A generalized Kowalevski Hamiltonian and new integrable cases on $e(3)$ and $so(4)$.

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1 A new integrable case for the Kirchhoff equation.

The Kirchhoff equations (see for example [1]) for the motion of a rigid body in an ideal fluid read as follows

$$M_t = M \times \frac{\partial H}{\partial M} + \Gamma \times \frac{\partial H}{\partial \Gamma}, \quad \Gamma_t = \Gamma \times \frac{\partial H}{\partial M},$$

(1)

where $M = (M_1, M_2, M_3)$ and $\Gamma = (\gamma_1, \gamma_2, \gamma_3)$ are three dimensional vectors, and $\times$ stands for the vector product. Without loss of generality the Hamiltonian $H$ can be taken in the form

$$H = \sum a_i M_i^2 + \sum b_{ij} (\gamma_i M_j + \gamma_j M_i) + \sum c_{ij} \gamma_i \gamma_j.$$

(2)

Apart from $H$, for arbitrary values of parameters $a_i, b_{ij}, c_{ij}$ equation (1) has the following first integrals

$$I_1 = \gamma_1^2 + \gamma_2^2 + \gamma_3^2, \quad I_2 = M_1 \gamma_1 + M_2 \gamma_2 + M_3 \gamma_3.$$

(3)
These integrals are Casimirs for the Poisson structure

\[
\{M_i, M_j\} = \varepsilon_{ijk} M_k, \quad \{M_i, \gamma_j\} = \varepsilon_{ijk} \gamma_k, \quad \{\gamma_i, \gamma_j\} = 0 \quad (4)
\]
on \epsilon(3), which corresponds (see [2]) to equations (1). Therefore for integrability of (1) we need one additional first integral \(I_3\), functionally independent of \(H, I_1, I_2\).

There are classical integrable cases found by Kirchhoff, Clebsch and Steklov-Lyapunov [3]. For all these cases the matrices \(B = \{b_{ij}\}\) and \(C = \{c_{ij}\}\) are diagonal and the Hamiltonian is of the form

\[
H = a_1 M_1^2 + a_2 M_2^2 + a_3 M_3^2 + 2b_{11} M_1 \gamma_1 + 2b_{22} M_2 \gamma_2 + 2b_{33} M_3 \gamma_3 + \\
\quad c_{11} \gamma_1^2 + c_{22} \gamma_2^2 + c_{33} \gamma_3^2.
\]
The Kirchhoff case is described by the relations

\[
a_1 = a_2, \quad b_{11} = b_{22}, \quad c_{11} = c_{22}.
\]
For the Clebsch and Steklov-Lyapunov cases the coefficients \(a_i\) are arbitrary and the remaining parameters satisfy the following conditions

\[
b_{11} = b_{22} = b_{33},
\]

\[
\frac{c_{11} - c_{22}}{a_3} + \frac{c_{33} - c_{11}}{a_2} + \frac{c_{22} - c_{33}}{a_1} = 0
\]

and

\[
\frac{b_{11} - b_{22}}{a_3} + \frac{b_{33} - b_{11}}{a_2} + \frac{b_{22} - b_{33}}{a_1} = 0,
\]

\[
c_{11} - \frac{(b_{22} - b_{33})^2}{a_1} = c_{22} - \frac{(b_{33} - b_{11})^2}{a_2} = c_{33} - \frac{(b_{11} - b_{22})^2}{a_3},
\]
respectively. For each of these three cases there exists an additional quadratic integral.

In the paper [4] the following Hamiltonian

\[
H = M_1^2 + M_2^2 + 2M_3^2 + 2M_5 (c_1 \gamma_1 + c_2 \gamma_2) + 2(c_1 M_1 + c_2 M_2) \gamma_3 + \\
\quad 4(c_2 \gamma_1 - c_1 \gamma_2)^2 - 4(c_1^2 + c_2^2) \gamma_3^2
\]
was considered. It turns out that (5) is in involution with some polynomial \(I_3\) of fourth degree with respect to brackets (4) or, equivalently, the corresponding Kirchhoff equations (1) possess an additional fourth degree integral.
Obviously, $I_3$ is defined up to a quadratic form of $I_1$, $I_2$, and $H$. In [4] this form has been chosen in such a way that $I_3$ depends on a minimal number of new variables. These variables

\begin{align*}
x_1 &= M_3 - c_1 \gamma_1 - c_2 \gamma_2, \quad x_2 = M_1 + c_1 \gamma_3, \quad x_3 = M_2 + c_2 \gamma_3, \\
x_4 &= 3(c_2 \gamma_1 - c_1 \gamma_2), \quad x_5 = M_3 + 2c_1 \gamma_1 + 2c_2 \gamma_2, \quad x_6 = \gamma_3
\end{align*}

(6)

are very useful for all computations related to the Hamiltonian (5).

In terms of these variables the additional integral for the Kirchhoff equations with the Hamiltonian (5) has the form

\[ I_3 = x_1^2 P + Q^2, \]

(7)

where

\[ P = (c_1^2 + c_2^2) x_5^2 + (c_2 x_2 - c_1 x_3)^2, \quad Q = c_2 x_2 x_4 + c_1 x_2 x_5 - c_1 x_3 x_4 + c_2 x_3 x_5. \]

It follows from these formulas that $I_3$ does not depend on the variable $x_6$.

Notice that one of the quadratic integrals becomes specially simple in the variables (5):

\[ H + \frac{5}{2}(c_1^2 + c_2^2) I_1 = \frac{1}{2}(x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2). \]

Simultaneous linear transformations of $M$ and $\Gamma$ with the orthogonal matrix of the form

\[ T = \begin{pmatrix} \cos \phi & \sin \phi & 0 \\ -\sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \]

(8)

preserve the form (5) of $H$ while changing the parameters as follows

\[ \bar{c}_1 = \cos \phi \ c_1 - \sin \phi \ c_2, \quad \bar{c}_2 = \sin \phi \ c_1 + \cos \phi \ c_2. \]

Therefore the only invariant is $\delta = c_1^2 + c_2^2$.

The parameter $\delta$ can be normalized by a rescaling of $\gamma_i$. If $\delta \neq 0$, we can take $c_1 = 1$, $c_2 = 0$. In addition, there exists a particular (complex) case

\[ c_2 = i c_1. \]

Here one can put $c_1 = 1$.

According to a classification theorem (see [4]), the Kirchhoff equation with Hamiltonian (5) is the only integrable case with $a_1 = a_2 \neq a_3$ and an additional first integral of fourth degree.

It was noticed by Borisov and Mamaev (see [5]) that the change of variables $\bar{\gamma}_i = \gamma_i$

\begin{align*}
\bar{M}_1 &= M_1 + c_1 \gamma_3, \quad \bar{M}_2 = M_2 + c_2 \gamma_3, \quad \bar{M}_3 = M_3 - c_1 \gamma_1 - c_2 \gamma_2,
\end{align*}

(9)
defined by the first three of the variables (3), preserves brackets (4) and
simplifies the Hamiltonian. Namely subtracting $4(c_1^2 + c_2^2)I_1$ from (5) and
performing transformation (9), we get a new form of the Hamiltonian

$$\bar{H} = M_1^2 + M_2^2 + 2M_3^2 + 2(a_1 \gamma_1 + a_2 \gamma_2)M_3 - (a_1^2 + a_2^2)\gamma_3^2,$$  \hspace{1cm} (10)

where we have put $a_1 = 3c_1$, $a_2 = 3c_2$. In the sequel we shall use this more
elegant form of the Hamiltonian.

Borisov and Mamaev [5] have observed also that the freedom in the defi-
nition of $I_3$ can be used to bring $I_3$ to a factorized form

$$I_3 = \bar{k}_1 \bar{k}_2,$$  \hspace{1cm} (11)

where we have put $a_1 = 3c_1$, $a_2 = 3c_2$. In the sequel we shall use this more
elegant form of the Hamiltonian.

It turns out that both $k_1 = 0$ and $k_2 = 0$ are invariant submanifolds. This
fact follows from the following formulas:

$$\dot{k}_1 = 2(a_2 \gamma_1 - a_1 \gamma_2)k_1, \quad \dot{k}_2 = -2(a_2 \gamma_1 - a_1 \gamma_2)k_2.$$  

The particular case $a_1^2 + a_2^2 = 0$ is simpler than the generic case. For
this complex case a Lax representation has been found in [4]. For the case
$a_1^2 + a_2^2 \neq 0$ Lax representations have not been found yet.

\section{The corresponding integrable case on $so(4)$.}

Let us consider the standard deformation

$$\{M_i, M_j\} = \varepsilon_{ijk} M_k, \quad \{M_i, \gamma_j\} = \varepsilon_{ijk} \gamma_k, \quad \{\gamma_i, \gamma_j\} = x \varepsilon_{ijk} M_k.$$  \hspace{1cm} (11)

of bracket (4), where $x$ is an arbitrary parameter. It is well known that if
$x > 0$ then (11) defines a Poisson bracket on $so(4)$. The Casimirs of this
bracket are given by

$$I_1 = x(M_1^2 + M_2^2 + M_3^2) + (\gamma_1^2 + \gamma_2^2 + \gamma_3^2), \quad I_2 = M_1 \gamma_1 + M_2 \gamma_2 + M_3 \gamma_3.$$  \hspace{1cm} (12)

The main result of the paper [5] consists in the following observation. The Hamiltonian (10) possesses a commuting integral of fourth degree $\bar{I}_3 = \bar{k}_1 \bar{k}_2$, where

$$\bar{k}_1 = k_1, \quad \bar{k}_2 = k_2 + x(a_2 M_1 - a_1 M_2)^2 M_3.$$  \hspace{1cm} 4
This integrable deformation of the integrable case from Section 1 has been found independently by Sokolov and Borisov-Mamaev. Here we present the most elegant form of the deformed additional integral proposed by Borisov and Mamaev.

Note that the dynamical system with Hamiltonian (10) and brackets (11) has just two one-parameter families of solutions of the form $M_i = X_i t^{-1}$, $\gamma_i = Y_i t^{-1}$. First of them is defined by

$$X_3 = Y_3 = 0, \quad Y_1^2 + Y_2^2 = 0, \quad 2a_1 Y_2 - 2a_2 Y_1 = 1, \quad X_1 = 2X_2 (a_1 Y_1 + a_2 Y_2).$$

For the second family we have

$$X_1 = a_1 Y_3, \quad X_2 = a_2 Y_3, \quad X_3 = -a_1 Y_1 - a_2 Y_2,$$

$$(1 + x (a_1^2 + a_2^2)) \left( Y_1^2 + Y_2^2 + Y_3^2 \right) = \frac{x}{4}, \quad 2a_1 Y_2 - 2a_2 Y_1 = -1.$$}

For both families the Kowalewski exponents are $\{-1, 0, 1, 2, 2\}$.

### 3 Generalizations.

This Section contains new results concerning possible integrable generalizations of the Hamiltonian (10).

It turns out that extra linear terms can be added to the Hamiltonian (10). The most general form of such terms is given by

$$H = M_1^2 + M_2^2 + 2 M_3^2 + 2 (a_1 \gamma_1 + a_2 \gamma_2) M_3 - (a_1^2 + a_2^2) \gamma_3^2 + k_1 (M_3 + a_1 \gamma_1 + a_2 \gamma_2) + k_2 (a_1 \gamma_2 - a_2 \gamma_1), \quad (13)$$

where $k_i$ are arbitrary constants. This Hamiltonian possesses an additional first integral of fourth degree on $so(4)$. Historically, the author found first that this is true for Hamiltonian (10) on $e(3)$ with $k_1 = 0$. After that, independently, V.Kuznetsov and A.Tsiganov have informed the author that the same Hamiltonian is integrable on $so(4)$. Finally, the term with non-zero $k_1$ was added by the author very recently.

Since the explicit expression for the additional integral $I_4$ is rather long, we put $a_1 = 0$ in the Hamiltonian (13) and rewrite it as follows

$$\tilde{H} = M_1^2 + M_2^2 + 2 M_3^2 + 2 \lambda_1 \gamma_2 M_3 - \lambda_1^2 \gamma_3^2 + 2 \lambda_2 (M_3 + \lambda_1 \gamma_2) + 2 \lambda_3 \gamma_1, \quad (14)$$

where $\lambda_i$ are arbitrary constants. If $\lambda_1 = \lambda_2 = 0$, then the Hamiltonian reduces just to the famous Kowalewski Hamiltonian. The case $\lambda_1 = 0$ corresponds to the Kowalewski Hamiltonian with the additional gyrostatic term.
For the shortened Hamiltonian (14) the additional integral has the form
\[ I_4 = M_3^2 (M_1^2 + M_2^2 + M_3^2) + \lambda_1 M_3 (2M_1 M_2 \gamma_1 + 2M_2^2 \gamma_2 + 2M_3^2 \gamma_2 + \lambda_1 M_3^2 \gamma_2^2 - 2\lambda_1 M_1 \gamma_1 \gamma_3 - 2\lambda_1 M_2 \gamma_2 \gamma_3 - \lambda_1 M_3 \gamma_3^2) + 2\lambda_1 (M_3 (M_1^2 + M_2^2 + M_3^2) + \lambda_1 M_1 M_2 \gamma_1 + \lambda_1 M_2^2 \gamma_2 + 2\lambda_1 M_3^2 \gamma_2 + \lambda_1^2 M_3^2 \gamma_2^2 - 2\lambda_1 \gamma_2^2 - 2\lambda_2 \lambda_3 M_1 \gamma_3 - 2\lambda_1 \lambda_2 \lambda_3 M_2 \gamma_3 + 2\lambda_1^2 \lambda_2 \lambda_3^2 \gamma_3^2 - 2\lambda_3^2 \gamma_3^2 - 2\lambda_3 \gamma_3 (\gamma_1 + \gamma_1 \gamma_2) + x (2M_1^2 M_3^2 + 2\lambda_1^2 \lambda_2 M_2^2 M_3 - 2\lambda_1 \gamma_1 M_1 M_2 M_3 + \lambda_1^2 \lambda_2 M_2^2 M_3 - 2\lambda_1 \gamma_1 M_1 M_2 M_3 - \lambda_1^2 M_3^2) \].

To get the integral \( I_4 \) on \( e(3) \) we can simply put \( x = 0 \) in this formula. The explicit form of the integral \( I_4 \) on \( so(4) \) for (13) can be easily reconstructed with the help of transformation (8).

For the next generalization let us consider the following Hamiltonian
\[ \tilde{H} = M_1^2 + M_2^2 + 2 M_3^2 + 2 (a_1 \gamma_1 + a_2 \gamma_2) M_3 - (a_1^2 + a_2^2) \gamma_3^2 + F(\gamma_1, \gamma_2). \] (15)
on \( e(3) \). It is not difficult to verify that the corresponding Kirchhoff equations possess an additional integral of fourth degree of the form \( \tilde{I}_3 = k_1 k_2 + T \), where \( T \) is a cubic polynomial in \( M_1, M_2, M_3 \) with coefficients dependent on \( \gamma_1, \gamma_2 \) iff
\[ F = 2\lambda_1 (a_2 \gamma_1 - a_1 \gamma_2) + \frac{\lambda_2}{\sqrt{\gamma_1^2 + \gamma_2^2}}, \]
where \( \lambda_i \) are arbitrary constants. In this case the function \( T \) reads as follows
\[ T = 2a_2 \lambda_1 \gamma_1 M_1^2 + 2\lambda_1 (a_2 \gamma_2 - a_1 \gamma_1) M_1 M_2 - 2a_1 \lambda_1 \gamma_2 M_2^2 + \left(2a_2 \lambda_1 \gamma_1 - 2a_1 \lambda_1 \gamma_2 + \frac{\lambda_2}{\sqrt{\gamma_1^2 + \gamma_2^2}}\right) M_3^2 + (a_1 \gamma_1 + a_2 \gamma_2) \left(2a_2 \lambda_1 \gamma_1 - 2a_1 \lambda_1 \gamma_2 + \frac{\lambda_2}{\sqrt{\gamma_1^2 + \gamma_2^2}}\right) M_3 - \lambda_1 \left(\lambda_1 (a_1 \gamma_1 + a_2 \gamma_2) + (a_1 \gamma_2 - a_2 \gamma_1) \frac{\lambda_2}{\sqrt{\gamma_1^2 + \gamma_2^2}}\right). \]
Performing the scaling $\gamma_i \rightarrow \varepsilon \gamma_i$ and redefining the constants $\lambda_i$, we get the following integrable Hamiltonian

\[
\tilde{H} = M_1^2 + M_2^2 + 2M_3^2 + 2\varepsilon (a_1 \gamma_1 + a_2 \gamma_2) M_3 - \varepsilon^2 (a_1^2 + a_2^2) \gamma_3^2 + 2\lambda_1 (a_2 \gamma_1 - a_1 \gamma_2) + \frac{\lambda_2}{\sqrt{\gamma_1^2 + \gamma_2^2}}.
\]

The case $\varepsilon = 0$ corresponds to a known generalization of the Kowalewski Hamiltonian found by H.M. Vehia \[6\].

Very recently A. Tsiganov and the author have found (see \[7\]) the following generalization of the Goryachev-Chaplygin Hamiltonian:

\[
H = M_1^2 + M_2^2 + 4M_3^2 + 2\lambda_1 \gamma_1 + 2\lambda_2 \gamma_2 + \lambda_3 M_3 + 4(a_1 \gamma_1 + a_2 \gamma_2) M_3 - (a_1^2 + a_2^2) \gamma_3^2.
\]  \hspace{1cm} (17)

On the fixed level $I_2 = 0$ of the Casimir $I_2$ the Hamiltonian $H$ commutes with

\[
I_3 = (M_1^2 + M_2^2) M_3 - 2(a_1 M_1 + a_2 M_2) M_3 \gamma_3 + (a_1^2 + a_2^2) M_3 \gamma_3^2 - 4\lambda_3 M_3^2 - 4\lambda_3 (a_1 \gamma_1 + a_2 \gamma_2) M_3 - (\lambda_1 M_1 + \lambda_2 M_2) \gamma_3 + (a_1 \lambda_1 + a_2 \lambda_2) \gamma_3^2 - 4\lambda_2 M_3 - 2\lambda_3 (a_1 \gamma_1 + a_2 \gamma_2).
\]

If $a_1 = a_2 = \lambda_3 = 0$, then (17) coincides with the Goryachev-Chaplygin Hamiltonian. The integrable case $a_1 = a_2 = 0$, $\lambda_3 \neq 0$ was found in (\[8\]).

4 Historical remarks.

A fourth degree integral for the Kirchhoff equation on $e(3)$ had been found by S.A. Chaplygin under the additional assumption that the scalar product $(M, \Gamma)$ is equal to zero \[9, 10\]. The corresponding partially integrable case on $so(4)$ was found by O.I. Bogoyavlenski \[11\].

An integrable case on $so(4)$ with an integral of fourth degree was found by M. Adler and P. van Moerbeke \[12\]. A Lax pair for that case has been constructed by A. Reyman and M. Semenov-Tian-Shansky \[13\]. The fact that $k_1 = 0$ is the invariant relation has been used in \[4\] to prove that the Kirchhoff equation with Hamiltonian (5) is not linearly equivalent to the Adler-van Moerbeke-Reyman-Semenov-tian-Shansky case on $so(4)$.

R. Liouville \[14\] has obtained necessary conditions for the existence of algebraic integrals for the Kirchhoff equation with non-diagonal matrix $B$. 

7
In recent investigations of algebraic integrability [15] it was assumed that all matrices $A, B, C$ are diagonal. In [16, 17] the matrix $A$ was assumed to be defined by the inertial tensor $I$ of a real rigid body: $A = I^{-1}$, and all entries $a_i$ to be distinct. The latter restriction has been also imposed in the papers [18, 19], devoted to integrability conditions of the equations on $so(4)$.

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