Dilogarithm Identities in Conformal Field Theory

Werner Nahm, Andreas Recknagel, Michael Terhoeven

Physikalisches Institut der Universität Bonn
Nussallee 12, D-53oo Bonn 1
Germany

Abstract
Dilogarithm identities for the central charges and conformal dimensions exist for at least large classes of rational conformally invariant quantum field theories in two dimensions. In many cases, proofs are not yet known but the numerical and structural evidence is convincing. In particular, close relations exist to fusion rules and partition identities. We describe some examples and ideas, and present conjectures useful for the classification of conformal theories. The mathematical structures seem to be dual to Thurston’s program for the classification of 3-manifolds.

e-mail: UNP01A or UNP044 at ibm.rhrz.uni-bonn.de November 1992
1. Introduction

Recently, the Rogers dilogarithm function has appeared in several places in physics. This function (cf. [20]), defined by

\[ L(z) = \text{Li}_2(z) + \frac{1}{2} \log(z) \log(1-z), \]

\[ \text{Li}_2(z) = \sum_{i=1}^{\infty} \frac{z^n}{n^2} = -\int_0^z \frac{\log(1-w)}{w} \, dw \] \hspace{1cm} (1.1)

for \( 0 \leq z \leq 1 \), is also characterized as the unique function that is three times differentiable and satisfies \( L(1) = \frac{\pi^2}{6} \) as well as the ”five term relation”

\[ L(w) + L(z) + L(1-wz) + L\left(\frac{1-w}{1-wz}\right) + L\left(\frac{1-z}{1-wz}\right) = 3L(1), \] \hspace{1cm} (1.2)

from which simpler functional equations can be derived, e.g.

\[ L(1-z) = L(1) - L(z) , \]

\[ L(z^2) = 2L(z) - 2L\left(\frac{z}{1+z}\right) . \] \hspace{1cm} (1.3)

\( L(z) \) may be extended consistently to real \( z > 1 \) with the help of

\[ L(z) = 2L(1) - L(1/z) . \] \hspace{1cm} (1.4)

Beyond these relations, the Rogers dilogarithm has other intriguing properties, which make it a very interesting function for various branches of mathematics such as number theory including algebraic \( K \)-theory, geometry of hyperbolic 3-manifolds, and even Grothendieck’s theory of motives. [26, 3, 8]

In physics, the dilogarithm has shown up in the context of integrable 2-dimensional quantum field theories and lattice models [2, 15, 16, 23]. More precisely, when studying the UV limit or the critical behaviour of such systems – typically by methods involving the Thermodynamic Bethe Ansatz (TBA) [15, 27] – it was observed that e.g. the central charge of the ”associated” conformal field theory can be expressed through the dilogarithm evaluated at certain algebraic numbers. One aim of this article is to show that these relations already emerge within a pure CFT context. Some of the results have been presented previously [21].

In the next section it is demonstrated how dilogarithm identities can be derived from the asymptotics of character identities of Rogers-Ramanujan type, mainly concentrating on the examples of non-unitary \( c(2, q) \) Virasoro minimal models. However, the method used, which is a slight modification of the one outlined in [24], is applicable to more general cases and should allow to prove some of the various conjectures on dilogarithm identities stated previously in the literature. Section 3 contains some formulas connecting the central charge and the conformal dimensions of a RCFT to the fusion rules by means
of the Rogers dilogarithm. The fusion rules here seem to replace the functional relations
from the TBA encountered in the above mentioned applications. We conclude with some
further conjectures on possible generalizations and give examples of their power in view of
the CFT classification problem. In addition, we sketch the geometric background of the
dilogarithm, which could well prove to be the most important feature about its appearance
in CFT.

2. Dilogarithm Identities from Character Asymptotics

We consider the non-unitary Virasoro minimal models with central charge

\[ c(2, k + 2) = 1 - \frac{3k^2}{k + 2} \]  (2.1)

\((k \geq 3, \text{ odd})\) and primary fields of conformal dimensions

\[ h_j = -\frac{j(k - j)}{2(k + 2)}, \quad j \in \{0, 1, \ldots, \lfloor k/2 \rfloor\}. \]  (2.2)

\(h_j\) is minimal for \(j = \lfloor k/2 \rfloor\), and the effective central charge \(c_{\text{eff}} = c - 24h_{\text{min}}\) thus takes the value

\[ c_{\text{eff}} = \frac{k - 1}{k + 2}. \]  (2.3)

The dilogarithm identities will arise from studying the asymptotic behaviour of the character functions

\[ \chi_j(\tau) = q^{h_j - c/24} \prod_{n \neq 0, \pm (j+1) \mod (k+2)} (1 - q^n)^{-1} \]  (2.4)

where \(q = e^{2\pi i \tau}\); they can also be written in sum-form \([1, 4]\)

\[ \prod_{n \neq 0, \pm (j+1) \mod (k+2)} (1 - q^n)^{-1} = \sum_{n_1, \ldots, n_{\lfloor k/2 \rfloor} \geq 0} q^{N_1^2 + \cdots + N_{\lfloor k/2 \rfloor}^2 + N_j + \cdots + N_{\lceil k/2 \rceil}} \frac{(q)_{n_1} \cdots (q)_{n_{\lfloor k/2 \rfloor}}}{(q)_{n_1} \cdots (q)_{n_{\lfloor k/2 \rfloor}}}, \]  (2.5)

where \(k\) and \(j\) satisfy the same constraints as above, \(N_i = n_i + \cdots + n_{\lfloor k/2 \rfloor}\) for \(i \in \{1, \ldots, \lfloor k/2 \rfloor\}\), and \((q)_n = (1 - q) \cdots (1 - q^n)\). (2.5) are generalized Rogers-Ramanujan identities, also known as Andrews-Gordon identities.

In the following, we study the asymptotic growth of the character functions. On the one hand, this follows quite abstractly from the modular properties of RCFT characters: Elaborating arguments given in [5] (cf. also [9]), one sees that in the limit \(t = -i\tau \to 0^+\) (i.e. \(q \to 1^-\))

\[ \chi_j(\tau) = \sum_l S_{j,l} \chi_l \left( \frac{1}{-\tau} \right) = S_{j,\min} e^{\tau c_{\text{eff}}/12t} \left[ 1 + \sum_{k \neq \min} \frac{S_{j,k}}{S_{j,\min}} e^{-2\pi (h_k - h_{\min})/t} + \ldots \right], \]  (2.6)
where $S$ represents the modular transformation $\tau \rightarrow -1/\tau$ on the character functions, in case of the $c(2, k + 2)$ models given by

$$S_{i,j} = \sqrt{\frac{4}{k + 2}}(-1)^{i+j+(k-1)/2}\sin\left(\frac{2\pi(i+1)(j+1)}{k + 2}\right). \quad (2.7)$$

Note that above quite naturally "generalized quantum dimensions" $D^{(i)}_j = \frac{S_{j,i}}{S_{j,min}}$ appear. On the other hand, the asymptotics of the coefficient $a_M$ in

$$\chi_j(\tau) = q^{h_j-c/24} \sum_{M=0}^{\infty} a_M q^M$$

(2.8)

can be calculated concretely from the sum-form, and if $a_M$ diverges e.g. as $\exp\{2\sqrt{ML}\}$ (which is just what we will find below), we can approximate

$$\sum_M a_M q^M \sim \int dM a_M q^M = 2 \exp\{L/2\pi t\} \int_0^{\infty} dx \exp\{-2\pi t(x - \sqrt{L}/2\pi t)^2\} \quad (2.9)$$

which by comparison with (2.6) leads to an equation for the effective central charge

$$c_{\text{eff}} = \frac{6}{\pi^2} L. \quad (2.10)$$

For the calculation of $L$ we start from the sum-side of the Andrews-Gordon identities (2.5). In particular, since the leading order of the asymptotic growth is the same for all characters of a given $c(2, 2 + k)$ model, we are free to choose the one of the $h_{\min} = h_{[k/2]}$ primary field

$$\sum_{M=0}^{\infty} a_M q^M = \sum_{n_1,\ldots,n_{[k/2]} \geq 0} q^{nBn^t} \frac{(q)_{n_1} \cdots (q)_{n_{[k/2]}}}{(q)_{n_1} \cdots (q)_{n_{[k/2]}}}, \quad (2.11)$$

where $n = (n_1, \ldots, n_{[k/2]})$, and $B$ is the inverse of the Cartan matrix of the tadpole graph $A_{k+1}/\mathbb{Z}_2$.

By Cauchy’s theorem, $a_{M-1}$ can be expressed as an integral

$$a_{M-1} = \oint \frac{dq}{2\pi i} \sum_{n \geq 0} \frac{q^{nBn^t-M}}{(q)_{n_1} \cdots (q)_{n_{[k/2]}}}. \quad (2.12)$$

The ideas how to obtain the large $M$ limit of (2.12) are given in [24]: The keyword is “saddle point approximation”, meaning that a crude estimate of the integral can be obtained from the integrand evaluated at its saddle point. In the following we present a simplified version of the procedure, which yields the correct leading term. For more details and for error estimates using the Euler-Maclaurin formula the reader is referred to [24].
First as in (2.9) we replace the summation by an integration over \( dn^{[k/2]} \), treating the \( n_i \)'s as continuous variables. Finding the saddle point amounts to setting the partial (logarithmic) derivatives of the integrand \( f(q, n) \) to zero. Before doing so, we use Euler-Maclaurin a second time for the denominator

\[
\log(q)_{n_i} \simeq \int_0^{n_i} dk \log(1 - q^k) \tag{2.13}
\]

so that the logarithm of the integrand is approximately given by

\[
\log f(q, n) \simeq (nBn^t - M) \log q - \sum_{i=1}^{[k/2]} \int_0^{n_i} dk \log(1 - q^k) \tag{2.14}
\]

Now the saddle point conditions \( \partial_{n_i} \log f = 0 \) give a set of algebraic equations on \( \xi_i := q^{n_i} \)

\[
1 - \xi_i = \prod_{j=1}^{[k/2]} \xi_j^{(B_{ij} + B_{ji})}. \tag{2.15}
\]

Note by the way that (2.15) is a parameter-independent variant of equations occurring in the TBA technique [15, 27, 23]. Moreover, as we will see below, the \( \xi_i \) are closely related to the quantum dimensions \( D_i \) of the representation with highest weight \( h_i \), defined as the ratio of the \( a_M \) for large \( M \), or [5, 7]

\[
D_i = \lim_{q \to 1^-} \frac{\chi_i(q)}{\chi_0(q)} = \frac{S_{i,0}}{S_{0,0}}. \tag{2.16}
\]

Before differentiating (2.14) with respect to \( q \) we plug these equations into \( \log f(q, n) \) and obtain – after having used several substitutions and the definition (1.1) of the Rogers dilogarithm –

\[
\int_0^{n_i} dk \log(1 - q^k) = \frac{n_i}{\log \xi_i} \left[ L(1 - \xi_i) + \frac{1}{2} \log(1 - \xi_i) \log(\xi_i) \right] \tag{2.17}
\]

and furthermore

\[
\log f(q, n) = -M \log q - \frac{1}{\log q} \sum_{i=1}^{[k/2]} L(\delta_i). \tag{2.18}
\]

Here \( \delta_i = 1 - \xi_i \) has been introduced for convenience. Now \( \partial_q \log f = 0 \) fixes \( q \) at the saddle point

\[
(\log q)^2 = \frac{1}{M} \sum_{i=1}^{[k/2]} L(\delta_i), \tag{2.19}
\]

so that finally the asymptotic behaviour of \( a_M \) is given by

\[
a_M \simeq f(q, n)|_{s.p.} = \exp\left\{ 2 \left( M \sum L(\delta_i) \right)^{\frac{1}{2}} \right\}. \tag{2.20}
\]
Together with (2.10) this allows us to express the effective central charge in terms of the dilogarithm.

\[ c_{\text{eff}} = \frac{1}{L(1)} \sum_{i=1}^{[k/2]} L(\delta_i). \]  

(2.21)

It remains to solve the algebraic equations (2.15), i.e. to determine \( \delta_i \) in

\[ \delta_i = \prod_{j=1}^{[k/2]} (1 - \delta_j)^{2B_{ij}}. \]  

(2.22)

After some manipulations, using \( B = (2 - I)^{-1} \), where \( I \) denotes the adjacency matrix of the graph \( A_{k+1}/\mathbb{Z}_2 \), and with the substitution \( \delta_i = 1/d_i^2 \), (2.22) can be brought into the more familiar form

\[ d_i^2 = 1 + \prod_{j=1}^{[k/2]} d_{ij} = \begin{cases} 1 + d_2, & i = 1, \\ 1 + d_{i-1}d_{i+1}, & i = 2, \ldots, [k/2] - 1, \\ 1 + d_{[k/2] - 1}d_{[k/2]}, & i = [k/2], \end{cases} \]  

(2.23)

which is just one way to recursively define the Čebyshev polynomials of the second kind, together with an additional "truncation condition"; put differently,

\[ d_i = \frac{\sin \left( \frac{(i+1)\pi}{k+2} \right)}{\sin \left( \frac{\pi}{k+2} \right)} \]  

(2.24)

solves (2.20). Rephrasing (2.21), we have derived formulas telling that the values of the dilogarithm at certain algebraic numbers (here: Jones indices) sum up to a rational result (except for a factor \( L(1) \))

\[ \sum_{i=1}^{[k/2]} L \left( \frac{\sin^2 \left( \frac{\pi}{k+2} \right)}{\sin^2 \left( \frac{(i+1)\pi}{k+2} \right)} \right) = \frac{\pi^2}{6} \frac{k-1}{k+2}. \]  

(2.25)

Of course, this method to extract dilogarithm identities from the asymptotics of the character functions can also be applied to other cases – provided a sum form of the characters similar to (2.5) is known [4, 12].

Furthermore, in view of the next-to-leading term in the expansion (2.6), it should be possible to obtain dilogarithm expressions for \( c - 24h_i \) as well, namely from the asymptotic behaviour of suitable linear combinations of characters.

### 3. On the Dilogarithm and Fusion Rules

There seems, however, to be another way to compute conformal dimensions with the help of the Rogers dilogarithm. To see this, let us once more consider formula (2.25). This
equation is already well known in the literature, and was derived previously by Kirillov [13] from the functional relations of \( L(z) \). As was stated by this author, the r.h.s. of (2.25) is also related to the central charge of the SU(2) level \( k \) WZW model. On the other hand, the arguments of the dilogarithm on the l.h.s. are just the inverse squares of the quantum dimensions of the primary fields of this WZW theory (or the \( c(2,k+2) \)-model), which can alternatively be characterized as the unique maximal eigenvalues \( \lambda_1(i) \) of the fusion matrices \( N_i \) associated to the primary fields. This observation suggests to replace the quantum dimensions by other eigenvalues \( \lambda_j(i) \) of the \( N_i \); then the following seems to hold

\[
\frac{1}{L(1)} \sum_{i=2}^{k+1} L \left( \frac{1}{\lambda_j^2(i)} \right) = c_k - 24h_{j-1}^{(k)} + 6(j - 1), \tag{3.1}
\]

where

\[
c_k = \frac{3k}{k+2} \quad \text{and} \quad h_{j-1}^{(k)} = \frac{j^2 - 1}{4(k+2)}, \quad j = 1, \ldots, k+1 \tag{3.2}
\]

are the central charge and the conformal dimensions of the SU(2) WZW primary fields, respectively, and the eigenvalues of the fusion matrices – closely related to the \( S \)-matrix or the ”generalized quantum dimensions” mentioned before – are given by

\[
\lambda_j(i) = \frac{S_{ij}}{S_{i1}} = \frac{\sin \frac{ij\pi}{k+2}}{\sin \frac{j\pi}{k+2}}, \quad i, j = 1, \ldots, k+1. \tag{3.3}
\]

We have checked (3.1) explicitly for low values of the level, and proved it for some special cases (e.g. for \( j = 2, k \) arbitrary) along the lines of Kirillov. An analogous formula holds for the \( c(2,k+2) \) models, with \( c \) replaced by \( c_{\text{eff}} \) and \( h_j \) by \( h_j - h_{\text{min}} \).

In the literature, formulas for other WZW models based on a Lie algebra \( X \) of rank \( r \) have appeared. To this end, one defines functions \( Q_m^{(a)}(z) \) on the complexified weight lattice of \( X \), which are subject to the following functional relations (for simplicity, we only deal with simply laced \( X \) in the following, for the general case see [13, 14, 18])

\[
1 - \frac{Q_{m-1}^{(a)}Q_{m+1}^{(a)}}{Q_m^{(a)}Q_{m}^{(a)}} = \prod_{b=1}^{r} Q_{m}^{(b)} - C_{ba}^X \tag{3.4}
\]

(here \( C^X \) is the Cartan matrix of \( X \)) and satisfy the initial conditions \( Q_{-1}^{(a)} = 0, Q_{0}^{(a)} = 1 \) for \( a = 1, \ldots, r \). The functions \( Q_m^{(a)} \) have been first introduced in [14] in the study of Lie group representations by means of Yangians, where it can be seen that they are essentially built up from classical Lie group characters. Using these quantities, Kirillov conjectured a dilogarithm formula for the central charge of the \( X \) WZW model [13]: Denoting the l.h.s. of (3.4) by \( f_m^{(a)}(z) \) and the central charge of the level \( l \) theory by \( c_{l}^X \), Kirillov claims that

\[
\sum_{a=1}^{r} \sum_{m=1}^{l-1} L(f_m^{(a)}(0)) = \frac{\pi^2}{6}(c_{l}^X - r). \tag{3.5}
\]
He proved this formula for the $X = A$ and $X = D$ cases. In [17, 18], conjecture (3.5) has been generalized to non-simply laced Lie algebras and also, more important, to formulas involving other entries of the $f_m^{(a)}(z)$: Taking $z = \Lambda$ to be a dominant highest weight satisfying the usual level restriction, these authors have claimed that via the dilogarithm also the conformal dimensions $h_i^X(\Lambda)$ of the Kac-Moody representation labelled by $\Lambda$ can be determined – if some correction terms are introduced

$$\frac{1}{L(1)} \sum_{a,m} \tilde{L}(f_m^{(a)}(\Lambda)) = (c_i^X - r - 24\tilde{h}_i^X(\Lambda)) \pmod{24}, \quad (3.6)$$

where the sums have the same ranges as in (3.5). $\tilde{L}(f_m^{(a)})$ is given by (with $\arg(z) \in (-\pi, \pi]$)

$$\tilde{L}(f_m^{(a)}) = \text{Li}_2(f_m^{(a)}) + \frac{1}{2} \log|f_m^{(a)}| \log|1 - f_m^{(a)}| + \frac{1}{2} \sum_{b,k} \arg(1 - f_m^{(a)}) C_{ab}^X (C_{mk}^{(l-1)})^{-1} \arg(1 - f_k^{(b)}) \quad (3.7)$$

- where $C^{(l-1)}$ is the Cartan matrix of $A_{l-1}$ - and the corrected highest weights are

$$\tilde{h}_i^X = h_i^X - \sum_{a,b=1}^r a^{(a)} C_{ab}^X (C_{b}^{(l-1)})^{-1} \arg(1 - f_k^{(b)}) \quad (3.8a)$$

with

$$a^{(a)} = l \arg Q_{l-1}^{(a)} - l \arg Q_{l}^{(a)} + \sum_{j=1}^{l-1} j \arg(1 - f_j^{(a)}) \quad (3.8b)$$

All the functions $f_m^{(a)}$ and $Q_m^{(a)}$ in (3.7,8) are evaluated at the highest weight $\Lambda$. Then, according to [18], “numerical checks indicate” that the $Q_m^{(a)}$ in addition to the functional relation (3.4) satisfy also

$$Q_m^{(a)} = Q_l^{(a)} Q_{l-m}^{(a)*}, \quad (3.9a)$$

which in particular enforces a truncation

$$Q_{l+1}^{(a)} = 0. \quad (3.9b)$$

In [18] the quantities $Q_m^{(a)}$ were regarded as arising from a TBA connected to the Lie algebra $X$. However, they can also be interpreted completely within the CFT framework, namely in terms of the $X$ WZW model fusion rules. Considering the simplest case $X = A_1$ first, one easily sees that the functional relations (3.4) together with (3.9) are just the SU(2) level $l$ fusion rules. For $X = A_r$, the $Q_m^{(a)}$ subject to (3.4,9) still can be identified with part of the WZW fusion generators, if the complex conjugation in (3.9) is translated into conjugation of the sector. In these cases, evaluating the functions $Q_m^{(a)}$ at various highest weights
corresponds to replacing the formal fusion ring generators by their various eigenvalues, which can be expressed through the $S$-matrix as 'generalized quantum dimensions'

$$Q_m^{(a)}(\Lambda) = \frac{S_m \Lambda_a; \Lambda}{S_{0; \Lambda}},$$

(3.10)

where $\Lambda_a$ are the fundamental weights of $X$. For the other algebras $X$, the situation is more complicated, but still the $Q_m^{(a)}(\Lambda)$ can be expressed as linear combination of such $S$-matrix ratios, i.e. of fusion matrix eigenvalues, with positive integer coefficients. This can be seen from the original definition of the $Q_m^{(a)}$ in terms of classical Lie algebra characters [14, 17, 18], which in turn give the WZW $S$-matrix elements – see e.g. [9], chapter 13. From all this we can conclude that the $f_m^{(a)}$ in the dilogarithm can be expressed as rational functions of the fusion ring generators, and thus within CFT the fusion rules are the algebraic structure underlying the dilogarithm identities (3.1,6).

This re-interpretation allows to stay within CFT, but unfortunately it gives no explanation of the argument correction terms entering the dilogarithm conjecture above. Note, however, that all these terms vanish if the entries $f_m^{(a)}$ are within the interval $(0,1)$ – which is the case for the original formulas given by Kirillov, $\Lambda = 0$: Then the $Q_m^{(a)}(0)$ are linear combinations of quantum dimensions of WZW primary fields so that $Q_m^{(a)}(0) \geq 1$ and due to (3.4) we have $0 \leq f_m^{(a)}(0) \leq 1$.

The arg-corrections in $\tilde{L}(z)$ at least appear understandable if one looks at the functional equations of the $f_m^{(a)}$, which are determined by the same matrix $C_X \otimes (C_{l-1})^{-1}$, see (3.15,17) below. In the last section, we will say a little more on their possible background.

On the other hand, the origin and the explicit form of the corrections in the conformal dimensions $\tilde{h}_l^X$ are more mysterious. According to [18] the $\tilde{h}_l^X$ coincide with a subclass (or including corrections depending on the path of analytic continuation with all) of the conformal dimensions of the corresponding parafermionic theory. However, from the CFT point of view it is unclear why the parafermion theory shows up, since we use the fusion rules of the full WZW theory to generate the arguments of the dilogarithm.

Note that the conjecture (3.1) given for SU(2) is not just a special case of (3.6), which means that one can hope to find – for each $X$ separately – a suitable modification of the dilogarithm leading to simpler dilogarithm formulas for the conformal dimensions. In other words: There might be a special "dilogarithm function" for each fusion ring yielding "rational invariants" of this algebra in the form of central charges and conformal dimensions. Therefore we extend the above conjectures to the following:

For any RCFT with central charge $c$ and conformal dimensions $h_j$ there are rational functions $f_i(N_1, \ldots, N_n)$ in the generators of the fusion ring such that if we insert the eigenvalues $\lambda_j(i)$ for $N_i$,

$$\frac{1}{L(1)} \sum_i \tilde{L}(f_i(\lambda_j(1), \ldots, \lambda_j(n))) = c - 24h_j + R_j,$$

(3.11)

holds with $R_j$ an integer remainder. $\tilde{L}$ is a modified dilogarithm function which for real arguments coincides with the Rogers dilogarithm.
There is an algebraic constraint on the rational functions $f_i$ which can appear as arguments of the modified dilogarithm in (3.11). In order to formulate it, we first have to introduce the $\mathbb{Z}$-module $\Lambda^2(F^\times)$, the second exterior power of the multiplicative group of a field $F$. $(F^\times = F - \{0\})$ is a vector space over $F$ with the field multiplication as ”summation operation”. For our applications, $F$ can be $\mathbb{C}$ or the quotient field of the fusion ring of some RCFT.) In $\Lambda^2(F^\times)$, the following rules hold (with $x, y, z, 1 \in F^\times$)

$$x \wedge y = -(y \wedge x), \ (xy) \wedge z = x \wedge z + y \wedge z, \ (\pm 1) \wedge z = 0.$$  \hfill (3.12)

Now consider the map $\beta : F^\times \rightarrow \Lambda^2(F^\times)$ defined by

$$\beta(z) = z \wedge (1 - z).$$ \hfill (3.13)

That this map is relevant for the dilogarithm can be expected from the differential

$$dL(z) = \frac{1}{2} (\log(z)d\log(1 - z) - \log(1 - z)d\log(z)).$$ \hfill (3.14)

$\beta(z)$ algebraically encodes the antisymmetry of $dL(z)$ as well as the functional properties of the logarithms in (3.14) – compare (3.12). Beyond that, it also plays a certain role in geometry of hyperbolic 3-manifolds, as well as in algebraic $K$-theory (especially for the so-called Bloch group, see [26]).

For our purposes, this construction is interesting because of the following theorem, which was pointed out to us by A. Goncharov:

If for a finite set of algebraic numbers $\{x_i\}$ one has $\sum_i L(x_i) = \frac{\pi^2}{6} q$ with $q$ rational, then $\sum_i \beta(x_i) = 0$ in $\Lambda^2(F^\times)$.

Because eigenvalues of fusion matrices always are algebraic numbers, so are the arguments $f_i$ of the dilogarithm in (3.11), and thus the condition $\sum \beta(f_i) = 0$ might lead, upon thorough investigation of its consequences, to a definition of the rational functions $f_i$ for arbitrary RCFT fusions rings.

Indeed one easily sees that the ”$\beta$-condition” holds for all the concrete cases listed above: Whenever the entries $f_i$ satisfy Bethe-Ansatz-like equations

$$f_i = \prod_j (1 - f_j)^{\hat{B}_{ij}}$$ \hfill (3.15)

with a symmetric matrix $\hat{B}$, then

$$\sum_i \beta(f_i) = \sum_i f_i \wedge (1 - f_i) = \sum_i \prod_j (1 - f_j)^{\hat{B}_{ij}} \wedge (1 - f_i) = \sum_{ij} \hat{B}_{ij} (1 - f_i) \wedge (1 - f_j) = 0$$ \hfill (3.16)

by (anti-)symmetry. In particular, for the ADE WZW models we have

$$\hat{B} = C^X \otimes (C^{(l-1)})^{-1}. $$ \hfill (3.17)
Beyond giving a criterion for finding the $f_i$, the physical meaning of the map $\beta$ as well of the structures connected to it, e.g. in algebraic $K$-theory, yet remains to be clarified.

4. Further Conjectures And Outlook

As already mentioned, the calculations of section 2 are by no means restricted to the Virasoro minimal $c(2, q)$ models, but are applicable to any rational CFT. This fact, and Kirillov’s identities for WZW models lead us to a general conjecture on the relation between (effective) central charges $c_{\text{eff}}$ and the Rogers’ dilogarithm:

Let $\mathcal{C}$ be the set of all possible values of $c_{\text{eff}}$ occuring in non-trivial RCFT. $\mathcal{C}$ is additive (since RCFT can be tensorized), countable and supposed to be closed and well-ordered. We conjecture that $\mathcal{C}$ is identical to the union $\mathcal{D}$ of the sets $\mathcal{D}^N$ of those rational numbers that can be expressed in the form $\sum_{i=1}^{N} L(x_i)/L(1)$ where the $x_i$ are algebraic numbers in the interval $(0, 1)$. Moreover, the $x_i$ producing $c_{\text{eff}}$ of a certain RCFT should lie in the field-extension of $\mathbb{Q}$ generated by the quantum dimensions of this RCFT.

If this conjecture is true, it would shed a new light on the task to classify all RCFT. Consider, e.g., $\mathcal{D}^1$, which is believed to be the set $\{\frac{1}{2}, \frac{3}{5}, \frac{5}{3}\}$. Whereas $\frac{1}{2} = L(\frac{1}{2})/L(1)$ corresponds to the free Majorana fermion, the other two values should belong to the $c(2, 5)$ and $c(3, 5)$ minimal models, where the only non-trivial quantum dimension $\tau$ is given by the golden ratio $\tau^2 = \tau + 1$. We have $L(1/\tau^2)/L(1) = \frac{\tau}{2}$ for $c(2, 5)$ (cf. section 2) and $L(1/\tau)/L(1) = \frac{2}{\tau}$ for $c(3, 5)$. This, in particular, implies that there is just one possibility for a fusion ring with only one generator except the identity, namely $\phi \times \phi = 1 + \phi$. From the mere axioms of fusion rings, also $\phi \times \phi = 1 + n\phi$ with any natural $n$ is allowed. Only the modular properties of RCFT enforce $n = 1$, which on the other hand comes out naturally from $\mathcal{D}^1$, i.e. from the dilogarithm. Examples in $\mathcal{D}^{k-1}$ are provided by eqs. (3.1) for $j = 1$ and (3.5).

The dilogarithm function also appears in the calculation of volumina of hyperbolic 3-manifolds (cf. [26, 22]). More precisely, any complete oriented hyperbolic 3-manifold $M$ of finite volume (possibly with cusps) can be triangulated into ”ideal” tetrahedra $\Delta_1, \ldots, \Delta_n$, each of which is specified by a complex number $z_\nu$. The volume of a tetrahedron of this type is given by $D(z_\nu)$ and consequently $\text{vol}(M) = \sum_{\nu=1}^{n} D(z_\nu)$, where $D$ is the so-called Bloch-Wigner function. This function up to some logarithmic correction terms equals the imaginary part of the dilogarithm. However, the function we have been looking at is definitely the real part of the dilogarithm, so the quest is for some geometrical interpretation of the latter. Indeed there is one: According to work of Thurston, Meyerhoff, Neumann, Zagier and Yoshida (cf. [25] and references therein) to $M$ one can associate an invariant $I(M) \in \mathbb{C}^\times$ having absolut value $\exp\{2 \text{vol}(M)/\pi\}$, whereas its argument is the Chern-Simons invariant $CS(M)$ of $M$. Having corrected this invariant by the length and torsion of certain geodesic loops, one obtains a function which is complex analytic on a neighborhood of the deformation space of $M$. In particular, for the hyperbolic manifolds $M_{(p, q)}$ obtained by performing Dehn surgery of type $(p, q)$ along the figure-eight knot in
It was proven in [25] that
\[ 12 \text{CS}(M_{\pm q}) + \frac{3}{\pi} \text{Tor} \gamma = -\frac{6}{\pi^2} \text{Re}(L(z) + L(w) - L(1)) \pmod{6}, \]
where the torsion \( \text{Tor} \gamma \) is given by
\[ \frac{3}{\pi} \text{Tor} \gamma = \left( \frac{6r}{p} - \frac{6}{\pi p} \text{arg}(z(1 - z)) \right) \pmod{6} \]
and \( r \in \{0, 1, \ldots, |p|\} \) by \( qr \equiv 1 \pmod{p} \); \( z \) and \( w \) are complex numbers with positive real part which satisfy
\[ \begin{align*}
\log z + \log(1 - z) + \log w + \log(1 - w) &= 0, \\
\frac{p}{q} \log(w(1 - z)) + q \log(z^2(1 - z)^2) &= 2\pi i.
\end{align*} \]

The similarity to the Bethe Ansatz type equations (3.15) is obvious. (4.3) indicates that the latter should be understood as equations on the universal covering of \( \mathbb{C}^\times \) – which might be responsible for the argument-corrections in (3.7). Of course one could speculate that the torsion, on the other hand, is connected to the corrections appearing in the conformal dimensions (3.8). Since topological field theories related to RCFT’s live on 3-manifolds, a pairing of the respective invariants is natural, including a prominent role for knots.

In this paper we have discussed a method which – in addition to a thorough exploitation of the dilogarithm’s functional equations – might be sufficient to prove at least some of the conjectures given here or previously. A deeper understanding, however, will certainly involve the above-mentioned mathematical branches where the dilogarithm – mysteriously enough – shows up.

There are more concrete problems, though. First, it would be nice if we could write at least one character of any CFT in a sum-form as in the Andrews-Gordon identities (see [19] for an interpretation of this in the language of affine Lie algebras) in order to generate dilogarithm expressions for the central charge and possibly for the conformal dimensions in any RCFT. Ways to achieve this might be to study the annihilating ideals [6] of the models and perhaps to find path representations of the highest weight modules [11, 12], or to introduce suitable filtrations on the space of fields. However, as these sum-formulas need not be at all unique, this could lead to further relations for the dilogarithm function.

Another open problem is to establish a direct connection between the dilogarithm equations obtained from character asymptotics to those involving the fusion rules. Probably such a link would bring in the modular properties of CFT and could also give some hints concerning the rather opaque role of the ”\( \beta \)-condition” in CFT.

Finally, the topics of integrable QFT and lattice models have only been touched very briefly – in agreement with the aim of this letter. Nevertheless, we believe that the study of the dilogarithm will reveal new connections between these subjects and CFT. At least in some cases CFT fusion rules appear as the (stationary) ”core” of the Thermodynamic Bethe Ansatz equations (”Baxterization of fusion rules”). We also expect that the ”canonical sum-form” of a character as propagated above will turn out to be related to perturbations of CFT (cf. [15]). Another interesting physical interpretation, however, has been given very recently in [10] in terms of quasi-particle excitations in quantum chains.
We would like to thank A. Goncharov, A.N. Kirillov, D. Zagier, J. Kellendonk, M. Rösgen and R. Varnhagen for various discussions on related subjects. We are indebted to A. Kuniba for pointing out a mistake in an earlier version of this paper.
A.R. is supported by the Max-Planck-Institut für Mathematik, Bonn.

References

[1] G.E. Andrews, *q-series: their development and application in analysis, number theory, combinatorics, physics, and computer algebra*, Conf. Board of the Math. Sciences, Reg. Conf. Ser. in Math. 66 (1886)
[2] V.V. Bazhanov, N.Yu. Reshetikhin, Int. J. Mod. Phys. A 4 (1989) 115
[3] A. Beilinson, A. Goncharov, V. Schechtman, A. Varchenko, in *The Grothendieck Festschrift*, vol. 1, Birkhäuser 1990, p. 135
[4] D.M. Bressoud, *Analytic and combinatorial generalizations of the Rogers-Ramanujan identities*, Memoirs of the Am. Math. Soc. No. 227, Vol. 24 (1980)
[5] R. Dijkgraaf, E. Verlinde, Nucl. Phys. B (Proc. Suppl.) 5B (1988) 87
[6] B.L. Feigin, T. Nakanishi, H. Ooguri, Int. J. Mod. Phys. A 7 Suppl. 1A (1992) 217
[7] J. Fuchs, Int. J. Mod. Phys. B6 (1992), 1951
[8] A. Goncharov, private communication
[9] V.G. Kac, *Infinite Dimensional Lie Algebras*, Cambridge 1990
[10] R. Kedem, B.M. McCoy, *Construction of modular branching functions from Bethe’s equations in the 3-state Potts chain*, Stony Brook preprint ITP-SB-92-56
[11] J. Kellendonk, A. Recknagel, *Virasoro representations on fusion graphs*, preprint BONN-HE-92-22, to appear in Phys. Lett. B
[12] J. Kellendonk, M. Rösgen, R. Varnhagen, in preparation
[13] A.N. Kirillov, J. Sov. Math. 47 (1989) 2450, (Zap. Nauch. Semin. LOMI 164 (1987) 121
[14] A.N. Kirillov, N.Yu. Reshetikhin, J. Sov. Math. 52 (1989) 3156, (Zap. Nauch. Semin. LOMI 160 (1987) 211)
[15] T.R. Klassen, E. Melzer, Nucl. Phys. B 338 (1990) 485, 370 (1992) 511
[16] A. Klümper, P.A. Pearce, J. Stat. Phys. 64 (1991) 13
[17] A. Kuniba, *Thermodynamics of the U_q(X_r^{(1)}) Bethe Ansatz System with q a root of unity*, Canberra preprint SMS-098-91
[18] A. Kuniba, T. Nakanishi, *Spectra in conformal field theories from the Rogers dilogarithm*, preprint [hep-th/9206033], SMS-042-92; *Rogers dilogarithm appearing in integrable systems*, Harvard preprint HUTP-92/A046
[19] J. Lepowsky, R. Wilson, Proc. Natl. Acad. Sci. U.S.A. 78 (1981) 7254; Invent. Math. 77 (1984) 199, 79, (1985) 417
[20] L. Lewin, *Polylogarithms and associated functions*, Elsevier North-Holland (1981)
[21] W. Nahm, Talk given at the Isaac Newton Institute, Cambridge, Sept. 1992;
to appear in the proceedings of the NATO workshop *Low-dimensional Topology and Quantum Field Theory*

[22] W.D. Neumann, D. Zagier, Topology 24 (1985) 307

[23] F. Ravanini (1992), Phys. Lett. B282 (1992) 73;  
    F. Ravanini, R. Tateo, A. Valleriani, *Dynkin TBA’s*, Bologna preprint DFUB-92-11

[24] B. Richmond, G. Szekeres, J. Austral. Math. Soc. Ser. A 31 (1981) 362

[25] T. Yoshida, Invent. Math. 81 (1985) 473

[26] D. Zagier, *The remarkable dilogarithm* (Number theory and related topics.  
    Papers presented at the Ramanujan colloquium, Bombay 1988, TIFR)

[27] Al.B. Zamolodchikov, Nucl. Phys. B 342 (1990) 695