DOA estimation under Bernoulli-Gaussian impulsive noise

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Abstract. Direction of arrival algorithms have ability to distinguish between directions of individual sources under additive white Gaussian noise. However, presence of the impulsive noise can corrupt the angles estimation. In this paper, the performance of MUSIC and ESPRIT, as estimation algorithms, is investigated in impulsive noise with white Gaussian noise environment. The impulsive noise is modelled using Bernoulli- Gaussian model and the effect of varying its statistical parameters on the estimation ability is tested. Three sources with different angles are simulated and a uniform linear array is utilized to capture the impinging signals. The performance analysis includes impulsive noise probability of occurrence and amplitude intensity which both control the impulsive power. The mean square error of the estimated angles is adopted as a metric to measure the performance. The simulation results provide a map of all the expected possible impulsive noise cases which influence the MUSIC and ESPRIT direction of arrival algorithms. The expected cases show that the performance degradation happening in ESPRIT is more than that of MUSIC. At 0 dB SNR, the impulsive noise increases the mean square error by almost 15 dB and 20 dB for MUSIC and ESPRIT, respectively.

1. Introduction

Signal processing has been utilized massively in several fields that serve humanity. One of the most important branches of signal processing is direction of arrival (DOA) estimation [1]. DOA estimation finds applications in radar, sonar, localization and wireless communication [2, 3]. In DOA, a uniform linear array (ULA) is exploited to estimate the direction of multiple simultaneous transmitting sources.

The distance \( d \) between each two consecutive elements in ULA should be less than half of the wavelength of the signal to avert the spatial aliasing. Usually, the signals received by the antennas are contaminated by noise which makes a real challenge in estimation the actual directions of the sources. Various DOA estimation algorithms have been proposed such as MUSIC [4], Capon algorithm [5], Maximum likelihood [6], Bartlett [7] and ESPRIT [8].

The additive white Gaussian noise (AWGN) is the most likely present noise in the received signals. Moreover, impulsive noise which is high amplitude pulses may happen suddenly and degrade DOA algorithms performance. To mitigate the degradation in estimation of DOA due to impulsive noise, researchers have developed many standard DOA estimation algorithms such as in [1], [9], and [10], while others have proposed new algorithms such as in [11], [12] and [13].

In this work, the Bernoulli-Gaussian impulsive noise (BGIN) effect on DOA estimation is investigated in terms of the impulsive noise statistical parameters. MUSIC and ESPRIT are highly
resolution, accurate, and fast algorithms [13, 14]. Therefore, these two techniques are adopted in the analysis.

The remaining part of this paper is organized as follows: Section 2 presents DOA algorithms. The impulsive noise statistical model is introduced in section 3, while section 4 clarifies the problem formulation and modelling. The simulation results are analysed in section 5. Finally, section 6 highlights the work conclusions.

2. DOA Algorithms

In general, consider an array of \( M \) elements as shown in figure 1, the array length \( D \) is \( (M - 1)d \). For a single snapshot of a signal impinging at time \( t \) at the middle of the ULA, the received signal captured by the array can be defined as [15]:

\[
X(t) = \alpha(t) \begin{bmatrix} e^{\frac{(M-1)\varphi}{2}} & \ldots & 1 & \ldots & e^{\frac{(M-1)\varphi}{2}} \end{bmatrix}^T + n(t)
\]

where \( \alpha(t) \) is the complex amplitude of impinging signal, \( \varphi = 2\pi d \sin(\theta)/\lambda \) and \( n(t) \) is the noise signal. Define the steering vector as \( a(\theta) = \begin{bmatrix} e^{\frac{(M-1)\varphi}{2}} & \ldots & 1 & \ldots & e^{\frac{(M-1)\varphi}{2}} \end{bmatrix}^T \). For \( N \) signals impinging simultaneously at the ULA, the model will be [15]:

\[
X(t) = A(\theta)S(t) + n(t)
\]

where \( A(\theta) = [a(\theta_1), \ldots, a(\theta_N)] \) is a matrix that includes the steering vectors of all the received signals, \( \theta_1, \ldots, \theta_N \) are the angels of arrival of the captured signals, and \( S(t) = [\alpha_1, \ldots, \alpha_N]^T \) is the matrix of signals amplitudes.

Different algorithms have been proposed to estimate the DOA of multiple sources. The algorithms based on subspace like MUSIC and ESPRIT show superior performance over the spectral ones [14]. This section presents the detailed steps of these algorithms.

2.1 MUSIC Algorithm

For simplicity, equation (2) can be written as:

\[
X = AS + n
\]

The covariance matrix of the received signals is [16]:

\[
R = E[XX^H] = A E[SS^H]A^H + E[nn^H]
\]

If the noise signal include AWGN as well as BGIN, the covariance matrix will be:

\[
R = ASA^H + \sigma^2I + \sigma_B^2I
\]

where \( \sigma^2, \sigma_B^2 \) are the variance of AWGN and BGIN, \( I \) is an identity matrix. The matrix \( S \) is:
Define $R_s = [A \Sigma A^H]$ which is a matrix of rank $N$ only. Hence, there will be $M - N$ zero eigenvalues in $R_s$ matrix.

Let $\mathbf{v} = A^H \mathbf{g}_n$, where $\mathbf{g}_n$ is the eigenvector corresponding to zero eigenvalue, then:

$$\mathbf{v}^H \Sigma \mathbf{v} = 0$$

The only situation that makes equation (6) valid is when $\mathbf{v} = 0$. Therefore, the solutions of $\theta_1, \theta_2, \ldots, \theta_N$ are $a^H(\theta) \mathbf{g}_n = 0$.

The eigen-decomposition of $R_s$ is [17]:

$$R_s = Q \Lambda_s Q^H$$

where $Q = [Q_s \quad Q_n]$, $Q_s$, $Q_n$ are related to signals and noise eigenvectors, respectively.

$$\Lambda_s = \begin{bmatrix} E[|s_1|^2] & & & & \\ & E[|s_2|^2] & & & \\ & & \ddots & & \\ & & & E[|s_N|^2] & \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

Finally, the locations of the peaks in the spatial pseudo-spectrum equation shown below are the DOAs [10].

$$P_{MUS}(\theta) = \frac{1}{a^H(\theta)Q_n Q_n^H a(\theta)}$$

2.2 ESPRIT Algorithm

ESPRIT technique is another algorithm which depends on eigen-decomposition. Re-write the matrix of the received signals steering vectors ($A$) as [18]:

$$A = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ z_1 & z_2 & \cdots & z_N \\ \vdots & \vdots & \ddots & \vdots \\ z_1^{M-1} & z_2^{M-1} & \cdots & z_N^{M-1} \end{bmatrix}$$

where $z_i = e^{j \frac{2\pi}{\lambda} d \sin(\theta_i)}$

Let $A_0 = A(1:M - 1,:)$ =

$$\begin{bmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ z_1^{M-2} & \cdots & z_N^{M-2} \end{bmatrix}$$

And $A_1 = A(2:M,:)$ =

$$\begin{bmatrix} z_1 & \cdots & z_N \\ \vdots & \ddots & \vdots \\ z_1^{M-1} & \cdots & z_N^{M-1} \end{bmatrix}$$

Hence, the relation between $A_0$ and $A_1$ is given be [18]:

$$\begin{bmatrix} 1 & \cdots & 1 \\ z_1 & \cdots & z_N \\ \vdots & \ddots & \vdots \\ z_1^{M-2} & \cdots & z_N^{M-2} \end{bmatrix}$$
where

\[
A_1 = A_0 \begin{bmatrix} z_1 & \cdots & 0 \\ 0 & \ddots & \vdots \\ 0 & \cdots & z_{M-1}^N \end{bmatrix} \text{ or } A_1 = A_0 \Phi
\]  

(13)

where

\[
\Phi = \begin{bmatrix} z_1 & \cdots & 0 \\ 0 & \ddots & \vdots \\ 0 & \cdots & z_{M-1}^N \end{bmatrix}
\]  

(14)

ESPIRT finds \( \Phi \), where diagonal elements automatically yields DOA. In this estimation method, no searching on \( \theta \) or root finding is needed. Recall that:

\[
R_x = A S A^H + \sigma^2 I + \sigma_B^2 I
\]  

(15)

where \( \hat{A} \) is a \((M \times N)\) matrix corresponding to the useful eigenvalues of \( A \) while \( \tilde{A} \) is a \((M \times (M - N))\) matrix matching the rest \( M - N \) eigenvalues of \( A \).

\[
R_x = \begin{bmatrix} S & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{A}^H & 0 \\ 0 & \hat{A}^H \end{bmatrix} + \sigma^2 I + \sigma_B^2 I
\]  

(16)

The same as in MUSIC algorithm, the eigen-decomposition of \( R_x \) is accompanied. Hence, \( Q \) is decomposed as \( Q = [Q_s \quad Q_n] \). Then \( Q_s \) and \( A \) span the same signal subspace. Therefore:

\[
Q_s = [A][T]
\]  

(18)

where \( T \) is a first-rank invertible matrix.

\[
Q_0 = Q(1: M - 1, :) = A_0 T \Rightarrow A_0 = Q_0 T^{-1}
\]

\[
Q_1 = Q(2: M, :) = A_1 T \Rightarrow A_1 = Q_1 T^{-1}
\]

Hence,

\[
Q_1 T^{-1} = Q_0 T^{-1} \Phi \quad \text{or} \quad Q_1 = Q_0 T^{-1} \Phi T = Q_0 \Phi \quad \text{where} \quad \Phi = T^{-1} \Phi T
\]

\[
\Phi = Q_0 T Q_1 = Q_0 T Q_0 \Phi \quad \text{or} \quad \Phi = (Q_0 T Q_0)^{-1} Q_0 T Q_1, \quad \Phi \text{ is not a diagonal matrix.}
\]

But \( \Phi = T^{-1} \Phi T = T^T \Phi T \) = eigen decomposition from which eigenvalues \( \lambda_i \) are obtained.

Finally, the angles of arrivals can be found from [18]:

\[
\theta_i = \sin^{-1} \left( \frac{\text{Im}(\lambda_i)}{\sqrt{\lambda_i}} \right)
\]  

(19)

3. BGIN Statistical model

Impulsive noise is short duration and high amplitude random pulses which can occur at any time. Different sources can initiate impulsive noise which is then reshaped by the communication channel. Impulsive noise can be modeled by Bernoulli-Gaussian statistical process [19] as follows: First, a
binary sequence of constant amplitudes is generated using Bernoulli process which includes randomness of occurrence time. For the discrete time index, \( l = 1, 2, \ldots \), the \( l^{th} \) bit of the binary sequence, \( b(l) \), could take one of two values; \( k \) (Bernoulli factor) or \( 0 \), with a probability given by [19]:

\[
P(b(l)) = \begin{cases} 
\rho & \text{for } b(l) = k \\
1 - \rho & \text{for } b(l) = 0
\end{cases}
\]  

(20)

Hence, the mean and variance of the Bernoulli process are given by equations below, respectively:

\[
m_B = E[b(l)] = kp, \quad l = 1, 2, \ldots, L
\]

(21)

\[
\sigma_B^2 = E[(b(l) - m_B)^2] = k^2 \rho (1 - \rho)
\]

(22)

where \( L \) is the length of the sequence.

Next step, the Bernoulli sequence is amplitude modulated by normally distributed Gaussian noise and the result sequence will have randomly varying amplitudes. The probability density function (pdf) of Bernoulli-Gaussian model is [19]:

\[
f(bg(l)) = (1 - \rho)\delta(bg(l)) + \rho f_\sigma(bg(l))
\]

(23)

where \( bg(l) \), \( l = 1, 2, \ldots, L \), is the Bernoulli-Gaussian sequence, \( \delta \) is Kronecker delta function and \( f_\sigma \) is the pdf of the Gaussian noise.

Finally, the impulse response of the communication channel plays a significant role in altering the shape of Bernoulli-Gaussian sequence; the pulse duration lasts for more than one sample and the amplitude changes also.

4. Problem Formulation and Modelling

This section presents the whole problem and the DOA algorithms performance mechanism. AWGN and BGIN signals are added to the signals of several sources located at various angles with respect to the receiver position. BGIN can be found in various scales and probabilities depending on the causative sources. In addition, the medium that conveys the noisy signal changes the shape of the impulsive pulses. The intensity and occurrence probability variation are considered in performance evaluation of DOA algorithms in the simulation results section.

| Binary sequence with occurrence probability \( \rho \) | Signals from sources at different angles |
| AWGN | Shaping filter |
| BGIN | AWGN |

Figure 2. Block diagram of DOA algorithms evaluation in presence of AWGN and BGIN.
The first step at the receiver is capturing the impinging signals using ULA. Two DOA algorithms which are MUSIC and ESPRIT process the captured signals trying to obtain the directions of the transmitting sources. The estimated angles are compared with the actual ones to obtain the estimation error. MUSIC and ESPRIT algorithms depend on eigen-decomposition. The impulsive noise can cause noteworthy eigenvalues which sometimes confuse the estimation process. Due to the randomness of the noise signals, many ensemble runs at each SNR. The block diagram which illustrates the evaluation of DOA algorithms in presence of AWGN and BGIN is shown in figure 2.

5. Simulation Results and Analysis

To investigate the influence of changing the statistical parameters of BGIN on the DOA algorithms, the following scenario is simulated: Three sources of angles \( \theta_1 = 30^\circ, \theta_2 = 45^\circ \) and \( \theta_3 = 60^\circ \) , with respect to the receiver, are sending signals simultaneously. ULA of 16 elements is utilized to capture the transmitted signals. 300 snapshots are generated for the simulation. Two types of noise are added to the signals; AWGN and BGIN. MUSIC and ESPRIT algorithms are applied to estimate DOA in noisy environment. To obtain more accurate results, 1000 iterations are conducted on each algorithm. MATLAB software is utilized to simulate the system.

For precise results analysis, two metrics, signal to noise ratio (SNR) and signal to impulsive noise ratio (SINR), are exploited to illustrate the DOA estimation with respect to mean square error (MSE). SNR is the ratio between the averages of signal power to the noise power. On the other hand, SINR represents the average signal power to the average power of the impulsive noise. Mathematically, SINR is written as [19]:

\[
\text{SINR} = \frac{P_{\text{signal}}}{P_{\text{impulse}}} 
\]

It is worth to mention that \( P_{\text{impulse}} \) is a function of both impulsive noise occurrence probability \( (\rho) \) and the average amplitude of that noise \( (\xi) \) which is related to \( k \), \( P_{\text{impulse}} = \rho \xi \). Hence, varying of these statistical parameters on the performance of the DOA estimation is tested.

First, the simulation is conducted using AWGN only. The performance as MSE versus SNR for the MUSIC and ESPRIT algorithms is shown in figure 3a and figure 3b, respectively, while figure 3c depicts the estimated angles of arrival. Both algorithms can estimate the directions of the sources correctly at moderate SNR values. For example, at SNR=0, the MSE are almost \( -7.9 \) and \( -6.7 \) for MUSIC and ESPRIT, respectively. In pure AWGN environment, the eigenvalues of impinging signals are easily distinguishable from the noise eigenvalues. Hence, the DOA algorithms can find the peaks positions which correspond to the angles values in equation (9) and equation (19) of MUSIC and ESPRIT algorithms, respectively. Starting from \(-17 \) dB SNR, the MUSIC algorithm can estimate the angles with low error (less than \( 10^{-3} \) degree), whereas the ESPRIT achieves the same estimation accuracy for the SNR values equal or more than \(-7 \) dB.

In the next step, a BGIN signal is generated according to the model illustrated in section 3. Figure 4 exhibits the generated BGIN signal with \( \rho = 0.05 \). From now onwards, BGIN will be added to AWGN in simulation. Using the same DOA algorithms, MUSIC and ESPRIT, the performance is re-evaluated under BGIN and AWGN and the results are shown in figure 5. It can be noticed that the DOA estimators performance worsens in presence of the impulsive noise. For instance, adding BGIN to AWGN at SNR=0 increases the MSE by almost \( 15 \) dB and \( 20 \) dB for MUSIC and ESPRIT, respectively. For MUSIC algorithm, the estimated angles deteriorate from \( 30.1^\circ, 45^\circ \) and \( 60.1^\circ \) to \( 30.2^\circ, 45.3^\circ \) and \( 59.5^\circ \) after adding BGIN, whereas BGIN degrades the DOA angles from \( 30^\circ, 45.1^\circ \) and \( 60^\circ \) to \( 29.7^\circ, 45.1^\circ \) and \( 58.8^\circ \) in ESPRIT technique. This performance deterioration is due to the existence of new significant eigenvalues caused by BIGN, and these eigenvalues lead to confusion in angles estimation process.
Figure 3. Performance with AWGN only, (a) MSE vs. SNR of MUSIC, (b) MSE vs. SNR of ESPRIT, (c) Estimated angles at SNR = 0 dB of MUSIC and ESPRIT.

Figure 4. BGIN signal with $\rho = 0.05$. 
Under the same conditions, MSE is recorded for both DOA algorithms while changing the SINR as shown in figure 6. SINR meter clearly shows the effect of the BGIN on the performance of MUSIC and ESPRIT algorithms. Along the SINR axis (from $-5$ dB to near $15$ dB), the differences in MSE values are almost $20$ dB and $30$ dB for MUSIC and ESPRIT algorithms, respectively. Another point can be noticed in figure 6 is that the MUSIC shows more robustness to impulsive noise than ESPRIT. In general, DOA performance of MUSIC is superior to ESPRIT in term of MSE in noisy environment. However, high speed estimation of the ESPRIT algorithm makes it recommended for the real time applications.

$F_{impulse}$ depends on both $\rho$ and $k$ factors. Figure 7 shows the impact of changing $\rho$ on the MSE evaluation of MUSIC and ESPRIT algorithms. The value of $\rho$ is increased from 0.01 to 0.5, while the value of $k$ is fixed. Generally, the algorithms performance represented by MSE deteriorates by almost $10$ dB and $13$ dB for MUSIC and ESPRIT, respectively.

In the next simulation, Bernoulli factor ($k$) is increased from 0.1 to 10 while maintaining the same value of $\rho$. Figure 8 shows MSE curves versus Bernoulli factor. It is evident that increasing Bernoulli factor leads to enormous increment in MSE, especially for ESPRIT. Figures 7 and 8 indicate an individual study for all possible BGIN cases, concerning the probability of occurrence and amplitude intensity.
The final conducted test includes study impact of BGIN on DOA algorithms in case of sources angles convergence. The angles are chosen to be 43°, 46° and 55°. The simulation is repeated with and without BGIN. The BGIN parameters are $\rho = 0.18$ and $k = 5$. Table 1 lists the estimated angles of MUSIC and ESPRIT at SNR=10 dB with and without BGIN. From the table, both algorithms lose at least one angle when BGIN is added. In DOA algorithms, the angles are estimated from the peaks in the spectrum which correspond to the greatest eigenvalues. In case of two close angles, only one peak may appear in the spectrum. Therefore, a DOA algorithm can be confused by the third greatest eigenvalue produced by BGIN as the second estimated angles and that what happened in Table 1.

Table 1. Performance comparison with and without BGIN.

| Actual source angle | MUSIC without BGIN | MUSIC with BGIN | ESPRIT without BGIN | ESPRIT with BGIN |
|---------------------|--------------------|----------------|---------------------|-----------------|
| 43°                 | 43.3°              | 30.6°          | 42.5°               | 44.6°           |
| 46°                 | 45.9°              | 44.3°          | 46°                 | 52.1°           |
| 55°                 | 55.1°              | 54.1°          | 55°                 | 54°             |
6. Conclusion
Signals transmitting in media often suffer from impulsive noise beside the additive white Gaussian noise. In this work, the performance of MUSIC and ESPRIT direction of arrival estimators is evaluated under various ferocity levels of impulsive noise. The impulsive noise is produced using Bernoulli-Gaussian model. Three different sources of angles $30^\circ$, $45^\circ$ and $60^\circ$ are used in the tests. The results show that increasing the probability of impulsive occurrence by 10 % or enlarging the Bernoulli factor by step of 2, leads to average increment of angle estimation mean square error by about 2 dB and 3 dB for MUSIC and ESPRIT, respectively. The last test involves repeating the performance evaluation in three close sources scenario in presence and absence of impulsive noise. Although the test is conducted at a high SNR which is $30 \text{ dB}$, both algorithms, MUSIC and ESPRIT, lose one angle and estimate the two others inaccurately. However, all the angles are correctly estimated in absence of the impulsive noise. In conclusion, the MUSIC technique shows more robustness to impulsive noise than ESPRIT technique.

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