FACES OF GENERALIZED CLUSTER COMPLEXES AND NONCROSSING PARTITIONS

ELENI TZANAKI

Abstract. Let Φ be a finite root system with corresponding reflection group W and let m be a nonnegative integer. We consider the generalized cluster complex ∆m(Φ) defined by S. Fomin and N. Reading and the poset NCm(W) of m-divisible noncrossing partitions defined by D. Armstrong. We give a characterization of the faces of ∆m(Φ) in terms of NCm(W), generalizing that of T. Brady and C. Watt given in the case m = 1. Making use of this, we give a case free proof of a conjecture of F. Chapoton and D. Armstrong, which relates a certain refined face count of ∆m(Φ) with the Möbius function of NCm(W).

1. Introduction

Let Φ be a finite root system of rank n with associated reflection group W. Let Φ+ be a positive system for Φ with corresponding simple system Π. Motivated by their theory of cluster algebras [16], S. Fomin and A. Zelevinsky introduced the cluster complex ∆(Φ) [15]. This is a pure (n−1)-dimensional simplicial complex on the vertex set Φ+ ∪ (−Π) which is homeomorphic to a sphere [15]. Later, S. Fomin and N. Reading [14] introduced the generalized cluster complex ∆m(Φ), where m is any nonnegative integer (see also [20]). This is a simplicial complex on the vertex set of colored almost positive roots Φm≥−1, that is the set consisting of m (colored) copies of each positive root and one copy of each negative simple root. In the case m = 1, the generalized cluster complex ∆m(Φ) reduces to ∆(Φ). The complex ∆m(Φ) has remarkable properties and surprising connections with other combinatorial objects like m-divisible noncrossing partitions NCm(W) [2] [13] [18] and Catalan hyperplane arrangements [2] [3] [7]. For instance, if Φ is irreducible the number of facets of ∆m(Φ) is equal to the generalized Catalan number Nm(Φ) = \prod_{i=1}^{n} \frac{e_i + m + 1}{e_i + 1} [2]. Moreover, the entries of the f and h-vector of ∆m(Φ), as well as those of the natural subcomplex ∆m+(Φ) called its positive part, have many interesting combinatorial interpretations [2] [7].

In his thesis [1] D. Armstrong defined the poset NCm(γ) where γ is a Coxeter element of W. It is proved that the isomorphism type of the poset NCm(γ) is independent of the Coxeter element γ. We denote this poset by NCm(W) when the choice of γ is irrelevant and call it the poset of m-divisible noncrossing partitions. The poset NCm(W) is a graded meet-semilattice and it reduces to the lattice of noncrossing partitions NCW [9] [11] associated to W in the case m = 1. If W is irreducible, the number of elements of NCm(W) is equal to Nm(Φ) and the

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h-polynomial of $\Delta^m(\Phi)$ is equal to the rank generating polynomial of $NC_m(W)$ \cite{13}.

The first main result of this paper is a new characterization of the faces of $\Delta^m(\Phi)$ in terms of $m$-divisible noncrossing partitions. More precisely, let $\Phi$ be an irreducible root system and let $\Pi = \Pi_+ \cup \Pi_-$ be a partition of the set of simple roots $\Pi$ into two disjoint sets such that the roots within each are pairwise orthogonal. Let $\gamma$ be a bipartite Coxeter element with respect to this partition of $\Pi$ (see \cite{5}). Consider a face $\sigma$ of $\Delta^m(\Phi)$ and let $\sigma^i$ be the subset of $\sigma$ consisting of positive roots of color $i$ and $\sigma_{\pm} = \sigma \cap (-\Pi_{\pm})$. If $\tau \subseteq \Phi_{\geq -1}$ such that either $\tau \subseteq (-\Pi)$ or $\tau$ consists of positive roots of the same color, we denote by $w_\tau$ the product of reflections through the roots in $\tau$ taken in a certain order (see \cite{9}). We make the convention that $w_\emptyset = 1$, where $1$ is the identity in $W$. The faces of $\Delta^m(\Phi)$ can be characterized by the following criterion.

**Theorem 1.1.** The set $\sigma \subseteq \Phi_{\geq -1}$ is a face of $\Delta^m(\Phi)$ if and only if the sequence $(w_{\sigma, +}, w_{\sigma, -}, \ldots, w_{\sigma, +}, w_{\sigma, -})$ is an element of $NC_m(\gamma)$ of rank $|\sigma|$.

Theorem \cite{161} specializes to the characterization of T. Brady and C. Watt \cite{12} Section 8 in the case $m = 1$. Moreover, making use of the above criterion one may discover many interesting properties of $\Delta^m(\Phi)$, for instance that $\Delta^m(\Phi)$ is shellable and $(m + 1)$-Cohen-Macaulay. This is discussed in the article \cite{8}.

Our second main result is the proof of a conjecture that relates the cluster complex $\Delta^m(\Phi)$ with the poset $NC_m(W)$. F. Chapoton \cite{13} conjectured a surprising enumerative relation between a refined face count of $\Delta(\Phi)$ and the M"obius function of $NC_m(W)$. This conjecture, which was proved by Athanasiadis \cite{11}, can be stated in the $m \geq 1$ case as follows $\cite{11}$ (see also \cite{17,18}). The $F$-triangle for $\Delta^m(\Phi)$ is defined by the generating function

$$F^m_{\Phi}(x, y) = \sum_{k=0}^{n} \sum_{l=0}^{n} f_{k,l}(\Phi, m) x^k y^l,$$

where $f_{k,l}(\Phi, m)$ is the number of faces of $\Delta^m(\Phi)$ consisting of $k$ colored positive roots and $l$ negative simple roots. The $M$-triangle for $NC_m(W)$ is defined similarly as

$$M^m_{W}(x, y) = \sum_{a \leq b \in NC_m(W)} \mu(b) x^{rk(b) - rk(a)} y^{rk(a)},$$

where $\leq$ denotes the order relation in $NC_m(W)$, $\mu$ stands for its M"obius function and $rk(a)$ is the rank of $a \in NC_m(W)$.

The following relation was formulated by F. Chapoton \cite{13} Conjecture 1] as a conjecture in the case $m = 1$ and was restated by D. Armstrong for any $m \geq 1$.

**Theorem 1.2.** Let $\Phi$ be a finite root system of rank $n$ with corresponding reflection group $W$ and let $m$ be a nonnegative integer. The $F$-triangle for $\Delta^m(\Phi)$ and the $M$-triangle for $NC_m(W)$ are related by the equality

$$(1 - y)^n F^m_{\Phi}(x + y, \frac{y}{1 - y}, \frac{y}{1 - y}) = M^m_{W}(-x, -y/x).$$
Theorem 1.2 has been proved in part by C. Kratthenthaler in a case by case fashion, for all finite root systems when \( m = 1 \) and for those that do not contain a copy of \( D_k \) for any \( k \geq 4 \), in the case \( m \geq 2 \). It was observed by C. Kratthenthaler that the relation implies the following interesting reciprocity \[ y^n M^{(-m)}_W(xy, 1/y) = M^{(m)}_W(x, y), \] for which there is no intrinsic explanation yet.

This paper is organized as follows. After providing the necessary background, we proceed with the proof of Theorem 1.2 in Section 3. In Section 4 we find an EL-labelling of the poset \( NC_{(m)}(\gamma) \), in which the falling chains are in bijection with facets of \( \Delta^m_+(\Phi) \). Given the above results, we conclude with the proof of Theorem 1.1 in Section 5, generalizing that of Athanasiadis for the case \( m = 1 \).

2. Preliminaries

In this section we introduce our main objects of study and state a few lemmas and theorems necessary in later sections.

The lattice \( NC_W \) and the absolute order: Let \( W \) be a finite Coxeter group of rank \( n \) and let \( T \) be the set of all reflections in \( W \). The group \( W \) is generated by \( T \) and one can define a length function \( l_T \) on \( W \) so that \( l_T(w) \) is the smallest \( k \) such that \( w \) can be written as a product of \( k \) reflections in \( T \). We define a partial order \( \leq \) on \( W \) by letting

\[ u \leq v \text{ if and only if } l_T(u) + l_T(u^{-1}v) = l_T(v), \]

in other words if there exists a shortest factorization of \( u \) into reflections in \( T \) which is a prefix of such a shortest factorization of \( v \). Note that all Coxeter elements are maximal in this partial order of \( W \). The group \( W \) acting by conjugation gives automorphisms of this partial order, since the set of reflections is stable under conjugation. We define the noncrossing partition lattice \( NC_W \) to be an interval between the identity \( 1 \) and any Coxeter element \( \gamma \). Since all Coxeter elements are conjugate, the isomorphism type of \( NC_W \) is independent of \( \gamma \). The poset \( NC_W \) is a self-dual lattice of rank \( n \) \[12\]. If we have chosen some particular Coxeter element \( \gamma \), then we denote the noncrossing partition lattice by \( NC_W(\gamma) \). If there is no fear of confusion we just write \( NC(\gamma) \) suppressing \( W \) in the notation.

For later reference we summarize a few definitions and general facts.

**Definition 2.1.** [1 Definition 3.1.1] The \( m \)-tuple \((w_1, \ldots, w_m) \in W^m \) is a minimal factorization of \( w \in W \) if \( w = w_1w_2 \cdots w_m \) and \( l_T(w) = \sum_{i=1}^{m} l_T(w_i) \).

**Lemma 2.2.**

(i) Conjugate elements have the same length.

(ii) [1 Lemma 3.1.2] If \((w_1, w_2, \ldots, w_k)\) is a minimal factorization of \( w \in W \), then so is the \( k \)-tuple

\[ (w_1, w_1^{-1}w_1, \ldots, w_k^{-1}w_1, w_{i_1+1}, \ldots, w_k), \]

for every \( 1 \leq i \leq k \). In other words, for every \( w_i \) there is a minimal factorization of \( w \) with \( w_i \) in the first place.

(iii) If \((w_1, w_2, \ldots, w_k)\) is a minimal factorization of \( w \), then \( w_i, w_{i+1} \cdots w_{i+\ell} \leq w \) and \( l_T(w_{i_1} \cdots w_{i_\ell}) = \sum_{j=1}^{\ell} l_T(w_{i_j}) \), for every \( 1 \leq i_1 < i_2 < \cdots < i_\ell \leq k \).
Lemma 2.3. [1] Lemma 2.1] Let $a, b, w$ be elements of $W$.

(i) $a \leq aw \leq b$ if and only if $w \leq a^{-1}b \leq b$.

(ii) $a \leq aw \leq b$ if and only if $a \leq bw^{-1} \leq b$.

(iii) $a \leq b$ if and only if $a^{-1}b \leq b$ and, in that case, the interval $[a, b]$ is isomorphic to $[1, a^{-1}b]$.

Lemma 2.4.

(i) [3] Lemma 2.1(iv)] If $a, b \leq c \leq w$ for some $w \in W$ and $ab \leq w$ then $ab \leq c$.

(ii) [12] Relation (3)] If $a \leq b \leq c$ then $a^{-1}b \leq a^{-1}c$ and $ba^{-1} \leq ca^{-1}$.

(iii) [12] Relation (7)] For distinct reflections $t_1, t_2$ and $w \in W$ we have $t_1 t_2 \leq w \iff t_2 \leq t_1 w \iff t_1 \leq wt_2$.

The following simple observation is immediate from the definitions and Lemma 2.2(i).

Lemma 2.5. For $u, v, w \in W$ we have $u \leq v$ if and only if $ww^{-1} \leq wv$. □

Posets and EL-labelings: Let $(P, \leq)$ be a finite graded poset. We say that $y$ covers $x$ and write $x \rightarrow y$ if $x < y$ and $x < z \leq y$ holds only for $z = y$. Let $\mathcal{E}(P)$ be the set of covering relations in $P$ and consider $\Lambda$ a totally ordered set. An edge labeling of $P$ with label set $\Lambda$ is a map $\lambda : \mathcal{E}(P) \rightarrow \Lambda$. If $C : x_0 \rightarrow x_1 \rightarrow \cdots \rightarrow x_r$ is an unrefinable chain we let $\lambda(C) = (\lambda(x_0 \rightarrow x_1), \lambda(x_1 \rightarrow x_2), \ldots, \lambda(x_{r-1} \rightarrow x_r))$ be the label of $C$ with respect to $\lambda$ and we call $C$ rising or falling with respect to $\lambda$ if the entries of $\lambda(C)$ strictly increase or weakly decrease respectively, in the total order of $\Lambda$. Since $P$ is graded, all maximal chains $C$ in every interval $I = [x, y]$ have the same length, equal to $rk(y) - rk(x)$. The edge labeling allows us to order the maximal chains in $I$ by ordering the labels $\lambda(C)$ lexicographically. That is, $C$ is lexicographically smaller than $C'$ if $\lambda(C)$ precedes $\lambda(C')$ in the lexicographic order induced by the total order of $\Lambda$.

Definition 2.6. [10] An edge labeling $\lambda$ of $P$ is called EL-labeling if for every non-singleton interval $[x, y]$ in $P$

(i) there is a unique rising maximal chain in $[x, y]$ and

(ii) this chain is lexicographically smallest among all maximal chains in $[x, y]$ with respect to $\lambda$.

If $P_1, \ldots, P_m$ are posets then their direct product $P = P_1 \times \cdots \times P_m$ is the poset on the set $\{(x_1, \ldots, x_m) : x_i \in P_i\}$, such that $(x_1, \ldots, x_m) \leq (x_1', \ldots, x_m')$ in $P$ if $x_i \leq x_i'$ in $P_i$ for every $1 \leq i \leq m$. Moreover $(x_1', \ldots, x_m')$ covers $(x_1, \ldots, x_m)$ in $P$ if there is some $1 \leq i_0 \leq m$ such that $x_i = x_i'$ for all $1 \leq i \neq i_0 \leq m$ and $x_{i_0}'$ covers $x_{i_0}$ in $P_{i_0}$. The following lemma is a reformulation of [10] Theorem 4.3).

Lemma 2.7. [10] If $P_1, P_2, \ldots, P_m$ are graded posets that admit an EL-labeling, then $P = P_1 \times P_2 \times \cdots \times P_m$ admits an EL-labeling as well.

The EL-labeling that $P$ inherits from $P_1, \ldots, P_m$ is the following. Assume that $\Lambda_i$ is the totally ordered set of the EL-labels of $P_i$. If $\lambda_i \in \Lambda_i$, we totally order the $m$-tuples $(\emptyset, \ldots, \emptyset, \lambda_i, \emptyset, \ldots, \emptyset)$ with $\lambda_i$ in the $i$-th place by letting

$$\lambda_i$$

if and only if

(4) $(\emptyset, \ldots, \emptyset, \lambda_i, \emptyset, \ldots, \emptyset) \leq (\emptyset, \ldots, \emptyset, \lambda_j, \emptyset, \ldots, \emptyset)$
If $x = (x_1, \ldots, x_m)$ is covered by $x' = (x'_1, \ldots, x'_m)$ in the poset $P$ then there is some $1 \leq i_0 \leq m$ such that $x_i = x'_i$ for all $1 \leq i \neq i_0 \leq m$ and $x_{i_0} \rightarrow x'_{i_0}$ in $P_{i_0}$. If $\lambda_{i_0} = \lambda(x_{i_0} \rightarrow x'_{i_0})$ is the edge label from the EL-labeling of $P_{i_0}$ then we label the edge $x \rightarrow x'$ in $P$ by the $m$-tuple $(\emptyset, \ldots, \emptyset, \lambda_{i_0}, \emptyset, \ldots, \emptyset)$ with $\lambda_{i_0}$ in the $i_0$-th entry. The edge labels of $P$ we obtain this way, totally ordered as in (4), form an EL-labeling of $P_1 \times P_2 \times \cdots \times P_m$.

The EL-labeling of $NC(\gamma)$: Let $\Phi$ be an irreducible root system of rank $n$ with positive part $\Phi^+$ and let $\Pi = \{\sigma_1, \sigma_2, \ldots, \sigma_r\}$ be a choice of simple system for $\Phi$ such that $\Pi_+ = \{\sigma_1, \sigma_2, \ldots, \sigma_r\}$ and $\Pi_- = \{\sigma_{r+1}, \ldots, \sigma_n\}$ are orthonormal sets. For each $\alpha \in \Phi$ we denote by $R(\alpha)$ the reflection in $\mathbb{R}^n$ through the hyperplane orthogonal to $\alpha$. A bipartite Coxeter element $\gamma = \gamma_+ \gamma_-$, where

$$\gamma_{\pm} = \prod_{\alpha \in \Pi_{\pm}} R(\alpha).$$

Thus, in the present case, the bipartite Coxeter element is $\gamma = R(\sigma_1) \cdots R(\sigma_n)$.

Let $N = nh/2$ be the number of positive roots of $\Phi^+$. For $1 \leq i \leq 2N$ we define $\rho_i = R(\sigma_1)R(\sigma_2) \cdots R(\sigma_{i-1})(\sigma_i)$ where the simple roots $\sigma_i$ are indexed cyclically modulo $n$ and we make the convention that $\rho_{i-1} = \rho_{2N-i}$ for $i \geq 0$. The following formula, which is easily verified, appears in [12, Section 3]

$$\rho_i = \begin{cases} 
\sigma_i & \text{for } i = 1, \ldots, r, \\
-\gamma(\sigma_i) & \text{for } i = r+1, \ldots, n, \\
\gamma(\rho_{i-n}) & \text{for } i > n.
\end{cases}$$

Then we have

$$\{\rho_1, \ldots, \rho_N\} = \Phi^+, \quad \{\rho_{N+i} : 1 \leq i \leq r\} = -\Pi_+,$$

$$\{\rho_{i-1} : 0 \leq i < n-r\} = \{\rho_{N-1} : 0 \leq i < n-r\} = -\Pi_-.$$

Moreover, the last $n-r$ positive roots are a permutation of $\Pi_-$. Let $\Phi_{\geq 1} = \Phi^+ \cup (-\Pi)$ be the set of almost positive roots. We totally order the roots in $\Phi_{\geq 1}$ as follows:

$$\rho_{n+r+1} < \cdots < \rho_0 < \rho_1 < \cdots < \rho_N < \rho_{N+1} < \cdots < \rho_{N+r}$$

Let $u, v \in NC(\gamma)$ such that $v$ covers $u$. Clearly $v = ut$ where $t$ a reflection in $W$. The natural edge labeling of the edge $u \rightarrow v$ is the reflection $t = u^{-1}v$.

**Theorem 2.8.** [5] Theorem 4.2 | If the set $T$ of positive roots $\Phi^+$ is totally ordered by [11] and $\gamma = R(\sigma_1)R(\sigma_2) \cdots R(\sigma_n)$ then the natural edge labeling of $NC(\gamma)$ with label set $T$ is an EL-labeling.

**The poset $NC_{(m)}(W)$** Here we recall some general facts for the poset $NC_{(m)}(W)$ of $m$-divisible noncrossing partitions. We refer the reader to [11] for more details.

**Definition 2.9.** [11] Definition 3.2.2 | Let $W$ be a reflection group and fix a Coxeter element $\gamma$. We denote by $NC_{(m)}(\gamma)$ the set of $m$-tuples $(w_1, \ldots, w_m) \in (NC(\gamma))^m$ for which
(i) $w = w_1 \cdots w_m \leq \gamma$ and  
(ii) $l_T(w) = \sum_{i=1}^m l_T(w_i)$.

Recall that a subset $I$ of a poset $(P, \leq)$ is an order ideal if $x, y \in P$, $x \in I$ and $y \leq x$ implies $y \in I$. The set $NC_{(m)}(\gamma)$ is a subposet of $(NC(\gamma))^m$. Moreover it is an order ideal of $(NC(\gamma))^m$ with maximal elements the minimal factorizations of $\gamma$ [Lemma 3.4.3]. Thus, the set $NC_{(m)}(\gamma)$ inherits the partial order as well as the rank function of $(NC(\gamma))^m$. More specifically, for $(u_1, \ldots, u_m), (w_1, \ldots, w_m) \in NC_{(m)}(\gamma)$ we have 

$$(u_1, u_2, \ldots, u_m) \leq (w_1, w_2, \ldots, w_m) \text{ if } u_i \leq w_i \text{ for all } 1 \leq i \leq m$$

and 

$$rk((w_1, \ldots, w_m)) = \sum_{i=1}^m l_T(w_i).$$

By the fact that all Coxeter elements are conjugate and Lemma 3.4.3, we deduce that the isomorphism type of $NC_{(m)}(\gamma)$ is independent of the choice of $\gamma$. Therefore, if we are not interested on the particular choice of $\gamma$ we write $NC_{(m)}(W)$ and we call this the poset of $m$-divisible noncrossing partitions. The poset $NC_{(m)}(W)$ is a ranked meet-semilattice [Lemma 3.4.4].

The cluster complex $\Delta(\Phi)$: Let $\Phi$ be an irreducible root system and let $\Phi_{\geq -1} = \Phi^+ \cup (\Pi_-)$ be the set of almost positive roots. We define the map $R$ on $\Phi_{\geq -1}$ as follows:

$$R(\alpha) = \begin{cases} 
\gamma^{-1}(\alpha) & \text{if } \alpha \not\in \Pi_+ \cup (\Pi_-) \\
-\alpha & \text{if } \alpha \in \Pi_+ \cup (\Pi_-)
\end{cases}$$

(8)

**Theorem 2.10.** [14] There is a unique symmetric binary relation on $\Phi_{\geq -1}$ called compatibility such that 

(i) $\alpha$ and $\beta$ are compatible if and only if $R(\alpha)$ and $R(\beta)$ are compatible,  
(ii) a negative simple root $-\sigma_i$ is compatible with a positive root $\beta$ if and only if the simple root expansion of $\beta$ does not involve $\sigma_i$.

The cluster complex $\Delta(\Phi)$ is the simplicial complex on the vertex set $\Phi_{\geq -1}$ such that a set $\sigma \subset \Phi_{\geq -1}$ is a face of $\Delta(\Phi)$ if every pair of roots in $\sigma$ is compatible. The maximal faces of $\Delta(\Phi)$ are called clusters. The simplicial complex $\Delta(\Phi)$ is pure, $(n-1)$-dimensional and homeomorphic to a sphere [15].

The generalized cluster complex $\Delta^m(\Phi)$: The generalized cluster complex $\Delta^m(\Phi)$ is a simplicial complex on the vertex set $\Phi^m_{\geq 0}$ of colored almost positive roots. More specifically, $\Phi^m_{\geq 0}$ consists of the set $\Phi^m_{\geq 0}$ of $m$ (colored) copies $\alpha^1, \ldots, \alpha^m$ of each positive root $\alpha \in \Phi^+$ and one copy of each negative simple root. We make the convention that each negative simple root is colored by 1. Thus, 

$$\Phi^m_{\geq 1} = \Phi^m_{\geq 0} \cup (\Pi)^m = \{\alpha^k : \alpha \in \Phi^+, k \in \{1, \ldots, m\}\} \cup \{\alpha^1 : \alpha \in (\Pi)\}.$$ 

The faces of $\Delta^m(\Phi)$ are the subsets of $\Phi^m_{\geq 1}$ whose elements are pairwise compatible in the sense we are going to describe just below. For every $\alpha \in \Phi_{\geq 1}$ we define the degree $d(\alpha)$ of $\alpha$ to be the smallest $d$ such that $R^d(\alpha)$ is a negative simple root. Clearly, $d(\alpha) = 0$ if $\alpha$ is a negative simple root.

**Definition 2.11.** [14] Definition 2.1] Two colored roots $\alpha^k, \beta^l \in \Phi^m_{\geq 1}$ are called $m$-compatible if and only if one of the following conditions is satisfied:
• $k > l$, $d(\alpha) \leq d(\beta)$ and the roots $\mathcal{R}(\alpha)$ and $\beta$ are compatible in the original noncolored sense (Theorem 2.17).
• $k < l$, $d(\alpha) \geq d(\beta)$ and the roots $\alpha$ and $\mathcal{R}(\beta)$ are compatible,
• $k > l$, $d(\alpha) > d(\beta)$ and the roots $\alpha$ and $\beta$ are compatible,
• $k < l$, $d(\alpha) < d(\beta)$ and the roots $\alpha$ and $\beta$ are compatible, and
• $k = l$ and the roots $\alpha$ and $\beta$ are compatible.

It is immediate from the definition that $m$-compatibility is a symmetric relation. There is another equivalent way to define this relation, by introducing the $m$-analogue of the map $\mathcal{R}$, as follows.

**Definition 2.12.** [14, Definition 2.3] For $\alpha^k \in \Phi_{\geq -1}^m$, we set

\[
\mathcal{R}_m(\alpha^k) = \begin{cases} 
\alpha^{k+1} & \text{if } \alpha \in \Phi_{>0}^m \text{ and } k < m \\
(\mathcal{R}(\alpha))^l & \text{otherwise.}
\end{cases}
\]

**Theorem 2.13.** [14, Theorem 2.4] The compatibility relation on $\Phi_{\geq -1}^m$ has the following properties:
(i) $\alpha^k$ is $m$-compatible with $\beta^l$ if and only if $\mathcal{R}_m(\alpha^k)$ is $m$-compatible with $\mathcal{R}_m(\beta^l)$,
(ii) $(-\sigma_1)^l$ is $m$-compatible with $\beta^l$ if and only if the simple root expansion of $\beta$ does not involve $\sigma_1$.

Furthermore, the above conditions uniquely determine this relation.

We denote by $\Delta_m^m(\Phi)$ the positive part of $\Delta_m^m(\Phi)$, which is the induced subcomplex of $\Delta_m^m(\Phi)$ on the vertex set $\Phi_{>0}^m$ of positive colored roots. If $\Phi$ is reducible with irreducible components $\Phi_1, \ldots, \Phi_l$ then $\Phi_{\geq -1}^m = \bigcup_{i=1}^l (\Phi_i^m)_{\geq -1}$. We declare two roots in $\Phi_{\geq -1}^m$ compatible if they either belong to different components, or belong to the same component and are compatible within it. Thus, the complex $\Delta_m^m(\Phi)$ is the simplicial join of the complexes $\Delta_m^m(\Phi_i)$.

### 3. Proof of Theorem 1.1

Throughout this section we assume that $\Phi$ is an irreducible root system and $\gamma$ a bipartite Coxeter element. Before proving Theorem 1.1, we need to establish a few lemmas.

Let $\mu : \mathbb{R}^n \to \mathbb{R}^n$ be the map introduced in [12] with

\[\mu(x) = 2(I - \gamma)^{-1}(x)\]

for all $x \in \mathbb{R}^n$. In what follows, we denote by $\alpha \cdot \beta$ the inner product of $\alpha, \beta \in \mathbb{R}^n$.

**Lemma 3.1.**
(i) [12, Section 8] Let $\alpha, \beta \in \Phi_{\geq -1}$ with $\alpha \prec \beta$. The roots $\alpha, \beta$ are compatible if and only if $R(\beta)R(\alpha) \leq \gamma$.
(ii) [10, Lemma 2.2] For nonparallel roots $\alpha, \beta$ we have $R(\alpha)R(\beta) \leq \gamma$ if and only if $\mu(\alpha) \cdot \beta = 0$.

**Lemma 3.2.** [12, Theorem 3.7] For the total order $\rho_i$ of roots in $\Phi_{\geq -1}$ we have $\mu(\rho_i) \cdot \rho_{i-n} = -\mu(\rho_i) \cdot \rho_j$ for all $i, j$.

**Lemma 3.3.** If $\rho_i$ is a positive root, then $\rho_{i-n} = \gamma^{-1}(\rho_i)$.
If Definition 2.11 this implies that \( \alpha \), \( \beta \) absurd. If \( R \) therefore \( \alpha \leq \beta \) and therefore \( R \) and therefore \( \alpha \).

positive roots \( \Phi \), \( \Phi \leq 1 \) and \( \Phi \leq N \) that the statement holds for \( \mu \) and \( t \). This completes our proof.

Proof. Part (i) is obvious from (10). To prove (ii), observe by (8) and (10) that if \( \Phi \leq \Phi \), \( \Phi \geq 1 \) by Lemma 3.3(ii). Moreover, since \( \Phi \leq \Phi \), \( \Phi \leq 1 \) by the inductive assumption and the fact that \( \Phi \leq \Phi \) of the same lemma. Setting \( a = t_2 \cdots t_k, b = t_{k+1}, c = t_1 w \) and applying Lemma 3.4(i) we get \( t_2 \cdots t_{k+1} \leq t_1 w \). Thus \( t_1 \cdots t_{k+1} \leq w \) by Lemma 3.4(ii).

To prove the statement for the length, note that since \( t_2 \cdots t_{k+1} \leq t_1 w \) then \( l_T(t_1 w) = l_T(t_2 \cdots t_{k+1}) + l_T(t_{k+1} \cdots t_1 w) \). Equivalently, \( l_T(w) = k + l_T(w) - l_T(t_1 \cdots t_{k+1}) \) by the inductive assumption and the fact that \( t_1 \cdots t_{k+1} \leq w \). Thus \( l_T(t_1 \cdots t_{k+1}) = k + 1 \), which completes our proof.

In view of (10) and the remarks following it, the total order (7) of the set of almost positive roots \( \Phi \geq 1 \) can be reformulated as follows:

\[
\begin{align*}
-\Pi_- &< \Pi_+ < \gamma(-\Pi_-) < \gamma(\Pi_+) < \gamma^2(-\Pi_-) < \gamma^2(\Pi_+) < \cdots < \gamma^{-1}(-\Pi_-) < \Pi_- < -\Pi_+.
\end{align*}
\]

Lemma 3.5.

(i) If \( \alpha, \beta \in \Phi^+ \) and \( d(\alpha) < d(\beta) \) then \( \alpha < \beta \).

(ii) If \( \alpha \in \Phi^+ \setminus \Pi_+ \) and \( d(\beta) \leq d(\alpha) \) then \( R(\beta) < \alpha \).

Proof. Part (i) is obvious from (11). To prove (ii), observe by (8) and (10) that if \( \beta \in \Phi^+ \setminus \Pi_+ \) then \( R(\beta) = \gamma^{-1}(\beta) \) so that \( d(R(\beta)) = d(\beta) - 1 \). Thus \( d(R(\beta)) < d(\alpha) \) and therefore \( R(\beta) < \alpha \) by (i).

Lemma 3.6. If \( \alpha^i, \beta^k \in \Phi^+ \) are m-compatible then \( \alpha \neq \beta \).

Proof. Assume on the contrary that \( \alpha^i, \alpha^k \) are m-compatible for some \( \alpha \in \Phi^+ \). By Definition 2.11, this implies that \( \alpha, R(\alpha) \) are compatible in the noncolored sense. If \( \alpha \in \Pi_+ \) then \( R(\alpha) = -\alpha \) by (8) and therefore \( \alpha, -\alpha \) are compatible which is absurd. If \( \alpha \in \Phi^+ \setminus \Pi_+ \) then set \( \alpha = \rho_i \) for some \( i \geq r + 1 \), so that \( R(\alpha) = R(\rho_i) \) \( = \gamma^{-1}(\rho_i) = \rho_{i-n} \) by (8) and Lemma 3.3. Moreover, since \( i \geq r + 1 \) then \( \rho_{i-n} < \rho_i \) and therefore \( R(\rho_i)R(\rho_{i-n}) \leq \gamma \) by Lemma 3.3(ii). Hence \( \mu(\rho_i) \cdot \rho_{i-n} = 0 \) by Lemma 3.4(ii) and thus \( \mu(\rho_i) \cdot \rho_i = 0 \) by Lemma 3.2 which is absurd since \( \mu(\rho_i) \cdot \rho_i = 1 \) \( \rho \). This completes our proof.

In what follows, for simplicity and since this does not affect the proof, we omit the color 1 from the negative simple roots in \( \Phi^+ \).

Proof of Theorem 1.4. We have to show that \( \sigma \in \Phi^+ \) is a face of \( \Delta^m(\Phi) \) if and only if \( w_\sigma = w_{\sigma_1}w_{\sigma_2} \cdots w_{\sigma_\ell} \geq \gamma \) and \( l_T(w_\sigma) = |\sigma| \). Recall that \( \sigma \subseteq \Phi^+ \) is a face of \( \Delta^m(\Phi) \) if and only if any two roots in \( \sigma \) are m-compatible. We claim that \( \alpha^i, \beta^j \in \sigma \) are m-compatible if and only if one of the following happens:
In this case, the roots \( w \not\in R(\beta)R(\alpha) \leq \gamma \), or \( \beta < \alpha \) and \( R(\alpha)R(\beta) \leq \gamma \). (ii) either \( \alpha \in -\Pi_+ \), \( \beta \in \Phi^{m}_{\alpha} \) and \( R(\alpha)R(\beta) \leq \gamma \), or \( \alpha \in -\Pi_- \), \( \beta \in \Phi^{m}_{\alpha} \) and \( R(\beta)R(\alpha) \leq \gamma \). (iii) \( i < j, \alpha \neq \beta \) and \( R(\beta)R(\alpha) \leq \gamma \).

In view of Lemma 3.4 with \( w \) replaced by \( \gamma \), our claim is equivalent to the fact that \( w_\sigma \leq \gamma \) and \( l_T(w_\sigma) = |\sigma| \).

To prove (i), recall by Definition 2.11 that roots of the same color are \( m \)-compatible if and only if they are compatible in the noncolored sense. Our claim then follows by Lemma 3.1(ii). For (ii) notice that \( \alpha \in -\Pi_\beta \in \Phi^{m}_{\alpha} \) are \( m \)-compatible if and only if \( \alpha \in -\Pi_\beta \in \Phi^+ \) are compatible in the noncolored sense (Theorem 2.11(ii) and 2.12(ii)). Since roots in \( -\Pi_+ (-\Pi_-) \) succeed (precede) all roots in \( \Phi^+ \), our claim is clear by Lemma 3.1(i).

For the proof of (iii) recall by Lemma 3.6 that if \( \alpha^i, \beta^j \) are \( m \)-compatible then \( \alpha \neq \beta \). We first assume that \( d(\alpha) < d(\beta) \) so that \( \alpha < \beta \) by Lemma 3.6(i). In this case, the roots \( \alpha^i, \beta^j \) are \( m \)-compatible if and only if \( \alpha, \beta \) are compatible. By Lemma 3.1(ii) the roots \( \alpha, \beta \) are compatible if and only if \( R(\beta)R(\alpha) \leq \gamma \), which proves our claim.

We next assume that \( d(\alpha) \geq d(\beta) \), in which case \( \alpha^i, \beta^j \) are \( m \)-compatible if and only if \( \alpha, R(\beta) \) are compatible. We distinguish cases.

**Case 1:** \( \alpha \in \Phi^+ \) and \( \beta \in \Pi_+ \). By (5) it is \( R(\beta) = -\beta \in -\Pi_+ \) and since the roots in \( \Phi^+ \) precede those in \( -\Pi_+ \) we have \( \alpha < \beta \). In view of Lemma 3.1(i) the roots \( \alpha, R(\beta) \) are compatible if and only if \( R(R(\beta))R(\alpha) \leq \gamma \). Since \( R(R(\beta)) = R(-\beta) = R(\beta) \) we have that \( \alpha, R(\beta) \) are compatible if and only if \( R(\beta)R(\alpha) \leq \gamma \), as desired.

**Case 2:** \( \alpha \in \Phi^+ \) and \( \beta \in \Phi^+ \setminus \Pi_+ \). Since \( d(\beta) \leq d(\alpha) \) we have that \( R(\beta) = \alpha^i \) by Lemma 3.6(ii). In view of Lemma 3.1(i) the roots \( \alpha, R(\beta) \) are compatible if and only if \( R(\alpha)R(R(\beta)) \leq \gamma \). Set \( \alpha = \rho_i, \beta = \rho_j \) with \( i \geq 0 \) and \( j \geq i + 1 \) and notice that \( R(\rho_j) = \gamma^{-1}(\rho_j) = \rho_{j-i} \) by (5) and Lemma 3.6. We have \( R(\alpha)R(R(\beta)) \leq \gamma \) \( \Leftrightarrow R(\rho_i)R(R(\rho_j)) \leq \gamma \Leftrightarrow R(\rho_i)R(\rho_{j-i}) \leq \gamma \Leftrightarrow \mu(\rho_i) \cdot \rho_{j-i} = 0 \Leftrightarrow \mu(\rho_j) \cdot \rho_{j-i} = 0 \Leftrightarrow R(\rho_j)R(\rho_i) \leq \gamma \Leftrightarrow R(\rho_j)R(\rho_i) \leq \gamma \leq R(\rho_i)R(\alpha) \leq \gamma \) by Lemmas 3.1(ii) and 3.3. Therefore \( \alpha, R(\beta) \) are compatible if and only if \( R(\alpha)R(\beta) \leq \gamma \) and this completes our proof.

4. The EL-labeling and falling chains in \( NC(m)(\gamma) \)

In this section we describe the EL-labeling that \( NC(m)(\gamma) \) inherits from \( NC(\gamma) \) and explain the relation of falling chains with respect to this EL-labeling to facets of \( \Delta^m_+(\Phi) \). The idea for the EL-labeling was suggested by D. Armstrong.

Since \( NC(\gamma) \) admits an EL-labeling so does \( (NC(\gamma))^m \), by Proposition 2.11. Moreover, recall that \( NC(m)(\gamma) \) is an order ideal in \( (NC(\gamma))^m \) so that we can restrict the EL-labeling of the latter to the former. The labeling that \( NC(m)(\gamma) \) inherits from \( (NC(\gamma))^m \) is the following. If \( (w_1, \ldots, w_m) \rightarrow (w'_1, \ldots, w'_m) \) in \( NC(m)(\gamma) \) then there exists some \( 1 \leq i_0 \leq m \) such that \( w_i = w'_i \) for all \( 1 \leq i \neq i_0 \leq m \) and \( w_{i_0} \leq w'_{i_0} = w_{i_0}t_{i_0} \) for some reflection \( t_{i_0} \) in \( W \). We label the edge \( (w_1, \ldots, w_m) \rightarrow (w'_1, \ldots, w'_m) \) by \( (1, \ldots, 1, t_{i_0}, 1, \ldots, 1) \) with \( t_{i_0} \) in the \( i_{0}\text{th} \) entry and we call this the natural edge labeling of \( NC(m)(\gamma) \). Let \( \Lambda \) be the set of \( m \)-tuples \( (1, \ldots, R(\alpha), \ldots, 1) \) where \( R(\alpha) \) is a reflection through the root \( \alpha \). Following (4), we totally order the elements of \( \Lambda \) by letting

\[
(1, \ldots, R(\alpha), \ldots, 1) \leq (1, \ldots, R(\alpha')', \ldots, 1)
\]
with the reflection on the \(i\)-th and \(i'\)-th entry respectively, if and only if

\[
i = i' \text{ and } \alpha \leq \alpha' \text{ or } i > i'.
\]

The preceding discussion leads us to the following result. We should point out that the following proposition is part of a stronger result \cite[Theorem 3.7.2]{1}, namely that the poset \(NC(m)\) of \(m\)-divisible noncrossing partitions with a maximal element adjoined is EL-shellable.

**Proposition 4.1.** If \(\Lambda\) is totally ordered as in \cite{11} then the natural edge labeling of \(NC(m)(\gamma)\) is an EL-labeling. \hfill \Box

Our next goal is to relate maximal falling chains in intervals \([0, w]\) of \(NC(m)(\gamma)\) to facets of certain subcomplexes of \(\Delta^m(\Phi)\). For a fixed \(w = (w_1, \ldots, w_m) \in NC(m)(\gamma)\) we define \(\Delta^m(w)\) as the induced subcomplex of \(\Delta^m(\Phi)\) on the vertex set of colored positive roots \(\alpha^i \in \Phi_+^{m}_0\) with \(R(\alpha) \leq w_i\), \(1 \leq i \leq m\). Since \(m\)-compatible roots are compatible and in view of Lemmas \cite{5, 4} we deduce that \(\Delta^m(w)\) is the simplicial complex whose faces are the sets \(\sigma \subseteq \Phi_+^m\) with \((w_{\sigma_0}, \ldots, w_{\sigma_1}) \leq (w_1, \ldots, w_m)\). The complex \(\Delta^m(w)\) is the simplicial join of \(\Delta^+_1(w_i)\) \(1 \leq i \leq m\), and therefore it is pure of dimension \(rk(w) - 1\).

Consider \(w = (w_1, \ldots, w_m) \in NC(m)(\gamma)\) with \(rk(w) = k\) and write \(\hat{\Phi} = (1, \ldots, 1)\). Let \(C\) be a maximal falling chain in \([0, w]\) and observe that we can decompose it into \(m\) unrefinable parts \(C_i\), such that \(C_i\) is the subchain of \(C\) whose edge labels have a reflection on the \(i\)-th entry. Clearly, \(C\) is the chain formed by arranging the \(C_i\) one after the other. Next, consider the subsets of \(\Phi^+\)

\[
\sigma(C_i) = \{ \alpha \in \Phi^+ : (1, \ldots, R(\alpha), \ldots, 1) \text{ is an edge label on } C_i \}
\]

for \(1 \leq i \leq m\). Let \(w_{\sigma(C_i)}\) be the product of reflections through the roots in \(\sigma(C_i)\) in decreasing order with respect to \(\Phi\). This is actually the order in which they appear on the edge labels of \(C\). Thus \(w_{\sigma(C_i)} = w_i\) and therefore \((w_{\sigma(C_1)}, \ldots, w_{\sigma(C_m)})\) \(\in NC(m)(\gamma)\). By Theorem \cite{11} if we color the roots in each \(\sigma(C_i)\) by \(m - i + 1\) we obtain a face of \(\Delta^m(\Phi)\) and in particular a facet of \(\Delta^m(w)\).

Conversely, let \(\sigma\) be a facet of \(\Delta^m(w)\), so that \(w_{\sigma_{m-i+1}} = w_i\) for all \(1 \leq i \leq m\). Consider the set

\[
\Lambda_C := \{ (1, \ldots, R(\alpha), \ldots, 1) : \alpha \in \sigma_i^{m-i+1}, R(\alpha) \text{ on the } i\text{-th entry}, 1 \leq i \leq m \}
\]

and order its elements in decreasing order with respect to \(\Phi\). It is clear that \(\Lambda_C\) is the label set of a maximal falling chain \(C\) in \([0, w]\), obtained by reversing the procedure described in the previous paragraph. We summarize the above facts in the following proposition.

**Proposition 4.2.** Let \(w = (w_1, \ldots, w_m) \in NC(m)(\gamma)\) and let \(C\) be a maximal falling chain in \([0, w]\). The map which sends \(C\) to \(\bigcup_{i=1}^{m} \sigma(C_i)^{m-i+1}\) is a bijection from the set of maximal falling chains of \([0, w]\) to the set of facets of \(\Delta^m(w)\). \hfill \Box

**Corollary 4.3.** The set of maximal falling chains in \(NC(m)(\gamma)\) bijects to the set of facets of \(\Delta^m(\Phi)\).

**Proof.** The set of maximal falling chains in \(NC(m)(\gamma)\) is the disjoint union of the sets of such chains within each \([0, w]\) with \(rk(w) = n\). Moreover, the set of facets
of $\Delta^m_+ (\Phi)$ is the disjoint union of the set of facets of each $\Delta^m_+ (w)$ where $rk(w) = n$. Our claim then follows from Proposition $\ref{prop:moebius}$.

The reader is invited to verify Corollary $\ref{cor:moebius}$ in the following example.

**Example 4.4.** Consider the root system $A_2$ with positive roots $\sigma_1$, $\sigma_2$ and $\alpha = \sigma_1 + \sigma_2$, where $\sigma_1$, $\sigma_2$ are simple roots. We set $\Pi_+ = \{ \sigma_1 \}$, $\Pi_- = \{ \sigma_2 \}$ so that $\sigma_1 \prec \alpha \prec \sigma_2$. We represent the elements of the reflection group $W_{A_2}$ as permutations in $S_3$. The reflection through $\sigma_1, \sigma_2$ and $\alpha$ acts on $\mathbb{R}^3$ by transposing coordinates, so that $R(\alpha_1) = (12)$, $R(\alpha_2) = (23)$ and $R(\alpha) = (13)$. The bipartite Coxeter element in this case is $\gamma = (123)$. Figure $\ref{fig:edge_labels}$ illustrates the natural edge labeling of $NC_{(2)}(\gamma)$. The set of maximal falling chains in $NC_{(2)}(\gamma)$ bijects to the set of positive clusters as shown in the following table.

| Edge labels of falling chains | positive clusters |
|------------------------------|-------------------|
| $(1, (23)) \rightarrow (1, (13))$ | $\sigma_2^0, \alpha^1$ |
| $(1, (13)) \rightarrow (1, (12))$ | $\sigma_3^0, \sigma_1^1$ |
| $(23, 1) \rightarrow (1, (13))$ | $\sigma_2^1, \alpha^2$ |
| $(23, 1) \rightarrow ((13), 1)$ | $\sigma_2^1, \alpha^2$ |
| $(12, 1) \rightarrow (1, (23))$ | $\sigma_1^2, \sigma_2^1$ |
| $(13, 1) \rightarrow ((12), 1)$ | $\alpha^2, \sigma_2^1$ |
| $(13, 1) \rightarrow (1, (12))$ | $\alpha^2, \sigma_2^1$ |

**Lemma 4.5.** For $w = (w_1, \ldots, w_m) \in NC_{(m)}(\gamma)$ the number $(-1)^{rk(w)} \mu(\hat{0}, w)$ is equal to the number of facets of $\Delta^m_+(w)$.

**Proof.** A standard fact on Möbius functions of EL-shellable posets is that the Möbius function on every interval $[a, b]$ is equal to $(-1)^{rk(b) - rk(a)}$ times the number of maximal falling chains in $[a, b]$ $\cite{el}$. Applying this in our case with $a = 0$ and $b = w$, we deduce that $(-1)^{rk(w)} \mu(\hat{0}, w)$ is equal to the number of maximal falling chains in $[\hat{0}, w]$ which, in view of Proposition $\ref{prop:moebius}$, is equal to the number of facets of $\Delta^m_+(w)$.

5. PROOF OF THEOREM $\ref{thm:main}$

As pointed out in $\cite{el}$ Proposition F(i)], for any finite root systems $\Phi$ and $\Phi'$ we have

$$F^{(m)}_{\Phi \times \Phi'} (x, y) = F^{(m)}_{\Phi} (x, y) \cdot F^{(m)}_{\Phi'} (x, y).$$

Moreover, by the multiplicativity of the Möbius function, the above relation holds for the $M$-triangle as well. Therefore, it suffices to prove Theorem $\ref{thm:main}$ in the case where $\Phi$ is irreducible.

The $h$-polynomial of an abstract $(n - 1)$-dimensional simplicial complex $\Delta$ is defined as

$$h(\Delta, y) = \sum_{i=0}^{n} f_i(\Delta) y^i (1 - y)^{n-i},$$

where $f_i(\Delta)$ is the number of faces of $\Delta$ of dimension $i - 1$. The link of a face $\sigma$ of $\Delta$ is the abstract simplicial complex $lk_\Delta (\sigma) = \{ \tau \in \Delta \setminus \sigma : \sigma \subseteq \tau \in \Delta \}$. One can check by comparing $\cite{el}$ Figure 3.4] and $\cite{el}$ Theorem 9.2] that for any irreducible root
For any $\sigma \in \Phi$ the $h$-polynomial of $\Delta^m(\Phi)$ coincides with the rank generating polynomial of $NC(m)(\gamma)$, thus
\[
h(\Delta^m(\Phi), y) = \sum_{w \in NC(m)(\gamma)} y^{rk(w)}.
\]

The following lemma will be used as in [4]. For $\alpha \in \Pi$ we denote by $\Phi_\alpha$ the standard parabolic root subsystem obtained by intersecting $\Phi$ with the linear span of $\Pi \setminus \{\alpha\}$.

**Lemma 5.1.** Let $\Phi$ be irreducible, $\alpha \in \Pi$ and $\sigma \subseteq \Phi^\alpha$. 
(i) For $\sigma \in \Delta^m(\Phi)$ we have $-\alpha \in \sigma$ if and only if $\sigma \cap \{-\alpha\} = \Delta^m(\Phi_\alpha)$.
(ii) For any $\beta^i \in \Phi^\alpha$ there exists $i$ such that $R^\alpha(\beta^i) \in (-\Pi)$.
(iii) $\sigma \in \Delta^m(\Phi)$ if and only if $R^\alpha(\sigma) \in \Delta^m(\Phi)$.

**Proof.** Part (i) can be verified from [14] Theorem 2.7. Part (ii) follows from Definition 2.12 and the fact that for every $\beta^i \in \Phi^\alpha$ there exists some $j$ such that $R^\alpha(\beta^i) \in (-\Pi)$. Finally, part (iii) is clear from Theorem 2.13 (i). \quad \square

We continue with some technical lemmas required in the proof of Theorem 1.2.

**Lemma 5.2.** If $\sigma$ is a face of $\Delta^m(\Phi)$ and $w_\sigma = w_{\sigma_+} w_{\sigma^-} w_{\sigma_{m-1}} \cdots w_{\sigma_1} w_{\sigma_-}$ then
\[
h(\Delta \sigma, y) = \sum_{\sigma \in NC(m)(\gamma w_{\sigma^{-1}})} y^{rk(\sigma)}.
\]

**Proof.** The proof is analogous to the proof of [3] Lemma 2.6, replacing $R$ by $R_m$ and using Lemma 5.1. \quad \square

**Lemma 5.3.** If $(a_1, \ldots, a_m) \leq (b_1, \ldots, b_m)$ in $NC(m)(\gamma)$ then $(a_1^{-1} b_1, \ldots, a_m^{-1} b_m) \in NC(m)(\gamma)$ and the intervals $I = [(a_1, \ldots, a_m), (b_1, \ldots, b_m)]$ and $I' = [(1, \ldots, 1), (a_1^{-1} b_1, \ldots, a_m^{-1} b_m)]$ are isomorphic.

**Proof.** It follows from Lemma 2.3 (iii) that since $a_i \leq b_i$ then $a_i^{-1} b_i \leq b_i$. Hence $(a_1, a_1^{-1} b_1, \ldots, a_m^{-1} b_m)$ is a minimal factorization of $b_1 \cdots b_m$ and in view of Lemma 2.3 iii we have that $(a_1^{-1} b_1, \ldots, a_m^{-1} b_m) \in NC(m)(\gamma)$. This proves the first part of our statement.

Next, we claim that the map $\varphi : I \rightarrow I'$ with $\varphi((w_1, \ldots, w_m)) = (a_1^{-1} w_1, \ldots, a_m^{-1} w_m)$ is an order preserving bijection and thus the intervals $I$ and $I'$ are isomorphic. We first have to check that $\varphi$ is well defined. Indeed, if $(w_1, \ldots, w_m) \in I$ then $(a_1, a_1^{-1} w_1, \ldots, a_m^{-1} w_m)$ is a minimal factorization of $w_1 \cdots w_m$ and therefore $(a_1^{-1} w_1, \ldots, a_m^{-1} w_m) \in NC(m)(\gamma)$ by Lemma 2.2 (ii). Conversely, let $(w_1, \ldots, w_m) \in I'$ so that $w_i \leq a_i^{-1} b_i \leq b_i$ for $1 \leq i \leq m$. In view of Lemma 2.3 (i) we have $a_i \leq w_i \leq b_i$ and therefore $(a_1 w_1, a_1^{-1} b_1, \ldots, a_m w_m, a_m^{-1} b_m)$ is a minimal factorization of $b_1 \cdots b_m$. Thus $(a_1 w_1, \ldots, a_m w_m) \in NC(m)(\gamma)$ as in the previous situation. That $\varphi$ is a bijection is immediate from the fact that $a_i \leq w_i \leq b_i$ and only if $a_i^{-1} w_i \leq a_i^{-1} b_i$ (Lemma 2.3 (i)). Moreover, $rk(\varphi(w)) = rk(w) - rk(a)$ and therefore $\varphi$ is order preserving. This completes our proof. \quad \square

**Lemma 5.4.** Let $(w_1, \ldots, w_m) \in NC(m)(\gamma)$. There is a rank preserving bijection between the sets $A = \{(a_1, \ldots, a_m) : (a_1, \ldots, a_m) \leq (a_1 w_1, \ldots, a_m w_m) \in NC(m)(\gamma)\}$ and $NC(m)(\gamma w_{m-1} \cdots w_1^{-1})$. 

indeed, $\phi(a_1, \ldots, a_m) = (a'_1, \ldots, a'_m)$. We will prove that $\phi$ is a rank preserving bijection between the sets $A$ and $\text{NC}_m(\gamma w_{r-1} \cdots w_i^{-1})$. Clearly, the map $\phi$ as well as its inverse $\phi^{-1}(a'_1, \ldots, a'_m) = (a_1, \ldots, a_m)$ with $a_i = w_{i-1}^{-1} \cdots w_1^{-1} a'_i w_1 \cdots w_{i-1}$ are injective. Moreover, since $a_i, a'_i$ are conjugate then $l_T(a_i) = l_T(a'_i)$ and therefore $\phi$ is rank preserving. So, it suffices to prove that $\phi$ is well defined. To this end let $(a_1, \ldots, a_m) \in A$ and note that $a'_1 \cdots a'_m = (a_1 w_1 \cdots a_m w_m)w_{r-1} \cdots w_i^{-1}$. Since $a_i \leq a_i w_i$ then $l_T(a_i w_i) = l_T(a_i) + l_T(w_i)$ and therefore $(a_1 w_1, \ldots, a_m w_m)$ is a minimal factorization of $a_1 w_1 \cdots a_m w_m$. Thus, by Lemma 2.4(iii) we have

\[ w_1 \cdots w_m \leq a_1 w_1 \cdots a_m w_m \leq \gamma, \]

with $l_T(w_1 \cdots w_m) = \sum_{i=1}^m l_T(w_i)$. In view of Lemma 2.4(ii) this implies that

\[ (a_1 w_1 \cdots a_m w_m)w_{r-1} \cdots w_i^{-1} \leq \gamma w_{r-1} \cdots w_i^{-1}, \]

with $l_T((a_1 w_1 \cdots a_m w_m)w_{r-1} \cdots w_i^{-1}) = \sum_{i=1}^m l_T(a_i)$. Equivalently, $a'_1 \cdots a'_m \leq \gamma w_{r-1} \cdots w_i^{-1}$ and $l_T(a'_1 \cdots a'_m) = \sum_{i=1}^m l_T(a'_i) = \sum_{i=1}^m l_T(a_i)$, which shows that indeed $(a'_1, \ldots, a'_m) \in \text{NC}_m(\gamma w_{r-1} \cdots w_i^{-1})$. For the converse let $(a'_1, \ldots, a'_m) \in \text{NC}_m(\gamma w_{r-1} \cdots w_i^{-1})$ and notice that $a_1 w_1 \cdots a_m w_m = a'_1 \cdots a'_m w_1 \cdots w_m$. Since $a'_1 \cdots a'_m \leq \gamma w_{r-1} \cdots w_i^{-1} \leq \gamma$ then by Lemma 2.4(ii) we have $a'_1 \cdots a'_m w_1 \cdots w_m \leq \gamma$ with $l_T(a'_1 \cdots a'_m w_1 \cdots w_m) = \sum_{i=1}^m l_T(a'_i) + \sum_{i=1}^m l_T(w_i)$. This forces $(a_1 w_1, \ldots, a_m w_m) \in \text{NC}_m(\gamma)$. We next prove that $(a_1, \ldots, a_m) \leq (a'_1, \ldots, a'_m)$. We have $a_i' \leq \gamma w_{r-1} \cdots w_i^{-1}$ or equivalently $w_1 \cdots w_{i-1} a_i w_{i-1} \cdots w_1^{-1} \leq \gamma w_{r-1} \cdots w_i^{-1}$. In view of Lemma 2.4 this implies that $a_i \leq a_i^{-1} w_{i-1}^{-1} \gamma w_{r-1} \cdots w_i^{-1}$. By Lemma 2.5 and elementary calculations one may check that $w_{i-1}^{-1} \cdots w_1^{-1} \gamma w_{r-1} \cdots w_i^{-1} \leq w_{i-1}^{-1} \cdots w_1^{-1} \leq \gamma w_{i-1}^{-1}$. Thus $a_i \leq \gamma w_{i-1}^{-1} \leq \gamma$ and therefore $a_i \leq a_i w_i \leq \gamma$ by Lemma 2.4(ii). This completes our proof.

Proof of Theorem 1.2 To simplify notation let us write $\Delta^m$, $\Delta^m_+$, $F(x, y)$ and $M(x, y)$ instead of $\Delta^m(\Phi)$, $\Delta^m_+(\Phi)$, $F_{\Phi}^m(x, y)$ and $M^m_W(x, y)$ respectively. We use the relation

\[ (1 - y)^n F\left(\frac{x+y}{1-y}, \frac{y}{1-y}\right) = \sum_{\sigma \in \Delta^m_+} x^{\ell(\sigma)} h(\text{lk}(\sigma), y), \]

which appears in the course of the proof of [11, Theorem 1.1] and can be generalized straightforward in the case where $m \geq 1$. We have

\[ M(-x, -y/x) = \sum_{a \leq b, a, b \in \text{NC}_{m+1}(\gamma)} \mu(a, b)(-x)^{rk(b) - rk(a)} y^{rk(a)}. \]

Let $\hat{\Theta} = (1, \ldots, 1)$, $w = (w_1, \ldots, w_m) = (a_1^{-1} b_1, \ldots, a_m^{-1} b_m)$ and note that $rk(w) = rk(b) - rk(a)$. In view of Lemmas 2.4 and 2.5 the last sum becomes

\[ \sum_{w \in \text{NC}_m(\gamma)} \mu(\hat{\Theta}, w)(-x)^{rk(w)} \sum_{a' \in \text{NC}_m(\gamma w_{r-1} \cdots w_i^{-1})} y^{rk(a')}, \]

which, by Lemma 2.4 is equal to

\[ \sum_{w \in \text{NC}_m(\gamma)} x^{rk(w)} \sum_{a' \in \text{NC}_m(\gamma w_{r-1} \cdots w_i^{-1})} y^{rk(a')}. \]
If $\sigma$ is a facet of $\Delta^m_+(w)$ then $w_{|\sigma|} = w_{|\sigma|} \cdots w_{|\sigma|} = w_1 \cdots w_m$ and thus the last expression is equal to
\[
\sum_{w \in NC(m)(\gamma)} \sum_{\sigma \in \Delta^m_+(w)} x^{||\sigma||}\sum_{a' \in NC(m)(\gamma w_{a' \sigma}^{-1})} y^{r k(a')},
\]
or, in view of Lemma \ref{lemma}:
\[
\sum_{w \in NC(m)(\gamma)} \sum_{\sigma \in \Delta^m_+(w)} x^{||\sigma||} h(\text{lk}_{\Delta^m_+}(\sigma), y).
\]
Observe that the sets of facets of the subcomplexes $\Delta^m_+(w)$ for $w \in NC(m)(\gamma)$ form a partition of $\Delta^m_+$, so that the last sum is
\[
\sum_{\sigma \in \Delta^m_+} x^{||\sigma||} h(\text{lk}_{\Delta^m_+}(\sigma), y)
\]
which, in view of (12), completes our proof.

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Department of Mathematics, University of Crete, 71409 Heraklion, Crete, Greece
E-mail address: etzanaki@math.uoc.gr