The holographic Weyl semi-metal

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We present a holographic model of a Weyl semi-metal. We show that upon varying a mass parameter the model undergoes a quantum phase transition from a topologically non-trivial state to a trivial one. The order parameter for this phase transition is the anomalous Hall effect (AHE). We give an interpretation of the results in terms of a holographic RG flow and compare to a weakly coupled field theoretical model. Since there are no quasiparticle excitations in the strongly coupled holographic model the topological phase can not be bound to notions of topology in momentum space.

Weyl semi-metals are an exciting new class of materials with exotic electronic transport properties (for reviews see [1, 2]). As a semi-metal their Fermi surface consists of isolated points. The electronic quasiparticle excitations around these points can be described by pairs of left- and right-handed Weyl spinors. The crucial property is that these singular points in the band structure of a crystal are separated by a spatial vector means that time reversal symmetry is broken. It is common to make the simplifying assumption that the Weyl cones are rotationally symmetric and that their opening angles are equal. Unfortunatly this assumption that the Weyl cones are rotationally symmetric is broken. It is common to make the simplifying assumption that the Weyl cones are rotationally symmetric and that their opening angles are equal. Unfortunatly this assumption is violated because of topological constraints there always have to appear pairs of left- and right-handed Weyl spinors [3, 4], see Fig. 1.

Figure 1: Locally around two band touching points separated by a vector \( b_{\text{eff}} \) the dispersion of the quasiparticles forms two Weyl cones of left- and right-handed chirality. Both nodes lie precisely at the Fermi energy \( \epsilon_F \).

The fact that the left- and right-handed Weyl nodes are separated by a spatial vector means that time reversal symmetry is broken. It is common to make the simplifying assumption that the Weyl cones are rotationally symmetric and that their opening angles are equal. Under these assumptions a quantum field theoretical model can be constructed. It takes the form of a “Lorentz breaking” Dirac system with Lagrangian [5]

\[
L = \bar{\Psi} \left( i\gamma^0 \partial^0 - e A^0 - \gamma_z \gamma^5 b_{\text{eff}} + M \right) \Psi .
\]  

At \( M = 0 \) the parameter \( b \) separates the left- and right-handed spinors by a distance of \( 2b \) in momentum space along the \( z \)-direction. On the other hand for \( b = 0 \) but \( M \neq 0 \) one is dealing with massive fermions. For arbitrary values of \( b \) and \( M \) the spectrum can be easily obtained and is sketched in Fig. 2. As long as \( |b| > |M| \) the spectrum is ungapped. It is characterized by band inversion and at the crossing points the wave function is well-described by Weyl fermions. The separation of the Weyl cones is given by \( 2\sqrt{b^2 - M^2} \). In this situation the quantum field theoretical model can be further reduced to an effective model with Lagrangian

\[
L = \bar{\psi} \left( i\gamma^0 \partial^0 - e A^0 - \gamma_z \gamma^5 b_{\text{eff}} \right) \psi
\]

with \( b_{\text{eff}} = \sqrt{b^2 - M^2} \).

If \( |b| < |M| \) then the system is gapped and the low energy description is simply one of a massive Dirac fermion

\[
L = \bar{\psi} \left( i\gamma^0 \partial^0 - e A^0 + \Delta \right) \psi,
\]

and \( \Delta = \sqrt{M^2 - b^2} \). Accordingly the system undergoes a quantum phase transition from the topologically non-trivial Weyl semi-metal phase to a trivial insulating phase.

Let us now give a quick derivation of the anomalous Hall effect using the effective field model (2). In this model the low energy axial symmetry is unbroken up to the axial anomaly. Therefore a redefinition of the low energy spinor \( \psi \rightarrow \exp(i\gamma_5 \theta)\psi \) will induce the anomaly term upon integrating over the redefined spinors. Choosing \( \theta = b_{\text{eff}} \cdot \mathbf{x} \) we find the anomaly term

\[
\Gamma_{\text{anom}} = e^2/(16\pi^2) \int d^4 x \left( b_{\text{eff}} \cdot \mathbf{x} \right) e^{i\nu \rho \lambda} F_{\nu \rho} F_{\rho \lambda}.
\]

The current can then be computed as variation with respect to

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the gauge field and we find the anomalous Hall effect [6–13]

\[ \mathbf{J} = \frac{e^2}{2\pi^2} \mathbf{b}_{\text{eff}} \times \mathbf{E}. \]  

(4)

On the other hand we can directly work with the “high energy” model (1). In this case the axial symmetry has both, an anomaly and a tree level breaking term

\[ \partial_\mu J_\mu^A = \frac{1}{16\pi^2} \epsilon^{\mu\nu\rho\lambda} F_{\nu\rho} F_{\mu\lambda} + 2M \bar{\Psi} \gamma_5 \Psi. \]  

(5)

The anomalous Hall effect can now be computed as a one-loop contribution to the polarization tensor. This calculation has a long history and is plagued by regularization ambiguities [14]. As we will see in the holographic model allows to resolve the ambiguities in a unique form.

The topological property in the Weyl semi-metal phase is intimately related to the fact that the wave function of a Weyl spinor can be understood as a monopole of the Berry curvature in momentum space, with the left-handed Weyl fermion having monopole charge $+1$ and the right-handed one having monopole charge $-1$ [3]. In an inherently strongly coupled system the status of the single particle wave function is not a priori clear. Moreover at strong coupling the concept of quasiparticle might not even be applicable. The question arises then if it is possible to construct a model at strong coupling that has the essential physical properties of a Weyl semi-metal. In particular does there exist a strongly coupled model in which the anomalous Hall effect and a quantum phase transition to a topological trivial phase persists even in the absence of the notion of singularities in the dispersion relations of fermionic quasiparticles? String theory inspired holography based on the AdS/CFT correspondence has arisen in the last few years as a singular and useful tool to address such questions. Holography has indeed already proved to be extremely useful for the understanding of transport properties of relativistic systems. In particular the modern understanding of anomaly related transport phenomena such as the chiral magnetic [15] and chiral vortical effects [16–19] is based to a considerable part on research using holographic models [27].

We consider the following holographic action which was studied in [22] to encode the axial charge dissipation effect in order to get a finite longitudinal DC magnetoconductivity [23]

\[
S = \int d^5x \sqrt{-g} \left[\frac{1}{2\kappa^2} (R + 12) - \frac{1}{4} F^2 - \frac{1}{4} F_5^2 + \frac{\alpha}{3} \epsilon^{\mu
u\rho\sigma\tau} A_\mu^5 F_{\nu\rho} F_{\sigma\tau} + 3F_{\nu\rho} F_{\sigma\tau} \right] - (D_\mu \Phi)^*(D^\mu \Phi) - m^2 \Phi^* \Phi .
\]  

(6)

As is well known, global symmetries correspond to gauge fields in AdS space. We will need two such gauge field, one representing the the electromagnetic $U(1)$ symmetry. Its AdS bulk gauge field is denoted by $V_\mu$ and its field strength is $F = dV$. The axial $U(1)$ symmetry is represented by the gauge field $A_\mu^5$ with field strength $F^5 = dA_5$. It is anomalous and the anomaly is represented in (6) by the Chern-Simons part of the action with coupling constant $\alpha$. Note that the choice of Chern-Simons term is the unique one that makes the electromagnetic symmetry non-anomalous [24]. The axial symmetry is also broken by the mass term. The mass deformation is introduced via a non-normalizable mode of a scalar field. This scalar field is charged only under the axial gauge transformation and its covariant derivative is $D_\mu \Phi = (\partial_\mu - i q A_\mu^5) \Phi$. The scalar bulk mass is chosen such that the dual operator has dimension three, i.e.

\[ m^2 L^2 = -3 \]  

where $L$ is the intrinsic length scale of AdS space [28]. The electromagnetic and axial currents are defined as

\[ J^\mu = \lim_{r \to \infty} \sqrt{-g} \left( F^{\mu\tau} + 4\alpha\epsilon^{\tau\mu\rho\sigma\tau} A_\rho^5 F_{\sigma\tau} \right) + \text{c.t.}, \]  

(7)

\[ J_5^\mu = \lim_{r \to \infty} \sqrt{-g} \left( F_5^{\mu\tau} + \frac{4\alpha}{3} r \epsilon^{\tau\mu\rho\sigma\tau} A_\rho^5 F_{\sigma\tau} \right) + \text{c.t.}. \]  

(8)

These are the consistent currents obtained by the variations of the on-shell boundary action with respect to the electromagnetic and axial boundary values of the field $V_\mu$ and $A_\mu^5$. It is also common to define covariant currents [25]. Their AdS definition is given by simply dropping the Chern-Simons parts in (7,8). The covariant currents are invariant even under the anomalous axial gauge transformations.

We will work in the following in the probe limit in which metric fluctuations are neglected. The metric background is fixed and taken as the AdS Schwarzschild solution

\[ ds^2 = -r^2 f(r) dt^2 + \frac{dr^2}{r^2 f(r)} + r^2 d\mathbf{x}^2 , \quad f(r) = 1 - \frac{r_h^4}{r^4}. \]  

(9)

Now we introduce the parameters $b$ and $M$ via boundary conditions on the fields $A_5^5$ and $\Phi = \phi(r)$

\[ b = \lim_{r \to \infty} A_5^5 (r) , \quad \phi (r) = \lim_{r \to \infty} r \phi (r) = M . \]  

(10)

The equations of motion for the background solution are

\[ (A_5^5)'' + \left( \frac{3}{r} + \frac{f'}{f} \right) (A_5^5)' - \frac{2q^2 \phi^2}{r^4 f} A_5^5 = 0 , \]  

(11)

\[ \phi'' + \left( \frac{5}{r} + \frac{f'}{f} \right) \phi' - \left( \frac{q^2 A_5^5}{r^4 f} + \frac{m^2}{r^2 f} \right) \phi = 0 . \]  

(12)

This system of differential equations can be solved by demanding that $A_5^5$ and $\phi$ are regular at the horizon $r = r_h$.

The Hall conductivity can be computed with the help of the Kubo formula

\[ \sigma_{xy} = \lim_\omega \frac{1}{i \omega} \langle J_x J_y \rangle_R . \]  

(13)

In holography the retarded correlation function of two currents can be computed by considering fluctuations
above the background. In particular we need fluctuations of the vector type bulk gauge field $V_\mu$ in $x$ and $y$ directions, i.e. $\delta V_x = v_x e^{-i\omega t}$, $\delta V_y = v_y e^{-i\omega t}$. The equations for the fluctuations are

$$v''_\pm + \left(\frac{3}{r} + \frac{f'}{f}\right) v'_\pm + \frac{\omega^2}{r^2 f^2} v_\pm \pm \frac{8\alpha \gamma}{r^3 f} (A_2^5) v_\pm = 0,$$

where $v_\pm = v_x \pm iv_y$.

In order to obtain the retarded correlator we impose infalling boundary conditions at the horizon. Furthermore we only need the leading behavior in an expansion around zero frequency. A convenient parametrization for the fluctuations is therefore

$$v_\pm = f^{-\frac{3\alpha}{2\pi}} (v^{(0)}_\pm + \omega v^{(1)}_\pm + \ldots).$$

To zeroth and first order in $\omega$

$$v^{(0)}''_\pm + \left(\frac{3}{r} + \frac{f'}{f}\right) v^{(0)}'_\pm = 0,$$

$$v^{(1)}''_\pm + \left(\frac{3}{r} + \frac{f'}{f}\right) v^{(1)}'_\pm = \left[\frac{i}{4r_h} \left(\frac{3f'}{f} + f''\right) \mp \frac{8\alpha}{r^3 f} (A_2^5)\right] v^{(0)}_\pm \pm \frac{i}{2r_h} f v^{(0)}_\pm.$$

We impose the regularity condition for $v^{(0)}_\pm$ and $v^{(1)}_\pm$ near horizon. From (16), we have $v^{(0)}_\pm = c_0$. From (17), we have

$$v^{(1)}_\pm = -\int_r^\infty dx \frac{c_0}{x^3 f} \left[\frac{i x^3 f'}{4r_h} - i r_h \mp 8\alpha (A_2^5 - A_2^5(r_h))\right].$$

Thus $G_\pm = \omega(\pm 8\alpha(b - A_2^5(r_h)) + ir_h)$, we have

$$\sigma_{xy} = \frac{G_+ - G_-}{2\omega} = 8\alpha(b - A_2^5(r_h)).$$

It is important to realize that this is the Hall conductivity in the covariant current. In order to obtain the total Hall conductivity we have to add the contribution from the Chern-Simons term [29],

$$\sigma_{\text{AHE}} = 8ab - \sigma_{xy} = 8\alpha A_2^5(r_h).$$

This is the main result.

We solved the equations numerically for different values of the the boundary parameters $M$ and $b$. In Fig. 3 we show the anomalous Hall response in the covariant current. Because of the underlying conformal symmetry we have fixed $b$ and vary $M/b$ and the temperature $T/b$. The plots show how the response of the covariant current builds up for high temperature (black curve) to low temperature (purple curve). At low temperature the response builds up quickly and saturates precisely at the value that cancels the Hall conductivity stemming from the Chern-Simons part of the current.

For low temperatures the change is rather drastic. This is consistent with the idea that the system undergoes a quantum phase transition from a topologically non-trivial to a trivial state. In Fig. 4 we have plotted the anomalous Hall conductivity in the total current at low temperature (we have chosen $\pi T/b = 0.125$). The total Hall conductivity drops off very quickly and basically vanishes at a critical value of the mass $M$. Due to the fact that we work in the probe limit we can not really reach the zero temperature limit. Accordingly we observe a smooth crossover instead of a sharp (quantum) phase transition. Nevertheless, already at $\pi T = 0.125 b$ the behavior of the anomalous Hall conductivity allows to estimate the critical value of the mass, which we find to be $M_c \approx 0.7b$. For comparison we also show the anomalous Hall conductivity of the simple free field model (1).

We note that the longitudinal conductivity is independent of $M$ and $b$ with $\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = \pi T$. A linear dependence on the temperature is natural for the ungapped topological phase. It represents the conductivity induced by the thermally activated pairs of charge fermions and anti-fermions (holes). In the topologically
trivial phase a linear dependence can be expected only in the case of small gap $\Delta/T < 1$. That we observe exact linear dependence even in the trivial phase has to attributed to the probe limit. Indeed it is expected that for large $\Delta/T$ the probe limit becomes unreliable. However, we do expect qualitatively similar behavior of the (quantum) phase transition in the backreacted case [26].

As we have argued the quantum phase transition in the quantum field theoretical model can be understood from the low energy perspective as a transition from the action (2) to (3). Both of them are special cases of (1) with the particular choices of parameters $M = 0, b = b_{\text{eff}}$ for the ungapped and $M = \Delta, b = 0$ for the gapped case. In fact our holographic model reproduces this low energy behavior with the limitations that arise by working at finite temperature. We remind the reader that the holographic direction has to be understood as an energy scale. The profile of the functions $A_5(z)$ and $\phi(r)$ represent therefore the running from high to low energies of the UV couplings $(M,b)$. We show typical profiles in Fig. 5.

In the topological phase ($b \gg M$) with significant Hall conductivity $A_5(z)$ stops running at some value $r > r_h$ and attains its IR value $A_5(z_{r_h})$. This value determines the Hall conductivity and therefore is the holographic analogue of $b_{\text{eff}}$. The scalar field first starts growing as one goes into the interior of AdS but turns around at a finite value of $r$ and becomes very small at the horizon. In the example of the plot for blue curve $\phi(r_h) \approx 10^{-3}$. It does not vanish exactly since a small thermally induced gap is naturally expected to exists for all values of $M,b$. Conversely, in the trivial phase the axial gauge field almost goes to zero at the horizon whereas the scalar field is monotonically increasing into the interior of AdS until it hits the black hole horizon.

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[1] P. Hosur and X. Qi, Comptes Rendus Physique 14, 857 (2013), [arXiv:1309.4464 [cond-mat.str-el]].
[2] O. Vafek and A. Vishwanat, Annual Review of Condensed Matter Physics, Vol. 5: 83-112, [arXiv:1305.2272 [cond-mat.str-el]].
[3] G. E. Volovik, The Universe in a Helium Droplet, (Oxford University Press, Oxford, 2003).
[4] H. B. Nielsen and M. Ninomiya, Phys. Lett. B 130 (1983) 389.
[5] D. Colladay and V. A. Kostelecky, Phys. Rev. D 58, 116002 (1998) [hep-ph/9809521].
[6] K. Y. Yang, Y. M. Lu and Y. Ran, Phys. Rev. B 84 (2011) 075129, [arXiv:1105.2353 [cond-mat.str-el]].
[7] G. Xu, H. Weng, Z. Wang, X. Dai and Z. Fang, Phys. Rev. Lett. 107 (2011) 186806, [arXiv:1106.3125 [cond-mat.mes-hall]]
[8] A. A. Burkov and L. Balents, Phys. Rev. Lett. 107, 127205, [arXiv:1105.5138 [cond-mat.mes-hall]].
[9] A. G. Grushin, Phys. Rev. D 86 (2012) 045001, [arXiv:1205.3722 [hep-th]].
[10] P. Goswami and S. Tewari, Phys. Rev. B 88 (2013) 245107, [arXiv:1210.6352 [cond-mat.mes-hall]].
[11] A. A. Zyuzin and A. A. Burkov, Phys. Rev. B 86 (2012) 115133, [arXiv:1206.1868 [cond-mat.mes-hall]].
[12] Y. Chen, S. Wu and A. A. Burkov, Phys. Rev. B 88 (2013) 125105 (2013), [arXiv:1306.5344 [cond-mat.mes-hall]].
[13] M. M. Vazifeh and M. Franz, Phys. Rev. Lett. 111, 027201 (2013), [arXiv:1303.5784 [cond-mat.mes-hall]].
[14] R. Jackiw, Int. J. Mod. Phys. B 14 (2000) 2011, [hep-
[15] K. Fukushima, D. E. Kharzeev and H. J. Warringa, Phys. Rev. D 78 (2008) 074033, [arXiv:0808.3382 [hep-ph]].
[16] J. Erdmenger, M. Haack, M. Kaminski and A. Yarom, JHEP 0901 (2009) 055 [arXiv:0809.2488 [hep-th]].
[17] N. Banerjee, J. Bhattacharya, S. Bhattacharyya, S. Dutta, R. Loganayagam and P. Surowka, JHEP 1101 (2011) 094, [arXiv:0809.2596 [hep-th]].
[18] D. T. Son and P. Surowka, Phys. Rev. Lett. 103 (2009) 191601, [arXiv:0906.5044 [hep-th]].
[19] K. Landsteiner, E. Megias, L. Melgar and F. Pena-Benitez, JHEP 1109 (2011) 121, [arXiv:1107.0368 [hep-th]].
[20] V. P. J. Jacobs, S. J. G. Vandoren and H. T. C. Stooft, Phys. Rev. B 90 (2014) 4, 045108, [arXiv:1403.3608 [cond-mat.str-el]].
[21] U. Gursoy, V. Jacobs, E. Plauschinn, H. Stoof and S. Vandoren, JHEP 1304 (2013) 127, [arXiv:1209.2593 [hep-th]].
[22] A. Jimenez-Alba, K. Landsteiner, Y. Liu and Y. W. Sun, arXiv:1504.06566 [hep-th].
[23] K. Landsteiner, Y. Liu and Y. W. Sun, JHEP 1503, 127 (2015), [arXiv:1410.6399 [hep-th]].
[24] A. Rebhan, A. Schmitt and S. A. Stricker, JHEP 1001 (2010) 026, [arXiv:0909.4782 [hep-th]]. A. Gynther, K. Landsteiner, F. Pena-Benitez and A. Rebhan, JHEP 1102 (2011) 110, [arXiv:1005.2587 [hep-th]].
[25] W. A. Bardeen and B. Zumino, Nucl. Phys. B 244 (1984) 421.
[26] K. Landsteiner, Y. Liu and Y. W. Sun, in preparation.
[27] Previous holographic treatments of Weyl semimetals [20, 21] differ from our approach in that they study the holographic fermionic spectral functions.
[28] In the following we will set $L = 1$ and also $q = 1$.
[29] We define the $\sigma_{\text{AHE}}$ via $J = \sigma_{\text{AHE}} e_b \times E$ where $e_b$ is the unit vector along the Weyl nodes separation direction $b$. Thus for free field model (1), we have $\sigma_{\text{AHE}} = \frac{e^2}{2\pi^2} b$. One may identify the anomaly constant $c = 8\alpha = \frac{e^2}{2\pi^2}$ [24].