Quasinormal modes, Superradiance and Area Spectrum for $2+1$ Acoustic Black Holes

Samuel Lepe and Joel Saavedra

Instituto de Física, Pontificia Universidad Católica de Valparaíso, Casilla 4950, Valparaíso, Chile.

We present an exact expression for the quasinormal modes of acoustic disturbances in a rotating $2+1$ dimensional sonic black hole (draining bathtub fluid flow) in the low frequency limit and evaluate the adiabatic invariant proposed by Kunstatter. We also compute, via Bohr-Sommerfeld quantization rule the equivalent area spectrum for this acoustic black hole, and we compute the superradiance phenomena for pure spinning $2+1$ black holes.

PACS numbers:

I. INTRODUCTION

Analog models of general relativity have received great attention in the last few years, since it is believed that these models are shedding light on possible experimental verifications of some fundamental problems in black hole physics, such as the evaporation of black holes and semiclassical quantities. The idea of using supersonic acoustic flows as analog systems to mimic some properties of black hole physics was proposed for the first time by Unruh [1] (for a comprehensive review of Analog Black Holes see [2] and references therein, and for a pedagogical review see [3][4][5]). The basis of the analogy between gravitational black hole and sonic black holes comes from considering the propagation of acoustic disturbances on a barotropic, inviscid, inhomogeneous and irrotational (at least locally) fluid flow. It is well known that the equation of motion for this acoustic disturbance (described by its velocity potential $\psi$)

is identical to the Klein-Gordon equation for a massless scalar field minimally coupled to gravity in a curved spacetime [3][4][5].

Therefore, this analogy gives us a powerful terrestrial laboratory to explore principal aspects of black hole physics, for example, the Hawking radiation with its significant consequence about the paradox of information loss, quasinormal modes (QNMs), known as the "ringing" of black holes, also play an important role in classical aspects of black holes physics. In particular, from Hod’s proposal [6], modes with high damping have received great attention, specially their relevance in the quantization of the area of black holes and the possibilities of fixing the Immirzi $\gamma$ parameter in loop quantum gravity [7][8]. In principle all these processes can be studied in a laboratory using the analog models of gravity.

In this work we analytically compute the QNMs or acoustic disturbances in a rotating $2+1$ dimensional sonic black hole (draining bathtub fluid flow), following the methodology developed in Refs. [9] and [10]. We also compute the adiabatic invariant proposed by Kunstatter [8], and we compute the equivalent area spectrum for this acoustic black hole via Bohr-Sommerfeld quantization rule. For a $2+1$ dimensional pure spinning black hole called perfect vortex [11] we show explicitly the absence of superresonance phenomena and the absence of QNMs. We conjecture that this acoustic space behaves like the BTZ extremal black hole [12].

The organization of the paper is as follows: In Sec. II we specify the Draining Vortex, the notation and some useful quantities for this acoustic geometry. In Sec. III we determine the QNMs, mass and the area spectrum in the large damping limits. In Sec. IV we treat the perfect vortex and explicity compute the absence of the superresonance and QNMs for this acoustic black hole. Finally, we conclude in Sec. V.
II. DRAINING VORTEX

The rotating $2+1$ acoustic black hole was proposed by Visser for the first time in Ref. [4] which corresponds to a draining bathtub fluid flow with a sink at the origin. The velocity profile for this configuration is given by

$$\mathbf{v}_0 = -\frac{A}{r} \hat{r} + \frac{B}{r} \hat{\phi},$$

(1)

where $A$ and $B$ are arbitrary real positive constants. If we assume the fluid to be locally irrotational, the background velocity potential becomes

$$\psi_0 = -A \log\left(\frac{r}{a}\right) + B \phi,$$

(2)

where $a$ is some length scale (we adopt $a = 1$).

The acoustic line element is given by

$$ds^2 = -c^2 dt^2 + \left(\frac{A}{r} dt + dr\right)^2 + r^2 \left(d\phi - \frac{B}{r^2} dt\right)^2,$$

(3)

being $c$ the constant velocity of sound. In order to study the general properties of this acoustic spacetime, we perform the coordinate transformations discussed in Ref. [10], namely

$$dt \to dt + \frac{1}{c^2} \frac{Ar}{r^2 - (A/c)^2} dr,$$

(4)

$$d\phi \to d\phi + \frac{1}{c^2} \frac{AB}{r^2 - (A/c)^2} dr,$$

(5)

so that the metric adopts a Kerr-like form

$$ds^2 = -c^2 \left(1 - \frac{(A/c)^2}{r^2}\right) dt^2 + \left(1 - \frac{(A/c)^2}{r^2}\right)^{-1} dr^2 + r^2 \left(d\phi - \frac{B}{r^2} dt\right)^2.$$

(6)

The radii of the horizon and ergosphere for this acoustic black hole are respectively

$$r_+ = \frac{A}{c},$$

(7)

$$r_e = \sqrt{r_+^2 + (B/c)^2}.$$

(8)

By considering the Komar [13] integrals we can calculate two “conserved charges”, conserved in the sense that they are generated by divergenceless currents

$$M = \frac{c^2}{2} \left[1 + (B/A)^2\right],$$

(9)

$$J = \frac{1}{4} c^2 B.$$

(10)

Though we have labelled these conserved charges $M$ and $J$ they do not represent the physical mass and angular momentum of the spacetime, such an identification would require the use of the Einstein equations, which do not apply in the current context. It is straightforward to take the limit of static acoustic black holes ($B = 0$) for these conserved charges. But if we take the limit of pure spinning acoustic black holes (perfect vortex $A = 0$ in (3)) the $M$ charge is undefined and we must use an independent method such an used for extremal black holes in general relativity.

Other interesting quantities for the study of black hole physics are the angular velocity of the horizon
\[ \Omega_+ = \frac{4}{c^2} J r_+^2 \]
\[ = \frac{c^4}{4J} \left( \frac{2M}{c^2} - 1 \right), \] (11)

and the surface gravity

\[ K_+ = c^2 r_+^{-1}, \] (13)

which allows to define the analogous to the Hawking temperature:

\[ T_H = \frac{1}{2\pi c} K_+. \] (14)

In the quantum version of this system we can hope that the acoustic black hole emits "acoustic Hawking radiation". This effect coming from the horizon of events is a pure kinematical effect that occurs in any Lorenzian geometry independent of its dynamical content[4]. It is well known that the acoustic metric does not satisfy the Einstein equations, due to the fact that the background fluid motion is governed by the continuity and the Euler equations. As a consequence of this fact, one should expect that the thermodynamic description of the acoustic black hole is ill defined. However, this powerful analogy between black hole physics and acoustic geometry allows to extend the study of many physical quantities associated to black holes, such as quasinormal modes and area spectrum. In the next section we consider this aspect.

### III. QNMS AND AREA SPECTRUM OF DRAINING VORTEX

The basis of the analogy between Einstein black holes and sonic black holes comes from considering the propagation of acoustic disturbances on a barotropic, inviscid, inhomogeneous and irrotational fluid flow. It is well known that the equation of motion for these acoustic disturbances (described by its velocity potential \( \psi \)) is identical to the Klein-Gordon equation for a massless scalar field minimally coupled to gravity in a curved spacetime [3][4][5], i.e.

\[ \frac{1}{\sqrt{g}} \partial_\mu \left( \sqrt{g} g^{\mu \nu} \partial_\nu \Phi \right) = 0. \] (15)

We make the Ansatz

\[ \Psi(t, r, \phi) = R(r) \ e^{-i\omega t} \ e^{im\phi}, \] (16)

where \( m \) is a real constant and in order to make \( \Psi(t, r, \phi) \) single valued, \( m \) must take on integer values. The radial function \( R(r) \) satisfies [10] the following equation:

\[ \frac{d^2 R(r)}{dr^2} + P_1(r) \frac{dR(r)}{dr} + Q_1(r) R(r) = 0 , \] (17)

where

\[ P_1(r) = \frac{1}{r(r^2 - r_+^2)} \left( r^2 + r_+^2 + 2 \frac{r_+}{c} (\Omega_+ r_+^2 m - \omega r^2) \right), \] (18)

and

\[ Q_1(r) = \frac{1}{r^2(r^2 - r_+^2)} \left( 2i \frac{\Omega_+}{c} r_+^2 m + \left( r^2 - \left( \frac{\Omega_+}{c} \right)^2 r_+^4 \right) m^2 + 2 \frac{\Omega_+}{c} \frac{\omega}{c} r_+^2 m - \left( \frac{\omega}{c} \right)^2 r_+^4 \right). \] (19)
Solutions of (17) were found in Ref.[10] using an adaptation of the matching procedure developed by Starobinsky. In order to solve the radial equation (17) they considered a new radial function defined by

\[ R(r) = \frac{r}{r_+} \exp \left[ \frac{i}{2} \left( \omega \log \left( \frac{r^2}{r_+^2} - 1 \right) - m\Omega \log \left( 1 - \frac{r^2}{r_+^2} \right) \right) \right] \ L(r) . \] (20)

and then, through a matching procedure for the region \( \omega \left( \frac{r^2}{r_+^2} - 1 \right) \ll m2\pi T_H \) and \( \omega \ll 2\pi T_H \), the radial equation for \( L(r) \) is transformed in the Riemann-Papparitz equation whose solutions in terms of hypergeometric functions read as follows

\[ L(r) = \left( \frac{r^2}{r_+^2} - 1 \right)^{\alpha'} \left( \frac{r}{r_+} \right)^{2\beta'} \left\{ C_1 \ \text{2F1} \left( \alpha; \beta; \gamma; 1 - \frac{r^2}{r_+^2} \right) \right. \]
\[ \left. - C_2 \left( \frac{r^2}{r_+^2} - 1 \right)^{1-\gamma} \ \text{2F1} \left( \alpha + 1 - \gamma; \beta + 1 - \gamma; 2 - \gamma; \frac{r^2}{r_+^2} - 1 \right) \right\} , \] (21)

where

\[ \alpha' = -i \frac{Q}{2} , \ \beta' = -\frac{1}{2} \] (22)
\[ \alpha = -\frac{S + iQ}{2} , \ \beta = \frac{S - iQ}{2} , \ \gamma = 1 - iQ , \]

and

\[ S^2 = m^2 - 2 \left( \frac{1}{2\pi T_H} \right)^2 \omega (\omega - m\Omega_+) , \] (23)
\[ Q^2 = \left( \frac{1}{2\pi T_H} \right)^2 (\omega - m\Omega_+)^2 . \] (24)

In order to compute the QNMs, we need to impose the boundary conditions upon the solution of the radial equation, meaning that only purely ingoing waves are allows at the horizon, that is, \( C_2 = 0 \) (since nothing comes out of the horizon) while \( L(r) \) is given by

\[ L(r) = C_1 \left( \frac{r^2}{r_+^2} - 1 \right)^{\alpha'} \left( \frac{r}{r_+} \right)^{2\beta'} \ \text{2F1} \left( \alpha; \beta; \gamma; 1 - \frac{r^2}{r_+^2} \right) . \] (25)

We also need to know the behavior of \( L(r) \) in the asymptotic region \( (r \to \infty) \)

\[ L(r) = D_1 (\frac{r^2}{r_+^2} - 1)^{(S-1)/2} + D_2 (\frac{r^2}{r_+^2} - 1)^{-(S+1)/2} , \]

where the constant coefficients \( D_1 \) and \( D_2 \) are given by

\[ D_1 = \frac{\Gamma (\gamma) \Gamma (\beta - \alpha)}{\Gamma (\beta) \Gamma (\gamma - \alpha)} C_1 , \] (26)
\[ D_2 = \frac{\Gamma (\gamma) \Gamma (\alpha - \beta)}{\Gamma (\beta) \Gamma (\gamma - \beta)} C_1 . \] (27)

We require that only outgoing waves should be present at the infinity. To see this condition we consider the possible values for \( S \):
\[ S > 1 \implies D_1 = 0 \ (\beta = -n, \gamma - \alpha = -n), \]  
\[ S < -1 \implies D_2 = 0 \ (\alpha = -n, \gamma - \beta = -n). \]  

This set of conditions can be summarized in the following condition:

\[ \frac{1}{2} (S - iQ) = -n, \]  

where \( n \) is a positive integer. Then, from (30) we obtain the frequencies of the QNMs

\[ \alpha = i2n + \sqrt{(m\beta - i2n)^2 + m^2}, \]  

being \( \alpha = \omega/2\pi T_H \) and \( \beta = \Omega_+/2\pi T_H \).

In particular, the case of highly damped modes (i.e., QNMs with a large imaginary part) has received great attention in the last time specially regarding their relevance in the quantization of the area of black holes and the possibilities of fixing the Immirzi \( \gamma \) parameter in loop quantum gravity through QNMs. In the present case and as a consequence of the Starobinsky matching procedure meaning \( \omega/2\pi T_H \ll 1 \), i.e., \( \alpha \ll 1 \), the spectrum of frequencies for \( n \gg 1 \) becomes

\[ \omega = m\Omega_+, \]  

and the imaginary part of the spectrum is vanishes, a result that differs from the gravitational case. Let us note that in order to satisfy \( \omega/2\pi T_H \ll 1 \), we also must have \( \beta \ll 1 \leftrightarrow \Omega_+ \ll 2\pi T_H/m \).

On the other hand, in the sense of Hod \cite{6, 14} the real part of the spectrum of QNMs of the black holes does not give the correct Schwarzschild limit \( 4T_H \ln3 + m\Omega_+ \), The absence of the Schwarzschild asymptotic limit was claimed in Ref.\cite{15} and its absence can be understood because the Hod’s conjecture is fundamentally based on black hole thermodynamics. At this point, we remark that the sonic metric does not satisfy the Einstein equations. Therefore, a thermodynamics scheme of acoustic black holes inspired from black hole physics in general relativity does not apply in the acoustic case.

Now, we consider Re(\( \omega \)) to be a fundamental vibrational frequency for a black hole of energy \( E = M \). Given a system with energy \( E \) and vibrational frequency \( \omega \) one can show that the quantity

\[ I = \int \frac{dE}{\omega(E)}, \]  

is an adiabatic invariant \cite{8} which via Bohr-Sommerfeld quantization, has an equally spaced spectrum in the semi-classical limit:

\[ I \approx \hbar k. \]  

Now by taking Re(\( \omega \)) \( \equiv \omega_{QNM} \) in this context we have \( (J \text{ fixed}) \)

\[ I = \int \frac{dM}{\omega_{QNM}}, \]  

\[ = \frac{2J}{c^2} \frac{1}{m} \ln \left( \frac{2M}{c^2} - 1 \right), \]  

and the mass spectrum becomes

\[ M(m,k) = \frac{c^2}{2} \left[ 1 + \exp \left( \frac{c^2}{2J} mk\hbar \right) \right]. \]  

On the other hand, the perimeter of the horizon of the black hole is given by
\[ A_+ = 2\pi r_+ , \tag{38} \]
\[ = \frac{8\pi J}{c^3} \left( \frac{2M}{c^2} - 1 \right)^{-1/2} , \tag{39} \]

and as a consequence of the mass spectrum, the area spectrum becomes
\[ A_+ (m, k) = \frac{8\pi}{c^3} J \exp \left[ -\left( \frac{c}{2} \right)^2 \frac{m}{J} (kh) \right] , \tag{40} \]

where we can see that both mass spectrum and area spectrum are not equally spaced against to the Schwarzschild black hole \[17\] \[18\]. The conserved charges \( M \) and \( J \) are not independent in the acoustic case \[9\] \[10\]. Since the constant \( B \) establishes their relationship. If we express these charges in terms of \( \Omega_+ \) and \( T_H \) (physical parameters of sonic black hole)
\[ M = \frac{c^2}{2} \left[ 1 + \left( \frac{\Omega_+}{2\pi T_H} \right)^2 \right] , \tag{41} \]
\[ J = \frac{1}{4} c^2 A \left( \frac{\Omega_+}{2\pi T_H} \right) , \tag{42} \]

then the parameter \( A \) which is an arbitrary constant, can be freely defined and hence as a consequence of this we are left with a unique effective degree of freedom \( (M) \) which allows an unambiguous computation of the adiabatic invariant. Let us recall however that the present situation is not applicable to the BTZ \[19\] \[20\] of Kerr black holes \[21\] \[22\] \[23\]. In these later ones, \( M \) and \( J \) are effectively independent charges with no relation between them as described above.

IV. PERFECT VORTEX

The perfect vortex is obtained when we replace \( A = 0 \) in \[3\]. In this case this spacetime (acoustic vortex) represents a fluid with a non-radial flow. This geometry was studied from a Riemannian geometry point of view in Ref. \[11\]. The metric and its inverse are, respectively, given by
\[ g_{\mu\nu} = \begin{pmatrix} -c^2 \left( 1 + r_e^2/r^2 \right) & 0 & -c r_e \\ 0 & 1 & 0 \\ -c r_e & 0 & r^2 \end{pmatrix} , \tag{43} \]
\[ g^{\mu\nu} = \begin{pmatrix} \frac{-1}{c^2} & 0 & -r_e/c r^2 \\ 0 & 1 & 0 \\ -r_e/c r^2 & 0 & (1 - r_e^2/r^2) / c r^2 \end{pmatrix} , \tag{44} \]

where
\[ r_e = \frac{B}{c} , \tag{45} \]
is the radius of the ergosphere and \( c \) is the constant velocity of sound as referenced in Refs. \[9\] \[10\].

Therefore using the Klein-Gordon equation with \[13\] and considering solutions of the form,
\[ \Psi(t, r, \phi) = R(r) e^{-i \omega t} e^{i m \phi} , \tag{46} \]
where \( m \) is an integer constant, we obtain the radial equation
\[ \frac{d^2 R(r)}{dr^2} + \frac{1}{r} \frac{dR(r)}{dr} + \frac{1}{r^2} \left[ \left( \frac{r_e}{r} \right)^2 - \left( m^2 + 2r_e \frac{\omega}{c} m \right) + \left( \frac{\omega r}{c} \right)^2 \right] R(r) = 0 . \tag{47} \]
With a change of variable

\[ R(r) = r^{-1/2} H(r), \]  

the radial equation (47) can be written as

\[ \frac{d^2 H(r)}{dr^2} + \left[ \frac{1}{c^2} \left( \omega - \frac{e c}{r^2} m \right)^2 - \left( m^2 - \frac{1}{4} \right) \frac{1}{r^2} \right] H(r) = 0. \]  

First, we shall study the super-resonance (analog to the super-radiance in black hole physics). For this goal we take the limit \( r \to \infty \), and (49) adopts the form

\[ \frac{d^2 H_{\infty}(r)}{dr^2} + \left( \frac{\omega}{c} \right)^2 H_{\infty}(r) = 0, \]  

whose solution is

\[ H_{\infty}(r) = \exp \left( -i \frac{\omega}{c} r \right) + \mathcal{R}(\omega, m) \exp \left( i \frac{\omega}{c} r \right), \]  

where \( \mathcal{R}(\omega, m) \) is the reflection coefficient in the sense of scattering potential.

Near to ergosphere \( (r \to r_{\text{e}}) \), the equation reads as follows

\[ \frac{d^2 H_0(r)}{dr^2} + \left( \frac{\omega}{c} \right)^2 \left( 1 - \frac{\omega_+}{\omega} \right) \left( 1 - \frac{\omega_-}{\omega} \right) H_0(r) = 0, \]  

where

\[ \omega_{\pm} = m \Omega_e \left[ 1 \pm \sqrt{1 - (4m^2)^{-1}} \right], \]  

and \( \Omega_e \) is the angular velocity of the ergosphere

\[ \Omega_e = \frac{c}{r_{\text{e}}}. \]  

In this region the solution reads as follow

\[ H_0(r) = \mathcal{T}(\omega, m) \exp \left( -i \frac{\omega}{c} \sqrt{\left( 1 - \frac{\omega_+}{\omega} \right) \left( 1 - \frac{\omega_-}{\omega} \right)} r \right), \]  

where \( \mathcal{T}(\omega, m) \) is the transmission coefficient. It is straightforward to check that the Wronskian of these approximate solutions is constant, such that

\[ 1 - |\mathcal{R}(\omega, m)|^2 = \left( 1 - \frac{\omega_+}{\omega} \right) \left( 1 - \frac{\omega_-}{\omega} \right) |\mathcal{T}(\omega, m)|^2. \]  

We can observe the absence of the super-resonance for the perfect vortex, meaning that it is impossible to have \( |\mathcal{R}(\omega, m)|^2 > 1 \), despite the fact that this geometry exhibits an ergosphere. To the best of our knowledge there is no gravitational black hole exhibits this behavior. By virtue of the results report in Refs.\[24,\, 25\] and \[12\], we could make the conjecture that the extreme BTZ black hole could be a possible candidate \[26\]. At this point we notice that our results are valid when the local velocity of the sound is a constant. Certainly this fact is different from the physical situation found in Bose-Einstein Condensates (BEC) where it is well known that the sound velocity is proportional to the density \[27,\, 28\]. The relevance of the perfect vortex in the case of \( c \) constant was discussed for the first time in the Ref.\[11\] where the acoustic perfect vortex leads to deflection of phonons like the photons bending due to the gravitational field. Recently, in the case \( c^2(r) \sim \rho \) Ref.\[29,\, 30\] it was shown that the superresonant scattering is possible if an ad hoc density profile is given.

In the following we explore the existence of QNMs. In order to do this, we need to solve (47) explicitly. Let us start with the change of variables \( x = r^2/r_{\text{e}}^2 \). Equation (47) becomes
\[
\frac{d^2 R(x)}{dx^2} + \frac{1}{x} \frac{dR(x)}{dx} + \frac{1}{4} \left[ \frac{m^2}{x} - \left( m^2 + \frac{2\omega}{\Omega_e} m \right) + \left( \frac{\omega}{\Omega_e} \right)^2 x \right] \frac{1}{x^2} R(x) = 0, \tag{57}
\]
and, in the spirit of the Starobinsky matching procedure, we consider the region described by
\[
\frac{\omega}{\Omega_e} x \ll m \quad \text{and} \quad \frac{\omega}{\Omega_e} \ll 1. \tag{58}
\]
In this region (57) reads as follows
\[
\frac{d^2 R(x)}{dx^2} + \frac{1}{x} \frac{dR(x)}{dx} + \frac{1}{4} \left[ \frac{m^2}{x} - \left( m^2 + \frac{2\omega}{\Omega_e} m \right) \right] \frac{1}{x^2} R(x) = 0, \tag{59}
\]
whose solutions are given by
\[
R(r) = C_1 J_{-S}(mr_e r^{-1}) + C_2 Y_{-S}(mr_e r^{-1}), \tag{60}
\]
where \(J_{-S}, Y_{-S}\) are Bessel functions, and \(S\) is given by
\[
S = m \sqrt{1 + \frac{2\omega}{m \Omega_e}}.
\]
From the asymptotic behaviors \[31\] of these Bessel functions, it is straightforward to show that
\[
\lim_{r \to \infty} R(r) \rightarrow \frac{1}{\Gamma(1 - S)} \left( \frac{2r}{mr_e} \right)^S \left[ C_1 - C_2 \cot \left( \frac{mr_e}{r} \right) \right] \tag{61}
\]
\[+ C_2 \frac{1}{\Gamma(1 + S)} \frac{1}{\sin \left( \frac{S mr_e}{2r} \right)} \left( \frac{mr_e}{2r} \right)^S. \tag{62}\]
Then, imposing the regularity conditions to find the QNMs, we obtain
\[
\Gamma(1 - S) = \infty \implies S = n = 1, 2, 3, \ldots \tag{63}
\]
Finally, we find
\[
\omega = \left( \frac{n^2 - m^2}{2m} \right) \Omega_e, \tag{64}
\]
and notice that, the imaginary part of the frequencies is vanishing, showing the absence of QNMs for the perfect vortex. This behavior is analogous to the case of BTZ extreme black hole \[12\]. Additionally, there is an infinite tower of ordinary well behaved normal modes in the limit of the Starobinsky matching procedure and it is straightforward to show that this spectrum contains only positive frequencies according to (58) (see Appendix).

V. CONCLUSIONS AND REMARKS

In this paper we have studied QNMs, mass spectrum and area spectrum for the 2 + 1 acoustic black hole, called ‘draining bathtub fluid flow’ or vortex geometry. We also studied superresonance phenomena and QNMs for the perfect vortex metric.

In the draining bathtub fluid flow we have found the explicit expression of the QNMs spectrum, and for highly damping modes this spectrum contains only real frequencies. We note that this last result is similar to the one proposed by Hod in Ref.\[32\] for the asymptotic real spectrum of the QNMs of rotating black holes (Kerr black holes). However, sonic black holes do not approach the Schwarzschild limit conjectured by Hod. This absence of the Schwarzschild limit was also pointed in Ref.\[15\] showing that the applications of Hod’s conjecture to the acoustic
black hole seems impossible. At this point we can note that this absence is natural, because Hod’s conjecture is deeper based on thermodynamics black hole physics. Finally, we showed that the area spectrum and mass spectrum are not equally spaced.

Along the same lines we computed the perfect vortex describes in Ref. [9]. Explicit solutions of the radial equations were obtained and we have showed that the superresonance modes can not be excited, despite the fact that the acoustic geometry has an ergosphere. We also demonstrated the absence of QNMs for the perfect vortex and we conjecture that this acoustic black hole behaves like an extremal BTZ black hole.

Acknowledgments

We are grateful to E. Ayon for many enlightening discussions. We also acknowledge the referee for useful suggestions in order to improve the presentation of the results of this paper. This work was supported by COMISION NACIONAL DE CIENCIAS Y TECNOLOGIA through FONDECYT Grant 1040229 (SL) and Postdoctoral Grant 3030025 (JS). This work was also partially supported by PUCV Grant No. 123.779/2005 (SL) and No. 123.778/2005 (JS). The authors wish to thank Centro de Estudios Científicos (CECS) and Departamentos de Física de la Universidad de La Frontera y de la Universidad de Concepción for its kind hospitality. We thank U. Raff for a careful reading of the manuscript.

VI. APPENDIX

From the equation (59)

\[
\frac{d^2 R(x)}{dx^2} + \frac{1}{x} \frac{dR(x)}{dx} + \frac{1}{4} \left[ \frac{m^2}{x} - S^2(m) \right] \frac{1}{x^2} R(x) = 0, \tag{65}
\]

where

\[
S^2(m) = m^2 \left[ 1 + 2 \frac{\omega}{\Omega_e} \frac{1}{|m|} \right], \tag{66}
\]

it can directly be shown that in order to satisfy the regularity condition for the QNMs, \( S^2(m) \) must be positive, such that only outgoing waves are allowed at infinity (the asymptotic solutions are given by (62)). For \( m < 0 \) we can write \( S^2(m) \) in the following form

\[
S^2(m < 0) = |m|^2 \left[ 1 - 2 \frac{\omega}{\Omega_e} \frac{1}{|m|} \right], \tag{67}
\]

where the condition

\[
0 < \omega < \frac{1}{2} |m| \Omega_e, \tag{68}
\]

is consistent with the range of validity of the Starobinsky matching procedure given by (58).

Now we study the behavior of the infinite tower of normal modes given by

\[
\omega = \frac{1}{2} \left( \frac{n^2 - m^2}{|m|} \right) \Omega_e. \tag{69}
\]

For \( m < 0 \) this equation can be rewritten as

\[
\omega = -\frac{1}{2} \left( \frac{n^2 - |m|^2}{|m|} \right) \Omega_e = \frac{1}{2} |m| \Omega_e - \frac{n^2}{2 |m|} \Omega_e, \tag{70}
\]
and in order to satisfy \( |m| > n \) it is necessary that \( |m| > n \).

The critical case \( m > n \) admits the existence of negative frequencies such that it can be associated with a possible instability. But according to Starobinsky, this instability does not appear. This shows the robustness of the matching procedure developed by Starobinsky from which we can obtain a well behaved tower of ordinary normal modes.

[1] W. G. Unruh, “Experimental black hole evaporation?”, Phys. Rev. Lett. 46, 1351–1353 (1981).
[2] M. Novello, M. Visser and G. Volovik, “Artificial black holes.”, River Edge, USA: World Scientific (2002) 391 p.
[3] M. Visser, “Acoustic black holes,” arXiv:gr-qc/9901047.
[4] M. Visser, “Acoustic black holes: Horizons, ergospheres, and Hawking radiation,” Class. Quant. Grav. 15, 1767 (1998) arXiv:gr-qc/9712010.
[5] M. Visser, “Acoustic propagation in fluids: An Unexpected example of Lorentzian geometry,” arXiv:gr-qc/9811028.
[6] S. Hod, “Bohr’s correspondence principle and the area spectrum of quantum black holes,” Phys. Rev. Lett. 81, 4293 (1998) arXiv:gr-qc/9812002.
[7] O. Dreyer, “Quasinormal modes, the area spectrum, and black hole entropy,” Phys. Rev. Lett. 90, 081301 (2003) arXiv:gr-qc/0211076.
[8] G. Kunstatter, “d-dimensional black hole entropy spectrum from quasi-normal modes,” Phys. Rev. Lett. 90, 161301 (2003) arXiv:gr-qc/0212014.
[9] S. Basak and P. Majumdar, “Reflection coefficient for superresonant scattering,” Class. Quant. Grav. 20, 2929 (2003) arXiv:gr-qc/0303012.
[10] S. Basak and P. Majumdar, “Superresonance’ from a rotating acoustic black hole,” Class. Quant. Grav. 20, 3907 (2003) arXiv:gr-qc/0203059.
[11] U. R. Fischer and M. Visser, “Riemannian geometry of irrotational vortex acoustics,” Phys. Rev. Lett. 88, 110201 (2002) arXiv:cond-mat/0110211.
[12] J. Cristófeno, S. Lepe and J. Saavedra, “Quasinormal modes of extremal BTZ black hole,” Class. Quant. Grav. 21, 2801 (2004), arXiv:hep-th/0402048.
[13] S. Basak, “Sound wave in vortex with sink,” arXiv:gr-qc/0310105.
[14] S. Hod, “Kerr black hole quasinormal frequencies,” Phys. Rev. D 67, 081501 (2003) arXiv:gr-qc/0301122.
[15] E. Berti, V. Cardoso and J. P. S. Lemos, “Quasinormal modes and classical wave propagation in analogue black holes,” Phys. Rev. D 70, 124006 (2004) arXiv:gr-qc/0408099.
[16] M. Bañados, C. Teitelboim and J. Zanelli, Phys. Rev. Lett. 69, 1819 (1992), arXiv:hep-th/9204099.
[17] T. Padmanabhan, Class. Quant. Grav. 21, L1 (2004) arXiv:gr-qc/0310027.
[18] T. R. Choudhury and T. Padmanabhan, Phys. Rev. D 69, 064033 (2004) arXiv:gr-qc/0311064.
[19] J. S. F. Chan and R. B. Mann, “Scalar wave falloff in asymptotically anti-de Sitter backgrounds,” Phys. Rev. D 55, 7546 (1997) arXiv:gr-qc/9612026.
[20] M. R. Setare, “Non-rotating BTZ black hole area spectrum from quasi-normal modes,” Class. Quant. Grav. 21, 1453 (2004) arXiv:hep-th/0311221.
[21] M. R. Setare, “Area spectrum of extremal Reissner-Nordstroem black holes from quasi-normal modes,” Phys. Rev. D 69, 044016 (2004) arXiv:hep-th/0312061.
[22] S. Das, H. Mukhopadhyay and P. Ramadevi, “Spectrum of rotating black holes and its implications for Hawking radiation,” Class. Quant. Grav. 22, 453 (2005) arXiv:hep-th/0407151.
[23] M. R. Setare and E. C. Vagenas, “Area spectrum of Kerr and extremal Kerr black holes from quasinormal modes,” arXiv:hep-th/0401187.
[24] J. Gamboa and F. Méndez, “Scattering in three dimensional extremal black holes,” Class. Quant. Grav. 18, 225 (2001), arXiv:hep-th/0008020.
[25] S. Lepe, F. Méndez, J. Saavedra and L. Vergara, “Fermions scattering in a three dimensional extreme black hole background,” Class. Quant. Grav. 20, 2417 (2003), arXiv:hep-th/0302035.
[26] S. Lepe and J. Saavedra, “Super-radiance in the three dimensional extreme black hole”, in preparation.
[27] N. Bogoliubov and A. Stegun, Handbook of mathematical functions, (Dover Publications, New York, 1970).