Non-dissipative decoherence bounds on quantum computation

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We investigate the capabilities of a quantum computer based on cold trapped ions in presence of non-dissipative decoherence. The latter is accounted by using the evolution time as a random variable and then averaging on a properly defined probability distribution. Severe bounds on computational performances are found.

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I. INTRODUCTION

A quantum computer processes quantum information that is stored in quantum bits (qubits) \[ |g\rangle \]. If a small set of fundamental operations, or “universal quantum logic gates” can be performed on the qubits, then a quantum computer can be programmed to solve an arbitrary problem \[ 3 \]. Essentially, the quantum computation can be viewed as a coherent superposition of digital computation proceeding in parallel. The explosion of interest in quantum computation can be traced to Shor’s demonstration that a quantum computer could efficiently factorize large integers \[ 3 \].

Cirac and Zoller \[ 4 \] have made one of the most promising proposal for the implementation of a quantum computer. A number of identical atoms are stored and laser cooled in a linear radio-frequency quadrupole trap to form a quantum register. The radio-frequency trap potential gives strong confinement of the ions in the directions transverse to the trap axis, while an electrostatic potential forces the ions to oscillate in an effective harmonic potential in the axial direction. After laser cooling the ions become localized along the trap axis with a spacing determined by their Coulomb repulsion and the confining axial potential. The normal mode of the ions’ collective oscillations which has the lowest frequency, is the axial center of mass (CM) mode, in which all the trapped ions oscillate together. A qubit is the electronic ground state \[ |g\rangle \] and a long-lived excited state \[ |e\rangle \] of the trapped ions. The electronic configuration of individual ions, and the quantum state of their collective CM vibrations can be manipulated by coherent interactions of the ion with a laser beam, in a standing wave configuration, which can be pointed at any of the ions. The CM mode of the axial vibrations may then be used as a “bus” to implement the quantum logical gates. Once the quantum computation has been completed, the readout is performed through the mechanism of quantum jumps. Several features of this scheme have been demonstrated experimentally, mostly using a single trapped ion \[ 5 \].

The implementation of a large-scale quantum computer is recognized to be a technological challenge of unprecedent proportions. In fact the qubits must be easily manipulated, but must be also well isolated from decohering influence which places considerable limitations on the capabilities of such devices \[ 6 \]. Practically, there are two fundamentally different types of decoherence during a computation: the dissipative one, due to the spontaneous decay of the metastable states \[ |e\rangle \], and the non-dissipative one due to random phase fluctuations of various nature. While the first has been considered carefully \[ 7 \], by also developing strategies to reduce spontaneous decay, like watchdog stabilization \[ 8 \], the second does not received much attention Ref. \[ 9 \].

On the other hand, non-dissipative decoherence seems actually dominant in trapped ion based experiments \[ 10 \], whose results have been recently well explained \[ 11 \], both qualitatively and quantitatively, by simply considering the time as a statistical variable and then averaging over a properly defined probability distribution \[ 12 \]. The developed approach results model-independent, hence quite ductile. Then, our aim here, is to study the performances of an ion-trap quantum information processor by using this approach. Practically, the latter allows us to account for the fluctuations of the Raman laser intensity \[ 13 \]. Instead, in \[ 9 \] a specific model is used, i.e. a random phase drift is added by hand at the end of each qubit rotation while in our case the stochasticity is already included in the time evolution. Moreover, our analysis is based on exact parameters values fitting experimental data \[ 11 \].

The paper is organized as follows. In Section II we shall review a general model for non-dissipative decoherence. In Section III we shall apply this theory to fundamental logic operations. In Section IV we evaluate the limits of a quantum computer due to this type of decoherence. Finally, Section V is for concluding remarks.
II. A GENERAL FORMALISM FOR NON-DISSIPATIVE DECOHERENCE

In this section we review the formalism describing non-dissipative decoherence derived in Refs. [12]. It is based on the idea that time is a random variable or, alternatively, that the system Hamiltonian (therefore its eigenvalues) fluctuates. This leads to random phases in the energy eigenstates representation. Then, the resulting evolution of the system must be averaged on a suitable probability distribution, and this leads to the decay of the off-diagonal elements of the density operator.

Let us consider an initial state \( \rho(0) \) and consider the case of a random evolution time. Then, the evolved state will be averaged over a probability distribution \( P(t, t') \), i.e.

\[
\bar{\rho}(t) = \int_0^\infty dt' \, P(t, t') \, \rho(t') ,
\]

where \( \rho(t') = \exp\{-iLt'\} \rho(0) \) is the usual solution of the Liouville-Von Neumann equation with \( L = \ldots = [H, \ldots]/\hbar. \)

One can write as well

\[
\bar{\rho}(t) = V_L(t) \rho(0) ,
\]

where

\[
V_L(t) = \int_0^\infty dt' \, P(t, t') \, e^{-iLt'} .
\]

In Ref. [12], the function \( P(t, t') \) has been determined to satisfy the following conditions: i) \( \bar{\rho}(t) \) must be a density operator, i.e. it must be self-adjoint, positive-definite, and with unit-trace. This leads to the condition that \( P(t, t') \) must be non-negative and normalized, i.e. a probability density in \( t' \), so that Eq. (3) is a completely positive map; ii) \( V_L(t) \) satisfies the semigroup property \( V_L(t_1 + t_2) = V_L(t_1) V_L(t_2) \), with \( t_1, t_2 \geq 0 \). These requirements are satisfied by

\[
V_L(t) = \frac{1}{(1 + iL \tau)^{t/\tau}} ,
\]

and

\[
P(t, t') = \frac{1}{\tau} \frac{e^{-t' / \tau}}{(t')^{(t/\tau)} - 1} ,
\]

where the parameter \( \tau \) naturally appears as a scaling time. The expression (5) is the so-called \( \Gamma \)-distribution function, well known in line theory. Its interpretation is particularly simple when \( t/\tau = k \) with \( k \) integer; in that case \( P(k, t') \) gives the probability density that the waiting time for \( k \) independent events is \( t' \) and \( \tau \) is the mean time interval between two events. Generally, the meaning of the parameter \( \tau \) can be understood by considering the mean of the evolution time \( \langle t' \rangle = t \), and its variance \( \langle t'^2 \rangle - \langle t' \rangle^2 = \tau t \).

When \( \tau \to 0 \), \( P(t, t') \to \delta(t - t') \) so that \( \bar{\rho}(t) \equiv \rho(t) \) and \( V_L(t) = \exp\{-iLt\} \) is the usual unitary evolution. However, for finite \( \tau \), the evolution operator \( V_L(t) \) describes a decay of the off diagonal matrix elements in the energy representation, whereas the diagonal matrix elements remain constants, i.e. the energy is still a constant of motion. In fact, in the energy eigenbasis, Eqs. (3) and (5) yield

\[
\bar{\rho}_{n,m}(t) = e^{-\gamma_{n,m} t} e^{-\nu_{n,m} t} \rho_{n,m}(0) ,
\]

where

\[
\gamma_{n,m} = \frac{1}{2 \tau} \log \left( 1 + \omega_{n,m}^2 \right) ,
\]

\[
\nu_{n,m} = \frac{1}{\tau} \arctan \left( \omega_{n,m} \tau \right) ,
\]

with \( \hbar \omega_{n,m} = (E_n - E_m) \) the energy difference. One can recognize in Eq. (5), beside the exponential decay, a frequency shift of every oscillating term.

The phase diffusion aspect of the present approach can also be seen in the evolution equation for the averaged density matrix \( \bar{\rho}(t) \). Indeed, by differentiating with respect to time Eq. (3) and using (5) one gets the following master equation for \( \bar{\rho}(t) \)
\[
\frac{d}{dt}\rho(t) = -\frac{1}{\tau} \log (1 + iL\tau) \rho(t).
\]

It is worth noting that by expanding the logarithm up to second order in \(\tau\), one obtains
\[
\frac{d}{dt}\mathcal{P}(t) = -\frac{i}{\hbar} [H, \mathcal{P}(t)] - \frac{\tau}{2\hbar^2} [H, [H, \mathcal{P}(t)]] ,
\]
which is the well known phase-destroying master equation [13]. Hence, Eq. (9) appears as a generalized phase destroying master equation taking into account higher order terms in \(\tau\). Nonetheless, the present approach is different from the usual master equation approach, in the sense that it is model independent, non perturbative and without specific statistical assumptions.

The theory well describes also Hamiltonian fluctuations. Let us consider, for instance, a Hamiltonian \(H(t) = \Omega(t)\hat{H}\), where \(\Omega(t)\) is a fluctuating parameter with mean value \(\Omega\). Whenever \(\hat{H} = H/\Omega\) induces oscillations, we can account for the fluctuations of the frequency \(\Omega(t)\) by writing [11]
\[
\rho(t) = \exp \left\{ -i\hat{L}A(t) \right\} \rho(0), \quad \Rightarrow \quad \mathcal{P}(t) = \int_0^\infty dA P(t, A)e^{-i\hat{L}A} \rho(0) ,
\]
where \(\hat{L} = [\hat{H}, \ldots]/\hbar\), and \(A(t) = \int_0^t d\xi \Omega(\xi)\) is a positive dimensionless random variable. Furthermore,
\[
P(t, A) = \frac{1}{\Omega^2(t/\tau)} \frac{e^{-A/\Omega\tau}}{\Gamma(t/\tau)} \left( \frac{A}{\Omega(t)} \right)^{(t/\tau)-1} .
\]
The first two moments of \(P(t, A)\) determine the properties of the fluctuating frequency \(\Omega(t)\),
\[
\langle A \rangle = \Omega t , \quad \sigma^2(A) = \langle A^2 \rangle - \langle A \rangle^2 = \Omega^2 t\tau ,
\]
that is, the frequency \(\Omega(t)\) is a white, non-gaussian (due to the non-gaussian form of \(P(t, A)\)) stochastic process. In fact, the semigroup assumption made, implies a Markovian treatment in which the spectrum of the fluctuations is flat in the relevant frequency range. This in particular implies that we are neglecting the dynamics at small times, of the order of the correlation time of the fluctuations. The non-gaussian character of \(P(t, A)\) can be traced back to the fact that \(P(t, A)\) must be definite and normalized in the interval \(0 < A < \infty\) and not in \(-\infty < A < \infty\). This is another fundamental difference with respect to Ref. [5], where a gaussian character for the random phase drift errors is assumed, while we make no a priori statistical assumptions. Indeed, the properties of the probability distribution \(P\) are derived only from the semigroup condition, and it is interesting to note that this condition yields a gaussian probability distribution as a limiting case. In fact, from Eq. (10) one can see that \(P(t, t')\) tends to become a gaussian in the large time limit \(t \gg \tau\).

In the case of Rabi oscillations, \(A\) becomes proportional to the pulse area, while \(\tau\) gives an estimate of the pulse area fluctuations, since it corresponds to a fractional error of the pulse area \(\sqrt{\sigma^2(A)/\langle A \rangle} = \sqrt{\tau/t}\).

### III. CHARACTERIZATION OF A QUANTUM PROCESS

Quantum computation ideally corresponds to a physical process \(|\Psi_{\text{out}}\rangle = U|\Psi_{\text{in}}\rangle\) where a given input state is mapped to an output state by a unitary transformation \(U\). It has been shown that any quantum computation can be decomposed into one-bit gates and a universal two-bit gate which involves entanglement operation on two qubits [15].

In order to characterize a physical process in a quantum system, we follow Ref. [14]. The aim is to characterize the process given as a “black box”, by a sequence of measurements in such a way that it is possible to predict what the output state will be for any input state.

We assume that the system is initially prepared in a pure state
\[
|\Psi_{\text{in}}\rangle = \sum_{i=0}^N c_i |i\rangle , \quad \Rightarrow \quad \rho_{\text{in}} = |\Psi_{\text{in}}\rangle \langle \Psi_{\text{in}}| ,
\]
where \(|0\rangle, |1\rangle, \ldots, |N\rangle\) are orthogonal states spanning the Hilbert space of allowed input states with dimension \(N+1\). Then, in the ideal case, the output state can be written as
\[ |\Psi_{\text{out}}\rangle\langle \Psi_{\text{out}}| \equiv \rho_{\text{out}} = \sum_{i,i'=0}^{N} c_i c_{i'}^* R_{i',i} , \]  

where

\[ R_{i',i} = U |i\rangle \langle i'| U^\dagger , \]  

are system operators not depending on the input state.

Whenever the unitary transformation is given by the time evolution operator \( U(t) \), the initial state \( \rho(0) \) represents the input state while the evolved state \( \rho(t) \) represents the output one. Since in Section II \( \rho(t) \) is replaced by \( \overline{\rho}(t) \), here we have to replace \( R_{i',i}(t) \) by

\[ \overline{R}_{i',i}(t) = \int_{0}^{\infty} dt' P(t,t') U(t') |i\rangle \langle i'| U^\dagger (t') . \]  

Now, in order to see to what extent the real physical process approaches the ideal one, we use as parameter the fidelity

\[ F(t) = \text{Tr} \{ \rho(t) \overline{\rho}(t) \}_{\text{ave}} , \]  

where the subscript ”ave” indicates the average overall possible input states. Obviously a gate fidelity close to one indicates that the gate was carried out almost ideally.

In what follows we will see that the fidelity \( F(t) \) can be expressed in terms of the tensor

\[ F_{i',i}^{i,j} \equiv \langle j'| U^\dagger \overline{R}_{i',i} U |j\rangle . \]  

A. One-bit gate

In the case of one bit \( N = 1 \), and the vector basis is given by \( \{|i=0\rangle \equiv |g\rangle , |i=1\rangle \equiv |e\rangle \} \). The individual rotation acting on a single ion can be performed using a laser frequency on resonance with the internal transition and with the equilibrium position of the ion coinciding with the antinode of the laser standing wave. The Hamiltonian describing such a interaction is

\[ H = \frac{\Omega}{2} [ |e\rangle \langle g| e^{-i\phi} + |g\rangle \langle e| e^{i\phi} ] , \]  

where \( \Omega \) is the Rabi frequency and \( \phi \) the laser phase. Eq.(20) leads to the following evolution

\[ U(t) |g\rangle = \cos [ \Omega t/2 ] |g\rangle - ie^{i\phi} \sin [ \Omega t/2 ] |e\rangle , \]  

\[ U(t) |e\rangle = \cos [ \Omega t/2 ] |e\rangle - ie^{-i\phi} \sin [ \Omega t/2 ] |g\rangle , \]  

which effectively corresponds to a single qubit rotation. It is clear from the above evolution equations that fluctuations in the Rabi frequency can be accounted by simply replacing \( \Omega t \) with \( A(t) = \int_{0}^{t} d\xi \frac{\Omega(\xi)}{\Omega} \) (where now \( \Omega \) indicates the mean Rabi frequency) accordingly to the arguments of Section II. Then, similarly to Eq.(17), it is possible to evaluate

\[ \overline{R}_{i',i}(t) = \int_{0}^{\infty} dA P(t,A) U(A) |i\rangle \langle i'| U^\dagger (A) , \]  

and the tensor elements (19) (see Appendix I). In terms of these elements the fidelity results

\[ F = \frac{3}{8} \sum_{i=0}^{1} F_{i,i}^{i,i} + \frac{1}{8} \sum_{i \neq j=0}^{1} ( F_{j,j}^{i,j} + F_{i,j}^{j,i} ) , \]  

In this case the fidelity depends on time which practically determines the amount of rotation, however, in the next Section we shall consider a \( \pi \) rotation, i.e. \( t = \pi/\Omega \).
B. Two-bit gate

In the case of two-bit $N = 3$, and the vector basis is given by \{ $|i = 0 \rangle \equiv |g_1g_2 \rangle$, $|i = 1 \rangle \equiv |g_1e_2 \rangle$, $|i = 2 \rangle \equiv |e_1g_2 \rangle$, $|i = 3 \rangle \equiv |e_1e_2 \rangle$ \}, where labels 1, 2 indicate the two bits. Beside that also the vibrational ground state $|0 \rangle$ is employed.

In this case the laser frequency is chosen tuned to the first motional sideband and the equilibrium position of the ion coincides with the node of the laser standing wave. The Hamiltonian describing such interaction is \cite{4}

$$H_{n,q} = \begin{cases} \frac{\Omega'}{2} \left[ |e \rangle_n \langle g| a e^{-i\phi} + |g \rangle_n \langle e| a^\dagger e^{i\phi} \right], & q = 0, \\ \frac{\Omega'}{2} \left[ |e' \rangle_n \langle g| a e^{-i\phi} + |g \rangle_n \langle e'| a^\dagger e^{i\phi} \right], & q = 1, \end{cases} \quad \Omega' = \frac{n\Omega}{\sqrt{N_a}}. \quad (25)$$

Here, $a^\dagger$ and $a$ are the creation and annihilation operators of the CM phonons, $\Omega$ is the Rabi frequency, $\phi$ is the laser phase, and $n$ is the Lamb-Dicke parameter. The index $q = 0, 1$ refers to the transitions excited by the laser, $|g \rangle \leftrightarrow |e \rangle$ or $|g \rangle \leftrightarrow |e' \rangle$, which depend on the laser polarization. Instead the index $n$ refers to the $n$-th ion on the trap (1, 2 in our case). Moreover, the factor $\sqrt{N_a}$, where $N_a$ indicates the number of trapped ions, appears as a consequence of the Mössbauer effect \cite{3}.

The universal two-bit gate, defined by \cite{212}

$$|\epsilon_1 \rangle_1 |\epsilon_2 \rangle_2 \rightarrow (-1)^{\epsilon_1\epsilon_2'} |\epsilon_1 \rangle_1 |\epsilon_2 \rangle_2, \quad (\epsilon_{1,2} = 0,1) \quad (26)$$

can be realized in three steps by means of Eq.(25):

- A $\pi$ laser pulse with polarization $q = 0$ and $\phi = 0$ excites e.g. the first ion. The evolution will be

$$U_I(t_1) = \exp \left[ -i \frac{\Omega'}{2} t_1 \left( |e \rangle_1 \langle g| a + |g \rangle_1 \langle e| a^\dagger \right) \right], \quad (27)$$

with $t_1 = \pi / \Omega'$.

- The laser directed on the second ion is then turned on for a time of a $2\pi$ pulse with polarization $q = 1$ and $\phi = 0$. The evolution will be

$$U_{II}(t_2) = \exp \left[ -i \frac{\Omega'}{2} t_2 \left( |e' \rangle_2 \langle g| a + |g \rangle_2 \langle e'| a^\dagger \right) \right], \quad (28)$$

with $t_2 = 2\pi / \Omega'$.

- A $\pi$ laser pulse with polarization $q = 0$ and $\phi = 0$ excites again the first ion. The evolution will be

$$U_{III}(t_3) = \exp \left[ -i \frac{\Omega'}{2} t_3 \left( |e \rangle_1 \langle g| a + |g \rangle_1 \langle e| a^\dagger \right) \right], \quad (29)$$

with $t_3 = \pi / \Omega'$.

Also in this case to account for the fluctuations in the Rabi frequency, we can introduce a stochastic variable $A'(t) = \int_0^t d\xi \Omega'(\xi) = \int_0^t d\xi \Omega(\xi)\eta / \sqrt{N_a}$ (where now $\Omega'$ is related to the mean Rabi frequency $\Omega$ as in (23)). Then, we have

$$\mathcal{R}_{\epsilon_1}' = \int_0^\infty dA_3' \int_0^\infty dA_2' \int_0^\infty dA_1' \int_0^\infty P(t_3, A_3')P(t_2, A_2')P(t_1, A_1') \times U_{III}(A_3')U_{II}(A_2')U_I(A_1') |i\rangle \langle i' | U_I^\dagger(A_1') U_{II}^\dagger(A_2') U_{III}^\dagger(A_3'), \quad (30)$$

which are used to calculate the tensor elements \cite{11} (see Appendix II). Finally, the fidelity results

$$\mathcal{F} = \frac{1}{8} \sum_{i=0}^3 F_{ii} + \frac{1}{24} \sum_{i \neq j=0}^3 \left( F_{ij}^2 + F_{ji}^2 \right). \quad (31)$$

In this case $\mathcal{F}$ does not depend on time since the gate operation is realized with the above definite steps.
IV. BOUNDS FOR QUANTUM COMPUTATION

If one accounts the recent breakthroughs in the real of fault tolerant quantum computation [17], an arbitrarily large quantum computations can be performed accurately provided to have an high degree of accuracy on the single gate operation. This sets an obvious figure of merit for quantum computation technology, namely, the expected probability of error in one quantum gate, which should be of the order of $10^{-6}$ [8].

In Fig.1 we show the deviation from the perfect gate fidelity as a function of the fractional error of the pulse area. In particular the lower straight line concerns the one-bit gate for a π rotation (24), and the upper straight line the universal two-bit gate (31). It is known [11] that the value of $τ \approx 10^{-8}$ s gives the best fit for the experimental data of Ref. [10], i.e. $Ω \approx 10^6$ s$^{-1}$, $η \approx 10^{-1}$. It means to have $Ωτ \approx 10^{-3}$ whose corresponding value of accuracy, i.e. $1 - F_τ$, is quite far from the desired one, i.e. $10^{-6}$. In the case of universal two-bit gate, we have a better value of accuracy (upper straight line) already for $Ωτ$ slightly less than $Ωτ$. It turns out that the main limitations rely on the single bit rotation instead on universal two-bit gate. This fact, though counterintuitive, can be easily understood if one consider the expression (33) for the two cases. It results $σ^2(A)/σ^2(A') = \sqrt{N_a}/η > 1$ for π-pulses. This means that the coupling of the qubits with the vibrational degree of freedom makes the two-bit gate less affected by the noise with respect to the one-bit gate. On the other hand, the latter is built up with on resonance pulse, hence it is more sensitive to noisy effects, differently from the two-bit gate where off resonant pulses are used. However, since actually $η/\sqrt{N_a} \approx 10^{-1}$, also in the two-bit gate the accuracy falls very far from the desirable value.

Based on these results, we can state that quantum information processing on a large scale is unrealistic. As a matter of fact, the fault tolerant quantum computation requires a value $Ωτ \approx 10^{-6}$, i.e. $τ \approx 10^{-11}$ s for $Ω \approx 10^5$ s$^{-1}$. Since the finite value of $τ$ is related to the Rabi frequency fluctuations (or in turn to laser intensity fluctuations) [11], this means to improve the laser stability by a factor $10^2$ at least!

It is also to remark that, within the presented non-dissipative decoherence theory, the linear behavior of the gate fidelity (Fig.1) is typical of the limit $Ωτ \ll 1$.

A rough estimation of the capabilities of a quantum computer in the presence of non-dissipative decoherence can be also made with the following arguments. Let us consider the optimistic case of a single run of the Shor’s algorithm, and the use of the universal two-bit gate operations. Then, the non-dissipative decoherence theory [12] shows that in the limit of $Ωτ \ll 1$ the decay rate can be written as $γ = 2Ωτ^2 (see e.g. Eq.(7))$. Let us now suppose to factorize a $L$-bit number. Then, the number of trapped ions $N_a$ should be at least $5L^3$ [7]. The time needed for a single run of the Shor’s algorithm is given by the time required for an elementary logical operation $4π\sqrt{5L}/ηΩ$ multiplied by the required number of elementary operations, approximately $(10L)^3$ [1]. Of course this product should be much less then the decoherence time $γ^{-1}$. Nevertheless, by using the actual experimental values for parameters (and the corresponding value of $τ \approx 10^{-8}$), it results clear that the factorization of even a four-bit number results impossible.

V. CONCLUSION

In conclusion we have studied the limitations imposed by non-dissipative decoherence on quantum computation. Practically, we have seen that non-dissipative decoherence actually constitute a serious impediment to realize quantum computer beside dissipative decoherence. We have used a model able to accurately describe the decoherence phenomena on ion trap based experiments caused by the fluctuations of classical quantities. We have shown that large scale computation seems impossible with the present proposals. Our results indicate that even a computationally modest goal will be extremely challenging experimentally [19].

Although the conclusions of this paper are rather pessimistic with regard to the practical application of quantum computers for actual computation, there are applications requiring much fewer operations which are worth considering [13].

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APPENDIX I

In this Appendix we explicitly calculate the tensor elements \([49]\) for one-bit gate which are useful to calculate the gate fidelity. We make use of Eqs.\((7)\), \((9)\) and \((2)\), \((2)\) obtaining

\[
F_{00}^0 = F_{11}^1 = F_{01}^0 = F_{10}^1 = \int_0^\infty dA \, P(t, A) \cos^2[(A - \Omega t)/2]
\]

\[
= \frac{1}{2(1 + \Omega^2 \tau^2)\tau^2} \left\{ (1 + \Omega^2 \tau^2)^{t/2\tau} \cos(\Omega t) \cos \left( \frac{t \arctan(\Omega \tau)}{\tau} \right) + \sin(\Omega t) \sin \left( \frac{t \arctan(\Omega \tau)}{\tau} \right) \right\},
\]

\[
F_{10}^0 = F_{01}^1 = \left( 1 - F_{00}^0 \right).
\]

The obtained results are valid in any regime even if we are interested in \(\Omega \tau \ll 1\) and \(t = \pi/\Omega\).

APPENDIX II

In this Appendix we explicitly calculate the tensor elements \([49]\) for two-bit gate which are useful to calculate the gate fidelity. Let us first examine the effects of evolution operators \((27)\), \((28)\), \((29)\) on the two-bit vector basis. For \(i = 0\), we have

\[
U_{1ii}U_{1i}U_i(g_1 | g_2) = | g_1 | g_2).
\]

For \(i = 1\), we have

\[
U_{1ii}U_{1i}U_i(g_1 | e_2) = | g_1 | e_2).
\]

For \(i = 2\), we have

\[
U_{1ii}U_{1i}U_i(e_1 | g_2) = \cos(\Omega t_1/2) \{ \cos(\Omega t_2/2) | e_1 | g_2) - i \sin(\Omega t_3/2) | g_1 | g_2) \}
\]

\[
- i \sin(\Omega t_1/2) \cos(\Omega t_2/2) \{ \cos(\Omega t_3/2) | g_1 | g_2) - i \sin(\Omega t_3/2) | e_1 | g_2) \}
\]

\[
- \sin(\Omega t_1/2) \sin(\Omega t_2/2) | g_1 | e_2).
\]

For \(i = 3\), we have

\[
U_{1ii}U_{1i}U_i(e_1 | e_2) = \cos(\Omega t_1/2) \{ \cos(\Omega t_2/2) | e_1 | e_2) - i \sin(\Omega t_3/2) | g_1 | e_2) \}
\]

\[
- i \sin(\Omega t_1/2) \cos(\Omega t_2/2) \{ \cos(\Omega t_3/2) | g_1 | e_2) - i \sin(\Omega t_3/2) | e_1 | e_2) \}.
\]

Then, by using the above results in Eqs.\((7)\) and \((19)\), we get

\[
F_{00}^0 = F_{11}^1 = F_{01}^0 = F_{10}^1 = 1,
\]

\[
F_{22}^2 = [C_2(\pi/\Omega')]^2 + [S_2(\pi/\Omega')]^2 C_2(2\pi/\Omega') - 2 \left[ Z(\pi/\Omega') \right]^2 C_1(2\pi/\Omega'),
\]

\[
F_{33}^3 = [C_2(\pi/\Omega')]^2 + [S_2(\pi/\Omega')]^2 - 2 \left[ Z(\pi/\Omega') \right]^2 C_1(2\pi/\Omega'),
\]

\[
F_{02}^0 = F_{20}^2 = F_{21}^1 = [C_1(\pi/\Omega')]^2 + [S_1(\pi/\Omega')]^2 C_1(2\pi/\Omega'),
\]

\[
F_{03}^0 = F_{30}^3 = F_{31}^1 = - [C_1(\pi/\Omega')]^2 + [S_1(\pi/\Omega')]^2 C_1(2\pi/\Omega'),
\]

\[
F_{22}^2 = [C_2(\pi/\Omega')]^2 + [S_2(\pi/\Omega')]^2 C_1(2\pi/\Omega') + \left[ Z(\pi/\Omega') \right]^2 C_1(2\pi/\Omega'),
\]

\[
F_{01}^0 = F_{22}^0 = F_{30}^3 = F_{11}^1 = F_{41}^{11} = F_{23}^{11} = F_{43}^{11} = F_{33}^3 = F_{23}^3 = 0,
\]

where

\[
C_2(t) = \int_0^\infty dA' \, P(t, A') \cos^2[A'/2] = \frac{1}{2(1 + \Omega^2 \tau^2)^{t/2\tau}} \left\{ (1 + \Omega^2 \tau^2)^{t/2\tau} \cos \left( \frac{t \arctan(\Omega \tau)}{\tau} \right) \right\},
\]

\[
Z(t) = \int_0^\infty dA' \, P(t, A') \sin[A'/2] \cos[A'/2] = \frac{1}{2(1 + \Omega^2 \tau^2)^{t/2\tau}} \left\{ \sin \left( \frac{t \arctan(\Omega \tau)}{\tau} \right) \right\},
\]

\[
S_2(t) = \int_0^\infty dA' \, P(t, A') \sin^2[A'/2] = 1 - C_2(t),
\]

\[
C_1(t) = \int_0^\infty dA' \, P(t, A') \cos[A'/2] = \frac{1}{(1 + \Omega^2 \tau^2)^{t/2\tau}} \left\{ \cos \left( \frac{t \arctan(\Omega \tau/2)}{\tau} \right) \right\},
\]

\[
S_1(t) = \int_0^\infty dA' \, P(t, A') \sin[A'/2] = \frac{1}{(1 + \Omega^2 \tau^2)^{t/2\tau}} \left\{ \sin \left( \frac{t \arctan(\Omega \tau/2)}{\tau} \right) \right\}.
\]
The obtained results are valid in any regime even if we are interested in $\Omega' \tau \ll 1$.

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\[\Omega' \tau \ll 1\]

**FIG. 1.** The quantity $1 - F$ vs the fractional error of the pulse area is shown in a log-log plot. The lower straight line concerns a one-bit gate (after a $\pi$ rotation). The upper straight line concerns the universal two-bit gate.