Strong averaging principle for stochastic Klein-Gordon equation with a fast oscillation *

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Abstract
This paper investigates an averaging principle for stochastic Klein-Gordon equation with a fast oscillation arising as the solution of a stochastic reaction-diffusion equation evolving with respect to the fast time. Stochastic averaging principle is a powerful tool for studying qualitative analysis of stochastic dynamical systems with different time-scales. To be more precise, the well-posedness of mild solutions of the stochastic hyperbolic-parabolic equations is firstly established by applying the fixed point theorem and the cut-off technique. Then, under suitable conditions, we prove that there is a limit process in which the fast varying process is averaged out and the limit process which takes the form of the stochastic Klein-Gordon equation is an average with respect to the stationary measure of the fast varying process. Finally, by using the Khasminskii technique we can obtain the rate of strong convergence for the slow component towards the solution of the averaged equation.

Keywords: Stochastic averaging principle; Stochastic Klein-Gordon equation; Effective dynamics; slow-fast SPDEs; Strong convergence

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1 Introduction

The nonlinear Klein-Gordon equation

\[ u_{tt} - u_{xx} + m^2 u + \mu |u|^2 u + \nu |u|^4 u = 0, \]

appears in the study of several problems of mathematical physics. For example, this equation arises in general relativity, nonlinear optics (e.g., the instability phenomena such as self-focusing), plasma physics, fluid mechanics, radiation theory or spin waves [23, 31, 38].

Stochastic Klein-Gordon equation is a stochastic wave equation, a large amount of work has been devoted to the study of the nonlinear stochastic wave equation:

- Existence and uniqueness of solution: [34] establishes the existence and uniqueness of solution for stochastic viscoelastic wave equations.

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• Explosive solution: [18], [46] and [7] invtisvities the explosive solution of stochastic wave equation.

• Large-time asymptotic properties of solutions: Large-time asymptotic properties of solutions to a class of semilinear stochastic wave equations with damping in a bounded domain are considered in [19]. In [39], relations between the asymptotic behavior for a stochastic wave equation and a heat equation are considered.

• Absolute continuity of the law of the solution: In [45], the authors prove some results concerning the existence of the density of the real valued solution of a 3D-stochastic wave equation.

• Invariant measure: The existence and uniqueness of an invariant measure for the transition semigroup associated with a nonlinear stochastic Klein-Gordon type are studied in [2] and [3], in [3], the authors consider the stochastic wave equations with nonlinear dissipative damping. In [6], the authors show the existence of a unique invariant measure associated with the transition semigroup under mild conditions.

• The corresponding Kolmogorov operator: In [2], the structure of the corresponding Kolmogorov operator associated with a stochastic Klein-Gordon equation is studied.

• Attractor: In [20], the existence of an attractor is proved, which implies the existence of an invariant measure. However, there is no a large overlap with the results obtained here and the methods are quite different. [18] deals with a class of non-autonomous stochastic linearly damped wave equations on Rd perturbed by multiplicative Stratonovich white noise of the form.

• Smoluchowski-Kramers approximation problem: The Smoluchowski-Kramers approximation problem for the nonlinear stochastic wave equation has been consider in [11, 12, 13, 14, 15, 16].

• Large deviation principle. In [40], by using a weak convergence method, a large deviation principle is built for the singularly perturbed stochastic nonlinear damped wave equations on bounded regular domains.

In this paper, we will be concerned with the averaging principle for stochastic Klein-Gordon equation with a fast oscillating perturbation

\[
\begin{align*}
\text{d}A^\varepsilon_t + \left[-A^\varepsilon_{xx} + \mu |A^\varepsilon|^2 A^\varepsilon + \nu |A^\varepsilon|^4 A^\varepsilon + f(A^\varepsilon(t), B(\frac{t}{\varepsilon}))\right]dt &= \sigma_1 dW_1 &\text{in } Q \\
A^\varepsilon(0, t) &= 0 = A^\varepsilon(1, t) &\text{in } (0, T) \\
A^\varepsilon(x, 0) &= A_0(x) &\text{in } I \\
A^\varepsilon_t(x, 0) &= A_1(x) &\text{in } I,
\end{align*}
\]

where \(B(t)\) is governed by the stochastic reaction-diffusion equation

\[
\begin{align*}
dB + [-B_{xx} + |B|^2 B + g(A, B)]dt &= \sigma_2 dW_2 &\text{in } Q \\
B(0, t) &= 0 = B(1, t) &\text{in } (0, T) \\
B(x, 0) &= B_0(x) &\text{in } I,
\end{align*}
\]

where \(T > 0, I = (0, 1), Q = I \times (0, T)\), the stochastic perturbations are of additive type, \(W_1\) and \(W_2\) are mutually independent Wiener processes on a complete stochastic basis \((\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})\), which will be specified later, denote by \(\mathbb{E}\) the expectation with respect to \(\mathbb{P}\). The coefficients \(\mu\) and \(\nu\) are positive constants, the noise coefficients \(\sigma_1\) and \(\sigma_2\) are positive constants.

Thus, we will be concerned with the averaging principle for multiscale stochastic Klein-
Gordon equation with slow and fast time-scales

\[
\begin{cases}
  dA^\varepsilon_t + \left[-A^\varepsilon_{xx} + \mu |A^\varepsilon|^2 A^\varepsilon + \nu |A^\varepsilon|^4 A^\varepsilon + f(A^\varepsilon, B^\varepsilon)\right]dt = \sigma_1 dW_1 & \text{in } Q \\
  dB^\varepsilon_t + \frac{1}{\varepsilon}[-B^\varepsilon_{xx} + |B^\varepsilon|^2 B^\varepsilon + g(A^\varepsilon, B^\varepsilon)]dt = \frac{1}{\sqrt{\varepsilon}} \sigma_2 dW_2 & \text{in } Q \\
  A^\varepsilon(0, t) = 0 = A^\varepsilon(1, t) & \text{in } (0, T) \\
  B^\varepsilon(0, t) = 0 = B^\varepsilon(1, t) & \text{in } (0, T) \\
  A^\varepsilon(x, 0) = A_0(x) & \text{in } I \\
  A^\varepsilon_t(x, 0) = A_1(x) & \text{in } I \\
  B^\varepsilon(x, 0) = B_0(x) & \text{in } I,
\end{cases}
\]

where the parameter \( \varepsilon \) is small and positive, which describes the ratio of time scale between the process \( A^\varepsilon \) and \( B^\varepsilon \). With this time scale the variable \( A^\varepsilon \) is referred as slow component and \( B^\varepsilon \) as the fast component.

The theory of stochastic averaging principle provides an effective approach for the qualitative analysis of stochastic systems with different time-scales and is relatively mature for stochastic dynamical systems. The theory of averaging principle serves as a tool in study of the qualitative behaviors for complex systems with multiscales, it is essential for describing and understanding the asymptotic behavior of dynamical systems with fast and slow variables. Its basic idea is to approximate the original system by a reduced system. The averaging principle is an important method to extract effective macroscopic dynamic from complex systems with slow component and fast component. The theory of averaging for deterministic dynamical systems, which was first studied by Bogoliubov [11], has a long and rich history.

The averaging principle in the stochastic ordinary differential equations setup was first considered by Khasminskii [41] which proved that an averaging principle holds in weak sense, and has been an active research field on which there is a great deal of literature. Recently, the averaging principle for stochastic differential equations has been paid much attention [29, 30, 32, 33, 36].

However, there are few results on the averaging principle for stochastic systems in infinite dimensional space. To this purpose we recall the recent results:

- parabolic-parabolic system: Cerrai and Freidlin [8], Cerrai [9, 10], Bréhier [4], Wang and Roberts [17], Fu and co-workers [24, 25, 27], Xu and co-workers [49, 50], Bao and co-workers [5];
- hyperbolic-parabolic system: Fu and co-workers [24, 28], Pei and co-workers [44];
- Burgers-parabolic system: Dong and co-workers [22];
- FitzHugh-Nagumo system: Fu and co-workers [26], Xu and co-workers [49].

However, as far as we know there are no results on the averaging principle for the stochastic Klein-Gordon equations with a fast oscillation \((\ast)\), a natural question is as follows:

Can we establish the averaging principle for the stochastic Klein-Gordon equations with a fast oscillation \((\ast)\)? To be more precise, can the slow component \( A^\varepsilon \) be approximated by the solution \( \bar{A} \) which governed by a stochastic Klein-Gordon equation?
These mathematical questions arise naturally which are important from the point of view of dynamical systems from both physical and mathematical standpoints. In this paper, the main object is to establish an effective approximation for slow process $A^\varepsilon$ with respect to the limit $\varepsilon \to 0$.

In this paper, we will take $
\mu = \nu = 1
$
for the sake of simplicity. All the results can be extended without difficulty to the general case.

We define

\[ L(u) = u_{xx}, \]
\[ F(u) = -|u|^2u - |u|^4u, \]
\[ G(u) = -|u|^2u, \]

then the stochastic Klein-Gordon equation (*) becomes

\[
\begin{align*}
DA^\varepsilon_t & = [L(A^\varepsilon) + F(A^\varepsilon) + f(A^\varepsilon, B^\varepsilon)]dt + \sigma_1dW_1 \\
DB^\varepsilon_t & = \frac{1}{\varepsilon}[L(B^\varepsilon) + G(B^\varepsilon) + g(A^\varepsilon, B^\varepsilon)]dt + \frac{1}{\sqrt{\varepsilon}}\sigma_2dW_2 \\
A^\varepsilon(0, t) & = 0 = A^\varepsilon(1, t) \\
B^\varepsilon(0, t) & = 0 = B^\varepsilon(1, t) \\
A^\varepsilon(x, 0) & = A_0(x) \\
A^\varepsilon(x, 0) & = A_1(x) \\
B^\varepsilon(x, 0) & = B_0(x)
\end{align*}
\]

in $Q$, in $Q$, in $(0, T)$, in $(0, T)$, in $I$, in $I$, in $I$, in $I$.

Multi-scale stochastic partial differential equations arise as models for various complex systems, such model arises from describing multi-scale phenomena in, for example, nonlinear oscillations, material sciences, automatic control, fluids dynamics, chemical kinetics and in other areas leading to mathematical description involving “slow” and “fast” phase variables. The study of the asymptotic behavior of such systems is of great interest. In this respect, the question of how the physical effects at large time scales influence the dynamics of the system is arisen. We focus on this question and show that, under some dissipative conditions on fast variable equation, the complexities effects at large time scales to the asymptotic behavior of the slow component can be omitted or neglected in some sense.

1.1 Mathematical setting

We introduce the following mathematical setting:

\( \diamond \) We denote by $L^2(I)$ the space of all Lebesgue square integrable functions on $I$. The inner product on $L^2(I)$ is

\[ (u, v) = \int_I uv dx, \]

for any $u, v \in L^2(I)$. The norm on $L^2(I)$ is

\[ \|u\| = (u, u)^{\frac{1}{2}}, \]

for any $u \in L^2(I)$.

$H^s(I)(s \geq 0)$ are the classical Sobolev spaces of functions on $I$. The definition of $H^s(I)$ can be found in [35], the norm on $H^s(I)$ is $\| \cdot \|_{H^s}$. 

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We set
\[ X_{p,\tau} = L^p(\Omega; C([0, \tau]; H^1(I))) \times L^p(\Omega; C([0, \tau]; L^2(I))) \times L^p(\Omega; C([0, \tau]; H^1(I))), \]
\[ Y_\tau = C([0, \tau]; H^1(I)) \times C([0, \tau]; L^2(I)) \times C([0, \tau]; H^1(I)), \]
where \( p \geq 1, \tau \geq 0. \)

\( \diamond \) For \( i = 1, 2 \), let \( \{e_{i,k}\}_{k \in \mathbb{N}} \) be eigenvectors of a nonnegative, symmetric operator \( Q_i \) with corresponding eigenvalues \( \{\lambda_{i,k}\}_{k \in \mathbb{N}} \), such that
\[ Q_i e_{i,k} = \alpha_{i,k} e_{i,k}, \quad \lambda_{i,k} > 0, \quad k \in \mathbb{N}. \]

Let \( W_i \) be an \( L^2(I) \)-valued \( Q_i \)-Wiener process with operator \( Q_i \) satisfying
\[ \text{Tr} Q_i = \sum_{k=1}^{+\infty} \alpha_{i,k} < +\infty, \quad k \in \mathbb{N} \]
and
\[ W_i = \sum_{k=1}^{+\infty} \alpha_{i,k}^{\frac{1}{2}} \beta_{i,k}(t) e_{i,k} < +\infty, \quad k \in \mathbb{N} \quad t \geq 0, \]
where \( \{\beta_{i,k}\}_{k \in \mathbb{N}}(i = 1, 2) \) are independent real-valued Brownian motions on the probability base \( (\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P}) \).

We denote \( \|\sigma_i\|_{Q_i}^2 \triangleq \sigma_{i}^2 \text{Tr} Q_i. \)

\( \diamond \) The functions \( f \) and \( g \) satisfy the global Lipschitz condition and the sublinear growth condition, specifically, there exist positive constants \( L_f \) and \( L_g \) such that
\[ \|f(u_1, v_1) - f(u_2, v_2)\| \leq L_f(\|u_1 - u_2\| + \|v_1 - v_2\|), \]
\[ \|g(u_1, v_1) - g(u_2, v_2)\| \leq L_g(\|u_1 - u_2\| + \|v_1 - v_2\|) \]
for all \( u_1, u_2, v_1, v_2 \in L^2(I) \).

\( \diamond \) Throughout the paper, the letter \( C \) denotes positive constants whose value may change in different occasions. We will write the dependence of constant on parameters explicitly if it is essential.

We adopt the following hypothesis (H) throughout this paper:

(H) \( \alpha \triangleq \lambda - L_g > 0 \), where \( \lambda > 0 \) is the smallest constant such that the following inequality holds
\[ \|u_x\|^2 \geq \lambda \|u\|^2, \]
where \( u \in H^1_0(I) \) or \( \int_I u \, dx = 0. \)

### 1.2 Main results

Asymptotical methods play an important role in investigating nonlinear dynamical systems. In particular, the averaging methods provide a powerful tool for simplifying dynamical systems, and obtain approximate solutions to differential equations arising from mechanics, mathematics, physics, control and other areas. In this paper, we use stochastic averaging principle to investigate stochastic Klein-Gordon equation \( \square \).

Now, we are in a position to present the main result in this paper.
Theorem 1.1. Suppose that the hypothesis (H) holds and $A_0, B_0 \in H^1_0(I), A_1 \in L^2(I)$, $(A^\varepsilon, B^\varepsilon)$ is the solution of (1.1) and $\tilde{A}$ is the solution of the effective dynamics equation

\[
\begin{cases}
  d\tilde{A}_t = [\mathcal{L}(\tilde{A}) + \mathcal{F}(\tilde{A}) + \tilde{f}(\tilde{A})]dt + \sigma_1 dW_1 & \text{in } Q \\
  \tilde{A}(0, t) = 0 = \tilde{A}(1, t) & \text{in } (0, T) \\
  \tilde{A}(x, 0) = A_0(x) & \text{in } I, \\
  \tilde{A}_t(x, 0) = A_1(x) & \text{in } I,
\end{cases}
\]

then we have for any $T > 0$, any $p > 0$,

\[
\lim_{\varepsilon \to 0} \left( \mathbb{E} \sup_{0 \leq t \leq T} \| A^\varepsilon(t) - \tilde{A}(t) \|^{2p} + \mathbb{E} \sup_{0 \leq t \leq T} \| A^\varepsilon_t(t) - \tilde{A}_t(t) \|^{2p} \right) = 0,
\]

where

\[
\tilde{f}(A) = \int_{L^2(I)} f(A, B) \mu^A(dB)
\]

and $\mu^A$ is an invariant measure for the fast motion with frozen slow component

\[
\begin{cases}
  dB = [\mathcal{L}(B) + \mathcal{G}(B) + g(A, B)]dt + \sigma_2 dW_2 & \text{in } Q \\
  B(0, t) = 0 = B(1, t) & \text{in } (0, T) \\
  B(x, 0) = B_0(x) & \text{in } I,
\end{cases}
\]

where $A \in L^2(I)$.

Moreover, if $p > \frac{5}{8}$, there exists a positive constant $C(p)$ such that

\[
\mathbb{E} \sup_{0 \leq t \leq T} \| A^\varepsilon(t) - \tilde{A}(t) \|^{2p} + \mathbb{E} \sup_{0 \leq t \leq T} \| A^\varepsilon_t(t) - \tilde{A}_t(t) \|^{2p} \leq C(p) \left( \frac{1}{1 - \ln \varepsilon} \right)^{\frac{1}{8p}};
\]

if $0 < p \leq \frac{5}{8}$, for any $\kappa > 0$, there exists a positive constant $C(p, \kappa)$ such that

\[
\mathbb{E} \sup_{0 \leq t \leq T} \| A^\varepsilon(t) - \tilde{A}(t) \|^{2p} + \mathbb{E} \sup_{0 \leq t \leq T} \| A^\varepsilon_t(t) - \tilde{A}_t(t) \|^{2p} \leq C(p, \kappa) \left( \frac{1}{1 - \ln \varepsilon} \right)^{\frac{8p}{(5+4\kappa)^2}}.
\]

This paper is organized as follows. In Sec. 2, we present some preliminary results and an exponential ergodicity of a fast motion equation (1.3) with the frozen slow component. In Sec. 3, we establish the well-posedness and a priori estimate for the slow-fast system (1.1) and averaged equation (1.2). In Sec. 4, we derive the stochastic averaging principle in sense of strong convergence for (1.1) by using the Khasminskii technique.

2 Preliminary results

2.1 Greens function for wave equation

For the deterministic wave equation

\[
u_{tt} - u_{xx} = 0,
\]
its **Greens function** is given by

\[
K(t, \xi, \zeta) = \sum_{k=1}^{\infty} \frac{\sin(\sqrt{\alpha_k}t)}{\sqrt{\alpha_k}} e_k(\xi) e_k(\zeta).
\]

It is easy to shown that the above series converge in \(L^2(I \times I)\) and the associated **Greens operator** is defined by, for any \(h(\xi) \in L^2(I)\),

\[
G(t)h(\xi) = \int_I K(t, \xi, \zeta)h(\zeta)d\zeta = \sum_{k=1}^{\infty} \frac{\sin(\sqrt{\alpha_k}t)}{\sqrt{\alpha_k}} e_k(\xi)(e_k, h).
\]

For Green operator \(G(t)\), it is easy to derive the following results:

**Lemma 2.1.** [74] P133, Lemma 3.1, Lemma 3.2] Green operator \(G(t)\) satisfies

1) Let \(k\) and \(m\) be nonnegative integers. Then, for any function \(h \in H^{k+m-1}\), the following estimates hold:

\[
\sup_{0 \leq t \leq T} \|G^{(k)}(t)h\|_{H^m}^2 \leq \|h\|^2_{H^{k+m-1}}, \text{ for } 0 \leq k + m \leq 2.
\]

2) Let \(f(\cdot, t) \in L^2(\Omega \times (0, T); L^2(I))\) satisfy

\[
\mathbb{E} \int_0^T \|f(\cdot, t)\|^2 dt < \infty.
\]

Then

\[
\int_0^t G(t - s)f(\cdot, s)ds
\]

is a continuous, adapted \(H^1\)-valued process and its time derivative is a continuous \(L^2(I)\)-valued process such that

\[
\mathbb{E} \sup_{0 \leq t \leq T} \left\| \int_0^t G(t - s)f(\cdot, s)ds \right\|_{H^k}^2 \leq C_k T \mathbb{E} \int_0^T \|f(\cdot, s)\|^2_{H^{k+1}} ds, \text{ for } k = 0, 1,
\]

and

\[
\mathbb{E} \sup_{0 \leq t \leq T} \left\| \left( \int_0^t G(t - s)f(\cdot, s)ds \right)' \right\|^2 \leq T \mathbb{E} \int_0^T \|f(\cdot, s)\|^2 ds, \text{ for } k = 0, 1.
\]

According to Lemma 2.1, we have

**Corollary 2.1.** Green operator \(G(t)\) satisfies: for any \(p > 0\),

1) Let \(k\) and \(m\) be nonnegative integers. Then, for any function \(h \in H^{k+m-1}\), the following estimates hold:

\[
\sup_{0 \leq t \leq T} \|G^{(k)}(t)h\|_{H^m}^p \leq \|h\|^p_{H^{k+m-1}}, \text{ for } 0 \leq k + m \leq 2.
\]
2) Let $f(\cdot,t) \in L^p(\Omega \times (0,T);L^2(I))$ satisfy
\[
\mathbb{E} \int_0^T \|f(\cdot,t)\|^p dt < \infty.
\]
Then
\[
\int_0^t G(t-s) f(\cdot,s) ds
\]
is a continuous, adapted $H^1$-valued process and its time derivative is a continuous $L^2(I)$-valued process such that
\[
\mathbb{E} \sup_{0 \leq t \leq T} \left\| \int_0^t G(t-s) f(\cdot,s) ds \right\|_{H^k}^p \leq C_k T^{p-1} \mathbb{E} \int_0^T \|f(\cdot,s)\|^p_{H^{k-1}} ds, \quad \text{for} \ k = 0, 1,
\]
and
\[
\mathbb{E} \sup_{0 \leq t \leq T} \left\| (\int_0^t G(t-s) f(\cdot,s) ds) \right\|_{L^p(I)}^p \leq T^{p-1} \mathbb{E} \int_0^T \|f(\cdot,s)\|^p ds, \quad \text{for} \ k = 0, 1.
\]

2.2 The heat semigroup $\{S(t)\}_{t \geq 0}$

According to [51, P83], the operator $-\mathcal{L}$ is positive, self-adjoint and sectorial on the domain $\mathcal{D}(-\mathcal{L}) = H^2(I) \cap H^1_0(I)$. By spectral theory, we may define the fractional powers $(-\mathcal{L})^\alpha$ of $-\mathcal{L}$ with the domain $\mathcal{D}((-\mathcal{L})^\alpha)$ for any $\alpha \in [0,1]$. We know that the semigroup $\{S(t)\}_{t \geq 0}$ generated by the operator $-\mathcal{L}$ is analytic on $L^p(I)$ for all $1 \leq p \leq \infty$ and enjoys the following properties [42]:

\[
\begin{aligned}
S(t)(-\mathcal{L})^\alpha &= (-\mathcal{L})^\alpha S(t), \\
\|(-\mathcal{L})^\alpha S(t)\varphi\|_{L^p(I)} &\leq Ct^{-\alpha} \|\varphi\|_{L^p(I)}, \quad \alpha \geq 0, \\
\|D^j S(t)\varphi\|_{L^q(I)} &\leq Ct^{-\frac{q}{p} \left(1 - \frac{q}{2} + j\right)} \|\varphi\|_{L^p(I)}, \quad q \geq p \geq 1, t \geq 0, 
\end{aligned}
\]  

(2.1)

where $D^j$ denotes the $j$-th order derivative with respect to the spatial variable.

2.3 Some useful inequalities

Lemma 2.2. Let $y(t)$ be a nonnegative function, if

\[
y' \leq -ay + f,
\]

we have

\[
y(t) \leq y(s)e^{-a(t-s)} + \int_s^t e^{-a(t-\tau)} f(\tau) d\tau.
\]

Lemma 2.3. If $a, b \in \mathbb{R}$, $p > 0$, it holds that

\[
(|a| + |b|)^p \leq \begin{cases} |a|^p + |b|^p & 0 < p \leq 1, \\ 2^{p-1}(|a|^p + |b|^p) & p > 1. \end{cases}
\]
2.4 Some useful estimates

The following lemmas are very useful in establishing a priori estimate for the slow-fast system.

**Lemma 2.4.** Let $A_1$ and $A_2$ be two real-valued numbers and $\sigma \geq \frac{1}{2}$. Then the following inequality is fulfilled

$$||A_1||^{2\sigma}A_1 - |A_2|^{2\sigma}A_2| \leq (4\sigma - 1)(|A_1|^{2\sigma} + |A_2|^{2\sigma})|A_1 - A_2|.$$

**Remark 2.1.** The same results can be found in [37, Lemma 7.2].

**Lemma 2.5.** [37, Lemma 7.3] Let $A_1$ and $A_2$ be two real-valued numbers and $\sigma > 0$. Then the following inequality is fulfilled

$$(A_1 - A_2)(|A_1|^{2\sigma}A_1 - |A_2|^{2\sigma}A_2) \geq 0.$$

**Remark 2.2.** The same results can be found in [37, Lemma 7.3].

Thus we have

**Corollary 2.2.** For any $A_1, A_2 \in \mathbb{R}$, we have

$$(A_1 - A_2, F(A_1) - F(A_2)) \leq 0,$$

$$(A_1 - A_2, G(A_1) - G(A_2)) \leq 0.$$

The following lemma is very useful in establishing a priori estimate for the slow-fast system.

**Lemma 2.6.** If $\sigma > 0$, we have

$$(-A_{xx}, -|A|^{2\sigma}A) \leq 0.$$

**Remark 2.3.** The same results can be found in [37, Lemma 7.4] and [51, Lemma 2.6].

2.5 Preliminary results on the fast motion equation (1.3)

First, we consider the stochastic heat equation, the solution of (1.3) will be denoted by $B^{A,B_0}$.

We could have the following property for the solution of (1.3):

**Lemma 2.7.** For $A \in L^2(I)$, let $B^{A,X}$ be the solution of

$$\begin{cases}
 dB = [L(B) + G(B) + g(A,B)]dt + \sigma dW_2 & \text{in } I \times (0, +\infty) \\
 B(0, t) = 0 = B(1, t) & \text{in } (0, +\infty) \\
 B(x, 0) = X(x) & \text{in } I.
\end{cases}
$$

(2.2)

1) There exists a positive constant $C$ such that $B^{A,X}$ satisfies:

$$\mathbb{E}||B^{A,X}(t)||^2 \leq e^{-ct}||X||^2 + C(||A||^2 + 1),$$

$$\mathbb{E}||B^{A,X}(t) - B^{A,Y}(t)||^2 \leq ||X - Y||^2 e^{-2ct},$$

(2.3)

for $t \geq 0$. 

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2) There is unique invariant measure $\mu^A$ for the Markov semigroup $P^A_t$ associated with the system (2.2) in $L^2(I)$. Moreover, we have
\[
\int_{L^2(I)} \|z\|^2 \mu^A(dz) \leq C(1 + \|A\|^2).
\]

3) There exists two positive constants $C$ such that $B^{A,X}$ satisfies:
\[
\|E(f(A, B^{A,X}) - \bar{f}(A))\|^2 \leq C(1 + X^2 + |A|^2)e^{-2\alpha t}
\]
for $t \geq 0$.

Proof. 1) By applying the generalized Itō formula with $\frac{1}{2}\|B^{A,X}\|^2$, we can obtain that
\[
\frac{1}{2}\|B^{A,X}\|^2 = \frac{1}{2}\|X\|^2 + \int_0^t (B^{A,X}, \mathcal{L}B^{A,X} + \mathcal{G}(B^{A,X}) + g(A, B^{A,X}))ds
\]
\[
+ \int_0^t (B^{A,X}, \sigma_dW_2) + \frac{1}{2} \int_0^t \|\sigma_2\|^2_{\mathcal{Q}_2}ds
\]
\[
= \frac{1}{2}\|X\|^2 - \int_0^t \|B^{A,X}_x\|^2 ds + \int_0^t (B^{A,X}, g(A, B^{A,X}))ds
\]
\[
+ \int_0^t (B^{A,X}, \mathcal{G}(B^{A,X}))ds + \int_0^t (B^{A,X}, \sigma_dW_2) + \frac{1}{2} \int_0^t \|\sigma_2\|^2_{\mathcal{Q}_2}ds.
\]
Taking mathematical expectation from both sides of above equation, we have
\[
E\|B^{A,X}\|^2 = \|X\|^2 - 2 \int_0^t E\|B^{A,X}_x\|^2 ds + 2 \int_0^t E(B^{A,X}, g(A, B^{A,X}))ds
\]
\[
+ 2 \int_0^t E(B^{A,X}, \mathcal{G}(B^{A,X}))ds + \int_0^t \|\sigma_2\|^2_{\mathcal{Q}_2}ds,
\]

namely,
\[
\frac{d}{dt} E\|B^{A,X}\|^2
\]
\[
= -2E\|B^{A,X}_x\|^2 + 2E(B^{A,X}, g(A, B^{A,X})) + 2E(B^{A,X}, \mathcal{G}(B^{A,X})) + \|\sigma_1\|^2_{\mathcal{Q}_1}.
\]

According to Corollary 2.2, we have
\[
(B^{A,X}, \mathcal{G}(B^{A,X})) \leq 0,
\]
thus,
\[
\frac{d}{dt} E\|B^{A,X}\|^2
\]
\[
\leq -2E\|B^{A,X}_x\|^2 + 2E(B^{A,X}, g(A, B^{A,X})) + \|\sigma_1\|^2_{\mathcal{Q}_1},
\]
\[
= -2E\|B^{A,X}_x\|^2 + 2E(B^{A,X}, g(A, B^{A,X}) - g(A, 0)) + 2E(B^{A,X}, g(A, 0)) + \|\sigma_1\|^2_{\mathcal{Q}_1}
\]
\[
\leq -2\lambda E\|B^{A,X}\|^2 + 2L_g E\|B^{A,X}\|^2 + \|\sigma_1\|^2_{\mathcal{Q}_1} + C
\]
\[
= -\lambda E\|B^{A,X}\|^2 + \|A\|^2 + \|\sigma_1\|^2_{\mathcal{Q}_1} + C
\]
\[
= -\alpha E\|B^{A,X}\|^2 + C\|A\|^2 + \|\sigma_1\|^2_{\mathcal{Q}_1} + C
\]
\[
= -\alpha E\|B^{A,X}\|^2 + C\|A\|^2 + \|\sigma_1\|^2_{\mathcal{Q}_1} + C.
\]
Hence, by applying Lemma 2.2 with $E\|B^A,X\|^2$, we have
\[ E\|B^A,X(t)\|^2 \leq e^{-\alpha t}\|X\|^2 + C(\|A\|^2 + 1). \]

• It is easy to see
\[
\begin{align*}
\frac{1}{2} &\|B^A,X - B^A,Y\|^2 \\
&= \frac{1}{2}\|X - Y\|^2 + \int_0^t (B^A,X - B^A,Y, L(B^A,X - B^A,Y) + G(B^A,X) - G(B^A,Y) + g(A, B^A,X) - g(A, B^A,Y))ds \\
&= \frac{1}{2}\|X - Y\|^2 - \int_0^t (B^A,X - B^A,Y) x ds + \int_0^t (B^A,X - B^A,Y, L(B^A,X - B^A,Y) + G(B^A,X) - G(B^A,Y))ds \\
&\quad + \int_0^t (B^A,X - B^A,Y, g(A, B^A,X) - g(A, B^A,Y))ds,
\end{align*}
\]

namely,
\[
\frac{d}{dt}\|B^A,X - B^A,Y\|^2 \\
= -2\|(B^A,X - B^A,Y) x\|^2 + 2(B^A,X - B^A,Y, G(B^A,X) - G(B^A,Y)) + 2(B^A,X - B^A,Y, g(A, B^A,X) - g(A, B^A,Y)).
\]

It follows from Lemma 2.2 we have
\[
(B^A,X - B^A,Y, G(B^A,X) - G(B^A,Y)) \leq 0,
\]
thus, we have
\[
\begin{align*}
\frac{d}{dt}\|B^A,X - B^A,Y\|^2 &\leq -2\|(B^A,X - B^A,Y) x\|^2 + 2Lg\|B^A,X - B^A,Y\|^2 \\
&\leq -2(\lambda - Lg)\|B^A,X - B^A,Y\|^2 \\
&= -2\alpha\|B^A,X - B^A,Y\|^2,
\end{align*}
\]
this yields
\[
\|B^A,X - B^A,Y\|^2 \leq \|X - Y\|^2 e^{-2\alpha t}.
\]
Thus, we have
\[
E\|B^A,X - B^A,Y\|^2 \leq \|X - Y\|^2 e^{-2\alpha t}.
\]

2) (2.3) imply for any $A \in L^2(I)$ that there is unique invariant measure $\mu^A$ for the Markov semigroup $P_t^A$ associated with the system (2.2) in $L^2(I)$ such that
\[
\int_{L^2(I)} P_t^A \varphi d\mu^A = \int_{L^2(I)} \varphi d\mu^A, \quad t \geq 0
\]
for any $\varphi \in B_b(L^2(I))$ the space of bounded functions on $L^2(I)$.

Then by repeating the standard argument as in [10, Proposition 4.2] and [8, Lemma 3.4], the invariant measure satisfies
\[ \int_{L^2(I)} \| z \|^2 \mu^A(dz) \leq C(1 + \| A \|^2). \]

3) According to the invariant property of $\mu^A$, (2) and (2.3), we have
\[
\| \mathbb{E}f(A, B^A,X) - \bar{f}(A) \|^2 \\
= \| \mathbb{E}f(A, B^A,X) - \int_{L^2(I)} f(A,Y)\mu^A(dY) \|^2 \\
= \| \mathbb{E}f(A, B^A,X) - \mathbb{E} \int_{L^2(I)} f(A, B^A,Y)\mu^A(dY) \|^2 \\
= \| \int_{L^2(I)} \mathbb{E}[f(A, B^A,X) - f(A, B^A,Y)]\mu^A(dY) \|^2 \\
\leq C \int_{L^2(I)} \mathbb{E}\| B^A,X - B^A,Y \|^2 \mu^A(dY) \\
\leq C \int_{L^2(I)} \| X - Y \|^2 e^{-2\alpha t} \mu^A(dY) \\
\leq C(1 + \| X \|^2 + \| A \|^2)e^{-2\alpha t}.
\]

3 Well-posedness and a priori estimate for the slow-fast system (1.1) and averaged equation (1.2)

We first establish the well-posedness for the slow-fast system (1.1). We consider the mild solution of (1.1). The Banach contraction principle is used as the main tool for proving the existence of mild solutions of SPDE in most of the existing papers. We first apply the fixed point theorem to the corresponding truncated equation and give the local existence of mild solutions to (1.1). Then, the energy estimate shows that the solution is also global in time.

3.1 Well-posedness and a priori estimate for the slow-fast system (1.1)

Definition 3.1. If $(A^\varepsilon, B^\varepsilon)$ is an adapted process over $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ such that $\mathbb{P}$-a.s. the integral equations
\[
A^\varepsilon(t) = G'(t)A_0 + G(t)A_1 + \int_0^t G(t-s)[\mathcal{F}(A^\varepsilon) + f(A^\varepsilon, B^\varepsilon)](s)ds + \int_0^t G(t-s)\sigma_1dB_1 \\
B^\varepsilon(t) = S(\frac{t}{\varepsilon})B_0 + \frac{1}{\varepsilon} \int_0^t S(\frac{t-s}{\varepsilon})[\mathcal{G}(B^\varepsilon) + g(A^\varepsilon, B^\varepsilon)](s)ds + \frac{1}{\sqrt{\varepsilon}} \int_0^t S(\frac{t-s}{\varepsilon})\sigma_2dB_2
\]
hold true for all $t > 0$, we say that it is a mild solution for Eqs. (1.1).

Proposition 3.1. For any $\varepsilon \in (0, 1)$, $T > 0$, $p \geq 1$, if $A_0, B_0 \in H^1_0(I), A_1 \in L^2(I)$, (1.1) admits a unique mild solution $(A^\varepsilon, B^\varepsilon) \in X_{2,T}$.

The proof of well-posedness for the slow-fast system (1.1) is divided into several steps.
3.1.1 Local existence

We can establish the local well-posedness for the slow-fast system (1.1) in \(X_{p,T}(p \geq 1)\).

**Lemma 3.1.** For any \(A_0, B_0 \in H^1_0(I), A_1 \in L^2(I),\) and \(p \geq 1, \varepsilon \in (0, 1)\) \(\text{(1.1)}\) admits a unique mild solution \((A^\varepsilon, B^\varepsilon) \in X_{p,T},\) where \(\tau_\infty\) is stopping time for \(p.\) Moreover, if \(\tau_\infty < +\infty,\) then \(\mathbb{P}-a.s.\)

\[
\lim_{t \to \tau_\infty} \|(A^\varepsilon, B^\varepsilon)\|_{Y_t} = +\infty.
\]

**Proof.** Inspired from [35], let \(\rho \in C_0^\infty(\mathbb{R})\) be a cut-off function such that \(\rho(r) = 1\) for \(r \in [0, 1]\) and \(\rho(r) = 0\) for \(r \geq 2.\) For any \(R > 0, y \in X_{p,T}\) and \(t \in [0, T],\) we set

\[
\rho_R(y)(t) = \rho\left(\frac{\|y\|_{C([0,\varepsilon];\mathcal{H}^1(t))}}{R}\right).
\]

The truncated equation corresponding to \(\text{(1.1)}\) is the following stochastic partial differential equation:

\[
\begin{cases}
  dA^\varepsilon_t = \mathcal{L}(A^\varepsilon) + \rho_R(A^\varepsilon)\mathcal{F}(A^\varepsilon) + f(A^\varepsilon, B^\varepsilon)\,dt + \sigma_1\,dW_1 & \text{in } Q, \\
  dB^\varepsilon_t = \rho_R(B^\varepsilon)\mathcal{G}(B^\varepsilon) + g(A^\varepsilon, B^\varepsilon)\,dt + \frac{1}{\sqrt{\varepsilon}}\sigma_2\,dW_2 & \text{in } Q, \\
  A^\varepsilon(0, t) = 0 = A^\varepsilon(1, t) & \text{in } (0, T), \\
  B^\varepsilon(0, t) = 0 = B^\varepsilon(1, t) & \text{in } (0, T), \\
  A^\varepsilon(x, 0) = A_0(x) & \text{in } I, \\
  A^\varepsilon(x, 0) = A_1(x) & \text{in } I, \\
  B^\varepsilon(x, 0) = B_0(x) & \text{in } I.
\end{cases}
\]

In this proof, we will take

\[
\varepsilon = 1
\]

for the sake of simplicity. All the results can be extended without difficulty to the general case. Thus, we consider the following system

\[
\begin{cases}
  dA_t = \mathcal{L}(A) + \rho_R(A)\mathcal{F}(A) + f(A, B)\,dt + \sigma_1\,dW_1 & \text{in } Q, \\
  dB = \mathcal{L}(B) + \rho_R(B)\mathcal{G}(B) + g(A, B)\,dt + \sigma_2\,dW_2 & \text{in } Q, \\
  A(0, t) = 0 = A(1, t) & \text{in } (0, T), \\
  B(0, t) = 0 = B(1, t) & \text{in } (0, T), \\
  A(x, 0) = A_0(x) & \text{in } I, \\
  A(x, 0) = A_1(x) & \text{in } I, \\
  B(x, 0) = B_0(x) & \text{in } I.
\end{cases}
\]

We define

\[
\Phi_R(A, B) = \left(\begin{array}{c}
\Phi^1_R(A, B) \\
\Phi^2_R(A, B)
\end{array}\right)
\]

\[
= \left(\begin{array}{c}
G(t)A_0 + G(t)A_1 + \int_0^t G(t-s)[\rho_R(A)\mathcal{F}(A) + f(A, B)](s)\,ds + \int_0^t G(t-s)\sigma_1\,dB_1 \\
S(t)B_0 + \int_0^t S(t-s)[\rho_R(B)\mathcal{G}(B) + g(A, B)](s)\,ds + \int_0^t S(t-s)\sigma_2\,dB_2
\end{array}\right).
\]

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It is easy to see the operator $\Phi_R(A, B)$ maps $X_{p, T_0}$ into itself.

The estimates of

$$
\mathbb{E} \sup_{0 \leq t \leq T_0} \| (\Phi^1_R(A_1, B_1) - \Phi^1_R(A_2, B_2))(t) \|_{H^1(I)}^p + \mathbb{E} \sup_{0 \leq t \leq T_0} \| ((\Phi^1_R(A_1, B_1) - \Phi^1_R(A_2, B_2))(t))' \|_p,
$$

$$
\mathbb{E} \sup_{0 \leq t \leq T_0} \| (\Phi^2_R(A_1, B_1) - \Phi^2_R(A_2, B_2))(t) \|_{H^1(I)}^p.
$$

Indeed, due to [51, P84], we have

$$
\| \rho_R(A_1)|A_1|^{2\sigma} A_1 - \rho_R(A_2)|A_2|^{2\sigma} A_2 \| \leq CR^{2\sigma} \| A_1 - A_2 \|_{H^1(I)}.
$$

It follows from Corollary 2.1 that

$$
\mathbb{E} \sup_{0 \leq t \leq T_0} \| \int_0^t G(t-s)(\rho_R(A_1)F(A_1) - \rho_R(A_2)F(A_2))(s)ds \|_{H^1(I)}^p + \mathbb{E} \sup_{0 \leq t \leq T_0} \| (\int_0^t G(t-s)(\rho_R(A_1)F(A_1) - \rho_R(A_2)F(A_2))(s)ds)' \|_p
\leq CT_0^{p-1}\mathbb{E} \sup_{0 \leq t \leq T_0} \| (\rho_R(A_1)F(A_1) - \rho_R(A_2)F(A_2))(s) \|_{H^1(I)}^p ds
\leq CT_0^{p-1}(R^{2p} + R^{4p})\mathbb{E} \sup_{0 \leq t \leq T_0} \| (A_1 - A_2)(s) \|_{H^1(I)}^p ds
\leq CT_0^{p}(R^{2p} + R^{4p})\mathbb{E} \sup_{0 \leq t \leq T_0} \| (A_1 - A_2)(t) \|_{H^1(I)}^p,
$$

and

$$
\mathbb{E} \sup_{0 \leq t \leq T_0} \| \int_0^t G(t-s)(f(A_1, B_1) - f(A_2, B_2))(s)ds \|_{H^1(I)}^p + \mathbb{E} \sup_{0 \leq t \leq T_0} \| (\int_0^t G(t-s)(f(A_1, B_1) - f(A_2, B_2))(s)ds)' \|_p
\leq CT_0^{p-1}\mathbb{E} \sup_{0 \leq t \leq T_0} \| (f(A_1, B_1) - f(A_2, B_2))(s) \|_{H^1(I)}^p ds
\leq CT_0^{p-1}\mathbb{E} \sup_{0 \leq t \leq T_0} \| (A_1 - A_2)(s) \|_p + \| (B_1 - B_2)(s) \|_p ds
\leq CT_0^{p}(\mathbb{E} \sup_{0 \leq t \leq T_0} \| (A_1 - A_2)(t) \|_p + \mathbb{E} \sup_{0 \leq t \leq T_0} \| (B_1 - B_2)(t) \|_p).
$$

Finally, collecting the above estimates (3.1)-(3.2), we get

$$
\mathbb{E} \sup_{0 \leq t \leq T_0} \| (\Phi^1_R(A_1, B_1) - \Phi^1_R(A_2, B_2))(t) \|_{H^1(I)}^p + \mathbb{E} \sup_{0 \leq t \leq T_0} \| ((\Phi^1_R(A_1, B_1) - \Phi^1_R(A_2, B_2))(t))' \|_p
\leq C[T_0^p(R^{2p} + R^{4p}) + T_0^p](\mathbb{E} \sup_{0 \leq t \leq T_0} \| (A_1 - A_2)(t) \|_{H^1(I)}^p + \mathbb{E} \sup_{0 \leq t \leq T_0} \| (B_1 - B_2)(t) \|_{H^1(I)}^p).
$$
By taking $p = q = 2, j = 1$ in the third inequality of (2.1), we have

\[
E \sup_{0 \leq t \leq T_0} \left\| \int_{0}^{t} S(t-s)(\rho_R(B_1)\mathcal{G}(B_1) - \rho_R(A_2)\mathcal{G}(B_2))(s)ds \right\|_{H^1}^p
\]
\[
\leq CE \sup_{0 \leq t \leq T_0} \left( \int_{0}^{t} (t-s)^{-\frac{1}{2}} \|\rho_R(B_1)\mathcal{G}(B_1) - \rho_R(B_2)\mathcal{G}(B_2)(s)\|ds \right)^p
\]
\[
\leq CE \sup_{0 \leq t \leq T_0} \left( \int_{0}^{t} (t-s)^{-\frac{1}{2}} R^2 \|B_1 - B_2(s)\|ds \right)^p
\]
\[
\leq CR^{2p} \sup_{0 \leq t \leq T_0} \left( \int_{0}^{t} (t-s)^{-\frac{1}{2}} ds \right)^p \sup_{0 \leq t \leq T_0} \|B_1 - B_2(t)\|_{H^1}^p
\]
\[
\leq C R^{2p} T_0^p \sup_{0 \leq t \leq T_0} \|B_1 - B_2(t)\|_{H^1}^p,
\]

and

\[
E \sup_{0 \leq t \leq T_0} \left\| \int_{0}^{t} S(t-s)(g(A_1, B_1) - g(A_2, B_2))(s)ds \right\|_{H^1(L)}^p
\]
\[
\leq E \sup_{0 \leq t \leq T_0} \left( \int_{0}^{t} \|S(t-s)(g(A_1, B_1) - g(A_2, B_2))(s)\|_{H^1(L)}ds \right)^p
\]
\[
\leq CE \sup_{0 \leq t \leq T_0} \left( \int_{0}^{t} (t-s)^{-\frac{1}{2}} \|g(A_1, B_1) - g(A_2, B_2)(s)\|ds \right)^p
\]
\[
\leq CE \sup_{0 \leq t \leq T_0} \left( \int_{0}^{t} (t-s)^{-\frac{1}{2}} \left( \|(A_1 - A_2)(s)\| + \|(B_1 - B_2)(s)\| \right)ds \right)^p
\]
\[
\leq C \sup_{0 \leq t \leq T_0} \left( \int_{0}^{t} (t-s)^{-\frac{1}{2}} ds \right)^p \left( E \sup_{0 \leq t \leq T_0} \|(A_1 - A_2)(t)\| + E \sup_{0 \leq t \leq T_0} \|(B_1 - B_2)(t)\| \right)
\]
\[
\leq CT_0^p \left( E \sup_{0 \leq t \leq T_0} \|(A_1 - A_2)(t)\| + E \sup_{0 \leq t \leq T_0} \|(B_1 - B_2)(t)\| \right).
\]

According to (3.3) and (3.5), we have

\[
E \sup_{0 \leq t \leq T_0} \left\| (\Phi_R^2(A_1, B_1) - \Phi_R^2(A_2, B_2))(t) \right\|_{H^1(L)}^p
\]
\[
\leq CR^{2p} T_0^p \left( E \sup_{0 \leq t \leq T_0} \|(A_1 - A_2)(t)\|_{H^1(L)}^p + E \sup_{0 \leq t \leq T_0} \|(B_1 - B_2)(t)\|_{H^1(L)}^p \right)
\]

It follows from (3.3) and (3.5) that

\[
E \sup_{0 \leq t \leq T_0} \left\| (\Phi_R^1(A_1, B_1) - \Phi_R^1(A_2, B_2))(t) \right\|_{H^1(L)}^p
\]
\[
+ E \sup_{0 \leq t \leq T_0} \left\| \left( (\Phi_R^1(A_1, B_1) - \Phi_R^1(A_2, B_2))(t) \right)' \right\|_{H^1(L)}^p
\]
\[
+ E \sup_{0 \leq t \leq T_0} \left\| (\Phi_R^2(A_1, B_1) - \Phi_R^2(A_2, B_2))(t) \right\|_{H^1(L)}^p
\]
\[
\leq C(T_0^{pR} R^{2p} + T_0^{pR} R^4 + T_0^{pR} R^{2p} + T_0^{pR}) \left( E \sup_{0 \leq t \leq T_0} \|(A_1 - A_2)(t)\|_{H^1(L)}^p + E \sup_{0 \leq t \leq T_0} \|(B_1 - B_2)(t)\|_{H^1(L)}^p \right),
\]

namely, we have

\[
||\Phi_R(A_1, B_1) - \Phi_R(A_2, B_2)||_{X_{p, T_0}}
\]
\[
\leq C(T_0(R^2 + R^4) + T_0^{\frac{pR}{2}} R^2 + T_0^{\frac{pR}{2}}) \|(A_1, B_1) - (A_2, B_2)\|_{X_{p, T_0}},
\]

(3.6)
• For a sufficiently small $T_0$, is $\Phi_R(A, B)$ a contraction mapping on $X_{p,T_0}$.

Hence, by applying the Banach contraction principle, $\Phi_R(A, B)$ has a unique fixed point in $X_{p,T_0}$, which is the unique local solution to (1.1) on the interval $[0, T_0]$. Since $T_0$ does not depend on the initial value $(A_0, B_0)$, this solution may be extended to the whole interval $[0, T]$.

We denote by $(A_R, B_R)$ this unique mild solution and let

$$
\tau_R = \inf \{ t \geq 0 : \|(A_R, B_R)\|_{X_{p,t}} \geq R \},
$$

with the usual convention that $\inf \emptyset = \infty$.

Since $R_1 \leq R_2$, $\tau_{R_1} \leq \tau_{R_2}$, we can put $\tau_\infty = \lim_{R \to +\infty} \tau_R$. We define a local solution to (1.1) as follows

$$
A(t) = A_R(t), \quad \forall t \in [0, \tau_R],
B(t) = B_R(t), \quad \forall t \in [0, \tau_R].
$$

Indeed, for any $t \in [0, \tau_{R_1} \wedge \tau_{R_2}]$

$$
A_{R_1}(t) - A_{R_2}(t)
= \int_0^t G(t-s)(\rho_{R_1}(A_{R_1})F(A_{R_1}) - \rho_{R_2}(A_{R_2})F(A_{R_2}) + f(A_{R_1}, B_{R_1}) - f(A_{R_2}, B_{R_2}))(s)ds,
$$

$$
B_{R_1}(t) - B_{R_2}(t)
= \int_0^t S(t-s)(\rho_{R_1}(B_{R_1})G(B_{R_1}) - \rho_{R_2}(B_{R_2})G(B_{R_2}) + g(A_{R_1}, B_{R_1}) - g(A_{R_2}, B_{R_2}))(s)ds.
$$

Proceeding as in the proof of (3.6), we can obtain

$$
\|(A_{R_1}, B_{R_1}) - (A_{R_2}, B_{R_2})\|_{X_{p,t}}
\leq C(t)\|(A_{R_1}, B_{R_1}) - (A_{R_2}, B_{R_2})\|_{X_{p,t}},
$$

where $C(t)$ is a monotonically increasing function and $C(0) = 0$. If we take $t$ sufficiently small, we can obtain

$$
A_{R_1}(t) = A_{R_2}(t),
B_{R_1}(t) = B_{R_2}(t).
$$

Repeating the same argument for the interval $[t, 2t]$ and so on yields

$$
A_{R_1}(t) = B_{R_2}(t),
A_{R_1}(t) = B_{R_2}(t).
$$

for the whole interval $[0, \tau]$. According to this, we can know the above definition of local solution to (1.1) is well defined.

If $\tau_\infty < +\infty$, the definition of $(A, B)$ yields $\mathbb{P}$–a.s.

$$
\lim_{t \to \tau_\infty} \|(A, B)\|_{X_{p,t}} = +\infty,
$$

which shows that $(A, B)$ is a unique local solution to (1.1) on the interval $[0, \tau_\infty)$.

This completes the proof of Lemma 3.1. \qed
3.1.2 Energy inequalities for the slow-fast system (1.1)

Now, we establish some energy inequalities for the slow-fast system (1.1).

**Proposition 3.2.** Let $\xi = \inf\{\tau_\infty, T\}$. If $A_0, B_0 \in H^1_0(I), A_1 \in L^2(I)$, for $\varepsilon \in (0, 1)$, $(A^\varepsilon, B^\varepsilon)$ is the unique solution to (1.1), then there exists a constant $C$ such that the solutions $(A^\varepsilon, B^\varepsilon)$ satisfy

$$
\sup_{\varepsilon \in (0, 1)} \mathbb{E} \sup_{0 \leq t \leq \xi} (\|A^\varepsilon_x(t)\|^2 + \|A^\varepsilon_t(t)\|^2 + \|A^\varepsilon(t)\|_{x^4(I)}^4 + \|A^\varepsilon(t)\|_{L^6(I)}^6) \leq C,
$$

$$
\sup_{\varepsilon \in (0, 1)} \mathbb{E} \sup_{t \in [0, \xi]} \|B^\varepsilon(t)\|_{H^1(I)}^2 \leq C \varepsilon,
$$

$$
\mathbb{E} \sup_{t \in [0, \xi]} \|B^\varepsilon(t)\|_{H^1(I)}^2 \leq \frac{C}{\varepsilon},
$$

$$
\sup_{\varepsilon \in (0, 1)} \mathbb{E} \int_0^\xi \|B^\varepsilon_{xx}(t)\|^2 dt \leq C.
$$

where $C$ depends on $T, A_0, B_0$ but independent of $\varepsilon \in (0, 1)$.

**Proof.** The proof of Proposition 3.2 is divided into several steps.

- The estimates of $\mathbb{E} \sup_{0 \leq t \leq \xi} (\|A^\varepsilon_x(t)\|^2 + \|A^\varepsilon_t(t)\|^2 + \|A^\varepsilon(t)\|_{x^4(I)}^4 + \|A^\varepsilon(t)\|_{L^6(I)}^6)$ and $\mathbb{E} \|B^\varepsilon(t)\|_{H^1(I)}^2$.

Indeed, it follows from [17, P137, Theorem 3.5] that

$$
\|A^\varepsilon_x(t)\|^2 + \|A^\varepsilon_t(t)\|^2
= \|A^\varepsilon_x(0)\|^2 + \|A^\varepsilon_t(0)\|^2 + 2 \int_0^t (A^\varepsilon_t, \mathcal{F}(A^\varepsilon) + f(A^\varepsilon, B^\varepsilon)) ds + 2 \int_0^t (A^\varepsilon_t, \sigma_1 dW_1) + \int_0^t \|\sigma_1\|_{Q_1}^2 ds,
$$

since

$$
\int_0^t (A^\varepsilon_t, \mathcal{F}(A^\varepsilon)) ds
= -\frac{1}{4} \|A^\varepsilon(t)\|_{x^4(I)}^4 + \frac{1}{4} \|A^\varepsilon(0)\|_{x^4(I)}^4 - \frac{1}{4} \|A^\varepsilon(t)\|_{L^6(I)}^6 + \frac{1}{6} \|A^\varepsilon(0)\|_{L^6(I)}^6,
$$

we have

$$
\|A^\varepsilon_x(t)\|^2 + \|A^\varepsilon_t(t)\|^2 + \frac{1}{4} \|A^\varepsilon(t)\|_{x^4(I)}^4 + \frac{1}{4} \|A^\varepsilon(t)\|_{L^6(I)}^6
= \|A^\varepsilon_x(0)\|^2 + \|A^\varepsilon_t(0)\|^2 + \frac{1}{4} \|A^\varepsilon(0)\|_{x^4(I)}^4 + \frac{1}{4} \|A^\varepsilon(0)\|_{L^6(I)}^6
+ 2 \int_0^t (A^\varepsilon_t, f(A^\varepsilon, B^\varepsilon)) ds + 2 \int_0^t (A^\varepsilon_t, \sigma_1 dW_1) + \int_0^t \|\sigma_1\|_{Q_1}^2 ds,
$$

it is easy to see

$$
\sup_{0 \leq t \leq \xi} (\|A^\varepsilon_x(t)\|^2 + \|A^\varepsilon_t(t)\|^2 + \frac{1}{4} \|A^\varepsilon(t)\|_{x^4(I)}^4 + \frac{1}{4} \|A^\varepsilon(t)\|_{L^6(I)}^6)
\leq C(\|A^\varepsilon_x(0)\|^2 + \|A^\varepsilon_t(0)\|^2 + \frac{1}{4} \|A^\varepsilon(0)\|_{x^4(I)}^4 + \frac{1}{4} \|A^\varepsilon(0)\|_{L^6(I)}^6)
+ \sup_{0 \leq t \leq \xi} \int_0^t (A^\varepsilon_t, f(A^\varepsilon, B^\varepsilon)) ds + \sup_{0 \leq t \leq \xi} \int_0^t (A^\varepsilon_t, \sigma_1 dW_1) + \sup_{0 \leq t \leq \xi} \int_0^t \|\sigma_1\|_{Q_1}^2 ds,
$$

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this implies that
\[
\sup_{0 \leq t \leq \xi} (\|A^\varepsilon_x(t)\|^2 + \|A^\varepsilon_t(t)\|^2 + \|A^\varepsilon(t)\|_{L^4(I)}^4 + \|A^\varepsilon(t)\|_{L^6(I)}^6)
\leq C(\|A^\varepsilon_x(0)\|^2 + \|A^\varepsilon_t(0)\|^2 + \|A^\varepsilon(0)\|_{L^4(I)}^4 + \|A^\varepsilon(0)\|_{L^6(I)}^6)
\]
\[+ \sup_{0 \leq t \leq \xi} \left| \int_0^t (A^\varepsilon_t, f(A^\varepsilon, B^\varepsilon))ds \right| + \sup_{0 \leq t \leq \xi} \left| \int_0^t (A^\varepsilon_t, \sigma_1 dW_1) \right| + \sup_{0 \leq t \leq \xi} \left| \int_0^t \|\sigma_1\|_{Q_1}^2 ds \right|.
\]

By taking mathematical expectation from both sides of above equation, we have
\[
E \sup_{0 \leq t \leq \xi} (\|A^\varepsilon_x(t)\|^2 + \|A^\varepsilon_t(t)\|^2 + \|A^\varepsilon(t)\|_{L^4(I)}^4 + \|A^\varepsilon(t)\|_{L^6(I)}^6)
\leq C(\|A^\varepsilon_x(0)\|^2 + \|A^\varepsilon_t(0)\|^2 + \|A^\varepsilon(0)\|_{L^4(I)}^4 + \|A^\varepsilon(0)\|_{L^6(I)}^6)
\]
\[+ E \sup_{0 \leq t \leq \xi} \left| \int_0^t (A^\varepsilon_t, f(A^\varepsilon, B^\varepsilon))ds \right| + E \sup_{0 \leq t \leq \xi} \left| \int_0^t (A^\varepsilon_t, \sigma_1 dW_1) \right| + E \sup_{0 \leq t \leq \xi} \left| \int_0^t \|\sigma_1\|_{Q_1}^2 ds \right|.
\]

In view of the Burkholder-Davis-Gundy inequality, it holds that
\[
E \sup_{0 \leq t \leq \xi} \left| \int_0^t (A^\varepsilon_t, \sigma_1 dW_1) \right|
\leq C \mathbb{E} \left( \int_0^\xi \|A^\varepsilon\|^2 \|\sigma_1\|^2_{Q_1} ds \right)^{1/2}
\leq C \mathbb{E} \left( \sup_{0 \leq t \leq \xi} \|A^\varepsilon\|^2 \int_0^\xi \|\sigma_1\|^2_{Q_1} ds \right)^{1/2}
\]
\[= C \mathbb{E} \left( \sup_{0 \leq t \leq \xi} \|A^\varepsilon\| \cdot \left( \int_0^\xi \|\sigma_1\|^2_{Q_1} ds \right)^{1/2} \right)
\leq \eta \mathbb{E} \sup_{0 \leq t \leq \xi} \|A^\varepsilon\|^2 + C(\eta) \mathbb{E} \left( \int_0^\xi \|\sigma_1\|^2_{Q_1} ds \right)
\]
\[= \eta \mathbb{E} \sup_{0 \leq t \leq \xi} \|A^\varepsilon\|^2 + C(\eta, T, \sigma_1).
\]
In view of the Hölder inequality, it holds that
\[
E \sup_{0 \leq t \leq \xi} |\int_0^t (A^\xi_t, f(A^\xi_t, B^\xi)) \, ds|
\leq C E(\int_0^\xi |(A^\xi_t, f(A^\xi_t, B^\xi))| \, ds)
\leq C E(\int_0^\xi \|A^\xi_t\||f(A^\xi_t, B^\xi)| \, ds)
\leq C E(\sup_{0 \leq t \leq \xi} \|A^\xi_t\| \int_0^\xi \|f(A^\xi_t, B^\xi)| \, ds)
= C E[\sup_{0 \leq t \leq \xi} \|A^\xi_t\| \cdot (\int_0^\xi \|f(A^\xi_t, B^\xi)| \, ds)]
\leq \eta E \sup_{0 \leq t \leq \xi} \|A^\xi_t\|^2 + C(\eta) E(\int_0^\xi \|f(A^\xi_t, B^\xi)|^2 \, ds)^2
= \eta E \sup_{0 \leq t \leq \xi} \|A^\xi_t\|^2 + C(\eta) E(\int_0^\xi \|f(A^\xi_t, B^\xi)|^2 \, ds)^2
\leq \eta E \sup_{0 \leq t \leq \xi} \|A^\xi_t\|^2 + C(\eta, T) E \int_0^\xi \|f(A^\xi_t, B^\xi)|^2 \, ds
\leq \eta E \sup_{0 \leq t \leq \xi} \|A^\xi_t\|^2 + C(\eta, T, \sigma_1) E \int_0^\xi \|A^\xi\|^2 \, ds + E \int_0^\xi \|B^\xi\|^2 \, ds + C(\eta, T, \sigma_1),
\]
by taking \(0 < \eta < 1\), it holds that
\[
E \sup_{0 \leq t \leq \xi} (\|A^\xi_t\|^2 + \|A^\xi_t\|^2 + \|A^\xi_t\|^4_{L^4(I)} + \|A^\xi_t\|^6_{L^6(I)})
\leq \tilde{C}(\|A^\xi_t(0)\|^2 + \|A^\xi_t(0)\|^2 + \|A^\xi(0)\|^4_{L^4(I)} + \|A^\xi(0)\|^6_{L^6(I)})
+ 2\eta E \sup_{0 \leq t \leq \xi} \|A^\xi_t\|^2 + C(\eta, T) E \int_0^\xi \|A^\xi\|^2 \, ds + E \int_0^\xi \|B^\xi\|^2 \, ds),
\]
thus, it follows from Gronwall inequality that
\[
E \sup_{0 \leq t \leq \xi} (\|A^\xi_t\|^2 + \|A^\xi_t\|^2 + \|A^\xi_t\|^4_{L^4(I)} + \|A^\xi_t\|^6_{L^6(I)})
\leq \tilde{C}(1 + \|A^\xi_t(0)\|^2 + \|A^\xi_t(0)\|^2 + \|A^\xi(0)\|^4_{L^4(I)} + \|A^\xi(0)\|^6_{L^6(I)})
+ E \int_0^\xi \|A^\xi\|^2 \, ds, \tag{3.7}
\]
moreover, we have
\[
E \sup_{0 \leq t \leq \xi} (\|A^\xi_t\|^2 + \|A^\xi_t\|^2 + \|A^\xi_t\|^4_{L^4(I)} + \|A^\xi_t\|^6_{L^6(I)})
\leq \tilde{C}(1 + \|A_0\|^2_{H^1} + \|A_1\|^2 + \|A_0\|^4_{L^4(I)} + \|A_0\|^6_{L^6(I)}) + E \int_0^\xi \|B^\xi\|^2 \, ds.
\]
by taking mathematical expectation from both sides of above equation, we have

\[
\frac{d}{dt} \mathbb{E} \|B^\varepsilon(t)\|^2 = \frac{2}{\varepsilon} (\mathcal{L}(B^\varepsilon) + \mathcal{G}(B^\varepsilon) + g(A^\varepsilon, B^\varepsilon), B^\varepsilon)dt + \frac{1}{\varepsilon} \|\sigma_2\|^2 dt + \frac{2}{\sqrt{\varepsilon}} (B^\varepsilon, \sigma_2 dW_2),
\]

this implies that

\[
\|B^\varepsilon(t)\|^2 = \|B^\varepsilon(0)\|^2 + \frac{2}{\varepsilon} \int_0^t (\mathcal{L}(B^\varepsilon) + \mathcal{G}(B^\varepsilon) + g(A^\varepsilon, B^\varepsilon), B^\varepsilon)ds + \frac{1}{\varepsilon} \int_0^t \|\sigma_2\|^2 ds + \frac{2}{\sqrt{\varepsilon}} \int_0^t (B^\varepsilon, \sigma_2 dW_2),
\]

by taking mathematical expectation from both sides of above equation, we have

\[
\mathbb{E} \|B^\varepsilon(t)\|^2 = \mathbb{E} \|B^\varepsilon(0)\|^2 + \frac{2}{\varepsilon} \mathbb{E} \int_0^t (\mathcal{L}(B^\varepsilon) + \mathcal{G}(B^\varepsilon) + g(A^\varepsilon, B^\varepsilon), B^\varepsilon)ds + \frac{1}{\varepsilon} \mathbb{E} \int_0^t \|\sigma_2\|^2 ds + \frac{2}{\sqrt{\varepsilon}} \mathbb{E} \int_0^t (B^\varepsilon, \sigma_2 dW_2),
\]

this implies that

\[
\frac{d}{dt} \mathbb{E} \|B^\varepsilon(t)\|^2 = \frac{2}{\varepsilon} \mathbb{E} (-\|B^\varepsilon\|^2 - \|B^\varepsilon\|_{L^4}^4 - (g(A^\varepsilon, B^\varepsilon), B^\varepsilon)) + \frac{1}{\varepsilon} \mathbb{E} \|\sigma_2\|^2.
\]

We consider this term

\[
\frac{2}{\varepsilon} \mathbb{E} (-\|B^\varepsilon\|^2 - \|B^\varepsilon\|_{L^4}^4 - (g(A^\varepsilon, B^\varepsilon), B^\varepsilon)) \\
\leq \frac{2}{\varepsilon} \mathbb{E} (-\|B^\varepsilon\|^2 - (g(A^\varepsilon, B^\varepsilon), B^\varepsilon)) \\
= \frac{2}{\varepsilon} \mathbb{E} (-\|B^\varepsilon\|^2) + \frac{2}{\varepsilon} \mathbb{E} (-g(A^\varepsilon, B^\varepsilon) + g(A^\varepsilon, 0, B^\varepsilon)) + \frac{2}{\varepsilon} \mathbb{E} (g(A^\varepsilon, 0, B^\varepsilon)) \\
\leq \frac{2}{\varepsilon} \mathbb{E} (-\|B^\varepsilon\|^2) + \frac{2}{\varepsilon} \mathbb{E} (g(A^\varepsilon, B^\varepsilon) - g(A^\varepsilon, 0)) \|B^\varepsilon\| + \frac{2}{\varepsilon} \mathbb{E} (g(A^\varepsilon, 0, B^\varepsilon)) \\
\leq \frac{-2\lambda}{\varepsilon} \mathbb{E} \|B^\varepsilon\|^2 + \frac{2}{\varepsilon} \mathbb{E} L_g \|B^\varepsilon\|^2 + \frac{2}{\varepsilon} \mathbb{E} (g(A^\varepsilon, 0, B^\varepsilon)) \\
= \frac{-2(\lambda - L_g)}{\varepsilon} \mathbb{E} \|B^\varepsilon\|^2 + \frac{2}{\varepsilon} (g(A^\varepsilon, 0, B^\varepsilon)),
\]

by using the Young inequality, we have

\[
\frac{2}{\varepsilon} \mathbb{E} (g(A^\varepsilon, 0, B^\varepsilon)) \\
\leq \frac{2}{\varepsilon} \mathbb{E} \|g(A^\varepsilon, 0)\| \|B^\varepsilon\| \\
\leq \frac{2}{\varepsilon} \mathbb{E} C \|\|A^\varepsilon\| + 1\| \|B^\varepsilon\| \\
= \frac{2}{\varepsilon} \mathbb{E} \|B^\varepsilon\| C(\|A^\varepsilon\| + 1) \\
\leq \frac{C}{\varepsilon} (\mathbb{E} \|B^\varepsilon\|\|A^\varepsilon\| + \mathbb{E} \|B^\varepsilon\|) \\
\leq \frac{2}{\varepsilon} \mathbb{E} \|B^\varepsilon\|^2 + \frac{2}{\varepsilon} \mathbb{E} \|A^\varepsilon\|^2 + \frac{C}{\varepsilon},
\]

it holds that

\[
\frac{2}{\varepsilon} \mathbb{E} (-\|B^\varepsilon\|^2 - \|B^\varepsilon\|_{L^4}^4 - (g(A^\varepsilon, B^\varepsilon), B^\varepsilon)) \\
\leq \frac{-2(\lambda - L_g)}{\varepsilon} \mathbb{E} \|B^\varepsilon\|^2 + \frac{2(\lambda - L_g)}{\varepsilon} \mathbb{E} \|B^\varepsilon\|^2 + \frac{C}{\varepsilon} \mathbb{E} \|A^\varepsilon\|^2 + \frac{C}{\varepsilon},
\]

\[
\frac{1}{\varepsilon} \mathbb{E} \|\sigma_2\|^2 \leq \frac{C}{\varepsilon},
\]

thus, we have

\[
\frac{d}{dt} \mathbb{E} \|B^\varepsilon(t)\|^2 \leq \frac{-2(\lambda - L_g)}{\varepsilon} \mathbb{E} \|B^\varepsilon\|^2 + \frac{C}{\varepsilon} \mathbb{E} \|A^\varepsilon\|^2 + \frac{C}{\varepsilon} = \frac{2}{\varepsilon} \mathbb{E} \|B^\varepsilon\|^2 + \frac{C}{\varepsilon} \mathbb{E} \|A^\varepsilon\|^2 + 1.
\]
Hence, by applying Lemma 2.2 with \( \mathbb{E}\|B^\varepsilon(t)\|^2 \), we have
\[
\mathbb{E}\|B^\varepsilon(t)\|^2 = \mathbb{E}\|B^\varepsilon(0)\|^2 e^{-\frac{2}{\varepsilon}t} + C \frac{\varepsilon}{\varepsilon^2} \int_0^t e^{-\frac{2}{\varepsilon}(t-s)} (\mathbb{E}\|A^\varepsilon(s)\|^2)^2 ds + C \frac{\varepsilon}{\varepsilon^2} \int_0^t e^{-\frac{2}{\varepsilon}(t-s)} \mathbb{E}\|A^\varepsilon(s)\|^2 ds
\]
\[
\leq \|B_0\|^2 e^{-\frac{2}{\varepsilon}t} + C \frac{\varepsilon}{\varepsilon^2} \int_0^t e^{-\frac{2}{\varepsilon}(t-s)} (\mathbb{E}\|A^\varepsilon(s)\|^2)^2 + C \frac{\varepsilon}{\varepsilon^2} \int_0^t e^{-\frac{2}{\varepsilon}(t-s)} \mathbb{E}\|A^\varepsilon(s)\|^2 ds
\]
\[
\leq \|B_0\|^2 e^{-\frac{2}{\varepsilon}t} + C \frac{\varepsilon}{\varepsilon^2} \int_0^t e^{-\frac{2}{\varepsilon}(t-s)} \mathbb{E}\|A^\varepsilon(s)\|^2 ds.
\]

Thus, plug (3.7) in the above inequality, we have
\[
\mathbb{E}\|B^\varepsilon(t)\|^2 \\
\leq C(\|B_0\|^2 + 1) + C \frac{\varepsilon}{\varepsilon^2} \int_0^t e^{-\frac{2}{\varepsilon}(t-s)} [1 + \|A_0\|^2 + \|A_1\|^2 + \|A_0\|^2 + \|A_0\|^2 + \|A_0\|^2 + \|B_0\|^2] ds
\]
\[
= C(1 + \|A_0\|^2 + \|A_1\|^2 + \|A_0\|^2 + \|A_0\|^2 + \|B_0\|^2) + C \frac{\varepsilon}{\varepsilon^2} \int_0^t e^{-\frac{2}{\varepsilon}(t-s)} \mathbb{E}\|B^\varepsilon(\tau)\|^2 ds
\]
\[
\leq CC(1 + \|A_0\|^2 + \|A_1\|^2 + \|A_0\|^2 + \|A_0\|^2 + \|B_0\|^2) + C \frac{\varepsilon}{\varepsilon^2} \int_0^t e^{-\frac{2}{\varepsilon}(t-s)} \mathbb{E}\|B^\varepsilon(\tau)\|^2 ds
\]
\[
= C(1 + \|A_0\|^2 + \|A_1\|^2 + \|A_0\|^2 + \|A_0\|^2 + \|B_0\|^2) + C \frac{\varepsilon}{\varepsilon^2} \int_0^t e^{-\frac{2}{\varepsilon}(t-s)} \mathbb{E}\|B^\varepsilon(\tau)\|^2 ds
\]
\[
= C(1 + \|A_0\|^2 + \|A_1\|^2 + \|A_0\|^2 + \|A_0\|^2 + \|B_0\|^2) + C \frac{\varepsilon}{\varepsilon^2} \int_0^t e^{-\frac{2}{\varepsilon}(t-s)} ds \cdot \mathbb{E}\|B^\varepsilon(\tau)\|^2 ds
\]
\[
= C(1 + \|A_0\|^2 + \|A_1\|^2 + \|A_0\|^2 + \|A_0\|^2 + \|B_0\|^2) + C \frac{\varepsilon}{\varepsilon^2} \int_0^t e^{-\frac{2}{\varepsilon}(t-s)} ds \cdot \mathbb{E}\|B^\varepsilon(\tau)\|^2 ds
\]
\[
\leq C(1 + \|A_0\|^2 + \|A_1\|^2 + \|A_0\|^2 + \|A_0\|^2 + \|B_0\|^2) + C \frac{\varepsilon}{\varepsilon^2} \int_0^t \mathbb{E}\|B^\varepsilon(\tau)\|^2 ds.
\]
thus, it follows from Gronwall inequality that
\[
\sup_{0 \leq t \leq \xi} \mathbb{E}\|B^\varepsilon(t)\|^2 \leq C(1 + \|A_0\|^2 + \|A_1\|^2 + \|A_0\|^2 + \|A_0\|^2 + \|B_0\|^2).
\]

Moreover, due to (3.7) and (3.8), it holds that
\[
\mathbb{E}\sup_{0 \leq t \leq \xi} (\|A^\varepsilon_x(t)\|^2 + \|A^\varepsilon_y(t)\|^2 + \|A^\varepsilon_z(t)\|^2 + \|A^\varepsilon(t)\|^2)
\]
\[
\leq C(1 + \|A_0\|^2 + \|A_1\|^2 + \|A_0\|^2 + \|A_0\|^2 + \|B_0\|^2).
\]

**The estimate of** \( \sup_{0 \leq t \leq \xi} \mathbb{E}\|B^\varepsilon_x(t)\|^2 \).

Indeed, we apply the generalized Itô formula (see [51, 17, 21, 43]) with \( \|B^\varepsilon_x\|^2 \) and obtain that
\[
d\|B^\varepsilon_x\|^2 = (-\frac{2}{\varepsilon}\|B^\varepsilon_x\|^2 + \frac{2}{\varepsilon}(-B^\varepsilon_{xx}, \mathcal{G}(B^\varepsilon_x) + g(A^\varepsilon, B^\varepsilon_x)) + \frac{1}{\varepsilon}\|\sigma_2\|^2_{Q_2}) dt + \frac{2}{\varepsilon}(-B^\varepsilon_{xx}, \sigma_2 dW_2),
\]
namely, it holds that
\[
\frac{1}{2} \|B^\varepsilon_x\|^2 = \frac{1}{2} \|B_{0x}\|^2 - \frac{1}{\varepsilon} \int_0^t \|B^\varepsilon_{xx}\|^2 ds + \frac{1}{\varepsilon} \int_0^t (-B^\varepsilon_{xx}, \mathcal{G}(B^\varepsilon) + g(A^\varepsilon, B^\varepsilon)) ds \\
+ \frac{1}{\sqrt{\varepsilon}} \int_0^t (-B^\varepsilon_{xx}, \sigma_2 dW_2) + \frac{1}{2\varepsilon} \int_0^t \|\sigma_2\|^2_{Q_2} ds,
\]  
(3.10)
by taking mathematical expectation from both sides of above equation, we have
\[
\frac{1}{2} \mathbb{E}\|B^\varepsilon_x\|^2 = \frac{1}{2} \mathbb{E}\|B_{0x}\|^2 - \frac{1}{\varepsilon} \int_0^t \mathbb{E}\|B^\varepsilon_{xx}\|^2 ds + \frac{1}{\varepsilon} \int_0^t \mathbb{E}(-B^\varepsilon_{xx}, \mathcal{G}(B^\varepsilon) + g(A^\varepsilon, B^\varepsilon)) ds \\
+ \frac{1}{2\varepsilon} \int_0^t \mathbb{E}\|\sigma_2\|^2_{Q_2} ds,
\]  
(3.11)
this implies that
\[
\frac{d}{dt} \mathbb{E}\|B^\varepsilon_x\|^2 = -\frac{2}{\varepsilon} \mathbb{E}\|B^\varepsilon_x\|^2 + \frac{2}{\varepsilon} \mathbb{E}(-B^\varepsilon_{xx}, \mathcal{G}(B^\varepsilon) + g(A^\varepsilon, B^\varepsilon)) + \frac{1}{\varepsilon} \mathbb{E}\|\sigma_2\|^2_{Q_2},
\]
according to Lemma 2.6, we have
\[
(-B^\varepsilon_{xx}, \mathcal{G}(B^\varepsilon)) \leq 0,
\]
thus, it holds that
\[
\frac{d}{dt} \mathbb{E}\|B^\varepsilon_x\|^2 \\
= -\frac{2}{\varepsilon} \mathbb{E}\|B^\varepsilon_x\|^2 + \frac{2}{\varepsilon} \mathbb{E}(-B^\varepsilon_{xx}, \mathcal{G}(B^\varepsilon) + g(A^\varepsilon, B^\varepsilon)) + \frac{1}{\varepsilon} \mathbb{E}\|\sigma_2\|^2_{Q_2} \\
\leq -\frac{2}{\varepsilon} \mathbb{E}\|B^\varepsilon_{xx}\|^2 + \frac{2}{\varepsilon} \mathbb{E}(-B^\varepsilon_{xx}, g(A^\varepsilon, B^\varepsilon)) + \frac{1}{\varepsilon} \mathbb{E}\|\sigma_2\|^2_{Q_2} \\
\leq -\frac{2}{\varepsilon} \mathbb{E}\|B^\varepsilon_{xx}\|^2 + \frac{1}{\varepsilon} \mathbb{E}\|B^\varepsilon_{xx}\|^2 + \frac{1}{\varepsilon} \mathbb{E}\|g(A^\varepsilon, B^\varepsilon)\|^2 + \frac{1}{\varepsilon} \mathbb{E}\|\sigma_2\|^2_{Q_2} \\
\leq -\frac{1}{\varepsilon} \mathbb{E}\|B^\varepsilon_{xx}\|^2 + \frac{1}{\varepsilon} \mathbb{E}\|A^\varepsilon\|^2 + \frac{1}{\varepsilon} \mathbb{E}\|B^\varepsilon\|^2 + \frac{1}{\varepsilon} \mathbb{E}\|\sigma_2\|^2_{Q_2} \\
= -\frac{1}{\varepsilon} \mathbb{E}\|B^\varepsilon_{xx}\|^2 + \frac{1}{\varepsilon} \mathbb{E}\|B^\varepsilon\|^2 + \frac{1}{\varepsilon} \mathbb{E}\|A^\varepsilon\|^2 + \frac{C}{\varepsilon} \\
\leq -\frac{1}{\varepsilon} \mathbb{E}\|B^\varepsilon_{xx}\|^2 + \frac{1}{\varepsilon} \mathbb{E}\|B^\varepsilon\|^2 + \frac{1}{\varepsilon} \mathbb{E}\|A^\varepsilon\|^2 + \frac{C}{\varepsilon} \\
= -\frac{1}{\varepsilon} \mathbb{E}\|B^\varepsilon_{xx}\|^2 + \frac{1}{\varepsilon} \mathbb{E}\|A^\varepsilon\|^2 + \frac{C}{\varepsilon} \\
= -\frac{1}{\varepsilon} \mathbb{E}\|B^\varepsilon_{xx}\|^2 + \frac{1}{\varepsilon} \mathbb{E}\|A^\varepsilon\|^2 + \frac{C}{\varepsilon} \\
= -\frac{1}{\varepsilon} \mathbb{E}\|B^\varepsilon_{xx}\|^2 + \frac{1}{\varepsilon} \mathbb{E}\|A^\varepsilon\|^2 + \frac{C}{\varepsilon} \\
= -\frac{1}{\varepsilon} \mathbb{E}\|B^\varepsilon_{xx}\|^2 + \frac{1}{\varepsilon} \mathbb{E}\|A^\varepsilon\|^2 + \frac{C}{\varepsilon}
\]
where \( \beta = \frac{\alpha(\lambda + L_0)}{\lambda} > 0 \).

Hence, by applying Lemma 2.2 with \( \mathbb{E}\|B^\varepsilon_x\|^2 \), we have
\[
\mathbb{E}\|B^\varepsilon_x\|^2 \\
\leq e^{-\frac{\beta}{\varepsilon} t} \mathbb{E}\|B_{0x}\|^2 + \frac{C}{\varepsilon} \int_0^t e^{-\frac{\beta}{\varepsilon} (t-s)} (1 + \mathbb{E}\|A^\varepsilon\|^2) ds \\
\leq C \|B_{0x}\|^2 + \frac{C}{\varepsilon} \int_0^t e^{-\frac{\beta}{\varepsilon} (t-s)} ds + \frac{C}{\varepsilon} \int_0^t e^{-\frac{\beta}{\varepsilon} (t-s)} \mathbb{E}\|A^\varepsilon\|^2 ds \\
\leq C (\|B_{0x}\|^2 + 1) + \frac{C}{\varepsilon} \int_0^t e^{-\frac{\beta}{\varepsilon} (t-s)} \mathbb{E}\|A^\varepsilon\|^2 ds.
\]
Combining this and (3.9), we have

\[
\mathbb{E}\|B^\varepsilon_x(t)\|^2 \\
\leq C(\|B_0x\|^2 + 1) + \frac{C}{\varepsilon} \int_0^t e^{\frac{\varepsilon}{2} (t-s)} \mathbb{E}\|A^\varepsilon\|^2 ds
\]

\[
\leq C(\|B_0x\|^2 + 1) + \frac{C}{\varepsilon} \int_0^t e^{\frac{\varepsilon}{2} (t-s)} (1 + \|A_0\|^2_{H^1} + \|A_1\|^2 + \|A_0\|^4_{L^4(I)} + \|A_0\|^6_{L^6(I)} + \|B_0\|^2) ds
\]

\[
\leq C(1 + \|A_0\|^2_{H^1} + \|A_1\|^2 + \|A_0\|^4_{L^4(I)} + \|A_0\|^6_{L^6(I)} + \|B_0\|^2_{H^1}).
\]

- The estimate of \( \mathbb{E} \sup_{0 \leq t \leq \xi} \|B^\varepsilon_x(t)\|^2 \).

Indeed, it follows from (3.10) that

\[
\frac{1}{2} \|B^\varepsilon_x\|^2 + \frac{1}{\varepsilon} \int_0^t \|B^\varepsilon_{xx}\|^2 ds = \frac{1}{2} \|B_0x\|^2 + \frac{1}{\varepsilon} \int_0^t (-B^\varepsilon_{xx}, G(B^\varepsilon) + g(A^\varepsilon, B^\varepsilon)) ds
\]

\[+ \frac{1}{\sqrt{\varepsilon}} \int_0^t (-B^\varepsilon_{xx}, \sigma_2 dW_2) + \frac{1}{2\varepsilon} \int_0^t \|\sigma_2\|_{Q_2}^2 ds,
\]

according to Lemma 2.6, we have

\((-B^\varepsilon_{xx}, G(B^\varepsilon)) \leq 0,\)

thus, we have

\[\frac{1}{2} \|B^\varepsilon_x\|^2 + \frac{1}{\varepsilon} \int_0^t \|B^\varepsilon_{xx}\|^2 ds \leq \frac{1}{2} \|B_0x\|^2 + \frac{1}{\varepsilon} \int_0^t (-B^\varepsilon_{xx}, g(A^\varepsilon, B^\varepsilon)) ds + \frac{1}{\sqrt{\varepsilon}} \int_0^t (-B^\varepsilon_{xx}, \sigma_2 dW_2) + \frac{1}{2\varepsilon} \int_0^t \|\sigma_2\|_{Q_2}^2 ds.
\]

In view of the Burkholder-Davis-Gundy inequality and the Young inequality, it holds that

\[\mathbb{E} \sup_{0 \leq t \leq \xi} \left| \frac{1}{\sqrt{\varepsilon}} \int_0^t (-B^\varepsilon_{xx}, \sigma_2 dW_2) \right| \leq \frac{1}{\sqrt{\varepsilon}} \mathbb{E} \int_0^\xi \|B^\varepsilon_{xx}\|^2 ds + \frac{C(T, \sigma_2)}{\varepsilon},\]

by the Cauchy inequality, we have

\[\mathbb{E} \sup_{0 \leq t \leq \xi} \left| \frac{1}{\varepsilon} \int_0^t (-B^\varepsilon_{xx}, g(A^\varepsilon, B^\varepsilon)) ds \right| \leq \frac{1}{\varepsilon} \mathbb{E} \int_0^\xi \|B^\varepsilon_{xx}\|^2 ds + \frac{C}{\varepsilon} \mathbb{E} \int_0^\xi \|g(A^\varepsilon, B^\varepsilon)\|^2 ds.
\]

Thus, we have

\[\frac{1}{2} \mathbb{E} \sup_{0 \leq t \leq \xi} \|B^\varepsilon_x\|^2 \leq \frac{1}{2} \|B_0x\|^2 + \frac{C}{\varepsilon} \mathbb{E} \int_0^\xi \|g(A^\varepsilon, B^\varepsilon)\|^2 dt + \frac{C}{\varepsilon},\]

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moreover, we have
\[
\mathbb{E} \sup_{0 \leq t \leq \xi} \|B_x^\varepsilon\|^2 \leq \frac{C}{\varepsilon}(1 + \|A_0\|_{H^1}^2 + \|A_1\|^2 + \|A_0\|_{L^4(I)}^4 + \|A_0\|_{L^6(I)}^6 + \|B_0\|_{H^1}^2).
\]

- The estimate of \(\mathbb{E} \int_0^\xi \|B_{xx}^\varepsilon(t)\|^2 dt\). Indeed, it follows from (3.11) that
\[
\frac{1}{\varepsilon} \mathbb{E} \int_0^t \|B_{xx}^\varepsilon\|^2 ds \leq \frac{1}{2}\|B_0\|^2 + \frac{1}{\varepsilon} \int_0^t \mathbb{E}(\nabla(B^\varepsilon), \mathcal{G}(B^\varepsilon) + g(A^\varepsilon, B^\varepsilon)) ds + \frac{1}{2\varepsilon} \mathbb{E} \int_0^t \|\sigma_2\|^2_{Q_2} ds.
\]

According to Lemma 2.6, we have
\[
(\nabla(B_x^\varepsilon, \mathcal{G}(B^\varepsilon)) \leq 0,
\]

it holds that
\[
\frac{1}{\varepsilon} \mathbb{E} \int_0^t \|B_{xx}^\varepsilon\|^2 ds \leq \frac{1}{2}\|B_0\|^2 + \frac{1}{\varepsilon} \int_0^t \mathbb{E}(\nabla(B_x^\varepsilon, \mathcal{G}(B^\varepsilon) + g(A^\varepsilon, B^\varepsilon)) ds + \frac{1}{2\varepsilon} \mathbb{E} \int_0^t \|\sigma_2\|^2_{Q_2} ds
\]

\[
\leq \frac{C}{\varepsilon}(1 + \|A_0\|_{H^1(I)}^2 + \|B_0\|_{H^1(I)}^2 + \|A_1\|^2 + \|A_0\|_{L^4(I)}^4 + \|A_0\|_{L^6(I)}^6 + \|B_0\|^2) + \frac{1}{\varepsilon} \int_0^t \mathbb{E}\|B_{xx}^\varepsilon\|^2 ds,
\]

it is easy to see that
\[
\frac{1}{\varepsilon} \mathbb{E} \int_0^t \|B_{xx}^\varepsilon\|^2 ds \leq \frac{C}{\varepsilon}(1 + \|A_0\|_{H^1(I)}^2 + \|B_0\|_{H^1(I)}^2 + \|A_1\|^2 + \|A_0\|_{L^4(I)}^4 + \|A_0\|_{L^6(I)}^6),
\]

thus, we have
\[
\mathbb{E} \int_0^\xi \|B_{xx}^\varepsilon\|^2 dt \leq C(1 + \|A_0\|_{H^1(I)}^2 + \|B_0\|_{H^1(I)}^2 + \|A_1\|^2 + \|A_0\|_{L^4(I)}^4 + \|A_0\|_{L^6(I)}^6). \tag{3.12}
\]

### 3.1.3 Proof of Proposition 3.1

Now, we prove Proposition 3.1.

**Proof of Proposition 3.1**. By the Chebyshev inequality, Proposition 3.2 and the definition of
(A^ε, B^ε), we have
\[
\mathbb{P}\{\{\omega \in \Omega | \tau_\infty(\omega) < +\infty\}\}
= \lim_{T \to +\infty} \mathbb{P}\{\{\omega \in \Omega | \tau_\infty(\omega) \leq T\}\}
= \lim_{T \to +\infty} \mathbb{P}\{\{\omega \in \Omega | \tau(\omega) = \tau_\infty(\omega)\}\}
= \lim_{T \to +\infty} \lim_{R \to +\infty} \mathbb{P}\{\{\omega \in \Omega | ||(A^ε, B^ε)||_{Y^R} \geq ||(A^ε, B^ε)||_{Y^R} \}\}
= \lim_{T \to +\infty} \lim_{R \to +\infty} \mathbb{P}\{\{\omega \in \Omega | ||(A^ε, B^ε)||_{Y^R} \geq R\}\}
\leq \lim_{T \to +\infty} \lim_{R \to +\infty} \mathbb{P}\{\{\omega \in \Omega | ||(A^ε, B^ε)||_{Y^R} \geq R\}\}
= \lim_{T \to +\infty} \lim_{R \to +\infty} \frac{||((A^ε, B^ε)||_{Y^R})^p}{R^p}
= 0,
\]
this shows that
\[
\mathbb{P}\{\{\omega \in \Omega | \tau_\infty(\omega) = +\infty\}\} = 1,
\]
namely, \(\tau_\infty = +\infty\) P-a.s. \[\square\]

### 3.1.4 Some a priori estimates for the slow-fast system (1.1)

Next, we establish some a priori estimates for the slow-fast system (1.1).

**Proposition 3.3.** If \(A_0, B_0 \in H^1_0(I), A_1 \in L^2(I), \) for \(\varepsilon \in (0,1), (A^ε, B^ε)\) is the unique solution to (1.1), then for any \(p > 0,\) there exists a constant \(C\) such that the solutions \((A^ε, B^ε)\) satisfy
\[
\mathbb{E} \sup_{0 \leq t \leq T} (||A^ε(t)||^p + ||A^ε(t)||^p + ||A^ε(t)||^p_{L^2(I)} + ||A^ε(t)||^p_{L^6(I)}) \leq C,
\]
\[
\mathbb{E} \sup_{0 \leq t \leq T} ||B^ε(t)||^p \leq C,
\]
\[
\mathbb{E} \int_0^T ||B^ε(t)||^p dt \leq C,
\]
where \(C\) dependent of \(p, T, A_0, B_0\) but independent of \(\varepsilon \in (0,1).\)

**Proof.** The proof of Proposition 3.3 is divided into several steps. It is also suffice to prove Proposition 3.3 holds when \(p\) is large enough. Here, the method of the proof is inspired from [22, 24, 25, 26, 27].

- The estimates of \(\mathbb{E} \sup_{0 \leq t \leq T} (||A^ε(t)||^p + ||A^ε(t)||^p + ||A^ε(t)||^p_{L^2(I)} + ||A^ε(t)||^p_{L^6(I)})\) and \(\mathbb{E} \sup_{0 \leq t \leq T} ||B^ε(t)||^p.\)
- Indeed, it follows from [17] P137, Theorem 3.5 that
\[
||A^ε(t)||^2 + ||A^ε(t)||^2
= ||A^ε(0)||^2 + ||A^ε(0)||^2 + 2 \int_0^t (A^ε, \mathcal{F}(A^ε + f(A^ε, B^ε))) ds + 2 \int_0^t (A^ε, \sigma_1 dW_1) + \int_0^t ||\sigma_1||^2_{Q^1} ds,
\]
25
since
\[ \int_0^t (A_t^\varepsilon, \mathcal{F}(A^\varepsilon)) ds = -\frac{1}{4} \| A^\varepsilon(t) \|_{L^4(I)}^4 + \frac{1}{4} \| A^\varepsilon(0) \|_{L^4(I)}^4 - \frac{1}{4} \| A^\varepsilon(t) \|_{H^6(I)}^6 + \frac{1}{6} \| A^\varepsilon(0) \|_{H^6(I)}^6, \]
we have
\[
\| A_t^\varepsilon(t) \|^2 + \| A_t^\varepsilon(t) \|^2 + \frac{1}{4} \| A^\varepsilon(t) \|^4_{L^4(I)} + \frac{1}{6} \| A^\varepsilon(t) \|^6_{H^6(I)}
= \| A_t^\varepsilon(0) \|^2 + \| A_t^\varepsilon(0) \|^2 + \frac{1}{4} \| A^\varepsilon(0) \|^4_{L^4(I)} + \frac{1}{6} \| A^\varepsilon(0) \|^6_{H^6(I)}
+ 2 \int_0^t (A_t^\varepsilon, f(A^\varepsilon, B^\varepsilon)) ds + 2 \int_0^t (A_t^\varepsilon, \sigma_1 dW_1) + \int_0^t \| \sigma_1 \|_{Q_1}^2 ds,
\]
it is easy to see
\[
\sup_{0 \leq t \leq T} \| A_t^\varepsilon(t) \|^2 + \| A_t^\varepsilon(t) \|^2 + \frac{1}{4} \| A^\varepsilon(t) \|^4_{L^4(I)} + \frac{1}{6} \| A^\varepsilon(t) \|^6_{H^6(I)}
\leq C(p)(\| A_t^\varepsilon(0) \|^2 + \| A_t^\varepsilon(0) \|^2 + \frac{1}{4} \| A^\varepsilon(0) \|^4_{L^4(I)} + \frac{1}{6} \| A^\varepsilon(0) \|^6_{H^6(I)}
+ \sup_{0 \leq t \leq T} | \int_0^t (A_t^\varepsilon, f(A^\varepsilon, B^\varepsilon)) ds |^p + \sup_{0 \leq t \leq T} | \int_0^t (A_t^\varepsilon, \sigma_1 dW_1) |^p + \sup_{0 \leq t \leq T} | \int_0^t \| \sigma_1 \|_{Q_1}^2 ds |^p,
\]
this implies that
\[
\begin{align*}
\sup_{0 \leq t \leq T} (\| A_t^\varepsilon(t) \|^2 + \| A_t^\varepsilon(t) \|^2 + \| A^\varepsilon(t) \|^4_{L^4(I)} + \| A^\varepsilon(t) \|^6_{H^6(I)}) \\
\leq C(p)(\| A_t^\varepsilon(0) \|^2 + \| A_t^\varepsilon(0) \|^2 + \| A^\varepsilon(0) \|^4_{L^4(I)} + \| A^\varepsilon(0) \|^6_{H^6(I)}) \\
+ \sup_{0 \leq t \leq T} | \int_0^t (A_t^\varepsilon, f(A^\varepsilon, B^\varepsilon)) ds |^p + \sup_{0 \leq t \leq T} | \int_0^t (A_t^\varepsilon, \sigma_1 dW_1) |^p + \sup_{0 \leq t \leq T} | \int_0^t \| \sigma_1 \|_{Q_1}^2 ds |^p.
\end{align*}
\]
By taking mathematical expectation from both sides of above equation, we have
\[
\begin{align*}
\mathbb{E} \sup_{0 \leq t \leq T} (\| A_t^\varepsilon(t) \|^2 + \| A_t^\varepsilon(t) \|^2 + \| A^\varepsilon(t) \|^4_{L^4(I)} + \| A^\varepsilon(t) \|^6_{H^6(I)}) \\
\leq C(p)(\| A_t^\varepsilon(0) \|^2 + \| A_t^\varepsilon(0) \|^2 + \| A^\varepsilon(0) \|^4_{L^4(I)} + \| A^\varepsilon(0) \|^6_{H^6(I)}) \\
+ \mathbb{E} \sup_{0 \leq t \leq T} | \int_0^t (A_t^\varepsilon, f(A^\varepsilon, B^\varepsilon)) ds |^p + \mathbb{E} \sup_{0 \leq t \leq T} | \int_0^t (A_t^\varepsilon, \sigma_1 dW_1) |^p + \mathbb{E} \sup_{0 \leq t \leq T} | \int_0^t \| \sigma_1 \|_{Q_1}^2 ds |^p.
\end{align*}
\]
In view of the Burkholder-Davis-Gundy inequality, it holds that
\[
\begin{align*}
\mathbb{E} \sup_{0 \leq t \leq T} | \int_0^t (A_t^\varepsilon, \sigma_1 dW_1) |^p \\
\leq C(p)\mathbb{E} \left( \int_0^T \| A_t^\varepsilon \|^2 \| \sigma_1 \|_{Q_1}^2 ds \right)^{\frac{p}{2}} \\
\leq C(p)\mathbb{E} \left( \sup_{0 \leq t \leq T} \| A_t^\varepsilon \|^2 \int_0^T \| \sigma_1 \|_{Q_1}^2 ds \right)^{\frac{p}{2}} \\
= C(p)\mathbb{E} \left( \sup_{0 \leq t \leq T} \| A_t^\varepsilon \|^p \cdot \left( \int_0^T \| \sigma_1 \|_{Q_1}^2 ds \right)^{\frac{p}{2}} \right) \\
\leq \eta \mathbb{E} \sup_{0 \leq t \leq T} \| A_t^\varepsilon \|^2 + C(p, \eta)\mathbb{E} \left( \int_0^T \| \sigma_1 \|_{Q_1}^2 ds \right)^p \\
= \eta \mathbb{E} \sup_{0 \leq t \leq T} \| A_t^\varepsilon \|^2 + C(p, \eta, T, \sigma_1).
\end{align*}
\]
In view of the Hölder inequality, it holds that
\[
\mathbb{E} \sup_{0 \leq t \leq T} \left( \int_0^t (A_x^\varepsilon(t), f(A^\varepsilon, B^\varepsilon)) ds \right)^p \\
\leq C \mathbb{E} \left( \int_0^T |(A_x^\varepsilon(t), f(A^\varepsilon, B^\varepsilon))| ds \right)^p \\
\leq C \mathbb{E} \left( \sup_{0 \leq t \leq T} \|A_x^\varepsilon\| \int_0^T \|f(A^\varepsilon, B^\varepsilon)\| ds \right)^p \\
\leq C \mathbb{E} \left( \sup_{0 \leq t \leq T} \|A_x^\varepsilon\|^p \cdot \left( \int_0^T \|f(A^\varepsilon, B^\varepsilon)\|^p ds \right) \right)^{1/p} \\
\leq \eta \mathbb{E} \sup_{0 \leq t \leq T} \|A_x^\varepsilon\|^{2p} + C(p, \eta) \mathbb{E} \left( \int_0^T \|f(A^\varepsilon, B^\varepsilon)\|^2 ds \right)^p \\
\leq \eta \mathbb{E} \sup_{0 \leq t \leq T} \|A_x^\varepsilon\|^{2p} + C(p, \eta, T) \mathbb{E} \int_0^T \|f(A^\varepsilon, B^\varepsilon)\|^2 ds \\
\leq \eta \mathbb{E} \sup_{0 \leq t \leq T} \|A_x^\varepsilon\|^{2p} + C(p, \eta, T)(\mathbb{E} \int_0^T \|A^\varepsilon\|^{2p} ds + \mathbb{E} \int_0^T \|B^\varepsilon\|^{2p} ds) + C(p, \eta, T).
\]
According to the above estimates, we have
\[
\mathbb{E} \sup_{0 \leq t \leq T} (\|A_x^\varepsilon(t)\|^{2p} + \|A_x^\varepsilon(t)\|^{2p} + \|A^\varepsilon(t)\|^{4p}_{L^4(I)} + \|A^\varepsilon(t)\|^{6p}_{L^6(I)}) \\
\leq C(p)(\|A_x^\varepsilon(0)\|^{2p} + \|A_x^\varepsilon(0)\|^{2p} + \|A^\varepsilon(0)\|^{4p}_{L^4(I)} + \|A^\varepsilon(0)\|^{6p}_{L^6(I)}) \\
+ 2\eta \mathbb{E} \sup_{0 \leq t \leq T} \|A_x^\varepsilon\|^{2p} + C(p, \eta, T)(\mathbb{E} \int_0^T \|A^\varepsilon\|^{2p} ds + \mathbb{E} \int_0^T \|B^\varepsilon\|^{2p} ds) + C(p, \eta, T, \sigma_1),
\]
by taking $0 < \eta << 1$, it holds that
\[
\mathbb{E} \sup_{0 \leq t \leq T} (\|A_x^\varepsilon(t)\|^{2p} + \|A_x^\varepsilon(t)\|^{2p} + \|A^\varepsilon(t)\|^{4p}_{L^4(I)} + \|A^\varepsilon(t)\|^{6p}_{L^6(I)}) \\
\leq C(1 + \|A_x^\varepsilon(0)\|^{2p} + \|A_x^\varepsilon(0)\|^{2p} + \|A^\varepsilon(0)\|^{4p}_{L^4(I)} + \|A^\varepsilon(0)\|^{6p}_{L^6(I)}) \\
+ \mathbb{E} \int_0^T \|A^\varepsilon\|^{2p} ds + \mathbb{E} \int_0^T \|B^\varepsilon\|^{2p} ds,
\]
thus, it follows from Gronwall inequality that
\[
\mathbb{E} \sup_{0 \leq t \leq T} (\|A_x^\varepsilon(t)\|^{2p} + \|A_x^\varepsilon(t)\|^{2p} + \|A^\varepsilon(t)\|^{4p}_{L^4(I)} + \|A^\varepsilon(t)\|^{6p}_{L^6(I)}) \\
\leq C(1 + \|A_x^\varepsilon(0)\|^{2p} + \|A_x^\varepsilon(0)\|^{2p} + \|A^\varepsilon(0)\|^{4p}_{L^4(I)} + \|A^\varepsilon(0)\|^{6p}_{L^6(I)}) + \mathbb{E} \int_0^T \|B^\varepsilon\|^{2p} ds,
\]
moreover, we have
\[
\mathbb{E} \sup_{0 \leq t \leq T} (\|A_x^\varepsilon(t)\|^{2p} + \|A_x^\varepsilon(t)\|^{2p} + \|A^\varepsilon(t)\|^{4p}_{L^4(I)} + \|A^\varepsilon(t)\|^{6p}_{L^6(I)}) \\
\leq C(1 + \|A_0\|_{H^1}^{2p} + \|A_1\|^{2p} + \|A_0\|^{4p}_{L^4(I)} + \|A_0\|^{6p}_{L^6(I)}) + \mathbb{E} \int_0^T \|B^\varepsilon\|^{2p} ds. \tag{3.13}
\]
\[ d\|B^\varepsilon\|^2_{2p} = \frac{2p}{\varepsilon} \|B^\varepsilon\|^{2p-2} (\mathcal{L}(B^\varepsilon) + \mathcal{G}(B^\varepsilon) + g(A^\varepsilon, B^\varepsilon, B^\varepsilon)) dt + \frac{2p}{\varepsilon} \|B^\varepsilon\|^{2p-2} \|\sigma_2\|^2 dt + \frac{2p(p-1)}{\varepsilon} \|B^\varepsilon\|^{2p-4} (B^\varepsilon, \sigma_2 dW_2)^2 + \frac{2p}{\sqrt{\varepsilon}} \|B^\varepsilon\|^{2p-2} (B^\varepsilon, \sigma_2 dW_2), \]

this implies that

\[ \|B^\varepsilon(t)\|_{2p}^{2p} = \|B^\varepsilon(0)\|_{2p}^{2p} + \frac{2p}{\varepsilon} \mathbb{E} \int_0^t \|B^\varepsilon\|^{2p-2} (\mathcal{L}(B^\varepsilon) + \mathcal{G}(B^\varepsilon) + g(A^\varepsilon, B^\varepsilon, B^\varepsilon)) ds \]

by taking mathematical expectation from both sides of above equation, we have

\[ \mathbb{E}\|B^\varepsilon(t)\|_{2p}^{2p} = \mathbb{E}\|B^\varepsilon(0)\|_{2p}^{2p} + \frac{2p}{\varepsilon} \mathbb{E} \int_0^t \|B^\varepsilon\|^{2p-2} (\mathcal{L}(B^\varepsilon) + \mathcal{G}(B^\varepsilon) + g(A^\varepsilon, B^\varepsilon, B^\varepsilon)) ds \]

this implies that

\[ \frac{d}{dt} \mathbb{E}\|B^\varepsilon(t)\|_{2p}^{2p} = \frac{2p}{\varepsilon} \mathbb{E}\|B^\varepsilon\|^{2p-2} (\mathcal{L}(B^\varepsilon) + \mathcal{G}(B^\varepsilon) + g(A^\varepsilon, B^\varepsilon, B^\varepsilon)) \]

We consider this term

\[ \frac{2p}{\varepsilon} \mathbb{E}\|B^\varepsilon\|^{2p-2} (-\|B^\varepsilon\|^2 - \|B^\varepsilon\|_{L^4}^4 - (g(A^\varepsilon, B^\varepsilon), B^\varepsilon)) \]

\[ \leq \frac{2p}{\varepsilon} \mathbb{E}\|B^\varepsilon\|^{2p-2} (-\|B^\varepsilon\|^2 - (A^\varepsilon, B^\varepsilon), B^\varepsilon) \]

\[ = \frac{2p}{\varepsilon} \mathbb{E}\|B^\varepsilon\|^{2p-2} (-\|B^\varepsilon\|^2) + \frac{2p}{\varepsilon} \mathbb{E}\|B^\varepsilon\|^{2p-2} (-g(A^\varepsilon, B^\varepsilon) + g(A^\varepsilon, 0, B^\varepsilon)) + \frac{2p}{\varepsilon} \mathbb{E}\|B^\varepsilon\|^{2p-2} (g(A^\varepsilon, 0, B^\varepsilon)) \]

\[ \leq \frac{2p}{\varepsilon} \mathbb{E}\|B^\varepsilon\|^{2p-2} (-\|B^\varepsilon\|^2) \]

by using the Young inequality, we have

\[ \frac{2p}{\varepsilon} \mathbb{E}\|B^\varepsilon\|^{2p-2} (g(A^\varepsilon, 0, B^\varepsilon)) \]

\[ \leq \frac{2p}{\varepsilon} \mathbb{E}\|B^\varepsilon\|^{2p-2} (g(A^\varepsilon, 0, B^\varepsilon)) \]

\[ \leq \frac{2p}{\varepsilon} \mathbb{E}\|B^\varepsilon\|^{2p-2} C(\|A^\varepsilon\| + 1) \|B^\varepsilon\| \]

\[ = \frac{2p}{\varepsilon} \mathbb{E}\|B^\varepsilon\|^{2p-2} C(\|A^\varepsilon\| + 1) \]

\[ = \frac{2p}{\varepsilon} \mathbb{E}\|B^\varepsilon\|^{2p-2} C(\|A^\varepsilon\| + 1) \]

\[ \leq \frac{2p}{\varepsilon} \mathbb{E}\|B^\varepsilon\|^{2p} + \frac{C(p)}{\varepsilon} \mathbb{E}\|A^\varepsilon\|^{2p} + \frac{C(p)}{\varepsilon}, \]

it holds that
\[
\frac{2p}{\varepsilon} \mathbb{E}\|B^\varepsilon\|_{2p-2}^2 (-\|B^\varepsilon\|_2^2 - \|B^\varepsilon\|_{2p}^4 - (g(A^\varepsilon, B^\varepsilon), B^\varepsilon)) \\
\leq \frac{-2p(\lambda - L_d)}{\varepsilon} \mathbb{E}\|B^\varepsilon\|_{2p}^2 + \frac{2p}{\varepsilon} \mathbb{E}\|B^\varepsilon\|_{2p}^2 + C(p) \mathbb{E}\|A^\varepsilon\|_{2p}^2 + C(p) \varepsilon,
\]
\[
\mathbb{E}\|B^\varepsilon\|_{2p-2}^2\|\sigma_2\|^2 + \frac{2p(p-1)}{\varepsilon} \mathbb{E}\|B^\varepsilon\|_{2p-4}\|\sigma_2\|^{2} \sqrt{\mathbb{E}\|B^\varepsilon\|_{2p}^2} \\
\leq \frac{2p(\lambda - L_d)}{\varepsilon} \mathbb{E}\|B^\varepsilon\|_{2p}^2 + C(p) \varepsilon,
\]

thus, we have
\[
\frac{d}{dt}\mathbb{E}\|B^\varepsilon(t)\|_{2p}^2 \leq \frac{-p(\lambda - L_d)}{\varepsilon} \mathbb{E}\|B^\varepsilon\|_{2p}^2 + C(p) \mathbb{E}\|A^\varepsilon\|_{2p}^2 + \frac{C(p)}{\varepsilon} \mathbb{E}\|A^\varepsilon\|_{2p}^2 + 1).
\]

Hence, by applying Lemma 2.2 with \(\mathbb{E}\|B^\varepsilon(t)\|_{2p}^2\), we have
\[
\mathbb{E}\|B^\varepsilon(t)\|_{2p}^2 \\
\leq \mathbb{E}\|B^\varepsilon(0)\|_{2p}^2 e^{-\frac{p}{\varepsilon}(t-s)} \mathbb{E}\|A^\varepsilon(s)\|_{2p}^2 + 1)ds \\
\leq \|B_0\|_{2p}^2 e^{-\frac{p}{\varepsilon}(t-s)} \mathbb{E}\|A^\varepsilon(s)\|_{2p}^2 + 1)ds \\
\leq \|B_0\|_{2p}^2 e^{-\frac{p}{\varepsilon}(t-s)} + \frac{C(p)}{\varepsilon} \mathbb{E}\|A^\varepsilon(s)\|_{2p}^2 + 1)ds \\
\leq C(\|B_0\|_{2p}^2 + 1) + \frac{C(p)}{\varepsilon} \mathbb{E}\|A^\varepsilon(s)\|_{2p}^2 ds.
\]

Thus, plug (3.13) in the above inequality, we have
\[
\mathbb{E}\|B^\varepsilon(t)\|_{2p}^2 \\
\leq C(\|B_0\|_{2p}^2 + 1) + \frac{C}{\varepsilon} \mathbb{E}\|A^\varepsilon(t)\|_{2p}^2 + \|A_1\|_{2p}^2 + || A_0 ||_{L^p(I)}^2 + || A_0 ||_{L^p(I)}^2 + \mathbb{E}\int_0^t \|B^\varepsilon(\tau)\|_{2p}^2 d\tau)ds \\
= C(1 + || A_0 ||_{H^1}^2 + || A_1 ||_{2p}^2 + || A_0 ||_{L^p(I)}^2 + \mathbb{E}\int_0^t \|B^\varepsilon(\tau)\|_{2p}^2 d\tau)ds \\
\leq C(1 + || A_0 ||_{H^1}^2 + || A_1 ||_{2p}^2 + || A_0 ||_{L^p(I)}^2 + \mathbb{E}\int_0^t \|B^\varepsilon(\tau)\|_{2p}^2 d\tau)ds \\
= C(1 + || A_0 ||_{H^1}^2 + || A_1 ||_{2p}^2 + || A_0 ||_{L^p(I)}^2 + \mathbb{E}\int_0^t \|B^\varepsilon(\tau)\|_{2p}^2 d\tau)ds \\
= C(1 + || A_0 ||_{H^1}^2 + || A_1 ||_{2p}^2 + || A_0 ||_{L^p(I)}^2 + \mathbb{E}\int_0^t \|B^\varepsilon(\tau)\|_{2p}^2 d\tau)ds \\
= C(1 + || A_0 ||_{H^1}^2 + || A_1 ||_{2p}^2 + || A_0 ||_{L^p(I)}^2 + \mathbb{E}\int_0^t \|B^\varepsilon(\tau)\|_{2p}^2 d\tau)ds \\
\leq C(1 + || A_0 ||_{H^1}^2 + || A_1 ||_{2p}^2 + || A_0 ||_{L^p(I)}^2 + \mathbb{E}\int_0^t \|B^\varepsilon(\tau)\|_{2p}^2 d\tau)ds \\
\leq C(1 + || A_0 ||_{H^1}^2 + || A_1 ||_{2p}^2 + || A_0 ||_{L^p(I)}^2 + \mathbb{E}\int_0^t \|B^\varepsilon(\tau)\|_{2p}^2 d\tau)ds \\
\leq C(1 + || A_0 ||_{H^1}^2 + || A_1 ||_{2p}^2 + || A_0 ||_{L^p(I)}^2 + \mathbb{E}\int_0^t \|B^\varepsilon(\tau)\|_{2p}^2 d\tau)ds \\
\leq C(1 + || A_0 ||_{H^1}^2 + || A_1 ||_{2p}^2 + || A_0 ||_{L^p(I)}^2 + \mathbb{E}\int_0^t \|B^\varepsilon(\tau)\|_{2p}^2 d\tau)ds \\
\leq C(1 + || A_0 ||_{H^1}^2 + || A_1 ||_{2p}^2 + || A_0 ||_{L^p(I)}^2 + \mathbb{E}\int_0^t \|B^\varepsilon(\tau)\|_{2p}^2 d\tau)ds.
\]

thus, it follows from Gronwall inequality that
\[
\sup_{0 \leq t \leq T} \mathbb{E}\|B^\varepsilon(t)\|_{2p}^2 \leq C(1 + || A_0 ||_{H^1}^2 + || A_1 ||_{2p}^2 + || A_0 ||_{L^p(I)}^2 + \mathbb{E}\int_0^t \|B^\varepsilon(\tau)\|_{2p}^2 d\tau).
\]
Moreover, due to (3.13) and (3.14), it holds that
\[
\mathbb{E} \sup_{0 \leq t \leq T} \left( \|A^\varepsilon(t)\|^{2p} + \|A^\varepsilon(t)\|^{2p} + \|A^\varepsilon(t)\|^{2p}_{L^4(I)} + \|A^\varepsilon(t)\|^{6p}_{L^6(I)} \right) \\
\leq C(1 + \|A_0\|^{2p}_{H^1} + \|A_1\|^{2p} + \|A_0\|^{4p}_{L^4(I)} + \|A_0\|^{6p}_{L^6(I)} + \|B_0\|^{2p}).
\]
(3.15)

- The estimate of \( \mathbb{E} \int_0^T \|B_x^\varepsilon(t)\|^{2p} dt \).

Indeed, it follows from (3.10) that
\[
d\|B_x^\varepsilon\|^2 = \left( -\frac{2}{\varepsilon}\|B_{xx}^\varepsilon\|^2 + \frac{2}{\varepsilon}(-B_{xx}^\varepsilon, \mathcal{G}(B^\varepsilon) + g(A^\varepsilon, B^\varepsilon)) + \frac{1}{\varepsilon}\|\sigma_2\|^2_{Q_2} \right) dt + \frac{2}{\sqrt{\varepsilon}}(-B_{xx}^\varepsilon, \sigma_2 dW_2),
\]
then,
\[
d\|B_x^\varepsilon\|^{2p} = p\|B_x^\varepsilon\|^{2p-2} \left( -\frac{2}{\varepsilon}\|B_{xx}^\varepsilon\|^2 + \frac{2}{\varepsilon}(-B_{xx}^\varepsilon, \mathcal{G}(B^\varepsilon) + g(A^\varepsilon, B^\varepsilon)) + \frac{1}{\varepsilon}\|\sigma_2\|^2_{Q_2} \right) dt \\
+ \frac{2p(p-1)}{\varepsilon}\|B_x^\varepsilon\|^{2p-4}(-B_{xx}^\varepsilon, \sigma_2 dW_2)^2 + \frac{2}{\varepsilon}\|B_x^\varepsilon\|^{2p-2}(-B_{xx}^\varepsilon, \sigma_2 dW_2),
\]
thus, we have
\[
\|B_x^\varepsilon(t)\|^{2p} = \|B_x^\varepsilon(0)\|^{2p} + p\int_0^t \|B_x^\varepsilon\|^{2p-2} \left( -\frac{2}{\varepsilon}\|B_{xx}^\varepsilon\|^2 + \frac{2}{\varepsilon}(-B_{xx}^\varepsilon, \mathcal{G}(B^\varepsilon) + g(A^\varepsilon, B^\varepsilon)) + \frac{1}{\varepsilon}\|\sigma_2\|^2_{Q_2} \right) ds \\
+ \frac{2p(p-1)}{\varepsilon}\int_0^t \|B_x^\varepsilon\|^{2p-4}(-B_{xx}^\varepsilon, \sigma_2 dW_2)^2 + \frac{2}{\varepsilon}\int_0^t \|B_x^\varepsilon\|^{2p-2}(-B_{xx}^\varepsilon, \sigma_2 dW_2).
\]

According to Lemma 2.9 we have
\[
(-B_{xx}^\varepsilon, \mathcal{G}(B^\varepsilon)) \leq 0,
\]
thus, it holds that
\[
\|B_x^\varepsilon(t)\|^{2p} \leq \|B_x^\varepsilon(0)\|^{2p} + p\int_0^t \|B_x^\varepsilon\|^{2p-2} \left( -\frac{2}{\varepsilon}\|B_{xx}^\varepsilon\|^2 + \frac{2}{\varepsilon}(-B_{xx}^\varepsilon, g(A^\varepsilon, B^\varepsilon)) + \frac{1}{\varepsilon}\|\sigma_2\|^2_{Q_2} \right) ds \\
+ \frac{2p(p-1)}{\varepsilon}\int_0^t \|B_x^\varepsilon\|^{2p-4}(-B_{xx}^\varepsilon, \sigma_2 dW_2)^2 + \frac{2}{\varepsilon}\int_0^t \|B_x^\varepsilon\|^{2p-2}(-B_{xx}^\varepsilon, \sigma_2 dW_2),
\]
Noting the fact that
\[
-\frac{2}{\varepsilon}\|B_{xx}^\varepsilon\|^2 + \frac{2}{\varepsilon}(-B_{xx}^\varepsilon, g(A^\varepsilon, B^\varepsilon)) \\
\leq -\frac{2}{\varepsilon}\|B_{xx}^\varepsilon\|^2 + \frac{1}{\varepsilon}\|B_{xx}^\varepsilon\|^2 + \frac{1}{\varepsilon}\|g(A^\varepsilon, B^\varepsilon)\|^2 \\
= -\frac{1}{\varepsilon}\|B_{xx}^\varepsilon\|^2 + \frac{1}{\varepsilon}\|g(A^\varepsilon, B^\varepsilon)\|^2,
\]
thus, it holds that
\[
\|B_x^\varepsilon(t)\|^{2p} \leq \|B_x^\varepsilon(0)\|^{2p} + p\int_0^t \|B_x^\varepsilon\|^{2p-2} \left( -\frac{1}{\varepsilon}\|B_{xx}^\varepsilon\|^2 \right) ds \\
+ \frac{2p(p-1)}{\varepsilon}\int_0^t \|B_x^\varepsilon\|^{2p-4}||\sigma_2\sqrt{Q_2}B_{xx}^\varepsilon||^2 ds \\
+ p\int_0^t \|B_x^\varepsilon\|^{2p-2} \left( \frac{1}{\varepsilon}\|g(A^\varepsilon, B^\varepsilon)\|^2 + \frac{1}{\varepsilon}\|\sigma_2\|^2_{Q_2} \right) ds \\
+ \frac{2p}{\varepsilon}\int_0^t \|B_x^\varepsilon\|^{2p-2}(-B_{xx}^\varepsilon, \sigma_2 dW_2),
\]
thus, by using the Young inequality, we have

\[
\|B_\varepsilon^x(t)\|^{2p}\n\leq \|B_\varepsilon^x(0)\|^{2p} + p \int_0^t \|B_\varepsilon^x\|^{2p-2} \left\{ -\frac{1}{\varepsilon} \|B_\varepsilon^{xx}\|^2 \right\} ds + \frac{pC(\sigma_2, Q_2)}{\varepsilon} \int_0^t (\rho \|B_\varepsilon^x\|^{2p-2} + C(p, \rho))\|B_\varepsilon^x\|^2 ds \\
+ \frac{p}{\sqrt{\varepsilon}} \|g(A_\varepsilon, B_\varepsilon)\| + \frac{1}{\varepsilon} \|\sigma_2\|^2_{Q_2} \|dx\| ds + \frac{2p}{\sqrt{\varepsilon}} \int_0^t \|B_\varepsilon^x\|^{2p-2}( -B_\varepsilon^{xx}, \sigma_2 dW_2 ) \\
+ \frac{p}{\sqrt{\varepsilon}} \|\sigma_2\|^2_{Q_2} \|dx\| ds + \frac{2p}{\sqrt{\varepsilon}} \int_0^t \|B_\varepsilon^x\|^{2p-2}( -B_\varepsilon^{xx}, \sigma_2 dW_2 ).
\]

By using the Young inequality again, we have

\[
\|B_\varepsilon^x\|^{2p-2} \left\{ -\frac{1}{\varepsilon} \|g(A_\varepsilon, B_\varepsilon)\| + \frac{1}{\varepsilon} \|\sigma_2\|^2_{Q_2} \right\} \\
\leq -\frac{p}{2} \|B_\varepsilon^x\|^{2p} + \frac{C}{\varepsilon}(\|A_\varepsilon\|^{2p} + \|B_\varepsilon\|^{2p} + 1),
\]

by taking mathematical expectation from both sides of above equation, we have

\[
\mathbb{E}\|B_\varepsilon^x(t)\|^{2p} \\
\leq \mathbb{E}\|B_\varepsilon^x(0)\|^{2p} + p\mathbb{E} \int_0^t \|B_\varepsilon^x\|^{2p-2} \left\{ -\frac{1}{\varepsilon} \|g(A_\varepsilon, B_\varepsilon)\| + \frac{1}{\varepsilon} \|\sigma_2\|^2_{Q_2} \right\} ds + \frac{C}{\varepsilon} \mathbb{E} \int_0^t \|B_\varepsilon^x\|^2 ds \\
+ \frac{p}{2} \mathbb{E} \int_0^t \|\sigma_2\|^2_{Q_2} \|dx\| ds + \frac{C}{\varepsilon} \mathbb{E} (\int_0^t \|A_\varepsilon\|^{2p} ds + \int_0^t \|B_\varepsilon\|^{2p} ds + 1).
\]

It follows from (3.12) that

\[
\mathbb{E}\|B_\varepsilon^x(t)\|^{2p} \\
\leq \|B_0\|_{L^1(\Omega)}^{2p} + p\mathbb{E} \int_0^t \|B_\varepsilon^x\|^{2p-2} \left\{ -\frac{1}{\varepsilon} \|g(A_\varepsilon, B_\varepsilon)\| + \frac{1}{\varepsilon} \|\sigma_2\|^2_{Q_2} \right\} ds \\
+ \frac{p}{2} \mathbb{E} \int_0^t \|\sigma_2\|^2_{Q_2} \|dx\| ds + \frac{C}{\varepsilon} \mathbb{E} (\int_0^t \|A_\varepsilon\|^{2p} ds + \int_0^t \|B_\varepsilon\|^{2p} ds + 1).
\]

If we take \(0 < \rho < 1\), we have

\[
-1 + \rho C(\sigma_2, Q_2) < -\frac{1}{2},
\]

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thus, it holds that
\[
\mathbb{E}\|B^\varepsilon_x(t)\|^{2p} \\
\leq \|B_0\|^{2p}_{H^1(I)} - \frac{p}{2\varepsilon}\mathbb{E}\int_0^t \|B^\varepsilon_x\|^{2p-2}\|B^\varepsilon_{xx}\|^2 ds \\
+ \frac{L}{2\varepsilon}\mathbb{E}\int_0^t \|B^\varepsilon_x\|^{2p} ds + \frac{C}{\varepsilon}\mathbb{E}\left(\int_0^t \|A^\varepsilon\|^{2p} ds + \int_0^t \|B^\varepsilon\|^{2p} ds + 1\right) \\
\leq \|B_0\|^{2p}_{H^1(I)} - \frac{p}{2\varepsilon}\mathbb{E}\int_0^t \|B^\varepsilon_x\|^{2p} ds \\
+ \frac{L}{2}\mathbb{E}\int_0^t \|B^\varepsilon_x\|^{2p} ds + \frac{C}{\varepsilon}\mathbb{E}\left(\int_0^t \|A^\varepsilon\|^{2p} ds + \int_0^t \|B^\varepsilon\|^{2p} ds + 1\right) \\
= \|B_0\|^{2p}_{H^1(I)} - \frac{p(\lambda - L\varepsilon)}{2\varepsilon}\mathbb{E}\int_0^t \|B^\varepsilon_x\|^{2p} ds + \frac{C}{\varepsilon}\mathbb{E}\left(\int_0^t \|A^\varepsilon\|^{2p} ds + \int_0^t \|B^\varepsilon\|^{2p} ds + 1\right) \\
\leq -\frac{p(\lambda - L\varepsilon)}{2\varepsilon}\mathbb{E}\int_0^t \|B^\varepsilon_x\|^{2p} ds + \frac{C}{\varepsilon}.
\]

due to (3.14) and (3.15), we have
\[
\mathbb{E}\|B^\varepsilon_x(t)\|^{2p} \leq -\frac{p(\lambda - L\varepsilon)}{2\varepsilon}\mathbb{E}\int_0^t \|B^\varepsilon_x\|^{2p} ds + \frac{C}{\varepsilon}.
\]

Hence, by applying Lemma 2.2 with \(\mathbb{E}\int_0^t \|B^\varepsilon_x\|^{2p} ds\), we have
\[
\mathbb{E}\int_0^t \|B^\varepsilon_x\|^{2p} ds \leq \int_0^t e^{-\frac{p\lambda}{2\varepsilon}(t-s)} C ds \leq C,
\]
thus, we have
\[
\mathbb{E}\int_0^T \|B^\varepsilon_x\|^{2p} dt \leq C.
\]

\[\square\]

3.2 Well-posedness for the averaged equation (1.2)

By the same method in Proposition 3.1 and Proposition 3.3, we can obtain the following proposition.

**Proposition 3.4.** If \(A_0 \in H^1_0(I), A_1 \in L^2_0(I)\), (1.2) has a unique solution \(\bar{A} \in L^2(\Omega, C([0, T]; H^1_0(I)))\).

Moreover, for any \(p > 0\), there exists a constant \(C\) such that the solution \(\bar{A}\) satisfies
\[
\mathbb{E}\sup_{t \in [0, T]} \|\bar{A}(t)\|^{2}_{H^1(I)} \leq C,
\]
where \(C_2\) dependent of \(p, T, A_0, B_0\) but independent of \(p > 0\).
4 Proof of Theorem 1.1

4.1 Hölder continuity of time variable for $A^\varepsilon$

The following proposition is a crucial step.

**Proposition 4.1.** There exists a constant $C(p, T)$ such that

$$
E\|A^\varepsilon(t + h) - A^\varepsilon(t)\|^2p \leq C(p, T)h^{2p}
$$

(4.1)

for any $t \in [0, T], h > 0$.

**Proof.** Since

$$
E\|A^\varepsilon(t + h) - A^\varepsilon(t)\|^2p = E\|\int_t^{t+h} A^\varepsilon(s)ds\|^2p
$$

$$
\leq E\left(\int_t^{t+h} \|A^\varepsilon(s)\|^2ds\right)^{2p}
$$

$$
\leq Eh^{2p-1}\left(\int_t^{t+h} \|A^\varepsilon(s)\|^2ds\right)
$$

$$
\leq C(p, T)h^{2p},
$$

we arrive at (4.1). \qed

4.2 Auxiliary process $(\hat{A}^\varepsilon, \hat{B}^\varepsilon)$

Next, we introduce an auxiliary process $(\hat{A}^\varepsilon, \hat{B}^\varepsilon) \in L^2(I) \times L^2(I)$ by Khasminskii in [41].

Fix a positive number $\delta$ and do a partition of time interval $[0, T]$ of size $\delta$. We construct a process $\hat{B}^\varepsilon \in L^2(I)$ by means of the equations

$$
\hat{B}^\varepsilon(t) = B^\varepsilon(k\delta) + \frac{1}{\sqrt{\varepsilon}} \int_{k\delta}^t \mathcal{L}\hat{B}^\varepsilon(s) + \mathcal{G}(\hat{B}^\varepsilon(s)) + g(A^\varepsilon(k\delta), \hat{B}^\varepsilon(s))ds
$$

$$
+ \frac{1}{\sqrt{\varepsilon}} \int_{k\delta}^t \sigma_2dW_2(s)
$$

for $t \in [k\delta, \min\{(k + 1)\delta, T\}), k \geq 0$.

Also define the process $\hat{A}^\varepsilon \in L^2(I)$ by

$$
\hat{A}^\varepsilon(t) = G'(t)A_0 + G(t)A_1 + \int_0^t G(t - s)[\mathcal{F}(A^\varepsilon(s(\delta))) + f(A^\varepsilon(s(\delta)), \hat{B}^\varepsilon(s))]ds
$$

$$
+ \int_0^t G(t - s)\sigma_1dW_1(s)
$$

for $t \in [0, T]$, where $s(\delta) = \left[\frac{s}{\delta}\right]\delta$ is the nearest breakpoint preceding $s$ and $[\cdot]$ is the integer function.
Thus \((\hat{A}^\varepsilon, \hat{B}^\varepsilon)\) satisfies
\[
\begin{cases}
  d\hat{A}^\varepsilon_t = [\mathcal{L}(\hat{A}^\varepsilon) + \mathcal{F}(A^\varepsilon(t(\delta))) + f(A^\varepsilon(t(\delta)), \hat{B}^\varepsilon)] ds + \sigma_1 dW_1 \\
  d\hat{B}^\varepsilon_t = \frac{1}{\varepsilon}[\mathcal{L}(\hat{B}^\varepsilon) + \mathcal{G}(\hat{B}^\varepsilon) + g(A^\varepsilon(k\delta), \hat{B}^\varepsilon)] dt + \frac{1}{\sqrt{\varepsilon}} \sigma_2 dW_2 \\
  \hat{A}^\varepsilon(0, t) = 0 = \hat{A}^\varepsilon(1, t) \\
  \hat{B}^\varepsilon(0, t) = 0 = \hat{B}^\varepsilon(1, t) \\
  \hat{A}^\varepsilon(x, 0) = A_0(x), \\
  \hat{A}^\varepsilon_t(x, 0) = A_1(x) \\
  \hat{B}^\varepsilon(x, k\delta) = B^\varepsilon(x, k\delta)
\end{cases}
\]
in \(Q\) in \((0, T)\) in \((k\delta, \min\{(k + 1)\delta, T\})\)
in \(I\) in \(I\) in \(I\).
\[\text{(4.2)}\]

4.3 Some priori estimates of \((\hat{A}^\varepsilon, \hat{B}^\varepsilon)\)

Because the proof almost follows the steps in Proposition 3.3, we omit the proof here.

**Proposition 4.2.** If \(A_0, B_0 \in H_0^1(I), A_1 \in L^2(I),\) for \(\varepsilon \in (0, 1), (\hat{A}^\varepsilon, \hat{B}^\varepsilon)\) is the unique solution to \((4.2),\) then there exists a constant \(C\) such that the solutions \((\hat{A}^\varepsilon, \hat{B}^\varepsilon)\) satisfy

\[
\begin{align*}
  &\sup_{\varepsilon \in (0, 1)} \mathbb{E} \sup_{t \in [0, T]} \|\hat{A}^\varepsilon(t)\|_{H^1(I)}^2 \leq C_1, \\
  &\sup_{\varepsilon \in (0, 1)} \mathbb{E} \sup_{t \in [0, T]} \|\hat{B}^\varepsilon(t)\|_{H^1(I)}^2 \leq C_1, \\
  &\sup_{\varepsilon \in (0, 1)} \mathbb{E} \int_0^T \|\hat{A}^\varepsilon_{xx}\|^2 dt \leq C_1, \\
  &\sup_{\varepsilon \in (0, 1)} \mathbb{E} \int_0^T \|\hat{B}^\varepsilon_{xx}\|^2 dt \leq C_1.
\end{align*}
\]

where \(C_1\) dependent of \(T, A_0, B_0\) but independent of \(\varepsilon \in (0, 1)\).

Moreover, for any \(p > 0,\) there exists a constant \(C_2\) such that

\[
\begin{align*}
  &\sup_{\varepsilon \in (0, 1)} \sup_{t \in [0, T]} \mathbb{E}\|\hat{A}^\varepsilon(t)\|^{2p} \leq C_2, \\
  &\sup_{\varepsilon \in (0, 1)} \sup_{t \in [0, T]} \mathbb{E}\|\hat{B}^\varepsilon(t)\|^{2p} \leq C_2, \\
  &\sup_{\varepsilon \in (0, 1)} \sup_{t \in [0, T]} \mathbb{E}\|\hat{A}^\varepsilon(t)\|_{H^1(I)}^{2p} \leq C_2, \\
  &\sup_{\varepsilon \in (0, 1)} \mathbb{E} \int_0^T \|\hat{B}^\varepsilon(t)\|_{H^1(I)}^{2p} dt \leq C_2;
\end{align*}
\]

where \(C_2\) dependent of \(p, T, A_0, B_0\) but independent of \(\varepsilon \in (0, 1), p > 0.\)

4.4 The errors of \(A^\varepsilon - \hat{A}^\varepsilon\) and \(B^\varepsilon - \hat{B}^\varepsilon\)

We will establish convergence of the auxiliary process \(\hat{B}^\varepsilon\) to the fast solution process \(B^\varepsilon\) and \(\hat{A}^\varepsilon\) to the slow solution process \(A^\varepsilon\), respectively.
Lemma 4.1. There exists a constant $C$ such that
\[
\sup_{0 \leq t \leq T} \mathbb{E} \| B^\varepsilon(t) - \hat{B}^\varepsilon(t) \|^2 \leq C^2 \varepsilon^{2p+1} \frac{2p}{\varepsilon},
\]
\[
\mathbb{E} \sup_{0 \leq t \leq T} \| A^\varepsilon(t) - \hat{A}^\varepsilon(t) \|_{L^2(\Omega; H^1)}^2 + \mathbb{E} \sup_{0 \leq t \leq T} \| A^\varepsilon(t) - \hat{A}^\varepsilon(t) \|^2 \leq C \varepsilon^{2p} + C^2 \varepsilon^{2p+1} \frac{2p}{\varepsilon},
\]
where $C$ is only dependent of $p, T, A_0, B_0$.

Proof. \textbullet{} We prove the first inequality.

Indeed, it is easy to see that $B^\varepsilon(t) - \hat{B}^\varepsilon(t)$ satisfies the following SPDE
\[
\begin{aligned}
&d(B^\varepsilon - \hat{B}^\varepsilon) = \frac{1}{\varepsilon} [L(B^\varepsilon - \hat{B}^\varepsilon) + \mathcal{G}(B^\varepsilon) - \mathcal{G}(\hat{B}^\varepsilon) + g(A^\varepsilon, B^\varepsilon) - g(A^\varepsilon(k\delta), \hat{B}^\varepsilon)]dt \\
&(B^\varepsilon - \hat{B}^\varepsilon)(0, t) = 0 = (B^\varepsilon - \hat{B}^\varepsilon)(1, t) \\
&(B^\varepsilon - \hat{B}^\varepsilon)(x, 0) = 0
\end{aligned}
\]

For $t \in [0, T]$ with $t \in [k\delta, (k+1)\delta)$, applying Itô formula to (4.3)
\[
\begin{align*}
&\| (B^\varepsilon - \hat{B}^\varepsilon)(t) \|_{H^1}^2 \\
&= \frac{2}{\varepsilon} \int_{k\delta}^{t} \| (B^\varepsilon - \hat{B}^\varepsilon)(s) \|_{H^1}^{2p-2} (B^\varepsilon - \hat{B}^\varepsilon, \frac{1}{\varepsilon} [L(B^\varepsilon - \hat{B}^\varepsilon) + \mathcal{G}(B^\varepsilon) - \mathcal{G}(\hat{B}^\varepsilon) + g(A^\varepsilon, B^\varepsilon) - g(A^\varepsilon(k\delta), \hat{B}^\varepsilon)])ds \\
&= \frac{2}{\varepsilon} \int_{k\delta}^{t} \| (B^\varepsilon - \hat{B}^\varepsilon)(s) \|_{H^1}^{2p-2} (B^\varepsilon - \hat{B}^\varepsilon, L(B^\varepsilon - \hat{B}^\varepsilon) + \mathcal{G}(B^\varepsilon) - \mathcal{G}(\hat{B}^\varepsilon) + g(A^\varepsilon, B^\varepsilon) - g(A^\varepsilon(k\delta), \hat{B}^\varepsilon))ds \\
&= -\frac{2}{\varepsilon} \int_{k\delta}^{t} \| (B^\varepsilon - \hat{B}^\varepsilon)(s) \|_{H^1}^{2p-2} (B^\varepsilon - \hat{B}^\varepsilon)_{x}^2 ds \\
&+ \frac{2}{\varepsilon} \int_{k\delta}^{t} \| (B^\varepsilon - \hat{B}^\varepsilon)(s) \|_{H^1}^{2p-2} (B^\varepsilon - \hat{B}^\varepsilon, \mathcal{G}(B^\varepsilon) - \mathcal{G}(\hat{B}^\varepsilon) + g(A^\varepsilon, B^\varepsilon) - g(A^\varepsilon(k\delta), \hat{B}^\varepsilon))ds.
\end{align*}
\]

By taking mathematical expectation from both sides of above equation, we have
\[
\begin{align*}
&\mathbb{E}\| (B^\varepsilon - \hat{B}^\varepsilon)(t) \|_{H^1}^2 \\
&= -\frac{2}{\varepsilon} \mathbb{E} \int_{k\delta}^{t} \| (B^\varepsilon - \hat{B}^\varepsilon)(s) \|_{H^1}^{2p-2} (B^\varepsilon - \hat{B}^\varepsilon)_{x}^2 ds \\
&+ \frac{2}{\varepsilon} \mathbb{E} \int_{k\delta}^{t} \| (B^\varepsilon - \hat{B}^\varepsilon)(s) \|_{H^1}^{2p-2} (B^\varepsilon - \hat{B}^\varepsilon, \mathcal{G}(B^\varepsilon) - \mathcal{G}(\hat{B}^\varepsilon) + g(A^\varepsilon, B^\varepsilon) - g(A^\varepsilon(k\delta), \hat{B}^\varepsilon))ds,
\end{align*}
\]
this implies that
\[
\begin{align*}
&\frac{d}{dt} \mathbb{E}\| (B^\varepsilon - \hat{B}^\varepsilon)(t) \|_{H^1}^2 \\
&= -\frac{2}{\varepsilon} \mathbb{E}\| (B^\varepsilon - \hat{B}^\varepsilon)(t) \|_{H^1}^{2p-2} (B^\varepsilon - \hat{B}^\varepsilon)_{x}^2 \\
&+ \frac{2}{\varepsilon} \mathbb{E}\| (B^\varepsilon - \hat{B}^\varepsilon)(t) \|_{H^1}^{2p-2} (B^\varepsilon - \hat{B}^\varepsilon, \mathcal{G}(B^\varepsilon) - \mathcal{G}(\hat{B}^\varepsilon) + g(A^\varepsilon, B^\varepsilon) - g(A^\varepsilon(k\delta), \hat{B}^\varepsilon)).
\end{align*}
\]
It follows from Lemma 2.2, we have
\[
(B^\varepsilon - \hat{B}^\varepsilon, \mathcal{G}(B^\varepsilon) - \mathcal{G}(\hat{B}^\varepsilon) \leq 0,
\]
it holds that
\[
\frac{d}{dt} E \| (B^\varepsilon - \hat{B}^\varepsilon)(t) \|^{2p} \\
\leq -\frac{2}{\varepsilon} E \| (B^\varepsilon - \hat{B}^\varepsilon)(t) \|^{2p-2} (B^\varepsilon - \hat{B}^\varepsilon)_x \|^2 \\
+ \frac{2}{\varepsilon} E \| (B^\varepsilon - \hat{B}^\varepsilon)(t) \|^{2p-2} (B^\varepsilon - \hat{B}^\varepsilon), g(A^\varepsilon, B^\varepsilon) - g(A^\varepsilon(k\delta), \hat{B}^\varepsilon) \| \\
\leq -\frac{2}{\varepsilon} E \| B^\varepsilon - \hat{B}^\varepsilon \|^{2p-2} (B^\varepsilon - \hat{B}^\varepsilon)_x \|^2 \\
+ \frac{2L_2}{\varepsilon} E \| B^\varepsilon - \hat{B}^\varepsilon \|^{2p-2} \| B^\varepsilon - \hat{B}^\varepsilon \| \| A^\varepsilon - A^\varepsilon(k\delta) \| \\
\leq -\frac{2L_2}{\varepsilon} E \| B^\varepsilon - \hat{B}^\varepsilon \|^{2p} + \frac{2L_2}{\varepsilon} E \| B^\varepsilon - \hat{B}^\varepsilon \|^{2p} \| A^\varepsilon - A^\varepsilon(k\delta) \| \\
= -\frac{2L_2}{\varepsilon} E \| B^\varepsilon - \hat{B}^\varepsilon \|^{2p} + \frac{2L_2}{\varepsilon} E \| B^\varepsilon - \hat{B}^\varepsilon \|^{2p-1} \| A^\varepsilon - A^\varepsilon(k\delta) \|.
\]

It follows from the Young inequality that
\[
\frac{d}{dt} E \| (B^\varepsilon - \hat{B}^\varepsilon)(t) \|^{2p} \\
\leq -\frac{2L_2}{\varepsilon} E \| B^\varepsilon - \hat{B}^\varepsilon \|^{2p} + \frac{2L_2}{\varepsilon} E \| B^\varepsilon - \hat{B}^\varepsilon \|^{2p} \| A^\varepsilon - A^\varepsilon(k\delta) \|^{2p} \\
= -\frac{2L_2}{\varepsilon} E \| B^\varepsilon - \hat{B}^\varepsilon \|^{2p} + \frac{2L_2}{\varepsilon} E \| A^\varepsilon - A^\varepsilon(k\delta) \|^{2p},
\]
due to Proposition 4.1, it holds that
\[
\frac{d}{dt} E \| (B^\varepsilon - \hat{B}^\varepsilon)(t) \|^{2p} \leq -\frac{2L_2}{\varepsilon} E \| B^\varepsilon - \hat{B}^\varepsilon \|^{2p} + \frac{C}{\varepsilon} \delta^{2p},
\]
hence, by applying Lemma 2.2 with $E \| (B^\varepsilon - \hat{B}^\varepsilon)(t) \|^{2p}$, we have
\[
E \| (B^\varepsilon - \hat{B}^\varepsilon)(t) \|^{2p} \\
\leq \int_k^t e^{-\frac{2}{\varepsilon} (t-s)} \frac{C}{\varepsilon} \delta^{2p} ds \\
= \frac{C}{\varepsilon} \delta^{2p} \int_k^t e^{-\frac{2}{\varepsilon} (t-s)} ds \\
= \frac{C}{\varepsilon} \delta^{2p+1} \int_k^t ds.
\]

\begin{itemize}
  \item We prove the second inequality.
  
  Indeed, noting $\dot{A}^\varepsilon, \dot{A}$ satisfy
  \[
  \left\{ \begin{array}{l}
  \frac{dA^\varepsilon}{dt} = [\mathcal{L}(A^\varepsilon) + \mathcal{F}(A^\varepsilon) + f(A^\varepsilon, B^\varepsilon)]dt + \sigma_1 dW_1, \\
  \frac{d\dot{A}}{dt} = [\mathcal{L}(\dot{A}) + \mathcal{F}(A(t(\delta))) + f(A(t(\delta)), \hat{B}^\varepsilon)]dt + \sigma_1 dW_1.
  \end{array} \right.
  \]

  According to Lemma 2.3, we have
  \[
  E \sup_{0 \leq t \leq T} \| A^\varepsilon(t) - \dot{A}^\varepsilon(t) \|^{2p}_{H^1} + E \sup_{0 \leq t \leq T} \| A^\varepsilon(t) - \dot{A}_t^\varepsilon(t) \|^{2p} \\
  \leq CT^{2p-1} \int_0^T \| \mathcal{F}(A(s)) - \mathcal{F}(A^\varepsilon(s(\delta))) + f(A^\varepsilon(s), B^\varepsilon(s)) - f(A(s(\delta)), \hat{B}^\varepsilon(s)) \|^{2p} ds.
  \]
\end{itemize}
It follows from Lemma 2.4, Proposition 4.1 and Proposition 3.3 that
\[
\begin{align*}
\mathbb{E} \int_0^T \| \mathcal{F}(A^\varepsilon(s)) - \mathcal{F}(A^\varepsilon(s(\delta))) \|^{2p} ds \\
&\leq C\mathbb{E} \int_0^T \| A^\varepsilon(s) - A^\varepsilon(s(\delta)) \| \left( \| A^\varepsilon(s) \|_{H^1}^2 + \| A^\varepsilon(s(\delta)) \|_{H^1}^2 \right)^{2p} ds \\
&= C\mathbb{E} \int_0^T \| A^\varepsilon(s) - A^\varepsilon(s(\delta)) \|^{2p} \left( \| A^\varepsilon(s) \|_{H^1}^2 + \| A^\varepsilon(s(\delta)) \|_{H^1}^2 \right)^{2p} ds \\
&\leq C(p,T)(\mathbb{E} \int_0^T \| A^\varepsilon(s) - A^\varepsilon(s(\delta)) \|^{2p} ds)^{\frac{1}{2}} \cdot \left( \mathbb{E} \int_0^T \| A^\varepsilon(s) \|_{H^1}^2 + \| A^\varepsilon(s(\delta)) \|_{H^1}^2 \right)^{\frac{1}{2}}^{4p}\frac{1}{2} \\
&\leq C\delta^{2p},
\end{align*}
\]
by the same method, we have
\[
\begin{align*}
\mathbb{E} \int_0^T \| f(A^\varepsilon(s), B^\varepsilon(s)) - f(A(s(\delta)), B^\varepsilon(s)) \|^{2p} ds \\
&\leq C\mathbb{E} \int_0^T \| A^\varepsilon(s) - A^\varepsilon(s(\delta)) \| + \| B^\varepsilon(s) - B^\varepsilon(s) \|^{2p} ds \\
&\leq C\mathbb{E} \int_0^T \| A^\varepsilon(s) - A^\varepsilon(s(\delta)) \|^{2p} + \| B^\varepsilon(s) - B^\varepsilon(s) \|^{2p} ds \\
&\leq C\delta^{2p} + C\frac{\delta^{2p+1}}{\varepsilon},
\end{align*}
\]
thus, we have
\[
\mathbb{E} \sup_{0 \leq t \leq T} \| A^\varepsilon(t) - \hat{A}^\varepsilon(t) \|_{H^1}^{2p} + \mathbb{E} \sup_{0 \leq t \leq T} \| \hat{A}^\varepsilon(t) - \hat{A}^\varepsilon(t) \|_{H^1}^{2p} \leq C\delta^{2p} + C\frac{\delta^{2p+1}}{\varepsilon}.
\]
\[
\square
\]

### 4.5 The errors of \( \hat{A}^\varepsilon - \bar{A} \)

Next we prove strong convergence of the auxiliary process \( \hat{A}^\varepsilon \) to the averaging solution process \( \bar{A} \).

**Lemma 4.2.** There exists a constant \( C(T,p) \) such that
\[
\begin{align*}
\mathbb{E} \sup_{0 \leq t \leq T} \| \hat{A}^\varepsilon(t) - \bar{A}(t) \|_{H^1}^{2p} + \mathbb{E} \sup_{0 \leq t \leq T} \| \hat{A}_t^\varepsilon(t) - \bar{A}_t(t) \|^{2p} \\
&\leq C(\varepsilon^p + \varepsilon^{p-\frac{1}{4}} + \varepsilon^{\frac{q}{4}} + \varepsilon^{p-\frac{1}{2}}) + C(1 - \ln T)^{\frac{1}{2p}}.
\end{align*}
\]

**Proof.** Noting \( \hat{A}^\varepsilon, \bar{A} \) satisfy
\[
\left\{ \begin{array}{l}
\dfrac{d\hat{A}_t^\varepsilon}{dt} = [\mathcal{L}(\hat{A}^\varepsilon) + \mathcal{F}(A^\varepsilon(t(\delta))) + f(A^\varepsilon(t(\delta)), \bar{B}^\varepsilon)] dt + \sigma_1 dW_1 \\
\dfrac{d\bar{A}_t}{dt} = [\mathcal{L}(\bar{A}) + \mathcal{F}(\bar{A}) + f(\bar{A})] dt + \sigma_1 dW_1.
\end{array} \right.
\]

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In mild sense, we introduce the following decomposition
\[
\hat{A}^\epsilon(t) - \tilde{A}(t)
\]
\[
= \int_0^t G(t - s)[\mathcal{F}(A^\epsilon(s(\delta))) - \mathcal{F}(\tilde{A}(s))] + f(A^\epsilon(s(\delta)), \hat{B}^\epsilon(s)) - \hat{f}(\tilde{A}(s))ds
\]
\[
= \int_0^t G(t - s)[\mathcal{F}(A^\epsilon(s(\delta))) - \mathcal{F}(\tilde{A}(s))]ds + \int_0^t G(t - s)[f(A^\epsilon(s(\delta)), \hat{B}^\epsilon(s)) - \hat{f}(\tilde{A}(s))]ds
\]
\[
= J_1 + J_2.
\]

- For \( J_1 \), according to Corollary 2.1, we have

\[
\mathbb{E} \sup_{0 \leq t \leq T} \|J_1(t)\|^2_{H^1} + \mathbb{E} \sup_{0 \leq t \leq T} \|J_{1t}(t)\|^2_{H^1}
\]
\[
\leq C T^{2p-1} \mathbb{E} \int_0^T \|\mathcal{F}(A^\epsilon(s(\delta))) - \mathcal{F}(\tilde{A}(s))\|^2_{2p} ds
\]
\[
\leq C \mathbb{E} \int_0^T \|\mathcal{F}(A^\epsilon(s(\delta))) - \mathcal{F}(\tilde{A}(s))\|^2_{2p} ds.
\]

we can rewrite it as
\[
\mathbb{E} \sup_{0 \leq t \leq T} \|J_1(t)\|^2_{H^1} + \mathbb{E} \sup_{0 \leq t \leq T} \|J_{1t}(t)\|^2_{H^1}
\]
\[
\leq C (\mathbb{E} \int_0^T \|\mathcal{F}(A^\epsilon(s(\delta))) - \mathcal{F}(A^\epsilon(s))\|^2_{2p} ds
\]
\[
\quad + \mathbb{E} \int_0^T \|\mathcal{F}(A^\epsilon(s)) - \mathcal{F}(\tilde{A}(s))\|^2_{2p} ds
\]
\[
\quad + \mathbb{E} \int_0^T \|\mathcal{F}(\tilde{A}(s)) - \mathcal{F}(\tilde{A}(s))\|^2_{2p} ds).
\]

* By using the Hölder inequality, we have

\[
\mathbb{E} \int_0^T \|\mathcal{F}(A^\epsilon(s(\delta))) - \mathcal{F}(A^\epsilon(s))\|^2_{2p} ds
\]
\[
\leq C \mathbb{E} \int_0^T \left( \|A^\epsilon(s(\delta)) - A^\epsilon(s)\|^2 + \|A^\epsilon(s(\delta))\|^2_{H^1} + \|A^\epsilon(s)\|^2_{H^1} \right)^{2p} ds
\]
\[
= C \mathbb{E} \int_0^T \|A^\epsilon(s(\delta)) - A^\epsilon(s)\|^2_{2p} \left( \|A^\epsilon(s(\delta))\|^2_{H^1} + \|A^\epsilon(s)\|^2_{H^1} \right)^{2p} ds
\]
\[
\leq C \mathbb{E} \int_0^T \|A^\epsilon(s(\delta)) - A^\epsilon(s)\|^4_{2p} ds \left( \mathbb{E} \int_0^T (\|A^\epsilon(s(\delta))\|^2_{H^1} + \|A^\epsilon(s(\delta))\|^2_{H^1})^{4p} ds \right)^{\frac{1}{2}}.
\]

It follows from Proposition 3.3 and Proposition 4.1 that

\[
\mathbb{E} \int_0^T \|\mathcal{F}(A^\epsilon(s(\delta))) - \mathcal{F}(A^\epsilon(s))\|^2_{2p} ds \leq C \delta^{2p}.
\]

* By using the Hölder inequality and the same method in above, we have

\[
\mathbb{E} \int_0^T \|\mathcal{F}(A^\epsilon(s)) - \mathcal{F}(\hat{A}(s))\|^2_{2p} ds
\]
\[
\leq C \mathbb{E} \int_0^T \|A^\epsilon(s) - \hat{A}(s)\|^4_{2p} ds \left( \mathbb{E} \int_0^T (\|A^\epsilon(s)\|^2_{H^1} + \|\hat{A}(s)\|^2_{H^1})^{4p} ds \right)^{\frac{1}{2}}.
\]
It follows from Proposition 3.3 and Proposition 4.1 that
\[
\mathbb{E} \int_0^T \| F(A^\varepsilon(s)) - F(\bar{A}^\varepsilon(s)) \|^{2p} ds \leq (C \delta^{4p} + C \delta^{4p+1}) \frac{1}{\varepsilon} \leq C \delta^{2p} + \frac{C}{\sqrt{\varepsilon}} \delta^{2p+\frac{1}{2}}.
\]

* By using the Hölder inequality and the same method in above, we have
\[
\mathbb{E} \int_0^T \| F(\hat{A}^\varepsilon(s)) - F(\bar{A}(s)) \|^{2p} ds \leq C \mathbb{E} \int_0^T \| \hat{A}^\varepsilon(s) - \bar{A}(s) \|^{2p} \| \hat{A}^\varepsilon(s) \|^{2p} \| \bar{A}(s) \|^{2p} ds.
\]

In order to deal with the above estimate, we will use the skill of stopping times, this is inspired from [22].

We define the stopping time
\[
\tau_n^\varepsilon = \inf\{ t > 0 : \| \hat{A}^\varepsilon(t) \|_{H^1} + \| \bar{A}(t) \|_{H^1} > n \}
\]
for any \( n \geq 1 \), and \( \varepsilon > 0 \).

We have
\[
\mathbb{E} \int_0^{T \wedge \tau_n^\varepsilon} \| \hat{A}^\varepsilon(s) - \bar{A}(s) \|^{2p} \| \hat{A}^\varepsilon(s) \|_{H^1}^{2p} + \| \bar{A}(s) \|_{H^1}^{2p} ds
\]
\[
\leq C n^{2p} \mathbb{E} \int_0^{T \wedge \tau_n^\varepsilon} \| \hat{A}^\varepsilon(s) - \bar{A}(s) \|^{2p} ds
\]
\[
\leq C n^{2p} \mathbb{E} \int_0^T \| \hat{A}^\varepsilon(s) - \bar{A}(s) \|^{2p} ds
\]
\[
= C n^{2p} \int_0^T \mathbb{E} \| \hat{A}^\varepsilon(s) - \bar{A}(s) \|^{2p} ds
\]
\[
\leq C n^{2p} \int_0^T \mathbb{E} \sup_{0 \leq r \leq s \wedge \tau_n^\varepsilon} \| \hat{A}^\varepsilon(r) - \bar{A}(r) \|^{2p} ds.
\]

* For \( J_2 \), we can rewrite \( J_2 \) as
\[
J_2 = \int_0^t G(t-s)[f(A^\varepsilon(s), \hat{B}^\varepsilon) - \bar{f}(A(s))] ds
\]
\[
= \int_0^t G(t-s)[f(A^\varepsilon(s), \hat{B}^\varepsilon) - \bar{f}(A^\varepsilon(s))] ds
\]
\[
+ \int_0^t G(t-s)[\bar{f}(A^\varepsilon(s)) - \bar{f}(\hat{A}^\varepsilon(s))] ds
\]
\[
+ \int_0^t G(t-s)[\bar{f}(\hat{A}^\varepsilon(s)) - \bar{f}(\bar{A}(s))] ds
\]
\[
\Delta = J_{21} + J_{22} + J_{23},
\]
where \( m_t = [\frac{t}{\delta}] \).

* For \( J_{21} \), it follows from [24] P3270,Lemma 6.2] that
\[
\mathbb{E} \sup_{0 \leq t \leq T} \| J_{21} \|_{H^1}^2 + \mathbb{E} \sup_{0 \leq t \leq T} \| (J_{21})_t \|^2 \leq C(\varepsilon^2 + \delta).
\]

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On the other hand, it follows from

\[
\begin{align*}
&\mathbb{E} \sup_{0 \leq t \leq T} \|J_{21}\|_{H^1}^{2p} + \mathbb{E} \sup_{0 \leq t \leq T} \|(J_{21})_{t}\|^{2p} \\
&= \mathbb{E} \sup_{0 \leq t \leq T} \left\| \int_0^t G(t - s) [f(A^e(s(\delta)), \hat{B}^e) - \tilde{f}(A^e(s))] ds \right\|_{H^1}^{2p} \\
&\quad + \mathbb{E} \sup_{0 \leq t \leq T} \left\| \int_0^t G(t - s) [f(A^e(s(\delta)), \hat{B}^e) - \tilde{f}(A^e(s))] ds \right\|^{2p} \\
&\leq \mathbb{E} \sup_{0 \leq t \leq T} \left\| \int_0^t G(t - s) [f(A^e(s(\delta)), \hat{B}^e) - \tilde{f}(A^e(s))] ds \right\|_{H^1}^{2p} \\
&\quad + \mathbb{E} \sup_{0 \leq t \leq T} \left\| \int_0^t G(t - s) [f(A^e(s(\delta)), \hat{B}^e) - \tilde{f}(A^e(s))] ds \right\|^{2p} \\
&\leq CE\left( \int_0^T \|f(A^e(s(\delta)), \hat{B}^e) - \tilde{f}(A^e(s))\| ds \right)^{2p} \\
&\leq CE (\int_0^T 1 ds)^{2p-1} \int_0^T \|f(A^e(s(\delta)), \hat{B}^e) - \tilde{f}(A^e(s))\|^{2p} ds \\
&\leq C(p, T),
\end{align*}
\]

thus, it holds that

\[
\begin{align*}
&\mathbb{E} \sup_{0 \leq t \leq T} \|J_{21}\|_{H^1}^{2p} \\
&\leq (\mathbb{E} \sup_{0 \leq t \leq T} \|J_{21}\|_{H^1}^{2(2p-1)})^{1/2} (\mathbb{E} \sup_{0 \leq t \leq T} \|J_{21}\|_{H^1}^{2})^{1/2} \\
&\leq C(p, T)(\sqrt{\frac{e}{\delta}} + \sqrt{\delta})
\end{align*}
\]

and

\[
\begin{align*}
&\mathbb{E} \sup_{0 \leq t \leq T} \|(J_{21})_{t}\|^{2p} \\
&\leq (\mathbb{E} \sup_{0 \leq t \leq T} \|(J_{21})_{t}\|^{2(2p-1)})^{1/2} (\mathbb{E} \sup_{0 \leq t \leq T} \|(J_{21})_{t}\|^{2})^{1/2} \\
&\leq C(p, T)(\sqrt{\frac{e}{\delta}} + \sqrt{\delta}),
\end{align*}
\]

thus, we have

\[
\begin{align*}
&\mathbb{E} \sup_{0 \leq t \leq T} \|J_{21}\|_{H^1}^{2p} + \mathbb{E} \sup_{0 \leq t \leq T} \|(J_{21})_{t}\|^{2p} \leq C(p, T)(\sqrt{\frac{e}{\delta}} + \sqrt{\delta}).
\end{align*}
\]

* For $J_{22}$, due to Lemma 4.1, it concludes that

\[
\begin{align*}
&\mathbb{E} \sup_{0 \leq t \leq T} \|J_{22}\|_{H^1}^{2p} + \mathbb{E} \sup_{0 \leq t \leq T} \|(J_{24})_{t}\|^{2p} \\
&\leq T^{2p-1} \mathbb{E} \int_0^T \|\tilde{f}(A^e(s)) - \tilde{f}(\hat{A}^e(s))\|^{2p} ds \\
&\leq C\delta^{2p} + C\delta^{2p+1}.\]
* For $J_{23}$, it concludes that

\[
E \sup_{0 \leq t \leq T} \|\hat{A}^\varepsilon(t) - A(t)\|_{H^1}^2 + E \sup_{0 \leq t \leq T} \|\hat{A}(t)\|_{H^1}^2 - \hat{f}(\hat{A}(s)) - \hat{f}(\hat{A}(s))\|_{L^2}^2 ds \\
\leq C \int_0^T E \sup_{0 \leq r \leq s \leq T} \|\hat{A}^\varepsilon(r) - \hat{A}(r)\|_{L^2}^2 ds.
\]

With the help of the above estimates, we have

\[
E \sup_{0 \leq t \leq T} \parallel \hat{A}^\varepsilon(t) - A(t) \parallel_{H^1}^2 + E \sup_{0 \leq t \leq T} \parallel \hat{A}(t)\parallel_{H^1}^2 \leq C(\delta^{2p} + \sqrt{\varepsilon} + \sqrt{\delta} + \frac{\varepsilon^{2p+1}}{\varepsilon}) e^{Cn^p},
\]

this implies that

\[
E( \sup_{0 \leq t \leq T} \parallel \hat{A}^\varepsilon(t) - A(t) \parallel_{H^1}^2 \cdot 1_{\{T \leq \tau_n^*\}} ) + E( \sup_{0 \leq t \leq T} \parallel \hat{A}(t)\parallel_{H^1}^2 \cdot 1_{\{T \leq \tau_n^*\}} ) \leq C(\delta^{2p} + \sqrt{\varepsilon} + \sqrt{\delta} + \frac{\varepsilon^{2p+1}}{\varepsilon}) e^{Cn^p}.
\]

On the other hand, due to Proposition 3.3, we have

\[
E( \sup_{0 \leq t \leq T} \parallel \hat{A}^\varepsilon(t) - A(t) \parallel_{H^1}^2 \cdot 1_{\{T > \tau_n^*\}} ) \leq C \frac{\sqrt{n}}{\varepsilon},
\]

and

\[
E( \sup_{0 \leq t \leq T} \parallel \hat{A}(t)\parallel_{H^1}^2 \cdot 1_{\{T > \tau_n^*\}} ) \leq C \frac{\sqrt{n}}{\varepsilon}.
\]

Hence, we have

\[
E( \sup_{0 \leq t \leq T} \parallel \hat{A}^\varepsilon(t) - A(t) \parallel_{H^1}^2 ) + E( \sup_{0 \leq t \leq T} \parallel \hat{A}(t)\parallel_{H^1}^2 ) \leq C(\delta^{2p} + \sqrt{\varepsilon} + \sqrt{\delta} + \frac{\varepsilon^{2p+1}}{\varepsilon}) e^{Cn^p} + C \frac{\sqrt{n}}{\varepsilon}.
\]
if we take \( n = \sqrt[4p]{\frac{1}{8\epsilon}} \ln \epsilon, \delta = \epsilon^{\frac{1}{8}} \), we obtain

\[
\mathbb{E}( \sup_{0 \leq t \leq T} \| A^\epsilon(t) - \tilde{A}(t) \|_{H^1}^2 ) + \mathbb{E}( \sup_{0 \leq t \leq T} \| A^\epsilon(t) - \tilde{A}_t(t) \|_{H^1}^2 ) \leq C(\epsilon^p + \epsilon^{p - \frac{1}{4}} + \epsilon^{\frac{1}{8}} + \epsilon^{p - \frac{1}{4}}) \epsilon^\frac{1}{8} + \frac{C}{\sqrt[8p]{-\frac{1}{ln \epsilon}}}
\]

\[= C(\epsilon^p + \epsilon^{p - \frac{1}{4}} + \epsilon^{\frac{1}{8}} + \epsilon^{p - \frac{1}{4}}) \epsilon^\frac{1}{8} + C(\frac{1}{-ln \epsilon}) \frac{1}{8p}.
\]

4.6 Proof of Theorem 4.1

By taking \( \delta = \epsilon^{\frac{1}{8}} \) in Lemma 4.1, we have

\[
\mathbb{E}( \sup_{0 \leq t \leq T} \| A^\epsilon(t) - \tilde{A}(t) \|_{H^1}^2 ) + \mathbb{E}( \sup_{0 \leq t \leq T} \| A^\epsilon(t) - \tilde{A}_t(t) \|_{H^1}^2 ) \leq C\epsilon^p + C\epsilon^{p - \frac{1}{4}},
\]

if \( p > \frac{5}{8} \), we have

\[
\mathbb{E}( \sup_{0 \leq t \leq T} \| A^\epsilon(t) - \tilde{A}(t) \|_{H^1}^2 ) + \mathbb{E}( \sup_{0 \leq t \leq T} \| A^\epsilon(t) - \tilde{A}_t(t) \|_{H^1}^2 ) \leq C\left(\frac{1}{-ln \epsilon}\right)^{\frac{1}{8p}},
\]

thus, we have

\[
\mathbb{E}( \sup_{0 \leq t \leq T} \| A^\epsilon(t) - \tilde{A}(t) \|_{H^1}^2 ) + \mathbb{E}( \sup_{0 \leq t \leq T} \| A^\epsilon(t) - \tilde{A}_t(t) \|_{H^1}^2 ) \leq C\left(\frac{1}{-ln \epsilon}\right)^{\frac{1}{8p}}.
\]

If \( 0 < p \leq \frac{5}{8} \), for any \( \kappa > 0 \), it holds that

\[
\mathbb{E}( \sup_{0 \leq t \leq T} \| A^\epsilon(t) - \tilde{A}(t) \|_{H^1}^2 ) + \mathbb{E}( \sup_{0 \leq t \leq T} \| A^\epsilon(t) - \tilde{A}_t(t) \|_{H^1}^2 ) \leq \left(\frac{1}{\frac{3+\kappa}{4}}\right)^{\frac{2p}{2+\kappa}} \frac{1}{-2p}
\]

\[
\left(\mathbb{E}( \sup_{0 \leq t \leq T} \| A^\epsilon(t) - \tilde{A}(t) \|_{H^1}^2 ) \|_{H^1}^2 ) \|_{H^1}^2 \right)^{\frac{2p}{2+\kappa}} \frac{1}{-2p}
\]

\[
= C(p, \kappa)(\frac{1}{-ln \epsilon})^{\frac{1}{\frac{3+\kappa}{4}+\kappa}}\frac{2p}{2+\kappa}
\]

\[
= C(p, \kappa)(\frac{1}{-ln \epsilon})^{\frac{1}{8+4\kappa}2p}.
\]

This completes the proof of Theorem 4.1.

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