On the limits of the Navier–Stokes equations

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Heuristic derivations of the Navier–Stokes equations are unable to reveal the applicability limits of these equations. In this paper we rederive the Navier–Stokes equations from kinetic theory, using a method that affords a step by step insight into the required simplifying assumptions. The major, but not the only task on this way is to find the conditions needed to truncate the resulting infinite system of transport equations at a finite level. The minimal obtainable closed set comprises three equations, for the particle number density $N$, the macroscopic velocity $v$, and the temperature $T$. It is verified that this 3–equation system conserves the total energy, i.e., the sum of kinetic and internal energy. As a consequence, the energy is not conserved if the integrity of this closed system is violated, as for instance in the case of the so–called incompressible Navier–Stokes equations (and likewise the incompressible Euler equations), where the equation for $T$ is discarded and replaced by $\nabla \cdot v = 0$. The Navier–Stokes system in its common form rests upon the tacit assumption that the particles are points without finite volume and without collective forces between them, whereby dense gases and liquids are excluded. In the final section of this paper, an attempt is made to generalize the equations for applicability to real fluids.

1. Introduction

The conversion of kinetic theory into transport theory gives rise to an infinite number of hierarchically ordered macroscopic transport equations. In order to obtain a tractable finite set, it is necessary to find truncation conditions allowing to disregard moments beyond a certain order. It turns out that the minimal achievable set comprises three equations, for the particle number density $N$, the macroscopic velocity $v$, and the temperature $T$ (see Refs. 1–3). The appertaining truncation conditions set limits to the applicability of these transport equations. A well–known condition is the Knudsen condition, demanding that the mean free path is short compared with a typical scale of the process. However, this condition alone is by far not sufficient.

We proceed as follows: In Sections 2 to 6 the equations belonging to the Navier–Stokes system will be rederived, using a method which affords a transparent view into the simplifications required to arrive at the known results. This will lead to a set of applicability conditions, allowing to decide whether a given mathematical solution is physically valid. A summarizing compilation of these conditions will be given in Section 9. In Section 7 it will be verified that the Navier–Stokes system conserves the total energy, i.e., the sum of kinetic and internal energy, and expressions will be given for the various energy transfer rates involved in the total energy balance. A critical discussion of the physical consequences of the frequently used incompressibility assumption will follow in Section 8. Finally, in Section 10, an attempt will be made to incorporate the effects of the finite size of particles and the collective forces between them, which are both lacking in the classical form of Navier–Stokes equations.

2. Basic equations

The mother equation for the system of macroscopic transport equations is the kinetic equation, describing the distribution function in phase space, $f(u, r, t)$. The meaning of $f$ is that $\int f d^3u d^3r$ gives the number of particles within the phase space volume element $d^3u d^3r$, so that $\int f d^3u = N$ is the particle number density.
Considering a one–constituent neutral fluid without particle production or loss, the kinetic equation is given by

\[
\frac{df}{dt} + (\mathbf{u} - \mathbf{v}) \cdot \nabla f + \frac{\mathbf{F}}{m} \cdot \nabla u f = \frac{\delta f}{\delta t}
\]  

(1)

where \(\delta f/\delta t\) symbolically denotes the temporal change of \(f\) due to the action of collisions, and \(\mathbf{F}\) is a force acting on the particles individually. This force may depend on \(r\) and \(t\), but not on \(u\) (unless \(\mathbf{F}\) is perpendicular to \(u\)). Furthermore it will be assumed that collisions are sufficiently soft to be non–ionizing, \(\delta \int f d^3 u/\delta t = \delta N/\delta t = 0\).

The common way to derive macroscopic equations from the kinetic equation (1), with \(\delta f/\delta t\) expressed by the Boltzmann collision integral, is to approximate the distribution function \(f\) by an expansion about the Maxwellian

\[
f = \frac{N}{(2\pi)^{3/2} V^3} \exp \left[ -\frac{(\mathbf{u} - \mathbf{v})^2}{2V^2} \right]
\]  

(2)

where \(V\) is the thermal velocity \((V^2 = KT/m)\), with \(K\) Boltzmann’s constant and \(m\) the particle mass. The two leading methods in this field are those of Chapman and Enskog (see Ref. 1) and Grad\(^2\). A comprehensive review is given by Schunk\(^3\).

The general problem in transforming the kinetic equation into macroscopic transport equations, independent of the method used, is that any moment of \(f\) of given order \(n\) is coupled with moments of order \(n + 1\), so that an infinite system of transport equations arises. The major task, therefore, will be to find conditions allowing to truncate the infinite sytem at a finite level in order to obtain a usable closed set of equations.

The truncation conditions set limits to the applicability of the resulting equations. To establish these applicability conditions, we will not follow the methods of Chapman–Enskog or Grad. For the given purpose, it will be more appropriate to stay entirely within the frame of transport equations, without knowledge of the distribution function. The price to be paid is that the occurring transport coefficients – viscosity and heat conductivity – will have to be determined by use of a collisional relaxation model. This price, however, does not appear too high in view of the fact that in practical applications of the transport equations empirical values for these transport coefficients will be used.

We define a complete set of moments of \(f\) by

\[
M^{(i,j,k)}(\mathbf{r}, t) = \int f(\mathbf{u}, \mathbf{r}, t) \phi^{(i,j,k)}(\mathbf{u}) d^3 \mathbf{u}
\]

(3)

with

\[
\phi^{(i,j,k)}(\mathbf{u}) = m (u_x - v_x)^i (u_y - v_y)^j (u_z - v_z)^k
\]

(4)

and

\[
\mathbf{v} = \frac{1}{N} \int f \mathbf{u} d^3 \mathbf{u}
\]

(5)

where \(i, j, k\) are non–negative integers, and \(n = i + j + k\) is the order of the moment. Multiplication of the terms in (1) by \(\phi^{(i,j,k)}\) with subsequent intergration over velocity space yields, after some elementary mathematical steps,

\[
T1 + T2 + T3 + T4 = 0
\]

(6)
with

\[ T_1 = \frac{dM^{(i,j,k)}}{dt} + M^{(i,j,k)} \left( \nabla \cdot \mathbf{v} + i \frac{\partial v_x}{\partial x} + j \frac{\partial v_y}{\partial y} + k \frac{\partial v_z}{\partial z} \right) - \frac{\delta M^{(i,j,k)}}{\delta t} \]  

(7)

\[ T_2 = i M^{(i-1,j,k)} \left( \frac{dv_x}{dt} - \frac{F_x}{m} \right) + j M^{(i,j-1,k)} \left( \frac{dv_y}{dt} - \frac{F_y}{m} \right) + k M^{(i,j,k-1)} \left( \frac{dv_z}{dt} - \frac{F_z}{m} \right) \]  

(8)

\[ T_3 = i \left( M^{(i-1,j+1,k)} \frac{\partial v_x}{\partial y} + M^{(i-1,j,k+1)} \frac{\partial v_x}{\partial z} \right) + j \left( M^{(i+1,j-1,k)} \frac{\partial v_y}{\partial x} + M^{(i,j-1,k+1)} \frac{\partial v_y}{\partial z} \right) + k \left( M^{(i,j+1,k-1)} \frac{\partial v_z}{\partial y} + M^{(i+1,j,k-1)} \frac{\partial v_z}{\partial x} \right) \]  

(9)

\[ T_4 = \frac{\partial M^{(i+1,j,k)}}{\partial x} + \frac{\partial M^{(i,j+1,k)}}{\partial y} + \frac{\partial M^{(i,j,k+1)}}{\partial z} \]  

(10)

For the conversion of the last left-hand side term in (1) into a moment term it has been necessary to postulate that the decrease of \( f \) is stronger than the increase of \( \phi^{(i,j,k)} \) as \( |u| \) goes to infinity.

The term \( T_1 \) contains the wanted moment \( M^{(i,j,k)} \) of order \( n = i + j + k \), the term \( T_2 \) moments of order \( n - 1 \), the term \( T_3 \) other moments of order \( n \), and the term \( T_4 \) moments of order \( n + 1 \). It is this fourth term that reveals the dilemma of transport theory.

The moment system (6)–(10) contains no terms to account for the finite size of the particles and for the cumulative action of intermolecular forces, and thus it relates to ideal gases. We will come back to this in Section 10.

Before making use of (6)–(10), we identify the moments of orders \( n = 0 \) to 3.

For \( n = 0 \):

\[ M^{(0,0,0)} = Nm \quad \text{(mass density)} \]  

(11)

For \( n = 1 \):

\[ M^{(1,0,0)} = M^{(0,1,0)} = M^{(0,0,1)} = 0 \]  

(12)

For \( n = 2 \):

\[ M^{(2,0,0)} = p_{xx} \quad M^{(0,2,0)} = p_{yy} \quad M^{(0,0,2)} = p_{zz} \]  

(13)

\[ M^{(1,1,0)} = p_{xy} = p_{yx} \quad M^{(1,0,1)} = p_{xz} = p_{zx} \quad M^{(0,1,1)} = p_{yz} = p_{zy} \]  

(14)

\[ p = \frac{1}{3} \left( M^{(2,0,0)} + M^{(0,2,0)} + M^{(0,0,2)} \right) \quad \text{(hydrostatic pressure)} \]  

(15)

\[ T = \frac{1}{3KN} \left( M^{(2,0,0)} + M^{(0,2,0)} + M^{(0,0,2)} \right) \]  

(16)

For \( n = 3 \):

\[ q_x = \frac{1}{2} \left( M^{(3,0,0)} + M^{(1,2,0)} + M^{(1,0,2)} \right) \]  

(17)

\[ q_y = \frac{1}{2} \left( M^{(2,1,0)} + M^{(0,3,0)} + M^{(0,1,2)} \right) \]  

(18)
\( q_z = \frac{1}{2} (M^{(2,0,1)} + M^{(0,2,1)} + M^{(0,0,3)}) \)  

(19)

These are the three components of the heat flux vector \( \mathbf{q} \).

The moment equations (6)–(10) in their complete form are unusable in practice because there are infinitely many. Let us speak of fluid theory when the infinite set of moment equations is reduced to a finite set. According to Stubbe\(^4\), a reduction of the infinite set of moment equations to a finite set of fluid equations is possible if the condition

\[
\frac{1}{\tau^2} + \frac{1}{\tau_c^2} \gg \frac{V^2}{l^2}
\]

(20)

is fulfilled, where \( \tau \) is a characteristic time and \( l \) a characteristic length of the process under consideration, and \( \tau_c \) the average time between collisions. Condition (20) comprises two limiting cases, viz. the inertial fluid limit

\[
\tau \ll \tau_c \quad \text{and} \quad \tau V \ll l
\]

(21)

and the collisional fluid limit

\[
\tau_c \ll \tau \quad \text{and} \quad \tau_c V \ll l
\]

(22)

Condition (21) typically applies to plasmas, whereas (22) is better suited for application to neutral fluids which are usually collision dominated. The second part of (22) corresponds to the Knudsen condition, and the first part of (22) is the precondition for the Knudsen condition. It must be realized that condition (20) merely justifies the general applicability of fluid theory. Specific applications, like the Navier–Stokes equations, will require further applicability conditions.

3. Transport equations for the fluid variables \( N, \mathbf{v} \) and \( T \)

Eqs. (6)–(10), together with the moment definitions (11) to (19), will now be used to extract equations for \( N, \mathbf{v} \) and \( T \). We obtain:

For \( n = 0 \):

\[
\frac{dN}{dt} = -N \nabla \cdot \mathbf{v}
\]

(23)

For \( n = 1 \):

\[
\frac{d\mathbf{v}}{dt} = \frac{\mathbf{F}}{m} - \frac{1}{Nm} \left( \nabla p + \nabla \cdot (\mathbf{p}^o) \right)
\]

(24)

where \( (\mathbf{p})^o \) is the traceless part of the pressure tensor \( \mathbf{p} \), with non–diagonal elements defined by (13), and diagonal elements \( p_{xx} - p, p_{yy} - p \) and \( p_{zz} - p \) defined by (14) and (15). Combination of (15) and (16) yields

\[
p = NKT
\]

(25)

so that (24) can be written as
\[
\frac{d\mathbf{v}}{dt} = \frac{\mathbf{F}}{m} - V^2 \left[ \frac{\nabla T}{T} + \frac{\nabla N}{N} \right] - \frac{1}{Nm} \nabla \cdot (p)^o \tag{26}
\]

For \( n = 2 \):

\[
\frac{dp}{dt} = -\frac{5}{3} p \nabla \cdot \mathbf{v} - \frac{2}{3} \left[ (p)^o \cdot \nabla \mathbf{v} + \nabla \cdot \mathbf{q} \right] \tag{27}
\]

or, using (23) and (25),

\[
\frac{dT}{dt} = -\frac{2}{3} T \nabla \cdot \mathbf{v} - \frac{2}{3K_N} \left[ (p)^o \cdot \nabla \mathbf{v} + \nabla \cdot \mathbf{q} \right] \tag{28}
\]

where \( \nabla \mathbf{v} \) denotes the dyadic product.

The derivation of these relations from (6)–(10) is straightforward and without approximations. Thus, with (6)–(10) given, the system [(23),(26),(28)] is exact, however useless at this stage since \((p)^o \) and \( \mathbf{q} \) are undetermined.

The reason why we see no collision terms in the above transport equations for \( N, \mathbf{v} \) and \( T \) is because we have assumed that collisions are non–ionizing \((\delta N/\delta t = 0)\), and because there exists no second constituent with which momentum and energy could be exchanged \((\delta \mathbf{v}/\delta t = 0 \) and \( \delta T/\delta t = 0 \) or \( \delta p/\delta t = 0 \)). However, the randomizing action of collisions will play a dominant role in the subsequent treatment of \((p)^o \) and \( \mathbf{q} \).

The transport equations (26) and (28) are open–ended since the moments \((p)^o \) and \( \mathbf{q} \) establish a link to an infinite number of higher order moments. It will be necessary, therefore, to find criteria allowing to express \((p)^o \) and \( \mathbf{q} \) in terms of \( N, \mathbf{v} \) and \( T \), in order to obtain a closed system of transport equations with no other variables than these. This task will be addressed in Section 5.

4. The Euler system

If the dominant role of randomizing collisions goes to the extreme that the system is completely isotropized, the \((p)^o \) and \( \mathbf{q} \) terms disappear, and we have

\[
\frac{d\mathbf{v}}{dt} = \frac{\mathbf{F}}{m} - V^2 \left[ \frac{\nabla T}{T} + \frac{\nabla N}{N} \right] \tag{29}
\]

\[
\frac{dT}{dt} = -\frac{2}{3} T \nabla \cdot \mathbf{v} \tag{30}
\]

The system [(23),(29),(30)] may be termed the Euler system because (29) corresponds to the Euler equation. We see, however, that the Euler equation is not an isolated equation, but an integral part within a three–equation system for the three fluid variables \( N, \mathbf{v} \) and \( T \). It is a questionable step, therefore, to violate the integrity of the system [(23),(29),(30)] by removing eq. (30) and using the condition \( \nabla \cdot \mathbf{v} = 0 \) instead, which is frequently done in the literature (see the review by Constantin\(^5\)). We will come back to this in Section 8.

Occasionally one finds the statement that the Euler equation is a special case of Newton’s second law, with the acting force expressed in terms of a pressure gradient, \( \mathbf{F} = -(1/N)\nabla p \). This is a simplistic view. In a dynamic system with internal motions the pressure has to be taken as what it is, namely a 9–element tensor, with non–diagonal elements unequal zero and diagonal elements
unequal among each other. It will require a thorough justification to reduce the full pressure tensor \( p \) to its isotropic limit \( p U \) (where \( U \) is the unit tensor).

5. Approximations for \( (p)^o \) and \( q \)

In order to extend the Euler system to a corresponding Navier–Stokes system, we are left with the task to express \( (p)^o \) and \( q \) in terms of \( N, v \) and \( T \). This will not be possible if all terms in (26) and (28) are treated as being on equal footing. It will be necessary to set up a hierarchy of terms, and this is done by treating as minor terms those which constitute the difference between (26) and (29), and between (28) and (30).

With respect to the \( (p)^o \) term, this means that \( |\nabla \cdot (p)^o| \) has to be much smaller than \( |\nabla p| \), resulting in the condition

\[
|p_{ij}| \ll p_{ii} \quad \text{and} \quad |p_{ii} - p| \ll p
\]

(31)

For the \( q \) term in (27) it follows that \( |\nabla \cdot q| \) must be much smaller than \( |p \nabla \cdot v| \), with the consequence that the term \( T4 \) (which contains the elements of the heat flux vector) has to be ignored when eqs. (6)–(10) are applied for \( n = 2 \). In other words, on this level of approximation \( (p)^o \) will not depend on \( q \).

In order to determine \( (p)^o \) for use in (26) and (28), we apply (6)–(10) to the pressure tensor elements \( p_{xx} \) (\( i = 2, j = k = 0 \)) and \( p_{xy} \) (\( i = j = 1, k = 0 \)). The term \( T2 \) vanishes due to (12). Since \( \delta p/\delta t = 0 \), the collision term \( \delta p_{xx}/\delta t \) in T1 can be altered into \( \delta (p_{xx} - p)/\delta t \), and from (6) with \( T2 = T4 = 0 \) we obtain

\[
\frac{\delta (p_{xx} - p)}{\delta t} = \frac{dp_{xx}}{dt} + p_{xx} \left( \nabla \cdot v + 2 \frac{\partial v_x}{\partial x} \right) + 2 p_{xy} \frac{\partial v_x}{\partial y} + 2 p_{xz} \frac{\partial v_x}{\partial z}
\]

(32)

Applying (31) to (32), we can ignore the last two terms and replace on the right-hand side \( p_{xx} \) by \( p \). Then, expressing \( dp/dt \) by the leading term in (27) \( (dp/dt = -(5/3) p \nabla \cdot v) \), and writing \( p = NKT \) according to (25), we obtain

\[
\frac{\delta (p_{xx} - p)}{\delta t} = NKT \left( 2 \frac{\partial v_x}{\partial x} - \frac{2}{3} \nabla \cdot v \right)
\]

(33)

With arguments fully corresponding to those leading to (33), \( \delta p_{xy}/\delta t \) is obtained as

\[
\frac{\delta p_{xy}}{\delta t} = NKT \left( \frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x} \right)
\]

(34)

Expressions corresponding to (33) and (34) are obtained for the other elements of the traceless pressure tensor, and the results can be written in compact form as

\[
\frac{\delta (p)^o}{\delta t} = NKT \left[ \nabla v + (\nabla v)^t - \frac{2}{3} (\nabla \cdot v) U \right]
\]

(35)

where \( (\nabla v)^t \) denotes the transposed dyad.

To finalize the derivation, we need a relation between \( \delta (p)^o/\delta t \) and \( (p)^o \). For this, we apply the 10–moment relaxation model of Stubbe\(^6\), which is a considerable extension of the frequently used model by Gross and Krook\(^7\), and we obtain
\[ \frac{\delta (p)^\alpha}{\delta t} = -2 \nu^{(r)} (p)^\alpha \] (36)

where the collision frequency \( \nu^{(r)} \) is a measure of the strength of the randomizing action of collisions, quantitatively defined by eqs. (28c) and (15) of Ref. 6. The other two collision frequencies involved in the model, \( \nu^{(m)} \) and \( \nu^{(e)} \), describing momentum and energy transfer, do not appear here due to the absence of a second constituent.

From (35) and (36)

\[ (p)^\alpha = -\eta \left[ \nabla v + (\nabla v)^t - \frac{2}{3} (\nabla \cdot v) U \right] \] (37)

where

\[ \eta = \frac{NKT}{2 \nu^{(r)}} \] (38)

is the dynamic viscosity. Since \( \nu^{(r)} \) is proportional to \( N \), \( \eta \) is independent of \( N \). The viscosity given by (38) agrees quantitatively with the corresponding result provided by the first approximation of the Chapman–Enskog theory (see Ref. 1), thereby showing that a properly designed relaxation model has more to offer than bare simplicity.

Next we have to derive a corresponding relation for \( q \). This turns out to be a highly intricate matter, which may appear surprising in view of the fact that the anticipated result, \( q = -\kappa \nabla T \) (with \( \kappa \) the heat conductivity), is as simple as a result can be. In fact, \( q \) depends on a wealth of terms, and it will be necessary to find conditions allowing to ignore the majority of them. One complication will be that T4 in this case cannot be ignored, so that moments of order \( n = 4 \) will come into play.

We consider the \( \alpha \)-component of \( q \) which, according to (17), requires knowledge of the three moments \( M^{(3,0,0)} \), \( M^{(1,2,0)} \) and \( M^{(1,0,2)} \). From (7)–(10):

For \((i,j,k) = (3,0,0)\)

\[ T_1 = \frac{dM^{(3,0,0)}}{dt} + M^{(3,0,0)} \left( \nabla \cdot v + 3 \frac{\partial v_x}{\partial x} \right) - \frac{\delta M^{(3,0,0)}}{\delta t} \] (39)

\[ T_2 = 3 p_{xx} \left( \frac{dv_x}{dt} - \frac{F_x}{m} \right) \] (40)

\[ T_3 = 3 \left( M^{(2,1,0)} \frac{\partial v_x}{\partial y} + M^{(2,0,1)} \frac{\partial v_x}{\partial z} \right) \] (41)

\[ T_4 = \frac{\partial M^{(4,0,0)}}{\partial x} + \frac{\partial M^{(3,1,0)}}{\partial y} + \frac{\partial M^{(3,0,1)}}{\partial z} \] (42)

For \((i,j,k) = (1,2,0)\)

\[ T_1 = \frac{dM^{(1,2,0)}}{dt} + M^{(1,2,0)} \left( \nabla \cdot v + \frac{\partial v_x}{\partial x} + 2 \frac{\partial v_y}{\partial y} \right) - \frac{\delta M^{(1,2,0)}}{\delta t} \] (43)

\[ T_2 = p_{yy} \left( \frac{dv_x}{dt} - \frac{F_x}{m} \right) + 2 p_{xy} \left( \frac{dv_y}{dt} - \frac{F_y}{m} \right) \] (44)
\[ T_3 = \left( M^{(0,3,0)} \frac{\partial v_x}{\partial y} + M^{(0,2,1)} \frac{\partial v_x}{\partial z} \right) + 2 \left( M^{(2,1,0)} \frac{\partial v_y}{\partial x} + M^{(1,1,1)} \frac{\partial v_y}{\partial z} \right) \] (45)

\[ T_4 = \frac{\partial M^{(2,2,0)}}{\partial x} + \frac{\partial M^{(1,3,0)}}{\partial y} + \frac{\partial M^{(1,1,2)}}{\partial z} \] (46)

For \((i,j,k) = (1,0,2)\)

\[ T_1 = \frac{dM^{(1,0,2)}}{dt} + M^{(1,0,2)} \left( \nabla \cdot \mathbf{v} + 2 \frac{\partial v_x}{\partial x} \right) - \frac{\delta M^{(1,0,2)}}{\delta t} \] (47)

\[ T_2 = p_{zz} \left( \frac{dv_x}{dt} - \frac{F_x}{m} \right) + 2 p_{xz} \left( \frac{dv_z}{dt} - \frac{F_z}{m} \right) \] (48)

\[ T_3 = \left( M^{(0,1,2)} \frac{\partial v_x}{\partial y} + M^{(0,0,3)} \frac{\partial v_x}{\partial z} \right) + 2 \left( M^{(2,0,1)} \frac{\partial v_y}{\partial x} + M^{(1,1,1)} \frac{\partial v_y}{\partial z} \right) \] (49)

\[ T_4 = \frac{\partial M^{(2,0,2)}}{\partial x} + \frac{\partial M^{(1,1,2)}}{\partial y} + \frac{\partial M^{(1,0,3)}}{\partial z} \] (50)

Our first approximative assumption will be that the collision term in the three expressions for \(T_1\) dominates the other terms, with the consequence that the collision term will be the only one to survive. By the same token, \(T_3\) vanishes altogether. Consequently, if we add the three \(T_1\) and \(T_3\) terms (the sum denoted by the \(\Sigma\)–sign), we obtain the simple result

\[ \Sigma T_1 + \Sigma T_3 = -2 \frac{\delta q_x}{\delta t} \] (51)

For the treatment of the \(T_2\)–terms we use condition (31) to neglect the \(p_{ij}\)–terms and replace the \(p_{ii}\)–terms by \(p\), giving

\[ \Sigma T_2 = 5 p \left( \frac{dv_x}{dt} - \frac{F_x}{m} \right) \] (52)

The terms \(T_4\), as they stand, open the path to an endless number of equations for moments of order \(n \geq 4\). To avoid this hopeless endeavour, we proceed as follows: We divide the moments of order \(n = 4\) into even and odd moments, \(M^{(i,j,k)}_{\text{even}}\) and \(M^{(i,j,k)}_{\text{odd}}\), where even means that all indices \(i, j, k\) are even (0 regarded as an even number), and odd that this is not the case. Further we introduce the moments \(M^{(i,j,k)}_{\text{maxw}}\) which are obtained by use of the Maxwellian (2). These definitions are the basis for the conditions

\[ |M^{(i,j,k)}_{\text{odd}}| \ll M^{(i,j,k)}_{\text{even}} \quad \text{and} \quad |M^{(i,j,k)}_{\text{even}} - M^{(i,j,k)}_{\text{maxw}}| \ll M^{(i,j,k)}_{\text{maxw}} \] (53)

which is similar to condition (31). The background for the conditions (53) is that odd moments occur only in the form of a deviation from equilibrium, whereas even moments also exist in equilibrium. Imposing (53) on the terms \(T_4\), we obtain

\[ \Sigma T_4 = \frac{\partial}{\partial x} \left( M^{(4,0,0)}_{\text{maxw}} + M^{(2,2,0)}_{\text{maxw}} + M^{(2,0,2)}_{\text{maxw}} \right) \] (54)
Putting the pieces together, we arrive at

\[ \frac{\delta q_x}{\delta t} = \frac{5}{2} p \left( \frac{dv_x}{dt} - \frac{F_x}{m} \right) + \frac{1}{2} \frac{\partial}{\partial x} \left( M^{(4,0,0)}_{\maxw} + M^{(2,2,0)}_{\maxw} + M^{(2,0,2)}_{\maxw} \right) \]  

(55)

With the useful relation

\[ M^{(i,j,k)}_{\maxw} = (i-1)!! (j-1)!! (k-1)!! NKT V^{n-2} \delta^{(i,j,k)} \]  

(56)

(where \( \delta^{(i,j,k)} = 1 \) if \( i, j, k \) are all even ; \( \delta^{(i,j,k)} = 0 \) otherwise), the terms in the second brackets in (55) add up to \( 5NK^2T^2/m \). Then, approximating \( (dv_x/dt - F_x/m) \) by the leading term in (24), \( -(1/Nm) \partial(NKT)/\partial x \), and generalizing from \( \delta q_x/\delta t \) to \( \delta q/\delta t \), we obtain without further approximation

\[ \frac{\delta q}{\delta t} = \frac{5}{2} NKV^2 \nabla T \]  

(57)

Treating the relation between \( \delta q/\delta t \) and \( q \) in the same way as we did for \( \delta (p)^o/\delta t \) and \( (p)^o \), we obtain the final result

\[ q = -\kappa \nabla T \]  

(58)

with the heat conductivity \( \kappa \) given by

\[ \kappa = \frac{5 NK V^2}{4 \nu^{(r)}} \]  

(59)

We note in passing that the result we would have obtained by neglecting moments of order \( n = 4 \) would be the same in magnitude as (58), but with opposite sign, i.e., heat would flow at the correct rate, but in the wrong direction from cold to hot.

Whereas the viscosity coefficient \( \eta \) given by (38) agreed fully with the first approximation of the Chapman–Enskog theory, (59) should be corrected by replacing the factor 5/4 by 15/8 to attain corresponding agreement for \( \kappa \). The reason for the discrepancy is that the collision model used (Ref. 6) employs a relaxation function which involves 10 moments of \( f \) (the first ten moments up to the elements of the pressure tensor), where it should involve 13 moments in order to be fully suited for a quantitative reproduction of the heat conductivity.

After having seen the full complexity of the equations (39)–(50), it appears like a miracle that such a simple result, eq. (58), could have been obtained in the end. But how is it possible that a result as plausible as (58) should require such a complicated derivation? Is it not obvious that heat will flow from hot to cold, at a rate proportional to the temperature gradient? The answer is that heuristic derivations, based on plausible arguments, may possibly lead to correct results, but will not be able to determine the applicability limits of these results. The importance of our treatment here is not that we have recovered the results (37) and (58), but that their applicability limits have been worked out thoroughly, and these are seen to be very narrow.

We have now reached our goal to obtain expressions for \( (p)^o \) and \( q \) which depend only on the fluid variables \( N, v \) and \( T \), thereby converting the open–ended system [(23),(26),(28)] into a closed system.
6. The Navier–Stokes system

Insertion of (37) and (58) in (26) and (28), treating \( \eta \) and \( \kappa \) as constants, yields

\[
\frac{dv}{dt} = \frac{F}{m} - v^2 \left[ \frac{\nabla T}{T} + \frac{\nabla N}{N} \right] + \frac{\eta}{Nm} \nabla \cdot \left[ \nabla v + (\nabla v)^t - \frac{2}{3} (\nabla \cdot v) \cdot U \right] \tag{60}
\]

\[
\frac{dT}{dt} = -\frac{2}{3} T \nabla \cdot v + \frac{2}{3} \frac{\eta}{KN} \left[ \nabla v : (\nabla v + (\nabla v)^t) - \frac{2}{3} (\nabla \cdot v)^2 \right] + \frac{2}{3} \frac{\kappa}{KN} \Delta T \tag{61}
\]

The system [(23),(60),(61)] may be called the Navier–Stokes system, since eq. (60), corresponding to the Navier–Stokes equation, is the central element in the system.

7. Energy equations

The system [(23),(60),(61)] must be energy conservative. In order to check this, we convert (60) and (61) into energy equations:

An ensemble of particles with individual velocities \( u \) has the energy density

\[
w = \frac{1}{2} m \int f u^2 d^3 u = \frac{1}{2} m \left( \int f v^2 d^3 u + \int f (u - v)^2 d^3 u + 2 \int f v (u - v) d^3 u \right) \tag{62}
\]

The first RHS term represents the kinetic energy density due to the bulk motion,

\[
w_K = \frac{1}{2} m \int f v^2 d^3 u = \frac{1}{2} N m v^2 \tag{63}
\]

the second the internal energy density,

\[
w_I = \frac{1}{2} m \int f (u - v)^2 d^3 u = \frac{3}{2} N K T \tag{64}
\]

and the third vanishes.

From (26) (multiplied by \( v \) and with \( F \) set to zero) and (28), both in conjunction with the continuity equation (23), the energy equations

\[
\frac{\partial w_K}{\partial t} + \nabla \cdot (w_K v) = -v \cdot [\nabla p + \nabla \cdot (p)^o] \tag{65}
\]

\[
\frac{\partial w_I}{\partial t} + \nabla \cdot (w_I v) = -[p \nabla \cdot v + (p)^o : \nabla v] - \nabla \cdot q \tag{66}
\]

are obtained. With the identity \( v \cdot [\nabla \cdot (p)^o] + (p)^o : \nabla v = \nabla \cdot [(p)^o \cdot v] \), the sum of \( w_K \) and \( w_I \), \( w = w_K + w_I \), is given by

\[
\frac{\partial w}{\partial t} + \nabla \cdot (w v) = -\nabla \cdot (p v) - \nabla \cdot [(p)^o \cdot v] - \nabla \cdot q \tag{67}
\]

Insertion of (37) and (58) in (65)–(67) yields
\[
\frac{\partial w_K}{\partial t} + \nabla \cdot (w_K \mathbf{v}) = - \mathbf{v} \cdot \nabla p + \eta \mathbf{v} \cdot \left( \nabla \cdot \left[ \nabla \mathbf{v} + (\nabla \mathbf{v})^t - \frac{2}{3} (\nabla \cdot \mathbf{v}) \mathbf{I} \right] \right) \quad (68)
\]

\[
\frac{\partial w_I}{\partial t} + \nabla \cdot (w_I \mathbf{v}) = - p \nabla \cdot \mathbf{v} + \eta \left[ \nabla \cdot [\nabla \mathbf{v} + (\nabla \mathbf{v})^t] - \frac{2}{3} (\nabla \cdot \mathbf{v})^2 \right] + \kappa \Delta T \quad (69)
\]

\[
\frac{\partial w}{\partial t} + \nabla \cdot (w \mathbf{v}) = - \nabla \cdot (p \mathbf{v}) + \eta \nabla \cdot \left( \mathbf{v} \cdot \left[ \nabla \mathbf{v} + (\nabla \mathbf{v})^t - \frac{2}{3} (\nabla \cdot \mathbf{v}) \mathbf{I} \right] \right) + \kappa \Delta T \quad (70)
\]

With Gauß’ integral theorem, a consequence of (67) or (70) is that the total energy
\[
E = \int w d^3 r
\]
inside a closed solid surface, or in entire space, is constant,
\[
\frac{dE}{dt} = 0 \quad (71)
\]
and this proves that the system [(23),(60),(61)] is indeed energy conservative. Naturally, the conservation of energy relates to \(E\) in total, not to its parts \(E_K = \int w_K d^3 r\) and \(E_I = \int w_I d^3 r\) individually. Thus, a violation of the integrity of the complete Navier–Stokes system would have the unacceptable consequence that energy is not conserved.

If we express the right–hand sides of (68) and (69) in component form, it will be difficult to attach a physical meaning to all the terms arising. We can, however, bundle terms in a useful way and get a better physical understanding thereby. We do this by introducing the following abbreviations:

\[
\dot{w}_{K \rightarrow P} = - \mathbf{v} \cdot \nabla p + 2 \eta \left[ v_x \frac{\partial^2 v_x}{\partial x^2} + v_y \frac{\partial^2 v_y}{\partial y^2} + v_z \frac{\partial^2 v_z}{\partial z^2} \right] - \frac{2}{3} \eta \mathbf{v} \cdot \nabla (\nabla \cdot \mathbf{v}) \quad (72)
\]

\[
\dot{w}_{I \rightarrow P} = - p \nabla \cdot \mathbf{v} + 2 \eta \left[ \left( \frac{\partial v_x}{\partial x} \right)^2 + \left( \frac{\partial v_y}{\partial y} \right)^2 + \left( \frac{\partial v_z}{\partial z} \right)^2 \right] - \frac{2}{3} \eta (\nabla \cdot \mathbf{v})^2 \quad (73)
\]

\[
\dot{w}_{K \rightarrow K} = \eta \left\{ \frac{\partial}{\partial x} \left[ v_y \left( \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial x} \right) + v_z \left( \frac{\partial v_z}{\partial x} + \frac{\partial v_y}{\partial z} \right) \right] + \ldots + \ldots \right\} \quad (74)
\]

\[
\dot{w}_{I \rightarrow I} = \kappa \nabla \cdot (\nabla T) \quad (75)
\]

\[
\dot{w}_{K \rightarrow I} = \eta \left[ \left( \frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x} \right)^2 + \left( \frac{\partial v_y}{\partial z} + \frac{\partial v_z}{\partial y} \right)^2 + \left( \frac{\partial v_z}{\partial x} + \frac{\partial v_x}{\partial z} \right)^2 \right] \quad (76)
\]

whereby the energy equations (68) and (69) adopt the short form

\[
\frac{\partial w_K}{\partial t} + \nabla \cdot (w_K \mathbf{v}) = \dot{w}_{K \rightarrow P} + \dot{w}_{K \rightarrow K} - \dot{w}_{K \rightarrow I} \quad (77)
\]

\[
\frac{\partial w_I}{\partial t} + \nabla \cdot (w_I \mathbf{v}) = \dot{w}_{I \rightarrow P} + \dot{w}_{I \rightarrow I} + \dot{w}_{K \rightarrow I} \quad (78)
\]

The energy transfer terms (72)–(76) have the following mathematical properties:
\[ \int \dot{w}_{K \leftrightarrow K} \, d^3 r = 0 \]  \hspace{1cm} (79)

\[ \int \dot{w}_{I \leftrightarrow I} \, d^3 r = 0 \]  \hspace{1cm} (80)

\[ \int \dot{w}_{K \rightarrow I} \, d^3 r > 0 \]  \hspace{1cm} (81)

Further, since \( \dot{w}_{K \leftrightarrow P} \) and \( \dot{w}_{I \leftrightarrow P} \) add up to

\[ \dot{w}_{K \leftrightarrow P} + \dot{w}_{I \leftrightarrow P} = -\nabla \cdot (p \mathbf{v}) + 2 \eta \left[ \frac{\partial}{\partial x} \left( \nu x \frac{\partial \nu}{\partial x} \right) + \ldots + \ldots \right] - \frac{2}{3} \eta \nabla \cdot \left[ \mathbf{v} \left( \nabla \cdot \mathbf{v} \right) \right] \]  \hspace{1cm} (82)

we have

\[ \int \left( \dot{w}_{K \leftrightarrow P} + \dot{w}_{I \leftrightarrow P} \right) \, d^3 r = 0 \]  \hspace{1cm} (83)

These relations show that all energy transfer rates occurring in (77) and (78) cancel out on integration over the volume within a closed solid surface, with the exception of \( \dot{w}_{K \rightarrow I} \) which is an unconditionally positive quantity everywhere in space. Since \( \dot{w}_{K \rightarrow I} \) acts as a loss term in (77), and as a gain term in (78), it is clear that \( \dot{w}_{K \rightarrow I} \) describes the irreversible loss of kinetic energy by transfer into internal energy, and this is indicated by the chosen nomenclature.

A few words about the physical meaning of the other energy transfer rates in (77) and (78). The meaning of the terms \( \dot{w}_{K \leftrightarrow P} \), \( \dot{w}_{I \leftrightarrow P} \) and \( \dot{w}_{I \leftrightarrow I} \) is obvious: \( \dot{w}_{K \leftrightarrow P} \) describes the mutual conversion of potential and kinetic energy of an ensemble of particles moving along or opposite to the pressure gradient, \( \dot{w}_{I \leftrightarrow P} \) describes the mutual conversion of potential and internal energy due to compression or dilatation, and \( \dot{w}_{I \leftrightarrow I} \) describes the spatial redistribution of internal energy due to conduction of heat. With this given, the remaining term \( \dot{w}_{K \leftrightarrow K} \) is left to describe the spatial redistribution of kinetic energy under the action of viscosity.

8. The assumption of incompressibility

The assumption of incompressibility is found in a vast number of publications, used with the intention to make the Navier–Stokes system simpler. Incompressibility means that the density of a fluid parcel cannot be changed. However, the Navier–Stokes system [(23),(60),(61)] contains no term that would prevent compressing a fluid parcel to an arbitrarily small volume. Thus, incompressibility here is not meant as a material property, but as the property of a process. Thereby, the assumption of incompressibility acts like a filter to sort out a great number of physical processes and concentrate on those which hopefully allow the desired simplification.

Under the assumption of incompressibility, the continuity equation (23) is replaced by

\[ N = \text{const.} \quad \text{and} \quad \nabla \cdot \mathbf{v} = 0 \]  \hspace{1cm} (84)

Application of (84) in (60) (with \( \mathbf{F} = 0 \)) and in (61) yields

\[ \frac{d\mathbf{v}}{dt} = -V^2 \frac{\nabla T}{T} + \frac{\eta}{Nm} \nabla \cdot [\nabla \mathbf{v} + (\nabla \mathbf{v})^t] \]  \hspace{1cm} (85)
Thereby, the closed 3–equation system \([23),(60),(61)\] is reduced to the closed 2–equation system \([85),(86)\], and the continuity equation (84) is no longer needed after it has done its duty to simplify (60) and (61). In (85), the temperature gradient \(\nabla T\) is left as the only driving agent after \(\nabla N\) has been removed due to (84). This fact is concealed if the first RHS term in (85) is written in the common form \(-\left(1/Nm\right)\nabla p\) (on account of \(\nabla p = \nabla (NK\nabla T) \rightarrow KN\nabla T = NmV^2 \nabla T/T\)).

Certainly, the system \([85),(86)\] is simpler than the complete system \([23),(60),(61)\], but still too complicated for an analytical treatment, and hence rather pointless considering that a lot of processes has been sacrificed for too little simplification. So the simplification goes on: The continuity equation (84) is reactivated for its second duty, namely to remove the thermal equation (86) and occupy its place. This leads to

\[
\frac{dv}{dt} = -V^2 \frac{\nabla T}{T} + \frac{\eta}{Nm} \Delta v \tag{87}
\]

\[
\nabla \cdot v = 0 \tag{88}
\]

where the identity \(\nabla \cdot [\nabla v + (\nabla v)'] = \Delta v + \nabla (\nabla \cdot v)\) has been used to come from (85) to (87).

The system \([87),(88)\] represents the so--called “incompressible Navier–Stokes equations”. A prominent example for its application is the Millennium Prize, endowed by the Clay Mathematics Institute (Fefferman\(^8\)) with the ambitious expectation to unlock the secrets hidden in the Navier–Stokes equations.

Despite its frequent use, the system \([87),(88)\] has to be rejected since it is physically deficient in several ways:

1. Eq. (87) calls for a closing equation for the temperature \(T\). It is obvious that this equation would have to comprise the basic thermal processes, i.e., heat transport by conduction and convection, and heat generation by viscous heating. The closing equation (88) is entirely useless in this regard.

2. The derivation of the Navier–Stokes system has shown that the thermal equation is an irremovable part of the system. A truncation of the infinite system (6)–(10) behind moments of first order is principally impossible. Thus, a neglect of the thermal equation leads to the inconsistency that terms of given order of magnitude are kept in one part of the system, but disregarded in the other.

3. Consequently, if one tries to enforce incompressibility, (88) can be used only as an additional equation, leaving (86) untouched, whereupon there would be three equations for the two unknowns \(v\) and \(T\). Or, in other words, solutions of \([87),(88)\] would be physically valid only if they are shown to agree with solutions of \([87),(86)\]. The aim to achieve simplification by assuming incompressibility would thus be missed and changed to the opposite. Incompressibility is a subordinate property, potentially following from the full set of transport equations, but unable to dictate the theoretical description from the beginning. The thermal equation does not lose its existence only because incompressibility is assumed.

4. When the thermal equation is disregarded, the loss of kinetic energy by conversion into internal energy, described by the term \(\dot{w}_{K\rightarrow I}\) in the energy equation (77), is not balanced anywhere in the system, with the consequence \(dE/dt < 0\). This non–conservation of energy is an absolute disqualifier for the system \([87),(88)\].

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In consequence, work based on the “incompressible Navier–Stokes equations”, and likewise on the “incompressible Euler equations”, has to be understood as mathematical work outside the range of justified physics.

9. Applicability conditions summarized

Fluid theory in general, and the Navier–Stokes system in particular, are bound to a number of applicability conditions which have been given above, scattered over the text. These will be summarized below:

1. In the kinetic equation (1), which is the basis for the system of macroscopic transport equations, particles are treated as points without finite size and without intermolecular forces (except those acting in close encounters). The applicability of (1), as well as anything following from (1), is thus restricted to ideal gases. In order to set a limit separating ideal and real gases, we employ van der Waals’ equation which provides a relation $p(N, T)$ for real fluids:

$$p = \frac{NKT}{1 - V_0N} - \frac{a}{A^2}N^2$$  \hspace{1cm} (89)

Here, $A$ is Avogadro’s number, $V_0$ is a measure of the volume occupied by a single particle, and $a$ is a tabulated quantity which determines the strength of the resultant intermolecular force, averaged over a large number of neighbouring particles. Eq. (89) shows that the ideal gas state is characterized by the conditions

$$N \ll \frac{1}{V_0} \quad \text{and} \quad N \ll \frac{KTA^2}{a}$$  \hspace{1cm} (90)

These conditions justify the use of the kinetic equation (1) and of the system of moment equations (6)–(10) following from (1).

2. To proceed from the infinite system of moment equations (6)–(10) to a finite set of fluid equations, condition (20) has to be satisfied. In the context of the present paper, it is the collisional limit of (20),

$$\tau_c \ll \tau \quad \text{and} \quad \tau_c V \ll l$$  \hspace{1cm} (22)

which has to be applied. This condition secures that different moments can be connected via local transport coefficients (e.g., $q = -\kappa \nabla T$). ‘Local’ means that the transport coefficient in question is a matter constant, independent of the characteristics of the process. A generalization from local to nonlocal is given in Ref. 4.

A second, but equivalent function of condition (22) is to justify the dominance of the collision term over other terms. This is seen most clearly in the derivation of $q = -\kappa \nabla T$ where all terms except the collision term have been neglected in $T_1$ and $T_3$ (see (51)).

3. A condition of equal importance concerns the different weights of even and odd moments. A moment is called even if all indices $(i,j,k)$ are even (zero taken as an even number), and odd otherwise. The corresponding condition, given above by eq. (53), reads

$$|M_{odd}^{(i,j,k)}| \ll M_{even}^{(i,j,k)} \quad \text{and} \quad |M_{even}^{(i,j,k)} - M_{maxw}^{(i,j,k)}| \ll M_{maxw}^{(i,j,k)}$$  \hspace{1cm} (53)

where $M_{maxw}^{(i,j,k)}$ corresponds to $M_{even}^{(i,j,k)}$ if the latter is obtained by using a Maxwell distribution. Condition (31), formulated for the elements of the pressure tensor, is physically equivalent to (53).
The physical background for the assumptions in (53) is that odd moments occur only in the form of a deviation from equilibrium, whereas even moments also exist in equilibrium. Thus, (53) rules out solutions far away from equilibrium. Since condition (53) is applied throughout the derivation of the Navier–Stokes system, it is an absolute necessity that solutions of the Navier–Stokes system obey (53) in order to be regarded as physically valid. One possibility to check this a posteriori is to calculate $p_{ii}$ and $p_{ij}$ following from (37) and ensure that $|p_{ij}|/p_{ii} \ll 1$.

4. Another validity check for solutions of the Navier–Stokes system consists in the requirement that the energy equation (70) is satisfied. This requirement is of special importance if the Navier–Stokes system [(23),(60),(61)] is not used in its complete form. In the case of the “incompressible Navier–Stokes equations” it is clear from the beginning that this check will have a negative outcome (see Section 8).

5. The conditions summarized above define another limit, viz. the limit between real and virtual hydrodynamics, where virtual hydrodynamics refers to the case that mathematical solutions without physical validity are obtained, and misinterpreted as physically real. Examples are solutions of the incompressible Navier–Stokes or Euler equations (see Section 8), and so–called blowup solutions which lead to unbounded values of the fluid variables. The Millennium Prize, too, pertains to virtual hydrodynamics. The prescribed task, challenging as it may be under mathematical aspects, lies outside the range of justified physics.

10. Tentative extension to real fluids

As noted before, the Navier–Stokes system [(23),(60),(61)] is not applicable to dense gases and liquids. An extension of these equations will have to take into account the finite size of particles and the intermolecular forces between them. It appears unlikely that this could be done in a rigorous fashion. This will limit our ambitions and suggest that we should proceed in a pragmatic way.

We do this by starting at van der Waal’s equation, given by eq. (89) above, which is a classical example for pragmatic physics. Being without stringent derivation, van der Waals’s equation ows its justification to its fair agreement with experimental results (outside the so–called Maxwell area), covering the range from ideal gases via dense gases (before reaching the Maxwell area) to liquids (after leaving the Maxwell area). In particular, the question of compressible vs. incompressible is self–consistently settled by (89): A compression beyond $NV_0 = 1$ is not possible.

The pressure gradient following from (89) reads

$$\nabla p = \nabla (NKT) + \nabla \left( \frac{V_0 N}{1 - V_0 N} NKT \right) - \frac{2a}{A^2} N \nabla N$$  \hspace{1cm} (91)

The first RHS term in (91) relates to a gas consisting of point particles without intermolecular forces, the second is a correction to account for the finite size of particles, and the third describes the accumulated action of intermolecular forces.

It is our intention to modify the kinetic equation (1) and the ensuing system of moment equations (6)–(10) in such a way that the following three conditions are fulfilled:

Condition 1: The result (91) must be reproduced.

Condition 2: The continuity equation (23) must be reproduced. The continuity equation, although derived here from the kinetic equation (1), can just as well be obtained without the aid of kinetic theory since it just counts the difference between the number of particles entering and leaving a volume element. This balance is independent of the size of particles or forces between them. Thus,
the continuity equation must remain the same, whether obtained from the kinetic equation (1) or from any modified version of (1).

Condition 3: The collisional invariances $\delta N/\delta t = 0$, $\delta v/\delta t = 0$ and $\delta T/\delta t = 0$, which do not depend on the kind of fluid, must be maintained, although $\delta f/\delta t$ will have to be altered. A modification of $\delta f/\delta t$ will affect the relation between $\delta (p)^o/\delta t$ and $(p)^o$, and between $\delta q/\delta t$ and $q$.

There exists a contradiction between $p$ given by (89) and $p$ defined by (15). The latter, in conjunction with the temperature definition (16), automatically leads to $p = NKT$. Now, realizing that the pressure gradient term in the equation of motion has its origin in $T^4$ (see eq. (10)), and that $T^4$ gives rise to $\nabla p = (1/3) \nabla (M^{(2,0,0)} + M^{(0,2,0)} + M^{(0,0,2)}) = \nabla (NKT)$, we see that the pressure is not the right place to accommodate the effects of the finite size of particles and of intermolecular forces. Needed is another place, but with the same effect.

As an attempt, one could replace in (1) $f$ by $f/(1 - V_0 N)$, thereby relating $f$ not to the full volume, but to the free volume left after subtracting the volume occupied by the particles. One consequence would be that the continuity equation would be changed to $dN/dt = -(1 - V_0 N) N \nabla \cdot v$, and so condition 2 would be violated.

Another attempt could be to modify $\delta f/\delta t$ in an appropriate way. However, since such a modification would have to generate an additional term in the equation of motion outside the viscosity term, whereas on the other hand condition 3 demands that $\delta v/\delta t = 0$, this attempt would be fruitless from the beginning.

In the end we see that the place left for a modification is the force term in (1) and (8), and by the replacement

$$ F = F_{ext} - \frac{1}{N} \nabla \left( \frac{V_0 N}{1 - V_0 N} NKT \right) + \frac{2a}{A^2} \nabla N $$

(92)

condition 1 is fulfilled, which is seen if the second and third terms in (92) are formally incorporated in $\nabla p$ by the correspondence $NF = \nabla p$. Here $F_{ext}$ is what $F$ has been before, viz. an external force acting on the particles individually. Inserting (92) in the term $T2$ (see eq. (8)) and following the derivation through to Section 3, we find (23) and (28) unaltered, whereas (26) is changed to

$$ \frac{dv}{dt} = \frac{1}{m} (F_{ext} + F_{int}) - V^2 \left[ \varphi \frac{\nabla T}{T} + \varphi^2 \frac{\nabla N}{N} \right] - \frac{1}{Nm} \nabla \cdot (p)^o $$

(93)

with

$$ \varphi(N) = \frac{1}{1 - V_0 N} \quad \text{and} \quad F_{int} = \frac{2a}{A^2} \nabla N $$

(94)

Since $\varphi > 1$, an important consequence of (93) is that density gradients have a higher weight than temperature gradients as driving agents for the velocity field.

The task left is to determine $(p)^o$ and $q$ for use in (93) and (28). Both will be affected by a modification of $\delta f/\delta t$. Starting with $(p)^o$ and following the derivation in Section 5, we find that nothing will be changed by the replacement (92) up to and including eq. (35). The problem is thus reduced to obtaining a connection between $\delta (p)^o/\delta t$ and $(p)^o$. In the case of an ideal gas this problem has been solved quantitatively, resulting in eq. (36). Here we will apply (36) in a qualitative way by assuming that $\delta (p)^o/\delta t$ is proportional to $(p)^o$, whereby (37) remains applicable, however with the caveat that $\eta$ is now an empirical quantity without theoretical backing. Insertion of (37) in (93) yields
\[ \frac{d\mathbf{v}}{dt} = \frac{1}{m} (\mathbf{F}_{\text{ext}} + \mathbf{F}_{\text{int}}) - V^2 \left[ \varphi \frac{\nabla T}{T} + \varphi^2 \frac{\nabla N}{N} \right] + \frac{\eta}{Nm} \nabla \cdot \left[ \nabla \mathbf{v} + (\nabla \mathbf{v})^t - \frac{2}{3} (\nabla \cdot \mathbf{v}) \mathbf{U} \right] \]  

(95)

By equivalent arguments, the thermal equation (61) is found to remain applicable unchanged, but with the corresponding reservation that the coefficient \( \kappa \) is an empirical quantity without a theoretical background.

A tacit assumption in the derivation leading from (1) to (61) has been that the particles have no internal degrees of freedom. Without this assumption, it would have been necessary to formulate separate kinetic equations for each excitation level, with transition terms between them. However, in the case that the potentially existing degrees of freedom are either fully activated, or not at all, we can correct (61) a posteriori (see Chapter 3 of Ref. 1) by translating the factors 5/3 and 2/3 occurring in (27) and (28) into \( \gamma \) and \( \gamma - 1 \), respectively, with \( \gamma \) the actual value of the adiabatic coefficient, subject to the number \( d \) of fully activated degrees of freedom (\( \gamma = (d + 2)/d \); \( d = 3 \) for monatomic particles, \( d = 5 \) for diatomic particles without vibrational excitation). In consequence, (61) is changed to

\[ \frac{dT}{dt} = (\gamma - 1) \left( -T \nabla \cdot \mathbf{v} + \frac{\eta}{KN} \left[ \nabla \mathbf{v} : (\nabla \mathbf{v})^t - \frac{2}{3} (\nabla \cdot \mathbf{v})^2 \right] + \frac{\kappa}{KN} \Delta T \right) \]  

(96)

The system [(23),(95),(96)] replaces the system [(23),(60),(61)]. We note that the sound velocity following from [(23),(95),(96)] increases with decreasing free volume (i.e., with decreasing mean distance between particles) due to the \( \varphi \)-terms in (95), in agreement with the experimental fact that sound is significantly faster in liquids than in gases, and in distinctive disagreement with [(23),(60),(61)].

As a final remark: In the main body of this paper the emphasis has been on a rigorous treatment. Here, in Section 10, we have to be satisfied with less, owing to the immanent intricacy of the subject. Our present attempt to extend the applicability of the Navier–Stokes system to real fluids is accompanied by the hope that further work will be stimulated thereby.

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