Quantum experiments and hypergraphs: Multiphoton sources for quantum interference, quantum computation, and quantum entanglement

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I. INTRODUCTION

Graph-theoretical concepts are widely used in multidisciplinary research involving physics, chemistry, neuroscience, and computer science, among others. Considerable progress has taken place in recent years in the direction of applying graph theory in quantum physics. Their connections have been carried out explicitly, leading to many interesting and complementary works. A well-known example is the so-called graph states, the structure of which can be described in a concise and fruitful way by mathematical graphs [1]. These states form a universal resource for quantum computing based on measurements [2,3] and later has been generalized to continuous-variable quantum computation [4], using an interesting connection between Gaussian states and graphs [5]. Graphs have also been used to study collective phases of quantum systems [6], characterize quantum correlations [7], and investigate quantum random networks [8,9].

A graph theoretical approach to quantum mechanics would also help to amalgamate visualization offered by graphs with the well-developed mathematical machinery of graph theory. For example, a very powerful pictorial tool for making calculations in quantum-mechanical theory is Feynman diagrams, which is indispensable in the context of quantum electrodynamics [10]. Moreover, quantum processes such as teleportation, logic-gate teleportation, entanglement swapping, etc. can be captured at a more abstract level [11,12]. A different graphical representation has been developed to describe quantum states and local unitaries [13]. Also, directed graphs have recently been investigated in order to simplify certain calculations in quantum optics, by representing creation and annihilation operators in a visual way [14].

Recently, an entirely different method of connecting graphs in quantum physics has been introduced, by showing that graphs can capture essential elements of quantum optical experiments [15–17]. The graph-experiment connection exploits the fact that the most common sources of photonic entanglement are spontaneous parametric down-conversion (SPDC) [18], which is a nonlinear process that probabilistically creates photon pairs. Those photon pairs are then interpreted as two vertices connected by an edge. This simple idea has been exploited in Refs. [15–17] to understand better the generalization of quantum states, quantum information protocols, and for gaining new insights towards quantum computation.

Can such a graph-experiment concept be extended and applied in the quantum experiments with other probabilistic photon sources that are not restricted to pairs of photons, such as SPDC? The answer is positive and this motivates us to provide a general description for quantum experiments by using different probabilistic photon sources, for example, experimental single-photon sources in form of the attenuated lasers or sources that can produce \( n > 2 \) photon tuples. For convenience, hereinafter we call all these probabilistic sources \( n \)-photon sources, which produce \( n \) correlated photons. Therefore, a single-photon source in the form of attenuated lasers is just a one-photon source, and the SPDC-based nonlinear crystal is a two-photon source.

Hypergraphs are generalizations of graphs [19,20], which have caused a significant interest in applications to real-world problems. Here we show that hypergraphs are a suitable mathematical model and effective tool for quantum experiments. Our contributions are the following: (1) We introduce a mapping between hypergraphs and quantum experiments, and properties of hypergraphs capture properties of the experiments (Sec. II). (2) We show how this mapping can
be used to design quantum experiments in an abstract and systematic way (Sec. III). (3) Thereby, we find concepts of setups for the generation of complex quantum states by using a combination of one- and two-photon sources. Those setups can be performed experimentally with standard quantum optics technology and use fewer resources than state-of-the-art techniques (Sec. III A). (4) With the mapping, we also find experimental configurations that overcome limitations from linear-optics experiments using SPDC crystals. It allows us to produce much-higher-dimensionally entangled Greenberger-Horne-Zeilinger (GHZ) states than is possible with SPDC (Sec. III B). (5) The abstract structure of hypergraphs for describing experiments let us identify mathematical challenges in hypergraphs and translate them to experimental setups. In particular, we show that hypergraph-generalizations of boson sampling (which is in different complexity classes than the standard approaches) can simply be designed using the hypergraph mapping (Sec. IV). (6) We show that hypergraphs can be used to understand interference structures in complex experiments intuitively with pictures. Thereby, we reinterpret the results of a previously introduced quantum interference experiments (Sec. V).

II. QUANTUM EXPERIMENTS AND HYPERGRAPHS

Here we briefly review some basic notation and terminology of hypergraphs that will be useful later. Formally speaking, a hypergraph \( H \) is a pair \( H = (V, E) \), where \( V = \{ v_i | i = 1, 2, \ldots, x \} \) is the set of \( x \) vertices (or nodes) and \( E = \{ e_i | i = 1, 2, \ldots, y \} \) is the set of \( y \) hyperedges. The number of vertices that a hyperedge \( e \) contains (or the degree of a hyperedge) is denoted \( d(e) \) and the number of hyperedges that a vertex \( v \) involves (or the degree of a vertex) is denoted \( d(v) \). A hypergraph is called \( k \)-uniform if every hyperedge contains exactly \( k \) vertices, i.e., \( d(e) = k \forall e \in E \). Clearly, an ordinary graph is a special case of hypergraphs where \( d(e) = 2 \), namely a \( 2 \)-uniform hypergraph. An example of graphs and hypergraphs is given in Fig. 1.

Now we start with a simple quantum experiment to illustrate how we update the graph-experiment link to hypergraphs in Fig. 2. There, all nonlinear crystals are pumped coherently. As the SPDC process is entirely probabilistic, it means that the probability of obtaining two pairs from one crystal or one pair from two crystals is the same. Such situations and even multiple pair emissions from SPDC are unavoidable. However, one can adjust the pump power such that these cases are in a sufficiently low probability which can be safely neglected. Therefore, we can adjust the laser power such that only two photon pairs are created with reasonable probabilities [21]. Every photon path represents a vertex and every nonlinear crystal corresponds to an edge [15,17]. An \( N \)-fold coincidence event (\( N \) detectors click simultaneously) is described as a perfect matching—a subset of edges that visit all vertices exactly once. Thus, the final quantum state under the condition of \( N \)-fold coincidences is given as the coherent superposition of perfect matchings in the graph. Because a regular graph is just a \( 2 \)-uniform hypergraph, we can reinterpret all the graph representations into hypergraphs. Thus a nonlinear crystal is a hyperedge with \( d(e) = 2 \) and an optical path is a vertex in a hyperedge (see the correspondence listed in Table I). The hypergraph interpretation of the experiment is shown in Fig. 2(b).

There are two perfect matchings in the hypergraph—a collection of edges covering all vertices only once, indicating that four-fold coincidences happen when crystals (I & II) or (III & VI) fire together. Thus the resulting quantum state is given as \( |\psi\rangle = \frac{1}{\sqrt{2}}(|0000\rangle + |1111\rangle) \), where numbers 0 and 1 denote photon mode numbers that correspond to the polarization of photons, or high-dimensional degrees of freedoms [22] such as the orbital angular momentum (OAM) [23–25], time bin [26,27], or frequency [28].

III. STATE GENERATION

The original graph description is only applicable to model pair correlations (two-photon source case, \( n = 2 \)), thus it cannot be used to describe other photon sources such as one-photon sources where \( n = 1 \). However, using our
TABLE I. The analogies between hypergraphs and quantum experiments.

| Hypergraphs                                      | Quantum experiments                      |
|-------------------------------------------------|------------------------------------------|
| Undirected hypergraphs                          | Quantum optical setup with \(n\)-photon sources |
| Vertex                                          | Optical output path                      |
| Hyperedge                                       | \(n\)-photon sources                    |
| Colors in the boundary of the hyperedge         | Photon-mode numbers                      |
| Colors in the region of the hyperedge           | Phases between photons                   |
| Transparency in the region of the hyperedge     | Amplitudes of photons                    |
| Perfect matchings                               | \(N\)-fold coincidences                  |

While a hypergraph is a mathematical structure, we can describe arbitrary probabilistic sources and their combinations. The power of our technique is that it allows us to design new quantum experiments, using clear mathematical structures. If one wants to find an experimental setup for a specific quantum entangled state, one can now rephrase the problem in terms of (hyper)graph theory. A specific state corresponds to a hypergraph with specific perfect matchings. Identifying perfect matchings is a task that is oftentimes easier than finding the experimental setup itself, for example, because it can be formalized as a simple mathematical question that can be solved by standard mathematical software. A solution to the mathematical question can then be directly translated to the solution of the experimental setup. Interestingly, if a solution cannot be found mathematically, there cannot be an experimental setup that produces the target state with the provided resources.

We show this design principle on several simple examples, which require hypergraphs. However, the formalism could apply more broadly in the same way.

Here we investigate state generations with \(n\)-photon sources where \(n\) is not necessarily \(n = 2\). First, we study the case \(n = 1\), which has been experimentally implemented several times in the form of attenuated lasers [29]. We find efficient and compact setups for high-dimensional multiphotonic state generations. Then we continue with the case \(n > 2\) and find that one can overcome limitations for the dimensionality of multiphotonic states.

A. \(n\)-photon sources: \(n < 3\)

1. Greenberger-Horne-Zeilinger states

GHZ states are the most prominent example for nonclassical correlations between more than two involved parties and have led to new understanding of the fundamental properties of quantum physics [32,33]. Such multipartite quantum entanglements are denoted

\[
|\text{GHZ}_{m}^{d}\rangle = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |i\rangle^{\otimes n},
\]

where \(m\) is the number of particles and \(d\) is the dimension for every particle.

For a two-dimensional three-particle GHZ state,

\[
|\text{GHZ}_{2}^{3}\rangle = \frac{1}{\sqrt{2}} (|000\rangle + |111\rangle),
\]

we show how to construct a hypergraph which describes the target state \(|\text{GHZ}_{2}^{3}\rangle\). Three particles indicate that there are three vertices in the hypergraph. There are two terms in the quantum state, meaning that there will be two perfect matchings in the hypergraph. One stands for term \(|000\rangle\), and the other represents term \(|111\rangle\). In Fig. 3(a), we show such a hypergraph with two perfect matchings consisted of hyperedges \(d(e) = 1\) and \(d(e) = 2\). These two perfect matchings contribute to threefold coincidences cases in the experiment, and their coherent superposition leads to the final entangled state, which is our expected two-dimensional three-particle GHZ state.

Equipped with the hypergraph-experiment connection from Table I, we can then transfer the hypergraph into its quantum optical setup with \(n\)-photon sources.
FIG. 4. General hypergraphs and setups for producing two-dimensional (odd) \(m\)-particle GHZ states using one-photon and two-photon sources. In Fig. 3(a), we show a hypergraph for \(m = 3\). One can arbitrarily extend that hypergraph without introducing new perfect matchings by adding more hyperedges \(d(e) = 2\), which means inserting several two-photon sources. Panels (a)–(c) show general hypergraphs and setups for creating two-dimensional five-, seven-, and \(m\)-particle GHZ states.

Corresponding experimental implementation. Hypergraphs of \(d(e) = 1\) and \(d(e) = 2\) in the setup, and the vertices represent photon output paths, see Fig. 3(b). There two beam splitters (BSs) are properly used to separate an infrared pulsed laser beam into three paths, where the single photon is approximated by an attenuated laser beam. The mostly transmitted laser beam undergoes up-conversion (UC) for producing blue light with high efficiency [30], which is used to coherently pump two nonlinear crystals to generate correlated photon pairs with reasonable probabilities. Then the laser-generated single photon and photon pair output paths are aligned identically (namely, \(\text{path identity}[21]\)) such that one cannot determine the “which source” information when observing the detectors.

One can experimentally adjust the power of the infrared laser and exploit attenuators to change the photon creation probability \(p_i\), \((i = 1, 2)\) in the setup, where subscript \(i\) stands for the photon source used. In the case of \(i = 1\), the photon source used is a one-photon source. In the case of \(i = 2\), that corresponds to a two-photon source. Instead of using heralded single photons via SPDC to produce three-particle entanglement with probability \(p_2\), we can obtain the expected quantum state \(|\Psi_{222}\rangle\) with probability \(p_1p_2\).

Now we generalize the hypergraph for producing two-dimensional \(m\)-particle GHZ states using \(n\)-photon sources. For even \(m\), we can use \(n = 2\) only [17] while for odd \(m\), we show that \(n = 1\) together with several \(n = 2\) works in Fig. 4. In addition to multiple GHZ states, we can also produce three-dimensional GHZ states, shown in Fig. 5. This is an

FIG. 5. Hypergraph-experiment link for producing three-dimensional three-particle GHZ states. (a) Analogous to Fig. 3(a), we add one more perfect matching which relates to quantum state term \(|012\rangle\) (maverick term). The probabilities of existence for hyperedges \(d(e) = 1\) and \(d(e) = 2\) are \(p_1\) and \(p_2\), respectively, which can be experimentally adjusted by attenuators. Thus one can make the maverick term in a sufficiently low probability or negligible if \(p_1^2 \ll p_1p_2\). (b) The corresponding experimental implementation. This setup is a special resource providing us a efficient and feasible technique to produce high-dimensional GHZ states.
alternative approach to the recent experimental implementation based on linear optics [34]. It is a very compact, reasonable experimental scheme for a high-dimensional GHZ state that can be realized in laboratories.

2. W states

We further illustrate our approach on W states, an important multiphoton entanglement class that is highly persistent against photon loss [35,36]. W states are one special case of Dicke states introduced by Dicke [37], which cannot be transformed into GHZ states with local operation and classical communication (LOCC) [38] and defined as

$$|W_m⟩ = \frac{1}{\sqrt{m}} \hat{S}(|0⟩^{⊗(n-1)}|1⟩), \quad (3)$$

where m stands for the number of particles and $\hat{S}$ is the symmetrical operator that sums over all distinct permutations of the m particles.

Let us consider the simplest case with a three-particle W state,

$$|W_3⟩ = \frac{1}{\sqrt{3}} (|001⟩ + |010⟩ + |100⟩), \quad (4)$$

and show how one can exploit the hypergraph-experiment connection. There are three terms in the quantum state, which correspond to three perfect matchings in the hypergraph.

Analogous to Fig. 5(a), we construct a hypergraph for the W state under the condition $p_1^3 \ll p_2$ in Fig. 6(a). Different colors of the boundaries stand for different photon mode numbers, for example, red means mode number 1. We observed that every perfect matching in the hypergraph involves only one red bounded hyperedge $d(e) = 1$, which means every term in the quantum state has exactly one excitation. The coherent superposition of perfect matchings describes a three-particle W state. The related experimental setup is shown below the hypergraph. Panels (b) and (c) describe general hypergraphs and setups for five- and $m$-particle W states under the condition $p_1^3 \ll p_2$.

3. Schmidt-rank vector states

When quantum entangled states go to higher dimensions, interesting properties [40] and nonclassical correlations
[41–43] show up. These new structures of multipartite high-dimensional entanglement are characterized by the Schmidt-rank vector (SRV) and give rise to new phenomena that only exist if both the number of particles and the number of dimensions are larger than two [41–43].

The SRV represents the rank of the reduced density matrices of each particle. In the three-particle pure state case (which is the case we consider and the three parties are $a$, $b$, and $c$), the ranks of the reduced density matrices

$$A = \text{rank}(\text{Tr}_c(|\psi\rangle\langle\psi|)), \quad B = \text{rank}(\text{Tr}_b(|\psi\rangle\langle\psi|)), \quad C = \text{rank}(\text{Tr}_a(|\psi\rangle\langle\psi|)),$$

together form the SRV $d_\psi = (A, B, C)$, where $A \geq B \geq C$. The values $A$, $B$, and $C$ stand for the dimensionality of entanglement (Schmidt rank) between every particle with the other two parties (for example $a \rightarrow bc$, $b \rightarrow ac$, and $c \rightarrow ab$ correspond to the entanglement between $a$ and $bc$, $b$ and $ac$, $c$ and $ab$). The classification with different SRVs provides an interesting insight that one can transform quantum states from higher classes to lower classes with LOCC, and not vice versa, which means that the dimensionality $i$ ($i = A, B, C$) cannot be increased with LOCC.

As an example, we show a maximally entangled state with $SRV = (4, 4, 3)$, which is

$$|\psi\rangle_{abc} = \frac{1}{2}(|000\rangle + |111\rangle + |222\rangle + |330\rangle).$$

There the first particle $a$ is four-dimensionally entangled with the other two particles $bc$, particle $b$ is four-dimensionally entangled with particles $ac$, whereas particle $c$ is only three-dimensionally entangled with the rest of the particles $ab$. Here we are only interested in maximally entangled states, which means that all amplitudes are the same. Furthermore, we want the quantum state with $SRV(A, B, C)$ to have $A$ terms. Thereby, the structure of the SRV is clearly visible in the computation basis, which is convenient experimentally. We call such a quantum entangled state a $SRV(A, B, C)$ state.

The generation of these SRV states has been well investigated theoretically [17,44] and experimentally [34,45,46]. To make future experimental investigations possible, we aim to find these $SRV(A, B, C)$ states without additional particles with probabilistic $n$-photon sources (in this case, $n = 1, 2$). We exploit our presented hypergraph-experiment connection for experimentally designing and find that the results in Ref. [17] using the graph-theoretical concept are applicable by using hypergraphs under the restriction $p_1^2 \ll p_2$. In Fig. 7, we show one example state described by Eq. (6).

Recent advances in integrated optics and silicon photonics have attracted increasing attention to entanglement generation on chips [47,48]. Many chip-based sources such as heralded single-photon sources [49,50], high-dimensional states [51,52], multiphoton states [53–55] and on-chip transverse-mode entangled photon pair sources [56] have been successfully demonstrated. Those platforms are ideal for the practical implementation of our approach.

**B. $n$-photon sources: $n \geq 3$**

So far, we have considered the cases of $n = 1$ and $n = 2$ photon sources. A general concept of an $n$-photon source has been theoretically and experimentally investigated [57–59], which can be perfectly interpreted by a hyperedge with $d(e) = n$ in the hypergraph. That means a setup only using $n$-photon sources can be translated into a $k$-uniform hypergraph where $k = n$. For simplicity, we only consider the case $n = 3$ and show an example in Fig. 8. There are ten disjoint perfect matchings (i.e., every hyperedge only appears in at most one of the perfect matchings) in the corresponding hypergraph, which are depicted in the different-colored backgrounds. The final quantum state is created conditioned on sixfold coincidences. It can be interpreted as a coherent superposition of perfect matchings that leads to a ten-dimensional six-photon GHZ state.

If we use nine output modes with three three-photon sources firing simultaneously, we can create a 13-dimensional nine-photon GHZ state. Interestingly, this indicates that the dimension of GHZ states grows when more crystals are added in the case of three-photon sources. This is in stark contrast to the case of two-photon sources where the maximum dimension $d = 3$ [15,60,61]. If we restrict ourselves to two $n$-photon sources firing simultaneously, then the maximal possible dimension for a GHZ state grows as $n = 2 - 7; d = 3, 10, 35, 126, 462, 1716, 6435$, which is potentially connected to the integer sequence in OEIS A001700 [62] (number of ways to put $n + 1$ indistinguishable balls into $n + 1$ distinguishable boxes).

Until now, our hypergraph-experiment connection mainly exploits the recently developed technique—entanglement by path identity [21], where one cannot determine the origin of every $N$-fold coincidence event. In realistic experimental scenarios, this method requires perfect output path overlapping, suitable temporal coherence, and indistinguishability which
can be obtained in Refs. [21,63]. If there are misalignments between the overlapping paths, it does not reduce the coherence between the different terms but changes their relative amplitudes. This is because misaligned beams do not arrive at the detectors and consequently do not lead to an N-fold coincidence count. Also, there is other unavoidable noise such as multiple photon emission, stimulated emission, loss of photons (including detection efficiencies). Similarly to the misalignment case, this noise also reduces the entanglement by unweighting the state. However, these effects could be compensated by adjusting the pump power before each photon source. Therefore, maximally entangled, arbitrary, high-dimensional entanglement states should be possibly created experimentally.

Our hypergraph technique can be applied to find experimental implementations for different high-dimensional multipartite quantum states, which will be interesting to study in more detail in the future—both for their fundamental properties as well as for their applications in novel quantum communication protocols.

**IV. COMPUTATION COMPLEXITY OF HYPERGRAPHS**

In a quantum experiment, the postselected quantum state is described as the coherent superposition of perfect matchings in its corresponding hypergraph, which means that the number of terms in the quantum state corresponds to the number of perfect matchings in the hypergraph. Detecting an N-fold coincidence event in a quantum experiment is thus equivalent to guaranteeing a perfect matching in the hypergraph.

The problem of finding a perfect matching in graphs is well understood. For instance, two well-known examples are Tutte’s Theorem [64] and Hall’s Marriage Theorem [65], which provide necessary and sufficient conditions for the existence of at least one perfect matching in a graph and a bipartite graph, respectively. Such graph perfect matching decision problems are efficiently solvable (i.e., Edmonds’ algorithm [66]). These mathematical tools can be employed to find out whether a certain quantum experiment with photon pair sources will produce an N-fold coincidence click.

Although the graph perfect matching problem is fairly well understood, and solvable in polynomial time, most of the problems related to hypergraph perfect matching tend to be very difficult and remain unsolved. Indeed, deciding if a k-uniform hypergraph (k ≥ 3) contains a perfect matching is among the historic 21 $\mathbf{NP}$-complete problems given by Karp [67].

The inability to efficiently decide whether a perfect matching exists means that no efficient classical algorithm can determine from the experimental setup whether there will be an N-fold coincidence click at all. This restriction is much stronger than experiments with two-photon sources. Interestingly, one could take advantage of the mathematical difficulty in quantum experiments. One could build the experiment corresponding to the hypergraph in a laboratory and observe whether there are N-fold coincidence clicks. It points toward the possibility of classically intractable problems that can be answered by using quantum resources. This difficulty is related to, but conceptually simpler than boson sampling [68–76]. Boson sampling exploits the fact that counting all perfect matchings is difficult classically (which is in #P-complete complexity class [77]), and obtains related properties using sampling techniques.

For hypergraphs already, deciding whether a perfect matching exists at all is classically difficult and is $\mathbf{NP}$-complete. Several theoretical studies identified sufficient algorithms for finding perfect matchings in hypergraphs, but the general problem is unsolved mathematically [78–80]. However, experimentally one could just record whether an N-fold coincidence count exists. It might be possible to exploit this classical difficulty for new quantum supremacy and quantum computation protocols, potentially by generalizing the system to a hypergraph version of boson sampling [81–83]. This could be done by sampling a subset of outputs from the experiment, which is produced by multiple multiphoton sources. In the standard boson sampling case, calculating one output requires the evaluation of the matrix function Hafnian (a generalization of Permanent, which lies in the complexity class #P). For multiphoton sources, calculating the output lies in the class #W[1] [84]. The general and systematic experimental designs for such setups motivates further theoretical investigation into this problem.

We show now in Fig. 9(a) an example that demonstrates the difficulty of identifying perfect matchings. This three-uniform hypergraph with nine vertices and 49 hyperedges corresponds...
to the experimental implementation in Fig. 9(b). It is very challenging to find whether there is a perfect matching in such a hypergraph. Interestingly, this hypergraph does not have any perfect matchings but adding any new hyperedges will necessarily create one perfect matching.

For realistic experimental situations, one needs to carefully consider the influence of multiple photon emissions, stimulated emission, loss of photons (including detection efficiencies), and amount of photon distinguishabilities in connection with statements of computation complexity. A full investigation of these very interesting questions is beyond the scope of the current paper.

V. MANY-PARTICLE INTERFERENCE

A general concept of many-particle interferometry and entanglement based on path identity has been presented recently by Lahiri [85]. How can our hypergraph-experiment connection describe such situations? First, we start with the simplest case—the Zou-Wang-Mandel experiment [86,87] in Fig. 10(a). There, two identical two-photon sources are aligned such that output paths connect to \{a, b, c\}, \{d, e, f\}, and \{g, h, i\}. Then there will be a ninefold coincidence event in the experiment which indicates that there are no perfect matchings in that hypergraph. However, one can just add more three-photon sources (for example, the three-photon sources’ output paths connect to \{a, b, c\}, \{d, e, f\}, and \{g, h, i\}), then there will be a ninefold coincidence event in the experiment which means there is one perfect matching in the hypergraph by adding more hyperedges. In general, deciding whether a hypergraph has a perfect matching, and thus whether a quantum experiment produces N-fold coincidences, is NP-complete. This could motivate a hypergraph version of the boson-sampling-like quantum supremacy algorithm, for which a setup similar to the one proposed here could act as the experimental implementation.

FIG. 9. An example showing the difficulty of finding a perfect matching in a three-uniform hypergraph and its corresponding quantum experiment. (a) The three-uniform hypergraph contains nine vertices and 49 hyperedges. Interestingly such a hypergraph has no perfect matchings while adding more hyperedges can create one perfect matching. (b) The three-uniform hypergraph is transferred into a quantum experiment with multiple three-photon sources. In this case, there will be no ninefold coincidence event in the experiment which indicates that there are no perfect matchings in that hypergraph. However, one can just add more three-photon sources (for example, the three-photon sources’ output paths connect to \{a, b, c\}, \{d, e, f\}, and \{g, h, i\}), then there will be a ninefold coincidence event in the experiment which means there is one perfect matching in the hypergraph by adding more hyperedges. In general, deciding whether a hypergraph has a perfect matching, and thus whether a quantum experiment produces N-fold coincidences, is NP-complete. This could motivate a hypergraph version of the boson-sampling-like quantum supremacy algorithm, for which a setup similar to the one proposed here could act as the experimental implementation.

Then we describe the general concept starting with three-photon sources case. Analogous to Fig. 10(a), two three-photon sources are pumped coherently and output paths \(d_1\) and \(d_2\) are aligned identically in Fig. 11(a). Such an experiment allows one to tune the entangled states and observe many-particle interference patterns without any interaction with the pair of particles. Furthermore, it also provides a new perspective of controlling the amount of entanglement in a quantum state [85].

Two-particle interference patterns are observed when photons are superposed by a 50 : 50 BS.

Until now, we have familiarized ourselves with the hypergraph-experiment connection for state generations. Now we introduce different complex weights in the hypergraphs, which can naturally describe quantum interference in the experiments. We utilize a color wheel scale of radius \(r = 1\) to represent the complex weight shown in Fig. 10(b). There we interpret the phase \(\phi\) and transition amplitudes into the color and transparency in the region of a hyperedge. The transparency ratio from 100% to 0% describes the related color in radial from \(r = 0\) to \(r = 1\). For example, white indicates that the amplitude is zero (no photon is in the paths) while red stands for the photon’s maximum amplitude and its phase is \(0\) or \(2\pi\).

We now translate the experiment into its hypergraph description in Fig. 10(b). For simplicity, we only consider two cases that the phase \(\phi\) is set to \(\pi/2\) and \(3\pi/2\). The actions of linear optics such as BS-Operation can be described in graphs [16], which is extensible in hypergraphs. All linear optical elements, such as mode shifter, are feasibly describable in hypergraphs with an internal vertex set. For further details about the graph description of linear optics, see Ref. [16]. Here we only show the corresponding initial and final hypergraphs in Fig. 10(b). Clearly, we find that the destructive and constructive interference happens when photons in path \(d_2\) are never detected.
Interestingly, the destructive and constructive interference leads to two different Bell states. Specifically speaking, in the case of $\varphi = \pi$, namely destructive interference for $(P_{d_1,d_2}, P_{d_1,d_2})$, the hypergraph indicates the Bell state

$$|\psi\rangle = \frac{1}{\sqrt{2}} (|d_1d_2\rangle - |d_1d_2\rangle).$$

Thus, a maximum and a minimum of an interference pattern can be attained for two different Bell states. Without interacting with the associated particles, one can modify the entangled states and the interference patterns by changing the phase. Our hypergraph description is well picturesque and visualized for understanding the general concept of many-particle interferometry and entanglement using path identity [21,85].
VI. CONCLUSION

In conclusion, we have introduced hypergraphs, a generalization of graphs, to reinterpret quantum optical experiments using probabilistic $n$-photon sources with the technique path identity [21] and linear optics. One striking feature is that the concept of hypergraphs is ideal to model different types of correlated photon sources and can therefore be used for designing new quantum experiments. It would be very interesting to understand whether even more complex sources, such as cascaded SPDC down-conversion [88,89] can also be described by using our hypergraph-experiment connection.

We also show many experimental implementations to create a vast array of well-defined entangled quantum states (i.e., GHZ states, $W$ states) with the hypergraph-experiment connection. The results indicate that the dimensionality of multiphotonic entangled states are going beyond those produced in the experiments only using two-photon sources. That advantage sheds light on the producibility of arbitrary quantum states using photonic technology with probabilistic $n$-photon sources.

Moreover, hypergraph states [90,91] as the generalizations of graph states [1] have become powerful resource states for measurement-based quantum computation [3,92], quantum algorithms [93], and quantum error correction [94]. Despite the similarity of names, graph states and hypergraph states are not related to the techniques we present. It will be of great interest to establish a connection between hypergraph states and the technology developed here.

Most multiphotonic entangled quantum states are created under the condition of $N$-fold coincidence detection, in which we have focused on the case with one photon per path. That is directly connected to perfect matchings of hypergraphs. Although there might be multiple photons per path, one can use a photon number filter based on quantum teleportation [95] in each output of the setup to solve. For arbitrary photons per path, it would be a very interesting question for future research exploiting not only perfect matchings, but also more general techniques in matching theory.

We have shown that one can investigate the striking properties of hypergraphs by implementing quantum experiments in laboratories. For example, classically intractable decision problems of perfect matchings in hypergraphs can be solved by experimentally detecting an $N$-fold coincidences case, indicating a potential advantage of quantum experiments and further related to quantum computational supremacy [76,81,83]. Our connection may enable future applications in quantum computation, especially in connection to already existing algorithms employing hypergraphs, e.g., the three-SAT problem [96]. It will be exciting to see the potential of quantum experiments as presented here to solve problems in hypergraph theory that classical computers cannot calculate.

Finally, we introduce complex weights in the hypergraphs for describing a general concept of many-photon interference using $n$-photon sources together with path identity [21,85]. The picturesque and visualized approach allows us to observe interference easily and provides us the ability to control entanglement without any interaction with the photons. It would potentially inspire new applications of the hypergraph theory in quantum experiments. Furthermore, description of quantum processes at a more abstract level [11,12] and calculations in quantum optics by representing creation and annihilation operators in a visual way [14] have recently been studied. A combination of these pictorial approaches with our methods could hopefully improve the abstraction and intuitive understanding of quantum processes.

Our hypergraph-experiment technique works very well with probabilistic $n$-photon sources. To escape the restrictions of our method, deterministic quantum sources [97,98] would need an adaption of the description, and it is not yet known how to describe active feed forward [99]. Can they be described with hypergraphs? What are the techniques that cannot be described in the way presented here?

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