COMPACT OBJECTS WITH SPIN PARAMETER $a_\ast > 1$

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In 4-dimensional General Relativity, black holes are described by the Kerr solution and are completely specified by their mass $M$ and by their spin angular momentum $J$. A fundamental limit for a black hole in General Relativity is the Kerr bound $|a_\ast| \leq 1$, where $a_\ast = J/M^2$ is the spin parameter. Future experiments will be able to probe the geometry around these objects and test the Kerr black hole hypothesis. Interestingly, if these objects are not black holes, the accretion process may spin them up to $a_\ast > 1$.

1 Introduction

Today we believe that the final product of the gravitational collapse is a black hole (BH). In 4-dimensional General Relativity, BHs are described by the Kerr solution and are completely specified by two parameters: the mass $M$, and the spin angular momentum, $J$. A fundamental limit for a BH in General Relativity is the Kerr bound $|a_\ast| \leq 1$, where $a_\ast = J/M^2$ is the spin parameter. For $|a_\ast| > 1$, the Kerr solution does not describe a BH, but a naked singularity, which is forbidden by the weak cosmic censorship conjecture.

From the observational side, we have at least two classes of astrophysical BH candidates: stellar-mass bodies in X-ray binary systems ($M \sim 5 - 20$ Solar masses) and super-massive bodies in galactic nuclei ($M \sim 10^5 - 10^{10}$ Solar masses). The existence of a third class of objects, intermediate-mass BH candidates ($M \sim 10^2 - 10^4$ Solar masses), is still controversial, because there are not yet reliable dynamical measurements of their masses. All these objects are commonly interpreted as BHs because they cannot be explained otherwise without introducing new physics. The stellar-mass objects in X-ray binary systems are too heavy to be neutron or quark stars. At least some of the super-massive objects in galactic nuclei are too massive, compact, and old to be clusters of non-luminous bodies.

2 Testing the Kerr Black Hole Hypothesis

In Newtonian gravity, the potential of the gravitational field, $\Phi$, is determined by the mass density of the matter, $\rho$, according to the Poisson’s equation, $\nabla^2 \Phi = 4\pi G_N \rho$. In the exterior region, $\Phi$ can be written as

$$\Phi(r, \theta, \phi) = -G_N \sum_{lm} \frac{M_{lm} Y_{lm}(\theta, \phi)}{r^{l+1}}, \quad (1)$$

where the coefficients $M_{lm}$ are the multipole moments of the gravitational field and $Y_{ml}$ are the Laplace’s spherical harmonics.
Because of the non-linear nature of the Einstein’s equations, in General Relativity it is not easy to define the counterpart of Eq. (1). However, in the special case of a stationary, axisymmetric, and asymptotically flat space-time, one can introduce something similar to Eq. (1) and define the mass-moments $M_n$ and the current-moments $S_n$. For a generic source, $M_n$ and $S_n$ are unconstrained. In the case of reflection symmetry, all the odd mass-moments and the even current-moments are identically zero. In the case of a Kerr BH, all the moments depend on $M$ and $J$ in a very specific way:

$$M_n + iS_n = M \left( \frac{iJ}{M} \right)^n,$$

where $i$ is the imaginary unit; that is, $i^2 = -1$. By measuring the mass, the spin, and at least one more non-trivial moment of the gravitational field of a BH candidate (e.g. the mass-quadrupole moment $Q = M_2 = -J^2/M$), one can test the Kerr BH hypothesis.

By considering the mean radiative efficiency of AGN, one can constrain possible deviations from the Kerr geometry. In term of the anomalous quadrupole moment $q$, defined by $Q = Q_{Kerr} - qM^3$, the bound is

$$-2.00 < q < 0.14.$$  

Let us notice that this bound is already quite interesting. Indeed, for a self-gravitating fluid made of ordinary matter, one would expect $q \sim 1 - 10$. In the case of stellar-mass BH candidates in X-ray binaries, $q$ can be potentially constrained by studying the soft X-ray component. The future detection of gravitational waves from the inspiral of a stellar-mass compact body into a super-massive object, the so-called extreme mass ratio inspiral (EMRI), will allow for putting much stronger constraints. LISA will be able to observe about $10^4 - 10^6$ gravitational wave cycles emitted by an EMRI while the stellar-mass body is in the strong field region of the supermassive object and the mass quadrupole moment of the latter will be measured with a precision at the level of $10^{-2} - 10^{-4}$.

### 3 Formation of Compact Objects with $a_* > 1$

If the current BH candidates are not the BHs predicted by General Relativity, the Kerr bound $|a_*| \leq 1$ does not hold and the maximum value of the spin parameter may be either larger or smaller than 1, depending on the metric around the compact object and on its internal structure and composition. In Ref. 8, 9, 10, 11, I studied some features of the accretion process onto objects with $|a_*| > 1$. However, an important question to address is if objects with $|a_*| > 1$ can form.

For a BH, the accretion process can spin the object up and the final spin parameter can be very close to the Kerr bound. In the case of a geometrically thin disk, the evolution of the spin parameter can be computed as follows. One assumes that the disk is on the equatorial plane and that the disk’s gas moves on nearly geodesic circular orbits. The gas particles in an accretion disk fall to the BH by loosing energy and angular momentum. After reaching the innermost stable circular orbit (ISCO), they are quickly swallowed by the BH, which changes its mass by $\delta M = \epsilon_{ISCO} \delta m$ and its spin by $\delta J = \lambda_{ISCO} \delta m$, where $\epsilon_{ISCO}$ and $\lambda_{ISCO}$ are respectively the specific energy and the specific angular momentum of a test-particle at the ISCO, while $\delta m$ is the gas rest-mass. The equation governing the evolution of the spin parameter is

$$\frac{da_*}{d\ln M} = \frac{1}{M} \frac{\lambda_{ISCO}}{\epsilon_{ISCO}} - 2a_*.$$

*For prolonged disk accretion, the timescale of the alignment of the spin of the object with the disk is much shorter than the time for the mass to increase significantly and it is correct to assume that the disk is on the equatorial plane.*
An initially non-rotating BH reaches the equilibrium $a_{*}^q = 1$ after increasing its mass by a factor $\sqrt{6} \approx 2.4^{12}$. Including the effect of the radiation emitted by the disk and captured by the BH, one finds $a_{*}^q \approx 0.998^{13}$, because radiation with angular momentum opposite to the BH spin has larger capture cross section.

As $\text{ISCO}$ and $\lambda_{\text{ISCO}}$ depend on the metric of the space-time, if the compact object is not a BH, the value of the equilibrium spin parameter $a_{*}^q$ may be different. The evolution of the spin parameter of a compact object with mass $M$, spin angular momentum $J$, and non-Kerr quadrupole moment $Q$ was studied in $^{14,15}$. In $^{15}$, I considered an extension of the Manko-Novikov-Sanabria Gómez (MMS) solution $^{16,17}$, which is a stationary, axisymmetric, and asymptotically flat exact solution of the Einstein-Maxwell’s equations. In Fig. 1, I show the evolution of the spin parameter $a_{*}$ for different values of the anomalous quadrupole moment $\tilde{q}$, defined by $Q = -(1 + \tilde{q})J^2/M$. For $\tilde{q} > 0$, the compact object is more oblate than a BH; for $\tilde{q} < 0$, the object is more prolate; for $\tilde{q} = 0$, one recovers exactly the Kerr metric. In Fig. 1, there are two curves for every value of $\tilde{q}$ because, for a given quadrupole moment $Q$, the MMS metric may have no solutions or more than one solution. In other words, two curves with the same $\tilde{q}$ represent the evolution of the spin parameter of two compact objects with the same mass, spin, and mass-quadrupole moment, but different values of the higher order moments.

As shown in Fig. 1, objects more oblate than a BH ($\tilde{q} > 0$) have an equilibrium spin parameter larger than 1. For objects more prolate than a BH ($\tilde{q} < 0$), the situation is more complicated, and $a_{*}^q$ may be either larger or smaller than 1. The origin of this fact is that for $\tilde{q} < 0$ the radius of the ISCO may be determined by the vertical instability of the orbits, while for $\tilde{q} \geq 0$ (which includes Kerr BHs) it is always determined by the radial instability.

Lastly, let us notice that Fig. 1 shows how, “in principle”, the accreting gas can spin a compact object with non-Kerr quadrupole moment up. It may happen that the compact object becomes unstable before reaching its natural equilibrium spin parameter. This depends on the internal structure and composition of the object. For example, neutron stars cannot rotate faster than about $\sim 1$ kHz, or $a_{*} \sim 0.7$. If the accretion process spins a neutron star up above its critical value, the latter becomes unstable and spins down by emitting gravitational waves. If the same thing happens to the super-massive BH candidates in galactic nuclei, they may be an unexpected source of gravitational waves for experiments like LISA.

4 Conclusions

The future gravitational wave detector LISA will be able to check if the super-massive objects at the center of most galaxies are the BHs predicted by General Relativity. A fundamental limit for a BH in General Relativity is the Kerr bound $|a_{*}| \leq 1$, which is the condition for the existence of the event horizon. If the current BH candidates are not the BHs predicted by General Relativity, the Kerr bound does not hold and the maximum value of the spin parameter may be either larger or smaller than 1. Here I showed that compact objects with $|a_{*}| > 1$ may form if they have a thin disk of accretion.

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References

1. R. Penrose, Riv. Nuovo Cim. Numero Speciale 1, 252 (1969).
2. R. Narayan, New J. Phys. 7, 199 (2005).
3. R. Hansen, J. Math. Phys. 15, 46 (1974).
4. F. D. Ryan, Phys. Rev. D 52, 5707 (1995).
5. C. Bambi, arXiv:1102.0616 [gr-qc].
6. C. Bambi and E. Barausse, Astrophys. J. 731, 121 (2011).
7. L. Barack and C. Cutler, Phys. Rev. D 75, 042003 (2007).
8. C. Bambi, et al., Phys. Rev. D 80, 104023 (2009).
9. C. Bambi, T. Harada, R. Takahashi and N. Yoshida, Phys. Rev. D 81, 104004 (2010).
10. C. Bambi and N. Yoshida, Phys. Rev. D 82, 064002 (2010).
11. C. Bambi and N. Yoshida, Phys. Rev. D 82, 124037 (2010).
12. J. M. Bardeen, Nature 226, 64 (1970).
13. K. S. Thorne, Astrophys. J. 191, 507 (1974).
14. C. Bambi, arXiv:1101.1364 [gr-qc].
15. C. Bambi, arXiv:1103.5135 [gr-qc].
16. V. S. Manko, E. W. Mielke and J. D. Sanabria-Gomez, Phys. Rev. D 61, 081501 (2000).
17. V. S. Manko, J. D. Sanabria-Gomez and O. V. Manko, Phys. Rev. D 62, 044048 (2000).