EULER-MARUYAMA SCHEME FOR SDES WITH DINI CONTINUOUS COEFFICIENTS

ZHEN WANG, YU MIAO, REN JIE

Abstract. In this paper, we show the convergence rate of Euler-Maruyama scheme for non-degenerate SDEs with Dini continuous coefficients, by the aid of the regularity of the solution to the associated Kolmogorov equation. We obtain the same conclusions using a simple and clever way to simplify the proof and weaken the conditions in [1] by the properties of Dini continuous and Taylor expansion.

Keywords: Non-degenerate, Stochastic differential equation, Euler-Maruyama scheme, Dini continuous, Kolmogorov equation.

1. Introduction

Let fix $T > 0$. Considering the following stochastic differential equation in $\mathbb{R}^d$:

$$X_t = X_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s, \quad X_0 \in \mathbb{R}^d,$$

(1.1)

where $b : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d$ are two Borel measurable functions, $\{W_t, t \in [0, T]\}$ is an $d$-dimensional standard Brown motion defined on a complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}; (\mathcal{F}_t)_{t \geq 0})$, and the initial value $X_0$ is $\mathcal{F}_0$-measurable $\mathbb{R}^d$-valued random variable.

The Euler-Maruyama scheme of (1.1) is

$$Y_t = Y_0 + \int_0^t b(\eta_\delta(s), Y_{\eta_\delta(s)})ds + \int_0^t \sigma(\eta_\delta(s), Y_{\eta_\delta(s)})dW_s, \quad X_0 = Y_0,$$

(1.2)

where $\delta := \frac{T}{N} \in (0, 1)$, $\eta_\delta(s) := \left[\frac{s}{\delta}\right] \delta$, $s \in \left[\left[\frac{s}{\delta}\right]\delta, \left[\frac{s}{\delta}\right]\delta + 1\right)\delta$, for the sufficiently large integer $N \in \mathbb{N}$. And the discretized scheme of (1.2) is analytically tractable on computer application of engineering, physical, finance, biology, etc..

If the coefficient of SDE is Lipschitz continuous, there are many previous research results. If $\sigma$ is an identity matrix and the coefficient $b$ is Lipschitz continuous in space and $\frac{1}{2}$-Hölder continuous in time then for any $p > 0$, there exists $C_p > 0$ such that the Euler-Maruyama scheme is the strong rate of $\frac{1}{2}$ (see for example [11]). Yan [15] proved the rate of convergence in $L^1$-norm sense for a range of SDEs, where

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the drift coefficient is Lipschitz and the diffusion coefficient is Hölder continuous, by means of the Meyer Tanaka formula. [7] extended [15] to the convergence rate in \( L^p \)-norm, by the Yamada-Watanabe approximation. Our finding partly improves upon recent results in [10] [12] and [7], as well as the well-known ones in [4] [5].

However, in many applications, the coefficients \( b \) and \( \sigma \) are not Lipschitz continuous. Zhang [17] is proved Euler-Maruyama approximation for SDE to converge uniformly to the solution in \( L^p \)-space with respect to the time and starting points under non-Lipschitz coefficients. If the drift coefficient is the Dini-Hölder continuous, Gyöngy and Rásonyi [7] implied the order of strong rate of convergence for one dimensional SDEs. And the case of \( d \)-dimension is introduced in [12], using a Yamada-Watanabe approximation technique (see [16]). Ngo and Taguchi [13] showed that the rate under the diffusion coefficient is bounded variation and Hölder continuous. In [14], the diffusion coefficient \( \sigma \) is an identity matrix, the drift coefficient \( b \) is bounded \( \beta \)-Hölder continuous with \( \beta \in (0, 1) \) in space and \( \eta \) -Hölder continuous in time with \( \eta \in [1/2, 1] \), then for any \( p \geq 1 \), the strong rate of convergence can be obtained. Bao, Huang and Yuan [1] discussed the strong convergence rate of Euler-Maruyama for non-degenerate SDE with rough coefficients, where the drift term is Dini-continuous and unbounded, by the regularity of non-degenerate Kolmogrov equation.

In this paper, we first study the convergence rate of the Euler-Maruyama scheme of (1.2), where the drift term \( b \) and the diffusion term \( \sigma \) are the uniformly bounded, \( b \) and \( \sigma \) satisfy correlated conditions of Dini-continuous (see Assumption 2.1), which is weaken the the conditions, simply the proof and obtains the same results in [1]. In addition, we also prove convergence rate for the non-degenerate SDEs with unbounded coefficients, which method is mainly based on the regularity of the solution to the Kolmogorov equation associated to the SDE (1.1).

This paper is structured as follows. In the next section, we introduce some notations and the main results. All proofs are deferred to Section 3.

2. Main results

2.1. Notations.

In this section, we recall the foundational definition and notations involved in the paper.

Let \( \mathcal{B}(\mathbb{R}^d) \) be the Borel-\( \sigma \)-algebra on \( \mathbb{R}^d \). Set \( \nabla := D = (\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_d})^* \), \( D^2 = (\frac{\partial^2}{\partial x_i \partial x_j})_{1 \leq i, j \leq d} \) and \( \Delta = \sum_{i=1}^d \frac{\partial^2}{\partial x_i^2} \), where \( * \) is the transpose of a vector or matrix. Take \( \| \cdot \| \) and \( \| \cdot \|_{HS} \) stand for the usual operator norm and the Hilbert-Schmidt norm, respectively.
Meanwhile, we introduce some space of function:

- $\|f\|_{T,\infty} = \sup_{t \in [0,T], x \in \mathbb{R}^d} \|f(t, x)\|$, where an operator-valued map $f$ is on $[0, T] \times \mathbb{R}^d$.
- $\mathbb{M}^d_{\text{non}}$ denotes the collection of all nonsingular $d \times d$-matrices.
- $C^\beta_b(\mathbb{R}^d, \mathbb{R}^k), \beta \in (0, 1)$ denotes the set of all function from $\mathbb{R}^d$ to $\mathbb{R}^k$ which are bounded and $\beta$-Hölder continuous functions. Hence if $f \in C^\beta_b(\mathbb{R}^d, \mathbb{R}^k)$, then

$$\sup_{x, y \in \mathbb{R}^d, x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\beta} < \infty.$$ 

- For $a < b$, we write $C^\beta_b([a, b])$ for $C([a, b]; C^\beta_b(\mathbb{R}^d, \mathbb{R}^d))$ and define the norm $\| \cdot \|_{C^\beta_b([a, b])}$ on $C^\beta_b([a, b])$ by

$$\|f\|_{C^\beta_b([a, b])} := \sup_{t \in [a, b], x \in \mathbb{R}^d} |f(t, x)| + \sup_{t \in [a, b], x \neq y} \frac{|f(t, x) - f(t, y)|}{|x - y|^\beta}.$$ 

Throughout the paper, we denote the constant as $C$, the shorthand notation $a \preceq b$ stands for $a \leq Cb$. And $C$ represents a positive constant although its value may change from one appearance to the next.

2.2. Main results.

In this paper, we study the convergence rate of Euler-Maruyama scheme, under the following non-Lipschitz condition. In this section, we state the related assumptions and main theorems of this paper.

Let $\mathcal{D}_0$ be the family of Dini function, i.e.,

$$\mathcal{D}_0 := \left\{ \phi : \phi : \mathbb{R}_+ \to \mathbb{R}_+ \text{ is increasing and } \int_0^1 \frac{\phi(s)}{s} \, ds < \infty \right\}.$$ 

A function $f : \mathbb{R}^d \to \mathbb{R}^d$ is called Dini-continuous if there exists $\phi \in \mathcal{D}_0$ such that $|f(x) - f(y)| \leq \phi(|x - y|)$ for any $x, y \in \mathbb{R}^d$. It is well known that every Dini-continuous function is continuous and every Lipschitz continuous function is Dini-continuous. Moreover, if $f$ is Hölder continuous, then $f$ is Dini-continuous, but not vice versa. And set

$$\mathcal{D} := \{ \phi \in \mathcal{D}_0 | \phi^2 \text{ is concave} \},$$

for instance, a function $f$ is Hölder-Dini continuous of order $\alpha \in (0, 1)$.

The non-Lipschitz assumptions is following:

**Assumption 2.1.**

(a) for every $t \in [0, T]$ and $x \in \mathbb{R}^n$, $\sigma(t, x) \in \mathbb{M}^n_{\text{non}}$, and

$$\|b\|_{T,\infty} + \|\sigma\|_{T,\infty} < +\infty,$$

where $\|\sigma\|_{T,\infty} := \sup_{0 \leq t \leq T} \|\sigma(t, x)\|_{\text{HS}}$. 

(b) For any $t \in [0, T]$, $\beta \in (0, 1)$ and $x, y \in \mathbb{R}^d$, there exists $\phi \in \mathcal{D}$ such that (regularity of $b$ and $\sigma$ w.r.t. spatial variables)

$$|b(t, x) - b(t, y)| \leq |x - y|^\beta \phi(|x - y|), \quad \|\sigma(t, x) - \sigma(t, y)\|_{HS} \leq \phi(|x - y|).$$

(c) For any $s, t \in [0, T]$ and $x \in \mathbb{R}^d$, there exists $\phi \in \mathcal{D}$ such that (regularity of $b$ and $\sigma$ w.r.t. time variables)

$$|b(s, x) - b(t, x)| + \|\sigma(s, x) - \sigma(t, x)\|_{HS} \leq \phi(|s - t|).$$

The following main results are stated including the convergence rate of SDEs.

**Theorem 2.1.** Suppose that Assumption 2.1 holds. For $p \geq 1$ and $\beta \in (0, 1)$, there exists the constant $C > 1$ depending on $T, p, d, \|\sigma\|_{T, \infty}$, $M, \beta$, then

$$\mathbb{E} \left( \sup_{0 \leq t \leq T} |X_t - Y_t|^p \right) \leq C \delta^{p/2}.$$

**Remark 2.1.** In [1], the diffusion term is the rough coefficient which refers to second order continuous differentiable. However, we check the results with the regularity of the solution to the Kolmogorov equation associated to the SDE (1.1), by the properties of Dini continuous and Taylor expansion instead of second order continuous differentiable, which is a simple and clever way to simplify the proof and weaken the conditions in [1].

**Remark 2.2.** In the article [1], the convergence rate is verified from the perspective of norm, while we not only get similar results but also do better conclusions using the properties of dimension. At the same time, we can also do the degenerate result with the help of the Hamiltonian system. Because the method is similar, we will not elaborate here.

It seems to be a little bit stringent that the coefficients are uniformly bounded, and the drift $b$ is global Dini-continuous, in Theorem 2.1. Therefore, the above conditions can be weakened by the means of uniform boundedness instead of local boundedness and global Dini-continuous instead of local Dini-continuous, respectively.

**Theorem 2.2.** Assume that for any $s, t \in [0, T]$, $\beta \in (0, 1)$ and for every $x \in \mathbb{R}^d$ and $\sigma(t, x) \in \mathbb{M}^d_{non}$, there exists the constant $C_T$, $b$ and $\sigma$ are Borel measurable functions such that

$$|b(t, x)| + \|\sigma(t, x)\|_{HS} \leq C_T(1 + x), \quad x \in \mathbb{R}^d,$$

And if $b$ and $\sigma$ satisfy

$$|b(t, x) - b(t, y)| \leq |x - y|^\beta \phi_k(|x - y|), \quad |x| \vee |y| \leq k,$$

$$\|\sigma(t, x) - \sigma(t, y)\|_{HS} \leq \phi_k(|x - y|), \quad |x| \vee |y| \leq k,$$

$$|b(s, x) - b(t, x)| + \|\sigma(s, x) - \sigma(t, x)\|_{HS} \leq \phi_k(|s - t|), \quad |x| \leq k,$$
where \( \phi_k \in D \). Then for all \( p \geq 1 \) and \( \mathbb{E}|X_0|^p < \infty \), it holds that

\[
\lim_{\delta \to 0} \mathbb{E} \left( \sup_{0 \leq t \leq T} |X_t - Y_t|^p \right) = 0
\]

**Remark 2.3.** We verify this conclusion by a method similar to Theorem 1.2 in [1]. In [1], the diffusion term is the uniformly bounded and second order continuous differentiable. However, in this paper, \( \sigma \) is the uniformly bounded, which is weaken the conditions of Theorem 1.2 in [1] and obtains the same conclusions.

3. Proofs of main results

We also need the following lemma for the proof.

**Lemma 3.1.** Let the coefficients \( b, \sigma \) is the uniformly bounded. For \( p \geq 1 \) and \( t \in [0, T] \), there exists a positive constant \( C > 0 \) depending on \( T, M, p, d \), it holds that

\[
\mathbb{E} \left[ \phi \left( \left| Y_t - Y_{\eta(t)} \right| \right)^p \right] \leq C \delta^{p/2}.
\]

**Proof.** Owing to \( \phi \in D \), based on Taylor expansion and the properties of Dini function, we have \( \phi(0) = 0 \), \( \phi' > 0 \) and \( \phi'' < 0 \), so that

\[
\phi(t) = \phi(0) + \phi'(0)t + \frac{\phi''(\theta t)}{2!}t^2, \quad \theta \in (0, 1), \quad t \in \mathbb{R}.
\]

Thus, for any \( t \in \mathbb{R} \),

\[
\phi(|t|) \leq M|t|.
\]

For \( p \geq 1 \), noticing that

\[
\phi \left( \left| Y_t - Y_{\eta(t)} \right| \right)^p \leq M^p \left| Y_t - Y_{\eta(t)} \right|^p.
\]

Using Assumption 2.1 (a)-(b), we deduce that

\[
\left| Y_t - Y_{\eta(t)} \right|^p \leq \left| \int_{\eta(t)}^t b(\eta(s), Y_{\eta(s)})ds \right|^p + \left| \int_{\eta(t)}^t \sigma(\eta(s), Y_{\eta(s)})dW_s \right|^p \leq \delta^p + |W_t - W_{\eta(t)}|^p.
\]

Hence there exists a positive constant \( C = C(T, M, p, d) \), such that, for \( p \geq 1 \),

\[
\mathbb{E} \left[ \phi \left( \left| Y_t - Y_{\eta(t)} \right| \right)^p \right] \leq M\mathbb{E} \left[ \left| Y_t - Y_{\eta(t)} \right|^p \right] \leq C\delta^{p/2}.
\]

The following lemma is taken from Theorem 2.8 in [3], which provides the regularity of solution to the Kolmogorov equation associated to the SDE (1.1).
Lemma 3.2. Let $T > 0$, for any $\varepsilon \in (0, 1)$, there exists $m \in \mathbb{N}$ such that $0 = t_0 < t_1 < \cdots < t_m = T$, for any $\varphi \in C([t_{j-1}, t_j]; C^{2\beta'}_b(\mathbb{R}^d, \mathbb{R}^d))$, $j = 1, \cdots, m$, there is at least one solution $u$ to the Backward Kolmogorov equation

$$\frac{\partial u}{\partial t} + \nabla u \cdot b + \frac{1}{2} \Delta u \cdot \sigma^2 = -\varphi, \quad \text{on} \quad [t_{j-1}, t_j] \times \mathbb{R}^d, \quad u(t_j, x) = 0$$

of class

$$u \in C \left( [t_{j-1}, t_j]; C^{2\beta'}_b(\mathbb{R}^d, \mathbb{R}^d) \right) \cap C^1 \left( [t_{j-1}, t_j], C^\beta_b(\mathbb{R}^d, \mathbb{R}^d) \right).$$

For some constant $K$ depending on $j$ and for all $\beta' \in (0, \beta)$, we have

$$\|D^2 u\|_{C^{\beta'}_b([t_{j-1}, t_j])} \leq K \|\varphi\|_{C^{\beta}_b([t_{j-1}, t_j])}$$

and for some constant $C_0$, it holds that

$$\|\nabla u\|_{C^{\beta}_b([t_{j-1}, t_j])} \leq C_0 (t_j - t_{j-1})^{1/2} \|\varphi\|_{C^{\beta}_b([t_{j-1}, t_j])}$$

At same time, we can obtain

$$\|\varphi\|_{C^{\beta}_b([0, T])} C_0 (t_j - t_{j-1})^{1/2} \leq \varepsilon.$$

Now we can give

Proof of Theorem 2.1. Let $T > 0$, for any $\varepsilon \in (0, 1)$, there is $m \in \mathbb{N}$, such that $0 = T_0 < T_1 < \cdots < T_m = T$. For $i = 1, \cdots, d$ and $j = 1, \cdots, m$, using Lemma 3.2 we can get

$$\frac{\partial u}{\partial t} + \nabla u \cdot b + \frac{1}{2} \Delta u \cdot \sigma^2 = -b, \quad \text{on} \quad [T_{j-1}, T_j] \times \mathbb{R}^d, \quad u(T_j, x) = 0,$$

and $u$ satisfies

$$\|\nabla u\|_{C^{\beta}_b([T_{j-1}, T_j])} \leq C_0 (T_j - T_{j-1})^{1/2} \|b\|_{C^{\beta}_b([T_{j-1}, T_j])} \leq \varepsilon. \quad (3.2)$$

For $t \in [T_{j-1}, T_j]$, by Itô’s formula and (3.1), we have

$$u(t, X_t) = u(T_{j-1}, X_{j-1}) + \int_{T_{j-1}}^t \frac{\partial u}{\partial t} (s, X_s) ds + \int_{T_{j-1}}^t \nabla u(s, X_s) dX_s$$

$$+ \frac{1}{2} \int_{T_{j-1}}^t \Delta u(s, X_s) d\langle X_s, X_s \rangle$$

$$= u(T_{j-1}, X_{j-1}) - \int_{T_{j-1}}^t b(s, X_s) ds + \int_{T_{j-1}}^t \langle \nabla u(s, X_s), \sigma(s, X_s) \rangle dW_s.$$
Similarly, we have

\[
\begin{align*}
&\quad u(t, Y_t) \\
&= u(T_{j-1}, Y_{j-1}) + \int_{T_{j-1}}^t \frac{\partial u}{\partial t}(s, Y_s) \, ds + \int_{T_{j-1}}^t \nabla u(s, Y_s) \, dW_s \\
&\quad + \frac{1}{2} \int_{T_{j-1}}^t \Delta u(s, Y_s) \, d\langle Y_s, Y_s \rangle \\
&= u(T_{j-1}, Y_{j-1}) - \int_{T_{j-1}}^t b(s, Y_s) \, ds + \int_{T_{j-1}}^t \langle \nabla u(s, Y_s), \sigma(\eta_b(s), Y_{\eta_b(s)}) \rangle dW_s \\
&\quad + \int_{T_{j-1}}^t \langle \nabla u(s, Y_s), b(\eta_b(s), Y_{\eta_b(s)}) - b(s, Y_s) \rangle ds.
\end{align*}
\]

Hence, we can get

\[
\int_{T_{j-1}}^t b(s, X_s) \, ds = u(T_{j-1}, X_{j-1}) - u(t, X_t) + \int_{T_{j-1}}^t \langle \nabla u(s, X_s), \sigma(s, X_s) \rangle dW_s, \quad (3.3)
\]

and

\[
\int_{T_{j-1}}^t b(s, Y_s) \, ds = u(T_{j-1}, Y_{j-1}) - u(t, Y_t) + \int_{T_{j-1}}^t \langle \nabla u(s, Y_s), \sigma(\eta_b(s), Y_{\eta_b(s)}) \rangle dW_s \\
\quad + \int_{T_{j-1}}^t \langle \nabla u(s, Y_s), b(\eta_b(s), Y_{\eta_b(s)}) - b(s, Y_s) \rangle ds. \quad (3.4)
\]

Combining with (3.3) and (3.4), we have

\[
\begin{align*}
X_t - Y_t &= X_{T_{j-1}} - Y_{T_{j-1}} \\
&\quad + \int_{T_{j-1}}^t (b(s, X_s) - b(\eta_b(s), Y_{\eta_b(s)})) \, ds + \int_{T_{j-1}}^t (\sigma(s, X_s) - \sigma(\eta_b(s), Y_{\eta_b(s)})) \, dW_s \\
&= X_{T_{j-1}} - Y_{T_{j-1}} + (u(T_{j-1}, X_{T_{j-1}}) - u(T_{j-1}, Y_{T_{j-1}})) - (u(t, X_t) - u(t, Y_t)) \\
&\quad + \int_{T_{j-1}}^t \left[ \langle \nabla u(s, X_s), \sigma(s, X_s) \rangle - \langle \nabla u(s, Y_s), \sigma(\eta_b(s), Y_{\eta_b(s)}) \rangle \right] dW_s \\
&\quad + \int_{T_{j-1}}^t \langle \nabla u(s, Y_s), b(s, Y_s) - b(\eta_b(s), Y_{\eta_b(s)}) \rangle ds + \int_{T_{j-1}}^t (b(s, Y_s) - b(\eta_b(s), Y_{\eta_b(s)})) \, ds \\
&\quad + \int_{T_{j-1}}^t (\sigma(s, X_s) - \sigma(\eta_b(s), Y_{\eta_b(s)})) \, dW_s.
\end{align*}
\]
By (3.2) and the mean-value theorem, we have:

\[ |X_t - Y_t| \]

\[ \leq |X_{T_j-1} - Y_{T_j-1}| + |u(T_{j-1}, X_{T_{j-1}}) - u(T_{j-1}, Y_{T_{j-1}})| + |u(t, X_t) - u(t, Y_t)| \]

\[ + \int_{T_{j-1}}^{t} \left[ \langle \nabla u(s, X_s), \sigma(s, X_s) \rangle - \langle \nabla u(s, Y_s), \sigma(\eta_\delta(s), Y_{\eta_\delta(s)}) \rangle \right] dW_s \]

\[ + \| \nabla u \|_{C^0_b[T_{j-1}, T_j]} \int_{T_{j-1}}^{t} |b(s, Y_s) - b(\eta_\delta(s), Y_{\eta_\delta(s)})| ds \]

\[ + \int_{T_{j-1}}^{t} |b(s, Y_s) - b(\eta_\delta(s), Y_{\eta_\delta(s)})| ds + \int_{T_{j-1}}^{t} (\sigma(s, X_s) - \sigma(\eta_\delta(s), Y_{\eta_\delta(s)})) dW_s \]

\[ \leq (1 + \varepsilon) |X_{T_j-1} - Y_{T_j-1}| + \varepsilon |X_t - Y_t| \]

\[ + \int_{T_{j-1}}^{t} \left[ \langle \nabla u(s, X_s), \sigma(s, X_s) \rangle - \langle \nabla u(s, Y_s), \sigma(\eta_\delta(s), Y_{\eta_\delta(s)}) \rangle \right] dW_s \]

\[ + (1 + \varepsilon) \int_{T_{j-1}}^{t} |Y_s - Y_{\eta_\delta(s)}|^\beta \phi \left( |Y_s - Y_{\eta_\delta(s)}| \right) ds + (1 + \varepsilon) \int_{T_{j-1}}^{t} \phi \left( |s - \eta_\delta(s)| \right) ds \]

\[ + \int_{T_{j-1}}^{t} (\sigma(s, X_s) - \sigma(\eta_\delta(s), Y_{\eta_\delta(s)})) dW_s. \]

For all \( p \geq 1 \), utilizing Jensen’s inequality, Hölder inequality and Lemma 3.1, we can obtain

\[ |X_t - Y_t|^p \]

\[ \leq 6^{p-1} (1 + \varepsilon)^p |X_{T_{j-1}} - Y_{T_{j-1}}|^p + 6^{p-1} \varepsilon^p |X_t - Y_t|^p \]

\[ + 6^{p-1} \left| \int_{T_{j-1}}^{t} \left[ \langle \nabla u(s, X_s), \sigma(s, X_s) \rangle - \langle \nabla u(s, Y_s), \sigma(\eta_\delta(s), Y_{\eta_\delta(s)}) \rangle \right] dW_s \right|^p \]

\[ + 6^{p-1} (1 + \varepsilon)^p (t - T_{j-1})^{p-1} M^p \int_{T_{j-1}}^{t} |Y_s - Y_{\eta_\delta(s)}|^{p(\beta+1)} ds \]

\[ + 6^{p-1} (1 + \varepsilon)^p (t - T_{j-1})^{p-1} M^p \int_{T_{j-1}}^{t} |s - \eta_\delta(s)|^p ds \]

\[ + 6^{p-1} \left| \int_{T_{j-1}}^{t} (\sigma(s, X_s) - \sigma(\eta_\delta(s), Y_{\eta_\delta(s)})) dW_s \right|^p. \]

Because \( \varepsilon \) is arbitrary, there exists \( c(p, \varepsilon) := 6^{p-1} \varepsilon^p < 1 \). Then we know

\[ |X_t - Y_t|^p \leq \frac{6^{p-1}(1 + \varepsilon)^p}{1 - c(p, \varepsilon)} |X_{T_{j-1}} - Y_{T_{j-1}}|^p. \]
\[
+ \frac{6^{p-1}}{1 - c(p, \varepsilon)} \int_{T_{j-1}}^{t} \mathbb{E} \left[ \sup_{T_{j-1} \leq u \leq s} \left| \left( \nabla u(s, X_u), \sigma(s, X_u) \right) - \left( \nabla u(s, Y_u), \sigma(\eta(s), Y_{\eta(s)}) \right) \right|^p \right] dW_s
\]

Taking the supremum, expectation on both sides of the above inequality, and using BDG’s inequality, for \( t \in (T_{j-1}, T_j) \), we have

\[
\mathbb{E} \left[ \sup_{T_{j-1} \leq u \leq t} \left| X_u - Y_u \right|^p \right] \leq \frac{6^{p-1}(1 + \varepsilon)^p}{1 - c(p, \varepsilon)} \mathbb{E} \left[ \left| X_{T_{j-1}} - Y_{T_{j-1}} \right|^p \right] + \frac{6^{p-1}C(p, d)T^{\frac{p}{2} - 1}}{1 - c(p, \varepsilon)}
\]

\[\times \int_{T_{j-1}}^{t} \mathbb{E} \left[ \sup_{T_{j-1} \leq u \leq s} \left| \nabla u(u, X_u)\sigma(u, X_u) - \nabla u(u, Y_u)\sigma(\eta(u), Y_{\eta(u)}) \right|^p \right] ds + \frac{6^{p-1}(1 + \varepsilon)^pT^{p-1}M^p}{1 - c(p, \varepsilon)}
\]

\[\times \int_{T_{j-1}}^{t} \mathbb{E} \left[ \sup_{T_{j-1} \leq u \leq s} \left| Y_u - Y_{\eta(u)} \right|^p \right] ds + T^{\varepsilon} \]

\[+ \frac{6^{p-1}C(p, d)T^{\frac{p}{2} - 1}}{1 - c(p, \varepsilon)} \int_{T_{j-1}}^{t} \mathbb{E} \left[ \sup_{T_{j-1} \leq u \leq s} \left| \sigma(u, X_u) - \sigma(\eta(u), Y_{\eta(u)}) \right|^p \right] ds \]

\[:= \sum_{i=1}^{4} \mathbb{I}_i, \]

where \( C(p, d) \) is the constant in BDG’s inequality. With the help of lemma 3.1 and the Assumption 2.1 (a), in \( \mathbb{I}_2 \), we have

\[
\mathbb{E} \left[ \sup_{T_{j-1} \leq u \leq s} \left| \nabla u(u, X_u)\sigma(u, X_u) - \nabla u(u, Y_u)\sigma(\eta(u), Y_{\eta(u)}) \right|^p \right] \leq 4^{p-2}2^p \varepsilon^p |\sigma|^p_{T, \infty} + 4^{p-1}2^p \varepsilon^p \mathbb{E} \left[ \sup_{T_{j-1} \leq u \leq s} \phi(|X_u - Y_u|)^p \right]
\]

\[+ 4^{p-1}2^p \varepsilon^p \mathbb{E} \left[ \sup_{T_{j-1} \leq u \leq s} \phi(|u - \eta(u)|)^p \right] + 4^{p-1}2^p \varepsilon^p \mathbb{E} \left[ \sup_{T_{j-1} \leq u \leq s} \phi(|Y_u - Y_{\eta(u)}|)^p \right] \]
such that

\[ \delta \leq 10 \]

In II, from the properties of Dini-function, it may be chosen the constant \( C_0, C_1 \), such that

\[
\mathbb{E} \left[ \sup_{T_{j-1} \leq u \leq s} \| \sigma(u, X_u) - \sigma(\eta_{\delta}(u), Y_{\eta_{\delta}(u)}) \|^p_{HS} \right] \\
= \mathbb{E} \left[ \sup_{T_{j-1} \leq u \leq s} \| \sigma(u, X_u) - \sigma(\eta_{\delta}(u), X_u) + \sigma(\eta_{\delta}(u), X_u) - \sigma(\eta_{\delta}(u), Y_u) + \sigma(\eta_{\delta}(u), Y_u) \|^p_{HS} \right] \\
\leq 3^{p-1} \mathbb{E} \left[ \sup_{T_{j-1} \leq u \leq s} \phi(|u - \eta_{\delta}(u)|)^p \right] + 3^{p-1} \mathbb{E} \left[ \sup_{T_{j-1} \leq u \leq s} \phi(|X_u - Y_u|)^p \right] \\
+ 3^{p-1} \mathbb{E} \left[ \sup_{T_{j-1} \leq u \leq s} \phi(|Y_u - \eta_{\eta_{\delta}(u)}|)^p \right] \\
\leq 3^{p-1} M^p \delta^p + 3^{p-1} M^p \mathbb{E} \left( \sup_{T_{j-1} \leq u \leq s} |X_u - Y_u|^p \right) + 3^{p-1} M^p C^p \delta^{p/2}. \]

Thus, for \( \delta \in (0, 1) \) and \( p \geq 1 \), there exists the constant \( C_2, C_3, C_4 \), we know

\[
\mathbb{E} \left[ \sup_{T_{j-1} \leq u \leq t} |X_u - Y_u|^p \right] \leq \frac{6^{p-1}(1 + \varepsilon)^p}{1 - c(p, \varepsilon)} \mathbb{E} \left[ |X_{T_{j-1}} - Y_{T_{j-1}}|^p \right] \\
+ \frac{6^{p-1} c(p, d) T^{\frac{p}{2} - 1} (\varepsilon^p M^p 4^{p-1} + 3^{p-1} M^p)}{1 - c(p, \varepsilon)} \int_{T_{j-1}}^t \mathbb{E} \left[ \sup_{T_{j-1} \leq u \leq s} |X_u - Y_u|^p \right] ds \\
+ \frac{6^{p-1} (1 + \varepsilon) T^{\frac{p}{2} - 1}}{1 - c(p, d, \varepsilon)} \left[ 4^{p-1} \varepsilon^p C^p \delta^{p/2} + 4^{p-1} \varepsilon^p M^p \delta^p + 4^{p-1} 2^p \varepsilon^p K^p + 3^{p-1} C^p \delta^{p/2} + 3^{p-1} M^p \delta^p \right] \\
+ \frac{6^{p-1} T^p (1 + \varepsilon)^p M^p}{1 - c(p, d, \varepsilon)} \left[ C \delta^{p(\beta+1)/2} + T \delta^p \right] \\
\leq C_2 \mathbb{E} \left[ |X_{T_{j-1}} - Y_{T_{j-1}}|^p \right] + C_3 \int_{T_{j-1}}^t \mathbb{E} \left[ \sup_{T_{j-1} \leq u \leq s} |X_u - Y_u|^p \right] ds + C_4 \delta^{p/2}. \]

Next, we prove by the Lemma 3.1 that for each \( j = 1, \cdots, m \),

\[
\mathbb{E} \left[ \sup_{T_{j-1} \leq u \leq t} |X_u - Y_u|^p \right] \leq A_j \delta^{p/2}, \quad t \in (T_{j-1}, T_j]. \tag{3.6} \]
where $A_1 = C_4 e^{C_3 T}$ and $A_j = (C_2 A_{j-1} + C_4) e^{C_3 T}$, for $j = 2, \cdots, m$. If $j = 1$, since $T_0 = 0, \forall t \in (0, T_1]$, we have
\[
\mathbb{E} \left[ \sup_{0 \leq u \leq t} |X_u - Y_u|^p \right] \leq C_3 \int_0^t \mathbb{E} \left[ \sup_{0 \leq u \leq s} |X_u - Y_u|^p \right] \, ds + C_4 \delta^{p/2}.
\]
Using Gronwall’s inequality, we can get
\[
\mathbb{E} \left[ \sup_{0 \leq u \leq t} |X_u - Y_u|^p \right] \leq C_4 e^{C_3 T} \delta^{p/2}, \quad t \in (0, T_1].
\]
We assume that (3.6) holds for $j = 1, 2, \cdots, i - 1$ with $2 \leq i \leq m$. Then $\forall t \in (T_{i-1}, T_i]$, we realize
\[
\mathbb{E} \left[ \sup_{T_{i-1} \leq u \leq t} |X_u - Y_u|^p \right] \leq C_2 \mathbb{E} \left[ |X_{T_{i-1}} - Y_{T_{i-1}}|^p \right]
\]
\[+ C_3 \int_{T_{i-1}}^t \mathbb{E} \left[ \sup_{T_{i-1} \leq u \leq s} |X_u - Y_u|^p \right] \, ds + C_4 \delta^{p/2}
\]
\[\leq C_3 \int_{T_{i-1}}^t \mathbb{E} \left[ \sup_{T_{i-1} \leq u \leq s} |X_u - Y_u|^p \right] \, ds + (C_2 A_{i-1} + C_4) \delta^{p/2}.
\]
By once more Gronwall’s inequality, it holds that
\[
\mathbb{E} \left[ \sup_{T_{i-1} \leq u \leq t} |X_u - Y_u|^p \right] \leq (C_2 A_{i-1} + C_4) e^{C_3 T} \delta^{p/2} = A_i \delta^{p/2}, \quad t \in (T_{i-1}, T_i].
\]
Hence $\forall j = 1, \cdots, m$, (3.6) is true. And we draw the conclusion that
\[
\mathbb{E} \left[ \sup_{0 \leq s \leq T} |X_u - Y_u|^p \right] \leq \sum_{j=1}^m \mathbb{E} \left[ \sup_{T_{j-1} \leq u \leq T_j} |X_u - Y_u|^p \right] \leq m A_j \delta^{p/2} := \beta \delta^{p/2}.
\]
So the proof is finished. \qed

**Proof of Theorem** Let $\chi \in C^\infty_b(\mathbb{R}_+)$ is the cut-off function, such that $0 \leq \chi \leq 1$, $\chi(r) = 1$ for $r \in (0, 1)$, and $\chi(r) = 0$, for $r \geq 2$. For any $t \in [0, T]$ and $k \geq 1$, let
\[
b^{(k)}(t, x) = b(t, x) \chi \left( \frac{|x|}{k} \right), \quad \sigma^{(k)}(t, x) = \sigma \left( t, \chi \left( \frac{|x|}{k} \right) \right), \quad x \in \mathbb{R}^n.
\]
Fixed $k \geq 1$, we have,
\[
X^{(k)}_t = x + \int_0^t b^{(k)}(s, X^{(k)}_s) \, ds + \int_0^t \sigma^{(k)}(s, X^{(k)}_s) \, dW_s, \quad t \in (0, T].
\]
The corresponding continuous-time Euler-Maruyama is
\[
Y^{(k)}_t = x + \int_0^t b^{(k)} \left( \eta(s), Y^{(k)}_{\eta(s)} \right) \, ds + \int_0^t \sigma^{(k)} \left( \eta(s), Y^{(k)}_{\eta(s)} \right) \, dW_s, \quad t \in (0, T]. (3.7)
\]
Using the BDG, Hölder and Gronwall inequality, for all \( p \geq 1 \), for some \( C_T \), we have (see the proof Theorem 1.2 in \([1]\))
\[
\mathbb{E}
\left(\sup_{0 \leq t \leq T} |X_t|^p\right) + \mathbb{E}
\left(\sup_{0 \leq t \leq T} |Y_t|^p\right) + \mathbb{E}
\left(\sup_{0 \leq t \leq T} |X_t^{(k)}|^p\right) + \mathbb{E}
\left(\sup_{0 \leq t \leq T} |Y_t^{(k)}|^p\right)
\leq C_T (1 + \mathbb{E}|X_0|^p) < +\infty.
\]

Since
\[
\mathbb{E}
\left(\sup_{0 \leq t \leq T} |X_t - Y_t|^p\right) \leq 3^{p-1}\mathbb{E}
\left(\sup_{0 \leq t \leq T} |X_t - X_t^{(k)}|^p\right) + 3^{p-1}\mathbb{E}
\left(\sup_{0 \leq t \leq T} |X_t^{(k)} - Y_t^{(k)}|^p\right) + 3^{p-1}\mathbb{E}
\left(\sup_{0 \leq t \leq T} |Y_t - Y_t^{(k)}|^p\right) =: \mathbb{I}_1 + \mathbb{I}_2 + \mathbb{I}_3,
\]
applying the Chebyshev inequality and (3.3), we can deduce
\[
\mathbb{I}_1 + \mathbb{I}_3 \leq \mathbb{E}
\left(\sup_{0 \leq t \leq T} |X_t - X_t^{(k)}|^p \mathbb{1}_{\{\sup_{0 \leq t \leq T} |X_t| \geq k\}}\right) + \mathbb{E}
\left(\sup_{0 \leq t \leq T} |Y_t - Y_t^{(k)}|^p \mathbb{1}_{\{\sup_{0 \leq t \leq T} |Y_t| \geq k\}}\right)
\leq \left(\mathbb{E}
\left(\sup_{0 \leq t \leq T} |X_t|^p\right) + \mathbb{E}
\left(\sup_{0 \leq t \leq T} |X_t^{(k)}|^p\right)\right) \frac{\mathbb{E}(\sup_{0 \leq t \leq T} |X_t|)}{k} + \frac{\mathbb{E}(\sup_{0 \leq t \leq T} |Y_t|)}{k}
\leq \frac{1}{k}.
\]

For the terms \( \mathbb{I}_2 \), by the Theorem 2.1, we have
\[
\mathbb{I}_2 \leq \beta_1 \delta^{p/2},
\]
where the constant \( \beta_1 > 1 \) depending on \( T, p, d, \|\sigma\|_{T, \infty}, M, k, \beta \). Consequently, we conclude that
\[
\mathbb{E}
\left(\sup_{0 \leq t \leq T} |X_t - Y_t|^p\right) \leq \frac{1}{k} + \beta_1 \delta^{p/2}.
\]

For any \( \varepsilon > 0 \), taking \( k = \frac{1}{\varepsilon} \) and \( \delta \to 0 \), implies that
\[
\lim_{\delta \to 0} \mathbb{E}
\left(\sup_{0 \leq t \leq T} |X_t - Y_t|^p\right) = 0.
\]

Thus, the proof of Theorem 2.2 can be complete. \(\square\)
EULER-MARUYAMA SCHEME FOR SDES

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