Primitive Values of Rational Functions at Primitive Elements of a Finite Field

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Abstract

Given a prime power $q$ and an integer $n \geq 2$, we establish a sufficient condition for the existence of a primitive pair $(\alpha, f(\alpha))$ where $\alpha \in \mathbb{F}_q$ and $f(x) \in \mathbb{F}_q(x)$ is a rational function of degree $n$. (Here $f = f_1/f_2$, where $f_1, f_2$ are coprime polynomials of degree $n_1, n_2$, respectively, and $n_1 + n_2 = n$.) For any $n$, such a pair is guaranteed to exist for sufficiently large $q$. Indeed, when $n = 2$, such a pair definitely does not exist only for 28 values of $q$ and possibly (but unlikely) only for at most 3911 other values of $q$.

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1 Introduction

Throughout this article let $q$ be a prime power and $n \geq 2$ be a positive integer. We use $\mathbb{F}_q$ to denote the finite field of order $q$ and $\mathbb{F}_q^*$ for the cyclic group of nonzero multiplicative elements of $\mathbb{F}_q$. A generator of the cyclic group $\mathbb{F}_q^*$ is called a primitive element of $\mathbb{F}_q$. For a rational function $f(x) \in ...$
In $\mathbb{F}_q(x)$ and $\alpha \in \mathbb{F}_q$, we call a pair $(\alpha, f(\alpha)) \in \mathbb{F}_q \times \mathbb{F}_q$ a \textit{primitive pair} in $\mathbb{F}_q$ if both $\alpha$ and $f(\alpha)$ are primitive elements in $\mathbb{F}_q$. Primitive elements have many applications in cryptography, see [11]. The security of many cryptographic schemes (e.g., Diffie-Hellman key exchange and ElGamal encryption scheme) relies on the computational intractability of finding solutions to Discrete Logarithm Problem, which uses primitive elements as its fundamental tool.

Broadly, our aim in this article is to classify finite fields for which there exists a primitive pair in $\mathbb{F}_q(x)$ for general rational functions $f(x) \in \mathbb{F}_q(x)$. In order to make this more precise, we introduce some terminology and conventions.

First, to say that a polynomial $f(x) \in \mathbb{F}_q[x]$ has degree $n \geq 0$ we mean that $f(x) = a_n x^n + \cdots + a_0$, where $a_n \neq 0$; in particular, $f$ is non-zero. Next, let $f(x) = f_1(x)/f_2(x)$ be a rational function in $\mathbb{F}_q(x)$, where $f_1, f_2$ are polynomials of degree $n_1, n_2$, respectively. In our study, we always assume $f$ is expressed in its lowest terms, i.e., $f_1$ and $f_2$ are coprime in which case call the function $f$ as described an $(n_1, n_2)$-function having whose degree is $\text{deg}(f) = n_1 + n_2$. Observe that $(\alpha, f(\alpha))$ is a primitive pair if and only if $(\alpha, (1/f)(\alpha))$ is a primitive pair. Hence, replacing $f$ by $1/f$, if necessary, we can suppose $n_1 \geq n_2$. Further, we can divide each of $f_1$ and $f_2$ by the leading coefficient of $f_2$ and suppose that $f_2$ is monic.

Finally, we introduce a minor restriction on the shape of $f$ to avoid some exceptional or awkward cases, namely, we suppose that $f$ is \textit{not exceptional}, i.e., \textit{not} of the form $f(x) = c x^j g^{d}(x)$, where $d > 1$ divides $q-1$ and $c \in \mathbb{F}_q^*$ for any rational function $g(x) \in \mathbb{F}_q(x)$. As way of explanation, we observe first that if $f(x) = g^d > 1(x)$, where $d > 1$ divides $q-1$, then $f(x)$ necessarily is a $d$th power and therefore cannot be primitive. Further, for example, if $f(x) = cx g^2(x)$, where $c$ is a non-square in $\mathbb{F}_q$, then, if $\alpha$ is primitive (and so a non-square), then $f(\alpha)$ is a square and so necessarily not primitive.

The question of the existence of primitive pairs has previously been considered in various cases of rational functions. For instance, Cohen [3] solved the existence problem for the specific $(1,0)$-function $x + 1$ and, in [5], Cohen et al. identified all finite fields for which there exists a primitive pair for every standard $(1,0)$-function (i.e., linear polynomial).

Recently, Booker et al. [1] classified all finite fields for which there exists a primitive pair for every (non-exceptional) $(2,0)$-function, i.e., quadratic polynomials (not of the form $c(x + \beta)^2$ for $c \in \mathbb{F}_q^*, \beta \in \mathbb{F}_q$).

Wang et al. [15] and Cohen [4] studied the existence problem for primitive
pairs in respect of the specific (2,1)-function \((x^2 + 1)/x\) for fields of even order and, more recently, Cohen et al. [6] (Corollary 2) provided a complete solution for the (2,1)-functions \((x^2 \pm 1)/x\).

Anju and Sharma [13] supplied a sufficient condition for the existence of primitive pairs for the general (2,1)-function. (See also [14].) Recently in [12], Sharma, Ambrish and Anju established a similar sufficient condition for the general (2,2)-function.

In this paper, we take \(f(x)\) to be a general rational function of degree \(n\) and prove the existence of primitive pairs \((\alpha, f(\alpha))\) in \(F_q\) for sufficiently large prime powers \(q\). To make this more precise, for each positive integer \(n\), let \(R_n\) be the set of non-exceptional rational functions \(f = f_1/f_2\), (with \(f_1, f_2\) coprime and \(f_2\) monic) of degree \(n\) (where \(n = n_1 + n_2\) and \(n_1 \geq n_2\)) and define \(Q_n\) as the set of prime powers \(q\) such that, for each \(f \in R_n\), there exists a primitive pair \((\alpha, f(\alpha)), \alpha \in \mathbb{F}_q\). For any positive integer define \(W(m) = 2^{\omega(m)}\), where \(\omega(m)\) is the number of distinct prime divisors of \(m\). (Thus, \(W(m)\) is the number of square-free divisors of \(m\).) The main theorem to be proved is the following:

**Theorem 1.1.** Let \(n \geq 2\) and \(q\) be a prime power. Suppose
\[
q^{\frac{n}{2}} > nW(q - 1)^2.
\] (1)

Then \(q \in Q_n\).

Hence, for each \(n \geq 2\), there exists \(C_n > 0\) such that, if \(q > C_n\), then \(q \in Q_n\).

Using a sieving modification of Theorem 3.1 we also give explicit values for \(C_n, n = 2, 3, 4,\) and 5, and conjecture that the best (least) value of \(C_2\) is 311.

We remark that, for a specific rational function \(f\) of degree \(n\) (for example, if \(f_1\) or \(f_2\) is not square-free), one could reduce the factor \(n\) on the right side of condition (1) by an appropriate amount.

We defer a study of those exceptional rational functions \(f\) for which there generally exists a primitive pair \((\alpha, f(\alpha))\) to another occasion.

## 2 Preliminaries

In this section, we state some related definitions and results required in the paper. For a divisor \(u\) of \(q - 1\), an element \(w \in \mathbb{F}_q^*\) is called \(u\)-free, if \(w = v^d\), for some \(v \in \mathbb{F}_q^*\).
where \( v \in \mathbb{F}_q \) and \( d | u \) implies \( d = 1 \). Note that an element \( w \in \mathbb{F}_q^* \) is \((q-1)\)-free if and only if it is primitive.

We refer [3] for basics on finite fields and characters of finite fields. Following Cohen and Huczynska [7], [8], it can be shown that for each divisor \( u \) of \( q-1 \)
\[
\sum_{d \mid u} \mu(d) \sum_{\chi_d} \chi_d(\alpha),
\]
where \( \theta(u) = \frac{\phi(u)}{u} \) (where \( \phi \) is Euler’s totient function), \( \mu \) is Möbius function and \( \chi_d \) denotes the multiplicative character of \( \mathbb{F}_q \) of order \( d \), gives a characteristic function for the subset of \( u \)-free elements of \( \mathbb{F}_q^* \).

We shall need the following result of Weil [16], as described in [2] at (1.2) and (1.3), for our main theorem.

**Lemma 2.1.** Let \( F(x) \in \mathbb{F}_q(x) \) be a rational function. Write \( F(x) = \prod_{j=1}^k F_j(x)^{r_j} \), where \( F_j(x) \in \mathbb{F}_q[x] \) are irreducible polynomials and \( r_j \) are non zero integers. Let \( \chi \) be a multiplicative character of \( \mathbb{F}_q \) of precise square-free order \( d \) (a divisor of \( q-1 \)). Suppose that \( F(x) \) is not of the form \( cG(x)^d \) for some rational function \( G(x) \in \mathbb{F}_q(x) \) and \( c \in \mathbb{F}_q^* \). Then we have
\[
\left| \sum_{\alpha \in \mathbb{F}_q, F(\alpha) \neq \infty} \chi(F(\alpha)) \right| \leq \left( \sum_{j=1}^k \deg(F_j) - 1 \right) q^{\frac{1}{2}}.
\]

A preliminary of another kind is subdivision of rational functions of degree \( n \) into the union of \( (n_1, n_2) \)-functions for every pairs \((n_1, n_2)\) with \( n_1 \geq n_2 \) and \( n_1 + n_2 = n \), as described in Section 1. Indeed, for each such pair \((n_1, n_2)\), define \( R_{n_1, n_2} \) as the set of non exceptional \((n_1, n_2)\)-rational functions, \( Q_{n_1, n_2} \) as the set of prime powers \( q \) such that for each \( f \in R_{n_1, n_2} \) there exists a primitive pair \( (\alpha, f(\alpha)) \) and \( C_{n_1, n_2} \) as a valid bound such that, if \( q > C_{n_1, n_2} \), then \( q \in Q_{n_1, n_2} \). Of course, our aim would be to find the least possible value for \( C_{n_1, n_2} \) in every case, whence \( C_n \) would be the maximum of the values of \( C_{n_1, n_2} \) over the pairs \((n_1, n_2)\) with \( n_1 \geq n_2 \) and \( n_1 + n_2 = n \). More generally, for a set of rational functions \( S \), define \( Q_S \) and \( C_S \) to say that \( q > C_S \) implies \( q \in Q_S \) in the above sense. For the present, simply observe the following. Suppose \( f = f_1/f_2 \) is a rational function with \( n_1 = n_2 = n/2 \). We always assume that \( f_1 \) and \( f_2 \) are coprime but suppose one of them is divisible by a positive
power of \( x \). In that case, the rational function \( f^*(x) = f(1/x) \) written in its lowest terms has degree \( n_0 < n \). Moreover, since \( \alpha \) a primitive element implies \( 1/\alpha \) is a primitive element, it follows that, if \( (\alpha, f^*(\alpha)) \) is a primitive pair, then \( (1/\alpha, f(1/\alpha)) \) is a primitive pair. Consequently, in effect, \( f \) can be considered as having degree \( n_0 < n \) and therefore, when considering rational functions of degree \( n \), if \( n_1 = n_2 \), we can suppose that both \( f_1 \) and \( f_2 \) have non zero constant terms. For example, suppose \( f(x) = a(x + b)/x, ab \neq 0 \) so that \( f \in R_{1,1} \). Then \( f^*(x) = ab(x + 1/b) \in R_{1,0} \) and hence we can deduce \( C(f) = 61 \), form \([5]\).

3 Sufficient conditions for the existence of primitive pairs in \( \mathbb{F}_q \)

For each \( m \in \mathbb{N} \), suppose \( \omega(m) \) denotes the number of prime divisors of \( m \) and \( W(m) \) denotes the number of square free divisors of \( m \). Let \( l_1, l_2 \in \mathbb{N} \) be such that if \( l_1, l_2 \) divide \( q - 1 \), then for each \( f(x) \in R_n \), \( N_f(l_1, l_2) \) denote the number of elements \( \alpha \in \mathbb{F}_q \) such that \( \alpha \) is \( l_1 \)-free and \( f(\alpha) \) is \( l_2 \)-free.

We now prove our one of the main results as follows.

**Theorem 3.1.** Let \( n \geq 2 \), and \( q \) be a prime power. Suppose that

\[
q^{\frac{1}{2}} > nW(q - 1)^2. \tag{2}
\]

Then \( q \in Q_n \).

**Proof.** To prove that \( q \in Q_n \), we need to show that \( N_f(q - 1, q - 1) > 0 \) for every (non-exceptional) \( f(x) \in R_n \). Now let \( f(x) \in R_n \) be any rational function. Let \( S \) be the set of poles of \( f(x) \) in \( \mathbb{F}_q \). Assume \( q > 2 \) (as we may) and \( l_1 > 1 \) and \( l_2 > 1 \) are divisors of \( q - 1 \). Then by definition we have

\[
N_f(l_1, l_2) = \sum_{\alpha \in \mathbb{F}_q \setminus S} \rho_{l_1}(\alpha)\rho_{l_2}(f(\alpha))
\]

and hence

\[
N_f(l_1, l_2) = \theta(l_1)\theta(l_2) \sum_{d_1|l_1, d_2|l_2} \frac{\mu(d_1)\mu(d_2)}{\phi(d_1)\phi(d_2)} \sum_{\chi d_1, \chi d_2} \chi_f(\chi d_1, \chi d_2), \tag{3}
\]

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where
\[ \chi_f(\chi_{d_1}, \chi_{d_2}) = \sum_{\alpha \in \mathbb{F}_q \setminus S} \chi_{d_1}(\alpha) \chi_{d_2}(f(\alpha)). \] \tag{4}\\

Let \( d_1 \) and \( d_2 \) be divisors of \( q - 1 \) (not both 1) and \( \chi_{d_1} \) and \( \chi_{d_2} \) be specific characters of orders \( d_1, d_2 \), respectively. In view of the Mőbius functions in (3) we can suppose that \( d_1 \) and \( d_2 \) are square-free.

First suppose that \( d_2 = 1 \), i.e., \( \chi_{d_2} = \chi_1 \) is the trivial character. Then \( |\chi_f(\chi_{d_1}, \chi_1)| \) is at most the sum of the number of zeros and poles of \( f \) and so does not exceed \( n \).

Accordingly, suppose \( d_2 > 1 \). Let \( d \) be the least common multiple of \( d_1 \) and \( d_2 \), and so a square-free divisor of \( q - 1 \). Moreover, \( d/d_1 \) and \( d_1 \) are coprime, as are \( d/d_2 \) and \( d_2 \). Further, there is a character \( \chi_d \) of order \( d \), such that \( \chi_{d_2} = \chi_d^{d/d_2} \). In that case, \( \chi_{d_1} = \chi_d^k \) for some integer \( k \) with \( 0 \leq k < q - 1 \).

From (4)
\[ \chi_f(\chi_{d_1}, \chi_{d_2}) = \sum_{\alpha \in \mathbb{F}_q \setminus S} \chi_d(\alpha^k f(\alpha)^{d/d_2}) = \sum_{\alpha \in \mathbb{F}_q \setminus S} \chi_d(F(\alpha)), \]
where \( F(x) = x^k f^{d/d_2}(x) \).

Now write \( f(x) = x^j f_0(x) \), where \( j \) is some integer (positive, negative or zero) and \( f_0 \) is a rational function such that \( x \) divides neither the numerator nor denominator of \( f_0(x) \). Thus \( F(x) = x^{k + \frac{ad}{d_2}} f_0^{d/d_2}(x) \). We can now apply Lemma 2.1 unless \( f_0^{d/d_2} = c^{d/d_2} G^d \) for some rational function \( G \) and \( c \in \mathbb{F}_q \). The latter, however, would imply that \( f(x) = cx^j G^{d_2}(x) \), where we have assumed \( d_2 > 1 \), which would mean that \( f \) is exceptional. Since \( f \) is not exceptional and the number of distinct zeros and poles of \( F \) in an algebraic closure of \( \mathbb{F}_q \) is at most \( n + 1 \), we conclude from Lemma 2.1 that
\[ |\chi_f(\chi_{d_1}, \chi_{d_2})| \leq n q^{\frac{1}{2}}. \] \tag{5}\\

Of course, (5) holds when \( d_2 = 1 \) (and \( d_1 > 1 \)). On the other hand, trivially,
\[ \chi_f(\chi_1, \chi_1) \geq q - 1 - (n + 1). \] \tag{6}\\

Combining (5) and (6) in (4), we obtain
\[ N_f(l_1, l_2) \geq \theta(l_1) \theta(l_2) \left\{ (q - (n + 1)) - n q^{\frac{1}{2}} (W(l_1) W(l_2) - 1) \right\} \]
\[ > \theta(l_1) \theta(l_2) \left\{ q - n q^{\frac{1}{2}} W(l_1) W(l_2) \right\}. \]
certainly, whenever \( q > nW(l_1)W(l_2) \). It follows that, if \( q > nW(l_1)W(l_2) \), then \( N_f(l_1, l_2) > 0 \). In particular, the theorem follows by taking \( l_1 = l_2 = q - 1 \). \( \square \)

For further calculation work we shall need following results. Their proofs have been omitted as they follow on ideas from [4] and [10].

**Lemma 3.2.** For each \( m \in \mathbb{N} \), \( W(m) \leq c_m m^{\frac{1}{6}} \), where \( c_m = \frac{2^r}{(p_1 \cdots p_s)^{\frac{1}{6}}} \), and \( p_1, \ldots, p_s \) are the distinct primes less than 64 which divide \( m \).

In particular, for all \( m \in \mathbb{N} \), \( c_m < 37.469 \), and for all odd \( m \), \( c_m < 21.029 \).

**Theorem 3.3.** Let \( l \mid (q - 1) \), and \( \{p_1, \ldots, p_s\} \) be the collection of all primes dividing \( q - 1 \) but not \( l \). Suppose \( \delta = 1 - 2 \sum_{i=1}^{s} \frac{1}{p_i} \), \( \delta > 0 \) and \( \Delta = \frac{(2s-1)}{\delta} + 2 \). If \( q^{\frac{3}{2}} > n\Delta W(l)^2 \) then \( q \in Q_n \).

## 4 Rational functions of degree 2

From Section 1, and the last paragraph of Section 2, we can classify rational functions of degree 2 as either \((2,0)\)-functions, i.e., quadratic polynomials \( ax^2 + bx + c \), where \( a(b^2 - 4ac) \neq 0 \), or \((1,1)\)-functions with non-zero constant terms, thus having the form \( a(x + b)/(x + c) \), \( abc(b - c) \neq 0 \). One can work with both the cases simultaneously. But it is appropriate to recall that by a demanding theoretical and computational analysis it has been established in [1], that \( C_{2,0} = 211 \) is a valid bound, and that this is the minimum possible. In this section, we shall find an explicit (though non-optimal) value for \( C_{1,1} \) and thereby one for \( C_2 \) by means of Theorems 3.1 and 3.3. However, our argument assumes merely that the functions we consider are in \( R_2 \) (rather than being restricted to \( R_{1,1} \)).

Suppose that \( q \) is a prime power and \( n = 2 \). From Lemma 3.2 \( W(q - 1) \leq 37.469q^{\frac{1}{6}} \) so that \( 2W(q - 1)^2 < 2807.852q^{\frac{3}{3}} \). Hence (2) holds whenever \( q > 4.901 \times 10^{20} \) in which case by Theorem 3.1 necessarily \( q \in Q_2 \). (Indeed when \( q \) is even, by Lemma 3.2 it suffices that \( q > 4.787 \times 10^{17} \).) Now suppose \( \omega(q - 1) \geq 17 \). Then \( q \geq 2 \times 3 \times 5 \times 7 \times \cdots \times 59 > 1.9 \times 10^{21} \) so that \( q \in Q_2 \). (When \( q \) is even and \( \omega(q - 1) \geq 15 \), then \( q \geq 3 \times 5 \times 7 \times \cdots \times 53 > 1.6 \times 10^{19} \), so that \( q \in Q_2 \).)

We can therefore assume that \( \omega(q - 1) \leq 16 \) and \( q \leq 4.901 \times 10^{20} \). To make further progress, we use the sieving Theorem 3.3 in place of Theorem
In Theorem 3.3, suppose \( 5 \leq \omega(q - 1) \leq 16 \) and take \( l \) as the product of the least 5 primes in \( q - 1 \), i.e. \( W(l) = 2^5 \). Then \( s \leq 11 \) and \( \delta \) will be at its least value when \( \{p_1, p_2, \cdots, p_{11}\} = \{13, 17, \cdots, 53\} \), i.e. the set of primes from 6th to 16th. This yields \( \delta > 0.173170 \) and \( \Delta < 123.267943 \), so that \( 2\Delta W(l)^2 < 2.52453 \times 10^5 \). From Theorem 3.3 provided \( q^\frac{1}{2} > 2.52453 \times 10^5 \) i.e. \( q > 6.3733 \times 10^{10} \), then \( q \in Q_2 \). In fact, if \( \omega(q - 1) \geq 11 \), then \( q > 2 \times 10^{11} \), which means we can assume \( \omega(q - 1) \leq 10 \).

Now repeat this procedure using Theorem 3.3 with \( 4 \leq \omega(q - 1) \leq 10 \) and \( W(l) = 2^4 \). Then \( s \leq 6 \), \( \delta > 0.2855034 \), \( \Delta < 40.5284367 \), \( 2\Delta W(l)^2 < 20751 \), whence \( q \in Q_2 \) provided \( q > 4.3061 \times 10^8 \) which is bound to be the case. But, \( w(q - 1) \geq 10 \), gives \( q > 6.46 \times 10^9 \). Hence the result holds for \( \omega(q - 1) = 10 \). Next, we assume \( 4 \leq \omega(q - 1) \leq 9 \), take \( W(l) = 2^4 \) so that \( s \leq 5 \), \( \delta > 0.3544689 \), \( \Delta < 27.3900959 \), \( 2\Delta W(l)^2 < 14024 \). Which proves the result for \( \omega(q - 1) = 9 \).

We apply the procedure when \( 3 \leq \omega(q - 1) \leq 8 \) with limited success. Take \( \omega(l) = 3 \) so that \( s \leq 5 \), \( \delta > 0.1557111 \), \( \Delta < 59.7993247 \) and \( 2W(l)^2 < 7655 \). Hence \( q \in Q_2 \) whenever \( q > 5.86 \times 10^7 \).

Finally, for \( q < 5.86 \times 10^7 \), we coded the criterion of Theorem 3.3 and obtained an explicit list of 3937 possible exceptions for which the criterion failed even when the exact prime factorization of \( q - 1 \) was used (see the appendix). The largest of these prime powers is 33093061. We summarise these results for rational functions in \( R_{1,1} \) in the next theorem.

**Theorem 4.1.** For rational functions in \( f(x) = \frac{a(x + b)}{(x + c)} \), where \( a, b, c \in \mathbb{F}_q^* \) with \( b \neq c \), then the bound 33093061 is a valid value for \( C_{1,1} \).

Of course, the value of \( C_{1,1} \) shown in Theorem 4.1 is not optimal. In the other direction we worked on the possible exceptions below 10000 computationally in GAP [9] and obtained a list of true exceptions as follows:

**Case 1.** \( (f(x) \in R_{1,1}) \)
\( q = 3, 4, 5, 7, 9, 11, 13, 16, 19, 23, 25, 29, 31, 37, 41, 43, 49, 61, 67, 71, 73, 79, 103, 121, 139, 151, 211 \) and 331.

**Case 2.** \( (f(x) \in R_{2,0}) \)
\( q = 3, 4, 5, 7, 11, 13, 19, 25, 31, 37, 41, 43, 61, 67, 71, 73, 79, 121, 151, \) and 211.

From [1], we know that the above is a complete list of genuine exceptions in Case 2, and \( C_{2,0} = 211 \). Analogously, we propose the following conjecture.

**Conjecture 1.** We have \( C_{1,1} = 331 \) and the list of prime powers not in \( Q_{1,1} \) is shown in Case 1, as above.
We complete this section with some remarks on the set \( S \) of exceptional quadratic polynomials, whose members comprise quadratics of the form \( f(x) = a(x + b)^2 \), where \( ab \neq 0 \). In the context of Lemma 2.1 their irreducible part is of degree 1 and hence the condition of Theorem 3.1 applies with \( n = 1 \). Here, if \((\alpha, f(\alpha))\) is primitive, then necessarily \( a \) is a non square, in which case it suffices that \( \alpha \) is primitive and \( a(\alpha + b)^2 \) is \( L \)-free, where \( L \) is the odd part of \( q - 1 \). Denote by \( R_{12,0} \) the subset of \( S \) for which \( a \) is a non-square. By methods of this section this will lead to a better (smaller) lower bound for \( C_{12,0} \) than the one shown in Theorem 4.1 for \( C_{1,1} \).

5 Case \( n=3, 4 \) and 5

In this section, we demonstrate how to get at least one value \( C_n \) for each \( n \in \mathbb{N} \) and \( n \geq 2 \). Further, we provide some calculated values to reduce the bound \( C_n \) for \( n = 3, 4 \) and 5.

As described above, Theorem 3.1 and Lemma 3.2 together imply that if \( n(37.469)^2q^\frac{1}{2} < q^\frac{1}{2} \) then \( q \in Q_n \) i.e. \( q > n^6(37.469)^{12} \) implies \( q \in Q_n \). Hence, for each \( n \in \mathbb{N} \), \( n \geq 2 \), one value of \( C_n \) is \( n^6(37.469)^{12} \approx n^6 \times 7.65713 \times 10^{18} \).

Thus \( q > 5.583 \times 10^{21}, q > 3.137 \times 10^{22} \) and \( q > 1.197 \times 10^{23} \) imply \( q \in Q_3, q \in Q_4, \) and \( q \in Q_5 \), respectively. If \( \omega(q-1) \geq 18 \) then \( q \geq 2 \times 3 \times 5 \times \cdots \times 61 > 1.1728 \times 10^{23} \), and if \( \omega(q-1) \geq 19 \) then \( q \geq 2 \times 3 \times 5 \times \cdots \times 67 > 7.858 \times 10^{24} \). Hence \( \omega(q-1) \geq 18 \) implies \( q \in Q_3, q \in Q_4, \) and \( \omega(q-1) \geq 19 \) implies \( q \in Q_5 \). The repeated application of Theorem 3.3 (as discussed above in the case \( n = 2 \)), with the values in Tables 1, 2 and 3, provide the bounds \( C_3 \approx 4.426 \times 10^8, C_4 \approx 7.867 \times 10^8, \) and \( C_5 \approx 1.23 \times 10^9 \), respectively.

### Table 1

| Sr. No. | \( a \leq \omega(q-1) \leq b \) | \( W(l) \) | \( \delta > \) | \( \Delta < \) | \( 3\Delta W(l)^2 < \) |
|---------|---------------------------------|----------|-------------|-------------|------------------|
| 1       | \( a = 5, b = 17 \)            | \( 2^5 \) | 0.1392719   | 167.1445296 | 513468          |
| 2       | \( a = 4, b = 11 \)            | \( 2^4 \) | 0.2209872   | 60.8269154  | 46716           |
| 3       | \( a = 4, b = 9 \)             | \( 2^4 \) | 0.3544689   | 27.3900959  | 21036           |
Table 2

| Sr. No. | \(a \leq \omega(q - 1) \leq b\) | \(W(l)\) | \(\delta >\) | \(\Delta <\) | \(4\Delta W(l)^2 <\) |
|---------|----------------|--------|--------|--------|----------------|
| 1       | \(a = 5, b = 17\) | \(2^5\) | 0.1392719 | 167.1445296 | 684624 |
| 2       | \(a = 4, b = 11\) | \(2^4\) | 0.2209872 | 60.8269154 | 62287 |
| 3       | \(a = 4, b = 9\) | \(2^4\) | 0.3544689 | 27.3900959 | 28048 |

Table 3

| Sr. No. | \(a \leq \omega(q - 1) \leq b\) | \(W(l)\) | \(\delta >\) | \(\Delta <\) | \(5\Delta W(l)^2 <\) |
|---------|----------------|--------|--------|--------|----------------|
| 1       | \(a = 5, b = 18\) | \(2^6\) | 0.1064850 | 236.7747170 | 1212287 |
| 2       | \(a = 4, b = 11\) | \(2^4\) | 0.2209872 | 60.8269154 | 77859 |
| 3       | \(a = 4, b = 9\) | \(2^4\) | 0.3544689 | 27.3900959 | 35060 |

Note that, similar reduction can be done for each \(n\). All the results of this section can be summarized in the following theorem.

**Theorem 5.1.** For each \(n \in \mathbb{N}, n \geq 2\), one of the value for \(C_n\) is \(n^6 \times 7.65713 \times 10^{18}\). For \(n = 3, 4\) and \(5\) it can be reduced to \(4.426 \times 10^8, 7.867 \times 10^8\) and \(1.23 \times 10^9\) respectively.

Theorem 3.1 and Theorem 5.1 together prove the main result of this article stated in Theorem 1.1.

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**References**

[1] A. Booker, S. D. Cohen, N. Sutherland, and T. Trudgian. Primitive values of quadratic polynomials in a finite field. *Math. Comp.*, 88(318):1903–1912, 2019.
[2] T. Cochrane and C. Pinner. Using Stepanov’s method for exponential sums involving rational functions. *J. Number Th.*, 116(2):270–292, 2006.

[3] S. D. Cohen. Consecutive primitive roots in a finite field. *Proc. Amer. Math. Soc.*, 93:189–197, 1985.

[4] S. D. Cohen. Pairs of primitive elements in fields of even order. *Finite Fields Appl.*, 28:22–42, 2014.

[5] S. D. Cohen, T. O. e Silva, N. Sutherland, and T. Trudgian. A proof of the conjecture of Cohen and Mullen on sums of primitive roots. *Math. Comp.*, 84(296):2979–2986, 2015.

[6] S. D. Cohen, T. O. e Silva, N. Sutherland, and T. Trudgian. Linear combinations of primitive elements of a finite field. *Finite Fields Appl.*, 51:388–406, 2018.

[7] S. D. Cohen and S. Huczynska. The primitive normal basis theorem—without a computer. *J. London Math. Soc.*, 67(1):41–56, 2003.

[8] S. D. Cohen and S. Huczynska. The strong primitive normal basis theorem. *Acta Arith.*, 143:299–332, 2010.

[9] The GAP Group. *GAP – Groups, Algorithms, and Programming, Version 4.10.1*, 2019.

[10] A. Gupta, R. K. Sharma, and S. D. Cohen. Primitive element pairs with one prescribed trace over a finite field. *Finite Fields Appl.*, 54:1–14, 2018.

[11] C. Paar and J. Pelzl. *Public-Key Cryptosystems Based on the Discrete Logarithm Problem*, pages 205–238. Springer Berlin Heidelberg, Berlin, Heidelberg, 2010.

[12] R. K. Sharma, A. Awasthi, and A. Gupta. Existence of pair of primitive elements over finite fields of characteristic 2. *J. Number Th.*, 193:386–394, 2018.

[13] R. K. Sharma and A. Gupta. Existence of some special primitive normal elements over finite fields. *Finite Fields Appl.*, 46:280–303, 2017.

[14] R. K. Sharma and A. Gupta. Pair of primitive elements with prescribed traces over finite fields. *Comm. Alg.*, 47(3):1278–1286, 2019.
[15] P. Wang, X. Cao, and R. Feng. On the existence of some specific elements in finite fields of characteristic 2. *Finite Fields Appl.*, 18(4):800–813, 2012.

[16] André Weil. On some exponential sums. *Proc. Nat. Acad. Sci.*, 34(5):204–207, 1948.