MODELLING AND ANALYSIS OF FRACTIONAL-ORDER REGULATED SYSTEMS IN THE STATE SPACE

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Abstract
In this paper we present the mathematical description and analysis of a fractional-order regulated system in the state space. A little historical background of our results in the analysis and synthesis of the fractional-order dynamical regulated systems is given. The methods and results of simulations of the fractional order system described by a state space equation equivalent to three-member fractional-order differential equation with a fractional-order $PD^\delta$ regulator are then presented. The possibility of investigating the stability of such systems is also considered.

Keywords: fractional calculus, fractional-order regulated system, model, state space.

1. Introduction
Since the first references on fractional-order derivatives in the 17-th century, the theory of fractional-order derivatives and integrals was highly developed by many mathematicians. In the last five decades, many authors made a great effort to apply this knowledge in practice. But only in the last decade can we find some significant works concerned with the description, analysis, and synthesis of fractional-order regulated systems.

Real objects are generally fractional-order, however, for many of them the fractionality is very low. But the control systems used so far were all considered as integer-order systems, regardless of the reality. Because of the higher complexity and the absence of adequate mathematical tools, fractional-order dynamical systems were only treated marginally in the theory and practice of control systems, e.g. [1, 2, 3]. In these works the first generalizations of analysis methods for fractional-order control systems were made (s-plane, frequency response, etc.).

At that time our research in this field began. On the basis of fundamental works [1, 2] we made at first an analysis of the fractional dimension of the controlled systems and properties of fractional regulators [3, 4, 5, 6]. Until then we did not make difference between the terms fractal and fractional regulator. Very significant theoretical works appeared in 1993 and later [7, 8, 9, 10, 11, 12, 13, 14]. This and the following results in this field were summarized in book [15]. The fundamentals for practical utilization of numerical methods [16, 17, 18], based on the relation for approximation of the fractional derivatives [19], and analytical methods [20] for fractional-order systems simulation were made. But the proposed iterative numerical methods did not converge to the correct solution. The first explicit noniterative numerical methods for simulation of fractional-order systems were presented in works [21, 22, 23]. In
works [17, 18] the first comparison was made of these noniterative numerical methods and corresponded analytical methods derived for fractional-order control system from analytical solutions published in [12]. An example of experimental design of a fractional-order P$D^\delta$ controller, with comparison of dynamic properties in fractional- and integer-order system with a fractional- and integer-order controller, designed for an integer-order system as the best approximation to given fractional-order system, was presented in works [17, 18] too. It followed from these results, that an application of integer-order regulator to fractional-order system is inadequate and with a change of system or regulator parameters can lead to system instability. This example was taken to many other works [13, 15, 20, 23, 24] etc. After experimental method of the fractional-order system parameters identification [17, 18], two exact methods [21, 22] were derived. A great effort was devoted to elaboration of methods of fractional-order controllers synthesis. From purely experimental methods [17, 18] we continued with methods based on minimization of regulated square [23, 25] etc., and methods based on stability and damping measure [24, 26, 29] etc. In these works the fundamentals of fractional-order P$T^\lambda D^\delta$ controllers synthesis are developed and now first works appear with application of these methods to the control of chaotic fractional-order systems [29, 30]. The development of the methods described in works [25, 24, 26, 29] required the elaboration of methods of fractional-order system analysis in the frequency domain [27, 28] etc. The following Bode analysis was utilized for the fractional-order controllers synthesis and such regulated control systems stability analysis.

This contribution is a continuation of the previous works and deals with the mathematical description and analysis of fractional-order regulated systems in the state space. We present results of simulations of a fractional-order system with the aim to investigate the stability of such system.

2. Example of the fractional-order control system in state space

For the definition of the control system we consider a simple unity feed-back control system illustrated in Fig.1, where $G_s(s)$ denotes the transfer function of the controlled system and $G_r(s)$ is the controller transfer function, both integer- or fractional-order.

![Feed-back control loop](image)

Fig.1: Feed-back control loop

The differential equation of the above closed regulation system for the transfer function of the controlled system $G_s(s) = 1/(a_2 s^\alpha + a_1 s^\beta + a_0)$ and the controller $G_r(s) = K + T_d s^\delta$ has the form

$$a_2 y^{(\alpha)}(t) + a_1 y^{(\beta)}(t) + T_d y^{(\delta)}(t) + (a_0 + K) y(t) = K w(t) + T_d w^{(\delta)}(t)$$

(1)

where $\alpha, \beta, \delta$, are generally real numbers and $a_0, a_1, a_2, K, T_d$ are arbitrary constants.

In works [31, 32, 29] was presented a state space model that expresses the fractional-order derivatives

$$x^{(\alpha)}(t) = A x(t) + B u(t),$$

$$y(t) = C x(t), \quad t \geq 0.$$  

(2)

This description is convenient for simple models of systems [31] with only one fractional-order derivation.

In this contribution we propose state space model of the linear time invariant one dimensional system which expresses the first derivatives in the state space equations (2) and which has the classical state space interpretation for the fractional-order system too. On the right side of these equations we can then transfer more than one fractional-order derivatives of the state space variables ($x^{(fr)}(t)$). A disadvantage of this expression is that we cannot describe the state space equations in vector and matrix relations as in previous description (2).

$$x'(t) = f(x^{(fr)}(t), u(t)),$$

$$y(t) = g(x^{(fr)}(t), u(t)), \quad t \geq 0.$$  

(3)
We verified the above methods on an example from [17, 18]. Assume the system described by differential equation (1) with coefficients:

\[ a_2 = 0.8, a_1 = 0.5, a_0 = 1, \alpha = 2.2, \beta = 0.9, K = 20.5, T_d = 3.7343, \delta = 1.15. \]

After its modification and with state space variables:

\[ x(t) = x_1(t), x'(t) = x_2(t), x''(t) = x_2(t) \]

we can derive the following state space model equivalent to model (1):

\[
\begin{align*}
x_1'(t) &= x_2(t), \\
x_2'(t) &= \frac{a_0 + K}{a_2} x_1^{(2-\alpha)}(t) - \frac{T_d}{a_2} x_1^{(1+\delta-\alpha)}(t) - \frac{a_1}{a_2} x_1^{(1+\beta-\alpha)}(t) - \frac{1}{a_2} w_1^{(2-\alpha)}(t), \\
y(t) &= K x_1(t) + T_d x_2^{(\delta-1)}(t), \quad t \geq 0.
\end{align*}
\] (4)

We can made other alternative state space models for the same system. The order of the integer-order system in the state space model was equivalent to the number of the state variables. In fractional-order systems it does not hold. We propose to consider the order of the fractional-order systems according to the order of the highest derivative (integer- or fractional-order) in the resulting differential equation, e.g. [1].

Under condition \( x'(t) = 0 \) we can obtain from state space model (1) the two equations of statics of this system, which are not algebraic equations as in integer-order linear systems, but differential equations, from which we can compute the coordinates of the equilibrium point, to which the state trajectories tend.

After discretisation of the first derivative by first differentiation in (4) we obtained the simple Euler methods for solving the state space model. For the fractional-order derivatives in (4) we can take the relation from e.g. [9, 10, 15, 17, 18]. In Fig.2 and Fig.3 is a comparison of the unit-step response of the classical numerical solution [17, 18] and the numerical solution of the state space model. The obtained state trajectories represent stable focal point for the above mentioned coefficients of the system (with stability measure \( S_t = -1.5 \) and damping measure \( T_l = 0.37 \), e.g. [27, 28]) and unstable focal point for only one changed coefficient \( T_d = 0.7343 \) of the same system.

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