Optical vortices in dispersive nonlinear Kerr type media

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Abstract

The applied method of slowly varying amplitudes of the electrical and magnet vector fields give us the possibility to reduce the nonlinear vector integro-differential wave equation to the amplitude vector nonlinear differential equations. It can be estimated different orders of dispersion of the linear and nonlinear susceptibility using this approximation. The critical values of parameters to observed different linear and nonlinear effects are determinate. The obtained amplitude equation is a vector version of 3D+1 Nonlinear Schrödinger Equation (VNSE) describing the evolution of slowly varying amplitudes of electrical and magnet fields in dispersive nonlinear Kerr type media. We show that VNSE admit exact vortex solutions with classical orbital momentum \( \ell = 1 \) and finite energy. Dispersion region and medium parameters necessary for experimental observation of these vortices, are determinate.

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1 Introduction

In present day, there are no difficulties for experimentalist in laser physics or nonlinear optics to obtain picosecond or femtosecond optical pulses with equal duration in \( x, y \) and \( z \) directions. The problems with so generated light bullets arise in the process of their propagation in a nonlinear media with
dispersion. In the transparency region of a dispersive Kerr type media, as it was established in [1, 2, 3], the scalar paraxial approximation (no dispersion in the direction of propagation), is in very good accordance with the experimental results. The paraxial approximation, used in the derivation of the scalar 2D+1 nonlinear Schrödinger equation (NSE), do not include a small (in first or second order of magnitude) second derivative of the amplitude function in the direction of propagation. The including of this term does not change the main result dramatically: in the case of a linear propagation (pulses with small intensity), as generated optical bullets at short distance are transformed in optical disks, with large transverse and small lengthwise dimension. Some of the experimental possibilities, this small term to become important, were discussed in [4]. In Ref. [5] it was shown, that this second derivative in the direction of propagation term is in the same order and with the same sign as the others only in some special cases: optical pulses near the Langmuir frequency or near some of the electronic resonances. In these regions the sign of the dispersion is negative and the scalar 2D+1 NSE becomes 3D+1 NSE ones [6, 7, 8]. Propagation of optical bullets under the dynamics of 2D+1 and 3D+1 NSE are investigated also in relation to different kind of nonlinearity [9, 10]. A generation of a new kind of 2D and 3D optical pulses, so called optical vortices, has recently become a topic of considerable interest. Generally, the optical vortices are such type of optical pulses, which admit angular dependence of electrical field or helical phase distribution. The electrical field or intensity is zero also in the center of the vortices. The original scalar theory of optical vortices was based on the well known 2D+1 NSE [11, 12, 13]. In a self focusing regime of propagation can be generated optical rings, but they are modulationally unstable [15, 16]. One alternative way of stabilizing of optical vortices in 2D and 3D case, using saturable [17, 18] or cubic-quintic [19, 20] nonlinearity, was discussed also. From the other hand, the experiments with optical vortices show, that the polarization, and the vector character of the electric field, play an important role in the dynamics and the stabilization of the vortices [21]. To investigate these cases we are going to a vector version of 3D+1 NSE. In Section 2 we derive this vector version of 3D+1 NSE (VNSE), using also the dispersion in the direction of propagation and envelope approximation in the standard way as it was constructed in [21, 3]. This vector generalization allows us to find exact vortex solutions with spin \( l=1 \) and finite energy. The question for shape and finiteness energy of these vortices is crucial, and will discuss largely in Section 5.
2 Maxwell’s equations for a source-free, dispersive, nonlinear Kerr type medium

The Maxwell’s equations for a source-free dispersive media with Kerr nonlinearity are:

\[ \nabla \times \vec{E} = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t}, \]

\[ \nabla \times \vec{H} = \frac{1}{c} \frac{\partial \vec{D}}{\partial t}, \]

\[ \nabla \cdot \vec{D} = 0, \]

\[ \nabla \cdot \vec{B} = \nabla \cdot \vec{H} = 0, \]

\[ \vec{B} = \vec{H}, \quad \vec{D} = \vec{P}_{\text{lin}} + 4\pi \vec{P}_{\text{nlin}}, \]

where \( \vec{E} \) and \( \vec{H} \) are the electric and magnetic intensity fields, \( \vec{D} \) and \( \vec{B} \) are the electric and magnetic induction fields, \( \vec{P}_{\text{lin}}, \vec{P}_{\text{nlin}} \) are the linear and nonlinear polarizations of the medium respectively. The linear electric polarization of an isotropic, disperse medium can be represented as:

\[ \vec{P}_{\text{lin}} = \int_{-\infty}^{t} \left( \delta(t - \tau) + 4\pi \chi^{(1)}(t - \tau) \right) \vec{E}(\tau, x, y, z) d\tau = \int_{-\infty}^{t} \varepsilon_0 (t - \tau) \vec{E}(\tau, x, y, z) d\tau, \]

where \( \chi^{(1)} \) and \( \varepsilon_0 \) are the linear electric susceptibility and the dielectric constant. In the following, we will study such polarization (linearly or circularly polarized light), where the nonlinear polarization in the case of one carrying frequency can be expressed as:
\[ \vec{P}_{nl}^{(3)} = \int_{-\infty}^{t} \int_{-\infty}^{t} \int_{-\infty}^{t} \chi^{(3)}(t - \tau_1, t - \tau_2, t - \tau_3) \times (\vec{E}(\tau_1, x, y, z) \cdot \vec{E}^*(\tau_2, x, y, z)) \vec{E}(\tau_3, x, y, z) \, d\tau_1 \, d\tau_2 \, d\tau_3. \]  

(7)

It is important to point here the remark of Akhmanov at all in [22], that the method slowly varying amplitudes of electrical and magnet fields applied to such nonstationary linear and nonlinear representation is valid as well when the optical pulse duration of the pulses \( t_0 \) is greater than the characteristic response time of the media \( \tau_0 \) \( (t_0 >> \tau_0) \), as when the time duration of the pulses are equal or less than the time response of the media \( (t_0 \leq \tau_0) \). This possibility is discussed in the process of deriving of the amplitude equations.

Taking the curl of equation (1) and using (2) and (5), we then obtain:

\[ \Delta \vec{E} - \nabla \left( \nabla \cdot \vec{E} \right) - \frac{1}{c^2} \frac{\partial^2 \vec{D}}{\partial t^2} = 0, \]  

(8)

where \( \Delta \equiv \nabla^2 \) is the Laplasian operator. Equation (8) is derived without the use of the third Maxwell’s equation. Using equation (4) and the expression for the linear and nonlinear polarizations (6) and (7), it can be estimate the second term in the equation (8) for arbitrary localized vector function of the electrical field. As is shown in [5], for such type of function \( \nabla \cdot \vec{E} \approx 0 \) and equation (8) is written:

\[ \Delta \vec{E} - \frac{1}{c^2} \frac{\partial^2 \vec{D}}{\partial t^2} = 0. \]  

(9)

We define a complex presentation of the real electrical field by the relation:

\[ \vec{E}(x, y, z, t) = \vec{A}'(x, y, z, t) \frac{1}{2i} \left( \exp \left( i(k_0 z - \omega_0 t) \right) - c.c \right), \]  

(10)

where \( \vec{A}' \) and \( \omega_0 \) and \( k_0 \) are the real vector amplitude, the optical frequency and the wave number of the optical field respectively. The real vector amplitude \( \vec{A}' \) is represented also by a complex vector amplitude field:
\[ \vec{A}'(x, y, z, t) = \frac{1}{2i} \left( \vec{A}(x, y, z, t) - \vec{A}^*(x, y, z, t) \right), \] 

where \( \vec{A} \) is the complex amplitude of the electrical field. In this way our real electrical field is presented with four complex fields of kind:

\[ \vec{E}(x, y, z, t) = -\frac{1}{4} \left( \vec{A} \exp (i(k_0 z - \omega_0 t)) + \vec{A}^* \exp (-i(k_0 z - \omega_0 t)) \right) \]

\[ + \frac{1}{4} \left( \vec{A} \exp (-i(k_0 z - \omega_0 t)) + \vec{A}^* \exp (i(k_0 z - \omega_0 t)) \right). \] 

This special kind of presentation of electrical field is connected with shape and finiteness energy of vortex solutions, and it is discussed in Section 5. Here are used also the Fourier representation of the complex amplitude function \( \vec{A} \) and their time derivatives to second order:

\[ \vec{A}(x, y, z, t) = \int_{-\infty}^{+\infty} \vec{A}(x, y, z, \omega - \omega_0) \exp (-i (\omega - \omega) t) d\omega, \] 

\[ \frac{\partial \vec{A}(x, y, z, t)}{\partial t} = -i \int_{-\infty}^{+\infty} (\omega - \omega_0) \vec{A}(x, y, z, \omega - \omega_0) \]

\[ \times \exp (-i (\omega - \omega_0) t) d\omega, \]

\[ \frac{\partial^2 \vec{A}(x, y, z, t)}{\partial t^2} = - \int_{-\infty}^{+\infty} (\omega - \omega_0)^2 \vec{A}(x, y, z, \omega - \omega_0) \]

\[ \times \exp (i (\omega - \omega_0) t) d\omega. \]

The principle of causality request the next conditions on the response functions:

\[ \varepsilon(t - \tau) = 0; \quad \chi^{(3)}(t - \tau_1, t - \tau_2, t - \tau_3) = 0, \]

\[ t - \tau < 0; \quad t - \tau_i < 0; \quad i = 1, 2, 3. \]
That because we can prolonged the upper integral boundary to infinity and to use standard Fourier presentation [3]:

\[
\int_{-\infty}^{t} \epsilon_0(\tau - t) \exp \left(i \omega \tau \right) d\tau = \int_{-\infty}^{+\infty} \epsilon_0(\tau - t) \exp \left(i \omega \tau \right) d\tau, \quad (17)
\]

\[
\int_{-\infty}^{t} \int_{-\infty}^{t} \int_{-\infty}^{t} \chi^{(3)}(t - \tau_1, t - \tau_2, t - \tau_3) d\tau_1 d\tau_2 d\tau_3 = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \chi^{(3)}(t - \tau_1, t - \tau_2, t - \tau_3) d\tau_1 d\tau_2 d\tau_3. \quad (18)
\]

The spectral presentation of linear optical susceptibility \( \hat{\epsilon}_0(\omega) \) is connected to the nonstationary optical response function by the next Fourier transform:

\[
\hat{\epsilon}_0(\omega) \exp \left(-i \omega t \right) = \int_{-\infty}^{+\infty} \epsilon_0(t - \tau) \exp \left(-i \omega \tau \right) d\tau. \quad (19)
\]

Similar is the expression for the spectral presentation of the non-stationary nonlinear optical susceptibility \( \hat{\chi}^{(3)} \):

\[
\hat{\chi}^{(3)}(\omega = 2\omega - \omega) \exp \left(-i \omega t \right) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \chi^{(3)}(t - \tau_1, t - \tau_2, t - \tau_3) \times \exp \left(-i(\omega(\tau_1 + \tau_2 + \tau_3))\right) d\tau_1 d\tau_2 d\tau_3. \quad (20)
\]

From (6), (10), (13), (17) and (19) for the linear polarization of one of the complex fields \( \vec{P}_{lin}' \left[ \vec{A} \exp \left(i (k_0 z - \omega t) \right) \right] \) is obtained:

\[
\vec{P}_{lin}' = \int_{-\infty}^{t} \epsilon_0(t - \tau) \exp \left(i (k_0 z - \omega_0 \tau) \right) \int_{-\infty}^{+\infty} \vec{A}(x, y, z, \omega - \omega_0) \times \exp \left(-i(\omega - \omega_0) \tau \right) d\omega d\tau \times \exp \left(i k_0 z \right) \int_{-\infty}^{+\infty} \vec{A}(x, y, z, \omega - \omega_0) \int_{-\infty}^{+\infty} \epsilon_0(t - \tau) \exp \left(-i \omega \tau \right) d\tau d\omega. \quad (21)
\]
The second integral in (21) is simply equation (19), and the linear polarization can be written:

\[
\vec{P}_{\text{lin}}'(x, y, z, t) = \exp \left( i(k_0z - \omega_0t) \right) \int_{-\infty}^{+\infty} \tilde{\varepsilon}_0(\omega) \vec{A}(x, y, z, \omega - \omega_0) \times \exp \left( -i(\omega - \omega_0)t \right) d\omega.
\]

(22)

It is important to point out that the similar expressions as (22) are written also for the polarization of the complex fields of kind:

\[
\vec{P}_{\text{lin}}' \left[ \vec{A} \exp \left( -i(k_0z - \omega t) \right) \right] : \vec{P}_{\text{lin}}' \left[ \vec{A}^* \exp \left( i(k_0z - \omega t) \right) \right].
\]

(23)

and

\[
\vec{P}_{\text{lin}}' \left[ \vec{A} \exp \left( i(k_0z - \omega t) \right) \right] : \vec{P}_{\text{lin}}' \left[ \vec{A}^* \exp \left( -i(k_0z - \omega t) \right) \right].
\]

(24)

The above procedures are used also to the nonlinear polarization. When the third harmonic term is neglected the result is:

\[
P_{\text{nl}}(x, y, z, t) = \frac{3}{4} \exp \left( i(k_0z - \omega_0t) \right) \int_{-\infty}^{+\infty} \tilde{\chi}^{(3)}(\omega) \exp \left( -i(\omega - \omega_0)t \right)
\]

\[
\times |\vec{A}(x, y, z, \omega - \omega_0)|^2 \vec{A}(x, y, z, \omega - \omega_0) d\omega.
\]

(25)

Multiply (22), (25) by $1/c^2$, and using the second derivatives in time, we obtain:

\[
\frac{1}{c^2} \frac{\partial^2 \vec{P}_{\text{lin}}'}{\partial t^2} (x, y, z, t) = - \exp \left( i(k_0z - \omega_0t) \right) \int_{-\infty}^{+\infty} \frac{\omega^2 \tilde{\varepsilon}_0(\omega)}{c^2} \vec{A}(x, y, z, \omega - \omega_0)
\]

\[
\times \exp \left( -i(\omega - \omega_0)t \right) d\omega.
\]

(26)

\[
\frac{4\pi}{c^2} \frac{\partial^2 \vec{P}_{\text{nl}}}{\partial t^2} (x, y, z, t) = - \exp \left( i(k_0z - \omega_0t) \right) \int_{-\infty}^{+\infty} \frac{3\pi^2 \tilde{\chi}^{(3)}(\omega)}{c^2}
\]

\[
\times |\vec{A}(x, y, z, \omega - \omega_0)|^2 \vec{A}(x, y, z, \omega - \omega_0) \exp \left( -i(\omega - \omega_0)t \right) d\omega.
\]

(27)
The spectrum of the amplitude function is restricted, by writing the wave vectors:

\[ k_{ln} = \sqrt{\frac{\omega^2 \varepsilon_0 (\omega)}{c^2}}, \quad (28) \]

and

\[ k_{nl} = \sqrt{\frac{3 \pi \omega^2 \chi^{(3)} (\omega)}{c^2}}, \quad (29) \]

near the carrying frequency in a Taylor series:

\[
k_{ln}^2 (\omega) = \frac{\omega^2 \varepsilon_0 (\omega)}{c^2} = k_{ln}^2 (\omega_0) + \frac{\partial (k_{ln}^2 (\omega_0))}{\partial \omega_0} (\omega - \omega_0) + \frac{1}{2} \frac{\partial^2 (k_{ln}^2 (\omega_0))}{\partial \omega_0^2} (\omega - \omega_0)^2 + ..., \quad (30)\]

\[
k_{nl}^2 (\omega) = \frac{\omega^2 \chi^{(3)} (\omega)}{c^2} = k_{nl}^2 (\omega_0) + \frac{\partial (k_{nl}^2 (\omega_0))}{\partial \omega_0} (\omega - \omega_0) + ... \quad (31)\]

It is convenient to express the nonlinear wave vector by linear wave vector and nonlinear refractive index:

\[
k_{nl}^2 (\omega_0) = 3 \pi \omega_0^2 \chi^{(3)} (\omega_0) / c^2 = \frac{\omega_0^2 \varepsilon (\omega_0)}{c^2} \frac{3 \pi \chi^{(3)} (\omega_0)}{\varepsilon (\omega_0)} = k_{nl}^2 n_2, \quad (32)\]

where

\[
n_2 (\omega_0) = \frac{3 \pi \chi^{(3)} (\omega_0)}{\varepsilon (\omega_0)}, \quad (33)\]

is the nonlinear refractive index. We note that any approximation of the response function no used. There is only one requirement of the spectral presentations (19) and (20) of the response functions: to admit first and second order derivatives in respect to frequency (to be smooth functions). So, this method is not limited from the time duration of the response function. This give us the possibility to apply it also when the half-max of the
pulses are in order of the time duration of the response function (femtosecond pulses). The restriction is only in respect of the relation between the main frequency $\omega_0$ and time duration of the envelope functions $t_0$ determinate from the relations (30) and (31) (conditions for slowly varying amplitudes). Putting eqn. (30) and (31) in (26) and (27) respectively and keeping in mind the expressions of time derivative of the amplitude functions (14)-(15) for the electric field, the second time derivative of the linear polarization of the optical component is represented in next truncated form:

$$\frac{1}{c^2} \frac{\partial^2 \mathbf{P}_{\text{lin}}(x, y, z, t)}{\partial t^2} = \left( -k_0^2 \mathbf{A} + \frac{2i k_0}{v} \frac{\partial \mathbf{A}}{\partial t} + \left( k_0 k_0' + \frac{1}{v^2} \right) \frac{\partial^2 \mathbf{A}}{\partial t^2} + \ldots \right) \times \exp \left( i (k_0 z - \omega_0 t) \right). \quad (34)$$

Similar result is obtained also for the linear polarization of the terms of kind (23) and (24). The nonlinear polarization term is:

$$\frac{4\pi}{c^2} \frac{\partial^2 \mathbf{P}_{\text{nl}}(x, y, z, t)}{\partial t^2} = \left( -k_0^2 n_2 |\mathbf{A}|^2 \mathbf{A} + 2 i k_0 n_l \frac{\partial n_l}{\partial \omega} \frac{\partial \left( |\mathbf{A}|^2 \mathbf{A} \right)}{\partial t} + \ldots \right) \times \exp \left( i (k_0 z - \omega_0 t) \right) =
$$

$$\left( -k_0^2 n_2 |\mathbf{A}|^2 \mathbf{A} + 2 i k_0 n_2 \frac{n_2}{v} + i k_0^2 \frac{\partial n_2}{\partial \omega} \right) \frac{\partial \left( |\mathbf{A}|^2 \mathbf{A} \right)}{\partial t} + \ldots \right) \times \exp \left( i (k_0 z - \omega_0 t) \right), \quad (35)$$

where $v = 1/(\partial k_{\text{lin}}(\omega)/\partial \omega)$ and $k_0'$ are the group velocity, and dispersion. From the wave equation (9), using (12), (34), (35) and (23), (24), the next two equations of complex vector amplitude and it complex conjugate are obtained:

$$i \left( \frac{\partial \mathbf{A}}{\partial t} + v \frac{\partial \mathbf{A}}{\partial z} + \left( n_2 + \frac{k_0 v}{2} \frac{\partial n_2}{\partial \omega} \right) \frac{\partial \left( |\mathbf{A}|^2 \mathbf{A} \right)}{\partial t} \right) =$$

$$\frac{v}{2k_0} \Delta \mathbf{A} - v \left( k_0 + \frac{1}{k_0 v^2} \right) \frac{\partial^2 \mathbf{A}}{\partial t^2} + \frac{n_2 k_0 v}{2} |\mathbf{A}|^2 \mathbf{A}, \quad (36)$$
\[
- i \left( \frac{\partial \tilde{A}^*}{\partial t} + v \frac{\partial \tilde{A}^*}{\partial z} + \left( n_2 + \frac{k_0 v}{2} \frac{\partial n_2}{\partial \omega} \right) \frac{\partial (|\tilde{A}|^2 \tilde{A}^*)}{\partial t} \right) = \\
\frac{v}{2k_0} \Delta \tilde{A}^* - \frac{v}{2} \left( k_0^2 + \frac{1}{k_0 v^2} \right) \frac{\partial^2 \tilde{A}^*}{\partial t^2} + \frac{n_2 k_0 v}{2} |\tilde{A}|^2 \tilde{A}^* = 0.
\]

The equations (36) and (37) are equal and solving (36) for \( \tilde{A} \) we can find also \( \tilde{A}^* \) and respectively the real amplitude vector field \( \tilde{A}' \). We investigate the case when:

\[ v^2 k_0 k_0'' = -1, \]

a condition that can only be fulfilled for materials possessing negative dispersion. The \( \frac{\partial^2 \tilde{A}}{\partial x^2} \) term in this case is neglected. Applying a Galilean transformation to the vector amplitude equation (36), where the new reference frame moves at the group velocity, \( t' = t; z' = z - vt \) (the primes are missed for clarify), we obtain our final vector amplitude equation:

\[
- i \left( \frac{\partial \tilde{A}}{\partial t} + \left( n_2 + \frac{k_0 v}{2} \frac{\partial n_2}{\partial \omega} \right) \frac{\partial (|\tilde{A}|^2 \tilde{A})}{\partial t} \right) + \\
\frac{v}{2k_0} \Delta_{\perp} \tilde{A} - \frac{v^3 k_0^2}{2} \frac{\partial^2 \tilde{A}}{\partial z^2} + \frac{n_2 k_0 v}{2} |\tilde{A}|^2 \tilde{A} = 0, \tag{39}
\]

where \( \Delta_{\perp} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \).

### 3 Norming VNSE. Exact vortex solutions

In this section we will obtain solutions in the case of amplitude vector equations of one carrying frequency. The case of two carrying frequencies is discussed in [5]. The case of three and more localized optical waves on different frequencies with additional conditions on the frequencies (parametric vortices) will be discussed in a next paper. Defining the rescaled variables:

\[
\tilde{A} = A_0 \tilde{A}; \quad x = r_0 x'; \quad y = r_0 y'; \quad z = r_0 z'; \quad t = t_0 t', \tag{40}
\]
and constants:

\[
\alpha = 2k_0v_0^2/t_0v; \quad \beta = v^2k''k_0; \quad \gamma = k_0^2r_0^2n_2 |A_0|^2; \\
\gamma_1 = k_0r_0n_2 |A_0|^2; \quad \gamma_2 = \frac{v}{c}k_0r_0n_2 |A_0|^2,
\]

(41)
equation (39) can be transformed in the following (the primes are not written):

\[
-i \left( \alpha \frac{\partial \vec{A}}{\partial t} + (\gamma_1 + \gamma_2) \frac{\partial}{\partial t} \left( |\vec{A}|^2 \vec{A} \right) \right) + \\
\Delta_\perp \vec{A} - \beta \frac{\partial^2 \vec{A}}{\partial z^2} + \gamma |\vec{A}|^2 \vec{A} = 0.
\]

(42)
The dispersion term (the second derivative in 'z' direction of the amplitude function) in transparency region of a media is usually one or two order of magnitude smaller than the diffractive term (transverse Laplasian). That because usually the linear (dispersion) parameter is \( \beta \sim 10^{-2} \). There is possibility to reach \( \beta = -1 \) only in some special cases, near to Lengmuire frequency in cold plasma, high-frequency transparency region of dielectrics and near to some of the electronic resonances [5]. The constant \( \alpha \) has a value of \( \alpha \approx 10^2 \) (\( \alpha \approx r_0k_0 \)) if the of slowly varying approximation is used. When the nonlinear constant admit typical, critical for self-focusing regime value \( \gamma = 1 \), the constants \( \gamma_1 \approx \gamma_2 \sim 10^{-2} \) are small. We point here that the effects of asymmetry of the pulses (asymmetry of their spectrum) due to nonlinear addition to the group velocity presented by second term in equation (42) is substantial when \( \gamma_1 \approx \gamma_2 \sim 1 \) or \( \gamma \sim 10^2 \). These terms depends from intensity of the fields and when \( \gamma \sim 10^2 \) or \( n_2 |A_0|^2 \sim 10^{-2} \) effects of self-steepening of the pulses, connected also with considerable self-focusing can be estimated. Such type of experiments [23] are provided with 80 femtosecond pulses and intensity \( I_0 \sim 10^{14} W t/cm^2 \). This correspond to the parameters discussed above (\( \gamma \sim 10^2 \) or \( n_2 |A_0|^2 \sim 10^{-2} \)). In this paper is investigated the case, when \( \beta = -1 \) (negative dispersion) and \( \gamma = 1 \). Neglecting the small terms the equation (42) becomes:

\[
-i \alpha \frac{\partial \vec{A}}{\partial t} + \Delta_\perp \vec{A} + \frac{\partial^2 \vec{A}}{\partial z^2} + |\vec{A}|^2 \vec{A} = 0.
\]

(43)
Now we going to the next step - the possible polarization of the amplitude vector field \([\vec{A}]\). Recently, this problem was discussed largely \([24, 25]\) and was pointed the different between the polarization of the optical waves with spectral bandwidth (slowly varying amplitudes), and the polarization of monochromatic fields. In case of monochromatic and quasi-monochromatic fields the Stokes parameters can be constructed from transverse components of the wave field \([24]\). This lead to two component vector fields in a plane, transverse to direction of propagation. For electromagnetic fields with spectral bandwidth (our case) the two dimensional coherency tensor cannot be used and the Stokes parameters cannot be found directly. As it was shown by T.Carozzi and all in \([25]\), using high order of symmetry (SU(3)), six independent Stokes parameters can be found. This corresponds to three component vector field. Here is investigate this case. The increasing of the spectral bandwidth of the vector wave, increasing also the depolarization term (component, normal to the standard Stokes coherent polarization plane). The amplitude vector function of electrical field \(\vec{A}\) is represented as sum of three orthogonal linearly polarized amplitudes:

\[
\vec{A}(x, y, z, t) = \sum_{j=x,y,z} \vec{j}A_j(x, y, z) \exp(i\Omega t). \tag{44}
\]

In a Cartesian coordinate system, for solutions of the kind of \((44)\), the vector equation is reduced to a scalar system of three nonlinear wave equations:

\[
\alpha \Omega A_l + \Delta A_l + \sum_{j=x,y,z} \left( |A_j|^2 \right) A_l^l = 0; \ l = x, y, z. \tag{45}
\]

As it was pointed in \([3]\), we may choose to express each of the components \(A_i\) in spherical coordinates \(A_i(r, \theta, \varphi)\) of the independent variables \(i = x, y, z\). In this way the linear unique (polarization) vector of each of the components hold down. The system \((45)\) in spherical variables is:

\[
\alpha \Omega A_l^l + \Delta_r A_l^l + \frac{1}{r^2} \Delta_{\theta, \varphi} A_l^l + \sum_{j=x,y,z} \left( |A_j|^2 \right) A_l^l = 0; \ l = x, y, z, \tag{46}
\]

where
\[ \Delta_r = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} \right); \quad \text{(47)} \]

\[ \Delta_{\theta,\varphi} = \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \varphi^2}, \quad \text{(48)} \]

are the radial and the angular operator respectively and \( r = \sqrt{x^2 + y^2 + z^2} \)
\( \theta = \arccos \frac{z}{r} \varphi = \arctan \frac{y}{x} \), are the moving spherical variables of the independent variables \( x, y, z \). The system of equations (46) are solved, using the method of separation of the variables. We present the components of the field as a product of a radial and an angular part:

\[ A_l(r, \theta, \varphi) = R_l(r) Y_l(\theta, \varphi); \quad l = x, y, z, \quad \text{(49)} \]

with the additional constraint on the angular parts:

\[ |Y_x(\theta, \varphi)|^2 + |Y_y(\theta, \varphi)|^2 + |Y_z(\theta, \varphi)|^2 = \text{const.} \quad \text{(50)} \]

Multiplying equation (46) by \( r^2 R Y_l \), and bearing in mind the constraint expressed in (50), we obtain:

\[ \frac{r^2 \Delta_r R}{R} + r^2 \left( \alpha \Omega + |R|^2 \right) = - \frac{\Delta_{\theta,\varphi} Y_l}{Y_l} = \ell(\ell + 1), \quad \text{(51)} \]

where \( \ell \) is a number. Thus the following equations for the radial and the angular parts of the wave functions are obtained:

\[ \Delta_r R + \alpha \Omega R + |R|^2 R - \frac{\ell(\ell + 1)}{r^2} R = 0 \quad \text{(52)} \]

\[ \Delta_{\theta,\varphi} Y_l + \ell(\ell + 1) Y_l = 0; \quad l = x, y, z. \quad \text{(53)} \]

Equations (52)-(53) shows that the nonlinear term occurs only in the radial components of the fields. As for the angular parts we have the usual linear
eigenvalue problem. The solutions of equations (53) of the angular parts are well known; each of them has exact solutions of the form:

\[ Y_l = Y^m_l = \Theta^m_l(\theta) \Phi_m(\varphi) = \sqrt{\frac{4\pi}{3}} \sqrt{\frac{2\ell + (\ell - m)!}{4\pi(\ell + m)!}} P^m_\ell(\cos \theta) \exp(i m \varphi), \quad (54) \]

where \( P^m_\ell \) are the Legendre’s functions for a discrete series of numbers: \( \ell = 0, 1, 2, \ldots; m = 0, \pm 1, \pm 2, \ldots \) and \( |m| < \ell \). Returning to equation (46), it is seen that separation of variables for spherical functions, which satisfy condition (50), is possible only for \( l = 1 \):

\[ Y_x = Y^{-1}_1 = \sin \theta \cos \varphi \]
\[ Y_y = Y^1_1 = \sin \theta \sin \varphi \]
\[ Y_z = Y^0_1 = \cos \theta, \quad (55) \]

or another appropriate combination of axes. By choosing one of these angular components for each of the field components we see that the eigenfunctions (55) are solutions to the angular part of the set of equations (53). It is straightforward to show that the radial part of the equations (52) admit “de Broglie soliton” solutions (27) of the form:

\[ R = \sqrt{2} \frac{\exp\left(\frac{i\sqrt{\alpha} \Omega r}{r}\right)}{r}. \quad (56) \]

In this way, we prove the existence of vortex solutions with angular momentum \( l = 1 \) in a Kerr type medium. The solutions of the vector amplitude equation (43) in a fixed basis are:

\[ A_x = \sqrt{2} \frac{\exp\left(\frac{i\sqrt{\alpha} \Omega r}{r}\right)}{r} \sin \theta \cos \varphi \exp(i\Omega t), \quad (57) \]
\[ A_y = \sqrt{2} \frac{\exp\left(\frac{i\sqrt{\alpha} \Omega r}{r}\right)}{r} \sin \theta \sin \varphi \exp(i\Omega t), \quad (58) \]
\[ A_z = \sqrt{2} \exp \left( \frac{i\sqrt{\alpha \Omega r}}{r} \right) \cos \theta \exp (i\Omega t). \]  
(59)

The equation (43) is normed and this lead to the next normed constants:

\[ \alpha \Omega = 1; \ \Omega = \sim 10^{-2}. \]  
(60)

Now we can turn back to the amplitude of real solutions \( \vec{A}' \), using equation (11):

\[ A'_x = \frac{1}{2i} (A_x - A_x^*) = \sqrt{2} \sin \left( \sqrt{\alpha \Omega r + \Omega t} \right) \frac{r}{\sin \theta \cos \varphi}, \]  
(61)

\[ A'_y = \frac{1}{2i} (A_y - A_y^*) = \sqrt{2} \sin \left( \sqrt{\alpha \Omega r + \Omega t} \right) \frac{r}{\sin \theta \sin \varphi}, \]  
(62)

\[ A'_z = \frac{1}{2i} (A_z - A_z^*) = \sqrt{2} \cos \left( \sqrt{\alpha \Omega r + \Omega t} \right) \frac{r}{\cos \theta}. \]  
(63)

The condition \( \nabla \cdot \vec{E} = 0 \) separates the nonlinear Maxwell’s system in two wave equation system, one nonlinear, of the electric field, and another nonlinear wave equation of the magnetic field. We define again a complex presentation of the real magnet field by the relation:

\[ \vec{H} (x, y, z, t) = \vec{C'} (x, y, z, t) \frac{1}{2i} \left( \exp (i(k_0 z - \omega_0 t)) - c.c. \right), \]  
(64)
where $\vec{C}'$ and $\omega_0$ and $k_0$ are the real vector amplitude, the optical frequency and the wave number of the optical field respectively. The real magnet vector amplitude $\vec{C}'$ is represented also by a complex vector amplitude field:

$$\vec{C}' (x, y, z, t) = \frac{1}{2i} \left( \vec{C} (x, y, z, t) - \vec{C}^* (x, y, z, t) \right), \quad (65)$$

where $\vec{C}$ is the complex amplitude of the magnet field. Applying similar procedures to those who were made for electric, namely; using the Fourier representation of the amplitude functions $\vec{C}$ and their time derivatives to second order, as were done for the electric, we obtain the vector equation of slowly varying amplitudes of the magnetic field $\vec{C}$:

$$i\alpha \frac{\partial \vec{C}}{\partial t} + \Delta_\perp \vec{C} + \frac{\partial^2 \vec{C}}{\partial z^2} + |\vec{A}|^2 \vec{C} = 0, \quad (66)$$

if the condition:

$$\nabla |\vec{A}|^2 \times \vec{A} = 0, \quad (67)$$

is satisfied. For solutions of the amplitudes of electrical field like (57) - (59), the condition (67) is satisfied. Equation (66) has a localized solution of the same kind (57) - (59), as the amplitude functions of the electric field, but with opposite phase:

$$C_x = \sqrt{2} \exp \left( i \sqrt{\alpha \Omega} r \right) \frac{1}{r} \sin \theta \cos \varphi \exp (-i \Omega t), \quad (68)$$

$$C_y = \sqrt{2} \exp \left( i \sqrt{\alpha \Omega} r \right) \frac{1}{r} \sin \theta \sin \varphi \exp (-i \Omega t), \quad (69)$$

$$C_z = \sqrt{2} \exp \left( i \sqrt{\alpha \Omega} r \right) \frac{1}{r} \cos \theta \exp (-i \Omega t). \quad (70)$$

The real dynamics of localized fields can be understood only by investigating both equation (42) and (66). The solutions of electrical (57) - (59) and magnet (68) - (70) fields are with opposite phase, and that because the corresponding electromagnetic localized wave oscillating in coordinate system moving with group velocity.
4 Experimental conditions

The results presented above require the conditions relating the linear parameter $\beta$ and the nonlinear parameter $\gamma$ to satisfy: namely $\beta = -1$ and $\gamma = 1$. The relation $\gamma = 1$ is a typical critical value for starting of self-focusing regime of a Gaussian pulse. The nonlinear parameter written again is:

$$\gamma = r_0^2 k_0^2 n_2 |A_0|^2 = \alpha^2 n_2 |A_0|^2 = 1.$$  \hspace{1cm} (71)

The constant $\alpha = r_0 k_i$ in the optical region ranges in value from:

$$\alpha \simeq 10^2 - 10^3. \hspace{1cm} (72)$$

Using this range in the condition (71), we obtain a required nonlinear refractive index change:

$$n_2 |A_0|^2 \simeq 10^{-4} - 10^{-6}. \hspace{1cm} (73)$$

It was shown, using relations (41), that if we investigate near to the critical value of self-focusing $\gamma = 1$, then $\gamma_1 \sim \gamma_2 \sim 10^{-2} - 10^{-3}$. This is one important result: The nonlinear addition to the group-velocity term in amplitude equation (42) is relatively small, even for (femtosecond) pulses with time duration equal or little to characteristic time response of the nonlinear media.

Using the condition of linear parameter $\beta$:

$$\beta = k_0 v^2 k'' = -1, \hspace{1cm} (74)$$

we see that the vortex propagation take place only in the negative-dispersion region. In [5] we find, that the constraint, given in equation (74) correspond to the next two experimental situations:

1. Cold plasma near the Langmuir frequency.
2. A region near an electronic resonance in an isotropic medium.

There is also a equivalence in high-frequency region between the spectral presentation of dielectric susceptibility of a cold plasma and this of a dielectric media. The expression is equal wish precise a (dielectric) constant.
Near these frequencies the dispersion parameter increases rapidly and admits values:
\[ |k''| \sim 10^{-24} - 10^{-25} \frac{v c k^2}{\omega} \]
This leads to the fact, that the dispersion term (the second derivative in z direction of the amplitude function) in normed amplitude equations of electric \(^{(42)}\) and magnet \(^{(63)}\) field is of the same order of the diffractive term (transverse Laplasian).

5 Finiteness of the energy for the vortex solutions

Now we come to the crucial point - energy and shape of the vortex solutions \((61)-(63)\) and \((68)-(70)\) of the electric and magnet fields. To prove the finiteness of energy of the vortex solutions \((61)-(63)\) and \((68)-(70)\), we start with the equations for averaged in time balance of energy density of electrical and magnet field \(^{(28)}\):

\[
\frac{\partial W}{\partial t} = \frac{1}{16\pi} \left( \vec{E} \cdot \frac{\partial \vec{D}^*}{\partial t} + \vec{E}^* \cdot \frac{\partial \vec{D}}{\partial t} + \vec{B} \cdot \frac{\partial \vec{B}^*}{\partial t} + \vec{B}^* \cdot \frac{\partial \vec{B}}{\partial t} \right), \tag{75}
\]

where \(\vec{D} = \vec{P}_{lin} + 4\pi \vec{P}_{nl}\) is a sum of the linear induction and the nonlinear induction of the electrical field. In more of the books the calculations of the averaged energy of the optical waves in a dispersive media are worked out bearing in mind only the first order of slowly varying amplitude approximation of the linear electrical induction. That is because the result comes to the old result of Brillouin 1921 for energy density of electrical field:

\[
W_{lin} = \frac{1}{8\pi} \left( \frac{\partial (\omega \varepsilon_0)}{\partial \omega} |\vec{A}|^2 + |\vec{C}|^2 \right)
= \frac{1}{8\pi} \left( \frac{c}{v} |\vec{A}|^2 + |\vec{C}|^2 \right). \tag{76}
\]

It is straightforward to show using this approximation that after integrating in space our vortex solutions they admit infinite energy. As seen from \(^{(76)}\), this approximation do not include the dispersion parameter \(k''\), which is present in the envelope equations \(^{(42)}\), \(^{(63)}\) as coefficient in front of the
second derivative in z direction. So there is no possibility to estimate the influence of $k''$ in the energy integral. Our vector amplitude equations are obtained, using the second order approximation of $\vec{P}_{lin}$. The right expression of the energy density in this case requires to expand the linear electrical inductions in the energy integral also up to the second order, and to include this dispersion parameter in the energy integral. Using truncated form for first derivative in time of the linear induction of electrical field we have:

$$\frac{\partial \vec{P}_{lin}}{\partial t} = -i\omega_0 \epsilon(\omega_0) \vec{A} + \frac{c}{v} \frac{\partial \vec{A}}{\partial t} + \frac{ick''}{2} \frac{\partial^2 \vec{A}}{\partial t^2}. \quad (77)$$

From (75), (77) and complex conjugate of (77) we obtain:

$$\frac{\partial W_{lin}}{\partial t} = \frac{c}{v} \left( \vec{A} \frac{\partial \vec{A}^*}{\partial t} + \vec{A}^* \frac{\partial \vec{A}}{\partial t} \right) + \frac{ick''}{2} \frac{\partial}{\partial t} \left( \vec{A}^* \frac{\partial \vec{A}}{\partial t} - \vec{A} \frac{\partial \vec{A}^*}{\partial t} \right) + \vec{B} \frac{\partial \vec{B}^*}{\partial t} + \vec{B}^* \frac{\partial \vec{B}}{\partial t}, \quad (78)$$

or

$$W_{lin} = \frac{c}{v} |\vec{A}|^2 + |\vec{C}|^2 + \frac{ick''}{2} \left( \vec{A}^* \frac{\partial \vec{A}}{\partial t} - \vec{A} \frac{\partial \vec{A}^*}{\partial t} \right), \quad (79)$$

where $A^*$ denote complex conjugate in time amplitude function. Rewriting the vortex solutions of electrical (61)-(63) and magnet (68)-(70) fields in not normed (dimension) coordinates and substituting in (79), for the case of negative dispersion finally we obtain the next expression of the linear part of averaged energy density:

$$W_{lin} = \left( 1 + \frac{c}{v} - c |k''| \Delta \omega \right) \vec{A}'^2, \quad (80)$$

where $\Delta \omega$ denote the spectral bandwidth of the vortices. We obtain one unexpected result: with the increasing of the spectral bandwidth of our solutions the linear part of energy density will decrease in the negative dispersion region. Conditions when the linear part of the energy density of electrical field is zero determine one critical spectral bandwidth $\Delta \omega_c$:
\[ \Delta \omega_c = \frac{v + c}{vc} |k''| \]  

(81)

Near to plasma frequency or some of the electronic resonances the dispersion parameter increases and values about: \( k'' \sim 10^{-24} - 10^{-25} \text{cm}^2 \). The critical spectral bandwidth in this case becomes: \( \Delta \omega_c \sim 10^{14} - 10^{15} \text{Hz} \). The envelope approximation request, that: \( \omega_0 > \Delta \omega_c \). This condition shows that the linear part of energy density of vortices is zero only when their main frequency is situated in optical region, or regions, which frequency is greater than the optical and admit spectral bandwidth equal to \( \Delta \omega_c \). The nonlinear part of averaged energy density is expressed in [29] and for the vortex solutions (57)-(59) it becomes:

\[
W_{\text{lin}} = n_2 \left| A' \right|^2 \left| A'' \right|^2 + \omega_0 \frac{\partial n_2}{\partial \omega_0} \left| A' \right|^2 \left| A'' \right|^2.
\]

(82)

These results give the conditions for the finiteness of energy of the vortices: the spectral bandwidth of vortices to be equal to \( \Delta \omega_c \). Integrating \( W_{\text{lin}} \) on the 3D space we obtained a finite value proportional to the main frequency \( \omega_0 \). The intensity part in this expression for the energy (82) is limited by the experimental condition (73) and it is a constant. In this way we obtain the result, that the localized energy of the solutions increasing linearly with the increasing of the main frequency. The numerical investigation of stability for such type vortices, using split-step Fourier method, is provided in [5]. Comparing the numerical calculations for vortices with these, of a standard Gaussian pulse, it can be clearly seen that the vortices propagate without any changing of their shape, whereas the Gaussian pulse, with the same initial amplitude, self-focuses rapidly in a much shorter distance.

In additional, we also investigate the shape and the behavior of electrical and magnet field in the origin and infinity. To obtain the correct result, the real part solutions of the amplitude equations (62)-(63) and (68)-(70) must be rewritten in the independent Cartesian coordinates x, y, z:

\[
A'_x = \frac{x \sin(\sqrt{\alpha \Omega} \sqrt{x^2 + y^2 + z^2} + \Omega t)}{x^2 + y^2 + z^2},
\]

(83)

\[
A'_y = \frac{y \sin(\sqrt{\alpha \Omega} \sqrt{x^2 + y^2 + z^2} + \Omega t)}{x^2 + y^2 + z^2},
\]

(84)
\[ A'_z = z \sin \left( \sqrt{\alpha \Omega} \sqrt{x^2 + y^2 + z^2 + \Omega t} \right) \frac{x^2 + y^2 + z^2}{x^2 + y^2 + z^2}. \] (85)

The limits of these functions in origin and in infinity are investigated, using the Haine criteria for limit of a multidimensional function. It is straightforward to show that the solutions (83)-(85) are odd functions, going to zero in infinity and admit very little value near to the origin. In the origin, using the same criteria, we find that the above functions have no limit. But as we must point out, the envelope function (83)-(85) is only image functions around the real electric field, expressed in our case by the relation (10):

\[ E_x = A'_x \sin (k_0 z - \omega_0 t) \] (86)
\[ E_y = A'_y \sin (k_0 z - \omega_0 t) \] (87)
\[ E_z = A'_z \sin (k_0 z) - \omega_0 t. \] (88)

Using the same criteria as in the previous case, it is seen, that the electrical field admit exact limit zero in the origin and infinity:

\[ \lim_{x,y,z \to 0} (E_i) = 0 \]
\[ \lim_{x,y,z \to \pm \infty} (E_i) = 0; \ i = x, y, z. \] (89)

6 Conclusion

The applied method of slowly varying amplitudes of the electrical and magnet vector fields give us the possibility to reduce the nonlinear vector integro-differential equations to vector nonlinear differential equations of amplitudes. Here is the place to explain more some of the differences between the solutions, obtained by separation of the variables of the usual linear scalar Schredinger equation with potential depending only on ‘r’ (for example hydrogen atom) and solutions of the vector version of NSE. For the linear Schredinger equations with potential, to higher order spherical functions correspond more higher order of radial spherical Bessel functions. While for vector version of NSE to higher order of spherical functions \((\ell = 1, 2, ..)\) correspond higher number of the fields components and higher value of localized energy. We have for all radial solutions the zero spherical Bessel function \(\sin or \frac{\sin \alpha r}{r}\) in nonlinear case. This is the reason to start with such special kind of complexification of our electrical field (10). It is important to highlight also
the equivalence in high frequency region between the linear dielectric susceptibility of a cold plasma and this of a dielectric media. A plasma frequency of dielectric media in this region can determine also. The expression is equal to this of cold plasma wish precise a (dielectric) constant. This discussion shows two possible regions for observing optical vortices: 1. High frequency (transparency) region of cubic dielectrics or cold plasma. 2. Near to some of the electronics resonances of $\chi^{(3)}$ media. Above mentioned investigations are provided only for the real part of linear and nonlinear susceptibility. That is why the first possibility, high frequency region, is more attractive for observing of optical vortices. The complex part of linear susceptibility is significant near to electronic resonances and should be get in mind in a more detailed analysis.
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