An Emptiness Algorithm for Regular Types with Set Operators

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Abstract. An algorithm to decide the emptiness of a regular type expression with set operators given a set of parameterised type definitions is presented. The algorithm can also be used to decide the equivalence of two regular type expressions and the inclusion of one regular type expression in another. The algorithm strictly generalises previous work in that tuple distributivity is not assumed and set operators are permitted in type expressions.

Keywords: type, emptiness, prescriptive type

1 Introduction

Types play an important role in programming languages [8]. They make programs easier to understand and help detect errors. Types have been introduced into logic programming in the forms of type checking and inference [16,21,28,31] or type analysis [16,21,28,31,12,32] or typed languages [16,21,28,31]. Recent logic programming systems allow the programmer to declare types for predicates and type errors are then detected either at compile time or at run time. The reader is referred to [27] for more details on types in logic programming.

A type is a possibly infinite set of ground terms with a finite representation. An integral part of any type system is its type language that specifies which sets of ground terms are types. To be useful, types should be closed under intersection, union and complement operations. The decision problems such as the emptiness of a type, inclusion of a type in another and equivalence of two types should be decidable. Regular term languages [14,8], called regular types, satisfy these conditions and have been used widely used as types [24,25,31,17,21,28,31,12,32,16,13,22,7,23].

Most type systems use tuple distributive regular types which are strictly less powerful than regular types [24,25,31,17,21,28,31,12,32,16,13,22,7,23].
Tuple distributive regular types are regular types closed under tuple distributive closure. Intuitively, the tuple distributive closure of a set of terms is the set of all terms constructed recursively by permuting each argument position among all terms that have the same function symbol [32].

This paper gives an algorithm to decide if a type expression denotes an empty set of terms. The correctness of the algorithm is proved and its complexity is analysed. The algorithm works on prescriptive types [28]. By prescriptive types, we mean that the meaning of a type is determined by a given set of type definitions. We allow parametric and overloading polymorphism in type definitions. Prescriptive types are useful both in compilers and other program manipulation tools such as debuggers because they are easy to understand for programmers. Type expressions may contain set operators with their usual interpretations. Thus, the algorithm can be used to decide the equivalence of two type expressions and the inclusion of one type expression in another. The introduction of set operators into type expressions allows concise and intuitive representation of regular types.

Though using regular term languages as types allow us to make use of theoretical results in the field of tree automata [14], algorithms for testing the emptiness of tree automata cannot be applied directly as type definitions may be parameterised. For instance, in order to decide the emptiness of a type expression given a set of type definitions, it would be necessary to construct a tree automaton from the type expression and the set of type definitions before an algorithm for determining the emptiness of an tree automaton can be used. When type definitions are parameterised, this would make it necessary to construct a different automaton each time the emptiness of a type expression is tested. Thus, an algorithm that works directly with type definitions is desirable as it avoids this repeated construction of automata.

Attempts have been made in the past to find algorithms for regular types [25,12,32,33,31,11,10]. To our knowledge, Dart and Zobel’s work [10] is the only one to present decision algorithms for emptiness and inclusion problems for prescriptive regular types without the tuple distributive restriction. Unfortunately, their decision algorithm for the inclusion problem is incorrect for regular types in general. See [24] for a counterexample. Moreover, the type language of Dart
and Zobel is less expressive than that considered in this paper since it doesn’t allow set operators and parameterised type definitions.

Set constraint solving has also been used in type checking and type inference \[1,2,20,11\]. However, set constraint solving methods are intended to infer descriptive types \[28\] rather than for testing emptiness of prescriptive types \[28\]. Therefore, they are useful in different settings from the algorithm presented in this paper. Moreover, algorithms proposed for set constraint solving \[3,4,2,1\] are not applicable to the emptiness problem we considered in this paper as they don’t take type definitions into account.

The remainder of this paper is organised as follows. Section 2 describes our language of type expressions and type definitions. Section 3 presents our algorithm for testing if a type expression denotes an empty set of terms. Section 4 addresses the of the algorithm. Section 5 presents the complexity of the algorithm and section 6 concludes the paper. Some lemmas are presented in the appendix.

2 Type Language

Let \( \Sigma \) be a fixed ranked alphabet. Each symbol in \( \Sigma \) is called a function symbol and has a fixed arity. It is assumed that \( \Sigma \) contains at least one constant that is a function symbol of arity 0. The arity of a symbol \( f \) is denoted as \( \text{arity}(f) \). \( \Sigma \) may be considered as the set of function symbols in a program. Let \( T(\Phi) \) be the set of all terms over \( \Phi \). \( T(\Sigma) \) is the set of all possible values that a program variable can take. We shall use regular term languages over \( \Sigma \) as types.

A type is represented by a ground term constructed from another ranked alphabet \( \Pi \) and \( \{\sqcap, \sqcup, \sim, 1, 0\} \), called type constructors. It is assumed that \( (\Pi \cup \{\sqcap, \sqcup, \sim, 1, 0\}) \cap \Sigma = \emptyset \). Thus, a type expression is a term in \( T(\Pi \cup \{\sqcap, \sqcup, \sim, 1, 0\}) \). The denotations of type constructors in \( \Pi \) are determined by type definitions whilst \( \sqcap, \sqcup, \sim, 1 \) and 0 have fixed denotations that will be given soon.

Several equivalent formalisms such as tree automata \[14,8\], regular term grammars \[14,10,8\] and regular unary logic programs \[32\] have been used to define regular types. We define types by type rules. A type rule is a production rule of the form \( c(\zeta_1, \ldots, \zeta_m) \rightarrow \tau \) where \( c \in \Pi \), \( \zeta_1, \ldots, \zeta_m \) are different type parameters and \( \tau \in T(\Sigma \cup \Pi \cup \Xi_m) \) where \( \Xi_m = \{\zeta_1, \ldots, \zeta_m\} \). The restriction that every type parameter in the righthand side of a type rule must occur in
the lefthand side of the type rule is often referred to as type preserving \[30\], and has been used in all the type definition formalisms.

Note that overloading of function symbols is permitted as a function symbol can appear in the righthand sides of many type rules. We denote by \(\Delta\) the set of all type rules and define \(\Xi \overset{\text{def}}{=} \bigcup_{c \in \Pi} \Xi_{\text{arity}(c)}\).

\(\langle \Pi, \Sigma, \Delta \rangle\) is a restricted form of context-free term grammar.

Example 1. Let \(\Sigma = \{0, s(), \text{nil}, \text{cons()},\}\) and \(\Pi = \{\text{Nat}, \text{Even}, \text{List}\}\). \(\Delta\) defines natural numbers, even numbers, and lists where

\[
\Delta = \left\{ \begin{array}{l}
\text{Nat} \rightarrow 0 | s(\text{Nat}), \\
\text{Even} \rightarrow 0 | s(\text{Even}), \\
\text{List}(\zeta) \rightarrow \text{nil} | \text{cons}(\zeta, \text{List}(\zeta))
\end{array} \right\}
\]

where, for instance, \(\text{Nat} \rightarrow 0 | s(\text{Nat})\) is an abbreviation of two rules \(\text{Nat} \rightarrow 0\) and \(\text{Nat} \rightarrow s(\text{Nat})\).

\(\Delta\) is called simplified if \(\tau\) in each production rule \(c(\zeta_1, \ldots, \zeta_m) \rightarrow \tau\) is of the form \(f(\tau_1, \ldots, \tau_n)\) such that each \(\tau_j\), for \(1 \leq j \leq n\), is either in \(\Xi_m\) or of the form \(d(\zeta_1', \ldots, \zeta_k')\) and \(\zeta_1', \ldots, \zeta_k' \in \Xi_m\). We shall assume that \(\Delta\) is simplified. There is no loss of generality to use a simplified set of type rules since every set of type rules can be simplified by introducing new type constructors and rewriting and adding type rules in the spirit of \[10\].

Example 2. The following is the simplified version of the set of type rules in example \[1\]. \(\Sigma = \{0, s(), \text{nil}, \text{cons()},\}\), \(\Pi = \{\text{Nat}, \text{Even}, \text{Odd}, \text{List}\}\) and

\[
\Delta = \left\{ \begin{array}{l}
\text{Nat} \rightarrow 0 | s(\text{Nat}), \\
\text{Even} \rightarrow 0 | s(\text{Odd}), \\
\text{Odd} \rightarrow s(\text{Even}), \\
\text{List}(\zeta) \rightarrow \text{nil} | \text{cons}(\zeta, \text{List}(\zeta))
\end{array} \right\}
\]

A type valuation \(\phi\) is a mapping from \(\Xi\) to \(T(\Pi \cup \{\sqcap, \sqcup, \sim, 1, 0\})\).

The instance \(\phi(R)\) of a production rule \(R\) under \(\phi\) is obtained by replacing each occurrence of each type parameter \(\zeta\) in \(R\) with \(\phi(\zeta)\). E.g., \(\text{List}(\text{Nat} \cap (\sim \text{Even})) \rightarrow \text{cons}(\text{Nat} \cap (\sim \text{Even}), \text{List}(\text{Nat} \cap (\sim \text{Even})))\) is the instance of \(\text{List}(\zeta) \rightarrow \text{cons}(\zeta, \text{List}(\zeta))\) under a type valuation that maps \(\zeta\) to \(\text{Nat} \cap (\sim \text{Even})\). Let

\[
\text{ground}(\Delta) \overset{\text{def}}{=} \{\phi(R) \mid R \in \Delta \land \phi \in (\Xi \mapsto T(\Pi \cup \{\sqcap, \sqcup, \sim, 1, 0\}))\}
\]

\(\cup \{1 \mapsto f(1, \ldots, 1) \mid f \in \Sigma\}\)
ground(Δ) is the set of all ground instances of grammar rules in Δ plus rules of the form \(1 \rightarrow f(1, \cdots, 1)\) for every \(f \in \Sigma\).

Given a set \(\Delta\) of type definitions, the type denoted by a type expression is determined by the following meaning function.

\[
\begin{align*}
[1]_\Delta & \overset{\text{def}}{=} T(\Sigma) \\
[0]_\Delta & \overset{\text{def}}{=} \emptyset \\
[E_1 \cap E_2]_\Delta & \overset{\text{def}}{=} [E_1]_\Delta \cap [E_2]_\Delta \\
[E_1 \cup E_2]_\Delta & \overset{\text{def}}{=} [E_1]_\Delta \cup [E_2]_\Delta \\
[\sim E]_\Delta & \overset{\text{def}}{=} T(\Sigma) - [E]_\Delta \\
[\omega]_\Delta & \overset{\text{def}}{=} \bigcup_{(\omega \rightarrow f(E_1, \cdots, E_n)) \in \text{ground}(\Delta)} \{f(t_1, \cdots, t_n) \mid \forall 1 \leq i \leq n. t_i \in [E_i]_\Delta\}
\end{align*}
\]

[\_]_\Delta gives fixed denotations to \(\cap, \cup, \sim, 1\) and 0. \(\cap, \cup \) and \(\sim\) are interpreted by [\_]_\Delta as set intersection, set union and set complement with respect to \(T(\Sigma)\). 1 denotes \(T(\Sigma)\) and 0 the empty set.

**Example 3.** Let \(\Delta\) be that in example 2. We have

\[
\begin{align*}
[Nat]_\Delta & = \{0, s(0), s(s(0)), \cdots\} \\
[Even]_\Delta & = \{0, s(s(0)), s(s(s(0))), \cdots\} \\
[Nat \cap \sim Even]_\Delta & = \{s(0), s(s(0)), s(s(s(0))), \cdots\} \\
[\text{List}(Nat \cap \sim Even)]_\Delta & = \{\text{cons}(s(0), \text{nil}), \text{cons}(s(s(0)), \text{nil}), \cdots\}
\end{align*}
\]

The lemma 5 in the appendix states that every type expression denotes a regular term language, that is, a regular type.

We extend [\_]_\Delta to sequences \(\theta\) of type expressions as follows.

\[
\begin{align*}
[\epsilon]_\Delta & \overset{\text{def}}{=} \{\epsilon\} \\
[\langle E \rangle \bullet \theta]_\Delta & \overset{\text{def}}{=} [E]_\Delta \times [\theta]_\Delta
\end{align*}
\]

where \(\epsilon\) is the empty sequence, \(\bullet\) is the infix sequence concatenation operator, \(\langle E \rangle\) is the sequence consisting of the type expression \(E\) and \(\times\) is the Cartesian product operator. As a sequence of type expressions, \(\epsilon\) can be thought of consisting of zero instance of 1. We use \(\Lambda\) to denote the sequence consisting of zero instance of 0 and define \([\Lambda]_\Delta = \emptyset\).

We shall call a sequence of type expressions simply a sequence. A sequence expression is an expression consisting of sequences of
the same length and $\cap$, $\cup$ and $\sim$. The length of the sequences in a sequence expression $\theta$ is called the dimension of $\theta$ and is denoted by $\|\theta\|$. Let $\theta, \theta_1$ and $\theta_2$ be sequence expressions of the same length.

$$\boxed{[\theta_1 \cap \theta_2]_\Delta \overset{\text{def}}{=} [\theta_1]_\Delta \cap [\theta_2]_\Delta}$$

$$\boxed{[\theta_1 \sqcup \theta_2]_\Delta \overset{\text{def}}{=} [\theta_1]_\Delta \cup [\theta_2]_\Delta}$$

$$\boxed{[\sim \theta]_\Delta \overset{\text{def}}{=} \mathcal{T}(\Sigma) \times \cdots \times \mathcal{T}(\Sigma) - [\theta]_\Delta \quad \|\theta\| \text{ times}}$$

A conjunctive sequence expression is a sequence expression of the form $\gamma_1 \land \cdots \land \gamma_m$ where $\gamma_i$ for, $1 \leq i \leq m$, are sequences.

3 Emptiness Algorithm

This section presents an algorithm that decides if a type expression denotes the empty set with respect to a given set of type definitions. The algorithm can also be used to decide if (the denotation of) one type expression is included in (the denotation of) another because $E_1$ is included in $E_2$ iff $E_1 \cap \sim E_2$ is empty.

We first introduce some terminology and notations. A type atom is a type expression of which the principal type constructor is not a set operator. A type literal is either a type atom or the complement of a type atom. A conjunctive type expression $C$ is of the form $\cap_{i \in I} l_i$ with $l_i$ being a type literal. Let $\alpha$ be a type atom. $\mathcal{F}(\alpha)$ defined below is the set of the principal function symbols of the terms in $[\alpha]_\Delta$.

$$\mathcal{F}(\alpha) \overset{\text{def}}{=} \{ f \in \Sigma \mid \exists \zeta_1 \cdots \zeta_k. ((\alpha \rightarrow f(\zeta_1, \cdots, \zeta_k)) \in \text{ground}(\Delta)) \}$$

Let $f \in \Sigma$. Define

$$\mathcal{A}_f^\alpha \overset{\text{def}}{=} \{ (\alpha, \cdots, \alpha_k) \mid (\alpha \rightarrow f(\alpha_1, \cdots, \alpha_k)) \in \text{ground}(\Delta) \}$$

We have $[\mathcal{A}_f^\alpha]_\Delta = \{ (t_1, \cdots, t_k) \mid f(t_1, \cdots, t_k) \in [\alpha]_\Delta \}$. Both $\mathcal{F}(\alpha)$ and $\mathcal{A}_f^\alpha$ are finite even though $\text{ground}(\Delta)$ is usually not finite.

The algorithm repeatedly reduces the emptiness problem of a type expression to the emptiness problems of sequence expressions and then reduces the emptiness problem of a sequence expression to the emptiness problems of type expressions. Tabulation is used to break down any possible loop and to ensure termination. Let $O$ be a type expression or a sequence expression. Define $\text{empty}(O) \overset{\text{def}}{=} ([O]_\Delta = \emptyset)$. 
3.1 Two Reduction Rules

We shall first sketch the two reduction rules and then add tabulation to form an algorithm. Initially the algorithm is to decide the validity of a formula of the form

\[ \text{empty}(E) \]  

(1)

where \( E \) is a type expression.

**Reduction Rule One.** The first reduction rule rewrites a formula of the form (1) into a conjunction of formulae of the following form.

\[ \text{empty}(\sigma) \]  

(2)

where \( \sigma \) is a sequence expression where \( \sim \) is applied to type expressions but not to any sequence expression.

It is obvious that a type expression has a unique (modulo equivalence of denotation) disjunctive normal form. Let \( \text{DNF}(E) \) be the disjunctive normal form of \( E \). \( \text{empty}(E) \) can written into \( \wedge C \in \text{DNF}(E) \text{empty}(C) \).

Each \( C \) is a conjunctive type expression. We assume that \( C \) contains at least one positive type literal. This doesn’t cause any loss of generality as \( [C] = [\{C\}] \) for any conjunctive type expression \( C \). We also assume that \( C \) doesn’t contain repeated occurrences of the same type literal.

Let \( C = \cap_{1 \leq i \leq m} \omega_i \cap \cap_{1 \leq j \leq n} \sim \tau_j \) where \( \omega_i \) and \( \tau_j \) are type atoms. The set of positive type literals in \( C \) is denoted as \( \text{pos}(C) \) and the set of complemented type atoms are denoted as \( \text{neg}(C) \).

\( \text{lit}(C) \) denotes the set of literals occurring in \( C \). By lemma 3 in the appendix, \( \text{empty}(C) \) is equivalent to

\[ \forall f \in \cap_{\alpha \in \text{pos}(C)} F(\alpha), \text{empty}((\cap_{\omega \in \text{pos}(C)} (\sqcup A_{\omega})) \cap (\cap_{\tau \in \text{neg}(C)} (\sim (\sqcup A_{\tau})))) \]  

(3)

The intuition behind the equivalence is as follows. \( [C]_\Delta \) is empty iff, for every function symbol \( f \), the set of the sequences \( \langle t_1, \ldots, t_k \rangle \) of terms such that \( f(t_1, \ldots, t_k) \in [C]_\Delta \) is empty. Only the function symbols in \( \cap_{\alpha \in \text{pos}(C)} F(\alpha) \) need to be considered.

We note the following two special cases of the formula (3).

(a) If \( \cap_{\alpha \in \text{pos}(C)} F(\alpha) = \emptyset \) then the formula (3) is true because \( \wedge \emptyset = \text{true} \). In particular, \( F(\emptyset) = \emptyset \). Thus, if \( \emptyset \in \text{pos}(C) \) then \( \cap_{\alpha \in \text{pos}(C)} F(\alpha) = \emptyset \) and hence the formula (3) is true.
(b) If $A_f^\tau = \emptyset$ for some $\tau \in neg(C)$ then $\bigcup A_f^\tau = \langle 0, \ldots, 0 \rangle$ and $\lnot (\bigcup A_f^\tau) = \langle 1, \ldots, 1 \rangle$. Thus, $\tau$ has no effect on the subformula for $f$ when $A_f^\tau = \emptyset$.

In order to get rid of complement operators over sequence sub-expressions, the complement operator in $\lnot (\bigcup A_f^\tau)$ is pushed inwards by the function $\text{push}$ defined in the following.

$$
\text{push}(\lnot (\bigcup_{i \in I} \gamma_i)) \overset{\text{def}}{=} \bigcap_{i \in I} \text{push}(\lnot \gamma_i)
$$

$$
\text{push}(\lnot \langle E_1, E_2, \ldots, E_k \rangle) \overset{\text{def}}{=} \bigcup_{1 \leq l \leq k} \langle 1, \ldots, 1, \lnot E_l, 1, \ldots, 1 \rangle \text{ for } k \geq 1
$$

$$
\text{push}(\lnot \epsilon) \overset{\text{def}}{=} \Lambda
$$

It follows from De Morgan’s law and the definition of $[,]_\Delta$ that $[\text{push}(\lnot (\bigcup A_f^\tau))]_\Delta = [\lnot (\bigcup A_f^\tau)]_\Delta$. Substituting $\text{push}(\lnot (\bigcup A_f^\tau))$ for $\lnot (\bigcup A_f^\tau)$ in the formula $[\ ]_\Delta$ gives rise to a formula of the form $[\ ]_\Delta$.

**Reduction Rule Two.** The second reduction rule rewrites a formula of the form $[\ ]_\Delta$ to a conjunction of disjunctions of formulae of the form $[\ ]_\Delta$. Formula $[\ ]_\Delta$ is written into a disjunction of formulae of the form.

$$
\text{empty}(\Gamma)
$$

where $\Gamma$ be a conjunctive sequence expression.

In the case $\|\Gamma\| = 0$, by lemma $[\ ]_\Delta$ in the appendix, $\text{empty}(\Gamma)$ can be decided without further reduction. If $\Lambda \in \Gamma$ then $\text{empty}(\Gamma)$ is true because $[\Lambda]_\Delta = \emptyset$. Otherwise, $\text{empty}(\Gamma)$ is false because $[\Gamma]_\Delta = \{\epsilon\}$.

In the case $\|\Gamma\| \neq 0$, $\text{empty}(\Gamma)$ is equivalent to

$$
\forall 1 \leq j \leq \|\Gamma\| \text{ empty}(\Gamma\downarrow j)
$$

where, letting $\Gamma = \gamma_1 \cap \cdots \cap \gamma_k$, $\Gamma\downarrow j \overset{\text{def}}{=} \cap_{1 \leq i \leq k} \gamma_i^j$ with $\gamma_i^j$ being the $j^{th}$ component of $\gamma_i$. Note that $\Gamma\downarrow j$ is a type expression and $\text{empty}(\Gamma\downarrow j)$ is of the form $[\ ]_\Delta$.

**3.2 Algorithm**

The two reduction rules in the previous section form the core of the algorithm. However, they alone cannot be used as an algorithm as a formula $\text{empty}(E)$ may reduce to a formula containing $\text{empty}(E)$.
as a sub-formula, leading to nontermination. Suppose $\Sigma = \{f(), a\}$, $\Pi = \{\text{Null}\}$ and $\Delta = \{\text{Null} \rightarrow f(\text{Null})\}$. Clearly, $\text{empty}(\text{Null})$ is true. However, by the first reduction rule, $\text{empty}(\text{Null})$ reduces to $\text{empty}(\langle \text{Null} \rangle)$ which then reduces to $\text{empty}(\text{Null})$ by the second reduction rule. This process will not terminate.

The solution, inspired by [10], is to remember in a table a particular kind of formulae of which truth is being tested. When a formula of that kind is tested, the table is first looked up. If the formula is implied by any formula in the table, then it is determined as true. Otherwise, the formula is added into the table and then reduced by a reduction rule.

The emptiness algorithm presented below remembers every conjunctive type expression of which emptiness is being tested. Thus the table is a set of conjunctive type expressions. Let $C_1$ and $C_2$ be conjunctive type expressions. We define $(C_1 \preceq C_2) \overset{\text{def}}{=} (\text{lit}(C_1) \supseteq \text{lit}(C_2))$. Since $C_i = \cap_{\alpha \in \text{lit}(C_i)} 1$, $C_1 \preceq C_2$ implies $[C_1]_{\Delta} \subseteq [C_2]_{\Delta}$ and hence $(C_1 \preceq C_2) \land \text{empty}(C_2)$ implies $\text{empty}(C_1)$.

Adding tabulation to the two reduction rules, we obtain the following algorithm for testing the emptiness of prescriptive regular types. Let $\mathcal{B}_C = (\cap_{\omega \in \text{pos}(C)} (\sqcup A^f_{\omega})) \cap (\cap_{\tau \in \text{neg}(C)} \text{push}(\neg (\sqcup A^f_{\tau})))$.

\begin{align*}
\text{etype}(E) & \overset{\text{def}}{=} \text{etype}(E, \emptyset) & (4) \\
\text{etype}(E, \Psi) & \overset{\text{def}}{=} \forall C \in \text{DNF}(E).\text{etype}_\text{conj}(C, \Psi) & (5) \\
\text{etype}_\text{conj}(C, \Psi) & \overset{\text{def}}{=} \\
& \begin{cases} 
\text{true,} & \text{if } \text{pos}(C) \cap \text{neg}(C) \neq \emptyset, \\
\text{true,} & \text{if } \exists C' \in \Psi. C \preceq C', \\
\forall f \in \cap_{\alpha \in \text{pos}(C)} \mathcal{F}(\alpha).\text{eseq}(\mathcal{B}_C, \Psi \cup \{C\}), & \text{otherwise.}
\end{cases} & (6) \\
\text{eseq}(\Theta, \Psi) & \overset{\text{def}}{=} \forall \Gamma \in \text{DNF}(\Theta).\text{eseq}_\text{conj}(\Gamma, \Psi) & (7) \\
\text{eseq}_\text{conj}(\Gamma, \Psi) & \overset{\text{def}}{=} \\
& \begin{cases} 
\text{true} & \text{if } \|\Gamma\| = 0 \land A \in \Gamma, \\
\text{false} & \text{if } \|\Gamma\| = 0 \land A \notin \Gamma, \\
\exists 1 \leq j \leq \|\Gamma\|.\text{etype}(\Gamma|j, \Psi) & \text{if } \|\Gamma\| \neq 0.
\end{cases} & (8)
\end{align*}

Equation 4 initialises the table to the empty set. Equations 5 and 6 implement the first reduction rule while equations 7 and 8 implement the second reduction rule. $\text{etype}(\cdot)$ and $\text{etype}_\text{conj}(\cdot)$ test the emptiness of an arbitrary type expression and that of a conjunctive type expression respectively. $\text{eseq}(\cdot)$ tests emptiness of a
sequence expression consisting of sequences and \( \sqcap \) and \( \sqcup \) operators while \( \text{eseq} \_\text{conj}(), \) tests the emptiness of a conjunctive sequence expression. The expression of which emptiness is to be tested is passed as the first argument to these functions. The table is passed as the second argument. It is used in \( \text{etype} \_\text{conj}(), \) to detect a conjunctive type expression of which emptiness is implied by the emptiness of a tabled conjunctive type expression. As we shall show later, this ensures the termination of the algorithm. Each of the four binary functions returns true iff the emptiness of the first argument is implied by the second argument and the set of type definitions.

Tabling any other kind of expressions such as arbitrary type expressions can also ensure termination. However, tabling conjunctive type expressions makes it easier to detect the implication of the emptiness of one expression by that of another because \( \text{lit}(C) \) can be easily computed given a conjunctive type expression \( C \). In an implementation, a conjunctive type expression \( C \) in the table can be represented as \( \text{lit}(C) \).

The first two definitions for \( \text{etype} \_\text{conj}(C, \Psi) \) in equation 6 terminates the algorithm when the emptiness of \( C \) can be decided by \( C \) and \( \Psi \) without using type definitions. The first definition also excludes from the table any conjunctive type expression that contains both a type atom and its complement.

3.3 Examples

We now illustrate the algorithm with some examples.

**Example 4.** Let type definitions be given as in example 2. The tree in figure 1 depicts the evaluation of \( \text{etype} (\text{Nat} \sqcap \lnot \text{Even} \sqcap \lnot \text{Odd}) \) by the algorithm. Nodes are labeled with function calls. We will identify a node with its label. Arcs from a node to its children are labeled with the number of the equation that is used to evaluate the node. Abbreviations used in the labels are defined in the legend to the right of the tree. Though \([A]_\Delta = [B]_\Delta\), \( A \) and \( B \) are syntactically different type expressions. The evaluation returns true, verifying \([\text{Nat} \sqcap \lnot \text{Even} \sqcap \lnot \text{Odd}]_\Delta = \emptyset\). Consider \( \text{etype} \_\text{conj}(B, \{A\}) \). We have \( B \preceq A \) as \( \text{lit}(A) = \text{lit}(B) \). Thus, by equation 6, \( \text{etype} \_\text{conj}(B, \{A\}) = \) true.
Example 5. Let type definitions be given as in example 3. The tree in figure 2 depicts the evaluation of $\text{etype}(\text{List}(\text{Even} \cap \sim \text{Nat}))$ by the algorithm. The evaluation returns false, verifying $[\text{List}((\text{Even} \cap \sim \text{Nat}))]_\Delta \neq \emptyset$. Indeed, $[\text{List}((\text{Even} \cap \sim \text{Nat}))]_\Delta = \{\text{nil}\}$. The rightmost node is not evaluated as its sibling returns false, which is enough to establish the falsity of their parent node.
Example 6. The following is a simplified version of the type definitions that is used in [24] to show the incorrectness of the algorithm by Dart and Zobel for testing inclusion of one regular type in another [10].

Let $\Pi = \{\alpha, \beta, \theta, \sigma, \omega, \zeta, \eta\}$, $\Sigma = \{a, b, g(), h()\}$ and

$$\Delta = \{\alpha \to g(\omega), \beta \to g(\theta) \mid g(\sigma), \theta \to a \mid h(\theta, \zeta), \sigma \to b \mid h(\sigma, \eta), \omega \to a \mid b \mid h(\omega, \zeta) \mid h(\omega, \eta), \zeta \to a, \eta \to b\}$$

Let $t = g(h(h(a, b), a)) \in [\alpha]_\Delta$ and $t \notin [\beta]_\Delta$, see example 3 in [24] for more details. So, $[\alpha]_\Delta \not\subseteq [\beta]_\Delta$. This is verified by our algorithm as follows. Let $\Psi_1 = \{\alpha\cap\beta\}$ and $\Psi_2 = \Psi_1 \cup \{\omega\cap\theta\cap\sigma\}$. By applying equations 4, 5, 6, 7, 8 and 5 in that order, we have $etype(\alpha\cap\beta) = etype_{conj}(\omega\cap\theta\cap\sigma, \Psi_1)$. By equation 6, we have $etype(\alpha\cap\beta) = eseq(\varepsilon\cap\Lambda\cap\varepsilon, \Psi_2) \land eseq(\varepsilon\cap\varepsilon\cap\varepsilon, \Psi_2) \land eseq(\Theta, \Psi_2)$ where $\Theta = ((\omega, \zeta)\cup(\omega, \eta))\cap((\sim\theta, 1)\cup\{1, \sim\zeta\})\cap((\sim\sigma, 1)\cup\{1, \sim\eta\})$. We choose not to simplify expressions such as $\varepsilon\cap\varepsilon\cap\sim\Lambda$ so as to make the example easy to follow. By applying equations 7 and 8, we have both $eseq(\varepsilon\cap\varepsilon\cap\varepsilon, \Psi_2) = true$ and $eseq(\varepsilon\cap\varepsilon\cap\Lambda, \Psi_2) = true$. So, $etype(\alpha\cap\beta) = eseq(\Theta, \Psi_2)$. Let $\Gamma = \langle\omega, \zeta\rangle\cap\langle\sim\theta, 1\rangle\cap\{1, \sim\eta\}$. To show $etype(\alpha\cap\beta) = false$, it suffices to show $eseq_{conj}(\Gamma, \Psi_2) = false$ by equation 6 because $\Gamma \in DNF(\Theta)$ and $etype(\alpha\cap\beta) = eseq(\Theta, \Psi_2)$.

Figure 3 depicts the evaluation of $eseq_{conj}(\Gamma, \Psi_2)$. The node that is linked to its parent by a dashed line is not evaluated because one of its siblings returns false, which is sufficient to establish the falsity of its parent. It is clear from the figure that $etype_{conj}(\Theta, \Psi_2) = false$ and hence $etype(\alpha\cap\beta) = false$.

4 Correctness

This section addresses the correctness of the algorithm. We shall first show that tabulation ensures the termination of the algorithm because the table can only be of finite size. We then establish the partial correctness of the algorithm.
4.1 Termination

Given a type expression $E$, a top-level type atom in $E$ is a type atom in $E$ that is not a sub-term of any type atom in $E$. The set of top-level type atoms in $E$ is denoted by $\text{TLA}(E)$. For instance, letting $E = \sim\text{List}(\text{Nat})\sqcup\text{Tree}(\text{Nat}\sqcap\sim\text{Even})$, $\text{TLA}(E) = \{\text{List}(\text{Nat}), \text{Tree}(\text{Nat}\sqcap\sim\text{Even})\}$. We extend $\text{TLA}(\cdot)$ to sequences by $\text{TLA}(\langle E_1, E_2, \ldots, E_k \rangle) \overset{\text{def}}{=} \bigcup_{1 \leq i \leq k} \text{TLA}(E_i)$.

Given a type expression $E_0$, the evaluation tree for $\text{etype}(E_0)$ contains nodes of the form $\text{etype}(E, \Psi)$, $\text{etype}\_\text{conj}(C, \Psi)$, $\text{eseq}(\Theta, \Psi)$ and $\text{eseq}\_\text{conj}(\Gamma, \Psi)$ in addition to the root that is $\text{etype}(E_0)$. Only nodes of the form $\text{etype}\_\text{conj}(C, \Psi)$ add conjunctive type expressions to the table. Other forms of nodes only pass the table around. Therefore, it suffices to show that the type atoms occurring in the first argument of the nodes are from a finite set because any conjunctive type expression added into the table is the first argument of a node of the form $\text{etype}\_\text{conj}(C, \Psi)$.

The set $\text{RTA}(E_0)$ of type atoms relevant to a type expression $E_0$ is the smallest set of type atoms satisfying

- $\text{TLA}(E_0) \subseteq \text{RTA}(E_0)$, and
- if $\tau$ is in $\text{RTA}(E_0)$ and $\tau \rightarrow f(\tau_1, \tau_2, \ldots, \tau_k)$ is in $\text{ground}(\Delta)$ then $\text{TLA}(\tau_i) \subseteq \text{RTA}(E_0)$ for $1 \leq i \leq k$.

Legend:

- $\Theta_1 = \langle \omega, \zeta \rangle \sqcup \langle \sim \theta, 1 \rangle \sqcup \langle 1, \sim \zeta \rangle$
- $\Psi_3 = \Psi_2 \cup \{\omega \sqcap \sim \theta\}$
- $\Psi_4 = \Psi_2 \cup \{\zeta \sqcap \sim \eta\}$
- $\Gamma = \langle \omega, \zeta \rangle \sqcup \langle \sim \theta, 1 \rangle \sqcup \langle 1, \sim \eta \rangle$
The height of $\tau_i$ is no more than that of $\tau$ for any $\tau \rightarrow f(\tau_1, \tau_2, \ldots, \tau_k)$ in $\text{ground}(\Delta)$. Thus, the height of any type atom in $\text{RTA}(E_0)$ is finite. There are only a finite number of type constructors in $\Pi_e$. Thus, $\text{RTA}(E_0)$ is of finite size. It follows by examining the algorithm that type atoms in the first argument of the nodes in the evaluation tree for $\text{etype}(E_0)$ are from $\text{RTA}(E_0)$ which is finite. Therefore, the algorithm terminates.

4.2 Partial Correctness

The partial correctness of the algorithm is established by showing $\text{etype}(E_0) = \text{true}$ iff $\text{empty}(E_0)$. Let $\Psi$ be a set of conjunctive type expressions. Define $\rho_\Psi \overset{\text{def}}{=} \land_{C \in \Psi} \text{empty}(C)$. The following two lemmas form the core of our proof of the partial correctness of the algorithm.

**Lemma 1.** Let $\Psi$ be a set of conjunctive type expressions, $E$ a type expression, $C$ a conjunctive type expression, $\Theta$ a sequence expression and $\Gamma$ a conjunctive sequence expression.

(a) If $\rho_\Psi \models \text{empty}(C)$ then $\text{etype}_{\text{conj}}(C, \Psi) = \text{true}$, and

(b) If $\rho_\Psi \models \text{empty}(E)$ then $\text{etype}(E, \Psi) = \text{true}$, and

(c) If $\rho_\Psi \models \text{empty}(\Gamma)$ then $\text{etype}(\Gamma, \Psi) = \text{true}$, and

(d) If $\rho_\Psi \models \text{empty}(\Theta)$ then $\text{etype}(\Theta, \Psi) = \text{true}$.

Proof. The proof is done by induction on the size of the complement of $\Psi$ with respect to the set of all possible conjunctive type expressions in which type atoms are from $\text{RTA}(E_0)$ where $E_0$ is a type expression.

**Basis.** The complement is empty. $\Psi$ contains all possible conjunctive type expressions in which type atoms are from $\text{RTA}(E_0)$. We have $C \in \Psi$ and hence $\text{etype}_{\text{conj}}(C, \Psi) = \text{true}$ by equation 6. Therefore, (a) holds. (b) follows from (a) and equation 5. (c) follows from (b), equation 8 and lemma 4 in the appendix, and (d) follows from (c) and equation 7.

**Induction.** By lemma 3 in the appendix, $\rho_\Psi \models \text{empty}(C)$ implies $\rho_\Psi \models \text{empty}(B_{C,f}^f)$ for any $f \in \cap_{\alpha \in \text{pos}(C)} F(\alpha)$. Thus, $\rho_{\Psi \cup \{C\}} \models \text{empty}(B_C)$. The complement of $\Psi \cup \{C\}$ is smaller than the complement of $\Psi$. By the induction hypothesis, we have $\text{eseq}(B_C^f, \Psi \cup \{C\}) = \text{true}$. By equation 6, $\text{etype}_{\text{conj}}(C, \Psi) = \text{true}$. Therefore, (a) holds. (b) follows from (a) and equation 3. (c) follows from (b), equation 8 and lemma 4 in the appendix and (d) follows from (c) and equation 7.

This completes the proof of the lemma.
Lemma 1 establishes the completeness of $\text{etype}(,)$, $\text{etype}_\text{conj}(,)$, $\text{eseq}(,)$ and $\text{eseq}_\text{conj}(,)$ while the following lemma establishes their soundness.

**Lemma 2.** Let $\Psi$ be a set of conjunctive type expressions, $E$ a type expression, $C$ a conjunctive type expression, $\Theta$ a sequence expression and $\Gamma$ a conjunctive sequence expression.

(a) $\rho_\Psi \models \text{empty}(C)$ if $\text{etype}_\text{conj}(C, \Psi) = \text{true}$, and
(b) $\rho_\Psi \models \text{empty}(E)$ if $\text{etype}(E, \Psi) = \text{true}$, and
(c) $\rho_\Psi \models \text{empty}(\Gamma)$ if $\text{etype}(\Gamma, \Psi) = \text{true}$, and
(d) $\rho_\Psi \models \text{empty}(\Theta)$ if $\text{etype}(\Theta, \Psi) = \text{true}$.

Proof. It suffices to prove (a) since (b), (c) and (d) follow from (a) as in lemma 1. The proof is done by induction on $\text{dp}(C, \Psi)$ the depth of the evaluation tree for $\text{etype}_\text{conj}(C, \Psi)$.

**Basis.** $\text{dp}(C, \Psi) = 1$. $\text{etype}_\text{conj}(C, \Psi) = \text{true}$ implies either (i) $\text{pos}(C) \cap \text{neg}(C) \neq \emptyset$ or (ii) $\exists C' \in \Psi.C \preceq C'$. In case (i), $\text{empty}(C)$ is true and $\rho_\Psi \models \text{empty}(C)$. Consider case (ii). By the definition of $\preceq$ and $\rho_\Psi$, we have $\text{etype}_\text{conj}(C, \Psi) = \text{true}$ implies $\rho_\Psi \models \text{empty}(C)$.

**Induction.** $\text{dp}(C, \Psi) > 1$. Assume $\text{etype}_\text{conj}(C, \Psi) = \text{true}$ and $\rho_\Psi \models \neg \text{empty}(C)$. By lemma 3, there is $f \in \cap_{\alpha \in \text{pos}(C)} F(\alpha)$ such that $\rho_\Psi \models \neg \text{empty}(B^f_C)$. We have $\rho_{\Psi \cup \{C\}} \models \neg \text{empty}(B^f_C)$. $\text{dp}(B^f_C, \Psi \cup \{C\}) < \text{dp}(C, \Psi)$. By the induction hypothesis, we have $\text{etype}(B^f_C, \Psi \cup \{C\}) = \text{false}$ for otherwise, $\rho_{\Psi \cup \{C\}} \models B^f_C$. By equation 4, $\text{etype}_\text{conj}(C, \Psi)$ is false which contradicts $\text{etype}_\text{conj}(C, \Psi) = \text{true}$. So, $\rho_\Psi \models \text{empty}(C)$ if $\text{etype}_\text{conj}(C, \Psi) = \text{true}$. This completes the induction and the proof of the lemma.

The following theorem is a corollary of lemmas 1 and 2.

**Theorem 1.** For any type expression $E$, $\text{etype}(E) = \text{true}$ if and only if $\text{empty}(E)$.

Proof. By equation 4, $\text{etype}(E) = \text{etype}(E, \emptyset)$. By lemma 1, (b) and lemma 2, (b), we have $\text{etype}(E, \emptyset) = \text{true}$ if $\rho_\emptyset \models \text{empty}(E)$. The result follows since $\rho_\emptyset = \text{true}$.
5 Complexity

We now address the issue of complexity of the algorithm. We only consider the worst-case time complexity of the algorithm. The time spent on evaluating $\text{etype}(E_0)$ for a given type expression $E_0$ can be measured in terms of the number of nodes in the evaluation tree for $\text{etype}(E_0)$.

The algorithm cycles through $\text{etype}(\cdot), \text{etype}\_\text{conj}(\cdot), \text{eseq}(\cdot)$ and $\text{eseq}\_\text{conj}(\cdot)$. Thus, children of a node of the form $\text{etype}(E, \Psi)$ can only be of the form $\text{etype}\_\text{conj}(C, \Psi)$, and so on.

Let $|S|$ be the number of elements in a given set $S$. The largest possible table in the evaluation of $\text{etype}(E_0)$ contains all the conjunctive type expressions of which type atoms are from $\text{RTA}(E_0)$. Therefore, the table can contain at most $2^{\text{|RTA}(E_0)|}$ conjunctive type expressions. So, the height of the tree is bounded by $O(2^{\text{|RTA}(E_0)|})$.

We now show that the branching factor of the tree is also bounded by $O(2^{\text{|RTA}(E_0)|})$. By equation 5, the number of children of $\text{etype}(E, \Psi)$ is bounded by two to the power of the number of type atoms in $E$ which is bounded by $|\text{RTA}(E_0)|$ because $E$ can only contain type atoms from $\text{RTA}(E_0)$. By equation 6, the number of children of $\text{etype}\_\text{conj}(C, \Psi)$ is bounded by $|\Sigma|$. The largest number of children of a node $\text{eseq}(\Theta, \Psi)$ is bounded by two to the power of the number of sequences in $\Theta$ where $\Theta = B_f C$. For each $\tau \in \text{neg}(C)$, $|\text{push}(\sim(\bot A_f))|$ is $O(\text{arity}(f))$ and $|C| < |\text{RTA}(E_0)|$. Thus, the number of sequences in $\Theta$ is $O(\text{arity}(f) \times |\text{RTA}(E_0)|)$ and hence the number of children of $\text{eseq}(\Theta, \Psi)$ is $O(2^{\text{|RTA}(E_0)|})$ since $\text{arity}(f)$ is a constant. By equation 8, the number of children of $\text{eseq}\_\text{conj}(\Gamma, \Psi)$ is bounded by $\max_{f \in \Sigma} \text{arity}(f)$. Therefore, the branching factor of the tree is bounded by $O(2^{\text{|RTA}(E_0)|})$.

The above discussion leads to the following conclusion.

Proposition 1. The time complexity of the algorithm is $O(2^{\text{|RTA}(E_0)|})$.

The fact that the algorithm is exponential in time is expected because the complexity coincides with the complexity of deciding the emptiness of any tree automaton constructed from the type expression and the type definitions. A deterministic frontier-to-root tree automaton recognising $[E_0]_{\Delta}$ will consist of $2^{\text{|RTA}(E_0)|}$ states as observed in the proof of lemma 4. It is well-known that the decision
of the emptiness of the language of a deterministic frontier-to-root tree automaton takes time polynomial in the number of the states of the tree automaton. Therefore, the worst-case complexity of the algorithm is the best we can expect from an algorithm for deciding the emptiness of regular types that contain set operators.

6 Conclusion

We have presented an algorithm for deciding the emptiness of prescriptive regular types. Type expressions are constructed from type constructors and set operators. Type definitions prescribe the meaning of type expressions.

The algorithm uses tabulation to ensure termination. Though the tabulation is inspired by Dart and Zobel [10], the decision problem we consider in this paper is more complex as type expressions may contain set operators. For that reason, the algorithm can also be used for inclusion and equivalence problems of regular types. The way we use tabulation leads to a correct algorithm for regular types while the Dart-Zobel algorithm has been proved incorrect for regular types [24] in general. To the best of our knowledge, our algorithm is the only correct algorithm for prescriptive regular types.

In addition to correctness, our algorithm generalises the work of Dart and Zobel [10] in that type expressions can contain set operators and type definitions can be parameterised. Parameterised type definitions are more natural than monomorphic type definitions [12,26,32] while set operators makes type expressions concise. The combination of these two features allows more natural type declarations. For instance, the type of the logic program append can be declared or inferred as append(List(α), List(β), List(α⊔β)).

The algorithm is exponential in time. This coincides with deciding the emptiness of the language recognised by a tree automaton constructed from the type expression and the type definitions. However, the algorithm avoids the construction of the tree automaton which cannot be constructed a priori when type definitions are parameterised.

Another related field is set constraint solving [3,20,18,11]. However, set constraint solving methods are intended to infer descriptive types [28] rather than for testing the emptiness of a prescriptive type [28]. Therefore, they are useful in different settings from the al-
algorithm presented in this paper. In addition, algorithms proposed for solving set constraints are not applicable to the emptiness problem we considered in this paper. Take for example the constructor rule in which states that emptiness of \( f(E_1, E_2, \ldots, E_m) \) is equivalent to the emptiness of \( E_i \) for some \( 1 \leq i \leq m \). However, \( \text{empty}(\text{List}(0)) \) is not equivalent to \( \text{empty}(0) \). The latter is true while the former is false since \( [\text{List}(0)]_\Delta = \{\text{nil}\} \). The constructor rule doesn’t apply because it deals with function symbols only but doesn’t take the type definitions into account.

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Appendix

Lemma 3. Let \( C \) be a conjunctive type expression. \( \text{empty}(C) \) iff
\[
\forall f \in \bigcap_{\alpha \in \text{pos}(C)} \mathcal{F}(\alpha).
\]
\( \text{empty}(\bigcap_{\omega \in \text{pos}(C)} (\bigcup \mathcal{A}_\omega) \cap (\bigcap_{\tau \in \text{neg}(C)} \sim (\bigcup \mathcal{A}_\tau))) \)

Proof. Let \( t \) be a sequence of terms and \( f \) a function symbol. By the definition of \( [\cdot]_\Delta \), \( f(t) \in [\cdot]_\Delta \) iff \( f \in \bigcap_{\alpha \in \text{pos}(C)} \mathcal{F}(\alpha) \) and \( t \in [\bigcap_{\omega \in \text{pos}(C)} (\bigcup \mathcal{A}_\omega)]_\Delta \setminus [\bigcap_{\tau \in \text{neg}(C)} (\bigcup \mathcal{A}_\tau)]_\Delta \). We have \( \forall f \in \bigcap_{\alpha \in \text{pos}(C)} \mathcal{F}(\alpha) \) and \( \exists f \) \( \text{empty}(\bigcap_{\omega \in \text{pos}(C)} (\bigcup \mathcal{A}_\omega) \cap (\bigcap_{\tau \in \text{neg}(C)} \sim (\bigcup \mathcal{A}_\tau))) \) for each \( f \in \bigcap_{\alpha \in \text{pos}(C)} \mathcal{F}(\alpha) \).

\[ \square \]

Lemma 4. Let \( \Gamma \) be a conjunctive sequence expression. Then
\[
\text{empty}(\Gamma) \iff \bigcup_{1 \leq j \leq |\Gamma|} \text{empty}(\Gamma \downarrow j)
\]

Proof. Let \( |\Gamma| = n \) and \( \Gamma = \gamma_1 \cap \gamma_2 \cap \cdots \cap \gamma_m \) with \( \gamma_i = \langle \gamma_{i,1}, \gamma_{i,2}, \cdots, \gamma_{i,n} \rangle \).
We have \( [\Gamma]_\Delta = \bigcap_{1 \leq j \leq m} [\gamma_j]_\Delta \). We have \( \Gamma \downarrow j = \gamma_{1,j} \cap \gamma_{2,j} \cap \cdots \cap \gamma_{m,j} \).
\( \exists 1 \leq j \leq n. \text{empty}(\Gamma \downarrow j) \iff \exists 1 \leq j \leq n. \cap_{1 \leq i \leq m} [\gamma_{i,j}]_\Delta = \emptyset \) iff \( [\Gamma]_\Delta = \emptyset \) iff \( \text{empty}(\Gamma) \).

\[ \square \]

Lemma 5. \([\mathcal{M}]_\Delta\) is a regular term language for any type expression \( \mathcal{M} \).

Proof. The proof is done by constructing a regular term grammar for \( \mathcal{M} \) \([\mathcal{F}]_\Delta\). We first consider the case \( \mathcal{M} \in T(\Pi \cup \{1, 0\}) \). Let \( R = \langle \text{RTA}(\mathcal{M}), \Sigma, \emptyset, \gamma, \mathcal{M} \rangle \) with
\[
\gamma = \{(\alpha \rightarrow f(\alpha_1, \cdots, \alpha_k)) \in \text{ground}(\Delta) \mid \alpha \in \text{RTA}(\mathcal{M})\}
\]
\( R \) is a regular term grammar. It now suffices to prove that \( t \in [\mathcal{M}]_\Delta \) iff \( \mathcal{M} \Rightarrow_\ast_R t \).

- Sufficiency. Assume \( \mathcal{M} \Rightarrow_\ast_R t \). The proof is done by induction on derivation steps in \( \mathcal{M} \Rightarrow_\ast_R t \).
  - Basis. \( \mathcal{M} \Rightarrow_\ast_R t \). \( t \) must be a constant and \( \mathcal{M} \rightarrow t \) is in \( \gamma \) which implies \( \mathcal{M} \rightarrow t \) is in \( \text{ground}(\Delta) \). By the definition of \( [\cdot]_\Delta \), \( t \in [\mathcal{M}]_\Delta \).

- Induction. Assume \( \mathcal{M} \Rightarrow_\ast_R \mathcal{M}_1 \rightarrow t \). By induction \( \mathcal{M}_1 \rightarrow t \) is in \( \text{ground}(\Delta) \). By the definition of \( [\cdot]_\Delta \), \( t \in [\mathcal{M}]_\Delta \).

- Concluding. Assume \( \mathcal{M} \Rightarrow_\ast_R \mathcal{M}_1 \rightarrow t \). By induction \( \mathcal{M}_1 \rightarrow t \) is in \( \text{ground}(\Delta) \). By the definition of \( [\cdot]_\Delta \), \( t \in [\mathcal{M}]_\Delta \).

\[ \square \]
• Induction. Suppose $M \Rightarrow f(M_1, \ldots, M_k) \Rightarrow_R^{(n-1)} t$. Then $t = f(t_1, \ldots, t_k)$ and $M_i \Rightarrow_R^{n_i} t$ with $n_i \leq (n - 1)$. By the induction hypothesis, $t_i \in \mathcal{M}_i$ and hence $t \in \mathcal{M}$ by the definition of $\Gamma$.

Necessity. Assume $t \in \mathcal{M}$. The proof is done by the height of $t$, denoted as height($t$).

- height($t$) = 0 implies that $t$ is a constant. $t \in \mathcal{M}$ implies that $M \Rightarrow t$ is in $\text{ground}(\Delta)$ and hence $M \Rightarrow t$ is in $\Upsilon$. Therefore, $M \Rightarrow_R t$.

- Let height($t$) = $n$. Then $t = f(t_1, \ldots, t_k)$, $t \in \mathcal{M}$ implies that $(M \Rightarrow f(M_1, \ldots, M_k)) \in \text{ground}(\Delta)$ and $t_i \in \mathcal{M}_i$. By the definition of $\Upsilon$, we have $(M \Rightarrow f(M_1, \ldots, M_k)) \in \Upsilon$. By the definition of $\text{RTA}(\cdot)$, we have $M_i \in \text{RTA}(M)$. By the induction hypothesis, $M_i \Rightarrow_R t_i$. Therefore, $M \Rightarrow_R f(M_1, \ldots, M_k) \Rightarrow_R f(t_1, \ldots, t_k) = t$.

Now consider the case $M \in T(\Pi \cup \{\sqcap, \sqcup, \sim, 1, 0\})$. We complete the proof by induction on the height of $M$.

- height($M$) = 0. Then $M$ doesn’t contain set operator. We have already proved that $\mathcal{M}_\Delta$ is a regular term language.

- Now suppose height($M$) = $n$. If $M$ doesn’t contain set operator then the lemma has already been proved. If the principal type constructor is one of set operators then the result follows immediately as regular term languages are closed under union, intersection and complement operators [14, 15, 8]. It now suffices to prove the case $M = c(M_1, \ldots, M_l)$ with $c \in \Pi$. Let $N = c(X_1, \ldots, X_l)$ where each $X_j$ is a different new type constructor of arity 0.

Let $\Pi’ = \Pi \{X_1, \ldots, X_l\}$, $\Sigma’ = \Sigma \cup \{x_1, \ldots, x_l\}$ and $\Delta’ = \Delta \cup \{X_j \rightarrow x_j | 1 \leq j \leq l\}$. $\mathcal{M}_\Delta$ is a regular term language on $\Sigma \cup \{x_1, \ldots, x_l\}$ because $N$ doesn’t contain set operators. By the induction hypothesis, $\mathcal{M}_j$ is a regular term language. By the definition of $\Gamma$, we have

$$\mathcal{M}_\Delta = \mathcal{M}_\Delta’[x_1 := [M_1]_\Delta, \ldots, x_l := [M_l]_\Delta]$$

which is a regular term language [14, 15, 8]. $S’, y_1 := S_{y_1}, \ldots,$ is the set of terms each of which is obtained from a term in $S$ by replacing each occurrence of $y_j$ with a (possibly different) term from $S_{y_j}$. This completes the induction and the proof.
The proof also indicates that a non-deterministic frontier-to-root tree automaton that recognises $[M]_{\Delta}$ has $|\text{RTA}(M)|$ states and that a deterministic frontier-to-root tree automaton that recognises $[M]_{\Delta}$ has $O(2^{|\text{RTA}(M)|})$ states.