True Functional Integrals in Algebraic Quantum Field Theory

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The familiar generating functional

\[ N^{-1} \exp \left[ i \int L(\phi) \right] \prod_x [d\phi(x)] \] (1)

in QFT is introduced usually as formal extrapolation of measures in finite-dimensional spaces. However, this fails to be a true measure since the Lebesgue measure \( \prod_x [d\phi(x)] \) in infinite-dimensional linear spaces is not defined in general. Therefore, the properties of the generating functional (1) remain elusive what becomes essential when describing non-perturbative phenomena. The problem lies in constructing representations of topological *-algebras of quantum fields which are not normed [1, 5].

These difficulties may be overcome if we restrict our consideration only to chronological forms on quantum field algebras. In this case, since chronological forms of boson fields are symmetric, the algebra of quantum fields can be replaced with the commutative tenzor algebra of the corresponding infinite-dimensional nuclear space \( \Phi \). This is the enveloping algebra of the abelian Lie group of translations in \( \Phi \). The generating functions of unitary representations of this group [4] play the role of Euclidean generating functionals in algebraic QFT. They are the Fourier transforms of measures in the dual \( \Phi' \) to the space \( \Phi \) [3, 4, 10]. To describe realistic field models, one can construct these measures in terms of multimomentum canonical variables when momenta correspond to derivatives of fields with respect to all world coordinates [11, 12, 13]. In the present work, we spell out the algebraic aspects of the above-mentioned approach. As a test case, scalar field is examined. By analogy with the case of boson fields, the corresponding anticommutative algebra of fermion fields is defined to be the algebra of functions taking their values into an infinite Grassman algebra [5].
1 Quantum field algebras

In accordance with the algebraic approach, a quantum field system can be characterized
by a topological *-algebra $A$ (with the unit element $1_A$) and a continuous positive form
(state) $f$ on $A$, that is,

$$f(aa^*) \geq 0, \quad f(1_A) = 1, \quad a \in A.$$ 

With reference to the field-particle dualism, realistic models are described by tensor al-
gebras as a rule, and states are treated as vacuum expectation form.

Let $\Phi$ be a real linear locally convex topological space endowed with an involution
operation

$$* : \phi \to \phi^*, \quad \phi, \phi' \in \Phi.$$ 

The linear space

$$A_\Phi = \bigoplus_0^\infty \Phi_n, \quad \Phi_0 = \mathbb{R}, \quad \Phi_{n>0} = \bigoplus_0^\infty \Phi_n$$

is a $\mathbb{Z}$-graded *-algebra with respect to the natural tensor product and the involution
operation

$$(\phi_1 \cdots \phi_n)^* = (\phi_n)^* \cdots (\phi_1)^*, \quad \phi^i \in \Phi.$$ 

The algebra $A_\Phi$ can be provided with the direct sum topology when the sequence

$$\{a^i = (\phi^i_0, \ldots, \phi^i_n, \ldots), \phi^i_n \in \Phi\}$$

converges to 0 if, whenever $n$, the sequence $\{\phi^i_n\}$ converges to 0 with respect to the
topology of $\Phi_n$ and there exists a number $N$ independent of $i$ such that $\phi^i_n = 0$ for all $i$
and $n > N$. This topology bring $A_\Phi$ into a topological *-algebra.

A state $f$ on the tensor algebra $A_\Phi$ is given by the collection of continuous forms $\{f_n\}$
on spaces $\Phi_n$.

We further assume that $\Phi$ is a nuclear space.

In the axiomatic quantum field theory of real scalar fields, the quantum field algebra
is the Borchers algebra $A_\Phi$ where $\Phi = \mathbb{R}S^4$. By $\mathbb{R}S_m$ is meant the real subspace of the
nuclear Schwartz space $S(\mathbb{R}^m)$ of complex $C^\infty$ functions $\phi(x)$ on $\mathbb{R}^m$ such that

$$\|\phi\|_{k,l} = \max_{|\alpha| \leq l, x \in \mathbb{R}^m} \sup_{x \in \mathbb{R}^m} (1 + |x|)^k \frac{\partial^{|\alpha|}}{(\partial x^1)^{\alpha_1} \cdots (\partial x^m)^{\alpha_m}} \phi(x),$$

is finite for any collection $(\alpha_1, \ldots, \alpha_m)$ and all $l, k \in \mathbb{Z}^{\geq 0}$. The space $\mathbb{R}S_m$ is endowed
with the set of seminorms $\|\phi\|_{k,l}$ and the associated topology. It is reflexive. The dual to
$S(\mathbb{R}^n)$ is the space of temperate distributions (or simply distributions) $S'(\mathbb{R}^{4n})$.

The following properties of $S(\mathbb{R}^n)$ motivates us to choose it in order to construct
quantum field algebras. The translation operator

$$x \to x + a, \quad x \in \mathbb{R}^n,$$
in \( S(\mathbb{R}^n) \) has the complete set of generalized eigen vectors \( \exp(ipx) \in S'(\mathbb{R}^n) \). Moreover, \( S(\mathbb{R}^n) \) and \( S'(\mathbb{R}^n) \) are isomorphic to their Fourier transforms \( S(\mathbb{R}_n) \) and \( S'(\mathbb{R}_n) \) respectively:

\[
\phi^F(p) = \int \phi(x) \exp(ipx) d^n x, \\
\phi(x) = \int \phi^F(p) \exp(-ipx) d_n p
\]

where \( \mathbb{R}_n = (\mathbb{R}^n)' \) and \( d_n = (2\pi)^{-2n} d^n \).

Remark that the subset \( \otimes S(\mathbb{R}^n) \) is dense in \( S(\mathbb{R}^{kn}) \) and every continuous form on it is uniquely extended to a continuous form on \( S(\mathbb{R}^{kn}) \). Every bilinear functional \( M(\phi_1, \phi_2) \) which is separately continuous with respect to \( \phi_1 \in S(\mathbb{R}^n) \) and \( \phi_2 \in S(\mathbb{R}^m) \) is expressed uniquely into the form

\[
M(\phi_1, \phi_2) = \int F(x, y)\phi_1(x)\phi_2(y)d^n x d^m y, \quad F \in S'(\mathbb{R}^{n+m}).
\]

When provided with the nondegenerate separately continuous Hermitian form

\[
\langle \phi_1 | \phi_2 \rangle = \int \phi_1(x) \overline{\phi_2}(x) d^n x, \quad (2)
\]

the nuclear space \( S(\mathbb{R}^n) \) is the rigged Hilbert space.

As a consequence, every state \( f = \{f_n\} \) on the Borchers algebra \( A_{\mathbb{R}^S_4} \) is represented by a family of temperate distributions \( W_n \in S'(\mathbb{R}^{4n}) \):

\[
f(\phi_1 \cdots \phi_n) = \int W_n(x_1, \ldots, x_n)\phi_1(x_1) \cdots \phi_n(x_n) d^4 x_1 \cdots d^4 x_n.
\]

In particular, if \( f \) obeys the Wightman axioms, \( W_n \) are the familiar \( n \)-point Wightman functions.

Let us consider the states on the Borchers algebra \( A_{\mathbb{R}^S_4} \) which describe free massive scalar fields.

Set the following complex positive bilinear form on \( A_{\mathbb{R}^S_4} \):

\[
\langle \phi | \phi' \rangle = \frac{2}{i} \int d^3 x d^4 y \phi(x) D^-(x - y) \phi(y) = \int d^3 \omega \frac{d^3 p}{\omega} \phi^F(\omega, \mathbf{p}) \overline{\phi^F}(\omega, \mathbf{p}), \quad (3)
\]

\[
D^-(x) = i(2\pi)^{-3} \int d^4 p \exp(-ipx) \theta(p_0) \delta(p^2 - m^2),
\]

where \( \theta \) is the step function, \( p^2 \) the Minkowski scalar and

\[
\omega = (p^2 + m^2)^{1/2}.
\]

Recall that, since the functions \( \phi \) are real, we have

\[
\phi^F(p) = \overline{\phi^F}(-p).
\]
The form (3) is degenerate, for $D^-$ satisfies the massive surface equation
\[(\square + m^2)D^-(x) = 0.\]

Let us consider the quotient
\[\gamma : \mathbb{R}S_4 \to \mathbb{R}S_4/I\]
of $\mathbb{R}S_4$ by the kernel
\[I := \{\phi \in \mathbb{R}S_4 : (\phi|\phi) = 0\}\]
of the form (3). The transformation $\gamma$ sends every element $\phi \in \mathbb{R}S_4$ whose Fourier transform is $\phi^F(p_0, \overline{p}) \in S(\mathbb{R}_4)$ to the pair of functions $(\phi^F(\omega, \overline{p}), \phi^F(-\omega, \overline{p}))$. Set the bilinear form
\[(\gamma\phi|\gamma\phi')_L = \Re(\phi|\phi') = \frac{1}{2} \int \frac{d^3p}{\omega} \phi(\omega, \overline{p})\phi'(-\omega, -\overline{p}) + \phi(-\omega, \overline{p})\phi'(\omega, -\overline{p}).\] (4)

The space $\mathbb{R}S_4/I$ is the direct sum of the subspaces
\[L^\pm = \{\phi^F(\omega, \overline{p}) = \pm \phi^F(-\omega, \overline{p})\}\]
which are orthonormal to each other with respect to the form (4).

There are exist continuous isometric mappings
\[\gamma_+ : \phi_+^F(\omega, \overline{p}) \to q(\overline{p}) = \omega^{-1/2}\phi_+^F(\omega, \overline{p}),\]
\[\gamma_- : \phi_-^F(\omega, \overline{p}) \to q(\overline{p}) = -i\omega^{-1/2}\phi_-^F(\omega, \overline{p})\]
of spaces $L^+$ and $L^-$ to the space $\mathbb{R}S_3$ endowed with the separately continuous positive nondegenerate Hermitian form
\[\langle q|q' \rangle = \int d^3pq^F(\overline{p})q^F(\overline{p}).\]

Note that the images of $\gamma_+(L^+)$ and $\gamma_-(L^-)$ in $\mathbb{R}S_3$ fail to be orthonormal. The superpositions of $\gamma$ and $\gamma_\pm$ yield the mappings
\[\tau_+(\phi) = \gamma_+(\gamma\phi)_+ = \frac{1}{2\omega^{1/2}} \int [\phi^F(\omega, \overline{p}) + \phi^F(-\omega, \overline{p})] \exp(-i\overline{p}\overline{x})d^3p,\]
\[\tau_-(\phi) = \gamma_+(\gamma\phi)_- = \frac{1}{2i\omega^{1/2}} \int [\phi^F(\omega, \overline{p}) - \phi^F(-\omega, \overline{p})] \exp(-i\overline{p}\overline{x})d^3p,\]
of the space $\mathbb{R}S_4$ to $\mathbb{R}S_3$.

Now, let us consider the algebra CCR
\[K_{\mathbb{R}S_3} = \{\pi(q), s(q), q \in \mathbb{R}S_3\}\]
and set the morphism
\[\mathbb{R}S_4 \ni \phi \to a(\phi) = s(\tau_+(\phi)) - \pi(\tau_-(\phi)).\]
One can think of the algebra $K_{RS_3}$ as being the algebra of simultaneous canonical commutation relations of fields $\phi \in \mathbb{R}S_4$. Every representation of this algebra CCR \cite{4} determines the positive form

$$f(\phi_1...\phi_n) = \int [s(\tau_+(\phi_1)) - \pi(\tau_-(\phi_1))] \cdots [s(\tau_+(\phi_n)) - \pi(\tau_-(\phi_n))]d\mu$$

(5)
on the Borchers algebra $A_{\mathbb{R}S_4}$. The corresponding distributions $W_n$ obey the massive surface equation, and the following commutation relations hold:

$$W_2(x, y) - W_2(y, x) = -iD(x - y)$$

where

$$D(x) = i(2\pi)^{-3} \int d^4p[\theta(p_0) - \theta(-p_0)]\delta(p^2 - m^2)\exp(-ipx)$$

is the Pauli-Iordan function. It follows that the form (3) describes massive scalar fields. In case of the Fock representation \{\pi(q), s(q)\} of the algebra CCR $K_{RS_3}$, the form (3) obeys the Wick relations

$$f_n(\phi_1 \cdots \phi_n) = \sum f_2(\phi_{i_1}\phi_{i_2}) \cdots f_2(\phi_{i_{n-1}}\phi_{i_n})$$

where the sum is over all pairs $i_1 < i_2, ..., i_{n-1} < i_n$ and $f_2$ is the Wightman function

$$W_2(x, y) = \frac{1}{i}D^-(x - y).$$

In this case, the form (3) describes free massive scalar fields.

2 Chronological forms

In QFT, particles created at some moment and destructed at another moment are described by the chronological forms $f^c$ on the Borchers algebra $A_{\mathbb{R}S_4}$. They are given by the expressions

$$W^c_n(x_1, \ldots, x_n) = \sum_{(i_1, \ldots, i_n)} \theta(x_{i_1}^0 - x_{i_2}^0) \cdots \theta(x_{i_{n-1}}^0 - x_{i_n}^0)W_n(x_1, \ldots, x_n)$$

(6)
where $(i_1 \ldots i_n)$ is a rearrangement of numbers $1, \ldots, n$. However, the forms (3) fail to be distributions in general and so, $f^c$ are not continuous forms on $A_{\mathbb{R}S_4}$. For instance, $W_1^c \in S'({\mathbb{R}})$ iff $W_1 \in S'({\mathbb{R}}_{\infty})$ where by $\mathbb{R}_{\infty}$ is meant the compactification of $\mathbb{R}$ by identification of the points $\{\infty\}$ and $\{-\infty\}$. Moreover, the chronological forms $f^c$ are not positive. Thus, they are not states on the Borchers algebra. At the same time, they issue from the Wick rotation of the Euclidean states on $A_{\mathbb{R}S_4}$ (see Appendix) which describe particles in the interaction zone.

Since chronological forms (3) are symmetric, the Euclidean states on the Borchers algebra $A_{\mathbb{R}S_4}$ can be introduced as states on the corresponding commutative tensor algebra.
$B_{RS_4}$. Note that they differ from the Swinger functions associated with the Wightman functions in the framework of the constructive Euclidean quantum field theory [14].

Let $B_{\Phi}$ be the complexified quotient of $A_{\Phi}$ by the ideal whose generating set is

$$J := \{ \phi \phi' - \phi' \phi, \phi, \phi' \in \Phi \}.$$ 

This algebra is exactly the enveloping algebra of the Lie algebra associated with the commutative Lie group $G_{\Phi}$ of translations in $\Phi$. We therefore can construct a state on the algebra $B_{\Phi}$ as a vector form of its cyclic representation induced by a strong-continuous unitary cyclic irreducible representation of $G_{\Phi}$. Such a representation is characterized by a positive-type continuous generating function $Z$ on $\Phi$, that is,

$$Z(\phi_i - \phi_j) \alpha^i \alpha^j \geq 0, \quad Z(0) = 1,$$

for all collections of $\phi_1, \ldots, \phi_n$ and complex numbers $\alpha^1, \ldots, \alpha^n$. If the function

$$\alpha \rightarrow Z(\alpha \phi)$$

on $\mathbb{R}$ is analytic at $0$ for each $\phi \in \Phi$, the state $F$ on $B_{\Phi}$ is given by the expressions

$$F_n(\phi_1 \cdots \phi_n) = i^{-n} \frac{\partial}{\partial \alpha^1} \cdots \frac{\partial}{\partial \alpha^n} Z(\alpha^i \phi_i)|_{\alpha^i = 0}.$$ 

In virtue of the well-known theorem [4], any function $Z$ of the above-mentioned type is the Fourier transform of a positive bounded measure $\mu$ in the dual $\Phi'$ to $\Phi$:

$$Z(\phi) = \int_{\Phi'} \exp[i \langle w, \phi \rangle] d\mu(w)$$

(7)

where $\langle, \rangle$ denotes the contraction between $\Phi'$ and $\Phi$. The corresponding representation of $G_{\Phi}$ is given by operators

$$\pi_\mu(g(\phi)) : u(w) \rightarrow \exp[i \langle w, \phi \rangle] u(w)$$

in the space $L^2(\Phi', \mu)$ of quadratically $\mu$-integrable functions $u(w)$ on $\Phi'$. We have

$$F_n(\phi_1 \cdots \phi_n) = \int \langle w, \phi_1 \rangle \cdots \langle w, \phi_n \rangle d\mu(w).$$

It should be noted that the representations $\pi_\mu$ and $\pi_{\mu'}$ are not equivalent if the measures $\mu$ and $\mu'$ are not equivalent.

For instance, free and quasi-free fields are described by Gaussian states $F$. A generating function $Z$ of a Gaussian state $F$ on $B_{\Phi}$ reads

$$Z(\phi) = \exp[-\frac{1}{2} M(\phi, \phi)]$$

(8)
where the covariance form $M(\phi_1, \phi_2)$ is a positive-definite Hermitian separately continuous bilinear form on $\Phi$. This generating function is the Fourier transform of a Gaussian measure in $\Phi$. The forms $F_{n>2}$ obey the Wick rules where

$$F_1 = 0, \quad F_2(\phi_1, \phi_2) = M(\phi_1, \phi_2).$$

In particular, if $\Phi = \mathbb{R}S_n$, the covariance form of a Gaussian state is uniquely defined by a distribution $W \in S'(\mathbb{R}^{2n})$:

$$M(\phi_1, \phi_2) = \int W(x_1, x_2)\phi_1(x_1)\phi_2(x_2) d^n x_1 d^n x_2.$$

In field models, a generating function $Z$ plays the role of a generating functional represented by the functional integral (7). If $Z$ is the Gaussian generating function (8), its covariance form $M$ defines Euclidean propagators. Propagators of fields on the Minkowski space are reconstructed by the Wick rotation of $M$.

### 3 Scalar fields

In particular, set $\Phi = \mathbb{R}S_4$. Let $F$ be a Gaussian state on the algebra $B_{\mathbb{R}S_4}$ such that its covariance form is represented by a distribution $M(\phi, \phi') \in S'(\mathbb{R}^8)$ which is the Green function of some positive elliptic differential operator

$$L_y M(y, y') = \delta(y - y').$$

This Gaussian state describes quasi-free Euclidean fields with the propagator $M(y, y')$. For instance, if

$$L_y = -\Delta_y + m^2, \quad M(y, y') = \int \frac{\exp(-iq(y - y'))}{q^2 + m^2} d_4 q,$$

where $q^2$ is the Euclidean scalar, the Gaussian state $F$ describes free Euclidean fields.

Note that, in case of the infinite-dimensional linear spaces, the measure $\mu$ in the expression (7) fails to be brought into explicit form. The familiar expression

$$\exp(-S[\phi] \prod_x [d\phi(x)])$$

is not a true measure. At the same time, the measure $\mu$ in $\Phi'$ is uniquely defined by the set of measures $\mu_N$ in the finite-dimensional spaces

$$\mathbb{R}_N = \Phi'/E$$

where $E \subset \Phi'$ denotes the subspace of forms on $\Phi$ which are equal to zero at some finite-dimensional subspace $\mathbb{R}_N^N \subset \Phi$. The measures $\mu_N$ are images of $\mu$ under the canonical
mapping $\Phi' \to \mathbb{R}_N$. For instance, every vacuum expectation $F(\phi_1 \cdots \phi_n)$ admits the representation by functional integral

$$F(\phi_1 \cdots \phi_n) = \int \langle w, \phi_1 \rangle \cdots \langle w, \phi_n \rangle d\mu_N(w)$$

(10)

for any finite-dimensional subspace $\mathbb{R}_N$ which contains $\phi_1, \ldots, \phi_n$. In particular, one can replace the generating function (8) by the generating function

$$Z_N(\lambda_i e^i) = \int \exp(i\lambda_i w^i)\mu_N(w^i)$$

on $\mathbb{R}_N$ where $\{e^i\}$ is a basis for $\mathbb{R}_N$ and $\{w^i\}$ are coordinates with respect to the dual basis for $\mathbb{R}_N$. If $F$ is a Gaussian state, we have the familiar expression

$$d\mu_N = (2\pi \det[M^{ij}])^{-N/2} \exp[-\frac{1}{2}(M^{-1})_{ij}w^iw^j]d^Nw$$

(11)

where $M^{ij} = M(e^i, e^j)$ is the non-degenerate covariance matrix.

The representation (10) however is not unique, and the measure $\mu_N$ depends on the specification of the finite-dimensional subspace $\mathbb{R}_N$ of $\Phi$.

Note that the Gaussian measure (11) admits the following countably infinite generalization.

Let $\{e_i, i \in I\}$ be an orthonormal basis of $\Phi$ with respect to some scalar form $\langle | \rangle$ on $\Phi$. Let $\mathbb{R}^I$ denotes a linear space of real functions $\phi$ on the set $I$ and $\mathbb{R}^I$ be provided with the simple convergence topology. The collection of forms

$$e^i : \mathbb{R}^I \ni \phi \to \phi(i) \in \mathbb{R}, \quad i \in I,$$

is an algebraic basis of the dual $(\mathbb{R}^I)'$ isomorphic to the subspace $\Phi_I \subset \Phi$ of vectors $\phi$ meeting finite decomposition

$$\phi = \lambda_i e^i$$

with respect to the basis $\{e^i\}$ for $\Phi$. Then, the restriction of a Gaussian generating function on $\Phi$ to $\Phi_I$ is the Fourier transform of some Gaussian premeasure $\mu_I$ in $\mathbb{R}^I$ which is a measure if the set $I$ is countable [2]. For example, if

$$M(e^i, e^j) = \frac{1}{2}\delta^{ij},$$

the corresponding measure $\mu_I$ in $\mathbb{R}^I$ is

$$\prod_{i \in I}(\pi^{-1/2}\exp[-(w^i)^2]dw^i).$$

(12)

The expression (9) in QFT is the formal continuum generalization of the measure (12).

In contrast to the formal expression (1) of perturbed QFT, the true integral representation (3) of generating functionals enables us to handle non-Gaussian and nonequivalent
Gaussian representations of the algebra $B_\Phi$. Here, we describe one of such a representation which is utilized to axiomatic theory of a Higgs vacuum [9].

In contrast with the finite-dimensional case, Gaussian measures in infinite-dimensional spaces fail to be quasi-invariant under translations as a rule. Let $\mu$ be a Gaussian measure in the dual $\Phi'$ to a nuclear space $\Phi$ and $\mu_a$ the image of $\mu$ under translation

$$\gamma : \Phi' \ni w \to w + a \in \Phi', \quad a \in \Phi'.$$

The measures $\mu$ and $\mu_a$ are equivalent iff the vector $a \in \Phi'$ belongs to the canonical image $\Phi^d$ of $\Phi$ in $\Phi'$ with respect to the scalar form

$$\langle | \rangle = M(,)$$

where $M$ is the covariance form corresponding to the measure $\mu$. In this case, we have

$$\langle a, \Phi \rangle = \langle \Phi | \phi_a \rangle$$

for a certain vector $\phi_a \in \Phi$. Then, the measures $\mu$ and $\mu_a$ define equivalent representations of the algebra $B_\Phi$. This equivalence is performed by the unitary operator

$$u(w) \to \exp(-\langle w | \phi_a \rangle) u(w + a), \quad u(w) \in L^2(\Phi', \mu).$$

This operator fails to be constructed if $a$ does not belong to $\Phi^d$.

### 4 Fermion fields

Chronological vacuum expectations of fermion fields are skew-symmetric. By analogy with the scalar field case, we therefore can construct their Euclidean images as forms on an anticommutative algebra.

Given a complex linear locally convex topological space $V$, let $C(V)$ be the quotient of the tensor algebra $A_V$ by the two-sided ideal with the generating set

$$J := \{vv' + v'v, \quad v, v' \in V\}.$$ 

All finite-dimensional subspaces $\mathbb{R}^N$ of $V$ yields the inductive family of subalgebras $\{C(\mathbb{R}^N)\}$ of $C(V)$. Every projective family of continuous forms $F_N$ on $C(\mathbb{R}^N)$ defines some projective limit form $F$ on $C(V)$, but this form fails to be continuous in general.

By analogy with the scalar field case, one can consider the homomorphism of the algebra $C(V)$ onto some algebra of functions in order to construct $F$.

We denote by $\Lambda$ a complex infinite-dimensional topological Grassmann algebra

$$\Lambda = \Lambda_0 \oplus \Lambda_1$$

equipped also with the $\mathbb{Z}$-graded structure $\Lambda = \prod \Lambda^i$. The algebra $C(V)$ has still the structure of a $\mathbb{Z}_2$-graded linear space. Let us construct the $\Lambda$ envelope $\Lambda C(V)$ of $C(V)$. We denote by

$$\Lambda V = \Lambda_1 \otimes V$$
the $\Lambda$ envelope of $V$ treated as the $\mathbb{Z}_2$-graded space containing only an odd part. Both $\Lambda V$ an $\Lambda C(V)$ have structures of $\Lambda_0$-linear spaces. Let $\Lambda V$ be a $\Lambda_0$-linear space of $\Lambda_0$-linear continuous mappings of $\Lambda V$ into $\Lambda$ which preserve $\mathbb{Z}_2$-gradation. We further call the $\Lambda$-valued form any linear mapping into $\Lambda$. The space $\Lambda V$ contains the $\Lambda$ envelope of $V$. To give analogy to the scalar field case, $\Lambda V$ is termed the $\Lambda$-dual to $V$ together with the pairing form $\langle ; \rangle$.

One can endow $\Lambda V$ with the topology such that, whenever $v \in V$, the form

$$\Lambda V \ni s \rightarrow \langle s; v \rangle \in \Lambda$$

is continuous. Let us consider the correspondence between elements $\{v^1 \cdots v^n\}$ of $C(V)$ and continuous functions

$$\langle s; v^1 \rangle \cdots \langle s; v^n \rangle$$

on $V$. This correspondence yields the homomorphisms

$$c \rightarrow c(s), \quad c \in C(V),$$

of the algebras $C(V)$ and $\Lambda C(V)$ into the algebras of continuous $\Lambda$-valued functions and $\Lambda_0$-valued functions on $\Lambda V$ respectively.

Various $\Lambda$-valued and numerical forms on different subalgebras of the algebras of such functions can be constructed, e.g., by means of real functions on $\Lambda$ and measures in $\Lambda V$. We here examine forms whose restrictions to $C(V)$ are numerical Gaussian forms. The corresponding forms $F_N$ on $C(\mathbb{R}^N)$ can be represented by means of the Berezin forms.

Given $\mathbb{R}^N \subset V$, there is homomorphism of $C(\mathbb{R}^N)$ onto the subalgebra of $\Lambda$-valued functions on

$$\mathbb{R}^N = \Lambda V / \Lambda E_N$$

where the linear subspace $\Lambda E_N \subset \Lambda V$ consists of the forms equal to zero on $\mathbb{R}^N \subset V$. Let $c_N$ be an annihilator element of $C(\mathbb{R}^N)$ and $F_{BN}$ the Berezin form on $C(\mathbb{R}^N)$ given by the expressions

$$F_{BN}(\{v^1 \cdots v^{<N}\})(s) = 0, \quad F_{BN}(c_N(s)) = 1. \quad (13)$$

Remark that, for different annihilator elements, the forms $F_{BN}$ on $C(\mathbb{R}^N)$ differs in numerical multipliers. For each element $z \in C(\mathbb{R}^N)$, we then can define the form

$$F_{zN}(c) = F_{BN}(cz) \quad (14)$$
on $C(\mathbb{R}^N)$. For instance, $F_{zN}$ is a nondegenerate Gaussian form if

$$z(s) = L \exp[S^{-1}_{ij} e^i(s) e^j(s)], \quad L \in \mathbb{R},$$

where $S_{ij}$ is the matrix of a nondegenerate skew-symmetric bilinear form on $\mathbb{R}^N$ (i.e., $N$ is even) with respect to a basis $\{e^i\}$ for $\mathbb{R}^N$. 

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Note that, given the basis \{e^i\}, one can represent the form \(F_{BN}\) by means of the Berezin-like integrals
\[
F_{BN} = \int cd^N e^i, \quad c_N = e^1 \cdots e^N.
\] (15)

We here are not concerned of different approaches to superintegration. Note only that the Berezin form (13) does not depend on specification of a Grassmann algebra \(\Lambda\). The Berezin forms do not preserve \(Z_2\)-gradation. They fail to form a projective family since, given \(\mathbb{R}^n \subset \mathbb{R}^N\), we observe that
\[
F_{BN}(c_n) \neq F_{BN}(jc_n), \quad j : C(\mathbb{R}^n) \to C(\mathbb{R}^N).
\]

The expressions (13) - (15) which contain \(F_{BN}\) therefore can not be extrapolated to the infinite dimension case. At the same time, Gaussian forms \(F_N\) can possess the continuous projective limit \(F\) on \(C(V)\).

One can think of the algebra \(C(V)\) as being the enveloping algebra of the commutative Lie superalgebra corresponding to the Lie supergroup of translations in the space \(\Lambda V\). This group can be represented by operators
\[
q(s) \to \exp[i\langle s; w \rangle]q(s), \quad w \in \Lambda V,
\]
in some subspace \(Q\) (\(\Lambda C(V) \subset Q\)) of \(\Lambda\)-valued continuous functions on \(V\). Given a continuous \(\Lambda\)-form \(F\) on \(Q\) with \(F(1) = 1\), this representation is characterized by the \(\Lambda\)-valued continuous function
\[
Z(w) = F(\exp[i\langle s; w \rangle]).
\]

Let the function
\[
\Lambda^n \ni (\lambda_1, \ldots, \lambda_m) \to Z(\lambda_i v^i) \subset \Lambda
\]
be holomorphic for every finite collection \(v^1, \ldots, v^m \in V\). Then, \(Z(w)\) is a generating function for a form \(F\) on \(C(V)\) given by the body parts of expressions
\[
F(v^1 \cdots v^m) = i^{-m} \frac{\partial}{\partial \lambda_1} \cdots \frac{\partial}{\partial \lambda_m} Z(\lambda_i v^i) = F(\langle s; v^1 \rangle \cdots \langle s; v^m \rangle).
\]

For instance, if \(V = \mathbb{R}^N\), a Gaussian form on \(C(\mathbb{R}^N)\) is defined by a generating function
\[
Z(\lambda_i e^i) = F_{BN}(L \exp[i\langle s; \lambda_i e^i \rangle + S^{-1}_{ij} \langle s; e^i \rangle \langle s; e^j \rangle]) = \exp[-S^{ij} \lambda_i \lambda_j], \quad L \in \mathbb{R}.
\]

Let \(V\) be a complex nuclear space endowed with an inner product \((, )\) and an involution operation such that
\[
(v^*, v'^*) = (v, v').
\]
Let \(V\) be split into two orthogonal isomorphic subspaces
\[
V = H \oplus H^*.
\]
Every continuous bilinear skew-symmetric form \( S(v, v') \) on \( V \) is equivalent to the form

\[
(v^*, S_-v')
\]

where \( S \) is some bounded linear operator on \( H \) and

\[
\begin{align*}
S_-v &= Sv, \quad v \in H, \\
S_-v &= -(S^+v^*)^*, \quad v \in H^*.
\end{align*}
\]

Note that \( S \) is invertible if \( S(v, v') \) is nondegenerate. A skew-symmetric form \( S(v, v') \) on \( V \) yields a symmetric form \( S(w, w') \) on \( \Lambda V \). A continuous Gaussian form \( F \) on \( C(V) \) is defined by a generating \( \Lambda_0 \)-valued function

\[
Z(\lambda_i v^i) = \exp[-\lambda_i\lambda_j(v^i, S_-v^j)].
\] (16)

For instance, let \( H \) be the space \( \Omega \oplus S(R^4) \) where by \( \Omega \) is meant the space of Euclidean 4-component spinors. Elements of \( V \) then can be treated as superdistributions (which however differ from that by ref \([7]\)). If \( S \) is the Green function of the Euclidean Dirac operator, the generating function \((16)\) describes free Euclidean fermion fields.

It should be noted that given a bilinear skew-symmetric form \( S(v, v') \) on \( V \), there exists the unique positive continuous form \( M(v, v') \) \([8]\) such that

\[
\begin{align*}
S(v, v') &= M(Jv, v') , \\
M(v, v') &= (x, A_1x') + (y^*, A_2y')
\end{align*}
\]

where

(i) \( v = x + y \) and \( v' = x' + y' \) with \( x, x' \in H \) and \( y, y' \in H^* \);

(ii) \( S =UA_1 = A_2U \) is the polar decomposition of \( S \) with a unitary operator \( U \) and positive operators \( A_1 \) and \( A_2 \);

(iii) \( Jv = (U^+)_-v^* \) is an antiunitary operator with \( J^2 = -\text{id} \).

For instance, if \( S = \text{id} \), we have

\[
M(v, v') = (v, v').
\]

Then, the Gaussian measure in \( V' \) with the Fourier transform

\[
\exp[-\frac{1}{2}M(v, v)]
\]

can be related with the generating function \((16)\).
5 Appendix

We use the Fourier-Laplace (F-L) transforms in order to construct strictly the Wick rotation.

Remark. By \( \mathbb{R}^n_+ \) and \( \mathbb{R}^n_+ \), we denote the subset of \( \mathbb{R}^n \) with the Descartes coordinates \( x^\mu > 0 \) and its closure respectively. Elements of \( S(\mathbb{R}^n_+) \) are \( \phi \in S(\mathbb{R}^n) \) such that \( \phi = 0 \) on \( \mathbb{R}^n \setminus \mathbb{R}^n_+ \). Elements of \( S(\mathbb{R}^n_+) \) correspond to distributions \( W \in S'(\mathbb{R}^n) \) with \( \text{supp} W \subset \mathbb{R}^n_+ \).

Given \( W \in S'(\mathbb{R}^n) \), let \( Q_W \) be the set of \( q \in \mathbb{R}^n \) such that

\[
\exp(-qx)W \in S'(\mathbb{R}^n).
\]

The L-F transform of \( W \) is defined to be the Fourier transform

\[
W^{FL}(k + iq) = (\exp(-qx)W)^F = \int \exp[i(k + iq)x]Wd^n x \in S'(\mathbb{R}^n).
\]

It is a holomorphic function on the tubular set \( \mathbb{R}^n_+ + iQ_W \subset \mathbb{C}_n \) over the interiority of \( Q_W \). Moreover, it defines the family of distributions \( W^{FL}_q(k) \in S'(\mathbb{R}^n) \) which is continuous in the parameter \( q \). In particular, if \( W \in S'(\mathbb{R}^n_+ ) \), then \( \mathbb{R}^n_+ \subset Q_W \) and \( W^{FL} \) is a holomorphic function on the tubular set \( \mathbb{R}^n_+ + i\mathbb{R}^n_+ \) so that \( W^{FL}(k + i0) = W^F(k) \), i.e.,

\[
\lim_{|q| \to 0, q \in \mathbb{R}^n_+} \langle W^{FL}_q, \phi \rangle = \langle W^F, \phi \rangle, \quad \phi(k) \in S(\mathbb{R}^n).
\]

Let \( W^{FL}(k + iq) \) be the L-F transform of some distribution \( W \in S'(\mathbb{R}^n_+) \). Then, the relation

\[
\int_{\mathbb{R}^n_+} W^{FL}(iq)\phi(q)d_nq = \int_{\mathbb{R}^n_+} W(x)\widehat{\phi}(x)d^nx, \quad \phi \in S(\mathbb{R}^n_+ ),
\]

\[
\widehat{\phi}(x) = \int_{\mathbb{R}^n_+} \exp(-qx)\phi(q)d_nq, \quad x \in \mathbb{R}^n_+, \quad \widehat{\phi} \in S(\mathbb{R}^n_+ ),
\]

(17)
defines the continuous linear functional \( W^L(q) = W^{FL}(iq) \) on \( S(\mathbb{R}^n_+) \). It is called the Laplace transform. The image of \( S(\mathbb{R}^n_+) \) under the continuous morphism \( \phi \to \widehat{\phi} \) is dense in \( S(\mathbb{R}^n_+) \), and the norms \( \|\phi\|_{k,l} = \|\widehat{\phi}\|_{k,l} \) induce the weakening topology in \( S(\mathbb{R}^n_+) \). The functional \( W^L(q) \) (17) is continuous with respect to this topology. There is the 1:1 correspondence between Laplace transforms of elements of \( S'(\mathbb{R}^n_+) \) and elements of \( S'(\mathbb{R}^n_+) \) continuous with respect to the weakening topology in \( S(\mathbb{R}^n_+) \). We use this correspondence in order to construct the Wick rotation.

If the Minkowski space is identified with the real subspace \( \mathbb{R}^4 \) of \( \mathbb{C}^4 \), its Euclidean partner is the subspace \( (iz^0, x^{1,2,3}) \) of \( \mathbb{C}^4 \). These spaces have the same spatial coordinate subspace \( (x^{1,2,3}) \). For the sake of simplicity, we further do not write spatial coordinate dependence. We consider the complex plane \( \mathbb{C}^1 = X \oplus iZ \) of time \( x \) and Euclidean time \( z \) and the complex plane \( \mathbb{C}_1 = K \oplus iQ \) of the associated momentum coordinates \( k \) and \( q \).
Let $W(q) \in S'(Q)$ be a distribution such that

$$W = W_+ + W_-, \quad W_+ \in S'(Q_+), \quad W_- \in S'(Q_-). \quad (18)$$

For every $\phi_+ \in S(X_+)$, we have

$$\int_{Q_+} W(q) \hat{\phi}_+(q) dq = \int \limits_{Q_+} dq \int \limits_{X_+} dx [W(q) \exp(-qx) \phi_+(x)] =$$

$$\int \limits_{Q_+} dq \int \limits_{k} dk \int \limits_{X_+} dx [W(q) \phi^F_+(k) \exp(-ikx - qx)] =$$

$$-i \int \limits_{Q_+} dq \int \limits_{k} dk [W(q) \phi^F_+(k)_{k - iq}] = \int \limits_{Q_+} W(q) \phi^L_+(iq) dq \quad (19)$$

due to the fact that the F-L transform $\phi^F_+(k + iq)$ of $\phi_+ \in S(X_+) \subset S'(X_+)$ exists and it is holomorphic on the tubular set $K + iQ_+, Q_+ \subset Q_{\phi_+}$, so that $\phi^F_+(k + i0) = \phi^F_+(k)$. The function $\hat{\phi}_+(q) = \phi^F_+(-q)$ can be regarded as the Wick rotation of $\phi_+(x)$. The relations (13) take the form

$$\int \limits_{Q_+} W(q) \hat{\phi}_+(q) dq = \int \limits_{X_+} \hat{W}_+(x) \phi_+(x) dx,$$

$$\hat{W}_+(x) = \int \limits_{Q_+} \exp(-qx) W(q) dq, \quad x \in X_+, \quad (20)$$

where $\hat{W}_+(x) \in S'(X_+)$ is continuous with respect to the weakening topology in $S(X_+)$.  

For every $\phi_- \in S(X_-)$, we have the similar relations

$$\int \limits_{Q_-} W(q) \hat{\phi}_-(q) dq = \int \limits_{X_-} \hat{W}_-(x) \phi_-(x) dx,$$

$$\hat{W}_-(x) = \int \limits_{Q_-} \exp(-qx) W(q) dq, \quad x \in X_- \quad (21)$$

The combination of (20) and (21) results in the relation

$$\int \limits_{Q} W(q) \hat{\phi}(q) dq = \int \limits_{X} \hat{W}(x) \phi(x) dx,$$

$$\hat{\phi} = \hat{\phi}_+ + \hat{\phi}_-, \quad \hat{\phi} = \hat{\phi}_+ + \hat{\phi}_-, \quad (22)$$

where $\hat{W}(x)$ is a linear functional on functions $\phi \in S(X)$ such that $\phi$ and all its derivatives are equal to zero at $x = 0$. This functional can be regarded as a functional on $S(X)$, but
it needs additional definition at $x = 0$. This is the well-known feature of chronological forms in quantum field theory. We can treat $\hat{W}$ as the Wick rotation of $W$.

For instance, let the covariance form $\Lambda$ of a Gaussian state on the commutative algebra of Euclidean scalar fields $\phi$ be given by a distribution $\tilde{\Lambda}(z_1 - z_2)$. We have

$$\int \tilde{\Lambda}(z_1 - z_2)\phi_1(z_1)\phi_2(z_2)dz_1dz_2 = \int \tilde{\phi}(z_1)\phi_2(z_1 - z)dz_1dz = \int \tilde{\phi}(z_1)\phi_2(z_1 - z)d\tilde{z}_1.$$

Let $\tilde{\phi}(q)$ satisfy the condition (18). Its Wick rotation (22) defines the functional

$$\tilde{\phi}(x) = \theta(x) \int_{\mathbb{R}^+} \tilde{\phi}(q) e^{-qx}dq + \theta(-x) \int_{\mathbb{R}^-} \tilde{\phi}(q) e^{-qx}dq$$

$$\int \tilde{\phi}(x)f(x)dx = \int \tilde{\Lambda}(x_1 - x_2)\phi(x_1)\phi(x_2)dx_1dx_2,$$

on scalar fields $\phi$ on the Minkowski space. For instance, if $\tilde{\phi}(q)$ is the Feynman propagator, $\tilde{\phi}$ is the familiar causal Green function $D_c$.

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