Abstract

We consider the twisted $N = 4$ SYM on $\Sigma \times S^2$. In the limit that $S^2$ shrinks to zero size the four dimensional theory reduces to a two dimensional SYM theory. We compute the correlation functions of a set of BRST cohomology classes in the reduced theory perturbed by mass.

Keywords: Topological field theory; Duality

1 Introduction

Topological field theories [1] have proven to be a useful tool in the investigation of the nonperturbative characteristics of supersymmetric gauge theories. There is an interplay between certain supersymmetric gauge theories and their corresponding topological versions: one can use topological results on smooth manifolds to learn about the underlying physical theory; conversely, one may use the physical arguments to gain new insight into the topological structure of the manifold on which fields are defined [2].

As an example of the first – i.e. , using the results
on the mathematical side to learn about physics – consider the $N = 4$ SYM theory. This theory has been conjectured to have an exact $SL(2, \mathbb{Z})$ duality \cite{3}. Since this duality relates the weak and strong coupling behaviour of the theory, to test the conjecture one needs quantities such as the partition function to be computed nonperturbatively. This is a formidable task and one actually does not know how to proceed in this direction. This is where topological field theory comes to provide an alternative approach to the problem. Instead of the physical theory, one considers the corresponding topological field theory obtained by a procedure called twisting. The basic characteristics of the theory, such as $SL(2, \mathbb{Z})$ invariance, remain intact under twisting, so one hopes to see the realization of this symmetry in the twisted model. In \cite{4} it has been shown, using the known facts about the structure of the moduli space of instantons and the associated Euler characteristic, the partition function of $N = 4$ twisted theory on some specific manifolds can be computed. So in this way it has become possible to make some exact and nonperturbative statements about the theory and its self-duality properties.

In this article, we will study the $N = 2$ reduction of the above theory obtained by mass perturbation for the hypermultiplet. This theory is still believed to be $S$-dual \cite{5}. We will compute the correlation functions of a set of specific operators using a method of calculation similar to that of \cite{3}.

Twisted $N = 2$ and $N = 4$ SYM theories on product manifolds $\Sigma \times C$, where $\Sigma$ and $C$ are both Riemann surfaces, have been considered in \cite{7}. There it was shown that, in the limit $C$ shrinks to zero, the four dimensional theory generically reduces to an effective two dimensional sigma model. However, when $C$ is a Riemann sphere – as is the case of interest in the present paper – things are a bit different. The dimension of the self-dual harmonic 2-forms, $b^+_2$, is one in this case. Hence the connection is reducible. It follows then that the path integral may get contribution from the so called $u$-plane \cite{8, 9}. Moreover, when $b^+_2 = 1$, there is a wall in the space of one parameter metrics. On crossing this wall the partition function may change its value.

Here we will compute the path integral in a chamber where $S^2$ shrinks to zero. We consider $SO(3)$ bundles such that the restricted bundle over $S^2$ is trivial. Bundles which restrict nontrivially on $S^2$ give zero contribution to the path integral. This is so because in the limit that $S^2$ shrinks to zero size, the path integral localizes on the moduli space of flat
connections in the $S^2$ direction. However, it can be shown that for a flat bundle over $S^2$, transition functions are trivial and the bundle must be trivial. Therefore nontrivial $SO(3)$ bundles on $S^2$ do not admit flat connections.

The organization of this paper is as follows: In section 2 we consider the twisted $N = 4$ Lagrangian on $\Sigma \times S^2$. In the limit where $S^2$ shrinks it is shown how the four dimensional theory reduces to an effective two dimensional theory. The fixed point equations imply, in the case of a nontrivial $SO(3)$ bundle over $\Sigma$, that the partition function of this reduced theory is in fact the Euler characteristic of the moduli space of flat connections on $\Sigma$. A mass perturbation makes the path integral calculation more tractable – particularly for the limiting two-dimensional theory. In section 3, we show how this comes about. Perturbing by the mass allows most fields to be integrated out, and reduces the path integral to a finite dimensional integral which can be easily performed. In section 4 we discuss the result. Although we have not yet managed to give an explicit check of $S$-duality, we have isolated the problems involved and hope to return to this in later work.

2 Twisted $N = 4$ on $\Sigma \times S^2$ and its reduction

The key point in twisting [1] is to redefine the global space-time symmetry such that at least one component of the supercharge becomes scalar under the new defined space-time symmetry. This procedure crucially depends on the existence of a suitable global R-symmetry. $N = 4$ SYM theory in four dimensions has a global $SU(4)$ symmetry such that the supercharges transform under the 4 of this symmetry. First one needs to see how this representation transforms under the space-time symmetry group $SU(2)_L \times SU(2)_R$. There are three possibilities for the decomposition which give rise to singlets under the twisting: (i) $(2, 1) \oplus (1, 2)$; (ii) $(1, 2) \oplus (1, 2)$; (iii) $(1, 2) \oplus (1, 1) \oplus (1, 1)$. As in [4], we will consider the case (ii) where, after twisting, two components of the supercharges turn out to be singlets and therefore square to zero. The scalar fields of the physical theory, which transform under the 6 of $SU(4)$, now transform under the new rotation group, $SU(2)_L \times SU(2)_R$, as $3(1, 1) \oplus (1, 3)$, three singlets and one self-dual 2-form.

Having determined how the new fields transform under the new symmetry group, what remains is to rewrite the Lagrangian in terms of these new fields on flat $\mathbb{R}^4$. This Lagrangian
can then be defined on an arbitrary smooth four manifold while preserving those two BRST-like symmetries.

Let us start our discussion with the twisted $N = 4$ Lagrangian in 4 dimensions \[\text{[4, 10]}\],

\[
\mathcal{L} = \frac{1}{e^2} \text{tr} \left\{ -D_\mu \lambda D^\mu \phi + \frac{1}{2} \tilde{H}^\mu (\tilde{H}_\mu - 2 \sqrt{2} D_\mu C + 4 \sqrt{2} D^\nu B_{\nu \mu} ) \\
+ \frac{1}{2} H^{\mu \nu} (H_{\mu \nu} - 2 F^+_{\mu \nu} - 4 i [B_{\mu \rho}, B^\rho_{\nu}] - 4 i [B_{\mu \nu}, C]) \\
+ 4 \psi_\mu D_\nu \chi^{\mu \nu} + 4 \bar{\chi}_\mu D_\nu \tilde{\psi}^{\mu \nu} + \bar{\chi}_\mu D^\mu \zeta - \psi_\mu D^\mu \eta \\
+ i \sqrt{2} \tilde{\psi}^{\mu \nu} \tilde{\psi}_{\mu \nu}, \lambda - i \sqrt{2} \chi^{\mu \nu} [\chi_{\mu \nu}, \phi] + i 2 \sqrt{2} \tilde{\psi}^{\mu \nu} [\chi_{\mu \nu}, C] + i 4 \sqrt{2} \tilde{\psi}^{\mu \nu} [\chi_{\mu \nu}, B_\nu^\rho] \\
- i \sqrt{2} \chi_{\mu \nu} [\zeta, B^{\mu \nu}] - i \sqrt{2} \tilde{\psi}_{\mu \nu} [\eta, B^{\mu \nu}] + i 4 \sqrt{2} \tilde{\psi}_{\mu \nu} [\bar{\chi}_{\nu}, B_{\nu}^\rho] - i \sqrt{2} \bar{\chi}_{\nu} [\bar{\chi}^{\nu}, \phi] \\
+ i \sqrt{2} \tilde{\psi}_{\mu \nu} [\psi^{\mu}, \lambda] - i 2 \sqrt{2} \tilde{\psi}_{\mu} [\bar{\chi}^{\mu}, C] + i \frac{i}{2 \sqrt{2}} \zeta [\zeta, \lambda] \\
- \frac{i}{\sqrt{2}} \zeta [\eta, C] + 2 [\phi, B^{\mu \nu}] [\lambda, B_{\mu \nu}] + 2 [\phi, C] [\lambda, C] \right\}.
\]

As mentioned, the action is invariant under two BRST transformations. However, for us it is enough to consider one of them, which reads \[\text{[10]}\]

\[
\delta A_\mu = -2 \psi_\mu \quad \delta \zeta = 4 i [C, \phi] \\
\delta \psi_\mu = -\sqrt{2} D_\mu \phi \quad \delta \lambda = \sqrt{2} \eta \\
\delta \phi = 0 \quad \delta \eta = 2 i [\lambda, \phi] \\
\delta B_{\mu \nu} = \sqrt{2} \tilde{\psi}_{\mu \nu} \quad \delta \bar{\chi}_\mu = \tilde{H}_\mu \\
\delta \tilde{\psi}_{\mu \nu} = 2 i [B_{\mu \nu}, \phi] \quad \delta \tilde{H}_\mu = 2 \sqrt{2} i [\bar{\chi}_\mu, \phi] \\
\delta C = \frac{1}{\sqrt{2}} \zeta \quad \delta \chi_{\mu \nu} = H_{\mu \nu} \\
\delta H_{\mu \nu} = 2 \sqrt{2} i [\chi_{\mu \nu}, \phi].
\]

In this article we choose $\phi$ and $\lambda$ to be two independent real scalars. This will render the Lagrangian to be hermitian and allow us to treat $\phi$ and $\lambda$ independently. The generators of the $SU(2)$ group are chosen to be hermitian $T^a = \frac{1}{\sqrt{2}} \sigma^a$ with $\text{tr} \left( T^a T^b \right) = \delta^{ab}$.

The theory enjoys an exact $U(1)$ ghost symmetry under which $\psi_\mu, \tilde{\psi}_{\mu \nu}, \zeta$ have charge 1, $\chi_{\mu \nu}, \eta, \bar{\chi}_\mu$ charge $-1$, while $\phi$ and $\lambda$ have charges 2 and $-2$ respectively. All other fields have ghost number zero.

\[\text{1The Lagrangian that we use is actually different from the one constructed in \[\text{[10]}\] by a BRST exact term } - \frac{i}{4} \delta (\eta [\phi, \lambda]).\]
Take the underlying manifold to be \( \Sigma \times S^2 \). Let us denote the indices on \( \Sigma \) by \( i, j, \cdots \) and those on \( S^2 \) by \( a, b, \cdots \). We define

\[
F_{ij} = \frac{1}{\sqrt{g_1}} \epsilon_{ij} f, \quad \chi_{ij} = \frac{1}{\sqrt{g_1}} \epsilon_{ij} \chi
\]
\[
B_{ij} = \frac{1}{2\sqrt{g_1}} \epsilon_{ij} b, \quad \tilde{\psi}_{ij} = \frac{1}{2\sqrt{g_1}} \epsilon_{ij} \tilde{\psi},
\]
and the same for indices on \( S^2 \)

\[
B_{ab} = \frac{1}{2\sqrt{g_2}} \epsilon_{ab} b', \quad \chi_{ab} = \frac{1}{\sqrt{g_2}} \epsilon_{ab} \chi', \quad \tilde{\psi}_{ab} = \frac{1}{2\sqrt{g_2}} \epsilon_{ab} \tilde{\psi}'.
\]

Here \( g_1 \) and \( g_2 \) denote the determinant of the metric on \( \Sigma \) and \( S^2 \) respectively.

The fields \( H_{\mu\nu}, B_{\mu\nu}, \chi_{\mu\nu} \) and \( \tilde{\psi}_{\mu\nu} \) are all self-dual. Note that

\[
B_{ij} = *B_{ij} \Rightarrow \frac{1}{2\sqrt{g_1}} \epsilon_{ij} b = \frac{1}{2\sqrt{g_2}} \epsilon_{ij} \epsilon^{ab} B_{ab} = \frac{1}{2\sqrt{g_2}} \epsilon_{ij} \epsilon^{ab} \left( \frac{1}{2\sqrt{g_2}} \epsilon_{ab} b' \right)
= \frac{1}{4g_1 g_2} 2g_2 \epsilon_{ij} b = \frac{1}{2\sqrt{g_1}} \epsilon_{ij} b',
\]

where we have used that \( \epsilon^{ab} \epsilon_{ab} = \epsilon^{a'b'} g_{aa'} g_{bb'} = 2g_2 \) and \( g^{aa'} g^{bb'} \epsilon_{a'b'} = \epsilon_{ab} \). Also we chose \( \epsilon^{12} = 1 \) and so \( \epsilon_{12} = g_2 \); thus, for example, we have \( B_{ab} = \frac{1}{2\sqrt{g_2}} \epsilon_{ab} b' \). Hence we conclude that

\[
b = b', \quad \chi = \chi', \quad \tilde{\psi} = \tilde{\psi}'.
\]

In [7] it was shown that upon shrinking the metric on \( \Sigma \), one gets an effective 2-dimensional sigma model governing the maps from \( S^2 \) to \( \mathcal{M} \), where \( \mathcal{M} \) is the moduli space of solutions to the Hitchin’s equations. Although the twisted theory is supposed to be topological, since the space of self-dual harmonic forms in this case is one-dimensional one may not get the same effective theory if one instead shrinks \( S^2 \). In that case we will see that the effective theory which emerges is a 2-dimensional twisted SYM theory, as conjectured in [7].

Thereto, we now scale the metric on \( S^2 \) by a factor of \( \epsilon \). Notice that the definitions (2) and (3) are consistent with this scaling, since both sides of the self-duality constraints scale with the same power of \( \epsilon \).

After integrating out the auxiliary fields, the bosonic part of the Lagrangian reads

\[
\mathcal{L}_B = \frac{1}{\epsilon^2} \text{tr} \left\{ -D_\mu \lambda D^\mu \dot{\phi} - (D_\mu C - 2 D^\nu B_{\mu\nu})^2 - \frac{1}{2} (F_{\mu\nu}^+ + 2i [B_{\mu\rho}, B_{\nu\rho}] + 2i [B_{\mu\nu}, C])^2 \right\},
\]

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where \( F^+ = \frac{1}{2}(F + *F) \) and \(*\) is the Hodge duality operation. Thus we can write
\[
-\frac{1}{2} \int \sqrt{g} F^+_{\mu\nu} F^{\mu\nu} = -\frac{1}{2} \int \sqrt{g} F_{\mu\nu} F^{\mu\nu} - \frac{1}{4} \int \sqrt{g} (*F)_{\mu\nu} F^{\mu\nu}.
\]
The last term is the instanton number and is metric independent. Using this, and the fact that \( B_{\mu\nu} \) is self-dual, we write the last term in (5) as
\[
-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - 2i F_{\mu\nu} ([B_{\mu\rho}, B_{\nu}^\rho] + [B_{\nu\rho}, C] + [B_{\mu\nu}, C])^2 - \frac{1}{4} (*F)_{\mu\nu} F^{\mu\nu}.
\]
In the last equality we noted that for a self-dual antisymmetric ten sor we have
\[
-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - 2i F_{\mu\nu} ([B_{\mu\rho}, B_{\nu}^\rho] + [B_{\nu\rho}, C] + [B_{\mu\nu}, C])^2 - \frac{1}{4} (*F)_{\mu\nu} F^{\mu\nu}.
\]
In the last equality we noted that for a self-dual antisymmetric tensor we have \( S_{ij}^2 = S_{ab}^2 \). In particular
\[
[B_{ab}, C]^2 = [B_{ij}, C]^2
\]
\[
\text{tr}([B_{ai}, B^i_{b}][B_{aj}, B^j_{b}]) = \text{tr}([B_{ai}, B^a_{j}][B^{ab}, B_{b}^j]).
\]
After scaling the metric, then, the Lagrangian splits to three parts;
\[
\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_0 + \mathcal{L}_{-1},
\]
where \( \mathcal{L}_n \) scales as \( \epsilon^n \). Specifically,
\[
\mathcal{L}_1 = \frac{\epsilon}{\epsilon^2} \left\{ -D_i \lambda D^i \phi - D_i C D^i C - D_i b D^i b + \frac{2}{\sqrt{g}_1} \epsilon^{ij} D_i b D_j C - \frac{1}{2} (f + 2i [b, C])^2 \right. \\
+ \frac{4}{\sqrt{g}_1} \epsilon^{ij} \tilde{c}_1 \psi_i D_j \chi + \frac{2}{\sqrt{g}_1} \epsilon^{ij} \tilde{c}_1 \psi_i D_j \psi + \tilde{c}_1 D^i \zeta - \psi_i D^i \eta \\
+ i \sqrt{2} \psi_\bar{\psi} [\tilde{\psi}, \lambda] - i 4 \sqrt{2} \psi \chi [\phi, \phi] + i 4 \sqrt{2} \psi [\chi, C] - 2 i \sqrt{2} \chi [\zeta, b] \\
- i \sqrt{2} \psi [\eta, b] + \frac{2 \sqrt{2}}{\sqrt{g}_1} \epsilon^{ij} \tilde{c}_1 \psi_i [\tilde{\psi}_j, b] - i \sqrt{2} \psi_\bar{\psi} [\phi, \phi] + i \sqrt{2} \psi_\bar{\psi} [\lambda, \chi] \\
- i \sqrt{2} \psi \bar{\psi} [\chi_i, C] + \frac{i}{2 \sqrt{2}} \zeta [\zeta, \lambda] - \frac{i}{\sqrt{2}} \zeta [\eta, C] \\
+ 2 \phi, b [\lambda, b] + 2 [\phi, C] [\lambda, C] \right\},
\]
(6)
\[ \mathcal{L}_0 = \frac{1}{\epsilon^2} \text{tr} \left\{ -D_a \nabla^a \phi - (D_a C + \frac{1}{\sqrt{g_2}} \epsilon_{ab} D^b \partial - 2 D_i B_{ai})^2 + 4 (D^a B_{ai}) (D^i C - 2 D^j B_{ji}) \right\} \]

\[ - (F_{ai}^i + 2i [B_{aj}, B^j_i] + 2i [B_{ab}, B^b_i] + 2i [B_{ai}, C])^2 - \frac{1}{4} (\ast F)_{ij} F^{ij} - \frac{1}{4} (\ast F)_{ab} F^{ab} \]

\[ - 2i (F_{ij} + 2i [B_{ij}, C]) [B^{ia}, B^a j] - 2i (F_{ab} + 2i [B_{ai}, B^i_b]) [B^{ab}, C] \]

\[ + 4 \psi_i D_a \chi^a + 4 \bar{\chi}_a \chi^a - \psi_a D^a \eta + 4 \psi_a D_b \chi^{ab} + 4 \bar{\chi}_a D_b \bar{\psi}^{ab} \]

\[ + 2i \sqrt{2} \bar{\psi}^{ai} [\bar{\psi}_{ai}, \lambda] - 2i \sqrt{2} \chi^{ai} [\chi_{ai}, \phi] + i 4 \sqrt{2} \bar{\psi}^{ai} [\chi_{ai}, C] + i 4 \sqrt{2} \bar{\psi}^{ab} [\chi_{ai}, B^i_b] \]

\[ + i 4 \sqrt{2} \bar{\psi}^{ia} [\chi_{ib}, B^b_a] + i 4 \sqrt{2} \bar{\psi}^{ai} [\chi_{ab}, B^b_i] + i 4 \sqrt{2} \bar{\psi}^{ai} [\chi_{aj}, B^{j}_{i}] + i 4 \sqrt{2} \bar{\psi}^{ia} [\chi_{ij}, B^{j}_a] \]

\[ + i 4 \sqrt{2} \bar{\psi}^{ij} [\chi_{ia}, B^{a}_j] - 2i \sqrt{2} \chi^{ai} [\chi_{ai}, B^{a}_j] - 2i \sqrt{2} \bar{\psi}_{ai} [\chi_{ai}, B^{a}_j] + i 4 \sqrt{2} \bar{\psi}_{ai} [\chi_{ib}, B^{ab}] \]

\[ + i 4 \sqrt{2} \bar{\psi}_{ai} [\chi_{ai}, B^{a}] + i 4 \sqrt{2} \bar{\psi}_{ai} [\chi_{ai}, B^{a}] - i \sqrt{2} \bar{\chi}_a [\chi^a, \phi] + i \sqrt{2} \bar{\psi}_{ai} [\psi^a, \lambda] \]

\[ - i 2 \sqrt{2} \bar{\psi}_{ai} [\chi^a, C] + 4 [\phi, B^a] [\lambda, B_{ai}] \}, \] (7)

and

\[ \mathcal{L}_{-1} = \frac{1}{\epsilon c^2 \text{tr} \left\{ -4 (D^a B_{ai})^2 - \frac{1}{4} (F_{ab} + 4i [B_{ai}, B^i_b])^2 \right\} \} \]. (8)

Now, in sending \( \epsilon \) to zero path integral localizes around the solutions of the following equations

\[ F_{ab} + 4i [B_{ai}, B^i_b] = 0 \]

\[ D^a B_{ai} = 0. \] (9)

In appendix A we show that these equations imply

\[ F_{ab} = B_{ai} = 0, \] (10)

and from \( F_{ab} = 0 \) it follows that the instanton number vanishes. A flat connection on sphere can be written globally as

\[ A_a = g^{-1} \partial_a g \]

for some gauge group element \( g \). Therefore, the connection \( A \) is

\[ A = A_i dx^i + (g^{-1} \partial_a g) dx^a. \]

We gauge transform \( A \) such that it lies in \( \Sigma \) direction

\[ A \rightarrow g A g^{-1} + g d g^{-1} = g (A_i dx^i) g^{-1} + g (\partial_i g^{-1}) d x^i = A'_i dx^i. \]
Setting $A_a = 0$ and $B_{ai} = 0$, $\mathcal{L}_0$ greatly simplifies. However, because of the zero modes of the operator $d_a$, one has to still keep the order $\epsilon$ terms in $\mathcal{L}_1$. We expand all fields in terms of eigen functions of $d_a$ and denote the zero modes by a 0 superscript. Effectively we do the following substitution

$$\Phi(z, \bar{z}; w, \bar{w}) \rightarrow \Phi^0(z, \bar{z}) + \Phi(z, \bar{z}; w, \bar{w})$$

where $\Phi(z, \bar{z}; w, \bar{w})$ on the RHS stands for the nonzero modes. The kinetic part of $\mathcal{L}_0$ then reads

$$\mathcal{L}_{0 \text{ kin}} = \frac{1}{e^2} \text{tr} \left\{ -\partial_a \lambda \partial^a \phi - (\partial_a C + \epsilon_{ab} \partial^b b)^2 - (\partial_a A_i)^2 \right\} + 4\psi [i \nabla_a \bar{\chi}^i a + 4\bar{\chi} [i \nabla_a \psi^i a + \bar{\chi} a \nabla^a \zeta - \psi_a \nabla^a \eta + 4\psi_a \nabla_b \chi^{ab} + 4\bar{\chi} a \nabla_b \bar{\psi}^{ab} \right\}.$$

(11)

Since $\chi_{ai}$ and $\bar{\psi}_{ai}$ are self-dual and since there are no holomorphic one forms on sphere (see the Appendix), $\mathcal{L}_{0 \text{ kin}}$ is nondegenerate. Thus in doing the integral over nonzero modes, one may drop the terms which are order of $\epsilon$. Keeping terms of order one, the integral over $\eta, \zeta, \chi, \bar{\psi}, \psi_i$ and $\bar{\chi} i$ results in a set of delta functions imposing the following constraints

$$\nabla_a \chi^{ai} = 0, \ \nabla_a \bar{\psi}^{ai} = 0$$

$$\nabla_a \psi^a = 0, \ \epsilon^{ab} \nabla_a \psi_b = 0$$

$$\nabla_a \bar{\chi}^a = 0, \ \epsilon^{ab} \nabla_a \bar{\chi} b = 0.$$  

(12)

As was mentioned, these equations have no nontrivial solutions on sphere. Setting these fields to zero, $\mathcal{L}_0$ reduces to

$$\mathcal{L}_0 = \frac{1}{e^2} \text{tr} \left\{ -\partial_a \lambda \partial^a \phi - (\partial_a C)^2 - (\partial_a b)^2 - (\partial_a A_i)^2 \right\}$$

where fields are all nonzero modes. Using the equation of motion for $A_i$ we obtain

$$d^i d A_i + \text{terms proportional to} \ \epsilon = 0$$

as $A_i$ is a nonzero mode this equation implies that, up to $\epsilon$ order, $A_i = 0$. The same happens for $\phi, b$ and $C$ fields. So in the limit $\epsilon \rightarrow 0$ all nonzero modes can be set to zero and one is left with a copy of $\mathcal{L}_1$ in which fields now depend only on coordinates on $\Sigma$. From now on we call this reduced Lagrangian $\mathcal{L}$ and drop the 0 superscript on zero modes.
The reduced Lagrangian, $\mathcal{L}$, which now describes a two-dimensional TFT, can be obtained by the BRST variation of $V$, where

$$V = \frac{1}{e^2} \int \Sigma \left\{ \frac{1}{2} \chi^i (\tilde{H}_i - 2\sqrt{2} D_i C + \frac{2\sqrt{2}}{\sqrt{g_1}} \epsilon_{ij} D^j b) + \chi (2H - 2f - 4i[b, C]) \right\}$$

and the BRST transformations of the two-dimensional fields are ($\delta \equiv \{ Q, \cdots \}$)

$$\delta A_i = -2\psi_i \quad \delta b = \sqrt{2} \tilde{\psi} \quad \delta C = \frac{1}{\sqrt{2}} \zeta$$
$$\delta \psi_i = -\sqrt{2} i D_i \phi \quad \delta \tilde{\psi} = -2[b, \phi] \quad \delta \zeta = -4[C, \phi]$$
$$\delta \chi_i = i \tilde{H}_i \quad \delta \chi = iH \quad \delta \lambda = \sqrt{2} \eta \quad \delta \phi = 0$$
$$\delta \tilde{H}_i = 2\sqrt{2} i [\tilde{\chi}_i, \phi] \quad \delta H = 2\sqrt{2} i [\chi, \phi] \quad \delta \eta = -2[\lambda, \phi].$$

The fixed points around which path integral localizes are those configurations that are BRST invariant. Thus, setting $\delta \chi = H = 0$ and $\delta \tilde{\chi}_i = \tilde{H}_i = 0$ and using the equation of motion for $H$ and $\tilde{H}_i$ we find the fixed point equations

$$s = f + 2i[b, C] = 0$$
$$k = D_i C + \frac{1}{\sqrt{g_1}} \epsilon_{ij} D^j b = 0. \quad (14)$$

Squaring these equations implies that

$$0 = \int \text{tr} \left\{ \frac{1}{2} |s|^2 + |k|^2 \right\}$$
$$= \int \text{tr} \left\{ \frac{1}{2} |f|^2 + 2|b, C|^2 + 2i f [b, C] + |D_i C|^2 + |D_i b|^2 + \frac{2}{\sqrt{g_1}} \epsilon_{ij} D^i C D^j b \right\}. $$

Using the definition of $f$ in (2), we can see that the third and the last term cancel against each other. Therefore this integral is zero if and only if

$$f = 0, \quad [b, C] = 0$$
$$D_i C = D_i b = 0. \quad (15)$$

Requiring that there are no reducible connections (as is the case for flat non-trivial $SO(3)$ bundles) it follows that the only solutions are $C = b = 0$. Therefore, following [4], it can be seen that in this case the partition function is nothing but the Euler characteristic of the moduli space of flat connections over $\Sigma$. 9
3 Perturbing by mass term

The theory discussed so far does not have a mass gap \[1\]. To make the calculations more feasible we perturb the theory such that it has a mass gap\[2\]. This enables us to integrate out most fields and reduce the path integral to a finite dimensional one.

The reduced 2-dimensional theory has a $U(1)$ ghost number symmetry coming directly from the nonanomalous $U(1)$ symmetry of the underlying 4-dimensional $N = 4$ SYM theory. Because of supersymmetry, the measure for nonzero modes is invariant under the $U(1)$ action. The ghost and the antighost zero modes, on the other hand, obey the same equations of motion such that there are equal number of ghost and antighost zero modes. This renders the measure to be invariant under the ghost symmetry of the action. Therefore the ghost symmetry is anomaly free.

As the measure is invariant under this symmetry, the correlation function of any operator that has a ghost charge is zero. Therefore, this symmetry allows us to perturb the Lagrangian, by adding gauge invariant terms with nonzero ghost number, without changing the partition function. Thus, for example, since the mass term for the hypermultiplet (as we will see presently) consists of a term with negative ghost number and a term which is BRST exact, one expects that the partition function is invariant under perturbing the Lagrangian by a mass term for the hypermultiplet. One can even go further to argue that an additional mass term for the chiral multiplet $\Phi$ (which contains $\phi$ and $\lambda$) still leaves the partition function invariant \[4, 12\].

In the following we are interested in the correlation functions of a set of BRST cohomology classes of the form

$$I(\varepsilon) = \frac{1}{4\pi^2} \int_{\Sigma} \text{tr} \left( \frac{i}{\sqrt{2}} \phi F + \frac{1}{2} \psi \wedge \psi \right) + \frac{\varepsilon}{32\pi^2} \int_{\Sigma} \text{tr} \phi^2.$$  

Part of this factor with an extra BRST exact term provides the mass for the chiral multiplet $\Phi$ \[2\]. The remaining part may have a nonvanishing expectation value in the mass deformed theory. This, in particular, implies that, in contrast to the partition function, the correlation functions of $I(\varepsilon)$ (in the perturbed theory by mass for the hypermultiplet) may depend on the mass parameter.

\[2\] A similar perturbation has been considered in \[4, 12\] for $N = 4$ SYM theory in four dimensions.
The next problem is to give a mass to $\chi, \eta, \lambda, \tilde{\psi}$ and $\zeta$. This can be achieved by adding $V'$ and $V''$ to $V$, where

$$V' = -\frac{2}{e^2} \int_{\Sigma} d\mu \, \text{tr} \{ \chi \lambda \}$$

$$V'' = \frac{1}{e^2} \int_{\Sigma} d\mu \, \text{tr} \{ \tilde{\psi} C - \frac{1}{2} \zeta b \}.$$  \hspace{1cm} (16)

To give a mass term to the bosonic fields $b, C$ and the fermionic one $\tilde{\chi}$ we change the BRST transformation rules for $\tilde{H}_i$, $\tilde{\psi}$ and $\zeta$ to the following ones

$$\delta_m \tilde{H}_i = 2\sqrt{2} i [\tilde{\chi}_i, \phi] + \frac{\sqrt{2}}{\sqrt{g_1}} m \epsilon_{ij} \tilde{\chi}_j$$

$$\delta_m \tilde{\psi} = -2 [b, \phi] + imC$$

$$\delta_m \zeta = -4 [C, \phi] - 2imb.$$  \hspace{1cm} (18)

Even though the metric is explicitly introduced via the above first BRST transformation rule, note that the extra term is still invariant under metric rescaling ($\epsilon_{ij} \sim g_1$).

Thus, in the following we will consider the theory defined by the deformed action

$$S = I(\epsilon) + i\delta_m (V + tV' + \frac{1}{2} m V'')$$

$$= I(\epsilon) + \frac{1}{e^2} \int_{\Sigma} d\mu \, \text{tr} \{ D_i \lambda D^i \phi + D_i C D^i C + D_i b D^i b - \frac{2}{\sqrt{g_1}} \epsilon^{ij} D_i b D_j C \}$$

$$+ \frac{1}{2} (f + 2i[b, C] + t\lambda)^2 + 2i\sqrt{2} t \chi \eta - \frac{1}{2} |m|^2 C^2 - \frac{1}{2} |m|^2 b^2$$

$$+ \frac{i m}{\sqrt{2} g_1} \sqrt{2} \tilde{\psi} - \frac{m}{\sqrt{2} g_1} \epsilon_{ij} \tilde{\chi}_j + 2i \tilde{\mu} \phi [b, C] - 2im \lambda [b, C]$$

$$+ \frac{4i}{\sqrt{g_1}} \epsilon^{ij} \psi_i D_j \chi + \frac{2i}{\sqrt{g_1}} \epsilon^{ij} \tilde{\chi}_i D_j \tilde{\psi} + i \tilde{\chi}_i D^i \zeta - i \tilde{\psi}_i D^i \eta$$

$$+ \sqrt{2} \tilde{\psi} [\tilde{\psi} \zeta, \lambda] + 4 \sqrt{2} \tilde{\chi} [\chi, \phi] - 4 \sqrt{2} \tilde{\psi} [\chi, C] + 2 \sqrt{2} \chi [\zeta, b]$$

$$+ \sqrt{2} \tilde{\psi}[\eta, b] - \frac{2 \sqrt{2}}{\sqrt{g_1}} \epsilon^{ij} \psi_i [\tilde{\chi}_j, b] + \sqrt{2} \tilde{\chi}_i [\tilde{\chi}^i, \phi] - \sqrt{2} \tilde{\psi}_i [\tilde{\psi}^i, \lambda]$$

$$+ 2 \sqrt{2} \psi_i [\tilde{\chi}^i, C] - \frac{1}{2 \sqrt{2}} \zeta [\zeta, \lambda] + \frac{1}{\sqrt{2}} \zeta [\eta, C] - 2 [\phi, b] [\lambda, b] - 2 [\phi, C] [\lambda, C].$$  \hspace{1cm} (19)

Notice that although the new BRST charge does not square to a gauge transformation (because of those new terms proportional to $m$), Lagrangian remains BRST invariant. This can be understood if we notice that $\delta_m^2$ acting on fields generates (up to a gauge transformation) a $U(1)$ action. Let $\delta_T \equiv \frac{1}{i \sqrt{2} m} \delta_m^2$ and $\beta \equiv b + i C$, $\psi \equiv \tilde{\psi} + \frac{i}{2} \zeta$, then $U(1)$ group acts as

$$\delta_T \beta = -i \beta,$$

$$\delta_T \psi = -i \psi.$$
\[ \delta_T \tilde{\chi}_i = \frac{1}{\sqrt{g_1}} \epsilon_{ij} \tilde{\chi}^j, \quad \delta_T \tilde{H}_i = \frac{1}{\sqrt{g_1}} \epsilon_{ij} \tilde{H}^j \]

thus the fields \( \beta, \psi, \tilde{\chi}_z \) and \( \tilde{H}_z \) all have charge \(-1\), with their complex conjugate having charge \(+1\). All other fields have zero charge under this \( U(1) \) group. The fact that \( S \) is invariant under \( \delta_m \) then follows since \( V, V' \) and \( V'' \) all have zero \( U(1) \) charge.

Before continuing the analysis, it is important to understand the relation between the perturbed and unperturbed theories. Since the perturbing terms proportional to \( t \) and \( \tilde{m} \) are BRST exact, one may expect that correlation functions are going to be independent of these two parameters, but actually this is not true in general: adding \( \delta_m V' \) and \( \delta_m V'' \) to the Lagrangian may result in some new set of fixed points flowing in from infinity and deforming the original moduli space of solutions \([6]\) such that the path integral gets contribution from these new fixed points. The theory will be independent of \( t \) and \( \tilde{m} \) if in varying these parameters Lagrangian remains nondegenerate and the perturbation does not introduce new components to the moduli space of fixed points.

We first discuss the situation for \( t = 0 \) with arbitrary \( m \) and \( \tilde{m} \). The perturbed Lagrangian (with \( t = 0 \)) can also be derived upon reducing the \( N = 4 \) theory broken to \( N = 2 \) by the mass term. Had we started with \( N = 2 \) theory with one massive hypermultiplet in the adjoint representation of the gauge group in four dimensions, we would have ended up with the same above perturbed Lagrangian after reduction.

The fixed point equations are those of (14) together with (setting \( \delta_m \tilde{\psi} = \delta_m \zeta = \delta_m \eta = \delta_m \psi_i = 0 \))

\[ [\beta, \phi] = \frac{1}{2} m \beta, \quad [\lambda, \phi] = 0, \quad D_i \phi = 0. \] (20)

If \( \phi \) is not identically zero then, being covariantly constant, it never vanishes and, in particular, can be diagonalized globally such that the bundle \( E \) splits as a sum of line bundles \([13]\). Moreover, if \( \beta \neq 0 \), the first equation in (20) fixes \( \phi \) (up to a sign)

\[ \phi = \frac{m}{4} \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right) \] (21)

with \( \beta \) as

\[ \beta = \tilde{\beta} \left( \begin{array}{cc} 0 & 0 \\ 1 & 0 \end{array} \right). \] (22)
Now the equations (14) become

\[ \tilde{f} + 2|\tilde{\beta}|^2 = 0 \]
\[ \bar{D}\tilde{\beta} = (\partial_\xi - iA_\xi)\tilde{\beta} = 0 \]

(notice \( f = \frac{1}{2}\tilde{f}\sigma_3 \), where \( \tilde{f} \) here is the \( U(1) \) curvature). Note that \( \phi = \frac{1}{4}m\sigma_3 \) corresponds to a point, in the classical moduli space of vacua, where a component of the hypermultiplet becomes massless.\(^3\) The relevant fixed points are then determined by the above equations. Clearly one can then argue that the path integral over massless modes computes the Euler characteristic of the moduli space of \( U(1) \) flat connections. However, to evaluate the contribution of this singular point to the path integral, one still has to do the integral over the massive modes.

This is not an easy task, but there is a special case where this point \( (\phi = \frac{1}{4}m\sigma_3) \) does not make any contribution. This occurs upon restricting to the nontrivial \( SO(3) \) bundles. As discussed above, a nonzero \( \phi \) breaks the gauge group down to \( U(1) \). In particular, \( SO(3) \) bundles split as

\[ E = L \oplus \mathcal{O} \oplus L^{-1}, \tag{23} \]

where \( L \) is the \( U(1) \) line bundle and \( \mathcal{O} \) is a trivial line bundle. In this case, \( w_2(E) \), which measures the nontriviality of the bundle \( E \), turns out to be the mod two reduction of \( c_1(L) \), the first Chern class of \( L \). Thus if \( f = 0 \), as is required by eqs. (14), \( w_2(E) \) has to be zero – implying that flat nontrivial \( SO(3) \) bundles do not admit reducible connections. Therefore, in this case, the point \( \phi = \frac{1}{4}m\sigma_3 \) does not contribute to the path integral.

Let now discuss the case that \( t \neq 0 \). The fixed point equations (14) turn into the following equations \( (\beta \equiv b + iC \) with \( \epsilon_{z\bar{z}} = i\sqrt{g_iz\bar{z}}) \)

\[ f + [\tilde{\beta},\beta] + t\lambda = 0 \]
\[ \bar{D}\beta = 0 , \quad D\tilde{\beta} = 0. \tag{24} \]

The vanishing argument now fails; \( f = \beta = 0 \) (and \( \lambda = 0 \)) are not the only solutions, there are new fixed points with \( f \neq 0 \) contributing to the partition function. Since the connection

\(^3\) As eq. (21) fixes \( \phi \) up to a sign, there are indeed two such singular points in the classical moduli space of vacua.
is not bounded to be flat any more, a set of $U(1)$ connections, in all classes of $U(1)$ bundles, appear in the moduli space of solutions. Moreover, the point $\phi = \frac{1}{4} m \sigma_3$ may contribute to the path integral even for nontrivial bundles. In the following we single out this point from our discussion and treat it independently.

Integrating $\lambda$, $\eta$ and $\chi$

Perturbing by $V'$ now allows us to integrate out the fields $\lambda$, $\eta$ and $\chi$. Using the equations of motion for $\lambda$ and $\eta$ we get

$$
t^2 \lambda = D^2 \phi - t(f + 2i[b, C]) + 2im[b, C] + \sqrt{2}[\psi_i, \psi^i] + \sqrt{2}[\tilde{\psi}, \tilde{\psi}] + \frac{1}{2\sqrt{2}}[\eta, \zeta] + 2[b, [\phi, b]] + 2[C, [\phi, C]]
$$

and

$$
\chi = \frac{1}{2\sqrt{2}t} \left\{ -D_i \psi^i + i\sqrt{2}[b, \tilde{\psi}] + \frac{i}{\sqrt{2}}[C, \zeta] \right\}.
$$

Putting these back into the Lagrangian yields

$$
S = I(\varepsilon) + \frac{1}{e^2} \int_{\Sigma} d\mu \text{ tr} \left\{ D_i C D^i C + D_i b D^i b - \frac{2}{\sqrt{g_1}} \epsilon^{ij} D_i b D_j C + \frac{2i}{\sqrt{g_1}} \epsilon^{ij} \tilde{\chi}_i D_j \tilde{\psi} + i D_i D^i \zeta \right\}
$$

$$
- \frac{2\sqrt{2}}{\sqrt{g_1}} \epsilon^{ij} \psi_i [\tilde{\chi}_j, b] + 2\sqrt{2} \psi_i [\tilde{\chi}^i, C] + \sqrt{2} \tilde{\chi}_i [\tilde{\chi}^i, \phi] - \frac{1}{2} \lambda m^2 C^2 - \frac{1}{2} m^2 b^2 + \frac{im}{\sqrt{2}} \zeta \tilde{\psi}
$$

$$
- \frac{m}{\sqrt{2}g_1} \epsilon_{ij} \tilde{\chi}^i \tilde{\chi}^j + 2im \phi [b, C] + \frac{1}{t} \left\{ (f + 2i[b, C]) \right\}
$$

$$
\times \left( D^2 \phi + 2im[b, C] + \sqrt{2}[\psi_i, \psi^i] + \sqrt{2}[\tilde{\psi}, \tilde{\psi}] + \frac{1}{2\sqrt{2}}[\eta, \zeta] + 2[b, [\phi, b]] + 2[C, [\phi, C]] \right)
$$

$$
+ \frac{i}{2\sqrt{2}} \left\{ -D_i \psi^i + i\sqrt{2}[b, \tilde{\psi}] + \frac{i}{\sqrt{2}}[C, \zeta] \right\} \left\{ -\frac{4}{g_1} \epsilon^{kl} D_k \psi_l - i4\sqrt{2}[b, \tilde{\psi} - 2i\sqrt{2}[\zeta, b]] \right\}
$$

$$
+ \frac{1}{2t^2} \left( D^2 \phi + 2im[b, C] + \sqrt{2}([\psi_i, \psi^i] + [\tilde{\psi}, \tilde{\psi}] + \frac{1}{4}[\eta, \zeta]) + 2[b, [\phi, b]] + 2[C, [\phi, C]] \right)
$$

$$
+ \sqrt{2} \left\{ -D_i \psi^i + i\sqrt{2}([b, \tilde{\psi}] + \frac{1}{2}[C, \zeta]) \right\} \left\{ -D_i \psi^i + i\sqrt{2}([b, \tilde{\psi}] + \frac{1}{2}[C, \zeta]) \right\} \right\}.
$$

Terms proportional to $1/t$ are indeed BRST trivial, and can be written

$$
\frac{i}{\sqrt{2}t} \delta_m \left\{ (f + 2i[b, C]) \left\{ -D_i \psi^i + i\sqrt{2}[b, \tilde{\psi}] + \frac{i}{\sqrt{2}}[C, \zeta] \right\} \right\}.
$$

Terms proportional to $1/t^2$ are also combining into

$$
\frac{i}{2\sqrt{2}t^2} \delta_m \left\{ \left\{ -D_i \psi^i + i\sqrt{2}[b, \tilde{\psi}] + \frac{i}{\sqrt{2}}[C, \zeta] \right\} \times
$$
\[
\left( D^2 \phi + 2im[b, C] + \sqrt{2}[\psi_1, \psi^j] + \sqrt{2}[\bar{\psi}, \bar{\psi}] + \frac{1}{2\sqrt{2}}[\zeta, \zeta] + 2[b, [\phi, b]] + 2[C, [\phi, C]] \right).
\]

In the effective Lagrangian (26), the kinetic terms are nondegenerate for all values of \( t \) and since those terms proportional to \( t \) are still in a BRST exact form, the path integral does not depend on \( t \).

**Large \( t \) Limit and The Integration over \( b, C, \zeta, \bar{\psi} \)**

As argued above, for nontrivial \( SO(3) \) bundles the point \( \phi = \frac{1}{4}m\sigma_3 \) does not contribute. For \( t \neq 0 \), because of the supersymmetry, even after integrating out \( \lambda, \eta \) and \( \chi \) the singularity still persists at \( \text{tr} \phi^2 = \frac{1}{8}m^2 \). As we have chosen \( \phi \) to be a real scalar field, reality of the action requires that \( m \) to be a real parameter. However, to regulate the contribution of the points in the neighborhood of \( \text{tr} \phi^2 = \frac{1}{8}m^2 \), we allow \( m \) to have a small imaginary part.

If there is going to be any singularity when \( \phi \) approaches \( m \), it has to show up in the final result when we take the limit \( \text{Im} \; m \to 0 \). This can be thought of as a kind of regularization by analytic continuation.

Now let us consider the large limit of \( t \). Since the kinetic terms remain nondegenerate we can actually take \( t \to \infty \). Using the auxiliary field \( \tilde{H}_i \), in this limit we are left with the action

\[
\begin{align*}
S &= \frac{1}{e^2} \int d\mu \text{tr} \left\{ -\frac{1}{2} \tilde{H}^i(\tilde{H}_i - 2\sqrt{2}D_iC + \frac{2\sqrt{2}}{\sqrt{g_1}}\varepsilon_{ji}D^j b) + \frac{2i}{\sqrt{g_1}}\varepsilon^{ij}\bar{\chi}_iD_j\bar{\psi} + i\bar{\chi}_iD^i\zeta \\
&\quad - \frac{1}{2}|m|^2C^2 - \frac{1}{2}|m|^2b^2 + \frac{i\bar{m}}{\sqrt{2}}\zeta\bar{\psi} - \frac{m}{\sqrt{2}g_1}\varepsilon^{ij}\bar{\chi}_i\bar{\chi}_j^i + 2im\phi[b, C] \\
&\quad - \frac{2\sqrt{2}}{\sqrt{g_1}}\varepsilon^{ij}\psi_i[\tilde{\chi}_j, b] + 2\sqrt{2}\psi_i[\tilde{\chi}_i, C] + \sqrt{2}\tilde{\chi}_i[\tilde{\chi}_i, \phi] \right\} + I(\varepsilon).
\end{align*}
\]

\( \mathcal{L} \) can still be written as a sum of BRST exact term

\[
i\delta_m \left\{ \frac{1}{e^2} \text{tr} \left\{ \frac{1}{2}\tilde{\chi}^i(\tilde{H}_i - 2\sqrt{2}D_iC + \frac{2\sqrt{2}}{\sqrt{g_1}}\varepsilon_{ji}D^j b) + \frac{1}{2}\bar{m}(\bar{\psi}C - \frac{1}{2}\zeta b) \right\} \right\}
\]

and \( I(\varepsilon) \). The integral over \( C \) gives a factor of \( \left( \det \left( \frac{1}{2\pi^2} |m|^2 \right) \right)^{-\frac{1}{4}} \) and leaves

\[
\begin{align*}
S &= \frac{1}{e^2} \int \text{tr} \left\{ -\frac{1}{2} \tilde{H}^i(\tilde{H}_i - 2\sqrt{2}[\tilde{\chi}_i, \phi] - \frac{m}{\sqrt{2}g_1}\varepsilon_{ij}\tilde{\chi}_i^j) + \frac{2i}{\sqrt{g_1}}\varepsilon^{ij}\bar{\chi}_iD_j\bar{\psi} + i\bar{\chi}_iD^i\zeta \\
&\quad + \frac{1}{|m|^2}(D_i\tilde{H}^i - 2[\tilde{\chi}_i, \psi^j])^2 - \frac{2\bar{m}}{m}[b, \phi]^2 + \frac{2i\sqrt{2}}{m}[b, \phi](D_i\tilde{H}^i - 2[\tilde{\chi}_i, \psi^j]) \right\}
\end{align*}
\]
Replacing $b$ and a factor of $b$ over equations of motion. In the evaluation of determinants, which appear in doing the integral $A$ where we have defined $(\ad \phi \psi)$, we finally get that the integral over the gauge fields constrains $\phi$ to be constant. The equation of motion for $b$ yields

$$b^A = \frac{\sqrt{2}}{|m|^2} K^{AB} \left( -\frac{1}{\sqrt{g_1}} \epsilon^{ij} D_j \bar{H}_i + \frac{2}{\sqrt{g_1}} \epsilon^{ij} [\bar{\chi}_i, \psi_j] - \frac{2i}{m} [(D_i \bar{H}^i - 2[\bar{\chi}_i, \psi^i]), \phi] \right)^B,$$

(27)

where we have defined $(A$ and $B$ are Lie algebra indices)

$$K^{AB} \equiv (1 - \frac{8}{m^2} \text{tr} \phi^2)^{-1} (\delta^{AB} - \frac{8}{m^2} \phi^A \phi^B).$$

Replacing $b$ in the action, we obtain

$$S = I(\varepsilon) + \frac{1}{e^2} \int d\mu \text{tr} \left\{ -\frac{1}{2} \bar{H}^i \bar{H}_i + \sqrt{2} \bar{\chi} i \bar{\chi}_i, \phi \right\}$$

$$- \frac{m}{\sqrt{2}g_1} \epsilon^{ij} (\bar{\chi}^j \bar{\chi}^j - \frac{4i}{|m|^2} D_i \bar{\chi}^i D^j \bar{\chi}^j) + \frac{1}{|m|^2} (D_i \bar{H}^i - 2[\bar{\chi}_i, \psi^i])^2 \right\}$$

$$+ \frac{1}{|m|^2} \left( -\frac{1}{\sqrt{g_1}} \epsilon^{ij} D_j \bar{H}_i + \frac{2}{\sqrt{g_1}} \epsilon^{ij} [\bar{\chi}_i, \psi_j] - \frac{2i}{m} [(D_i \bar{H}^i - 2[\bar{\chi}_i, \psi^i]), \phi] \right)^A K^{AB}$$

$$\times \left( -\frac{1}{\sqrt{g_1}} \epsilon^{kl} D_k \bar{H}_l + \frac{2}{\sqrt{g_1}} \epsilon^{kl} [\bar{\chi}_k, \psi_l] - \frac{2i}{m} [(D_l \bar{H}^l - 2[\bar{\chi}_l, \psi^l]), \phi] \right)^B,$$

and a factor of

$$\left( \det \left( \frac{1}{\sqrt{2e^2 \bar{m}}} \right) \right) \equiv \frac{\sqrt{2m}}{\sqrt{g_1}} \epsilon^{ij} D_i \bar{\chi}_j D_i \bar{\chi}_j,$$

where $(\text{ad} \phi)_{AB} = -f_{ABCD} \phi^C$ and $\Omega^0$ indicates the space of zero-forms.

The following are easily derived,

$$\delta \phi \left\{ (D_i \bar{\chi}^i)(D_i \bar{H}^i - 2[\bar{\chi}_i, \psi^i]) \right\} = i(D_i \bar{H}^i - 2[\bar{\chi}_i, \psi^i])^2 - 2\sqrt{2i}(D_i \bar{\chi}^i)[D_i \bar{\chi}_i, \phi]$$

$$+ \frac{\sqrt{2m}}{\sqrt{g_1}} \epsilon^{ij} D_i \bar{\chi}_j D_i \bar{\chi}_j,$$
and
\[
\delta_m \left\{ \frac{1}{\sqrt{g_1}} \epsilon_{ij} D_j \bar{\chi}_i + \frac{2i}{m} [D_i \bar{\chi}^i, \phi] \right\} = \frac{i \epsilon_{ij}}{\sqrt{g_1}} D_j \bar{H}_i - \frac{2i}{\sqrt{g_1}} \epsilon_{ij} [\bar{\chi}_i, \psi_j] - \frac{2}{m} [(D_i \bar{H}^i - 2[\bar{\chi}_i, \psi^i]), \phi] \\
\delta_m^2 \left\{ \frac{1}{\sqrt{g_1}} \epsilon_{ij} D_j \bar{\chi}_i + \frac{2i}{m} [D_i \bar{\chi}^i, \phi] \right\} = i \sqrt{2/m} D_i \bar{\chi}^i - \frac{4i \sqrt{2}}{m} [D_i \bar{\chi}^i, \phi, \phi].
\]

Using these, the action can be written as
\[
S = I(\varepsilon) + \frac{1}{e^2} \int_{\Sigma} d\mu \, tr \left( -\frac{1}{2} \bar{H}^i \bar{H}_i + \sqrt{2} \bar{\chi}^i [\bar{\chi}_i, \phi] - \frac{m}{\sqrt{2g_1}} \epsilon_{ij} \bar{\chi}^i \bar{\chi}^j \right) \\
- \frac{i}{e^2 |m|^2} \int_{\Sigma} d\mu \, \delta_m \left\{ tr \left( (D_i \bar{\chi}^i)(D_i \bar{H}^i - 2[\bar{\chi}_i, \psi^i]) \right) \right\}
+ \left( \frac{\epsilon_{ij}}{\sqrt{g_1}} D_j \bar{\chi}_i + \frac{2i}{m} [D_i \bar{\chi}^i, \phi] \right)^A K^{AB} \frac{1}{\sqrt{2g_1}} (D_i \bar{H}_k - 2[\bar{\chi}_k, \psi^i]) + \frac{2i}{m} [(D_i \bar{H}^i - 2[\bar{\chi}_i, \psi^i]), \phi] \right\} \right\}.
\]

Note that the integration over \(b, C, \zeta\) and \(\tilde{\psi}\) has not destroyed the manifest BRST exactness of the action, in particular, the variation of \(S\) with respect to \(\bar{m}\) is still a BRST commutator.

Large \(\bar{m}\) Limit and The Final Reduction

We note the partition function is formally independent of \(\bar{m}\) (since the variation of the partition function with respect to \(\bar{m}\) gives an BRST exact expression) and is really independent of \(\bar{m}\) if in varying \(\bar{m}\) the Lagrangian remains nondegenerate with a good behaviour at infinity in field space. The mass term for \(\bar{\chi}_i\), the term \(\bar{H}^i \bar{H}_i\), and the form of the cohomology classes that we have added by hand, guarantee that this is actually the case. Having this freedom in the value of \(\bar{m}\), we simply set \(\bar{m} = \infty\). This leaves us with the action
\[
S = I(\varepsilon) + \frac{1}{e^2} \int_{\Sigma} d\mu \left\{ -\frac{1}{2} \bar{H}^i A \bar{H}_i A - \bar{\chi}^i A \left( \frac{1}{\sqrt{2g_1}} m \epsilon_{ij} \delta_{AB} - 2i f_{ABC} \phi g_{ij} \right) \bar{\chi}^B \right\},
\]
and the partition function reads
\[
Z[\varepsilon, m] = \int D(A_i, \psi_i, \phi, H_i, \bar{\chi}_i) \left( \frac{\det \left( \frac{1}{\sqrt{2\varepsilon^2}} \bar{m} \right)}{(\det \frac{1}{2\varepsilon^2} |m|)(\det (1 + \frac{8}{m^2} (ad \phi)^2))^{\frac{1}{2}}} \right)^{\Omega_{\phi \phi E}} e^{-S}. 
\]
The explicit appearance of \(m\) on the LHS reminds us that, although independent of \(\bar{m}\), \(Z\) does depend on \(m\). This is so because \(m\) was introduced through the BRST transformation laws. This is reminiscent of holomorphicity of \(N = 1\) theories in four dimensions.
Doing the integral over $\bar{\chi}^i$ gives a similar determinant, but this time over the space of one-forms. Putting all pieces together one gets

$$Z[\varepsilon, m] = \int \mathcal{D}(A_i, \psi_i, \phi) \left( \det m(1 - 2i\sqrt{\frac{2}{m}} \text{ad} \phi) \right)_{\Omega^0 \otimes E} \left( \frac{\det m(1 - 2i\sqrt{\frac{2}{m}} \text{ad} \phi)}{\Omega^0 \otimes E} \right) e^{\left( \frac{-i}{2} \int_{\Sigma} \text{tr} \left( \frac{i}{\sqrt{2}} \phi F + \frac{1}{2} \psi \wedge \psi \right) - \frac{\varepsilon}{32\pi^2} \int_{\Sigma} \text{tr} \phi^2 \right)} (29)$$

Notice that, as expected, $\bar{m}$ cancels out between the fermionic and bosonic determinants. The integral over $\psi_i$ provides a symplectic measure for the gauge fields $A_i$ [6]. Performing the path integral over $\phi$ and $A_i$ is now straightforward. In appendix B, using the Faddeev-Popov gauge fixing technique, it has been shown that the integral over the gauge fields constrains $\phi$ to be constant and hence the path integral calculation reduces to a finite dimensional integral over constant $\phi$ [15]. Explicitly, for $SO(3)$ gauge group we have

$$Z[\varepsilon, m] = m^{3(g-1)} \sum_{n \in \mathbb{Z}} \int d\phi \phi^{2-2g} (1 - \frac{8}{m^2} \phi^2)^{g-1} \left( \frac{m - 2\sqrt{2} \phi}{m + 2\sqrt{2} \phi} \right)^{2n+1} \times \exp \left( -i \sqrt{2} \frac{\phi(2n+1)}{4\pi} - \frac{\varepsilon \phi^2}{32\pi^2} \right). (30)$$

4 Discussion

We have reduced the calculation of the correlation functions in the mass deformed theory to a finite dimensional integral in (30). We can now perform the sum over $n$ which results in a delta function restricting $\phi$ to obey the following equation

$$\exp \left( \frac{i \sqrt{2} \phi}{2\pi} \right) = \left( \frac{m - 2\sqrt{2} \phi}{m + 2\sqrt{2} \phi} \right)^2. (31)$$

Therefore

$$Z[\varepsilon, m] = m^{3(g-1)} \sum_{\phi_s} \phi_s^{2-2g} (1 - \frac{8}{m^2} \phi_s^2)^{g-1} \left( \frac{m - 2\sqrt{2} \phi_s}{m + 2\sqrt{2} \phi_s} \right) \exp \left( -i \sqrt{2} \frac{\phi_s}{4\pi} - \frac{\varepsilon \phi_s^2}{32\pi^2} \right), (32)$$

where $\phi_s$ is a solution to the eq. (31). A similar result for the correlation functions of a topological field theory corresponding to the Hitchin equations has been derived in [14].

To this one still has to add the contribution of the point $\phi = \frac{1}{4} m \sigma_3$. However, note that from the discussion we had in section 3, for nontrivial $SO(3)$ bundles, this point contributes
only if we perturb to \( t \neq 0 \)\(^4\). Thus if we are interested in the limit of \( t = 0 \), we can just ignore the contribution of this point.

In conclusion we note two observations. Firstly, the result is \( m \)-dependent as might be expected from the discussion in section 3. Note in particular that the expression (30) has the right behavior when \( m \to \infty \); in this limit, the solutions of eq. (31) are reduced to

\[
\phi_s = 2\sqrt{2}\pi^2 l
\]

and therefore eq. (30) reduces to the expression for the corresponding correlation functions in the say pure \( N = 2 \) theory \([6]\). The extra factor, \( m^{3(g-1)} \), is left from the integration over the heavy fields in that limit. The power of \( m \) is in accord with the dimension of the moduli space of flat connections which is

\[
\dim\mathcal{M} = 6g - 6.
\]

Any two zero modes of \( \chi_i \) are absorbed by the corresponding mass term in the Lagrangian and gives a power of \( m \).

Secondly, we recall that, in general, \( S \)-duality relates the strong and weak couplings and swaps the gauge group with its dual group. However, as in the limit where \( S^2 \) shrinks only instantons with \( k = 0 \) contribute to the path integral, unlike \([4]\), the correlators in the effective theory do not depend on the modular parameter “\( \tau \)”. Hence the action of \( S \)-duality is now simply to exchange the gauge group \( SU(2) \) with \( SO(3) \). Thus to derive that \( S \)-duality holds in this calculations, we must extend it for the \( SU(2) \) case; in particular, the contribution of the point \( \phi = \frac{1}{4} m \sigma_3 \) must be taken into account. Amusingly, one can infer properties of this contribution by demanding \( S \)-duality.

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\(^4\) As is discussed in \([3]\) the contribution of the original moduli space is invariant under perturbing to \( t \neq 0 \).
A The Vanishing Argument

In this appendix we want to discuss the solutions to eqs. (9):

\[ k = F_{ab} + 4i[B_{ai}, B_{bi}^i] = 0 \]
\[ s = D^a B_{ai} = 0. \]

Let first analyze the second equation. After squaring we get

\[
\int \text{tr} \left( D^a B_{ai} \right)^2 = - \int \text{tr} B^{ai}(D_a D_b B^b_i) \\
= - \int \text{tr} \left( B^{ai} D_b D_a B^b_i + B^{ai}[D_a, D_b] B^b_i \right) \\
= \int \text{tr} \left( (D_a B^a_i)(D_b B^b_i) + R_{ab} B^{ai} B^b_i - i B^{ai}[F_{ab}, B^b_i] \right) \\
= \int \text{tr} \left( (D_a B^a_i)^2 - \frac{1}{2}(D_{a B_{[ai}}) + \frac{1}{2} R B^{ai} B_{ai} - i B^{ai}[F_{ab}, B^b_i] \right) \tag{33}
\]

where we used the fact that in two dimensions, Ricci tensor takes a simple form

\[ R_{ab} = \frac{1}{2} g_{ab} R \]

and

\[ [D_a, D_b] B^c = R_{d ab} B^d_i + i [F_{ab}, B^c] \]
\[ [D_a, D_b] B^{ai} = R_{ab} B^{ai} + i [F_{ab}, B^{ai}] . \tag{34} \]

Since \( B_{\mu \nu} \) is self-dual, we have \( B_{\mu \nu} = B_{\nu \mu} = 0 \), hence

\[
(D_{[a B_{b]i}})(D^{[a B^{b]}_i}) = (D_\mu B_{\nu z})(D^\nu B^{\mu z}) + (D_\mu B_{\nu \bar{z}})(D^\nu B^{\mu \bar{z}}) = (D^\mu B_{\nu z})(D_\mu B_{\nu z}) + (D^\mu B_{\nu \bar{z}})(D_\mu B_{\nu \bar{z}}) = (D^a B_{ai})(D_b B^{bi}) . \tag{35}
\]

Putting this back into (33) we get

\[
\frac{3}{2} \int \text{tr} \left( D^a B_{ai} \right)^2 = \int \text{tr} \left( (D_b B_{ai})^2 + \frac{1}{2} R B^{ai} B_{ai} - i B^{ai}[F_{ab}, B^b_i] \right) . \tag{36}
\]
Upon adding the squares of the sections $k$ and $s$, we have
\[
\int \text{tr} \left( \frac{1}{4}k^2 + 3s^2 \right) = \int \text{tr} \left\{ \frac{1}{4}(F_{ab})^2 - 4[B_{ai}, B^i_{a}]^2 + 2iF_{ab}[B^a_{ai}, B^b_{bi}] + 2(D_bB_{ai})^2 
+ R B^a_{ai}B^a_{ai} - 2iB^a_{ai}[F_{ab}, B^b_{ai}] \right\}
= \int \text{tr} \left\{ \frac{1}{4}(F_{ab})^2 - 4[B_{ai}, B^i_{a}]^2 + 2(D_bB_{ai})^2 + R B^a_{ai}B_{ai} \right\}
\]
the right hand side vanishes if and only if $k = s = 0$. However, for sphere ($R > 0$) all terms on the RHS are positive definite so a solution to $k = s = 0$ has necessarily $B^a_{ai} = 0$. This leaves us with the equation
\[ F_{ab} = 0 \]
this equation implies that the connection is locally a pure gauge $A_a = u^{-1}d_uu$ for some $SU(2)$ matrix $u$. However, as the transition functions for $SU(2)$ bundles on sphere are trivial, the connection can be written globally as a pure gauge and be gauged away. Moreover, one can argue that this can be done continuously all over $\Sigma$. Thus we can set $A_a = 0$ everywhere. More rigorously if $\{U_a\}$ is an open covering of $\Sigma$ by contractible sets and $\{V_i\}$ is an open covering of $S^2$ by such sets, the sets $U_a \times V_i$ give an open cover of $\Sigma \times S^2$ by contractible sets. On the intersection of two patches, the connection $A$ now satisfies
\[ A_{ai} = g^{-1}_{ai\beta j}A_{bj}g_{ai\beta j} + g^{-1}_{ai\beta j}dg_{ai\beta j}, \]
or
\[ dg_{ai\beta j} + A_{bj}g_{ai\beta j} - g_{ai\beta j}A_{ai} = 0. \]
Since the $S^2$ component of the curvature is zero we have that $(A_a)_{ai} = u^{-1}_{ai}d_uu_{ai}$. Putting this in the above equation yields
\[ d_u(u_{ai}g_{ai\beta j}u^{-1}_{\beta j}) = 0. \]
Therefore $g_{ai\beta j} \equiv u_{ai}g_{ai\beta j}u^{-1}_{\beta j}$ does not depend on the coordinates of $S^2$. This implies that $g_{ai\beta j}$’s are a set of locally constant transition functions equivalent to $g_{ai\beta j}$ and for a fixed point on $\Sigma$ define a map from $S^1$ to $SU(2)$. This map is trivial so $g_{ai\alpha j}$ belongs to the conjugacy class of identity
\[ g_{ai\alpha j} = g_{ai\alpha j}^{-1} = u_{ai}g_{ai\alpha j}u_{ai}^{-1}. \]
or \((g^{-1}_{\alpha i} u_{\alpha i}) g^{\alpha i \alpha j} (g^{-1}_{\alpha j} u_{\alpha j})^{-1} = 1\). Now consider \((g^{-1}_{\alpha i} u_{\alpha i}) g^{\alpha i \beta j} (g^{-1}_{\beta j} u_{\beta j})^{-1}\). This is a constant matrix in the \(S^2\) direction. Since \(g_{\alpha i \beta j} = g_{\alpha i \beta i} g_{\beta i \beta j}\) it is equal to \((g^{-1}_{\alpha i} u_{\alpha i}) g^{\alpha i \beta i} (g^{-1}_{\beta i} u_{\beta i})^{-1}\), and since \(g_{\alpha i \beta j} = g_{\alpha i \alpha j} g_{\alpha j \beta j}\) it is equal to \((g^{-1}_{\alpha j} u_{\alpha j}) g^{\alpha j \beta j} (g^{-1}_{\beta j} u_{\beta j})^{-1}\). Thus it is in fact independent of the index \(i\) and therefore defines a matrix \(\bar{g}_{\alpha \beta}\) depending only on \(x \in U_{\alpha \beta}\) and satisfying the cocycle condition.

Since the transition functions are independent of \(i\), therefore \((A_{\Sigma})_{\alpha i}\) do not depend on \(i\) index and \(A_a\) can be gauged away.

It is now easy to see that the flatness condition, \(F_{ab} = 0\), necessarily requires the instanton number to be zero. The curvature locally takes the form

\[ F = dA + A \wedge A \]

therefore locally we can write

\[ \text{tr} (F \wedge F) = d \text{ tr} (A \wedge dA + \frac{2}{3} A \wedge A \wedge A), \]

but since \(A_a = 0\), instanton number reads

\[ k = \frac{1}{8\pi^2} \int_{\Sigma \times S^2} \text{tr} F \wedge F = \frac{1}{8\pi^2} \int_{\Sigma \times S^2} d_C \text{ tr} (A_{\Sigma} \wedge d_C A_{\Sigma}) \]

where the subindex \(C\) indicates differentiating with respect to the coordinates on \(S^2\). Note that the integrand is still a local one. However, we showed that the transition functions are independent of the local coordinates on \(S^2\). Therefore, for a fixed point on \(\Sigma\), \(A_{\Sigma}\) is globally defined on \(S^2\). This means that the integral over \(S^2\) is a total divergence and gives zero for the instanton number. In summary, we have learned that if the bundle \(E\) admits a flat connection in \(S^2\) direction then it has to be trivial (for those bundles that are classified only by instanton number) and \(k\), the instanton number, is zero.

\[ ^5 \text{The proof of this part was provided by Nicholas Buchdahl}. \]
B  Faddeev-Popov gauge fixing

In this appendix we want to show how the eq. (30) is obtained starting from (29). To evaluate the path integral over gauge fields and $\phi$, following [15], we choose the so called unitary gauge in which one rotates the lie algebra valued field $\phi^a$ to the Cartan subalgebra by conjugation, i.e. we choose $\phi_{\pm} = 0$, where

$$\phi = \phi_3 \tau_3 + \phi_+ \tau_+ + \phi_- \tau_-.$$  

This gauge can always be achieved at least locally, but there might be some topological obstruction to impose it globally [15]. Implementing this gauge in the path integral requires to introduce the Faddeev-Popov ghosts $c$ and antighosts $\bar{c}$ together with a bosonic auxiliary field $b$. These fields transform under a BRST operator $\delta$ like

$$\delta \phi_{\pm} = \pm i c_{\pm} \phi_3, \quad \delta \phi_3 = 0, \quad \delta c_{\pm} = 0,$$

$$\delta \bar{c}_{\pm} = b_{\pm}, \quad \delta b_{\pm} = 0. \tag{37}$$

The Faddeev-Popov prescription consists of adding a BRST-trivial term

$$i \delta (\bar{c}_- \phi_+ + \bar{c}_+ \phi_-) = ib_- \phi_+ + ib_+ \phi_- + \bar{c}_- \phi_3 c_+ - \bar{c}_+ \phi_3 c_-$$

to the action in (29). It is now clear that the integration over $b$ will impose the gauge condition; $\phi_{\pm} = 0$. We have

$$\text{tr} \, \phi F = \phi_3 F_3 = \phi_3 (dA_3 + (A \wedge A)_3) = \phi_3 (dA_3 + i \sqrt{2} A_1 \wedge A_2),$$

therefore, defining $\phi \equiv \phi_3, A \equiv A_3$ and $F \equiv F_3$, the action in (29) turns into

$$S = \int_{\Sigma} \left( \frac{i}{\sqrt{2}} \phi \, dA - \phi A_1 \wedge A_2 + \frac{\varepsilon}{8} \phi^2 \right) + \int_{\Sigma} d\mu \, (\bar{c}_- \phi \, c_+ - \bar{c}_+ \phi \, c_-).$$

Integration over Faddeev-Popov ghosts gives

$$[\det \phi^2]_{\Omega^0(\Sigma_\phi)},$$

while over $A_1$ and $A_2$ results in

$$[\det \phi^2]_{\Omega^1(\Sigma_\phi)}^{-1/2}.$$
Using the Hodge decomposition theorem we can express the product of these two determinants as
\[
\frac{[\det \phi^2]_{H^0(\Sigma_g)}}{[\det \phi^2]_{H^1(\Sigma_g)}^{1/2}}.
\]
When \( E \) is a nontrivial \( SO(3) \) bundle we write the curvature of the reduced \( U(1) \) bundle as
\[
F = 2\pi(2n + 1)\omega + dA,
\]
where \( \omega \) is the volume form \( (\int_\Sigma \omega = 1) \) and
\[
2n + 1 = \frac{1}{2\pi} \int_\Sigma F
\]
is the first Chern class which characterizes the \( U(1) \) bundle. To gauge fix the residual \( U(1) \) symmetry
\[
A \to A + d\alpha,
\]
we again appeal to the Faddeev-Popov prescription. We demand that a selected slice be normal to the gauge orbit,
\[
\langle d\alpha, A \rangle = 0,
\]
which implies that \( d^\dagger A = 0 \). Imposing this gauge, the action is
\[
\frac{1}{4\pi^2} \int_\Sigma \left( i\sqrt{2}\pi(2n + 1)\phi \omega + \frac{\varepsilon}{8} \phi^2 + \frac{i}{\sqrt{2}}(\phi dA + bd * A + c d * d\phi) \right).
\]
The kinetic term for \( A \) vanishes for \( A \) a harmonic one-form, i.e. when \( dA = 0 \) and \( d^\dagger A = 0 \). Hence there is still a residual symmetry under
\[
A \to A + \gamma
\]
\[
b \to b + \text{constant}
\]
\[
c \to c + \text{constant},
\]
where \( \gamma \) is a harmonic one-form. Integration over the zero modes of \( b \) and \( c \) and over the harmonic one-forms gives an unspecified constant factor that can be simply absorbed in the normalization. Therefore we need only be concerned about the nonzero modes. Dropping the harmonic part of \( A \), it can be written globally and uniquely as
\[
A = d\alpha + *d\beta,
\]
for some zero-forms $\alpha$ and $\beta$. The action then looks like
\[
\frac{1}{4\pi^2} \int_{\Sigma} \left( i\sqrt{2}\pi(2n+1)\phi + \frac{\varepsilon}{8}\phi^2 + \frac{i}{\sqrt{2}}(\phi d\ast d\beta + bd\ast d\alpha + \bar{cd} \ast dc) \right),
\]
and the measure is
\[
DA = D\alpha D\beta \det [dd^1]_{\Omega_0}.
\]
Note that $\ast^2 = (-1)^p$ when acting on a $p$-form and $d^\dagger = - \ast d\ast$. The integral over $b$ and $\alpha$ results in a determinant, $\det [dd^1]_{\Omega_0}^1$, which cancels the jacobian in (38). Also the integral over $\beta$ gives a delta function
\[
\delta(dd^1\phi) = \det [dd^1]_{\Omega_0}^1 \delta(\phi).
\]
Notice that since we are integrating over nonzero modes the delta function on the right hand side is a delta function on nonconstant $\phi$'s. The determinant in eq. (38) gets cancelled against the determinant coming from the ghosts. At the end we are left with a finite dimensional integral over constant $\phi$ fields
\[
Z[\varepsilon, m] = \sum_{n \in \mathbb{Z}} \int d\phi \left( \frac{\det m(1 - \frac{2i\sqrt{2}}{m}\text{ad}\phi)}{\det m(1 - \frac{2i\sqrt{2}}{m}\text{ad}\phi)} \right) \frac{[\det \phi^2]_{H^0}}{[\det \phi^2]_{H^1}} \exp \left( -i\sqrt{2}\phi(2n+1) \frac{4}{4\pi} - \frac{\varepsilon\phi^2}{32\pi^2} \right).
\]
Using the Riemann-Roch formula
\[
\dim \Omega^1 \otimes L - \dim \Omega^0 \otimes L = g - 1 - c_1(L)
\]
and the definition of Euler characteristic of a Riemann surface $\chi(\Sigma_g) = 2b^0 - b^1 = 2 - 2g$, and the fact that $\phi$ is now a constant, we can write the partition function as
\[
Z[\varepsilon, m] = m^{3(g-1)} \sum_{n \in \mathbb{Z}} \int d\phi \phi^{2-2g}(1 - \frac{8}{m^2}\phi^2)^{g-1} \left( \frac{m - 2\sqrt{2}\phi}{m + 2\sqrt{2}\phi} \right)^{2n+1} \times \exp \left( -i\sqrt{2}\phi(2n+1) \frac{4}{4\pi} - \frac{\varepsilon\phi^2}{32\pi^2} \right),
\]
which is the equation (30).

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