A symmetric Finsler space with Chern connection

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Abstract
We define a symmetry for a Finsler space with Chern connection and investigate its implementation and properties and find a relation between them and flag curvature.

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1 Introduction
It is well known that a Riemannian space is locally symmetric if and only if $\nabla R = 0$, for Levi-Civita connection. Therefore we define a symmetric Finsler space to be a Finsler space whose h-h-curvature is parallel with respect to the Chern connection. As the h-h-curvature is the Riemannian curvature in Riemannian case, this generalizes the definition of Riemannian symmetric space to Finsler spaces.

2 Preliminaries
We will follow verbatim the notation of [1]. Let $M$ be a manifold of dimension $n$, a local system of coordinate $(x^i), i = 1...n$ on $M$ gives rise to a local system of coordinate $(x^i, y^i)$ on the tangent bundle $TM$ through $y = y^i \frac{\partial}{\partial x^i}$.

A Finsler structure on $M$ is a function $F : TM \rightarrow [0, \infty)$ satisfying the following condition:

(i) $F$ is differentiable away from the origin.

(ii) $F$ is homogeneous of degree one in $y$ i.e for all $\lambda > 0$

$$F(x, \lambda y) = \lambda F(x, y)$$

(iii) the $n \times n$ matrix

$$g_{ij} = \frac{1}{2} \frac{\partial^2 F^2}{\partial y^i \partial y^j}$$

is positive-definite at every point of $TM \setminus 0$. 
Denote the natural projection $T M \setminus 0 \to M$ by $\pi$. The pullback bundle of $T^*M$ is defined by the commutative diagram

$$
pullback{\pi^*\pi} \to \pi^*T M
$$

The components $g_{ij}$ in $(iii)$ define a section $g = g_{ij}dx^i \otimes dx^j$ of the pulled back bundle $\pi^*(T^*M) \otimes \pi^*(T^*M)$, where $g$ is called the fundamental tensor, and usually depends on both $x, y$.

To simplify the computation, we introduce adapted bases for the bundles $T^*(TM \setminus 0)$ and $T(TM \setminus 0)$, these are:

$\{dx^i, \delta y^i\}$

$\{\delta x^i, F \partial \partial y^i\}$

where $N^i_m = \frac{1}{4} \frac{\partial}{\partial y^m}(g^{is}(\frac{\partial g_{sk}}{\partial x^j} + \frac{\partial g_{sj}}{\partial x^k} - \frac{\partial g_{sk}}{\partial x^j}g^{ji}y^j)g^{kl}y^l)$

In fact they are dual to each other. The vector space spanned by $\delta x^i$ (resp. $F \partial \partial y^i$) is called horizontal (resp. vertical) subspace of $T(TM)$.

Let $(M,F)$ be a Finsler manifold. There exists a symmetric connection

$$\nabla : \Gamma(T(TM)) \times \Gamma(\pi^*(T M)) \to \Gamma(\pi^*(T M))$$

whose Christoffel symbols are given by:

$$\Gamma^i_{jk} = \frac{\delta g_{ij}}{2}(\frac{\delta g_{sk}}{\partial x^j} - \frac{\delta g_{sj}}{\partial x^k} + \frac{\delta g_{ks}}{\partial x^l})$$

this connection is called Chern connection and has the following properties:

(i) the connection 1-form does not depend on $dy$;

(ii) the connection $\nabla$ is almost $g$-compatible in the sense that;

$$\nabla_{\delta x^i} \frac{g_{ij}}{\partial} = 0 \quad \text{and} \quad \nabla_{F \partial \partial y_i} \frac{g_{ij}}{\partial} = 2A_{ij}$$

where $A_{ij}$ is the component of Cartan tensor.

The curvature of Chern connection can be splitted into two components according to the vector argument being horizontal or vertical. The first is the hh-curvature tensor

$$(\nabla_{\delta x^k} \nabla_{\delta x^l} - \nabla_{\delta x^l} \nabla_{\delta x^k} - \nabla_{[\delta x^k, \delta x^l]} \delta x^j) \partial / \partial x^j = R^i_{jkl} \partial / \partial x^i$$

where
\[
R^i_{jkl} = \frac{\delta \Gamma^i_{jl}}{\delta x^k} - \frac{\delta \Gamma^i_{jk}}{\delta x^l} + \Gamma^i_{hk} \Gamma^h_{jl} - \Gamma^i_{hl} \Gamma^h_{jk}
\]

the second is hv-curvature tensor

\[
(\nabla_F \frac{\partial}{\partial y_k} \nabla_{\delta/\delta x^l} \nabla_F \frac{\partial}{\partial y^i} - \nabla_{\delta/\delta x^l} \nabla_F \frac{\partial}{\partial y^i} \nabla_F \frac{\partial}{\partial y^j}) \partial/\partial x^j = P^i_{jkl} \partial/\partial x^j
\]

where

\[
P^i_{jkl} = -F \frac{\partial \Gamma^i_{jl}}{\partial y_k}
\]

3 Symmetric Finsler Space

**Definition 3.1** A Finsler space throughout which the hh-curvature \( R \) possesses vanishing covariant derivative with respect to horizontal vector field will be called a symmetric Finsler space:

\[
\nabla_{\delta/\delta x^h} R^i_{jkh} = 0
\]

**Example 3.1** Evidently a locally Minkowski space is symmetric space, as the Chern connection coefficients \( \Gamma^i_{jk} \) vanish identically.

Let

\[
R^i_{kl} = l^j R^i_{jkl} \quad \text{and} \quad R^i_j = l^h R^i_{jk}
\]

where \( l^i = \frac{\partial}{\partial y^i} \), then we have:

\[
\nabla_h R^i_{kl} = 0 \quad (1)
\]

and

\[
\nabla_h R^i_j = 0
\]

where the index \( h \) denotes \( \delta/\delta x^h \).

Moreover the horizontal covariant derivative of the following contraction will be zero.

\[
R^i_{kh} = R^i_{kh} \quad , \quad R^i_{ik} = R^i_k \quad , \quad R^i_i = (n - 1) R
\]

In order to obtain further consequences in a symmetric Finsler space we compute

\[
R^i_{hkl} = \frac{\partial}{\partial y^h} R^i_{kl} + y^j \frac{\partial}{\partial y^k} [\dot{A}^i_{jl}] \dot{A}^l_{ik} + \dot{A}^i_{jl} \dot{A}^l_{sk} - \dot{A}^i_{sl} \dot{A}^l_{jk}
\]

where \( \dot{A}^i_{jl} \) is the horizontal covariant derivative of \( A^i_{jl} \) with respect to \( \delta/\delta x^s \) and \( \dot{A}^i_{jl} = A^i_{jl} |_{s} \) and denote

\[
y^j \frac{\partial}{\partial y^h} [\dot{A}^i_{jl}] \dot{A}^l_{ik} + \dot{A}^i_{jl} \dot{A}^l_{sk} - \dot{A}^i_{sl} \dot{A}^l_{jk}
\]
by \( D^i_{hkl} \), thus we have

\[
R^i_{hkl} = \frac{\partial}{\partial y^h} R^i_{kl} + D^i_{hkl}
\]

then

\[
\nabla_p R^i_{hkl} = \nabla_p \frac{\partial}{\partial y^h} R^i_{kl} + \nabla_p D^i_{hkl}
\]

and if

\[
\nabla_p D^i_{hkl} = 0 \quad (2)
\]

then

\[
\nabla_p R^i_{hkl} = \nabla_p \frac{\partial}{\partial y^h} R^i_{kl}
\]

Now we can prove:

**Theorem 3.2** Let \( M \) be a Finsler space for which \( \nabla_p D^i_{hkl} = 0 \). Then \( M \) is symmetric if and only if

(i) \( \nabla_h R^i_{kl} = 0 \)

(ii) \( \frac{\partial R^i_{kl}}{\partial y^h} \dot{A}^m_{hp} + R^m_{kl} \Gamma^i_{mph} - R^i_{ml} \Gamma^m_{kph} - R^i_{km} \Gamma^m_{lph} = 0 \)

**Proof**: For a symmetric space we have \( \nabla_h R^i_{kl} = 0 \) and if moreover \( \nabla_p D^i_{hkl} = 0 \) holds then with the help of the commutation formula:

\[
\frac{\partial}{\partial y^h} \nabla_p R^i_{kl} - \nabla_p \frac{\partial}{\partial y^h} R^i_{kl} = \frac{\partial R^i_{kl}}{\partial y^h} \dot{A}^m_{hp} + R^m_{kl} \Gamma^i_{mph} - R^i_{ml} \Gamma^m_{kph} - R^i_{km} \Gamma^m_{lph} \quad (3)
\]

where \( \Gamma^i_{mph} \) is used to denote \( \frac{\partial}{\partial y^h} \Gamma^i_{mp} \), we have

\[
\frac{\partial R^i_{kl}}{\partial y^h} \dot{A}^m_{hp} + R^m_{kl} \Gamma^i_{mph} - R^i_{ml} \Gamma^m_{kph} - R^i_{km} \Gamma^m_{lph} = 0 \quad (4)
\]

Conversely let us suppose that (i) and (ii) satisfy then from the commutation formula we have the following equation

\[
\nabla_p R^i_{hkl} = -\frac{\partial R^i_{kl}}{\partial y^h} \dot{A}^m_{hp} + R^m_{kl} \Gamma^i_{mph} + R^m_{ml} \Gamma^m_{kph} + R^i_{km} \Gamma^m_{lph}
\]

therefore \( \nabla_p R^i_{hkl} = 0 \) . Q.E.D.

As for a Landsberg space \( \dot{A}_{ijk} = 0 \) we have the following corollary.

**Corollary 3.3** A Landsberg space is symmetric if and only if (1) and (4) hold.

**Definition 3.4** A Finsler structure \( F \) is said to be of Berwald type if the Chern connection coefficient \( \Gamma^i_{jk} \), in natural coordinates, have no \( y \) dependence.

Landsberg spaces include Berwald type spaces, now we have:
Theorem 3.5 Let (1) holds in a Berwald type space then it is symmetric.

If $X$ is a nowhere zero vector field defined on an open subset $O$ of a Finsler manifold $(M, F)$, then we may associate to $X$ a Riemannian metric on $O$. A particularly interesting case of this construction is when the integral curves of the vector field are geodesics of the Finsler metric on $M$.

Definition 3.6 Let $M$ be a Finsler manifold and let $v_m \in T_m M$ be a nonzero vector. If $P \subset T_m M$ is a two-dimensional subspace containing $v_m$, and $X$ is a geodesic vector field on a neighborhood of $m$ such that $X(m) = v_m$, then the sectional curvature of the Riemannian metric associated to $X$ at the plan $P$ is called flag curvature of $M$ at the flag $(P, v_m)$.

Theorem 3.7 Let $(M, F)$ be a connected Finsler manifold with constant flag curvature of dimension at least 3, if (2) and (4) hold then $M$ is symmetric.

Proof: Let $(M, F)$ has constant flag curvature $\lambda$, and its dimension $n$ is at least 3, then from [1] we have

$$R^i_{kl} = \lambda (\delta^i_k l_l - \delta^i_l l_k)$$

thus $\nabla_k R^i_{kl} = 0$. Q.E.D.

Corollary 3.8 A Landsberg space with constant flag curvature and dimension greater than 2 which satisfies in (4) is symmetric.

Corollary 3.9 Any Berwald type space with constant flag curvature and dimension greater than 2 is symmetric.

Remark 3.10 Traditionally Riemannian symmetric spaces were first defined and studied by E.Cartan [2], by means of symmetries i.e involutive isometries fixing a point. It is shown that locally it is equivalent to having parallel curvature tensor field. Affine(locally and globally) symmetric space are defined by replacing isometries with affine transformation, and is equivalent to having a torsionfree connection with parallel curvature tensor field. O.Loos considered affine symmetric space as a smooth manifold with an appropriate operation and produced the required connection. Therefore it seem reasonable to consider a space together with a nice operation and find a suitable connection.

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