PERSPECTIVES IN STATISTICAL MECHANICS

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Dedicated to Barry Simon on the occasion of his sixtieth birthday

ABSTRACT. Without attempting to summarize the vast field of statistical mechanics, we briefly mention some of the progress that was made in areas which have enjoyed Barry Simon’s interests. In particular, we focus on rigorous non-perturbative results which provide insight on the spread of correlations in Gibbs equilibrium states and yield information on phase transitions and critical phenomena. Briefly mentioned also are certain spinoffs, where ideas which have been fruitful within the context of statistical mechanics proved to be of use in other areas, and some recent results which relate to previously open questions and conjectures.

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1. AN APPRECIATION

In this chapter of the Festschrift celebrating Barry Simon’s contributions to the fields which have enjoyed his attention, we focus on statistical mechanics. I shall list here some of the subject’s core topics, and mention a selection of results related to issues which have attracted Barry’s interests. These represent only a small part of the results which were derived in this vibrant field during the period addressed here.
The results presented below came from a community of people who throughout their work have stimulated and informed each other, activities in which Barry Simon has excelled. He has done that with flair and in his unique style: with remarkable energy, mathematical skill, eye for the essence of the argument, and intellectual generosity towards students, colleagues, and predecessors. Thinking of a way to convey Barry Simon’s impact, I am reminded of a question which one is occasionally asked to comment upon: In what way would the world have been different without the contribution of this individual? I usually find this to be a rather humbling and somewhat troubling question. However, if this is in regard to Barry Simon, my answer is:

Most likely the world would have been poorer, my ignorance much greater, and our accomplishments fewer.

And I believe that this could also be said by all of Barry’s generation in our field, some of his elders, and the many students and postdocs he has generously informed and inspired.

Thank you - Barry!

2. STATISTICAL MECHANICS IN RELATION TO FIELD THEORY

Statistical mechanics is a subject originating in some profound observations tying the orderly behavior which is the subject of thermodynamics with an underlying chaos at the lower scales. A similar claim can be made about the origins of the order seen in most of physics, as we now see in hindsight, of quantum mechanics and quantum field theory. Looking more closely at statistical mechanics, one finds a rich collection of interesting phenomena, challenging questions, and lessons for other disciplines. While Barry Simon has embraced the subject close enough to eventually write a book about it [110], his perspective has, at least initially, been driven by the relation of statistical mechanics with constructive field theory. The challenges of the latter (54) have carried a sense of urgency, a glimpse of which can be found in the introduction to Simon’s “The $P(\phi)^2$ Euclidean (Quantum) Field Theory” (102). The two subjects are related in a number of ways:

i. Similarity

The Gell-Mann low formula,

$$
\tau(x_1, \ldots, x_n) = \frac{\langle 0 | T \left[ e^{(i \int H(x) d^4x)} \phi(x_1) \cdots \phi(x_n) \right] | 0 \rangle}{\langle 0 | T \left[ e^{(i \int H(x) d^4x)} \right] | 0 \rangle},
$$

(2.1)
displays a formal similarity with the Gibbs state expression for the correlation function of local (spin) variables in a thermal equilibrium state:

$$
\langle \sigma_1 \cdots \sigma_n \rangle = \frac{\sum_{\sigma_i = \pm 1} \sigma_1 \cdots \sigma_n e^{-\beta H(\sigma)}}{\sum_{\sigma_i = \pm 1} e^{-\beta H(\sigma)}},
$$

(2.2)

As discussed in [102], the formal similarity is made even more compelling by the observation that, in Bosonic field theory, under analytic continuation into imaginary time vacuum, expectation values of products of field operators are transformed into amplitudes associated with functional integrals over non-negative measures. The integrals are rendered convergent through suitable ultraviolet and infrared cutoffs, and the corresponding Schwinger
functions then fall within the realm of classical statistical mechanics. In that situation, the general perspective of this field and some of its specific tools become applicable. Some of the dramatic consequences which this reduction has had on the program of constructive field theory can be seen in [60, 62, 102], and are described in the contribution of Rosen in this volume.

ii. Criticality underlying the FT path integrals

One of the lessons of statistical mechanics is that in an extensive system of variables, with short range interactions, at generic choices of parameters the correlations decay exponentially. The correlation length is typically not much greater than the range of the interaction, and it is only in the vicinity of critical points that correlations exhibit structures of much greater scales. For a field theory based on local interactions, the interaction range vanishes on the scale of the continuum limit. Hence, in a constructive approximation, for which convergence is accomplished through ultraviolet cutoffs, the underlying system of the local variables needs to be very near a critical point. Some familiarity with critical phenomena is, therefore, essential for the understanding of a continuum field theory.

iii. Statistical mechanics as a constructive tool for field theory

Scaling limits of the fluctuating component of the local order parameter in statistical mechanics are described by Euclidean fields. To some extent, this relation has fueled the interest in statistical mechanics within the community of constructive field theorists, in particular through the hope to construct the Euclidean $\phi^4$ field theory through limits of critical Ising-type models. This can indeed be done, but to the chagrin of some, only below the upper critical dimension, which in the above case is $d_c = 4$. The “no-go” theorems related to the phenomenon of the upper-critical dimension ([1, 42, 51]) may have somewhat diminished the interest in the field on the part of those who came to it looking for help with the tasks of constructive FT. Nevertheless, statistical mechanics has continued to serve as a rich source of interesting challenges and insights which enrich other fields. Among the new challenges one could find the effects of frustration and disorder in the parameters (Imry–Ma effect and spin glass phenomena). Ideas which were developed in the context of statistical mechanics have affected developments in various other areas. Such “spinoffs” have included topics of discrete mathematics with relation to computer science, as well as techniques for addressing the spectral and dynamical properties of random Schrödinger operators.

3. Aspects of Equilibrium Statistical Mechanics

Without trying to provide a proper summary of the essential results in this subject, and as a prelude to a selected few presented next, let me mention a non-exhaustive list of themes which have drawn the attention of mathematical physicists working in this area.

I. Derivation of Thermodynamics

- Convergence of the free energy density
- Entropy and its properties
- Free energy for the long range Coulomb interaction
- Finite size effects

One of the first accomplishments of Boltzmann’s and Gibbs’ statistical mechanics has been in presenting an intellectually coherent basis for the laws of thermodynamics and, in particular expanding our understanding of entropy—the concept at the root of thermodynamics. The book by Ruelle on the subject [98] was an invitation extended to
mathematically-minded researchers and, indeed, in short order a slew of interesting results have followed. Among the general and foundational results of the subject is convergence of the free energy density for extensive systems with short range interactions. Key contributors have included van Hove [69], Ruelle [96], Fisher [38], and Griffiths [55]. Further studies were needed to address the question for long range forces, such as the Coulomb interaction ([78]), which of course is of deep interest. The proof of convergence for quantum Coulomb systems was accomplished in a fundamental paper of Lieb and Lebowitz [82]. The work benefitted from the input of Simon, who contributed a technical appendix [100]. Further studies were needed to extend our understanding of entropy to the quantum domain. A major accomplishment was the proof of subadditivity of entropy by Lieb and Ruskai [83], in another article featuring an appendix by Simon. It was also noted that the von Neumann notion of entropy does not offer a fully satisfactory quantum counterpart to the Kolmogorov–Sinai entropy of classical dynamical systems, and neither is it fully satisfactory in the setting of statistical mechanics, where the dynamics correspond to translations. After some search, other proposals were developed [27].

II. Mapping the Phase Diagrams

- High temperature and low temperature regimes
- Phase transitions for long range interactions
- Conditions for the existence of symmetry breaking
- Establishing the criticality of a phase transition

One of the next tasks for statistical mechanics is to produce the tools for mapping the phase diagram and description of the distinct phase regimes in the thermodynamic space, whose parameters include the temperature $T = (k\beta)^{-1}$, the overall strength of the coupling, and control parameters such as the magnetic field. Of particular interest is understanding the conditions under which there will be phase transitions associated with discrete or continuous symmetry breaking ([90, 56, 86, 48, 35, 49, 52]).

The structure of the Gibbs equilibrium states at high temperatures can be approached through fairly general and robust methods, at least for systems with rapidly decaying interactions. The tools include cluster expansions, such as the improved Meyer expansion [91, 97, 118], Dobrushin’s uniqueness-of-state technique (bounds for a generally defined “influence kernel”) [28, 104], and the generalized polymer expansion of Kotecký and Preiss [75]. For low temperatures, the basic tools include the Peierls argument [90] and the corresponding bounds and expansion, the more general Pirogov–Sinai theory [94], and occasionally a duality map converting low to high temperature regime.

Naturally, the analysis gets to be somewhat trickier near the boundaries of the distinct phases, which is where critical phenomena are found. In particular, a non-perturbative argument is needed for the important issue of establishing that a phase transition is critical, which means here that the correlation length diverges as the transition point is approached. More is said on this topic below.

III. Structure of the Gibbs States

- Correlation functions: bounds, inequalities, and relations
- Extremal state decomposition: applications for issues of uniqueness
- Effects of long range interactions, and in particular $1/|x - y|^2$ interactions in $1D$

Some could find it disappointing that in dimensions $d > 2$ most systems of interest are not solvable. In lieu of solutions, for some systems useful information can be obtained
through correlation inequalities through which much can be learned about the phase structure, properties of the Gibbs states, and the critical behavior.

Among the general results of statistical mechanics is the statement that, in one dimension, short range interactions do not yield phase transitions, but phase transitions and symmetry breaking are possible if the interactions decay slowly enough. The threshold case, which is of interactions falling as \(1/|x - y|^2\), has attracted attention for a number of reasons. The question of the exact condition for the threshold decay rate has attracted a number of different ideas, leading to the conclusion that it is \(J_{x,y} \approx J/|x - y|^2\) with \(\beta J = 1\) \([117, 33, 108]\). Through an intriguing energy-vs.-entropy argument, Thouless has suggested that in the borderline case of Ising spins with the \(1/|x - y|^2\) ferromagnetic interaction, there should be found an unusual discontinuity of the spontaneous magnetization at the transition temperature. Anderson, Yuval, and Hamman \([15]\) have pointed out that this particular model is quite significant for the Kondo problem, in which context the one-dimensional parameter is time. They also introduced a very insightful renormalization group analysis which explains the system’s essential features. The curious prediction of Thouless was eventually established rigorously \([9, 7]\). However, related analysis also showed that the phenomenon is not accounted for by the original argument \([21]\).

IV. Critical Phenomena

- Critical exponents
- Universality
- The “renormalization group” perspective
- The phenomenon of upper critical dimension(s)

Among the major notions to emerge in the late sixties and early seventies of the twentieth century have been the realization of “universality” in critical phenomena, and the “renormalization group” perspective on the subject \([39, 119]\). The translation of the latter into proper mathematical statement is not easy \([26]\), and to large extent still remains to be done. In this situation, it was deemed of value to have even partial rigorous results, to either correct or to offer supportive evidence to the validity of the sometimes far-reaching claims and, by implication, build confidence in the heuristic methods. Various results have been obtained, often with a characteristic time delay of about a decade and usually by means which have been very different from the earlier physicists’ heuristic arguments.

V. Structures Emerging from Local Interactions

- Random path / random cluster representations
- Random surface models, and their phase transitions

The spin correlations which exist in the equilibrium states of ferromagnetic models can often be presented as the result of correlated excitations which are associated with random paths, as in \([44, 7]\) below, or with random clusters—as seen in the Fortuin and Kasteleyn representation of the Q-state Potts models \([40]\). Subtle correlations may then be robustly expressed in terms of properties of the associated stochastic geometric models. This has led to a rather fruitful line of research, with progress made through both analogies and exact relations of spin systems with percolation type models (see \([11, 8, 21, 58]\) and references therein).

In gauge models one finds relevant excitations associated with random surfaces. In this context, an interesting extension of the percolation transition is found in a random plaquette model in three dimensions \([6]\). The standard percolation transition is related there by duality to a confinement-deconfinement transition, which has a natural definition
for a system of randomly occupied plaquettes. This is but one example of interesting issues associated with random surface models [14].

VI. Scaling Limits

- The emergence of stochastic geometry
- Insights and challenges of the quantum gravity method
- Conformal invariance and SLE

It is generally understood that the scaling limit of the fluctuations of the local order parameter would naturally be described by fields. More could be said about the limiting distribution of the fractal stochastic geometric structures which are associated with the long range correlations exhibited in these models at criticality.

The theory of the critical fluctuations has been developed most effectively in two dimensions where conformal symmetry applies and has particularly strong consequences [17]. An intriguing tool has been the “quantum gravity” method, which in effect means the study of statistical mechanics on fluctuating surfaces. In that setup, various characteristic exponents can be calculated through the asymptotics of random matrices. Curiously, the resulting scaling laws are predicted to bear an explicit relation to the corresponding ones on a rigid plane [32]. Another approach has recently been enabled through the introduction, by Schramm [92], of the SLEκ family of processes which are endowed with the “conformal Markov property” [76, 95]. There have been many interesting mathematical results in this area, including the description of some of the scaling limits [113, 25], as well as various results on critical exponents [76]. In general, the latter fit very well with the quantum gravity predictions, although a broad statement still remains to be established.

VII. Disorder Effects

- Effects of randomness in the coupling strengths; the Imry–Ma phenomenon
- Spin glass phenomena

In addition to its intrinsic interest, the Imry–Ma phenomenon is remarkable in providing a rare example where a rigorous result was derived before a consensus has emerged in the physics community concerning two conflicting predictions ([70, 23, 13]).

On the last listed topic—spin glass models—significant progress [65, 116] was recently enabled through a surprisingly effective interpolation argument which was introduced by Guerra and Toninelli. Curiously, Barry Simon’s interest in statistical mechanics was stimulated through a different, yet similarly surprising and stimulating, contribution which Guerra made at the beginning of the time frame which we bear in mind writing these notes [59].

4. Some Cherries from the Pie: Essential Results Derived Through Non-Perturbative Methods

Rather than review the body of results derived on the topics listed above, which may take the talents of Barry Simon to write, I shall present here only some examples. These are selected by few common themes:

i. They demonstrate that often “soft arguments,” such as inequalities which by their nature can be viewed as an imprecise tool, can address questions which are beyond the reach of hard analytical methods;

ii. The examples are related to some of the basic models which have driven the interests of the intellectual community in which Barry Simon was active; and

iii. For many of these results there is an underlying random walk perspective. Various insights have sprung from this perspective on a variety of topics.
4.1. Absence of Phase Transitions for the Ising Model and the $\phi^4$ Euclidean Field Theory at $h \neq 0$. The determination of the phase diagram of a model is of course of fundamental importance. For ferromagnetic Ising models, the task can be greatly simplified through correlation inequalities. In particular, the combination of the GHS and FKG inequalities (Griffiths–Hurst–Sherman [57] and Fortuin–Kasteleyn–Ginibre [41]) allows one to conclude that a first order phase transition can occur only along the line of symmetry $h = 0$.

The GHS inequality states that in spin models, with $\sigma_x = \pm 1$ and the Hamiltonian
\[
H(\sigma) = \sum_{\{x,y\}} J_{x,y} \sigma_x \sigma_y + \sum_x h_x \sigma_x
\] (4.1)
with $J_{x,y} \geq 0$, the cluster functions ($u$ for Ursel)
\[
u_\ell(x_1, \ldots, x_\ell) := \frac{\partial^\ell}{\partial h_{x_1} \cdots \partial h_{x_\ell}} \ln Z(\Lambda; h, J) = \langle \sigma_{x_1}; \ldots; \sigma_{x_\ell} \rangle
\] satisfy
\[
\text{sgn } h \cdot u_3(x_1, x_2, x_3) \leq 0
\] (4.2) and, at $h = 0$:
\[
u_4(x_1, \ldots, x_4) \leq 0.
\]

Given that the mean magnetization, $M(\beta, h) = \langle \sigma_0 \rangle_{\beta, h}$, obeys
\[
\frac{\partial^2 M(\beta, h)}{\partial h^2} = \beta^2 \sum_{x,y} u_3(0, x, y),
\] (4.3)
the inequality (4.2) allows one to conclude that $M(\beta, h)$ is concave as function of $h$ at $h > 0$, and convex for $h < 0$. The important—and otherwise difficult to reach—implication is that for $h \neq 0$ the magnetization is continuous in $h$. The FKG inequality allows one in such case to rule out first order phase transitions (which require the existence of more than one Gibbs state, in the infinite volume limit). Related results for the line $h = 0$, $\beta \in [\beta_c, \infty)$ were derived by yet different inequalities of Lebowitz [77].

A yet stronger statement is allowed by the Lee–Yang theorem [80, 84] which implies that in this class of models the infinite volume free energy density is analytic in $h$ for $h \neq 0$. An important implication is that at $h \neq 0$ the correlation functions decay exponentially in the distance [79].

The results described next have allowed to extend the above results also to the Euclidean $\phi^4$ field theory, for which exponential decay of correlations translates into a mass gap [103, 63].

4.2. Construction of $\phi^4$ Variables out of Ising Spins. The analyticity methods and the correlation inequalities which were initially derived for systems of Ising spins (on arbitrary graphs) admit natural extensions to broader families of systems of ferromagnetically coupled variables. Following up on the observation of Griffiths that these methods extend to systems whose spin variables can be presented as linear combinations of ferromagnetically coupled Ising spins and that various examples of interest can be obtained by taking limits of such variables, Simon and Griffiths [111] showed that the class also includes the important case, historically and conceptually, of the continuous $\phi^4$ measure
\[
\rho(d\phi) = e^{-\lambda \phi^4 + b\phi^2} d\phi,
\] (4.4)
at $\lambda > 0, b \geq 0$. 
Thus, the observations made above about Ising models also extend to functional integrals of the form

$$
\langle F(\phi) \rangle := \int \cdots \int F(\phi) e^{-\sum J_{x,y}(\phi_x - \phi_y)^2 + \sum_x h_x \phi_x} e^{-\sum_x \lambda \phi_x^4 + h_0 \phi_x^2} \Pi_x d\phi_x/Z(\Lambda; h, J, \lambda, b).
$$

Of particular interest has been the scaling (continuum) limit, for which the lattice spacing is taken as \( a \to 0 \), and one considers the limiting probability measure for \( \Psi(x)/a := \zeta(a)^{-1} \phi/a \), which is to be interpreted in the distributional sense. For the construction of a meaningful limit, both the coupling constants and the field strength renormalization \( \zeta(a) \) are adjusted while \( a \to 0 \), so as to guarantee convergence, in a weak sense, and regular values (neither \( \equiv 0 \), nor \( \infty \)) for

$$
W_n(x_1, \ldots, x_n) = \lim_{a \to 0} \langle \Psi(x_1) \cdots \Psi(x_n) \rangle_a.
$$

4.3. Insights from Lee–Yang Theory for the Scaling Limits. An important consequence, noted by Newman [87], is that any scaling limit of the lattice \( \phi^4 \) model will be non-gaussian if and only if the suitably scaled limit of \( u^4(x_1, \ldots, x_4) \equiv \langle \phi_{x_1} \cdots \phi_{x_4} \rangle \) does not vanish. The inequalities presented in [1] and [42] yield such a conclusion through direct bounds, yet the insight from the Lee–Yang theory was instructive.

4.4. A Random Path Perspective. A new range of insights and tools become available through random path representations, and/or random cluster representations, of the correlation functions in some of the essential models. For Ising ferromagnetic spin systems, after suitable expansion of the Gibbs factor and summation of the spin variables, one finds [57, 1], for \( h = 0 \):

$$
\langle \sigma_{x_1} \cdots \sigma_{x_k} \rangle = \frac{\sum_{\partial m = \{x_1, \ldots, x_k\}} w(m)}{\sum_{\partial m = \emptyset} w(m)}
$$

where \( m \equiv \{m_b\} \) ranges over “random current” configurations, each described by a collection of (integer) fluxes defined over the lattice bonds. The weight function \( w(m) \) is a product of local terms. The sum is restricted by a constraint on the set of sites at which \( m \) has odd flux, which is denoted here by \( \partial m \). Configurations with a prescribed set of such “sources” consist of an assortment of many current loops and few “current lines” linking pairs of sources, as is indicated in Figure 1. In other words, in this representation the spin-spin correlations are expressed through the amplitudes associated with source-insertion operators in a loop-soup integral. Under partial summation over the background loops, one obtains a random-path representation:

$$
\langle \sigma_{x_1} \cdots \sigma_{x_{2n}} \rangle = \sum_{\text{paths} \gamma} \sum_{\gamma_{x_1 \cdots x_{2n}}} \rho(\gamma_{x_1 \cdots x_{2n}}) \langle \rho(\gamma_{11}, \ldots, \gamma_{nn}) \rangle
$$

where \( \gamma_j \) ranges over paths connecting the designated sources. The weight for a collection of paths \( \rho(\gamma_{1}, \ldots, \gamma_{n}) \) factorizes, approximately, if the paths are remote from each other.

This representation captures the fact that without the interactions, the spin variables are decoupled and the correlations are built through pair interactions. The representation is made particularly effective through some convenient identities, which allow one to reduce various truncated correlations to path intersection amplitudes [11]. As will be indicated below, such geometrization of correlations has far reaching consequences.

The path representation for the correlation functions is not unique, though some have particular technical advantages. Related representations have also been worked out for
\( \phi^4 \) correlators in terms which are native to the FT functional integrals \(4.5\), \(42\). Their implications are extensively discussed in the monograph \(37\).

4.5. Proofs of the Criticality of (Certain) Phase Transitions. Phase transitions at which the correlation length diverges are referred to as critical—a condition which is not met at the usual first order transitions. Criticality plays an important role for the emergence of universality in critical phenomena, and for the existence of scaling limits. It is therefore of value to be able to establish the criticality of phase transitions of interest. The question seems to fall beyond the reach of the standard perturbative methods, as these tend to diverge at phase transitions. It was therefore gratifying to find a useful tool for this purpose in the form of the family of inequalities which has evolved from the “Simon inequality” \(106\) for ferromagnetic Ising spin systems. Particularly effective is the Lieb improved version \(81\) which, after further improvement, states that for any pair of sites \(\{x, y\}\) within the set on which the model is defined and any domain \(D\) which includes \(x\) but not \(y\), the correlations of the Ising ferromagnetic model, \(G(x, y) \equiv \langle \sigma_x \sigma_y \rangle\), satisfy:

\[
0 \leq G(x, y) \leq \sum_{u \in D, v \in \mathbb{Z}^d \setminus D} G(x, u)_D \beta J_{u,v} G(v, y)
\]  

(4.8)

where \(G(x, u)_D\) is the correlation function for the system restricted to \(D\). Applying the inequality with \(D\) chosen as finite boxes of increasing size, \(4.8\) allows one to conclude the correlation length has to diverge as the temperature approaches the transition point.

It may be added that the spark which led to \(4.8\) originated in a discussion of the work by Dobrushin and Pechersky \(29\), where it was shown that, quite generally, fast enough power law decay of generalized correlation functions implies exponential decay. The extraction of a simple, pedagogical, and useful principle, at least for a subclass of models, is a good example of Barry at work. It is also an example of a robust idea: a number of inequalities of this genre, valid for different spin models, have then appeared in close succession—close enough to be published in the same issue of CMP.

The inequality \(4.8\) can be understood through the random walk perspective on the spread of correlations in the model. (Although the original derivations did not make use of the representation \(4.7\), equation \(4.8\), which was not in the original batch of the similar inequalities, has an easy derivation using the technique of \(1\)). Useful variants of this relation occur also in various other models: a very similar statement is valid for the connectivity function in percolation models \(8\), and a related inequality is satisfied by the (fractional) moments of the Green function of the discrete Schrödinger operators with random potentials \(11\).
4.6. Systematic Criteria for Mapping the High Temperature Regime. Another reason for interest in inequalities of the Simon–Lieb type, is that (4.8) yields a sequence of finite volume criteria for the high temperature phase. For each $L < \infty$, let $\beta_L$ be defined by the condition

$$\beta_L = \max \left\{ \beta : \sum_{u \in [-L,L]^d, v \in \mathbb{Z}^d \setminus [-L,L]^d} G_\beta(0, u) D \beta J_{u,v} \leq 1 \right\}. \tag{4.9}$$

The inverse temperatures $\beta_L$ can in principle be evaluated to the desired accuracy through finite computations. The inequality (4.8) allows one to conclude that any $\beta_L$ provides a rigorous bound on the actual transition temperature $\beta_c$, and it can also be shown that these bounds actually converge:

$$\text{for each } L < \infty: \quad \beta_L < \beta_c, \quad \text{and} \quad \lim_{L \to \infty} \beta_L = \beta_c. \tag{4.10}$$

A more general method for such bounds which, in principle, permits one to successively map the entire high temperature regime, is provided by the work of Dobrushin and Shlosman [31]. While it was only very rarely used as a tool for practical computations, the finite volume principle has interesting implications for the nature of the high temperature regime.

The method has also inspired similar statements for the study of the regime of Anderson localization for the aforementioned random Schrödinger operators [50, 11, 53].

4.7. Continuous Symmetry Breaking (in $d > 2$ Dimensions). A serious challenge, whose robust resolution could provide a tool for addressing a number of issues, is to establish that under suitable conditions there is continuous symmetry breaking. An outstanding development was the series of results which started with the work of Fröhlich, Simon, and Spencer [48]. Their argument proceeds through the gaussian domination bound, which says that for reflection positive spin models with a pair interaction and periodic boundary conditions, for dual momenta $p \in \Lambda^* \setminus 0$:

$$\langle |\widehat{S}(p)|^2 \rangle \leq \frac{d}{2\beta \mathcal{E}(p)}. \tag{4.11}$$

Here, $\widehat{S}(p) = \Lambda^{-1/2} \sum_{x \in \Lambda} e^{i px} S_x$, and for $p \neq 0$, $\mathcal{E}(p) = -\sum_{x \in \Lambda} e^{i px} J_x$. It helps to note that the quantity on the left side in (4.11) plays a dual role: the Fourier transform of the two point function $G(x, y) = \langle S_x S_y \rangle$ is given by $\langle |\widehat{S}(p)|^2 \rangle$, and $\mathcal{E}(p) \langle |\widehat{S}(p)|^2 \rangle$ gives the mean value of energy in the $p$ mode. By the latter observation, (4.11) says that the “equipartition law” provides a rigorous bound. The former observation is used to show that in dimensions $d > 2$, at low temperatures there is symmetry breaking, which can be attributed to Bose–Einstein like macroscopic occupation of the $p = 0$ mode, proven by an estimate which resembles the BE calculation [37].

The gaussian domination bound was proven using the “Chessboard Inequality,” which is based on the reflection positivity. The inequality is a remarkable tool. Among its other applications are bounds on expectation values of products of local observable through thermodynamic quantities [43]. In particular, it permits one to establish the existence of phase transitions though arguments of thermodynamic flavor.

Reflection positivity (RP) is a tool with contradictory aspects, and limitations which are not always intuitive. For instance, there is still no rigorous proof of the existence of symmetry breaking in the quantum Heisenberg ferromagnet, although such a result was established for the antiferromagnetic model for the suitable dimensions [34]. When RP
applies, its results are spectacular and physically well motivated, but when its exacting condition is not satisfied even to a small degree, it provides no information. One could say it is a gem of an argument.

4.8. The Mermin–Wagner Phenomenon and the Kosterlitz–Thouless Phase for Plane Rotors in Two Dimensions.

A celebrated general statement, known as the Mermin–Wagner theorem [86], is that in two dimension there is no continuous symmetry breaking. The rigorous proof of that for the compact symmetry group of rotations was initially provided by Dobrushin and Shlosman [30]. The phenomenon derives from the fact that in $d \leq 2$ dimensions, the ground state configuration of a large system, with rotation-invariant interactions of short-range, can be rotated against the ordered boundary conditions with only small energy cost. The energy penalty vanishes in the infinite volume limit, just as the related variational quantity, for $d \leq 2$:

$$\inf \left\{ \int_{1 \leq |x| \leq L} |\nabla \theta(x)|^2 d^d x : \theta(x) = \begin{cases} 0 & |x| = 1 \\ 1 & |x| = L \end{cases} \right\} \xrightarrow{L \to \infty} 0,$$  \hspace{1cm} (4.12)

which for two dimensions vanishes as $1/\log L$. However, the argument outlined above is incomplete, as what is needed for the proof is an estimate of the free energy associated with the rotation of a state with thermal disorder. Surprisingly, the harder part to deal with is the first order term, which vanishes for the totally aligned configurations and thus does not show up in the ground state calculation. A particularly simple and non-perturbative argument for the case of compact symmetry group was devised by Pfister [93] (see also [52]). Altogether, the so-called Mermin–Wagner Theorem is one of the essential results of statistical mechanics.

It is of interest that at the thresholds for the feasibility of symmetry breaking one finds borderline models with unusual behavior: low temperature phases at which there is neither long range order nor rapid decay of correlations, where the correlation functions decay by temperature-dependent power laws. For continuous symmetry such behavior is encountered in the two component X-Y ($O(2)$) model in two dimensions, in what is known as the Kosterlitz–Thouless (KT) phase [74] (long range order is ruled out there by the Mermin–Wagner Theorem). The rigorous proof of the existence of this phase, by Fröhlich and Spencer [49] (see also [85]), was a notable accomplishment. The techniques which were developed for this purpose have included elements of the multiscale analysis which has found many other applications since, in particular, in the theory of Anderson localization [50, 45, 53].

At the threshold for discrete symmetry breaking is the one dimensional Ising model with $J_{x,y} = 1/|x - y|^2$. This system also exhibits temperature-dependent power law decay of correlations, which here occurs before the onset of long range order [71]. The existence of this unusual low temperature behavior shows that the Thouless phenomenon, discontinuity of the spontaneous magnetization, which is exhibited there [21] is not accounted for by the argument which were initially used for its prediction [14]. Related energy-entropy arguments, and their limitations, were discussed by Simon and Sokal [12].

It may also be added here that KT-like phases are not expected to occur for $O(N)$ models with $N > 2$, in two dimensions. The reasoning is interesting: a calculation shows that under a renormalization group scheme at both very low and very high temperatures, the temperature flows upward, and it was surmised that this flow connects the two regimes. However, the analysis gets to be more complicated at intermediate temperatures. The renormalization group map is somewhat ill defined there, and that leaves a serious gap in the argument. It was argued in [89] that the question which is left open is an interesting one,
and that its resolution may be significant also for clarifying other assertions of “asymptotic freedom,” which is an issue of great significance.

As a prelude to the next topic, it may be added that an intermediate phase is expected to occur in two dimensions for discrete two-component “clock” models with discrete rotational symmetry. Its essential characteristic is the existence of power law decay of correlations at power which varies with the temperature, and without long range order. Such a phase does not occur for translation invariant Ising spin systems, in any dimension.

4.9. The Coincidence $\beta_T = \beta_H$ for Ising and Percolation Models. For ferromagnetic Ising spin models in $d > 1$ dimensions, the edge of the high temperature regime, which is characterized by the exponential-decay of correlations, coincides with the threshold for the non-vanishing of the long range order parameter. That is, such a system does not exhibit an intermediate phase which shares the characteristics of the KT phase. (The two transition temperatures were initially designated $\beta_T^{-1}$ and $\beta_H^{-1}$, for Temperly and Hammersley, correspondingly).

Such statements are of basic interest, as they reflect on both the phase diagram and on the critical behavior, yet the proofs are beyond the reach of the available expansions. Nevertheless, it was found that the above assertion can be addressed at certain generality, of the translation invariant Ising spin models. An analogous statement also holds for percolation models on transitive graphs (with application also to the contact process). As it turned out, one understands each of the models better by considering the two simultaneously.

It is somewhat remarkable that the above statement, and other information about the behavior at the critical point, can be proven by means of soft looking partial differential inequalities which bear no specific reference to the critical temperature. Of course, the nonlinearity of the expressions contains the seeds for the information about possible critical behavior, as the analysis shows [8, 26, 4].

The proof is enabled by a pair of partial differential inequalities which are valid for any homogeneous (i.e., invariant under a transitive group of lattice symmetries) system of Ising spins. Similar relations are valid for percolation models, for which $h$ is an auxiliary parameter which requires some explanation, $\beta$ controls the bond density ($p_{x,y} = 1 - e^{-J_{x,y} \beta}$), and $M(\beta, h = 0+)$ is the percolation probability. These are:

$$\frac{\partial}{\partial \beta} M \leq |J| M \frac{\partial}{\partial h} M \quad \text{("Burgers inequality") (4.13)}$$

$$M \leq h \frac{\partial}{\partial h} M + |J| M^{n-2} \frac{\partial}{\partial \beta} M + M^{n-1} \quad \text{("$\phi^n$ bound") (4.14)}$$

with $n = 4$ for Ising ($\phi^4$) and $n = 3$ for percolation ($\phi^3$).

The observation that the combination of the two inequalities yields $\beta_T = \beta_H$, or $p_T = p_H$, was made first in the context of percolation models [4, 5]. The inequality (4.13) appeared earlier in the context of Ising systems, for which it follows from the GHS inequality. Newman [88] has noted that the relation resembles the Burgers equation for the evolution of the profile of a velocity field on $\mathbb{R}$, under the correspondence: $\{M, h, \beta\} \mapsto \{V, x, t\}$. The Burgers equation provides a simple example of evolution yielding shocks, which in our case correspond to first order phase transitions.

An interesting aspect of the relation (4.14) is that it contains hints of the $\phi^n$ structure of the relevant diagrams in the two models. The point, which is better understood by looking at the proof, also suggests other insights, including that the upper critical dimensions may be $d_c = 4$ for short range Ising models, and $d_c = 6$ for percolation, which is indeed the case, as is mentioned below.
It may be added that the equality \( p_T = p_H \) has a further implication in two dimensions, for which the model is self-dual under a transformation which exchanges the high and low temperature regimes. There, the knowledge that there is no intermediate phase permits one to determine the critical temperature, or density, as the unique self-dual point. Before the availability of the general result, the case of 2D percolation was addressed by Kesten [73] through other arguments.

4.10. **Mean Field Critical Exponent Bounds.** The differential inequalities presented above also permit one to conclude bounds on the critical exponents which are expected to be associated with the singular behavior in the vicinity of the critical point, as in:

\[
\chi = \partial_{\beta} M(\beta, h = 0) \approx |\beta - \beta_c|^{-\gamma} \\
M(\beta, h) \approx |h|^{-1/\delta} \\
M(\beta, h = 0) \approx |\beta - \beta_c|^\hat{\beta}.
\] (4.15)

The interest in such exponents is greatly enhanced by their universality. The latter is the statement of the independence of critical behavior from many of the local details of the model. The conceptual explanation of this experimentally observed fact was provided by the renormalization group picture. A particular implication is that the exponents measured in real phase transitions may agree exactly with those associated with the simpler mathematical models. Nevertheless, even seemingly simple models are not solvable in dimensions \( d > 2 \), and the mathematical expression of the renormalization group ideas is still an incomplete task. In this situation, even partial results are of value.

While the proof of strict power law behavior is still incomplete, partial differential inequalities allow one to conclude meaningful bounds. For instance, the inequality

\[
\frac{\partial \chi}{\partial \beta}|_{h=0} \leq |J| \chi^2
\] (4.16)

which for Ising systems follows from GHS, allows one to conclude that \( \gamma_- \geq 1 \), where the subscript indicates that the exponent pertains to the approach of \( \beta_c \) from below. The full argument requires that attention be paid to suitable cutoffs, as is explained in [8]. With somewhat more involved integration and interpolation [5] (see also [115, 37]), the inequalities (4.13) and (4.14) permit one to complete the list to:

\[
\hat{\beta} \leq (n-2)^{-1} \quad \gamma_- \geq 1 \quad \delta \geq n - 1
\] (4.17)

(with \( n \) as explained below (4.14)) A notable feature here is that the exponents are shown to be bounded by the values they assume in the corresponding mean-field models. The reader may find a detailed discussion, and a far more comprehensive list of references, in the monograph [37].

4.11. **Upper Critical Dimensions.** A high point of the random walk methods was reached when it was realized that they permit one to establish the existence of the phenomenon of the upper critical dimension. By that we mean the existence of \( d_c \) such that in dimensions \( d > d_c \) the critical exponents assume their mean-field values, and in particular the relations (4.17) hold as equalities [1] [32] [5] [37] (this is usually also true at \( d = d_c \) apart from logarithmic corrections [51]).
The value of the upper critical dimension depends on the model, and the interaction range. For short-range models:

\[
d_c = \begin{cases} 
4 & \text{weakly self repelling walks (presumably also strongly SAW) [24, 66]} \\
4 & \text{ferromagnetic systems of Ising spins, or } \phi^4 \text{ variables [11, 42]} \\
6 & \text{percolation, at least with spread enough connections [8, 16, 66]} \\
8 & \text{“lattice animals” [66]}
\end{cases}
\]

The phenomenon is driven by the fact that in high dimensions, loop effects are of diminishing significance. That is exemplified through the asymptotic vanishing, for \( d > 4 \), of the probability of mutual intersection for the paths of random walks whose end points are sampled randomly within a region of increasing diameter. The same turns out to be true for the paths with the weights \( \rho(\gamma, \gamma') \) in \( \phi^4 \).

For the proofs of the above statements, it has been very helpful to first establish that a sufficiency criterion for the mean-field behavior of the critical exponents is the finiteness at the critical point of a certain model dependent diagram. The relevant diagrams tend to be in the form

\[
D_k := \frac{G_{\beta}^*(k)}{(0,0)} = \sum_{u_1, \ldots, u_{k-1}} G_{\beta}(0, u_1) G_{\beta}(u_1, u_2) \cdots G_{\beta}(u_{k-1}, 0)
\]

\[
= \int_{[-\pi/2, \pi/2]^d} \hat{G}(p) \, dp
\]

(4.19)

where \( \hat{G}(\cdot) \) is the Fourier transform of \( G(o, \cdot) \). The diagrammatic condition is:

\[
\limsup_{\beta \to \beta_c} D_k < \infty
\]

(4.20)

with \( k = 2 \) for Ising/\( \phi^4 \) systems [113, 11, 52], \( k = 3 \) for percolation [8, 16, 20], and \( k = 4 \) for “lattice animals” [66] (which are mentioned here mainly for variety). The proof that the diagrammatic condition is satisfied under suitable conditions usually needs to be provided by other methods. That task can be accomplished through either the infrared bound (4.11)—in the few cases where it is applicable, or through the technique of the “lace expansion” which was initiated by Brydges and Spencer [24]. Through a number of developments, this method has grown into a robust and versatile tool [66].

4.12. Scaling Limits of Critical Models. Our brief tour now returns to the topic which was mentioned at the beginning of this account. A striking feature of the upper critical dimension is that for the Ising/\( \phi^4 \) systems, in the scaling (continuum) limit of the infinite system, the fluctuations of the local order parameter have the distribution of a gaussian field [11, 52, 51].

For an explicit statement, one may take the model as formulated on the lattice \((a\mathbb{Z})^d\) with periodic boundary conditions at distance \( L \), and consider the distribution of the block spin variables

\[
S_R = \frac{1}{N_a} \sum_{x \in [0, R] \cap (a\mathbb{Z})^d} \sigma_x, \quad (4.21)
\]

or more generally

\[
S(f) = \frac{1}{N_a} \sum_{x \in [0, 1] \cap (a\mathbb{Z})^d} f(x) \sigma_x \quad (4.22)
\]
with some positive, compactly supported, $F \in C_0(\mathbb{R}^d)$. The normalizing constant $N_a$ is to be determined through the second moment condition:

$$\langle S_1^2 \rangle_{L,a} = 1. \quad (4.23)$$

If the distributional limits for variables of the form $S(f)$ exist, one may define a random field $\Psi(\cdot)$, in the sense of a random distribution for which variables of the form $\int f(x)\Psi(x)dx$ have the limiting distribution of $S(f)$.

The question is then what are the limiting distributions of the variables $S_1$ and $S(f)$, in the continuum limit, for which $a \to 0$ and the temperature is either fixed at $T_c$ or taken to approach that value (at a rate for which the localization length does not vanish on the continuum scale). The results of [1, 42] state that in $d > 4$ dimensions, if $T$ is fixed at $T_c$, and the limit $L \to \infty$ taken first, then one gets the normal gaussian distribution:

$$D- \lim_{a \to 0} \lim_{L \to \infty} S_1 = \mathcal{N}(0,1) \quad (4.24)$$

with suitably adjusted normal law, $\mathcal{N}(0,b_f)$, for limits of $S(f)$. Related statements are available for the cases where $T$ is allowed to be adjusted with $a$ without leaving residual long-range order.

This phenomenon is nicely explained by the tendency, in high dimensions, of the random paths to stay out of each other’s way. The resulting factorization of the joint amplitude (4.7) reduces to Wick’s formula, i.e., gaussian structure for the correlation function. A more complete discussion of this observation, and its consequences, can be found in [1, 42, 37].

5. RELATED RECENT DEVELOPMENTS

There have recently been a number of developments which tie in with the topics discussed above. Of these, let me mention certain results which address a seeming contradiction concerning the critical behavior in high dimensions, $d > d_c$. From experience we know that some of it may appear surprising.

Once one is informed that above the upper critical dimension the critical exponents no longer vary with the dimension, two different paradigms suggest themselves for their calculation. One approach is to carry the calculation on the tree graph version of the model. Calculations on tree graphs are usually more approachable. As graphs, trees can be viewed as infinite dimensional, and it is natural to expect that a homogeneous graph with no loops provides a good approximation for a situation in which loop effects are no longer relevant. The other path towards the exponent calculation is to employ the mean field approximation. The two paradigms lead to common values for many of the exponents, including those listed in (4.17) (where the inequalities are then saturated). However, they also differ on some of the predictions. In particular, for Ising/$\phi^4$ spin systems there are drastic differences in the suggested behavior of the scaling limits. The following results shed some light on these differences.

The mean-field ferromagnetic interaction for a finite system is

$$H_{mf} = \frac{1}{2|\Lambda_{L,a}|} \sum_{x,y \in \Lambda_{L,a}} \sigma_x \sigma_y. \quad (5.1)$$

A calculation, which in effect was already mentioned—being at the basis of the Griffiths–Simon construction [42]—implies that at the critical point the limiting distribution of the full block spin variable $S_L$ is not gaussian. From this perspective there is room to wonder
about consistency with (4.24), which asserts normal limit for the distribution of the block spin variable when that is sampled from an infinite system, for \( d > 4 \).

In a joint work with Papathanakos (10, in progress) we prove that if one takes the local (non-mean field) Hamiltonian with the periodic boundary conditions, the scaling limit of the total block spin variables is also not gaussian. Furthermore, even the “local” block spin \( S_1 \) will have non-gaussian scaling limits, when \( L \) is taken to infinity while \( a \to 0 \), provided the increase of \( L \) is not too fast. (The limit is, nevertheless, gaussian if \( L \to \infty \) is taken first, as in (4.24).)

A familiar and convenient measure of the deviation from the gaussian distribution is provided by the renormalized coupling constant, which can be formulated with \( S \equiv S_1 \) as:

\[
g = \left| \frac{\langle S^4 \rangle - 3 \langle S^2 \rangle \langle S^2 \rangle}{\langle S^2 \rangle^2} \right|. \tag{5.2}
\]

Here \( \langle \ldots \rangle \equiv \langle \ldots \rangle_{L,a} \), and thus the renormalized coupling constant depends on \( \{L, a\} \). The new result is that for periodic boundary conditions and \( T = T_c \):

\[
\liminf_{a \to 0} g_{L,a} > 0 \tag{5.3}
\]

where the limit is taken at fixed \( L \), or also at \( L = L(a) \to \infty \) provided the increase is limited to a sufficiently low power of \( 1/a \).

The explanation of this phenomenon requires the discussion of the winding paths which appear in the random path expansion (4.7) under the periodic boundary conditions. (One can find related effects of the periodic boundary conditions in the rather simpler context of loop erased walks, for which interesting results were presented recently in [18, 92].)

A similar phenomenon was conjectured to occur also for percolation models, for which one may define an effective \( \phi^3 \) coupling constant along the lines which were explored in [8, 3]. In a recent work, Heydenreich and van der Hofstad [68] have proven, up to logarithmic terms, the percolation version of the conjecture which was formulated by this author concerning that case. The boundary conditions affect there both the effective coupling constant and the size of the maximal connected cluster in a large finite system at \( p = p_c \) (thus reconciling the results and predictions of [3, 20]). Although the technique is quite different, the coupling algorithm which was introduced in [68] has been of value for the above results for the Ising case (10).

**Epilogue**

In concluding, I hope that this brief article has served to remind Barry Simon of some good times of constructive and joyful jostling and camaraderie. I would like to thank my collaborators for the pleasure of learning through joint works, and to express my regret at being able to mention here only a small part of the many interesting works on the subject.

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