{\infty} \binom{m+s}{m} (-x)^m, \tag{20}
\]

we see

\[
tr\rho_s (\gamma) = \gamma^{s+1} \sum_{m=0}^{\infty} \binom{m+s}{m} (1-\gamma)^m = 1. \tag{21}
\]

Then the unity decomposition in Eq. (17) becomes

\[
1 = \frac{1-\gamma}{\gamma} \sum_{s=0}^{\infty} \rho_s (\gamma). \tag{22}
\]

This indicates that unity in Fock space can be partitioned using the negative binomial states.

Our next task is to see how NBS can be generated theoretically.

5 Generation of the negative binomial state

We use \(a |n\rangle = \sqrt{n} |n-1\rangle\) and
to convert $\rho_s$ in (29) to

$$\rho_s(\gamma) = \gamma^{s+1} \sum_{n'=s}^{\infty} \binom{n'}{n'-s} (1-\gamma)^{n'-s} |n'-s\rangle \langle n'-s|$$

(24)

$$= \frac{\gamma^{s+1}}{(1-\gamma)^s} \sum_{n=0}^{\infty} \frac{n!}{(n-s)!} |n-s\rangle \langle n-s|$$

$$= \frac{\gamma^{s}}{(1-\gamma)^s} \sum_{n=0}^{\infty} \gamma (1-\gamma)^n |n\rangle \langle a^\dagger|^s$$

$$= \frac{1}{s!(n_c)} a^s \rho_c a^\dagger|^s$$

where

$$n_c \equiv \frac{1-\gamma}{\gamma}.$$  

(25)

Recalling that

$$\rho_c = \sum_{n=0}^{\infty} \gamma (1-\gamma)^n |n\rangle \langle n| = \gamma (1-\gamma)^a^\dagger a = \gamma e^{a^\dagger a \ln(1-\gamma)} = \gamma : e^{-\gamma a^\dagger a} :$$

(26)

describes a thermo light field (chaotic light), we can prove that $n_c$ is just the average photon number of the chaotic light field. Indeed, using the photon number operator $N = a^\dagger a$, we have

$$Tr(\rho_c N) = \sum_{n=0}^{\infty} \gamma (1-\gamma)^n \langle n| a^\dagger a |n\rangle = \gamma \sum_{n=0}^{\infty} n (1-\gamma)^n$$

(27)

$$= \frac{1-\gamma}{\gamma} = n_c$$

thus $\gamma = \frac{1}{n_c+1}$. Employing the formula converting an operator to its anti-normally ordered form

$$\rho(a,a^\dagger) = \int \frac{d^2\beta}{\pi} :\langle -\beta| \rho(a,a^\dagger) |\beta\rangle e^{\beta a^\dagger + \beta^* a} :$$

(28)

where the symbol $:\ :$ denotes anti-normally ordering, and $|\beta\rangle = \exp[-|\beta|^2/2+\beta a^\dagger] |0\rangle$ is a coherent state, $\langle -\beta| \beta\rangle = e^{-2|\beta|^2}$, then $\rho_c$ can be put into the following form

$$\rho_c = \gamma \int \frac{d^2\beta}{\pi} :\langle -\beta| : e^{-\gamma a^\dagger a} : |\beta\rangle e^{\beta a^\dagger + \beta^* a} :$$

(29)

$$= \frac{\gamma}{1-\gamma} : e^{-\gamma a^\dagger a} :$$

thus the density operator of negative binomial state in Eq. (24) becomes

$$\rho_s(\gamma) = \frac{1}{s!(n_c)^{s+1}} a^s e^{\frac{\gamma}{1-\gamma} a^\dagger a} a^\dagger|^s.$$

(30)
Noticing that operators $a^\dagger$ and $a$ are also permutable within the anti-normally ordering symbol $::$, we can perform the summation over $s$ and again obtain

$$\sum_{s=0}^{\infty} \rho_s (\gamma) = \frac{1}{n_c} \sum_{s=0}^{\infty} \frac{1}{s!} (n_c)^s e^{\frac{s}{n_c}} a^\dagger a;$$

which coincides with Eq. (22). Thus the conclusion of Fock space partitioned by the negative binomial states (22) is confirmed. Moreover from $\rho_s = \frac{1}{s!} a^s \rho a^\dagger$ we may predict that the negative binomial state can be generated by absorbing $s$ photons from the chaotic field $\rho_c$, and this may happen in a light-atom interaction governed by the interacting Hamiltonian $H = g a^\dagger \sigma_+ + g^* a \sigma_-$, here $\sigma_\pm$ are atom's hopping operators.

### 6 Derivation of the generalized negative binomial theorem involving Laguerre polynomial

As an application of Eq. (30) we can use the form of $\rho_s (\gamma)$ to derive generalized negative binomial theorem involving Laguerre polynomial. For this purpose, we need to reform (30) as in normal ordering. Introducing so-called two-variable Hermite polynomial through its generating function

$$\sum_{m,n=0}^{\infty} \frac{t^m \tau^n}{m!n!} H_{m,n}(x,y) = \exp (tx + \tau y - t\tau),$$

then

$$H_{m,n}(x,y) = \frac{\partial^{n+m}}{\partial^m \partial^n} \exp (tx + \tau y - t\tau) |_{t=\tau=0}$$

$$= \frac{\partial^m}{\partial^m} e^{tx} \frac{\partial^n}{\partial^n} \exp (\tau (y - t)) |_{t=\tau=0}$$

$$= \frac{\partial^m}{\partial^m} \left[ e^{tx} (y - t)^n \right] |_{t=0}$$

$$= \sum_{l=0}^{\min(m,n)} \binom{m}{l} \frac{\partial^l}{\partial t^l} (y - t)^n \frac{\partial^{m-l}}{\partial t^{m-l}} e^{tx} |_{t=0}$$

$$= \sum_{l=0}^{\min(m,n)} \frac{m!n!(-1)^l}{l!(m-l)!(n-l)!} x^{m-l} y^{n-l}$$

Comparing it with the Laguerre polynomial in Eq.(11), we identify

$$L_n (xy) = \frac{(-1)^n}{n!} H_{n,n}(x,y).$$

Then we employ the completeness relation of the coherent state representation

$$\int \frac{d^2 z}{\pi} |z\rangle \langle z| = \int \frac{d^2 z}{\pi} e^{-|z|^2 + za^\dagger + z^* a - a^\dagger} : 1,$$

where $|z\rangle = \exp \left[ -\frac{|z|^2}{2} + za^\dagger \right] |0\rangle$, and the method of integration within ordered product of operators (IWOP) [2] to derive
Further, using and the generating function of the Laguerre polynomial in (9) we have

\[ \langle e^{\lambda a^\dagger} \rangle = \sum_{l=0}^{\infty} \lambda^l L_l (-a^\dagger a) :. \]  

(37)

It follows from

\[ \int \frac{d^2 \beta}{\pi} \beta^m \beta^* e^{-|\beta|^2} \exp \left[ -2 \beta \alpha^* + \beta^* \alpha \right] = (-i)^{m+n} H_{m,n} (i\alpha^*, i\alpha) e^{|\alpha|^2} \]  

(38)

and the IWOP method that

\[ a^\dagger e^{\lambda a^\dagger} : a^m = \int \frac{d^2 z}{\pi} z^n e^{\lambda |z|^2} \langle z | z^m \rangle \]  

(39)

\[ = \int \frac{d^2 z}{\pi} z^n z^m e^{-|z|^2 + z^* a^\dagger z a + a^\dagger a z} :. \]  

\[ = \frac{1}{(1 - \lambda)^{(n+m)/2+1}} \int \frac{d^2 z}{\pi} z^n z^m e^{-|z|^2 + z^* a^\dagger z a + a^\dagger a} :. \]  

\[ = (-i)^{m+n} (1 - \lambda)^{-(n+m)/2-1} : e^{\lambda a^\dagger a/(1-\lambda)} H_{m,n} (\frac{i a^\dagger}{\sqrt{1-\lambda}}, \frac{i a}{\sqrt{1-\lambda}}) :. \]  

On the other hand (see the Appendix),

\[ a^n e^{\lambda a^\dagger} : a^m = \sum_{l=0}^{\infty} \frac{\lambda^l}{l!} (a^\dagger)^{m+n} : H_l^{m+n} (ia^\dagger, ia) :. \]  

(40)

Comparing (39) with (40) we see

\[ \sum_{l=0}^{\infty} \frac{\lambda^l}{l!} (-i)^{m+n+2l} : H_{l+m, l+n} (ia^\dagger, ia) :. \]  

(41)

\[ = (-i)^{m+n} (1 - \lambda)^{-(n+m)/2-1} : e^{\lambda a^\dagger a/(1-\lambda)} H_{m,n} (\frac{i a^\dagger}{\sqrt{1-\lambda}}, \frac{i a}{\sqrt{1-\lambda}}) :. \]  

noting \( a^\dagger \) and \( a \) are commutable within \( :. \); so replacing \( ia^\dagger \to x, ia \to y, \lambda \to -\lambda \), we obtain

\[ \sum_{l=0}^{\infty} \frac{\lambda^l}{l!} H_{l+m, l+n} (x, y) = (1 + \lambda)^{-(n+m)/2-1} e^{\lambda xy/(1+\lambda)} H_{m,n} (\frac{x}{\sqrt{1+\lambda}}, \frac{y}{\sqrt{1+\lambda}}) \]  

(42)

this is a new important generating function of \( H_{l+m, l+n} (x, y) \). Especially, when \( m = n \),

\[ \sum_{l=0}^{\infty} \frac{\lambda^l}{l!} H_{l+n, l+n} (x, y) = (1 + \lambda)^{-n-1} e^{\lambda xy/(1+\lambda)} H_{n,n} (\frac{x}{\sqrt{1+\lambda}}, \frac{y}{\sqrt{1+\lambda}}) \]  

(43)

Using (41) we can recast (43) into

\[ \sum_{l=0}^{\infty} \frac{(n+l)!}{l!n!} L_{n+l} (z) = (1 + \lambda)^{-n-1} e^{\lambda z/(1+\lambda)} L_n \left( \frac{z}{1+\lambda} \right). \]  

(44)
This is the generalized negative binomial theorem involving Laguerre polynomial. When \( z = 0 \), it reduces to the negative-binomial formula

\[
\sum_{l=0}^{\infty} \frac{(n+l)!}{l!n!}(-\lambda)^l = (1 + \lambda)^{-n-1}
\]

(45)
as expected.

7 The normally ordered form of negative binomial state

when \( n = m = s \), Eq. (40) reduces to

\[
a^s e^{\lambda a^t} a^s = (-1)^s (1 - \lambda)^{-s-1} : e^{\lambda a^t a/(1-\lambda)} H_{s,s}(\frac{ia^t}{\sqrt{1-\lambda}}, \frac{ia}{\sqrt{1-\lambda}}) : .
\]

(46)

Taking \( \lambda = \frac{\gamma}{1-\gamma} \), \( 1 - \lambda = \frac{1}{1-\gamma} \frac{1}{\gamma} - 1 = n_c \), then

\[
\rho_s(\gamma) = \frac{1}{s!(n_c)^{s+1}} : a^s e^{\gamma a^t a^s} H_{m,n}(\frac{ia^t}{\sqrt{1-\gamma}}, \frac{ia}{\sqrt{1-\gamma}}) : .
\]

(47)

This is the normally ordered form of negative binomial state, which is Laguerre polynomial weighted. We can re-derive Eq. (22) by using Eq. (47) and (9), i.e.,

\[
1 = (1 - \gamma) \sum_{s=0}^{\infty} \rho_s(\gamma) = (1 - \gamma) e^{\gamma a^t a} \sum_{s=0}^{\infty} \gamma^s L_s([- (1 - \gamma) a^t a]) :
\]

(48)

In summary, we have found that the complete quantum state space can be divided into mixed states in two ways, one is according to binomial states, and another is according to negative binomial states. We now understand the structure of Fock space more deeply. The new partition in negative binomial states helps to build the generalized negative binomial theorem involving Laguerre polynomial. The preparation possibility of these mixed states is pointed out.

8 Appendix

We begin with presenting the operator identity

\[
a^n a^m = (-i)^m a^n : H_{m,n} (ia^t, ia) :
\]

(49)

where : : denoted normal ordering.

In fact, using the Baker-Hausdorff formula and Eq. (52) we have

\[
e^{t^a} e^{t a^t} = e^{t a^t} e^{t^a} = e^{(-i t^a) a^t + (-it) a^t a^t (-it')} : = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-it)^m (-it')^n}{n!m!} : H_{m,n} (ia^t, ia) :
\]

(50)
Comparing Eq. (50) with

\[ e^{\alpha^\dagger \alpha} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{t^m t^n}{n! m!} a^n a^m \]  

we see

\[ a^n a^m = (-i)^{m+n} H_{m,n} (\alpha^\dagger, \alpha). \]  

Thus

\[ \sum_{l=0}^{\infty} \lambda^l \frac{1}{l!} (-i)^{m+n+2l} H_{l+m, l+n} (\alpha^\dagger, \alpha) = \sum_{l=0}^{\infty} \lambda^l \frac{1}{l!} a^l a^{l+m} = a^n e^{\lambda \alpha^\dagger \alpha} a^m. \]  

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