The Angular Distribution of Clusters in Skewed CDM Models

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Abstract

We perform a detailed investigation of the statistical properties of the projected (angular) distribution of galaxy clusters obtained in Cold Dark Matter (CDM) models with both Gaussian and skewed (i.e. non–Gaussian) primordial density fluctuations. We use large numerical simulations of these skewed CDM models to construct a set of simulations of the Lick catalogue. An objective cluster–finding algorithm is used to identify regions where the projected number–density of galaxies in the catalogues exceeds some density threshold criterion. In this way we can construct catalogues containing the angular position and richness of real and simulated clusters which are suitable for statistical analysis. For Gaussian models, the overall number of clusters is too small in the standard CDM case compared to observations, but a model with higher normalisation is in much better agreement; non–Gaussian models with negative initial skewness also fit the observed numbers fairly well. We compute the angular correlation function of clusters of different richness and find a strong dependence of the clustering amplitude with richness in all models. Even with a higher normalization, the Gaussian CDM model fails at producing sufficient large–scale cluster clustering. We also find that the Lick data are better reproduced only by a CDM model with negative initial skewness; initially skew–positive models fail to produce enough large–scale clustering. This conclusion is confirmed by two other statistical analyses; the properties of the minimal spanning tree and the multifractal scaling of the real clusters are much better reproduced by skew–negative CDM models. In particular, the small–scale self–similarity in the distribution of richest real clusters turns out to be a crucial test, which is only passed by skew–negative models. We show that a skewness–variance relation of hierarchical type is followed by skew–positive models, as well as by the more evolved Gaussian model.

Key Words: Galaxies: formation, clustering – large-scale structure of the Universe – early Universe – dark matter.
1 Introduction

Clusters of galaxies have been recognised for some time to be efficient tracers of the large–scale structure of the Universe, especially on scales $> 10 h^{-1}\text{Mpc}$ where the clustering of galaxies themselves is hardly detectable above the noise (see, e.g., Bahcall 1988). Rich galaxy systems are strongly correlated even on scales where the gravitational dynamics of the underlying matter distribution is still in the linear regime, so that their spatial distribution should give useful information about the spectrum and probability distribution function of the primordial density fluctuations, which are thought to be the seeds of galaxy formation.

A classical result in the analysis of cluster clustering is the power–law shape of the 2–point correlation function of clusters,

$$\xi(r) = \left(\frac{r_o}{r}\right)^\gamma,$$

which declines with an exponent $\gamma \simeq 1.8$, remarkably similar to that relevant for galaxies, but with a much larger correlation length, $r_o \simeq 20 h^{-1}\text{Mpc}$ (Bahcall & Soneira 1983; Klypin & Kopylov 1983; Postman, Huchra & Geller 1992). Well–defined relations have also been found between cluster richness and correlation amplitude, the richest clusters being much more strongly correlated than relatively poor ones (Postman, Geller & Huchra 1986; Bahcall & Burgett 1986). It is possible that this is part of an even wider relation which involves single galaxies and perhaps even huge superclusters. At least qualitatively, this behaviour can be interpreted in hierarchical clustering theories, such as CDM, as a consequence of the fact that larger structures arise from higher peaks in the primordial density fluctuations which are intrinsically more strongly clustered than typical points (Kaiser 1984). But even allowing for this effect, the large observed amplitude of the cluster correlation is difficult to account for quantitatively in CDM models. This lack of large–scale power represented one of the first pitfalls of the ‘standard’ CDM model, which predicts a correlation length smaller than the observed value by a factor $\sim 2$ (White et al. 1987). Some authors have taken this line of argument to an extreme and argued that the observed richness–clustering relation is the consequence of a kind of fractal clustering, extending up to arbitrarily large mass scales with no evidence of homogeneity (Coleman & Pietronero 1992). Independent analyses of the scaling properties of the two– and three–dimensional cluster distributions (Borgani, Plionis & Valdarnini 1993; Borgani, Martínez & Valdarnini 1993, in preparation), together with the convergence of the cluster dipole anisotropy at $\sim 150 h^{-1}\text{Mpc}$ (Plionis & Valdarnini 1991; Scaramella, Vettolani & Zamorani 1991), show that this explanation is not correct.

Serious doubts about the reality of the strong spatial correlations of clusters have been raised by some authors (Sutherland 1988; Sutherland & Efstathiou 1991). They claim that the inclusion of background and/or foreground galaxies can spuriously amplify the small–scale clustering in the line–of–sight direction for a richness–limited sample such as the Abell catalogue (Abell 1958). This claim is, however, still controversial. Different authors have pointed out that,
although present, this effect should not be significant (Dekel et al. 1989; Olivier et al. 1990; Jing, Plionis & Valdarnini 1992). The availability of more recent cluster samples based on automated procedures using plate scanning devices such as the APM (Dalton et al. 1992) and COSMOS (Collins, Nichol & Lumsden 1992), should help to clarify this point. Nevertheless, different groups still reach different conclusions using these new samples, even though they consistently give a correlation length in the range $13–16 h^{-1}\text{Mpc}$. Efstathiou et al. (1992) conclude that APM clusters display a quite low correlation length, similar to that of the Abell clusters, after correcting the clustering anisotropy in the line–of–sight direction. By contrast, Bahcall & West (1992) observe that APM clusters are generally poorer than Abell’s, so that the weaker clustering can be interpreted in terms of a general relation between richness and clustering strength (Bahcall & Burgett 1986; Postman, Geller & Huchra 1986). The large–scale coherence of the cluster distribution has also been confirmed by recent analyses of the Postman, Huchra & Geller (1992) redshift sample. It should be stressed, however, that even allowing for a correlation length as small as $13 h^{-1}\text{Mpc}$, both analytical arguments based on linear theory (Coles 1988; Borgani 1990) and $N$–body simulations (White et al. 1987) indicate that cluster clustering is still a problem for the standard CDM scenario. More recently, Scaramella (1992), Peacock & West (1992) and Jing & Valdarnini (1993) have found that the power–spectrum traced by clusters possesses much more large–scale power than the power–spectrum predicted in the CDM model. The problem may even be more deeply rooted than the choice of primordial fluctuation spectrum: Plionis, Valdarnini & Jing (1992) have discovered features in the spatial cluster distribution which are not accounted for by Gaussian models constructed to reproduce the observed 2– and 3–point cluster correlation functions.

In a series of papers, we have performed a detailed analysis of the statistical properties of angular cluster samples, obtained from the Lick map by Plionis, Barrow & Frenk (1991; hereafter PBF samples). These clusters are selected by the application of an objective overdensity criterion to the underlying galaxy cell–counts provided by the Lick map (see Section 3, below). Plionis, Barrow & Frenk (1991; Paper I) analysed the projected shapes of clusters. Plionis & Borgani (1991; Paper II) and Borgani, Jing & Plionis (1992; Paper III) worked out the correlation properties of the PBF cluster distributions, devoting particular care to the relation between cluster richness and clustering strength. Borgani, Plionis & Valdarnini (1993; Paper IV) performed a detailed multifractal analysis of the PBF samples, finding that, at least for the richest clusters, a self–similar clustering develops at the scales of non–linearity (i.e., $\lesssim 20 h^{-1}\text{Mpc}$).

In this paper, we apply a number of statistical analyses to angular cluster samples obtained from $N$–body simulations with skewed CDM initial conditions (Moscardini et al. 1991; Matarrese et al. 1991; Messina et al. 1992; Lucchin et al. 1993). Cluster samples are selected by applying the same criteria as in Paper I to the simulated Lick maps (Coles et al. 1993b).

It is important to consider the effect of skewed CDM initial conditions for at least two main reasons. First, as already observed, the large–scale clustering of rich galaxy systems represents
one of the drawbacks of the standard CDM model. It is interesting to ask whether dropping the assumption of initially Gaussian density fluctuations enables one to add sufficient large-scale cluster correlation to reconcile the CDM model with observational data. Indeed, previous statistical analyses of the simulations of the models considered in this work have shown that models with negative initial skewness are at least as good as Gaussian models at reproducing the observed clustering data (Moscardini et al. 1991; Messina et al. 1992; Lucchin et al. 1993; Coles et al. 1993a). In particular they succeed in producing coherence on large scales and high bulk motions by the slow non-linear process of merging of voids and disruption of low-density bridges. The 2-point correlation function and the topological analysis applied to projected catalogues extracted from these models give support to the same result (Moscardini et al. 1993; Coles et al. 1993b). Moreover Matarrese et al. (1991) showed that grouping properties are strongly dependent on the statistics of the underlying mass distribution. Second, the observed scale invariance of cluster distributions on scales $R \lesssim 20 \, h^{-1}\text{Mpc}$ (Borgani, Plionis & Valdarnini 1993) bears all the hallmarks of strongly non-Gaussian statistics. The question is whether such non-Gaussian statistics can be generated on a scale where the gravitational evolution of the matter distribution is still linear, just by the cluster selection procedure. If this proves to be impossible we need to consider another source for this non-Gaussian signature.

The plan of this paper is as follows. In Section 2 we briefly describe the considered non-Gaussian CDM models and the procedure for extracting the Lick samples from the $N$-body simulations. In Section 3 we review the PBF cluster identification algorithm and comment on some simple properties of the resulting simulated cluster catalogues. Detailed quantitative statistical analyses of the simulated cluster distributions are contained in Sections 4, 5, 6 and 7, where we also compare them with analogous results obtained for the real cluster samples studied in Papers I–IV. The aim of this work is to constrain our models for the statistics of the primordial CDM density fluctuations. The main elements of our statistical analysis are as follows: in Section 4 we evaluate the two-point cluster correlation function for the different models and cluster richness; in Section 5 we investigate the higher-order correlation statistics using a graph-theoretical construction called the minimal-spanning-tree (MST); Section 6 is devoted to a fractal analysis of the synthetic cluster samples; in Section 7 we apply the skewness test to cell counts and analyze the skewness–variance relation. A detailed discussion of the main results and a summary of our conclusions is contained in Section 8.

2 The Lick Simulations

For this analysis we need to generate realistic simulations of the angular spatial distribution of galaxies resulting from CDM models with skewed (i.e. non-Gaussian) initial fluctuations. The first step in obtaining simulated projected catalogues is to perform $N$-body computations of the spatial distribution of galaxies in such scenarios (e.g., Messina et al. 1992). We have generated
such angular simulations already in the context of a study of the topology of projected catalogues in non–Gaussian models by Coles et al. (1993b). The details of our simulation techniques are described fully in that paper, so we shall only give a brief outline here.

We consider the same types of non–Gaussian initial fluctuation statistics as Moscardini et al. (1991), Messina et al. 1992, Lucchin et al. (1993) and Coles et al. (1993a,b), namely the Lognormal ($LN$) and Chi–squared of order unity ($\chi^2$). We construct simulations such that these distributions apply to the peculiar gravitational potential $\Phi$, before the fluctuations are modulated by the CDM transfer function. The distributions split into two different types of model – positive ($LN_p$ and $\chi^2_p$) and negative ($LN_n$ and $\chi^2_n$) – according to whether the linear mass fluctuations have positive or negative skewness. The models are constructed so that $\Phi$ has the CDM power–spectrum

$$P_\Phi(k) = \frac{9}{4} P_0 k^{-3} T^2(k),$$

(2)

where $P_0$ is the primordial Zel’dovich spectrum of density fluctuations and $T(k)$ is the CDM transfer function (e.g. Davis et al. 1985)

$$T(k) = [1 + 6.8k + 72.0k^{3/2} + 16.0k^2]^{-1},$$

(3)

having considered a flat universe with Hubble constant $h = 1/2$ in units of 100 km sec$^{-1}$ Mpc$^{-1}$. Standardizing the spectrum in such a way allows a direct comparison with the standard, i.e. Gaussian, CDM (hereafter $G$) model.

We used a particle–mesh code with $N_p = 128^3$ particles on $N_g = 128^3$ grid–points [more details are given by Messina et al. (1992)]. Computations were performed at the CINECA Centre (Bologna) on a Cray YMP/432 running under UNICOS. The box–size of our simulations is $L = 260 h^{-1}$ Mpc; each particle has mass $m = 4.7 \times 10^{12} M_\odot$. We evolve our models starting from the same amplitude up to the ‘present time’ $t_0$. We define $t_0$ as the time when the galaxy two–point function is best fitted by the power–law $\xi(r) \propto r^{-\gamma}$, with $\gamma = 1.8$ in a suitable interval. To obtain the galaxies in a given simulation we proceed as follows: we smooth the initial density field with a Gaussian filter of radius $1 h^{-1}$ Mpc and identify as galaxies the set of particles inside excursion sets obtained by a threshold fixed in order to have a galaxy number density equal to $3 \times 10^{-2} h^3$ Mpc$^{-3}$, corresponding to a total number of $\sim 530,000$ galaxies in the whole box (suitable for the construction of our Lick look–a–like maps).

Different epochs will be parameterized by the bias factor $b$ defined, as usual, from the variance of linear mass–fluctuations on a sharp–edged sphere of radius $R_8 = 8 h^{-1}$ Mpc, i.e.

$$\sigma^2(R_8) = \frac{P_0}{2\pi^2} \int_0^\infty dk k^3 T^2(k) W_{TH}^2(k R_8) = \frac{1}{b^2},$$

(4)

where $W_{TH}(x) = (3/x) j_1(x)$ is a top–hat window function and $j_1$ is the Bessel function of order 1. The present time $t_0$ corresponds to $b = 1.5$ for the Gaussian model, $b = 2$ for both the positive models, $b = 0.5$ for the negative $\chi^2$ and $b = 0.4$ for the negative Lognormal. For this
study we use a subset of the simulations, comprising 6 in total. We use two different times for
the Gaussian simulation, \( G_{1.5} \) and \( G_1 \), defined such that \( b = 1.5 \) (i.e. at the present epoch)
and \( b = 1 \) respectively. The four skewed simulations, one of each type described above, are all
considered at the relevant present epoch.

The primordial gravitational potential, \( \Phi(x) \) is obtained by the convolution of a real function
\( \tau(x) \) with a random field \( \varphi(x) \),

\[
\Phi(x) = \int d^3y \; \tau(y - x) \varphi(y). \tag{5}
\]

The field \( \varphi \) is obtained by a non–linear transformation on a zero–mean Gaussian process \( w \),
with unit variance and flicker–noise power–spectrum; the function \( \tau \) is fixed by its Fourier
transform,

\[
\tilde{\tau}(k) \equiv \int d^3xe^{-i k \cdot x} \tau(x) = T(k) F(k), \tag{6}
\]

where \( T(k) \) is the CDM transfer function of eq.(3) and \( F(k) \) a positive correction factor which
we applied to have the exact CDM initial power–spectrum of eq.(2) in all our models. The
precise forms of the non–linear transformation from \( w \) to \( \varphi \) are

\[
\varphi(x) \propto e^{w(x)}, \tag{7}
\]

and

\[
\varphi(x) \propto w^2(x) \tag{8}
\]

for LN and \( \chi^2 \) respectively (Moscardini et al. 1991).

The Lick map has a characteristic depth of \( D_* \sim 210 \ h^{-1} \ Mpc \) (Groth & Peebles 1977), but
galaxies with \( D \gg D_* \) are also included in the catalogue. In order to simulate the overall extent
of the Lick catalogue we therefore need to replicate our original simulation box exploiting its
periodic boundary conditions. As the box–side is \( 260 \ h^{-1} \ Mpc \) and the solid angle we want to
study is such that \( b^{\mu} \geq 45^\circ \), we have to consider the superposition, in the \( z \)–direction, of three
levels of replicated boxes while each level, starting from \( z = 0 \), has an increasing number of
boxes; 4 at the lowest level, 16 at the intermediate and 36 at the highest level (i.e. 56 boxes).
Defining the \( z \)–cartesian axis in the direction of the Galactic north pole, we choose the origin
of our coordinate system to be at \( z = 0 \) and at the central point in the \( x, y \) coordinates of the
lowest series of connected boxes.

In order to generate our Lick look–a–like maps we assign to the \( \sim 530,000 \) galaxies an
absolute magnitude according to the Schechter (1976) luminosity function and then determine
its apparent magnitude taking into account \( K \)–corrections and expansion effects. We then
select galaxies whose apparent magnitude exceeds the corresponding value for the Lick map
\( (m_{lim} \leq 18.8) \). If the number of selected galaxies is larger than that of the Lick catalogue
\( (316,000 \ for \ b^{\mu} \geq 45^\circ) \) we perform a sparse sampling in the set of the included objects. We
have tested the robustness of our simulated catalogues to variations of the luminosity function in Coles et al. (1993b).

A possible problem, deriving from the use of replications of the original box, could be an artificial and periodic magnification of the structures present in the box. Note, however, that all the galaxies, even those which are selected more than once, are assigned a different absolute magnitude and since we are interested in the projected distribution of galaxies, most of the artificial periodicity will be probably washed out. In Coles et al. (1993b), we performed quantitative tests of any residual spurious superposition and found the effect to be very small. Grey-scale plots of the resulting angular distributions are displayed in Coles et al. (1993b). The models differ widely in visual appearance; none appears visually to match the behaviour of the Lick map in detail, but the ‘most similar’ appears to be the $\chi^2$ (negative) model which seems to reproduce qualitatively the observed bubbly appearance of the Lick counts. This is confirmed by the topology study which demonstrates that the $\chi^2_n$ model provides a reasonable fit to the quantitative measures of pattern used in that study.

3 The cluster samples

We extract cluster samples from the synthetic Lick catalogues (described in the previous section) using the same procedure adopted in Paper I by Plionis, Barrow & Frenk (1991). The cluster finding algorithm is based on identifying clusters as high peaks of the underlying galaxy cell counts, after smoothing on a suitable angular scale.

If $n_{ij}$ is the unsmeothed galaxy count in the $10 \times 10$ arcmin cell, labelled by the indices $i, j$, the corresponding smoothed count is

$$n_{ij}^* = \sum_{l=i-1}^{i+1} \sum_{k=j-1}^{j+1} w_{lk} n_{lk}. \quad (9)$$

Here, the weights $w_{lk}$ are assigned so that $w_{ij} = 1/4$, $w_{i\pm 1,j} = w_{i,j\pm 1} = 1/8$, $w_{i\pm 1,j\pm 1} = 1/16$ and their sum is unity in order to preserve the total galaxy count. This procedure is roughly equivalent to smoothing the projected galaxy distribution with a Gaussian window on a 30 arcmin scale.

After applying this smoothing procedure, we identify those cells whose smoothed count is larger than a fixed threshold value:

$$n_{ij}^* \geq \kappa \bar{n}, \quad (10)$$

$\bar{n}$ being the average cell count, which is obviously preserved after smoothing. Connected cells whose galaxy count satisfies eq.(10) form a cluster. We identify the cluster centre with the centre of the member cell having the highest galaxy density. This definition is different from that adopted in Paper I, where the cluster centre is identified as the centre of mass. We expect this difference to have no significant influence on the statistics of the cluster distribution on
scales exceeding the 30 arcmin smoothing scale. Selected clusters are found to have a quite well–defined shape, with a central peak surrounded by cells with decreasing density (see, e.g., Figure 2 of Paper I). In fact, we checked the effect of taking the two definitions of cluster centres for the real samples and found no differences between the respective clustering properties.

By choosing different values for the threshold parameter $\kappa$ in eq.(10), we construct samples of clusters having different richness, the richer corresponding to higher $\kappa$ values. Following Paper II, we take four different thresholds, corresponding to $\kappa = 1.8, 2.5, 3$ and $3.6$ (C18, C25, C30 and C36 samples, respectively). Clusters identified at higher thresholds are also included in lower $\kappa$ samples. It is worth remembering that such a cluster identification algorithm is objective and, thus, does not introduce biases arising from a visual inspection of photographic plates, as it is believed to happen for the Abell (1958) and Zwicky et al. (1961) samples.

Our simulations have been set up to mimic only the northern galactic hemisphere. We shall therefore compare them only with the clusters seen in the northern part of the actual Lick map. Furthermore, there is a strong dependence of the projected cluster number density upon the galactic latitude at low $b^\mu$ values (see, e.g., Paper II). We shall therefore apply a cutoff in galactic latitude, using both real and simulated clusters only for $b^\mu > 50^\circ$ clusters.

We apply the above procedure to the 6 Lick simulations described in the previous section, so that we end up with 24 cluster samples, whose statistical properties are compared to that of the 4 observational data sets. As an example of the resulting cluster distribution, we plot in Figure 1 the observed as well as the simulated C25 samples in quasi equal–area coordinates

$$
X = (b^\mu - 90^\circ) \sin l^\mu \\
Y = (b^\mu - 90^\circ) \cos l^\mu.
$$

(11)

As usual, $b^\mu$ and $l^\mu$ represent the galactic latitude and longitude, respectively. Also for the real data we plot only $b^\mu > 45^\circ$ clusters. Already from a visual inspection, it is apparent the remarkable difference in the number of selected clusters and in their clustering pattern between different models. In particular, some models like $G_{1.5}$, $LN_p$ and $\chi^2_p$, produce many fewer clusters than observed. In this sense, Figure 1a contains the ‘bad’ models, i.e. those that fail to account for the zero–th order statistics of cluster number density. Figure 1b contains those models ($G_1$, $\chi^2_n$ and $LN_n$) that produce a larger number of clusters. As far as the texture of the projected cluster distribution is concerned, we note that the $LN_n$ model seems to produce too large features (voids, filaments and cluster condensations) that, even after projection to a $\sim 210 h^{-1}\text{Mpc}$ depth, involve angular scales comparable to the width of the observational cone. Our Figure 1 is to be compared with Figure 2 of Coles et al. (1993b), which reports the whole galaxy distributions of the Lick map simulations. By comparing these two pictures, it is easy to recognize that cluster identification through peak selection acts as an amplifier of the underlying clustering features and makes more clear the existing difference between model and real data sets. The following sections contain a more quantitative description of the statistics of the cluster distributions.
More information about the selected cluster samples are presented in Figure 2 and Table I, where we report the number of clusters and of active (selected) cells, as well as the cluster richness, for both observational data and simulations and for each adopted $\kappa$ value. Following Paper II, we take two different definitions of cluster richness: either the smoothed number count inside the most populated cell ($R_1$), or the average count between the cells belonging to each cluster ($R_2$). Since $R_1$ values are found to have a smaller spread within the same sample, we plot only this in Figure 2, while both $R_1$ and $R_2$, together with the respective $rms$ values, are given in Table I.

The left column of Figure 2 is for the ‘bad’ models, while the right column is for the ‘good’ models. We note that the same richness–threshold relation holds more or less for all the models, so it cannot be used to discriminate between them. This is quite easy to understand because, according to our definition of richness, $R_1$ essentially depends upon the threshold chosen for cluster identification. On the other hand, $G_{1.5}$, $LN_p$ and $\chi^2_p$ generate too few very dense cells, i.e. fewer clusters than observed. Although $G_1$, $LN_n$ and $\chi^2_n$ are much better, there is a systematic tendency to produce a number of clusters which is smaller by a factor $\sim 2$ with respect to observations. It seems therefore that none of these CDM models, either Gaussian or non–Gaussian, succeeds in generating as many clusters as observed. We do not believe that this means that no CDM model is capable of generating the number of clusters seen in the Lick map. Instead, we think that the behaviour in the simulations can be ascribed to the finite resolution of our $N$–body simulations. Since the parent simulation cubes have a size of $260 h^{-1}$Mpc with $128^3$ grid points, we are not able to resolve scales below $2–3 h^{-1}$Mpc, which, at the depth of the Lick map ($\sim 210 h^{-1}$Mpc), corresponds to an angular size of $\sim 50$ arcmin. Thus, even after smoothing the cell counts over the 30 arcmin scale used in the clustering identification procedure, we expect there still to be a residual numerical smoothing, whose effect is to suppress the projected density fluctuations and so to suppress the number of cells belonging to high–threshold excursion sets.

In order to check whether this is the correct explanation and to see the size of the effect of finite numerical resolution, we also applied the same cluster identification procedure after binning the projected galaxy distribution in $20 \times 20$ arcmin cells. This amounts to take an effective smoothing scale of $1^\circ$, thus larger than (or at least comparable to) the scale of numerical smoothing. The results for $\kappa = 1.8$ and 2.5 are summarized in Table II, where we report for real data and for each model the number of selected clusters and of active cells. In this case we get qualitatively the same trends in the results at higher angular resolution, but a much better agreement with real data is obtained. The $G_{1.5}$, $LN_p$ and $\chi^2_p$ models still produce too few clusters, so that they are virtually ruled out already by this test. Although better, the $LN_n$ model gives a quite high number of C25 clusters and, vice versa, a quite low number of C18 clusters. This is not surprising, since the $LN_n$ model generates large coherent structures, so that, as the threshold decreases, many clusters percolate to form larger structures. Therefore, their total number increases less than for a model producing more isolated structures. Even in
this case, the best models are represented by the $G_1$ and $\chi_2^n$ models, which produce an adequate number of clusters. On the one hand, these results confirm those obtained by selecting clusters from $10\times10$ arcmin cell counts; on the other, they warn us that we must pay attention to the limited numerical resolution. Nevertheless, we still prefer to adopt the original cluster identification from counts in $10\times10$ arcmin cells, since we wish to compare the results we obtain in this paper with the previously–published results on observational data sets, which are based on it. Only in those cases where the comparison between data and simulations becomes particularly problematic we will more carefully investigate the effect of decreasing the angular resolution.

Obviously the large number of simulated samples we use generates a large amount of information. To keep the paper down to a manageable size, we shall discuss only the main results and concentrate on showing figures for the ‘good’ models (i.e., $G_1$, $LN_n$ and $\chi_2^n$), while results for the ‘bad’ models will be summarized in tables.

4 The 2–point correlation analysis

The angular 2–point correlation function, $w(\vartheta)$, for the PBF clusters has been investigated in Paper II. Here we will use the estimator

$$w(\vartheta) = \lambda^2 \frac{DD(\vartheta)}{RR(\vartheta)} - 1,$$

(12)

where $DD(\vartheta)$ and $RR(\vartheta)$ represent the number of cluster pairs at separation $\vartheta$ in the real sample and in a random sample having the same boundaries as the real one. The random sample contains a number of points which is $\lambda$ times larger than the real one. We find that $\lambda = 5$ is enough to get stable results. (Note that in Paper II, $RR(\vartheta)$ was evaluated by averaging over 20 different random samples, all having the same number of points as the real data set. Since the total number of pairs increases as $\lambda^2$, we expect that the procedure adopted here amounts to take 25 random realizations.) A comparison of the results presented in Figure 3 for real clusters with those in Figure 3 of Paper II shows a complete agreement.

In Figures 3a, b and c we show $w(\vartheta)$ for the $G_1$, $\chi_2^n$ and $LN_n$ models, as compared to real data, at each $\kappa$ threshold value. The error bars have been estimated through the bootstrap resampling technique (Ling, Frenk & Barrow 1986). Also plotted are the best least–square fit models

$$w(\vartheta) = A \vartheta^{1-\gamma}$$

(13)

for both real clusters (dotted lines) and simulations (dashed lines). In Paper II we fixed $\gamma = 2$ so to leave the amplitude $A$ as the only parameter to be fitted. Here we prefer to leave free both $A$ and $\gamma$, since different models develop significantly different slopes. In Table III we show the results for all the models considered, along with the corresponding $rms$ uncertainties and the angular scale range where eq.$(13)$ provides a satisfactory fit.
As for the $G_1$ model, we note that it gives a systematically smaller correlation amplitude at scales $> 1^\circ$, while providing consistent results at small angles. We observed that, since the simulated clusters are identified from smoother distributions than the real ones, they correspond at a fixed $\kappa$ value to relatively higher, and consequently more strongly clustered, peaks. This makes even more significant the lack of correlation displayed by the $G_1$ clusters, at any $\kappa$ value. Therefore, we conclude that Gaussian CDM models are not able to account for the large–scale power traced by the cluster distribution, even when identifying the present dynamical time in $N$–body simulations with more evolved (i.e., $b \simeq 1$) configurations.

In contrast, the $LN_n$ model always produces systematically higher large–scale correlations, with a slope of $w(\vartheta)$ which is smaller than observed (see Figure 3c). This can be easily interpreted by looking at the distribution of $LN_n$ clusters, as shown in Figure 1: the presence of huge coherent structures generates an excess of large–scale clustering and does not allow $w(\vartheta)$ to decline rapidly. We also investigated the effect of taking a less evolved $LN_n$ configuration. Although in this case a correct large–scale clustering amplitude is recovered, large–scale phase coherence due to the highly non–Gaussian primordial statistics generates an even flatter profile of $w(\vartheta)$.

Better results are obtained from the $\chi^2_n$ model. In this case (see Figure 3b) both the clustering amplitude and the slope are better reproduced. The still existing discrepancy could well be due to the fact that our simulated clusters correspond to relatively higher peaks than real ones. As a consequence, we expect to have a higher $w(\vartheta)$, with a slightly larger slope.

As far as the ‘bad’ models are concerned, we see from Table III that they give unacceptably steep profiles of $w(\vartheta)$, thus confirming that all of them fail at accounting for the clustering in the distribution of galaxy clusters.

As already observed in Paper II, the increasing trend of the correlation amplitude with the $\kappa$ threshold translates into a clustering–richness dependence. In Figure 4 we plot this relation for both the ‘good’ models and the observational data. A strong correlation is always generated, although the details depend on the model. In order to compare data and simulations, we shall remember again that our simulated clusters have for a given $\kappa$ (thus for a given richness) an excess clustering amplitude. This could be the explanation for the discrepancy between the $\chi^2_n$ and $LN_n$ models and real data, while it should make even less reliable the $G_1$ model, which already produces too a low clustering strength.

As a brief summary of the correlation analysis of the cluster samples, we can say that the $G_{1.5}$, $LN_p$ and $\chi^2_p$ clearly fail to account for observational results. Even within the ‘good’ models, the $G_1$ one does not produce enough large–scale clustering. Both the $\chi^2_n$ and the $LN_n$ gives reasonable results, although the former seems to be better suited to provide an adequate amount of large–scale correlation amplitude.
5 The MST analysis

The minimal spanning tree (MST) for a given point distribution is defined as the unique graph connecting all the points, with no closed loops and having minimal length (e.g., Ore 1962). The construction of the MST graph proceeds as follows. For our angular samples, let us start with a randomly chosen cluster and connect it to its nearest neighbour. At this first step, the tree $T_1$ has only one branch of length $\vartheta_1$. At the $k$-th step, we define the distance of the $i$-th point, still not belonging to the MST, from the $T_{k-1}$ tree as

$$\vartheta_{i,T_{k-1}} = \min_{j \in T_{k-1}} \vartheta_{ij}. \quad (14)$$

Thus, for a distribution of $N$ points the MST is given by $T_{N-1}$ and contains the set of branch lengths $\{\vartheta_i\}_{i=1}^{N-1}$. From its definition, it follows that the MST construction is unique and independent of the point which is used to start building the tree. There are several reasons why the MST is a useful tool in clustering analysis. First of all, it is completely determined only once the position of each single point is known, so that it conveys informations about correlations of any order. Moreover, when one branch is added to the tree, its position does not depend on that of the previously added branch, so that we can say that the MST construction is delocalized. For the above reasons, the MST is particularly suited to emphasize the main features of the global texture of a point distribution, such as its connectivity, filamentarity, etc.

The MST has been applied in a cosmological context by Barrow, Bhavsar & Sonoda (1985) who showed that it is efficient at discriminating between different kinds of models. Bhavsar & Ling (1988) used the MST to study the filamentarity of the spatial galaxy distribution in the CfA redshift survey, while Plionis, Valdarnini & Jing (1992) performed a similar analysis using a redshift sample of Abell and ACO clusters. The construction of the MST has also been recognised as a useful tool for estimating the fractal dimension of a point distribution (van de Weygaert, Jones & Martínez 1992).

In Figure 5 we show the edge–length frequency distribution, $F(\vartheta)$, for the $\kappa = 1.8$ (left column) and $\kappa = 3$ (right column) clusters generated by the ‘good’ models. The $F(\vartheta)$ distribution is defined in such a way that $F(\vartheta) \, d\vartheta$ is the fraction of branches in the tree with length between $\vartheta$ and $\vartheta + d\vartheta$. Barrow, Bhavsar & Sonoda (1985) have shown that a Poissonian distribution is characterized by a Gaussian $F(\vartheta)$, with the maximum occurring at the average branch length, $\langle \vartheta \rangle$. In Figure 5, the solid histograms are for real data, while the dashed ones are for the simulations. By comparing the results for the two different thresholds, one can see again the increasing clustering strength with $\kappa$ value, which is reflected in the increasing skewness of the $F(\vartheta)$ profile. The spanning trees of the different models are all different, but the really striking differences are between the model trees and the tree of the real cluster distribution. In particular, all the simulations have broader and more skewed $F(\vartheta)$ distributions, with an excess of both very short and very long branches. Once more, this is probably due to the fact that the simulated clusters represent higher peaks in the background galaxy distributions than the
real clusters. In order to check the consistency of the $F(\phi)$ distributions in a more quantitative way, we applied a non-parametric Kolmogorov–Smirnov (KS) test to the various distributions. In Table IV we summarize the results for the ‘good’ models. All the models are rejected at a quite high confidence level, probably because the simulated samples contain systematically higher peaks.

In order to avoid such problems, we repeated the MST analysis for the C18 clusters, identified from the $1^\circ$ smoothed count (see §3). The resulting $F(\phi)$ shows a much better agreement with real data, although even in this case the $\chi_n^2$ model fares marginally better than the others. This is also confirmed by the results of the KS test, reported in Table IV. This shows that the capability of the MST to enhance high-order statistical information makes it a useful instrument for detecting differences between models, even in presence of rather limited data sets.

6 A fractal analysis

Statistical methods based on fractal analysis are particularly suited to investigate the scaling properties of point distributions, in several physical contexts (e.g., Mandelbrot 1982). A simple concept which is useful in understanding scaling properties is the fractal, or Hausdorff, dimension. For a given point distribution in a $d$-dimensional ambient space ($d = 2$ for our angular cluster distributions), let $N_c(\epsilon)$ be the number of cells of size $\epsilon$, which are required to completely cover the set. For a fractal distribution, we expect that at very small $\epsilon$ values, $N_c(\epsilon)$ should scale according to a pure power-law,

$$N_c(\epsilon) \propto \epsilon^{-D_o}.$$  \hfill (15)

Here, $D_o$ is defined as the box-counting or capacity dimension, which, in most practical cases, gives a close estimate of the Hausdorff dimension. According to the definition (15), it must be $0 < D_o \leq d$. Note that the capacity dimension depends only on the geometry of the distribution via the number of non-empty cells. It does not measure anything to do with the clustering properties of the set as measured by the correlation functions. To characterize this sort of information, a continuous set of scaling indices must be introduced, which in some sense are equivalent to the whole sequence of correlation functions, required to describe any statistical system. To this purpose, let us define $p_i(\epsilon) = n_i(\epsilon)/N$, as the probability measure in the $i$-th cell, containing $n_i(\epsilon)$ out of the total number $N$ of points. Accordingly, the generalized Renyi dimensions (Renyi 1970) are defined as

$$D_q = \frac{1}{q-1} \lim_{\epsilon \to 0} \frac{\log \sum_i [p_i(\epsilon)]^q}{\log \epsilon}; \quad D_1 = \lim_{q \to 1} D_q.$$  \hfill (16)

Note that for $q = 0$ the above equation reduces to the definition (15) of capacity dimension. The $q = 2$ case is for the correlation dimension, which, at small scales, is related to the slope of
the 2-point correlation function according to $D_2 = 3 - \gamma$. The *multifractal* dimension spectrum of eq.\( \text{(16)} \) gives a comprehensive description of the scaling properties of a point distribution; positive-order dimensions are sensitive to the statistics inside the overdense regions, while the negative–$q$ ones account for the underdensities. It is also possible to show that, under general conditions, $D_q$ must be a non-increasing function of $q$. A particularly simple case occurs when $D_q = \text{const}$, independent of $q$. In this case, the distribution is said to be monofractal and a single scaling index completely specifies the statistics. More in general, the shape of $D_q$ gives a clear indication of the nature of the clustering (e.g., Jones, Coles & Martínez 1992) and precise relations can be found between the hierarchy of correlation functions and the multifractal dimension spectrum (Balian & Schaeffer 1989; Borgani 1993). A detailed review of multifractal concepts is given by Paladin & Vulpiani (1987).

The formal definition \( \text{(16)} \) of multifractal dimensions is given in the limit of infinitesimally small scales. Obviously, in practical cases only a finite number of points is available and only a limited scale range can be studied. In addition, in most physical situations, different scaling regimes are expected at different scales, thus giving rise also to an upper limit to the scale range where self-similarity develops. This is just what happens in the study of the large-scale structure, where the distributions of galaxies and galaxy systems develop a well-defined scaling only at small scales, while homogeneity is recovered at large scales.

Fractal analysis methods have been applied in recent years in cosmology to describe the scaling properties of galaxy clustering, using both observational data and $N$-body simulations. The emerging picture is that any self-similar behaviour is confined to small scales ($< 5 \, h^{-1} \text{Mpc}$), where clustering is non-linear (e.g., Martínez et al. 1990; Valdarnini, Borgani & Provenzale 1992; Colombi, Bouchet & Schaeffer 1992; Yepes, Domínguez–Tenreiro & Couchman 1992).

In Paper IV we realized a detailed multifractal analysis of the PBF cluster samples by applying several dimension estimators. We found that the C36 clusters develop a good scaling behaviour up to angular scales $\sim 5^\circ-6^\circ$, which correspond to a physical scale $\sim 20 \, h^{-1} \text{Mpc}$ at the depth of the Lick map. It is clear that, although non-linear gravitational clustering furnishes a dynamical mechanism for generating self-similarity, it only does so up to scales of a few Mpc. Thus, at the larger scales of cluster clustering, the scaling behaviour detected cannot have a dynamical origin. At such scales, the non-linearity of the clustering has a statistical origin and resides in the identification of clusters as exceptionally high peaks of the underlying density field. The question thus arises as to whether the scaling detected up to $\sim 20 \, h^{-1} \text{Mpc}$, which is the unique imprint of a strong non-Gaussian statistics, is generated just by means of a high-peak selection on a Gaussian background or requires something more. Here we apply the same analysis as in Paper IV to our synthetic cluster samples in order to check whether the existence of the scaling behaviour in the real data constrains the models with primordial non-Gaussian fluctuations.

A number of different methods have been devised to provide estimates of scaling dimensions from finite data sets; these rely on different definitions of dimension and use different kinds of
approximation. One must be very careful to use appropriate methods when dealing with poor statistics or when self–similarity develops only over a limited scale–range (e.g., Borgani at al. 1993).

The first method we use is based on the evaluation of the moments of counts of neighbours (Grassberger & Procaccia 1983; Paladin & Vulpiani 1984). For our angular cluster distributions, we estimate the partition function

$$Z(q, \vartheta) = \frac{1}{N} \sum_{i=1}^{N} [p_i(\vartheta)]^{q-1},$$

where $p_i(\vartheta) = n_i(\vartheta)/N$, with $n_i(\vartheta)$ the number of neighbours within the angular distance $\vartheta$ from the $i$–th object. For a fractal structure, we expect

$$Z(q, \vartheta) \propto \vartheta^{\tau_q},$$

with the resulting multifractal dimension spectrum given by

$$D_q = \frac{\tau_q}{q-1}.$$  

Though it converges rapidly for $q > 1$, this method is extremely sensitive to discreteness effects when $q$ is negative, and one must use a different method to probe that regime.

A second method, which is better suited in the $q < 1$ regime represents a kind of inversion of the previous one; instead of counting the number of points within a given radius, it is based on measuring the radius of the sphere containing a fixed number of points (Grassberger, Badii & Politi 1988). Let $\vartheta_i(p)$ be the smallest radius of the disk centred at the $i$–th point and containing $n = pN$ objects. Then, the partition function

$$W(\tau, p) = \frac{1}{N} \sum_{i=1}^{N} [\vartheta_i(p)]^{-\tau}$$

describes the scaling properties of the distribution. For a fractal structure, $W(\tau, p) \propto p^{1-q\tau}$, and the corresponding multifractal spectrum is given by eq. (19).

A further characterization of a multifractal set can be given in terms of the singularity spectrum (Hentschel & Procaccia 1983). Let us define the local dimension $\alpha_i$, such that the scaling inside the $i$–th cell is $n_i(\vartheta) \propto \vartheta^{\alpha_i}$. Moreover, let $S(\alpha)$ be the subset containing all the points around which the local dimension lies in the range $[\alpha, \alpha + d\alpha]$. We define the singularity spectrum, $f(\alpha)$, as the Hausdorff dimension of $S(\alpha)$. It is possible to show that the two pairs of variables, $(q, \tau_q)$ and $(\alpha, f(\alpha))$, are related by a Legendre transform:

$$\alpha(q) = \frac{d\tau_q}{dq}; \quad \tau_q = q\alpha - f(\alpha),$$

15
so that they give equivalent descriptions of a fractal structure. According to eq. (21), if a distribution is monofractal, then only one $\alpha$ value is allowed and the singularity spectrum degenerates into a single point, $(D_o, f(D_o) = D_o)$. In general, multifractal behaviour is reflected in a spread of $\alpha$ values within a finite interval, with its smallest and largest values determined by the scaling inside the most overdense and underdense regions, respectively:

$$\alpha_{\text{min}} = \lim_{q \to +\infty} D_q \quad ; \quad \alpha_{\text{max}} = \lim_{q \to -\infty} D_q. \quad (22)$$

In this sense, we can say that a multifractal structure is characterized by local scaling properties (different local dimensions around different points), while a monofractal structure has global scaling. Furthermore, the decreasing slope of $D_q$ constrains $f(\alpha)$ to be a convex function, with its maximum value equal to the Hausdorff dimension.

In Figure 7 and 8 we plot the $Z(q, \theta)$ partition function for the C36 and C25 clusters, respectively. (We have analysed the C30 and C18 samples with this method but we do not show the results here; they represent intermediate cases.) From left to right, we display the results for the $G_1$, $\chi^2_n$ and $LN_n$ models, while upper and lower figures are for $q = 0$ and $q = 4$, respectively. In each panel we plot both $Z(q, \theta)$ and the corresponding local dimension, $D_q(\theta)$, evaluated from a three point linear regression on the partition function slope. For an ideal fractal structure, $D_q(\theta)$ should remain constant. In general, the $\theta$ range where it stays flat gives a precise idea of the scales at which self–similarity (if any) takes place. According to the definition (17), the amplitude of the partition function is not normalized to be the same for two distributions having the same scaling properties but a different number of points. For this reason, we are only interested to compare the slopes of $Z(q, \theta)$ for data and simulations and not their amplitudes; only the slopes are related to the corresponding fractal dimensions. In order to correct for boundary effects in the computation of $Z(q, \theta)$ we take as centres only those clusters whose angular distance from the sample boundaries is $< \theta$.

For $q = 0$, which corresponds to the estimate of the Hausdorff dimension, for both the C36 and the C25 samples the local dimension shows a smooth transition from $D_o \approx 0$ at small scales, to $D_q \approx 2$ at $\theta \sim 6^\circ$, which is the signature of large–scale homogeneity (at least in the projected distribution). Note that all the three models generate a homogeneous geometry of the distributions roughly at the same scale as observational data. At smaller scales, the best model is the $LN_n$ one, which correctly reproduces the partition function slope. The other two models slightly underestimate $D_q(\theta)$, although $\chi^2_n$ seems to be more successful than $G_1$, especially at small angles. This small–scale behaviour can be interpreted by saying that the low number of objects does not allow to resolve the underlying distribution, so that the algorithm measures nothing but the dimension of each single point, which is indeed zero.

More interesting is the $q = 4$ case for the C36 clusters. As already found in Paper IV, the real cluster distribution displays a rather flat $D_q(\theta)$ shape at $\theta < 6^\circ$, which indicates small–scale scaling. In contrast, the $G_1$ model, does not succeed in generating any small–scale self–similarity. Instead, at $\theta < 2^\circ$ the partition function rapidly flattens, so that the local
dimension declines to zero. Unlike the \( q = 0 \) case, we do not believe that this is only an effect of discreteness for two main reasons. First, for \( q > 1 \) eq. (17) assigns most weight to the overdense parts of the distribution, where the sampling on the scales we are looking at should be good. Second, if the limited statistics were the reason, we should expect to find a similar behaviour even for the other two models, which produces a comparable number of C36 clusters. However, this is not the case for both the \( \chi^2_n \) and \( LN_n \) models. Instead, they better reproduces the small–scale flat shape of the local dimension, with a resulting fractal dimension \( D_q \ll 1 \) to characterize the clustering inside the overdense regions. In particular, note the remarkable good agreement between data and \( LN_n \) model at \( \vartheta \sim 6^\circ \), although at larger scales it generates too much clustering and, consequently, too small a dimension.

As already observed in Paper IV, as lower thresholds are considered any self–similarity disappears. This is confirmed by Figure 8, which is the same as Figure 7, but for the C25 clusters. For both data and simulations no scale range exists where \( D_q(\vartheta) \) remains nearly constant. Even in this case, the \( G_1 \) cluster distribution again underestimates the dimensions at \( \vartheta \ll 2^\circ \), while both \( \chi^2_n \) and \( LN_n \) fare much better.

Although we have also realized a similar scaling analysis using the \( W(\tau, p) \) partition function, for reasons of space we prefer not to show the partition function results here. We note, however, that these results confirm those coming from the \( Z(q, \vartheta) \) function and extend them even into the regime where \( q < 1 \); only the \( \chi^2_n \) and \( LN_n \) models develop a self–similar clustering up to \( \vartheta \sim 5^\circ–6^\circ \), similar to that detected for real data (see Figures 2 and 10 in Paper IV for the results of the \( W(\tau, p) \) analysis on the PBF samples).

In Figure 9 we plot the \( D_\tau \) multifractal dimension spectrum for the two non–Gaussian models and for real data. The dimensions have been evaluated by a log–log linear regression of the \( W(\tau, p) \) values in the \( p \) range where self–similarity develops. Also plotted is the \( f(\alpha) \) singularity spectrum defined by eq.(21). Error bars are standard deviations in the linear regression. No similar plot has been produced for the \( G_1 \) model, since it does not produce any fractality in the cluster distribution and no fractal dimension can be defined. Both the \( \chi^2_n \) and the \( LN_n \) models produce slightly lower dimensions. While this difference is not significant for \( \tau < 0 \), it is for \( \tau > 0 \). This is also reflected by the values taken by the local dimension \( \alpha \); note that the \( \alpha_{min} \) value is always smaller for the non–Gaussian simulations than for the real data, thus indicating the presence of stronger singularities. This difference could be partly due to the fact that the clusters selected in the simulations correspond to relatively higher peaks than real clusters. In fact, in a multifractal structure, the dimension of the distribution of high peaks is always smaller than that for the distribution of lower peaks.

In brief, the main result of this section is that the self–similar clustering displayed by the C36 cluster sample at \( \vartheta \ll 6^\circ \) (corresponding to \( R \ll 20 \, h^{-1}\text{Mpc} \) in the spatial distribution) is not generated by the Gaussian model, instead it requires the presence of initial phase correlations, such that provided by the \( \chi^2_n \) and \( LN_n \) models.
7 Cell–count Skewness

The analysis of the skewness of cell counts has been widely advocated in recent times as a probe of the large–scale structure of the galaxy distribution (Efstathiou et al. 1990; Saunders et al. 1991; Loveday et al. 1992). Coles & Frenk (1991) described in detail the physical motivation for the usefulness of skewness as a diagnostic for the distribution of cosmic structures. In particular, because the skewness is the lowest–order imprint of non–Gaussian statistics, it is likely to be a powerful test of the nature of initial conditions. Coles et al. (1993a) used the skewness of the three–dimensional distribution to discriminate between different non–Gaussian models for initial conditions; in this section we apply the skewness analysis to our projected cluster distributions, for both real data and ‘good’ models.

Let us divide the surveyed area of the sky into \( N_c \) cells of side \( \vartheta \) and let \( N_i \) be the object number count within the \( i \)–th cell. Then, if \( \bar{N} = \sum_{i=1}^{N_c} N_i / N_c \) is the average count within such cells, we define \( \delta_i = N_i / \bar{N} - 1 \) to be the relative fluctuation in the number count. The statistics of the distribution can be described in terms of the moments

\[
\langle \delta^n \rangle = \frac{1}{N_c} \sum_{i=1}^{N_c} \delta_i^n .
\]

(23)

For \( n = 2 \), eq.(23) defines the variance, \( \Sigma^2 = \langle \delta^2 \rangle \); \( n = 3 \) gives the skewness, \( \Gamma = \langle \delta^3 \rangle \). A serious limitation in the analysis of the moments of cell counts occurs when small cells are considered, which contain only a small number of objects. In cases where most cells contain only one cluster, the distribution is said to be dominated by shot–noise. In less extreme circumstances, the discrete nature of the distribution always imposes a non–vanishing skewness. It is common use to account for such discreteness effects by assuming that the point distribution represents a Poissonian sampling of a continuous density field. In this case, it can be shown that the variance \( \sigma^2 \) and the skewness \( \gamma \) of the continuous density field are related to those of the discrete realization according to

\[
\sigma^2 = \Sigma^2 - \frac{1}{N}, \quad \gamma = \Gamma - \frac{3\sigma^2}{N} - \frac{1}{N^2}
\]

(24)

(Peebles 1980; Saunders et al. 1991; Coles & Frenk 1991). It must be stressed, however, that this is only a model of the discreteness effects; the data need not be well represented by such a Poissonian sampling, so the statistics obtained by subtracting off the discreteness terms in eq. (24) need not correspond to anything physical. In our case we are dealing with clusters defined as peaks of the underlying galaxy density field and this is far from a Poisson sample. The equations (24) are not expected to provide a valid discreteness correction for our case. We have verified this suspicion with our data: the ‘raw’ cell–counts give very stable variance and
skewness behaviour while the ‘corrected’ results display only noise. For the remainder of this section we will not consider any kind of discreteness correction.

Coles & Frenk (1991) discussed the hierarchical variance–skewness relation

$$\gamma = 3Q\sigma^4.$$  \hspace{1cm} (25)

An expression of this form is expected to hold in a variety of clustering scenarios: hierarchical clustering in the non–linear regime; quasi–linear growth of Gaussian perturbations; lognormal density distributions. Even biased CDM models follow this form to some degree of accuracy. The exact value of the $Q$ parameter depends on the details of the model; observational results suggest $Q \simeq 1$ for galaxies and $Q \simeq 0.6–0.8$ for clusters.

In order to test the reliability of eq. (25), we plot in Figure 10 log $\Gamma$ vs. log $\Sigma^2$ for the C30 and C18 cluster samples, for both real and simulated data. Error bars correspond to one standard deviation, as evaluated by the bootstrap resampling procedure. The dashed lines correspond to the relation (24), with $Q = 0.6$. The hierarchical model provides a remarkably good fit in all the cases, independent of the initial conditions and the richness of the clusters selected, at least at the scales of non–linear clustering. On the other hand, in the weak clustering regime eq. (25) no longer applies and the skewness rapidly declines. The stability of the result is also confirmed by Table 5, where we report the best–fit values for $Q$, along with the respective standard deviations. Although Figure 10 gives the impression of a remarkably good fit, the uncertainties in the $Q$ values are quite large. We do find, however, that a non–vanishing skewness is always detected at a 2$\sigma$ level. The corresponding variance–skewness relation closely follows eq. (25), with $Q \simeq 0.6$. Although this result demonstrates the remarkable stability of the hierarchical model for the projected cell–counts, it does show that the skewness is not effective at distinguishing the details of the different models. In the case studied in this paper, a combination of projection and limited statistics acts to weaken the usefulness of this test compared to its three–dimensional analogue.

8 Discussion and Conclusions

In this paper we have analysed the statistical clustering properties of clusters obtained by applying an objective overdensity criterion to the Lick map and various simulations of it. Analysing clusters in this way, rather than the individual galaxies, is a very efficient way to pick out the essential features of the underlying clustering pattern. The calculation of clustering properties such as the two–point correlation function for the entire mock Lick maps is of course possible and indeed has already been done (Moscardini et al. 1993), but it is a laborious task. Using only the high–density regions allows one to describe the projected pattern at less cost in terms of work. Of course, some of the clusters selected by our criterion may not represent true physical associations of galaxies in three–dimensional space, but this does not matter.
Whatever they are, we compare objects selected in precisely the same way in both the real data and the simulations so it is irrelevant to know whether the clusters are bound structures or chance projections.

The clusters we analyse show up differences between the different models even at the ‘zero’ order level. The number of clusters selected by applying the same criterion to the different models depends very strongly on the nature of the model and we can rule some of our models out with extremely high confidence using just this simple statistic on its own. In particular, the standard CDM model with Gaussian initial conditions and $b = 1.5$ generates a projected galaxy distribution which is too smooth and consequently fails to account for the observed number of PBF clusters. Although this finding seems to be at variance with respect to the findings of White et al. (1987) about the number of Abell clusters produced in a biased CDM model, nevertheless we should bear in mind that we are not selecting precisely the same kind of objects. It has been suggested that Gaussian CDM models might be satisfactory if galaxy formation proceeds in such a way that galaxies are not significantly more clustered than the dark matter. In the ‘low–bias’ ($b \simeq 1$) CDM model (e.g., Couchman & Carlberg 1992) the present epoch is identified with a configuration which is dynamically much more evolved that the standard, biased, version. The greater dynamical evolution allows the model more time to build up coherence on larger scales. Moreover, the recent detection of temperature anisotropy in the cosmic microwave background (Smoot et al. 1992) seems to require a low bias parameter if the temperature anisotropy is interpreted as the imprint of primordial fluctuations in the dark matter density. We have found that this model does indeed generate a larger number of clusters such that, taking into account the numerical smoothing in our simulated Lick maps, is in reasonable agreement with real data.

Although this model does survive the zeroth–order analysis, it runs into trouble when we apply more detailed statistical descriptors. In particular, the 2–point cluster correlation function has a much lower amplitude than the observed correlation function and has a much steeper slope on large scales. This is a consequence of the shape of the CDM primordial fluctuation spectrum and is an unavoidable consequence of Gaussian models which have no phase coherence capable of generating large–scale power on scales where there is little primordial power.

Given the apparent failure of the Gaussian CDM models to account for all the data, it is natural to ask whether the CDM hypothesis can be rescued by the introduction of non–Gaussian primordial fluctuations. The initially skew–positive CDM models we have considered fail in a similar way to the Gaussian model in both number and correlation strength of clusters. These models introduce a phase coherence only on small scales and cannot alleviate the lack of large–scale power in the CDM spectrum. On the other hand initially skew–negative models are generally successful at accounting for the cluster number–richness function and the two–point correlation length. This is particularly true for $\chi^2_n$ which is the most successful of all the models we have considered.

Although the two–point correlation analysis obviously gives us useful information about
the nature of clustering in the models and the real data, the visual texture, which is also strikingly different in the models, is dominated by higher–order correlations which are difficult to measure correctly. We therefore decided to subject our models to three further statistical analyses, which are more sensitive to high–order correlations and could therefore provide more effective discrimination between the models and the data than the simpler statistics.

We used the Minimal Spanning Tree in an attempt to measure the intrinsic ‘filamentariness’ of the cluster distribution. The analysis shows up nicely a number of quantitative differences between all the models and, again, the $\chi^2_n$ model emerges as the one that agrees best with the real data.

The multifractal spectrum of a data set reveals information about the scaling behaviour of higher–order moments and is related closely to the non–Gaussian character of the distribution (Balian & Schaeffer 1989; Jones, Coles & Martínez 1992; Borgani 1993). We again find that this kind of analysis can distinguish between our models and the scaling behaviour we see, particularly in the high density regions, clearly favours the skew–negative models. A remarkable aspect of the multifractal analysis is the wide difference between Gaussian and skew–negative models in the produced scaling properties: for the C36 clusters, the Gaussian model clearly fails to generate the self–similar behaviour displayed by real data at the scales of non–linear clustering ($\lesssim 20\, h^{-1}\text{Mpc}$; see Paper IV), while both $\chi^2_n$ and $LN_n$ are successful. The sensitivity of this test suggests that the observed self–similarity at scales where the gravitational clustering is still in the linear regime can be justified on the ground of initial phase correlations.

As a final test we also applied the skewness analysis, which represents the lowest–order statistics to detect a non–Gaussian behaviour. We find that a variance–skewness relation of hierarchical type is always satisfied, for both the unbiased Gaussian model and the skew–negative models.

All these results are in good agreement with a previously–published analysis of the same Lick map simulations using two–dimensional topological characteristics by Coles et al. (1993b). We have also performed a correlation analysis of individual galaxies in the simulated Lick maps, comparing them with the results from the APM data (Moscardini et al. 1993). This analysis also shows that Gaussian or skew–positive CDM models suffer from a lack of large–scale power (regardless of the bias parameter employed), whereas the skew–negative models have no problem to produce copious large–scale power. A consistent picture thus emerges: the CDM model with Gaussian fluctuations cannot account for the properties of clusters in the Lick map, as measured by a number of independent statistical tests, and, in order to reconcile CDM with the data, one needs a distribution of fluctuations with negative initial skewness.

The qualitative agreement between these various statistical descriptions (number–richness, correlation functions, MST, multifractal, topology, skewness) demonstrates the usefulness of these methods at quantifying the large-scale clustering. They all emphasize different aspects of the clustering pattern – different descriptors show up certain distinguishing features of different models – so they are not all displaying the same information as contained in the two–point
correlation function. The agreement between the results of our analysis of projected catalogues with similar analyses of the spatial distribution of matter in these models (Moscardini et al. 1991; Messina et al. 1992; Coles et al. 1993a; Lucchin et al. 1993) also shows that using projected data is still a very worthwhile exercise given the enormous size of the data sets available. This points is even more remarkable in view of the recent compilation of extended angular cluster samples, also based on overdensity criteria, selected from the APM (Dalton et al. 1992) and COSMOS (Lumsden et al. 1992) galaxy surveys.

In the introduction to this paper, we asked two questions concerning non–Gaussian primordial fluctuations. The first was whether using skewed initial data could add sufficient large–scale power to reconcile the CDM hypothesis with observations. We have only explored a small (infinitesimal) subset of all possible non–Gaussian models, but even so we have found that we can make models that agree much more closely with the observations than the Gaussian model. We believe therefore that we have answered this question with a firm ‘yes’.

The second question that arises is whether it is necessary to consider non–Gaussian fluctuations to solve the large–scale structure problem. Of course, the two–point correlation properties of galaxies and clusters could be brought into agreement with the data by simply adding more primordial power to the power–spectrum, either by changing the background cosmology to have a low value of $\Omega h^2$ or by adding a source of large–scale fluctuations such as a Hot Dark Matter component. Gravitational evolution generates phase correlations in the non–linear regime in a manner which is coupled to the primordial power spectrum. It is consequently difficult to distinguish the effects of extra power from those of intrinsic phase correlations. This is particularly a problem with angular galaxy catalogues where projection induces a mixing of length scales which further clouds the issue. All our simulations incorporate the CDM power spectrum as initial data and we have not looked at spectra with more primordial power so we cannot say for certain that non–Gaussian fluctuations are indicated by the observations regardless of the form of the initial power–spectrum. It is clear, however, that non–Gaussian CDM models are a viable alternative to the other solutions to the large–scale structure problem and one should not discard them until the observations unambiguously rule them out.

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Figure captions

Figure 1. The distribution of C25 clusters in the quasi equal area coordinates defined by eq.(11), for both the ‘bad’ (Fig.1a) and the ‘good’ (Fig.1b) models. In both cases, the upper left panel is for the real clusters. The remarkable difference in the number density of objects produced by the different models is readily apparent. It is also clear how clusters can amplify the clustering pattern of the underlying galaxy distribution (cf. Figure 2 of Coles et al. 1993b).

Figure 2. The number of selected clusters $N_{cl}$, the number of ‘active’ cells belonging to clusters $N_{cells}$ and the cluster richness $R_1$ (see text) are plotted from top to bottom, as functions of the cluster identification threshold $\kappa$ (see eq.[10]). Left and right panels are for ‘bad’ and ‘good’ models, respectively. Filled circles are for the real clusters, the open circles for the Gaussian models, the filled squares for the $\chi^2$ models and the open squares for the Lognormal models.

Figure 3. The 2–point correlation functions, $w(\vartheta)$, for the ‘good’ models, for each $\kappa$ value. Figure 3a, b and c are for the $G_1$, $\chi^2_n$ and $LN_n$ models, respectively. Filled circles are for the real data, while open circles are for simulated samples. The dashed and the dotted straight lines are the power–law best fits to observational and simulated data, respectively. Error bars represent 1$\sigma$ uncertainties, obtained through the bootstrap resampling technique.

Figure 4. The relation between the correlation amplitude $A$ and the cluster richness $R_1$. We plot the results for the $G_1$ (open circles), $\chi^2_n$ (filled squared) and $LN_n$ models (open squares), as well as for real clusters (filled circles). Error bars are 1$\sigma$ standard deviations from the log–log linear regression of $w(\vartheta)$ in the scale range where the power–law model is valid (see also Table 2).

Figure 5. The frequency distribution of MST branch lengths, $F(\vartheta)$, for the C30 and C18 clusters (left and right columns, respectively). Solid histograms are for real clusters, while the dashed ones are for the simulated samples. From top to bottom we plot results for $G_1$, $\chi^2_n$ and $LN_n$ models.

Figure 6. The same as in Figure 5, but for clusters selected from galaxy counts in $20\times20$ arcmin cells. We only consider the $\kappa = 1.8$ case, since higher thresholds give too few objects at such a larger smoothing scale.

Figure 7. The correlation–integral partition function, $Z(q, \vartheta)$ (see eq.[17]), for C36 clusters for $q = 0$ (upper panels) and $q = 4$ (lower panels). From left to right we report the results for the $G_1$, $\chi^2_n$ and $LN_n$ models. Also plotted is the local dimension, $D_q(\vartheta)$, obtained according to eq.(19), from a three–point log–log linear regression on the partition function. Filled circles are for PBF clusters (see also Borgani, Plionis & Valdarnini 1993), while open squares are for simulations. It is apparent in the $q = 4$ case the difference between the Gaussian model and
the non–Gaussian ones at reproducing a small–scale flat shape of the local dimension.

**Figure 8.** The same as Figure 7, but for the C25 clusters.

**Figure 9.** The multifractal dimension spectrum, \( D_\tau \), and the singularity spectrum, \( f(\alpha) \), for the \( \chi^2_n \) (left panels) and the \( LN_n \) (right panels) C36 clusters (open circles), as compared to the PBF C36 sample (filled circles). The dimension values are obtained from the slope of the \( W(\tau, p) \) partition function (see eq.\[20\]) in the \( p \) range of values where a good scaling is observed for both real and simulated data. Error bars are 1\( \sigma \) standard deviations for the log–log linear regression on the partition function.

**Figure 10.** The variance–skewness relation for the ‘good’ models is plotted for C30 (left panels) and C18 (right panels) clusters. Filled triangles are for real clusters, while filled dots are for simulations. The dashed lines correspond to the hierarchical expression of eq.\[23\], with \( Q = 0.6 \).
Table 1: Characteristics of the cluster samples. Column 2: number of selected clusters. Column 3: number of active cells belonging to clusters. Columns 4 and 5: cluster richness according to two different definitions (see text) along with the respective rms values.

| Real clusters |          |          |          |          |
|---------------|----------|----------|----------|----------|
|               | $N_{cl}$ | Cells    | $R_1$    | $R_2$    |
| C36           | 285      | 1098     | 5.81 ± 0.72 | 6.57 ± 1.94 |
| C30           | 626      | 2720     | 4.69 ± 0.59 | 5.34 ± 1.67 |
| C25           | 1159     | 5849     | 3.91 ± 0.49 | 4.49 ± 1.50 |
| C18           | 2685     | 20020    | 2.84 ± 0.36 | 3.34 ± 1.27 |

| $G_{1.5}$ |          |          |          |          |
|-----------|----------|----------|----------|----------|
| $N_{cl}$  | Cells    | $R_1$    | $R_2$    |           |
| C36       | 45       | 122      | 5.82 ± 0.40 | 6.15 ± 0.79 |
| C30       | 104      | 413      | 4.93 ± 0.40 | 5.35 ± 0.87 |
| C25       | 306      | 1279     | 4.05 ± 0.37 | 4.39 ± 0.86 |
| C18       | 1569     | 8014     | 2.94 ± 0.25 | 3.20 ± 0.68 |
|           |          |          |          |          |

| $G_{1}$ |          |          |          |          |
|---------|----------|----------|----------|----------|
| $N_{cl}$ | Cells    | $R_1$    | $R_2$    |           |
| C36     | 44       | 202      | 6.01 ± 0.61 | 6.68 ± 1.61 |
| C30     | 108      | 523      | 4.98 ± 0.53 | 5.53 ± 1.44 |
| C25     | 230      | 1244     | 4.13 ± 0.44 | 4.62 ± 1.28 |
| C18     | 1464     | 6875     | 1.92 ± 0.26 | 3.17 ± 0.76 |

| $\chi^2_p$ |          |          |          |          |
|------------|----------|----------|----------|----------|
| $N_{cl}$   | Cells    | $R_1$    | $R_2$    |           |
| C36        | 46       | 202      | 6.01 ± 0.61 | 6.68 ± 1.61 |
| C30        | 108      | 523      | 4.98 ± 0.53 | 5.53 ± 1.44 |
| C25        | 230      | 1244     | 4.13 ± 0.44 | 4.62 ± 1.28 |
| C18        | 1464     | 6875     | 1.92 ± 0.26 | 3.17 ± 0.76 |

| $\chi^2_n$ |          |          |          |          |
|------------|----------|----------|----------|----------|
| $N_{cl}$   | Cells    | $R_1$    | $R_2$    |           |
| C36        | 44       | 202      | 6.01 ± 0.61 | 6.68 ± 1.61 |
| C30        | 108      | 523      | 4.98 ± 0.53 | 5.53 ± 1.44 |
| C25        | 230      | 1244     | 4.13 ± 0.44 | 4.62 ± 1.28 |
| C18        | 1464     | 6875     | 1.92 ± 0.26 | 3.17 ± 0.76 |

| $LN_{p}$ |          |          |          |          |
|----------|----------|----------|----------|----------|
| $N_{cl}$ | Cells    | $R_1$    | $R_2$    |           |
| C36      | 44       | 242      | 6.08 ± 0.72 | 6.90 ± 1.98 |
| C30      | 85       | 513      | 5.06 ± 0.65 | 5.82 ± 1.89 |
| C25      | 190      | 1002     | 4.15 ± 0.58 | 4.70 ± 1.61 |
| C18      | 1658     | 7093     | 2.85 ± 0.25 | 3.07 ± 0.81 |

| $LN_{n}$ |          |          |          |          |
|----------|----------|----------|----------|----------|
| $N_{cl}$ | Cells    | $R_1$    | $R_2$    |           |
| C36      | 44       | 242      | 6.08 ± 0.72 | 6.90 ± 1.98 |
| C30      | 85       | 513      | 5.06 ± 0.65 | 5.82 ± 1.89 |
| C25      | 190      | 1002     | 4.15 ± 0.58 | 4.70 ± 1.61 |
| C18      | 1658     | 7093     | 2.85 ± 0.25 | 3.07 ± 0.81 |
Table 2: Clusters from 20×20 arcmin cells. Column 2: number of selected clusters. Column 3: number of active cells belonging to clusters.

| Real clusters | $N_{cl}$ | Cells   | $N_{cl}$ | Cells   |
|---------------|----------|---------|----------|---------|
| C25$^{20}$    | 96       | 445     | 26       | 73      |
| C18$^{20}$    | 369      | 2503    | 188      | 886     |

| $G_{1.5}$ | $N_{cl}$ | Cells | $N_{cl}$ | Cells |
|-----------|----------|-------|----------|-------|
| C25$^{20}$ | 26       | 73    | 86       | 394   |
| C18$^{20}$ | 188      | 886   | 311      | 2095  |

| $\chi^2_p$ | $N_{cl}$ | Cells | $\chi^2_n$ | $N_{cl}$ | Cells |
|------------|----------|-------|-------------|----------|-------|
| C25$^{20}$ | 23       | 125   | 126         | 625      |       |
| C18$^{20}$ | 130      | 756   | 333         | 3653     |       |

| $LN_p$ | $N_{cl}$ | Cells | $LN_n$ | $N_{cl}$ | Cells |
|--------|----------|-------|--------|----------|-------|
| C25$^{20}$ | 24       | 107   | 146    | 1483     |       |
| C18$^{20}$ | 117      | 621   | 288    | 5467     |       |
Table 3: Best-fit parameters for the 2-point correlation function, $w(\vartheta) = A\vartheta^{1-\gamma}$, and the scale range $[\vartheta_1, \vartheta_2]$ where the power-law is well defined. The errors are 1σ standard deviations. No results are reported for the C36 clusters of ‘bad’ models, since they are too few to realize a meaningful correlation analysis.

|   | $A$       | $\gamma$   | $[\vartheta_1, \vartheta_2]$ |
|---|-----------|------------|------------------------------|
| C36 | $1.66 \pm 0.25$ | $2.02 \pm 0.26$ | $0.6$–$2.7$ |
| C30 | $0.96 \pm 0.10$ | $2.04 \pm 0.09$ | $0.6$–$8.3$ |
| C25 | $0.58 \pm 0.04$ | $1.98 \pm 0.05$ | $0.9$–$8.3$ |
| C18 | $0.19 \pm 0.01$ | $1.58 \pm 0.06$ | $0.9$–$8.3$ |

|   | $G_{1.5}$ | $G_1$ |
|---|-----------|-------|
| A   | $\gamma$ | $[\vartheta_1, \vartheta_2]$ | $A$ | $\gamma$ | $[\vartheta_1, \vartheta_2]$ |
| C36 | 1.26 | $0.6$–$8.3$ | 1.26 | $0.6$–$8.3$ |
| C30 | 2.13 | $0.6$–$8.3$ | 2.13 | $0.6$–$8.3$ |
| C25 | 2.53 | $0.6$–$8.3$ | 2.53 | $0.6$–$8.3$ |
| C18 | 2.12 | $0.6$–$8.3$ | 2.12 | $0.6$–$8.3$ |

|   | $\chi^2_p$ | $\chi^2_n$ |
|---|-----------|------------|
| A   | $\gamma$ | $[\vartheta_1, \vartheta_2]$ | $A$ | $\gamma$ | $[\vartheta_1, \vartheta_2]$ |
| C36 | 2.17 | $0.6$–$2.7$ | 2.17 | $0.6$–$2.7$ |
| C30 | 2.04 | $0.6$–$8.3$ | 2.04 | $0.6$–$8.3$ |
| C25 | 2.12 | $0.6$–$8.3$ | 2.12 | $0.6$–$8.3$ |
| C18 | 1.89 | $0.6$–$8.3$ | 1.89 | $0.6$–$8.3$ |

|   | $LN_p$ | $LN_n$ |
|---|-------|-------|
| A   | $\gamma$ | $[\vartheta_1, \vartheta_2]$ | $A$ | $\gamma$ | $[\vartheta_1, \vartheta_2]$ |
| C36 | 3.88 | $0.6$–$8.3$ | 3.88 | $0.6$–$8.3$ |
| C30 | 1.74 | $0.6$–$8.3$ | 1.74 | $0.6$–$8.3$ |
| C25 | 1.52 | $0.6$–$8.3$ | 1.52 | $0.6$–$8.3$ |
| C18 | 1.53 | $0.6$–$8.3$ | 1.53 | $0.6$–$8.3$ |
Table 4: Kolmogorov–Smirnov (KS) test for the MST branch–length frequency distribution, $F(\vartheta)$ (see text). Column 2: the KS statistics, $\bar{D}$, measuring the maximum difference between the $F(\vartheta)$ cumulative distributions for real and simulated cluster samples. Column 3: probability to measure a larger difference from a statistically equivalent realization; it is the significance level of the difference; smaller values correspond to more significative differences. We also present results for the C18$^{20}$ samples, which contains clusters selected from smoothed counts in $20 \times 20$ arcmin cells (see also Table 2).

| Cluster | $G_{1.5}$ | $G_{1}$ |
|---------|-----------|---------|
|         | $\bar{D}$ | $P(D > \bar{D})$ | $\bar{D}$ | $P(D > \bar{D})$ |
| C36     | 0.10      | 0.86     | 0.15      | 0.04         |
| C30     | 0.18      | 6.2 $10^{-3}$ | 0.12      | 7.5 $10^{-3}$ |
| C25     | 0.10      | 0.02     | 0.10      | 5.2 $10^{-4}$ |
| C18     | 0.07      | 7.3 $10^{-5}$ | 0.06      | 2.6 $10^{-3}$ |
| C18$^{20}$ | 0.08 | 0.44     | 0.05      | 0.88         |

| Cluster | $\chi^2_p$ | $\chi^2_n$ |
|---------|------------|------------|
|         | $\bar{D}$ | $P(D > \bar{D})$ | $\bar{D}$ | $P(D > \bar{D})$ |
| C36     | 0.12      | 0.62     | 0.12      | 0.09         |
| C30     | 0.11      | 0.17     | 0.09      | 0.04         |
| C25     | 0.12      | 8.3 $10^{-3}$ | 0.11      | 1.1 $10^{-4}$ |
| C18     | 0.08      | 8.3 $10^{-6}$ | 0.09      | 1.4 $10^{-6}$ |
| C18$^{20}$ | 0.10 | 0.36     | 0.03      | 0.99         |

| Cluster | $LN_p$ | $LN_n$ |
|---------|--------|--------|
|         | $\bar{D}$ | $P(D > \bar{D})$ | $\bar{D}$ | $P(D > \bar{D})$ |
| C36     | 0.22    | 0.09     | 0.12      | 0.08         |
| C30     | 0.14    | 0.11     | 0.05      | 0.67         |
| C25     | 0.11    | 0.05     | 0.12      | 3.4 $10^{-5}$ |
| C18     | 0.08    | 3.2 $10^{-5}$ | 0.09      | 1.6 $10^{-6}$ |
| C18$^{20}$ | 0.10   | 0.45     | 0.06      | 0.75         |
Table 5: The coefficient $Q$ in the variance–skewness relation (see eq.[25]), for real data and ‘good’ models. The errors are 1σ standard deviations in the log $\Gamma$–log $\Sigma^2$ linear regression.

|       | Real clusters | $G_1$     |
|-------|--------------|-----------|
| C36   | 0.54 ± 0.28  | 0.53 ± 0.27 |
| C30   | 0.71 ± 0.37  | 0.60 ± 0.32 |
| C25   | 0.65 ± 0.36  | 0.63 ± 0.35 |
| C18   | 0.83 ± 0.50  | 0.70 ± 0.42 |

|       | $\chi^2_n$ | $LN_n$ |
|-------|-----------|--------|
| C36   | 0.58 ± 0.30 | 0.59 ± 0.31 |
| C30   | 0.65 ± 0.33 | 0.57 ± 0.31 |
| C25   | 0.59 ± 0.30 | 0.62 ± 0.32 |
| C18   | 0.66 ± 0.38 | 0.68 ± 0.40 |