LOCAL WELL-POSEDNESS FOR THE TROPICAL CLIMATE MODEL WITH FRACTIONAL VELOCITY DIFFUSION

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Abstract. This paper deals with the Cauchy problem for tropical climate model with the fractional velocity diffusion which was derived by Frierson-Majda-Pauluis in [16]. We establish the local well-posedness of strong solutions to this generalized model.

1. Introduction. This paper considers the following tropical climate model with fractional velocity diffusion:

\begin{align*}
\partial_t u + (u \cdot \nabla) u + \Lambda^{2\alpha} u + \nabla p + \text{div}(v \otimes v) &= 0, \\
\partial_t v + (u \cdot \nabla) v + \Lambda^{2\beta} v + \nabla \theta + (v \cdot \nabla) u &= 0, \\
\partial_t \theta + (u \cdot \nabla) \theta + \text{div}v &= 0, \\
\text{div}u &= 0, \\
u(x, 0) &= u_0(x), \quad v(x, 0) = v_0(x), \quad \theta(x, 0) = \theta_0(x),
\end{align*}

where \(x \in \mathbb{R}^d\) with \(d \geq 2\), \(u = (u_1(x, t), u_2(x, t), \ldots, u_d(x, t))\) and \(v = (v_1(x, t), v_2(x, t), \ldots, v_d(x, t))\) are vector fields representing the barotropic mode and the first baroclinic mode of the velocity, respectively, while \(p = p(x, t)\) and \(\theta = \theta(x, t)\) denote the scalar pressure and temperature, respectively. \(\alpha \geq 0, \beta \geq 0\) are real parameters. We identify the ideal case \(\alpha = \beta = 0\) as the original system derived in [16] stands for none of the diffusion terms of the barotropic mode and the first baroclinic mode of the velocity. Fractional Laplacian operator \(\Lambda = (-\Delta)^{1/2}\) is defined in terms of the Fourier transform,

\[\hat{\Lambda}^\alpha f(\xi) = |\xi|^\alpha \hat{f}(\xi)\]

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The tropical climate model is derived from the primitive equations by performing a Galerkin truncation to the first baroclinic mode. In 1990s, Lions, Temam and Wang originally in [24]-[26] established the global existence of weak solutions for the viscous primitive equations (but the uniqueness is still unknow for 3D case). Meanwhile, the global existence of strong solutions for the viscous primitive equations is given in [9] and [20]. For the case of the inviscid primitive equations, in [21] and [22] the authors established the local existence of solutions on a bounded domain. [10] and [36] showed that the corresponding smooth solutions of the inviscid primitive equations blow up in finite time for certain class of initial data. Very recently, global well-posedness of strong solutions for the 2D equations (1)-(5) with $\alpha = \beta = 1$ were obtained in [23]. The problem of global well-posedness of the $n$-dimensional ($n \geq 3$) equations (1)-(5) is an outstanding challenge problem for obvious reason that it is the model includes the Navier-Stokes (or Euler when $\alpha = \beta = 0$) equations as a special case. Thus, it is a natural question to consider the local well-posedness for this model.

We survey the local existence and uniqueness of strong solutions to (1)-(5) with any $\alpha \geq 1$, $\beta \geq 0$ or $1 > \alpha \geq 0$, $\beta \geq 1$. More precisely, our main result is the following theorem.

**Theorem 1.1.** For any $\alpha \geq 1$, $\beta \geq 0$ or $1 > \alpha \geq 0$, $\beta \geq 1$. Assume $(u_0, v_0, \theta_0) \in H^s(\mathbb{R}^d)$ with $s > \max\left\{\frac{d}{2} + 1 - \alpha, 1\right\}$, and $\text{div}u_0 = 0$. Then there exists a time $T_0 = T_0(\|u_0, v_0, \theta_0\|_{H^s}) > 0$ and a unique solution $(u, v, \theta)$ of equations (1)-(5) on $[0, T_0]$ such that $(u, v, \theta) \in C([0, T_0]; H^s(\mathbb{R}^d))$.

**Remark 1.** In recent years there has been a surge of activity focused on the nonlocal (especially fractional diffusion) operators to replace the standard Laplace operator because of their connection with many real-world phenomena. The new operators do not act by point wise differentiation but by a global integration with respect to a singular kernel.

If $0 < \alpha < 2$, we can also use the integral representation

$$\Lambda^\alpha f = C_{d, \alpha} \text{P.V.} \int_{\mathbb{R}^d} \frac{g(x) - g(z)}{|x - z|^{d+\alpha}} dz,$$

where P.V. denotes principal value and $C_{d, \alpha} = \frac{2^{\alpha-1}\alpha\Gamma((d+\alpha)/2)}{\pi^{d/2}\Gamma(1-\alpha/2)}$ is a normalization constant. In the limits $C_{d, \alpha} \approx \alpha$ as $\alpha \to 0$ and $C_{d, \alpha} \approx 2 - \alpha$ as $\alpha \to 2$ it is possible to recover respectively the identity or the standard Laplacian, cf. [5].

Indeed, the interest in these fractional operators has a long history in Probability and other applied sciences. Motivation from Mechanics appears in the famous Signorini problem (with $\alpha = 1/2$) (cf. [32, 7]). And there are applications in Fluid Mechanics, (cf. [8, 19]). There is a wide literature for further motivation, fractional operators arise in the optimization [13], in finance [11], in anomalous diffusion [28], in crystal dislocation [34], in conservation laws [4], in the ultrarelativistic limit of quantum mechanics [14], in materials science [1] and in water waves [33, 12, 30].

**Remark 2.** The motivation of this paper comes from [15], [17] for MHD, because of $\text{div}v \neq 0$, we have to split the nonlinear term $\text{div}(v \otimes v)$ in (1) into two parts $(v \cdot \nabla)v$ and $\text{edive}$. Then the difficulty occurs when we estimate the high order norm of $\text{edive}$.
Remark 3. The primary idea of proving Theorem 1.1 is to suitably shift the derivatives of \( v, \theta \) in the nonlinear term to \( u \).

The rest of this paper is divided into two sections followed by an appendix. In Section 2, we review some elementary results and prove the result for the local \textit{a priori} estimates. In Section 3, we complete the proof of Theorem 1.1. The definition and related proposition of Littlewood-Paley decomposition Besov spaces used in this paper are provided in the appendix.

2. Preliminaries and local \textit{a priori} estimates. In this section we recall some elementary results which will be used in this paper. Then we establish a local \textit{a priori} estimates for smooth solutions of (1)-(5), which play a key role in the proof of Theorem 1.1.

The following lemma is about Fourier truncation estimates, where the inequalities are proved in [15].

Lemma 2.1. Let \( S_R \) be the Fourier truncation operator and it is defined as follows:

\[
\hat{S_R}(\xi) = 1_{B_R}(\xi) \hat{f}(\xi) = \begin{cases} 
\hat{f}(\xi), & |\xi| \leq R, \\
0, & |\xi| > R,
\end{cases}
\]

where \( B_R \) denotes the closed ball of radius \( R \) centered at 0 and \( 1_{B_R} \) denotes the characteristic functions on \( B_R \). Then the following estimations are satisfied

\[
\| S_R f - f \|_{H^s} \leq C \frac{1}{R^k} \| f \|_{H^{s+k}},
\]

and

\[
\| S_R f - S_R f \|_{H^s} \leq C \max \left\{ \frac{1}{R^k}, \frac{1}{R^k} \right\} \| f \|_{H^{s+k}}.
\]

For self-consistency of our paper, we will give another proof of the Lemma 2.2 in [17] by the Littlewood-Paley decomposition and Besov space techniques in the appendix.

Lemma 2.2. (Commutator Estimates) Let \( \alpha \geq 0 \), for any \( s > \max \left\{ \frac{d}{2} + 1 - \alpha, 1 \right\}, \) we have

\[
|\langle \Lambda^s ([u \cdot \nabla] v) - (u \cdot \nabla) \Lambda^s v, \Lambda^s u \rangle| \leq C \| u \|_{H^{s+\alpha}} \| v \|_{H^s}^2,
\]

and

\[
|\langle \Lambda^s ([v \cdot \nabla] v) - (v \cdot \nabla) \Lambda^s u, \Lambda^s u \rangle| \leq C \| u \|_{H^{s+\alpha}} \| v \|_{H^s}^2.
\]

Here \( \langle \cdot, \cdot \rangle \) denotes the \( L^2(\mathbb{R}^d) \) inner product and \( C \) is a constant depending on \( d, s \) only.

The result for the local \textit{a priori} bound can be stated as follows

Proposition 1. For any \( \alpha \geq 1, \beta \geq 0 \) or \( 1 > \alpha \geq 0, \beta \geq 1 \). Assume the initial data \((u_0, v_0, \theta_0) \in H^s(\mathbb{R}^d)\) with \( s > \max \left\{ \frac{d}{2} + 1 - \alpha, 1 \right\} \). Let \((u, v, \theta)\) be the corresponding solution of equations (1)-(5). Then, there exists a time \( T_0 = T_0(\| (u_0, v_0, \theta_0) \|_{H^s}) > 0 \) such that

\[
\sup_{t \in [0, T_0]} \left( \| (u(t)) \|_{H^s} + \| v(t) \|_{H^s} + \| \theta(t) \|_{H^s} \right) \leq C(\alpha, T_0, \| (u_0, v_0, \theta_0) \|_{H^s})
\]

and

\[
\int_0^{T_0} \left( \| u(t) \|_{H^{s+\alpha}}^2 + \| v(t) \|_{H^{s+\beta}}^2 \right) ds \leq C(\alpha, T_0, \| (u_0, v_0, \theta_0) \|_{H^s}).
\]
Proof of Proposition 1. We first prove the basic energy estimate. Taking the $L^2(\mathbb{R}^d)$ inner product to equations (1), (2), (3) with $(u, v, \theta)$, after integrating by parts and taking the divergence free property into account, we have
\[
\frac{1}{2} \frac{d}{dt} \left( \|u\|_{L^2}^2 + \|v\|_{L^2}^2 + \|\theta\|_{L^2}^2 \right) + \|\Lambda^\alpha u\|_{L^2}^2 + \|\Lambda^\beta v\|_{L^2}^2 = 0. \tag{10}
\]

Next, we prove the $H^s$ estimate. Applying the operator $\Lambda^s$ to (1), (2), (3) and taking the $L^2(\mathbb{R}^d)$ inner product to the resultants with $(\Lambda^s u, \Lambda^s v, \Lambda^s \theta)$, after integrating by parts and using the cancellation property
\[
\langle \Lambda^s \theta, \text{div} \Lambda^s v \rangle - \langle \Lambda^s \text{div} v, \Lambda^s \theta \rangle = 0,
\]
we have
\[
\frac{1}{2} \frac{d}{dt} \left( \|\Lambda^s u\|_{L^2}^2 + \|\Lambda^s v\|_{L^2}^2 + \|\Lambda^s \theta\|_{L^2}^2 \right) + \|\Lambda^{s+\alpha} u\|_{L^2}^2 + \|\Lambda^{s+\beta} v\|_{L^2}^2 = I_1 + I_2 + I_3 + I_4. \tag{11}
\]

where
\[
I_1 = -\langle \Lambda^s [(u \cdot \nabla)] u, \Lambda^s u \rangle, \quad I_2 = -\langle \Lambda^s [(u \cdot \nabla)] v, \Lambda^s v \rangle, \quad I_3 = -\langle \Lambda^s [(u \cdot \nabla)] \theta, \Lambda^s \theta \rangle, \quad I_4 = -\langle \Lambda^s [(v \cdot \nabla)] v, \Lambda^s u \rangle - \langle \Lambda^s [(v \cdot \nabla)] u, \Lambda^s v \rangle - \langle \Lambda^s [\text{div} v], \Lambda^s u \rangle.
\]

Using the fact that $u$ is divergence free, we have
\[
\langle (u \cdot \nabla) \Lambda^s u, \Lambda^s u \rangle = 0, \quad \langle (u \cdot \nabla) \Lambda^s v, \Lambda^s v \rangle = 0, \quad \langle (u \cdot \nabla) \Lambda^s \theta, \Lambda^s \theta \rangle = 0, \quad \langle (u \cdot \nabla) \Lambda^s \theta, \Lambda^s \theta \rangle = 0,
\]
combining with commutator estimate (8) and (9) yields
\[
|I_1| = |\langle \Lambda^s [(u \cdot \nabla)] u, (u \cdot \nabla) \Lambda^s u, \Lambda^s u \rangle| \leq C \|u\|_{H^{s+\alpha}} \|u\|_{H^s}^2,
|I_2| = |\langle \Lambda^s [(u \cdot \nabla)] v, (u \cdot \nabla) \Lambda^s v, \Lambda^s v \rangle| \leq C \|u\|_{H^{s+\beta}} \|v\|_{H^s}^2,
|I_3| = |\langle \Lambda^s [(u \cdot \nabla)] \theta, (u \cdot \nabla) \Lambda^s \theta, \Lambda^s \theta \rangle| \leq C \|u\|_{H^{s+\alpha}} \|\theta\|_{H^s}^2,
\]
and
\[
|I_4| = |\langle \Lambda^s [(v \cdot \nabla)] v, (v \cdot \nabla) \Lambda^s v, \Lambda^s u \rangle + \langle \Lambda^s [(v \cdot \nabla)] u, (v \cdot \nabla) \Lambda^s u, \Lambda^s v \rangle + \langle \Lambda^s [\text{div} v], \text{div} \Lambda^s v, \Lambda^s u \rangle| \leq I_{41} + I_{42} + I_{43},
\]
where
\[
I_{41} = |\langle \Lambda^s [(v \cdot \nabla)] v, (v \cdot \nabla) \Lambda^s v, \Lambda^s u \rangle|, \quad I_{42} = |\langle \Lambda^s [(v \cdot \nabla)] u, (v \cdot \nabla) \Lambda^s u, \Lambda^s v \rangle|, \quad I_{43} = |\langle \Lambda^s [\text{div} v], \text{div} \Lambda^s v, \Lambda^s u \rangle|.
\]
The estimate of $I_{41}$ is given by (9).

We recall the Kato-Ponce type commutator estimate (refer to [18] for details)
\[
\|\Lambda^s (fg) - f \Lambda^s g\|_{L^p} \leq C(\|\nabla f\|_{L^{q_1}} \|\Lambda^{s-1} g\|_{L^{r_1}} + \|\Lambda^s f\|_{L^{q_2}} \|g\|_{L^{r_2}}),
\]
where $\frac{1}{p} = \frac{1}{q_i} + \frac{1}{r_i}$, $i = 1, 2$ and $p_i, q_i, r_i \in [1, \infty]$.

Using the Kato-Ponce type commutator estimate together with the Sobolev inequality, the $I_{42}$ can be estimated as follows
\[
I_{42} = \|\Lambda^s [(v \cdot \nabla)] u, (v \cdot \nabla) \Lambda^s u\|_{L^2} \|\Lambda^s v\|_{L^2} \leq C(\|\nabla v\|_{L^{q_1}} \|\Lambda^s u\|_{L^{r_1}} + \|\Lambda^s v\|_{L^2} \|\nabla u\|_{L^{r_2}}) \|\Lambda^s v\|_{L^2} \leq C(\|v\|_{H^{s+\alpha}} \|\Lambda^s u\|_{H^s} + \|v\|_{H^{s+\alpha}} \|\Lambda^s v\|_{H^{s+\beta}}) \|\Lambda^s v\|_{L^2} \leq C \|u\|_{H^{s+\alpha}} \|v\|_{H^s}^2.
\]
Here we used the following Sobolev embedding

\[ H^\alpha \rightarrow L^{r_1}, \quad \frac{1}{r_1} + \frac{1}{q_1} = \frac{1}{2}, \quad \begin{cases} r_1 = \frac{2d}{d+2\alpha}, & q_1 = \frac{d}{\alpha}, \quad \text{if } \alpha < \frac{d}{2}, \\ r_1 > \frac{d}{\alpha}, & q_1 < \frac{2d}{d+2\alpha}, \quad \text{if } \alpha = \frac{d}{2}, \\ r_1 = \infty, & q_1 = 2, \quad \text{if } \alpha > \frac{d}{2}, \end{cases} \]

(12)

and the following Gagliardo-Nirenberg interpolation inequalities

\[ \|\nabla v\|_{L^q} \leq C\|v\|_{L^2}^{1-\lambda_1}\|\Lambda^s v\|_{L^2}^{\lambda_1}, \quad \begin{cases} \lambda_1 = \frac{d/2+1-\alpha}{s}, & \text{if } \alpha < \frac{d}{2}, \\ \lambda_1 = \frac{d/\alpha+1}{s} < 1, & \text{if } \alpha = \frac{d}{2}, \end{cases} \]

(13)

The most difficult term \( I_{43} \) can be split into three parts

\[
I_{43} = |\langle \Lambda^s [v \text{div} v] - \text{div} \Lambda^s v, \Lambda^s u \rangle| \\
\leq |\langle \Lambda^s [v \text{div} v] - v \Lambda^s \text{div} v, \Lambda^s u \rangle| + |\langle \text{div} \Lambda^s v, \Lambda^s u \rangle| + |\langle v \Lambda^s \text{div} v, \Lambda^s u \rangle| \\
:= I_{431} + I_{432} + I_{433}.
\]

Similar to (9), \( I_{431} \) can be estimated as follows

\[
I_{431} = |\langle \Lambda^s [v \text{div} v] - v \Lambda^s \text{div} v, \Lambda^s u \rangle| \\
\leq C\|u\|_{H^{s+\alpha}}\|v\|_{H^s}^2.
\]

Using the Sobolev inequality (12), Gagliardo-Nirenberg interpolation inequalities (13), we have

\[
I_{432} = |\langle \text{div} \Lambda^s v, \Lambda^s u \rangle| \\
\leq C\|\text{div} \Lambda^s v\|_{L^{p_3}}\|\Lambda^s u\|_{L^{q_3}} \\
\leq C\|\nabla v\|_{L^{p_3}}\|\Lambda^s v\|_{L^2}\|\Lambda^s u\|_{H^s} \\
\leq C\|u\|_{H^{s+\alpha}}\|v\|_{H^s}^2,
\]

(14)

where

\[
\frac{1}{p_3} + \frac{1}{q_3} = 1, \quad \frac{1}{p_3} = \frac{1}{r_3} + \frac{1}{2} \quad \text{with}
\]

\[
\begin{cases} p_3 = \frac{2d}{d+2\alpha}, & q_3 = \frac{2d}{d+2\alpha}, \quad r_3 = \frac{d}{\alpha}, \quad \text{if } \alpha < \frac{d}{2}, \\ p_3 > \frac{2d}{d+2\alpha-2}, & q_3 < \frac{2d}{d+2\alpha-2}, \quad r_3 > \frac{d}{\alpha-1}, \quad \text{if } \alpha = \frac{d}{2}, \\ p_3 = 1, & q_3 = \infty, \quad r_3 = 2, \quad \text{if } \alpha > \frac{d}{2}. \end{cases}
\]

When \( \alpha \geq 1 \), integration by part, we have

\[
I_{433} = |\langle v \Lambda^s \text{div} v, \Lambda^s u \rangle| \\
\leq \sum_{i,j=1}^{d} |\langle \partial_i v_j \Lambda^s v_i, \Lambda^s u_j \rangle| + \sum_{i,j=1}^{d} |\langle v_j \Lambda^s v_i, \Lambda^s \partial_i u_j \rangle|,
\]

similar to (14), the first term can be estimated as follows

\[
\sum_{i,j=1}^{d} |\langle \partial_i v_j \Lambda^s v_i, \Lambda^s u_j \rangle| \\
\leq C\|u\|_{H^{s+\alpha}}\|v\|_{H^s}^2.
\]
and the second term can be estimated as follows
\[
\sum_{i,j=1}^{d} |\langle v_j \Lambda^s v_i, \Lambda^s \partial_i u_j \rangle| \\
\leq C \| v \otimes \Lambda^s v \|_{L^4} \| \Lambda^s \nabla u \|_{L^4} \\
\leq C \| v \|_{L^4} \| \Lambda^s v \|_{L^2} \| \Lambda^s \nabla u \|_{H^{s-1}} \\
\leq C \| u \|_{H^{s+\alpha}} \| v \|_{H^{s+\beta}}.
\]
where we have used the Sobolev inequality (12), Gagliardo-Nirenberg interpolation inequalities
\[
\| v \|_{L^4} \leq C \| v \|_{L^2}^{1-\lambda_3} \| \Lambda^s v \|_{L^2}^{\lambda_3}
\]
with
\[
\begin{align*}
\lambda_3 &= \frac{d/2+1-\alpha}{s}, \quad \text{if } \alpha - 1 < \frac{d}{2}, \\
\lambda_3 &= \frac{d}{2s}, \quad \text{if } \alpha - 1 = \frac{d}{2},
\end{align*}
\]
and
\[
\frac{1}{p_4} + \frac{1}{q_4} = 1, \quad \frac{1}{p_4} = \frac{1}{r_4} + \frac{1}{2}
\]
with
\[
\begin{align*}
p_4 &= \frac{2d}{d+2(1-\alpha)}, \quad q_4 = \frac{2d}{d-2(\alpha-1)}, \quad r_4 = \frac{d}{(\alpha-1)}, \quad \text{if } \alpha - 1 < \frac{d}{2}, \\
p_4 &= \frac{2d}{d+2,} \quad q_4 < \frac{2d}{d-2}, \quad r_4 > \frac{d}{s}, \quad \text{if } \alpha - 1 = \frac{d}{2}, \\
p_4 &= 1, \quad q_4 = \infty, \quad r_4 = 2, \quad \text{if } \alpha - 1 > \frac{d}{2}.
\end{align*}
\]
When \(1 > \alpha \geq 0, \beta \geq 1\), we have
\[
I_{13} = \langle (v \Lambda^s \text{div}, \Lambda^s u) \rangle \\
\leq \| v \Lambda^s \text{div} \|_{L^2} \| \Lambda^s u \|_{L^2} \\
\leq C \| v \|_{L^\infty} \| \Lambda^s \text{div} \|_{L^2} \| \Lambda^s u \|_{L^2} \\
\leq C \| v \|_{H^s} \| v \|_{H^{s+\beta}} \| u \|_{H^s},
\]
\[
\leq C \| v \|_{H^{s+\beta}} (\| v \|_{H^s}^2 + \| u \|_{H^s}^2).
\]
Inserting all the estimates above in (11) and combining this with basic energy estimate (10), we obtain
\[
\frac{1}{2} \frac{d}{dt} \| u \|_{H^s}^2 + \| v \|_{H^s}^2 + \| \theta \|_{H^s}^2 + \| \Lambda^{s+\alpha} u \|_{L^2}^2 + \| \Lambda^{s+\alpha} v \|_{L^2}^2 \\
\leq C (\| u \|_{H^{s+\alpha}} + \| v \|_{H^{s+\beta}}) (\| u \|_{H^s}^2 + \| v \|_{H^s}^2 + \| \theta \|_{H^s}^2),
\]
(15)
Using Young’s inequality and Gronwall’s inequality, we deduce that
\[
\| u(t) \|_{H^s}^2 + \| v(t) \|_{H^s}^2 + \| \theta(t) \|_{H^s}^2 \leq \frac{\| u_0 \|_{H^s}^2 + \| v_0 \|_{H^s}^2 + \| \theta_0 \|_{H^s}^2}{1 - CT (\| u_0 \|_{H^s}^2 + \| v_0 \|_{H^s}^2 + \| \theta_0 \|_{H^s}^2)}
\]
for all \(t \in [0, T]\). So we can choose \(T_0 \). Thus we obtain
\[
\int_0^{T_0} \| (u(t), v(t), \theta(t)) \|_{H^s} \leq C (\alpha, T_0, \| (u_0, v_0, \theta_0) \|_{H^s})
\]
and
\[
\int_0^{T_0} \| (u(t), v(t), \theta(t)) \|_{H^{s+\beta}} \leq C (\alpha, T_0, \| (u_0, v_0, \theta_0) \|_{H^s}).
\]
This completes the proof of Proposition 1.

From (15) we can immediately obtain the following results.
Corollary 1. (Blow-up Criterion) Let \( s > \max \left\{ \frac{d}{2} + 1 - \alpha, 1 \right\} \), \( \alpha \geq 1 \), \( \beta \geq 0 \) or \( 1 > \alpha \geq 0 \), \( \beta \geq 1 \) and initial data \( (u_0, v_0, \theta_0) \in H^s(\mathbb{R}^d) \) with \( \text{div} u_0 = 0 \). Then for the first blow-up time \( T_0 < \infty \) of the classical solution to equations (1)-(5), its holds that
\[
\limsup_{t \to T_0} (\|u(t)\|_{H^s} + \|v(t)\|_{H^s} + \|\theta(t)\|_{H^s}) = \infty,
\]
if and only if
\[
\int_0^{T_0} (\|u(t)\|_{H^s}^2 + \|v(t)\|_{H^s}^2 + \|\theta(t)\|_{H^s}^2) \, dt = \infty,
\]
or
\[
\int_0^{T_0} \|u(t)\|_{H^{s+\beta}(\mathbb{R}^d)} + \|v\|_{H^{s+\beta}(\mathbb{R}^d)} \, dt = \infty.
\]

Remark 4. The blow-up criteria is useful in proving global existence and other blow-up criteria. According to the Corollary, it is enough to give the \( m \)-th order estimation for \( m > \frac{d}{2} + 1 - \alpha \).

3. Local existence and uniqueness. In this section, we prove Theorem 1.1 through an approximation procedure that is given by Friedrichs.

Now, we consider the following truncated tropical climate model
\[
\begin{align*}
\partial_t u^R + S_R[(u^R \cdot \nabla)u^R] + \Lambda^2 u^R + \nabla p^R + S_R[\text{div}(v^R \otimes v^R)] &= 0, \\
\partial_t v^R + S_R[(u^R \cdot \nabla)v^R] + \Lambda^2 v^R + \nabla \theta^R + S_R[(v^R \cdot \nabla)u^R] &= 0, \\
\partial_t \theta^R + S_R[(u^R \cdot \nabla)\theta^R] + \text{div} v^R &= 0, \\
\text{div} u^R &= 0,
\end{align*}
\]
\[
u^R(x, 0) = S_R u_0(x), \quad v^R(x, 0) = S_R v_0(x), \quad \theta^R(x, 0) = S_R \theta_0(x).
\]
We introduce the function space
\[
V^s(\mathbb{R}^d) \equiv \left\{ f \in H^s(\mathbb{R}^d); \text{div} f = 0, \text{ supp } \hat{f} \subset B_R \right\},
\]
\[
W^s(\mathbb{R}^d) \equiv \left\{ f \in H^s(\mathbb{R}^d); \text{ supp } \hat{f} \subset B_R \right\}.
\]
Set
\[
X^R = (u^R, v^R, \theta^R) \in V^s(\mathbb{R}^d) \times W^s(\mathbb{R}^d) \times W^s(\mathbb{R}^d).
\]
Thus we can reduce the truncated tropical climate model (16)-(20) to the following abstract ODE in the Banach space \( V^s(\mathbb{R}^d) \times W^s(\mathbb{R}^d) \times W^s(\mathbb{R}^d) \)
\[
\begin{align*}
\frac{d}{dt} X^R &= F(X^R), \\
(u^R, v^R, \theta^R)(x, 0) &= (S_R u_0, S_R v_0, S_R \theta_0),
\end{align*}
\]
(21)
where
\[
F(X^R) = \begin{pmatrix}
-S_R[(u^R \cdot \nabla)u^R] - \Lambda^2 u^R - \nabla p^R - S_R[\text{div}(v^R \otimes v^R)] \\
-S_R[(u^R \cdot \nabla)v^R] - \Lambda^2 v^R - \nabla \theta^R - S_R[(v^R \cdot \nabla)u^R] \\
-S_R[(u^R \cdot \nabla)\theta^R] - \text{div} v^R
\end{pmatrix}.
\]
(22)
For each fixed \( R \), Using (6), (7) and the fact of \( \|S_R f\|_{H^s} \leq C \|f\|_{L^2} \), we can verify
\[
\|F(X^R_1) - F(X^R_2)\|_{H^s} \leq C(\|X_0\|_{L^2}) \|X^R_1 - X^R_2\|_{H^s},
\]
(23)
Therefore, $F$ is locally Lipschitz continuous. Thus by Picard’s theorem (see Theorem 3.1 in Majda & Bertozzi [27](2002), for example), there exists a unique solution $X^R \in C^1([0,T^R]; V^*(\mathbb{R}^d) \times W^*(\mathbb{R}^d) \times W^*(\mathbb{R}^d))$, for some $T^R$ depending on $R$.

We denote the solution of (16)-(20) by $(u^R, v^R, \theta^R)$. Similarly as the proof of Proposition 1 and due to $\|(u^R, v^R, \theta^R)\|_{H^*} \leq \|(u_0, v_0, \theta_0)\|_{H^*}$, we can obtain a uniform local bounds of the solution as follows

$$\sup_{t \in [0,T_0]} \left( \|u^R(t)\|_{H^*}^2 + \|v^R(t)\|_{H^*}^2 + \|\theta^R(t)\|_{H^*}^2 \right) \leq C(\alpha, T_0, \|(u_0, v_0, \theta_0)\|_{H^*})$$

and

$$\int_0^{T_0} \left( \|u^R(t)\|_{H^{*+\beta}}^2 + \|v^R(t)\|_{H^{*+\beta}}^2 \right) ds \leq C(\alpha, T_0, \|(u_0, v_0, \theta_0)\|_{H^*}).$$

The above inequalities are uniform in $R$.

Furthermore, these uniform bounds allow us to show that $(u^R, v^R, \theta^R)$ converge strongly in $L^\infty([0,T_0]; L^2(\mathbb{R}^d))$, and one has the following proposition.

**Proposition 2.** The family $(u^R, v^R, \theta^R)$ of solutions of equations (16)-(20) are Cauchy sequence (as $R \to \infty$) in $L^\infty([0,T_0]; L^2(\mathbb{R}^d))$.

**Proof of Proposition 2.** Taking the difference between the equations (16)-(18) for $R$ and $R'$, we have

$$\partial_t (u^R - u^{R'}) = -S_R[(u^R \cdot \nabla)u^R] + S_R[(u^{R'} \cdot \nabla)u^{R'}] - \Lambda^{2\alpha} (u^R + \Lambda^{2\alpha} u^{R'}) - \nabla \rho^R + \nabla \rho^{R'} - S_R[(v^R \cdot \nabla)u^R] + S_R[(v^{R'} \cdot \nabla)u^{R'}]$$

$$- \nabla \theta^R + \nabla \theta^{R'} - S_R[(v^R \cdot \nabla)v^R] + S_R[(v^{R'} \cdot \nabla)v^{R'}],$$

$$\partial_t (v^R - v^{R'}) = -S_R[(u^R \cdot \nabla)v^R] + S_R[(u^{R'} \cdot \nabla)v^{R'}] - \Lambda^{2\beta} (v^R + \Lambda^{2\beta} v^{R'}) - \nabla \theta^R + \nabla \theta^{R'} - S_R[(v^R \cdot \nabla)v^R] + S_R[(v^{R'} \cdot \nabla)v^{R'}],$$

$$\partial_t (\theta^R - \theta^{R'}) = -S_R[(u^R \cdot \nabla)\theta^R] + S_R[(u^{R'} \cdot \nabla)\theta^{R'}] - \nabla v^R + \nabla v^{R'}.$$

Taking the $L^2(\mathbb{R}^d)$ inner product to equations (24)-(26) with $u^R - u^{R'}, v^R - v^{R'}$ and $\theta^R - \theta^{R'}$ respectively, summing the resultants up and using the cancellation property

$$\langle \nabla (\theta^R - \theta^{R'}), v^R - v^{R'} \rangle + \langle \nabla (v^R - v^{R'}), \theta^R - \theta^{R'} \rangle = 0,$$

we have

$$\frac{1}{2} \frac{d}{dt} \left( \|u^R - u^{R'}\|_{L^2}^2 + \|v^R - v^{R'}\|_{L^2}^2 + \|\theta^R - \theta^{R'}\|_{L^2}^2 \right) = -\langle S_R[(u^R \cdot \nabla)u^R] - S_R[(u^{R'} \cdot \nabla)u^{R'}], u^R - u^{R'} \rangle$$

$$- \langle S_R[(v^R \cdot \nabla)v^R] - S_R[(v^{R'} \cdot \nabla)v^{R'}], v^R - v^{R'} \rangle$$

$$- \langle S_R[(u^R \cdot \nabla)\theta^R] - S_R[(u^{R'} \cdot \nabla)\theta^{R'}], \theta^R - \theta^{R'} \rangle$$

$$= -\langle S_R[(u^R \cdot \nabla)u^R] - S_R[(u^{R'} \cdot \nabla)u^{R'}], u^R - u^{R'} \rangle$$

$$- \langle S_R[(v^R \cdot \nabla)v^R] - S_R[(v^{R'} \cdot \nabla)v^{R'}], v^R - v^{R'} \rangle$$

$$- \langle S_R[(u^R \cdot \nabla)\theta^R] - S_R[(u^{R'} \cdot \nabla)\theta^{R'}], \theta^R - \theta^{R'} \rangle$$

$$:= \sum_{i=1}^{6} I_i.$$
We can split each term into three parts and without loss of generality, we assume \( R > R' \geq 1 \). For \( I_4 \), we have
\[
I_4 = - \langle S_R[(u^R \cdot \nabla)v^R] - S_{R'}[(u^{R'} \cdot \nabla)v^{R'}], v^R - v^{R'} \rangle \\
= - \langle (S_R - S_{R'})[(u^R \cdot \nabla)v^R], v^R - v^{R'} \rangle \\
- \langle S_{R'}[(u^R - u^{R'}) \cdot \nabla)v^R], v^R - v^{R'} \rangle \\
- \langle S_{R'}[(u^{R'} \cdot \nabla)(v^R - v^{R'}), v^R - v^{R'} \rangle \\
:= \sum_{i=1}^{3} I_{4i}.
\]
For the first term \( I_{41} \)
\[
|I_{41}| = |\langle (S_R - S_{R'})[(u^R \cdot \nabla)v^R], v^R - b^{R'} \rangle| \\
\leq \frac{1}{R^4} \|(u^R \cdot \nabla)v^R\|_{H^3} \|v^R - v^{R'}\|_{L^2} \\
\leq \frac{1}{R^4} \|u^R\|_{H^{3+\alpha}} \|v^R\|_{H^3} \|v^R - v^{R'}\|_{L^2} \\
\leq \frac{1}{R^4} \|u^R\|_{H^{1+\alpha}}^2 \|v^R - v^{R'}\|_{L^2}^2,
\]
here we have used the following inequality (refer to [6] for details)
\[
\|fg\|_{H^{\lambda-1}} \leq C(\|f\|_{\infty} \|g\|_{H^{\lambda-1}} + \|g\|_{2\lambda}(\|f\|_{H^{\lambda}} + \|f\|_{\infty})), \lambda > 1.
\]
Let \( \lambda = 1 + \epsilon \), \( f = u^R \), \( g = \nabla v^R \), we have
\[
\|(u^R \cdot \nabla)v^R\|_{H^r} \leq C(\|u^R\|_{\infty} \|v^R\|_{H^{1+\epsilon}} + \|\nabla v^R\|_{2(1+\epsilon)}(\|u^R\|_{H^{1+\epsilon}} + \|u^R\|_{\infty})).
\]
Then use the Gagliardo-Nirenberg inequalities
\[
\|\nabla v\|_{L^p(1+\epsilon)} \leq C\|v\|^{\frac{1}{2} - \theta_3}\|\Lambda^\alpha v\|_{L^2}^{\theta_3},
\]
\[
12(1+\epsilon) = \frac{1}{d} + \left( \frac{1}{2} - \frac{s}{d} \right) \theta_3 + \frac{1 - \theta_3}{2}, \quad 0 \leq \theta_3 \leq 1,
\]
\[
\theta_3 = \frac{1 + \frac{d-s}{2}}{d},
\]
here we need \( s > 1 \) and \( \epsilon \) suitably small.
Using the Sobolev inequality (12), Gagliardo-Nirenberg interpolation inequalities (13) and Young’s inequality, we have
\[
|I_{42}| = |\langle S_{R'}[(u^R - u^{R'}) \cdot \nabla)v^R], v^R - v^{R'} \rangle| \\
\leq \|(u^R - u^{R'}) \cdot \nabla)v^R\|_{L^2} \|v^R - v^{R'}\|_{L^2} \\
\leq \|u^R - u^{R'}\|_{L^{p_1}} \|\nabla v^R\|_{L^{q_1}} \|v^R - v^{R'}\|_{L^2} \\
\leq \|u^R - u^{R'}\|_{H^{\alpha}} \|v^R\|_{H^3} \|v^R - v^{R'}\|_{L^2} \\
\leq \frac{1}{12} \|u^R - u^{R'}\|_{H^{\alpha}}^2 + C \|v^R\|_{H^3}^2 \|v^R - v^{R'}\|_{L^2}^2,
\]
where
\[
\frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{2}, \quad p_1 = \frac{2d}{d-2\alpha}, q_1 = \frac{d}{2}, \quad \text{if} \ \alpha < \frac{d}{2},
\]
\[
p_1 > \frac{d}{2}, q_1 < \frac{2d}{d-2\alpha+2}, \quad \text{if} \ \alpha = \frac{d}{2},
\]
\[
p_1 = \infty, q_1 = 2, \quad \text{if} \ \alpha > \frac{d}{2}.
\]
After integrating by parts and taking the divergence free property into account, we have \( I_{43} = 0 \).
Hence we obtain that

$$I_4 \leq \frac{1}{12} \| u^R - u^{R'} \|^2_{H^s} + \frac{1}{R^6} \| v^R \|^2_{H^s} + C \| u^R \|^2_{H^{s+\alpha}} \| v^R - v^{R'} \|^2_{L^2} + C \| u^R \|^2_{H^s} \| u^R - u^{R'} \|^2_{L^2}.$$  

Similarly, the $J_1$ and $J_2$ can be estimated as follows

$$I_1 \leq \frac{1}{12} \| u^R - u^{R'} \|^2_{H^s} + \frac{1}{R^6} \| u^R \|^2_{H^s} + C \| u^R \|^2_{H^{s+\alpha}} \| u^R - u^{R'} \|^2_{L^2},$$

$$I_2 \leq \frac{1}{12} \| u^R - u^{R'} \|^2_{H^s} + \frac{1}{R^6} \| v^R \|^2_{H^s} + C \| u^R \|^2_{H^{s+\alpha}} \| v^R - v^{R'} \|^2_{L^2} + C \| v^R \|^2_{H^s} \| v^R - v^{R'} \|^2_{L^2},$$

Now, we estimate other terms

$$I_2 + I_3 + I_5 = - \langle (S_R - S_{R'})[(v^R \cdot \nabla)v^R], u^R - u^{R'} \rangle$$

$$- \langle S_{R'}[(v^R - v^{R'}) \cdot \nabla)v^R], u^R - u^{R'} \rangle$$

$$- \langle S_{R'}[(v^R') \cdot \nabla](v^R - v^{R'}), u^R - u^{R'} \rangle$$

$$- \langle (S_R - S_{R'})[v^R \div v^R], u^R - u^{R'} \rangle$$

$$- \langle (S_{R'})(v^R - v^{R'})\div(v^R - v^{R'}), u^R - u^{R'} \rangle$$

$$= \sum_{i=1}^3 I_{2i} + \sum_{i=1}^3 I_{3i} + \sum_{i=1}^3 I_{5i},$$

where

$$|I_{21}| = |\langle (S_R - S_{R'})[(v^R \cdot \nabla)v^R], u^R - u^{R'} \rangle|$$

$$= \int_{\mathbb{R}^3} |\xi|^{-\alpha} (S_R - S_{R'})[(v^R \cdot \nabla)v^R]|\xi|^\alpha u^R - u^{R'} d\xi$$

$$\leq |||\xi|^{-\alpha} (S_R - S_{R'})[(v^R \cdot \nabla)v^R]|\xi|^\alpha u^R - u^{R'} ||_{L^2}$$

$$\leq \|\|\|\xi|^{-\alpha} (S_R[(v^R \cdot \nabla)v^R] - (v^R \cdot \nabla)v^R)]\|_{H^s} \| u^R - u^{R'} \|_{H^s}$$

$$\leq \|\|\|\xi|^{-\alpha} (S_R[(v^R \cdot \nabla)v^R] - (v^R \cdot \nabla)v^R)]\|_{L^2} \| u^R - u^{R'} \|_{H^s}$$

$$\leq \frac{C}{R^6} \|\|\|\xi|^\alpha (v^R \cdot \nabla)v^R||_{L^{\frac{2}{1+2\alpha}}} \| u^R - u^{R'} \|_{H^s}$$

$$\leq \frac{C}{R^6} \|\|\|\Lambda^\epsilon[(v^R \cdot \nabla)v^R]|\|_{L^{\frac{2}{1+2\alpha}}} \| u^R - u^{R'} \|_{H^s}$$

$$\leq \frac{C}{R^6} \| v^R \|_{H^s}^2 \| u^R - u^{R'} \|_{H^s}$$

$$\leq \frac{C}{R^6} \| v^R \|_{H^s}^4 + \frac{1}{12} \| u^R - u^{R'} \|_{H^s}^2.$$
Here choosing $\epsilon > 0$ so small such that $0 < 1 + \epsilon < s$, and we have used Gagliardo-Nirenberg interpolation inequalities (13) and the following estimates

$$\|\Lambda' (fg)\|_{L^{p'}} \leq C (\|f\|_{L^{p_1}} \|\Lambda' g\|_{L^{p_1'}} + \|\Lambda' f\|_{L^{q_1'}} \|g\|_{L^{q_1'}}),$$

with $s > 0, 1 + \frac{1}{p'} = \frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{q_2}, p', q', q_1' \in [1, \infty]$, and

$$\|\|\xi\|^{-\alpha} (S_R (u^R \cdot \nabla) u^R) - (v^R \cdot \nabla) v^R)\|_{L^2} \leq C \left( \int_{B_R} |\xi|^{-2\alpha - 2} |\xi|^{2s} |(v^R \cdot \nabla) v^R|^2 d\xi \right)^{\frac{1}{2}} \leq C \left( \int_{B_R} |\xi|^{-q_1(\alpha + \epsilon)} d\xi \right)^{\frac{1}{2}} \|\xi (v^R \cdot \nabla) v^R\|_{L^{p_2}} \leq \frac{C}{R^2} \|\Lambda' [(v^R \cdot \nabla) v^R]\|_{L^{p_2}},$$

where we have used the Hausdorff-Young inequality, and choosing $\epsilon > 0$ so small such that

$$\|\Lambda'^{s+1} v\|_{L^{p_2}} \leq C \|v\|_{L^{p_2}}^{1-s} \|\Lambda^{s} v\|_{L^{p_2}}^{s} \text{ with } \begin{cases} 
\lambda_2 = \frac{d + 1 - \alpha \epsilon}{s} < 1, & \text{if } \alpha < \frac{d}{2}, \\
\lambda_2 = \frac{d + 1 + \epsilon}{s} < 1, & \text{if } \alpha = \frac{d}{2}, \\
\end{cases}$$

with

$$\frac{1}{p_2} + \frac{1}{q_2} = \frac{1}{2}, \quad \begin{cases} 
p_2 = \frac{2d}{d - 2\alpha}, & q_2 = d, & \text{if } \alpha < \frac{d}{2}, \\
p_2 > \frac{d}{s - 1}, & q_2 < \frac{2d}{s - 2\alpha}, & \text{if } \alpha = \frac{d}{2}, \\
p_2 = \infty, & q_2 = 2, & \text{if } \alpha > \frac{d}{2}. 
\end{cases}$$

Similarly, the $I_{31}$ and $I_{51}$ can be estimated as follows

$$|I_{31}| = |\langle (S_R - S_{R'}) (v^R \mathrm{div} v^R), u^R - u^{R'} \rangle| \leq C \frac{1}{R^2} \|v^R\|_{H^{s}}^2 \|u^R - u^{R'}\|_{H^{\alpha}} \leq C \frac{1}{R^2} \|v^R\|_{H^{s}}^2 \left\| \frac{1}{12} u^R - u^{R'} \right\|_{H^{\alpha}}^2,$$

$$|I_{51}| = |\langle (S_R - S_{R'}) [(v^R \cdot \nabla) u^R], v^R - u^{R'} \rangle| \leq C \frac{1}{R^2} \|u^R\|_{H^{s}} \|v^R\|_{H^{s}} \|u^R - u^{R'}\|_{H^{\alpha}} \leq C \frac{1}{R^2} \|u^R\|_{H^{s}}^2 \|v^R\|_{H^{s}}^2 \left\| \frac{1}{12} u^R - u^{R'} \right\|_{H^{\alpha}}^2.$$

Using the Sobolev inequality (12), Gagliardo-Nirenberg interpolation inequalities (13) and Young’s inequality, we have

$$|I_{22}| = |\langle (v^R - u^{R'}) \cdot \nabla) v^R, u^R - u^{R'} \rangle| \leq \|(v^R - u^{R'}) \cdot \nabla) v^R\|_{L^{p_2}} \|u^R - u^{R'}\|_{L^{q_2}} \leq \|v^R - u^{R'}\|_{L^{2}} \|\nabla v^R\|_{L^{q_2}} \|u^R - u^{R'}\|_{H^{\alpha}} \leq \|v^R - u^{R'}\|_{L^{2}} \|v^R\|_{H^{s}} \|u^R - u^{R'}\|_{H^{\alpha}} \leq \frac{1}{12} \|u^R - u^{R'}\|_{H^{\alpha}}^2 + C \|v^R\|_{H^{s}}^2 \|v^R - u^{R'}\|_{L^{2}}^2,$$
where

\[
\frac{1}{p_3} + \frac{1}{q_3} = 1, \quad \frac{1}{p_3} = \frac{1}{r_3} + \frac{1}{2}
\]

with

\[
\begin{cases}
  p_3 = \frac{2d}{d+2\alpha}, & q_3 = \frac{2d}{d-2\alpha}, & r_3 = \frac{d}{\alpha}, & \text{if } \alpha < \frac{d}{2}, \\
p_3 > \frac{2d}{d+2\alpha-2}, & q_3 < \frac{2d}{d-2\alpha+2}, & r_3 > \frac{d}{\alpha}, & \text{if } \alpha = \frac{d}{2}, \\
p_3 = 1, & q_3 = \infty, & r_3 = 2, & \text{if } \alpha > \frac{d}{2}.
\end{cases}
\]

Similarly, the \(I_{32}\) can be estimated as follows

\[
|I_{32}| = |\langle S_{R} [(v^R - v'^R) \text{div} v^R], u^R - u'^R \rangle| \\
\leq \frac{1}{12} \| u^R - u'^R \|^2_{H^\alpha} + C \| v^R \|^2_{H^\alpha} \| v^R - v'^R \|^2_{L^2},
\]

and

\[
|I_{52}| = |\langle S_{R} [(v^R - v'^R) \cdot \nabla]v^R], v^R - v'^R \rangle| \\
\leq \| (v^R - v'^R) \cdot \nabla \|_{L^2} \| v^R - v'^R \|_{L^2} \\
\leq \| \nabla u^R \|_{L^\infty} \| v^R - v'^R \|_{L^2} \\
\leq C \| u^R \|_{H^{\alpha+\delta}} \| v^R - v'^R \|^2_{L^2}.
\]

Integration by parts, we estimate other terms

\[
I_{23} + I_{53} = - \langle S_{R} [(v'^R \cdot \nabla]v^R), u^R - u'^R \rangle \\
- \langle S_{R} [(v'^R \cdot \nabla]v^R), u^R - u'^R \rangle \\
= \langle (\text{div} v'^R)(v^R - v'^R), S_{R} (u^R - u'^R) \rangle \\
\leq \frac{1}{12} \| u^R - u'^R \|^2_{H^\alpha} + C \| v'^R \|^2_{H^\alpha} \| v^R - v'^R \|^2_{L^2}.
\]

When \(\alpha \geq 1\), integration by part, we have

\[
I_{33} = - \langle S_{R} [v'^R \text{div} (v^R - v'^R)], u^R - u'^R \rangle \\
= - \langle v'^R \text{div} (v^R - v'^R), S_{R} (u^R - u'^R) \rangle \\
= \sum_{i,j=1}^{d} \langle \partial_i v'^R (v_i^R - v_i'^R), S_R (u_j^R - u_j'^R) \rangle \\
= \sum_{i,j=1}^{d} \langle \partial_i v_j'^R (v_i^R - v_j'^R), S_R (u_j^R - u_j'^R) \rangle \\
+ \sum_{i,j=1}^{d} \langle v_j'^R (v_i^R - v_j'^R), S_R \partial_i (u_j^R - u_j'^R) \rangle
\]

\[= I_{331} + I_{332}.
\]

Similar to \(I_{22}\), the \(I_{331}\) can be estimated as follows

\[
|I_{331}| = |\sum_{i,j=1}^{d} \langle \partial_i v_j'^R (v_i^R - v_j'^R), S_R (u_j^R - u_j'^R) \rangle| \\
\leq \frac{1}{12} \| u^R - u'^R \|^2_{H^\alpha} + C \| v'^R \|^2_{H^\alpha} \| v^R - v'^R \|^2_{L^2}.
\]
Next, we estimate the last term $I_{332}$
\[
|I_{332}| = \left| \sum_{i,j=1}^{d} \langle v^R_i (v^R_i - v^R_j), S_R \partial_t (u^R_i - u^R_j) \rangle \right|
\leq C \| (v^R - v^R) \otimes v^R \|_{L^{p_4}} \| \nabla (u^R - u^R) \|_{L^{q_4}}
\leq C \| v^R - v^R \|_{L^2} \| v^R \|_{L^{p_4}} \| u^R - u^R \|_{H^s}
\leq C \| v^R - v^R \|_{L^2} \| v^R \|_{H^s} \| u^R - u^R \|_{H^s}
\leq \frac{1}{12} \| u^R - u^R \|_{H^s}^2 + C \| v^R \|_{H^s}^2 \| v^R - v^R \|_{L^2}^2,
\]
where we used the Sobolev inequality (12) and Gagliardo-Nirenberg interpolation inequalities
\[
\| v \|_{L^{p_4}} \leq C \| v \|_{L^2}^{-\lambda_3} \| \Lambda^s v \|_{L^2}^{\lambda_3}
\]
with
\[
\lambda_3 = \frac{d/2+1-\alpha}{s}, \quad \text{if } \alpha - 1 < \frac{d}{2},
\lambda_3 = \frac{d}{s \alpha}, \quad \text{if } \alpha - 1 = \frac{d}{2}.
\]
with
\[
\frac{1}{p_4} + \frac{1}{q_4} = 1, \quad \frac{1}{p_4} = \frac{1}{r_4} + \frac{1}{2}
\]
and
\[
\begin{cases}
    p_4 = \frac{2d}{\alpha + 1}, & q_4 = \frac{2d}{\alpha - 1}, \quad r_4 = \frac{d}{\alpha - 1}, \quad \text{if } \alpha - 1 < \frac{d}{2},
    \\
p_4 > \frac{2d}{\alpha + 2}, & q_4 < \frac{2d}{\alpha - 2}, \quad r_4 > \frac{d}{\alpha}, \quad \text{if } \alpha - 1 = \frac{d}{2},
    \\
p_4 = 1, & q_4 = \infty, \quad r_4 = 2, \quad \text{if } \alpha - 1 > \frac{d}{2}.
\end{cases}
\]
When $1 > \alpha \geq 0, \beta \geq 1$, we have
\[
I_{33} = - \left( S_R [ v^R \text{div}(v^R - v^R)], u^R - u^R \right)
\leq \| v^R \|_{L^{p_4}} \| \text{div}(v^R - v^R) \|_{L^2} \| u^R - u^R \|_{L^2}
\leq \| v^R \|_{H^{s+\alpha}} \| \text{div}(v^R - v^R) \|_{L^2} \| u^R - u^R \|_{H^s}
\leq \| v^R \|_{H^{s+\alpha}}^2 \| u^R - u^R \|_{H^s}^2 + \frac{1}{2} \| v^R - v^R \|_{H^s}^2.
\]
Inserting all the estimates above in (27) and using the bound
\[
\sup_{t \in [0,T_0]} \left( \| (u^R(t))_{t} \|_{H^s} + \| u^R(t) \|_{H^s} + \| \theta^R(t) \|_{H^s} \right) \leq C,
\]
and
\[
\int_0^{T_0} \left\{ \| u^R(t) \|_{H^{s+\alpha}}^2 + \| v^R(t) \|_{H^{s+\beta}}^2 \right\} ds \leq C,
\]
we obtain
\[
\frac{d}{dt} \left( \| u^R - u^R \|_{L^2}^2 + \| v^R - v^R \|_{L^2}^2 + \| \theta^R - \theta^R \|_{L^2}^2 \right)
\leq \frac{C}{R^s} + C \left( \| u^R(t) \|_{H^{s+\alpha}}^2 + 1 \right) \left( \| u^R - u^R \|_{L^2}^2 + \| v^R - v^R \|_{L^2}^2 + \| \theta^R - \theta^R \|_{L^2}^2 \right).
\]
Setting $Y(t) = \|u^R - u^R''\|_{L^2}^2 + \|v^R - v^R''\|_{L^2}^2 + \|\theta^R - \theta^R''\|_{L^2}^2$, we have
\[
\frac{dY(t)}{dt} \leq C + CY(t)(1 + \|u^R\|_{H_{s+\alpha}}^2).
\]
By the Gronwall’s inequality, we get
\[
\sup_{t \in [0, T_0]} Y(t) \leq \frac{M(\alpha, C, T_0)}{R^\epsilon}.
\]
Thus $(u^R, v^R, \theta^R)$ are Cauchy sequence in $L^\infty([0, T_0]; L^2(\mathbb{R}^d))$ so that they converge strongly to values $(u, v, \theta) \in L^\infty([0, T_0]; L^2(\mathbb{R}^d))$. This completes the proof of Proposition 2.

Proof of Theorem 1.1. It is straightforward to use the estimate in the proof above to show that $\Lambda^\alpha u^R \rightarrow \Lambda^\alpha u$, $\Lambda^\beta v^R \rightarrow \Lambda^\beta v$ strongly in $L^2([0, T_0]; L^2(\mathbb{R}^d))$. Combining Propositions 1 and 2 and using Sobolev interpolation inequality, for any $0 < s' < s$,
\[
\|f\|_{H^{s'}} \leq C\|f\|_{L^2}^{1 - \frac{s'}{s}}\|f\|_{H^s}^{\frac{s'}{s}},
\]
we further obtain the strong convergence
\[
\sup_{t \in [0, T_0]} \|(u^R, v^R, \theta^R) - (u, v, \theta)\|_{H^{s'}} \rightarrow 0 \quad \text{as} \quad R \rightarrow \infty.
\]
Furthermore, $\Lambda^\alpha u^R \rightarrow \Lambda^\alpha u$, $\Lambda^\beta v^R \rightarrow \Lambda^\beta v$ strongly in $L^2([0, T_0]; H^{s'}(\mathbb{R}^d))$ and thus $\text{div} u^R \rightarrow \text{div} u$, $\Lambda^\alpha u^R \rightarrow \Lambda^\alpha u$ and $\Lambda^\beta v^R \rightarrow \Lambda^\beta v$ strongly in $L^2([0, T_0]; H^{s'-1}(\mathbb{R}^d))$, $L^2([0, T_0]; H^{s'-\alpha}(\mathbb{R}^d))$ and $L^2([0, T_0]; H^{s'-\beta}(\mathbb{R}^d))$, respectively. Nonlinear terms are also strongly convergent in a suitable space. Thus $(u, v, \theta)$ is indeed a strong solution of (1).

Using the uniform bounds proposition, one may use the Banach-Alaoglu theorem to extract a weakly-* convergent subsequence such that
\[
\Lambda^\alpha u^R \rightharpoonup u, \quad \Lambda^\beta v^R \rightharpoonup v, \quad \Lambda^\alpha u^R \rightharpoonup \theta, \quad \text{in} \quad L^\infty([0, T_0]; H^s(\mathbb{R}^d)),
\]
\[
\Lambda^\alpha u^R \rightharpoonup \Lambda^\alpha u, \quad \Lambda^\beta v^R \rightharpoonup \Lambda^\beta v, \quad \text{in} \quad L^2([0, T_0]; H^s(\mathbb{R}^d)),
\]
\[
\text{hence the limit satisfies}
\]
\[
\begin{align*}
&u \in L^\infty([0, T_0]; H^s(\mathbb{R}^d)) \cap L^2([0, T_0]; H^{s+\alpha}(\mathbb{R}^d)), \\
v &\in L^\infty([0, T_0]; H^s(\mathbb{R}^d)) \cap L^2([0, T_0]; H^{s+\beta}(\mathbb{R}^d)), \\
\theta &\in L^\infty([0, T_0]; H^s(\mathbb{R}^d)).
\end{align*}
\]
The proof of uniqueness for (1) is similar to the proof of Proposition 2 and is omitted.

In addition, the uniform bound of $(u, v, \theta)$ allows us to show the weak time continuity. We have $u, v, \theta \in C_W([0, T]; H^s(\mathbb{R}^n))$.
\[
(u, v, \theta) \in C_W([0, T]; H^s) \quad \text{i.e.} \quad t \mapsto \int (u(x, t), v(x, t), \theta(x, t))\phi(x) \, dx
\]
is continuous for any $\phi \in H^{-s}$.

Using the standard argument (see Theorem 3.5 in Majda & Bertozzi [27] (2002), for example), we have $u, v, \theta \in C([0, T_1]; H^s(\mathbb{R}^n))$ (If $\alpha = 0$, we apply the method for Euler equations to obtain $u \in C([0, T_1]; H^s(\mathbb{R}^n))$). If $\alpha > 0$, we use the method for Navier-Stokes equation to get it. Similarly, we can obtain $v, \theta \in C([0, T_1]; H^s(\mathbb{R}^n))$. This completes the proof of Theorem 1.1. \qed
Appendix. In this appendix, we will prove Lemma 2.2. Firstly, we recall the definitions and some properties of the Besov spaces. This kind of space plays an important role in studying nonlinear partial differential equations (see [2, 3, 29, 31, 35] and references therein).

Let $\mathcal{B} = \{\xi \in \mathbb{R}^d, |\xi| \leq \frac{3}{4}\}$ and $\mathcal{C} = \{\xi \in \mathbb{R}^d, \frac{3}{4} \leq |\xi| \leq \frac{5}{4}\}$. Choose two nonnegative smooth radial functions $\chi$, $\varphi$ supported, respectively, in $\mathcal{B}$ and $\mathcal{C}$ such that

$$
\chi(\xi) + \sum_{j \geq 0} \varphi(2^{-j} \xi) = 1, \quad \xi \in \mathbb{R}^d,
$$

$$
\sum_{j \in \mathbb{Z}} \varphi(2^{-j} \xi) = 1, \quad \xi \in \mathbb{R}^d \setminus \{0\}.
$$

We denote $\varphi_j = \varphi(2^{-j} \xi)$, $h = \tilde{\mathfrak{g}}^{-1} \varphi$ and $\tilde{h} = \tilde{\mathfrak{g}}^{-1} \chi$, where $\tilde{\mathfrak{g}}^{-1}$ stands for the inverse Fourier transform. Then the dyadic blocks $\Delta_j$ and $S_j$ can be defined as follows

$$
\Delta_j f = \varphi(2^{-j} D) f = 2^j \int_{\mathbb{R}^d} \tilde{h}(2^j y) f(x - y) dy,
$$

$$
S_j f = \sum_{k \leq j - 1} \Delta_k f = \chi(2^{-j} D) f = 2^j \int_{\mathbb{R}^d} \tilde{\chi}(2^j y) f(x - y) dy.
$$

Formally, $\Delta_j = S_j - S_{j-1}$ is a frequency projection to annulus $\{\xi : C_1 2^j \leq |\xi| \leq C_2 2^j\}$, and $S_j$ is a frequency projection to the ball $\{\xi : |\xi| \leq C 2^j\}$. One can easily verify that with our choice of $\varphi$:

$$
\Delta_j \Delta_k f = 0 \text{ if } |j - k| \geq 2 \quad \text{and} \quad \Delta_j (S_{k-1} f \Delta_k f) = 0 \text{ if } |j - k| \geq 5.
$$

With the introduction of $\Delta_j$ and $S_j$, let us recall the definition of the Besov space.

Let $s \in \mathbb{R}$, $(p, q) \in [1, \infty]^2$, the homogeneous space $\dot{B}^{s}_{p,q}$ is defined by

$$
\dot{B}^{s}_{p,q} = \{ f \in \mathcal{S}'; \| f \|_{\dot{B}^{s}_{p,q}} < \infty \},
$$

where

$$
\| f \|_{\dot{B}^{s}_{p,q}} = \begin{cases} 
\left( \sum_{j \in \mathbb{Z}} 2^{sjq} \| \Delta_j f \|_{L^p}^q \right)^{\frac{1}{q}}, & \text{for } 1 \leq q < \infty, \\
\sup_{j \in \mathbb{Z}} 2^{sj} \| \Delta_j f \|_{L^p}, & \text{for } q = \infty,
\end{cases}
$$

and $\mathcal{S}'$ denotes the dual space of $\mathcal{S} = \{ f \in \mathcal{S}(\mathbb{R}^d); \partial^\alpha \hat{f}(0) = 0; \forall \alpha \in \mathbb{N}^d \text{ multi-index} \}$ and can be identified by the quotient space of $\mathcal{S}'/\mathcal{P}$ with the polynomials space $\mathcal{P}$.

Let $s > 0$, and $(p, q) \in [1, \infty]^2$, the inhomogeneous Besov space $B^{s}_{p,q}$ is defined by

$$
B^{s}_{p,q} = \{ f \in \mathcal{S}'(\mathbb{R}^d); \| f \|_{B^{s}_{p,q}} < \infty \},
$$

where

$$
\| f \|_{B^{s}_{p,q}} = \| f \|_{L^p} + \| f \|_{\dot{B}^{s}_{p,q}}.
$$

Additionally, we have the following equivalence relations.

$$
\| f \|_{B^{s}_{p,q}} \approx \| f \|_{\dot{H}^{s}}, \quad \| f \|_{B^{s}_{2,2}} \approx \| f \|_{H^{s}}.
$$

Bernstein’s inequalities are useful tools when dealing with Fourier localized functions and these inequalities trade integrability for derivatives. The following proposition provides Bernstein type inequalities for fractional derivatives.
**Proposition 3.** Let $\alpha \geq 0$. Let $1 \leq p \leq q \leq \infty$.

1) If $f$ satisfies

$$\text{supp } \hat{f} \subset \{ \xi \in \mathbb{R}^d : |\xi| \leq K2^j \},$$

for some integer $j$ and a constant $K > 0$, then

$$\|(-\Delta)\alpha f\|_{L^p(\mathbb{R}^d)} \leq C_1 2^{2\alpha j + j(d(\frac{1}{2} - \frac{1}{p})}) \|f\|_{L^p(\mathbb{R}^d)}.$$ 

2) If $f$ satisfies

$$\text{supp } \hat{f} \subset \{ \xi \in \mathbb{R}^d : K_1 2^j \leq |\xi| \leq K_2 2^j \}$$

for some integer $j$ and constants $0 < K_1 \leq K_2$, then

$$C_1 2^{2\alpha j} \|f\|_{L^p(\mathbb{R}^d)} \leq \|(-\Delta)\alpha f\|_{L^p(\mathbb{R}^d)} \leq C_2 2^{2\alpha j + j(d(\frac{1}{2} - \frac{1}{p}))} \|f\|_{L^p(\mathbb{R}^d)},$$

where $C_1$ and $C_2$ are constants depending on $\alpha, p$ and $q$ only.

Now, we prove the Lemma 2.2 by the Littlewood-Palay decomposition and Besov space techniques.

**Proof of Lemma 2.2.** Firstly, we prove the commutator estimate (8). Using inhomogeneous Bony’s decomposition, we have

$$\{|\Lambda^*[u \cdot \nabla]v - (u \cdot \nabla)\Lambda^*v, \Lambda^*v|\}
= \left| \sum_{j \in \mathbb{Z}} 2^{2js} \langle \Delta_j [(u \cdot \nabla)v] - (u \cdot \nabla)\Delta_j v, \Delta_j v \rangle \right|
\leq \sum_{j \in \mathbb{Z}} 2^{2js} \|\Delta_j, (u \cdot \nabla)v\|_{L^2} \|\Delta_j v\|_{L^2}
\leq \sum_{j \in \mathbb{Z}} 2^{2js} \sum_{|k-j| \leq 4} \|\Delta_j, S_{k-1}(u \cdot \nabla)\Delta_k v\|_{L^2} \|\Delta_j v\|_{L^2}
+ \sum_{j \in \mathbb{Z}} 2^{2js} \sum_{|k-j| \leq 4} \|\Delta_j (\Delta_k (u \cdot \nabla) S_{k-1} v)\|_{L^2} \|\Delta_j v\|_{L^2}
+ \sum_{j \in \mathbb{Z}} 2^{2js} \sum_{k \geq j-2} \|\Delta_k (u \cdot \nabla) \Delta_j S_{k+2} v\|_{L^2} \|\Delta_j v\|_{L^2}
+ \sum_{j \in \mathbb{Z}} 2^{2js} \sum_{k \geq j-3} \|\Delta_j (\Delta_k (u \cdot \nabla) \tilde{\Delta}_k v)\|_{L^2} \|\Delta_j v\|_{L^2}
:= I_1 + I_2 + I_3 + I_4,$$

where $\tilde{\Delta}_k := \Delta_{k-1} + \Delta_k + \Delta_{k+1}$.

By Hölder inequality and Bernstein’s inequality, we estimate $I_1, I_2, I_3, I_4$ as follows

$$I_1 = \sum_{j \in \mathbb{Z}} 2^{2js} \sum_{|k-j| \leq 4} \|\Delta_j, S_{k-1}(u \cdot \nabla)\Delta_k v\|_{L^2} \|\Delta_j v\|_{L^2}
\leq \sum_{j \in \mathbb{Z}} 2^{2js} \sum_{|k-j| \leq 4} \|\nabla S_{k-1} u\|_{L^\infty} \|\Delta_k v\|_{L^2} \|\Delta_j v\|_{L^2}
\leq C \|\nabla u\|_{L^\infty} \sum_{j \in \mathbb{Z}} 2^{2js} \|\Delta_j v\|_{L^2}^2
\leq C \|\nabla u\|_{L^\infty} \|v\|_{H^s}^2,
\leq C \|u\|_{H^{s+\alpha}} \|v\|_{H^s}^2,$$
\[ I_2 = \sum_{j \in \mathbb{Z}} 2^{2js} \sum_{|k-j| \leq 4} \| \Delta_j (\Delta_k (u \cdot \nabla) S_{k-1} v) \|_{L^2} \| \Delta_j v \|_{L^2} \]
\[ \leq \sum_{j \in \mathbb{Z}} 2^{2js} \sum_{|k-j| \leq 4} \| \Delta_k u \|_{L^2} \| \nabla S_{k-1} v \|_{L^\infty} \| \Delta_j v \|_{L^2} \]
\[ \leq C \sum_{j \in \mathbb{Z}} 2^{2js} \| \Delta_j u \|_{L^2} \| \nabla S_{j-1} v \|_{L^\infty} \| \Delta_j v \|_{L^2} \]
\[ = C \sum_{j \in \mathbb{Z}} 2^{2js(s+\alpha)} \| \Delta_j u \|_{L^2} 2^{-2js} \| \nabla S_{j-1} v \|_{L^\infty} \| \Delta_j v \|_{L^2} \]
\[ \leq C \| \nabla v \|_{B^{s-\alpha}_{\infty, \infty}} \| u \|_{H^{s+\alpha}} \| v \|_{H^s} \]
\[ \leq C \| \nabla v \|_{B^{\frac{s}{2}+1}_{\infty, \infty}} \| u \|_{H^{s+\alpha}} \| v \|_{H^s} \]
\[ \leq C \| u \|_{H^{s+\alpha}} \| v \|_{H^s}, \]

\[ I_3 = \sum_{j \in \mathbb{Z}} 2^{2js} \sum_{k \geq j-2} \| \Delta_k (u \cdot \nabla) \Delta_j S_{k+2} v \|_{L^2} \| \Delta_j v \|_{L^2} \]
\[ \leq C \sum_{j \in \mathbb{Z}} 2^{2js} \sum_{k \geq j-2} \| \Delta_k u \|_{L^\infty} \]
\[ \leq C \sum_{j \in \mathbb{Z}} 2^{2js} \| \Delta_j v \|_{L^2} \sum_{k \geq j-2} 2^{-j-k} 2^{-k(\frac{s}{2}+1)} \| \Delta_k u \|_{L^2} \]
\[ \leq C \| u \|_{H^{\frac{s}{2}+1}} \| v \|_{H^s} \]
\[ \leq C \| u \|_{H^{s+\alpha}} \| v \|_{H^s}, \]

and
\[ I_4 = \sum_{j \in \mathbb{Z}} 2^{2js} \sum_{k \geq j-3} \| \Delta_j (\Delta_k (u \cdot \nabla) \tilde{S}_k v) \|_{L^2} \| \Delta_j v \|_{L^2} \]
\[ \leq C \sum_{j \in \mathbb{Z}} 2^{2js} \| \Delta_j v \|_{L^2} \sum_{k \geq j-3} 2^{j-k} 2^{k \frac{s}{2}} \| \Delta_k u \|_{L^2} \| \tilde{S}_k v \|_{L^2} \]
\[ \leq C \sum_{j \in \mathbb{Z}} 2^{2js} \| \Delta_j v \|_{L^2} \sum_{k \geq j-3} 2^{(j-k)(s+\frac{s}{2})} 2^{-k(s+\frac{s}{2}+1)} \| \Delta_k u \|_{L^2} \| \tilde{S}_k v \|_{L^2} \]
\[ \leq C \| u \|_{H^{s+\alpha}} \| v \|_{H^s}. \]

Finally, we prove the commutator estimate (9). Using inhomogeneous Bony’s decomposition, we have
\[ |(\Lambda^s [(v \cdot \nabla) v] - (v \cdot \nabla) \Lambda^s v, \Lambda^s u)| \]
\[ = |\sum_{j \in \mathbb{Z}} 2^{2js} \langle \Delta_j [(v \cdot \nabla) v] - (v \cdot \nabla) \Delta_j v, \Delta_j u \rangle | \]
\[ = |\sum_{j \in \mathbb{Z}} 2^{2js} \langle [\Delta_j, (v \cdot \nabla)] v, \Delta_j u \rangle | \]
\[ \leq \sum_{j \in \mathbb{Z}} 2^{2js} \sum_{|k-j| \leq 4} \| [\Delta_j, S_{k-1} (v \cdot \nabla)] \Delta_k v \|_{L^2} \| \Delta_j v \|_{L^2} \]
\[ + \sum_{j \in \mathbb{Z}} 2^{2js} \sum_{|k-j| \leq 4} \| \Delta_j (\Delta_k (v \cdot \nabla) S_{k-1} v) \|_{L^2} \| \Delta_j v \|_{L^2} \]
\[ + \sum_{j \in \mathbb{Z}} 2^{2js} \sum_{k \geq j-2} \| \Delta_k (v \cdot \nabla) \Delta_j S_{k+2} v \|_{L^1} \| \Delta_j u \|_{L^\infty} \]
This completes the proof of Lemma 2.2.

By Hölder inequality and Bernstein’s inequality, we estimate $J_1$, $J_2$, $J_3$, $J_4$ as follows:

$J_1 = \sum_{j \in \mathbb{Z}} 2^{2j} \sum_{|k-j| \leq 4} \|\Delta_j (\Delta_k (v \cdot \nabla) \Delta_k v)\|_{L^1} \|\Delta_j u\|_{L^\infty}$

$\leq \sum_{j \in \mathbb{Z}} 2^{2j} \sum_{|k-j| \leq 4} \|\nabla S_k \cdot v\|_{L^\infty} \|\Delta_k v\|_{L^2} \|\Delta_j u\|_{L^2}$

$\leq \sum_{j \in \mathbb{Z}} 2^{2j} \|\nabla S_{j-1} v\|_{L^\infty} \|\Delta_j v\|_{L^2} \|\Delta_j u\|_{L^2}$

$= \sum_{j \in \mathbb{Z}} 2^{-j} \|\nabla S_{j-1} v\|_{L^\infty} 2^{j+\alpha} \|\Delta_j v\|_{L^2} 2^{j(s+\alpha)} \|\Delta_j u\|_{L^2}$

$\leq C \|\nabla v\|_H \|\Delta_j S_{j-1} v\|_{H^{s+\alpha}} \|\Delta_j u\|_{H^{s+\alpha}}$

By Hölder inequality and Bernstein’s inequality, we estimate $J_1$, $J_2$, $J_3$, $J_4$ as follows:

$J_1 = \sum_{j \in \mathbb{Z}} 2^{2j} \sum_{|k-j| \leq 4} \|\Delta_j (\Delta_k (v \cdot \nabla) \Delta_k v)\|_{L^1} \|\Delta_j u\|_{L^2}$

$\leq \sum_{j \in \mathbb{Z}} 2^{2j} \sum_{|k-j| \leq 4} \|\nabla S_k \cdot \nabla \Delta_j \Delta_k v\|_{L^2} \|\Delta_j u\|_{L^\infty}$

$\leq \sum_{j \in \mathbb{Z}} 2^{2j} \|\Delta_k (v \cdot \nabla) \Delta_j \Delta_k v\|_{L^2} \sum_{k \geq j-2} 2^{2j} \|\Delta_k v\|_{L^2}$

$\leq \sum_{j \in \mathbb{Z}} 2^{2j} \|\nabla u\|_{L^\infty} \left( \sum_{j \in \mathbb{Z}} 2^{2j} \|\Delta_j v\|_{L^2}^2 \right)^{\frac{1}{2}} \left( \sum_{k \geq j-2} 2^{(j-k)s} 2^{k} \|\Delta_k v\|_{L^2}^2 \right)^{\frac{1}{2}}$

$\leq C \|\nabla u\|_{H^{s+\alpha}} \|v\|_{H^s}^2$

and

$J_4 = \sum_{j \in \mathbb{Z}} 2^{2j} \sum_{k \geq j-3} \|\Delta_j (\Delta_k (v \cdot \nabla) \Delta_k v)\|_{L^1} \|\Delta_j u\|_{L^\infty}$

$\leq C \|\nabla u\|_{L^\infty} \sum_{j \in \mathbb{Z}} 2^{2j} \sum_{k \geq j-3} \|\Delta_k v\|_{L^2} \|\Delta_k v\|_{L^2}$

$\leq C \|\nabla u\|_{L^\infty} \left( \sum_{j \in \mathbb{Z}} \left( \sum_{k \geq j-3} 2^{(j-k)s} 2^{k} \|\Delta_k v\|_{L^2} \|\Delta_k v\|_{L^2} \right)^2 \right)^{\frac{1}{2}}$

$\leq C \|\nabla u\|_{H^{s+\alpha}} \|v\|_{H^s}^2$

This completes the proof of Lemma 2.2.

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