Quantum Thermodynamics of a Quantum Sized AdS Black Hole

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Abstract

In this paper, we investigate the effects of non-perturbative quantum gravitational corrections on a quantum sized AdS black hole. It will be observed that these non-perturbative quantum gravitational corrections modify the stability of this black hole. We will use the non-equilibrium quantum thermodynamics to investigate the evaporation of this black hole between two states. We will analyze the effects of non-perturbative quantum gravitational corrections on this non-equilibrium quantum thermodynamics. We will explicitly obtain the quantum work distribution for this black hole, as it evaporates between two states. It will be observed that this quantum work distribution is modified due to non-perturbative quantum gravitational corrections.

Key-Words: AdS Black Hole, Quantum Thermodynamics

1 Introduction

It can be demonstrated using quantum field theory in curved spacetime that black holes emit a thermal radiation called Hawking radiation\textsuperscript{1,4}. The temperature of the Hawking radiation is inversely proportional to its surface gravity. The black holes also have an entropy, which scales with the area of the horizon. This scaling property of the entropy has led to the development of the holographic principle\textsuperscript{5,6}. As this analysis is based on quantum field theory in curved spacetime, it is a semi-classical approximation. Thus, it is expected that the black hole thermodynamics will be corrected due to quantum gravitational corrections. In fact, it has been explicitly demonstrated that the relation between the entropy and the area of a black hole would be modified by quantum...
gravitational corrections \[7\text{–}10\]. So, these quantum corrections to the structure of spacetime can produce thermal fluctuations \[11\] to the thermodynamics of black holes. However, even these correction terms are functions of the area, rather than the volume, and so the holographic principle can still be used to analyze them. In fact, as AdS/CFT is a concrete realization of the holographic principle, it has been used to obtain such quantum gravitational corrections to the entropy of a black hole \[12\text{–}15\]. It is possible to use various different approaches to quantum gravity to obtain corrections to the black hole thermodynamics \[16\text{–}20\]. The effects of such quantum gravitational corrections on rotating black hole in AdS spacetime \[21\], charged black hole solution in Rastall theory \[22, 23\], AdS black hole with a monopole \[24\], Skyrmion black holes \[25\] and black hole in a hyperscaling violation background \[26\] have been investigated. It was observed that these corrections produce important modifications to the equilibrium thermodynamics of these black holes.

The connection between quantum corrections to the structure of spacetime, and thermodynamics of black holes can be clearly seen from the Jacobson formalism \[27\]. In this formalism, Einstein equation can be derived from thermodynamical considerations \[27\]. It has also been observed using the Jacobson formalism, the quantum fluctuations to the structure of spacetime can be an obtained from thermal fluctuations \[28\]. At large scales, the quantum fluctuations can be neglected, and the geometry can be described by a classical geometry. As the temperature of a black hole scales inversely with its mass, for large black holes, such thermal fluctuations can also be neglected, and the system can be described by using equilibrium thermodynamics. However, for sufficiently small black hole, the perturbative quantum corrections produce thermal fluctuations to the equilibrium thermodynamics of the system \[29\text{–}34\]. The effects of both the leading order thermal corrections \[35\text{–}38\], next-to-the leading order thermal corrections \[40, 41\] on black hole thermodynamics have been investigated. However, such corrections only only at a scale, where the perturbative corrections to the equilibrium entropy are still valid. Now, for a quantum sized black hole, whose size is comparable to the Planck scale, this perturbative treatment is expected to breakdown. At such a small scale, we have to include non-perturbative quantum gravitational corrections. It has been argued that the non-perturbative quantum gravitational corrections would modify the original equilibrium entropy of a black hole by an exponential function \[42\]. In fact, it is known that the such corrections can also be obtained using supergravity functional integral \[43\text{–}45\]. The effect of such corrections on the thermodynamical stability of spherical symmetric black holes has been investigated \[46\]. It has also been demonstrated that such corrections modify the behavior of a Born-Infeld black hole in a spherical cavity \[47\]. These non-perturbative corrections to black branes thermodynamics have been used to construct a quantum corrected geometry for black branes \[48\].

The AdS black holes has been constructed as solutions to supergravity approximation of string theory \[49\text{–}53\]. The thermodynamics of AdS black holes is important as it can be investigated using the thermodynamics of the conformal field theory dual to it \[60, 61\]. Thus, the corrections to the thermodynamics of the dual theory can be used to obtain the corrected thermodynamics of the AdS black hole \[12\text{–}15\]. In this paper, we will analyze the effects of non-perturbative exponential corrections \[43\text{–}45\] on the thermodynamics of an AdS black hole \[62\text{–}65\]. It is known that in an AdS black hole the cosmological constant can be treated as the thermodynamic pressure, and its thermodynamics can be studied in an extended phase space \[54\text{–}57\]. It has been observed that this extended phase space thermodynamics of an AdS black hole gets modified due to perturbative quantum gravitational corrections \[58, 59\]. Here, we will also analyze the effect of non-perturbative quantum corrections to the thermodynamics of an AdS black hole in an extended phase space.

As these corrections are non-perturbative, they would correspond to non-equilibrium thermody-
namic. Thus, we will use the formalism of non-equilibrium quantum thermodynamics \[78, 79\] to investigate the behavior of a quantum sized AdS black hole. We will calculate the quantum work distributions between two states of such a black hole using this non-equilibrium quantum thermodynamics \[80, 81\]. This quantum work in quantum thermodynamics is an analog of classical work, and is obtained using the Crooks fluctuation theorem \[67\]. The quantum work distributions between two states of a system can be directly related to the free energies between them using the Jarzynski inequality \[68\]. The free energy of a black hole can be obtained from standard black hole thermodynamics, and this can be used to obtain quantum work distribution for a black hole \[69, 70\]. It may be noted that the quantum work distribution is only significant at very small scales, and at such scales we cannot neglect the quantum gravitational corrections to the system. To properly analyze the quantum work distribution for AdS black holes, we need to use the non-perturbative quantum gravitational corrected free energies. Thus, we have to use the non-perturbative quantum gravitational corrections \[42–45\] to obtain corrected free energies for an AdS black hole. Then the quantum work distribution for an evaporating quantum sized AdS black hole \[69, 70\] can be obtained using these quantum corrected free energies. It may be noted that the corrections to the quantum work distribution from non-perturbative quantum gravitational corrections have been obtained for a system of M2-M5 branes \[71\], and a quantum sized Myers-Perry black hole \[72\]. Here, we will use this formalism \[71, 72\] to obtain the quantum work distribution for a quantum sized AdS black hole.

2 Quantum Gravitational Corrections

In this section, we will analyze the quantum gravitational corrections \[42, 43\] to the thermodynamics of a \(n\)-dimensional Schwarzschild-Tangherlini AdS black hole \[62–65\]. These quantum gravitational corrections will modify the original entropy of this AdS black hole \[42–45\], and these modifications will in turn change the thermodynamic behavior of other thermodynamics quantities. The metric of a \(n\)-dimensional Schwarzschild-Tangherlini AdS black hole can be written as \[62–65\]

\[
ds^2 = -f(r) \, dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_{n-2}^2,
\]

where \(f(r)\) in the metric can be expressed as

\[
f(r) = 1 - \frac{16\pi M}{(n-2) \omega r^{n-3}} + \frac{r^2}{l^2},
\]

with \(M\) as the ADM mass of the black hole, \(l\) as the AdS radius, and \(\omega = \frac{2\pi^{\frac{n+1}{2}}}{\Gamma \left(\frac{n+1}{2}\right)}\) as the area of a \(n-1\)-dimensional unit sphere. In Fig. 1, we investigate the structure of the horizon for a \(n\)-dimensional Schwarzschild-Tangherlini AdS black hole. We observe there is a positive real root for \(f(r) = 0\), which can be denoted by \(r_h\).

As the thermodynamics of black holes can be studied using a conformal field theory, thus it is possible to obtain perturbative corrections to the thermodynamics of black holes from such a conformal field theory \[84–87\]. The modular invariance of the partition functions for a conformal field theory \[84–87\], can be directly used to demonstrate that the corrections to the entropy of a black hole can be expressed in terms of the original equilibrium entropy of the black hole \(S_0\) as \(S_{\text{per}} \sim \ln(S_0)\) \[29, 34\]. This is done by investigating the effects of small fluctuations around the equilibrium on the original black hole entropy. So, using the modular invariance of the partition function of the
conformal field theory \cite{84–87}, it is possible to express this corrected entropy as \( S(\beta) = a\beta^n + b\beta^m \), where \( a, b, n, m, > 0 \) are constants. At extremum for \( S(\beta) \), \( \beta_0 = (nb/ma)^{1/m+n} = T^{-1} \) represents the original equilibrium temperature \cite{29–32}. The perturbative corrections can then be expressed in terms of this equilibrium temperature, and the equilibrium entropy \( S_0 = S(\beta)|_{\beta=\beta_0} \). These perturbative corrections can explain the behavior of small black holes, which are still large enough to neglect the full non-perturbative quantum gravitational corrections. However, for quantum sized black holes, we have to consider the effects of non-perturbative quantum gravitational corrections to the black hole entropy. These non-perturbative quantum gravitational corrections to the black hole entropy again can be expressed in terms of the original equilibrium entropy of a black hole as \( S_{\text{non-per}} \sim \exp -S_0 \) \cite{42–45, 47}. As this form of the correction has been proposed to be universal \cite{42, 45, 47}, we will analyze the effects of such corrections on the thermodynamics of an AdS black hole. We will specifically introduce a control parameter \( \eta \) to control the effects of such non-perturbative corrections, and write the corrected entropy as \( S = S_0 + \eta e^{-S_0} \). This introduction of a control parameter has been motivated from the use of such a control parameter for perturbative corrections \cite{33, 34}. Thus, using the original entropy of a \( n \)-dimensional Schwarzschild-Tangherlini AdS black hole, we can write the corrected entropy of a quantum sized AdS black hole as

\[
S = \frac{\omega}{2} r_h^{n-2} + \eta \exp \left( -\frac{\omega}{2} r_h^{n-2} \right)
\]

(3)

It should be observed that this corrected entropy term reduces the original entropy of an AdS black hole, when the AdS black hole is large enough for the quantum gravitational corrections to be neglected. Only at very small scales, at which non-perturbative quantum gravitational corrections cannot be neglected, we have to use this corrected AdS entropy.

It is important to observe that these corrections are expressed in terms of the original temperature and original equilibrium entropy. Thus, we will use the original temperature for this AdS black hole, which can be written as

\[
T = \frac{1}{4\pi} \left( \frac{df(r)}{dr} \right)_{r=r_h} = \frac{(n-1)r_h^2 + (n-3)l^2}{4\pi l^2 r_h}.
\]

(4)
where in the last equality we remove ADM mass using horizon radius. Now for \( n = 3 \) and \( n = 5 \), the horizon radius \( r_h \), can be written as

\[
r_h = l \sqrt{\frac{(16\pi M - \omega)}{\omega}}; \quad n = 3, \quad \tag{5}
\]

\[
r_h = \sqrt{-\frac{l^2}{2} + \sqrt{(3l^2\omega)^2 + 12(16\pi M l^2)}}; \quad n = 5. \quad \tag{6}
\]

For these cases, the temperatures can be expressed as

\[
T = \frac{1}{2\pi l} \sqrt{\frac{(16\pi M - \omega)}{\omega}}; \quad n = 3, \quad \tag{7}
\]

\[
T = \frac{1}{2\pi \sqrt{-\frac{l^2}{2} + \sqrt{(3l^2\omega)^2 + 12(16\pi M l^2)}}} + \frac{1}{\pi l^2} \sqrt{-\frac{l^2}{2} + \sqrt{(3l^2\omega)^2 + 12(16\pi M l^2)}}; \quad n = 5. \quad \tag{8}
\]

In Fig. 2 we plot the temperature in terms of the radius of the horizon. For \( n \geq 4 \), we can observe that there is a minimum value for the temperature. However, for \( n = 3 \) temperature is a linear function of \( r_h \). We can use this original temperature, along with the corrected entropy to analyze the effect of quantum gravitational corrections to on the thermodynamics of this quantum sized AdS black hole.

![Figure 2: Temperature of \( n \)-dimensional Schwarzschild-Tangherlini AdS black hole for \( l = 1 \).](image)

### 3 Stability

The first thing that we will analyze is the effect of these corrections to the stability of this system. This can be done by studying the effect of these non-pertubative quantum gravitational corrections
on the specific heat of this system. Thus, we will use the corrected entropy $S$ given by Eq. (3), to write the corrected specific heat as

$$C_v = T \left( \frac{\partial S}{\partial T} \right)_V$$

This quantum gravitationally corrected specific heat of an AdS black hole can be explicitly written as

$$C_v = \left( \frac{(n-3)}{4\pi r_+} + \frac{(n-1)}{4\pi l^2} r_+^2 \right) \left( \frac{\omega(n-2)}{4\pi r_+^2} + \frac{\omega(n-1)}{4\pi l^2} r_+^2 \right) (1 - \eta e^{-S_0})$$

In Fig. 3 we observe the behavior of specific heat in terms of radius of the horizon, and we find that there occurs a phase transition for a black hole with $n \geq 4$, and such a black hole will become unstable at a sufficient small scales. In Fig. 3 (a), we observe that the specific heat reduces due to quantum corrections. However, this three-dimensional AdS black hole is stable for all values of $r_h$. Thus, the stability of a AdS black hole depends on $n$ for such a black hole.

It is known that for an AdS black hole, the cosmological constant can be treated as the thermodynamic pressure, and a volume conjugate to it can be used to construct in an extended phase space \[54\]. So, we can investigate the effect of the non-perturbative quantum corrections \[42, 45, 47\] on the thermodynamics of an AdS black hole in this extended phase space. The pressure and volume of this AdS black hole are given by \[62, 65\],

$$P = \frac{(n-1)(n-2)}{16\pi l^2}, \quad \text{and} \quad V = \frac{\omega}{n-1} r_h^{n-1}.$$
We can write the equation of state for this AdS black hole as

\[ \frac{Pv}{T} = 1 - \frac{n-3}{(n-2)\pi T} \frac{1}{v}, \quad \text{with} \quad v = \frac{4}{n-2} \left( \frac{(n-1)V}{\omega_{n-2}} \right)^{\frac{1}{n-1}} \]

Comparing Eq. (12) with the virial expansion, we observe

\[ \frac{Pv}{T} = 1 + \frac{B(T)}{v} + \frac{C(T)}{v^2} + \frac{D(T)}{v^3} + \cdots \]

We can write the non-vanishing virial coefficient as

\[ B(T) = -\frac{n-3}{(n-2)\pi T}, \]

here \( C(T) = D(T) = 0 \) (however, if we assume a four dimensional charged black hole, then \( D(T) \neq 0 \)). In that case, Boyle temperature (the temperature where the virial coefficient \( B(T) \) approaches zero) approaches infinity. It should be noted that this equation of state, given by Eq. (12) is a special case of van der Waals equation of state

\[ P v = T v - b + a \frac{v^2}{v^3}, \]

with \( b = 0 \). For such an equation of state, we observe that \( a = n - 3/(n-2)\pi \).

Now we can analyze the behavior of a very small black hole with a quantum scale radius. For such an AdS black hole, we can write

\[ S \approx \frac{\omega (1-\eta)}{2} \left( \frac{n-2}{4} v \right)^{n-2}. \]

The equation of state given by Eq. (12), along with the condition for the critical points, can be used to observe that

\[ \frac{\partial P}{\partial v} = 0, \quad \text{and} \quad \frac{\partial^2 P}{\partial v^2} = 0 \]

This indicates that there is an absence of critical points, except at \( v = 0 \). Thus, we can calculate the modification to stability of the system using the exponential corrections to the entropy from quantum gravitational effects.

### 4 Corrected Thermodynamics

In this section, we will analyze the effects of these non-perturbative quantum gravitational corrections on other thermodynamic quantities of this system. We can express the quantum gravitationally corrected internal energy of this AdS black hole as

\[ E = \int T dS \]

\[ = \frac{\omega (n-2)}{8\pi} \left( r_n^{n-3} + \frac{r_{h-1}^{n-1}}{t^2} \right) + \eta \frac{(n-3)}{8\pi} 2^{\frac{n-3}{n-2}} \omega^{\frac{1}{n-2}} \Gamma \left( \frac{n-3}{n-2}, \frac{1}{2} r_{h-2}^{n-2} \omega \right) \]

\[ + \eta \frac{(n-1)}{8\pi t^2} 2^{\frac{n-1}{n-2}} \omega^{\frac{1}{n-2}} \Gamma \left( \frac{n-1}{n-2}, \frac{1}{2} r_{h-2}^{n-2} \omega \right), \]
Figure 4: Internal energy of $n$-dimensional Schwarzschild-Tangherlini AdS black hole for $l = 1$.  

where $\Gamma(z, x)$ is the Gamma-function

$$
\Gamma(z, x) = x^z e^{-x} \sum_{n=0}^{\infty} \frac{L_n^{(z)}(x)}{n+1}.
$$

Here $L_n^{(z)}(x)$ the generalized Laguerre polynomial are defined as

$$
L_0^{(z)}(x) = 0, \\
L_1^{(z)}(x) = 1 + z - x, \\
L_{k+1}^{(z)}(x) = \frac{(2k+1 + z - x)L_k^{(z)}(x) - (k+z)L_{k-1}^{(z)}(x)}{k+1} \quad k \geq 1.
$$

In Fig.4 we plot the internal energy and observe that it increases with the increase in the radius of the horizon. We also observe that the quantum gravitational corrections can be neglected for large black holes. However, at small quantum scales, these corrections change the internal energy of the AdS black hole. We can obtain the free energy for this AdS black hole using the quantum corrected internal energy, and entropy $F = E - TS$. Thus, for this AdS black hole the quantum corrected free
energy can be written as

\[ F = \omega \left( (n-2)r_h (l^2 r_h^{n-3} + r_h^{n-1}) - r_h^{n-2} (l^2 (n-3) + (n-1)r_h^2) \right) \]

\[ + \frac{\eta}{8\pi l^2} \left( \frac{n-1}{n-2} \omega \Gamma \left( \frac{n-1}{n-2}, \frac{1}{2} r_h^{n-2} \right) + l^2 \omega (n-3) \Gamma \left( \frac{n-3}{n-2}, \frac{1}{2} r_h^{n-2} \omega \right) \right) \]

\[ - \frac{\eta}{4\pi r_h l^2} (l^2 (n-3) + (n-1)r_h^2) \exp \left( -\frac{\omega r_h^{n-2}}{2} \right). \]  

(21)

We can now explicitly use quantum corrected free energy to calculate the quantum gravitationally corrections to Gibbs free energy. Thus, we can write the quantum gravitationally corrected Gibbs free energy \( G = F + PV \) as

\[ G = \frac{\omega (n-2)}{8\pi} \left( r_h^{n-3} + \frac{3r_h^{n-1}}{l^2} \right) - \frac{\omega}{8\pi l^2} \left( (n-1)r_h^2 + (n-3)l^2 \right) r_h^{n-3} \]

\[ + \frac{\eta}{8\pi l^2} \left( \frac{n-1}{n-2} \omega \Gamma \left( \frac{n-1}{n-2}, \frac{1}{2} r_h^{n-2} \omega \right) + l^2 \omega (n-3) \Gamma \left( \frac{n-3}{n-2}, \frac{1}{2} r_h^{n-2} \omega \right) \right) \]

\[ - \frac{\eta}{4\pi r_h l^2} (l^2 (n-3) + (n-1)r_h^2) \exp \left( -\frac{\omega r_h^{n-2}}{2} \right). \]

(22)

In Fig. 5, we plot the behavior of the quantum gravitationally corrected Gibbs free energy. It is again observed that the quantum gravitational corrections to the Gibbs free energy can be neglected.
at large scales. However, for quantum sized AdS black holes, the behavior of Gibbs free energy is significantly changed due to these quantum gravitational corrections. We also observe that the corrections to the Gibbs free energy depend on the dimensions of the AdS black hole.

5 Quantum Work Distribution

In this section, we analyze the quantum work distribution for an AdS black hole, as it evaporates between two states. Thus, let us assume that the microstates of the AdS black hole change from $\Omega_1$ to $\Omega_2$ during evaporation. Then the partition function of the black hole will change from $Z_1[\Omega_1]$ to $Z_2[\Omega_2]$. This will change different thermodynamic quantities between these two states. As the quantum gravitational corrections become important only at small horizon radius, we can express the difference in the corrected entropy of a quantum sized AdS black hole as

$$\Delta S = \frac{\omega}{2} (1 - \eta)(r_{h2}^{n-2} - r_{h1}^{n-2}).$$

This difference between the quantum gravitationally corrected entropy can be used to obtain the difference between the internal energy of a black hole, as it evaporates between from $\Omega_1$ to $\Omega_2$

$$\Delta E = \frac{\omega (n-2)}{8\pi} \left( r_{h2}^{n-3} - r_{h1}^{n-3} + \frac{r_{h2}^{n-1} - r_{h1}^{n-1}}{l^2} \right)$$

$$+ \frac{\eta (n-3)}{8\pi} \frac{2^{n-1}}{\omega} \omega^{\frac{1}{n-2}} \left[ \Gamma \left( \frac{n-3}{n-2}, \frac{1}{2} \frac{r_{h2}^{n-2}\omega}{l} \right) - \Gamma \left( \frac{n-3}{n-2}, \frac{1}{2} \frac{r_{h1}^{n-2}\omega}{l} \right) \right]$$

$$+ \frac{\eta (n-1)}{8\pi l^2} \frac{2^{n-2}}{\omega} \omega^{\frac{1}{n-2}} \left[ \Gamma \left( \frac{n-1}{n-2}, \frac{1}{2} \frac{r_{h2}^{n-2}\omega}{l} \right) - \Gamma \left( \frac{n-1}{n-2}, \frac{1}{2} \frac{r_{h1}^{n-2}\omega}{l} \right) \right].$$

These corrections to the internal energy will produce quantum gravitational corrections to the Hawking radiation, for quantum sized AdS black holes. We can denote this total heat obtained from quantum gravitationally corrected Hawking radiation by $Q$. Now at such scales, we cannot neglect the effects of quantum work, as the size of the black hole reduces. So, we obtain finite quantum work as the black hole evaporates from $\Omega_1$ to $\Omega_2$. If we denote the average quantum work by $\langle W \rangle$, we can write

$$\Delta E = Q - \langle W \rangle$$

Now let us assume that an AdS black hole with a partition function $Z_1[\Omega_1]$ evaporates to an AdS black hole with a partition function $Z_2[\Omega_2]$. The term $Z_2/Z_1$ can be related to the quantum work distribution, using the Jarzynski equality \cite{82, 83}

$$\langle \exp - \beta W \rangle = \frac{Z_2}{Z_1}$$

We can also express the relative weights of the partition function to the difference between the equilibrium free energies as $\exp \beta \Delta F = Z_2/Z_1$. Thus, we can use the Jarzynski equality \cite{82, 83} to express the quantum work in terms of difference of the equilibrium free energies of these two states for an AdS black hole

$$\langle \exp - \beta W \rangle = \exp \beta \Delta F$$

We can use this quantum corrected free energy to calculate the average quantum work between these two black hole states. It is possible to use the Jensen inequality to relate the average of the exponential of quantum work to the exponential of the average of quantum work as as
Figure 6: $e^{-\Delta F}$ for $n$-dimensional Schwarzschild-Tangherlini AdS black hole, with $r_{h1} = 1$, $\eta = 1$ and $l = 1$.

\[
\exp \langle -\beta W \rangle \leq \langle \exp -\beta W \rangle. \quad \text{Using this inequality, we can express the quantum work distribution during the evaporation of an AdS black hole as}
\]

\[
\langle W \rangle \approx -\frac{\omega}{8\pi r_{h2}^2} \left( (n-2)r_{h2}(l^2r_{h2}^{n-3} + r_{h2}^{n-1}) - r_{h2}^{n-2}(l^2(n-3) + (n-1)r_{h2}^2) \right) \\
+ \frac{\omega}{8\pi r_{h1}^2} \left( (n-2)r_{h1}(l^2r_{h1}^{n-3} + r_{h1}^{n-1}) - r_{h1}^{n-2}(l^2(n-3) + (n-1)r_{h1}^2) \right) \\
- \frac{\eta}{8\pi l^2} \left( 2^{\frac{n-4}{2}} \omega^{\frac{1}{2}} (n-1)\Gamma \left( \frac{n-1}{n-2}, \frac{1}{2} r_{h2}^{n-2} \omega \right) + l^2 2^{\frac{n-3}{2}} \omega^{\frac{1}{2}} (n-3)\Gamma \left( \frac{n-3}{n-2}, \frac{1}{2} r_{h2}^{n-2} \omega \right) \right) \\
+ \frac{\eta}{8\pi l^2} \left( 2^{\frac{n-4}{2}} \omega^{\frac{1}{2}} (n-1)\Gamma \left( \frac{n-1}{n-2}, \frac{1}{2} r_{h1}^{n-2} \omega \right) + l^2 2^{\frac{n-3}{2}} \omega^{\frac{1}{2}} (n-3)\Gamma \left( \frac{n-3}{n-2}, \frac{1}{2} r_{h1}^{n-2} \omega \right) \right) \\
- \frac{2\eta}{8\pi r_{h2}l^2} (l^2(n-3) + (n-1)r_{h2}^2) \left( 1 - \frac{\omega r_{h2}^{n-2}}{2} \right) \\
+ \frac{2\eta}{8\pi r_{h1}l^2} (l^2(n-3) + (n-1)r_{h1}^2) \left( 1 - \frac{\omega r_{h1}^{n-2}}{2} \right) \quad \text{(27)}
\]

This is plotted in Fig. 6. We can use this expression to express the quantum work in terms of the partition functions $Z_1$ and $Z_2$ as $\langle e^{-\beta W} \rangle = Z_2/Z_1$ \cite{68}. So, the relative weights of the partition functions of a black hole depends on the quantum work done between those two states. This is expected as the quantum work is dependent on the difference between equilibrium free energies between these two states, which also depends on the microstates of the black hole. Now, as the black hole evaporates, it emits Hawking radiation. However, we have to also consider average quantum work between these two states. The terms related to quantum work becomes important only at small scales, and so we had to consider the effects of quantum gravitational corrections on it. These corrections modified the expression for equilibrium free energies, and this in turn modified the expression for quantum work. We had to use this modified expression for free energies to analyze the effect of quantum gravitational corrections on quantum work distribution.
6 Conclusion

In this paper, we have studied the thermodynamics of a quantum sized Schwarzschild-Tangherlini AdS black hole in higher dimension. We used quantum gravitationally corrected entropy to analyze the corrections to the thermodynamic behavior of such a black hole. We considered the effect of such non-perturbative quantum gravitational corrections on the stability of such a black hole. This was done by analyzing quantum corrections to the specific heat for this AdS black hole. The effect of such corrections on the Gibbs free energy was also investigated. Finally, we observed that at such a small scale, we cannot neglect the average quantum work between two black hole states. The difference in free energies between two states was used to obtain quantum work distribution. This was done using the Jarzynski equality. We also discussed the relation between quantum work distribution and relative weights of the partition for an evaporating AdS black hole.

It seems important that the effect of such non-perturbative quantum gravitational correction should be investigated for other quantum sized black holes. It would be interesting to analyze such corrections to extra-dimensional black objects, such as black strings. This formalism can be used to analyze the corrections to the thermodynamics of black strings. We can also calculate the average quantum work for the black strings using the difference between free energies. In the context of singularity theorems, we can use such quantum gravitational corrections to obtain modified quantum Raychaudhuri equation for those effective geometries. It is expected that the geometrical flow could be effected by such non-perturbative quantum gravitational corrections. It would be interesting to investigate such effects.

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