Charged Vector Particles Tunneling
From 5D Black Hole and Black Ring

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Abstract

In this paper, we investigate the Hawking radiation process as a semiclassical quantum tunneling phenomenon from black ring and Myers-Perry black holes in 5-dimensional (5D) spaces. Using Lagrangian of Glashow-Weinberg-Salam model with background electromagnetic field (for charged W-bosons) and the WKB approximation, we will evaluate the tunneling rate/probability of charged vector particles through horizons by taking into account the electromagnetic vector potential. Moreover, we investigate the corresponding Hawking temperature values by considering Boltzmann factor for both cases and analyze the whole spectrum generally.

Keywords: Black rings; Myers-Perry black holes; Charged vector particles; Semiclassical quantum tunneling phenomenon; Hawking radiation.

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1 Introduction

In the early universe, by assuming the quantum background, Hawking observed that black hole (BH) emits particles and the spectrum of the passed

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off particles is purely thermal. Hawking expressed the emission spectrum for all kinds of particles e.g. neutrinos, photons, gravitons, electrons, positrons etc. and accounted the emission rate for these particles. The vector particles or vector bosons (with spin equals to 1), i.e., $W^\pm$ and $Z$ play a fundamental role as important component of the standard model for electroweak interaction, so that the emission of such particles should also be of more significance in the analysis of Hawking radiation. Different methods have been suggested for the analysis of Hawking radiation as a tunneling spectrum across the horizons of BHs. The tunneling method is based on the fundamental physical particles action which allows BH radiation.

Existence of the positive and negative energy pair of virtual particles is similar to the existence of particle and anti-particle pair creation, respectively. At instant, the negative and positive energy virtual particles are created and annihilated in form of pairs. The positive energy virtual particle disappears by tunnel through the horizon and it is emitted as a depart of the Hawking radiation. While, the negative energy particle goes inside the BH or absorbed by the BH. The rule of conservation of energy is adapted in the procedure. The specific value of temperature at which particles emitted from the BH is known as Hawking temperature. For the Schwarzschild back ground geometry the Hawking temperature is obtained for scalar particle emission.

Emparan and Recall [1] discussed a rotating 5D black ring. Lim and Teo [2] calculated the particles motion in the spacetime of the rotating black ring. Chen and Teo [3] discussed rotating black rings on Taub-NUT space by applying the inverse-scattering procedure. Matsumoto, Yoshino and Kodama [4] talked about the loss of BH’s mass and angular momenta through Hawking radiation. Cvetic and Guber [5] discussed R-charged BHs (particular BHs in gauged super-gravity in $D = 5$ and $D = 7$). Aliev [6] considered that a BH may posses a small electric charged and constructed a 5-vector potential for electromagnetic field in the background of Myers-Perry metric. Sharif and Wajiha [7] have computed the Hawking radiation spectrum for charged fermions tunneling from charged accelerating and rotating BHs with NUT parameter. Jan and Gohar [8] applied the Hamilton-Jacobi method to the Klein-Gordon equation by using WKB approximation. They determined the tunneling probability of outgoing charged scalar particles from the event horizon of BH and also detect the corresponding Hawking temperature. Kruglov [9] studied the Hawking radiation of spin-1 particles from BHs in (1+1)-dimensions by using Proca equation. The procedure is viewed
as quantum tunneling of bosons through the event horizon. They established the result that the emission temperature with the Schwarzschild background geometry is similar to the Hawking temperature of scalar particles emission. Kruglov also found the Hawking temperature for the de Sitter, Rindler and Schwarzschild de Sitter spacetimes. Li and Zu [10] have studied the tunneling rate and Hawking temperature of scalar particles for Gibbons-Maeda-Dilation BH.

Feng, Chen and Zu [11] analyzed the Hawking radiation of vector particles from 4D and 5D BHs. Saleh, Thomas and Kofane [12] obtained the Hawking radiation from a 5D Lovelock BH by applying the Hamilton-Jacobi procedure. Saifullah and Yau [13] studied the outgoing and ingoing probabilities and temperature of the spin-$\frac{3}{2}$ particles. Chen and Huang [14] studied the Hawking radiation of vector particles for Vaidya BHs. Li and Chen [15] looked into vector particles tunneling (uncharged and charged bosons) from the Kerr and Kerr-Newman BHs. Singh, Meitei and Singh [16] have used the WKB approximation and Hamiliton-Jacobi ansatz to the Proca equation and calculated the tunneling rate and Hawking temperature of vector bosons for Kerr-Newman BH. In this paper, the Hawking temperature is discussed for three coordinate systems. Gursel and Sakalli [17] studied the Hawking radiation of massive vector particles from rotating warped anti-de Sitter BH in 3D and showed that the radial mapping gives the tunneling rate for outgoing particles. Different authors [18, 19] have found the Hawking temperature for various types of particles from BHs. Li and Zhao [20] analyzed the massive vector particles tunneling and found the Hawking temperature from the neutral rotating anti-de Sitter BHs in conformal gravity by applying tunneling procedure. Ali and Jusufi [21] studied the quantum gravity effects on the Hawking radiation of charged spin-1 particles for noncommutative charged BHs, RN BHs and charged BHs.

It is to be observed that the kinetics of charged vector particles are ruled along the Proca field equations by using WKB approximation and the tunneling rate and Hawking temperature for the passed off particles can be deduced. It is observed that the tunneling effects are not associated to the mass of BHs but depends on the mass/energy of the outgoing particles. In this paper we have extended the work of charged vector particles tunneling for 5D spacetimes with electromagnetic background. This paper is determined as follows: We discuss in the section 2, the tunneling rate and Hawking temperature of charged vector $W^{\pm}$-bosons for black rings. Section 3 is based on the analysis for Myers-Perry metric. Section 4 is explained the outlook.
2 Rotating Charged Black Ring

In this section, we analyze Hawking radiation process as tunneling of charged vector particles from 5D charged black rings. The discussion of tunneling phenomenon for Dirac particles from 5D charged black rings is studied in Ref. [22]. The black rings can go around in the azimuthal focus of the $S^2$. The result is identifying a revolving black ring, which takes essential conic singularity, because at that place no centrifugal force exists to produce equilibrium in self-gravity of black rings. Revolving black rings in Taub-NUT have significance in Kaluza-Klein theory. The black ring reduces to the electrically charged BH, as the NUT parameter reduces to the similar Kaluza-Klein monopole [3].

The study of BH thermodynamics has significance in gravitational physics. Laws of BH thermodynamics have suggested that BHs have finite temperature (known as Hawking temperature) which is proportional to their surface gravity and BH entropy is proportional to the horizon area, which corresponds to the first law of thermodynamics. The thermodynamical relationships for 5D physical objects at horizons are more complicated as compare to BHs, in particular for spinning black rings and black saturns. Matsumoto et al. [4] have studied the time evolution of thin black ring via Hawking radiation. The black ring is considered to emit massless scalar particles. They observed the energy and angular momentum of emitted scalar particles. The minimum (or zero) velocity is deduced for black ring which allows a limit for jump particles to be bounded inside the black ring. The analysis is based on geodesics to analyze the complicated mathematical structure for particles to go across through the ring [23].

In this section, we focus on studying Hawking radiation of charged vector particles via tunneling from 5D rotating charged black ring, which is special solution of the Einstein-Maxwell-Dilaton gravity model (EMD) in 5D. The line element of black rings in a unit electric charged can be written in the following form [22]

\[
\begin{align*}
    ds^2 &= -\frac{F(y)}{F(x)K^2(x,y)} \left( dt - C(\nu, \lambda)R \frac{1 + y}{F(y)} \cosh^2 \alpha d\phi \right)^2 \\
    &+ \frac{R^2}{(x-y)^2}F(x) \left[ -\frac{G(y)}{F(y)}d\phi^2 - \frac{dy^2}{G(y)} + \frac{dx^2}{G(x)} + \frac{G(x)}{F(x)}d\varphi^2 \right], \quad (2.1)
\end{align*}
\]
where

\[ C(\nu, \lambda) = \sqrt{\lambda(\lambda - \nu) \frac{1 + \lambda}{1 - \lambda}} \]

\[ K(x, y) = 1 + \lambda(x - y) \sinh^2 \alpha / F(x) \]

\[ F(\zeta) = 1 + \lambda \zeta, \quad G(\zeta) = (1 - \zeta^2)(1 + \nu \zeta) \]

and parameters \( \lambda \) and \( \nu \) are dimensionless and takes values in the range of \( 0 < \nu \leq \lambda < 1 \) and not takes the conical singularity at \( x = 1, \lambda \) and \( \nu \) associated to each other say \( \lambda = 2\nu/(1 + \nu^2) \) and \( \alpha \) is the parameter acting as the electric charge. The coordinate \( \varphi \) and \( \phi \) are two cycles of the black ring and \( x \) and \( y \) the values range \(-1 \leq x \leq 1\) and \(-\infty \leq y \leq -1\) and \( R \) has the dimensional length. The explicit calculation of the electric charged is \( Q = 2M \sinh 2\alpha/(3(1 + \frac{4}{3} \sinh^2 \alpha)) \). The mass of the black ring is \( M = 3\pi R^2 \lambda/[4(1 - \nu)] \), and its angular momentum takes the form \( J = \pi R^3 \rho \sqrt{\lambda(\lambda - \nu)(1 + \lambda)/[2(1 - \nu)^2]} \).

The line element given by Eq.(2.1) can be rewritten as

\[ ds^2 = Adt^2 + Bd\phi^2 + Cdy^2 + Ddx^2 + Ed\varphi^2 + 2Fdtd\phi \]  \hspace{1cm} (2.2)

The charged bosons \((W^\pm)\) act differently as from the uncharged bosons \((Z)\) in the presence of electromagnetic field. The \( W^+ \) bosons behave similarly as \( W^- \) bosons and their tunneling processes are similar too. In the background of electromagnetic field, for anti-symmetric tensor \( \psi_{\mu\nu} \), the Lagrangian equation implies

\[ \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} \psi_{\nu}^{\mu}) + \frac{m^2}{\hbar^2} \psi_{\nu} + \frac{i}{\hbar} e A_\mu \psi_{\nu}^{\mu} + \frac{i}{\hbar} e F_{\nu}^{\mu} \psi_{\mu} = 0, \]  \hspace{1cm} (2.3)

where

\[ \psi_{\nu\mu} = \partial_\nu \psi_\mu - \partial_\mu \psi_\nu + \frac{i}{\hbar} e A_\nu \psi_\mu - \frac{i}{\hbar} e A_\mu \psi_\nu \]  \hspace{1cm} and \[ F^{\mu\nu} = \nabla^{\mu} A^{\nu} - \nabla^{\nu} A^{\mu}. \]
The values of the components of $\psi^\mu$ and $\psi^{\nu\mu}$ are given as follows

\[
\psi^0 = \frac{B}{AB - F^2} \psi_0 - \frac{F}{AB - F^2} \psi_1, \quad \psi^1 = \frac{-F}{AB - F^2} \psi_0 + \frac{A}{AB - F^2} \psi_1,
\]
\[
\psi^2 = \frac{1}{C} \psi_2, \quad \psi^3 = \frac{1}{D} \psi_3, \quad \psi^4 = \frac{1}{E} \psi_4, \quad \psi^{01} = \frac{F^2 \psi_{10} + AB \psi_{01}}{(AB - F^2)^2},
\]
\[
\psi^{02} = \frac{B \psi_{02} - F \psi_{12}}{C(AB - F^2)}, \quad \psi^{03} = \frac{B \psi_{03} - F \psi_{13}}{D(AB - F^2)}, \quad \psi^{04} = \frac{B \psi_{04} - F \psi_{14}}{E(AB - F^2)},
\]
\[
\psi^{12} = \frac{-F \psi_{02} + A \psi_{12}}{C(AB - F^2)}, \quad \psi^{13} = \frac{-F \psi_{03} + A \psi_{13}}{D(AB - F^2)}, \quad \psi^{14} = \frac{-F \psi_{04} + A \psi_{14}}{E(AB - F^2)},
\]
\[
\psi^{23} = \frac{\psi_3}{D}, \quad \psi^{24} = \frac{\psi_4}{E}, \quad \psi^{34} = \frac{\psi_4}{D}.
\]

The electromagnetic vector potential is given by

\[ A_\mu = A_t dt + A_\phi d\phi, \quad (2.4) \]

where

\[ A_t = \frac{\lambda (x - y) \sinh \alpha \cosh \alpha}{F(x) K(x, y)}, \quad A_\phi = \frac{C(\nu, \lambda) R(1 + y) \sinh \alpha \cosh \alpha}{F(x) K(x, y)}. \]

Using, WKB approximation [23], i.e.,

\[ \psi_\nu = c_\nu \exp \left[ \frac{i}{\hbar} S_0(t, r, \theta, \phi) + \Sigma h^n S_n(t, r, \theta, \phi) \right], \quad (2.5) \]

to the Lagrangian (2.3), where $c_\nu$ are arbitrary constants, $S_0$ and $S_n$ are arbitrary functions. By ignoring the terms for $n = 1, 2, 3, 4, \ldots$, we obtain the following set of equations

\[
\frac{F^2}{AB - F^2} \left[ C_0 (\partial_1 S_0)^2 - C_1 (\partial_1 S_0)(\partial_0 S_0) + C_0 (\partial_1 S_0) e A_1 - C_1 (\partial_1 S_0) e A_0 \right]
\]
\[
+ \frac{AB}{AB - F^2} \left[ C_1 (\partial_1 S_0)(\partial_0 S_0) - C_0 (\partial_1 S_0) + C_1 (\partial_1 S_0) e A_0 - C_0 (\partial_1 S_0) e A_1 \right]
\]
\[
+ \frac{B}{C} \left[ C_2 (\partial_0 S_0)(\partial_2 S_0) - C_0 (\partial_2 S_0)^2 + C_2 (\partial_0 S_0) e A_0 \right] - \frac{F}{C} [C_2 (\partial_1 S_0)(\partial_2 S_0)
\]
\[
- C_1 (\partial_2 S_0)^2 + C_2 (\partial_2 S_0) e A_1] + \frac{B}{D} [C_3 (\partial_0 S_0)(\partial_3 S_0) - C_0 (\partial_3 S_0)^2 + C_3 (\partial_3 S_0)]
\]
\[ eA_0 - \frac{F}{D}[C_3(\partial_1 S_0)(\partial_3 S_0) - C_1(\partial_3 S_0)^2 + C_3(\partial_3 S_0)eA_1] + \frac{B}{E}[C_4(\partial_0 S_0)(\partial_4 S_0) - C_0(\partial_4 S_0)^2 + C_4(\partial_4 S_0)eA_0] - \frac{F}{E}[C_4(\partial_1 S_0)(\partial_4 S_0) - C_1(\partial_4 S_0)^2] + C_4(\partial_4 S_0)eA_1] - m^2 BC_0 + m^2 FC_1 + \frac{eA_1 F^2}{AB - F^2}[C_0(\partial_1 S_0) - C_1(\partial_0 S_0) + eA_1 C_0 - eA_0 C_1]
\]
\[ = 0, \quad (2.6) \]
\[ -\frac{F^2}{AB - F^2}[C_0(\partial_1 S_0)(\partial_0 S_0) - C_1(\partial_0 S_0)^2 + C_0(\partial_0 S_0)eA_1 - C_0(\partial_0 S_0)eA_0] - \frac{AB}{AB - F^2}[C_1(\partial_0 S_0)^2 - C_0(\partial_1 S_0)(\partial_0 S_0) + C_1(\partial_0 S_0)eA_1 - C_0(\partial_0 S_0)eA_1] - \frac{F}{C}[C_2(\partial_0 S_0)(\partial_2 S_0) - C_0(\partial_2 S_0)^2 + C_2(\partial_2 S_0)eA_0] + \frac{A}{C}[C_2(\partial_1 S_0)(\partial_2 S_0) - C_1(\partial_2 S_0)^2 + C_2(\partial_2 S_0)eA_1] - \frac{F}{D}[C_3(\partial_0 S_0)(\partial_3 S_0) - C_0(\partial_3 S_0)^2 + C_3(\partial_3 S_0)eA_1] - \frac{F}{E}[C_4(\partial_0 S_0)(\partial_4 S_0) - C_0(\partial_4 S_0)^2 + C_4(\partial_4 S_0)eA_1] + m^2 FC_0 - m^2 AC_1 - \frac{eA_0 F^2}{AB - F^2}[C_0(\partial_1 S_0) - C_1(\partial_0 S_0) + C_2(\partial_0 S_0)eA_1] - eA_0 C_0 + eA_0 C_1 + \frac{eA_0 AB}{AB - F^2}[C_3(\partial_1 S_0) - C_1(\partial_3 S_0) + eA_1 C_3] = 0, \quad (2.7) \]
\[ eA_0 - \frac{F}{D}[C_3(\partial_1 S_0)(\partial_3 S_0) - C_1(\partial_3 S_0)^2 + C_3(\partial_3 S_0)eA_1] + \frac{B}{E}[C_4(\partial_0 S_0)(\partial_4 S_0) - C_0(\partial_4 S_0)^2 + C_4(\partial_4 S_0)eA_0] - \frac{F}{E}[C_4(\partial_1 S_0)(\partial_4 S_0) - C_1(\partial_4 S_0)^2] + C_4(\partial_4 S_0)eA_1] - m^2 BC_0 + m^2 FC_1 + \frac{eA_1 F^2}{AB - F^2}[C_0(\partial_1 S_0) - C_1(\partial_0 S_0) + eA_1 C_0 - eA_0 C_1] \]
\[-C_0(\partial_2 S_0) + C_2 e A_0 + \frac{e A_1 F}{AB - F^2}[C_2(\partial_0 S_0) - C_0(\partial_2 S_0) + e A_0 C_2] - \frac{e A_1 A}{AB - F^2}[C_2(\partial_1 S_0) - C_1(\partial_2 S_0) + e A_1 C_2] = 0, \quad (2.8)\]
\[-B[C_3(\partial_0 S_0)^2 - C_0(\partial_0 S_0)(\partial_3 S_0) + C_3(\partial_0 S_0)e A_0] + F[C_3(\partial_0 S_0)(\partial_1 S_0) - C_1(\partial_3 S_0) + e A_0 C_3] + e A_1 F[C_3(\partial_0 S_0) - C_0(\partial_3 S_0) + e A_1 C_3] = 0, \quad (2.9)\]
\[-C_0(\partial_0 S_0)(\partial_4 S_0) + C_4(\partial_0 S_0)e A_0] + F[C_4(\partial_0 S_0)(\partial_1 S_0) - C_0(\partial_1 S_0)(\partial_4 S_0) + C_4(\partial_1 S_0)e A_0] - A[C_4(\partial_2 S_0)^2 - C_1(\partial_3 S_0)(\partial_4 S_0) + C_4(\partial_1 S_0)e A_1] - \frac{AB - F^2}{C}[C_4(\partial_2 S_0)^2 - C_2(\partial_3 S_0)(\partial_4 S_0)] - \frac{AB - F^2}{D}[C_4(\partial_3 S_0)^2 - C_3(\partial_3 S_0)(\partial_4 S_0)] - \frac{AB - F^2}{D}[C_4(\partial_3 S_0)^2 - C_3(\partial_3 S_0)(\partial_4 S_0)] - \frac{AB - F^2}{D}[C_4(\partial_3 S_0)^2 - C_3(\partial_3 S_0)(\partial_4 S_0)] - \frac{AB - F^2}{D}[C_4(\partial_3 S_0)^2 - C_3(\partial_3 S_0)(\partial_4 S_0)] = 0. \quad (2.10)\]

Using the following rule for separation of variables \cite{23}, i.e.,
\[S_0 = -(E - \sum_{i=1}^{2} j_i \hat{\Omega}_i) t + J \phi + W(x, y) + L \varphi, \quad (2.11)\]

where \(E\) is the energy of the particle, \(J\) and \(L\) represent the particle’s angular momentums corresponding to the angles \(\phi\) and \(\varphi\), respectively, while \(K\) is the complex constant. From Eqs.\((2.6)-(2.10)\), we can obtain a $5 \times 5$ matrix equation
\[G(C_0, C_1, C_2, C_3, C_4)^T = 0,\]
where the matrix is labeled as \( \mathbf{G} \), whose components are given as follows:

\[
G_{00} = \frac{1}{AB - F^2} \left[ F^2 J^2 + eA_1 \right] - AB[J - eA_1J] - \frac{1}{C}(\dot{W}^2 - \frac{1}{D}(\partial_3 W)^2
- \frac{L^2}{E} - m^2 B + \frac{eA_1}{AB - F^2}[F^2 J - eA_1] - AB[J + eA_1]
\]

\[
G_{01} = \frac{1}{AB - F^2} \left[ (E - \sum_{i=1}^{2} j_i \dot{\Omega}_i)J - eA_0J \right] - AB[(E - \sum_{i=1}^{2} j_i \dot{\Omega}_i)J - eA_0J]
+ F[(\partial_2 W)^2 + eA_2(\partial_2 W)] - F[(\partial_3 W)^2 - L^2] + \frac{eA_1}{AB - F^2}[(E - \sum_{i=1}^{2} j_i \dot{\Omega}_i)
- eA_0] - AB[(E - \sum_{i=1}^{2} j_i \dot{\Omega}_i) - eA_0]
\]

\[
G_{02} = \frac{B}{C}[(E - \sum_{i=1}^{2} j_i \dot{\Omega}_i)(\partial_2 W) + eA_0(\partial_2 W)] - F[J(\partial_3 W) + eA_1(\partial_2 W)]
\]

\[
G_{03} = \frac{B}{D}[(E - \sum_{i=1}^{2} j_i \dot{\Omega}_i)(\partial_3 W) + eA_0(\partial_3 W)] - F[J(\partial_3 W) + eA_1(\partial_3 W)]
\]

\[
G_{04} = \frac{B}{E}[(E - \sum_{i=1}^{2} j_i \dot{\Omega}_i)L + eA_0L] + F[(E - \sum_{i=1}^{2} j_i \dot{\Omega}_i)L - eA_1L]
\]

\[
G_{10} = \frac{F^2}{AB - F^2} \left[ (E - \sum_{i=1}^{2} j_i \dot{\Omega}_i)J + eA_1(E - j\dot{\Omega}) \right] + AB[(E - \sum_{i=1}^{2} j_i \dot{\Omega}_i)J
- eA_1(E - \sum_{i=1}^{2} j_i \dot{\Omega}_i)] + \frac{F}{C}(\partial_2 W)^2 + \frac{F}{D}(\partial_3 W)^2 + \frac{F}{E}L^2 + m^2 F
- \frac{eA_0 F^2}{AB - F^2}[J - eA_0]
\]

\[
G_{11} = \frac{1}{AB - F^2} \left[ (E - \sum_{i=1}^{2} j_i \dot{\Omega}_i)^2 - eA_0(E - \sum_{i=1}^{2} j_i \dot{\Omega}_i) \right] + AB[(E - \sum_{i=1}^{2} j_i \dot{\Omega}_i)^2
- eA_0(E - \sum_{i=1}^{2} j_i \dot{\Omega}_i)] - \frac{A}{C}(\partial_2 W) + \frac{1}{D}(\partial_3 W)^2 A - \frac{1}{E}AL^2 - m^2 A
- \frac{eA_0 F^2}{AB - F^2}[(E - \sum_{i=1}^{2} j_i \dot{\Omega}_i) - eA_0 - (\partial_3 W)AB]
\]
\[ G_{12} = \frac{F}{C}[\left( E - \sum_{i=1}^{2} j_i \tilde{\Omega}_i \right) + eA_0(\partial_2 W)] + \frac{A}{C}[J(\partial_2 W) + eA_1(\partial_2 W)] \]

\[ G_{13} = \frac{F}{D}[\left( E - \sum_{i=1}^{2} j_i \tilde{\Omega}_i \right)(\partial_3 W) - eA_0(\partial_3 W)] + \frac{A}{D}[J(\partial_3 W) + eA_1(\partial_3 W)] + \frac{eA_0}{AB - F^2}[ABJ + AB eA_1] \]

\[ G_{14} = \frac{F}{E}[\left( E - \sum_{i=1}^{2} j_i \tilde{\Omega}_i \right)L - eA_0L] + \frac{A}{E}[JL + eA_1L] \]

\[ G_{20} = -\frac{B}{AB - F^2}(E - \sum_{i=1}^{2} j_i \tilde{\Omega}_i)(\partial_2 W) - \frac{1}{AB - F^2}(\partial_2 W) - \frac{eA_0B}{AB - F^2}(\partial_2 W) \]

\[ G_{21} = \frac{F}{AB - F^2}(E - \sum_{i=1}^{2} j_i \tilde{\Omega}_i)(\partial_2 W) + \frac{A}{AB - F^2}J(\partial_2 W) - \frac{eA_0}{AB - F^2}(\partial_2 W) \]

\[ G_{22} = -\frac{F}{AB - F^2}\left[ J(E - \sum_{i=1}^{2} j_i \tilde{\Omega}_i) + eA_1(E - \sum_{i=1}^{2} j_i \tilde{\Omega}_i) \right] - \frac{B}{AB - F^2}[(E - \sum_{i=1}^{2} j_i \tilde{\Omega}_i)] - \frac{F}{AB - F^2}[(E - \sum_{i=1}^{2} j_i \tilde{\Omega}_i)J - eA_0J] \]

\[ G_{23} = \frac{A}{AB - F^2}[J^2 + eA_1J] - \frac{1}{D}(\partial_3 W)^2 - \frac{L^2}{E} - m^2 + \frac{eA_0F}{AB - F^2}[J + eA_1] \]

\[ G_{24} = \frac{eA_1A}{AB - F^2}[J + eA_1], \quad G_{23} = \frac{1}{D}(\partial_2 W)(\partial_3 W), \quad G_{24} = \frac{1}{E}(\partial_2 W)L \]

\[ G_{30} = -B(\partial_3 W)[E - \sum_{i=1}^{2} j_i \tilde{\Omega}_i) - FJ(\partial_3 W) - eA_0(\partial_3 W) + eA_1F(\partial_3 W) \]

\[ G_{31} = F(E - \sum_{i=1}^{2} j_i \tilde{\Omega}_i)(\partial_3 W) + AJ - eA_0F(\partial_3 W) + eA_1A(\partial_3 W) \]

\[ G_{32} = \frac{AB - F^2}{C}(\partial_2 W)(\partial_3 W) \]

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\[ G_{33} = -B[(E - \sum_{i=1}^{2} j_i \tilde{\Omega}_i)^2 - eA_0(E - j \tilde{\Omega})] - F[(E - \sum_{i=1}^{2} j_i \tilde{\Omega}_i)J] - eA_1(E - \sum_{i=1}^{2} j_i \tilde{\Omega}_i)] - F[(E - \sum_{i=1}^{2} j_i \tilde{\Omega}_i)J - eA_0J] - \\
A[J^2 + eA_1J] - \frac{AB - F^2}{C} (\partial_2 W)^2 - \frac{AB - F^2}{E} L - \\
\frac{AB - F^2}{D} m^2 - eA_0B[(E - \sum_{i=1}^{2} j_i \tilde{\Omega}_i) - eA_0] + eA_0F[J + eA_1] - \\
eA_1F[(E - \sum_{i=1}^{2} j_i \tilde{\Omega}_i) - eA_0] - AeA_1[J + eA_1] \\
G_{34} = \frac{AB - F^2}{E} (\partial_3 W)L \\
G_{40} = -B(E - \sum_{i=1}^{2} j_i \tilde{\Omega}_i)L - FJL + eA_0BL - eA_1FL \\
G_{41} = F(E - \sum_{i=1}^{2} j_i \tilde{\Omega}_i)L + AJL - eA_0FL + eA_1A \\
G_{42} = \frac{AB - F^2}{C} (\partial_2 W)L, \quad G_{43} = \frac{AB - F^2}{D} (\partial_3 W)L \\
G_{44} = -F(E - \sum_{i=1}^{2} j_i \tilde{\Omega}_i)J - FeA_1(E - j \tilde{\Omega}) - B[(E - \sum_{i=1}^{2} j_i \tilde{\Omega}_i)^2 - \\
eA_0(E - \sum_{i=1}^{2} j_i \tilde{\Omega}_i)] - F[(E - \sum_{i=1}^{2} j_i \tilde{\Omega}_i) - eA_0J] - A[J^2 + eA_1J] - \\
\frac{AB - F^2}{C} (\partial_2 W)^2 - \frac{AB - F^2}{D} (\partial_3 W)^2 - m^2(AB - F^2) + eA_0F \\
[J + eA_1] + eA_0B[(E - \sum_{i=1}^{2} j_i \tilde{\Omega}_i) - eA_0] - eA_1F[(E - \sum_{i=1}^{2} j_i \tilde{\Omega}_i) \\
+ eA_0] - eA_1A[J + eA_1] \\
\]

For the non-trivial solution, the absolute value |G| is equal to zero, so that
one can yield

\[ ImW^\pm = \pm \int \sqrt{(E - \sum_{i=1}^{2} j_i \tilde{\Omega}_i - eA_0 - \Omega_h e A_1)^2 + X}, \]

\[ = \pm i\pi \frac{E - eA_0 - \sum_{i=1}^{2} j_i \tilde{\Omega}_i - \Omega_h e A_1}{2\kappa(r_+)}, \]  

(2.12)

where \( -\Omega_h = \frac{F^2}{B^2} \), while + and − represent the outgoing and incoming particles, respectively. The function \( X \) can be defined as

\[ X = \frac{FJ}{B}(E - j\tilde{\Omega}) - e^2 A_1^2 \left[ \frac{F}{B} - 1 \right] + \frac{FJ}{B} \left[ (E - j\tilde{\Omega}) - eA_0 \right] + \frac{A J}{B} \left[ (J + eA_1) + m^2 (AB - F^2) \right] - eA_0 \frac{F}{B} J + eA_1 \frac{A}{B} (J + eA_1) \]  

(2.13)

and the surface gravity is

\[ \kappa(r_+) = \frac{1}{2} \sqrt{K_y(x, y) H_y(x, y)}, \]

where we have chosen \( -C_y(x, y) = K_y(x, y) \) and \( D_y^{-1}(x, y) = H_y(x, y) \). The tunneling probability for charged vector particles is given by

\[ \Gamma = \frac{\Gamma_{(emission)}}{\Gamma_{(absorption)}} = \exp \left[ -4\pi \frac{(E - \sum_{i=1}^{2} j_i \tilde{\Omega}_i - eA_0 - \Omega_h e A_1)}{\sqrt{K_y(x, y) H_y(x, y)}} \right]. \]  

(2.14)

By comparing the above expression with the Boltzmann factor, \( \exp[-\beta E] \), one can derive the Hawking temperature, which is \( T_H = \frac{1}{\beta} \) at the outer horizon \( r_+ \). For this case, we can obtain the following Hawking temperature as

\[ T_H = \sqrt{\frac{K_y(x, y) H_y(x, y)}{4\pi}}, \]

\[ = \sqrt{\frac{(x - y)^2(1 - x^2)(1 + vx)[2 + vy + vy^3 + xv - 2xy - 3xy^2v]}{4\pi \sqrt{(1 - y^2)(1 + yv)}}}. \]  

(2.15)

The Hawking temperature of electrically charged black ring is depending on \( x, y \) and \( v \). The resulting Hawking temperature at which vector particles tunnel through the horizon is similar to the Hawking temperature for scalar and Dirac particles at which they tunnel through the horizon of black ring [22].
3 Myers-Perrry Black Hole

Black holes are most valuable astrophysical objects predicted in general relativity \[25\]. Kaluza’s theory (1921) along with Klein’s version (1929) known as Kaluza-Klein theory which provides the proposal to unify the theory of general relativity and electromagnetic theory in 5D vacuum spacetime. The Einstein field equations for 5D spacetime are equivalent to 4D Einstein’s equation on with matter comprising of scalar and electromagnetic fields. Abdolrahimi et al. \[26\] have studied distorted Myers-Perry BH in an external gravitational field with a single angular momentum as exact solution of the 5D Einstein equations in vacuum.

The Myers-Perry BH \[27\] is a solution of vacuum Einstein field equations in an arbitrary dimension. Here, we consider the 5D case which represents a regular rotating BH (with two rotation parameters) which is a generalization of the Kerr solution. The corresponding line element in the Boyer-Lindquist coordinates \((t, r, \theta, \phi, \varphi)\) is defined as follows \[28\]

\[
\begin{align*}
{ds}^2 &= \frac{\rho^2 r^2}{\Delta} dr^2 + \rho^2 d\theta^2 - dt^2 + (r^2 + a^2) \sin^2 \theta d\phi^2 + (r^2 + b^2) \cos^2 \theta d\varphi^2 \\
&+ \frac{r_0^2}{\rho^2} [dt + a \sin^2 \theta d\phi + b \cos^2 \theta d\varphi]^2 
\end{align*}
\]  

(3.1)

where

\[\rho^2 = r^2 + a^2 \cos^2 \theta + b^2 \sin^2 \theta, \quad \Delta = (r^2 + a^2)(r^2 + b^2) - r_0^2 r^2,\]

and the angle \(\phi\) and \(\varphi\) assume measures from the interval \([0, \frac{\pi}{2}]\); \(a\) and \(b\) are the two angular momenta, \(r_0\) associated to the BH mass. The BH horizons are turned as

\[
r_{+}^2 = \frac{1}{2} [r_0^2 - a^2 - b^2 \pm \sqrt{(r_0^2 - a^2 - b^2)^2 - 4a^2b^2}] \quad \text{for} \quad (r_0^2 - a^2 - b^2)^2 > 4a^2b^2
\]

The Myers-Perry BH is regular except if \(a = b = 0\) that corresponds to 5D Schwarzschild-Tangherlini BH solution \[29\] and only in this case it has a singularity at \(r = 0\).

The line element given by Eq.(3.1) can be rewritten as

\[
{ds}^2 = \tilde{A}dt^2 + \tilde{B}dr^2 + \tilde{C}d\theta^2 + \tilde{D}d\phi^2 + \tilde{E}d\varphi^2 + 2\tilde{F}dtd\phi + 2\tilde{G}dtd\varphi + 2\tilde{H}d\phi d\varphi.
\]

(3.2)
The electromagnetic vector potential is defined as \( A_\mu = (A_1, 0, 0, A_4, 0) \), where
\[
A_1 = \frac{b}{\kappa \phi^2} b \cos^2 \theta \frac{r_0^2}{\rho^2} \quad \text{and} \quad A_4 = \frac{r_0^2}{\rho^2 \kappa \phi^2} ab \sin^2 \theta \cos^2 \theta
\]

In Lagrangian Eq. (2.3) the components of \( \psi^\nu \) and \( \psi^{\mu \nu} \) are given by
\[
\begin{align*}
\psi^0 &= J \psi_0 + M \psi_3 + N \psi_4, \quad \psi^1 = T \psi_1, \quad \psi^2 = U \psi_2, \\
\psi^3 &= M \psi_0 + P \psi_3 + R \psi_4, \quad \psi^4 = \bar{E}^{-1} \psi_4, \\
\psi^{01} &= J T \psi_{01} + M T \psi_{31} + N T \psi_{41}, \quad \psi^{02} = J U \psi_{02} + M U \psi_{32} + N U \psi_{42}, \\
\psi^{03} &= (J P - M^2) \psi_{03} + (J R - M N) \psi_{04} + (M R - N P) \psi_{34}, \\
\psi^{04} &= M N \psi_{30} + (M^2 - J S) \psi_{10} + J R \psi_{03} + N R \psi_{33}, \quad \psi^{12} = T U \psi_{12}, \\
\psi^{13} &= T M \psi_{10} + T P \psi_{13} + T R \psi_{14}, \quad \psi^{14} = T N \psi_{10} + T R \psi_{13} + T S \psi_{14}, \\
\psi^{23} &= M U \psi_{20} + U P \psi_{23} + U R \psi_{24}, \quad \psi^{24} = M U \psi_{20} + U P \psi_{23} + U R \psi_{24}, \\
\psi^{34} &= (M R - P N) \psi_{03} + P S \psi_{34},
\end{align*}
\]

where
\[
\begin{align*}
J &= \frac{DE - H^2}{-ADE + AH^2 + F^2 E - 2HFG + DG^2}, \\
M &= \frac{DE - H^2}{-ADE + AH^2 + F^2 E - 2HFG + DG^2}, \\
N &= \frac{DE - H^2}{-ADE + AH^2 + F^2 E - 2HFG + DG^2}, \\
P &= \frac{EA - G^2}{-ADE + AH^2 + F^2 E - 2HFG + DG^2}, \\
R &= \frac{-HA - FG}{-ADE + AH^2 + F^2 E - 2HFG + DG^2}, \\
S &= \frac{-DE + F^2}{-ADE + AH^2 + F^2 E - 2HFG + DG^2}, \\
T &= \frac{1}{B}, \quad U = \frac{1}{C}.
\end{align*}
\]

Using Eq. (2.3) and WKB approximation, by neglecting the terms of order
\[ n = 1, 2, 3, 4... \] we obtain the following set of equations

\[ JT[C_1(\partial S_0)(\partial S_0) - C_0(\partial S_0)^2 + eA_0C_1(\partial S_0)] + MT[C_1(\partial S_0)(\partial S_0) - C_3(\partial S_0)^2 + eA_3C_1(\partial S_0)] + NT[C_1(\partial S_0)(\partial S_0) - C_4(\partial S_0)^2] + JU[C_2(\partial S_0)(\partial S_0) - C_0(\partial S_0)^2 + eA_0C_2(\partial S_0)] + MU[C_2(\partial S_0)(\partial S_0) - C_3(\partial S_0)^2 + eA_3C_2(\partial S_0)] + [J]\]

\[ [C_3(\partial S_0)(\partial S_0) - C_0(\partial S_0)^2 + eA_3C_2(\partial S_0) + [J]\]

\[ = 0 \quad (3.3) \]

\[ -JT[C_1(\partial S_0)^2 - C_0(\partial S_0)(\partial S_0) + eA_0C_1(\partial S_0)] - MT[C_1(\partial S_0)^2 - C_3(\partial S_0)^2 + eA_3C_1(\partial S_0)] - NT[C_1(\partial S_0)(\partial S_0) - C_4(\partial S_0)(\partial S_0)] + TU[C_2(\partial S_0)(\partial S_0) - C_1(\partial S_0)^2] + TM[C_0(\partial S_0)(\partial S_0) - C_3(\partial S_0)(\partial S_0)] - eA_0C_1(\partial S_0) + TP[C_2(\partial S_0)(\partial S_0) - C_1(\partial S_0)^2 - eA_3C_1(\partial S_0)] + TR[C_4(\partial S_0)(\partial S_0) - C_0(\partial S_0)(\partial S_0)] + TN[C_0(\partial S_0)(\partial S_0) - eA_0C_1(\partial S_0)] + TS[C_4(\partial S_0)(\partial S_0) - C_1(\partial S_0)^2 - m^2TC_1 - eA_0JT[C_0(\partial S_0) - C_1(\partial S_0)^2] - eA_0C_1 - MT[eA_0[C_1(\partial S_0) - C_3(\partial S_0) - eA_3C_1] - eA_0NT[C_1(\partial S_0) - C_4(\partial S_0)] + TM[eA_3[C_0(\partial S_0) - C_1(\partial S_0)] - eA_0C_1 + eA_3TP[C_3(\partial S_0) - C_1(\partial S_0) - eA_3C_1] + TReA_3[C_4(\partial S_0) - C_1(\partial S_0)] = 0. \quad (3.4) \]

\[ JU[C_0(\partial S_0)^2 - C_2(\partial S_0)^2 - eA_0C_2(\partial S_0)] - MU[C_2(\partial S_0)^2 - eA_3C_2(\partial S_0)] - NU[C_0(\partial S_0)^2 - C_4(\partial S_0)^2] - TU[C_2(\partial S_0)^2 - C_1(\partial S_0)^2] + MU[C_0(\partial S_0)^2 - C_3(\partial S_0)^2 - C_2(\partial S_0)^2 - eA_3C_2(\partial S_0)] + UR[C_4(\partial S_0)^2 - C_0(\partial S_0)^2 - C_2(\partial S_0)(\partial S_0)] + MN[C_0(\partial S_0)^2 - C_4(\partial S_0)^2 - C_2(\partial S_0)(\partial S_0)] - eA_0C_2(\partial S_0) + UR[C_4(\partial S_0)^2 - C_0(\partial S_0)^2 - C_2(\partial S_0)(\partial S_0)] + UM[eA_3[C_0(\partial S_0) - C_2(\partial S_0) - eA_0C_2] + eA_3UP[C_3(\partial S_0) - C_2(\partial S_0)] \]
Using separation of variables technique, we can define the particles action for this BH as (also defined in Eq. (2.11))

\[ (JP - M^2)[C_3(\partial_0 S_0)^2 - C_0(\partial_0 S_0)(\partial_3 S_0) + eA_0C_3(\partial_0 S_0) - eA_3C_0(\partial_0 S_0)] \]

\[ -(JJ - MN)[C_4(\partial_0 S_0)^2 - C_0(\partial_0 S_0)(\partial_2 S_0) + eA_0C_4(\partial_0 S_0)] - NU[C_2(\partial_0 S_0)(\partial_3 S_0) - C_4(\partial_0 S_0)(\partial_2 S_0)] - (MR - NP)[C_4(\partial_0 S_0)(\partial_3 S_0) - C_3(\partial_0 S_0)(\partial_4 S_0)] \]

\[ -TM[C_0(\partial_1 S_0)^2 - C_1(\partial_0 S_0)(\partial_1 S_0) - eA_0C_1(\partial_1 S_0)] - TP[C_3(\partial_1 S_0)^2 - C_1(\partial_0 S_0)(\partial_1 S_0) - eA_3C_1(\partial_1 S_0)] - TR[C_4(\partial_1 S_0)^2 - C_1(\partial_0 S_0)(\partial_1 S_0) - eA_3C_1(\partial_1 S_0)] - MU \]

\[ [C_0(\partial_2 S_0)^2 - C_2(\partial_0 S_0)(\partial_2 S_0) - eA_0C_2(\partial_2 S_0)] - UP[C_3(\partial_2 S_0)^2 - C_2(\partial_2 S_0)(\partial_4 S_0) + (MR - PN) \]

\[ [C_3(\partial_0 S_0)(\partial_4 S_0) - C_0(\partial_0 S_0)(\partial_4 S_0) + eA_0C_3(\partial_1 S_0) - eA_3C_0(\partial_1 S_0)] + PS \]

\[ [C_4(\partial_0 S_0)(\partial_3 S_0) - C_3(\partial_0 S_0) + eA_3C_4(\partial_0 S_0)] - eA_0(JP - M^2)[C_3(\partial_0 S_0) - C_0(\partial_3 S_0) + eA_0C_3 - eA_3C_0 - eA_3(JR - MN)[C_4(\partial_0 S_0) - C_0(\partial_4 S_0)] \]

Using separation of variables technique, we can define the particles action for this BH as (also defined in Eq. (2.11))

\[ S_0 = -(E - \sum_{i=1}^{2} j_i \hat{\Omega}_i) t + W(r, \theta) + L\phi + K(\varphi). \]
matrix provide in the following form

\[
\Lambda_{00} = -JT(\partial_1 W)^2 - JU(\partial_2 W) - (JP - M^2)L^2 - (JP - M^2)eA_3L - (JR - MN)L(\partial_3 K) - (JR - MN)L(\partial_3 K) - (JR - MN)[L(\partial_3 K) + eA_3 (\partial_4 K)] - (JS - M^2)(\partial_1 K) - m^2J - eA_3L - e^2A_3^2 - eA_3(JR - MN) (\partial_4 K)
\]

\[
\Lambda_{01} = -(E - \sum_{i=1}^{2} j_i \tilde{\Omega}_i)JT(\partial_1 W) + JTeA_0(\partial_1 W) + JTeA_0(\partial_1 W) + MT(\partial_1 W)L + eA_3(\partial_1 W) + NT(\partial_1 W)(\partial_4 K)
\]

\[
\Lambda_{02} = -(E - \sum_{i=1}^{2} j_i \tilde{\Omega}_i)JU(\partial_2 W) + JUeA_0(\partial_2 W) + MU(\partial_1 W)L + MUeA_3(\partial_2 W) + NU(\partial_2 W)(\partial_4 K)
\]

\[
\Lambda_{03} = -MT(\partial_1 W)^2 - MU(\partial_2 W)^2 - (JP - M^2)(E - \sum_{i=1}^{2} j_i \tilde{\Omega}_i)L + (JP - M^2)eA_0 L - (MR - NP)L(\partial_4 K) - (JR - MN)[(E - \sum_{i=1}^{2} j_i \tilde{\Omega}_i)(\partial_4 K) - eA_0(\partial_4 K)] + NR(\partial_4 K)^2 - m^2M - (JP - M^2)[(E - \sum_{i=1}^{2} j_i \tilde{\Omega}_i)eA_3 - e^2A_3^2] - eA_3(M R - NP)(\partial_4 K)
\]

\[
\Lambda_{04} = -NT(\partial_1 W) - NU(\partial_2 W) - (JR - MN)[(E - \sum_{i=1}^{2} j_i \tilde{\Omega}_i) - eA_0L] + (MR - NP)[L^2 - eA_3L] - (JS - M^2)[(E - \sum_{i=1}^{2} j_i \tilde{\Omega}_i)(\partial_4 K) - eA_0(\partial_4 K)] - NR [(\partial_4 K) + eA_3(\partial_4 K)] - m^2N - eA_3(JR - MN)[(E - \sum_{i=1}^{2} j_i \tilde{\Omega}_i) - eA_0] + eA_3 (MR - NP)[L + eA_3]
\]

\[
\Lambda_{10} = -(E - \sum_{i=1}^{2} j_i \tilde{\Omega}_i)(\partial_1 W)JT + TM(\partial_1 W)L + TN(\partial_1 W)(\partial_4 K) - eA_0JT(\partial_1 W) + TM eA_3(\partial_1 W)
\]
\[ \Lambda_{11} = -JT[(E - \sum_{i=1}^{2} j_i \tilde{\Omega}_i)^2 - eA_0(E - \sum_{i=1}^{2} j_i \tilde{\Omega}_i)] - MTL[(E - \sum_{i=1}^{2} j_i \tilde{\Omega}_i) + eA_3] + NT(E - \sum_{i=1}^{2} j_i \tilde{\Omega}_i)(\partial_4 K) - TU(\partial_2 W) + TML[(E - \sum_{i=1}^{2} j_i \tilde{\Omega}_i) - eA_0] - TPL[L + eA_3] - TRL(\partial_4 K) + TN[(E - \sum_{i=1}^{2} j_i \tilde{\Omega}_i)(\partial_4 K) + eA_0(\partial_4 K)] - TR[(\partial_4 K)L + eA_3(\partial_4 K)] - TS(\partial_4 K) - m^2 T - eA_0JT[(E - \sum_{i=1}^{2} j_i \tilde{\Omega}_i)] - eA_0 - MTeA_0[L + eA_3] - eA_0 NT(\partial_4 K) + TMeA_3[(E - \sum_{i=1}^{2} j_i \tilde{\Omega}_i) - eA_0] - TP[L + eA_3] - eA_3 TR(\partial_4 K) \]

\[ \Lambda_{12} = TU(\partial_1 W)(\partial_2 W) \]

\[ \Lambda_{13} = -(E - \sum_{i=1}^{2} j_i \tilde{\Omega}_i)(\partial_1 W) MT + TP(\partial_1 W)L + TR(\partial_1 W)(\partial_4 K) + TMeA_0 (\partial_1 W) + TPeA_3(\partial_1 W) \]

\[ \Lambda_{14} = -(E - j \tilde{\Omega})(\partial_1 W) NT + TR(\partial_1 W)L + TS(\partial_1 W)(\partial_4 K) + eA_0 NT(\partial_1 W) + eA_3 TR(\partial_1 W) \]

\[ \Lambda_{20} = -UJ(E - \sum_{i=1}^{2} j_i \tilde{\Omega}_i)(\partial_2 W) + MUL(\partial_2 W) + MU(\partial_2 W)(\partial_4 K) - JUeA_0 (\partial_2 W) + MUeA_3(\partial_2 W) \]

\[ \Lambda_{21} = TU(\partial_2 W)(\partial_1 W) \]

\[ \Lambda_{22} = -UJ[(E - \sum_{i=1}^{2} j_i \tilde{\Omega}_i)[(E - \sum_{i=1}^{2} j_i \tilde{\Omega}_i) - eA_0] + MU(E - \sum_{i=1}^{2} j_i \tilde{\Omega}_i)[L + eA_3] + NU(E - \sum_{i=1}^{2} j_i \tilde{\Omega}_i)(\partial_4 K) - TU(\partial_1 W) + MUL[(E - \sum_{i=1}^{2} j_i \tilde{\Omega}_i) - eA_0] - UPL L + eA_3] - URL(\partial_4 K) + MU(\partial_4 K)[(E - \sum_{i=1}^{2} j_i \tilde{\Omega}_i) - eA_0] - UPL (\partial_4 K) - UR(\partial_4 K) - m^2 U + JUeA_0[eA_0 - (E - \sum_{i=1}^{2} j_i \tilde{\Omega}_i)] - MULEA_0 \]
\[-NUeA_0(\partial_1K) + MU eA_3[(E - \sum_{i=1}^{2} j_i \tilde{\Omega}_i) - eA_0] - UPL eA_3 - UReA_3 (\partial_4K)\]

\[\Lambda_{23} = -(E - \sum_{i=1}^{2} j_i \tilde{\Omega}_i)(\partial_2W)MU + UPL(\partial_2W) + UP(\partial_2W)(\partial_4K) + MU eA_0 (\partial_2W) + eA_3 UPL(\partial_2W)\]

\[\Lambda_{24} = -NU(E - \sum_{i=1}^{2} j_i \tilde{\Omega}_i)(\partial_2W) + UR(\partial_2W)L + UR(\partial_2W)(\partial_4K)NU eA_0(\partial_2W) + UReA_3(\partial_2W)\]

\[\Lambda_{30} = -(JP - M^2)(E - \sum_{i=1}^{2} j_i \tilde{\Omega}_i)[L + eA_3] - (JR - MN)(E - \sum_{i=1}^{2} j_i \tilde{\Omega}_i)(\partial_4K) - TM(\partial_1W) - MU(\partial_2W)^2 - (MR - PN)(\partial_4K)[L + eA_3] + eA_0(JP - M^2)[L + eA_3] + eA_0(JR - MN)(\partial_4K)\]

\[\Lambda_{31} = TM(\partial_1W)[(E - \sum_{i=1}^{2} j_i \tilde{\Omega}_i) - eA_0] + TP(\partial_1W)[L + eA_3] + TR(\partial_1W)(\partial_4K)\]

\[\Lambda_{32} = -MU(\partial_2W)[(E - \sum_{i=1}^{2} j_i \tilde{\Omega}_i) - eA_0] + UP(\partial_2W)[L + eA_3] + UR(\partial_2W)(\partial_4K)\]

\[\Lambda_{33} = -(JP - M^2)(E - \sum_{i=1}^{2} j_i \tilde{\Omega}_i)[(E - \sum_{i=1}^{2} j_i \tilde{\Omega}_i) - eA_0] - (MR - NP)(E - \sum_{i=1}^{2} j_i \tilde{\Omega}_i)(\partial_4K) - TP(\partial_1W) + UP(\partial_2W) - (MR - PN)(\partial_4K)[(E - \sum_{i=1}^{2} j_i \tilde{\Omega}_i) - eA_0] - PS(\partial_4K) + eA_0(JP - M^2)[(E - \sum_{i=1}^{2} j_i \tilde{\Omega}_i) - eA_0] + eA_0(\partial_4K)\]

\[\Lambda_{34} = -(JR - MN)(E - \sum_{i=1}^{2} j_i \tilde{\Omega}_i)[(E - \sum_{i=1}^{2} j_i \tilde{\Omega}_i) - eA_0] + (MR - NP)(E - \sum_{i=1}^{2} j_i \tilde{\Omega}_i)L - TR(\partial_1W) - UR(\partial_2W) + PS(\partial_4K)[L + eA_3] + eA_0(JR - MN)(\partial_4K)\]
\[-MN \left[ (E - \sum_{i=1}^{2} j_i \tilde{\Omega}_i) - eA_0 \right] - eA_0 (MR - NP)(L + eA_3) \]

\[\Lambda_{40} = - (MN - JR) \left( E - \sum_{i=1}^{2} j_i \tilde{\Omega}_i \right) [L + eA_3] - (M^2 - JS) \left( E - \sum_{i=1}^{2} j_i \tilde{\Omega}_i \right) \]

\[\left( \partial_4 K \right) + TN \left( \partial_1 W \right)^2 - MN \left( \partial_2 W \right)^2 + (MR - PN) L \left[ L + eA_3 \right] + eA_0 \]

\[\left( MN - JR \right) \left[ L + eA_3 \right] + eA_0 \left( M^2 - JS \right) \left( \partial_4 K \right) + eA_3 (MR - PN) \left[ L + eA_3 \right] \]

\[\Lambda_{41} = TN \left( \partial_1 W \right) \left[ (E - \sum_{i=1}^{2} j_i \tilde{\Omega}_i) + eA_1 \right] - TR \left( \partial_1 W \right) [L + eA_3] + TS \left( E - \sum_{i=1}^{2} j_i \tilde{\Omega}_i \right) \left( \partial_1 W \right) \]

\[\Lambda_{42} = - MN \left( \partial_2 W \right) \left[ (E - j \tilde{\Omega}) - eA_0 \right] + UP \left( \partial_2 W \right) [L + eA_3] + UR \left( \partial_2 W \right) \]

\[\left( \partial_4 K \right) \]

\[\Lambda_{43} = - (MN - JR) \left( E - \sum_{i=1}^{2} j_i \tilde{\Omega}_i \right) \left[ \left( E - \sum_{i=1}^{2} j_i \tilde{\Omega}_i \right) - eA_0 \right] - NR \left( E - \sum_{i=1}^{2} j_i \tilde{\Omega}_i \right) \]

\[\left( \partial_4 K \right) + TR \left( \partial_1 W \right)^2 - UP \left( \partial_2 W \right)^2 + (MR - PN) L \left[ (E - \sum_{i=1}^{2} j_i \tilde{\Omega}_i) - eA_0 \right] + PSL \left( \partial_4 K \right) + eA_0 (MN - JR) \left[ \left( E - \sum_{i=1}^{2} j_i \tilde{\Omega}_i \right) - eA_0 \right] \]

\[\Lambda_{44} = \left( JS - M^2 \right) \left[ \left( E - \sum_{i=1}^{2} j_i \tilde{\Omega}_i \right) \left[ \left( E - \sum_{i=1}^{2} j_i \tilde{\Omega}_i \right) - eA_0 \right] + NR \left( E - \sum_{i=1}^{2} j_i \tilde{\Omega}_i \right) \right] \]

\[j_i \tilde{\Omega}_i \left[ L + eA_3 \right] + TS \left( \partial_1 W \right) - UR \left( \partial_2 W \right)^2 - PSL \left[ L + eA_3 \right] \]

\[-e^2 A_0^2 \left( M^2 - JS \right) - eA_0 NR \left[ L + eA_3 \right] \]

For the non-trivial solution, the determinant \( \Lambda \) is equal to zero provides the
following expression

\[ ImW^\pm = \pm \int \frac{\sqrt{(E - eA_0 - \Omega_1 eA_3 - \sum_{i=1}^{2} j_i \dot{\Omega}_i)^2 + \dot{X}}}{JR_{MN - JR}}, \]

where

\[ X = \frac{NR}{MN - JR} (E - \sum_{i=1}^{2} j_i \dot{\Omega}_i)(\partial_4 K) + \frac{UP}{MN - JR} (\partial_2 W)^2 - \]

\[ \frac{(MR - PN) L}{MN - JR} [(E - \sum_{i=1}^{2} j_i \dot{\Omega}_i) - eA_0] - \frac{PSL}{MN - JR} (\partial_4 K) - \]

\[ eA_0 \frac{NR}{MN - JR} (\partial_4 K) - eA_3 \frac{(MR - PN)}{MN - JR} [(E - \sum_{i=1}^{2} j_i \dot{\Omega}_i) - eA_0] - eA_0 - e^2 A_3^2 + 2(E - \sum_{i=1}^{2} j_i \dot{\Omega}_i)eA_3 - 2e^2 A_3 A_0 \]

and the surface gravity is

\[ \kappa(r_+) = \frac{1}{2} \sqrt{\tilde{M}_r(r, \theta)\tilde{N}_r(r, \theta)}, \]

where \(-\tilde{B}_r(r, \theta) = \tilde{M}_r(r, \theta)\) and \(C^{-1}_r(r, \theta) = \tilde{N}_r(r, \theta)\). The required tunneling probability is

\[ \tilde{\Gamma} = \frac{\bar{\Gamma}_{\text{emission}}}{\Gamma_{\text{absorption}}} = \exp[-4ImW^+] = \exp \left[ -4\pi \frac{(E - eA_0 - \Omega_1 eA_3 - \sum_{i=1}^{2} j_i \dot{\Omega}_i)}{\sqrt{\tilde{M}_r(r, \theta)\tilde{N}_r(r, \theta)}} \right]. \]  

The Hawking temperature for BH in 5D by using Boltzmann factor \( \beta = \frac{1}{T_H} \), is given by

\[ T_H = \sqrt{\frac{\tilde{M}_r(r, \theta)\tilde{N}_r(r, \theta)}{4\pi}} = \frac{\sqrt{2r^2[r_0^2 r^3 - b^2 - r b^2 \sin^2 \theta(b^2 + 2a^2 - 2r^2 - 1)]}}{4\pi(r^2 + a^2 \cos^2 \theta + b^2 \sin^2 \theta)(r^4 + b^2 r^2 + a^2 r^2 - r_0^2 r^2)}. \] 

The tunneling probability related to \( E, A_0, A_3, \dot{\Omega} \) angular momentum, surface gravity of a BH. The Hawking temperature depends on parameters \( r_0, a \) and \( b \).
4 Outlook

In this paper, we have used Lagrangian equation to investigate the tunneling of charged particles from electrically charged black ring and Myers-Perry BHs. In 5D, black rings have many unusual properties not shared by Myers-Perry BHs with spherical topology, e.g., their event horizon topology is not spherical for the cases of neutral, dipole and charged black rings.

For black rings, the tunneling spectrum of scalar particles has already been discussed by using the Hamilton-Jacobi method [30]. While, the Dirac particles tunneling phenomenon for black ring has also been discussed [22]. Recently, the anomalous derivation of Hawking radiation has attempted to recover the Hawking temperature of black rings via gauge and gravitational anomalies at the horizon [31]. As far as I know, till now, there is no references to report Hawking radiation of charged vector particles across single electrically charged black ring. So it is interesting to see if charged bosons tunneling process is still applicable in such exotic spacetime. In our analysis, we have found that the tunneling probability given by Eq. (2.14) depends on vector potential components, i.e., $A_0$ and $A_1$, energy, angular momentum, particle’s charge and surface gravity of black ring. While, the Hawking temperature (2.15) depends on parameter $\alpha$, i.e., charge of a black ring. We have found that the recovered Hawking temperature is same as already obtained in the literature for various particles.

For Myers-Perry BH, the tunneling spectrum of massive scalar and vector particles tunneling have been discussed in different coordinate systems [25]. They investigated the Hawking temperature in the Painlevé coordinates and in the corotating frames and showed that the coordinate system do not affect the Hawking temperature. Here, we have evaluated charged vector particles tunneling from Myers-Perry BH by solving Proca equations by applying the WKB approximation to the Hamilton-Jacobi method. For this 5D BH, the tunneling probability given by Eq. (3.9) is associated to the energy, vector potential, angular momentum and surface gravity of BH. While, the Hawking temperature (3.10) depends on parameters $r_0$, $a$ and $b$, which is similar to the temperature as given in [25]. It is to be noted that the effect of electromagnetism appear only on the tunneling probability of these vector particles which tunnel through the horizon but not on the Hawking temperature.

In this paper, we have found that the Hawking temperature related to the emission rate is similar for every type of particles, i.e., scalars, fermions, vectors (bosons either charged or uncharged). It is to be mentioned that
the obtained Hawking temperature is independent of the background BH geometry, coordinate system, particles species and the temperature must be same either emitted particles are charged or uncharged. The Hawking temperature calculated recovered by various methods is same and agrees with the temperature generally calculated in the literature [32]. However, the tunneling probably is different for different cases.

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