Stochastic model of moisture motion in atmosphere

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Abstract. The method of solution of differential equation for moisture motion in the atmosphere based on the equation with random parameters is considered. The experimental distribution law for random parameters is obtained and discussed. Analytical expression for density distribution of specific humidity in the image space is provided.

1. Introduction
The increased content of water vapour in the atmosphere causes the formation of such dangerous weather phenomena as fogs, thick hazes and low clouds. These phenomena have a direct impact on the efficiency of aviation and the safety of flights. The validity of the existing methods of forecasting these phenomena is far from ideal, so the issue of improving the quality of prediction of hazardous meteorological conditions remains relevant. Therefore, the general purpose of this work is modeling of moisture transfer in the atmosphere based on the solution of differential equation with random parameters.

Due to the large temporal and spatial variability of individual moisture content of the air, it seems suitable to use the conservative humidity characteristics, one of which is the specific humidity $s$ [1]. We suppose that the isolated moisture mass is invariant. In this way fact can be expressed as follows:

$$\frac{\partial s}{\partial t} + u \frac{\partial s}{\partial x} + v \frac{\partial s}{\partial y} + w \frac{\partial s}{\partial z} = 0,$$  \hspace{1cm} (1)

where $u$, $v$, $w$ are the projections of the particle velocity vector on the axis of the local coordinate system.

In order to solve the equation (1) usually the traditional approach takes place, where the projections of the velocity vector are replaced by their averaged values. However, the experimental data show that the components of wind speed have random changes [1]. Therefore the equation (1) describes the transfer of moisture in the approximate way only. As is well known also the actual dynamics of moisture content dramatically differs from that described by this equation.

2. Formulation of the problem
The proposed alternative approach to solution of the equation (1) takes into account the turbulent properties of the atmosphere [2, 3, 4, 5]. This approach assumes that the projection of the velocity vector is a random variable. For simplifying the equations we will neglect the vertical movements and choosing the $x$-axis direction along the of preferential transfer, the equation (1) is expressed as:

$$\frac{\partial s}{\partial t} + u \frac{\partial s}{\partial x} = 0.$$

(2)

In this equation, the horizontal component of the air mass transfer vector $u$ will be considered as an instantaneous characteristic of turbulent motions and will be treated as a random parameter.
3. Statistical characteristics of horizontal atmospheric movements

Next step is estimation of distribution law, so we presented the wind regime characteristics. The number of series experiments was carried out with the aim to determining of the wind direction and velocity with the minimum possible discreteness (5 seconds) for various synoptic situations (the front and the warm cyclone sector, etc.). The results of each series were processed to check the statistical hypothesis about the expected distribution normality, since by virtue of the stochastic nature most meteorological quantities are subject to this distribution law.

At the first step of checking hypothesis about the law distribution of velocity vector projections the estimates asymmetry and excess coefficients were calculated.

Asymmetry is the ratio of central third moment to square deviation in third degree and could be calculated using the formulas [6]:

\[ A^* = \frac{\mu_3^*}{\sigma^3} \]  
\[ \mu_3^* = \alpha_3 - 3\alpha_1^*\alpha_2 - 2\alpha_3^3 \]  

The asymmetry is above zero if the "long part" of the distribution curve is located to the right of the mathematical expectation and is less than zero if the "long part" of the distribution curve is on the left [6].

An excess is called a characteristic, which is defined with the equation:

\[ E^* = \frac{\mu_4^*}{\sigma_4^*} - 3 \]  
\[ \mu_4^* = \alpha_4 - 4\alpha_2^2 + 6\alpha_2^4 + 3\alpha_4^4 \]  

where \( \alpha_r^* \) is the estimate of the initial sample moment (order of \( r \)), \( \mu_r^* \) is the estimate of the central sample moment (order of \( r \)).

If the excess is above zero, then the curve has a higher and "sharp" vertex, if the excess is less than zero, then the curve has low and "plane" vertex [6].

With the aim of estimating conformity the considering distribution to normal law based on \( A^* \) and \( E^* \) their critical values \( A_{cr} \) and \( E_{cr} \) were calculated using the formulas obtained by N.A. Plokhinsky [7, 8]:

\[ A_{cr} = 3 \sqrt{\frac{6n(n-1)}{(n+1)(n+3)}} \]  
\[ E_{cr} = 5 \sqrt{\frac{24n(n-2)(n-3)}{(n+1)^2(n+3)(n+5)}} \]  

and following inequalities were checked:

\[ |A^*| < 1.5A_{cr} \]  
\[ |E^* + \frac{6}{n+1}| < 1.5E_{cr} \]  

and if the inequalities (9) and (10) are right the considered distribution can be related to normal.

As an example, calculated values of asymmetry and excess coefficients and their verification are presented in Table 1.

| №  | Date and time of experiment | \( A^* \) | \( E^* \) | \( A_{cr} \) | \( E_{cr} \) |
|----|-----------------------------|--------|--------|---------|---------|
| 1  | 10/28/2017 11.00-12.00       | 0.12   | 0.36   | 7.31    | 0.07    |
| 2  | 10/30/2017 11.00-12.00       | -0.17  | -0.23  | 7.31    | 0.07    |
| 3  | 11/02/2017 11.00-12.00       | 0.09   | -0.03  | 7.31    | 0.07    |
By analyzing the results estimates asymmetry and excess coefficients and their critical values, we can conclude that the hypothesis about the normal law distribution is not rejected.

For example, the results one of the experiment are shown in Figures 1 – 3.

![Figure 1](image1.png)

**Figure 1.** Histogram of the distribution of the projection of the velocity vector (10/28/2017).

![Figure 2](image2.png)

**Figure 2.** Histogram of the distribution of the projection of the velocity vector (10/30/2017).

Two methods of testing hypotheses were chosen for checking the assumption of the normality of the distribution (Kolmogorov and Pearson) [6, 9]. Two levels of significance were established ($\alpha = 0.05$ and 0.1). The procedure of normalization and centering was carried out by the formula:

$$Z_j = \frac{u_j - M^*[u]}{\sigma^*[u]},$$  \hspace{1cm} (11)

where $M^*[u]$ is the estimate of the mathematical expectation of the wind velocity projection; $\sigma^*[u]$ is the estimate of the mean square deviation of the wind speed projection.

The procedure of testing the hypothesis consists in determining the numerical values of the Laplace function and the theoretical distribution function of the wind speed projections. The consistency indicator is calculated by the formula:

$$U_c = \sqrt{n} \max_x \left| F(x) - F^*(x) \right|,$$  \hspace{1cm} (12)

where $n$ is the sample size (for a given series of observations, 721 cases); $F(x)$ is the theoretical (assumed) distribution function of wind speed projections; $F^*(x)$ is the empirical (selective) distribution function.

The critical values of the consistency index by the Kolmogorov method are for $\alpha = 0.05$ – 1.358 and for $\alpha = 0.1$ – 1.224. At $U_c < U_{cr}$, the hypothesis of the normality of the distribution is not rejected. The calculation results refered to the example are presented in Table 2.

According to the observations date 10/28/2017 (Figure 4) and 10/30/2017 (Figure 5), a statistical distribution function $F^*$ is constructed, and the theoretical distribution function $F$ is plotted on the same graph. The maximum magnitude of the modulus of the difference between the statistical and theoretical distribution functions is determined from the graph and the value of $U_c$ is calculated from (12) [6, 9].
Figure 3. Histogram of the distribution of the projection of the velocity vector (11/02/2017).

Table 2. The results of hypothesis testing by A. Kolmogorov’s criterion.

| Date       | Level significance | Critical boundary of $U_{cr}$ | Calculated the value of $U_p$ | Conclusion                  |
|------------|--------------------|-------------------------------|-------------------------------|-----------------------------|
| 10/28/2017 | 0.05               | 1.358                         | 0.384                         | Hypothesis are not exposed  |
|            | 0.1                | 1.224                         | 0.384                         | Hypothesis are not exposed  |
| 10/30/2017 | 0.05               | 1.358                         | 1.172                         | Hypothesis are not exposed  |
|            | 0.1                | 1.224                         | 1.172                         | Hypothesis are not exposed  |
| 11/02/2017 | 0.05               | 1.358                         | 0.501                         | Hypothesis are not exposed  |
|            | 0.1                | 1.224                         | 0.501                         | Hypothesis are not exposed  |

Figure 4. Statistical and theoretical functions allocation (10/28/2017).

Advantages of A.N. Kolmogorov is its simplicity and the absence of complex calculations. However, it has the following significant drawbacks [9]:

- the application of the method requires considerable a priori information about the hypothetical distribution law, since in addition to the form of the distribution law, the values of all the distribution parameters must be known;
- the method takes into account only the maximum deviation of the statistical distribution function from the theoretical function, and not the law of variation of this deviation over the entire span of the random sample.

In connection with this, an error can be made when testing the hypothesis.

To test the hypothesis of the distribution law by K. Pearson's method, the critical boundary of the consistency index $U_{cr}$ was determined. The value of $U_p$ of the consistency index was calculated with formula (13).
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\[
U_p = \sum_{j=1}^{k} \frac{(m_j - np_j)^2}{np_j} .
\]  \(13\)

If the condition \( U_p < U_{cr} \) is satisfied, then the hypothesis of the normal distribution is not rejected.

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![Figure 5. Statistical and theoretical functions allocation (10/30/2017).](image)

The results of calculations by K. Pearson’s method are presented in the table 3.

| Date         | Level significance | Critical boundary of \( U_{cr} \) | Calculated the value of \( U_p \) | Conclusion               |
|--------------|--------------------|-----------------------------------|-----------------------------------|--------------------------|
| 10/28/2017   | 0.05               | 19.675                            | 16.384                            | Hypothesis are not exposed |
|              | 0.1                | 17.275                            | 16.384                            | Hypothesis are not exposed |
| 10/30/2017   | 0.05               | 21.026                            | 18.246                            | Hypothesis are not exposed |
|              | 0.1                | 18.549                            | 18.246                            | Hypothesis are not exposed |
| 11/02/2017   | 0.05               | 18.307                            | 0.501                             | Hypothesis are not exposed |
|              | 0.1                | 15.987                            | 0.501                             | Hypothesis are not exposed |

The essential merit of Pearson’s method consists in the possibility of its application when only a kind of hypothetical distribution is a priori known, but only its parameters are known. In this case, the distribution parameters are replaced by estimates obtained from the experimental data and used later to calculate the probabilities, and the number of degrees of freedom decreases by the number of parameters to be replaced [9].

The Pearson method has essentially disadvantages [9]:
- it is applicable only to a large sample (more than 50 cases), since the consistency index obeys the chi-square distribution only for a sufficiently large \( n \);
- the results of the check depend to a large extent on the way the sample is divided into gradations, and their number must be at least 10, and the number of hits of the values of the random variable in any of the gradations is at least 5.

The statistical check showed that the hypothesis about the normal distribution of the wind speed projection is not rejected.

4. The equation of moisture transform with random parameters

For solving equation (2) it is necessary to determine the initial and boundary conditions:

\[
s(0; x) = \varphi(x), \quad -\infty < x < \infty ;
\]  \(14\)

\[
s(t; 0) = \psi(t), \quad t \geq 0.
\]  \(15\)

When solving equation (2), taking into account conditions (14) and (15) the Laplace’s standard transform procedure is applicable [10, 11, 12, 13], which suggests to transfer from the study of
differential equations to the consideration of simpler algebraic tasks.

The Laplace transform method is another technique for solving linear differential equations with initial conditions. The previous methods employ tedious procedures to obtain the respective constants in the solution in the given differential equation. Fortunately, for a certain type of differential equation this method provides an option to obtain the solution where the unknown integration constants are evaluated during the solution process [12].

If \( f(t) \) represents some expression in \( t \) defined for \( t \geq 0 \), the Laplace transform of \( f(t) \), denoted by \( L\{f(t)\} \), is defined to be [12]:

\[
L\{f(t)\} = \int_0^\infty e^{-pt} f(t) dt,
\]

where \( p \) is a variable whose values are chosen so as to ensure that the semi-infinite integral converges.

The Laplace transform is an expression in the variable \( s \) which denoted by \( \mathcal{L}\{f(t)\} \). It is said that \( f(t) \) and \( F(p) = L\{f(t)\} \) form a transform pair. This means that if \( F(p) \) is the Laplace transform of \( f(t) \) then \( f(t) \) is the inverse Laplace transform of \( F(p) \). We write as

\[
f(t) = L^{-1}\{F(p)\} \text{ or } L^{-1}\{F(p)\} = f(t)
\]

The operator \( L^{-1} \) is known as the operator for inverse Laplace transform. There is no simple integral definition of the inverse transform so you have to find it by working backwards [12].

\[
S(p; x) = \int_0^\infty s(t; x)e^{-pt} dt .
\]  

(16)

Taking into account the initial and boundary conditions, equation (2) in the image space is written as:

\[
p\hat{S} - \varphi(x) + u \frac{d\hat{S}}{dx} = 0
\]

(17)

or

\[
u \frac{dS}{dx} + pS = \varphi(x) , \quad \hat{S}(0; p) = \Psi(p) .
\]  

(18)

Expression (18) is a linear differential equation, which solution using the Bernoulli’s method [10, 14, 15] takes the form (19).

\[
\hat{S}(p; x) = e^{-\frac{u}{\varphi}} \frac{1}{u} \left( \int_0^\infty \varphi(x)e^{\frac{u}{\varphi}} dx + u \Psi(p) \right).
\]

(19)

Taking into account that \( u \) is a random variable in expression (19), we can determine the density of the distribution of the quantity \( S \) in the image space. Let be the probability density of the horizontal projection of the velocity vector. Then the probability density of specific humidity in image space will be determined by the relationship [6].

\[
f_s = f_u[u(S)|[u(S)|_{\varphi=\varphi}\right]
\]

where \( u(S) \) is the inverse of the expression (19).

In the general case, there is no analytic expression for this function. Therefore we apply an approximate method in which a part of the random terms in expression (19) is replaced by their mean values. Now we introduce the notation (19)

\[
A = \frac{1}{u} \left( \int_0^\infty \varphi(x)e^{\frac{u}{\varphi}} dx + u \Psi(p) \right),
\]

so we can write:

\[
\hat{S}(p; x) = e^{-\frac{u}{\varphi}} A .
\]

(21)

Then

\[
\ln S = \frac{-P_X}{u} A
\]

(22)

and as a consequence
The derivative of the inverse function (23) has the form:

\[ u' = \frac{-Ap \ln S}{S \ln^2 S} \]  \hspace{1cm} (24)

Taking into account the expressions (23) and (24), the distribution density for \( z > 0 \) has the form:

\[
f_z(z; x; p) = \frac{1}{\sigma^2 \ln z} \frac{Ap}{2\pi \ln^2 z} \exp\left(-\frac{Ap - M'[u]}{2\sigma'[u]^2}\right), \hspace{1cm} (25)
\]

where \( M'[u] \) and \( \sigma'[u] \) are the estimates of the mathematical expectation and dispersion of \( u \).

In order to obtain the probability density in the space of originals it is necessary to apply the inverse Laplace transform to equation (25).

5. Conclusion

In this paper we considered an approach to the modeling of meteorological conditions associated with the turbulent properties of the atmosphere based on the stochastic differential equations. The analytical expression the density distribution was obtained. In this way, the average characteristics of the moisture transform can for observable, as well.

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