Counteraexmles on Jumarie’s three basic fractional calculus formulae for non-differentiable continuous functions

Cheng-shi Liu  
Department of Mathematics  
Northeast Petroleum University  
Daqing 163318, China  
Email: chengshiliu-68@126.com

March 10, 2022

Abstract

Jumarie proposed a modified Riemann-Liouville derivative definition and gave three so-called basic fractional calculus formulae (u(t)v(t))^(α) = u^(α)(t)v(t) + u(t)v^(α)(t), (f(u(t)))^(α) = f’u^(α)(t) and (f(u(t)))^(α) = (f(u))^(α)(u’(t))^α where u and v are required to be non-differentiable and continuous for the first formula, f is assumed to be differentiable for the second formula, while in the third formula f is non-differentiable and u is differentiable, at the point t. I once gave three counterexamples to show that Jumarie’s three formulae are not true for differentiable functions (Cheng-shi Liu. Counterexamples on Jumarie’s two basic fractional calculus formulae. Communications in Nonlinear Science and Numerical Simulation, 2015, 22(1): 92-94.). However, these examples cannot directly become the suitable counterexamples for the case of non-differentiable continuous functions. In the present paper, I first provide five counterexamples to show directly the Jumarie’s formulae are also not true for non-differentiable continuous functions. Then I prove that essentially non-differentiable cases can be transformed to the differentiable cases. Therefore, those counterexamples in the above paper are indirectly right. In summary, the Jumarie’s formulae are not true. This paper can be considered as the corrigendum and supplement to the above paper.

Keywords: counterexample, fractional calculus, modified Riemann-Liouville’s derivative

1 Introduction

Jumarie proposed a modified Riemann-Liouville fractional derivative[1-5]:

\[ f^{(α)}(t) = \frac{1}{\Gamma(1-α)} \frac{d}{dt} \int_0^t (t-x)^{-α}(f(x) - f(0))dx. \]
and gave some basic fractional calculus formulae (see, for example, formulae (3.11)-(3.13) in Ref. [4] or formulae (4.3), (4.4) and (4.5) in Ref. [5]):

\[ (u(t)v(t))^{(\alpha)} = u^{(\alpha)}(t)v(t) + u(t)v^{(\alpha)}(t), \]  
\[ (f(u(t)))^{(\alpha)} = f'_u u^{(\alpha)}(t), \]  

where Jumarie requires the functions \( u \) and \( v \) are non-differentiable and continuous, while \( f \) is differentiable, at the point \( t \). Jumarie’s third formula is given by

\[ (f(u(t)))^{(\alpha)} = (f(u))^{(\alpha)}(u'(t))^\alpha, \]

where \( f \) is non-differentiable and \( u \) is differentiable at the point \( t \).

The formula (3) has been applied to solve the exact solutions to some non-linear fractional order differential equations (see, for example, Refs. [6-9]). In [10], I once gave three counterexamples to show that Jumarie’s so-called basic formulae are not correct. But, I neglected the conditions of the Jumarie’s formulae. Indeed, for example, the formula (2) requires that the functions \( u \) and \( v \) are non-differentiable and continuous, and the formula (3) requires that \( f \) is differentiable while \( u \) is non-differentiable and continuous, at the point \( t \). Therefore, the examples in [10] cannot be considered as suitable direct counterexamples to Jumarie’s formulae under the conditions of non-differentiable functions. However, I will show that these counterexamples do hold indirectly. In addition, since the functions in [6-9] need to be differentiable, my counterexamples are still right for these applications. In [11], Jumarie emphasizes that it is just at some point that his formulae do hold. At such point, the function is non-differentiable and continuous, and the fractional derivative exists. In the present paper, I further provide five counterexamples which satisfy all conditions in Jumarie’s formulae to show directly that Jumarie’s formulae are incorrect for the cases of non-differentiable continuous functions. Finally, I prove that essentially non-differentiable cases can be transformed to the differentiable cases. Therefore, in other words, those counterexamples in the paper [10] are indirectly valid.

Recently, some problems about the rules of fractional derivatives have been discussed by some authors (see, for example, [10-13]). For instance, Tarasov [12] gave an important result for Leibniz rule. For local fractional derivative of nowhere differentiable continuous functions on open intervals, some detailed discussions can be found in [13].

2 Counterexamples to formula (2)

As in [10], we need the \( \frac{1}{2} \)-order derivatives of the following four functions \( f(t) = t, f(t) = \sqrt{t}, f(t) = t^2 \) and \( f(t) = t^{\frac{3}{2}} \) with \( f(0) = 0 \):

\[ (t)^{(1/2)} = 2\sqrt{\frac{t}{\pi}}, \]  

(5)
(\sqrt{t})^{(1/2)} = \frac{\sqrt{\pi}}{2}, \quad (6)

(t^2)^{(1/2)} = \frac{8t^{3/2}}{3\sqrt{\pi}}, \quad (7)

(t^{\frac{3}{2}})^{(1/2)} = \frac{3\sqrt{\pi}t}{4}. \quad (8)

**Counterexample 1** (The counterexample of formula (2)). Take \( \alpha = \frac{1}{2} \) and

\[ u(t) = \begin{cases} \sqrt{t}, & 0 \leq t \leq 1, \\ \sqrt{t} + t - 1, & t > 1. \end{cases} \quad (9) \]

It is easy to see that \( u(t) \) is continuous, and is non-differentiable at \( t = 1 \). Further, we have

\[ H(t) = \int_0^t (t-x)^{-\alpha}(f(x)-f(0))dx = \begin{cases} \int_0^t \frac{\sqrt{t}}{\sqrt{t-x}}dx, & 0 \leq t \leq 1, \\ \int_0^1 \frac{\sqrt{t}}{\sqrt{t-x}}dx + \int_1^t \frac{\sqrt{t+x-1}}{\sqrt{t-x}}dx, & t > 1. \end{cases} \quad (10) \]

And then, we have

\[ H(t) = \begin{cases} \int_0^t \frac{\sqrt{t}}{\sqrt{t-x}}dx, & 0 \leq t \leq 1, \\ \int_0^1 \frac{\sqrt{t}}{\sqrt{t-x}}dx + \int_1^t \frac{x-1}{\sqrt{t-x}}dx, & t > 1. \end{cases} \quad (11) \]

Let \( t - x = s^2 \). Then we have

\[ K(t) = \int_1^t \frac{x-1}{\sqrt{t-x}}dx = 2 \int_0^{\sqrt{t-1}} (t-1-s^2)dx = \frac{4}{3} (t-1)^{\frac{3}{2}}. \quad (12) \]

Therefore, if \( 0 \leq t < 1 \),

\[ u^{(1/2)}(t) = (\sqrt{t})^{(1/2)} = \frac{\sqrt{\pi}}{2}; \quad (13) \]

If \( t > 1 \),

\[ u^{(1/2)}(t) = (\sqrt{t})^{(1/2)} + \frac{1}{\sqrt{\pi}} K'(t) = \frac{\sqrt{\pi}}{2} + \frac{2}{\sqrt{\pi}} (t-1)^{\frac{1}{2}}, \quad (14) \]

where we use \( \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \). And hence, at \( t = 1 \), it follows that \( u^{(1/2)}(1) \) exists and

\[ u^{(1/2)}(1) = \frac{\sqrt{\pi}}{2}. \quad (15) \]

Further, by taking \( v(t) = u(t) \), we have

\[ u^{(1/2)}(1)v(1) + u(1)v^{(1/2)}(1) = \sqrt{\pi}. \quad (16) \]

On the other hand, we have

\[ u(t)v(t) = \begin{cases} t, & 0 \leq t \leq 1, \\ (\sqrt{t} + t - 1)^2, & t > 1. \end{cases} \quad (17) \]
Hence, if \( t < 1 \), we have

\[
(uv)^{(1/2)}(t) = (t)^{\frac{1}{2}} = 2\sqrt{\frac{t}{\pi}}. 
\]  
(18)

If \( t > 1 \), we have

\[
(uv)^{(1/2)}(t) = \frac{1}{\sqrt{\pi}} \frac{d}{dt} \left\{ \int_0^t \frac{x}{\sqrt{t-x}} \, dx + \int_1^t \frac{(\sqrt{t} + x - 1)^2}{\sqrt{t-x}} \, dx \right\}. 
\]  
(19)

Further, we have

\[
(uv)^{(1/2)}(t) = (t)^{\frac{1}{2}} + \frac{1}{\sqrt{\pi}} \int_1^t \frac{3\sqrt{x} + 2(x - 1) - x^{-\frac{1}{2}}}{\sqrt{t-x}} \, dx. 
\]  
(20)

By computing the last integral, we get

\[
(uv)^{(1/2)}(t) = 2\sqrt{\frac{t}{\pi}} + \frac{1}{\sqrt{\pi}} \left\{ \frac{8}{3} (t-1)^{\frac{3}{2}} + 3\sqrt{t-1} + 3t(\frac{\pi}{2} - \arcsin \frac{1}{\sqrt{t}}) + 2 \arcsin \frac{1}{\sqrt{t}} - \frac{1}{\sqrt{t}} \right\}. 
\]  
(21)

Therefore, at \( t = 1 \), \((uv)^{(1/2)}(t)\) does exist and \((uv)^{(1/2)}(1) \neq \sqrt{\pi}.\) From (16), it turns out that we have

\[
(uv)^{(1/2)}(1) \neq u^{(1/2)}(1)v(1) + u(1)v^{(1/2)}(1). 
\]  
(22)

This example shows that Jumarie’s formula (2) is not true for the non-differentiable continuous functions.

Next, we give a more simple example.

**Counterexample 2.** Take \( \alpha = \frac{1}{2} \) and

\[
u(t) = \begin{cases} 1-t, & t \leq 1, \\ t-1, & t > 1. \end{cases}
\]  
(23)

It is easy to see that \( u(t) \) is continuous, and is non-differentiable at \( t = 1 \). Further, we have

\[
H(t) = \int_0^t (t-x)^{-\alpha} (u(x) - u(0)) \, dx = \begin{cases} \int_0^t \frac{-x}{\sqrt{t-x}} \, dx, & t \leq 1, \\ \int_0^1 \frac{-x}{\sqrt{t-x}} \, dx + \int_1^t \frac{x-2}{\sqrt{t-x}} \, dx, & t > 1. \end{cases}
\]  
(24)

And then, we have

\[
H(t) = \begin{cases} \int_0^t \frac{-x}{\sqrt{t-x}} \, dx, & t \leq 1, \\ \int_0^1 \frac{-x}{\sqrt{t-x}} \, dx + 2 \int_1^t \frac{x-1}{\sqrt{t-x}} \, dx, & t > 1. \end{cases}
\]  
(25)

Therefore, if \( t < 1 \),

\[
(uv)^{(1/2)}(t) = -(t)^{\frac{1}{2}} = -2\sqrt{\frac{t}{\pi}}. 
\]  
(26)
If $t > 1$,
\[ u^{(1/2)}(t) = \frac{1}{\sqrt{\pi}} H'(t) = -2\sqrt{\frac{t}{\pi}} + 4\sqrt{\frac{t-1}{\pi}}. \] (27)

It follows that
\[ u^{(1/2)}(1) = -\frac{2}{\sqrt{\pi}}. \] (28)

Hence, from $u(1) = 0$ we have
\[ 2u(1)u^{(1/2)}(1) = 0. \] (29)

On the other hand, we have $u^2(t) = (t - 1)^2 = t^2 - 2t + 1$, and then
\[ (u^2)^{(1/2)}(t) = (t^2)^{(1/2)} - 2(t)^{(1/2)} = \frac{8}{3\sqrt{\pi}} t^{3/2} - \frac{4}{\sqrt{\pi}} t^{1/2}. \] (30)

Therefore, we get
\[ (u^2)^{(1/2)}(1) = -\frac{4}{3\sqrt{\pi}} \neq 0. \] (31)

So we give
\[ (u^2)^{(1/2)}(1) \neq 2u(1)u^{(1/2)}(1). \] (32)

Therefore, if we take $v(t) = u(t)$, we have equivalently from (32)
\[ (uv)^{(1/2)}(1) \neq u^{(1/2)}(1)v(1) + u(1)v^{(1/2)}(1). \] (33)

This shows again that Jumarie’s formula (2) is not true for the non-differentiable continuous functions.

3 Counterexamples to formula (3)

Counterexample 3 (The counterexample of formula (3)). Take $f(u) = u^2$ and $u(t)$ is also given by (9), and $\alpha = \frac{1}{2}$. Then $f(u(t))$ is given by
\[ f(u(t)) = \begin{cases} 
    t, & 0 \leq t \leq 1, \\
    (\sqrt{t} + t - 1)^2, & t > 1.
\end{cases} \] (34)

Firstly, therefore, we get the $\frac{1}{2}$ order derivative of $f(u(t))$ at $t = 1$
\[ (f(u(t)))^{(1/2)}|_{t=1} = (u^2(t))^{(1/2)}|_{t=1} = \frac{2}{\sqrt{\pi}}. \] (35)

Secondly, from (15) and $u(1) = 1$, at $t = 1$, we have
\[ f'_u(1)u^{(1/2)}(1) = \sqrt{\pi}. \] (36)

So we find
\[ (f(u(t)))^{(1/2)}|_{t=1} \neq f'_u(1)^{1/2}(t)|_{t=1}. \] (37)
This shows that the Jumarie’s formula (3) is not correct for non-differentiable continuous functions.

**Counterexample 4.** Take \( u(t) \) as the same as (23) in counterexample 2, and \( f(u) = u^2 \). According to the counterexample 2, we have

\[
(f(u(t)))^{(1/2)}|_{t=1} = (u^2)^{(1/2)}(1) = -\frac{4}{3\sqrt{\pi}},
\]

and

\[
f'(u)u^{(1/2)}|_{t=1} = 2u(1)u^{(1/2)}(1) = 0.
\]

It turns out that

\[
(f(u(t)))^{(1/2)}|_{t=1} \neq f'(u)u^{(1/2)}(t)|_{t=1}.
\]

This shows again that the Jumarie’s formula (3) is not correct for non-differentiable continuous functions.

## 4 Counterexamples to formula (4)

**Counterexample 5.** We take \( \alpha = \frac{1}{2} \), \( u(t) = t^2 \). \( f(u) \) is taken as the following form

\[
f(u) = \begin{cases} \sqrt{u}, & 0 \leq u \leq 1, \\ \sqrt{u} + u - 1, & u > 1. \end{cases}
\]

(41)

It is easy to see that \( f(u) \) is non-differentiable at \( u = 1 \) and \( f'(\frac{1}{2})(1) = \frac{\sqrt{\pi}}{2} \) from the formula (15). Therefore, we have

\[
f'(\frac{1}{2})(u(u')(t))^{1/2}|_{t=1} = \frac{\sqrt{\pi}}{2}.
\]

(42)

On the other hand, since \( u(t) = t^2 \), we have

\[
f(u(t)) = \begin{cases} t, & 0 \leq t \leq 1, \\ t + t^2 - 1, & t > 1. \end{cases}
\]

(43)

Therefore, when \( 0 \leq t < 1 \), we have

\[
(f(u(t)))^{(1/2)} = (t)^{(1/2)} = 2\sqrt{\frac{t}{\pi}}.
\]

(44)

When \( t > 1 \), we have

\[
(f(u(t)))^{(1/2)} = (t)^{(1/2)} = 2\sqrt{\frac{t}{\pi}} + \frac{2}{\sqrt{\pi}}(t\sqrt{t-1} - \frac{1}{3}(t-1)^{3/2}).
\]

(45)

In fact, when \( t > 1 \), we have

\[
H(t) = \int_0^t (t-x)^{-\alpha} (f(u(x)) - f(u(0)))dx = \int_0^1 \frac{x}{\sqrt{t-x}}dx + \int_1^t \frac{x + x^2 - 1}{\sqrt{t-x}}dx.
\]

(46)
And then, we have
\[ H(t) = \int_0^t \frac{x}{\sqrt{t-x}} \, dx + \int_1^t \frac{x^2-1}{\sqrt{t-x}} \, dx. \]  
(47)

hence, we have
\[ (f(u(t)))^{(1/2)} = \frac{1}{\sqrt{\pi}} H'(t) = (t)^{(1/2)} + \frac{1}{\sqrt{\pi}} \int_1^t \frac{2x}{\sqrt{t-x}} \, dx. \]  
(48)

By taking \( x = t - s^2 \), we have
\[ \int_1^t \frac{x}{\sqrt{t-x}} \, dx = \int_0^{\sqrt{t-1}} (t-s^2) \, ds = t^{1/2} - \frac{1}{3} (t-1)^{3/2}. \]  
(49)

It follows that
\[ (f(u(t)))^{(1/2)} = 2\sqrt{\frac{t}{\pi}} + 2t^{1/2} - \frac{2}{3} (t-1)^{3/2}. \]  
(50)

Therefore, we get
\[ (f(u(t)))^{(1/2)}|_{t=1} = \frac{2}{\sqrt{\pi}}. \]  
(51)

By (42) and (51), it turns out that
\[ (f(u(t)))^{(1/2)}|_{t=1} \neq (f(u))(1/2)(u'(t))^{1/2}|_{t=1}. \]  
(52)

This shows that the Jumarie’s formula (4) is not true for non-differentiable continuous functions.

5 Explanation

**Theorem.** Let \( u_1(t) \) and \( u_2(t) \) be two continuous differentiable functions on \([0, +\infty)\), that is, belong to \( C^1[0, +\infty) \). Take the function \( u(t) \) as

\[ u(t) = \begin{cases} 
  u_1(t), & t \leq t_0, \\
  u_2(t), & t > t_0,
\end{cases} \]  
(53)

where \( t_0 > 0 \) is a fixed point, \( u_1(t_0) = u_2(t_0) \) and \( u'_1(t_0) \neq u'_2(t_0) \), that is, \( u(t) \) is continuous and non-differentiable at the point \( t_0 \). If \( u_1^{(\alpha)}(t) \) is continuous, and \( u_1^{(\alpha)}(t_0) \) exists and is finite, then \( u^{(\alpha)}(t_0) \) also exists and \( u^{(\alpha)}(t_0) = u_1^{(\alpha)}(t_0) \), where \( 0 < \alpha < 1 \).

**Proof.** By the definition of \( u(t) \) and the conditions of the theorem, we have

\[ u^{(\alpha)}(t) = \begin{cases} 
  u_1^{(\alpha)}(t), & t < t_0, \\
  \frac{1}{\Gamma(1-\alpha)} \left( \int_0^t \frac{u_1(x)-u_1(0)}{(t-x)^\alpha} \, dx + \int_0^t \frac{u_2(x)-u_2(0)}{(t-x)^\alpha} \, dx \right), & t > t_0.
\end{cases} \]  
(54)
Further, we have
\[ u^{(\alpha)}(t) = \begin{cases} 
    u_1^{(\alpha)}(t), & t < t_0, \\
    \frac{1}{(1-\alpha)} \frac{d}{dt} \left\{ \int_0^t \frac{u_2(x)-u_1(0)}{(t-x)\alpha} \, dx + \int_{t_0}^t \frac{u_2(x)-u_1(x)}{(t-x)\alpha} \, dx \right\}, & t > t_0. 
\end{cases} \]  
(55)

Therefore, we get
\[ u^{(\alpha)}(t) = \begin{cases} 
    u_1^{(\alpha)}(t), & t < t_0, \\
    u_1^{(\alpha)}(t) + \frac{1}{(1-\alpha)} \frac{d}{dt} \int_{t_0}^t \frac{u_2(x)-u_1(x)}{(t-x)\alpha} \, dx, & t > t_0. 
\end{cases} \]  
(56)

Since \( u_2(x) - u_1(x) \) is differentiable, we have
\[ \frac{d}{dt} \int_{t_0}^t \frac{u_2(x)-u_1(x)}{(t-x)\alpha} \, dx = \int_{t_0}^t \frac{u_2'(x)-u_1'(x)}{(t-x)\alpha} \, dx. \]  
(57)

Because \( u_2'(x) - u_1'(x) \) is continuous from the conditions of the theorem, and \( \frac{1}{(t-x)\alpha} \) is integrable in \([t_0,t]\), by the mean value theorem of integral (see, for example, [14]), we have
\[ \int_{t_0}^t \frac{u_2'(x)-u_1'(x)}{(t-x)\alpha} \, dx = (u_2'(\xi) - u_1'(\xi)) \frac{(t-t_0)^{1-\alpha}}{1-\alpha}, \]  
(58)

where \( \xi \in (t_0,t) \). Hence, we obtain
\[ \lim_{t \to t_0} \frac{d}{dt} \int_{t_0}^t \frac{u_2(x)-u_1(x)}{(t-x)^\alpha} \, dx = \lim_{t \to t_0} (u_2'(\xi) - u_1'(\xi)) \frac{(t-t_0)^{1-\alpha}}{1-\alpha} = 0. \]  
(59)

From the continuity of \( u_1^{(\alpha)}(t) \) at the point \( t_0 \), it follows that
\[ u^{(\alpha)}(t_0) = u_1^{(\alpha)}(t_0). \]  
(60)

The proof is completed.

According to the above theorem, we know that \( u^{(\alpha)}(t_0) \) doesn’t depend on the function \( u_2 \), that is, \( u^{(\alpha)}(t_0) \) doesn’t depend on the values of \( u(t) \) on \([t_0,\infty)\). On the other hand, from the definition (1), we can also see the fact. Therefore, to compute \( u^{(\alpha)}(t_0) \), we can smoothly continue the function \( u(t) \) from \( u_1(t) \) on \([0,t_0]\) to \((t_0,\infty)\), such that \( u(t) \) is differentiable at the point \( t_0 \). In other words, essentially, those counterexamples in [10] are also right. For instance, the function \( u(t) = \sqrt{t} \) in the counterexample 1 in [10] is just the smooth continuation of the function \( \sqrt{t} \) defined on \([0,1]\) in the counterexample 1 in the present paper.

6 conclusion

By the above counterexamples and theory, we have showed that the Jumarie’s three basic formulae for fractional derivative are not correct for non-differentiable continuous functions.

Acknowledgements. Thanks to Prof.Jumarie and Prof.Kamocki for their pointing out to me the negligence in my paper[12] on the conditions of the Jumarie’s formulae.
References

[1] G Jumarie. On the representation of fractional Brownian motion as an integral with respect to $(dt)^\alpha$. Applied Mathematics Letters, 2005, 18(7): 739-748.

[2] G Jumarie. On the solution of the stochastic differential equation of exponential growth driven by fractional Brownian motion. Applied Mathematics Letters, 2005, 18(7): 817-826.

[3] G Jumarie. New stochastic fractional models for Malthusian growth, the Poissonian birth process and optimal management of populations. Mathematical and computer modelling, 2006, 44(3): 231-254.

[4] G Jumarie. Modified Riemann-Liouville derivative and fractional Taylor series of nondifferentiable functions further results. Computers and Mathematics with Applications, 2006, 51(9): 1367-1376.

[5] G Jumarie. Table of some basic fractional calculus formulae derived from a modified Riemann-Liouville derivative for non-differentiable functions. Applied Mathematics Letters, 2009, 22(3): 378-385.

[6] Y Pandir, Y Gurefe, E Misirli. The Extended Trial Equation Method for Some Time Fractional Differential Equations. Discrete Dynamics in Nature and Society, 2013.

[7] H Bulut, H M Baskonus, Y Pandir. The Modified Trial Equation Method for Fractional Wave Equation and Time Fractional Generalized Burgers Equation. Abstract and Applied Analysis. Hindawi Publishing Corporation, 2013.

[8] Z B Li, W H Zhu, J H He. Exact solutions of time-fractional heat conduction equation by the fractional complex transform. Thermal Science, 2012, 16(2): 335-338.

[9] H A Ghany, A S O El Bab, A M Zabel, A A Hyder. The fractional coupled KdV equations: Exact solutions and white noise functional approach. Chinese Physics B, 2013, 22(8): 080501.

[10] Cheng-shi Liu. Counterexamples on Jumarie’s two basic fractional calculus formulae. Communications in Nonlinear Science and Numerical Simulation, 2015, 22(1): 92-94.

[11] G Jumarie. The Leibniz rule for fractional derivatives holds with nondifferentiable functions. Math. Stat, 2013, 1(2): 50-52.

[12] V E Tarasov. No violation of the Leibniz rule. No fractional derivative. Communications in Nonlinear Science and Numerical Simulation, 2013.
[13] Cheng-shi Liu. On the local fractional derivative of everywhere non-differentiable continuous functions on intervals. Communications in Nonlinear Science and Numerical Simulation, 2017, 42: 229-235. 18(11): 2945-2948.

[14] W Rudin. Principles of mathematical analysis. New York: McGraw-Hill, 1964.