An explicit Lyapunov function for reflection symmetric parabolic partial differential equations on the circle

B. Fiedler, C. Grotta-Ragazzo, and C. Rocha

Abstract. An explicit Lyapunov function is constructed for scalar parabolic reaction-advection-diffusion equations under periodic boundary conditions. The non-linearity is assumed to be even with respect to the advection term. The method followed was originally suggested by H. Matano for, and limited to, separated boundary conditions.

Bibliography: 20 titles.

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1. Introduction and main result

We consider real scalar semilinear parabolic partial differential equations of the form

\[ u_t = u_{xx} + f, \]  

in one space dimension \( 0 < x < 1 \) and with \( C^1 \) non-linearities \( f \).

Heeding Mark I. Vishik’s advice “nicht zu eilen” (not to rush), we only focus on existence versus non-existence of Lyapunov functions in the present paper. This is but one crucial element in our ongoing quest to clarify and classify the dynamics on the global attractors of these parabolic equations, in detail and in their simplest form.

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scalar form. For example, see [7], [8] and the references there. That entire project, in turn, is just one modest attempt to explore a tiny part of the richness of global PDE attractors as they have been studied, for example, in the groundbreaking and monumental work of Babin and Vishik [3] and their many followers worldwide.

Under Dirichlet or Neumann \emph{separated boundary conditions}

\[ u = 0 \quad \text{or} \quad u_x = 0 \]  

(1.2)

at \( x = 0, 1 \) and for non-linearities

\[ f = f(x, u) \]  

(1.3)

it is well known that there exists an explicit \emph{Lyapunov function}

\[ V(u) := \int_0^1 L(x, u, u_x) \, dx \]  

(1.4)

with the \emph{Lagrange function} integrand

\[ L(x, u, u_x) := \frac{1}{2} u_x^2 - F(x, u). \]  

(1.5)

Here \( F \) is a primitive function of \( f \) with respect to \( u \). The Lyapunov function \( V \) indeed satisfies the equation

\[ \dot{V} = \frac{d}{dt} V(u(t, \cdot)) = - \int_0^1 (u_t)^2 \, dx \]  

(1.6)

along any classical solution \( u = u(t, x) \) of (1.1). By LaSalle’s invariance principle this forces convergence to equilibria for bounded solutions as \( t \to +\infty \). By adding suitable boundary terms to \( V \) the result can be extended to separated non-linear boundary conditions

\[ u_x = \beta(x, u), \quad x = 0, 1, \]  

(1.7)

of Robin type. Upon passage from strong solutions to weak solutions, similar statements remain valid and identify the semiflow (1.1) as the \( L^2 \)-gradient semiflow of the Lyapunov function \( V \). See [10] and [15] for a general background, and [13] for the case of \( C^1 \) non-linearities.

It is a little less well known how [19] and [11] extended this classical result to non-linearities

\[ f = f(x, u, u_x), \]  

(1.8)

which also depend on the \emph{advection term} \( u_x \), again under separated boundary conditions. For the convenience of the reader we recall in §2 the beautiful argument by Matano in [11]. For a suitable Lagrange function \( L = L(x, u, p) \) replacing (1.5), the Lyapunov decay property (1.6) is replaced by

\[ \dot{V} = \frac{d}{dt} V(u(t, \cdot)) = - \int_0^1 L_{pp}(x, u, u_x)(u_t)^2 \, dx, \]  

(1.9)

with strict convexity of the function \( p \mapsto L(x, u, p) \), that is, with positive second partial derivative

\[ L_{pp} > 0. \]  

(1.10)
Therefore, $L_{pp}$ provides the appropriate non-homogeneous $L^2$-metric to view (1.1), (1.2), (1.8) as a gradient semiflow.

Under *periodic boundary conditions* $x \in S^1 := \mathbb{R}/\mathbb{Z}$, that is,

$$[u]^1_0 = [u_x]^1_0 = 0,$$  \hspace{1cm} (1.11)

the parabolic PDE (1.1) retains its gradient character (1.1)–(1.6) for non-linearities $f = f(x, u)$. The presence of advection terms $u_x$ can produce non-equilibrium *time-periodic solutions* $u(t, x)$, however. For example, consider the $SO(2)$-equivariant case

$$f = f(u, u_x),$$  \hspace{1cm} (1.12)

where $u(t, x)$ is a solution of the PDE (1.1) if and only if $u(t, x + \vartheta)$ is, for any fixed $\vartheta \in S^1 = SO(2)$. It was observed already in [2] that spatially non-homogeneous *rotating wave solutions*

$$u = U(x - ct)$$  \hspace{1cm} (1.13)

with non-vanishing wave speeds $c \neq 0$ may then occur. Indeed, this only requires non-stationary 1-periodic solutions $U$ of the travelling wave equation

$$U'' + cU' + f(U, U') = 0$$  \hspace{1cm} (1.14)

to exist. In general, convergence to equilibria as $t \to +\infty$ is then augmented by the possibility of convergence to rotating waves. For a specific example consider the non-linearity

$$f(u, p) := \lambda u(1 - u^2) - cp$$  \hspace{1cm} (1.15)

for $\lambda > \pi$. This amounts to viewing solutions with the cubic non-linearity

$$f_0(u) := \lambda u(1 - u^2),$$  \hspace{1cm} (1.16)

known as the Chafee–Infante problem [4], in coordinates which rotate at constant speed $c$ around $x \in S^1$. The non-homogeneous equilibria $U(x)$ of the Chafee–Infante problem (1.1), (1.16) then provide non-equilibrium rotating wave solutions $U(x - ct)$ of (1.14). Of course, this argument extends to any non-linearity of the form $f(u, p) = f_0(u) - cp$. Other examples include non-linearities $f = f(u, p)$ with travelling wave equations (1.14) of van der Pol type. For general not necessarily $SO(2)$-equivariant non-linearities $f = f(x, u, p)$, time-periodic solutions $u = u(t, x)$ may arise which are not rotating waves. Still, a Poincaré–Bendixson theorem holds which emphasizes the dichotomy between equilibria and periodic solutions for $t \to +\infty$ (see [5]).

With this motivation we consider the $O(2)$-equivariant case of the PDE (1.1) with periodic boundary conditions (1.11) in the present paper. We therefore assume the non-linearity $f$ to be even with respect to $p = u_x$ in order also to accommodate reflections $x \mapsto -x \in S^1$ on the circle. Specifically, we assume that

$$f = f(u, p) := \bar{f}\left(u, \frac{1}{2}p^2\right),$$  \hspace{1cm} (1.17)

with $C^1$ non-linearity

$$\bar{f} = \bar{f}(u, q), \quad q = \frac{1}{2}p^2.$$  \hspace{1cm} (1.18)
Arguments based on Sturm nodal properties and the zero numbers as in [2], [3] then show that all rotating waves are frozen to become equilibria, that is, ‘rotate’ with wave speed $c = 0$ (see also [7]). Instead we construct an explicit Lyapunov function, in the O(2)-case, which forces convergence to equilibria directly by LaSalle’s invariance principle. Convergence to single equilibria, in that case, was established already in [11]. Those arguments essentially excluded the alternative of rotating waves and were based on Sturm nodal properties. They did not use the explicit Lyapunov function, which we now construct to explore the gradient flow variational character of the PDE (1.1) on the circle.

To state our main result, Theorem 1.1 below, we assume that the O(2)-equivariant non-linearity $f = \tilde{f}(u, q)$ of (1.17), (1.18) is such that the non-autonomous ODE

$\frac{d}{du} q = -\tilde{f}(u, q), \quad q(u_0) = q_0, \quad (1.19)$

possesses a global solution

$q(u_1) = \Psi^{u_1, u_0}(q_0) \quad (1.20)$

for all real $q_0, u_0, u_1$. This assumption is satisfied if $\tilde{f}$ grows at most linearly with respect to $q$: a one-sided condition like $u\tilde{f}(u, q) \leq c_1(u) + c_2(u)q$ in the relevant region $q \geq 0$ with continuous functions $c_1$ and $c_2$, for example, prevents blow-up of solutions of equation (1.19) in finite ‘time’ $u$ (see also §2 in [16]).

We define the Lagrange function $L$—the integrand of the Lyapunov function $V$ in (1.4) —as

$L(u, p) := \int_0^P \int_0^{P_1} \exp \left( F_q(u, \frac{1}{2}p_2^2) \right) dp_2 dp_1 - F(u), \quad (1.21)$

with the abbreviations

$F(u) := \int_0^u \tilde{f}(u_1, 0) \exp(F_q(u_1, 0)) du_1, \quad (1.22)$

$F_q(u, q) := \int_0^u \tilde{f}_q(u_1, \Psi^{u_1, u}(q)) du_1.$

Here $\tilde{f}_q = \tilde{f}_q(u_1, q_1)$ denotes the partial derivative with respect to the second argument $q_1 = \Psi^{u_1, u}(q)$, and not the chain rule total derivative with respect to $q$ in the function $q \mapsto \tilde{f}_q(u_1, \Psi^{u_1, u}(q))$.

**Theorem 1.1.** Let $\tilde{f} \in C^1$ be such that the solution (1.20) of ODE (1.19) exists globally.

Then the functional

$V(u) := \int_0^1 L(u, u_x) dx \quad (1.23)$

with the Lagrange function $L$ of (1.21) is a Lyapunov function for the parabolic PDE (1.1) with O(2)-equivariant non-linearity $f = \tilde{f}(u, u_x^2/2)$ under periodic boundary conditions (1.11). More precisely, the equality

$\dot{V} = \frac{d}{dt} V(u(t, \cdot)) = -\int_0^1 L_{pp} \left( u_x, \frac{1}{2}(u_x)^2 \right) (u_t)^2 dx \quad (1.24)$
holds on classical solutions $u = u(t, x)$ of (1.1), with strict convexity of $L(x, u, p)$ with respect to $p$, that is, with positive metric coefficient

$$L_{pp} = \exp\left(-F_q\left(u, \frac{1}{2}p^2\right)\right).$$ (1.25)

In case $\bar{f}(u, u_x^2/2) = f(u)$ is independent of $q = p^2/2 = u_x^2/2$, we have $\bar{f}_q \equiv 0$, $F_q \equiv 0$, $L_{pp} \equiv 1$, and $L(u, p) = p^2/2 - F(u)$ with a primitive function such that $F' = f$. We therefore recover the classical Lyapunov function (1.4)–(1.6) in Theorem 1.1.

The formulation (1.21), (1.22) of the Lagrange function $L(u, p)$ still involves multiple integrals in terms of the evolution $\Psi^{u_1, u_0}(q_0)$ of the characteristic ODE (1.19), (1.20) and the non-linearity $\bar{f}$. To eliminate some of these integrals and provide a more direct expression for $L$, we define an auxiliary function $\varphi = \varphi(u, p)$ such that

$$\varphi_p(u, p) = \Psi_q^{0, u}(\frac{1}{2}p^2),$$ (1.26)

where $\Psi_q^{0, u}(q)$ denotes the partial derivative of the evolution $\Psi^{0, u}(q)$ with respect to $q$. Of course, (1.26) amounts to the simple integration,

$$\varphi(u, p) := \int_0^p \Psi_q^{0, u}(\frac{1}{2}p^2) \, dp_2.$$ (1.27)

**Corollary 1.2.** The Lagrange function $L$ defined in (1.21), (1.22) can be written equivalently as

$$L(u, p) = p\varphi(u, p) - \Psi_q^{0, u}(\frac{1}{2}p^2).$$ (1.28)

Again, the trivial case $\bar{f}_q = F_q = 0$, $\Psi^{0, u}(q) = q + F(u)$, $\Psi_q^{0, u}(q) = 1$, $\varphi(u, p) = p$ implies that $L(u, p) = p^2 - p^2/2 - F(u) = p^2/2 - F(u)$. An explicit construction of the Lyapunov function $V$ is also possible when $f(u, p) = a(u) + b(u)p^2/2$. Then equation (1.19) is linear and can be integrated. After some computations we can express the Lagrangian $L$ of $V$ in terms of integrals of the functions $a$ and $b$. For linear $a(u)$ and constant $b$ an explicit Lyapunov function was also constructed in Proposition 5.8 of [9] using ideas in [20] (Chap. 2).

In §2 we reproduce Matano’s elegant construction of the Lagrange function for $f = f(x, u, u_x)$ and indicate where the argument fails at a technical level, as it must, under periodic boundary conditions. In §3 we prove Theorem 1.1, based on Matano’s construction. An alternative approach can be based on the fact that under the hypothesis (1.17) the equilibrium equation $u_{xx} + f = 0$ admits a first integral (see [16]). Section 4 proves Corollary 1.2. In §5 we provide an example which shows how our Lyapunov function fails on $x \in S^1$, as it must, for non-linearities $f(x, u, p) = f(-x, u, -p)$ which admit only a single reflection rather than full $O(2)$-equivariance: $f(u, p) = f(u, -p)$, that is, $f = f(u, p^2/2)$. Again, this is due to the occurrence of non-stationary time-periodic orbits. Section 6 collects comments on the associated PDE global attractors, on quasi-linear equations, and on negative $q = u_x^2/2$, that is, imaginary $u_x$. 
The author Bernold Fiedler is much indebted to Hiroshi Matano for drawing
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for non-linearities $f = f(x, u, u_x)$ under separated boundary conditions many years
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2. Matano’s construction

In this section we recall Matano’s construction [11] of a Lagrange function $L = L(x, u, p)$ such that

$$V(u) := \int_0^1 L(x, u, u_x) \,dx$$

(2.1)

becomes a Lyapunov function for the PDE (1.1) under separated Dirichlet or Neu-
mann boundary conditions (1.2) and for general non-linearities $f = f(x, u, u_x)$. We
show that

$$\dot{V} = \frac{d}{dt} V(u(t, \cdot)) = -\int_0^1 L_{pp}(x, u, u_x) \cdot (u_t)^2 \,dx$$

(2.2)

with the condition that the function $p \mapsto L(x, u, p)$ is strictly convex, so that

$$L_{pp} > 0.$$ (2.3)

See Lemma 1 in [19] for the closely related original construction due to Zelenyak.
Twenty years later the original construction was retrieved from oblivion, clarified,
and slightly modified under less restrictive regularity assumptions in the appendix
to [11]. The differences become more apparent in the more general quasi-linear
parabolic case (6.1), which we review briefly in the discussion of §6. For clarity of
presentation we follow Matano here, rather than the somewhat convoluted original
argument by Zelenyak. Again, we emphasize that both constructions of a Lyapunov
function are limited to separated boundary conditions, albeit of slightly more gen-
eral form, and must fail for periodic boundary conditions.

The construction proceeds as follows. For classical solutions $u = u(t, x)$ we
integrate (2.1) by parts and substitute $u_{xx} = u_t - f$ to get from (1.1) that

$$\dot{V} = \int_0^1 \left( L_u u_t + L_p u_{tx} \right) \,dx$$

$$= \int_0^1 \left( L_u - \frac{d}{dx} L_p(x, u, u_x) \right) u_t \,dx$$

$$= \int_0^1 (L_u - L_{px} - L_{pu} u_x - L_{pp} u_{xx}) u_t \,dx$$

$$= \int_0^1 ((L_u - L_{px} - L_{pu} u_x + L_{pp} f)u_t - L_{pp}(u_t)^2) \,dx$$

$$= -\int_0^1 L_{pp}(u_t)^2 \,dx,$$

(2.4)
as required. Here for simplicity we have assumed Dirichlet boundary conditions
$u = 0$, and hence $u_t = 0$, at $x = 0, 1$. Neumann boundary conditions or more
general non-linear boundary conditions (1.7) of Robin type can be covered by adding suitable boundary terms to $V$. To satisfy the last equality, of course, the Lagrange function $L$ is required to satisfy the first-order linear PDE

$$L_u - L_{xp} - pL_{up} + fL_{pp} = 0$$

(2.5)

for all real arguments $u$, $p$, and $0 \leq x \leq 1$. To reduce the order and guarantee the convexity condition $L_{pp} > 0$, Matano assumes the Ansatz

$$L_{pp} =: \exp(g).$$

(2.6)

Upon taking the partial derivative of (2.5) with respect to $p$, the terms $L_{up}$ cancel and he obtains the first-order linear PDE

$$g_x + pg_u - fg_p = f_p$$

(2.7)

for $g = g(x, u, p)$. This PDE can be solved for $g$ by the method of characteristics: along the solutions $(u, p)(x)$ of the ODE

$$\frac{du}{dx} = p, \quad \frac{dp}{dx} = -f(x, u, p)$$

(2.8)

the function $x \mapsto g = g(x, u(x), p(x))$ must have the total derivative

$$\frac{d}{dx}g = f_p(x, u, p).$$

(2.9)

For example, we may assume that

$$g(0, u, p) \equiv 0$$

(2.10)

and obtain $g$ globally in this way, provided that the solutions of the characteristic ODE (2.8) exist for all $0 \leq x \leq 1$ and for all real initial conditions $u$, $p$ at $x = 0$. In other words, ascending from (2.7) · $L_{pp} = (2.5)_p$ to (2.5) itself via (2.6), we then define

$$L(x, u, p) := \int_0^p \int_0^{p_1} \exp(g(x, u, p_2)) \, dp_2 \, dp_1 - F(x, u),$$

$$F(x, u) := \int_0^u f(x, u_1, 0) \exp(g(x, u_1, 0)) \, du_1.$$  

(2.11)

Indeed, the left-hand side of (2.5) is independent of $p$, by this construction. Therefore, (2.5) holds for all $p$ if we verify that (2.5) holds for $p = 0$. At $p = 0$ the definition (2.11) implies that $0 \equiv L_p \equiv L_{px}$ and $L_u = -F_u = -f \exp g = -fL_{pp}$. This proves (2.5) and completes Matano’s construction of the Lyapunov function $V$.

Of course this correct construction should fail when abused to cover periodic boundary conditions. And it does. Suppose that the characteristic equation (2.8) possesses a periodic orbit $(u, p)(x)$ of period one, that is,

$$[(u, p)(x)]_0^1 = 0.$$  

(2.12)
Then 1-periodicity of the map $x \mapsto g(x,u,p)$ implies that
\begin{align}
0 &= [g(x,u(0),p(0))]_0^1 = [g(x,u(x),p(x))]_0^1 \\
&= \int_0^1 \frac{d}{dx} g(x,u(x),p(x)) \, dx = \int_0^1 f_p(x,u(x),p(x)) \, dx
\end{align}
(2.13)
in view of (2.9). But this integrability condition for $f_p$ may easily be violated while keeping the periodic orbit $(u,p)(x)$ unaffected. Thus, Matano’s construction must fail in general whenever time-periodic orbits appear in the PDE (1.1). The non-locality of the ill-posed compatibility condition
\begin{align}
[g(x,u(x),p(x))]_0^1 &= \int_0^1 f_p(x,u(x),p(x)) \, dx
\end{align}
(2.14)
along the characteristics (2.8) can be used to modify Matano’s construction in the $O(2)$-equivariant case.

3. Proof of Theorem 1.1

The proof of Theorem 1.1 consists of a slight adaptation of Matano’s construction in §2, to the case of $O(2)$-equivariant non-linearities
\begin{align}
f = f(u,p) = \tilde{f}(u,q), \quad q := \frac{1}{2} p^2.
\end{align}
(3.1)
The non-linearity $f$ is even with respect to $p = u_x$ due to reflection invariance, and does not depend on $x$ explicitly due to rotation invariance. Consequently, we consider $x$-independent Lagrange functions $L = L(u,p)$ and seek $O(2)$-invariant Lyapunov functions of the form
\begin{align}
V(u) := \int_0^1 L(u,u_x) \, dx,
\end{align}
(3.2)
where
\begin{align}
L_{pp} = \exp(g) > 0,
\end{align}
(3.3)
and $g$ takes the reflection symmetric form
\begin{align}
g = g(u,p) = \bar{g}(u,q), \quad q := \frac{1}{2} p^2.
\end{align}
(3.4)
The Matano calculation (2.4)–(2.7) then leads to the first-order linear PDE
\begin{align}
p(\bar{g}_u - \bar{f} \bar{g}_q) = \bar{p} \bar{f}_q.
\end{align}
(3.5)
Here we have used the chain rule, and we substituted the definitions (3.1) and (3.4) of $f$ and $g$ in (2.7). We divide by $p$ and solve the equation
\begin{align}
\bar{g}_u - \bar{f} \bar{g}_q = \bar{f}_q
\end{align}
(3.6)
by the method of characteristics along the global solutions $q(u_1) = \Psi^{u_1,u_0}(q_0)$ of the equation
\begin{equation}
\frac{d}{du}q = -\bar{f}(u,q),
q(u_0) = q_0,
\end{equation}
defined in (1.19), (1.20). Then any $\bar{g}$ which satisfies
\begin{equation}
\frac{d}{du}\bar{g}(u,q(u)) = \bar{f}_q(u,q(u))
\end{equation}
along the characteristics, say with initial condition
\begin{equation}
\bar{g}(0,q) := 0,
\end{equation}
solves the first-order PDE (3.5). With the help of the evolution $q(u_1) = \Psi^{u_1,u_0}(q_0)$ this implies that
\begin{equation}
\bar{g}(u,q) = \int_0^u \bar{f}_q(u_1,\Psi^{u_1,u}(q))
\end{equation}
(see also the abbreviation (1.22)).

Again, we ascend from (3.10) to (2.5), which now takes the form
\begin{equation}
L_u - pL_{up} + fL_{pp} = 0.
\end{equation}
With the definition $L_{pp} := \exp(\bar{g}) = \exp(F_q)$ we have
\begin{equation}
L(u,p) := \int_0^p \int_0^{p_1} \exp\left(F_q\left(u,\frac{1}{2}p_2^2\right)\right) dp_2 dp_1 - F(u).
\end{equation}
Here $F(u)$ is a suitable integration constant. To determine $F(u)$ we only have to evaluate (3.11) at $p = 0$ to get that
\begin{equation}
\frac{d}{du}F(u) = -L_u = fL_{pp} = f(u,0)\exp(F_q(u,0)).
\end{equation}

The requirements (3.12), (3.13) are satisfied for the Lagrange function $L$ defined in (1.21), (1.22). This proves Theorem 1.1.

4. Proof of Corollary 1.2

The proof of Corollary 1.2 proceeds by explicitly computing the integrals in the definition (1.20), (1.21) of the Lagrange function $L(u,p)$. Alternatively, of course, it is possible to verify directly that $V(u) = \int_0^1 L(u,u_x) \, dx$ is a Lyapunov function. This would not motivate the construction of $L$, however, in contrast to Matano’s elegant approach.

To evaluate the integrals (1.21) and (1.22), we first observe that the derivative $\eta(u_1)$ of the evolution $\Psi^{u_1,u_0}(q_0)$ of the characteristic ODE (1.19) with respect to the initial condition $q_0$,
\begin{equation}
\eta(u_1) := \Psi^{u_1,u_0}(q_0),
\end{equation}
satisfies the linearized characteristic equation

\[
\frac{d}{du_1} \eta(u_1) = - \bar{f}_q(u_1, \Psi^{u_1, u_0}(q_0)) \eta(u_1), \quad \eta(u_0) = 1.
\] (4.2)

Explicit integration of (4.2) gives us that

\[
\eta(u_1) = \exp \left( - \int_{u_0}^{u_1} \bar{f}_q(u_2, \Psi^{u_2, u_0}(q_0)) \, du_2 \right).
\] (4.3)

Inserting \( u_0 = u, \ u_1 = 0, \ q_0 = q \), we get from (1.22) that

\[
\exp(F_q(u, q)) = \eta(0) = \Psi^{0, u}_q(q).
\] (4.4)

Insertion of (4.4) with \( q = p_2^2 / 2 \) in the double integral \( L_1 \) in the definition (1.21) of the Lagrange function \( L(u, p) \) and subsequent integration by parts then yields

\[
L_1(u, p) := \int_{p_0}^{p} \int_{p_0}^{p_1} \exp \left( F_q \left( u, \frac{1}{2} p_2^2 \right) \right) \, dp_2 \, dp_1
= \int_{p_0}^{p} 1 \cdot \int_{p_0}^{p_1} \Psi^{0, u}_q \left( \frac{1}{2} p_2^2 \right) \, dp_2 \, dp_1
= \left[ p_1 \int_{p_0}^{p_1} \Psi^{0, u}_q \left( \frac{1}{2} p_2^2 \right) \, dp_2 \right]_{p_0}^{p} - \int_{p_0}^{p} p_1 \Psi^{0, u}_q \left( \frac{1}{2} p_1^2 \right) \, dp_1
= p \varphi(u, p) - \int_{p_0}^{\frac{1}{2} p_2^2} \Psi^{0, u}_q(q) \, dq
= p \varphi(u, p) - \Psi^{0, u}_q \left( \frac{1}{2} p_2^2 \right) + \Psi^{0, u}_q(0).
\] (4.5)

Here we have used the definition (1.26) of the auxiliary function \( \varphi \) and we have substituted \( q = p_2^2 / 2 \) in the integral.

To compute the remaining term \(-F(u)\) in \( L = L_1 - F \) (see (1.21)), we first observe that the evolution property of \( \Psi^{u_1, u_0} \) trivially implies that the partial derivative \( \Psi^{u_1, u_0}(q) \) satisfies the equation

\[
\Psi^{u_2, u_0}_u(q) = \lim_{\varepsilon \to 0} \varepsilon^{-1} \left( \Psi^{u_2, u_0 + \varepsilon}(q) - \Psi^{u_2, u_0}(q) \right)
= \Psi^{u_2, u_0}_q(q) \bar{f}(u, q).
\] (4.6)

by the chain rule and the definition (1.19) of the evolution \( \Psi \). Therefore, (1.22), (4.4), and (4.6) with \( u = u_1 \) and \( u_2 = q = 0 \) imply that

\[
-F(u) = - \int_{0}^{u} \bar{f}(u_1, 0) \exp(F_q(u_1, 0)) \, du_1
= - \int_{0}^{u} \bar{f}(u_1, 0) \Psi^{0, u_1}_q(0) \, du_1 = - \int_{0}^{u} \Psi^{0, u_1}_q(0) \, du_1
= - [\Psi^{0, u_1}(0)]_0^u = - \Psi^{0, u}(0).
\] (4.7)
Addition of (4.5) and (4.7) gives us that
\[ L(u, p) = L_1(u, p) - F(u) = p\varphi(u, p) - \Psi^{0,u}\left(\frac{1}{2}p^2\right), \]
as claimed in (1.28). This proves the corollary.

### 5. Reflection symmetry

In this section we study the parabolic PDE (1.1) under periodic boundary conditions \( x \in S^1 = \mathbb{R}/2\pi\mathbb{Z} \), as in (1.11). We consider non-linearities \( f = f(x, u, u_x) \) which are required to possess only the single reflection symmetry \( x \mapsto -x \), that is,
\[ f(-x, u, p) = f(x, u, p). \tag{5.1} \]

In the spirit of the old flow embedding result [17] we show that any planar flow
\[ \dot{a} = g(a, b), \quad \dot{b} = h(a, b) \tag{5.2} \]
can be realized by an embedding (5.5) (see below) in this class of PDEs under the condition that (5.2) is also reflection symmetric, that is,
\[ g(a, -b) = g(a, b), \]
\[ h(a, -b) = -h(a, b). \tag{5.3} \]

Since there exist reflection symmetric planar vector fields with non-stationary periodic orbits, PDEs (1.1) with the associated non-linearity \( f \) do not possess Lyapunov functions of the form (1.4), (1.9), (1.10).

Our realization of the ODE flow (5.2) will be in the invariant subspace
\[ E = \text{span}\{c, s\} \]
spanned by the first Fourier modes \( c = \cos x \), \( s = \sin x \). In fact, we define
\[ f(x, ac + bs, -as + bc) := (a + g(a, b))c + (b + h(a, b))s \tag{5.4} \]
for all \( x \in S^1 \) and \( a, b \in \mathbb{R} \). It is easy to see that this definition is correct, since in view of the relation
\[ \begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} c & s \\ -s & c \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \tag{5.5} \]
it is equivalent to the following:
\[ f(x, u, p) := (uc - ps + g(uc - ps, us + pc))c + (us + pc + h(uc - ps, us + pc))s \]
for all \( x, u, p \). Note how the reflection symmetry (5.3) of \( g, h \) implies the reflection symmetry (5.1) of \( f \). Substituting the Ansatz
\[ u(t, x) := a(t)c + b(t)s \in E \tag{5.6} \]
into the PDE (1.1), we see from (5.4) that such a \( u \) solves (1.1) if and only if the coefficients \( a(t) \) and \( b(t) \) satisfy the planar ODE (5.2). This proves our realization assertion and establishes the possibility of non-stationary periodic orbits.

An analogous construction based instead on the span of \( \cos(nx) \) and \( \sin(nx) \) shows the possibility of non-stationary periodic orbits in the presence of any finite number of reflection symmetries of the PDE (1.1) with respect to \( x \in S^1 \).
6. Concluding remarks

We briefly comment on the related problem of global attractors for the PDE (1.1), on generalizations to quasi-linear and non-linear equations, on finite-time blow-up of solutions, and finally, on the hidden extension to imaginary $p = u_x$ in our construction of the Lyapunov function $V = \int_0^1 L(x, u, u_x) \, dx$.

One purpose of Lyapunov functions is to reveal the gradient flow variational character of the PDE (1.1) on the circle. In particular, we prove convergence to equilibria. Under a dissipativeness assumption on $f$, the global attractor $\mathcal{A}_f$, that is, the bounded set of solutions which exist and stay uniformly bounded for all positive and negative times, has received much attention recently. In the presence of a Lyapunov function (1.4), (1.9), (1.10) the global attractor consists solely of equilibria and their heteroclinic orbits. In contrast, consider the SO(2)-equivariant case $f = f(u, u_x)$ on the circle $x \in S^1$, which does not admit a Lyapunov function. The global attractor $\mathcal{A}_f$ in this case consists of equilibria, rotating waves, and the heteroclinic orbits connecting them (see [12]). In [7] the heteroclinic connections were studied by, first, freezing all the rotating waves to become circles of non-homogeneous equilibria and, second, symmetrizing $f$ to become even with respect to $p = u_x$ by using suitable homotopies. The present paper then provides an explicit Lyapunov function for dealing with the symmetrized case of frozen waves. The main tool in [7] for studying the remaining heteroclinic orbits between equilibria was a Sturm nodal property going back to Sturm [18] (1836) (see also [1] and the references there). Hence we call such global attractors $\mathcal{A}_f$ Sturm attractors.

Matano in fact studied quasi-linear parabolic PDEs of the form

$$u_t = a(x, u, u_x)u_{xx} + f(x, u, u_x)$$

in [11], where $a$ is assumed to be uniformly positive. The derivation (2.4)–(2.11) then remains valid if we replace the substitution $u_{xx} = u_t - f$ by $u_{xx} = a^{-1}u_t - a^{-1}f$ there. In particular,

$$\dot{V} = -\int_0^1 a^{-1}L_{pp}(u_t)^2 \, dx,$$

and we just have to replace $f$ by $f/a$ in (2.5)–(2.11). Similarly, Theorem 1.1 remains valid for O(2)-equivariant

$$a = a(u, p) = \bar{a}\left(u, \frac{1}{2}p^2\right)$$

if we replace $\bar{f}$ by $\bar{f}/\bar{a}$ in (1.19)–(1.22) and replace $L_{pp}$ in (1.24), (1.25) by $L_{pp}/a$.

For fully non-linear parabolic equations

$$u_t = f(x, u, u_x, u_{xx})$$

and their equivariant variants a Lyapunov function is not known. Under separated boundary conditions convergence of bounded solutions to single equilibria may still be possible to prove, based on Sturm nodal properties. However, the technical ingredients are not yet sufficiently developed at present to provide a short proof here.
Returning to the $O(2)$-equivariant semilinear case (1.1) with $f = \bar{f}(u, u^{2}/2)$, we remark that it may be interesting to explore the consequences of our Lyapunov function for blow-up on the circle $x \in S^{1}$ (see also [6] for the case of separated boundary conditions). Basically, two different effects may occur. First, the Lagrangian integrand $L$ of the Lyapunov function in (1.23) may become unboundedly negative with respect to $u$. Second, the characteristics $q = q(u)$ in (1.19), (1.20) may already explode for finite values of $u$, terminating our very definition of the Lagrangian integrand $L$ in (1.21), (1.22). It would be interesting to compare this second phenomenon, which may occur for non-linearities $f$ which grow superquadratically with respect to the gradient $u_{x}$, with the gradient blow-up described in [14].

We conclude with a curious complex phenomenon in our construction of the Lagrangian integrand $L$ via the characteristics (1.19), (1.20). Let us first interpret the characteristic

$$\frac{dq}{du} = -\bar{f}(u, q). \quad (6.5)$$

As long as $q$ remains positive, it is easy to see that $q = q(u)$ solves (6.5) if and only if any solution of the equation

$$u_{x} = \pm \sqrt{2q(u(x))}, \quad (6.6)$$

with $u(x)$ in the indicated domain of positivity of $q$, is a solution of the equilibrium ODE

$$0 = u_{xx} + \bar{f}\left(u, \frac{1}{2}u^{2}_{x}\right) \quad (6.7)$$

of the PDE (1.1). For a proof we just multiply (6.7) by $u_{x}$ and compare the result with (6.5) using the chain rule applied to $\frac{d}{dx}q(u(x))$:

$$0 = \frac{d}{dx}\left(\frac{1}{2}u^{2}_{x}\right) + \bar{f}\left(u, \frac{1}{2}u^{2}_{x}\right)u_{x}$$

versus

$$0 = \frac{d}{dx}q(u) + \bar{f}(u, q(u))u_{x}. \quad (6.8)$$

A trivial example again arises if $\bar{f} = f(u)$ is independent of $q$, where

$$q = -F(u) + E, \quad (6.9)$$

with the primitive $F$ of $f$ and the energy $E$ of the second-order pendulum equation $u_{xx} + f(u) = 0$. Indeed, (6.6) integrates the pendulum equation, taking the form

$$u_{x} = \pm \sqrt{2(E - F(u))}. \quad (6.10)$$

Our evolution $\Psi^{u_{1},u_{0}}$ of the characteristic equation in (1.19), (1.20), however, does not stop at $q = 0$, but instead happily proceeds through negative $q = p^{2}/2$, that is, imaginary $p = u_{x}$, to re-emerge as positive in other regions of the phase plane $(u, q)$. It may therefore become a fascinating speculation to ponder the significance
of our simple Lyapunov function for extensions to complex, rather than just real, values of \( u \) and \( u_x \).

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