Intersection number of a map with the set of matrices of positive corank

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Abstract
The definition of the intersection number of a map with a closed manifold can be extended to the case of a closed stratified set such that the difference between dimensions of its two biggest strata is greater than 1. The set $\Sigma$ of matrices of positive corank is an example of such a set. It turns out that the intersection number of a map from an $(n-k+1)$–dimensional manifold with boundary into the set of $n \times k$ real matrices with $\Sigma$ coincides with a homotopy invariant associated with a map going to the Stiefel manifold $\tilde{V}_k(\mathbb{R}^n)$. In a polynomial case, we present an effective method to compute this intersection number. We also show how to use it to count the number mod 2 or the algebraic sum of cross–cap singularities of a map from an $m$–dimensional manifold with boundary to $\mathbb{R}^{2m-1}$.

1 Introduction

In [8] Hirsch defined the intersection number of a map $M \to W$ with a closed $n$–dimensional submanifold $N \subset W$, where $M$ and $W$ are manifolds, $M$ is compact, $\dim M = m$, $\dim W = n+m$. We extend this definition (see Section 2) to the case where $N$ is a closed stratified subset of $W$ such that the difference between dimensions of the two biggest strata is greater than 1. We denote this
intersection number of \( f : M \to W \) with \( N \) by \( \text{I}(f, N) \) or \( \text{I}_2(f, N) \), depending on the orientability of manifolds.

In case of a mapping \( a \) from an \((n-k+1)\)–dimensional compact manifold \( M \) with boundary to \( M_k(\mathbb{R}^n) \) (the set of \( n \times k \) real matrices), the intersection number with the set \( \Sigma \) of matrices of corank \( \geq 1 \) is well defined. In Section 3 we present a nice characterization of this intersection number. We show that for \( a|\partial M \) it coincides with invariants presented in [10] (when \( M \) is a closed ball, \( n - k \) odd) and in [11] (\( n - k \) even), where the authors defined a homotopy invariant \( \Lambda \) associated with a map from an \((n-k)\)–dimensional boundaryless manifold into the Stiefel manifold \( \tilde{V}_k(\mathbb{R}^n) \). When the domain is a sphere, \( \Lambda \) induces an isomorphism between \( \mathbb{Z}_2 \) or \( \mathbb{Z} \) (depending on the parity of \( n - k \)) and \((n-k)\)–th homotopy group of \( \tilde{V}_k(\mathbb{R}^n) \).

In Section 4 we present effective methods to compute the intersection number modulo 2 of a polynomial map \( a : M \to M_k(\mathbb{R}^n) \). We express \( I_2(a, \Sigma) \) in terms of signs of determinants of matrices of some quadratic forms (see Theorem 4.2).

We show some applications of the intersection number in Section 5. In [16, 17] Whitney studied general singularities (cross–caps) of mappings from an \( m \)–dimensional manifold to \( \mathbb{R}^{2m-1} \). When \( m \) is odd, he associated a sign with a cross–cap singularity. In [17], Whitney proved that if \( M \) is closed and \( f \) has only cross–caps as singularities then the number of cross–caps is even, moreover if \( m \) is odd, then the algebraic sum (i.e. the sum of signs) of them equals zero. If \( M \) has a boundary then following [17, Theorem 4], for a homotopy \( f_1 : M \to \mathbb{R}^{2m-1} \) regular in some open neighbourhood of \( \partial M \), if the only singular points of \( f_0 \) and \( f_1 \) are cross–caps then the numbers of cross–caps of \( f_0 \) and \( f_1 \) are congruent mod 2, and if \( m \) is odd, then the algebraic sums of cross–caps of \( f_0 \) and \( f_1 \) are the same. We present methods of verifying if \( f \) has only cross–caps as singular points, so in the polynomial case using [3, 14] one can compute the number of them. We also express the number of cross–caps modulo 2, and for \( m \) odd the algebraic sum of them, in terms of the intersection number presented in this paper. So in the polynomial case, using [11], and resp. the results presented in Section 4 one can compute them using for example SINGULAR.

Section 6 is devoted to the index \( I_2(s) \) of a section \( s \) of a vector bundle. We show that with a map \( a : M \to M_k(\mathbb{R}^n) \) one may associate a vector bundle \( \xi \) and its section \( s_a \) such that \( I_2(s_a) = I_2(a, \Sigma) \).

In this article we will use some properties of the topological degree \( \text{deg} g \) (resp. topological degree modulo 2 — \( \text{deg}_2 g \)) of a map \( g \) between two compact manifolds.

Let us recall that if \( M \) is an oriented \( m \)–manifold, \( f : M \to \mathbb{R}^m \), and \( p \in M \setminus \partial M \) is isolated in \( f^{-1}(0) \), then there exists a compact \( m \)–manifold \( K \subset M \setminus \partial M \) with boundary such that \( f^{-1}(0) \cap K = \{ p \} \) and \( f^{-1}(0) \cap \partial K = \emptyset \). By \( \text{deg}(f, p) \) we will denote the local topological degree of \( f \) at \( p \), i.e. the topological degree of the mapping \( \partial K \ni x \mapsto f(x)/|f(x)| \in S^{m-1} \) (for properties of \( \text{deg}(f, p) \) see for example [13]).
If \( g: M \rightarrow \mathbb{R}^m \) is close enough to \( f \), then \( g^{-1}(0) \cap \partial K \) is also empty, and the topological degree of the mapping \( \partial K \ni x \mapsto g(x)/|g(x)| \in S^{n-1} \) is equal to \( \deg(f, p) \).

If \( M \) is not oriented, then the local topological degree modulo 2 — \( \deg_2(f, p) \) of \( f \) at \( p \) is defined similarly using \( \deg_2 f \).

If \( f: (M, \partial M) \rightarrow (\mathbb{R}^m, \mathbb{R}^m \setminus \{0\}) \), then by \( \deg(f|\partial M) \) (resp. \( \deg_2(f|\partial M) \)) we denote the topological degree (resp. the topological degree modulo 2) of the map \( \partial M \ni x \mapsto f(x)/|f(x)| \in S^{n-1} \).

In this article manifold means a smooth manifold. By \( \text{id} \) we mean an identity map, \# denotes the number of elements. For the mapping \( f \) depending on variables \((x, y) \in \mathbb{R}^m \times \mathbb{R}^l \) we denote by \( \frac{\partial f}{\partial x} \) the matrix of partial derivatives of \( f \) with respect to the variables \( x \).

## 2 Intersection number of a map with a closed stratified set

Let \( W \) be an \((m+n)\)-dimensional manifold and \( N = \bigcup_{i=0}^{n} N_i \) be a closed stratified subset of \( W \), where \( N_n \) is an \( n \)-dimensional submanifold of \( W \), \( N_{n-1} = \emptyset \) and \( N_i \) is either an \( i \)-dimensional submanifold of \( W \) or an empty set, for \( i = 0, \ldots, n-2 \). In this Section we will define the intersection number of a mapping from an \( m \)-dimensional compact manifold to \( W \) with the set \( N \), which coincide with the definition from [8, Chapter 5.] in case when \( N \) is a submanifold of \( W \).

For any manifold \( S \) with or without boundary, and a smooth map \( f: S \rightarrow W \) we will say that \( f \) is transversal to \( N \) \( (f \pitchfork N) \) if \( f \pitchfork N_i \) for \( i = 0, \ldots, n \). Let us see that the set \( \{ f \in C^\infty(S, W) \, | \, f \pitchfork N \} = \bigcap \{ f \pitchfork N_i \} \). According to [5, II.4.9, II.4.12] each set \( \{ f \pitchfork N_i \} \) contains a residual subset, and since the set of smooth maps from \( S \) to \( W \) is a Baire space, \( \{ f \in C^\infty(S, W) \, | \, f \pitchfork N \} \) is a dense subset of \( C^\infty(S, W) \).

Take an \( m \)-dimensional compact manifold \( M \) without boundary and a smooth map \( f: M \rightarrow W \). Then \( f \pitchfork N \) if and only if \( f \pitchfork N_i \) and \( f(M) \cap N_i = \emptyset \) for \( i = 0, \ldots, n - 1 \). In that case \( f^{-1}(N) = f^{-1}(N_n) \), moreover \( \text{codim } f^{-1}(N_n) = \text{codim } N_n = m \), so \( f^{-1}(N_n) \) is a 0-dimensional submanifold of a compact manifold and so \( f^{-1}(N) \) is finite.

Let \( M, W, \) and \( N_n \) be oriented. Take a smooth \( f: M \rightarrow W \) such that \( f \pitchfork N \).

**Definition 2.1.** For \( x \in f^{-1}(N) \) we define \( \text{I}(f, N)_x \) as \(+1\) or \(-1\) depending on whether the mapping

\[
\eta: T_x M \xrightarrow{\frac{\partial f}{\partial x}} T_{f(x)} W \rightarrow T_{f(x)} W/T_{f(x)} N_n
\]

preserves or reverses orientation.
There the orientation of $T_{f(x)}W/T_{f(x)}N_n$ is the orientation of the normal bundle of $N_n$ at $f(x)$ chosen in such a way, that the orientation of the direct sum of the tangent space and the normal bundle of $N_n$ at $f(x)$ coincides with the orientation of $T_{f(x)}W$.

**Definition 2.2.** The intersection number of $f$ (with $N$) is given by

$$I(f, N) = \sum_{x \in f^{-1}(N)} I(f, N)_x.$$  

**Theorem 2.3.** If $M, W, N_n$ are oriented, $f, g : M \to W$ are smooth maps such that $f \pitchfork N$, $g \pitchfork N$ and $f$ and $g$ are homotopic, then $I(f, N) = I(g, N)$.

**Proof.** Take $H : M \times [0, 1] \to W$ such that $H(\cdot, 0) = f$ and $H(\cdot, 1) = g$. We may assume that $H$ is smooth and $H \pitchfork N$. Since $N_{n+1} = \emptyset$, we have $H \pitchfork N_n$ and $H^{-1}(N_i) = \emptyset$ for $i = 1, \ldots, n - 1$.

Let us see that $H^{-1}(N) = H^{-1}(N_n)$ is a compact 1-dimensional submanifold of $M \times [0, 1]$ and $H|_{M \times \{0\}} \pitchfork N$ and $H|_{M \times \{1\}} \pitchfork N$. According to [8, p. 60] we have

$$\partial H^{-1}(N) = H^{-1}(N) \cap \partial(M \times [0, 1]) = f^{-1}(N) \times \{0\} \cup g^{-1}(N) \times \{1\}.$$

Similarly as in [8], using methods from [8, Lemma 5.1.1, Lemma 5.1.2], one may show that

$$I(H|\partial(M \times [0, 1]), N) = 0.$$  

Since $I(H|\partial(M \times [0, 1]), N) = I(f, N) - I(g, N)$, we get $I(f, N) = I(g, N)$. \hfill $\square$

Now we can define the intersection number of a continuous map $g : M \to W$, as $I(g, N) := I(f, N)$, where $f \pitchfork N$ and $f$ is smooth and homotopic to $g$.

If $W, N_n$ or $M$ is not oriented, one can define the intersection number of the mapping modulo 2.

**Definition 2.4.** For $f \pitchfork N$ we define the intersection number of $f$ (with $N$) modulo 2 as

$$I_2(f, N) = \# f^{-1}(N) \mod 2.$$  

Since the boundary of a compact 1-dimensional manifold has an even number of points, we get the following.

**Theorem 2.5.** If $f, g : M \to W$ are smooth maps such that $f \pitchfork N$, $g \pitchfork N$ and $f$ and $g$ are homotopic, then $I_2(f, N) = I_2(g, N)$.

Then the intersection number modulo 2 of a continuous map $g : M \to W$ is $I_2(g, N) := I_2(f, N)$, where $f \pitchfork N$ and $f$ is smooth and homotopic to $g$.

If $M$ is allowed to have a boundary, $I(f, N)$ and $I_2(f, N)$ are defined in the same way, but only for maps fulfilling $f(\partial M) \cap N = \emptyset$. In this case we should take the map $g : M \to W$ homotopic to $f$ by a homotopy that takes $\partial M$ into $W \setminus N$ at each stage. Of course $I(f, N)$ and $I_2(f, N)$ are invariants only of this kind of homotopy.
Remark 2.6. Of course if $M$ is a manifold without boundary, and $W = \mathbb{R}^{n+m}$, then the appropriate intersection number of a map $M \to W$ with any $N$ is equal to 0.

3 Intersection number of a map into $M_k(\mathbb{R}^n)$

Take $n-k > 0$. Let $M_k(\mathbb{R}^n)$ be the space of $n \times k$ real matrices. We identify it with $\mathbb{R}^{kn}$ writing down elements by columns. By $\Sigma^i \subset M_k(\mathbb{R}^n)$ we denote the set of matrices of corank $i$. Put $\Sigma = \bigcup_{i=1}^k \Sigma^i$.

Note that $\Sigma$ is closed and each $\Sigma^i$ is a submanifold of $M_k(\mathbb{R}^n)$ of codimension $(n-k+i)i$ (see [5] Chapter II, Proposition 5.3]). Moreover $\Sigma^i$’s are orientable for $n-k$ even, and non–orientable for $n-k$ odd (see [2]). Then the stratification $\Sigma = \bigcup_{i=1}^k \Sigma^i$ has properties described at the beginning of Section 2 where $N_{nk-(n-k+i)i} = \Sigma^i$, and other $N_j = \emptyset$.

Let now $M$ be an $(n-k+1)$–dimensional compact manifold with boundary. Take a smooth mapping $a : M \to M_k(\mathbb{R}^n)$. By $a_i$ we denote its $i$–th column, and by $a^i_j(x)$ — the elements of the matrix $a(x)$ (standing in the $j$–th row and $i$–th column). From now on we will assume that

\[(1) \quad a(\partial M) \cap \Sigma = \emptyset.\]

In this case the intersection number $I(a, \Sigma)$ or $I_2(a, \Sigma)$ of the map $a$ is well defined (see Section 2).

With the mapping $a$ we may associate (as in [10, 11]) the mapping $\tilde{a} : S^{k-1} \times M \to \mathbb{R}^n$ given by

$\tilde{a}(\beta, x) = \beta_1 a_1(x) + \ldots + \beta_k a_k(x),$

where $\beta = (\beta_1, \ldots, \beta_k) \in S^{k-1}$. There $S^{k-1}$ denotes the $(k-1)$–dimensional sphere with radius 1 centered at the origin. Note that $\tilde{a}(-\beta, x) = -\tilde{a}(\beta, x)$, and $a(x) \in \Sigma$ if and only if there is $\beta \in S^{k-1}$ such that $\tilde{a}(\beta, x) = 0$. Moreover with each $x \in a^{-1}(\Sigma^i)$ correspond exactly two elements $(\beta, x), (-\beta, x) \in \tilde{a}^{-1}(0)$.

According to [10] Theorem 2.3, $a \cap \Sigma^1$ if and only if 0 is a regular value of the mapping $\tilde{a}$.

3.1 The case $n-k$ even

Let $n-k > 0$ be an even number and let $M$ be an $(n-k+1)$–dimensional oriented compact manifold with boundary. In this case $\Sigma^i$ are orientable (see [2]).

In [11] the authors defined a homotopy invariant $\Lambda$ associated with a map from an $(n-k)$–dimensional boundaryless manifold into the Stiefel manifold $\tilde{V}_k(\mathbb{R}^n)$. When this manifold is a sphere, $\Lambda$ induces an isomorphism between $\mathbb{Z}$ and $(n-k)$–th homotopy group of $\tilde{V}_k(\mathbb{R}^n)$. In our case $\Lambda(a|\partial M)$ is well defined. According to [11] Section 2

$\Lambda(a|\partial M) = \frac{1}{2} \deg(\tilde{a}|S^{k-1} \times \partial M),$
where $\tilde{a}|S^{k-1} \times \partial M : S^{k-1} \times \partial M \to \mathbb{R}^n \setminus \{0\}$.

**Theorem 3.1.** Let $n-k > 0$ be even. There is an orientation of $\Sigma^1$ such that for each smooth manifold $M$ ($(n-k+1)$-dimensional, oriented, compact, with boundary) and smooth $a : M \to M_k(\mathbb{R}^n)$ with $a(\partial M) \cap \Sigma = \emptyset$ we have

$$I(a, \Sigma) = \Lambda(a|\partial M).$$

In the case where $a^{-1}(\Sigma) = a^{-1}(\Sigma^1)$ is a finite set,

$$I(a, \Sigma) = \frac{1}{2} \sum_{(\beta, x) \in \tilde{a}^{-1}(0)} \deg(\tilde{a}, (\beta, x)).$$

**Proof.** At the beginning of the proof we fix the orientation of $\Sigma^1$. In the second step we prove that for the mapping $a$ transversal to $\Sigma^1$ the intersection number of $a$ and the local topological degree of $\tilde{a}$ locally coincide. At the end we have to use a technical lemma saying that each point of the intersection of $a$ with $\Sigma^1$ can be transformed to the form to which we can apply the second step of the proof.

One may represent any matrix $S \in M_k(\mathbb{R}^n)$ as

$$S =\begin{bmatrix} A_{(n-k+1) \times 1} & B_{(n-k+1) \times (k-1)} \\ C_{(k-1) \times 1} & D_{(k-1) \times (k-1)} \end{bmatrix}. \tag{2}$$

If $\det D \neq 0$, then by [3] Chapter II, Lemma 5.2, Proposition 5.3 $S \in \Sigma^1$ if and only if $A - BD^{-1}C = 0$. Moreover the map $f(S) = A - BD^{-1}C \in \mathbb{R}^{n-k+1}$ is such that $f^{-1}(0)$ is locally a complete intersection and coincides with $\Sigma^1$. Then in a neighbourhood of matrices such that $\det D > 0$ the map $f$ gives a natural orientation of $\Sigma^1$ in the following way: vectors $v_{n-k+2}, \ldots, v_{nk}$ are well-oriented in the tangent space $T_S \Sigma^1$ if and only if vectors $\nabla f_1(S), \ldots, \nabla f_{n-k+1}(S), v_{n-k+2}, \ldots, v_{nk}$ are well-oriented in $\mathbb{R}^{nk}$ (there $\nabla f_i$ denotes the gradient of $i$-th coordinate of $f$). Note that $\Sigma^1$ is connected, and since $n-k$ is even, it is also orientable. Let $\theta$ be the orientation of $\Sigma^1$ that agrees locally with the orientation defined above. From now on we treat $\Sigma^1$ as an oriented manifold $(\Sigma^1, (-1)^{k-1}\theta)$.

By elementary column and row operations each matrix $S \in \Sigma^1$ can be transformed to the form (2), where $A$, $B$, $C$ are zero matrices, and $\det D > 0$.

It is sufficient to show the conclusion for $a \pitchfork \Sigma$.

Let us assume that $a \pitchfork \Sigma$. Then $a^{-1}(\Sigma) = a^{-1}(\Sigma^1)$ is a finite set. With each point $x \in a^{-1}(\Sigma)$ we can associate $(\beta, x), (-\beta, x) \in \tilde{a}^{-1}(0)$. According to [10] Theorem 2.3 $(\beta, x), (-\beta, x)$ are regular points. By [11] Proposition 2.4

$$\deg(\tilde{a}, (\beta, x)) = \deg(\tilde{a}, (-\beta, x)),$$

moreover $\Lambda(a|\partial M) = \frac{1}{2} \sum_{(\beta, x) \in \tilde{a}^{-1}(0)} \deg(\tilde{a}, (\beta, x)).$
Assume that \( x \in a^{-1}(\Sigma) \) is such that
\[
a(x) = \begin{bmatrix} 0 \\ \vdots \\ 0_{(n-k+1) \times (k-1)} \\ 0 \\ \vdots \\ D_{(k-1) \times (k-1)} \end{bmatrix},
\]
where \( \det D > 0 \).

Since \( x \) is fixed, we may treat \( a \) near \( x \) as a mapping \((\mathbb{R}^{n-k+1}, 0) \to M_k(\mathbb{R}^n)\), using a local coordinate system.

Let us remind that \( I(a, \Sigma)_x \) depends on \( \eta: T_x M \to T_{a(x)} M_k(\mathbb{R}^n)/T_{a(x)} \Sigma^1 \). It is easy to verify that the orientation of \( T_{a(x)} M_k(\mathbb{R}^n)/T_{a(x)} \Sigma^1 \) is given by \( n - k + 1 \) vectors \((-1)^{k-1}, 0, \ldots, 0, 0, 1, \ldots, 0, \ldots, 0, 1, 0, \ldots, 0\). So
\[
I(a, \Sigma)_x = (-1)^{k-1} \sgn \det \left[ \frac{\partial(a_1^1, \ldots, a_i^{n-k+1})}{\partial x} \right] (x).
\]
Note that \((\pm 1, 0, \ldots, 0, x) \in \tilde{a}^{-1}(0)\). As in [15] Lemma 3.2 we obtain
\[
\deg(\tilde{a}, ((1, 0, \ldots, 0), x)) = \deg((\beta_1^2 + \ldots + \beta_k^2 - 1, \tilde{a}), ((1, 0, \ldots, 0), x)) = \sgn \det \left[ \begin{array}{cccc} 1 & 0 & \ldots & 0 \\ & a(x) & \vdots & 0 \\ & \frac{\partial(a_1^1, \ldots, a_i^k)}{\partial x} (x) & \end{array} \right] .
\]
Since \( \det D > 0 \), we can reformulate the last term in the following way:
\[
\sgn \det \left[ \begin{array}{cccc} 1 & 0 & \ldots & 0 \\ & a(x) & \vdots & 0 \\ & \frac{\partial(a_1^1, \ldots, a_i^k)}{\partial x} (x) & \end{array} \right] = (-1)^{k-1} \sgn \det \left[ \frac{\partial(a_1^1, \ldots, a_i^{n-k+1})}{\partial x} \right] (x) = I(a, \Sigma)_x.
\]
To prove that \( I(a, \Sigma) = \sum_{x \in a^{-1}(\Sigma)} I(a, \Sigma)_x = \frac{1}{2} \sum_{(\beta, x) \in \tilde{a}^{-1}(0)} \deg(\tilde{a}, (\beta, x)) \) it is sufficient to show, that the composition of elementary column and row operations preserves the equality between \( \deg(\tilde{a}, (\beta, x)) \) and \( I(a, \Sigma)_x \) for \( (\beta, x) \in \tilde{a}^{-1}(0) \), as is proved in the next Lemma.

**Lemma 3.2.** Let \( \Phi: M_k(\mathbb{R}^n) \to M_k(\mathbb{R}^n) \) be an elementary column or row operation, and \( x \in a^{-1}(\Sigma^1) \), \( (\beta, x) \in \tilde{a}^{-1}(0) \). Assume that \( I(a, \Sigma)_x = \deg(\tilde{a}, (\beta, x)) \). Then there exists such \( \tilde{\beta} \) that \( \Phi(a)(\tilde{\beta}, x) = 0 \), and
\[
I(\Phi(a), \Sigma)_x = \deg(\tilde{\Phi}(a), (\tilde{\beta}, x)).
\]
Proof. It is obvious that \( \Phi \) and \( \Phi|\Sigma^1 : \Sigma^1 \to \Sigma^1 \) are diffeomorphisms. Note that the following diagram commutes.

\[
\begin{array}{c}
T_x M & \xrightarrow{\eta_1} & T_{a(x)} M_k(\mathbb{R}^n) / T_{a(x)} \Sigma^1 \\
\downarrow & & \downarrow \\
T_x M & \xrightarrow{\eta_2} & T_{\Phi(a)(x)} M_k(\mathbb{R}^n) / T_{\Phi(a)(x)} \Sigma^1
\end{array}
\]

So \( I(a, \Sigma)_x = (\text{sgn } s) \cdot I(\Phi(a), \Sigma)_x \), where \( s \) depends on the way \( \Phi \) acts on \( M_k(\mathbb{R}^n) \) and \( \Sigma^1 \).

One can check, that

1. if \( \Phi \) multiplies one column or row by \( c \neq 0 \), then
   - \( \Phi \) reverses the orientation on \( M_k(\mathbb{R}^n) \) if and only if \( c < 0 \) and \( n \) is odd,
   - \( \Phi|\Sigma^1 \) reverses the orientation on \( \Sigma^1 \) if and only if \( c < 0 \) and \( n \) is even,

   and in this case \( s = c \);

2. if \( \Phi \) interchanges two subsequent columns or rows, then
   - \( \Phi \) reverses the orientation on \( M_k(\mathbb{R}^n) \) if and only if \( n \) is odd,
   - \( \Phi|\Sigma^1 \) reverses the orientation on \( \Sigma^1 \) if and only if \( n \) is even,

   and in this case \( s = -1 \);

3. if \( \Phi \) replaces the second column (or row) by the sum of the first and second column (or row), then \( \Phi \) and \( \Phi|\Sigma^1 \) always preserves the orientation, and in this case \( s = 1 \).

To verify whether \( \Phi|\Sigma^1 \) preserves or reverses the orientation, it is enough to check it at some fixed point of \( \Phi|\Sigma^1 \).

Note that there exist two diffeomorphisms \( \Psi_1 : S^{k-1} \to S^{k-1}, \Psi_2 : \mathbb{R}^n \to \mathbb{R}^n \) such that the following diagram commutes

\[
\begin{array}{c}
S^{k-1} \times M & \xrightarrow{\bar{a}} & \mathbb{R}^n \\
(\Psi_1, \text{id}) \downarrow & & \downarrow \Psi_2 \\
S^{k-1} \times M & \xrightarrow{\hat{\Phi}(a)} & \mathbb{R}^n,
\end{array}
\]

and \( \bar{a}(\beta, x) = 0 \iff \hat{\Phi}(a)(\Psi_1(\beta), x) = 0 \). Note that if \( \Phi \) is a column operation, then \( \Psi_2 = \text{id} \), and if \( \Phi \) is a row operation, then \( \Psi_1 = \text{id} \).

So \( \text{deg}(\bar{a}, (\beta, x)) = (\text{sgn } t) \cdot \text{deg}(\hat{\Phi}(a), (\Psi_1(\beta), x)) \), where \( t \) depends on the way \( \Psi_1 \) and \( \Psi_2 \) acts on \( S^{k-1} \) and \( \mathbb{R}^n \). One can verify that \( \text{sgn } t \) and \( \text{sgn } s \) coincide. Since \( I(a, \Sigma)_x = \text{deg}(\bar{a}, (\beta, x)) \), we have \( I(\Phi(a), \Sigma)_x = \text{deg}(\hat{\Phi}(a), (\hat{\beta}, x)) \).
3.2 The case \( n - k \) odd

Let \( n - k > 0 \) be an odd number and let \( M \) be an \((n - k + 1)\)-dimensional compact manifold with boundary, not necessarily orientable. In this case \( \Sigma^i \) are non–orientable (see [2]).

**Theorem 3.3.** Let us assume that \( a : M \to M_k(\mathbb{R}^n) \) is such that \( a(\partial M) \cap \Sigma = \emptyset \). The set \( \tilde{a}^{-1}(0) \) is finite if and only if \( a^{-1}(\Sigma) = a^{-1}(\Sigma^1) \) and \( a^{-1}(\Sigma^1) \) is finite. If that is the case, then

\[
I_2(a, \Sigma) = \sum_{(\beta, x)} \deg_2(\tilde{a}, (\beta, x)) \mod 2,
\]

where \((\beta, x) \in \tilde{a}^{-1}(0)\) and we choose only one from each pair \((\beta, x), (\beta, x) \in \tilde{a}^{-1}(0)\).\((\beta, x) \) runs through half of the zeros of \( \tilde{a} \).

**Proof.** It is sufficient to show the conclusion for \( a \cap \Sigma \). Note that then 0 is a regular value of \( \tilde{a} \) (see [10]).

Take \( a \cap \Sigma \), then \( I_2(a, \Sigma) = \#a^{-1}(\Sigma^1) = \frac{1}{2} \#\tilde{a}^{-1}(0) \mod 2 \). Since 0 is a regular value of \( \tilde{a} \), \( \deg_2(\tilde{a}, (\beta, x)) = 1 \) for each \((\beta, x) \in \tilde{a}^{-1}(0)\). Hence \( I_2(a, \Sigma) = \sum_{(\beta, x)} \deg_2(\tilde{a}, (\beta, x)) \mod 2 \), where \((\beta, x) \) runs through half of the zeros of \( \tilde{a} \). \( \square \)

Note that \( \partial(S^{k-1} \times M) = S^{k-1} \times \partial M, \tilde{a}|_{S^{k-1} \times \partial M} : S^{k-1} \times \partial M \to \mathbb{R}^n \setminus \{0\} \)
and the modulo 2 topological degree of \( \tilde{a}|_{S^{k-1} \times \partial M} \) is well defined, but for \( n - k \) odd it is always equal to 0, see [11].

**Remark 3.4.** If \( n - k \) is even and \( M \) is non–orientable, then \( I_2(a, \Sigma) \) is defined, and Theorem 3.3 still holds true.

In [10] the authors defined a homotopy invariant \( \Lambda \) associated with a map from an \((n - k)\)-dimensional sphere \( S^{k-1} \) into the Stiefel manifold \( \tilde{V}_k(\mathbb{R}^n), n - k \) odd. This \( \Lambda \) induces an isomorphism between \((n - k)\)-th homotopy group of \( \tilde{V}_k(\mathbb{R}^n) \) and \( \mathbb{Z}_2 \).

Now let us assume that \( M = B^{n-k+1} \), the \((n - k + 1)\)-dimensional ball. Then \( a(S^{n-k}) \cap \Sigma = \emptyset \), and \( \Lambda(a|_{S^{n-k}}) \) is well defined. According to Theorem 3.3 and [10, Section 2.], we get

**Corollary 3.5.**

\[
\Lambda(a|_{S^{n-k}}) = I_2(a, \Sigma).
\]

4 Counting the intersection number for polynomial mappings

In this Section we present the method to compute \( I_2(a, \Sigma) \) in a polynomial case. Since \( I(a, \Sigma) = \Lambda(a|\partial M) \), to compute \( I(a, \Sigma) \) for polynomial mappings one may use [11, Theorem 3.3.].
Take polynomial mappings \( h = (h_1, \ldots, h_l) : \mathbb{R}^{n-k+1+l} \to \mathbb{R}^l \) and \( g : \mathbb{R}^{n-k+1+l} \to \mathbb{R} \), \( n - k > 0 \) odd. Let us assume that \( h^{-1}(0) \) is a complete intersection (i.e. \( h^{-1}(0) \neq \emptyset \) and \( \text{rank } dh(x) = l \) for \( x \in h^{-1}(0) \)) and \( M = \{g \geq 0\} \cap h^{-1}(0) \). Take a polynomial mapping \( a : M \to M_k(\mathbb{R}^n) \), such that \( a(\partial M) \cap \Sigma = \emptyset \). As in [15] Lemma 3.2 for an isolated zero \((\beta, x) \in \tilde{a}^{-1}(0)\) we have

\[
\deg(\bar{a}, (\beta, x)) = \deg((h, \bar{a}), (\beta, x)),
\]

where by \((h, \bar{a})\) we mean a mapping from \( S^{k-1} \times \mathbb{R}^{n-k+1+l} \) to \( \mathbb{R}^l \times \mathbb{R}^n \), and so the same equality holds for modulo 2 local topological degrees.

Let us take an ideal \( J \) in \( \mathbb{R}[x] = \mathbb{R}[x_1, \ldots, x_{n-k+1+l}] \) generated by \( h_1, \ldots, h_l \) and all \( k \times k \) minors of matrix \( a(x) \), and let the ideal \( J' \) be generated by \( h_1, \ldots, h_l \) and all \( (k-1) \times (k-1) \) minors of \( a(x) \). Put \( \mathcal{A} = \mathbb{R}[x] / J \). From now on we will assume that \( \dim_\mathbb{R} \mathcal{A} < \infty \), \( J + J' = \mathbb{R}[x] \), and \( \langle g \rangle = \mathbb{R}[x] \). From the first assumption we get that the zero set \( V(J) \) of \( J \) is finite, from the second one — that \( a^{-1}(\Sigma) = a^{-1}(\Sigma') \), and so \( \tilde{a}^{-1}(0) \) and \( a^{-1}(\Sigma) \) are finite sets. The last assumption implicates that \( a(\partial M) \cap \Sigma = \emptyset \).

Denote by \( \mathcal{O}_x^\mathcal{A} \) the ring of germs at \( x \in \mathbb{R}^n \) of analytic functions \( \mathbb{R}^n \to \mathbb{R} \), by \( \mathcal{O}_x^S \) the ring of germs at \( x \in S \) of analytic functions \( S \to \mathbb{R} \).

The following Proposition contains some results from [11] Section 3, adapted to our case.

**Proposition 4.1.** Take \( p \in V(J) \) and \((\beta, p) \in \tilde{a}^{-1}(0)\). One can construct a polynomial mapping \( F = (F_1, \ldots, F_n) : \mathbb{R}^{k-1} \times \mathbb{R}^{n-k+1+l} \to \mathbb{R}^n \) such that there is such \( \lambda \in \mathbb{R}^{k-1} \) that \((\lambda, p)\) is an isolated zero of \((h, F)\) with following properties:

\[
\begin{align*}
\bullet & \quad \deg_2(\bar{a}, (\beta, p)) = \deg_2((h, F), (\lambda, p)), \\
\bullet & \quad \mathcal{O}_p^{n-k+1+l} / J \simeq \mathcal{O}_p^{n+l} / \langle h, F \rangle.
\end{align*}
\]

**Proof.** Take \( p \in V(J) \) and \((\beta, p) \in \tilde{a}^{-1}(0)\). Analogically as in [11], one can construct a polynomial mapping \( F \) such that \( \deg_2(\bar{a}, (\beta, p)) = \deg_2(F|_{(\mathbb{R}^{k-1} \times h^{-1}(0))}) \), for some \( \lambda \). Then by applying [15] Lemma 3.2 we get \( \deg_2(F|_{(\mathbb{R}^{k-1} \times h^{-1}(0))}) \), \((\lambda, p)\)) = \deg_2((h, F), (\lambda, p))\).

Moreover in the neighbourhood of \( p \) there is \( \lambda(x) = (\lambda_2(x), \ldots, \lambda_k(x)) \) such that \( F_{i_1}(\lambda(x), x) = \ldots = F_{i_k}(\lambda(x), x) = 0, 1 \leq i_1 < \ldots < i_k \leq n \). As in [11] \( \Gamma = \{ (\lambda(x), x) \mid x \in h^{-1}(0) \} \) is an \((n-k+1)\)-dimensional manifold, and we put \( \Omega(x) = (\Omega_1(x), \ldots, \Omega_n(x)) = F(\lambda(x), x) \).

It is easy to see that

\[
\mathcal{O}_p^n / \langle h, F \rangle \simeq \mathcal{O}_p^{\Omega} / \langle F \rangle \simeq \mathcal{O}_p^{h^{-1}(0)} / \langle \Omega \rangle \simeq \mathcal{O}_p^{n-k+1+l} / \langle h, \Omega \rangle.
\]

From [11] Lemma 3.7 we get that \( J = \langle h, \Omega \rangle \) in \( \mathcal{O}_p^{n-k+1+l} \), and then

\[
\mathcal{O}_p^{n+l} / \langle h, F \rangle \simeq \mathcal{O}_p^{n-k+1+l} / J.
\]

\( \square \)
Theorem 4.2. Take polynomial mappings $h = (h_1, \ldots, h_l) : \mathbb{R}^{n-k+1+l} \rightarrow \mathbb{R}^l$ and $g : \mathbb{R}^{n-k+1+l} \rightarrow \mathbb{R}$, $n-k > 0$ odd, such that $h^{-1}(0)$ is a complete intersection and $M = \{ g > 0 \} \cap h^{-1}(0)$. For a polynomial $a : M \rightarrow M_k(\mathbb{R}^n)$, let us assume that $\dim \mathbb{R} A < \infty$, $J + J' = \mathbb{R}[x]$, $J + \langle g \rangle = \mathbb{R}[x]$. Then for any linear functional $\varphi : A \rightarrow \mathbb{R}$ and $\Phi, \Psi$ - the bilinear symmetric forms on $A$ given by $\Phi(f_1, f_2) = \varphi(f_1 f_2)$, $\Psi(f_1, f_2) = \varphi(g f_1 f_2)$ such that $\det[\Psi] \neq 0$, we have $\det[\Phi] \neq 0$ and

$$I_2(a, \Sigma) = \dim \mathbb{R} A + 1 + \frac{1}{2}(\text{sgn det}[\Phi] + \text{sgn det}[\Psi]) \mod 2,$$

where $[\Phi]$ denotes the matrix of the form $\Phi$.

Proof. Let $V(J) = \{p_1, \ldots, p_m\}$. There is an even–dimensional algebra $D$ such that the natural projection

$$A \rightarrow \bigoplus_{i=1}^m O_{p_i}^{n-k+1+l} / J \oplus D$$

is an isomorphism of $\mathbb{R}$–algebras (see e.g. [15, Section 1.]), and so $\dim \mathbb{R} A = \sum_i \dim \mathbb{R} O_{p_i}^{n-k+1+l} / J \mod 2$.

By Proposition 4.1 we have an isomorphism

$$\bigoplus_{i=1}^m O_{(\lambda_i, p_i)}^{n+l} / (h, F) \oplus D \simeq A.$$

According to [15, Theorem 2.3.] we get

$$\sum_{i : g(p_i) > 0} \deg_2((h, F), (\lambda_i, p_i)) = \dim \mathbb{R} \left( \bigoplus_{i=1}^m O_{(\lambda_i, p_i)}^{n+l} / (h, F) \right) + 1 + \frac{1}{2}(\text{sgn det}[\Phi] + \text{sgn det}[\Psi]) \mod 2 = \dim \mathbb{R} A + 1 + \frac{1}{2}(\text{sgn det}[\Phi] + \text{sgn det}[\Psi]) \mod 2.$$

Since by Propositions 3.3, 4.1 $I_2(a, \Sigma) = \sum_{i : g(p_i) > 0} \deg_2((h, F), (\lambda_i, p_i))$, we get the conclusion.

Remark 4.3. If we take $g = 1$ then $M = h^{-1}(0)$ is a manifold without boundary, $\Phi = \Psi$, and we get that $I_2(a, \Sigma) = \dim \mathbb{R} A = 0 \mod 2$.

Remark 4.4. One can also use Theorem 4.2 as a simple way to compute $I(a, \Sigma)$ (and so the invariant from [11]) modulo 2 in the case where $n-k$ is even (because $I(a, \Sigma) \mod 2 = I_2(a, \Sigma)$).

Using SINGULAR ([4]) and previous Theorem we present the following examples.
Example 4.5. Let $M$ be a half of a 2-dimensional sphere $(g(x, y, z) = -z, h(x, y, z) = x^2 + y^2 + z^2 - 1: \mathbb{R}^3 \to \mathbb{R})$. Take $a: M \to M_2(\mathbb{R}^3)$ as

$$a(x) = \begin{bmatrix}
10x^2z + 4xyz + 10x & 2x^2y + 7x^2 + 6y \\
6xz^2 + 4y^2 + 7y & 3z^2 + 10z^2 + 9z \\
4x^3 + 2z^2 + 7x & 7xz^2 + 7xy + 2y
\end{bmatrix}.$$ 

Then the dimension of the algebra $A$ equals 48, and $I_2(a, \Sigma) = 0 \mod 2$.

Example 4.6. Let $M$ be a half of a 2-dimensional torus $(g(x, y, z, w) = z, h(x, y, z, w) = (1 - x^2 - w^2, 1 - y^2 - z^2): \mathbb{R}^4 \to \mathbb{R}^2)$. Take $a: M \to M_2(\mathbb{R}^3)$ as

$$a(x) = \begin{bmatrix}
w^3x + 7w^2x + 8y^2 + 8z & y^4 + 7xz^2 + 6z^2 + 3x \\
xyz^2 + 5wxy + 10wx + 4y & x^2y^2 + 8wy^2 + 2wy + 9w \\
2x^2y^2 + 7x^3 + z^2 + 10y & 5w^4 + 8wxy + 6w^2 + 8y
\end{bmatrix}.$$ 

Then the dimension of the algebra $A$ equals 184, and $I_2(a, \Sigma) = 1 \mod 2$.

5 Applications. Counting the number of cross-caps modulo 2 and the algebraic sum of them

At the beginning of this section we will present three technical lemmas and then we switch to the applications.

Lemma 5.1. Let $U \subset \mathbb{R}^{n-k+1}$ be an open set, $n - k > 0$, $a: U \to M_k(\mathbb{R}^n)$. Take $x \in U$ such that $a(x) \in \Sigma^1$. Let us assume that $a(x)$ has the following form:

$$a_1(x) = (0, \ldots, 0) \text{ and } a_i(x) = (0, \ldots, 0, a_i^{n-k+2}(x), \ldots, a_i^n(x))$$

for $i = 2, \ldots, k$ (here $a_i^j$ is the element standing in the $j$-th row and $i$-th column). Then $a \cap \Sigma^1$ at $x$ if and only if

$$\text{rank } \left[ \frac{\partial(a_1, \ldots, a_1^{n-k+1})}{\partial(x_1, \ldots, x_{n-k+1})}(x) \right] = n - k + 1.$$

Proof. As in the proof of Theorem 3.1 one can show that the tangent space $T_{a(x)}\Sigma^1$ is spanned by vectors $v_i = (0, \ldots, 0, 1, \ldots, 0)$, where 1 stands at $(i + n - k + 1)$-th place, $i = 1, \ldots, nk - (n - k + 1)$.

Let us observe that $a \cap \Sigma^1$ at $x$ if and only if rank $da(x)$ is maximal (i.e. equals $n - k + 1$) and $T_{a(x)}\Sigma^1 \cap da(x)\mathbb{R}^{n-k+1} = \{0\}$. It is equivalent to the condition:

$$\text{rank } \left[ \frac{\partial(a_1, \ldots, a_1^{n-k+1})}{\partial(x_1, \ldots, x_{n-k+1})}(x) \right] = n - k + 1.$$ 

$\square$
Lemma 5.2. Let $U \subset \mathbb{R}^{n-k+1}$ be an open set, $n-k > 0$, $a: U \to M_k(\mathbb{R}^n)$, $b: U \to M_k(\mathbb{R}^s)$, $c: U \to M_s(\mathbb{R}^{n+s})$. We define $e: U \to M_{k+s}(\mathbb{R}^{n+s})$ as

$$e(x) = \begin{bmatrix} b(x) \\ a(x) \\ c(x) \end{bmatrix}.$$ Let us assume that for each $x \in U$ we have $\text{rank } a(x) = \text{rank } \begin{bmatrix} b(x) \\ a(x) \end{bmatrix}$ and $s + \text{rank } a(x) = \text{rank } e(x)$. Then $a \cap \Sigma^1 \subset M_k(\mathbb{R}^n)$ at $x$ if and only if $e \cap \Sigma^1 \subset M_{k+s}(\mathbb{R}^{n+s})$ at $x$.

Proof. Take $x \in U$ such that $a(x) \in \Sigma^1$. Note that then also $e(x) \in \Sigma^1$. By elementary column and row operations we may transform mappings $a$ and $e$ to such forms, that the first $n-k+1$ elements of the first column of both of them coincide, and at the point $x$ matrices $a(x)$ and $e(x)$ have the following forms:

$$a(x) = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \\ D_{(k-1) \times (k-1)} \end{bmatrix}, \quad e(x) = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ E_{s \times s} \end{bmatrix},$$

where $D$ and $E$ have maximal rank. From the previous Lemma we obtain the conclusion. \hfill \Box

Lemma 5.3. Let $U \subset \mathbb{R}^{n-k+1}$ be an open set, $n-k > 0$, $a: U \to M_k(\mathbb{R}^n)$, $b: U \to M_k(\mathbb{R}^k)$. We define $ab: U \to M_k(\mathbb{R}^n)$ as $ab(x) = a(x) \cdot b(x)$. Let us assume that for each $x \in U$ we have $\text{rank } b(x) = k$. Then $a \cap \Sigma^1$ at $x$ if and only if $ab \cap \Sigma^1$ at $x$.

For $\bar{x}$ such that $a \cap \Sigma^1$ at $\bar{x}$, if $\det b(\bar{x}) > 0$, then $I(a, \Sigma)_{\bar{x}} = I(ab, \Sigma)_{\bar{x}}$, resp. $I_2(a, \Sigma)_{\bar{x}} = I_2(ab, \Sigma)_{\bar{x}}$.

Proof. Take $x \in U$ such that $a(x) \in \Sigma^1$. Note that then also $ab(x) \in \Sigma^1$. By elementary column and row operations we may transform mappings $a$ and $ab$ to such forms, that at the point $x$ matrices $a(x)$ and $ab(x)$ have the following forms:

$$a(x) = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \\ D_{(k-1) \times (k-1)} \end{bmatrix}, \quad ab(x) = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \\ \vdots \\ \bar{D}_{(k-1) \times (k-1)} \end{bmatrix},$$
where $D$ and $\tilde{D}$ have maximal rank.

Note that $(ab)^i_j = \sum_{r=1}^{k} a^i_r b^r_j$. From the forms of the matrices $a(x)$ and $ab(x)$ we get $b^1_j(x) \neq 0$, $b^k_j(x) = \ldots = b^k_k(x) = 0$ for $j = 1, \ldots, n - k + 1$, $r = 1, \ldots, k$. Hence

$$
\left[\frac{\partial((ab)^1_1, \ldots, (ab)^{n-k+1}_{1-1})}{\partial(x_1, \ldots, x_{n-k+1})}(x)\right] = b^1_1(x) \cdot \left[\frac{\partial(a^1_1, \ldots, a^{n-k+1}_{1-1})}{\partial(x_1, \ldots, x_{n-k+1})}(x)\right].
$$

From Lemma 5.1 we obtain the conclusion.

There is $r > 0$ such that $\bar{B}(\bar{x}, r) \subset U$ and $a^{-1}(\Sigma) \cap \bar{B}(\bar{x}, r) = \{\bar{x}\}$. Since the set of $k \times k$–matrices with positive determinant is path–connected, then there exists a homotopy $h: \bar{B}(\bar{x}, r) \times [0, 1] \to M_k(\mathbb{R}^k)$ between the constant map equal the identity matrix and $b$. Then $ah: \bar{B}(\bar{x}, r) \times [0, 1] \to M_k(\mathbb{R}^n)$ given by

$$
ah(x, t) = a(x) \cdot h(x, t)
$$

is a homotopy between $a$ and $ab$ such that $ah^{-1}(\Sigma) \subset B(\bar{x}, r) \times [0, 1]$. Since $I$ and $I_2$ are homotopy invariants, we get the conclusion. 

Let $M$ be a smooth $m$–dimensional manifold. According to [5, 16, 17], a point $p \in M$ is a cross–cap of a smooth mapping $f : M \to \mathbb{R}^{2m-1}$, if there is a coordinate system near $p$, such that in some neighbourhood of $p$ the mapping $f$ has the form

$$(x_1, \ldots, x_m) \mapsto (x_1^2, x_2, \ldots, x_m, x_1 x_2, \ldots, x_1 x_m).$$

Take $f : M \to \mathbb{R}^{2m-1}$ with only cross–caps as singularities. By [17] Theorem 3], if $m$ is even and $M$ is a closed manifold, then $f$ has an even number of cross–caps.

Let $(M, \partial M)$ be an $m$–dimensional smooth compact manifold with boundary, $m$ even. Take a continuous mapping $f : [0, 1] \times M \to \mathbb{R}^{2m-1}$ such that there exists a neighbourhood of $\partial M$ in which all the $f_i$’s are regular, and $f_0, f_1$ have only cross–caps as singularities. According to [17] Theorem 4], mappings $f_0$ and $f_1$ have the same number of cross–caps $\mod 2$.

Let $(M, \partial M)$ be an $m$–dimensional smooth compact manifold with boundary, $m$ odd. Take a smooth mapping $f : M \to \mathbb{R}^{2m-1}$ and let $p \in M$ be a cross–cap of $f$. According to [17], there are coordinate systems near $p$ and $f(p)$, such that

$$
\frac{\partial f}{\partial x_1}(p) = 0
$$

and vectors

$$
\frac{\partial^2 f}{\partial x_1^2}(p), \frac{\partial f}{\partial x_2}(p), \ldots, \frac{\partial f}{\partial x_m}(p), \frac{\partial^2 f}{\partial x_1 \partial x_2}(p), \ldots, \frac{\partial^2 f}{\partial x_1 \partial x_m}(p)
$$

are linearly independent. The cross–cap $p$ is called positive (negative) if the vectors (5) determine the negative (positive) orientation of $\mathbb{R}^{2m-1}$. According
going from $R$ have the same algebraic sum of cross–caps. According to [17, Theorem 4], mappings $f_0$ and $f_1$ have the same algebraic sum of cross–caps.

For any smooth map $g: \mathbb{R}^n \to \mathbb{R}^s$ we consider its tangent map $dg$ as a map going from $\mathbb{R}^n$ to $M_s(\mathbb{R}^n)$.

Let $h = (h_1, \ldots, h_l): \mathbb{R}^{m+l} \to \mathbb{R}^l$ be a smooth map such that $h^{-1}(0) \neq \emptyset$ and rank $dh(x) = l$ for $x \in h^{-1}(0)$, i.e. $h^{-1}(0)$ is a complete intersection and so a smooth manifold of dimension $m$.

**Proposition 5.4.** Let $f = (f_1, \ldots, f_{2m-1}): \mathbb{R}^{m+l} \to \mathbb{R}^{2m-1}$ be a smooth map. Then $p \in h^{-1}(0)$ is a cross–cap of $f|h^{-1}(0)$ if and only if rank $d(h, f)(p) = m + l - 1$ and $(d(h, f))|h^{-1}(0) \cap \Sigma^1$ at $p$.

**Proof.** Note that if $p$ is a cross–cap of $f|h^{-1}(0)$, then rank $d(f|h^{-1}(0))(p) = m - 1$, and this holds if and only if rank $d(h, f)(p) = m + l - 1$. Let $p$ be such that rank $d(h, f)(p) = m + l - 1$.

Let us take a local coordinate system $\varphi: (\mathbb{R}^{m+l}, 0) \to (\mathbb{R}^{m+l}, p)$ such that $(h^{-1}(0), 0) = \varphi(\mathbb{R}^m \times \{0\}, 0)$. Note that $h \circ \varphi|\mathbb{R}^m \times \{0\}, 0) \equiv 0$.

The point $p$ is a cross–cap of $f|h^{-1}(0)$ if and only if $0$ is a cross–cap of $f \circ \varphi|(\mathbb{R}^m \times \{0\}, 0)$. By [10] Remark, p. 14 this takes place if and only if $d((f \circ \varphi)|\mathbb{R}^m \times \{0\}, 0)) \cap \Sigma^1$ at $0$. From Lemma 5.2 it holds if and only if $(d(h, f) \circ \varphi)|\mathbb{R}^m \times \{0\}, 0) \cap \Sigma^1$ at $0$.

Let us observe that $dh(\varphi(x)) \cdot \left[ \frac{\partial \varphi}{\partial y}(x) \right]$ is a square matrix with maximal rank, and

$$d((h, f) \circ \varphi)(0) = d(h, f)(p) \cdot d\varphi(0) = \begin{bmatrix} 0 & \frac{\partial dh}{\partial x}(0) \\ df(p) \cdot \frac{\partial \varphi}{\partial x}(0) & \ast \end{bmatrix},$$

where $(x, y)$ are coordinates in $\mathbb{R}^m \times \mathbb{R}^l = \mathbb{R}^{m+l}$.

Since $d((h, f) \circ \varphi)(x, 0) = d(h, f)(\varphi(x, 0)) \cdot d\varphi(x, 0)$, from Lemma 5.3 we obtain that $(d((h, f) \circ \varphi))|\mathbb{R}^m \times \{0\}, 0) \cap \Sigma^1$ at $0$ if and only if $(d(h, f)) \circ \varphi|(\mathbb{R}^m \times \{0\}, 0) \cap \Sigma^1$ at $0$, which is equivalent to $(d(h, f))|h^{-1}(0) \cap \Sigma^1$ at $p$.

**Proposition 5.5.** Take $p \in h^{-1}(0)$ such that rank $d(f|h^{-1}(0))(p) = m - 1$ (note that this occurs if and only if rank $d(h, f)(p) = m - 1 + l$). The point $p$ is a cross–cap of $f|h^{-1}(0)$ if and only if $p$ is a regular zero of the mapping $\mu|h^{-1}(0)$, where $\mu: \mathbb{R}^{m+l} \to \mathbb{R}^s$ is given at the point $(x, y)$ by all the $(m + l) \times (m + l)$–minors of $d(h, f)(x, y)$.

**Proof.** By Proposition 5.4 we get that $p$ is a cross–cap if and only if $d(h, f)|h^{-1}(0) \cap \Sigma^1 \subset M_{m+l}(\mathbb{R}^{2m-1+l})$ at $p$.

Take $\Phi: (M_{m+l}(\mathbb{R}^{2m-1+l}), d(h, f)(p)) \to \mathbb{R}^s$ such that $\Phi(A)$ is given by all $(m + l) \times (m + l)$–minors of $A$. Note that $\Phi^{-1}(0) = (\Sigma^1, d(h, f)(p))$. Similarly as
in [9] proof of Lemma 2] one can show that \( \text{rank } d\Phi(d(h, f)(p)) = m \). According to [9] Lemma 1 we obtain, that \( d(h, f)|h^{-1}(0) \cap \Sigma^1 \neq \emptyset \) at \( p \) if and only if \( \text{rank } d(\Phi \circ d(h, f)|h^{-1}(0))(p) = m \). Note that \( \Phi \circ d(h, f)|h^{-1}(0) = \mu|h^{-1}(0) \), so we get that \( p \) is a cross–cap of \( f|h^{-1}(0) \) if and only if \( p \) is a regular zero of \( \mu|h^{-1}(0) \). \( \square \)

Let us take a smooth map \( g: \mathbb{R}^{m+l} \rightarrow \mathbb{R} \) such that \( M = h^{-1}(0) \cap \{ g \geq 0 \} \) is an \( m \)–dimensional manifold with boundary. We fix an orientation of \( M \) as follows. Let \( x \in M \). We say, that vectors \( v_1, \ldots, v_m \in \mathbb{R}^{m+l} \) are well oriented in \( T_x M \) if and only if \( \nabla h_1(x), \ldots, \nabla h_{j}(x), v_1, \ldots, v_m \) are well oriented in \( \mathbb{R}^{m+l} \). From Proposition 5.4 we obtain the following.

**Theorem 5.6.** Let \( f: \mathbb{R}^{m+l} \rightarrow \mathbb{R}^{2m-1} \) be a smooth map such that \( f|M \) has no singular points near \( \partial M \). If \( m \) is **even** then the number of cross–caps in \( M \) of every smooth mapping \( \hat{f}: M \rightarrow \mathbb{R}^{2m-1} \) close enough to \( f|M \) with only cross–caps as singular points is congruent to \( I_2(d(h, f)|M, \Sigma) \mod 2 \).

**Proposition 5.7.** Let \( f: \mathbb{R}^{m+l} \rightarrow \mathbb{R}^{2m-1} \) be a smooth map such that \( f|M \) has only cross–caps as singular points and finitely many of them, moreover no singular points belongs to \( \partial M \). If \( m \) is **odd** then the algebraic sum of cross–caps of \( f|M \) is equal to 
\[
\frac{(-1)^{l+1}}{2} \sum_{\beta, p} \deg(d(h, f)|M, (\beta, p)),
\]
where \((\beta, p)'s\) are the zeroes of \( d(h, f)|M \).

**Proof.** In this proof we will use [10] Theorem 1 and Lemma 5.3. We will define some mappings to the space of matrices, and we will present connections between local invariants associated with them.

Let us take \( p \in M \) — a cross-cap of \( f|M \).

Take an orientation–preserving diffeomorphism \( \varphi: (\mathbb{R}^{m+l}, 0) \rightarrow (\mathbb{R}^{m+l}, p) \) such that \( \varphi|\mathbb{R}^m \times \{ 0 \}, 0 = (M, p) \). Put \( \varphi|\mathbb{R}^m(x) = \varphi(x, 0), x \in (\mathbb{R}^m, 0) \). Note that \( h \circ \varphi|\mathbb{R}^m \equiv 0 \).

One can choose \( \varphi \) (composing, if necessary, with an orientation–preserving linear diffeomorphism in the domain) such that the first \( m \) columns of the matrix \( d\varphi(0) \) are orthogonal. Then it is easy to check that the local coordinate system \( \varphi|\mathbb{R}^m \) of \( M \) is orientation–preserving if and only if \( \text{sgn } \det \left[ dh(p) \frac{\partial \varphi}{\partial y}(0) \right] = (-1)^l \).

According to [9] Theorem 1 the sign of \( p \) is equal to
\[
-\frac{1}{2} \left( \deg(d(f \circ \varphi|\mathbb{R}^m), (\gamma, 0)) + \deg(d(f \circ \varphi|\mathbb{R}^m), (-\gamma, 0)) \right),
\]
where \((\gamma, 0), (-\gamma, 0)\) are the only zeros of \( d(f \circ \varphi|\mathbb{R}^m) \).

Let us define \( a, c: (\mathbb{R}^m, 0) \rightarrow M_{m+l}(\mathbb{R}^{2m-l+1}) \) as \( c(x) = d(h, f)(\varphi(x, 0)) \) and \( a(x) = c(x) \cdot d\varphi(x, 0) \). Then both \( \tilde{a} \) and \( \tilde{c} \) have only two zeros, \( \tilde{a}^{-1}(0) = \{((\gamma, 0), 0), ((-\gamma, 0), 0)\} \), and from Lemma 5.3 all the local topological degrees of them at these zeros are equal.
Using the definition of the local topological degree, after some computations one can show that
\[-\frac{1}{2} \text{sgn det} \left[ dh(p) \frac{\partial \varphi}{\partial y}(0) \right] \left( \deg(d(f \circ \varphi)[\mathbb{R}^m], (\gamma, 0)) + \deg(d(f \circ \varphi)[\mathbb{R}^m], (-\gamma, 0)) \right) =\]
\[-\frac{1}{2} \left( \deg(\tilde{a}, ((\gamma, 0), 0)) + \deg(\tilde{a}, ((-\gamma, 0), 0)) \right).\]
We get that the sign of the cross-cap \(p\) equals
\[-\frac{1}{2} \text{sgn det} \left[ dh(p) \frac{\partial \varphi}{\partial y}(0) \right] \sum_{(\delta, 0) \in \tilde{c}^{-1}(0)} \deg(\tilde{c}, (\delta, 0)).\]

Note that \(\tilde{c}\) is \(d(h, f)|M\) composed with the local coordinate system \(\varphi|\mathbb{R}^m\), so local topological degrees of \(\tilde{c}\) and \(d(h, f)|M\) at its zeros \((\delta, 0)\) and \((\beta, 0)\) are equal if and only if \(\varphi|\mathbb{R}^m\) is orientation-preserving. Then we obtain the conclusion of this Proposition.

From Propositions 5.4 and 5.7 and Theorem 3.1 we obtain the following.

**Theorem 5.8.** Let \(m\) be odd. There is such an orientation of \(\Sigma^1 \subset M_{m^0}(\mathbb{R}^{2m-1})\) that the following is true.

Let \(f : \mathbb{R}^{m+1} \to \mathbb{R}^{2m-1}\) be a smooth map such that \(f|\mathring{M}\) has no singular points near \(\partial \mathring{M}\). The algebraic sum of cross-caps in \(M\) of every smooth mapping \(\hat{f} : M \to \mathbb{R}^{2m-1}\) close enough to \(f|\mathring{M}\) with only cross-caps as singular points is equal to \(1(d(h, f)|\mathring{M}, \Sigma)\).

Using SINGULAR (4), Theorems 4.2 and 5.6 and Proposition 5.5 we present the following examples.

**Example 5.9.** Let \(M\) be a half of the surface of genus 2 (double torus), where \(g(x, y, z) = z, h(x, y, z) = (x(x - 1)^2(x - 2) + y^2)^2 + z^2 - 0, 01 : \mathbb{R}^3 \to \mathbb{R}\). Take \(f(x, y, z) = (6yz + 2x, 6yz + 4y, 8z^2 + 5z) : \mathbb{R}^3 \to \mathbb{R}^3\). Then \(f|\mathring{M}\) has only cross-caps as singular points and its number equals \(0\) mod \(2\).

**Example 5.10.** Let \(M\) be as in the previous example. Take \(f(x, y, z) = (x^2 + 8x, 10xy + 10x, 9z^2 + 9z) : \mathbb{R}^3 \to \mathbb{R}^3\). The map \(f|\mathring{M}\) has not only cross-caps as singularities, although every map close enough to \(f|\mathring{M}\) with only cross-caps as singularities has the number of cross-caps congruent to \(1\) modulo \(2\).

**Example 5.11.** Let \(h(x, y, z, w) = x^2 + y^2 + z^2 - w^2 - 1, \) and \(g_r(x, y, z, w) = r^2 - x^2 - y^2 - z^2 - w^2\). Let us define \(M_r = h^{-1}(0) \cap \{g_r \geq 0\}\). Then for \(r > 1\), \(M_r\) is a 3-dimensional manifold with boundary, and the boundary has \(2\) connected components. Put \(f : \mathbb{R}^4 \to \mathbb{R}^5\) as \(f(x, y, z, w) = (8z^2 - 4x + 4y, 4xy + 4z, xz - 7y + 3w, z^2 - 3x + 8y, zw^2 + y^2)\). Then mappings \(f|M_2, f|M_3, f|M_{10}\) have only cross-caps as singular points, and no singular points on the boundary. Moreover the algebraic sum of cross-caps of \(f|M_2\) equals \(1\), of \(f|M_3\) equals \(0\), and of \(f|M_{10}\) equals \(-3\).
6 \ I(\alpha, \Sigma) as the index of section of vector bundle

In this Section we will show, that \( I_2(\alpha, \Sigma) \) can be represented as an intersection number of some section of a vector bundle associated with \( \alpha \).

Let \( B \) be a smooth compact manifold with boundary, \( \dim B = n \). Let \( \xi = (p, E, B) \) be an \( n \)-dimensional vector bundle. Then \( E \) is also a manifold with boundary, and if \( s: B \to E \) is a section of \( \xi \) then the boundary \( \partial s(B) = s(\partial B) = \partial E \cap s(B) \).

Let us assume that \( s: B \to E \) is a continuous section of \( \xi \) that has the property
\[
\forall x \in \partial B \quad s(x) \notin z(B),
\]
where \( z: B \to E \) is a zero section of \( \xi \). Then we have
\[
s(\partial B) \subset \partial E \setminus z(B).
\]

We define the index \( I_2(s) \) of a section \( s \) as \( I_2(s, z(B)) \) — the intersection number mod 2 of \( s \) and \( z(B) \) (see [8, Chapter 5] or Section 2.5). More precisely, if \( \tilde{s}: B \to E \) is such a section of \( \xi \) that is a small smooth perturbation of \( s \) which is transversal to \( z(B) \), then the index of \( s \) is defined as
\[
I_2(s) = \# \tilde{s}^{-1}(z(B)) \mod 2.
\]

Since the inverse image of \( z(B) \) under \( \tilde{s} \) is a 0-dimensional manifold, and \( B \) is compact, it is finite.

**Remark 6.1.** If also the inverse image \( s^{-1}(z(B)) \) is finite, say \( s^{-1}(z(B)) = \{x_1, x_2, \ldots, x_r\} \), then one can choose open pairwise disjoint neighbourhoods \( U_i \ni x_i \) such that there are local trivializations \( \varphi_i: p^{-1}(U_i) \to U_i \times \mathbb{R}^n \). Let us define a map \( F_i = \pi_2 \circ \varphi_i \circ s|_{U_i}: U_i \to \mathbb{R}^n \), so
\[
F_i: U_i \xrightarrow{s} p^{-1}(U_i) \xrightarrow{\varphi_i} U_i \times \mathbb{R}^n \xrightarrow{\pi_2} \mathbb{R}^n.
\]
Then \( F_i^{-1}(0) = \{x_i\} \), and mod 2 topological degree \( \deg_2 F_i \) of \( F_i \) is defined.

**Proposition 6.2.** If the inverse image \( s^{-1}(z(B)) \) is finite then
\[
I_2(s) = \sum_{i=1}^{r} \deg_2 F_i \mod 2.
\]

**Proof.** If we choose a section \( \tilde{s} \) sufficiently close to \( s \), then the maps \( \tilde{F}_i = \pi_2 \circ \varphi_i \circ \tilde{s}|_{U_i} \) and \( F_i \) are also close enough, and
\[
\deg_2 F_i = \deg_2 \tilde{F}_i \mod 2.
\]
The origin is a regular value of $\hat{F}_i$, so
\[
\sum_{i=1}^{r} \deg_2 \hat{F}_i = \# \hat{s}^{-1}(z(B)) \mod 2.
\]

We get $I_2(s) = \sum_{i=1}^{r} \deg_2 F_i \mod 2$. \qed

By [8, Chapter 5, Theorem 2.1] we obtain

**Corollary 6.3.** If $s_1, s_2 : B \to E$ are sections of $\xi$ that fulfill the condition (6) and are homotopic by a homotopy that fulfills (4) at every stage, then
\[
I_2(s_1) = I_2(s_2) \mod 2.
\]

Let $n - k > 0$ be an odd number, and let $M$ be a smooth compact manifold with boundary, possibly non-orientated, dim $M = n - k + 1$. Take $a : M \to M_k(\mathbb{R}^n)$ — a continuous mapping such that the restriction $a|\partial M : \partial M \to \tilde{V}_k(\mathbb{R}^n)$ (i.e. $a(\partial M) \cap \Sigma = \emptyset$), and define $\tilde{a} : S^{k-1} \times M \to \mathbb{R}^n$ as before.

Let us take a trivial vector bundle $\xi = (p, S^{k-1} \times M \times \mathbb{R}^n, S^{k-1} \times M)$, where $p$ is the projection. Let $z : S^{k-1} \times M \to S^{k-1} \times M \times \mathbb{R}^n$, $z(\beta, x) = (\beta, x, 0)$ be a zero section of $\xi$.

We have $p(\beta, x, \tilde{a}(\beta, x)) = (\beta, x)$, so
\[
(id, \tilde{a}) : S^{k-1} \times M \to S^{k-1} \times M \times \mathbb{R}^n
\]
is a section of $\xi$. Moreover it fulfills the condition (6) — if $(\beta, x) \in S^{k-1} \times \partial M$ then $(id, \tilde{a})(\beta, x) = (\beta, x, \tilde{a}(\beta, x)) \notin z(S^{k-1} \times M)$, because $a(x) \in \tilde{V}_k(\mathbb{R}^n)$.

We define the action of the group $\mathbb{Z}_2 = \{-1, 1\}$ on $S^{k-1} \times M \times \mathbb{R}^n$ and on $S^{k-1} \times M$ as
\[
1 \cdot (\beta, x, v) = (\beta, x, v), (-1) \cdot (\beta, x, v) = (-\beta, x, -v)
\]
\[
1 \cdot (\beta, x) = (\beta, x), (-1) \cdot (\beta, x) = (-\beta, x)
\]

These actions are continuous, free, and we have $p(\pm 1 \cdot (\beta, x, v)) = \pm 1 \cdot p(\beta, x, v)$. So $\xi$ is a $\mathbb{Z}_2$–bundle, and $S^{k-1} \times M \times \mathbb{R}^n$ and $S^{k-1} \times M$ are $\mathbb{Z}_2$–free.

We define the corresponding mapping
\[
\tilde{p} : (S^{k-1} \times M \times \mathbb{R}^n)/\mathbb{Z}_2 \to (S^{k-1} \times M)/\mathbb{Z}_2
\]
on orbit spaces by $[\beta, x, v] \mapsto [\beta, x]$. It is obvious that $\tilde{p}$ is well-defined.

Then $\tilde{\xi} = (\tilde{p}, (S^{k-1} \times M \times \mathbb{R}^n)/\mathbb{Z}_2, (S^{k-1} \times M)/\mathbb{Z}_2)$ is an $n$–dimensional vector bundle (see [11 1.6.1]).

We define a section $s_\alpha : (S^{k-1} \times M)/\mathbb{Z}_2 \to (S^{k-1} \times M \times \mathbb{R}^n)/\mathbb{Z}_2$ by $[\beta, x] \mapsto [\beta, x, \tilde{a}(\beta, x)]$. 

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Both \((S^{k-1} \times M)/\mathbb{Z}_2\) and \((S^{k-1} \times M \times \mathbb{R}^n)/\mathbb{Z}_2\) are smooth manifold of dimension respectively \(n\) and \(2n\) (see for instance \[12\] Thm. 21.10), and

\[
\partial \left( (S^{k-1} \times M \times \mathbb{R}^n)/\mathbb{Z}_2 \right) = (S^{k-1} \times \partial M \times \mathbb{R}^n)/\mathbb{Z}_2, \\
\partial \left( (S^{k-1} \times M)/\mathbb{Z}_2 \right) = (S^{k-1} \times \partial M)/\mathbb{Z}_2.
\]

Let \(\tilde{z} : (S^{k-1} \times M)/\mathbb{Z}_2 \to (S^{k-1} \times M \times \mathbb{R}^n)/\mathbb{Z}_2\) be defined as \(\tilde{z}([\beta, x]) = \left[\beta, x, 0\right]\). Then \(\tilde{z}\) is a zero section of \(\tilde{\xi}\) and

\[
[\beta, x] \in \partial \left( (S^{k-1} \times M)/\mathbb{Z}_2 \right) \Rightarrow \\
\Rightarrow \tilde{a}(\beta, x) \neq 0 \Rightarrow s_a([\beta, x]) = (\beta, x, \tilde{\alpha}(\beta, x)) \not\in \tilde{z}(\left(S^{k-1} \times M\right)/\mathbb{Z}_2),
\]

so \(s_a\) fulfills the condition \((\mathbf{3})\), and then \(I_2(s_a)\) is well defined.

**Proposition 6.4.** The set \(s_a^{-1}(\tilde{z}(\left(S^{k-1} \times M\right)/\mathbb{Z}_2))\) is finite if and only if \(\tilde{a}^{-1}(0)\) is finite. If that is the case, and \(\tilde{a}^{-1}(0) = \{(\pm \beta, x_i) \mid i = 1, \ldots, r\}\), then

\[
I_2(s_a) = \sum_{i=1}^{r} \deg_2(\tilde{a}, (\beta, x_i)) \mod 2.
\]

**Proof.** We have the equality

\[
s_a^{-1}(\tilde{z}(\left(S^{k-1} \times M\right)/\mathbb{Z}_2)) = (id, \tilde{\alpha})^{-1}(z(\left(S^{k-1} \times M\right))/\mathbb{Z}_2).
\]

Then the sets \(s_a^{-1}(\tilde{z}(\left(S^{k-1} \times M\right)/\mathbb{Z}_2))\) and \((id, \tilde{\alpha})^{-1}(z(\left(S^{k-1} \times M\right))) = \tilde{a}^{-1}(0)\) are either both finite or both infinite (each element \([\beta, x] \in s_a^{-1}(\tilde{z}(\left(S^{k-1} \times M\right)/\mathbb{Z}_2))\) is associated with exactly two elements \((\beta, x), (-\beta, x) \in (id, \tilde{\alpha})^{-1}(z(\left(S^{k-1} \times M\right)))).

Let us assume that \(s_a^{-1}(\tilde{z}(\left(S^{k-1} \times M\right)/\mathbb{Z}_2))\) is finite and let \(s_a^{-1}(\tilde{z}(\left(S^{k-1} \times M\right)/\mathbb{Z}_2)) = \{[\beta_1, x_1], \ldots, [\beta_r, x_r]\}\). Then \((id, \tilde{\alpha})^{-1}(z(\left(S^{k-1} \times M\right))) = \tilde{a}^{-1}(0) = \{(\beta_1, x_1), (-\beta_1, x_1), \ldots, (\beta_r, x_r), (-\beta_r, x_r)\}\). The vector bundles \(\xi\) and \(\tilde{\xi}\) are locally diffeomorphic (see \[11\] p.36), so for every \((\beta, x_i)\) there exists a sufficiently small open neighbourhood \(U_i \subset S^{k-1} \times M\) such that the diagram

\[
\begin{array}{ccc}
p^{-1}(U_i) & \xrightarrow{pr_1} & (U_i \times \mathbb{R}^n)/\mathbb{Z}_2 \\
\downarrow & & \downarrow \tilde{p} \\
U_i & \xrightarrow{pr_2} & U_i/\mathbb{Z}_2
\end{array}
\]

commutes. There \(pr_1\) and \(pr_2\) are the natural projection mappings and \(pr_1|_{U_i \times \mathbb{R}^n}\), \(pr_2|_{U_i}\) are diffeomorphisms.

We define \(\widetilde{F}_i = \pi_2 \circ (pr_2|_{U_i \times \mathbb{R}^n}) \circ (pr_1|_{U_i \times \mathbb{R}^n})^{-1} \circ s_a\), so

\[
\widetilde{F}_i : U_i/\mathbb{Z}_2 \xrightarrow{s_a} \tilde{p}^{-1}(U_i/\mathbb{Z}_2) = (U_i \times \mathbb{R}^n)/\mathbb{Z}_2 \xrightarrow{pr_1^{-1}} p^{-1}(U_i) = U_i \times \mathbb{R}^n \xrightarrow{(pr_2, id)} (U_i/\mathbb{Z}_2) \times \mathbb{R}^n \xrightarrow{\pi} \mathbb{R}^n.
\]

Then we obtain \(\widetilde{F}_i([\beta, x]) = \tilde{\alpha}(\beta, x)\) if \((\beta, x) \in U_i\) and \(\widetilde{F}_i([-\beta, x]) = -\tilde{\alpha}(\beta, x)\) if \((-\beta, x) \in U_i\). By Proposition 6.2 we have

\[
I_2(s_a) = \sum_{i=1}^{r} \deg_2 \widetilde{F}_i \mod 2.
\]
If we take $F_i = \tilde{F}_i \circ pr_2: U_i \to \mathbb{Z}_2 \to \mathbb{R}^n$, then $\deg_2 F_i = \deg_2 \tilde{F}_i$, and $F_i$ is diffeomorphic to $\tilde{a}|_{U_i}$, so

$$\deg_2 F_i = \deg_2 \tilde{a}|_{U_i} = \deg_2 (\tilde{a}, (\beta_i, x_i)) \mod 2.$$ \hfill $\Box$

According to Theorem 3.3 and Proposition 6.4 we get

**Corollary 6.5.**

$$I_2(s_a) = I_2(a, \Sigma).$$

Using Corollary 6.3, we obtain the following result, which covers [10, Theorem 2.2].

**Proposition 6.6.** Let $a, b: M \to M_k(\mathbb{R}^n)$ be smooth mappings such that the restrictions $a|_{\partial M}, b|_{\partial M}: \partial M \to \tilde{V}_k(\mathbb{R}^n)$ and $a|_{\partial M}$ and $b|_{\partial M}$ are homotopic. Then $I_2(s_a) = I_2(s_b) \mod 2$. In particular, when $M$ is a closed ball, $\Lambda(a|S^{n-k}) = \Lambda(b|S^{n-k}) \mod 2$.

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