Models of Bars II: Exponential Profiles

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ABSTRACT

We present a new model for galactic bars with exponentially falling major axis luminosity profiles and Gaussian cross-sections. This is based on the linear superposition of Gaussian potential-density pairs with an exponential weight function, using an extension of the method originally introduced by Long & Murali (1992). We compute the density, potential and forces, using Gaussian quadrature. These quantities are given as explicit functions of position. There are three independent scaled bar parameters that can be varied continuously to produce be-spoke bars of a given mass and shape. We categorise the effective potential by splitting a reduced parameter space into six regions. Unusually, we find bars with three stable Lagrange points on the major axis are possible. Our model reveals a variety of unexpected orbital structure, including a bifurcating $x_1$ orbit coexisting with a stable $x_2$ orbit. Propeller orbits are found to play a dominant role in the orbital structure, and we find striking similarities between our bar configuration and the model of Kaufmann & Contopoulos (1996). We find a candidate orbital family, sired from the propeller orbits, that may be responsible for the observed high velocity peaks in the Milky Way’s bar. As a cross-check, we inspect, for the first time, the proper motions of stars in the high velocity peaks, which also match our suggested orbital family well. This work adds to the increasing body of evidence that real galactic bars may be supported at least partly by propeller orbits rather than solely the $x_1$ family.

Key words: galaxies: dynamics and kinematics – galaxies: structure – Galaxy: bulge

1 INTRODUCTION

A central bar-shaped concentration of stars is a common feature of spiral galaxies. There is ample evidence to suggest that the Milky Way is a late-type barred spiral galaxy from infrared photometry, starcounts, stellar and gas kinematics and microlensing (e.g., Weinberg 1992; Paczynski et al. 1994; Dwek et al. 1995; Wegg et al. 2015; Sanders et al. 2019a). This is no surprise, as it has been known since the 1960s that bars form readily in $N$-body simulations, and are long-lived, robust stellar dynamical equilibria (Hohl 1971; Toomre 1981).

Analyses of observational evidence from photometry of barred galaxies suggests there are in fact two main types of galactic bars - “flat” and “exponential” (Elmegreen & Elmegreen 1985; Sellwood & Wilkinson 1993). In early-type barred galaxies, the luminosity along the major axis falls slowly, and is sometimes almost flat all the way to the end of the bar. In contrast, late-type barred galaxies have bars with an almost exponentially falling luminosity profile along the major axis. This dichotomy may even be an evolutionary sequence, as early-type barred galaxies tend to be more massive. In this picture, exponential bars may gradually redistribute their mass and angular momentum to become flatter in profile. By contrast, there have been few systematic studies of the density profiles along the minor and intermediate axes, which of course requires deprojection of the surface photometry. The sparse information that we possess suggests that their profiles appear to be close to Gaussian (Blackman 1983) or exponential (Gadotti et al. 2007).

As measured against the endpoints of $N$-body experiments, the analytic models used to describe bars are often unrealistic. For example, the analytic Ferrers (1877) ellipsoids have zero density outside a given elliptical radius (Binney & Tremaine 2008), namely

$$
\rho(x, y, z) = \begin{cases} 
\rho_0(1 - m^2)^n & m < 1 \\
0 & m \geq 1,
\end{cases}
$$

with $m^2 = x^2/a^2 + y^2/b^2 + z^2/c^2$. Here $a$, $b$ and $c$ are the constant semi-axes of the ellipsoidal density contours, whilst $n$ is an integer. The gravitational potential within the bar is a polynomial of order $2n + 2$ in $x$, $y$ and $z$. Their tractability means there have been extensive investigations of the orbital structure of Ferrers bars (e.g., Pfenniger 1984a; Athanassoula 1992; Skokos et al. 2002). On the other end of the spectrum, the purely numerical Cazes bar (Cazes & Tohline 2000; Barnes & Tohline 2001) is constructed from realistic hydrodynamical simulations, but has a potential defined only on an $800 \times 800$ Cartesian grid, and so is not simple to investigate. A huge amount of insight into orbital properties of bars has been discovered through investigation of these models, and their useful...
ness should not be underestimated. Despite this, the inventory
of realistic and simply calculable bar models is small, and a gap in
the market remains. This is especially the case for exponential bars,
for which there are no simple models in the literature. For example,
although Ferrers bars with large values of $n$ have more rapid density
fall-off, it is always polynomial, and never exponential.

Long & Murali (1992) introduced a convenient and versa-
tile algorithm for producing flexible barred potentials. They con-
voluted a simple spherical or axisymmetric background potential
with a needle-like weight function. They applied their method to
the Plummer and Miyamoto-Nagai models to produce a variety of
prolate and triaxial bars. Their method was then used successfully
in Williams & Evans (2017) to produce a model for a flat bar by
convolving a logarithmic density with a needle-like weight func-
tion. Here, we will extend the Long & Murali (1992) algorithm,
but with an exponential weight function with a view to producing
exponential bars.

The paper is arranged as follows. Section 2 presents some sim-
ple properties of our new model and calibrates it against an
$N$-body simulation that mimics the Milky Way bar. The orbital structure is
discussed in some detail in Section 3 via characteristic diagrams
and Poincaré surfaces. Section 4 presents an application to the pres-
sence of high velocity peaks in the Milky Way bulge. We sum up in
Section 5, outlining some future challenges.

2 MODEL BUILDING

Long & Murali (1992) introduced a method for constructing barred
potential-density pairs from a general axisymmetric potential-
density pair ($\Phi_\text{d}, \rho_\text{d}$) via convolution with a weight function $w(x)$:

$$\rho_{\text{bar}} = \int \frac{w(x')}{\rho_\text{d}(x-x', y, z)} dx',$$
$$\Phi_{\text{bar}} = \int w(x') \Phi_\text{d}(x-x', y, z) dx'. \quad (2)$$

The weight function is a one-dimensional function of the major
axis coordinate, such that the new density is preferentially stretched
along the $x$ direction.

Our aim is to build a triaxial bar with a major axis density pro-
file that is roughly exponential, and then derive the potential and the
forces acting on a particle due to the density distribution. Taking the
minor and intermediate axis profiles as Gaussian in cross-section,
we start with the density ansatz

$$\Phi_{\text{Gaussian}} = -\frac{M}{q(2\pi\sigma^2)^{3/2}} \exp \left[ -\left( \frac{x^2 + y^2 + z^2/q^2}{2\sigma^2} \right) \right]. \quad (3)$$

with total mass $M$, variance $\sigma^2$ and flattening ratio $q$. The gravita-
tional potential of the Gaussian is (Cappellari 2008)

$$\Phi_{\text{Gaussian}} = -\frac{3}{\pi\sigma^2} GM \int_0^1 H(m) dm, \quad (4)$$

where

$$H(m) = \frac{1}{\sqrt{1-e^{-m^2}}} \exp \left[ -\frac{m^2}{2\sigma^2} \left( R^2 + \frac{z^2}{1-e^{-2m^2}} \right) \right]. \quad (5)$$

with $e$ given by $e^2 = 1 - q^2$. Here, $G$ is the gravitational con-
tant. This result follows from the general potential theory of elip-
soidal density distributions (Chandrasekhar 1987). In constructing
their bar, Williams & Evans (2017) used constant weight functions
($w(x) = 1$ for $-a < x < a$), producing flattish major-axis den-
sity profiles, suitable for early-type bars. Here, we instead use an
exponential weight function, to produce near-exponential profiles
suitable for late-type bars. We use the weight function

$$w(x) = \frac{1}{2b(1 - e^{-a/b})} \exp \left[ -\frac{|x|}{b} \right]. \quad (6)$$

in the region $-a < x < a$, where our pre-factor ensures that

$$\int_{-a}^a w(x') dx' = 1. \quad (7)$$

The parameter $a$ therefore specifies the length of our bar, while $b$
controls the rate of exponential decay along the major axis.

In what follows, we streamline the algebraic expressions by de-
noting intermediate auxiliary functions to reduce clutter. Addi-
tionally, we adopt the notation that for a function $f(x, \ldots)$

$$f_s(x) \equiv f(x) + f(-x), \quad f_e(x) \equiv f(x) - f(-x), \quad (8)$$

are the symmetrisation and antisymmetrisation respectively of $f$
with respect to $x$.

2.1 Bar density

The density of the bar is the convolution

$$\rho_{\text{bar}} = \int_{-a}^a w(x') \rho_{\text{Gaussian}}(x-x', y, z) dx'. \quad (9)$$

We define a function $G(x)$ as

$$G(x) = e^{-\frac{a}{bx}} \left( \frac{abx + \sigma^2}{\sqrt{2b}\sigma} \right) - erf \left( \frac{\sigma^2}{\sqrt{2b}\sigma} \right). \quad (10)$$

with the standard error function given as

$$erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt. \quad (11)$$

We can compute the integral in $\rho_{\text{bar}}$ to find

$$\rho_{\text{bar}} = M \exp \left[ \frac{2ab\sigma^2}{2b^2} \right] \exp \left[ -\frac{y^2 + z^2/q^2}{2\sigma^2} \right] G(x), \quad (12)$$

where $G_s(x)$ is the symmetrisation of $G(x)$ with respect to $x$. The
density profile is exactly Gaussian along the intermediate and min-
or axes of the triaxial figure, whilst it is exponential to a good
approximation along the major axis.

The model bar has in total four parameters: $a$, $b$, $\sigma$ and $e$.
Here, $e$ is an overall length scale, which we set to unity in our plots
unless otherwise stated, whilst $a$ controls the length of the bar and
$b$ its flatness, as shown in the panels of Fig. 1. Together, these two
parameters prescribe the shape of the density in the $(x, y)$ plane, as
shown in the density contour plots of Fig. 2. $e$ then controls the $z$
flattening of the bar, and has no effect on the $(x, y)$ density profiles
other than scaling. We set $e = 0$ ($q = 1$) unless otherwise stated.

2.2 Surface density

To find the surface densities, we integrate the density along a given
line of sight. For simplicity, we consider the lines of sight to be
the axis directions, and obtain three surface brightness functions
$\Sigma_s(y, z)$, $\Sigma_y(x, z)$ and $\Sigma_y(\bar{x}, \bar{y})$, namely

$$\Sigma_s(y, z) \propto \frac{M}{4\pi q\sigma^2 (e^{a/b} - 1)} \exp \left[ \frac{-y^2 + z^2/q^2}{2\sigma^2} \right]. \quad (13)$$
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Figure 1. Left: Major axis density profiles, showing that as \(a\) increases, the length of the bar increases, but the central peaked shape remains present. The model has \((b, \sigma, \epsilon) = (1, 1, 0)\). Right: Major axis density profiles, showing that as \(b\) increases, the bar’s density profile becomes flatter. The model has \((a, \sigma, \epsilon) = (2, 1, 0)\).

\[
\Sigma_y(x, z) = \frac{M}{4b(e^a/b - 1)\sqrt{2\pi}q\sigma^2} \exp \left[ -\frac{r^2}{2q^2\sigma^2} \right] G_s(x), \quad (14)
\]

and

\[
\Sigma_z(x, y) = \frac{M}{4a(e^a/b - 1)\sqrt{2\pi}q\sigma^2} \exp \left[ -\frac{r^2}{2q^2\sigma^2} \right] G_s(x). \quad (15)
\]

We therefore have Gaussian luminosity profiles along the minor and intermediate axes. The luminosity profile along the major axis is exponentially falling to a very good approximation (as opposed to almost constant along the bar).

Figure 2. Four logarithmically-spaced contour plots of \(\rho\) in the \((x, y)\) plane, for varying \(a\) and \(b\). Top left shows \((a, b) = (3, 1)\), top right \((3, 10)\), bottom left \((5, 1)\), bottom right \((5, 10)\). It can be seen that increasing \(a\) (going from the top line to the bottom) increases the length of the bar, whereas increasing \(b\) (going from left to right) makes the bar flatter but does not affect the length.

\[
\Sigma_y(x, z) = \frac{M}{4b(e^a/b - 1)\sqrt{2\pi}q\sigma^2} \exp \left[ -\frac{r^2}{2q^2\sigma^2} \right] G_s(x), \quad (14)
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2.3 Bar potential

To find the potential \(\phi_{\text{bar}}\) for this density, we follow eqs (4) and (5), and write

\[
\Phi_{\text{bar}} = -\frac{GM}{\sqrt{2\pi}q\sigma} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{e^{-\frac{r^2}{2q^2\sigma^2}}}{b(1 - e^{-a/b})} \ dx' \ dm - \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{e^{-\frac{r^2}{2q^2\sigma^2}}}{b(1 - e^{-a/b})} \ dx' \ dm. \quad (16)
\]

The inner integral can be evaluated analytically. Defining functions \(F(m, x)\) and \(E(m, y, z)\) by

\[
F(m, x) = e^{-\frac{r^2}{2q^2\sigma^2}} \left( \text{erf} \left( \frac{-bm^2(a - x) + \sigma^2}{\sqrt{2}bm\sigma} \right) - \text{erf} \left( \frac{-bm^2 + \sigma^2}{\sqrt{2}bm\sigma} \right) \right) \quad (17)
\]
and
\[ E(m, y, z) = \exp\left[ -\frac{m^2 y^2}{2\sigma^2} - \frac{m^2 z^2}{2\sigma^2(1 - m^2 \epsilon^2)} + \frac{\sigma^2}{2b^2 m^2} \right], \]  
(18)
we can write our potential as
\[ \Phi_{\text{bar}} = -\frac{GM}{2b(1 - e^{-a/b})} \int_0^1 H_{\text{conv}}(x, y, z) \, dm, \]  
(19)
where
\[ H_{\text{conv}} = \frac{1}{m^2(1 - m^2 \epsilon^2)} E(m, y, z) F_2(m, x). \]  
(20)
The numerical computation of \( H_{\text{conv}} \) needs care close to \( m = 0 \). In order to integrate accurately, we build a Taylor approximation \( H_{\text{conv}}^{TS} \) of \( H_{\text{conv}} \) around \( m = 0 \). We find a functional form for the value of \( m \) at which the switchover occurs, \( m = \text{Osc}(r, b) \). In other words, we write our integral as
\[ \int_0^{\text{Osc}(r, b)} H_{\text{conv}}^{TS}(x, y, z) \, dm + \int_{\text{Osc}(r, b)}^1 H_{\text{conv}}(x, y, z) \, dm. \]  
(21)
To find a useful expansion, we factorise out \( \exp\left[ (-m^2/(2\sigma^2))(R^2 + z^2/(1 - m^2 \epsilon^2)) \right] \) (the coordinate dependence of \( H \) before convolution with our weight function, see eq (5)) from \( H_{\text{conv}} \) before expanding as a Taylor series. This term is \( O(1) \) as \( m \to 0 \), and so is well-behaved. This ensures that our approximation remains valid as \( |x| \to \infty \). Our Taylor approximation is
\[ H_{\text{conv}}^{TS} = \frac{2}{\pi \sigma^2} \exp\left[ -\frac{m^2 x^2 + y^2}{2\sigma^2} - \frac{m^2 z^2}{2\sigma^2(1 - m^2 \epsilon^2)} \right] \left( 2b \left( 1 - e^{-a/b} \right) + m^2 \frac{b}{\sigma^2} e^{-a/b} \left( a^2 + 2ab + (1 - e^{-a/b})/(2b^2 - \sigma^2 \epsilon^2) \right) \right). \]  
(22)
The integration is performed using Gauss-Legendre quadrature (Abramowitz & Stegun 1972) throughout. To find the onset of numerical instability as \( m \to 0 \), we plot \( H_{\text{conv}} \) and vary the bar parameters continuously. We find a useful guide to choose
\[ \text{Osc}(r, b) = 0.2 \frac{\sigma}{b}. \]  
(23)
This is a slight overestimate of the switchover point, which ensures that the results of our integration are secure.

2.4 Forces

To find the force acting on a test particle due to the bar, we need to evaluate
\[ \mathbf{F}_{\text{bar}} = -\nabla \Phi_{\text{bar}} = \nabla \int_0^1 \frac{GM}{2b(1 - e^{-a/b})} H_{\text{conv}} \, dm. \]  
(24)
We take the gradient inside the integral to write
\[ \mathbf{F}_{\text{bar}} = \frac{GM}{2b(1 - e^{-a/b})} \int_0^1 \nabla H_{\text{conv}} \, dm \]  
(25)
and
\[ = \frac{GM}{2b(1 - e^{-a/b})} \int_0^1 \mathbf{F}_{\text{conv}} \, dm. \]
Defining a function \( I(m, x) \) by
\[ I(m, x) = \sqrt{\frac{2}{\pi \sigma^2}} \exp\left[ \frac{x}{b} \left( \frac{b m^2 (a + x) + \sigma^2}{2b^2 m^2 \sigma^2} \right) \right], \]  
(26)
we can calculate our integrand to be
\[ \mathbf{F}_{\text{conv}} = \left( \frac{E(m, y, z)}{\sqrt{1 - m^2 \epsilon^2}} \left[ I(m, x) - \frac{1}{b m^2} G_3(m, x) \right] \right) \]  
(27)
using the notation given in eq. (8).

We encounter the same problem in evaluating components of

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure3}
\caption{Logarithm of the deviation of our force from Poisson’s equation, \( \log([|\nabla \mathbf{F}_{\text{bar}}| - 4\pi G \rho_{\text{bar}}]|/|\mathbf{F}_{\text{bar}}|) \), for bar parameters \((a, b, e, \epsilon) = (5, 2, 1, 0)\). We use 20 abscissae in our numerical integration and a step size of \( 10^{-4} \) to evaluate derivatives with the central difference method. We note that it is everywhere very small, and so our forces are accurately computed.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure4}
\caption{Major, intermediate and minor axis profiles for an \( N \)-body simulation and our best-fit model (which has the bar parameters: \((r, e, a, b) = (0.1865, 0.3657 \text{kpc}, 2.8887 \text{kpc}, 0.3513 \text{kpc})\). Note the different scale for the topmost plot.}
\end{figure}
$\mathbf{F_{\text{conv}}}$ as we faced in integrating $H_{\text{conv}}$. Again, we use a Taylor approximation, splitting our integral into two regions. While the forces and potential are not easily computed by hand, and their analytical forms are not elegant, we can evaluate them speedily using a Gaussian quadrature, allowing for rapid investigation of the bar and its properties.

To check the accuracy of our quadrature, we ensure that the forces satisfy Poisson’s equation
\[
\nabla \cdot \mathbf{F}_{\text{bar}} = -4\pi G \rho_{\text{bar}}.
\]

Using a simple finite difference method verifies that this is the case, as shown in Fig. 3, and so we can be confident that our integration method is faithful. Reducing $b/\sigma$ increases the proportion of the force computed using the Taylor approximation, as is evident from eq. (23). This can also be a source of inaccuracy in our integration. Our numerical investigation suggests that this becomes a problem for models with $b/\sigma \lesssim 1/7$. However, this regime is unrealistic, as the “bar” density is very spread out in the $y$ and $z$ directions but has a steeply falling $x$ profile. This is not bar-like at all, and so failure in this unphysical regime is not a serious cause for concern.

### 2.5 Validation

To ensure that our model can represent realistic bars, we match to an $N$-body simulation which is believed to mimic approximately the properties of the Milky Way. It uses the initial condition generation $\text{mkgalaxy}$ from McMillan & Dehnen (2007). The simulation is described in Sanders, Smith & Evans (2019b) and contains three components: a disc, a bulge and a dark halo (which we do not consider, as our model is for only the barred stellar component). The disc contains 200,000 particles and the bulge 40,000. The Toomre $Q$ for the disc is chosen to make a bar form rapidly.

We fit our bar model to the endpoint of this simulation. We find a set of parameters $(\sigma, e, a, b)$ that matches the profile along the three principal axes simultaneously by least squares fitting. The simulation density and our best fit are shown in Fig. 4. It is clear that our model is capable of reproducing bars in $N$-body simulations well. We note that the model does underestimate the density of luminous matter at large radii. This is not surprising, as the simulation contains both bar and disc, whereas our model is only expected to be a reasonable match to the bar.

### 3 ORBITAL STRUCTURE

With our expressions for the forces of our new model, we now turn to inspecting the orbital structure for different parameter choices.

#### 3.1 The Effective Potential

We restrict our attention to orbits with $z = 0$. As the bar rotates, we transform to a frame of reference which corotates with the bar clockwise at a constant pattern speed of $\Omega_b$. This is consistent with standard convention for the Milky Way’s bar. The equations of motion in this frame are (e.g., Binney & Tremaine 2008):
\[
\dot{\mathbf{r}} = -\nabla \phi - 2(\Omega_b \times \mathbf{r}) - \Omega_b \times (\Omega_b \times \mathbf{r}),
\]
where $\Omega_b = -\Omega_0 \mathbf{e}_z$, with a negative sign ensuring clockwise rotation when in a right-handed coordinate frame. We can view this as a motion in an effective potential:
\[
\dot{\mathbf{r}} = -\nabla \Phi_{\text{eff}} - 2(\Omega_b \times \mathbf{r}),
\]

where $\Phi_{\text{eff}}$ is
\[
\Phi_{\text{eff}} = \Phi_{\text{bar}} - \frac{1}{2} \Omega_b^2 (x^2 + y^2).
\]

The quantity $H_J = \frac{1}{2} |\mathbf{r}|^2 + \Phi_{\text{eff}}$, known as the Jacobi integral or the energy in the rotating frame, is conserved.

Corotation occurs near the end of the bar, which in our case is $x = a$. We can use this to fix the pattern speed as (c.f. Williams & Evans 2017)
\[
\Omega_b = \sqrt{-\frac{F(a, 0)}{a}}.
\]

Each combination of parameters $(\sigma, e, a, b)$ gives a distinct bar, with considerable scope for different behaviour as these parameters vary. In what follows, we fix $\sigma = 0.1865$ and $e = 0.3657$ (our best fit values in Fig. 4) unless otherwise specified, and treat $a$ and $b$ as independent parameters. A useful discriminant is the profile of
Figure 7. Poincaré surface of section for a type 1 potential (single potential well), with \(a = 1\) and \(b = 1\) at energy \(H_J = -2.5\). Two distinct periodic orbits can be seen – they are labelled as A (which sires the \(x_3\) family) and B (which sires the \(x_1\) family). These labels correspond to the orbits shown in Figure 8. The quasi-periodic orbits librating around the stable orbits map out the invariant curves around each fixed point. All displayed orbits have energy conservation better than \(|\Delta E/E| = 10^{-8.5}\).

Figure 8. Left: Characteristic diagram showing where periodic orbits with energy \(H_J\) intersect the \(x = 0\) plane. Each periodic orbit is labelled with a capital letter. Orbits with \(y > 0\) are prograde (rotating with the rotation of the bar), whereas orbits with \(y < 0\) are retrograde. Right: The periodic orbits present in our bar for \(a = 1\), \(b = 1\), \(\sigma = 0.2\), \(\epsilon = 0.4\). Orbits B, F, J are the \(x_1\) backbone of the bar, with other orbits bifurcating away from this.

The effective potential shape, and hence the number of stable and unstable Lagrange points, on the major axis. There are six different types of possible profile, as shown in Fig. 5 and categorized below:

0 - Here the pattern speed is too large for any bound orbits to exist,
1 - The origin is the only stable fixed point,
2 - The origin is unstable, but two stable fixed points appear away from the origin,
2.i - The effective potential is small enough at the origin that orbits can encircle both stable fixed points,
2.ii - No orbits can encircle both fixed points, and orbits originating at the origin are unbound,
3 - The origin and two other fixed points are all stable,
3.i - Orbits can exist encircling all three fixed points, as the maximum of the effective potential is outside the three fixed points,
3.ii - The three stable fixed points are isolated - orbits with enough energy to travel from the origin to the other fixed points are unbound.

In Fig. 6, we label the regions in \((b/\sigma, a/\sigma)\) parameter space in which each type of effective potential can be found. We note also the effect of varying \(\epsilon\) on our configuration space. Increasing \(\epsilon\) corresponds to making the bar’s mass more concentrated in the \((x, y)\) plane. This causes Region 1 grow at the expense of the other regions.

Types 1 and 2.i are seen in many types of bar, such as the numerical Cazes bars (Barnes & Tohline 2001), and the prolate \(n = 2\)
Poincaré sections and orbital structure

A valuable tool, which has been used extensively to visualise the orbital structure in bar potentials, is the Poincaré surface of section (e.g., Henon & Heiles 1964; Lichtenberg & Lieberman 1992; Contopoulos 2002). This means its Jacobian has determinant equal to unity. If a periodic orbit is stable, then its two eigenvalues are complex and lie on the unit circle (Hénon 1965). This means a nearby orbit librates around the periodic orbit. For an unstable periodic orbit, the eigenvalues are real and one lies outside the unit circle. Those orbits starting nearby will therefore quickly diverge away. We are particularly interested in the stable periodic orbits – they will sire orbital families, which librate around them.

To integrate the orbits, we use two methods: a fourth order Runge-Kutta integration method with a fixed timestep, or, where more accuracy is required, the adaptive time-stepping routine LSODA in scipy. We integrate over 200 time units, corresponding to approximately 1 Gyr for the Milky Way-like bar in Section 4. We test the accuracy of our orbits by ensuring that the energy $H_f$ is conserved for a random orbit sample – this is true to one part in $10^{-16}$ in most cases (the energy conservation deteriorates for orbits that are marginally bound and move out to very large radius). For situations that require an accurate intersection of a trajectory with a surface, such as the Poincaré sections later in this Section, we use a secondary integration to refine our solution in a given interval. An example is shown in Fig. 7 where all displayed orbits have energy conservation better than $|AE/E| = 10^{-8.5}$. To calculate the force, we typically use 20 abscissae in our Gauss-Legendre quadrature. Increasing the number of abscissae from 20 to 200 reduces the error in conservation of energy by roughly a factor of 40. Further improvements are possible by increasing the order of the Taylor expansion employed and in turn increasing the factor used in $\text{Osc}(\sigma, b)$ slightly.

### 3.2.1 Type 1 potentials

We first investigate a short, flattish bar with $a = 1$ and $b = 1$ and compare this to the bars investigated by Williams & Evans (2017). In Fig. 7, we see two different periodic orbits at $y = -0.11$ and 0.09. The invariant curves enclosing the fixed points on the surface of section $S$ to the next point at which its orbit will intersect the surface: $P(x_0) = x_1$. It is Hamiltonian and area preserving (e.g., Lichtenberg & Lieberman 1992; Contopoulos 2002). This means its Jacobian has determinant equal to unity. If a periodic orbit is stable, then its two eigenvalues are complex and lie on the unit circle (Hénon 1965). This means a nearby orbit librates around the periodic orbit. For an unstable periodic orbit, the eigenvalues are real and one lies outside the unit circle. Those orbits starting nearby will therefore quickly diverge away. We are particularly interested in the stable periodic orbits – they will sire orbital families, which librate around them.

To integrate the orbits, we use two methods: a fourth order Runge-Kutta integration method with a fixed timestep, or, where more accuracy is required, the adaptive time-stepping routine LSODA in scipy. We integrate over 200 time units, corresponding to approximately 1 Gyr for the Milky Way-like bar in Section 4. We test the accuracy of our orbits by ensuring that the energy $H_f$ is conserved for a random orbit sample – this is true to one part in $10^{-16}$ in most cases (the energy conservation deteriorates for orbits that are marginally bound and move out to very large radius). For situations that require an accurate intersection of a trajectory with a surface, such as the Poincaré sections later in this Section, we use a secondary integration to refine our solution in a given interval. An example is shown in Fig. 7 where all displayed orbits have energy conservation better than $|AE/E| = 10^{-8.5}$. To calculate the force, we typically use 20 abscissae in our Gauss-Legendre quadrature. Increasing the number of abscissae from 20 to 200 reduces the error in conservation of energy by roughly a factor of 40. Further improvements are possible by increasing the order of the Taylor expansion employed and in turn increasing the factor used in $\text{Osc}(\sigma, b)$ slightly.

### 3.2.1 Type 1 potentials

We first investigate a short, flattish bar with $a = 1$ and $b = 1$ and compare this to the bars investigated by Williams & Evans (2017). In Fig. 7, we see two different periodic orbits at $y = -0.11$ and 0.09. The invariant curves enclosing the fixed points on the surface of section $S$ to the next point at which its orbit will intersect the surface: $P(x_0) = x_1$. It is Hamiltonian and area preserving (e.g., Lichtenberg & Lieberman 1992; Contopoulos 2002). This means its Jacobian has determinant equal to unity. If a periodic orbit is stable, then its two eigenvalues are complex and lie on the unit circle (Hénon 1965). This means a nearby orbit librates around the periodic orbit. For an unstable periodic orbit, the eigenvalues are real and one lies outside the unit circle. Those orbits starting nearby will therefore quickly diverge away. We are particularly interested in the stable periodic orbits – they will sire orbital families, which librate around them.

To integrate the orbits, we use two methods: a fourth order Runge-Kutta integration method with a fixed timestep, or, where more accuracy is required, the adaptive time-stepping routine LSODA in scipy. We integrate over 200 time units, corresponding to approximately 1 Gyr for the Milky Way-like bar in Section 4. We test the accuracy of our orbits by ensuring that the energy $H_f$ is conserved for a random orbit sample – this is true to one part in $10^{-16}$ in most cases (the energy conservation deteriorates for orbits that are marginally bound and move out to very large radius). For situations that require an accurate intersection of a trajectory with a surface, such as the Poincaré sections later in this Section, we use a secondary integration to refine our solution in a given interval. An example is shown in Fig. 7 where all displayed orbits have energy conservation better than $|AE/E| = 10^{-8.5}$. To calculate the force, we typically use 20 abscissae in our Gauss-Legendre quadrature. Increasing the number of abscissae from 20 to 200 reduces the error in conservation of energy by roughly a factor of 40. Further improvements are possible by increasing the order of the Taylor expansion employed and in turn increasing the factor used in $\text{Osc}(\sigma, b)$ slightly.
Figure 11. Poincaré surface of section for our exponential bar, with $a = 2.89$ and $b = 0.35$ at energy $H_J = -1.2$. 4 periodic orbits can be seen, at $y = -0.24, -0.15, -0.03$ and $0.6$, and are labelled, along with the quasi-periodic orbits that they sire.

Figure 12. Left: Periodic orbit structure for the exponential bar. Orbits are again labelled with capital letter. Orbits with $y > 0$ are prograde, whereas orbits with $y < 0$ are retrograde. The zero velocity surface is given by $H_J = \phi_{\text{eff}}$, such that orbits on this surface have zero kinetic energy. Right: The periodic orbits present in our bar for $a = 2.89, b = 0.35$.

Poincaré section are the quasi-periodic orbits that librate around the periodic orbits. A single surface of section is only a snapshot of the phase space at a fixed energy. To build a full picture, we search for periodic orbits as fixed points of the Poincaré map $P$, and plot their intersection $y$ coordinate against the energy at which they are present. This is known as a characteristic diagram (e.g., Contopoulos 2002; Kaufmann & Patsis 2005; Williams & Evans 2017).

Fig. 8 shows the characteristic diagram for the bar with $a = 1$ and $b = 1$. We can see a variety of different orbital families emerging as the periodic families bifurcate and connect onto either the zero-velocity surface or another family (some tracks are artificially truncated in the figure due to the resolution of our periodic orbit search). Each family is labelled with a letter – from A through to J in this instance. Our phase space is dominated by the $x_1$ (elongated prograde) and $x_4$ (retrograde) orbits, using the nomenclature of Contopoulos (2002). The $x_1$ sequence is $B \to F \to J$, with successive orbits bifurcating away from the main sequence. The $x_4$ orbit is $A$. It sires orbits that are retrograde and so unlikely to be highly populated in a real bar, as significant counter-streaming is not generally observed (e.g. Sellwood & Wilkinson 1993; Kaufmann & Patsis 2005). Although the majority of the phase space in Fig. 7 is indeed taken up by the $x_1$ family, this does not by itself ensure that such orbits are populated. For example, in Figure 21 of Patsis, Athanassoula & Quillen (1997), the area occupied by $x_4$
orbits is much greater than $x_1$, even though any actual bar must be
-dominated by the prograde orbits.

The phenomenon of $x_1$ orbits producing orbits through suc-
cessive bifurcations has been witnessed smaller for type 2 and 3 po-
tentials (e.g., Contopoulos & Grosbøl 1989; Contopoulos 2002).
This is a generic feature of rotating Hamiltonians, driven by the
existence of resonances. Another very common feature visible in
Fig. 7 is the co-existence of the bifurcating $x_1$ sequence with a sta-
bile $x_4$ orbit (A). This too has been seen before, first by Pfenniger
(1984a) and subsequently by others (Contopoulos 2002; Kaufmann &
Patsis 2005; Contopoulos & Harsoula 2013).

3.2.2 Type 2 and 3 potentials

We now focus on type 2.i and 3.1 potentials. Of course, types 2.ii
and 3.ii are unable to be self-consistent or resemble real bars, as
bound orbits do not exist. We take as our type 2.i potential $(a, b) =
(0.9, 5)$, and as our type 3.i potential $(a, b) = (1.25, 3)$.

As all orbits intersect the line $y = 0$, we construct our Poincaré
section by looking at intersection with the $(x, \dot{x})$ plane. The range
of energies for bound orbits is much smaller for type 2 and 3 po-
tentials. The orbital structure shown in Figs. 9 and 10 is generic:
$x_4$ orbits around each Lagrange point, each taking up a roughly
equal portion of phase space, with no other major orbital families
(Patis, Kalapotharakos & Grosbøl (2010) dub these off-centre $x_4$
orbits, $x_4$-like or $x_4'$ and $x_4''$). This behaviour is similar to the “double
well” potential investigated in Williams & Evans (2017).

3.3 Orbital structure in a Milky Way-like bar

We now investigate the bar that closely resembles the simulation of
Sanders, Smith & Evans (2019b), which itself is a reasonable match
to the Milky Way’s bar. We take the bar parameters $(a, b) =
(2.89, 0.35), which corresponds to a long, bar with an exponential
profile. We first show a representative Poincaré section in Fig. 11.
We see that there are portions of phase space in which the invariant
curves have broken up and chaotic orbits occur and no discernible
invariant curves can be seen.

The characteristic diagram is shown in Figure 12. The orbital
structure here is markedly different. We no longer have a prograde
$x_1$ orbit and a retrograde $x_4$ orbit present at all energies. Instead, the
dominant orbital family is a propeller orbit, B (Kaufmann & Patsis
2005; Williams & Evans 2017). These are so called because they are
long and thin, and can appear as an elongated figure 8 shape,
similar to a propeller. This orbit only undergoes one bifurcation,
giving two offshoots, and persists at all energies. This propeller
family coexists with an $x_4$ orbit A, with an $x_1$ orbit only appearing
at high energies (C). The dominance of the propeller orbits in this
analogue of the Milky Way bar suggests that this orbital family may
be responsible for its thinness and morphology.

The orbital structure is in fact strikingly similar to the “Model
6” bar in Kaufmann & Patsis (2005) and Kaufmann & Contopoulos
(1996). This bar was constructed as a best fit to NGC 1073, an
SB(rs)c galaxy possessing an exponential bar with half-length 2.95
kpc. Their model consists of a $n = 2$ Ferrers ellipsoid, exponential
disc and a spiral perturbation. This suggests that propeller orbits are
a generic feature of many galactic bars, not just the Milky Way’s.

3.4 Observational evidence

The phenomenon of $x_1$ orbits producing orbits through suc-
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velocity corrected for the solar motion: \( V_{\text{solar}} \) and the bar viewing angle as \( \hat{v} \). The radial and transverse velocities at this point. In the heliocentric frame, the positions and velocities are

$$
x = Rx' + \begin{pmatrix} 0 \\ \ell \end{pmatrix}, \quad v = \Omega_b R \hat{v} - v_{\odot}.
$$

where \( R \) is the distance of the star from the centre of the bar, \( \hat{v} \) is the unit vector in the direction of increasing \( \theta \), and \( v_{\odot} \) is the motion of the Sun in relation to the Galactic centre. The matrix \( R \) is a rotation counterclockwise by \( 90^\circ - \alpha \). The radial and transverse velocities are then

$$
v_{\text{rad}} = -v_{y} \sin \ell + v_{x} \cos \ell,$n_{\text{trans}} = -v_{x} \cos \ell - v_{y} \sin \ell.
$$

We can extract the proper motion \( \mu_{l} \), measured in milliarcseconds per year, from the transverse velocity using \( \mu_{l} = v_{\text{trans}}/(4.74\alpha) \), where \( s \) is the heliocentric distance. We also work with the radial velocity corrected for the solar motion: \( V_{\text{gsr}} \).

We take the distance to the Galactic centre as \( d = 8.2 \) kpc and the bar viewing angle as \( \alpha = 27^\circ \) (Bland-Hawthorn & Gerhard 2016). For the velocity of the Sun with respect to the Galactic centre, we use the fact that the longitudinal proper motion of Sgr A* is \(-6.379 \) mas yr\(^{-1}\) (Reid & Brunthaler 2004), which, assuming Sgr A* is at rest with respect to the Galaxy, gives the Sun's tangential velocity relative to the Galactic centre as \( 4.74 \times -6.379 \times 8.2 = -248 \) km s\(^{-1}\). The radial velocity of the Sun towards the Galactic centre is \( 11 \) km s\(^{-1}\) (Schönrich et al. 2010). We therefore have

$$
v_{\odot} = \begin{pmatrix} 248 \\ -11 \end{pmatrix} \text{ km s}^{-1}.
$$

A depiction of the geometry of our observation is shown in Fig. 13. We note that the orbit shown is not the true orbit of any one star as viewed from the Sun, as the bar itself would rotate many times as the star orbits in the bar potential. Instead, the orbit plotted represents an entire family of stars, all on the same orbit, crossing our line of sight simultaneously.

### 4.3 Searching for peaks

To search for high velocity peaks, we look for orbits that have a large proportion of their radial velocities above a certain threshold. To do this, we integrate orbits with initial conditions \( x_{0} = y_{0} = 0, \)

| Main stars | High velocity stars |
|------------|---------------------|
| \((l, b) = (4^\circ, 0^\circ)\) | \((l, b) = (6^\circ, 0^\circ)\) |
| \(\langle V_{\text{gsr}} \rangle\) | \(\langle V_{\text{gsr}} \rangle\) |
| \(\sigma_{v}\) | \(\sigma_{v}\) |
| 39.4 | 203.6 |
| 100.9 | 234.3 |

Table 1. Selection of observations of high velocity peaks, listing mean and variance of the double Gaussian fit, in units of kms\(^{-1}\).

### 4.2 Galactic geometry and coordinate transformations

We aim to find candidate orbital families responsible for these high velocity peaks, using our exponential bar with parameters \((\sigma, e, a, b) = (0.1865, 0.3657, 2.8887, 0.3513)\), which is the best fit to the Milky Way simulation from Sanders, Smith & Evans (2019b). We integrate orbits in the frame of reference corotating with the bar using a pattern speed \(\Omega_p = 40 \) km s\(^{-1}\)kpc\(^{-1}\) (Sanders, Smith & Evans 2019b; Bovy et al 2019), and perform a coordinate transformation so that we have the orbital position and velocity from the point of view of the Sun. We then find each time the orbit crosses our line of sight, and extract the radial and tangential velocities at this point. In the heliocentric frame, the positions and velocities are

$$
\begin{align*}
x & = Rx' + \begin{pmatrix} 0 \\ \ell \end{pmatrix}, \\
v & = \Omega \hat{v} - v_{\odot}.
\end{align*}
$$

where \( R \) is the distance from the point of view of the Sun, \( v_{\odot} \) is the velocity of the Sun relative to the Galactic centre. To search for high velocity peaks, we look for orbits that have a significant variation of the double Gaussian fit, in units of kms\(^{-1}\).

We take all stars in three APOGEE DR14 fields centred on \((l, b) = (4, 0)\) deg, available from Gaia DR2 (Gaia Collaboration et al. 2016, 2018). We cross-match APOGEE DR14 fields centred on \(b = 0\) and \(\ell = (8, 6, 4)\) deg (Abolfathi, et al. 2018). Following Zhou et al. (2017), we remove contaminating foreground stars (in particular disc red clump) by using only \( T_{\text{eff}} < 4000 \) K. We cross-match to the Gaia DR2 catalogue and the VIREC catalogue (using a cross-match radius of 1 arcsec), and combine the proper motions from the two sources using inverse variance weighting. The resulting (solar-corrected) radial velocity against longitudinal proper motion diagrams are shown in Fig. 15. We see all three fields have significant high velocity peaks centred around \( \approx 200 \) km s\(^{-1}\). The high velocity peak of our candidate orbit produces a peak which coincides well with the peak in the data.
Figure 15. A sample propeller orbit showing a characteristic high velocity peak as seen in APOGEE data. This orbit has initial conditions $y = 0.544$ kpc and $\dot{y} = 235$ km s$^{-1}$ ($H_J = \frac{1}{1.207} \sqrt{GM/\dot{a}}$). The left panel shows the orbit coloured by the radial velocity (solid grey line is the $\ell = 4$ deg line and the grey shading is an equi-density contour). The right three panels are kernel density estimates of the radial velocity against proper motion space for three APOGEE fields at $\ell = 8$, 6 and 4 deg (the number of stars used is given in each panel). Each field shows a high velocity peak. In the $\ell = 4$ deg panel we show points from the sample orbit which fall within 0.2 deg of the line-of-sight, finding this orbit has a high velocity peak that matches in both radial velocity and proper motion.

5 CONCLUSIONS

Analytically tractable bar models are few and far between. This is especially the case for bar models with realistic density profile. We have introduced a novel model for exponential bars, extending the original algorithm of Long & Murali (1992). Our bar model has Gaussian density profiles along the minor and intermediate axes. It has a roughly exponentially falling profile along the major axis, as indicated by the observational date (Elmegreen & Elmegreen 1985; Gadotti, et al. 2007). It adds extra versatility to the library of analytic bar models in the literature, all of which have density fall-offs that behave roughly like power-laws (e.g., the widely-used Ferrers bars have $\rho \propto (1 - R^2/a^2)^b$ along the major axis, so that when $n$ is small the density is more homogeneous than shown by real bars).

Our bar has four key parameters: $\sigma$ is the variance of the underlying Gaussian density and sets the length scale; $a/\sigma$ is the half-length of the bar; $b/\sigma$ its “flatness” along the major axis; and $e$ relates the variances of the intermediate- and minor-axis profiles. We have found analytic formulae for the density, potential and forces arising from this bar, and noted the numerical difficulties that arise. These numerical difficulties do not prevent successful computation of the bar quantities – our model satisfies $\nabla \cdot \mathbf{F} = -4\pi G \rho$ to a high degree of accuracy. Furthermore, our model resembles well a Milky Way-like bar obtained from the N-body simulation of Sanders, et al. (2019a). We have therefore succeeded in constructing a bar model that is realistic and customisable, with explicit analytical formulae for the potential and forces.

The bar has a large parameter space ($e, a/\sigma, b/\sigma$) to be explored. We note that our model can give rise to a “triple well” effective potential, a feature not seen in other bars. Using Poincaré sections and by categorising periodic orbits, we have investigated in detail the orbital structure for two different bars. Analysing only these two bars has revealed a variety of novel orbital structure. We have displayed the first bar in which a bifurcating $x_1$ sequence coexists with a stable $x_1$ orbit at all energies. In another bar, the best fit to a simulation of the Milky Way, propeller orbits such as those seen in Kaufmann & Patsis (2005) form the bar backbone. Our analysis suggests that bars with propeller orbits playing a central role are much more common than previously believed.

We have used this model to investigate the observation of Nidever et al. (2012), namely that radial velocity histograms for stars in the Milky Way’s bar show distinctive high velocity peaks. We have identified the propeller orbit family as being the likely candidate for such peaks, lending further importance to their role in the orbital structure of exponential bars. As a test, we have, for the first time, inspected the high velocity peaks distribution in proper motion space, finding a good match with our proposed family. We note that this investigation shows both the value of this model in producing a realistic facsimile of the Milky Way itself, and the ease with which investigations can be conducted using this bar.

A natural extension of the work in this paper is to investigate self-consistency; that is, whether superpositions of different orbital families can reproduce the density distribution. The Schwarzschild (1979) method was used to show self-consistency of several bar models by Pfenniger (1984b) and Hänker, et al. (2000). The key advantage of self-consistency is that it would tell us what proportion of the bar’s stellar mass occupies each different orbital family, and explicitly demonstrate the importance of propeller orbits as the major family building the bar. Another fruitful avenue for future exploration is to look for analytic bar models whose vertical profile exhibits the peanut-structure characteristics of buckled disks (e.g., Raha, et al. 1991; Lütticke et al. 2000) and evident in the Milky Way’s deprojected density and stellar populations (Fragkoudi, et al. 2018; Sanders, et al. 2019a).

Furthermore, while we originally aimed to model those bars with exponentially falling luminosity profiles along the major axis, we have seen that by increasing the parameter $b$, we can obtain very flat luminosity profiles, similar to the flat bar investigated in Williams & Evans (2017). In fact, the short flat bar we investigated (with $(a, b) = (1, 1)$) gave similar orbital structure to the strong bar in Williams & Evans (2017). Our bar has the advantage that it represents only luminous matter in the bar, and not the disc or halo, unlike the bar in Williams & Evans (2017), which has a logarithmic
potential. This means it can be used in multi-component galaxy models more readily, giving it an extra level of versatility.

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REFERENCES

Abolfathi B., et al., 2018, ApJS, 235, 42
Abramowicz, M., & Stein, I. A., 1972, Handbook of Mathematical Functions, New York: Dover, 1972.
Athanassoula E., 1992, MNRAS, 259, 328
Athanassoula E., Romero-Gómez M., Masdemont J. J., 2009, MNRAS, 394, 67
Aumer M., Schönrich R., 2015, MNRAS, 454, 3166
Barnes, E. I., & Tohline, J. E. 2001, ApJ, 551, 80
Binney, J., & Tremaine, S., 2008, Galactic Dynamics: Second Edition, Princeton University Press, Princeton, NJ USA, 2008
Blackman, C. P., 1983, MNRAS, 202, 379
Bland-Hawthorn, J., & Gerhard, O. 2015, ARA&A, 54, 529
Bovy J., Leung H. W., Hunt J. A. S., Mackereth J. T., Garcia-Hernandez D. A., Roman-Lopes A., 2019, arXiv e-prints, arXiv:1905.11404
Cappellari, M. 2008, MNRAS, 390, 71
Cazes, J. E., & Tohline, J. E. 2000, ApJ, 532, 1051
Chandrasekhar, S. 1987, Ellipsoidal Figures of Equilibrium, New York: Dover, 1987.
Contopoulos G., Grosbol P., 1989, A&A & RV, 1, 261
Contopoulos, G. 2002, Order and chaos in dynamical astronomy, Springer Verlag, New York
Contopoulos G., Harsoula M., 2013, MNRAS, 436, 1201
Debattista V. P., Ness M., Eap S. W. F., Cole D. R., 2015, ApJ, 812, L16
Dwek, E., Arendt, R. G., Hauser, M. G., et al. 1995, ApJ, 445, 716
Elmegreen B. G., & Elmegreen, D. M. 1985, ApJ, 288, 438
Ferrers, N. M., 1877, Quart. J. Pure and Applied Math., 1, 1
Gadotti D. A., Athanassoula E., Carrasco L., Bosma A., de Souza R. E., Recillas E., 2007, MNRAS, 381, 943
Fragkoudi F., Di Matteo P., Haywood M., Schultheis M., Khoperskov S., Gómez A., Combes F., 2018, AA, 616, A180
Gaia Collaboration, Prusti, T., de Bruijne, J. H. J., et al. 2016, A&A, 595, A1
Gaia Collaboration, Brown, A. G. A., Vallenari, A., et al. 2018, A&A, 616, A1
Häfner R., Evans N. W., Dehnen W., Binney J., 2000, MNRAS, 314, 433
Hénon M., 1965, AnAp, 28, 499
Henon, M., & Heiles, C. 1964, AJ, 69, 73
Hohl, F. 1971, ApJ, 168, 343
Kaufmann, D. E., & Contopoulos, G. 1996, A&A, 309, 381
Kaufmann, D. E., & Patsis, P. A. 2005, ApJ, 624, 693
Li Z.-Y., Shen J., Rich R. M., Kunder A., Mao S., 2014, ApJL, 785, L17
Lichtenberg, A., & Lieberman, M. 1992, Regular and Chaotic Dynamics, Springer Verlag, New York
Long, K., & Muraki, C. 1992, ApJ, 397, 44
Lütticke, R., Deitmar, R.-J., & Pohlen, M. 2000, A&A, 145, 405
McMillan, P. J., & Dehnen, W. 2007, MNRAS, 378, 541
Molloy M., Smith M. C., Evans N. W., Shen J., 2015, ApJ, 812, 146
Nidever, D. L., Zasowski, G., Majewski, S. R., et al. 2012, ApJ, 755, L25
Paczynski, B., et al. 1994, ApJ, 435, L113
Patsis P. A., Athanassoula E., 2019, MNRAS, 490, 2740
Patsis P. A., Athanassoula E., Quillen A. C., 1997, ApJ, 483, 731
Patsis P. A., Kalapotharakos C., Grosbol P., 2010, MNRAS, 408, 22
Pfenniger D. 1984a, A&A, 134, 373
Pfenniger D. 1984b, A&A, 141, 171
Raha N., Sellwood J. A., James R. A., Kahn F. D., 1991, Natur, 352, 411
Reid, M. J., & Brunthaler, A. 2004, ApJ, 616, 872
Sanders J. L., Smith L., Lucas N. W., Lucas P., 2019a, MNRAS, 487, 5188
Sanders J. L., Smith L., Evans N. W., 2019b, MNRAS, 488, 4552
Schönrich, R., Binney, J., & Dehnen, W. 2010, MNRAS, 403, 1829
Schönrich R., Aumer M., Sale S. E., 2015, ApJ, 812, L21
Schwarzschild M., 1979, ApJ, 232, 236
Sellwood J. A., & Wilkinson, A. 1993, Reports on Progress in Physics, 56, 173
Skokos, C., Patsis, P. A., & Athanassoula, E. 2002, MNRAS, 333, 847
Smith, L. C., Lucas P. W., Kurtev, R., et al. 2018, MNRAS, 474, 1826
Sparks, L. S., & Sellwood, J. A. 1987, MNRAS, 225, 653
Toomre, A. 1964, ApJ, 139, 1217
Toomre, A. 1981, Structure and Evolution of Normal Galaxies, 111
Wegg, C., Gerhard, O., & Portail, M. 2015, MNRAS, 450, 4050
Weinberg, M. D. 1992, ApJ, 384, 81
Williams, A. A., & Evans, N. W. 2017, MNRAS, 469, 4414
Zhou, Y., Shen, J., Liu, C., et al. 2017, ApJ, 847, 74

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