Non-symmetric transition probability in generalized qubit models

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Abstract

The quantum mechanical transition probability is symmetric. A probabilistically motivated and more general quantum logical definition of the transition probability was introduced in two preceding papers without postulating its symmetry, but in all the examples considered there it remains symmetric. Here we present a class of binary models where the transition probability is not symmetric, using the extreme points of the unit interval in an order unit space as quantum logic. We show that their state spaces are strictly convex smooth compact convex sets and that each such set \( K \) gives rise to a quantum logic of this class with the state space \( K \). The transition probabilities are symmetric iff \( K \) is the unit ball in a Hilbert space. In this case, the quantum logic becomes identical with the projection lattice in a spin factor which is a special type of formally real Jordan algebra.

Keywords: quantum transition probability; qubit; convexity theory; spin factors; Jordan algebras; quantum logics

1 Introduction

A typical feature of common quantum mechanics is the symmetry of the transition probability. All physical experiments and observations are in line with it, and attempts to find an axiomatic access to quantum theory often postulate it a priori \([3, 18]\). It is also a part of Alfsen and Shultz’s so-called pure state properties \([2]\). However, doubts that there is any reasonable physical reason for this postulate are as old as the early axiomatic attempts \([10, 11, 15, 16]\). A probabilistically motivated and more general definition of the transition probability in the quantum logical framework was introduced in two preceding papers \([17, 19]\) without postulating the symmetry, but in all the examples considered there the transition probability remains symmetric.

A crucial question thus becomes whether there are any quantum logical structures with a non-symmetric transition probability and how this probability might behave then. Here we identify such structures and explore some explicit examples. Models from convexity theory are reused, which have already been studied by Alfsen and Shultz \([2]\) and Berdikulov \([5, 6, 7, 8]\), but with a different focus; these authors have not explicitly examined the transition probability and we do this here. Moreover, using the extreme points of the unit interval in an order unit space, the approach presented here provides a simpler and directer access to the quantum logics and the transition probabilities than Alfsen and...
Shultz’s theory. We shall see that our transition probabilities become identical with those introduced by Mielnik in a different way \[15, 16\].

With one exception, the quantum logics which we consider are binary. They represent the classical bit, the quantum bit (qubit) and generalized versions thereof. We use the transition probabilities to define the new type of binary quantum logics. Our main result then becomes an interesting one-to-one relationship between the quantum logics of this type and the strictly convex smooth compact convex sets. The characteristics of the quantum logic are determined by those of the convex set. Only if this set is the unit ball in a (pre-)Hilbert space, the transition probabilities become symmetric. In this case the quantum logic is the projection lattice in a spin factor. The spin factors are formally real Jordan algebras \[12\] and are known from the canonical anti-commutator relations for the fermions. A certain generalization of these spin factors will be studied first here and will later motivate the definition of the new type of binary quantum logic.

The paper is organized as follows. Sections 2 and 3 briefly sketch those definitions from the preceding papers \[17, 19\] that will be needed subsequently. The generalization of the spin factors is then introduced in section 4. The quantum logics, state spaces and transition probabilities of these generalized spin factors are studied in sections 5, 6 and 7. After a brief intermezzo on the spectral decomposition in section 8, we come in section 9 to our main results. The transition probabilities are used to characterize the generalized spin factors in a new way and to define the new type of binary quantum logic. Moreover, the relationship to the strictly convex and smooth compact convex sets is elaborated. In section 10 we turn to the special generalized spin factors arising from the spaces \(l^p\) and \(L^p\) and examine explicit numerical examples with non-symmetric transition probabilities. Further examples are discussed in section 11.

Some basic knowledge of convex analysis (extreme points, smoothness, strict convexity, order unit spaces \[1, 2\]) and the theory of normed linear spaces \[9, 13\] will be helpful for the reader.

## 2 Quantum logics and states

The quantum logical framework that was used in the preceding papers \[17, 19\] shall be recapitulated here first.

A quantum logic is an orthomodular partially ordered set \(L\) with order relation \(\leq\), smallest element \(0\), largest element \(I\) and an orthocomplementation \('\). This means that the following conditions are satisfied by the \(e, f \in L\):

(a) \(e \leq f\) implies \(f' \leq e'\).

(b) \((e')' = e\).

(c) \(e \leq f'\) implies that \(e \lor f\), the supremum of \(e\) and \(f\), exists.

(d) Orthomodular law: \(f \leq e\) implies \(e = f \lor (e \land f')\).
Here, $e \wedge f$ denotes the infimum of $e$ and $f$, which exists iff $e' \vee f'$ exists. Condition (d) is the orthomodular law; it implies that $e \vee e' = 1$ for any $e \in L$.

An element $e \in L$ with $e \neq 0$ is called minimal or an atom if there is no $f \in L$ with $f \leq e$ and $0 \neq f \neq e$. Two elements $e$ and $f$ in $L$ are orthogonal, if $e \leq f'$ or, equivalently, $f \leq e'$; in this case, $e \vee f$ exists and shall be noted by $e + f$ in the following. In a lattice, $e \wedge f$ and $e \vee f$ would exist for any elements $e$ and $f$.

A state $\mu$ shall allocate probabilities to the elements of the quantum logic. Therefore it becomes a map from $L$ to the unit interval $[0, 1] \subseteq \mathbb{R}$ with $\mu(1) = 1$ and $\mu(e + f) = \mu(e) + \mu(f)$ for any two orthogonal elements $e$ and $f$ in $L$. A set $S$ of states on $L$ is called strong if, for any $e, f \in L$,

$$\{\mu \in S \mid \mu(e) = 1\} \subseteq \{\mu \in S \mid \mu(f) = 1\} \Rightarrow e \leq f.$$  

Note that this implies that there is a state $\mu \in S$ with $\mu(e) = 1$ for each $e \in L$ with $e \neq 0$.

### 3 Transition probability in quantum logics

We can now restate the following definition of the transition probability from the preceding papers [17][19].

**Definition 3.1** Let $L$ be a quantum logic and $S$ a strong set of states on $L$. If a pair $e, f \in L$ with $e \neq 0$ and some $s \in [0, 1]$ satisfies the condition

$$\mu(f) = s \text{ for all } \mu \in S \text{ with } \mu(e) = 1,$$

then $s$ is called the transition probability from $e$ to $f$ and is denoted by $\mathbb{P}(f|e)$.

The identity $\mathbb{P}(f|e) = s$ is equivalent to the set inclusion

$$\{\mu \in S \mid \mu(e) = 1\} \subseteq \{\mu \in S \mid \mu(f) = s\}$$

and means that, whenever the probability of $e$ is 1, the probability of $f$ is fixed and its numerical value is $s$; particularly in the situation after a quantum measurement that has provided the outcome $e$, the probability of $f$ becomes $s$, independently of any initial state before the measurement.

Two elements $e$ and $f$ in $L$ are orthogonal iff $\mathbb{P}(f|e) = 0$ and $e \leq f$ holds iff $\mathbb{P}(f|e) = 1$. The second part here holds since $S$ is a strong set of states, and the first part follows by considering $f'$.

In a preceding paper [17] it was elaborated that, in the case of a Hilbert space quantum logic with its usual state space, this transition probability becomes identical with the usual quantum mechanical transition probability:

$$\mathbb{P}(f|e) = |\langle \varphi | \psi \rangle|^2,$$
where \( \varphi \) and \( \psi \) are normalized vectors in the Hilbert space and the atoms \( e \) and \( f \) are the one-dimensional subspaces generated by \( \varphi \) and \( \psi \) (or the corresponding self-adjoint projection operators). Therefore the \textit{symmetry condition}

\[
P(f|e) = P(e|f)
\]

holds. The quantum logics formed by the projection lattices in the von Neumann algebras and formally real Jordan algebras (with their usual state spaces) satisfy the same symmetry condition [19]. The following lemma demonstrates an interesting consequence of this symmetry.

**Lemma 3.2** Let \( L \) be a quantum logic and \( S \) a strong set of states on \( L \). Moreover, suppose that the transition probability \( P(f|e) \) exists for every atom \( e \) and every element \( f \) in \( L \) and that \( P(f|e) = P(e|f) \) holds for any two atoms \( e \) and \( f \). If \( e_1 + e_2 + \ldots + e_m = f_1 + f_2 + \ldots + f_n \) is satisfied for two families of atoms \( e_1, e_2, \ldots, e_m \) and \( f_1, f_2, \ldots, f_n \) such that the atoms in the same family are pairwise orthogonal, then \( m = n \) must hold.

**Proof.** Under the above assumptions we have \( f_k \leq \sum_l e_l \) and \( P(\sum_l e_l | f_k) = 1 \) for each \( k \) as well as \( e_l \leq \sum_k f_k \) and \( P(\sum_k f_k | e_l) = 1 \) for each \( l \). Then

\[
\sum_k \sum_l P(e_l | f_k) = \sum_k P(\sum_l e_l | f_k) = n
\]

and

\[
\sum_k \sum_l P(e_l | f_k) = \sum_l \sum_k P(f_k | e_l) = \sum_k P(\sum_k f_k | e_l) = m.
\]

Theorem 3.2 means that a kind of dimension function exists, if the transition probabilities are symmetric.

The existence of the transition probability \( P(f|e) \) for two elements \( e \) and \( f \) in a quantum logic \( L \) does not require that \( e \) is an atom [17, 19]. However, if \( P(f|e) \) exists for all \( f \in L \), \( e \) must be an atom [19] and there is only one unique state \( \mu_e \) with \( \mu_e(e) = 1 \); \( \mu_e(f) = P(f|e) \) for \( f \in L \). If there is only one unique state \( \mu_e \) with \( \mu_e(e) = 1 \) for some \( e \in L \), \( P(f|e) \) exists for all \( f \in L \) with \( P(f|e) = \mu_e(f) \) and \( e \) must be an atom. In this case, \( \mu_e \) is an extreme point in the state space \( S \) \( \{ \mu_e = t \nu_1 + (1 - t) \nu_2 \} \) with \( \nu_1, \nu_2 \in S \) and \( 0 < t < 1 \) implies \( 1 = \mu_e(e) = t \nu_1(e) + (1 - t) \nu_2(e) \), thus \( \nu_1(e) = 1 = \nu_2(e) \) and \( \nu_1 = \nu_2 = \mu_e \).

The extreme points in the state space \( S \) are usually called \textit{pure} states.

**4 Definition of the generalized spin factors**

A special class of quantum logics arises from the so-called generalized spin factors, which were studied by Berdikulov [5, 6, 7, 8]. The following notions from the theory of normed linear spaces will be needed to define them.

A normed linear space \( X \) is called \textit{smooth} if its unit ball is smooth. This means that, for every \( x \in X \), \( ||x|| = 1 \), there exists a unique bounded linear functional \( \rho_x \) in its dual such that \( ||\rho_x|| = 1 \) and \( \rho_x(x) = 1 \). A normed linear space \( X \) is called \textit{strictly convex}, if its unit ball is strictly convex. This means that we have \( ||tx + (1 - t)y|| < 1 \) for all \( x, y \in X \), \( x \neq y \), \( ||x|| = 1 = ||y|| \) and
Examples of normed linear spaces that are both smooth and strictly convex are the Hilbert spaces, the $L^p$ and $L^p$ spaces with the norm $\| \cdot \|_p$, $1 < p < \infty$ [9, 13]. Concerning their mathematical structure, the smooth and strictly convex normed spaces are closer to the Hilbert spaces than the general normed spaces.

Now let $X$ be any normed $\mathbb{R}$-linear space and consider the direct sum $A := X \oplus \mathbb{R}$ with the following order relation: $0 \leq x \oplus s$ if $\|x\| \leq s$ for $x \in X$, $s \in \mathbb{R}$. Moreover, define $I := 0 \oplus 1$. It is not hard to verify that $A$ becomes an order unit space with order unit $I$ in this way and that $A$ has the following norm: $\|x \oplus s\| = \|x\| + |s|$ for $x \in X$ and $s \in \mathbb{R}$.

In the case of a Hilbert space $X$ over the real numbers $\mathbb{R}$ with the inner product $( \cdot , \cdot )$, $A$ becomes a formally real Jordan algebra [2, 12], when it is equipped with the following product: $(x \oplus s) \circ(y \oplus t) := (tx + sy) \oplus ((x,y) + st)$ for $x,y \in X$ and $s,t \in \mathbb{R}$. This special type of Jordan algebra is usually called spin factors [2, 12]. They represent the canonical anti-commutator relations for the fermions, since we have $(x_k \oplus 1) \circ (x_l \oplus 1) = \delta_{kl}1$ for any orthonormal basis $x_k$ in the Hilbert space $H$ and since the Jordan product $\circ$ becomes identical with the anti-commutator $[ , ]^\ast$, when $A$ is represented as operator algebra (which is generally possible for the spin factors [2, 12]).

Therefore Berdikulov chose the name generalized spin factor for the order unit spaces $A = X \oplus \mathbb{R}$, arising from the smooth and strictly convex normed linear spaces $X$, but required that $X$ is the dual of some other space [5, 6, 7, 8]. Note that, in this paper, we do neither require that $X$ is complete nor that $X$ is a dual space, and we call $A = X \oplus \mathbb{R}$ a generalized spin factor (spin factor) also in those cases when $X$ is just a smooth and strictly convex normed $\mathbb{R}$-linear space (pre-Hilbert space). Moreover we include the case $X = \mathbb{R}$ which is usually ruled out, because $A = X \oplus \mathbb{R}$ is decomposable then and does not satisfy the criteria of a factor.

5 The quantum logic of a generalized spin factor

Instead of using Alfsen and Shultz’s compressions, projective faces or projective units [2] of the state space, we provide a rather direct and simple access to the quantum logics of the generalized spin factors. Consider the unit interval $[0,1]$ in $A = X \oplus \mathbb{R}$ with a smooth and strictly convex normed $\mathbb{R}$-linear space $X$.

$$[0,1] := \{ a \in A : 0 \leq a \leq 1 \} = \{ a \in A : 0 \leq a \text{ and } \|a\| \leq 1 \} \quad (1)$$

$$= \{ x \oplus s : x \in X, s \in \mathbb{R}, \|x\| \leq s \text{ and } \|x\| \leq 1 - s \} \quad (2)$$

$$= \{ x \oplus s : x \in X, s \in \mathbb{R}, \|x\| \leq s \leq 1 - \|x\| \} \quad (3)$$

$$\subseteq \left\{ x \oplus s : x \in X, \|x\| \leq \frac{1}{2} \text{ and } 0 \leq s \leq 1 \right\} \quad (4)$$

The projection lattice in a von Neumann algebra or formally real Jordan algebra and particularly in a spin factor is identical with the extreme points of the the
positive part of the unit ball $\mathbb{I}$ and therefore the extreme points of $[0, \mathbb{I}]$ shall become our quantum logic $L_X$:

$$L_X := \text{ext}([0, \mathbb{I}]).$$

The order relation is inherited from the one on $A$. The orthocomplement of $e \in L_X$ is $e' := \mathbb{I} - e$. Note that $e$ is extremal iff $\mathbb{I} - e$ is extremal in $[0, \mathbb{I}]$.

**Proposition 5.1** $L_X = \text{ext}([0, \mathbb{I}]) = \{0, \mathbb{I}\} \cup \left\{ \frac{1}{2} (x \oplus 1) : x \in X, \|x\| = 1 \right\}$.

**Proof.** In any order unit space, 0 and I are extreme points of $[0, \mathbb{I}]$. Now suppose $x \in X, \|x\| = 1$ and

$$\frac{1}{2} (x \oplus 1) = t(y_1 \oplus s_1) + (1 - t)(y_2 \oplus s_2) = (ty_1 + (1 - t)y_2) \oplus (ts_1 + (1 - t)s_2)$$

with $y_1 \oplus s_1, y_2 \oplus s_2 \in [0, \mathbb{I}]$ and $0 < t < 1$. Then $\frac{1}{2} x = ty_1 + (1 - t)y_2$ and $\frac{1}{2} = ts_1 + (1 - t)s_2$. Since $0 \leq \|y_1\|, \|y_2\| \leq \frac{1}{2}$ by (4), the strict convexity implies $y_1 = y_2 = \frac{1}{2} x$. Since $\|y_1\| = \|y_2\| = \frac{1}{2}$ and $y_1 \oplus s_1, y_2 \oplus s_2 \in [0, \mathbb{I}]$, we get by (3) $s_1 = s_2 = \frac{1}{2}$. Therefore $\frac{1}{2} (x \oplus 1)$ is an extreme point in $[0, \mathbb{I}]$.

Now let $x \oplus s$ be any element in $[0, \mathbb{I}]$ with $x \neq 0$ and $\|x\| < \frac{1}{2}$. It becomes a convex combination of the three elements 0, I and $\frac{1}{2} (\frac{x}{\|x\|} \oplus 1)$ in $[0, \mathbb{I}]$ with the parameters $1 - s - \|x\|, s - \|x\|$ and $2 \|x\|$. These three parameters are non-negative real numbers and their sum becomes 1. Since $0 < 2 \|x\| < 1$, $x \oplus s$ is not an extreme point in $[0, \mathbb{I}]$. Moreover, $0 \oplus s$ is not an extreme point unless $s = 0$ or $s = 1$, since in the other cases we have $0 \oplus s = (1 - s)0 + s\mathbb{I}$. In the remaining cases we have $\|x\| = \frac{1}{2}$; then $s = \frac{1}{2}$ by (3) and $x \oplus s$ becomes an extreme point as shown above. \[\square\]

Note that only the strict convexity is used in the proof of Proposition 5.1; the smoothness of $X$ is not needed here.

With $x \in X, \|x\| = 1$, the orthogonal complement of $\frac{1}{2} (x \oplus 1) \in L_X$ is $\frac{1}{2} (-x \oplus 1) \in L_X$. All these elements are atoms in the quantum logic $L_X$; only 0 and I are not atoms. The set of atoms in $L_X$ thus becomes isomorphic to the unit sphere in $X$.

The quantum logic $L_X$ is a lattice, since the supremum $e \vee f$ and the infimum $e \wedge f$ exist for all $e,f \in L_X$: $0 \vee f = f, \mathbb{I} \vee f = \mathbb{I}, 0 \wedge f = 0, \mathbb{I} \wedge f = f, f \vee f = f \wedge f$ for all $f \in L_X$, and $e \vee f = \mathbb{I}, e \wedge f = 0$ for any two atoms $e$ and $f$ with $e \neq f$. The orthomodular law is satisfied since $e \leq f$ is possible for $e,f \in L_X$ only if either $e = 0$ or $f = \mathbb{I}$ or $e = f$.

Since the maximum number of pairwise orthogonal atoms is two, each quantum logic $L_X$ with any smooth and strictly convex normed $\mathbb{R}$-linear space $X$ represents a binary model. In the cases $X = \mathbb{R}^n$ with the Euclidean norm we get the quantum logic of the usual complex qubit with $n = 3$ (Bloch sphere), the real version of the qubit with $n = 2$ and the classical bit with $n = 1$. 

6
6 The state space of a generalized spin factor

The state space $S$ of an order unit space $A$ consists of those $\mathbb{R}$-linear functionals $\mu : A \to \mathbb{R}$ that satisfy $\mu(1) = 1$ and $\mu(a) \geq 0$ for all $a \in A$ with $a \geq 0$. For the quantum logic $L_X$, defined in the previous section, we consider the state space $S_X$ that consists of the restrictions of the states on $A = X \oplus \mathbb{R}$ to $L_X$. The restriction of a state on $A$ to $X$ becomes a bounded linear functional $\rho$ on $X$ with $\|\rho\| \leq 1$. Vice versa, each bounded linear functional $\rho$ on $X$ with $\|\rho\| \leq 1$ extends to a unique state $\mu$ on $A$ by $\mu(x + s) := \rho(x) + s$. The state space of $A$ and $L_X$ thus becomes isomorphic to the unit ball of the dual of $X$.

The state resulting from $\rho = 0$ is called the trace state and is denoted by $\mu_\tau$ here. It allocates the same number $(1/2)$ to each atom and $2\mu_\tau$ becomes a kind of a dimension function.

Lemma 6.1 (i) For each atom $e$ in $L_X$ there is one unique state $\mu_e$ in $S_X$ with $\mu_e(e) = 1$. With $e = \frac{1}{2}(x \oplus 1)$, $x \in X$ and $\|x\| = 1$, the restriction of $\mu_e$ to $X$ is identical with the unique $\rho_x$ which results from the smoothness of $X$ and which satisfies $\|\rho_x\| = 1$ and $\rho_x(x) = 1$.

(ii) The state space $S_X$ of the quantum logic $L_X$ is strong.

Proof. (i) Suppose $e = \frac{1}{2}(x \oplus 1)$ with $x \in X$ and $\|x\| = 1$. Then $\mu(e) = 1$ with $\mu \in S_A$ iff the restriction of $\mu$ to $X$ allocates $1$ to $x$. Due to the smoothness of $X$, there is one unique such functional $\rho_x$.

(ii) To show that $S_X$ is strong, it is sufficient to consider $e, f \in L_X$, both different from $0$ and $1$, with $\{\mu \in S_X \mid \mu(e) = 1\} \subseteq \{\mu \in S_X \mid \mu(f) = 1\}$. However, by (i) there is only one state $\mu_e$ with $\mu_e(e) = 1$ and only one state $\mu_f$ with $\mu_f(f) = 1$. Therefore $\mu_e = \mu_f$. From $e = \frac{1}{2}(x \oplus 1)$ and $f = \frac{1}{2}(y \oplus 1)$ with $x, y \in X$ and $\|x\| = 1 = \|y\|$ we get by (i) $\rho_x = \rho_y =: \rho$; this means $\rho(x) = 1 = \rho(y)$. For all $t \in [0, 1]$ then $1 = \rho(tx + (1 - t)y) \leq \|tx + (1 - t)y\| \leq t\|x\| + (1 - t)\|y\| = 1$ and therefore $\|tx + (1 - t)y\| = 1$. From the strict convexity of $X$ we get $x = y$ and $e = f$, particularly $e \leq f$. □

7 Transition probability in a generalized spin factor

By Lemma 6.1 (i), the transition probability $P(f|e)$ exists for each atom $e$ and all elements $f$ in the quantum logic $L_X$, arising from any smooth and strictly convex normed space $X$ as explained in the previous three sections. Moreover, using the same notation as in Lemma 6.1, with $e = \frac{1}{2}(x + 1)$, $f = \frac{1}{2}(y + 1)$, $x, y \in X$ and $\|x\| = 1 = \|y\|$, we have

$$P(f|e) = \mu_e(f) = \frac{1}{2}(\rho_x(y) + 1).$$

(5)

Since $e' = \frac{1}{2}(-x + 1)$ and $\rho_{-x} = -\rho_x$, this implies the identity

$$P(f|e) + P(f|e') = 1.$$

(6)
In sections 9 and 11, we shall encounter other quantum logics where (6) does not hold.

**Theorem 7.1** The following two conditions are equivalent:

(i) \( \mathbb{P}(f|e) = \mathbb{P}(e|f) \) holds for the atoms \( e \) and \( f \) in the quantum logic \( L_X \) (this means that the transition probability is symmetric).

(ii) \( X \) is a pre-Hilbert space (this means that \( A = X \oplus \mathbb{R} \) is a spin factor).

Proof. (i) \( \Rightarrow \) (ii): Suppose that \( \mathbb{P}(f|e) = \mathbb{P}(e|f) \) for the atoms \( e \) and \( f \) in the quantum logic \( L_X \). For \( x = 0 \) or \( y = 0 \) in \( X \) we define \( \langle x|y \rangle := 0 \).

For \( x \neq 0 \neq y \) in \( X \) we define \( \langle x|y \rangle := \rho_x(y) \), which is linear in the second argument \( y \). Again \( \rho_x \) denotes the unique functional on \( X \) with \( ||\rho_x|| = 1 \) and \( \rho_x(x) = ||x|| \), which exists due to the smoothness of \( X \). Now \( e_x := \frac{1}{2}(\frac{x}{||x||} \oplus 1) \) and \( e_y := \frac{1}{2}(\frac{y}{||y||} \oplus 1) \) are atoms and

\[
\langle x|y \rangle = \rho_x(y) = ||x|| \ ||y|| \ \rho_{\frac{x}{||x||}} \left( \frac{y}{||y||} \right) \\
= ||x|| \ ||y|| \ (2\mathbb{P}(e_y|e_x) - 1)
\]

by (5). Then \( \langle x|x \rangle = ||x||^2 > 0 \). The symmetry of the transition probability implies \( \langle x|y \rangle = \langle y|x \rangle \) and the linearity in the first argument. In this way \( X \) becomes a pre-Hilbert space.

(ii) \( \Rightarrow \) (i): Use equation (5) and note that \( \rho_x(y) = \langle x|y \rangle \) and \( \langle x|y \rangle = \langle y|x \rangle \) for the elements \( x, y \) in the real pre-Hilbert space \( X \) with \( ||x|| = 1 = ||y|| \). \( \square \)

### 8 Spectral decomposition in a generalized spin factor

If \( 0 \neq x \in X \) and \( s \in \mathbb{R} \), the element \( x \oplus s \) in the order unit space \( A = X \oplus \mathbb{R} \) with any smooth and strictly convex normed space \( X \) can be written as

\[ x \oplus s = (s + ||x||) e + (s - ||x||) e' \]

with the orthogonal atoms

\[ e := \frac{1}{2} \left( \frac{x}{||x||} \oplus 1 \right) \text{ and } e' := \frac{1}{2} \left( \frac{-x}{||x||} \oplus 1 \right). \]

This representation can be considered as the spectral decomposition of \( x \oplus s \), and \( x \oplus s \) can be interpreted as an observable with the potential measurement outcomes \( s + ||x|| \) and \( s - ||x|| \). The spectral decomposition is unique; this will be shown in Theorem 9.1 (i) under more general assumptions.

Any functions and particularly the power functions can then be applied to \( x \oplus s \) in the usual way:

\[ (x \oplus s)^n := (s + ||x||^n) e + (s - ||x||^n) e'. \]
Then \( 0 \leq (x \oplus s)^2 \), and the quantum logic \( L_X \) becomes identical with the idempotent elements in \( A \). The product in a Jordan algebra satisfies the identity

\[
a \circ b = \frac{1}{4} ((a + b)^2 - (a - b)^2)
\]

and therefore the question arises whether this identity can be used to derive a product in \( A = X \oplus \mathbb{R} \). The bilinearity of this product again requires that \( X \) is a pre-Hilbert space (and the transition probabilities become symmetric), since an inner product is then defined on \( A \) and particularly on the subspace \( X \) by \( \langle a | b \rangle := \mu_{tr}(a \circ b) \) for \( a, b \in A \). Here \( \mu_{tr} \) denotes the trace state on \( A \) (see section 6) and by (7) we get \( \mu_{tr}(x^2) = \|x\|^2 \) for \( x \in X \); therefore \( \langle x | x \rangle = \mu_{tr}(x^2) \geq 0 \) and \( \langle x | x \rangle = 0 \) iff \( x = 0 \).

It is interesting to note that any generalized spin factor with the power function defined by (7) satisfies Segal’s postulates for a system of observables [20]. Other examples, which demonstrate that Segal’s postulates are not sufficient to get a bilinear product for the observables, were presented very early by Lowdenslager [14] and Sherman [21], but so far nobody has studied the connection with the transition probabilities. The above considerations together with Theorem 7.1 and the fact that spin factors are Jordan algebras show that, in the case of the generalized spin factors, the symmetry of the transition probabilities is equivalent to the bilinearity of the product (8).

Generally, in Lowdenslager’s and Sherman’s examples, the idempotent elements (their quantum logic) do not coincide with the extreme points of the unit interval (our quantum logic), the state space is not strong and the transition probabilities do not exist for their quantum logic. This means that these examples are more bizarre than the generalized spin factors and further cases which we are going to consider in sections 9 and 11.

9 Axiomatic characterization

Instead of starting from a smooth and strictly convex normed linear space \( X \) and constructing the direct sum \( X \oplus \mathbb{R} \), we shall now identify those characteristics of an order unit space that make it a generalized spin factor. We will not use Alfsen and Shultz’s theory of spectral convex sets [2] as Berdikulov does in his different characterization [8]. In our approach the transition probabilities will play an important role.

**Theorem 9.1** Let \( A \) be an order unit space with order unit \( \mathbb{I} \). As our quantum logic we consider the extreme points \( L_A := \text{ext}[0, \mathbb{I}] \) of the unit interval and the state space \( S_A \) consists of the normalized positive \( \mathbb{R} \)-linear functionals on \( A \). Moreover, we assume that

(a) the state space \( S_A \) is strong for the quantum logic \( L_A \) with the order relation inherited from \( A \) and \( e' = \mathbb{I} - e \) for \( e \in L_A \),

(b) \( 0 \) and \( \mathbb{I} \) are the only elements in \( L_A \) that are not atoms,
(c) each $a \in A$ has a spectral decomposition $a = se + te'$ with $s, t \in \mathbb{R}$, an atom $e$ and its orthogonal complement $e'$ (which is also an atom),

(d) the transition probability $\mathbb{P}(f|e)$ exists for each atom $e$ and $f \in L_A$.

Then:

(i) The spectral decomposition is unique for $a \notin \mathbb{R}$.

(ii) The state space $S_A$ is a strictly convex and smooth $w^*$-compact convex subset of the hyperplane \( \{ \rho \in A^* : \rho(\mathbb{I}) = 1 \} \) in the dual $A^*$ of $A$. Moreover, $\partial S_A = \text{ext}(S_A) = \{ \mu_e : e \text{ is an atom} \}$. Here $\partial S_A$ and $\text{ext}(S_A)$ denote the topological boundary (in the hyperplane) and the extreme boundary of $S_A$.

Moreover:

(iii) $A$ is a generalized spin factor iff equation (6) holds for the transition probabilities of any atoms $e$ and $f$.

(iv) $A$ is a spin factor iff the transition probabilities are symmetric.

(v) $A$ is a spin factor iff the spectral decomposition results in a bilinear product via $a_1 \circ a_2 := \frac{1}{4}((a_1 + a_2)^2 - (a_1 - a_2)^2)$ for $a_1, a_2 \in A$ and $a^2 := s^2 e + t^2 e'$ for $a = se + te'$, $s, t \in \mathbb{R}$ and an atom $e$.

Proof. For each atom $e$ let $\mu_e \in S_A$ denote the unique state with $\mu_e(e) = 1$. This means $\mu_e(f) = \mathbb{P}(f|e)$. First we show that $0 \leq se + te'$ iff $0 \leq s, t$ and $\|se + te'\| = \max \{|s|, |t|\}$ for each atom $e$.

If $0 \leq se + te'$, use $\mu_e$ and $\mu_{e'}$ to get $0 \leq s, t$. The only-if part is obvious. From

$$\min \{s, t\} \mathbb{I} = \min \{s, t\} e + \min \{s, t\} e' \leq se + te' \leq \max \{s, t\} e + \max \{s, t\} e' = \max \{s, t\} \mathbb{I}$$

we get $\|se + te'\| \leq \max \{|s|, |t|\}$ and then use $\mu_e$ and $\mu_{e'}$ again to get

$$\|se + te'\| = \max \{|s|, |t|\}.$$
\[ \nu \in S_A; \ a = s_1 e + s_2 e' \] with an atom \( e, s_1, s_2 \leq 1 \) and \( s_1 < 1 \) or \( s_2 < 1 \). Then \( 1 = s_1 \nu_o(e) + s_2 \nu_o(e') = s_1 \nu_o(e) + s_2(1 - \nu_o(e)) \) and therefore either \( \nu_o(e) = 0, \nu_o(e') = 1 \) and \( \nu_o = \mu_{e'} \) or \( \nu_o(e) = 1, \nu_o = \mu_e \). Thus we have \( \partial S_A \subseteq \{ \mu_{e} : e \text{ is an atom} \} \).

Since \( \{ \mu_{e} : e \text{ is an atom} \} \subseteq \text{ext}(S_A) \subseteq \partial S_A \) generally holds, these three sets coincide. Particularly each topological boundary point is an extreme point and therefore \( S_A \) is strictly convex in the hyperplane.

We now prove that \( S_A \) is smooth in the hyperplane at the point \( \mu_f \) for each atom \( f \). Let \( a \neq 1 \) be an element in \( A \) with \( \mu_f(a) = 1 \) and \( \nu(a) \leq 1 \) for all \( \nu \in S_A \). Then \( a = s_1 e + s_2 e' \) with an atom \( e, s_1, s_2 \leq 1 \) and \( s_1 < 1 \) or \( s_2 < 1 \). We have to show the uniqueness of \( \{ \rho \in A^* : \rho(\|) = 1 = \rho(a) \} \).

Suppose \( s_1 < 1 \). From \( 1 = s_1 \mu_f(e) + s_2 \mu_f(e') \) and \( 1 = \mu_f(e) + \mu_f(e') \) we get that \( \mu_f(e) = 0 \) and \( \mu_f(e') = 1 = s_2 \). Hence \( \mu_f = \mu_{e'} \). Since the state space is strong, we have \( f = e' \) and \( a = s_1 f + f \). In the case \( s_1 = 0 \) we get \( a = f \). In the case \( s_1 \neq 0 \) we get for \( \rho \in A^* \) with \( \rho(\|) = 1 \) that \( 1 = \rho(a) \text{ iff } \rho(f') = \rho(a) - \rho(f) = s_1 \rho(f') \text{ iff } 0 = \rho(f') \text{ iff } 1 = \rho(f) \). Therefore \( \{ \rho \in A^* : \rho(\|) = 1 = \rho(a) \} = \{ \rho \in A^* : \rho(\|) = 1 = \rho(f) \} \) in all cases with \( s_1 < 1 \).

The same follows in the case \( s_2 < 1 \) with exchanged roles of \( e \) and \( e' \) and of \( s_1 \) and \( s_2 \).

(iii) Assume that equation (6) holds for all atoms \( e \) and \( f \). This means \( \mu_e(f) + \mu_{e'}(f) = 1 = \mu_e(f') + \mu_{e'}(f') \). Select any atom \( e \) and consider

\[ X := \{ a \in A : \mu_e(a) + \mu_{e'}(a) = 0 \}. \]

Then \( \| X \) and \( A = X \oplus \mathbb{R} \).

If \( f \) is any atom, \( f = \frac{1}{2}(x + \|) \) with \( x \in X \) and \( \| x \| = 1 \); choose \( x := f - f' \)
Vice versa, if \( x \in X \subseteq A \) and \( \| x \| = 1 \), let \( x = sf + tf' \) with an atom \( f \) and \( s, t \in \mathbb{R} \) be the spectral decomposition of \( x \). With \( \mu_e + \mu_{e'} \) we get \( 0 = s + t \). Therefore \( x = sf - f' \) and \( 1 = \| x \| = |s| \). This means that either \( s = 1, x = f - f' \) and \( \frac{1}{2}(x + \|) = f \) or \( s = -1, x = f' - f \) and \( \frac{1}{2}(x + \|) = f' \). Therefore, the atoms in \( L_A \) are identical with the \( \frac{1}{2}(x + \|), x \in X, \| x \| = 1 \). For \( 0 \neq x \in X \) and \( s \in \mathbb{R} \) the spectral decomposition of \( x + s \) is

\[ x + s\| = (s + \| x \|)f + (s - \| x \|)f' \]

with the atom

\[ f := \frac{1}{2} \left( \frac{1}{\| x \|} x + \| \right). \]

Then \( 0 \leq x + s\| \) iff \( 0 \leq s - \| x \| \). Use \( \mu_f \) for the only-if direction.

Assume \( x \in X, \| x \| = 1 \), and choose an element \( \rho \) in the dual of \( X \) with \( 1 = \rho(x) = \| \rho \| \). Consider the atom \( f = \frac{1}{2}(x + \|) \) and the following extension \( \tilde{\rho} \) of \( \rho \) to a state on \( A \) and \( L_A, \tilde{\rho} : x + s\| \to \rho(x) + s \) for \( x \in X \) and \( s \in \mathbb{R} \). Then \( \tilde{\rho}(f) = 1 \) and thus \( \tilde{\rho} = \mu_f \). Therefore \( \tilde{\rho} \) and thus \( \rho \) are uniquely determined and \( X \) is smooth.
Now assume \( \|tx + (1-t)y\| = 1 \) with \( x, y \in X, \|x\| = 1 = \|y\| \) and \( 0 < t < 1 \). Select an element \( \rho \) in the dual of \( X \) with \( \rho(tx + (1-t)y) = 1 = \|\rho\| \). Then \( 1 = tp(x) + (1-t)p(y) \) with \( |p(x)| \leq 1 \) and \( |p(y)| \leq 1 \). Therefore \( p(x) = 1 = p(y) \).

Use the same extension \( \tilde{\rho} \) of \( \rho \) to a state on \( A \) as above and the atoms \( f_1 = \frac{1}{2}(x + I) \) and \( f_2 = \frac{1}{2}(y + I) \). Then \( \tilde{\rho}(f_1) = 1 = \tilde{\rho}(f_2) \) and \( \mu_{f_1} = \tilde{\rho} = \mu_{f_2} \).

Since \( S_A \) is strong, we get \( f_1 = f_2 \) and \( x = y \). We have thus shown that \( X \) is strictly convex, which completes the proof that \( A \) is a generalized spin factor.

The only-if parts of (iii) and (iv) have been proven in section 7. For the proof of the remaining part of (iv) assume that the transition probabilities are symmetric. For any two atoms \( e \) and \( f \) then \( \mathbb{P}(f|e) + \mathbb{P}(f|e') = \mathbb{P}(e|f) + \mathbb{P}(e'|f) = \mathbb{P}(I|f) = 1 \). This means that (6) holds. By (iii) \( A \) is a generalized spin factor, and by Theorem 7.1 \( A \) is a spin factor.

(v) If \( A \) is a spin factor, \( A \) is a Jordan algebra and the product is bilinear. Now suppose that the product is bilinear. Choose any atom \( f \) and define an inner product by \( \langle a|b \rangle := \mu_f(a\circ b) + \mu_{f'}(a\circ b) \) for \( a, b \in A \). With \( a = se + te' \), \( s, t \in \mathbb{R} \) and an atom \( e \) we then have \( \langle a|a \rangle = s^2(\mu_f(e) + \mu_{f'}(e)) + t^2(\mu_f(e') + \mu_{f'}(e')) \).

Since \( 0 = \mu_f(e) + \mu_{f'}(e') \) would imply \( \mu_f(e) = 0 = \mu_{f'}(e), \mu_f(e') = 1 = \mu_{f'}(e') \) and thus \( \mu_f(\mu) = \mu_{f'}(\mu) \), which contradicts \( \mu_f \neq \mu_{f'} \), we have \( 0 < \mu_f(e) + \mu_{f'}(e) \).

Therefore \( \langle a|a \rangle = 0 \) iff \( s = 0 = t \) iff \( a = 0 \). Choose \( X := \{ x \in A : \langle I|x \rangle = 0 \} = \{ x \in A : \mu_f(x) + \mu_{f'}(x) = 0 \} \) to see that \( A \) is a spin factor.

We shall now show that the reverse of Theorem 9.1 (ii) also holds and that each strictly convex and smooth compact convex set is the state space of some binary quantum logic.

**Theorem 9.2** Let \( K \) be a strictly convex and smooth compact convex set in a locally convex space \( V \). Then there is a complete order unit space \( A_K \) with quantum logic \( L_K = \text{ext}[0, I] \) and state space \( S_K \) such that the conditions (a), (b), (c) and (d) of Theorem 9.1 are satisfied and the state space \( S_K \) is isomorphic to \( K \).

Moreover, \( A_K \) is a generalized spin factor iff \( K \) is the unit ball in some Banach space, and \( A_K \) is a spin factor iff \( K \) is the unit ball in some Hilbert space.

Proof. Suppose that \( K \) is a strictly convex and smooth compact convex set in a locally convex space \( V \). Then the topological boundary \( \partial K \) of \( K \) coincides with the extreme boundary \( \text{ext}(K) \). Let \( A_K \) consist of the continuous affine real functions on \( K \). Equipped with the usual pointwise ordering, \( A_K \) becomes a complete order unit space; we can choose the constant function that allocates 1 to each element of \( K \) as order unit \( I \). Besides \( I \) and 0, the extreme points of the unit interval are those functions \( e_\omega \) that assume the value 1 at one point \( \omega \) of the boundary and the value 0 at the antipodal point \( \omega' \).
For $\omega \in K$ define the state $\delta_\omega$ by $\delta_\omega(a) := a(\omega)$, $a \in A_K$. The map $\omega \to \delta_\omega$ is an isomorphism between $K$ and the state space $S_K$ of $A_K$ \cite{12} and $S_K$ is strong for the quantum logic $L_K(K)$. Each $a \in A_K$ assumes its maximum $s$ at a point $\omega$ on the boundary and its minimum $t$ at the antipodal point $\omega'$; then $a = se_\omega + te_{\omega'}$.

Moreover, $\mathbb{P}(\epsilon_{\omega_1}, \epsilon_{\omega_2}) = \delta_{\omega_1}(\epsilon_{\omega_2})$ for $\omega_1, \omega_2 \in \partial K$.

Suppose that $K$ is the unit ball in some Banach space. Then $-\omega_1$ is the antipodal point to a boundary point $\omega_1$ and $\frac{1}{2}(\delta_{\omega_1} + \delta_{-\omega_1}) = \delta_0$ on $A_K$. For all boundary points $\omega_2$ we have $\delta_0(\epsilon_{\omega_2}) = 1/2$ and $\delta_{\omega_1}(\epsilon_{\omega_2}) + \delta_{-\omega_1}(\epsilon_{\omega_2}) = 2\delta_0(\epsilon_{\omega_2}) = 1$. Therefore equation (6) is satisfied and, by Theorem 9.1 (iii), $A_K$ becomes a generalized spin factor $X \oplus \mathbb{R}$ with a smooth and strictly convex normed linear space $X$. The unit ball of the dual $X^*$ is isomorphic to the state space of $A_K$ and thus to $K$. If $K$ is the unit ball in a Hilbert space space, $X^*$ becomes a Hilbert space. Then the second dual $X^{**}$ is a Hilbert space as well. The canonical embedding of $X$ in $X^{**}$ shows that $X$ is a pre-Hilbert space. Moreover, since $A_K$ is complete, $X$ must be complete and we get a Banach space $X$ in the first case here and a Hilbert space $X$ in the second case.

If $A_K$ is a generalized spin factor, arising from the smooth and strictly convex normed linear space $X$, its state space is isomorphic to the unit ball of the dual of $X$ and this dual is a Banach space. If $A_K$ is a spin factor, $X$ is a pre-Hilbert space and its dual is a Hilbert space.

With Theorem 9.1, we have not only characterized the generalized spin factors in a new axiomatic way, but we have found a more general type of binary model. Beyond that, we have identified a one-to-one correspondence between the models of this type and the strictly convex and smooth compact convex sets. A model of this type arises from each such set $K$ by Theorem 9.2. The characteristics of $K$ determine those of the transition probabilities; they are symmetric iff $K$ is the unit ball in a Hilbert space, and they satisfy equation (6) for all atoms $e$ and $f$ iff $K$ is the unit ball of a Banach space.

The proof of Theorem 9.2 reveals that our transition probability becomes identical with the one introduced by Mielnik \cite{13} \cite{14} in a different way. He considers a convex set and directly defines the transition probability between its extreme points (the pure states).

### 10 Examples arising from the spaces $L^p$ and $L^p$

By Theorem 7.1 non-symmetric transition probabilities arise when we construct a generalized spin factor from a smooth and strictly convex normed space $X$ that is not a Hilbert space. Such spaces are the $L^p$ and $L^p$ spaces with the norm $\|\cdot\|_p$, $1 < p < \infty$ and $p \neq 2$. With $p = 2$, $L^p$ and $L^p$ become Hilbert spaces and the transition probabilities are symmetric. The dual spaces of $L^p$ and $L^p$ are $L^q$ and $L^q$ with $\frac{1}{p} + \frac{1}{q} = 1$. For $(\alpha_k) \in L^p$ and $(\beta_k) \in L^q$ we have the duality

$$\langle (\beta_k) | (\alpha_k) \rangle = \sum_k \alpha_k \beta_k$$

and $\| (\alpha_k) \|^p_p = \langle (\beta_k) | (\alpha_k) \rangle$ with $(\beta_k) = (\text{sign}(\alpha_k) | \alpha_k |^{p-1}) \in L^q$. 

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Here, $\text{sign}(\alpha) := 1, 0, -1$ for the three cases $\alpha > 0, = 0, < 0$ with any real number $\alpha$.

For $x, y \in \ell^p$ with $\|x\|_p = \|y\|_p = 1$ and the corresponding atoms $e = \frac{1}{2}(x \oplus 1)$, $f = \frac{1}{2}(y \oplus 1)$ in the quantum logic $L_\mathcal{X}$ with $X = \ell^p$, we get by (5)

$$P(f|e) = \frac{1}{2} \left( 1 + \sum_k \text{sign}(\alpha_k) |\alpha_k|^{p-1} \beta_k \right)$$

and

$$P(e|f) = \frac{1}{2} \left( 1 + \sum_k \alpha_k \text{sign}(\beta_k) |\beta_k|^{p-1} \right),$$

when $x = (\alpha)_k$, $y = (\beta)_k$. In the case $\alpha_1 = 1$ and $\alpha_k = 0$ for $k \neq 1$ this becomes

$$P(f|e) = \frac{1}{2}(1 + \beta_1) \quad \text{and} \quad P(e|f) = \frac{1}{2}(1 + \text{sign}(\beta_1) |\beta_1|^{p-1}).$$

Figure 1 shows this transition probability $P(e|f)$ as a function of the parameter $\beta_1$ for some different $p$. The diagonal line represents not only the symmetric case $p = 2$ for $P(e|f)$, but as well the transition probability $P(f|e)$, which does not depend on $p$ in the case considered here. The deviation of $P(e|f)$ from
the transition probability $P(f|e)$ increases, when $p$ moves away from 2, either towards larger numbers or closer to 1. The difference between $P(e|f)$ and $P(f|e)$ can come close to $\frac{1}{2}$, when $p$ approaches $\infty$ or 1. In the limiting case $p \to \infty$, $P(e|f)$ would become $\frac{1}{2}$ with the three exceptions $\beta_1 = -1, 0, 1$ where we always have $P(e|f) = 0, \frac{1}{2}, 1$. Note that $\beta_1 = 1$ means $e = f$ and that $\beta_1 = -1$ means $e' = f$. In the other limiting case $p \to 1$, $P(e|f)$ would become 0 for $-1 < \beta_1 < 0$ and 1 for $0 < \beta_1 < 1$.

The striking symmetry of each curve with respect to the point $(0, \frac{1}{2})$ in Figure 1 is due to equation (6); a swap of $f$ and $f'$ means a change of sign for $\beta_1$.

We complete this section with a brief look at the spaces $L^p, 1 < p < \infty$, with any measure $\lambda$. In the same way as above we get the transition probabilities

$$P(f|e) = \frac{1}{2} \left( 1 + \int g \, \text{sign}(h) |h|^{p-1} d\lambda \right)$$

and

$$P(e|f) = \frac{1}{2} \left( 1 + \int \text{sign}(g) |g|^{p-1} h \, d\lambda \right)$$

for any $g, h \in L^p$ with $\|g\|_p = \|h\|_p = 1$ and the corresponding atoms $e = \frac{1}{2}(g \oplus 1), f = \frac{1}{2}(h \oplus 1)$ in the quantum logic $ext[0, I]$ of the generalized spin factor $A = L^p \oplus \mathbb{R}$.

The spaces $l^1, L^1, l^\infty$ and $L^\infty$ are neither smooth nor strictly convex and the transition probabilities do not exist for the atoms in the quantum logic $ext[0, I]$ in $A = l^1 \oplus \mathbb{R}, A = L^1 \oplus \mathbb{R}, A = l^\infty \oplus \mathbb{R}$ or $A = L^\infty \oplus \mathbb{R}$.

11 Further examples

Binary examples where (6) is violated can easily be constructed using Theorem 9.2 and a strictly convex and smooth compact convex set $K$ that is not isomorphic to the unit ball of some normed space $X$.

A non-binary example with non-symmetric transition probability becomes Alfsen and Shultz’s triangular pillow shown in Figure 2 (more precisely: the quantum logic is not the pillow itself, but the orthomodular lattice consisting of the projective faces, the compressions or projective units; the triangular pillow is its state space). We avoid to go into the details of Alfsen and Shultz’s theory and sketch this example only briefly.

The vertexes of the equilateral triangle represent three pairwise orthogonal atoms $f_1, f_2, f_3$ with $f_1 + f_2 + f_3 = I$. A boundary point on the curved surface off the triangle and its antipodal boundary point represent two orthogonal atoms the sum of which is $I$; this means that the orthogonal complement of a boundary point on the curved surface off the triangle is its antipodal point. An example of two such atoms are the north-pole point $e$ and the south-pole point $e'$.

The triangular pillow is not strictly convex at the edges and not smooth at the vertexes of the triangle. However, it is a spectral convex set and, for each atom, there is only one single state such that the atom carries the
probability 1 in this state [2]. Therefore, the transition probabilities exist for the atoms. Lemma 3.2 then rules out that they are symmetric, since we have $I = f_1 + f_2 + f_3 = e + e'$ for the three pairwise orthogonal atoms $f_1, f_2, f_3$ and the orthogonal pair of atoms $e, e'$.

Moreover, the symmetries of the triangular pillow imply $P(f_1|e) = P(f_2|e) = P(f_3|e) = P(f_1|e') = P(f_2|e') = P(f_3|e')$ and $P(e|f_k) = P(e'|f_k)$ for $k = 1, 2, 3$. Since $P(f_1|e) + P(f_2|e) + P(f_3|e) = P(I|e) = 1$ and $P(e|f_k) + P(e'|f_k) = P(I|e) = 1$, we get for $k = 1, 2, 3$ the non-symmetric transition probabilities

$$P(f_k|e) = P(f_k|e') = \frac{1}{3}$$ and $P(e|f_k) = P(e'|f_k) = \frac{1}{2}$

and furthermore

$$\sum_k P(e|f_k) = \sum_k P(e'|f_k) = \frac{3}{2}$$ and $P(f_k|e) + P(f_k|e') = \frac{2}{3}$,

which violates equation (6) in section 7.

The triangular pillow does not satisfy the conditions (b) and (c) of Theorem 9.1, since $f_1 + f_2$ is not an atom, but $0 \neq f_1 + f_2 \neq 1$ and $f_2 \neq f_1'$. It is one very demonstrative three-dimensional example in a larger class which Alfsen and Shultz constructed [2] and which includes spectral convex sets with higher dimensions.

Figure 2: The triangular pillow [2]

The examples of this section yield further cases, where Segal’s postulates [20] are satisfied, but a bilinear product that is in line with the spectral decomposition does not exist (see section 8). This follows for the first examples from Theorem 9.1 (v) and for the triangular pillow from Corollary 9.44 in Ref. [2]. The product is bilinear then only for compatible elements of the order unit space; these are linear combinations of the same pairwise orthogonal elements $e_k$ of the quantum logic and then $(\sum s_k e_k) \circ (\sum t_k e_k) = \sum s_k t_k e_k$ for $s_k, t_k \in \mathbb{R}$.

A bilinear product for elements that are not compatible may be very appealing from the mathematical point of view, but convincing physical reasons to make it a general postulate are hard to find.
12 Conclusions

We have seen that there is an abundance of mathematical structures with non-symmetric transition probabilities, and our main result is a complete axiomatic characterization of the binary models, which represent the classical bit, the quantum bit (qubit) and generalized versions thereof. Their state spaces are smooth and strictly convex compact convex sets and, vice versa, for each such set \( K \) there is a model with state space \( K \). This reveals an interesting bidirectional relation between the binary models and the smooth strictly convex compact convex sets. The characteristics of \( K \) determine those of the transition probabilities. They satisfy equation (6) for all atoms \( e \) and \( f \) if and only if \( K \) is the unit ball of a Banach space, and they become symmetric if and only if \( K \) is the unit ball of a Hilbert space. The first case results in a generalized spin factor and the second case in a genuine spin factor.

Only the genuine spin factors possess a bilinear product and become formally real Jordan algebras. They include the real version of the qubit, the usual complex qubit and many further cases. Some favorable features distinguishing the usual complex qubit (and, more generally, quantum mechanics with the Hilbert spaces over the complex numbers) have been identified: the generator of every continuous reversible time evolution can be associated to an observable \[4\], and the usual qubit permits a reasonable model of a multiple bit system with local accessibility of the state [22, 23]; nowadays local accessibility has become more familiar as local tomography [4]. Nevertheless the other spin factors play an important role in quantum mechanics, since they represent the canonical anti-commutator relations for the fermions.

It would be interesting to know whether there are any further quantum logics with strong state spaces and non-symmetric transition probabilities beyond those considered here (and their direct sums), perhaps with a richer structure and an infinite family of pairwise orthogonal atoms. A next step might be to study generalized ternary models along the lines of Theorem 9.1, which means that the assumptions (b) and (c) in Theorem 9.1 are replaced by:

(b') The maximum number of pairwise orthogonal atoms is three.

(c') Each \( a \in A \) has a spectral decomposition \( a = s_1e_1 + s_2e_2 + s_3e_3 \) with \( s_1, s_2, s_3 \in \mathbb{R} \) and three pairwise orthogonal atoms \( e_1, e_2, e_3 \).

This includes the decomposable cases \( \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R} \), \( A \oplus \mathbb{R} \), where \( A \) is an order unit space satisfying the assumptions (a), (b), (c) and (d) in Theorem 9.1, and the four non-decomposable formally real Jordan algebras formed by the self-adjoint \( 3 \times 3 \)-matrices over the real numbers, complex number, quaternions or octonions [12, 19]. The first case is the classical one, the complex \( 3 \times 3 \)-matrices represent the usual quantum mechanical ternary model (the complex qutrit). In the four Jordan algebras, the transition probability is symmetric [19]. An open issue is whether these are already all or whether there are more non-decomposable ternary cases, possibly with non-symmetric transition probabilities, and whether they can be classified. Note that the triangular pillow does
not satisfy \((c')\); it represents neither a generalized binary nor ternary model, but a strange non-decomposable hybrid between these two. Alfsen and Shultz’s further examples in the class where the triangular pillow belongs to \([2]\) are hybrids with higher dimensions.

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