AN OPERATOR MODEL IN THE ANNULUS

GLENIER BELLO AND DMITRY YAKUBOVICH

Abstract. For an invertible linear operator $T$ on a Hilbert space $H$, put
\[ \alpha(T^*, T) := -T^* T^2 + (1 + r^2)T^* T - r^2 I, \]
where $I$ stands for the identity operator on $H$ and $r \in (0, 1)$; this expression comes from applying Agler’s hereditary functional calculus to the polynomial $\alpha(t) = (1 - t)(t - r^2)$. We give a concrete unitarily equivalent functional model for operators satisfying $\alpha(T^*, T) \geq 0$. In particular, we prove that the closed annulus $r \leq |z| \leq 1$ is a complete $K$-spectral set for $T$. We explain the relation of the model with the Sz.-Nagy–Foias one and with the observability gramian and discuss the relationship of this class with other operator classes related to the annulus.

1. Introduction

Given a bounded subset $\Omega$ of the complex plane, and a Hilbert space operator $T$ with spectrum in $\Omega$, the closure $\Omega^{cl}$ of $\Omega$ is said to be a $K$-spectral set for $T$, for some constant $K \geq 1$, if
\[ \|f(T)\| \leq K \max_{z \in \Omega^{cl}} \|f(z)\| \]
for any rational function $f$ with poles outside $\Omega^{cl}$. The notion of spectral sets (i.e., $K$-spectral for $K = 1$) was introduced by von Neumann in [18]. If the same inequality holds for any rational $n \times n$ matrix-valued function $f$ with poles outside $\Omega^{cl}$, for any size $n$, then $\Omega^{cl}$ is said to be completely $K$-spectral for $T$. Arveson [3] proved that $\Omega^{cl}$ is a complete $K$-spectral set for $T$ for some $K \geq 1$ if and only if $T$ is similar to an operator which has a normal dilation with spectrum contained in the boundary of $\Omega$. The case when $\Omega$ is the unit disc $D$ is deeply related with the Sz.-Nagy–Foias theory of Hilbert space contractions. An excellent reference for this theory is the book [15]. In the landmark paper [2], Agler studied the case when $\Omega$ is an annulus
\[ \mathbb{A} := \{ r < |z| < 1 \}, \]
for some $0 < r < 1$. He proved that $\mathbb{A}^{cl}$ is 1-spectral for $T$ if and only if $\mathbb{A}^{cl}$ is completely 1-spectral for $T$. Moreover, it is well-known that operators for which $\mathbb{A}^{cl}$ is a 1-spectral set, that we will denote by $\text{Sp} \mathbb{A}$, admit the following model:

Theorem A (cf. [4, Theorem 2.1]). The set $\mathbb{A}^{cl}$ is 1-spectral for $T$ if and only if there exist unitary operators $U_1, U_2$ and a weight $\omega$ on a Hilbert space...
Let $E$ be such that $T$ is unitarily equivalent to a compression of the operator

$$(M_z \text{ on } H^2(\mathbb{A}, E, \omega)) \oplus U_1 \oplus rU_2$$

to its coinvariant subspace.

Recall that, given a Hilbert space $E$, the vector valued Hardy space $H^2(\mathbb{A}, E)$ consists of all analytic functions $f : \mathbb{A} \to E$ such that

$$\sup_{r < \rho < 1} \int_{\mathbb{T}} \|f(\rho z)\|^2 |dz| < \infty.$$ 

There is no unique canonical way of choosing a Hilbert space on this space. For any bounded, positive and invertible operator $\omega$ on $E$, $H^2(\mathbb{A}, E, \omega)$ is defined as the vector space $H^2(\mathbb{A}, E)$ equipped with the Hilbert space norm

$$\|f\|_{H^2(\mathbb{A}, E, \omega)}^2 := \int_{\mathbb{T}} \|f(z)\|^2_E |dz| + \int_{\mathbb{T}} \|\omega f(z)\|^2_E |dz| \quad (1.1)$$

All these norms are equivalent. The operator $M_z$ acts on $H^2(\mathbb{A}, E, \omega)$ by $M_z f(z) := z f(z)$. These operators are purely subnormal and mutually similar, but, in general, not unitarily equivalent to each other.

For a general finitely connected domain $\Omega$, Abrahamse and Douglas [1] showed the importance of what they called bundle shifts; they form a subclass of pure subnormal operators with spectrum in $\Omega^c$. In the case of an annulus, the set of bundle shifts coincides with the family of operators $M_z$ on spaces $H^2(\mathbb{A}, E, \omega)$.

Consider the polynomial $\alpha(t) := (1 - t)(t - r^2)$. For any Hilbert space operator $T$, we define

$$\alpha(T^*, T) := -T^* T^2 + (1 + r^2) T^* T - r^2 I, \quad (1.2)$$

where $I$ stands for the identity operator on the space where $T$ acts. Here we put all expressions $T^*$ on the left, because we are applying the so-called Agler’s hereditary functional calculus, whose relevance in constructing operator models is well-known. We denote by $C_\alpha$ the class of all invertible bounded linear operators $T$ such that the operator $\alpha(T^*, T)$ is non-negative. Let $C_{1, r}$ stand for the family of all bounded invertible operators $T$ such that $\|T\| \leq 1$ and $\|T^{-1}\| \leq 1/r$. We have the following strict inclusions of classes of operators.

**Theorem 1.1.** $\text{Sp } \mathbb{A} \subset C_\alpha \subset C_{1, r}$.

In this paper we focus on the study of the class $C_\alpha$. It is natural to try an approach in the spirit of Sz.-Nagy–Foias theory. Suppose we are given an operator $T$ in $C_\alpha$. Denote by $D$ the non-negative square root of $\alpha(T^*, T)$ and by $D$ the closure of the range of $D$; these will be called the defect operator and the defect space of $T$, respectively. We will construct an explicit model where $D$ plays the role of the abstract defect operator. More precisely, the lifting of the model will involve the output transform

$$O_{T, D} : H \to H^2_r(B, D), \quad O_{T, D} x(z) = D(z - T)^{-1} x, \quad x \in H, z \in B.$$
Here $\mathcal{B}$ denotes the complementary of $\mathcal{A}$. For any Hilbert space $E$, the space $H^2_r(\mathcal{B}, E)$ consists of all functions $f : \mathcal{B} \to E$ representable as
\[
\sum_{n=0}^{\infty} f_n z^n \quad \text{for } |z| < r, \quad \sum_{n=-\infty}^{-1} f_n z^n \quad \text{for } |z| > 1,
\]
where $\{f_n\}_{n \in \mathbb{Z}}$ is a sequence in $E$, with finite norm
\[
\|f\|_{H^2_r(\mathcal{B}, E)}^2 := \frac{1}{1-r^2} \sum_{n=0}^{\infty} r^{2n} \|f_n\|^2 + \frac{1}{1-r^2} \sum_{n=-\infty}^{-1} \|f_n\|^2.
\]
On these spaces the operator $M_z$ acting by
\[
M_z f(z) = z f(z) - (zf(z))|_{z=\infty}
\]
is well-defined. If we identify a function $f$ in $H^2_r(\mathcal{B}, E)$ with the two-sided vector sequence $(\ldots, f_{-3}, f_{-2}, f_{-1}, f_0, f_1, f_2, \ldots)$, then $M_z$ takes the form of the bilateral shift. Our model theorem for operators in $\mathcal{C}_\alpha$ is the following.

**Theorem 1.2 (Model theorem).** Suppose that $T$ is invertible. The following statements are equivalent.

(i) $\alpha(T^*, T) \geq 0$.

(ii) $T$ is unitarily equivalent to a part of an operator of the form
\[
(M_z^* \text{ on } H^2_r(\mathcal{B}, E)) \oplus S,
\]
where $E$ is a Hilbert space and $S$ is a subnormal operator whose minimal unitary extension has spectrum contained in the union of the circles $\{ |z| = r \}$ and $\{ |z| = 1 \}$.

If (i) holds, one can take $E = \mathcal{D}$ in (ii).

Using a certain duality between models, in Theorem 3.2 we obtain a model for $T^*$ with the structure of Theorem A. The explicit model permits to obtain a concrete value of $K$ such that $\mathcal{A}^{cl}$ is completely $K$-spectral for operators in $\mathcal{C}_\alpha$ (see Theorem 4.2).

Let us give a few comments on the literature concerning the annulus. Apart from Agler’s paper cited above, another classic work is that by Sarason [13]. He obtained for $H^p(\mathcal{A})$ analogous results to the well-established theory for $H^p(\mathbb{D})$. The transition is not always smooth: for example, Blaschke products cannot be implemented in the annulus. Sarason overcame this obstacle by introducing what he called *modulus automorphic functions*. In the second part of the paper, he studied invariant and doubly invariant subspaces for the multiplication operator on $L^2(\partial \mathcal{A})$. Recent papers dealing with the annulus are for example [10, 12, 11] (see also the references therein). In [10], McCullough and Sultanic proved a kind of commutant lifting theorem for the annulus. Earlier, in a different context, a result on commutant lifting for finitely connected domains has been obtained by Ball in [4]. In [12], Pietrzycki obtained an analytic model on an annulus for left-invertible operators. This model allowed him to extend in [11] the notion of generalized multipliers for left-invertible analytic operators, introduced in [8], to left-invertible operators. These works were motivated by weighted shifts on directed trees.
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Our forthcoming work [5] is devoted to operator theory corresponding to
general multiply connected domains, in a somewhat different context. Some
consequences of the results of the present paper will be derived there.

The paper is organized as follows. In Section 2 we obtain basic properties
of operators in \( C_\alpha \) and prove our model theorem. In Section 3 we study
the duality inherent to this work and prove Theorem 1.1. Finally, in Section 4
we derive consequences from the explicit model obtained.

2. AN EXPLICIT FUNCTIONAL MODEL FOR OPERATORS IN \( C_\alpha \)

This section is devoted to proving Theorem 1.2.

Fix a bounded linear operator \( T \) acting on a Hilbert space \( H \).

Notice that

\[
\alpha(T^*, T) \geq 0 \text{ if and only if } (1 + r^2)\|Tx\|^2 - \|T^2x\|^2 - r^2\|x\|^2 \geq 0 \tag{2.1}
\]

for all \( x \) in \( H \). Observe that the left hand side of (2.1) is precisely \( \|Dx\|^2 \).

In fact, for all \( x \in H \) and all \( n \in \mathbb{Z} \) we have

\[
\|DT^n x\|^2 = (1 + r^2)\|T^{n+1}x\|^2 - \|T^{n+2}x\|^2 - r^2\|T^nx\|^2 \geq 0. \tag{2.2}
\]

Simple computations using (2.1) reveal that the membership of operators
\( T \) and \( rT - 1 \) in the class \( C_\alpha \) are closely related. More precisely:

Proposition 2.1. Suppose that \( T \) is invertible. Then \( T \) is in \( C_\alpha \) if and
only if \( rT - 1 \) is in \( C_\alpha \).

We next obtain the inclusion \( C_\alpha \subset C_{1,r} \), stated in Theorem 1.1. The
complete proof of that theorem will be given at the end of Section 3.

Proposition 2.2. If \( T \) is in \( C_\alpha \), then \( \|T\| \leq 1 \) and \( \|T^{-1}\| \leq 1/r \).

Proof. Note that the inequality in (2.2) can be written as

\[
\frac{\|T^{n+1}x\|^2 - \|T^nx\|^2}{r^{2n+2} - r^{2n}} \leq \frac{\|T^{n+2}x\|^2 - \|T^{n+1}x\|^2}{r^{2n+4} - r^{2n+2}}.
\]

If we consider \( x \in H \) fixed, this means that the broken line with vertices in
\((r^{2n}, \|T^nx\|^2)_{n \in \mathbb{Z}} \) is concave. Then, for any its edge, the straight line
containing it is above the whole broken line. Since \( x \)-coordinates \( r^{2n} \) of the
vertices go to \( \infty \) as \( n \to -\infty \) and the broken line is above the \( x \)-axis, all edges
have non-negative slope. In particular, \( \|Tx\|^2 \leq \|x\|^2 \), so \( T \) is a contraction.
The second part of the statement follows using Proposition 2.1. \( \square \)

As usual, let \( \sigma(T) \) denote the spectrum of \( T \). The following result is an
immediate consequence of Proposition 2.2

Corollary 2.3. If \( T \) is in \( C_\alpha \), then \( \sigma(T) \subset \mathbb{H}^1 \), and the limits

\[
L^+(T, x) := \lim_{n \to \infty} \|T^nx\|^2, \quad L^-(T, x) := \lim_{n \to \infty} \|r^nT^{-n}x\|^2
\]

exist for all \( x \) in \( H \).

Notice that \( L^+(T, Tx) = L^+(T, x) \) and \( L^-(T, Tx) = r^2L^-(T, x) \) for all \( x \in H \). This obvious fact will be implicitly used below.
Lemma 2.4. If $T$ is in $C_\alpha$, then
\[
\sum_{n=0}^{\infty} \|DT^n x\|^2 = \|Tx\|^2 - r^2\|x\|^2 + (r^2 - 1)L^+(T, x)
\] (2.3)
for all $x$ in $H$. In particular, the series on the left hand side converges.

Proof. Note that
\[
\|DT^n x\|^2 = -\|T^{n+2}x\|^2 + (1 + r^2)\|T^{n+1}x\|^2 - r^2\|Tx\|^2.
\]
for all $x \in H$ and all $n \in \mathbb{Z}$. Therefore
\[
\sum_{n=0}^{N} \|DT^n x\|^2 = -\sum_{n=2}^{N+2} \|T^n x\|^2 + (1 + r^2)\sum_{n=1}^{N+1} \|T^n x\|^2 - r^2\sum_{n=0}^{N} \|T^n x\|^2
\]
for all $N \in \mathbb{N}$. The statement follows letting $N \to \infty$. \hfill \Box

Lemma 2.5. If $T$ is in $C_\alpha$, then
\[
-\sum_{n=-\infty}^{-1} r^{-2n-2}\|DT^n x\|^2 = -\|Tx\|^2 + \|x\|^2 + (r^2 - 1)L^-(T, x)
\] (2.4)
for all $x$ in $H$. In particular, the series on the left hand side converges.

Proof. By Proposition (2.1) $rT^{-1} \in C_\alpha$. Let $\tilde{D}$ denote the defect operator of $rT^{-1}$. A straightforward computation gives that
\[
r^{-2}\||\tilde{D}T^2 x\|^2 = \|Dx\|^2
\] (2.5)
for all $x \in H$. Using Lemma (2.4) for $rT^{-1}$, we obtain
\[
\sum_{n=0}^{\infty} \|\tilde{D}r^n T^{-n} y\|^2 = \|rT^{-1} y\|^2 - r^2\|y\|^2 + (r^2 - 1)L^+(rT^{-1}, y)
\] (2.6)
for all $y \in H$. Set $y = r^{-1}Tx$. Then
\[
L^+(rT^{-1}, y) = \lim_{n \to \infty} \|r^n T^{-n} y\|^2 = \lim_{n \to \infty} \|r^{-n-1} T^{-n+1} x\|^2 = L^-(T, x).
\]
Hence the right hand side of (2.6) is equal to the right hand side of (2.4). Using (2.4) we get
\[
\sum_{n=0}^{\infty} \|\tilde{D}r^n T^{-n} y\|^2 = \sum_{n=-\infty}^{-1} r^{-2n-2}\|DT^n x\|^2.
\]
Therefore (2.4) follows. \hfill \Box

Theorem 2.6. If $T$ is in $C_\alpha$, then
\[
\|x\|^2 = \frac{1}{1 - r^2} \sum_{n=0}^{\infty} \|DT^n x\|^2 + \frac{1}{1 - r^2} \sum_{n=-\infty}^{-1} r^{-2n-2}\|DT^n x\|^2
\]
\[+ L^+(T, x) + L^-(T, x)
\]
for all $x$ in $H$. In particular, both series on the right hand side converge.

Proof. Add up equations (2.3) and (2.4), and rearrange the terms. \hfill \Box
The function $O$ foreshadows its structure. Let $T \in C$. Consider the operator
$$O_{T}: H \to H^2_{\mathfrak{D}} \oplus \tilde{H}, \quad O_{T}x(z) = D(z-T)^{-1}x, \quad x \in H, z \in \mathfrak{D}.$$ The function $O_{T,D}x(z)$ can be defined as
$$- \sum_{n=0}^{\infty} (DT^{-n-1}x)z^n \quad \text{for } |z| < r, \quad \sum_{n=-\infty}^{-1} (DT^{-n-1}x)z^n \quad \text{for } |z| > 1.$$ Hence
$$\|O_{T,D}x\|_{H^2_{\mathfrak{D}}}^2 = \frac{1}{1-r^2} \sum_{n=0}^{\infty} r^{2n}\|DT^{-n-1}x\|^2 + \frac{1}{1-r^2} \sum_{n=-\infty}^{-1} r^{-2n-2}\|DT^{-n}x\|^2$$
for all $x \in H$. Therefore, by Theorem 2.6 $O_{T,D}$ is a contraction.

We construct the second component of the lifting using a well-known argument (cf. [9, 14]), that we also employed in [6, Section 4]. By Proposition 2.2 and the polarization identity, one can define on $H$ the sesquilinear form
$$\langle x, y \rangle_+ := \lim_{n \to \infty} \langle T^nx, T^ny \rangle.$$
Let $H_0$ be the subspace of vectors $x$ in $H$ such that $L^+(T,x) = \langle x, x \rangle_+ = 0$. Denote by $\tilde{H}_+$ the Hilbert space obtained as the completion of the quotient space $H/H_0$ in the pre-Hilbert norm $x \mapsto \langle x, x \rangle^{1/2}_+$. Let $W_+: H \to \tilde{H}_+$ be the operator that maps each vector $x$ to its class $[x]_+$. In the same way, thanks to Proposition 2.2 starting from the sesquilinear form
$$\langle x, y \rangle_- := \lim_{n \to \infty} \langle r^nT^{-n}x, r^nT^{-n}y \rangle$$
on $H$, we define the Hilbert space $\tilde{H}_-$ and the operator $W_- : H \to \tilde{H}_-$ mapping each vector $x$ to its class $[x]_-$. Denote by $W$ the operator $(W_+, W_-)$, and let $\tilde{H}$ be the closure of its range. That is, $\tilde{H}$ is a closed subspace of $\tilde{H}_+ \oplus \tilde{H}_-$, and
$$W : H \to \tilde{H}, \quad Wx = W_+x \oplus W_-x.$$ Notice that
$$\|Wx\|^2 = \|W_+x\|^2 + \|W_-x\|^2 = L^+(T,x) + L^-(T,x)$$for all $x \in H$. Therefore, as a consequence of Theorem 2.6 we obtain our lifting theorem.

Theorem 2.7 (Lifting of the model). If $T$ is in $C$, the transformation
$$(O_{T,D}, W) : H \to H^2_{\mathfrak{D}} \oplus \tilde{H}, \quad (O_{T,D}, W)(x) = O_{T,D}x \oplus Wx$$is an isometry.

In order to obtain the model theorem for operators in $C$, we first need to analyze the particular case of operators $T$ such that $\alpha(T^*, T) = 0$.

Theorem 2.8. Suppose that $T$ is invertible. The following statements are equivalent.
Proposition 2.9. For any Hilbert space $E$, the operator

$$\mathcal{M}_f^T : H_\mathcal{B}^2(\mathbb{B}, E) \to H_\mathcal{B}^2(\mathbb{B}, E)$$

given by (1.3) belongs to $\mathcal{C}_\alpha$.

Proof. Identifying $f \in H_\mathcal{B}^2(\mathbb{B}, E)$ with the sequence $\{f_n\}_{n \in \mathbb{Z}}$ of its coefficients, a straightforward computation gives that

$$(1 + r^2) \| \mathcal{M}_f \|^2 - \| \mathcal{M}_f^T \|^2 - r^2 \| f \|^2 r^2 \| f \|^2 = \| f_0 \|^2 \geq 0.$$ 

Hence $\alpha(\mathcal{M}_f^T)$, $\mathcal{M}_f^T \geq 0$. If we view $\mathcal{M}_f^T$ as a bilateral shift, we get that it is invertible. \hfill $\square$

Proof of Theorem 1.2. Assume that $\alpha(T^*, T) \geq 0$, that is, $T \in \mathcal{C}_\alpha$. Then we can obtain a normal operator $U = U_+ \oplus U_-$ as at the beginning of the proof of Theorem 2.8. Let $S$ be the restriction of $U$ to $\tilde{H}$. Note that $S$ has the desired properties of (ii). Moreover, $SW = WT$. Since $\mathcal{M}_f^T$, acting on $H_\mathcal{B}^2(\mathbb{B}, \mathcal{D})$, satisfies $M_f^T \mathcal{O}_T \mathcal{D} = \mathcal{O}_T \mathcal{D} T$, using Theorem 2.7 we obtain that $T$ is unitarily equivalent to a part of $\mathcal{M}_f^T \oplus S$.

Now assume (ii). Let $U$ be the normal minimal extension of $S$. Since $\sigma(U)$ is contained in the union of the circles $\{ |z| = r \}$ and $\{ |z| = 1 \}$, $U$ can be written as an orthogonal sum $U_+ \oplus U_-$, where $U_+$ and $r^{-1}U_-$ are unitaries. By Theorem 2.8 $\alpha(U^*, U) = 0$, so we also have $\alpha(S^*, S) = 0$.

(i) $\alpha(T^*, T) = 0$.
(ii) $T$ is a subnormal operator whose minimal normal extension can be written as a sum $U_+ \oplus U_-$, where $U_+$ and $r^{-1}U_-$ are unitary operators.

In this case, either $T$ is normal or $\sigma(T) = \mathbb{A}^d$. 

Proof. Assume that $\alpha(T^*, T) = 0$. In particular $T \in \mathcal{C}_\alpha$. Note that $W_+ : H \to \tilde{H}_+$ is onto. Since $W_+ x = W_+ T x$ for all $x \in H$, the formula $U_+ W_+ x = W_+ T x$ defines an isometric operator $U_+$ on $\tilde{H}_+$. Since the range of $U_+$ is $\tilde{H}_+$, $U_+$ is unitary. Analogously, the operator $U_-$ on $\tilde{H}_-$ given by $U_- W_- x = W_- T x$ satisfies that $r^{-1} U_-$ is a unitary operator. Set $U := U_+ \oplus U_-$, which acts on $\tilde{H}_+ \oplus \tilde{H}_-$. Then $U$ is a normal operator with

$$\sigma(U) \subset \{|z| = r \} \cup \{|z| = 1 \}.$$ 

Since $\mathcal{D} = 0$, Theorem 2.7 gives that $W : H \to \tilde{H}$ is a unitary operator. Therefore $T$ is unitarily equivalent to the restriction of $U$ to $\tilde{H}$, which has the desirable properties of (ii).

Now suppose that $T$ satisfies (ii). Let $U = U_+ \oplus U_-$ be its minimal unitary extension. Then $U$ is bounded and $\sigma(U)$ is contained in the union of the circles $\{|z| = r \}$ and $\{|z| = 1 \}$. Write each vector $x$ in the space where $U$ acts as a pair $(x_+, x_-)$, so that $U x = U_+ x_+ + U_- x_-$. Then

$$\| U x \|^2 = \| U_+ x_+ \|^2 + \| U_- x_- \|^2 = \| x_+ \|^2 + r^2 \| x_- \|^2.$$ 

Hence, a straightforward computation shows that

$$\alpha(U^*, U) = 0.$$ 

The last statement follows from [7, Theorem II.2.11 (c)]. \hfill $\square$

The last result we need to prove Theorem 1.2 is the following.

Proposition 2.9. For any Hilbert space $E$, the operator

$$\mathcal{M}_f^T : H_\mathcal{B}^2(\mathbb{B}, E) \to H_\mathcal{B}^2(\mathbb{B}, E)$$

given by (1.3) belongs to $\mathcal{C}_\alpha$. 

Proof. Identifying $f \in H_\mathcal{B}^2(\mathbb{B}, E)$ with the sequence $\{f_n\}_{n \in \mathbb{Z}}$ of its coefficients, a straightforward computation gives that

$$\| \mathcal{M}_f \|^2 - \| \mathcal{M}_f^T \|^2 - r^2 \| f \|^2 r^2 \| f \|^2 = \| f_0 \|^2 \geq 0.$$ 

Hence $\alpha(\mathcal{M}_f^T)$, $\mathcal{M}_f^T \geq 0$. If we view $\mathcal{M}_f^T$ as a bilateral shift, we get that it is invertible. \hfill $\square$
Since \( M^t_z \) is in \( C_\alpha \) (see Proposition 2.9) and \( T \) is a part of \( M^t_z \oplus S \), we obtain (i).

Notice that Theorem 2.8 can be seen as a particular case of Theorem 1.2 when the structure involving the defect operator disappears.

3. Dual models

In this section we discuss the duality behind the models involving operators \( M_z \) acting on spaces of functions on \( \mathbb{A} \), on one hand, and the models in terms of operators \( M^t_z \), acting on spaces of functions on \( \mathbb{B} \), on the other hand. In \[3\], we exploit this duality in the context of multiply connected domains.

**Proposition 3.1.** The operator \( M^t_z \) acting on \( H^2(\mathbb{B}, E) \) is dual to the operator \( M_z \) acting on \( H^2(\mathbb{A}, E) \) via the duality

\[
(f, g) = \int_{\mathbb{T}} \langle f(z), g(\bar{z}) \rangle_E \, dz + \int_{\mathbb{T}} \langle f(z), g(z) \rangle_E \, dz,
\]

where \( f \) is in \( H^2(\mathbb{B}, E) \) and \( g \) is in \( H^2(\mathbb{A}, E) \).

**Proof.** It is easy to see that \( M^t_z \) acting on \( H^2(\mathbb{B}, E) \) is unitarily equivalent to the backward shift

\[
B : \ell^2_t(\mathbb{B}, E) \rightarrow \ell^2_t(\mathbb{B}, E), \quad B(\{f_n\}_{n \in \mathbb{Z}}) = \{f_{n+1}\}_{n \in \mathbb{Z}},
\]

where \( \ell^2_t(\mathbb{B}, E) \) is the space of sequences \( \{f_n\}_{n \in \mathbb{Z}} \) in \( E \) with finite norm

\[
\| \{f_n\}_{n \in \mathbb{Z}} \|_{\ell^2_t(\mathbb{B}, E)} := \frac{1}{1 - r^2} \left( \sum_{n=0}^{\infty} \| f_n \|^2 + \sum_{n=-\infty}^{-1} r^{-2n-2} \| f_n \|^2 \right).
\]

In the same way, \( M_z \) acting on \( H^2(\mathbb{A}, E) \) is unitarily equivalent to the forward shift

\[
F : \ell^2_{r,-}(\mathbb{A}, E) \rightarrow \ell^2_{r,-}(\mathbb{A}, E), \quad F(\{g_n\}_{n \in \mathbb{Z}}) = \{g_{n-1}\}_{n \in \mathbb{Z}},
\]

where \( \ell^2_{r,-}(\mathbb{A}, E) \) is the space of sequences \( \{g_n\}_{n \in \mathbb{Z}} \) in \( E \) with finite norm

\[
\| \{g_n\}_{n \in \mathbb{Z}} \|_{\ell^2_{r,-}(\mathbb{A}, E)} := (1 - r^2) \left( \sum_{n=0}^{\infty} \frac{1}{r^{2n}} \| g_n \|^2 + \sum_{n=-\infty}^{-1} \| g_n \|^2 \right).
\]

Finally, the duality can be written as

\[
\langle \{f_n\}, \{g_n\} \rangle = \sum_{n \in \mathbb{Z}} \langle f_n, g_{n-1} \rangle_E,
\]

where \( \{f_n\}_{n \in \mathbb{Z}} \in \ell^2_t(\mathbb{B}, E) \) and \( \{g_n\}_{n \in \mathbb{Z}} \in \ell^2_{r,-}(\mathbb{A}, E) \). Now an easy computation shows that the statement holds.

Therefore, as an immediate consequence of Theorem 1.2 and Proposition 3.1, we obtain a dual model for operators in the class \( C_\alpha \).

**Theorem 3.2** (Dual model theorem). Suppose that \( T \) is invertible. Then \( T \) is in \( C_\alpha \) if and only if \( T^* \) is a compression of an operator of the form

\[
(M_z \text{ on } H^2(\mathbb{A}, E)) \oplus U_0 \oplus rU_1,
\]

where \( E \) is a Hilbert space, and \( U_0 \) and \( U_1 \) are unitary operators.
In what follows, we will use the Hilbert spaces $H^2(\mathbb{A}, E, \omega)$, defined in (1.1), in the scalar-valued case when $E = \mathbb{C}$. Putting $\omega = a > 0$, we get the Hilbert norms
\[ \|f\|_{H^2(\mathbb{A}, \mathbb{C}, a)}^2 = \int_T |f(z)|^2 |dz| + a \int_{rT} |f(z)|^2 |dz|. \]
Notice that $\{z^n\}_{n \in \mathbb{Z}}$ is an orthogonal basis in $H^2(\mathbb{A}, \mathbb{C}, a)$ with
\[ \|z^n\|_{H^2(\mathbb{A}, \mathbb{C}, a)}^2 = 1 + a^2 r^{2n} \]
for all $n \in \mathbb{Z}$. Therefore, in terms of the coefficients $\{f_n\}_{n \in \mathbb{Z}}$ of the Laurent series of $f$ in $\mathbb{A}$ (that is, $f(z) = \sum_{n \in \mathbb{Z}} f_n z^n$), we have
\[ \|f\|_{H^2(\mathbb{A}, \mathbb{C}, a)}^2 = \sum_{n=-\infty}^{\infty} (1 + a^2 r^{2n}) |f_n|^2. \]

Now consider the space $H^2(\mathbb{B}, \mathbb{C}, a)$ of all functions $f: \mathbb{B} \to \mathbb{C}$ given by
\[ \sum_{n=0}^{\infty} f_n z^n \text{ for } |z| < r, \quad \sum_{n=-\infty}^{-1} f_n z^n \text{ for } |z| > 1, \]
equipped with the Hilbert space norm
\[ \|f\|_{H^2(\mathbb{B}, \mathbb{C}, a)}^2 := \sum_{n=-\infty}^{\infty} \frac{1}{1 + a^2 r^{-2n-2}} |f_n|^2. \]
For any Hilbert space $E$, set
\[ H^2(\mathbb{B}, E, a) := H^2(\mathbb{B}, \mathbb{C}, a) \otimes E, \quad H^2(\mathbb{A}, E, a) := H^2(\mathbb{A}, \mathbb{C}, a) \otimes E. \]

**Proposition 3.3.** The operator $M_z^t$ acting on $H^2(\mathbb{B}, E, a)$ is dual to the operator $M_z$ acting on $H^2(\mathbb{A}, E, a)$.

*Proof.* The argument is essentially the same as in the proof of Proposition 3.1. Indeed, the same duality works. Hence we omit it. \( \square \)

**Lemma 3.4.** The operator $M_z^t$ acting on $H^2(\mathbb{B}, E, a)$ is in $C_\alpha$.

*Proof.* Identifying functions $f$ in $H^2(\mathbb{B}, E, a)$ with the sequence $\{f_n\}_{n \in \mathbb{Z}}$ of its coefficients, it is immediate that $M_z^t$ can be identified with the forward shift $F$ acting on the weighted space $\ell_\omega^2(\mathbb{Z})$ of bilateral sequences with
\[ \omega_n = \frac{1}{1 + a^2 r^{-2n-2}}. \]
Note that $F$ satisfies (2.1) if
\[ \frac{1 + r^2}{1 + ar^{-2n-4}} = \frac{1}{1 + ar^{-2n-6}} - \frac{r^2}{1 + ar^{-2n-2}} \geq 0 \]
for all $n \in \mathbb{Z}$. Setting $r^2 = \rho$ and $ar^{-2n-6} = x$, this follows from
\[ \frac{1 + \rho}{1 + x\rho} - \frac{1}{1 + x} - \frac{\rho}{1 + x\rho^2} = \frac{(\rho - 1)^2(\rho + 1)}{(1 + x)(1 + x\rho)(1 + x\rho^2)} \geq 0. \]
Hence $F$ is in $C_\alpha$, as we wanted to prove. \( \square \)
We remark that a special role of spaces $H^2(B, E, a)$ for the values $a = i^{2m}, \ m \in Z$, was observed in [10] (see Proposition 2.2 of that work). Namely, for these (and only these) values, the corresponding reproducing kernel $k(z, w)$ does not vanish on $\mathbb{A} \times \mathbb{A}$. The commutant lifting theorem given in [10] involves the operator class, defined in terms of the operator $M_z$ on $H^2(B, E, a)$, where $a$ takes one of these special values.

**Proof of Theorem 4.7** First, notice that the inclusion $C_{\alpha} \subset C_{1,r}$ has already been proved in Proposition 2.2. Now take $T \in \text{Sp}(\mathbb{A})$ and let us see that $T \in C_{\alpha}$. Since $\mathbb{A}^{cl}$ is also 1-spectral for $T^*$, by Theorem A we know that $T^*$ is unitarily equivalent to a compression of the operator

$$(M_z \text{ on } H^2(\mathbb{A}, E, \omega)) \oplus U_1 \oplus rU_2$$

to its coinvariant subspace, where $\omega \in L(E)$ is a weight and $U_1$, $U_2$ are unitaries. Therefore $T$ is unitarily equivalent to a part of the operator

$$(M_z \text{ on } H^2(\mathbb{A}, E, \omega))^* \oplus U_1^* \oplus rU_2^*.$$ 

Hence, it suffices to check that any operator of this form is in $C_{\alpha}$. It is clear that the operator $N = U_1^* \oplus rU_2^*$ is in $C_{\alpha}$ (indeed $\alpha(N^*, N) = 0$). Let us see now that the operator $M_z$ on $H^2(\mathbb{A}, E, \omega)$ is also in $C_{\alpha}$. By the spectral theorem, $\omega \in L(E)$ is unitarily equivalent to the multiplication operator $M_\alpha f(a) = af(a)$, acting on a direct integral of Hilbert spaces

$$\int\int H(a) d\mu(a),$$

where $\mu$ is a finite Borel measure, concentrated on $\sigma(\omega)$. In this representation, the operator $M_z$ on $H^2(\mathbb{A}, E, \omega)$ rewrites as

$$\int H(a) d\mu(a).$$

Therefore, it is enough to check only the scalar case. That is, we need to prove that the adjoint to $M_z$ on the scalar space $H^2(\mathbb{A}, C, a)$ is in $C_{\alpha}$ for all $a > 0$, which follows from Proposition 3.3 and Lemma 3.3. Hence, we have proved the inclusion $\text{Sp} \mathbb{A} \subset C_{\alpha}$.

Next, we show that the inclusion $C_{\alpha} \subset C_{1,r}$ is strict for all $0 < r < 1$. Consider the matrices

$$T_1 := \begin{pmatrix} 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1 \end{pmatrix}, \quad T_2 := \begin{pmatrix} \sqrt{r} & 0 \\ 1 - r & \sqrt{r} \end{pmatrix}.$$ 

It is easy to check that $\frac{1}{2} \|x\| \leq \|T_1 x\| \leq \|x\|$ and $r \|x\| \leq \|T_2 x\| \leq \|x\|$ for all $x \in H$. Now we test the left hand side of (2.1) with $x = (1,0)$ for both $T_1$ and $T_2$. For $T_1$ we obtain the result $-r^2/4$. For $T_2$ we get $(r - 1)^2(r^2 - 3r + 1)$. Therefore, if $0 < r \leq 1/2$ then $T_1$ is in $C_{1,r}$ but not in $C_{\alpha}$, while if $1/2 < r < 1$ then $T_2$ is in $C_{1,r}$ but not in $C_{\alpha}$.

Finally, let us see that the inclusion $\text{Sp} \mathbb{A} \subset C_{\alpha}$ is strict. Consider the shifts $B$ and $F$ given in the proof of Proposition 3.1. There we established the duality between these two operators. Testing the left hand side of (2.1) for the operator $F$ and the vector $f = \{f_n\}_{n \in Z}$, with $f_0 = 1$ and $f_n = 0$ if
\( n \neq 0 \), we have
\[
(1 + r^2)\|Ff\|^2 - \|F^2f\|^2 - r^2\|f\|^2 = (1 - r^2)\left(\frac{1 + r^2}{r^2} - \frac{1}{r^4} - r^2\right)
= \frac{(1 - r^2)^2}{r^4}(r^4 - 1),
\]
which is negative for all \( 0 < r < 1 \). Hence, \( B \in \mathcal{C}_\alpha \) (see Proposition 2.9) but \( F \notin \mathcal{C}_\alpha \). Since the class \( \mathcal{S}_{\alpha} \) is invariant by passing to the adjoint operator (i.e., \( T \in \mathcal{S}_{\alpha} \) if and only if \( T^* \in \mathcal{S}_{\alpha} \)), we obtain that \( B \in \mathcal{C}_\alpha \setminus \mathcal{S}_{\alpha} \). □

**Remark 3.5.** In an earlier version of this paper (published in arxiv), we asked whether it is true that \( \mathcal{S}_{\alpha} = \{ T \in \mathcal{C}_\alpha : T^* \in \mathcal{C}_\alpha \} \). Recently G. Tsikalas gave a concrete example, showing that \( \mathcal{S}_{\alpha} \) is strictly smaller than the set \( \{ T \in \mathcal{C}_\alpha : T^* \in \mathcal{C}_\alpha \} \).

4. **Consequences of the model**

In this section we present some results derived from the explicit model obtained in Theorem 1.2 for operators in \( \mathcal{C}_\alpha \). For instance, we give a concrete value \( K \) such that \( A_{\text{cl}} \) is completely \( K \)-spectral for all operators in \( \mathcal{C}_\alpha \), and we establish a characterization of the inclusion of classes \( \mathcal{C}_\alpha \subset \mathcal{C}_\beta \).

**Proposition 4.1.** The inclusion map
\[
J_a : H^2(\mathbb{B}, E) \hookrightarrow H^2(\mathbb{B}, E, a), \quad J_a f = f
\]
is well-defined. Moreover, it is a bijection and
\[
\|J_a\|^2 = \frac{1 - r^2}{\min\{a^2r^2 - 2, 1\}}, \quad \|J_a^{-1}\|^2 = \frac{1 + a^2r^2}{1 - r^2}.
\]

**Proof.** For the orthogonal basis \( \{z^n\}_{n \in \mathbb{Z}} \) we have
\[
\|z^n\|^2_{H^2(\mathbb{B}, E, a)} = \frac{1}{1 + a^2r^2n^2}
\]
and
\[
\|z^n\|^2_{H^2(\mathbb{B}, E)} = \begin{cases} 
 1/(1 - r^2) & \text{if } n \geq 0 \\
 1/(1 - r^2) & \text{if } n \leq -1 
\end{cases}
\]
Therefore
\[
\|J_a\|^2 = \max \left\{ \sup_{n \geq 0} \frac{1 - r^2}{(1 + a^2r^2n^2)}, \sup_{n \leq -1} \frac{1 - r^2}{1 + a^2r^2n^2} \right\}, \quad (4.1)
\]
and
\[
\|J_a^{-1}\|^2 = \max \left\{ \sup_{n \geq 0} \frac{1 + a^2r^2n^2}{1 - r^2}, \sup_{n \leq -1} \frac{1 + a^2r^2n^2}{1 - r^2} \right\}. \quad (4.2)
\]
Recall that \( 0 < r < 1 \). Hence, in (4.1) we need to compare the cases \( n \to \infty \) and \( n \to -\infty \), while in (4.2) the maximum is reached for \( n = 0 \). □

**Theorem 4.2.** \( \mathcal{A}^{\text{cl}} \) is completely \( \sqrt{2} \)-spectral for all operators in \( \mathcal{C}_\alpha \).
**Proof.** Let $T \in \mathbb{C}_\alpha$. Given $a > 0$, the operator $M_z$ acting on $H^2(\mathbb{B}, \mathcal{D}, a)$ has $K_a$ as a complete 1-spectral set (see Theorem A). Therefore, its adjoint operator, $M_tz$ acting on $H^2(\mathbb{B}, \mathcal{D}, a)$, also has $K_a$ as a complete 1-spectral set. By Theorem 1.2 and Proposition 4.1 it follows that $K_a$ is a complete $K_a$-spectral set for $T$, where $K_a := \|J_a\| \cdot \|J_a^{-1}\|$. Using Proposition 4.1 again, we have $\inf_{a>0} K_a = \sqrt{2}$. This infimum is indeed attained when $a = r$. Now the statement follows. □

In the recent preprint [17], Tsikalas also obtained the constant $\sqrt{2}$ of Theorem 4.2, and proved that, in fact, this constant is the best possible. That work contains an alternative proof of Proposition 2.2 (see [17, Lemma 4.1]). In his other recent preprint [16], he showed that the corresponding constant for the class $C_{1,r}$ is at least 2, which, in particular, provides an alternative proof of the fact that $C_{\alpha}$ is strictly smaller than $C_{1,r}$.

**Theorem 4.3.** Let $\beta(t) = (1-t)(t-s^2)$ for some $0 < s < 1$. Then $C_{\alpha} \subset C_{\beta}$ if and only if $s \leq r$.

**Proof.** Let $U$ be a unitary operator. Using (2.1), note that $rU$ is in $C_\beta$ if and only if

\[(1-r^2)(r^2-s^2) = (1+s^2)r^2 - r^4 - s^2 \geq 0.\]

Hence, $C_{\alpha} \subset C_{\beta}$ implies $s \leq r$. Now suppose that $s \leq r$. Using the model theorem for operators in $C_\alpha$, it remains to prove that $M_z^t$ acting on the space $H^2_\mathbb{B}(E)$ is in $C_\beta$, for any Hilbert space $E$. Equivalently, we want to prove that the backward shift $B$ acting on $\ell^2_\mathbb{B}(E)$ is in $C_\beta$. For any sequence $f = \{f_n\}_{n \in \mathbb{Z}}$ we have

\[(1+s^2)\|Bf\|^2 - \|B^2f\|^2 - s^2\|f\|^2 = \|f_0\|^2 + (r^2 - s^2) \sum_{n=-\infty}^{-1} r^{-2n-2}\|f_n\|^2,

which clearly is non-negative. Therefore the statement follows using (2.1) again. □

**Acknowledgments**

The first author was supported by National Science Centre, Poland grant UMO-2016/21/B/ST1/00241. The second author acknowledges partial support by Spanish Ministry of Science, Innovation and Universities (grant no. PGC2018-099124-B-I00) and the ICMAT Severo Ochoa project SEV-2015-0554 of the Spanish Ministry of Economy and Competitiveness of Spain and the European Regional Development Fund, through the “Severo Ochoa Programme for Centres of Excellence in R&D”. The second author also acknowledges financial support from the Spanish Ministry of Science and Innovation, through the “Severo Ochoa Programme for Centres of Excellence in R&D” (SEV-2015-0554) and from the Spanish National Research Council, through the “Ayuda extraordinaria a Centros de Excelencia Severo Ochoa” (20205CEX001).

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G. Bello
Institute of Mathematics of the Polish Academy of Sciences, 00-656 Warszawa, ul. Śniadeckich 8, Poland
Email address: gbello@impan.pl

D. V. Yakubovich
Departamento de Matemáticas, Universidad Autónoma de Madrid, Cantoblanco, 28049 Madrid, Spain, and Instituto de Ciencias Matemáticas (CSIC-UAM-UC3M-UCM)
Email address: dmitry.yakubovich@uam.es