Chapter 1

STABILITY OF D-WAVE SUPERCONDUCTIVITY IN THE $t-J$ MODEL

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Abstract

We use a recently developed technique, which allows to perform few Lanczos steps on a given wavefunction even for large system sizes, to investigate the $t-J$ model in the physical parameter region and to check the stability of the BCS d-wave variational wavefunction [1]. Our statistical Lanczos algorithm, which extends and improves the one Lanczos step proposed in Ref. [2], has been extensively tested on the small $L = 18$ sites cluster where many Lanczos iterations can be performed exactly. In this case, at doping $\delta \sim 10\%$ the BCS wavefunction represents a very good initial state to achieve extremely accurate energies and correlation functions with few Lanczos iterations. For large sizes ($L \leq 98$) the behavior is similar: the low-energy d-wave order parameter $P_d$ is weakly affected by a couple of Lanczos iterations in the low doping $\delta \sim 10\%$ region, whereas the energy is considerably lowered. As a further test of our calculation we have computed the variance of the Hamiltonian

$$\Delta E_p = \frac{\langle \hat{H}^2 \rangle - \langle \hat{H} \rangle^2}{L^2}$$

on the BCS wavefunction with $p = 0, 1, 2$ Lanczos steps. For large $p$, when the Lanczos algorithm converges to the exact ground state, the variance vanishes exponentially with increasing $p$. The remarkable reduction of the variance, observed for $p = 1, 2$ Lanczos steps even for the largest lattice size considered, suggests a smooth and rapid convergence to the exact ground state. These results support the existence of off-diagonal long-range d-wave superconducting order in the two-dimensional $t-J$ model.

1. INTRODUCTION

One of the most important question raised after the discovery of high-Tc superconductivity is whether a simple model of strongly correlated electrons can capture the low-energy physics of real materials. In particular it is still a very much debated issue whether a purely repulsive electronic interaction can give rise to a d-wave superconducting ground state by doping an antiferromagnetic Mott insulator with a small amount of holes. The simplest model that has been
proposed immediately after the discovery of high-Tc superconductors is the $t-J$ model \[3, 4\]

$$
\hat{H} = J \sum_{\langle i,j \rangle} \left( \hat{S}_i \cdot \hat{S}_j - \frac{1}{4} \hat{n}_i \hat{n}_j \right) - t \sum_{\langle i,j \rangle, \sigma} \tilde{c}^\dagger_{i,\sigma} \tilde{c}_{j,\sigma}, \tag{1.1}
$$

where $\tilde{c}^\dagger_{i,\sigma} = \hat{c}^\dagger_{i,\sigma} (1 - \hat{n}_{i,\sigma})$, $\hat{n}_i = \sum_{\sigma} \hat{n}_{i,\sigma}$ is the electron density on site $i$, $\hat{S}_i = \sum_{\sigma,\sigma'} \tilde{c}^\dagger_{i,\sigma} \tau_{\sigma,\sigma'} \tilde{c}_{i,\sigma'}$ is the spin operator and $\tau_{\sigma,\sigma'}$ are Pauli matrices. In the following we put $t = 1$.

After many years of intense numerical and theoretical efforts there is no general consensus on the properties of this simple Hamiltonian and of the related Hubbard model. From the numerical point of view, the density matrix renormalization group (DMRG) \[5\] predicts that a charge density wave instability \[6\], nowadays called striped-phase for its one dimensional character, is strongly competing with pairing and superconductivity \[7\]. Within DMRG it appears therefore difficult to explain Copper-Oxide superconductors with a simple one-band model, especially because, in order to be consistent with photoemission experiments \[8\], a negative next-nearest-neighbor hopping amplitude $t'$ has to be included in the model. In fact, the negative $t'$ suppresses even further superconductivity, so that the $t-J$ model becomes unrealistic to describe the low-energy physics of high-Tc superconductors. However, the DMRG results, though quite accurate, are not exact in two dimensions. Moreover, for technical reasons it is possible to consider only particular boundary conditions (open in one direction and periodic in the other), which certainly make the DMRG calculation still far to be representative of the thermodynamic limit.

Quantum Monte Carlo (QMC) is an appealing alternative numerical approach. This numerical method is still severely limited for two-dimensional fermionic systems by the well-known sign problem and is consequently biased by the initial guess of the ground state used to control this numerical instability \[9, 10, 11\]. However this technique has the important advantage to work very well with periodic boundary conditions since translation invariance can be explicitly used to improve the efficiency of QMC algorithms. In particular, an approximate ground state can be obtained starting from a translation invariant wavefunction $|\psi_G\rangle$ by applying exactly few powers of the Hamiltonian $(-\hat{H})^p$ \[12\] or many approximate ones (by using for instance the Fixed-Node FN approximation \[9\]). However, within various QMC schemes the situation is still controversial. First Heeb \[2\] and Khono \[13\] have found d-wave superconductivity in a reasonable parameter range. Later Shih and co-workers \[14\], using a very similar method, have excluded drastically this possibility. The latter results were obtained within the monotonic-behavior assumption of the off-diagonal superconducting order parameter as a function of the number $p$ of Hamiltonian
powers applied to the initial wavefunction. This assumption, although reasonable, is highly questionable. By contrast, in a recent QMC work [15], a clear tendency to d-wave superconductivity in the $t-J$ model was found. Moreover, very recently, an almost realistic phase diagram with a corresponding high-Tc d-wave superconducting transition has been obtained for the Hubbard model within the Dynamical Mean Field Approximation [16, 17]. Furthermore, in the contest of the Hubbard model, also a weak-coupling renormalization group approach [18, 19] gives rise to a d-wave order parameter in a large region of the phase diagram. The latter results strongly support the relevance of a single band model for the explanation of high-Tc superconductivity.

2. NUMERICAL METHOD

In this work we make a further attempt to clarify the controversial numerical findings on the issue of d-wave superconductivity in the $t-J$ model, using the statistical few Lanczos-step technique (FLST), efficiently implemented by means of the stochastic reconfiguration (SR) [10, 11]. Within the latter scheme all kind of correlation functions can be computed efficiently without any mixed average [11] approximation: an enormous advantage compared to the FN or to the original SR technique [10, 11]. In these cases this bias can be removed at the price of adding a small field coupled to the desired correlation function [15]. However, it turns out that it is extremely difficult and computationally demanding to work in the small field limit when unbiased correlation functions can be obtained. FLST is instead a very good compromise that solves efficiently this numerical problem of QMC methods.

The wavefunctions that we are able to sample statistically read:

$$|\psi_p\rangle = \left(1 + \sum_{k=1}^{p} \alpha_k \hat{H}^p \right)|\psi_G\rangle$$  \hspace{1cm} (1.2)

with parameters $\{\alpha_k\}$ for $k = 1, \cdots, p$ minimizing the energy expectation value $\langle \psi_p | \hat{H} | \psi_p \rangle / \langle \psi_p | \psi_p \rangle$. For any $p$ it is simple to show that the wavefunction (1.2) corresponds exactly to apply $p$ Lanczos step iterations to the initial wavefunction $|\psi_G\rangle$. This wavefunction is sampled statistically with the SR technique, by using in the reconfiguration scheme the first $p$ powers of the Hamiltonian. Unlike the previous method [10, 11] the reference wavefunction $|\psi_f\rangle$ is not evolved during the Markov iteration. $|\psi_f\rangle$ is instead kept statistically equal to the initial wavefunction $|\psi_G\rangle$ using the variational scheme proposed by Hellberg and Manousakis [20], which highly reduce the statistical fluctuations related to the SR technique. The equivalence of FLST to the standard Lanczos algorithm will be discussed in a forthcoming paper [21].
The initial wavefunction to which FLST will be applied can be written as follows [1]:

$$|\psi_G\rangle = |\psi_{p=0}\rangle = \hat{P}_0 \hat{P}_N \hat{J} |D\rangle. \tag{1.3}$$

where $|D\rangle$ is a BCS wavefunction, which is an exact eigenstate of the following Hamiltonian:

$$\hat{H}_{BCS} = \hat{H}_0 + \frac{\Delta_{BCS}}{2} (\hat{\Delta}^\dagger + \hat{\Delta}) \tag{1.4}$$

$$\hat{\Delta}^\dagger = \sum_{\langle i,j \rangle} M_{i,j} (\hat{c}_{i,\uparrow}^\dagger \hat{c}_{j,\downarrow}^\dagger + \hat{c}_{j,\uparrow}^\dagger \hat{c}_{i,\downarrow}^\dagger) \tag{1.5}$$

where $\hat{H}_0 = \sum_{k,\sigma} \epsilon_k \hat{c}_{k,\sigma}^\dagger \hat{c}_{k,\sigma}$ is the free electron tight binding nearest-neighbor Hamiltonian, $\epsilon_k = -2(\cos k_x + \cos k_y) - \mu$, $\mu$ is the free-electron chemical potential and $\hat{\Delta}^\dagger$ creates all possible nearest-neighbor singlet bonds with d-wave symmetry being $M_{i,j} = 1$ or $-1$ if the bond $\langle i, j \rangle$ is in the $x$ or $y$ direction, respectively. $\hat{P}_N$ and $\hat{P}_0$ are the projectors over the subspaces with a fixed number $N$ of particles and no doubly occupied states. Finally the Jastrow factor $\hat{J} = \exp \left( \frac{\gamma}{2} \sum_{i,j} v(i - j) \hat{n}_i \hat{n}_j \right)$ couples the holes via the density operators $\hat{n}_i$ and contains another variational parameter $\gamma \sim 1$ which scales an exact analytic form, obtained by approximating the holes with hard-core bosons at the same density, and applying the spin-wave theory to the corresponding XY model [22]. We note here that by performing a particle-hole transformation on the spin down $\hat{c}_{i,\downarrow}^\dagger \rightarrow (-1)^i \hat{c}_{i,\downarrow}$, the ground state of the BCS Hamiltonian is just a Slater-determinant with $N = L$ particles [23]. This is the reason why this variational wavefunction can be considered of the generic Jastrow-Slater form, a standard variational wavefunction used in QMC. Using the particle-hole transformation, it is also possible to control exactly the spurious finite system divergences related to the nodes of the d-wave order parameter.

## 3. NUMERICAL TESTS

In this section we show the accuracy of FLST applied to the BCS wavefunction (1.3) on a small 18-site cluster, where exact results are available.

Our main task is to compute the order parameter at finite system size

$$P_d = \frac{1}{L} \langle \psi_p^{N+2} | \hat{\Delta}^\dagger | \psi_p^N \rangle, \tag{1.6}$$

where $|\psi_p^N\rangle$ and $|\psi_p^{N+2}\rangle$ are the states with $N$ and $N + 2$ particles, respectively. If $P_d$ is finite in the thermodynamic limit this necessarily implies off-diagonal long-range order in the ground state. Following Ref. [15], it is convenient with an approximate technique to calculate a short-range quantity like $P_d$, instead
Figure 1.1  Ground-state energy per site $E_0/L$ and d-wave order parameter $P_d$ as a function of the variance for $N = 16$, $L = 18$, $J = 0.4$: full dots (optimal $\Delta_{BCS}$), empty dots ($\Delta_{BCS} \to 0$). Dashed lines are quadratic fits of the estimates with $p = 0, 1, 2$.

of the more conventional long-range expectation value $\langle \psi_p^N | \hat{\Delta} \hat{\Delta}^\dagger | \psi_p^N \rangle / L^2$. In Table 1.1 we show a comparison between FLST and the exact results for 18 and 16 electrons at $J = 0.4$.

In Table 1.2 we show $P_d$ as a function of the number of Lanczos step iterations for the 18-site cluster at $J = 0.4$. In the same Table we have computed also the variance $\Delta E_p = (\langle \psi_p | \hat{H}^2 | \psi_p \rangle - \langle \psi_p | \hat{H} | \psi_p \rangle^2)/L^2$, the overlap squared $Z_p = |\langle \psi_p | \psi_0 \rangle|^2$ of the FLST wavefunction with the true ground state $|\psi_0\rangle$, and the average sign of the FLST wavefunction:

$$\langle S_p \rangle = \sum_x \langle x | \psi_0 \rangle^2 \text{Sgn} (\langle x | \psi_p \rangle \langle x | \psi_0 \rangle),$$

(1.7)

where $|x\rangle$ denotes configurations with definite electron positions and spins. For an exact calculation, namely $p >> 1$, both $Z_p \to 1$ and $S_p \to 1$, whereas
\[ \Delta E_p \rightarrow 0. \] The variance thus represents a very important tool to estimate the ‘distance’ from the exact ground state when the latter one is not known. In particular whenever \( Z_p \simeq 1 \) the energy approaches the exact result linearly with the variance \( \Delta E_p \), allowing us to estimate the error in the variational energy.

This can be achieved by plotting the variational energies \( E_p \) as a function of the corresponding variance \( \Delta E_p \), and performing a very stable linear or quadratic fit to the \( \Delta E_p = 0 \) exact limit (see Fig. 1.1). Similar fits can be attempted for correlation functions though, in this case, also a term \( \propto \sqrt{\Delta E_p} \) is expected for \( \Delta E_p \rightarrow 0 \). This term is however negligible for quantities like \( P_d \) that are averaged bulk correlation functions in a large system size (see the Appendix). In practice even in the small 18-site cluster the non-linear term turns out to be negligible (see Fig. 1.1). We believe that, being the convergence of the Lanczos algorithm particularly well behaved and certainly unbiased, the variance extrapolation method is in this case particularly useful and reliable. However for bad initial wavefunction (e.g., randomly generated) or very large sizes the approach to zero of the variance may behave rather wildly, requiring many Lanczos steps to reach the regime where the extrapolation is possible.

As shown in Table 1.2 the quality of the variational BCS wavefunction (1.2) is exceptionally good, especially in the doped \( N = 16 \) case. Here \( Z_p \) is larger than 0.9 even at the simplest \( p = 0 \) variational level, and is drastically improved with really few Lanczos step iterations. Remarkable is also the behavior of the average sign \( S_p \) which measures directly the accuracy of the BCS wavefunction phases, without caring about the amplitudes. In the undoped case the signs of the BCS wavefunction \( \langle S_0 \rangle \) can be proven to be exact, i.e., \( \langle S_0 \rangle = 1 \), having the BCS state the well-known Marshall signs, i.e., the phases of the exact ground state of Heisenberg model. For the two-hole case, the BCS nodes change in a non trivial way. Nevertheless, \( \langle S_0 \rangle \) remains very close to 1 and it is much higher than the average sign of the corresponding Gutzwiller wavefunction (\( \Delta_{BCS} \rightarrow 0 \)), also shown in the table for comparison.

| \( N \) | \( \Delta_{BCS} \) | \( p \) | \( E_0/L \) (FLST) | \( P_d \) (FLST) | \( E_0/L \) (Exact) | \( P_d \) (Exact) |
|---|---|---|---|---|---|---|
| 18 | 0.55 | 1 | -0.4765(1) | -0.47749 |
| 18 | 0.55 | 2 | -0.4775(1) | -0.47668 |
| 16 | 0.20 | 1 | -0.6541(1) | 0.10730 |
| 16 | 0.55 | 2 | -0.6583(1) | 0.12135 |

Table 1.1 Comparison between the estimates of the ground-state energy per site \( E_0/L \) and of the d-wave order parameter \( P_d \) obtained with the exact and the statistical (FLST) application of \( p \) Lanczos steps on the variational wavefunction of Eq. (1.3). \( L = 18, N = 16, 18 \) and \( J = 0.4 \).
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| $N$ | $\Delta_{BCS}$ | $p$ | $\langle S_p \rangle$ | $Z_p$ | $\Delta E_p \times L^2$ | $E_0/L$ |
|-----|-----------------|-----|-------------------------|------|-------------------------|--------|
| 18  | 0.00            | 0   | 1.0000                  | 0.6898 | 1.194                  | -0.43833 |
| 16  | 0.00            | 0   | 0.9656                  | 0.8306 | 1.174                  | -0.63847 |
| 18  | 0.80            | 0   | 1.0000                  | 0.8850 | 0.335                  | -0.46639 |
| 18  | 0.80            | 1   | 1.0000                  | 0.9915 | 0.042                  | -0.47662 |
| 18  | 0.80            | 2   | 1.0000                  | 0.9995 | 0.004                  | -0.47752 |
| 18  | 0.80            | 3   | 1.0000                  | 1.0000 | 0.0003                 | -0.47759 |
| 18  | 0.80            | 4   | 1.0000                  | 1.0000 | 0.00002                | -0.47759 |
| 18  | 0.80            | $\infty$ | 1.0000                  | 1.0000 | 0.0                     | -0.47759 |
| 16  | 0.55            | 0   | 0.9881                  | 0.9260 | 0.940                  | -0.64559 |
| 16  | 0.55            | 1   | 0.9988                  | 0.9814 | 0.191                  | -0.65638 |
| 16  | 0.55            | 2   | 0.9999                  | 0.9942 | 0.060                  | -0.65826 |
| 16  | 0.55            | 3   | 1.0000                  | 0.9983 | 0.018                  | -0.65882 |
| 16  | 0.55            | 4   | 1.0000                  | 0.9995 | 0.005                  | -0.65898 |
| 16  | 0.55            | 5   | 1.0000                  | 0.9999 | 0.002                  | -0.65902 |
| 16  | 0.55            | 6   | 1.0000                  | 0.9999 | 0.0005                 | -0.65904 |
| 16  | 0.55            | 7   | 1.0000                  | 1.0000 | 0.0001                 | -0.65904 |
| 16  | 0.55            | $\infty$ | 1.0000                  | 1.0000 | 0.0                     | -0.65904 |

Table 1.2: Average sign $\langle S_p \rangle$, overlap squared on the exact ground state $Z_p$ and variance times the volume squared $\Delta E_p \times L^2$ obtained applying exactly $p$ Lanczos steps on the variational wavefunction of Eq. (1.3). $L = 18$, $N = 16, 18$ and $J = 0.4$. 
These results suggest that there is a tendency to d-wave BCS pairing in the $t-J$ model at $\sim 10\%$ doping and $J \sim 0.4$, and that the BCS wavefunction is a particularly accurate wavefunction to describe the small and even zero doping region of the $t-J$ model.

4. LARGER SIZE CALCULATIONS

Though few Lanczos steps may appear inadequate for large system size, this simple scheme is instead providing us very good variational energies up to $L \sim 100$ sites in the $t-J$ model, even when this variational energies are compared with more complicated schemes like the FN. In order to show that the Lanczos scheme remains effective for larger sizes it is useful to consider first a relevant case where a numerically exact solution is possible: the zero doping limit of the $t-J$ model, i.e., the Heisenberg model. In Fig. 1.2 we plot the energy results calculated for few Lanczos steps as a function of the variance. The fact that the energy is approaching the exact result smoothly with the variance both in the 50- and 98-site clusters (with a slightly larger curvature in the latter case) indicates that the projected BCS wavefunction should have a substantially large overlap squared $Z_p$ with the exact ground state of the Heisenberg model even on these large system sizes. This is also confirmed by the behavior of the square order parameter (Fig. 1.3), which is considerably improved by few Lanczos steps. This result is particularly important since the starting BCS wavefunction has not antiferromagnetic long-range order, whereas the two-dimensional Heisenberg model is widely believed to be antiferromagnetically ordered.

The above test represents also a further strong evidence that the ground-state wavefunction of the Heisenberg model is smoothly connected to a d-wave BCS superconducting wavefunction. This circumstance represents a very interesting numerical fact that clearly supports the experimental observation of high-Tc d-wave superconductivity coming just upon a small doping of a quantum antiferromagnet.

Within the Lanczos approximate states $|\psi_p\rangle$ ($p = 0, 1, 2$) acting on the best variational BCS wavefunction, we have performed a finite-size scaling of the order parameter $P_d$ as defined in Eq. (1.6), at a fixed doping $\delta = 13.3\%$ (corresponding to $N = 84$ on the largest size $L = 98$). Since $P_d$ is computed between two states with $N$ and $N + 2$ particle numbers, we assume that it refers to the intermediate doping $\delta = 1 - (N + 1)/L$. For smaller sizes the doping $\delta = 13.3\%$ is not possible and we have interpolated linearly $P_d$ between the two fillings closest to this doping. The main results of this paper is then shown in the Fig. 1.4. Here the size scaling for $p = 0, 1, 2$ Lanczos steps and for their zero-variance extrapolation clearly indicates a finite $P_d$ in the thermodynamic limit. The stability of the BCS variational wavefunction is evident from this
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Figure 1.2  Energy per site of the finite-size Heisenberg model. Comparison of exact results (indicated by arrows) and the approximate $p = 0, 1, 2$ Lanczos step iterations over the projected d-wave wavefunction. Continuous lines are quadratic fit of the data.

figure: $P_d$ remains much larger than the corresponding value of the Gutzwiller ($\Delta_{BCS} \to 0$) metallic state. Instead, as shown in Fig. 1.5 and similarly to the undoped case, the energy is very much lowered by FLST, suggesting that the method remains effective even for large sizes and finite doping.

5. CONCLUSIONS

In conclusion we have provided evidences in favor of the stability of d-wave pairing correlations in the $t-J$ model. Our result disagrees with a recent QMC calculation [14], for which, however, the self-consistent assumption mentioned in the introduction leads to a very poor, high-energy variational state. Furthermore, also the DMRG results provide a qualitatively different scenario, namely that stripes suppress d-wave superconductivity [7]. In order to clarify this disagreement, we have performed a simulation at $J = 0.4$ for a cluster with $L = 6 \times 12$, 8 holes, and open boundary conditions on the long direction, as usually done within DMRG. Our accuracy is in this case worse than with periodic boundary condition, especially for quantities that are not bulk average correlation functions (see the Appendix). For $P_d$ instead we have found that, even at the simplest variational level open boundary conditions strongly suppress $P_d$ by more than 30%, much more than the difference found
in Fig. (1.4). Thus we find that boundary conditions play a very important role and may strongly destabilize the uniform BCS d-wave ground state, as obtained within DMRG. This is consistent with experimental findings [24], showing that any small asymmetry in the CuO planes, acting in our point of view as a distorted boundary condition, may enhance the tendency to stripe formation and then suppressing superconductivity. Our results instead support the recent numerical works [16, 17, 18, 19], indicating that d-wave superconductivity can be obtained in a one band model with repulsive electron interaction.

**Appendix: Variance estimate of the error on bulk correlation functions**

In this Appendix we estimate the error on correlation functions assuming that the ground state $|\psi_0\rangle$ is approximated with the wavefunction $|\psi_p\rangle$:

$$|\psi_0\rangle = |\psi_p\rangle + \epsilon_p |\psi'\rangle$$  \hspace{1cm} (1.A.1)

where $\langle \psi_p | \psi_p \rangle = \langle \psi' | \psi' \rangle = 1$, and $|\psi'\rangle$ represents a normalized wavefunction orthogonal to the exact one, $\langle \psi_0 | \psi' \rangle = 0$. We restrict our analysis to thermo-
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Figure 1.4  Superconducting d-wave order parameter $P_d$ in the $t-J$ model as defined in the text in Eq. (1.6). These results were obtained starting by the variational wavefunction (triangles) defined in Eq. (1.3) and by applying to it one (squares) or two (dots) Lanczos steps. Black dots are obtained by quadratic extrapolation to the zero variance exact limit in order to estimate the size dependent error of the approximate variational calculations. The stars refer to the Gutzwiller wavefunction with $\Delta_{BCS} \rightarrow 0$ and $\gamma = 1$ in Eq. (1.3). Continuous lines are quadratic fit of the data.

We show here that the expectation value of bulk-averaged operators $\hat{O}$ on the approximate state $|\psi_p\rangle$ satisfy the following relation:

$$\langle \psi_p | \hat{O} | \psi_p \rangle = C + O(\epsilon_p^2, \epsilon_p/\sqrt{L}), \quad (1.3)$$

thus implying that for large enough size the expectation value (1.3) approaches the exact correlation function $C$ linearly with the variance. The
validity of the above statement is very simple to show under very general grounds. In fact by definition:

$$\langle \psi_p | \hat{O} | \psi_p \rangle = C + 2\epsilon_p \langle \psi' | \hat{O} | \psi_0 \rangle + \epsilon_p^2 \langle \psi' | \hat{O} | \psi' \rangle. \quad (1.A.4)$$

The term proportional to $\epsilon_p$ in the above equation can be easily bounded by use of the Schwartz inequality:

$$|\langle \psi' | \hat{O} | \psi_0 \rangle|^2 = |\langle \psi' | \hat{O} - C | \psi_0 \rangle|^2 \leq \langle \psi_0 | (\hat{O} - C)^2 | \psi_0 \rangle . \quad (1.A.5)$$

The final term in the latter inequality can be estimated under the general assumption that correlation functions $C(d) = \langle \psi_0 | (\hat{O}_i - C)(\hat{O}_{i+d} - C) | \psi_0 \rangle / \langle \psi_0 | \psi_0 \rangle$ decay sufficiently fast with the distance $|d|$, as a consequence of the cluster property:

$$\langle \psi_0 | (\hat{O} - C)^2 | \psi_0 \rangle = (1 + \epsilon_p^2) \frac{1}{L} \sum_d C(d).$$

This concludes the proof of the statement of this Appendix, provided $\sum_d C(d)$ is finite for $L \to \infty$. 

Figure 1.5  Energy per site of the $t - J$ model for 44 electrons on 50 sites (triangles) and 86 electrons on 98 sites (squares), $J = 0.4$, for the approximate $p = 0, 1, 2$ Lanczos step iterations over the projected d-wave wavefunction Eq. (1.3). Continuous lines are quadratic fit of the data.
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