Part of the Unipotency of Free Group Generated by Two Elements Whose Maximum Jordan Matrix Does Not Exceed 8 Times

Chaoran Zhen and Xinsong Yang*
Harbin University of Science and Technology, Harbin, Heilongjiang province, China

*Corresponding author email: yangxinsong2005@163.com

Abstract. Intelligent calculations can already assist in the completion of mathematical proofs. We have completed a partial verification of an important theory with data calculation as the main method. When the dimension is ten, free group generated by two elements with linear representation is unipotent when the primitive elements are unipotent and the maximum Jordan block does not exceed the eighth order, at the same time there is a primitive element with the maximum Jordan block not exceeding the second order.

Keywords: Free group; Unipotent group; Primitive element.

1. Introduction
Groups are an important structure in algebra. In recent years, there have been many studies on the structure of groups, and it is also an important way to study the structure of groups by studying the Unipotency of free groups[1]-[11]. In the research process, as the number of representation dimensions increases, the amount of calculation will increase exponentially. Especially when the representation dimension is high enough, it is very difficult to calculate directly, so it is necessary to use calculation software to simplify the calculation process. In recent years, many researchers have used computer programming to assist in proof[11]-[14], which shows that the use of computer programming for scientific research is a very important part of the future. It is foreseeable that with the rapid development of computer science, many scientific research tasks that were difficult to complete in the past will be completed with the help of computers in the future. We will try to prove a result of unipotency of matrix group by calculating trail of matrix with the help of computer. We have proved theoretically that the tenth-order matrix group G generated by two unipotent matrices which Jordan blocks does not accordingly exceed 2-orders and 4-orders is unipotent when each primitive element \(M\) of G satisfies \(\left( A - B \right)^2 = 0\) (This result is under reviewer.). Now we will prove it maybe true when each primitive element \(M\) of G satisfies \(\left( A - B \right)^3 = 0\).

Throughout the text, \(trX\) denotes the trace of the matrix X. \(A_{[i,j]}\) denotes the element of matrix A which lies in ith-row and jth-column. E denotes the unite matrix. Other signal used in this text is usual means which can be found in any text book for masters. We say an element a in group G is associated with b if G is generated by a and b.

2. Lemmas

Lemma 2.1[15] Let \(p, q\) be the a couple of generator in the free group generated by two elements, Then all primitive elements associated with \(p\) are of the form \(p^\alpha q^\beta p^\gamma (\alpha, \beta \in N^*, \gamma = \pm 1)\).
Lemmas 2.2 If every primitive element of group generated by $A = E + H$ and $B$ is unipotent, then (1) $tr(HBBH) = 0$; (2) $trH^nB = 0$, $n \in Z^+$; (3) $trHB^n = 0$, $n \in Z^+$.

Lemmas 2.3 If every primitive element of group generated by matrices $A = E + H$ and $B = T + E$ is unipotent, and the largest Jordan block of primitive elements does not exceed the eighth order, then there must be equation $T^7F + T^6FT + T^5FT^2 + T^4FT^3 + T^3FT^4 + T^2FT^5 + TFBNT^6 + FBT^7 = 0$ where $H^T = 0, T^B = 0$, and $F = H - \frac{H^2}{2} + \frac{H^3}{3}$.

Proof: It is obvious that $F^4 = 0$ and $A^m = E + mF + \frac{1}{2} m^2 F^2 + \frac{1}{6} n^3 F^3$. Discussing with primitive element $A^mB$, we get $(A^mB - E)^B = 0$ thus,

$$
\left(B - E + mFB + \frac{1}{2} m^2 F^2B + \frac{1}{6} n^3 F^3B\right)^B = 0.
$$

Expand the above formula, we have $W_0 + mW_1 + nm^2W_2 + \cdots + m^2nW_{24} = 0$ that is $W_0 = W_1 = W_2 = \cdots = W_{24} = 0$ Thus $W_1 = T^7FB + T^6FBT + T^5FBT^2 + T^4FBT^3 + T^3FBT^4 + T^2FBT^5 + TFBNT^6 + FBT^7 = 0$. Reducing B from left because $BT = TB$ and B is invertible, we get

$$
T^7F + T^6FT + T^5FT^2 + T^4FT^3 + T^3FT^4 + T^2FT^5 + TFB^6 + FBT^7 = 0.
$$

Lemmas 2.4 Assume that each primitive element of the tenth-order matrix group $G$ generated by $A = E + H$ and $B = E + T$ is unipotent. If $tr(H^BHB^I)$ when $k < i + j$, then

$$
\sum_{i=0}^{k} \frac{tr(H^BHB^i)}{i!(k-i)!} = 0, \sum_{i=1}^{k} \frac{tr(H^BHB^i)}{(i-1)!(k-i-1)!} = 0.
$$

Proof: $BA^m$ is the primitive element associated with $A$ in the sense of lemma2.1, so it is unipotent. Thus $trA^mBAB^m = 10$. This formula can be expands to $tr\left(B + mHB + \frac{m(m-1)}{2!} H^2B + \cdots + \frac{m(m-1)\cdots(m-b)}{9!} H^9B\right)^2 = 10$. The expansion of the above formula has $tr\left(H^B\sqrt{m} + \frac{H^B}{9!} - \frac{36H^B}{9!}\right)^2 = 10$. Let the above formula be

$$
tr(a_0 + a_1 m + a_2 m^2 + a_3 m^3 + a_4 m^4 + a_5 m^5 + a_6 m^6 + a_7 m^7 + a_8 m^8 + a_9 m^9)^2 = 10.
$$

Expansion it we have $F_0 = F_1 = F_2 = \cdots = F_{18} = 0$. $F_{18} = tr\left(H^B\sqrt{m} \times H^B_{9!}\right) = 0$ and $F_{17} = tr\left(H^B_{8!} - \frac{36H^B}{9!}\right) \times H^B_{9!} + \frac{H^B}{9!} \times \left(H^B_{8!} - \frac{36H^B}{9!}\right) = 0$, we havetr\left(H^B_{8!} \times H^B_{9!} \times H^B_{8!}\right) = 0$. Substitute $F_i$ into $F_{i-1}$ in turn, and notice that $trH^BHB^I = 0$ when $k < i + j$, we get

$$
\sum_{i=0}^{k} \frac{tr(H^BHB^i)}{i!(k-i)!} = 0.
$$

For primitive elements $A$ and $B$, we have $trAB^2 = trB^2 = tr(AB)^2 = 10$. After unfolding and reorganizing, $trHBHB = 0$ can be obtained. Similarly, for primitive elements $A$ and $BA^m$. Use the same method to expand and get

$$
\sum_{i=1}^{k} \frac{tr(H^BHB^i)}{(i-1)!(k-i-1)!} = 0.
$$

The following lemma proof is the same as the above method, so it won’t be repeated here.

Use $tr(A^mB^mB^m) = 10$, $tr(HA^mBHA^mB^m) = 0$ we can proof

Lemmas 2.5 Assume that each primitive element of the tenth-order matrix group $G$ generated by $A = E + H$ and $B = E + T$ is unipotent. If $trH^BHB^kB^I = 0$ when $m < i + j + k$, then

$$
\sum_{i+j+k=m} \frac{tr(H^BHB^kB^I)}{i!j!k!} = 0, \sum_{i+j+k=m} \frac{tr(H^BHB^kB^I)}{(i-1)!(j-1)!(k-1)!} = 0.
$$

2
**Lemmas 2.6** Assume that each primitive element of the tenth-order matrix group \( G \) generated by \( A = E + H \) and \( B = E + T \) is unipotent. If \( trH^iBH^jB = 0 \) when \( m < i + j + k + l \), then
\[
\sum_{i+j+k+l=m} \frac{tr(H^iBH^jB)}{i!j!k!l!} = 0, \quad \sum_{i+j+k+l=m} \frac{tr(H^iBH^jB)}{(i-1)!(j-1)!(k-1)!(l-1)!} = 0.
\]

Here we remark the method using lemma 2.4-2.6: calculating with computer to find positive integer \( m \) which satisfies \( tr(H^iBH^jB) \) when \( k < i + j \) (or \( trH^iBH^jB = 0 \) when \( m < i + j + k \), or \( trH^iBH^jB = 0 \) when \( m < i + j + k + l \)), then getting equations we need. We call this method “calculating-finding” in the sense of lemma 2.4-2.6.

**Lemmas 2.7** The group \( G \) generated by unipotent matrices \( A \) and \( B \) must be unipotent if there exists invertible matrix \( P \) such that both \( PAP^{-1} \) and \( PBP^{-1} \) are quasi-upper triangular matrices.

3. Theorem and Proof

**Theorem 3.1** The tenth-order matrix group \( G \) generated by two unipotent matrices which Jordan blocks does not accordingly exceed 2-orders and 4-orders is unipotent when each primitive element \( M \) of \( G \) satisfies \( (M - E)^8 = 0 \).

Proof: Let the matrices \( A,B \) be associative primitive elements. If there is no primitive element which Jordan block is of 8-orders, then we can finish the proof which we have remarked in section 1. So we may assume that \( B \) is a primitive element which has 8-order Jordan block. Since tenth-order matrix \( B \) has 8-order Jordan block, so there two cases: firstly, \( B = diag(J_{b1}, J_{b2}) \), secondly, \( B = diag(J_{b}, J_{c}) \). These two cases can be proved in same method, so we only show the proof of first case. We denote
\[
A = \begin{bmatrix} x_{ij} \end{bmatrix}_{10 \times 10}, \quad E = \begin{bmatrix} 1 \end{bmatrix}_{10 \times 10}, \quad B = diag(J_{b1}, J_{b2})
\]

Where \((A - E)^4 = 0 \) or \((A - E)^2 = 0 \).

By “calculating-finding”, it is found (after calculating by computer) that \( trT^iAT^jA \) is 0 except for \( trT^iAT^jA \) when \( i + j \geq 14, i,j \in [0,7] \). So we get \( trT^iAT^jA = 0 \) by lemma 2.4, that is \( x_{b1} = 0 \). Now, after calculating by computer we found by calculating-finding that \( trT^iAT^jA = 0 \) \( i + j > 2 \) \( i,j \in [0,7] \). So we get following system of equations by lemma 2.2 and lemma 2.4
\[
tr(T^6A) = 0, \quad 2tr(T^5AT^7A) + \frac{tr(T^6AT^6A)}{6!6!} = 0, \quad \frac{2tr(T^5AT^7A)}{4!6!} + \frac{tr(T^6AT^6A)}{5!5!} = 0
\]
which implies that \( x_{b2} = 0 \). \( x_{71} = 0 \).

Again by calculating-finding in the sense of lemma 2.4, we get equations
\[
\frac{2tr(T^5AT^3A)}{7!3!} + \frac{2tr(T^6AT^4A)}{6!4!} + \frac{tr(T^5AT^5A)}{5!5!} = 0, \quad \frac{2tr(T^7AT^3A)}{6!2!} + \frac{2tr(T^6AT^4A)}{5!3!} + \frac{tr(T^5AT^5A)}{4!4!} = 0
\]

It implies \( x_{61} = 0 \), \( x_{72} = 0 \), \( x_{83} = 0 \) from the solution of this systems.

By calculating-finding in the sense of lemma 2.4, 2.5 we can get equations
\[
\frac{2tr(T^5AT^3A)}{5!3!} + \frac{2tr(T^6AT^6A)}{7!} + \frac{tr(T^4AT^4A)}{4!4!} = 0, \quad \frac{2tr(T^5AT^3A)}{4!2!} + \frac{2tr(T^6AT^6A)}{6!} + \frac{tr(T^4AT^4A)}{3!3!} = 0
\]

\[
\frac{3tr(T^3AT^3AT^6A)}{3!3!6!} + \frac{3tr(T^4AT^5AT^3A)}{4!3!5!} + \frac{3tr(T^5AT^4AT^3A)}{4!3!5!} + \frac{3tr(T^4AT^6AT^2A)}{4!2!6!}
\]
+ \frac{3 \text{tr}(7^4 A T^2 A T^6 A)}{4! 2! 6!} + \frac{3 \text{tr}(T A T^3 A T^2 A)}{3! 2! 7!} + \frac{3 \text{tr}(T^3 A T A T^2 A)}{3! 2! 7!} + \frac{3 \text{tr}(T^5 A T^5 A T^2 A)}{5! 2! 5!} \\
+ \frac{3 \text{tr}(T^5 A T^6 ATA)}{5! 6!} + \frac{3 \text{tr}(T^6 A T^5 ATA)}{5! 6!} + \frac{3 \text{tr}(T^4 A T^7 ATA)}{7! 4!} + \frac{3 \text{tr}(T A T^4 ATA)}{7! 4!} \\
+ \frac{\text{tr}(T^4 A T^4 A T^4 A)}{4! 4! 4!} = 0

And \ \text{tr}(T^4 A) = 0 \ is \ follows \ from \ lemma 2.2. \ Solving \ this \ systems, \ we \ get \ x_{5,1} = 0, x_{6,2} = 0, x_{7,3} = 0, x_{8,4} = 0.

We \ can \ not \ get \ precise \ solutions \ from \ system \ obtained \ by \ calculating-finding. \ So \ we \ need \ simplify \ A \ by \ isomorphism. \ Let

\[
P = \begin{bmatrix}
\alpha & b & c & d & e & f & g & h & x & y \\
0 & a & b & c & d & e & f & g & 0 & 0 \\
0 & 0 & a & b & c & d & e & f & 0 & 0 \\
0 & 0 & 0 & a & b & c & d & e & 0 & 0 \\
0 & 0 & 0 & 0 & a & b & c & d & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & a & b & c & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & a & b & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & a & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & v & s1 \ s2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & u & s3 \ s4
\end{bmatrix}
\]

Since \ G \ is \ isomorphic \ to \ PGP^{-1}, \ and \ B = PB \ P^{-1}, \ we \ may \ assume \ that \ A \ itself \ be \ the \ isomorphism \ image \ A_1 = P A P^{-1}. \ Now \ we \ find \ that

\[
A_{[9,1]} = \frac{s1 x_{9,1} + s2 x_{10,1}}{a}, \ A_{[9,2]} = -\frac{(s1 x_{9,1} + s2 x_{10,1}) b}{a^2} + \frac{s1 x_{9,2} + s2 x_{10,2}}{a}
\]

\[
A_{[10,1]} = \frac{s3 x_{9,1} + s4 x_{10,1}}{a}, \ A_{[10,2]} = -\frac{(s3 x_{9,1} + s4 x_{10,1}) b}{a^2} + \frac{s3 x_{9,2} + s4 x_{10,2}}{a}
\]

Denote \ Q = \begin{pmatrix} A_{[9,1]} & A_{[9,2]} \\ A_{[10,1]} & A_{[10,2]} \end{pmatrix}, \ then \ the \ proof \ can \ be \ divided \ into \ the \ following \ three \ cases:

I: \ the \ rank \ of \ Q \ is \ 0, \ II: \ the \ rank \ of \ Q \ is \ 1, \ III: \ the \ rank \ of \ Q \ is \ 2.

Since \ the \ method \ of \ proof \ is \ similar, \ so \ we \ only \ show \ the \ proof \ of \ first \ case \ I. \ Now \ we \ have

\[
A = \begin{bmatrix}
x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} & x_{1,5} & x_{1,6} & x_{1,7} & x_{1,8} & x_{1,9} & x_{1,10} \\
x_{2,1} & x_{2,2} & x_{2,3} & x_{2,4} & x_{2,5} & x_{2,6} & x_{2,7} & x_{2,8} & x_{2,9} & x_{2,10} \\
x_{3,1} & x_{3,2} & x_{3,3} & x_{3,4} & x_{3,5} & x_{3,6} & x_{3,7} & x_{3,8} & x_{3,9} & x_{3,10} \\
x_{4,1} & x_{4,2} & x_{4,3} & x_{4,4} & x_{4,5} & x_{4,6} & x_{4,7} & x_{4,8} & x_{4,9} & x_{4,10} \\
0 & x_{5,2} & x_{5,3} & x_{5,4} & x_{5,5} & x_{5,6} & x_{5,7} & x_{5,8} & x_{5,9} & x_{5,10} \\
0 & 0 & x_{6,3} & x_{6,4} & x_{6,5} & x_{6,6} & x_{6,7} & x_{6,8} & x_{6,9} & x_{6,10} \\
0 & 0 & 0 & x_{7,4} & x_{7,5} & x_{7,6} & x_{7,7} & x_{7,8} & x_{7,9} & x_{7,10} \\
0 & 0 & 0 & 0 & x_{8,5} & x_{8,6} & x_{8,7} & x_{8,8} & x_{8,9} & x_{8,10} \\
0 & 0 & x_{9,3} & x_{9,4} & x_{9,5} & x_{9,6} & x_{9,7} & x_{9,8} & x_{9,9} & x_{9,10} \\
0 & 0 & x_{10,3} & x_{10,4} & x_{10,5} & x_{10,6} & x_{10,7} & x_{10,8} & x_{10,9} & x_{10,10}
\end{bmatrix}
\]

By calculating-finding in the sense of lemma 2.4, 2.5 and based on lemma 2.2, we get

\[
\frac{2 \text{tr}(T^5 A)}{4!} + \frac{2 \text{tr}(T^2 A T^4 A)}{3!} + \frac{\text{tr}(T^3 A T^3 A)}{2! 2!} + \frac{\text{tr}(T^3 A T^3 A)}{3! 3!} = 0
\]

\[
\frac{2 \text{tr}(T A T^5 A)}{5!} + \frac{2 \text{tr}(T^2 A T^4 A)}{3!} + \frac{\text{tr}(T^3 A T^3 A)}{2! 2!} + \frac{\text{tr}(T^3 A T^3 A)}{3! 3!} = 0
\]

\[
\frac{2 \text{tr}(T A T^5 A)}{4!} + \frac{2 \text{tr}(T^2 A T^4 A)}{3!} + \frac{\text{tr}(T^3 A T^3 A)}{2! 2!} + \frac{\text{tr}(T^3 A T^3 A)}{3! 3!} = 0
\]
\[ + \frac{3\text{tr}(T^3 AAT^6 A)}{3!} + \frac{3\text{tr}(AT^3 AT^6 A)}{3!} + \frac{3\text{tr}(T^4 AT^5 AA)}{4!5!} + \frac{3\text{tr}(T^5 AT^4 AA)}{4!5!} + \frac{3\text{tr}(TAT^6 AT^2 A)}{2!6!} + \frac{3\text{tr}(TAT^2 AT^6 A)}{2!6!} + \frac{3\text{tr}(T^5 AT^2 AT^2 A)}{5!2!2!} + \frac{3\text{tr}(T^5 AT^3 ATA)}{5!3!} + \frac{3\text{tr}(T^3 AT^5 ATA)}{5!3!} + \frac{3\text{tr}(TAT^7 ATA)}{7!} + \frac{3\text{tr}(TAT^4 AT^4 A)}{4!4!} + \frac{3\text{tr}(T^3 AT^4 AT^2 A)}{4!3!2!} + \frac{3\text{tr}(T^4 AT^3 AT^2 A)}{4!3!2!} + \frac{3\text{tr}(T^3 AT^3 AT^3 A)}{3!3!} = 0, \]

\[ + \frac{3\text{tr}(TAT^6 AT^2 A)}{5!} + \frac{3\text{tr}(TAT^2 AT^6 A)}{3!} + \frac{3\text{tr}(T^5 AT^2 AT^2 A)}{5!2!2!} + \frac{3\text{tr}(T^5 AT^3 ATA)}{4!} + \frac{3\text{tr}(T^3 AT^5 ATA)}{4!2!} + \frac{3\text{tr}(T^5 AT^5 ATA)}{4!2!} + \frac{3\text{tr}(TAT^7 ATA)}{4!2!} + \frac{3\text{tr}(T^3 AT^3 AT^3 A)}{3!3!} = 0, \]

\[ + \frac{6!}{2!2!} \frac{\text{tr}(T^3 AT^3 AT^3 A)}{3!3!} = 0, \text{tr}(T^3 A) = 0 \]

Which implies \( x_{4,1} = 0, x_{5,2} = 0, x_{6,3} = 0, x_{7,4} = 0, x_{8,5} = 0. \) At this point, we find

\[ A_{[9,3]} = \frac{s1x_{9,3} + s2x_{10,3}}{a}, A_{[9,4]} = -\left(\frac{s1x_{9,3} + s2x_{10,3}}{a}\right)b + \frac{s1x_{9,4} + s2x_{10,4}}{a} \]

\[ A_{[10,3]} = \frac{s3x_{9,3} + s4x_{10,3}}{a}, A_{[10,4]} = -\left(\frac{s3x_{9,3} + s4x_{10,3}}{a}\right)b + \frac{s3x_{9,4} + s4x_{10,4}}{a} \]

Denote \( Q_1 = \frac{A_{[9,3]}}{A_{[10,3]}}, \frac{A_{[9,4]}}{A_{[10,4]}} \), then the proof of case 1 can be divided into the following three cases:

I-1: the rank of \( Q_1 \) is 0, I-2: the rank of \( Q_1 \) is 1, I-3: the rank of \( Q_1 \) is 2.

I-1: the rank of \( Q_1 \) is 0. It is true that \( (A - E)^4 = 0 \) when \( (A - E)^2 \neq 0 \). Denote \( N = (A - E)^4 \), we have

\[ N_{[i,j]} = 0. \]

Let \( F = (A - E)^2 + \frac{(A - E)^3}{2} + \frac{(A - E)}{3}, Y = T^7 F + T^6 F^T + T^5 F^2 + T^4 F^3 + T^3 F^4 + T^2 F^5 + T^7 F^6 + F^7, M = T^2 + \frac{T^3}{3}, T^4 + \frac{T^5}{5}, T^6 + \frac{T^7}{7}, H = A - E \) and \( W = T^6 F F^T + T^6 F^2 F^T + T^6 F^3 F^T + T^6 F^4 F^T + T^6 F^5 F^T + T^6 F^6 F^T \)

\[ + T^6 F^7 F^T + T^6 F^8 F^T + T^6 F^9 F^T + T^6 F^{10} F^T + T^6 F^{11} F^T + T^6 F^{12} F^T + T^6 F^{13} F^T + T^6 F^{14} F^T + T^6 F^{15} F^T \]

\[ + T^6 F^{16} F^T + T^6 F^{17} F^T + T^6 F^{18} F^T + T^6 F^{19} F^T + T^6 F^{20} F^T + T^6 F^{21} F^T + T^6 F^{22} F^T + T^6 F^{23} F^T + T^6 F^{24} F^T + T^6 F^{25} F^T + T^6 F^{26} F^T + T^6 F^{27} F^T + T^6 F^{28} F^T \]

\[ + T^6 F^{29} F^T + T^6 F^{30} F^T + T^6 F^{31} F^T + T^6 F^{32} F^T + T^6 F^{33} F^T + T^6 F^{34} F^T + T^6 F^{35} F^T + T^6 F^{36} F^T + T^6 F^{37} F^T + T^6 F^{38} F^T \]

\[ + T^6 F^{39} F^T + T^6 F^{40} F^T + T^6 F^{41} F^T + T^6 F^{42} F^T + T^6 F^{43} F^T + T^6 F^{44} F^T + T^6 F^{45} F^T + T^6 F^{46} F^T + T^6 F^{47} F^T + T^6 F^{48} F^T \]

\[ + T^6 F^{49} F^T + T^6 F^{50} F^T + T^6 F^{51} F^T + T^6 F^{52} F^T + T^6 F^{53} F^T + T^6 F^{54} F^T + T^6 F^{55} F^T + T^6 F^{56} F^T + T^6 F^{57} F^T + T^6 F^{58} F^T \]

we have \( Y_{[i,j]} = 0, W_{[i,j]} = 0 \). So \( Y_{[10,8]}, Y_{[9,9]}, Y_{[7,8]}, W_{[8,2]}, W_{[9,1]} \) and \( W_{[10,1]} \) are both 0. we can get the following seven situations and discuss them separately.

I-1-1: \( x_{5,3} = 0, x_{8,6} = 0 \); I-1-2: \( x_{5,3} = 0, x_{6,4} = 0 \); I-1-3: \( x_{4,2} = 0, x_{9,3} = 0 \); I-1-4: \( x_{3,1} = 0, x_{8,6} = 0 \); I-1-5: \( x_{3,1} = 0, x_{6,4} = 0 \); I-1-6: \( x_{3,1} = 0, x_{4,2} = 0 \); I-1-7: \( x_{9,5} = 0, x_{10,5} = 0 \).

I-1-1: when \( x_{5,3} = 0, x_{8,6} = 0, \) let \( N, F, Y, M, H, W \) as mentioned above. we have \( N_{[i,j]} = 0, W_{[i,j]} = 0 \) and \( Y_{[i,j]} = 0 \). So \( W_{[7,3]} = 0, W_{[6,2]} = 0, W_{[5,1]} = 0, W_{[9,3]} = 0, W_{[10,3]} = 0, N_{[8,2]} = 0, N_{[7,4]} = 0, N_{[10,1]} = 0, N_{[9,1]} = 0, Y_{[6,8]} = 0 \) Add these equations to following equations

\[ \text{tr}(T^2 A) = 0, \frac{\text{tr}(T^4 AA)}{4!} + \frac{\text{tr}(TAT^3 A)}{3!} + \frac{\text{tr}(T^2 AT^2 A)}{8} = 0, \frac{2\text{tr}(TAT^3 A)}{2!} + \text{tr}(T^2 AT^2 A) \]
\[
\begin{align*}
&= 0, \frac{tr(T^4 A^2)}{2!} + \frac{tr(TAT^3 A)}{2} + \frac{tr(T^2 A T^2 A)}{2!} + \frac{tr(T^3 A T^2 A)}{3!} = 0, \frac{3T r(T^6 A^3)}{6!} + \frac{3T r(T^5 A T A^2)}{5!} \\
&+ \frac{3tr(TAT^5 A^3)}{5!} + \frac{3tr(T^4 A^2 T^2 A)}{4!} + \frac{3tr(T^2 A T^4 A^2)}{3!} + \frac{3tr(T^3 A T^3 A^2)}{3!} + \frac{3tr(T^4 TAT^2 A)}{4!} = 0, \frac{Tr(T^5 A T^3 A^2)}{4!} + \frac{Tr(T^5 A T^2 A)}{3!} \\
&+ \frac{3tr(T^2 A T^3 A TA)}{3!} + \frac{tr(T^2 A T^3 A TA)}{2} + \frac{tr(T^2 A T^3 A TA)}{2} + \frac{tr(T^3 A T^2 A TA)}{2} + \frac{tr(T^3 A T^2 A TA)}{2} = 0, \frac{3tr(TAT^4 A^2)}{3!} + \frac{3tr(T^2 AT^3 A)}{2} \\
&+ \frac{3tr(T^3 AT^2 A TA)}{2} + \frac{tr(T^2 AT^2 A TA)}{2} = 0, \frac{tr(T^6 A^3)}{5!} + \frac{tr(T^5 A T A^2)}{4!} + \frac{tr(T^5 A T A^2)}{5!} \\
&+ \frac{tr(T^2 AT^4 A^2)}{4!} + \frac{tr(TAT^5 A^2)}{3!} + \frac{tr(T^2 AT^4 A^2)}{4!} + \frac{tr(T^3 A T^3 A^2)}{3!} + \frac{tr(T^3 A T^3 A^2)}{3!} + \frac{tr(T^4 A T^2 A TA)}{4!} + \frac{tr(T^4 A T^2 A TA)}{4!} \\
&+ \frac{tr(T^4 TAT^2 A)}{3!} + \frac{tr(T^2 A T^2 A TA)}{2!} + \frac{tr(T^2 A T^2 A TA)}{2!} = 0, \frac{4tr(TATAT^5 A)}{4!} + \frac{4tr(TATAT^5 A)}{4!} + \frac{4tr(TATAT^4 A)}{3!} + \frac{3tr(TATAT^4 A)}{3!} \\
&+ \frac{4tr(TAT^3 AT^2 A TA)}{3!} + \frac{2tr(TATAT^3 AT^3 A)}{2!} + \frac{2tr(TATAT^3 AT^3 A)}{2!} + \frac{4tr(TAT^2 AT^3 A^2)}{2!} + \frac{4tr(TAT^2 AT^3 A^2)}{2!} = 0, \frac{5tr(TATATAT^6 A)}{5!} + \frac{5tr(TATAT^2 AT^2 A^2)}{2!} + \frac{5tr(TATAT^2 AT^2 A^2)}{2!} + \frac{5tr(TATAT^2 AT^2 A^2)}{2!} = 0, \frac{5tr(TATAT^5 A T^2 A)}{4!} + \frac{5tr(TATAT^5 A T^2 A)}{4!} + \frac{5tr(TATAT^5 A T^2 A)}{4!} + \frac{5tr(TATAT^5 A T^2 A)}{4!} \\
&+ \frac{5tr(TAT^2 AT^2 AT^2 A)}{3!} + \frac{5tr(TAT^2 AT^2 AT^2 A)}{3!} + \frac{5tr(TAT^2 AT^2 AT^2 A)}{3!} + \frac{5tr(TAT^2 AT^2 AT^2 A)}{3!} = 0, \frac{5tr(TAT^3 AT^3 A^2)}{3!} + \frac{5tr(TAT^3 AT^3 A^2)}{3!} + \frac{5tr(TAT^3 AT^3 A^2)}{3!} + \frac{5tr(TAT^3 AT^3 A^2)}{3!} \\
&+ \frac{5tr(TAT^3 AT^3 AT^2 A)}{4} + \frac{5tr(TAT^3 AT^3 AT^2 A)}{4} + \frac{5tr(TAT^3 AT^3 AT^2 A)}{4} + \frac{5tr(TAT^3 AT^3 AT^2 A)}{4} = 0, \frac{5tr(TAT^3 AT^3 AT^2 A)}{4} + \frac{5tr(TAT^3 AT^3 AT^2 A)}{4} + \frac{5tr(TAT^3 AT^3 AT^2 A)}{4} + \frac{5tr(TAT^3 AT^3 AT^2 A)}{4}.
\end{align*}
\]
theoretically that the tenth-order matrix group \( G \) generated by two unipotent matrices which Jordan blocks does not accordingly exceed 2-orders and 4-orders is unipotent when each primitive element representation dimension is ten with the help of calculation software. Although we have not proved

This article mainly discusses some special cases of unipotency of free group when its linear representation dimension is ten we have found that

\[
\frac{5 \text{tr}(T^2 A T^2 A T^3 A T)}{2} + \frac{5 \text{tr}(T^2 A T^3 A T^2 A T)}{2} + \frac{5 \text{tr}(T^2 A T^3 A T^2 A T A T)}{2} + \text{tr}(T^2 A T^2 A T^2 A T^2 A T^2 A T^2 A T) = 0
\]

Which implies \( x_{3,1} = 0, x_{4,2} = 0, x_{6,4} = 0, x_{7,5} = 0 \).

We have found \( \text{tr} TA = 0, \text{tr} TATA = 0, \text{tr} TATATA = 0, \text{tr} TATATATATA = 0 \), \( \text{tr} TATATATATATATA = 0 \) by calculating – finding. Which implies \( x_{2,1} = 0, x_{3,2} = 0, x_{5,4} = 0, x_{6,5} = 0, x_{7,6} = 0, x_{8,7} = 0 \), thus this situation ends.

I-1-1: The rank of \( Q \) is 0, \( x_{10,5} = 0 \). Let \( P \) as mentioned above and \( G = PAP^{-1} \) we have

\[
G_{[2,1]} = \frac{ax_{2,1} + bx_{3,1}}{a}, G_{[3,1]} = x_{3,1}, G_{[3,2]} = -\frac{x_{3,1}b}{a} + \frac{ax_{3,2} + bx_{4,2}}{a}, G_{[8,6]} = x_{8,6},
\]

\[
G_{8,7} = -\frac{x_{8,6}b}{a} + x_{8,7}, G_{[9,6]} = \frac{s1x_{9,6} + s2x_{10,6} + u x_{8,6}}{a}
\]

\[
G_{[10,6]} = \frac{s3x_{9,6} + s4x_{10,6} + u x_{8,6}}{a}
\]

If \( G_{[8,6]} = 0 \), Solve the same equations which is in I-1-1.

It is implies \( x_{3,1} = 0, x_{4,2} = 0, x_{5,3} = 0, x_{6,4} = 0, x_{7,5} = 0 \). End of this situation.

If \( x_{9,6} \neq 0 \) Then \( x_{8,7} = 0, x_{9,6} = 0, x_{10,6} = 0 \), let \( N = (A - E)^4 \), \( F, Y \) as mentioned above.

We have \( N_{[i,j]} = 0 \) that \( N_{[10,1]} = 0 \) and \( N_{[9,1]} = 0 \) We have \( x_{10,7}x_{7,5}x_{5,3}x_{3,1} = 0 \) and \( x_{9,7}x_{7,5}x_{5,3}x_{3,1} = 0 \) if \( x_{7,5}x_{5,3}x_{3,1} = 0 \) thus \( x_{8,6} = 0 \), End of this situation.

If \( x_{7,5}x_{5,3}x_{3,1} \neq 0 \) we have \( x_{10,7} = 0 \) and \( x_{9,7} = 0 \) at this time Let \( N = (A - E)^4 \), We have \( N_{[i,j]} = 0 \), that is \( N_{[10,1]} = 0, N_{[9,1]} = 0 \). That is \( x_{10,8}x_{8,6}x_{6,4}x_{4,2} = 0 \) and \( x_{9,8}x_{8,6}x_{6,4}x_{4,2} = 0 \).

If one of \( x_{8,6}, x_{6,4} \) and \( x_{4,2} \) is zero. End of this situation. Thus \( x_{10,8} = 0 \) and \( x_{9,8} = 0 \). This situation is over.

So we finished the proof of I-1. And I-2, I-3 can be proved in similar method.

The theorem 3.1 to be proved in this article can be divided into the following situations

I: \( B = \text{diag}(I_8, E_2) \) \( II: B = \text{diag}(I_8, I_2) \).

In case I, according to the rank of matrix \( Q \) in the article, it can be divided into the following three cases

I.1: the rank of \( Q \) is 0, I.2: the rank of \( Q \) is 1, I.3: the rank of \( Q \) is 2.

This article mainly elaborates the method to prove situation I.1. The method of other proofs is similar to this article, but the calculation program has not been run yet. We can use the methods used in this article to study the rest of the situation.

4. Conclusion

This article mainly discusses some special cases of unipotency of free group when its linear representation dimension is ten with the help of calculation software. Although we have not proved theoretically that the tenth-order matrix group \( G \) generated by two unipotent matrices which Jordan blocks does not accordingly exceed 2-orders and 4-orders is unipotent when each primitive element \( M \) of \( G \) satisfies \( (M - E)^8 = 0 \), we can now prove it when the representation dimension is ten with the help of calculation software. It is shown here that the calculation is a necessary supplement to the proof of theory.

References

[1] TAVGEN OI, SAMSONOV YU. B. Unipotency of the Image of \( F_2(x, y) \) Representation by Matrices of \( GL(n, C) \), \( n = 1,2,3,4 \) Under the Condition of Mapping of Generating and Primitive Elements into Unipotent Matrices [J]. Report NAS of Belarus, 2001, 45(6): 29-32(Russian)
[2] YANG XINSONG, LIU HUAN. Unipotency of Bivariate Generated Free Groups [J]. Journal of Harbin University of Science and Technology, 2014, 19(6): 16-25.

[3] TAVGEN O I, DU JUNHUA, LIU CHUNYAN. The Representation Image Unipotency of the Group by Mapping Primitive Elements into Unipotent Matrices with Small Jordan Blocks [J]. Problems of Physics, Mathematics and Technics, 2011, 3(8): 81-83 (Russian).

[4] YANG XINSONG. Linear Structures of Free Groups [M]. RIVS: Minsk 2011: 11-61 (Russian).

[5] TAN PENGSHUN. Unipotency of Bivariate Generated Free Groups with Primitive unipotent [D]. hust, 2013, 3-5 (Chinese).

[6] DU JUNHUA, TAN PENGSHUN, YANG XINSONG. Unipotency of Eighth Order Linear Groups [J]. Journal of Natural Science of Harbin Commercial University, 2012, 28(4): 461-464 (Chinese).

[7] YANG XINSONG, WANG NAYING. The unipotency of a group of normative matrices not higher than the fifth order Jordan block matrices [J]. Journal of Harbin University of Science and Technology, 2016, 21(2): 112-117 (Chinese).

[8] Matringe Nadir. Distinction for Unipotent p-Adic Groups. 2020, 46(6): 1571-1582.

[9] P. Gvozdevsky. Overgroups of Levi subgroups I. The case of abelian unipotent radical. 2020, 31(6): 969-999.

[10] Brunat Olivier, Dudas Olivier, Taylor Jay. Unitriangular shape of decomposition matrices of unipotent blocks. 2020, 192(2): 583-663.

[11] Justin Campbell. The Bernstein center of a p-adic unipotent group. 2020, 560: 521-537.

[12] MEIJING SHAN. Mechanical Proving for ERDMS-SZEKERES Problem. 2016.

[13] JOHN HARDING. Wigner's theorem for an infinite set. 2018, 68(5): 1173-1222.

[14] WILAYAT KHAN, MUHAMMAD KAMRAN, SYED RAMEEZ NAQVI, et al. Formal Verification of Hardware Components in Critical Systems. 2020, 2020.

[15] PLATONOV V P, POTAPCHIK A E. New Combinatorial Properties of Linear Group Algebra [J]. Waterloo: Journal of Algebra, 2001: 3-451.