Bifurcations of thresholds in essential spectra of elliptic operators under localized non-Hermitian perturbations

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Abstract
We consider the operator

$$\mathcal{H} = \mathcal{H}' - \frac{\partial^2}{\partial x_d^2} \quad \text{on} \quad \omega \times \mathbb{R}$$

subject to the Dirichlet or Robin condition, where a domain $\omega \subseteq \mathbb{R}^{d-1}$ is bounded or unbounded. The symbol $\mathcal{H}'$ stands for a second-order self-adjoint differential operator on $\omega$ such that the spectrum of the operator $\mathcal{H}'$ contains several discrete eigenvalues $\Lambda_j, j = 1, \ldots, m$. These eigenvalues are thresholds in the essential spectrum of the operator $\mathcal{H}$. We study how these thresholds bifurcate once we add a small localized perturbation $\varepsilon \mathcal{L}(\varepsilon)$ to the operator $\mathcal{H}$, where $\varepsilon$ is a small positive parameter and $\mathcal{L}(\varepsilon)$ is an abstract, not necessarily symmetric operator. We show that these thresholds bifurcate into eigenvalues and resonances of the operator $\mathcal{H}$ in the vicinity of $\Lambda_j$ for sufficiently small $\varepsilon$. We prove effective simple conditions determining the existence of these resonances and eigenvalues and find the leading terms of their asymptotic expansions. Our analysis applies to generic nonself-adjoint perturbations and, in particular, to perturbations characterized by the parity-time ($PT$) symmetry. Potential applications of our result embrace a broad class of physical systems governed by
dispersive or diffractive effects. As a case example, we
employ our findings to develop a scheme for a control-
lable generation of non-Hermitian optical states with
normalizable power and real part of the complex-valued
propagation constant lying in the continuum. The corre-
spanding eigenfunctions can be interpreted as an optical
generalization of bound states embedded in the contin-
um. For a particular example, the persistence of asym-
totic expansions is confirmed with direct numerical
evaluation of the perturbed spectrum.

1 | INTRODUCTION

1.1 | Physical context and motivation

Physics of non-Hermitian Hamiltonians is attracting steadily growing attention both on the funda-
damental level in the development of complex formulations of quantum mechanics\(^1\)–\(^3\) and in sev-
eral applied and experimental fields, such as optics and photonics, Bose–Einstein condensates of
atoms or exciton-polaritons, acoustics, and in other areas where diffractive or dispersive effects
are governed by Schrödinger-like elliptic operators, see Refs. \(^4\)–\(^8\) for recent reviews. Prominent
examples of essentially non-Hermitian phenomena that were introduced in mathematical lit-
erature long ago but entered various areas of physics much more recently include exceptional
points\(^9\) and spectral singularities.\(^10\)–\(^12\) In particular, unusual effects associated with exceptional
points are being extensively discussed in optics and photonics (e.g., Refs. \(^13\)–\(^15\)), whereas spec-
tral singularities\(^10\)–\(^12\) are now understood to play an important role in wave scattering\(^16\),\(^17\) and
are used to implement coherent perfect absorption of electromagnetic,\(^18\) sound,\(^19\) and matter\(^20\)
waves. Another prototypical behavior, which is forbidden in Hermitian physics but is of the utmost
importance in non-Hermitian systems, is the transition from the entirely real spectrum of eigen-
values to a partially complex one.

In a real-world system, the non-Hermiticity usually corresponds to the presence of an energy
gain or absorption, which creates an effective complex potential for propagating waves.\(^21\) An espe-
cially interesting situation, where a judicious balance between amplification and losses results in
rich physics, corresponds to the so-called parity-time (\(PT\)) symmetric systems famous for their
property to robustly preserve reality of all eigenvalues in spite of the absence of Hermiticity.\(^1\),\(^22\)
Tuning a control parameter of a \(PT\)-symmetric system, one can realize various qualitative
changes in its spectral structure, and these changes are typically associated with rich and intrigu-
ing behaviors. The simplest of those behaviors, which was observed in a series of experiments,\(^14\),\(^15\)
corresponds to the collision of a pair of real discrete eigenvalues in an exceptional point with
a subsequent splitting in a complex-conjugate pair. In systems with a continuous spectrum, the
situation is further enriched. In particular, the bifurcation of an isolated eigenvalue from the bot-
tom of the essential spectrum can be accompanied by a so-called jamming anomaly, i.e., a non-
monotonous dependence of the energy flux through the gain-to-loss interface on the parameter
characterizing the strength of the non-Hermiticity.\(^23\) The bifurcations of a complex-conjugate
pair of eigenvalues from an internal point in the essential spectrum are even more interesting and have recently been discussed as an unconventional mechanism of $\mathcal{PT}$-symmetry breaking\textsuperscript{24–27} distinctively different from the better studied $\mathcal{PT}$-symmetry breaking through an exceptional point. Complex eigenfunctions associated with bifurcated eigenvalues are $L^2$-integrable, and the real parts of these eigenvalues belong to the continuous spectrum. This enables a physical interpretation of such eigenfunctions in terms of non-Hermitian generalizations\textsuperscript{28,29} of bound states embedded in the continuum, well-known in quantum mechanics,\textsuperscript{30–33} optics, and other fields.\textsuperscript{34} It should be noticed at the same time that most of the activity devoted to non-Hermitian optical bound states in the continuum is being carried out for one-dimensional systems. For multidimensional geometries, most of the available results are numerical in nature.\textsuperscript{29} From the practical point of view, it is also important that eigenfunctions associated with emerging from the essential spectrum eigenvalues are extremely weakly localized in the vicinity of the bifurcation, which hinders their efficient numerical evaluation. Naturally, this problem is even more pronounced in multidimensional geometries, where many more computational resources are necessary to approximate the eigenfunctions. Therefore, any analytical information on the properties of such states is highly desirable.

1.2 Mathematical context

The phenomenon that a small localized perturbation of a self-adjoint differential operator can generate discrete eigenvalues from the edges in the essential spectrum is known for about a hundred of years. Its rigorous mathematical study was initiated by classical works by B. Simon, M. Klaus, R. Blankenbecler, and M. L. Goldberger,\textsuperscript{35–38} and since that time, hundreds of papers on this subject were written. While classical works were devoted to the Schrödinger operator on an axis and plane perturbed by a small localized potentials, in further works, the studies were made for plenty of other models, like waveguide-like structures, see, for instance, Refs.\textsuperscript{39–42} for periodic operators, see, for instance, Ref.\textsuperscript{43} for operator with distant perturbations\textsuperscript{44} and many others. All these works treated symmetric perturbations of self-adjoint operators, and the perturbed operators were self-adjoint as well.

Nonsymmetric perturbations of self-adjoint operators were studied in essentially less details. In Refs.\textsuperscript{45,46}, there was considered the Laplacian on the axis perturbed by a small abstract localized operator, which was not assumed to be symmetric. The main result was sufficient conditions ensuring the existence and absence of the emerging eigenvalues from the bottom of the essential spectrum and if they exist, the leading terms in the asymptotic expansions for the emerging eigenvalues were found. These results were essentially extended in Refs.\textsuperscript{47,48}. Here, an unperturbed operator was an arbitrary periodic self-adjoint operator on the line\textsuperscript{47} or on the plane.\textsuperscript{48} A perturbation was a small abstract operator not necessarily symmetric and localized in a much weaker sense than in Refs.\textsuperscript{45,46}. The structure of the spectra of such operators was studied in details. Qualitative properties such as stability of the essential spectrum, the countability of the point spectrum, the absence of the residual spectrum, and the existence of embedded eigenvalues were addressed. Sufficient conditions ensuring the existence and absence of the eigenvalues emerging from edges of internal gaps in the essential spectrum were established and if they exist, the leading terms in their asymptotic expansions were obtained. Eigenvalues emerging from the bottom of the essential spectrum were also studied in Refs.\textsuperscript{49–51} for waveguides with $\mathcal{PT}$-symmetric Robin-type boundary condition. In Ref.\textsuperscript{49}, a planar waveguide was considered with a locally perturbed coefficient in the $\mathcal{PT}$-symmetric boundary condition. In Refs.\textsuperscript{50,51}, similar
two- and three-dimensional waveguides were considered and the perturbation was a small width of these waveguides. The main obtained results were sufficient conditions ensuring the existence of the emerging eigenvalues and the leading terms of their asymptotic expansions. We also mention work, 52 where the Dirichlet or Neumann Laplacian in a multidimensional cylinder was considered and it was perturbed by a small localized nonsymmetric perturbations. The eigenvalues bifurcating from the bottom and the internal thresholds in the essential spectrum were studied. There were obtained certain sufficient conditions ensuring the existence of such eigenvalues and the leading terms of their asymptotic expansions were calculated. However, there was a gap in calculations in Ref. 52, which made the results of this work true but incomplete. Namely, while working with operators providing meromorphic continuations for the resolvent in the vicinity of internal thresholds, the author of Ref. 52 considered only one continuation, while, as we show in the present work, even in our more general setting two continuations exist and complex eigenvalues are the poles just for one of these continuations. This is why the results of Ref. 52 described, roughly speaking, only half of emerging eigenvalues.

The bifurcations of the thresholds in the essential spectrum can also be studied for perturbations of nonself-adjoint operators, provided that the spectral structure of the limiting operator is known in sufficiently great details. As examples, we mention works 53,54, where an evolutionary nonlinear Schrödinger equation was considered with both linear and nonlinear perturbations. The linearization of this equation on solitary wave solutions gave rise to a spectral problem for a linear nonself-adjoint operator. The essential spectral of such operator consists of two real semi-axes; the main results of Refs. 53, 54 provided conditions, under which the end points of the essential spectrum bifurcated into eigenvalues. If the latter existed, their two-terms asymptotic expansions were calculated.

An important feature of the eigenvalues emergence is that usually, the total multiplicity of the emerging eigenvalues does not exceed the multiplicity of edge in the essential spectrum from which they emerge. The multiplicity of the edge is to be treated in the sense of some appropriate generalized eigenfunctions. However, there were found examples, when this commonly believed rule failed. The earliest work on this subject we know is paper, 55 where the Schrödinger operator on \( \mathbb{R}^3 \) perturbed by a small localized potential was considered. It was found that in certain cases, an \( n \)-multiple bottom of the essential spectrum can generate \( n \) eigenvalues and \( n \) antibound states or \( 2n \) resonances. In Ref. 56, a similar phenomenon was found for the Dirichlet Laplacian in a pair of three-dimensional layers coupled by a window, when the perturbation was a small variation of the window shape. Very recently, we succeeded to find an even more impressive example of infinitely many eigenvalues and/or resonances emerging from the bottom of an essential spectrum. This was done in papers Refs. 57, 58, where we considered a one-dimensional Schrödinger operator on the axis with two complex localized potentials, the supports of which were separated by a large distance. It was found that as this distance increases, more and more resonances and eigenvalues appear in the vicinity of the bottom of the essential spectrum, while the multiplicity of this bottom is at most one. The location and asymptotic behavior of these emerging eigenvalues and resonances were analyzed in details.

Emerging eigenvalues were also studied not only in the context of classical eigenvalue problems, but also for more complicated operator pencils. In Ref. 59, there was considered a special quadratic operator pencil on the line with a special small periodic \( PT \)-symmetric perturbation. The structure of the gaps in the essential spectrum and complex eigenvalues in the vicinities of the edges of these gaps were analyzed in great details. In Ref. 60, there was considered a similar quadratic operator pencil with a special small localized \( PT \)-symmetric potential and there were studied eigenvalues emerging from thresholds in the essential spectrum. Sufficient existence
conditions were established and the leading terms of the asymptotic expansions of the emerging eigenvalues were obtained.

1.3 Model and results

In the present paper, we carry out a rigorous analysis of bifurcations of isolated eigenvalues and resonances from the essential spectrum of a multidimensional operator under a small localized general abstract perturbation. Namely, we consider a self-adjoint operator of the form

$$\mathcal{H} = \mathcal{H}' - \frac{\partial^2}{\partial x^2_d} \quad \text{on} \quad \omega \times \mathbb{R}$$

subject to the Dirichlet or Robin condition, where $\omega \subseteq \mathbb{R}^{d-1}$ is some domain, which can be both bounded and unbounded and $\mathcal{H}'$ is a self-adjoint second-order differential operator on $\omega$ subject to the same boundary condition as $\mathcal{H}$. We assume that the spectrum of the operator $\mathcal{H}'$ contains several discrete eigenvalues $\Lambda_1 \leq \Lambda_2 \leq \ldots \leq \Lambda_m$ below the essential spectrum. Then the essential spectrum of the operator $\mathcal{H}$ is the half-line $[\Lambda_1, +\infty)$ and the mentioned eigenvalues become thresholds in this essential spectrum. We add a small localized perturbation to the operator $\mathcal{H}$. This perturbation reads as $\varepsilon \mathcal{L}(\varepsilon)$, where $\varepsilon$ is a small positive parameter and $\mathcal{L}(\varepsilon)$ is an abstract not necessarily symmetric operator acting from a weighted Sobolev space $W^2_\omega(\Omega, e^{-\delta |x_d|}dx)$ into a weighted Lebesgue space $L^2_\omega(\Omega, e^{\delta |x_d|}dx)$. Exact definitions of these spaces will be given in the next section and now we just say that these weights and the operator $\mathcal{L}(\varepsilon)$ are designed so that this operator maps exponentially growing functions into exponentially decaying ones. The latter fact is exactly how we understand the localization of this operator in a generalized sense.

The main result of our paper describes how the thresholds $\Lambda_j$ in the essential spectrum of the operator $\mathcal{H}$ bifurcate under the presence of the perturbation $\varepsilon \mathcal{L}(\varepsilon)$. We show that if the bottom of the essential spectrum $\Lambda_j$ is an $m$-multiple eigenvalue of the operator $\mathcal{H}'$, then there can be at most $m$ eigenvalues and resonances of the operator $\mathcal{H}$ in the vicinity of $\Lambda_j$ for sufficiently small $\varepsilon$. The vicinity of an internal threshold $\Lambda_j > \Lambda_1$ in the essential spectrum being an $m$-multiple eigenvalue of the operator $\mathcal{H}'$ can contain at most $2m$ eigenvalues and resonances of the operator $\mathcal{H}$. The eigenvalues and resonances are identified via an analysis of the poles of an appropriate meromorphic continuation of the resolvent in the vicinity of each threshold $\Lambda_j$, $j \geq 1$. Each such pole generates either an eigenvalue or a resonance, and we provide simple sufficient conditions allowing one to identify whether a considered pole is an eigenvalue or a resonance. We also construct two-terms asymptotic expansions for the emerging eigenvalues and resonances.

1.4 Applications to specific models

While our result is rather general and applies to a broad range of physical models, where elliptic operators play the prominent role, we will exemplify applications of the work using some particular physical models. Namely, we discuss various examples of the unperturbed operator $\mathcal{H}$ and of the perturbation $\mathcal{L}(\varepsilon)$. Then we consider models of two- and three-dimensional waveguides and models of two- and three-dimensional quantum oscillators. As a perturbation, we choose a small localized complex potential. Such choice is motivated by physical models of optical
waveguides filled with a homogeneous medium, when the refractive index of the waveguide is locally modulated. This creates a small, generally, non-Hermitian perturbation in the form of an effective complex-valued optical potential. In particular, this potential can be \( PT \)-symmetric. Another physical model motivating the above examples is a two-dimensional Bose–Einstein condensate trapped in a harmonic potential in one dimension and without any trapping in the second dimension. The nonlinear interactions between particles of the condensate are assumed to be negligible such that its evolution can be described by the linear Schrödinger operator. As a perturbation, a localized non-Hermitian defect such as a localized dissipation serves. The similar approach can be applied to a three-dimensional condensate, where a localized perturbation can trigger formation of fully localized structures with internal vorticity.

For such examples, we show that given an internal threshold of a multiplicity \( n \), by tuning appropriately the perturbing potential, we can make the threshold to bifurcate into \( n \) pairs of complex-conjugated eigenvalues. This example demonstrates that the total multiplicity of the emerging eigenvalues can exceed the multiplicity of the internal threshold.

1.5 Organization of the paper

The rest of this paper is organized as follows. In Section 2, we elaborate rigorous mathematical formulation of the problem, and then present and discuss the main results that are formulated in several theorems. Section 3 is dedicated to examples, including a case study of optical bound states in the continuum emerging under a small \( PT \)-symmetric perturbation. Sections 4 and 5 contain the proofs of theorems.

2 PROBLEM AND RESULTS

2.1 Problem

Let \( x' = (x_1, ..., x_{d-1}), x = (x', x_d) \) be Cartesian coordinates in \( \mathbb{R}^{d-1} \) and \( \mathbb{R}^d \), respectively, where \( d \geq 2 \), and \( \omega \subseteq \mathbb{R}^{d-1} \) be an arbitrary domain. The domain \( \omega \) can be bounded or unbounded, the case \( \omega = \mathbb{R}^{d-1} \) is also possible. If the boundary of the domain \( \omega \) is nonempty, we assume that \( \partial \omega \in C^2 \). We let \( \Omega := \omega \times \mathbb{R} \) and we suppose that the domain \( \omega \) is such that

\[
\|u\|_{L^2(\partial\omega)} \leq C\|u\|_{W^{1,2}_2(\omega)}
\]

for all \( u \in W^{1,2}_2(\omega) \) with a constant \( C \) independent of \( u \). This inequality implies that

\[
\|u\|_{L^2(\partial\Omega)} \leq C\|u\|_{W^{1,2}_2(\Omega)}
\]

for all \( u \in W^{1,2}_2(\Omega) \) with a constant \( C \) independent of \( u \). This means that on the boundary of the domain \( \Omega \), the traces of the functions in \( W^{1,2}_2(\Omega) \) are well defined and the trace operator is bounded. This fact is employed below in definitions of various operators and sesquilinear forms without explicit mentioning.

By \( A_{ij} = A_{ij}(x'), A_j = A_j(x'), A_0 = A_0(x') \), \( i, j = 1, ..., d - 1 \), we denote real functions defined on \( \omega \) and with the following smoothness: \( A_{ij}, A_j \in C^1(\omega), A_0 \in C(\omega) \). The functions \( A_{ij} \) satisfy
the usual uniform ellipticity condition, that is, \( A_{ij} = A_{ji} \) and
\[
\sum_{i,j=1}^{d-1} A_{ij}(x')\xi_i\xi_j \geq c_0 \sum_{i=1}^{d-1} |\xi_i|^2 \quad \text{for all } x' \in \overline{\omega}, \quad \xi_i \in \mathbb{C},
\]
where \( c_0 > 0 \) is a positive constant independent of \( x' \) and \( \xi_i \). The functions \( A_{ij} \) and \( A_j \) are assumed to be uniformly bounded on \( \omega \), while for \( A_0 \), only an uniform lower bound is supposed. By \( i \) we denote the imaginary unit.

In terms of the introduced functions, we define an operator
\[
\mathcal{H} = - \sum_{i,j=1}^{d-1} \frac{\partial}{\partial x_i} A_{ij} \frac{\partial}{\partial x_j} - \frac{\partial^2}{\partial x_d^2} + i \sum_{j=1}^{d-1} \left( A_j \frac{\partial}{\partial x_j} + \frac{\partial}{\partial x_j} A_j \right) + A_0 \quad \text{in } \Omega
\]
subject to the Dirichlet condition or Robin condition:
\[
u = 0 \quad \text{on } \partial \Omega \quad \text{or} \quad \frac{\partial \nu}{\partial \nu} - a \nu = 0 \quad \text{on } \partial \Omega.
\]

In the case of Robin condition, the conormal derivative is defined as
\[
\frac{\partial u}{\partial \nu} := \sum_{i,j=1}^{d-1} A_{ij} \nu_i \frac{\partial u}{\partial x_j} - i \sum_{j=1}^{d-1} A_j \nu_j u + \nu_d \frac{\partial u}{\partial x_d},
\]
where \( \nu = (\nu_1, ..., \nu_d) \) is the unit outward normal to \( \partial \Omega \) and \( a = a(x') \) is a real function defined on \( \partial \Omega \). We assume that \( a \in C(\partial \Omega) \) and that this function is uniformly bounded on \( \partial \Omega \). We define a chosen boundary operator in (2) by \( B \), that is, \( Bu = u \) or \( Bu = \frac{\partial u}{\partial \nu} - au \).

Rigorously, we introduce the operator \( \mathcal{H} \) as follows. In the space \( \mathcal{L}_2(\Omega) \), we define a sesquilinear form
\[
\mathcal{H}(u, v) := \sum_{i,j=1}^{d-1} \left( A_{ij} \frac{\partial u}{\partial x_j}, \frac{\partial v}{\partial x_i} \right)_{\mathcal{L}_2(\Omega)} + \left( \frac{\partial u}{\partial x_d}, \frac{\partial v}{\partial x_d} \right)_{\mathcal{L}_2(\Omega)} + i \sum_{j=1}^{d-1} \left( A_j \frac{\partial u}{\partial x_j}, v \right)_{\mathcal{L}_2(\Omega)}
\]
\[
- i \sum_{j=1}^{d-1} \left( u, A_j \frac{\partial v}{\partial x_j} \right)_{\mathcal{L}_2(\Omega)} + (A_0 u, v)_{\mathcal{L}_2(\Omega)}
\]
on the domain \( \mathcal{D}(\mathcal{H}) := \mathcal{W}_2^1(\Omega) \cap \mathcal{L}_2(\Omega, (1 + |A_0|)dx) \) if the Dirichlet condition is chosen in (2), and
\[
\mathcal{H}(u, v) := \sum_{i,j=1}^{d-1} \left( A_{ij} \frac{\partial u}{\partial x_j}, \frac{\partial v}{\partial x_i} \right)_{\mathcal{L}_2(\Omega)} + \left( \frac{\partial u}{\partial x_d}, \frac{\partial v}{\partial x_d} \right)_{\mathcal{L}_2(\Omega)} + i \sum_{j=1}^{d-1} \left( A_j \frac{\partial u}{\partial x_j}, v \right)_{\mathcal{L}_2(\Omega)}
\]
\[
- i \sum_{j=1}^{d-1} \left( u, A_j \frac{\partial v}{\partial x_j} \right)_{\mathcal{L}_2(\Omega)} + (A_0 u, v)_{\mathcal{L}_2(\Omega)} - (au, v)_{\mathcal{L}_2(\partial \Omega)}
\]
on the domain $\mathfrak{D}(h) := W^1_2(\Omega) \cap L_2(\Omega, (1 + |A_0|) dx)$. Here $\dot{W}^1_2(\Omega)$ is a subspace of the space $W^1_2(\Omega)$ consisting of the functions with a zero trace on $\partial \Omega$. Given a positive function $\phi$ on $\Omega$, by $L_2(\Omega, \phi dx)$, we denote a weighted space formed by the functions in $L^2_{2,loc}(\Omega)$ with a finite norm $\| \cdot \|_{L_2(\Omega, \phi dx)}$ defined as

$$\|u\|_{L_2(\Omega, \phi dx)}^2 = \int_{\Omega} |u(x)|^2 \phi \, dx.$$ 

Thanks to the above assumptions on the functions $A_{ij}$, $A_{j}$, $A_{0}$, and $a$, the form $h$ is closed, symmetric, and lower semibounded. The self-adjoint operator in $L_2(\Omega)$ associated with this form is exactly the operator $H$.

We introduce one more weighted space $W^2_2(\Omega, e^{-\vartheta |x_d|} dx)$ as a subspace of $W^2_{2,loc}(\Omega)$ formed by the functions with finite norms $\| \cdot \|_{W^2_2(\Omega, e^{-\vartheta |x_d|} dx)}$ is defined as follows:

$$\|u\|_{W^2_2(\Omega, e^{-\vartheta |x_d|} dx)}^2 = \int_{\Omega} \sum_{\alpha \in \mathbb{Z}^2, \, |\alpha| \leq 2} |\partial^\alpha u(x)|^2 e^{-\vartheta |x_d|} \, dx.$$ 

Here $\vartheta > 0$ is some fixed constant. By $\varepsilon$ we denote a small positive parameter and the symbols $\mathfrak{H}_1$, $\mathfrak{H}_2$, $\mathfrak{H}_3 = \mathfrak{H}_3(\varepsilon)$ stand for operators mapping the space $W^2_2(\Omega, e^{-\vartheta |x_d|} dx)$ into $L_2(\Omega, e^{\vartheta |x_d|} dx)$. These operators are assumed to be bounded; the operator $\mathfrak{H}_3$ is bounded uniformly in $\varepsilon$. We stress that the operators $\mathfrak{H}_1$, $\mathfrak{H}_2$, $\mathfrak{H}_3$ are not supposed to be necessarily symmetric.

The main object of our study is a perturbed operator

$$H_\varepsilon = H + \varepsilon \mathfrak{L}(\varepsilon), \quad \mathfrak{L}(\varepsilon) := \mathfrak{H}_1 + \varepsilon \mathfrak{H}_2 + \varepsilon^2 \mathfrak{H}_3(\varepsilon)$$

on $\mathfrak{D}(H)$. The operator $H_\varepsilon$ is well defined since

$$\mathfrak{D}(H) \subseteq W^2_2(\Omega) \subseteq W^2_2(\Omega, e^{-\vartheta |x_d|} dx).$$

Moreover, it is clear that the operator $\mathfrak{L}(\varepsilon)$ is relatively bounded with respect to the operator $H$ and this is why, for sufficiently small $\varepsilon$, the operator $H_\varepsilon$ is closed.

Our main aim is to study the behavior of the eigenvalues of the operator $H_\varepsilon$ emerging from certain internal points in its essential spectrum. We denote the latter by $\sigma_{\text{ess}}(\cdot)$ and define it in terms of a characteristic sequences. Namely, a point $\lambda$ belongs to an essential spectrum $\sigma_{\text{ess}}(A)$ of some operator $A$ if there exists a bounded noncompact sequence $u_d \in \mathfrak{D}(A)$ such that

$$\inf_d \|u_d\| > 0 \quad \text{and} \quad (A - \lambda)u_d \rightarrow 0, \quad d \rightarrow \infty.$$ 

To describe the essential spectrum $\sigma_{\text{ess}}(H_\varepsilon)$, we introduce two auxiliary operators $H'$ and $H_0$. The former is a self-adjoint operator in $L_2(\mathbb{R}^{d-1})$ associated with a lower semibounded symmetric sesquilinear form

$$h'(u, v) := \sum_{i,j=1}^{d-1} A_{ij} \left( \frac{\partial u}{\partial x_j}, \frac{\partial v}{\partial x_i} \right)_{L_2(\omega)} + i \sum_{j=1}^{d-1} \left( A_j \frac{\partial u}{\partial x_j}, v \right)_{L_2(\omega)}.$$
\[-i \sum_{j=1}^{d-1} \left( u, A_j \frac{\partial v}{\partial x_j} \right)_{L_2(\omega)} + (A_0 u, v)_{L_2(\omega)} \]

on the domain $\mathcal{D}(\mathfrak{h}') := \tilde{W}^1_2(\omega) \cap L_2(\omega, (1 + |A_0|)dx')$ if the Dirichlet condition is chosen in (2) and

\[
\mathfrak{h}'(u, v) := \sum_{i,j=1}^{d-1} \left( A_{ij} \frac{\partial u}{\partial x_j}, \frac{\partial v}{\partial x_i} \right)_{L_2(\omega)} + i \sum_{j=1}^{d-1} \left( u, A_j \frac{\partial u}{\partial x_j} \right)_{L_2(\omega)}
\]

\[-i \sum_{j=1}^{d-1} \left( u, A_j \frac{\partial v}{\partial x_j} \right)_{L_2(\omega)} + (A_0 u, v)_{L_2(\omega)} - (a u, v)_{L_2(\partial \omega)} \]

on the domain $\mathcal{D}(\mathfrak{h}') := W^1_2(\omega) \cap L_2(\omega, (1 + |A_0|)dx')$ if the Robin condition is chosen in (2). This is the operator

\[
\mathcal{H}' = -\sum_{i,j=1}^{d-1} \frac{\partial}{\partial x_i} A_{ij} \frac{\partial}{\partial x_j} + i \sum_{j=1}^{d-1} \left( A_j \frac{\partial}{\partial x_j} + \frac{\partial}{\partial x_j} A_j \right) + A_0 \text{ in } \omega
\]

subject to the Dirichlet condition or Robin condition:

\[
u = 0 \text{ on } \partial \omega \quad \text{or} \quad \frac{\partial u}{\partial \nu} = au \text{ on } \partial \Omega,
\]

\[
\frac{\partial u}{\partial \nu'} := \sum_{i,j=1}^{d-1} A_{ij} \nu \frac{\partial u}{\partial x_j} - i \sum_{j=1}^{d-1} A_j \nu_j.
\]

The operator $\mathcal{H}_0$ is a one-dimensional Schrödinger operator

\[
\mathcal{H}_0 := -\frac{d^2}{dx^2}
\]

in $L_2(\mathbb{R})$ on the domain $W^2_2(\mathbb{R})$. Its spectrum is pure essential and coincides with $[0, +\infty)$. We assume that there exists a constant $c_0$ such that the spectrum of the operator $\mathcal{H}'$ below this constant consists of finitely many discrete eigenvalues, which we denote by $\Lambda_j$ and we arrange them in an ascending order counting multiplicities:

\[
\Lambda_1 \leq \Lambda_2 \leq \ldots \leq \Lambda_m < c_0.
\]

The associated orthonormalized in $L_2(\mathbb{R}^{d-1})$ eigenfunctions are denoted by $\psi_j = \psi_j(x')$, $j = 1, \ldots, m$.

The essential spectrum of the operator $\mathcal{H}_\varepsilon$ is described in the following lemma.

**Lemma 1.** The essential spectrum of the operator $\mathcal{H}_\varepsilon$ coincides with that of the operator $\mathcal{H}$ for all sufficiently small $\varepsilon$ and is given by the identity:

\[
\sigma_{\text{ess}}(\mathcal{H}_\varepsilon) = \sigma_{\text{ess}}(\mathcal{H}) = \sigma(\mathcal{H}) = [\Lambda_1, +\infty),
\]
where $\sigma(\cdot)$ denotes a spectrum of an operator.

According to this lemma, the points $\Lambda_j$, $j = 1, \ldots, m$, belong to the essential spectrum of the operator $\mathcal{H}_\varepsilon$. The point $\Lambda_1$ is the bottom of such spectrum, while other points $\Lambda_j$ are internal thresholds.

### 2.2 Main results

Our results describe a meromorphic continuation of the resolvent of the operator $\mathcal{H}_\varepsilon$ in the vicinity of the points $\Lambda_j$ as well as eigenvalues and resonances emerging from these points due to the presence of the perturbation $\varepsilon \mathcal{L}(\varepsilon)$. Before presenting our main results, we introduce some auxiliary constants and notations.

By $B_\delta$ we denote a ball of radius $\delta$ centered at the origin in the complex plane. We fix $p \in \{1, \ldots, m\}$ and assume that $\Lambda_p = \ldots = \Lambda_{p+n-1}$ is an $n$-multiple eigenvalue of the operator $\mathcal{H}'$, where $n \geq 1$. Then we consider a new complex parameter $k$ ranging in a small neighborhood of the origin and we introduce auxiliary functions:

$$
K_j(k) := -i \sqrt{\Lambda_p - \Lambda_j - k^2} \quad \text{as } j < p,
$$

$$
K_j(k) := k \quad \text{as } j = p, \ldots, p+n-1,
$$

$$
K_j(k) := \sqrt{\Lambda_j - \Lambda_p + k^2} \quad \text{as } j \geq p+n.
$$

Hereinafter the branch of the square root is fixed by the condition $\sqrt{1} = 1$ with the branch cut along the negative real semiaxis. Given $R > 0$, we let $\Omega_R^\pm := \Omega \cap \{x : \pm x_d > R\}$.

Now we are in position to formulate our first main result.

**Theorem 1.** Fix $p \in \{1, \ldots, m\}$, and $\tau \in \{-1, +1\}$ and let $\Lambda_p = \ldots = \Lambda_{p+n-1}$ be an $n$-multiple eigenvalue of the operator $\mathcal{H}'$, where $n \geq 1$. For all sufficiently small $\varepsilon$, the resolvent $(\mathcal{H}_\varepsilon - \Lambda_p + k^2)^{-1}$ admits a meromorphic continuation with respect to a complex parameter $k$ ranging in a sufficiently small neighborhood of the origin. Namely, there exists a bounded operator

$$
\mathcal{R}_{\varepsilon, \tau}(k) : L_2(\Omega, e^{\varphi| x_d |} \, dx) \rightarrow W_2^2(\Omega, e^{-\varphi| x_d |} \, dx)
$$

meromorphic with respect to complex $k \in B_\delta$ for a sufficiently small fixed $\delta$ independent of $\varepsilon$. If $p = 1$, then the operator $\mathcal{R}_{\varepsilon, \tau}(k)$ is independent of the choice of $\tau$ and for $\text{Re} k > 0$, this operator coincides with the resolvent $(\mathcal{H}_\varepsilon - \Lambda_1 + k^2)^{-1}$ restricted on $L_2(\Omega, e^{\varphi| x_d |} \, dx)$. If $p > 1$, then the operator $\mathcal{R}_{\varepsilon, \tau}(k)$ does depend on the choice of $\tau$ and for $\text{Re} k > 0$ and $\text{Im} k^2 < 0$, this operator coincides with the resolvent $(\mathcal{H}_\varepsilon - \Lambda_p + k^2)^{-1}$ restricted on $L_2(\Omega, e^{\varphi| x_d |} \, dx)$.

For all $f \in L_2(\Omega, e^{\varphi| x_d |} \, dx)$, the function $u_\varepsilon := \mathcal{R}_{\varepsilon, \tau}(k)f$ solves the boundary value problem

\[
\left( - \sum_{i,j=1}^{d-1} \frac{\partial}{\partial x_i} A_{ij} \frac{\partial}{\partial x_j} + \frac{\partial^2}{\partial x_d^2} + i \sum_{j=1}^{d-1} \left( A_{j} \frac{\partial}{\partial x_j} + \frac{\partial}{\partial x_j} A_{j} \right) + A_0 + \varepsilon \mathcal{L}(\varepsilon) - \Lambda_p + k^2 \right) u_\varepsilon = f \quad \text{in} \quad \Omega,
\]

\[
Bu_\varepsilon = 0 \quad \text{on} \quad \partial \Omega,
\]

(3)
and for sufficiently large $x_d$, it can be represented as follows:

$$u_\varepsilon(x, k) = \sum_{j=1}^{m} u_{\varepsilon,j}^\pm (x_d, k) \psi_j(x') + u_{\varepsilon,\perp}^\pm (x, k), \quad \pm x_d > R, \quad (4)$$

where $R$ is some fixed number, $u_{\varepsilon,j}^\pm \in L_2(I_\pm, e^{\mp \delta x_d} dx_d)$ are some meromorphic in $k \in B_\delta$ functions, $I_+ := (R, +\infty), I_- := (-\infty, -R)$, possessing the asymptotic behavior

$$u_{\varepsilon,j}^\pm(x_d) = e^{-R K_j(k)} \left( C_{\varepsilon,j}^\pm (k) + O(e^{-\delta |x_d|}) \right), \quad |x_d| \to \infty, \quad j = 1, \ldots, p - n + 1,$$

$$u_{\varepsilon,j}^\pm(x_d) = e^{-K_j(k)} \left( C_{\varepsilon,j}^\pm (k) + O(e^{-\delta |x_d|}) \right), \quad |x_d| \to \infty, \quad j = p, \ldots, m, \quad (5)$$

$C_{\varepsilon,j}^\pm(k)$ are some meromorphic in $k \in B_\delta$ functions, $0 < \delta < \delta$ is some fixed constant independent of $k$ and $x$, and $u_{\varepsilon,\perp}^\pm \in W_{2,loc}^2(\Omega^\pm_R)$ are some functions meromorphic in $k \in B_\delta$ and obeying the identities

$$\left( u_{\varepsilon,\perp}^\pm (\cdot, x_d), \psi_j \right)_{L_2(\omega)} = 0$$

for almost each $x_d \in I_\pm$ and for each $j = 1, \ldots, m$.

If $k_\varepsilon \in B_\delta$ is a pole of the operator $R_{\varepsilon,\tau}(k)$, for $k = k_\varepsilon$, problem (3) with $f = 0$ has a nontrivial solution $\psi_\varepsilon$ in $W_{2,loc}^2(\Omega)$, which satisfies a representation similar to (4):

$$\psi_\varepsilon(x) = \sum_{j=1}^{m} \phi_{\varepsilon,j}^\pm (x_d, k) \psi_j(x') + \psi_{\varepsilon,\perp}^\pm (x, k), \quad \pm x_d > R, \quad (7)$$

where $R$ is some fixed number, $\phi_{\varepsilon,j}^\pm \in L_2(I_\pm, e^{\mp \delta x_d} dx_d)$ are functions with the asymptotic behavior

$$\phi_{\varepsilon,j}^\pm(x_d) = e^{-R K_j(k)} \left( c_{\varepsilon,j}^\pm (k) + O(e^{-\delta |x_d|}) \right), \quad |x_d| \to \infty, \quad j = 1, \ldots, p - n + 1,$$

$$\phi_{\varepsilon,j}^\pm(x_d) = e^{-K_j(k)} \left( c_{\varepsilon,j}^\pm (k) + O(e^{-\delta |x_d|}) \right), \quad |x_d| \to \infty, \quad j = p, \ldots, m, \quad (8)$$

c_{\varepsilon,j}^\pm(k)$ are some constants, and $\psi_{\varepsilon,\perp}^\pm \in W_{2,loc}^2(\Omega^\pm_R)$ are some functions obeying the identities

$$\left( \psi_{\varepsilon,\perp}^\pm (\cdot, x_d), \psi_j \right)_{L_2(\omega)} = 0$$

for almost each $x_d \in I_\pm$ and for each $j = 1, \ldots, m$.

We define a subspace $L_\perp$ in $L_2(\Omega)$ as a set of functions $\nu \in L_2(\Omega)$ such that

$$\left( \nu(\cdot, x_d), \psi_j \right)_{L_2(\omega)} = 0$$

for almost each $x_d \in \mathbb{R}$ and for all $j = 1, \ldots, m$. The space $L_\perp$ is a Hilbert one. By $H_\perp$ we denote the restriction of the operator $H$ on $\mathfrak{D}(H) \cap L_\perp$. The following lemma will be proved in Section 4.1.
**Lemma 2.** The space $L^\perp$ is invariant for the operator $\mathcal{H}^\perp$, that is, this operator maps $\mathfrak{D}(H) \cap L^\perp$ into $L^\perp$. This is an unbounded self-adjoint operator in $L^\perp$ and its spectrum is located in $[c_0, +\infty)$.

The above lemma means that the resolvent $(\mathcal{H}^\perp - \Lambda_p)^{-1}$ is well defined for all $p = 1, \ldots, m$ as an operator from $L_2(L^\perp)$ into $\mathfrak{D}(H) \cap L^\perp$. As above, we fix $p \in \{1, \ldots, m\}$ and assume that $\Lambda_p = \ldots = \Lambda_{p+n-1}$, where $n \geq 1$, and in terms of the latter resolvent, we introduce an auxiliary operator mapping $L_2(\Omega, e^{\vartheta|x_d|} dx)$ into $W_2^2(\Omega, e^{-\vartheta|x_d|} dx)$:

\begin{equation}
(G_{p,\tau}f)(x) := \sum_{j=1}^{p-1} \frac{\psi_j(x')}{2\tau K_j(0)} \int_\Omega e^{-\tau K_j(0)|x_d-y_d|} \overline{\psi_j(y')} f(y) dy - \frac{1}{2} \sum_{j=p}^{p+n-1} \psi_j(x') \int \Omega |x_d-y_d| \overline{\psi_j(y')} f(y) dy + \sum_{j=p+n}^{m} \frac{\psi_j(x')}{2K_j(0)} \int_\Omega e^{-K_j(0)|x_d-y_d|} \overline{\psi_j(y')} f(y) dy + \left((\mathcal{H}^\perp - \Lambda_p)^{-1} f^\perp\right)(x),
\end{equation}

\begin{equation}
f^\perp(x) := f(x) - \sum_{j=1}^{m} f_j(x_d) \psi_j(x').
\end{equation}

As above, here $\tau \in \{-1, +1\}$. In the case $p = 1$, the first sum in the above definition is missing and the operator $G_{p,\tau}$ becomes independent of the choice of $\tau$.

We define the matrix $M_1$ with entries

\begin{equation}
M_1^{ij} := -\frac{1}{2} \int_\Omega \psi_{i+p-1} \ell_1 \psi_{j+p-1} dx, \quad i, j = 1, \ldots, n,
\end{equation}

where $i$ counts the rows and $j$ does the columns in the matrix $M_1$; as the operator $\ell_1$ acts from $W_2^2(\Omega, e^{-\vartheta|x_d|} dx)$ into $L_2(\Omega, e^{\vartheta|x_d|} dx)$, the functions $\ell_1 \psi_{j+p-1}$ belong to $L_2(\Omega, e^{\vartheta|x_d|} dx)$ and this obviously ensures the convergence of the integrals in the above identity.

By $\mu_i, i = 1, \ldots, N$, we denote different eigenvalues of the matrix $M_1$ of multiplicities $q_1, \ldots, q_N$. It is clear that $N \leq n$ and $q_1 + \ldots + q_N = n$.

**Theorem 2.** Fix $p \in \{1, \ldots, m\}$, $\tau \in \{-1, +1\}$ and let $\Lambda_p = \ldots = \Lambda_{p+n-1}$ be an $n$-multiple eigenvalue of the operator $\mathcal{H}'$, where $n \geq 1$. There are exactly $N$ poles, counting their orders, of the operator $R_{\varepsilon,\tau}(k)$ converging to zero as $\varepsilon \to +0$. These poles, denoted by $k_{ij}(\varepsilon)$, have the asymptotic behavior

\begin{equation}
k_{ij}(\varepsilon) = \varepsilon \mu_i + O\left(\varepsilon^{1+\frac{1}{q_i}}\right), \quad i = 1, \ldots, N, \quad j = 1, \ldots, q_i.
\end{equation}
Asymptotic expansion (13) for the poles \( k_{ij} \) can be specified in more details and this will be done in terms of one more matrix \( M_{2,\tau} \) with entries

\[
M_{2,\tau}^{ij} := \frac{1}{2} \int_{\Omega} \psi_{i+p-1}(L_2 - L_1 \psi_{j+p-1}) dx \quad i, j = 1, \ldots, n,
\]

where \( i \) counts the rows and \( j \) does the columns in the matrix \( M_{2,\tau} \). Due to the definition of the operators \( L_1 \) and \( L_2 \), the second term in the integrand in (14) belongs to \( L_2(\mathbb{R}, e^{\delta|x|}dx_d) \) and this ensures the convergence of the integral. We denote

\[
Q_{i,\tau}(z) := \frac{\partial}{\partial \varepsilon} \det \left( zE - M_1 + \varepsilon M_{2,\tau} \right) \bigg|_{\varepsilon=0}.
\]

We stress that if \( \Lambda_p = \Lambda_1 \), the matrix \( M_{2,\tau} \) and the function \( Q_{i,\tau} \) become independent of \( \tau \).

**Theorem 3.** Under the assumptions of Theorem 2, we fix \( i \in \{1, \ldots, N\} \). If \( Q_{i,\tau}(z) \) vanishes identically, then

\[
k_{ij}(\varepsilon) = \varepsilon \mu_i + O\left( \varepsilon^{1 + \frac{2}{q_i}} \right), \quad i = 1, \ldots, N, \quad j = 1, \ldots, q_i.
\]

(16)

If \( Q_{i,\tau} \) is not identically zero, then there exists a fixed nonnegative integer \( r_{i,\tau} < q_i \) such that

\[
\gamma_{i,\tau} := \frac{r_{i,\tau}!}{N} \prod_{j=1}^{N} \frac{d^{r_{i,\tau}} Q_{i,\tau}}{dz^{r_{i,\tau}}} (\mu_i) \neq 0.
\]

(17)

If \( 2r_{i,\tau} \geq q_i \), then

\[
k_{ij}(\varepsilon) = \varepsilon \mu_i + O\left( \varepsilon^{1 + \frac{1}{r_{i,\tau}}} \right), \quad i = 1, \ldots, N, \quad j = 1, \ldots, q_i.
\]

(18)

If \( 2r_{i,\tau} \leq q_i - 1 \), then exactly \( r_{i,\tau} \) poles \( k_{ij}, j = 1, \ldots, r_{i,\tau} \) have the asymptotic behavior

\[
k_{ij}(\varepsilon) = \varepsilon \mu_i + O\left( \varepsilon^{1 + \frac{1}{r_{i,\tau}}} \right), \quad i = 1, \ldots, N, \quad j = 1, \ldots, r_{i,\tau},
\]

(19)

while other poles \( k_{ij}, j = r_{i,\tau} + 1, \ldots, q_i \), have the asymptotic behavior

\[
k_{ij}(\varepsilon) = \varepsilon \mu_i + \varepsilon^{1 + \frac{1}{q_i-r_{i,\tau}}} \left( -\gamma_{i,\tau} \right) \frac{1}{z^{q_i-r_{i,\tau}}} e^{\frac{2\pi i}{q_i-r_{i,\tau}}(j-r_{i,\tau})} + O\left( \varepsilon^{1 + \frac{2}{q_i-r_{i,\tau}}} \right),
\]

(20)

where the branch of the fractional power \( z^{1/q_i-r_{i,\tau}} \) is fixed by the condition \( 1^{1/q_i-r_{i,\tau}} = 1 \) with the branch cut along the negative real semiaxis.
We give some definitions before we formulate our next result. A pole \( k \in B \) of an operator \( R_{\varepsilon,\tau}(k) \) corresponds to an eigenvalue \( \Lambda_p - k^2 \) of an operator \( H_{\varepsilon} \) if an associated nontrivial solution \( \psi_{k,\varepsilon} \) to (3) and (7) belongs to \( W^2_2(\Omega) \). Otherwise it corresponds to a resonance \( \Lambda_p - k^2 \).

Our further results provide conditions allowing to determine whether resonances or eigenvalues are associated with the poles described in two previous theorems. We first present the main result on the poles emerging from the bottom of the essential spectrum.

**Theorem 4.** Let \( p = 1 \) and make the assumptions of Theorems 2 and 3. If \( \text{Re} \mu_i > 0 \), then the poles \( k_{ij}, j = 1, \ldots, q_i \) correspond to the eigenvalues \( \lambda_{ij}(\varepsilon) = \Lambda_p - k_{ij}^2(\varepsilon) \) with the asymptotic behavior

\[
\lambda_{ij}(\varepsilon) = \Lambda_p - \varepsilon^2 \mu_i^2 + O\left(\varepsilon^{2+\frac{1}{\alpha_i}}\right) \tag{21}
\]

with \( j = 1, \ldots, q_i \), where

\[
\alpha_i := \begin{cases} \frac{q_i}{2} & \text{if } Q_{i,\tau} \text{ vanishes identically,} \\ r_{i,\tau} & \text{if } 2r_{i,\tau} \geq q_i. \end{cases} \tag{22}
\]

If \( 2r_{i,\tau} \leq q_i - 1 \), then the eigenvalues \( \lambda_{ij} \) still have asymptotic behavior (21) with \( \alpha_i = r_{i,\tau} \) for \( j = 1, \ldots, r_{i,\tau} \), while the asymptotic behaviors for the other eigenvalues read as

\[
\lambda_{ij}(\varepsilon) = \Lambda_p - \varepsilon^2 \mu_i^2 - 2\varepsilon^{2+\frac{1}{q_i-r_{i,\tau}}} \frac{1}{q_i-r_{i,\tau}} \left(-\gamma_{i,\tau}\right)^{\frac{2}{q_i-r_{i,\tau}}} e^{\frac{2\pi i}{q_i-r_{i,\tau}} (j-r_{i,\tau})} + O\left(\varepsilon^{2+\frac{2}{q_i-r_{i,\tau}}}\right), \quad j = r_{i,\tau} + 1, \ldots, q_i, \tag{23}
\]

If \( \text{Re} \mu_i < 0 \), then the poles \( k_{ij}, j = 1, \ldots, q_i \) correspond to the resonances \( \lambda_{ij}(\varepsilon) = \Lambda_p - k_{ij}^2(\varepsilon) \) with asymptotic expansions (21)–(23).

Let \( \text{Re} \mu_i = 0, Q_{i,\tau} \) be not identically zero and \( 2r_{i,\tau} \leq q_i - 1 \). As \( j = r_{i,\tau} + 1, \ldots, q_i \), if

\[
\text{Re}\left(-\gamma_{i,\tau}\right)^{\frac{1}{q_i-r_{i,\tau}}} e^{\frac{2\pi i}{q_i-r_{i,\tau}} (j-r_{i,\tau})} > 0, \tag{24}
\]

then the pole \( k_{ij} \) corresponds to an eigenvalue, while if

\[
\text{Re}\left(-\gamma_{i,\tau}\right)^{\frac{1}{q_i-r_{i,\tau}}} e^{\frac{2\pi i}{q_i-r_{i,\tau}} (j-r_{i,\tau})} < 0, \tag{25}
\]

the pole \( k_{ij} \) corresponds to a resonance. The asymptotic expansion for this eigenvalue/resonance is given by (23) if \( \mu_i \neq 0 \) and in the case \( \mu_i = 0 \), it reads as

\[
\lambda_{ij}(\varepsilon) = \Lambda_p - \varepsilon^{2+\frac{2}{q_i-r_{i,\tau}}} \left(-\gamma_{i,\tau}\right)^{\frac{2}{q_i-r_{i,\tau}}} e^{\frac{4\pi i}{q_i-r_{i,\tau}} (j-r_{i,\tau})} + O\left(\varepsilon^{2+\frac{3}{q_i-r_{i,\tau}}}\right). \tag{26}
\]

The next theorem concerns the poles emerging from the internal thresholds in the essential spectrum. Given \( p > 1 \) such that \( \Lambda_p > \Lambda_1 \), and \( \tau \in \{-1,+1\} \), by \( k_{ij,\tau} = k_{ij,\tau}(\varepsilon) \), we redenote the corresponding poles \( k_{ij} \) of the operator \( R_{\varepsilon,\tau} \) described in Theorems 2 and 3.
Theorem 5. Let $\Lambda_p > \Lambda_1$ and fix $i \in \{1, \ldots, N\}$, $j \in \{1, \ldots, q_i\}$, $\tau \in \{-1, +1\}$. Let

$$\text{Re} \mu_i > 0$$

or

$$\text{Re} \mu_i = 0, \quad Q_{i,\tau} \neq 0, \quad 2r_{i,\tau} \leq q_i - 1,$$

$$j \in \{r_i + 1, \ldots, q_i\}, \quad \text{Re}(\gamma_{i,\tau}) e^{\gamma_{i,\tau} (j-r_{i,\tau})} > 0,$$

and

$$\tau \text{Im} \mu_i < 0$$

or

$$\text{Im} \mu_i = 0, \quad Q_{i,\tau} \neq 0, \quad 2r_{i,\tau} \leq q_i - 1,$$

$$j \in \{r_i + 1, \ldots, q_i\}, \quad \tau \text{Im}(\gamma_{i,\tau}) e^{\gamma_{i,\tau} (j-r_{i,\tau})} < 0.$$

Then the pole $k_{i,j,\tau}(\varepsilon)$ corresponds to an eigenvalue $\lambda_{i,j,\tau}(\varepsilon) = \Lambda_p - k_{i,j,\tau}^2(\varepsilon)$ with asymptotic expansions (21)–(23) if $\mu_i \neq 0$ and asymptotic expansion (26) if $\mu_i = 0$.

Let

$$\text{Re} \mu_i < 0$$

or

$$\text{Re} \mu_i = 0, \quad Q_{i,\tau} \neq 0, \quad 2r_{i,\tau} \leq q_i - 1,$$

$$j \in \{r_i + 1, \ldots, q_i\}, \quad \text{Re}(\gamma_{i,\tau}) e^{\gamma_{i,\tau} (j-r_{i,\tau})} < 0.$$

Then the pole $k_{i,j,\tau}(\varepsilon)$ corresponds to a resonance $\lambda_{i,j,\tau}(\varepsilon) = \Lambda_p - k_{i,j,\tau}^2(\varepsilon)$ with asymptotic expansions (21)–(23) if $\mu_i \neq 0$ and asymptotic expansion (26) if $\mu_i = 0$.

Let $\mu_i$ be a simple eigenvalue of the matrix $M_1$ with an associated eigenvector $e_i := (e_{i,1}, \ldots, e_{i,n})$, $j = 1$, condition (27) or (28) hold, and

$$\tau \text{Im} \mu_i < 0$$

or

$$\text{Im} \mu_i = 0, \quad Q_{i,\tau} \neq 0, \quad r_{i,\tau} = 0, \quad \tau \text{Im} \gamma_{i,\tau} < 0,$$

and there exist $s \in \{1, \ldots, p - 1\}$ such that

$$\sum_{l=1}^n \int_{\Omega} e^{-K_l(0) x d} \overline{\psi_s(x')} L_1 e_{i,l} \psi_{l-p+1} \, dx \neq 0 \quad \text{or} \quad \sum_{l=1}^n \int_{\Omega} e^{K_l(0) x d} \overline{\psi_s(x')} L_1 e_{i,l} \psi_{l-p+1} \, dx \neq 0.$$

Then the pole $k_{i1,\tau}(\varepsilon)$ corresponds to a resonance $\lambda_{i,\tau}(\varepsilon) = \Lambda_p - k_{i1,\tau}^2(\varepsilon)$ with the asymptotic behavior

$$\lambda_{i,\tau}(\varepsilon) = \Lambda_p - \varepsilon^2 \mu_i^2 + O(\varepsilon^4)$$
if \( Q_{i,\tau} \) vanishes identically, and

\[
\lambda_{i,\tau}(\epsilon) = \Lambda_p - \epsilon^2 (\mu_i - \epsilon \gamma_{i,\tau})^2 + O(|\mu_i| \epsilon^4 + \epsilon^5)
\]

otherwise.

2.3 Discussion of the results

In this subsection, we discuss the main results formulated in Theorems 1–5. The first of them, Theorem 1, describes a meromorphic continuation of the resolvent of the perturbed operator. This continuation is local and is constructed in the vicinity of the points \( \Lambda_p, \ p = 1, \ldots, m \). The point \( \Lambda_1 \) is the bottom of the essential spectrum, see Lemma 1 and in vicinity of this point, just one meromorphic continuation is possible. It is introduced as a solution to problem (3) with a specified behavior at infinity, see (4), in terms of an auxiliary spectral parameter \( k \). The right-hand side in Equation in (3) is not in the class of compactly supported functions as it is usually assumed for meromorphic continuations, but an element of a wider space \( L_2(\Omega, e^{\theta|x_d|} dx) \). Here the presence of the weight \( e^{\theta|x_d|} \) means that the elements of latter space in certain sense decay exponentially as \( x_d \to \pm \infty \), namely, they are represented as \( f = e^{-\theta|x_d|/2} \tilde{f} \), where \( \tilde{f} \in L_2(\Omega) \). The final operator providing the meromorphic continuation is \( R_{\epsilon,\tau} \) and for \( \Lambda_1 \), it is independent of \( \tau \).

In the vicinity of internal thresholds \( \Lambda_p > \Lambda_1 \) in the essential spectrum, there are two different meromorphic continuations given by the operators \( R_{\epsilon,-1} \) and \( R_{\epsilon,+1} \). The former describes a meromorphic continuation from the lower complex half-plane into the upper one, while the latter does from the upper half-plane into the lower one. In the theory of self-adjoint operators, usually, only the latter continuation from the upper half-plane into the lower one is studied because it is physically meaning and it arises while considering a corresponding Cauchy problem for an evolutionary Schrödinger equation. However, as our perturbing operator is not assumed to be symmetric, the operator is not necessary self-adjoint. As a result, it can possess complex eigenvalues in the vicinity of the threshold \( \Lambda_p \). These eigenvalues are the poles of the resolvent of the perturbed operator. And as we see below, once we continue meromorphically the resolvent from the upper half-plane into the lower one, the eigenvalues in the lower half-plane can become “invisible” for the continuation in the sense that this continuation has no poles at such eigenvalues. A similar situation can hold once we continue analytically the resolvent from the lower half-plane into the upper one. A clear explanation of this phenomenon is due to representations (4)–(9). Namely, as \( \Lambda_p > \Lambda_1 \), the functions \( \phi_{\epsilon,j}, j = 1, \ldots, p - 1 \), in (8) behave at infinity as \( \phi_{\epsilon,j}(x_d) \sim e^{-\tau K_j(k)|x_d|} \). In view of obvious identities

\[
K_j(k) = -i\sqrt{\Lambda_p - \Lambda_j} + \frac{i}{2\sqrt{\Lambda_p - \Lambda_j}} k^2 + O(k^4), \quad k \to 0,
\]

the exponents \( e^{-\tau K_j(k)|x_d|} \) decay only if \( \tau \text{ Im } k^2 < 0 \). Depending on \( \tau \), the latter condition means that in general only the eigenvalues either in the upper or lower complex half-plane can serve as poles of the meromorphic continuation of the resolvent of the operator \( H_\epsilon \). This is a main reason why we deal with both meromorphic continuations, in contrast to the case of self-adjoint operators with symmetric perturbations.
Theorems 2 and 3 describe the poles of the meromorphic continuations of the resolvent in the vicinity of the thresholds $\Lambda_p$ in the essential spectrum. The first theorem states that in the vicinity of an $n$-multiple threshold $\Lambda_p = \cdots = \Lambda_{p+n-1}$, there exist exactly $n$ poles of the operator $R_{\varepsilon, \tau}$ counting their orders. We stress that here we count the orders of the poles and not their multiplicities, that is, not the number of associated linear independent solutions to problem (3) with $f = 0$, $k = k_{ij}(\varepsilon)$. The multiplicity of each pole does not exceed its order; this can be shown by the technique used in the proofs of Lemmata 6.2 and 6.3 in Ref. 61 and Lemmata 6.2 and 6.3 in Ref. 62. However, in general, the multiplicities and the orders coincide only if the perturbation $\mathcal{L}(\varepsilon)$ is symmetric. The reason is that in the general case of a nonsymmetric perturbation, in a certain matrix controlling the structure of the poles $k_{ij}$, a nondiagonal Jordan block can arise and this gives rise to adjoint vectors instead of the eigenvectors, see Section 5 and the calculations involving matrix $M_{\varepsilon, \tau}$. Of course, the multiplicity of each pole $k_{ij, \tau}$ is at least one. In particular, if all poles $k_{ij, \tau}$ are different for a fixed $\tau$, the total multiplicity is equal to $n$. Theorem 2 provides leading terms in the asymptotic expansions for the poles $k_{ij, \tau}$, while Theorem 3 specifies these expansions. In some cases, it just improves the estimate for the error terms, see (16), (18), and (19), while in some cases, a next-to-leading term in the expansions can be found, see (20). Theorems 2 and 3 treat a general case, when the eigenvalues of the matrix $M_1$ are of arbitrary multiplicities and no extra assumptions are made for the matrix $M_{2, \tau}$. In an important particular case, when $\mu_i$ is a simple eigenvalue of the matrix $M_1$, we have $q_i = 1$ and $r_{ij} = 0$. In this case, there exists just one pole $k_{i1, \tau}$ with asymptotic behavior (13) and expansion (16), (20) can be applied, which yields that

$$k_{i1, \tau}(\varepsilon) = \varepsilon \mu_i - \varepsilon^2 \gamma_{i1, \tau} + O(\varepsilon^3).$$

If $n = 1$, that is, $\Lambda_p$ is a simple eigenvalue of the operator $\mathcal{H}'$, the above expansions can be specified as follows:

$$k_{i1, \tau}(\varepsilon) = -\frac{\varepsilon}{2} \int_{\Omega} \frac{\psi_p}{\lambda_p} \lambda_p \psi_p \, dx - \frac{\varepsilon^2}{2} \int_{\Omega} \frac{\psi_p}{\mathcal{L}_2 - \lambda_p \mathcal{L}_1} \psi_p \, dx + O(\varepsilon^3).$$

(37)

We also observe that as the operator $R_{\varepsilon, \tau}$ is independent of $\tau$ if $p = 1$, in the general situation, there are only $n$ poles in the vicinity of the bottom $\Lambda_1$ of the essential spectrum. In the vicinity of internal thresholds $\Lambda_p > \Lambda_1$, the operators $R_{\varepsilon, \tau}$ depend on $\tau$ and this is why there are $2n$ poles in the vicinity of $\Lambda_p$. In particular, if $\Lambda_p$ is an $n$-multiple eigenvalue of the operator $\mathcal{H}'$, there can be $2n$ different simple eigenvalues of the operator $\mathcal{H}_\varepsilon$ converging to $\Lambda_p$, see examples in Subsection 3.3.

The above-discussed poles of the operators $R_{\varepsilon, \tau}$ correspond either to the eigenvalues or resonances depending on the behavior of the associated nontrivial solutions. This behavior is completely described by formulas (7) and (8), and we just need to identify whether the function $\psi_\varepsilon$ decays exponentially at infinity or not. In the former case, we deal with an eigenvalue, otherwise with a resonance. As we see, the functions $\phi_{\varepsilon, j}(x_d)$ decay exponentially as $j \geq p + n$ no matter how the corresponding pole looks like. However, for $j = p, \ldots, p + n - 1$, these functions behave at infinity as $\phi_{\varepsilon, j}(x_d, k) \sim e^{-k_{ij, \tau}|x_d|}$. These functions decay exponentially as $\text{Re} \, k_\varepsilon > 0$, is periodic as $\text{Re} \, k_\varepsilon = 0$ and grows exponentially as $\text{Re} \, k_\varepsilon < 0$. If $\Lambda_p > \Lambda_1$, we also have to control the behavior of the functions $\phi_{\varepsilon, j}(x_d)$ with $j = 1, \ldots, p - 1$. This is easily done by identities (36): the functions $\phi_{\varepsilon, j}(x_d)$, $j = 1, \ldots, p - 1$, decay exponentially if $\text{Im} \, k_\varepsilon^2 < 0$, are periodic if $\text{Im} \, k_\varepsilon^2 = 0$ and grow exponentially if $\text{Im} \, k_\varepsilon^2 > 0$. All discussed conditions can be checked by means of asymptotic expansions provided by Theorems 2 and 3 for a given pole. And exactly this is done in the proof.
of Theorems 4 and 5. Conditions in Theorem 4 are aimed at checking the sign of the real part of a given pole and proving at the same time that at least one of the coefficients $c_{\epsilon,j}^{\pm}, j = 1, \ldots, n$, in (8) is nonzero. Similar conditions (27)–(32) ensure that the functions $\phi_{\epsilon,j}^{\pm}, j = 1, \ldots, p - 1$ decay exponentially, that is, $\tau \text{Im} k_\epsilon^2 < 0$, while for $j = p, \ldots, p + n - 1$, these functions demonstrate either an exponential decay or an exponential growth. Conditions (33)–(35) describe a more gentle situation. Namely, here the real part of the pole is negative and the functions $\phi_{\epsilon,j}^{\pm}, j = p, \ldots, p + n - 1$, decay exponentially. However, $\tau \text{Im} k_\epsilon^2 > 0$ and this means that the functions $\phi_{\epsilon,j}^{\pm}, j = 1, \ldots, p - 1$, can grow exponentially. This is true, once we guarantee that at least one of the coefficients $c_{\epsilon,j}^{\pm}, j = 1, \ldots, p - 1$, is nonzero. This is indeed the case thanks to condition (35). The asymptotic expansions for the eigenvalues and the resonances provided in Theorems 4 and 5 are implied immediately by the formula $\lambda_\epsilon = \Lambda_p - k_\epsilon^2$ relating the eigenvalues/resonances with a pole $k_\epsilon$ and the asymptotic expansions for the poles stated in Theorems 2 and 3.

Let us briefly discuss the main ideas underlying our main results. First, we rather straightforwardly construct the meromorphic continuation for the resolvent of the unperturbed operator. Namely, we find explicitly the projection of the solution to problem (3) on the eigenfunctions $\psi_j, j = 1, \ldots, m$, and study then the properties of the coefficients in this projection and of the remaining orthogonal part in the solution. Once such continuation is constructed, for proving our main results, we apply an approach being a modification of the technique suggested in Refs. 45–47, 63. The idea is to regard the perturbation as a right-hand side and to apply then the meromorphic continuation of the unperturbed operator. After some simple calculations, this leads us to an operator equation with a certain finite rank perturbation. Resolving this equation, we rather easily succeed to construct the meromorphic continuation for the perturbed operator and identify its poles as solutions to a nonlinear eigenvalue problem for some explicitly calculated matrix depending also on the small parameter. Analyzing then this problem by means of methods from the theory of complex functions, we study the existence of the poles and their asymptotic behavior.

Although we restrict ourselves by considering second-order differential operators, our approach can also be adapted for certain operators of higher order. However, general higher order operators can have a richer spectral structure of the edges in the essential spectrum and there can be more complicated scenarios of their bifurcations under perturbations, see, for instance, Ref. 64. This is why the case of second-order operators deserves a separate study, what is done in the present work. Our results are of general nature and are applicable to wide classes of unperturbed operators and perturbations. In the next section, we discuss some possible examples of both unperturbed operators and perturbations as well as some specific examples motivated by physical models.

3 | EXAMPLES

In this section, we provide examples demonstrating our main results.

3.1 | Unperturbed operator

Here we discuss some examples of the unperturbed operator, namely, of the operator $H$. This is a general self-adjoint second-order differential operator and it includes such classical operators as
a Schrödinger operator:

\[ \mathcal{H} = -\Delta + A_0, \quad A_0 = A_0(x'), \]

a magnetic Schrödinger operator:

\[ \mathcal{H} = (i \nabla_{x'} + A)^2 - \frac{\partial^2}{\partial x_d^2} + A_0, \quad A = (A_1, \ldots, A_{d-1}), \quad A_j = A_j(x'), \quad A_0 = A_0(x'), \]

a Schrödinger operator with metric:

\[ \mathcal{H} = -\sum_{i,j=1}^{d-1} \frac{\partial}{\partial x_i} A_{ij} \frac{\partial}{\partial x_j} - \frac{\partial^2}{\partial x_d^2} + A_0, \quad A_{ij} = A_{ij}(x'), \quad A_0 = A_0(x'). \]

All these operators are considered in a tubular domain \( \Omega = \omega \times \mathbb{R} \). If \( \omega = \mathbb{R}^{d-1} \), then the domain \( \Omega \) becomes an entire space \( \mathbb{R}^d \). If \( \omega \) is a bounded domain, not necessarily connected, then \( \Omega \) is an infinite cylinder, which is to be regarded as a quantum waveguide if the Dirichlet condition is imposed on its boundary and as an acoustic waveguide if the boundary is subject to the Neumann condition. Further examples of unbounded domains \( \omega \) are also possible. For instance, if \( \omega \) is the half-space \( \omega = \{ x' : x_j > 0 \} \) in \( \mathbb{R}^{d-1} \) for some \( j = 1, \ldots, d-1 \), the domain \( \Omega \) becomes the half-space \( \{ x : x_j > 0 \} \) in \( \mathbb{R}^d \). We can also consider a more complicated domain \( \omega = \{ x' : x_{d-1} < h(x_1, \ldots, x_{d-2}) \} \) for some smooth function \( h \), then \( \Omega = \{ x : x_{d-1} < h(x_1, \ldots, x_{d-2}), x_d \in \mathbb{R} \} \).

### 3.2 Perturbation

In this subsection, we discuss possible examples of the perturbing operator \( \mathcal{L}(\varepsilon) \). The first example is a second-order differential operator:

\[
\mathcal{L}(\varepsilon) = \sum_{i,j=1}^{n} Y_{ij}(x, \varepsilon) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{j=1}^{n} Y_j(x, \varepsilon) \frac{\partial}{\partial x_j} + Y_0(x, \varepsilon). \tag{38}
\]

Here \( Y_{ij}, Y_j, Y_0 \in L_\infty(\Omega) \) are some functions, not necessarily real-valued, satisfying the representations

\[
Y_{ij}(x, \varepsilon) = Y_{ij}^{(1)}(x) + \varepsilon Y_{ij}^{(2)}(x) + \varepsilon^2 Y_{ij}^{(3)}(x, \varepsilon), \quad \| Y_{ij}^{(s)} \|_{L_\infty(\Omega)} < C, \tag{39}
\]

where \( Y_{ij}^{(s)} \in L_\infty(\Omega) \) are some functions obeying the estimates:

\[
\| Y_{ij}^{(s)} \|_{L_\infty(\Omega)} < C.
\]
and $\mathcal{L}$ and $C$ are some fixed positive constant independent of $\epsilon$. In this case, the operators $\mathcal{L}_s$ read as

$$\mathcal{L}_s := \sum_{i,j=1}^{n} Y_{ij}^{(s)} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{j=1}^{n} Y_j^{(s)} \frac{\partial}{\partial x_j} + Y_0^{(s)}, \quad s = 1, 2, 3.$$  

Particular cases of this example are small potential, small magnetic field, and small metric. The second example is an integral operator of the form

$$(\mathcal{L}(\epsilon)u)(x, \epsilon) = \int_{\Omega} J(x, y, \epsilon) u(y) dy,$$

where $J \in L^2(\Omega \times \Omega)$ is some kernel, not necessarily real-valued and symmetric and satisfying the representation:

$$J(x, y, \epsilon) = J_1(x, y) + \epsilon J_2(x, y) + \epsilon^2 J_3(x, y, \epsilon),$$

where $L_i$ are some functions obeying the estimates:

$$\int_{\Omega \times \Omega} |J_i| e^{\vartheta (|x_d| + |y_d|)} dx dy < C, \quad i = 1, 2, 3,$$

and $\vartheta$ and $C$ are some fixed positive constant independent of $\epsilon$.

The third example is a localized $\delta$-interaction with a complex-valued density. Namely, let $S \subset \Omega$ be a manifold of codimension 1 and of smoothness $C^3$. We assume that it is compact and has no edge. The perturbed operator in question is

$$\mathcal{H}_\epsilon = -\Delta + \epsilon \beta \delta(x - S),$$

which acts as $\mathcal{H}_\epsilon u = -\Delta u$ on the domain $\mathcal{D}(\mathcal{H}_\epsilon)$ formed by the functions $u \in W^2_2(\Omega \setminus S) \cap W^1_2(\Omega)$ obeying the boundary condition $Bu = 0$ on $\partial \Omega$ and the boundary conditions

$$[u]_S = 0, \quad \left[\frac{\partial u}{\partial \nu}\right]_S = \epsilon \beta u.$$  

Here $[u]_S$ denotes the jump of the function on $S$, namely,

$$[u]_S := \lim_{t \to 0^+} (u(\cdot + tv) - u(\cdot - tv)),$$

and $\nu$ is the unit normal to $S$ directed outside the domain enveloped by $S$. By $\beta$ we denote some complex-valued function defined on $S$, uniformly bounded and belonging to $C^2(S)$. Such operator does not satisfy our assumptions for $\mathcal{L}(\epsilon)$ as now the perturbation changes the domain. However, it is possible to reduce the perturbed operator to another one obeying needed assumptions and having the same eigenvalues and resonances. Namely, thanks to the made assumptions on the manifold $S$, in a small vicinity of the $S$, we can introduce a new variable $\rho$ being the distance from a point to $S$ measured along the normal $\nu$. This variable is well defined at least in the neighborhood $\{x : \text{dist}(x, S) < \rho_0\}$ of $S$, where $\rho_0$ is some fixed number. Let $\chi = \chi(\rho)$ be an infinitely
differentiable function such that \( \chi(\rho) = \frac{|\rho|}{2} \) as \( |\rho| < \frac{\rho_0}{3} \) and \( \chi(\rho) = 0 \) as \( |\rho| > \frac{2\rho_0}{3} \). By \( U_\varepsilon \) we denote the multiplication operator \( U_\varepsilon u := (1 + \varepsilon \beta \chi)^{-1} u \). It is straightforward to check that this operator maps the domain of the operator \( H_\varepsilon \) onto the space \( \{ u \in W^2_2(\Omega) : Bu = 0 \text{ on } \partial\Omega \} \). This space serves as the domain for an operator \( \tilde{H}_\varepsilon := U_\varepsilon^* H_\varepsilon U_\varepsilon^{-1} \). It is straightforward to confirm that the differential operator for the latter operator reads as

\[
\tilde{H}_\varepsilon = -\Delta + \varepsilon L(\varepsilon), \quad L(\varepsilon) := -2(1 + \varepsilon \beta \chi)^{-1} \nabla \beta \chi \cdot \nabla - (1 + \varepsilon \beta \chi)^{-1} \Delta \beta \chi
\]

and we see that a first-order differential operator \( L(\varepsilon) \) is a particular case of operator (38). It is also clear that the operators \( H_\varepsilon \) and \( \tilde{H}_\varepsilon \) have the same eigenvalues and resonances because the operator \( U_\varepsilon \) does not change the behavior of the functions at infinity. Hence, we can study the eigenvalues and resonances of the operator \( \tilde{H}_\varepsilon \) and transfer then the results to the operator \( H_\varepsilon \).

Our fourth example is a geometric perturbation. Namely, let \( \omega \) have a nonempty boundary, then the same is true for \( \Omega \). By \( \Gamma \) we denote a bounded subset of the boundary \( \partial\Omega \). Let \( \rho \) be a distance to a point measured along the outward normal to \( \partial\Omega \) and \( h \in C^2(\partial\Omega) \) be some real function defined on \( \Gamma \) and compactly supported in \( \Gamma \). Then, we consider a domain \( \Omega_\varepsilon \) obtained by a small variation of the part \( \Gamma \) of the boundary \( \partial\Omega \). Namely, \( \Omega_\varepsilon \) is a domain with the following boundary:

\[
\partial\Omega_\varepsilon := (\partial\Omega \setminus \Gamma) \cup \{ x : \rho = \varepsilon h \}.
\]

In such domain, we consider an operator with differential expression (1) subject to the Dirichlet boundary condition or Neumann condition. We assume that all the coefficients in the differential expression depend on \( x' \) only and are infinitely differentiable. Such perturbed operator does not fit our scheme since here the domain \( \Omega_\varepsilon \) depends on \( \varepsilon \). However, as in the previous example, it is possible to transform such operator to another one fitting our assumptions. Namely, let \( \chi = \chi(x) \) be an infinitely differentiable cutoff function equaling to one in some fixed sufficiently small \( d \)-dimensional neighborhood of \( \Gamma \) and vanishing outside some bigger neighborhood. In this bigger neighborhood, we introduce local coordinates \( (P, \rho) \), where \( P \in \partial\Omega \). A point \( x \) is recovered from \( (P, \rho) \) by measuring the distance \( \rho \) along the outward normal to \( \partial\Omega \) at the point \( P \). Then, we define a mapping \( P \) by the following rule: for each point \( x \), we find corresponding \( (P, \rho) \) and the action of the mapping is a point corresponding to \( (P, \rho - \varepsilon h(P)) \). We introduce new coordinates by the formula \( x := x(1 - \chi(x)) + \varepsilon \chi(x) P(x) \). It is easy to see that these coordinates are well defined, provided that \( \varepsilon \) is small enough and after passing to these new coordinates, the domain \( \Omega_\varepsilon \) transforms into \( \Omega \), while the operator \( H_\varepsilon \) becomes \( H + L(\varepsilon) \), where \( L(\varepsilon) \) is some second-order differential operator of form (38) with compactly supported coefficients obeying (39).

### 3.3 Emerging poles for particular models

In this section, we apply Theorems 4 and 5 to some simple two- and three-dimensional operators motivated by an interesting physical background.
3.3.1 Planar waveguide

The first model is an infinite planar waveguide modeled by the Dirichlet Laplacian. Namely, we let $d = 2$, $\omega := (0, \pi)$, and $H' = -\frac{d^2}{dx_2^2}$ subject to the Dirichlet boundary condition. Then $\Omega := \{x : 0 < x_1 < \pi\}$ is an infinite strip and $H = -\Delta$ is the Dirichlet Laplacian in $\Omega$. As a perturbation, we choose a complex-valued potential of the form $V(\varepsilon) := V_1 + \varepsilon V_2$, where $V_i = V_i(x)$ are some continuous compactly supported complex-valued functions. The operator $H'$ has a purely discrete spectrum formed by simple eigenvalues $\Lambda_{p} := p^2$, $p \in \mathbb{N}$, and the associated eigenfunctions normalized in $L_2(0, \pi)$ are $\psi_p(x_d) := \sqrt{\frac{2}{\sqrt{\pi}}} \sin px_1$.

This situation models a slab optical waveguide of a finite width, where the cladding in direction $x_1$ imposes zero boundary conditions at $x_1 = 0$ and $x_1 = \pi$. Assuming that the waveguide is infinite in the second direction, a paraxial diffraction of an incident beam can be described using the normalized equation in the form (see, e.g., Ref. 65):

$$i \partial_z \Phi + \Delta \Phi - \varepsilon V_\varepsilon(x)\Phi = 0,$$

where $\Phi(x_1, x_2, z)$ corresponds to complex amplitude of the electrical field, the optical potential $V_\varepsilon(x)$ describes a weak localized modulation of the complex-valued refractive index, and $z$ is the direction of propagation of the pulse. For stationary modes $\Phi = e^{-i\lambda z} \psi$, where $-\lambda$ has the meaning of propagation constant, Equation (40) reduces to the eigenvalue problem in the above-described planar waveguide for the equation

$$-\Delta \psi + \varepsilon V_\varepsilon \psi = \lambda \psi.$$

This is exactly the mathematical model we formulated above once we let $V_\varepsilon = V_1 + \varepsilon V_2$. Let us consider the bifurcation of the thresholds $p^2$ under the presence of a small localized potential $V_\varepsilon$.

For $p = 1$, there is just one pole $k_\varepsilon$ and according to formula (37), its asymptotic behavior reads as

$$k_\varepsilon = -\frac{\varepsilon}{\pi} \int_\Omega V_1(x) \sin^2 x_1 \, dx + \frac{\varepsilon^2}{\pi} \int_\Omega \left( V_2(x) \sin^2 x_1 - V_1(x) U(x) \sin x_1 \right) \, dx + O(\varepsilon^3),$$

where $U := g_{1,\varepsilon}(V_1 \sin x_1)$. This function is given by formula (10). The term $U^\perp := ((H^\perp - \Lambda_1)^{-1} f^\perp)(x)$ with $f = V_1 \sin x_1$ and $f^\perp$ defined by (11) solves the boundary value problem

$$(-\Delta - 1)U^\perp = V_1 \sin x_1 - \frac{2}{\pi} \sin x_1 \int_0^\pi V_1(t_1, x_2) \sin^2 t_1 \, dt_1 \quad \text{in} \quad \Omega, \quad U^\perp = 0 \quad \text{on} \quad \partial \Omega.$$

This problem can be solved explicitly by the separation of variables and this gives the final formula for $U$:

$$U(x) = -\frac{1}{\pi} \sin x_1 \int_\Omega |x_2 - t_2| V_1(t_1, t_2) \sin^2 t_1 \, dt_1$$

$$+ \sum_{j=2}^\infty \frac{\sin jx_1}{\pi \sqrt{j^2 - 1}} \int_\Omega e^{-\sqrt{j^2 - 1}|x_2 - t_2|} V_1(t_1, t_2) \sin t_1 \sin jt_1 \, dt_1.$$
Hence,
\[
\int_{\Omega} \left( V_2(x) \sin^2 x_1 - V_1(x)U(x) \sin x_1 \right) dx \\
= \int_{\Omega} V_2(x) \sin^2 x_1 \, dx + \frac{1}{\pi} \int_{\Omega^2} |x_2 - t_2| V_1(x)V_1(t) \sin^2 t_1 \sin^2 x_1 \, dt \, dx \\
- \sum_{j=2}^{\infty} \frac{1}{\pi \sqrt{j^2 - 1}} \int_{\Omega^2} e^{-\sqrt{j^2 - 1}|x_2 - t_2|} V_1(t) V_1(x) \sin t_1 \sin t_1 \sin jx_1 \sin jx_1 \, dt \, dx.
\]

Now we apply Theorem 4 and we see that if
\[
\text{Re} \int_{\Omega} V_1(x) \sin^2 x_1 \, dx < 0
\]
or
\[
\text{Re} \int_{\Omega} V_1(x) \sin^2 x_1 \, dx = 0 \text{ and } \text{Re} \int_{\Omega} \left( V_2(x) \sin^2 x_1 - V_1(x)U(x) \sin x_1 \right) dx < 0,
\]
then the pole $k_\varepsilon$ corresponds to an eigenvalue $\lambda_\varepsilon = 1 - k_\varepsilon^2$. And if
\[
\text{Re} \int_{\Omega} V_1(x) \sin^2 x_1 \, dx > 0
\]
or
\[
\text{Re} \int_{\Omega} V_1(x) \sin^2 x_1 \, dx = 0 \text{ and } \text{Re} \int_{\Omega} \left( V_2(x) \sin^2 x_1 - V_1(x)U(x) \sin x_1 \right) dx > 0,
\]
then the pole $k_\varepsilon$ corresponds to a resonance $\lambda_\varepsilon = 1 - k_\varepsilon^2$. The asymptotic expansion for this eigenvalue/resonance is given by (23) and (26), but it is more straightforward to find it by (41) and the above formula for $\lambda_\varepsilon$.

We proceed to the case $p > 1$. Here we again apply formula (37) to obtain
\[
k_{\varepsilon,\tau} = -\frac{\varepsilon}{\pi} \int_{\Omega} V_1(x) \sin^2 px_1 \, dx - \frac{\varepsilon^2}{\pi} \int_{\Omega} \left( V_2(x) \sin^2 px_1 - V_1(x)U_\tau(x) \sin px_1 \right) dx + O(\varepsilon^3),
\]
where $U_\tau := G_\tau V_1 \sin px_1$ is given by formula (10). The term $U_\tau^\perp := (\mathcal{H}_\tau^\perp - \Lambda_p)^{-1} f^\perp(x)$ with $f = V_1 \sin px_1$ and $f^\perp$ defined by (11) solves the boundary value problem
\[
(-\Delta - p^2)U_\tau^\perp = V_1 \sin px_1 - \frac{2}{\pi} \sin px_1 \int_{0}^{\pi} V_1(t_1, x_2) \sin^2 pt_1 \, dt_1 \text{ in } \Omega, \quad U_\tau^\perp = 0 \text{ on } \partial\Omega.
\]

The solution is again given by the separation of variables and a final formula for $U_\tau$ reads as
\[
U_\tau(x) = \sum_{j=1}^{\nu} \frac{i \tau \sin jx_1}{\pi \sqrt{p^2 - j^2}} \int_{\mathbb{R}} e^{i \tau \sqrt{p^2 - j^2} |x_2 - t_2|} U_j(t_2) \, dt_2.
\]
\[- \frac{1}{\pi} \sin px_1 \int_{\mathbb{R}} |x_2 - t_2| U_p(t_2) dt_2 \]
\[+ \sum_{j=p+1}^{\infty} \frac{\sin jx_1}{\pi \sqrt{j^2 - p^2}} \int_{\mathbb{R}} e^{-\sqrt{j^2 - p^2}|x_2 - t_2|} U_j(t_2) dt_2. \]

\[U_j(x_2) := \int_{0}^{\pi} V_1(t_1, x_2) \sin pt_1 \sin jt_1 dt, \quad j \neq p, \quad (43)\]

Then, we get:

\[
\int_{\Omega} V_1(x) U_\tau(x) \sin px_1 \, dx = \sum_{j=1}^{p-1} \int_{\mathbb{R}^2} \frac{i\tau e^{i\tau \sqrt{p^2 - j^2}|x_2 - t_2|}}{\pi \sqrt{p^2 - j^2}} U_j(x_2) U_j(t_2) \, dx_2 \, dt_2
\]
\[- \frac{1}{\pi} \int_{\mathbb{R}^2} |x_2 - t_2| U_p(x_2) U_p(t_2) \, dx_2 \, dt_2 \]
\[+ \sum_{j=p+1}^{\infty} \int_{\mathbb{R}^2} e^{-\sqrt{j^2 - p^2}|x_2 - t_2|} U_j(x_2) U_j(t_2) \, dt_2 \, dx_2. \quad (44)\]

Now we can apply Theorem 5 for \(\tau = +1\) and \(\tau = -1\) and to determine whether the poles \(k_{\varepsilon, \tau}\) correspond to eigenvalues or resonances. As we see, in a general situation, we can have two eigenvalues or two resonances or one eigenvalue and one resonance. Let us show that each of these situations is possible.

First of all, we observe that in notations of Theorem 5, we have \(N = 1, q_i = 1, r_i = 0,\)

\[\mu_1 = -\frac{1}{\pi} \int_{\Omega} V_1(x) \sin^2 px_1 \, dx,\]

\[-\gamma_{1,\tau} \frac{1}{\eta_{\gamma,1,\tau}} e^{\frac{\pi i}{\eta_{\gamma,1,\tau}} (j-\tau,\tau)} = -\frac{1}{\pi} \int_{\Omega} \left( V_2(x) \sin^2 px_1 - V_1(x) U_{1,\tau}(x) \sin px_1 \right) \, dx.\]

Assume now that \(V_1\) is a complex-valued potential such that

\[\text{Re} \int_{\Omega} V_1(x) \sin^2 px_1 \, dx > 0.\]

Then condition (31) is satisfied and both poles \(k_{\varepsilon, \tau}, \tau = \{-1, +1\}\) correspond to resonances. If

\[\text{Re} \int_{\Omega} V_1(x) \sin^2 px_1 \, dx < 0, \quad \text{Im} \int_{\Omega} V_1(x) \sin^2 px_1 \, dx \neq 0,\]

then conditions (27) and (29) are satisfied with

\[\tau = \text{sgn} \text{Im} \int_{\Omega} V_1(x) \sin^2 px_1 \, dx\]
and for such \( \tau \), the pole \( k_{\varepsilon, \tau} \) corresponds to an eigenvalue. If in addition, 
\[
\int_{\Omega} e^{-i\sqrt{p^2 - s^2} x_2} V_1(x) \sin sx_1 \sin px_1 \, dx \neq 0 \quad \text{or} \quad \int_{\Omega} e^{i\sqrt{p^2 - s^2} x_2} V_1(x) \sin sx_1 \sin px_1 \, dx \neq 0,
\]
for some \( s \in \{1, \ldots, p - 1\} \), then conditions (33) and (35) are satisfied and the pole \( k_{\varepsilon, \tau} \) with
\[
\tau = -\text{sgn} \text{Im} \int_{\Omega} V_1(x) \sin^2 px_1 \, dx
\]
corresponds to a resonance.

To realize a situation with two eigenvalues, we assume that \( V_2 = 0 \) and consider a special class of \( \mathcal{PT} \)-symmetric potentials \( V_1 \). Namely, we suppose that
\[
V_1(x) = W_1(x) + iW_2(x), \quad (45)
\]
where \( W_1 \) and \( W_2 \) are real-valued compactly supported potentials with certain parity:
\[
W_1(x_1, -x_2) = W_1(x_1, x_2), \quad W_2(x_1, -x_2) = -W_2(x_1, x_2). \quad (46)
\]

These assumptions yield that the operator \( \mathcal{H}_\varepsilon \) is \( \mathcal{PT} \)-symmetric (or partially \( \mathcal{PT} \)-symmetric using the terminology from Ref. 66). They also imply immediately that
\[
\int_{\Omega} V_1(x) \sin^2 px_1 \, dx = \int_{\Omega} W_1(x) \sin^2 px_1 \, dx
\]
and we assume that
\[
\int_{\Omega} W_1(x) \sin^2 px_1 \, dx < 0. \quad (47)
\]

It follows from assumptions (46) and the definition of the functions \( U_j \) in (43) that these functions are given by the formulas
\[
U_j(x_2) := W_{1,j}(x_2) + iW_{2,j}(x_2), \quad W_{s,j}(x_2) := \int_0^\pi W_s(t_1, x_2) \sin pt_1 \sin jt_1 \, dt, \quad s = 1, 2,
\]
and the functions \( W_{1,j} \) are even, while \( W_{2,j} \) are odd. As \( j \geq p + 1 \), by making the change of the variables \( x_2 \mapsto -x_2, t_2 \mapsto -t_2 \) in the integrals in the second sum in (44), we get:
\[
\int_{\mathbb{R}^2} \frac{e^{-\sqrt{j^2 - p^2} |x_2 - t_2|}}{\sqrt{j^2 - p^2}} U_j(x_2)U_j(t_2) \, dt_2 \, dx_2 = \int_{\mathbb{R}^2} \frac{e^{-\sqrt{j^2 - p^2} |x_2 - t_2|}}{\sqrt{j^2 - p^2}} \frac{U_j(x_2)U_j(t_2)}{U_j(x_2)U_j(t_2)} \, dt_2 \, dx_2
\]
and hence,
\[
\text{Im} \sum_{j=p+1}^{\infty} \int_{\mathbb{R}^2} \frac{e^{-\sqrt{j^2 - p^2} |x_2 - t_2|}}{\sqrt{j^2 - p^2}} U_j(x_2)U_j(t_2) \, dt_2 \, dx_2 = 0.
\]
In the same way, we confirm that
\[
\text{Im} \sum_{j=1}^{p-1} \frac{\tau \sin \tau \sqrt{p^2 - j^2} |x_2 - t_2|}{\pi \sqrt{p^2 - j^2}} U_j(x_2)U_j(t_2) \, dt_2 \, dx_2 = 0,
\]
\[
\text{Im} \frac{1}{\pi} \int_{\mathbb{R}^2} |x_2 - t_2| U_p(x_2)U_p(t_2) \, dx_2 \, dt_2 = 0.
\]
Hence, by two above identities and (44),
\[
\text{Im} \int_{\Omega} V_1(x)U_\tau(x) \sin px_1 \, dx = \text{Im} \sum_{j=1}^{p-1} \frac{\tau \cos \tau \sqrt{p^2 - j^2} |x_2 - t_2|}{\pi \sqrt{p^2 - j^2}} U_j(x_2)U_j(t_2) \, dt_2 \, dx_2
\]
\[
= \tau \text{Re} \sum_{j=1}^{p-1} \frac{\cos \sqrt{p^2 - j^2} |x_2 - t_2|}{\pi \sqrt{p^2 - j^2}} U_j(x_2)U_j(t_2) \, dt_2 \, dx_2
\]
\[
= \tau \sum_{j=1}^{p-1} \frac{\cos \sqrt{p^2 - j^2} |x_2 - t_2|}{\pi \sqrt{p^2 - j^2}} (W_{1,j}(x_2)W_{1,j}(t_2) - W_{2,j}(x_2)W_{2,j}(t_2)) \, dt_2 \, dx_2.
\]
By straightforward calculations, for an arbitrary compactly supported function $W(x_2)$, we obtain:
\[
\int_{\mathbb{R}^2} \cos \sqrt{p^2 - j^2} (x_2 - t_2)W(x_2)W(t_2) \, dt_2 \, dx_2 = \left( \int_{\mathbb{R}} W(x_2) \cos \sqrt{p^2 - j^2} x_2 \, dx_2 \right)^2
\]
\[
+ \left( \int_{\mathbb{R}} W(x_2) \sin \sqrt{p^2 - j^2} x_2 \, dx_2 \right)^2.
\]
Hence, by two latter identities and (46),
\[
\tau \text{Im} \int_{\Omega} V_1(x)U_\tau(x) \sin px_1 \, dx = \frac{1}{\pi} \sum_{j=1}^{p-1} \frac{1}{\sqrt{p^2 - j^2}} \left( \left( \int_{\mathbb{R}} W_{1,j}(x_2) \cos \sqrt{p^2 - j^2} x_2 \, dx_2 \right)^2 \right)
\]
\[
- \left( \int_{\mathbb{R}} W_{2,j}(x_2) \sin \sqrt{p^2 - j^2} x_2 \, dx_2 \right)^2.< 0.
\]
The latter formula implies that the sign of its left-hand side is the same for both $\tau \in \{-1, +1\}$ and we can make this sign being $-1$ by choosing appropriately $W_2$ once we fix $W_1$ satisfying (47), namely, we can satisfy the condition
\[
\sum_{j=1}^{p-1} \frac{1}{\sqrt{p^2 - j^2}} \left( \left( \int_{\mathbb{R}} W_{1,j}(x_2) \cos \sqrt{p^2 - j^2} x_2 \, dx_2 \right)^2 \right)
\]
\[
- \left( \int_{\mathbb{R}} W_{2,j}(x_2) \sin \sqrt{p^2 - j^2} x_2 \, dx_2 \right)^2 < 0.
\]
For instance, this can be done by letting \( W_2 = \alpha W_2 \) with a sufficient large \( \alpha \), where \( W_2 = W_2(x) \) is a real odd compactly supported function such that

\[
\sum_{j=1}^{p-1} \frac{1}{\sqrt{p^2 - j^2}} \left( \int_{\Omega} W_2(x) \sin px_1 \sin jx_1 \sin \sqrt{p^2 - j^2}x_2 \, dx \right)^2 > 0.
\]

Once conditions (47) and (48) hold, Theorem 5 states that both poles \( k_{\varepsilon,\tau}, \tau \in \{-1, +1\} \) correspond to the eigenvalues located in the vicinity of the internal threshold \( \Lambda_p \).

The above analytic results, namely, the discussed asymptotic expansions, approximate well the true eigenvalues and resonances for sufficiently small \( \varepsilon \). To demonstrate how small \( \varepsilon \) is to be chosen, we make some numerical computations.

For numerics, we use

\[
W_1(x) = -\sum_{j=1}^{3} a_j \sin jx_1 \cos \frac{x_2}{2}, \quad W_2(x) = \sum_{j=1}^{3} b_j \sin jx_1 \sin x_2,
\]

where \( a_j \) and \( b_j \) are real coefficients, and we additionally let \( W_i(x) \equiv 0 \) as \( |x_2| > \pi, i = 1, 2 \). The corresponding eigenvalue problem is approximated using a second-difference numerical scheme with Dirichlet boundary conditions at \( x_1 = 0 \) and \( x_1 = \pi \) and a decay condition at \( x_2 \to \pm \infty \). To achieve a numerically efficient approximation of decay condition at \( x_2 \to \pm \infty \), a quasi-equidistant grid is used with a step size gradually increasing toward \( x_2 \to \pm \infty \). For small \( \varepsilon \ll 0.2 \), where the localization of the eigenfunctions in \( x_2 \)-direction is extremely weak, and an adequate approximation of the decay condition as \( x_2 \to \pm \infty \) is practically impossible, we use the Neumann condition to approximate slowly decaying oscillating tails of the eigenfunctions:

\[
\partial_{x_2} \Psi(x_1, \pm X_0) = 0, \quad \text{where} \quad X_0 \gg 1.
\]

In Figure 1, we plot the dependencies \( \Lambda_1(\varepsilon) \) obtained from the asymptotic expansions, when only the leading term is taken into account and both terms are used, for the particular set of parameters \( a_1 = 1, b_1 = 0.1, \) and \( a_2 = b_2 = 0 \). For \( \varepsilon \ll 0.2 \), the agreement between the analytical predictions and numerical results is rather good, while for large values of \( \varepsilon \), it is only qualitative. In Figure 2, we plot the same data for two eigenvalues bifurcating from the internal threshold with \( p = 2 \); here only the eigenvalue with positive imaginary part is shown. The following set of parameters is used: \( a_1 = 1, b_1 = 0, a_2 = 0, \) and \( b_2 = 3 \). Again, the numerical results are in a good agreement with asymptotic expansions for weak perturbations, and in a qualitative agreement for stronger ones. Real part of the eigenvalue is found to decrease with the increase of \( \varepsilon \). Nevertheless, for all values of \( \varepsilon \) shown in Figure 2, the real part of eigenvalue \( \Lambda_2(\varepsilon) \) is larger than the lower edge of the essential spectrum. Therefore, in the optical context, the corresponding eigenfunction, whose three-dimensional modulus plot is shown in Figure 2(D), indeed represents a non-Hermitian generalization of a bound state in the continuum.

### 3.3.2 Two-dimensional Bose–Einstein condensate with parabolic trapping

In the previous example, a choice of \( \omega \) was not really important and the discussed results are of a more general nature. For instance, we can choose the operator \( \mathcal{H}' \) being a quantum harmonic
oscillator. Namely, let $\omega = \mathbb{R}$ and

$$
\mathcal{H} := -\frac{d^2}{dx_1^2} + x_1^2 \quad \text{on} \quad \mathbb{R}.
$$

Then $\Omega = \mathbb{R}^2$ and the operator $\mathcal{H}$ becomes

$$
\mathcal{H} = -\Delta + x_1^2 \quad \text{on} \quad \mathbb{R}^2.
$$

As a perturbation, we choose $\mathcal{L}(\varepsilon) = V_1$, where $V_1$ is a complex-valued compactly supported potential on $\mathbb{R}^2$. Here,

$$
\Lambda_p = 2p + 1, \quad \psi_p(x_1) = \frac{e^{-x_1^2/2}}{\sqrt{2^p p! \sqrt{\pi}}} H_p(x_1), \quad H_n(t) := (-1)^n e^{t^2} \frac{d^n}{dt^n} e^{-t^2},
$$

i.e., $H_p$ are Hermite polynomials, and $p = 0, 1, \ldots$. Then, all calculations and results from previous example can be easily reproduced, just in all formulas, the functions $\frac{\sqrt{2}}{\sqrt{\pi}} \sin j x_1$ are to be replaced by the functions $\psi_j(x_1)$ introduced above.

The described situation corresponds to a two-dimensional cloud of Bose–Einstein condensate with a parabolic confinement in $x_1$ direction. Assuming that the interparticle interactions are negligible, that is, the condensate is effectively linear, we can model its dynamics by the Schrödinger-like equation, which in the theory of Bose–Einstein condensates is known as Gross–Pitaevskii equation:

$$
i \partial_t \Phi + \Delta \Phi - x_1^2 \Phi - \varepsilon V(x) \Phi = 0,
$$

Figure 1. (A) Eigenvalue $\Lambda_1(\varepsilon)$ emerging from the bottom of the spectrum computed using only the first term in expansion (41) (blue curve) and two terms in (41) (green curve). Red points connected by red lines are obtained from direct numerical solution of the eigenvalue problem. (B) The real and imaginary parts of the eigenfunction at $\varepsilon = 0.1$ plotted as a function of $x_2$ for $x_1 = \pi/2$. Here $a_1 = 1$, $a_3 = 4$, $b_1 = 0.5$, and all other coefficients are zero.
FIGURE 2 The real and imaginary parts of the eigenvalue $\Lambda_2(\varepsilon)$ emerging from an internal threshold in the essential spectrum computed using only the first term in expansion (42) (blue curve) and two terms in (42) (green curve). Red points connected by red lines are obtained from direct numerical evaluation of the spectrum. (C) The real and imaginary parts of the eigenfunction at $\varepsilon = 0.3$ plotted as a function of $x_2$ for $x_1 = \pi/4$. Two insets show the plots of the squared amplitude of the eigenfunction $|\Psi|^2 = (\text{Re}\, \Psi)^2 + (\text{Im}\, \Psi)^2$, using larger domains in the horizontal axes and linear (left inset) and logarithmic (right inset) scales in the vertical axes. (D) Full plot of the modulus of the eigenfunction. In all panels $a_1 = 1$, $b_2 = 3$, and all other coefficients $a_i$ and $b_i$ are zero. Only the eigenvalue with a positive imaginary part is shown. There also exists a complex-conjugate eigenvalue with a negative imaginary part and a $\mathcal{PT}$-conjugate eigenfunction.
where $\Phi(x_1, x_2, t)$ stands for the macroscopic wave function of the condensate. Again, for stationary states in the form $\Phi = e^{-i\lambda t}\Psi$, where $\lambda$ has the meaning of the chemical potential, the problem is reduced to the described eigenvalue problem. Positive and negative imaginary parts of the perturbation correspond to inject the particles from an external source and absorption of the particles, respectively.

### 3.3.3 Three-dimensional circular waveguide and three-dimensional Bose–Einstein condensate

Here we consider two examples of a three-dimensional waveguide and a three-dimensional quantum oscillator, which extend the above examples of the planar waveguide and the harmonic oscillator adduced above.

In first example, we deal with a circular waveguide assuming that $\omega = \{ x' = (x_1, x_2) : |x'| < 1 \}$ and the operator $\mathcal{H}'$ is introduced as the Schrödinger operator with a radially symmetric potential subject to the Dirichlet condition:

$$\mathcal{H}' = -\Delta_{x'} + V_0 \quad \text{in} \quad \omega, \quad V_0 = V_0(|x'|).$$

Then, the domain $\Omega$ is a straight cylinder along the axis $x_3$ with the cross-section $\omega$ and

$$\mathcal{H} = -\Delta + V_0(|x'|) \quad \text{in} \quad \Omega$$

subject to the homogeneous Dirichlet condition. The operator $\mathcal{H}'$ has a purely discrete spectrum and thanks to the assumed radial symmetricity for the potential $V_0$, it possesses double eigenvalues $\Lambda_p = \Lambda_{p+1} > \Lambda_1$ such that the associated eigenfunctions, orthonormalized in $L^2(\omega)$, read as $\psi_p(x') = \Psi(|x'|) \cos s \theta$, $\psi_{p+1}(x') = \Psi(|x'|) \sin s \theta$, where $s$ is some fixed integer number, $\Psi$ is some real function, and $\theta$ is a polar angle associated with $x'$. Such eigenvalues are degenerate internal thresholds in the essential spectrum. We define then a $\mathcal{PT}$-symmetric potential $V_1$ by formula (45) and in addition, we assume that $W_1$ is even in $x_1$. Then, it is easy to see that the corresponding matrix $M_1$ is diagonal:

$$M_1 = \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix}, \quad \mu_1 := -\frac{1}{2} \int_\Omega W_1 \psi_p^2 \, dx, \quad \mu_2 := -\frac{1}{2} \int_\Omega W_1 \psi_{p+1}^2 \, dx.$$

Then, we assume that $\mu_1 \neq \mu_2$ and $\mu_1 < 0, \mu_2 < 0$; these conditions can be easily satisfied by choosing appropriately $W_1$. Since for both $\mu_i$, we have $q_i = 1$, for each corresponding pole, we can apply asymptotic expansion (37):

$$k_{i,\tau}(\varepsilon) = \varepsilon \mu_i + \frac{\varepsilon^2}{2} \int_\Omega V_1(x) U_{i,\tau}(x) \psi_{i-1+p} \, dx + O(\varepsilon^3),$$

where $U_{i,\tau} := G_{i,\tau} V_1 \psi_{i-1+p}$ are solutions to the boundary value problem

$$(-\Delta - \Lambda_p) U_{i,\tau} = V_1 \psi_{i-1+p} - \psi_{i-1+p} \int_\Omega V_1(t', x_3) \psi_p^2(t') \, dt' \, dx_3 \quad \text{in} \quad \Omega, \quad U_{i,\tau} = 0 \quad \text{on} \quad \partial \Omega,$$
and can be found by a separation of variables similar to (43). Then, we can reproduce calculations from the first example and obtain that choosing appropriately the function $W_2$, we can satisfy conditions (27) and (30) for all poles $k_{1,\tau}$, $i = 1, 2$, $\tau \in \{-1, +1\}$, and this means that these poles correspond to eigenvalues. In other words, this means that for appropriately chosen $W_1$ and $W_2$, we can generate four different simple eigenvalues of the operator $H_\varepsilon$ in the vicinity of the double eigenvalue $\Lambda_1$ of the operator $H'$ serving as an internal threshold in the essential spectrum.

A similar situation can also be realized in other three-dimensional models. For instance, we can let $\omega = \mathbb{R}^2$ and $H' = -\Delta x' + V_0(|x'|)$, where $V_0 = V_0(t)$ is some function growing unboundedly at infinity. Such operator $H'$ again can have double eigenvalues with the eigenfunctions of form $\Psi(|x'|) \cos s\theta$ and $\Psi(|x'|) \sin s\theta$. Linear combinations $\Psi(|x'|) \cos s\theta \pm i\Psi(|x'|) \sin s\theta$ correspond to vortex states with integer $s$ being the vorticity or topological charge. Therefore, in the context of Bose–Einstein condensates, this mechanism can be potentially applied for generation of localized in all three spatial dimensions vortex rings (see, e.g., Ref. 69).

We also observe that in both discussed three-dimensional examples, the operator $H'$ can have not only double eigenvalues, but also ones of higher multiplicities $n$. And in such cases, it is possible to find a $\mathcal{PT}$-symmetric potential $V_1$ generating $2n$ eigenvalues of the operator $H_\varepsilon$ in the vicinity of the considered multiple eigenvalue.

4 | MEROMORPHIC CONTINUATION

In this section, we prove Theorem 1 on the meromorphic continuation of the resolvent of the operator $H_\varepsilon$. First, we prove some auxiliary statements in a separate subsection, and then, we prove the theorem.

4.1 | Auxiliary lemmata

In this subsection, we prove three auxiliary lemmata, which will be employed in the proof of Theorem 1. The first statement is Lemma 1.

Proof of Lemma 1. Reproducing literally the proof of Lemma 2.3 in Ref. 48, one can check easily the identity $\sigma_{\text{ess}}(H_\varepsilon) = \sigma_{\text{ess}}(H)$. The operator $H$ can be represented as a sum of tensor products $H = H' \otimes I + I \otimes H_0$. And since

$$\sigma_{\text{ess}}(H_0) = \sigma(H_0) = [0, +\infty), \quad \inf \sigma(H') = \Lambda_1,$$

we immediately get:

$$\sigma(H) = \sigma_{\text{ess}}(H) = [\Lambda_1, +\infty).$$

This completes the proof. ■

The next statement is Lemma 2.
Proof of Lemma 2. For each \( u \in \mathfrak{D}(H) \cap L^\perp \), each \( \psi_j \), and almost each \( x_d \in \mathbb{R} \), we have:

\[
\left( (H^\perp u)(\cdot, x_d), \psi_j \right)_{L_2(\omega)} = \mathfrak{h}'(u(\cdot, x_d), \psi_j) - \left( \frac{\partial^2 u}{\partial x_d^2}(\cdot, x_d), \psi_j \right)_{L_2(\omega)} = -\Lambda_j(u(\cdot, x_d), \psi_j)_{L_2(\omega)} - \frac{d^2}{dx_d^2} \left( u(\cdot, x_d), \psi_j \right)_{L_2(\omega)} = 0.
\]

Hence, the operator \( H^\perp \) maps \( \mathfrak{D}(H) \cap L^\perp \) into \( L^\perp \). This is an unbounded self-adjoint operator in \( L^\perp \) associated with the restriction of the form \( \mathfrak{h} \) on \( \mathfrak{D}(\mathfrak{h}) \cap L^\perp \). Moreover, for each \( u \in \mathfrak{D}(\mathfrak{h}) \cap L^\perp \), we have:

\[
\mathfrak{h}(u, u) = \int_\mathbb{R} \mathfrak{h}'(u(\cdot, x_d), u(\cdot, x_d)) dx_d + \int_\Omega \left| \frac{\partial u}{\partial x_d} \right|^2 dx \geq c_0 \int_{\mathbb{R}^d} \| u(\cdot, x_d) \|^2_{L_2(\omega)} dx_d = c_0 \| u \|^2_{L_2(\Omega)}.
\]

Hence, the spectrum of the operator \( H^\perp \) is located in \([c_0, +\infty)\). The proof is complete. \( \blacksquare \)

The third lemma provides a meromorphic continuation for the resolvent of the unperturbed operator.

**Lemma 3.** Fix \( p \in \{1, \ldots, m\} \) and \( \tau \in \{-1, +1\} \) and let \( \Lambda_p = \ldots = \Lambda_{p+n-1} \) be an \( n \)-multiple eigenvalue of the operator \( H' \), where \( n \geq 1 \). There exists a sufficiently small fixed \( \delta > 0 \) such that for all complex \( k \in B_\delta \) and all \( f \in L_2(\Omega, e^{\delta|x|} dx) \), the boundary value problem

\[
\left( -\sum_{i,j=1}^{d-1} \frac{\partial}{\partial x_i} A_{ij} \frac{\partial}{\partial x_j} + i \sum_{j=1}^{d-1} \left( A_{ij} \frac{\partial}{\partial x_i} + \frac{\partial}{\partial x_j} A_{ij} \right) + A_0 - \Lambda_p + k^2 \right) u = f \ \text{in} \ \Omega, \quad Bu = 0 \ \text{on} \ \partial\Omega, \quad (50)
\]

is solvable in \( W^{2,\infty}_{2,loc}(\Omega) \) and possesses a solution, which can be represented as

\[
u = A_{1,\tau}(k)f, \quad (51)
\]

where \( A_{1,\tau} \) is a linear operator mapping \( L_2(\Omega, e^{\delta|x|} dx) \) into \( W^{2,\infty}_{2,\infty}(\Omega, e^{-\delta|x|} dx) \). This operator is bounded and meromorphic in \( k \in B_\delta \), which is at zero and simple:

\[
A_{1,\tau}(k) = \frac{1}{k} A_2 + G_{p,\tau} + k A_{4,\tau}(k), \quad (52)
\]

\[
A_2 f := \sum_{j=p}^{p+n-1} \psi_j \ell_j f, \quad \ell_j f := \frac{1}{2} \int_\Omega \psi_j(x') f(x) dx, \quad (53)
\]
where $A_{±,x}(k) : L_2(Ω, e^{δ|x_d|}dx) → W^2_2(Ω, e^{−δ|x_d|}dx)$ is an operator bounded uniformly in $k ∈ B_δ$ and holomorphic in $k ∈ B_δ$, and, we recall, $G_{p,τ}$ is the operator defined in (10).

For sufficiently large $x_d$, the solution $u$ given by (51) can be represented as

$$u(x, k) = \sum_{j=1}^{m} u_j^+(x_d, k)\psi_j(x_d) + u_+^-(x, k), \quad ±x_d > R, \quad (54)$$

where $R$ is some fixed number, $u_j^± ∈ L_2(I_±, e^{±δx_d}dx_d)$ are some meromorphic in $k ∈ B_δ$ functions possessing the asymptotic behavior

$$u_j^+(x_d) = e^{−τK_j(k)|x_d|}\left(C_j^±(k) + O(e^{−δ|x_d|})\right), \quad |x_d| → \infty, \quad j = 1, ..., p − 1,$$

$$u_j^+(x_d) = e^{−K_j(k)|x_d|}\left(C_j^±(k) + O(e^{−δ|x_d|})\right), \quad |x_d| → \infty, \quad j = p, ..., m, \quad (55)$$

$C_j^±(k)$ are some meromorphic in $k ∈ B_δ$ functions, $0 < ̄δ < δ$ is some fixed constant independent of $k$ and $x$, and $u_+^± ∈ W^2_2(Ω_±)$ are some functions meromorphic in $k ∈ B_δ$ and obeying the identities

$$(u_+^±(⋅, x_d), \psi_j)_{L_2(Ω)} = 0 \quad (56)$$

for almost each $x_d ∈ I_±$ and for each $j = 1, ..., m$.

**Proof.** The functions

$$f_j(x_d) := (f(⋅, x_d), \psi_j)_{L_2(Ω)}$$

obviously belong to $L_2(ℝ, e^{δ|x_d|}dx_d)$ since $f ∈ L_2(Ω, e^{δ|x_d|}dx)$. It is straightforward to check that the functions

$$u_j(x_d, k) := \frac{1}{2τK_j(k)} \int_ℝ e^{−τK_j(k)|y_d|} e^{−K_j(k)|x_d|} f_j(y_d) dy_d \quad \text{as} \quad j < p − 1,$$

$$u_j(x_d, k) := \frac{1}{2K_j(k)} \int_ℝ e^{−K_j(k)|y_d|} e^{−K_j(k)|x_d|} f_j(y_d) dy_d \quad \text{as} \quad j ≥ p, \quad (57)$$

solve the equations

$$-u_j'' + (Λ_j − Λ_p + k^2)u_j = f_j \quad \text{in} \quad ℝ, \quad j = 1, ..., m.$$ 

Employing these facts, we seek a solution to problem (50) as

$$u(x, k) = \sum_{j=1}^{m} u_j(x'_d, k)\psi_j(x_d') + u_+^-(x, k) \quad (58)$$

and for $u_+$, we immediately get problem (50) with $f$ replaced by the function $f_±$ defined in (11). We see easily that $f_± ∈ L_+$. 


Thanks to Lemma 2, the resolvent \((H^\perp - \Lambda_p + k^2)^{-1}\) is well defined for all sufficiently small complex \(k \in B_5\), provided that \(\delta\) is small enough. This resolvent is holomorphic in \(k \in B_5\) as an operator in \(L_2(\Omega)\) and is uniformly bounded in \(k \in B_5\). It can be expanded via the standard Neumann series:

\[
(H^\perp - \Lambda_p + k^2)^{-1} = \sum_{j=0}^{\infty} (-k^2)^j ((H^\perp - \Lambda_p)^{-1})^{j+1}.
\] (59)

This expansion implies that the resolvent \((H^\perp - \Lambda_p + k^2)^{-1}\) is also holomorphic as an operator from \(L_2(\Omega)\) into \(W^2_2(\Omega)\). We define then

\[
u_\perp := (H^\perp - \Lambda_p + k^2)^{-1} f^\perp,
\]

and this obviously gives a solution to problem (50). We denote the operator mapping \(f\) into the described solution by \(A_{\perp}(k)\), and let us show that it possesses all stated properties.

First of all, we observe that the introduced operator is independent of the choice of \(\tau\) as \(\Lambda_p = \Lambda_1\) because in this case, the functions \(u_j\) with \(\Lambda_j < \Lambda_p\) are missing and the above constructions become independent of \(\tau\).

Just by the embeddings \(L_2(\Omega, e^{\vartheta|x|}dx) \subset L_2(\Omega)\) and \(W^2_2(\Omega, e^{-\vartheta|x|}dx), W^2_2(\Omega, e^{-\vartheta|x|}dx)\) holomorphic in \(k \in B_5\) for sufficiently small \(\delta\) and bounded uniformly in \(k \in B_5\). An operator mapping \(f\) into \(\sum_{j=1}^{m} u_j(x_d, k)\psi_j(x')\) is given explicitly and by straightforward calculations, we can check that this is also a bounded operator from \(L_2(\Omega, e^{\vartheta|x|}dx)\) into \(W^2_2(\Omega, e^{-\vartheta|x|}dx)\) holomorphic in \(k \in B_5\). The calculations are based on estimates of the following kind:

\[
\|u_j \psi_j\|^2_{L_2(\Omega, e^{\vartheta|x|}dx)} = \|u_j\|^2_{L_2(\mathbb{R}, e^{\vartheta|x|}dx)} \leq \frac{\|f\|^2_{L_2(\Omega, e^{\vartheta|x|}dx)}}{2|K_j|^2} \int_{\Omega^2} e^{-|x_d-y_d| \tau \Re K_j(k)-a(|x_d|+|y_d|)} dx_d dy_d \leq C(k)\|f\|^2_{L_2(\Omega, e^{\vartheta|x|}dx)},
\]

where \(C(k)\) is some constant independent of \(f\). As the functions \(f\) belong to \(L_2(\Omega, e^{\vartheta|x|}dx_d)\), the functions \(u_j\) can be expanded into the Laurent series with respect to the small parameter \(k\); in particular, this means that the functionals \(\ell_j : L_2(\Omega, e^{\vartheta|x|}dx) \rightarrow \mathbb{C}\) are well defined and bounded. Substituting these expansions and (59) into (58), we immediately get representations (52), (53), and (10).

In view of formula (58), to prove representations (54), (55), and (56), it is sufficient to analyze the behavior of the functions \(u_j\) at infinity. This can be done easily by an obvious identity

\[
u_j(x_d, k) = \frac{e^{\pm \tau K_j(k)x_d}}{2\tau K_j(k)} \int_{\mathbb{R}} e^{\pm \tau K_j(k)y_d} f_j(y_d) dy_d = \tilde{u}_j^\pm(x_d, k), \quad \pm x_d > 0, \quad (60)
\]

\[
\tilde{u}_j^\pm(x_d, k) := \frac{1}{2K_j(k)} \int_{x_d}^{\pm \infty} (e^{K_j(k)(x_d-y_d)} - e^{-K_j(k)(x_d-y_d)}) f_j(y_d) dy_d,
\]
and an estimate
\[
|\tilde{u}_j(x_d, k)| \leq \frac{1}{2|K_j(k)|} \left( \int_{x_d}^{\pm \infty} e^{K_j(k)(x_d - y_d)} - e^{-K_j(k)(x_d - y_d)} \right)^{\frac{1}{2}} \left| f_j \right|_{L_2(\mathbb{R}, e^{\tilde{\vartheta}|x_d|} dx_d)}^{\frac{1}{2}}
\]
\[
\leq C(k) e^{-\tilde{\vartheta}|y_d|} \left| f_j \right|_{L_2(\mathbb{R}, e^{\tilde{\vartheta}|x_d|} dx_d)}, \quad \pm x_d > 0,
\]
where \( C(k) > 0 \) and \( 0 < \tilde{\vartheta} < \vartheta \) are some constants independent of \( x_d \) and \( f \). The proof is complete.

4.2 | Meromorphic continuation for the resolvent of the perturbed operator

This section is devoted to the proof of Theorem 1. We begin with rewriting the operator equation
\[
(\mathcal{H}_\varepsilon - \Lambda_p + k^2) u_\varepsilon = f,
\]
as a boundary value problem (3). Then, we denote
\[
g_\varepsilon := f - \varepsilon \mathcal{L}(\varepsilon) u_\varepsilon,
\]
and rewrite (3) as problem (50) with the function \( f \) replaced by \( g_\varepsilon \). According to Lemma 3, such problem is solvable and there exists a solution given by formula (51):
\[
u_\varepsilon = \mathcal{A}_{1,\tau}(k) g_\varepsilon.
\]
(62)

We substitute this formula into corresponding boundary value problem (3), and this leads to an operator equation in the space \( L_2(\Omega, e^{\tilde{\vartheta}|x_d|} dx) \):
\[
g_\varepsilon + \varepsilon \mathcal{L}(\varepsilon) \mathcal{A}_{1,\tau}(k) g_\varepsilon = f.
\]

Then we substitute representations (52), (53), and (10) into this equation:
\[
g_\varepsilon + \frac{\varepsilon}{k} \sum_{j=p}^{p+n-1} (\ell_j g_\varepsilon) \mathcal{L}(\varepsilon) \psi_j + \varepsilon \mathcal{L}(\varepsilon) (G_{p,\tau} + k \mathcal{A}_{4,\tau}(k)) g_\varepsilon = f.
\]

(63)

Thanks to the properties of the operators \( G_{p,\tau} \) and \( \mathcal{A}_{4,\tau}(k) \) described in Lemma 3, the operator \( \mathcal{L}(\varepsilon)(G_{p,\tau} + k \mathcal{A}_{4,\tau}(k)) \) is bounded uniformly in \( \varepsilon \) and \( k \) as an operator in \( L_2(\Omega, e^{\tilde{\vartheta}|x_d|} dx) \) and is holomorphic in \( k \in \mathbb{B}_\delta \). Hence, for sufficiently small \( \varepsilon \), the operator \( (I + \varepsilon \mathcal{L}(\varepsilon)(G_{p,\tau} + k \mathcal{A}_{4,\tau}(k))) \) is boundedly invertible and the inverse is holomorphic in \( k \in \mathbb{B}_\delta \) as an operator in \( L_2(\Omega, e^{\tilde{\vartheta}|x_d|} dx) \); hereinafter the symbol \( I \) stands for the identity mapping. We denote
\[
\mathcal{A}_{5,\tau}^\varepsilon(k) := \left( I + \varepsilon \mathcal{L}(\varepsilon)(G_{p,\tau} + k \mathcal{A}_{4,\tau}(k)) \right)^{-1}
\]
(64)
and apply this operator to Equation (63). This gives:

\[ g_\varepsilon + \frac{\varepsilon}{k} \sum_{j=p}^{p+n-1} (\ell_j g_\varepsilon) A_{5,\tau}^\varepsilon (k) \mathcal{L}(\varepsilon) \psi_j = A_{5,\tau}^\varepsilon (k)f. \]  

(65)

Our next step is to find \( \ell_j g_\varepsilon \). Once we do this, we shall be able to find \( g_\varepsilon \) from Equation (63) as

\[ g_\varepsilon = -\frac{\varepsilon}{k} \sum_{j=p}^{p+n-1} (\ell_j g_\varepsilon) A_{5,\tau}^\varepsilon (k) \mathcal{L}(\varepsilon) \psi_j + A_{5,\tau}^\varepsilon (k)f \]

(66)

and recover the function \( u_\varepsilon \) by formula (62).

We apply the functionals \( \ell_i \) to Equation (65) with all \( i \) such that \( \Lambda_i = \Lambda_p \). As a result, we arrive to a system of linear equations for the vector \( \ell g_\varepsilon := (\ell_j g_\varepsilon)_{j=p,...,p+n-1} \); this is a vector column. The system reads as

\[ \left( E + \frac{\varepsilon}{k} M_{\varepsilon,\tau}(k) \right) \ell g_\varepsilon = F_\varepsilon(k). \]  

(67)

Here \( E \) is the unit matrix and \( M_{\varepsilon,\tau}(k) \) is a square matrix with entries \( M_{ij}^\varepsilon (k) := \ell_i A_{5,\tau}^\varepsilon (k) \mathcal{L}(\varepsilon) \psi_j \), where \( i \) counts the rows and \( j \) does the columns in the matrix \( M_{\varepsilon,\tau}(k) \), while the symbol \( F_\varepsilon(k) \) denotes a vector column with coordinates \( \ell_i A_{5,\tau}^\varepsilon (k)f, \) \( i = p,...,p+n-1 \). As it was shown in the proof of Lemma 3, the functionals \( \ell_i : L_2(\Omega, e^{\delta |x_d|}dx) \rightarrow \mathbb{C} \) are bounded and since the operator \( A_{5,\tau}^\varepsilon (k) \) is a bounded one in \( L_2(\Omega, e^{\delta |x_d|}dx) \), the introduced matrix \( M_{\varepsilon,\tau}(k) \) and vector \( F_\varepsilon(k) \) are well defined.

In view of the holomorphy of the operator \( A_{5,\tau}^\varepsilon \), the entries of the matrix \( M_{\varepsilon,\tau}(k) \) are holomorphic in \( k \) and this implies that the matrix \( M_{\varepsilon,\tau}(k) \) is holomorphic in \( k \in B_\delta \). Hence, the determinant of the matrix \( kE + \varepsilon M_{\varepsilon,\tau}(k) \) is a holomorphic in \( k \) function, and therefore, the matrix \( (kE + \varepsilon M_{\varepsilon,\tau}(k))^{-1} \) is well defined as a meromorphic in \( k \) matrix. This allows us to solve system (67):

\[ \ell g_\varepsilon = k(kE + \varepsilon M_{\varepsilon,\tau}(k))^{-1}F_\varepsilon(k) = k \left( \ell_j^\varepsilon (k)f \right)_{j=p,...,p+n-1}, \quad \ell_j g_\varepsilon = k \ell_j^\varepsilon (k)f, \]

(68)

where \( \ell_j^\varepsilon (k) \) are functionals on \( L_2(\Omega, e^{\delta |x_d|}dx) \) meromorphic in \( k \in B_\delta \). Substituting these formulas into (66), we can find \( g_\varepsilon \) and determine \( u_\varepsilon \) by formula (62):

\[ u_\varepsilon = \sum_{j=p}^{p+n-1} \left( \ell_j^\varepsilon f \right) \left( \psi_j - \varepsilon (G_{p,\tau} + kA_{4,\tau}(k)) A_{5,\tau}^\varepsilon (k) \mathcal{L}(\varepsilon) \psi_j \right) + (G_{p,\tau} + kA_{4,\tau}(k)) A_{5,\tau}^\varepsilon (k)f =: \mathcal{R}_{\varepsilon,\tau}(k)f. \]

(69)

The function \( u_\varepsilon \) solves problem (3) and satisfies representations (4), (5), and (6). The latter are implied by similar representations (54), (55), and (56) and formula (62). Since for sufficiently small \( k \), the functions \( K_j, \Lambda_j \subset \Lambda_p \), satisfy asymptotic identities (36), the functions \( e^{-\tau K_j(k)|x_d|}, \Lambda_j < \Lambda_p \) and \( e^{-k|x_d|} \) decay exponentially at infinity as \( \text{Re} \ k > 0 \) and \( \text{Im} \ k^2 < 0 \). Hence, under the same
conditions, we have \( u_\varepsilon = (\mathcal{H}_\varepsilon - \Lambda_p + k^2)^{-1} f \) and this means that the operator \( \mathcal{R}_{\varepsilon, \tau} \) provides a meromorphic continuation of the resolvent \( (\mathcal{H}_\varepsilon - \Lambda_p + k^2)^{-1} \).

We observe that the introduced operator \( \mathcal{R}_{\varepsilon, \tau} \) is meromorphic in \( k \in B_\delta \) as an operator from \( L^2(\Omega, e^{\varepsilon |x_d|} \, dx) \) into \( W^2_2(\Omega) \cap L^2(\Omega, e^{-\varepsilon |x_d|} \, dx) \) due to formulas (62), (66), and (68) and representations (52), (53), and (10).

According to formula (69), the poles of the operator \( \mathcal{R}_{\varepsilon, \tau}(k) \) defined in this formula coincide with the poles of the functionals \( \ell^\varepsilon_j(k) \). Let at least one of these functionals have a pole at a point \( k_\varepsilon \in B_\delta \). In view of the definition of the functionals in (68), this means that the matrix \( (k_\varepsilon E + \varepsilon M_{\varepsilon, \tau}(k_\varepsilon)) \) is degenerate and by the Cramer’s rule, system (67) with \( F_\varepsilon(k) = 0 \), \( k = k_\varepsilon \) has a nontrivial solution, which we denote by \( \ell^\varepsilon = (\ell^\varepsilon_j)_{j=\Lambda_j=\Lambda_p} \). We observe that by the definition,

\[
k_\varepsilon \ell^\varepsilon_i = -\varepsilon \sum_{j=p}^{p+n-1} \ell^\varepsilon_i \ell_j A_{\varepsilon, \tau}^\varepsilon(k) L(\varepsilon) \psi_j. \tag{70}
\]

Employing these identities and (52) and (53), it is straightforward to check that the formula

\[
\psi_\varepsilon := -\varepsilon \lim_{k \to k_\varepsilon} A_{1, \tau}(k) \sum_{i=1}^{n} \ell^\varepsilon_i A_{\varepsilon, \tau}^\varepsilon(k) L(\varepsilon) \psi_{i+p-1}
\]

\[
= \sum_{i=1}^{n} \ell^\varepsilon_i \psi_{i+p-1} - \varepsilon (G_{p, \tau} + k_\varepsilon A_{4, \tau}(k_\varepsilon)) \sum_{i=1}^{n} \ell^\varepsilon_i A_{\varepsilon, \tau}^\varepsilon(k) L(\varepsilon) \psi_{i+p-1}
\]

\[
= \sum_{i=1}^{n} \ell^\varepsilon_i (I + \varepsilon (G_{p, \tau} + k_\varepsilon A_{4, \tau}(k_\varepsilon)) L(\varepsilon))^{-1} \psi_{i+p-1}
\]

defines a nontrivial solution to problem (3) with \( f = 0 \), \( k = k_\varepsilon \). Thanks to formulas (58), (57), and (61), this function can also be represented as

\[
\psi_\varepsilon = \sum_{j=1}^{m} \psi_j(x') \varphi_j(x_d) + (H^\perp - \Lambda_p + k^2)^{-1} f \perp,
\]

\[
\varphi_j(x_d) := \frac{1}{2\tau K_j(k_\varepsilon)} \int_{\Omega} e^{-\tau K_j(k_\varepsilon)|x_d-y_d|} f(y) \overline{\psi_j(y')} \, dy \quad \text{as } j < p - 1,
\]

\[
\varphi_j(x_d) := \frac{1}{2K_j(k_\varepsilon)} \int_{\Omega} e^{-K_j(k_\varepsilon)|x_d-y_d|} f(y) \overline{\psi_j(y')} \, dy \quad \text{as } j \geq p + n - 1,
\]

\[
\varphi_j(x_d) := \frac{1}{2k_\varepsilon} \int_{\Omega} e^{-k_\varepsilon|x_d-y_d|} f(y) \overline{\psi_j(y')} \, dy \quad \text{as } j = p, \ldots, p + n - 1, \quad k_\varepsilon \neq 0,
\]

\[
\varphi_j(x_d) := -\frac{1}{2} \int_{\Omega} |x_d-y_d| f(y) \overline{\psi_j(y')} \, dy \quad \text{as } j = p, \ldots, p + n - 1, \quad k_\varepsilon = 0,
\]

\[
(72)
\]
with

\[ f := -\varepsilon \sum_{i=1}^{n} I_i^\varepsilon \mathcal{A}_5^\varepsilon (k^\varepsilon) \mathcal{L}(\varepsilon) \psi_{i+p-1} \]

and \( f^\perp \) defined by (11). We also observe that as \( k^\varepsilon = 0 \), it follows from (70) that

\[ (f(\cdot, x_d), \psi_j)_{L_2(\Omega)} = 0 \]

for almost each \( x_d \in I_\varepsilon \) and for each \( j = 1, \ldots, m \). Representations (71) and (72) and relations (60) and (61) imply representations (7), (8), and (9). This completes the proof of Theorem 1.

## 5 | POLES OF MEROMORPHIC CONTINUATIONS

In this section, we study the asymptotic behavior of poles of the operators \( \mathcal{R}_{\varepsilon, \tau} \) and we prove Theorems 2 and 3. As it was shown in the proof of Theorem 1 in the previous section, the poles of the operator \( \mathcal{R}_{\varepsilon, \tau}(k) \) coincide with poles of the matrix \((kE + \varepsilon \mathcal{M}_{\varepsilon, \tau}(k))^{-1}\). The orders of the mentioned poles of the operator \( \mathcal{R}_{\varepsilon, \tau}(k) \) and the orders of the zeroes of \( \det(kE + \varepsilon \mathcal{M}_{\varepsilon, \tau}(k)) \) obviously coincide.

### 5.1 | Proof of Theorem 2

Let us study the solvability of the equation

\[ \det(kE + \varepsilon \mathcal{M}_{\varepsilon, \tau}(k)) = 0 \]  

(73)

in \( B_\delta \). It is straightforward to check that

\[ \det(kE + \varepsilon \mathcal{M}_{\varepsilon, \tau}(k)) = k^n + P(k, \varepsilon), \quad P(k, \varepsilon) := \sum_{i=1}^{n} \varepsilon^i k^{n-i} P_i(k, \varepsilon), \]

where \( P_i(k, \varepsilon) \) are some functions holomorphic in \( k \in B_\delta \) and uniformly bounded in \( k \in B_\delta \) and sufficiently small \( \varepsilon \). We fix arbitrary \( \delta' \in (0, \delta) \) and for sufficiently small \( \varepsilon \), we have an uniform estimate:

\[ |P(k, \varepsilon)| \leq C \varepsilon \quad \text{on} \quad \partial B_{\delta'}. \]  

(74)

Since \( |k^n| = (\delta')^n \) on \( \partial B_{\delta'} \) and the function \( k \mapsto k^n \) has the only zero in \( B_\delta \) of order \( n \) at the origin, estimate (74) allows us to apply the Rouché theorem and to conclude that Equation (73) has exactly \( n \) zeroes counting their orders and these zeroes converge to the origin as \( \varepsilon \to +0 \).

Our next step is to show that all zeroes of Equation (73) are located in a ball \( B_{b\varepsilon} \) for some fixed \( b \). For arbitrary \( b \) and sufficiently small \( \varepsilon \) such that \( b\varepsilon < d \), by the definition of the function \( P \), we have:

\[ |P(k, \varepsilon)| \leq C \varepsilon^n(1 + b + \ldots + b^{n-1}), \quad |k^n| = \varepsilon^n b^n \quad \text{on} \quad \partial B_{b\varepsilon}. \]
Then, we choose \( b \) large enough and \( \varepsilon \) small enough so that

\[
b^n > C(1 + b + \ldots + b^{n-1}), \quad b\varepsilon < \delta,
\]

and applying the Rouché theorem once again, we see that the zeroes of Equation (73) are located inside the ball \( B_{b\varepsilon} \). This fact allows us to seek the zeroes of Equation (73) as \( k = z\varepsilon \), where \( |z| < b \) for all sufficiently small \( \varepsilon \).

The operator \( A_{5,\tau}^\varepsilon \) defined in (64) can be expanded into the standard Neumann series and this yields:

\[
A_{5,\tau}^\varepsilon (k) = I - \varepsilon \mathcal{L}(\varepsilon)(G_{p,\tau} + kA_{4,\tau}(k)) + \varepsilon^2(\mathcal{L}(\varepsilon)(G_{p,\tau} + kA_{4,\tau}(k)))^2 A_{5,\tau}^\varepsilon (k). \tag{75}
\]

In view of the above identity, the change \( k = z\varepsilon \), \( |z| < b \), and the definition of the matrix \( M_{\varepsilon,\tau} \), we can represent the latter as

\[
M_{\varepsilon,\tau}(k) = -M_1 + \varepsilon \tilde{M}_2(z, \varepsilon), \tag{76}
\]

where \( \tilde{M}_2(z, \varepsilon) \) is some uniformly bounded matrix holomorphic in \( z \), and we recall that the matrix \( M_1 \) is defined in (12). Both these matrices are well defined because they arise as the terms in the expansion for \( M_{\varepsilon,\tau} \).

Then we can rewrite Equation (73) as

\[
\det (zE - M_1 + \varepsilon \tilde{M}_2(z, \varepsilon)) = 0. \tag{77}
\]

We recall that \( \mu_j, j = 1, \ldots, N \), are different eigenvalues of the matrix \( M_1 \) of the multiplicities \( q_j \), and hence,

\[
\det(zE - M_1) = \prod_{j=1}^{N} (z - \mu_j)^{q_j}.
\]

This allows us to represent Equation (77) as

\[
\prod_{i=1}^{N} (z - \mu_i)^{q_i} + \varepsilon \tilde{P}(z, \varepsilon) = 0, \tag{78}
\]

where \( \tilde{P} \) is some function holomorphic in \( z \) and bounded uniformly in \( z \) and \( \varepsilon \):

\[
|\tilde{P}(z, \varepsilon)| \leq C. \tag{79}
\]

By \( T_r(z_0) \) we denote a ball of radius \( r \) in the complex plane centered at the point \( z_0 \). We introduce the balls \( T_{\bar{b}\varepsilon q_i} (\mu_i) \), where \( \bar{b} \) is some fixed number. Then for sufficiently small \( \varepsilon \), each ball \( T_{\bar{b}\varepsilon q_i} (\mu_i) \) contains the only zero of the function \( z \mapsto \prod_{j=1}^{N} (z - \mu_j)^{q_j} \), this zero is \( \mu_i \) and the order of this zero is \( q_i \). On the boundary of each ball \( T_{\bar{b}\varepsilon q_i} (\mu_i) \), for sufficiently small \( \varepsilon \), an obvious estimate holds
true:

$$\left| \prod_{j=1}^{N} (z - \mu_j)^{q_j} \right| \geq C \tilde{b}^{q_i} \varepsilon,$$

(80)

where $C$ is some fixed constant independent of $\varepsilon$ and $\tilde{b}$. Hence, in view of estimate (79), we can choose $\tilde{b}$ large enough and $\varepsilon$ small enough and apply the Rouche theorem to the left-hand side of Equation (78). This yields that each ball $T_{\frac{\tilde{b} \varepsilon}{q_i}}(\mu_i)$ contains exactly $q_i$ zeroes of Equation (78), which we denote by $z_{ij} = z_{ij}(\varepsilon)$. These zeroes satisfy the asymptotic identities

$$z_{ij}(\varepsilon) = \mu_i + O(\varepsilon^{q_i}), \quad j = 1, \ldots, q_i.$$  

(81)

Returning back to Equation (73), we see that it has exactly $n$ zeroes counting their orders and these zeroes obey asymptotic expansion (13). The proof is complete.

5.2 Proof of Theorem 3

Identity (75) allows us to specify the form of the matrix $M_{\varepsilon, \tau}$ in more details than in (76). Namely, this identity implies that

$$M_{\varepsilon, \tau}(k) = -M_1 + \varepsilon M_2 + \varepsilon^2 \tilde{M}_3(z, \varepsilon),$$

where $\tilde{M}_3(z, \varepsilon)$ is some uniformly bounded matrix holomorphic in $z$, and we recall that the matrix $M_2$ is defined in (14). All these matrices are well defined as the terms in the expansion for $M_{\varepsilon, \tau}$. Equation (77) is replaced by a more detailed one:

$$\det (zE - M_1 + \varepsilon M_2 + \varepsilon^2 \tilde{M}_3(z, \varepsilon)) = 0.$$  

(82)

We fix $i \in \{1, \ldots, N\}$ and we are going to study the asymptotic behavior of the zeroes $z_{ij}(\varepsilon)$, $j = 1, \ldots, q_i$. Let $S$ be a matrix reducing the matrix $M_1$ to its Jordan canonical form, which we denote by $J := S M_1 S^{-1}$. Then Equation (82) can be rewritten as

$$\det (zE - J + \varepsilon S M_2 S^{-1} + \varepsilon^2 \tilde{S} \tilde{M}_3(z, \varepsilon) S^{-1}) = 0.$$  

Taking into consideration the structure of the matrix $J$, we rewrite this equation as follows:

$$\prod_{j=1}^{N} (z - \mu_j)^{q_j} + \varepsilon(z - \mu_i)^{r_{i,\varepsilon}} Y_1(z) + \varepsilon^2 Y_2(z, \varepsilon) = 0,$$  

(83)

where $Y_1$ and $Y_2$ are some functions holomorphic in $z$ and bounded uniformly in $\varepsilon$ and $z$ and $r_{i,\varepsilon} < q_i$ is some nonnegative integer number. These functions arise as some polynomials in $z$ and $\varepsilon$ and the entries of the matrices $S M_2 S^{-1}$ and $\tilde{M}_3(z, \varepsilon) S^{-1}$. It is clear that $(z - \mu_i)^{r_{i,\varepsilon}} Y_1(z) = Q_{i,\varepsilon}(z),$
where \( Q_i \) is defined in (15). And if \( Y_1 \) is not identically zero, then
\[
Y_1(z) = \gamma_{i,\tau} \prod_{j=1}^{N}(\mu_i - \mu_j)^{q_j} \neq 0,
\]
(84)
where \( \gamma_{i,\tau} \) is defined in (17). By \( T_r(z_0) \) we denote a ball of radius \( r \) in the complex plane centered at the point \( z_0 \).

We first consider the case, when \( Y_1 \) vanishes identically. Then the term \( \varepsilon Y_1 \) disappears in Equation (83). Here we consider the ball \( T_{\frac{1}{\varepsilon}}^{\frac{1}{\varepsilon}}(\mu_i) \) and for sufficiently small \( \varepsilon \), on the boundary of this ball, we have the estimates
\[
\left| \prod_{j=1}^{N}(z - \mu_j)^{q_j} \right| \geq C\tilde{b}^q\varepsilon^2,
|\varepsilon^2 Y_2(z, \varepsilon)| \leq \tilde{C}\varepsilon^2,
\]
where \( C \) and \( \tilde{C} \) are some fixed constants independent of \( \varepsilon \) and \( \tilde{b} \). Then we apply the Rouché theorem proceeding as in (78), (79), (80), and (81) and we arrive immediately to (16).

Now we proceed to the case, when \( Y_1 \) is not identically zero. Here we divide Equation (83) by \( \prod_{j=1}^{N}(z - \mu_j)^{q_j} \) and in view of (84), we get:
\[
(z - \mu_i)^{q_i} + \varepsilon(z - \mu_i)^{r_{i,\tau}} \gamma_{i,\tau} + \varepsilon(z - \mu_i)^{r_{i,\tau}+1} Y_3(z) + \varepsilon^2 Y_4(z, \varepsilon) = 0,
\]
(85)
where \( Y_3, Y_3 \) are some functions holomorphic in \( z \) in the vicinity of the point \( \mu_i \) and bounded uniformly in \( \varepsilon \) and \( z \).

Suppose that \( q_i \leq 2r_{i,\tau} \) and consider the ball \( T_{\frac{1}{\varepsilon}}^{\frac{1}{\varepsilon}}(\mu_i) \). For sufficiently small \( \varepsilon \), on the boundary of this ball, we have
\[
|\varepsilon(z - \mu_i)^{r_{i,\tau}+1} Y_3(z) + \varepsilon^2 Y_4(z, \varepsilon)| \leq C(\tilde{b}^{r_{i,\tau}+1} \varepsilon^2 + \varepsilon^2) < C\varepsilon^2,
\]
(87)
where \( C \) is some constant independent of \( \varepsilon \) and \( \tilde{b} \). Since \( q_i - r_{i,\tau} \geq r_{i,\tau} + 1 \), it is also straightforward to check that for sufficiently small \( \varepsilon \) and sufficiently large \( \tilde{b} \), all zeroes of the function \( z \mapsto (z - \mu_i)^{q_i} + \varepsilon(z - \mu_i)^{r_{i,\tau}} \gamma_{i,\tau} \) belong to the ball \( T_{\frac{1}{\varepsilon}}^{\frac{1}{\varepsilon}}(\mu_i) \). Taking this fact and estimates (86) and (87) into consideration, we apply the Rouché theorem and arrive at asymptotic expansion (18).
Suppose that $q_i - 2r_{i,\tau} \geq 1$. The zeroes of the function $z \mapsto (z - \mu_i)^{q_i} + \varepsilon(z - \mu_i)^{r_{i,\tau}} \gamma_{i,\tau}$ can be found explicitly. The first of them is a $r_{i,\tau}$-multiple zero $z_0 = \mu_i$ and $q_i - r_{i,\tau}$ simple zeroes

$$
 z_j(\varepsilon) := \mu_i - \varepsilon^{q_i - r_{i,\tau}} (z - \mu_i) \frac{1}{q_i - r_{i,\tau}} e^{2\pi i j \frac{1}{q_i - r_{i,\tau}}}, \quad j = 1, \ldots, q_i - r_{i,\tau}.
$$

(88)

First, we consider a ball $T \frac{1}{b \varepsilon^{q_i - r_{i,\tau}}}(\mu_i)$ and we observe that under our assumption, $q_i - r_{i,\tau} \geq r_{i,\tau} + 1$. Hence, for sufficiently small $\varepsilon$, the ball $T \frac{1}{b \varepsilon^{q_i - r_{i,\tau}}}(\mu_i)$ does not contain the zeroes $z_j(\varepsilon)$. Moreover, on the boundary of this ball, the estimate holds true:

$$
 |(z - \mu_i)^{q_i} + \varepsilon(z - \mu_i)^{r_{i,\tau}} \gamma_{i,\tau}| \geq \varepsilon |\gamma_{i,\tau}| |(z - \mu_i)^{r_{i,\tau}}| - |(z - \mu_i)^{q_i}| = \frac{2}{r_{i,\tau}} \varepsilon |z_0|^{q_i - r_{i,\tau}} > \frac{1}{2} \tilde{b}_{r_{i,\tau}} \varepsilon^2.
$$

We also observe that estimate (87) remains true in the considered case. Then we apply Rouché theorem to the ball $T \frac{1}{b \varepsilon^{q_i - r_{i,\tau}}}(\mu_i)$ and conclude that this ball contains exactly $r_{i,\tau}$ zeroes of Equation (85) counting their orders. This implies asymptotic expansion (19).

Now we consider another ball $T \frac{1}{b \varepsilon^{q_i - r_{i,\tau}}}(z_j(\varepsilon))$ for $z_j$ defined in (88) with some fixed $j$. For sufficiently small $\varepsilon$, this ball does not contain the origin and other points $z_s$ with $s \neq j$. On the boundary of this ball, we have the following estimate:

$$
 |(z - \mu_i)^{q_i} + \varepsilon(z - \mu_i)^{r_{i,\tau}} \gamma_{i,\tau}| \geq C \varepsilon^{q_i - r_{i,\tau}} |(z - \mu_i)^{q_i - r_{i,\tau}} + \varepsilon \gamma_{i,\tau}| = C \varepsilon^{q_i - r_{i,\tau}} \prod_{s=1}^{q_i - r_{i,\tau}} |z - z_s| \geq C \varepsilon^{q_i - r_{i,\tau}},
$$

where $C$ is some constant independent of $\varepsilon$ and $\tilde{b}$. We observe that under our assumptions,

$$
 \frac{q_i + 1}{q_i - r_{i,\tau}} \leq 2.
$$

In view of the latter inequality, on the boundary of the ball $T \frac{2}{b \varepsilon^{q_i - r_{i,\tau}}}(z_j(\varepsilon))$, the estimates

$$
 |\varepsilon(z - \mu_i)^{r_{i,\tau} + 1} Y_3(z) + \varepsilon^2 Y_4(z, \varepsilon)| \leq C(\varepsilon^{q_i - r_{i,\tau}} + \varepsilon^2) < C \varepsilon^{q_i - r_{i,\tau}}
$$

hold true, where $C$ is some constant independent of $\varepsilon$ and $\tilde{b}$. This estimate and (89) allow us to apply the Rouché theorem, and we conclude that the ball $T \frac{2}{b \varepsilon^{q_i - r_{i,\tau}}}(z_j(\varepsilon))$ contains exactly one zero of Equation (83). This gives asymptotic expansion (20). The proof is complete.
EMERGING RESONANCES AND EIGENVALUES

In this section, we determine whether the poles of the operators $R_{\varepsilon, \tau}$ described in the previous section correspond to the eigenvalues or resonances of the operator $H_{\varepsilon}$. In this way, we prove Theorems 4 and 5.

We fix $p \in \{1, \ldots, m\}$ and assume that $\Lambda_p = \cdots = \Lambda_{p+n-1}$, where $n \geq 1$ is a multiplicity of the eigenvalue $\Lambda_p$ of the operator $H'$. In what follows, we analyze the nature of poles $k_{ij}(\varepsilon)$ from Theorem 2 with asymptotic behavior (13). To do this, we analyze the behavior of the associated nontrivial solutions to problem (3) with $f = 0$.

6.1 | Bottom of the spectrum

In this subsection, we assume that $p = 1$ and we prove Theorem 4. In the considered case, the meromorphic continuation of the resolvent given the operator $R_{\varepsilon, \tau}$ is independent of $\tau$ and the same is true for the corresponding poles $k_{ij}(\varepsilon)$ from Theorem 2 with asymptotic behavior (13). The behavior of the associated nontrivial solutions to problem (3) with $f = 0$ is provided by representation (7). In this representation, the first sum over $j = 1, \ldots, p - 1$ is obviously missing and the leading term at infinity is the sum over $j = p, \ldots, p + n - 1$.

We fix one of the poles $k_{ij}(\varepsilon)$ with some $i \in \{1, \ldots, n\}$, $j = 1, \ldots, q_i$. It follows from (70), (71), (72), (52), and (53) that the coefficients $c_{s, \varepsilon}^{\pm}$, $s = 1, \ldots, n$, in representation (7) coincide with $l_{\varepsilon}^s$ determined by (70). Since we deal with a nontrivial solution of system (70), this implies that at least one of the coefficients $c_{s, \varepsilon}^{\pm}$ is nonzero. Then representation (7) for a nontrivial solution to problem (3) defined in (71) and formulas (36) implies that if

$$\text{Re} k_{ij}(\varepsilon) > 0,$$

(90)

then a nontrivial solution $\psi_{\varepsilon}$ decays exponentially at infinity, and thus, is an eigenfunction, while the opposite inequality

$$\text{Re} k_{ij}(\varepsilon) \leq 0$$

(91)

ensures that $\psi_{\varepsilon}$ is not in $W_2^2(\Omega)$.

Asymptotic expansions for $k_{ij}$ established in Theorems 2 and 3 allow us to check effectively the above conditions. Namely, if $\text{Re} \mu_i > 0$, this ensures inequality (90) for sufficiently small $\varepsilon$. Then the poles $k_{ij}$, $j = 1, \ldots, q_i$, correspond to the eigenvalues $\lambda_{ij}(\varepsilon) = \Lambda_p - k_{ij}^2(\varepsilon)$ with asymptotic expansions (21), (22), and (23) as described in the formulation of Theorem 4. If $\text{Re} \mu_i < 0$, this guarantees inequality (91) for sufficiently small $\varepsilon$ and the poles $k_{ij}$ correspond to the resonances $\lambda_{ij}(\varepsilon) = \Lambda_p - k_{ij}^2(\varepsilon)$ with same asymptotic expansions (21)–(23).

If $\text{Re} \mu_i = 0$, to understand which of conditions (90) and (91) is realized, we need to check the next term in the asymptotic expansion for $k_{ij}$. According to Theorem 3, this can be done under an additional assumption $2r_{i, \tau} \leq q_i - 1$ and only for poles $k_{ij, \tau}$ with $j = r_i + 1, \ldots, q_i$. Then, asymptotic expansion (20) implies that under condition (24), the pole $k_{ij}$, $j = r_i + 1, \ldots, q_i$ corresponds to an eigenvalue, while under condition (25), the pole $k_{ij}$ corresponds to a resonance. The asymptotic behavior for this eigenvalue/resonance is given by (23) if $\mu_i \neq 0$, and it is given by (26) if $\mu_i = 0$. 
6.2 Internal thresholds in the spectrum

In this subsection, we study the nature of the poles emerging from internal points \( \Lambda_p \) in the essential spectrum and we prove Theorem 5. As in the previous subsection, here we again analyze the behavior at infinity of nontrivial solutions \( \psi_\varepsilon \) associated with poles \( k_{ij} \), and this analysis will be based on representation (7). However, there are important differences. The matter is that now the representation involves also a sum over \( j = 1, \ldots, p - 1 \). Its terms can decay or grow at infinity. As it has been mentioned in the proof of Theorem 1, according to identity (36), the functions \( e^{-\tau K_j(k)|x_d|}, j = 1, \ldots, p - 1 \), decay exponentially at infinity if \( \tau \text{ Im } k^2 < 0 \) and grow exponentially or vary periodically as \( \tau \text{ Im } k^2 \geq 0 \). This means that apart of the sign of \( \text{Re } k_{ij} \), we should also control the sign of \( \tau \text{ Im } k_{ij}^2 \).

One more point is that now we have two meromorphic continuations of the resolvent, the operators \( \mathcal{R}_{\tau,\varepsilon}, \tau \in \{-1, +1\} \), respectively, and two sets of their poles \( k_{ij,\tau}(\varepsilon) \) converging to zero as \( \varepsilon \to +0 \). We observe that according to Theorems 2 and 3, the first terms in asymptotic expansions for poles \( k_{ij,\tau} \) are the eigenvalues \( \mu_i \) of the matrix \( M_1 \) and they are independent of \( \tau \). In fact, we can see the influence of \( \tau \) on the asymptotic behavior of \( k_{ij} \) only in formulas (20), for \( j = r_{i,\tau} + 1, \ldots, q_i \) under an additional assumption \( 2r_{i,\tau} \leq q_i - 1 \).

We choose \( i \in \{1, \ldots, N\}, j \in \{1, \ldots, q_i\}, \tau \in \{-1, +1\} \) and consider the pole \( k_{ij,\tau}(\varepsilon) \). As in the above proof of Theorem 4, it is easy to see that at least one of the coefficients \( c_{s,\varepsilon}^\pm \), \( s = 1, \ldots, n \), in representation (7) for the associated nontrivial functions \( \psi_\varepsilon \) is nonzero. Then, it follows from asymptotic expansions (16), (18), (19), and (20) that conditions (27)–(30) ensure that \( \text{Re } k_{ij,\tau}(\varepsilon) > 0 \) and \( \tau \text{ Im } k_{ij,\tau}(\varepsilon) < 0 \), and hence, the pole \( k_{ij,\tau}(\varepsilon) \) corresponds to an eigenvalue \( \lambda_{ij,\tau}(\varepsilon) = \Lambda_p - k_{ij,\tau}^2(\varepsilon) \). The stated asymptotic behavior for this eigenvalue is implied by (16), (18), (19), and (20). In the same way, conditions (31) and (32) ensure that \( \text{Re } k_{ij,\tau}(\varepsilon) < 0 \), and hence, the pole \( k_{ij,\tau} \) corresponds to a resonance \( \lambda_{ij,\tau}(\varepsilon) = \Lambda_p - k_{ij,\tau}^2(\varepsilon) \) and it possesses the stated behavior.

If \( \mu_i \) is a simple eigenvalue of the matrix \( M_1 \), then \( q_i = 1 \). In this case, a nontrivial solution \( \xi \) to system (70) associated with \( k_{1,\tau}(\varepsilon) \) converges to the vector \( e_1 \). Indeed, it follows from representation (76) that the corresponding zero \( z_{ij,\tau}(\varepsilon) \) of Equation (77) is an eigenvalue of the matrix \( M_1 - \varepsilon M_2(z_{ij,\tau}(\varepsilon), \varepsilon) \) and \( \xi_\varepsilon \) is an associated eigenvector. And since \( z_{ij,\tau}(\varepsilon) \) converges to \( \mu_i \) and both these eigenvalues are simple, the eigenvector associated with \( k_{1,\tau}(\varepsilon) \) converges to \( e_1 \) as well. Now by representations (71) and (72), definition (10) of the operator \( \varphi_{\mu,\tau} \), and formula (60), we conclude that the coefficients \( c_{s,\varepsilon}^\pm \) in representation (7) satisfy the identities:

\[
c_{s,\varepsilon}^\pm = \varepsilon \sum_{i=1}^n \int_\Omega e^{\tau K_i(0)|x_d|} \varphi_\delta(\varepsilon') \varphi_{\delta,i}(\varepsilon') \psi_{r_{i,\tau}}(\varepsilon') e_{i,\tau} \psi_{r_{i,\tau}+1} \, dx + o(\varepsilon).
\]

Hence, by condition (35), at least one of the coefficients \( c_{s,\varepsilon}^\pm \) is nonzero. As above, it is easy to see that one of conditions (27) or (28) and one of conditions (33) or (34) ensures that the functions \( e^{-K_\varepsilon(k_{ij,\tau}(\varepsilon)|x_d|) \text{ Im } k_{ij,\tau}(\varepsilon)} \) grows exponentially at infinity, and hence, the same is true for the function \( \psi_\varepsilon \). Therefore, the pole \( k_{ij,\tau}(\varepsilon) \) corresponds to a resonance. The asymptotic expansion for this resonance can be established as above. The proof of Theorem 5 is complete.

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DATA AVAILABILITY STATEMENT
The data that support the findings of this study are available from the corresponding author upon reasonable request.

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