NONCOMMUTATIVE HAMILTONIAN DYNAMICS
ON FOLIATED MANIFOLDS

YURI A. KORDYUKOV

Dedicated to Stephen Smale on his 80th birthday

Abstract. First, we review the notion of a Poisson structure on a noncommutative algebra due to Block-Getzler and Xu and introduce a notion of a Hamiltonian vector field on a noncommutative Poisson algebra. Then we describe a Poisson structure on a noncommutative algebra associated with a transversely symplectic foliation and construct a class of Hamiltonian vector fields associated with this Poisson structure.

1. Introduction

The class of Hamiltonian flows on a symplectic manifold is an important class of dynamical systems. It naturally arises as a geometric manifestation of Hamilton’s equations in classical mechanics. Hamiltonian flows can be defined more generally on an arbitrary Poisson manifold.

The purpose of this paper is to discuss the notion of a Hamiltonian flow in the framework of Poisson geometry on a particular class of singular symplectic manifolds, namely, on the leaf space of a transversely symplectic compact foliated manifold. In this case, it is natural to use the notions and methods of noncommutative differential geometry initiated by Alain Connes [3].

There are several fundamental ideas, which lie in the base of noncommutative geometry. The first of them is to pass from geometric spaces to algebras of functions on these spaces and translate basic geometric and analytic notions and constructions into the algebraic language. An application of this idea in Poisson geometry leads to a notion of Poisson algebra.

The next idea is that, in many cases (especially, in those cases when the classical commutative algebra of functions is small or has a bad structure), it is useful to consider as its analogue some noncommutative algebra. This gives rise to the need to extend the basic geometric and analytic definitions to general noncommutative algebras. A noncommutative analogue of a Poisson structure was introduced independently by Block and Getzler [1] and Xu [20]. They used some ideas from deformation theory of associative algebras.

In [2], Connes associated with an arbitrary foliated manifold (M, F) the C*-algebra C*(M, F), which can be naturally considered as a noncommutative analogue of the algebra of continuous functions on the leaf space M/F of the foliation.
When the foliation has a transverse symplectic structure, there is a natural noncommutative Poisson structure defined on a dense subalgebra of the $C^*$-algebra $C^*(M, \mathcal{F})$, which was constructed in [1].

In this paper, we introduce the notion of a Hamiltonian vector field associated with a noncommutative Poisson structure and construct a class of Hamiltonian vector fields on the $C^*$-algebra $C^*(M, \mathcal{F})$ associated with a transversally symplectic foliation $(M, \mathcal{F})$. This study was partially motivated by our investigations of transversally elliptic operators on foliated manifolds, related trace formulae and the corresponding classical dynamics [10, 11, 12]. In particular, it follows from the results of the paper that the dynamical systems on foliation algebras, which appear in the Egorov type theorems for transversally elliptic operators proved in [11, 12], are Hamiltonian flows.

We remark that there is a notion of a noncommutative symplectic manifold introduced by Kontsevich [9] and further developed by Ginzburg in [5, 6]. One can define a notion of a Hamiltonian vector field on a noncommutative symplectic manifold. Moreover, one can show that a transversally symplectic foliation gives rise to a noncommutative symplectic manifold, and the noncommutative vector fields constructed in our paper are Hamiltonian vector fields on this noncommutative symplectic manifold. These results will be discussed elsewhere.

The paper is organized as follows. First, we review the notions of a Poisson structure on an associative algebra and a Hamiltonian vector field associated with a noncommutative Poisson structure. Next, we describe the noncommutative geometry of the leaf space of a foliated manifold and the noncommutative Poisson structure of a transversely symplectic foliated manifolds and construct a class of noncommutative Hamiltonian flows on these manifolds.

We refer the reader to [18] for basic notions of Poisson geometry and to the survey paper [13] for information and references on noncommutative geometry of foliations.

2. Preliminaries on noncommutative Poisson geometry

In this Section, we review some basic notions related with noncommutative Poisson structures on associative algebras, following [11, 20].

2.1. Noncommutative Poisson structures. Let $A$ be an associative algebra over $\mathbb{C}$. The space of Hochschild $k$-cochains on $A$ is $C^k(A, A) = \text{Hom}(A^\otimes k, A)$. The differential $b : C^k(A, A) \to C^{k+1}(A, A)$ is given by

$$(bc)(a_1, \cdots, a_{k+1}) = a_1 c(a_2, \cdots, a_{k+1})$$
$$+ \sum_{i=1}^{k} (-1)^i c(a_1, \cdots, a_i a_{i+1}, \cdots, a_{k+1}) + (-1)^k c(a_1, \cdots, a_k) a_{k+1}.$$ 

The cohomology of the complex $(C^*(A, A), b)$ is called the Hochschild cohomology $H^*(A, A)$ of $A$. For example, $H^0(A, A)$ is just the center of $A$, and $H^1(A, A)$ is the space $\text{Out}(A) = \text{Der}(A)/\text{Inn}(A)$ of outer derivations of $A$. 

We define a pre-Lie product on \( C^*(A, A) \). For any \( U \in C^u(A, A) \) and \( V \in C^v(A, A) \), \( U \ast V \in C^{u+v-1}(A, A) \) is given by
\[
(U \ast V)(a_1, \ldots, a_{u+v-1}) = \sum_{i=1}^u (-1)^{(i-1)(v-1)} U(a_1, \ldots, a_{i-1}, V(a_i, \ldots, a_{i+v-1}), a_{i+v}, \ldots, a_{u+v-1}).
\]

The Gerstenhaber bracket \( \Pi \) is defined to be the commutator of the pre-Lie bracket: for any \( U \in C^u(A, A) \) and \( V \in C^v(A, A) \), \([U, V] \in C^{u+v-1}(A, A) \) is given by
\[
[U, V] = U \ast V - (-1)^{(u-1)(v-1)} V \ast U.
\]

The Gerstenhaber bracket is a generalization of the usual Schouten-Nijenhuis brackets of multivector fields.

**Definition 2.1.** A Poisson structure on \( A \) is a Hochschild two-cocycle \( \Pi \in Z^2(A, A) \) such that \( \Pi^{0, 1} \) is a three-boundary, that is, \( \Pi \) is a homomorphism \( \Pi : A \otimes A \rightarrow A \) such that
\begin{enumerate}
  \item for any \( a_1, a_2, a_3 \in A \),
  \begin{equation}
  a_1 \Pi(a_2, a_3) - \Pi(a_1 a_2, a_3) + \Pi(a_1, a_2 a_3) - \Pi(a_1, a_2) a_3 = 0;
  \end{equation}
  \item there is a homomorphism \( \Pi_1 : A \otimes A \rightarrow A \) such that, for any \( a_1, a_2, a_3 \in A \),
  \begin{equation}
  \Pi_1(a_1, \Pi(a_2, a_3)) - \Pi(\Pi(a_1, a_2), a_3) - \Pi(a_1, a_2 a_3) + \Pi(a_1, a_2 a_3) - \Pi_1(a_1, a_2) a_3 = 0.
  \end{equation}
\end{enumerate}

**Example 2.2.** Let \( M \) be a compact smooth manifold. Recall that any Poisson bracket \( \{\cdot, \cdot\} \) on \( M \) is determined by a Poisson bivector \( \Lambda \in C^\infty(M, \Lambda^2 TM) \):
\[
\{f, g\} = \langle \Lambda, df \wedge dg \rangle = \sum_{ij} \Lambda^{ij} \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial x^j}, \quad f, g \in C^\infty(M),
\]
where \( \{x^i\} \) are local coordinates on \( M \). In local coordinates, \( \Lambda \) has to satisfy the following condition
\[
\sum_\alpha \left( \Lambda^{\alpha i} \frac{\partial \Lambda^{jk}}{\partial x^\alpha} + \Lambda^{\alpha j} \frac{\partial \Lambda^{ki}}{\partial x^\alpha} + \Lambda^{\alpha k} \frac{\partial \Lambda^{ij}}{\partial x^\alpha} \right) = 0
\]
for any \( i, j, k \).

An invariant meaning of this identity is provided by the Schouten-Nijenhuis bracket, which is a bilinear local type extension of the Lie derivative \( L_X \) to an operation
\[
[\cdot, \cdot] : C^\infty(M, \Lambda^p TM) \times C^\infty(M, \Lambda^q TM) \rightarrow C^\infty(M, \Lambda^{p+q-1} TM).
\]

A bivector \( \Lambda \in C^\infty(M, \Lambda^2 TM) \) is a Poisson bivector if and only if \([\Lambda, \Lambda] = 0\).

Consider the commutative algebra \( A = C^\infty(M) \) of smooth functions on \( M \). Then there is an isomorphism \([19]\)
\[
H^*(C^\infty(M), C^\infty(M)) = C^\infty(M, \Lambda^* TM).
\]
For any $p$-vector field $X_1 \wedge \ldots \wedge X_p \in C^\infty(M, \Lambda^p TM)$, the corresponding cocycle $c \in H^p(C^\infty(M), C^\infty(M))$ is given by

$$c(f_1, \ldots, f_p) = \det \|X_i f_j\|_{j=1}^p, \quad f_1, \ldots, f_p \in C^\infty_c(M).$$

The bilinear map

$$\Pi(f, g) = \{f, g\}, \quad f, g \in C^\infty(M),$$

is a Poisson structure on the algebra $C^\infty(M)$. The corresponding homomorphism $\Pi_1$ is described as follows. Let $\nabla$ be a torsion-free connection on $M$. It induces the covariant derivative $\nabla : C^\infty(M, T^* M) \to C^\infty_c(M, T^* M \otimes T^* M)$ on $T^* M$. We can consider the composition of this operator with the de Rham differential $d : C^\infty(M) \to C^\infty(M, T^* M)$:

$$\nabla^2 = \nabla \circ d : C^\infty(M) \to C^\infty(M, S^2 T^* M).$$

The operator $\nabla^2$ takes values in $C^\infty(M, S^2 T^* M)$ since $\nabla$ is torsion-free.

The corresponding two-cochain $\Pi_1$ is defined by

$$\Pi_1(f, g) = \langle \Lambda \otimes \Lambda, \nabla^2 f \otimes \nabla^2 g \rangle, \quad f, g \in C^\infty(M),$$

where the pairing of the tensor $\Lambda \otimes \Lambda$ with $\alpha \otimes \beta \in S^2 T^* M \otimes S^2 T^* M$ is given by the formula

$$\langle \Lambda \otimes \Lambda, \alpha \otimes \beta \rangle = \Lambda^{ij} \Lambda^{kl} \alpha_{ik} \beta_{jl}.$$  

For any $f \in C^\infty(M)$, the map $g \mapsto \{f, g\}$ is a derivation of $C^\infty(M)$, Therefore, there exists a well defined vector field $X_f$ on $M$ such that

$$\{f, g\} = X_f g = -X_g f = dg(X_f) = -df(X_g).$$

$X_f$ is called the Hamiltonian vector field of $f$.

**Example 2.3.** One of the basic examples in noncommutative differential geometry is the noncommutative two-torus $A_\theta$. The algebra $A_\theta$ is generated by two elements $U$ and $V$, satisfying the relation

$$VU = e^{2\pi i \theta} UV.$$  

A generic element of $A_\theta$ can be represented as a formal power series

$$a = \sum_{(n,m) \in \mathbb{Z}^2} a_{nm} U^n V^m,$$

where $a_{nm} \in S(\mathbb{Z}^2)$ is a rapidly decreasing sequence (that is, for any natural $k$ we have $\sup_{(n,m) \in \mathbb{Z}^2} (|n| + |m|)^k |a_{nm}| < \infty$).

$A_\theta$ is a locally convex topological algebra under the topology generated by the seminorms

$$p_k(a) = \sup_{(n,m) \in \mathbb{Z}^2} (|n| + |m|)^k |a_{nm}|, \quad k \in \mathbb{N}.$$  

There are two canonical derivations $\delta_1$ and $\delta_2$ on $A_\theta$ given, respectively, by

$$\delta_1(U^n V^m) = 2\pi i n U^n V^m, \quad \delta_2(U^n V^m) = 2\pi i m U^n V^m.$$  

It is easy to see that $[\delta_1, \delta_2] = 0$.

By [20] Theorem 4.1, there is a canonical Poisson structure $\Pi \in Z^2(A_\theta, A_\theta)$ on $A_\theta$ defined by

$$\Pi(a_1, a_2) = \delta_1(a_1) \delta_2(a_2), \quad a_1, a_2 \in A_\theta.$$
In particular, the Jacobi rule \((2)\) holds with
\[
\Pi_{1}(a_1, a_2) = -\frac{1}{2} \delta_1^2(a_1) \delta_2^2(a_2), \quad a_1, a_2 \in A_\theta.
\]

More examples of noncommutative Poisson structures can be found in \([1, 20, 16, 8]\).

2.2. Hamiltonian dynamics. For a given noncommutative Poisson structure \(\Pi\) on an associative algebra \(A\) over \(\mathbb{C}\), we denote the center of \(A\) by \(C = H^0(A, A)\).

**Definition 2.4.** For any element \(c\) of \(C\), the Hamiltonian derivation of \(A\) associated to \(c\) is defined as
\[
X_c = \frac{1}{2} \left[ \Pi, c \right] \in H^1(A, A),
\]
or equivalently
\[
X_c(a) = \frac{1}{2} \left( \Pi(c, a) - \Pi(a, c) \right), \quad a \in A.
\]

**Remark 2.5.** It is impossible, in general, to associate a Hamiltonian derivation to an arbitrary element of \(A\) due to the lack of outer derivations in \(A\).

One can introduce a bracket on the center \(C\) as follows: For any \(c\) and \(e\) in \(C\)
\[
\{c, e\} = [X_c, e] \in H^0(A, A) = C.
\]
We have the following properties (see \([20\), Proposition 2.1]).

**Proposition 2.6.** For any \(c\) and \(e\) in \(C\)
\begin{enumerate}
\item \(L_{X_c} \Pi = 0\);
\item \([X_c, X_e] = -X_{\{c, e\}}\);
\item \(C\) together with the bracket \(\{\cdot, \cdot\}\) introduced above becomes a Poisson algebra in the usual sense.
\end{enumerate}

**Remark 2.7.** It is easy to see that, for a compact Poisson manifold \(M\), the bracket on the commutative algebra \(C^\infty(M)\) defined by \((3)\) coincides with the bracket on \(C^\infty(M)\) given by the Poisson structure, and the Hamiltonian derivation \(X_f\) of \(C^\infty(M)\) associated to \(f \in C^\infty(M)\) by Definition \((2.4)\) is determined by the classical Hamiltonian vector field with Hamiltonian \(f\).

3. Transverse geometry of foliations

Throughout in this Section, \((M, \mathcal{F})\) is a compact foliated manifold, \(\dim M = n, \dim \mathcal{F} = p, p + q = n\). We will consider foliated charts \(\phi : U \subset M \to I^p \times I^q\) on \(M\) with coordinates \((x, y) \in I^p \times I^q\) \((I\) is the open interval \((0, 1))\) such that the restriction of \(\mathcal{F}\) to \(U\) is given by the level sets \(y = \text{const}\). We will use the following notation: \(T\mathcal{F}\) is the tangent bundle of \(\mathcal{F}\); \(\tau = TM/T\mathcal{F}\) is the normal bundle of \(\mathcal{F}\); \(N^*\mathcal{F} = \{\xi \in T^*M : \langle \xi, X \rangle = 0 \ \forall X \in T\mathcal{F}\}\) is the conormal bundle of \(\mathcal{F}\).
3.1. **Transverse symplectic structures.** A transverse symplectic structure on a foliated manifold \((M, \mathcal{F})\) is given by a covering \(\{U_i, \phi_i\}\) by foliated charts, \(\phi_i : U_i \to I^p \times I^q\), and by a family of symplectic forms \(\omega_i\) on local bases \(I^q\) such that for any coordinate transformation
\[
\phi_{ij}(x, y) = (\alpha_{ij}(x, y), \gamma_{ij}(y)), \quad x \in I^p, y \in I^q,
\]
the map \(\gamma_{ij}\) preserves the symplectic structure, \(\omega_j = \gamma_{ij}^* \omega_i\).

A manifold \(M\) is called presymplectic, if it is endowed with a closed two-form \(\omega\) of constant rank.

One can show \([1]\) that presymplectic structures are essentially the same as transverse symplectic structures. More precisely, if \(M\) is a presymplectic manifold and \(F \subset TM\) is the subbundle on which \(\omega\) vanishes, then \(F\) is integrable and thus defines a foliation \(\mathcal{F}\) on \(M\). The restrictions of \(\omega\) to the local bases of foliated charts on \(M\) define a transverse symplectic structure on the foliated manifold \((M, \mathcal{F})\). On the other hand, a transverse symplectic structure on a foliated manifold \((M, \mathcal{F})\) determines in a unique manner a presymplectic structure on \(M\) such that \(F = TF\) is the kernel of \(\omega\).

3.2. **Foliation algebras.** Here we will describe a noncommutative algebra, which can be considered, according to noncommutative geometry, as an algebra of functions on the leaf space \(M/\mathcal{F}\) of a foliation \((M, \mathcal{F})\). As a vector space, this algebra is the space \(C_c^\infty(G)\) of smooth compactly supported functions on the holonomy groupoid \(G\) of the foliation. Therefore, we recall first the notion of holonomy groupoid.

Consider the equivalence relation \(\sim_h\) on the set of continuous leafwise paths \(\gamma : [0, 1] \to M\), setting \(\gamma_1 \sim_h \gamma_2\), if \(\gamma_1\) and \(\gamma_2\) have the same initial and final points and the same holonomy maps: \(h_{\gamma_1} = h_{\gamma_2}\). The holonomy groupoid \(G\) is the set of \(\sim_h\)-equivalence classes of leafwise paths. The set of units \(G^{(0)}\) is \(M\). The multiplication in \(G\) is given by the product of paths. The corresponding range and source maps \(s, r : G \to M\) are given by \(s(\gamma) = \gamma(0)\) and \(r(\gamma) = \gamma(1)\). Finally, the diagonal map \(\Delta : M \to G\) takes any \(x \in M\) to the element in \(G\) given by the constant path \(\gamma(t) = x, t \in [0, 1]\). To simplify the notation, we will identify \(x \in M\) with \(\Delta(x) \in G\).

For any \(x \in M\) the map \(s\) takes the set \(G^x = r^{-1}(x)\) onto the leaf \(L_x\) through \(x\). The group \(G^x = s^{-1}(x) \cap r^{-1}(x)\) coincides with the holonomy group of \(L_x\). The map \(s : G^x \to L_x\) is the covering map associated with the group \(G^x\), called the holonomy covering.

The holonomy groupoid \(G\) has a structure of a smooth (in general, non-Hausdorff and non-paracompact) manifold of dimension \(2p + q\) \([2]\). A local coordinate system on \(G\), denoted by \(W(\phi, \phi')\), is determined by a pair of compatible foliated charts \(\phi\) and \(\phi'\) on \(M\). The coordinates in \(W(\phi, \phi')\) will denote by \((x, x', y) \in I^p \times I^p \times I^q\).

Let us fix a positive smooth leafwise density \(\alpha \in C_c^\infty(M, |T\mathcal{F}|)\). For any \(x \in M\), we define a positive Radon measure \(\nu^x\) on \(G^x\) to be the lift of the restriction of \(\alpha\) to \(L_x\) by the holonomy cover \(s : G^x \to L_x\). The structure of an involutive algebra on \(C_c^\infty(G)\) is defined by
\[
k_1 \ast k_2(\gamma) = \int_{G^x} k_1(\gamma_1)k_2(\gamma_1^{-1}\gamma) \, d\nu^x(\gamma_1), \quad \gamma \in G^x,
k^*(\gamma) = k(\gamma^{-1}), \quad \gamma \in G.
\]
where \( k, k_1, k_2 \in C_c^\infty(G) \).

There are natural left and right actions of the commutative algebra \( C^\infty(M) \) on \( C_c^\infty(G) \) given by the formulas

\[
a \cdot \sigma(\gamma) = a(r(\gamma))\sigma(\gamma), \quad \sigma \cdot a(\gamma) = a(s(\gamma))\sigma(\gamma), \quad \gamma \in G,
\]

for any \( a \in C^\infty(M) \) and \( \sigma \in C_c^\infty(G) \).

We enlarge the algebra \( C_c^\infty(G) \), introducing the unital algebra

\[
\hat{C}_c^\infty(G) = C_c^\infty(G) + C^\infty(M)
\]

with the multiplication given by

\[
(k_1 + a_1)(k_2 + a_2) = k_1 \cdot k_2 + a_1 \cdot k_2 + k_1 \cdot a_2 + a_1 a_2,
\]

where \( a_1 a_2 \) is the product of the functions \( a_1 \) and \( a_2 \).

We will also need the noncommutative analogue of the algebra of differential forms on the leaf space of the foliation. Denote \( \Omega_{\gamma}^k(G) = C_c^\infty(G, r^*N^*\mathcal{F}) \). There is a product

\[
\Omega_{\gamma}^k(G) \times \Omega_{\gamma}^k(G) \to \Omega_{\gamma}^{k+k}(G)
\]

given, for any \( \omega \in \Omega_{\gamma}^k(G) \) and \( \omega_1 \in \Omega_{\gamma}^k(G) \), by

\[
(\omega \wedge \omega_1)(\gamma) = \int_{G^s} \omega(\gamma_1) \Lambda H_{\gamma_1} [\omega_1(\gamma_1^{-1}\gamma)] d\nu(\gamma_1), \quad \gamma \in G, \quad r(\gamma) = y.
\]

Here \( H_{\gamma} : N_{s(\gamma)}^* \mathcal{F} \to N_{s(\gamma)}^* \mathcal{F} \) is the linear holonomy map associated with \( \gamma_1 \).

One can also define natural left and right actions of the algebra \( \Omega^*_H(M) = C^\infty(M, \Lambda^* N^* \mathcal{F}) \) of transverse differential forms on \( M \) on \( \Omega^*_{\gamma}(G) \) by the formulas

\[
a \wedge \omega(\gamma) = r^* a(\gamma) \wedge \omega(\gamma), \quad \omega \wedge a(\gamma) = \omega(\gamma) \Lambda H_{\gamma}(s^* a(\gamma)), \quad \gamma \in G,
\]

for any \( a \in \Omega^*_H(M) \) and \( \omega \in \Omega^*_{\gamma}(G) \).

We enlarge the algebra \( \Omega^*_{\gamma}(G) \), introducing the unital algebra

\[
\hat{\Omega}^*_{\gamma}(G) = \Omega^*_{\gamma}(G) + \Omega^*_H(M),
\]

with the multiplication given by

\[
(\omega_1 + a_1) \wedge (\omega_2 + a_2) = \omega_1 \wedge \omega_2 + a_1 \wedge \omega_2 + \omega_1 \wedge a_2 + a_1 \wedge a_2,
\]

where \( \omega_1 \wedge \omega_2 \) is the product of the forms \( \omega_1 \) and \( \omega_2 \).

### 3.3. Transverse differential

Let \( H \subset TM \) be a \( q \)-dimensional distribution such that \( TM = F \oplus H \). There is \( \mathbf{3, 15} \) the transverse differentiation, which is a linear map

\[
D_H : \Omega_{\gamma}^0(G) = C^\infty_c(G) \to \Omega^1_{\gamma}(G) = C^\infty_c(G, r^*N^* \mathcal{F}),
\]

satisfying the condition

\[
D_H(k_1 \cdot k_2) = D_H k_1 \cdot k_2 + k_1 \cdot D_H k_2, \quad k_1, k_2 \in C^\infty_c(G).
\]

In this Section, we recall its definition.

The transverse distribution \( H \) naturally defines a transverse distribution \( HG \cong r^*H \) on the foliated manifold \( (G, \mathcal{G}) \) and the corresponding transversal de Rham differential \( d_{H} : C^\infty_c(G) \to C^\infty_c(G, r^*N^* \mathcal{F}) \). For any \( X \in H_y \), there is a unique vector \( \hat{X} \in T_yG \) such that \( ds(\hat{X}) = dh_{\gamma}^{-1}(X) \) and \( dr(\hat{X}) = X \), where \( dh_{\gamma} : H_y \to H_y \) is the linear holonomy map associated with \( \gamma \). The space \( H_yG \) consists of all vectors of the form \( \hat{X} \in T_yG \) for different \( X \in H_y \). In any coordinate chart
\[ W(\phi, \phi') \) on \( G, \) the distribution \( H_iG \) consists of vectors \( X \phi + X' \phi + Y \phi \) such that \( X \frac{\partial}{\partial x} + Y \frac{\partial}{\partial y} \in H(x, y) \) and \( X' \frac{\partial}{\partial x'} + Y \frac{\partial}{\partial y'} \in H(x', y'). \)

For any \( f \in C^c_c(G), \) define \( d_H f \in C^c_c(G, r^*N^*F) \) by
\[
d_H f(X) = df(\tilde{X}), \quad X \in (r^*\tau)_{\gamma}, \quad \gamma : x \to y,
\]
where \( \tilde{X} \in H_iG \subset T_IG \) is a unique vector such that \( ds(\tilde{X}) = dh^{-1}_{\gamma}(X) \) and \( dr(\tilde{X}) = X. \)

For the fixed smooth leafwise density \( \alpha \in C^c_c(M, |TF|), \) we define a transverse 1-form \( k(\alpha) \in C^c_c(M, H^*) \cong C^c_c(M, N^*F) \) as follows. Take an arbitrary point \( m \in M \) and \( X \in H_m. \) Let \( \tilde{X} \) be an arbitrary local projectable vector field, that coincides with \( X \) at \( m. \) In a foliated chart \( \phi : U \to I^p \times I^q \) near \( m \) such that \( \phi(m) = (x^0, y^0), \) one can write
\[
\tilde{X}(x, y) = \sum_{i=1}^p X^i(x, y) \frac{\partial}{\partial x_i} + \sum_{j=1}^q Y^j(y) \frac{\partial}{\partial y_j}.
\]
Then we put
\[
k(\alpha)(X) = \sum_{i=1}^p X^i(x^0, y^0) \frac{\partial f}{\partial x_i}(x^0, y^0) + \sum_{j=1}^q Y^j(y) \frac{\partial f}{\partial x_j}(x^0, y^0)
+ \sum_{i=1}^p \frac{\partial X^i}{\partial x_i}(x^0, y^0)f(x^0, y^0).
\]
It can be checked that this definition is independent of the choice of a foliated chart \( \phi \) and an extension \( \tilde{X}. \) If \( M \) is Riemannian, \( \alpha \) is given by the induced leafwise Riemannian volume form, and \( H = F^\perp, \) then \( k(\alpha) \) coincides with the mean curvature 1-form of \( F \) (cf., for instance, [17]).

For \( f \in C^c_c(G), \) define \( D_H f \in C^c_c(G, r^*N^*F) \) as
\[
D_H f(\gamma) = d_H f(\gamma) + \frac{1}{2}(H_f[s^* k(\alpha)(\gamma)] + r^* k(\alpha)(\gamma))f(\gamma), \quad \gamma \in G.
\]
Finally, note that the operator \( D_H \) has a unique extension to a differentiation of the differential graded algebra \( Q_\infty(G) \) (see [3, 15]).

4. Noncommutative Poisson geometry of foliations

4.1. Noncommutative Poisson structures. Let \( (M, F) \) be a transversely symplectic compact foliated manifold, and \( \omega \) the corresponding closed two-form of constant rank on \( M. \) In this Section, we describe a Poisson structure on the algebra \( C^c_c(G) \) associated to the foliation \( F. \)

First, we need some facts about connections on foliated manifolds. Recall that there is a canonical flat connection
\[
\tilde{\nabla} : C^c_c(M, TF) \times C^c_c(M, \tau) \to C^c_c(M, \tau)
\]
in the normal bundle \( \tau, \) defined along the leaves of \( F \) (the Bott connection). It is given by
\[
\tilde{\nabla} X N = \theta(X) N = P_\tau[X, \tilde{N}], \quad X \in C^c_c(M, TF), \quad N \in C^c_c(M, \tau),
\]
where \( P_\tau : TM \to \tau \) is the natural projection and \( \tilde{N} \in C^c_c(M, TM) \) is any vector field on \( M \) such that \( P_\tau(\tilde{N}) = N. \) Thus, the restriction of \( \tau \) to any leaf of \( F \) is a
flat vector bundle. The parallel transport in $\tau$ along any leafwise path $\gamma : x \to y$ defined by $\tilde{\nabla}$ coincides with the linear holonomy map $dh_\gamma : \tau_x \to \tau_y$.

A connection $\nabla : C^\infty(M,TM) \times C^\infty(M,\tau) \to C^\infty(M,\tau)$ in the normal bundle $\tau$ is called adapted, if its restriction to $C^\infty(M,TF)$ coincides with the Bott connection $\tilde{\nabla}$.

One can construct an adapted connection, starting with an arbitrary Riemannian metric $g_M$ on $M$. Denote by $\nabla^g$ the Levi-Civita connection, defined by $g_M$. An adapted connection $\nabla$ is given by

$$\nabla X N = P_\tau[X, \tilde{N}], \quad X \in C^\infty(M,TF), \quad N \in C^\infty(M,\tau)$$

$$(5) \quad \nabla X N = P_\tau \nabla^g X \tilde{N}, \quad X \in C^\infty(M,TF^\perp), \quad N \in C^\infty(M,\tau),$$

where $\tilde{N} \in C^\infty(M,TM)$ is any vector field such that $P_\tau(\tilde{N}) = N$. One can show that the adapted connection $\nabla$ described above is torsion-free.

An adapted connection $\nabla$ in the normal bundle $\tau$ is called holonomy invariant, if, for any $X \in C^\infty(M,TF)$, $Y \in C^\infty(M,TM)$, and $N \in C^\infty(M,\tau)$, we have

$$(\theta(X)\nabla)Y N := \theta(X)[\nabla Y N] - \nabla_{\theta(X)Y} N - \nabla_Y [\theta Y N] = 0.$$

A holonomy invariant adapted connection in $\tau$ is called a basic (or projectable) connection.

If the foliation $\mathcal{F}$ is Riemannian and $g_M$ is a bundle-like metric, then the connection $\nabla$ defined by (5) is a basic connection. There are topological obstructions for the existence of basic connections for an arbitrary foliations found independently by Kamber-Tondeur and Molino.

We will assume the following:

**Hypotheses 4.1.** There exists a basic connection on the normal bundle $\tau$ of $\mathcal{F}$.

The two-form $\omega$ induces an isomorphism $\#_\omega$ between the bundle $\tau$ and $\tau^*$:

$$\langle \#_\omega X, Y \rangle = \omega(\tilde{X}, \tilde{Y}), \quad X, Y \in \tau,$$

where $\tilde{X} \in TM$ and $\tilde{Y} \in TM$ are such that $P_\tau(\tilde{X}) = X$. Thus, we have a skew-symmetric form on $\tau^*$, which we denote by $\Lambda$:

$$\Lambda(\#_\omega X, \#_\omega Y) = \omega(X, Y), \quad X, Y \in \tau.$$

It is shown in [1] that, under Hypotheses 4.1, $M$ has a presymplectic connection, that is, a basic connection $\nabla$ on $\tau$ such that $\nabla^\perp = 0$. From now on, we will assume that $\nabla$ is a presymplectic connection.

The definition of a noncommutative Poisson structure depends on a choice of a $q$-dimensional distribution $H \subset TM$ such that $TM = F \oplus H$. The Poisson bracket of two functions $k_1, k_2 \in C^\infty_c(G)$ is defined by the formula

$$\Pi_H(k_1, k_2) = \Lambda(D_H k_1, D_H k_2)$$

or

$$\Pi_H(k_1, k_2)(\gamma) = \int_{G^y} \langle \Lambda_{y}, D_H k_1(\gamma_1) \wedge H_{\gamma_1} [D_H k_2(\gamma_1^{-1} \gamma_1)] \rangle dv^y(\gamma_1), \quad \gamma \in G^y.$$
Denote by
\[ D^2 = \nabla \circ D_H : C^\infty_c(G) \to C^\infty_c(G, r^*N^*F \otimes r^*N^*F) \]
the composition of \( D_H : C^\infty_c(G) \to C^\infty_c(G, r^*N^*F) \) with \( \nabla ; \) \( D^2 \) takes values in \( C^\infty_c(G, S^2r^*N^*F) \) since \( \nabla \) is torsion-free. Then \( \Pi_1 \) is a two-chain on \( C^\infty_c(G) \) defined by the formula
\[ \Pi_1(k_1, k_2) = \Lambda \otimes \Lambda(D^2k_1 \ast D^2k_2), \quad k_1, k_2 \in C^\infty_c(G). \]

We extend \( \Pi_H \) to the algebra \( \hat{C}^\infty(G) \) in the sense of Definition 2.1.

4.2. Transverse Hamiltonian flows. As above, we suppose that \((M, F)\) is a transversely symplectic compact foliated manifold and \(\omega\) is the corresponding closed two-form of constant rank on \(M\). A Hamiltonian on the singular symplectic manifold \(M/F\) is given by a \(C^\infty\) function \(h\) on \(M\), which is constant on each leaf of the foliation \(F\). It is easy to see that \(h\) belongs to the center of the algebra \(\hat{C}^\infty(G)\). The purpose of this section is to give an explicit geometric description of the associated Hamiltonian derivation \(X_h\).

First, we recall (see [7]) that for any presymplectic manifold \((M, \omega)\) there is a symplectic manifold \((\Phi, \eta)\) and an embedding \(i : M \to \Phi\) such that \(\omega = i^*\eta\) and \(M\) is a coisotropic submanifold of \(\Phi\). Moreover, such a coisotropic embedding is unique up to local symplectomorphism about \(M\). Its construction makes use of an auxiliary choice of a distribution \(H \subset TM\) such that \(TM = H \oplus T^*F\). Such a distribution yields an embedding \(j\) of \(T^*F\) in \(T^*M\). Let \(\pi : T^*F \to M\) be the natural projection. Let \(j^*\omega_{T^*M}\) be the pull-back of the canonical symplectic form \(\omega_{T^*M}\) on \(T^*M\) to \(T^*F\). Then one can take
\[ \eta = \pi^*\omega + j^*\omega_{T^*M}. \]

It is easy to see that the restriction of \(\eta\) to \(M\) equals \(\omega\). The manifold \(\Phi\) is defined to be a tubular neighborhood of the zero section \(M\) in \(T^*F\) so that \(\eta\) restricted to \(\Phi\) is non-degenerate.

Remark that the restricted tangent bundle \(T_M\Phi\) has the canonical decomposition
\[ T_M\Phi \cong TM \oplus T^*F. \]

Thus, we have
\[ T_M\Phi \cong H \oplus TF \oplus T^*F. \]
Denote by \(p\) the induced projection \(T_M\Phi \to TF \oplus T^*F\). For \(m \in M\), let \(\omega_F\) denote the canonical symplectic structure on \(TF \oplus T^*F\):
\[ \omega_F(f_1 + f_1^*, f_2 + f_2^*) = \langle f_2^*, f_1 \rangle - \langle f_1^*, f_2 \rangle. \]
Then the restriction of \(\eta\) to \(T_M\Phi\) is described as
\[ \eta = \pi^*\omega + \omega_F \circ (p \times p). \]
Thus, for any \( X = \pi_*(X) + f^x_\gamma \in T_\gamma \Phi \) and \( Y = \pi_*(Y) + f^y_\gamma \in T_\gamma \Phi \), we have

\[
\eta(X, Y) = \omega(\pi_*(X), \pi_*(Y)) + \langle f^y_\gamma, p_F(\pi_*(X)) \rangle - \langle f^x_\gamma, p_F(\pi_*(Y)) \rangle.
\]

Given a \( C^\infty \) function \( h \) on \( M \), which is constant on each leaf of \( \mathcal{F} \), we extend it to a smooth function \( \tilde{h} \) on \( \Phi \). Let \( v^\gamma_h \) be the Hamiltonian vector field of the function \( \tilde{h} \) on \( \Phi \). Recall that by definition we have

\[
i_{v^\gamma_h} \eta = d\tilde{h}.
\]

Then (see, for instance, [14]) the submanifold \( M \) of \( \Phi \) is invariant under the flow of \( v^\gamma_h \). Indeed, for \( Y = \pi_*(Y) \in TM \subset T_\gamma \Phi, Y = p_F(Y) \), we have

\[
0 = d\tilde{h}(Y) = \eta(v^\gamma_h, Y) = -\langle v^\gamma_h - \pi_*(v^\gamma_h), p_F(\pi_*(Y)) \rangle.
\]

Therefore, we conclude that \( v^\gamma_h = \pi_*(v^\gamma_h) \).

It is easy to see that \( v^\gamma_h \) depends only on \( h \) and \( d\tilde{h} \) restricted to \( T^* \mathcal{F} \subset T_\gamma \Phi \). For any \( Y = f^y_\gamma \in T^* \mathcal{F} \subset T_\gamma \Phi \), we have

\[
d\tilde{h}(Y) = \eta(v^\gamma_h, Y) = \langle f^y_\gamma, p_F(v^\gamma_h) \rangle.
\]

Thus, we see that

\[
d\tilde{h}|_{T^* \mathcal{F}} = p_F(v^\gamma_h) \in C^\infty(M, T^* \mathcal{F}).
\]

Finally, if we denote by \( v_h \) the restriction of \( v^\gamma_h \) to \( M \), then one can show that the flow of \( v_h \) on \( M \) preserves the foliation \( \mathcal{F} \), that is, it takes a leaf of \( \mathcal{F} \) into a leaf. Therefore, there is a natural lift of \( v_h \) to a vector field \( \hat{v}_h \) on \( G \) such that for any \( \gamma \in G \), \( s_*(\hat{v}_h(\gamma)) = v_h(s(\gamma)) \) and \( r_*(\hat{v}_h(\gamma)) = d\tilde{h}_\gamma[v_h(s(\gamma))] = v_h(r(\gamma)) \), where \( d\tilde{h}_\gamma \) is the differential of the holonomy map along \( \gamma \). In local foliated coordinates, \( v_h \) has a form

\[
v_h(x, y) = \sum_{j=1}^p X^j(x, y) \frac{\partial}{\partial x_j} + \sum_{k=1}^q Y^k(y) \frac{\partial}{\partial y_k}, \quad (x, y) \in I^p \times I^q.
\]

and \( \hat{v}_h \) is given by

\[
\hat{v}_h(x, x', y) = \sum_{j=1}^p X^j(x, y) \frac{\partial}{\partial x_j} + \sum_{j=1}^p X^j(x', y) \frac{\partial}{\partial x_j'} + \sum_{k=1}^q Y^k(y) \frac{\partial}{\partial y_k},
\]

\[(x, x', y) \in I^p \times I^p \times I^q.
\]

Define an operator \( \mathcal{L}_{\hat{v}_h} \) on the space \( C^\infty_c(G) \) by the formula

\[
\mathcal{L}_{\hat{v}_h} f = \hat{v}_h f + \frac{1}{2} (s^* [k(\alpha)(\hat{v}_h)] + r^* [k(\alpha)(\hat{v}_h)]) f, \quad f \in C^\infty_c(G).
\]

It coincides with the Lie derivative by \( \hat{v}_h \) on the space \( C^\infty_c(G, |TG|^{1/2}) \) of leafwise half-densities on the holonomy groupoid \( G \). In a foliated chart, for any \( k \in C^\infty_c(G) \), we have

\[
\mathcal{L}_{\hat{v}_h}(k)
\]

\[
= (\hat{v}_h k(x, x', y) + \frac{1}{2} \sum_{j=1}^p \frac{\partial X^j}{\partial x_j}(x, y) k(x, x', y) + \frac{1}{2} \sum_{j=1}^p \frac{\partial X^j}{\partial x_j'}(x', y) k(x, x', y)),
\]

\[(x, x', y) \in I^p \times I^p \times I^q.
\]

We arrive at the main result of the paper.
Theorem 4.2. Suppose that \((M, F)\) is a transversely symplectic compact foliated manifold such that there exists a basic connection on the normal bundle \(\tau\) of \(F\). Let \(H\) be a \(q\)-dimensional distribution on \(M\) such that \(TM = F \oplus H\), \(\Pi_H\) the noncommutative Poisson structure on the algebra \(\mathcal{C}^{\infty}(G)\) defined by \(\hat{\mathcal{L}}\), and \(i_H : M \rightarrow \Phi_H\) the corresponding coisotropic embedding into a symplectic manifold \((\Phi_H, \eta_H)\).

Let \(h\) be a \(C^\infty\) function on \(M\), which is constant on each leaf of \(F\), and \(\hat{h}\) its extension to a smooth function on \(\Phi_H\) such that \(d\hat{h}|_{T^*F} = 0\).

The Hamiltonian derivation \(X_h\) of the algebra \(\mathcal{C}^{\infty}(G)\) associated to the Hamiltonian \(h\) and the noncommutative Poisson structure \(\Pi_H\) coincides with \(\mathcal{L}_{\hat{\phi}_h} + \hat{v}_h:\)

\[
X_h(k + a) = \mathcal{L}_{\hat{\phi}_h}(k) + \hat{v}_h(a), \quad k \in C^{\infty}_c(G), \quad a \in C^{\infty}(M).
\]

Proof. The key step in the proof is the following lemma.

Lemma 4.3. For any \(a \in C^{\infty}(M)\), we have

\[
\Lambda(d_H h, d_H a) = v_h(a).
\]

Proof. Denote by \(#_\eta : T\Phi \rightarrow T^*\Phi\) the isomorphism induced by the two-form \(\eta:\)

\[
\langle \#_\eta X, Y \rangle = \eta(X, Y), \quad X, Y \in T\Phi.
\]

and by \(\Lambda_{\Phi}\) the induced two-form on \(T^*\Phi\)

\[
\Lambda_{\Phi}(\#_\eta X, \#_\eta Y) = \eta(X, Y), \quad X, Y \in T\Phi.
\]

It is easy to see that \(#_\eta\) maps \(TM\) to \(N^*F\), and the kernel of the map \(#_\eta : TM \rightarrow N^*F\) coincides with \(TF\). Thus, we have the induced map \(\#_{\#_\eta} : TM/TF \rightarrow N^*F\), which is equal to \(#_\eta\).

Using these facts, we easily derive that

\[
\Lambda_{\Phi}(\nu_1, \nu_2) = \Lambda(\nu_1, \nu_2), \quad \nu_1, \nu_2 \in N^*F.
\]

On the other hand, for a given function \(a \in C^{\infty}(M)\) take its extension \(\hat{a} \in C^{\infty}(\Phi)\) to \(\Phi\) such that \(d\hat{a}|_{T^*F} = 0\). Then by definition we have

\[
\Lambda_{\Phi}(d_H h, d\hat{a}) = d\hat{a}(v_h).
\]

Let us restrict both sides of this identity to \(M\). Then by assumption the restriction of \(d_H h \in C^{\infty}(\Phi, T^*\Phi)\) to \(M\) coincides with \(dh \in C^{\infty}(M, T^*M) \subset C^{\infty}(M, T^*_M\Phi)\). Moreover, we have \(dh = d_H h\). Similarly, the restriction of \(d\hat{a} \in C^{\infty}(\Phi, T^*\Phi)\) to \(M\) coincides with \(d\hat{a} \in C^{\infty}(M, T^*M) \subset C^{\infty}(M, T^*_M\Phi)\). Since \(v_h|_M = v_h \in C^{\infty}(M, TM) \subset C^{\infty}(M, T_M\Phi)\), in particular, this implies that \(d\hat{a}(v_h)|_M = da(v_h)\).

We arrive at the identity

\[
\Lambda_{\Phi}(d_H h, da) = da(v_h).
\]

It remains to show that

\[
\Lambda_{\Phi}(d_H h, d_F a) = 0.
\]

Given \(X \in T_M F\), \(Y = \pi_*(Y) + f_Y^* \in T_M \Phi\), we have

\[
\langle \#_\eta X, Y \rangle = \eta(X, Y) = \langle f_Y^*, X \rangle.
\]

Therefore, \(#_\eta X \in T^* F\), and \(#^{-1}_\eta : T^* F \rightarrow T\Phi\). So \(#^{-1}_\eta d_H h \in TM\), \(#^{-1}_\eta d_F a \in T^* F\), and we obtain

\[
\Lambda_{\Phi}(d_H h, d_F a) = \eta(\#^{-1}_\eta d_H h, \#^{-1}_\eta d_F a) = \omega(\#^{-1}_\eta d_H h, \#^{-1}_\eta d_F a) = 0,
\]

that completes the proof of the lemma. \(\square\)
By this lemma, it follows easily that, for any \( a \in C^\infty(M) \),
\[
\frac{1}{2}(\Pi(h, a) - \Pi(a, h)) = \nu_h(a)
\]
and, for any \( k \in C^\infty_c(G) \),
\[
\frac{1}{2}(\Pi(h, k) - \Pi(k, h)) = \mathcal{L}_{\nu_h}(k),
\]
that completes the proof. \( \square \)

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