Gap probabilities in the bulk of the Airy process

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Abstract

We consider the probability that no points lie on $g$ large intervals in the bulk of the Airy point process. We make a conjecture for all the terms in the asymptotics up to and including the oscillations of order 1, and we prove this conjecture for $g = 1$.

1 Introduction and statement of results

The Airy point process is a determinantal point process on $\mathbb{R}$ which plays an important role in many seemingly unrelated topics: for example, it models the largest eigenvalues of certain large random matrices [10, 15, 29], the fluctuations at the solid-liquid boundary of certain tiling models [25], the largest parts of Young diagrams with respect to the Plancherel measure [4, 9], the fluctuations of the length of the longest increasing subsequence of a random permutation of $\{1, 2, \ldots, N\}$, $N \gg 1$ [3], and the Brownian particles near the edge-curve in non-intersecting Brownian paths [1]. It has also recently been shown to appear at the liquid-gas boundary of the two-periodic Aztec diamond [5]. The correlation kernel of the Airy point process is given by

$$K^{\text{Ai}}(u, v) = \frac{\text{Ai}(u)\text{Ai}'(v) - \text{Ai}'(u)\text{Ai}(v)}{u - v}, \quad u, v \in \mathbb{R},$$

where $\text{Ai}$ denotes the Airy function. There are infinitely many random points which are almost surely all distinct, and there is almost surely a largest point. The Airy process is rigid, in the sense that given a bounded Borel set $A$, the restriction of any point configuration to $\mathbb{R} \setminus A$ almost surely determines the number of points in $A$, see [11]. We also mention that the maximum fluctuations of the points have been studied in [14, 32] and in [12, Theorem 1.4].

A gap probability refers to the probability that a given region is free from points. It is well-known from the general theory of determinantal point processes (see e.g. [28]) that gap probabilities are Fredholm determinants. In particular, if $I \subset \mathbb{R}$ is a finite union of intervals, then the probability of finding no points in $I$ for the Airy process is given by

$$F(I) := \mathbb{P}(\text{no points lie in } I) = \det(I - K^{\text{Ai}} \chi_I),$$

where $K^{\text{Ai}}$ is the integral operator whose kernel is $K^{\text{Ai}}$, and $\chi_I$ is the projection operator. The properties of $F(I)$ depend on the structure of $I$: given $g \in \mathbb{N} = \{0, 1, 2, \ldots\}$ and $\tilde{x}_0, x_1, \ldots, x_{2g} \in \mathbb{R}$ such that $x_{2g} < \ldots < x_1 < \tilde{x}_0$, we distinguish the cases

$$I_g^{(c)} = (x_{2g}, x_{2g-1}) \cup \ldots \cup (x_2, x_1) \cup (\tilde{x}_0, +\infty),$$

$$I_g^{(b)} = (x_{2g}, x_{2g-1}) \cup \ldots \cup (x_2, x_1).$$

If $x_1 < 0$, then $F(I_g^{(b)})$ is the probability of finding no points on a union of $g$ intervals in the bulk, while $F(I_g^{(c)})$ is the probability of finding no points on these $g$ intervals and a gap at the edge. Note that the

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Case $\mathcal{I}_0^{(b)} = \emptyset$ leads trivially to $F(\mathcal{I}_0^{(b)}) = 1$. The distribution $F(x_0^{(c)}) = F((\bar{x}_0, +\infty))$ is the well-known Tracy–Widom distribution of the largest point which is naturally expressed in terms of the solution to a Painlevé II equation [29]. More generally, $F(\mathcal{I}_0^{(b)})$ (resp. $F(x_g^{(b)})$) can be expressed in terms of a solution to a system of $2g + 1$ (resp. $2g$) coupled Painlevé II equations [13], see also [31].

Gap probabilities are transcendental functions, but in certain cases they possess convenient approximations. For example, for small values of $r > 0$, $F(rz)$ represents the probability of a small gap, and can be estimated from its Fredholm series representation. A classical and much harder problem is to find large gap asymptotics, which are asymptotics for $F(rz)$ as $r \to +\infty$. Large $r$ asymptotics for $F(\mathcal{I}_0^{(c)})$ and $F(x_g^{(b)})$ have been analysed in [2, 16, 29] and [6, 26], respectively. In this work, we focus on $F(x_g^{(b)})$ with $g \geq 1$ (the case $g = 0$ is trivial, since $F(rx_g^{(b)}) = 1$ for all $r$).

**Large gap asymptotics on $g$ intervals in the bulk**

In Conjecture 1.1 below, we formulate a conjecture for the large $r$ asymptotics of $F(rx_g^{(b)})$ for any finite number of intervals $g \geq 1$, up to and including the oscillations of order 1. This conjecture has been verified numerically for $g = 1, 2, 3$, and we prove it rigorously for $g = 1$.

To formulate the conjecture, let $g \geq 1$ be an integer and let $\bar{x} = (x_1, \ldots, x_{2g}) \in \mathbb{R}^{2g}$ be such that $x_{2g} < \ldots < x_1 < x_0$. Define the square root $\sqrt{R(z)}$ by

$$\sqrt{R(z)} = \prod_{j=0}^{2g} \sqrt{z - x_j},$$

where the principal branch is taken for each of the square roots, and $x_0 = x_0(\bar{x})$ is defined below. Let $q_g = -1$ and define \( \{m_{ij}\}_{j=1}^g \) and \( \{\tilde{m}_i\}_{i=1}^g \) by

$$m_{ij} = \int_{x_{2i}}^{x_{2i-1}} \frac{s^{j-1}}{\sqrt{R(s)}} ds, \quad \tilde{m}_i = -\int_{x_{2i}}^{x_{2i-1}} q_s + q_g s q_g \frac{ds}{\sqrt{R(s)}}, \quad \text{where} \quad q_g = \frac{1}{2} \sum_{j=0}^{2g} q_j.$$

The conjecture involves the solution $(x_0, q_0, ..., q_{g-1}) = (x_0(\bar{x}), q_0(\bar{x}), ..., q_{g-1}(\bar{x})) \in \mathbb{R}^{g+1}$ of the following system of equations:

$$\begin{pmatrix}
(m_{11} & m_{12} & \cdots & m_{1g}) & (q_0) & (\tilde{m}_1) \\
(m_{21} & m_{22} & \cdots & m_{2g}) & (q_1) & (\tilde{m}_2) \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
(m_{g1} & m_{g2} & \cdots & m_{gg}) & (q_{g-1}) & (\tilde{m}_g)
\end{pmatrix} = 0,$$

with $x_0 = \max\{x_1, 0\}$.

Let $\mathcal{A} = \{\bar{x} \in \mathbb{R}^{2g} : x_{2g} < \ldots < x_2 < x_1 \text{ and there exists } x_0 \text{ satisfying (1.3)}\}$. For all $g \geq 1$, we expect that $\{\bar{x} \in \mathbb{R}^{2g} : x_{2g} < \ldots < x_2 < x_1 < 0\} \subset \mathcal{A}$, and we completely characterize $\mathcal{A}$ for $g = 1$ in Theorem 1.2 below (see also Proposition 3.1). The conjecture also involves the two-sheeted Riemann surface $X$ of genus $g$ associated to $\sqrt{R(z)}$ where $\sqrt{R(z)} > 0$ for $z \in (x_0, +\infty)$ on the first sheet. We define $A$-cycles and $B$-cycles on $X$ as in Figure 1 and consider the matrix $\hat{A} \in \mathbb{R}^{g \times g}$ given by

$$A_{jk} := \oint_{A_j} \frac{s^{k-1}}{\sqrt{R(s)}} ds = 2 \sum_{l=1}^{g} \int_{x_{2l-1}}^{x_{2l}} \frac{s^{k-1}}{\sqrt{R(s)}} ds \in \mathbb{R}, \quad 1 \leq j, k \leq g.$$

It follows from the general theory of Riemann surfaces [22] that $\hat{A}$ is invertible. Consider

$$\begin{pmatrix}
(\omega_1 & \omega_2 & \cdots & \omega_g)
\end{pmatrix} = \frac{dz}{\sqrt{R(z)}} \begin{pmatrix} 1 & \cdots & z^{g-1} \end{pmatrix} \hat{A}^{-1}.$$  (1.4)
The period matrix \( \tau \) is symmetric with positive definite imaginary part.

Assume that there exist \( \nu_j \), \( j = 1, \ldots, g \), such that

\[
\nu_j = -\frac{\Omega_j}{2\pi i}, \quad \text{with} \quad \Omega_j = 2i \pi \sum_{k=1}^{g+1} \frac{q_{jk} \xi_k}{\sqrt{\Omega(\xi)_+}} \, d\xi > 0, \quad j = 0, \ldots, g - 1.
\]

Assume that there exist \( \delta_1, \delta_2 > 0 \) such that

\[
\left| \sum_{j=1}^{g} m_j \Omega_{j-1} \right| \geq \delta_1 \left( \sum_{j=1}^{g} m_j^2 \right)^{-\delta_2}, \quad \text{for all} \ m_1, \ldots, m_g \in \mathbb{Z} \text{ such that } \sum_{j=1}^{g} m_j \Omega_{j-1} \neq 0.
\]
Remark 1. If condition (1.8) fails, then the error term \( O(r^{-\frac{7}{2}}) \) in (1.10) must be replaced by \( o(\log r) \) as in [19]. On the other hand, if condition (1.9) does not hold, from [19] and [7, Theorems 1.1–1.4] we expect that the log-coefficient \(-\frac{7}{2}\) should be replaced by a rather complicated hyperelliptic integral. Again from [19, 7], we expect the map \((x_1, \ldots, x_{2g}) \mapsto (\Omega_0, \ldots, \Omega_{g-1})\) to contain open balls in \((0, +\infty)^g\).

Since the set \(\{(\Omega_0, \ldots, \Omega_{g-1})\} \subset (0, +\infty)^g\) for which either (1.8) or (1.9) does not hold has Lebesgue measure zero, the asymptotics (1.10) are expected to hold for generic \((x_1, \ldots, x_{2g}) \in \mathcal{A}\). Note that for \(g = 1\), both (1.8) and (1.9) are satisfied for all \((x_1, x_2) \in \mathcal{A}\).

Remark 2. The fact that the oscillations are described in terms of the Riemann \(\theta\)-function related to a genus \(g\) Riemann surface is not surprising; this happens also in the case of the sine process [19, 21, 30].

Remark 3. The above conjecture has been verified numerically for \(g = 1, 2, 3\) for several choices of \(\vec{x}\) by using the Bornemann Linear Algebra package [8].

Theorem 1.2. Conjecture 1.1 holds for \(g = 1\). Furthermore, if \(g = 1, 2, 3\) for several choices of \(\vec{x}\) by using the Bornemann Linear Algebra package [8].

Outline of the paper. In Section 2, we use the method developed by Its, Izergin, Korepin and Slavnov [24], combined with a result of Claeyts and Doeraene [13], to express \(\partial_t \log F(r^2) = \partial_t \log F(r(x_2, x_1))\) in terms of the solution, denoted \(\Psi\), of a \(2 \times 2\) matrix Riemann-Hilbert (RH) problem. In Sections 3-6, we obtain the asymptotics of \(\Psi\) as \(r \to +\infty\) via the Deift-Zhou steepest descent method [17, 20]. As a consequence of our RH analysis, we obtain large \(r\) asymptotics for \(\partial_t \log F(r(x_2, x_1))\). The simplification of these asymptotics is the most challenging part of this work; this is done in Section 7. Finally, we prove Theorem 1.2 by integrating the asymptotics of \(\partial_t \log F(r(x_2, x_1))\) with respect to \(r\).

2 Differential identity for \(F\)

The method of Its, Izergin, Korepin and Slavnov [24] applies to kernels of the integrable form \(K(x, y) = \frac{\bar{f}(x)\bar{h}(y)}{x-y}\), where \(\bar{f}(x)\) and \(\bar{h}(y)\) are column vectors satisfying \(\bar{f}(x)\bar{h}(x) = 0\). Given a subset \(\mathcal{A} \subset \mathbb{R}\), we write \(\chi_{\mathcal{A}}\) for the characteristic function of \(\mathcal{A}\). It is easy to see that

\[
K^{\chi_{\mathcal{A}}}((x, y) = K^{\chi_{\mathcal{A}}}(x, y)\chi_{r(x_2, x_1)}(y), \quad x, y \in \mathbb{R},
\]

is integrable with \(\bar{f}\) and \(\bar{h}\) given by

\[
\bar{f}(x) = \left( \begin{array}{c} \text{Ai}(x) \\ \text{Ai}'(x) \end{array} \right), \quad \bar{h}(y) = \left( \begin{array}{c} \text{Ai}'(y)\chi_{r(x_2, x_1)}(y) \\ -\text{Ai}(y)\chi_{r(x_2, x_1)}(y) \end{array} \right).
\]

The associated integral operator \(\mathcal{K}_r\), acting on \(L^2(rx_2, +\infty)\), is given by

\[
\mathcal{K}_r\phi(x) = \int_{rx_2}^{+\infty} K^{\chi_{\mathcal{A}}}(x, y)\chi_{r(x_2, x_1)}(y)\phi(y)dy, \quad \phi \in L^2(rx_2, +\infty).
\]  

(2.1)

By definition, \(F(r(x_2, x_1)) = \text{det}(I - \mathcal{K}_r) = \mathbb{P}(\text{no points lie in } (rx_2, rx_1)) > 0\). Hence, using standard properties of trace class operators, we obtain

\[
\partial_t \log \text{det}(I - \mathcal{K}_r) = -\text{Tr} \left[ (I - \mathcal{K}_r)^{-1} \partial_t \mathcal{K}_r \right] = \sum_{j=1}^{2} (-1)^j x_j \text{Tr} \left[ (I - \mathcal{K}_r)^{-1} \mathcal{R}_r \delta_{rx_j} \right] \\
= \sum_{j=1}^{2} (-1)^j x_j \text{Tr} \left[ \mathcal{R}_r \delta_{rx_j} \right] = \sum_{j=1}^{2} (-1)^j x_j \lim_{u \to rx_j} R_r(u, u),
\]

(2.2)

where the limits \(u \to rx_j, \ j = 1, 2\), are taken from the interior of \((rx_2, rx_1)\), \(\delta_{rx_j}\) is the Dirac delta operator, the integral operator \(\mathcal{R}_r\) is given by

\[
\mathcal{R}_r := (I - \mathcal{K}_r)^{-1} \mathcal{K}_r = (I - \mathcal{K}_r)^{-1} - I,
\]

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and $R_r$ is the kernel of $\mathcal{R}_r$. Now, we invoke a result of Claeys and Doeraene [13]: for $u \in (rx_2, rx_1)$, we have

$$R_r(u, u) = \frac{1}{2\pi i} \left( \Psi_{+}^{-1}\Psi'_{+} \right)_{21} (u; rx_1, rx_2),$$

where $\Psi$ is the solution to the following RH problem.

**RH problem for $\Psi = \Psi(\cdot; x_1, x_2)$**

(a) $\Psi : \mathbb{C} \setminus \Gamma \to \mathbb{C}^{2 \times 2}$ is analytic, where

$$\Gamma = e^{\pm \frac{2\pi i}{3}} (x_2, +\infty) \cup (-\infty, x_2] \cup [x_1, +\infty)$$

is oriented as in Figure 2.

(b) $\Psi(z)$ has continuous boundary values as $\Gamma \setminus \{x_1, x_2\}$ is approached from the left (+ side) and from the right (− side) and they are related by

$$\Psi_+(z) = \Psi_-(z) \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \quad \text{for } z \in e^{\pm \frac{2\pi i}{3}} (x_2, +\infty),$$

$$\Psi_+(z) = \Psi_-(z) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{for } z \in (-\infty, x_2),$$

$$\Psi_+(z) = \Psi_-(z) \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{for } z \in (x_1, +\infty).$$

(c) As $z \to \infty$, we have

$$\Psi(z) = (I + \mathcal{O}(z^{-1})) z^{rac{1}{2} \sigma_3} M^{-1} e^{-\frac{2}{3} z^{rac{1}{2}} \sigma_3},$$

where principal branches of $z^{rac{1}{2}}$ and $z^{rac{1}{3}}$ are taken, and

$$M = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(d) $\Psi(z) = \mathcal{O}(\log(z - x_j))$ as $z \to x_j$, $j = 1, 2$.

By combining (2.2) with (2.3), we arrive at

$$\partial_r \log F(r(x_2, x_1)) = \sum_{j=1}^{2} \frac{(-1)^j x_2}{2\pi i} \lim_{z \to x_j} \left( \Psi_{+}^{-1}\Psi'_{+} \right)_{21} (rz; rx_1, rx_2),$$

where the limits as $z \to x_j$, $j = 1, 2$ are taken with $z \in (x_2, x_1)$.

Figure 2: The jump contour $\Gamma$ for the RH problem for $\Psi$. 
3 Asymptotic analysis of $\Psi$: first steps

The goal of this section is to implement the first steps of the steepest descent method for the RH problem for $\Psi$. In Subsection 3.1, we construct the $g$-function and list its important properties. In Subsection 3.2, we use $g$ to normalize the RH problem at $\infty$, and then proceed with the so-called opening of the lenses. We henceforth assume that $g = 1$.

3.1 $g$-function

Our first goal is to prove that the system of equations (1.3) has a unique solution $(x_0, q_0) \in \mathbb{R}^+ \times \mathbb{R}$, where $\mathbb{R}^+ = (0, +\infty)$. According to the last equation in (1.3), the degree two polynomial $q(z) := \sum_{j=0}^{x_0} q_j z^j = q_2 z^2 + q_1 z + q_0$ satisfies $q(x_0) = 0$, and is therefore of the form

$$q(z) = -z^2 + \frac{x_0}{2} (x_0 + x_1 + x_2) + q_0, \quad q_0 = \frac{x_0}{2} (x_0 - x_1 - x_2), \quad (3.1)$$

for a certain $x_0$ that will be determined so that the first equation in (1.3) holds, i.e., so that $m_{11} q_0 = \tilde{m}_1$.

Defining the function $\mathcal{F} : [x_1, \infty) \to \mathbb{R}$ by

$$\mathcal{F}(x) := \int_{x_2}^{x_1} \frac{\sqrt{x - s}}{\sqrt{(x_1 - s)(s - x_2)}} \, ds, \quad (3.2)$$

the equation $m_{11} q_0 = \tilde{m}_1$ can be rewritten as $\mathcal{F}(x_0) = 0$. Hence the following proposition implies that (1.3) has a unique solution $(x_0, q_0) \in \mathbb{R}^+ \times \mathbb{R}$.

**Proposition 3.1.**

(a) Given $x_2 < x_1 < 0$, there is a unique $x_0 > 0$ such that $\mathcal{F}(x_0) = 0$. Moreover, $x_0 \in (0, x_1 - x_2)$.

(b) Given $x_1 \geq 0$, the equation $\mathcal{F}(x_0) = 0$ admits a solution $x_0 \in (x_1, +\infty)$ if and only if $x_2 < -2x_1$. Moreover, the solution $x_0$ is unique and satisfies $x_0 \in (x_1, x_1 - x_2)$.

**Proof.** (a) It is clear that $\mathcal{F}(x)$ is continuous and differentiable on its domain. We will show that $\mathcal{F}(0) < 0$, $\mathcal{F}(x_1 - x_2) > 0$, and that $\mathcal{F}'(x) > 0$ for all $x \in (0, \infty)$ and $\mathcal{F}'(x) \to +\infty$ as $x \to +\infty$, which implies the stated result. The inequality $\mathcal{F}(x_1 - x_2) > 0$ is easy to establish from a direct inspection of (3.2). Also, we note that

$$\mathcal{F}(x) = \int_{x_2}^{x_1} \frac{3x - 3x_1 + x_2}{4(x - s)(x_1 - s)(s - x_2)} \, ds, \quad (3.3)$$

which is clearly positive for $x > 0$ because $x_2 < x_1 < 0$, and $\mathcal{F}(x) \sim \frac{3x}{4} \sqrt{x} \to +\infty$ as $x \to +\infty$. To show that $\mathcal{F}(0) < 0$, let $x_1 = \frac{1}{2}(x_1 + x_2)$, and note that $\mathcal{F}(0)$ can be written as

$$\mathcal{F}(0) = \left( \int_{x_2}^{x'} + \int_{x'}^{x_1} \right) \frac{\sqrt{-s} (s - x_1)}{\sqrt{(x_1 - s)(s - x_2)}} \, ds < 0.$$ 

Since $f(s) = \frac{3s - 3x_1 + x_2}{\sqrt{(x_1 - s)(s - x_2)}}$ satisfies $f(s) = -f(s) = -f(s)$ for all $s \in (-\frac{2x_1 + x_2}{3}, \frac{2x_1 + x_2}{3})$ and since $\sqrt{-s}$ is positive and decreases as $s \in (x_2, x_1)$ increases, this implies that $\mathcal{F}(0) < 0$.

(b) The numerator in (3.3) is positive if and only if $x > \frac{2x_1 + x_2}{3}$. Since $x_1 \geq 0$, we have $\frac{2x_1 + x_2}{3} < x_1$ and therefore $\mathcal{F}'(x) > 0$ for all $x \in (x_1, +\infty)$. By a direct computation we get

$$\mathcal{F}(x_1) = \int_{x_2}^{x_1} \frac{s - \frac{x_2}{3}}{\sqrt{s - x_2}} \, ds = \frac{1}{3} \sqrt{x_1 - x_2}(2x_1 + x_2).$$

This shows that $\mathcal{F}(x_1) < 0$, and therefore that there exists $x_0 \in (x_1, +\infty)$ such that $\mathcal{F}(x_0) = 0$, if and only if $x_2 < -2x_1$. Since $x_2 < 0$, the inequality $\mathcal{F}(x_1 - x_2) > 0$ still holds, which implies that $x_0 \in (x_1, x_1 - x_2)$. \qed
Throughout the remainder of this paper, \( x_0 \) is defined as in Proposition 3.1. Since
\[ q(z) = \bar{q}(z)(z - x_0), \quad \bar{q}(z) := -z - \frac{i}{2}(x_0 - x_1 - x_2), \]
and \( x_2 < -\frac{i}{2}(x_0 - x_1 - x_2) < x_1 \), we have
\[ q(x_2) < 0, \quad q(x_1) > 0, \quad q(x_0) = 0, \quad q'(x_0) = \bar{q}(x_0) < 0. \] (3.4)

The square root \( \sqrt{R(z)} = \sqrt{z - x_0}\sqrt{z - x_1}\sqrt{z - x_2} \) is analytic on \( C \setminus ((-\infty, x_2] \cup [x_1, x_0]) \), behaves as \( \sqrt{R(z)} \sim z^2 \) as \( z \to \infty \), and satisfies the jump relation
\[ \sqrt{R(z)}_+ + \sqrt{R(z)}_- = 0, \quad z \in (-\infty, x_2) \cup (x_1, x_0). \] (3.5)

We define the \( g \)-function by
\[ g(z) = \int_{x_0}^{z} \frac{q(s)}{\sqrt{R(s)}} ds, \] (3.6)
where \( q \) is given by (3.1), and the path of integration does not cross \( (-\infty, x_0] \).

**Lemma 3.2.** The \( g \)-function satisfies the following properties:

1. \( g \) is analytic in \( C \setminus (-\infty, x_0] \) and satisfies \( g(z) = \bar{g}(\bar{z}) \) for \( z \in C \setminus (-\infty, x_0) \).

2. \( g \) satisfies the jump conditions
\[ g_+(z) + g_-(z) = 0, \quad z \in (-\infty, x_2) \cup (x_1, x_0), \] (3.7)
\[ g_+(z) - g_-(z) = i\Omega, \quad z \in (x_2, x_1), \] (3.8)
where \( \Omega = 2i \int_{x_1}^{x_0} g_+(s) ds = -2ig_+(x_1) > 0. \)

3. As \( z \to \infty \), we have
\[ g(z) = -\frac{2}{3}z^2 + \frac{1}{4}\left((x_1 - x_2)^2 - x_0(3x_0 - 2x_1 - 2x_2)\right)z^{-\frac{1}{2}} + O(z^{-\frac{3}{2}}). \] (3.9)

In particular, \( \text{Re} g(z) \to +\infty \) as \( z \to \infty \) along either of the two rays \( \arg(z - x_2) = \pm 2\pi/3 \).

4. There exists an open neighborhood \( V \subset C \) of \( (-\infty, x_0) \) and an \( M > 0 \) such that
\[ \left\{ z \in C : \arg(z - x_2) \in \left(-\frac{2\pi}{3}, \frac{2\pi}{3}\right) \cup (\frac{2\pi}{3}, \pi) \text{ and } |z| \geq M \right\} \subset V \] (3.10)
and such that
\[ \text{Re} g(z) \geq 0 \text{ for all } z \in V, \] (3.11)
where equality holds in (3.11) if and only if \( z \in (-\infty, x_2] \cup [x_1, x_0] \).

**Proof.** The analyticity of \( g \) follows from (3.6). The symmetry \( g(z) = \bar{g}(\bar{z}) \) follows from the fact that \( q \) has real coefficients. The jumps (3.5) combined with \( \int_{x_2}^{x_1} \frac{\bar{q}(s)}{\sqrt{R(s)}} ds = -\mathcal{F}(x_0) = 0 \) imply (3.7). The jump (3.8) is a consequence of (3.5) and (3.6); the fact that \( \Omega > 0 \) follows from (3.4) and (3.6). Expanding \( g'(z) \) as \( z \to \infty \), and then integrating these asymptotics gives
\[ g(z) = -\frac{2}{3}z^2 + g_0 + \frac{1}{4}\left((x_1 - x_2)^2 - x_0(3x_0 - 2x_1 - 2x_2)\right)z^{-\frac{1}{2}} + O(z^{-\frac{3}{2}}) \quad \text{as } z \to \infty, \]
for some \( g_0 \in C \). Equation (3.7) implies that \( g_0 = 0 \), which proves (3.9). Equation (3.9) implies that there exists an \( M > 0 \) such that \( \text{Re} g(z) \geq 0 \) for all \( z \in C \) with \( |z| \geq M \) and \( \arg z \in (-\frac{2\pi}{3}, \frac{2\pi}{3}) \cup (\frac{2\pi}{3}, \pi) \). Also, from (3.7) and the symmetry \( g(z) = \bar{g}(\bar{z}) \), we deduce that \( \text{Re} g_+(z) = \text{Re} g_-(z) = 0 \) for \( z \in (-\infty, x_2] \cup [x_1, x_0] \). Since \( g(z) < 0 \) for \( z \in (-\infty, x_2) \) and \( g(z) > 0 \) for \( z \in (x_1, x_0) \) by (3.4), the imaginary part of \( g_+(x) \) is decreasing as \( x \in (-\infty, x_2) \cup (x_1, x_0) \) increases. Hence, by the Cauchy-Riemann equations,
for each \( x \in (-\infty, x_2) \cup (x_1, x_0) \), there exists \( \epsilon = \epsilon(x) > 0 \) such that \( \text{Re} \, g(x + iu\epsilon(x)) > 0 \) for all \( u \in (0,1] \). Assertion 4 will follow if we can show that \( \epsilon \) can be chosen independently of \( x \). This can be achieved by a local analysis of \( g \) near each of the points \( x_0, x_1, x_2 \). As \( z \to x_0 \), since \( q(x_0) = 0 \), we have \( g(z) \sim \frac{2q(x_0)(z - x_0)^{3/2}}{\sqrt{(x_0 - x_1)(x_0 - x_2)}} \). Since \( q'(x_0) < 0 \), there exists a small neighborhood \( V_0 \) of \( x_0 \) such that \( \text{Re} \, g(z) \geq 0 \) for \( z \in V_0 \). As \( z \to x_1 \), we have \( g(z) \sim 2q(x_1)\sqrt{z - x_1}/\sqrt{(x_0 - x_1)(x_0 - x_1)} \). Recalling that \( q(x_1) > 0 \), \( \text{Re} \, g(z) \geq 0 \) in a full open neighborhood \( V_1 \) of \( x_1 \), with equality only if \( z \in [x_1, x_0] \cap V_1 \). The local analysis near \( x_2 \) is similar. \( \Box \\

\textbf{Remark 4.} \) For future reference, we note that the equation \( F(x_0) = 0 \) can be rewritten as

\[
\frac{E(k)}{K(k)} = -\frac{2(x_0 - x_1)}{x_0 + x_1 + x_2}, \quad \text{where} \quad k := \sqrt{\frac{x_1 - x_2}{x_0 - x_2}},
\]

(3.12)

and \( K, E \) are the complete elliptic integrals of the first and second kind (see [23, Eqs. 8.111.2 and 8.111.3])

\[
K(k) := \int_0^{\pi/2} \frac{dt}{\sqrt{1 - k^2 \sin^2 t}}, \quad E(k) := \int_0^{\pi/2} \sqrt{1 - k^2 \sin^2 t} \, dt.
\]

Also, using [23, Eqs. 3.131.6, 3.131.11, 3.132.5 and 3.141.23] and (3.12), the constant \( \Omega \) can be written as

\[
\Omega = \frac{2}{3} \sqrt{x_0 - x_2(x_0 + x_1 + x_2)} \left[ K(k') \left( 1 - \frac{E(k)}{K(k)} \right) - E(k') \right],
\]

(3.13)

where \( k' := \sqrt{\frac{x_0 - x_1}{x_0 - x_2}} = \sqrt{1 - k^2} \).

\section*{3.2 Rescaling and opening of the lenses}

The first transformation \( \Psi \to T \) of the steepest descent analysis is defined by

\[
T(z) = \left( r^{-\frac{1}{4}} 0 \right) \left( 2x_0(x + x_2) - 3x_0^2 + (x_1 - x_2)^2 \right) r^{-\frac{1}{4}} \Psi(r; r_1, r_2) e^{-i\frac{\pi}{4} g(z) \sigma_3}.
\]

(3.14)

The asymptotics of \( T(z) \) as \( z \to \infty \) can be computed using the asymptotics (2.5) of \( \Psi \) and (3.9) of \( g \). The first matrix on the right-hand side of (3.14) is chosen to compensate for the behavior (3.9) of \( g(z) \) as \( z \to \infty \). It follows that

\[
T(z) = (I + O(z^{-1})) \left[ z^{\frac{1}{4}} M^{-1} \right] \quad \text{as} \quad z \to \infty,
\]

(3.15)

where the complex powers are defined using the principal branch. Using (3.7), we note the following factorization of the jump \( T_-(z)^{-1}T_+(z) \) for \( z \in (x_1, x_0) \):

\[
\begin{pmatrix}
1 & 0 \\
e^{-2i\frac{\pi}{4} g_\pm(z)} & 1
\end{pmatrix} = \begin{pmatrix}
1 & 0 \\
e^{-2i\frac{\pi}{4} g_\mp(z)} & 1
\end{pmatrix} \begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix} \begin{pmatrix}
1 & 0 \\
e^{-2i\frac{\pi}{4} g_\pm(z)} & 1
\end{pmatrix}.
\]

(3.16)

Let \( \tilde{\gamma}_+ \) and \( \tilde{\gamma}_- \) be two simple curves oriented from \( x_1 \) to \( x_0 \) lying in the upper and lower half-planes, respectively, see Figure 3. The second transformation \( T \to S \) is defined by

\[
S(z) = T(z) \begin{cases}
1 & \text{if} \quad \text{z is below} \quad \tilde{\gamma}_+, \text{Im} \, z > 0, \\
-1 & \text{otherwise},
\end{cases}
\]

(3.17)

\( S \) satisfies the following RH problem, whose properties can be deduced from those of \( \Psi \), combined with Lemma 3.2, the definition (3.14) of \( T \) and the factorization (3.16).
RH problem for $S$

(a) $S : \mathbb{C} \setminus \Sigma_S \to \mathbb{C}^{2 \times 2}$ is analytic, where $\Sigma_S := \mathbb{R} \cup \gamma_+ \cup \gamma_-$ and $\gamma_\pm := \tilde{\gamma}_\pm \cup (e^{\pm 2\pi i/3}(-\infty, x_2))$. The orientation of $\Sigma_S$ is shown in Figure 3.

(b) The jumps for $S$ are given by

$$S_+(z) = S_-(z) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad z \in (-\infty, x_2) \cup (x_1, x_0),$$

$$S_+(z) = S_-(z) \begin{pmatrix} 1 & 0 \\ e^{-2\pi z/3}g(z) & 1 \end{pmatrix}, \quad z \in \gamma_+ \cup \gamma_-,$$

$$S_+(z) = S_-(z)e^{-i\Omega r_3/2} \sigma_3, \quad z \in (x_2, x_1),$$

$$S_+(z) = S_-(z) \begin{pmatrix} 1 & e^{2\pi z/3}g(z) \\ 0 & 1 \end{pmatrix}, \quad z \in (x_0, \infty).$$

(c) As $z \to \infty$, we have $S(z) = (I + \mathcal{O}(z^{-1})) z^{3/2} M^{-1}$.

(d) As $z \to x_j$, $j = 0, 1, 2$, we have $S(z) = \mathcal{O}(\log(z-x_j))$.

Choose $\mathcal{V}$ as in Lemma 3.2. Deforming the contours $\gamma_+$ and $\gamma_-$ if necessary, we may assume that they lie in $\mathcal{V}$. Since $\text{Reg}(z) \geq 0$ for all $z \in \mathcal{V}$ with equality only if $z \in (-\infty, x_2] \cup [x_1, x_0]$, the jumps for $S$ are exponentially close to $I$ as $r \to +\infty$ on $\gamma_+ \cup \gamma_-$. This convergence is uniform except for $z$ in small neighborhoods of $x_j$, $j = 0, 1, 2$.

4 Global parametrix

Ignoring the exponentially small jumps for $S$, and ignoring small neighborhoods of $x_0, x_1, x_2$, we are led to consider the following RH problem.

RH problem for $P^{(\infty)}$

(a) $P^{(\infty)} : \mathbb{C} \setminus (-\infty, x_0] \to \mathbb{C}^{2 \times 2}$ is analytic.

(b) The jumps for $P^{(\infty)}$ are given by

$$P^{(\infty)}_+(z) = P^{(\infty)}_-(z) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad z \in (-\infty, x_2) \cup (x_1, x_0),$$

$$P^{(\infty)}_+(z) = P^{(\infty)}_-(z) e^{-i\Omega r_3/2} \sigma_3, \quad z \in (x_2, x_1).$$

Figure 3: The jump contour $\Sigma_S$. 
For convenience; its solution is denoted by $\Phi_A$.

As $z \to x_j$, $j = 0, 1, 2$, we have $P^{(\infty)}(z) = O((z - x_j)^{-\frac{1}{2}})$.

The above RH problem was solved in [6] (the points $y_1$, $y_2$ and 0 in [6] should be identified with $x_0$, $x_1$ and $x_2$ in the present paper, respectively). Consider the function $\varphi : \mathbb{C} \setminus (-\infty, x_0) \to \mathbb{C}$ defined by

$$\varphi(z) = \int_{x_0}^{z} \omega, \quad \omega = \frac{c_0 dz}{\sqrt{\mathcal{K}(z)}}, \quad c_0 = \frac{\sqrt{x_0 - x_2}}{4K(k)}, \quad (4.2)$$

where the path of integration lies in $\mathbb{C} \setminus (-\infty, x_0)$, and $k$ has been defined in (3.12). Note that the holomorphic differential $\omega$ of (1.4) with $g = 1$ coincides with $\omega$. By [6], $P^{(\infty)}$ is given by

$$P^{(\infty)}(z) = \begin{pmatrix} \frac{1}{G'(\frac{1}{2})} & -iG(z) \\ 0 & \frac{1}{G(0)} \end{pmatrix} \begin{pmatrix} \beta(z)G(-\varphi(z)) & -iG(\varphi(z)) \\ -iG(-\varphi(z) - \frac{1}{2}) & G(\varphi(z) - \frac{1}{2}) \end{pmatrix} \quad (4.3)$$

with

$$G(z) = \frac{\Theta(z + \nu)}{\Theta(z)}, \quad \nu = -\frac{\Omega}{2\pi} \tau^\frac{1}{2}, \quad c_G = 2c_0(\log G)'(\frac{1}{2}), \quad \beta(z) = \frac{(z - x_0)^{1/4}(z - x_2)^{1/4}}{(z - x_1)^{1/4}} \quad (4.4)$$

where the principal branch is taken for the roots. The function $\theta$ is defined by (1.6) with $g = 1$, and is associated to the parameter $\tau \in i\mathbb{R}^+$ defined in (1.5).

5 Local parametrices

The goal of this section is to construct local approximations (called “parametrices”) of $S$ in small open disks $\mathbb{D}_{x_j}$ centered at $x_j$, $j = 0, 1, 2$. The local parametrix $P^{(x_j)}$ possesses the same jumps as $S$ inside $\mathbb{D}_{x_j}$, $P^{(x_j)} = O(\log(z - x_j))$ as $z \to x_j$, and satisfies $P^{(x_j)}(z)P^{(\infty)}(z)^{-1} = I + o(1)$ as $r \to +\infty$ uniformly for $z \in \partial\mathbb{D}_{x_j}$. In our case, the local parametrices are standard: $P^{(x_0)}$ can be built in terms of Airy functions (as in [17]), and $P^{(x_1)}$ and $P^{(x_2)}$ in terms of Bessel functions (as in [19]).

5.1 Parametrix at $x_0$

The function $f_0(z) := (-\frac{d}{dz})(\frac{1}{2})\frac{1}{\sqrt{\mathcal{K}(z)}}$ is a conformal map from $\mathbb{D}_{x_0}$ to a neighborhood of 0, and satisfies

$$f_0(z) = c_{x_0}(z - x_0) + c_{x_0}^{(2)}(z - x_0)^2 + O((z - x_0)^3) \quad \text{as} \quad z \to x_0, \quad (5.1)$$

$$c_{x_0} = \frac{(-\dot{q}(x_0))^\frac{1}{2}}{(x_0 - x_1)^\frac{1}{2}(x_0 - x_2)^\frac{1}{2}} > 0, \quad c_{x_0}^{(2)} = \frac{-2x_0 + x_1 + x_2}{5(x_0 - x_1)(x_0 - x_2)} - \frac{2}{5\dot{q}(x_0)} \quad (5.2)$$

$$175c_{x_0}^{(3)} = \frac{43x_0^2 - 43x_0x_1 + 17x_1^2 - 43x_0x_2 + 9x_1x_2 + 17x_2^2}{(x_0 - x_1)^2(x_0 - x_2)^2} + \frac{18(2x_0 - x_1 - x_2)}{(x_0 - x_1)(x_0 - x_2)\dot{q}(x_0)} - \frac{7}{\dot{q}(x_0)^2},$$

where we recall that $-\dot{q}(x_0) > 0$. The model RH problem of [17], which is needed for the construction of $P^{(x_0)}$, is presented in Appendix A for convenience; its solution is denoted by $\Phi_A$. Deforming $\gamma_{\pm}$ if necessary, we may assume that $f_0(\gamma_{\pm} \cap \mathbb{D}_{x_0}) \subset e^{\pm i\pi} \mathbb{R}^+$. It can be verified that

$$P^{(x_0)}(z) = E_{x_0}(z)\Phi_A(r f_0(z))e^{-\frac{1}{2}\varphi(z)\sigma_3}, \quad E_{x_0}(z) = P^{(\infty)}(z)M^{-1}(r f_0(z))^\frac{1}{2}, \quad (5.3)$$

where $E_{x_0}(z)$ is analytic for $z \in \mathbb{D}_{x_0}$. Furthermore, due to (5.1) and (A.2),

$$P^{(x_0)}(z)P^{(\infty)}(z)^{-1} = I + \frac{P^{(\infty)}(z)\Phi_{A,1}P^{(\infty)}(z)^{-1}}{r^{\frac{1}{2}}f_0(z)^\frac{1}{2}} + O(r^{-3}) \quad (5.4)$$

as $r \to +\infty$ uniformly for $z \in \partial\mathbb{D}_{x_0}$.
Again, deforming the lenses if necessary, we may assume that \( \tilde{P} \) by definition of \( e_{R} \), \( \Sigma_{R} \) for \( S \) Let us define follows from the steepest descent analysis that
\[
E \to z_{\infty}
\]
\[
E \to z_{1}
\]
\[
E \to z_{0}
\]

\[\begin{aligned}
\tilde{f}(z) &= -c_{x_{1}}(z - x_{1})(1 + c^{(2)}_{x_{1}}(z - x_{1}) + O((z - x_{1})^{2})), \quad c_{x_{1}} = \frac{q^{2}(x_{1})}{(x_{1} - x_{2})(x_{0} - x_{1})}, \quad \text{as } z \to x_{1}, \quad (5.5) \\
\tilde{f}(z) &= c_{x_{2}}(z - x_{2})(1 + c^{(2)}_{x_{2}}(z - x_{2}) + O((z - x_{2})^{2})), \quad c_{x_{2}} = \frac{q^{2}(x_{2})}{(x_{0} - x_{2})(x_{1} - x_{2})}, \quad \text{as } z \to x_{2}. \quad (5.6)
\end{aligned}\]

Again, deforming the lenses if necessary, we may assume that \( \tilde{f}(\gamma_{\pm} \cap D_{x_{1}}) \subset e^{\pm \frac{\pi i}{3}} \mathbb{R}^{+} \) and \( \tilde{f}(\gamma_{\pm} \cap D_{x_{2}}) \subset e^{\pm \frac{\pi i}{3}} \mathbb{R}^{+} \). The local parametrices \( P^{(x_{1})} \) and \( P^{(x_{2})} \) are given by
\[
\begin{aligned}
P^{(x_{1})}(z) &= E_{x_{1}}(z)\sigma_{3}\Phi_{\text{Be}}(r^{3}\tilde{f}(z))\sigma_{3}e^{-\frac{\pi i}{2}g(z)\sigma_{3}}, \quad E_{x_{1}}(z) = P^{(\infty)}(z)e^{\mp \frac{\pi i}{3}r^{3}z_{x_{1}}}M\left(2\pi r^{2}\tilde{f}(z)^{\frac{1}{2}}\right)^{\frac{\pi i}{2}}, \quad (5.7) \\
P^{(x_{2})}(z) &= E_{x_{2}}(z)\Phi_{\text{Be}}(r^{3}\tilde{f}(z))e^{-\frac{\pi i}{2}g(z)\sigma_{3}}, \quad E_{x_{2}}(z) = P^{(\infty)}(z)e^{\mp \frac{\pi i}{3}r^{3}z_{x_{2}}}M^{-1}\left(2\pi r^{2}\tilde{f}(z)^{\frac{1}{2}}\right)^{\frac{\pi i}{2}}, \quad (5.8)
\end{aligned}\]

where \( E_{x_{j}}(z) \) is analytic in \( D_{x_{j}}, \ j = 1, 2, \) and \( \Phi_{\text{Be}} \) is the solution of the Bessel model RH problem recalled in Appendix B. As \( r \to +\infty \), we have \( P^{(x_{j})}(z)P^{(\infty)}(z)^{-1} = I + J^{(1)}_{R}(z)r^{-\frac{\pi i}{3}} + O(r^{-3}) \) uniformly for \( z \in \partial D_{x_{j}}, \) for a certain matrix \( J^{(1)}_{R}(z) \) that will be computed in (6.3).

## 6 Small norm problem

Let us define
\[
R(z) = \begin{cases} 
S(z)P^{(x_{j})}(z)^{-1}, & z \in D_{x_{j}}, \ j = 0, 1, 2, \\
S(z)P^{(\infty)}(z)^{-1}, & z \in \mathbb{C} \setminus \bigcup_{j=0}^{2} \partial D_{x_{j}}.
\end{cases} \quad (6.1)
\]

From the asymptotics of \( S(z) \) and \( P^{(\infty)}(z) \) as \( z \to \infty \), we have \( R(z) = I + O(z^{-1}) \) as \( z \to \infty \). Also, by definition of \( P^{(x_{j})}, j = 0, 1, 2, \) \( R(z) \) is analytic for \( z \in \bigcup_{j=0}^{2} \partial D_{x_{j}} \). Therefore, the jump contour for \( R \), denoted by \( \Sigma_{R} \), is given by
\[
\Sigma_{R} = \left( (x_{0}, +\infty) \cup \gamma_{+} \cup \gamma_{-} \cup \bigcup_{j=0}^{2} \partial D_{x_{j}} \right) \setminus \bigcup_{j=0}^{2} \partial D_{x_{j}},
\]
where we orient (for convenience) the boundaries of the disks in the clockwise direction, see Figure 4. It follows from the steepest descent analysis that
\[
J_{R}(z) := R_{-}(z)^{-1}R_{+}(z) = \begin{cases} 
I + O(e^{-\frac{\pi i}{3}r|x|^{2}}), & \text{uniformly for } z \in \Sigma_{R} \cup_{j=0}^{2} \partial D_{x_{j}}, \\
I + \frac{J^{(1)}_{R}(z)}{r^{\frac{\pi i}{3}}} + O(r^{-3}), & \text{uniformly for } z \in \bigcup_{j=0}^{2} \partial D_{x_{j}},
\end{cases} \quad (6.2)
\]
where \( \hat{c} > 0 \) is a sufficiently small constant, and \( J_R^{(1)}(z) \) is given by

\[
J_R^{(1)}(z) = \begin{cases} \frac{p^{(\infty)}(z)\Phi_{\mathbb{A}_1}p^{(\infty)}(z)^{-1}}{f_0(z)} & \text{if } z \in \partial \mathbb{D}_{x_0}, \\ \frac{p^{(\infty)}(z)e^{\Xi^1_{21}}}{f(z)^{1/2}} & \text{if } z \in \partial \mathbb{D}_{x_j}, \ j = 1, 2. \end{cases}
\]  

(6.3)

Since \( \Omega \in \mathbb{R} \), \( J_R^{(1)}(z) = \mathcal{O}(1) \) as \( r \to +\infty \) uniformly for \( z \in \cup_{j=0}^2 \partial \mathbb{D}_{x_j} \). The jump matrix \( J_R \) is uniformly close to the identity, thus \( R(z) \) exists for sufficiently large \( r \) and [17, 18, 20]

\[
R(z) = I + \frac{R^{(1)}(z)}{r^2} + \mathcal{O}(r^{-3}), \quad \text{where } R^{(1)}(z) = \frac{1}{2\pi i} \int_{j=0}^2 \frac{\mathcal{J}_R^{(1)}(\xi)}{\xi - z} \, d\xi,
\]  

(6.4)

as \( r \to +\infty \) uniformly for \( z \in \mathbb{C} \setminus \Sigma_R \), and these asymptotics can be differentiated with respect to \( z \) without changing the error term. In Section 7, we will need explicit expressions for

\[
\left[ E_{x_j}(x_j)^{-1}R^{(1)}(x_j)E_{x_j}(x_j) \right]_{21}, \quad j = 1, 2.
\]  

(6.5)

These quantities can be computed from (6.3) and (6.4) by residue calculations. Let us define \( \mathcal{J}_{x_j}(z) := [E_{x_j}(x_j)^{-1}J_R^{(1)}(z)x_j(x_j)]_{21} \) for \( j = 1, 2 \). By (6.3), as \( z \to x_j', \ j' = 0, 1, 2 \), we have

\[
\mathcal{J}_{x_j}(z) = \sum_{k=-2}^1 (\mathcal{J}_{x_j})^{(k)}_{x_j}(z-x_j')^k + \mathcal{O}((z-x_j')^2), \quad (\mathcal{J}_{x_j})^{(-2)}_{x_j} = (\mathcal{J}_{x_j})^{(-2)}_{x_j} = 0,
\]  

(6.6)

for certain matrices \( (\mathcal{J}_{x_j})^{(k)}_{x_j} \) that can be computed if needed. Since the circles \( \partial \mathbb{D}_{x_j}, \ j = 0, 1, 2 \), have clockwise orientation in (6.4), we obtain

\[
\left[ E_{x_j}(x_j)^{-1}R^{(1)}(x_j)E_{x_j}(x_j) \right]_{21} = -(\mathcal{J}_{x_j})^{(1)}_{x_j} + \frac{2(\mathcal{J}_{x_j})^{(-2)}_{x_j}}{(x_0 - x_j)^3} - \sum_{j' \neq j} \frac{2(\mathcal{J}_{x_j})^{(-1)}_{x_j}}{(x_j - x_j')^2}, \quad j = 1, 2.
\]  

(6.7)

### 7 Proof of Theorem 1.2

Substituting the transformations \( \Psi \to T \to S \to R \) of Sections 3-6 into the differential identity (2.6) and using the expansion (6.4) of \( R \), we obtain the following asymptotics:

\[
\partial_r \log F(r(x_2, x_1)) = I_1(r) + I_2(r) + I_3(r) + \mathcal{O}(r^{-\hat{c}}) \quad \text{as } r \to +\infty,
\]  

(7.1)

where

\[
I_1(r) := r^2 \sum_{j=1}^2 (-1)^j x_j c_{x_j}, \quad I_2(r) := \frac{1}{2\pi i r^2} \sum_{j=1}^2 (-1)^j x_j \left[ E_{x_j}(x_j)^{-1}R^{(1)}(x_j)E_{x_j}(x_j) \right]_{21},
\]  

\[
I_3(r) := \frac{1}{2\pi i r^2} \sum_{j=1}^2 (-1)^j x_j \left[ E_{x_j}(x_j)^{-1}E_{x_j}^t(x_j) \right]_{21},
\]

with \( c_{x_j}, j = 1, 2 \), given by (5.5)-(5.6). A straightforward calculation shows that

\[
I_1(r) = c \partial_r r^3,
\]  

(7.2)

where

\[
c = \frac{1}{12} \left[ x_0^3 + x_1^3 + x_2^3 - (x_0 + x_1)(x_0 + x_2)(x_1 + x_2) \right] - \frac{g_0}{3}(x_0 + x_1 + x_2),
\]  

(7.3)

and the constant \( g_0 \) is defined in (3.1). By a simple rearrangement of the terms, it is easy to see that \( c \) in (7.3) coincides with the constant \( c \) defined in (1.11) for \( g = 1 \). This establishes the leading term in (1.10) for \( g = 1 \); the two subleading terms are more complicated to evaluate.
7.1 Explicit expression for $I_3(r)$

From the definition of $E_{x_i}$, $j = 1, 2$, it is possible to obtain explicit expressions for $E_{x_1}(x_1)$ and $E'_{x_1}(x_1)$. These expressions are rather long, so we only write down $E_{x_1}(x_1)$ and $E'_{x_1}(x_1)$ (the expressions for $E_{x_2}(x_2)$ and $E'_{x_2}(x_2)$ are similar):

$$E_{x_1}(x_1) = e^{i\nu}e^{-\frac{1}{2}x_3^2} \left[ \frac{\beta_{x_1}^{(-\frac{1}{2})}}{\mathcal{G}(\frac{1}{2})} \left( \begin{array}{c} \mathcal{G}(\frac{1}{2}) \varphi_{x_1}^{(\frac{1}{2})} G'(\frac{1}{2}) \\ 0 \\ \varphi_{x_1}^{(\frac{1}{2})} G'(\frac{1}{2}) \\ 0 \end{array} \right) \right] + \frac{1}{\beta_{x_1}^{(-\frac{1}{2})} \mathcal{G}(0)} \left( \begin{array}{c} c_g \\ 1 \\ \varphi_{x_1}^{(\frac{1}{2})} G'(\frac{1}{2}) \\ 0 \end{array} \right) \left( \begin{array}{c} \varphi_{x_1}^{(\frac{1}{2})} G'(\frac{1}{2}) \\ 0 \end{array} \right) \right],$$

$$E'_{x_1}(x_1) = e^{i\nu}e^{-\frac{1}{2}x_3^2} \left[ \frac{\beta_{x_1}^{(-\frac{1}{2})}}{\mathcal{G}(\frac{1}{2})} \left( \begin{array}{c} \mathcal{G}(\frac{1}{2}) \varphi_{x_1}^{(\frac{1}{2})} G'(\frac{1}{2}) \\ 0 \\ \varphi_{x_1}^{(\frac{1}{2})} G'(\frac{1}{2}) \\ 0 \end{array} \right) \right] + \frac{1}{\beta_{x_1}^{(-\frac{1}{2})} \mathcal{G}(0)} \left( \begin{array}{c} c_g \\ 1 \\ \varphi_{x_1}^{(\frac{1}{2})} G'(\frac{1}{2}) \\ 0 \end{array} \right) \left( \begin{array}{c} \varphi_{x_1}^{(\frac{1}{2})} G'(\frac{1}{2}) \\ 0 \end{array} \right) \right],$$

where $\mathcal{G}(z) = \mathcal{G}(z - \frac{1+i}{2})$, the coefficients $\beta_{x_1}^{(-\frac{1}{2})}$, $\beta_{x_1}^{(\frac{1}{2})}$, $\varphi_{x_1}^{(\frac{1}{2})}$, $\varphi_{x_1}^{(\frac{1}{2})}$, $c_{x_1}^{(2)}$ are defined through the expansions

$$\beta(z) = \beta_{x_1}^{(-\frac{1}{2})}(z - x_1)^{-\frac{1}{2}} + \beta_{x_1}^{(\frac{1}{2})}(z - x_1)^{\frac{1}{2}} + O((z - x_1)^{\frac{3}{2}}), \quad \text{as } z \to x_1, \text{ Im } z > 0,$$

$$\varphi(z) = \varphi_{x_1}^{(\frac{1}{2})}(z - x_1)^{\frac{1}{2}} + \varphi_{x_1}^{(\frac{1}{2})}(z - x_1)^{-\frac{1}{2}} + O((z - x_1)^0), \quad \text{as } z \to x_1, \text{ Im } z > 0,$$

and via (5.5), and they are given by

$$\beta_{x_1}^{(-\frac{1}{2})} = \frac{e^{\frac{i}{2}x_1^2}}{\sqrt{2}(x_0 - x_1)^{\frac{1}{2}}(x_0 - x_2)^{\frac{1}{2}}}, \quad \beta_{x_1}^{(\frac{1}{2})} = \frac{e^{\frac{i}{2}x_1^2}}{\sqrt{2}(x_0 - x_1)^{\frac{1}{2}}}, \quad \beta_{x_1}^{(-\frac{1}{2})} = \frac{e^{\frac{i}{2}x_1^2}}{\sqrt{2}(x_0 - x_1)^{\frac{1}{2}}}, \quad \varphi_{x_1}^{(\frac{1}{2})} = \frac{-2ic_0}{\sqrt{(x_0 - x_1)^{\frac{1}{2}}(x_0 - x_2)^{\frac{1}{2}}}},$$

$$\varphi_{x_1}^{(\frac{1}{2})} = \frac{3ic_0(x_0 - x_1)^{\frac{1}{2}}(x_0 - x_2)^{\frac{1}{2}}}{3(x_0 - x_1)^{\frac{1}{2}}(x_0 - x_2)^{\frac{1}{2}}}.$$

Each of the expressions for $E_{x_1}(x_1)$, $E_{x_2}(x_2)$, $E'_{x_1}(x_1)$ and $E'_{x_2}(x_2)$ contain the $\theta$-function and its derivatives at the points $0, \frac{1}{2}, 1$ and $\frac{1+i}{2}$. We use $\theta$-function identities to simplify these expressions following the strategy of [6].

The function $\varphi$ can be analytically continued to the Riemann surface $X$ defined in the introduction, and its quotient $\varphi_A(z) := \varphi(z) \mod (\mathbb{Z} + i\mathbb{Z})$ is a bijection from $X$ to $\mathbb{C} / (\mathbb{Z} + i\mathbb{Z})$ [22]. The following proposition can be taken straight from [6] after replacing $y_1$, $y_2$ and $z$ with $x_0 - x_2$, $x_1 - x_2$ and $z - x_2$, respectively, and noting that $\varphi_A(z - x_2)$ in [6] equals $\varphi_A(z)$ here.

**Proposition 7.1** (Some $\theta$-function identities). For all $\hat{\nu} \in X$, we have

$$\theta(\hat{\nu} + \frac{1}{2}) = e^{-2i\pi \nu} \frac{D_1^2}{a - x_2} \theta(\hat{\nu})^2, \quad D_1 = (x_0 - x_2)^\frac{1}{2}(x_1 - x_2)^\frac{1}{2}e^{-\frac{\pi i}{4}},$$

$$\theta(\hat{\nu} + \frac{1+i}{2}) = e^{-2i\pi \nu} \frac{D_2^2}{(a - x_0)} \theta(\hat{\nu})^2, \quad D_2 = i(x_1 - x_2)^\frac{1}{2} e^{-\frac{\pi i}{4}} (x_0 - x_1)^\frac{1}{2}.$$
where \( \dot{a} = \varphi_0^{-1}(\dot{\nu}) \). These expressions make it possible to express \( \theta^{(j)}(\frac{1}{2}), \theta^{(j)}(\frac{2}{3}), \theta^{(j)}(\frac{j+1}{4}) \) for \( j \geq 0 \) in terms of \( \theta^{(j)}(0), j \geq 0 \). For example,

\[
\theta\left(\frac{\tau}{2}\right) = e^{-\frac{i\pi}{2}(x_1-x_2)_2^+} \theta(0), \quad \theta\left(\frac{1}{2}\right) = (x_0-x_1)_2^+ \theta(0), \quad \theta\left(\frac{\tau}{2}\right) = -i\pi e^{-\frac{i\pi}{2}(x_1-x_2)_2^+} \theta(0),
\]

\[
\theta'\left(\frac{1+\tau}{2}\right) = ie^{-\frac{i\pi}{2}(x_0-x_1)_2^+}(x_1-x_2)_2^+ \theta(0), \quad \theta''\left(\frac{1+\tau}{2}\right) = \pi e^{-\frac{i\pi}{2}(x_0-x_1)_2^+}(x_1-x_2)_2^+ \theta(0),
\]

\[
\theta''\left(\frac{\tau}{2}\right) = e^{-\frac{i\pi}{2}(x_1-x_2)_2^+} \theta''(0) - \left[ \pi^2 + \frac{x_0-x_1}{4c_0^2} \right] \theta(0),
\]

\[
\theta''\left(\frac{1}{2}\right) = -\frac{(x_0-x_1)_2^+}{4c_0^2(x_0-x_2)_2^+} \left( x_1-x_2 \theta(0) + 4c_0^2 \theta''(0) \right).
\]

As previously mentioned, we compute \( E_{x_j}(x_j), E'_{x_j}(x_j), j = 1, 2 \), via \( (5.7), (5.8) \) and now use Proposition 7.1 to obtain

\[
\frac{-x_1}{2\pi r} \left[ E_{x_1}(x_1)^{-1}E'_{x_1}(x_1) \right]_{21} = -\frac{8\pi_2 c_1 q_1(x_1)}{3\Omega(x_0-x_1)(x_1-x_2)} \frac{d}{dr} \left[ \log(\theta(0)) \right],
\]

\[
\frac{x_2}{2\pi r} \left[ E_{x_2}(x_2)^{-1}E'_{x_2}(x_2) \right]_{21} = \frac{8\pi_0 c_0 q_2(x_2)}{3\Omega(x_0-x_2)(x_1-x_2)} \frac{d}{dr} \left[ \log(\theta(0)) \right],
\]

where \( \nu \) is defined in \( (4.4) \). It follows that \( I_3(r) = c_3 \partial_r \log(\theta(\nu)) \), where the constant \( c_3 \) is given by

\[
c_3 = \frac{-8\pi_2 c_1}{3\Omega(x_0-x_2)} \left( \frac{x_1 q_1(x_1)}{x_0-x_1} - \frac{x_2 q_2(x_2)}{x_0-x_2} \right) = \frac{4\pi_0}{3\Omega} \left( x_0 + x_1 + x_2 \right).
\]

Using \( (3.13), (4.2) \), and the property \([23, Eq. 8.122]\) of complete elliptic integrals, we find

\[
\frac{\Omega}{c_0} = \frac{8}{3} (x_0 + x_1 + x_2) \left( K(k) - E(k) \right) = \frac{4\pi}{3} (x_0 + x_1 + x_2),
\]

which shows that \( c_3 = 1 \). This proves that

\[
I_3(r) = \partial_r \log(\theta(\nu)). \quad \text{(7.5)}
\]

### 7.2 Analysis of \( I_2(r) \)

By \( (6.7) \), we have

\[
I_2(r) = \frac{1}{2\pi r^2} \left\{ \sum_{j=1}^2 (-1)^{j+1} x_j (J_{x_j})^{(1)} \right\} + \frac{2}{(x_0-x_1)^2} \sum_{j=1}^2 (-1)^{j+1} x_j (J_{x_j})^{(2)}
\]

\[
+ \frac{2}{x_0-x_1} \sum_{j=1}^2 \sum_{j'=0}^{2} \frac{(-1)^{j+1} x_j (J_{x_j})^{(1)} x_{j'}}{(x_j-x_{j'})^2} \right\}. \quad \text{(7.6)}
\]

To simplify the right-hand side, we need Proposition 7.1 as well as the following identities.

**Proposition 7.2.** We have

\[
\frac{\theta''(0)}{\theta(0)} = -\frac{(x_1-x_2)(\gamma x_0 + \delta)}{4c_0^2 (x_0 + x_1 + x_2)}, \quad \frac{\theta''(\varphi(z))}{\theta(\varphi(z))} = \frac{\gamma z + \delta}{z - x_2} \varphi'(z), \quad z \in X,
\]

where

\[
\gamma = \frac{(x_0-x_2)(3x_0-x_1+x_2)}{4c_0^2 (x_0 + x_1 + x_2)}, \quad \delta = \frac{(x_0-x_2)(x_0(x_1+2x_2)+x_1(x_1-x_2))}{4c_0^2 (x_0 + x_1 + x_2)}.
\]
Proof. We will first show that there exist $\gamma, \delta \in \mathbb{C}$ such that
\[
\frac{\theta''(\varphi(z))\theta(\varphi(z)) - \theta'(\varphi(z))^2}{\theta(\varphi(z))^2} = \frac{\gamma z + \delta}{z - x_2}, \tag{7.9}
\]

Let $h(z)$ denote the left-hand side of (7.9). Using (1.7) and the fact that $\frac{1+z}{2}$ is a simple zero of $\theta(u)$, we verify that $\left(\frac{\theta'(u)}{\theta(u)}\right)' = \frac{\theta''(u)\theta(u) - \theta'(u)^2}{\theta(u)^2}$ is well-defined on $\mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$ with a double pole at $\frac{1+z}{2}$ and no other poles. Since the Abel map $\varphi_A : X \rightarrow \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$ is an isomorphism, it follows that $h$ is a meromorphic function on $X$ with a double pole at $\varphi^{-1}(\frac{1+z}{2}) = x_2$ and no other poles. In fact, since $\theta(u)$ is an even function of $u$ and $\varphi(z) = -\varphi(-z)$, where $i : X \rightarrow X$ denotes the sheet-changing involution, $h$ descends to a meromorphic function on the Riemann sphere with a simple pole at $x_2$ and no other poles. This shows that (7.9), and hence also (7.8), holds for some choice of $\gamma, \delta \in \mathbb{C}$.

Taking $z \rightarrow \infty$ in (7.9) and using Proposition 7.1, we see that
\[
\gamma = \frac{\theta''(\frac{1}{j})}{\theta(\frac{1}{j})} = \frac{x_1 - x_2}{4c_0^2} + \frac{\theta''(0)}{\theta(0)}. \tag{7.10}
\]

Multiplying (7.9) by $\varphi'(z)$, integrating from $x_0$ to $x > x_0$, and using (4.2) and [23, Eqs. 3.131.7, 3.133.17], we obtain
\[
\theta'(\varphi(x)) = \frac{\gamma x_1 + \delta}{x_1 - x_2} \varphi(x) + \frac{2(\gamma x_2 + \delta)(x - x_0)}{x_0 - x_2} \varphi'(x) = \frac{\gamma x_2 + \delta}{2(x_1 - x_2)} \int_0^x \left(\sin^{-1}\frac{x-x_0}{2(\gamma x_1 + \delta)} \kappa \right), \tag{7.11}
\]

where $k = \sqrt{\frac{x_1 - x_2}{x_0 - x_2}}$, and $\mathcal{E}(\varphi, k) := \int_0^\pi \sqrt{1 - k^2 \sin^2 t} \, dt$ is the elliptic integral of the second kind. Sending $x \rightarrow \infty$ in (7.11) and using (3.12), we obtain $\delta$ in terms of $\gamma$. Then, taking $z \rightarrow x_0$ in (7.9), we find (7.7). Substituting (7.7) into (7.10), we find the asserted expression for $\gamma$ and as an immediate consequence we find the asserted expression for $\delta$. \hfill \square

The quantity $a := \varphi_A^{-1}(\nu) \in X$ appears in the large $r$ asymptotics of (7.6). From (4.2), we note that $\varphi(x)$ is monotone for $x \in (x_0, +\infty)$, and satisfies $\varphi_A(x_0) = 0$ and $\varphi_A(\infty) = \frac{1}{r}$. Hence, as $r$ increases, $a$ oscillates between $x_0$ and $+\infty$ (and changes sheet). More precisely, $a$ belongs to the upper sheet and satisfies $\sqrt{\kappa(a)} > 0$ if $\nu \mod 1 \in (0, \frac{1}{r})$, while $a$ belongs to the lower sheet and satisfies $\sqrt{\kappa(a)} < 0$ if $\nu \mod 1 \in (\frac{1}{r}, 1)$.

Proposition 7.3. We have
\[
I_2(r) = \frac{d}{dr} \left\{ \tilde{c}_0 \log(r) + \int_0^r \left[ \frac{\tilde{c}_2}{\bar{r}(\bar{a} - x_2)^2} + \frac{\tilde{c}_1}{\bar{r}(\bar{a} - x_2)} + \frac{\tilde{c}_3}{\bar{r}} \left( \frac{\theta'(\tilde{\nu})}{\theta(\tilde{\nu})} \right)^3 \right] \, d\tilde{\nu} \right\}, \tag{7.12}
\]

where $\tilde{\nu} = -\frac{\bar{r}}{r} \bar{a}, \bar{a} = \varphi_A^{-1}(\nu)$, and
\[
\begin{align*}
\tilde{c}_2 &= \frac{17(x_0 - x_2)(x_1 - x_2)(x_1 \bar{q}(x_2) - x_2 \bar{q}(x_1))}{48q(x_0)(x_0 - x_1)}, \\
\tilde{c}_1 &= \frac{3x_2q(x_2) - 3x_1 \bar{q}(x_1) - 29x_2^2 - 20x_0x_1 + 21x_0x_2 - 11x_1^2 + 9x_1x_2 + 30x_2^2}{48q(x_0)(x_0 - x_1)(x_0 + x_1 + x_2)} \\
&+ \frac{x_2 \bar{q}(x_2)(29x_2^2 + 13x_1x_1 - 14x_0x_2 + 4x_1^2 - 9x_1x_2 - 23x_2^2)}{q(x_0)(x_0 - x_1)(x_0 + x_1 + x_2)} + \frac{3(x_0 - x_2)(x_1 \bar{q}(x_1) - x_2 \bar{q}(x_2))}{q(x_0)^2},
\end{align*}
\]
\[ \tilde{e}_0 = \frac{1}{48} \left( \frac{2x_2q(x_2)}{q(x_2)} \left( -11x_0^2 + 2x_0(x_1 + x_2) + 3x_1^2 + x_1x_2 + 3x_2^2 \right) + 7x_1q(x_1)(x_0 - x_2)(x_0 + x_1 + x_2) \right. \\
+ \frac{3x_2q(x_2)}{q(x_2)} + \frac{3x_1q(x_1)}{q(x_1)} - \frac{3x_2q(x_2)}{q(x_1)(x_1 - x_2)} + 3x_0 \left( \frac{6}{x_0 + x_1 + x_2} + \frac{1}{x_0 - x_1} + \frac{1}{x_0 - x_2} \right) \\
- \frac{3x_1q(x_1)}{q(x_1)} - \frac{3x_2q(x_2)}{q(x_2)} \right), \\
\tilde{e}_1(\tilde{a}) = \frac{c_0}{q(x_0)(x_0 - x_1)} \left( x_0q(x_0) - x_2q(x_2) \right), \\
\tilde{e}_2(\tilde{a}) = -\frac{\pi^2 a_0^2}{9\Omega^2 q(x_0)(x_0 - x_1)} \left( \frac{5x_1q(x_1) - x_2q(x_2)}{\tilde{a} - x_2} + \frac{5x_2q(x_2) - 9x_0q(x_0)}{x_0 - x_2} \right), \\
\tilde{e}_3 = -\frac{8\pi a_0^3}{9\Omega^2 q(x_0)(x_0 - x_1)} \frac{1}{(x_0 - x_1)(x_0 - x_2)}. \\
\]

Proof. First, using (6.3), we compute the coefficients in the expansion of \( J^{(1)}_R(z) \) as \( z \to x_j, \ j = 1, 2 \). By combining these coefficients with the expressions for \( E_{x_j}(x_j) \) (already needed in Subsection 7.1), this gives us expressions for the quantities \( (J_{x_1})^{(1)}_{x_1} \), \( (J_{x_2})^{(1)}_{x_2} \) and \( (J_{x_0})^{(1)}_{x_0} \) for \( j = 1, 2, j' = 0, 1, 2, j \neq j' \). These expressions contain the \( \theta \)-function and its derivatives evaluated at the eight points \( 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, \frac{2}{3}, \frac{1}{3}, \frac{1}{2}, \frac{3}{2} \). Second, we use Proposition 7.1 to express the quantities \( \theta^{(1)}(\frac{2}{3}), \theta^{(1)}(\frac{1}{3}), \theta^{(1)}(1), \theta^{(1)}(\frac{2}{1}), \theta^{(1)}(\frac{1}{1}), \theta^{(1)}(\frac{2}{2}) \) and \( \theta^{(1)}(\nu + \frac{1}{2}) \) in terms of only \( a, \theta^{(1)}(0) \) and \( \theta^{(1)}(\nu) \), \( \nu \geq 0 \). This allows us to obtain the following simplified expressions:

\[
(J_{x_1})^{(1)}_{x_1} = \frac{i\pi \nu}{8} \left[ \frac{q'(x_1)}{q(x_1)} + \frac{4\nu^2}{\nu} \left( \frac{\theta^{(1)}(\nu)}{\theta^{(1)}(0)} \right)^2 \right] + 12\nu \left( \frac{\theta^{(1)}(\nu)}{\theta^{(1)}(0)} - \frac{\theta^{(1)}(0)}{\theta^{(1)}(0)} \right) + x_0 - x_2 + 2x_1 + x_2, \\
(J_{x_2})^{(1)}_{x_2} = \frac{i\pi \nu}{8} \left[ \frac{q'(x_2)}{q(x_2)} - \frac{4\nu^2}{\nu} \left( \frac{\theta^{(1)}(\nu)}{\theta^{(1)}(0)} \right)^2 \right] + 12\nu \left( \frac{\theta^{(1)}(\nu)}{\theta^{(1)}(0)} - \frac{\theta^{(1)}(0)}{\theta^{(1)}(0)} \right) + x_0 - x_2 + 2x_1 + x_2, \\
(J_{x_0})^{(1)}_{x_0} = \frac{5i\pi \nu}{8} \left( x_0 - x_2 \right) q(x_1), \\
(J_{x_2})^{(1)}_{x_2} = \frac{5i\pi \nu}{24} \left( a - x_0 \right) q(x_0). \\
\]

The expressions for \( (J_{x_j})^{(1)}_{x_0} \), \( j = 1, 2 \), are significantly larger and also contain the quantities \( \frac{\theta^{(2)}(\nu)}{\theta^{(1)}(0)}, \frac{\theta^{(1)}(\nu)}{\theta^{(2)}(\nu)} \) and \( a \). In addition, they contain terms involving \( \sqrt{R(a)^{\nu}(\nu)} \). Third, we use (7.7) to express \( \theta^{(2)}(\nu) \) in terms of the \( x_j \), and we also use the relation \( \frac{\theta^{(2)}(\nu)}{\theta^{(1)}(\nu)^2} = \left( \frac{\theta^{(1)}(\nu)}{\theta^{(1)}(\nu)} \right)^2 + \frac{\theta^{(2)}(\nu)}{\theta^{(1)}(\nu)^2} \). Collecting all of the terms involving \( \frac{\theta^{(2)}(\nu)}{\theta^{(1)}(\nu)^2} \), we obtain

\[
3\tilde{e}_3 \theta^{(1)}(\nu)^2 \gamma a + \frac{\delta}{\nu^2} \theta^{(1)}(\nu)^2 a - x_2 \theta^{(1)}(\nu)^2, \\
\]

which can be rewritten as \( \frac{\gamma}{\nu^2} \theta^{(1)}(\nu)^2 \left( \frac{\theta^{(1)}(\nu)}{\theta^{(1)}(\nu)} \right)^2 \) by using (7.8). □

The next proposition generalizes [6, Proposition 8.6].

Proposition 7.4. Let \( M > 0 \) and let \( H \in L^1([0, 1]) \) be a periodic function of period 1 satisfying \( H(-u) = H(u) \) for all \( u \in [0, 1] \). As \( r \to +\infty \),

\[
\int_M H(\tilde{u}) \frac{d\tilde{u}}{r} = \log(r) \int_0^1 H(s)ds + \mathcal{C} + O(r^{-\frac{2}{3}}), \\
\]

where \( \tilde{u} = -\frac{\Omega}{2\pi r^\frac{1}{3}} \) and \( \mathcal{C} \) is a constant independent of \( r \). In particular, there are no oscillations of order 1.
Proof. Recall that $\Omega > 0$. Hence,
\[
\int_M^r \mathcal{H} \left( -\frac{\Omega\mathbb{P}^2}{2\pi} \right) \frac{d\nu}{\nu} = \frac{2}{3} \int_M^r \mathcal{H}(\nu) \frac{d\nu}{\rho} = \frac{2}{3} \left( \int_{n_0}^{n_r} + \int_{n_r}^{\nu} \right) \mathcal{H}(\nu) \frac{d\nu}{\rho} + \frac{2}{3} \sum_{j=n_0}^{n_r-1} \int_0^{1} \mathcal{H}(s) \frac{ds}{s+j},
\]
where $M' = -\frac{\partial}{\partial r} \mathbb{P}^2 < 0$, $n_0 = [M']$ and $n_r = [\nu]$. Because $\mathcal{H} \in L^1([0,1])$ and $j \leq n_0 < 0$, we have
\[
\frac{2}{3} \left( \int_{n_0}^{n_r} + \int_{n_r}^{\nu} \right) \mathcal{H}(\nu) \frac{d\nu}{\rho} = \tilde{C}_1 + O(r^{-\frac{1}{2}}),
\]
\[
\sum_{j=n_0}^{n_r-1} \int_0^{1} \frac{\mathcal{H}(s)}{s+j} ds = \sum_{j=n_0}^{n_r-1} \int_0^{1} \mathcal{H}(s) ds + \tilde{C}_2 + O(r^{-3}),
\]
as $r \to +\infty$, for certain constants $\tilde{C}_1$, $\tilde{C}_2$. Finally, since $\mathcal{H}(-s) = \mathcal{H}(s)$ and $n_r < 0$, we deduce that
\[
\int_0^{1} \mathcal{H}(s) ds \sum_{j=n_0}^{n_r-1} \frac{1}{j} = \frac{3}{2} \log r \int_0^{1} \mathcal{H}(s) ds + \tilde{C}_3 + O(r^{-\frac{1}{2}}), \quad \text{as } r \to +\infty,
\]
for some constant $\tilde{C}_3$.

We next extract the leading order behavior from Proposition 7.3.

Proposition 7.5. Fix $M > 0$. As $r \to +\infty$,
\[
\int_M^r I_2(\hat{r}) d\hat{r} = c_2 \log(r) + \tilde{C}_4 + O(r^{-\frac{1}{2}}),
\]
where $\tilde{C}_4$ is independent of $r$ and
\[
c_2 = \tilde{c}_{-1}(3x_0 - x_1 + x_2) + \tilde{c}_2 \frac{2x_0^2 + x_0(x_1 - 3x_2) - (x_1 - x_2)^2}{(x_0 - x_2)(x_1 - x_2)^2(x_0 + x_1 + x_2)} - \frac{7}{6} \int_{x_0}^\infty \tilde{c}_1(x) \frac{x}{\theta(x)} \frac{\varphi'(x)}{\varphi(x)} dx,
\]
with the integration contour from $x_0$ to $\infty$ taken on the upper sheet of $X$.

Proof. The first term on the right-hand side of (7.14) follows trivially from (7.12). Since $\nu \mapsto \varphi_A^{-1}(\nu)$ is even and periodic of period 1, we can apply Proposition 7.4 with $\mathcal{H}(\hat{\nu}) = \frac{x}{\varphi_A^{-1}(\hat{\nu}) - x_2} + \frac{d}{\varphi_A^{-1}(\hat{\nu}) - x_2}$. Moreover, with the help of [23, Eqs. 3.133.18, 3.134.18, (3.12), and (4.2), we compute
\[
\int_0^{1} \frac{ds}{\varphi_A^{-1}(s) - x_2} = 2 \int_{x_0}^{\infty} \frac{\varphi'(x)}{x} dx = \frac{3x_0 - x_1 + x_2}{(x_1 - x_2)(x_0 + x_1 + x_2)},
\]
where the integration contours in the integrals with respect to $d\xi$ lie on the upper sheet. Using these identities to evaluate $\int_0^{1} \mathcal{H}(s) ds$ explicitly, we obtain the second and third terms on the right-hand side of (7.14). Next, we use Proposition 7.4 with $\mathcal{H}(\hat{\nu}) = \tilde{c}_1(\hat{\nu}) \frac{\varphi'(\nu)}{\varphi(\nu)}$ to obtain
\[
\int_M^r \tilde{c}_1(\hat{\nu}) \frac{\varphi'(\nu)}{\varphi(\nu)} \frac{d\nu}{\rho} = -2 \log(r) \int_{x_0}^\infty \tilde{c}_1(x) \frac{\varphi'(x)}{\varphi(x)} \varphi'(x) dx + \tilde{C}_5 + O(r^{-\frac{1}{2}}),
\]
where the contour in the $x$-integral lies on the upper sheet. Furthermore, integrating by parts and then using Proposition 7.4 again, we find
\[
\int_M^r \tilde{c}_2(a) \frac{d}{d\hat{\nu}} \frac{\varphi'(\nu)}{\varphi(\nu)} \frac{d\nu}{\rho} = -\frac{5}{12} \int_M^r \tilde{c}_2(a) \frac{\varphi'(\nu)}{\varphi(\nu)} \frac{d\nu}{\rho} + \frac{3}{2} \int_M^r \tilde{c}_2(a) \frac{\varphi'(\nu)}{\varphi(\nu)} \frac{d\nu}{\rho} = \frac{5}{6} \log(r) \int_{x_0}^\infty \tilde{c}_1(x) \frac{\varphi'(x)}{\varphi(x)} \varphi'(x) dx + \tilde{C}_6 + O(r^{-\frac{1}{2}}),
\]
where again the contour in the $x$-integral lies on the upper sheet. We obtain the last term on the right-hand side of (7.14) by adding (7.16) and (7.17). Lastly, an integration by parts shows that
\[
\int_{0}^{r} \frac{e_3}{r^2} \frac{d}{dr} \left[ \frac{\theta'(\overline{\nu})^3}{\theta(\overline{\nu})} \right] \, dr = \tilde{C}_7 + \mathcal{O}(r^{-\frac{7}{2}}),
\]
which completes the proof.

We finally simplify the constant $c_2$.

**Proposition 7.6.** We have $c_2 = -\frac{3}{4}$.

**Proof.** We start by evaluating the integral in (7.14), which can be written as
\[
\int_{x_0}^{\infty} \tilde{c}_1(x) \frac{\theta'(\varphi(x))}{\theta(\varphi(x))} \varphi'(x) \, dx = \frac{c_2^2(x_1 \varphi(x_1) - x_2 \varphi(x_2))}{\varphi(x_0)(x_0 - x_1)} \int_{x_0}^{\infty} \frac{\theta'(\varphi(x))}{\theta(\varphi(x))} \frac{dx}{x - x_2^2}.
\]
Using Proposition 7.2, an integration by parts gives
\[
\int_{x_0}^{\infty} \frac{\theta'(\varphi(x))}{\theta(\varphi(x))} \frac{dx}{x - x_2^2} = \int_{x_0}^{\infty} \frac{\gamma x + \delta}{\sqrt{\tilde{R}(x)}} \frac{dx}{(x - x_2^2)^2}.
\]
Computing the integral on the right-hand side as in (7.15), we infer that
\[
\int_{x_0}^{\infty} \tilde{c}_1(x) \frac{\theta'(\varphi(x))}{\theta(\varphi(x))} \varphi'(x) \, dx = \frac{x_0(7x_0 - 2x_1 - 2x_2) - (x_1 - x_2)^2}{8(x_0 + x_1 + x_2)(3x_0 - x_1 - x_2)}
\]
and substituting this into (7.14), we obtain the result.

Theorem 1.2 follows by combining (7.1), (7.2), (7.5), (7.13), and Proposition 7.6.

## A Airy model RH problem

The Airy model RH problem consists of finding a function $\Phi_{Ai}$ satisfying the following properties.

(a) $\Phi_{Ai} : \mathbb{C} \setminus \Sigma_{Ai} \to \mathbb{C}^{2 \times 2}$ is analytic, where $\Sigma_{Ai}$ is shown in Figure 5.

(b) $\Phi_{Ai}$ has the jump relations
\[
\begin{align*}
\Phi_{Ai,+}(z) &= \Phi_{Ai,-}(z) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & \text{on } \mathbb{R}^-, \\
\Phi_{Ai,+}(z) &= \Phi_{Ai,-}(z) \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, & \text{on } \mathbb{R}^+, \\
\Phi_{Ai,+}(z) &= \Phi_{Ai,-}(z) \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, & \text{on } e^{\frac{1}{2} \pi i} \mathbb{R}^+, \\
\Phi_{Ai,+}(z) &= \Phi_{Ai,-}(z) \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, & \text{on } e^{-\frac{1}{2} \pi i} \mathbb{R}^+.
\end{align*}
\]

(c) As $z \to \infty$, $z \notin \Sigma_{Ai}$, we have
\[
\Phi_{Ai}(z) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} M \left( 1 + \sum_{k=1}^{\infty} \frac{\Phi_{Ai,k}}{z^{\frac{k+1}{2}}} \right) e^{-\frac{i}{2}z^{\frac{3}{2}} \sigma_3},
\]
where $M = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$ and $\Phi_{Ai,1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}$.

(d) As $z \to 0$, we have $\Phi_{Ai}(z) = \mathcal{O}(1)$.

The unique solution $\Phi_{Ai}$ can be written in terms of Airy functions [17], but its exact expression is unimportant for us. The quantity we need is the explicit expression for $\Phi_{Ai,1}$. 

\[\text{18}\]
Bessel model RH problem

The Bessel model RH problem consists of finding a function \( \Phi_{Be} \) satisfying the following properties.

(a) \( \Phi_{Be} : \mathbb{C} \setminus \Sigma_{Be} \rightarrow \mathbb{C}^{2 \times 2} \) is analytic, where \( \Sigma_{Be} \) is shown in Figure 6.

(b) \( \Phi_{Be} \) satisfies the jump conditions

\[
\Phi_{Be,+}(z) = \Phi_{Be,-}(z) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad z \in \mathbb{R}^-, \\
\Phi_{Be,+}(z) = \Phi_{Be,-}(z) \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad z \in e^{\frac{-2\pi i}{3}} \mathbb{R}^+, \\
\Phi_{Be,+}(z) = \Phi_{Be,-}(z) \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad z \in e^{\frac{2\pi i}{3}} \mathbb{R}^+.
\]  

(B.1)

(c) As \( z \to \infty, z \notin \Sigma_{Be} \), we have

\[
\Phi_{Be}(z) = (2\pi z^\frac{1}{2})^{-\frac{2\pi}{3}} M \left( I + \frac{\Phi_{Be,1}}{z^{-\frac{2\pi}{3}}} \right) e^{2\pi i \sigma_3},
\]  

(B.2)

where \( \Phi_{Be,1} = \frac{1}{16} \begin{pmatrix} -1 & -2i \\ 2i & 1 \end{pmatrix} \) and \( M = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \).

(d) As \( z \to 0 \), we have

\[
\Phi_{Be}(z) = \begin{cases} 
\begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(|\log z|) \\ \mathcal{O}(z) & \mathcal{O}(|\log z|) \end{pmatrix}, & |\arg z| < \frac{2\pi}{3}, \\
\begin{pmatrix} \mathcal{O}(|\log z|) & \mathcal{O}(|\log z|) \\ \mathcal{O}(|\log z|) & \mathcal{O}(|\log z|) \end{pmatrix}, & \frac{2\pi}{3} < |\arg z| < \pi.
\end{cases}
\]  

(B.3)

The unique solution \( \Phi_{Be} \) is given in terms of Bessel functions [19], but its exact expression is unimportant for our purposes. Again, the quantity we need is the explicit expression for \( \Phi_{Be,1} \).

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References

[1] M. Adler, J. Delépine, and P. van Moerbeke, Dyson’s nonintersecting Brownian motions with a few outliers, Comm. Pure Appl. Math. 62, (2008), 334–395.

[2] J. Baik, R. Buckingham, and J. Di Franco, Asymptotics of Tracy-Widom distributions and the total integral of a Painlevé II function, Comm. Math. Phys. 280 (2008), 463–497.

[3] J. Baik, P. Deift, and K. Johansson, On the distribution of the length of the longest increasing subsequence of random permutations, J. Amer. Math. Soc. 12 (1999), 1119–1178.

[4] J. Baik, P. Deift, and E. Rains, A Fredholm determinant identity and the convergence of moments for random Young tableaux, Comm. Math. Phys. 223 (2001), 627–672.

[5] V. Beffara, S. Chhita, and K. Johansson, Airy point process at the liquid-gas boundary, Ann. Probab. 46 (2018), 2973–3013.

[6] E. Blackstone, C. Charlier, and J. Lenells, Oscillatory asymptotics for the Airy kernel determinant on two intervals, Int. Math. Res. Not. (2020), https://doi.org/10.1093/imrn/rnaa205.

[7] E. Blackstone, C. Charlier, and J. Lenells, The Bessel kernel determinant on large intervals and Birkhoff’s ergodic theorem, preprint.

[8] F. Bornemann, On the numerical evaluation of Fredholm determinants, Math. Comp. 79 (2010), 871–915.

[9] A. Borodin, A. Okounkov, and G. Olshanski, Asymptotics of Plancherel measures for symmetric groups, J. Amer. Math. Soc. 13 (2000), 481–515.

[10] P. Bourgade, L. Erdős, and H.-T. Yau, Edge universality of beta ensembles, Comm. Math. Phys. 332 (2014), 261–353.

[11] A. Bufetov, Rigidity of determinantal point processes with the Airy, the Bessel and the gamma kernel, Bull. Math. Sci. 6 (2016), 163–172.

[12] C. Charlier and T. Claessens, Global rigidity and exponential moments for soft and hard edge point processes, arXiv:2002.03833.

[13] T. Claessens and A. Doeraene, The generating function for the Airy point process and a system of coupled Painlevé II equations, Stud. Appl. Math. 140 (2018), 403–437.

[14] I. Corwin and P. Ghosal, Lower tail of the KPZ equation, Duke Math. J. 169 (2020), 1329–1395.

[15] P. Deift, D. Gioev, Universality at the edge of the spectrum for unitary, orthogonal and symplectic ensembles of random matrices, Comm. Pure Appl. Math. 60 (2007), 867–910.

[16] P. Deift, A. Its, and I. Krasovsky, Asymptotics for the Airy-kernel determinant, Comm. Math. Phys. 278 (2008), 643–678.

[17] P. Deift, T. Kriecherbauer, K. McLaughlin, S. Venakides, and X. Zhou, Strong asymptotics of orthogonal polynomials with respect to exponential weights, Comm. Pure Appl. Math. 52 (1999), 1491–1552.

[18] P. Deift, T. Kriecherbauer, K.T.-R. McLaughlin, S. Venakides and X. Zhou, Uniform asymptotics for polynomials orthogonal with respect to varying exponential weights and applications to universality questions in random matrix theory, Comm. Pure Appl. Math. 52 (1999), 1335–1425.

[19] P. Deift, A. Its and X. Zhou, A Riemann-Hilbert approach to asymptotic problems arising in the theory of random matrix models, and also in the theory of integrable statistical mechanics, Ann. of Math. 146 (1997), 149–235.

[20] P. Deift and X. Zhou, A steepest descent method for oscillatory Riemann–Hilbert problems. Asymptotics for the MKdV equation, Ann. Math. 137 (1993), 295–368.
[21] B. Fábs and I. Krasovsky, Sine-kernel determinant on two large intervals, preprint, arXiv:2003.08136.

[22] H. Farkas and I. Kra, *Riemann Surfaces*, second edition, Springer, Berlin (1980).

[23] I.S. Gradshteyn and I.M. Ryzhik, Table of integrals, series, and products. Seventh edition. Elsevier Academic Press, Amsterdam, 2007.

[24] A. Its, A.G. Izergin, V.E. Korepin, and N.A. Slavnov, Differential equations for quantum correlation functions, *In proceedings of the Conference on Yang-Baxter Equations, Conformal Invariance and Integrability in Statistical Mechanics and Field Theory*, Volume 4, (1990) 1003–1037.

[25] K. Johansson, The arctic circle boundary and the Airy process, *Ann. Prob.* 33 (2005), 1–30.

[26] I. Krasovsky and T. Maroudas, Airy-kernel determinant on two large intervals, preprint.

[27] F.W.J. Olver, D.W. Lozier, R.F. Boisvert and C.W. Clark, NIST handbook of mathematical functions (2010), *Cambridge University Press.*

[28] A. Soshnikov, Determinantal random point fields, *Russian Math. Surveys* 55 (2000), 923–975.

[29] C. Tracy and H. Widom, Level-spacing distributions and the Airy kernel, *Comm. Math. Phys.* 159 (1994), 151–174.

[30] H. Widom, Asymptotics for the Fredholm determinant of the sine kernel on a union of intervals, *Comm. Math. Phys.* 171 (1995), 159–180.

[31] S.-X. Xu and D. Dai, Tracy-Widom distributions in critical unitary random matrix ensembles and the coupled Painlevé II system, *Comm. Math. Phys.* 365 (2019), 515–567.

[32] C. Zhong, Large deviation bounds for the Airy point process, arXiv:1910.00797.