Topologically Twisted $\mathcal{N} = (2, 2)$ Supersymmetric Yang-Mills Theory on Arbitrary Discretized Riemann Surface

So Matsuura$^1$, Tatsuhiro Misumi$^1$ and Kazutoshi Ohta$^2$

$^1$ Department of Physics in Hiyoshi Campus, Keio University, 4-1-1 Hiyoshi, Yokohama, 223-8521, Japan

$^2$ Institute of Physics, Meiji Gakuin University, Yokohama 244-8539, Japan

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Abstract

We define supersymmetric Yang-Mills theory on an arbitrary two-dimensional lattice (polygon decomposition) with preserving one supercharge. When a smooth Riemann surface $\Sigma_g$ with genus $g$ emerges as an appropriate continuum limit of the generic lattice, the discretized theory becomes topologically twisted $\mathcal{N} = (2, 2)$ supersymmetric Yang-Mills theory on $\Sigma_g$. If we adopt the usual square lattice as a special case of the discretization, our formulation is identical with Sugino’s lattice model. Although the tuning of parameters is generally required while taking the continuum limit, the number of the necessary parameters is at most two because of the gauge symmetry and the supersymmetry. In particular, we do not need any fine-tuning if we arrange the theory so as to possess an extra global $U(1)$ symmetry ($U(1)_R$ symmetry) which rotates the scalar fields.
1 Introduction

Since the middle of 80’s after the first success of numerical QCD simulations based on the lattice regularization, extension of the lattice technique to supersymmetric gauge theories has been pursued with great interest [1–4]. The hindrance encountered there was the fact that the regularization breaks the Poincaré invariance to its discrete subgroup and the supersymmetry can not be straightforwardly realized on the lattice. To date, however, several lattice formulations of supersymmetric gauge theories have been developed by bypassing this difficulty. In particular, for one or two-dimensional theories with extended supersymmetries, there are such lattice formulations that are free from fine-tuning in taking the continuum limit thanks to partially preserved supercharges on lattice.

In [5–18], some of supercharges are exactly preserved on hypercubic lattice by applying the so-called orbifolding procedure to supersymmetric matrix theory (mother theory)\(^1\). In these formulations, the bosonic link variables are not unitary but complex matrices, which restricts gauge groups to \(U(N)\) rather than \(SU(N)\). In numerical simulations, therefore, we must introduce a large mass in the \(U(1)\) part of the complex link variables in order to fix the lattice spacing and take care of the fermionic zero modes in computing the Dirac matrix [20–22]. In [23–28], the authors discretized topologically twisted gauge theories with preserving one or two supercharges. In these formulations, lattice gauge fields are expressed by compact link variables on the hypercubic lattice as in the conventional lattice gauge theories and we can choose the gauge group \(SU(N)\), which will be more convenient for numerical simulations [29–31]. In addition, the problem of vacuum degeneracy of lattice gauge fields pointed out in these models [24] has been recently solved without using an admissibility condition [32].

As for three- and four-dimensional supersymmetric theories, apart from the formulations [33,34] with exact chiral symmetry enabling the whole supersymmetry restoration in the continuum limit, lattice regularized gauge theories require parameter tunings in taking the continuum limit even if a part of supersymmetry is exactly preserved, since the symmetries on the lattice are generally insufficient to forbid relevant operators which break the rest symmetries\(^2\).

As a common feature of lattice gauge theories so far, the topology of the spacetime has

\(^1\) For a review, see [19].

\(^2\) As another approach to circumvent this issue, a hybrid regularization has been proposed for four-dimensional \(\mathcal{N} = 2\), 4 supersymmetric Yang-Mills theories [35–37], where two different discretizations by lattice and matrix [38–40] are combined. Regarding to the planar limit, four-dimensional \(\mathcal{N} = 4\) supersymmetric Yang-Mills theory can be obtained by using a large \(N\) reduction technique [11,12]. For recent development of direct numerical simulations of \(\mathcal{N} = 4\) SYM, see [43,47].
not been paid attention. Indeed, all the previous lattice formulations of supersymmetric theories are discretized on a periodic hypercubic lattice thus the topology is always torus. Although this is natural because the main interest in conventional lattice gauge theories is in the UV nature; the topology of the spacetime is usually irrelevant there, it is also true that the topology is sometimes quite important for supersymmetric gauge theories especially in the context of topological field theory [48]. Importance of such theories has been recently re-increasing, in relation to the height of the localization technique in supersymmetric gauge theories [49].

In this paper, we consider topologically twisted two-dimensional $\mathcal{N} = (2, 2)$ supersymmetric Yang-Mills theory on a generic Riemann surface. We discretize the Riemann surface to an arbitrary lattice (polygons) and propose a way to define the supersymmetric gauge theory on it with preserving a supercharge. We show that we can define the theory on any decomposition of the two-dimensional surface and the tree-level continuum limit reproduces the continuum theory. We see that, if we consider the usual square lattice as a special case of the discretization, our formulation coincides with Sugino’s formulation [23–26]. We discuss that there are two types of theories depending on the hermiticity of the scalar fields; the theories with and without an extra global $U(1)$ symmetry ($U(1)_R$ symmetry). If the theory has this symmetry, we can take the continuum limit without any fine-tuning, while we need one-parameter (two-parameter) tuning in taking the continuum limit if the theory does not have this symmetry and the gauge group is $SU(N)$ ($U(N)$).

This paper is organized as follows. In the next section, we briefly review the continuum topologically twisted two-dimensional $\mathcal{N} = (2, 2)$ supersymmetric Yang-Mills theory on a curved background. In section 3, we define the theory on a general lattice and discuss the continuum limit and possible radiative corrections. The section 4 is devoted to the conclusion and discussion. In appendix A, we calculate the continuum limit of a face variable in detail.

2 Continuum two-dimensional $\mathcal{N} = (2, 2)$ supersymmetric Yang-Mills theory

We start with the two-dimensional $\mathcal{N} = (2, 2)$ supersymmetric Yang-Mills theory on a flat Euclidean spacetime, which is obtained from a dimensional reduction of four-dimensional
\( N = 1 \) supersymmetric Yang-Mills theory:

\[
S = \frac{1}{2g_{2d}^2} \int d^2x \, \text{Tr} \left\{ \frac{1}{2} F_{\mu\nu}^2 + (D_\mu \Phi) (D_\mu \bar{\Phi}) + \frac{1}{4} [\Phi, \bar{\Phi}]^2 \\
+ i\bar{\Psi} \Gamma_\mu D_\mu \Psi - \frac{1}{2} \bar{\Psi} \Gamma_+ [\Phi, \Psi] - \frac{1}{2} \bar{\Psi} \Gamma_- [\Phi, \Psi] \right\}, \tag{2.1}
\]

where \( \mu, \nu = 1, 2, \Gamma_\mu, \Gamma_\pm = \Gamma_3 \pm i\Gamma_4 \) are four-dimensional Dirac matrices satisfying \( \{\Gamma_M, \Gamma_N\} = -2\delta_{MN} \) \((M, N = 1, \ldots, 4)\), \( \Psi \) is a four-component spinor, \( \bar{\Psi} = -i\Psi^T \Gamma_4 \), \( F_{\mu\nu} \) is the field strength of a gauge field \( A_\mu \), and \( \Phi \) and \( \bar{\Phi} \) are scalar fields. We assume that the gauge group \( G \) is \( U(N) \) or \( SU(N) \) in the following.

We fix the notation of the gamma matrices by

\[
\Gamma_1 = \begin{pmatrix} i\sigma_3 & i\sigma_3 \\ i\sigma_3 & i\sigma_3 \end{pmatrix}, \quad \Gamma_2 = \begin{pmatrix} i\sigma_1 & i\sigma_1 \\ i\sigma_1 & i\sigma_1 \end{pmatrix}, \quad \Gamma_3 = \begin{pmatrix} \sigma_2 & -\sigma_2 \\ -\sigma_2 & \sigma_2 \end{pmatrix}, \quad \Gamma_4 = \begin{pmatrix} -i\sigma_2 & -i\sigma_2 \\ -i\sigma_2 & -i\sigma_2 \end{pmatrix}, \tag{2.2}
\]

and express the components of the spinor \( \Psi \) as

\[
\Psi = (\lambda_1, \lambda_2, \chi, \eta/2)^T. \tag{2.3}
\]

Then (2.1) reduces to

\[
S = \frac{1}{2g_{2d}^2} \int d^2x \, \text{Tr} \left\{ \frac{1}{2} F_{\mu\nu}^2 + (D_\mu \Phi) (D_\mu \bar{\Phi}) + \frac{1}{4} [\Phi, \bar{\Phi}]^2 \\
+ i\eta D_\mu \lambda_\mu + 2i\chi (D_1 \lambda_2 - D_2 \lambda_1) + \lambda_\mu \left[ \Phi, \lambda_\mu \right] - \chi [\Phi, \chi] - \frac{1}{4} \eta [\Phi, \eta] \right\}. \tag{2.4}
\]

We see that (2.1) (and of course (2.4)) is invariant under the supersymmetric transformation,

\[
\delta \Phi = -i\bar{\xi} \Gamma_+ \Psi, \quad \delta \bar{\Phi} = -i\bar{\xi} \Gamma_- \Psi, \quad \delta A_\mu = -i\bar{\xi} \Gamma_\mu \Psi, \\
\delta \Psi = -F_{12} \Gamma_{12} \xi - \frac{i}{2} (D_\mu \bar{\Phi}) \gamma_{\mu+} \xi - \frac{i}{2} (D_\mu \Phi) \Gamma_{\mu-} \xi - \frac{i}{4} \left[ \Phi, \bar{\Phi} \right] \Gamma_{+} \xi, \tag{2.5}
\]

where \( \xi \) is a four-component spinor parameter and \( \Gamma_{MN} \equiv \frac{1}{2} \left[ \Gamma_M, \Gamma_N \right] \).

Now let us consider a specific SUSY transformation associated with the parameter \( \xi = (0, 0, 0, \epsilon)^T \) and define the corresponding supercharge \( \hat{Q} \) as\(^3\)

\[
\delta \phi \equiv -i\epsilon \left( \hat{Q} \phi \right), \tag{2.6}
\]

\(^3\) Here we have put a hat on the supercharge \( Q \) in order to distinguish it from the one appeared in the discretized theory in the next section.
for an arbitrary field $\phi$. We can read off the $\hat{Q}$-transformation of the fields as

\[
\begin{align*}
\hat{Q}\Phi &= 0, \\
\hat{Q}\bar{\Phi} &= \eta, \\
\hat{Q}A_\mu &= \lambda_\mu, \\
\hat{Q}\lambda_\mu &= iD_\mu\Phi, \\
\hat{Q}Y &= [\Phi, \chi], \\
\hat{Q}\chi &= Y,
\end{align*}
\]

where $Y$ is an auxiliary field. Then the action (2.4) can be expressed in the $\hat{Q}$-exact or topologically twisted form \[48, 50\] by

\[
S = \frac{1}{2g_{2d}^2} \int d^2x \sqrt{g} \text{Tr} \left[ \frac{1}{4} \eta [\Phi, \bar{\Phi}] - i\lambda^\mu D_\mu\Phi + \chi (Y - 2iF_{12}) \right].
\]

(2.8)

It is important that the $\hat{Q}^2$ is equal to the infinitesimal gauge transformation with a parameter $\Phi$. Since $\hat{Q}$ is acting on a gauge invariant expression in the action (2.8), the $\hat{Q}$-invariance of the action is manifest.

We next extend the above theory to that on a curved background. One of the motivations to consider topological twist is to preserve a partial supersymmetry in a curved background \[51\]. The supersymmetry we usually use is completely broken on a curved background because there is no covariantly constant spinor in general. However, by twisting the local Lorentz symmetry with R-symmetry, there can appear “scalar supercharges” which are preserved in any curved background. The supercharge $\hat{Q}$ in (2.8) becomes the scalar supercharge as it is, and thus we can define topological Yang-Mills theory on the curved spacetime with keeping $\hat{Q}$ as

\[
S = \hat{Q} \frac{1}{2g_{2d}^2} \int d^2x \sqrt{g} \text{Tr} \left[ \frac{1}{4} \eta [\Phi, \bar{\Phi}] - i\lambda^\mu D_\mu\Phi + \chi (H - 2iF) \right],
\]

(2.9)

where the covariant derivative $D_\mu$ now includes not only the gauge field but also the spacetime connection, $\hat{Q}$ is the same with (2.7), $\Sigma_g$ is an oriented or un-oriented two-dimensional manifold with the metric $g_{\mu\nu}$ and $f(x) = \frac{1}{2} \frac{i\omega^\mu}{\sqrt{g(x)}} F^\mu_\nu(x)$ is the Poincaré dual of the field strength. Because of the deformation of the background, the other three supersymmetries are broken in general.

Here we make some comments. First, the operations of twisting and curving do not commute. The action (2.9) is obtained by twisting the theory on the flat spacetime followed by curving the background. This theory differs from the one obtained by first curving the background followed by twisting (or renaming the fermionic fields). In the following section, we discretize the former (topological) theory. Therefore, even if we take $^4\Sigma_g$ can have even boundaries. In that case, we take the free boundary condition for simplicity.
the continuum limit, we do not obtain the latter (physical) theory. We note that it does
not conflict with the fact that the continuum limit of Sugino’s lattice formulation is the
physical supersymmetric gauge theory \[23–26\]. This is because Sugino’s formulation is
defined on a flat spacetime where the physical theory and the topological theory coincide
and twisting is merely a renaming of the fields.

Second, we can choose the hermiticity of the scalar fields $\Phi(x)$ and $\bar{\Phi}(x)$. They are
usually regarded as hermitian conjugate with each other from the construction; they are
originally related with the components of the gauge fields of the four-dimensional theory
as $\Phi = A_3 + iA_4$ and $\bar{\Phi} = A_3 - iA_4$. In this case, the theory possesses the $U(1)_R$ symmetry,

$$
\Phi \rightarrow e^{2i\alpha}\Phi, \quad \bar{\Phi} \rightarrow e^{-2i\alpha}\bar{\Phi}, \quad A_\mu \rightarrow A_\mu, \\
\eta \rightarrow e^{-i\alpha}\eta, \quad \lambda_\mu \rightarrow e^{i\alpha}\lambda_\mu, \quad \chi \rightarrow e^{-i\alpha}\chi.
$$

On the other hand, as often adopted in the context of the topological field theory, we
can instead regard $\Phi(x)$ and $\bar{\Phi}(x)$ as independent hermitian variables. As a result, it
is impossible to impose the $U(1)_R$ rotation (2.10). This choice quite changes the theory.
For example, the expectation value $\langle \int d^2x \sqrt{g(x)} \text{Tr}(\Phi(x)^n) \rangle$ is zero in the former theory
because of the $U(1)_R$ symmetry (2.11) but it takes some non-trivial value in the latter
theory. We can consider both the theories depending of the purpose and we can use the
same discretization explaining in the next section.

3 $\mathcal{N} = (2, 2)$ supersymmetric Yang-Mills theory on arbitrary discretized Riemann surface

In this section, we discretize the continuum theory described in the previous section on a
given decomposition of the two-dimensional surface, that is, a set of sites, links and faces.
As mentioned in the previous section, we can use the same discretization if we regard the
scalar field $\Phi$ as either complex or hermitian so we do not specify it in constructing the
discretized formulation. We will see, however, that this choice is crucial in considering
radiative corrections.
3.1 Definition of the model

A polygon decomposition of the two-dimensional surface consists of a set of sites $S$, links $L$ and faces $F$, respectively:

$$S \equiv \{s|s = 1, \ldots, N_S\},$$

$$L \equiv \{(st)|s, t \in S\},$$

$$F \equiv \{(s_1, \ldots, s_n)|s_1, \ldots, s_n \in S, (s_i, s_{i+1}) \in L \text{ or } (s_{i+1}, s_i) \in L\},$$

where $N_S$ is the number of sites, a link $⟨st⟩$ possesses a direction from $s$ to $t$, and a face $(s_1, \ldots, s_n)$ is a surface surrounded by the links $⟨s_is_{i+1}⟩$ $(i = 1, \ldots, n)$.

We sometimes call the first site $s_1$ of the face $f$ as the representative point (site) of the face $f$. This is apparently a generalization of the usual square lattice which is given by the data,

$$S = \{\vec{X} = (x, y)|1 \leq x \leq L_x, 1 \leq y \leq L_y\},$$

$$L = \\{\langle \vec{X} \vec{X} + \hat{x} \rangle, \langle \vec{X} \vec{X} + \hat{y} \rangle | \vec{X} \in S\},$$

$$F = \\{(\vec{X}, \vec{X} + \hat{x}, \vec{X} + \hat{x} + \hat{y}, \vec{X} + \hat{y}) | \vec{X} \in S\}.$$

We next consider the following “fields” associated with the sites, links and faces of a given decomposition, respectively:

$$\Phi_s, \bar{\Phi}_s, \eta_s : \text{site variables } (s \in S),$$

$$U_{st}, \Lambda_{st} : \text{link variables } (⟨st⟩ \in L),$$

$$Y_f, \chi_f : \text{face variables } (f \in F),$$

where $\Phi_s, \bar{\Phi}_s, U_{st}$ and $Y_f$ are bosonic variables and $\eta_s, \Lambda_{st}$ and $\chi_f$ are fermionic variables. We assume that the site variables $\Phi_s, \bar{\Phi}_s$ and $\eta_s$ live on the site $s$, the link variables $U_{st}$ and $\Lambda_{st}$ live on the link $⟨st⟩$, and the face variables $Y_f$ and $\chi_f$ live on the representative point of the face $f$. We often express the link fermion $\Lambda_{st}$ as

$$\Lambda_{st} \equiv \lambda_{st}U_{st}, \quad (3.4)$$

where $\lambda_{st}$ lives on the site $s$. We assume that $U_{st} \in G$ and the other fields including $\lambda_{st}$ are in the adjoint representation of $G$. For a given link $⟨st⟩$, we sometimes use the

\[\text{Only the sites } s_i \text{ and } s_{i+1} (i = 1, \ldots, n) \text{ must be connected by a link.} \]
notation $U_{ts} \equiv U_{st}^{-1}$. Then the gauge transformation of the fields are

\begin{align*}
\Phi_s &\to g_s \Phi_s g_s^{-1}, \\
\bar{\Phi}_s &\to g_s \bar{\Phi}_s g_s^{-1}, \\
\eta_s &\to g_s \eta_s g_s^{-1}, \\
U_{st} &\to g_s U_{st} g_t^{-1}, \\
\Lambda_{st} &\to g_s \Lambda_{st} g_t^{-1}, \\
Y_f &\to g_f Y_f g_f^{-1}, \\
\chi_f &\to g_f \chi_f g_f^{-1},
\end{align*}

(3.5)

where $g_s \in G$ ($s \in S$) and we have used the same symbol $f$ to describe a face and the representative point in the last line. It is easy to see that $\lambda_{st}$ transforms as $\lambda_{st} \to g_s \lambda_{st} g_s^{-1}$ under the gauge transformation.

Corresponding to the SUSY transformation (2.7), we consider the following transformation of the fields on the general lattice:

\begin{align*}
Q\Phi_s &= 0, \\
Q\bar{\Phi}_s &= \eta_s, \\
Q\eta_s &= [\Phi_s, \bar{\Phi}_s], \\
QU_{st} &= i\lambda_{st} U_{st}, \\
Q\Lambda_{st} &= i \left( U_{st} \Phi_t U_{st}^{-1} - \Phi_s + \lambda_{st} \Lambda_{st} \right), \\
QY_f &= [\Phi_f, \chi_f], \\
Q\chi_f &= Y_f.
\end{align*}

(3.6)

Note that the third line can be rewritten by

\begin{align*}
QU_{st} &= i\Lambda_{st}, \\
Q\Lambda_{st} &= i \left( U_{st} \Phi_t U_{st}^{-1} - \Phi_s + \lambda_{st} \Lambda_{st} \right),
\end{align*}

(3.7)

in terms of $\Lambda_{st}$ instead of $\lambda_{st}$. It is easy to see that $Q^2$ is equal to the infinitesimal gauge transformation with the parameter $\Phi_s$ thus $Q$ is nilpotent if it acts on a gauge invariant expression. Using this supercharge, we define the action,

\begin{align*}
S &= S_S + S_L + S_F \\
&\equiv Q \sum_{s \in S} \alpha_s \Xi_s + Q \sum_{(st) \in L} \alpha_{(st)} \Xi_{(st)} + Q \sum_{f \in F} \alpha_f \Xi_f,
\end{align*}

(3.8)

with

\begin{align*}
\Xi_s &\equiv \frac{1}{2g_0^2} \text{Tr} \left\{ \frac{1}{4} \eta_s [\Phi_s, \bar{\Phi}_s] \right\}, \\
\Xi_{(st)} &\equiv \frac{1}{2g_0^2} \text{Tr} \left\{ -i \lambda_{st} \left( U_{st} \Phi_t U_{st}^{-1} - \bar{\Phi}_s \right) \right\}, \\
\Xi_f &\equiv \frac{1}{2g_0^2} \text{Tr} \left\{ \chi_f \left( Y_f - i \beta_f \mu(U_f) \right) \right\},
\end{align*}

(3.9), (3.10), (3.11)

where $\alpha_s$, $\alpha_{(st)}$, $\alpha_f$ and $\beta_f$ are constants which will be fixed later so that the theory has
an appropriate continuum limit, $\mu(U_f)$ is given by

$$
\mu(U_f) = \begin{cases} 
2i \left[ (U_f - U_f^{-1})^{-1} (2 - U_f - U_f^{-1}) + (2 - U_f - U_f^{-1}) (U_f - U_f^{-1})^{-1} \right] & \text{for } G = U(N), \\
\frac{2i}{M} \left[ (U_f^M - U_f^{-M}) (2 - U_f^M - U_f^{-M}) + (2 - U_f^M - U_f^{-M}) (U_f^M - U_f^{-M}) \right] & \text{for } G = SU(N),
\end{cases}
$$

with $2M > N$, and $U_f$ is the “plaquette variable” defined by

$$
U_f \equiv \prod_{i=1}^{n} U_{s_is_{i+1}},
$$

for $f = (s_1, \cdots, s_n)$. Note that the form of $\mu(U_f)$ is determined in order that the theory possesses unique vacuum at $U_f = 1$ (see [32] for detail). The explicit expression of the action is

$$
S = S_b + S_f,
$$

with

$$
S_b = \frac{1}{2g_0^2} \sum_{s \in S} \alpha_s \text{Tr} \left\{ \frac{1}{4}[\Phi_s, \bar{\Phi}_s]^2 \right\} + \frac{1}{2g_0^2} \sum_{(st) \in L} \alpha_{(st)} \text{Tr} \left\{ (U_{st}\Phi_t U_{st}^{-1} - \Phi_s)(U_{st}\bar{\Phi}_t U_{st}^{-1} - \bar{\Phi}_s) \right\} + \frac{1}{2g_0^2} \sum_{f \in F} \alpha_f \text{Tr} \left\{ Y_f(Y_f - i\beta_f \mu(U_f)) \right\},
$$

$$
S_f = \frac{1}{2g_0^2} \sum_{s \in S} \alpha_s \text{Tr} \left\{ -\frac{1}{4} \eta_s [\Phi_s, \eta_s] \right\} + \frac{1}{2g_0^2} \sum_{(st) \in L} \alpha_{(st)} \text{Tr} \left\{ -i\lambda_st(U_{st}\eta_t U_{st}^{-1} - \eta_s) - \lambda_{st}\lambda_{st}(U_{st}\Phi_t U_{st}^{-1} + \bar{\Phi}_s) \right\} + \frac{1}{2g_0^2} \sum_{f \in F} \alpha_f \left\{ -\chi_f[\Phi_f, \chi_f] + i\beta_f \chi_f \left( Q\mu(U_f) \right) \right\}.
$$

If we consider the torus discretization corresponding to the square lattice and set $\alpha_s = \alpha_{(st)} = \alpha_f = \beta_f = 1$, this action reproduces that of the lattice formulation of two-dimensional $N = (2, 2)$ supersymmetric Yang-Mills theory given in [23][32].

We make a comment before closing this subsection. The construction of the discretized theory given above is based on the abstract data which includes such polygons
that cannot be interpreted as discretization of any Riemann surface. Since our main purpose in this paper is to discretize the two-dimensional topological gauge field theory, we will implicitly restrict the polygons to discretized Riemann surfaces in the next section. However, it would be worth noting that our construction is applicable to a wider class of discretized objects in principle.

### 3.2 Classical continuum limit

Let us next consider the tree-level continuum limit. To this end, we assume that the given decomposition is sufficiently fine to approximate a Riemann surface $\Sigma_g$. We first define the “lattice spacing” through the relation,

$$a^2 N_F = \int_{\Sigma_g} \sqrt{g(x)} \, d^2x,$$

where $N_F$ is the number of faces. In other words, $a^2$ is equal to an average area of the faces. The continuum limit is defined by the limit of $a \to 0$ and $N_F \to \infty$ with fixing (3.17). We also define the area of each face as

$$a^2 A_f = \int_{\sigma_f} \sqrt{g(x)} \, d^2x,$$

where the integration is taken over the region (simplex) $\sigma_f$ corresponding to the face $f$. In particular, we see

$$a^2 \sum_{f \in F} A_f \rightarrow \int_{\Sigma_g} \sqrt{g(x)} \, d^2x,$$

in the continuum limit.

Since we assume that the given decomposition sufficiently well approximates the Riemann surface $\Sigma_g$, we can identify the index $s$ of a site with a two-dimensional coordinate $x_s$. Then, corresponding to the link $\langle st \rangle$, we can define a covariant vector,

$$e^\mu_{st} \equiv \frac{1}{a} (x^\mu_t - x^\mu_s),$$

where $x_s$ and $x_t$ are the two-dimensional coordinates corresponding to the sites $s$ and $t$, respectively. Here let $L_f$ denote a set of links which construct the face $f$. From the definition of the continuum limit, it is natural to identify a face as a tangent space of the Riemann surface. Thus we assume that all the vectors $e^\mu_{st}$ for $\langle st \rangle \in L_f$ are in the same two-dimensional plane.

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6The 3D cubic lattice is a typical example.
Here we should note that all the fields on a general lattice are defined as dimensionless quantities, thus we must supply appropriate powers of $a$ in order to define the corresponding continuum fields. We must also require that the correspondence must be consistent with the $Q$-transformation. From these requirement, it is natural to consider the following correspondence between the discrete and the continuum fields:

$$
\Phi_s = a\Phi(x_s), \quad \tilde{\Phi}_s = a\tilde{\Phi}(x_s), \quad \eta_s = a^{\frac{2}{3}}\eta(x_s),
$$

$$
U_{st} = e^{ia\epsilon_{st}^\mu A_\mu(x_s + \epsilon_{st}^\nu)},
$$

$$
\lambda_{st} = a^{\frac{3}{2}}e^{\frac{i}{2}a\epsilon_{st}^\mu A_\mu(x_s + \frac{1}{2}\epsilon_{st}^\nu)}e^{\frac{i}{2}a\epsilon_{st}^\mu A_\mu(x_s + \frac{1}{2}\epsilon_{st}^\nu)}e^{-\frac{i}{2}a\epsilon_{st}^\mu A_\mu(x_s + \frac{1}{2}\epsilon_{st}^\nu)},
$$

$$
Y_f = a^2Y(x_f), \quad \chi_f = a^{\frac{2}{3}}\chi(x_f).
$$

Not only the fields but also the supercharge $Q$ and the coupling constant $g_0$ on the lattice are dimensionless as well. Therefore they must be also rescaled as

$$
Q = a^{1/2}\hat{Q}, \quad \frac{1}{g_0^2} = \frac{1}{a^2g_0^2}. \tag{3.22}
$$

Let us now evaluate the action (3.8) in the continuum limit. Substituting (3.21) and (3.22) in the action (3.8), we obtain

$$
S_S = \frac{\hat{Q}}{2g_0^2} \sum_{f \in F} a^2 A_f \left( \sum_{s \in S_f} \frac{\alpha_s^f}{A_f} \text{Tr} \left( \frac{1}{2} \eta(x_s)[\Phi(x_s), \tilde{\Phi}(x_s)] \right) \right), \tag{3.23}
$$

$$
S_L = \frac{\hat{Q}}{2g_0^2} \sum_{f \in F} a^2 A_f \left( \sum_{\langle st \rangle \in L_f} \frac{\alpha_{(st)}^f}{A_f} \epsilon_{st}^\mu \epsilon_{st}^\nu \text{Tr} \left\{ -i\lambda_{x_s}(x_s)D_\nu \tilde{\Phi}(x_s) + O(a) \right\} \right), \tag{3.24}
$$

$$
S_F = \frac{\hat{Q}}{2g_0^2} \sum_{f \in F} a^2 A_f \left( \frac{\alpha_f}{A_f} \text{Tr} \left\{ \chi(x_f) \left( Y(x_f) - i\beta_f A_f \frac{\epsilon^\mu \nu}{\sqrt{g(x_f)}} F_{\mu\nu} + O(a) \right) \right\} \right), \tag{3.25}
$$

where $S_f$ is the set of the sites which construct the face $f$, $F_s$ is the set of the faces which meet at the site $s$, $\alpha_s^f$ and $\alpha_{(st)}^f$ are constants satisfying $\alpha_s = \sum_{f \in F_s} \alpha_s^f$ and $\alpha_{(st)} = \sum_{f \in F_{(st)}} \alpha_{(st)}^f$, respectively, and we have used

$$
\mu(U_f) = ia^2 \frac{A_f}{\sqrt{g(x_f)}} \epsilon^\mu \nu F_{\mu\nu} + O(a^3), \tag{3.26}
$$

while evaluating (3.25) (see the appendix $A$). Here $F_{(st)}$ is the set of the faces which share the link $\langle st \rangle$.

7 If the link $\langle st \rangle$ is a component of the boundary, if it exists, of the surface, only one face shares it. Otherwise two faces share it.
the face action \((3.25)\) becomes the corresponding part of the continuum action \((2.9)\) by setting the parameters \(\alpha_s\), \(\alpha_f\) and \(\beta_f\) as

\[
\alpha_s = \sum_{f \in F_s} \frac{A_f}{|S_f|}, \quad \alpha_f = A_f, \quad \beta_f = \frac{1}{A_f}.
\]

\[(3.27)\]

The link part \((3.24)\) is slightly more complicated; in order to reproduce the continuum action, \(\alpha_{(st)}\) must satisfy

\[
\sum_{(st) \in L_f} \alpha_{(st)}^I e^\mu_{st} e^\nu_{st} = A_f g^{\mu\nu}(x_f).
\]

\[(3.28)\]

It is easy to see we can determine the value of \(\alpha_{(st)}\) for any given Riemann surface by solving \((3.28)\). In fact, when the face \(f\) consists of \(n\) links, \(l_i (i = 1, \ldots, n)\), the rank of the \(3 \times n\) matrix \(M^I_l \equiv e^\mu_{li} e^\nu_{li} (I = (\mu, \nu) = (1, 1), (2, 2), (1, 2))\) is three since we assume that all the vectors \(\vec{e}_l\) are all in the same two-dimensional plane. In particular, if we consider triangulation, \(\alpha_{(st)}^I\) are uniquely determined through the equation \((3.28)\). Therefore we see that the classical continuum limit of the discretized theory \((3.14)\) becomes two-dimensional topological field theory on the Riemann surface \(\Sigma_g\) by setting \(\alpha_s\), \(\alpha_{(st)}\), \(\alpha_f\) and \(\beta_f\) as \((3.27)\) and \((3.28)\).

### 3.3 Radiative corrections

We next discuss possible radiative corrections which appear in taking the continuum limit. The discussion is completely parallel with that for Sugino’s formulation given in \([23–26]\).

From the power counting, we see that possible relevant or marginal operators which can appear radiatively are \(B_1(x)\) or \(B_1(x)B_2(x)\) with bosonic fields \(B_1(x)\) and \(B_2(x)\). From the gauge symmetry and the \(\hat{Q}\)-symmetry, only the possible terms are \(\text{Tr} \Phi(x)\) and \(\text{Tr} \Phi(x)^2\) up to constant factors.

As announced, the situation differs depending on whether the scalar fields \(\Phi(x)\) and \(\bar{\Phi}(x)\) are complex conjugate with each other or not. When \(\Phi(x)\) and \(\bar{\Phi}(x)\) are complex conjugate with each other as in Sugino’s formulation, both of \(\text{Tr} \Phi(x)\) and \(\text{Tr} \Phi(x)^2\) are forbidden by the \(U(1)\) symmetry \((2.10)\). Therefore, we do not need any fine-tuning in taking the continuum limit in this case. On the other hand, when \(\Phi(x)\) and \(\bar{\Phi}(x)\) are independent hermitian variables, there is no symmetry that forbids to appear these operators radiatively. Therefore we need to add counter terms,

\[
S_C = \begin{cases} 
\sum_{s \in S} \text{Tr} (c_1 \Phi_s^2 + c_2 \Phi_s) & \text{for } G = U(N), \\
\sum_{s \in S} \text{Tr} (c_1 \Phi_s^2) & \text{for } G = SU(N),
\end{cases}
\]

\[(3.29)\]
to the action and tune the parameters $c_1$ ($c_1$ and $c_2$) for $G = SU(N)$ ($G = U(N)$) in taking the continuum limit.\footnote{Because of the $Q$-symmetry, we see that the expectation values of some operators in $Q$-cohomology can be exactly evaluated even in the lattice theory\cite{53}. In simulation, therefore, we will be able to use this exact result in tuning $c_1$ and $c_2$.}

4 Conclusion and discussion

In this paper, we have constructed the discrete formulation of the topologically twisted $\mathcal{N} = (2,2)$ supersymmetric Yang-Mills theory on an arbitrary two-dimensional lattice with preserving a supercharge. When the polygon decomposition (general lattice) is the discretization of the Riemann surface $\Sigma_g$, the continuum limit of this theory becomes the topologically twisted $\mathcal{N} = (2,2)$ supersymmetric Yang-Mills theory on $\Sigma_g$. If we consider the usual square lattice as an example of the decomposition, our model reproduces Sugino’s lattice formulation of $\mathcal{N} = (2,2)$ supersymmetric Yang-Mills theory on the torus.

We have also shown that we can take the continuum limit without any fine-tuning if the theory possesses the $U(1)_R$ symmetry, that is, we regard the two scalar fields in the vector multiplet as complex conjugate with each other. On the other hand, if the scalar fields are independent hermitian variables and the gauge group is $SU(N)$ (or $U(N)$), there is no $U(1)_R$ symmetry in the model and we need one parameter (or two parameters) tuning in the continuum limit.

A natural question would arise as to whether there is fermion doubler in this model or not. In order to answer this question, we have to examine if the kinetic terms of the fermions have no non-trivial zero, which depends on the structure of the discretization. However, we should recall that the origin of fermion doubler is the periodicity in the momentum space, which is associated with the discrete translational invariance of the lattice. Since a general lattice has less discrete translational symmetry than the usual square lattice, there is less chance to appear fermion doublers. In addition, even if we consider the square lattice, it is shown that fermion doubler is absent \cite{23}. Although it is still possible that fermion doublers appear by discretizing the Riemann surface by a highly symmetric tiling, we can conclude that there is no fermion doubler in most case.

In the continuum theory, the so-called localization is used to examine the topological nature of the two-dimensional gauge theory\cite{52}. Since our model preserves the scalar supersymmetry, which is the crucial symmetry in order that localization works, we can use the same technique in the discretized theory, which will be discussed separately in \cite{53}.

It will be straightforward to apply our method to the two-dimensional $\mathcal{N} = (4,4)$
and (8, 8) supersymmetric Yang-Mills theories or two-dimensional supersymmetric QCD. Furthermore our method is also applicable to the orbifold lattice theory \cite{5-8}. In the original orbifold lattice theory is based on the concept of deconstruction and constructed by dividing a matrix theory (mother theory) by a discrete subgroup of the mother theory. The only background we can obtain in this way is the torus: it seems to be impossible that the standard orbifold projection constructs a theory on an arbitrary Riemann surface. On the other hand, by using our method, we can construct the theory on the arbitrary lattice and we can embed the fields in sparse matrices. In this sense, our method can be regarded as a non-trivial extension of deconstruction, which will be connected with network theory. It might give a novel way to examine the topological nature of gauge theory.

Including the fluctuation of polygons like Regge calculus \cite{54} or dynamical triangulation \cite{55} will be apparently a fascinating next step. To this end, our set up given in the section 3 would be insufficient in order to generate Riemann surface dynamically because it includes too wide discretized objects. One plausible idea is to restrict the discretization to simplicial complex. It will be interesting question if the diffeomorphism invariance is recovered in the continuum limit under such restriction.

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\section{A Continuum limit of the plaquette variable}

Let us consider a face \((s_1, \ldots, s_n)\) and the corresponding plaquette variable,

\[ U_f = \prod_{i=1}^{n} U_{s_i s_{i+1}} \quad (s_{n+1} = s_1). \]  

(A.1)

We here assume that the vectors \(e_{s_k s_{k+1}}\) constructing this face span the same two-dimensional plane. Recalling that it is reasonable to think that the continuum gauge field is living at the middle point of the link:

\[ U_{s_k s_{k+1}} = \exp \left\{ iae_{s_k s_{k+1}}^\mu A_\mu(s_k + \frac{a}{2} e_{s_k s_{k+1}}) \right\}, \]  

(A.2)
and the argument of $A_\mu$ is rewritten as
\[
s_k + \frac{a}{2} e_{s_k s_{k+1}} = s_1 + \frac{a}{2} \left( e_{s_1 s_2} + e_{s_2 s_3} + \cdots + e_{s_{k-1} s_k} - e_{s_k s_{k+2}} - \cdots - e_{s_n s_1} \right),
\]  
we can rewrite (A.2) as
\[
U_{s_k s_{k+1}} = \exp \left\{ i a e^{\mu}_{st} A_\mu(s_1) + \frac{i}{2} a^2 e^{\mu}_{s_k s_{k+1}} \left( \sum_{l<k} e^{\nu}_{s_l s_{l+1}} - \sum_{l>k} e^{\nu}_{s_{k-l} s_{k+1}} \right) \partial_\nu A_\mu(s_1) + O(a^3) \right\}.
\]  
Substituting (A.4) to (A.1) and using Campbell-Baker-Hausdorff formula,
\[
e^{M_1} e^{M_2} \cdots e^{M_n} = e^{\sum_{i=1}^n M_i + \frac{1}{2} \sum_{i<j} [M_i, M_j] + \cdots},
\]
we see
\[
U_f = \exp \left\{ \frac{i}{2} a^2 C_{f}^{\mu\nu} F_{\mu\nu}(s_1) + O(a^3) \right\},
\]
where
\[
F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu],
\]
and
\[
C_{f}^{\mu\nu} = \frac{1}{2} \sum_{k=1}^n e^{\mu}_{s_k s_{k+1}} \left( -\sum_{l<k} e^{\nu}_{s_l s_{l+1}} + \sum_{l>k} e^{\nu}_{s_{k-l} s_{k+1}} \right).
\]
In order to see the geometrical meaning of $C_{f}^{\mu\nu}$, it is convenient to rewrite it as
\[
C_{f}^{\mu\nu} = \frac{1}{2} \sum_{i=3}^n \left( e^{\mu}_{s_i s_{i-1}} e^{\nu}_{s_i s_1} - e^{\nu}_{s_i s_{i-1}} e^{\mu}_{s_i s_1} \right),
\]
where $e_{s_i s_1} \equiv -e_{s_i s_{i+1}} - e_{s_{i+1} s_{i+2}} - \cdots - e_{s_n s_1}$. Since $\frac{1}{2} (e_{s_i s_{i-1}}^1 e^{2}_{s_i s_1} - e_{s_i s_{i-1}}^2 e^{1}_{s_i s_1})$ is the area of the triangle with the vertices $s_1, s_{i-1}, s_i$, we see
\[
C_{f}^{\mu\nu} = \frac{A_f}{\sqrt{g(x_f)}} e^{\mu\nu},
\]
which is proportional to a unit area of the polygon made up of $e_{s_i s_{i+1}}$’s.

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