On Accuracy Function and Distance Measures of Interval-valued Pythagorean Fuzzy Sets with Application in Decision Making

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Abstract

In the present communication, a new accuracy function is being provided to overcome the limitations of the existing score/accuracy functions for interval valued Pythagorean fuzzy sets. The proposed accuracy function has been validated and discussed in detail through the illustrative examples. Further, a new distance measure for interval valued Pythagorean fuzzy numbers has been proposed and used in context with the existing weighted averaging operators. Finally, in view of the proposed accuracy function, distance measure and weighted averaging operators, a numerical example of multi-criteria decision making process has been presented to validate the methodology.

Keywords: Interval-valued Pythagorean Fuzzy Numbers, Score Function, Accuracy Function, Distance Measures and Weighted Averaging Operators.

1 Introduction

Fuzzy sets [1] have been used to express imprecise or vague information in various fields of real world application. Intuitionistic Fuzzy Set (IFS) by Atanassov’s [2] has been found to be highly adjustable framework to grapple the uncertainty with certain amount of hesitation arising from imperfect or vague information. The concept of Intuitionistic Fuzzy set has been widely studied and applied to deal with uncertainties and hesitancy inherent in practical circumstances. The most significant characteristic of an IFS is that it assigns a number from the unit interval [0, 1] to every element in the domain of discourse, a degree of membership and
a degree of non-membership along with the degree of indeterminacy whose total sum equals unity. In literature, IFSs and interval-valued IFSs comprehensively span applications in the field of decision making problems [3], pattern recognition, sales analysis financial services, medical diagnosis etc.

Pythagorean Fuzzy Set [4], is an efficient generalization of Intuitionistic Fuzzy Set (IFS) characterized by the inequality that the squared sum of a membership cum non-membership value is less than or equal to 1. Yager and Abbasov [5] well stated that in some practical multiple criteria decision making problems, it is viable that sum of the degree of the membership and the degree of non-membership value of a particular alternative provided by a decision maker may be in such a way that their sum is bigger than 1, where it would not be feasible to use Intuitionistic Fuzzy Set. Therefore, Pythagorean Fuzzy Set (PFS) proves to be proficiently more capable to represent and handle vagueness, impreciseness and uncertainties than IFS in various decision making processes. It may be noted that PFS is more generalized than IFS as the span of membership degree of PFS is more than span of membership degree of IFS which enables wider applicability.

Yager [4] also developed some aggregation operations under the PFS environment. Further, Yager and Abbasov [5] investigated the correspondence between the degrees of Pythagorean membership and complex numbers with an observation that Pythagorean membership degrees are only the subsets of complex numbers called \( \mathbb{P} \) numbers. Utilizing PFSs for solving multi-criteria decision-making problems, [6] enhanced the existing technique of order preference by similarity to ideal solution. Recently, Pythagorean fuzzy sets have been generalized to interval-valued Pythagorean fuzzy sets by [7] and provided some interval-valued Pythagorean fuzzy aggregation operators for handling related applications. Various other new operators have also been initiated by Peng [8] along with different properties and application in the field of decision making. Ranking method for Pythagorean fuzzy numbers and interval valued Pythagorean fuzzy numbers has been proposed by Zhang [9] by taking the idea of closeness index into account.

In literature, the notion of distance measures play a key role in the fuzzy set theory and in the application fields such as MCDM problems [10], [11], pattern recognition, medical diagnosis, financial services, etc. In recent past, various researchers have proposed different types of distance measures for different types of sets - viz. fuzzy sets, IFSs, Pythagorean fuzzy sets [12]. The Hamming distance, Euclidean distance and Hausdorff distance measures are some popular and widely utilized distance measures in the application and research world of soft computing ([13],[14],[15],[16],[17],[18],[19]). Zhang and Xu [6] provided a new measure of Pythagorean fuzzy numbers and applied it in multi-criteria decision making problem. Further, Li and Zeng [20] pointed that the strength and direction of commitment also play an important role for well description of PPFNs.
Therefore, they considered the four fundamental parameters - membership, non-membership a, strength and direction of commitment of PFNs and provided some distance measures. Most recently, Liu et al. [21] introduced important distance measures for IVPFNs along with its generalized, weighted and ordered weighted version. In addition to this, some generalized probabilistic distance measures and various important operators have been proposed by them and used in MCDM problems. The present work is extended by incorporating a new accuracy function for IVPFNs and overcome the limitations of the score/accuracy function in the existing methodologies while using IVPFNs for solving a MCDM problem.

The manuscript is organized as follows. We first present some basic notions of Pythagorean fuzzy numbers along with their corresponding score and accuracy functions in section 2. We discuss the notion of interval-valued Pythagorean fuzzy numbers and investigate about the shortcomings in the existing score/accuracy functions of IVPFNs in section 3. Also, a new accuracy function for IVPFNs has been proposed to handle the stated shortcomings in this section. In section 4, we are first listing some exiting distance measures and weighted averaging operators for IVPFNs and then proposing a new interval-valued Pythagorean fuzzy $p$-distance measure for IVPFNs in order to overcome the shortcomings in the existing methodology. In section 5, procedural steps of the proposed algorithm are provided for a MCDM problem. Subsequently a numerical example is being solved for the validation of the algorithm which incorporates the applicability of our proposed accuracy function, IVPF $p$-distance measure and the weighted averaging operators. Based on the illustrative example of multi-criteria decision making problem, some important remarks on the limitations of the existing methods have been listed in section 6. The paper is finally concluded in section 7 by providing the concluding remarks and scope for future work.

2 Pythagorean Fuzzy Numbers with their Score and Accuracy Functions

We present some fundamentals related to Pythagorean fuzzy sets/numbers along with their score/accuracy functions in this section.

A Pythagorean fuzzy set over $U$ (domain) is defined by Yager [4] as $P = \{< u, \mu_P(u), \nu_P(u) > | u \in U \}$, where $\mu_P : U \to [0, 1]$ and $\nu_P : U \to [0, 1]$ are the membership and non-membership functions such that $0 \leq (\mu_P(u))^2 + (\nu_P(u))^2 \leq 1$. The numbers $\mu_P(u)$ and $\nu_P(u)$ denote the degree of membership and non-membership of $u \in U$ in $P$, respectively. For each PFS $P \in U$, the quantity $\pi_P(u) = \sqrt{1 - \mu_P^2(u) - \nu_P^2(u)}$ represents the degree of indeterminacy of $u \in U$. 

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For simplicity, Zhang and Xu [6] called the pair \((\mu_P(u), \nu_P(u))\) as a Pythagorean fuzzy number denoted by \(p = (\mu_P, \nu_P)\). [7] defined an interval-valued Pythagorean fuzzy set in \(U\) is given by \(P = \{u, \mu_p(u) = [\mu_p(u), \bar{\mu}_p(u)], \nu_p(u) = [\nu_p(u), \bar{\nu}_p(u)]|u \in U\}\), where \([\mu_p(u), \bar{\mu}_p(u)]\) and \([\nu_p(u), \bar{\nu}_p(u)]\) denote the membership and non-membership degree of \(u\) in \(P\), respectively, with the condition \(0 \leq (\bar{\nu}_p(u))^2 + (\bar{\mu}_p(u))^2 \leq 1\). Here, \(\mu_p(u) = \inf \mu_p(u), \bar{\mu}_p(u) = \sup \mu_p(u)\) and \(\nu_p(u) = \inf \nu_p(u), \bar{\nu}_p(u) = \sup \nu_p(u)\) for all \(u \in U\). The degree of indeterminacy \(\pi_p(u) = [\pi_p(u), \bar{\pi}_p(u)]\) for all \(u \in U\) is called the interval-valued Pythagorean fuzzy index of \(u\) in \(P\), where \(\pi_p(u) = \sqrt{1 - (\bar{\pi}_p(u))^2 - (\bar{\nu}_p(u))^2}\) and \(\bar{\pi}_p(u) = \sqrt{1 - (\bar{\mu}_p(u))^2 - (\bar{\nu}_p(u))^2}\).

Yager and Abbasov [5] used another representation for PFN as \(p = (r_p, d_p)\), where \(r_p\), and \(d_p \in [0, 1]\) called the strength and direction of the commitment of \(p\) respectively. There is one to one correspondence between \((\mu_p, \nu_p)\) and \((r_p, d_p)\), given by \(\mu_p = r_p \cos(\theta_p), \nu_p = r_p \sin(\theta_p)\), where \(\theta_p = \arccos(\mu_p/r_p)\) and \(d_p = 1 - \frac{2\theta_p}{\pi}\).

Further, Yager and Abbasov [5] revealed that the Pythagorean membership degrees are contained in the class of complex numbers, denoted by \(\prod - i\) numbers. Therefore, they denoted PFN \(p = (\mu_p, \nu_p)\) as \(p = re^{-i\theta}\), where \(\mu_p = r_p \cos(\theta)\) and \(\nu_p = r_p \sin(\theta)\).

In order to compare two PFNs, Yager [4] proposed the following formula:

\[
V(p) = \frac{1}{2} + r_p \left( d_p - \frac{1}{2} \right) = \frac{1}{2} + r_p \left( \frac{1}{2} - \frac{2\theta_p}{\pi} \right) . \tag{2.1}
\]

Based on the above stated formula (2.1), Yager [4] gave the following rule for comparison:
Let \(p_1 = (r_{p_1}, d_{p_1})\) and \(p_2 = (r_{p_2}, d_{p_2})\) be two PFNs, then

- If \(V(p_1) > V(p_2)\), then \(p_1 \succ p_2\);
- If \(V(p_1) = V(p_2)\), then \(p_1 \sim p_2\).

Further, for comparing two Pythagorean fuzzy numbers, Zhang and Xu [6] proposed a score function of \(p = (\mu_p, \nu_p)\) given as:

\[
s(p) = (\mu_p)^2 - (\nu_p)^2, \tag{2.2}
\]

where \(s(p) \in [-1, 1]\).

Based on the above score function (2.2), Zhang and Xu [6] gave the following comparison rule:
Let \(p_1 = (\mu_{p_1}, \nu_{p_1})\) and \(p_2 = (\mu_{p_2}, \nu_{p_2})\) be two PFNs, then
\* s(p_1) < s(p_2) \Rightarrow p_1 \prec p_2;
\* s(p_1) > s(p_2) \Rightarrow p_1 \succ p_2;
\* s(p_1) = s(p_2) \Rightarrow p_1 \sim p_2.

It has also been pointed out by Peng and Yang [22] that the score function defined by Zhang and Xu [6] is not reasonable in some cases. For instance, suppose that for two PFNs \( p_1 = (0.6, 0.6) \) and \( p_2 = (0.7, 0.7) \), then by using score function (2.2), we have \( p_1 \sim p_2 \), but \( p_1 \) and \( p_2 \) are different. Thus, in view of the shortcoming of score function (2.2), The idea of accuracy function of a PFN has been proposed in literature and revised rules for comparison as follows: Let \( p = (\mu_p, \nu_p) \) be a PFN, then the accuracy function of \( p \) is given by

\[ a(p) = (\mu_p)^2 + (\nu_p)^2, \quad (2.3) \]

where \( a(p) = [0, 1] \). Based on the above accuracy function (2.3), following comparison rules are provided:

Let \( p_1 = (\mu_{p_1}, \nu_{p_1}) \) and \( p_2 = (\mu_{p_2}, \nu_{p_2}) \) be two PFNs, then

1. \( s(p_1) < s(p_2) \Rightarrow p_1 \prec p_2; \)
2. If \( s(p_1) = s(p_2) \), then,
   \* \( a(p_1) < a(p_2) \Rightarrow p_1 \prec p_2; \)
   \* \( a(p_1) = a(p_2) \Rightarrow p_1 \sim p_2. \)

3 Proposed Score and Accuracy Functions

Next, we discuss the basics of interval-valued fuzzy numbers along with their score and accuracy functions. A new accuracy function for the interval-valued fuzzy numbers is being proposed and studied in contrast with the existing accuracy functions.

For an interval valued Pythagorean fuzzy set \( P \), consider the pair

\[ \left( [\mu_p(u), \mu_p(u)], [\nu_p(u), \nu_p(u)] \right) \]

as an interval-valued Pythagorean fuzzy number and denote it by

\[ p = \left( [\mu_p(u), \mu_p(u)], [\nu_p(u), \nu_p(u)] \right). \]
Then for convenience, we represent IVPFN as $p = ((r_p, \tau_p), (\delta_p, \overline{\delta}_p))$, where the pair $(r_p, \tau_p)$ is called the lower and upper strength of $p$ and the pair $(\delta_p, \overline{\delta}_p)$ is called the lower and upper direction of the lower and upper strength of $p$, respectively. Moreover, the pairs $(r_p, \tau_p)$ and $(\delta_p, \overline{\delta}_p)$ are connected with interval-valued membership degree $[\mu_p(u), \overline{\mu}_p(u)]$ and non-membership degrees $[\nu_p(u), \overline{\nu}_p(u)]$, indicating the support for membership/belongingness and the support against membership of $u \in P$, respectively.

The relationship between $((\mu_p(u), \overline{\mu}_p(u)), [\nu_p(u), \overline{\nu}_p(u)])$ and $((r_p, \tau_p), (\delta_p, \overline{\delta}_p))$ is as follows:

$$
\begin{align*}
\mu_p(u) &= r_p \cos(\theta_p), \quad \nu_p(u) = r_p \sin(\theta_p); \\
\overline{\mu}_p(u) &= \tau_p \cos(\overline{\theta}_p), \quad \overline{\nu}_p(u) = \tau_p \sin(\overline{\theta}_p)
\end{align*}
$$

and

$$(\delta_p, \overline{\delta}_p) = \left(1 - \frac{2\theta_p}{\pi}, 1 - \frac{2\overline{\theta}_p}{\pi}\right),$$

where $\theta_p = \arccos(\mu_p(u)/r_p), \overline{\theta}_p = \arccos(\overline{\mu}_p(u)/\tau_p)$.

Further, we can easily show that the Pythagorean membership and non-membership degrees of IVPFN $p = ([\mu_p(u), \overline{\mu}_p(u)], [\nu_p(u), \overline{\nu}_p(u)])$ can be viewed as a radial length $p = [r_p e^{i\theta_p}, r_p e^{i\overline{\theta}_p}]$ in the complex plane, where

$$
\begin{align*}
r_p &= \sqrt{(\mu_p(u))^2 + (\nu_p(u))^2}, \\
\tau_p &= \sqrt{(\overline{\mu}_p(u))^2 + (\overline{\nu}_p(u))^2}.
\end{align*}
$$

To resolve the issue of comparison of interval valued Pythagorean fuzzy numbers, Peng and Yang [7] utilized the notion of score and accuracy functions, which are given as below:

$$
\begin{align*}
s(p) &= \frac{1}{2} \left((\mu_p(u))^2 + (\overline{\mu}_p(u))^2 - (\nu_p(u))^2 - (\overline{\nu}_p(u))^2\right), \\
n(p) &= \frac{1}{2} \left((\mu_p(u))^2 + (\overline{\mu}_p(u))^2 + (\nu_p(u))^2 + (\overline{\nu}_p(u))^2\right),
\end{align*}
$$

where $s(p) \in [-1, 1]$.

$$
\begin{align*}
a(p) &= \frac{1}{2} \left((\mu_p(u))^2 + (\overline{\mu}_p(u))^2 - (\nu_p(u))^2 - (\overline{\nu}_p(u))^2\right), \\
a(p) &= \frac{1}{2} \left((\mu_p(u))^2 + (\overline{\mu}_p(u))^2 + (\nu_p(u))^2 + (\overline{\nu}_p(u))^2\right),
\end{align*}
$$

where $a(p) \in [0, 1]$.

Based on these functions, they provided the following comparison rules:

Let $p_1$ and $p_2$ be two IVPFNs, then

1. $s(p_1) < s(p_2) \Rightarrow p_1 \prec p_2$.
2. $s(p_1) > s(p_2) \Rightarrow p_1 \succ p_2$. 


3. If \( s(p_1) = s(p_2) \), then

- \( a(p_1) < a(p_2) \Rightarrow p_1 \prec p_2 \).
- \( a(p_1) > a(p_2) \Rightarrow p_1 \succ p_2 \).
- \( a(p_1) = a(p_2) \Rightarrow p_1 \sim p_2 \).

In some cases, it may be observed that the score and accuracy functions given by equations (3.1) and (3.2) may not be able to rank IVPFNs accurately. In the following example, it has been shown that the existing score and the accuracy functions are not sufficiently appropriate to set the correct order preference of the objects involved in the MCDM problem.

**Example 1.** Consider two interval-valued Pythagorean fuzzy numbers given by

\[
p_1 = ([0.3, 0.6], [0.4, 0.8]) \quad \text{and} \quad p_2 = ([\sqrt{0.20}, \sqrt{0.25}], [\sqrt{0.35}, \sqrt{0.45}]).
\]

Using equation (3.1), we get \( s(p_1) = -0.1750 \) and \( s(p_2) = -0.1750 \). Now, we compute the value of accuracy function by using equation (3.2) and get \( a(p_1) = 0.6250 \) and \( a(p_2) = 0.6250 \). Therefore, based on the comparison rule, we get \( p_1 \sim p_2 \). But it may clearly be noted that \( p_1 \neq p_2 \). Hence, the existing score and accuracy functions of the IVPFNs are not capable enough to give the correct order preference.

Further, to overcome the shortcomings of the accuracy function given by equation (3.2), Garg [23] proposed a new improved accuracy function by considering the hesitation degree while the formulation.

For any IVPFN \( p = ([\mu_p(u), \mu_p(u)], [\mu_p(u), \mu_p(u)]) \), the improved accuracy function \( K(p) \) of \( p \) is defined as

\[
K(p) = \frac{1}{2} \left( (\mu_p(u))^2 + (\mu_p(u))^2 \sqrt{1 - (\mu_p(u))^2} - (\mu_p(u))^2 \right) + (\mu_p(u))^2 \sqrt{1 - (\mu_p(u))^2} - (\mu_p(u))^2 \right).
\]  

(3.3)

Garg [23] gave the following comparison rule on the basis of (3.3):

Let \( p_1 \) and \( p_2 \) be two IVPFNs.

- \( K(p_1) < K(p_2) \Rightarrow p_1 \prec p_2 \).
- \( K(p_1) > K(p_2) \Rightarrow p_1 \succ p_2 \).
- \( K(p_1) = K(p_2) \Rightarrow p_1 \sim p_2 \).

If we apply equation (3.3) in the above Example 1, then we get \( K(p_1) = 0.3809 \) and \( K(p_2) = 0.3636 \). Since \( K(p_1) > K(p_2) \) then \( p_1 \) has high preference than \( p_2 \).
**Example 2.** Let \( p_1 = ([0.1, 0.2], \sqrt{0.05}, 0.6]) \) and \( p_2 = ([0.1, 0.2], \sqrt{0.04}, \sqrt{0.37}) \) be two IVPFNs, then by using equations (3.1) and (3.2), we obtain the values
\[
\begin{align*}
s(p_1) &= -0.1400, \quad s(p_2) = -0.1400 \quad \text{and} \quad a(p_1) = 0.2700, \quad a(p_2) = 0.2700.
\end{align*}
\]
Thus, this shows that \( p_1 \) and \( p_2 \) are equivalent but both \( p_1 \) and \( p_2 \) are different. However, by using equation (3.3), we get
\[
\begin{align*}
K(p_1) &= 0.04826, \\
K(p_2) &= 0.04833,
\end{align*}
\]
i.e., \( K(p_1) < K(p_2) \) which implies that \( p_1 \succ p_2 \). However, if we take the value of the accuracy function up to four decimal places, then the improved accuracy function is unable to differentiate between the \( p_1 \) and \( p_2 \). Hence, both existing accuracy functions (3.2) and (3.3) are unable to differentiate between the IVPFNs.

Hence, to resolve the above stated comparison issues between IVPFNs, we have proposed the extended Yager’s [4] accuracy function (2.1) of PFNs to IVPFNs as follows:

For each IVPFN \( p = ([\underline{r}_p, \overline{r}_p], [\underline{d}_p, \overline{d}_p]) \), we define a new accuracy function as
\[
T(p) = \frac{2(\underline{r}_p \overline{d}_p + \overline{r}_p \underline{d}_p) - (\underline{r}_p + \overline{r}_p) + 2}{4},
\]
where \( \underline{r}_p = \sqrt{(\underline{r}_p(u))^2 + (\overline{r}_p(u))^2} \), \( \overline{r}_p = \sqrt{(\underline{r}_p(u))^2 + (\overline{r}_p(u))^2} \) and \( (\underline{d}_p, \overline{d}_p) = \left(1 - \frac{2p}{z^2}, 1 - \frac{2p}{z^2}\right) \).

Based on the proposed accuracy function (3.4), we give the following comparison rules:

Let \( p_1 \) and \( p_2 \) be two IVPFNs, then

- \( T(p_1) < T(p_2) \Rightarrow p_1 \prec p_2 \).
- \( T(p_1) > T(p_2) \Rightarrow p_1 \succ p_2 \).
- \( T(p_1) = T(p_2) \Rightarrow p_1 \sim p_2 \).

If we apply the proposed accuracy function (3.4) in the above two examples, we obtain the following values:

- In Example 1, \( T(p_1) = 0.4322, T(p_2) = 0.4288 \).
- In Example 2, \( T(p_1) = 0.3782, T(p_2) = 0.3818 \).
- In light of the revised comparison rules for the proposed accuracy function (3.4), in Example 1 we have \( T(p_1) > T(p_2) \) which indicate that \( p_1 \succ p_2 \) and in Example 2 we have \( T(p_1) < T(p_2) \) which indicate that \( p_1 \prec p_2 \).
Table 1: Comparative Analysis w.r.t. Example 1 and Example 2

| Methodology      | Score & Accuracy functions | Preference Order |
|------------------|-----------------------------|------------------|
| Peng and Yang [7] |                     | $p_1 \sim p_2$   |
|                  | $s(p_1) = -0.175$, $s(p_2) = -0.17$, $a(p_1) = 0.625$, $a(p_2) = 0.625$ |                  |
| Garg [23]        | $K(p_1) = 0.381$, $K(p_2) = 0.364$ | $p_1 \succ p_2$ |
| Proposed method  | $T(p_1) = 0.432$, $T(p_2) = 0.429$ | $p_1 \succ p_2$ |

Example 2

| Methodology      | Score & Accuracy functions | Preference Order |
|------------------|-----------------------------|------------------|
| Peng and Yang [7] |                     | $p_1 \sim p_2$   |
|                  | $s(p_1) = -0.140$, $s(p_2) = -0.140$, $a(p_1) = 0.270$, $a(p_2) = 0.270$ |                  |
| Garg [23]        | $K(p_1) = 0.0483$, $K(p_2) = 0.0483$ | $p_1 \sim p_2$ |
| Proposed method  | $T(p_1) = 0.378$, $T(p_2) = 0.382$ | $p_1 \prec p_2$ |

In order to demonstrate the effectiveness of the proposed accuracy function, the findings and observations in contrast to the existing accuracy function with their preference order is shown in Table 1.

From Table 1, it is clear that in Example 1, Peng and Yang [7] score and accuracy functions are not appropriate to set the correct preference order between the IVPFNs and preference ordering obtained using the proposed accuracy functions is consistent with the Garg [23] accuracy functions. In Example 2, both existing accuracy functions of Peng and Yang [7] and Garg [23] are not sufficiently appropriate to set the correct preference order between the IVPFNs but the proposed accuracy function is capable to set the correct ordering between the IVPFNs. Therefore, the ranking results obtained by the proposed accuracy function may be more appropriate than that accomplished by the existing score and the accuracy functions.

Example 3. Consider two interval-valued Pythagorean fuzzy numbers given by

$p_1 = \left( \left[ \sqrt{0.3}, \sqrt{0.6} \right], \left[ \sqrt{0.2}, \sqrt{0.3} \right] \right)$ and $p_2 = \left( \left[ \sqrt{0.3}, \sqrt{0.4} \right], \left[ \sqrt{0.1}, \sqrt{0.2} \right] \right)$.

We compute $r_{p_1} = \sqrt{0.5}$, $r_{p_2} = \sqrt{0.9}$, $d_{p_1} = 0.5641$, $d_{p_2} = 0.6082$ and $r_{p_2} = \sqrt{0.4}$, $r_{p_2} = \sqrt{0.6}$, $d_{p_2} = 0.6082$, $d_{p_2} = 0.6667$, $s(p_1) = 0.2$, $s(p_2) = 0.2$, and $a(p_1) = 0.07$, $a(p_2) = 0.05$. In view of the rules for comparison given by [7],
we obtain \( p_1 > p_2 \). Also, by the comparison rules provided by Garg [23], we get \( K(p_1) = 0.7096 \), \( K(p_2) = 0.5998 \) i.e., \( K(p_1) > K(p_2) \) which implies that \( p_1 > p_2 \). However, by Equation (3.4), we get \( T(p_1) = 0.5740 \), \( T(p_2) = 0.5946 \), i.e., \( T(p_1) < T(p_2) \) which implies \( p_1 < p_2 \). Hence, by utilizing the comparison rules of proposed accuracy function (3.4), we have \( p_1 < p_2 \), which is in contradiction with that of \( p_1 > p_2 \).

In the above example, the direction of commitment \( d_{p_1}(> \pi/4) \) is greater than the direction of commitment \( d_{p_2}(< \pi/4) \) and \( d_{p_1} = d_{p_2} = 0.6082 < \pi/4 \) which indicates that \( p_2 \) supports the membership more than \( p_1 \), i.e., \( p_2 \) is more preferred than \( p_1 \) (refer Table 2). Since the proposed accuracy function (3.4) takes the direction of commitment of IVPFN as an input, therefore, the ranking result may be more appropriate than that accomplished by the existing score and the accuracy functions.

### Table 2: Comparative Analysis w.r.t. Example 3

| Methodology       | Score and Accuracy functions       | Preference Order |
|-------------------|------------------------------------|-----------------|
| Peng and Yang [7] | \( s(p_1) = 0.2 \), \( s(p_2) = 0.2 \), \( a(p_1) = 0.07 \), \( a(p_2) = 0.05 \) | \( p_1 > p_2 \) |
| Garg [23]         | \( K(p_1) = 0.7096 \), \( K(p_2) = 0.5998 \) | \( p_1 > p_2 \) |
| Proposed method   | \( T(p_1) = 0.5740 \), \( T(p_2) = 0.5946 \) | \( p_1 < p_2 \) |

### 4 IVPF Distance Measures and Weighted Averaging Operators

In this section, firstly, we present the distance measures and weighted averaging operators proposed by Liu et al. [21] for interval-valued Pythagorean fuzzy numbers. We propose a new distance measure for interval valued Pythagorean fuzzy numbers and study the shortcomings in the existing distance measure proposed by Liu et al. [21].

Liu et al. [21] proposed the following IVPF \( p \)-distance measure, generalized IVPF weighted distance measure, generalized probabilistic IVPF OWA distance
operators and its further proceedings based on IVPF $p$-distance measure for interval valued Pythagorean fuzzy numbers.

**Definition 1. IVPF $p$-Distance Measure:** Consider any two IVPFNs

$$
\xi_1 = (\mu_{\xi_1}, \nu_{\xi_1}) = ([\mu_{\xi_1}, \bar{\mu}_{\xi_1}], [\nu_{\xi_1}, \bar{\nu}_{\xi_1}]) \quad \text{and} \quad \xi_2 = (\mu_{\xi_2}, \nu_{\xi_2}) = ([\mu_{\xi_2}, \bar{\mu}_{\xi_2}], [\nu_{\xi_2}, \bar{\nu}_{\xi_2}] ).
$$

The IVPF $p$-distance between $\xi_1$ and $\xi_2$ is denoted by $d_p(\xi_1, \xi_2)$ and defined as follows:

$$
d_p(\xi_1, \xi_2) = \frac{1}{4} \left[ |(\mu_{\xi_1})^2 - (\mu_{\xi_2})^2|^p + |(\bar{\mu}_{\xi_1})^2 - (\bar{\mu}_{\xi_2})^2|^p + |(\nu_{\xi_1})^2 - (\nu_{\xi_2})^2|^p + |(\bar{\nu}_{\xi_1})^2 - (\bar{\nu}_{\xi_2})^2|^p \right]. \tag{4.1}
$$

**Definition 2. Generalized IVPF Weighted Distance Measure:** Let $A = (\xi_1, \xi_2, \ldots, \xi_n)$ and $B = (\eta_1, \eta_2, \ldots, \eta_n)$ be two $n$-tuples of IVPFNs, where $(\mu_{\xi_i}, \nu_{\xi_i}) = ([\mu_{\xi_i}, \bar{\mu}_{\xi_i}], [\nu_{\xi_i}, \bar{\nu}_{\xi_i}])$ and $(\mu_{\eta_i}, \nu_{\eta_i}) = ([\mu_{\eta_i}, \bar{\mu}_{\eta_i}], [\nu_{\eta_i}, \bar{\nu}_{\eta_i}])$. Then, the generalized IVPF weighted distance (GIVPFWD) measure is a function $\Phi : IVPFN^n \times IVPFN^n \rightarrow \mathbb{R}$ defined as

$$
\Phi(A, B) = \left( \sum_{i=1}^{n} \omega_i d_p(\xi_i, \eta_i) \right)^{1/p}, \tag{4.2}
$$

where $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ is a weight vector with $\omega_i > 0$, $\sum_{i=1}^{n} \omega_i = 1$ and $d_p(\xi_i, \eta_i)$ is the IVPF $p$-distance between IVPFNs $\xi_i$ and $\eta_i$ defined by equation (4.1).

In addition to this, we have the following points to mention:

- If $d^p(\xi_i, \eta_i)$ is the $i$th largest value of $d^p(\xi_j, \eta_j), j = 1, 2, \ldots, n$ in the equation (4.2), then the distance measure (4.2) is called the generalized IVPF ordered weighted distance (GIVPFOWD) measure for IVPFNs.

- If we take $p = 1$ in equation (4.2), then it becomes the IVPF weighted averaging distance (IVPFWAD) measure given by

$$
\Phi(A, B) = \sum_{i=1}^{n} \omega_i d^1(\xi_i, \eta_i). \tag{4.3}
$$

- If we take $p = 2$ in equation (4.2), then it becomes the IVPF weighted Euclidean distance (IVPFWED) measure given by

$$
\Phi(A, B) = \left( \sum_{i=1}^{n} \omega_i d^2(\xi_i, \eta_i) \right)^{1/2}. \tag{4.4}
$$
Definition 3. **Generalized Probabilistic IVPF-OWA Distance Operators:** Let \( A = (\xi_1, \xi_2, \ldots, \xi_n) \) and \( B = (\eta_1, \eta_2, \ldots, \eta_n) \) be two \( n \)-tuples of IVPFNs, where

\[
(\mu_{\xi_i}, \nu_{\xi_i}) = \left( [\underline{\mu}_{\xi_i}, \bar{\mu}_{\xi_i}], [\underline{\nu}_{\xi_i}, \bar{\nu}_{\xi_i}] \right) \quad \text{and} \quad (\mu_{\eta_i}, \nu_{\eta_i}) = \left( [\underline{\mu}_{\eta_i}, \bar{\mu}_{\eta_i}], [\underline{\nu}_{\eta_i}, \bar{\nu}_{\eta_i}] \right).
\]

Then, the generalized probabilistic IVPF weighted averaging distance \((P\text{-GIVPFWAD})\) operator is a function \( \Phi : \text{IVPFN}^n \times \text{IVPFN}^n \rightarrow \mathbb{R} \) defined as

\[
\Phi(A, B) = \left( \sum_{i=1}^{n} \rho_i d^p(\xi_i, \eta_i) \right)^{1/p}, \quad (4.5)
\]

where \( \omega = (\omega_1, \omega_2, \ldots, \omega_n)^T \) is a weight vector with \( \omega_i > 0, \sum_{i=1}^{n} \omega_i = 1 \); \( \rho_i = \lambda_i w_i + (1 - \lambda_i) p_i \) and \( p_i \) is the associated probability of IVPF \( p \)-distance \( d^p(\xi_i, \eta_i) \). \( \lambda_i \in [0, 1] \) and \( (1 - \lambda_i) \) represent the degree of weight and the degree of probabilistic information respectively.

**Remarks:**

- If \( \lambda_i = 0 \), then GPWIVPF distance measure (4.5) is called the Generalized Probabilistic Interval-valued Pythagorean Fuzzy Distance (P-GIVPFD) measure.
- If \( \lambda_i = 1 \), then it reduces to GIVPFWD distance measure (4.2).
- If we take \( p = 1 \) in equation (4.5), then it becomes probabilistic interval-valued Pythagorean fuzzy weighted averaging distance \((P\text{-IVPFWAD})\) operator, given by

\[
\Phi(A, B) = \sum_{i=1}^{n} \rho_i d^1(\xi_i, \eta_i). \quad (4.6)
\]
- If we take \( p = 2 \) in equation (4.5), then it becomes the probabilistic interval-valued Pythagorean fuzzy weighted Euclidean distance \((P\text{-IVPFWED})\) operator, given by

\[
\Phi(A, B) = \left( \sum_{i=1}^{n} \rho_i d^2(\xi_i, \eta_i) \right)^{1/2}. \quad (4.7)
\]
- If \( d^p(\xi_i, \eta_i) \) as the \( i \)th largest of \( d^p(\xi_j, \eta_j), j = 1, 2, \ldots, n \) and \( p_i \) is the corresponding probability of the \( i \)th largest of \( d^p(\xi_j, \eta_j) \), then the distance measure (4.5) is called P-GIVPFOWAD operator.
If we take $\rho_i = \frac{w_ip_i}{\sum_{i=1} w_ip_i}$ and $d^p(\xi_i, \eta_i)$ as the $i$th largest of $d^p(\xi_j, \eta_j), j = 1, 2, \ldots, n$ with $p_i$ is the corresponding probability of the $i$th largest of $d^p(\xi_j, \eta_j)$, then the distance measure (4.5) is called IP-GIVPFOWAD operator.

It may be observed that the above distance measures or operators may result some unreasonable output sometimes. For instance:

**Example 4.** Let $\xi_1 = ([0.5, 0.6], [0.6, 0.7])$, $\xi_2 = ([0.6, 0.7], [0.5, 0.6])$ and $\xi_3 = ([0.3, 0.4], [0.4, 0.5])$ be three IVPFNs, then by using equation (4.1) for $p = 1$, we have $d(\xi_1, \xi_3) = d(\xi_2, \xi_3) = 0.40$ and $L_{\xi_1} = L_{\xi_2} = 0.7810$; $\tau_{\xi_1} = \tau_{\xi_2} = 0.9220$; $\vartheta_{\xi_1} = 0.8761, \vartheta_{\xi_2} = 0.8622; \theta_{\xi_1} = 0.6947, \theta_{\xi_2} = 0.7086; d_{\xi_1} = 0.4423, d_{\xi_3} = 0.4511; d_{\xi_2} = 0.5577, d_{\xi_3} = 0.5489$. It is observed that $\xi_1$ and $\xi_2$ have same lower and upper length but different lower and upper direction of commitment (strength). Hence, the distance between $\xi_1$ and $\xi_3$ should be different from that the distance between $\xi_2$ and $\xi_3$. Hence, in this case IVPF $p$-distance measure (4.1) is not appropriate to use.

Since the strength and the direction of commitment are important parameters of IVPFNs, therefore, their ignorance may lead to inappropriate results. Hence, by taking all the four parameters into account for IVPFNs, $[\mu_p(u), \pi_p(u)], [\nu_p(u), \nu_p(u)], (r_p, \tau_p)$ and $(d_p, \vartheta_p)$, we propose the IVPF $p$-distance measure between two IVPFNs $\xi_1$ and $\xi_2$ as follows:

$$d^p_{T}(\xi_1, \xi_2) = \frac{1}{4}\left[|\mu_{\xi_1} - \mu_{\xi_2}|^p + |\pi_{\xi_1} - \pi_{\xi_2}|^p + |\nu_{\xi_1} - \nu_{\xi_2}|^p + |\tau_{\xi_1} - \tau_{\xi_2}|^p + |L_{\xi_1} - L_{\xi_2}|^p + |\tau_{\xi_1} - \tau_{\xi_2}|^p + |d_{\xi_1} - d_{\xi_2}|^p + |\vartheta_{\xi_1} - \vartheta_{\xi_2}|^p\right].$$  

(4.8)

Next, by using the proposed IVPF $p$-distance measure (4.8) for $p = 1$, in the above Example 4, we have $d_T(\xi_1, \xi_3) = 0.3542$ and $d_T(\xi_2, \xi_3) = 0.4075$.

| Example 4 | Methodology       | IVPF $p$-distance measure |
|-----------|-------------------|---------------------------|
| Liu et al. [21] | $d(\xi_1, \xi_3) = d(\xi_2, \xi_3) = 0.40$ |
| Proposed method | $d_T(\xi_1, \xi_3) = 0.3542$, $d_T(\xi_2, \xi_3) = 0.4075$ |

Therefore, from the above Table 3, we can say that the proposed IVPF $p$-distance measure (4.8) is more reasonable distance measure to use in applications than the distance measure (4.1).
5 Application in Multi-criteria Decision Making Problem

In this section, we provide an application of the proposed accuracy function, proposed distance measures and OWA operators to deal with the multi-criteria decision-making (MCDM) problem under interval-valued Pythagorean fuzzy information. Consider the following setup of a MCDM problem in IVPFNs environment:

Consider the set of \( m \) possible alternatives, say, \( X = \{x_1, x_2, \ldots, x_m\} \) and the set of \( n \)-criteria on which the performance of alternatives are measured, say, \( C = \{C_1, C_2, \ldots, C_n\} \). Let \( \omega = \{\omega_1, \omega_2, \ldots, \omega_n\} \) be the weight vector of all criteria such that \( 0 \leq \omega_i \leq 1 \) and \( \sum_{i=1}^{n} \omega_i = 1 \). Assume that the performance of an alternative \( x_i (i = 1, 2, \ldots, m) \) with respect to the criteria \( C_j, j = 1, 2, \ldots, n \) are measured by IVPFNs \( C_j(x_i) = ([\mu_{i,j}, \nu_{i,j}], [\mu_{i,j}, \nu_{i,j}]) \), \( j = 1, 2, \ldots, n; i = 1, 2, \ldots, m \), where \( [\mu_{i,j}, \mu_{i,j}] \) represents the degree that alternative \( x_i \) satisfies the criterion \( C_j \) and \( [\nu_{i,j}, \nu_{i,j}] \) represents the degree that alternative \( x_i \) does not satisfy the criterion \( C_j \). Let \( D_{m \times n} = C_j(x_i)_{m \times n} \) is an interval valued Pythagorean fuzzy decision matrix. The procedural steps of the proposed MCDM algorithm are as follows:

Step 1: Compute the accuracies of each IVPFN of the obtained decision matrix \( M_{m \times n} = C_j(x_i)_{m \times n} \) by applying the proposed accuracy function (3.4).

Step 2: Determine the IVPF-Positive Ideal Solution (PIS) \( x^+ = \{\langle x_i, \max_j T(C_j(x_i)) \rangle\} \) and the IVPF-Negative Ideal Solution (NIS) \( x^- = \{\langle x_i, \min_j T(C_j(x_i)) \rangle\} \), for \( j = 1, 2, \ldots, n; i = 1, 2, \ldots, m \) with the help of the accuracies of IVPFNs obtained in Step 1.

Step 3: Evaluate the distance of each alternative \( x_i, i = 1, 2, \ldots, m \) from \( x^+ \) and \( x^- \) using the proposed IVPF \( p \)-distance measure (4.8).

Step 4: Using the values obtained in step 3 and rearranging the probability weight, we evaluate the new weights either by using

\[
\rho_i = \lambda_i w_i + (1 - \lambda_i)p_i \quad \text{or} \quad \rho_i = \frac{w_ip_i}{\sum_{i=1}^{n} w_ip_i}.
\]

Step 5: Determine the P-GIVPFOWAD or IP-GIVPFOWAD of the alternative \( x_i \) from the positive ideal IVPFN solution \( x^+ \) and the negative ideal IVPFN solution \( x^- \).
Step 6: Finally, we compute the coefficient of relative closeness for each alternative $x_i$ as follows:

$$r(x_i) = \frac{D(x_i, x^-)}{D(x_i, x^+) + D(x_i, x^-)}, \ i = 1, 2, \ldots, m.$$ (5.1)

where $D(\cdot)$ is an IP-GIVPFOWAD or P-GIVPFOWAD.

Step 7: Rank all the alternatives based on the coefficient of relative closeness $r(x_i)$ and the optimal alternative is chosen.

5.1 Numerical Example

In order to illustrate the implementation of the steps of the proposed algorithm stated above, we consider the following multi-criteria decision making problem about the selection of the strategy for an optimal production referring the related literature and undertakings completed by ([24], [25], [21]):

Suppose that a firm desires to manufacture a new product and looking for an optimal target of having the maximum benefits. Based upon a survey analysis of the market, they lay down the following 5 possible strategies (alternatives):

- $x_1$: Creating a new product aligning the rich customers;
- $x_2$: Creating a new product aligning the mid-level customers;
- $x_3$: Creating a new product aligning the low-level customers;
- $x_4$: Creating a new product suited to all customers;
- $x_5$: No manufacturing of any product.

After a detailed investigation of the information received from sources, the decision makers go for the following general criteria for the adaptability of strategies for production:

- $C_1$: Short term benefits;
- $C_2$: Mid term benefits;
- $C_3$: Long term benefits;
- $C_4$: Production risk;
- $C_5$: Various other factors.
Construct the decision matrix $D_{5 \times 5} = C_j(x_i)_{5 \times 5} = ([\mu_{i,j}, \mu_{i,j}], [\nu_{i,j}, \nu_{i,j}]), j = 1, 2, \ldots, 5; i = 1, 2, \ldots, 5$ as shown in the Table 4.

Suppose that with reference to the problem under consideration, the decision makers find the probabilistic information $p = (0.3, 0.3, 0.2, 0.1, 0.1)$ and weight vector $w = (0.2, 0.25, 0.15, 0.3, 0.1)$ which represents the degree of importance/weightage of each criteria. Then, to get the most desirable alternative, we apply the steps of the proposed algorithm. First, we compute the accuracies of each IVPFN of the decision matrix shown in Table 4 by applying the proposed accuracy function (3.4) and the computed values are tabulated in Table 5.

| $x_1$ | $C_1$ | $C_2$ | $C_3$ | $C_4$ | $C_5$ |
|-------|-------|-------|-------|-------|-------|
| 0.4549 | 0.451 | 0.4773 | 0.5451 | 0.4773 |
| 0.451 | 0.4549 | 0.5227 | 0.4549 | 0.5227 |
| 0.4322 | 0.5905 | 0.6603 | 0.5000 | 0.3162 |
| 0.5225 | 0.6164 | 0.3357 | 0.6403 | 0.5451 |
| 0.5925 | 0.5225 | 0.3619 | 0.4774 | 0.6603 |

Based on the accuracies obtained in Table 5, we find the IVPF-positive ideal solution $x^+$ and the IVPF-negative ideal solution $x^-$ as follows:

$$x^+ = \{(C_1, ([0.3, 0.4], [0.1, 0.2])), (C_2, ([0.4, 0.5], [0.1, 0.3])), (C_3, ([0.6, 0.8], [0.3, 0.4])), (C_4, ([0.4, 0.5], [0.1, 0.2])), (C_5, ([0.6, 0.8], [0.3, 0.4]))\}$$

(5.2)

and

$$x^- = \{(C_1, ([0.3, 0.6], [0.4, 0.8])), (C_2, ([0.4, 0.5], [0.5, 0.6])), (C_3, ([0.1, 0.2], [0.4, 0.6])), (C_4, ([0.4, 0.5], [0.5, 0.6])), (C_5, ([0.3, 0.4], [0.7, 0.8]))\}$$

(5.3)

respectively. Further, we evaluate the P-GIVPFOWAD or IP-GIVPFOWAD of each alternative $x_i, i = 1, 2, \ldots, m$ from the IVPF-positive ideal solution $x^+$ and the IVPF-negative ideal solution $x^-$ and their respective results are shown in Tables 6 to 9. Then, we find the relative closeness coefficient by the equation (5.1) of each alternative $x_i$ and their respective results are shown in Tables 10 and 11. Also, the respective results have been illustrated through corresponding figures - Figure 1 and Figure 2. From Table 10, Table 11 and Table 12, we observe
that for different value of parameter \( p \), the ranking order of alternatives remains unchanged by applying either P-GIVPFOWAD or IP-GIVPFOWAD and all the results show that \( x_5 \) is the optimal alternative.

Table 6: Distances between \( x_i \) and \( x^+ \) obtained by P-GIVPFOWAD

| Value of \( p \) | \( D(x_1, x^+) \) | \( D(x_2, x^+) \) | \( D(x_3, x^+) \) | \( D(x_4, x^+) \) | \( D(x_5, x^+) \) |
|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| 1               | 0.4934          | 0.4783          | 0.4152          | 0.3468          | 0.2807          |
| 2               | 0.3924          | 0.3717          | 0.3331          | 0.2635          | 0.2234          |
| 4               | 0.3681          | 0.3402          | 0.3138          | 0.2421          | 0.2125          |
| 6               | 0.3690          | 0.3452          | 0.3150          | 0.2405          | 0.2145          |
| 8               | 0.3728          | 0.3470          | 0.3183          | 0.2417          | 0.2171          |
| 10              | 0.3766          | 0.3490          | 0.3214          | 0.2434          | 0.2194          |

Table 7: Distances between \( x_i \) and \( x^- \) obtained by P-GIVPFOWAD

| Value of \( p \) | \( D(x_1, x^-) \) | \( D(x_2, x^-) \) | \( D(x_3, x^-) \) | \( D(x_4, x^-) \) | \( D(x_5, x^-) \) |
|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| 1               | 0.2320          | 0.2609          | 0.2245          | 0.3103          | 0.3258          |
| 2               | 0.1912          | 0.2229          | 0.1787          | 0.2519          | 0.2666          |
| 4               | 0.1902          | 0.2178          | 0.1695          | 0.2385          | 0.2539          |
| 6               | 0.1965          | 0.2280          | 0.1700          | 0.2398          | 0.2555          |
| 8               | 0.2017          | 0.2339          | 0.1712          | 0.2423          | 0.2583          |
| 10              | 0.2057          | 0.2386          | 0.1723          | 0.2444          | 0.2607          |

Table 8: Distances between \( x_i \) and \( x^+ \) obtained by IP-GIVPFOWAD

| Value of \( p \) | \( D(x_1, x^+) \) | \( D(x_2, x^+) \) | \( D(x_3, x^+) \) | \( D(x_4, x^+) \) | \( D(x_5, x^+) \) |
|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| 1               | 0.4715          | 0.4568          | 0.3903          | 0.3662          | 0.2453          |
| 2               | 0.3815          | 0.3510          | 0.3122          | 0.2750          | 0.1963          |
| 4               | 0.3619          | 0.3281          | 0.2918          | 0.2501          | 0.1871          |
| 6               | 0.3647          | 0.3279          | 0.2910          | 0.2475          | 0.1889          |
| 8               | 0.3694          | 0.3301          | 0.2929          | 0.2483          | 0.1914          |
| 10              | 0.3738          | 0.3325          | 0.2951          | 0.2499          | 0.1935          |
Table 9: Distances between $x_i$ and $x^-$ obtained by IP-GIVPFOWAD

| Value of $p$ | $D(x_1, x^-)$ | $D(x_2, x^-)$ | $D(x_3, x^-)$ | $D(x_4, x^-)$ | $D(x_5, x^-)$ |
|-------------|----------------|----------------|----------------|----------------|----------------|
| 1           | 0.2213         | 0.2629         | 0.1995         | 0.3122         | 0.2902         |
| 2           | 0.1842         | 0.2238         | 0.1561         | 0.2545         | 0.2372         |
| 4           | 0.1843         | 0.2215         | 0.1466         | 0.2423         | 0.2241         |
| 6           | 0.1910         | 0.2275         | 0.1466         | 0.2445         | 0.2243         |
| 8           | 0.1965         | 0.2331         | 0.1475         | 0.2476         | 0.2260         |
| 10          | 0.2008         | 0.2377         | 0.1483         | 0.2501         | 0.2277         |

Table 10: Relative closeness coefficient obtained by P-GIVPFOWAD

| Value of $p$ | $r(x_1)$ | $r(x_2)$ | $r(x_3)$ | $r(x_4)$ | $r(x_5)$ |
|-------------|----------|----------|----------|----------|----------|
| 1           | 0.3198   | 0.3529   | 0.3509   | 0.4722   | 0.5372   |
| 2           | 0.3276   | 0.3749   | 0.3492   | 0.4887   | 0.5441   |
| 4           | 0.3407   | 0.3904   | 0.3507   | 0.4963   | 0.5444   |
| 6           | 0.3475   | 0.3978   | 0.3505   | 0.4993   | 0.5436   |
| 8           | 0.3511   | 0.4027   | 0.3497   | 0.5006   | 0.5433   |
| 10          | 0.3533   | 0.4061   | 0.3490   | 0.5010   | 0.5430   |

Figure 1: Ranking Order based on P-GIVPFOWAD

Table 11: Relative closeness coefficient obtained by IP-GIVPFOWAD

| Value of $p$ | $r(x_1)$ | $r(x_2)$ | $r(x_3)$ | $r(x_4)$ | $r(x_5)$ |
|-------------|----------|----------|----------|----------|----------|
| 1           | 0.3194   | 0.3653   | 0.3383   | 0.4602   | 0.5419   |
| 2           | 0.3256   | 0.3894   | 0.3333   | 0.4806   | 0.5472   |
| 4           | 0.3474   | 0.4030   | 0.3344   | 0.4921   | 0.5450   |
| 6           | 0.3477   | 0.4096   | 0.3350   | 0.4970   | 0.5428   |
| 8           | 0.3472   | 0.4139   | 0.3349   | 0.4993   | 0.5414   |
| 10          | 0.3495   | 0.4169   | 0.3345   | 0.5002   | 0.5406   |

Table 12: Distances between $x_1$ and $x_2$ from $x^+$

|        | $D(x_1, x^+)$ | $D(x_2, x^+)$ |
|--------|----------------|----------------|
| $D(x_1, x^+)$ | 0.5800         | 0.2550         |
| $D(x_2, x^+)$ | 0.5800         | 0.2550         |

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6 Remarks on Limitations of Existing Methods

Based on the numerical example and the computed values obtained in the last section, we put forward some remarks on the limitations of the existing methods:

- If we proceed the MCDM problem with the proposed accuracy function (3.4) and Liu et al. [21] distance measure (4.1), then from Table 9, we observe that the distances of $x_1$ and $x_2$ from the IVPF-PIS $x^+$ for $p = 1$ are same, but the alternatives $x_1$ and $x_2$ w.r.t. all the criteria $C_j; j = 1, 2, \ldots, n$ are taken as different IVPFNs.

- Also, $P - GIVPFOWAD(x_1, x^+) = P - GIVPFOWAD(x_2, x^+) = 0.3561$ and $IP - GIVPFOWAD(x_1, x^+) = IP - GIVPFOWAD(x_2, x^+) = 0.33340$ which indicates that $x_1$ and $x_2$ have same preference. Hence, in such cases, Liu et al., [21] distance measure (4.1) is not much reasonable to apply.

- The scoring of alternatives $x_1$ and $x_2$ with respect to the criteria $C_3$ and $C_4$ by using the Peng and Yang [7]'s score and accuracy functions (3.1) and (3.2) are as follows: $s(C_3(x_1)) = s(C_3(x_2)) = -0.0250$, but both $C_3(x_1)$ and $C_3(x_2)$ are represented by different IVPFNs. Similarly, $s(C_5(x_1)) = s(C_5(x_2)) = 0.0250$, but both $C_5(x_1)$ and $C_5(x_2)$ are represented by different IVPFNs.

- Further, if we calculate their accuracies, then we have $a(C_3(x_1)) = s(C_5(x_1)) = 0.4250$ and $a(C_3(x_2)) = s(C_5(x_2)) = 0.5550$ but both $C_3(x_1) \neq C_5(x_1)$ and $C_3(x_2) \neq C_5(x_2)$. Hence, we can not proceed for decision making in the right direction by the Liu et al. [21] approach.
• If the performance of alternatives \(x_3\) and \(x_4\) with respect to the criterion \(C_4\) in the decision matrix shown by Table 4 are represented by the IVPFNs \(C_3(x_4) = (0.1, 0.2, \sqrt{0.04, \sqrt{0.37}})\) and \(C_3(x_5) = (0.2, 0.3, \sqrt{0.05, 0.6})\), respectively, then by using the Peng and Yang [7]'s score and accuracy functions (3.1) and (3.2), we have \(s(C_3(x_4)) = -0.01400; s(C_3(x_5)) = -0.1400\) and \(a(C_3(x_4)) = 0.2700; a(C_3(x_5)) = 0.2700\). Thus, this shows that \(C_3(x_4)\) and \(C_3(x_5)\) are equivalent but both \(C_3(x_4)\) and \(C_3(x_5)\) are different.

• Therefore, by the Peng and Yang [7]'s score and accuracy functions, we can not determine the scoring and accuracies of \(C_3(x_4)\) and \(C_3(x_5)\) and may fail to find the positive or negative ideal solutions. Hence, we can not proceed for decision making by the Liu et al. [21] approach.

7 Conclusions and Scope for future work

In order to overcome the stated shortcomings, we have successfully incorporated the important four parameters - membership/non-membership/strength & direction of commitment and introduced a new accuracy function. Furthermore, a new IVPF \(p\)-distance measure for interval-valued Pythagorean fuzzy numbers has been well proposed and used along with existing weighted averaging operators to accomplish the task carried out in an example of the MCDM problem. In the process of any model evaluation and quantifying the quality of prediction, the estimator score function and accuracy function may suitably be utilized in subsequent future work. Also, the proposed distance measure may be used in multi-label ranking metrics, regression metrics and clustering metrics.

Compliance with ethical standards

Conflict of interest: Authors declare that they have no conflict of interest. Ethical approval: This article does not contain any studies with human participants or animals performed by any of the authors.

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