Ehrhart quasi-polynomials of almost integral polytopes

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Abstract

A lattice polytope translated by a rational vector is called an almost integral polytope. In this paper, we study Ehrhart quasi-polynomials of almost integral polytopes. We study the connection between the shape of polytopes and the algebraic properties of the Ehrhart quasi-polynomials. In particular, we prove that lattice zonotopes and centrally symmetric lattice polytopes are characterized by Ehrhart quasi-polynomials of their rational translations.

1 Introduction

A polytope $P$ is the convex hull of finitely many points in $\mathbb{R}^d$. A polytope $P$ is called a lattice polytope (resp. rational polytope) if all its vertices are contained in $\mathbb{Z}^d$ (resp. $\mathbb{Q}^d$). The set of lattice points in a rational polytope $P$ is a fundamental object of study in enumerative combinatorics (§3, §4). Notably, it is known that the function $\mathbb{Z}_{>0} \ni t \mapsto L_P(t) := \#(tP \cap \mathbb{Z}^d)$ is a quasi-polynomial [5]. In other words, there exists a positive integer $\rho > 0$ and polynomials $f_1(x), f_2(x), \ldots, f_\rho(x) \in \mathbb{Z}[x]$ such that

$$L_P(t) = \begin{cases} f_1(t), & \text{if } t \equiv 1 \mod \rho, \\ f_2(t), & \text{if } t \equiv 2 \mod \rho, \\ \vdots \\ f_\rho(t), & \text{if } t \equiv \rho \mod \rho. \end{cases}$$

(1)

Here, $\rho$ is referred to as the period, and $f_1(x), \ldots, f_\rho(x)$ are called the constituents. (We identify the $\rho$-th constituent $f_\rho(x)$ with the 0-th one $f_0(x)$.) $L_P(t)$ is called the Ehrhart quasi-polynomial of $P$.

It is obvious that a multiple of a period of $L_P(t)$ is again a period of $L_P(t)$. We define the minimal period as the smallest possible period, and therefore, every period is a multiple of this minimal period. It is a well-known fact that the minimal period divides the least common multiple (LCM) of the denominators of the coordinates of the vertices of $P$. For the sake of simplicity, we will primarily focus on the minimal period. However, it is important to note that this restriction is not essential for our purposes (See Proposition [4] and Remark [5].

Generally, the constituents of $L_P(t)$ are usually distinct from each other. However, in numerous examples, some of the constituents may be found to be identical. As a result, the number of distinct constituents becomes strictly less than the minimal period $\rho$. Typical examples are as follows.

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Example 1.1. Consider the following polytopes:

- \( P_1 = \frac{1}{9} \cdot [0, 1]^3 \), which is a 3-cube with side length \( \frac{1}{9} \),
- \( P_2 = \left( \frac{3}{9}, \frac{5}{9}, \frac{2}{9} \right)^t + \text{Conv}\{ \pm e_i \mid i = 1, 2, 3 \} \), an octahedron translated by a rational vector (where \( e_1, e_2, e_3 \) are the standard basis vectors of \( \mathbb{R}^3 \)),
- \( P_3 = \left( \frac{4}{9}, \frac{2}{9}, \frac{1}{9} \right)^t + [0, 1]^3 \), a unit cube translated by a rational vector.

The Ehrhart quasi-polynomials \( L_{P_1}(t) \), \( L_{P_2}(t) \), and \( L_{P_3}(t) \) all share the same minimal period \( \rho = 9 \). The constituents of \( L_{P_1}(t) \) are \( f_k(t) = \left( \frac{4k+1}{9} \right)^3 \) for \( k = 0, 1, \ldots, 8 \), they are mutually distinct. In contrast, the constituents of \( L_{P_2}(t) \) and \( L_{P_3}(t) \) are as follows.

\[
L_{P_2}(t) = \begin{cases} 
\frac{4}{9}t^3 - \frac{4}{3}t, & (t \equiv 1, 8 \mod 9), \\
\frac{4}{9}t^3 + \frac{2}{3}t, & (t \equiv 2, 7 \mod 9), \\
\frac{4}{9}t^3 + t^2 + \frac{2}{3}t, & (t \equiv 3, 6 \mod 9), \\
\frac{4}{9}t^3 - \frac{1}{9}t, & (t \equiv 4, 5 \mod 9), \\
\frac{4}{9}t^3 + 2t^2 + \frac{5}{9}t + 1, & (t \equiv 9 \mod 9),
\end{cases}
\]

\[
L_{P_3}(t) = \begin{cases} 
t^3 & (t \equiv 1, 2, 4, 5, 7, 8 \mod 9), \\
t^3 + t^2 & (t \equiv 3, 6 \mod 9), \\
(t + 1)^3 & (t \equiv 9 \mod 9).
\end{cases}
\]

Note that in \( L_{P_2}(t) \) and \( L_{P_3}(t) \) some of constituents coincide.

The motivation for our paper is to investigate the connection between coincidences of constituents and the shape of the polytope \( P \). To formalize these “coincidences among constituents”, we introduce the following notion.

**Definition 1.2.** Let \( L(t) \) be a quasi-polynomial with period \( \rho \) and constituents \( f_1, \ldots, f_{\rho-1}, f_{\rho} (= f_0) \).

1. We say that \( L(t) \) is symmetric if \( f_k = f_{\rho-k} \) for \( 0 \leq k \leq \rho \).
2. We say that \( L(t) \) has the GCD-property if \( f_k = f_\ell \) whenever \( \text{GCD}(\rho, k) = \text{GCD}(\rho, \ell) \).

Clearly, if a quasi-polynomial satisfies the GCD-property, then it is symmetric.

**Remark 1.3.** Quasi-polynomials with the GCD-property naturally appear in the theory of hyperplane arrangements. See §6.2 further details.

In Example 1.1, \( L_{P_2}(t) \) is symmetric and \( L_{P_3}(t) \) satisfies the GCD-property. Surprisingly, these properties of Ehrhart quasi-polynomials are closely related to the fact that “\( P_2 \) is centrally symmetric” and “\( P_3 \) is a zonotope”, respectively. In fact, the primary objective of this paper is to establish the correspondence between two columns of the following table.

| Shape of \( P \) | Property of \( L_P(t) \) |
|------------------|--------------------------|
| General polytope | General quasi-polynomial |
| \( \cup \)       | \( \cup \)                |
| Centrally symmetric | Symmetric                 |
| \( \cup \)       | \( \cup \)                |
| Zonotope         | GCD-property              |
The main results (Theorem 4.2 and Theorem 5.5) establish that for almost integral polytopes (rationally translated lattice polytopes), the properties of the left column imply those of the right. Furthermore, we can also characterize the left column using the properties of the right column.

The paper is organized as follows:

In §2, after introducing basic notions, we recall a formula by Ardila-Beck-McWhirter, which expresses the Ehrhart quasi-polynomial of an almost integral zonotope. Applying this formula, we demonstrate that the Ehrhart quasi-polynomial satisfies the GCD-property.

In §3 we introduce the concept of the translated lattice point enumerator, denoted as

$$L_{(P,c)}(t) := \#((c + tP) \cap \mathbb{Z}^d),$$

where $P$ is a lattice polytope in $\mathbb{R}^d$ and $c \in \mathbb{R}^d$. We prove that $L_{(P,c)}(t)$ is a polynomial in $t \in \mathbb{Z}_{>0}$ (Theorem 3.2). Furthermore, we show that the constituents of an almost integral polytope can be described in terms of $L_{(P,c)}(t)$ (Corollary 3.4). This result readily implies that if $P$ is a centrally symmetric almost integral polytope, then $L_P(t)$ is symmetric (Corollary 3.5).

In §4 we present the first main result. Specifically, we prove that a lattice polytope $P$ is centrally symmetric if and only if the Ehrhart quasi-polynomial $L_{c+P}(t)$ is symmetric for every rational vector $c$ (Theorem 4.2). The “only if” part has already been established in §3. To complete the proof, we show that if $P$ is not centrally symmetric, then there exists a rational vector $c$ such that $L_{(P,c)}(t) \neq L_{(P,-c)}(t)$. For this, we employ Minkowski’s result, which characterizes a polytope based on normal vectors and volumes of facets.

In §5 we present the second main result. Specifically, we prove that a lattice polytope $P$ is a zonotope if and only if the Ehrhart quasi-polynomial $L_{c+P}(t)$ satisfies the GCD-property for every rational vector $c$ (Theorem 5.5). The “only if” part has already been established in §2. To complete the proof, we employ an involved argument using McMullen’s characterization of zonotopes in terms of central symmetricity of faces. We establish that if $P$ is not a zonotope, then there exists a rational vector $c$ with odd denominators such that $L_{(P,c)}(t) \neq L_{(P,2c)}(t)$, which implies $L_{c+P}(t)$ does not satisfy the GCD-property.

In §6 we will examine related problems. First, we discuss the minimal periods of almost integral polytopes and explore the relationship between the Ehrhart quasi-polynomials of almost integral zonotopes and the characteristic quasi-polynomials of hyperplane arrangements. We pose several related questions.

We conclude this section with an elementary proposition concerning the periods of quasi-polynomials. This proposition justifies our convention that we may always assume a period is minimal.

**Proposition 1.4.** Let $Q$ be a quasi-polynomial with the minimal period $\rho_0$ and the $i$-th constituent $f_i$. Let $k \in \mathbb{Z}_{>0}$. Then $Q$ has the GCD-property for $\rho_0$ if and only if $Q$ has the GCD-property for $k\rho_0$.

**Proof.** Suppose that $Q$ has the GCD-property for the minimal period $\rho_0$. Note that $\gcd(\rho_0, i) = \gcd(\rho_0, \gcd(k\rho_0, i))$. Hence if $\gcd(k\rho_0, i) = \gcd(k\rho_0, j)$, then $\gcd(\rho_0, i) = \gcd(\rho_0, j)$. Therefore, it satisfies the GCD-property for $k\rho_0$.

Conversely, suppose $Q$ has the GCD-property for $k\rho_0$. Suppose $\gcd(i, \rho_0) = \gcd(j, \rho_0) = d$. We shall prove $f_i = f_d = f_j$. Without loss of generality, we may assume that $i = d = \gcd(j, \rho_0)$. It suffices to show that there exists an integer $m \in \mathbb{Z}$ such that $i = \gcd(i, k\rho_0) = \gcd(j + mp_0, k\rho_0)$. This means, by the GCD-property for $k\rho_0$, that $f_i = f_{j + mp_0} = f_j$. Define

$$m = \frac{\text{rad}(k)}{\text{rad}(\gcd(\frac{d}{k}, k))},$$

where $\text{rad}(a)$ is the product of all primes dividing $a$. 

3
By using that definition, it follows from \( \text{GCD}(\frac{j}{m} + m\mathbb{Q}, \mathbb{Q}) = \text{GCD}(\frac{j}{m}, \mathbb{Q}) = 1 \) that \( \text{GCD}(\frac{j}{m} + m\mathbb{Q}, k\mathbb{Q}) = \text{GCD}(\frac{j}{m} + m\mathbb{Q}, k) \). Now let \( r \in \mathbb{N} \) be a prime such that \( r|k \). By the definition of \( m \), \( r \) divides \( m \) if and only if \( r \) does not divide \( \frac{j}{m} \). Hence, \( \text{GCD}(\frac{j}{m} + m\mathbb{Q}, k) = 1 \), and we also have \( \text{GCD}(\frac{j}{m} + m\mathbb{Q}, k\mathbb{Q}) = 1 \). Therefore, \( \text{GCD}(j + m\rho_0, k\rho_0) = i \).

\[ \square \]

**Remark 1.5.** Similarly, being symmetric is independent of the period.

## 2 Ehrhart quasi-polynomials for rational polytopes

### 2.1 Notation

A polytope \( P \subset \mathbb{R}^d \) is called centrally symmetric if there exists \( c \in \mathbb{R}^d \) such that \( P = c + (-P) \). The point \( \frac{j}{m} \in P \) serves as the center of \( P \). A zonotope \( Z(u_1, \ldots, u_n) \), formed by vectors \( u_1, \ldots, u_n \in \mathbb{R}^d \), is the Minkowski sum of the line segments \([0, u_i] \). In other words,

\[
Z(u_1, \ldots, u_n) = \{ \lambda_1 u_1 + \cdots + \lambda_n u_n \mid 0 \leq \lambda_i \leq 1, i = 1, \ldots, n \}.
\]

It is easily seen that zonotopes are centrally symmetric.

For a polytope \( P \subset \mathbb{R}^d \), we denote the minimal affine subspace containing \( P \) by \( \text{aff}(P) \). Additionally, we denote by \( \text{aff}_0(P) \) the linear subspace of \( \mathbb{R}^d \) parallel to \( \text{aff}(P) \) and contains the origin. The dimension of a polytope \( P \) is defined as \( \dim \text{aff}_0(P) \) and is denoted by \( \dim P \). Let \( P \) be a lattice polytope. If \( X \subset P \) is an \( m \)-dimensional face, then \( \text{aff}(X) \cap \mathbb{Z}^d \cong \mathbb{Z}^m \).

The relative volume of the polytope \( X \) is the volume of \( X \) rescaled so that the unit cube in \( \text{aff}(X) \cap \mathbb{Z}^d \cong \mathbb{Z}^m \) has volume 1. We denote the relative volume of \( X \) by \( \text{relvol}(X) \). The \( k \)-dimensional Euclidean volume is denoted by \( \text{vol}_k \). Note that if \( P \subset \mathbb{R}^d \) is a \( d \)-polytope (a polytope of dimension \( d \)), then \( \text{relvol}(P) = \text{vol}_d(P) \). For a rational polytope \( P \) of dimension \( m \leq d \), the leading coefficient of every constituent (the coefficient in degree \( m \)) of \( L_P(t) \) is the relative volume \( \text{relvol}(P) \) of \( P \).

**Definition 2.1.** A polytope \( P \subset \mathbb{R}^d \) is called almost integral, if there exists a lattice polytope \( P' \subset \mathbb{R}^d \) and a translation vector \( c \in \mathbb{Q}^d \), such that \( P = c + P' \).

A period of the Ehrhart quasi-polynomial \( L_P \) of an almost integral polytope \( P = c + P' \) with the translation vector \( c = (c_1, \ldots, c_d) \in \mathbb{Q}^d \) is \( \text{den}(c) := \text{lcm}\{\text{den}(c_i) \mid i = 1, \ldots, d\} \}, \) where \( \text{den}(c_i) \) denotes the denominator of the reduced fraction of \( c_i \). It is expected that \( \text{den}(c) \) is the minimal period of \( L_P \). See §6.1 for a related discussion.

### 2.2 Ehrhart quasi-polynomials for almost integral zonotopes

The following proposition by Ardila, Beck, and McWhirter describes the Ehrhart quasi-polynomial of almost integral zonotopes.

**Proposition 2.2.** [1 Proposition 3.1] Let \( U \subset \mathbb{Z}^d \) be a finite set of integer vectors, and let \( c \in \mathbb{Q}^d \) be a rational vector. Then the Ehrhart quasi-polynomial of the almost integral zonotope \( c + Z(U) \) is given by

\[
L_{c+Z(U)}(t) = \sum_{\substack{W \subseteq U \mid W \text{ lin. indep.}}} \chi_W(t) \cdot \text{relvol}(Z(W)) \cdot t^{|W|},
\]

where

\[
\chi_W(t) = \begin{cases} 
1, & \text{if } (tc + \text{aff}(W)) \cap \mathbb{Z}^d \neq \emptyset \\
0, & \text{otherwise}
\end{cases}
\]
Let \( \rho_0 = \text{den}(c) \). Then \( \rho_0 \) is a period of \( L_{c+Z(U)}(t) \). Proposition 2.2 says that the \( k \)-th constituent of \( L_{c+Z(U)}(t) \) is given by

\[
f_k(t) = \sum_{W \subseteq U, W \in \text{lin. indep.}} \chi_W(k) \cdot \text{relvol}(Z(W)) \cdot t^{|W|},
\]

(5)

The first result of this paper is the following.

**Theorem 2.3.** Let \( P = c + Z(U) \subset \mathbb{R}^d \) be an almost integral zonotope. Then \( L_{P} \) satisfies the GCD-property.

**Proof.** In order to prove the GCD-property for the Ehrhart quasi-polynomials of almost integral zonotopes, it is sufficient to show that the function \( \chi_W(t) \) in Proposition 2.2 satisfies the GCD-property.

Let \( W = \{u_1, \ldots, u_k\} \subseteq U \) be linearly independent subset. Denote by

\[
\langle W \rangle = \left( \sum_{i=1}^{k} \mathbb{R}u_i \right) \cap \mathbb{Z}^d,
\]

the intersection of the linear subspace generated by \( W \) with \( \mathbb{Z}^d \). By extending a \( \mathbb{Z} \)-basis of \( \langle W \rangle \) to that of \( \mathbb{Z}^d \), we have \( u_{k+1}, \ldots, u_d \in \mathbb{Z}^d \) such that \( \mathbb{Z}^d = \langle W \rangle \oplus \oplus_{i=k+1}^{d} \mathbb{Z}u_i \). Decompose \( c = (c_1, \ldots, c_d) \in \mathbb{Q}^d \) as \( c = c' + a_{k+1}u_{k+1} + \cdots + a_du_d \), where \( c' \in \text{aff}_0(W) \cap \mathbb{Q}^d \) and \( a_i \in \mathbb{Q} \). Then \( (tc + \text{aff}(W)) \cap \mathbb{Z}^d = \emptyset \) if and only if \( ta_{k+1}, \ldots, ta_d \in \mathbb{Z} \). This is also equivalent to \( t \) being divisible by \( \text{lcm}(\text{den}(a_{k+1}), \ldots, \text{den}(a_d)) \). Note that \( \text{lcm}(\text{den}(a_{k+1}), \ldots, \text{den}(a_d)) \) is a divisor of \( \rho_0 = \text{den}(c) \). Thus \( \chi_W(t) \) depends only on \( \text{GCD}(\rho_0, t) \).

\( \square \)

### 3 Translated lattice point enumerator

To verify the symmetricity or the GCD-property for quasi-polynomials, it is necessary to compare different constituents of a quasi-polynomial. Hence, we introduce a new function for this purpose.

**Definition 3.1.** Let \( P \subset \mathbb{R}^d \) be a polytope and \( c \in \mathbb{R}^d \). We define the function \( L_{(P,c)}(t) = \#((c + tP) \cap \mathbb{Z}^d) \) for \( t \in \mathbb{Z}_{\geq 0} \), which we shall call the translated lattice point enumerator.

**Theorem 3.2.** (1) If \( P \subseteq \mathbb{R}^d \) is a lattice polytope of dimension \( d \) and \( c \in \mathbb{R}^d \), then \( L_{(P,c)}(t) \in \mathbb{Q}[t] \). Furthermore, \( \deg L_{(P,c)}(t) = d \) and the leading coefficient of \( L_{(P,c)}(t) \) is \( \text{relvol}(P) \).

(2) Let \( P \subseteq \mathbb{R}^d \) be a lattice polytope (not necessarily of dimension \( d \)) and \( c \in \mathbb{R}^d \). Then \( L_{(P,c)}(t) \in \mathbb{Q}[t] \). Furthermore, if \( L_{(P,c)}(t) \neq 0 \), then it is polynomial of degree \( d \) \( P \) with the leading coefficient \( \text{relvol}(P) \).

**Proof.** We begin by proving that (1) implies (2). Suppose that \( \dim P = k < d \). If \( (c + \text{aff}(P)) \cap \mathbb{Z}^d = \emptyset \), then clearly \( L_{(P,c)}(t) = 0 \). Now, assume that \( (c + \text{aff}(P)) \cap \mathbb{Z}^d \neq \emptyset \). Let \( c' \in (c + \text{aff}(P)) \cap \mathbb{Z}^d \). Then we can express it as

\[
P + c = (P + c') + (c - c'),
\]

where \( P + c' \) is a lattice polytope in \( (c + \text{aff}(P)) \cap \mathbb{Z}^d \) and \( (c - c') \in \text{aff}_0(P) \) serves as a translation vector. Since \( c' + P \) is of full dimension in \( \text{aff}(c' + P) \), which is isomorphic to \( \mathbb{R}^k \), we can apply (1) to conclude (2).

Now, let us prove (1). We define two sets as follows.

\[
L(t) = \left( (tP + [0,c]) \setminus (c + tP) \right) \cap \mathbb{Z}^d \quad \text{“lost points”}
\]

\[
N(t) = \left( (tP + [0,c]) \setminus tP \right) \cap \mathbb{Z}^d \quad \text{“new points”}
\]

5
Denote by \( \ell(t) \) the number of lost points, that is, \( \ell(t) = \#L(t) \), and by \( n(t) \) the number of new points, that is, \( n(t) = \#N(t) \). It is easily seen that

\[
\#((tP + [0, c]) \cap \mathbb{Z}^d) = L_{(P,c)}(t) + \ell(t) = L_P(t) + n(t).
\]

(6)

To prove \( L_{(P,c)}(t) \) is a polynomial, it is sufficient to show that \( \ell(t) \) and \( n(t) \) are polynomials. We shall prove this fact by induction on \( d \).

For \( d = 0 \) there exists just one polytope \( P = \{0\} \) with one translation vector \( c = 0 \). Hence, \( L_{(P,c)}(t) = L_P(t) = 1 \).

Next, let \( P \subset \mathbb{R}^d \) be a lattice polytope of dimension \( d \) with \( c \in \mathbb{R}^d \). If \( c = 0 \), then \( L_{(P,c)}(t) = L_P(t) \), which is equal to the Ehrhart polynomial.

Suppose \( c \neq 0 \). We call a face \( F \) of \( P \) an “upper face” if, \((tc + F) \cap P = \emptyset\) for any \( t \in \mathbb{R}_{>0} \). Similarly, a face \( F \) of \( P \) is a “lower face” if, \((tc + F) \cap P = \emptyset\) for any \( t \in \mathbb{R}_{<0} \). Then we have

\[
N(t) = \bigcup_{F: \text{ upper face of } P, \ 0<s\leq1} ((sc + tF) \cap \mathbb{Z}^d).
\]

When \((c + \text{aff}(F)) \cap \mathbb{Z}^d \neq \emptyset\), choose \( c' \in (c + \text{aff}(F)) \cap \mathbb{Z}^d \). Then \( F + c = F + c' + (c-c') \) with \( c-c' \in \text{aff}_0(F) \). By inductive assumption, we have that \( L_{(F+c',c-c')}(t) \) is a polynomial in \( t \in \mathbb{Z}_{>0} \).

By definition of the volume of \( P \), we have

\[
\lim_{t \to \infty} \frac{L_{(P,c)}(t)}{t^d} = \text{relvol}(P).
\]

Therefore, \( \deg L_{(P,c)}(t) = d \) and the leading coefficient is \( \text{relvol}(P) \). \( \square \)

**Example 3.3.** Let \( P = \text{conv}\{(1,0)^t, (0,1)^t, (0,2)^t, (1,3)^t, (2,1)^t\} \subset \mathbb{R}^2 \) be a lattice polytope and \( c = (\frac{2}{7}, \frac{2}{7})^t \in \mathbb{Q}^d \) be a rational translation vector (Figure 1a). Then, the upper faces of \( P \) with respect to \( c \) are \([1,3)^t, (2,1)^t\], \{ (1,3)^t \}, \{ (2,1)^t \}, \) while the lower faces can be described as \([(0,2)^t, (0,1)^t], [(0,1)^t, (1,0)^t] \) with their corresponding vertices, which is illustrated in Figure 1a.

The set of lost points is \( L(1) = \{(P + [0, c]) \cap (c + P)\} \cap \mathbb{Z}^2 = \{(1,0)^t, (0,1)^t, (0,2)^t, (1,1)^t\} \) and the newly obtained points \( N(1) = \{(P + [0, c]) \setminus P\} \cap \mathbb{Z}^2 = \{(2,2)^t, (2,3)^t\} \), as seen in Figure 1b.

In that manner we obtain the following number of lattice points.

| \( t \) | 0 | 1 | 2 |
|-----|---|---|---|
| \( L_{(P,c)}(t) \) | 0 | 5 | 17 |
| \( L_P(t) \) | 1 | 7 | 20 |
| \( l_{c+P}(t) \) | 1 | 4 | 7 |
| \( n_{c+P}(t) \) | 0 | 2 | 4 |

From this, we calculate the polynomials as follows.

\[
L_{(P,c)}(t) = \frac{7}{2} t^2 + \frac{3}{2} t, \quad L_P(t) = \frac{7}{2} t^2 + \frac{5}{2} t + 1, \\
\ell(t) = 3t + 1, \quad n(t) = 2t.
\]

We can express constituents of the Ehrhart quasi-polynomial using the translated lattice point enumerator.

**Corollary 3.4.** Let \( P \in \mathbb{R}^d \) be a lattice polytope, and \( c \in \mathbb{Q}^d \). Then the \( k \)-th constituent of \( L_{c+P}(t) \) is \( L_{(P,kc)}(t) \).
Proof. Let $f_k$ be the $k$-th constituent of the Ehrhart quasi-polynomial $L_{c+P}$. Let $\rho = \text{den}(c)$. Then $L_{c+P}$ has a period $\rho$. Since a translation by a vector in $\mathbb{Z}^d$ does not affect the number of lattice points, we get

$$f_k(t) = \#(t(P + c) \cap \mathbb{Z}^d) \quad \text{for } t \equiv k \mod \rho$$

$$= \#((tP + tc) \cap \mathbb{Z}^d) \quad \text{for } t \equiv k \mod \rho$$

$$= \#((tP + kc) \cap \mathbb{Z}^d) \quad \text{for } t \equiv k \mod \rho$$

$$= L_{(P, kc)}(t) \quad \text{for } t \equiv k \mod \rho$$

We now prove that the Ehrhart quasi-polynomial of any almost integral centrally symmetric polytope is symmetric.

Corollary 3.5. Let $P \in \mathbb{R}^d$ be a centrally symmetric lattice polytope. Then, for every $c \in \mathbb{Q}^d$, the Ehrhart quasi-polynomial $L_{c+P}(t)$ is symmetric.

Proof. Let $f_0, \ldots, f_{\rho-1}$ be the constituents of $L_{c+P}(t)$. Note that since $P$ is centrally symmetric, $c + P$ and $-c + P$ contain the same number of lattice points. For $k \in \{1, \ldots, \rho-1\}$, by Corollary 3.4 we have

$$f_k(t) = L_{(P, kc)}(t)$$

$$= \#(kc + tP) \cap \mathbb{Z}^d$$

$$= \#((-kc + tP) \cap \mathbb{Z}^d$$

$$= \#((\rho - k)c + tP) \cap \mathbb{Z}^d$$

$$= L_{(P, (\rho - k)c)}(t) = f_{\rho - k}(t).$$

Therefore, $L_{c+P}(t)$ is symmetric.

\[ \Box \]

4 Characterizing centrally symmetric polytopes

Recall that a polytope $P$ is characterized up to translation by the normal vectors and the $(d-1)$-volumes of its facets (this fact was first proved by Minkowski [11]. See also [6, 7, 13]). From this fact, it follows.

Lemma 4.1. Let $P \subset \mathbb{R}^d$ be a $d$-polytope. Then $P$ is centrally symmetric if and only if for each facet $F$ there exists a parallel facet $F^{\text{op}}$, such that $\text{vol}_{d-1}(F) = \text{vol}_{d-1}(F^{\text{op}})$, where $\text{vol}_{d-1}$ is the $(d-1)$-dimensional Euclidean volume.
Lemma 4.1, we find a facet $F$ and $L$, because the former case can be considered as the latter case with $\text{vol}^m$.

The implication ($\Rightarrow$) was proved in Corollary [3]. Now, let $P \subset \mathbb{R}^d$ be a non-centrally symmetric polytope of dimension $m$. We will prove that there exists a translation vector $c \in \mathbb{Q}^d$ such that $L_{(P,c)}(t) \neq L_{(P,-c)}(t)$ for some $t \in \mathbb{Z}_{\geq 0}$. Since $P$ is not centrally symmetric, by using Lemma [4.1], we find a facet $F$ that has either no parallel facet or a parallel facet $F'$ with different $(m-1)$-dimensional Euclidean volumes $\text{vol}_{m-1}(F) \neq \text{vol}_{m-1}(F')$. We only consider the latter case, because the former case can be considered as the latter case with $\text{vol}_{m-1}(F') = 0$. Since $F$ and $F'$ are parallel, the unit cubes in $\text{aff}(F) \cap \mathbb{Z}^d$ and $\text{aff}(F') \cap \mathbb{Z}^d$ have the same $(m-1)$-dimensional Euclidean volume. On the other hand $F$ and $F'$ have different volumes, so their relative volumes are different. This means for the Ehrhart polynomials $L_F(t) = c_{m-1}t^{m-1} + \ldots + c_0$ and $L_{F'}(t) = c'_{m-1}t^{m-1} + \ldots + c'_0$ that $c_{m-1} \neq c'_{m-1}$. Without loss of generality, we may assume that $c_{m-1} > c'_{m-1}$. Let $c \in \text{aff}(F) = \text{aff}(F')$ be a nonzero vector. By choosing $c$ generically (we also note that $c$ can be chosen arbitrarily small), we may assume $(\text{aff}(X) + c) \cap \mathbb{Z}^d = \emptyset$ for every proper face $X \neq F$, $F'$. Then $(tX + c) \cap \mathbb{Z}^d = \emptyset$ for every positive integer $t$. From the argument above and Theorem [3], the leading coefficients of $L_{(F,c)}(t)$ and $L_{(F',c)}(t)$ are different. Hence, we have $L_{(F,c)}(t) \neq L_{(F',c)}(t)$ and $L_{(X,c)}(t) = 0$ for other proper faces $X$. Note that by genericity of $c$, $X$ is either an upper or a lower face.

Next let $c' \in \text{aff}_0(P)$ be constructed by inclining $c$ slightly so that $F$ becomes an upper face (see Figure [2] and $F'$ becomes a lower face and every other face remains in its status (in particular, $c' + \partial P$ does not contain lattice points). Then we have

$$L_{(P,c')} (t) = L_{(P,c)}(t) - L_{(F',c)}(t),$$

$$L_{(P,-c')} (t) = L_{(P,-c)}(t) - L_{(F,-c)}(t).$$

Next let $c'' \in \mathbb{Q}^d$ be obtained by inclining $c$ slightly inward (Figure [2]), such that $F$ becomes a lower face and $F'$ becomes an upper face. Then we have

$$L_{(P,c'')} (t) = L_{(P,c)}(t) - L_{(F,c)}(t),$$

$$L_{(P,-c'')} (t) = L_{(P,-c)}(t) - L_{(F',-c)}(t).$$

Now suppose that both the equations

(a) $L_{(P,c')} (t) = L_{(P,-c')} (t)$ and

(b) $L_{(P,c'')} (t) = L_{(P,-c'')} (t)$

hold. Then [7] and [8] deduce

$$L_{(P,c)}(t) - L_{(P,-c)}(t) = L_{(F',c)}(t) - L_{(F,-c)}(t) = L_{(F,c)}(t) - L_{(F',c)}(t).$$

Recall that the leading coefficients of $L_{(F,\pm c)}(t)$ and $L_{(F',\pm c)}(t)$ are $c_{m-1}$ and $c'_{m-1}$, respectively. Therefore, the leading coefficient of $L_{(F',c)}(t) - L_{(F,-c)}(t)$ is negative, while that of $L_{(F,c)}(t) - L_{(F',-c)}(t)$ is positive. This is a contradiction. 

Proof.
In the proof of the above theorem, we may suppose (a) does not hold, i.e., \( L_{(P,c)}(t) \neq L_{(P,c')}\). Then \( c' \) can be perturbed slightly. That is, there exists a small open neighborhood \( U \subset \text{aff}_0(P) \) of \( c' \) such that every replacement of \( c' \) by an element of \( U \cap \mathbb{Q}^d \) works similarly. Thus we have the following.

**Corollary 4.3.** Let \( P \subset \mathbb{R}^d \) be a non-centrally symmetric lattice polytope. Then there exists an open set \( U \subset \text{aff}_0(P) \) such that \( U \) intersects with any open neighborhood of \( 0 \in \text{aff}_0(P) \) and \( L_{(P,c)} \neq L_{(P,-c)} \) for every translation vector \( c \in U \).

![Figure 2: Translation by \( c, c' \) and \( c'' \).](image)

## 5 Characterizing zonotopes

This section deals with a characterization of lattice zonotopes in a similar way to Theorem 4.2. Specifically, if the Ehrhart quasi-polynomial of every rational shift of a lattice polytope satisfies the GCD-property, then the polytope is a zonotope. Notice that we have already proved this statement for non-centrally symmetric polytopes. In fact, \( \text{GCD}(1, \rho) = \text{GCD}(\rho - 1, \rho) \) for all periods \( \rho \geq 1 \), but by Theorem 4.2 we find a \( c \in \mathbb{Q}^d \) such that \( L_{(P,c)} \neq L_{(P,-c)} \). In order to construct a translation vector for the centrally symmetric polytopes that are no zonotopes, we consider almost constant functions:

**Definition 5.1.** A function \( f: \mathbb{R} \to \mathbb{R} \) is called **almost locally constant**, if it is locally constant except for a discrete set of points. A function \( f: \mathbb{R} \to \mathbb{R} \) is called **almost constant**, if it is constant except for a discrete set of points.

Let \( P \subset \mathbb{R}^d \) be a polytope and \( c \in \mathbb{R}^d \). Consider the function \( L^P_c: \mathbb{R} \to \mathbb{R} \) defined by \( x \mapsto \#((xc + P) \cap \mathbb{Z}^d) \). It is an almost locally constant function. If furthermore \( c \in \mathbb{Q}^d \), then \( L^P_c(t) \) is periodic with a period \( \rho_0 = \text{den}(c) \).

**Proposition 5.2.** Let \( P \subset \mathbb{R}^d \) be a lattice \( d \)-polytope and \( c \in \mathbb{Q}^d \). If \( L^P_c(x) \) is not almost constant, then there exists \( c' \in \mathbb{Q}^d \) such that \( L^{c'}_c(t) \) does not satisfy the GCD-property.

**Proof.** Since \( L^P_{kc}(x/k) = L^P_c(x) \) for \( k \in \mathbb{Z}_{>0} \), we may assume that \( c \in \mathbb{Z}^d \). Consider the function \( \delta(x) := L^P_c(x) - L^P_c(2x) \). Then \( \delta(x) \) is clearly an almost locally constant function, which is 0 near \( x = 0 \). That is, there exists \( \varepsilon > 0 \) such that \( L^P_c(x) - L^P_c(2x) \) is constantly 0 on the interval \((0, \varepsilon)\). We shall prove that \( \delta(x) \) is not almost constant. Suppose the contrary. Then \( \delta(x) = 0 \) except for a discrete set of points. By induction on \( n > 0 \), we have \( L^P_c(x) = L^P_c(2^n x) \) for almost all \( x \in (0, \varepsilon) \). Since \( L^P_c(x) \) is periodic, \( L^P_c(x) \) is almost constant, which contradicts the assumption. Hence \( \delta(x) \)
is not almost constant, and there exists an interval \((a, b) \subset \mathbb{R}\) such that \(\delta(x) \neq 0\) for all \(x \in (a, b)\). Let \(x \in (a, b) \cap \mathbb{Q}\) be a rational number such that \(	ext{den}(xc)\) is odd. Then \(L_{c}(x) \neq L_{c}(2x)\), which is equivalent to \(L_{(x,c)}(1) \neq L_{(2x,c)}(1)\).

Consider the polytope \(xc + P\). Since coordinates of \(xc \in \mathbb{Q}^d\) have odd denominators, the minimal period \(\rho_0\) is an odd integer. Therefore \(\text{GCD}(1, \rho_0) = \text{GCD}(2, \rho_0) = 1\). However, the argument above implies \(L_{(x,c)}(t) \neq L_{(2x,c)}(t)\). Hence the first and the second constituents of the Ehrhart quasi-polynomial \(L_{xc+P}(t)\) are different, and \(L_{xc+P}(t)\) does not have the GCD-property.

\[\square\]

**Example 5.3.** Let \(P_1 = \text{conv}(0,1) \subset \mathbb{R}^2\) and \(c_1 = (\frac{1}{2}, \frac{1}{2}) \in \mathbb{Q}^2\). Note that \(P_1\) is a zonotope. Then,

\[
L_{c_1}(x) = \begin{cases} 
4 & \text{if } x \in 4\mathbb{Z}_{\geq 0} \\
2 & \text{if } x \in 2 + 4\mathbb{Z}_{\geq 0} \\
1 & \text{else}
\end{cases}
\]

is almost constant, as illustrated in Figure 3a.

In contrast to \(P_1\), consider the 3-dimensional cross-polytope \(P_2 = \text{conv}(\pm e_i | i = 1, 2, 3) \subset \mathbb{R}^3\) and \(c_2 = \frac{1}{3} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \in \mathbb{Q}^3\). We observe that \(L_{c_2}(x)\), see Figure 3b, is not almost constant:

\[
L_{c_2}(x) = \begin{cases} 
7 & \text{if } x \in 3\mathbb{Z}_{\geq 0} \\
1 & \text{if } x \in (k, k + 1] \text{ for } k \in 3\mathbb{Z}_{\geq 0} \text{ or } x \in [k - 1, k) \text{ for } k \in 3\mathbb{Z}_{> 0} \\
0 & \text{else}
\end{cases}
\]

Therefore, we can find, for example, \(x = \frac{2}{5}\) for that \(L_{xc+P_2}\) does not satisfy the GCD-property. More explicitly, the Ehrhart quasi-polynomial of the octahedron \(P_2\) translated by \(xc = (\frac{1}{5}, \frac{1}{5}, \frac{1}{5})^t\) is given by

\[
L_{xc+P_2}(n) = \begin{cases} 
\frac{1}{3} n^3 + 2n^2 + \frac{8}{3} n + 1 & \text{if } n \equiv 0 \mod 5 \\
\frac{4}{3} n^3 - \frac{4}{3} n & \text{if } n \equiv 1 \mod 5 \\
\frac{4}{3} n^3 - \frac{4}{3} n & \text{if } n \equiv 2 \mod 5 \\
\frac{4}{3} n^3 - \frac{4}{3} n & \text{if } n \equiv 3 \mod 5 \\
\frac{4}{3} n^3 - \frac{4}{3} n & \text{if } n \equiv 4 \mod 5 
\end{cases}
\]

for which the first and second constituents are different, but \(\text{GCD}(1, 5) = \text{GCD}(2, 5)\).

We will use the following characterization of a zonotope.

**Proposition 5.4.** \([\text{UI}]\) Let \(P \subset \mathbb{R}^d\) be a polytope of dimension \(m \leq d\). Then, \(P\) is a zonotope if and only if all faces of dimension \(j\) are centrally symmetric for some \(2 \leq j \leq m - 2\).

The second main result of this paper is the following.

**Theorem 5.5.** Let \(P \subset \mathbb{R}^d\) be a lattice polytope. Then the following are equivalent

\[(i)\] \(P\) is a zonotope.

\[(ii)\] \(L_{c+P}(t)\) satisfies the GCD-property for every \(c \in \mathbb{Q}^d\).

\[\text{Proof.}\] The implication \((i) \Rightarrow (ii)\) was proved in Theorem 2.3. Now, let \(P \subset \mathbb{R}^d\) be an \(m\)-polytope that is not a zonotope. We aim to prove the existence of \(c \in \mathbb{Q}^d\) such that the Ehrhart quasi-polynomial \(L_{c+P}(t)\) does not satisfy the GCD-property.

We can categorize polytopes that are not zonotopes into three types:
symmetric, be a translation of \(-n\) odd. This process terminates at some of \(G\) let \(c\) be the facet of \(i\) such that \(\pi(c)\) does not separate \(F\), \(F\) and let \(X\) be an integral vector such that \(d\) is centrally symmetric. Let \(X\) \(\not\in\) \(P\) be a non-centrally symmetric polytope. As mentioned earlier, this case has been addressed in Theorem 4.2. Then, the image \(\pi(c)\) of \(c\) gives a Hausdorff distance between \(G\) and \(P\) which itself is a translate of \(G\). Observe that by this construction, \(G\) is a translation of \(G\). In particular, \(G\) is parallel to \(G\). Continue like this and let \(F\) be the facet of \(P\) such that \(G_{i-1} = F_{i-1} \cap F_i\) and let \(G_i\) be the opposite facet of \(G_{i-1}\) in \(F_i\). Observe that by this construction, \(G_i\) is a translation of \(G\) (resp. \(G^0\)), if \(i\) is even (resp. odd). This process terminates at some \(n\) \(\in\) \(N\). Then \(n\) is even. Because, if \(n\) is odd, \(G\) would be a translation of \(-G\), which contradicts the assumption. Furthermore, since \(P\) is centrally symmetric, \(G^0\) is opposite of \(G\) and thus, a translation of \(-G\). Hence, \(\frac{P}{2}\) is odd.

Let \(\pi : \mathbb{R}^d \to \mathbb{R}^2\) be the orthogonal projection to the orthogonal complement \(\text{aff}(G)\perp\) of \(\text{aff}(G)\). Then, the image \(\pi(P)\) is an \(n\)-gon with vertices \(v_1, \ldots, v_n(=v_0)\) and edges \(u_1, \ldots, u_n\). We choose the numbering so that \(G_i = \pi^{-1}(v_i) \cap P\) and \(F_i = \pi^{-1}(u_i) \cap P\).

We claim that \(\text{aff}_0(G)\) is transversal to \(\text{aff}_0(F)\) for all facets \(F \neq F_i, i = 1, \ldots, n\). Since \(F\) is of codimension 1, we have \(\text{aff}_0(G) + \text{aff}_0(F)\) is either \(\text{aff}_0(F)\) or \(\text{aff}_0(G) + \text{aff}_0(F) = \text{aff}_0(P)\). The latter case means that \(\text{aff}_0(G)\) and \(\text{aff}_0(F)\) are transversal. The former case is equivalent to \(\text{aff}_0(G) \subseteq \text{aff}_0(F)\). Then the image \(\pi(F) \subseteq \pi(P)\) is 1-dimensional. However, since \(F\) does not separate \(P\), \(\pi(F)\) does not separate \(\pi(P)\). Thus, \(\pi(F) = u_i\) for some \(i \in \{1, \ldots, n\}\) and \(F = F_i\).

(a) non-centrally symmetric polytopes,
(b) centrally symmetric polytopes with at least one non-centrally symmetric facet, and
(c) centrally symmetric polytopes whose facets are all centrally symmetric.

First, let \(P\) be a non-centrally symmetric polytope. As mentioned earlier, this case has been addressed in Theorem 1.2.

Second, suppose \(P\) is a centrally symmetric polytope with at least one facet \(F\) that is not centrally symmetric. Let \(F \subseteq P\) be a non-symmetric face with its opposite face \(F^0\). By Corollary 1.3, there exists an open subset \(U \subseteq \text{aff}(F)\) such that for every \(c' \in U \cap \mathbb{Q}^d\), there exists \(t \in \mathbb{Z}_{>0}\) such that \(L_{c,F'}(t) \neq L_{c,F''}(t)\). Now choose \(c' \in U \cap \mathbb{Q}^d\) with odd denominators and generic enough so that we may assume that \(c' + \text{aff}(X)\cap \mathbb{Z}^d = \emptyset\) for all faces \(X \neq F, F^0\) of \(P\). Let \(c'' \in \mathbb{Z}^d\) be an integral vector such that \(F\) is an upper face and \(F^0\) is a lower face. Consider \(c = c' + c''\). Then, \(L_{c,F'}(x)\) is not almost constant. Because, at \(x = 1\), the number of lattice points in the upper faces \#((c + tF)\cap \mathbb{Z}^d)\) and that of the lower faces \#((c + tF^0)\cap \mathbb{Z}^d)\) are different. Hence the function \(L_{c,F'}(x)\) takes different values on the interval \(x \in (1 - \varepsilon, 1)\) and \(x \in (1, 1 + \varepsilon)\). Thus, by Proposition 5.2, the Ehrhart quasi-polynomial \(L_{c+P}(t)\) does not satisfy the GCD-property.

Finally, consider a centrally symmetric polytope \(P\) whose facets are also centrally symmetric, but that is not a zonotope. By using Proposition 5.4, there exists a face \(G\) of dimension \((m - 2)\) that is not centrally symmetric. Let \(F_0\) and \(F_1\) be facets of \(P\) such that \(G_0 := G = F_0 \cap F_1\). Let \(G_1\) be the opposite facet of \(G_0\) in \(F_1\). Since \(F_1\) is centrally symmetric, \(G_1\) is a translation of \(G^0\) which itself is a translate of \(-G\). In particular, \(G_1\) is parallel to \(G_0\). Continue like this and let \(F_i\) be the facet of \(P\) such that \(G_{i-1} = F_{i-1} \cap F_i\) and let \(G_i\) be the opposite face of \(G_{i-1}\) in \(F_i\). Observe that by this construction, \(G_i\) is a translation of \(G\) (resp. \(G^0\)), if \(i\) is even (resp. odd). This process terminates at some \(n\) \(\in\) \(N\). Then \(n\) is even. Because, if \(n\) is odd, \(G\) would be a translation of \(-G\), which contradicts the assumption. Furthermore, since \(P\) is centrally symmetric, \(G^0\) is opposite of \(G\) and thus, a translation of \(-G\). Hence, \(\frac{P}{2}\) is odd.

Let \(\pi : \mathbb{R}^d \to \mathbb{R}^2\) be the orthogonal projection to the orthogonal complement \(\text{aff}(G)\perp\) of \(\text{aff}(G)\). Then, the image \(\pi(P)\) is an \(n\)-gon with vertices \(v_1, \ldots, v_n(=v_0)\) and edges \(u_1, \ldots, u_n\). We choose the numbering so that \(G_i = \pi^{-1}(v_i) \cap P\) and \(F_i = \pi^{-1}(u_i) \cap P\).
By Corollary [13], we are able to choose a sufficiently small rational vector \( c \in \text{aff}_0(G) \cap \mathbb{Q}^d \) such that \(#((c + G) \cap \mathbb{Z}^d) \neq #((-c + G) \cap \mathbb{Z}^d)\) and \((c + (\partial P \setminus (F_1 \cup F_2 \cup \ldots \cup F_n))) \cap \mathbb{Z}^d = \emptyset.\) The latter is possible since \( \text{aff}_0(G) \) is transversal to \( \text{aff}_0(F) \) for all facets \( F \neq F_i, \ i = 1, \ldots, n.\)

![Figure 4: projection \((n = 10)\)](image)

Lastly, take a vector \( v \in \text{aff}_0(F_1) \cap \mathbb{Z}^d = \text{aff}_0(F_{\frac{n}{2} + 1}) \cap \mathbb{Z}^d \) such that \( F_2, \ldots, F_{\frac{n}{2}} \) are upper faces, and \( F_{\frac{n}{2} + 2}, \ldots, F_n \) are lower faces with respect to \( v.\) Now, we count the number of lattice points lost and newly obtained by translating \( P \) by \( c' = c + v.\) The opposite face of \( F_i \) is \( F_{i+\frac{n}{2}}, \) where the indices are considered modulo \( n.\) In addition, \( F_{i+\frac{n}{2}} \) is a translation of \(-F\) which itself is a translation of \( F,\) since all facets are symmetric. In order to simplify notation, let \((X)_Z\) denote \( X \cap \mathbb{Z}^d \) for a set \( X \subset \mathbb{R}^d.\) Since \( \frac{n}{2} \) is odd, the number of lattice points on the upper faces of \( P \) with respect to \( c' \) is

\[
\int(c + F_2)_Z + \ldots + \int(c + F_{\frac{n}{2}})_Z + (c + G_1)_Z + \ldots + (c + G_{\frac{n}{2}}) = \sum_{i=2}^{\frac{n}{2}} \int(c + F_i)_Z + \frac{n+2}{4}(-c + G)_Z + \frac{n-2}{4}(c + G)_Z.
\]

Whereas the number of lattice points on the lower faces is

\[
\int(c + F_{\frac{n}{2} + 2})_Z + \ldots + \int(c + F_n)_Z + (c + G_{\frac{n}{2} + 1})_Z + \ldots + (c + G_n) = \sum_{i=2}^{\frac{n}{2}} \int(c + F_{i+\frac{n}{2}})_Z + \frac{n-2}{4}(-c + G)_Z + \frac{n+2}{4}(c + G)_Z.
\]

From \(#(c + G) \cap \mathbb{Z}^d \neq #(-c + G) \cap \mathbb{Z}^d,\) it follows that these numbers are different. Therefore, we have \( \#((xc' + P) \cap \mathbb{Z}^d) \neq \#((x'c' + P) \cap \mathbb{Z}^d)\) for \( x \in (1 - \epsilon, 1), x' \in (1, 1 + \epsilon)\) with \( \epsilon \in \mathbb{R} \) small enough. Thus, \( L_{c'}^{P} \) is not almost constant. By Proposition [5.2], the Ehrhart quasi-polynomial \( L_{c'}^{P}(t) \) does not satisfy the GCD-property. \( \square \)

6 Discussions and further problems

6.1 Minimal periods

In Ehrhart theory, the minimal period of an Ehrhart quasi-polynomial sometimes becomes strictly smaller than the LCM of denominators of vertices. This is known as the period collapse phenomenon. For almost integral polytopes, it is natural to ask the following.

**Problem 6.1.** Let \( P \) be a lattice polytope in \( \mathbb{R}^d \) and \( c \in \mathbb{Q}^d.\) Is the minimal period of \( L_{c+P}(t) \) equal to \( \text{den}(c)? \)
As a partial result, we have the following for zonotopes.

**Proposition 6.2.** Let $P \subset \mathbb{R}^d$ be a lattice zonotope and $c \in \mathbb{Q}^d$.

1. The minimal period of $L_{c+P}(t)$ is $\text{den}(c)$.
2. The inequality
   \[ \#((c + P) \cap \mathbb{Z}^d) \leq \#(P \cap \mathbb{Z}^d) \quad (10) \]
   holds. Furthermore, in (10), the equality holds if and only if $c \in \mathbb{Z}^d$.

**Proof.** Consider the term $W = \emptyset$ in Ardila-Beck-McWhirtner’s formula (Proposition 2.2). Then $\chi_W(t) = 1$ if and only if $tc \in \mathbb{Z}^d$, which is equivalent to $t$ is divisible by $\text{den}(c)$. This yields (1).

(2) is the special case $t = 1$.

One of the difficulties of Problem 6.1 is that Proposition 6.2 (2) does not hold for general polytopes. The next example shows that $c + P$ can contain more lattice points than the original lattice polytope $P$. It seems of interest in its own right.

**Example 6.3.** Let $n > 7$. Define the lattice polytope $P_n$ in $\mathbb{R}^3$ by
\[
\text{conv}\{0, ne_1, ne_2, n(e_1 + e_2), e_3, ne_2 + e_3, (1 - n)e_3\},
\]
where $e_1, e_2, e_3$ are the standard basis of $\mathbb{R}^3$. For $0 < k < n$, let $c_k = \frac{k}{n}e_3$. Then straightforward computation shows
\[
|P_n \cap \mathbb{Z}^3| = \frac{2n^3 + 3n^2 + 19n + 12}{6},
\]
and we have
\[
\alpha(n, k) := |(c_k + P_n) \cap \mathbb{Z}^3| - |P_n \cap \mathbb{Z}^3| = k(n + 1) - k^2 - 2n - 1,
\]
which becomes positive for some $k$, e.g., $k = 3$. If $n$ is odd, $\alpha(n, k)$ attains maximum at $k = \frac{n+1}{2}$ and
\[
\alpha(n, \frac{n+1}{2}) := \frac{n^2 - 6n - 3}{4}. \quad (14)
\]

**Problem 6.4.** For which lattice $d$-polytope $P$ and $c \in \mathbb{Q}^d$ does the inequality
\[
|P \cap \mathbb{Z}^d| < |(c + P) \cap \mathbb{Z}^d| \quad (15)
\]
hold? For a lattice polytope $P$, what is $\max\{(|(c + P) \cap \mathbb{Z}^d| : c \in \mathbb{Q}^d\}$?

**Problem 6.5.** Do there exist certain constants $\kappa, \lambda > 0$ such that the following holds for every $P$ and $c$?
\[
|(c + P) \cap \mathbb{Z}^d| \leq |P \cap \mathbb{Z}^d| + \kappa|P \cap \mathbb{Z}^d|^\lambda \quad (16)
\]
(The formula (14) in Example 6.3 may suggest that $\lambda \geq \frac{d-1}{d}$.)

Let us fix a lattice polytope $P$ in $\mathbb{R}^d$. Then the translation $P + c$ ($c \in \mathbb{Q}^d$) defines infinitely many rational polytopes. It seems natural to ask what kind of polynomials appear as constituents of the Ehrhart quasi-polynomial $L_{P+c}(t)$ of a translation $P + c$ ($c \in \mathbb{Q}^d$). We pose the following.

**Problem 6.6.** Let $P \subset \mathbb{R}^d$ be a lattice polytope. We can define a map (see Theorem 3.2)
\[
\Gamma_P : (\mathbb{R}/\mathbb{Z})^d \to \mathbb{Q}[t] \quad c \mapsto L_{P+c}(t).
\]

What can we say about the map? For example, is the image a finite set? (Note that the finiteness of the image implies that the set of all constituents of the Ehrhart quasi-polynomials of all possible translations $P + c$ is finite.)
6.2 Rational polytopes with the GCD-property and hyperplane arrangements

In our main results, Theorem 4.2 and 5.5, we assumed that $P$ is an almost integral polytope. The authors are uncertain whether these assumptions are necessary or not. Thus we pose the following question.

**Problem 6.7.** Let $P \subset \mathbb{R}^d$ be a rational polytope.

1. If $L_{c+P}(t)$ is symmetric for all $c \in \mathbb{Q}^d$, is $P$ a centrally symmetric (almost integral) polytope?
2. If $L_{c+P}(t)$ satisfies the GCD-property for all $c \in \mathbb{Q}^d$, is $P$ a (almost integral) zonotope?

It is worth noting that almost integral zonotopes are not the only examples of polytopes whose Ehrhart quasi-polynomials satisfy the GCD-property. For instance, some interesting rational polytopes (simplices) have this property. For example, Suter [15] observed that the fundamental alcove (a certain rational simplex) of a root system has an Ehrhart quasi-polynomial with the GCD-property. The following are examples corresponding to the root systems of type $E_6, E_7, E_8, F_4$ and $G_2$.

**Example 6.8.** Let $e_1, \ldots, e_\ell$ be the standard basis of $\mathbb{R}^\ell$.

1. Consider the 6-dimensional rational simplex $P_{E_6} \subset \mathbb{R}^6$ defined by
   \[ P_{E_6} = \text{conv} \left\{ 0, e_2, \frac{1}{2}e_3, \frac{1}{2}e_4, \frac{1}{2}e_5, \frac{1}{3}e_6 \right\}. \] (17)
   The Ehrhart quasi-polynomial $L_{P_{E_6}}(t)$ has the minimal period $\rho = 6$ and satisfies the GCD-property.
2. Consider the 7-dimensional rational simplex $P_{E_7} \subset \mathbb{R}^7$ defined by
   \[ P_{E_7} = \text{conv} \left\{ 0, e_2, \frac{1}{2}e_3, \frac{1}{2}e_4, \frac{1}{3}e_5, \frac{1}{3}e_6, \frac{1}{4}e_7 \right\}. \] (18)
   The Ehrhart quasi-polynomial $L_{P_{E_7}}(t)$ has the minimal period $\rho = 12$ and satisfies the GCD-property.
3. Consider the 8-dimensional rational simplex $P_{E_8} \subset \mathbb{R}^8$ defined by
   \[ P_{E_8} = \text{conv} \left\{ 0, \frac{1}{2}e_1, \frac{1}{3}e_2, \frac{1}{3}e_3, \frac{1}{4}e_4, \frac{1}{4}e_5, \frac{1}{5}e_6, \frac{1}{6}e_7, \frac{1}{6}e_8 \right\}. \] (19)
   The Ehrhart quasi-polynomial $L_{P_{E_8}}(t)$ has the minimal period $\rho = 60$ and satisfies the GCD-property.
4. Consider the 4-dimensional rational simplex $P_{F_4} \subset \mathbb{R}^4$ defined by
   \[ P_{F_4} = \text{conv} \left\{ 0, \frac{1}{2}e_1, \frac{1}{2}e_2, \frac{1}{3}e_3, \frac{1}{4}e_4 \right\}. \] (20)
   The Ehrhart quasi-polynomial $L_{P_{F_4}}(t)$ has the minimal period $\rho = 12$ and satisfies the GCD-property.
5. Consider the 2-dimensional rational simplex $P_{G_2} \subset \mathbb{R}^2$ defined by
   \[ P_{G_2} = \text{conv} \left\{ 0, \frac{1}{2}e_1, \frac{1}{3}e_2 \right\}. \] (21)
   The Ehrhart quasi-polynomial $L_{P_{G_2}}(t)$ has the minimal period $\rho = 6$ and satisfies the GCD-property.
As we observed in Example 1.1 not all rational polytopes possess the GCD-property. It appears to be a relatively rare phenomenon. This naturally leads to the following question.

**Problem 6.9.** Which rational polytopes exhibit Ehrhart quasi-polynomials with the GCD-property (or symmetric quasi-polynomial)?

The notion of the GCD-property for quasi-polynomials was first formulated in the theory of hyperplane arrangements. Let us recall briefly.

Consider non-zero group homomorphisms \( \alpha_i : \mathbb{Z}_\ell \rightarrow \mathbb{Z} \) \((i = 1, \ldots, n)\). These homomorphism lead to the definition of a hyperplane arrangement as the set of hyperplanes defined by \( \text{Ker}(\alpha_i \otimes \mathbb{R} : \mathbb{R}_\ell \rightarrow \mathbb{R})\), \(i = 1, \ldots, n\).

On the other hand, for any positive integer \( q > 0 \), \( \alpha_i \) induces a homomorphism \( \alpha_i \otimes (\mathbb{Z}/q\mathbb{Z}) : (\mathbb{Z}/q\mathbb{Z})^\ell \rightarrow \mathbb{Z}/q\mathbb{Z} \). Kamiya, Takemura and Terao [8] established that the number of points in the “mod \( q \) complement”

\[
\left| (\mathbb{Z}/q\mathbb{Z})^\ell \setminus \bigcup_{i=1}^n \text{Ker}(\alpha_i \otimes (\mathbb{Z}/q\mathbb{Z})) \right|
\]

is a quasi-polynomial in \( q \) with the GCD-property, which is called the characteristic quasi-polynomial of the arrangement.

One of the most important properties is that the first constituent of the characteristic quasi-polynomial is equal to the characteristic polynomial of a hyperplane arrangement [12]. Moreover, the characteristic quasi-polynomial plays a significant role in the context of toric arrangements [4, 9, 16].

It is worth noting that Suter’s computations (Example 6.8) is closely related to characteristic quasi-polynomials. In fact, the Ehrhart quasi-polynomials in Example 6.8 up to scalar multiples, are identical to characteristic quasi-polynomials of corresponding reflection arrangements [2, 17].

Since the relationship between zonotopes and hyperplane arrangements is an actively studied research topic (e.g., see [18, Chap 7]), it would be interesting to explore the following question.

**Problem 6.10.** Are there direct connections between characteristic quasi-polynomials of hyperplane arrangements and Ehrhart quasi-polynomials of almost integral zonotopes? (Note that both are quasi-polynomials with the GCD-property.)

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