PARTIALLY HYPERBOLIC DYNAMICS AND 3-MANIFOLD TOPOLOGY

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1. INTRODUCTION

Partially hyperbolic dynamics is a young and exciting subject that has in the recent years played a central role, directly or indirectly, in some spectacular achievements both in dynamics and geometry as well as other nearby fields. Partially hyperbolic diffeomorphisms cover large classes of examples: they form an open class, containing important classes of homogeneous dynamics and many geometric flows and play a fundamental role in the study of robust dynamical phenomena, such as stable ergodicity and robust transitivity.

The success of the field may be attributed to the fact that it is a good weakening of hyperbolic dynamics, a subject which has already seen an immense number of applications. Partial hyperbolicity is more flexible and therefore better suited to model more diverse phenomena, but rigid enough to get strong and deep results, especially when looking for the interaction between the topology/geometry of the manifold and this structure. A natural caveat is that being more general, the understanding of its dynamical consequences as well as its general implications is definitely weaker; this makes its direct study also very challenging and many important questions remain that we will partially comment here.

In this note we plan to explain in some detail a beautiful sample result due to Margulis and Plante-Thurston which exhibits the interaction between the existence of geometric structures invariant under some dynamics and the topology of the underlying manifold. This will serve as an introduction to explain the problem of topological classification of partially hyperbolic systems, particularly the problem of finding topological obstructions for their existence. At the end we discuss on some further directions and connections with the mainstream of the subject as well as suggest some survey references for further reading.

2. DYNAMICS OF FLOWS AND Diffeomorphisms

Dynamical systems theory is concerned in understanding long term behaviour of transformation laws of a phase space given by some deterministic rule. Smooth dynamics studies this problem in the case where the fase space is a smooth manifold and the transformation laws admit some regularity with respect to this smooth structure. For technical simplicity and since it will be the case we will be mostly concerned with,
we will assume throughout that \( M \) is a closed manifold, meaning that it is compact, connected and without boundary.

For instance, if one is given a smooth vector field \( X \) in a smooth manifold \( M \) (that is, one assigns to each point \( x \) in \( M \) a tangent vector \( X(x) \in T_xM \)) it is well known that \( X \) integrates into a flow, this means that there exists a one parameter family of diffeomorphisms \( \phi_t : M \to M \) which verify the flow relation \( \phi_{t+s} = \phi_t \circ \phi_s \) and whose orbits are tangent to \( X \), meaning that \( \frac{\partial}{\partial t} \phi_t(x) \big|_{t=0} = X(x) \). We say in this case that the deterministic rule \( \dot{x} = X(x) \) determines the transformation rule \( \phi_t \), and our goal is to understand the asymptotic structure of orbits \( \phi_t(x) \) as \( t \to \infty \). Ideally, this should be based on a priori or static knowledge of the equation (this means, information that one can get without integrating the equation).

Examples of such dynamics abound and appear in very diverse fields of mathematics and mathematical physics. Several objects and quantities in differential geometry and classical mechanics are governed by differential equations in smooth phase spaces, but flows in manifolds arise also in a diversity of different situations, for instance the understanding of orbit closures in certain dynamical systems of algebraic nature is crucial in some problems in Diophantine approximation, and in the understanding of moduli spaces of certain geometric structures in surfaces or other manifolds.

For diverse reasons it sometimes makes sense to understand discrete versions of these. In this case, the rule is given by a single diffeomorphism \( f : M \to M \) and the dynamics is given by the iterates \( f^n \) of \( f \) where we denote \( f^0 = \text{id} \) and \( f^n = f \circ f^{n-1} \) for every \( n \in \mathbb{Z} \). The object of study here are the orbits \( \{ f^n(x) \} \) for \( x \in M \).

The fact that our transformations are smooth (in contraposition to continuous or measurable) allows us to use differential topology in their study. In particular, diffeomorphisms or smooth flows allow us to consider its derivative to analyse the local structure. If \( f : M \to M \) is a diffeomorphism, its derivative \( Df \) acts on several vector bundles over \( M \) constructed using the tangent \( TM \) or cotangent \( T^*M \) bundle of \( M \). This way one can say that \( f \) preserves e.g. a symplectic or volume form, or that it preserves a subbundle \( E \subset TM \). Stability questions also require some smoothness in general, and many natural dynamics appearing in other sciences or other fields of mathematics naturally possess some amount of regularity.

One can think of a flow as an action of \( \mathbb{R} \) in \( M \) and a diffeomorphism as an action of \( \mathbb{Z} \). In many cases more general group actions arise naturally in the understanding of certain phenomena. We will not touch upon this interesting topic, but mention that general group actions incorporate several actions by simpler groups such as \( \mathbb{R} \) or \( \mathbb{Z} \).

Since our main goal is to understand long term behaviour from static information, we will restrict to a subclass of dynamics. This subclass has the property that can be detected with static information. Moreover, the information that we get is strong enough to give some tools to understand its asymptotic behaviour, particularly in low dimensions. This class is an extension of what is called hyperbolic systems and we call them partially hyperbolic.

3. Anosov flows

Anosov systems are a particularly important example of partially hyperbolic systems. Beyond some algebraic constructions, these arise quite naturally when studying spaces of negative curvature: for instance, the geodesic flow of a surface of negative curvature is well known to be an Anosov flow.
A flow $\phi_t : M \to M$ generated by a (smooth) vector field $X$ on a closed manifold $M$ is said to be an Anosov flow if there is a continuous $D\phi_t$-invariant splitting of the tangent bundle $TM = E^s \oplus \mathbb{R}X \oplus E^u$ verifying that there is $t_0 > 0$ so that for every $v^\sigma \in E^\sigma$ ($\sigma = s, u$) a unit vector we have that $\|D\phi_{t_0} v^s\| < 1 < \|D\phi_{t_0} v^u\|$. This immediately implies that stable vectors (i.e. those in $E^s$) are contracted exponentially by $D\phi_t$ while unstable vectors (i.e. those in $E^u$) are expanded exponentially fast by $D\phi_t$.

One could write a complete note about equivalent definitions and reasons these systems do appear but let us just comment that being Anosov is an open property among smooth flows and that it can be detected with finite precision by the use of cone-fields. Indeed, the reason one defines these kind of systems by use of some geometric structure preserved by the differential is really because of the robust nature of the condition as well as its easy checkable nature. The reason the definition is useful is that one can construct objects in the manifold whose dynamics mimic those of the invariant bundle of the differential.

**Theorem 3.1** (Stable manifold theorem). The bundles $E^s$ and $E^s \oplus \mathbb{R}X$ are uniquely integrable.

We need to say some words to explain what we mean. Let $E \subset TM$ be a $k$-dimensional subbundle of the tangent bundle of $M$. We say that $E$ is uniquely integrable if through every point $x \in M$ there is a $k$-dimensional submanifold $S_x$ everywhere tangent to $E$ such that every curve tangent to $E$ through $x$ must be completely contained in $S_x$.

The same is true for $E^u$ and $\mathbb{R}X \oplus E^u$ by applying the theorem to $\phi_{-t}$. Notice that even when $\dim(E^s) = 1$ showing its unique integrability is not obvious since these bundles are typically no better than Hölder continuous, so one needs to appeal to dynamics to get this kind of results.

However, the proof of this result when $\dim(E^s) = 1$ is not too complicated. Notice that if there are two different curves $\gamma_1, \gamma_2$ everywhere tangent to $E^s$ through a point $x \in M$, then iterating forward by $\phi_t$ the length of the curves gets exponentially contracted, while the distance along the transverse direction cannot decrease. To
show unique integrability of $E^s \oplus \mathbb{R}X$ one just needs to flow the integral curves of $E^s$ by the flow (whose defining vector field is smooth, so uniquely integrable).

The curves tangent to $E^s$ and the surfaces tangent to $E^s \oplus \mathbb{R}X$ form what we call the strong stable and weak stable foliations, which together with their dual strong unstable and weak unstable foliations are one of the main tools to understand the dynamics and geometry of Anosov systems. Let us just state an easy fact about these that we will use later:

**Proposition 3.2.** There are no compact submanifolds $N$ of $M$ tangent to $E^s \oplus \mathbb{R}X$.

This is a direct consequence of the fact that the time $t_0$ map $\phi_{t_0}$ of the flow would be a diffeomorphism of the compact manifold $N$ whose derivative is everywhere contracting volume, which is just impossible.

### 4. Margulis/Plante-Thurston’s result

In the late 60’s Margulis showed the following beautiful result:

**Theorem 4.1.** Let $M$ be a closed 3-dimensional manifold admitting an Anosov flow $\phi_t$, then, the fundamental group of $M$ grows exponentially.

A finitely generated group $\Gamma$ has exponential growth if for some finite generator $F \subset \Gamma$ it follows that the number of different group elements that can be written as a product of at most $n$ elements of $F \cup F^{-1}$ grows exponentially with $n$. This is independent of the finite generator $F$. Let us give an equivalent definition of exponential growth of the fundamental group of a closed manifold $M$: we say that a closed manifold $M$ has exponential growth of fundamental group if and only if the volume of a ball or radius $R$ in $\tilde{M}$, the universal cover of $M$, growth exponentially with respect to $R$. This means, if $\pi : \tilde{M} \to M$ is the universal covering map, and we consider in $\tilde{M}$ the metric induced by $\pi$, then there is a point $x \in \tilde{M}$ and constants $c, \delta > 0$ so that:

$$\text{vol}(B(x, R)) > ce^{\delta R},$$  \hspace{1cm} (4.1)$$

where $B(x, R)$ denotes the ball of center $x$ and radius $R$ in $\tilde{M}$. It is an easy exercise to show that, up to changing the constants $c, \delta$, the definition is independent on the
point \( x \in \tilde{M} \) as well as on the metric one pulls back from \( M \), so that this is indeed a topological property of \( M \) which in fact only depends on its fundamental group.

The proof by Margulis [Mar] is direct and independent of any deep result in foliation theory (even if the foliations are used crucially). Later, Plante and Thurston [PT] gave a more conceptual proof that works for general codimension one Anosov flows\(^2\) and uses some deeper results in foliation theory. The proof we shall present here has ingredients from both organised in a way that will lead us naturally to the generalisation of these arguments to the classification problem of partially hyperbolic diffeomorphisms in dimension 3.

We emphasize the following fact: up to double cover, every closed 3-manifold has trivial tangent bundle. That is, \( TM \cong M \times \mathbb{R}^3 \), therefore the existence of the splitting cannot be an obstruction by itself. It will be finer properties of the foliations, namely, the non-existence of compact leafs, that will come handy for this issue.

**Remark 4.2.** At the time that Margulis proved this theorem, the only known examples of Anosov flows in closed 3-manifolds where the geodesic flows in negative curvature (and its finite lifts), and the suspension flows of linear hyperbolic automorphisms of tori. Later, new examples started to appear, specially in dimension 3 (see [Bar]).

**Remark 4.3.** Theorem 4.1 implies the following result which also admits a more elementary proof just using Lefschetz index. If \( f : T^2 \to T^2 \) is an Anosov diffeomorphism\(^3\) of a two-torus, then, the action of \( f \) in homology is hyperbolic, meaning that it has no eigenvalue of modulus 1. An interesting challenge could be to prove this statement after (or before!) reading the proof below.

### 5. The proof

We provide here a quick proof of Theorem 4.1 based on the original arguments, but probably with a more modern viewpoint. The goal is motivating tools that allow understanding the interaction between topology and dynamics.

An easy consequence of Theorem 4.1 is the non-existence of Anosov flows in the sphere \( S^3 \). This can also be shown quite directly by a shortcut in the same argument: Assume that \( \phi_t : S^3 \to S^3 \) is an Anosov flow. Consider \( \mathcal{F}^{ws} \) the weak stable foliation of \( \phi_t \) given by Theorem 3.1 by Novikov’s compact leaf theorem we know that every foliation by surfaces in \( S^3 \) must have a compact leaf, this contradicts Proposition 3.2.

Novikov’s compact leaf theorem can be stated in a more general way as follows. Recall that a foliation by surfaces of a 3-manifold \( M \) is a partition of \( M \) by injectively immersed \( C^1 \)-surfaces (called leaves) that locally look like horizontal planes \( \mathbb{R}^2 \times \{ t \} \) sitting inside \( \mathbb{R}^3 \) (i.e. there are charts sending leaves to horizontal planes, see figure 3). An example of a foliation by surfaces would be the weak stable foliation of an Anosov flow in a 3-manifold.

A transversal to a foliation is an embedded circle which is everywhere transverse to the leafs of \( \mathcal{F} \), notice that if \( M \) is compact there are always transversals since a transverse curve intersecting a foliation box twice can be closed into a transversal. See figure 4.

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\(^2\)i.e. those whose stable or unstable bundle is one-dimensional

\(^3\)An Anosov diffeomorphism \( g : M \to M \) is such that \( Dg \) preserves a splitting \( TM = E^s \oplus E^u \) so that vectors in \( E^s \) are uniformly contracted and vectors in \( E^u \) are uniformly expanded as in § 5.
**Theorem 5.1.** Let $\mathcal{F}$ be a foliation by surfaces in a closed 3-manifold. Assume that there is a transversal $\gamma$ to $\mathcal{F}$ which is homotopically trivial. Then $\mathcal{F}$ has a compact leaf.

We will not prove this beautiful result which has several expositions. In fact, Novikov’s result is much stronger and implies the existence of what are known as *Reeb components*. One should think that in 3-manifolds compact leafs (or Reeb components) of foliations play the role that singularities play in vector fields in surfaces, and therefore Novikov’s theorem acts as the Poincare-Bendixon’s theorem in this setting.

With this elements in hand, we are ready to give the proof. The reader not familiar with the basics of algebraic topology can use the 3-torus $T^3 = S^1 \times S^1 \times S^1 = \mathbb{R}^3 / \mathbb{Z}^3$ as a model for this proof. The theorem implies that $T^3$ does not admit Anosov flows. Here, one will have that $\tilde{T}^3 = \mathbb{R}^3$ with the Euclidean metric (so balls do not have exponential growth of volume by a direct computation).

**Proof of Theorem 4.1.** Let $\pi : \tilde{M} \to M$ be the universal cover and lift $\phi_t$ to a flow $\tilde{\phi}_t : \tilde{M} \to \tilde{M}$. Let $\mathcal{F}^{w_s}$ the lift of the weak stable foliation to $\tilde{M}$.

Consider an arc $J$ tangent to the bundle $\tilde{E}^u$ (the lift of $E^u$). The arc $J$ is transverse to $\mathcal{F}^{w_s}$. Since the foliation is invariant under $\tilde{\phi}_t$ and the arc $J$ maps to another arc tangent to $\tilde{E}^u$ we deduce that the arc $\tilde{\phi}_t(J)$ cannot intersect the same foliation box twice, since that would allow to construct a transversal to $\mathcal{F}^{w_s}$ which is homotopically trivial, contradicting Theorem 5.1 and Proposition 3.2. See figure 4.

Foliation boxes have uniform size since they can be pulled back from $M$ which is compact. One deduces that there exists a uniform constant $c_0 > 0$ so that:

$$\text{vol}(B(\tilde{\phi}_t(J), 1)) > c_0 \text{length}(\tilde{\phi}_t(J)).$$

where $B(X, r)$ denotes the set of points in $\tilde{M}$ at distance less than $r$ from $X$. Moreover, since $J$ is tangent to $E^u$ there are positive constants $c_1, \delta > 0$ so that $\text{length}(\tilde{\phi}_t(J)) > c_1 e^{\delta t}$. Putting this together, one gets:

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4Even if much deeper, there is a part of the proof of Novikov’s theorem (which is indeed enough to rule out homotopically trivial transversal loops for Anosov flows) that is very much modelled in the proof of Poincare-Bendixon’s theorem. It is known as *Haefliger’s argument*: using the transverse loop, one constructs a disk whose boundary is transverse to the foliation and which is in general position; studying the induced flow on the disk is enough to find a configuration which is not compatible with Anosov flows, and such that with enough work produces a Reeb component. We note that for the partially hyperbolic case to be treated later, the full version of Novikov’s theorem is important.
Figure 4. If a positively transverse curve intersects a leaf twice one can construct a closed transversal.

\[ \text{vol}(B(\tilde{\phi}_t(J), 1)) > c_0 c_1 e^{\hat{\delta}t}. \]

We will now show that there is a constant \( c_2 > 0 \) so that if \( x_0 \in J \), then \( \tilde{\phi}_t(J) \) is contained in \( B(x_0, R_t) \) where \( R_t \leq c_2 t + \text{diam}(J) \). This is obtained by computing, for \( x \in J \)

\[ d(x_0, \tilde{\phi}_t(x)) \leq d(x_0, x) + d(x, \tilde{\phi}_t(x)) \leq \text{diam}(J) + c_2 t, \tag{5.1} \]

where \( c_2 \) is a bound of the norm for the vector field generating \( \phi_t \).

This implies that \( B(\tilde{\phi}_t(J), 1) \subset B(x_0, R_t + 1) \) and therefore, taking \( c_3 = e^{-\frac{\text{diam}(J)+1}{c_2}} \) and \( \hat{\delta} = \frac{\hat{\delta}}{c_2} \) we get

\[ \text{vol}(B(x_0, R_t + 1)) > c_0 c_1 c_3 e^{\hat{\delta}(R_t+1)} \]

which gives (4.1) and completes the proof. \( \Box \)

Margulis proof is more elementary since it does not use any deep result about foliations, however, it depends crucially on the fact that the weak stable/unstable foliation is complete in the sense that a weak stable/unstable leaf is the union of the strong stable/unstable manifolds through points of a given orbit. This fact fails when one goes to the partially hyperbolic setting. This property is used by Margulis to construct by hand the universal cover of \( M \) and compute its volume growth.

The proof of Plante and Thurston is much more similar to the one we present here, only that instead of computing volume they construct many loops that they show not to be pairwise homotopic. For this, they use Haefliger’s argument (cf. footnote 4). In particular, as the proof presented here and in contrast with Margulis proof it only needs one of the two foliations and that is why it extends to codimension one Anosov flows. But the importance here is that this line of reasoning does not depend on understanding the internal structure of the codimension one foliation, and so is well suited to be extended in other contexts.
6. Classification of partially hyperbolic systems

We will now come to the problem of understanding the structure of general partially hyperbolic systems in 3-dimensional manifolds by modelling the questions and ideas in the work done in the previous section.

Let us first define partially hyperbolic diffeomorphisms. We say that a diffeomorphism \( f : M \to M \) is partially hyperbolic if the tangent space \( TM \) splits as a direct sum of non-trivial continuous subbundles \( E^s \oplus E^c \oplus E^u = TM \) which are \( Df \)-invariant and verify that there is some \( \ell > 1 \) so that for every \( x \in M \), if \( v^\sigma \in E^\sigma(x) \) (\( \sigma = s, c, u \)) are unit vectors, then:

\[
\|Df^\ell v^s\| < \min\{1, \|Df^\ell v^c\|\} \quad \text{and} \quad \|Df^\ell v^u\| > \max\{1, \|Df^\ell v^c\|\}.
\]

Examples include many homogeneous dynamics (all of those with positive entropy) such as linear automorphisms of tori given by matrices \( A \in \text{SL}(d, \mathbb{Z}) \) with some eigenvalue of modulus different from one, or certain translations in compact quotients of simple Lie groups (for instance, the time one map of the geodesic flow in constant negative curvature surfaces). But also other geometric constructions belong to this class (e.g. time one maps of frame flows in negatively curved manifolds), as well as time one maps of Anosov flows defined before. Partially hyperbolic diffeomorphisms form a \( C^1 \)-open class of diffeomorphisms and some robust dynamical behaviour typically forces some sort of partial hyperbolicity. We refer the reader to the surveys provided in §7 for discussions on these aspects.

Here we shall concentrate on the following questions of current research interest which can be considered as continuations of the problem discussed above for Anosov flows:

**Question 6.1.** Which 3-manifolds admit partially hyperbolic diffeomorphisms? Which isotopy classes? Do these resemble in some way to the known examples?

6.1. Basic properties. We refer the reader to [CP] for a general expositions on the basic facts about partially hyperbolic systems as well as a long list of examples. Here we will concentrate in a few relevant aspects specific to 3-dimensions.

A main difference which makes studying partially hyperbolic diffeomorphisms much harder than Anosov flows is that eventhough the strong bundles \( E^s \) and \( E^u \) integrate uniquely into \( f \)-invariant foliations (essentially by the same argument as in the Anosov flow case), this is no longer true for the center stable and center unstable bundles \( E^{cs} = E^s \oplus E^c \) nor \( E^{cu} = E^c \oplus E^u \). This makes the study of partially hyperbolic diffeomorphisms much harder, though at the beginning of its exploration, the topological study of these systems assumed the existence of such foliations under the concept of dynamical coherence. We say that a partially hyperbolic diffeomorphism is dynamically coherent if there are \( f \)-invariant foliations tangent respectively to \( E^{cs} \) and \( E^{cu} \).

A recent breakthrough result by Burago and Ivanov [BI] provided a tool for avoiding such an undesirable hypothesis.

**Theorem 6.2.** Up to finite cover, there is a Reebless foliation \( \mathcal{F} \) transverse to the unstable direction \( E^u \).

This implies by iterating backwards that one can choose the foliation to be as close to tangent to \( E^{cs} \) as desired, but does not imply dynamical coherence, as in the limit
the leaves could merge together forming what is called a *branching foliation* which is an incredibly useful tool for the study of partially hyperbolic diffeomorphisms but that we will avoid to discuss here. We note here that the proof of Theorem 6.2 depends very strongly on the fact that $E^c$ and $E^u$ are one dimensional; indeed, one can not expect a similar result if $E^c$ has higher dimensions.

In fact, to show the result it is enough to show that there exists such a foliation since the non-existence of Reeb components follows from the fact that there are no closed curves tangent to $E^u$, which is a beautiful observation from [BI] which allows to treat the problem locally and obtain this global information. This just follows from the fact that a flow transverse to a Reeb component must have a closed orbit. This observation together with their amazing result allowed them to obtain the first obstruction for the existence of partially hyperbolic diffeomorphisms in a direct way:

**Corollary 6.3** (Burago-Ivanov). *The sphere $S^3$ does not admit partially hyperbolic diffeomorphisms.*

6.2. **Exponential growth.** The proof of Theorem 4.1 in §5 has as a moral that to expand a one-dimensional foliation transverse to a two-dimensional foliation in a 3-manifold one needs space. This moral quite extends to the diffeomorphism case, only that diffeomorphisms can wrap the manifold onto themselves and then obtain expansion without much space.

For instance, a matrix in $\text{SL}(3, \mathbb{Z})$ with real eigenvalues and at least one larger than 1 induces a partially hyperbolic diffeomorphism on $T^3$. The volume growth of the universal cover $\mathbb{R}^3$ of $T^3$ is just polynomial. The reason is that the action of $f$ itself already gives the foliation space to expand. In the proof of Theorem 4.1 this appears in the crucial use of the fact that $\phi_t$ is a flow (or equivalently, that its time one map is homotopic to the identity) which gives equation (5.1).

With essentially the same proof as for Theorem 4.1 by replacing the stable manifold theorem with Theorem 6.2 one can obtain the following result which provides obstructions for the mapping classes which admit partially hyperbolic diffeomorphisms:

**Theorem 6.4.** If $f : M \to M$ is a partially hyperbolic diffeomorphism of a closed 3-dimensional manifold and $\hat{M}_f$ is the mapping torus of $f$, then the fundamental group $\pi_1(\hat{M}_f)$ of $\hat{M}_f$ has exponential growth.

Recall that the *mapping torus* of a map $F : X \to X$ is the space $X \times [0, 1]/\sim$ where one identifies $(x, 1) \sim (F(x), 0)$ for all $x$. It depends only on the homotopy class of the map $F$, and produces a smooth manifold if $X$ is a manifold and $F$ a diffeomorphism (so that the equivalence with volume growth still holds).

But Theorem 6.2 is indeed stronger, since it can also provide further obstructions thanks to the well developed theory of Reebless foliations. There are manifolds with exponential growth of fundamental group known not to admit foliations without compact leaves, these provide also obstructions to the existence of partially hyperbolic diffeomorphisms. Up to recently, these were more or less all the known obstructions to the existence of partially hyperbolic diffeomorphisms. At the moment of this writing, we do not know any manifold with exponential growth of fundamental group which admits a partially hyperbolic diffeomorphism but does not admit an Anosov flow. But lots of developments have been made recently that give hope that the understanding of partially hyperbolic diffeomorphisms is not far from the understanding of Anosov flows.
6.3. **Examples.** As mentioned, the obstruction given by Theorem 6.4 is not sharp, so it makes sense to see to what extent one can characterise the homotopy classes of diffeomorphisms of 3-manifolds admitting partially hyperbolic diffeomorphisms. It turns out that only very recently examples in new isotopy classes where found [BGHP]. In these examples, new features of partially hyperbolic systems were exposed, in particular, the global nature of dynamical coherence is now better understood.

But somehow, all examples we know build in some way or the other on some Anosov system. The examples in [BGHP] are constructed by using the cone-field criterium to guarantee partial hyperbolicity together with a careful understanding of the global structure of the invariant bundles. This way, it is possible to construct diffeomorphisms of the manifold which respect transversalities between the bundles, and this allows to create new partially hyperbolic diffeomorphisms in new isotopy classes. These kind of constructions are still in their infancy, and it is likely that new examples can be created using this ideas. Nonetheless, there are some manifolds and isotopy classes of diffeomorphisms where the partially hyperbolic dynamics seem amenable to classification, notably hyperbolic and Seifert 3-manifolds [BFFP, BFP].

A notion of collapsed Anosov flow has been proposed recently that may account for all new examples, and which needs to be tested against new potential constructions [BFP].

In higher dimensions, Anosov systems are far from being classified, and new ways to construct partially hyperbolic examples have been devised [CHO], which depend to some extent on Anosov systems, but seem likely to be more flexible and maybe combinable with the techniques in [BGHP]. Even the most basic questions in high dimensions remain quite open.

7. **Further reading**

For more information on the theory of foliations we recommend [Ca, Chapter 4] for an intuitive and geometric introduction. The topological structure of Anosov flows have received a lot of attention and development since the pioneering work of Margulis and Plante-Thurston, we refer the reader to [Bar] for a survey of the main results in dimension 3 together with several of the key ideas.

Partially hyperbolic diffeomorphisms appeared not only as a generalisation of time one maps of Anosov flows and have played a prominent role in smooth dynamics in the recent years. We refer the reader to [BDV] for a general overview of smooth dynamics and to [Wil] for a recent account on partial hyperbolicity. In [CHHU] the reader can find a survey on the dynamics of partially hyperbolic diffeomorphisms specialized to dimension 3 which also touches upon the classification problem.

If the reader wishes to know more about the classification problem of partially hyperbolic diffeomorphisms in dimension 3, then the following references could be a useful introduction [CHHU, HP, Pot, BFFP, BFP].

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