Physical Acceptability of Isolated, Static, Spherically Symmetric, Perfect Fluid Solutions of Einstein’s Equations.

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Abstract

We ask the following question: Of the exact solutions to Einstein’s equations extant in the literature, how many could represent the field associated with an isolated static spherically symmetric perfect fluid source? The candidate solutions were subjected to the following elementary tests: i) isotropy of the pressure, ii) regularity at the origin, iii) positive definiteness of the energy density and pressure at the origin, iv) vanishing of the pressure at some finite radius, v) monotonic decrease of the energy density and pressure with increasing radius, and vi) subluminal sound speed. A total of 127 candidate solutions were found. Only 16 of these passed all the tests. Of these 16, only 9 have a sound speed which monotonically decreases with radius. The analysis was facilitated by use of the computer algebra system GRTensorII.

1 Introduction

There are a fair number of static, spherically symmetric exact solutions of Einstein’s equations which have been written down in closed form. (A detailed discussion of methods for solving Einstein’s equations in this context in both curvature and isotropic coordinate systems can be found in Kuchowicz [1]. For a partial list of solutions see Kramer et al. [2], Finch [3] and Finch and Skea [4].) In this paper we explore the question as to how many of these solutions satisfy the most elementary criteria for physical acceptability. Here we consider only those solutions which can be considered isolated in the sense that the boundary (where the pressure vanishes) occurs at a finite radius. This is equivalent to demanding that the solution match onto the exterior Schwarzschild solution [5] as it can be shown that a necessary and sufficient condition for matching in the present case is that the pressure equal zero at a finite radius. Cosmological solutions, where the pressure does not vanish or vanishes only as \( r \to \infty \), are not considered [3]. In addition, solutions giving only partial metrics or inexact metric functions are considered incomplete as the complete spacetime cannot be examined. Finally we only consider solutions described by a single metric and have excluded from our search all interior solutions derived from joining two or more exact solutions together. The elementary criteria for physical acceptability that we have used are as follows:

1. Since all the solutions found purport to represent a perfect fluid, we start by verifying the isotropy of the pressure. In notation as explained in Appendix A, \( G^r_r = G^\theta_\theta \), \( G^a_b \) the Einstein tensor. The symmetry of course guarantees that \( G^0_\theta = G^\theta_0 \).

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\[ \text{2The review by Finch and Skea is closely related to the present paper in that it considers essentially the same problem. Almost all of the present paper was completed before we learned of this work. The present contribution extends this work to verify the physical conditions for a wider class of solutions. It is a pleasure to thank Malcolm MacCallum who first pointed out this work to us.} \]

\[ \text{3The cosmological solutions of Einstein and de Sitter are included only as they will often be referenced in the analysis. A few other solutions were found to be cosmological after analysis and are included in the tables. Solutions which were originally stated to be cosmological are not included.} \]
2. Since the solutions must integrate from a regular origin, we check the regularity of the scalars polynomial in the Riemann tensor. Appendix A contains the conditions for regularity.

3. We require that both the pressure \( (p) \) and energy density \( (\rho) \) be positive definite at the origin. Appendix B summarizes the standard equilibrium conditions.

4. To be isolated we require that the pressure reduce to zero at some finite boundary radius \( r_b > 0 \).

5. We require that both the pressure and energy density be monotonically decreasing to the boundary.

6. We require a subluminal sound speed \( (c_s^2 = \frac{dp}{d\rho} < 1) \). Whereas at very high densities the adiabatic sound speed may not equal the actual propagation speed of the signal, we do not distinguish between these cases in this paper.

It was found that all solutions that were studied which satisfied criteria one through three also satisfied the dominant energy condition \( (p/\rho < 1 \text{ for all } r < r_b \text{ here}) \).

2 Metrics

The following tables give the name and reference of the solution studied, along with the corresponding metric. We have named the solutions with the authors name (with a number to differentiate between multiple publications by one author), followed by a solution number. If an author rederived a previously known solution, we have given that solution a number, but not included it in the tables. (For example, Kuchowicz wrote a series of five papers on static, spherically symmetric solutions. The first solution in his first paper in the series is identical to Tolman VI. The second solution of that paper is derived in the same manner, but is apparently a new solution. We named the first solution of this paper Kuch1 Ia, but did not include it in the tables, while Kuch1 Ib is found below.) The metric names are repeated with the references and are presented in chronological order.

Appendix C contains a brief list of rediscoveries which were not recognized by the authors. Rediscoveries such as Kuch1 Ia=Tolman VI which were properly acknowledged are not included in this Appendix. We make no claim that Appendix C is comprehensive.

All but six of the spacetimes that we have examined turn out to be in either isotropic or curvature coordinates (as explained in Appendix A). The choice of coordinates can be read immediately from the following tables: If the coefficient of \( d\Omega^2 \) is \( r^2 \) then the coordinates are curvature coordinates.

It is worth noting that our search was not restricted by choice of coordinates.

Finally, note that the style in which some of the solutions are presented has not been optimized, but rather for the most part reflects the forms presented in the original papers.

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4 Buch3 and R-R III-VIII all of which fail the tests.
| name [ref.]   | metric                                                                 |
|-------------|------------------------------------------------------------------------|
| Schw Int. 5 | \(-\left( A - B \sqrt{1 - \frac{\rho_2}{\rho_r}} \right)^2 dt^2 + \left( 1 - \frac{\rho_2}{\rho_r} \right)^{-1} dr^2 + r^2 d\Omega^2 \) |
| Einstein 6  | \(-c^2 dt^2 + \left( 1 - \frac{\rho_2}{\rho_r} \right)^{-1} dr^2 + r^2 d\Omega^2 \) |
| de Sitter 7 | \(- \left( 1 - \frac{\rho_2}{\rho_r} \right) dt^2 + \left( 1 - \frac{\rho_2}{\rho_r} \right)^{-1} dr^2 + r^2 d\Omega^2 \) |
| Kottler 8   | \(-c^2 \left( 1 - \frac{2m}{r} - \frac{\rho_2}{\rho_r} \right) dt^2 + \left( 1 - \frac{2m}{r} - \frac{\rho_2}{\rho_r} \right)^{-1} dr^2 + r^2 d\Omega^2 \) |
| Tolman IV 9 | \(-B^2 \left( 1 + \frac{\rho_2}{\rho_r} \right) dt^2 + \frac{1 + 2n}{\rho_r} \left( 1 + \frac{1}{\rho_r} \right)^2 dr^2 + r^2 d\Omega^2 \) |
| Tolman V 10 | \(-B^2 r^{2n} dt^2 + \frac{1 + 2n - n^2}{\rho_r} \left( 1 + \frac{1}{\rho_r} \right)^2 dr^2 + r^2 d\Omega^2 \) |
| Tolman VI 11 | \(-(Ar^{1/n} - Br^{1/n})^2 dt^2 + (2 - n^2) dr^2 + r^2 d\Omega^2 \) |
| Tolman VII 12 | \(-B^2 \sin^2 \left( \frac{1}{\sqrt{2}} \frac{\sqrt{1 + \frac{1}{2\rho}} + \frac{2\sqrt{2}}{\rho} - \frac{\rho_2}{\rho_r} \rho_2}{4} \right)^{-1} dt^2 + \left( 1 - \frac{\rho_2}{\rho_r} + \frac{4\rho_2}{\rho_r} \right)^{-1} dr^2 + r^2 d\Omega^2 \) |
| Tolman VIII 13 | \(-B^2 r^{2n} \left( \frac{2}{(a-b)(a+2b-1)} - \frac{2m}{\rho_r} \right) a^{a+2b-1} - \left( \frac{r}{\rho_r} \right)^{a-b} \right)^{-1} dt^2 + \left( \frac{2}{(a-b)(a+2b-1)} - \frac{2m}{\rho_r} \right) a^{a+2b-1} - \left( \frac{r}{\rho_r} \right)^{a-b} \right)^{-1} dr^2 + r^2 d\Omega^2 \) |
| N-P-V Ia 14 | \(\sqrt{2} < n \leq 2 - \left( ar^{1/x} + br^{1/x} \right)^2 \left( Ar^{1/x} + Br^{1/x} \right)^{-2} dt^2 + \left( Ar^{1/x} + Br^{1/x} \right)^{-2} \left( dr^2 + r^2 d\Omega^2 \right), \) |
| N-P-V Ib 15 | \(n = \sqrt{2} - \left( ar^{1/x} + br^{1/x} \right)^2 \left( Ar^{1/x} + Br^{1/x} \right)^{-2} \left( dr^2 + r^2 d\Omega^2 \right) \) |
| N-P-V Ic 16 | \(0 < n < \sqrt{2} - \left( ar^{1/x} + br^{1/x} \right)^2 \left( Ar^{1/x} + Br^{1/x} \right)^{-2} \left( dr^2 + r^2 d\Omega^2 \right) \) |

5 Originally given in the form \(-c^2 \left( \frac{\rho_2}{\rho_r} \right)^2 dt^2 + \frac{1}{2\rho_r} \left[ d\chi^2 + \sin^2 \chi d\theta^2 + \sin^2 \chi \sin^2 \theta d\phi^2 \right]\) where \(\sin^2 \chi = \frac{\rho_2}{\rho_r}\). Throughout this paper we use the notation \(d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2\).

6 Originally given in the form of \(g_{tt} = -1, g_{ab} = \left( \delta_{ab} + \frac{\rho_{ab}}{R^2} \sum \frac{1}{r^2} \right)\), \(a, b \neq t\) and is a cosmological solution with the cosmological constant not zero for positive pressure and density.

7 Originally given in the form of \(g_{\mu\nu} = - \left( \delta_{\mu\nu} + \frac{\rho_{\mu\nu}}{R^2} \sum \frac{1}{r^2} \right)\), and is also a cosmological solution with the cosmological constant not zero.

8 Otherwise known as the Schwarzschild-de Sitter solution.
\begin{align*}
\text{name [ref.]} & \quad \text{metric} \\
\text{N-P-V Ila}^{\text{[13, 14]}} & \quad - \left( \frac{1 - \alpha}{1 + \alpha} \right)^{\frac{k}{2}} \left( \frac{n}{\pi} \right)^{2 - 2n + k} \left[ (c + d) - (c - d) \left( \frac{n}{\pi} \right)^{2n} \right]^{2} dt^{2} + \\
0 \geq k > -2 + \sqrt{2} & \quad (1 + \alpha)^{1} \left( \frac{\pi}{2} \right)^{k} \left( dr^{2} + \frac{r^{2}}{2} d\Omega^{2} \right), \quad \text{where } c = 2 + 3k + \frac{3}{2}k^{2}, d = n(k + 2), \quad n = \sqrt{1 + 2k + \frac{1}{2}k^{2}} \text{ and } \alpha = \frac{k}{k + 1} = \frac{m}{2n} \\
\text{N-P-V Iib}^{\text{[13]}} & \quad - \left( \frac{1 - \alpha}{1 + \alpha} \right)^{\frac{k}{2}} \left( \frac{n}{\pi} \right)^{2} \left[ 1 + \frac{\alpha - \ln r}{\sqrt{2} \pi} \right]^{2} dt^{2} + \\
k = -2 + \sqrt{2} & \quad (1 + \alpha)^{1} \left( \frac{\pi}{2} \right)^{2 - 2\sqrt{2}} \left( dr^{2} + r^{2} d\Omega^{2} \right), \quad \text{where } \alpha = \frac{2 - \sqrt{2}}{2 + \sqrt{2}} \\
\text{N-P-V Iic}^{\text{[13]}} & \quad - \left( \frac{1 - \alpha}{1 + \alpha} \right)^{\frac{k}{2}} \left( \frac{n}{\pi} \right)^{2 + k} \cos^{2} \Phi \cos^{2} \left[ n_{1} \ln \left( \frac{n}{\pi} \right) + \Phi \right] dt^{2} + \\
-2 + \sqrt{2} > k \geq -2 & \quad (1 + \alpha)^{1} \left( \frac{\pi}{2} \right)^{k} \left( dr^{2} + r^{2} d\Omega^{2} \right), \quad \text{where } n_{1} = \sqrt{1 - 2k - \frac{k^{2}}{2}}, \tan \Phi = \frac{2 + 3k + \frac{3}{2}k^{2}}{n_{1}(k + 2)} \quad \text{and } \alpha = \frac{k}{k + 1} = \frac{m}{2n} \\
\text{P-V 9Ia}^{\text{[17]}} & \quad - \left\{ A \cos \left[ \frac{1}{2} \arctan \left( \frac{b^{2}r^{2} - c}{\sqrt{b^{2} - c^{2}}} \right) \right] + d \right\}^{2} dt^{2} + \\
b > c & \quad \left( b^{2}r^{4} - 2cr^{2} + 1 \right)^{-1} dr^{2} + r^{2} d\Omega^{2}, \\
\text{P-V Iib}^{\text{[17]}} & \quad - \left\{ A \cos \left[ \frac{1}{2} \arccosh \left( \frac{b^{2}r^{2} - c}{\sqrt{b^{2} - c^{2}}} \right) \right] + d \right\}^{2} dt^{2} + \\
b < c & \quad \left( b^{2}r^{4} - 2cr^{2} + 1 \right)^{-1} dr^{2} + r^{2} d\Omega^{2}, \\
\text{P-V Iic}^{\text{[17]}} & \quad - \left\{ A \exp \left[ \frac{1}{2} \arcsin \left( \frac{b^{2}r^{2} - c}{\sqrt{b^{2} - c^{2}}} \right) \right] + d \right\}^{2} dt^{2} + \\
\text{P-V IV}^{\text{[17]}} & \quad \left( b^{2}r^{4} + B \sin \frac{1}{\sqrt{2}} \right)^{2} dt^{2} + \left( k^{2}r^{2} \right)^{-1} dr^{2} + r^{2} d\Omega^{2}, \\
\text{P-V V}^{\text{[17]}} & \quad \left( A\xi^{\frac{1}{2}}(\xi - 1)^{\frac{1}{4}} + B\xi^{\frac{1}{2}}(\xi - 1)^{\frac{3}{4}} \right)^{2} dt^{2} + \frac{7}{4(\sqrt{r^{2} + 1})} dr^{2} + r^{2} d\Omega^{2}, \quad \text{where } \xi = \frac{1}{2} \left( 1 + \sqrt{1 + \sqrt{r^{2}}} \right) \\
\text{Wyman}^{\text{[15]}} & \quad - \left\{ A F(a, b; \frac{1}{2}; x^{2}) + Bx F(a + \frac{1}{2}, b + \frac{1}{2}; \frac{3}{2}; x^{2}) \right\}^{2} dt^{2} + \\
x^{-2} dr^{2} + r^{2} d\Omega^{2}, \\
\text{Wyman II}^{\text{[18]}} & \quad - \left( Ar^{1-n} - Br^{1+n} \right)^{2} dt^{2} + \\
n \neq 2 & \quad \left( \frac{1}{2-n} + x \right)^{-1} dr^{2} + r^{2} d\Omega^{2}, \\
\text{Wyman II}^{\text{[18]}} & \quad \text{and } n \neq 2 \\
\text{ Wyman II}^{\text{[18]}} & \quad \text{where } b = 2a^{2-n}_{n-2}, c = 2a^{2-n}_{n-3}^{2} \\
\text{Wyman II}^{\text{[18]}} & \quad \text{Wyman II (a,b,c) is a generalization of Tolman VI solution. As a specific example, Wyman evaluates for } n = 1 \text{ while Kuchowicz [14] also gives specific solutions for } n = \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right). 
\end{align*}
Wyman IIb 18
\( n = 2 \)
\[- \left( \frac{4}{r} - Br^2 \right)^2 dt^2 + \left( -\frac{1}{r} + \frac{\text{arcsin} \sqrt{\frac{r}{R}}}{r} \right)^{-1} dr^2 + r^2 d\Omega^2 \]

Wyman IIIc 18
\( n = \sqrt{2} \)
\[- \left( A r^{1-\sqrt{2}} - Br^{1+\sqrt{2}} \right)^2 dt^2 + \left( \frac{1}{r} + \frac{\text{ln}(A(2-\sqrt{2})-B(2+\sqrt{2})r^{2-\sqrt{2}})}{r(2+\sqrt{2})} \right)^{-1} dr^2 + r^2 d\Omega^2 \]

Wyman III 18
\[ - \frac{2}{r} \frac{\text{ln} r}{(36r - 1)} + 2 \frac{\text{ln} r}{36r - 1} + A e^{-2r} + B e^{r-3b} \]
\[ \left( 1 + 2 \sqrt{2} \right) \left( \sqrt{2} + 1 \right) \]

Wyman IVa 18
\[ \frac{1}{r} \frac{\text{ln} r}{2 + \frac{2}{r} \text{ln} r} + 2 \frac{\text{ln} r}{2 + \frac{2}{r} \text{ln} r} + A e^{-2r} + B e^{r-3b} \]
\[ \left( 1 + 2 \sqrt{2} \right) \left( \sqrt{2} + 1 \right) \]

Wyman IVb 18
\[ - \frac{fdr}{r} + \left[ A \left( 1 - \frac{r^2}{p^2} \right) \frac{f}{r} \right]^{1/2} dr^2 + r^2 d\Omega^2 \]
where \( f = A \left( 1 - \frac{r^2}{p^2} - 1 \right) \text{arcsin} \frac{r}{p} + B \sqrt{1 - \frac{r^2}{p^2} + 1} \),
where \( a > 0 \) and \( b > 0 \)

Nariai III 19
\[ - A e^{-2r} + B \left( 1 - \frac{2}{r} \text{ln} r \right)^2 \frac{1}{2} \left( \frac{A + B}{q} \right) \]
\[ \left( \frac{2}{r} \text{ln} r \right)^{1/2} \left( \frac{A + B}{q} \right)^{1/2} \]
where \( k \neq -1 \), \( k \neq 0 \), \( q = \frac{2(1+k)}{k^2 - 2k - 1} \)

Nariai IV 19
\[ - A e^{-2r} + B \left( 1 - \frac{2}{r} \text{ln} r \right)^2 \frac{1}{2} \left( \frac{A + B}{q} \right) \]
\[ \left( \frac{2}{r} \text{ln} r \right)^{1/2} \left( \frac{A + B}{q} \right)^{1/2} \]
where \( k \neq -1 \), \( k \neq 0 \), \( q = \frac{2(1+k)}{k^2 - 2k - 1} \)

Nariai VI 19
\( \alpha > -1 \)
\[ - D \left( \frac{A e^{-2r}}{A + B} \right)^2 dt^2 + \]
\[ D(\alpha + q + Br)^{-2}(dr^2 + r^2 d\Omega^2) \]
where \( p = \sqrt{1 + \alpha} \) and \( q = \sqrt{1 + \alpha} \)

Nariai VII 19
\(-1 > \alpha > -2 \)
\[ - D \left( \frac{A e^{-2r}}{A + B} \right)^2 dt^2 + \]
\[ D(\alpha + q + Br)^{-2}(dr^2 + r^2 d\Omega^2) \]
where \( p = \sqrt{1 + \alpha} \) and \( q = \sqrt{1 + \alpha} \)

Nariai VIII 19
\( \alpha < -2 \)
\[ - D \left( \frac{A e^{-2r} \sin \theta}{A + B \sin \theta} \right)^2 dt^2 + \]
\[ D(\alpha + q + Br)^{-2}(dr^2 + r^2 d\Omega^2) \]
where \( p = \sqrt{1 + \alpha} \) and \( q = \sqrt{1 + \alpha} \)

12Kuchowicz [19] also corrects Wyman IIc however, neither metric satisfies Einstein’s equations; the metric given by Wyman is stated above.

13This is a generalization of Tolman V. Wyman gives the specific example of \( n = 1 \), while Kuchowicz [19] also gives specific solutions for \( n = \{2, -\frac{1}{2}, 3\} \).

14Nariai VI through IX resemble those for slowly rotating cylinders.
| Name            | Ref. | Metric                                                                 |
|-----------------|------|----------------------------------------------------------------------|
| Narai IX        | [20] | \[-D \left( \frac{a^2 + b^2 \ln r + c^2}{A + B r + c^2} \right)^2 \, dt^2 + \]
|                 |      | \[\alpha = -1 \]
|                 |      | \[D r^{-2} \left( A r^\frac{1}{2} + B r^\frac{1}{3} \right)^{-2} \left( d\theta^2 + \frac{1}{r^2} \, d\Omega^2 \right) \]
| Narai X         | [20] | \[-D \left( \frac{a \cos \ln r + b \sin \ln r}{A + B \ln r} \right)^2 \, dt^2 + \]
|                 |      | \[\alpha = -2 \]
|                 |      | \[D r^{-2} (A + B \ln r)^{-2} \left( r^2 + d\Omega^2 \right) \]
| Buch1           | [21] | \[-A \left[ (1 + C r^2)^{3/2} + B \sqrt{2 - C r^2 (5 + 2 C r^2)} \right]^2 \, dt^2 + \]
|                 |      | \[\frac{2(1 + C r^2)}{A r^2} \, dr^2 + r^2 d\Omega^2 \]
| Buch2           | [22] | \[-\frac{(1-f)^2}{(1+f)^2} \, dt^2 + (1 + f)^4 \left( d\theta^2 + \frac{1}{r^2} \, d\Omega^2 \right) \]
|                 |      | where \( f = \frac{\rho}{2\sqrt{1 + \xi r^2}} \)
| Mehra           | [23] | \[-\left[ \sqrt{1 - \frac{10 \pi^2 \rho c}{15}} \cos \left( \frac{2 \pi a}{2} \right) - \frac{2a}{3} \sqrt{\frac{2 \pi \rho c}{15}} \sin \left( \frac{2 \pi a}{2} \right) \right]^2 \, dt^2 + \]
|                 |      | \[\left[ 1 - \frac{8 \pi \rho c}{15} \left( \frac{b^2}{a^2} - \frac{3 x^4}{a^4} \right) \right]^{-1} \, dr^2 + r^2 d\Omega^2, \]
|                 |      | where \( a^2 < \frac{9}{10 \pi \rho c}, \quad z = \ln \left( \frac{a^2 - b}{b - 0} + \sqrt{\frac{a^2 - b}{b - 0}} \right)^2 \)
|                 |      | and \( z_1 = \ln \left( \frac{b}{b + \sqrt{5 \pi^2 \rho c} - 5} \right) \)
| Buch3           | [24] | \[-\frac{(b - \eta)}{b \eta} \, dt^2 + \frac{(b + \eta)}{b \eta} \, dr^2 + r^2 d\Omega^2, \]
|                 |      | where \( \eta = \beta \sin(Ar), \beta = \sqrt{b^2 - 4ac}, a > 0, b > 0 \) and \( c > 0 \)
| Kucha           | [25] | \[-\frac{1}{\rho c} \left[ A \cos \left( m \ln \left( \frac{R}{r} \right) \right) + B \sin \left( m \ln \left( \frac{R}{r} \right) \right) \right]^2 \, dt^2 + \left\{ \frac{1}{x^2 + m^2} + \frac{C R^2}{x^2 + m^2} \left[ (2A + mB) \cos \left( m \ln \left( \frac{R}{r} \right) \right) + (2B - mA) \sin \left( m \ln \left( \frac{R}{r} \right) \right) \right]^{-1} \right\}^{-1} \, dr^2 + \]
|                 |      | \[r^2 d\Omega^2, \]
|                 |      | where \( a = \frac{2(2 + m^2)}{1 + m^2} \)
| Kuch1 Ia        | [25] | \[-x^2 \left[ A + B \ln \left( \frac{1 + \sqrt{1 + bx^2}}{1 - \sqrt{1 + bx^2}} \right) \right]^2 \, dt^2 + \frac{2}{1 + bx^2} \, dr^2 + r^2 d\Omega^2, \]
|                 |      | where \( x = \frac{r}{\rho}, a = \sqrt{2 - \frac{b}{a}} \) and \( b = \frac{C}{\rho} R^2 \)
| Kuch1 Ib        | [25] | \[-x^2 \left[ A + B \ln \left( \frac{1 + \sqrt{1 + bx^2}}{1 - \sqrt{1 + bx^2}} \right) \right]^2 \, dt^2 + \frac{2}{1 + bx^2} \, dr^2 + r^2 d\Omega^2, \]
|                 |      | where \( x = \frac{r}{\rho}, b = 2CR^2 \neq 0 \)
| Kuch1 Ic        | [25] | \[-x^2 \left[ A \sin \left( a \ln \left( \frac{1 + \sqrt{1 + bx^2}}{x} \right) \right) + B \cos \left( a \ln \left( \frac{1 + \sqrt{1 + bx^2}}{x} \right) \right) \right]^2 \, dt^2 + \]
|                 |      | \[\left( D + C r^2 \right)^{-1} \, dr^2 + r^2 d\Omega^2, \]
|                 |      | where \( x = \frac{r}{\rho}, a = \sqrt{2 - \frac{b}{a}} \) and \( b = \frac{C}{\rho} R^2 \)
| Kuch1 Id        | [25] | \[-x^2 \left[ A \sin \left( a \ln \left( \frac{1 + \sqrt{1 + bx^2}}{x} \right) \right) + B \cos \left( a \ln \left( \frac{1 + \sqrt{1 + bx^2}}{x} \right) \right) \right]^2 \, dt^2 + \]
|                 |      | \[\left( D + C r^2 \right)^{-1} \, dr^2 + r^2 d\Omega^2, \]
|                 |      | where \( x = \frac{r}{\rho}, a = \sqrt{2 - \frac{b}{a}} \) and \( b = \frac{C}{\rho} R^2 \)

\( J_{\pm \nu} \) are Bessel functions with \( \nu = \frac{\sqrt{(1+b)^2 + 4b}}{2(b-1)} \).
| Name      | Metric                                                                 |
|-----------|------------------------------------------------------------------------|
| Kuch1 IIIb | $- \left[A x \sqrt[3]{3} + x + \sqrt{3} \frac{\sqrt{3}}{2} x + \sqrt{3} \frac{\sqrt{3}}{2} x - 2 \sqrt{3} \frac{\sqrt{3}}{2} x \right] +$ |
|           | \(B x^{1-\sqrt[3]{3}} F \left( \frac{\sqrt{3}}{4} - \sqrt{3} + \sqrt{3} \frac{\sqrt{3}}{2} x - \sqrt{3} \frac{\sqrt{3}}{2} x - 2 - 2 \sqrt{3} \frac{\sqrt{3}}{2} x \right) \right)^2 \right] \right]^{2} dt^2 +$ |
|           | \(1 + \frac{2C r^2}{4} \right)^{2} dr^2 + r^2 d\Omega^2, \text{ where } x = \frac{r}{r_0}$ |
| Kuch1 IV  | $-r^{2(1-\sqrt[3]{3})} e^{C(\sqrt[3]{3}-1)} \left[ A F \left( \frac{1}{4} - \frac{1}{2} \sqrt{3} \frac{\sqrt{3}}{2} x \right) \right] +$ |
|           | \(B \sqrt{2(2n r - C)} \right] \right]^{2} dt^2 +$ |
|           | \(C - 2 \ln r)^{2} dr^2 + r^2 d\Omega^2$ |
| Kuch1 V   | $-r^{2(1-\sqrt[3]{3})} F(A(x - b) - bCr^{2b}) +$ |
|           | \(B^{2b(1-\gamma)} F(\alpha - \gamma + 1, \beta - \gamma + 1; 2 - \gamma - bCr^{2b}) \right]^{2} dt^2 +$ |
|           | \(\frac{1}{4} + C r^{2b} \right)^{2} dr^2 + r^2 d\Omega^2,$ |
|           | where $d = b + \sqrt{b^2 - b + 1}, \gamma = 1 + \frac{1}{2} \sqrt{b^2 - b + 1}, \gamma \in I$ |
|           | $\alpha = \frac{1}{2\pi} \left( \frac{3b}{4} - 1 + \sqrt{b^2 - b + 1} + \sqrt{b^2 - b + 2} \right)$ |
|           | $\beta = \frac{1}{2\pi} \left( \frac{3b}{4} - 1 + \sqrt{b^2 - b + 1} - \sqrt{b^2 - b + 2} \right)$ |
| Kuch2 I   | $-B r A dr^2 + \left( \frac{C r^{-a} - \frac{1}{A x - 1}}{x - 1} \right)^{2} dr^2 + r^2 d\Omega^2,$ |
|           | where $r = A^{-4/2} - 1 x - 1 \right)^{2} dr^2 + r^2 d\Omega^2,$ |
| Kuch2 III | $-B e^{\frac{4}{2 \pi} dr} \left\{ 1 + r^2 e^{-\frac{4}{2 \pi} A r^2} \left[ C - \frac{4}{2 \pi} Ei \left( 1 + \frac{4 r^2}{2 \pi} \right) \right] \right\}^{1} dr^2 + r^2 d\Omega^2,$ |
| Kuch2 IV  | $-B e^{\frac{4}{2 \pi} dr^2} + \frac{1}{C e^{2} + 8(4 + 14x^3 + 96x^2 + 384x + 784)} dr^2 + r^2 d\Omega^2,$ |
|           | where $x = \frac{A}{r}$ |
| Kuch2 VI  | $-B(1 - A)^2 dr^2 + \left[ \left( C - \frac{1}{A x} \right) r^2 - \frac{2x}{A x} + 1 + \frac{4r^2}{A x} \right]^{2} dt^2 +$ |
|           | $r^2 d\Omega^2$ |
| Kuch2 VII | $-A r^{2(1+\sqrt[3]{3})} dr^2 + \left( C - \frac{r}{A x} \right)^{2} dr^2 + r^2 d\Omega^2$ |
| Whittaker  | $-n \left( 1 + B - \frac{2}{\sqrt{2}} \right)^{3/2} \left( \frac{1}{\sqrt{2}} - a^{2} r^2 \right)^{3/2} \left( \frac{1}{\sqrt{2}} - a^{2} r^2 \right) \right]^{2} dt^2 +$ |
|           | \((1 + B)(1 - a^{2} r^2)^{-1} dr^2 + r^2 d\Omega^2,$ |
|           | where $B = \frac{1}{\sqrt{2}} \right)^{2} dr^2 + r^2 d\Omega^2,$ |
| B-L       | $-C e^{2} dr^2 + \left( \frac{1}{x} - \frac{C e^{2}}{x} \right)^{2} dr^2 + r^2 d\Omega^2$ |
| Kuch68 I  | $-\left( A \sqrt{1 + \frac{1}{2} B} - \frac{4x}{2 \pi} - \frac{b e^{2}}{6} + \frac{15}{6} \sqrt{1 + \frac{1}{2} B} \ln \left( 1 + \frac{1}{2} \sqrt{r} + \frac{1}{2} \sqrt{r} \right) \right) \right]^{2} dt^2 +$ |
|           | \((1 + \frac{1}{2} B)^{2} dr^2 + r^2 d\Omega^2$ |

\(^{15}\text{Kuch2 I is only a slightly more general version of Tolman V, obtained by setting } C = 1. \text{ There are many metrics which are interconnected with one another. In this paper we have noted some generalizations and duplications but we have not exhausted all the possible interconnections between metrics.}\)

\(^{16}\text{Kuchowicz does not state that Kuch2 II is a special case of Kuch2 VII.}\)
| name [ref.] | metric |
|-------------|---------|
| Kuch68 II   | \((-1 + \frac{a}{r}) dt^2 + \left[ (1 + \frac{a}{r}) (1 + C(a + 2r^2)) \right]^{-1} dr^2 + r^2 d\Omega^2\) |
| Leib I      | \((-A - Bx) dt^2 + \frac{r F(x)}{r^2 + a^2} dr^2 + r^2 d\Omega^2,\) where \( F(x) = [x(A - Bx) + B(1 - a^2)] \left[ \frac{(x-y)^2}{(x-z)^2} \right]^{2/(x-y)}, x = \sqrt{1 - \frac{\alpha^2}{r^2}}, y = \frac{A - \sqrt{A^2 + \beta B^2}}{1/2} \) and \( z = \frac{A + \sqrt{A^2 + \beta B^2}}{1/2} \) |
| Leib IV     | \(-g^{-1} dt^2 + r^2 y^2 \left[ \frac{(x-y)^2}{(x-z)^2} \right]^{2/(x-y)} dr^2 + r^2 d\Omega^2,\) where \( g = 1 - \alpha r \) |
| Heint IIa   | \(-A^2 \left( 1 + a r^2 \right)^3 dt^2 + \left( 1 - 3 a r^2 \frac{1 + C(1 + 4ar^2)}{1 + 4ar^2} \right)^{-1} dr^2 + r^2 d\Omega^2\) |
| Heint IIb   | \(-B^2 \left( 5 + a r^2 \right)^2 \left( 2 - ar^2 \right) dt^2 + \left( 1 - 3 a r^2 \frac{1 + C(1 + 4ar^2)}{1 + 4ar^2} \right)^{-1} dr^2 + r^2 d\Omega^2\) |
| Heint III a | \(-r \left( 2 + a r^2 \right)^2 dt^2 + \left[ 1 - 2 r^2 \left( \frac{2a}{r^2 - a - \frac{a^2}{8}} + C e^{-2B/3} \right) \right]^{-1} dr^2 + r^2 d\Omega^2\) |
| Heint III b | \(-r \left( 3 + b r^2 \right)^2 dt^2 + \left[ 1 - 2 r^2 \left( \frac{3b r + C}{4r^2 + 3b} \right) \right]^{-1} dr^2 + r^2 d\Omega^2\) |
| Heint III c | \(-\rho^2 \left( 1 + a r^2 \right)^2 \left( 1 + b r^2 \right)^2 \right)^{-1} dr^2 + r^2 d\Omega^2,\) where \( D = \frac{1}{2} \left( 4 + 2 \sqrt{3A} \right), \alpha = \frac{1}{2} (5 + \sqrt{3A}) \) and \( \beta = \frac{1}{2} (3 - \sqrt{3A}) \) |
| Kuch3 I a   | \(-\rho^2 \left( 1 + a r^2 \right)^2 \left( 1 + b r^2 \right)^2 \right)^{-1} dr^2 + r^2 d\Omega^2,\) where \( D = \frac{1}{2} \left( 4 + 2 \sqrt{3A} \right), \alpha = \frac{1}{2} (5 + \sqrt{3A}) \) and \( \beta = \frac{1}{2} (3 - \sqrt{3A}) \) |
| Kuch3 Ic    | \(-\rho^2 \left( 1 + a r^2 \right)^2 \left( 1 + b r^2 \right)^2 \right)^{-1} dr^2 + r^2 d\Omega^2,\) where \( D = \frac{1}{2} \left( 4 + 2 \sqrt{3A} \right), \alpha = \frac{1}{2} (5 + \sqrt{3A}) \) and \( \beta = \frac{1}{2} (3 - \sqrt{3A}) \) |
| Kuch3 II a  | \(-\rho^2 \left( 1 + a r^2 \right)^2 \left( 1 + b r^2 \right)^2 \right)^{-1} dr^2 + r^2 d\Omega^2,\) where \( D = \frac{1}{2} \left( 4 + 2 \sqrt{3A} \right), \alpha = \frac{1}{2} (7 + \sqrt{7A}) \) and \( \beta = \frac{1}{2} (5 - \sqrt{7A}) \) |
| name [ref.] | metric |
|------------|--------|
| Kuch3 III | $-\left(A r^\alpha + B r^\beta\right)^2 d^2 t^2 + \left\{ C r^{-2D} - \frac{1}{D|n+(n-2)D|} + 2\frac{n r^{-2D}}{|n+(n-2)D|}\right.$ |
|           | $\left[\sum_{k=1}^n \left. \left(\frac{1}{2}\right)^{k-1} \left[\frac{1}{D|n+(n-2)D|}\right]^{k-1} + \sum_{n+1-k}^{n-1} \delta(n+1-k) + \right. \right.$ |
|           | $\left. -\left(\frac{1}{2}\right)^n \left[\frac{1}{D|n+(n-2)D|}\right]^n \ln\left(\frac{1}{D|n+(n-2)D|} + B|n+(n-2)D|r^{2D}\right)\right]\right\}^{-1} d^2 r^2 + r^2 d\Omega^2$, where $\alpha = 1 + D\left(\frac{1}{2} - n\right)$, $\beta = 1 + D\left(\frac{1}{2} + n\right)$, |
|           | $D_\pm = -2n^2 \frac{2}{n^2 - 1}$ and $n \in I > 3$ |
| Kuch3 IV  | $-r^{2+D}(A + B \ln r)^2 d^2 t^2 + \left\{ C r^{-2D} - \frac{1}{D(1 + \alpha)} + \frac{2}{(1 + \nu)^2}\right.$ |
|           | $\exp\left[-2D\left(\frac{1}{F} + \frac{1}{1 + \nu}\right)\right] r^{-2D} E\left[2D\left(\frac{1}{F} + \frac{1}{1 + \nu}\right) + 2D\ln r\right]\right\}^{-1} d^2 r^2 +$ |
|           | $r^2 d\Omega^2$, where $D_\pm = \pm 2\sqrt{2}$ and $\alpha = 1 + D/2$ |
| Kuch3 V   | $-B r^r e^{2Ar} d^2 t^2 + |F(r)|^{-1} d^2 r^2 + r^2 d\Omega^2$, where $F(r) = C r^r e^{-Ar} + \frac{1 + Ar}{Ar} + A r^2 e^{-2Ar} \left(\frac{1}{x} E(2Ar + 4) - \frac{1}{5} Ei(2Ar)\right)$ |
| Kuch71 I  | $-\left(\frac{1}{2}\right)^{k-1} \left[\frac{1}{D|n+(n-2)D|}\right]^{k-1} (A e^{\sqrt{c}r^n} + Be^{-\sqrt{c}r^n})\right] d^2 t^2 +$ |
|           | $\frac{(k-2)!}{10^3!} (4 r^k)^{-1} d^2 r^2 + r^2 d\Omega^2$, where $\zeta = (ar^4)^{\frac{1}{2-k}}$, |
|           | $b = \frac{1}{2} + \frac{2n+1}{2n+3}, k = \frac{2}{2n+2}, c = \frac{2}{n+1}$ and $m = \frac{1}{2n+1}$ |
| Kuch71 II | $-\left(A e^{\sqrt{c}r^n} + Be^{-\sqrt{c}r^n}\right) d^2 t^2 + Ce^{\sqrt{c}r^n} (d^2 r^2 + r^2 d\Omega^2)$ |
| Kuch71b  | $-ar^2 \sqrt{2\sqrt{2}} d^2 t^2 + \frac{1}{\sqrt{c}+2\sqrt{2}+d} (d^2 r^2 + r^2 d\Omega^2)$ |
| Kuch5 I  | $-r^4 e^{Ar} (1 - \sqrt{2}) \left\{ CF(\delta; 3; A \sqrt{2}r) + D \left[\frac{1}{2A} + \frac{2}{4\sqrt{2}} + \frac{1}{2} (\delta - 1)(\delta - 2) F(\delta; 3; A \sqrt{2}r) \times \ln(A \sqrt{2}r) + \right.$ |
|           | $\left. + \sum_{k=1}^{\infty} \frac{\delta(\delta-1)(\delta-k+1)\cdots(\delta-k+1)}{2^3 \cdots (2k+1)\cdots k!} (A \sqrt{2}r)^k \times \sum_{n=0}^{k-1} \left(\frac{1}{2^n} - \frac{1}{3^n} + \frac{1}{3^n} \right)\right]\right\}^2 d^2 t^2 + \frac{2}{Be^{Ar} (d^2 r^2 + r^2 d\Omega^2)}$, where $\delta = \frac{1}{2} (3 + \sqrt{2})$ |
| Kuch5 IV  | $-r^4 e^{-\sqrt{2}Ar} \left\{ CF\left(\frac{1}{2\sqrt{2}}; 3; \frac{1}{\sqrt{2}}r^4\right) + D_\delta F\left(\frac{1}{2\sqrt{2}}; \frac{1}{2}; \frac{1}{\sqrt{2}}r^4\right)\right\} d^2 t^2 + \frac{2}{Be^{Ar} (d^2 r^2 + r^2 d\Omega^2)}$ |
| Kuch5 VIII| $-C e^{ar^2} d^2 t^2 + De^{-ar^2} \cosh^{-2}\left(-\frac{2\sqrt{2}}{2} r^2 + A\right) (d^2 r^2 + r^2 d\Omega^2)$ |
| Kuch5 IX  | $-A r^2 \sqrt{2\sqrt{2}} d^2 t^2 + \frac{2}{(B r^2 - C)} (d^2 r^2 + r^2 d\Omega^2)$, where $s > \frac{1}{2}$ |
| Kuch5 XI  | $-C F(r) d^2 t^2 + D \left(B + \frac{2^{2A}}{2A + 1}\right)^2 F(r) (d^2 r^2 + r^2 d\Omega^2)$, where $F(r) = \exp\left(2\sqrt{\frac{2A}{2A + 1}} \arctan(\sqrt{2B(1-A)} r^{2A+2} - 1)\right)$ and $A = 1$ |
| Kuch5 XII | $-C \left(\frac{\sqrt{2}A F(r)|^{\frac{1}{2}}}{\sqrt{2A} F(r)_{|+}}\right)^2 (d^2 r^2 + r^2 d\Omega^2)$, where $F(r) = \frac{1}{\sqrt{2}} + Be^{Ar^2}$ |
| Kuch5 XIII| $-C \left(|a^2 - \frac{1}{2}r^2\right) (d^2 r^2 + D \left(|a^2 - \frac{1}{2}r^2\right) (d^2 r^2 + r^2 d\Omega^2)$ |
| Name | Ref. | Metric |
|------|------|--------|
| Kuch5 XV | 98 | \(-C \left| \sqrt{2} F(r) + Be^{Ar} - A \right|^2 e^{\frac{1}{2} F(r)} dt^2 + \right| B \sqrt{2} F(r) + Be^{Ar} - A \right|^2 e^{\frac{1}{2} F(r)} (dr^2 + r^2 d\Omega^2), \text{ where } F(r) = \sqrt{\frac{1}{2} B^2 e^{2Ar} - AB e^{Ar}}. |
| Kuch5 XVI | 98 | \(-C \left| B \sqrt{2} F(r) + B^2 r^2 + AB \right|^2 e^{\frac{1}{2} B^2 r^2} F(r) dt^2 + \right| B \sqrt{2} F(r) + B^2 r^2 + AB \right|^2 e^{\frac{1}{2} B^2 r^2} F(r) (dr^2 + r^2 d\Omega^2), \text{ where } F(r) = \sqrt{\frac{B^2 r^4 + AB r^2 + \frac{A^2}{2} - B}{}}. |
| R-R I | 99 | \(-\frac{1}{c^2} dt^2 + \frac{C^2}{(1 - \frac{r^2}{R^2})} dr^2 + C^2 r^2 d\Omega^2 |
| R-R II | 99 | \(-f^{-1} dt^2 + C^2 (1 - \frac{r^2}{R^2}) \right| f = C^2 (1 - \frac{2m}{r} - \frac{r^2}{R^2}) |
| R-R III | 99 | \(-f^{-1} dt^2 + f^2 \left( 1 - \frac{r^2}{R^2} \right)^{-1} dr^2 + f^2 r^2 d\Omega^2, \text{ where } f = \left( A - B \sqrt{1 - \frac{r^2}{R^2}} \right)^2 |
| R-R IV | 99 | \(-f^{-1} dt^2 + B^2 f^{-1} \left( 1 - \frac{r^2}{R^2} \right) \frac{1}{R^4} dr^2 + f^2 r^2 d\Omega^2, \text{ where } f = B^2 \left( 1 + \frac{r^2}{R^2} \right) |
| R-R V | 99 | \(-\frac{1}{B^2} dt^2 + B^2 r^2 \left( \frac{4}{R^2} \right)^{\frac{1}{2}} dr^2 + B^2 r^2 d\Omega^2 |
| R-R VI | 99 | \(-f^{-1} dt^2 + f^2 (2 - \frac{n^2}{r^2}) dr^2 + f^2 r^2 d\Omega^2, \text{ where } f = (Ar^{1-n} - B^{1+n})^2 |
| R-R VII | 99 | \(-f^{-1} dt^2 + f^2 \left( 1 - \frac{r^2}{R^2} + \frac{4r^2}{A^2} \right)^{-1} dr^2 + f^2 r^2 d\Omega^2, \text{ where } f = B^2 \sin^2 \ln \sqrt{\frac{\sqrt{1 - \frac{r^2}{R^2} + \frac{4r^2}{A^2} + \frac{2r^2}{B^2} - \frac{A^2}{4}}}{\frac{2r^2}{B^2} (\frac{a+b}{a+b}) - \frac{2m}{T}} - \left( \frac{a+b}{a+b} \right)^2 - \frac{1}{\frac{a+b}{a+b}} |
| R-R VIII | 99 | \(-f^{-1} dt^2 + B^2 r^2 \left( \frac{2m}{T} \right)^{\frac{1}{2}} + \left( \frac{a+b}{a+b} \right) - \frac{2m}{T} \right)^{\frac{1}{2}} (dr^2 + r^2 d\Omega^2), \text{ where } f = B^2 r^2 \left( \frac{2m}{T} \right)^{\frac{1}{2}} \left( \frac{a+b}{a+b} \right) - \frac{2m}{T} \right)^{\frac{1}{2}} (dr^2 + r^2 d\Omega^2) |
| Kuch73 I | 10 | \(-\frac{a}{c} e^{-\frac{r^2}{c}} dt^2 + \frac{e^{-\frac{r^2}{c}}}{\sqrt{1 + e^{-\frac{r^2}{c}} \sqrt{1 - \frac{y^2}{c^2}}}} dr^2 + \frac{e^{-\frac{r^2}{c}}}{\sqrt{1 + e^{-\frac{r^2}{c}} \sqrt{1 - \frac{y^2}{c^2}}}} \right| y = \frac{1}{c} \frac{dr^2}{\sqrt{1 + e^{-\frac{r^2}{c}} \sqrt{1 - \frac{y^2}{c^2}}}} |
| Kuch73 II | 10 | \(-\frac{a}{e^{-\frac{r^2}{c}}} dt^2 + \frac{e^{-\frac{r^2}{c}}}{\sqrt{1 + e^{-\frac{r^2}{c}} \sqrt{1 - \frac{y^2}{c^2}}}} \right| y = \frac{1}{c} \frac{dr^2}{\sqrt{1 + e^{-\frac{r^2}{c}} \sqrt{1 - \frac{y^2}{c^2}}}} |
| K-N-B | 10 | \(-c \exp \left( \frac{\sin^{-1} \left( \frac{a^2 r^2 + 2b}{2\sqrt{a^2 + 2b}} \right) \right) dr^2 + \left( 1 - br^2 - \frac{a^2}{4r^4} \right)^{-1} dr^2 + r^2 d\Omega^2 |
| G-G | 10 | \(-A \left( \frac{2r^2 - (b^2 d + 1) r^2}{2r^2 - (b^2 d + 1) r^2} \right) \right| B \left( \frac{2r^2 - (b^2 d + 1) r^2}{2r^2 - (b^2 d + 1) r^2} \right)^{-1} (dr^2 + r^2 d\Omega^2) |
| where c = \sqrt{\frac{b^2 d - 1}{2} - 8b^2} |
| name          | metric                                                                                                                                                                                                 |
|---------------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| Bayin III     | $-A^2 \left( r + \sqrt{r^2 - \frac{2}{M^2}} \right)^{2/\sqrt{B}} dt^2 + \left( \frac{2}{M^2} - B \right)^{-1} dr^2 + r^2 d\Omega^2$ |
| Bayin VI      | $-AC^{-a}e^{-d\tau^2} dr^2 + B C^{-a}e^{d\tau^2}(dr^2 + r^2 d\Omega^2)$, where $a^2 + 2ab - b^2 = 0$                                                                                                      |
| Gold I        | $-Af \tau dt^2 + B c^{(1+2ag+2g^2)} f^{-1/2} (dr^2 + r^2 d\Omega^2)$, where $f = \frac{1}{1+(a+b)^2}, g = \frac{c}{1-d\tau}$ and $b = \sqrt{a^2 - 2} $                                                                 |
| Gold II       | $-A \left[ \frac{(1+c)a^2+1-c}{2a} \right]^{1/2} u^{-1} \sqrt{\tau} dt^2 + B \left[ \frac{2y}{(1+c)a^2+1-c} \right]^{1/2} u^{-1} \sqrt{\tau} (dr^2 + r^2 d\Omega^2)$, where $u = e^{a-b\tau}$ |
| Gold III      | $-A \left( \frac{2-1}{9+1} \right) dt^2 + B \left( 1 + \frac{1}{9} \right)^2 (dr^2 + r^2 d\Omega^2)$, where $g = \cosh(a + b\tau)$                                                                                   |
| M-W I         | $-\left( 1 + a\frac{\tau}{2\tau} \right) \left( A \sin z + B \cos z \right)^2 dt^2 + \left( 1 + 2a\frac{\tau}{2\tau} \right) dr^2 + r^2 d\Omega^2$, where $z = \sqrt{1 + 2a\frac{\tau}{2\tau}} - \tan^{-1} \sqrt{1 + 2a\frac{\tau}{2\tau}}$ |
| M-W II        | $-\left( \frac{1}{2} + a\frac{\tau}{2\tau} \right) - \sqrt{1 + a\frac{\tau}{2\tau}} \cos^{-1} \sqrt{\frac{1}{2} \left( 1 + a\frac{\tau}{2\tau} \right)} \bigg] + B \sqrt{1 + a\frac{\tau}{2\tau}} \bigg) dt^2 + \frac{1+2a\frac{\tau}{2\tau}}{1+a\frac{\tau}{2\tau}} dr^2 + r^2 d\Omega^2 $ |
| M-W III       | $-\left( \frac{2\tau}{3\tau} - r\frac{\tau}{9\tau} \right) dt^2 + \frac{63}{36-2\tau^2} dr^2 + r^2 d\Omega^2$                                                                                             |
| K-K           | $-\left( \frac{2}{\tau} \right) \left( 1 + ar^{-k} \right)^{1-2\alpha} dt^2 + \left( 1 - 2A - A^2 \right) \xi \frac{r^2+b}{(1+2\xi)r^2} dr^2 + r^2 d\Omega^2$ |
| Stewart       | $-\left( 1 - \frac{M}{2\alpha} \right)^2 \left( 1 - \frac{Mr^2}{2\alpha^2} \right)^2 \left( 1 + \frac{M}{a} - \frac{Mr^2}{a^2} - \frac{M^2r^2}{4a^4} \right)^2 dt^2 + \left( 1 - \frac{M}{2\alpha} \right)^2 \left( 1 - \frac{Mr^2}{2\alpha^2} \right)^2 \left( 1 + \frac{M}{a} - \frac{Mr^2}{a^2} - \frac{M^2r^2}{4a^4} \right)^2 (dr^2 + r^2 d\Omega^2)$ |
| P-S Ia        | $-B^2 \tau^{2\left( 1 + \frac{c-b}{1 - kr^2} \right)} dt^2 + \left( 1 + \frac{a+3}{1 - kr^2} \right) \xi d\tau^2 + \frac{1+kr(1+c+b)}{\left( 1 - kr^2 \right) \left( 1 + kr^2 \right)} \xi ^2 d\tau^2 + r^2 d\Omega^2$, where $b = \sqrt{a^2 + 10a + 32}, c = \frac{a+4}{1}$ and $a \neq 0$ |
| P-S Ib        | $-B^2 \tau^{2\left( 1 - \frac{c-b}{1 - kr^2} \right)} dt^2 + \frac{2\sqrt{2} - 2\sqrt{2}}{\left( 1 - kr^2 \right)} \xi d\tau^2 + \frac{\sqrt{2} - 2\sqrt{2}}{\left( 1 - kr^2 \right)} \xi d\tau^2 + r^2 d\Omega^2$                                                   |
| Durg IV       | $-A(1 + Cr^2)^4 dt^2 + \left( \frac{7-10Cr^2+C^2r^4}{H(1+Cr^2)} \right) + \frac{KCr^2}{(1+Cr^2)^2 (1+5Cr^2)} \xi d\tau^2 + r^2 d\Omega^2$                                                                       |
| Durg V        | $-A(1 + Cr^2)^5 dt^2 + \frac{1-Cr^2+2Cr^2(1-Cr^2)+2Cr^2+2Cr^2(1+5Cr^2)}{(1+Cr^2)^2} \xi d\tau^2 + r^2 d\Omega^2$                                                                                       |

\(^{17}\)Bayin III may be contained within Kuch1 V and Bayin VI within Kuch5 IV although the specific values of the hypergeometric equation are not given.  
\(^{18}\)The general form was originally found by Korkina but the only solution he presented was Heint IIa.
| Name       | Ref. | Metric                                                                 |
|------------|------|------------------------------------------------------------------------|
| Whitman II | [51] | \(- \left( A^2 \left( 1 + \frac{a^2}{R^2} \right) + B \frac{r}{R \sqrt{2a}} \sinh \beta \right) \left( 1 + \frac{R^2 + 2ar^2}{2a^2} \right) \times \left( 1 - \sqrt{10a} \frac{r}{\sqrt{R^2 + 2a^2}} \coth \beta \right) \right) \, \mathrm{d}t^2 + B^2 \left( 1 + 2\alpha \frac{r^2}{R^2} \right) \times \left( 1 + \frac{R^2 + 2ar^2}{2a^2} \right) \times \left( 1 - \sqrt{10a} \frac{r}{\sqrt{R^2 + 2a^2}} \coth \beta \right) \right) ^{-1} \, \mathrm{d}r^2 + r^2 \mathrm{d}\Omega^2 \)  
   where \( \beta = \sqrt{3} \arcsinh \frac{R^2}{2a^2} \) |
| Whitman III | [51] | \(- f \, \mathrm{d}t^2 + a f \left( 1 - \frac{b^2 r^2}{R^2} \right) \, \mathrm{d}r^2 + r^2 \, \mathrm{d}\Omega^2 \)  
   where \( f = \frac{R^2}{R^2} \left( a \left( 1 - \frac{30b^2 r^2}{7R^2} + \frac{32b^2 r^4}{7R^4} + \frac{64b^4 r^6}{49R^6} - \frac{640b^4 r^8}{147R^8} + \frac{256b^4 r^{10}}{147R^{10}} \right) + + A \left( 1 - \frac{b^2 r^2}{R^2} \right) ^{-1} \left( 1 - \frac{b^2 r^2}{R^2} \right) \left( 1 - \frac{b^2 r^2}{R^2} \right) + \frac{\sqrt{R^2 - b^2 r^2}}{\sqrt{b^2 r^2 - 1}} \arctan \left( \frac{b^2 r^2}{R^2 - b^2 r^2} \right) \)  
   and \( \beta = 2 \sqrt{3} \arcsin \frac{b^2 r^2}{R^2 - b^2 r^2} \) |
| Whitman IV | [51] | \(- f \, \mathrm{d}t^2 + \frac{a^2}{f} \left( 1 - \frac{b^2 r^2}{R^2} \right) ^2 \, \mathrm{d}r^2 + r^2 \, \mathrm{d}\Omega^2 \)  
   where \( f = \frac{R^2}{R^2} \left( a \left( 1 - \frac{30b^2 r^2}{7R^2} + \frac{32b^2 r^4}{7R^4} + \frac{64b^4 r^6}{49R^6} - \frac{640b^4 r^8}{147R^8} + \frac{256b^4 r^{10}}{147R^{10}} \right) + + A \left( 1 - \frac{b^2 r^2}{R^2} \right) ^{-1} \left( 1 - \frac{b^2 r^2}{R^2} \right) \left( 1 - \frac{b^2 r^2}{R^2} \right) + \frac{\sqrt{R^2 - b^2 r^2}}{\sqrt{b^2 r^2 - 1}} \arctan \left( \frac{b^2 r^2}{R^2 - b^2 r^2} \right) \)  
   and \( \beta = 2 \sqrt{3} \arcsin \frac{b^2 r^2}{R^2 - b^2 r^2} \) |
| D-P-P^19 | I | \(- B \frac{\mathrm{d}t^2}{\sqrt{1 - c^2 r^2} \sqrt{1 - e^2 r^2} \sqrt{1 - f^2 r^2}} + \frac{\mu}{(2 - e)(2 - c)(2 - f)} \, \mathrm{d}r^2 + r^2 \, \mathrm{d}\Omega^2 \)  
| V-P II | [53] | \(- F^2 \cos^{-1} \left( C \sqrt{1 - Ar^2} \right) \, \mathrm{d}t^2 + \left( 1 - Ar^2 \right) \, \mathrm{d}r^2 + r^2 \, \mathrm{d}\Omega^2 \)  
| P-S2 | [54] | \(- A^2 \left( 1 + \frac{\delta}{1 + \delta} \right) ^2 \, \mathrm{d}t^2 + \left( 1 + \frac{\delta}{1 + \delta} \right) ^2 \, \mathrm{d}r^2 + r^2 \, \mathrm{d}\Omega^2 \)  
   where \( \delta = k \sqrt{\frac{1 + \frac{\delta}{1 + \delta}}{1 + \frac{\delta}{1 + \delta} \frac{r^2}{R^2}}} \) |
| D-F | [55] | \(- A \left( 1 + cr^2 \right) ^2 + B \left( 7 + 3cr^2 \right) f + 2 \ln \left( \frac{1 + cr^2}{1 + cr^2} \right) \left( 1 + cr^2 \right) \right) \, \mathrm{d}t^2 + + \left( 1 - \frac{8}{3} \frac{cr^2}{1 + cr^2} \right) ^{-1} \, \mathrm{d}r^2 + r^2 \, \mathrm{d}\Omega^2 \)  
   where \( f = \sqrt{7 - 10cr^2 - c^2 r^4} \) |
| H-B I | [56] | \(- D^2 (Ar^2 + B) \left( \frac{Ar^2 + B}{24A^2} \right) \left( 1 + \frac{\alpha \beta}{\pi} \right) \, \mathrm{d}t^2 + (Ar^2 + B) \left( \frac{dr^2 + r^2 \, \mathrm{d}\Omega^2}{4A^2} \right) , \)  
   where \( \alpha = \frac{4a - 2b - 2c}{b - a} \) and \( \beta = \frac{a + b}{a - b} + 1 \) |

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\(^{19}\)Eq. (38) of [12] has an obvious misprint in \( y \).

\(^{20}\)As with Bayin VII (for which this is an extension) H-B I is probably contained within Kuch5 XIII.
| name  | ref. | metric                                                                 |
|-------|------|----------------------------------------------------------------------|
| H-B II | [74] | \(-\left(\frac{1-f}{1+f}\right)^2 dt^2 + (1+f)^4B^2 \left[\frac{A^2}{2K} \left(-\frac{1}{27r^2} + \frac{f^2}{2} - 2\ln f\right) + C\right]^{-2} (dr^2 + r^2d\Omega^2), \) where \(f = \frac{A}{\sqrt{1+Kr}}.\) |
| F-S  | [8]  | \(-D^2 \left((B-\sqrt{1+Cr^2})\cos \sqrt{1+Cr^2} + (A+\sqrt{1+Cr^2})\sin \sqrt{1+Cr^2}\right)^2 dt^2 + (1+Cr^2)dr^2 + r^2d\Omega^2\) |
| P-P I | [62] | \(-\left(a + br\frac{r^2}{b+yr^2} \right) dt^2 + \left(\frac{a+br\frac{r^2}{b+yr^2}}{a}\right)^{\frac{3}{2}} × \left(\frac{a+br\frac{r^2}{b+yr^2}}{a+(\sqrt{3}-1)b}\left(a+(\sqrt{3}-1)b\right)\left(a+br\frac{r^2}{b+yr^2}\right)\right)^{-1} dr^2 + r^2d\Omega^2,\) where \(f = a + \frac{\sqrt{17+7}r}{8} br\frac{r^2}{b+yr^2}\) |
| P-P II| [62] | \(-\left(a + br\frac{r^2}{b+yr^2} \right) dt^2 + \left(\frac{a+br\frac{r^2}{b+yr^2}}{a}\right)^{\frac{3}{2}} × \left(1 - \frac{\sqrt{17+7}r}{8} f + \frac{f^2}{2a}\right) + Av^2 f^{\frac{3}{2}+\frac{1}{2}} \right)^{-1} dr^2 + r^2d\Omega^2,\) |
| K-O III | [59] | \(-A(1+ar^2)^2 dt^2 + dr^2 + r^2d\Omega^2\) |
| K-O VI | [60] | \(-A(1+y)^2 dt^2 + [(1+br^2)(1+Cr^2)M]^{-1} dr^2 + r^2d\Omega^2,\) where \(M = \frac{(4+\sqrt{D^2+8+4b})^{2n}}{(2n+6+D+4b)^{2n+4}}, D = bd - 1, a = \frac{D}{\sqrt{D^2+y}}\) and \(y = \frac{1}{1+\sqrt{1+br^2}}.\) |
| K-O VII | [60] | \(-A(1+br^2)(D+N)^2 dt^2 + \frac{1+2br}{1+br} dr^2 + r^2d\Omega^2,\) where \(N = -\frac{\sqrt{17+7}r}{8} + \sqrt{2} ln(\sqrt{1+2br^2} + \sqrt{2} + 2br^2)\) |
| Burl I | [21] | \(-A(1+r^2)^{a\frac{r^2}{a+br^2}+1} dt^2 + (1+r^2)^{a\frac{r^2}{a+br^2}+1} dr^2 + r^2d\Omega^2,\) |
| Burl II| [21] | \(-A \sqrt{\frac{a+r^2}{a+\frac{a+r^2}{a+\frac{a+2r^2}{a+\frac{a+2r^2}{a+\frac{a+2r^2}{a+2r^2}}}}} dt^2 + B \frac{a+\frac{a+2r^2}{a+\frac{a+2r^2}{a+\frac{a+2r^2}{a+2r^2}}}}{a+\frac{a+2r^2}{a+\frac{a+2r^2}{a+\frac{a+2r^2}{a+2r^2}}}} dr^2 + r^2d\Omega^2.\) |
| Pant I | [21] | \(-\left(\frac{c+2r^2+f(r^2-2b)}{(c+f(r-4b))}\right)^2 dt^2 + \frac{2a+2b+m}{c+f(r-4b)} dr^2 + r^2d\Omega^2, where \(a = \frac{4+12m-3m^2}{8(m+2)}, \) and \(b = -\frac{1}{8(m+2)} \sqrt{16 - 96m + 152m^2 - 24m^4 + 16}\) |
| Pant II| [21] | \(-\left(\frac{c+2r^2+f(r^2-2b)}{(c+f(r-4b))}\right)^2 dt^2 + \frac{2a+2b+m}{c+f(r-4b)} dr^2 + r^2d\Omega^2,\) where \(3 \leq l \leq 4,\) and \(b = -\frac{1}{4}\sqrt{2^2 + 2l} \) |

\(21\) The metric of Duorah and Ray \[57\] did not satisfy Einstein's equations and this metric is the correction by \[8\].
## 3 Analysis

The following tables contain the metric name and state whether or not the metric satisfies the criteria explained in the introduction. Each metric is analyzed until a failure occurs. There is no column for criterion four. If a solution was found to be cosmological, a footnote states this, and no further analysis was completed. Although not a requirement, a plus sign in the last column is used to identify a solution where the sound speed monotonically decreases to the boundary. The metrics were analyzed with the package `GRTensorII` in conjunction with MapleV.

| name          | $G_r^r = G_\theta^\theta$ (iff) | Regular at the origin (iff) | $p$ and $\rho$ positive def. | $p$ and $\rho$ M.D. (example) | $\frac{dp}{d\rho} < 1$ |
|---------------|---------------------------------|-----------------------------|-----------------------------|-----------------------------|-------------------------|
| Schw Int.     | Y                               | Y                           | Y                           | Y ($\{A, B, R\}$ = $\{1, \frac{1}{2}, 2\}$) | N                       |
| Einstein      | Y                               | Y                           | N                           |                             |                         |
| de Sitter     | Y                               | Y                           | N                           |                             |                         |
| Kottler       | Y                               | Y $(m = 0)$                 | N                           |                             |                         |
| Tolman IV     | Y                               | Y                           | Y                           | Y $(\{A, R\}$ = $\{5, 10\}$) | Y                       |
| Tolman V      | Y                               | Y $(n = 0)$                 | N                           |                             |                         |
| Tolman VI     | Y                               | Y $(n = \pm 1)$             | N                           |                             |                         |
| Tolman VII    | Y                               | Y                           | Y $(\{A, R, B, C\}$ = $\{1, 0.54, 1, 20\}$) | Y+                       |
| Tolman VIII   | Y                               | Y $(\{a, b, m\}$ = $\{2, 0, 0\}$ $\rightarrow$ Tolman II) |                             |                             |                         |
| N-P-V Ia      | Y                               | Y $(n = 2, B = 1 \rightarrow$ Sch. Int.) |                             |                             |                         |
| N-P-V Ib      | Y                               | N                           |                             |                             |                         |
| N-P-V Ic      | N                               |                             |                             |                             |                         |
| N-P-V IIa     | Y                               | Y $(k = 0 \rightarrow$ flat space) |                             |                             |                         |

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22 In the Table, (iff) means if and only if the condition in the bracket is satisfied, positive def. means positive definite, and M.D. (example) means monotonically decreasing and a specific example is given in the bracket.

23 `GRTensorII` is a package which runs within MapleV. It is entirely distinct from packages distributed with MapleV and must be obtained independently. The `GRTensorII` software and documentation is distributed freely on the World-Wide-Web from the address [http://www.astro.queensu.ca/~grtensor/GRHome.html](http://www.astro.queensu.ca/~grtensor/GRHome.html) or [http://www.maths.soton.ac.uk/~dp/grtensor/](http://www.maths.soton.ac.uk/~dp/grtensor/). Worksheets which demonstrate some calculations reported here, and the associated metric files, can be downloaded from these sites.
| name  | $G^*_r = G^*_d$ (iff) | Regular at the origin (iff) | $p$ and $\rho$ positive def. | $p$ and $\rho$ M.D. (example) | $\frac{\partial p}{\partial \rho} < 1$ |
|-------|----------------------|---------------------------|-------------------------------|-----------------------------|--------------------------|
| N-P-V IIb | Y | N | | | |
| N-P-V IIc | Y | N | | | |
| P-V IIa | Y | Y | Y | $Y \{A, B, b, c, d\} = \{1, 3, 2, 1, 1\}$ | Y+ |
| P-V IIb | Y | Y | N | | |
| P-V IIc | Y ($c = 0$ or $b = 0$) or Sch. Int. | Y | Y | N | |
| P-V IV | Y | N | | | |
| P-V V | Y | N | | | |
| Wyman I | Y ($N = 2 \rightarrow$ Sch. Int.) | | | | |
| Wyman IIa | Y | $Y (n = 1)$ | Y | $Y \{(a, A, B) = \{-1, 1, -\frac{1}{2}\}\}$ | Y |
| | | $Y (n = -1)$ | Y | $Y \{(n, A, B) = \{-1, \frac{1}{2}, -1\}\}$ | Y |
| Wyman IIb | Y ($a = 0$) | N | | | |
| Wyman IIc | N | N | | | |
| Wyman III | Y ($n = 1$) | N | | | |
| Wyman IVa | Y | N | | | |
| Wyman IVb | Y ($A = 0$) | N | | | |
| Nariai III | Y ($k = 1 \rightarrow$ Sch. Int.) | | | | |
| Nariai IV | Y | $Y (A = \cos^2(b))$ | Y | $Y \{(a, b, M) = \{1, 2, 4, 1\}\}$ | Y+ |
| Nariai VI | Y | $Y (\alpha = 0 \rightarrow$ Sch. Int.) | | | |
| Nariai VII | Y | N | | | |
| Nariai VIII | Y | N | | | |
| Nariai IX | Y | N | | | |
| Nariai X | Y | N | | | |
| Buch1 | Y | Y | Y | $Y \{\{A, B, C\} = \{1, \frac{1}{2}, 1\}\}$ | Y |
| Buch2 | Y | $Y (a = 0 \rightarrow$ flat space) | | | |
| Mehra | Y | Y | Y | $Y \{(a, \rho_c) = \{0.99, 0.1\}\}$ | Y+ 24 |
| Buch3 | Y | Y | N | | |

24 Mehra’s solution is the only one where the pressure, density and speed of sound all equal zero at the boundary $r_b = a$.

25 Buch3 is regular iff $\xi = \frac{b - \sqrt{b^2 - 4ac}}{b + \sqrt{b^2 - 4ac}}$. 
| name     | \( G^*_r = G^*_θ \) (iff) | Regular at the origin (iff) | \( p \) and \( ρ \) positive def. | \( p \) and \( ρ \) M.D. (example) | \( \frac{∂p}{∂ρ} < 1 \) |
|----------|--------------------------|-----------------------------|-----------------------------------|---------------------------------|---------------------|
| Kucha    | \( Y \ (C = 0) \)       | N                           |                                   |                                 |                     |
| Kuchb Ia | \( Y \)                  | \( Y \ (B = 1 \rightarrow \text{Sch. Int.}) \) |                                   |                                 |                     |
| Kuchb Ib | \( Y \ (B = 0) \)       | N                           |                                   |                                 |                     |
| Kuchb Ic | N                        |                             |                                   |                                 |                     |
| Kuch1 Ia | \( Y \ (C = 2) \)       | N                           |                                   |                                 |                     |
| Kuch1 Ic | N                        |                             |                                   |                                 |                     |
| Kuch1 Ic | N                        |                             |                                   |                                 |                     |
| Kuch1 II | \( Y \)                  |                             |                                   |                                 |                     |
| Kuch1 IV | N                        |                             |                                   |                                 |                     |
| Kuch1 V  | N                        |                             |                                   |                                 |                     |
| Kuch2 I  | \( Y \)                  | \( Y \ (A = 0 \rightarrow \text{Tolman I}) \) |                                   |                                 |                     |
| Kuch2 III| \( Y \)                  | \( Y \)                     | \( Y \)                           | \( \{ A, C \} = \{ 5, -3 \} \) | \( Y+ \)            |
| Kuch2 IV | N                        |                             |                                   |                                 |                     |
| Kuch2 VI | \( Y \)                  | N                           |                                   |                                 |                     |
| Kuch2 VII| \( Y \)                  | N                           |                                   |                                 |                     |
| Whittaker| \( Y \)                  | \( Y \)                     | \( Y \)                           | N                               |                     |
| B-L      | \( Y \)                  | \( Y \)                     |                                   | N                               |                     |
| Kuch68 I | \( Y \)                  | N                           |                                   |                                 |                     |
| Kuch68 II| \( Y \)                  | N                           |                                   |                                 |                     |
| Leib I   | \( Y \ (a = 0 \rightarrow \text{Toman III}) \) |                   |                                   |                                 |                     |
| Leib IV  | \( Y \ (a = 0 \rightarrow \text{Tolman I}) \) |                   |                                   |                                 |                     |
| Heint IIa| \( Y \)                  | \( Y \)                     | \( Y \)                           | \( \{ a, A, C \} = \{ 1, 1, 1 \} \) | \( Y \)            |
| Heint IIb| \( Y \ (a = 0 \rightarrow \text{flat space or } C = 0) \) |                   | \( Y \)                           | \( N \)                        |                     |
| Heint IIIa| \( Y \ (b = 0 \rightarrow \text{flat space}) \) |                   |                                   |                                 |                     |
| Heint IIIb| \( Y \ (a = C = 0) \)   | N                           |                                   |                                 |                     |
| Heint IIIc| \( Y \)                  | \( Y \)                     | \( Y \)                           | \( N \)                        |                     |
| Kuch3 Ia | N                        |                             |                                   |                                 |                     |
| Kuch3 Ib | \( Y \)                  | N                           |                                   |                                 |                     |
| Kuch3 Ic | N                        |                             |                                   |                                 |                     |
| Kuch3 II | \( Y \ (B = 0) \)       | N                           |                                   |                                 |                     |
| Kuch3 III| N                        |                             |                                   |                                 |                     |
| name     | $G'_r = G_\theta^r$ (iff) | Regular at the origin (iff) | $p$ and $\rho$ positive def. | $p$ and $\rho$ M.D. (example) | $\frac{\partial p}{\partial \rho} < 1$ |
|----------|---------------------------|----------------------------|-------------------------------|-------------------------------|-------------------------------|
| Kuch3 IV | N                         |                            |                              |                               |                               |
| Kuch3 V  | N                         |                            |                              |                               |                               |
| Kuch71 I | N                         |                            |                              |                               |                               |
| Kuch71 II| Y                         | Y                          | Y                            | N                             |                               |
| Kuch71b  | Y                         |                             | N                             |                               |                               |
| Kuch5 I  | N                         |                            |                              |                               |                               |
| Kuch5 IV | Y ($A = 0 \to$ Sch. Int.)|                            |                              |                               |                               |
| Kuch5 VIII| Y ($a = 0 \to$ flat space)|                            |                              |                               |                               |
| Kuch5 IX | Y                         |                             | Y                             |                               |                               |
| Kuch5 X| N                         |                            |                              |                               |                               |
| Kuch5 XII| N                         |                            |                              |                               |                               |
| Kuch5 XIII| Y                         | Y                          | Y                             | Y²⁰ |                               |
| Kuch5 XV | Y ($B = 0 \to$ flat space)|                            |                              |                               |                               |
| Kuch5 XVI| N                         |                            |                              |                               |                               |
| R-R I    | Y                         | Y                          | Y²⁷                          |                               |                               |
| R-R II   | Y                         |                             | N                             |                               |                               |
| R-R III  | Y                         | Y ($A = B \pm 1$)          | N                             |                               |                               |
| R-R IV   | Y                         | Y ($B = 1$)                | N                             |                               |                               |
| R-R V    | Y                         |                             | N                             |                               |                               |
| R-R VI   | Y                         | Y ($n = 1, A = 1$ or $n = -1, B = 1$) | N                             |                               |                               |
| R-R VII  | Y                         | Y²⁸                         | N                             |                               |                               |
| R-R VIII | Y                         | Y ($\{a, b, m, B\} = \{2, 0, 0, 1\}$) | N                             |                               |                               |
| Kuch73 I | Y                         |                             | N                             |                               |                               |
| Kuch73 II| N                         |                            |                              |                               |                               |
| K-N-B    | Y                         | Y                          | Y²⁹                          | Y                             | N                             |
| G-G      | Y                         | Y²⁰                         | Y                             | N                             |                               |

²⁰Kuch5 XIII is a cosmological solution.
²⁷All these solutions have either a negative pressure or density, as was expected since Roy-Rao used the Buchdahl Transformation (BT) [65, 66]. A generalization of the BT that allows for positive pressure and density was completed by Stewart [47].
²⁸Regular iff $B \sin \left( \frac{1 + R^2 - A^2}{2R \cdot C} \right) = 1$.
²⁹G-G is regular iff $\frac{B}{B \cdot (2 + d)} \cdot \left( \frac{-b^2 \cdot d - 1 - c}{b^2 \cdot d - 1 + c} \right) \cdot (-b^2 \cdot d - 2b + 1)/c = 1$. 
| name          | $G_r^c = G_\theta^p$ (iff) | Regular at p and $\rho$ positive def. | $p$ and $\rho$ M.D. (example) | $\frac{dp}{d\rho} < 1$ |
|---------------|---------------------------|--------------------------------------|-------------------------------|-------------------|
| Bayin III     | N                         |                                      |                               |                   |
| Bayin VI      | Y (BC$^b = 1$)            | Y                                    | Y (N)                         |                   |
| Gold I        | Y                         |                                      | Y (Y$^{30}$)                  |                   |
| Gold II       | Y (BC$^b = 1$)            |                                      | Y                             |                   |
| Gold III      | Y (BC$^b = 1$)            |                                      | Y                             |                   |
| M-W I         | Y                         |                                      | Y                             |                   |
| M-W II        | Y (A = 0 $\rightarrow$ Tolman IV) |                              | Y                             |                   |
| M-W III       | N                         |                                      | Y                             |                   |
| K-K           | N                         |                                      | Y                             |                   |
| Stewart       | Y                         |                                      | Y                             |                   |
| P-S Ia        | Y (a = -2 $\rightarrow$ Tolman IV) |                              | Y                             |                   |
| P-S Ib        | Y                         |                                      | Y                             |                   |
| Durg IV       | Y                         |                                      | Y                             |                   |
| Durg V        | Y (a = -2 $\rightarrow$ Tolman IV) |                              | Y                             |                   |
| Whitman II    | Y                         |                                      | Y                             |                   |
| Whitman III   | N                         |                                      | Y                             |                   |
| Whitman IV    | Y                         |                                      | Y                             |                   |
| D-P-P I       | Y                         |                                      | Y                             |                   |
| V-P II        | Y (A = 0 $\rightarrow$ flat sp) |                              | Y                             |                   |
| P-S2          | Y                         |                                      | Y                             |                   |
| D-F           | Y (B = 0)                 |                                      | Y                             |                   |
| H-B I         | Y (B$^b/(1-a) = 1$)       |                                      | Y                             |                   |

30Gold I is regular iff $\frac{B}{a+2c+2a} \left( \frac{1+ac-bc}{1+c+c-\alpha} \right)^{-\alpha} = 1$.

31Gold II is regular iff $Be(-2-\sqrt{3})a/(c^2-1) \left( \frac{2a}{c^2(1+c)+1-\alpha} \right)^{-\alpha} = 1$. 

18
| name    | $G^r_\nu = G^\nu_\rho$ (iff) | Regular at the origin (iff) | $p$ and $\rho$ positive def. | $p$ and $\rho$ M.D. (example) | $\frac{dp}{dp} < 1$ |
|---------|-------------------------------|-----------------------------|-----------------------------|-------------------------------|------------------|
| H-B II  | Y                            | Y                           | N                           | Y                             |                  |
| F-S     | Y                            | Y                           | Y                           | Y                             |                  |
| P-P I   | Y                            | Y                           | $Y \{(a,b) = \{1,0\}$ → Einstein |                  |
| P-P II  | Y                            | $Y (b = 0)$                 | N                           |                  |
| K-O III | Y                            | Y                           | N                           |                  |
| K-O VI  | Y                            | Y                           | N                           |                  |
| K-O VII | $Y (b = 0 \rightarrow$ flat space) |                    | N                           |                  |
| Burl I  | N                            |                            |                             |                  |
| Burl II | N                            |                            |                             |                  |
| Pant I  | Y                            | Y                           | $Y (m = 2 \rightarrow$ Tolman IV |                  |
| Pant II | Y                            | N                           |                             |                  |

$^{32}$H-B II if regular iff $\frac{16b^2K^2(A+1)^4}{A^4+4b^2K^2-4A^2\ln A-1} = 1$. 

19
4 Discussion

Of the 127 solutions studied, only 16 satisfy all the criteria set out in section I, and 9 of these 16 were found to have the additional property of having the sound speed monotonically decrease with radius. About one solution in five failed to give isotropic pressure, and of those that did one quarter failed to be regular at the origin. Of the solutions that were regular at the origin one quarter failed to give positive energy density and pressure there. Of the solutions that formally allow an integration from the origin about one in five failed to produce monotone decreasing functions $\rho(r)$ and $p(r)$.

It is fair to say then that most of the spherically symmetric perfect fluid “exact solutions” of Einstein’s field equations that are in the literature are of no physical interest. The reason for this can be traced to the Tolman-Oppenheimer-Volkoff equation (11). Whereas it is now known that for a given value of $0 < p(0) < \infty$ there exists a unique global solution to equation (11) [68], this does not mean that closed-form solutions are easy to find. Indeed “one soon has to use numerical methods” [2]. For the most part the spacetimes examined in this paper were obtained by various simplifying assumptions developed in order to obtain an “exact solution”. What we have shown here is that this procedure almost never leads to a physically interesting conclusion.

The content of this work would make a useful addition to a database of solutions of Einstein’s field equations. The pioneering effort in this field is the “On-Line Invariant Classification Database” developed by Jim Skea.

Acknowledgments

We would like to thank M. MacCallum for his comments, suggestions and corrections, J. Skea for a preprint [4] with M. Finch which contained a number of references we had not found, and H. Knutsen, A. Krasinski and H. Stephani for comments. This work was supported in part by grants (to KL) from the Natural Sciences and Engineering Research Council of Canada and the Advisory Research Committee of Queen’s University.

33 Since the Einstein tensor is diagonal for the metrics under consideration, any such metric can be considered as an anisotropic “solution”.

34 The main site in Brazil is at edradour.symbcomp.uerj.br/, and mirrors are at www.astro.queensu.ca/~jimsk/ and www.maths.soton.ac.uk/~rdi/database. A keyword search with “static sphere” returns 7 Petrov type D and 2 Petrov type O spacetimes. A search with “perfect fluid” returns 31 type D and 10 type O spacetimes.
Appendix A

Consider the general diagonal, static, spherically symmetric metric

\[ ds^2 = -A(r)dt^2 + B(r)dr^2 + C(r)r^2d\Omega^2. \]  

(1)

The regularity conditions are

\[ A(0) = \text{const.}, B(0) = C(0) = 1, \]  

(2)

and

\[ A'(0) = B'(0) = C'(0) = 0. \]  

(3)

Note that the metric (1), if regular, is conformally flat at the origin so that the only non-vanishing invariants there are the Ricci invariants. In terms of the trace-free Ricci tensor \( S_a^a \) defined by

\[ S_a^b = R_a^b - \delta_a^b R/4, \]  

(4)

where \( R_a^b \) is the Ricci tensor, and \( R \) the Ricci scalar, the Ricci invariants at the origin reduce to

\[ R = 3(B''(0) - A''(0) - 3C''(0)), \]  

(5)

\[ r_1 \equiv S_a^b S_b^a/4 = 3(B''(0) + A''(0) - 3C''(0))^2/2^4, \]  

(6)

\[ r_2 \equiv -S_a^b S_b^c S_c^d/8 = 3(B''(0) + A''(0) - 3C''(0))^3/2^6, \]  

(7)

\[ r_3 \equiv S_a^b S_b^c S_c^d S_d^a = 21(B''(0) + A''(0) - 3C''(0))^4/2^{10}. \]  

(8)

The energy density and pressure at the origin reduce to

\[ 8\pi \rho(0) = 3B''(0)/2 - 9C''(0)/2, \]  

(9)

and

\[ 8\pi p(0) = A''(0) - B''(0)/2 + 3C''(0)/2 \]  

(10)

respectively. In isotropic coordinates \( C(r) = B(r) \), and \( C(r) = 1 \) in curvature coordinates.

Appendix B

The equilibrium is governed by the well known Tolman-Oppenheimer-Volkoff equation

\[ \frac{dp(r)}{dr} = -\frac{(\rho(r) + p(r))(m(r) + 4\pi p(r)r^3)}{r(r - 2m(r))} \]  

(11)

where \( m(r) \) is the effective gravitational mass

\[ m(r) = 4\pi \int_0^r \rho(s)s^2ds. \]  

(12)

It follows that

\[ \frac{dp}{dr}|_{r=0} = 0, \]  

(13)

35 Any static spherically symmetric line element can be brought into this form. 

21
and that
\[ d^2p/dr^2|_{r=0} = -4\pi(p(0) + 3p(0))(\rho(0) + p(0))/3, \quad (14) \]
so that the pressure is locally maximal with respect to \( r \) at the origin. If we assume that there is a smooth \((C^2)\) equation of state \( p(\rho) \) then we have
\[ dp/dr = (dp/d\rho)(d\rho/dr), \quad (15) \]
and
\[ d^2p/dr^2 = (d^2p/d\rho^2)(d\rho/dr)^2 + (dp/d\rho)(d^2\rho/dr^2). \quad (16) \]
As a result, \( \rho \) is locally maximal with respect to \( r \) at \( r = 0 \) or (not and) \( p \) is locally maximal with respect to \( \rho \) at \( r = 0 \).

If \( dp/d\rho > 0 \) \( (17) \) for \( p > 0 \), it is known that for a given value of \( 0 < p(0) < \infty \) there exists a unique global solution to equation (11) \[68\]. More recently this has been established without condition (17) \[69\]. Without an equation of state, specification of \( \rho(r) \) (or \( m(r) \)) reduces the problem to a single differential equation (11). The resultant implied equation of state is almost always of no interest.

**Appendix C**

The following table contains the original solution in the first column. The second column contains rediscoveries where the authors were unaware of the previous work. For example, Kuchowicz [7] rederived Wyman IIa [18], but gave credit to Adler [74] as the originator, while Leibovitz [33] also rederived Wyman IIa but was unaware of Wyman’s work. We do not claim that this table is complete.

| original         | rederived by         |
|------------------|----------------------|
| Tolman IV [14]   | K-GT-P [70]          |
| Tolman V [14]    | D-G [71]             |
| Tolman VI [14]   | P-V [17], Klein [72] |
| Tolman VII [14]  | D-P-P II [53]        |
| Wyman IIa \( (n=1) \) [18] | Leib II [33], Heint IIIc [34], DurgII [49], Adler [74], A-C [75], Kuch75 [76], Whitman [77] |
| Buch1 [21]       | D-B [73]             |
| Kuch2 I [29]     | Krori [78], I-S II [79] |
| Kuch2 VI [29]    | Heint IIIc [34]      |
| Kuch2 III [29]   | Leib III [33]        |
| Leib I [33]      | Heint IIIc [34]      |
| Heint IIa [34]   | Korkina [50], Durg III [49] |
| Heint IIIe [34]  | Bayin IV [43]        |
| Kuch5 XIII [38]  | Bayin VII [43]       |
| K-N-B [41]       | Bayin II [43]        |

\[36\] The metric given by Leibovitz is incorrect, although the correct form was already given by Kuchowicz.
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