Extremal equation for optimal completely-positive maps

Jaromír Fiurášek

Department of Optics, Palacký University, 17. listopadu 50, 77200 Olomouc, Czech Republic

We derive an extremal equation for optimal completely-positive map which most closely approximates a given transformation between pure quantum states. Moreover, we also obtain an upper bound on the maximal mean fidelity that can be attained by the optimal approximate transformation. The developed formalism is applied to universal-NOT gate, quantum cloning machines, quantum entanglers, and qubit $\phi$-shifter.

PACS number(s): 03.67.-a

I. INTRODUCTION

It is well known that certain transformations are forbidden in quantum theory. For example, it is impossible to exactly clone an unknown quantum state [1] or to prepare a qubit orthogonal to a given unknown qubit [1]. These transformations can be implemented only approximately [1–8]. A convenient measure of the quality of such an approximate transformation is the mean fidelity and the transformation is optimal if it reaches the maximum possible fidelity.

In quantum theory, the class of allowed transformations consists of trace-preserving completely-positive maps (CP-maps) $\mathcal{E}$ between input and output density matrices. In this paper we derive a generic extremal equation for the optimal CP-map which approximates a given desired transformation. This equation may easily be solved numerically by means of repeated iterations which yields the optimal CP-map. We also derive an upper bound on the maximum attainable fidelity. If we find a CP-map which saturates this upper bound then such a transformation is, by definition, the optimal one. We illustrate the application of our formalism on universal-NOT gate, quantum cloning machine, quantum entanglers, and quantum $\phi$-shifter.

We shall consider only the unconditional transformations which always provide an output. This should be contrasted with conditional transformations whose output is either accepted or rejected in dependence on the result of some measurement on the ancilla. Note that the conditional transformations can achieve higher fidelity than the unconditional ones, but only at the expense of possibly very low probability of success.

The paper is organized as follows. In Sec. II we introduce a useful parametrization of the CP-maps and derive the extremal equation for the optimal CP-map. We also obtain an upper bound on the maximum fidelity. In Sec. III we briefly describe how any CP-map can be implemented as a unitary transformation on larger Hilbert space. In Sec. IV we present examples of application of our method. Finally, Sec. V contains conclusions.

II. EXTREMAL EQUATION AND BOUND ON FIDELITY

Suppose we would like to implement a transformation between pure states $|\psi_{\text{in}}\rangle \in \mathcal{H}$ and $|\psi_{\text{out}}\rangle \in \mathcal{K}$

$$|\psi_{\text{in}}(\vec{x})\rangle \rightarrow |\psi_{\text{out}}(\vec{x})\rangle.$$  \hspace{1cm} (1)

Here $\vec{x}$ denotes a set of numbers parametrizing all pure states in Hilbert space of input states $\mathcal{H}$. Note that the dimensions of input and output Hilbert spaces may differ in general, $\dim\mathcal{H} \neq \dim\mathcal{K}$. It may happen that the desired transformation $|\psi_{\text{in}}\rangle \rightarrow |\psi_{\text{out}}\rangle$ cannot be implemented exactly because $|\psi_{\text{in}}\rangle$ is not a linear CP-map and we thus want to find a CP-map which most closely approximates the transformation $|\psi_{\text{in}}\rangle \rightarrow |\psi_{\text{out}}\rangle$.

Let us begin by introducing a convenient representation of the CP-maps. We say that map $\hat{\rho} \rightarrow \mathcal{E}(\hat{\rho})$ is positive if it preserves the positivity of operators $\hat{\rho}$. The map $\mathcal{E}$ is completely positive if and only if the extension $\mathcal{E}_{\mathcal{H}} \otimes \mathcal{I}_\mathcal{K}$ is a positive map for any Hilbert space $\mathcal{H}'$, where $\mathcal{I}$ is an identity map. Any CP-map can be represented by a positive operator $\hat{\chi}$. Consider the maximally entangled state on $\mathcal{H} \otimes \mathcal{K}$,

$$|\varphi\rangle = \sum_{j=1}^{\dim\mathcal{H}} |j\rangle_1 |j\rangle_2$$  \hspace{1cm} (2)

and define the operator

$$\hat{\chi} = \mathcal{E}_{\mathcal{H}} \otimes \mathcal{I}_\mathcal{K}(|\varphi\rangle \langle \varphi|).$$  \hspace{1cm} (3)

Note that $\hat{\chi}$ acts on a Hilbert space $\mathcal{H} \otimes \mathcal{K}$. It is easy to show that the CP-map $\hat{\rho}_{\text{out}} = \mathcal{E}(\hat{\rho}_{\text{in}})$ can be written as

$$\hat{\rho}_{\text{out}} = \text{Tr}[\hat{\chi} \hat{\rho}_{\text{in}}^T \otimes \hat{1}_\mathcal{K}],$$  \hspace{1cm} (4)

where $T$ stands for the transposition and $\hat{1}_\mathcal{K}$ denotes the identity operator on $\mathcal{K}$. The requirement that the CP-map $\mathcal{E}$ should preserve the trace imposes the following constraint on $\hat{\chi}$,

$$\text{Tr}_\mathcal{K}[\hat{\chi}] = \hat{1}_\mathcal{H}. \hspace{1cm} (5)$$

We would like to quantify how well the CP map $\hat{\chi}$ approximates the desired transformation $|\psi_{\text{in}}\rangle \rightarrow |\psi_{\text{out}}\rangle$. To this end we define the mean fidelity as
\[ F = \int d\vec{x} (|\psi_{\text{out}}(\vec{x})\rangle\langle \psi_{\text{in}}(\vec{x})| \mathcal{E} (|\psi_{\text{in}}(\vec{x})\rangle\langle \psi_{\text{in}}(\vec{x})|) |\psi_{\text{out}}(\vec{x})\rangle), \]  

where \( d\vec{x} \) denotes the proper measure on space of pure states \( |\psi_{\text{in}}(\vec{x})\rangle \).

With the help of Eq. (4) we may rewrite the expression (3) as

\[ F = \text{Tr}[\hat{R}], \]

where the positive operator \( \hat{R} \) acting on \( \mathcal{H} \otimes \mathcal{K} \) is given by

\[ \hat{R} = \int d\vec{x} \left(|\psi_{\text{in}}(\vec{x})\rangle\langle \psi_{\text{in}}(\vec{x})| \right)^T \otimes |\psi_{\text{out}}(\vec{x})\rangle\langle \psi_{\text{out}}(\vec{x})|. \]

Our task is to find a trace-preserving CP-map \( \hat{\chi} \) which maximizes the fidelity (7). We take into account the constraint (5) by introducing an operator Lagrange multiplier \( \hat{\Lambda} = \hat{\lambda} \otimes 1_{\mathcal{K}} \) and we look for the maximum of the functional

\[ \hat{F}[\hat{\chi}] = \text{Tr}[\hat{\chi}\hat{R}] - \text{Tr}[\hat{\chi}\hat{\Lambda}]. \]

We expand \( \hat{\chi} \) in eigenstate basis

\[ \hat{\chi} = \sum_j r_j |\pi_j\rangle \langle \pi_j|, \]

and rewrite the functional (9) as

\[ \hat{F}[\hat{\chi}] = \sum_j r_j \langle \pi_j | \hat{R} - \hat{\Lambda} |\pi_j\rangle. \]

A variation of \( \hat{F}[\hat{\chi}] \) with respect to \( \langle \pi_j | \) yields the extremal equations,

\[ (\hat{R} - \hat{\Lambda}) r_j |\pi_j\rangle = 0. \]

We note that recently a similar equation has been derived for elements of positive operator valued measure representing an optimal quantum measurement [10].

We multiply Eq. (11) by \( \langle \pi_j | \) and sum over \( j \). After some manipulations we obtain

\[ \hat{\chi} = \hat{\Lambda}^{-1} \hat{R}\hat{\chi}. \]

Further we take Hermitian conjugate of this formula, \( \hat{\chi} = \hat{\chi}^{\dagger} \hat{R} \hat{\Lambda}^{-1} \), and insert it back to the right-hand side of Eq. (12). Thus we arrive at a symmetrized extremal equation

\[ \hat{\chi} = \hat{\Lambda}^{-1} \hat{R}\hat{\chi}^{\dagger} \hat{R} \hat{\Lambda}^{-1}. \]

The Lagrange multiplier \( \hat{\Lambda} = \hat{\lambda} \otimes 1_{\mathcal{K}} \) can be determined from the constraint (5) which provides expression for \( \hat{\lambda} \),

\[ \hat{\lambda} = \left( \text{Tr}_{\mathcal{K}}[\hat{R}\hat{\chi}^{\dagger} \hat{R}] \right)^{1/2}. \]

We fix the square root by postulating that \( \hat{\lambda} \) is positive Hermitian operator. The system of coupled nonlinear extremal equations (13) and (14) can be conveniently solved by means of repeated iterations, starting from some initial ‘unbiased’ CP-map, for example a map which transforms every input density matrix to the maximally mixed state on \( \mathcal{K} \), \( \hat{\rho}_{\text{in}} \rightarrow 1_{\mathcal{K}} / \text{dim} \mathcal{K} \). Note that the iterations preserve the positivity of \( \hat{\chi} \) and the constraint (5) is exactly satisfied at each iteration step.

From the formula (15) we can obtain an upper bound on the optimal fidelity \( F \),

\[ F \leq \text{Tr}[\hat{\chi}] \rho_{\max} = \text{dim} \mathcal{H} \text{dim} \mathcal{K}, \]

where \( \rho_{\max} \) is the largest eigenvalue of the operator \( \hat{R} \). If we find a CP-map which reaches the upper bound on fidelity (15) then such transformation is, by definition, optimal one. We shall use this theorem below to prove the optimality of quantum cloning machine and universal NOT gate.

III. IMPLEMENTATION OF CP-MAPS

Before turning to explicit examples of application we should briefly comment on the possibility of experimental realization of the CP-map \( \hat{\chi} \). It holds that any trace-preserving CP-map can be accomplished as a unitary evolution on an extended Hilbert space. Any CP-map can be written in the form of Kraus decomposition [12]

\[ \hat{\rho}_{\text{out}} = \sum_l \hat{A}_l \hat{\rho}_{\text{in}} \hat{A}_l^\dagger \]

and the trace-preservation condition gives

\[ \sum_l \hat{A}_l^\dagger \hat{A}_l = \hat{1}_\mathcal{H}. \]

Rewritten in terms of the matrix elements \( A^{(l)}_{ki} \equiv \langle k|\hat{A}_l|i\rangle \) this constraint reads

\[ \sum_{k,l} A^{(l)}_{ki} A^{(l)}_{kj} = \delta_{ij}. \]

The number of necessary operators \( \hat{A}_l \) is equal to the number of nonzero eigenvalues of matrix \( \hat{\chi} \) which we denote by \( C \). In fact, the operators \( \hat{A}_l \) may be associated with the eigenstates \( |\pi_i\rangle \) of the operator \( \hat{\chi} \),

\[ A^{(l)}_{ki} = \sqrt{\delta_{ki}} \langle \pi_i | \hat{\chi} |\pi_l\rangle, \]

where \( |i\rangle \in \mathcal{H} \) and \( |k\rangle \in \mathcal{K} \) are states in input and output Hilbert spaces, respectively.

In order to implement \( \hat{\chi} \) as a unitary transformation, we must work in Hilbert space with dimension \( D = C \text{dim} \mathcal{K} \), where we define the transformation

\[ |i\rangle |0\rangle \rightarrow \sum_{k,l} A^{(l)}_{ki} |k\rangle |l\rangle. \]

Here \( |0\rangle \) denotes an initial state of the ancilla and \( |l\rangle \) are orthogonal output states of the ancilla. With the help of the formula (18) it is easy to check that the transformation (20) is indeed unitary.
Having established the general formalism, we may proceed to explicit examples. In all the examples, \( \mathcal{H} \) is a Hilbert space of single qubit, all pure states in \( \mathcal{H} \) can be visualized as points on the surface of Bloch sphere parametrized by angles \( \theta \) and \( \phi \),

\[
|\psi(\theta, \phi)\rangle = \cos \frac{\theta}{2} |0\rangle + e^{i\phi} \sin \frac{\theta}{2} |1\rangle,
\]
and the proper integral measure is

\[
\int d\vec{x} \equiv \frac{1}{4\pi} \int_0^{2\pi} \int_0^{\pi} \sin \theta d\theta d\phi.
\]

**IV. APPLICATIONS**

After a straightforward integration we arrive at

\[
|\psi(\theta, \phi)\rangle = \cos \frac{\theta}{2} |0\rangle + e^{i\phi} \sin \frac{\theta}{2} |1\rangle,
\]
and the proper integral measure is

\[
\int d\vec{x} \equiv \frac{1}{4\pi} \int_0^{2\pi} \int_0^{\pi} \sin \theta d\theta d\phi.
\]

**A. Universal NOT gate**

Suppose that we have \( N \) copies of an unknown qubit \( |\psi\rangle \) and we would like to prepare an inverted qubit \( |\psi\rangle \),

\[
|\psi\rangle = 0, \quad \langle \psi| = 0
\]

and the proper integral measure is

\[
\int d\vec{x} \equiv \frac{1}{4\pi} \int_0^{2\pi} \int_0^{\pi} \sin \theta d\theta d\phi.
\]

The matrix \( \hat{R}_{\text{NOT}} \) is block diagonal and the calculation of the eigenvalues of \( \hat{R}_{\text{NOT}} \) boils down to evaluation of eigenvalues of \( 2 \times 2 \) matrices. The largest eigenvalue reads \( R_{\text{max}} = 1/(N+2) \) and it is \( N+2 \)-fold degenerate. Taking into account that \( \dim \mathcal{H}^{\otimes N} = N+1 \) we get immediately from Eq. 13 the upper bound on the fidelity of U-NOT gate,

\[
F_{\text{NOT}} \leq \frac{N+1}{N+2}.
\]

Since the fidelity of U-NOT gate proposed by Bužek et al. reaches the bound \( (N+1)/(N+2) \) it is optimal. Note also that our determination of the upper bound on fidelity of the U-NOT gate is conceptually similar to the derivation of the fidelity of the optimal quantum measurement

\[
\hat{R}_{\text{NOT}} = \frac{1}{3}(|0\rangle\langle 0| + |10\rangle\langle 10| + |\Phi_-\rangle\langle \Phi_-|),
\]

where

\[
|\Phi_-\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle).
\]

We calculate the operator \( \hat{R}_{\text{NOT}} \) as

\[
\hat{R}_{\text{NOT}} = \frac{1}{4\pi} \int_0^{2\pi} d\phi \int_0^{\pi} \sin \theta d\theta \int_0^{2\pi} d\phi |\langle \psi| \psi\rangle| \cos \frac{\theta}{2} |0\rangle \langle 0| \cos \frac{\phi}{2} |1\rangle \langle 1|.
\]

We calculate the operator \( \hat{R}_{\text{NOT}} \) as

\[
\hat{R}_{\text{NOT}} = \frac{1}{4\pi} \int_0^{2\pi} d\phi \int_0^{\pi} \sin \theta d\theta \int_0^{2\pi} d\phi |\langle \psi| \psi\rangle| \cos \frac{\theta}{2} |0\rangle \langle 0| \cos \frac{\phi}{2} |1\rangle \langle 1|.
\]

After a straightforward integration we arrive at

\[
\hat{R}_{\text{NOT}} = \sum_{k=0}^{N} \frac{N-k+1}{(N+1)(N+2)} |N,k\rangle \langle N,k| \otimes |0\rangle \langle 0|
\]

\[
+ \sum_{k=0}^{N} \frac{k+1}{(N+1)(N+2)} |N,k\rangle \langle N,k| \otimes |1\rangle \langle 1|
\]

\[
- \sum_{k=1}^{N} \sqrt{k(N-k+1)} \frac{1}{(N+1)(N+2)} |N,k\rangle \langle N,k-1| \otimes |0\rangle \langle 1|
\]

\[
- \sum_{k=1}^{N} \sqrt{k(N-k+1)} \frac{1}{(N+1)(N+2)} |N,k-1\rangle \langle N,k| \otimes |1\rangle \langle 0|.
\]
\[
\begin{pmatrix}
\rho_{00} & \rho_{01} \\
\rho_{10} & \rho_{11}
\end{pmatrix} - \frac{1}{3} \begin{pmatrix}
2\rho_{11} + \rho_{00} & -\rho_{01} \\
-\rho_{10} & 2\rho_{00} + \rho_{11}
\end{pmatrix}.
\]

(31)

B. Quantum cloning machine

An ideal \(1 \rightarrow N\) cloning machine would prepare \(N\) exact clones of an unknown qubit \(|\psi\rangle\),

\[|\psi\rangle \rightarrow |\tilde{\psi}\rangle.\]

(32)

We restrict ourselves to the symmetric quantum cloners which produce \(N\) identical approximate copies. In this case the output Hilbert state is the bosonic space \(\mathcal{H}^{\otimes N}\) and the ideal output state \(|\tilde{\psi}\rangle\) is given by Eq. (26). The construction of the operator \(\hat{R}_c\) is straightforward,

\[\hat{R}_c = \frac{1}{4\pi} \int_0^\pi \sin \vartheta d\vartheta \int_0^{2\pi} d\phi \langle |\psi\rangle|\psi\rangle^T \otimes |\tilde{\psi}\rangle\langle \tilde{\psi}|,
\]

which yields

\[
\hat{R}_c = \sum_{k=0}^N \frac{k+1}{(N+1)(N+2)} |0\rangle\langle 0| \otimes |N,k\rangle\langle N,k| \\
+ \sum_{k=0}^N \frac{N-k+1}{(N+1)(N+2)} |1\rangle\langle 1| \otimes |N,k\rangle\langle N,k| \\
+ \sum_{k=1}^N \sqrt{k(N-k+1)} |0\rangle\langle 1| \otimes |N,k\rangle\langle N,k-1| \\
+ \sum_{k=1}^N \sqrt{k(N-k+1)} |1\rangle\langle 0| \otimes |N,k-1\rangle\langle N,k|.
\]

(33)

The largest eigenvalue of the block-diagonal matrix \(\hat{R}_c\) reads \(1/(N+1)\). Consequently, we have the constraint on maximum cloning fidelity,

\[F_c \leq \frac{2}{N+1}.
\]

The symmetric \(1 \rightarrow N\) cloning machines proposed by Bužek and Hillery and Gisin and Massar saturate the bound \(F_c = 2/(N+1)\) and are thus optimal, as shown earlier in Refs. [4,5].

For \(N = 2\) we solved numerically the extremal Eqs. (13) and (14) by means of repeated iterations and we obtained a CP-map which describes the optimal universal quantum cloner designed by Bužek and Hillery. The convergence was very fast and we obtained the desired CP-map with 10-digit precision after several tens of iterations. Since the optimal CP-map is unique the numerical solution does not depend on the choice of initial CP-map \(\chi_0\), provided that all eigenvalues of \(\chi_0\) are positive.

C. Quantum entangler

Recently, Bužek and Hillery investigated the possibility of entangling an unknown qubit \(|\psi\rangle\) with a qubit in known state \(|0\rangle\). They showed that the fidelity of this transformation

\[|\psi\rangle \rightarrow \mathcal{N}(|\psi\rangle|0\rangle + |0\rangle|\psi\rangle) = |\psi_{\text{out}}\rangle,
\]

(34)
cannot be carried out exactly. The ideal output state \(|\psi_{\text{out}}\rangle\) can be expressed as

\[|\psi_{\text{out}}\rangle = \frac{\sqrt{2} e^{i\phi} \sin(\vartheta/2) |\psi⟩ + e^{i\vartheta} |\psi⟩ }{\sqrt{1 + \cos^2(\vartheta/2)}},\]

(35)

and

\[|\Psi_{+}\rangle = \frac{1}{\sqrt{2}}(|01⟩ + |10⟩).
\]

(36)

Bužek and Hillery constructed a universal entangler whose state-independent fidelity is \(F \approx 0.946\). Here we show that this entangler is not optimal and we find an entangling machine which achieves a higher fidelity.

We proceed along the same lines as in the two previous examples and determine the operator \(R_{\text{ent}}\). After a straightforward integration we find,

\[
\hat{R}_{\text{ent}} = (2\ln 2 - 1)|0\rangle\langle 0| \otimes |00\rangle\langle 00| \\
+ (3 - 4\ln 2)|1\rangle\langle 1| \otimes |00\rangle\langle 00| \\
+ (3/2 - 2\ln 2)|0\rangle\langle 0| \otimes |\Psi_{+}\rangle\langle \Psi_{+}| \\
+ (4\ln 2 - 5/2)|1\rangle\langle 1| \otimes |\Psi_{+}\rangle\langle \Psi_{+}| \\
+ \sqrt{2}(3/2 - 2\ln 2)|0\rangle\langle 1| \otimes |00\rangle\langle \Psi_{+}| \\
+ \sqrt{2}(3/2 - 2\ln 2)|1\rangle\langle 0| \otimes |\Psi_{+}\rangle\langle 00|.
\]

(37)

Since the largest eigenvalue of \(\hat{R}_{\text{ent}}\) is 1/2 we have from (33) that \(F_{\text{ent}} \leq 1\). Although the upper bound 1 cannot be reached we can get closer than in Ref. [14]. When we numerically solve the extremal equations for optimal CP-map, we find that the optimal map is a unitary transformation on the space of two qubits, where the second qubit is initially prepared in the state \(|0\rangle\),

\[|00\rangle \rightarrow |00\rangle, \quad |10\rangle \rightarrow |\Psi_{+}\rangle.
\]

(38)

For an arbitrary input qubit \(|\psi\rangle\) we thus have

\[|\psi\rangle|0\rangle \rightarrow \cos \frac{\vartheta}{2} |00\rangle + e^{i\vartheta} \sin \frac{\vartheta}{2} |\Psi_{+}\rangle.
\]

(39)

The fidelity of this transformation depends on \(\vartheta\),

\[F_{\text{ent}}(\vartheta) = \frac{\sqrt{2} \cos^2(\vartheta/2) + \sin^2(\vartheta/2)}{1 + \cos^2(\vartheta/2)}.
\]

(40)

We plot the fidelity \(F_{\text{ent}}(\vartheta)\) in Fig. 1. The mean fidelity reads
which is clearly higher than the fidelity 0.946 obtained in [14]. Since $F_{\text{ent}}(\theta)$ is state dependent, the entangler is not universal. Nevertheless, for any state $|\psi\rangle$, the fidelity of the transformation (33) is higher than the fidelity of the universal entangler proposed in [14] and the minimum of the fidelity (43) is

$$F_{\text{min}} = 4\sqrt{2}(\sqrt{2} - 1)^2 \approx 0.9706.$$  \hspace{1cm} (42)

It is easy to see why the non-universal entangler outperforms the universal one. Since we a-priori know the second state $|0\rangle$, this information breaks down the symmetry, and there is no reason to expect that the universal entangler should be the best one. Indeed, there exists a preferred basis spanned by $|0\rangle$ and its orthogonal counterpart $|1\rangle$. Note that the transformation (33) realizes ideally the desired transformation (34) for the basis states $|0\rangle$ and $|1\rangle$.

Let us also study another type of quantum entangler considered in [14]. Suppose we have an unknown qubit $|\psi\rangle$ and we would like to prepare a maximally entangled state

$$|\psi\rangle \rightarrow \frac{1}{\sqrt{2}}(|\psi\rangle|\psi_\perp\rangle + |\psi_\perp\rangle|\psi\rangle).$$ \hspace{1cm} (43)

On inserting the desired output state (43) into (8), we find

$$\hat{R}_\perp = \frac{1}{8}I_1 \otimes (\hat{I}_2 \otimes I_3 + \frac{1}{3} \sum_{j=x,y,z} \hat{\sigma}_{2,j} \otimes \hat{\sigma}_{3,j}),$$ \hspace{1cm} (44)

where $\hat{\sigma}_j$ denote Pauli matrices and the subscripts 1 and 2, 3, refer to input Hilbert space $H$ and output space $K = H \otimes H$, respectively. The largest eigenvalue of $\hat{R}_\perp$, which is equal to 1/6, limits the fidelity of the entangler (43) to

$$F_\perp \leq 1/3.$$ \hspace{1cm} (45)

This result was already obtained in [14], but our proof is much simpler than that presented in [14]. When we iterate Eqs. (13) and (14), we obtain the optimal CP-map which reaches the maximum fidelity 1/3. It turns out that the optimal transformation is strikingly simple. For any input state, the entangler should prepare the same mixed output state

$$\hat{\rho}_{\text{out}} = \frac{1}{3}(|00\rangle\langle 00| + |\Psi_+\rangle\langle \Psi_+| + |11\rangle\langle 11|).$$ \hspace{1cm} (46)

The fidelity is 1/3 and does not depend on the input state. Moreover, it can be easily shown with the help of the Peres-Horodecki criterion [15,16] that the state (46) is separable. Thus we conclude that this optimal universal “entangler” exhibits rather poor performance because it does not prepare an entangled state.

D. Optimal $\vartheta$-shifter

Recently, Hardy and Song [17,18] studied an approximate implementation of single-qubit transformation where the angle $\vartheta$ is shifted by $\alpha \in [0, \pi]$,

$$|\psi(\vartheta, \phi)\rangle \rightarrow |\psi(\vartheta + \alpha, \phi\rangle).$$ \hspace{1cm} (47)

Hardy and Song found that the optimal universal CP-map whose fidelity does not depend on $\vartheta$ is identity map for $\alpha < \pi/2$ and a U-NOT gate for $\alpha > \pi/2$. Remarkably, the lowest fidelity $F = 1/2$ is achieved for $\alpha = \pi/2$. However, the universal transformation is not the best one and one can find a CP-map which for certain range of angles $\alpha$ approximates the transformation (47) much better (i.e. attains much higher mean fidelity) than the optimal universal transformation [13].

In what follows we show that our approach naturally and straightforwardly leads to the optimal non-universal CP-map. On inserting the output state (47) into Eq. (8) we obtain the operator $\hat{R}_\vartheta(\alpha)$,

$$\hat{R}_\vartheta(\alpha) = \left(\frac{1}{4} + \frac{1}{12} \cos \alpha + \frac{\pi}{16} \sin \alpha\right) |00\rangle\langle 00|,$$

$$+ \left(\frac{1}{4} - \frac{1}{12} \cos \alpha + \frac{\pi}{16} \sin \alpha\right) |01\rangle\langle 01|,$$

$$+ \left(\frac{1}{4} - \frac{1}{12} \cos \alpha + \frac{\pi}{16} \sin \alpha\right) |10\rangle\langle 10|,$$

$$+ \left(\frac{1}{4} + \frac{1}{12} \cos \alpha + \frac{\pi}{16} \sin \alpha\right) |11\rangle\langle 11|,$$

$$+ \frac{1}{6} \cos \alpha (|00\rangle\langle 11| + |11\rangle\langle 00|).$$ \hspace{1cm} (48)

We have solved numerically the extremal Eq. (13), (14) for various values of $\alpha$. An analysis of the structure of resulting CP-maps reveals that they represent a simple damping process and we can make for them the following ansatz,
\[ |0\rangle\langle 0 | \rightarrow \cos^2 \beta |0\rangle\langle 0 | + \sin^2 \beta |1\rangle\langle 1 | ,
|1\rangle\langle 0 | \rightarrow \cos \beta |1\rangle\langle 0 | ,
|0\rangle\langle 1 | \rightarrow \cos \beta |0\rangle\langle 1 | ,
|1\rangle\langle 1 | \rightarrow |1\rangle\langle 1 | . \tag{49} \]

The mean fidelity is a function of \( \alpha \) and \( \beta \),
\[
F(\alpha, \beta) = \frac{1}{2} + \frac{\cos \alpha}{6} (\cos^2 \beta + 2 \cos \beta) + \frac{\pi}{8} \sin \alpha \sin^2 \beta . \tag{50} \]

The optimal \( \beta \) is obtained from the extremal Eq.
\[
\frac{\partial}{\partial \beta} F(\alpha, \beta) = 0. \tag{51} \]

The solution
\[
\beta_1 = 0 \tag{52} \]
exists for all \( \alpha \). A second solution
\[
\cos \beta_2 = \left( \frac{3\pi}{4} \tan \alpha - 1 \right)^{-1} \tag{53} \]
exists only if
\[
\alpha \geq \arctan \frac{8}{3\pi} = \alpha_0 . \tag{54} \]

The second root always leads to higher mean fidelity than the first one. Thus for \( \alpha < \alpha_0 \) it seems optimal to apply an identity operation. However, as soon as the angle shift \( \alpha \) overcomes the threshold \( \alpha_0 \) one should rather apply a damping process (49) with angle \( \beta \) given by Eq. (53). The resulting fidelity \( F(\alpha) \) can be expressed as
\[
F(\alpha) = \begin{cases} 
\frac{1}{2}(1 + \cos \alpha), & \alpha \leq \alpha_0, \\
F(\alpha, \beta_2), & \alpha > \alpha_0,
\end{cases} \tag{55} \]
and is plotted in Fig. 2. Notice that \( F(\alpha) \) is not a monotonic function of \( \alpha \) \([18]\). In Fig. 2 we also show the upper bound on fidelity obtained from the maximum eigenvalue of \( R_0(\alpha) \). We can see that the fidelity \([53]\) of the approximate \( \vartheta \)-shifter (49) is lower than the upper bound. This bound is reached only at three discrete points. The most trivial case is \( \alpha = 0 \) where the identity transformation achieves fidelity 1. For \( \alpha = \pi \) the U-NOT gate reaches maximum possible fidelity 2/3. The third point \( \alpha = \pi/2 \), where the fidelity of the \( \vartheta \)-shifter reads
\[
F\left( \frac{\pi}{2} \right) = \frac{4 + \pi}{8} ,
\]
is perhaps most interesting. Since for \( \alpha = \pi/2 \) we have from Eq. (53) \( \beta_2 = \pi/2 \), the output of the \( \vartheta \)-shifter (49) is the state \( |1\rangle \) regardless of the input state. This means that the transformation \( \vartheta \rightarrow \vartheta + \pi/2 \) is implemented exactly for all pure states on the equator of Bloch sphere because all these states should be mapped to the south pole of this sphere, i.e. to the state \( |1\rangle \).

\section{V. CONCLUSIONS}

We have derived an extremal equation for optimal completely positive map \( \tilde{\chi} \) which most closely approximates certain prescribed transformation between pure states. The nonlinear extremal equation can be conveniently numerically solved by means of repeated iterations, which provides very straightforward and simple means for deriving the optimal CP-map. Moreover, we have obtained an upper bound on the mean fidelity which, in some cases, allows us to simply prove the optimality of the calculated CP-map. We have applied our results to universal NOT gate, quantum cloning machine, quantum entanglers, and qubit \( \vartheta \)-shifter. The examples discussed in this paper illustrate that the present method is very general and may find further applications in quantum information processing, for example in design of optimal CP maps for quantum teleportation \([19]\).

\section{ACKNOWLEDGMENTS}

I would like to thank J. Řeháček, Z. Hradil, R. Filip, and M. Ježek for valuable comments and stimulating discussions. This work was supported by Grant No LN00A015 of the Czech Ministry of Education.

[1] W.K. Wootters and W.H. Zurek, Nature(London) \textbf{299}, 802 (1982).
[2] V. Bužek, M. Hillery, and R.F. Werner, Phys. Rev. A \textbf{60}, R2626 (1999); J. Mod. Opt. \textbf{47}, 211 (2000).
[3] V. Bužek and M. Hillery, Phys. Rev. A \textbf{54}, 1844 (1996); V. Bužek, S. Braunstein, M. Hillery, and D. Druss, Phys. Rev. A \textbf{56}, 3446 (1997).
[4] N. Gisin and S. Massar, Phys. Rev. Lett. \textbf{79}, 2153 (1997).
[5] C.S. Niu and R.B. Griffiths, Phys. Rev. A 58, 4377 (1998).
[6] D. Bruss, D.P. DiVincenzo, A. Ekert, C.A. Fuchs C. Macchiavello, and J.A. Smolin, Phys. Rev. A 57, 2368 (1998).
[7] D. Bruss, A. Ekert, and C. Macchiavello, Phys. Rev. Lett. 81, 2598 (1998).
[8] R.F. Werner, Phys. Rev. A 58, 1827 (1998).
[9] G.M. D’Ariano and P. Lo Presti, quant-ph/0101106; M. Sacchi, Phys. Rev. A 63, 054104 (2001).
[10] R. Derka, V. Bužek, and A.K. Ekert, Phys. Rev. Lett. 80, 1571 (1998).
[11] S. Massar and S. Popescu, Phys. Rev. A 61, 062303 (2000).
[12] K. Kraus, Ann. Phys. 64, 311 (1971).
[13] S. Massar and S. Popescu, Phys. Rev. Lett. 74, 1259 (2001).
[14] V. Bužek and M. Hillery, Phys. Rev. A 62, 022303 (2000).
[15] A. Peres, Phys. Rev. Lett. 77, 1413 (1996).
[16] M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Lett. A 223, 1 (1996).
[17] L. Hardy and D.D. Song, Phys. Rev. A 63, 032304 (2001).
[18] L. Hardy and D.D. Song, arXiv:quant-ph/0102100.
[19] J. Reháček, Z. Hradil, J. Fiurášek, and C. Brukner, arXiv:quant-ph/0105119.