Sampling Random Group Fair Rankings

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Abstract

In this paper, we consider the problem of randomized group fair ranking that merges given ranked list of items from different sensitive demographic groups while satisfying given lower and upper bounds on the representation of each group in the top ranks. Our randomized group fair ranking formulation works even when there is implicit bias, incomplete relevance information, or when only ordinal ranking is available instead of relevance scores or utility values.

We take an axiomatic approach and show that there is a unique distribution $D$ to sample a random group fair ranking that satisfies a natural set of consistency and fairness axioms. Moreover, $D$ satisfies representation constraints for every group at every rank, a characteristic that cannot be satisfied by any deterministic ranking. We propose three algorithms to sample a random group fair ranking from $D$. Our first algorithm samples rankings from $D$ exactly, in time exponential in the number of groups. Our second algorithm samples random group fair rankings from $D$ exactly and is faster than the first algorithm when the gap between upper and lower bounds on the representation for each group is small. Our third algorithm samples rankings from a distribution $\epsilon$-close to $D$ in total variation distance, and has expected running time polynomial in all input parameters and $1/\epsilon$, when there is a large gap between upper and lower bound representation constraints for all the groups. We experimentally validate the above guarantees of our algorithms for group fairness in top ranks and representation in every rank on real-world data sets.

1 Introduction

Machine learning models used in critical decision-making scenarios can potentially amplify biases and have an adverse social and economic impact on individuals and protected demographic groups, e.g., race, gender (Barocas et al., 2019). Classification, scoring, and ranking are typical use cases for machine learning models to automate and assist decisions. In this paper, we investigate fairness in ranking.

A utilitarian view of ranking is to order a given set of items according to their individual merit or utility. Group fairness in ranking demands fairness to different protected demographic groups while maximizing utility. From defining fair ranking metrics and their evaluation (Kulshrestha et al. [2017] Yang and Stoyanovich [2017], Diaz et al. [2020]) to designing algorithms for maximizing ranking utility while satisfying group fairness (Celis et al. [2018], Geyik et al. [2019], Singh and Joachims [2018], Wu et al. [2018], Zehlike et al. [2017]), there is a large body of recent work on group fairness in ranking.

A natural approach to obtain group fair ranking is via post-processing: first, get the unconstrained (or utility-maximizing) ranking and then re-arrange the items to satisfy group fairness constraints. Previous works have proposed deterministic post-processing algorithms such that the ranking they output satisfies group fairness measured in terms of the representation of each group in the top few ranks. For example, Zehlike et al. (2017)
consider group fairness constraints that require each prefix of the ranking to have at least $p^*$ proportion of ranks assigned to the protected or underprivileged group, where $p^*$ is its true proportion in the total population. Celis et al. (2018b) generalize this to allow group fairness constraints consisting of both lower and upper bounds on the number of items from each group in each prefix of the ranking. Both these works give algorithms to output a deterministic ranking that maximizes the ranking utility subject to their respective group fairness constraints.

Post-processing algorithms for group fair ranking can be viewed as performing two steps: first, selecting a sufficient number of items from each group, and second, ranking them together so that utility is maximized while satisfying group fairness constraints. Both these steps require accurate and unbiased observation of the merit of all items. Only then can one select the items with the highest observed merit from each group and finally merge their group-wise ranked lists to create a combined ranking while satisfying group fairness constraints. However, accurate and unbiased observations of merit are difficult in the real world and could contain implicit bias towards social groups. In the presence of implicit bias, Celis et al. (2020b) show that maximizing the observed utility of ranking based on the observed merit can be suboptimal for the true or latent utility. Celis et al. (2020b) justify group fair ranking by proving that when the latent scores of all the candidates are drawn i.i.d. from a uniform distribution, the ranking of maximum latent utility is retrieved by maximizing the observed ranking utility subject to bias-independent lower bound constraints on the representation of the protected group in every prefix of the ranking. The above assumption about latent scores or merit being i.i.d. may not always hold in real-world applications. We proceed with a weaker assumption that only an ordinal ranking of items within each group is available, and we do not have any scores or utility values to compare items (especially items across different groups). Our assumption circumvents implicit bias and allows us to consider group fair ranking even under incomplete or biased data about pairwise comparisons.

Any deterministic ranking algorithm is restricted to assign each rank to only one item, hence, only one group. This can lead to a loss of opportunities for other groups. For example, consider multiple companies with each having a few similar open positions and suppose they use the same recruitment system to rank a common candidate pool for job interviews. If the ranking is deterministic and the number of groups is more than the number of candidates interviewed by each company, representation constraints for group fairness cannot be satisfied. Moreover, if the candidates from certain protected groups are sufficiently represented but systematically left out of the top ranks, they have fewer opportunities to be interviewed and hired even in a deterministic and group fair ranked list of candidates.

Randomized ranking is a way to create opportunities for items in a way that a deterministic ranking cannot. Deterministic fair ranking has an inevitable trade-off with underranking of individuals in the original ranking without fairness constraints (Gorantla et al., 2021). Recent work by Singh et al. (2021) studies randomized rankings under uncertainty in the merit scores of items, where the observed features give rise to posterior merit distributions. Given these merit distributions, Singh et al. (2021) give a randomized ranking to optimally trade-off a notion of approximate individual fairness to the true merit of each item against the overall ranking utility.

On a different note from Singh et al. (2021) but intending to combine the benefits of randomized ranking and group fairness, we take an axiomatic approach to define randomized group fair rankings and design algorithms to sample random group fair rankings.

1.1 Our contributions

We assume that we are given a total order of the items belonging to the same group. For simplicity, we ignore any additional information, such as any comparison of items from different groups or any estimates of merit for an individual item. In order to sample a random group fair ranking that can work even in the presence of biased or missing comparisons across groups, we assume that the order of items belonging to the same group is given to
us and is reliable, whereas we do not use any information about the comparison of items from different groups as that may be unreliable.

Our main theoretical contributions are as follows: Our first contribution is a mathematical formulation of the above assumption using a set of consistency and fairness axioms. Our second contribution is to prove that there is exactly one distribution over the set of all feasible rankings that satisfies all our axioms of consistency and fairness (details in Section 3). Our third contribution is to give efficient algorithms to sample a random group fair ranking from the above unique distribution (details in Section 4).

We show that a random group fair ranking can be sampled in three steps: First, we sample the group-wise representation, that is, the number of items from each group to be present in the top \( k \) ranks. The second step is to use a random permutation of \( k \) to take the above representation and assign group memberships to each position in the top \( k \) ranks. The first two steps ensure group fairness. Finally, the third step assigns individual items from each group to the positions allotted to their group in a way that is consistent with their given ordering within the group. Following the above recipe, we give an algorithm to sample a random group fair ranking efficiently, mainly solving the first step mentioned above. We take the rational convex polytope formed by the group fairness constraints. We expand this polytope by a suitable factor, sample a rational point from this expanded polytope from a distribution close to uniform, using an efficient oracle, and then deterministically round it to an integral point in the polytope. We show that our algorithm samples integral points from a distribution close to uniform, from the polytope of group fairness constraints. This algorithm works for any configuration of the parameters, that is, given any lower and upper bounds on the representation for each group. When the difference between these bounds is sufficiently large, our algorithm has a faster expected running time. We show that our algorithm is practical and efficient on real-world datasets, even when the above condition does not hold. When the difference between the upper and lower bounds is very small, we give an alternate efficient but brute-force algorithm. We also give an exact sampling algorithm that runs in time exponential in the number of groups.

We validate our theoretical results on the real-world datasets using our random walk-based algorithm (details in Section 5). Our experimental results show that the algorithm achieves group fairness in the top \( k \) ranks while maintaining sufficient representation from each of the groups in each rank. Due to space constraints, we moved all the paper’s proofs to the appendices.

2 Related work

Algorithmic fairness has been an important area of study in the past decade. In fair classification literature, individual fairness is defined as similar outcomes for similar individuals (Dwork et al., 2012), whereas group fairness is defined as equal outcomes (e.g., demographic parity, equalized odds) for different demographic groups (Barocas et al., 2019). Combining individual fairness and group fairness, Castillo (2019) lays down the principles of fair ranking as treating similar items consistently, sufficient presence of items from socially salient groups, and proportional representation from every group. Even in fair ranking literature, the dichotomy between individual fairness and group fairness exists, and both have been studied along with their trade-off (Singh and Joachims, 2019; Bower et al., 2021; Singh et al., 2021; García-Soriano and Bonchi, 2020). Our work is focused on randomized post-processing algorithms for group fairness in ranking. However, we discuss related work from the broader fair ranking literature below to put it in perspective.

**Group fairness in ranking.** Yang and Stoyanovich (2017) propose a group fairness metric for ranking based on statistical parity; they take the difference between the proportion of a protected group in each prefix of the ranking with its true proportion in the overall population, with a logarithmic discount and normalization similar to
nDCG (Normalized Discounted Cumulative Gain) to emphasize fairness in the top ranks more than fairness in the bottom ranks. They propose a multi-objective optimization objective to maximize ranking utility while satisfying group fairness constraints based on the above metric. Geyik et al. (2019) propose a different group fairness metric based on the skew of group-wise representation in the top ranks and show its relation to other group fairness constraints. They give a deterministic post-processing or re-ranking algorithm with provable guarantee of proportional representation for at most 3 demographic groups. Subsequently, deterministic post-processing algorithms have been proposed to guarantee sufficient or proportional group-wise representation in each prefix of the top $k$ ranks (Celis et al., 2018b; Zehlike et al., 2017) or in blocks of $k$ consecutive ranks (Gorantla et al., 2021).

Other definitions of fair ranking. Beutel et al. (2019) and Narasimhan et al. (2020) propose separate metrics for intra-group and inter-group fairness. Intra-group fairness requires that two items from the same group must be treated consistently, whereas inter-group fairness requires that items from the protected groups should not be treated differently when compared with items from the favored/majority group. Kuhlman et al. (2019) propose group fairness metrics called rank equality, rank calibration, and rank parity based on pairwise comparison errors, and propose bias mitigation techniques for fairness with respect to these metrics. When different demographic groups have incomparable qualities, Kearns et al. (2017) propose a notion of meritocratic fairness for cross-population selection that requires each item to be treated according to its relative performance in its group.

Randomized fair rankings. Recent work by Diaz et al. (2020) proposes a randomized ranking to ensure equal expected exposure across all groups. Similar to our work, they observe that a deterministic ranking cannot distribute exposure fairly amongst relevant items. They propose stochastic versions of post-processing algorithms to achieve equality of expected exposure. Singh and Joachims (2018) give a randomized ranking that maximizes the ranking utility subject to group fairness constraints of statistical parity, disparate treatment, and disparate impact restated in terms of exposure. Biega et al. (2018) propose that individuals should receive equal amortized attention/exposure when ranked repeatedly and propose re-ranking algorithms to achieve this. The above results on randomized fair ranking focus on group fairness metrics derived for equality of exposure. Gao and Shah (2020) proposed a randomized ranking algorithm called fair $\epsilon$-greedy that is neither optimized to be close to the unconstrained ranking nor uses the biased relevance scores. It still empirically achieves group fairness without losing a lot in the ranking utility. The group fairness notions considered are statistical parity fairness – a ranking is considered to be fair if the representation of documents from different groups is equal; these constraints are similar to our constraints. Moreover, similar to our setting, the algorithm does not compare two items from different groups, hence avoiding adverse effects of bias in the merit scores of the protected groups, but entirely relies on the comparison of items from the same group given by the merit scores. More details in Section 5.

Previous work has also used randomization in ranking, recommendations, and summarization of ranked results to demonstrate its other benefits such as controlling polarization (Celis et al., 2019), mitigating data bias (Celis et al., 2020a), and promoting diversity (Celis et al., 2018a).

3 Group fairness in ranking

Given a set $N$ of $n$ items, a top-$k$ ranking is a selection of $k < n$ items followed by assignment of each rank in $[k]$ to exactly one of the selected items. Let $a, a' \in N$ be two different items such that the item $a$ is assigned to rank $i$ and item $a'$ is assigned to rank $i'$. Whenever, $i < i'$ we say that item $a$ is ranked lower than item $a'$. Throughout
the paper, we refer to a top-\(k\) ranking by just ranking. We use index \(i\) to refer to a rank, index \(j\) to refer to a group, and \(a\) to refer to elements in the set \(N\). The set \(N\) can be partitioned into \(\ell\) disjoint groups of items depending on their characteristic features. A group fair ranking is any ranking that satisfies a set of group fairness constraints. Whenever the constraints can be represented as lower and upper bounds, \(L_j, U_j \in [k]\) respectively, on the number of top \(k\) ranks assigned to group \(j\), for each group \(j \in [\ell]\), we call them the representation constraints. Throughout the paper, we assume that we are given a ranking of the items within the same group for all groups. We call these rankings in-group rankings. The goal of the group fair ranking algorithm is to output a ranking, given all the in-group rankings, such that the representation constraints are satisfied. This paper takes an axiomatic approach to characterize a random group fair ranking.

### 3.1 Random group fair ranking

The three axioms we state below are natural consistency and fairness requirements for a distribution over all the rankings.

The first axiom states that for any ranking sampled from the distribution, for any two items \(a, a'\) from the same group, their order in the randomly sampled ranking should be consistent with their order in their in-group ranking. This abides with our assumption that the comparisons of the items from the same group are reliable.

**Axiom 3.1 (In-group consistency).** For any ranking sampled from the distribution, for all items \(a, a'\) belonging to the same group \(j \in [\ell]\), item \(a\) is ranked lower than item \(a'\) if and only if item \(a\) is ranked lower than item \(a'\) in the in-group ranking of group \(j\).

The support of any distribution that satisfies Axiom 3.1 consists only of rankings with items within the same group ranked in the order of their in-group ranking. In all these rankings, once we fix which group to be assigned to a rank \(i\), there exists exactly one item that can be ranked at rank \(i\). Hence, for the next axioms we look at the group assignments instead of rankings. A group assignment is an assignment of each rank in the top \(k\) ranking to exactly one of the \(\ell\) groups. Let \(Y_i\) be a random variable representing the group \(i\)th rank is assigned to. Therefore \(Y = (Y_1, Y_2, \ldots, Y_k)\) is a random vector representing a group assignment. Let \(y = (y_1, y_2, \ldots, y_k)\) represent an instance of a group assignment. A group fair assignment is a group assignment that satisfies the representation constraints. Therefore the set of group fair assignments is,

\[
\left\{ y \in [\ell]^k : \sum_{i \in [k]} I[y_i = j] \leq U_j, \forall j \in [\ell] \right\},
\]

where \(I[\cdot]\) is an indicator function. The ranking can then be obtained by assigning the items within the same group according to their in-group ranking, to the ranks assigned to the group. We use \(Y_0\) to represent a dummy group assignment of length 0 for notational convenience when no group assignment is made to any group (e.g. in Axiom 3.3).

Let \(X_j\) be a random variable representing the number of ranks assigned to group \(j\) in a group assignment, for all \(j \in [\ell]\). Therefore \(X = (X_1, X_2, \ldots, X_{\ell})\) represents a random vector for a group representation. Let \(x = (x_1, x_2, \ldots, x_{\ell})\) represent an instance of a group representation. Then the set of group fair representations is,

\[
\left\{ x \in \mathbb{Z}_{\geq 0}^{\ell} : \sum_{j \in [\ell]} x_j = k \text{ and } L_j \leq x_j \leq U_j, \forall j \in [\ell] \right\}.
\]
Notice that if we have a distribution over the group assignments, this also gives us a distribution over the group representations such that,

\[ \Pr[X = x] = \sum_{y : \sum_i I[y_i = j] = x_j, \forall j \in [\ell]} \Pr[Y = y]. \]

Recall that ranking is a two-step process where we first select a set of \( k \) items and then rank them in the order preferred. The following axiom asks for fairness in the selection step. It states that each group fair representation should be equally likely.

**Axiom 3.2 (Representation Fairness).** All the non-group fair representations should be sampled with probability zero, and all the group fair representations should be sampled uniformly at random.

The third axiom asks for fairness in the second step of ranking – assigning the selected set of items to the top \( k \) ranks.

**Axiom 3.3 (Ranking Fairness).** For any two groups \( j, j' \in [\ell] \), for all \( i \in \{0, \ldots, k - 2\} \), conditioned on the top \( i \) ranks and a group representation \( x \), the \( (i + 1) \)-th and the \( (i + 2) \)-th ranks can be assigned to the groups \( j \) and \( j' \) interchangeably with equal probability. That is,

\[ \Pr[Y_{i+1} = j, Y_{i+2} = j' \mid Y_0, Y_1, \ldots, Y_i, X] = \Pr[Y_{i+1} = j', Y_{i+2} = j \mid Y_0, Y_1, \ldots, Y_i, X], \]

\( \forall j, j' \in [\ell], \forall i \in \{0, \ldots, k - 2\} \).

Let \( U \) represent a uniform distribution. In the result below, we show that there exists a unique distribution over the rankings that satisfies all three axioms.

**Theorem 3.4.** Let \( D \) be a distribution from which a ranking is sampled as follows,

1. Sample a group representation \( x \) as, \( X \sim U \left\{ x \in \mathbb{Z}_{\geq 0}^\ell : \sum_{j \in [\ell]} x_j = k \text{ and } L_j \leq x_j \leq U_j, \forall j \in [\ell] \right\} \).
2. Sample a group assignment \( y \), given \( x \), as, \( Y \mid X \sim U \left\{ y \in [\ell]^k : \sum_{i \in k} I[y_i = j] = x_j, \forall j \in [\ell] \right\} \).
3. Rank the items within the same group in the order consistent with their in-group ranking, in the ranks assigned to the groups in the group assignment \( y \).

Then \( D \) is the unique distribution that satisfies all three axioms.

**Proof.** Recall that \( x = (x_1, x_2, \ldots, x_k) \) is defined as group representation where \( x_j \) is the number of ranks assigned to group \( j \) for all \( j \in [\ell] \), and \( y = (y_1, y_2, \ldots, y_k) \) is defined as group assignment where \( y_i \) is the group assigned to rank \( i \) for all \( i \in [k] \).

For Axiom 3.1 to be satisfied, the distribution should consist only of rankings where the items from the same group are ranked in the order of their merit. Clearly \( D \) satisfies Axiom 3.1.

To satisfy Axiom 3.2 all the group fair representations need to be sampled uniformly at random, and all the non-group fair rankings need to be sampled with probability zero. Hence, \( D \) also satisfies Axiom 3.2.

We now use strong induction on the prefix length \( i \) to show that any distribution over group assignments that satisfies Axiom 3.3 has to sample each group assignment \( y_i \), conditioned on a group representation \( x \), with equal probability. We note that whenever we say common prefix, we refer to the longest common common prefix.
Therefore we can conclude that for a fixed $x$, any two group assignments with group representation $x$ have to be sampled with equal probability. Therefore we can conclude that for a fixed $x$, any two group assignments with the same prefix of length $k - 2$ have to be sampled with equal probability. We note here that there do not exist two or more group assignments with group representation $x$ and common prefix of length exactly $k - 1$.

**Induction hypothesis.** Any two rankings with a common prefix of length $i$, for some $0 \leq i \leq k - 2$, have to be sampled with equal probability.

**Base case ($i = k - 2$).** Let $y$ and $y'$ represent a pair of group assignments with fixed group representation $x$ and common prefix till ranks $k - 2$. Then there exist exactly two groups $j, j' \in [\ell]$ such that

$$y_{k-1} = y'_{k} = j \quad \text{and} \quad y_{k} = y'_{k-1} = j'.$$

Therefore, to satisfy Axiom $3.3$ these two group assignments $y$ and $y'$ need to be sampled with equal probability. Therefore we can conclude that for a fixed $x$, any two group assignments with the same prefix of length $k - 2$ have to be sampled with equal probability. We note here that there do not exist two or more group assignments with group representation $x$ and common prefix of length exactly $k - 1$.

**Induction step.** Assume that for some $i < k - 2$, any two group assignments with group representation $x$ and common prefix of length $i' \in \{i + 1, i + 2, \ldots, k - 2\}$ are equally likely. Then we want to show that any two group assignments with group representation $x$ and common prefix of length $i$ are also equally likely. Let $y^{(s)}$ and $y^{(t)}$ be two different group assignments with group representation $x$ and common prefix of length $i$. Let $w = (w_1, w_2, \ldots, w_i)$ represent this common prefix of length $i$, that is,

$$w_1 := y^{(s)}_1 = y^{(t)}_1, w_2 := y^{(s)}_2 = y^{(t)}_2, \ldots, w_i := y^{(s)}_i = y^{(t)}_i.$$

Observe that if $x'_j$ represents the number of ranks assigned to group $j$ in ranks $(i + 1, i + 2, \ldots, k)$ in $y^{(s)}$, then the number of ranks assigned to group $j$ in ranks $(i + 1, i + 2, \ldots, k)$ in $y^{(t)}$ is also $x'_j$ for all $j \in [\ell]$, since $y^{(s)}$ and $y^{(t)}$ have common prefix of length $i$, and both have group representation $x$.

Since $w$ is of length exactly $i$ we also have that $y^{(s)}_{i+1} \neq y^{(t)}_{i+1}$. But the observation above give us that the group assigned to rank $i + 1$ in $y^{(t)}$ appears in one of the ranks between $i + 2$ and $k$ in $y^{(s)}$. Let $P$ be the set of all permutations of the elements in the multi-set

$$\left\{ y^{(s)}_{i+2}, y^{(s)}_{i+3}, \ldots, y^{(s)}_{k} \right\} \setminus \left\{ y^{(t)}_{i+1} \right\},$$

that is, we remove one occurrence of the group assigned to rank $i + 1$ in the group assignment $y^{(t)}$ from the multi-set $\left\{ y^{(s)}_{i+2}, y^{(s)}_{i+3}, \ldots, y^{(s)}_{k} \right\}$. We then have that each element of $P$ is a tuple of length $k - i - 2$. We now construct two sets of group assignments $M^{(s)}$ and $M^{(t)}$ as follows,

$$M^{(s)} := \left\{ \begin{pmatrix} w_1, w_2, \ldots, w_i, y^{(s)}_{i+1}, y^{(s)}_{i+2}, \ldots, y^{(s)}_{k-i-2} \end{pmatrix} : \forall \hat{w} \in P \right\},$$

$$M^{(t)} := \left\{ \begin{pmatrix} w_1, w_2, \ldots, w_i, y^{(t)}_{i+1}, y^{(t)}_{i+2}, \ldots, y^{(t)}_{k-i-2} \end{pmatrix} : \forall \hat{w} \in P \right\}.$$

For a fixed $\hat{w} \in P$ there is exactly one group assignment in $M^{(s)}$ and one group assignment in $M^{(t)}$ such that their $i + 1$st and $i + 2$nd coordinates are interchanged, and their first $i$ and last $k - i - 2$ coordinates are same. Therefore, $|M^{(s)}| = |M^{(t)}|$. 

7
We also have from the induction hypothesis that all the group assignments in \( M^{(s)} \) are equally likely since they have a common prefix of length \( i + 2 \). Similarly all the group assignments in \( M^{(t)} \) are equally likely. For any group assignment in \( M^{(s)} \) let \( \delta^{(s)} \) be the probability of sampling it. Similarly, for any group assignment in \( M^{(t)} \) let \( \delta^{(t)} \) be the probability of sampling it. Then,

\[
\Pr \left[ Y_{i+1} = y^{(s)}_{i+1}, Y_{i+2} = y^{(t)}_{i+1} \mid Y_0, (Y_1, \ldots, Y_i) = w, X = x \right] = \Pr \left[ \text{sampling a group assignment from } M^{(s)} \right]
\]

\[
= \left| M^{(s)} \right| \delta^{(s)}, \quad (3)
\]

\[
\Pr \left[ Y_{i+1} = y^{(t)}_{i+1}, Y_{i+2} = y^{(s)}_{i+1} \mid Y_0, (Y_1, \ldots, Y_i) = w, X = x \right] = \Pr \left[ \text{sampling a group assignment from } M^{(t)} \right]
\]

\[
= \left| M^{(t)} \right| \delta^{(t)}. \quad (4)
\]

Fix two group assignments \( y^{(s')} \in M^{(s)} \) and \( y^{(t')} \in M^{(t)} \). By the induction hypothesis \( y^{(s)} \) and \( y^{(s')} \) are equally likely since they have a common prefix of length \( i + 1 \). Similarly \( y^{(t)} \) and \( y^{(t')} \) are also equally likely. Therefore, for \( y^{(s)} \) and \( y^{(t)} \) to be equally likely we need \( y^{(s')} \) and \( y^{(t')} \) to be equally likely.

Comparing \( y^{(s')} \) and \( y^{(t')} \) instead of \( y^{(s)} \) and \( y^{(t)} \). We know from above that \( y^{(s')} \) and \( y^{(t')} \) are sampled with probability \( \delta^{(s)} \) and \( \delta^{(t)} \) respectively. Therefore for any distribution satisfying Axiom 3.3 we have,

\[
\Pr \left[ Y_{i+1} = y^{(s)}_{i+1}, Y_{i+2} = y^{(t)}_{i+1} \mid Y_0, (Y_1, \ldots, Y_i) = w, X = x \right]
\]

\[
= \Pr \left[ Y_{i+1} = y^{(t)}_{i+1}, Y_{i+2} = y^{(s)}_{i+1} \mid Y_0, (Y_1, \ldots, Y_i) = w, X = x \right]
\]

\[
\implies \left| M^{(s)} \right| \delta^{(s)} = \left| M^{(t)} \right| \delta^{(t)} \quad \text{from Equations (3) and (4)}
\]

\[
\implies \delta^{(s)} = \delta^{(t)}.
\]

Note that the converse is also easy to show, which means that Axiom 3.3 is satisfied if and only if \( y^{(s')} \) and \( y^{(t')} \) are equally likely. Therefore, Axiom 3.3 is satisfied if and only if \( y^{(s)} \) and \( y^{(t)} \) are equally likely.

For a fixed group representation \( x \), for any two group assignments with corresponding group representation \( x \), there exists an \( i \in \{0, 1, \ldots, k - 2\} \) such that they have a common prefix of length \( i \). Therefore, any two group assignments, for a fixed group representation \( x \), have to be equally likely. Therefore \( D \) is the unique distribution that satisfies all three axioms.

We also have the following additional characteristic of the distribution in Theorem 3.4. It guarantees that every rank in a randomly sampled group assignment is assigned to group \( j \) with probability at least \( \frac{L_j}{k} \) and at most \( \frac{U_j}{k} \). Hence, every rank gets a sufficient representation of each group. Note that no deterministic group fair ranking can achieve this.

**Theorem 3.5.** For any group assignment \( Y \) sampled from a distribution \( D \) that satisfies all three axioms, for every group \( j \in [\ell] \) and for every rank \( i \in [k] \), the following always holds,

\[
\frac{L_j}{k} \leq \Pr_{D} \left[ Y_i = j \right] \leq \frac{U_j}{k}.
\]
Proof of Theorem 3.5  Fix a group \( j \in [\ell] \) and a rank \( i \in [k] \). Let \( \mathcal{X} \) be the set of all group fair representations for given constraints, \( L_j, U_j, \forall j \in [\ell] \). Then,

\[
\Pr [Y_i = j] = \sum_{x \in \mathcal{X}} \Pr [X = x] \Pr [Y_i = j | X]
\]

by the law of total probability

\[
= \sum_{x \in \mathcal{X}} \frac{1}{|\mathcal{X}|} \Pr [Y_i = j | X]
\]

due to Axiom 3.2

\[
= \frac{1}{|\mathcal{X}|} \sum_{x \in \mathcal{X}} \Pr [Y_i = j | X]
\]

due to step 2 in \( D \) in Theorem 3.4

\[
= \frac{1}{|\mathcal{X}|} \sum_{x \in \mathcal{X}} \frac{x_j}{k}
\]

Since \( L_j \leq x_j \leq U_j, \forall j \in [\ell] \), the theorem follows.

Given a group representation \( x \), one can find a uniformly random \( y \), for a given \( x \), by sampling a random binary string of length at most \( \log k! \). Since \( x \) is fixed, this gives us a uniform random sample of \( y \) conditioned on \( x \). This takes time \( O(k \log k) \). This sampling takes care of Step 2. Step 3 simply takes \( O(k) \) time, given in-group rankings of all the groups. The main difficulty is to provide an efficient algorithm to perform Step 1. Therefore, in the next section, we focus on sampling a uniform random group fair representation in Step 1.

4 Sampling a uniform random group fair representation

We first note that each group fair representation corresponds to a unique integral point in the convex rational polytope \( K \) defined below,

\[
K = \left\{ (x_1, x_2, \ldots, x_\ell) \in \mathbb{R}^\ell \mid \sum_{j \in [\ell]} x_j = k \quad \text{and} \quad L_j \leq x_j \leq U_j, \forall j \in [\ell] \right\}.
\]

Therefore, sampling a uniform random group fair representation is equivalent to sampling an integral or a lattice point uniformly at random from the convex set \( K \). This problem is known to be \( \#P \)-Hard (Valiant, 1979).

4.1 Exact uniform sampling

In Equation (5) the convex rational polytope \( K \) is described using an \( \mathcal{H} \) description defined as follows,

**Definition 4.1 (\( \mathcal{H} \)-description of a polytope).** A representation of the polytope as the set of solutions of finitely many linear inequalities.

We can also have a representation of the polytope as described by its vertices, defined as follows,

**Definition 4.2 (\( \mathcal{V} \)-description of a polytope).** The representation of the polytope by the set of its vertices.

Barvinok (2017) gave an algorithm to count exactly, the number of integral points in \( K \), as re-stated below.
Theorem 4.3 (Theorem 7.3.3 in [Barvinok 2017]). Let us fix the dimension \( \ell \). Then there exists a polynomial time algorithm that, for any given rational \( \mathcal{V} \)-polytope \( P \subset \mathbb{R}^\ell \), computes the number \( |P \cap \mathbb{Z}^\ell| \). The complexity of the algorithm in terms of the dimension \( \ell \) is \( \mathcal{O}(\ell^d) \).

We also have the algorithm by Pak [2000] gives us an exact uniform random sampler for the integral points in \( K \).

Theorem 4.4 (Theorem 1 in [Pak 2000]). Let \( P \subset \mathbb{R}^\ell \) be a rational polytope, and let \( B = P \cap \mathbb{Z}^\ell \). Assume an oracle can compute \( |B| \) for any \( P \) as above. Then there exists a polynomial-time algorithm for sampling uniformly from \( B \), which calls this oracle \( \mathcal{O}(\ell^2L^2) \) times where \( L \) is the bit complexity of the input.

Using the counting algorithm given by Theorem 4.3 as the counting oracle in Theorem 4.4 gives us our first algorithm that samples a uniform random group representation exactly.

Theorem 4.5. For given fairness parameters \( L_j, U_j \in \mathbb{Z}_{\geq 0} \) and an integer \( k > 0 \), there is an algorithm that samples an exact uniform random integral point in \( K \) and runs in time \( \mathcal{O}(\ell^d)O(\log^2 k) \).

Proof of Theorem 4.5. The proof essentially follows from the proof of Theorem 1 in [Pak 2000]. They assume access to an oracle that counts the number of integral points in any convex polytope that their algorithm constructs. We show that Barvinok’s algorithm can be used as an oracle for all the polytopes that are constructed in the algorithm to sample a uniform random integral point from our convex rational polytope \( K \).

The algorithm in Theorem 4.4 intersects the polytope by an axis-aligned hyperplane and recurses on one of the smaller polytopes (to be specified below). In the deepest level of recursion where the polytope in that level contains only one integral point, the algorithm terminates the halving process and outputs that point. The proof of their theorem shows that this gives us a uniform random integral point from the polytope we started with.

Let us consider the dimension 1 w.l.o.g. The algorithm finds a value \( c \) such that \( L_1 < c < U_1, |H_+ \cap B|/|B| \leq 1/2, |H_- \cap B|/|B| \leq 1/2 \), where \( H_+ \) and \( H_- \) are two halves of the space separated by the hyperplane \( H \) defined by \( x_1 = c \). That is, \( H_+ \) is the halfspace \( x_1 \geq c \) and \( H_- \) is the halfspace \( x_1 \leq c \). Therefore, there are three possible polytopes for the algorithm to recurse on, \( H_+ \cap B, H_- \cap B, \) and \( H \cap B \). Here \( |H \cap B|/|B| \) can be \( \geq 1/2 \). Let

\[
\begin{align*}
&f_+ = |H_+ \cap B|/|B|, \quad f_- = |H_- \cap B|/|B|, \quad \text{and} \quad f = |H \cap B|/|B|.
\end{align*}
\]

Then the algorithm recurses on the polytope \( H_+ \cap B \) with probability \( f_+ \), on \( H_- \cap B \) with probability \( f_- \), and on \( H \cap B \) with probability \( f \).

Observe that \( K \) is also defined by the axis aligned hyperplanes, \( x_1 = L_1 \) and \( x_1 = U_1 \), amongst others. Therefore, \( x_1 \geq L_1 \) will become a redundant constraint if the algorithm recurses on \( H_+ \cap B \). Else \( x_1 \leq U_1 \) will become a redundant constraint if the algorithm recurses on \( H_- \cap B \). In both these cases, the number of integral points reduces by more than 1/2. If the algorithm recurses on \( H \cap B \), it fixes the value of \( x_1 \) to \( c \), and the dimension of the problem reduces by 1. Since the number of integral points is \( \exp(dL) \), the number of halving steps performed by the algorithm is at most \( O(dL) \).

Observe that in all levels of recursion, the polytopes constructed are of \( d \) dimensions (\( 1 \leq d \leq \ell \)) and are of the following form,

\[
\left\{ (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d \mid \sum_{j \in [d]} x_j = k' \quad \text{and} \quad c'_j \leq x_j \leq c''_j, \forall j \in [\ell] \right\},
\]

where \( k', c', c'' \) are some constants. This gives us the \( \mathcal{H} \)-description of each of the polytopes the algorithm constructs in each level of recursion. The vertices of such a polytope are formed by the intersection of \( d \) hyperplanes.
Therefore, there could be at most $\binom{2^{d+2}}{d} = 2^{O(d)}$ number of vertices for such a polytope in $d$ dimensions, which gives us that the $\mathcal{V}$-description can be computed from the $\mathcal{H}$-description in time $2^{O(d)}$. Therefore, for all these intermediate polytopes, we can use the counting algorithm given by Theorem 4.3 whose run time depends on $d^{O(d)}$. Using $d \leq \ell$, we get that the counting algorithm given by Theorem 4.3 takes time $\ell^{O(\ell)}$ for all the polytopes constructed by the algorithm. Further, in each step of recursion, the algorithm makes at most $O(dL)$ calls to this counting algorithm.

Since the input to the algorithm consists of a number $k > 0$ and fairness parameters $0 \leq L_j \leq U_j \leq k, \forall j \in [\ell]$, where all these parameters are integers, we have that the bit complexity of the input is $L = O(\ell \log k)$.

Therefore, the total running time of our algorithm is $O(d^2L^2)\ell^{O(\ell)} = O(\ell^4\log^2 k)\ell^{O(\ell)} = O(\log^2 k)\ell^{O(\ell)}$.

\section{Faster exact uniform sampling for small gap instances}

Our second algorithm is a brute-force algorithm that also samples a group fair representation uniformly at random; however, its running time is much faster than our first algorithm when the gap between the lower and upper bounds for most groups is small. This brute-force algorithm first enumerates the set of all integral points in $K$. Then it samples a point from this set with probability equal to the inverse of the size of this set. This gives us the following result.

**Theorem 4.6.** For any given representation constraints, $L_j, U_j \in \mathbb{Z}_{\geq 0}$, for all $j \in [\ell]$, for each group $j \in [\ell]$, the number of feasible group representations $x$ defined in (2) is at most $O\left(\prod_{j \in [\ell]}(U_j - L_j + 1)\right)$.

**Proof of Theorem 4.6.** Note that $x_j$ for group $j \in [\ell]$ can take integer values between $L_j$ and $U_j$. Therefore, there are at most $U_j - L_j + 1$ integral values possible for $x_j$. Therefore the total number of feasible integral values of $x = (x_1, x_2, \ldots, x_\ell)$ is at most $\prod_{j \in [\ell]}(U_j - L_j + 1)$. The brute-force algorithm checks all these values and adds to set $\mathcal{X}$ the feasible points. Hence the statement of the theorem follows.

Therefore uniformly randomly sampling from the set of all group fair representations can be performed in time $O\left(\log \left(\prod_{j \in [\ell]}(U_j - L_j + 1)\right)\right)$.

When there are a constant number of groups with a larger gap between the upper and the lower bounds and the rest of the groups with equal upper and lower bounds, we get the following result as a corollary to Theorem 4.6

**Corollary 4.7.** Given representation constraints, $L_j, U_j \in \mathbb{Z}_{\geq 0}$, for each group $j \in [\ell]$, such that $L_j = U_j$ for all $j \in S \subseteq [\ell]$ and $\ell - |S| = O(1)$, the brute force algorithm runs in time $k^{O(1)}$.

\section{Approximate uniform sampling}

We now give our third algorithm (Algorithm 1) that outputs integral point from $K$, defined in Equation (5), from a density that is close to the uniform distribution over the set of integral points in $K$, with respect to the total variation distance. We have given an overview of the algorithm and the proof in Section 4.3.1.

There is a long line of work on polynomial-time algorithms to sample a point approximately uniformly from a given convex polytope or a convex body \cite{Dyer et al., 1991, Lovász and Vempala, 2006, Cousins and Vempala, 2018}. We restate and use the following theorem by Cousins and Vempala \cite{Cousins and Vempala, 2018}.

**Theorem 4.8** (Theorem 1.2 in \cite{Cousins and Vempala, 2018}). There is an algorithm that, for any $\epsilon > 0$, $p > 0$, and any convex body $C \subseteq \mathbb{R}^d$ that contains the unit ball and has $\mathbb{E}_C(||X||^2) = \mathbb{R}^2$, with probability $1 - p$, gener-
Algorithm 1 Sampling an approximately uniform random group fair representation

Input: Fairness constraints $L_j, U_j$ for all the groups $j \in [\ell]$, numbers $k \in \mathbb{Z}_{\geq 0}$ and $\epsilon$

1. $H := \left\{ (x_1, x_2, \ldots, x_\ell) \in \mathbb{R}^\ell \mid \sum_{j \in [\ell]} x_j = k \right\}$.
2. $P := \left\{ (x_1, x_2, \ldots, x_\ell) \in \mathbb{R}^\ell \mid L_j \leq x_j \leq U_j, \forall j \in [\ell] \right\}$. // Note that $K = H \cap P$
3. $\Delta := \min \left\{ k - \left( \sum_{j \in [\ell]} k_i \right), (\sum_{j \in [\ell]} U_j) - k, \min_{j \in [\ell]} \frac{U_j - L_j}{2} \right\}$.

/* Find an $x^* \in H \cap \mathbb{Z}^\ell_{\geq 0}$ s.t. $B(x^*, \Delta) \subseteq P$. */

4. $x_j^* := L_j + \Delta, \forall j \in [\ell]$.
5. for $j := 1, 2, \ldots, \ell$ do
6.   if $\sum_{j' \in [\ell]} x_{j'}^* < k$ then
7.     $x_j^* := \min \left\{ k - \sum_{j' \neq j} x_j^*, U_j - \Delta \right\}$.
8. end
9. $K' := K - x^*$. // translate the polytope
10. $z := \text{SAMPLING ORACLE} \left( \left( 1 + \frac{\sqrt{\ell}}{\Delta} \right) K', \epsilon \right)$. // it samples $z$ from a distribution $\epsilon$-close to the uniform distribution in total variation distance, over $\left( 1 + \frac{\sqrt{\ell}}{\Delta} \right) K'$
11. /* Round $z$ to an integer point on $H$. */
12. Let $m = \left\lceil \sum_{j \in [\ell]} \lfloor z_j \rfloor \right\rceil$.
13. $x_j = \begin{cases} \lfloor z_j \rfloor ; & \text{if } j \in [m] \\ \lceil z_j \rceil ; & \text{otherwise} \end{cases}$.
14. if $x \in K'$ then
15.   Return $x + x^*$.
16. else
17.   Reject $x$ and go to Step [11]
18. end

ates random points from a density $\nu$ that is within total variation distance $\epsilon$ from the uniform distribution on $C$. In the membership oracle model, the complexity of each random point, including the first, is $\mathcal{O}^* \left( \max \left\{ R^2 d^2, d^3 \right\} \right)$.

Using the algorithm by Cousins and Vempala (2018) as SAMPLING ORACLE in Algorithm 1 we get an algorithm with expected running time $\text{poly}(k, \ell, 1/\epsilon)$ to sample a close to uniform random group fair representation.

Theorem 4.9. Let $L_j, U_j \in \mathbb{Z}_{\geq 0}, \forall j \in [\ell]$ be the fairness constraints and $k \in \mathbb{Z}_{\geq 0}$ be the size of the ranking. Let $\Delta := \min \left\{ k - \left( \sum_{j \in [\ell]} k_i \right), (\sum_{j \in [\ell]} U_j) - k, \min_{j \in [\ell]} \frac{U_j - L_j}{2} \right\}$. Then for any non-negative number $\epsilon < e^{-2\Delta / \sqrt{\ell}}$, Algorithm 1 samples a random point from a density that is within total variation distance $\epsilon$ from the uniform distribution on the integral points in $K$ by making $1/ \left( e^{-2\Delta / \sqrt{\ell}} - \epsilon \right)$ calls to the oracle in expectation. When $\epsilon$ is a non-negative constant such that $\epsilon < e^{-2}$, and $\Delta = \Omega \left( \ell^{1.5} \right)$, Algorithm 1 calls the oracle only a constant number of times in expected.

1The $\mathcal{O}^*$ notation suppresses error terms and logarithmic factors.
Remark. The polytope \((1 + \frac{\sqrt{2}}{N}) K'\) is an \(\ell - 1\) dimensional polytope given to us by the \(\mathcal{H}\) description in \(\ell\) dimensions. The random walk-based algorithms used as \textsc{Sampling Oracle} in Step \(\text{[11]}\) require the polytope they sample from to be full-dimensional. Below we describe a rotation operation such that the rotated polytope, that is, the polytope formed after applying the rotation on \((1 + \frac{\sqrt{2}}{N}) K'\), is full-dimensional in \(\ell - 1\) dimensions. This is a well-known transformation used as a pre-processing step to make a polytope full-dimensional. Let \(u_1, u_2, \ldots, u_\ell\) be orthonormal basis of \(\mathbb{R}^\ell\) such that \(u_\ell := (1, 1, \ldots, 1)^T\). We now construct a matrix \(R\) such that \(Ru_j = e_j, \forall j \in [\ell]\), where \(e_1, e_2, \ldots, e_\ell\) are the standard basis vectors in \(\ell\) dimensions. Fix \(j \in [\ell]\). We know that \(e_j = \sum_{j' \in [\ell]} \alpha_{j'}^{(j)} u_{j'}\), where \(\forall j' \in [\ell], \alpha_{j'}^{(j)} = e_j^T u_{j'}\). Thus, we get that \(Re_j = \sum_{j' \in [\ell]} \alpha_{j'}^{(j)} Ru_{j'} = \sum_{j' \in [\ell]} \alpha_{j'}^{(j)} e_{j'}\), which implies that the vector \((\alpha_1^{(j)}, \alpha_2^{(j)}, \ldots, \alpha_\ell^{(j)})^T\) forms the \(j\)th column of \(R\). It is also easy to verify that \(R\) is orthogonal. Therefore, the rotation matrix \(R\) can be computed efficiently. This rotation maps the hyperplane \(\sum_{j \in [\ell]} x_j = 0\) into the \(\ell-1\) dimensional space spanned by \(e_1, e_2, \ldots, e_{\ell-1}\). Therefore, the rotated polytope is an \(\ell - 1\) dimensional polytope in \(\ell - 1\) dimensions. For any point \((x_1, x_2, \ldots, x_{\ell-1}) \in \mathbb{R}^{\ell-1}\) we check the membership of \(R^{-1}(x_1, x_2, \ldots, x_{\ell-1}, 0)\) in \((1 + \frac{\sqrt{2}}{N}) K'\). This gives us the membership oracle for the rotated polytope. We can then sample a rational point from this rotated polytope in Step \(\text{[11]}\) apply \(R^{-1}\) on the point sampled to get a point in \((1 + \frac{\sqrt{2}}{N}) K'\), and proceed with our algorithm.

4.3.1 Overview of the algorithm and the proof of Theorem 4.9

Let \(H, P,\) and \(\Delta\) be as defined in Steps \(\text{[1]}\) to \(\text{[3]}\) respectively. Clearly \(K = H \cap P\). We first find an integral center in \(x^* \in H \cap P\) (Steps \(\text{[4]}\) to \(\text{[9]}\)) such that there is a ball of radius \(\Delta\) in \(P\) (see Lemma 4.10). We then translate the origin to this point \(x^*\). This ensures that the translated polytope \(K'\) (see Step \(\text{[10]}\)) contains a ball of radius \(\Delta\) with center at origin. Moreover, \(x^*\) being an integral point ensures that there exists a bijection between the set of integral points in the translated polytope \(K'\) and the original polytope \(K\) (see proof of Theorem 4.9).

Now consider the expanded polytope \((1 + \frac{\sqrt{2}}{N}) K'\). Note that for any scalar \(\alpha\), the polytope \(\alpha K'\) is constructed by adding to \(\alpha K'\) a point \(x\) whenever \(\frac{1}{\alpha} x\) belongs to \(K'\). Let \(H' = H - x^*\) and \(P' = P - x^*\). Then \(K' = K - x^* = H \cap P - x^* = (H - x^*) \cap (P - x^*) = H' \cap P'\). Let \(B(x, \Delta)\) represent an \(l_2\) ball of radius \(\Delta\) centered at \(x\).

We show in Lemma 4.11 that our deterministic rounding algorithm (in Steps \(\text{[12]}\) to \(\text{[13]}\)) is designed such that the set of points in the expanded polytope that get rounded to an integral point on \(H'\) is contained inside a cube of side length 2 around this point. In Lemma 4.12 we show this cube of side length 2 is fully contained in this expanded polytope. Lemma 4.14 gives us that for any two integral points \(x\) and \(x'\), there is a bijection between the set of points that get rounded to these points. Therefore, every integral point is sampled from distribution close to uniform, given the \textsc{Sampling Oracle} samples any rational point in the expanded polytope from a distribution close to uniform. Therefore, in Step \(\text{[11]}\) we sample a random point from a distribution close to uniform, using a \textsc{Sampling Oracle}, from the expanded polytope. We then round the point sampled to an integer point on \(H'\).

If it belongs to \(K'\) we accept, else we reject and continue. Our algorithm has expected running time polynomial in \(k, 1/\epsilon\) and exponential in \(\ell\) in expectation. However, if \(\Delta = \Omega \left(\ell^{1.5}\right)\) it has expected running time polynomial in \(k, 1/\epsilon\) and \(\ell\).

Note that in Algorithm \(\text{[1]}\) \(K' = K - x^*\) where \(x^*\) is an integral point such that \(\sum_{j \in [\ell]} x_j^* = k\). Therefore, there is a one-to-one correspondence between the integral points in \(K\) and those in \(K'\). Therefore, we consider the polytope \(K'\) in the above theorem. The value of \(R^2\) in Theorem 4.8 for the polytope \(K'\) is \(k^2\). Therefore, the algorithm by Cousins and Vempala (2018) gives a rational point from \(K'\), from a distribution close to uniform,
in time $O^* \left( k^2 \ell^2 \right)$. Therefore each oracle call in Theorem 4.9 takes time $O^* \left( k^2 \ell^2 \right)$ when we use Theorem 4.8 as the \textsc{Sampling Oracle}.

Algorithm 1 is inspired from the algorithm by Kannan and Vempala (1997) to sample integral points from a convex polytope, from a distribution close to uniform. On a high level, their algorithm on polytope $K'$ works slightly differently when compared to our algorithm on $K'$. They first expand $K'$ by $O \left( \sqrt{T \log \ell} \right)$, sample a rational point from a distribution close to uniform, over this expanded polytope (similar to Step 1), and use a probabilistic rounding method to round it to an integral point. If the integral point is in $K'$ they accept, otherwise reject and repeat from the sampling step. Their algorithm requires that a ball of radius $\Omega \left( \ell^{1.5} \sqrt{\log \ell} \right)$ lies entirely inside $K'$. We expand the polytope $K'$ by $O \left( \sqrt{\ell} \right)$, sample a rational point from this polytope from a distribution close to uniform, and then deterministically round it to an integral point. If the integral point is in $K'$ we accept, otherwise reject and repeat from the sampling step. Our algorithm only requires that a ball of radius $\Omega \left( \ell^{1.5} \right)$ lies inside $P - x^*$ with center on $H - x^*$, where $P$, $H$ and $x^*$ are as defined in Algorithm 1. As a result, we get an algorithm with expected running time $\text{poly}(k, \ell, 1/\epsilon)$ for a larger set of fairness constraints. We also note here that the analysis of the success probability of Algorithm 1 is the same as that of the algorithm by Kannan and Vempala (1997).

### 4.3.2 Proof of Theorem 4.9

**Lemma 4.10.** $B(0, \Delta) \subseteq P'$.

**Proof.** From the definition of $\Delta$ we have the following inequalities.

$$\Delta \leq \left[ \frac{k - \left( \sum_{j \in [\ell]} L_j \right)}{\ell} \right] \implies \Delta \leq \frac{k - \left( \sum_{j \in [\ell]} L_j \right)}{\ell} \implies \ell \cdot \Delta + \sum_{j \in [\ell]} L_j \leq k \implies \sum_{j \in [\ell]} (L_j + \Delta) \leq k, \tag{6}$$

$$\Delta \leq \left[ \frac{\left( \sum_{j \in [\ell]} U_j \right) - k}{\ell} \right] \implies \Delta \leq \frac{\left( \sum_{j \in [\ell]} U_j \right) - k}{\ell} \implies \ell \cdot \Delta + \sum_{j \in [\ell]} U_j \geq k \implies \sum_{j \in [\ell]} (U_j - \Delta) \geq k, \tag{7}$$

and

$$\Delta \leq \left[ \frac{U_j - L_j}{2} \right] \implies \Delta \leq \frac{U_j - L_j}{2} \implies 2 \Delta \leq U_j - L_j \implies L_j + \Delta \leq U_j - \Delta. \tag{8}$$

To show that Steps 4 to 9 find the correct center we use the following loop invariant.

**Loop invariant.** At the start of every iteration of the for loop $x^*$ is an integral point such that $L_j + \Delta \leq x^*_j \leq U_j - \Delta, \forall j \in [\ell]$.

**Initialization:** In Step 4 each $x^*_j$ is initialized to $L_j + \Delta$. From Equation (8) we know that $L_j + \Delta \leq U_j - \Delta$. Moreover, $L_j, U_j, \text{ and } \Delta$ are all integers. Therefore, $x^*$ is integral and satisfies $L_j + \Delta \leq x^*_j \leq U_j - \Delta, \forall j \in [\ell]$.

**Maintenance:** If the condition in Step 6 fails, the value of $x^*$ is not updated. Therefore the invariant is maintained. If the condition succeeds we have that,

$$\sum_{j \in [\ell]} x^*_j < k \tag{9}$$
The value $x_j^*$ is set to min $\left\{ k - \sum_{j' \in [\ell]:j' \neq j} x_{j'}^*, \ U_j - \Delta \right\}$ in Step 7. The following two cases arise based on the minimum of the two quantities.

- **Case 1:** $k - \sum_{j' \in [\ell]:j' \neq j} x_{j'}^* \leq U_j - \Delta$.

  In this case $x_j^*$ is set to $k - \sum_{j' \in [\ell]:j' \neq j} x_{j'}^* \leq U_j - \Delta$, which is an integer value since both $x_j^*$ and $k - \sum_{j' \in [\ell]:j' \neq j} x_{j'}^*$ are integers before the iteration. From (9) we have that

  \[ k - \sum_{j' \in [\ell]:j' \neq j} x_{j'} = x_j^* = k - \sum_{j' \in [\ell]} x_{j'} > x_j^*. \tag{10} \]

  Since $x_j^* \geq L_j + \Delta$ before the iteration, (10) gives us that $x_j^*$ is greater than $L_j + \Delta$ even after the update.

- **Case 2:** $k - \sum_{j' \in [\ell]:j' \neq j} x_{j'}^* > U_j - \Delta$.

  Since $U_j - \Delta \geq L_j + \Delta$ from Equation (8) and since $U_j - \Delta$ is an integer, the value of $x_j^*$ after the update is an integer such that $U_j - \Delta \geq x_j^* \geq L_j + \Delta$.

Therefore in both the cases the invariant is maintained.

**Termination:** At termination $j = \ell$. The invariant gives us that $x^*$ is an integral point such that $L_j + \Delta \leq x_j^* \leq U_j - \Delta$, $\forall j \in [\ell]$.

From Equation (6) we have that before the start of the for loop $\sum_{j \in [\ell]} x_j^* = \sum_{j \in [\ell]} L_j + \Delta \leq k$. After the termination of the for loop we have that $x_j^* = U_j - \Delta$, for all $j \in [\ell]$, when the if condition in Step 6 fails for all $j \in [\ell]$, or the if condition in Step 6 succeeds for some $j$, in which case $\sum_{j \in [\ell]} x_j^* = k$, and the value of $x^*$ does not change after this iteration. Therefore, after the for loop we get $\sum_{j \in [\ell]} x_j^* = \min \left\{ \sum_{j \in [\ell]} U_j - \Delta, k \right\}$. But Equation (7) gives us that $\sum_{j \in [\ell]} U_j - \Delta \geq k$. Therefore, the for loop finds an integral point $x^*$ such that $L_j + \Delta \leq x_j^* \leq U_j - \Delta$, $\forall j \in [\ell]$, and $\sum_{j \in [\ell]} x_j^* = k$.

There is an $l_1$ ball of radius $\Delta$ in $P$ centered at the integral point $x^* \in H$ (that is, $\sum_{j \in [\ell]} x_j^* = k$). Consequently there exists an $l_1$ ball of radius $\Delta$ centered at origin encloses an $l_2$ ball of radius $\Delta$ centered at origin we get that an $l_2$ ball of radius $\Delta$ centered at the origin, $B(0, \Delta)$, is in the polytope $P'$.

Let $C(x, \beta) \subseteq \mathbb{R}^\ell$ represent a cube of side length $\beta$ centered at $x$. For any integral point $x \in K'$ let $F_x$ represent the set of points in $\left(1 + \frac{\sqrt{\ell}}{\beta}\right) K'$ that are rounded to $x$.

**Lemma 4.11.** For any integral point $x \in K'$, $F_x \subseteq H' \cap C(x, 2)$.

**Proof.** Let $z$ be the point sampled in Step 11. Since $z \in \left(1 + \frac{\sqrt{\ell}}{\beta}\right) K'$ we have that $\sum_{j \in [\ell]} z_j = 0$. Therefore,

\[ \sum_{j \in [\ell]} [z_j] \leq 0 \quad \text{and} \quad \sum_{j \in [\ell]} \lfloor z_j \rfloor \geq 0. \]

Then,

\[ m = \left| \sum_{j \in [\ell]} [z_j] \right| = \left| \sum_{j \in [\ell]} [z_j] - \sum_{j \in [\ell]} z_j \right| = \sum_{j \in [\ell]} (\lfloor z_j \rfloor - z_j) \leq \sum_{j \in [\ell]} \lfloor |z_j| - z_j | \leq \ell, \]

15
where the second equality is because $\sum_{j \in [\ell]} z_j = 0$. Hence, starting from $x_j = \lfloor z_j \rfloor$, $\forall j \in [\ell]$, the algorithm has to round at most $x$ coordinates to $x_j = \lfloor z_j \rfloor$. Since $j \in [\ell]$ this is always possible. Therefore, the rounding in Step 13 always finds an integral point $x$ that satisfies the following,

$$\sum_{j \in [\ell]} x_j = 0 \quad \text{and} \quad (\forall j \in [\ell], x_j = \lfloor z_j \rfloor \quad \text{or} \quad x_j = \lceil z_j \rceil). \quad (11)$$

Therefore, the set of points $z \in \left(1 + \frac{\sqrt{\ell}}{2}\right)K'$ that are rounded to the integral point $x \in K'$ satisfying (11) is a strict subset of

$$\left\{ z : (\forall j \in [\ell], x_j = \lfloor z_j \rfloor \lor x_j = \lceil z_j \rceil) \land \sum_{j \in [\ell]} z_j = 0 \right\},$$

which is contained in $H' \cap C(x, 2)$ since $|z_j - \lfloor z_j \rfloor| \leq 1$ and $\lfloor z_j \rfloor - z_j \leq 1, \forall j \in [\ell]$.\qed

**Lemma 4.12.** For any $x \in K'$, $H' \cap C(x, 2) \subseteq \left(1 + \frac{\sqrt{\ell}}{2}\right)K'$.

**Proof.** Fix a point $x \in P'$. Then for any $x' \in C(x, 2), \|x' - x\|_2 \leq \sqrt{\ell}$. Lemma 4.10 gives us that the translated polytope $P'$ contains a ball of radius $\Delta$ centered at the origin. Then the polytope $\frac{\sqrt{\ell}}{2}P'$ contains every vector of length at most $\sqrt{\ell}$. Therefore, $x' - x \in \frac{\sqrt{\ell}}{2}P'$. Now since $x \in P'$ we get that $x' \in \left(1 + \frac{\sqrt{\ell}}{2}\right)P'$. Therefore, $C(x, 2) \subseteq \left(1 + \frac{\sqrt{\ell}}{2}\right)P'$. Consequently, $H' \cap C(x, 2) \subseteq H' \cap \left(1 + \frac{\sqrt{\ell}}{2}\right)P' = \left(1 + \frac{\sqrt{\ell}}{2}\right)(H' \cap P')$ since $\alpha H' = H'$ for any scalar $\alpha \neq 0$. Hence, $H' \cap C(x, 2) \subseteq \left(1 + \frac{\sqrt{\ell}}{2}\right)K'$.\qed

**Lemma 4.13.** For any point $z \in \frac{1}{\left(1 + \frac{\sqrt{\ell}}{2}\right)}K'$ the integral point it is rounded to belongs to the polytope $K'$.

**Proof.** From Lemma 4.11 we know that for any integral point $x \in K'$, $F_x \subseteq H' \cap C(x, 2)$. Due to convexity of $K'$, $\frac{1}{\left(1 + \frac{\sqrt{\ell}}{2}\right)}K'$ is contained entirely inside $K'$. Therefore, Lemma 4.11 is true for all the points in $\frac{1}{\left(1 + \frac{\sqrt{\ell}}{2}\right)}K'$. By arguing similarly as in the proof of Lemma 4.12 we can show that for any any $x \in \frac{1}{\left(1 + \frac{\sqrt{\ell}}{2}\right)}K'$, $H' \cap C(x, 2) \subseteq K'$. This proves the lemma.\qed

Let $\mu$ be a uniform probability measure on the convex rational polytope $\left(1 + \frac{\sqrt{\ell}}{2}\right)K'$.

**Lemma 4.14.** Fix any two distinct integral points $x, x' \in K'$, $\mu(F_x) = \mu(F_{x'})$.

**Proof.** Given two distinct integral points $x, x' \in K'$, let $c = x' - x$. Clearly $c$ is an integral point and $\sum_{j \in [\ell]} c_j = \sum_{j \in [\ell]} x'_j - \sum_{j \in [\ell]} x_j = 0 - 0 = 0$. Let $z \in F_x$ and $z' = z + c$. Then

$$\left| \sum_{j \in [\ell]} \lfloor z'_j \rfloor \right| = \sum_{j \in [\ell]} \lfloor z_j \rfloor + c_j = \sum_{j \in [\ell]} \lfloor z_j \rfloor + \sum_{j \in [\ell]} c_j = \sum_{j \in [\ell]} \lfloor z_j \rfloor = m.$$ 

Therefore, for both $z$ and $z'$ the first $m$ coordinates are rounded up, and the remaining are rounded down, in Step 13. Since $\lfloor z'_j \rfloor = \lfloor z_j \rfloor + c_j$ and $\lceil z'_j \rceil = \lceil z_j \rceil + c_j$, the point $z'$ is rounded to is nothing but $x'$. Therefore, for every point $z \in F_x$ there is a unique point $z' \in F_{x'}$ such that they are rounded to $x$ and $x'$ respectively. This gives us a bijection between the sets $F_x$ and $F_{x'}$. Therefore, $\mu (F_x) = \mu (F_{x'})$.\qed

16
Proof of Theorem 4.9. Let $K'' = \left(1 + \frac{\sqrt[3]{\ell}}{\Delta}\right) K'$.

Let $\nu$ be the distribution from which the SAMPLING ORACLE samples a point from $K''$. That is for a given $\epsilon > 0$,

$$\sup_{A \subseteq K''} |\nu(A) - \mu(A)| \leq \epsilon. \quad (12)$$

Close to uniform sample. From Lemmas 4.11 and 4.12 we know that for any point $x \in K'$, $F_x \subseteq K''$. Therefore, from (12) we have that

$$|\nu(F_x) - \mu(F_x)| \leq \epsilon.$$

Let $\nu'$ be the density from which Algorithm 1 samples an integral point from $K$. We know that $\forall x, x \in K' \iff x + x^* \in K$. Since $x^*$ is an integral point, for any integral point $x$ we also have that $x \in K' \iff x + x^* \in K$. This gives us a bijection between the integral points in $K'$ and $K$.

Therefore for any integral point $x \in K'$, $\nu'(x) = \nu(F_x)$. Let $\mu'(x) := \mu(F_x)$ for any integral point $x$ in $K'$.

For any two integral points $x, x' \in K'$, Lemma 4.14 gives us that $\mu(F_x) = \mu(F_{x'})$. Moreover, since $F_x$ and $F_{x'}$ are both fully contained in $K''$, we get that $\mu'(F_x) = \mu(F_x) = \mu(F_{x'}) = \mu'(F_{x'})$. Therefore, $\mu'$ is a uniform measure over all the integral points in $K'$.

Moreover, for any subset of integral points in $K$, say $I$, we have that

$$\nu'(I) = \nu(\cup_{x \in I} F_x).$$

From Equation (12) we have that

$$|\nu(\cup_{x \in I} F_x) - \mu(\cup_{x \in I} F_x)| \leq \epsilon.$$

Consequently,

$$|\nu'(I) - \mu'(I)| \leq \epsilon.$$

Therefore $\nu'$ over the integral points in $K'$ is at a total variation distance of at most $\epsilon$ from the uniform probability measure $\mu'$ over the integral points in $K'$.

Probability of acceptance. Algorithm 1 samples points from $\left(1 + \frac{\sqrt[3]{\ell}}{\Delta}\right) K'$ in each iteration. Due to Lemma 4.13 we know for sure that whenever the algorithm samples a point from $\left(1 + \frac{\sqrt[3]{\ell}}{\Delta}\right) K'$, it will be rounded to an integral point in $K'$. Therefore, the probability of sampling an integral point in $K'$ is

$$\geq \nu\left(\frac{1}{\left(1 + \frac{\sqrt[3]{\ell}}{\Delta}\right)} K\right) \geq \mu\left(\frac{1}{\left(1 + \frac{\sqrt[3]{\ell}}{\Delta}\right)} K\right) - \epsilon \quad \text{from Equation (12)}$$

$$= \frac{\text{Vol}_{\ell-1}\left(\frac{1}{\left(1 + \frac{\sqrt[3]{\ell}}{\Delta}\right)} K\right)}{\text{Vol}_{\ell-1}\left(K''\right)} - \epsilon \quad : \mu \text{ is a uniform distribution over } K''$$

$$= \frac{\text{Vol}_{\ell-1}\left(\frac{1}{\left(1 + \frac{\sqrt[3]{\ell}}{\Delta}\right)} K\right)}{\text{Vol}_{\ell-1}\left(1 + \frac{\sqrt[3]{\ell}}{\Delta}\right) K'} - \epsilon \quad \text{by the definition of } K''$$
\[
\left(1 + \frac{\sqrt{\ell}}{\Delta}\right)^{-2(\ell-1)} - \epsilon \\
\geq e^{-\frac{\sqrt{\ell}}{\Delta}2(\ell-1)} - \epsilon
\]
Vol_{\ell-1} represents the volume in \(\ell-1\) dimensions using \((1 + x) \leq e^x, \forall x\).

Therefore the expected running time of the algorithm before it outputs an acceptable point is inversely proportional to the probability of acceptance, that is, \(1\left(e^{-\frac{\sqrt{\ell}}{\Delta}2(\ell-1)} - \epsilon\right)\). When \(\epsilon < e^{-2}\) and is a non-negative constant, and if \(\Delta = \Omega(\ell^{1.5})\), this probability is at least a constant. Hence, repeating the whole process a polynomial number of times in expectation guarantees we sample an integral point from \(K'\).

5 Experimental Results

In this section, we apply our algorithms to various real-world datasets and validate the guarantees provided by our algorithm. We implement our algorithm using the tool called PolytopeSampler\(^2\) to sample a point from a distribution close to uniform, on the convex rational polytope \(K\). This tool implements a constrained Riemannian Hamiltonian Monte Carlo for sampling from high dimensional distributions on polytopes (Kook et al., 2022). We call the implementation of Algorithm 1 using this tool as ‘Random walk’ in the plots.

Datasets. We evaluate our results on the German Credit Risk dataset\(^3\) which consists of credit risk scoring of 1000 adult German residents (Dua and Graff, 2017). The dataset also contains demographic information about the individuals (e.g., gender, age, etc.) We use the Schufa scores of these individuals are used to get the in-group rankings. A ranking based on the Schufa scores on the entire dataset is biased against the adults of age \(< 25\) as observed by (Castillo, 2019). Their representation in the top 100 ranks is 10% even though their true representation in the whole dataset is 15%. Similarly the adults of age \(< 35\) are also consistently underrepresented in the top 100 ranks according to the Schufa scores. Therefore, we evaluate our algorithm on the grouping based on age \(< 25\) (see Figure 1 and grouping based on age \(< 35\) similar to (Zehlike et al., 2017; Gorantla et al., 2021).

We also evaluate our algorithm on the IIT-JEE 2009 dataset, also used in (Celis et al., 2020b). The dataset\(^4\) consists of the student test results of the joint entrance examination (JEE) conducted for undergrad admissions at the Indian Institutes of Technology (IITs). A total of 384,977 students’ total scores (sum of the scores in Math, Physics, and Chemistry) ranging from \(-105\) to \(480\) marks\(^5\) gives us the merit scores of the students, and hence the score-based in-group rankings. Additional information about the students has also been released under the “Right to Information Act” (Kumar, 2009). This includes students’ gender\(^6\) (25% women and 75% men). However, female students are consistently underrepresented, as observed in (Celis et al., 2020b) in a score-based ranking on the entire dataset. That is, in the top 100 ranks, 0.04%, even though their representation is 25% in the dataset. We evaluate our algorithm with female as the protected group.

We compare our experimental results with fair \(\epsilon\)-greedy proposed by Gao and Shah (2020). As the name suggests, this is a greedy algorithm with \(\epsilon\) as a parameter. From \(i = 1,2,\ldots,k\) the algorithm assigns to rank \(i\) a uniform random group, with probability \(\epsilon\), or a group that pushes the ranking to satisfy the representation

\(^1\)github.com/ConstrainedSampler/PolytopeSamplerMatlab
\(^2\)taken from Gorantla et al. repository
\(^3\)taken from Celis et al. 2020b repository
\(^4\)average = 28.36, maximum = 424 and minimum = -86.
\(^5\)only binary gender information was annotated in the dataset.
Figure 1: Results on the German Credit Risk dataset with age < 25 as the protected group in the first row and age < 35 as the protected group in the second row. For Fair ϵ-greedy we use ϵ = 0.3 (see Figures 5 and 8 for other values of ϵ).

constraints (lower bound on the fractions of ranks assigned to the group in the top-k ranks), with probability 1 − ϵ. To the best of our knowledge, this is the only other randomized ranking algorithm that does not rely on comparing the scores of two candidates from different groups. It only relies on the given scores for ranking items within the same group. Hence, it is the closest state-of-the-art baseline to our setting. We also compare our results with a recent deterministic re-ranking algorithm given by Gorantla et al. (2021) that achieves the best balance of both group fairness and underranking of individual items compared to their original ranks in top-k. We call this algorithm GDL21 in our plots.

We plot our results for the protected groups in each dataset (see Figures 1 and 2 for details).

We use the representation constraints $L_j = \lceil (p_j^* - 0.05)k \rceil$ and $U_j = \lfloor (p_j^* + 0.05)k \rfloor$ for group j where $p_j^*$ is the total fraction of items from group j in the dataset.

The “representation” (on the y-axis) plot shows the fraction of ranks assigned to the protected group in a top k' ranks (on the x-axis). For randomized algorithms, we sample 1000 rankings and output the mean and the standard deviation. The dashed green line is the true representation of the protected group in the dataset, which we call $p^*$, dropping the subscript. The “fraction of rankings” (on the y-axis) plot for randomized ranking algorithms represents the fraction of 1000 rankings that assign rank i (on the x-axis) to the protected group.
Figure 2: Results on the JEE 2009 dataset with gender as the protected group. For Fair $\epsilon$-greedy we use $\epsilon = 0.3$ (see Figures 7 and 9 for other values of $\epsilon$).

|                | $\ell = 2$    | $\ell = 5$    |
|----------------|---------------|---------------|
| Random walk    | 7.11 ± 0.15   | 19.55 ± 0.28  |
| Fair $\epsilon$-greedy | 1.82 ± 0.03   | 14.1 ± 0.13   |

Table 1: Mean and standard deviation of running time in seconds, over 5 runs, to sample 1000 rankings.

5.1 Experimental observations

The rankings sampled by our algorithm have the property that in a sufficient fraction of rankings, rank $i$ is assigned to the protected group, for any rank $i$ (see plots with “fraction of rankings” on the y-axis). This experimentally validates our Theorem 3.5. Moreover, this fraction is stable across the ranks. Hence the line looks almost flat. Whereas fair $\epsilon$-greedy fluctuates a lot unless the proportion of the protected group in the whole dataset is close to half, which is the case with the protected group age < 35 with 54% in the dataset.

Our algorithm also closely satisfies representation constraints for the protected group in the top $k'$ ranks for $k' = 20, 40, 60, 80, 100$, in expectation (see plots with “representation” on the y-axis). Fair $\epsilon$-greedy overcompensates for representing the protected group. The deterministic algorithm GDL21 achieves very low representation for smaller values of $k'$, although all run with similar representation constraints. This is because the deterministic algorithm uses comparisons based on the scores, hence putting most of the protected group items in higher ranks. With larger value of $\epsilon$, fair $\epsilon$-greedy get much higher “representation” of the protected group than necessary (see Figure 8, Figure 9), whereas with a smaller value of $\epsilon$, it fluctuates a lot in the “fraction of rankings” (see Figure 6, Figure 7).

We also run experiments on the JEE 2009 dataset with birth category defining the demographic groups (GE=60%, OC=4%, ON=23%, SC=9.3%, ST=3.2%). We plot the results in Figure 3. Additionally, we plot the results for the ranking utility metric, normalized discounted cumulative gain, defined as

$$nDCG@k = \frac{\sum_{i\in[k]} (2^{\hat{s}_i} - 1)}{\sum_{i\in[k]} (2^{s_i} - 1)},$$

where $\hat{s}_i$ and $s_i$ are the scores of the items assigned to rank $i$ in the group fair and the score-based ranking, respectively. See Figure 4 for the performance of the algorithms with respect to this metric.
Figure 3: Results on the JEE 2009 dataset with birth category as the protected group (with 5 groups). For Fair $\epsilon$-greedy we use $\epsilon = 0.3$. 

Protected group = age < 25, $k = 100$ 

Group = GE, $k = 100$ 

Group = ON, $k = 100$ 

Group = SC, $k = 100$ 

Group = OC, $k = 100$ 

Group = ST, $k = 100$ 

Group = GE, $k = 100$ 

Group = ON, $k = 100$ 

Group = SC, $k = 100$ 

Group = OC, $k = 100$ 

Group = ST, $k = 100$
We define a group-wise normalized discounted cumulative gain as follows,

$$\text{group-nDCG}@k = \frac{\min_{j \in [\ell]} \frac{1}{n_j} \sum_{i \in [k]} \frac{(2^{s_i} - 1) I[y_i = j]}{\log_2(i+1)}}{\max_{j \in [\ell]} \frac{1}{\hat{n}_j} \sum_{i \in [k]} \frac{(2^{s_i} - 1) I[\hat{y}_i = j]}{\log_2(i+1)}}$$

where $y_i$ and $\hat{y}_i$ represent the group assigned to rank $i$ in the scored-based and group-fair ranking respectively. Similarly $n_i$ and $\hat{n}_i$ represent the total number of top $k$ ranks assigned to group $j$ in the scored-based and group-fair ranking, respectively. This definition is similar to the group ndcg metric defined in (Shang et al., 2020). See Figure 5 for the performance of the algorithms with respect to this metric.

The experiments were run on a Quad-Core Intel Core i5 processor consisting of 4 cores, with a clock speed of 2.3 GHz and DRAM of 8GB. See Table 1 for details of the running time. Our random walk-based algorithm also runs as fast as the $O(k\ell)$ time greedy algorithm.

6 Conclusion

We take an axiomatic approach to define randomized group fair rankings and show that it leads to a unique distribution over all feasible rankings that satisfy lower and upper bounds on the group-wise representation in the top ranks. We propose practical and efficient algorithms to exactly and approximately sample a random group fair ranking from this distribution. Our approach requires merging a given set of ranked lists, one for each group, and can help circumvent implicit bias or incomplete comparison data across groups.
Figure 6: Results on the German Credit Risk dataset with $age < 25$ as the protected group in the first row and $age < 35$ as the protected group in the first row. For Fair $\epsilon$-greedy we use $\epsilon = 0.15$.

The natural open problem is to extend our method to work even for noisy, uncertain inputs about rankings, scores, or comparisons within each group. Another open problem is investigating the possibility of polynomial-time algorithms to sample random group fair rankings under representation-based constraints for each group in every prefix.

A limitation of our work as a post-processing method is that it cannot fix all sources of bias, e.g., bias in data collection and labeling. Our guarantees for group fairness may not necessarily reflect the right fairness metrics for downstream applications for reasons including biased, noisy, incomplete data and legal or ethical considerations in quantifying the eventual adverse impact on individuals and groups.

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Figure 8: Results on the German Credit Risk dataset with \( \text{age} < 25 \) as the protected group in the first row and \( \text{age} < 35 \) as the protected group in the first row. For Fair \( \varepsilon \)-greedy we use \( \varepsilon = 0.5 \).

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