Gravitational Correction in Neutrino Oscillations

Yasufumi Kojima

Department of Physics, Hiroshima University,
Higashi-Hiroshima 739, JAPAN

Abstract

We investigate the quantum mechanical oscillations of neutrinos propagating in weak gravitational field. The correction to the result in the flat space-time is derived.

Recently, the neutrino oscillation phase due to the presence of gravity is much discussed [1], [2], [3]. If the neutrinos are massive particles and mixed, the neutrino oscillation between different flavor states occurs. Suppose that neutrinos are created as weak flavor eigenstate, say, at \( \vec{r}_A \), and propagate to \( \vec{r}_B \). The state is a linear superposition of mass eigenstates and each phase of mass eigenstate evolves in different way. As a result, the mixing phase angle for the relativistic neutrinos propagating in a flat space is given by

\[
\phi_0 = \frac{\Delta m^2 L}{(4\bar{\hbar}E)},
\]

where \( E \) is energy, \( L = |\vec{r}_B - \vec{r}_A| \), and \( \Delta m^2 = m_1^2 - m_2^2 \). (See e.g., [4].)

Gravitational effect was not seriously studied so far. Gravity can be eliminated by choosing appropriate inertial frame locally. The propagating distance of the neutrinos is so long in some case, that gravitational effect may become important. Ahluwalia and Burgard [1] considered the gravitational effect on the neutrino oscillation. They showed that the external weak gravitational field of a star with mass \( M \) adds a new contribution to the phase difference, denoted by

\[
\phi_G = -\frac{\Delta m^2 L \langle \phi \rangle}{(4\bar{\hbar}E)},
\]

where \( \langle \phi \rangle \) is defined by the average of gravitational potential over the semi-classical path, i.e., \( \langle \phi \rangle = -\int_{\vec{r}_A}^{\vec{r}_B} dLGM/r/\bar{\hbar}E \). This gravitationally induced phase can be estimated as \( \phi_G = -\langle \phi \rangle \phi_0 \). The phase becomes to the extent of roughly 20% of \( \phi_0 \), near neutron stars. They suggested that the new oscillation phase may be significant effect on the supernova explosions, since the extremely large fluxes of neutrinos are produced with different energies corresponding to the flavor states.

The gravitationally induced oscillation phase may have the important astrophysical consequences. However, their derivation and even the definitions such as energy were not clear in their original paper. Bhattacharya, Habib and Mottola [2] critically re-examined the quantum mechanical phase mixing. They calculated the phase difference for radially propagating particles, and found that the term of \( \Delta m^2 L \langle \phi \rangle/\bar{\hbar}E \sim GM\Delta m^2 \log(r_B/r_A)/\bar{\hbar}E \) is canceled out. They showed that the possible gravitational effect appears at the higher order, \( \Delta m^4/E^3 \), and that the phase difference for radially propagating particles is \( GM\Delta m^4 \log(r_B/r_A)/(4\hbar E^3) \). Numerically its magnitude is equal to \( \sim 10^{-9} \frac{(M/M_\odot)}{(\Delta m^4/eV^4)} \frac{(E/MeV)^{-3}}{(r_B/r_A)} \), which is completely negligible in typical astrophysical applications. Therefore, the conclusion of Ahluwalia and Burgard [1] seems to be incorrect for radially propagating case. Natural question is what happens in more general case. Does the term of \( GM\Delta m^2/E \) always disappear?

\[1\] E-mail address: kojima@theo.phys.sci.hiroshima-u.ac.jp
Furthermore, several authors have discussed the possibility of the violation of equivalence principle in the neutrino oscillation, e.g., [5], [6] and reference therein. If the universality of the gravitational couplings to different flavors breaks down, additional phase difference appears. It is, therefore, an important matter to understand the modification of the neutrino oscillation phase due to the presence of gravity, even within the equivalence principle. This question must be settled first. In this paper, we will consider the quantum mechanical phase mixing of the neutrinos propagating in the weak gravitational field in detail.

Let us consider a weak flavor eigenstate \( |\nu_\alpha \rangle (\alpha = e, \mu, \text{ or } \tau) \), created at \( \vec{r}_A \) with a certain energy. The value can be specified by the energy at infinity \( E \). The flavor eigenstate is a linear superposition of mass eigenstates, which are represented by \( |\nu_a \rangle \),

\[
|\nu_\alpha \rangle = \sum_a U_{\alpha a} |\nu_a \rangle, \tag{1}
\]

where \( U_{\alpha a} \) is unitary mixing matrix. The different mass eigenstates propagate with different velocity. The classical paths deviate reciprocally in the gravitational field. We use wave packet formalism ([7], [8]) to examine the evolution of mass eigenstates. We assume that each mass eigenstate is created with the same energy as the Gaussian waveform in momentum space. The direction of the momentum is the same, but the mean value is different due to different mass. The wave packet for mass eigenstate, \( a \) can be written in coordinate space as

\[
\Psi_a(\vec{x}, t) = (2\pi\sigma^2)^{-3/4} \exp \left( \frac{i S_a}{\hbar} \right) \exp \left( -\frac{(|\vec{x} - \vec{z}_a|^2)}{4\sigma^2} \right), \tag{2}
\]

where \( S \) is phase function and should satisfy \( g^{\mu\nu} S_\mu S_\nu + m^2 = 0 \) in geometrical optics limit. The size of the wave packet, \( \sigma \) is assumed to be much smaller than the curvature radius of the external gravitational field. The center of the wave packet, \( \vec{z}_a(t) \) is determined by classical geodesic.

The quantum mechanical transition probability from \( |\nu_\alpha \rangle \) to \( |\nu_\beta \rangle \) is

\[
P_{\alpha \rightarrow \beta}(\vec{x}, t) = | \sum_a U_{\beta a}^* \Psi_a(\vec{x}, t) U_{aa} |^2
\]

\[
= (2\pi\sigma^2)^{-3/2} \sum_{ab} U_{\beta a}^* U_{aa} U_{\beta b} U_{ab}^* \exp \left( \frac{i}{\hbar} (S_a - S_b) \right) F, \tag{3}
\]

where

\[
F(\vec{x}, t) = \exp \left( -\frac{1}{4\sigma^2} (|\vec{x} - \vec{z}_a|^2 + |\vec{x} - \vec{z}_b|^2) \right)
\]

\[
= \exp \left( -\frac{1}{2\sigma^2} |\vec{x} - \frac{1}{2}(\vec{z}_a + \vec{z}_b)|^2 \right) \exp \left( -\frac{1}{8\sigma^2} |\vec{z}_a - \vec{z}_b|^2 \right). \tag{4}
\]

The function \( F \) has a peak at \( (\vec{z}_a + \vec{z}_b)/2 \), so that the phase factor can be evaluated by taking the dominant contribution in the stationary point. The factor \( \exp(-|\vec{z}_a - \vec{z}_b|^2/(8\sigma^2)) \) represents that the coherence will be lost if two orbits separate much more than the size of the wave packet. The unitary matrix
for the mixing between two generations is parameterized by $\theta$. If the deviation of the orbit is negligible, then we have

$$P_{\alpha \to \beta}(t) = \sin^2(2\theta) \sin^2(\varphi),$$

where $\varphi$ is the phase angle between two states, $\varphi = -\Delta S/(2\hbar) = (S_2 - S_1)/(2\hbar)$.

In the opposite case, we have time-averaged one,

$$P_{\alpha \to \beta} = \frac{1}{2} \sin^2(2\theta).$$

We now estimate the split of the orbit, $\vec{z}_1 - \vec{z}_2$, and the phase difference $\Delta S = -2\hbar \varphi$, between two mass eigenstates. The classical trajectory of a particle with mass $m$ around non-rotating star with mass $M$ can be described by the energy at infinity $E$ and angular momentum $L$. The orbital plane may be chosen as the equatorial plane of the spherical coordinate. Then the trajectory leaving at $t = 0$ from $(r, \chi) = (r_A, 0)$ can be written as

$$\chi = \int_{r_A}^r \frac{1}{\sqrt{B}} \frac{L}{r^2} dr,$$

$$t = \int_{r_A}^r \frac{E}{\sqrt{B}} \left(1 - \frac{2GM}{r}\right)^{-1} dr,$$

where

$$B = E^2 - \left(1 - \frac{2GM}{r}\right) \left(m^2 + \frac{L^2}{r^2}\right).$$

We will consider unbounded orbit, i.e., $E \gg m$. It is convenient to use the turning point, $r_0$, of the radial direction to calculate the above integrals. Eliminating $L$ and expanding by $G$, we have

$$\chi = \left[-\sin^{-1}\frac{r_0}{r} + \frac{GM}{1 - q} \frac{2r + r_0}{rr_0} \sqrt{\frac{r - r_0}{r + r_0}} - \frac{qGM}{1 - q} \sqrt{\frac{r^2 - r_0^2}{rr_0}} \right]_{r_A}^r,$$

$$\sqrt{1 - qt} = \left[\sqrt{r^2 - r_0^2} + \frac{GM}{1 - q} \frac{r - r_0}{r + r_0} + \frac{(2 - 3q)GM}{1 - q} \log \left(\frac{r + \sqrt{r^2 - r_0^2}}{r_0}\right) \right]_{r_A}^r,$$

where $q = m^2/E^2$, and $[f(r)]_{r_A}^r = f(r) - f(r_A)$. The first terms in eqs. (10) and (11) represent straight lines in the spherical coordinate. The first order terms in $G$ are well known effects on the propagation of massless particles. The second term in (10) is relevant to deflection of light, and the logarithmic term in (11) is relevant to the time delay in the radar echo experiment. (See, e.g., [9].)

The phase $S$ is given by the four momentum, $p_\mu$, conjugate to $x^\mu$ as,

$$S = \int p_\mu dx^\mu = \int (-Edt + p_i dx^i).$$
Both mass eigenstate are assumed to have the same conserved energy $E$, so that the first term $\int E \, dt$ is canceled out. We only calculate the contribution from the second term. The phase function can be evaluated for the classical path of the particle with mass $m$ and energy $E$ as

$$S = \sqrt{E^2 - m^2} \times \left[ r^2 - r_0^2 + GM \left( \frac{2 - q}{1 - q} \log \left( \frac{r + \sqrt{r^2 - r_0^2}}{r_0} \right) + \frac{1}{1 - q} \sqrt{\frac{r - r_0}{r + r_0} - 2 \sqrt{r^2 - r_0^2}} \right) \right]_{r_A}^{r}$$

(13)

Expanding by $m^2/E^2$ for the relativistic particles, we have

$$S \approx E \left[ r^2 - r_0^2 + GM \left( 2 \log \left( \frac{r + \sqrt{r^2 - r_0^2}}{r_0} \right) + \frac{r - r_0}{r + r_0} - 2 \sqrt{r^2 - r_0^2} \right) \right]_{r_A}^{r}$$

$$- \frac{m^2}{2E} \left[ r^2 - r_0^2 - GM \left( \frac{r - r_0}{r + r_0} + 2 \sqrt{r^2 - r_0^2} \right) \right]_{r_A}^{r}$$

$$- \frac{m^4}{8E^3} \left[ r^2 - r_0^2 - GM \left( 2 \log \left( \frac{r + \sqrt{r^2 - r_0^2}}{r_0} \right) + 3 \frac{r - r_0}{r + r_0} + 2 \sqrt{r^2 - r_0^2} \right) \right]_{r_A}^{r}$$

$$+ \cdots.$$  

(14)

It is clear that the logarithmic term never appears in order $GMm^2/E$, but in order $GMm^4/E^3$. However, there is still another term in order $GMm^2/E$. We only consider the gravitational effect up to $GMm^2/E$, and neglect the higher order terms hereafter. The particle with $m^2 \pm \Delta m^2/2$ arrives at $(r \pm \Delta r, \chi \pm \Delta \chi/2)$ at the same coordinate time $t$. The split of the orbit can be calculated from eqs. (10) and (11). The phase difference $\Delta S$ can be calculated from eq. (14) at $r$. We explicitly show the differences in the path and phase due to mass square difference for the relativistic particles emitted to the radial and azimuthal directions, i.e., $\dot{\chi} = 0$ and $\dot{r} = 0$ initially. The turning points for these orbits can be chosen as $r_0 = 0$, and $r_0 = r_A$, respectively.

(1) radial orbits

The orbital deviation and phase difference in the radial direction are given by,

$$\Delta r \approx - \frac{\Delta m^2}{2E^2} \left\{ 1 - \frac{2GM}{r} \right\} L,$$

(15)

$$\Delta S \approx - \frac{\Delta m^2}{2E} L,$$

(16)

where $L$ is the path length in flat space, $L = r - r_A$. 

---

1 If the mixed state is a superposition of states with different energy, there is additional phase difference, $\Delta S = - \Delta Et = - \Delta m^2 t/(2E)$. We have the logarithmic term in $\Delta S$ by eq. (11), in converting from the coordinate time to the propagating distance. We will not consider such a term from the transformation between $t$ and $r$ hereafter.
(2) *transverse orbits*

For the particles emitted transversely, we have

\[
\Delta r \simeq -\frac{\Delta m^2}{2E^2} \left\{ 1 + \frac{4GM}{r + r_A} - \frac{3GM}{r} \right\} \frac{L}{r}, \tag{17}
\]

\[
r\Delta \chi \simeq -\frac{\Delta m^2}{2E^2} \left\{ 1 + \frac{3GM}{r_A} + \frac{GM}{r + r_A} - \frac{2GM}{r} - \frac{2GMr}{r_A^2} \right\} \frac{r_AL}{r}, \tag{18}
\]

\[
\Delta S \simeq -\frac{\Delta m^2}{2E} \left\{ 1 - \frac{GM}{r + r_A} - \frac{2GM}{r} \right\} L, \tag{19}
\]

where \( L = \sqrt{r^2 - r_A^2} \) in this case. The differences increase with the propagating distance \( L \), as expected. Gravitational corrections are shown in the braces \( \{ \} \). The coherent wave packet with the initial size \( \sigma \) will survive until \( L \sim E^2\sigma/\Delta m^2 \). Gravitationally induced oscillation phase appears in order \( G\Delta m^2/E \) except the purely radial motion, in which the gravitational correction is accidentally canceled as seen in eq.(14). The magnitude of the oscillation phase can be written as \( \varphi_G \sim \langle \phi \rangle \varphi_0 \), from the dimensional argument. However, the factor \( \langle \phi \rangle \) is not the average of the gravitational potential over the semi-classical path. Rather, the correction can be regarded as the modification of the propagating distance.

**ACKNOWLEDGMENT**

I would like to thank T. Morozumi for the stimulating conversation. This work was supported in part by the Grant-in-Aid for Scientific Research Fund of the Ministry of Education, Science and Culture of Japan (No.08640378).

**References**

[1] Ahluwalia D. V. and Burgard C. (1996) preprint: gr-qc/9603008, 1996 Gravity Research Foundation Essay, Gen. Rel and Grav. (in press).

[2] Bhattacharya T., Habib S., and Mottola E. (1996) preprint: gr-qc/9605074.

[3] Ahluwalia D. V. and Burgard C. (1996) preprint: gr-qc/9606031.

[4] Mohapatra R. N. and Pal P. B. (1991) *Massive Neutrinos in Physics and Astrophysics* (World Scientific Publishing, Singapore)

[5] Iida K., Minakata H., and Yasuda O. (1993) Mod. Phys. Lett. A8, 1037.

[6] Halprin A., Leung C. N., and Pantaleone J. (1996) Phys. Rev. D53, 5365.

[7] Kayser B. (1981) Phys. Rev. D24, 110.

[8] Giunti C., Kim C. W., and Lee U. W. (1991) Phys. Rev. D44, 3635.

[9] Weinberg, S. (1972) *Gravitation and Cosmology* (John Wiley & Sons, USA)