SCHOENBERG’S PROBLEM ON POSITIVE DEFINITE FUNCTIONS

ALEXANDER KOLDOSKY

UNIVERSITY OF MISSOURI–COLUMBIA
DEPARTMENT OF MATHEMATICS
COLUMBIA, MO 65211

1. Introduction

In 1938, I. J. Schoenberg [28] posed the following problem: For which numbers \( \beta > 0 \) is the function \( \exp(-\|x\|_q^\beta) \) positive definite on \( \mathbb{R}^\times \)? Here \( q > 2 \) and \( \|x\|_q = (|x_1|^q + \cdots + |x_n|^q)^{1/q} \). Denote by \( B_n(q) \) the set of such numbers \( \beta \).

We prove in this article that the functions \( \exp(-\|x\|_q^\beta) \) are not positive definite for all \( n \geq 3 \), \( q > 2 \) and \( \beta > 0 \), i.e. \( B_n(q) = \emptyset \) for every \( q > 2 \) and \( n \geq 3 \).

Besides, \( B_2(q) = (0, 1] \) for every \( q > 2 \). In the case \( n = 2 \) we have to prove only that \( B_2(q) \cap (1, \infty) = \emptyset \). In fact, it is well–known that \( \exp(-\|x\|^\beta) \) is a positive definite function for every two–dimensional norm and every \( \beta \in (0, 1] \), see Ferguson [9], Hertz [12], Lindenstrauss and Tzafriri [20], Dor [1], Misiewicz and Ryll–Nardziewski [23], Yost [31], Koldobsky [14] for different proofs.

Thus we give a complete answer to Schoenberg’s question.
The case \( q \in (0, 2] \) was settled by Schoenberg [28]. Here one has \( B_n(q) = (0, q] \) for every \( n \geq 2 \).

Let us mention some facts which are valid for all \( q > 0 \), see Schoenberg [28]:

(i) \( B_1(q) = (0, 2] \);
(ii) \( B_n(q) \supset B_{n+1}(q) \) for every \( n \in \mathbb{N} \);
(iii) if \( \beta \in B_n(q) \) then \( (0, \beta] \subset B_n(q) \).

Thanks to (ii) and (iii) it is enough to prove that \( B_3(q) \cap (0, 2) = \emptyset \) to verify that \( B_n(q) = \emptyset \) for every \( n \geq 3 \).

However, we treat here a more general situation. Denote by \( \phi_n(q) \) the class of all even functions \( f : \mathbb{R} \to \mathbb{R} \) which are such that \( f(\|x\|_q) \) is a characteristic function, i.e. there exists a probability measure \( \mu \) on \( \mathbb{R}^\times \) with \( \hat{\mu}(x) = f(\|x\|_q) \) for all \( x \in \mathbb{R}^\times \).

It is clear that \( \phi_n(q) \supset \phi_{n+1}(q) \). If \( f \in \phi_n(q) \) then by Bochner’s theorem the function \( f(\|x\|_q) \) is positive definite, as well as \( f \) itself. Hence there exists a probability measure \( \nu \) on \( \mathbb{R} \) with \( \hat{\nu} = f \).

We prove the following

**Theorem.** Let \( q > 2, n \in \mathbb{N}, f \in \phi_n(q), f \not\equiv 1 \) and \( \hat{\nu} = f \). Then in every one of the cases:

\( a) \) \( n \geq 3, \beta \in (0, 2) \);
\( b) \) \( n = 2, \beta \in (1, 2) \);
\( c) \) \( n \geq 4, \beta \in (-1, 0) \)

the \( \beta \)-th moment of the measure \( \nu \) is infinite, i.e. \( \int_{\mathbb{R}} |t|^\beta d\nu(t) = \infty \).

Using a) one can immediately get an answer to Schoenberg’s question for \( n \geq 3 \). Indeed, for every \( \beta \in (0, 2) \), \( \exp(-|t|^\beta) \) is the characteristic function of the \( \beta \)-stable measure on \( \mathbb{R} \). This measure has finite moments of all positive orders less than \( \beta \), see Zolotarev [30] or the text
preceding Lemma 4 below. By a) the function \(\exp(-|t|^\beta) \not\in \phi_3(q)\) for every \(q > 2\). By Bochner’s theorem the function \(\exp(-\|x\|^\beta_q)\) is not positive definite on \(\mathbb{R}^d\). Thus \(B_3(q) \cap (0, 2) = \emptyset\) for every \(q > 2\) and we are done.

For the same reason, it follows from b) that \(B_2(q) \cap (1, 2) = \emptyset\) for every \(q > 2\).

The statements a) and c) show that the measure \(\nu\) corresponding to a function \(f \in \phi_n(q), n \geq 4, q > 2\) must have a very special behavior at infinity and at zero. Nevertheless, the following question is open: For \(n \geq 3\) and \(q > 2\), are there any functions in the class \(\phi_n(q)\) besides the function \(f \equiv 1\)?

The classes \(\phi_n(q)\) have been investigated by several authors. Schoenberg [29] described the classes \(\phi_n(2)\) and \(\phi_\infty(2) = \bigcap_n \phi_n(2)\) completely. A similar result was obtained for the classes \(\phi_n(1)\) by Cambanis et al [2]. Bretagnolle et al [4] described the classes \(\phi_\infty(q)\) for all \(q > 0\). In particular, for very \(q > 2\) the class \(\phi_\infty(q)\) contains no functions besides \(f \equiv 1\). For some partial results on the classes \(\phi_n(q), 0 < q < 2\), see Richards [26, 27] and Misiewicz [21]. Misiewicz [22] proved that for \(n \geq 3\) a function \(f(\max(|x_1|, \ldots, |x_n|))\) is positive definite only if \(f \equiv 1\). This result gives an answer to Problem 2 from Schoenberg [28].

References related to the topic include also Askey [3], Berg and Ressel [8], Kuelbs [16], Eaton [8], Christensen and Ressel [4], Kuritsyn [17], Kuritsyn and Shestakov [18], Misiewicz and Scheffer [24], Aharoni et al [1].

The well known theorem of Bretagnolle et al [1] states that a Banach space \((E, \| \cdot \|)\) is isometric to a subspace of \(L_\beta([0, 1])\), \(\beta \in [1, 2]\) iff the function \(\exp(-\|x\|_q^\beta)\) is positive definite. Dor [7] pointed out all the pairs of numbers \(\beta, q \in [1, \infty)\) for which the space \(l^n_q\) for every \(q > 2\).
is isometric to a subspace of $L_\beta([0,1])$. Combining these results one can prove that $B_n(q) \cap (1,\infty) = \emptyset$ for every $n \geq 2$ and $q > 2$. So Schoenberg’s problem was open only for $\beta \in (0,1)$.

In spite of the connection between Schoenberg’s problem and isometries, in this paper we don’t deal with isometries directly. Our main tool is the Fourier transform of distributions.

2. Some Applications of the Fourier Transform

Let $(E,\|\cdot\|)$ be an $n$–dimensional Banach space and $f : \mathbb{R} \to \mathbb{R}$ be an even function such that there exists a probability measure $\mu$ on $\mathbb{R}^\times$ with $\hat{\mu}(x) = f(\|x\|)$ for all $x \in \mathbb{R}^\times$. Measures with characteristic functions of the form $f(\|x\|)$ are called $E$–stable or pseudo–isotropic, since all one–dimensional projectional of such measures are equal up to a scale parameter. We give an easy proof of this fact.

**Lemma 1** (see Levy [19], Eaton [8]). If $\mu$ is a probability measure on $\mathbb{R}^\times$ with the characteristic function $f(\|x\|)$ then:

a) there exists a probability measure $\nu$ on $\mathbb{R}$ with $\hat{\nu} = f$;

b) for every $x \in \mathbb{R}^\times$, $x \neq 0$, the image of the measure $\mu$ under the mapping $\xi \to \langle x,\xi \rangle/\|x\|$ from $\mathbb{R}^\times$ to $\mathbb{R}$ coincides with the measure $\nu$ (here $\langle x,\xi \rangle$ stands for the scalar product).

**Proof.** Fix an element $x \in \mathbb{R}^\times$, $x \neq 0$. Let $\nu$ be the image of the measure $\mu$ under the mapping $\xi \to \langle x,\xi \rangle/\|x\|$. Put $y = \langle x,\xi \rangle/\|x\|$.

Then for every $k \in \mathbb{R}$

$$f(k\|x\|) = \hat{\mu}(kx) = \int_{\mathbb{R}^\times} \exp(-i\langle kx,\xi \rangle) d\mu(\xi)$$

$$= \int_{\mathbb{R}^\times} \exp(-ik\|x\|\langle x,\xi \rangle/\|x\|) d\mu(\xi) = \int_{\mathbb{R}} \exp(-ik\|x\|y) d\nu(y) = \hat{\nu}(k\|x\|).$$

Hence $\hat{\nu} = f$ and $\nu$ doesn’t depend on the choice of $x \in \mathbb{R}^\times$, $x \neq 0$.
Let $S(\mathbb{R}^\kappa)$ be, as usual, the space of rapidly decreasing infinitely differentiable functions. Denote by $S'(\mathbb{R}^\kappa)$ the space of distribution over $S(\mathbb{R}^\kappa)$.

Let $\Omega$ be an open subset of $\mathbb{R}^\kappa$. A distribution $g \in S'(\mathbb{R}^\kappa)$ is called positive (negative) on $\Omega$ if $\langle g, \psi \rangle \geq 0$ ($\langle g, \psi \rangle \leq 0$) for every non-negative function $\psi \in S(\mathbb{R}^\kappa)$ with $\text{supp} \psi \subset \Omega$.

**Lemma 2** (cf. Koldobsky [13]). Let $\beta \in (-1, \infty)$, $\beta \neq 0, 2, 4, \ldots$. Let $\psi \in S(\mathbb{R}^\kappa)$ be a function with $0 \not\in \text{supp} \psi$. Then for every $\xi \in \mathbb{R}^\kappa, \xi \neq 0$,

$$
\int_{\mathbb{R}^\kappa} |\langle x, \xi \rangle|^{\beta} \hat{\psi}(x) \, dx = c_\beta \int_{\mathbb{R}} |t|^{-\beta} \psi(t\xi) \, dt, \quad c_\beta = \frac{2^{\beta+1} \pi^{1/2} \Gamma((\beta+1)/2)}{\Gamma(-\beta/2)}.
$$

**Proof.** It is well-known that $(|x|^\beta)^\wedge(t) = c_\beta |t|^{-\beta}, t \neq 0$, for all $\beta \in (-1, \infty), \beta \neq 0, 2, 4, \ldots$, see Gelfand and Shilov [11]. By the Fubini theorem

$$
\int_{\mathbb{R}^\kappa} |\langle x, \xi \rangle|^{\beta} \hat{\psi}(x) \, dx = \int_{\mathbb{R}} |z|^\beta \left( \int_{\langle x, \xi \rangle = z} \hat{\psi}(x) \, dx \right) \, dz = \left\langle |z|^\beta, \int_{\langle x, \xi \rangle = z} \hat{\psi}(x) \, dx \right\rangle.
$$

The function $t \to 2\pi \psi(-t\xi)$ is the Fourier transform of the function $z \to \int_{\langle x, \xi \rangle = z} \hat{\psi}(x) \, dx$ (it is a simple property of the Radon transform, see Gelfand et al [11]). Therefore, we can continue the equality (2.1):

$$
= c_\beta \langle |t|^{-\beta}, \psi(t\xi) \rangle = c_\beta \int_{\mathbb{R}} |t|^{-\beta} \psi(t\xi) \, dt. \quad \Box
$$

**Lemma 3.** Let $f, \mu$ and $\nu$ be as in Lemma [4]. Additionally, assume that $f \neq 1$ and $\int_{\mathbb{R}} |t|^\beta \, d\nu(t) < \infty$ for a number $\beta \in (-1, 2)$, $\beta \neq 0$. Then, if $\beta \in (-1, 0)$ the distribution $\|(x|\beta)^{\wedge}$ is positive on $\mathbb{R}^\kappa \setminus \{\nu\}$. If $\beta \in (0, 2)$ the distribution $\|(x|\beta)^{\wedge}$ is negative on $\mathbb{R}^\kappa \setminus \{\nu\}$.
Proof. Since \( f \neq 1 \), the measure \( \nu \) is not supported at zero and \( \int_\mathbb{R} |t|^\beta \, d\nu(t) > 0 \). By Lemma 1 we have

\[
\int_{\mathbb{R}^\times} |\langle x, \xi \rangle|^\beta \, d\mu(\xi) = \|x\|^\beta \int_{\mathbb{R}^\times} \left| \frac{\langle x, \xi \rangle}{\|x\|} \right|^\beta \, d\mu(\xi) = \|x\|^\beta \int_{\mathbb{R}} |t|^\beta \, d\nu(t)
\]

for every \( x \in \mathbb{R}^\times \). It follows from Lemma 2 and the Fubini theorem that

\[
\langle \|x\|^\beta, \hat{\psi} \rangle = \langle \|x\|^\beta, \psi \rangle = \frac{1}{\int_{\mathbb{R}} |t|^\beta \, d\nu(t)} \int_{\mathbb{R}^\times} d\mu(\xi) \int_{\mathbb{R}^\times} |\langle x, \xi \rangle|^\beta \hat{\psi}(x) \, dx
\]

\[
= \frac{c_\beta}{\int_{\mathbb{R}} |t|^\beta \, d\nu(t)} \int_{\mathbb{R}} d\mu(\xi) \int_{\mathbb{R}} |t|^{-1-\beta} \psi(t\xi) \, dt
\]

for every function \( \psi \in S(\mathbb{R}^\times) \) with \( 0 \notin \text{supp} \psi \). If the function \( \psi \) is non-negative, the quantity in the right-hand side has the same sign as the number \( c_\beta \). It suffices to remark that \( c_\beta > 0 \) if \( \beta \in (-1,0) \) and \( c_\beta < 0 \) if \( \beta \in (0,2) \). \( \square \)

For arbitrary \( q > 0 \), define the function \( \gamma_q \) on \( \mathbb{R} \) by \( \gamma_q(t) = (\exp(-|x|^q))^\wedge(t) \), \( t \in \mathbb{R} \). Then

\[
\lim_{t \to \infty} t^{1+q} \gamma_q(t) = 2\Gamma(q+1) \sin(q\pi/2),
\]

see Polya and Szego [25], Part 3, Problem 154. Therefore, the integral

\[
S_q(\alpha) = \int_{\mathbb{R}} |t|^\alpha \gamma_q(t) \, dt
\]

converges absolutely for every \( \alpha \in (-1,q) \).

**Lemma 4** (for the case \( q \in (0,2) \), see Zolotarev [30] or Koldobsky [13]).

*Let

\( q > 2 \). Then for every \( \alpha \in (-1,q), \alpha \neq 0,2,\ldots,2[q/2] \),

\[
S_q(\alpha) = 2^{\alpha+2}\pi^{1/2}(-\alpha/q)\Gamma((\alpha+1)/2)/(q\Gamma(-\alpha/2)).
\]

In particular, \( S_q(\alpha) > 0 \) if \( \alpha \in (-1,2), \alpha \neq 0 \), and \( S_q(\alpha) < 0 \) if \( \alpha \in (2,\min(4,q)) \).
Proof. Assume $-1 < \alpha < 0$. By the Parseval theorem
\[
S_q(\alpha) = \int_{\mathbb{R}} |t|^\alpha \gamma_q(t) \, dt = c_\alpha \int_{\mathbb{R}} |z|^{-1-\alpha} \exp(-|z|^q) \, dz
= \frac{2^{\alpha+1} \pi^{1/2} \Gamma((\alpha+1)/2)}{\Gamma(-\alpha/2)} \cdot \frac{2\Gamma(-\alpha/q)}{q}.
\] (2.2)

If we allow $\alpha$ to assume complex values then all functions of $\alpha$ in (2.2) are analytic in the domain \((-1 < \Re \alpha < q, \alpha \neq 0, 2, \ldots, 2[q/2])\). Since analytic continuation from the interval \((-1, 0)\) is unique, the equality (2.2) remains valid for all $\alpha \in (-1, q)$, $\alpha \neq 0, 2, \ldots, 2[q/2]$. To complete the proof, note that $\Gamma(z) > 0$ if $z > 0$ or $z \in (-2, -1)$, and $\Gamma(z) < 0$ if $z \in (-1, 0)$.

Let us compute the Fourier transform of the function $\|x\|_q^\beta$.

Lemma 5. Let $q > 0$, $n \in \mathbb{N}$, $-n < \beta < qn$, $\beta/q \notin \mathbb{N} \cup \{2\}$, $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$, $\xi_k \neq 0$, $1 \leq k \leq n$. Then
\[
((|x_1|^q + \cdots + |x_n|^q)^{\beta/q})^\wedge(\xi) = \frac{q}{\Gamma(-\beta/q)} \int_0^\infty t^{n+\beta-1} \prod_{k=1}^n \gamma_q(t \xi_k) \, dt
\] (2.3)

Proof. Assume $-1 < \beta < 0$. By definition of the $\Gamma$–function
\[
(|x|^q + \cdots + |x_n|^q)^{\beta/q} = \frac{q}{\Gamma(-\beta/q)} \int_0^\infty y^{-1-\beta} \exp(-y^q(|x_1|^q + \cdots + |x_n|^q)) \, dy.
\]

For every fixed $y > 0$,
\[
(\exp(-y^q(|x_1|^q + \cdots + |x_n|^q))^\wedge(\xi) = y^{-n} \prod_{k=1}^n \gamma_q(t \xi_k).
\]

Put $t = 1/y$. We have
\[
((|x_1|^q + \cdots + |x_n|^q)^{\beta/q})^\wedge(\xi) = \frac{q}{\Gamma(-\beta/q)} \int_0^\infty y^{-n-\beta-1} \prod_{k=1}^n \gamma_q(\xi_k/y) \, dy
= \frac{q}{\Gamma(-\beta/q)} \int_0^\infty t^{1+\beta-1} \prod_{k=1}^n \gamma_q(t \xi_k) \, dt.
\]
The latter integral converges if \(-n < \beta < qn\) because the function \(t \to \prod_{k=1}^{n} \gamma_q(t \xi_k)\) decreases at infinity like \(t^{-n-q}\) (remind that \(\xi_k \neq 0, 1 \leq k \leq n\); see the text preceding Lemma 4).

If \(\beta\) is allowed to assume complex values then the both sides of (2.3) are analytic functions of \(\beta\) in the domain \(\{-n < \Re \beta < nq, \beta/q \notin \mathbb{N} \cup \{\mathcal{R}\}\}\). These two functions admit unique analytic continuation from the interval \((-1, 0)\). Thus the equality (2.3) remains valid for all \(\beta \in (-n, qn), \beta/q \notin \mathbb{N} \cup \{\mathcal{R}\}\) (see Gelfand and Shilov [10] for details of analytic continuation in such situations).

For \(q \in [1, 2]\), Lemma 5 was proved in Koldobsky [15].

3. Proof of Theorem

Let \(n \in \mathbb{N}, n \geq 2, -n < \beta < qn, q > 2\). Consider the integral

\[
J_n(\alpha_1, \ldots, \alpha_{n-1}) = \int_{\mathbb{R}^n-\mathcal{K}} |\xi_1|^{\alpha_1} \cdots |\xi_{n-1}|^{\alpha_{n-1}} \cdot \left( \int_0^{\infty} t^{n+\beta-1} \gamma_q(t) \prod_{k=1}^{n-1} \gamma_q(t \xi_k) \, dt \right) d\xi_1 \cdots d\xi_{n-1}
\]

\[
= \int_0^{\infty} t^{n+\beta-1} \gamma_q(t) \left( \prod_{k=1}^{n-1} |\xi_k|^{\alpha_k} \gamma_q(t \xi_k) \, d\xi_k \right) \, dt
\]

\[
= \int_0^{\infty} t^{-\alpha_1-\cdots-\alpha_{n-1}+\beta} \gamma_q(t) \, dt \cdot \prod_{k=1}^{n-1} \int_{\mathbb{R}} |z|^{\alpha_k} \gamma_q(z) \, dz
\]

\[
= S_q(\alpha_1) \cdot S_q(\alpha_2) \cdots \cdot S_q(-\alpha_1 - \alpha_2 - \cdots - \alpha_{n-1} + \beta).
\]

If all the numbers \(\alpha_1, \ldots, \alpha_{n-1}, -\alpha_1 - \cdots - \alpha_{n-1} + \beta\) belong to the interval \((-1, q)\) then the integrals in (3.1) converge absolutely. Therefore, the Fubini theorem is applicable and the integral \(J_n(\alpha_1, \ldots, \alpha_{n-1})\) converges.

Case a): \(n \geq 3, \beta \in (0, 2)\).
Note that $\phi_n(q) \supset \phi_{n+1}(q)$, so it is enough to prove the theorem for $n = 3$.

Let $n = 3$ and assume that there exists a function $f \in \phi_3(q)$, $f \not\equiv 1$, with $\int_R |t|^\beta d\nu(t) < \infty$, where $\hat{\nu} = f$ and $\beta \in (0, 2)$.

Put $\alpha_1 = \alpha_2 = -1 + \delta$ with $\delta \in (0, 1)$ and $(2 + \beta - \min(4, q)) < \delta < \beta/2$. Then $\alpha_1, \alpha_2 \in (-1, 0)$ and $-\alpha_1 - \alpha_2 + \beta \in (2, \min(4, q))$. By Lemma 4, $J_3(\alpha_1, \alpha_2) = S_q(\alpha_1) \cdot S_q(\alpha_2) \cdot S_q(-\alpha_1 - \alpha_2 + \beta) < 0$.

On the other hand, by Lemma 3 the function $\|x\|^{\beta}$ is negative on $\mathbb{R}^\times \setminus \{0\}$. Since $\Gamma(-\beta/q) < 0$, it follows from Lemma 5 that $h(\xi_1, \xi_2) = \int_0^\infty t^{\beta+2} \gamma_q(t\xi_1) \gamma_q(t\xi_2) \gamma_q(t) dt \geq 0$ for every $\xi_1, \xi_2 \neq 0$. Hence

$$J_3(\alpha_1, \alpha_2) = \int_{\mathbb{R}^\times} |\xi_1|^{\alpha_1} \cdot |\xi_2|^{\alpha_2} h(\xi_1, \xi_2) d\xi_1 d\xi_2 > 0$$

and we get a contradiction.

In the cases b) and c) proofs are similar. In the case b) we put $\alpha_1 = -1 + \delta$ with $\delta \in (0, 1)$ and $1 + \beta - \min(4, q) < \delta < \beta - 1$. In the case c) we may restrict ourselves to $n = 4$ and put $\alpha_1 = \alpha_2 = \alpha_3 = -1 + \delta$ with $\delta \in (0, 1)$ and $(3 + \beta - \min(4, q))/3 < \delta < (\beta + 1)/3$. Then the numbers $J_2(\alpha_1)$ and $J_4(\alpha_1, \alpha_2, \alpha_3)$ are negative. On the other hand, if there exists a function $f$ for which the measure $\nu$ has finite $\beta$-th moment, then Lemma 4 and Lemma 3 imply $J_2(\alpha_1) \geq 0$ and $J_4(\alpha_1, \alpha_2, \alpha_3) \geq 0$ and we get a contradiction (note that $\Gamma(-\beta/q) > 0$ if $\beta \in (-1, 0)$ and $\Gamma(-\beta/q) < 0$ if $\beta \in (1, 2)$.)

Acknowledgment . I am grateful to Jolanta Misiewicz for helpful discussions during my stay in Technical University of Wroclaw in 1989.
References

[1] Aharoni, I., Maurey, B., Mitiagin, B. (1985). Uniform embeddings of metric spaces and of Banach spaces into Hilbert spaces. Israel J. Math. 52, 251–265.

[2] Askey, R. (1973) Radial characteristic functions. Tech. Report. No. 1262, Math. Research Centre, University of Wisconsin–Madison.

[3] Berg C., Ressel P. (1978). Une forme abstraite du théorème de Schoenberg. Arch. Math. 30, 55–61.

[4] Bretagnolle, J., Dacunha–Castelle, D. and Krivine, J. L. (1966). Lois stables et espaces $L_p$. Ann. Inst. H. Poincare. Ser. B, 2, 231–259.

[5] Cambanis, S., Keener, R., and Simons, G. (1983). On $\alpha$–symmetric multivariate distributions. J. Multivariate Anal. 13, 213–233.

[6] Christensen, J. P. R. and Ressel, P. (1983). Norm dependent positive definite functions on $B$–spaces. Lect. Notes in Math. 990, 47–53.

[7] Dor, L. (1976). Potentials and isometric embeddings in $L_1$. Israel J. Math. 24, 260–268.

[8] Eaton, M. L. (1981). On the projections of isotropic distributions. Ann. Stat. 9, 1159–1164.

[9] Ferguson, T. S. (1962). A representation of the symmetric bivariate Cauchy distribution. Ann. Math. Stat. 33, 1256–1266.

[10] Gelfand, I. M. and Shilov, G. E. (1959). Generalized functions I, Fizmatgiz, Moscow (Russian).

[11] Gelfand, I. M. and Graev, M. I. and Vilenkin, N. Ya. (1966). Generalized functions V, Academic Press, New York.

[12] Hertz, C. S. (1963). A class of negative definite functions. Proc. Amer. Math. Soc. 14, 670–676.

[13] Koldobsky, A. L. (1991). Convolution equations in certain Banach spaces. Proc. Amer. Math. Soc. 111, 755–765.

[14] Koldobsky, A. L. (1992) Generalized Levy representation of norms and isometric embeddings into $L_p$–spaces, Ann. Inst. H. Poincare (Prob. and Stat). 28, #3.

[15] Koldobsky, A. L. (1991). The Fourier transform technique for convolution equations in the infinite dimensional $l_q$–spaces, Math. Ann. 291, 403–407.
[16] Kuelbs, J. D. (1973). Positive definite symmetric functions on linear spaces. J. Math. Anal. and Appl. 42, 413–426.
[17] Kuritsyn, Yu. G. (1989). Multidimensional versions and two Schoenberg’s problems. Stability Problems for Stochastic Models. Moscow, Inst. for Systems Studies, 72–79 (Russian).
[18] Kuritsyn, Yu. G. and Shestakov, A. V. (1984). On α–symmetric distributions. Theory Prob. and Appl. 29, 769–772.
[19] Levy, P. (1937). Theorie de l’addition de variable aleatoires, Paris.
[20] Lindenstrauss, J. and Tzafriri, L. (1973). Classical Banach spaces. Lect. Notes in Math. 338.
[21] Misiewicz, J. (1988). On norm dependent positive definite functions. Bull. Acad. Sci. Georgian SSR 130, 253–256 (Russian).
[22] Misiewicz, J. (1989). Positive definite functions on $l_\infty$. Statist. and Prob. Letters 8, 255–260.
[23] Misiewicz, J. and Ryll–Nardzewski, Cz. (1987). Norm dependent positive definite functions and measures on vector spaces. Lect. Notes in Math. 1391, 284–292.
[24] Misiewicz, J. and Scheffer, C. L. (1990). Pseudo isotropic measures. Nieuw Archief voor Wiskunde 8/2, 111–152.
[25] Polya, G. and Szego, G. (1964). Aufgaben und lehrsae ze aus der analysis, Springer–Verlag, Berlin–New York.
[26] Richards, D. St. P. (1986). Positive definite symmetric functions on finite dimensional spaces. I. J. Multivariate Anal. 19, 280–298.
[27] Richards, D. St. P. (1985). Positive definite symmetric functions on finite dimensional spaces. II. Statistics and Probab. Letters 3, North–Holland, 325–329.
[28] Schoenberg, I. J. (1938). Metric spaces and positive definite functions. Trans. Amer. Math. Soc. 44, 522–536.
[29] Schoenberg, I. J. (1938). Metric spaces and completely monotone functions. Ann. of Math. 39, 811–841.
[30] Zolotarev, V. M. (1986). One–dimensional stable distributions, Amer. Math. Soc. Providence.
[31] Yost, D. (1988). $L_1$ contains every two-dimensional normed spaces. Ann. Polonica Math. 49, 17–19.