ON NODE DISTRIBUTIONS FOR INTERPOLATION AND SPECTRAL METHODS

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Abstract. A scaled Chebyshev node distribution is studied in this paper. It is proved that the node distribution is optimal for interpolation in $C_{M}^{s+1}[-1,1]$, the set of $(s+1)$-time differentiable functions whose $(s+1)$-th derivatives are bounded in absolute value by a constant $M > 0$. Node distributions for computing spectral differentiation matrices and integration matrices are proposed and studied. Numerical experiments have shown that the proposed node distributions can yield results of higher accuracy than those obtained by the most commonly used Chebyshev-Gauss-Lobatto node distribution.

1. Introduction

Choosing node distribution is important in interpolating a function and solving differential and integral equations by pseudospectral methods. Given a sufficiently smooth function, if nodes are not suitably chosen, then the interpolation polynomials do not converge to the function as the number of nodes tends to infinity. A well-known example is Runge’s phenomenon. In particular, if one uses equi-spaced nodes to interpolate the Runge function $f(x) = \frac{1}{1+25x^2}$ over the interval $[-1,1]$, then the errors of Lagrange polynomial interpolation blow up to infinity as the number of nodes increases (see, e.g., [2]).

Let $f$ be a continuous function on $[-1,1]$, let $c := (c_i)_{i=0}^{s}$, $c_i \in [-1,1]$, and let $L_c(f)(x)$ be the Lagrange interpolation polynomial of $f$ over the nodes $(c_i)_{i=0}^{s}$. It is well known from interpolation theory that

$$\max_{x \in [-1,1]} |L_c(f)(x) - f(x)| \leq (1 + \Lambda_c) \max_{x \in [-1,1]} |P^*(x) - f(x)|,$$

where $P^*(x)$ is the best polynomial approximation of degree $s$ and $\Lambda_c$ is the Lebesgue constant corresponding to the node distribution $c = (c_i)_{i=0}^{s}$. The Lebesgue constant $\Lambda_c$ indicates how far the Lagrange interpolation polynomial $L_c(x)$ is from the best polynomial approximation of degree $s$. Lebesgue constants have been studied extensively in the literature (see, e.g., [1], [2], [4], [5], [7], [8], [9], and the references therein). It is of interest to find a node distribution of which the Lebesgue constant is minimal among Lebesgue constants of all node distributions with the same number of nodes. This node distribution if existing is called an optimal node distribution. It is known that for a given number of nodes, the optimal node distribution may not be unique. If one wants these nodes to include
boundary points, then such optimal node distribution is unique (cf. [4]). However, finding these node distributions is not an easy task.

The most commonly used node distribution in practice is Chebyshev-Gauss-Lobatto points which will be denoted by \((c_L^i)_{i=0}^s\). These points are extrema of \(T_s(x)\) the Chebyshev polynomial of the first kind of order \(s\) over \([-1,1]\), i.e.,

\[
\begin{align*}
  c_L^i &= -\cos\left(\frac{i\pi}{s}\right), & i = 0, \ldots, s.
\end{align*}
\]

This node distribution is also referred to as Chebyshev points. In [5] the Lebesgue constant of this node distribution was studied. It was proved that the Lebesgue constant for Chebyshev-Gauss-Lobatto nodes in (1.2) satisfies the following estimate (see, e.g., [2], [5]):

\[
\Lambda^{\text{CGL}}(n) = \frac{2}{\pi} \left( \ln n + \gamma + \ln \frac{8}{\pi} \right) + O\left(\frac{1}{n^2}\right),
\]

where \(\gamma \approx 0.577215\) is the Euler constant and \(n\) is the number of nodes.

Although Chebyshev-Gauss-Lobatto node distribution works well in practice, it is not optimal in the sense that the Lebesgue constant for this node distribution is minimal among Lebesgue constants of all node distributions for the same number of nodes. It is well known that for each function \(f\) there is an optimal node distribution for interpolating \(f\). This optimal node distribution varies when \(f\) varies. When \(f\) is known, there are algorithms for finding an optimal node distribution for interpolating \(f\). However, these algorithms are not efficient in practice. In many cases, these algorithms are not applicable since the function \(f\) is not known. This is the case when \(f\) is a solution to a differential or an integral equation.

It was proved that the optimal Lebesgue constant satisfies the following estimate (see, e.g., [10])

\[
\Lambda^{\text{min}}(n) = \frac{2}{\pi} \left( \ln n + \gamma + \ln \frac{4}{\pi} \right) + O\left(\frac{1}{(\ln n)^{1/3}}\right).
\]

Again, \(n\) denotes the number of nodal points. From equations (1.3) and (1.4) one can see that the Lebesgue constants of Chebyshev-Gauss-Lobatto points are close to the optimal one.

In [3] the Lebesgue constant for a scaled Chebyshev node distribution was studied. These nodes are obtained by scaling zeros of the Chebyshev polynomial \(T_{s+1}(x)\) so that the smallest and largest values of these nodes are \(-1\) and \(1\), respectively. In particular, the scaled Chebyshev nodes \((c_s^i)_{i=0}^s\) in [3] are defined as follows:

\[
\begin{align*}
  c_s^i &= -\cos\left(\frac{2i+1}{2(s+1)}\pi\right)\lambda_s^{-1}, & i = 0, \ldots, s, & \lambda_s &= \cos\left(\frac{1}{2(s+1)}\pi\right).
\end{align*}
\]

The Lebesgue constant of the scaled Chebyshev node distribution satisfies the following estimate (see, e.g., [3], [5]):

\[
\Lambda^{\text{sC}}(n) = \frac{2}{\pi} \left( \ln n + \gamma + \ln \frac{8}{\pi} - \frac{2}{3} \right) + O\left(\frac{1}{\ln n}\right).
\]

Note that \(\ln \frac{8}{\pi} - \frac{2}{3} \approx \ln \frac{4}{\pi} + 0.0265\) and \(\ln \frac{8}{\pi} \approx \ln \frac{4}{\pi} + 0.24\). Thus, for “large” \(n\), the Lebesgue constants of the scaled Chebyshev points are closer to the optimal Lebesgue constants compared to the Lebesgue constants of Chebyshev-Gauss-Lobatto points. The scaled Chebyshev nodes are often mentioned as the optimal
choice in practice for interpolation (cf. [4]). However, to the author’s knowledge, there is no justification for the optimality of this choice in any sense.

In practice one often uses Chebyshev-Gauss-Lobatto nodes and scaled Chebyshev nodes for interpolation and pseudospectral methods. The reason for choosing Chebyshev-Gauss-Lobatto nodes is often because of small Lebesgue constants of this node distribution. Thus, this explanation makes sense only for interpolation.

To the author’s knowledge there is no reasonable explanation why one should choose this node distribution for solving differential and integral equations.

In this paper, we study node distributions for interpolation and pseudospectral methods over the class of functions

\[ C_{s+1}^M[-1, 1] := \{ f \in C_{s+1}[-1, 1] : \max_{x \in [-1, 1]} |f^{(s+1)}(x)| \leq M \}. \]

We prove in Theorem 2.3 that the scaled Chebyshev nodes are optimal for interpolation over \( C_{s+1}^M[-1, 1] \). We also construct node distributions for computing differentiation matrices and integration matrices. Numerical experiments with the new node distributions in Section 5 in this paper have shown that these nodes yield better results than do Chebyshev-Gauss-Lobatto points for some certain cases.

The paper is organized as follows. In Section 2 we study node distributions for interpolation. We prove that the scaled Chebyshev nodes are “optimal” for interpolation over \( C_{s+1}^M[-1, 1] \). In Section 3 node distributions for calculating differentiation matrices are proposed and justified. In Section 4 two node distributions for computing integration matrices are proposed and their properties are investigated.

Section 5 presents numerical experiments with the new node distributions.

2. Interpolation

Let \( L_c(f) \) denote the Lagrange interpolation polynomial of a sufficiently smooth function \( f \) over the nodes \( c = (c_i)_{i=0}^s, -1 \leq c_0 < c_1 < \cdots < c_s \leq 1 \). The error of Lagrange interpolation is given by the formula (see, e.g., [6])

\[
(2.1) \quad f(x) - L_c(f)(x) = \frac{f^{(s+1)}(\xi(x))}{(s + 1)!} \prod_{i=0}^{s} (x - c_i), \quad \xi(x) \in [-1, 1].
\]

We are interested in finding a node distribution \( c \) so that the interpolation error \( \|L_c(f) - f\|_\infty \) is as small as possible. Here, \( \|g\|_\infty \) denotes the sup-norm of a continuous function \( g \) over the interval \([-1, 1]\), i.e., \( \|g\|_\infty := \sup_{x \in [-1, 1]} |g(x)| \). Note that the element \( \xi(x) \) in (2.1) depends on \( x \) and \((c_i)_{i=0}^s \) in a nontrivial manner. Therefore, to minimize \( \|L_c(f) - f\|_\infty \) one often tries to find a distribution of \((c_i)_{i=0}^s \) so that

\[
(2.2) \quad \max_{-1 \leq x \leq 1} \left| \prod_{i=0}^{s} (x - c_i) \right| \rightarrow \min_{-1 \leq c_0 < c_1 < \cdots < c_s \leq 1}.
\]

It is well known that the zeros of \( T_{s+1}(x) \), the Chebyshev polynomial of order \( s + 1 \) of the first kind over \([-1, 1]\), are the solution to (2.2). These zeros are given by the formula

\[
(2.3) \quad c_i = x_i^G := -\cos \left( \frac{2i + 1}{2(s + 1)} \pi \right), \quad i = 0, \ldots, s.
\]
In practice one often wants to have boundary points as interpolation nodes, i.e., \(c_0 = -1\) and \(c_s = 1\). Let
\[
C := \{c = (c_i)_{i=0}^s : -1 = c_0 < c_1 < \cdots < c_{s-1} < c_s = 1\}.
\]
The following question arises: For which set of points \((c_i)_{i=0}^s\in C\), do we have
\[
\max_{-1 \leq x \leq 1} \left| \prod_{i=0}^s (x - c_i) \right| \rightarrow \min_{c \in C}?
\]

For simplicity of notation, let us denote the scaled Chebyshev points by
\[
c_i^{sC} := x_i^G \lambda_s^{-1}, \quad 0 \leq i \leq s, \quad \lambda_s := \cos \frac{\pi}{2(s+1)},
\]
where \((x_i^G)_{i=0}^s\) are Chebyshev–Gauss points (cf. (2.3)).

The answer to the question in (2.5) is given in the following result:

**Theorem 2.1.** The unique solution to problem (2.5) is \(\bar{c} = (c_i^{sC})_{i=0}^s\), the scaled Chebyshev points determined by (2.6).

**Proof.** Let
\[
P(x) := \prod_{i=0}^s (x - c_i^{sC}) = \prod_{i=0}^s (x - x_i^G \lambda_s^{-1}).
\]

Then
\[
P(x\lambda_s^{-1}) = [\lambda_s^{-1}]^{s+1} \prod_{i=0}^s (x - x_i^G)
\]
\[
= \frac{\lambda_s^{-(s+1)}}{2^s} T_{s+1}(x),
\]
where \(T_{s+1}(x)\) is the Chebyshev polynomial of the first kind over \([-1,1]\) of degree \(s + 1\). Therefore,
\[
P(x) = \frac{\lambda_s^{-(s+1)} T_{s+1}(x\lambda_s)}{2^s}.
\]

Note that \(\left(\cos\left(\frac{ix\pi}{s+1}\right)\right)_{i=1}^s\) are all critical points of the Chebyshev polynomial \(T_{s+1}(x)\) and that \(|T_{s+1}(x)| \leq 1, \forall x \in [-1,1]\). This and equation (2.8) imply that all critical points of \(P(x)\) are
\[
d_i := x_i \lambda_s^{-1}, \quad x_i := \cos \frac{i\pi}{s+1}, \quad i = 1, \ldots, s,
\]
and we have
\[
P(d_i) = \frac{\lambda_s^{-(s+1)} T_{s+1}(x_i)}{2^s} = \frac{\lambda_s^{-(s+1)}}{2^s} (-1)^i, \quad i = 1, \ldots, s.
\]

It follows from equations (2.8) and (2.10) that
\[
\max_{-1 \leq x \leq 1} |P(x)| = \frac{\lambda_s^{-(s+1)}}{2^s} \min_{-1 \leq x \leq 1} |T_{s+1}(x\lambda_s)| = \frac{\lambda_s^{-(s+1)}}{2^s}.
\]

Therefore,
\[
\min_{\bar{c} \in \bar{C}} \max_{-1 \leq x \leq 1} \left| \prod_{i=0}^s (x - c_i) \right| \leq \max_{x \in [-1,1]} |P(x)| = \frac{\lambda_s^{-(s+1)}}{2^s}.
\]
Let \((\tilde{c}_i)_{i=0}^s \in C\) be a solution to \((2.3)\). Let us prove that \(\tilde{c}_i = c_i^{sC}\) where \(c_i^{sC}\), \(i = 0, \ldots, s\), are defined by \((2.6)\). Let

\[
Q(x) := \prod_{i=0}^{s} (x - \tilde{c}_i)
\]

and

\[
R(x) := Q(x) - P(x).
\]

Since \(P(x)\) and \(Q(x)\) are monic polynomials of degree \(s + 1\), one concludes from \((2.9)\) that \(R(x)\) is a polynomial of degree at most \(s\).

From \((2.10)\), \((2.14)\), and \((2.15)\), one obtains

\[
|Q(x)| \leq \frac{\lambda_{s}^{-(s+1)}}{2^s} = \max_{-1 \leq x \leq 1} |P(x)|, \quad \forall x \in [-1, 1].
\]

From \((2.10)\), \((2.14)\), and \((2.15)\), one obtains

\[
R(d_i)(-1)^i \geq 0, \quad i = 1, \ldots, s.
\]

This implies that the polynomial \(R(x)\) has at least \(s - 1\) zeros on the interval \([-d_1, d_s] \subset (-1, 1)\). Since \(c_0^{sC} = c_0 = -1\) and \(c_s^{sC} = \tilde{c}_s = 1\), it is clear that \(-1\) and \(1\) are zeros of both \(Q(x)\) and \(P(x)\). Thus, \(-1\) and \(1\) are also zeros of \(R(x)\). Therefore, \(R(x)\) has at least \(s + 1\) zeros on the interval \([-1, 1]\). This and the fact that \(R(x)\) is a polynomial of degree at most \(s\) imply that \(R(x) = 0\). Thus, \(Q(x) \equiv P(x)\) and this implies \(\tilde{c}_i = c_i^{sC}, \ i = 0, \ldots, s\). This conclusion implies two things: First, \((c_i^{sC})_{i=0}^s\) is a solution to problem \((2.5)\); Second, there is no solution other than \((c_i^{sC})_{i=0}^s\), i.e., \((c_i^{sC})_{i=0}^s\) is the unique solution to problem \((2.5)\).

Theorem \((2.1)\) is proved. \(\square\)

**Remark 2.2.** From the proof of Theorem \((2.1)\) one gets

\[
\min_{c \in C} \max_{-1 \leq x \leq 1} \left| \prod_{i=0}^{s} (x - c_i) \right| = \max_{-1 \leq x \leq 1} \left| \prod_{i=0}^{s} (x - c_i^{sC}) \right| = \frac{\lambda_{s}^{-(s+1)}}{2^s}.
\]

Let

\[
C_{M+1}^{s+1} := \{ f \in C^{s+1}[-1, 1] : \max_{x \in [-1, 1]} |f^{(s+1)}(x)| \leq M \}, \quad M > 0.
\]

Let \(L_c(f)\) denote the Lagrange interpolation polynomial of \(f\) over the nodes \(c := (c_i)_{i=0}^s\). We are interested in solving the following problem:

\[
\min_{c \in C} \sup_{f \in C_{M+1}^{s+1}} \|f - L_c(f)\|_{\infty}.
\]

Here \(\|g\|_{\infty} := \sup_{x \in [-1, 1]} |g(x)|\).

We have the following result:

**Theorem 2.3.** The node distribution \(\bar{c} := (c_i^{sC})_{i=0}^s\) is the unique solution to problem \((2.19)\).

**Proof.** Let \(c = (c_i)_{i=0}^s \in C\) be an arbitrary node distribution over \([-1, 1]\). The error of Lagrange interpolation is given by the formula (see, e.g., [3])

\[
f(x) - L_c(f)(x) = \frac{f^{(s+1)}(\xi(x))}{(s + 1)!} \prod_{i=0}^{s} (x - c_i), \quad \xi(x) \in [-1, 1].
\]
From equation (2.20) one gets

\[
(2.21) \quad \sup_{f \in C^{s+1}_M} \| f - L_c(f) \|_\infty \leq M \left( \frac{s}{s+1} \right) \max_{x \in [-1, 1]} \left| \prod_{i=0}^{s} (x - c^s_i) \right| = \frac{M \lambda^{-(s+1)}_s}{(s+1)! 2^s}.
\]

This equation implies

\[
(2.22) \quad \min_{c \in C} \sup_{f \in C^{s+1}_M} \| f - L_c(f) \|_\infty \leq \sup_{f \in C^{s+1}_M} \| f - L_c(f) \|_\infty = \frac{M \lambda^{-(s+1)}_s}{(s+1)! 2^s}.
\]

Let \( P_0(x) \) be a polynomial of degree \( s + 1 \) such that \( P_0^{(s+1)}(x) \equiv M \). Thus, \( P_0(x) \in C^{s+1}_M \). Using formula (2.20) for \( f(x) = P_0(x) \), one gets

\[
(2.23) \quad P_0(x) - L_c(P_0)(x) = P_0^{(s+1)}(\xi(x)) \prod_{i=0}^{s} (x - c_i) = \frac{M}{(s+1)!} \prod_{i=0}^{s} (x - c_i).
\]

From equations (2.23) and (2.17) we have

\[
\min_{c \in C} \sup_{f \in C^{s+1}_M} \| f - L_c(f) \|_\infty \geq \min_{c \in C} \| P_0 - L_c(P_0) \| \infty
\]

\[
(2.24) \quad \min_{c \in C} \sup_{f \in C^{s+1}_M} \| f - L_c(f) \|_\infty = \frac{M}{(s+1)!} \min_{c \in C} \max_{x \in [-1, 1]} \left| \prod_{i=0}^{s} (x - c_i) \right| = \frac{M \lambda^{-(s+1)}_s}{(s+1)! 2^s}.
\]

From equation (2.24) and (2.22), we conclude that

\[
(2.25) \quad \min_{c \in C} \sup_{f \in C^{s+1}_M} \| f - L_c(f) \|_\infty = \frac{M \lambda^{-(s+1)}_s}{(s+1)! 2^s}.
\]

This and (2.21) imply that \( \hat{c} := (c^s_i)_{i=0}^{s} \) is the solution to (2.19). The uniqueness of solution to (2.19) follows from the uniqueness of the solution to (2.5).

Theorem 2.3 is proved. \( \square \)

Remark 2.4. Theorem 2.3 says that the scaled Chebyshev node distribution (cf. (1.5)) is optimal in the sense of (2.19). Namely, the scaled Chebyshev node distribution is optimal for interpolating functions in \( C^{s+1}_M \). The uniqueness of solution to (2.19) follows from the uniqueness of the solution to (2.5).

Remark 2.5. For the Chebyshev-Gauss-Lobatto node distribution \( (c^L_i)_{i=0}^{s} \), we have

\[
(2.26) \quad \prod_{i=0}^{s} (x - c^L_i) = (x^2 - 1) \prod_{i=1}^{s-1} (x - c^L_i) = (x^2 - 1) \frac{T_s'(x)}{2s-1}.
\]

Since \( T_s(x) = \cos(s \arccos x) \), one has

\[
T_s'(x) = \frac{s \sin(s \arccos x)}{\sqrt{1 - x^2}}.
\]

This and equation (2.26) imply

\[
(2.27) \quad \prod_{i=0}^{s} (x - c^L_i) = \frac{-\sqrt{1 - x^2} \sin(s \arccos x)}{2^s - 1}.
\]
Table 1. Values of $\lambda_s^{-(s+1)}$.

| $s$ | 4   | 6   | 8   | 10  | 12  | 14  | 16  |
|-----|-----|-----|-----|-----|-----|-----|-----|
| $\lambda_s^{-(s+1)}$ | 1.285 | 1.195 | 1.148 | 1.119 | 1.010 | 1.086 | 1.075 |

This implies

$$\max_{-1 \leq x \leq 1} \left| \prod_{i=0}^{s} (x - c_i^L) \right| \leq \frac{1}{2^{s-1}}.$$  \hfill (2.28)

Inequality (2.28) becomes equality when $s$ is odd and the maximum value is attained at $x = 0$. When $s$ is even, inequality (2.28) becomes a strict inequality. When $s$ is even and is not ‘small’, then the left-hand side of (2.28) is very close to the right-hand side of (2.28). It follows from (2.20) and (2.28) that for the Chebyshev-Gauss-Lobatto node distribution $c_L = (c_i^L)_{s=0}^s$ one has

$$\sup_{f \in C_s^{M+1}} \| f - L_{c_L}(f) \|_{\infty} \leq \frac{M}{(s+1)!} \frac{1}{2^{s-1}}.$$  \hfill (2.29)

Inequality (2.29) becomes equality when $s$ is odd. When $s$ is even, inequality (2.29) is a strict inequality. However, if $s$ is even and $s$ is not small, then the relative difference between two sides of (2.29) is very small. One can verify that

$$\lim_{s \to \infty} \lambda_s^{-(s+1)} = \lim_{s \to \infty} \cos \left( \frac{\pi}{2(s+1)} \right)^{-(s+1)} = 1,$$

and $\lambda_s^{-(s+1)}$ is very close to 1 when $s \geq 12$ (see Table 1). This, equation (2.28), and relation (2.29) imply that the error $\sup_{f \in C_s^{M+1}} \| f - L_c(f) \|_{\infty}$ for scaled Chebyshev node distribution is nearly half as small as that of Chebyshev-Gauss-Lobatto node distribution of the same number of nodes $s$ when $s \geq 12$.

One can also consider the problems

$$\max_{-1 \leq x \leq 1} \left| \prod_{i=0}^{s} (x - c_i) \right| \to \min_{c \in C_{a,b}}$$  \hfill (2.30)

and

$$\sup_{f \in C_s^{M+1}} \| f - L_c(f) \|_{\infty} \to \min_{c \in C_{a,b}},$$  \hfill (2.31)

where

$$C_{a,b} = \{(c_i)_{i=0}^s | a = c_0 < c_1 < \cdots < c_s = b, -1 \leq a \leq -\lambda_s, \lambda_s \leq b \leq 1 \}.$$  

By using similar arguments as in Theorems 2.1 and 2.3, one can prove the following result:

**Theorem 2.6.** The unique solution to problem (2.30) and problem (2.31) is $c = (c_i^{a,b})_{i=0}^s$ where

$$c_i^{a,b} = \frac{b-a}{2} c_i^{a} - \frac{a+b}{2}, \quad i = 0, \ldots, s.$$
3. Spectral differentiation matrices

In many problems one is interested in estimating the first derivative $f'$ of a function $f \in C^1[-1,1]$ based on the values of $f$ at $(c_i)_{i=0}^s$, $c_i \in [-1,1]$. One of the approaches is to use $(L_c(f))'$ as an approximation to $f'$ where $L_c(f)$ is the Lagrange interpolation polynomial of the function $f$ over the nodes $c = (c_i)_{i=0}^s$. Thus, the following problem arises

$$
\min_{c \in C} \max_{x \in [-1,1]} |f'(x) - (L_c(f))'(x)|, \quad f \in C^1[-1,1].
$$

Unfortunately, a solution $c$ to (3.1) if existing is not independent of $f$, in general, and is not easy to find even when $f$ belongs to the class of functions $C^{s+1}_M[-1,1]$.

To find a node distribution $c$ which minimizes the interpolation error $\|f' - (L_c(f))'\|_\infty$ we consider the following problem:

$$
\max_{-1 \leq x \leq 1} \left| \frac{d}{dx} \prod_{i=0}^s (x - c_i) \right| \rightarrow \min_{c \in C}.
$$

The motivation for considering problem (3.2) comes from the following analysis. Let $f \in C^{s+1}_M[-1,1]$ and $c = (c_i)_{i=0}^s$ be a node distribution over $[-1,1]$. It follows from (2.1) that

$$
f'(x) - (L_c(f))'(x) = \prod_{i=0}^s (x - c_i) \frac{d}{dx} \frac{f^{(s+1)}(\xi(x))}{(s+1)!} + \frac{f^{(s+1)}(\xi(x))}{(s+1)!} \prod_{i=0}^s (x - c_i).
$$

This implies

$$
f'(c_k) - (L_c(f))'(c_k) = \left. \frac{f^{(s+1)}(\xi(x))}{(s+1)!} \frac{d}{dx} \prod_{i=0}^s (x - c_i) \right|_{x=c_k}, \quad 0 \leq k \leq s.
$$

In addition, fix $\bar{x} \in [-1,1]$ and $\bar{x} \neq c_i$, $i = 0, \ldots, s$. Let

$$
R(x) := f(x) - L_c(f)(x) - \frac{f^{(s+1)}(\xi(\bar{x}))}{(s+1)!} \prod_{i=0}^s (x - c_i).
$$

Then $R(x)$ has $s+2$ zeros which are $(c_i)_{i=0}^s$ and $\bar{x}$ (cf. (2.1)). Thus, by Rolle’s Theorem, the function $R'(x)$ has at least $s+1$ zeros on $[-1,1]$. Let $(\eta_i)_{i=0}^s$ be zeros of $R'(x)$. Then one gets

$$
f'(\eta_k) - (L_c(f))'(\eta_k) = \left. \frac{f^{(s+1)}(\xi(x))}{(s+1)!} \frac{d}{dx} \prod_{i=0}^s (x - c_i) \right|_{x=\eta_k}, \quad k = 0, \ldots, s.
$$

Thus, by using the node distribution $c = (c_i)_{i=0}^s$ which solves problem (3.2) we hope to be able to minimize the error $|f'(x) - (L_c(f))'(x)|$ for $x = c_i$, $i = 0, \ldots, s$ and $x = \eta_i$, $i = 0, \ldots, s$, and for other values of $x \in [-1,1]$.

Remark 3.1. From (3.4) and (3.6) one may ask whether or not there exists a constant $C > 0$ such that

$$
|f'(x) - (L_c(f))'(x)| \leq C \left| \frac{d}{dx} \prod_{i=0}^s (x - c_i) \right|, \quad x \in [-1,1], \quad f \in C^{s+1}_M.
$$
Unfortunately, the answer to this question is negative. It is because zeros of the left-hand side of (3.7) are, in general, not zeros of the right-hand side of (3.7). Thus, if $\xi$ is a zero of the right-hand side of (3.7) but is not a zero of the left-hand side of (3.7), then equation (3.7) does not hold for any $C$ when $x = \xi$.

Consider problem (3.2), i.e.,

\begin{equation}
\max_{-1 \leq x \leq 1} \left| \frac{d}{dx} \prod_{i=0}^{s} (x - c_i) \right| \rightarrow \min_{c \in C}.
\end{equation}

Since $\frac{d}{dx} \prod_{i=0}^{s} (x - c_i)$ is a polynomial of degree $s$ and has $s + 1$ as the leading coefficient, the set of points $(c_i)_{i=0}^{s} \in C$ is a solution to problem (3.8) if

\begin{equation}
\frac{d}{dx} \prod_{i=0}^{s} (x - c_i) = \frac{(s + 1)T_s(x)}{2^{s-1}}.
\end{equation}

Thus, we want to find $(c_i)_{i=0}^{s} \in C$ so that

\begin{equation}
\prod_{i=0}^{s} (x - c_i) = \int_{0}^{x} \left( \frac{(s + 1)T_s(\xi)}{2^{s-1}} \right) d\xi + C,
\end{equation}

where $C = (-1)^{s+1} \prod_{i=0}^{s} c_i$. Equivalently, we want to find a constant $C$ so that the polynomial $\int_{0}^{x} \left( \frac{(s + 1)T_s(\xi)}{2^{s-1}} \right) d\xi + C$ has $(c_i)_{i=0}^{s} \in C$ as its roots.

Remark 3.2. It is well known that the Chebyshev polynomial $T_s$ satisfies the following relation:

\begin{equation}
\int T_s(x) \, dx = \frac{1}{2} \left( \frac{T_{s+1}(x)}{s + 1} - \frac{T_{s-1}(x)}{s - 1} \right) + C,
\end{equation}

where $C = \text{const}$. Actually, the integral of $T_s$ expressed in (3.11) has been used before in studying superconvergence property of the Chebyshev spectral method; see (2.4) in [11].

From (3.10) and (3.11) we want to find $(c_i)_{i=0}^{s} \in C$ so that

\begin{equation}
\prod_{i=0}^{s} (x - c_i) = \int_{0}^{x} \left( \frac{(s + 1)T_s(\xi)}{2^{s-1}} \right) d\xi + C,
\end{equation}

where $C = (-1)^{s+1} \prod_{i=0}^{s} c_i$. Equivalently, we want find $C$ so that the polynomial

\begin{equation}
\frac{s + 1}{2^{s-1}} \left[ \frac{1}{2} \left( \frac{T_{s+1}(x)}{s + 1} - \frac{T_{s-1}(x)}{s - 1} \right) + C \right]
\end{equation}

has $(c_i)_{i=0}^{s} \in C$ as its roots. However, it is not clear if such a constant $C$ exists. In fact, if $s$ is an even integer, then such a constant $C$ does not exist as we will prove later.

Consider the case when $s$ is odd. Let

\begin{equation}
P_{s+1}(x) := \frac{s + 1}{2^{s-1}} \left[ \frac{1}{2} \left( \frac{T_{s+1}(x)}{s + 1} - \frac{T_{s-1}(x)}{s - 1} \right) + \frac{1}{s^2 - 1} \right].
\end{equation}

We have the following result:
Theorem 3.4. Let \( s > 0 \) be an odd integer. The polynomial \( P_{s+1}(x) \) defined in (3.14) has \( s+1 \) distinct zeros \( \{c_i\}_{i=0}^{s+1} \) on the interval \([-1, 1]\), \(-1 = c_0 < c_1 < \cdots < c_s = 1\). Moreover, these zeros are symmetric about 0.

Proof. When \( s \) is odd, the polynomials \( T_{s+1}(x) \) and \( T_{s-1}(x) \) are even functions on \( \mathbb{R} \). Thus, \( P_{s+1}(x) \) is an even function and its zeros are symmetric about 0.

Since \( s+1 \) and \( s-1 \) are even, one has \( T_{s+1}(\pm 1) = T_{s-1}(\pm 1) = 1 \). From (3.14) one gets

\[
(3.15) \quad P_{s+1}(-1) = P_{s+1}(1) = \frac{s+1}{2^{s-1}} \left( \frac{1}{s+1} - \frac{1}{s-1} + \frac{1}{s^2 - 1} \right) = 0.
\]

From (3.11) and (3.14) one gets \( P'_{s+1}(x) = \frac{s+1}{2s} T_s(x) \). Thus, \( P'_{s+1}(x) \) and \( T_s(x) \) share the same zeros which are \( \{x_i\}_{i=0}^{s-1}, x_i = \cos \left( \frac{2i+1}{2s} \pi \right) \). Since \( T_{s+1}(x) + T_{s-1}(x) = 2x T_s(x) \) and \( \{x_i\}_{i=0}^{s-1} \) are zeros of \( T_s(x) \), one gets \( T_{s+1}(x_i) + T_{s-1}(x_i) = 2x_i T_s(x_i) = 0 \). Thus, \( T_{s+1}(x_i) = -T_{s-1}(x_i), i = 0, \ldots, s-1 \), and from (3.14) one gets

\[
(3.16) \quad P_{s+1}(x_i) = \frac{s+1}{2^{s-1}} \left[ \frac{1}{2} \left( \frac{T_{s+1}(x_i)}{s+1} - \frac{T_{s-1}(x_i)}{s-1} \right) + \frac{1}{s^2 - 1} \right], \quad i = 0, \ldots, s-1.
\]

From the relation \( \arccos(x_i) = \frac{2i+1}{2s} \pi \), one gets

\[
(3.17) \quad T_{s+1}(x_i) = \cos ((s+1) \arccos(x_i)) = \cos \left( \frac{2i+1}{2} \pi + \frac{2i+1}{2s} \pi \right) = (-1)^{i+1} \sin \left( \frac{2i+1}{2s} \pi \right).
\]

Note that

\[
x - \sin \left( \frac{\pi x}{2} \right) < 0, \quad \forall x \in (0, 1).
\]

Thus,

\[
(3.18) \quad \sin \left( \frac{2i+1}{2s} \pi \right) \geq \sin \left( \frac{\pi}{2s} \right) > \frac{1}{s}, \quad i = 0, \ldots, s-1, \quad s > 1.
\]

From (3.17), (3.18) one obtains

\[
(3.19) \quad (-1)^{i+1} P_{s+1}(x_i) = (-1)^{i+1} \frac{s}{(s-1)2^{s-1}} \left( T_{s+1}(x_i) + \frac{1}{s} \right)
\]

\[
= \frac{s}{(s-1)2^{s-1}} \left( \sin \left( \frac{2i+1}{2s} \pi \right) + \frac{(-1)^{i+1}}{s} \right)
\]

\[
> \frac{s}{(s-1)2^{s-1}} \left( \frac{1}{s} + \frac{(-1)^{i+1}}{s} \right) \geq 0, \quad i = 0, \ldots, s-1.
\]

The inequality \( (-1)^{i+1} P_{s+1}(x_i) > 0 \), \( i = 0, \ldots, s-1 \), implies that \( P_{s+1}(x) \) has at least \( s-1 \) distinct zeros on the interval \([x_0, x_{s-1}) \subset (-1, 1) \). This and (3.15) imply that \( P_{s+1}(x) \) has \( s+1 \) distinct zeros on the interval \([-1, 1]\). Theorem 3.4 is proved. \( \square \)
Remark 3.5. It follows from Theorem 3.4 and the analysis above that, when \( s \) is an odd integer, a solution to problem (3.2) is the node distribution \( \mathbf{c} = (c_i)_{i=1}^s \) which consists of zeros of the polynomial \( P_{s+1}(x) \) defined by (3.14).

Consider the case when \( s \) is an even integer.

**Theorem 3.6.** Let \( 0 < s \) be an even integer. For any constant \( C \) the polynomial

\[
g_{s+1}(x) = \frac{s+1}{2s-1} \left( \frac{T_{s+1}(x)}{s+1} - \frac{T_{s-1}(x)}{s-1} \right) + C
\]

has at most \( s \) zeros on the interval \([-1,1]\).

**Proof.** From Remark 3.2 and (3.20) one gets \( g_{s+1}'(x) = \frac{s+1}{2s-1} T_s(x) \). Thus, all zeros of \( g_{s+1}'(x) \) are zeros of \( T_s(x) \) which are \( (x_i)_{i=0}^{s-1}, \ x_i = \cos(\frac{2i+1}{2s} \pi), \ i = 0, \ldots, s - 1 \).

Since \( g_{s+1}'(x) \) does not change sign on intervals \([x_i, x_{i+1}], \ i = 0, \ldots, s - 2\), the function \( g_{s+1}(x) \) has at most \( s - 1 \) zeros on \([x_0, x_{s-1}]\) for any given \( C \).

We have

\[
g_{s+1}'(\pm 1) = \frac{s+1}{2s-1} T_s(\pm 1) = \frac{s+1}{2s-1} > 0,
\]

when \( s \) is even. This and the fact that the polynomial \( g_{s+1}'(x) \) does not change sign on \( x \in [-1, x_0] \cup [x_{s-1}, 1] \) imply

\[
g_{s+1}'(x) = \frac{s+1}{2s-1} T_s(x) \geq 0, \quad x \in [-1, x_0] \cup [x_{s-1}, 1].
\]

Therefore,

\[
\min_{[-1,x_0]} g_{s+1}(x) = g_{s+1}(-1) = \frac{s+1}{2s-1} \frac{1}{s^2 - 1} + C
\]

\[
> - \frac{s+1}{2s-1} \frac{1}{s^2 - 1} + C = g_{s+1}(1) = \max_{x \in [x_{s-1}, 1]} g_{s+1}(x).
\]

Thus,

\[
\left\{ g_{s+1}(x) \mid x \in [-1, x_0] \right\} \cap \left\{ g_{s+1}(x) \mid x \in [x_{s-1}, 1] \right\} = \emptyset.
\]

Hence, for any given \( C \) there exists at most one zero of \( g_{s+1}(x) \) on \([-1, x_0] \cup [x_{s-1}, 1]\). Therefore, the function \( g_{s+1}(x) \) has at most \( s \) zeros on the interval \([-1,1]\) for any given \( C \). Theorem 3.6 is proved.

**Remark 3.7.** Theorem 3.6 says that when \( s \) is even, there does not exist a constant \( C \) and \( (c_i)_{i=0}^s, \ c_i \in [-1,1], \) so that equation (3.12) holds. Thus, when \( s \) is even, there does not exist a node distribution \( (c_i)_{i=0}^s \in C \) satisfying equation (3.3).

Let us propose two possible node distributions \((c_i)_{i=0}^s\) for computing differentiation matrices when \( s \) is even. Let

\[
Q_{s+1}(x) := \frac{s+1}{2s} \left( \frac{T_{s+1}(x)}{s+1} - \frac{T_{s-1}(x)}{s-1} + \frac{2x}{s^2 - 1} \right).
\]

Let us prove that \( Q_{s+1}(x) \) has \( s + 1 \) zeros \((c_i)_{i=0}^s\) on the interval \([-1,1], -1 = c_0 < c_1 < \cdots < c_s = 1\). Thus, one can use this node distribution for the computation of differential matrices when \( s \) is even. We have the following result:

**Theorem 3.8.** Let \( s \) be an even integer. The polynomial \( Q_{s+1}(x) \) has \( s + 1 \) zeros \((c_i)_{i=0}^s\) on the interval \([-1,1], -1 = c_0 < c_1 < \cdots < c_s = 1\).
Proof. When \( s \) is even, \( T_{s+1}(x) \) and \( T_{s-1}(x) \) are odd functions and we have \( T_{s+1}(1) = T_{s-1}(1) = 1 \). Thus, \( Q_{s+1}(x) \) is an odd function and we have

\[
Q_{s+1}(-1) = Q_{s+1}(1) = \frac{s + 1}{2^s} \left( \frac{1}{s + 1} - \frac{1}{s - 1} + \frac{2}{s^2 - 1} \right) = 0.
\]

Let \( x_i = \cos(\frac{2i+1}{2s+1}\pi) \), \( i = 0, \ldots, s-1 \). Then \( (x_i)_{i=0}^{s-1} \) are zeros of \( T_s(x) \). By similar arguments as in Theorem 3.21 (cf. (3.21)) one gets

\[
(-1)^{i+1}Q_{s+1}(x_i) = (-1)^{i+1} \frac{s}{(s-1)2^{s-1}} (T_{s+1}(x_i) + x_i).
\]

Let \( d \) be chosen as zeros of \( Q_{s+1}(x) \). Thus, for this choice of \( (c_i)_{i=0}^{s} \) one concludes that \( Q_{s+1}(x) \) has \( s + 1 \) zeros on the interval \([0, 1]\). Taking into account (3.24), one gets

\[
\frac{d}{dx} \prod_{i=0}^{s} (x - c_i) = \frac{s + 1}{2^{s-1}} \left( T_s(x) + \frac{1}{(s-1)(s+1)} \right).
\]

Thus, for this choice of \( (c_i)_{i=0}^{s} \) equation (3.9) does not hold. However, when \( s \) is large, the fraction \( \frac{1}{(s-1)(s+1)} \) is small compared to 1, the maximum value of \( |T_s(x)| \) on \([-1, 1]\). Therefore, for large even integer \( s \) the node distribution based on zeros of the polynomial \( Q_{s+1}(x) \) defined by (3.25) is almost as good as the node distribution based on zeros of the polynomial \( P_{s+1}(x) \) defined by (3.14).

Let us discuss another possible choice for \( (c_i)_{i=0}^{s} \) when \( s \) is even. Consider the polynomial

\[
\tilde{Q}_{s+1}(x) := \frac{s + 1}{2^s} \left( \frac{T_{s+1}(x)}{s + 1} - \frac{T_{s-1}(x)}{s - 1} \right).
\]

The functions \( T_{s-1}(x) \) and \( T_{s+1}(x) \) are odd functions when \( s \) is an even integer. Thus, \( \tilde{Q}_{s+1}(s) \) is an odd function when \( s \) is even. Therefore, zeros of \( \tilde{Q}_{s+1}(s) \) are symmetric about 0. By similar arguments as in Theorem 3.21 one can show that \( \tilde{Q}_{s+1}(s) \) has \( s + 1 \) zeros. Note that not all these \( s + 1 \) zeros are in \([-1, 1]\). Let \( d_0 < d_1 < \cdots < d_s \) be zeros of \( \tilde{Q}_{s+1}(s) \) and let \( c_i := d_i \). Then

\[
-1 = c_0 < c_1 < \cdots < c_s = 1, \quad c_i = d_i \quad \text{for} \quad i = 0, \ldots, s.
\]

Note that if \( (c_i)_{i=0}^{s} \) are chosen by (3.28), then equation (3.9) does not hold. In fact, for this choice of \( (c_i)_{i=0}^{s} \) one has

\[
\frac{d}{dx} \prod_{i=0}^{s} (x - c_i) = \frac{1}{d_s} \left( \frac{s + 1}{2^{s-1}} T_s(d_s x) \right).
\]
Remark 3.9. For the Chebyshev-Gauss-Lobatto node distribution \((c_i^L)_i=0^s\), we have (cf. (2.27))

\[
\prod_{i=0}^{s}(x - c_i^L) = \frac{-\sqrt{1-x^2} \sin(s \arccos x)}{2^{s-1}}.
\]

This implies

\[
\frac{d}{dx} \prod_{i=0}^{s}(x - c_i^L) = \frac{1}{2^{s-1}} \left( \frac{x \sin(s \arccos x)}{\sqrt{1-x^2}} + s \cos(s \arccos x) \right) + \frac{1}{2^{s-1}} \left( xT_s'(x) + sT_s(x) \right).
\]

(3.31)

Note that \(T_s'(c_i^L) = 0, i = 1, \ldots, s - 1\). This and equation (3.31) imply

\[
\left| \frac{d}{dx} \prod_{i=0}^{s}(x - c_i^L) \right|_{x=c_i} = \frac{sT_s(c_i)}{2^{s-1}} = \frac{s}{2^{s-1}} < \frac{s + 1}{2^{s-1}}, \quad i = 1, \ldots, s - 1.
\]

(3.32)

It follows from equations (3.39) and (3.32) that the Chebyshev-Gauss-Lobatto nodes yield better results at interior nodes than the node distribution based on zeros of \(P_{s+1}(x)\) in equation (3.14) and zeros of \(Q(x)\) in equation (3.23). However, the difference is insignificant for large \(s\) as the relative difference of two sides of inequality (3.32) is very small when \(s\) is large.

4. Spectral integration matrices

In this section we consider the problem of evaluating antiderivatives of a function \(f\) using the values of \(f\) at nodes \((c_i^s)_i=0^s\), \(c_i \in [-1, 1]\). Using Lagrange interpolation polynomial \(L_c(f)\) of \(f\) over the nodes \((c_i^s)_i=0^s\), we approximate the function \(\int_a^x f(t) dt, \ a \in [-1, 1]\) by \(\int_a^x L_c(f)(t) dt\). When the parameter \(a\) can be freely chosen, we consider the following problem:

\[
\min_{c} \max_{a \in [-1, 1]} \left| \int_a^x (L_c(f)(t) - f(t)) dt \right| \to \min_c.
\]

(4.1)

From equation (2.1), we have

\[
\int_a^x \left(L_c(f)(t) - f(t)\right) dt = \int_a^x f^{(s+1)}(\xi(t)) \prod_{i=0}^{s}(t - c_i) dt.
\]

(4.2)

Since the term \(f^{(s+1)}(\xi(t))\) in the above equation depends on \(t\) in a nontrivial manner, it is not easy to find a node distribution \(c\) which solves problem (4.1).

Thus, we consider the following problem instead of problem (4.1):

\[
\min_{a \in [-1, 1]} \max_{x \in [-1, 1]} \left| \int_a^x \prod_{i=0}^{s}(t - c_i) dt \right| \to \min_a.
\]

(4.3)

We do not claim that the solution to problem (4.3) solves problem (4.1). We hope that by solving problem (4.3), we will be able to ‘minimize’ the error for approximating antiderivatives of \(f\) on the interval \([-1, 1]\).

Since \(\prod_{i=0}^{s}(t - c_i)\) is a monic polynomial of order \(s + 1\), the integral \(\int_a^x \prod_{i=0}^{s}(t - c_i) dt\) is a polynomial of order \(s + 2\) and has the constant \(1/(s + 2)^{s+2}\) as its leading coefficient. From the theory of Chebyshev polynomials one concludes that a set of nodes \((c_i^s)_i=0^s\) is the solution to problem (4.3) if \(\int_a^x \prod_{i=0}^{s}(t - c_i) dt = \frac{1}{(s + 2)^{s+2}} T_{s+2}(x)\). This is
the case if \((c_i)_{i=0}^s\) are critical points of the Chebyshev polynomial \(T_{s+2}(x)\) and \(a\) is a root of \(T_{s+2}(x)\). Hence, we have the following result:

**Theorem 4.1.** The solution to problem \((4.3)\) is \(c = (c_i)_{i=0}^s\) where \((c_i)_{i=0}^s\) are the critical points of the Chebyshev polynomial \(T_{s+2}(x)\), i.e., the zeros of \(T_{s+2}'(x)\).

In some cases, one may have \(a = 1\) and may want to use the end points \(±1\) as two nodes for approximating \(\int_{-1}^{1} f(t) \, dt\). Thus, we consider the following problem:

\[
(4.4a) \quad \max_{x \in [-1,1]} \left| \int_{-1}^{x} \prod_{i=0}^{s} (t - c_i) \, dt \right| \rightarrow \min_{c \in \mathbb{C}},
\]

subject to

\[
(4.4b) \quad \int_{-1}^{1} \prod_{i=0}^{s} (t - c_i) \, dt = 0.
\]

Note that equation \((4.4b)\) is included as a constraint for problem \((4.4a)\) because we want the error at the end point \(x = 1\) to be minimal. We have the following result:

**Theorem 4.2.** Let \((\tilde{c}_i)_{i=0}^s \in \mathbb{C}\) be such a node distribution that the polynomial

\[
I(x) = \int_{-1}^{x} \prod_{i=0}^{s} (t - \tilde{c}_i) \, dt
\]

satisfies the relations

\[
(4.5) \quad I(\tilde{c}_i) = (-1)^{s+1-i}M, \quad i = 1, \ldots, s - 1, \quad |I(x)| \leq M, \quad x \in [-1,1],
\]

where \(M\) is a positive constant. Then the unique solution to problem \((4.4)\) is \((c_i)_{i=0}^s = (\tilde{c}_i)_{i=0}^s\).

**Proof.** Assume to the contrary that \((c_i)_{i=0}^s\) is a solution to problem \((4.4)\) and \((c_i)_{i=0}^s \neq (\tilde{c}_i)_{i=0}^s\). Define

\[
(4.6) \quad P(x) = \int_{-1}^{x} \prod_{i=0}^{s} (t - c_i) \, dt.
\]

Since \((c_i)_{i=0}^s\) is a solution to problem \((4.4)\) we have

\[
(4.7) \quad \max_{-1 \leq x \leq 1} |P(x)| \leq M, \quad P(1) = \int_{-1}^{1} \prod_{i=0}^{s} (t - c_i) \, dt = 0.
\]

Let

\[
(4.8) \quad R(x) := I(x) - P(x).
\]

It follows from equations \((4.5)\) and \((4.7)\) that

\[
(4.9) \quad (-1)^{s+1-i} R(c_i) := (-1)^{s+1-i} I(c_i) - (-1)^{s+1-i} P(c_i)
\]

\[
\geq M - (-1)^{s+1-i} P(c_i) \geq 0, \quad i = 1, \ldots, s - 1.
\]

This implies that the polynomial \(R(x)\) has at least \(s - 2\) zeros on the interval \([c_1, c_{s-1}]\). Since \(P(±1) = I(±1) = 0\), one has \(R(±1) = I(±1) - P(±1) = 0\). Thus, the total zeros of \(R(x)\) on the interval \([-1,1]\) is at least \(s - 2 + 2 = s\). By Rolle’s theorem, one concludes that the function \(R'(x)\) has at least \(s - 1\) zeros on the interval \((-1,1)\). Since \(c_0 = \tilde{c}_0 = -1\) and \(c_s = \tilde{c}_s = 1\), both \(-1\) and \(1\) are zeros of \(I'(x)\) and \(P'(x)\). Thus, \(-1\) and \(1\) are zeros of \(R'(x)\). This and the arguments above imply that \(R'(x)\) has at least \(s - 1 + 2 = s + 1\) zeros on the interval \([-1,1]\). It follows
from their definitions that $I'(x)$ and $P'(x)$ are monic polynomials of degree $s + 1$. Thus, $R'(x) = I'(x) - P'(x)$ is a polynomial of degree at most $s$. This and the fact that $R'(x)$ has at least $s + 1$ zeros imply that $R(x) \equiv 0$. Therefore, $I(x) \equiv P(x)$ and $(c_i)_{i=0}^s = (\tilde{c_i})_{i=0}^s$. Thus, $(\tilde{c_i})_{i=0}^s$ is the unique solution to problem (4.4). Theorem 4.2 is proved. \qed

Remark 4.3. It is not known if there are explicit formulas for the polynomial $I(x)$ and the constants $(\tilde{c_i})_{i=0}^s$ in Theorem 4.2. In our experiments, we found approximate values for $(\tilde{c_i})_{i=0}^s$ for small $s$ by looking for polynomials that satisfy condition (4.5) in Theorem 4.2 and then computing zeros of these polynomials.

One can also modify the condition $c \in \mathbb{C}$ in (4.4a) and consider the following problem:

\begin{align*}
\text{(4.10a)} & \quad \max_{x \in [-1,1]} \left| \int_{-1}^{x} \prod_{i=0}^{s} (t - c_i) \, dt \right| \longrightarrow -1 \leq c_0 < c_1 < \cdots < c_s \leq 1, \\
\text{(4.10b)} & \quad \text{subject to } \int_{-1}^{1} \prod_{i=0}^{s} (t - c_i) \, dt = 0.
\end{align*}

We have the following result:

Theorem 4.4. The unique solution to problem (4.10) is $c = (d_i)_{i=0}^s$, where

\begin{equation}
\tag{4.11}
d_i = -\cos \left( \frac{(i+1)\pi}{s+2} \right) \lambda_{s+1}^{-1}, \quad i = 0, \ldots, s, \quad \lambda_{s+1} := \cos \left( \frac{\pi}{2(s+2)} \right).
\end{equation}

Proof. It is clear that $d_i \lambda_{s+1} = -\cos \frac{(i+1)\pi}{s+2}$, $i = 0, \ldots, s$, are all $s + 1$ zeros of the polynomial $T_{s+2}'(x)$ and

$$T_{s+2}(\lambda_{s+1}d_i) = \cos \left( (s+2) \arccos \left( -\cos \frac{(i+1)\pi}{s+2} \right) \right) = (-1)^{s+1-i}.$$ Here $T_{s+2}(x)$ is the Chebyshev polynomial of the first kind of order $s + 2$. Thus, we have

\begin{equation}
\tag{4.12}
\prod_{i=0}^{s} (x - d_i) = \frac{1}{\lambda_{s+1}^{s+1}(s+2)^{2s+1}} T_{s+2}'(\lambda_{s+1}x).
\end{equation}

This implies that

\begin{equation}
\tag{4.13}
J(x) := \int_{-1}^{x} \prod_{i=0}^{s} (\xi - d_i) \, d\xi = \frac{1}{\lambda_{s+1}^{s+1}(s+2)^{2s+1}} \int_{-1}^{x} T_{s+2}'(\lambda_{s+1}\xi) \, d\xi \\
= \frac{1}{\lambda_{s+1}^{s+2}(s+2)^{2s+1}} T_{s+2}(\lambda_{s+1}x).
\end{equation}

Here, we have used the relation that

$$T_{s+2}(-\lambda_{s+1}) = \cos \left[ (s+2) \arccos \left( -\cos \frac{\pi}{2(s+2)} \right) \right] = \cos \left( \frac{\pi}{2} + (s+2)\pi \right) = 0.$$
From equation (4.13) one gets
\[ J(d_i) = \frac{1}{\lambda_{s+1}^2(s+2)2s+1}T_{s+2}(\lambda_{s+1}d_i) = \frac{(-1)^{s+1-i}}{\lambda_{s+1}^2(s+2)2s+1}, \quad i = 0, \ldots, s, \]
(4.14)
\[ |J(x)| \leq \frac{1}{\lambda_{s+1}^2(s+2)2s+1}, \quad x \in [-1, 1], \quad J(-1) = J(1) = 0. \]

Let \( c = (c_i)_{i=0}^s \) be a solution to problem (4.10). Define
(4.16)
\[ H(x) = \int_{-1}^{x} \prod_{i=0}^{s} (\xi - c_i) \, d\xi. \]

Since \( c = (c_i)_{i=0}^s \) is a solution to (4.10), we have
(4.17)
\[ H(-1) = H(1) = 0, \quad |H(x)| \leq |J(x)|, \quad \forall x \in [-1, 1]. \]

Define \( R(x) = J(x) - H(x) \). By similar arguments and in Theorem 2.1, one can show that the polynomial \( R'(x) \) has at least \( s + 1 \) zeros. However, \( R'(x) \) is a polynomial of degree at most \( s \). Therefore, \( R'(x) \equiv 0 \) or \( J'(x) \equiv H'(x) \). This implies that \( c_i = d_i \), and, therefore, the node distribution \( c = (d_i)_{i=0}^s \) is the unique solution to problem (4.10). Theorem 4.4 is proved.

\[ \square \]

5. Numerical experiments

5.1. Interpolation. In this section we will carry out numerical experiments to compare the Lebesgue constants of the following node distributions:

1. Chebyshev-Gauss-Lobatto points
(5.1)
\[ c^L_i = -\cos \left( \frac{i\pi}{s} \right), \quad i = 0, \ldots, s. \]

2. Scaled Chebyshev points
(5.2)
\[ c^{SC}_i = -\frac{\cos \left( \frac{2i+1}{2s+2} \pi \right)}{\cos \left( \frac{1}{2s+2} \pi \right)}, \quad i = 0, \ldots, s. \]

3. Equidistant nodes
(5.3)
\[ c_i = -1 + \frac{2i}{s}, \quad i = 0, \ldots, s. \]

The Lebesgue constant \( \Lambda_c \) can be computed by the formula (see, e.g., [2])
(5.4)
\[ \Lambda_c = \max_{x \in [-1, 1]} F_c(x), \quad F_c(x) := \sum_{k=0}^{s} |F_k(x)|, \]
where
(5.5)
\[ F_k(x) = \prod_{j=0}^{s} (x - c_j) / \prod_{j=0, j \neq k}^{s} (c_k - c_j), \quad k = 0, \ldots, s. \]

In all experiments, we denote by CGL the numerical solutions obtained by using Chebyshev-Gauss-Lobatto node distribution.

Figure plots the function \( F_c(x) \) based on equidistant nodes, Chebyshev-Gauss-Lobatto nodes, and the scaled Chebyshev nodes studied in this paper. From Figure we can see that the scaled Chebyshev node distribution yields a function \( F_c(x) \) with minimal sup-norm among the three node distributions.
Table 2 presents Lebesgue constants for the three node distributions for various $n$ where $n = s + 1$, the number of nodes in each distribution. From Table 2, one concludes that the scaled Chebyshev nodes yield the smallest Lebesgue constants among the three node distributions. One can also see that the Lebesgue constant $\Lambda_c(n)$ of equidistant node distribution increases very fast when $n$ increases.

**Table 2.** Lebesgue constants $\Lambda_c(n)$.

| Node distribution          | $n = 6$ | $n = 8$ | $n = 10$ | $n = 12$ | $n = 14$ | $n = 16$ | $n = 18$ |
|---------------------------|---------|---------|----------|----------|----------|----------|----------|
| Equi-spaced               | 3.1     | 6.9     | 17.8     | 51.2     | 158.1    | 512.3    | 1716.4   |
| Chebyshev-Gauss-Lobatto   | 2.0     | 2.2     | 2.4      | 2.5      | 2.6      | 2.7      | 2.8      |
| Scaled Chebyshev          | 1.7     | 1.9     | 2.0      | 2.1      | 2.2      | 2.3      | 2.4      |

Figure 2 plots absolute values of errors of Lagrange interpolation $|f(x) - L_c(f)(x)|$ for equidistant nodes, Chebyshev-Gauss-Lobatto nodes, and the scaled Chebyshev nodes when $f(x) = e^x$ (left) and $f(x) = \cos x$ (right). From Figure 2, one concludes that the scaled Chebyshev node distribution is the best among the three node distributions in this experiment.

Figure 3 plots the absolute values of errors for interpolating the function $f(x) = x^{41}$ for the scaled Chebyshev nodes and CGL nodes when $s = 40$. From the figure, we can see that the scaled Chebyshev nodes yield a better result than do CGL nodes. In particular, the CGL node distribution yields better results at nodes close to the boundary and worse results at nodes close to the center than does the scaled Chebyshev node distribution.

Figure 4 presents the absolute values of errors for interpolating the function $f(x) = \cos(4\pi x)^{20}$ for the scaled Chebyshev nodes and CGL nodes when $s = 200$. From Figure 4, we can see that the scaled Chebyshev nodes yield better results than do CGL nodes at nodes close to the center. However, at nodes close to the boundary, the CGL node distribution yields better results than does the scaled
Chebyshev node distribution. Overall, the scaled Chebyshev nodes yield better results in sup-norm on the interval [-1,1].

5.2. Numerical differentiation. The Lagrange interpolation polynomial $L_c(f)$ of $f$ over the nodes $(c_i)_{i=0}^s$ is given by

$$L_c(f)(x) = \sum_{i=0}^s f(c_i)\ell_{c,i}(x), \quad \ell_{c,i}(x) = \frac{\prod_{j=0}^s (x - c_j)}{\prod_{j=0, j \neq i}^s (c_i - c_j)}.$$

Figure 2. Plots of errors for interpolating $f(x) = e^x$ (left) and $f(x) = \cos x$ (right).

Figure 3. Plots of errors for interpolating $f(x) = x^{41}$. 

(5.6)
Figure 4. Plots of errors for interpolating $f(x) = \cos(4\pi x)^2$.

Therefore,

\[(L_c(f))'(x) = \sum_{i=0}^{s} f(c_i) \ell'_{c,i}(x).\]

This implies

\[(L_c(f))'(c_k) = \sum_{i=0}^{s} f(c_i) \ell'_{c,i}(c_k), \quad k = 0, \ldots, s.\]

These equations can be rewritten as

\[
\begin{pmatrix}
(L_c(f))'(c_0) \\
(L_c(f))'(c_1) \\
\vdots \\
(L_c(f))'(c_s)
\end{pmatrix}
= \begin{pmatrix}
d_{00} & d_{01} & \cdots & d_{0s} \\
d_{10} & d_{11} & \cdots & d_{1s} \\
\vdots & \vdots & \ddots & \vdots \\
d_{s0} & d_{s1} & \cdots & d_{ss}
\end{pmatrix}
\begin{pmatrix}
f(c_0) \\
f(c_1) \\
\vdots \\
f(c_s)
\end{pmatrix},
\quad d_{ij} := \ell'_{c,j}(c_i).
\]

The matrix $\begin{pmatrix} d_{ij} \end{pmatrix}_{i,j=0}^{s}$ is called a differentiation matrix. The derivatives $f'(c_i)$, $i = 0, \ldots, s$, are approximated by $(L_c(f))'(c_i)$ which are computed by (5.9).

Let us derive formulas for computing the differentiation matrix $D = \begin{pmatrix} d_{ij} \end{pmatrix}_{i,j=0}^{s}$. From (5.6), one gets

\[
(\ell'_{c,i}(x) = \ell_{c,i}(x) \sum_{j=0}^{s} \frac{1}{x - c_j}, \quad x \neq c_j, \quad j = 0, \ldots, s.
\]

Thus,

\[
\begin{align*}
\ell'_{c,i}(c_j) &= \prod_{k=0}^{s} (c_i - c_k) / \prod_{k=0}^{s} (c_j - c_k), \quad i \neq j, \quad i, j = 0, \ldots, s, \\
\ell'_{c,i}(c_i) &= \sum_{k=0}^{s} \frac{1}{(c_i - c_k)}, \quad i = 0, \ldots, s.
\end{align*}
\]

One can find similar formulas in [9].
When \((c_i)_{i=0}^s\) are Chebyshev-Gauss-Lobatto points, the differentiation matrix \(D = (d_{ij})_{i,j=0}^s\) is given by (see, e.g., [9])

\[
    d_{00} = \frac{2s^2 + 1}{6}, \quad d_{ss} = \frac{2s^2 + 1}{6},
\]

(5.13)

\[
    d_{jj} = \frac{-c_j}{2(1 - c_j^2)}, \quad j = 1, \ldots, s - 1,
\]

(5.14)

\[
    d_{ij} = \frac{a_i (-1)^{i+j}}{a_j (c_i - c_j)}, \quad i \neq j, \quad i, j = 0, \ldots, s,
\]

(5.15)

where

\[
    a_i = \begin{cases} 
        2, & i = 0 \text{ or } i = s, \\
        1, & \text{otherwise}. 
    \end{cases}
\]

(5.16)

Let us do some numerical experiments with the computation of the first derivative of a function \(f\) using different types of node distributions. These node distributions are Chebyshev-Gauss-Lobatto points, the scaled Chebyshev points, and the node distribution developed in Section 3. In our experiments, the node distributions from Theorem 3.4 and Theorem 3.8 are denoted by DND.

Figure 5 plots the errors \(|f'(c_i) - (L_c(f))'(c_i)|\) for the three node distributions for the function \(f(x) = e^x\) when \(s = 11\) and \(s = 12\). From Figure 5 one can see that the node distribution DND studied in this paper yields the best results in the sup-norm. The approximation for \((f'(c_i))_{i=0}^s\) with scaled Chebyshev nodes are very good when \(c_i\) is close to 0 but are not good when \(c_i\) is close to the boundary \(x = -1\) or \(x = 1\). The accuracy of numerical solutions from all node distributions in this experiment is high even with twelve nodes.

![Figure 5. Plots of \(|f'(c_i) - (L_c(f))'(c_i)|\), \(i = 0, \ldots, s\), for \(f(x) = e^x\).](image)

Figure 6 plots the errors \(|f'(c_i) - (L_c(f))'(c_i)|\) for the two node distributions CGL and DND for the function \(f(x) = e^{x^2}\) when \(s = 38\). From Figure 6 one can see that the result obtained from the node distribution DND is the best in the sup-norm.
5.3. **Numerical integration.** Let us derive formulas for computing integration matrices. From equation (5.6) one gets

$$ \int_{0}^{x} L_c(f)(t) \, dt = \sum_{i=0}^{s} f(c_i) \int_{0}^{x} \ell_{c,i}(t) \, dt. $$

**Figure 6.** Plots of $|f'(c_i) - (L_c(f))'(c_i)|$, $i = 0, \ldots, s$, for $f(x) = x^{39}$.

**Figure 7.** Plots of $|f'(c_i) - (L_c(f))'(c_i)|$, $i = 0, \ldots, s$, for $f(x) = \cos^{5}(11\pi x)$.
This implies
\[
\int_0^{c_j} L_c(f)(t) \, dt = \sum_{i=0}^{s} f(c_i) \int_0^{c_j} \ell_{c,i}(t) \, dt, \quad j = 0, \ldots, s.
\]

These equations can be written as
\[
\begin{pmatrix}
\int_0^{c_0} L_c(f)(t) \, dt \\
\int_0^{c_1} L_c(f)(t) \, dt \\
\vdots \\
\int_0^{c_s} L_c(f)(t) \, dt
\end{pmatrix}
= \begin{pmatrix}
a_{00} & a_{01} & \cdots & a_{0s} \\
a_{10} & a_{11} & \cdots & a_{1s} \\
\vdots & \vdots & \ddots & \vdots \\
a_{s0} & a_{s1} & \cdots & a_{ss}
\end{pmatrix}
\begin{pmatrix}
f(c_0) \\
f(c_1) \\
\vdots \\
f(c_s)
\end{pmatrix},
\]
where
\[
a_{ij} := \int_0^{c_i} \ell_{c,j}(t) \, dt.
\]

The matrix \( A = (a_{ij})_{i,j=0}^{s} \) is called an integration matrix. The definite integrals \( \int_0^{c_i} f(t) \, dt, \ i = 0, \ldots, s \), are approximated by \( \int_0^{c_i} L_c(f)(t) \, dt, \ i = 0, \ldots, s \), which are computed by (5.17). The elements \( (a_{ij})_{i,j=0}^{s} \) in equation (5.17) can be computed by means of quadrature formulas. In our experiments, we used the function \texttt{quad} in MATLAB to compute these coefficients.

Let us do numerical experiments to approximate the following integral given the values of \( f \) at \( (c_i)_{i=0}^{s} \):
\[
\int_{-1}^{1} f(\xi) \, d\xi, \quad -1 \leq x \leq 1.
\]

In this section, the node distribution in Theorem 4.2 is denoted by IND and the node distribution in Theorem 4.4 is denoted by IND2. Figure 8 plots the numerical results for approximating the integral (5.18) when \( f(x) = \cos(\pi x) \) and \( s = 7 \) and \( s = 9 \). It follows from Figure 8 that the node distributions IND and IND2 yield the best results. We also see from Figure 8 that the scaled Chebyshev nodes perform better than the Chebyshev-Gauss-Lobatto nodes. The node distribution DND is the worst among the four node distribution. This is a consequence of the fact that the DND node distribution is developed for approximating derivatives of functions not for approximating antiderivatives of functions. We have also carried out numerical experiments with other types of functions \( f \) and get similar results, i.e., the IND and IND2 node distributions are the best and the DND is the worst among the four node distributions for approximating integral (5.18).

We have also carried out numerical experiments to solve the following initial value problem:
\[
y'(x) = f(x, y(x)), \quad 0 \leq x \leq 1, \quad y(0) = y_0.
\]

For simplicity of calculations we have chosen \( f(x, y) = 2y + \cos x, \ y(0) = 0 \). For this particular choice, equation (5.19) becomes
\[
y' = 2y + \cos x, \quad y(0) = 0, \quad 0 \leq x \leq 1.
\]

The solution to (5.20) is
\[
y(x) = \frac{2}{5}e^{2x} + \frac{1}{5}(\sin x - 2 \cos x).
\]

The numerical results for solving equation (5.20) when \( s = 9 \) is presented in Figure 9. From Figure 9 one can see that the Chebyshev-Gauss-Lobatto node distribution yields the best result. The second best node distribution is the DND which
is developed in Section 3. As we have discussed in Remark 3.9, the Chebyshev-Gauss-Lobatto node distribution yields better results than the DND node distribution does at interior nodes. Note that in equation (5.20), the value at one end of the interval $x = 0$ is known exactly. This would suggest that the CGL node distribution is better than the DND node distribution for solving boundary value problems. The scaled Chebyshev and the IND node distributions are the two worst node distributions.

Let us look at a different approach to solve equation (5.19). Equation (5.19) can be rewritten as an integral equation as follows:

$$y(x) = y_0 + \int_0^x f(t, y(t)) \, dt, \quad 0 \leq x \leq 1.$$
Let us solve equation (5.21) with our earlier choice \( f(x, y) = 2y + \cos x, \ y(0) = 0 \). Numerical results for the four node distributions are presented in Figure 10. In Figure 10, we also include the numerical result from the previous experiment, i.e., the result obtained from solving equation (5.20) using Chebyshev-Gauss-Lobatto node distribution, for ease of comparison. This result is denoted by CGL-dif.

From Figure 10 one can see that the IND and IND2 node distributions yield the best numerical results. The scaled Chebyshev node distribution is the third best while the Chebyshev-Gauss-Lobatto node and the DND node distribution rank forth and fifth, respectively. Although the Chebyshev-Gauss-Lobatto node distribution yields the best result in Figure 9, this result which is denoted by CGL-dif in Figure 10 is much worse than the other results in the figure.

![Figure 10](image)

**Figure 10.** Plots of absolute values of errors for solving equation (5.21) when \( y(x) = \frac{2}{5} e^{2x} + \frac{1}{5} (\sin x - 2 \cos x) \).

The conclusion from Figures 9 and 10 is: Although equation (5.21) and (5.19) are analytically equivalent, equation (5.21) yields numerical solutions of higher accuracy than does equation (5.19). This could be a consequence of the fact that differentiation is an ‘ill-posed’ process while integration is a ‘well-posed’ one. Moreover, when solving integral equations, the node distributions IND and IND2 are much better than the most commonly used Chebyshev-Gauss-Lobatto node distribution as we have seen from our experiments. However, the cost for this improvement is computationally expensive. It is because computing differentiation matrices are much faster than computing integration matrices.

We have also carried out numerical experiments to solve equation (5.21) with \( f(x, y) = 2y + g(x) \) where \( g(x) \) is chosen so that the function \( y(x) = \sin^{12}(3\pi \xi) \) is the solution to equation (5.21). Beside the three nodes distributions scaled Chebyshev, CGL, and IND2, we have also used a node distribution denoted by CG which consists of critical points of the polynomial \( T_{s+2}(x) \), i.e., zeros of the polynomial \( T_{s+2}(x) \) in our experiments. The node distribution IND hasn’t been used in these experiments as we haven’t been able compute this node distribution for large \( s \). Numerical results are reported in Figure 11.
It can be seen from Figure 11 that the IND2 is the best node distribution while the scaled Chebyshev node distribution is the second best in this experiment. The errors of the result from the CG nodes oscillate much more than the errors of the results from other node distributions.

![Figure 11. Plots of absolute values of errors for solving equation (5.21) when \( y(x) = \sin^{12}(3\pi x) \).](#)

6. Concluding remarks

Several node distributions have been proposed and studied in the paper. It has been shown in our study that having smaller Lebesgue constants does not necessarily make a node distribution yield better numerical results in pseudospectral methods. The scaled Chebyshev nodes have smaller Lebesgue constants than do the most frequently used Chebyshev-Gauss-Lobatto nodes. However, for approximating derivatives of functions using collocation spectral methods, Chebyshev-Gauss-Lobatto nodes are often better than the scaled Chebyshev nodes. Chebyshev-Gauss-Lobatto node distribution often yields the best results among the node distributions studied in this paper for solving initial value problems, and possibly, for solving boundary value problems. However, for approximating functions, derivatives of functions, and antiderivatives of functions Chebyshev-Gauss-Lobatto node distribution is not always the best choice. We have constructed nodes distributions which could be better than Chebyshev-Gauss-Lobatto node distribution for each of the above problems. In particular, the scaled Chebyshev node, the DND node distribution, and the IND and IND2 node distributions could be better than Chebyshev-Gauss-Lobatto node distribution for approximating functions, derivatives of functions, and antiderivatives of functions, respectively.

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