The Navier-Stokes equations: 
on the existence of a weak solution enjoying the energy equality

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Abstract - Under the assumption of an initial datum divergence free and in $L^2$, we prove the existence of a weak solution to the Navier-Stokes initial boundary value problem enjoying the energy equality on $(0, t)$, almost everywhere in $t > 0$, in particular, for all $t \in [\theta, \infty)$, with $\theta := \theta(\|v_0\|_2)$. Also, the result allows us to refine some others.

Keywords: Navier-Stokes equations, weak solutions, energy equality.
AMS Subject Classifications: 35Q30, 35B65, 76D05.

1 Introduction

We consider the Navier-Stokes initial boundary value problem:

$$
v_t + v \cdot \nabla v + \nabla \pi = \Delta v, \quad \nabla \cdot v = 0, \quad \text{in } (0, T) \times \Omega, 
$$

$$
v = 0 \quad \text{on } (0, T) \times \partial \Omega, \quad v = v_0 \quad \text{on } \{0\} \times \Omega,
$$

where $\Omega \subseteq \mathbb{R}^3$ can be a bounded or an exterior domain, whose boundary $\partial \Omega$ for simplicity is assumed smooth, a half-space or the whole space as well. We set $w_t := \frac{\partial}{\partial t} w$ and $w \cdot \nabla w := (w \cdot \nabla) w$.

We investigate the existence of a weak solution $v$, in the sense of Leray-Hopf, to problem (1) enjoying the energy equality in the following form:

$$
\|v(t)\|_2^2 + 2 \int_0^t \|\nabla v(\tau)\|_2^2 d\tau = \|v_0\|_2^2, \quad t > 0.
$$

In 2D the result is true for all $t > 0$. Instead, this is an open problem in nD, $n \geq 3$. Actually, limiting ourselves to the three-dimensional case, a weak solution $a priori$ satisfies an energy inequality. The following one is in the strong form and is due to Leray (1934) [24]:

$$
\|v(t)\|_2^2 + 2 \int_s^t \|\nabla v(\tau)\|_2^2 d\tau \leq \|v(s)\|_2^2, \quad \text{for all } t > s \text{ and a.e. in } s > 0 \text{ and for } s = 0.
$$

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A priori there is no reason for the validity of a strong inequality in (2). To date, it appears as a consequence of the weakness of the weak solution v. In particular, it should be due to the presence of possible instants of singularity of v.

In Leary’s paper [24], the possible instants of singularity are heuristically interpreted as possible phenomenas of turbulence in a fluid motion, which are expected in the 3D model of the fluid.

We know, in part also thanks to inequality (2), that the set $S$ of the possible instants of singularity of a Leray weak solution has at least $H^a(S) = 0$, $H^a$ is the a-dimensional Hausdorff measure, [5, 34].

To date, with regard to the question, we detect two different trains of thoughts. One tries to justify the inequality by means of turbulence arguments. An other looks for sufficient conditions in order to obtain the energy equality.

Regarding the former, recently, the authors of [11, 12] support the following ideas. In the case of “less regularity”, the idea is that the presence of turbulences in fluid motions causes an extra dissipation, the one that balances the gap in the energy inequality (2). Actually, they consider as possible an analogy between the gap in (2) and the “anomalous dissipation” conjectured by Onsager\(^1\) (1949) in [32] for weak solutions to the Euler equations. In [11, 12], the dissipation $\varepsilon[u^\nu](t, x)$ proposed in the energy relation is defined, for all $\nu > 0$ (viscosity coefficient),

$$\varepsilon[u^\nu](t, x) := \nu |\nabla u^\nu(t, x)|^2 + D[u^\nu(t, x)],\tag{3}$$

with $D[u^\nu] \geq 0$. So that, in place of (2) the energy relation becomes

$$\|v^\nu(t)\|_2^2 + 2\nu \int_0^t \|\nabla v^\nu(\tau)\|_2^2 d\tau + 2 \int_0^t \|D[v^\nu]\|_1 d\tau = \|v_0\|_2^2, \text{ a.e. in } t > 0.$$

Regarding the latter train of thoughts, it is based on a wide literature originated\(^2\) by Prodi (1959) in [33], that required a weak solution $v$ to belong to $L^4(0, T; L^4(\Omega))$ (more properly an extra condition) in order to obtain the energy equality for $v$. In the setting of extra conditions (Prodi’s kind), we have recently found some new ones that concern the derivatives of the weak solution. The goal is to weaken “Prodi’s condition”. Actually, in [6] the authors assume $v \in L^3(0, T; D(A^{\frac{n}{2}}_{2}))$ and in [14] the authors assume $v \in L^3(0, T; D(A^{\frac{1}{n}}_{2}))$. In papers [1, 2] and in Theorem 1.3 Ch. I of [30] (PhD thesis), the authors assume $\nabla v \in L^r(0, T; L^s(\Omega))$ with $\frac{2n}{s} + \frac{n+2}{s} = n + 2$ ($n \geq 3$ is the Euclidean dimension of $\Omega$)\(^3\). However, the extra conditions, as a matter of course, force the initial

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\(^1\) The Onsager conjecture is not part of the goals of this note. We refer to the fundamental results obtained in [3, 7, 36].

\(^2\) Prodi’s result on the energy equality is in Lemma 2 of the quoted paper, which is devoted to the uniqueness of weak solutions enjoying an extra condition. The extra condition for the uniqueness is the same used by Serrin [38], that is $L^{\frac{2n}{n-3}}(0, T; L^p(\Omega)), p > 3$.

\(^3\) For $\beta \in (0, 1)$, $A_{\beta}$ is the fractional power of the Stokes operator $A_4 := -P_{\lambda} \Delta$. If $n = 3$, due to variability of the exponents $r, s$, the extra assumptions in the papers [1, 2, 30] are not in all comparable with the ones of [6, 14].
The Navier-Stokes equations: on the existence of a weak solution enjoying the energy equality

Datum to be suitably more regular than the minimal assumption \( v_0 \in J^2(\Omega) \). Therefore, these results can only concern the weak solutions corresponding to a subset of initial data in \( J^2(\Omega) \).

Conversely, recently, some papers have been devoted to study the compatibility between the initial datum only in \( L^2 \) and the validity of the energy equality of a weak solution, see \([8, 9, 10, 28]\).

In particular, the result found in \([28]\) suggests that there is no incompatibility\(^4\) between the validity of the energy equality and the assumption \( v_0 \in J^2(\Omega) \). Here, we go beyond a compatibility question, because we completely achieve the statement.

We detect some interesting factors in the validity of (4) of Theorem 1 (see Remarks 1-3), and, in order to better state the result, we commence by giving some notation and the definition of a weak solution to problem (1).

We denote by \( \mathcal{C}_0(\Omega) \) we mean the set of functions divergence free and belonging to \( C_\infty^0(\Omega) \). We indicate by \( J^2(\Omega) \) and by \( J^{1,2}(\Omega) \) the completion of \( C_0(\Omega) \) in \( L^2(\Omega) \) and in \( W^{1,2}(\Omega) \), respectively.

We denote by \( \mu_n, n \geq 1 \), the Lebesgue measure in \( \mathbb{R}^n \).

Definition 1. For all \( v_0 \in J^2(\Omega) \), a field \( v : (0, \infty) \times \Omega \to \mathbb{R}^3 \) is said a weak solution corresponding to datum \( v_0 \) if

1) for all \( T > 0 \), \( v \in L^\infty(0,T; J^2(\Omega)) \cap L^2(0,T; J^{1,2}(\Omega)) \),

2) for all \( T > 0 \) and \( t, s \in (0, T) \), the field \( v \) satisfies the equation:

\[
\int_s^t \left[ (v, \varphi_\tau) - (\nabla v, \nabla \varphi) + (v \cdot \nabla \varphi, v) \right] d\tau + (v(s), \varphi(s)) = (v(t), \varphi(t)),
\]

for all \( \varphi(t, x) \in \mathcal{C}_0([0, T) \times \Omega) \) with \( \nabla \cdot \varphi(t, x) = 0 \),

3) \( \lim_{t \to 0^+} (v(t), \varphi) = (v_0, \varphi) \), for all \( \varphi \in \mathcal{C}_0(\Omega) \).

Our chief result is

Theorem 1. For all \( v_0 \in J^2(\Omega) \) there exists a weak solution \( v(t, x) \) to problem (1) enjoying the energy inequality (2) for all \( t > s, s \in [0, \infty) - I \), with \( I \subset (0, \theta) \), \( \mu_1(I) = 0 \) and \( \theta \leq c \| v_0 \|_2^4 \). Moreover, \( v \) enjoys the energy equality

\[
\| v(t) \|_2^2 + 2 \int_0^t \| \nabla v(\tau) \|_2^2 d\tau = \| v_0 \|_2^2, \text{ for all } t \in (0, \infty) - I. \hspace{1cm} (4)
\]

In particular, we get

\[
2 \int_0^\infty \| \nabla v(t) \|_2^2 dt = \| v_0 \|_2^2. \hspace{1cm} (5)
\]

\( ^4 \)In \([28]\) it is proved that also the extra conditions for the regularity can be relaxed in such a way to be compatible with the assumption \( v_0 \) just in \( J^2(\Omega) \). This fact points out that the characterization between the initial data and some extra-conditions of Serrin’s kind, investigated by some authors, see e.g. \([20, 21]\), concerns the only conditions and cannot be regarded as a characterization of the regularity properties of a Leray-Hopf weak solution.
Finally, for all $\varphi \in J^2(\Omega)$, the function $(v(t), \varphi)$ belongs to $C([0,T))$, $v(t,x)$ is a continuous function in the $L^2$-norm in all instants $t$ for which (4) holds, and $v(t,x)$ is right continuous in the $L^2$-norm in $t = 0$.

Remark 1.

- We set the viscosity $\nu = 1$. Theorem 1 holds for all $\nu > 0$ provided that $\theta \leq \frac{c_1}{\nu} \|v_0\|^2_2$.
- An interesting implication of the energy equality (4) is that the possible instants of singularity of $\|\nabla v(t)\|_2$ do not invalidate the energy equality (4) and the energy equality (5).
- From (4), by difference, one easily deduces that
  \[ \|v(t)\|^2_2 + 2 \int_s^t \|\nabla v\|^2_2 d\tau = \|v(s)\|^2_2 , \]
  for all $t, s \in [0, \theta] - I$, for all $t \geq \theta$ and $s \in [0, \theta] - I$, and for all $t, s \geq \theta$.
- We stress that in Theorem 1 the claim related to the validity of (2) has not to be considered pleonastic. Actually, assuming $s = 0$, estimate (2) holds for all $t > 0$, in contrast to the validity almost everywhere of the energy equality (4).
- The energy equality (4) improves the regularity of our Leray’s weak solution in all instants $s$ in which (2) is true. Actually, we know that estimate (2) furnishes the right continuity in all $s$ for which (2) holds. In Theorem 1 in such instants there is the continuity in the $L^2$-norm.
- If the equality (4) remedies to a possible strong inequality in (2), by means of equality (5) we prove that the dissipation of viscosity completely fulfills its role as in 2D case.

Remark 2.

- Our result in particular proves that the weak solution $v$ of Theorem 1 is the strong limit in $L^2(0,T;W^{1,2}(\Omega))$ of a sequence of solutions $\{v_m\}$ to problem (23). In this connection, it is interesting to stress that we initially prove the validity of (4) and subsequently the strong convergence. So that, the energy equality (4) is a necessary and sufficient condition in order to obtain the strong convergence of the sequence $\{\nabla v^m\}$ in $L^2(0,T;L^2(\Omega))$ where $v_m$, for all $m \in \mathbb{N}$, is solution to problem (23).
- The strong convergence allows us to show that there is the continuity in the $L^2$-norm of our weak solution $v$ at any $t$ where (4) holds. This improves the classical result ensured by (2) (see 5th item of Remark 1).
- It is worth to point out that, as consequence of the strong convergence, considering the same sequence $\{v^m\}$, we are able to obtain the localized energy relation furnished in [5] as equality,
that is, a.e. in $t \in (0, \theta)$ and for all $t \geq \theta$, and for all the non-negative $\phi \in C^{1,2}_0([0, T) \times \Omega)$,

$$\|v(t)\phi_{t}^\frac{1}{2}\|_2^2 + 2 \int_0^t \|\nabla v\phi_{x}^\frac{1}{2}\|_2^2 \, d\tau = \|v_0\phi_{t}^\frac{1}{2}\|_2^2 + \int_0^t (\phi_{t} + \Delta \phi)v^2 \, d\tau + \int_0^t (v^2 + 2\pi) v \cdot \nabla \phi \, d\tau. \quad (6)$$

**Remark 3.**

- Removing the possible strong inequality in (2), our result proves the existence of at least a weak solution such that in (3) $D[u^\nu] = 0$, almost everywhere in $(t, x)$.

- We can see the connection between turbulence and singularity in a different way, that is, in terms of the continuity of $\|v(t)\|_2$. Since in our approach a right or left continuity of $\|v(t)\|_2$, for $t \in I$, is enough to achieve the energy equality, then the possible turbulence, without denying the energy equality on $(0, t)$, $t > \theta$, could cause a loss of the right and left continuity of $\|v(t)\|_2$ in all the instants $t \in I$.

- We stress that, in order to achieve the validity of (4) of Theorem 1, the initial datum $v_0 \in L^2$ has not to be burdened by further assumptions, which would be inessential. This is in contrast with the extra conditions.

- Finally, stating (4), we better delineate the case of validity of a possible example of non-regular or non-unique weak solution. For example, in the paper [4] the non-uniqueness is proved for very weak solutions, that is, solutions satisfying a very weak variational formulation of the Navier-Stokes equations and simply belonging to $C([0, T); L^2(\Omega))$. The result of non-uniqueness does not hold for a little more regular solution, but a priori it also does not work for a weak solution verifying an energy inequality. Another example is the conjecture by Scheffer, [35], which is based on an localized energy inequality, that is (6) with $\leq$. Now, since (6) is true for our weak solution, the Scheffer example has to be reformulated in such a way that relation (6) holds.

**Remark 4.**

- One can consider with no difficulty a data force $f \in L^2(0, T; L^2(\Omega))$. We omit the details since we think to the question in another context, like the one of time periodic solutions, where the energy equality has a special interest. The case of a domain $\Omega$ with unbounded smooth boundary will also be discussed later.

**Outline of the proof.**

Our result is an existence result, so that we suitably construct our weak solution $v$. The starting point is a sequence $\{v^m\}$, whose $m$th element is a smooth solution to the mollified Navier-Stokes IBVP (see (23)).

The first step is to prove that, for all $q \in [1, 2)$, the sequence $\{\nabla v^m\}$ is strongly convergent in $L^q(0, T; L^2(\Omega))$ (Lemma 7). Subsequently, by means of an auxiliary function, and making use of
the energy relation for the smooth solutions \( \{v^m\} \), we are able to prove a first result of convergence that leads to the following relation (formula (46)):

\[
\frac{2}{\pi} \lim_{\alpha \to 1} \frac{1}{1 - \alpha^\gamma} \lim_m \int_{J(\alpha^\gamma, m)} \frac{\|v^m\|_2^2}{1 + \|\nabla v^m\|_2^2} d\tau \|\nabla v^m\|_2^2 d\tau = \|v(s)\|_2^2 - \|v(t)\|_2^2 - \frac{2}{\pi} \lim_m \int_{J(\alpha^\gamma, m)} \|\nabla v^m\|_2^2 d\tau.
\]

\( J(\alpha^\gamma, m) \subset (s, t), \gamma \in (0, 1], \) see (45). Thanks always to special properties of our auxiliary function and to the energy relation valid for the sequence \( \{v^m\} \), we prove that the term of on the left-hand side is equal (formula (52)) to

\[
\frac{2}{\pi} \lim_{\alpha \to 1} \frac{1}{1 - \alpha^\gamma} \lim_m \int_{J(\alpha^\gamma, m)} \frac{\|v^m\|_2^2}{1 + \|\nabla v^m\|_2^2} d\tau \|\nabla v^m\|_2^2 d\tau = \lim_{\alpha \to 1} \lim_m \int_{J(\alpha^\gamma, m)} \|\nabla v^m\|_2^2 d\tau.
\]

Finally, we prove that the term on the right hand side is zero for \( \gamma = \frac{1}{3} \) (formula (61)). This last allow us to deduce the thesis. \( \square \)

Some lemmas of this paper were stated in \[8, 10\]. However, we again propose their proofs as they are slightly or completely different and because we make the paper self-contained.

The plan of the paper is the following. After recalling and proving some preliminary results, in sect. 3. we introduce the Navier-Stokes initial boundary value problem with a mollified non-linear term, in order to work with a sequence \((v^m, \pi^m)\) of smooth approximating solutions, whose limit in “metric of the energy” gives our weak solution. Finally, in sect. 4 we furnish the proof of Theorem 1.

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### 2 Some preliminary lemmas

We start by recalling the \( L^q \)-Helmholtz decomposition, that is

\[
L^q(\Omega) \equiv J^q(\Omega) \oplus G^q(\Omega),
\]

where \( J^q(\Omega) \) is the completion of \( \mathcal{C}_0(\Omega) \) in \( L^q(\Omega) \) and \( G^q(\Omega) := \{ w \in L^q(\Omega) : w \equiv \nabla h \text{ with } h \in W^{1,q}_{\text{loc}}(\Omega) \} \). By the symbol \( P_q \) we denote the projection from \( L^q(\Omega) \) onto \( J^q(\Omega) \). In the case of \( q = 2 \), we just write \( P \). For details on the Helmholtz decomposition see the monograph \[16\].

By the symbol \(-P_q \Delta\) we mean the Stokes operator defined on \( J^{1,q}(\Omega) \cap W^{2,q}(\Omega) \) with range \( J^q(\Omega) \). Here the symbol \( J^{1,q}(\Omega) \) denotes the completion of \( \mathcal{C}_0(\Omega) \) in \( W^{1,q}(\Omega) \). For further results on the Stokes operator we refer to the monograph \[16\].

**Lemma 1.** There exists a constant \( c > 0 \) such that

\[
\|w\|_\infty \leq c \|P \Delta w\|_2^{\frac{1}{2}} \|\nabla w\|_2^{\frac{1}{2}}, \text{ for all } w \in J^{1,2}(\Omega) \cap W^{2,2}(\Omega).
\]
Proof. Estimate (7) is an inequality of the Gagliardo-Nirenberg kind, whose right-hand side has the Stokes operator as max order of derivatives. For the proof see [27, 29]. □

Lemma 2 (Friedrichs’s lemma). Let \( \Omega \) be a bounded domain, and \( \{ a_p \} \) an orthogonal basis in \( L^2(\Omega) \). For all \( \varepsilon > 0 \) there exists \( N \in \mathbb{N} \) such that

\[
\| u \|_2 \leq (1 + \varepsilon) \left[ \sum_{k=1}^{N} (u, a^k)^2 \right]^{\frac{1}{2}} + \varepsilon \| \nabla u \|_q, \text{ for all } u \in W^{1,q}(\Omega),
\]

provided that \( q > \frac{6}{5} \).

Proof. This lemma is a generalization of the well known Friedrichs’s lemma stated for \( q = 2 \). The proof is given in [23] Ch.II Lemma 2.4. □

Lemma 3. Let \( \{ h_m(t) \} \subset L^1(0,T) \) be a sequence of non-negative functions such that \( \| h_m \|_1 \leq M < \infty \) for all \( m \in \mathbb{N} \). Also, assume that \( h_m(t) \to h(t) \) a.e. in \( t \in (0,T) \) with \( h(t) \in L^1(0,T) \). Then we get,

\[
\text{for all } \alpha \in (0,\alpha_0), \quad \lim_{m} \int_{0}^{T} h_m(t)p(\alpha, h_m(t))dt = \int_{0}^{T} h(t)p(\alpha, h(t))dt,
\]

and

\[
\lim_{\alpha \to \alpha_0} \lim_{m} \int_{0}^{T} h_m(t)p(\alpha, h_m(t))dt = \int_{0}^{T} h(t)dt,
\]

provided that, for all \( \alpha \in (0,\alpha_0) \), function \( p(\alpha, r) \) is continuous in \( r > 0 \) and

\[
p(\alpha, r) := \begin{cases} 
1, & \text{if } r \in [0, g(\alpha)], \\
\text{is decreasing, with } \lim_{r \to \infty} p(\alpha, r) = 0, & \text{if } r > g(\alpha),
\end{cases}
\]

where \( g(\alpha) \) denotes a strictly increasing and continuous function with \( \lim_{\alpha \to \alpha_0} g(\alpha) = \infty \).

Proof. We have

\[
\int_{0}^{T} h_m(t)p(\alpha, h_m(t))dt = \int_{0}^{T} (h_m(t) - h(t))p(\alpha, h_m(t))dt + \int_{0}^{T} h(t)p(\alpha, h_m(t))dt
\]

\[=: I_1(\alpha, m) + I_2(\alpha, m).\]

Since \( h_m(t) \to h(t) \) a.e. in \( t \in (0,T) \), then, for all \( \alpha \in (0,\alpha_0) \), \( p(\alpha, h_m(t)) \to p(\alpha, h(t)) \) a.e. in \( t \in (0,T) \). Since \( p(\alpha, h_m(t)) \leq 1 \) for all \( t \in (0,T) \), recalling that \( h(t) \in L^1(0,T) \), then the following limit holds:

\[
\lim_{m} I_2(\alpha, m) = \int_{0}^{T} h(t)p(\alpha, h(t))dt, \text{ for all } \alpha \in (0,\alpha_0).
\]

(11)
Now, we consider $I_1$ that, in our assumptions, is uniformly bounded in $\alpha \in (0, \alpha_0)$ and $m \in \mathbb{N}$. Our goal is to prove that $\lim_{m} I_1(\alpha, m) = 0$, for all $\alpha \in (0, \alpha_0)$. We point out that, for $\varepsilon \in (0, \alpha_0 - \alpha)$ and for all $m \in \mathbb{N}$,

$$(0, T) = \{t : h_m(t) \leq g(\alpha_0 - \varepsilon)\} \cup \{t : g(\alpha_0 - \varepsilon) < h_m(t)\} =: J_m^1(\varepsilon) \cup J_m^2(\varepsilon).$$

Hence, for all $m \in \mathbb{N}$, we write

$$I_1(\alpha, m) = \int_0^T \chi_{J_m^1}(t)(h_m(t) - h(t))p(\alpha, h_m(t))dt + \int_0^T \chi_{J_m^2}(t)(h_m(t) - h(t))p(\alpha, h_m(t))dt$$

$$= I_1^1(\alpha, m, \varepsilon) + I_1^2(\alpha, m, \varepsilon).$$

Letting $m \to \infty$ we get $\chi_{J_m^1}(t)|h_m(t) - h(t)|p(\alpha, h_m(t)) \leq |h_m(t) - h(t)| \to 0$ a.e. in $t \in (0, T)$. Recalling that $p(\alpha, h_m(t)) \leq 1$, we have $|\chi_{J_m^2}(t)(h_m(t) - h(t))p(\alpha, h_m(t))| \leq g(\alpha_0 - \varepsilon) + h(t)$ for all $t \in (0, T)$. Hence, by virtue of Lebesgue’s dominated convergence theorem, for all $\alpha \in (0, \alpha_0)$ and for all $\varepsilon \in (0, \alpha_0 - \alpha)$, we arrive at

$$\lim_{m} I_1^1(\alpha, m, \varepsilon) = 0. \quad (12)$$

For the second integral we point out that $\chi_{J_m^2}(t)p(\alpha, h_m(t)) \leq p(\alpha, g(\alpha_0 - \varepsilon))$ for all $t \in (0, T)$. Recalling that the sequence $\{h_m\}$ is bounded in $L^1(0, T)$ and that $h \in L^1(0, T)$, for all $\alpha \in (0, \alpha_0)$ and for all $\varepsilon \in (0, \alpha_0 - \alpha)$, uniformly in $m \in \mathbb{N}$, we deduce

$$|I_1^2(\alpha, m, \varepsilon)| \leq cp(\alpha, g(\alpha_0 - \varepsilon)). \quad (13)$$

Hence, via $(12)$ and $(13)$, for all $\alpha \in (0, \alpha_0)$ and $\varepsilon \in (0, \alpha_0 - \alpha)$, we arrive at

$$\lim_{m} |I_1(\alpha, m)| \leq cp(\alpha, g(\alpha_0 - \varepsilon)).$$

By the assumptions, $p(\alpha, r)$ tends to zero for $r \to \infty$, for all $\alpha \in (0, \alpha_0)$, and since $\varepsilon \in (0, \alpha_0 - \alpha)$ is arbitrary, we arrive at

$$\lim_{m} I_1(\alpha, m) = 0.$$

The last one and the limit property $(11)$ prove $(9)$. Thereafter, since $\rho \in L^1(0, T)$, $p(\alpha, \rho) \leq 1$ and has limit 1 for $\alpha \to \alpha_0$, via Lebesgue’s dominated convergence theorem, from $(9)$ we get

$$\lim_{\alpha \to \alpha_0} \int_0^T h(t)p(\alpha, h(t))dt = \int_0^T h(t)dt,$$

that proves $(10)$.

In order to state the following lemmas, we set

a) $\{p_m\}$ denotes a sequence of measurable non negative functions defined on $(s, t)$, bounded interval of $\mathbb{R},$
b) given \( \alpha \in (0, 1) \) and \( m \in \mathbb{N} \)

\[
J(\alpha, m) := \bigcup_{h \in \mathbb{N}(\alpha, m)} (s_h, t_h), \text{ where, for all } h \in \mathbb{N}(\alpha, m), (s_h, t_h) := \{ \tau \in (s, t) : \tan \frac{\alpha \pi}{2} < \rho_m \},
\]

the set \( \mathbb{N}(\alpha, m) \) is at most countable, and, for any \( h \neq k \), \((s_h, t_h) \cap (s_k, t_k) = \emptyset\),

c) for all \( \alpha \in (0, 1) \) and \( m \in \mathbb{N} \)

\[
J(\alpha, m) := \{ \tau \in J(\alpha, m) : \tan \frac{\alpha \pi}{2} < \rho_m \},
\]

d) for all \( \alpha \in (0, 1) \) and \( m \in \mathbb{N} \)

\[
J(\alpha, m) - J(\alpha, m) := \{ \tau \in J(\alpha, m) : \tan \frac{\pi}{2} < \rho_m(\tau) \leq \tan \frac{\alpha \pi}{2} \}.
\]

**Lemma 4.** Assume that the sequence \( \{\rho_m\} \) of item a) converges a.e. on \((s, t)\) to \( \rho \in L^1(s, t) \). Assume that for a constant \( c \), independent of \( \alpha \in (0, 1) \), the set defined in item b) is such that

\[
\mu_1(J(\alpha, m)) \leq \frac{c}{\tan \frac{\alpha \pi}{2}}, \text{ uniformly in } m \in \mathbb{N}.
\]

Then, we get

\[
\lim_{\alpha \to 1^-} \frac{1}{1 - \alpha} \lim_m \int_{J(\alpha, m)} \frac{\rho_m^2}{1 + \rho_m^2} d\tau = 0.
\]

**Proof.** We initially remark that the left-hand side of (15) is well posed for all \( \alpha \in (0, 1) \) and \( m \in \mathbb{N} \). Actually, \( \rho_m^2/(1 + \rho_m^2) \in (0, 1) \) and, by virtue of assumption (14), \( \lim_{\alpha \to 1^-} \frac{|J(\alpha, m)|}{1 - \alpha} \leq \lim_{\alpha \to 1^-} \frac{c}{(1 - \alpha) \tan \frac{\alpha \pi}{2}} \leq c \frac{\pi}{2} \). By means of a simple computation, we get

\[
\lim_m \int_{J(\alpha, m)} \frac{\rho_m^2}{1 + \rho_m^2} d\tau = \lim_m \left[ \int_{J(\alpha, m)} \frac{\rho_m^2}{1 + \rho_m^2} - \frac{\rho^2}{(1 + \rho_m^2)^\frac{1}{2}(1 + \rho^2)^\frac{1}{2}} \right] d\tau + \int_{J(\alpha, m)} \frac{\rho^2}{(1 + \rho_m^2)^\frac{1}{2}(1 + \rho^2)^\frac{1}{2}} d\tau
\]

\[
=: \lim_m \left[ I_1(\alpha, m) + I_2(\alpha, m) \right].
\]

For \( I_1 \) we have

\[
\frac{\rho_m^2}{1 + \rho_m^2} - \frac{\rho^2}{(1 + \rho_m^2)^\frac{1}{2}(1 + \rho^2)^\frac{1}{2}} \to 0, \text{ a.e. in } \tau \in (s, t),
\]

and

\[
\left| \frac{\rho_m^2}{1 + \rho_m^2} - \frac{\rho^2}{(1 + \rho_m^2)^\frac{1}{2}(1 + \rho^2)^\frac{1}{2}} \right| \leq (1 + \rho), \text{ a.e. in } \tau \in (s, t).
\]
Hence, applying Lebesgue’s dominated convergence theorem, we obtain

$$\lim_{m} I_1(\alpha, m) = \int_s^t \left[ \frac{\rho_m^2}{1 + \rho_m^2} - \frac{\rho^2}{(1 + \rho_m^2)^{\frac{1}{2}}(1 + \rho^2)^{\frac{1}{2}}} \right] d\tau = 0,$$

for all $\alpha \in (0, 1)$.

By virtue of item b), $\rho_m > \tan(\alpha \frac{\pi}{2})$ for all $t \in J(\alpha, m)$. Hence, for the second integral we have

$$I_2(\alpha, m) \leq \frac{1}{\tan(\alpha \frac{\pi}{2})} \int_{J(\alpha, m)} \rho d\tau.$$

We set

$$X(\alpha) := \{ Y \subset (s, t) : |Y| \leq \frac{c}{\tan(\alpha \frac{\pi}{2})} \}.$$

Since we assumed $|J(\alpha, m)| \leq \frac{c}{\tan(\alpha \frac{\pi}{2})}$, then $J(\alpha, m) \in X(\alpha)$ and

$$I_2(\alpha, m) \leq \frac{1}{\tan(\alpha \frac{\pi}{2})} \sup_{Y \in X(\alpha)} \int_Y \rho d\tau.$$

So that we get

$$\lim_{m} \int_{J(\alpha, m)} \frac{\rho_m^2}{1 + \rho_m^2} d\tau \leq \frac{1}{(1 - \alpha) \tan(\alpha \frac{\pi}{2})} \sup_{Y \in X(\alpha)} \int_Y \rho d\tau.$$

Hence, we arrive at

$$\frac{1}{1 - \alpha} \lim_{m} \int_{J(\alpha, m)} \frac{\rho_m^2}{1 + \rho_m^2} d\tau \leq \frac{1}{(1 - \alpha) \tan(\alpha \frac{\pi}{2})} \sup_{Y \in X(\alpha)} \int_Y \rho d\tau.$$

Being $\lim_{\alpha \to 1^-} (1 - \alpha) \tan(\alpha \frac{\pi}{2}) = \frac{2}{\pi}$, we are going to prove that

$$\lim_{\alpha \to 1^-} \sup_{Y \in X(\alpha)} \int_Y \rho d\tau = 0. \quad (16)$$

Thereafter, the thesis of the lemma holds. By virtue of the theorem on the absolute continuity of the integral, for all $\varepsilon > 0$ there exists $\alpha_0 \in (0, 1)$ such that

$$\mu_1(Y) < \frac{c}{\tan(\alpha_0 \frac{\pi}{2})} \Rightarrow \int_Y \rho d\tau < \varepsilon,$$

in particular, for all $\alpha \in (\alpha_0, 1)$: $\mu_1(Y) \leq \frac{c}{\tan(\alpha \frac{\pi}{2})} < \frac{c}{\tan(\alpha_0 \frac{\pi}{2})} \Rightarrow \int_Y \rho d\tau < \varepsilon \quad (17)$

For the same $\varepsilon$, by virtue of the $sup$-property, we get the existence of $\overline{Y} \in X(\alpha)$ such that

$$\sup_{Y \in X(\alpha)} \int_Y \rho d\tau < \int_{\overline{Y}} \rho d\tau + \varepsilon < 2\varepsilon,$$

where in the last step we took (17) into account. Since $\varepsilon$ is arbitrary, we have proved (16).  \qed
Lemma 5. Assume that the sequence \( \{\rho_m\} \) of item a) converges to \( \rho \) in \( L^{1-\delta}((s,t)) \), for some \( \delta \in (0,1) \). Assume that items b)-d) hold. Assume that

\[
\mu_1(J(\alpha, m)) \leq \frac{c}{\tan \alpha \frac{\pi}{2}}, \quad \text{uniformly in } m \in \mathbb{N}.
\]  

Then we get

\[
\lim_{\alpha \to 1^-} \lim_{m} \int_{J(\alpha, m)-J(\alpha^\frac{1}{3}, m)} \rho_m d\tau = 0.
\]  

Proof. For all \( m \in \mathbb{N} \), taking the definition of \( J(\alpha, m) - J(\alpha^\frac{1}{3}, m) \) into account, for \( \delta \in (0,1) \), we get

\[
\lim_{m} \int_{J(\alpha, m)-J(\alpha^\frac{1}{3}, m)} \rho_m d\tau \leq |\tan \alpha \frac{\pi}{2}| \lim_{m} \int_{J(\alpha, m)-J(\alpha^\frac{1}{3}, m)} \rho^{1-\delta} d\tau.
\]  

Moreover, for all \( \alpha \), we deduce

\[
\lim_{m} \int_{J(\alpha, m)-J(\alpha^\frac{1}{3}, m)} \rho^{1-\delta} d\tau \leq \lim_{m} \int_{J(\alpha, m)-J(\alpha^\frac{1}{3}, m)} |\rho_m - \rho|^{1-\delta} d\tau + \lim_{m} \int_{J(\alpha, m)-J(\alpha^\frac{1}{3}, m)} \rho^{1-\delta} d\tau
\]

\[
\leq \lim_{m} \int_{s}^{t} |\rho_m - \rho|^{1-\delta} d\tau + \lim_{m} \int_{J(\alpha, m)-J(\alpha^\frac{1}{3}, m)} \rho^{1-\delta} d\tau = \lim_{m} \int_{J(\alpha, m)-J(\alpha^\frac{1}{3}, m)} \rho^{1-\delta} d\tau,
\]

where we employed the strong convergence of \( \{\rho_m\} \) to \( \rho \) in \( L^{1-\delta}((s,t)) \).

We denote by \( X(\alpha) := \{Y \subset (s,t) : \mu_1(Y) < \frac{c}{\tan \alpha \frac{\pi}{2}}\} \), with \( c \) independent of \( \alpha \). Via our assumptions (18), for all \( m \in \mathbb{N} \), \( \mu_1(J(\alpha, m) - J(\alpha^\frac{1}{3}, m)) < \frac{c}{\tan \alpha \frac{\pi}{2}} \) holds too. Then we have \( J(\alpha, m) - J(\alpha^\frac{1}{3}, m) \in X(\alpha) \) for all \( m \in \mathbb{N} \). For all \( \alpha \in (0,1) \) and uniformly in \( m \in \mathbb{N} \), one gets

\[
\int_{J(\alpha, m)-J(\alpha^\frac{1}{3}, m)} \rho^{1-\delta} d\tau \leq \sup_{Y \in X(\alpha)} \int_{Y} \rho^{1-\delta} d\tau.
\]

Applying Hölder’s inequality, uniformly in \( m \in \mathbb{N} \), we also get

\[
\int_{J(\alpha, m)-J(\alpha^\frac{1}{3}, m)} \rho^{1-\delta} d\tau \leq \left[ \mu_1(Y) \right]^{\delta} \left[ \sup_{Y \in X(\alpha)} \int_{Y} \rho d\tau \right]^{1-\delta} \leq c \tan^{-\delta}(\alpha \frac{\pi}{2}) \left[ \sup_{Y \in X(\alpha)} \int_{Y} \rho d\tau \right]^{1-\delta}. \]  

Collecting estimates (20)-(21), since \( \lim_{\alpha \to 1^-} \frac{\tan \alpha \frac{\pi}{2}}{\tan \alpha \frac{\pi}{2}} = 3 \), we obtain

\[
\lim_{\alpha \to 1^-} \lim_{m} \int_{J(\alpha, m)-J(\alpha^\frac{1}{3}, m)} \rho_m d\tau \leq \lim_{\alpha \to 1^-} \left[ \frac{\tan \alpha \frac{\pi}{2}}{\tan \alpha \frac{\pi}{2}} \right]^{\delta} \left[ \sup_{Y \in X(\alpha)} \int_{Y} \rho d\tau \right]^{1-\delta} = 3^{\delta} \lim_{\alpha \to 1^-} \left[ \sup_{Y \in X(\alpha)} \int_{Y} \rho d\tau \right]^{1-\delta}.
\]
To conclude the proof it remains to show that the last limit is equal to zero. We employ the same argument of the previous lemma, that for the convenience of the reader we report here. Since \( \rho \in L^1((s,t)) \), by virtue of the theorem on the absolute continuity of the integral, for all \( \varepsilon > 0 \) there exists \( \alpha_0 \in (0,1) \) such that,

\[
\mu_1(Y) < \frac{c}{\tan(\alpha_0 \pi/2)} \Rightarrow \int_Y \rho d\tau < \varepsilon,
\]

in particular, for all \( \alpha \in (\alpha_0,1) : \mu_1(Y) \leq \frac{c}{\tan(\alpha_0 \pi/2)} < \frac{c}{\tan(\alpha \pi/2)} \Rightarrow \int_Y \rho d\tau < \varepsilon
\] (22)

For the same \( \varepsilon \), by virtue of the \( \sup \)-property, we get the existence of \( \Upsilon \in X(\alpha) \) such that

\[
\sup_{Y \in X(\alpha)} \int_Y \rho d\tau < \int_{\Upsilon} \rho d\tau + \varepsilon < 2\varepsilon,
\]

where in the last step we took (22) into account. Since \( \varepsilon \) is arbitrary, we arrive at

\[
\lim_{\alpha \to 1^-} \left[ \sup_{Y \in X(\alpha)} \int_Y \rho d\tau \right]^{1-\delta} = 0.
\]

The proof of (19) is achieved. \( \square \)

3 Mollified Navier-Stokes equations

We introduce an auxiliary Navier-Stokes initial boundary value problem:

\[
v_m^t + \mathbb{J}_m[v_m] \cdot \nabla v_m + \nabla \pi_m = \Delta v_m, \quad \nabla \cdot v_m = 0, \quad \text{on } (0,T) \times \Omega,
\]

\[
v_m = 0 \quad \text{on } (0,T) \times \partial \Omega, \quad v_m(v_0^m) \text{ on } \{0\} \times \Omega,
\] (23)

where \( \mathbb{J}_m[\cdot] \equiv J^m_{\mathbb{J}}[\cdot] \) and \( J^m_{\mathbb{J}}[\cdot] \) is the Friedrichs (space) mollifier, and \( \{v_0^m\} \subset C_0(\Omega) \) converges to \( v_0 \) in \( L^2 \)-norm.

**Theorem 2.** For all \( v_0^m \in C_0(\Omega) \) there exists a unique smooth solution, for \( t > 0 \), to problem (23) defined for all \( T > 0 \). In particular the following holds:

\[
v_m \in C^1((0,T); J^{1,2}(\Omega)) \cap L^2(0,T; W^{2,2}(\Omega)), \quad v_m^t \in L^2(0,T; L^2(\Omega)),
\] (24)

and, for all \( t > s \geq 0 \),

\[
\|v_m(t)\|_2^2 + 2 \int_s^t \|\nabla v_m\|_2^2 d\tau = \|v_m(s)\|_2^2 \leq \|v_0\|_2^2.
\] (25)
Moreover, for all \( t \geq \theta \) and uniformly in \( m \in \mathbb{N} \), the following holds:

\[
\|\nabla v^m(t)\|_2^2 + \frac{1}{2} \int_0^t \left[ \|P \Delta v^m\|_2^2 + \|v_t^m\|_2^2 \right] d\tau \leq (2c\|v_0\|_2^2)^{-1}, \tag{26}
\]

where we have \( \theta \leq c\|v_0\|_2^2 \) with the constant \( c \) independent\(^5\) of \( t, m \) and \( v_0 \).

\( \text{Proof.} \) For the existence and the regularity of a solution \((v^m, \pi^m)\) we can employ the well known Faedo-Galerkin method as proposed in [19, 39] (one founds developed this idea, e.g., in Appendix of [5] or in sect. 2 of [17]). In particular one arrives at proving (24)-(25).

We prove estimate\(^6\) (26). We apply the projection operator \( P \) to equation (23), then, we consider the \( L^2 \)-norm of both sides:

\[
\|v_t^m - P \Delta v^m\|_2^2 = \|P(J_m(v^m) \cdot \nabla v^m)\|_2^2.
\]

Since \((v_t^m, P \Delta v^m) = -\frac{d}{dt} \|\nabla v^m\|_2^2\), we get the following

\[
\frac{d}{dt} \|\nabla v^m\|_2^2 + \frac{1}{2} \|P \Delta v^m\|_2^2 + \|v_t^m\|_2^2 = \|P(J_m(v^m) \cdot \nabla v^m)\|_2^2. \tag{27}
\]

We estimate the right-hand side by means of (7):

\[
\|P(J_m(v^m) \cdot \nabla v^m)\|_2^2 \leq \|J_m(v^m) \cdot \nabla v^m\|_2^2 \leq \|v^m\|_{\infty}^2 \|\nabla v^m\|_2^2 \leq c \|\nabla v^m\|_2^2 \|P \Delta v^m\|_2. \tag{28}
\]

Hence, we deduce

\[
\frac{d}{dt} \|\nabla v^m\|_2^2 + \frac{1}{2} \|P \Delta v^m\|_2^2 + \|v_t^m\|_2^2 \leq c \|\nabla v^m\|_2^2. \tag{29}
\]

From energy estimate (25), for all \( m \in \mathbb{N} \), we deduce the existence of a \( \theta_m \leq c\|v_0\|_2^2 \) such that \( \|\nabla v^m(\theta_m)\|_2^2 \leq (2c\|v_0\|_2^2)^{-1} \). Actually, if the estimate does not hold, we get

\[
\frac{1}{2} \|v_0\|_2^2 = (2c\|v_0\|_2^2)^{-1} \|v_0\|_2^2 \leq \int_0^{\frac{c\|v_0\|_2^2}{2}} \|\nabla v^m(\tau)\|_2^2 d\tau \leq \frac{1}{2} \|v_0\|_2^2 \leq \frac{1}{2} \|v_0\|_2^2,
\]

which is an \textit{absurdum}. Since the differential energy relation furnishes \( \|\nabla v^m\|_2^2 \leq \|v^m\|_2 \|v_t^m\|_2 \leq \|v_0\|_2 \|v_t^m\|_2 \), from (29) we obtain

\[
\frac{d}{dt} \|\nabla v^m\|_2^2 + \frac{1}{2} \|P \Delta v^m\|_2^2 + \|v_t^m\|_2^2 < \|\nabla v^m\|_2^2 \|v_0\|_2^2 \|\nabla v^m\|_2^2 (2c\|v_0\|_2^2 \|\nabla v^m\|_2^2 - 1) \big|_{t=\theta_m} \leq 0.
\]

This last easily leads to (26). \( \square \)

---

\(^5\)If we consider the viscosity coefficient different from 1, then the instant \( \theta \) depends on \( \nu \).

\(^6\) Estimate (26) is generally given on the weak solution \( v \), it furnishes the regularity of \( v \) for \( t \geq \|v_0\|_2^2 \). This is a result related to the Leray’s partial regularity, called structure theorem of a Leray’s weak solution. Here we furnish the proof on \( \{v^m\} \). One gets the same regularity for the weak solution \( v \) with, of course, the instant \( \theta \) as end point of the regularity interval.
Remark 5. In place of $J[-]$ mollifier one can construct an approximating system by means of the Yosida Approximation. Then one arrives at the same result, that is, estimates (24)-(25) hold. For this result we quote [18, 31].

Lemma 6. For all $T > 0$, the sequence $\{v^m\}$ furnished by Theorem 2 weakly converges to $v \in L^2(0,T;J^{1,2}(\Omega))$ and strongly converges to $v$ in $L^2(0,T;L^2(\Omega))$.

Proof. The result is well known, it is a part of existence theorem of a weak solution enjoying the energy inequality in strong form. So that we omit the details and we limit ourselves to sketch the idea of the proof. The weak convergence to $v$ is a consequence of the energy relation (25). Moreover, for all $t > 0$, $v$ is also a weak limit in $J^2(\Omega)$ with $(v(t), \varphi)$, for all $\varphi \in C_0(\Omega)$, continuous function, e.g. see [22]. This is enough to apply, the Friedrich Lemma 2 for the sequence $\{v^m\} \subset L^2(0,T;L^2(\Omega))$ in the way suggested in [22], that is, for all $R > 0$,

$$
\int_0^T \|v^m - v\|^2_{L^2(\Omega \cap B_R)} d\tau \leq (1 + \varepsilon) \sum_{p=1}^N \int_0^T (v^m - v, a^k)^2 + \varepsilon \int_0^T \|\nabla v^m - \nabla v\|^2 d\tau. \tag{30}
$$

Hence, the result of convergence is immediate in the case of $\Omega$ bounded. In the case of $\Omega$ unbounded, in accord with our assumptions, the result can be proved in the way proposed by Leray in [24]. The idea of the proof is to achieve for the sequence $\{v^m\}$ the conditions of the compactness theorem in $L^p(\mathbb{R}^n)$ for all $t > 0$. Assume that the following holds:

$$
\|v^m(t)\|^2_{L^2(|x|>R)} \leq \|v_0^m\|^2_{L^2(|x|>\frac{R}{4})} + c(t)\psi(R) \quad \text{for any } t > 0, R > 2R \text{ and } m \in \mathbb{N}, \tag{31}
$$

with $c(t) \in L^\infty(0,T)$ and $\psi(R,v_0) = o(1)$. Set

$$
H(R,T) := T\|v_0\|^2_{L^2(|x|>R)} + \psi(R) \int_0^T c(t) dt + \int_0^T \|v(t)\|^2_{L^2(|x|>R)} dt \equiv o(1),
$$

then, for all $T > 0$, we get

$$
\int_0^T \|v^m(t) - v(t)\|^2 dt = \int_0^T \|v^m(t) - v(t)\|^2_{L^2(\Omega \cap B_R)} dt + \int_0^T \|v^m(t) - v(t)\|^2_{L^2(|x|>R)} dt
\leq 2 \int_0^T \|v^m(t) - v(t)\|^2_{L^2(\Omega \cap B_R)} dt + 2\|v^m_0 - v_0\|^2_{L^2(|x|>\frac{R}{4})} + 2H(R,t).
$$

Employing (30) for the first term on the right-hand side, and recalling that $\{v^m_0\}$ converges to $v_0$ in the $L^2$-norm, letting $m \to \infty$, the first two terms on the right hand-side tend to zero. Secondary, being $H(R,T) = o(1)$, letting $R \to \infty$, we get that the right hand side approaches zero.

Estimate (31) is proved in several papers concerning the question of the energy inequality (2). For this reason we do not give the proof, but, for the interested reader, we quote the proof furnished in sect. 6.4.1 of the paper [8].
The following lemma was proved for the first time in [8]

**Lemma 7.** Let \( \{v^m\} \) be the sequence furnished by Theorem 2 and let \( v \) be the limit ensured by Lemma 6. Then, for all \( q \in [1, 2) \), the sequence \( \{v^m\} \) strongly converges to \( v \) in \( L^q(0, T; J^{1,2}(\Omega)) \).

**Proof.** In order to prove the strong convergence, we initially prove that \( \{P\Delta v^m\} \) is bounded in \( L^{\frac{3}{2}}(0, T; L^2(\Omega)) \). We consider (29) again. Hence, we trivially deduce

\[
\frac{d}{dt}\|\nabla v^m\|_2^2 + \frac{1}{2}\|P\Delta v^m\|_2^2 + \|v^m_t\|_2^2 \leq c\|\nabla v^m\|_2^6 \leq c(1 + \|\nabla v^m\|_2^2)^2\|\nabla v^m\|_2^2.
\]

This last is integrated in the following way:

\[
\frac{1}{1 + \|\nabla v^m(0)\|_2^2} + \int_0^t \left[ \frac{1}{2}\|P\Delta v^m\|_2^2 + \|v^m_t\|_2^2 \right] d\tau \leq \frac{1}{1 + \|\nabla v^m(t)\|_2^2} + c \int_0^t \|\nabla v^m\|_2^2 d\tau.
\]

(32)

Applying Hölder’s reverse inequality with “complementary” exponents \( q = -\frac{1}{2} \) and \( q' = \frac{1}{3} \), uniformly in \( m \in \mathbb{N} \), we get

\[
\left[ \int_0^t (\|P\Delta v^m\|_2^2 + \|v^m_t\|_2^2) d\tau \right]^{\frac{3}{2}} \leq \left[ \int_0^t (1 + \|\nabla v^m\|_2^2) d\tau \right]^{\frac{2}{3}} \left[ 1 + c \int_0^t \|\nabla v^m\|_2^2 d\tau \right]^{\frac{1}{3}}
\]

\[
\leq \left[ t + \|v_0\|_2^2 \right]^{\frac{3}{2}} \left[ 1 + c\|v_0\|^2 \right] := \mathcal{A}(t, \|v_0\|_2), \text{ for all } t > 0.
\]

For any pair \( (m, p) \) and for all \( t > 0 \), by means of an integration by parts and Hölder’s inequality, we obtain

\[
\int_0^t \|\nabla v^m - \nabla v^p\|_2^2 d\tau \leq \int_0^t \|P\Delta(v^m - v^p)\|_2^2 \|v^m - v^p\|_2^\frac{3}{2} d\tau \leq \left[ \int_0^t \|\nabla^2(v^m - v^p)\|_2^2 d\tau \right]^{\frac{3}{4}} \left[ \int_0^t \|v^m - v^p\|_2^2 d\tau \right]^{\frac{1}{4}}
\]

\[
\leq 2^{\frac{3}{4}} \mathcal{A}(t, \|v_0\|_2) \left[ \int_0^t \|v^m - v^p\|_2^2 d\tau \right]^{\frac{1}{4}}.
\]

By virtue of Lemma 6, we obtain the Cauchy condition for \( \{v^m\} \) in \( L^1(0, T; J^{1,2}(\Omega)) \). Via (25), the sequence \( \{v^m\} \) is bounded in \( L^2(0, T; J^{1,2}(\Omega)) \). Hence, by interpolation we realize the Cauchy condition in \( L^q(0, T; J^{1,2}(\Omega)) \), for all \( q \in [1, 2) \). So that, the sequence admits the strong limit in \( L^q(0, T; J^{1,2}(\Omega)) \), for all \( q \in [1, 2) \), which coincides with \( v \in L^2(0, T; J^{1,2}(\Omega)) \). The lemma is proved.

**Corollary 1.** The sequence of solutions \( \{v^m\} \) to problem (23) strongly converges to \( v \) in \( J^{1,2}(\Omega) \), uniformly in \( t \geq \theta \).

---

7 An analogous integrability property for weak solutions has been obtained both in [15] and in [13]. But our proof, very short, is original with regards to the ones of the quoted papers and is furnished on the sequence \( \{v^m\} \), we are not interested to the property on the weak solution.
Proof. By virtue of Lemma 6 and Lemma 7 we have that \( \{v^m\} \) strongly converges to \( v \) in \( J^{1,2} \) a.e. in \( t > 0 \). Without invalidating the thesis, we can assume that the strong convergence holds for \( t = \theta \). Thanks to estimates (25) and (26) we can state the following estimates:

employing (7), we get

\[
\int_0^t \|v^m(\tau)\|^2_{L^\infty}\ d\tau \leq c \left[ \int_0^t \|P \Delta v^m(\tau)\|^2_2\ d\tau \right]^{\frac{1}{2}} \left[ \int_0^t \|\nabla v^m(\tau)\|^2_2\ d\tau \right]^{\frac{1}{2}} \leq \frac{1}{4c^2}, \text{ uniformly in } m \in \mathbb{N} \text{ and } t > \theta;
\]  
(33)

employing Young’s inequality, we get

\[
\int_0^t \|\nabla v^m(\tau)\|^2_2\ d\tau \leq \max_{[\theta, t]} \|\nabla v(\tau)\|^2_2\int_0^t \|\nabla v^m(\tau)\|^2_2\ d\tau \leq \frac{1}{4c^2}, \text{ uniformly in } m \in \mathbb{N} \text{ and } t > \theta.
\]  
(34)

In the following we set \( \frac{1}{4c^2} =: \mathcal{D} \). Set \( w := v^m - v^p \), from system (23) by difference, we deduce

\[
\frac{d}{dt} \|\nabla w\|^2_2 + \|P \Delta w\|^2_2 + \|w_t\|^2_2 = \|P(v^m \cdot \nabla w) + (v^m \cdot \nabla v^p)\|^2_2 \quad \text{for all } t > \theta.
\]  
(35)

Applying Hölder’s inequality to the term of the right-hand side, we get

\[
\|P(v^m \cdot \nabla w) + (v^m \cdot \nabla v^p)\|^2_2 \leq 2 (\|v^m\|^2_{L^\infty} \|\nabla w\|^2_2 + \|w\|^2_{L^\infty} \|\nabla v^p\|^2_2) \leq c (\|v^m\|^2_{L^\infty} \|\nabla w\|^2_2 + \|w\|^2_{L^2} \|P \Delta w\|^2_2 \|\nabla v^p\|^2_2),
\]

where in the last estimate we employed (7) again. Increasing the right-hand side of (35) via the last estimate, and employing the Young inequality, we deduce the differential equation

\[
\frac{d}{dt} \|\nabla w\|^2_2 \leq c \|\nabla w\|^2_2 (\|v^m\|^2_{L^\infty} + \|\nabla v^p\|^4_2), \text{ for all } t > \theta,
\]

that, by means of estimates (33) and (34), via an integration, uniformly in \( t \geq \theta \), furnishes

\[
\|\nabla v^m(t) - \nabla v^p(t)\|^2_2 = \|\nabla w(t)\|^2_2 \leq c \exp[\mathcal{D}] \|\nabla w(\theta)\|^2_2 \leq c \exp[\mathcal{D}] \|\nabla v^m(\theta) - \nabla v^p(\theta)\|^2_2, \text{ for all } m, p \in \mathbb{N}.
\]  
(36)

In the case of the \( L^2 \)-norm, for all \( t > \theta \), we consider the energy relation related to \( w \):

\[
\frac{d}{dt} \|w(t)\|^2_2 + 2 \|w(t)\|^2_2 = (w \cdot \nabla w, v^m) \leq \|w\|^2_{L^2} \|\nabla w\|^2_2 \|v^m\|^2_2 \|\nabla v^m\|^2_2 \leq c \|v_0\|^2_2 \|\nabla w\|^2_2 \|\nabla v^m\|^2_2 \leq c \|v_0\|^2_2 \|\nabla w\|^2_2 \left[ \|\nabla v^m\|^2_2 + \|\nabla v^p\|^2_2 \|\nabla v^m\|^2_2 \right],
\]
where we have increased by means of Hölder’s inequality, Sobolev inequality and the Gagliardo-Nirenberg inequality. Taking estimate (25) into account, an integration on \( t > \theta \), uniformly in \( t \geq \theta \), furnishes

\[
\|v^m(t) - v^p(t)\|_2^2 = \|w(t)\|_2^2 \leq \|w(\theta)\|_2^2 + c\|v_0\|_2^\frac{5}{2} \exp[D] \|\nabla v^m(\theta) - \nabla v^p(\theta)\|_2^\frac{1}{2} \\
\leq \|v^m(\theta) - v^p(\theta)\|_2^2 + c\|v_0\|_2^\frac{5}{2} \exp[D] \|\nabla v^m(\theta) - \nabla v^p(\theta)\|_2^\frac{1}{2},
\]

(37)

Now, from our assumption of strong convergence of \( \{v^m(\theta)\} \) in \( J^{1,2}(\Omega) \), via estimates (36) and (37), the thesis of the lemma holds.

\[ \square \]

**Lemma 8.** The sequence \( \{v^m\} \) furnished by Theorem 2 enjoys the following estimate:

\[
\int_0^t \frac{1}{(1 + \|v^m\|_2^2)^2} \left| \frac{d}{dt} \|v^m\|_2^2 \right| d\tau \leq 1 + c\|v_0\|_2^2, \text{ uniformly in } m \in \mathbb{N} \text{ and for all } t > 0.
\]

(38)

**Proof.** From (27) we are able to get

\[
\frac{1}{(1 + \|v^m\|_2^2)^2} \left| \frac{d}{dt} \|v^m\|_2^2 \right| = \left| \frac{\|P(\|v^m\| \cdot \nabla v^m)\|_2^2}{(1 + \|v^m\|_2^2)^2} - \frac{\|P \Delta v^m\|_2^2 + \|v^m\|_2^2}{(1 + \|v^m\|_2^2)^2} \right| \\
\leq \frac{\|\nabla v^m\|_2^2 \|P \Delta v^m\|_2^2 + \|v^m\|_2^2}{(1 + \|v^m\|_2^2)^2} \\
\leq \frac{\|\nabla v^m\|_2^2 + c \|P \Delta v^m\|_2^2 + \|v^m\|_2^2}{(1 + \|v^m\|_2^2)^2} \\
\leq \frac{\|\nabla v^m\|_2^2 + c \|P \Delta v^m\|_2^2 + \|v^m\|_2^2}{(1 + \|v^m\|_2^2)^2},
\]

(39)

where estimating the right hand-side we took (28) into account and applied the Young inequality. Integrating (39) on \((0, t)\), via estimate (32) and via energy relation (25), we arrive at (38).

\[ \square \]

## 4 Proof of Theorem 1

We start by proving the existence of a weak solution.

**Theorem 3.** For all \( v_0 \in J^2(\Omega) \) there exists a sequence \( \{v^m\} \), where, for all \( m \in \mathbb{N} \), \( v^m \) is the solution to problem (23) furnished by Theorem 2. The sequence \( \{v^m\} \) enjoys the following limit properties:

\[
\text{for all } T > 0, \quad \{v^m\} \rightharpoonup v \text{ in } L^2(0, T; J^{1,2}(\Omega)), \\
\{v^m\} \to v \text{ in } L^2(0, T; L^2(\Omega)) \cap L^q(0, T; J^{1,2}(\Omega)), \quad q \in [1, 2), \quad (40) \\
\{v^m\} \to v \text{ in } J^{1,2}(\Omega), \text{ a.e. in } t \in [0, \theta) \text{ and uniformly in } t \geq \theta, \text{ with } \theta \leq c\|v_0\|_2^4, \\
\]
where the constant $c$ is independent of $v_0$. Moreover, the limit $v$ is a weak solution to problem (1) and enjoys the properties:

$$\|v(t)\|_2^2 + 2 \int_0^t \|\nabla v\|_2^2 \, dt \leq \|v(s)\|_2^2,$$

for all $t > s$, and for all $s \in [0, \infty) - I$, $I \subset (0, \theta)$, $\mu_1(I) = 0$,

for all $s \in I^c$, \(\lim_{t \to s^+} \|v(t) - v(s)\|_2 = 0\), for all $\varphi \in J^2(\Omega)$, $(v(t), \varphi)$ is a continuous function of $t$,

$$v \in C([\theta, T); J^{1,2}(\Omega)) \cap L^2(\theta, T; W^{2,2}(\Omega)), \quad v_t, \nabla \pi \in L^2(\theta, T; L^2(\Omega)), \quad \text{for all } T > \theta,$$

and

$$\lim_{t \to \infty} \|v(t)\|_2 = 0. \quad (42)$$

**Proof.** The existence result of $v$ is achieved following the approach furnished by Leray in [24]. Actually, we consider the limit $v$ stated in Lemma 6, Lemma 7 and Corollary 1, then the existence of $\theta$ and the related estimate are deduced from Theorem 2. For all $t \in [0, T)$, $v$ is also a weak limit in $J^2(\Omega)$ with $(v(t), \varphi) \in C([0, T))$, for all $\varphi \in J^2(\Omega)$. One easily verifies that the limit $v$ is a weak solution. Estimate (41) is the so-called energy inequality in strong form introduced by Leray. In order to deduce (41)1, we consider an instant $s \geq 0$ in which $\lim_m \|v^m(s)\|_2 = \|v(s)\|_2$. In the case of $s \neq 0$, the existence is ensured, almost everywhere in $s \in (0, \theta)$, by Lemma 6, and for all $s \geq \theta$ by Corollary 1. We denote by $I$ the set of all possible instants $s$ for which the strong convergence fails to hold. By our arguments, $I \subset (0, \theta)$ and $\mu_1(I) = 0$. Then, for all $s \in I^c$, we consider (25), and we perform the lower limit of the left-hand side and the limit of the right-hand side of (25). Thereafter, one easily achieves the energy inequality for the weak limit $v$ as stated in (41). Properties (41)2 are an immediate consequence of the weak limit properties and of (41)1. Property (41)3 is the regularity of $v$ achieved for $t > \theta$, that in our proof is a consequence of the uniform estimate (26). Finally, the limit property (42) is also a well-known result. We are not able to quote all the contributions on the question, so that we limit ourselves to the following: for bounded domains, a first systematic theory for weak solution is given in [37]; for unbounded domains of the kind considered in this paper we quote, e.g., [25, 26, 31].

Now, our goal is to prove that the weak solution $v$ furnished in Theorem 3 enjoys the energy equality (4). For this task we are going to prove that some strong convergences hold. These convergences are a consequence of the strong convergences furnished in sect. 2 and in sect. 3, of some auxiliary functions and of the differential energy equality (25) deduced for the elements of the sequence $\{v^m\}$.

We denote by

$$\mathcal{T} := \{s \geq 0 : \{v^m(s, x)\} \text{ is strongly convergent to } v \text{ in } J^{1,2}(\Omega)\}.$$

By virtue of (40)3, the set $\mathcal{T}$ is not empty, and

$$\mu_1([0, \infty) - \mathcal{T}) = \mu_1([0, \theta) - (\mathcal{T} \cap [0, \theta])) = 0.$$
Also, we denote by

\[ I^C := \{ s \geq 0 : \{ v^m(s, x) \} \text{ is strong convergent to } v \text{ in } L^2(\Omega) \} \supset \mathcal{T}, \]  

which has been detected in the proof of Theorem 3. Of course, \([0, \infty) – I^C \text{ coincides with } I\), where \(I\) is introduced in (41)\(_1\).

For \(s, t \in \mathcal{T}\), we set

\[ A_m := \max_{[s, t]} \| \nabla v^m(\tau) \|_2^2 \], and \(A := \sup_{m \in \mathbb{N}} A_m \).

**Lemma 9.** Let \(\{ v^m \}\) be the sequence of solutions to (23) and \(v\) the weak solution to (1) furnished in Theorem 3. Let \(t, s \in \mathcal{T}\). Assume that in (44) \(A = \infty\). Let \(\gamma \in (0, 1]\). Then there exists a family \(\{ J(\alpha^\gamma, m) \}\) of sets, where, for all \(\alpha \in (\alpha_1, 1)\), \(\alpha_1 > 0\), and \(m \geq m_0\),

\[ J(\alpha^\gamma, m) := \bigcup_{h \in \mathbb{N}(\alpha^\gamma, m)} (s_h, t_h), \mathbb{N}(\alpha^\gamma, m) \text{ countable set, and } |J(\alpha^\gamma, m)| \leq \| v_0 \|_2^2 \left[ \tan \alpha^\gamma \pi \right]^{-1}, \]

for all \(h \in \mathbb{N}(\alpha^\gamma, m)\), \((s_h, t_h) := \{ \tau : \tan \alpha^\gamma \pi \frac{2}{\gamma} < \| \nabla v^m(\tau) \|_2^2 \leq A_m \}\),

with \(\| \nabla v^m(s_h) \|_2^2 = \| \nabla v^m(t_h) \|_2^2 = \tan \alpha^\gamma \pi \frac{2}{\gamma}\), and, for \(h \neq k\), \((s_h, t_h) \cap (s_k, t_k) = \emptyset\),

and the weak solution \(v\) enjoys the following special “energy relation”:

\[ \frac{2}{\pi} \lim_{\alpha \to 1^-} \left[ \frac{1}{1 - \alpha^\gamma} \lim_{m \to \infty} \int_{J(\alpha^\gamma, m)} \| v^m \|_2^2 \frac{d\tau}{1 + \| \nabla v^m \|_2^4} \right] \int d\tau = \| v(s) \|_2^2 - \| v(t) \|_2^2 - 2 \int_s^t \| \nabla v \|_2^2 d\tau. \]  

**Proof.** To prove the claim (46) we develop a suitable construction.

For all \(\alpha \in (0, 1)\), we consider the function

\[ p(\alpha^\gamma, \rho) := \begin{cases} 1, & \text{if } \rho \in [0, \tan \alpha^\gamma \frac{\pi}{2}] \\ \frac{\pi - \arctan \rho}{(1 - \alpha^\gamma)^2}, & \text{if } \rho \in (\tan \alpha^\gamma \frac{\pi}{2}, \infty) \end{cases} \]  

Choosing \(g(\alpha) = \tan \alpha^\gamma \frac{\pi}{2}\), the function \(p(\alpha^\gamma, \rho)\) enjoys the same properties of the function introduced in Lemma 3 with \(\alpha_0 := 1\). We consider the sequence \(\{ v^m \}\) of solutions. For all \(m \in \mathbb{N}\), the energy equation (25), that for the convenience of the reader we reproduce, holds:

\[ \frac{d}{dt} \| v^m(t) \|_2^2 + 2 \| \nabla v^m(t) \|_2^2 = 0 \iff \| v^m(t) \|_2^2 + 2 \int_s^t \| \nabla v^m \|_2^2 d\tau = \| v^m(s) \|_2^2. \]  

We set \(\rho_m(t) := \| \nabla v^m(t) \|_2^2\), and we consider \(p(\alpha, \rho_m(t))\).

Since \(s, t \in \mathcal{T}\), there exists \(\alpha_1\) such that

\[ \max\{ \| \nabla v(s) \|_2^2, \| \nabla v(t) \|_2^2 \} < \tan \alpha^\gamma \frac{\pi}{2}, \text{ for all } \alpha \in (\alpha_1, 1). \]
Hence, by virtue of the convergence in $J^{1,2}(\Omega)$-norm, we claim the existence of $m_0$ such that
\[
\max\{\|\nabla v^m(s)\|^2_2, \|\nabla v^m(t)\|^2_2\} < \tan \alpha \pi \over 2, \quad \text{for all } m \geq m_0 \text{ and } \alpha \in (\alpha_1, 1).
\]
We denote by $J(\alpha^\gamma, m) := \{\tau \in (s, t) : \rho_m(\tau) \in (\tan \alpha^\gamma \over 2, A^m]\}$. If $A_m \leq \tan \alpha^\gamma \over 2$, then $J(\alpha^\gamma, m)$ is an empty set. If $A_m > \tan \alpha^\gamma \over 2$ holds, since $\rho_m(s) < \tan \alpha^\gamma \over 2$, by continuity there exists the minimum $\overline{s} > s$ such that $\rho_m(\overline{s}) = \tan \alpha^\gamma \over 2$, as well, being $\rho_m(t) < \tan \alpha^\gamma \over 2$, there exists the maximum $\overline{t} < t$ such that $\rho_m(\overline{t}) = \tan \alpha^\gamma \over 2$. Thus, if $J(\alpha^\gamma, m)$ is a non-empty set, by the regularity of $\rho_m(t)$, we get that $J(\alpha^\gamma, m)$ is at most the union of a countable family of maximal intervals $(s_h, t_h)$ as indicated in (45).

We set $E^m := (s, t) - \bigcup_{h \in \mathbb{N}} (s_h, t_h)$.

For all $\tau \in E^m$ we have $\rho_m(\tau) \leq \tan \alpha^\gamma \over 2$. Thus, by the continuity of $\rho_m$, we get $\rho_m(s_h) = \tan \alpha^\gamma \over 2 = \rho_m(t_h)$ for all $h \in \mathbb{N}$. For the measure of $J(\alpha^\gamma, m) \subset (s, t)$, we get
\[
|J(\alpha^\gamma, m)| \tan \alpha \pi \over 2 \leq \int_{J(\alpha^\gamma, m)} \rho_m(\tau)d\tau \leq \int_s^t \rho_m d\tau \leq \frac{1}{2} \|v_0\|^2_2 ,
\]
where we took the energy relation (48) into account. Estimate (49) completes (45). Recalling the definition of $p(\alpha^\gamma, \rho_m(t))$, we have
\[
\frac{d}{d\tau} p(\alpha^\gamma, \rho_m(\tau)) = \begin{cases} 0 & \text{a.e. in } \tau \in E^m, \\ -\frac{2}{\pi} \frac{1}{1 - \alpha^\gamma} \frac{\hat{\rho}_m(\tau)}{1 + \hat{\rho}_m^2(\tau)} & \text{for all } \tau \in J(\alpha^\gamma, m), \end{cases}
\]
where took into account that, for all $\alpha \in (0, 1)$, the function $p$ is a Lipschitz’s function in $\rho_m$, and $\rho_m(t) \in C^1([s, t])$. Hence, $p(\alpha^\gamma, \rho_m(t))$ is a Lipschitz’s function in $t$. We multiply equation (48) by $p(\alpha^\gamma, \rho_m(t))$ and we integrate by parts on $(s, t)$:
\[
\frac{2}{\pi} \int_{J(\alpha^\gamma, m)} \frac{e_m}{1 + \rho_m^2(\tau)} \hat{\rho}_m(\tau)d\tau = e_m(s) - e_m(t) - 2 \int_s^t \rho_m p(\alpha^\gamma, \rho_m)d\tau ,
\]
where we set $e_m := \|v_m\|^2_2$. Since the hypotheses of Lemma 3 are satisfied, and since $s, t \in T$, by virtue of (9), letting $m \to \infty$, each term on the right-hand side of (50) admits limit. So that we arrive at
\[
\frac{2}{\pi} \int_{J(\alpha^\gamma, m)} \frac{e_m}{1 + \rho_m^2(\tau)} \hat{\rho}_m(\tau)d\tau = e(s) - e(t) - 2 \int_s^t \rho p(\alpha^\gamma, \rho)d\tau ,
\]
where we set $e := \|v\|_2$ and $\rho := \|\nabla v\|^2_2, v$ weak solution. Thereafter, letting $\alpha \to 1^-$, via (10), we arrive at
\[
\frac{2}{\pi} \lim_{\alpha \to 1^-} \int_{J(\alpha^\gamma, m)} \frac{e_m}{1 + \rho_m^2(\tau)} \hat{\rho}_m d\tau = e(s) - e(t) - 2 \int_s^t \rho d\tau ,
\]
that is equivalent to (46). \qed
Now, keeping the assumption \( A = \infty \) in (44), our task is to prove that the right-hand side of (46) is equal to zero. We achieve the result in three steps.

**Step 1.**

\[
\frac{2}{\pi} \lim_{\alpha \to 1^-} \frac{1}{1 - \alpha^\gamma} \lim_{m \to \infty} \frac{e_m}{1 + \rho_m^2} \dot{\rho}_m d\tau = \lim_{\alpha \to 1^-} \lim_{m \to \infty} \int_{J(\alpha^\gamma, m)} \rho_m d\tau. \tag{52}
\]

**Proof of Step 1.** Recalling (45) related to \( J(\alpha^\gamma, m) \) given in Lemma 9, by means of an integration by parts, we get

\[
\int_{J(\alpha^\gamma, m)} \frac{e_m}{1 + \rho_m^2} \dot{\rho}_m d\tau = \sum_{h \in N(\alpha^\gamma, m)} \sum_{s_h}^{t_h} \frac{e_m(s_h) - e_m(t_h)}{1 + \rho_m^2} \rho_m d\tau + \sum_{h \in N(\alpha^\gamma, m)} \sum_{s_h}^{t_h} \frac{\dot{e}_m \rho_m}{1 + \rho_m^2} d\tau.
\]

Hence, via the energy relation (48), we arrive at

\[
- \sum_{h \in N(\alpha^\gamma, m)} \sum_{s_h}^{t_h} \frac{e_m}{1 + \rho_m^2} \dot{\rho}_m d\tau + 2 \sum_{h \in N(\alpha^\gamma, m)} \sum_{s_h}^{t_h} \frac{e_m \rho_m^2}{1 + \rho_m^2} d\tau - \sum_{h \in N(\alpha^\gamma, m)} \sum_{s_h}^{t_h} \frac{\dot{\rho}_m}{1 + \rho_m^2} d\tau.
\]

This last is equivalent to

\[
\frac{1 + \tan^2 \alpha^\gamma \pi}{\tan \alpha^\gamma \pi} \int_{J(\alpha^\gamma, m)} \frac{e_m}{1 + \rho_m^2} \dot{\rho}_m d\tau + \frac{1 + \tan^2 \alpha^\gamma \pi}{\tan \alpha^\gamma \pi} \int_{J(\alpha^\gamma, m)} \frac{\rho_m^2}{1 + \rho_m^2} d\tau - \frac{1 + \tan^2 \alpha^\gamma \pi}{\tan \alpha^\gamma \pi} \int_{J(\alpha^\gamma, m)} \frac{e_m}{1 + \rho_m^2} \dot{\rho}_m d\tau
\]

Letting \( m \to \infty \), we get

\[
\lim_{m \to \infty} \int_{J(\alpha^\gamma, m)} \rho_m d\tau = A(\alpha) + B(\alpha), \tag{53}
\]

where we set

\[
A(\alpha) := \frac{1 + \tan^2 \alpha^\gamma \pi}{\tan \alpha^\gamma \pi} \lim_{m \to \infty} \int_{J(\alpha^\gamma, m)} \frac{e_m}{1 + \rho_m^2} \dot{\rho}_m d\tau,
\]
and

\[
B(\alpha) := \lim_{m} \left[ \frac{1 + \tan^2 \alpha \gamma \pi}{\tan \alpha \gamma \pi} \int_{J(\alpha, m)} \frac{\rho_m^2}{1 + \rho_m^2} d\tau - \frac{1 + \tan^2 \alpha \gamma \pi}{\tan \alpha \gamma \pi} \int_{J(\alpha, m)} \frac{e_m}{(1 + \rho_m^2)^2} \dot{\rho}_m d\tau \right].
\]  

(54)

The terms \(A(\alpha)\) and \(B(\alpha)\) are both well posed. For \(A(\alpha)\), via (51), we have the existence of the limit. For \(B(\alpha)\) both the terms on the right-hand side of (54) are bounded, uniformly in \(m \in \mathbb{N}\) (the former being \(J(\alpha, m) \subset (s, t)\), the latter via (38)). We recall the following elementary limit property:

\[
\lim_{\alpha \to 1^-} (1 - \alpha) \tan \alpha \gamma \pi = \frac{2}{\pi}.
\]

Hence, we get

\[
\lim_{\alpha \to 1^-} A(\alpha) = \frac{2}{\pi} \lim_{\alpha \to 1^-} \frac{1}{1 - \alpha} \lim_{m} \int_{J(\alpha, m)} \frac{e_m}{1 + \rho_m^2} \dot{\rho}_m d\tau.
\]  

(55)

This limit is well posed by virtue of (46). Moreover, estimate (38) and the lower bound of \(\rho_m\), implicit in (45), allow us to deduce that

\[
\lim_{\alpha \to 1^-} \frac{1 + \tan^2 \alpha \gamma \pi}{\tan \alpha \gamma \pi} \lim_{m} \int_{J(\alpha, m)} \frac{e_m}{(1 + \rho_m^2)^2} \dot{\rho}_m d\tau \leq \lim_{\alpha \to 1^-} \frac{1}{\tan \alpha \gamma \pi} \lim_{m} \int_{J(\alpha, m)} \frac{e_m}{1 + \rho_m^2} \dot{\rho}_m d\tau \leq \frac{1 + \|\bar{v}_0\|^2_2}{\tan(\alpha \gamma \pi)}.
\]

Hence, one deduces that

\[
\lim_{\alpha \to 1^-} \frac{1 + \tan^2 \alpha \gamma \pi}{\tan \alpha \gamma \pi} \lim_{m} \int_{J(\alpha, m)} \frac{e_m}{(1 + \rho_m^2)^2} \dot{\rho}_m d\tau = 0.
\]

These last and the limit property (15) lead to

\[
\lim_{\alpha \to 1^-} B(\alpha) = 0.
\]  

(56)

Estimates (55) and (56), via (53), ensure (52).

\[ \square \]

\textbf{Step 2.}

Keeping \(A = \infty\) in (44), we are going to consider the values \(\gamma := 1\) and \(\gamma = \frac{1}{3}\) in Lemma 9 and in (52).

Via (45), for \(\gamma = 1\), we have

\[
J(\alpha, m) = \bigcup_{h \in \mathbb{N}(\alpha, m)} (s_h, t_h),
\]

where \(\rho_m(s_h) = \rho_m(t_h) = \tan \alpha \frac{\pi}{2}\) and \((s_h, t_h) := \{\tau : \tan \alpha \frac{\pi}{2} < \rho_m\}\).

(57)

with \(\mathbb{N}(\alpha, m)\) countable set, and, for any \(h \neq k \in \mathbb{N}(\alpha, m), (s_h, t_h) \cap (s_k, t_k) = \emptyset\) holds.
Analogously, via (45), for $\gamma = \frac{1}{3}$, we have

$$J(\alpha^{\frac{1}{3}}, m) = \bigcup_{i \in \mathbb{N}(\alpha^{\frac{1}{3}}, m)} (\sigma_i, \tau_i),$$

(58)

where $\rho_m(\sigma_i) = \rho_m(\tau_i) = \tan \alpha^{\frac{1}{3}} \frac{\pi}{2}$ and $(\sigma_i, \tau_i) := \{\tau : \tan \alpha^{\frac{1}{3}} \frac{\pi}{2} < \rho_m\}.$

with $\mathbb{N}(\alpha^{\frac{1}{3}}, m)$ countable set, and, for any $i \neq j \in \mathbb{N}(\alpha^{\frac{1}{3}}, m)$, $(\sigma_i, \tau_i) \cap (\sigma_j, \tau_j) = \emptyset$.

Hence, by virtue of (57) and (58), we have

for all $\alpha \in (0, 1)$ and $m \in \mathbb{N}$, $J(\alpha^{\frac{1}{3}}, m) \subset J(\alpha, m)$,

and $J(\alpha, m) - J(\alpha^{\frac{1}{3}}, m) = \{\tau \in (s, t) : \rho_m(\tau) \in (\tan(\alpha \frac{\pi}{2}), \tan(\alpha^{\frac{1}{3}} \frac{\pi}{2}))\}$.

Finally, thanks to (52), we have

for $\gamma = 1$,

$$\frac{2}{\pi} \lim_{\alpha \to 1^-} \frac{1}{1 - \alpha} \lim_{m \to \infty} \int_{J(\alpha, m)} \frac{e_m}{1 + \rho_m^2} \rho_m d\tau = \lim_{\alpha \to 1^-} \lim_{m \to \infty} \int_{J(\alpha, m)} \rho_m d\tau,$$

(60)

for $\gamma = \frac{1}{3}$,

$$\frac{2}{\pi} \lim_{\alpha \to 1^-} \frac{1}{1 - \alpha} \lim_{m \to \infty} \int_{J(\alpha^{\frac{1}{3}}, m)} \frac{e_m}{1 + \rho_m^2} \rho_m d\tau = \lim_{\alpha \to 1^-} \lim_{m \to \infty} \int_{J(\alpha^{\frac{1}{3}}, m)} \rho_m d\tau.$$

Step 3.

$$\lim_{\alpha \to 1^-} \lim_{m \to \infty} \int_{J(\alpha^{\frac{1}{3}}, m)} \rho_m d\tau = 0.$$

(61)

Proof of Step 3. For all $h \in \mathbb{N}(\alpha, m)$, we evaluate the energy relation on the interval $(s_h, t_h)$:

$$\frac{1}{2} \frac{d}{dt} e_m(t) + \rho_m(t) = 0, \quad t \in (s_h, t_h), \implies e_m(t_h) + 2 \int_{s_h}^{t_h} \rho_m d\tau = e_m(s_h).$$

(62)

After multiplying this last by $\arctan \rho_m$, we integrate by parts on $(s_h, t_h)$. In virtue of the values of $\rho_m$ in the end points of intervals $(s_h, t_h)$, stated in (57), we get

$$\alpha \frac{\pi}{2} \frac{1}{2} [e_m(t_h) - e_m(s_h)] + \int_{s_h}^{t_h} \arctan \rho_m \rho_m d\tau = \frac{1}{2} \int_{s_h}^{t_h} \frac{e_m \rho_m}{1 + \rho_m^2} d\tau,$$

that, via (62), is equivalent to

$$\int_{s_h}^{t_h} \left[ \arctan \rho_m - \alpha \frac{\pi}{2} \right] \rho_m d\tau = \frac{1}{2} \int_{s_h}^{t_h} \frac{e_m \rho_m}{1 + \rho_m^2} d\tau,$$

(63)
that has the left-hand side non-negative, for all \( h \in \mathbb{N}(\alpha, m) \) and \( m \in \mathbb{N} \). We sum (63) on index \( h \in \mathbb{N}(\alpha, m) \). Hence, the following holds:

\[
\int_{J(\alpha,m)} \left[ \arctan \rho_m - \frac{\alpha \pi}{2} \right] \rho_m d\tau = \frac{1}{2} \int_{J(\alpha,m)} \frac{e_m \rho_m}{1 + \rho_m^2} d\tau.
\]

Thereafter, letting \( m \to \infty \), we obtain

\[
\lim_m \int_{J(\alpha,m)} \left[ \arctan \rho_m - \frac{\alpha \pi}{2} \right] \rho_m d\tau = \frac{1}{2} \lim_m \int_{J(\alpha,m)} \frac{e_m \rho_m}{1 + \rho_m^2} d\tau,
\]

which is right, being equivalent to (51), except for the multiplicative factor \( \frac{2}{\pi} \frac{1}{1 - \alpha} \). After multiplying by \( \frac{1}{1 - \alpha} \), letting \( \alpha \to 1^- \), we get

\[
I_1 := \lim_{\alpha \to 1^-} \frac{1}{1 - \alpha} \lim_m \int_{J(\alpha,m)} \left[ \arctan \rho_m - \frac{\alpha \pi}{2} \right] \rho_m d\tau = \frac{1}{2} \lim_{\alpha \to 1^-} \lim_m \int_{J(\alpha,m)} \frac{e_m \rho_m}{1 + \rho_m^2} d\tau,
\]

where we took (52) into account. We look for a lower bound of \( I_1 \). By virtue of (59), we get

\[
\int_{J(\alpha,m)} \left[ \arctan \rho_m - \frac{\alpha \pi}{2} \right] \rho_m d\tau = \int_{J(\alpha,m)} \left[ \arctan \rho_m - \frac{\alpha \pi}{2} \right] \rho_m d\tau + \int_{J(\alpha,m) - J(\alpha, \frac{1}{3}, m)} \left[ \arctan \rho_m - \frac{\alpha \pi}{2} \right] \rho_m d\tau + \int_{J(\alpha, \frac{1}{3}, m)} \left[ \arctan \rho_m - \frac{\alpha \pi}{2} \right] \rho_m d\tau \geq \int_{J(\alpha, \frac{1}{3}, m)} \left[ \arctan \rho_m - \frac{\alpha \pi}{2} \right] \rho_m d\tau \geq \frac{\pi}{2} \cdot \frac{1}{3} (1 - \alpha^2) \lim_m \int_{J(\alpha, \frac{1}{3}, m)} \rho_m d\tau.
\]

Since (64) ensures that the left side admits limit on \( m \), we get

\[
\lim_m \int_{J(\alpha,m)} \left[ \arctan \rho_m - \frac{\alpha \pi}{2} \right] \rho_m d\tau \geq \frac{\pi}{2} \cdot \frac{1}{3} (1 - \alpha^2) \lim_m \int_{J(\alpha, \frac{1}{3}, m)} \rho_m d\tau,
\]

and, trivially,

\[
\frac{1}{1 - \alpha} \lim_m \int_{J(\alpha,m)} \left[ \arctan \rho_m - \frac{\alpha \pi}{2} \right] \rho_m d\tau \geq \frac{\pi}{2} \cdot \frac{1}{3} (1 - \alpha^2) \lim_m \int_{J(\alpha, \frac{1}{3}, m)} \rho_m d\tau,
\]

Via (65) and (60)\(_2\), both the sides of this last inequality admit a limit on \( \alpha \). Hence, we obtain the following lower bound for \( I_1 \):

\[
I_1 \geq \frac{\pi}{3} \lim_{\alpha \to 1^-} \lim_m \int_{J(\alpha, \frac{1}{3}, m)} \rho_m d\tau,
\]

(66)
where in the penultimate step, we took\(\lim_{\alpha \to 1^-} \frac{1-\alpha^2}{1-\alpha} = \frac{2}{3}\) into account.

Now, we estimate the right-hand side of (65)\(\text{2}\). The statements (57)-(59) coincide with items b)-d) of sect.2. Since \{\rho_m\} converges almost everywhere to \rho, and (45)\(\text{2}\) is equivalent to (18), we can apply Lemma 5. Hence, we get

\[
\lim_{\alpha \to 1^-} \lim_{m} \int_{J(\alpha,m)} \rho_m d\tau = \lim_{\alpha \to 1^-} \lim_{m} \left[ \int_{J(\alpha,m)-J(\alpha,\frac{1}{2},m)} \rho_m d\tau + \int_{J(\alpha,\frac{1}{2},m)} \rho_m d\tau \right]
\]

\[
\leq \lim_{\alpha \to 1^-} \lim_{m} \int_{J(\alpha,\frac{1}{2},m)} \rho_m d\tau.
\]

This last estimate and estimate (66), via (65), lead to an absurdum (that is, \(\frac{\pi}{3} \leq \frac{\pi}{4}\)). Hence, we conclude that

\[
\lim_{\alpha \to 1^-} \lim_{m} \int_{J(\alpha,\frac{1}{2},m)} \rho_m d\tau = 0.
\]

\(\square\)

**Proof of Theorem 1.** We start from (44). Assume that \(A < \infty\). By virtue of Lemma 7, we know that \{\|\nabla v^m(\tau)\|_2\} converges to \(\|\nabla v(\tau)\|_2\) a.e. in \(\tau \in (0,T)\). Being \(A < \infty\) on \((s,t)\), for the integral on \((s,t)\), we can apply the dominated convergence theorem to the sequence \{\|\nabla v^m\|_2\}. Since \(s,t \in \mathcal{T}\), we also have the convergence of \{\|v^m(t)\|_2\} and \{\|v^m(t)\|_2\} to \(\|v(t)\|_2\) and \(\|v(s)\|_2\), respectively. Therefore, (4) follows from the energy relation (25) by letting \(m \to \infty\).

Assume that \(A = \infty\) in (44). Then, matching formulas (46), (52) considered with \(\gamma = \frac{1}{3}\), and (61), we prove the energy equality for all \(s,t \in \mathcal{T}\), that is

\[
\|v(t)\|_2^2 + 2 \int_s^t \|\nabla v\|_2^2 d\tau = \|v(s)\|_2^2, \text{ for all } t, s \in \mathcal{T}.
\]

(68)

As we pointed out in (43), a priori, there is the possibility of \(\mathcal{T} \subset I^C\). In particular, there is the possibility that \(0 \notin \mathcal{T}\). Let \(t \in I^C - \mathcal{T} \subseteq (0,\infty) - \mathcal{T}\). Then, by virtue of (41)\(\text{2}\), for all \(t \in I^C\), we have \(\lim_{t \to t^+} \|v(\tau)\|_2 = \|v(t)\|_2\), and the existence of a sequence \(\{t_p\} \subset \mathcal{T}\) such that \(t_p \to t^+\). Let us consider \(0 \text{ and } t \in I^C - \mathcal{T}\). Then from (68), via a suitable \(\sigma \in (0,t) \cap \mathcal{T}\), we get

\[
\|v(t)\|_2^2 - \|v_0\|_2^2 = \lim_{t_p \to t^+} \|v(t_p)\|_2^2 - \lim_{s_p \to 0^+} \|v(s_p)\|_2^2 = -2 \lim_{t_p \to t^+} \int_{s_p}^t \|\nabla v\|_2^2 d\tau - 2 \lim_{s_p \to 0^+} \int_{s_p}^\sigma \|\nabla v\|_2^2 d\tau
\]

\[
= -2 \int_0^t \|\nabla v\|_2^2 d\tau,
\]

which proves the extension of the validity of the energy equality for \(0\) and \(t \in I^C\).
The sequence \( \{\nabla v^m\} \) is now strongly convergent in \( L^2(0,T;L^2(\Omega)) \) for all \( T > 0 \). Actually, considering any \( T \in (\theta, \infty) \), recalling that, for such a \( T \), (40)_3 ensures the strong convergence of \( \{v^m(T)\} \) to \( v(T) \) in \( L^2 \)-norm, letting \( m \to \infty \) in (25) with \( t = T \) and \( s = 0 \), we get

\[
\lim_m \int_0^T \|\nabla v^m\|_2^2 = \|v_0\|_2^2 - \|v(T)\|_2^2 = \int_0^T \|\nabla v\|_2^2 d\tau ,
\]

where in the last equality we employ (68) for \( s = 0 \) and \( t = T \). This last and the weak convergence imply the strong convergence in \( L^2(0,T;J^{1,2}(\Omega)) \) for all \( T > 0 \).

Now, we are going to prove the continuity in the \( L^2 \)-norm of the weak solution \( v \), for all \( t \in I^C \). We already know the right continuity at instant \( t \in I^C \), see (41)_2. Consider \( t > s \) with \( t \in I^C \). By definition of the set stated in (43), we have the strong convergence of \( \{v^m(t,x)\} \) to \( v(t,x) \) in \( L^2(\Omega) \), and, by the previous result, the one of \( \{\nabla v^m\} \) to \( \nabla v \) in \( L^2(s,t;L^2(\Omega)) \). Then, via (25), we get

\[
\|v(t)\|_2 + 2 \int_s^t \|\nabla v\|_2^2 d\tau = \lim_m \|v^m(s)\|_2^2.
\]

Hence, we have

\[
\|v(t) - v(s)\|_2^2 = \|v(t)\|_2^2 + \|v(s)\|_2^2 - 2(v(t),v(s)) \leq \|v(t)\|_2^2 + \lim_m \|v^m(s)\|_2^2 - 2(v(t),v(s))
\]

\[
= 2\|v(t)\|_2^2 + 2 \int_s^t \|\nabla v\|_2^2 d\tau - 2(v(t),v(s)).
\]

Letting \( s \to t^- \), then we deduce at the instant \( t \) the left continuity in the \( L^2 \)-norm of the weak solution \( v \). Hence, we have proved that \( v(t,x) \) is continuous in the \( L^2 \)-norm at instants \( t \) for which (4) holds.

Finally, considering (4) for \( t > \theta \), where \( \theta \) is furnished in Theorem 3, and letting \( t \to \infty \), by virtue of the asymptotic behavior (42), we arrive at (5).

Theorem 1 is completely proved.

**Declaration:**

**Conflict of interests** The author declares that he has no conflict of interest.

**References**

[1] H. Beirão da Veiga and J. Yang, *On the energy equality for solutions to Newtonian and non-Newtonian fluids*, Nonlinear Anal. 185 (2019) 388-402.

[2] L.C. Berselli and E. Chiodaroli, *Remarks on the energy equality for the 3D Navier-Stokes equations*, Waves in flows-the 2018 Prague-Sum Workshop lectures, 91-107, Adv. Math. Fluid Mech., Birkhäuser/Springer, Cham, (2021).
The Navier-Stokes equations: on the existence of a weak solution enjoying the energy equality

[3] T. Buckmaster, C. de Lellis, L. Székelyhidi and V. Vicol, *Onsager’s conjecture for admissible weak solutions*, Comm. Pure Appl. Math. **72** (2019) 229-274.

[4] T. Buckmaster and V. Vicol, *Nonuniqueness of weak solutions to the Navier-Stokes equation*, Ann. of Math. (2) **189** (2019), no. 1, 101–144.

[5] L. Caffarelli, R. Kohn, and L. Nirenberg, *Partial regularity of suitable weak solutions of the Navier-Stokes equations*, Comm. Pure Appl. Math. **35** (1982), no.6, 771-831.

[6] A. Cheskidov, S. Friedlander and R. Shvydkoy, *On the energy equality for weak solutions of the 3D Navier-Stokes equations*, In Rannacher, R., Sequeira, A. (eds.) Advances in Mathematical Fluid Mechanics, pp. 171-175. Springer, Berlin (2010)

[7] P. Constantin, W. E and E.S. Titi, *Onsager’s conjecture on the energy conservation for solutions of Euler’s equation*, Comm. Math. Phys. **165** (1994) 207-209.

[8] F. Crispo, C.R. Grisanti and P. Maremonti, *Some new properties of a suitable weak solution to the Navier-Stokes equations*, in Waves in Flows: The 2018 Prague-Sum Workshop Lectures, series: Lecture Notes in Mathematical Fluids Mechanics, editors: G.P.Galdi, T. Bodnar, S. Necasova, Birkhauser.

[9] F. Crispo, C.R. Grisanti and P. Maremonti, *Navier-Stokes equations: an analysis of a possible gap to achieve the energy equality*, Ricerche di Matematica, **70** (2021) 235-249, https://doi.org/10.1007/s11587-020-00525-5

[10] F. Crispo, C.R. Grisanti and P. Maremonti, *Navier-Stokes equations: a new estimate of a possible gap related to the energy equality of a suitable weak solution*, to appear in Meccanica, ArXive:2204.11359.

[11] T.D. Drivas and G.L. Eyink, *An Onsager singularity theorem for Leray solutions of incompressible Navier-Stokes*, Nonlinearity **32** (2019) 4465-4482.

[12] J. Duchon and R. Robert, *Inertial energy dissipation for weak solutions of incompressible Euler and Navier-Stokes equations*, Nonlinearity **13** (2000) 249-255.

[13] G.F.D. Duff, *Derivative estimates for the Navier-Stokes equations in a three-dimensional region*, Acta Math. **164** (1990) 145-210.

[14] R. Farwig and Y. Taniuchi, *On the energy equality of Navier-Stokes equations in general unbounded domains*, Arch. Math. **95** (2010) 447-456.

[15] C. Foias, C. Guillopé, and R. Temam, *New a priori estimates for Navier-Stokes equations in dimension 3*, Comm. Partial Differential Equations **6** (1981) 329-359.

[16] G.P. Galdi, *An introduction to the mathematical theory of the Navier-Stokes equations*, (I) **38** Springer Tracts in N.P., (1994).

[17] G.P. Galdi and P. Maremonti, *Monotonic decreasing and asymptotic behavior of the kinetic energy for weak solutions of the Navier-Stokes equations in exterior domains*, Arch. Rational Mech. Anal. **94** (1986) 253-266.

[18] Y. Giga and H. Sohr, *Abstract Lp estimates for the Cauchy problem with applications to the Navier-Stokes equations in exterior domains*, J. Funct. Analysis, **102**(1991) 72-94.

[19] J.G. Heywood, *The Navier-Stokes equations: existence, regularity and decay of solutions*, Indiana Univ. Mathem. J., **29** (1980) 639-681.
[20] H. Kozono, A. Okada and S. Shimizu, Characterization of initial data in the homogeneous Besov space for solutions in the Serrin class of the Navier-Stokes equations, J. Funct. Analysis, 278 n.5 (2020), https://doi.org/10.1016/j.jfa.2019.108390

[21] H. Kozono, A. Okada and S. Shimizu, Necessary and sufficient condition on initial data in the Besov space for solutions in the Serrin class of the Navier-Stokes equations, 21 (2021), 3015-3033, https://doi.org/10.1007/s00028-020-00614-w

[22] O.A. Ladyženskaja, The mathematical theory of viscous incompressible flow, Gordon and Breach Sc. Publisher, (1969).

[23] O.A. Ladyženskaja, V.A. Solonnikov and N.N. Ural’ceva, Linear and quasi-linear equations of parabolic type, AMS (1968).

[24] J. Leray, Sur le mouvement d’un liquide visqueux emplissant l’espace, Acta Math. 63 (1934), no. 1, 193–248.

[25] P. Maremonti, On the energy asymptotic decay of viscous incompressible fluids in exterior domains, Proc. of the Workshop on Mathematical Aspects of Fluid and Plasma Dynamics, Univ. degli Studi di Trieste May 30 - June 2, 1984.

[26] P. Maremonti, Asymptotic stability in the mean for viscous fluid motion in exterior domains (Italian), Annali di Mat. pura e applicata, 142 (1985), 57-75.

[27] P. Maremonti, Some interpolation inequalities involving Stokes operator and first order derivatives, Ann. Mat. Pura Appl., 175 (1998), 59–91.

[28] P. Maremonti, A note on Prodi-Serrin conditions for the regularity of a weak solution to the Navier-Stokes equations, J. Math. Fluid Mech. 20 (2018), no. 2, 379–392.

[29] P. Maremonti, On an interpolation inequality involving the Stokes operator, Mathematical analysis in fluid mechanics—selected recent results, Contemp. Math., vol. 710, Amer. Math. Soc., Providence, RI, 2018, pp. 203–209.

[30] J.A. Mauro, Some analytic questions in Mathematical Physics Problem, (2010), https://etd.adm.unipi.it/t/etd-12232009-161531/

[31] T. Miyakawa and H. Sohr, On energy inequality, smoothness and large time behavior in $L^2$ for weak solutions of the Navier-Stokes equations, Math. Z., 199 (1988) 455-478.

[32] L. Onsager, Statistical hydrodynamics, Nuovo Cimento 6 (1949) 279-287.

[33] G. Prodi, Un teorema di unicità per le equazioni di Navier-Stokes, Annali di Mat. Pura e Appl. 48, (1959) 173-182.

[34] V. Scheffer, Turbulence and Hausdorff dimension, in Turbulence and navier-Stokes equations, L.N. in Mathematics 565 (1976), Springer.

[35] V. Scheffer, A solution to the Navier-Stokes inequality with an internal singularity, Comm. Mathem. Physics, 101 (1985) 47-85.

[36] V. Scheffer, An inviscid flow with compact support in space-time, J. Geom. Anal. 3 (1993) 343–401.

[37] D.H. Sattinger, The mathematical problem of hydrodynamic stability, J. Math. and Mechanics, 19 (1970) 797-817.
[38] J. Serrin, *The Initial Value Problem for the Navier-Stokes Equations*, The University Wisconsin Press, Madison (1963).

[39] R. Temam, *Navier-Stokes equations*, North-Holland, 1979.