Neutrino Mass Matrix Model with Only Three Adjustable Parameters

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Abstract

Stimulated by a successful quark mass matrix model based on $U(3) \times U(3)'$ family symmetry, a phenomenological neutrino mass matrix for the Majorana neutrinos ($\nu_L, \nu_R, N_L, N_R$) is proposed. The model has only three adjustable parameters. Nevertheless, the model gives reasonable predictions for the neutrino masses, mixings, and CP violating phases in the neutrino mixing matrix.

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1. Introduction

The greatest concern in the flavor physics is how to understand the origin of the observed structures of masses and mixings of quarks and leptons. Recently, we have proposed a quark mass matrix model based on the $U(3) \times U(3)'$ flavor symmetry \cite{1, 2}, which originates from the following $2 \times 2$ blocks mass matrix model:

\begin{equation}
\begin{pmatrix}
(\bar{f}_L, \bar{F}_L) \\
(\Phi_f)_{ij} \rho_{ij} \frac{S_f \conjugate\alpha \beta}{(S_f)_{ij} \conjugate} \frac{f_{Rj} \rho_{Rj}}{(S_f)_{ij} \conjugate} \\
\end{pmatrix},
\end{equation}

Here, we consider hypothetical heavy fermions $F_\alpha$ ($\alpha = 1, 2, 3$), which belong to $(1, 3)$ of $U(3) \times U(3)'$, in addition to quarks and charged leptons $f_i = (u_i, d_i, e_i)$ ($i = 1, 2, 3$) which belong to $(3, 1)$. The fields $\Phi_f$ and $S_f$ are scalars which belong to $(3, 3^{*})$ and $(1, 8 + 1)$ of $U(3) \times U(3)'$, respectively. According to a seesaw-like mechanism, we obtain quarks and charged lepton mass matrices

\begin{equation}
(M_f)_{ij} = \langle (\Phi_f)_{ij} \rangle \langle (S_f^{-1})_{ij} \rangle \langle (S_f)_{ij} \rangle,
\end{equation}

where $\alpha$ and $\beta$ are indexes of $U(3)'$ and $i$ and $j$ are indexes of $U(3)$. (Here, exactly speaking, the matrix $(M_f)_{ij}^{\beta}$ in Eq.(1.2) represents the Yukawa coupling constant $(Y_f)_{ij}^{\alpha}$ of the fermion $f$. However, for convenience, we will call it as "mass matrix".)

Furthermore, we assume the vacuum expectation values (VEVs) of those scalars as follows:

\begin{equation}
\langle S_f^{-1} \rangle = \left[v_S \left(1 + b_f X_3\right)\right]^{-1} = v_S^{-1} \left(1 + a_f X_3\right),
\end{equation}
where $1$ and $X_3$ are defined by

$$1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad X_3 = \frac{1}{3} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad (1.4)$$

and $a_f$ and $b_f$ are complex parameters with the relation

$$a_f = -\frac{b_f}{1 + b_f}. \quad (1.5)$$

On the other hand, the VEV form of $\langle \Phi_f \rangle$ are chosen as

$$\langle \Phi_f \rangle = v \Phi \text{diag}(z_1 e^{i\phi_1}, z_2 e^{i\phi_2}, z_3 e^{i\phi_3}), \quad (1.6)$$

where

$$z_i = \frac{\sqrt{m_{ei}}}{\sqrt{m_e + m_\mu + m_\tau}} \quad (1.7)$$

Here, $m_{ei} = (m_e, m_\mu, m_\tau)$ are charged lepton masses.

The mass matrix form (1.2) with the form (1.3) is known as the “democratic seesaw mass matrix” form [3]. The term “democratic” was named by Jarlskog [4]. In this model, parameters which we can adjust are only the generation-independent parameters $a_f$ ($f = u$ and $d$).

Under this mass matrix model (1.2), we have successfully obtained [1] a unified description of quark masses and mixing by using the observed charged lepton masses. The models for quarks were proposed in the previous century, but we consider that the basic idea should be still inherited to the neutrino mass matrix model.

A naive extension of the quark mass matrix model (1.1) to the $2 \times 2$ blocks mass matrix for neutrino $\nu$ and heavy neutrino $N$ will be as follows:

$$\begin{pmatrix} \nu^c_L \\ \bar{N}^c_L \end{pmatrix} \begin{pmatrix} (0)_{ij} & (\Phi_\nu)_{ij} \\ (\bar{\Phi}_\nu^c)_{\alpha j} & -(S_\nu)^{-1} \end{pmatrix} \begin{pmatrix} (\nu^c_L)^j \\ (\bar{N}^c_L)^j \end{pmatrix}, \quad (1.8)$$

which leads to

$$(M_\nu)_{ij} = \langle (\Phi_\nu)^\alpha_i \rangle \langle (S_\nu^{-1})_{\alpha j} \rangle \langle (\Phi_\nu)^\beta j \rangle. \quad (1.9)$$

However, this neutrino mass matrix (1.9) is too simplified. We think that the Majorana mass terms must be taken into consideration.

In this paper, we consider a $4 \times 4$ blocks mass matrix [5] for the neutrino states $(\nu_L, \nu^c_R, N_L, N^c_R)$ ($f^c$ denote a charge conjugate state of a fermion $f$). The explicit form will be given in the next
section. Our model is a three parameter model, and those parameters are fixed by three of the four observed values in the neutrino oscillation data. Thereby, we can make good fitting for the rest observed values, and we predict the CP violation phase factor which will be soon observed rigidly.

2. Majorana neutrino mass matrix

In the present paper, we assume the following mass matrix:

\[
\begin{pmatrix}
(0)_{\circ \circ} & (1)_{\circ \circ} & (\Phi)_{\circ \circ} & (0)_{\circ \circ} \\
(1)_{\circ \circ} & (0)_{\circ \circ} & (1)_{\circ \circ} & (\Phi)_{\circ \circ} \\
\Phi_{\circ \circ} & (1)_{\circ \circ} & S_{\circ \circ} & S_{\circ \circ} \\
(0)_{\circ \circ} & (0)_{\circ \circ} & S_{\circ \circ} & (1)_{\circ \circ}
\end{pmatrix}
\times
\begin{pmatrix}
(\nu_L)_{\circ \circ} \\
(\nu_R)_{\circ \circ} \\
(\nu_L)_{\circ \circ} \\
(\nu_R)_{\circ \circ}
\end{pmatrix}.
\] (2.1)

Here, for convenience, we put \( \circ \) for indexes \( i, j \) \( \cdots \) and \( \bullet \) for \( \alpha, \beta, \cdots \). And also, for convenience, we dropped the symbols "(" and ")". In the mass matrix, the element \( (0) \) shows an empty element. \( \Phi_{\circ \circ}, \Phi_{\circ \circ}, S_{\circ \circ} \) and \( S_{\circ \circ} \) were already introduced in the quark mass matrix model \([1, 2]\) .

As characteristic Majorana mass terms, we have introduced \((1)_{\circ \circ}, (1)_{\circ \circ}, S_{\circ \circ} \) and \((1)_{\circ \circ}, \) so that we take the same form as in Eq.\((1.9)\). Here, the term \( S_{\circ \circ} \) is a Majorana version of the Dirac mass terms \( S_{\circ \circ} \) and \( S_{\circ \circ} \).

Since we demand that the number of free parameters in the model is as small as possible, for the other Majorana mass terms \((1)_{\circ \circ}, (1)_{\circ \circ}, (1)_{\circ \circ}, \) we assumed that those are structureless, e.g. a unit matrix. Also, we assumed unit matrix forms for mass terms with lower energy scale \((1)_{\circ \circ} \) and \((1)_{\circ \circ} \). We will refer the mass matrix given in Eq.\((2.1)\) as \( M_{4 \times 4} \).

In the assignments of the scalars in the mass matrix \( M_{4 \times 4} \), there is no theoretical inevitability. We demand that the main term takes the familiar form \((1.8)\), and that the second term takes a simple and plausible form.

We obtain the neutrino mass matrix \( M_{\nu} \) from the generalized mass matrix \( M_{4 \times 4} \) by using the following seesaw approximation as follows:

\[
M_{4 \times 4} \Rightarrow M_{3 \times 3} = \begin{pmatrix}
(0)_{\circ \circ} & 1^\circ & \Phi_{\circ \circ} \\
1^\circ & (0)_{\circ \circ} & 1^\circ \\
\Phi_{\circ \circ} & 1^\circ & S_{\circ \circ}
\end{pmatrix}
- \begin{pmatrix}
(0)_{\circ \circ} \\
(0)_{\circ \circ} \\
S_{\circ \circ}
\end{pmatrix}
(1_{\circ \circ})^{-1}
(0)_{\circ \circ}, (0)_{\circ \circ}, S_{\circ \circ}.
\] (2.2)

\[
= \begin{pmatrix}
(0)_{\circ \circ} & 1^\circ & \Phi_{\circ \circ} \\
1^\circ & (0)_{\circ \circ} & 1^\circ \\
\Phi_{\circ \circ} & 1^\circ & S_{\circ \circ} - S_{\circ \circ}(1_{\circ \circ})^{-1}S_{\circ \circ}
\end{pmatrix},
\]
where
\[ (S_{\text{eff}})^{\star \star} \equiv S^{\star \star} - S^{\star}(1_{\star \star})^{-1}S^{\star}. \] (2.4)

Then, we obtain the neutrino mass matrix
\[ M_{2 \times 2} \Rightarrow (M_{\nu})_{\star \star} = -\Phi^{\star \star}(S_{\text{eff}})^{-1} \Phi^{\star \star} + [1_{\star \star} - \Phi^{\star \star}(S_{\text{eff}})^{-1}] \Phi^{\star \star} \]
\[ \times [1_{\star \star} - \Phi^{\star \star}(S_{\text{eff}})^{-1}]^{-1} (1_{\star \star})^{-1} [1_{\star \star} - \Phi^{\star \star}(S_{\text{eff}})^{-1}] \Phi^{\star \star}. \] (2.5)

This neutrino mass matrix (2.5) is too complicated for a numerical analysis. Therefore, we take the following approximation:
\[ |1_{\star \star}| \gg |1_{\star \star}^{\star \star}(S_{\text{eff}})^{-1} \Phi^{\star \star}|, \] (2.6)
in Eq.(2.5). Then, we obtain a simple form
\[ (M_{\nu})_{\star \star} = -\Phi^{\star \star}(S_{\text{eff}})^{-1} \Phi^{\star \star} + 1_{\star \star}^{\star \star}(1_{\star \star}^{\star \star})^{-1}(S_{\text{eff}})^{\star \star \star}(1_{\star \star}^{\star \star})^{-1}(1_{\star \star}). \] (2.7)

3. Numerical estimates

According to Eq.(2.7), we estimate the following mass matrix
\[ M_{\nu} = k_{\nu}\{\Phi_{\nu}(1 + a_{\text{eff}}X_{3})\Phi_{\nu} + \xi(1 + a_{\text{eff}}X_{3})^{-1}\}. \] (3.1)

Here, the parameter \( a_{\text{eff}} \) is defined by \( (S_{\text{eff}})^{-1} = v_{S_{\text{eff}}}^{-1}(1 + a_{\text{eff}}X_{3}) \), and \( k_{\nu} \) is an overall factor determined by the VEV scales. Since we are interested only in the mass ratios and mixing matrix, hereafter, we put \( k_{\nu} = 1 \), and we use dimensionless expressions for \( \Phi_{\nu} \) given by
\[ \Phi_{\nu} = \text{diag}(z_{1}e^{i\phi_{1}}, z_{2}e^{i\phi_{2}}, z_{3}e^{i\phi_{3}}). \] (3.2)

where \( \phi_{1} = \phi_{2} = \phi_{3} = 0 \).

We have only two (complex) parameters, \( a_{\text{eff}} \) and \( \xi \) in (3.1). As we discuss later, since we have only four neutrino data, the number of parameters must be smaller than three to avoid falling in a mere parameter-fitting model. Therefore, one of parameters \( a_{\text{eff}} \) and \( \xi \) must be real. In our previous study for the quark mass matrix, we have taken the parameter \( a_{f} \) as complex. In the present neutrino mass matrix model, too, we assume \( a_{\text{eff}} \) is complex, so that we take \( \xi \) as real. So we have only three real parameters \( a_{\text{eff}} = |a_{\text{eff}}|e^{i\alpha} \) and \( \xi \).

Since the neutrino mass matrix \( M_{\nu} \) is a symmetric, i.e. \( M_{\nu}^{T} = M_{\nu} \), the mass matrix is diagonalized by an unitary matrix \( U \) as
\[ U^{T}M_{\nu}U = D_{\nu} \equiv \text{diag}(m_{1}, m_{2}, m_{3}), \] (3.3)
where $m_i$ are the Majorana neutrino masses.

For convenience, hereafter, we use the following standard form $U_{\nu}$ given by

$$U_{\nu} = \begin{pmatrix}
  c_{13}c_{12} & c_{13}s_{12} & s_{13}e^{-i\delta_{CP}} \\
-c_{23}s_{12} - s_{23}s_{13}e^{i\delta_{CP}} & c_{23}c_{12} - s_{23}s_{13}e^{i\delta_{CP}} & s_{23}c_{13} \\
 s_{23}s_{12} - c_{23}s_{13}e^{i\delta_{CP}} & -s_{23}c_{12} - c_{23}s_{13}e^{i\delta_{CP}} & c_{23}c_{13}
\end{pmatrix} \times \text{diag}(1, e^{i\beta}, e^{i\gamma}), \tag{3.4}
$$

where $c_{ij} = \cos \theta_{ij}$ and $s_{ij} = \sin \theta_{ij}$ with the mixing angles $\theta_{ij}$. Note that since the Majorana neutrino fields have no freedom of rephasing invariance, so that we can use only the rephasing freedom of the charged lepton fields to transform the form of $U$ to $U_{\nu}$. Hereafter, we call the mixing matrix (3.4) as the Maki-Nakagawa-Sakata (MNS) mixing matrix [6].

The MNS mixing angles $\theta_{12}$, $\theta_{23}$ and $\theta_{13}$ are calculable from mixing matrix $U_{\nu}$ as,

$$\sin^2 \theta_{12} = |U_{12}|^2/(1 - |U_{13}|^2), \quad \sin^2 \theta_{23} = |U_{23}|^2/(1 - |U_{13}|^2), \quad \sin^2 \theta_{13} = |U_{13}|^2. \tag{3.5}
$$

The $CP$-violating phase $\delta_{CP}$, the additional Majorana phase $\beta$ and $\gamma$ in the representation Eq.(3.4) are also calculable and obtained as

$$\delta_{CP} = \arg \left[ \frac{U_{12}U_{22}^*}{U_{13}U_{23}} + \frac{|U_{12}|^2}{1 - |U_{13}|^2} \right], \tag{3.6}
$$

$$\beta = \arg \left( \frac{U_{12}}{U_{11}} \right), \quad \gamma = \arg \left( \frac{U_{13}}{U_{11}} e^{i\delta_{CP}} \right). \tag{3.7}
$$

The ratio of neutrino mass square differences $R_\nu$ is also calculable from the mass eigenvalues of $M_{\nu}$ as

$$R_\nu \equiv \frac{\Delta m_{21}^2}{\Delta m_{32}^2} = \frac{m_2^2 - m_1^2}{m_3^2 - m_2^2}. \tag{3.8}
$$

The effective neutrino mass $\langle m \rangle$ of the neutrinoless double beta decay is defined by

$$\langle m \rangle = |m_1U_{11}^2 + m_2U_{12}^2 + m_3U_{13}^2|. \tag{3.9}
$$

At present, we have the four observed values[7] as

$$R_\nu \equiv \frac{m_2^2 - m_1^2}{m_3^2 - m_2^2} = \frac{7.39^{+0.21}_{-0.20} \times 10^{-5}eV^2}{2.525^{+0.033}_{-0.031} \times 10^{-3}eV^2} = (2.93 \pm 0.12) \times 10^{-2}, \tag{3.10}
$$
\[
\sin^2 \theta_{12} = 0.310^{+0.013}_{-0.012}, \quad \sin^2 \theta_{13} = 0.02240^{+0.00065}_{-0.00066}, \quad \sin^2 \theta_{23} = 0.582^{+0.015}_{-0.019}. \tag{3.11}
\]

Let us give our strategy of parameter fitting to the observables. In our model we have three parameters, \( |a_{eff}|, \alpha, \) and \( \xi \) in \( M_\nu \). Therefore the MNS mixing angles \( \theta_{12}, \theta_{23}, \) and \( \theta_{13} \) and the ratio of neutrino mass square differences \( R_\nu \) and so on are functions of \( |a_{eff}|, \alpha, \) and \( \xi \) in this model.

The values of our three free parameters can be fixed by using the three experimental values (center values) of \( R_\nu \), \( \sin^2 \theta_{12} \) and \( \sin^2 \theta_{13} \) in (3.10) and (3.11) as follows:

\[
a_{eff} = -4.069, \quad \alpha = \mp 0.408^o, \quad \xi = 0.756. \tag{3.12}
\]

There are two solutions \( \alpha = -0.408 \) and \( \alpha = +0.408^o \). Hereafter, we call Case (A) for \( \alpha = -0.408^o \) and Case (B) for \( \alpha = +0.408^o \). The observed values versus parameter choices are given in Table 1.

Table 1: Observed values vs. our parameter choices

|          | \( \sin^2 \theta_{12} \) | \( \sin^2 \theta_{13} \) | \( R_\nu [10^{-2}] \) |
|----------|--------------------------|--------------------------|--------------------------|
| Obs      | 0.310                    | 0.02240                  | 2.93                     |
|          | \( \pm 0.013 \)          | \( \pm 0.00065 \)        | \( \pm 0.12 \)           |
| Our choice | 0.314                    | 0.02242                  | 2.91                     |

For the input parameters in cases (A) and (B), we predict

\[
\sin^2 \theta_{23} = 0.672, \quad \delta_{CP} = \mp 119^o, \quad \beta = \pm 10.8^o, \quad \gamma = \pm 7.15^o. \tag{3.13}
\]

When we use \( \Delta m^2_{32} = m_3^2 - m_2^2 = 2.525 \times 10^{-3} \text{eV}^2 \), we obtain the prediction of the neutrino mass \( m_i \) and effective neutrino mass \( \langle m \rangle \) for the both (A) and (B) as follows:

\[
m_1 = 0.0486 \text{eV}, \quad m_2 = 0.0494 \text{eV}, \quad m_3 = 0.0698 \text{eV}, \quad \langle m \rangle = 0.0328 \text{eV}. \tag{3.14}
\]

The predicted values of observables are listed in Table 2.

In this three parameter model, the three parameters are fixed so as to reproduce the observed values of \( R_\nu, \sin^2 \theta_{12} \) and \( \sin^2 \theta_{13} \). The predictions of the \( \delta_{CP}, \beta, \) and \( \gamma \) in (3.13) and of the neutrino masses \( m_i \) and effective neutrino mass \( \langle m \rangle \) in (3.14) will be checked in near future experiments.

We have constructed the neutrino mass matrix model with the number of free parameters as small as possible in order to have high predictability in our unified description approach for the quarks and leptons. It seems impossible to build a neutrino mass matrix model with furthermore few parameters, e.g. a two parameter model.
Table 2: Predicted values vs. observed values.

|       | \(\sin^2 \theta_{23}\) | \(\delta_{CP}\) | \(\beta\) | \(\gamma\) | \(m_1\) [eV] | \(m_2\) [eV] | \(m_3\) [eV] | \(\langle m \rangle\) [eV] |
|-------|----------------|---------------|--------|--------|------------|------------|------------|----------------|
| Pred  | 0.672         | \(\pm 119^\circ\) | \(\pm 10.8^\circ\) | \(\pm 7.15^\circ\) | 0.0486     | 0.0494     | 0.0698     | 0.0328          |
| Obs   | 0.582         | -             | -      | -      | -          | -          | -          | < O(10^{-1})     |
|       | +/-0.015      |               |        |        |            |            |            |                |
|       | +/-0.019      |               |        |        |            |            |            |                |

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