A Girsanov Result through Birkhoff Integral

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Abstract. A vector-valued version of the Girsanov theorem is presented, for a scalar process with respect to a Banach-valued measure. Previously, a short discussion about the Birkhoff-type integration is outlined, as for example integration by substitution, in order to fix the measure-theoretic tools needed for the main result, Theorem 6, where a martingale equivalent to the underlying vector probability has been obtained in order to represent the modified process as a martingale with the same marginals as the original one.

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1 Introduction

In probability theory, the so-called Girsanov Theorem is a well-known result, whose interest lies both in its theoretical features and in its technical consequences, see for example [19]. The original formulation of this theorem is related to the Wiener measure, i.e. the distribution of the standard Brownian Motion, $(B_t)_{t \in [0, +\infty]}$ as a stochastic process on a probability space $(\Omega, \mathcal{A}, P)$.

The Girsanov Theorem is a fundamental tool for Stochastic Calculus and Random Walks; this last mathematical model has many uses as a simulation tool: Brownian Motion of Molecules, stock prices and behavior of investors, modeling of cascades of neuron firings in brain and it has important practical uses in the internet: Twitter uses random walks to suggest who to follow, Google uses random walks to order pages which match a search phrase.

In many concrete situations, particularly in stochastic calculus, the resultant processes $(\tilde{B}_t)_{t}$ are usually obtained as suitable transformations of $(B_t)_{t}$, and so their distribution is different from the Wiener measure. A typical situation is the

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following: assume that \( a(t, \omega) \) is a stochastic process adapted to the Brownian Motion \((B_t)_{t} \), and define:

\[
(\tilde{B}_t)_t := \int_0^t a(t, \omega)dt + B(t).
\]

Though the distribution of this process is different by the Wiener measure, the Girsanov theorem states that it is possible to endow the basic probability space, \( Q \) (which turns out to be absolutely continuous w.r.t. \( P \)), in such a way that the distribution of \((\tilde{B}_t)_{t} \) under the new probability \( Q \) is the Wiener measure, i.e. the same as \((B_t)_{t} \) under the original probability \( P \). This clearly simplifies all calculations involving just the distribution of \((\tilde{B}_t)_{t} \), since in the new probability space this process is the same as \((B_t)_{t} \).

In the example outlined above, the measure \( Q \) can be described by its derivative w.r.t. \( P \):

\[
\frac{dQ}{dP}(\omega) = \exp \left\{ -\int_0^T a(s, \omega)dB_s - \frac{1}{2} \int_0^T a^2(s, \omega)dt \right\}.
\]

We also point out that, in this example, the process

\[
\exp \left\{ -\int_0^t a(s, \omega)dB_s - \frac{1}{2} \int_0^t a^2(s, \omega)dt \right\}
\]

is a martingale (Novikov condition).

Our research in this paper is motivated by the fact that, when the distributions involved are conditioned by some initial information (that can be represented as a particular sub-\(\sigma\)-algebra \( \mathcal{F} \) of \( \mathcal{A} \), then they should be evaluated with respect to \((P|\mathcal{F}) \), which is a Banach space-valued measure.

So, in our setting here, continuing the study started in [8], changing a bit the notations, the basic space is \((T, \mathcal{A}, \mu) \) where \( \mu : \mathcal{A} \to \mathbb{R}^+ \) a non-negative countably additive measure. Let \( (X, \| \cdot \|) \) be a Banach space with the origin \( 0 \). We also will consider measures, taking values in \( X \); in this case measures will be usually denoted with letters like \( m \) or \( M \), while functions with capital letters, like \( F \) or \( W \).

2 Preliminaries

Let \( T \) be an abstract, non-empty set, \( \mathcal{A} \) a \( \sigma \)-algebra of subsets of \( T \), \( \mathcal{B} \) the Borel \( \sigma \)-algebra in the real line and \( \mu : \mathcal{A} \to \mathbb{R}^+_0 \) a non-negative countably additive measure. Let \( (X, \| \cdot \|) \) be a Banach space with the origin \( 0 \). We also will consider measures, taking values in \( X \); in this case measures will be usually denoted with letters like \( m \) or \( M \), while functions with capital letters, like \( F \) or \( W \).
Definition 1. A partition of $T$ is a finite or countable family of nonempty sets $P := \{A_n\}_{n \in \mathbb{N}} \subset \mathcal{A}$ such that $A_i \cap A_j = \emptyset$, $i \neq j$ and $\cup_{n \in \mathbb{N}} A_n = T$.

If $P$ and $P'$ are two partitions of $T$, then $P'$ is said to be finer than $P$, denoted by $P' > P$, if every set of $P'$ is included in some set of $P$. The common refinement of two partitions $P$ and $P'$ is the partition $P \vee P'$.

We shall make use of the Birkhoff integral, for two different cases. According with the results obtained in [4], and taking into account that we need measurable scalar functions $f$ and strongly measurable vector functions $F$, we can adopt the following version:

Definition 2. Given two pairs $(F, \mu)$ and $(f, m)$ with $F : T \to X$ and $f : T \to \mathbb{R}$, while $\mu$ and $m$ denote two countably additive measures with values in $\mathbb{R}_0^+$ and $X$ respectively, then

$\alpha)$ a strongly measurable vector function $F$ is $B_1$-integrable on $T$ w.r.t $\mu$

$\beta)$ a measurable scalar function $f$ is $B_2$-integrable on $T$ w.r.t $m$

if $\exists I \in X$ with the following property: $\forall \varepsilon > 0$, $\exists P_{\varepsilon}$ partition of $T$ so that $\forall P = \{A_n\}_{n \in \mathbb{N}}$ of $T$, with $P \geq P_{\varepsilon}$ and $\forall t_n \in A_n, n \in \mathbb{N}$, one has (respectively)

$$\lim_{n} \left\| \sum_{i=1}^{n} F(t_i)\mu(A_i) - I \right\| \leq \varepsilon; \quad (1)$$

$$\lim_{n} \left\| \sum_{i=1}^{n} f(t_i)m(A_i) - I \right\| \leq \varepsilon. \quad (2)$$

The set $I$ is called the $B_1$ ($B_2$) integral of $F$ ($f$) on $T$ with respect to $\mu$ ($m$) and is denoted by $\int_T F d\mu$, $\int_T f dm$; the corresponding spaces of $B_1$-integrable functions are denoted with $L_{B_1}(\mu, X)$ and $L_{B_2}(m, \mathbb{R}_0^+)$. 

As proved in [4, Theorem 3.18], even if $\mu$ is $\sigma$-finite then this notion of integrability is equivalent to the classic Birkhoff integrability for Banach-valued strongly measurable mappings. Moreover in [7] $B_1$-integrability is named strong Birkhoff integrability. One can easily deduce, by means of a Cauchy criterion, that the $B_1$- or $B_2$-integrability on $T$ implies the same in every subset $A \in \mathcal{A}$.

For an extensive literature on the Birkhoff or non absolute integrals see for example [1][7][18][20][23].

3 Some Properties of the Birkhoff Integrals

We will deduce now some useful formulas for the notions of Birkhoff integral previously introduced. These formulas will also give a link between the $B_1$- and $B_2$-integral. First, let us mention a stronger result concerning the $B_1$-integrability.
Theorem 1. (Theorem 3.14) Let $F \in L_{B_1}(\mu, X)$, then $\forall \varepsilon > 0$ there exists a countable partition $P := \{A_n, n \in \mathbb{N}\} \subset \mathcal{A}$, such that

$$\sum_j \left\| F(t_j)\mu(E_j) - \int_{E_j} F \, d\mu \right\| \leq \varepsilon$$

holds true, for every partition $P' := \{E_j, j \in \mathbb{N}\} > P$ and $\forall t_j \in E_j$.

Remark 1. If $f$ is measurable and $F$ is $B_1$-integrable then the product $t \mapsto f(t)F(t)$ is strongly measurable. Moreover, thanks to the measurability of $f$, we can define a countable measurable partition of $T$, $(H_j)_j := (\{t \in T : j - 1 \leq |f(t)| < j\})_j$. Since $F$ is $B_1$-integrable, according with Theorem 1 for every $\varepsilon > 0$ and for each integer $j$ there exists a measurable countable partition $\{E^j_k, k \in \mathbb{N}\}$ of $H_j$ such that

$$\sum_r \left\| F(t^*_r)\mu(E^*_r) - \int_{E^*_r} F \, d\mu \right\| \leq \frac{\varepsilon}{2^j}$$

holds true, for every finer partition $\{E^r_j, r \in \mathbb{N}\}$ and every choice of points $t^*_r \in E^*_r$. Then, we have also

$$\sum_j \sum_r \left\| F(t^*_r)f(t^*_r)\mu(E^*_r) - f(t^*_r)\mu(E^*_r) \right\| \leq 2\varepsilon. \quad (3)$$

Theorem 2. (integration by substitution) Given $f : T \to \mathbb{R}$ and $F \in L_{B_1}(\mu, X)$, the product $t \mapsto f(t)F(t) \in L_{B_1}(\mu, X)$ iff $f \in L_{B_2}(M, \mathbb{R})$, where $M(A) := \int_A F \, d\mu$ and

$$\int_T f(t)F(t) \, d\mu = \int_T f(t) \, dM. \quad (4)$$

Another useful formula comes from probability theory. We just state it in a particular situation. Given a measurable $f : T \to \mathbb{R}$ and a countably additive measure $m : \mathcal{A} \to X$, we can set $m_f(B) = m(f^{-1}(B))$ for every Borel set $B \in \mathcal{B}$. Of course, $m_f$ is a countably additive measure, called the distribution of $f$ (with respect to $m$). We have the following result.

Theorem 3. For any measurable function $g : \mathbb{R} \to \mathbb{R}$, one has

$$\int_T g(f) \, dm = \int_{\mathbb{R}} g(t) \, dm_f$$

provided that both $B_2$-integrals exist.

We shall denote by $\sigma_f$ the sub-$\sigma$-algebra of $\mathcal{A}$ induced by $f : T \to \mathbb{R}$, i.e. the family of all sets of the type $f^{-1}(B)$, $B \in \mathcal{B}$.

Definition 3. Let $F \in L_{B_1}(\mu, X)$. Given any sub-$\sigma$-algebra $\mathcal{E}$ of $\mathcal{A}$, the conditional expectation $\mathbb{E}(F|\mathcal{E})$ (if it exists) is a strongly $\mathcal{E}$-measurable mapping $Z$, in $L_{B_1}(\mu, X)$, such that

$$\int_E F \, d\mu = \int_E Z \, d\mu$$
for every $E \in \mathcal{E}$.

In case $\mathcal{E} = \sigma_f$, then we write $Z = \mathbb{E}(F|\mathcal{E}) = \mathbb{E}(F|f)$, and in this case $Z$ turns out to be a measurable function of $f$, say $Z = h(f)$: then

$$\int_{f^{-1}(B)} Fd\mu = \int_{f^{-1}(B)} h(f)d\mu$$

for every Borel set $B$.

The conditional expectation enjoys several properties, easy to deduce, among which linearity with respect to $F$, and the so-called tower property, i.e., whenever $\mathcal{E} \subset \mathcal{G} \subset \mathcal{A}$

$$\mathbb{E}(F|\mathcal{E}) = \mathbb{E}(\mathbb{E}(F|\mathcal{G})|\mathcal{E}),$$

provided that all the involved quantities exist.

The next theorem states another important property of the conditional expectation.

**Theorem 4.** Let us assume that $\mathbb{E}(F|\mathcal{E})$ exists. Then, for every $\mathcal{E}$-measurable mapping $g : T \to \mathbb{R}$ it holds: $\mathbb{E}(F(t)g(t)|\mathcal{E}) = g(t)\mathbb{E}(F|\mathcal{E})$ provided that $F(t)g(t) \in L_{B_1}(\mu, X)$.

### 4 Girsanov Theorem

We shall now state an analogous result as the well-known Girsanov Theorem. With this purpose, we shall assume that in the space $(T, A)$ a $\sigma$-additive measure $M : A \to X$ is fixed.

**Definition 4.** A scalar process $(w_s)_s$ is said to be a **Martingale** in itself, if for every $s, v \in [0, S], s < v$, it holds $\mathbb{E}(w_v|\mathcal{E}_s) = w_s$, i.e.

$$\int_E w_v dM = \int_E w_s dM$$

holds true, $\forall v, s \in [0, S], s < v$, and $\forall E \in \mathcal{E}_s$, where $\mathcal{E}_s$ is the least $\sigma$-algebra contained in $\mathcal{A}$ such that all $w_r, r \leq s$, are measurable.

**Assumption 1** Let us assume that a scalar-valued process $(w_s)_s \in [0, S]$ is defined, in the space $(T, A, M)$, with the property that

- **(1a)** $w_s \in L_{B_2}(M, \mathbb{R})$ for each $s$, with null integral, and that its distribution $M_s := M(w_s^{-1}(B)), \forall B \in \mathcal{B}$ has a density $f_s \in L_{B_1}(\lambda, X)$

- **(1b)** let $\tilde{w}_s = w_s + sq$, with $q \in \mathbb{R}^+$; $\forall s$ there exists a measurable mapping $g_s : T \to \mathbb{R}$ such that $f_s(x) = g_s(x)f_s(x - qs)$, so that $\forall B \in \mathcal{B}$

$$M_s(B) = \int_B g_s(x) dM_{\tilde{w}_s};$$  \hspace{1cm} (5)

(We observe that, since $g_s(x)f_s(x - qs) = f_s(x)$ is in $L_{B_1}(\lambda, X)$, from Theorem 2 it follows that $g_s$ is $B_2$-integrable w.r.t. $M_{\tilde{w}_s}$)
\( \{ g_s(\tilde{w}_s) \} \) is a Martingale.

As a consequence, we have

**Theorem 5.** Set for every \( A \in \mathcal{A} \), \( Q(A) := \int_A g_S(\tilde{w}_S) dM \). Under Assumptions 1 it turns out that \( Q(\tilde{w}_s) = M(\tilde{w}_s) \), for every \( s \in [0, S] \).

The previous theorem shows that, under the new measure \( Q \), every random variable \( \tilde{w}_s = w_s + sq \) has the same distribution as the corresponding \( w_s \) under \( M \).

Our next step is to prove that the process \( \{ \tilde{w}_s \} \) is a martingale, under \( Q \).

(This property is usually formulated by saying that \( Q \) is a Martingale equivalent measure). To this aim, we shall assume also the following:

**Assumption 2** The scalar process \( \{ \tilde{w}_t g_t(\tilde{w}_t) \} \) is a martingale w.r.t. \( M \).

Concerning the last assumption, we remark that, in case \( \{ w_t \} \) is the classical (scalar) Brownian Motion, then the process \( \{ \tilde{w}_t g_t(\tilde{w}_t) \} \) reduces to \( \{ w_t e^{-qw_t - \frac{1}{2} q^2 t} + qte^{-qw_t - \frac{1}{2} q^2 t} \} \), which shows that the classical Brownian Motion satisfies the Assumption 2. So we have

**Theorem 6.** Under Assumptions 1, 2 the process \( (\tilde{w}_s) \) is a martingale with respect to \( Q \).

**Proof.** Fix arbitrarily \( s \) and \( v \), with \( s < v \), and fix \( E \in \mathcal{E}_s \). We observe that

\[
\int_E \tilde{w}_v dQ = \int_E w_v g_v(\tilde{w}_v) dM + qvQ(E).
\]

Since \( E \in \mathcal{E}_s \), it is clear that \( Q(E) = \int_E g_s(\tilde{w}_s) dM \). Therefore,

\[
\int_E \tilde{w}_v dQ = \int_E (w_v g_v(\tilde{w}_v) + qv g_v(\tilde{w}_v)) dM = \int_E \tilde{w}_v g_v(\tilde{w}_v) dM.
\]

By the Assumption 2 it follows then

\[
\int_E \tilde{w}_v dQ = \int_E (w_s g_s(\tilde{w}_s) dM + qsQ(E).
\]

But

\[
\int_E w_s g_s(\tilde{w}_s) dM = \int_E w_s g_S(\tilde{w}_S) dM = \int_E w_s dQ
\]

and in conclusion

\[
\int_E \tilde{w}_v dQ = \int_E \tilde{w}_s dQ,
\]

which shows the martingale property.
Conclusion

We have studied some theoretical aspects of the Birkhoff integral, both for scalar valued functions with respect to Banach-valued measures and for the dual situation of vector-valued functions with respect to scalar measures. These previous results are then used in order to state an abstract version of the Girsanov Theorem, where the underlying probability measure $M$ is Banach-valued. The main results state that, under suitable conditions, a Martingale Equivalent to $M$ is found, under which the transformed process is a martingale with the same marginals as the original one.

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