Two-particle pairing and phase separation in a two-dimensional Bose-gas with one or two sorts of bosons

M.Yu. Kagan and D.V. Efremov
Max-Planck-Institut für Physik Komplexer Systeme, Nöthnitzer Str. 38, 01187 Dresden, Germany
and Kapitza Institute for Physical Problems, Kosygin str. 2, 117334 Moscow, Russia

We present a phase diagram for a dilute two-dimensional Bose-gas on a lattice. For one sort of boson we consider a realistic case of the van der Waals interaction between particles with a strong hard-core repulsion $U$ and a van der Waals attractive tail $V$. For $V < 2t$, $t$ being a hopping amplitude, the phase diagram of the system contains regions of the usual one-particle Bose-Einstein condensation (BEC). However for $V > 2t$ we have total phase separation on a Mott-Hubbard Bose solid and a dilute Bose gas. For two sorts of structureless bosons described by the two band Hubbard model an $s$-wave pairing of the two bosons of different sort $\langle b_1 b_2 \rangle \neq 0$ is possible. The results we obtained should be important for different Bose systems, including submonolayers of $^4$He, excitons in semiconductors, Schwinger bosons in magnetic systems and holons in HTSC. In the HTSC case a possibility of two-holon pairing in the slave-bosons theories of superconductivity can restore a required charge $2e$ of a Cooper pair.

I. INTRODUCTION

It is well known that, in contrast with two-particle Cooper pairing in Fermi systems, the essence of a superfluidity in Bose systems is one particle Bose-Einstein condensation (BEC). This asymmetry between Fermi and Bose (two-particle versus one-particle condensation) was challenged in a pioneering paper by Valatin and Butler. They proposed a BCS-like variational function for the description of an attractive Bose gas. The most difficult problem with the validity of their description is connected with a tendency toward phase separation which arises in attractive Bose systems. Later on Nozieres and Saint-James conjectured that in a Bose system with a short-ranged, hard core repulsion and a van der Waals attractive tail, in principle, it is possible to create a two-particle bosonic bound state and to escape collapse. Unfortunately their calculations in a three-dimensional system showed that at least for one sort of structureless boson either standard one-particle BEC is more energetically beneficial, or that phase separation takes place earlier than the two-particle condensation. Note that the same result was obtained earlier by Jordanski for the case of weak van der Waals attraction.

The important development of the ideas of Nozieres and Saint James belongs to Rice and Wang. The authors claimed that in two-dimensional (where already an infinitely small attraction leads to the bound state in a symmetrical potential well) it is possible to realize a two-particle boson pairing. Moreover, this two-particle pairing results, for small momenta $p \xi_0 < 1$ in a linear, soundlike, dispersion law of quasiparticles at $T = 0$ in analogy with a standard one-particle Bose-condensation.

To escape a collapse in a 2D attractive Bose gas, the authors of Ref. introduced in their model a Hartee-Fock shift of the chemical potential $\mu_B \sim U n$, connected with the short-range repulsion $U$. This shift in the case of $U > 2V$, where $V$ is the magnitude of the van der Waals tail, leads to a positive compressibility in the system $\kappa^{-1} = d\mu_B / dn = U - 2V > 0$.

The main goal of a present paper is to construct a phase diagram of a 2D dilute Bose gas with the van der Waals interaction between particles, by taking into account on equal grounds the full contribution of a hard-core repulsion $U$ and a van der Waals tail $V$. Throughout the paper we will consider the lattice model, and will base our results on the exact solution of the two-particle $T$-matrix problem presented in Refs. and . We will study the possibility of different two-boson pairings, as well as the possibility of a total phase separation in the system. We will also consider the two sorts of structureless bosons described by the two-band Hubbard model. In the case of attraction between bosons of two different sorts, we will find a possibility of an $s$-wave two-boson pairing $\langle b_1 b_2 \rangle \neq 0$.

II. THEORETICAL MODEL

The model under consideration is described by the following Hamiltonian on the 2D square lattice:
\[
H = -t \sum_{ij} b_i^\dagger b_j + \frac{U}{2} \sum_i n_i^2 - \frac{V}{2} \sum_{ij} n_i n_j, \tag{1}
\]

where \(n_i = b_i^\dagger b_i\) is a 2D boson density. We will work in the limit of strong hard-core repulsion \(U \gg \{V; t\}\), and restrict ourselves mostly to a low-density limit \(n_p d^2 \ll 1\), \(d\) being the interatomic distance. Further on we put \(d = 1\).

Note that in the case of \(V = 0\) model \((3)\) is just the Bose-Hubbard model, extensively studied in the literature for the case of 2D \(^4\)He submonolayers, as well as for the flux lattices and Josephson arrays in the type-II superconductors (see Refs. 10—13).

We would like to emphasize that model \((3)\) is, to some extent, a Bose analog of the fermionic \(t - J\) model with released constraint, considered by Kagan and Rice. After the Fourier transformation from Eq. \((1)\) we obtain:

\[
H = \sum_p \epsilon_p b_p^\dagger b_p + \frac{U}{2} \sum_{k_1 k_2 q} b_{k_1}^\dagger b_{k_2}^\dagger b_{k_2 - q} b_{k_1 + q}
- \sum_{k_1 k_2} V(q) b_{k_1}^\dagger b_{k_1 + q} b_{k_2}^\dagger b_{k_2 - q}, \tag{2}
\]

where \(\epsilon_p = -2t(\cos p_x + \cos p_y)\) is a bosonic spectrum on the square lattice and \(V(q) = V(\cos(q_x) + \cos(q_y))\) is a Fourier transform of the Van der Waals tail. As a result, a total interaction in the momentum space is given by the formula:

\[
V_{eff}(q) = \frac{U}{2} - V(q). \tag{3}
\]

### III. THE \(T\)-MATRIX PROBLEM.

An instability toward a two-particle boson pairing manifests itself in the appearance of a pole at a temperature \(T = T_c\) in the solution of the Bethe-Salpeter equation for the two-particle vertex \(\Gamma\) for zero total momentum of the two bosons \((\mathbf{p}, -\mathbf{p})\). To proceed to the solution of this equation, we must solve at first the \(T\)-matrix problem for the two particles in vacuum. Here we can use the results of Ref. 10, because the solution of the two-particle problem does not depend upon statistics of colliding particles. For the \(T\)-matrix problem it is convenient to expand \(V_{eff}(q)\) in Eq. \((3)\) in series with the eigenfunctions of the irreducible representation of the lattice symmetry group \(D_4\). This yields

\[
\begin{align*}
V_{eff}(\text{extended } s\text{-wave}) &= \frac{U}{2} - \frac{V}{2} (\cos p_x + \cos p_y)(\cos p'_x + \cos p'_y), \\
V_{eff}(p\text{-wave}) &= -\frac{V}{2} (\sin p_x \sin p'_x + \sin p_y \sin p'_y), \\
V_{eff}(d_{x^2-y^2}) &= -\frac{V}{2} (\cos p_x - \cos p_y)(\cos p'_x - \cos p'_y),
\end{align*}
\tag{4}
\]

where \(q = p - p'\) is a transferred momentum. Note that, for spinless bosons, which we formally consider in Eq. \((1)\), the total spin of the Bose pair is zero. Hence only \(s\) - and a \(d\)-wave pairings are allowed by the symmetry of the pair \(\Psi\) function. A \(p\)-wave pairing is allowed only for an odd total spin of the two bosons. Nevertheless we will conserve the results for the \(p\)-wave pairing in our paper because the generalization of Eq.(1) for the case of bosons with internal degrees of freedom is straightforward. The \(T\)-matrix problems for \(p\)- and \(d\)-wave channels are very simple. Solutions of these problems for the two particles with a total momentum zero and a total energy \(E\) yield

\[
T_{d,p}(E) = -\frac{1}{2} \frac{V}{1 + \frac{1}{2} \frac{V}{2} I_{d,p}}, \tag{5}
\]

where

\[
I_{d,p} = \int_0^{2\pi} \int_0^{2\pi} \frac{|\phi_{d,p}|^2}{E + 4t(\cos p_x + \cos p_y)}, \tag{6}
\]

and the functions \(\phi_d = \cos p_x - \cos p_y\) and \(\phi_p = \sin p_x + i \sin p_y\) correspond, respectively, to the \(d\)- and \(p\)-wave channels.

Let us find the thresholds for the bound states in the \(d\)- and the \(p\)-wave channels. The appearance of a bound state means that \(E = -W - |\bar{E}|\), where \(W = 8t\) is a bandwidth. For the threshold \(\bar{E} = 0\). An exact solution of Eqs. \((4)\) and \((5)\), which involves the calculation of elliptic integrals of first and second order, yields
\[
\left( \frac{V_c}{4t} \right)_{p\text{-wave}} = \frac{1}{1 - \frac{2}{\pi}} \approx 2.8,
\]
\[
\left( \frac{V_c}{4t} \right)_{d\text{-wave}} = \frac{1}{\frac{4}{\pi} - 1} \approx 3.7.
\] (7)

Note that a threshold for a \( p \)-wave pairing is lower. Now let us proceed to an \( s \)-wave channel. Here an ordinary \( s \)-wave pairing is suppressed by large hard-core repulsion \( U \), however an extended \( s \)-wave pairing with a symmetry of the order parameter \( \Delta_s = \Delta_0 (\cos p_x + \cos p_y) \) is allowed. In real space this pairing corresponds to the particles on the neighboring sites. Moreover the pair \( \Psi \) function is zero in the region of a hard core \( r < r_0 \) and is centered (has a maximum) in the region of a van der Waals attraction for \( r \sim r_1 \) (see Fig. 1). On the lattice \( r_0 \sim d/2 \) and \( r_1 \sim d \).

One can see that the \( \Psi \) function has a region of zero value zero. But it has no nodes because it does not change its sign for all values of \( r \). The rigorous calculation of the threshold for an extended \( s \)-wave pairing gives\(^\text{4, 5}\)

\[
\left( \frac{V_c}{4t} \right)_{s\text{-wave}} = 1.
\] (8)

Moreover for \( V > V_c \) an energy of the bound state has the form

\[
|E_b| = |\tilde{E}| = 8W \exp \left\{ -\frac{\pi V}{(V - V_{\text{crit}})} \right\}.
\] (9)

Of course, in a strong coupling case \( V \gg W \)

\[
|E_b| \approx V.
\]

The \( T \) matrix in an \( s \)-wave channel for small and intermediate values of \( V \) is given by

\[
T_s(|\tilde{E}|) = \frac{W \left( 1 - \frac{V}{4t} \right)}{\frac{1}{\pi} \left( 1 - \frac{V}{4t} \right) \ln \frac{8W}{|E|} - \frac{V}{4t}}.
\] (10)

The most important is that a strong Hubbard repulsion \( U \) acts only as an excluded volume and effectively drops out from Eq. (10) at low energies. For \( V \ll 4t \) the \( T \) matrix

\[
T_s(|\tilde{E}|) \approx \frac{\pi W}{\ln \frac{8W}{|E|}}
\]

corresponds to repulsion and coincides with the \( T \) matrix for the 2D Bose-Hubbard model at low density. Of course, at very high energies the \( T \) matrix contains an additional pole \( E \approx U \) which corresponds to an antibound state in a total analogy with the fermionic Hubbard model. For \( V = 4t \): \( T_s(|\tilde{E}|) = 0 \) and there is no interaction at all. Finally, for \( V > 4t \), \( T_s(|\tilde{E}|) < 0 \) corresponds to an attraction and reflects the appearance of the bound state.

### IV. BETHE-SALPETER EQUATION FOR AN S-WAVE PAIRING OF THE TWO Bosons

Let us consider at first the most interesting case of \( V > 4t \) and find the critical temperature for an extended \( s \)-wave pairing of the two bosons. The solution of the Bethe-Salpeter equation for bosonic systems reads\(^\text{4, 7}\)

\[
\Gamma_s = \frac{T_s}{1 + T_s \int \int dp_x dp_y \coth \frac{\varepsilon_p - \mu}{2t} \frac{\varepsilon_p - \mu}{2(\varepsilon_p - \mu)}.
\] (11)

For a low density of bosons \( n_B \ll 1 \) one has

\[
\varepsilon_p \approx -4t + \frac{p^2}{2m},
\]
where \( m = 1/2t \) is a boson mass. Accordingly \( \mu = -4t + \tilde{\mu} \) and \( \xi_p = \varepsilon_p - \mu = p^2/2m - \tilde{\mu} \).

The most substantial difference of Eq. (11) from an analogous fermionic equation is the replacement of \( \tanh \xi/2T \) by \( \coth \xi/2T \) in its kernel. Moreover, as shown in Ref. [12] for the 2D Fermi gas \( \tilde{\mu} = \varepsilon_F - |E_b|/2 \). So, in a weak-coupling case, when \( \varepsilon_F \gg |E_b| \), the chemical potential \( \tilde{\mu} \approx \varepsilon_F > 0 \) is positive. In contrast to this we shall see below that a bosonic chemical potential \( \tilde{\mu} \) is always negative even in the weak-coupling case, when a binding energy is much smaller than a degeneracy temperature:

\[
|E_b| \ll T_0 = \frac{2\pi n}{m}.
\]

Another very important point is that the \( T \) matrix, which enters into the Bethe-Salpeter equation, must be calculated for a total energy \( \tilde{E} = 2\tilde{\mu} \) of colliding bosons. The chemical potential \( \tilde{\mu} \) can be determined from the requirement of the number of particle conservation. This requirement yields

\[
n_B = \int \int \frac{d^2 p}{(2\pi)^2} \frac{1}{\exp \left\{ p^2/2m - \tilde{\mu} T \right\} - 1}.
\]

From Eq.(12) for the temperatures \( |E_b| < T < T_0 < W \) we obtain

\[
\tilde{\mu} = -T \exp \left( -\frac{T_0}{T} \right) < 0.
\]

Note that a standard Hartree-Fock shift \( nU \) drops out from the expression for \( \xi_p = \varepsilon_p - \mu \) both in the Bethe-Salpeter equation (11) and in the equation for the number of particle conservation (12). Now we are ready to solve the Bethe-Salpeter equation (11). The critical temperature \( T_c \) corresponds to the pole in Eq. (11)

\[
1 + \frac{mT_s(2\tilde{\mu})}{2\pi I} = 0,
\]

where

\[
I = \int_0^{W/T_c} dy \frac{\coth \left( y + |\tilde{\mu}| \right) 2T_c}{y + |\tilde{\mu}| 2T_c},
\]

and

\[
y = \frac{p^2}{4mT_c}.
\]

An analysis of Eq.(15) shows that the main contribution to the integral comes from the lower limit of integration. Hence providing \( |\tilde{\mu}|/T_c \ll 1 \) we have

\[
I \approx \int_0^{W/T_c} dy \frac{dy}{\left( y + |\tilde{\mu}| \right)^2} \approx \frac{2T_c}{|\tilde{\mu}|}.
\]

As a result Eq. (14) can be represented in the following form:

\[
\frac{T_c}{|\tilde{\mu}|} = -\frac{\pi}{mT_s(2\tilde{\mu})}.
\]

It is useful now to represent \( T_s(2\tilde{\mu}) \) in terms of the binding energy \( E_b \). Utilizing Eqs. (9) and (10) we can write

\[
T_s(2\tilde{\mu}) = -\frac{\pi W}{\ln \left( \frac{2|\tilde{\mu}|}{|E_b|} \right)} = -\frac{4\pi}{m \ln \left( \frac{2|\tilde{\mu}|}{|E_b|} \right)}.
\]
It is important to mention here that $\tilde{\mu} < 0$, and hence the $T$ matrix in Eq. (18) does not contain an imaginary part. In the fermionic case $\tilde{\mu} = \varepsilon_F > 0$, and the $T$ matrix contains an imaginary part corresponding to the resonant scattering. As a result, from Eq. (18) we obtain:

$$4 \ln \left( \frac{2|\tilde{\mu}|}{E_b} \right) = \frac{|\tilde{\mu}|}{T_c}.$$  \hspace{1cm} (19)

Assuming that $|E_b| \ll T_c \ll T_0$, we obtain

$$\tilde{\mu}(T_c) = -T_c \exp \left( -\frac{T_0}{T_c} \right)$$

and

$$\frac{|\tilde{\mu}|}{T_c} = \exp \left\{ -\frac{T_0}{T_c} \right\}.$$  \hspace{1cm} (20)

Later on we will justify this assumption.

As a result from Eq. (19) we will obtain

$$T_c = \frac{T_0}{\ln \left( \frac{1}{4} \ln \left( \frac{2|\tilde{\mu}|}{|E_b|} \right) \right)}.$$  \hspace{1cm} (20)

Recall that in the case of the fermionic pairing in two dimensions a critical temperature reads

$$T_c = \sqrt{2\varepsilon_F |E_b|}.$$  \hspace{1cm} (20)

Let us analyze expression (20). As we already know

$$|E_b| = 8W \exp \left\{ -\frac{1}{\lambda} \right\},$$

where

$$\lambda = \frac{(V - V_{cs})}{\pi V}.$$  \hspace{1cm} (21)

Then a condition $|E_b| \ll T_0$ means

$$\lambda \ll \frac{1}{\ln \frac{T_0}{T_0}} \approx 1.$$  \hspace{1cm} (22)

Hence $\ln T_0/|E_b| = 1/\lambda - \ln W/T_0 \approx 1/\lambda$, and

$$T_c \approx \frac{T_0}{\ln \frac{1}{4} \ln \frac{T_0}{|E_b|}} \approx \frac{T_0}{\ln \frac{1}{\lambda}},$$  \hspace{1cm} (23)

which is in agreement with Ref. 4. Note that $T_c$ from Eq. (23) satisfies the conditions $|E_b| \ll T_c \ll T_0$, so an assumption used for the derivation of $T_c$ is justified.

For $T < T_c$ the spectrum of the quasiparticles acquires a gap:

$$E_p = \sqrt{(\varepsilon_p + |\tilde{\mu}|)^2 - \Delta^2}.$$  \hspace{1cm} (24)

Note that at low densities of bosons a gap $\Delta$ becomes isotropic in the principal approximation.

The gap $\Delta$ together with the chemical potential $\tilde{\mu}$ must be defined self-consistently from the two coupled equations

$$1 = \frac{\lambda}{4} \int_{|\tilde{\mu}|}^{\infty} dz \coth \sqrt{z^2 - \Delta^2} \frac{\sqrt{z^2 - \Delta^2}}{4z^2}.$$  \hspace{1cm} (25)
\[ n_B = \frac{\lambda}{4} \int_{|\tilde{\mu}|}^{W} \frac{1}{\exp\left(\frac{\sqrt{\xi^2 - \Delta^2}}{T}\right) - 1} \, d\xi, \quad (26) \]

where \( \xi = \varepsilon + |\tilde{\mu}| \) and \( z = \xi/2T \).

Of course, the solution of the system of equations Eqs. (25) and (26) exists only if \( |\tilde{\mu}| \geq \Delta \), or, in other words, only if \( E^2 - |\tilde{\mu}|^2 > 0 \). The exact solution of these equations yields for zero temperature in an agreement with

\[ |\tilde{\mu}(T = 0)| = \Delta = \frac{|E_b|}{2}. \quad (27) \]

This result is very important. It justifies our scenario, leading to a linear, soundlike spectrum of the quasiparticles for a small momenta \( p \). Indeed

\[ E_p = \sqrt{\varepsilon_p^2 + 2\varepsilon_p |\tilde{\mu}|} = \sqrt{\varepsilon_p^2 + \varepsilon_p |E_b|}. \quad (28) \]

From Eq. (28) for the case \( p\xi_0 \ll 1 \), where \( \xi_0 = 1/\sqrt{2m|E_b|} \) is the coherence length of the boson pair, we immediately obtain a linear dispersion law:

\[ E_p = cp. \quad (29) \]

In (29) \( c^2 = |E_b|/2m \) is a sound velocity squared. This means that an inverse compressibility of the system \( \kappa^{-1} = c^2 \) is positive. This fact proves the stability of a superfluid paired state and excludes the possibility of the collapse of the pairs in the system. Note also that close to \( T_c \) one has

\[ \Delta(T) \approx \Delta(0) \sqrt{\frac{T_c - T}{T_c}}, \quad (30) \]

which is similar to the BCS theory. We would like to mention that bosonic pairs in the limit \( |E_b| \ll T_0 \) are extended in full analogy with the BCS theory. That is the coherence length in this limit,

\[ \xi_0 \gg \frac{1}{\sqrt{n}} \gg 1, \quad (31) \]

is larger than the mean distance between the bosons. The Bose pairs are strongly overlapping in this limit. The pairing takes place in the momentum space in an analogy with the Cooper pairing in the BCS picture of superconductivity.

In the opposite limit \( |E_b| \gg T_0 \), the situation closely resembles the bipolaronic limit for the fermionic systems\(^{13,14}\). That is, the creation of the bosonic bound pairs is associated with the crossover temperature\(^{13}\)

\[ T^* = \frac{|E_b|}{\ln(1/n)}. \quad (32) \]

The Bose condensation of the pairs occurs at lower temperature\(^{13,14}\)

\[ T_c = \frac{T_0}{\ln(1/n)}. \quad (33) \]

Note that this temperature is obtained from the condition \( \mu_b(T_c) = -T_c \exp(-T_0/T_c) + f_0 T_c = 0 \), where \( f_0 = 1/\ln(1/n) \) is a repulsive interaction between the local pairs. So the superfluid transition takes place only for a residual repulsion between the pairs. Also note that in a dilute Bose gas the Berezinski-Kosterlitz-Thouless contribution of vortices\(^{13}\) is important only very close to \( T_c \), so the mean field expression \(^{(33)}\) gives a very good estimate for the exact BKT critical temperature:

\[ (T_c - T_{BKT})/T_c \sim 1/\ln(1/n) \ll 1. \]

In the case of the local pairs the coherence length is small:

\[ \xi_0 \ll \frac{1}{\sqrt{n}}. \quad (34) \]

The pairs are compact, and the pairing takes place in the real space.
V. POSSIBILITY OF $P$-WAVE AND $D$-WAVE PAIRING OF THE TWO BOSONS

Now let us analyze the solution of the Bethe-Salpeter equation for $p$- and $d$-wave boson pairings. Here the critical temperatures should be found from the conditions

$$1 + T_{p,d}(2\mu)\tilde{I}_{p,d} = 0,$$

(35)

where

$$\tilde{I}_{p,d} = \int \frac{d\mathbf{p}}{2\pi^2} \frac{d\mathbf{p}'}{2\pi^2} \coth \frac{\varepsilon - \varepsilon'}{2T} |\phi_{p,d}|^2.$$

(36)

In a low-density limit the $\phi$ functions can be approximated by the following expressions:

$$\phi_p \approx px + ipy = pe^{i\phi},$$

$$\phi_d \approx \frac{1}{2}(p_x^2 - p_y^2) = \frac{1}{2}p^2 \cos 2\phi.$$

Hence after an angular integration we obtain:

$$\tilde{I}_p = \frac{m}{2\pi} \int p dp \coth \frac{\xi}{2T} p^2,$$

$$\tilde{I}_d = \frac{m}{16\pi} \int p dp \coth \frac{\xi}{2T} p^4,$$

(37)

where again $\xi = p^2/2m + |\tilde{\mu}|$.

Additional factors $p^2$ and $p^4$ in the integral expressions for $\tilde{I}_p$ and $\tilde{I}_d$ reflect a well-known fact, that for slow 2D particles in vacuum an $s$-wave harmonics of the scattering amplitude behaves as $f_0 \sim \ln 1/p^2$, whereas for a magnetic number $m \neq 0$, the scattering amplitude vanishes for $p$ goes to zero as $f_m \sim p^{2m}$. The additional factor $p^4$ leads to the absence of an infra-red singularity for $\varepsilon \to 0$ in $\tilde{I}_d$:

$$\tilde{I}_d \sim \int \frac{d\varepsilon \varepsilon^2}{\varepsilon^2} \sim \varepsilon \to 0.$$

(38)

For the $p$-wave channel the infra-red singularity becomes logarithmically weak:

$$\tilde{I}_p \sim \int \frac{d\varepsilon \varepsilon}{\varepsilon^2} \sim \ln \varepsilon.$$

(39)

This means that the Bethe-Salpeter equation has no solutions in $p$- and $d$-wave channels for $|V|/t < 1$.

Hence the boson pairing with a large coherence length $\xi_0 > 1/\sqrt{n}$ is absent in a $p$-wave channel as well as in a $d$-wave channel. Here only the limit of the local pairs is possible. For $p$- and $d$-wave channels local pairs are created at the crossover temperatures given by Eq. (32). In this case the binding energies for $p$- and $d$-wave channels read, correspondingly,

$$|E_{p}^b| \approx W \frac{(V - V_{cp})}{V} \frac{1}{\ln (V - V_{cp})},$$

(40)

$$|E_{d}^b| \approx W \frac{(V - V_{cd})}{V}.$$

(41)

Note that $V_{cp}$ and $V_{cd}$ in Eqs. (40) and (41) are the thresholds for the $p$- and $d$-wave pairings given by Eq. (7). We would also like to mention that for a fixed $V$: $|E_{p}^b| > |E_{d}^b|$. Providing that the interaction between the local pairs is repulsive, the temperature of the Bose condensation of the local pairs in these two cases is again given by Eq. (33). Summarizing this case we see that, while the crossover temperatures $T^*$ are different for the $s$, $p$- and $d$-wave local pairs, their critical temperatures coincide with each other in a principal order.
VI. TOTAL PHASE SEPARATION

As we discussed in the Sec. 1, the real collapse is prohibited in our system by large Hubbard repulsion $U$. However the phase separation on the two large clusters is allowed. The first cluster corresponds to the Mott-Hubbard Bose solid. In this cluster $n_0 \to 1$, that is, each site is practically occupied by one boson. Such a cluster is localized due to Mott-Hubbard considerations. It has no kinetic energy. However it has a potential energy of the order of $-2V$ for one particle. A second cluster has a very small boson density $n \to 0$. In this cluster for $V < 4t$ the energy per particle is $-W/2 + f_0 n$, where $f_0 = m/4\pi \ln(1/n)$ is a two-particle $T$ matrix in the absence of a bound state. Rigorously speaking, at a given bosonic density $n$ the phase separation results in the formation of the two clusters with the densities $n_1 > n$ and $n_2 < n$, where $n_1$ is close to or identically equal to 1. The phase separation for $V < 4t$ takes place if the energy per particle in the cluster with the density $n_1$ becomes smaller than the energy per particle in the cluster with the density $n_2$

$$-2\eta V \leq -W/2,$$

(42)

where $\eta$ is an unknown numerical coefficient of the order of 1. Note that in the fermionic $t - J$ model, considered in Ref. 18, the Mott-Hubbard cluster with $n = 1$ has an antiferromagnetic order. Hence instead of $\eta V$ in (42) one should write $1.18J$ — the energy per spin of the 2D antiferromagnetic. As a result, in a fermionic case $J_{ps} = 3.8t$. In our system $V_{ps} \approx 2t$, due to the absence of kinetic energy and zero point energy in the case of structureless bosons. In the same time for $n \to 1$ the phase separation between the Bose solid and the one-particle BEC takes place. According to Ref. 19 for $n_B \to 1$, the phase separation takes place already for small values of $|V|/t$.

In principle, another scenario of the phase separation connected with the creation of quartets is also possible in our system. It requires an evaluation of the four-particle vertex which is impossible to do analytically. However we think that our scenario of total phase separation takes place for smaller values of $V/t$ than the quartet formation. This is in agreement with numerical calculations for the 2D fermionic $t - J$ model.

VII. PHASE DIAGRAM OF THE SYSTEM

In this section we will complete the phase diagram of the system. At first, note that for $V < V_{ps}$ (when the $T$ matrix for an $s$-wave channel is repulsive) we have at low density a standard Bose gas with a hard-core repulsion. It will be unstable toward a standard one-particle BEC at a critical temperature given again by Eq. (33). For $V > 2t$ a total phase separation on two large clusters takes place in our system. One of these clusters contains a Mott-Hubbard Bose solid, another one contains a Bose gas with one particle condensation.

For large densities $n = n_c \leq 1$ ($n_c \equiv 1$ in Ref. 18 for structureless bosons) the system will undergo a transition to the Mott-Hubbard Bose solid. As a result, on a qualitative level the phase diagram for our system has the form, presented in Fig. 2. Note that our phase diagram should be important for the understanding of the physics of the gas of kinks and steps on a solid interface of $^4$He. It will be interesting in this context to obtain phase separation in a system of kinks on a quantum atomically rough surface of $^4$He. Another possible application of our results can be connected with the study of biexcitonic pairing in semiconductors. In this context we must mention an interesting recent paper of Lozovik et al.

VIII. TWO-BAND HUBBARD MODEL FOR THE TWO SORTS OF BOSONS

Let us consider the two-band Hubbard model for the two sorts of structureless bosons. The Hamiltonian of the system has the form

$$H = -t_a \sum_{ij} a_i^\dagger a_j - t_b \sum_{ij} b_i^\dagger b_j + \frac{U_{aa}}{2} \sum_i n_{ia}^2 + \frac{U_{bb}}{2} \sum_i n_{ib}^2 - \frac{U_{ab}}{2} \sum_i n_{ia} n_{ib},$$

(43)

where $t_a$ and $t_b$ and $n_a$ and $n_b$ are, respectively, the hopping matrix elements and densities for bosons of sorts $a$ and $b$. For simplicity we will consider the case $t_a = t_b$, corresponding to the equal masses $m_a = 1/(2t_a) = m_b$. We also assume that the bottoms of the bands coincide. In the Hamiltonian $U_{aa}$ and $U_{bb}$ are Hubbard onsite repulsions for bosons of sorts $a$ and $b$. Finally $U_{ab}$ is an onsite attraction between bosons of two different sorts.
Let us consider the low-density limit, when both \( n_a \ll 1 \) and \( n_b \ll 1 \). In this limit we must replace the Hubbard interaction \( U_{ab} \) by the corresponding \( T \) matrix. The relevant expression for the \( T \) matrix \( T_{ab} \) is given by

\[
T_{ab}(\tilde{E}) = \frac{U_{ab}}{1 - \frac{U_{ab}}{2\pi^2 m}\frac{1}{\pi^2/|E|}},
\]

where \( \tilde{E} \) is given again by \( \tilde{E} = E + W \). The \( T \) matrix has the pole for the energy

\[
|E| = |E_b| = W \exp \left\{ \frac{-4\pi}{mU_{ab}} \right\}.
\]

In the extremely strong coupling case \( U_{ab} > W \) the pole corresponds to the energy \( |E| = |E_b| = U_{ab} \). The pole in the \( T \) matrix reflects the appearance of the bound state of the two bosons of different sorts.

Now we can solve the two-particle problem in the presence of the bosonic background. A simple analysis shows that only local bosonic pairs (bipolarons) are possible in our case. They are formed at a high crossover temperature \( T^* = |E_b|/\ln(W/T_0) \), where \( T_0 = \min\{T^a_0, T^b_0\} \), and \( T^a_0 \) and \( T^b_0 \) are degeneracy temperatures for bosons of the two sorts. Correspondingly the pairs are Bose condensed at lower temperature \( T^0_{ab} = T_0/\ln(1/\lambda) \), where \( \lambda = mU_{ab}/(4\pi) \).

Our results are valid in the case \( |E_b| > \{T^a_0, T^b_0\} \). In the opposite case of higher densities, when at least one of the temperatures \( T^a_0 \) or \( T^b_0 \) is larger then \( |E_b| \), we have at first a standard one particle condensation for bosons with higher density. As a result the two-particle pairing between bosons of different sort can take place only as a second superfluid transition inside the superfluid phase with one particle BEC. Let us now analyze the stability of our system with respect to collapse. For simplicity we consider an extremely strong coupling case \( \{U_{aa}, U_{bb}, U_{ab}\} > W \). In this case the local pairs of two bosons of different sorts have onsite character \( \langle a_i b_j \rangle \neq 0 \). To escape collapse in this case we must satisfy the following stability criteria: namely, we must prevent the collapse of \( \langle ab \rangle \) pairs on one site (a creation of the quartets \( \langle aabb \rangle \)). This requires the inequality \( U_{aa} + U_{bb} - 4U_{ab} > 0 \). These results can be important for slave-boson and Schwinger-boson theories.

**IX. SLAVE-BOSON FORMULATION OF THE T – J MODEL.**

**APPLICATION TO HTSC.**

We conclude our paper with a brief discussion of the problems which arise in a slave-boson formulation of the \( t – J \) model. In this formulation close to half-filling \( n \rightarrow 1 \) an electron is represented as a product of holon and spinon:

\[
c_i^{\dagger} = \bar{f}_i^{\dagger} b_i.
\]

A superconductive \( d \)-wave gap

\[
\Delta_d = \langle c_i^{\dagger} c_j \rangle - c_i^{\dagger} c_j \rangle
\]

is a direct product

\[
\Delta_d = \Delta_{sp} \Delta_{h}^{\dagger}
\]

of a spinon \( d \)-wave gap

\[
\Delta_{sp} = \langle f_i^{\dagger} f_j \rangle - f_i^{\dagger} f_j \rangle
\]

and a holon \( s \)-wave gap

\[
\Delta_h = \langle b_i b_j \rangle.
\]

Then a natural question arises whether \( \langle b_i \rangle \neq 0 \) and, accordingly, \( \Delta_h = \langle b_i \rangle \langle b_j \rangle \), or \( \langle b_i \rangle = 0 \) but \( \langle b_i b_j \rangle \neq 0 \). In other words, whether a one particle or two-particle condensations of the holons takes place in our system.

This problem is a very difficult one and, surely deserves a very careful analysis. Our preliminary considerations show, however, that the more beneficial conditions for the two-particle condensation may arise in the SU(2) formulation of the \( t – J \) model. In the standard U(1) formulation of the model an effective potential of the two-holon interaction on neighboring sites appearing after the Hubbard-Stratonovich transformation has a form.
\[
\left(\frac{8t^2}{J} - \frac{J}{4}\right) \sum_{ij} b_i^\dagger b_j^\dagger b_ib_j, \tag{46}
\]

and thus corresponds to the repulsion for \( t > J \). This observation excludes the possibility of the two-holon pairing in the U(1) formulation of the \( t-J \) model.

In the SU(2) case it will be desirable to derive conditions when \( \langle b_1 \rangle = \langle b_2 \rangle = 0 \) but \( \langle b_1 b_2 \rangle \neq 0 \). For such a nondiagonal pairing, as already discussed above it is easier to satisfy the stability criteria. Note also that the same situation with two sorts of bosons and a possible attraction between them can be realized for 2D magnetic systems. We obtain the corresponding bosonic Hamiltonian here after a Schwinger transformation of spins\(^{29,30}\) in extended Heisenberg models. Work along these lines is now in progress.

X. CONCLUSIONS

In conclusion we analyzed the possibility of the formation of boson pairs with \( s \)-wave symmetry and an appearance of total phase separation in a 2D Bose gas. In addition we considered the case of boson pairs with symmetry of \( p \)- and \( d \)-wave type. We also constructed the qualitative phase diagram for the 2D Bose gas with the van der Waals interaction between the particles, which, besides a standard one particle BEC, contains regions of the Mott-Hubbard Bose solid and a total phase separation. We also consider the situation for two sorts of bosons described by the two-band Hubbard model, and found the conditions for the two-particle pairing between bosons of different sorts. We discuss the applicability of our results for the different physical systems ranging from submonolayers of \(^4\)He and excitons in the semiconductors till Schwinger bosons in magnetic systems and holons in the slave-boson theories of HTSC.

The authors acknowledge helpful discussions with P. Fulde, O. Fischer, Yu. Kagan, N.M. Plakida, G. Khaliullin, S.H.S. Salk, M.A. Baranov, D. Ivanov, G. Jackeli, G.V. Shlyapnikov and V.I. Marchenko. This work was also supported by Russian Foundation for Basic Research (grant 02-02-17520), Russian President Program for Science Support (grant 96-15-9694) and by Grant of Russian Academy of Sciences for young scientists.

1. J.G. Valatin and D. Butler, Nuovo Cimento \textbf{10}, 37 (1958).
2. P. Nozieres and D. Saint James, J. Phys. (Paris) \textbf{43}, 1133 (1982) and references therein.
3. S.V. Jordanski, Zh. Eksp. Teor. Fiz. \textbf{47}, 167 (1964) [Sov. Phys. JETP, \textbf{20}, 112 (1965)].
4. M.J. Rice and Y.R. Wang, Phys. Rev. B \textbf{37}, 5893 (1988).
5. Y.J. Emery, S.A. Kivelson, and H.Q. Lin, Phys. Rev. Lett. \textbf{64}, 475 (1990).
6. M. Yu, Kagan and T.M. Rice, J. Phys.: Condens. Matt. \textbf{6}, 3771 (1994).
7. M.P.A. Fischer, P.B. Weichman, G. Grinstein, and D.S. Fisher, Phys. Rev. B \textbf{40}, 546 (1989); P.B. Weichman, M. Rasolt, M.E. Fisher, and M.J. Stephen, Phys. Rev. B \textbf{33}, 4632, 1986.
8. J.K. Frenzicks and H. Monien, Europhys. Lett. \textbf{26}, 545 (1994).
9. D.S. Rokhsar and B.G. Kotliar, Phys. Rev. B \textbf{44}, 10328 (1991).
10. A.P. Kampf and G.T. Zimanyi, Phys. Rev. B \textbf{47}, 279 (1993); K. Sheshardi et al, Europhys. Lett. \textbf{22}, 257 (1993).
11. H. Shi and A. Griffin, Phys. Rep. \textbf{304}, 1 (1998).
12. K. Miyake, Prog. Theor. Phys \textbf{69}, 1794 (1983).
13. A. Alexandrov, J. Ranninger, Phys. Rev. B \textbf{25}, 1796 (1981).
14. P. Nozieres, S. Schmitt-Rink, J. Low Temp. Phys. \textbf{59}, 195 (1985).
15. M. Yu, Kagan, R. Fresard, M. Capezzali, and H. Beck, Phys. Rev. B \textbf{57}, 5995 (1998), Physica B \textbf{284-288}, 447 (2000).
16. D.S. Fisher and P.C. Hohenberg, Phys. Rev. B \textbf{37}, 4936 (1988).
17. V.N. Popov, Theor. Math. Phys. \textbf{11}, 565 (1972).
18. J.M. Kosterlitz and D.J. Thouless, J. of Phys. C \textbf{6}, 1181 (1973); V.L. Berezinski, Zh. Eksp. Teor. Fiz. \textbf{61}, 610 (1972), [Sov. Phys. JETP \textbf{34}, 610 (1972)].
19. E. Dagotto and J. Riera, Phys. Rev. Lett. \textbf{70}, 682 (1993); E. Dagotto, J. Riera et al, Phys. Rev. B \textbf{49}, 3548 (1994), Appendix B.
20. G. Röpke, A. Schnell, P. Schuck, and P. Nozieres, Phys. Rev. Lett \textbf{80}, 3177 (1998).
21. L.V. Keldysh, Solid State Commun. \textbf{84}, 37 (1992), V.S. Babichenko, Zh. Eksp. Teor. Fiz. \textbf{64}, 612 (1973) [Sov. Phys. JETP \textbf{37}, 311 (1973)].
22. Yu.E. Lozovik, O.L. Berman, M. Villander, Zh. Eksp. Teor. Fiz. \textbf{115}, 1786 (1999) [Sov. Phys. JETP, \textbf{88}, 980 (1999)].
FIG. 1. The $\Psi$-function of an extended $s$-wave pairing. $r_0$ is a radius of a hard core repulsion, $r_1$ is a radius of a Van der Waals attraction. On the lattice $r_0 \sim d/2$ and $r_1 \sim d$. 

23 P.W. Anderson, Science 235, 1196 (1987).
24 X.G. Wen and P.A. Lee, Phys. Rev Lett. 80, 2193 (1998).
25 G.Kotliar and J. Liu, Phys. Rev. B 38, 5142 (1988).
26 P.A. Lee, N. Nagaosa, T.-K. Ng, X.G. Wen, Phys. Rev. B 57, 6003 (1998).
27 S.-S. Lee, S.H.S. Salk, cond-mat/0001218 (2000).
28 P.A. Lee, N. Nagaosa, Phys. Rev. B 46, 5621 (1992).
29 D.P. Arovas, A.Auerbach, Phys.Rev.B 38, 316 (1988); A. Chubukov, Phys.Rev.B 44, 12318 (1991).
30 V.Yu.Irkhin, A.A.Katanin, M.I.Katsnelson, Phys.Rev.B 60, 1082 (1999).
FIG. 2. Qualitative phase diagram of the 2D Bose gas with the Van der Waals interaction on the square lattice.