YANG–MILLS DUALITY AS ORIGIN OF GENERATIONS, QUARK MIXING, AND NEUTRINO OSCILLATIONS

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The origin of fermion generations is one of the great mysteries in particle physics. We consider here a possible solution within the Standard Model framework based on a nonabelian generalization of electric–magnetic duality. First, nonabelian duality says that dual to the colour (electric) symmetry SU(3), there is a “colour magnetic symmetry” SU(3), which by a result of ’t Hooft is spontaneously broken and can thus play the role of the “horizontal symmetry” of generations. Second, nonabelian duality suggests the manner this symmetry is broken with frame vectors in internal symmetry space acting as Higgs fields. As a result, mass matrices factorize leading to fermion mass hierarchy. At the tree level, there is no mixing but with loop corrections, the mass matrices rotate and mixing occurs. A calculation to first order gives mixing (CKM and MNS) matrices in general agreement with experiment. In particular, quark mixing is seen naturally to be weak compared with leptons, while within the lepton sector, \( \mu - \tau \) mixing turns out near maximal but \( e - \tau \) mixing small, just as seen in recent \( \nu \) oscillation experiments. In addition, the scheme leads to many testable predictions ranging from rare FCNC meson decays and \( \mu - e \) conversion in nuclei to cosmic ray air showers above \( 10^{20} \) eV, which will be detailed in the following talk by Chan.

1 Yang–Mills Duality

In electromagnetism, the dual field tensor \( *F_{\mu \nu} \) is given in terms of the field tensor \( F_{\mu \nu} \) by a duality transformation which is just the Hodge star operation:

\[
*F_{\mu \nu} = -\frac{i}{2} \epsilon_{\mu \nu \rho \sigma} F^{\rho \sigma}.
\]

Under this operation:

\[
\begin{align*}
E &\leftrightarrow B \\
electric charge &\leftrightarrow magnetic monopole \\
e &\leftrightarrow \tilde{e}
\end{align*}
\]

where \( e \tilde{e} = 2\pi \). As is well-known, electromagnetism is symmetric with respect to this duality transform.

A natural question to ask is whether the symmetry persists in nonabelian Yang–Mills theory. If we again use the Hodge star as our duality transform, then the question was answered in the negative by Gu and Yang [8], who gave counter-examples where \( F_{\mu \nu} \) satisfies the source-free Yang–Mills equation but \( *F_{\mu \nu} \), although defined, is not a gauge field.

So if we want to generalize electric–magnetic duality to the nonabelian case, then we have to look for a generalization of the Hodge star. Preferably this generalized dual transform \((\cdot)^*\) should satisfy the following properties:

1. \( (\cdot)^* = \pm (\cdot) \),
2. electric field \( F_{\mu \nu} \leftrightarrow magnetic fields \tilde{F}_{\mu \nu} \),
3. both \( A_\mu \) and \( \tilde{A}_\mu \) exist as potentials (away from charges and monopoles),
4. magnetic charges are monopoles of \( A_\mu \), and electric charges are monopoles of \( \tilde{A}_\mu \),
5. \( * \) reduces to \( * \) in the abelian case.

A hint of what we should be looking for comes from a result of Wu and Yang [4], namely that what describes gauge theory exactly is neither the gauge potential \( A_\mu \) nor the gauge field \( F_{\mu \nu} \), but the Dirac phase factor or Wilson loop

\[
\Phi(C) = P \exp i g \int_C A_\mu(x) dx^\mu,
\]

where \( P \) denotes path ordering with respect to the loop \( C \). This leads to a loop space formulation of Yang–Mills theory [9], as originally proposed by Polyakov [12]. And it is in terms of these loop variables that we were able to formulate a generalized transform satisfying the above 5 conditions, and to show that Yang–Mills theory is completely dual symmetric in this sense.

What is relevant for what follows is that as a result of this dual symmetry the gauge symmetry is doubled, so that if \( F_{\mu \nu} \) is the gauge field of an \( SU(N) \) Yang-Mills theory, then the total symmetry is \( SU(N) \times SU(N) \), where \( SU(N) \) is abstractly the same group \( SU(N) \) but has the opposite parity.

2 Generation Puzzle

Broadly speaking, the generation puzzle consists of three questions with as yet no generally accepted answers:

- Why are there 3 and only 3 generations?
- Why is there mass hierarchy?
- Why is there nontrivial mixing?
The 3 generations are: for the up-quarks $t, c, u$; for the down-quarks $b, s, c$; for the neutrinos $\nu_{e}, \nu_{\mu}, \nu_{\tau}$; and for the charged leptons $\tau, \mu, e$. As far as we can observe, these triples differ only in their masses, but then to such an extent that they are ‘hierarchical’. Thus:

\[
m_t \sim 180 \text{ GeV} \quad m_c \sim 1.1 \text{ GeV} \quad m_u \sim 4 \text{ MeV} \\
m_b \sim 4.2 \text{ GeV} \quad m_s \sim 120 \text{ MeV} \quad m_d \sim 7 \text{ MeV} \\
m_\tau \sim 1.8 \text{ GeV} \quad m_\mu \sim 100 \text{ MeV} \quad m_e \sim 0.5 \text{ MeV}
\]

By mixing we mean that up and down states are not aligned but related by the empirical matrices (absolute values only), for quarks and leptons respectively:

\[
|V_{\text{CKM}}| = \begin{pmatrix}
0.975 & 0.222 & 0.003 \\
0.222 & 0.974 & 0.040 \\
0.009 & 0.039 & 0.999
\end{pmatrix}, \quad (3)
\]

\[
|U_{\text{MNS}}| = \begin{pmatrix}
? & 0.4 & -0.7 \\
? & 0.0 & -0.15 \\
? & 0.45 & -0.85
\end{pmatrix}, \quad (4)
\]

with the following three noticeable features:

- mixing is smaller for quarks and larger for leptons,
- the corner elements in both are very much smaller,
- the $(23)$ element for leptons is much larger than for quarks.

3 Dualized Standard Model

The Dualized Standard Model (DSM) tries to answer all these questions by making use of the dual symmetry we proved (§1) and applying it to the Standard Model. One immediate consequence is that the gauge group is now doubled to

\[
SU(3) \times SU(2) \times U(1) \times SU(3) \times SU(2) \times U(1).
\]

In the application we are concerned with in this article, we need only focus on the colour and dual colour symmetries, that is: $SU(3) \times SU(3)$. Now by a theorem of ‘t Hooft, we know that the fact that $SU(3)$ is confined implies that dual colour $SU(3)$ is broken. We now make the crucial assumption that this broken $SU(3)$ is the generation symmetry, in other words, a very specific type of horizontal symmetry. The only other assumption that DSM makes is about the role of Higgs fields as frame vectors. For want of space, we shall not elaborate on this.

As a result, the fermion mass matrix at tree level is of rank 1:

\[
m = m_T \begin{pmatrix} x \\ y \\ z \end{pmatrix} (x, y, z), \quad (5)
\]

where $(x, y, z)$ is a normalized vector and $m_T$ is essentially the mass of the heaviest generation. With only one nonzero eigenvalue we already have a good tree-level approximation to the mass hierarchy. Under renormalization (to one loop) $m$ remains factorized, but

\[
\frac{d}{d\mu} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{5}{32\pi^2} \rho \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}, \quad (6)
\]

where $\rho$ is a (fitted) constant and

\[
x_1 = \frac{x(x^2 - y^2)}{x^2 + y^2} + \frac{x(x^2 - z^2)}{x^2 + z^2}, \quad \text{cyclic.} \quad (7)
\]

So we see that the vector $(x, y, z)$ rotates with the energy scale where the rotation depends on the fermion-type, so that up and down states become disoriented with respect to each other leading to nontrivial mixing matrices. At the same time, mass starts to “leak” from the top generation into the two lower generations giving them small but nonzero masses. As the energy changes, the vector $(x, y, z)$ traces out a trajectory on the unit sphere, starting at high energy from near the fixed point $(1, 0, 0)$ and moving towards the low-energy fixed point $(1, 1, 1)$. Although the trajectories can in principle be different for different fermion-types, the data demand, for reasons yet theoretically unclear, that they coincide to a very good approximation. The 12 different fermion states thus occupy only different points on this single trajectory (Figure 1 of reference [3]), from which we are already able to deduce, qualitatively and even quantitatively, the three features of mixing listed above.

This follows from elementary differential geometry. For a given fermion type, the state vector for the top (i.e. heaviest) generation is the radial vector $(x, y, z)$ at the appropriate scale, and the state vector for the second generation is approximately the tangent vector to both curve and surface. These together with the state vector for the lightest generation form the Darboux triad at the position of the top generation. So we have two such triads at the positions of the up and down fermions, and the direction cosines between them give the corresponding mixing matrix. To first order in the separation $\Delta s$ of the two top fermion positions (i.e. at $t$ and $b$ for quarks and $\nu_3$ and $\tau$ for leptons), this is given by the Serret-Frenet-Darboux formulae as

\[
\begin{pmatrix}
1 & -\kappa_g \Delta s & -\tau_g \Delta s \\
\kappa_g \Delta s & 1 & \kappa_n \Delta s \\
\tau_g \Delta s & -\kappa_n \Delta s & 1
\end{pmatrix}, \quad (8)
\]

with $\kappa_n$ the normal curvature, $\kappa_g$ the geodesic curvature, and $\tau_g$ the geodesic torsion of a curve on a surface. For the unit sphere, $\kappa_n = 1$ and $\tau_g = 0$. From this we deduce first that the corner elements are of second order in
\( \Delta s \) and therefore small compared with the others. Secondly, we conclude that the 23 and 32 elements are given approximately just by the separation \( \Delta s \). And indeed, if one takes the trouble to measure with a bit of string these distances on the the trajectory in Figure 1 of reference 1, one will find values very close to the experimental numbers given for these elements in 3 and 4.

Of course, having actually done the calculation 1, 2, 3, one can make much a more detailed comparison with data than the above purely geometric picture. One finds that all the mixing parameters overlap with the present experimental limits, except for the solar neutrino mixing element \( U_{e2} \), which being related to the trajectory-dependent geodesic curvature according to 4 is particular difficult for our calculation to get correct. We note that all these numbers have been obtained by adjusting only one parameter to the Cabibbo angle \( V_{us} \sim V_{cd} \), the other two parameters in the calculation having already been fitted to fermion masses 20.

4 Further Results

By making the assumption that dual colour, which exists by virtue of Yang–Mills duality, is generation symmetry, DSM is able to account for the origin of exactly 3 generations with generally correct mixing. In addition, as reported in the same volume 1, the scheme leads to many testable predictions, including rare FCNC meson decays, \( \mu \rightarrow e \) conversion in nuclei, and even cosmic ray air showers above the GZK bound 22.

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