Sampling and cubature on sparse grids based on a B-spline quasi-interpolation

Dinh Dũng

Vietnam National University, Hanoi, Information Technology Institute
144 Xuan Thuy, Cau Giay, Hanoi, Vietnam
dinhzung@gmail.vn

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Abstract

Let $X_n = \{x_j\}_{j=1}^n$ be a set of $n$ points in the $d$-cube $\mathbb{I}^d := [0,1]^d$, and $\Phi_n = \{\phi_j\}_{j=1}^n$ a family of $n$ functions on $\mathbb{I}^d$. We consider the approximate recovery functions $f$ on $\mathbb{I}^d$ from the sampled values $f(x^1), \ldots, f(x^n)$, by the linear sampling algorithm

$$L_n(X_n, \Phi_n, f) := \sum_{j=1}^n f(x^j)\phi_j.$$ 

The error of sampling recovery is measured in the norm of the space $L_q(\mathbb{I}^d)$-norm or the energy norm of the isotropic Sobolev space $W^\gamma_q(\mathbb{I}^d)$ for $0 < q \leq \infty$ and $\gamma > 0$. Functions $f$ to be recovered are from the unit ball in Besov type spaces of an anisotropic smoothness, in particular, spaces $B^{a}_{p,\theta}$ of a nonuniform mixed smoothness $a \in \mathbb{R}^d_+$, and spaces $B^{\alpha,\beta}_{p,\theta}$ of a “hybrid” of mixed smoothness $\alpha > 0$ and isotropic smoothness $\beta \in \mathbb{R}$. We constructed optimal linear sampling algorithms $L_n(X_n, \Phi_n, \cdot)$ on special sparse grids $X_n^*$ and a family $\Phi_n^*$ of linear combinations of integer or half integer translated dilations of tensor products of B-splines. We computed the asymptotic of the error of the optimal recovery. This construction is based on a B-spline quasi-interpolation representations of functions in $B^{a}_{p,\theta}$ and $B^{\alpha,\beta}_{p,\theta}$. As consequences we obtained the asymptotic of optimal cubature formulas for numerical integration of functions from the unit ball of these Besov type spaces.

Keywords and Phrases Linear sampling algorithm; Cubature formula; Sparse grid; Optimal sampling recovery; Optimal cubature; Besov type space of anisotropic smoothness; B-spline quasi-interpolation.

Mathematics Subject Classifications (2010) 41A15; 41A05; 41A25; 41A58; 41A63.
1 Introduction

The aim of the present paper is to investigate linear sampling algorithms and cubature formulas on sparse grids based on a B-spline quasi-interpolation, and their optimality for functions on the unit $d$-cube $\mathbb{I}^d := [0,1]^d$, having an anisotropic smoothness. The error of sampling recovery is measured in the norm of the space $L_q(\mathbb{I}^d)$-norm or the (generalized) energy norm of the isotropic Sobolev space $W_q^\gamma(\mathbb{I}^d)$ for $0 < q \leq \infty$ and $\gamma > 0$. For convenience, we use somewhere the notation $W_0^q(\mathbb{I}^d) := L_q(\mathbb{I}^d)$.

Let $X_n = \{x^j\}_{j=1}^n$ be a set of $n$ points in $\mathbb{I}^d$, $\Phi_n = \{\varphi_j\}_{j=1}^n$ a family of $n$ functions on $\mathbb{I}^d$. If $f$ a function on $\mathbb{I}^d$, for approximately recovering $f$ from the sampled values $f(x^1),\ldots,f(x^n)$, we define the linear sampling algorithm $L_n(X_n, \Phi_n, f)$ by

$$L_n(X_n, \Phi_n, f) := \sum_{j=1}^n f(x^j)\varphi_j.$$

(1.1)

Let $B$ be a quasi-normed space of functions on $\mathbb{I}^d$, equipped with the quasi-norm $\|\cdot\|_B$. For $f \in B$, we measure the recovery error by $\|f - L_n(X_n, \Phi_n, f)\|_B$. Let $W \subset B$. To study optimality of linear sampling algorithms of the form (1.1) for recovering $f \in W$ from $n$ their values, we will use the quantity

$$r_n(W, B) := \inf_{X_n, \Phi_n} \sup_{f \in W} \|f - L_n(X_n, \Phi_n, f)\|_B.$$

A general nonlinear sampling algorithm of recovery can be defined as

$$R_n(X_n, P_n, f) := P_n(f(x^1),\ldots,f(x^n)),$$

where $P_n : \mathbb{R}^n \to B$ is a given mapping. To study optimal nonlinear sampling algorithms of recovery for $f \in W$ from $n$ their values, we can use the quantity

$$\varrho_n(W, B) := \inf_{P_n, X_n} \sup_{f \in W} \|f - R_n(X_n, P_n, f)\|_B.$$

Further, let $\Lambda_n = \{\lambda_j\}_{j=1}^n$ a sequence of $n$ numbers. For a $f \in C(\mathbb{I}^d)$, we want to approximately compute the integral

$$I(f) := \int_{\mathbb{I}^d} f(x) \, dx$$

by the cubature formula

$$I_n(X_n, \Lambda_n, f) := \sum_{j=1}^n \lambda_j f(x^j).$$

To study the optimality of cubature formulas for $f \in W$, we use the quantity

$$i_n(W) := \inf_{X_n, \Lambda_n} \sup_{f \in W} |I(f) - I_n(X_n, \Lambda_n, f)|.$$
Recently, there has been increasing interest in solving approximation and numerical problems that involve functions depending on a large number of variables. The computation time typically grows exponentially in \(d\), and the problems become intractable already for mild dimensions \(d\) without further assumptions. This is so called the curse of dimensionality [2]. In sampling recovery and numerical integration, a classical model in attempt to overcome it which has been widely studied, is to impose certain mixed smoothness or more general anisotropic smoothness conditions on the function to be approximated, and to employ sparse grids for construction of approximation algorithms for sampling recovery or integration. We refer the reader to [6, 31, 39, 40] for surveys and the references therein on various aspects of this direction.

Sparse grids for sampling recovery and numerical integration were first considered by Smolyak [43]. He constructed the following grid of dyadic points

\[
\Gamma(m) := \{2^{-k} s : k \in D(m), s \in I^d(k)\},
\]

where \(D(m) := \{k \in \mathbb{Z}_+^d : |k| \leq m\}\) and \(I^d(k) := \{s \in \mathbb{Z}_+^d : 0 \leq s_i \leq 2^{k_i}, i \in [d]\}\). Here and in what follows, we use the notations: \(x_y := (x_1 y_1, ..., x_d y_d)\); \(2^s := (2^{s_1}, ..., 2^{s_d})\); \(|x|_1 := \sum_{i=1}^d |x_i|\) for \(x, y \in \mathbb{R}^d\); \([d]\) denotes the set of all natural numbers from 1 to \(d\); \(x_i\) denotes the \(i\)th coordinate of \(x \in \mathbb{R}^d\), i.e., \(x := (x_1, ..., x_d)\). Observe that \(\Gamma(m)\) is a sparse grid of the size \(2^m m^{d-1}\) in comparing with the standard full grid of the size \(2^{dm}\).

In Approximation Theory, Temlyakov [44] – [46] and the author of the present paper [16] – [18] developed Smolyak’s construction for studying the asymptotic order of \(r_n(W, L_q(\mathbb{T}^d))\) for periodic Sobolev classes \(W^p_{\alpha, 1}\) and Nikol’skii classes \(H^\alpha_p\) having mixed smoothness \(\alpha\), where \(1 := (1, 1, ..., 1) \in \mathbb{R}^d\) and \(\mathbb{T}^d\) denotes the \(d\)-dimensional torus. Recently, Sickel and Ullrich [41] have investigated \(r_n(U_{p, \theta}^{\alpha, 1}, L_q(\mathbb{T}^d))\) for periodic Besov classes. For non-periodic functions of mixed smoothness linear sampling algorithms have been recently studied by Triebel [47] \((d = 2)\), Dinh Đùng [23], Sickel and Ullrich [42], using the mixed tensor product of B-splines and Smolyak grids \(\Gamma(m)\). In [24], we have constructed methods of approximation by arbitrary linear combinations of translates of the Korobov kernel \(\kappa_{r, d}\) on Smolyak grids of functions from the Korobov space \(K^{\alpha}_r(\mathbb{T}^d)\) which is a reproducing kernel Hilbert space with the associated kernel \(\kappa_{r, d}\). This approximation can have applications in Machine Learning. Smolyak grids are a counterpart of hyperbolic crosses which are frequency domains of trigonometric polynomials widely used for approximations of functions with a bounded mixed smoothness. These hyperbolic cross trigonometric approximations are initiated by Babenko [1]. For further surveys and references on the topic see [15, 46], the references given there, and the more recent contributions [41, 48].

In Computational Mathematics, the sparse grid approach was first considered by Zenger [52] in parallel algorithms for numerical solving PDEs. Numerical integration was investigated in [30]. For non-periodic functions of mixed smoothness of integer order, linear sampling algorithms on sparse grids have been investigated by Bungartz and Griebel [6] employing hierarchical Lagrangian polynomials multilevel basis and measuring the approximation error in the \(L_2\)-norm and energy \(H^1\)-norm. There is a very large number of papers on sparse grids in various problems of approximations, sampling recovery and integration with applications in data mining, mathematical finance, learning theory, numerical solving of PDE and stochastic PDE, etc. to mention all of them. The reader can see the surveys in [6, 36] and the references therein. For recent further developments and
necessary to emphasize that any sampling algorithm on Smolyak grids always gives a lower bound for functions on $\mathbb{I}^d$ having mixed smoothness $\alpha$. For various $0 < p, \theta, q \leq \infty$ and $\alpha > 1/p$, we proved upper bounds for $r_n(U_{p,\theta}^\alpha, L_q(\mathbb{I}^d))$ which in some cases, coincide with the asymptotic order

$$r_n(U_{p,\theta}^\alpha, L_q(\mathbb{I}^d)) \asymp n^{-\alpha+(1/p-1/q)+\log_2(d-1)b} n,$$

where $b = b(\alpha, p, \theta, q) > 0$ and $x_+ := \max(0, x)$ for $x \in \mathbb{R}$. By using a quasi-interpolation representation of functions $f \in B_{p,\theta}^\alpha$ by mixed B-spline series, we constructed optimal linear sampling algorithms on Smolyak grids $\Gamma(m)$.

In the paper [26], we obtained the asymptotic order of optimal sampling recovery on Smolyak grids in the $L_q(\mathbb{I}^d)$-quasi-norm of functions from $U_{p,\theta}^\alpha$ for $0 < p, \theta, q \leq \infty$ and $\alpha > 1/p$. It is necessary to emphasize that any sampling algorithm on Smolyak grids always gives a lower bound of recovery error of the form as in the right side of (1.2) with the logarithm term $\log_2(b) n$, $b > 0$. Unfortunately, in the case when the dimension $d$ is very large and the number $n$ of samples is rather mild, the main term becomes $\log_2(b) n$ which grows fast exponentially in $d$. To avoid this exponential grow we impose to functions other anisotropic smoothnesses and construct appropriate sparse grids for functions having them. Namely, we extend the above study to functions on $\mathbb{I}^d$ from the classes $U_{p,\theta}^\alpha$ for $\alpha \in \mathbb{R}_+$, and $U_{p,\theta}^{\alpha,\beta}$ for $\alpha > 0, \beta \in \mathbb{R}$, which are defined as the unit ball of the Besov type spaces $B_{p,\theta}^\alpha$ and $B_{p,\theta}^{\alpha,\beta}$. The space $B_{p,\theta}^0$ and $B_{p,\theta}^{\alpha,\beta}$ are certain sets of functions with bounded mixed modulus of smoothness. Both of them are generalizations in different ways of the space $B_{p,\theta}^\alpha$ of mixed smoothness $\alpha$. The space $B_{p,\theta}^\alpha$ is $B_{p,\theta}^\alpha$ for $\alpha = \alpha\mathbf{1}$. The space $B_{p,\theta}^{\alpha,\beta}$ is a “hybrid” of the space $B_{p,\theta}^\alpha$ and the classical isotropic Besov space $B_{p,\theta}^\beta$ of smoothness $\beta$ (see Section 2). Hyperbolic cross approximations and sparse grid sampling recovery of functions from a space $B_{p,\theta}^\alpha$ with uniform and nonuniform mixed smoothness $\alpha$ were studied in a large number of works. We refer the reader to [15, 16] as well to recent papers [23, 25] for surveys and bibliography. These problems were extended to functions from an intersection of spaces $B_{p,\theta}^\alpha$, see [14, 15, 19, 27, 28].

The space $B_{p,\theta}^{\alpha,\beta}$ is a Besov type generalization of the Sobolev type space $H^{\alpha,\beta} = B_{2,2}^{\alpha,\beta}$. The latter space has been introduced in [30] for solutions of the following elliptic variational problems:

$$a(u, v) = (f, v) \text{ for all } v \in H^\gamma,$$

where $f \in H^{-\gamma}$ and $a : H^{\gamma} \times H^\gamma \to \mathbb{R}$ is a bilinear symmetric form satisfying the conditions $a(u, v) \leq \lambda\|u\|_{H^\gamma}\|v\|_{H^\gamma}$ and $a(u, u) \geq \mu\|u\|_{H^\gamma}^2$. By use of tensor-product biorthogonal wavelet bases, the authors of these papers constructed so-called optimized sparse grid subspaces for finite element approximations of the solution having $H^{\alpha,\beta}$-regularity, whereas the approximation error is measured in the energy norm of isotropic Sobolev space $H^\gamma$. They generalized the construction of [3] for a hyperbolic cross approximation of the solution of Poisson’s equation to elliptic variational problems. The necessary dimension $n_\varepsilon$ of the optimized sparse grid space for the finite element approximation of the solution with accuracy $\varepsilon$ does not exceed $C(d, \alpha, \gamma, \beta)\varepsilon^{-(\alpha+\beta-\gamma)}$ if $\alpha > \gamma, \beta > 0$. A generalization $H^{\alpha,\beta}((\mathbb{R}^3)^N)$ of the space $H^{\alpha,\beta}$ of functions on $(\mathbb{R}^3)^N$, based on isotropic
Sobolev smoothness of the space $H^1(\mathbb{R}^3)$, has been considered by Yserentant \[39–51\] for solutions $u : (\mathbb{R}^3)^N \to \mathbb{R} : (x_1, \ldots, x_N) \to u(x_1, \ldots, x_N)$ of the electronic Schrödinger equation $Hu = \lambda u$ for eigenvalue problem where $H$ is the Hamilton operator. He proved that the eigenfunctions are contained in the intersection of spaces $H^{1,0}(\mathbb{R}^3)^N \cap \bigcap_{\vartheta < 3/4} H^{\vartheta,1}(\mathbb{R}^3)^N$. In numerical solving by hyperbolic cross approximations the error is measured in the norm of the space $L_2(\mathbb{R}^3)^N$ and the energy norm of the isotropic Sobolev space $H^1(\mathbb{R}^3)^N$. See also \[32–35, 37\] for further results and developments.

In the present paper, we study the problem of computing the asymptotic orders of $r_n(U_{p,\vartheta}^a, L_q(\mathbb{I}^d))$, $q_n(U_{p,\vartheta}^a, L_q(\mathbb{I}^d))$ for the case of nonuniform mixed smoothness $a$, $r_n(U_{p,\vartheta}^{a,\beta}, W_q^\gamma(\mathbb{I}^d))$, $q_n(U_{p,\vartheta}^{a,\beta}, W_q^\gamma(\mathbb{I}^d))$ for the case $\beta \neq \gamma$, and $i_n(U_{p,\vartheta}^a)$, $i_n(U_{p,\vartheta}^{a,\beta})$ and constructing asymptotically optimal linear algorithms for them. The main results of this paper are the following.

(i) Let $0 < p, \theta, q \leq \infty$ and $a \in \mathbb{R}^d$ with $1/p < a_1 < a_2 \leq \ldots \leq a_d$. Then we have

$$r_n(U_{p,\vartheta}^a, L_q(\mathbb{I}^d)) \asymp q_n(U_{p,\vartheta}^a, L_q(\mathbb{I}^d)) \asymp n^{-a_1+(1/p-1/q)_+};$$

$$i_n(U_{p,\vartheta}^a) \asymp n^{-a_1+(1/p-1)_+}.$$  \hspace{1cm} (1.3)

(ii) Let $0 < p, \theta, q, \tau \leq \infty$, $\alpha, \gamma \in \mathbb{R}_+$, $\beta \in \mathbb{R}$ satisfying the conditions $\min(\alpha, \alpha + \beta) > 1/p$ and $\alpha > (\gamma - \beta)/d$ if $\beta > \gamma$, and $\alpha > \gamma - \beta$ if $\beta < \gamma$. Then we have

$$r_n(U_{p,\vartheta}^{a,\beta}, W_q^\gamma(\mathbb{I}^d)) \asymp q_n(U_{p,\vartheta}^{a,\beta}, W_q^\gamma(\mathbb{I}^d)) \asymp \begin{cases} n^{-\alpha+(\beta-\gamma)/d+(1/p-1/q)_+}, & \beta > \gamma; \\ n^{-\alpha-(\gamma-\beta)/d+(1/p-1/q)_+}, & \beta < \gamma; \end{cases}$$

$$i_n(U_{p,\vartheta}^{a,\beta}) \asymp \begin{cases} n^{-\alpha-(\gamma-\beta)/d+(1/p-1)_+}, & \beta > 0, \\ n^{-\alpha-(\gamma-\beta)/d+(1/p-1)_+}, & \beta < 0. \end{cases}$$  \hspace{1cm} (1.4)

It is remarkable that the asymptotic orders in (1.3), (1.4) and in (1.5) for $\beta < \gamma$, (1.6) for $\beta < 0$, do not contain any exponent in $d$ and moreover, do not depend on $d$.

For a set $\Delta \subset \mathbb{Z}_+^d$, we define the grid $G(\Delta)$ of points in $\mathbb{I}^d$ by

$$G(\Delta) := \{2^{-k}s : k \in \Delta, s \in I^d(k)\}.$$  \hspace{1cm} (1.5)

For the quantities of optimal recovery in (1.5) and (1.3), asymptotically optimal linear sampling algorithms of the form

$$L_n(X_n^*, \Phi_n^*, f) = \sum_{k \in \Delta_n} \sum_{j \in I^d(k)} f(2^{-k}j)\psi_{k,j}$$  \hspace{1cm} (1.6)

are constructed where $X_n^* := G(\Delta_n)$, $\Phi_n^* := \{\psi_{k,j}\}_{k \in \Delta_n, j \in I^d(k)}$ and $\psi_{k,j}$ are explicitly constructed as linear combinations of at most at most $N$ B-splines $M_{k,s}^{(r)}$ for some $N \in \mathbb{N}$ which is independent of $k, j, m$ and $f$, $M_{k,s}^{(r)}$ are tensor products of either integer or half integer translated dilations of the centered B-spline of order $r$. The set $\Delta_n$ is specially constructed for each class of $U_{p,\vartheta}^a$ and
depending on the relationship between $0 < p, \theta, q \leq \infty$ and $a$ or $0 < p, \theta, q, \tau \leq \infty$ and $\alpha, \beta$ respectively. The grids $G(\Delta_n)$ are sparse and have much smaller number of sample points than the corresponding standard full grids and the Smolyak grids, but give the same error of the sampling recovery on the both latter ones. The asymptotically optimal linear sampling algorithms $L_n(X_n^*, \Phi_n^*, \cdot)$ are based on quasi-interpolation representations by B-spline series of functions in spaces $B_{p,\theta}^\alpha$ and $B_{p,\theta}^{\alpha,\beta}$. Moreover, if the error of sampling recovery is measured in the $L_1$-norm, $L_n(X_n^*, \Phi_n^*, \cdot)$ generates an asymptotically optimal cubature formula (see Section 6 for details).

We are restricted to compute the asymptotic order of $r_n$ and $\varrho_n$ with respect only to $n$ when $n \to \infty$, not analyzing the dependence on the number of variables $d$. Recently, in [25] Kolmogorov $n$-widths $d_n(U, H^\gamma)$ and $\varepsilon$-dimensions $n_\varepsilon(U, H^\gamma)$ in space $H^\gamma$ of periodic multivariate function classes $U$ have been investigated in high-dimensional settings, where $U$ is the unit ball in $H^{\alpha,\beta}$ or its subsets. We computed the accurate dependence of $d_n(U, H^\gamma)$ and $n_\varepsilon(U, H^\gamma)$ as a function of two variables $n, d$ or $\varepsilon, d$. Although $n$ is the main parameter in the study of convergence rate with respect to $n$ when $n \to \infty$, the parameter $d$ may affect this rate when $d$ is large. It is interesting and important to investigate optimal sampling recovery and cubature in terms of $r_n, \varrho_n$ and $i_n$ in such high-dimensional settings. We will discuss this problem in a forthcoming paper.

The present paper is organized as follows.

In Section 2 we give definitions of Besov type spaces $B_{p,\theta}^\Omega$ of of functions with bounded mixed modulus of smoothness, in particular, spaces $B_{p,\theta}^\alpha$ and $B_{p,\theta}^{\alpha,\beta}$, and prove theorems on quasi-interpolation representation by B-spline series, with relevant discrete equivalent quasi-norms. In Section 3 we construct linear sampling algorithms on sparse grids of the form (1.7) for function classes $U_{p,\theta}^\alpha$ and $U_{p,\theta}^{\alpha,\beta}$, and prove upper bounds for the error of recovery by these algorithms. In Section 4 we prove the sparsity and asymptotic optimality of the linear sampling algorithms constructed in Section 3 for the quantities $\varrho_n(U_{p,\theta}^\alpha, L_q(\mathbb{I}^d))$, $r_n(U_{p,\theta}^\alpha, L_q(\mathbb{I}^d))$ and $\varrho_n(U_{p,\theta}^{\alpha,\beta}, L_q(\mathbb{I}^d))$, $r_n(U_{p,\theta}^{\alpha,\beta}, L_q(\mathbb{I}^d))$, and establish their asymptotic orders. In Section 5 we extend the investigations of Sections 3 and 4 to the quantities $r_n(U_{p,\theta}^{\alpha,\beta}, W_q^\gamma(\mathbb{I}^d))$ and $\varrho_n(U_{p,\theta}^{\alpha,\beta}, W_q^\gamma(\mathbb{I}^d))$ for $\gamma > 0$. In Section 6 we discuss the problem of optimal cubature formulas for numerical integration in terms of $i_n(U_{p,\theta}^\alpha)$ and $i_n(U_{p,\theta}^{\alpha,\beta})$.

## 2 Function spaces and B-spline quasi-interpolation representations

Let us first introduce spaces $B_{p,\theta}^\Omega$ of functions with bounded mixed modulus of smoothness and Besov type spaces $B_{p,\theta}^\alpha$ and $B_{p,\theta}^{\alpha,\beta}$ of functions with anisotropic smoothness and give necessary knowledge of them.

Let $G$ be a domain in $\mathbb{R}$. For univariate functions $f$ on $G$ the $r$th difference operator $\Delta_h^r$ is
defined by
\[ \Delta^r_h(f, x) := \sum_{j=0}^{r} (-1)^{r-j} \binom{r}{j} f(x + jh). \]

If \( e \) is any subset of \([d]\), for multivariate functions on \( \mathbb{G}^d \) the mixed \((r, e)\)th difference operator \( \Delta^r_{h, e} \) is defined by
\[ \Delta^r_{h, e} := \prod_{i \in e} \Delta^r_{h_i}, \quad \Delta^r_{h_i} = I, \]
where the univariate operator \( \Delta^r_{h_i} \) is applied to the univariate function \( f \) by considering \( f \) as a function of variable \( x_i \) with the other variables held fixed.

Denote by \( L_p(\mathbb{G}^d) \) the quasi-normed space of functions on \( \mathbb{G}^d \) with the \( p \)th integral quasi-norm \( \| \cdot \|_{p, \mathbb{G}^d} \) for \( 0 < p < \infty \), and the sup norm \( \| \cdot \|_{\infty, \mathbb{G}^d} \) for \( p = \infty \).

Let
\[ \omega^e_r(f, t)_{p, \mathbb{G}^d} := \sup_{|h_i| < t_i, i \in e} \| \Delta^r_{h, e}(f) \|_{p, \mathbb{G}^d(h, e)}, \quad t \in \mathbb{G}^d, \]
be the mixed \((r, e)\)th modulus of smoothness of \( f \), where \( \mathbb{G}^d(h, e) := \{ x \in \mathbb{G}^d : x_i, x_i + rh_i \in \mathbb{G}, i \in e \} \) (in particular, \( \omega^e_r(f, t)_{p, \mathbb{G}^d} = \| f \|_{p, \mathbb{G}^d} \)).

For \( x, x' \in \mathbb{R}^d \), the inequality \( x' \leq x \) (\( x' < x \)) means \( x_i' \leq x_i \) (\( x_i' < x_i \), \( i \in [d] \)). Denote: \( \mathbb{R}_+ := \{ x \in \mathbb{R} : x \geq 0 \} \). Let \( \Omega : \mathbb{R}^d_+ \rightarrow \mathbb{R}_+ \) be a function satisfying conditions
\[ \Omega(t) > 0, \quad t > 0, \quad t \in \mathbb{R}^d_+, \quad (2.1) \]
\[ \Omega(t) \leq C\Omega(t'), \quad t \leq t', \quad t, t' \in \mathbb{R}^d_+, \quad (2.2) \]
and for a fixed \( \gamma \in \mathbb{R}^d_+ \), \( \gamma \geq 1 \), there is a constant \( C' = C'(\gamma) \) such that for every \( \lambda \in \mathbb{R}^d_+ \) with \( \lambda \leq \gamma \),
\[ \Omega(\lambda t) \leq C'\Omega(t), \quad t \in \mathbb{R}^d_+. \quad (2.3) \]

For \( e \subset [d] \), we define the function \( \Omega_e : \mathbb{R}^d_+ \rightarrow \mathbb{R}_+ \) by
\[ \Omega_e(t) := \Omega(t^e), \]
where \( t^e \in \mathbb{R}^d_+ \) is given by \( t^e_j = t_j \) if \( j \in e \), and \( t^e_j = 1 \) otherwise.

If \( 0 < p, \theta \leq \infty \), we introduce the quasi-semi-norm \( |f|_{B^{\Omega_e}_{p, \theta}(\mathbb{G}^d)} \) for functions \( f \in L_p(\mathbb{G}^d) \) by
\[ |f|_{B^{\Omega_e}_{p, \theta}(\mathbb{G}^d)} := \begin{cases} \left( \int_{\mathbb{G}^d} \left\{ \omega^e_r(f(t)_{p, \mathbb{G}^d}/\Omega_e(t)) \cdot \prod_{i \in e} t^{-1}_i dt \right\}^{\theta} \right)^{1/\theta}, & \theta < \infty, \\ \sup_{t \in \mathbb{G}^d} \omega^e_r(f(t)_{p, \mathbb{G}^d}/\Omega_e(t)), & \theta = \infty, \end{cases} \quad (2.4) \]
(in particular, \( |f|_{B^{\Omega_e}_{p, \theta}(\mathbb{G}^d)} = \| f \|_{p, \mathbb{G}^d} \)).
Another alternative definition of the quasi-semi-norm $|f|_{\mathcal{B}_{p,\theta}^\Omega(e)}$ is obtained by replacing the integral or supremum over $I^d$ in (2.4) by one over $\mathbb{R}^d_+$. In what follows, we preliminarily assume that the function $\Omega$ satisfies the conditions (2.1)–(2.3).

For $0 < p, \theta \leq \infty$, the Besov type space $\mathcal{B}_{p,\theta}^\Omega(\mathbb{G}^d)$ is defined as the set of functions $f \in L_p(I^d)(\mathbb{G}^d)$ for which the quasi-norm

$$
\|f\|_{\mathcal{B}_{p,\theta}^\Omega(\mathbb{G}^d)} := \sum_{e \subset [d]} |f|_{\mathcal{B}_{p,\theta}^\Omega(e)}
$$

is finite. Since in the present paper we consider only functions defined on $I^d$, for simplicity we somewhere drop the symbol $I^d$ in the above notations.

We use the notations: $A_n(f) \ll B_n(f)$ if $A_n(f) \leq CB_n(f)$ with $C$ an absolute constant not depending on $n$ and/or $f \in W$, and $A_n(f) \asymp B_n(f)$ if $A_n(f) \ll B_n(f)$ and $B_n(f) \ll A_n(f)$. Put $\mathbb{Z}_+ := \{s \in \mathbb{Z} : s \geq 0\}$ and $\mathbb{Z}_+^d(e) := \{s \in \mathbb{Z}_+^d : s_i = 0, i \notin e\}$ for a set $e \subset [d]$.

**Lemma 2.1** Let $0 < p, \theta \leq \infty$. Then we have the following quasi-norm equivalence

$$
\|f\|_{\mathcal{B}_{p,\theta}^\Omega} \asymp B_1(f) := \sum_{e \subset [d]} \left( \sum_{k \in \mathbb{Z}_+^d(e)} \left\{ \omega_e^\alpha(f, 2^{-k})/\Omega(2^{-k}) \right\}^\theta \right)^{1/\theta},
$$

with the corresponding change to sup when $\theta = \infty$.

**Proof.** This lemma follows from properties of mixed modulus of smoothness $\omega_e^\alpha(f, t)^p$ and the properties (2.1)–(2.3) of the function $\Omega$.

Let us define the Besov type spaces $\mathcal{B}_{p,\theta}^\alpha$ and $\mathcal{B}_{p,\theta}^{\alpha,\beta}$ of functions with anisotropic smoothness as particular cases of $\mathcal{B}_{p,\theta}^\Omega$.

For $a \in \mathbb{R}^d_+$, we define the space $\mathcal{B}_{p,\theta}^\alpha$ of mixed smoothness $a$ as follows.

$$
\mathcal{B}_{p,\theta}^\alpha := \mathcal{B}_{p,\theta}^\Omega, \text{ where } \Omega(t) = \prod_{i=1}^d t_i^{a_i}, \ t \in \mathbb{R}_+^d. \tag{2.5}
$$

Let $\alpha \in \mathbb{R}_+$ and $\beta \in \mathbb{R}$ with $\alpha + \beta > 0$. We define the space $\mathcal{B}_{p,\theta}^{\alpha,\beta}$ as follows.

$$
\mathcal{B}_{p,\theta}^{\alpha,\beta} := \mathcal{B}_{p,\theta}^\Omega, \text{ where } \Omega(t) = \begin{cases} 
\prod_{i=1}^d t_i^{\alpha \inf_{j \in [d]} t_{j,i}^\beta}, & \beta \geq 0, \\
\prod_{i=1}^d t_i^{\alpha \sup_{j \in [d]} t_{j,i}^\beta}, & \beta < 0.
\end{cases} \tag{2.6}
$$
The definition (2.6) seems different for $\beta > 0$ and $\beta < 0$. However, it can be well interpreted in terms of the equivalent discrete quasi-norm $B_1(f)$ in Lemma 2.1. Indeed, the function $\Omega$ in (2.6) for both $\beta \geq 0$ and $\beta < 0$ satisfies the assumptions (2.1)–(2.3) and moreover, $1/\Omega(2^{-x}) = 2^\alpha|x|_1 + \beta|x|_\infty$, $x \in \mathbb{R}^d$, where $|x|_\infty := \max_{j \in [d]} |x_j|$ for $x \in \mathbb{R}^d$. Hence, by Lemma 2.1 we have the following quasi-norm equivalence

$$\|f\|_{B^{\alpha,\beta}_{p,\theta}} \asymp \sum_{e \subset [d]} \left( \sum_{k \in \mathbb{Z}^d(e)} \left\{ 2^\alpha|k|_1 + \beta|k|_\infty \omega^e_r(f, 2^{-k}) \right\}^{\theta} \right)^{1/\theta} \quad (2.7)$$

with the corresponding change to sup when $\theta = \infty$. The notation $B^{\alpha,\beta}_{p,\theta}$ becomes explicitly reasonable if we take the right side of (2.7) as a definition of the quasi-norm of the space $B^{\alpha,\beta}_{p,\theta}$.

The definition of $B^{\alpha,\beta}_{p,\theta}$ includes the well known classical isotropic Besov space and its mixed smoothness modifications. Thus, we have $B^{\alpha,\beta}_{p,\theta} = B^{\alpha,0}_{p,\theta}$ for $\beta = 0$, and $B^{\alpha,\beta}_{p,\theta} = B^{0,\beta}_{p,\theta}$ for $\alpha = 0$, where $B^{\beta}_{p,\theta}$ is the classical isotropic Besov space of smoothness $\beta$. From Lemma 2.1 and (2.7) we derive that for $\alpha \geq 0$ and $\beta \geq 0$,

$$B^{\alpha,\beta}_{p,\theta} = \bigcap_{j=1}^d B^{a_j}_{p,\theta},$$

and for $\alpha + \beta \geq 0$ and $\beta < 0$,

$$B^{\alpha,\beta}_{p,\theta} = \bigoplus_{j=1}^d B^{a_j}_{p,\theta}, \quad (2.8)$$

where $a_j = \alpha 1 + \beta e^j$, $e^j$ is the $j$th unit vector in $\mathbb{R}^d$ and

$$\sum_{j=1}^d B^{a_j}_{p,\theta} := \{ f \in L_p(\mathbb{T}^d) : f = \sum_{j=1}^d f_j, f_j \in B^{a_j}_{p,\theta} \}.$$

Next, we introduce quasi-interpolation operators for functions on $\mathbb{T}^d$. For a given natural number $r$, let $M$ be the centered B-spline of order $r$ with support $[-r/2, r/2]$ and knots at the points $-r/2, -r/2 + 1, \ldots, r/2 - 1, r/2$. Let $\Lambda = \{ \lambda(s) \}_{j \in P(\mu)}$ be a given finite even sequence, i.e., $\lambda(-j) = \lambda(j)$, where $P(\mu) := \{ j \in \mathbb{Z} : |j| \leq \mu \}$ and $\mu \geq r/2 - 1$. We define the linear operator $Q$ for functions $f$ on $\mathbb{R}$ by

$$Q(f, x) := \sum_{s \in \mathbb{Z}} \Lambda(f, s) M(x - s), \quad (2.9)$$

where

$$\Lambda(f, s) := \sum_{j \in P(\mu)} \lambda(j) f(s - j). \quad (2.10)$$

The operator $Q$ is local and bounded in $C(\mathbb{R})$ (see [3], p. 100–109], where $C(G)$ denotes the normed space of bounded continuous functions on $G$ with sup-norm $\| \cdot \|_{C(G)}$. Moreover,

$$\|Q(f)\|_{C(\mathbb{R})} \leq \|\Lambda\| \|f\|_{C(\mathbb{R})}.$$
for each \( f \in C(\mathbb{R}) \), where \( \| \Lambda \| = \sum_{j \in P(\mu)} |\lambda(j)| \). An operator \( Q \) of the form (2.9)–(2.10) reproducing \( P_{r-1} \), is called a quasi-interpolation operator in \( C(\mathbb{R}) \).

There are many ways to construct quasi-interpolation operators. A method of construction via Neumann series was suggested by Chui and Diamond [9] (see also [3] p. 100–109). A necessary and sufficient condition of reproducing \( P_{r-1} \) for operators \( Q \) of the form (2.9)–(2.10) with even \( r \) and \( \mu \geq r/2 \), was established in [7]. De Bore and Fix [10] introduced another quasi-interpolation operator based on the values of derivatives.

We give some examples of quasi-interpolation operator. The simplest example is a piecewise constant quasi-interpolation operator

\[
Q(f, x) := \sum_{s \in \mathbb{Z}} f(s)M(x - s),
\]

where \( M \) is the symmetric piecewise constant B-spline with support \([-1/2, 1/2]\) and knots at the half integer points \(-1/2, 1/2\). A piecewise linear quasi-interpolation operator is defined as

\[
Q(f, x) := \sum_{s \in \mathbb{Z}} f(s)M(x - s),
\]

where \( M \) is the symmetric piecewise linear B-spline with support \([-1, 1]\) and knots at the integer points \(-1, 0, 1\). It is related to the classical Faber-Schauder basis of the hat functions (see, e.g., [23], [17], for details). A quadric quasi-interpolation operator is defined by

\[
Q(f, x) := \sum_{s \in \mathbb{Z}} \frac{1}{8} \{-f(s - 1) + 10f(s) - f(s + 1)\}M(x - s),
\]

where \( M \) is the symmetric quadric B-spline with support \([-3/2, 3/2]\) and knots at the half integer points \(-3/2, -1/2, 1/2, 3/2\). Another example is the cubic quasi-interpolation operator

\[
Q(f, x) := \sum_{s \in \mathbb{Z}} \frac{1}{6} \{-f(s - 1) + 8f(s) - f(s + 1)\}M(x - s),
\]

where \( M \) is the symmetric cubic B-spline with support \([-2, 2]\) and knots at the integer points \(-2, -1, 0, 1, 2\).

If \( Q \) is a quasi-interpolation operator of the form (2.9)–(2.10), for \( h > 0 \) and a function \( f \) on \( \mathbb{R} \), we define the operator \( Q(\cdot; h) \) by

\[
Q(f; h) := \sigma_h \circ Q \circ \sigma_{1/h}(f),
\]

where \( \sigma_h(f, x) = f(x/h) \). From the definition it is easy to see that

\[
Q(f, x; h) = \sum_k \Lambda(f, k; h)M(h^{-1}x - k),
\]

where

\[
\Lambda(f, k; h) := \sum_{j \in P(\mu)} \lambda(j)f(h(k - j)).
\]
The operator $Q(\cdot; h)$ has the same properties as $Q$: it is a local bounded linear operator in $C(\mathbb{R})$ and reproduces the polynomials from $\mathcal{P}_{r-1}$. Moreover, it gives a good approximation for smooth functions [11, p. 63–65]. We will also call it a quasi-interpolation operator for $C(\mathbb{R})$. However, the quasi-interpolation operator $Q(\cdot; h)$ is not defined for a function $f$ on $\mathbb{R}$, and therefore, not appropriate for an approximate sampling recovery of $f$ from its sampled values at points in $\mathbb{R}$.

An approach to construct a quasi-interpolation operator for functions on $\mathbb{R}$ is to extend it by interpolation Lagrange polynomials. This approach has been proposed in [21] for the univariate case. Let us recall it.

For a non-negative integer $k$, we put $x_j = j2^{-k}, j \in \mathbb{Z}$. If $f$ is a function on $\mathbb{R}$, let $U_k(f)$ and $V_k(f)$ be the $(r-1)$th Lagrange polynomials interpolating $f$ at the $r$ left end points $x_0, x_1, ..., x_{r-1}$, and $r$ right end points $x_2^{k-r+1}, x_2^{k-r+3}, ..., x_2^k$, of the interval $\mathbb{R}$, respectively. The function $\bar{f}_k$ is defined as an extension of $f$ on $\mathbb{R}$ by the formula

$$
\bar{f}_k(x) := \begin{cases} 
U_k(f, x), & x < 0, \\
f(x), & 0 \leq x \leq 1, \\
V_k(f, x), & x > 1.
\end{cases}
$$

If $f$ is continuous on $\mathbb{R}$, then $\bar{f}_k$ is a continuous function on $\mathbb{R}$. Let $Q$ be a quasi-interpolation operator of the form (2.9)–(2.10) in $C(\mathbb{R})$. If $k \in \mathbb{Z}_+$, we introduce the operator $Q_k$ by

$$
Q_k(f, x) := Q(\bar{f}_k, x; 2^{-k}), \quad x \in \mathbb{R},
$$

for a function $f$ on $\mathbb{R}$.

We define the integer translated dilation $M_{k,s}$ of $M$ by

$$
M_{k,s}(x) := M(2^k x - s), \quad k \in \mathbb{Z}_+, \; s \in \mathbb{Z}.
$$

Then we have for $k \in \mathbb{Z}_+$,

$$
Q_k(f, x) = \sum_{s \in J(k)} a_{k,s}(f)M_{k,s}(x), \quad \forall x \in \mathbb{R},
$$

where

$$
J(k) := \{ s \in \mathbb{Z} : -r/2 < s < 2^k + r/2 \}
$$

is the set of $s$ for which $M_{k,s}$ do not vanish identically on $\mathbb{R}$, and the coefficient functional $a_{k,s}$ is defined by

$$
a_{k,s}(f) := \Lambda(\bar{f}_k, s; 2^{-k}) = \sum_{|j| \leq \mu} \lambda(j)\bar{f}_k(2^{-k}(s - j)).
$$

For $k \in \mathbb{Z}_d^d$, let the mixed operator $Q_k$ be defined by

$$
Q_k := \prod_{i=1}^d Q_{k_i}, \quad (2.11)
$$

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where the univariate operator \( Q_{k_i} \) is applied to the univariate function \( f \) by considering \( f \) as a function of variable \( x_i \) with the other variables held fixed.

We define the \( d \)-variable B-spline \( M_{k,s} \) by

\[
M_{k,s}(x) := \prod_{i=1}^{d} M_{k_i,s_i}(x_i), \quad k \in \mathbb{Z}_+^d, \quad s \in \mathbb{Z}^d.
\]  

(2.12)

Then we have

\[
Q_k(f, x) = \sum_{s \in J^d(k)} a_{k,s}(f) M_{k,s}(x), \quad \forall x \in \mathbb{I}^d,
\]

where \( M_{k,s} \) is the mixed B-spline defined in (2.12),

\[
J^d(k) := \{ s \in \mathbb{Z}^d : -r/2 < s_i < 2^{k_i} + r/2, \ i \in [d] \}
\]

is the set of \( s \) for which \( M_{k,s} \) do not vanish identically on \( \mathbb{I}^d \),

\[
a_{k,s}(f) := a_{k_1,s_1}(a_{k_2,s_2}(...a_{k_d,s_d}(f)));
\]

(2.13)

and the univariate coefficient functional \( a_{k_i,s_i} \) is applied to the univariate function \( f \) by considering \( f \) as a function of variable \( x_i \) with the other variables held fixed.

The operator \( Q_k \) is a local bounded linear mapping in \( C(\mathbb{I}^d) \) for \( r \geq 2 \) and in \( L_\infty(\mathbb{I}^d) \) for \( r = 1 \), and reproducing \( \mathcal{P}_{r-1}^d \) the space of polynomials of order at most \( r - 1 \) in each variable \( x_i \). In particular, we have for every \( f \in C(\mathbb{I}^d) \),

\[
\|Q_k(f)\|_\infty \leq C \|\Lambda\|_d^d \|f\|_{C(\mathbb{I}^d)},
\]

(2.14)

For \( k \in \mathbb{Z}_+^d \), we write \( k \to \infty \) if \( k_i \to \infty \) for \( i \in [d] \).

**Lemma 2.2** We have for every \( f \in C(\mathbb{I}^d) \),

\[
\|f - Q_k(f)\|_\infty \leq C \sum_{e \in [d], e \neq \emptyset} \omega^e(f, 2^{-k})_\infty,
\]

(2.15)

and, consequently,

\[
\|f - Q_k(f)\|_\infty \to 0, \ k \to \infty.
\]

(2.16)

**Proof.** For \( d = 1 \), the inequality (2.15) is of the form

\[
\|f - Q_k(f)\|_\infty \leq C \omega(r, 2^{-k})_\infty.
\]

(2.17)

This inequality is derived from the inequalities (2.29)–(2.31) in [22] and the inequality (2.14). For simplicity, let us prove the the inequality (2.15) for \( d = 2 \) and \( r \geq 2 \). The general case can be proven in a similar way. Let \( I \) be the identity operator and \( k = (k_1, k_2) \). From the equation

\[
I - Q_k = (I - Q_{k_1}) + (I - Q_{k_2}) - (I - Q_{k_1})(I - Q_{k_2})
\]

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and the inequality 2.17 applied to $f$ as an univariate in each variable, we obtain
\[
\|f - Q_k(f)\|_\infty \leq \|(I - Q_{k_1})(f)\|_\infty + \|(I - Q_{k_2})(f)\|_\infty + \|(I - Q_{k_1})(I - Q_{k_2})(f)\|_\infty
\]
\[
\ll \omega_r^{(1)}(f, 2^{-k}) + \omega_r^{(2)}(f, 2^{-k}) + \omega_r^{[2]}(f, 2^{-k})_\infty.
\]

If $\tau$ is a number such that $0 < \tau \leq \min(p, 1)$, then for any sequence of functions $\{g_k\}$ there is the inequality
\[
\left\| \sum g_k \right\|_p^\tau \leq \sum \|g_k\|_p^\tau.
\tag{2.18}
\]

Further, we define the half integer translated dilation $M_{k, s}^*$ of $M$ by
\[
M_{k, s}^*(x) := M(2^k x - s/2), \quad k \in \mathbb{Z}_+, \quad s \in \mathbb{Z},
\]
and the $d$-variable B-spline $M_{k, s}^*$ by
\[
M_{k, s}^*(x) := \prod_{i=1}^d M_{k_i, s_i}^*(x_i), \quad k \in \mathbb{Z}_+^d, \quad s \in \mathbb{Z}^d.
\]

In what follows, the B-spline $M$ will be fixed. We will denote $M_{k, s}^{(r)} := M_{k, s}$ if the order $r$ of $M$ is even, and $M_{k, s}^{(r)} := M_{k, s}^*$ if the order $r$ of $M$ is odd. Let $J_{r}^d(k) := J_r^d(k)$ if $r$ is even, and $J_{r}^d(k) := \{s \in \mathbb{Z}^d : -r < s < 2^k + 1 + r, \quad i \in [d]\}$ if $r$ is odd. Notice that $J_{r}^d(k)$ is the set of $s$ for which $M_{k, s}^{(r)}$ do not vanish identically on $\mathbb{I}^d$. Denote by $\Sigma_r^d(k)$ the span of the B-splines $M_{k, s}^{(r)}$, $s \in J_{r}^d(k)$. If $0 < p \leq \infty$, for all $k \in \mathbb{Z}_+^d$ and all $g \in \Sigma_r^d(k)$ such that
\[
g = \sum_{s \in J_{r}^d(k)} a_s M_{k, s}^{(r)},
\tag{2.19}
\]
there is the quasi-norm equivalence
\[
\|g\|_p \asymp 2^{-|k|_1/p} \|\{a_s\}\|_{p, k},
\tag{2.20}
\]
where
\[
\|\{a_s\}\|_{p, k} := \left( \sum_{s \in J_{r}^d(k)} |a_s|^p \right)^{1/p}
\]
with the corresponding change when $p = \infty$.

For convenience we define the univariate operator $Q_{-1}$ by putting $Q_{-1}(f) = 0$ for all $f$ on $\mathbb{I}$. Let the operator $q_k$, $k \in \mathbb{Z}_+^d$, be defined in the manner of the definition 2.11 by
\[
q_k := \prod_{i=1}^d (Q_{k_i} - Q_{k_{i-1}}).
\tag{2.21}
\]
We have
\[ Q_k = \sum_{k' \leq k} q_k'. \] (2.22)

From (2.22) and (2.16) it is easy to see that a continuous function \( f \) has the decomposition
\[ f = \sum_{k \in \mathbb{Z}^d_+} q_k(f) \]
with the convergence in the norm of \( L_\infty(\mathbb{I}^d) \).

From the definition of (2.21) and the refinement equation for the B-spline \( M \), we can represent the component functions \( q_k(f) \) as
\[ q_k(f) = \sum_{s \in J^r_{\mathbb{I}}(k)} c_{k,s}(f) \cdot M_{k,s} \] (2.23)
where \( c_{k,s}(f) \) are certain coefficient functionals of \( f \), which are defined as follows (see [23] for details).

We first define \( c_{k,s}(f) \) for univariate functions \((d = 1)\). If the order \( r \) of the B-spline \( M \) is even,
\[ c_{k,s}(f) := a_{k,s}(f) - a'_{k,s}(f), \quad k \geq 0, \] (2.24)
where
\[ a'_{k,s}(f) := 2^{-r+1} \sum_{(m,j) \in C_r(k,s)} \binom{r}{j} a_{k-1,m}(f), \quad k > 0, \quad a'_{0,s}(f) := 0. \]
and
\[ C_r(k,s) := \{(m,j) : 2m + j - r/2 = s, \ m \in J(k-1), \ 0 \leq j \leq r \}, \quad k > 0, \quad C_r(0,s) := \{0\}. \]

If the order \( r \) of the B-spline \( M \) is odd,
\[ c_{k,s}(f) := \begin{cases} 0, & k = 0, \\ a_{k,s/2}(f), & k > 0, \ s \text{ even}, \\ 2^{-r+1} \sum_{(m,j) \in C_r(k,s)} \binom{j}{s} a_{k-1,m}(f), & k > 0, \ s \text{ odd}, \end{cases} \]
where
\[ C_r(k,s) := \{(m,j) : 4m + 2j - r = s, \ m \in J(k-1), \ 0 \leq j \leq r \}, \quad k > 0, \quad C_r(0,s) := \{0\}. \]

In the multivariate case, the representation (2.23) holds true with the \( c_{k,s}(f) \) which are defined in the manner of the definition of (2.13) by
\[ c_{k,s}(f) = c_{k_1,s_1}^{(r)}(c_{k_2,s_2}^{(r)}(...c_{k_d,s_d}^{(r)}(f))). \] (2.25)

Thus, we have proven the following
Lemma 2.3 Every continuous function $f$ on $\mathbb{I}^d$ is represented as B-spline series

$$f = \sum_{k \in \mathbb{Z}_+^d} q_k(f) = \sum_{k \in \mathbb{Z}_+^d} \sum_{s \in J(k)} c_{k,s}^{(r)}(f) M_{k,s}^{(r)},$$

converging in the norm of $L_\infty(\mathbb{I}^d)$, where the coefficient functionals $c_{k,s}^{(r)}(f)$ are explicitly constructed by formula (2.24)–(2.25) as linear combinations of at most $N$ function values of $f$ for some $N \in \mathbb{N}$ which is independent of $k,s$ and $f$.

We now prove theorems on quasi-interpolation representation of functions from $B_{p,\theta}^\alpha$ and $B_{p,\theta}^\beta$ by series (2.26) satisfying a discrete equivalent quasi-norm. We need some auxiliary lemmas.

Let us use the notations: $x_+ := ((x_1)_+,\ldots,(x_d)_+)$ for $x \in \mathbb{R}^d$.

Lemma 2.4 ([23]) Let $0 < p \leq \infty$ and $\tau \leq \min(p,1)$. Then for any $f \in C(\mathbb{I}^d)$ and $k \in \mathbb{N}^d(e)$, we have

$$\|q_k(f)\|_p \leq C \sum_{v \in e} \left( \sum_{s \in \mathbb{Z}_+^d(v), \ s \geq k} \{2|s-k|\frac{1}{|1/p,\omega^v_r(f,2^{-s})_p\}^{\tau} \right)^{1/\tau}$$

with some constant $C$ depending at most on $r,\mu,p,d$ and $\|\Lambda\|$, whenever the sum in the right-hand side is finite.

Lemma 2.5 Let $0 < p \leq \infty$, $0 < \tau \leq \min(p,1)$, $\delta = \min(r, r-1 + 1/p)$. Let $g \in L_p(\mathbb{I}^d)$ be represented by the series

$$g = \sum_{k \in \mathbb{Z}_+^d} g_k, \ g_k \in \Sigma^d_r(k)$$

converging in the norm of $L_\infty(\mathbb{I}^d)$. Then for any $k \in \mathbb{Z}_+^d(e)$, there holds the inequality

$$\omega^e_r(g,2^{-k})_p \leq C \left( \sum_{s \in \mathbb{Z}_+^d(e)} \left\{ 2^{-\delta|k-s|_1} \|g_s\|_p \right\}^\tau \right)^{1/\tau}$$

with some constant $C$ depending at most on $r,\mu,p,d$ and $\|\Lambda\|$, whenever the sum on the right-hand side is finite.

Proof. This lemma can be proven in a way similar to the proof of [23 Lemma 2.3].

Let $0 < p, \theta \leq \infty$ and $\psi : \mathbb{Z}_+^d \to \mathbb{R}$. If $\{g_k\}_{k \in \mathbb{Z}_+^d}$ is a sequence whose component functions $g_k$ are in $L_p(\mathbb{I}^d)$, we define the “quasi-norm” $\|\{g_k\}\|_{b_{p,\theta}}$ by

$$\|\{g_k\}\|_{b_{p,\theta}} := \left( \sum_{k \in \mathbb{Z}_+^d} \left( 2^{\psi(k)} \|g_k\|_p^\theta \right)^{1/\theta} \right)
with the usual change to a supremum when \( \theta = \infty \). When \( \{g_k\}_{k \in \mathbb{Z}^d_+} \) is a positive sequence, we replace \( \|g_k\|_p \) by \( |g_k| \) and denote the corresponding quasi-norm by \( \|\{g_k\}\|_{b^\theta_p} \).

We will need the following generalized discrete Hardy inequality (see, e.g., [12] for the univariate case with \( \psi(k) = \alpha k, \alpha > 0 \)).

**Lemma 2.6** Let \( \{a_k\}_{k \in \mathbb{Z}^d_+} \) and \( \{b_k\}_{k \in \mathbb{Z}^d_+} \) be two positive sequences and let for some \( A > 0, \tau > 0, \delta > 0 \)

\[
b_k \leq \frac{A}{\tau} \left( \sum_{s \in \mathbb{Z}^d_+} \left( 2^{\delta |s-k|_1} \right)_{\tau} \right)^{1/\tau} \tag{2.27}
\]

Let the function \( \psi : \mathbb{Z}^d_+ \to \mathbb{R} \) satisfy the following. There are numbers \( c_1, c_2 \in \mathbb{R}, \epsilon > 0 \) and \( 0 < \zeta < \delta \) such that

\[
\psi(k) - \epsilon |k|_1 \leq \psi(k') - \epsilon |k'|_1 + c_1, \; k \leq k', \; k, k' \in \mathbb{Z}^d_+, \tag{2.28}
\]

and

\[
\psi(k) - \zeta |k|_1 \geq \psi(k') - \zeta |k'|_1 + c_2, \; k \leq k', \; k, k' \in \mathbb{Z}^d_+, \tag{2.29}
\]

Then for \( 0 < \theta \leq \infty \), we have

\[
\|\{b_k\}\|_{b^\theta_p} \leq CA \|\{a_k\}\|_{b^\theta_p} \tag{2.30}
\]

with \( C = C(c_1, c_2, \epsilon, \theta, d) > 0 \).

**Proof.** Because the right side of (2.27) becomes larger when \( \tau \) becomes smaller, we can assume \( \tau < \theta \). From (2.27) we have

\[
b_k \ll A \sum_{e \subset [d]} B_k(e), \; k \in \mathbb{Z}^d_+, \tag{2.31}
\]

where

\[
B_k(e) := 2^{-\delta |k(e)|_1} \left( \sum_{s \in Z(e,k)} \left( 2^{\delta |s(e)|_1} \right)_{\tau} \right)^{1/\tau}
\]

and

\[
Z(e,k) := \{ s \in \mathbb{Z}^d_+: s_j \leq k_j, \; j \in e; s_j > k_j, \; j \notin e \}.
\]

For \( e \subset [d] \) and \( s \in \mathbb{Z}^d \), let \( \bar{e} := [d] \setminus e \) and \( s(e) \in \mathbb{Z}^d \) be defined by \( s(e)_j = s_j \) if \( j \in e \) and \( s(e)_j = 0 \) if \( j \notin e \). Take numbers \( \epsilon', \zeta', \theta' \) with the conditions \( 0 < \epsilon' < \epsilon, \zeta < \zeta' < \delta \) and
\(\tau/\theta + \tau'/\theta' = 1\), respectively. Applying Hölder's inequality with exponents \(\theta/\tau, \theta'/\tau\), we obtain

\[
B_k(e) \leq 2^{-\delta|k(e)|_1} \left( \sum_{s \in Z(e,k)} \left( 2\zeta'|s(e)|_1 + \varepsilon'|s(e)|_1 a_s \right)^\theta \right) \left( \sum_{s \in Z(e,k)} \left( 2(\delta - \zeta')|s(e)|_1 - \varepsilon'|s(e)|_1 \right)^{\theta'} \right)^{1/\theta'} \\
\leq 2^{-\delta|k(e)|_1} \left( \sum_{s \in Z(e,k)} \left( 2\zeta'|s(e)|_1 + \varepsilon'|s(e)|_1 a_s \right)^\theta \right) \left( \sum_{s \in Z(e,k)} \left( 2(\delta - \zeta')|k(e)|_1 - \varepsilon'|k(e)|_1 \right) \left( \sum_{s \in Z(e,k)} \left( 2\zeta'|s(e)|_1 + \varepsilon'|s(e)|_1 a_s \right)^\theta \right) \right)^{1/\theta'} \\
\leq 2^{-\zeta'|k(e)|_1 - \varepsilon'|k(e)|_1} \left( \sum_{s \in Z(e,k)} \left( 2\zeta'|s(e)|_1 + \varepsilon'|s(e)|_1 a_s \right)^\theta \right) \left( \sum_{s \in Z(e,k)} \left( 2(\delta - \zeta')|k(e)|_1 - \varepsilon'|k(e)|_1 \right) \right)^{1/\theta'}.
\]

Hence,

\[
\| \{ B_k(e) \} \|^\theta_{b^\theta} \leq \sum_{k \in \mathbb{Z}^d_+} 2^\theta(\psi(k) - \zeta'|k(e)|_1 - \varepsilon'|k(e)|_1) \sum_{s \in Z(e,k)} \left( 2\zeta'|s(e)|_1 + \varepsilon'|s(e)|_1 a_s \right)^\theta \leq \sum_{k \in \mathbb{Z}^d_+} 2^\theta(\psi(k) - \zeta'|k(e)|_1 - \varepsilon'|k(e)|_1) \sum_{k \in X(e,s)} 2^\theta(\psi(k) - \zeta'|k(e)|_1 - \varepsilon'|k(e)|_1),
\]

where

\[
X(e,s) := \{ k \in \mathbb{Z}^d_+ : k_j \geq s_j, j \in e; k_j < s_j, j \notin e \}.
\]

By (2.28) and (2.29) we have for \( k \in X(e,s) \),

\[
\psi(k) = \psi(k) - \zeta|k|_1 + \zeta(|k(e)|_1 + |k(\overline{e})|_1) \\
\leq \psi(s(e),k(\overline{e})) - \zeta(|s(e)|_1 + |k(\overline{e})|_1) + \zeta(|k(e)|_1 + |k(\overline{e})|_1) \\
= \psi(s(e),k(\overline{e})) - \zeta|s(e)|_1 + \zeta|k(e)|_1,
\]

and

\[
\psi(s(e),k(\overline{e})) = \psi(s(e),k(\overline{e})) - \varepsilon(|s(e)|_1 + |k(\overline{e})|_1) + \varepsilon(|s(e)|_1 + |k(\overline{e})|_1) \\
\leq \psi(s(e),s(\overline{e})) - \varepsilon(|s(e)|_1 + |s(\overline{e})|_1) + \varepsilon(|s(e)|_1 + |k(\overline{e})|_1) \\
= \psi(s) - \varepsilon|s(\overline{e})|_1 + \varepsilon|k(\overline{e})|_1.
\]

Consequently,

\[
\psi(k) - \zeta'|k(e)|_1 - \varepsilon'|k(\overline{e})|_1 \leq \psi(s) - \zeta|s(e)|_1 - \varepsilon|s(e)|_1 - (\zeta' - \zeta)|k(e)|_1 + (\varepsilon' - \varepsilon')|k(\overline{e})|_1,
\]

and therefore, we can continue the estimation (2.32) as

\[
\| \{ B_k(e) \} \|^\theta_{b^\theta} \leq \sum_{s \in \mathbb{Z}^d_+} 2^\theta(\psi(s) + (\zeta' - \zeta)|s(e)|_1 - (\varepsilon' - \varepsilon')|s(e)|_1 a_s^\theta \sum_{k \in X(e,s)} 2^\theta(-\zeta'|k(e)|_1 - \varepsilon'|k(\overline{e})|_1) \\
\leq \sum_{s \in \mathbb{Z}^d_+} 2^\theta(\psi(s) + (\zeta' - \zeta)|s(e)|_1 - (\varepsilon' - \varepsilon')|s(e)|_1 a_s^\theta \sum_{s \in \mathbb{Z}^d_+} 2^\theta(-\zeta'|k(e)|_1 - \varepsilon'|k(\overline{e})|_1) \\
= \sum_{s \in \mathbb{Z}^d_+} 2^\theta \psi(s) a_s^\theta = \| \{ a_k \} \|^\theta_{b^\theta}.
\]
Hence, by (2.31) we prove (2.30). □

We now are able to prove quasi-interpolation B-spline representation theorems for functions from $B^p_\Omega$ and $B^{\alpha,\beta}_p$, $B^a_\Omega$. For functions $f$ on $\mathbb{R}^d$, we introduce the following quasi-norms:

$$B^2(f) := \left( \sum_{k \in \mathbb{Z}^d_+} \left\{ \|q_k(f)\|_{p,\Omega(2^{-k})} \right\}^\theta \right)^{1/\theta};$$

$$B^3(f) := \left( \sum_{k \in \mathbb{Z}^d_+} \left\{ 2^{-|k|/p} \|c_{k,s}(f)\|_{p,k,\Omega(2^{-k})} \right\}^\theta \right)^{1/\theta}.$$ 

Observe that by (2.20) the quasi-norms $B^2(f)$ and $B^3(f)$ are equivalent.

**Theorem 2.1** Let $0 < p, \theta \leq \infty$ and $\Omega$ satisfy the additional conditions: there are numbers $\mu, \rho > 0$ and $C_1, C_2 > 0$ such that

$$\Omega(t) \prod_{i=1}^d t_i^{-\mu} \leq C_1 \Omega(t') \prod_{i=1}^d t_i'^{-\mu}, \ t \leq t', \ t, t' \in \mathbb{R}^d,$$

(2.33)

$$\Omega(t) \prod_{i=1}^d t_i^{-\rho} \geq C_2 \Omega(t') \prod_{i=1}^d t_i'^{-\rho}, \ t \leq t', \ t, t' \in \mathbb{R}^d.$$

(2.34)

Then we have the following.

(i) If $\mu > 1/p$ and $\rho < r$, then a function $f \in B^\Omega_p$ can be represented by the B-spline series (2.26) satisfying the convergence condition

$$B^2(f) \ll \|f\|_{B^\Omega_p}.$$

(2.35)

(ii) If $\rho < \min(r, r - 1 + 1/p)$, then a function $g$ on $\mathbb{R}^d$ represented by a series

$$g = \sum_{k \in \mathbb{Z}^d_+} g_k = \sum_{k \in \mathbb{Z}^d_+} \sum_{s \in J^d_k} c_{k,s} M_k^{(r)},$$

(2.36)

satisfying the condition

$$B^4(g) := \left( \sum_{k \in \mathbb{Z}^d_+} \left\{ \|g_k\|_{p,\Omega(2^{-k})} \right\}^\theta \right)^{1/\theta} < \infty,$$

belongs the space $B^\Omega_p$. Moreover,

$$\|g\|_{B^\Omega_p} \ll B^4(g).$$
(iii) If \( \mu > 1/p \) and \( \rho < \min(r, r - 1 + 1/p) \), then a function \( f \) on \( \mathbb{R}^d \) belongs to the space \( B^{\Omega}_{p,\theta} \) if and only if \( f \) can be represented by the series \( \sum_{n \in \mathbb{Z}^d} c_n r^n \) satisfying the convergence condition \( (2.35) \). Moreover, the quasi-norm \( \|f\|_{B^{\Omega}_{p,\theta}} \) is equivalent to the quasi-norm \( B_2(f) \).

Proof. Put \( \phi(x) := \log_2[1/\Omega(2^{-x})] \). Due to \( (2.33) - (2.34) \), the function \( \phi \) satisfies the following conditions

\[
\phi(x) - \mu|x|_1 \leq \phi(x') - \mu|x'|_1 + \log_2 C_1, \quad x \leq x', \quad x, x' \in \mathbb{R}^d, \quad (2.37)
\]

and

\[
\phi(x) - \rho|x|_1 \geq \phi(x') - \rho|x'|_1 + \log_2 C_2, \quad x \leq x', \quad x, x' \in \mathbb{R}^d. \quad (2.38)
\]

We also have

\[
B_1(f) = \sum_{e \in [d]} \left( \sum_{k \in \mathbb{Z}^d_e} \left\{ \phi(2^k) \omega_r^e(f, 2^{-k})_p \right\}^\theta \right)^{1/\theta}, \quad (2.39)
\]

with the corresponding change to sup when \( \theta = \infty \). Fix a number \( 0 < \tau \leq \min(p, 1) \). Let

\[
N^d(e) := \{ s \in \mathbb{Z}^d : s_i > 0, \; i \in e, \; s_i = 0, \; i \notin e \} \quad \text{for} \; e \subset [d], \quad \text{in particular,} \quad N^d(\emptyset) = \{0\} \quad \text{and} \quad N^d([d]) = N^d.
\]

We have \( N^d(u) \cap N^d(v) = \emptyset \) if \( u \neq v \), and the following decomposition of \( \mathbb{Z}^d_+ \):

\[
\mathbb{Z}^d_+ = \bigcup_{e \subset [d]} N^d(e).
\]

Assertion (i): From \( (2.37) \) we derive \( \mu|k|_1 \leq \phi(k) + c, \quad k \in \mathbb{Z}^d_+ \), for some constant \( c \). Hence, by Lemma \( (2.1) \) and \( (2.39) \) we have

\[
\|f\|_{B^{p,1}_p} \leq C\|f\|_{B^{\Omega}_{p,\theta}}, \quad f \in B^{\Omega}_{p,\theta},
\]

for some constant \( C \). Since for \( \mu > 1/p \), \( B^{p,1}_p \) is compactly embedded into \( C(\mathbb{R}^d) \), by the last inequality so is \( B^{\Omega}_{p,\theta} \). Take an arbitrary \( f \in B^{\Omega}_{p,\theta} \). Then \( f \) can be treated as an element in \( C(\mathbb{R}^d) \). By Lemma \( (2.3) \) \( f \) is represented as B-spline series \( \sum_{n \in \mathbb{Z}^d} a_n r^n \) converging in the norm of \( L_\infty(\mathbb{R}^d) \). For \( k \in \mathbb{Z}^d_+ \), put

\[
b_k := 2^{|k|_1/p} \|q_k(f)\|_p, \quad a_k := \left( \sum_{e \subset [d]} \left\{ 2^{|k|_1/p \omega_r^e(f, 2^{-k})}_p \right\}^\tau \right)^{1/\tau}.
\]

if \( k \in N^d(e) \). By Lemma \( (2.4) \) we have for \( k \in \mathbb{Z}^d_+ \),

\[
b_k \leq C \left( \sum_{s \geq k} a_s^\tau \right)^{1/\tau} \leq C \left( \sum_{s \in \mathbb{Z}^d_+} \left( 2^{|(k-s)+1|_1} a_s \right)^\tau \right)^{1/\tau}, \quad k \in \mathbb{Z}^d_+,
\]

for a fixed \( \delta > \rho + 1/p \). Let the function \( \psi \) be defined by \( \psi(k) = \phi(k) - |k|_1/p, \quad k \in \mathbb{Z}^d_+ \). By the inequality \( \mu > 1/p \), \( (2.37) \) and \( (2.38) \), it is easy to see that

\[
\psi(k) - |k|_1 \leq \psi(k') - |k'|_1 + \log_2 C_1, \quad k \leq k', \quad k, k' \in \mathbb{Z}^d_+.
\]
and
\[ \psi(k) - \zeta |k|_1 \geq \psi(k') - \zeta |k'|_1 + \log_2 C_2, \quad k \leq k', \quad k, k' \in \mathbb{Z}_+^d, \]
for \( \epsilon < \mu - 1/p \) and \( \zeta = \rho + 1/p \). Hence, applying Lemma 2.6 gives
\[ B_2(f) = \|\{b_k\}\|_{b_0^\psi} \leq C \|\{a_k\}\|_{b_0^\psi} \times B_1(f) \times \|f\|_{B_0^{p,\theta}}. \]

**Assertion (ii):** For \( k \in \mathbb{Z}_+^d \), define
\[ b_k := \left( \sum_{\nu \in e} \left\{ \omega_{\nu}^p(g, 2^{-k})_p \right\}^\tau \right)^{1/\tau}, \quad a_k := \|g_k\|_p \]
if \( k \in \mathbb{N}^d(e) \). By Lemma 2.5 we have for any \( k \in \mathbb{Z}_+^d(e) \),
\[ \omega_{\nu}^p(g, 2^{-k})_p \leq C_3 \left( \sum_{s \in \mathbb{Z}_+^d} \left\{ 2^{-\delta(k-s)+1} \|g_s\|_p \right\}^\tau \right)^{1/\tau}, \]
where \( \delta = \min(r, r - 1 + 1/p) \). Therefore,
\[ b_k \leq C_4 \left( \sum_{s \in \mathbb{Z}_+^d} \left( 2^{\delta(k-s)+1} a_s \right)^\tau \right)^{1/\tau}, \quad k \in \mathbb{Z}_+^d. \]

Taking \( \zeta = \rho \) and \( 0 < \epsilon < \mu \), we obtain by (2.37) and (2.38)
\[ \phi(k) - \epsilon |k|_1 \leq \phi(k') - \epsilon |k'|_1 + \log_2 C_1, \quad k \leq k', \quad k, k' \in \mathbb{Z}_+^d, \]
and
\[ \phi(k) - \zeta |k|_1 \geq \phi(k') - \zeta |k'|_1 + \log_2 C_2, \quad k \leq k', \quad k, k' \in \mathbb{Z}_+^d. \]
Applying Lemma 2.6 we get
\[ \|g\|_{B_0^{p,\theta}} \times B_1(g) \times \|\{b_k\}\|_{b_0^\psi} \leq C \|\{a_k\}\|_{b_0^\psi} = B_4(g). \]

**Assertion (ii) is proven.**

**Assertion (iii):** This assertion follows from Assertions (i) and (ii). \( \square \)

From Assertion (ii) in Theorem 2.1 we obtain

**Corollary 2.1** Let \( 0 < p, \theta \leq \infty \) and \( \Omega \) satisfy the assumptions of Assertion (ii) in Theorem 2.1. Then for every \( k \in \mathbb{Z}_+^d \), we have
\[ \|g\|_{B_0^{p,\theta}} \ll \|g\|_p / \Omega(2^{-k}), \quad g \in \Sigma_\nu^d(k). \]
Theorem 2.2 Let $0 < p, \theta \leq \infty$ and $a \in \mathbb{R}^d_+$. Then we have the following.

(i) If $1/p < \min_{j \in [d]} a_j \leq \max_{j \in [d]} a_j < r$, then a function $f \in B^a_{p,\theta}$ can be represented by the mixed B-spline series (2.26) satisfying the convergence condition

$$B_2(f) = \left( \sum_{k \in \mathbb{Z}^d_+} \{2^{(a,k)\|q_k(f)\|_p}\}^\theta \right)^{1/\theta} \ll \|f\|_{B^a_{p,\theta}}. \quad (2.40)$$

(ii) If $0 < \min_{j \in [d]} a_j \leq \max_{j \in [d]} a_j < \min(r, r - 1 + 1/p)$, then a function $g$ on $I^d$ represented by a series (2.36) satisfying the condition

$$B_4(g) := \left( \sum_{k \in \mathbb{Z}^d_+} \{2^{(a,k)\|g_k\|_p}\}^\theta \right)^{1/\theta} < \infty,$$

belongs to the space $B^a_{p,\theta}$. Moreover,

$$\|g\|_{B^a_{p,\theta}} \ll B_4(g).$$

(iii) If $1/p < \min_{j \in [d]} a_j \leq \max_{j \in [d]} a_j < \min(r, r - 1 + 1/p)$, then a function $f$ on $I^d$ belongs to the space $B^a_{p,\theta}$ if and only if $f$ can be represented by the series (2.26) satisfying the convergence condition (2.40). Moreover, the quasi-norm $\|f\|_{B^a_{p,\theta}}$ is equivalent to the quasi-norms $B_2(f)$.

Proof. For $\Omega$ as in (2.5), we have $1/\Omega(2^{-x}) = 2^{(a,x)}$, $x \in \mathbb{R}^d_+$. One can directly verify the conditions (2.1)–(2.3) and the conditions (2.33)–(2.34) with $1/p < \mu < \min_{j \in [d]} a_j$ and $\rho = \max_{j \in [d]} a_j$, for $\Omega$ defined in (2.5). Applying Theorem 2.1(i), we obtain the assertion (i).

The assertion (ii) can be proven in a similar way. The assertion (iii) follows from the assertions (i) and (iii). \(\square\)

Theorem 2.3 Let $0 < p, \theta \leq \infty$ and $\alpha \in \mathbb{R}_+, \beta \in \mathbb{R}$. Then we have the following.

(i) If $1/p < \min(\alpha, \alpha + \beta) \leq \max(\alpha, \alpha + \beta) < r$, then a function $f \in B^{\alpha,\beta}_{p,\theta}$ can be represented by the mixed B-spline series (2.26) satisfying the convergence condition

$$B_2(f) = \left( \sum_{k \in \mathbb{Z}^d_+} \{2^{\alpha|k|_1 + \beta|k|_\infty}\|q_k(f)\|_p\}^\theta \right)^{1/\theta} \ll \|f\|_{B^{\alpha,\beta}_{p,\theta}}. \quad (2.41)$$

(ii) If $0 < \min(\alpha, \alpha + \beta) \leq \max(\alpha, \alpha + \beta) < \min(r, r - 1 + 1/p)$, then a function $g$ on $I^d$ represented by a series (2.36) satisfying the condition

$$B_4(g) := \left( \sum_{k \in \mathbb{Z}^d_+} \{2^{\alpha|k|_1 + \beta|k|_\infty}\|g_k\|_p\}^\theta \right)^{1/\theta} < \infty,$$
\[ \|g\|_{B_{p,\theta}^{\alpha,\beta}} \ll B_4(g). \]

(iii) If \( 1/p < \min(\alpha, \alpha + \beta) \leq \max(\alpha, \alpha + \beta) < \min(r, r - 1 + 1/p) \), then a function \( f \) on \( \mathbb{R}^d \) belongs to the space \( B_{p,\theta}^{\alpha,\beta} \) if and only if \( f \) can be represented by the series (2.26) satisfying the convergence condition (2.31). Moreover, the quasi-norm \( \|f\|_{B_{p,\theta}^{\alpha,\beta}} \) is equivalent to the quasi-norms \( B_2(f) \).

\[ \phi(x) := \log_2 \{1/\Omega(2^{-x})\} = \alpha |x|_1 + \beta |x|_\infty, \ x \in \mathbb{R}^d. \]

Then the conditions (2.22)–(2.23) and (2.33)–(2.34) are equivalent to the following conditions for the function \( \phi \),

\[ \phi(x) \leq \phi(x') + \log_2 C, \ x \leq x', \ x, x' \in \mathbb{R}^d; \] (2.42)

for every \( b \leq \log_2 \gamma := (\log_2 \gamma_1, ..., \log_2 \gamma_d) \),

\[ \phi(x + b) \leq \phi(x) + \log_2 C', \ x, x + b \in \mathbb{R}^d; \] (2.43)

\[ \phi(x) - \mu |x|_1 \leq \phi(x') - \mu |x'|_1 + \log_2 C_1, \ x \leq x', \ x, x' \in \mathbb{R}^d; \] (2.44)

\[ \phi(x) - \rho |x|_1 \geq \phi(x') - \rho |x'|_1 + \log_2 C_2, \ x \leq x', \ x, x' \in \mathbb{R}^d. \] (2.45)

We first consider the case \( \beta \geq 0 \). Take \( \mu \) and \( \rho \) with the conditions \( 1/p < \mu < \alpha \) and \( \rho = \alpha + \beta \). The conditions (2.42)–(2.43) can be easily verified. From the inequality \( \alpha - \mu > 0 \) and the equation

\[ \phi(x) - \mu |x|_1 = (\alpha - \mu) |x|_1 + \beta |x|_\infty, \ x \in \mathbb{R}^d. \] (2.46)

follows (2.44). We have

\[ \phi(x) - \rho |x|_1 = \beta (|x|_\infty - |x|_1) = -\beta \min_{j \in [d]} \sum_{i \neq j} x_i, \ x \in \mathbb{R}^d. \]

Hence, we deduce (2.45).

Let us next consider the case \( \beta < 0 \). The condition (2.43) is obvious. Take \( \mu \) and \( \rho \) with the conditions \( 1/p < \mu < \alpha + \beta \) and \( \alpha < \rho < r \). Let \( x \leq x', \ x, x' \in \mathbb{R}^d \). Assume that \( |x|_\infty = x_j \) and \( |x'|_\infty = x'_j \). By using the inequalities \( x_j \leq x_j \leq x'_j \leq x'_j \) and \( \alpha - \mu > \alpha + \beta - \mu > 0 \) from (2.46), we get

\[ \phi(x) - \mu |x|_1 = (\alpha - \mu) \sum_{i \neq j} x_i + (\alpha + \beta - \mu) x_j + (\alpha - \mu) x_j \]

\[ \leq (\alpha - \mu) \sum_{i \neq j} x'_i + (\alpha + \beta - \mu) x'_j + (\alpha - \mu) x'_j \]

\[ = \phi(x') - \mu |x'|_1. \]
The inequality (2.44) is proven. The inequality (2.42) and (2.45) can be proven analogously. Instead the inequalities \( \alpha - \mu > \alpha + \beta - \mu > 0 \), in the proof we should use \( \alpha > \alpha + \beta > 0 \) and \( \alpha + \beta - \rho < \alpha - \rho < 0 \), respectively. Thus, the assertion (i) is proven.

The assertion (ii) can be proven in a similar way. The assertion (iii) follows from the assertions (i) and (iii).

**Remark** Theorem 2.2 for \( a = \alpha 1 \) and Theorem 2.3 for \( \beta = 0 \) coincide. This particular case has been proven in [23]. Some modifications of the space \( B_{p,\theta}^{\Omega} \) were introduced in [15] where approximations by trigonometric polynomials with frequencies from hyperbolic cross, \( n \)-widths were investigated from functions from these spaces.

There are many examples of function \( \Omega \) for the space \( B_{p,\theta}^{\Omega} \) for which Theorem 2.1 is true with some light natural restrictions and to which it is interesting to extend the results of the present paper. Let us give some important ones of them.

For bounded sets \( A, B \) with \( B \subset A \subset \mathbb{R}^d_+ \),

\[
\Omega(t) = \inf_{x \in A} \prod_{i=1}^{d} t_{x_i}^\alpha \sup_{y \in B} \prod_{i=1}^{d} t_{y_i}^{\beta}.
\]

For univariate functions \( \Omega_j, \ j \in [d] \), satisfying the conditions (2.1)–(2.3),

\[
\Omega(t) = \prod_{j \in [d]} \Omega_j(t_j).
\]

For \( a \in \mathbb{R}^d_+ \) and a univariate function \( \Omega^* \) satisfying the conditions (2.1)–(2.3),

\[
\Omega(t) = \Omega^*(\prod_{i=1}^{d} t_{i}^{a_i}).
\]

### 3 Sampling recovery

Let \( \Delta \subset \mathbb{Z}^d_+ \) be given. Put \( K(\Delta) := \{(k, s) : k \in \Delta, \ s \in I^d(k)\} \) and denote by \( M^d(\Delta) \) the set of B-spines \( M_{k,s}^{(r)}, k \in \Delta, s \in J^d_r(k) \). We define the operator \( R_\Delta \) for functions \( f \) on \( \mathbb{I}^d \) by

\[
R_\Delta(f) := \sum_{k \in \Delta} q_k(f) = \sum_{k \in \Delta} \sum_{s \in J^d_r(k)} c_{k,s}^{(r)}(f) M_{k,s}^{(r)},
\]

and the grid \( G(\Delta) \) of points in \( \mathbb{I}^d \) by

\[
G(\Delta) := \{2^{-k}s : (k, s) \in K(\Delta)\}.
\]
Lemma 3.1 The operator $R_{\Delta}$ defines a linear sampling algorithm of the form (1.1) on the grid $G(\Delta)$. More precisely,

$$R_{\Delta}(f) = L_n(X_n, \Phi_n, f) = \sum_{(k,s) \in K(\Delta)} f(2^{-k}j)\psi_{k,s},$$

where $X_n := G(\Delta) = \{2^{-k}s\}_{(k,s) \in K(\Delta)}$, $\Phi_n := \{\psi_{k,j}\}_{(k,s) \in K(\Delta)}$,

$$n := |G(\Delta)| = \sum_{k \in \Delta} \prod_{j=1}^{d} (2^{kj} + 1),$$

and $\psi_{k,j}$ are explicitly constructed as linear combinations of at most $N$ B-splines $M_{k,s}^{(r)} \in M_{\psi}^{d}(\Delta)$ for some $N \in \mathbb{N}$ which is independent of $k, j, \Delta$ and $f$.

Proof. This lemma can be proven in a way similar to the proof of [23 Lemma 3.1]. \qed

Let $\psi : \mathbb{Z}_+^d \to \mathbb{R}_+$. Denote by $B_{p,\theta}^{(\psi)}$ the space of all functions $f$ on $\mathbb{I}^d$ for which the following quasi-norm is finite

$$\|f\|_{B_{p,\theta}^{(\psi)}} := \left( \sum_{k \in \mathbb{Z}_+^d \setminus \Delta} \{2^{\psi(k)}\|q_k(f)\|_p\}^\theta \right)^{1/\theta}.

Lemma 3.2 Let $0 < p, \theta, q \leq \infty$ and $\psi : \mathbb{Z}_+^d \to \mathbb{R}_+$. Then for every $f \in B_{p,\theta}^{(\psi)}$, we have the following.

(i) For $p \geq q$,

$$\|f - R_{\Delta}(f)\|_q \ll \|f\|_{B_{p,\theta}^{(\psi)}} \left\{ \sup_{k \in \mathbb{Z}_+^d \setminus \Delta} 2^{-\psi(k)}, \quad \theta \leq \min(q,1) \right\} \left( \sum_{k \in \mathbb{Z}_+^d \setminus \Delta} \{2^{\psi(k)}\}^{\theta^*} \right)^{1/\theta^*}, \quad \theta > \min(q,1),$$

where $\theta^* := \frac{1}{1/\min(q,1) - 1/\theta}$.

(ii) For $p < q < \infty$,

$$\|f - R_{\Delta}(f)\|_q \ll \|f\|_{B_{p,\theta}^{(\psi)}} \left\{ \sup_{k \in \mathbb{Z}_+^d \setminus \Delta} 2^{-\psi(k)+(1/p-1/q)|k|_1}, \quad \theta \leq q \right\} \left( \sum_{k \in \mathbb{Z}_+^d \setminus \Delta} \{2^{\psi(k)+(1/p-1/q)|k|_1}\}^{q^*} \right)^{1/q^*}, \quad \theta > q,$

where $q^* := \frac{1}{1/q - 1/\theta}$. 

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(iii) For $p < q = \infty$,

$$
\|f - R_\Delta(f)\|_\infty \ll \|f\|_{B^{(\psi)}_{p,\theta}} \left\{ \begin{array}{ll}
\sup_{k \in \mathbb{Z}_+^d \setminus \Delta} 2^{-\psi(k) + |k|/p}, & \theta \leq 1 \\
\left( \sum_{k \in \mathbb{Z}_+^d \setminus \Delta} \{2^{-\psi(k) + |k|/p}\}^{\theta'} \right)^{1/\theta'}, & \theta > 1,
\end{array} \right.
$$

where $\theta' := \frac{1}{1 - 1/\theta}$.

**Proof.**

**Case (i):** $p \geq q$. For an arbitrary $f \in B^{(\psi)}_{p,\theta}$, by the representation (2.26) and (2.18) we have

$$
\|f - R_\Delta(f)\|_q^\tau \ll \sum_{k \in \mathbb{Z}_+^d \setminus \Delta} \|q_k(f)\|_q^\tau
$$

with any $\tau \leq \min(q, 1)$. Therefore, if $\theta \leq \min(q, 1)$, then by Theorem 2.3 and the inequality $\|q_k(f)\|_q \leq \|q_k(f)\|_p$ we get

$$
\|f - R_\Delta(f)\|_q \ll \left( \sum_{k \in \mathbb{Z}_+^d \setminus \Delta} \|q_k(f)\|_q^\theta \right)^{1/\theta} \ll \sup_{k \in \mathbb{Z}_+^d \setminus \Delta} 2^{-\psi(k)} \left( \sum_{k \in \mathbb{Z}_+^d \setminus \Delta} \{2^{\psi(k)}\|q_k(f)\|_p\}^{\theta} \right)^{1/\theta} \ll \|f\|_{B^{(\psi)}_{p,\theta}} \sup_{k \in \mathbb{Z}_+^d \setminus \Delta} 2^{-\psi(k)}.
$$

If $\theta > \min(q, 1)$, then

$$
\|f - R_\Delta(f)\|_q^\nu \ll \sum_{k \in \mathbb{Z}_+^d \setminus \Delta} \|q_k(f)\|_q^\nu = \sum_{k \in \mathbb{Z}_+^d \setminus \Delta} \{2^{\psi(k)}\|q_k(f)\|_q\}^\nu \{2^{-\psi(k)}\}^\nu,
$$

where $\nu = \min(q, 1)$. Since $\nu/\theta + \nu/\theta^* = 1$, by Hölder’s inequality with exponents $\theta/\nu, \theta^*/\nu$, the inequality $\|q_k(f)\|_q \leq \|q_k(f)\|_p$ and Theorem 2.3 we obtain

$$
\|f - R_\Delta(f)\|_q \ll \left( \sum_{k \in \mathbb{Z}_+^d \setminus \Delta} \{2^{\psi(k)}\|q_k(f)\|_q\}^{\theta} \right)^{1/\theta} \left( \sum_{k \in \mathbb{Z}_+^d \setminus \Delta} \{2^{-\psi(k)}\}^{\theta^*} \right)^{1/\theta^*} \ll \|f\|_{B^{(\psi)}_{p,\theta}} \left( \sum_{k \in \mathbb{Z}_+^d \setminus \Delta} \{2^{-\psi(k)}\}^{\theta^*} \right)^{1/\theta^*}.
$$

This and (3.1) prove Case (i).
Case (ii): $p < q < \infty$. For an arbitrary $f \in B_{p,\theta}^{(\psi)}$, by the representation (2.26) and Lemma 5.3, we have

$$
\|f - R_\Delta(f)\|_q^q \leq \sum_{k \in \mathbb{Z}_+^d \setminus \Delta} \{2^{(1/p-1/q)|k_1|} \|q_k(f)\|_p \}^q.
$$

Therefore, if $\theta \leq q$, then

$$
\|f - R_\Delta(f)\|_q \leq \left( \sum_{k \in \mathbb{Z}_+^d \setminus \Delta} \{2^{(1/p-1/q)|k_1|} \|q_k(f)\|_p \}^\theta \right)^{1/\theta}
$$

$$
\leq \sup_{k \in \mathbb{Z}_+^d \setminus \Delta} 2^{-\psi(k) + (1/p-1/q)|k_1|} \left( \sum_{k \in \mathbb{Z}_+^d \setminus \Delta} \{2^{\psi(k)} \|q_k(f)\|_p \}^\theta \right)^{1/\theta}
$$

$$
\leq \|f\|_{B_{p,\theta}^{(\psi)}} \sup_{k \in \mathbb{Z}_+^d \setminus \Delta} 2^{-\psi(k) + (1/p-1/q)|k_1|}.
$$

If $\theta > q$, then

$$
\|f - R_\Delta(f)\|_q^q \leq \sum_{k \in \mathbb{Z}_+^d \setminus \Delta} \{2^{(1/p-1/q)|k_1|} \|q_k(f)\|_p \}^q
$$

$$
= \sum_{k \in \mathbb{Z}_+^d \setminus \Delta} \{2^{\psi(k)} \|q_k(f)\|_p \}^q \{2^{-\psi(k) + (1/p-1/q)|k_1|} \}^q.
$$

Hence, similarly to (3.2), we get

$$
\|f - R_\Delta(f)\|_q \leq \|f\|_{B_{p,\theta}^{(\psi)}} \left( \sum_{k \in \mathbb{Z}_+^d \setminus \Delta} \{2^{-\psi(k) + (1/p-1/q)|k_1|} \}^q \right)^{1/q^*}.
$$

This completes the proof of Case (ii).

Case (iii): $p < q = \infty$. Case (iii) can be proven analogously to Case (ii) by using the inequality

$$
\|f - R_\Delta(f)\|_\infty \leq \sum_{k \in \mathbb{Z}_+^d \setminus \Delta} 2^{|k_1|/p} \|q_k(f)\|_p.
$$

Denote by $\mathbb{R}_+^3$ the set of triples $(p, \theta, q)$ such that $0 < p, \theta, q \leq \infty$. According to Lemma 3.2, depending on the relationship between $p, \theta, q$ for $(p, \theta, q) \in \mathbb{R}_+^3$, the error $\|f - R_\Delta(f)\|_q$ of the approximation of $f \in B_{p,\theta}^{(\psi)}$ has an upper bound of two different forms: either

$$
\|f - R_\Delta(f)\|_q \leq \|f\|_{B_{p,\theta}^{(\psi)}} \sup_{k \in \mathbb{Z}_+^d \setminus \Delta} 2^{-\psi(k) + (1/p-1/q)_+|k_1|},
$$

(3.3)
Theorem 3.1
Let \( 0 < p, \theta, q \leq \infty \), and define the set \( \Delta(\xi) \) as
\[
\| f - R_{\Delta}(f) \|_q \ll \| f \|_{B^{(p,\theta)}_p} \left( \sum_{k \in \mathbb{Z}^d_+ \setminus \Delta} \{2^{-\psi(k) + (1/p - 1/q)_+ |k|_1}\}^{1/\tau} \right). \tag{3.4}
\]

Let us decompose \( \mathbb{R}^3_+ \) into two sets \( A \) and \( B \) with \( A \cap B = \emptyset \) as follows. A triple \((p, \theta, q) \in \mathbb{R}^3_+ \) belongs to \( A \) if and only if for \((p, \theta, q) \) there holds (3.3), and belongs to \( B \) if and only if for \((p, \theta, q) \) there holds (3.4). By Lemma 3.2, \( A \) consists of all \((p, \theta, q) \in \mathbb{R}^3_+ \) satisfying one of the following conditions

- \( p \geq q, \theta \leq \min(q, 1) \);
- \( p < q, \theta \leq q \);
- \( p < q = \infty, \theta \leq 1 \),

and \( B \) consists of all \((p, \theta, q) \in \mathbb{R}^3_+ \) satisfying one of the following conditions

- \( p \geq q, \theta > \min(q, 1) \);
- \( p < q, \theta > q \);
- \( p < q = \infty, \theta > 1 \).

We construct special sets \( \Delta(\xi) \) parametrized by \( \xi > 0 \), for the recovery of functions \( f \in U^{\alpha,\beta}_{p,\theta} \) by \( R_{\Delta(\xi)}(f) \). Let \( 0 < p, \theta, q \leq \infty \) and \( \alpha \in \mathbb{R}_+, \beta \in \mathbb{R} \) be given. We fix a number \( \varepsilon \) so that
\[
0 < \varepsilon < \min(\alpha - (1/p - 1/q)_+, |\beta|),
\]
and define the set \( \Delta(\xi) \) for \( \xi > 0 \) by
\[
\Delta(\xi) := \begin{cases}
\{ k \in \mathbb{Z}^d_+ : (\alpha - (1/p - 1/q)_+) |k|_1 + \beta |k|_\infty \leq \xi \}, & (p, \theta, q) \in A, \\
\{ k \in \mathbb{Z}^d_+ : (\alpha - (1/p - 1/q)_+ + \varepsilon/d) |k|_1 + (\beta - \varepsilon) |k|_\infty \leq \xi \}, & (p, \theta, q) \in B, \beta > 0, \\
\{ k \in \mathbb{Z}^d_+ : (\alpha - (1/p - 1/q)_+ - \varepsilon) |k|_1 + (\beta + \varepsilon) |k|_\infty \leq \xi \}, & (p, \theta, q) \in B, \beta < 0.
\end{cases}
\]

Preliminarily note that for \((p, \theta, q) \in A, \Delta(\xi) \) is defined as the set \( \{ k \in \mathbb{Z}^d_+ : (\alpha - (1/p - 1/q)_+) |k|_1 + \beta |k|_\infty \leq \xi \} \), but for \((p, \theta, q) \in B, \Delta(\xi) \) is defined as an extension of the last one parametrized by \( \varepsilon \). We will give a detailed comment on this substantial difference in a remark at the end of the next section where the optimality and sparsity are investigated.

Theorem 3.1 Let \( 0 < p, \theta, q \leq \infty \) and \( \alpha \in \mathbb{R}_+, \beta \in \mathbb{R}, \beta \neq 0 \), such that
\[
1/p < \min(\alpha, \alpha + \beta) \leq \max(\alpha, \alpha + \beta) < r.
\]
Then we have the following upper bound
\[
\sup_{f \in U_{p,\theta}} \|f - R_\Delta(f)\|_q \ll 2^{-\xi}.
\] (3.5)

**Proof.** If \((p, \theta, q) \in A\), by Lemma 3.2, we have
\[
\sup_{f \in U_{p,\theta}} \|f - R_\Delta(f)\|_q \ll \sup_{k \in Z_+^d \setminus \Delta(\xi)} 2^{-\tau(\alpha/(1-p-1/q)_+)|k|_1 + \beta|k|_\infty} \ll 2^{-\xi}.
\]

We next consider the case \((p, \theta, q) \in B\). In this case, by Lemma 3.2, we have
\[
\sup_{f \in U_{p,\theta}} \|f - R_\Delta(f)\|_q \ll \sum_{k \in Z_+^d \setminus \Delta(\xi)} 2^{-\tau(\alpha/(1-p-1/q)_+)|k|_1 + \beta|k|_\infty} =: \Sigma(\xi).
\] (3.6)

We first assume that \(\beta > 0\). It is easy to verify that for every \(\xi > 0\),
\[
\Sigma(\xi) \asymp \int_{W(\xi)} 2^{-(\alpha(1,x) - \beta M(x))} dx,
\] (3.7)
where \(M(x) := \max_{j \in [d]} x_j\) for \(x \in \mathbb{R}^d\), and
\[
W(\xi) := \{x \in \mathbb{R}^d_+ : \alpha(1 + \epsilon/d)(1, x) + (\beta - \epsilon)M(x) > \xi\}.
\]

Putting
\[
V(\xi, s) := \{x \in W(\xi) : \xi + s - 1 \leq \alpha(1, x) + \beta M(x) < \xi + s\}, \quad s \in \mathbb{N},
\]
from (3.7) we have
\[
\Sigma(\xi) \asymp 2^{-\xi} \sum_{s=1}^{\infty} 2^{-s}|V(\xi, s)|.
\] (3.8)

Let us estimate \(|V(\xi, s)|\). Put \(V^*(\xi, s) := V(\xi, s) - x^*\), where \(x^* := (\nu d)^{-1}\xi 1\) and \(\nu := \alpha + \beta/d\). For every \(y = x - x^* \in V^*(\xi, s)\), from the equation \((1, x^*) = \xi/\nu\) and the inequality \(\alpha(1, x) + \beta M(x) < \xi + s\) we get
\[
\alpha(1, y) + \beta M(y) < s.
\] (3.9)

On the other hand, for every \(x \in V(\xi, s)\), from the inequality \(\alpha(1, x) + \beta M(x) < \xi + s\) and \((\alpha + \epsilon/d)(1, x) + (\beta - \epsilon)M(x) > \xi\) we get \(M(x) - (1, x)/d < \epsilon^{-1}s\). This inequality together with
the inequality $\alpha(1, x) + \beta M(x) \geq s-1$ gives $(1, x) \geq \xi/\nu + (1-\varepsilon-1)\beta + 1)/\nu$ for every $x \in V(\xi, s)$. Hence, for every $y = x - x^* \in V^*(\xi, s)$,

$$(1, y) \geq ((1-\varepsilon-1)\beta + 1)/\nu. \quad (3.10)$$

This means that $V^*(\xi, s) \subset V'(s)$ for every $\xi > 0$, where $V'(s) \subset \mathbb{R}^d$ is the set of all $y \in \mathbb{R}^d$ given by the conditions (3.9) and (3.10). Since $V'(s)$ is a bounded polyhedron and consequently,

$$|V(\xi, s)| = |V^*(\xi, s)| \leq |V'(s)| \approx s^d,$$

combining (3.6) and (3.8), we obtain

$$\sup_{f \in U_{p, \theta}^a} \|f - R_{\Delta (\xi)}(f)\|_q \ll 2^{-\xi} \sum_{s=1}^{\infty} 2^{-s} s^d \times 2^{-\xi}. \quad (3.12)$$

If $\beta < 0$, similarly to (3.6) and (3.7), we have for every $\xi > 0$,

$$\Sigma(\xi) \approx \int_{W'(\xi)} 2^{-(a_1, x) - \beta M(x)} dx,$$

where

$$W'(\xi) := \{x \in \mathbb{R}^d_+: (a - \varepsilon)(1, x) + (\beta + \varepsilon) M(x) > \xi\}.$$ 

From the last relation, similarly to the proof for the case $\beta > 0$, we prove (3.5) for the case $\beta < 0$. $\Box$

We construct special sets $\Delta'(\xi)$ parametrized by $\xi > 0$, for the recovery of functions $f \in U_{p, \theta}^a$ by $R_{\Delta'(\xi)}(f)$. Let $0 < p, q, \theta \leq \infty$ and $a \in \mathbb{R}^d_+$ be given. In what follows, we assume the following restriction on the smoothness $a$ of $B_{p, \theta}^a$:

$$1/p < a_1 < a_2 < \ldots < a_d < r. \quad (3.11)$$

We fix a number $\varepsilon$ so that

$$0 < \varepsilon < a_2 - a_1,$$

and define the set $\Delta'(\xi)$ for $\xi > 0$, by

$$\Delta'(\xi) := \{k \in \mathbb{Z}_+^d : (a(k) - (1/p - 1/q)_+|k|_1 \leq \xi \}, \quad (p, \theta, q) \in A,$

$$\{k \in \mathbb{Z}_+^d : (a(\varepsilon), k) - (1/p - 1/q)_+|k|_1 \leq \xi \}, \quad (p, \theta, q) \in B,$$

where $a(\varepsilon) = (a_1, a_2 - \varepsilon, \ldots, a_d - \varepsilon)$.

**Theorem 3.2** Let $0 < p, \theta, q \leq \infty$ and $a \in \mathbb{R}^d_+$ satisfying the condition (3.11) and

$$1/p < a_1 < a_d < r.$$

Then we have the following upper bound

$$\sup_{f \in U_{p, \theta}^a} \|f - R_{\Delta'(\xi)}(f)\|_q \ll 2^{-\xi}. \quad (3.12)$$
Proof. Let us first consider the case \((p, \theta, q) \in A\). In this case, by Lemma 3.2 we have
\[
\sup_{f \in U_{p, \theta}^a} \| f - R_{\Delta^\prime}(\xi)(f) \|_q \ll \sup_{k \in \mathbb{Z}^d_+ \setminus \Delta^\prime(\xi)} 2^{-(a, k) - (1/p - 1/q)_+ |k|_1} \ll 2^{-\xi}.
\]

We next treat the case \((p, \theta, q) \in B\). In this case, by Lemma 3.2 we have
\[
\sup_{f \in U_{p, \theta}^a} \| f - R_{\Delta^\prime}(\xi)(f) \|_q^\tau \ll \sum_{k \in \mathbb{Z}^d_+ \setminus \Delta^\prime(\xi)} 2^{-\tau((a, k) - (1/p - 1/q)_+ |k|_1)}
\]
for \(\tau = \theta^*, q^*, \theta^\prime\). For simplicity we prove the case \((p, \theta, q) \in B\) for \(\tau = 1\) and \((1/p - 1/q)_+ = 0\), the general case can be proven similarly. In this particular case, we get
\[
\sup_{f \in U_{p, \theta}^a} \| f - R_{\Delta^\prime}(\xi)(f) \|_q \ll \sum_{k \in \mathbb{Z}^d_+ \setminus \Delta^\prime(\xi)} 2^{-(a, k)} =: \Sigma(\xi). \tag{3.13}
\]
It is easy to verify that for every \(\xi > 0\),
\[
\Sigma(\xi) \asymp \int_{W(\xi)} 2^{-(a, x)} dx, \tag{3.14}
\]
where
\[W(\xi) := \{x \in \mathbb{R}^d_+: (a', x) > \xi\}\].

We put
\[V(\xi, s) := \{x \in W(\xi) : \xi + s - 1 \leq (a, x) < \xi + s\}, s \in \mathbb{N}\]
then from (3.14) we have
\[
\Sigma(\xi) \asymp 2^{-\xi} \sum_{s=1}^{\infty} 2^{-s} |V(\xi, s)|. \tag{3.15}
\]

Let us estimate \(|V(\xi, s)|\). Put \(V^*(\xi, s) := V(\xi, s) - x^*, \) where \(x^* := (a_1)^{-1}\xi e^1\). For every \(y = x - x^* \in V^*(\xi, s)\), from the equation \((a, x^*) = \xi\) and the inequality \((a, x) < \xi + s\) we get \((a, y) < s\) and therefore,
\[
y_j < s/a_j, j \in [d]. \tag{3.16}
\]

On the other hand, for every \(x \in V(\xi, s)\), from the inequality \((a, x) < \xi + s\) and \((a, x) - \varepsilon(1', x) = (a', x) > \xi\) we get \((1', x) < \varepsilon^{-1}s\), where \(1' := (0, 1, 1, ..., 1) \in \mathbb{R}^d\). This inequality together with the inequality \(a_1 x_1 + a_d(1', x) \geq (a, x) \geq \xi + s - 1\) gives \(x_1 \geq \xi/a_1 + ((1 - 1/a_d)s + 1)/a_1\) for every \(x \in V(\xi, s)\). Hence, for every \(y = x - x^* \in V^*(\xi, s)\),
\[
y_1 \geq ((1 - 1/a_d)s + 1)/a_1, \quad y_j \geq 0, j = 2, ..., d. \tag{3.17}
\]
This means that \(V^*(\xi, s) \subset V'(s)\) for every \(\xi > 0\), where \(V'(s) \subset \mathbb{R}^d\) is the box of all \(y \in \mathbb{R}^d\) given by the conditions (3.16) and (3.17). Since
\[|V(\xi, s)| = |V^*(\xi, s)| \leq |V'(s)| \asymp s^d,
\]
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by (3.13) and (3.15), we obtain
\[
\sup_{f \in U_{p,\theta}} \| f - R_{\Delta(\xi)} \|_q \ll 2^{\xi} \sum_{s=1}^{\infty} 2^{-s} s^d \times 2^{-\xi}.
\]

Remark The grids \( G(\Delta(\xi)) \) and \( G(\Delta'(\xi)) \) defined for \((p, \theta, q) \in A \) or \((p, \theta, q) \in B \) with \( \beta > 0 \), were employed in [16, 17, 18] for sampling recovery of periodic functions from an intersection of spaces of different mixed smoothness. The grids \( G(\Delta(\xi)) \) defined for \( \beta = -1, \theta = 1, p = q \geq 1 \), were used in [6] for sampling recovery of non-periodic functions based on a hierarchical Lagrangian basis polynomials representation, with the approximation error measured in the energy \( H^1 \)-norm.

4 Sparsity and optimality

Lemma 4.1 Let \( 0 < p, \theta, q \leq \infty \) and \( \alpha \in \mathbb{R}_+, \beta \in \mathbb{R}, \beta \neq 0 \), such that
\[
1/p < \min(\alpha, \alpha + \beta) \leq \max(\alpha, \alpha + \beta) < r.
\]
Then we have
\[
|G(\Delta(\xi))| \asymp \sum_{k \in \Delta(\xi)} 2^{k_1} \asymp 2^{\xi/\nu},
\]
where
\[
\nu := \begin{cases} 
\alpha + \beta/d - (1/p - 1/q)_+, & \beta > 0, \\
\alpha + \beta - (1/p - 1/q)_+, & \beta < 0.
\end{cases}
\]

Proof. The first asymptotic equivalence in (4.1) follows from the definitions. Let us prove the second one. For simplicity we prove it for the case where \( p \geq q \), the general case can be proven similarly.

Let us first consider the case \((p, \theta, q) \in B, \beta > 0 \). It is easy to verify that for every \( \xi > 0 \),
\[
\sum_{k \in \Delta(\xi)} 2^{k_1} \asymp \int_{W(\xi)} 2^{(1,x)} \, dx,
\]
where
\[
W(\xi) := \{ x \in \mathbb{R}^d_+ : (\alpha + \varepsilon/d) (1, x) + (\beta - \varepsilon) M(x) \leq \xi \}
\]
and \( M(x) := \max_{j \in [d]} x_j \) for \( x \in \mathbb{R}^d \). We put
\[
V(\xi, s) := \{ x \in W(\xi) : \xi/\nu + s - 1 \leq (1, x) < \xi/\nu + s \}, \quad s \in \mathbb{Z}_+.
\]
From the inequalities \( \beta > \varepsilon \) and \( M(x) - (1, x)/d \geq 0, \quad x \in \mathbb{R}^d_+ \), one can verify that for every \( x \in W(\xi), (1, x) \leq \xi/\nu \). Hence, we have
\[
\int_{W(\xi)} 2^{(1,x)} \, dx \ll 2^{\xi/\nu} \sum_{s=0}^{[\xi/\nu]} 2^{-s} |V(\xi, s)|.
\]
Let us estimate \(|V(\xi, s)|\). Put \(V^*(\xi, s) := V(\xi, s) - x^*\), where \(x^* := (\nu d)^{-1}\xi 1\). From the equation \((1, x^*) = \xi/\nu\), we get for every \(y = x - x^* \in V^*(\xi, s)\),

\[
s - 1 \leq (1, y) < s. \tag{4.5}
\]

and

\[
(\alpha + \varepsilon/d)(1, y) + (\beta - \varepsilon)M(y) \leq 0. \tag{4.6}
\]

This means that \(V^*(\xi, s) \subset V'(s)\) for every \(\xi > 0\), where \(V'(s) \subset \mathbb{R}^d\) is the set of all \(y \in \mathbb{R}^d\) given by the conditions \([4.5]\) and \([4.6]\). Notice that \(V'(s)\) is a bounded polyhedron and \(|V'(s)| \asymp s^{d-1}\).

Hence, by the inequality

\[
|V(\xi, s)| = |V^*(\xi, s)| \leq |V'(s)|, \tag{4.3}
\]

and

\[
(\alpha + \varepsilon/d)(1, y) + (\beta - \varepsilon)M(y) \leq 0. \tag{4.4}
\]

we prove the upper bound in \((4.1)\):

\[
\sum_{k \in \Delta(\xi)} 2^{k_1} \ll 2^{\xi/\nu} \sum_{s=0}^{\infty} 2^{-s} s^{d-1} \asymp 2^{\xi/\nu}. \tag{4.7}
\]

To prove the lower bound for this case, we take \(k^* := \lfloor \xi/\nu \rfloor 1 \in \mathbb{Z}_+^d\). It is easy to check \(k^* \in \Delta(\xi)\) and consequently,

\[
\sum_{k \in \Delta(\xi)} 2^{k_1} \geq 2^{k^*_1} \gg 2^{\xi/\nu}.
\]

The case \((p, \theta, q) \in B, \beta < 0\) can be proven similarly with a slight modification. To prove the case \((p, \theta, q) \in A\) it is enough to put \(\varepsilon = 0\) in the proof of the case \((p, \theta, q) \in B\). \(\square\)

**Lemma 4.2** Let \(0 < p, \theta, q \leq \infty\) and \(a \in \mathbb{R}_+^d\) satisfying the condition \((3.11)\) and

\[
1/p < a_1 < a_d < r.
\]

Then we have

\[
|G(\Delta'(\xi))| \asymp \sum_{k \in \Delta'(\xi)} 2^{k_1} \asymp 2^{\xi/(a_1 - (1/p - 1/q) s)}. \tag{4.7}
\]

**Proof.** The first asymptotic equivalence in \((4.7)\) follows from the definitions. Let us prove the second one. For simplicity we prove it for the case where \(p \geq q\), the general case can be proven similarly.

Let us first consider the case \((p, \theta, q) \in B\). It is easy to verify that for every \(\xi > 0\),

\[
\sum_{k \in \Delta'_{\xi}(\xi)} 2^{k_1} \asymp \int_{W(\xi)} 2^{(1, x)} dx, \tag{4.8}
\]

where

\[
W(\xi) := \{ x \in \mathbb{R}_+^d : (a', x) \leq \xi \}.
\]

We put

\[
V(\xi, s) := \{ x \in W(\xi) : \xi/a_1 + s - 1 \leq (1, x) < \xi/a_1 + s \}, s \in \mathbb{Z}_+.
\]
One can verify that for every $x \in W(\xi)$, $(1, x) \leq \xi/a_1$. Hence, we have

$$\int_{W(\xi)} 2^{(1,x)} dx \ll 2^{\xi/a_1} \sum_{s=0}^{[\xi/a_1]} 2^{-s}|V(\xi, s)|. \quad (4.9)$$

Let us estimate $|V(\xi, s)|$. Put $V^*(\xi, s) := V(\xi, s) - x^*$, where $x^* := (a_1)^{-1}\xi e_1$. From the equation $(1, x^*) = \xi/a_1$, we get for every $y = x - x^* \in V^*(\xi, s)$,

$$s - 1 \leq (1, y) < s. \quad (4.10)$$

and

$$(a', y) \leq 0. \quad (4.11)$$

This means that $V^*(\xi, s) \subset V'(s)$ for every $\xi > 0$, where $V'(s) \subset \mathbb{R}^d$ is the set of all $y \in \mathbb{R}^d$ given by the conditions $(4.10)$ and $(4.11)$. Notice that $V'(s)$ is a bounded polyhedron and $|V'(s)| \asymp s^{d-1}$. Hence, by the inequality $|V(\xi, s)| = |V^*(\xi, s)| \leq |V'(s)|$, $(4.8)$ and $(4.9)$, we obtain the upper bound in $(4.7)$:

$$\sum_{k \in \Delta'(\xi)} 2^{|k|_1} \ll 2^{\xi/a_1} \sum_{s=0}^{\infty} 2^{-s}s^{d-1} \asymp 2^{\xi/a_1}.$$

To prove the lower bound, we take $k^* := [\xi/a_1]e_1 \in \mathbb{Z}_+^d$. It is easy to check $k^* \in \Delta'(\xi)$ and consequently,

$$\sum_{k \in \Delta'(\xi)} 2^{|k|_1} \geq 2^{|k^*|_1} \gg 2^{\xi/a_1}.$$ 

\[\square\]

**Remark** The grids of sample points $G(\Delta(\xi))$ and $G(\Delta'(\xi))$ are sparse and have much less elements than the standard dyadic full grids $G(\Delta_1(\xi))$ and Smolyak grids $G(\Delta_2(\xi))$ which give the same recovery error, where $\Delta_1(\xi) := \{k \in \mathbb{Z}_+^d : \lambda|k|_\infty \leq \xi\}$ and $\Delta_2(\xi) := \{k \in \mathbb{Z}_+^d : \lambda|k|_1 \leq \xi\}$ and the number $\lambda := \nu$ is as in $(4.2)$ for $G(\Delta(\xi))$ and $\lambda := a_1 - (1/p - 1/q)_+$ for $G(\Delta'(\xi))$. For instance, the linear sampling algorithms $R_{\Delta_i}(\xi)$, $i = 1, 2$, on the grids $G(\Delta_i(\xi))$ gives the worst case error

$$\sup_{f \in U_{p,q}} \|f - R_{\Delta_i}(\xi)(f)\|_q \asymp 2^{-\xi}.$$ 

The number of sample points in $G(\Delta_1(\xi))$ is $|G(\Delta_1(\xi))| \asymp 2^{d\xi/\nu}$, and in $G(\Delta_2(\xi))$ is $|G(\Delta_2(\xi))| \asymp 2^{\xi/\nu}2^{d-1}$. Whereas, due to Theorem $(4.1)$ and Lemma $(4.1)$ we can get the same error by the linear sampling algorithm $R_{\Delta(\xi)}$ on the grids $G(\Delta(\xi))$ with the number of sample points $|G(\Delta(\xi))| \asymp 2^{\xi/\nu}$.

The following two theorems show that the linear sampling sampling algorithms $R_{\Delta(\xi)}$ on sparse grids $G(\Delta(\xi))$, and $R_{\Delta'(\xi)}$ on sparse grids $G(\Delta'(\xi))$ are asymptotically optimal in the sense of the quantities $r_n$ and $g_n$. 

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Theorem 4.1 Let $0 < p, \theta, q \leq \infty$ and $\alpha \in \mathbb{R}_+$, $\beta \in \mathbb{R}$, $\beta \neq 0$, such that

$$1/p < \min(\alpha, \alpha + \beta) \leq \max(\alpha, \alpha + \beta) < r.$$ 

Assume that for a given $n \in \mathbb{Z}_+$, $\xi_n$ is the largest nonnegative number such that

$$|G(\Delta(\xi_n))| \leq n. \quad (4.12)$$

Then $R_{\Delta(\xi_n)}$ defines an asymptotically optimal linear sampling algorithm for $r_n := r_n(U^\alpha, \theta, L_q)$ and $\rho_n := \rho_n(U^\alpha, \theta, L_q)$ by

$$R_{\Delta(\xi_n)}(f) = L_n(X^*_n, \Phi_n, f) = \sum_{(k,s) \in K(\Delta(\xi_n))} f(2^{-k}s)\psi_{k,s}, \quad (4.13)$$

where $X^*_n := G(\Delta(\xi_n)) = \{2^{-k}s\}_{(k,s) \in K(\Delta(\xi_n))}$, $\Phi_n := \{(\psi_{k,s}\}_{(k,s) \in K(\Delta(\xi_n))}$, and we have the following asymptotic orders

$$\sup_{f \in U^\alpha, \theta} \|f - R_{\Delta(\xi_n)}(f)\|_q \asymp r_n \asymp \rho_n \asymp \left\{ \begin{array}{ll} n^{-\alpha - \beta/(1/p - 1/q)} + , & \beta > 0, \\ n^{-\alpha + \beta/(1/p - 1/q)} + , & \beta < 0. \end{array} \right. \quad (4.14)$$

Proof. 

Upper bounds. Due to Lemma 4.1 we have

$$n \asymp 2^{\xi_n/\nu} \asymp |G(\Delta(\xi_n))| \leq n,$$

where $\nu$ is as in (1.2). Hence, we find

$$2^{-\xi_n} \asymp \left\{ \begin{array}{ll} n^{-\alpha + \beta/(1/p - 1/q)} + , & \beta > 0, \\ n^{-\alpha - \beta/(1/p - 1/q)} + , & \beta < 0. \end{array} \right. \quad (4.15)$$

By Lemma 3.1 and (4.12), $R_{\Delta(\xi_n)}$ is a linear sampling algorithm of the form (1.1) as in (4.13) and consequently, from Theorem 3.5 we get

$$\rho_n \leq r_n \leq \sup_{f \in U^\alpha, \theta} \|f - R_{\Delta(\xi_n)}(f)\|_q \ll 2^{-\xi_n}.$$

These relations together with (4.15) proves the upper bounds of (4.14).

Lower bounds. We need the following auxiliary result. If $W \subset L_q$, then we have

$$r_n(W, L_q(\mathbb{R}^d)) \gg \inf_{X_n = \{x_i\}_{i=1}^n \subset \mathbb{R}^d} \sup_{f \in W: f(x_j) = 0, j = 1, \ldots, n} \|f\|_q. \quad (4.16)$$

For the proof of this inequality see [38, Proposition 19]. Since $\|f\|_q \geq \|f\|_p$ for $p \leq q$, it is sufficient to prove the lower bound for the case $p \leq q$. Fix a number $r' = 2^m$ with integer $m$ so that $\max(\alpha, \alpha + \beta) < \min(r', r' - 1 + 1/p)$. 

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We first treat the case $\beta > 0$. Put $k^* = k^*(\eta) := \eta 1$ for integer $\eta > m$. Consider the boxes 
\[ J(s) \subset \mathbb{I}^d \]
\[ J(s) := \{ x \in \mathbb{I}^d : 2^{-\eta+m}s_j \leq x_j < 2^{-\eta+m}(s_j + 1), \ j \in [d] \}, \ s \in Z(\eta), \]
where 
\[ Z(\eta) := \{ s \in \mathbb{Z}_+^d : 0 \leq s_j \leq 2^{\eta-m} - 1, \ j \in [d] \}.
For a given $n$, we find $\eta$ satisfying the relations 
\[ n \times 2^{|k^*|_1} \times 2^{d(\eta-m)} = |Z(\eta)| \geq 2n. \tag{4.17} \]
Let $X_n = \{ x^i \}_{j=1}^n$ be an arbitrary subset of $n$ points in $\mathbb{I}^d$. Since $J(s) \cap J(s') = \emptyset$ for $s \neq s'$, and $|Z(\eta)| \geq 2n$, there is $Z^*(\eta) \subset Z(\eta)$ such that $|Z^*(\eta)| \geq n$ and 
\[ X_n \cap \{ \cup_{s \in Z^*(\eta)} J(s) \} = \emptyset. \tag{4.18} \]
Consider the function $g^* \in \Sigma_d^d(k^*)$ defined by 
\[ g^* := \lambda 2^{-\alpha |k^*|_1 - \beta |k^*|_\infty + |k^*|_1/p} \sum_{s \in Z^*(\eta)} M_{k^*,s,r'/2}, \tag{4.19} \]
where $M_{k^*,s,r'/2}$ are B-splines of order $r'$. Since $|Z^*(\eta)| \geq 2^{|k^*|_1}$, by (2.20) we have 
\[ \|g^*\|_q \asymp \lambda 2^{-\alpha |k^*|_1 - \beta |k^*|_\infty + (1/p - 1/q)|k^*|_1}, \tag{4.20} \]
and 
\[ \|g^*\|_p \asymp \lambda 2^{-\alpha |k^*|_1 - \beta |k^*|_\infty}. \]
Hence, by Corollary 2.21 there is $\lambda > 0$ independent of $\eta$ and $n$ such that $g^* \in U_{p,\theta}^{\alpha,\beta}$. Notice that $M_{k^*,s,m-1}(x), x \notin J(s)$, for every $s \in Z^*(\eta)$, and consequently, by (4.18) $g^*(x^j) = 0, j = 1, \ldots, n$. From the inequality (4.15) (4.20) and (4.17) we obtain 
\[ \varrho_n \gg \|g^*\|_q \asymp n^{-\alpha - \beta/d + 1/p - 1/q}. \]
This proves the lower bound of (4.14) for the case $\beta > 0$.

We now consider the case $\beta < 0$. We will use some notations which coincide with those in the proof of the case $\beta > 0$. Put $k^* = k^*(\eta) := (\eta,m,\ldots,m)$ for integer $\eta > m$. Consider the boxes 
\[ J(s) \subset \mathbb{I}^d \]
\[ J(s) := \{ x \in \mathbb{I}^d : 2^{-\eta+m}s_1 \leq x_1 < 2^{-\eta+m}(s_1 + 1), \ s \in Z(\eta), \]
where 
\[ Z(\eta) := \{ s \in \mathbb{Z}_+^d : 0 \leq s_j \leq 2^{\eta-m} - 1, \ s_j = 0, \ j = 2, \ldots, d \}.
For a given $n$, we find $\eta$ satisfying the relations 
\[ n \times 2^{|k^*|_1} \times 2^{n-m} = |Z(\eta)| \geq 2n. \tag{4.21} \]
Let $X_n = \{ x^i \}_{j=1}^n$ be an arbitrary subset of $n$ points in $\mathbb{I}^d$. Since $J(s) \cap J(s') = \emptyset$ for $s \neq s'$, and $|Z(\eta)| \geq 2n$, there is $Z^*(\eta) \subset Z(\eta)$ such that $|Z^*(\eta)| \geq n$ and 
\[ X_n \cap \{ \cup_{s \in Z^*(\eta)} J(s) \} = \emptyset. \tag{4.22} \]
Consider the function \( g^* \in \Sigma^d(\kappa^*) \) defined by
\[
g^* := \lambda 2^{-(\alpha + \beta - 1/p)k^*_1} \sum_{s \in Z^*(\eta)} M_{k^*,s+r'/2},
\] (4.23)
where \( M_{k^*,s+r'/2} \) are B-splines of order \( r' \). Since \( |Z^*(\eta)| \asymp 2^{k^*_1} \), by (2.20) we have
\[
\|g^*\|_q \asymp \lambda 2^{-(\alpha + \beta - 1/p + 1/q)k^*_1},
\] (4.24)
and
\[
\|g^*\|_p \asymp \lambda 2^{-(\alpha + \beta)k^*_1}.
\]
Hence, by Corollary 2.1 there is \( \lambda > 0 \) independent of \( \eta \) and \( n \) such that \( g^* \in U_{\alpha,\beta}^p,\gamma \). Notice that
\[
M_{k^*,s+r'/2}(x), x \notin J(s),
\]
for every \( s \in Z^*(\eta) \), and consequently, by (4.22) \( g^*(x^j) = 0, j = 1, ..., n \).

From the inequality (4.16) (4.24) and (4.21) we obtain
\[
\eta_n(U_{\alpha,\beta}^p,\gamma, L_q) \gg \|g^*\|_q \asymp n^{-\alpha - \beta + 1/p - 1/q}.
\] (4.25)
This proves the lower bound of (4.14) for the case \( \beta < 0 \). \( \blacksquare \)

**Theorem 4.2** Let \( 0 < p, \theta, q \leq \infty \) and \( a \in \mathbb{R}^d_+ \) satisfying the condition (3.11) and
\[
1/p < a_1 < a_2 \leq ... \leq a_d < r.
\]
Assume that for a given \( n \in \mathbb{Z}_+ \), \( \xi_n \) is the largest nonnegative number such that
\[
|G(\Delta'(\xi_n))| \leq n.
\] (4.25)
Then \( R_{\Delta(\xi_n)} \) defines an asymptotically optimal linear sampling algorithm for \( r_n := r_n(U_{p,\theta}^\alpha,\beta, L_q(\mathbb{I}^d)) \) and \( \varrho_n := \varrho_n(U_{p,\theta}^\alpha,\beta, L_q(\mathbb{I}^d)) \) by
\[
R_{\Delta'(\xi_n)}(f) = L_n(X^*_n, \Phi^*_n, f) = \sum_{(k,s) \in K(\Delta'(\xi_n))} f(2^{-k}s)\psi_{k,s},
\]
where \( X^*_n := G(\Delta'(\xi_n)) = \{2^{-k}s\}_{(k,s) \in K(\Delta'(\xi_n))}, \Phi^*_n := \{\psi_{k,s}\}_{(k,s) \in K(\Delta'(\xi_n))} \), and we have the following asymptotic order
\[
\sup_{f \in U_{p,\theta}^\alpha} \|f - R_{\Delta'(\xi_n)}(f)\|_q \asymp r_n \asymp \varrho_n \asymp n^{-a_1 + (1/p - 1/q)_+}.
\] (4.26)

**Proof.**
**Upper bounds.** For a given \( n \in \mathbb{Z}_+ \) (large enough), due to Lemma 4.2 we can define \( \xi = \xi_n \) as the largest nonnegative number such that
\[
n \asymp 2^{\xi_n/(a_1 - (1/p - 1/q)_+)} \asymp |G(\Delta'(\xi_n))| \leq n.
\]
Hence, we find
\[
2^{-\xi_n} \asymp n^{-a_1 + (1/p - 1/q)_+}.
\] (4.27)
By Lemma 3.1 and (4.25) $R_{\Delta'}(\xi_n)$ is a linear sampling algorithm of the form (1.1) and consequently, from Theorem 3.12 we get

$$g_n \leq r_n \leq \sup_{f \in U_{p,\theta}} \|f - R_{\Delta'}(\xi_n)(f)\|_q \leq 2^{-\xi_n}.$$  

These relations together with (4.27) proves the upper bounds for (4.26).

**Lower bounds.** As in the proof of Theorem 4.1 it is sufficient to prove the lower bound for the case $p \leq q$. Fix a number $r = 2^m$ with integer $m$ so that $r_d < \min(r, r - 1 + 1/p)$. In the next steps, the proof is similar to the proof of the lower bound for the case $\beta < 0$ in Theorem 4.1. Indeed, we can repeat almost all the details in it with replacing $\alpha + \beta$ by $a_1$.

**Remark** Concerning the asymptotically optimal sparse grids of sampling points $G(\Delta(\xi_n))$ and $G(\Delta'(\xi_n))$ for $r_n(U_{p,\theta}^{\alpha,\beta}, L_q(\mathbb{I}^d))$, $\varrho_n(U_{p,\theta}^{\alpha,\beta}, L_q(\mathbb{I}^d))$ and $r_n(U_{p,\theta}^{\alpha,\beta}, L_q(\mathbb{I}^d))$, $\varrho_n(U_{p,\theta}^{\alpha,\beta}, L_q(\mathbb{I}^d))$, it is worth to notice the following. Let set $A$ and $B$ be the sets of triples $(p, \theta, q)$ introduced in Section 3. For every triple $(p, \theta, q) \in A$, we can define the best choice of family of asymptotically optimal sparse grids $G(\Delta(\xi_n))$ and $G(\Delta'(\xi_n))$. Whereas, for a triple $(p, \theta, q) \in B$, there are many families of asymptotically optimal sparse grids $G(\Delta(\xi_n))$ and $G(\Delta'(\xi_n))$ depending on parameter $\varepsilon > 0$, for $r_n(U_{p,\theta}^{\alpha,\beta}, L_q(\mathbb{I}^d))$, $\varrho_n(U_{p,\theta}^{\alpha,\beta}, L_q(\mathbb{I}^d))$ and $r_n(U_{p,\theta}^{\alpha,\beta}, L_q)$, $\varrho_n(U_{p,\theta}^{\alpha,\beta}, L_q(\mathbb{I}^d))$, respectively.

Moreover, the parameter $\varepsilon > 0$ plays a crucial role in the construction asymptotically optimal sparse grids for $(p, \theta, q) \in B$. Indeed, to understand the substance let us consider, for instance, the problem of asymptotically optimal sparse grids for even the simplest case $r_n(U_{2,2}^{\alpha,\beta}, L_2(\mathbb{I}^d))$ and $\varrho_n(U_{2,2}^{\alpha,\beta}, L_2(\mathbb{I}^d))$ with $\beta < 0$. Suppose that for this case instead the set

$$\Delta(\xi) := \{k \in \mathbb{Z}_+^d : (\alpha - \varepsilon)|k|_1 + (\beta + \varepsilon)|k|_\infty \leq \xi\},$$

we take the set

$$\tilde{\Delta}(\xi) := \{k \in \mathbb{Z}_+^d : \alpha|k|_1 + \beta|k|_\infty \leq \xi\}.$$  

Then $\tilde{\Delta}(\xi)$ is a proper subset of $\Delta(\xi)$, i.e., the grid $G(\Delta(\xi))$ is essentially extended from $G(\tilde{\Delta}(\xi))$ by parameter $\varepsilon$. However, $|G(\Delta(\xi))| \asymp |G(\tilde{\Delta}(\xi))|$. On the other hand, the grid $G(\Delta(\xi))$ cannot be asymptotically optimal for $r_n(U_{2,2}^{\alpha,\beta}, L_2(\mathbb{I}^d))$ and $\varrho_n(U_{2,2}^{\alpha,\beta}, L_2(\mathbb{I}^d))$, because (3.5) is replaced by

$$\sup_{f \in U_{2,2}^{\alpha,\beta}} \|f - R_{\tilde{\Delta}(\xi)}(f)\|_2 \asymp 2^{-\xi d - 1}.$$  

## 5 Sampling recovery in energy norm

In this section, we extend the results on sampling recovery in space $L_q(\mathbb{I}^d)$ of functions from $B_{p,\theta}^{\alpha,\beta}$ in Sections 3 and 4 to sampling recovery in the energy norm of the isotropic Sobolev space $W_q^\gamma(\mathbb{I}^d)$ with $\gamma > 0$. We preliminary study the sampling recovery in the norm of $W_q^\gamma(\mathbb{I}^d)$, and the receive the results on sampling recovery in the norm of $W_q^\gamma(\mathbb{I}^d)$ as consequences of those in the norm of $B_{q,\tau}^\gamma$.

Put $\tau^* := \min(\tau, 1)$ and $\theta^* := (1/\tau^* - 1/\theta)^{-1}$. 

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Lemma 5.1 Let $0 < p, \theta, q, \tau \leq \infty$, $0 < \gamma < \min(r, r - 1 + 1/p)$ and $\psi : \mathbb{Z}_+^d \rightarrow \mathbb{R}_+$. Then for every $f \in B_{p, \theta}^{(\psi)}$, we have

$$
\|f - R_\Delta(f)\|_{B^\gamma_{q,r}} \ll \|f\|_{B^\psi_{p,\theta}} \left\{ \sup_{k \in \mathbb{Z}_+^d \setminus \Delta} 2^{-\psi(k) + \gamma}, \quad \theta \leq \tau^*, \\
\left( \sum_{k \in \mathbb{Z}_+^d \setminus \Delta} \left( 2^{-\psi(k) + \gamma} \right)^{\theta^*} \right)^{1/\theta^*}, \quad \theta > \tau^*.
\right.
$$

Proof. Let $g$ be a function of the form (2.19). We have the following Bernstein type inequality

$$
\|g\|_{B^\gamma_{q,r}} \ll 2^{\gamma \|k\|_1\|g\|_p}
$$

which can be proven in a way similar to the proof of [13, Corollary 5.2]. This inequality together with (2.20) gives

$$
\|g\|_{B^\gamma_{q,r}} \ll 2^{\gamma \|k\|_1 + (1/p - 1/q)\|k\|_1\|g\|_p}.
$$

Hence, we obtain for every $f \in B_{p,\theta}^{\psi}$,

$$
\|f - R_\Delta(f)\|_{B^\gamma_{q,r}} \ll \sum_{k \in \mathbb{Z}_+^d \setminus \Delta} 2^{\gamma \|k\|_1 + (1/p - 1/q)\|k\|_1\|q_k(f)\|_p} \tau^*.
$$

By use of this inequality, in a way similar to the proof of Lemma 3.2(i) we prove the lemma. □

Let $0 < p, \theta, q, \tau \leq \infty$ and $\alpha, \gamma \in \mathbb{R}_+, \beta \in \mathbb{R}$ be given. We fix a number $\varepsilon$ so that

$$
0 < \varepsilon < \min(\alpha - (1/p - 1/q)_+, |\gamma - \beta|),
$$

and define the set $\Delta''(\xi)$ for $\xi > 0$ by

$$
\Delta''(\xi) := \left\{ \begin{array}{ll}
\{k \in \mathbb{Z}_+^d : (\alpha - (1/p - 1/q)_+)\|k\|_1 - (\gamma - \beta)\|k\|_\infty \leq \xi\}, & \theta \leq \tau^*, \\
\{k \in \mathbb{Z}_+^d : (\alpha - (1/p - 1/q)_+ + \varepsilon/d)\|k\|_1 - (\gamma - \beta - \varepsilon)\|k\|_\infty \leq \xi\}, & \theta > \tau^*, \beta > \gamma, \\
\{k \in \mathbb{Z}_+^d : (\alpha - (1/p - 1/q)_+ - \varepsilon)\|k\|_1 - (\gamma - \beta + \varepsilon)\|k\|_\infty \leq \xi\}, & \theta > \tau^*, \beta < \gamma.
\end{array} \right.
$$

The following theorems and lemma are counterparts of the corresponding results on sampling recovery in space $L_q(\mathbb{R}^d)$ of functions from $B_{p,\theta}^{\alpha,\beta}$ in Sections 3 and 4. They can be proven in a similar way with slight modifications.

Theorem 5.1 Let $0 < p, \theta, q, \tau \leq \infty$, $\alpha, \gamma \in \mathbb{R}_+$ and $\beta \in \mathbb{R}$, $\beta \neq \gamma$, satisfy the conditions

$$
\alpha > \left\{ \begin{array}{ll}
(\gamma - \beta)/d, & \beta > \gamma, \\
\gamma - \beta, & \beta < \gamma,
\end{array} \right.
$$

and

$$
1/p < \min(\alpha, \alpha + \beta) \leq \max(\alpha, \alpha + \beta) < r, \quad 0 < \gamma < \min(r, r - 1 + 1/p).
$$

Then we have the following upper bound

$$
\sup_{\Delta''(\xi), \xi > 0} \|f - R_\Delta''(\xi)(f)\|_{B^\gamma_{q,r}} \ll 2^{-\xi}.
$$

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Lemma 5.2 Under the assumptions of Theorem 5.1 we have
\[ |G(\Delta''(\xi))| \leq \sum_{k \in \Delta''(\xi)} 2^{|k|_1} \times 2^{|k|_\nu}, \]
where \( \nu := \begin{cases} \alpha - (\gamma - \beta)/d - (1/p - 1/q)_+, & \beta > \gamma, \\ \alpha - (\gamma - \beta) - (1/p - 1/q)_+, & \beta < \gamma. \end{cases} \)

Theorem 5.2 Under the assumptions of Theorem 5.1, let for a given \( n \in \mathbb{Z}_+ \), \( \xi_n \) be the largest nonnegative number such that
\[ |G(\Delta''(\xi_n))| \leq n. \]
Then \( R_{\Delta''(\xi_n)} \) defines an asymptotically optimal linear sampling algorithm for \( r_n := r_n(U_{p,q}^{\alpha,\beta}, B_{q,\tau}^\gamma) \) and \( \varrho_n := \varrho_n(U_{p,q}^{\alpha,\beta}, B_{q,\tau}^\gamma) \) by
\[ R_{\Delta''(\xi_n)}(f) = L_n(X_n^*, \Phi_n^*, f) = \sum_{(k,s) \in K(\Delta''(\xi_n))} f(2^{-k}s)\psi_{k,s}, \]
where \( X_n^* := G(\Delta''(\xi_n)) = \{2^{-k}s\}_{(k,s) \in K(\Delta''(\xi_n))} \), \( \Phi_n^* := \{\psi_{k,s}\}_{(k,s) \in K(\Delta''(\xi_n))} \), and we have the following asymptotic orders
\[ \sup_{f \in U_{p,q}^{\alpha,\beta}} \|f - R_{\Delta''(\xi_n)}(f)\|_{B_{q,\tau}^\gamma} \asymp r_n \asymp \varrho_n \asymp \begin{cases} n^{-\alpha-(\beta-\gamma)/d+(1/p-1/q)_+}, & \beta > \gamma, \\ n^{-\alpha-\beta+(1/p-1/q)_+}, & \beta < \gamma. \end{cases} \]

Let \( W_q^\gamma(\mathbb{I}^d) , \gamma > 0, 1 < q < \infty \), be the isotropic Sobolev space of functions on \( \mathbb{I}^d \) (see, e.g. [3] for a definition).

Theorem 5.3 Under the assumptions of Theorem 5.1, we have the following asymptotic orders for \( 1 < q < \infty \),
\[ r_n(U_{p,q}^{\alpha,\beta}, W_q^\gamma(\mathbb{I}^d)) \asymp \varrho_n(U_{p,q}^{\alpha,\beta}, W_q^\gamma(\mathbb{I}^d)) \asymp \begin{cases} n^{-\alpha-(\beta-\gamma)/d+(1/p-1/q)_+}, & \beta > \gamma, \\ n^{-\alpha-\beta+1/p-1/q}_+), & \beta < \gamma. \end{cases} \]

Proof. This theorem is follows from Theorem 5.2 and the inequality for \( f \in B_{q,\min(p,2)}^\gamma \):
\[ \|f\|_{W_q^\gamma(\mathbb{I}^d)} \leq C \|f\|_{B_{q,\min(p,2)}^\gamma}. \]
The last inequality can be proven in a way similar to the proof of the inequality [20] (14) on the basis of a generalization of the well-known Littlewood-Paley theorem for the norm \( \| \cdot \|_{W_q^\gamma(\mathbb{I}^d)} \).

Remark Asymptotically optimal linear sampling algorithms for \( r_n(U_{p,q}^{\alpha,\beta}, W_q^\gamma(\mathbb{I}^d)) \) and \( \varrho_n(U_{p,q}^{\alpha,\beta}, W_q^\gamma(\mathbb{I}^d)) \) are the same as for \( r_n(U_{p,q}^{\alpha,\beta}, B_{q,\min(p,2)}^\gamma) \) and \( \varrho_n(U_{p,q}^{\alpha,\beta}, B_{q,\min(p,2)}^\gamma) \). Theorem 5.3 is true also for \( \gamma \in \mathbb{N}, 0 < q \leq \infty \).
6 Optimal cubature

Every linear sampling algorithm $L_n(X_n, \Phi_n, \cdot)$ of the form (1.1) generates the cubature formula $I_n(X_n, \Lambda_n, f)$ where

$\Lambda_n = \{\lambda_j\}_{j=1}^n, \quad \lambda_j = \int_\Omega \varphi_j(x) \, dx.$

Hence, it is easy to see that

$|I(f) - I_n(X_n, \Lambda_n, f)| \leq \|f - L_n(X_n, \Phi_n, f)\|_1,$

and consequently, from the definitions we have the following inequality

$i_n(W) \leq r_n(W)_1. (6.1)$

**Theorem 6.1** Let $0 < p, \theta \leq \infty$ and $\alpha \in \mathbb{R}_+, \beta \in \mathbb{R}$ such that

$1/p < \min(\alpha, \alpha + \beta) \leq \max(\alpha, \alpha + \beta) < r.$

Assume that for a given $n \in \mathbb{Z}_+$, $\xi_n$ is the largest nonnegative number such that

$|G(\Delta(\xi_n))| \leq n.$

Then $R\Delta(\xi_n)$ defines an asymptotically optimal cubature formula for $i_n(U^\alpha,\beta_{p,\theta})$ by

$I_n(X^*_n, \Phi^*_n, f) = \sum_{(k,s) \in K(\Delta(\xi_n))} \lambda_{k,s} f(2^{-k}s),$

where

$X^*_n := G(\Delta(\xi_n)) = \{2^{-k}s\}_{(k,s) \in K(\Delta(\xi_n))}, \quad \Lambda^*_n := \{\lambda_{k,s}\}_{(k,s) \in K(\Delta(\xi_n))} \quad \lambda_{k,s} := \int_\Omega \psi_{k,s}(x) \, dx,$

and we have the following asymptotic orders

$\sup_{f \in U^\alpha,\beta_{p,\theta}} |I(f) - I_n(X^*_n, \Phi^*_n, f)| \asymp i_n(U^\alpha,\beta_{p,\theta}) \asymp \begin{cases} n^{-\alpha-\beta/d+(1/p-1)_+}, & \beta > 0, \\ n^{-\alpha-\beta+(1/p-1)_+}, & \beta < 0. \end{cases} (6.2)$

**Proof.** The upper bound of (6.2) follows from (6.1) and Theorem 4.1. To prove the lower bound of (6.2) we observe that

$i_n(W) \geq \inf_{X_n = \{x^j\}_{j=1}^n \subset \mathbb{R}^d} \sup_{f \in W: f(x^j) = 0, j = 1, \ldots, n} |I(f)|,$

and for the functions $g^*$ given in (4.19) and (4.23) we have $I(g^*) = \|g^*\|_1$. Hence, we can see that the lower bound is derived from the proof of the lower bound of Theorem 4.1.

In a similar way, we can prove the following
Theorem 6.2 Let $0 < p, \theta \leq \infty$ and $a \in \mathbb{R}^d_+$ satisfying the condition (3.11) and $a_1 > 1/p$. Assume that for a given $n \in \mathbb{Z}_+$, $\xi_n$ is the largest nonnegative number such that

$$|G(\Delta'(\xi_n))| \leq n.$$  

Then $R_{\Delta'(\xi_n)}$ defines an asymptotically optimal cubature formula for $i_n(U^a_{p,\theta})$ by

$$I_n(X^*_n, \Phi^*_n, f) = \sum_{(k,s) \in K(\Delta'(\xi_n))} \lambda_{k,s} f(2^{-k}s),$$

where

$$X^*_n := G(\Delta'(\xi_n)) = \{2^{-k}s\}_{(k,s) \in K(\Delta'(\xi_n))}, \quad \Lambda^*_n := \{\lambda_{k,s}\}_{(k,s) \in K(\Delta'(\xi_n))} \quad \lambda_{k,s} := \int_{\mathbb{R}^d} \psi_{k,s}(x) \, dx,$$

and we have the following asymptotic order

$$\sup_{f \in U^a_{p,\theta}} |I(f) - I_n(\Lambda^*_n, X^*_n, f)| \asymp i_n(U^a_{p,\theta}) \asymp n^{-a_1 + (1/p - 1)_+}.$$  

Remark If in Theorems 6.1 and 6.2 we assume $1 \leq p \leq \infty$, then

$$i_n(U^{\alpha, \beta}_{p,\theta}) \asymp \begin{cases} n^{-\alpha - \beta/d}, & \beta > 0, \\ n^{-\alpha - \beta}, & \beta < 0, \end{cases}$$

and

$$i_n(U^a_{p,\theta}) \asymp n^{-a_1}.$$  

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