Multiparametric Quantum Algebras
and the Cosmological Constant

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Abstract

With a view towards applications for de Sitter, we construct the multi-parametric $q$-deformation of the $so(5,\mathbb{C})$ algebra using the Faddeev-Reshetikhin-Takhtadzhyan (FRT) formalism.
1 Introduction

There are reasons based on arguments of holography and finiteness of entropy, to believe that the Hilbert space for quantum theory in a de Sitter background is finite dimensional \([1, 3, 4]\). Since the isometry group of de Sitter, SO\((4,1)\), has to be represented on this Hilbert space, and since we expect that quantum theory is unitary, this gives rise to an immediate problem: SO\((4,1)\) cannot have finite-dimensional unitary representations, because it is a non-compact group. It is in this context that the possibility of considering a deformed de Sitter space with a \(q\)-deformed version of its isometry group, becomes interesting \([5, 6, 7, 8]\) (Some of these references work in the context of dS/CFT). It is a well-known fact that for certain values of the deformation parameter, (non-compact) quantum groups have unitary, finite dimensional representations \([9, 10, 11]\).

But recently it was shown \([12]\) that single-parameter quantum deformation can give rise to deformed de Sitter space only when the deformation parameter is real. This throws a spanner in the above program because finite dimensional representations for one-parameter deformations exist only when the deformation parameter \(q\) is a root
of unity. One obvious way to work around this problem is to consider multi-parametric
deformations of the de Sitter isometry group, and the aim of this paper is to take a
first step in that direction by writing down the algebra for this case explicitly.

Another reason why multi-parametric deformations are interesting is because in
the coordinate system of a static observer in de Sitter, the full $SO(4,1)$ isometry
group is not visible: the manifest isometries are $SO(3)$ and a time-translation (see
the Appendix for an elementary demonstration of this fact). So one of the questions
we need to answer when we quantize in de Sitter, is to understand how the static
observer and the full isometry group are related to each other. One hope behind the
construction of multi-parametric deformations of $SO(4,1)$, is that finding represen-
tations of such an algebra will shed some light on the states of the observer and their
relation to the representations of the full isometry group. We will be working at the
level of complexified algebras, so what we refer to as the algebra of $SO(4,1)$ or $SO(5)$
is in fact $B_2$ in the Cartan scheme.

The usual one-parameter $q$-deformation for a Lie Algebra is the Drinfeld-Jimbo
(DJ) Algebra. We will be interested in a construction of this algebra starting with a
dual description in terms of $R$-matrices: using the Faddeev-Reshetikhin-Takhtadzhyan
[14] approach. What we will do in this paper is to take the DJ algebras to be defined
by the FRT method, and then extend the definition by using a generalized, multi-
parametric $R$-matrix [15, 16]. We will do this explicitly for $SO(5, \mathbb{C})$ and the result
will be a multi-parametric generalization of the DJ algebra.

In the next section we will provide an introduction to the DJ algebra and how it
can be derived from a dual description. In section 4, we will write down the explicit
form of the multi-parametric $R$-matrix for $SO(5)$ from [16], and use that in the dual
description to construct the multi-parametric algebra for $SO(5)$. We conclude with
some speculations and possibilities for future research.

Finite-dimensionality of de Sitter Hilbert space has also been discussed in [19, 20],
and $q$-deformation in the context of AdS/CFT has been considered in [21, 22].

\section{One-parameter DJ Algebra and its Dual Description}

Drinfeld-Jimbo algebra is a deformation of the universal enveloping algebra of the
Lie algebra of a classical group. A universal enveloping algebra is the algebra spanned
by polynomials in the generators, modulo the commutation relations. When we deform it, we mod out by a set of deformed relations, instead of the usual commutation relations. These relations are what define the DJ algebra. When the deformation parameter tends to the limit unity, the algebra reduces to the universal enveloping algebra of the usual Lie algebra.

We will write down the algebra relations in the so-called Chevalley-Cartan-Weyl basis. The rest of the generators of the Lie Algebra can be generated through commutations between these. The Drinfeld-Jimbo algebra is constructed as a deformation of the relations between the Chevalley generators. So without any further ado, lets write down the form of the DJ algebra \[13\] for a generic semi-simple Lie algebra \( g \) of rank \( l \) and Cartan matrix \((a_{ij})\). In what follows, \( q \) is a fixed non-zero complex number (the deformation parameter), and \( q_i = q^{d_i} \), with \( d_i = (\alpha_i, \alpha_i) \) where \( \alpha_i \) are the simple roots of the Lie algebra. The norm used in the definition of \( d_i \) is the norm defined in the dual space of the Cartan sub-algebra, through the Killing form. These are all defined in standard references \[17, 13\]. The indices run from 1 to \( l \).

With these at hand we can define the Drinfeld-Jimbo algebra \( U_q(g) \) as the algebra generated by \( E_i, F_i, K_i, K_i^{-1}, 1 \leq i \leq l \), and the defining relations,

\[
K_i K_j = K_j K_i, \quad K_i K_i^{-1} = K_i^{-1} K_i, \tag{2.1}
\]

\[
K_i E_j = q_i^{a_{ij}} E_j K_i, \quad K_i F_j = q_i^{-a_{ij}} F_j K_i, \tag{2.2}
\]

\[
E_i F_j - F_j E_i = \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}, \tag{2.3}
\]

\[
\sum_{r=0}^{1-a_{ij}} (-1)^r [1 - a_{ij}; r]_q E_i^{1-a_{ij}-r} E_j E_i^r = 0, \quad i \neq j, \tag{2.4}
\]

\[
\sum_{r=0}^{1-a_{ij}} (-1)^r [1 - a_{ij}; r]_q F_i^{1-a_{ij}-r} F_j F_i^r = 0, \quad i \neq j, \tag{2.5}
\]

with

\[
[n; r]_q = \frac{[n]_q!}{[r]_q! [n-r]_q!} \tag{2.6}
\]

and

\[
[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}, \quad [n]_q! = [1]_q [2]_q ... [n]_q, \quad [0]_q \equiv 1. \tag{2.7}
\]

The relations containing only the \( E_i \)s or the \( F_i \)s are called Serre relations and they should be thought of as the price that we have to pay in order to write the algebra
relations entirely in terms of the Chevalley generators. Sometimes, it is useful to write $K_i$ as $q^{H_i}$. In the limit of $q \to 1$, the DJ algebra relations reduce to the Lie Algebra relations written in the Chevalley basis, with $H_i$'s the generators in the Cartan subalgebra and the $E_i$'s and $F_i$'s the raising and lowering operators.

We will be interested in the specific case of $SO(5)$ (Cartan’s $B_2$), and we will rewrite the DJ algebra $U_q^{1/2}(\mathfrak{so}(5))$ for that case in a slightly different form for later convenience:

\begin{align*}
k_1k_2 &= k_2k_1, \quad k_1^{-1} = q^{H_1+H_2/2}, \quad k_2^{-1} = q^{H_2/2} \\
k_1E_1 &= q^{-1}E_1k_1, \quad k_2E_1 = qE_1k_2, \\
k_1E_2 &= E_2k_1, \quad k_2E_2 = q^{-1}E_2k_2, \\
k_1F_1 &= qF_1k_1, \quad k_2F_1 = q^{-1}F_1k_2, \\
k_1F_2 &= F_2k_1, \quad k_2F_2 = qF_2k_2, \\
[E_1, F_1] &= \frac{k_2k_1^{-1} - k_2^{-1}k_1}{q - q^{-1}}, \\
[E_2, F_2] &= \frac{k_2^{-1} - k_2}{q^{1/2} - q^{-1/2}}.
\end{align*}

The Serre relations take the form:

\begin{align*}
E_1^2E_2 - (q + q^{-1})E_1E_2E_1 + E_2E_1^2 &= 0 \quad (2.14) \\
E_1E_2^3 - (q + q^{-1} + 1)E_2E_1E_2^2 + (q + q^{-1} + 1)E_2^2E_1E_2 - E_2^3E_1 &= 0 \quad (2.15)
\end{align*}

with analogous expressions for the $F$s.

Drinfeld-Jimbo algebra is one way to describe a “quantum group”. Another way to do this is to work with the groups directly and deform the group structure using the so-called $R$-matrices, rather than to deform the universal envelope of the Lie algebra. It turns out that both these approaches are dual to each other, and one can obtain the DJ algebra by starting with $R$-matrices. Faddeev-Reshetikhin-Takhtadzhyan have constructed a formalism for working with the $R$-matrices, and to construct the DJ algebra starting from the dual approach. So, a natural place to look for when trying to generalize the DJ algebra of $SO(5)$ is to look at this dual construction and try to see whether it admits any generalizations.

In the rest of this section, we will review the construction of the DJ algebra starting with the $R$-matrices. In the next section, we will start with a multi-parametric generalization of the $R$-matrix for $SO(5)$ and follow an analogous procedure to obtain the multi-parametric $SO(5)$ DJ algebra.
As already mentioned, the deformation of the group structure is done in the dual picture through the introduction of the R-matrix. The duality between the two approaches is manifested through the so-called L-functionals \[13\]. If one defines the L-functionals as certain matrices constructed from the DJ algebra generators, then the R-matrix and the L-functionals would together satisfy certain relations (which we will call the duality relations), as a consequence of the fact that the generators satisfy the DJ algebra. Conversely, we could start with L-functionals thought of as matrices with previously unconstrained matrix elements, and then the duality relations would be the statement that the matrix elements should satisfy the DJ algebra. Thus, the L-functionals, together with the duality relations is equivalent to the DJ algebra.

For any R-matrix\[i\] we can define an algebra \(A(R)\), with \(N(N + 1)\) generators \(l^+_{ij}\), \(l^-_{ij}\), \(i \leq j, j = 1, 2, ..., N\), and defining relations

\[
L^+_1 L^+_2 R = RL^+_2 L^+_1, \quad L^-_1 L^+_2 R = RL^+_2 L^-_1
\]

(2.16)

\[
l^+_i l^-_i = l^-_i l^+_i = 1, \quad i = 1, 2, ..., N
\]

(2.17)

where the matrices \(L^\pm \equiv (l^\pm_{ij})\) and \(l^+_i = 0 = l^-_i\), for \(i > j\) (that is, they are upper or lower triangular). The subscripts 1 and 2 have the following meaning: \(L^+_1\) stands for \(L^+\) tensored with the \(N \times N\) identity matrix, and \(L^+_2\) stands for the \(N \times N\) identity matrix tensored with \(L^+\). So the matrix multiplication with \(R\) is well-defined because the \(R\)-matrix is an \(N^2 \times N^2\) matrix. The above relations will be referred to as the duality relations. It turns out that this algebra has a Hopf algebra structure with

\[
\Delta(l^\pm_{ij}) = \sum_k l^\pm_{ik} \otimes l^\pm_{kj},
\]

(2.18)

\[
\epsilon(l^\pm_{ij}) = \delta_{ij},
\]

(2.19)

\[
S(L^\pm) = (L^\pm)^{-1}.
\]

(2.20)

Now, let’s choose the R-matrix in the above case to be the one-parameter R-matrix for \(SO(N)\), with \(N = 2n + 1\).

\[
R = q \sum_{i \neq j}^{2n} E_{ii} \otimes E_{ii} + q^{-1} \sum_{i \neq j}^{2n} E_{ii} \otimes E_{ij} + E_{n+1,n+1} \otimes E_{n+1,n+1} +
\]

\[
+ \sum_{i \neq j, j'}^{2n} E_{ii} \otimes E_{jj} + (q - q^{-1}) \left[ \sum_{i > j}^{2n} E_{ij} \otimes E_{ji} - \sum_{i > j}^{2n} q^{n-n_j} E_{ij} \otimes E_{ij'} \right].
\]

(2.21)

\[1\]It is useful here to keep in mind that R-matrices are \(N^2 \times N^2\) matrices.
Here \( E_{ij} \) is the \( 2n \times 2n \) matrix with 1 in the \((i,j)\)-position and 0 everywhere else, and the symbol \( \otimes \) stands for tensoring of two matrices. \( i' = 2n + 2 - i \) and similarly for \( j' \). The deformation parameter is \( q \). Finally, \((\rho_1, \rho_2, ..., \rho_{2n}) = (n - 1/2, n - 3/2, ..., 1/2, 0, -1/2, ..., -n + 1/2)\).

Let \( I(\text{so}(N)) \) be the two-sided ideal in \( A(R) \) generated by

\[
L^\pm C^t(L^\pm)^t(C^{-1})^t = I = C^t(L^\pm)^t(C^{-1})^tL^\pm
\]

(2.22)

where \( I \) is the identity matrix, and the metric \( C \) defines a length in the vector space where the quantum matrices are acting. \( C \) provides the constraint arising from the fact that the underlying classical group is an orthogonal group: \( TC^{-1}T^tC = I = C^{-1}T^tCT \) for quantum matrices \( T \) (see [14]). For \( \text{SO}(N) \),

\[
C = (C^a_i), \quad C^a_j = \delta_{ij}q^{-\rho_i}
\]

(2.23)

with \( j' \) and \( \rho_i \) are as defined above.

Now, \( I(\text{so}(N)) \) is a Hopf ideal of \( A(R) \) [13], so the quotient \( A(R)/I(\text{so}(N)) \) is also a Hopf algebra which we will call \( U^L_q(\text{so}(N)) \). Now, there is a theorem (see, for example [13] or [14] for a proof) which says that \( U^L_q(\text{so}(N)) \) is isomorphic to \( U_{q^{1/2}}(\text{so}(2n + 1)) \), which is the DJ algebra for \( \text{SO}(2n + 1) \) with deformation parameter \( q^{1/2} \). Explicitly, this isomorphism can be written down as,

\[
\begin{align*}
l^+_i &= q^{-H'_i}, \quad l^+_{ij'} = q^{H'_j}, \\
l^+_{k,k+1} &= (q - q^{-1})q^{-H'_k}E_k, \\
l^+_{k+1,k} &= -(q - q^{-1})F_k q^{H'_k}, \\
l^+_{n,n+1} &= (q^{1/2} + q^{-1/2})^{1/2}(q^{1/2} - q^{-1/2})q^{-H'_n}E_n, \\
l^+_{n+1,n+2} &= -q^{-1/2}(q^{1/2} + q^{-1/2})^{1/2}(q^{1/2} - q^{-1/2})E_n, \\
l^+_{n+1,n} &= -(q^{1/2} + q^{-1/2})^{1/2}(q^{1/2} - q^{-1/2})F_n q^{H'_n}, \\
l^+_{n+2,n+1} &= q^{1/2}(q^{1/2} + q^{-1/2})^{1/2}(q^{1/2} - q^{-1/2})F_n
\end{align*}
\]

(2.24)

Here, \( i = 1, 2, ..., n \) as always, and \( 1 \leq k \leq n - 1 \). \( H'_i = H_i + H_{i+1} + ... + H_{n-1} + H_n/2 \). The above relations (which we will call the isomorphism relations) define the relations between elements of the \( L \) matrices and the Chevalley-Cartan-Weyl generators. Sometimes it will be convenient to call \( q^{-H'_i} \) as \( k_i \) because it makes comparison with \( \text{SO}(5) \) DJ algebra (written earlier) more direct.
3 The Multi-parametric Algebra

Our procedure for constructing the multi-parametric algebra is straightforward. Instead of using the usual one-parametric $R$-matrices in the duality relations, we use the multi-parametric $R$-matrices that Schirrmacher has written down [16]. We keep the isomorphism relations to be the same as above and use the duality relations to define the new multi-parametric algebra.

In principle this procedure could be done for all the multi-parametric $R$-matrices of all the different Cartan groups using their associated isomorphism relations. We have endeavored to do this procedure for only the case of $SO(5)$, but at least for the smaller Cartan groups, the exact same procedure can be performed on a computer using the appropriate $R$-matrices. To write down the form of the multi-parametric DJ algebra for a generic semisimple Lie algebra is an interesting problem which we have not attempted to tackle here.

The multi-parametric $R$-matrix for $SO(2n + 1)$ (which for our purposes is the same thing as $B_n$) looks like

$$R = r \sum_{i \neq i'}^{2n} E_{ii} \otimes E_{ii} + r^{-1} \sum_{i \neq i'}^{2n} E_{ii} \otimes E_{i'i'} + E_{n+1,n+1} \otimes E_{n+1,n+1} +$$

$$+ \sum_{i<j, i \neq j}^{2n} \frac{r}{q_{ij}} E_{ii} \otimes E_{jj} + \sum_{i>j, i \neq j'}^{2n} \frac{q_{ij}}{r} E_{ii} \otimes E_{jj} +$$

$$+ (r - r^{-1}) \left[ \sum_{i>j}^{2n} E_{ij} \otimes E_{ji} - \sum_{i>j}^{2n} r^{\rho_i - \rho_j} E_{ij} \otimes E_{i'j'} \right]. \quad (3.1)$$

The deformation parameters are $r$ and $q_{ij}$ and they are not all independent: $q_{ii} = 1$, $q_{ji} = r^2/q_{ij}$ and $q_{ij} = r^2/q_{ij} = r^2/q_{ij} = q_{i'j'}$. These relations basically imply that $q_{ij}$ with $i < j \leq n$ determine all the deformation parameters. It should be noted that when all the independent deformation parameters are set equal to each other ($= q$), then the $R$-matrix reduces to the usual one parametric version. In the case of $SO(5)$, the multi-parametric $R$-matrix has only two independent parameters, which we will call $r$ and $q$.

We extensively used a Mathematica package called NCALGEBRA (version 3.7)[18] to do the computations, since the matrix elements (being generators of an algebra) are not commuting objects. The first task is to obtain the duality relations between
the matrix elements explicitly. The $L$ matrices are chosen to be upper and lower triangular. The task is straightforward but tedious because the duality relations are 25 by 25 matrix relations for the case of $SO(5)$. So one has to scan through the resulting output to filter out the relations that are dual to the relations between the Chevalley-Cartan-Weyl generators. Doing the calculation for the single-parameter case will give a hint about which relations are relevant in writing down the algebra.

The first line of the isomorphism relations (for the specific case of $SO(5)$) implies that we can use $k_1, k_2, 1, k_2^{-1}$ and $k_1^{-1}$ instead of $l_{11}, l_{22}, l_{33}, l_{44}$ and $l_{55}$ respectively. With this caveat, the algebra looks like the following in terms of the relevant $L$ matrix elements:

\[
\begin{align*}
    k_1k_2 &= k_2k_1, \\
    k_1l_{12}^+ &= \frac{r}{q^2}l_{12}^-k_1, & k_2l_{12}^+ &= \frac{q^2}{r}l_{12}^+k_2, \\
    k_1l_{23}^+ &= \frac{q}{r}l_{23}^-k_1, & k_2l_{23}^+ &= q^{-1}l_{23}^+k_2, \\
    k_1l_{21}^- &= rl_{21}^-k_1, & k_2l_{21}^- &= \frac{r}{q^2}l_{21}^-k_2, \\
    k_1l_{32}^- &= \frac{q}{r}l_{32}^-k_1, & k_2l_{32}^- &= ql_{32}^-k_2, \\
    [l_{45}^+, l_{21}^-] &= (q - q^{-1})(k_1^{-2} - k_2^{-2}), \\
    [l_{23}^+, l_{32}^-] &= (q - q^{-1})(k_2 - k_2^{-1}), \\
    l_{12}^{+2}l_{23}^- - \left(\frac{q}{r}\right)^2l_{12}^{+2}l_{23}^-l_{12}^+ + \frac{1}{q^2}l_{12}^+l_{12}^{+2} &= 0, \\
    q\frac{q^2}{r^2}l_{23}^- - \left(\frac{q^2}{r^2} + \frac{q^5}{r^3} + \frac{q}{r}\right)l_{23}^{+2}l_{12}^+ + \left(q + \frac{q^4}{r^2} + \frac{q^5}{r^2}\right)l_{23}^+l_{12}^{+2} - \frac{q^4}{r}l_{12}^+l_{23}^- &= 0.
\end{align*}
\]

The last two equations correspond to the Serre relations. (We write them down only for the $L^+$ matrix elements.). As an example of the general procedure for obtaining these algebra relations from the duality relations (i.e., the Mathematica output), we will demonstrate the derivation of the first Serre relation. The relevant expressions that one gets from Mathematica are:

\[
\begin{align*}
    l_{12}^+l_{23}^- - \frac{q}{r}l_{23}^-l_{12}^+ &= -(q - q^{-1})l_{13}^+k_2, \\
    l_{12}^+l_{13}^- &= \frac{1}{q}l_{13}^+l_{12}^+.
\end{align*}
\]
Solving for $l_{13}^+$ from the first equation by multiplying by $k_2^{-1}$ on the right, plugging it back into the second equation and using the commutation rules for $k_2$, we get our Serre relation. This kind of manipulation is fairly typical in the derivation of the above algebra.

As a next step, we use the isomorphism relations defined at the end of the last section to rewrite the above algebra in terms of the Chevalley-Cartan-Weyl type generators. The result is

\begin{align}
  k_1k_2 &= k_2k_1, \\
  k_1E_1 &= \frac{r}{q^2}E_1k_1, \quad k_2E_1 = \frac{q^2}{r}E_1k_2, \\
  k_1E_2 &= \frac{q}{r}E_2k_1, \quad k_2E_2 = \frac{1}{q}E_2k_2, \\
  k_1F_1 &= rF_1k_1, \quad k_2F_1 = \frac{r}{q^2}F_1k_2, \\
  k_1F_2 &= \frac{q}{r}F_2k_1, \quad k_2F_2 = qF_2k_2, \\
  \frac{q}{r}E_1F_1 - \frac{r}{q}F_1E_1 &= \frac{k_2k_1^{-1} - k_2^{-1}k_1}{q - q^{-1}}, \\
  [E_2, F_2] &= \frac{k_2^{-1} - k_2}{q^{1/2} - q^{-1/2}}, \\
  E_1^2E_2 - \left(\frac{q^2}{r} + \frac{r}{q^2}\right)E_1E_2E_1 + E_2E_1^2 &= 0, \\
  E_2^3E_1 - \left(\frac{r}{q} + \frac{q^2}{r} + \frac{r}{q^2}\right)E_2^3E_1E_2 + \left(\frac{r^2}{q^3} + 1 + q\right)E_2E_2E_2^2 - \frac{r}{q}E_1E_2^3 &= 0.
\end{align}

This is our final form for the multi-parametric version of $SO(5)$ Drinfeld-Jimbo algebra. Together with the Hopf algebra relations from (2.18)-(2.20), these relations complement our definition of the multi-parametric algebra. Notice that they reduce to the one-parameter DJ algebra of $SO(5)$ in the limit of $r \to q$.

4 Results and Outlook

We have constructed the multi-parametric version of the Drinfeld-Jimbo algebra for the case of $SO(5)$ with the intention of possible applications in de Sitter quantum
mechanics and quantum gravity. As physicists, we are more interested in working with the algebra directly than working with the groups and the $R$-matrix, because presumably, finding representations of the algebra would be more direct (even though still non-trivial). Finding representations is interesting because that could be a first step in embedding the Hilbert space of the static patch of an observer, in the Hilbert space of the full de Sitter space. It might be the case that embedding the $SO(3)_q$ of the observer is easier to accomplish, in the added luxury of two parameters. Also, if it turns out that this embedding is possible only when there is a relationship between the parameters, it could translate into a statement about the surprising smallness of the cosmological statement in terms of scales which are more readily accessible to the observer. Of course, at this stage, this is pure speculation. The bottomline is that it seems like there is the exciting possibility of addressing the problem of the smallness of the positive cosmological constant using the multi-parametric deformation\textsuperscript{2}. Some of these issues are currently being investigated.

It is also interesting as a pure mathematical problem to write down the multi-parametric DJ algebra for a generic Lie algebra. To the best of our knowledge, this is still an open problem.

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6 Appendix

In this appendix we want to give an elementary demonstration that the boosts in $SO(4, 1)$ correspond to time translations for the static observer. The metric for the static patch is,

$$ds^2 = -(1 - r^2)dt^2 + \frac{dr^2}{1 - r^2} + r^2 d\Omega_2^2$$  \hspace{1cm} (6.1)

We take $\Lambda/3 = 1$, where $\Lambda$ is the Cosmological constant.

The easiest way to think about the de Sitter isometry group ($SO(4, 1)$) is to think of it as being embedded in a 5D Minkowski space. In terms of these Minkowski coordinates, the static patch can be written as,

$$X^0 = \sqrt{1 - r^2} \sinh t,$$  \hspace{1cm} (6.2)
$$X^3 = r \cos \theta,$$  \hspace{1cm} (6.3)
$$X^1 = r \sin \theta \cos \phi,$$  \hspace{1cm} (6.4)
$$X^2 = r \sin \theta \sin \phi,$$  \hspace{1cm} (6.5)
$$X^4 = \sqrt{1 - r^2} \cosh t.$$  \hspace{1cm} (6.6)

It’s easy to check that $-(X^0)^2 + (X^i)^2 = 1$ and that $-dX^0^2 + dX^i^2$ is equal to the metric on the static patch. Boosts in $SO(4, 1)$ look like,

$$
\begin{pmatrix}
X'^0 \\
X'^4
\end{pmatrix} =
\begin{pmatrix}
\cosh \beta & \sinh \beta \\
\sinh \beta & \cosh \beta
\end{pmatrix}
\begin{pmatrix}
X^0 \\
X^4
\end{pmatrix}
$$  \hspace{1cm} (6.7)

Plugging in the expressions for $X^0$ and $X^4$ in terms of $r$ and $t$, multiplying out the matrices and simplifying, we end up with,

$$
\begin{pmatrix}
X'^0 \\
X'^4
\end{pmatrix} =
\begin{pmatrix}
\sqrt{1 - r^2} \sinh(t + \beta) \\
\sqrt{1 - r^2} \cosh(t + \beta)
\end{pmatrix},
$$  \hspace{1cm} (6.8)

which is just the time-translated version of the original expressions.
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