New refinements of the McKay conjecture for arbitrary finite groups

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Abstract

Let $G$ be an arbitrary finite group and fix a prime number $p$. The McKay conjecture asserts that $G$ and the normalizer in $G$ of a Sylow $p$-subgroup have equal numbers of irreducible characters with degrees not divisible by $p$. The Alperin-McKay conjecture is a version of this as applied to individual Brauer $p$-blocks of $G$. We offer evidence that perhaps much stronger forms of both of these conjectures are true.

1. Introduction and Conjecture A

Let $G$ be an arbitrary finite group and fix a prime number $p$. As is well known, there seem to be some mysterious and unexplained connections between the representation theory of $G$ and that of certain of its $p$-local subgroups. For example, it appears to be true that if $P$ is a Sylow $p$-subgroup of $G$ and $N = N_G(P)$, then equal numbers of the irreducible (complex) characters of $G$ and of $N$ have degrees not divisible by $p$. This “McKay Conjecture” was first proposed by J. McKay in [9], where it was stated in the case where $G$ is simple and $p = 2$. (See also [10].) The more general formulation of the conjecture, for arbitrary finite groups and arbitrary primes, was stated by J. L. Alperin in [1], although it was first suggested by the first author in [6], where it was proved for (solvable) groups of odd order. (In fact, in the odd-order case considered in [6], a natural bijection was constructed between the sets of irreducible characters of $p'$-degree of $G$ and of $N_G(P)$, but it is known that no natural bijection exists in general.) The McKay conjecture has been verified for several additional types of groups including $p$-solvable groups, symmetric groups and the sporadic simple groups. As yet, however, no one has given a proof, or has even proposed an explanation for why this conjecture might hold in the general case.

One could argue that the more precisely a conjecture can be stated, the better it will be understood and thus, perhaps, the easier it might become to discover a proof. In fact, the McKay conjecture was generalized and strength-
ened by Alperin, who formulated a version that applies to the Brauer $p$-blocks of $G$. (The Alperin-McKay conjecture was first proposed in [1]. We will review its statement in Section 2, where we discuss blocks.)

In this note we propose several further refinements of the McKay conjecture and the Alperin-McKay conjecture. To state the first of these, we define the integer $M_k(G)$, where $G$ is an arbitrary finite group and $k$ is an integer not divisible by our fixed prime $p$. We write $M_k(G)$ to denote the total number of irreducible characters of $G$ having degree congruent modulo $p$ to $\pm k$. For example, if $p = 5$ and $G$ is the alternating group $A_5$, we see that $M_1(G) = 2$ since $A_5$ has one irreducible character of degree 1 and one of degree 4. Also, $M_2(G) = 2$ since $A_5$ has two irreducible characters of degree 3. (Note that for odd primes $p$, the only values of $k$ that we need to consider are $1 \leq k \leq (p-1)/2$.)

**Conjecture A.** Let $G$ be an arbitrary finite group and let $N = N_G(P)$, where $P \in \text{Syl}_p(G)$. Then for each integer $k$ not divisible by $p$, we have $M_k(G) = M_k(N)$.

For example, if $p = 5$ and $G = A_5$, we saw that $M_1(G) = 2 = M_2(G)$. In this case, we know that $N$ is dihedral of order 10, and thus $N$ has two irreducible characters of degree 1 and two of degree 2 and we see that $M_1(N) = 2 = M_2(N)$, as predicted by Conjecture A.

Note that if $p = 2$ or $p = 3$, then $M_1(G)$ is the number of irreducible characters of $G$ of degree not divisible by $p$, and so for these two primes, Conjecture A is exactly equivalent to the McKay conjecture. For $p > 3$, however, we see that the number of irreducible characters of $G$ of degree not divisible by $p$ is the sum of the numbers $M_k(G)$ for $1 \leq k \leq (p-1)/2$, and so for these primes, Conjecture A is strictly stronger than the McKay conjecture.

But what is the evidence that our conjecture is true? In the case where $G$ has odd order, the bijection constructed in [6] carries a $p'$-degree character $\chi \in \text{Irr}(G)$ to a character $\xi \in \text{Irr}(N_G(P))$ such that $\chi(1) \equiv \pm \xi(1) \mod p$, and hence Conjecture A certainly holds in this case. More generally, but using some deep theory, the conjecture can be proved for all $p$-solvable groups. Also, P. Fong [5] has recently succeeded in proving it for all symmetric groups (for all primes). Furthermore, Conjecture A holds if the Sylow $p$-subgroup is cyclic. (This follows from E. C. Dade’s cyclic defect theory [3], and we shall have more to say about that in Section 2.)

If $G = \text{GL}(n, q)$, where $q$ is a power of the prime $p$, then it is known that all irreducible characters of $G$ of degree not divisible by $p$ have degrees congruent to $\pm 1 \mod p$. (In fact, these degrees are congruent to $\pm 1 \mod q$; this fact follows from results in [4].) Also, it is not too hard to see that all of the irreducible $p'$-degrees for the relevant Sylow normalizer of this group are powers
of \( q - 1 \), and hence they too are all congruent to \( \pm 1 \mod p \). Since the McKay conjecture is known to be true for \( \text{GL}(n, q) \) in the defining characteristic, (see [1]), it follows that Conjecture A is also true in this case. (In fact, with some additional work, the conjecture can also be checked for \( \text{SL}(n, q) \) in the defining characteristic.)

Finally, we mention that Conjecture A is also true for all primes for all of the sporadic simple groups. Since the McKay conjecture is known to hold for these groups [11] and since Conjecture A is automatically true when the Sylow \( p \)-subgroup is cyclic, we see that it suffices to check the conjecture for primes exceeding 3 and for which a Sylow subgroup is not cyclic. We have carried out this check, relying on the ATLAS for the irreducible character degrees and the paper [11] of R. A. Wilson for the Sylow normalizers of these groups. (However, Wilson’s paper has an error: the normalizer of a Sylow 5-subgroup in \( F_{i23} \) is incorrectly described. When we reported this to Wilson, he provided us with a corrected version.)

The following table gives the relevant data. The third column lists the numbers \( M_k(G) \) for \( 1 \leq k \leq (p - 1)/2 \). The McKay conjecture, of course, predicts only the sum of these numbers, while Conjecture A predicts each of the numbers in the third column.

| Group | Prime | Prime |
|-------|-------|-------|
| \( J_2 \) | 5     | 12    |
| \( HS \) | 5     | 9     |
| \( McL \) | 5     | 9     |
| \( He \) | 5     | 8     |
|       | 7     | 12    |
| \( Ru \) | 5     | 10    |
| \( Suz \) | 5     | 8     |
| \( O'N \) | 7     | 12    |
| \( CO_3 \) | 5     | 10    |
| \( CO_2 \) | 5     | 10    |
| \( Fi_{22} \) | 5     | 10    |
| \( HN \) | 5     | 10    |
| \( Ly \) | 5     | 25    |
| \( Th \) | 5     | 10    |
|       | 7     | 9     |
| \( Fi_{23} \) | 5     | 20    |
| \( Co_1 \) | 5     | 25    |
|       | 7     | 9     |
| \( J_4 \) | 11    | 12    |
| \( Fi_{24} \) | 5     | 28    |
| \( B \) | 5     | 25    |
|       | 7     | 27    |
| \( M \) | 5     | 40    |
|       | 7     | 49    |
|       | 11    | 10    |
|       | 13    | 12    |

Conjecture A has an amusing consequence for symmetric groups. (As we mentioned, this case of our conjecture has been proved by Fong.) The fact is that if \( n \geq p \), then all of the numbers \( M_k(S_n) \) are multiples of the prime \( p \). To prove this, of course, it suffices to check that the numbers \( M_k(N) \) are
multiples of $p$, where $N$ is the normalizer of a Sylow $p$-subgroup $P$ of the symmetric group $S_n$. It is not very hard to establish this fact using known information about the Sylow normalizers in symmetric groups. (See Fong’s paper [5] for more detail.)

Finally, we mention that there is a significant difference between Conjecture A and the many other conjectures that relate the representation theory of an arbitrary finite group to the $p$-local structure of the group. (In addition to the McKay conjecture, these include the Brauer height conjecture, the Alperin weight conjecture and the several variations and strengthenings of the latter that were formulated by E. C. Dade.) All of these conjectures deal only with the $p$-parts of irreducible character degrees while our Conjecture A, of course, is concerned also with $p'$-parts of character degrees. In the following section, we discuss Conjecture B, which to an even greater extent is also concerned with $p'$-parts of character degrees.

2. Blocks

Let $B$ be a $p$-block of an arbitrary finite group $G$ and let $D$ be a defect group for $B$. (Recall that $D$ is a $p$-subgroup of $G$ that is uniquely determined up to $G$-conjugacy by $B$.) As is customary, we will write $\text{Irr}(B)$ to denote the subset of $\text{Irr}(G)$ consisting of those characters that belong to the block $B$. Recall that the degrees of the characters in $\text{Irr}(B)$ are all divisible by $|P|/|D|$, where $P$ is a Sylow $p$-subgroup of $G$, and that those members of $\text{Irr}(B)$ whose $p$-part is exactly equal to $|P|/|D|$ are the “height zero” characters of $B$.

Now let $N = N_G(D)$. Recall that R. Brauer’s famous First Main Theorem asserts that there is a certain natural bijection between the set of $p$-blocks of $G$ having defect group $D$ and the set of $p$-blocks of $N$ having defect group $D$. If the block $b$ of $N$ corresponds to the block $B$ of $G$ under this bijection, we say that $b$ is the “Brauer correspondent” of $B$ with respect to the defect group $D$.

Consider the case where $D \in \text{Syl}_p(G)$, so that $B$ and $b$ are blocks of “maximal defect”. It is easy to see that the members of $\text{Irr}(G)$ having degree not divisible by $p$ are exactly all of the height zero characters in all $p$-blocks of $G$ that have maximal defect. Similarly, the members of $\text{Irr}(N)$ with degree not divisible by $p$ are just the height zero characters in the maximal defect $p$-blocks of $N$. The McKay conjecture asserts, therefore, that the total number of height zero characters in $p$-blocks of maximal defect of $G$ is equal to the total number of height zero characters in $p$-blocks of maximal defect of $N$. Since the latter blocks are exactly the Brauer correspondents of the former, it is reasonable to guess that for each $p$-block $B$ of maximal defect, the number of height zero characters in $\text{Irr}(B)$ is equal to the number of height zero characters in $\text{Irr}(b)$, where $b$ is the Brauer correspondent of $B$. 
In fact, it may be unnecessary to limit ourselves to blocks of maximal defect. Perhaps it is true for every $p$-block $B$ of $G$ that the numbers of height zero characters in $\text{Irr}(B)$ and $\text{Irr}(b)$ are equal, where $b$ is the Brauer correspondent of $B$ with respect to some defect group. This is precisely the Alperin-McKay conjecture, which appears as Conjecture 3 of Alperin's paper [1]. This conjecture has been shown to be valid for many families of groups.

Our Conjecture B strengthens the Alperin-McKay conjecture in the same way that Conjecture A strengthens the McKay conjecture. To state it, we need to define the integer $M_k(B)$, where $B$ is a $p$-block of $G$ and $k$ is an integer not divisible by $p$. We write $M_k(B)$ to denote the number height zero characters in $\text{Irr}(B)$ for which the $p'$-part of the degree is congruent modulo $p$ to $\pm k$. Note that if we hold $k$ fixed and sum $M_k(B)$ over all $p$-blocks $B$ of $G$ of maximal defect, we obtain the number $M_k(G)$. Also, it is clear that if $p > 2$ and we sum $M_k(B)$ for $1 \leq k \leq (p - 1)/2$, we get the total number of height zero characters in $\text{Irr}(B)$.

At this point, one might guess that it is always true that $M_k(B) = M_k(b)$, where $k$ is any integer not divisible by $p$ and $b$ is the Brauer correspondent of $B$ with respect to some defect group $D$. To see that this cannot be correct, however, consider the situation where $G$ is $p$-solvable. In this case, there is a subgroup $G_0$ of $G$ and a block $B_0$ of $G_0$ having defect group $D$ and such that the members of $\text{Irr}(B)$ are exactly the induced characters $\psi^G$, where $\psi$ runs over $\text{Irr}(B_0)$. Now let $N = N_G(D)$, write $N_0 = N \cap G_0$ and let $b_0$ be the Brauer correspondent of $B_0$. All of this can be done so that $D \in \text{Syl}_p(G_0)$ and the members of $\text{Irr}(b)$ are exactly the induced characters $\xi^N$, where $\xi$ runs over $\text{Irr}(b_0)$. Note that in this situation, the height zero characters in $\text{Irr}(B)$ are induced from the height zero characters in $\text{Irr}(B_0)$ and similarly, the height zero characters in $\text{Irr}(b)$ are induced from the height zero characters in $\text{Irr}(b_0)$.

Suppose now that we knew that $M_k(B_0) = M_k(b_0)$, for all integers $k$ not divisible by $p$. (And in fact, this can be proved using deep facts from the representation theory of $p$-solvable groups.) Since the height zero characters of $B$ and $b$ are obtained by induction from $G_0$ and $N_0$, respectively, it follows that $M_{rk}(B) = M_{rk}(b)$, where $r$ and $s$ are respectively the $p'$-parts of the indices $|G : G_0|$ and $|N : N_0|$. But in general, $r$ and $s$ are not congruent modulo $p$, and so we cannot expect that $M_k(B) = M_k(b)$ for all choices of $k$.

In this situation, let $c$ be the $p'$-part of $|G : N|$. Since $D \in \text{Syl}_p(G_0)$, we know by Sylow's theorem that $|G_0 : N_0| \equiv 1 \mod p$, and thus

$$r \equiv |G : G_0|_{p'} \equiv |G : N_0|_{p'} = |G : N|_{p'}|N : N_0|_{p'} \equiv cs \mod p.$$ 

We now have $M_{sk}(b) = M_{rk}(B) = M_{ckk}(B)$ for all integers $k$ that are not divisible by $p$, or equivalently, $M_{sk}(B) = M_k(b)$ for all such integers $k$. We conjecture that this fact about $p$-solvable groups holds in general.
Conjecture B.  Let $B$ be a $p$-block of an arbitrary finite group $G$ and suppose that $b$ is the Brauer correspondent of $B$ with respect to some defect group $D$. Then for each integer $k$ not divisible by $p$, we have $M_{ck}(B) = M_k(b)$, where $c = |G : N_G(D)|_{p'}$.

Observe that Conjecture B implies Conjecture A. To see why this is so, recall that $M_k(G)$ is the sum of the quantities $M_k(B)$ as $B$ runs over all $p$-blocks of $G$ of maximal defect. For each such block, however, Conjecture B asserts that $M_k(B) = M_k(b)$, where $b$ is the Brauer correspondent of $B$ with respect to some fixed Sylow $p$-subgroup $P$ of $G$. (This is because by Sylow’s theorem, the constant $c$ that appears in Conjecture B is congruent to 1 modulo $p$ in this case.) Conjecture A then follows since the sum of the quantities $M_k(b)$ is exactly $M_k(N)$, where $N = N_G(P)$.

As we have already indicated, Conjecture B is true for $p$-solvable groups. Also, Fong’s paper [5] establishes this conjecture for all symmetric groups (for all primes). There is one other situation where we know that our Conjecture B is valid.

(2.1) Theorem.  Conjecture B holds for all $p$-blocks that have cyclic defect groups.

Since Conjecture B implies Conjecture A, it follows that Conjecture A is valid for all groups $G$ having a cyclic Sylow $p$-subgroup. (This was mentioned in Section 1.)

To establish Theorem 2.1, we need to appeal to the deep theory of blocks with cyclic defect groups that was developed by Dade. A consequence of this theory in [3] is the following useful fact.

(2.2) Theorem.  Suppose that $B$ is a $p$-block of $G$ with cyclic defect group $D = \langle x \rangle$ and let $b$ be the Brauer correspondent of $B$ with respect to $D$. Then for each character $\chi \in \text{Irr}(B)$ there is a sign $\epsilon_{\chi} = \pm 1$ and a character $\tilde{\chi} \in \text{Irr}(b)$ such that
$$
\chi(xy) = \epsilon_{\chi} \tilde{\chi}(xy)
$$
for all $p$-regular elements $y \in C_G(x)$. Furthermore, the map $\chi \mapsto \tilde{\chi}$ defines a bijection from $\text{Irr}(B)$ onto $\text{Irr}(b)$.

Sketch of proof. Write $C = C_G(D) = N = N_G(D)$. According to Dade’s paper [3], there is a certain uniquely determined subgroup $E$ with $C \subseteq E \subseteq N$, where $|E : C| = e$ is a divisor of $p - 1$. (The uniqueness of $E$ depends on the fact that $N/C$ is abelian.) Dade shows in Theorem 1, Part 1, that the members of $\text{Irr}(B)$ are of two types. There are $e$ characters of the form $\chi_j$, where $j$ is an integer $1 \leq j \leq e$ and there are $(|D| - 1)/e$ characters $\chi_\lambda$, where $\lambda$ is a nonprincipal linear character of $D$. (Our notation, which differs slightly from
Dade’s, is set up so that \( \chi_\lambda = \chi_\mu \) if and only if \( \lambda \) and \( \mu \) are conjugate in \( E \).
Similarly, if we apply this reasoning to \( N \) in place of \( G \), with \( b \) in place of \( B \), we
get the same subgroup \( E \), and thus the members of \( \text{Irr}(b) \) can be parametrized
as the characters \( \psi_j \) with \( 1 \leq j \leq e \) and \( \psi_\lambda \), where \( \lambda \) is a nonprincipal linear
character of \( D \) and \( \psi_\lambda = \psi_\mu \) if and only if \( \lambda \) and \( \mu \) are conjugate in \( E \). We can
now define the bijection \( \chi \mapsto \tilde{\chi} \) by \( \chi_j \mapsto \psi_j \) for \( 1 \leq j \leq e \) and \( \chi_\lambda \mapsto \psi_\lambda \) if \( \lambda \) is
a linear character of \( D \).

Dade’s Corollary 1.9 gives formulas for the evaluation of \( \chi_j(x y) \) and \( \chi_\lambda(x y) \), and of course similar formulas can be used to compute \( \psi_j(x y) \) and \( \psi_\lambda(x y) \) by working in \( N \) instead of \( G \). (Note that since \( D = \langle x \rangle \), we must
set \( i = 0 \) in Corollary 1.9.) Examination of the right sides of the formulas in
Corollary 1.9 shows that everything is determined inside the group \( N \) except
for a constant \( \gamma_0 \) (which is always equal to 1 by Dade’s Equation 1.10) and
certain signs \( \epsilon_j \), which depend on the character. (All of the characters \( \chi_\lambda \) use
the same sign, \( \epsilon_0 \).) It follows that \( \chi(x y) \) and \( \tilde{\chi}(x y) \) agree, except possibly for
a sign depending on \( \chi \). This completes the proof. \( \square \)

Proof of Theorem 2.1. Let \( B \) be a \( p \)-block of \( G \) with cyclic defect group
\( D = \langle x \rangle \) and let \( b \) be the Brauer correspondent of \( B \) with respect to \( D \), so
that \( b \) is a \( p \)-block of \( N = N_G(D) \). We will show that the bijection \( \chi \mapsto \tilde{\chi} \)
of Theorem 2.2 maps the height zero characters in \( \text{Irr}(B) \) onto the height zero
characters in \( \text{Irr}(b) \). (Actually, it is true that in this case all of the members of
\( \text{Irr}(B) \) and \( \text{Irr}(b) \) have height zero, but we will not need that fact.) Also, we
will show that if \( \chi \) and \( \tilde{\chi} \) are height zero characters, then \( \chi(1)_p' \equiv \pm e\tilde{\chi}(1)_p' \)
mod \( p \), where \( c = |G : N|_p' \). The result will then follow.

Let \( K \) be a defect class for \( B \). In particular, \( D \) is a defect group for \( K \),
which means that there exists \( y \in K \) such that \( D \in \text{Syl}_p(C_G(y)) \), and thus
\( y \in C \), where \( C = C_G(D) = C_G(x) \). Also, because \( K \) is a defect class for \( B \),
we know that \( y \) is \( p \)-regular and that \( \lambda_B(\tilde{K}) \neq 0 \), where \( \lambda_B \) is the “central
homomorphism” corresponding to \( B \) and \( \tilde{K} \) is the sum of the elements of \( K \)
in the appropriate group ring. (Recall that for every class \( L \) of \( G \), we have
\( \lambda_B(\tilde{L}) = \omega_\chi(\tilde{L})^* \), where \( \chi \) is any member of \( \text{Irr}(B) \) and \( ( )^* \) is the canonical
homomorphism from the ring of \( p \)-local integers to its residue class field modulo
some fixed maximal ideal \( M \) containing \( p \).

Now let \( L = \text{cl}_G(x y) \) and note that \( D \in \text{Syl}_p(C_G(x y)) \), so that
\( |L|_p = \frac{|G|_p}{|D|} = |K|_p \). We claim now that \( \lambda_B(\tilde{L}) \neq 0 \). To see why this is so, let
\( \chi \in \text{Irr}(B) \) have height zero. Then \( |K|_p = \chi(1)_p = |L|_p \), and thus \( |K|/|\chi(1) \) and
\( |K|/|L| \) are \( p \)-local integers. Also, since \( \chi(x y) \equiv \chi(x y) \), where we are working
modulo the maximal ideal \( M \), it follows that
\[
\omega_\chi(\tilde{K}) = \frac{\chi(y)|K|}{\chi(1)} = \frac{\chi(x y)|K|}{\chi(1)} = \frac{\chi(x y)|L|}{|K|} = \omega_\chi(\tilde{L}) \frac{|K|}{|L|}.
\]
We now have
\[ 0 \neq \lambda_B(\hat{K}) = \omega_\chi(\hat{K})^* = \omega_\chi(\hat{L})^* \left( \frac{|K|}{|L|} \right)^* = \lambda_B(\hat{L}) \left( \frac{|K|}{|L|} \right)^* , \]
and it follows that \( \lambda_B(\hat{L}) \neq 0 \), as claimed.

By Lemma 15.46 of [7], we know that \( L \cap C \) is a class of \( N \), and thus \( L \cap C = c_{\lambda N}(xy) \). Since \( C_G(xy) \subseteq C \subseteq N \), we see that \( C_G(xy) = C_N(xy) \), and from this, we compute that \( |L| = |G : C_G(xy)| = |G : N|N : C_N(xy)| = |G : N||L \cap C| \). Observe that since \( b^G = B \), we have \( \lambda_B(\hat{L}) = \lambda_b(\hat{L} \cap C) \), and we write \( \alpha \) to denote this nonzero element of the residue class field of the \( p \)-local integers.

Now let \( \chi \in \text{Irr}(B) \) be arbitrary and write \( \psi = \chi \), in the notation of Theorem 2.2. We thus have
\[ \omega_\chi(\hat{L})^* = \alpha = \omega_\psi(L \cap C)^* . \]
Since \( L \cap C \) is a class of \( N \) with defect group \( D \), we see that \( \chi(1)/|L| \) and \( \psi(1)/|L \cap C| \) are \( p \)-local integers. Also, we observe that \( \chi \) has height zero in \( B \) precisely when \( \chi(1)/|L| \neq 0 \), and similarly, \( \psi \) has height zero in \( b \) if and only if \( \psi(1)/|L \cap C| \neq 0 \).

By Theorem 2.2, we have
\[ \frac{\chi(1)}{|L|} \omega_\chi(\hat{L}) = \chi(xy) = \pm \psi(xy) = \pm \frac{\psi(1)}{|L \cap C|} \omega_\psi(L \cap C) . \]
Since \( \chi(1)/|L| \) and \( \psi(1)/|L \cap C| \) are \( p \)-local integers, we deduce that
\[ \left( \frac{\chi(1)}{|L|} \right)^* \alpha = \pm \left( \frac{\psi(1)}{|L \cap C|} \right)^* \alpha , \]
and thus since \( \alpha \neq 0 \), we have \( \chi(1)/|L| \equiv \pm \psi(1)/|L \cap C| \). In particular, \( \chi \) has height zero if and only if \( \psi \) has height zero. If we multiply both sides by \( |L|p' = |G : N|p'|L \cap C|p'| = c|L \cap C|p' \), we obtain
\[ \frac{\chi(1)}{|L|p'} \equiv \pm c \frac{\psi(1)}{|L \cap C|p} \]
and since both sides of this congruence are rational integers, these numbers are actually congruent modulo \( p \). In the case where \( \chi \) and \( \psi \) have height zero, the integers \( \chi(1)/|L|_p \) and \( \psi(1)|L \cap C|_p \) are exactly the \( p' \)-parts of the degrees of \( \chi \) and \( \psi \), and so the result follows.

Our Conjecture B is related to some conjectures and results of M. Broué in [2]. As usual, suppose that \( B \) is a \( p \)-block of \( G \) and that \( b \) is its Brauer correspondent with respect to the defect group \( D \). Broué conjectures that in the case where \( D \) is abelian, there exists a “perfect isometry” between \( B \)
and $b$, and he proves this in the case where $D$ is cyclic. (It is known that a perfect isometry need not exist in the case where $D$ is nonabelian.) A perfect isometry, if it exists, would imply the existence of a certain bijection $\chi \mapsto \tilde{\chi}$ from $\text{Irr}(B)$ onto $\text{Irr}(b)$. In addition, Broué shows that if a perfect isometry exists, there would be some constant $c$, depending on the block $B$, such that $\chi(1) \equiv \pm c\tilde{\chi}(1) \mod p$ for all $\chi \in \text{Irr}(B)$. Of course, it follows in this case that for each integer $k$ not divisible by $p$, we would have (in our notation) $M_{ck}(B) = M_k(b)$. Furthermore, Broué shows that his constant $c$ is equal to 1 if $B$ is the principal block of $G$, but he does not evaluate the constant in other cases. (Note that according to Conjecture B, this constant should be 1 for every block of maximal defect, and not just for the principal block.) We mention that if $G$ has an abelian self-centralizing Sylow $p$-subgroup, then the principal block is the only $p$-block of maximal defect, and in that case, Broué’s perfect isometry conjecture would imply our Conjecture A.

3. Field automorphisms

There are other directions in which the McKay conjecture might be extended. Suppose, for example, that $G$ is a group for which the McKay conjecture holds in the strong sense that there is a canonical bijection $\chi \mapsto \tilde{\chi}$ from $\text{Irr}(G)$ onto $\text{Irr}(N)$. (Here $N = N_G(P)$, where $P \in \text{Syl}_p(G)$, and we are using the notation $\text{Irr}_p(X)$ to denote the subset of $\text{Irr}(X)$ consisting of characters of $p'$-degree.) In this case, we see that if $\sigma$ is any automorphism of the cyclotomic field $\mathbb{Q}[G]$, then $(\tilde{\chi})^\sigma = \tilde{\chi^\sigma}$, and thus in particular, the sets $\text{Irr}_p(G)$ and $\text{Irr}_p(N)$ would have equal numbers of $\sigma$-fixed members.

If $G$ is (solvable) of odd order, then there is such a canonical bijection, and the numbers of $\sigma$-fixed characters in $\text{Irr}_p(G)$ and $\text{Irr}_p(N)$ are equal for all choices of the field automorphism $\sigma$. But this fails for solvable groups in general. (For example, if $G = \text{GL}(2, 3)$ and $p = 3$, then all members of $\text{Irr}_p(N)$ are rational valued, but the same is not true for $\text{Irr}_p(G)$.) If we impose some conditions on the field automorphism $\sigma$, however, then the equality of the numbers of $\sigma$-fixed characters is known to hold for all $p$-solvable groups. (See Corollary C of [8].) We conjecture that under these conditions on $\sigma$, equality holds for all groups.

**Conjecture C.** Let $G$ be an arbitrary finite group and fix a prime $p$. Let $\sigma$ be an automorphism of the cyclotomic field $\mathbb{Q}[G]$ and assume that $\sigma$ has $p$-power order and that it fixes all $p'$-roots of unity in $\mathbb{Q}[G]$. Then $\sigma$ fixes equal numbers of characters in $\text{Irr}_p(G)$ and $\text{Irr}_p(N)$, where $N = N_G(P)$ and $P \in \text{Syl}_p(G)$. 
Of course, if we take \( \sigma \) to be the identity automorphism, we recover the McKay conjecture from Conjecture C. Another consequence of the conjecture is that the character table of a group \( G \) determines the exponent of the abelian group \( P/P' \), where, \( P \in \text{Syl}_p(G) \). To see why this is true, let \( N = N_G(P) \) and fix a positive integer \( n \). Let \( \sigma_n \) be the unique automorphism of \( Q_{|G|} \) that fixes all \( p' \)-roots of unity and maps every \( p \)-power root of unity \( \epsilon \) to \( \epsilon^{p^n+1} \). Then \( \sigma_n \) has \( p \)-power order and it fixes roots of unity of order \( p^n \) but not those of order \( p^{n+1} \) or higher. It is not hard to see from this that a necessary and sufficient condition for \( \sigma_n \) to fix every member of \( \text{Irr}(P/p')(N) \) is that \( P/P' \) has exponent at most \( n \). If Conjecture C is true, therefore, it follows that \( P/P' \) has exponent at most \( n \) if and only if \( \sigma_n \) fixes every member of \( \text{Irr}(G) \), and thus the exponent of \( P/P' \) is determined from the character table of \( G \), as claimed.

We also propose a block version of Conjecture C that generalizes the Alperin-McKay conjecture. To state it, we observe that if \( \sigma \) is a field automorphism fixing \( p' \)-roots of unity and \( \chi \in \text{Irr}(G) \), then \( \chi \) and \( \chi^\sigma \) necessarily belong to the same \( p \)-block of \( G \). (This is because these characters agree on all \( p' \)-elements of \( G \).) Such a field automorphism, therefore, permutes the set of height zero characters in \( \text{Irr}(B) \) for each \( p \)-block \( B \) of \( G \).

**Conjecture D.** Let \( B \) be a \( p \)-block for an arbitrary finite group \( G \) and suppose that \( b \) is the Brauer correspondent of \( B \) with respect to some defect group. Let \( \sigma \) be an automorphism of the cyclotomic field \( Q_{|G|} \) and assume that \( \sigma \) has \( p \)-power order and that it fixes all \( p' \)-roots of unity in \( Q_{|G|} \). Then \( \sigma \) fixes equal numbers of height zero characters in \( \text{Irr}(B) \) and \( \text{Irr}(b) \).

By Theorem G of [8], Conjecture D is known to hold for \( p \)-solvable groups. We present a proof of the conjecture in the case where the defect group of \( B \) is cyclic.

**Theorem.** Conjecture D is valid for blocks with cyclic defect group.

**Sketch of proof.** We suppose that the defect group \( D \) of \( B \) is cyclic and that \( b \) is the Brauer correspondent of \( B \) with respect to \( D \). As we observed in the proof of the Theorem 2.2, the members of \( \text{Irr}(B) \) are of two types. There are \( e \) characters \( \chi_j \), where \( 1 \leq j \leq e \) and there are \((|D|-1)/e\) different characters of the form \( \chi_\lambda \), where \( \lambda \) is a nonprincipal linear character of \( D \) and \( \chi_\lambda = \chi_\mu \) if and only if \( \lambda \) and \( \mu \) lie in the same \( E \)-orbit. Also, there is a similar parametrization of the members of \( \text{Irr}(b) \).

We will see that all of the characters \( \chi_j \) and \( \psi_j \) are fixed by \( \sigma \) and that \( \chi_\lambda \) and \( \psi_\lambda \) are fixed by \( \sigma \) if and only if the linear character \( \lambda \) is \( \sigma \)-fixed. The result will then follow.
Suppose that $\chi \in \text{Irr}(B)$ and that $g \in G$ is arbitrary. If $g$ is $p$-regular, we know that $\chi(g) = p$-rational, and thus $\chi(g) = \chi(g)^\sigma$. Also, if the $p$-part of $g$ is not conjugate to an element of $D$, then $\chi(g) = 0 = \chi(g)^\sigma$. It follows that to determine whether or not $\chi$ is $\sigma$-fixed, it suffices to consider only the values $\chi(xy)$, where $1 \neq x \in D$ and $y$ is a $p$-regular element in $C_G(x)$. We are thus exactly in the situation where Corollary 1.9 of [3] applies.

It is immediate from Dade’s Corollary 1.9 that if $\chi = \chi_j$ with $1 \leq j \leq e$, then $\chi(xy)$ is $p$-rational, and it follows that all of the characters $\chi_j$ are $\sigma$-fixed, as claimed. Also from Corollary 1.9, we see that $\chi_\lambda(xy)^\sigma = \chi_\lambda^e(xy)$.

If $\lambda$ is $\sigma$-fixed, it is now immediate that $\chi_\lambda$ is $\sigma$-fixed. Conversely, if $\chi_\lambda$ is $\sigma$-fixed and we write $\mu = \lambda^\sigma$, then we have $\chi_\lambda = \chi_\mu$, and thus the $E$-orbit containing $\lambda$ is invariant under $\sigma$. Since $\sigma$ has $p$-power order, however, and the $E$-orbit containing $\lambda$ has size $e$, which is not divisible by $p$, it follows that every member of this $E$-orbit is $\sigma$-fixed, and in particular $\lambda$ is $\sigma$-fixed, as desired.

We have now shown that as claimed, the $\sigma$-fixed members of $\text{Irr}(B)$ are exactly the characters $\chi_j$ with $1 \leq j \leq e$ and the characters $\chi_\lambda$, where $\lambda$ is $\sigma$-fixed. Exactly the same reasoning applies to the block $b$, and the proof is complete.

Of course, one could combine our Conjectures A and C. Perhaps, for example, it is true that for each integer $k$ not divisible by $p$ and each appropriate field automorphism $\sigma$, the groups $G$ and $N = N_G(P)$ always have equal numbers of $\sigma$-fixed characters with degrees congruent modulo $p$ to $\pm k$. Similarly, one could combine our Conjectures B and D, but we will refrain from stating these composite conjectures formally.

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