A BAYESIAN LEVEL SET METHOD FOR AN INVERSE MEDIUM SCATTERING PROBLEM IN ACOUSTICS

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Abstract. In this work, we are interested in the determination of the shape of the scatterer for the two dimensional time harmonic inverse medium scattering problems in acoustics. The scatterer is assumed to be a piecewise constant function with a known value inside inhomogeneities and its shape is represented by the level set functions for which we investigate the information using the Bayesian method. In the Bayesian framework, the solution of the geometric inverse problem is defined as a posterior probability distribution. The well-posedness of the posterior distribution is discussed and the Markov chain Monte Carlo (MCMC) method is applied to generate samples from the posterior distribution. Numerical experiments are presented to demonstrate the effectiveness of the proposed method.

1. Introduction. The inverse scattering problems have been extensively investigated because of their great importance and broad applications in radar and sonar, geophysical exploration, medical imaging and to name a few [11, 12]. One of the main goals of the inverse scattering problems is to determine the unknown scatterer, such as location, geometry, or material property etc. [25, 44, 51]. This kind of inverse scattering problems can be regarded as inverse medium scattering problems (IMSP). The IMSP are ill-posed and nonlinear admitting great theoretical and computational challenges, which have attracted attention of many researchers in the past decades.

Many numerical methods have been proposed to solve the IMSP [1, 3, 47]. Classical methods for the IMSP can be roughly classified into two categories: direct methods and indirect methods. The direct methods mainly recover the support or the shape of the scatterer, such as linear sampling methods [8, 9, 10], multiple signal classification methods [24, 26] and factorization methods [34, 35]. The indirect methods attempt to determine the unknown representation of the scatterer by applying regularization techniques, including recursive linearization methods [2, 4], level set methods [18, 21] and Gauss-Newton methods [36, 37]. Among these indirect methods, the level set method is a great tool for the computation of evolving boundaries and interfaces [17]. The level set method was originally designed to track propagating interfaces through topological changes [45] and more recently it has been expanded to the research of inverse problems involving obstacles [21, 30, 50].

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In addition to the deterministic methods, another kind of methods lies in the class of statistical methods and one of those is known as the Bayesian approach. The Bayesian method has attracted considerable attention in inverse problems due to its ability of characterizing uncertainty quantification [15, 33, 49]. Recently, it has been widely applied to solve the inverse scattering problems [5, 28, 31, 39, 40, 42]. In the Bayesian setting, the Gaussian measures are favorable options of the prior distributions, which play central roles in the theory of the Bayesian approach [49]. Samples from the Gaussian priors are generated by solving a related stochastic differential equation or using the Karhunen-Loève expansion based on the eigenfunctions and eigenvalues of covariance operators of the prior distributions [19, 29, 32, 38]. The solution to the Bayesian inversion, i.e. the original problem, is a posterior distribution. To explore the information of the posterior distribution, sampling methods such as the Markov chain Monte Carlo (MCMC) methods are usually employed [6, 7, 13, 22, 43, 46].

In this work, we are mainly interested in solving the IMSP by the Bayesian level set method, which is a coupling of the Bayesian method and the level set method. Assuming that the scatterer is a piecewise constant function with known values, we characterize the shape of the scatterer by the level set functions. There are few literatures on the numerical solution of the inverse scattering problems by using the Bayesian level set method. In [29], the authors have established the mathematical foundations of the Bayesian level set method and its hierarchical extension has been developed in [19]. In [16, 52], the Bayesian method and the ensemble Kalman filter approach based on level set parameterization are introduced for acoustic source identification using multi-frequency information, respectively. Actually, when the level set method is coupled with the Bayesian approach, there are several advantages for the shape reconstruction. First of all, the proposed method can be used to approximate the solution to the inverse problem without the implementation of the Fréchet derivative of the forward map as well as the corresponding adjoint operator. Secondly, the method do not lead to an additional computation involving the evolution of the level set functions governed by a Hamilton-Jacobi type equation. Finally, the Bayesian level set method not only can provide point estimates of the solution, such as the maximum a posterior (MAP) estimate and the conditional mean (CM) estimate, but also can deliver the important uncertainty information of the reconstruction results. In this paper, we consider the Whittle-Matérn Gaussian random fields as the prior[41, 48], with which the level sets of the Gaussian random fields have zero Lebesgue measure [29]. We will also discuss the well-posedness of the posterior distribution based on Bayes’ theorem. Applying the MCMC methods, we will show the numerical results via the CM estimates.

The rest of the paper is organized as follows. In Section 2, we simply describe the forward model and employ the Dirichlet-to-Neumann finite element method (DtN-FEM) as the forward solver [23, 27]. We discuss the Bayesian level set approach solving the IMSP with the proposed prior and the well-posedness theory of the posterior distribution in Section 3. In Section 4, the numerical results are presented to illustrate the effectiveness of the proposed method.

2. Direct scattering problem. In this section, we introduce the propagation of time harmonic acoustic waves in two dimensions. The scatterer formed by an inhomogeneous medium is embedded in an infinite homogeneous background medium.
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2.1. A model problem. The scattering problem under consideration is modeled by

\[
\begin{align*}
\Delta u + k^2(1 + q(x))u &= 0, \quad \text{in } \mathbb{R}^2, \\
\lim_{r \to \infty} r^{\frac{1}{2}}(\frac{\partial u^s}{\partial r} - iku^s) &= 0, \quad r = |x|,
\end{align*}
\]

where \( k > 0 \) denotes the wavenumber, \( u = u^r + u^i \) is the total field, \( u^i \) is the plane incident field and \( u^s \) is the scattered field which satisfies the Sommerfeld radiation condition (1b) uniformly in all directions. \( q(x) > -1 \) is assumed to be a piecewise constant function describing the scatterer. Moreover, we assume that the scatterer has a compact support contained in \( B_R := \{ x \in \mathbb{R}^2 : |x| < R \} \), which is bounded by an artificial boundary \( \Gamma_R := \{ x \in \mathbb{R}^2 : |x| = R \} \) with \( R \) being sufficiently large to enclose the scatterer inside (see Fig. 1). In particular, considering the plane incident wave \( u^i = e^{ikx \cdot d} \) with the incident direction \( d \in \{ x \in \mathbb{R}^2 : |x| = 1 \} \), we can rewrite the equation (1a) as

\[
\Delta u^s + k^2(1 + q(x))u^s = -k^2 q(x)u^i \quad \text{in } \mathbb{R}^2.
\]

2.2. Equivalent formulation. In the following, let \( L^2(B_R) \) be the usual Hilbert space of all square integrable functions and let \( H^1(B_R) \) be the Sobolev space equipped with the norm

\[
\|u\|_{H^1(B_R)} = (\|u\|^2_{L^2(B_R)} + \|\nabla u\|^2_{L^2(B_R)})^{\frac{1}{2}}.
\]

Define the trace space \( H^s(\Gamma_R), s \in \mathbb{R}, \) as

\[
H^s(\Gamma_R) = \{ u \in L^2(\Gamma_R) \mid \|u\|_{H^s(\Gamma_R)} < \infty \}
\]
equipped with the norm

\[
\|u\|_{H^s(\Gamma_R)} = (\frac{|a_0|^2}{2} + \sum_{n \in \mathbb{Z}} (1 + n^2)^s(|a_n|^2 + |b_n|^2))^{\frac{1}{2}},
\]

where \( a_n \) and \( b_n \) are Fourier coefficients of \( u \in H^s(\Gamma_R) \).

In the domain \( \mathbb{R}^2 \setminus B_R \), the solution of equation (2) has the form in the polar coordinates as follows [12]:

\[
u^s(r, \theta) = \sum_{n \in \mathbb{Z}} \frac{H_n^{(1)}(kr)}{H_n^{(1)}(kR)} \hat{u}^s_ne^{in\theta},
\]
where
\[ \hat{u}_n = (2\pi)^{-1} \int_0^{2\pi} u^s(R, \theta)e^{-in\theta} d\theta, \]
and \( H_n^{(1)} \) is the Hankel function of the first kind with order \( n \).

On the artificial boundary \( \Gamma_R \), we can define the Dirichlet-to-Neumann (DtN) operator \( T : H^{1/2}(\Gamma_R) \to H^{-1/2}(\Gamma_R) \) as follows: for any \( u^s \in H^{1/2}(\Gamma_R) \),
\[ \frac{\partial u^s}{\partial \nu} \bigg|_{\Gamma_R} = Tu^s := k \sum_{n \in \mathbb{Z}} \frac{H_n^{(1)'}(kR)}{H_n^{(1)}(kR)} \hat{u}_n e^{in\theta}, \]
where \( \nu \) is the unit outward normal to \( \Gamma_R \). Alternatively, the DtN operator \( T \) can be expressed as
\[ \frac{\partial u^s}{\partial \nu} \bigg|_{\Gamma_R} = Tu^s := \sum_{n=0}^{\infty} \frac{kH_n^{(1)'}(kR)}{2\pi H_n^{(1)}(kR)} \int_0^{2\pi} u^s(R, \varphi) \cos(n(\theta - \varphi)) d\varphi. \]

The original scattering problem (1a)-(1b) defined on \( \mathbb{R}^2 \) can be equivalently reduced to the following problem defined on the bounded domain \([1]\),
\[ \begin{align*}
\Delta u^s + k^2 (1 + q(x)) u^s &= -k^2 q(x)u^i \quad \text{in} \ B_R, \\
\frac{\partial u^s}{\partial \nu} &= Tu^s \quad \text{on} \ \Gamma_R.
\end{align*} \]
Then, we have the weak formulation of the boundary value problem (6): find \( u^s \in H^1(B_R) \) such that
\[ A(u^s, v) = \ell(v) \quad \forall v \in H^1(B_R), \]
where the bilinear form \( A(\cdot, \cdot) : H^1(B_R) \times H^1(B_R) \to \mathbb{C} \) is defined by
\[ A(u^s, v) = \int_{B_R} \nabla u^s \cdot \nabla \overline{v} dx - k^2 \int_{B_R} (1 + q(x))u^s \overline{v} dx - \int_{\Gamma_R} Tu^s \overline{v} dS, \]
and the linear functional \( \ell(\cdot) : H^1(B_R) \to \mathbb{C} \) is defined by
\[ \ell(v) = k^2 \int_{B_R} q(x)u^i \overline{v} dx. \]

Finally, we point out that given the incident field \( u^i \) and the scatterer \( q(x) \), the direct scattering problem is to determine the scattered field \( u^s \).

3. Bayesian level set inversion. In this section, we combine Bayesian inference with the level set method to reformulate the inverse medium scattering problem as a problem in statistical inference.

3.1. The inverse problem.

Definition 3.1. The scatterer \( q(x) \in L^\infty(B_R) \) is said to be admissible if there exists a compact domain \( D \subset \subset B_R \) such that
\[ q(x) = \begin{cases} 
\ b, & \text{for } x \in D, \\
\ 0, & \text{for } x \in \mathbb{R}^2 \setminus D,
\end{cases} \]
where \( b > 0 \) is a constant. The set of all such scatterers will be denoted by \( \mathcal{A}(B_R) \).
Given the incident field $u^i$ and measured scattered field $u^s$ on $\Gamma$, the inverse problem considered here is to determine the shape of the scatterer. For the problem (6), we assume that $M$ different wavenumbers $k := k_m, m = 1, \ldots, M$, are given. For each of these wavenumbers, there correspond to the incident waves $u^i_mj = e^{ik_mx \cdot dj}, j = 1, \ldots, J$. Thus, for a given wavenumber $k_m$ and a given incident wave $u^i_mj$, we define the forward operator $G_mj : X \rightarrow Y$ by $u^s_mj(x) = G_mj(q(x))$, where $q(x) \in A(B_R) =: X$, $u^s_mj(x) \in H^1(B_R) =: Y$. On the other hand, we denote the observation data with noise by

$$y_{mj} = O_{mj} \circ G_{mj}(q(x)) + \eta_{mj},$$

where $y_{mj} \in \mathbb{C}^N$, $\eta_{mj} \sim \mathcal{N}(0, \Sigma_0)$ is the additive Gaussian noise with covariance matrix $\Sigma_0 \in \mathbb{R}^{N \times N}$, and $O_{mj} : Y \rightarrow \mathbb{C}^N$ is the collection of $N$ continuous linear functions given by

$$O_{mj}(u^s_mj(\cdot)) := (\phi^1_m(u^s_mj(\cdot)), \ldots, \phi^n_m(u^s_mj(\cdot)))^T.$$

Here, $\phi^i_m : Y \rightarrow \mathbb{C}$ denotes the continuous linear functionals, $n = 1, \ldots, N$. Gathering all the observations, one can rewrite (8) as

$$y = O \circ G(q(x)) + \eta,$$

where $y := (y_{11}, \ldots, y_{MJ})^T \in \mathbb{C}^{MJN} =: \mathcal{Y}$. Replacing $y_{ij}$ by $O_{mj}$ (resp. $G_{ij}$ and $\eta_{ij}$), we likewise define $O$ (resp. $G$ and $\eta$). Here, $\eta \sim \mathcal{N}(0, \Sigma)$ with covariance matrix $\Sigma = \text{diag}(\Sigma_0, \ldots, \Sigma_0) \in \mathbb{R}^{MJN \times MJN}$.

### 3.2. Level set parameterization.

The scatterer $q(x)$ is characterized by

$$q(x) = \sum_{i=1}^L b_i I_{B_i}(x),$$

where $\{B_i\}_{i=1}^L$ are $L$ subdomains such that $B_i \cap B_j \neq \emptyset, \forall i \neq j$ and $\bigcup_{i=1}^L \overline{B_i} = \overline{B_R}$, $I$ denotes the indicator function of a set and the $\{b_i\}_{i=1}^L$ are known constants, $b_i \in \{0, 1\}$. In this setting, the unknown scatterer would be determined by the domains $B_i, i = 1, \ldots, L$. It is natural to make use of the level set representation of the domains through a continuous real-valued function $\phi : B_R \rightarrow \mathbb{R}$. To this purpose, we define $B_i \subseteq B_R$ by a real-valued continuous level set function $\phi$,

$$B_i = \{ x \in B_R | c_i-1 < \phi(x) < c_i \}, \quad i = 1, \ldots, L,$$

where $c_i$ are constants with $-\infty = c_0 < c_1 < \cdots < c_L = \infty, i \in \mathbb{N}$. We define the level sets as

$$B_i^0 = \overline{B_i} \cap \overline{B_{i+1}} = \{ x \in B_R | \phi(x) = c_i \}, \quad i = 1, \ldots, L-1.$$

It is evident that the same domain $B_i$ can be represented by different level set functions $\phi_1$ and $\phi_2$ and however, different domains can not be determined by the same level set representation. Therefore, we can use the level set representation to uniquely specify the domain $B_i$ by an associated level set function, $i = 1, \ldots, L$.

Let $\mathcal{X} = C(\overline{B_R}, \mathbb{R})$, $F : \mathcal{X} \rightarrow X$ is the level set map described by

$$(F\phi)(x) = q(x) = \sum_{i=1}^L b_i I_{B_i}(x).$$

Then we modify our operator $O \circ G$ into $\mathcal{G} = O \circ G \circ F : \mathcal{X} \rightarrow \mathcal{Y}$. As a result, the inverse problem can be reformulated as: for given $y$, find $\phi$ such that

$$y = \mathcal{G}(\phi) + \eta.$$
3.3. Bayesian inference. In the Bayesian framework, all quantities in (14) are viewed as random variables. Since it is assumed that \( \eta \in \mathbb{R}^{MN} \) is additive Gaussian, it is typically straightforward to write the likelihood function, i.e. the probability density of \( y \) given \( \phi \),

\[
\pi(y|\phi) \propto \exp\left(-\frac{1}{2}\|G(\phi) - y\|_\Sigma^2\right),
\]

where \( \|\cdot\|_\Sigma := \|\Sigma^{-\frac{1}{2}}\cdot\| \) denotes the weighted norm in terms of the Euclidean norm \( \|\cdot\| \).

In the following, we denote \( \frac{1}{2}\|G(\phi) - y\|_\Sigma^2 \) by the potential \( \Phi(\phi; y) \). We assume that the prior measure of the unknown \( \phi \) is \( \mu_0 \), the posterior measure \( \mu_y \) is represented as the Radon-Nikodym derivative with respect to the prior measure \( \mu_0 \):

\[
\frac{d\mu_y}{d\mu_0}(\phi) = \frac{1}{Z} \exp(-\Phi(\phi; y)),
\]

where \( Z = \int_X \pi(y|\phi)\mu_0(d\phi) \) is a normalization constant. The equation (16) can be viewed as the Bayes' rule in the infinite-dimensional setting.

3.3.1. Whittle-Matérn Gaussian random field prior. We now introduce Gaussian prior of Whittle-Matérn type with covariance \[48\]

\[
c(x, y) = \sigma^2 \frac{2^{2-\alpha}}{\Gamma(\alpha - 1)} \left(\frac{|x - y|}{l}\right)^{\alpha-1} K_{\alpha-1}\left(\frac{|x - y|}{l}\right), \quad x, y \in \mathbb{R}^2,
\]

where \( K_{\alpha-1} \) is the modified Bessel function of the second kind of order \( \alpha - 1 \), \( \sigma^2 > 0 \) is the variance, \( l > 0 \) is the characteristic length scale and \( \Gamma(\cdot) \) is the Gamma function. We generate the samples from the Whittle-Matérn prior by solving a stochastic partial differential equation

\[
(I - l^2\Delta)^{\frac{\alpha}{2}} \phi = l\sqrt{\zeta},
\]

where \( (I - l^2\Delta)^{\frac{\alpha}{2}} \) is a pseudo-differential operator defined by its Fourier transform, \( \zeta \) is a Gaussian white noise and \( \zeta = \sigma^2 \frac{4\pi(\alpha)}{\Gamma(\alpha - 1)} \) is a constant. Set \( \tau = l^{-1} > 0 \), we have the stochastic partial differential equation

\[
C_{\alpha, \tau}^{\frac{1}{2}} \phi = \xi,
\]

where \( C_{\alpha, \tau} = \zeta \tau^{2\alpha-2}(\tau^2 I - \Delta)^{-\alpha} \) denotes the covariance operator of prior distribution \( \mu_0 \), \( \alpha \) controls the regularity of the samples and \( \tau \) represents the inverse length scale of the samples. In what follows, assume that \( \mathcal{A} := \Delta \) is Laplacian with Neumann boundary conditions on \( B_R \) and its domain is given by

\[
B_R(A) := \{ \phi : B_R \to \mathbb{R} | \phi \in H^2(B_R; \mathbb{R}), \frac{\partial \phi}{\partial \nu} = 0 \text{ on } \partial B_R \}.
\]

In Fig. 2 and Fig. 3, we display random samples obtained from (19) regarding to different values of the inverse length scale \( \tau \) and the regularity \( \alpha \). These samples are constructed in the domain \( B_R \) with \( R = 1 \).

3.3.2. Well-posedness of the posterior distribution. We now discuss the well-posedness of the posterior distribution for the IMSP. It is clear that the level set map is discontinuous and due to this fact, we get that the map \( \mathcal{G} \) is discontinuous. Although the well-posedness of Bayesian inversion relies on the continuity of the map \( \mathcal{G} \), it demonstrates ([29]) that the discontinuity set is a probability zero event under the Gaussian prior. As a result, \( F \) will be almost surely continuous under the prior and it will be given in the following Lemma 3.2. Thus, we are able to get the measurability required in the Bayes' theorem [49]. Furthermore, we need to verify
some regularity properties of the potential $\Phi(\phi; y)$ which satisfy the assumptions of the Bayes’ theorem. Prior to the presentation, we define a complete probability space $(\mathcal{X}, \Xi, \mu_0)$ for the unknown $\phi$, where $\mathcal{X}$ denotes a separable Banach space and $\Xi$ is the $\sigma$-algebra.

**Lemma 3.2.** Define the map $F : \mathcal{X} \rightarrow \mathcal{X}$ given by (13). Let $\phi \in \mathcal{X}$ be such that $m(B^0_i) = 0$, for $i = 1, \cdots, L - 1$. Assume that $\{\phi_{\epsilon}\}_{\epsilon > 0} \subseteq C(B_R)$ denotes an approximate sequence of level set functions such that $\|\phi_{\epsilon} - \phi\|_{\infty} \rightarrow 0$. Then $F(\phi_{\epsilon}) \rightarrow F(\phi)$ in measure.

**Remark 1.** Here $m(B^0_i)$ denotes the Lebesgue measure of the set $B^0_i$. The proof of this lemma is almost identical to that of Proposition 3.5 in [20] and we omit the details here.

**Proposition 1.** The potential $\Phi(\phi; y)$ and probability measure $\mu_0$ on the measure space $(\mathcal{X}, \Xi)$ satisfy the following properties:

1. for every $r > 0$, there is a $K_1 = K_1(r)$ such that, for all $\phi \in \mathcal{X}$ and $y \in \mathcal{Y}$ with $|y|_{\Sigma} < r$,
   \begin{equation}
   0 \leq \Phi(\phi; y) \leq K_1;
   \end{equation}

2. for any fixed $y \in \mathcal{Y}$, $\Phi(\cdot; y) : \mathcal{X} \rightarrow \mathbb{R}$, is continuous $\mu_0$-almost surely on the probability space $(\mathcal{X}, \Xi, \mu_0)$;

3. for every $r > 0$, there exists a $K_2 = K_2(r)$ such that, for all $\phi \in \mathcal{X}$ and $y_1, y_2 \in \mathcal{Y}$ with $\max\{|y_1|_{\Sigma}, |y_2|_{\Sigma}\} < r$,
   \begin{equation}
   |\Phi(\phi; y_1) - \Phi(\phi; y_2)| \leq K_2|y_1 - y_2|_{\Sigma}.
   \end{equation}

**Proof.** (1) From the problem (6), it can be observed that the map $G$ is nonlinear with respect to $q$. We know that the following estimate holds [1]

\begin{equation}
\|u^s\|_{H^1(B_R)} = \|G(q)\|_{H^1(B_R)} \leq C\|q\|_{L^\infty(B_R)}\|u^d\|_{L^2(B_R)}.
\end{equation}
\[ O : Y \to \mathcal{Y} \] is the bounded linear map and \( \| F(\phi) \|_\infty \) is bounded uniformly over \( \phi \in \mathcal{X} \). Hence
\[
|G(\phi)|_\Sigma = |O \circ G \circ F(\phi)|_\Sigma \leq C.
\]
Note that
\[
\Phi(\phi; y) = \frac{1}{2}|y - G(\phi)|^2 \leq |y|^2 + |G(\phi)|^2.
\]
Then, for any \( y \in \mathcal{Y} \) with \( |y|_\Sigma < r \), we obtain the bound
\[
\Phi(\phi; y) \leq C(1 + r^2) =: K_1.
\]
(2) It is known that the map \( G : X \to \mathcal{Y} \) is continuous \([1]\), i.e.
\[
\| G(q_1) - G(q_2) \|_{H^1(B_R)} \leq C \| q_1 - q_2 \|_{L^\infty(B_R)} \| u^1 \|_{L^2(B_R)},
\]
and \( O : Y \to \mathcal{Y} \) is the linear continuous map. Therefore, the discontinuity set of \( G \) is determined by the discontinuity set of the level set map \( F \). However, since we assume that \( \phi \sim \mu_0 \) is a Gaussian measure, it follows from the Proposition 2.8 in \([29]\) that \( m(B_0) = 0 \), \( \mu_0 \)-almost surely, \( i = 1, \cdots, L - 1 \). In brief, the level sets of the Gaussian random field have zero Lebesgue measure. By Lemma 3.2, we can obtain that \( \| \phi_x - \phi \|_\infty \to 0 \) implies that \( F(\phi_x) \to F(\phi) \) in measure. Therefore, \( \Phi(\cdot; y) \) is continuous \( \mu_0 \)-almost surely.

(3) Let \( \phi \in \mathcal{X} \) and \( y_1, y_2 \in \mathcal{Y} \) with \( \max\{|y_1|_\Sigma, |y_2|_\Sigma\} < r \). It follows that
\[
| \Phi(\phi; y_1) - \Phi(\phi; y_2) | = \frac{1}{2} |y_1 - G(\phi)|^2 - \frac{1}{2} |y_2 - G(\phi)|^2
\]
\[
= \frac{1}{2} |(y_1 - y_2, y_1 + y_2 - 2G(\phi))|_\Sigma
\]
\[
\leq \left( |y_1|_\Sigma + |y_2|_\Sigma + 2|G(\phi)|_\Sigma \right) |y_1 - y_2|_\Sigma
\]
\[
\leq (r + 2|G(\phi)|_\Sigma) |y_1 - y_2|_\Sigma
\]
\[
=: K_2 |y_1 - y_2|_\Sigma.
\]

**Definition 3.3.** Let \( \nu_0 \) be a common reference measure. The Hellinger distance between \( \mu \) and \( \mu' \) with common reference measure \( \nu_0 \) is
\[
d_{\text{Hd}}(\mu, \mu') = \sqrt{\frac{1}{2} \int \left( \frac{d\mu}{d\nu_0} - \frac{d\mu'}{d\nu_0} \right)^2 d\nu_0}.
\]

**Theorem 3.4.** Assume that \( \phi \sim \mu_0 := \mathcal{N}(0, \mathcal{C}_{\alpha, \tau}) \). Then the following results hold:
(1) The posterior measure \( \mu_y \) exists and is absolutely continuous with respect to \( \mu_0 \), i.e. \( \mu_y \ll \mu_0 \), with Radon-Nikodym derivative given by \((16)\).
(2) \( \mu_y \) is locally Lipschitz in the data \( y \), with respect to the Hellinger distance: if \( \mu_y \) and \( \mu_{y'} \) are two measures with data \( y \) and \( y' \), then for all \( y \) and \( y' \) with \( \max\{|y|_\Sigma, |y'|_\Sigma\} < r \), there exists a constant \( C = C(r) \) such that
\[
d_{\text{Hd}}(\mu_y, \mu_{y'}) \leq C|y - y'|_\Sigma.
\]

**Proof.** From the Proposition 1 (2), we get that \( \Phi(\cdot; y) \) is continuous \( \mu_0 \)-almost surely. Using the \( \mu_0 \)-almost surely continuity, it establishes the measurability in Lemma 6.1 \((29)\). Then, the first result follows from the Theorem 6.29 in \([49]\). For the Lipschitz continuity of the \( \mu_y \), it could be proved by the Theorem 4.5 in \([15]\). Therefore, we omit the details here. 

\[ \square \]
4. **Numerical experiments.** In this section, some numerical results are presented to demonstrate the efficiency and accuracy of the proposed method. In particular, we compare the results of the Bayesian level set method with the regular Bayesian approach. All experiments were performed on a Windows 10 (64 bit) PC-Intel(R) Core(TM) i7- 6700k CPU 3.40 GHz, 8 GB of RAM using MATLAB R2017b.

4.1. **Sampling algorithm.** The Markov chain Monte Carlo (MCMC) methods are usually applied to draw the samples from the posterior distribution $\mu_y$ defined above. In this work, we employ the preconditioned Crank-Nicolson (pCN) algorithm [13], which is described in Algorithm 1. We adopt the proposal variance parameter $\beta \in (0, 1]$ to control the stepsize in numerical implementations. We take the choice of $\beta = 0.001$ for trial and error to balance the acceptance rate and size of the proposed move. The proposed pCN-MCMC algorithm is performed with samples $N_s = 10^4$. We take the last $3 \times 10^3$ samples to compute the conditional mean (CM) estimation. The pCN-MCMC method requires the implementation of the forward problem for each iteration. Thus the main computational cost of the proposed method lies in the evaluation of the forward map with an amount of $MJN_s$.

Algorithm 1 The pCN-MCMC algorithm.

1: Collect the scattered field measured data over all frequencies $k_m$, $m = 1, \cdots, M$ and the incident direction $d_j$, $j = 1, \cdots, J$.
2: Set $s = 0$. Choose an initial state $\phi(0) \in \mathcal{X}$.
3: for $s = 0$ to $N_s$ do
4: Propose $\psi(s) = \sqrt{1 - \beta^2} \phi(s) + \beta \xi(s), \quad \xi(s) \sim N(0, C_{\alpha, \tau})$;
5: Draw $\theta \sim U[0, 1]$;
6: Let $a(\phi(s), \psi(s)) := \min\{1, \exp(\Phi(\phi(s)) - \Phi(\psi(s)))\}$;
7: if $\theta \leq a$ then
8: $\phi(s + 1) = \psi(s)$;
9: else
10: $\phi(s + 1) = \phi(s)$;
11: end if
12: end for

4.2. **Data and parameters.** We consider the case of $R = 1$ and discretize the domain with 16512 elements uniformly with meshsize $h = 2.45 \times 10^{-2}$. Meanwhile, we adopt a uniform mesh with meshsize $\tilde{h} = 2h$ for the application of the pCN-MCMC algorithm. The synthetic data is generated by solving the forward model with the noise being added and the data are measured on the boundary $\Gamma_R$. We assume that the noise is Gaussian, $\eta \sim N(0, \gamma^2 I)$, where $\gamma = 0.005$. The number of incident directions $d_j$ is taken as $J = 6$ and the incident directions $d_j$ are equally distributed around $\Gamma_R$ with $d_1 = (1, 0)$. The wavenumber varies from $k = 0.5\pi$ to $k = 3\pi$ with $M = 5$.

4.3. **Numerical results.** We test the regular Bayesian approach and the Bayesian level set method in the following examples. For Example 1 to 3, the prior is taken to be a zero mean Gaussian with Matérn covariance, i.e. $N(0, C_{\alpha, \tau})$ with fixing $\alpha = 3$. For Example 4, the prior $N(0, C_{\alpha, \tau})$ is chosen with fixing $\tau = 10$. 
**Example 1.** The true scatterer has the form of

\begin{equation}
q^\dagger(x) = \begin{cases} 
1, & x \in D, \\
0, & x \in B_R \setminus D,
\end{cases}
\end{equation}

where \(D = \{(x,y) \in \mathbb{R}^2 : x^2 + (y - \sqrt{x^2})^2 \leq 1\}\) is a love-shaped scatterer. In the level set context, we parameterize \(D\) in terms of the level set function given by \(D = \{x \in D | \phi(x) \geq 0\}\) and the corresponding level set map is \(F(\phi) = \mathbb{I}_D\). We apply the regular Bayesian approach and the Bayesian level set method to recover the shape of the scatterer with different inverse length scales, respectively. The reconstructed results are presented in Fig. 4. One can see from Fig. 4 that both methods are effective to recover the shape of the scatterer. In addition, the numerical results in Fig. 4 present that the posterior CM estimates obtained by the Bayesian level set method are clearly of better quality than those of the regular Bayesian method, suggesting that coupling the level set method can significantly improve the performance of the regular Bayesian method. Moreover, it can be observed that the choice of the inverse length scale \(\tau\) takes a significant effect on the performance of the two methods. The spatial correlation of the samples depends on the spatial correlation of the covariance function, which is controlled by the parameter \(\tau\). It is then clear that the spatial correlation of the samples increases as decreasing the inverse length scale \(\tau\). Intuitively, posterior samples generated by small spatial correlation corresponding to large inverse length scale \(\tau\) is not appropriate to accurately describe the scatterer.

A main advantage of the Bayesian inversion is that it can quantify the uncertainty of the reconstructed results. To show this, we plot the standard deviations of the CM estimations in Fig. 5. We can observe that the major uncertainties quantified through the standard deviations are almost concentrated around the boundary by using the Bayesian level set method. In Fig. 6, we present the trace plots and the corresponding autocorrelation functions for the negative log-likelihood \(\Phi(\phi)\) (or ‘data misfit’), respectively. Moreover, the relative errors with respect to the truth \(q^\dagger\) are displayed in Fig. 6. The relative error is defined as follows

\[
\text{error} = \frac{\|q - q^\dagger\|_{L^2(B_R)}}{\|q^\dagger\|_{L^2(B_R)}}.
\]

For the regular Bayesian method, we observe that data-misfit functions increase with reducing \(\tau\). The corresponding autocorrelation functions (ACF) decay to zero after around a lag of 1000 and it appears that the ACF have slower decay with \(\tau\). However, the autocorrelation functions present a much faster decay of the autocorrelation with the Bayesian level set method than with the regular Bayesian level set method within a lag of 1000. These results indicate that \(\tau = 10\) would be an appropriate choice.

**Example 2.** The expression of the scatterer is the same as that of Example 1, where \(D\) is a cross-shaped scatterer plotted in Fig. 7. We define the level set map \((F\phi)(x) = \mathbb{I}_D\) with \(D = \{x \in D | \phi(x) \geq 0\}\) and the zero level set presents the unknown boundary \(\partial D\). In Fig. 7, we display the samples of the posterior CM estimates with different inverse length scale \(\tau\) by the two methods. As mentioned earlier, we can see that, compared to the regular Bayesian method, the Bayesian level set method can much better identify the boundary of the scatterer, thanks to the level set function. In Fig. 8, we show the standard deviations of the CM estimations. In Fig. 9, we present the trace plots, the autocorrelation functions and the relative error with different \(\tau\) for the two methods. From these results, it seems
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Figure 4. The Top: true scatterer, Second Row: the reconstructions of the scatterer for the regular Bayesian method with $\tau = 20, 10, 5$. Third Row: the reconstructions of the scatterer for the Bayesian level set method with $\tau = 20, 10, 5$.

Figure 5. The standard deviation for the regular Bayesian method (top block) and the Bayesian level set method (bottom block) with $\tau = 20, 10, 5$.

that the inverse length scale $\tau = 10$ is a good choice for our Bayesian inversion. The CPU time spent on $10^4$ MCMC steps with the regular Bayesian method and the Bayesian level set method are $8.75 \times 10^4$ seconds and $8.89 \times 10^4$ seconds, respectively.
Figure 6. The trace plots, the autocorrelation functions of the data misfit and the relative error for the regular Bayesian method (top block) and the Bayesian level set method (bottom block) with $\tau = 20, 10, 5$.

Figure 7. The Top: true scatterer, Second Row: the reconstructions of the scatterer for the regular Bayesian method with $\tau = 20, 10, 5$. Third Row: the reconstructions of the scatterer for the Bayesian level set method with $\tau = 20, 10, 5$.

Example 3. The scatterers are two disjoint domains satisfying

$$q^i(x) = \begin{cases} 
3, & x \in D_1 \cup D_2 \\
0, & x \in B_R \setminus (D_1 \cup D_2),
\end{cases}$$
where $D_1 = \{(x, y) \in \mathbb{R}^2 : (x + 0.3)^2 + (y - 0.3)^2 \leq 0.04\}$ and $D_2 = \{(x, y) \in \mathbb{R}^2 : (x - 0.3)^2 + (y + 0.3)^2 \leq 0.04\}$ are shown in Fig. 10. For the Bayesian level set method, $D$ is characterized by the level set function and the corresponding level set map is $F(\phi) = 3I_{D_1} + 3I_{D_2}$ with $D_i = \{x \in D_i | \phi(x) \geq 0\}, i = 1, 2$. It should be noted here that we do not know a priori about the number of the scatterers. We can see from Fig. 10 that both of the methods are effective to recover the shape and the number of the scatterers. It implies that the Bayesian inversion has the potential capability to address the topological changes. The figures shown in Fig. 11 are the standard deviations. In Fig. 12, the trace plots and the autocorrelation functions for the data misfit functions are plotted with different $\tau$. The relative errors are
presented in Fig. 12. As we can see from the results, the parameter $\tau = 10$ can be a suitable choice.

**Figure 10.** The Top: true scatterer, Second Row: the reconstructions of the scatterer for the regular Bayesian method with $\tau = 10, \frac{20}{3}, 5$. Third Row: the reconstructions of the scatterer for the Bayesian level set method with $\tau = 10, \frac{20}{3}, 5$.

**Figure 11.** The standard deviation for the regular Bayesian method (top block) and the Bayesian level set method (bottom block) with $\tau = 10, \frac{20}{3}, 5$. 
Example 4. The scatterer is the same as that of Example 2. We fix $\tau = 10$ and show the effect of the choice of the parameter $\alpha$ on the numerical results. The reconstructed results are displayed in Fig. 13. Indeed, the parameter $\alpha$ can affect the regularity of the samples. The results from Fig. 14 present the standard deviations which are able to quantify the uncertainties. Similarly, we obtain a reasonable shape identification of the scatterer and the area of the largest uncertainties are around the boundary for Bayesian level set method. Fig. 15 displays the trace plots and the autocorrelation functions for the data-misfit function. In particular, the autocorrelation functions have no obvious change with different $\alpha$. The relative errors shown in Fig. 15 are stabilized with different $\alpha$. These results indicate that $\alpha = 3$ would be a suitable choice.

Example 5. In this example, we take $\alpha = 3$ and $\tau = 10$. We reconstruct the shape of the scatterer by using the two methods from multi-frequency data and single-frequency data, respectively. The numerical results are displayed in Fig. 16. We find that the single-frequency data used for the regular Bayesian method can hardly capture the shape of the true scatterer. However, the Bayesian level set method seems can identify the shape of the scatterer in the single-frequency data case. We demonstrate that the results obtained from multi-frequency scattered data are better than those obtained from single frequency data for the two methods.

Remark 2. The pCN algorithm has been introduced in [13]. In [13], it is shown that the pCN-MCMC method is independent of discretization dimensionality. It constructs a Crank-Nicolson discretization of a stochastic partial differential equation (SPDE) that preserves the reference measure. It is relatively easy to approximate the solution for nonlinear PDE-constrained inverse problems and to show the potential application for addressing a wide range of parameter identification problems. Moreover, we can avoid the implementation of the Fréchet derivative of the forward map as well as the corresponding adjoint operators.
Figure 13. The Top: true scatterer, Second Row: the reconstructions of the scatterer for the regular Bayesian method with $\alpha = 2, 3, 4$. Third Row: the reconstructions of the scatterer for the Bayesian level set method with $\alpha = 2, 3, 4$.

Figure 14. The standard deviation for the regular Bayesian method (top block) and the Bayesian level set method (bottom block) with $\alpha = 2, 3, 4$.

We note that, an alternative class of methods can improve the sampling efficiency by guiding the proposal with the local derivative information of the likelihood function. Such derivative-based methods include: the stochastic Newton MCMC
Stochastic Newton can be interpreted as a Hessian-preconditioned Langevin MCMC method. In [43], the stochastic Newton MCMC method requires the gradient and the Hessian information of the forward map. Although the method can speed up convergence of the sampling process and use a low-rank approximation of the Hessian, it is necessary to pay additional cost for the computation of gradient and Hessian information at every proposed sample point. In [46], a modified stochastic Newton MCMC method with MAP-based Hessian has been proposed to relieve the computational cost based on the local gradient information as well as Hessian information computed initially at the maximum a posterior (MAP) point and then reused at every sample point. The purpose of the stochastic Newton MCMC is to construct fast convergence both in terms of the number of samples as well as the number of PDE solves. The pCN method can be a derivative-free computational framework for approximating solutions to nonlinear PDE-constrained inverse problems. It is easy to implement in applications where the PDE (forward) model is complex. Besides, as we have highlighted, the pCN-MCMC algorithm is independent of discretization dimensionality while the standard approaches may not, such as the random walk Metropolis-Hastings.

Reference [7] mainly introduces two classes of MCMC methods: the preconditioned Metropolis-adjusted Langevin algorithm and the hybrid Monte Carlo (also known as the Hamiltonian Monte Carlo) method. Metropolize-then-discretize and discretize-then-Metropolize are presented to explore PDE-constrained Bayesian inverse problems. The authors utilize finite element discretization schemes so that both commutativity and discretization-invariant can be attained. It is possible to construct discretization-invariant discretize-then-Metropolize MCMC for large-scale inverse problems. In addition, the two methods incorporate the local gradient information of the likelihood function. In our work, we have used the pCN-MCMC method without the implementation of the Fréchet derivative as well as the corresponding adjoint equation. In future works, we would like to investigate and modify...
Figure 16. The Top: true scatterer. Second Row: reconstruct the scatterer with single-frequency data (left) and multi-frequency data (right) by using the regular Bayesian method, respectively. Third Row: reconstruct the scatterer with single-frequency data (left) and multi-frequency data (right) by using the Bayesian level set method, respectively.

the above algorithm for the problem being considered in the current manuscript and other more general problems.

Remark 3. For Examples 2 and 4, the same scatterer was considered in [3] (see Example 2 in [3]) in which a combination of recursive linearization and level set method was applied. There is no essential difference in accuracy between these two methods. As we have classified in the manuscript, the main advantage of the Bayesian method is that we are able to quantify the uncertainties, such as the standard deviation. In addition, the proposed method can avoid the calculation of the Fréchet derivative or its adjoint. But it should be pointed out that, since the Bayesian type method requires a large number of samples ($N_s = 10^4$ in the numerical examples) of the posterior distribution, it implies that it requires more computing time. Use of other optional forward models for accelerating the Bayesian inference could be considered in future work.

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