Stability and Stabilization of Coupled Jump Diffusions and Applications

Dang H. Nguyen*, Duy Nguyen†, Nhu N. Nguyen‡, George Yin§

Abstract

This paper develops stability and stabilization results for systems of fully coupled jump diffusions. Such systems frequently arise in numerous applications where each subsystem (component) is operated under the influence of other subsystems (components). This paper derives sufficient conditions under which the underlying coupled jump diffusion is stable. The results are then applied to investigate the stability of linearizable jump diffusions, fast-slow coupled jump diffusions. Moreover, weak stabilization of interacting systems and consensus of leader-following systems are examined.

Keywords. Coupled switching jump diffusion, Lyapunov exponent, stability, stabilization.

AMS subject classifications. 60J60, 60J76, 93E03, 93E15.
1 Introduction

Nowadays, networked systems have posed unprecedented opportunities as well as challenges. Such systems have numerous applications in control engineering, wireless communications, mathematical biology, financial engineering, and actuarial science. In many stochastic networked systems, subsystems and/or components are intertwined and highly coupled. Moreover, empirical studies reveal that there exists sudden rapid moments in the mid quotes of stock prices, i.e., jumps during trading periods [5, 27]. Treating competitive Lotka-Volterra populations, as observed in [6], the population may suffer sudden environmental shocks such as earthquakes, hurricanes, epidemics, etc. The commonly used diffusion-type stochastic Lotka-Volterra models cannot explain such phenomena. To allow sudden changes, systems of stochastic differential equations involving Lévy process are often used to capture fluctuations as well as jumps caused by rare events.

When a system has been operated for a long time, its long-time behavior becomes an important feature. Stability of such systems has therefore been studied extensively. Given a system with coupled components/subsystems, can we derive the stability of one specific component based on the information of the fast subsystem? In addition, for a coupled system of jump diffusions, can we design a feedback control so as to obtain the desired stability? This work aims to examine these questions.

Previous works on stability of jump-diffusion processes can be found in [38] for multidimensional jump-diffusion processes, [8] for constrained jump-diffusion processes, [19] for jump diffusions in a Hilbert space, [7, 11, 12, 41, 43] for regime-switching jump diffusions, [4] for jump diffusions with state-dependent densities. In contrast to the existing works in the literature, this work focuses on stability of fully coupled jump diffusions, where two jump-diffusion components interact with each other. Such systems have a wide range of applications to numerous physical, engineering, and biological problems such as chemical reactors [32], power transmission lines [14], flow regulation in deep mines [37], elastic beams linked to rigid bodies [29], blood flow model [13, 18], mitochondrial swelling [17], to mention just a few among others. In such a situation, we are interested in the stability, averaging phenomena under the influence of the interacting processes in the environment. From a technical point of view, not only do the coupled systems possess many interesting properties, but also present many challenges; see coupled ordinary differential and partial differential equations (ODEs-PDEs) [1, 24, 26, 35], coupled diffusion or stochastic differential equations [20, 21, 34], and coupled stochastic reaction-diffusion or stochastic partial differential equations [3, 9, 22]. The motivations and urgent need in both theory and applications motivate the current work.

Let \( \mathbb{R}^{\ell_1} \) and \( \mathbb{R}^{\ell_2} \) be two Euclidean spaces of dimensions \( \ell_1 > 0 \) and \( \ell_2 > 0 \), respectively. Denote by \( \mathbf{0} \) a zero vector with appropriate dimension (which will be clear from the context). We assume \( \mathbf{X}_1(t) \) and \( \mathbf{X}_2(t) \) are coupled jump-diffusion processes in \( \mathbb{R}^{\ell_1} \), \( \mathbb{R}^{\ell_2} \), respectively. More specifically, the pair \((\mathbf{X}_1(t), \mathbf{X}_2(t))\) is the solution of the system

\[
\begin{align*}
\frac{d\mathbf{X}_1(t)}{dt} &= b_1(\mathbf{X}_1(t), \mathbf{X}_2(t))dt + \sigma_1(\mathbf{X}_1(t), \mathbf{X}_2(t))d\mathbf{W}_1(t) + \int_{\mathbb{R}^{\ell_1}} \gamma_1(\mathbf{X}_1(t^-), \mathbf{X}_2(t^-), \phi) \tilde{N}_1(\mathbf{t}dt, d\phi), \\
\frac{d\mathbf{X}_2(t)}{dt} &= b_2(\mathbf{X}_1(t), \mathbf{X}_2(t))dt + \sigma_2(\mathbf{X}_1(t), \mathbf{X}_2(t))d\mathbf{W}_2(t) + \int_{\mathbb{R}^{\ell_2}} \gamma_2(\mathbf{X}_1(t^-), \mathbf{X}_2(t^-), \phi) \tilde{N}_2(\mathbf{t}dt, d\phi), \\
\mathbf{X}_1(0) &= \mathbf{x}_1, \quad \mathbf{X}_2(0) = \mathbf{x}_2,
\end{align*}
\]

where, for \( i = 1, 2 \), \( \mathbf{W}_i(t) \) are standard \( \mathbb{R}^{d_i} \)-valued Brownian motions; \( \mathbf{N}_i(dt, d\phi) \)'s are Poisson random measures independent of \( \mathbf{W}_1(t), \mathbf{W}_2(t) \), and \( \mathbf{N}_i(dt, d\phi) = \mathbf{N}_i(dt, d\phi) - \nu_i(dt, d\phi)dt \) are the
and thus
$$0 \gamma_0 \text{ which the equilibrium point }$$
argin
 is assumed that
argin
\begin{align*}
b_2(x_1, 0) &= 0, \quad \sigma_2(x_1, 0) = 0, \quad \gamma_2(x_1, 0, \phi) = 0,
\end{align*}
argin
and thus 0 is an equilibrium point of $X_2(t)$. We wish to derive conditions that are mild under which the equilibrium point 0 or the trivial solution of $X_2(t)$ is stable.

In what follows, the assumptions for the stability are given in term of Lyapunov functions but the insight and intuition are derived from a dynamic system point of view combined with the averaging principle and ergodicity of coupled systems. We demonstrate that the conditions are easily applicable. The intuition and idea are that if the interacting process $X_1(t)$ on the boundary (i.e., when $X_2(t) = 0$) admits a unique invariant measure and the corresponding decoupled (averaging) equations (of the process $X_2(t)$) at the stationary distribution of $X_1(t)$ satisfies some appropriate stability conditions then $X_2(t)$ is also stable. Taking the idea from a dynamic system point of view, the stability conditions are obtained by considering the Lyapunov exponent of the process $X_2(t)$ corresponding to the invariant measure of the interacting process $X_1(t)$ on the boundary. Such conditions coincide with the intuition that when the main process $X_2(t)$ is close to the equilibrium 0, the interacting process is close to the solution on the boundary. Thus the stability conditions of the main process only need to be based on the information of the interacting process on the boundary, which however, poses great challenges. We need to reveal the behavior of the system around the boundary. Since two components are fully coupled, handling their interactions and analyzing their behavior require a careful analysis. To overcome the difficulties, one of the main tools used in this paper is the coupling method \cite{10}.

With the stability results at our hands, we study systems that commonly arise in applications. In particular, one of the main questions one would like to answer is: What are the relationships between nonlinear systems and the associated linearized systems for jump diffusions? We address this question and provide sufficient conditions for the stability of linearizable jump diffusions. In various applications, the subsystems and/or components often display different time-scales. It is often necessary to treat fast-slow coupled jump diffusions. We provide sufficient conditions for stability of the slow component based only on the limit system. Next, we design stabilizing strategies in a coupled system when only the interacting process is available to be controlled. Finally, leader-following systems are studied and conditions for the consensus controllability of the underlying systems are given.

The rest of paper is arranged as follows. Section 2 presents the main results on stability. Section 3 deals with linearizable systems and systems with fast and slow components. Section 4 focuses on stabilization and treats consensus problems. Finally, Section 5 concludes the paper and issues further remarks.

## 2 Stability of Coupled Jump Diffusions

In this paper, we use $| \cdot |$ to denote the Euclidean norm for either vectors or matrices, and $A^\top$ the transpose of a vector or a matrix $A$. For two real numbers $a$ and $b$, $a \lor b = \max(a, b)$, and $a \land b = \min(a, b)$. Denote $Z = (X_1^\top, X_2^\top)^\top$, $z = (x_1^\top, x_2^\top)^\top$, $b = (b_1^\top, b_2^\top)^\top$, $\sigma = (\sigma_1^\top, \sigma_2^\top)^\top$, $\gamma = (\gamma_1^\top, \gamma_2^\top)^\top$, $W = (W_1^\top, W_2^\top)^\top$, and $N = (N_1^\top, N_2^\top)^\top$. We will use $Z$ and $(X_1, X_2)$ exchangeably. Moreover, a function of $(x_1, x_2)$ can often be written as a function of $z$ with $z = (x_1^\top, x_2^\top)^\top$, which will be clear.
from the context. Note that the equation (1.1) in vector form becomes

\[ d\mathbf{Z}(t) = b(\mathbf{Z}(t))dt + \sigma(\mathbf{Z}(t))d\mathbf{W}(t) + \int_{\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}} \gamma(\mathbf{Z}(t^-), \phi)\tilde{N}(dt, d\phi), \quad \mathbf{Z}(0) = \mathbf{z}. \]

We will use \( \mathbb{P}_\mathbf{z} \) (and/or \( \mathbb{E}_\mathbf{z} \)) to indicate the initial conditions of the system.

Next, define the operator \( \mathcal{L} \) by

\[ \mathcal{L}g(z) := [\partial_z g(z)]^T b(z) + \frac{1}{2} \text{tr}[\sigma(z)\sigma^T(z)\partial_z^2 g(z)] \]

\[ + \int_{\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}} [g(z + \gamma(z, \phi)) - g(z) - (\partial_z g)\gamma(z, \phi)] \nu(d\phi), \]

for \( g \in \mathcal{D}_\mathcal{L} \), where \( \partial_z \) and \( \partial_z^2 \) denote the gradient and Hessian matrix with respect to \( z \), respectively, and

\[ \mathcal{D}_\mathcal{L} := \{ g : \mathbb{R}^{\ell_1} \times \mathbb{R}^{\ell_2} \to \mathbb{R} : g(z) \text{ is twice continuously differentiable and} \]

\[ \int_{\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}} |g(z + \gamma(z, \phi)) - g(z) - \partial_z g \cdot \gamma(z, \phi)| \nu(d\phi) < \infty \}. \]

It is noted that \( z \) in \( \mathcal{L}g(z) \) represents the variable of \( \mathcal{L}g \) rather than the variable of \( g \). Indeed, later \( g \) can be plugged in by either functions of \( x_1 \) or functions of \( x_2 \). For example, if \( g \) is a function of \( x_1 \) only, the gradient of \( g \) (with respect to \( z \)) will be \(([\partial_{x_1} g(x_1)]^T, 0^T)^T \). However, because the coefficients are fully coupled, \( \mathcal{L}g \) is a function of \( z \); we still write it as \( \mathcal{L}g(z) \). [This example is in fact associated to the generalized Itô formula (given below) for the first component.]

The following result is known as the generalized Itô formula (see e.g., [36, 41]):

\[ g(\mathbf{Z}(t)) - g(\mathbf{Z}(0)) = \int_0^t \mathcal{L}g(\mathbf{Z}(s^-)) ds + \int_0^t \partial_{z} g(\mathbf{Z}(s^-)) \sigma(\mathbf{Z}(s^-)) d\mathbf{W}(s) \]

\[ + \int_0^t \int_{\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}} [g(\mathbf{Z}(s^-) + \gamma(\mathbf{Z}(s^-), \phi)) - g(\mathbf{Z}(s^-))] \tilde{N}(ds, d\phi). \]

To ensure the existence and uniqueness of the solution, we impose the following assumption.

**Assumption 2.1.** There are some constants \( K_1, K_2 > 0 \) such that \( \forall \mathbf{z}_1, \mathbf{z}_2 \in \mathbb{R}^{\ell_1} \times \mathbb{R}^{\ell_2} \)

\[ |b(\mathbf{z}_1) - b(\mathbf{z}_2)|^2 + |\sigma(\mathbf{z}_1) - \sigma(\mathbf{z}_2)|^2 + \int_{\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}} |\gamma(\mathbf{z}_1, \phi) - \gamma(\mathbf{z}_2, \phi)|^2 \nu(d\phi) \leq K_1 |\mathbf{z}_1 - \mathbf{z}_2|^2, \quad (2.1) \]

and

\[ \int_{\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}} |\gamma(z, \phi)|^2 \nu(d\phi) \leq K_2 (1 + |z|^2). \quad (2.2) \]

**Remark 2.1.** Assumption 2.1 can be replaced with local Lipschitz conditions together with a suitable condition imposed on a Lyapunov function.

To investigate the stability of the coupled jump diffusion system, we need the following assumptions.

**Assumption 2.2.** The following conditions hold.

(i) There exist positive functions \( V_0, V_1 : \mathbb{R}^{\ell_1} \mapsto \mathbb{R}_+ \) satisfying

\[ \mathcal{L}V_0(\mathbf{z}) \leq K_3 - K_4 V_1(\mathbf{x}_1), \quad \forall \mathbf{z} = (\mathbf{x}_1, 0), \quad (2.3) \]

for some constants \( K_3, K_4 > 0 \).
(ii) There exist a function $U : \mathbb{R}^{\ell_2} \mapsto \mathbb{R}_+$ and a constant $m_0 > 0$ such that
\[
\lim_{x_2 \to 0} U(x_2) = \infty, \quad U(x_2) - U(x_2') \leq m_0 \ln \frac{|x_2'|}{|x_2|}, \quad \forall x_2, x_2' \neq 0. \tag{2.4}
\]
Moreover, assume that there are a function $f_1 : \mathbb{R}^{\ell_1} \mapsto \mathbb{R}$, and a Lipschitz function $f_2 : \mathbb{R}^{\ell_1} \mapsto \mathbb{R}_+$, constants $\Delta_0 > 0$, $\alpha_0 > 0$, and $K_5 > 0$ such that
\[
LU(z) \geq f_1(x_1), \forall z = (x_1, x_2), |x_2| \leq \Delta_0, \tag{2.5}
\]
and
\[
|U_{x_2} \sigma_2(z)|^2 + \int_{\mathbb{R}^{\ell_2}} \left[ \exp \left\{ -\alpha_0 (U(x_2 + \gamma_2(z, \phi)) - U(x_2)) \right\} \right] \nu_2(d\phi)
\leq f_2(x_1), \quad \forall z = (x_1, x_2), |x_2| \leq \Delta_0, \tag{2.6}
\]
where $U_{x_2}$ denotes the gradient of $U$; and
\[
|f_1(x_1)| + f_2(x_1) < K_5 V_1(x_1). \tag{2.7}
\]
(iii) When $x_2 = 0$ (yielding $X_2(t) = 0$), the corresponding system for $X_1$
\[
dX_1(t) = b_1(X_1(t), 0)dt + \sigma_1(X_1(t), 0)dW_1(t) + \int_{\mathbb{R}^{\ell_1}} \gamma_1(X_1(t), 0, \phi) \tilde{N}_1(dt, d\phi),
\]
admits a unique invariant measure $\mu^*$ and
\[
\Lambda_1 := \int f_1(x_1) \mu^*(dx_1) > 0.
\]

**Assumption 2.3.** Suppose that $\sigma_1(x_1, x_2)$ admits a right inverse $\sigma^{-1}_1(x_1, x_2)$ such that
\[
\|\sigma^{-1}_1(x_1, x_2)\| \leq c_\sigma < \infty \text{ for all } |x_2| \leq \Delta_0,
\]
where $\Delta_0$ is as in Assumption 2.2. Moreover, assume that there is a constant $K_6$ such that
\[
\forall z = (x_1^T, x_2^T)^T, z' = ((x_1')^T, (x_2')^T)^T \in \mathbb{R}^{\ell_1} \times \mathbb{R}^{\ell_2}
\]
\[
(b_1(z) - b_1(z'))^T (x_1 - x_1') + |\sigma_1(z) - \sigma_1(z')|^2 \leq K_6 |z - z'|^2.
\]
In addition, $\gamma_1$ and $\gamma_2$ are Lipschitz in $z$ and uniformly in $\phi$.

**Remark 2.2.** Assumption 2.2 is the main assumption for stability. This assumption is rather mild and not restrictive. The functions $U$ and $f_1$ in Assumption 2.2(ii) are used to estimate the Lyapunov exponent of $X_2(t)$; $\Lambda_1$ in Assumption 2.2(iii) is a bound of the Lyapunov exponent. A simple but promising candidate of $U(\cdot)$ that satisfies the proposed conditions is $U(x_2) = (-\ln |x_2|) \vee 0$. Assumption 2.2(i) and 2.7 guarantee that the bound $\Lambda_1$ of the Lyapunov exponent is well-defined. Assumption 2.3 collects a form of strong non-degenerate condition of the diffusion and some technical conditions. It can be seen that the conditions are applicable to many systems in applications; see Example 2.1 below as well as Section 3.

Now, we state our main results.
Theorem 2.1. For any $x_1 \in \mathbb{R}^{d_1}$ and $\varepsilon > 0$, there exists $\theta_{x_1, \varepsilon} > 0$ such that if $|x_1 - \bar{x}_1| + |\bar{x}_2| \leq \theta_{x_1, \varepsilon}$,
\begin{equation}
\mathbb{P}_{\bar{z}} \left\{ \lim_{t \to \infty} \frac{U(X_2(t))}{t} \geq m_0 \gamma_0 \right\} \geq 1 - \varepsilon,
\end{equation}
where $\bar{z} = (\bar{x}_1^\top, \bar{x}_2^\top)^\top$; and thus,
\begin{equation}
\mathbb{P}_{\bar{z}} \left\{ \limsup_{t \to \infty} \frac{\ln |X_2|}{t} \leq -\gamma_0 \right\} \geq 1 - \varepsilon.
\end{equation}

Example 2.1. To illustrate our results, let us provide a simple example. Consider a stochastic SIR epidemic model with Beddington-DeAngelis incidence rate given by the following SDEs with jumps,
\begin{equation}
\begin{cases}
    ds(t) = \left[ c_0 - c_1 S(t) - \frac{c_3(S(t))I(t)}{c_4 + c_5 S(t) + c_6 I(t)} \right] dt + \sigma_1(S(t), I(t))dW_1(t) + \int_{\mathbb{R}^n_1} \gamma_1(S(t), I(t), \phi)\tilde{N}_1(dt, d\phi), \\
    dS(t) = \left[ -c_2 I(t) + \frac{c_3(S(t))I(t)}{c_4 + c_5 S(t) + c_6 I(t)} \right] dt + c_7 I(t)dW_2(t) + I(t) \int_{\mathbb{R}^n_1} \gamma_2(S(t), \phi)\tilde{N}_1(dt, d\phi).
\end{cases}
\end{equation}

In the above, $\sigma_1, \gamma_1, \gamma_2$ are bounded functions and are such that $\sigma_1(s,i), \gamma_1(s,i,\phi), \gamma_2(s,i,\phi) = \tilde{\gamma}_2(\phi)$ satisfy the technical conditions in Assumptions 2.1 and 2.2. Now, we check the stability conditions (Assumption 2.2). It is easily verified that Assumption 2.2(i) is satisfied with $V_0(s) = V_1(s) = s$; Assumption 2.2(ii) is satisfied with
\begin{equation*}
    U(i) = (-\ln i) \vee 0, \quad f_1(s) = c_2 + \frac{c_3}{2} + \int_{\mathbb{R}^n_2} |\tilde{\gamma}_2(\phi)|^2 \nu_2(d\phi) - \frac{c_3 s}{c_4 + c_5 s},
\end{equation*}
and $f_2(s)$ is some large constant. Under certain conditions, the corresponding system when $I(t) = 0$
\begin{equation*}
    d\tilde{S}(t) = (c_0 - c_1 \tilde{S}(t))dt + \sigma_1(\tilde{S}(t), 0)dW_1(t) + \int_{\mathbb{R}^n_1} \gamma_1(\tilde{S}(t), 0, \phi)\tilde{N}_1(dt, d\phi)
\end{equation*}
has a unique invariant measure $\mu^*$. Therefore, if $A_1 := \int f_1(s)\mu^*(ds) > 0$, applying Theorem 2.1 $I(t)$ is stable at 0. Without jump, this result is consistent with the longtime characterization in stochastic SIR epidemic models in [15, 16]; it generalizes the results in the aforementioned references.

To prove Theorem 2.1, we begin with some auxiliary lemmas. Lemma 2.1 provides a local boundedness (uniform in finite intervals) in probability of the solution and Lemma 2.2 illustrates the continuity on initial value (at 0) in probability of $\sup_{t \in [0,T]} |X_2(t)|$, for any finite time $T$.

Lemma 2.1. For any $T > 0$, $\varepsilon > 0$, $R > 0$, there exists $H_{1,T,\varepsilon,R} > 0$ such that
\begin{equation*}
    \mathbb{P}_{z} \left\{ \sup_{t \in [0,T]} |Z(t)| \leq H_{1,T,\varepsilon,R} \right\} > 1 - \varepsilon, \text{ for all } |z| \leq R.
\end{equation*}

Proof. Under Assumption 2.1 using a standard argument (see e.g., [30] Lemma 6.9), we obtain the following local boundedness
\begin{equation*}
    \mathbb{E}_z \sup_{t \in [0,T]} |Z(t)| < C_T(z),
\end{equation*}
where $C_T(z)$ is some finite constant depending on $T$ and $z$ that is locally bounded in $z$. As a result, the Markov inequality implies that for any $H > 0$,
\begin{equation*}
    \mathbb{P}_z \left\{ \sup_{t \in [0,T]} |Z(t)| > H \right\} \leq \frac{\mathbb{E}_z \sup_{t \in [0,T]} |Z(t)|}{H} < \frac{C_T(z)}{H},
\end{equation*}
which yields that for any $T > 0$, $\varepsilon > 0$, and $R > 0$, there is an $H_{1,T,\varepsilon,R} > 0$ such that
\[
\mathbb{P}_{z}\left\{ \sup_{t \in [0,T]} |Z(t)| \leq H_{1,T,\varepsilon,R} \right\} > 1 - \varepsilon, \text{ for all } |z| \leq R.
\]

The proof is complete. \hfill \Box

**Lemma 2.2.** For any $T > 0$, $\varepsilon > 0$, $R > 0$, and $\delta_1 > 0$, there exists $\delta_2 = \delta_2(T,\varepsilon,R,\delta_1) > 0$ such that
\[
\mathbb{P}_{z}\left\{ \sup_{t \in [0,T]} |X_2(t)| \leq \delta_1 \right\} > 1 - \varepsilon, \text{ for all } |x_1| \leq R, |x_2| \leq \delta_2.
\]

**Proof.** Because of (2.4), there is an $L = L(\delta_1)$ such that whenever $U(x_2) > L(\delta_1)$, $|x_2| < \delta_1$. By the generalized Itô formula, we have
\[
U(X_2(t)) = U(x_2) + \int_0^t \mathcal{L}U(z(u))du + \int_0^t U_{x_2}\sigma_2(Z(u))dW_2(u) + \int_0^t \int_{\mathbb{R}^2} [U(x_2(u) + \gamma(Z(u), \phi)) - U(x_2(u))] \tilde{N}_2(du, d\phi).
\]
(2.11)

Let $H(z, \phi) = U(x_2 + \gamma_2(z, \phi)) - U(x_2)$, $z = (x_1, x_2)^T$, and $\alpha$ be such that $0 < \alpha < \frac{\alpha_0}{2}$, where $\alpha_0$ is in Assumption 2.2. From the exponential martingale inequality (see e.g., [2] Theorem 5.2.9), we have that with probability greater $1 - \varepsilon$ the following inequality holds for all $t \in [0, T]$,
\[
- \int_0^t U_{x_2}(X_2(u))\sigma_2(Z(u))dW_2(u) - \int_0^t \int_{\mathbb{R}^2} H(Z(u), \phi)\tilde{N}_2(du, d\phi)
\leq \frac{\alpha}{2} \int_0^t |U_{x_2}(X_2(u))\sigma_2(Z(u))|^2 du + \frac{1}{\alpha} \int_0^t \int_{\mathbb{R}^2} \left( e^{\alpha H(Z(u), \phi)} - 1 - \alpha H(Z(u), \phi) \right) \nu_2(\phi)du - \frac{\ln \varepsilon}{\alpha}
\leq \frac{\alpha}{2} \int_0^t |U_{x_2}(X_2(u))\sigma_2(Z(u))|^2 du + \frac{\alpha}{2} \int_0^t \int_{\mathbb{R}^2} H^2(Z(u), \phi) e^{\frac{\alpha}{2} H(Z(u), \phi)} \nu_2(\phi)du - \frac{\ln \varepsilon}{\alpha}
\leq \frac{\alpha}{2} \int_0^t |U_{x_2}(X_2(u))\sigma_2(Z(u))|^2 du + \frac{C_{\alpha_0}\alpha}{2} \int_0^t \int_{\mathbb{R}^2} e^{\alpha H(Z(u), \phi)} \nu_2(\phi)du - \frac{\ln \varepsilon}{\alpha},
\]

where $C_{\alpha_0}$ is some finite constant such that $\tilde{t}^2 e^{\frac{\alpha_0\tilde{t}}{2}} \leq C_{\alpha_0} e^{\alpha_0 \tilde{t}}, \forall \tilde{t} \geq 0$. As a result, we have
\[
\mathbb{P}_{\{ \int_0^t U_{x_2}(X_2(u))\sigma_2(Z(u))dW_2(u) + \int_0^t \int_{\mathbb{R}^2} H(Z(u), \phi)\tilde{N}_2(du, d\phi)
\geq - \frac{\alpha}{2} \int_0^t |U_{x_2}(X_2(u))\sigma_2(Z(u))|^2 du - \frac{C_{\alpha_0}\alpha}{2} \int_0^t \int_{\mathbb{R}^2} e^{\alpha H(Z(u), \phi)} \nu_2(\phi)du + \frac{\ln \varepsilon}{\alpha}, \forall t \in [0, T] \}
\geq 1 - \varepsilon.
\]
(2.12)

We obtain from Assumption 2.2 (ii) that
\[
- \int_0^t \mathcal{L}U(Z(u))du + \frac{\alpha}{2} \int_0^t |U_{x_2}(X_2(u))\sigma_2(Z(u))|^2 du + \frac{C_{\alpha_0}\alpha}{2} \int_0^t \int_{\mathbb{R}^2} e^{\alpha H(Z(u), \phi)} \nu_2(\phi)du
\leq C_{\alpha,\alpha_0} \int_0^t V_1(X_1(u), 0)du.
\]
In addition, by Lemma 2.1 with probability greater than $1 - \varepsilon$, $|Z(t)| \leq H_{1,T,\varepsilon,R}$, $\forall t \in [0,T]$. As a result, with probability greater $1 - 2\varepsilon$, we have

$$- \int_0^t [L \cdot (Z(u))] du + \frac{\alpha}{2} \int_0^t |U_{X_2}(X_2(u))\sigma_2(Z(u))|^2 du + \frac{C_{\alpha_0}\alpha}{2} \int_0^t \int_{\mathbb{R}^n} e^{\alpha_0|H(Z(u),\phi)|} \nu_2(d\phi) du \leq C_{T,R,\varepsilon,\alpha,\alpha_0},$$

(2.13)

for some finite constant $C_{T,R,\varepsilon,\alpha,\alpha_0}$. The combination of (2.11), (2.12), and (2.13) yields that with probability greater $1 - 2\varepsilon$,

$$U(X_2(t)) \geq U(x_2) - C_{T,R,\varepsilon,\alpha,\alpha_0} + \frac{\ln \varepsilon}{\alpha}, \quad \forall t \in [0,T].$$

Because of (2.4), there exists a $\delta_2 > 0$ such that for all $|x_2| < \delta_2$,

$$U(x_2) - C_{T,R,\varepsilon,\alpha,\alpha_0} + \frac{\ln \varepsilon}{\alpha} > L(\delta_1).$$

Therefore, with probability greater $1 - 2\varepsilon$, $\sup_{t \in [0,T]} U(X_2(t)) \geq L(\delta_1)$, for all $|x_1| < R, |x_2| < \delta_2$, and thus, for all $|x_1| < R, |x_2| < \delta_2$, $\sup_{t \in [0,T]} |X_2(t)| \leq \delta_1$, $\forall t \in [0,T]$.

\[\square\]

As was mentioned, one of the main challenges in this work is the coupled interaction of $X_1(t)$ and $X_2(t)$. To overcome this difficulty, we apply and modify the coupling method [10]. Let $\lambda > 0$ such that $\lambda > 2(1 + K_2)$ with $K_2$ being the Lipschitz constant in Assumption 2.1. Consider the following coupled equations

$$\begin{cases}
\d X_1(t) = b_1(X_1(t), 0) dt + \sigma_1(X_1(t), 0) d\mathbf{W}_1(t) + \int_{\mathbb{R}^n_+} \gamma_1(X_1(t), 0, \phi) \mathbf{N}_1(dt, d\phi), \\
\d \tilde{X}_1(t) = b_1(\tilde{Z}(t)) dt + \lambda(X_1(t) - \tilde{X}_1(t)) dt + \sigma_1(\tilde{Z}(t)) d\mathbf{W}_1(t) + \int_{\mathbb{R}^n_+} \gamma_1(\tilde{Z}(t), \phi) \mathbf{N}_1(dt, d\phi), \\
\d \tilde{X}_2(t) = b_2(\tilde{Z}(t)) dt + \sigma_2(\tilde{Z}(t)) d\mathbf{W}_2(t) + \int_{\mathbb{R}^n_2} \gamma_2(\tilde{Z}(t), \phi) \mathbf{N}_2(dt, d\phi), \\
X_1(0) = x_1, \tilde{X}_1(0) = \tilde{x}_1, \tilde{X}_2(0) = \tilde{x}_2,
\end{cases}$$

(2.14)

where $\tilde{Z}(t) = ([X_1(t)]^T, [X_2(t)]^T)^T$, $\tilde{z} = ([x_1]^T, [x_2]^T)^T$. We will use $\mathbb{P}_{X_1,\tilde{z}}$ to indicate the initial conditions of the coupled equations (2.14).

Denote $A_2 := \int f_2(x_1)\mu^*(dx_1) > 0$, where $\mu^*$ is the unique invariant measure of $X_1(t)$ when $X_2(t) = 0$ as in Assumption 2.2 (iii). Let $C_{\alpha_0} > 1$ be a constant as in Lemma 2.2, i.e., a constant satisfying $t^2 e^{\frac{\alpha t^2}{2}} \leq C_{\alpha_0} e^{\alpha_0 t}, \forall t \geq 0$. In the remaining of this section, let

$$\gamma_0 \in (0, \frac{A_1}{m_0}), \quad \sigma_0 = \frac{A_1 - m_0 \gamma_0}{3} > 0, \quad \alpha = \frac{\sigma_0}{2C_{\alpha_0} A_2}, \quad \lambda_0 \in (0, \frac{\gamma_0}{4}),$$

and for each $\delta > 0$,

$$\tilde{\tau}_\delta := \inf\{t > 0 : |\tilde{X}_2(t)| \geq \delta e^{-\gamma_0 t}\}.$$

To proceed, we present the following lemmas. Lemma 2.3 provides estimates of the coupling time of the coupled processes, while Lemma 2.4 handles the diffusion and jump parts.
Lemma 2.3. There is a universal constant $\bar{C} > 1$ such that
\[
\mathbb{E}\left( \sup_{t \leq \tau_{\tilde{\eta}}} e^{\lambda t} |X_1(t) - \tilde{X}_1(t)|^2 \right) \leq \bar{C}(|x_1 - \tilde{x}_1| + \delta)^2.
\] (2.15)

As a result, for any $\varepsilon > 0$ one has
\[
\mathbb{P}\left\{ \int_0^{t \wedge \tau_{\tilde{\eta}}} |v(s)|^2 ds \geq \frac{(|x_1 - \tilde{x}_1| + \delta)^2}{\varepsilon} \quad \text{for some } t \geq 0 \right\} \leq \frac{\bar{C}\lambda c_\varepsilon}{\lambda_0},
\]
where $c_\varepsilon$ is in Assumption \[2.2\] and 
\[
v(t) := \lambda \sigma_1^{-1}(X_1(t), 0)(X_1(t) - \tilde{X}_1(t)).
\] (2.16)

Proof. Use $C$ to denote a finite constant, whose values may change at difference appearances. By the generalized Itô formula for jump diffusions, we have that
e^{\lambda t} |X_1(t) - \tilde{X}_1(t)|^2
= |x_1 - \tilde{x}_1|^2 + \int_0^t e^{\lambda s} (\lambda_0 - \lambda) |X_1(s) - \tilde{X}_1(s)|^2 ds
+ 2 \int_0^t e^{\lambda s} \left( (\sigma_1(X_1(s), 0) - \sigma_1(\tilde{X}_1(s), \tilde{X}_2(s))) (\sigma_1(X_1(s), 0) - \sigma_1(\tilde{X}_1(s), \tilde{X}_2(s))) \right)^T ds
+ \int_0^t e^{\lambda s} \int_{\mathbb{R}^n} \left[ \left| X_1(s) - \tilde{X}_1(s) + \gamma_1(X_1(s), 0, \phi) - \gamma_1(\tilde{X}_1(s), \tilde{X}_2(s), \phi) \right|^2 - |X_1(s) - \tilde{X}_1(s)|^2 
- 2(\tilde{X}_1(s) - \tilde{X}_1(s))^T (\gamma_1(\tilde{X}_1(s), 0, \phi) - \gamma_1(\tilde{X}_1(s), \tilde{X}_2(s), \phi)) \nu_1(d\phi) ds
+ 2 \int_0^t e^{\lambda s} \left( \sigma_1(X_1(s), 0) - \sigma_1(\tilde{X}_1(s), \tilde{X}_2(s)) \right)^T dW_1(s)
+ \int_0^t e^{\lambda s} \int_{\mathbb{R}^n} \left[ \left| X_1(s) - \tilde{X}_1(s) + \gamma_1(X_1(s), 0, \phi) - \gamma_1(\tilde{X}_1(s), \tilde{X}_2(s), \phi) \right|^2
- |X_1(s) - \tilde{X}_1(s)|^2 \right] dN_1(ds, d\phi).
\] (2.17)

By virtue of Kunita’s first inequality [21 Theorem 4.4.23, p. 265], for all $T \geq 0$,
\[
\mathbb{E}\sup_{t \in [0, T]} \left[ \int_0^{t \wedge \tau_{\tilde{\eta}}} e^{2\lambda s} \left[ \left| X_1(s) - \tilde{X}_1(s) + \gamma_1(X_1(s), 0, \phi) - \gamma_1(\tilde{X}_1(s), \tilde{X}_2(s), \phi) \right|^2
- |X_1(s) - \tilde{X}_1(s)|^2 \right] dN_1(ds, d\phi) \right] \leq C \mathbb{E} \int_0^{T \wedge \tau_{\tilde{\eta}}} e^{2\lambda s} \left[ \left| X_1(s) - \tilde{X}_1(s) + \gamma_1(X_1(s), 0, \phi) - \gamma_1(\tilde{X}_1(s), \tilde{X}_2(s), \phi) \right|^2
- |X_1(s) - \tilde{X}_1(s)|^2 \right] \nu_1(d\phi) ds
\leq C E \left( \int_0^{T \wedge \tau_{\tilde{\eta}}} e^{2\lambda s} |X_1(s) - \tilde{X}_1(s)|^4 ds + \delta^4 \int_0^{T \wedge \tau_{\tilde{\eta}}} e^{(-4\gamma_0 + 2\lambda_0)s} ds \right)
\leq C \left( \delta^4 + \mathbb{E} \int_0^{T \wedge \tau_{\tilde{\eta}}} e^{2\lambda s} |X_1(s) - \tilde{X}_1(s)|^4 ds \right).
\] (2.18)
Taking the expectation on both sides of (2.21), using the Lipschitz continuity of again, we have
\[ \mathbb{E} \sup_{t \in [0,T]} \left| e^{\lambda_0 s} (X_1(s) - \bar{X}_1(s)) \right|^4 \leq \mathbb{E} \left[ \int_0^{T \wedge \bar{\tau}_3} e^{2\lambda_0 s} |X_1(s) - \bar{X}_1(s)|^4 ds + \delta^4 \int_0^{T \wedge \bar{\tau}_3} e^{(-4\delta_0+2\lambda_0)s} ds \right] \]
(2.19)

Now, applying (2.18) and (2.19) to (2.17), and using the Lipschitz continuity of \( b_1(\cdot, \cdot), \sigma_1(\cdot, \cdot), \) and \( \gamma_1(\cdot, \cdot, \cdot) \), we obtain that
\[ \mathbb{E} \sup_{t \leq T \wedge \bar{\tau}_3} e^{2\lambda_0 t} |X_1(t) - \bar{X}_1(t)|^4 \leq \left| x_1 - \bar{x}_1 \right|^4 + C \left( \delta^2 + \int_0^T e^{(2\lambda_0 - \gamma_0)s} ds + \left[ \mathbb{E} \int_0^{T \wedge \bar{\tau}_3} e^{2\lambda_0 s} |X_1(s) - \bar{X}_1(s)|^4 ds \right]^{\frac{1}{2}} \right). \]
(2.20)

To proceed, we estimate \( \mathbb{E} \int_0^{T \wedge \bar{\tau}_3} e^{2\lambda_0 s} |X_1(s) - \bar{X}_1(s)|^4 ds \). Using the generalized Itô formula again, we have
\[ e^{2\lambda_0 t} |X_1(t) - \bar{X}_1(t)|^4 \]
\[ = \left| x_1 - \bar{x}_1 \right|^4 + \int_0^t e^{2\lambda_0 (2\lambda_0 - \lambda)} |X_1(s) - \bar{X}_1(t)|^4 ds \]
\[ + \int_0^t 4e^{2\lambda_0 s} |X_1(s) - \bar{X}_1(s)|^2 (X(s) - \bar{X}_1(s))^\top (b_1(X_1(s), 0) - b_1(\bar{X}_1(s), \bar{X}_2(s))) ds \]
\[ + 3 \int_0^t e^{2\lambda_0 s} |X_1(s) - \bar{X}_1(s)|^2 \times tr \left[ (\sigma_1(X_1(s), 0, \phi) - \sigma_1(\bar{X}_1(s), \bar{X}_2(s), \phi) - \sigma_1(\bar{X}_1(s), \bar{X}_2(s), \phi))^\top \right] ds \]
\[ + \int_0^t \int_{\mathbb{R}^n} e^{2\lambda_0 s} \left[ |X_1(s) - \bar{X}_1(s) + \gamma_1(X_1(s), 0, \phi) - \gamma_1(\bar{X}_1(s), \bar{X}_2(s), \phi)|^4 - |X_1(s) - \bar{X}_1(s)|^4 \right. \\
- 4 |X_1(s) - \bar{X}_1(s)|^2 (X_1(s) - \bar{X}_1(s))^\top (\gamma_1(X_1(s), 0, \phi) - \gamma_1(\bar{X}_1(s), \bar{X}_2(s), \phi)) \right] \nu_1(d\phi) ds \]
\[ + 4 \int_0^t e^{2\lambda_0 s} |X_1(s) - \bar{X}_1(s)|^2 \times (X_1(s) - \bar{X}_1(s))^\top (\sigma_1(X_1(s), 0) - \sigma_1(\bar{X}_1(s), \bar{X}_2(s))) dW_1(s) \]
\[ + \int_0^t \int_{\mathbb{R}^n} e^{2\lambda_0 s} \left[ |X_1(s) - \bar{X}_1(s) + \gamma_1(X_1(s), 0, \phi) - \gamma_1(\bar{X}_1(s), \bar{X}_2(s), \phi)|^4 \right. \\
- |X_1(s) - \bar{X}_1(s)|^4 \left. \bar{N}_1(d\phi, d\phi) \right. \]
(2.21)

Taking the expectation on both sides of (2.21), using the Lipschitz continuity of \( b_1(\cdot, \cdot), \sigma_1(\cdot, \cdot), \gamma_1(\cdot, \cdot, \cdot) \), and noting \( \lambda \) being chosen to be sufficiently large, we obtain
\[ d\mathbb{E} e^{2\lambda_0 (t \wedge \bar{\tau}_3)} |X_1(t \wedge \bar{\tau}_3) - \bar{X}_1(t \wedge \bar{\tau}_3)|^4 \]
\[ \leq \mathbb{E} \left[ - D_1 e^{2\lambda_0 (t \wedge \bar{\tau}_3)} |X_1(t) - \bar{X}_1(t)|^4 + D_2 \delta^4 e^{(-4\delta_0+2\lambda_0)(t \wedge \bar{\tau}_3)} \right] dt, \]
(2.22)
for some finite positive constants $D_1$ and $D_2$. Eq. (2.22) implies that
\[
\mathbb{E} \int_0^{\bar{T} \wedge \bar{\zeta}} e^{2\lambda_0 s} |X_1(s) - \bar{X}_1(s)|^4 \, ds \leq C \left( |x_1 - \bar{x}_1|^4 + \delta^4 \right) \text{ for all } t \geq 0,
\]
and thus,
\[
\left[ \mathbb{E} \int_0^{\bar{T} \wedge \bar{\zeta}} e^{2\lambda_0 s} |X_1(s) - \bar{X}_1(s)|^4 \, ds \right]^{\frac{1}{2}} \leq C \left( |x_1 - \bar{x}_1|^2 + \delta^2 \right) \text{ for all } t \geq 0. \tag{2.23}
\]
Combining (2.23) and (2.20), we get that
\[
\mathbb{E} \sup_{t \leq T \wedge \bar{\zeta}} e^{\lambda_0 t} |X_1(t) - \bar{X}_1(t)|^2 \leq C \left( |x_1 - \bar{x}_1|^2 + \delta^2 \right) \text{ for all } T \geq 0. \tag{2.24}
\]
Therefore, (2.15) is proved.

Now, we consider the second part. By virtue of the definition of $v(t)$ in (2.16) and Assumption 2.2 (iv),
\[
P \left\{ \int_0^{T \wedge \bar{\zeta}} |v(s)|^2 \, ds \geq \frac{(|x_1 - \bar{x}_1| + \delta)^2}{\varepsilon} \text{ for some } t \geq 0 \right\}
\leq P \left\{ \int_0^{\bar{T} \wedge \bar{\zeta}} \lambda \sigma_1^{-1}(X_1(s), 0)(X_1(s) - \bar{X}_1(s))^2 \, ds \geq \frac{(|x_1 - \bar{x}_1| + \delta)^2}{\varepsilon} \right\} \tag{2.25}
\leq P \left\{ \int_0^{\bar{T} \wedge \bar{\zeta}} |X_1(s) - \bar{X}_1(s)|^2 \, ds \geq \frac{(|x_1 - \bar{x}_1| + \delta)^2}{\lambda \sigma_1 \varepsilon} \right\}.
\]
A standard calculation shows that for the integrable function $h(s)$,
\[
\int_0^t h(s) \, ds = \int_0^t e^{\lambda_0 s} h(s) \, ds \leq \sup_{s \in [0,t]} e^{\lambda_0 s} h(s) \int_0^t e^{-\lambda_0 s} \, ds \leq \frac{1}{\lambda_0} \sup_{s \in [0,t]} e^{\lambda_0 s} h(s).
\]
Therefore, it follows from (2.25) that
\[
P \left\{ \int_0^{T \wedge \bar{\zeta}} |v(s)|^2 \, ds \geq \frac{(|x_1 - \bar{x}_1| + \delta)^2}{\varepsilon} \text{ for some } t \geq 0 \right\}
\leq P \left\{ \sup_{t \leq \bar{T} \wedge \bar{\zeta}} e^{\lambda_0 t} |X_1(t) - \bar{X}_1(t)|^2 \geq \frac{\lambda_0 (|x_1 - \bar{x}_1| + \delta)^2}{\lambda \sigma_1 \varepsilon} \right\} \tag{2.26}
\leq \frac{\lambda \sigma_1 \varepsilon}{\lambda_0} \mathbb{E} \sup_{t \leq \bar{T} \wedge \bar{\zeta}} e^{\lambda_0 t} |X_1(t) - \bar{X}_1(t)|^2 \leq \frac{\tilde{C} \lambda \sigma_1 \varepsilon}{\lambda_0} \text{ (due to (2.15)).}
\]
As a result, the proof is complete. \hfill \Box

**Lemma 2.4.** Let $\tilde{C}$ be as in Lemma 2.1 and $L$ be a Lipschiz constant of $f_1(\cdot)$ and $f_2(\cdot)$. Suppose $|x_1 - \bar{x}_1| + \delta < 1$. Then we have
\[
P_{x_1, \bar{x}_2} \left\{ \int_0^t U_{x_2} (\tilde{Z}(u)) dW_2(u) + \int_0^t \int_{\mathbb{R}^2} H(\tilde{Z}(u), \phi) \tilde{N}_2(du, d\phi) \right. \\
\left. \geq -C_0 \alpha \int_0^t f_2(X_1(u)) \, du + \frac{\ln \varepsilon}{\alpha} - \frac{2LC}{\varepsilon \lambda_0}, \forall 0 \leq t \leq \bar{T} \right\} \geq 1 - 2\varepsilon.
Proof. Let \( H(z, \phi) = U(x_2 + \gamma_2(z, \phi)) - U(x_2) \). From the exponential martingale inequality (see e.g., [2, Theorem 5.2]), we have with probability greater than \( 1 - \varepsilon \),

\[
- \int_0^t U_{x_2}(\bar{X}_2(u))\sigma_2(\bar{Z}(u))dW_2(u) - \int_0^t \int_{\mathbb{R}_+^2} H(\bar{Z}(u), \phi)\bar{N}_2(du, d\phi)
\]

\[
\leq \frac{\alpha}{2} \int_0^t |U_{x_2}(\bar{X}_2(u))\sigma_2(\bar{Z}(u))|^2 du + \frac{1}{\alpha} \int_0^t \int_{\mathbb{R}_+^2} \left( e^{\alpha H(\bar{Z}(u), \phi)} - 1 - \alpha H(\bar{Z}(u), \phi) \right) \nu_2(du)du - \frac{\ln \varepsilon}{\alpha}
\]

\[
\leq \frac{\alpha}{2} \int_0^t |U_{x_2}(\bar{X}_2(u))\sigma_2(\bar{Z}(u))|^2 du + \frac{\alpha}{2} \int_0^t \int_{\mathbb{R}_+^2} H^2(\bar{Z}(u), \phi)e^{\frac{\alpha}{2} |H(\bar{Z}(u), \phi)|} \nu_2(du)du - \frac{\ln \varepsilon}{\alpha}
\]

\[
\leq \frac{\alpha}{2} \int_0^t |U_{x_2}(\bar{X}_2(u))\sigma_2(\bar{Z}(u))|^2 du + \frac{C_{\alpha, \alpha}}{2} \int_0^t \int_{\mathbb{R}_+^2} e^{\alpha |H(\bar{Z}(u), \phi)|} \nu_2(du)du - \frac{\ln \varepsilon}{\alpha}.
\]

As a result, \( \mathbb{P}(\Omega_1) \geq 1 - \varepsilon \), where

\[
\Omega_1 := \left\{ \int_0^t U_{x_2}(\bar{X}_2(u))\sigma_2(\bar{Z}(u))dW_2(u) + \int_0^t \int_{\mathbb{R}_+^2} H(\bar{Z}(u), \phi)\bar{N}_2(du, d\phi) \right\}
\]

\[
\leq \frac{\alpha}{2} \int_0^t |U_{x_2}(\bar{X}_2(u))\sigma_2(\bar{Z}(u))|^2 du + \frac{C_{\alpha, \alpha}}{2} \int_0^t \int_{\mathbb{R}_+^2} e^{\alpha |H(\bar{Z}(u), \phi)|} \nu_2(du)du - \frac{\ln \varepsilon}{\alpha}
\].

On the other hand, by (2.13), one has \( \mathbb{P}(\Omega_2) \geq 1 - \varepsilon \), where

\[
\Omega_2 := \left\{ \sup_{t \leq \bar{t}_\delta} e^{\lambda_0 t}|X_1(t) - \bar{X}_1(t)|^2 \geq \frac{C}{\varepsilon} \right\}.
\]

For \( t \leq \bar{t}_\delta \) and \( \omega \in \Omega_1 \cap \Omega_2 \), we have

\[
- \int_0^t U_{x_2}(\bar{X}_2(u))\sigma_2(\bar{Z}(u))dW_2(u) - \int_0^t \int_{\mathbb{R}_+^2} H(\bar{Z}(u), \phi)\bar{N}_2(du, d\phi)
\]

\[
\leq \frac{\alpha}{2} \int_0^t |U_{x_2}(\bar{X}_2(u))\sigma_2(\bar{Z}(u))|^2 du + \frac{C_{\alpha, \alpha}}{2} \int_0^t \int_{\mathbb{R}_+^2} e^{\alpha |H(\bar{Z}(u), \phi)|} \nu_2(du)du - \frac{\ln \varepsilon}{\alpha}
\]

\[
\leq C_{\alpha, \alpha} \int_0^t f_2(\bar{X}_1(u))du - \frac{\ln \varepsilon}{\alpha} + L \int_0^t |X_1(u) - \bar{X}_1(u)|du
\]

\[
\leq C_{\alpha, \alpha} \int_0^t f_2(X_1(u))du - \frac{\ln \varepsilon}{\alpha} + \frac{L \sqrt{C}}{\varepsilon} \int_0^t e^{-\frac{\lambda_0 u}{2}} du
\]

\[
\leq C_{\alpha, \alpha} \int_0^t f_2(X_1(u))du - \frac{\ln \varepsilon}{\alpha} + \frac{2L \sqrt{C}}{\varepsilon \lambda_0}.
\]

Therefore, the proof is complete. \( \square \)

**Proof of Theorem 2.7.** We can assume that \( \varepsilon \in (0, \frac{1}{2}) \) and \( e^{-3\varepsilon} \geq 1 - 4\varepsilon \). Let

\[
\delta \leq \frac{\varepsilon^3}{\ln \varepsilon} \wedge \frac{\varepsilon^2}{2} \wedge \frac{\lambda_0}{2C \lambda_\sigma},
\]

(2.27)

where \( \tilde{C} \) is in Lemma 2.1. We have from the definitions of \( \Lambda_1, \Lambda_2 \) that

\[
\mathbb{P}_{X_1, 0} \left( \limsup_{t \to \infty} \frac{1}{t} \int_0^t f_1(X_1(u))du = \Lambda_1 \right) = \mathbb{P}_{X_1, 0} \left( \limsup_{t \to \infty} \frac{1}{t} \int_0^t f_2(X_1(u))du = \Lambda_2 \right) = 1.
\]
It is noted that \( m_0 \gamma_0 + 2 \sigma_0 < \Lambda_1 \) so that there exists \( T_{x_1, \varepsilon} \geq \frac{1}{\sigma_0} \left( \frac{-\ln \varepsilon}{\alpha} + \frac{4L \tilde{C}}{\varepsilon \lambda_0} \right) \) such that \( \mathbb{P}_{x_1, 0}(\Omega_3) \geq 1 - \varepsilon \), where

\[
\Omega_3 := \left\{ \frac{1}{t} \int_0^t f_1(X_1(u))du \geq m_0 \gamma_0 + 2 \sigma_0 \ , \ \frac{1}{t} \int_0^t f_2(X_1(u))du \leq \Lambda_2 \ , \ t \geq T_{x_1, \varepsilon} \right\}.
\]

Let \( M_{x_1, \varepsilon} > 0 \) be such that \( \mathbb{P}_{x_1, 0}(\Omega_4) \geq 1 - \varepsilon \), where

\[
\Omega_4 := \left\{ \frac{1}{t} \int_0^t f_1(X_1(u))du - \frac{C_{a_0} \alpha}{t} \int_0^t f_2(X_1(u))du + \frac{\ln \varepsilon}{\alpha} - \frac{4L \tilde{C}}{\varepsilon \lambda_0} \geq -M_{x_1, \varepsilon}, \ t \leq T_{x_1, \varepsilon} \right\},
\]

and \( \Omega_1, \Omega_2 \) be as in Lemma 2.4. By the generalized Itô formula, we have

\[
U(\bar{X}_2(t)) = U(x_2) + \int_0^t \mathcal{L}U(\bar{Z}(u))du + \int_0^t U_{x_2} \sigma_2(\bar{Z}(u-))d\mathcal{W}_2(u)
+ \int_0^t \int_{\mathbb{R}^n} [U(\bar{X}_2(u-)) + \gamma_2(\bar{Z}(u-), \phi)] - U(\bar{X}_2(u-))]\bar{N}_2(du, d\phi).
\]

Definitions of \( \Omega_1, \Omega_2, \) and \( \Omega_4 \) lead to

\[
U(\bar{X}_2(t)) \leq U(x_2) - M_{x_1, \varepsilon}, \quad \forall t \leq T_{x_1, \varepsilon} \cap \bar{\tau}_\delta, \quad \omega \in \Omega_1 \cap \Omega_2 \cap \Omega_4.
\]

Let \( \theta_{x_1, \varepsilon} \in (0, \frac{1}{\delta}) \) such that \( U(x_2) - U(x_2') > M_{x_1, \varepsilon} \) if \( |x_2| \leq \theta_{x_1, \varepsilon}, |x_2'| \geq \delta e^{-T_{x_1, \varepsilon}} \). Such a \( \theta_{x_1, \varepsilon} \) exists owing to (2.7). For \( |x_2| < \theta_{x_1, \varepsilon} \) and \( \omega \in \Omega_2 \cap \Omega_3 \), we must have \( \bar{\tau}_\delta > T_{x_1, \varepsilon} \). Otherwise, if \( \bar{\tau}_\delta \leq T_{x_1, \varepsilon} \), we have \( U(\bar{X}_2(\bar{\tau}_\delta)) \geq U(\bar{X}_2) - M_{x_1, \varepsilon} \), which implies that \( |\bar{X}_2(\bar{\tau}_\delta)| < \delta e^{-T_{x_1, \varepsilon}} \), and again contradict to the definition of \( \bar{\tau}_\delta \).

For \( \omega \in \cap_{i=1}^1 \Omega_i \), we have \( \bar{\tau}_\delta > T_{x_1, \varepsilon} \). From (2.28), Assumption 2.2 and Lemma 2.4 one has

\[
U(\bar{X}_2(t)) \geq U(\bar{X}_2) + \int_0^t f_1(\bar{X}_1(u))du - C_{a_0} \alpha \int_0^t f_2(X_1(u))du + \frac{\ln \varepsilon}{\alpha} - \frac{2L \tilde{C}}{\varepsilon \lambda_0}
\geq U(\bar{X}_2) + \int_0^t f_1(\bar{X}_1(u))du - L \int_0^t |X_1(u) - \bar{X}_1(u)|du
- \frac{C_{a_0} \alpha}{t} \int_0^t f_2(X_1(u))du + \frac{\ln \varepsilon}{\alpha} - \frac{2L \tilde{C}}{\varepsilon \lambda_0}
\geq U(\bar{X}_2) + m_0 \gamma_0 t + 2 \sigma_0 \gamma_0 t - \frac{L \tilde{C}}{\varepsilon} \int_0^t e^{\lambda_0 u \frac{\ln \varepsilon}{\alpha}}du + \frac{\ln \varepsilon}{\alpha} - \frac{2L \tilde{C}}{\varepsilon \lambda_0}
\geq U(\bar{X}_2) + m_0 \gamma_0 t + 2 \sigma_0 \gamma_0 t + \frac{\ln \varepsilon}{\alpha} - \frac{4L \tilde{C}}{\varepsilon \lambda_0}
\geq U(\bar{X}_2) + m_0 \gamma_0 t, \quad \forall t \in [T, \bar{\tau}_\delta].
\]

The combination of (2.31) and (2.4) implies that \( \ln |\bar{X}_2(t)| \leq \ln |\bar{X}_2| - \gamma_0 t, \forall t \in [T, \bar{\tau}_\delta] \), so that \( |\bar{X}_2(t)| \leq \frac{\delta}{2} e^{-\gamma_0 t}, t \in [T, \bar{\tau}_\delta] \). As a result, if \( \bar{X}_2 < \theta_{x_1, \varepsilon}, \bar{\tau}_\delta = \infty \) for all \( \omega \in \cap_{i=1}^1 \Omega_i \). It yields that for all \( \omega \in \cap_{i=1}^1 \Omega_i \),

\[
\liminf_{t \to \infty} \frac{U(\bar{X}_2(t))}{\gamma_0} \geq m_0 \gamma_0 \text{ and } \limsup_{t \to \infty} \frac{\ln |\bar{X}_2(t)|}{t} \leq -\gamma_0.
\]

An application of Lemma 2.3 leads to \( \mathbb{P}_{x_1, \bar{X}_-(\Omega_5)} \geq 1 - \varepsilon \), where

\[
\Omega_5 := \left\{ \int_0^{\bar{\tau}_\delta} |x_1|^2 ds \leq \frac{C \lambda \sigma_0 (|x_1 - \bar{x}_1| + \delta)^2}{\lambda_0 \varepsilon} \quad \forall t \geq 0 \right\}.
\]
If $|x_1 - \bar{x}_1| \leq \delta$, we have from (2.27) that for all $\omega \in \Omega_5$

$$\int_0^{t \wedge \bar{\tau}_5} |v(s)|^2 ds \leq \frac{4C\lambda_0 \delta^2}{\lambda_0 \epsilon} \leq \frac{2\delta}{\epsilon} \forall t \geq 0.$$  

By the exponential martingale inequality [30, Theorem 7.4, page 44], we have $P_{x_1, \tilde{z}}(\Omega_6) \geq 1 - e^{\epsilon \delta} = 1 - \epsilon$ because $\delta \leq \frac{\epsilon^3}{m_\epsilon}$, where

$$\Omega_6 := \left\{ - \int_0^t v(s)dW_1(s) \leq \frac{\epsilon^2}{2\delta} \int_0^t |v(s)|^2 ds + \epsilon, t \leq \bar{\tau}_\delta \right\}.$$  

Let $\hat{Z}(t) = ([\hat{X}_1(t)]^\top, [\hat{X}_2(t)]^\top)^\top$ be solution of following coupled system

$$\begin{cases}
  d\hat{X}_1(t) = b_1(\hat{X}_1(t), 0)dt + \sigma_1(\hat{X}_1(t), 0)dW_1(t) + \int_{\mathbb{R}_+^1} \gamma_1(\hat{X}_1(t), 0, \phi)\hat{N}_1(dt, d\phi), \\
  d\hat{X}_1(t) = b_1(\hat{Z}(t))dt + 1_{\{t \leq \tilde{\tau}_\delta\}} \lambda(\hat{X}_1(t) - \hat{X}_1(t))dt + \sigma_1(\hat{Z}(t))dW_1(t) + \int_{\mathbb{R}_+^1} \gamma_1(\hat{Z}(t), \phi)\hat{N}_1(dt, d\phi), \\
  d\hat{X}_2(t) = b_2(\hat{Z}(t))dt + \sigma_2(\hat{Z}(t))dW_2(t) + \int_{\mathbb{R}_+^2} \gamma_2(\hat{Z}(t), \phi)\hat{N}_2(dt, d\phi), \\
  X_1(0) = x_1, \hat{X}_1(0) = \bar{x}_1, \hat{X}_2(0) = \bar{x}_2,
\end{cases}$$  

(2.31)

The Cameron-Martin-Girsanov theorem implies that there exist a measure $Q_{x_1, \tilde{z}}$ such that under $Q_{x_1, \tilde{z}}$, $W_1(t) + \int_0^{t \wedge \bar{\tau}_5} v(s)ds$ is a Wiener process so that $\hat{Z}(t) = ([\hat{X}_1(t)]^\top, [\hat{X}_2(t)]^\top)^\top$ under $Q_{x_1, \tilde{z}}$ is the solution to (1.1) with initial value $\tilde{z}$. Because $\hat{Z}(t) = \tilde{Z}(t)$ for all $t$ if $\bar{\tau}_\delta = \infty$,

$$\lim_{t \to \infty} \frac{U(\hat{X}_2(t))}{t} = \lim_{t \to \infty} \frac{U(\hat{X}_2(t))}{t} \geq m_0 \gamma_0 > 0$$

in the set $\cap_{i=1}^4 \Omega_i$.

On the other hand, if $|x_1 - \bar{x}_1| + |\bar{x}_2| \leq \theta_{x_1, \epsilon}$, for $\omega \in \Omega_5 \cap \Omega_6$, we have

$$\frac{dQ_{x_1, \tilde{z}}}{dp_{x_1, \tilde{z}}} = \exp \left\{ \int_0^{\bar{\tau}_5} v(s)dW_1(s) - \int_0^{\bar{\tau}_5} |v(s)|^2 ds \right\} \geq e^{-\epsilon - \epsilon - \epsilon} \geq 1 - 4\epsilon.$$  

Since

$$P_{x_1, \tilde{z}}(\cap_{i=1}^6 \Omega_i) \geq 1 - 6\epsilon,$$

we have

$$Q_{x_1, \tilde{z}}(\cap_{i=1}^6 \Omega_i) \geq (1 - 6\epsilon)(1 - 4\epsilon) = 1 - 24\epsilon,$$

which implies

$$Q_{x_1, \tilde{z}} \left\{ \lim_{t \to \infty} \frac{U(\hat{X}_2(t))}{t} \geq m_0 \gamma_0 \right\} \geq 1 - 24\epsilon.$$  

Therefore, we obtain that if $|x_1 - \bar{x}_1| + |\bar{x}_2| \leq \theta_{x_1, \epsilon}$ then

$$P_{x_1, \tilde{z}} \left\{ \lim_{t \to \infty} \frac{U(\hat{X}_2(t))}{t} \geq m_0 \gamma_0 \right\} \geq 1 - 24\epsilon.$$  

Hence, scaling $\epsilon$ by $\frac{\epsilon}{24}$, we obtain (2.28), which together with (2.4) implies (2.9). The proof is complete.

\qed
3 Stability of Linearizable Systems and Fast-Slow Systems

In this section, we consider the system of equations

\[
\begin{cases}
    dY_1(t) = b_1(Y_1(t), Y_2(t))dt + \sigma_1(Y_1(t), Y_2(t))dW_1(t) + \int_{\mathbb{R}_+} \gamma_1(Y_1(t^-), Y_2(t^-), \phi)\tilde{N}_1(dt, d\phi), \\
    dY_2(t) = b_2(Y_1(t), Y_2(t))dt + \sigma_2(Y_1(t), Y_2(t))dW_2(t) + \int_{\mathbb{R}_+} \gamma_2(Y_1(t^-), Y_2(t^-), \phi)\tilde{N}_2(dt, d\phi).
\end{cases}
\]

(3.1)

Under the condition that the second equation can be linearized (see Assumption 3.1 below), we examine the stability of $Y_2(\cdot)$. Then we consider the case when the two components have different time scales.

3.1 Stability of Linearizable Systems

Assumption 3.1. Assume that the assumptions in Theorem 2.1 for $b_1, \sigma_1, \gamma_1$ still hold and that $b_2, \sigma_2, \gamma_2$ are linearizable in $y_2$. That is, there exist matrices $B_2(y_1), \Sigma_{21}(y_1), \ldots, \Sigma_{2d_2}(y_1), \Gamma_2(y_1)$ bounded in $y_1$ such that $\Sigma_{2i}(y_1)$ has bounded right inverse and

\[
\begin{align*}
    \|b_2(z) - B_2(y_1)z_2\| &\leq o(y_2)V(y_1), \\
    \|\sigma_2(z) - [\Sigma_{21}(y_1)z_2, \ldots, \Sigma_{2d_2}(y_1)z_2]\| &\leq o(y_2)\sqrt{V(y_1)}, \\
    \gamma_2(z, \phi) &\leq \Gamma_2(y_1, \phi)z_2 + o(y_2)V(y_1), \\
    \|b_2(z)\| + \|\sigma_2(z)\sigma_2^\top(z)\| &\leq KV(y_1),
\end{align*}
\]

where $o(y_2)$ is a matrix or vector depending on $z$ satisfying $\lim_{y_2 \to 0} \frac{\sup_{y_2 \in \mathbb{R}^{d_2}} \{o(y_2)\}}{|y_2|} = 0$.

Let $\Theta(t) = \frac{Y_2(t)}{|Y_2(t)|^2}$, $R(t) = |Y_2(t)|^2$, by the generalized Itô formula for jump diffusions we have the following equations for $Y_1(t), \Theta(t), R(t)$

\[
\begin{cases}
    dY_1(t) = b_1(Z(t))dt + \sigma_1(Z(t))dW_1(t) + \int_{\mathbb{R}_+} \gamma_1(Z(t^-), \phi)\tilde{N}_1(dt, d\phi), \\
    d\Theta(t) = g_1(Y_1(t), \Theta(t), R(t))dt + g_2(Y_1(t), \Theta(t), R(t))dW_2(t) + \int_{\mathbb{R}_+} g_3(Y_1(t), \Theta(t), R(t), \phi)\tilde{N}_2(dt, d\phi), \\
    dR(t) = h_1(Y_1(t), \Theta(t), R(t))dt + h_2(Y_1(t), \Theta(t), R(t))dW_2(t) + \int_{\mathbb{R}_+} h_3(Y_1(t), \Theta(t), R(t), \phi)\tilde{N}_2(dt, d\phi),
\end{cases}
\]

(3.2)

where $Z(t) = ([Y_1(t)]^\top, [Y_2(t)]^\top)^\top = ([Y_1(t)]^\top, \sqrt{R(t)}[\Theta(t)]^\top)^\top$. In (3.2), $g_i$ and $h_i$ with $i = 1, 2, 3$...
are given as follows. If we denote \( z = (y_1^T, y_2^T)^T = (y_1^T, \sqrt{r} \theta^T)^T \) then

\[
g_1(y_1, \theta, r) = \frac{b_2(z)}{|y_2|} - \sigma_2(z) \sigma_2^T(z) y_2 + \left( -y_2^T b_2(z) - \frac{1}{2} \text{tr}(\sigma_2^T(z) \sigma_2(z)) + \frac{3|y_2^T \sigma_2(z)|^2}{2|y_2|^2} \right) \frac{y_2}{|y_2|^3} + \int_{\mathbb{R}_{+}^2} \left( \frac{y_2 + \gamma_2(z, \phi)}{|y_2 + \gamma_2(z, \phi)|} - \frac{|y_2|^2 \gamma_2(z, \phi) - (y_2^T \gamma_2(z, \phi)) y_2}{|y_2|^3} \right) \nu_2(d\phi),
\]

\[
= B_2(y_1) \theta - \sum_{l=1}^{d_2} \left[ \theta^T \Sigma_2(y_1) \right] \left[ \Sigma_2(y_1) \right]^{-1} \theta + \left( -\theta^T B_2(y_1) \theta + \frac{1}{2} \sum_{l=1}^{d_2} \left[ \left| \Sigma_2(y_1) \right| + 3 \left| \theta^T \Sigma_2(y_1) \theta \right|^2 \right] \right) \theta + \int_{\mathbb{R}_{+}^2} \left( \frac{\theta + \Gamma_2(y_1, \phi)}{|\theta + \Gamma_2(y_1, \phi)|} - \theta - \left| \theta \right|^2 \gamma_2(y_1, \phi) + \left( \theta^T \gamma_2(y_1, \phi) \right) \nu_2(d\phi) + o(1) \sqrt{V(y_1)},
\]

\[
(3.3)
\]

\[
g_2(y_1, \theta, r) = \frac{\sigma_2(z)}{|y_2|} - \frac{y_2 y_2^T \sigma_2(z)}{|y_2|^3} + \left( (\Sigma_2(y_1) \theta, \ldots, \Sigma_2(y_1) \theta) - \theta^T (\Sigma_1(y_1) \theta, \ldots, \Sigma_2(y_1) \theta), \right),
\]

\[
g_3(y_1, \theta, r, \phi) = \frac{\sigma_2(z)}{|y_2|} - \frac{y_2 y_2^T \sigma_2(z)}{|y_2|^3} = \frac{\theta + \Gamma_2(y_1, \phi) \theta}{|\theta + \Gamma_2(y_1, \phi)|} - \theta,
\]

\[
(3.4)
\]

\[
h_1(y_1, \theta, r) = 2 y_2 b_2(z) + \text{tr}(\sigma_2^T(z) \sigma_2(z)) + \int_{\mathbb{R}_{+}^2} |\gamma_2(z, \phi)|^2 \nu_2(d\phi)
\]

\[
= r \left( 2 \theta^T B_2(y_1) \theta + \sum_{l=1}^{d_2} \left| \Sigma_2(y_1) \theta \right|^2 + \int_{\mathbb{R}_{+}^2} |\Gamma_2(y_1, \phi) \theta|^2 \nu_2(d\phi) \right) + o(r) V_1(y_1),
\]

\[
h_2(y_1, \theta, r) = 2 y_2^T \sigma_2(z) = 2 r \sum_{l=1}^{d_2} \theta^T \Sigma_2(y_1) \theta + o(r) \sqrt{V_1(y_1)},
\]

\[
h_3(y_1, \theta, r, \phi) = |y_2 + \gamma_2(z, \phi)|^2 - |y_2|^2 = r (|\theta + \Gamma_2(y_1, \phi) \theta|^2 - 1).
\]

(3.5)

Let \( X_1(t) = ([Y_1(t)]^T, [\Theta(t)]^T)^T \in \mathbb{R}^{d_1} \times S_d \) and \( X_2(t) = R(t) \in \mathbb{R}_{+} \). We have that

\[
\ln R(t) = h_4(Y_1(t), \Theta(t), R(t)) dt + h_5(Y_1(t), \Theta(t), R(t)) dW_2(t)
\]

\[
+ \int_{\mathbb{R}_{+}^2} h_6(Y_1(t), \Theta(t), R(t), \phi) \tilde{N}_2(dt, d\phi),
\]

where

\[
h_4(y_1, \theta, r) = 2 \theta^T B_2(y_1) \theta + \sum_{l=1}^{d_2} \left| \Sigma_2(y_1) \theta \right|^2 - 2 \theta^T \Sigma_2(y_1) \theta
\]

\[
+ \int_{\mathbb{R}_{+}^2} \left( \ln |\theta + \Gamma_2(y_1, \phi) \theta|^2 - |\theta + \Gamma_2(y_1, \phi) \theta|^2 + 1 \right) \nu_2(d\phi) + o(1) V_1(y_1),
\]

\[
h_5(y_1, \theta, r) = \frac{2 y_2^T \sigma_2(z)}{|y_2|^2} = 2 \sum_{l=1}^{d_2} \theta^T \Sigma_2(y_1) \theta + o(1) \sqrt{V(y_1)},
\]

\[
h_6(y_1, \theta, r, \phi) = \ln \frac{|y_2 + \gamma_2(z, \phi)|^2}{|y_2|^2} = \ln |\theta + \Gamma_2(y_1, \phi) \theta|^2 + o(1) \sqrt{V(y_1)}.
\]
When $r = 0$, the equation for $X_1(t) := ([Y_1(t)]^T, [\Theta(t)]^T)^T$ is

\[
\begin{aligned}
&dY_1(t) = b_1(Y_1(t), 0)dt + \sigma_1(Y_1(t), 0)dW_1(t) + \int_{\mathbb{R}^{n_1}} \gamma_1(Y_1(t^-), 0, \phi)\tilde{N}_1(dt, d\phi), \\
&d\Theta(t) = g_1(Y_1(t), \Theta(t), 0)dt + g_2(Y_1(t), \Theta(t), 0)dW_2(t) + \int_{\mathbb{R}^{n_2}} g_3(Y_1(t), \Theta(t), 0, \phi)\tilde{N}_2(dt, d\phi).
\end{aligned}
\]

Assume that $\nu_1$ and $\nu_2$ are finite measures. Then the system has a unique invariant measure on $\mathbb{R}^{d_1} \times \mathcal{S}_d$ denoted by $\Pi$, due to the non-degeneracy of the diffusion, which follows from the explicit formula for $g_i$ given above, and the boundedness of $\Theta(t)$ and the Assumption 2.2(iv).

Now, let $U(r) = (\ln |r|) \vee 0$. Then we can apply Theorem 2.1 to show that if

\[
\begin{aligned}
&\int_{\mathbb{R}^{d_1} \times \mathcal{S}_d} (2\Theta^T B_2(y_1) \theta + \sum_{i=1}^d (|\Sigma_{2i}(y_1)\theta|^2 - 2|\Theta^T \Sigma_{2i}(y_1)\theta|^2) \\
&\quad + \ln |\theta + \Gamma_2(y_1, \phi)\theta| - |\theta + \Gamma_2(y_1, \phi)\theta|^2 + 1) \Pi(dy_1, d\theta) < 0,
\end{aligned}
\]

then $R(t)$ converges to 0 in probability. Note that the component $\Theta$ lives in a compact manifold $\mathcal{S}_d$ but Theorem 2.1 can be applied here because the coupling method in Lemma 2.3 and Theorem 2.1 can be done through the $(d-1)$-dimensional Euclidean coordinates of $\mathcal{S}_d$ and the right inverse of the diffusion coefficient for $(Y_1(t), \Theta(t))$ is bounded.

### 3.2 Stability of the Slow Component in a Fast-Slow System

We study the fast-slow coupled jump diffusions as follows

\[
\begin{aligned}
&dY_1^\varepsilon(t) = \frac{1}{\varepsilon} b_1(Y_1^\varepsilon(t), Y_2^\varepsilon(t))dt + \frac{1}{\sqrt{\varepsilon}} \sigma_1(Y_1^\varepsilon(t), Y_2^\varepsilon(t))dW_1(t) + \int_{\mathbb{R}^{n_1}} \gamma_1(Y_1^\varepsilon(t^-), Y_2^\varepsilon(t^-), \phi)\tilde{N}_1(dt, d\phi), \\
&dY_2^\varepsilon(t) = b_2(Y_1^\varepsilon(t), Y_2^\varepsilon(t))dt + \sigma_2(Y_2^\varepsilon(t))dW_2(t) + \int_{\mathbb{R}^{n_2}} \gamma_2(Y_2^\varepsilon(t^-), \phi)\tilde{N}_2(dt, d\phi),
\end{aligned}
\]

where $\tilde{N}_i(dt, d\phi) = N_1(dt, d\phi) - \frac{1}{\varepsilon} \nu_1(dt, d\phi)$, and $\sigma_2$ and $\gamma_2$ are assumed to be functions of $Y_2^\varepsilon$ only. We will study the stability of the slow component for small $\varepsilon$ based on the stability of the limit system. This problem is very important in applications. Typically, the limit system is often much easier to analyze and compute.

We assume that Assumption 3.1 holds with $\Sigma_{21}, \ldots, \Sigma_{2d_2}$ and $\Gamma_2(\phi)$ are now independent of $y_1$. Let $g_i, h_i$ be functions defined as in Section 3. Using the change of variable as in Section 3 by the generalized Itō formula for jump diffusions, we have the following equations for $Y_1(t)$, $\Theta^\varepsilon(t) = \frac{Y_1^\varepsilon(t)}{|Y_2^\varepsilon(t)|}$, $R^\varepsilon(t) = |Y_2^\varepsilon(t)|$

\[
\begin{aligned}
&dY_1^\varepsilon(t) = \frac{1}{\varepsilon} b_1(Z^\varepsilon(t))dt + \frac{1}{\sqrt{\varepsilon}} \sigma_1(Z^\varepsilon(t))dW_1(t) + \int_{\mathbb{R}^{n_1}} \gamma_1(Z^\varepsilon(t^-), \phi)\tilde{N}_1(dt, d\phi), \\
&d\Theta^\varepsilon(t) = g_1(Y_1^\varepsilon(t), \Theta^\varepsilon(t), R^\varepsilon(t))dt + g_2(\Theta^\varepsilon(t), R^\varepsilon(t))dW_2(t) + \int_{\mathbb{R}^{n_2}} g_3(\Theta^\varepsilon(t), R^\varepsilon(t), \phi)\tilde{N}_2(dt, d\phi), \\
&dR^\varepsilon(t) = h_1(Y_1^\varepsilon(t), \Theta^\varepsilon(t), R^\varepsilon(t))dt + h_2(\Theta^\varepsilon(t), R^\varepsilon(t))dW_2(t) + \int_{\mathbb{R}^{n_2}} h_3(\Theta^\varepsilon(t), R^\varepsilon(t), \phi)\tilde{N}_2(dt, d\phi),
\end{aligned}
\]

where $Z^\varepsilon(t) = ([Y_1^\varepsilon(t)]^T, [Y_2^\varepsilon(t)]^T)^T = ([Y_1^\varepsilon(t)]^T, \sqrt{R^\varepsilon(t)}[\Theta^\varepsilon(t)]^T)^T$. 

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Let $\Pi^\varepsilon$ be the family of invariant measures of the system

\[
\begin{aligned}
dY^\varepsilon_1(t) &= \frac{1}{\varepsilon} b_1(Y^\varepsilon_1, 0)dt + \int_{\mathbb{R}^1} \sigma_1(Y^\varepsilon_1(t), 0) dW_1(t) + \int_{\mathbb{R}^1} \gamma_1(Y^\varepsilon_1(t), 0, \phi) \tilde{N}_1^\varepsilon(dt, d\phi), \\
d\Theta^\varepsilon(t) &= g_1(Y^\varepsilon_1(t), \Theta^\varepsilon(t), 0)dt + g_2(\Theta^\varepsilon(t), 0) dW_2(t) + \int_{\mathbb{C}^2} g_3(\Theta^\varepsilon(t), 0, \phi) \tilde{N}_2(dt, d\phi),
\end{aligned}
\]

(3.9)

By a standard averaging principle (see e.g., [40]), we can show that $\Pi^\varepsilon$ converges weakly to $\Pi_1 \times \Pi_2$ where $\Pi_1$ is the invariant measure of the system (due to the fast component of (3.9) is decoupled from the slow component),

\[
dY_1(t) = b_1(Y_1, 0)dt + \sigma_1(Y_1, 0) dW_1(t) + \int_{\mathbb{R}^1} \gamma_1(Y_1(t), 0, \phi) \tilde{N}_1(dt, d\phi),
\]

and $\Pi_2$ is the invariant measure of the averaged system

\[
d\Theta(t) = \bar{g}_1(\Theta(t))dt + g_2(\Theta(t), 0) dW_2(t) + \int_{\mathbb{C}^2} g_3(\Theta(t), 0, \phi) \tilde{N}_2(dt, d\phi),
\]

with

\[
\begin{aligned}
\bar{g}_1(\theta) &= \int_{\mathbb{R}^1} g_1(y_1, \theta, 0) \Pi_1(dy_1) \\
&= \left( -\theta^\top \overline{B}_2 \theta + \frac{1}{2} \sum_{l=1}^d \left[ -|\Sigma_{2l}\theta|^2 + 3\theta^\top \Sigma_{2l}\theta|2 \right] \right) \theta \\
&\quad + \int_{\mathbb{R}^2} \left( \frac{\theta + \Gamma_2(\phi)\theta}{|\theta + \Gamma_2(\phi)|} - \theta - |\theta|^2 \Gamma_2(\phi) + (\theta^\top \Gamma_2(\phi)\theta) \nu_2(d\phi),
\end{aligned}
\]

where

\[
\overline{B}_2 = \int_{\mathbb{R}^1} B_2(y_1) \Pi_1(dy_1).
\]

In view of Section 3, the condition for the stability of $Y^\varepsilon_2$ is

\[
\lambda_\varepsilon := \int_{\mathbb{R}^1 \times \mathbb{S}_d} \left( 2\theta^\top \overline{B}_2(y_1)\theta + \sum_{l=1}^d \left( |\Sigma_{2l}\theta|^2 - 2|\theta^\top \Sigma_{2l}\theta|2 \right) \\
+ \ln |\theta + \Gamma_2(\phi)\theta|^2 - |\theta + \Gamma_2(\phi)\theta|^2 + 1 \right) \Pi^\varepsilon(dy_1, d\theta) < 0.
\]

Since $\Pi^\varepsilon(dy_1, d\theta)$ converges weakly to $\Pi_1 \times \Pi_2$ as $\varepsilon \to \infty$, we have

\[
\lim_{\varepsilon \to 0} \lambda_\varepsilon = \lambda_* := \int_{\mathbb{R}^1 \times \mathbb{S}_d} \left( 2\theta^\top \overline{B}_2(y_1)\theta + \sum_{l=1}^d \left( |\Sigma_{2l}\theta|^2 - 2|\theta^\top \Sigma_{2l}\theta|2 \right) \\
+ \ln |\theta + \Gamma_2(\phi)\theta|^2 - |\theta + \Gamma_2(\phi)\theta|^2 + 1 \right) (\Pi_1 \times \Pi_2)(dy_1, d\theta).
\]

In view of [23], it is easy to check that $\lambda_* < 0$ is the necessary and sufficient condition for the following linear system to be exponentially stable

\[
\begin{aligned}
d\tilde{Y}_2(t) &= \overline{B}_2 \tilde{Y}_2(t)dt + \sum_{l=1}^d \Sigma_{2l} \tilde{Y}_2(t) dW_{2l}(t) + \int_{\mathbb{R}^1} \Gamma_2(\phi) \tilde{Y}_2(t) \tilde{N}_2(dt, d\phi),
\end{aligned}
\]

(3.10)
In fact,
\[
\lim_{t \to \infty} \frac{\ln |Y_2(t)|}{t} = \lambda_s < 0.
\]
Thus, if \(\lambda_s < 0\) then
\[
\lim_{y_2 \to 0} \mathbb{P}_{y_1, y_2} \left\{ \lim_{t \to \infty} \frac{\ln |Y_2(t)|}{t} < \frac{\lambda_s}{2} < 0 \right\} = 1,
\]
as \(\varepsilon\) is sufficiently small. So, we can obtain the condition for the stability of fast-slow coupled jump-diffusion systems.

**Remark 3.1.** Treating two time-scale systems, one often uses the so-called freezing component argument; see [23] and [25, pp. 88-90]. Here we use a somewhat different argument. Using polar decomposition (3.9), whether \(R^\varepsilon\) converges to 0 depends on the invariant probability measure \(\Pi^\varepsilon\) of \((Y_1^\varepsilon, \Theta^\varepsilon)\) for each \(\varepsilon\). Then the Lyapunov exponent that determines stability and that is given by \(\lambda^\varepsilon\), is computed based on \(\Pi^\varepsilon\). Finally, we show \(\lambda^\varepsilon\) converges to \(\lambda^*\), which is obtained based on the limit system.

### 4 Stabilization and Consensus Problems

#### 4.1 Stabilization

In this section, we consider the controlled jump-diffusion system given by the following equations,

\[
\begin{align*}
\begin{cases}
    dX_1(t) = & b_1(X_1(t), X_2(t))dt + \sigma_1(X_1(t), X_2(t))dW_1(t) + \int_{\mathbb{R}^n_+} \gamma_1(X_1(t-), X_2(t-), \phi)N_1(dt, d\phi), \\
    & +u(t)dt \\
    dX_2(t) = & b_2(X_1(t), X_2(t))dt + \sigma_2(X_1(t), X_2(t))dW_2(t) + \int_{\mathbb{R}^n_+} \gamma_2(X_1(t-), X_2(t-), \phi)N_2(dt, d\phi), \\
    X_1(0) = & x_1, \quad X_2(0) = x_2,
\end{cases}
\end{align*}
\]

(4.1)

where \(u(t)\) is a control. We want to construct a control so as to stabilize the process \(X_2\). However, we cannot act directly to \(X_2\) but only the interacting process \(X_1\) can be controlled. We will apply our result to show that under certain conditions, we can control the interacting process \(X_1\) to have the stability of the process \(X_2\). A system may or may not have an invariant probability measure. The weak stabilization essentially means that we construct a control so that the resulting system is weakly stable. That is, the resulting system has an invariance measure. The term weak stability was originated from the work of Wonham [39].

**Assumption 4.1.** There exists a function \(U : \mathbb{R}^{l_2} \mapsto [0, \infty)\) such that

\[
\lim_{x_2 \to 0} U(x_2) = \infty, \quad \sup_{|x_2| \geq \theta x_1, \varepsilon} U(x_2) < \infty, \quad U(x_2) - U(x_1) \leq c_0 \ln \frac{|x_1|}{|x_2|};
\]

and there are a constant \(\Delta_0\) and functions \(f_1, f_2 : \mathbb{R}^{l_1} \to \mathbb{R}\) so that

\[
|LU(x)| \leq f_1(x_1), \quad (U_{x_2} \sigma_2(x))^2 \leq f_2(x_1), \quad \forall |x_2| \leq \Delta_0.
\]

Moreover, we suppose that \(f_1\) and \(f_2\) are bounded above by \(K(1 + |x_1|^2)\) and \(f_1(0) < 0\) and \(\lim_{x_1 \to \infty} f_1(x_1) > 0\). Finally, suppose that there is a matrix \(Q\) such that

\[
b_1^\top(z)Q + \text{tr} \sigma_1^\top(z)Q\sigma_1(z) + \int_{\mathbb{R}^n_+} \gamma_1^\top(z, \phi)Q\gamma_1(z, \phi)\nu_1(d\phi) \leq c_1 + c_2|x_1|, \quad \text{if } |x_2| \leq \Delta_0.
\]
From the assumption on \( f_1 \), we can write \( f_1(x_1) \) as \( f_1(x_1) \leq -K_1 + K_2|x_1|^2 \) for some constants \( K_1, K_2 \). Consider the control \( u(t) = AX_1(t) \), where \( A \) is a matrix satisfying

\[
-\lambda_A := \max_{x_1 \in \mathbb{R}^n} \frac{x_1^T Q x_1}{|x_1|^2} < \frac{K_2 c_1 + K_1 c_2}{K_1}.
\]

Now, when \( X_2 = 0 \) the corresponding system for \( X_1 \) is

\[
dX_1(t) = \left( b_1(X_1(t), 0) + u(t) \right) dt + \sigma_1(X_1(t), 0) dW_1(t) + \int_{\mathbb{R}^n} \gamma_1(X_1(t), 0, \phi) \tilde{N}_1(dt, d\phi).
\]

We have for \( V(x_1) := x_1^T Q x_1 \) that

\[
\mathcal{L} V(x_1) = b_1^T(x, 0)Q + tr \sigma_1^T(x_1, 0)Q \sigma_1(x_1, 0) + \int_{\mathbb{R}^n} \gamma_1^T(x_1, 0, \phi)Q \gamma_1(x_1, 0, \phi) \nu_1(d\phi) + x_1^T Q A x_1 \\
\leq c_1 - (\lambda_A - c_2)|x_1|^2.
\]

As a result, when \( x_2 = 0 \), there exists a unique invariant measure \( \Pi_1 \) for \( X_1(t) \) and

\[
c_1 - (\lambda_A - c_2) \int_{\mathbb{R}^1} |x_1|^2 \Pi_1(dx_1) \geq 0.
\]

That yields

\[
\int_{\mathbb{R}^1} f_1(x_1) \Pi_1(dx_1) \leq -K_1 + K_2 \frac{c_1}{\lambda_A - c_2} < 0,
\]

which implies the stability of the controlled system by an application of our main result (Theorem 2.1).

### 4.2 Leader-Following Consensus Problems

In this section, we apply our results to the leader-following consensus problems. We consider a network with a leader and \( N \) identical followers. The dynamics of the leader is described by

\[
dx_0(t) = f(x_0(t))dt + dW(t),
\]

and the dynamics of the \( i \)-th follower is described by

\[
dx_i(t) = f(x_i(t))dt + Bu_i(t)dt + dW(t), \quad i = 1, \ldots, N,
\]

where \( x_i(t) \in \mathbb{R}^n, i = 0, \ldots, N \), \( f : \mathbb{R}^n \to \mathbb{R}^n \), \( W(t) \) is an \( n \)-dimensional Brownian motion, \( u = [u_1^T, \ldots, u_N^T] \) \( (u_i \in \mathbb{R}^n, i = 1, \ldots, N) \) is the control to be designed, \( B \in \mathbb{R}^{n \times n} \).

Now, we model the information flow structure among different agents as follows. Let \( G = \{\mathcal{V}, \mathcal{E}, \mathcal{A}\} \) be a connected directed graph, where \( \mathcal{V} = \{0, \ldots, N\} \) denotes the set of nodes with \( k \) representing the \( k \)-th agent, \( \mathcal{E} \) is the set of directed edges, \( \mathcal{A} = [a_{kl}] \in \mathbb{R}^{N \times N} \) is the adjacency matrix of \( G \) with \( a_{kl} = 1 \) or 0 indicating whether or not there is a directed information flow between agents \( l \) and \( k \). Also, denoted by \( N_k \) the set of node \( k \)'s neighbors, i.e., \( N_k := \{l \in \{1, \ldots, N\} : a_{kl} = 1\} \), and \( \deg_k := \sum_{l=1}^{N} a_{kl} \) the degree of \( k \)-th agent. The Laplacian matrix of \( G \) is defined as \( \mathcal{H} = \mathcal{D} - \mathcal{A} \), where \( \mathcal{D} = \text{diag}(\deg_1, \ldots, \deg_N) \).

Suppose that the leader may send information to the followers, but does not receive information from any one of the followers, and that \( G \) contains a spanning tree rooted at the leader; and that
the communication between followers are undirected. Then we obtain the Laplacian matrix \( H \) of \( G \) as follows

\[
H = \begin{bmatrix}
0 & 0^T_N \\
a_0 & H_N
\end{bmatrix},
\]

where \( a_0 = (a_{10}, \ldots, a_{N0})^T \), \( 0_N := (0, \ldots, 0)^T \in \mathbb{R}^N \) and

\[
\tilde{H} = \begin{bmatrix}
\sum_{j=0, j \neq 1}^{N} a_{1j} & -a_{12} & \cdots & -a_{1N} \\
-a_{21} & \sum_{j=0, j \neq 2}^{N} a_{2j} & \cdots & -a_{2N} \\
\vdots & \vdots & \ddots & \vdots \\
-a_{N1} & -a_{N2} & \cdots & \sum_{j=0, j \neq N}^{N} a_{Nj}
\end{bmatrix}.
\]

Since we assume that the communication between the followers is undirected in this paper (for simplicity of notation), \( \tilde{H} \) is symmetric and the eigenvalues of \( \tilde{H} \) are real numbers.

For the \( i \)-th follower, we consider the following leader-following consensus protocol

\[
u_i(t) = K \sum_{j=0, j \neq i}^{N} z_{ji}(t), \quad i = 1, \ldots, N,
\]

(4.4)

where the symmetric matrix \( K \in \mathbb{R}^{n \times n} \) is the control gain to be designed, and

\[
z_{ji}(t) = x_j(t) - x_i(t) + (x_i(t) - x_j(t))\xi_{ji}(t),
\]

is the measurement of the agent \( j \) from its neighbor agent \( i \), and \( \xi_{ji} \)'s are some random noises. Denoted by

\[U = \{u(t) = ([u_1(t)]^T, \ldots, [u_N(t)]^T)^T \mid u_i(t) \text{ is given by (4.4), } t \geq 0, \text{ and } i = 1, \ldots, N\},\]

the collection of all admissible distributed protocols. We refer the reader to [33, 42] and references therein for motivation of the above system.

**Assumption 4.2.** We assume the following:

(i) \( f(y) \) satisfies \( |f(y)| \leq -c|y| \), for all \( y \in \mathbb{R}^n \) for some constant \( c > 0 \).

(ii) The noise \( \xi_{ji}(t) \) satisfies that

\[
\int_0^t (x_j(s) - x_i(s))\xi_{ji}(s)ds = \int_0^t (x_j(s) - x_i(s))\left(\sigma_{ji}dw_{ij}(s) + \int_{\mathbb{R}^n_+} \gamma_{ji}(\phi)\tilde{N}_{ji}(ds, d\phi)\right),
\]

where \( w_{ij}(s) \) are independent standard Brownian motions, \( \tilde{N}_{ji}(s, \phi) \) are jump processes.

**Definition 4.1.** System (4.2) and (4.3) is said to be exponentially consentable in probability with respect to \( U \) if there exists a protocol \( u \in U \) so that for any \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that for all \( i = 1, \ldots, N \)

\[
P \{|x_i(t) - x_0(t)| \text{ converges exponentially fast to } 0 \} \geq 1 - \varepsilon,
\]

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whenever the initial values \((y_0, \ldots, y_n) \in \mathbb{R}^{nN}\) of (4.2) and (4.3) satisfying that
\[
\sum_{i=1}^{N} |y_i - y_0|^2 < \delta.
\]

To proceed, let \(X_i(t) = x_i(t) - x_0(t), i = 1, \ldots, N, x(t) := [x_1^T(t), \ldots, x_n^T(t)]^T, F_i(x_0(t), x_i(t)) := f(x_i(t)) - f(x_0(t)), \) and \(F(x_0(t), \ldots, x_N(t)) := [F_1^T(x_0(t), x_1(t)), \ldots, F_N^T(x_0(t), x_N(t))]^T, X(t) := [X_1^T(t), \ldots, X_N^T(t)]^T.\) For simplicity of notation, we will write \(F(x_0(t), \ldots, x_N(t))\) as \(F(x_0(t), X(t))\) by the identity \(F(x_0(t), X(t)) = F(x_0(t), x_0(t) + X_1(t), \ldots, x_0(t) + X_N(t))\). Then, one can obtain
\[
dx_0(t) = f(x_0(t))dt + dW(t),
\]
\[
dX(t) = \left(F(x_0(t), X(t)) - \mathcal{H} \otimes B K\delta(t)\right) dt + dM_1(t) + dM_0(t) + dM_0^N(t) + dM_1^N(t),
\]
where \(A \otimes B\) denotes the Kronecker product of \(A\) and \(B\), and
\[
M_1(t) = \sum_{i,j=1}^{N} \int_0^t \sigma_{ji} \left[S_{ij} \otimes B K\right] X(s) dw_{ji}(s),
\]
\[
M_0(t) = - \sum_{i=1}^{N} \int_0^t \sigma_{0i} \left[S_i \otimes B K\right] X(s) dw_{0i}(s),
\]
\[
M_1^N(t) = \sum_{i,j=1}^{N} \int_0^t \int_{\mathbb{R}_+^1} \left[S_{ij} \otimes B K\right] X(s) \gamma_{ji}(\phi) \bar{N}_{ji}(ds, d\phi),
\]
\[
M_0^N(t) = - \sum_{i=1}^{N} \int_0^t \int_{\mathbb{R}_+^1} \left[S_i \otimes B K\right] X(s) \gamma_{ji}(\phi) \bar{N}_{ji}(ds, d\phi),
\]
where \(S_{ij} = [s_{kl}]_{N \times N}\) is an \(N \times N\) matrix with \(s_{ii} = -a_{ii}\) and \(s_{ij} = a_{ij}\) and all other elements being 0, for \(i, j = 1, \ldots, N,\) and \(\bar{S}_i = [\bar{s}_{kl}]_{N \times N}\) is an \(N \times N\) matrix with \(\bar{s}_{ii} = a_{i0}\) and all other entries being 0. It is easily seen that the consensus problem of (4.2) and (4.3) is equivalent to the stability of (1.5).

**Assumption 4.3.** There exists a matrix \(K \in \mathbb{R}^{n \times n}\) such that there exists a function \(U : \mathbb{R}^{nN} \mapsto [0, \infty)\) satisfying the following conditions,
\[
\lim_{X \to 0} U(X) = \infty, \quad U(X) - U(X') \leq c_0 \ln \frac{|X'|}{|X|},
\]
and there is \(\Delta_0\) such that
\[
|LU(X)| \leq c_1(x_0), |X| \leq \Delta_0,
\]
\[
\sum_{i,j=1}^{N} \left(U_X^T(\sigma_{ji} [S_{ij} \otimes B K] X)\right)^2 + \sum_{i=1}^{N} \left(U_X^T(\sigma_{0i} [S_i \otimes B K] X)\right)^2 \leq c_2(x_0), |X| \leq \Delta_0,
\]
\[
\int_{\mathbb{R}_+^2} \left[\exp \left\{ - \alpha_0(U(X_2 + \gamma_2(X, \phi)) - U(X))_+ \right\} \nu_2(\phi) \right] \leq c_3(x_0), |X| \leq \Delta_0,
\]
and
\[
\int (c_1(x_0) + c_2(x_0) + c_3(x_0)) \mu^*(dx_0) > 0.
\]
In above, \(\mu^*(\cdot)\) is the invariant measure of (4.2). Such a \(\mu^*(\cdot)\) always exists because of Assumption 4.2.
Theorem 4.1. Under Assumptions \ref{assumption1} and \ref{assumption2}, system \ref{system} is exponentially stable in probability. As a consequence, the leader-following system \ref{leader-following-system} and \ref{follower-system} is exponentially consensual in probability.

In fact, our above results are verifiable. To illustrate that, we provide the following explicitly computational example.

Example 4.1. In this example, assume that $f(x) = Ax$ where $A \in \mathbb{R}^{n \times n}$. Assume that $B$ is invertible. Let $U(\delta) = -\ln|\delta|$, by directed calculations, we have

$$LU(X) = -\frac{X \left( A \otimes I_N - \tilde{H} \otimes BK \right) \delta^T}{|X|^2}.$$ 

Then, it is easy to check the remaining conditions.

5 Conclusion

We studied stability and stabilization of a fully coupled system of jump diffusions. Sufficient conditions for stability are derived. We then investigate the stability of linearizable jump diffusions and fast-slow coupled jump diffusions. Next, we develop strategies for weak stabilization of a coupled system in which only one component can be controlled. Also considered are consensus problem of leader-following systems. The consideration of this paper can be readily extended to systems with many components. There are many interesting important problems remain to be investigated. Future research could be extended for regime-switching with state-dependent diffusions or hidden Markov systems. Efforts can also be directed to studying systems with mean-fields interactions. These and other topics deserve to be carefully examined.

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