Relative Computability and Uniform Continuity of Relations

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Abstract. A type-2 computable real function is necessarily continuous; and this remains true for relative, i.e. oracle-based computations. Conversely, by the Weierstrass Approximation Theorem, every continuous \( f : [0, 1] \rightarrow \mathbb{R} \) is computable relative to some oracle. In their search for a similar topological characterization of relatively computable multi-valued functions \( f : [0, 1] \rightrightarrows \mathbb{R} \) (aka relations), Brattka and Hertling (1994) have considered two notions: weak continuity (which is weaker than relative computability) and strong continuity (which is stronger than relative computability). Observing that uniform continuity plays a crucial role in the Weierstrass Theorem, we propose and compare several notions of uniform continuity for relations. Here, due to the additional quantification over values \( y \in f(x) \), new ways arise of (linearly) ordering quantifiers—yet none turns out as satisfactory. We are thus led to a notion of uniform continuity based on the Henkin quantifier; and prove it necessary for relative computability of compact real relations. In fact iterating this condition yields a strict hierarchy of notions each necessary, and the \( \omega \)-th level also sufficient, for relative computability.

1 Introduction
A simple counting argument shows that not every (total) integer function \( f : \mathbb{N} \rightarrow \mathbb{N} \) can be computable; on the other hand, each such function can be encoded into an oracle \( O \subseteq \{0, 1\}^* \) that renders it relatively computable. Over real numbers, similarly, not every total \( f : [0, 1] \rightarrow \mathbb{R} \) can be computable for cardinality reasons; and this remains true for oracle machines. In fact it is folklore in Recursive Analysis that any function \( f \) computably mapping approximations of real numbers \( x \) to approximations of \( f(x) \) must necessarily be continuous; and the same remains true for oracle computations. Even more surprisingly, this implication can be reversed: If a (say, real) function \( f \) is continuous, then there exists an
oracle which renders \( f \) computable\(^5\). This can for instance be concluded from the Weierstrass Approximation Theorem. A far reaching generalization from the reals to so-called *admissibly represented spaces* is the Kreitz-Weihrauch Theorem, cf. e.g. [Weih00, 3.2.11] and compare the Myhill-Shepherdson Theorem in Domain Theory. The equivalence between continuity and relative computability has led DANA SCOTT to consider continuity as an approximation to computability.

Now many computational problems are more naturally expressed as relations (i.e. multivalued) rather than as (single-valued) functions. For instance when diagonalizing a given real symmetric matrix, one is interested in some basis of eigenvectors, not a specific one. It is thus natural to consider computations which, given \( x \), intensionally choose and output *some* value \( y \in f(x) \). Indeed, a multifunction may well be computable yet admit no continuous single-valued *selection*; cf. e.g. [Weih00, EXERCISE 5.1.13] or [Luck77]. Hence multivaluedness avoids some of the topological restrictions of single-valued functions—but of course not all of them. Specifically it is easy to see that a multifunction \( f \) is relatively computable iff it admits a continuous so-called *realizer*, that is a function mapping any infinite binary string encoding some \( x \) to an infinite binary string encoding some \( y \in f(x) \).

However the single-valued case raises the hope for an intrinsic characterization of relative computability of \( f \), without referral to Cantor space. Such an investigation has been pursued in [BrHe94], yielding both necessary and sufficient conditions for a relation to be computable relative to some oracle (which, there, is called *relative continuity* and we shall denote as *relative computability*). BRATTKA and HERTLING have established what remains to-date the best counterpart to the Kreitz-Weihrauch Theorem for the multivalued case:

**Fact 1.** Let \( X, Y \) be separable metric spaces and \( Y \) in addition complete. Then a pointwise closed relation \( f : X \rightrightarrows Y \) is relatively computable iff it has a strongly continuous tightening\(^6\).

Here, being *pointwise closed* means that \( f(x) := \{y \in Y : (x, y) \in f\} \) is a closed subset for every \( x \in X \). We shall freely switch between the viewpoint of \( f : \subseteq X \rightrightarrows Y \) being a relation \( (f \subseteq X \times Y) \) and being a set-valued partial mapping \( f : \subseteq X \rightarrow 2^Y \). \( x \mapsto f(x) \).

Such \( f \) is considered *total* (written \( f : X \rightrightarrows Y \)) if \( \text{dom}(f) := \{x \in X : f(x) \neq \emptyset\} \) coincides with \( X \). Following [Weih08, DEFINITION 7], \( g \) is said to *tighten* \( f \) (and \( f \) to *loosen* \( g \)) if both \( \text{dom}(f) \subseteq \text{dom}(g) \) and \( \forall x \in \text{dom}(f) : g(x) \subseteq f(x) \) hold; see Figure 1a) and note that tightening is obviously reflexive and transitive. Furthermore write \( f[S] := \bigcup_{x \in S} f(x) \) for \( S \subseteq X \) and \( \text{range}(f) := f[X] \); also \( f|_S := f \cap (S \times Y) \) and \( f|^T := f \cap (X \times T) \) for \( T \subseteq Y \).

Finally let \( f^{-1} := \{(y, x) : (x, y) \in f\} \) denote the inverse of \( f \), i.e. such that \( (f^{-1})^{-1} = f \) and \( \text{range}(f) = \text{dom}(f^{-1}) \).

## 2 Continuity for Relations

For multivalued mappings, the literature knows a variety of easily confusable notions of continuity like [KlTh84, 5.7] or [ScNe07]. Some of them capture the intuition that, upon input \( x \), all \( y \in f(x) \) occur as output for some ‘nondeterministic’ choice [Brat03, SECTION 7]; or that the ‘value’ \( f(x) \) be produced extensionally as a set [Spre09]. Here we pursue the original conception that, upon input \( x \), *some* value \( y \) be output subject to the condition \( y \in f(x) \).

\(^5\) It has been observed that a continuous function \( f : [0,1] \rightarrow [0,1] \) will usually not have a *least* oracle rendering it computable [MiIl04]

\(^6\) We reserve the original term “restriction” to denote either \( f|_A := f \cap (A \times Y) \) or \( f|^B := f \cap (X \times B) \) for some \( A \subseteq X \) or \( B \subseteq Y \).
Definition 2. Let \((X,d)\) and \((Y,e)\) denote metric spaces and abbreviate \(B(x,r) := \{x' \in X : d(x,x') < r\} \subseteq X\) and \(\overline{B}(x,r) := \{x' \in X : d(x,x') \leq r\}\); similarly for \(Y\).

Now fix some \(f : X \Rightarrow Y\) and call \((x,y) \in f\) a point of continuity of \(f\) if the following formula holds (cf. Figure 1b):

\[
\forall \varepsilon > 0 \ \exists \delta > 0 \ \forall x' \in B(x,\delta) \cap \text{dom}(f) \ \exists y' \in B(y,\varepsilon) \cap f(x') .
\]

a) Call \(f\) strongly continuous if every \((x,y) \in f\) is a point of continuity of \(f\); equivalently:

\[
\forall x \in \text{dom}(f) \ \exists y \in f(x) \ \forall \varepsilon > 0 \ \exists \delta > 0 \ \forall x' \in B(x,\delta) \cap \text{dom}(f) \ \exists y' \in B(y,\varepsilon) \cap f(x') .
\]

b) Call \(f\) weakly continuous if the following holds:

\[
\forall x \in \text{dom}(f) \ \exists y \in f(x) \ \forall \varepsilon > 0 \ \exists \delta > 0 \ \forall x' \in B(x,\delta) \cap \text{dom}(f) \ \exists y' \in B(y,\varepsilon) \cap f(x') .
\]

c) Call \(f\) uniformly weakly continuous if the following holds:

\[
\forall \varepsilon > 0 \ \exists \delta > 0 \ \forall x \in \text{dom}(f) \ \exists y \in f(x) \ \forall x' \in B(x,\delta) \cap \text{dom}(f) \ \exists y' \in B(y,\varepsilon) \cap f(x') .
\]

d) Call \(f\) nonuniformly weakly continuous if the following holds:

\[
\forall \varepsilon > 0 \ \forall x \in \text{dom}(f) \ \exists y \in f(x) \ \forall x' \in B(x,\delta) \cap \text{dom}(f) \ \exists y' \in B(y,\varepsilon) \cap f(x') .
\]

e) Call \(f\) uniformly strongly continuous if the following holds:

\[
\forall \varepsilon > 0 \ \exists \delta > 0 \ \forall x \in \text{dom}(f) \ \exists y \in f(x) \ \forall x' \in B(x,\delta) \cap \text{dom}(f) \ \exists y' \in B(y,\varepsilon) \cap f(x') .
\]

f) Call \(f\) semi-uniformly strongly continuous if the following holds:

\[
\forall \varepsilon > 0 \ \forall x \in \text{dom}(f) \ \exists \delta > 0 \ \forall y \in f(x) \ \forall x' \in B(x,\delta) \cap \text{dom}(f) \ \exists y' \in B(y,\varepsilon) \cap f(x') .
\]

Items a) and b) are quoted from [BrHe94, DEFINITION 2.1]. In the single-valued case, quantifications over \(y \in f(x)\) and \(y' \in f(x')\) drop out. Here, all a),b),d),f) collapse to classical continuity; and both c) and e) to uniform continuity. In the multivalued case, however, these notions are easily seen distinct. Note for instance that in f), \(\delta\) may depend on \(x\) but not on \(y\); whereas \(y\) may depend on \(\varepsilon\) in c) but not in b). Logical connections between the various notions are collected in the following.
Lemma 3. a) Strong continuity implies weak continuity
b) but not vice versa.
c) Weak continuity implies nonuniform weak continuity.
d) Uniform weak continuity implies nonuniform weak continuity.
e) Let \( f \) be uniformly weakly continuous and suppose that \( f(x) \subseteq Y \) is compact for every \( x \in X \). Then \( f \) is weakly continuous.
f) Uniform strong continuity implies semi-uniform strong continuity which in turn implies strong continuity.
g) For compact \( \text{dom}(f) \subseteq X \), nonuniform weak continuity implies uniform weak continuity.
h) If \( f(x) \subseteq Y \) is compact for every \( x \in X \), then strong continuity implies semi-uniform strong continuity.
i) If \( f \subseteq X \times Y \) is compact and strongly continuous, it is uniformly strongly continuous.
j) If \( f \subseteq X \times Y \) is compact, then so are \( \text{dom}(f) \subseteq X \) and \( f[S] \subseteq Y \), for every closed \( S \subseteq X \); in particular \( f(x) \) is compact.
k) If \( f \subseteq X \times Y \) is compact, then its inverse \( f^{-1} \subseteq Y \times X \) are compact.
l) If \( X \) is compact and single-valued total \( f : X \to Y \) is continuous, then both \( f \subseteq X \times Y \) and its inverse \( f^{-1} \subseteq Y \times X \) are compact.

Note that the (classically trivial) implication from (weak) uniform continuity to (weak) continuity in e) is based on the (again, classically trivial) hypothesis that \( f(x) \subseteq Y \) be compact. Similarly, the classical fact that continuity on a compact set classically yields uniform continuity is generalized in g)+c).

![Fig. 2.](image)

Fig. 2. a) Example of a uniformly weakly continuous but not weakly continuous relation. b) A semi-uniformly strongly continuous relation which is not uniformly strongly continuous. c) A compact, weakly and uniformly weakly continuous relation which is not computable relative to any oracle.

Proof. Items a), c), d), and f) are obvious.

b) is due to [BrHe94, PROPOSITION 2.3(3)]; cmp. Example 4d).

e) Fix \( x \in \text{dom}(f) \). By hypothesis there exists, to every \( \varepsilon = 1/n \), some \( \delta_n \) and \( y_n \in f(x) \) with: \( \forall x' \in B(x, \delta_n) \cap \text{dom}(f) \exists y' \in B(y_n, 1/n) \cap f(x') \). Now since \( f(x) \) is compact, there some subsequence \( y_{nm} \) of \( y_n \) converges to, say, \( y_0 \in f(x) \) with \( d(y_{nm}, y_0) \leq 1/m \). We claim that this \( y_0 \) (which does not depend on \( \varepsilon \) anymore) satisfies

\[
\forall \varepsilon = 2/m > 0 \exists \delta := \delta_{nm} > 0 \forall x' \in B(x, \delta) \cap \text{dom}(f) \exists y' \in B(y_0, \varepsilon) \cap f(x').
\]

Indeed, to arbitrary \( x' \in B(x, \delta_{nm}) \cap \text{dom}(f) \), the hypothesis yields some \( y' \in B(y_0, 1/m) \cap f(x') \). Then, by triangle inequality, it follows \( y' \in B(y_0, 2/m) \).

Note that a different \( x \) may require a different subsequence \( n_m \); hence \( \delta \) may become dependent on \( x \) even if it did not before.
g) We claim that Definition 2d) is equivalent to the formula
\[ \forall \varepsilon > 0 \ \forall x \in \text{dom}(f) \ \exists \delta > 0 : \ \Phi(f, \varepsilon, x, \delta) \] (1)
where \( \Phi(f, \varepsilon, x, \delta) \) abbreviates the predicate
\[ \forall x' \in B(x, \delta) \cap \text{dom}(f) \ \exists y' \in f(x') \ \forall x'' \in B(x, \delta) \cap \text{dom}(f) \ \exists y'' \in f(x'') : \ \varepsilon(y', y'') < \varepsilon \]
Indeed, \( x', x'' \in B(x, \delta) \) yield \( y' \in f(x') \cap B(y, \varepsilon) \) and \( y'' \in f(x'') \cap B(y, \varepsilon) \), hence \( \varepsilon(y', y'') < 2\varepsilon \) by triangle inequality; and, conversely, \( x' := x \) yields \( y \in f(x) \). Next observe that, again by triangle inequality, \( \Phi(f, \varepsilon, x, \delta) \) implies \( \Phi(f, \varepsilon, z, \delta/2) \) for all \( z \in B(x, \delta/2) \cap \text{dom}(f) \).
Now for arbitrary but fixed \( \varepsilon \) and to every \( x \in \text{dom}(f) \) there exists by hypothesis some \( 0 < \delta = \delta(x) \) such that \( \Phi(f, \varepsilon, x, \delta(x)) \) holds. The open sets \( B(x, \delta(x)/2) \) cover \( \text{dom}(f) \); and by compactness, finitely many of them suffice to do so: say, \( B(x, \delta(x_i)/2), i = 1, \ldots, I \). Now take \( \delta > 0 \) as the minimum over these finitely many \( \delta(x_i)/2 \); it will satisfy \( \Phi(f, \varepsilon, y, \delta) \) for all \( y \in \text{dom}(f) \).

h) Similarly to g), consider the predicate
\[ \forall \varepsilon > 0 \ \forall x \in \text{dom}(f) \ \forall y \in f(x) \ \exists \delta \in (0, \varepsilon) \ \exists x', x'' \in B(x, \delta) \cap \text{dom}(f) \ \exists y' \in f(x') \cap B(y, \delta) \ \exists y'' \in f(x'') \cap B(y', \varepsilon) \]
and note that it is equivalent to strong continuity: The restriction to \( \delta < \varepsilon \) is no loss of generality; \( y' \in B(y, \delta) \) and \( y'' \in f(x'') \cap B(y, \varepsilon) \) according to b) implies \( \varepsilon(y', y'') < \delta + \varepsilon < 2\varepsilon \) arbitrary; whereas, conversely, strong continuity is recovered with \( x' := x \) and \( y' := y \). Finally, \( \Phi(f, \varepsilon, x, y, \delta) \) implies \( \Phi(f, \varepsilon, x, y, \delta/2) \) for all \( y \in B(y, \delta/2) \). Now the balls \( B(y, \delta(y)/2), y \in f(x), \) cover \( f(x) \); and by compactness, finitely many of them suffice to do so.

j) This time abbreviate
\[ \Phi(f, x, y, \varepsilon, \delta) := \forall x' \in B(x, \delta) \cap \text{dom}(f) \ \exists y' \in f(x') \cap B(y, \varepsilon) \]
and observe that strong continuity \( \forall \varepsilon > 0 \ \forall (x, y) \in f \ \exists \delta > 0 \ \Phi(f, x, y, \varepsilon, \delta/2) \) is equivalent to \( \forall \varepsilon > 0 \ \forall (x, y) \in f \ \exists \delta > 0 \ \Phi(f, x, y, \varepsilon, \delta/2) \). Moreover, \( \Phi(f, x, y, \varepsilon, \delta/2) \) and \( (x, y) \in f \cap (B(x, \delta/2) \times B(y, \varepsilon/2)) \) together imply \( \Phi(f, x, y, \varepsilon, \delta/2) \). For fixed \( \varepsilon > 0 \) there exists by hypothesis to each \( (x, y) \in f \) some \( \delta = \delta(x, y) \) such that \( \Phi(f, x, y, \varepsilon/2, \delta) \). The open balls \( B(x, \delta(x)/2) \times B(y, \varepsilon/2), (x, y) \in f, \) thus cover \( f; \) and by compactness, already finitely many of them suffice to do so. Taking \( \delta \) as the minimum of their corresponding \( \delta(x, y) \), we conclude that \( \Phi(f, x, y, \varepsilon, \delta/2) \) holds for all \( (x, y) \in f: \) uniform strong continuity.

k) Let \( U_i \subseteq X \) \( (i \in I) \) denote an open covering of \( \text{dom}(f) \). Then \( U_i \times Y \) is an open covering of \( f \), hence contains a finite subcover: whose projection onto the first component is a finite subcover of \( U_i \).
Similarly, let \( V_j \subseteq Y \) \( (j \in J) \) denote an open covering of \( f[S] \subseteq Y \). Then \( X \times V_j \), together with \( (X \setminus S) \times Y \), constitutes an open covering of \( f; \) hence contains a finite subcover: and the corresponding \( V_j \) yield a finite subcover of \( f[S] \).
Finally, \( S := \{x\} \) is closed and thus also \( f[S] = f(x) \).

l) Let \( (x_n, y_n) \subseteq f \) be a sequence. Since \( (x_n) \subseteq X \) compact, it has a converging subsequence; w.l.o.g. \( (x_n) \) itself. Now by continuity and single-valuedness, \( y_n = f(x_n) \rightarrow f(x) \) converges. Thus, \( f \) is compact; and homeomorphic to \( f^{-1} \).
We say that $f$ is **pointwise compact** if $f(x) \subseteq Y$ is compact for every $x \in \text{dom}(f)$. Any single-valued $f$ automatically satisfies this condition; which in turn implies being **pointwise closed** as required in Fact 1. Pointwise compactness is essential for uniform weak continuity to imply weak continuity in Lemma 3c):

**Example 4.** a) The multifunction from [Zieg09, Example 27c], namely
\[
f : [-1, +1] \Rightarrow [0, 1], \quad 0 \geq x \mapsto [0, 1], \quad 0 < x \mapsto \{1\}
\]
depicted in Figure 2a), is uniformly weakly continuous but not weakly continuous.

b) The multifunction $g : [0, 1] \Rightarrow [0, 1]$ with graph($g$) = $([0, 2/3] \times \{0\}) \cup ([1/3, 1] \times \{1\})$ depicted in Figure 2b) has compact $\text{dom}(g)$ and $g(x)$ for every $x$ but graph($g$) is not compact. Moreover, $g$ is semi-uniformly strongly continuous but not uniformly strongly continuous.

c) The relation $(\mathbb{Q} \times (\mathbb{R} \setminus \mathbb{Q})) \cup ((\mathbb{R} \setminus \mathbb{Q}) \times \mathbb{Q})$ from [BrHe94, Example 7.2] is uniformly strongly continuous.

d) Inspired by [BrHe94, Proposition 2.3(3)], the relation $g : [-1, +1] \Rightarrow [-1, +1]$ depicted in Figure 2c) with graph
\[
\{(x, 0) : x \leq 0\} \cup \{(x, -1) : x > 0\} \cup \{(x, 1 + (-1)^n) : n \in \mathbb{N}, 1/(n + 1) \leq x \leq 1/n\}
\]
is compact and both weakly continuous and uniformly weakly continuous but not strongly continuous.

**Proof.** a) To assert uniform weak continuity, consider $\delta = \delta(\varepsilon) := \varepsilon$. Moreover let $y = y(x, \varepsilon) := 1$ for $x > 0$ and $y(x, \varepsilon) := 1 - \varepsilon/2$ for $x \leq 0$. Then, in case $x' > 0$, choose $y' := 1$; and in case $x' \leq 0$, chose $y' := 1 - \varepsilon/2$.

Suppose $f$ is weakly continuous at $x := 0$, i.e. there exists some appropriate $y \in f(x) =\{0, 1\}$. The consider $\varepsilon := 1 - y$ and the induced $\delta > 0$ as well as $x' := \delta/2$: No $y' \in f(x') = \{1\}$ can satisfy $\varepsilon > |y' - y| = 1 - y$, contradiction.

b) Note $\text{dom}(g) = [0, 1]$ and $g(x) = \{0\}$ for $x \leq 1/3$, $g(x) = \{0, 1\}$ for $1/3 < x < 2/3$, and $g(x) = \{1\}$ for $x \geq 2/3$: all compact. Concerning semi-uniform strong continuity, for $x \leq 1/3$ let $\delta := 1/3$ and $y' := 0 = y$; for $x \geq 2/3$ let $\delta := 1/3$ and $y' := 1 = y$; whereas for $1/3 < x < 2/3$, choose $\delta := \min(2/3 - x, x - 1/3)$ and $y' := y$. Uniform strong continuity leads to a contradiction when considering $x := 1/3 + \delta/2$ and $y := 1$ and $x' := 1/3$.

c) Let $\delta := 1$; then observe that $\mathbb{Q}$ is dense in $\mathbb{R} \setminus \mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$ is dense in $\mathbb{R}$.

d) Concerning weak continuity, in case $x \leq 0$ choose $y := 0$ and $\delta := \varepsilon$: then, to $x' \in B(x, \delta)$, $y' := 0$ will do for $x' \leq 0$ as well as for every $x' \in [1/(n + 1), 1/n]$ with $n$ odd; and $y' := 2/(n + 1)$ for $x' \in [1/(n + 1), 1/n]$ with even $n$. In case $x > 0$ choose $y := -1$ and $\delta := x$; then $x' \in B(x, \delta)$ implies $x' > 0$ and $y' := -1$ works.

Regarding uniform weak continuity, let $\delta := \varepsilon$ and distinguish cases $x < \varepsilon$ and $x \geq \varepsilon$. In the former case, $y := 0$ will do for $x \leq 0$ and for $x \in [1/(n + 1), 1/n]$ with $n$ odd; and $y := 2/(n + 1)$ for $x \in (0, \varepsilon) \cap [1/(n + 1), 1/n]$ with even $n$. In the latter case, $y := -1$ works.

Strong continuity is violated, e.g., at $(x, y) = (1/2, 2/3)$ for $\varepsilon := 1/4$. \qed

### 2.1 Continuity and Computability of Relations

Recall that (relative) computability of a multifunction $f : \mathbb{R} \Rightarrow \mathbb{R}$ means that some (oracle) Turing machine can, upon input of any sequence of integer fractions $a_n/b_n$ with $|x - a_n/b_n| \leq$
$2^{-n}$ for every $n \in \mathbb{N}$ and some $x \in \text{dom}(f)$, output a sequence $u_m/v_m$ of integer fractions with $|y - u_m/v_m| \leq 2^{-m}$ for every $m \in \mathbb{N}$ and some $y \in f(x)$. More generally, a multifunction $f : \subseteq A \Rightarrow B$ between represented spaces $(A, \alpha)$ and $(B, \beta)$ is considered (relatively) computable if it admits a (relatively) computable $(\alpha, \beta)$–realizer, that is a function $F : \subseteq \{0, 1\}^\omega \Rightarrow \{0, 1\}^\omega$ mapping every $\alpha$–name of some $a \in \text{dom}(a)$ to a $\beta$–name of some $b \in f(a)$ [Weih00, Definition 3.1.3].

**Lemma 5.** Define the composition of multifunction $f : \subseteq X \Rightarrow Y$ and $g : \subseteq Y \Rightarrow Z$ as

$$g \circ f := \{(x, z) \mid x \in X, z \in Z, f(x) \subseteq \text{dom}(g), \exists y \in Y : (x, y) \in f \land (y, z) \in g\}.$$  \hspace{1cm} (3)

a) id$_X$ tightens $f^{-1} \circ f$; if $f$ is single-valued, then $f \circ f^{-1} = \text{id}_{\text{range}(f)}$.

b) If $f'$ tightens $f$ and $g'$ tightens $g$, then $g' \circ f'$ tightens $g \circ f$.

c) If $\text{range}(f) \subseteq \text{dom}(g)$ holds and both $f$ and $g$ are compact, then so is $g \circ f$.

d) If $\text{range}(f) \subseteq \text{dom}(g)$ holds and if both $f$ and $g$ map compact sets to compact sets, then so does $g \circ f$.

e) Fix representations $\alpha$ for $X$ and $\beta$ for $Y$. A multifunction $F : \subseteq \{0, 1\}^\omega \Rightarrow \{0, 1\}^\omega$ tightens $\beta^{-1} \circ f \circ \alpha$ iff $\beta \circ F \circ \alpha^{-1}$ tightens $f$.

f) A function $F : \subseteq \{0, 1\}^\omega \Rightarrow \{0, 1\}^\omega$ is an $(\alpha, \beta)$–realizer of $f$ iff $F$ tightens $\beta^{-1} \circ f \circ \alpha$ iff $\beta \circ F \circ \alpha^{-1}$ tightens $f$.

Motivated by f), let us call a multifunction $F$ as in e) an $(\alpha, \beta)$–multirealizer of $f$.

**Proof.**  a) Note $f(x) \subseteq \text{dom}(f^{-1})$ and $f^{-1} \circ f = \{(x, x') : \exists y : (x, y), (x', y) \in f\}$.

b) Note $\text{dom}(g \circ f) = \{x \in \text{dom}(f) : f(x) \subseteq \text{dom}(g)\}$; hence $\text{dom}(f) \subseteq \text{dom}(f') \land \text{dom}(g) \subseteq \text{dom}(g') \land f(x) \supseteq f'(x) \land g(y) \supseteq g'(y)$ implies $\text{dom}(g \circ f) \subseteq \text{dom}(g' \circ f')$ as well as $(g' \circ g')(x) = \{z : \exists y \in f'(x) \land y \in g'(y)\} \subseteq (g \circ f)(x)$; cmp. [Weih08, Lemma 8.3].

c) Since $\text{range}(f) \subseteq \text{dom}(g)$, $g \circ f$ is the image of compact $(f \times \text{range}(g)) \cap (\text{dom}(f) \times g) \subseteq X \times Y \times Z$ under the continuous projection $\Pi_{1,3} : X \times Y \times Z \ni (x, y, z) \mapsto (x, z) \in X \times Z$.

d) immediate from $(g \circ f)[S] = g[f[S]]$, holding under the hypothesis $\text{range}(f) \subseteq \text{dom}(g)$.

e) If $F$ tightens $\beta^{-1} \circ f \circ \alpha$, then $\beta \circ F \circ \alpha^{-1}$ tightens $\beta \circ \beta^{-1} \circ f \circ \alpha \circ \alpha^{-1}$ due to b); which in turn coincides with $\text{id}_X \circ f \circ \text{id}_Y = f$ according to a).

Conversely, $F = \text{id}_{\{0, 1\}^\omega} \circ F \circ \text{id}_{\{0, 1\}^\omega}$ tightens $\beta \circ \beta^{-1} \circ F \circ \alpha \circ \alpha^{-1}$ by a); which in turn tightens $\beta^{-1} \circ f \circ \alpha$ by hypothesis and by b).

f) $F$ being an $(\alpha, \beta)$–realizer of $f$ means $\text{dom}(F) \supseteq \text{dom}(f \circ \alpha)$ and $\beta(F(\sigma)) \subseteq \text{dom}(f \circ \alpha)$ for every $\sigma \in \text{dom}(f \circ \alpha)$; now apply e). \hspace{1cm} $\square$

The above notion composition for relations is, like that of ‘tightening’, from [Weih08, Section 3]. Mapping compact sets to compact sets is a property which turns out useful below. It includes both compact relations (Lemma 3k) and continuous functions:

**Example 6.** a) Let $f : X \rightarrow Y$ be a single-valued continuous function. Then $f$ maps compact sets to compact sets.

b) The inverse $(p_{sd}^d)^{-1}$ of the $d$-dimensional signed digit representation maps compact set to compact sets.

c) The functions $\text{id} : x \rightarrow x$ and $\text{sgn} : \mathbb{R} \rightarrow \{-1, 0, 1\}$ both map compact sets to compact sets; however their Cartesian product $\text{id} \times \text{sgn}$ does not map compact $\{(x, x) : -1 \leq x \leq 1\}$ to a compact set.
Indeed, the signed digit representation $\rho_{\text{sd}}$ is well-known proper \cite[pp.209-210]{Weih00}, i.e. preimages of compact sets are compact.

Focusing on complete separable metric spaces and pointwise compact multifunctions, strong continuity is in view of Fact 1 (in general strictly) stronger than relative computability; whereas weak continuity is (again in general strictly) weaker than relative computability:

Example 7. a) The relation (2) from Example 4d) is not computable relative to any oracle.
b) The relation from Example 4c) is (uniformly strongly continuous but, lacking pointwise compactness) not computable relative to any oracle.
c) The closure of the relation from Example 4b), that is with graph $([0,2/3] \times \{0\}) \cup ([1/3,1] \times \{1\})$, is computable but not strongly continuous.

\textbf{Proof.} a) by contradiction: Suppose some oracle machine $M$ computes this relation. On input of the rational sequence $(0,0,0,\ldots)$ as a $\rho$-name of $x := 0$ it thus outputs a $\rho$-name of $y = 0$, i.e. a rational sequence $(p_m)$ with $|p_m| < 2^{-m}$. In particular it prints $p_1 > -1/2$ after having read only finitely many elements from the input sequence; say, up to the $(N-1)$-st element. Now consider the behavior of $M$ on the input sequence $(0,0,\ldots,0,2^{-N},2^{-N},\ldots)$ as $\rho$-name of $x' := 2^{-N}$: Its output sequence $(p_m')$ will, again, begin with $p'_1 = p_1 > -1/2$ and thus cannot be a $\rho$-name of $-1$. Since $g(x') = \{-1,0,2/(1+2^N)\}$, it must therefore satisfy $|p'_m - y| < 2^{-m}$ for all $m$ and for one of $y = 0 =: y_0$ or $y = 2/(1+2^N) =: y_1$. In particular, $p'_{N+1}$ satisfies $y_j \in B(y_j, 2^{-N-1}) \ni y_1-j$ for the unique $j \in \{0,1\}$ with $y = y_j$ and is printed upon reading only the first, say, $N' \geq N$ elements of $(0,0,\ldots,2^{-N},2^{-N},\ldots)$. Finally it is easy to extend this finite sequence to a $\rho$-name of some $x''$ close to $x'$ with $y_j \notin g(x'') \ni y_{1-j}$ and upon this input $M$ will now, again, output elements $p'_1, \ldots, p'_{N+1}$ which, however, cannot be extended to a $\rho$-name of any $y'' \in g(x'')$: contradiction.

b) see \cite[p.24]{BrHe94}.

c) Immediate. \hfill $\square$

For relations with discrete range, on the other hand, we have

\textbf{Theorem 8.} Let $X, Y$ be computable metric spaces \cite[Definition 8.1.2]{Weih00}.
If $Y$ is discrete and $f : X \entails Y$ weakly continuous, then $f$ is relatively computable.

\textbf{Proof.} Since $Y$ is discrete, $\varepsilon := \min_{y \neq y'} d(y,y') > 0$. Now to $y \in Y$ consider the set

$$U_y := \{ x \in \text{dom}(f) : \exists \delta > 0 \forall x' \in B(x,\delta) \cap \text{dom}(f) \exists y' \in f(x') \cap B(y,\varepsilon) \}$$

and note that it is open in $\text{dom}(f)$ because $y' \in B(y,\varepsilon)$ requires $y' = y$. Hence $U_y = \text{dom}(f) \cap \bigcup_{j \in \mathbb{N}} B(q_{j,y},1/n_{j,y})$ for certain $n_{j,y} \in \mathbb{N}$ and $q_{j,y}$ from the fixed dense subset of $X$. Now consider an encoding of (names of) these $q_{j,y}$ and $n_{j,y}$ as oracle. Then, given $x \in \text{dom}(f)$, search for some $(j,y)$ with $x \in B(q_{j,y},1/n_{j,y}) \subseteq U_y$: when found, such $y$ by construction belongs to $f(x)$ and, conversely, weak continuity asserts $x$ to belong to $U_y$ for some $y$. \hfill $\square$

\subsection{2.2 Motivation for Uniform Continuity}

Many proofs of uncomputability of relations or of topological lower bounds \cite{Zieg09} apply weak continuity as a necessary condition: merely necessary, in view of the above example, and thus of limited applicability. The rest of this work thus explores topological conditions stronger than weak continuity yet necessary for relative computability.
Relative Computability and Uniform Continuity of Relations

Uniform continuity of functions is such a stronger notion — and an important concept of its own in mathematical analysis — yet does not straightforwardly (or at least not unanimously) extend to multifunctions. Guided by the equivalence between uniform continuity and relative computability for functions with compact graph, our aim is a topological characterization of oracle-computable compact real relations. One such characterization is Fact 1; however we would like to avoid (second-order) quantifying over tightenings.

To this end observe that every (relatively) computable function \( f \) is (relatively) effectively locally uniformly continuous [Weih00, Theorem 6.2.7], that is, uniformly continuous on every compact subset \( K \subseteq \text{dom}(f) \) [KrWe87]:

\[
\forall \varepsilon > 0 \; \exists \delta > 0 \; \forall x \in K \; \forall x' \in B(x, \delta) \cap K : \; d(f(x), f(x')) < \varepsilon .
\]

This suggests to look for related concepts for multifunctions, i.e. where \( \delta \) does not depend on \( x \). Uniform weak continuity in the sense of Definition 2c), however, fails to strengthen weak continuity because it allows \( y \) to depend on \( \varepsilon \).

3 Henkin-Continuity

In view of the above discussion, we seek for an order on the four quantifiers

\[
\forall x \in \text{dom}(f), \; \exists y \in f(x), \; \forall \varepsilon > 0, \; \exists \delta > 0
\]

such that \( y \) does not depend on \( \varepsilon \) and \( \delta \) does not depend on \( x \). This cannot be expressed in classical first-order logic and has spurred the introduction of the non-classical so-called Henkin Quantifier [Vaan07]

\[
Q_H(x, y, \varepsilon, \delta) = \left( \begin{array}{c}
\forall x \exists y \\
\forall \varepsilon \exists \delta
\end{array} \right)
\]

where the suggestive writing indicates that very condition: that \( y \) may depend on \( x \) but not on \( \varepsilon \) while \( \delta \) may depend on \( \varepsilon \) but not on \( x \). We thus adopt from [Bees85, p.380] the following\footnote{Its generalization from metric to uniform spaces is immediate but beyond our purpose.}

**Definition 9.** Call \( f \) Henkin-continuous if the following holds:

\[
\left( \begin{array}{c}
\forall \varepsilon > 0 \; \exists \delta > 0 \\
\forall x \in \text{dom}(f) \; \exists y \in f(x)
\end{array} \right) \; \forall x' \in B(x, \delta) \cap \text{dom}(f) \; \exists y' \in B(y, \varepsilon) \cap f(x') . \tag{4}
\]

Observe that uniform strong continuity implies Henkin-continuity; from which in turn follows both weak continuity and uniform weak continuity. In fact, Henkin-continuity is strictly stronger than the latter two:

**Example 10.** a) The relation \( g \) from Examples 4d) and 7a) is (compact and both weakly continuous and uniformly weakly continuous but) not Henkin-continuous.

b) It does, however, satisfy \( \left( \begin{array}{c}
\forall \varepsilon > 0 \; \exists \delta > 0 \\
\forall x, x' \exists y \in g(x)
\end{array} \right) \; \exists y' \in g(x') \left( x' \in B(x, \delta) \rightarrow y' \in B(y, \varepsilon) \right) . \)

c) The relations from Examples 4b) and 7c) are (computable and) Henkin-continuous.
Proof. a) by contradiction: Suppose \( y = y(x) \) satisfies Equation (4). Now let \( \varepsilon := 1/2 \) and consider \( \delta := \delta(\varepsilon) \) according to Equation (4). Then \( y(x) = -1 \) is impossible for all \( 0 < x < \delta \), as \( x' := (x - \delta)/2 < 0 \) implies \( g(x') = \{0\} \) which is disjoint to \( B(y, \varepsilon) \). Now consider \( \varepsilon' := \delta \cdot 2/3 \) and \( \delta' := \delta(\varepsilon') \). We claim that \( y(x) = -1 \) is necessary for all \( x > \varepsilon' \), this leading to a contradiction for \( \delta \cdot 2/3 < x < \delta \). Indeed, in case \( y(x) = x \), rational \( x' \in B(x, \min\{\delta', \delta/3\}) \) implies \( g(x') = \{0\} \) which is disjoint to \( B(y, \varepsilon') \); whereas in case \( y(x) = 0 \), irrational \( x' \in B(x, \min\{\delta', \delta/3\}) \) implies \( g(x') = \{x'\} \) which is disjoint to \( B(y, \varepsilon') \).

b) Let \( \delta := \varepsilon \) and take \( y := -1 \) in case \( x, x' > 0 \); \( y := 0 \) in case \( x \leq 0 \); and \( \{y\} := g(x) \cap [0, 1] \) in case \( x' \leq 0 \).

c) For \( x \leq \frac{1}{2} \) choose \( y := 0 \) and for \( x > \frac{1}{2} \) choose \( y := 1 \); independently, choose \( \delta := \frac{1}{6} \). □

3.1 Further Examples and Some Properties

Recall that, for single-valued functions, Henkin-continuity coincides with uniform continuity.

**Example 11.** Recall from the Type-2 Theory of Effectivity (TTE) the Cauchy representation \( \rho_{\text{C}} \) [Weih00, Definition 4.1.5] and the signed digit representation \( \rho_{\text{sd}} \) [Weih00, Definition 7.1.4] of real numbers.

a) \( \rho_{\text{sd}} : \mathbb{B}^\omega \to \mathbb{R} \) is not uniformly continuous

b) nor is the restriction \( \rho_{\text{C}}^{[0,1]} : \mathbb{B}^\omega \to [0, 1] \); cmp. [Weih00, Example 7.2.3].

c) However for every compact \( K \subseteq \mathbb{R} \), the restriction \( \rho_{\text{sd}}|_K : \mathbb{B}^\omega \to K \) is uniformly (i.e. Henkin-) continuous;

d) and so are the restrictions \( \rho_{\text{C}}|_C : C \to \mathbb{R} \) and \( \rho_{\text{sd}}|_C : C \to \mathbb{R} \) for any compact \( C \subseteq \mathbb{B}^\omega \).

e) \( \rho_{\text{C}}^{-1} : \mathbb{B}^\omega \to \mathbb{R} \), \( \mathbb{R} \ni x \to \{\bar{\sigma} : \rho_{\text{C}}(\bar{\sigma}) = x\} \), the inverse of the Cauchy representation, is Henkin-continuous.

f) Let \( \langle \cdot, \cdot \rangle : \mathbb{N} \times \mathbb{N} \to \mathbb{N} \) be an integer pairing function with \( \langle n, m \rangle \geq n + m \) for every \( n, m \in \mathbb{N} \). Then the string pairing function \( \{0, 1\}^{\omega \times \omega} \to \{0, 1\}^\omega \), \( (b_{\langle n, m \rangle})_{n,m \in \omega} \to (b_k)_{k \in \omega} \) is 1-Lipschitz (and thus uniformly) continuous.

Proof. a) Consider some large integer \( x = 2^k \in \mathbb{N} \) with \( \rho_{\text{sd}} \)-name \( 10 \cdots 0 \cdots \) (each digit 0, 1, \( \bar{1} \), and the point , encoded as a constant-length string over \( \{0, 1\}^* \)). Then modifying this name \( \bar{\sigma} \) at the \( k \)-th position affects the value \( \rho_{\text{sd}}(\bar{\sigma}) \) by an absolute value of 1. In particular, to \( \varepsilon := 1, \delta > 0 \) satisfying

\[
d(\bar{\sigma}, \tau) < \delta \quad \Rightarrow \quad d(\rho_{\text{sd}}(\bar{\sigma}), \rho_{\text{sd}}(\tau)) < \varepsilon
\]

must depend on the value of \( x = 2^k \), i.e. on \( \bar{\sigma} \).

b) Fix \( k \in \mathbb{N} \), and consider integers \( a_n := 2^{k+n} \) and \( b_n := 3 \cdot 2^{k+n} \). Hence the concatenation \( \bar{\sigma} \) of binary-encoded numerators \( a_n \) and denominators \( b_n \) constitutes a \( \rho_{\text{C}} \)-name of \( x := 1/3 \). Note that the secondmost-significant digit of \( b_k \) resides roughly at position \( \# k \) in \( \bar{\sigma} \). Hence switching to \( a'_n := a_n \) and \( b'_n := 2 \cdot 2^{k+n} \) yields \( \bar{\sigma}' \) of metric distance to \( \bar{\sigma} \) of order \( \delta = 2^{-k} \); whereas the value \( x' = \rho_{\text{C}}(\bar{\sigma}') = 1/2 \) changes by \( \varepsilon = 1/6 \).

c) First consider the case \( K = [0, 1] \). Then, modifying the \( k \)-th digit \( b_k \in \{0, +1, -1\} \) of a signed digit expansion \( \sum_{n=0}^{\infty} b_n 2^{-n} \) affects its value by no more than \( 2^{-k} \). In the general case, let \( 2^\ell \) denote a bound on \( K \). Then, similarly, modifying the \( k \)-th position of a signed digit expansion \( \sum_{n=-N}^{\infty} b_n 2^{-n} \) affects its value by no more than \( 2^{\ell-k} \).
d) Like any admissible representation, $\rho_C$ and $\rho_{ad}$ are continuous; hence uniformly continuous on compact subsets.

e) To $\varepsilon = 2^{-k} > 0$ let $\delta := 2^{-k}$. Now consider arbitrary $x \in \mathbb{R}$ and as $\rho_C$-name $\bar{\sigma}$ the (binary encodings of numerators and denominators of the) dyadic sequence $q_n := \lfloor x \cdot 2^{n+1} \rfloor / 2^{n+1}$. In fact it holds $|x - q_n| \leq 2^{-n-1} \leq 2^{-n}$. Now $x' \in B(x', \delta)$ has $|x' - q_n| \leq 2^{-k} + 2^{-n-1} \leq 2^{-n}$ for $n \leq k - 1$. Therefore the first $k - 1$ elements of $(q_n)$, and in particular the first $k - 1$ symbols of $\bar{\sigma}$, extend to a $\rho_C$-name $\bar{\tau}$ of $x'$; i.e. such that $d(\bar{\sigma}, \bar{\tau}) < \varepsilon$.

f) Modifying the the argument at index $(n, m)$ affects the image at index $(n, m) \geq n + m$, i.e. the metric at weight $\leq 2^{-(n+m)}$. □

A classical property both of continuity and uniform continuity is closure under restriction and under composition. Also Henkin-continuity passes these (appropriately generalized) sanity checks:

**Observation 12.** a) Let $f : \subseteq X \times Y$ be Henkin-continuous and tighten $g : \subseteq X \times Y$. Then $g$ is Henkin-continuous, too.

b) If $f : \subseteq X \times Y$ and $g : \subseteq Y \times Z$ are Henkin-continuous, then so is $g \circ f : \subseteq X \times Z$.

**Proof.** a) For $g$ loosening $f$ and in the definition of Henkin-continuity of $g$, the universal quantifiers range over a subset, and the existential quantifiers range over a superset, of those in the definition of Henkin-continuity of $f$.

b) By hypothesis, we have

$$
\forall \varepsilon > 0 \quad \exists \delta > 0 \quad \forall y \in \text{dom}(g) \quad \exists z \in g(y) \quad \forall y' \in B(y, \delta) \cap \text{dom}(g) \quad \exists z' \in B(z, \varepsilon) \cap g(y')
$$

(5)

$$
\forall \delta > 0 \quad \exists \gamma > 0 \quad \forall x \in \text{dom}(f) \quad \exists y \in f(x) \quad \forall x' \in B(x, \gamma) \cap \text{dom}(f) \quad \exists y' \in B(y, \delta) \cap f(x')
$$

(6)

Thus, to $\varepsilon > 0$, take $\delta > 0$ according to Equation (5) and in turn $\gamma > 0$ according to Equation (6). Similarly, to $x \in \text{dom}(g \circ f) \subseteq \text{dom}(f)$, take $y \in f(x) \subseteq \text{dom}(g)$ according to Equations (6) and (3); and in turn $z \in g(y)$ according to Equation (5). This $z$ thus belongs to $(g \circ f)(x)$ and was obtained independently of $\varepsilon$, nor does $\gamma$ depend on $x$. Moreover to $x' \in B(x, \gamma) \cap \text{dom}(g \circ f)$ there is a $y' \in B(y, \delta) \cap f(x') \subseteq B(y, \delta) \cap \text{dom}(g)$; to which in turn there is a $z' \in B(z, \varepsilon) \cap g(y')$, i.e. $z' \in B(z, \varepsilon) \cap (g \circ f)(x')$. □

The following further example in Item b) turns out as rather useful:

**Proposition 13.** a) Every $x \in \mathbb{R}$ has a signed digit expansion

$$
x = \sum_{n=-N}^{\infty} a_n 2^{-n}, \quad a_n \in \{0, 1, \bar{1}\}
$$

(7)

with no consecutive digit pair $1\bar{1}$ nor $\bar{1}1$ nor $1\bar{1}$ nor $\bar{1}1$.

b) For $k \in \mathbb{N}$, each $|x| \leq \frac{2}{3} \cdot 2^{-k}$ admits an expansion with $a_n = 0$ for all $n \leq k$. And, conversely, $x = \sum_{n=k+1}^{\infty} a_n 2^{-n}$ with $(a_n, a_{n+1}) \in \{10, \bar{1}0, 0\bar{1}, 00\}$ for every $n$ requires $|x| \leq \frac{2}{3} \cdot 2^{-k}$.

c) Let $x = \sum_{n=-N}^{\infty} a_n 2^{-n}$ be a signed digit expansion and $k \in \mathbb{N}$ such that $(a_n, a_{n+1}) \in \{10, \bar{1}0, 0\bar{1}, 00\}$ for each $n > k$. Then every $x' \in [x - 2^{-k}/3, x + 2^{-k}/3]$ admits a signed digit expansions $x' = \sum_{n=-N}^{\infty} b_n 2^{-n}$ with $a_n = b_n \forall n \leq k$. 

Let us call a mapping \( \lambda : \mathbb{N} \rightarrow \mathbb{N} \) a modulus; and say that a multifunction \( f : \subseteq X \rightrightarrows Y \) is \( \lambda \)-continuous in \( (x, y) \in f \) if, to every \( m \in \mathbb{N} \) and every \( x' \in \text{dom}(f) \cap \overline{B}(x, 2^{-\lambda(m)}) \) there exists a \( \lambda \)-continuous in \( (x', y) \) for \( y \in f(x') \).
some \( y' \in f(x') \cap \overline{B}(y, 2^{-m}) \). Here, \( \overline{B}(x, r) := \{ x' \in X : d(x, x') \leq r \} \) denotes the closed ball of radius \( r \) around \( x \). Now Skolemization of “\( \forall \varepsilon > 0 \exists \delta > 0 \)” yields

**Observation 14.** A multifunction \( f : \subseteq X \Rightarrow Y \) is Henkin-continuous iff there exists a modulus \( \lambda \) such that, for every \( x \in \text{dom}(f) \), there exists \( y \in f(x) \) such that \( f \) is \( \lambda \)-continuous in \((x, y)\);
equivalently: if, for every \( x \in \text{dom}(f) \), \( f \) admits some single-valued total selection \( f_x : X \to Y \) \( \lambda \)-continuous in \((x, f_x(x))\) (but possibly not continuous anywhere else, see Example 16 below).

**Definition 15.** a) For \( L > 0 \), a multifunction \( f : \subseteq X \rightrightarrows Y \) is \( L \)-Lipschitz if

\[
\forall x \in \text{dom}(f) \ \exists y \in f(x) \ \forall x' \in \text{dom}(f) \ \exists y' \in B(y, L \cdot d(x, x')) \cap f(x'). \tag{8}
\]

b) Call a family \( f_i : \subseteq X_i \rightrightarrows Y_i \ (i \in I) \) of multifunctions equicontinuous if they share a common modulus in the sense that the following holds:

\[
\left( \forall \varepsilon > 0 \ \exists \delta > 0 \ \forall i \in I \ \forall x \in \text{dom}(f_i) \ \exists y \in f_i(x) \right) \ \forall x' \in B(x, \delta) \ \exists y' \in B(y, \varepsilon) \cap f(x'). \tag{9}
\]

So every Lipschitz relation is Henkin-continuous; and every family of total \( L \)-Lipschitz relations is equicontinuous. The proof of Proposition 13d) reveals Item a) of the following

**Example 16.** a) For \( \Sigma = \{0, 1, \bar{1}, .\} \), the inverse \( \rho^{-1}_{sd} : \mathbb{R} \Rightarrow \Sigma^\omega \) of the signed digit representation**, is \( \frac{3}{2} \)-Lipschitz.

b) The relation

\[
f := \{(0, 0)\} \cup \bigcup_{k \in \mathbb{N}} \left[ 2^{-k}, \max\{1, 3 \cdot 2^{-k}\} \right] \times \{2^{-k}\} \subseteq [0, 1] \times [0, 1].
\]

depicted in Figure 3 is compact and \( 1 \)-Lipschitz. Moreover, \( f \) is computable but has no locally continuous selection in \( x_0 = 0 \).

Fig. 3. Computable compact relation with no locally continuous selection in \( x_0 = 0 \).

Concerning Example 16b), the ratio \( \min\{|y - y'| : y \in f(x), y' \in f(x')\}/|x - x'| \) becomes worst for \( x = 2 \cdot 2^{-k-1} - \varepsilon \) (hence \( f(x) = \{2^{-k-1}\} \), i.e. \( y = 2^{-k-1} \)) and \( x' = 3 \cdot 2^{-k-1} + \varepsilon \)

\** Note that proceeding from alphabet \( \Sigma \) to \( \{0, 1\}^2 \) affects the Lipschitz constant by a factor of 2.
(hence $f(x') = \{2^{-k}\}$, i.e. $y = 2^{-k}$). Moreover every $(x, y) \in f$ satisfies $x/3 \leq y \leq x$. Thus the following algorithm computes $f$: Given $x \in [0, 1]$ in form of a nested sequence $[a_n, b_n]$ of intervals with rational endpoints $b_n - a_n \leq 2^{-n-1}$, test whether $[a_n, b_n] \subseteq [2^{-n}, 3 \cdot 2^{-n}]$ holds: if not, output $[a_n/3, b_n]$ and proceed to interval $\#n + 1$, otherwise switch to outputting the constant sequence $[2^{-n}, 2^{-n}]$. Note that for $x = 0$, the output sequence $[a_n/3, b_n]$ will indeed converge to $y = 0$. In case $3 \cdot 2^{-k-1} \leq x \leq 2 \cdot 2^{-k}$ on the other hand, $[a_k, b_k] \subseteq [2^{-k}, 3 \cdot 2^{-k}]$ holds and will result in the output of $y = 2^{-k} \in f(x)$, compliant with possible previous intervals $[a_n/3, b_n] \supseteq [x/3, x] \supseteq f(x)$. In the final case $2 \cdot 2^{-k-1} \leq x \leq 3 \cdot 2^{-k-1}$, at least one of $[a_k, b_k] \subseteq [2^{-k}, 3 \cdot 2^{-k}]$ and $[a_k, b_k] \subseteq [2^{-k-1}, 3 \cdot 2^{-k-1}]$ holds; hence the algorithm will produce $2^{-n}$ either for $n = k$ or for $n = k + 1$.

**Proposition 17.**  
a) I denote an ordinal and $f_i : X \Rightarrow Y$ ($i \in I$) an equicontinuous family of pointwise compact multifunctions and decreasing in the sense that $f_j$ tightens $f_i$ whenever $j > i$. Then $f(x) := \bigcap_{i \in I, f_i(x) \neq \emptyset} f_i(x)$ is again pointwise compact and Henkin-continuous a tightening of each $f_i$.  
Moreover, if all $f_i$ are $\lambda$-continuous, then so is $f$.

b) Let $f : X \Rightarrow Y$ be $\lambda$-continuous and pointwise compact for some modulus $\lambda$. Then $f$ has a minimal $\lambda$-continuous pointwise compact tightening.

**Proof.**  
a) Since the case of a finite $I$ is trivial, it suffices to treat the case $I = \mathbb{N}$ of a sequence; the general case then follows by transfinite induction. Let $x \in \text{dom}(f_i)$. Then $f_j(x) \subseteq f_i(x)$ for each $j > i$, and hence $f(x) = \bigcap_{j \geq i} f_j(x) \subseteq f_i(x)$ is (compact and) the intersection of non-empty compact decreasing sets: $f(x) \neq \emptyset$, $x \in \text{dom}(f)$. Moreover let $\varepsilon > 0$ be arbitrary and consider an appropriate $\delta$ according to Equation (9) independent of $x$; similarly take $y_j \in f_j(x)$ independent of $\varepsilon$ as asserted by equicontinuity. Then the sequence $(y_j)_{j > i}$ belongs to compact $f_j(x)$ and thus has some accumulation point $y \in f_j(x) \subseteq f_i(x)$ for each $j$: thus yields $y \in f(x)$ independent of $\varepsilon$. W.l.o.g. $y_j \to y$ by proceeding to a subsequence. Now let $d(x, x') \leq \delta$. Then by hypothesis there exists $y'_j \in f_j(x')$ with $d(y_j, y'_j) \leq \varepsilon$; and, again, an appropriate subsequence of $(y'_j)$ converges to some $y' \in f(x')$. Moreover, $d(y, y') \leq d(y, y_j) + d(y_j, y'_j) + d(y'_j, y') \leq d(y, y_j) + \varepsilon + d(y'_j, y') \to \varepsilon$.

b) Consider the family $\mathcal{F}$ of all $\lambda$-continuous and pointwise compact tightenings of $f$. According to a), these form a directed complete partial order (dcpo) with respect to total restriction. More explicitly, apply Zorn’s Lemma to get a maximal chain $(f_i)_i \subseteq I$. Then a) asserts that $g(x) := \bigcap_{i, f_i(x) \neq \emptyset} f_i(x)$ defines a $\lambda$-continuous and pointwise compact tightening of $f$. In fact a minimal one: If $h \in \mathcal{F}$ tightens $g$, then $h = f_j$ for some $j \in I$ because of the maximality of $(f_i)_{i \in I}$; hence $g$ tightens $f_j$.

**3.3 Relative Computability requires Henkin-Continuity**

With the above examples and tools, it is now easy to establish

**Theorem 18.** Let $K \subseteq \mathbb{R}$ be compact.

a) If $f : K \Rightarrow \mathbb{R}$ is computable relative to some oracle, then it is Henkin-continuous.

b) More precisely suppose $F : \{0, 1\}^\omega \Rightarrow \{0, 1\}^\omega$ is a Henkin-continuous $(\rho_{sd}, \rho_{sd})$–multirealizer of $f : K \Rightarrow \mathbb{R}$ (recall Lemma 5) which maps compact sets to compact sets. Then $f$ itself must be Henkin-continuous, too; and has a Henkin-continuous tightening $g : K \Rightarrow \mathbb{R}$ mapping compact sets to compact sets.
c) Conversely, if \( f : K \rightarrow \mathbb{R} \) is Henkin-continuous and maps compact sets to compact sets, then \( F := \rho_{sd}^{-1} \circ f \circ \rho_{sd}[K] \) is a Henkin-continuous \((\rho_{sd}, \rho_{sd})\)-multirealizer of \( f \) which maps compact sets to compact sets.

**Proof.** a) Recall [Weih00, SECTION 3] that a real relation is relatively computable iff it has a continuous \((\rho, \rho)\)-realizer; equivalently [Weih00, THEOREM 7.2.5.1]: a continuous \((\rho_{sd}, \rho_{sd})\)-realizer \( F \). In particular, single-valued \( F \) maps compact sets to compact sets. Moreover, \( F \) is a \((\rho_{sd}, \rho_{sd})\)-multirealizer according to Lemma 5f); and has \( \text{dom}(F) = \text{dom}(\rho_{sd}[K]) \) compact [Weih00, pp.209-210], hence is even uniformly continuous, i.e. Henkin-continuous. Now apply b).

b) Proposition 13d) asserts \( \rho_{sd}^{-1} \) to be Henkin-continuous; and so is \((\rho_{sd}[K])^{-1} = (\rho_{sd}^{-1})[K]\), cmp. Observation 12a). Now range \( (\rho_{sd}[K])^{-1} = \rho_{sd}^{-1}[K] \) is compact; which \( F \) maps by hypothesis to some compact set \( C \subseteq \{0,1\}^\omega \). Therefore \( \rho_{sd}|_C \) is uniformly (i.e. Henkin-) continuous (Example 11d); and so is \( \rho_{sd}|_C \circ F \circ (\rho_{sd}[K])^{-1} \) (Observation 12b); which, because of \( C = \text{range}(F \circ (\rho_{sd}[K])^{-1}) \), coincides with \( g := \rho_{sd} \circ F \circ \rho_{sd}^{-1} \). Now this \( g \) by hypothesis tightens \( f \); hence \( f \) is also Henkin-continuous (Observation 12a). Moreover, \( g \) maps compact sets to compact sets according to Lemma 5d) because each subterm \( \rho_{sd}^{-1} \) [Weih00, pp.209-210], \( F \) (hypothesis), and \( \rho_{sd} \) (continuous) does so.

c) Again, \( \rho_{sd}[K] \) and \( \rho_{sd}^{-1} \) are Henkin-continuous by Example 11c) and Proposition 13d); hence so is the composition \( F \) (Observation 12a). \( F \) maps compact sets to compact sets according to Lemma 5d); note that \( \text{range}(f) \subseteq \mathbb{R} = \text{dom}(\rho_{sd}^{-1}) \) and \( \text{range}(\rho_{sd}[K]) = K = \text{dom}(f) \). Finally, Lemma 5a+b) shows \( f \) to tighten \( \rho_{sd} \circ F \circ \rho_{sd}^{-1} \).

**3.4 Henkin-Continuity does not imply Relative Computability**

The relation from Example 4c) is Henkin-continuous but not relatively computable. On the other hand, it violates the natural condition of (pointwise) compactness. Instead, we modify Example 16 to obtain (counter-)

**Example 19.** Let

\[
\begin{align*}
f_+ &:= \left( (-\infty,0] \times \{0\} \right) \cup \left\{ (x,(-1)^n/(n+1)) : n \in \mathbb{N}, 1/(n+1) \leq x \leq 1/n \right\} \\
f_- &:= \left( [0,\infty) \times \{1\} \right) \cup \left\{ (-x,1+(-1)^n/(n+1)) : n \in \mathbb{N}, 1/(n+1) \leq x \leq 1/n \right\}
\end{align*}
\]

Then \( f_1 := f_+ \cup f_- : [-1,+1] \Rightarrow [-1,+2] \) is compact, total, and 1-Lipschitz (hence Henkin-continuous), but not relatively computable; see Figure 4.

**Proof.** Both \( f_+ \) and \( f_- \) are closed and bounded and total. Moreover, the restriction \( f_+|_{[-1,0]} \) is 1-Lipschitz: To \( x \leq 0 \) set \( y := 0 \) and \( \delta := \varepsilon \) (1-Lipschitz); now if \( x' \leq 0, y' := 0 \) will do; and if \( 0 < x' < \delta \), consider \( n \in \mathbb{N} \) with \( 1/(n+1) \leq x' \leq 1/n, y' := (-1)^n/(n+1) \in f_+(x') \) has \( |y'-y| = 1/(n+1) \leq x' < \delta = \varepsilon \). Similarly, \( f_-|_{[0,1]} \) is 1-Lipschitz; hence \( f_1 \) is 1-Lipschitz—but not relatively computable: Given a name of \( x = 0 \), the putative realizer has the choice of producing either a name of \( y_+ = 0 \) or of \( y_- = 1 \): knowing \( x \) only up to some \( \delta = 1/n, n \in \mathbb{N} \). In the first case, i.e. already tied to \( f_+ \), switch to an input \( x' := 1/(n+1) \); clearly a point of discontinuity of \( f_+ \). A similar contradiction arises in the second case. \( \square \)
4 Iterated Henkin-Continuity

(Counter-)Example 19 suggests to strengthen Definition 9:

**Definition 20.** Call a total†† multifunction $f : X \Rightarrow Y$ doubly Henkin-continuous iff the following holds:

$$
\left(\forall \epsilon > 0 \ \exists \delta > 0 \ \forall x \in X \ \exists y \in f(x) \ \forall x' \in B(x, \delta) \ \exists y' \in f(x') \cap B(y, \epsilon) \right) \ \forall x'' \in B(x', \delta') \ \exists y'' \in f(x'') \cap B(y', \epsilon')
$$

Even more generally, $\ell$-fold Henkin-continuity ($\ell \in \mathbb{N}$) is to mean

$$
\left(\forall \epsilon_1 > 0 \ \exists \delta_1 > 0 \ \forall x_1 \in X \ \exists y_1 \in f(x_1) \ \forall x_2 \in B(x_1, \delta_1) \ \exists y_2 \in f(x_2) \cap B(y_1, \epsilon_1) \right) \ \ldots
$$

$$
\ldots \left(\forall \epsilon_\ell > 0 \ \exists \delta_\ell > 0 \ \forall x_\ell \in B(x_{\ell-1}, \delta_{\ell-1}) \ \exists y_\ell \in f(x_\ell) \cap B(y_{\ell-1}, \epsilon_{\ell-1}) \right)
$$

$$
\forall x_{\ell+1} \in B(x_\ell, \delta_\ell) \ \exists y_{\ell+1} \in B(y_\ell, \epsilon_\ell) \ \cap f(x_{\ell+1}).
$$

(10)

Generalizing Example 19, we observe that this notion indeed gives rise to a proper hierarchy:

**Example 21 (Hierarchy).** To every $\ell \in \mathbb{N}$ there exists a compact total relation $f_\ell : [-1,1] \Rightarrow [-1,2]$ which is $\ell$-fold Henkin-continuous but not $(\ell + 1)$-fold Henkin-continuous.

To this end, consider $\ell = 1$ and recall that the relation in Figure 4 is (1-fold) Henkin-continuous. To $x = 0$ w.l.o.g. suppose $y = 0$ is chosen and to $\epsilon := 1/4$ some $\delta > 0$. Now consider $x' := 1/n < \delta$: Since $f_+\mid$ is discontinuous at $x'$, both choices $y' = s(-1)^n/(n+1)$ and $y' = -(-1)^n/(n+2)$ from $f(x')$ contradict 2-fold Henkin-continuity for some $x'' = x' \pm \epsilon'$.

Figure 5 depicts an iteration $f_2$ of Figure 4 which, similarly, can be seen 2-fold Henkin-continuous but not 3-fold. Repeating this iteration, one obtains a fractal sequence $f_\ell$ with the claimed properties.

†† This requirement is employed only for notational convenience and can always be satisfied by proceeding to the restriction $f|_{\text{dom}(f)}$. 
Many properties of Henkin-continuity translate to the iterated case:

**Lemma 22.** Fix $\ell \in \mathbb{N}$.

a) If $f$ is $(\ell + 1)$-fold Henkin-continuous, it is also $\ell$-fold Henkin-continuous; but not necessarily vice versa.

b) If $f : X \rightrightarrows Y$ is uniformly strongly continuous (and in particular if $f : X \rightarrow Y$ is uniformly continuous), it is $\ell$-fold Henkin-continuous for every $\ell$.

c) If $f : X \times Y$ is $\ell$-fold Henkin-continuous and tightens $g : \subseteq X \times Y$, then $g$ is $\ell$-fold Henkin-continuous (on $\text{dom}(g)$) as well.

d) If $f : X \times Y$ and $g : Y \times Z$ are both $\ell$-fold Henkin-continuous, then so is $g \circ f$ (on $\text{dom}(g \circ f)$).

**Proof.** a) The first claim is obvious; failure of the converse is demonstrated in Example 21.

b) Immediate induction.

c) As in the proofs of Observation 12a), $g$ restricts the range of the universal quantifiers occurring in Equations (10) and extends the range of the existential quantifiers.

d) By hypothesis we have Equation (10) for $f$ and the following for $g$:

\[
\begin{align*}
&\left( \forall \delta_1 > 0 \hspace{1em} \exists \gamma_1 > 0 \right) \left( \forall y_1 \in Y \hspace{1em} \exists z_1 \in g(y_1) \right) \left( \forall \delta_2 > 0 \hspace{1em} \exists \gamma_2 > 0 \hspace{1em} \forall y_2 \in B(y_1, \gamma_1) \hspace{1em} \exists z_2 \in f(y_2) \cap B(z_1, \delta_1) \right) \cdots \\
&\cdots \left( \forall \delta_\ell > 0 \hspace{1em} \exists \gamma_\ell > 0 \hspace{1em} \forall y_\ell \in B(y_{\ell-1}, \gamma_{\ell-1}) \hspace{1em} \exists z_\ell \in f(y_\ell) \cap B(z_{\ell-1}, \delta_{\ell-1}) \right) \\
&\forall y_{\ell+1} \in B(y_\ell, \gamma_\ell) \hspace{1em} \exists z_{\ell+1} \in B(z_\ell, \delta_\ell) \cap f(y_{\ell+1}) .
\end{align*}
\]

Now inductively, to $\varepsilon_{k+1} > 0$ and to $x_{k+1} \in \text{dom}(g \circ f) \cap B(x_k, \delta_k)$, there exist $\delta_{k+1} > 0$ independent of $x_{k+1}$ and $y_{k+1} \in f(x_{k+1}) \cap B(y_k, \varepsilon_k)$ independent of $\varepsilon_{k+1}$; to which in turn there exist $\gamma_{k+1} > 0$ independent of $y_{k+1}$ and $z_{k+1} \in g(y_{k+1}) \cap B(z_k, \delta_k)$ independent of $\delta_k$.  \qed
4.1 Examples and Properties

Note that $\delta_2$ in Equation (10), although independent of $x_2$, may well depend on $x_1$: which perhaps does not entirely express what might be expected from a notion of uniform continuity for relations. On the other hand, just like continuity on a compact set is in the single-valued case equivalent to uniform continuity, we establish

**Lemma 23.** For compact $X$, total $f : X \rightarrow Y$, and $\ell \in \mathbb{N}$, the following are equivalent:

i) $f$ is $\ell$-fold Henkin-continuous

\[
\forall \epsilon > 0 \quad \exists \delta > 0
\]
\[
i \in X \exists y_1 \in f(x_1) \quad \forall x_2 \in X \exists y_2 \in Y \quad \forall x_\ell \exists y_\ell \quad \forall x_{\ell+1} \exists y_{\ell+1}
\]
\[
\bigwedge_{k=1}^{\ell} (x_{k+1} \in \overline{B}(x_k, \delta) \rightarrow y_{k+1} \in f(x_{k+1}) \cap B(y_k, \epsilon))
\]  

(11)

ii) There exists a total function $\lambda : \mathbb{N} \rightarrow \mathbb{N}$ such that

\[
\forall x_1 \exists y_1 \in f(x_1) \quad \forall m_1 \in \mathbb{N} \forall x_2 \in \overline{B}(x_1, 2^{-\lambda(m_1)}) \exists y_2 \in f(x_2) \cap \overline{B}(y_1, 2^{-m_1})
\]
\[
\forall m_2 \in \mathbb{N} \forall x_3 \in \overline{B}(x_2, 2^{-\lambda(m_2)}) \exists y_3 \in f(x_3) \cap \overline{B}(y_2, 2^{-m_2})
\]
\[
\ldots
\]
\[
\forall m_\ell \in \mathbb{N} \forall x_{\ell+1} \in \overline{B}(x_\ell, 2^{-\lambda(m_\ell)}) \exists y_{\ell+1} \in f(x_{\ell+1}) \cap \overline{B}(y_\ell, 2^{-m_\ell})
\].  

(12)

For non-compact $X$, it still holds ‘i) $\iff$ ii) $\iff$ iii)’.

We call $\lambda$ as in iii) a modulus of $\ell$-fold Henkin-continuity of $f$.

**Proof.** Note that $\delta_k$ in Equation (10) may depend on $x_1, \ldots, x_{k-1}$; and $y_k$ on $\epsilon_1, \ldots, \epsilon_{k-1}$.

ii)$\implies$i): Apply Equation (11) to $\epsilon := \min\{\epsilon_1, \ldots, \epsilon_{\ell}\}$ and take $\delta_1 := \cdots := \delta_\ell := \delta$ in (10).

i)$\implies$ii): Recall that $(\forall x_k \exists y_k)$ clearly implies $\forall x_k \forall y_k \exists \delta_k y_k$. Moreover we may replace the open balls $B(x_k, \delta_k)$ with their topological closures $\overline{B}(x_k, \delta_k)$ by reducing $\delta_k$ a bit. Now exploit compactness and slightly extend (the proof of) Lemma 3g to see that $\delta_k$ can be chosen independent of $x_1, \ldots, x_k$, that is, $\forall x_k \exists \delta_k \forall y_k \exists y_k \forall x_{k+1} \exists y_{k+1}$ implies $\forall x_k \exists \delta_k \forall y_k \in \overline{B}(x_k, \delta_k) \exists \delta_k \forall y_{k+1} \forall x_{k+1} \exists y_{k+1}$ for every $1 \leq j \leq k \leq \ell$. More formally, let $\Phi(\delta_j, x_k, \delta_k)$ denote the formula

\[
\exists y_k \in f(x_k) \cap \overline{B}(y_{k+1}, \epsilon_{k+1}) \quad \forall x_{k+1} \in \overline{B}(x_{k+1}, \delta_{k+1})
\]

Then, by hypothesis, to $\epsilon_j > 0$ and arbitrary but fixed $x_k \in \overline{B}(x_{k-1}, \delta_{k-1})$, there exists $\delta_j = \delta_j(x_k) > 0$ such that $\Phi(\delta_j, x_k, \delta_k)$ holds. Now by triangle inequality, every $x_k' \in B(x_k, \delta_k/2) \cap \overline{B}(x_{k-1}, \delta_{k-1})$ satisfies $\Phi(\delta_j(x_k), x_k', \delta_k/2)$. The relatively open balls $B(x_k, \delta_k/2) \cap \overline{B}(x_{k-1}, \delta_{k-1})$ cover compact $\overline{B}(x_{k-1}, \delta_{k-1}) \subseteq X$, hence finitely many of them suffice to do so. And these induce finitely many $\delta_j(x_k)$, such that their minimum $\delta_j$ satisfies $\Phi(\delta_j, x_k', \delta_k/2)$ for every $x_k' \in \overline{B}(x_{k-1}, \delta_{k-1})$.

Inductively swapping quantifiers as justified above, we deduce

\[
\left( \forall x_1 > 0 \quad \exists \delta_1 > 0 \right)
\]
\[
\left( \forall x_1 \in X \exists y_1 \in f(x_1) \right)
\]
\[
\forall x_2 \in \overline{B}(x_1, \delta_1) \exists y_2 \in f(x_2) \cap B(y_1, \epsilon_1) \quad \ldots
\]
\[
\forall x_\ell \in \overline{B}(x_{\ell-1}, \delta_{\ell-1}) \exists y_\ell \in f(x_\ell) \cap B(y_{\ell-1}, \epsilon_{\ell-1})
\]
\[
\forall x_{\ell+1} \in \overline{B}(x_{\ell+1}, \delta_{\ell+1}) \exists y_{\ell+1} \in B(y_\ell, \epsilon_\ell) \cap f(x_{\ell+1})
\]
and, by one further step, obtain independence of $\delta_2, \ldots, \delta_\ell$ even from $x_1 \in X$:

$$(\forall \epsilon > 0 \exists \delta_1 \forall \epsilon_2 \exists \delta_2 \cdots \forall \epsilon_{\ell} \exists \delta_{\ell}) \quad \forall x_1 \in X \quad \exists y_1 \in f(x_1)$$

$$(\forall x_2 \in \overline{B}(x_1, \delta_1) \exists y_2 \in f(x_2) \cap B(y_1, \epsilon_1) \cdots \forall x_{\ell-1} \in \overline{B}(x_{\ell-1}, \delta_{\ell-1}) \exists y_{\ell-1} \in f(x_{\ell-1}) \cap B(y_{\ell-1}, \epsilon_{\ell-1})$$

$$\forall x_\ell \in \overline{B}(x_\ell, \delta_\ell) \exists y_\ell \in f(x_\ell) \cap B(y_\ell, \epsilon_\ell) \cap f(x_{\ell+1})$$

Apply this to given $\epsilon > 0$ by choosing let $\epsilon_1 := \cdots := \epsilon_\ell := \epsilon$ and taking $\delta := \min\{\delta_1, \ldots, \delta_{\ell}\}$.

ii) $\Rightarrow$ iii): For $m \in \mathbb{N}$ set $\epsilon := 2^{-m}$, apply ii) to obtain some $\delta = \delta(m)$, and define $\lambda(m) := [\log_2(1/\delta)]$. We show inductively that this satisfies Equation (12). To $x_1 \in X$, ii) yields some $y_1 \in f(x_1)$ independent of $\epsilon$; now given furthermore $m_1 \in \mathbb{N}$, apply ii) to $\epsilon := 2^{-m_1}$ and obtain some $\delta > 0$ (which by construction dominates $2^{-\lambda(m_1)}$) and to every $x_2 \in B(x_1, 2^{-\lambda(m_1)})$ some $y_2 \in f(x_2) \cap B(y_1, 2^{-m_1})$; next, to $m_2 \in \mathbb{N}$, ii) with $\epsilon := 2^{-m_2}$ yields some $\delta \geq 2^{-\lambda(m_2)}$ and to every $x_3 \in B(x_2, 2^{-\lambda(m_2)})$ some $y_3 \in f(x_3) \cap B(y_2, 2^{-m_2})$; and so on.

iii) $\Rightarrow$ ii): To $\epsilon > 0$, take $m := [\log_2(1/\epsilon)]$ and $\delta := 2^{-\lambda(m)}$ with $\lambda : \mathbb{N} \to \mathbb{R}$ according to iii). Then by Equation (12) inductively, to every $m_k := m$ and every $x_{k+1} \in B(x_k, \delta) = B(x_k, 2^{-\lambda(m_k)})$, there exists some $y_{k+1} \in f(x_{k+1}) \cap B(y_k, 2^{-m_k}) \subseteq B(y_k, \epsilon)$.

**Observation 24.** If the family $f_i : X_i \Rightarrow Y_i$ ($i \in I$) is $\ell$-fold Henkin-equicontinuous in the sense of have a common modulus $\lambda$ of $\ell$-fold Henkin-continuity, this will also be a modulus of $\ell$-fold Henkin-continuity for $\prod_{i \in I} f_i : \prod_{i \in I} X_i \Rightarrow \prod_{i \in I} Y_i$ with respect to the maximum metric $d((x_i), (x'_i)) = \max_{i \in I} d_i(x_i, x'_i)$ and $d((y_i), (y'_i)) = \max_{i \in I} d_i(y_i, y'_i)$.

Note also that equivalence of the Cauchy representation $\rho$ to the signed digit representation $\rho_{sd}$ means that its inverse $\rho_{sd}^{-1} : \mathbb{R} \Rightarrow \Sigma^\omega$ be computable. Hence Fact 1 asserts that $\rho_{sd}^{-1}$ has a strongly continuous (and w.l.o.g. pointwise compactly) tightening. We now strengthen this as well as Proposition 13(c)+d):

**Proposition 25.** a) Let $x = \sum_{n=-N}^{\infty} a_n 2^{-n}$ be a signed digit expansion and $k \in \mathbb{N}$ such that $(a_n, a_{n+1}) \in \{10, \overline{10}, 01, 0\overline{1}, 00\}$ for each $n > k$. Then every $x' = \sum_{n=-N}^{\infty} b_n 2^{-n}$ satisfying $a_n = b_n \forall n \leq k$ and $(b_n, b_{n+1}) \in \{10, \overline{10}, 01, 0\overline{1}, 00\}$ for all $n > k+1$.

b) Let $D := \{\sigma \in \text{dom}(\rho_{sd}) : \sigma_N = \ldots(\sigma_n, \sigma_{n+1}) \in \{10, \overline{10}, 01, 0\overline{1}, 00\} \forall n > N\}$. Then $(\rho_{sd}|D)^{-1} : \mathbb{R} \Rightarrow D$ tightens the signed digit representation and is uniformly strongly continuous with $\delta(2^{-n-1}) := 2^{-n}/6$.

c) In particular, $\rho_{sd}^{-1}$ is $\ell$-fold Henkin-continuous for every $\ell \in \mathbb{N}$ with modulus $\lambda : m \mapsto m+2$.

**Proof.** a) First consider the case $a_{k+1} = 0$. Then $x'' = \sum_{n=-N}^{k} a_n 2^{-n} = \sum_{n=-N}^{k+1} a_n 2^{-n}$ has $0 \leq x - x'' \leq x'' - x'' = (x'' - (x - x''))$ and $x' - x'' = x'' = \sum_{n=-N}^{k} a_n 2^{-n} = \sum_{n=-N}^{\infty} a_n 2^{-n} + \sum_{n=k+1}^{\infty} b_n 2^{-n}$ with $(b_n, b_{n+1}) \in \{10, \overline{10}, 01, 0\overline{1}, 00\}$ for all $n$. This yields $x' = (x' - x'') + x'' = \sum_{n=-N}^{k} a_n 2^{-n} = \sum_{n=-N}^{\infty} a_n 2^{-n}$ with the claimed properties.

It remains to consider the case $a_{k+1} = 1$ (and $a_{k+1} = \overline{1}$ proceeds analogously). Here the hypothesis on $(a_n, a_{n+1})$ asserts $a_{k+2} = 0$. Therefore $x'' = \sum_{n=-N}^{k+1} a_n 2^{-n} = \sum_{n=-N}^{\infty} a_n 2^{-n}$.
has \(0 \leq x - x'' \leq 2^{-k}/6\) due to Proposition 13b). Hence \(x' - x'' = (x' - x) + (x - x'') \in [-2^{-k}/6, 2^{-k}/3] \subseteq [-\frac{1}{2}, 2^{-k+1}]\) has, again according to Proposition 13b), a signed digit expansion \(x' - x'' = \sum_{n=k+2}^{\infty} b_n 2^{-n}\) with \((b_n, b_{n+1}) \in \{10, \overline{01}, 01, 00, 0\}\) for all \(n\). This yields \(x' = (x' - x'') + x'' = \sum_{n=1-k}^{\infty} a_n 2^{-n} + \sum_{n=k+2}^{\infty} b_n 2^{-n}\) an expansion with the claimed properties.

b) According to a), every \(x'\) admits a signed digit expansion \(x' = \sum_{n=-N}^{\infty} b_n 2^{-n}\) with \((b_n, b_{n+1}) \in \{10, 0, 01, 0\}\), i.e. encoding a \(\rho_{sd}\)-name \(\sigma \in D\). Moreover, to each expansion \(x = \sum_{n=-N}^{\infty} a_n 2^{-n}\) with \((a_n, a_{n+1}) \in \{10, 0, 01, 0\}\) corresponding to a \(\rho_{sd}\)-name \(\sigma \in D\) and each \(k \in \mathbb{N}\), a) asserts that also every \(x' \in B(x, 2^{-k}/6)\) admits a \(\rho_{sd}\)-name \(\sigma' \in D \cap \overline{B}(\sigma, 2^{-k-1})\); the \(-1\) arising because the digit \(\cdot\) is also shared by both \(\sigma\) and \(\sigma'\).

c) follows from b) in view of Lemma 22b).

4.2 Infinitary Henkin Continuity and the Main Result

Lemma 26. For a total, pointwise compact multifunction \(f : X \Rightarrow Y\), the following are equivalent:

i) \(f\) admits a modulus \(\lambda\) of \(\ell\)-fold Henkin-continuity independent of \(\ell \in \mathbb{N}\)

ii) the following infinitary formula holds:

\[
\exists \delta_1, \delta_2, \ldots, \delta_\ell, \ldots > 0 \quad \forall x_1 \in X \exists y_1 \in Y \forall x_2 \in X \exists y_2 \in Y \ldots \forall x_\ell \in X \exists y_\ell \in Y \ldots:
\]

\[
y_1 \in f(x_1) \land \bigwedge_{\ell \in \omega} (x_{\ell+1} \in B(x_\ell, \delta_\ell) \rightarrow y_{\ell+1} \in f(x_{\ell+1}) \cap \overline{B}(y_\ell, 2^{-\ell}))
\]

(13)

Naturally, Formula (13) is endowed with the semantics of an infinite two-player game (and we make sure not to rely on determinacy). For a more in-depth background on infinitary logics, the reader may refer to [Keis65,KeKn04].

Proof. i)\(\Rightarrow\)ii): For each \(m \in \mathbb{N}\) let \(\delta_m := \lambda(m)\). Now apply Equation (12) to \(m_1 := 1, m_2 := \ldots, m_\ell := \ell \ldots\): Fix \(\ell\); then, to \(x_1 \in X\) there exists \(y_1^{(\ell)} \in f(x_1)\); to \(x_2 \in B(x_1, \delta_1) = B(x_1, 2^{-\lambda(m_1)})\) there exists \(y_2^{(\ell)} \in f(x_2) \cap \overline{B}(y_1, 2^{-1})\); and, inductively, to \(x_{\ell+1} \in B(x_{\ell}, \delta_\ell) = B(x_\ell, 2^{-\lambda(m_\ell)})\) there exists \(y_{\ell+1}^{(\ell)} \in f(x_{\ell+1}) \cap \overline{B}(y_\ell, 2^{-\ell})\). Note that the \(y_k^{(\ell)}\) indeed depend on \(\ell\) since the hypothesis asserts \(\lambda\) to be a modulus of \(\ell\)-fold Henkin-continuity for every fixed \(\ell\) only. On the other hand, for each such \(\ell\), the sequence \((y_k^{(\ell)})_k\) \(\ell\)-lives’ in \(\chi_\ell f(x_k)\); which is compact according to Tychonoff: recall our hypothesis that \(f\) be pointwise compact. Hence the sequence of sequences \(((y_k^{(\ell)})_k)_\ell\) has a subsequence converging to some \((y_k)_k \in \chi_\ell f(x_k)\); and \(y_{k+1}^{(\ell)} \in B(y_k^{(\ell)}, 2^{-k})\) implies \(y_{k+1} \in B(y_k, 2^{-k})\).

ii)\(\Rightarrow\)i): For each \(m \in \mathbb{N}\) let \(\lambda(m) := \lfloor \log_2 (1/\delta_m) \rfloor\).

We first assert this to be a modulus of 2-fold Henkin-continuity: For \(x_1 \in X\), apply Equation (13) to \(x_1 := x'_1 := x'_2 := \ldots := x'_{m_1}\) and obtain \((y'_1, \ldots, y'_{m_1})\) as well as a \(y'_{m_1} := y_1 \in f(x)\) such that for every \(x'_{m_1+1} := x_2 \in B(x_1, 2^{-\lambda(m_1)}) \subseteq B(x'_{m_1}, \delta_{m_1})\) there exists some \(y_2 := y'_{m_1+1} \in f(x_2) \cap \overline{B}(y_1, 2^{-m_1})\).

Now iterating this argument inductively shows \(\lambda\) to be a modulus of \(\ell\)-fold Henkin-continuity for every \(\ell \in \mathbb{N}\). \(\square\)

Let us say that \(f\) is \(\omega\)-fold Henkin-continuous if it satisfies Equation (13). On Cantor space, this may be regarded as a uniform version of König’s Lemma; cmp. [Kohl02]. And indeed we have
Proposition 27. Suppose \( F \subseteq \{0,1\}^\omega \rightarrow \{0,1\}^\omega \) maps compact sets to compact sets and is \( \omega \)-fold Henkin-continuous. Then \( F \) admits a uniformly continuous total selection \( G : \text{dom}(F) \rightarrow \{0,1\}^\omega \).

More precisely if \( \lambda \) is a modulus of \( \ell \)-fold Henkin-continuity of \( F \) for every \( \ell \), then \( \lambda \) is also a modulus of continuity of \( G \).

Proof. Note that the triangle inequality in \( \{0,1\}^\omega \) strengthens to \( d(\bar{x}, \bar{z}) \leq \max\{d(\bar{x}, \bar{y}), d(\bar{y}, \bar{z})\} \). Moreover it is no loss of generality to suppose \( \delta_\ell = 2^{-\lambda(\ell)} > \delta_{\ell+1} \) for each \( \ell \) in Equation (13).

Now with [Weih00, Lemma 2.1.11.2] in mind, we first construct a ‘block-monotone’ partial mapping \( g : \subseteq \{0,1\}^* \rightarrow \{0,1\}^* \); more specifically: \( g : \{0,1\}^{\lambda(\ell)} \rightarrow \{0,1\}^\ell \) for every \( \ell \in \mathbb{N} \) such that \( g(\alpha) \) is (defined and) an initial substring of \( g(\alpha \beta) \) whenever \( \alpha \in \{0,1\}^{\lambda(\ell)} \) and \( \beta \in \{0,1\}^{\lambda(\ell+1)-\lambda(\ell)} \) satisfy \( \alpha \beta \in \text{dom}(g) \). The construction proceeds inductively as follows:

For \( x_1 \in \{0,1\}^{\lambda(1)} \), consider some \( \bar{x}_1 \in \text{dom}(F) \) extending \( x_1 \), i.e. \( \bar{x}_1 \in \{0,1\}^\omega \). If no such \( \bar{x}_1 \) exists, \( g(x_1) \) shall be undefined; otherwise there is by hypothesis some \( \bar{y}_1 \in F(\bar{x}_1) \) satisfying the matrix of Equation (13): then define \( g(x_1) := y_1 := \bar{y}_1_{|_1} \), the first symbol of \( \bar{y}_1 \). For \( x_2 \in \bar{x}_1 \circ \{0,1\}^{\lambda(2)-\lambda(1)} \), if there exists some \( \bar{x}_2 \in (x_2 \circ \{0,1\}^\omega) \cap \text{dom}(F) \), it holds \( \bar{x}_2 \in \overline{B}(\bar{x}_1,2^{-\lambda(1)}) \) and we may set \( g(x_2) := y_2 := \bar{y}_2_{|_2} \) with \( \bar{y}_2 \in F(\bar{x}_1) \cap (y_1 \circ \{0,1\}^\omega) \) according to Equation (13). Inductively, for \( x_{\ell+1} \in x_\ell \circ \{0,1\}^{\lambda(\ell+1)-\lambda(\ell)} \), if \( \emptyset \neq (x_{\ell+1} \circ \{0,1\}^\omega) \cap \text{dom}(F) \ni \bar{x}_{\ell+1} \), set \( g(x_{\ell+1}) := y_{\ell+1} := \bar{y}_{\ell+1}_{|_{\ell+1}} \) with \( \bar{y}_{\ell+1} \in F(\bar{x}_\ell) \cap (y_\ell \circ \{0,1\}^\omega) \) according to Equation (13).

Now observe that \( \emptyset \neq (x_{\ell+1} \circ \{0,1\}^\omega) \cap \text{dom}(F) \) implies \( \emptyset \neq (x_\ell \circ \{0,1\}^\omega) \cap \text{dom}(F) \); hence, for \( \bar{x} \in \text{dom}(F) \), \( g(\bar{x}_{|_{\leq \lambda(\ell)}}) \) is defined for every \( \ell \). Since \( g \) is ‘block-monotone’ in the above sense, \( G(x) := \lim_\ell (g(\bar{x}_{|_{\leq \lambda(\ell)}}) \circ 0^\omega) \) is well-defined on \( \text{dom}(F) \); and continuous with modulus \( \lambda \) via its construction through \( g \). Moreover, \( \bar{y} := G(\bar{x}) \) satisfies by definition \( \bar{y} = \lim_\ell \bar{y}_\ell \) with \( \bar{y}_{\ell+1} \in \overline{B}(\bar{y}_\ell,2^{-\ell}) \cap F(\bar{x}_{\ell+1}) \) for some \( \bar{x}_{\ell+1} \in \overline{B}(\bar{x},2^{-\ell}) \); hence \( (\bar{x}_\ell, \bar{y}_\ell) \) is a sequence in \( F \) converging to \( (\bar{x}, \bar{y}) \) with \( \bar{x} \in \text{dom}(F) \). By hypothesis, \( F \) maps compact \( \{\bar{x}_\ell : \ell \} \cup \{\bar{x}\} \) to a compact set containing \( \{\bar{y}_\ell\} \), requiring \( (\bar{x}, \bar{y}) \in F : G \) is a selection of \( F \).

We can now strengthen Theorem 18:

**Theorem 28.** Fix compact \( K \subseteq \mathbb{R}^d \).

a) Let \( f : K \rightarrow \mathbb{R} \) be computable relative to oracle \( \mathcal{O} \). Then there exists \( g : K \rightarrow \mathbb{R} \) tightening \( f \) which is still computable relative to \( \mathcal{O} \) and maps compact sets to compact sets.

b) If \( f : K \rightarrow \mathbb{R} \) is relatively computable, it is \( \omega \)-fold Henkin-continuous.

c) Suppose \( f : K \rightarrow \mathbb{R} \) maps compact sets to compact sets and is \( \omega \)-fold Henkin-continuous.

Then \( f \) is relatively computable.

This theorem provides the desired topological characterization of relative computability:

**Corollary 29.** For \( X := [0,1]^d \), a total relation \( f : X \rightarrow \mathbb{R} \) mapping compact sets to compact sets (and in particular one with compact graph) is relatively computable iff it satisfies Equation (13).

**Proof (Theorem 28).**

a) By hypothesis, \( f \) admits an \( \mathcal{O} \)-computable (and thus continuous) \( (\rho^d_{\text{sd}}, \rho_{\text{sd}}) \)-realizer \( F : \subseteq \{0,1\}^\omega \rightarrow \{0,1\}^\omega \) on compact \( \text{dom}(F) = \text{dom}(\rho^d_{\text{sd}}|_K) \), i.e. mapping compact sets to compact sets. And so does \( (\rho^d_{\text{sd}})_{-1} \) (Example 6b) and continuous \( \rho_{\text{sd}} \). Thus, again according to Lemma 5d), also \( g := \rho_{\text{sd}} \circ F \circ (\rho^d_{\text{sd}})^{-1} : K \rightarrow \mathbb{R} \) maps compact sets to compact sets; and tightens \( f \) (Lemma 5f); and is computable relative to \( \mathcal{O} \).
b) According to a) and Lemma 22c) we may w.l.o.g. suppose that \( f \) maps compact sets to compact sets and in particular that \( C := f[K] \) is compact. Combining Proposition 25c) with Observation 24 and Example 11f) shows \((\rho_{sd}^d)^{-1} : \mathbb{R}^d \rightarrow \{0,1\}^{\omega} \) to be \( \omega \)-fold Henkin-continuous. By hypothesis, \( f \) admits a continuous \((\rho_{sd}^d, \rho_{sd})\)-realizer \( F : \subseteq \{0,1\}^{\omega} \rightarrow \{0,1\}^{\omega} \) on compact \( \text{dom}(F) = \text{dom}(\rho_{sd}^d|K) \); in particular, \( F \) is uniformly continuous. Moreover, \( \rho_{sd}^d|C \circ F \circ (\rho_{sd}^d|K)^{-1} : K \Rightarrow C \subseteq \mathbb{R} \) tightens \( f \) (Lemma 5f) with \( \text{dom}(\rho_{sd}^d|C) \) compact, hence \( \rho_{sd}^d|C : \subseteq \{0,1\} \rightarrow C \) is uniformly continuous. Now apply Lemma 22b)+c)+d) to conclude that both \( \rho_{sd}^d|C \circ F \circ (\rho_{sd}^d)^{-1} \) and \( f \) are \( \omega \)-fold Henkin-continuous.

c) As in the proof of Theorem 18c), observe that \( F := \rho_{sd}^{-1} \circ f \circ \rho_{sd}^d|K \) is \( \omega \)-fold Henkin-continuous according to Proposition 25c) and Lemma 22b)+c)+d). And \( F \) maps compact sets to compact sets (Lemma 5d). Hence \( F \) admits a continuous selection \( G \) on \( \text{dom}(F) = \text{dom} \left( \rho_{sd}^d|K \right) \) due to Proposition 27. This is a continuous (and hence relatively computable) \((\rho_{sd}^d, \rho_{sd})\)-realizer of \( f \).

\[ \square \]

5 Conclusion

We have proposed a hierarchy of notions of uniform continuity for real relations based on the Henkin quantifier; and shown its \( \omega \)-th level to characterize relative computability in the compact case.

Our condition may be considered descriptionally simpler than the previous characterization from [BrHe94]. Indeed, although Equation (13) does employ countably infinitary logic, Fact 1 even quantifies over subsets of uncountable \( \mathbb{R} \).

**Question 30.** Does Theorem 28 extend from compact subsets \( K \) of \( \mathbb{R}^d \) to general compact metric spaces?

A promising candidate replacement for \( \rho_{sd}^d|K \) is provided in [BdBP10, PROPOSITION 4.1]. But is its inverse \( \omega \)-fold Henkin-continuous (or does even admit a uniformly strongly continuous tightening) ?

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