On uniqueness and existence of conformally compact Einstein metrics with homogeneous conformal infinity

Gang Li

Received: 18 April 2021 / Accepted: 27 December 2021 / Published online: 7 February 2022
© The Author(s), under exclusive licence to Springer-Verlag GmbH Germany, part of Springer Nature 2022

Abstract
In this paper we show that for a generalized Berger metric \( \hat{g} \) on \( S^3 \) close to the round metric, the conformally compact Einstein (CCE) manifold \((M, g)\) with \((S^3, [\hat{g}])\) as its conformal infinity is unique up to isometries. For the high-dimensional case, we show that if \( \hat{g} \) is an SU\((k + 1)\)-invariant metric on \( S^{2k+1} \) for \( k \geq 1 \), the non-positively curved CCE metric on the \((2k + 1)\)-ball \( B_1(0) \) with \((S^{2k+1}, [\hat{g}])\) as its conformal infinity is unique up to isometries. In particular, since in Li (Trans Amer Math Soc 369(6): 4385–4413, 2017), we proved that if the Yamabe constant of the conformal infinity \( Y(S^{2k+1}, [\hat{g}]) \) is close to that of the round sphere then any CCE manifold filled in must be negatively curved and simply connected, therefore if \( \hat{g} \) is an SU\((k + 1)\)-invariant metric on \( S^{2k+1} \) which is close to the round metric, the CCE metric filled in is unique up to isometries. Using the continuity method, we prove an existence result of the non-positively curved CCE metric with prescribed conformal infinity \((S^{2k+1}, [\hat{g}])\) when the metric \( \hat{g} \) is SU\((k + 1)\)-invariant.

Mathematics Subject Classification
Primary 53C25; Secondary 58J05 · 53C30 · 34B15

1 Introduction
This is a continuation of our previous work ([33]) on uniqueness of conformally compact Einstein (CCE) metrics (see Definition 2.1) with prescribed homogeneous conformal infinity. Let \( B_1 \) be the unit ball in the Euclidean space \( \mathbb{R}^{n+1} \) of dimension \( (n + 1) \), with its boundary \( S^n \) the unit sphere. For a given homogeneous metric \( \hat{g} \) on \( S^n \), in this paper we mainly focus on the uniqueness and existence of non-positively curved CCE metrics \( g \) on \( B_1 \) with \((S^n, [\hat{g}])\) as its conformal infinity.
In [22], for any Riemannian metric \( \hat{g} \) on the \( n \)-sphere \( S^n \) which is \( C^{2,\alpha} \) close to the round metric, Graham and Lee proved the seminal existence result that there exists a CCE metric on the \((n+1)\)-ball \( B_1(0) \) with \((S^n, [\hat{g}])\) as its conformal infinity, and the solution is unique in a small neighborhood of the asymptotic solution they constructed in some weighted space by the implicit function theorem. Later Lee [32] generalized this existence result to a small neighborhood of more general CCE manifolds which include the case when they are non-positively curved. Biquard [7] used harmonic analysis on homogeneous spaces on \( S^n \) to give an elementary proof of the perturbation result when \( \hat{g} \) is a homogeneous metric near the round metric. It is interesting to understand whether the solution is globally unique with the prescribed conformal infinity. On the other hand, in light of LeBrun’s local construction in [31], when the conformal infinity is a Berger metric on \( S^3 \) or a generalized Berger metric which is left invariant under the \( SU(2) \) action, Pedersen [40] and Hitchin [28] could fill in a global CCE metric on the 4-ball, which has self-dual Weyl curvature, and the metric is unique under the self-duality assumption. If both the conformal infinity \((S^n, [\hat{g}])\) and the non-local term in the expansion (see Theorem 2.3) of the Einstein metric at infinity are given, Anderson [1] and Biquard [8] proved that the CCE metric is unique up to isometry using unique continuation method at infinity. When \( \hat{g} \) is the round metric, it is proved that the CCE metric filled in must be the hyperbolic space; see [4,18,35,42] (see also [14]). In general, given the conformal infinity, the CCE filling are not necessarily unique; see [27] and [1]. For instance, there are CCE metrics on \( S^2 \times \mathbb{R}^2 \) which are not isometric, with the same conformal infinity \( S^2 \times S^1(\lambda) \) with \( 0 < \lambda < \frac{\sqrt{3}}{2} \).

Based on [44] and [18], in [35] we proved that given a conformal infinity \((S^n, [\hat{g}])\) with its Yamabe constant close to that of the round sphere metric, the CCE filling must be an Hadamard manifold (a simply connected non-positively curved complete non-compact Riemannian manifold). Indeed, we showed that in this case the CCE filling \((M^{n+1}, g)\) is negatively curved, with its sectional curvature close to \(-1\) uniformly; on the other hand, Theorem 3.3 in [46] showed that \( \pi_1 (M) = 0 \) since \( \pi_1 (S^n) = 0 \) and \( Y(S^n, [\hat{g}]) > 0 \) (the same conclusion holds when \( Y(S^n, [\hat{g}]) \geq 0 \); see [12]). So it is natural to consider the uniqueness problem under the assumption that the CCE metric is non-positively curved. Let \((M^4, g)\) be a non-positively curved CCE with \((S^3, [\hat{g}])\) as its conformal infinity, where \( \hat{g} \) is any homogeneous metric (a generalized Berger metric) on \( S^3 \) of the form \((1.1)\). In [33], inspired by X. Wang’s work in [45] (see also [4] and [3]) we deformed the Einstein equation with prescribed conformal infinity into a two-point boundary value problem of a system of ODEs \((2.4)–(2.8)\) of the unknown functions \((y_1, y_2, y_3)\) on \( x \in [0, 1] \), with \((y_2, y_3)\) the conformal components and \(y_1\) the determinant component.

For the Berger metric case, either \( y_2 \) or \( y_3 \) vanishes identically on \( x \in [0, 1] \). For that, in [33] we were able to use the monotonicity of the solutions \( y_i(x) \) on \( x \in [0, 1] \) and an integral version of comparison theorem to show the uniqueness of the solution to the boundary value problem i.e., when \( \hat{g} \) is a Berger metric on \( S^3 \), up to isometry there exists at most one CCE metric on \( B_1 \) which is non-positively curved with \((S^3, [\hat{g}])\) its conformal infinity, and hence it is the metric constructed by Pedersen in [40]. Moreover, by [35], for the Berger metric \( \hat{g} \) in \((S^3, [\hat{g}])\) close enough to the round sphere metric, the CCE manifold filled in is automatically simply connected and non-positively curved, and therefore it is unique up to isometry, which is Pedersen’s metric in [40] and also Graham-Lee’s metric in [22].
1.1 Uniqueness results

For a generalized Berger metric $\hat{g}$ on $\mathbb{S}^3$, by the interaction of $y_1(x)$, $y_2(x)$ and $y_3(x)$ in the system (2.4)–(2.7), the integral comparison argument fails to give uniqueness of the solution to this boundary value problem. In this paper, we consider the uniqueness of the solution in this case by a contradiction argument. Based on a monotonicity argument and an a priori estimate of the solution, we consider the total variation of the difference of two solutions and show that

Theorem 1.1 Let $(\mathbb{S}^3, \hat{g})$ be a generalized Berger sphere with the standard form

$$\hat{g} = \lambda_1 \sigma_1^2 + \lambda_2 \sigma_2^2 + \lambda_3 \sigma_3^2,$$

with $\sigma_1$, $\sigma_2$ and $\sigma_3$ three $SU(2)$-invariant 1-forms, such that $\lambda_1$, $\lambda_2$ and $\lambda_3$ differ from one another and without loss of generality we assume $\lambda_1 > \max\{\lambda_2, \lambda_3\}$. Assume also that $\phi_1(0) = \frac{\lambda_1}{2}$ and $\phi_2(0) = \frac{\lambda_2}{2}$ satisfy the inequality $1 < \phi_1(0) + \phi_1(0)\phi_2(0)$. Then there exists some constant $0 < \eta_0 < 1 - 3 \times 10^{-8}$, such that if $\eta_0 < \phi_1(0) < 1$ and $\eta_0 < \phi_2(0)$, then the conformally compact Einstein filling $(M^4, g)$ which is an Hadamard manifold, with $(\mathbb{S}^3, \hat{g})$ as its conformal infinity, must be unique up to isometry. Therefore, it coincides with Hitchin’s metric in [28], and for $\eta_0$ sufficiently close to 1 it is also identical to Graham-Lee’s metric in [22].

The estimates of the solution are based on the monotonicity of the solutions and an integral argument by the choice of different types of integrating factors for different equations, on different parts of the interval $x \in [0, 1]$, due to the degeneration of the elliptic equations. We should remark that the upper bound of the norm of Weyl tensor $|W|_g$ is used in the estimate of the solutions near $x = 1$. It turns out that for the difference of any given two solutions $(z_1, z_2, z_3) = (y_{11}, y_{12}, y_{13}) - (y_{21}, y_{22}, y_{23})$, the total variation of $z_2$ and $z_3$ are well controlled only on some special intervals of monotonicity on $x \in [0, 1]$, by a certain type of integration of the equation, which is enough for our argument. If we could have an estimate that $\sup_M |W|_g$ is small for the given conformal infinity, then the lower bound $\eta_0 > 0$ in Theorem 1.1 could be much smaller, by the estimates in the proof. On the other hand, by continuity of the Yamabe constant at a positive scalar curvature metric, for $0 < \eta_0 < 1$ sufficiently close to 1, $Y(\mathbb{S}^3, [\hat{g}])$ in Theorem 1.1 is sufficiently close to that of the round metric, and hence by Theorem 2.2, $(M^4, g)$ must be an Hadamard manifold and the quantity $\sup_M |W|_g$ is sufficiently small. Therefore, as a corollary of Theorem 1.1, we have

Theorem 1.2 Let $(\mathbb{S}^3, \hat{g})$ be a generalized Berger sphere with $\hat{g}$ satisfying (1.1) such that $\lambda_1$, $\lambda_2$ and $\lambda_3$ differ from one another and $\lambda_1 > \max\{\lambda_2, \lambda_3\}$. Assume that $\phi_1(0) = \frac{\lambda_2}{2}$ and $\phi_2(0) = \frac{\lambda_3}{2}$. Then there exists some constant $0 < \eta_0 < 1$ sufficiently close to 1, such that if $\eta_0 < \phi_1(0) < 1$ and $\eta_0 < \phi_2(0)$, then the conformally compact Einstein manifold $(M^4, g)$ with $(\mathbb{S}^3, [\hat{g}])$ as its conformal infinity is unique up to isometry, and hence it must be isometric to the (anti-)self-dual metric constructed in [28] and also the metric obtained from perturbation in [22].

The uniqueness result can be generalized to high dimensions. Homogeneous metrics on $\mathbb{S}^n$ have been classified by D. Montgomery and H. Samelson ([38]), and A. Borel ([9,10]); see [5] (p.179) and also [48] for instance. Up to a scaling factor and isometry, homogeneous metrics on spheres are in one of the three classes: a one parameter family of $SU(k+1)$-invariant metrics on $\mathbb{S}^{2k+1} \cong SU(k+1)/SU(k)$ ($k \geq 1$), a three parameter family of $Sp(k+1)$-invariant metrics on $\mathbb{S}^{4k+3} \cong Sp(k+1)/Sp(k)$ (containing the SU(2k+1)-invariant metrics as a subset
in these dimensions) with \( k \geq 0 \), and a one parameter family of Spin(9)-invariant metrics on \( S^{15} \cong \text{Spin}(9)/\text{Spin}(7) \). For more details, see Sect. 4, and also [48]. As the Berger metric case, for the first two classes of homogeneous metric \( \hat{g} \), the prescribed conformal infinity problem of the CCE metrics is deformed into a two-point boundary value problem of a system of ODEs on the interval \( x \in [0, 1] \); see (4.19)–(4.22) (with two functions \( (y_1(x), y_2(x)) \)) and (4.25)–(4.30) (with four functions \( (y_1(x), ..., y_4(x)) \)) correspondingly. The third class could be done similarly. Using the same approach of the uniqueness argument for the Berger metric case in [33] based on a monotonicity argument and an integral version comparison theorem, we show that

**Theorem 1.3** Let \( \hat{g} \) be a homogeneous metric on \( S^n \cong SU(k+1)/SU(k) \) with \( n = 2k+1 \) for \( k \geq 1 \) so that \( \hat{g} \) has the standard diagonal form

\[
\hat{g} = \lambda_1 \sigma_1^2 + \lambda_2 (\sigma_2^2 + \cdots + \sigma_n^2),
\]

(1.2)

at a point where \( \lambda_1 \) and \( \lambda_2 \) are two positive constants and \( \sigma_1, ..., \sigma_n \) are the 1-forms with respect to the basis vectors in \( p \), in the Ad\(_{SU(k)}\)-invariant splitting \( su(k+1) = su(k) \oplus p \).

Assume that \( \frac{1}{n+1} < \frac{\lambda_1}{\lambda_2} < n+1 \), then up to isometry there exists at most one non-positively curved conformally compact Einstein metric on the \((n+1)\)-ball \( B_1(0) \) with \((S^n, [\hat{g}])\) as its conformal infinity. In particular, it coincides with the metric obtained from perturbation in [22] and the metric constructed in [36] when \( \frac{\lambda_1}{\lambda_2} \) is close to 1. Moreover, by Theorem 2.2, for \( \frac{\lambda_1}{\lambda_2} \) sufficiently close to 1, any conformally compact Einstein manifold filling is automatically negatively curved and simply connected, and therefore it is unique up to isometry.

Notice that the assumption in Theorem 1.3 that the CCE filling is a Hadamard manifold does not imply \( Y(S^n, [\hat{g}]) > 0 \), and hence the inequality (4.33) does not always give a positive lower bound estimates of the volume growth. Instead, the assumption \( \phi(0) > \frac{1}{n+1} \) combined with the inequality (4.35) derived from the first order equation (4.34) gives a positive lower bound of \( K(0) = \lambda_1 \lambda_2^{n-1} \) and hence a lower bound of the volume growth.

**1.2 Existence results**

Recall that for given real analytic data: the conformal infinity \((\partial M, [\hat{g}])\) and the non-local term in the expansion in Theorem 2.3, existence of CCE metrics in a neighborhood of the boundary \( \partial M \) is proved by Fefferman and Graham in [20] for \( \partial M \) of odd dimension and by Kichenassamy in [30] for even dimensional boundary, and for \( C^\infty \) data at conformal infinity; see Gursky and Székelyhidi [26]. Anderson [2] studied the existence of CCE metrics on \( B^4_1 \) with general prescribed conformal infinity \((S^3, [\hat{g}])\) using the continuity method. Recently, Gursky and Han [24] showed that there are infinitely many Riemannian metrics \( \hat{g} \) on \( S^7 \) lying in different connected components of the set of positive scalar curvature metrics such that there exists no CCE metrics on the unit Euclidean ball \( B^8 \) with \((S^7, [\hat{g}])\) as its conformal infinity and with Stolz they showed in [25] that a similar phenomena holds for some more general Spin manifolds in high dimensions.

By [33], the CCE manifold which is Hadamard with homogeneous conformal infinity, is of cohomogeneity one. Calculations of the curvature tensors on manifolds of cohomogeneity one can be found in [23]. Recently, using Schauder degree theory, Buttsworth [11] showed that for two \( G \)-invariant Riemannian metrics \( \hat{g}_1 \) and \( \hat{g}_2 \) on a compact homogeneous space \( G/H \), if the isotropy representation of \( G/H \) consists of pairwise inequivalent irreducible summands, then there exists an Einstein metric on \( G/H \times [0, 1] \) such that when restricted on \( G/H \times [0, 1] \) and \( G/H \times \{1\} \), \( g \) coincides with \( \hat{g}_1 \) and \( \hat{g}_2 \) respectively.
In Sect. 4, we prove a compactness result of a sequence of non-positively curved CCE metrics with their conformal infinity \((S^n, [\hat{g}_j]) (j \geq 1)\), where \(\hat{g}_j\) is of the form \((1.2)\) and uniformly bounded. Using this compactness result and Graham-Lee and Lee’s perturbation result in \([22]\) and \([32]\), we show the following existence theorem by the continuity method.

**Theorem 1.4** Let \(B_1 \subseteq \mathbb{R}^{n+1}\) be the unit ball on the Euclidean space with the boundary \(S^n\). Assume that \(n = 2k + 1\) for some integer \(k \geq 1\). Let \(\hat{g}^\lambda\) be a homogeneous metric on the boundary \(S^n \cong SU(k + 1)/SU(k)\) so that \(\hat{g}^\lambda\) has the standard diagonal form

\[
\hat{g}^\lambda = \sigma_1^2 + \lambda(\sigma_2^2 + \cdots + \sigma_n^2),
\]

at a point where \(\lambda\) is a positive constant and \(\sigma_1, \ldots, \sigma_n\) are the 1-forms with respect to the basis vectors in \(p\) in the \(Ad_{SU(k)}\)-invariant splitting \(su(k + 1) = su(k) \oplus p\). Then as the parameter \(\lambda\) varies from \(\lambda = 1\) continuously on the interval \((\frac{1}{n+1}, 1)\) (resp. on the interval \([1, n + 1]\)) one of the following holds:

- there exists a conformally compact Einstein metric on \(B_1\) which is non-positively curved with \((S^n, [\hat{g}^\lambda])\) as its conformal infinity for each \(\lambda \in (\frac{1}{n+1}, 1)\) (resp. \(\lambda \in [1, n + 1]\));
- there exists \(\lambda_1 \in (\frac{1}{n+1}, 1)\) (resp. \(\lambda_1 \in [1, n + 1]\), such that for each \(\lambda \in [\lambda_1, 1]\) (resp. \(\lambda \in [1, \lambda_1]\)) there exists a conformally compact Einstein metric \(g^\lambda\) on \(B_1\) which is non-positively curved with \((S^n, [\hat{g}^\lambda])\) as its conformal infinity and there exists \(\rho \in B_1\) such that the sectional curvature of \(g^\lambda_1\) is zero in some direction at \(p\) and moreover, for any \(\epsilon > 0\) small there exists \(\lambda_2 \in (\lambda_1 - \epsilon, \lambda_1)\) (resp. \(\lambda_2 \in (\lambda_1, \lambda_1 + \epsilon)\)) such that there exists a conformally compact Einstein metric \(g^\lambda_2\) on \(B_1\) with \((S^n, [\hat{g}^\lambda_2])\) as its conformal infinity and the sectional curvature of \(g^\lambda_2\) is positive in some direction at some point \(p \in B_1\).

This existence result can be viewed as a generalization of Pedersen’s result in [40] to higher dimensions. For \(n = 3\), the constant \(\lambda_1\) in Theorem 1.4 was calculated explicitly in [16] by Pedersen’s explicit solutions. Indeed, in the Euclidean unit ball \(B_1 \subseteq \mathbb{R}^4\), Pedersen’s metric is defined as

\[
g_m = \frac{1}{(1 - \rho^2)^2} \left( \frac{1 + m^2 \rho^2}{1 + m^2 \rho^4} d\rho^2 + \rho^2(1 + m^2 \rho^2)(\sigma_2^2 + \sigma_3^2) + \frac{\rho^2(1 + m^2 \rho^4)}{1 + m^2 \rho^2} \sigma_1^2 \right),
\]

where the boundary is the sphere at \(\rho = 1\) and \(\sigma_1, \sigma_2, \sigma_3\) are three left-invariant 1-forms on \(S^3\) satisfying \(d\sigma_i = \sum_{j,k} \epsilon_{ijk} \sigma_j \wedge \sigma_k\). \(g_m\) has non-positive curvature iff \(0 \leq m^2 \leq 1\) (which is \(1 \leq \lambda \leq 2\)) and hence \(\lambda_1 = 2\). Similarly when \(-1 < m^2 < 0\), it can be calculated that \(g_m\) is negatively curved for \(0 < \lambda \leq 1\). Notice that the conformal infinity of \(g_m\) satisfies that \(Y(S^3, [\hat{g}_m]) < 0\) for \(-1 < m^2 < -\frac{3}{4}\). Also, a similar existence result holds for the generalized Berger metric case (which should be the Hitchin’s metric for \(\hat{g}\) close to the round sphere by uniqueness), since we have the same compactness result for it.

Based on the technique developed in this paper, similar uniqueness and existence results of the conformally compact Einstein metrics are established in [34] for the prescribed conformal infinity \((S^n, [\hat{g}])\) of general dimension \(n\), where \(\hat{g}\) is a homogeneous metric.

In Sect. 3, we first show that each function \(y_j(x)\) in the solution \((y_1, y_2, y_3)\) to the boundary value problem \((2.4) - (2.8)\) is monotonic on \([0, 1]\); see Lemma 3.1. Based on this and the elliptic system, we obtain the interior estimate of the solution in \(x \in (0, 1)\). Then we employ certain integrating factors to deal with the degeneration of the equations at \(x = 0\) and obtain an a priori estimate of the solutions on \(x \in [0, \frac{3}{4}]\); see Lemma 3.2. That fails to work near \(x = 1\). We have to use the boundedness of the Weyl tensor for the non-positively curved Einstein metric and the Einstein equation to give an estimate of the solution near \(x = 1\); see
Lemma 3.3. Then we go to the proof of Theorem 1.1 by a contradiction argument. Assume we have two solutions \((y_{11}, y_{12}, y_{13})\) and \((y_{21}, y_{22}, y_{31})\), we show that the total variation of each function \(z_i\) in the difference \((z_1, z_2, z_3)\) of the two solutions vanishes.

In Sect. 4, for conformal infinity \((\mathbb{S}^n, [\hat{g}])\) of general dimensions, with \(\hat{g}\) a homogeneous metric, we use the symmetry extension to reduce the Einstein equation with the prescribed conformal infinity to a two-point boundary value problem of a system of ODEs on \(x \in [0, 1]\); see (4.19)–(4.22) when \(\hat{g}\) is SU\((k + 1)\)-invariant and (4.25)–(4.30) when \(\hat{g}\) is Sp\((k + 1)\)-invariant. Then by the same argument as the Berger metric case in [33], we prove Theorem 1.3. Based on the monotonicity of the solutions proved in Lemma 4.1, we give a uniform C\(^3\) estimate of the solution to the boundary value problem (4.19)–(4.22) on \(x \in [0, \frac{3}{4}]\) in Lemma 4.4. Then combining that with the interior estimate of the Einstein metric based on the uniform bound of the Weyl tensor, we get a compactness result of the non-positively curved conformally compact Einstein metrics with uniformly bounded conformal infinity. Hence with the aid of the perturbation result in [22] and [32], we are able to use the continuity method to prove the existence result in Theorem 1.4. Notice that the global compactness estimate on CCE metrics is difficult to obtain for general conformal infinity; see [2] and [13].

Recently, in [36], based on the method of Page and Pope, the author constructed the CCE metrics explicitly on the unit ball for SU\((n)\) invariant conformal classes on \(S^{2n-1}\). Here the proof of Theorem 1.4 shows the well-posedness of (4.19)–(4.22) and has its own interests. In [13], based on a compactness result for certain conformal compactified metrics of the CCE metrics when the conformal infinity is sufficiently close to that of the round metric, the authors proved the uniqueness of the CCE metrics for the conformal infinity on \(S^3\) close enough to that of the round metric. Our method here shows the possibility that once the Weyl tensor is not large pointwisely, the uniqueness result still holds.

\section{2 Preliminaries}

\textbf{Definition 2.1} Suppose \(M\) is the interior of a smooth compact manifold \(\overline{M}\) of dimension \(n + 1\) with boundary \(\partial M\). A defining function \(x\) on \(\overline{M}\) is a smooth function \(x\) on \(\overline{M}\) such that

\[ x > 0 \text{ in } M, \quad x = 0 \text{ and } dx \neq 0 \text{ on } \partial M. \]

A complete Riemannian metric \(g\) on \(M\) is said to be conformally compact if there exists a defining function \(x\) such that \(x^2g\) extends by continuity to a Riemannian metric (of class at least \(C^0\)) on \(\overline{M}\). The rescaled metric \(\tilde{g} = x^2g\) is called a conformal compactification of \(g\). If for some smooth defining function \(x\), \(\tilde{g}\) is in \(C^k(\overline{M})\) or the Holder space \(C^{k,\alpha}(\overline{M})\), we say \(g\) is conformally compact of class \(C^k\) or \(C^{k,\alpha}\). Moreover, if \(g\) is also Einstein:

\[ \text{Ric} \tilde{g} = -ng, \quad (2.1) \]

we call \(g\) a conformally compact Einstein (CCE) metric. Also, for the restricted metric \(\hat{g} = \tilde{g}\big|_{\partial M}\), the conformal class \((\partial M, [\hat{g}]\)) is called the conformal infinity of \((M, g)\). A defining function \(x\) is called a geodesic defining function about \(\hat{g}\) if \(\hat{g} = \tilde{g}\big|_{\partial M}\) and \(|dx|_{\tilde{g}} = 1\) in a neighborhood of the boundary.

Let \((M^{n+1}, g)\) be an Hadamard manifold, and we also assume that \((M, g)\) is a CCE manifold with its conformal infinity \((\partial M, [\hat{g}]\)). By the non-positivity of the sectional curvature of \(g\) and (2.1), we have \(-n \leq K \leq 0\) for any sectional curvature \(K\) and \(|W|_g \leq \sqrt{(n^2 - 1)n}\) at any point \(p \in M\). It is shown in [33] that if \((M, g)\) is not the hyperbolic space, there exists a unique point \(p_0 \in M\) which is the center of the unique closed geodesic ball of the
smallest radius that contains the set \( S = \{ p \in M \mid |W|_g(p) = \sup_M |W|_g \} \). We call \( p_0 \) the (spherical) center of gravity of \((M, g)\). Each conformal Killing vector field \( Y \) on \((\partial M, [\hat{g}] )\) can be extended continuously to a Killing vector field \( X \) on \((M, g)\), with each geodesic sphere centered at \( p_0 \) an invariant subset of the action generated by \( X \) (see [33]). Let \( r \) be the distance function to \( p_0 \) on \((M, g)\). Then \( g \) has the orthogonal splitting

\[
g = dr^2 + g_r, \tag{2.2}
\]

with \( g_r \) the restriction of \( g \) on the \( r \)-geodesic sphere centered at \( p_0 \). If moreover, \((\partial M, \hat{g})\) is a homogeneous space, then the function \( x = e^{-r} \) is a geodesic defining function about \( C \hat{g} \) with \( C > 0 \) some constant. That is to say, \( \hat{g} = C \lim_{x \to 0} (x^2 g_r) \). We also have the form

\[
g = dr^2 + \sinh^2(r)\tilde{h} = x^{-2} \left( dx^2 + \frac{(1-x^2)^2}{4} \tilde{h} \right), \tag{2.3}
\]

for \( 0 \leq x \leq 1 \). Let \((r, \theta)\) be the polar coordinates centered at \( p_0 \).

Let \( \hat{g} \) be a generalized Berger metric on \( S^3 \) of the standard form

\[
\hat{g} = \lambda_1 \sigma_1^2 + \lambda_2 d\sigma_2^2 + \lambda_3 d\sigma_3^2,
\]

with \( \sigma_1, \sigma_2 \) and \( \sigma_3 \) three \( \text{SU}(2) \)-invariant 1-forms, where \( \lambda_1, \lambda_2 \) and \( \lambda_3 \) differ from one another. It is shown in [33] that the metrics \( \tilde{h} \) in (2.3) on the geodesic spheres have the diagonal form

\[
\tilde{h} = I_1(x) d(\theta^1)^2 + I_2(x) d(\theta^2)^2 + I_3(x) d(\theta^3)^2,
\]

at a point \((x, \theta_0)\) under the coordinates \((x, \theta) = (x, \theta^1, \theta^2, \theta^3)\) such that \( d\theta^i = \sigma_i \) (with \( \sigma_i \) an \( \text{SU}(2) \)-invariant 1-form satisfying \( d\sigma_i = \sum_{j, k} \varepsilon_{ijk} \sigma_j \wedge \sigma_k \)) at \( \theta = \theta_0 \) for \( 0 \leq x \leq 1 \), with some positive functions \( I_i \in C^\infty([0, 1]) \) satisfying \( I_i(1) = 1, i = 1, 2, 3 \).

Denote \( K = I_1 I_2 I_3, \phi_1 = \frac{I_2}{I_1}, \phi_2 = \frac{I_3}{I_2}, y_1 = \log(K), y_2 = \log(\phi_1) \) and \( y_3 = \log(\phi_2) \), and hence

\[
I_1 = (K \phi_1^{-2} \phi_2^{-1})^{\frac{1}{3}}, \quad I_2 = (K \phi_1 \phi_2^{-1})^{\frac{1}{3}}, \quad I_3 = (K \phi_1 \phi_2^{-2})^{\frac{1}{3}}.
\]

It was shown in [33] that for a generalized Berger metric \( \hat{g} \) on \( S^3 \) with \( \lambda_1, \lambda_2, \lambda_3 \) different from one another, the Einstein equation (2.1) with the prescribed conformal infinity (\( S^3, [\hat{g}] \)) is equivalent to

\[
y_1'' - x^{-1}(1 + 3x^2)(1 - x^2)^{-1}y_1' + \frac{1}{6} (y_1')^2 + \frac{1}{3} (y_2')^2 + y_2'y_3 + (y_3')^2 = 0, \tag{2.4}
\]

\[
y_2'' - x^{-1}(5 + 7x^2)(1 - x^2)^{-1}y_1' + \frac{1}{2} (y_2')^2 + 16(1 - x^2)^{-2}(3 - 2K^{-\frac{1}{3}} (\phi_1' \phi_2)^{-\frac{1}{3}} - 2K^{-\frac{1}{3}} (\phi_1^{-1} \phi_2)^{-\frac{1}{3}}
\]

\[
- 2K^{-\frac{1}{3}} (\phi_1 \phi_2)^{-\frac{1}{3}} + K^{-\frac{1}{3}} \phi_1^{-\frac{1}{3}} \phi_2^{-\frac{2}{3}} + K^{-\frac{1}{3}} \phi_1^{-2} \phi_2^{-\frac{1}{3}} + K^{-\frac{1}{3}} \phi_1^{-2} \phi_2^{-2} \right) = 0, \tag{2.5}
\]

\[
y_3'' - 2x^{-1}(1 + 2x^2)(1 - x^2)^{-1}y_2' + \frac{1}{2} y_1'y_2' + 32(1 - x^2)^{-2} K^{-\frac{1}{3}} \left[ \phi_1^{-\frac{1}{3}} \phi_2^{-\frac{1}{3}} - \phi_1^{-\frac{1}{3}} \phi_2^{-\frac{2}{3}} \right] = 0, \tag{2.6}
\]

\[
y_3'' - 2x^{-1}(1 + 2x^2)(1 - x^2)^{-1}y_3' + \frac{1}{2} y_1'y_3' + 32(1 - x^2)^{-2} K^{-\frac{1}{3}} \left[ \phi_1^{-\frac{1}{3}} \phi_2^{-\frac{1}{3}} - \phi_1^{-\frac{1}{3}} \phi_2^{-\frac{2}{3}} \right]
\]

\[
- \phi_1^{\frac{2}{3}} \phi_2^{-\frac{2}{3}} + \phi_1^{-\frac{2}{3}} \phi_2^{\frac{2}{3}} \right) = 0, \tag{2.7}
\]

for \( y_i(x) \in C^\infty([0, 1]) \) for \( i = 1, 2, 3 \) with the boundary condition

\[
\]
\[ \phi_1(0) = \frac{\lambda_2}{\lambda}, \quad \phi_2(0) = \frac{\lambda_3}{\lambda_2}, \quad K(1) = \phi_1(1) = \phi_2(1) = 1, \quad y_i(0) = y_i'(1) = 0, \quad \text{for } i = 1, 2, 3. \] (2.8)

Combining (2.4) and (2.5) we have

\[ (y_i')^2 - [(y_2')^2 + y_2'y_3' + (y_3')^2] - 12x^{-1}(1 + x^2)(1 - x^{-2})^{-1}y_1' + 48(1 - x^2)^{-2}[3 - 2K^{-\frac{1}{4}}(\phi_2^2)\frac{1}{4} - 2K^{-\frac{1}{4}}(\phi_1\phi_2^{\frac{1}{4}}) - 2K^{-\frac{1}{4}}(\phi_2^3)\frac{1}{4} + K^{-\frac{1}{4}}\phi_1^{-\frac{1}{2}}\phi_2^{-\frac{3}{2}} + K^{-\frac{1}{4}}\phi_1^\frac{1}{2}\phi_2^{-\frac{3}{2}} + K^{-\frac{1}{4}}\phi_1^\frac{3}{2}\phi_2^\frac{1}{2}] = 0, \]

(2.9)

Recall that any three equations in the system of five equations (2.4)–(2.7) and (2.9) containing at least one of (2.6) and (2.7), combining with the initial data imply the other two equations.

The idea of symmetry extension approach in [33] is inspired by [45]; see also [4] and [2]. For \( n = 3 \), assume \((M^4, g)\) is a CCE manifold with its conformal infinity \((\partial M, [\hat{g}])\). If we replace the non-positive sectional curvature condition by the condition that there exists a constant \( \frac{1}{2} < \lambda \leq 1 \) such that

\[ Y(\partial M, [\hat{g}]) \geq \lambda^2 Y(S^3, [g_0]), \]

with \( g_0 \) the round sphere metric, then the upper bound estimate

\[ \sup_M |W|_g \leq T \] (2.10)

still holds with some constant \( T \) depending on \( \lambda \); see Corollary 1.7 in [35]. The proof there is based on the control of the lower bound of the relative volume growth of geodesic balls by the Yamabe constant of the conformal infinity and a blowing up argument relating to the relative volume growth. Moreover, we have the following curvature pinching estimates:

**Theorem 2.2** (Theorem 1.6, [35]) For any \( \epsilon > 0 \), there exists \( \delta = \delta(n, \epsilon) > 0 \) such that for any conformally compact Einstein manifold \((M^{n+1}, g) \) \( (n \geq 3) \), one gets

\[ |K[g] + 1| \leq \epsilon, \] (2.11)

for all sectional curvature \( K \) of \( g \), provided that

\[ Y(\partial M, [\hat{g}]) \geq (1 - \delta)Y(S^n, [g_0]). \]

Particularly, any conformally compact Einstein manifold with its conformal infinity of Yamabe constant sufficiently close to that of the round sphere is necessarily negatively curved, and also \((M, g)\) is simply connected provided that \( \partial M \) is simply connected.

Recall that for any smooth metric \( h \in [\hat{g}] \) at the conformal infinity, there exists a unique geodesic defining function \( x \) about \( h \) in a neighborhood of \( \partial M \); see [21]. For a CCE metric of \( C^2 \), based on [21], in [15] the authors proved the following regularity result.

**Theorem 2.3** Assume \( \overline{M} \) is a smooth compact manifold of dimension \( n + 1, \ n \geq 3, \) with \( M \) its interior and \( \partial M \) its boundary. If \( g \) is a conformally compact Einstein metric of class \( C^2 \) on \( M \) with conformal infinity \((\partial M, [\gamma])\), and \( \hat{g} \in [\gamma] \) is a smooth metric on \( \partial M \). Then there exists a smooth coordinates cover of \( \overline{M} \) and a smooth geodesic defining \( x \) corresponding to \( \hat{g} \). Under this smooth coordinates cover, the conformal compactification \( \tilde{g} = x^2g \) is smooth up to the boundary for \( n \) odd and has the expansion

\[ \tilde{g} = dx^2 + g_x = dx^2 + \hat{g} + x^2g^{(2)} + \text{(even powers)} + x^{n-1}g^{(n-1)} + xng^{(n)} + \cdots \] (2.12)
with \( g^{(k)} \) smooth symmetric \((0, 2)\)-tensors on \( \partial M \) such that for \( 2k < n \), \( g^{(2k)} \) can be calculated explicitly and inductively using the Einstein equation and \( g^{(n)} \) is a smooth trace-free nonlocal term; while for \( n \) even, \( \bar{g} \) is of class \( C^{n-1} \), and more precisely it is polyhomogeneous and has the expansion
\[
\bar{g} = dx^2 + g_x = dx^2 + \bar{g} + x^2 g^{(2)} + \text{(even powers)} + x^n \log(x) \bar{g} + x^n g^{(n)} + \cdots
\]
with \( \bar{g} \) and \( g^{(k)} \) smooth symmetric \((0, 2)\)-tensors on \( \partial M \), such that for \( 2k < n \), \( g^{(2k)} \) and \( \bar{g} \) can be calculated explicitly and inductively using the Einstein equation, \( \bar{g} \) is trace-free and \( g^{(n)} \) is a smooth nonlocal term with its trace locally determined.

3 Uniqueness of non-positively curved conformally compact Einstein metrics with generalized Berger sphere conformal infinity

The main goal of this section is to prove Theorem 1.1. Recall that for a given Berger metric \( \hat{g} \) on \( S^3 \), we have proved the uniqueness of the non-positively curved CCE metrics \( g \) with \((S^3, [\hat{g}])\) as its conformal infinity ([33]). In this section, for a given generalized Berger metric \( \hat{g} \) on \( S^3 \), we study the uniqueness of the non-positively curved CCE metrics \( g \) satisfying \((2.1)\) with \((S^3, [\hat{g}])\) as its conformal infinity. By [33], it is equivalent to show the uniqueness of the solution to the boundary value problem \((2.4)-(2.8)\).

We start with the monotonicity of \( y_i(x) \) \( (i = 1, 2, 3) \) for a global solution \((y_1, y_2, y_3)\) on \([0, 1]\). If either \( \phi_1(0) = 1 \), or \( \phi_2(0) = 1 \), or \( \phi_1(0) \phi_2(0) = 1 \), then the metric \( \hat{g} \) is a Berger metric, and the uniqueness of the solution to the boundary value problem has been proved in [33]. In this section we consider the generalized Berger metric \( \hat{g} \) with \( \phi_1(0) \neq 1 \), \( \phi_2(0) \neq 1 \) and \( \phi_1(0) \phi_2(0) \neq 0 \). By the volume comparison theorem,
\[
K(0) = \lim_{x \to 0} \frac{\det(h)}{\det(h_{s^4}^*)} = \lim_{r \to +\infty} \frac{\det(g_r)}{\det(g_r^{s^4})} < 1,
\]
where
\[
g_{s^4} = dr^2 + g_{s^4}(r) = x^{-2} \left( dx^2 + \frac{(1-x^2)^2}{4} r_{s^4} \right)
\]
is the hyperbolic metric. Moreover it is proved in [35] that
\[
\left( \frac{Y(S^3, [\hat{g}])}{Y(S^3, [g^{s^3}])} \right)^2 \leq K(0) = \lim_{r \to +\infty} \frac{\det(g_r)}{\det(g_r^{s^4})},
\]
where \( Y(S^3, [\hat{g}]) \) is the Yamabe constant of \((S^3, [\hat{g}])\) and \( g^{s^3} \) is the round sphere metric.

Now we give an outline of the proof of Theorem 1.1 in this section. To prove the uniqueness result, we deal with the equations in a point of view of a system of elliptic PDEs and first derive an a priori estimate on \( y_i^{(k)} \); see Lemma 3.1, Lemma 3.2 and Lemma 3.3. Indeed, for \( L^\infty \) estimates on \( y_i \), we show that \( y_i(x) \) is monotone on \((0, 1)\) (see Lemma 3.1), and use the first order equation \((2.9)\) to get a lower bound of \( K(0) \), see \((3.13)\) and \((3.14)\). Notice that for estimates on higher order derivatives of \( y_i \) near \( x = 1 \), standard boundary estimates near \( x = 1 \) based on the \( L^\infty \) bound of \( y_i \) is not enough for later use, and we have to employ the condition \(|W| \leq C \) for some constant \( C > 0 \) (which is satisfied by the non-positively curved CCE metrics) to obtain enough decay ratio near \( x = 1 \); see Lemma 3.3. Based on the a priori estimates, we then prove the uniqueness result by a contradiction argument.
Assume that we have two different solutions \((y_{i1}, y_{i2}, y_{i3})\) with \(i = 1, 2\) and consider their difference \((z_1, z_2, z_3)\). We find that the integral type comparison argument fails to work due to the interaction of the three functions \(y_i\) in the system. To overcome this difficulty, we employ a global argument on the total variation of the functions \(z_i\) based on the a priori estimates and certain kinds of integrations of the Einstein equations on certain intervals on \([0, 1]\). Interestingly, it turns out that for each conformal component \(z_i\) \((i = 2, 3)\), in the first monotonic interval of each peak and trough, the total variation of \(z_i\) is well controlled under the condition of Theorem 1.1; while these kinds of estimates can not be obtained from the equation on the remaining intervals. But on the other hand, the summation of the total variations of \(z_i\) on these intervals is just half of the total variation of \(z_i\) on \([0, 1]\) for \(i = 2, 3\), by the boundary conditions. Hence we obtain the control of the total variations of the conformal components \(z_2\) and \(z_3\) on \([0, 1]\). For the determinant component \(z_1\), the summation of the total variations of \(z_1\) on such intervals fails to be half of the total variation of \(z_1\) on \([0, 1]\), since \(z_1(0)\) might not vanish. Instead, we use certain integration of the equation (2.4) on each monotonic interval of \(z_1\) and the a priori estimates to derive a control of the total variation of \(z_1\). These three derived linear inequalities about the total variation of \(z_i\) on \([0, 1]\) finally lead to a contradiction.

From now on, without loss of generality, we assume \(\lambda_1 > \max\{\lambda_2, \lambda_3\}\), and hence \(\phi_1(0) < 1\) and \(\phi_1(0)\phi_2(0) < 1\).

**Lemma 3.1** For the initial data \(\phi_1(0), \phi_2(0) > 0\), we have \(y_1'(x) > 0\) for \(x \in (0, 1)\). Moreover, if \(\phi_1(0) < 1\), \(\phi_1(0)\phi_2(0) < 1\) and \(1 < \phi_1(0) + \phi_1(0)\phi_2(0)\), then \(y_2', y_3'\) and \(y_2' + y_3'\) have no zero on \(x \in (0, 1)\). That is to say, \(K, \phi_1, \phi_2\) and \(\phi_1\phi_2\) are monotonic on \((0, 1)\).

**Proof** The proof is a modification of Lemma 5.1 in [33]. Notice that the zeroes of \(y_i' (1 \leq i \leq 3)\) are discrete on \(x \in [0, 1]\). Assume that there exists a zero of \(y_1'\) on \(x \in (0, 1)\). Let \(x_1\) be the largest zero of \(y_1'\) on \(x \in (0, 1)\). Multiplying \(x^{-1}(1 - x^2)^2\) on both sides of (2.4) and integrating the equation on \(x \in [x_1, 1]\), we have

\[
(x^{-1}(1 - x^2)^2 y_1')' + x^{-1}(1 - x^2)^2[\frac{1}{6}(y_1')^2 + \frac{1}{3}((y_2')^2 + y_2'y_3' + (y_3')^2)] = 0,
\]

\[
\int_{x_1}^1 x^{-1}(1 - x^2)^2[\frac{1}{6}(y_1')^2 + \frac{1}{3}((y_2')^2 + y_2'y_3' + (y_3')^2)]dx = 0. \tag{3.1}
\]

Therefore, \(y_1' = 0\) on \(x \in [x_1, 1]\). Since \(y_1\) is analytic, \(y_1' = 0\) for \(x \in [0, 1]\), contradicting with the fact \(y_1(0) < y_1(1)\). Therefore, there is no zero of \(y_1'\) on \(x \in (0, 1)\). Therefore, \(y_1' > 0\) for \(x \in (0, 1)\).

Denote \(\phi_3 = \phi_1\phi_2\) and \(y_4 = \log(\phi_3)\). Summarizing (2.6) and (2.7), we have

\[
y_4'' - 2x^{-1}(1 + 2x^2)(1 - x^2)^{-1}y_4' + \frac{1}{2}y_1'y_4' + 32(1 - x^2)^{-2}K^{-\frac{1}{2}} = \frac{2}{3}\phi_3^\frac{1}{3}\phi_2^\frac{2}{3} - \phi_1^\frac{1}{3}\phi_2^\frac{2}{3} - \phi_1^\frac{2}{3}\phi_2^\frac{4}{3} + \phi_1^\frac{4}{3}\phi_2^\frac{2}{3} = 0. \tag{3.2}
\]
Let \( x_i \in (0, 1) \) be a zero of \( y_i' \) for \( i = 2, 3, 4 \). We multiplying \( x^{-2}(1 - x^2)^3 \) on both sides of (2.6), and integrate the equation on \( x \in [x_2, 1] \) to obtain

\[
(x^{-2}(1 - x^2)^3 y_2')' + \frac{1}{2} x^{-2}(1 - x^2)^3 y_1 y_2' + 32 x^{-2}(1 - x^2) K^{-\frac{3}{2}} [\frac{1}{4} \phi_2^\frac{1}{2} - \frac{1}{4} \phi_2^\frac{3}{2} - \frac{1}{4} \phi_1^\frac{1}{2} - \frac{1}{4} \phi_1^\frac{3}{2}] = 0,
\]

\[
\int_{x_2}^{1} \frac{1}{2} x^{-2}(1 - x^2)^3 y_1 y_2' dx = - \int_{x_2}^{1} 32 x^{-2}(1 - x^2) K^{-\frac{3}{2}} (1 - \phi_1) (1 + \phi_1 - \phi_3) dx.
\tag{3.3}
\]

That is,

\[
\int_{x_2}^{1} \frac{1}{2} x^{-2}(1 - x^2)^3 y_1 y_2' dx = - \int_{x_2}^{1} 32 x^{-2}(1 - x^2) K^{-\frac{3}{2}} \phi_1^\frac{1}{2} \phi_2^\frac{3}{2} (1 - \phi_2) (-1 + \phi_1 + \phi_3) dx.
\tag{3.4}
\]

Similarly,

\[
\int_{x_3}^{1} \frac{1}{2} x^{-2}(1 - x^2)^3 y_1 y_3' dx = - \int_{x_3}^{1} 32 x^{-2}(1 - x^2) K^{-\frac{3}{2}} \phi_1^\frac{1}{2} \phi_2^\frac{3}{2} (1 - \phi_2) (-1 + \phi_1 + \phi_3) dx,
\tag{3.5}
\]

\[
\int_{x_4}^{1} \frac{1}{2} x^{-2}(1 - x^2)^3 y_1 y_4' dx = - \int_{x_4}^{1} 32 x^{-2}(1 - x^2) K^{-\frac{3}{2}} \phi_1^\frac{1}{2} \phi_2^\frac{3}{2} (1 - \phi_2) (-1 + \phi_1 + \phi_3) dx.
\tag{3.6}
\]

We assume that \( y_2' \) achieves the largest zero \( x_2 \in (0, 1) \) in \( \{ y_2', y_3', y_4' \} \). Notice that \( y_i'(1) = 0 \) and \( y_i(1) = 0 \) for \( i = 2, 3, 4 \). We have that \( y_i'(1 - \phi_{i-1}) > 0 \) on \( x \in (x_2, 1) \) for \( i = 2, 3, 4 \). By (3.4), there exists a point \( x \in (x_2, 1) \) such that

\[
1 + \phi_1(x) - \phi_3(x) < 0.
\tag{3.7}
\]

We claim that there is no zero of \( y_3' \) and \( y_4' \) on \( x \in (0, 1) \). If that is not the case, assume \( y_3' \) achieves the largest zero \( x_3 \in (0, x_2] \) in \( \{ y_3', y_4' \} \). Since \( y_4' \) keeps the sign on \( (x_3, 1) \), by (3.7) we have \( \phi_3(x) > 1 \) and \( y_4' < 0 \) on \( (x_3, 1) \), contradicting with (3.5). Otherwise, assume \( y_4' \) achieves the largest zero \( x_4 \in (0, x_2] \) in \( \{ y_3', y_4' \} \). Then \( y_3' \) keeps the sign on \( (x_2, 1) \) and by (3.7) we have \( \phi_3 = \phi_1 \phi_2 > \phi_1 \) on \( (x_2, 1) \) and therefore \( \phi_2 > 1 \) on \( (x_4, 1) \), which implies \( 1 - \phi_1 + \phi_3 > 0 \) on \( (x_4, 1) \), contradicting with (3.6). That proves the claim. By (3.7), \( \phi_2 > 1 \) and \( \phi_3 = \phi_1 \phi_2 > 1 \) on \( x \in (0, 1) \). Therefore, for \( x \in (x_2, 1) \),

\[
1 + \phi_1(x) - \phi_3(x) > 1 + \phi_2^{-1} - \phi_3 > 1 + \phi_2^{-1}(0) - \phi_3(0) > 0,
\tag{3.8}
\]

where the last inequality is by the condition of the lemma, contradicting with (3.4). Therefore, \( y_2' \) could not achieve the largest zero on \( x \in (0, 1) \) in \( \{ y_2', y_3', y_4' \} \).

Similar argument yields that neither \( y_3' \) nor \( y_4' \) could achieve the largest zero on \( x \in (0, 1) \) in \( \{ y_2', y_3', y_4' \} \). Therefore, there is no zero of \( y_i' \) on \( x \in (0, 1) \) for \( i = 2, 3, 4 \). This completes the proof of the lemma.

\[\square\]

Using the monotonicity of \( y_i \) (1 ≤ \( i \) ≤ 4), we now give a uniform estimate of \( y_i \) for \( x \in [0, 1] \) under the condition in Lemma 3.1.
By (2.9) and the initial value condition, we have
\[
y_1' = 6x^{-1}(1 - x^2)^{-1}[1 + x^2 - \sqrt{(1 + x^2)^2 + \frac{1}{36}x^2(1 - x^2)2((y_2')^2 + y_2'y_3' + (y_3')^2) - \frac{4}{3}x^2(3 - \Upsilon(x))}]
\]
\[
= 6x^{-1}(1 - x^2)^{-1}[1 + x^2 - \sqrt{(1 + x^2)^2 + \frac{1}{36}x^2(1 - x^2)2((y_2')^2 + y_2'y_3' + (y_3')^2) + \frac{4}{3}x^2\Upsilon(x)}].
\]
where
\[
\Upsilon(x) = K^{-\frac{1}{3}}[2(\phi_1^2\phi_2)^{\frac{3}{2}} + 2(\phi_1^{-1}\phi_2)\frac{3}{2} + 2(\phi_1\phi_2^2)^{-\frac{1}{3}} - \phi_1^{-\frac{1}{3}}\phi_2^{-\frac{1}{2}} - \phi_1^{\frac{1}{2}}\phi_2^{-\frac{3}{2}} - \phi_1^{-\frac{2}{3}}\phi_2^{\frac{1}{2}}].
\]
Since \(y_1' > 0\) for \(x \in (0, 1)\), by (3.9) it is clear that
\[
3 - \Upsilon(x) > 0 \text{ for } x \in (0, 1).
\]
Recall that \(\Upsilon \in C^\infty([0, 1]), \) hence we have \(3 - \Upsilon(0) \geq 0\). Notice that by (2.5)–(2.7),
\[
y''(0) = 4(3 - \Upsilon(0)),
\]
\[
y''(0) = 32K(0)^{-\frac{1}{3}}[\phi_1(0)^{\frac{3}{2}}\phi_2(0)^{\frac{1}{2}} - \phi_1(0)\phi_2(0)^{-\frac{1}{2}} - \phi_1(0)^{-\frac{1}{2}}\phi_2(0) - \phi_1(0)^{-\frac{2}{3}}\phi_2(0)^{-\frac{1}{2}}],
\]
\[
y''(0) = 32K(0)^{-\frac{1}{3}}[\phi_1(0)^{-\frac{1}{3}}\phi_2(0)^{\frac{1}{2}} - \phi_1(0)^{-\frac{1}{2}}\phi_2(0)^{-\frac{1}{2}} - \phi_1(0)^{-\frac{2}{3}}\phi_2(0)^{-\frac{1}{2}} + \phi_1(0)^{-\frac{1}{3}}\phi_2(0)^{\frac{1}{2}}].
\]
Therefore,
\[
\frac{d^2}{dx^2}(3 - \Upsilon) = \left(\frac{1}{3}y''(0)\Upsilon(0) - \frac{2}{3}K(0)^{-\frac{1}{3}}((2(\phi_1^2\phi_2)^{\frac{3}{2}} - (\phi_1^{-1}\phi_2)^{\frac{1}{2}} - (\phi_1\phi_2^2)^{-\frac{1}{3}} + 2\phi_1^{-\frac{1}{3}}\phi_2^{-\frac{1}{2}} - \phi_1^{\frac{1}{2}}\phi_2^{-\frac{3}{2}})^{\frac{1}{3}}\Upsilon(0) + \left(\phi_1\phi_2^2\right)^{\frac{1}{3}} + \phi_1^{-\frac{1}{3}}\phi_2^{\frac{1}{2}} + \phi_1^{-\frac{1}{2}}\phi_2^{\frac{3}{2}} - 2\phi_1^{-\frac{2}{3}}\phi_2^{\frac{1}{2}}]y''(0)\right).
\]
We claim that \((3 - \Upsilon(0)) > 0\) and hence \(y''(0) > 0\). Otherwise, \((3 - \Upsilon(0)) = 0\) and hence \(y''(0) = 0\). After substituting the expression of \(y''(0)\), we have
\[
\frac{d^2}{dx^2}(3 - \Upsilon) = -128K^{-\frac{1}{3}}\phi_1^{-\frac{1}{3}}\phi_2^{-\frac{4}{3}}[(1 + \phi_1)^2(1 + \phi_2 + \phi_1\phi_2^2) + \phi_1^3\phi_2^2(1 - \phi_1\phi_2)(1 - \phi_2)]\bigg|_{x=0}
\]
\[
= -128K^{-\frac{1}{3}}\phi_1^{-\frac{1}{3}}\phi_2^{-\frac{4}{3}}[(1 - \phi_1\phi_2)^2(1 + \phi_1\phi_2^2) + \phi_1^3\phi_2^2](1 - \phi_1 - 1(1 - \phi_1))\bigg|_{x=0}.
\]
Using the first identity when \(\phi_2(0) < 1\), while using the second identity when \(\phi_2(0) > 1\), combining with the fact \(\phi_1(0) < 1\) and \(\phi_1(0)\phi_2(0) < 1\), we have
\[
\frac{d^2}{dx^2}(3 - \Upsilon) < 0.
\]
Since \(\frac{d}{dx}(3 - \Upsilon) < 0\), we have \(\frac{d}{dx}(3 - \Upsilon(x)) < 0\) for \(x > 0\) small, and therefore \((3 - \Upsilon(x)) < 0\) for \(x > 0\) small, contradicting with (3.10). The claim is proved. This gives a lower bound of \(K(0)\) under the condition of Lemma 3.1:
\[
K(0) > \left(\frac{2(\phi_1^2\phi_2)^{\frac{3}{2}} + 2(\phi_1^{-1}\phi_2)^{\frac{1}{2}} + 2(\phi_1\phi_2^2)^{-\frac{1}{3}} - \phi_1^{-\frac{1}{3}}\phi_2^{-\frac{1}{2}} - \phi_1^{\frac{1}{2}}\phi_2^{-\frac{3}{2}} - \phi_1^{-\frac{2}{3}}\phi_2^{\frac{1}{2}}}{3}\right)^3\bigg|_{x=0}.
\]
and hence for $|1 - \phi_1(0)|$ small, there exists a constant $C > 0$ independent of the initial data and the solution, such that

$$|1 - K(0)| \leq C(|1 - \phi_1(0)| + |1 - \phi_2(0)|).$$ \hspace{1cm} (3.12)

By the condition in Lemma 3.1, we have

$$K(0) > \frac{1}{9} \phi_1(0)^{-1} \phi_2^{-2}(0)(1 + \phi_1(0)\phi_2(0))^3,$$ \hspace{1cm} (3.13)

when $\phi_2(0) \geq 1$; while for $\phi_2(0) < 1$,

$$K(0) > \frac{1}{9} \phi_1(0)^{-1} \phi_2(0)(1 + \phi_1(0))^3.$$ \hspace{1cm} (3.14)

Now we give an estimate of the solution away from $x = 1$.

**Lemma 3.2** Let $\phi_1(0) < 1$, $\phi_1(0)\phi_2(0) < 1$ and $1 < \phi_1(0) + \phi_1(0)\phi_2(0)$. If moreover we have $\phi_1(0), \phi_2(0) > \delta_0$ for some constant $\delta_0 \in (0, 1)$, then there exists a constant $C = C(\delta_0) > 0$ independent of the solution and the initial data $\phi_1(0)$ such that

$$|y^{(k)}_i(x)| \leq C(|1 - \phi_1(0)| + |1 - \phi_2(0)|)x^{2-k},$$ \hspace{1cm} (3.15)

$$|y^{(k)}_i(x)| \leq C|1 - \phi_{i-1}(0)|x^{2-k},$$ \hspace{1cm} (3.16)

with $y^{(k)}_i$ the $k$-th order derivative of $x$, for $k = 1, 2, 3$ and $x \in (0, \frac{3}{4})$. The control still holds on the interval $(0, 1 - \epsilon)$ for any $\epsilon > 0$ small with some constant $C = C(\delta_0, \epsilon) > 0$.

**Proof** By Lemma 3.1 and the equation (3.9), we have that for $x \in (0, 1)$,

$$y'_1 < 6x^{-1}(1 - x^2)^{-1}[1 + x^2 - \sqrt{1 - x^2}] = 12x(1 - x^2)^{-1}.$$ \hspace{1cm} (3.17)

Notice that

$$\min\{\phi_i(0), 1\} < \phi_i(x) < \max\{\phi_i(0), 1\},$$

for $x \in (0, 1)$ and $1 \leq i \leq 3$, and $K(0) < K(x) < 1$ for $x \in (0, 1)$. By the interior estimates of the second order elliptic equations (2.4)–(2.7) and the inequality (3.12), there exists some constant $C = C(\delta_0) > 0$ independent of the initial data and the solution so that

$$|y^{(k)}_i(x)| \leq C(|1 - \phi_1(0)| + |1 - \phi_2(0)|), \quad \text{and}$$

$$|y^{(k)}_i(x)| \leq C|1 - \phi_{i-1}(0)|,$$

for $2 \leq i \leq 4$, $0 \leq k \leq 4$ and $x \in \left[\frac{1}{4}, \frac{3}{4}\right]$. To get global estimates, we multiply $x^{-5}(1 - x^2)^6K^\frac{1}{2}$ to (2.5) and do integration on the interval $[x, \frac{3}{4}]$ to obtain

$$(x^{-5}(1 - x^2)^6K^\frac{1}{2}y'_i)' + 16x^{-5}(1 - x^2)^4K^\frac{1}{2}[3 - \gamma] = 0,$$

$$x^{-5}(1 - x^2)^6K(x)^\frac{1}{2}y'_i(x) = \left(\frac{4}{3}\right)^5\left(\frac{7}{16}\right)^6K^\frac{1}{2}\left(\frac{3}{4}\right)y'_i(x) + \int_x^\frac{3}{4} 16s^{-5}(1 - s^2)^4K^\frac{1}{2}[3 - \gamma(s)]ds,$$

for $x \in \left(0, \frac{3}{4}\right)$, and hence

$$0 < y'_i(x) \leq C(|1 - \phi_1(0)| + |1 - \phi_2(0)|)x,$$ \hspace{1cm} (3.18)

for $x \in \left(0, \frac{3}{4}\right)$, with some constant $C = C(\delta_0) > 0$ independent of the initial data and the solution.
Similarly, by multiplying \( x^{-2}(1 - x^2)^3 K^{\frac{1}{2}} \) to (2.6) and (2.7) and doing integration on \([x, \frac{3}{4}]\) correspondingly, we have

\[
|y'_2(x)| \leq C|1 - \phi_1(0)|x, \quad \text{(3.19)}
\]

\[
|y''_2(x)| \leq C|1 - \phi_2(0)|x, \quad \text{(3.20)}
\]

for \( x \in (0, \frac{3}{4}) \), with some constant \( C = C(\delta_0) > 0 \) independent of the solution and the initial data. Substituting these inequalities to (2.5)–(2.7), we then have

\[
|y''_1(x)| \leq C(|1 - \phi_1(0)| + |1 - \phi_2(0)|), \quad \text{(3.21)}
\]

\[
|y''_3(x)| \leq C|1 - \phi_{i-1}(0)|, \quad \text{(3.22)}
\]

with \( C = C(\delta_0) > 0 \) some constant independent of the initial data and the solutions, for \( x \in (0, \frac{3}{4}) \) and \( i = 2, 3 \).

For \( x \in (\frac{1}{2}, 1) \), multiplying \((x - x^3)\) on both sides of (2.6), we have

\[
((x - x^3)y'_2)' - (3 + x^2)y'_2 + \frac{1}{2}(x - x^3)y'_1y'_2 + 32x(1 - x^2)^{-1}K^{-\frac{1}{2}}\phi_1^{-\frac{1}{2}}\phi_2^{-\frac{3}{2}}(1 - \phi_1(1 + \phi_1 - \phi_3) = 0. \quad \text{(3.23)}
\]

Since \( y'_1 \) and \( y'_2 \) keep the sign, we integrate the equation on \((\frac{1}{2}, x)\) and combine it with (3.17) to have

\[
(x - x^3)|y'_2(x)| \leq C|y'_2(\frac{1}{2})| + C|y_2(x) - y_2(\frac{1}{2})| - C|1 - \phi_1(0)|\log(1 - x^2),
\]

\[
|y'_2(x)| \leq C|1 - \phi_1(0)|(1 - x^2)^{-1}(1 - \log(1 - x^2)),
\]

with some constant \( C = C(\delta_0) > 0 \) independent of the initial data and the solution, for \( \frac{1}{2} \leq x < 1 \). Similarly, we have

\[
|y'_3(x)| \leq C|1 - \phi_2(0)|(1 - x^2)^{-1}(1 - \log(1 - x^2)),
\]

with some constant \( C = C(\delta_0) > 0 \) independent of the initial data and the solution, for \( \frac{1}{2} \leq x < 1 \).

Now we use the bound of the Weyl tensor to give better estimates on \( y^{(k)}_i \) with \( k = 1, 2 \) and \( 1 \leq i \leq 3 \) near \( x = 1 \). Under the polar coordinates \((\theta^0, \theta^1, \theta^2, \theta^3) = (r, \theta^1, \theta^2, \theta^3)\),

\[
g = d(\theta^0)^2 + \sum_{1 \leq i, j \leq 3} g_{ij}d\theta^id\theta^j,
\]

and along the geodesic \( \theta = \theta_0 = (\theta_0^1, \theta_0^2, \theta_0^3) \),

\[
g = d(\theta^0)^2 + \sum_{1 \leq i \leq 3} g_{ii}d(\theta^i)^2.
\]

It is easy to obtain the calculations:

\[
\frac{d}{dr}g_{ii} = -\frac{x}{4}\frac{d}{dx}((x^{-1} - x)^2I_i) = \frac{1}{4x^2}(-x(1 - x^2)^2 \frac{d}{dx}I_i + 2(1 - x^4)I_i),
\]

\[
g_{ii}^{-1}\frac{d}{dr}g_{ii} = (-xI_i^{-1} \frac{d}{dx}I_i + 2(1 + x^2)(1 - x^2)^{-1})I_i,
\]

for \( i = 1, 2, 3 \). Recall that by the non-positivity of the sectional curvature of \( g \) and the Einstein equation (2.1), we have that \( |W|_g \leq T = 2\sqrt{6} \). Using the boundedness of the Weyl tensor, we give an estimate of the solution on \( x \in [\frac{3}{4}, 1] \).
Lemma 3.3 Assume that $|W|_g \leq \varepsilon$ with some constant $0 < \varepsilon \leq T$. Under the condition of Lemma 3.1, and moreover we assume there exists a constant $\delta_0 \in (0, 1)$ such that $\phi_1(0), \phi_2(0) \geq \delta_0$. Then we have

$$|y_1^{(k)}| \leq C \varepsilon^2 (1 - x^2)^{4-k}$$

$$(3.24)$$

$$|y_i^{(k)}| \leq C \varepsilon (1 - x^2)^{2-k}$$

$$(3.25)$$

for $k = 1, 2$ and $x \in [\frac{3}{4}, 1]$, with $C = C(\delta_0)$.

**Proof** By the condition in the lemma, the following components of the Weyl tensor has the bound

$$|(g_{ii})^{-\frac{1}{2}}(g_{qq})^{-\frac{1}{2}}(g_{pp})^{-\frac{1}{2}}W_{piq0}(g)| \leq \varepsilon,$$

which is expressed as

$$|(g_{ii})^{-\frac{1}{2}}(g_{qq})^{-\frac{1}{2}}(g_{pp})^{-\frac{1}{2}}W_{piq0}(g)|$$

$$= |(g_{ii})^{-\frac{1}{2}}(g_{qq})^{-\frac{1}{2}}(g_{pp})^{-\frac{1}{2}}[\frac{1}{2} \frac{dg_{ii}}{dr} (-g_{ii})^{-1} + (g_{pp})^{-1} g_{qq} (g_{ii})^{-1}) + \frac{1}{2} \frac{dg_{pp}}{dr} ((g_{pp})^{-1} - (g_{pp})^{-2} g_{qq} - (g_{pp})^{-1} \frac{dg_{pp}}{dr})|$$

$$= \frac{x^2}{(1 - x^2)} I_p^{\frac{1}{2}} I_q^{\frac{1}{2}} I_i^{\frac{1}{2}} \left[(I_p^{-1} \frac{dI_p}{dx}) (-1 + I_p I_q^{-1} + I_q^{-1} I_q) + I_p^{-1} \frac{dI_p}{dx} (1 - I_p I_q^{-1} + I_q^{-1} I_q)ight]$$

$$- \frac{2}{(1 - x^2)} I_k^{\frac{1}{2}} I_l^{\frac{1}{2}} I_m^{\frac{1}{2}} \frac{d}{dx} \frac{d}{dx} \left[ I_p^{\frac{1}{2}} I_q^{-\frac{1}{2}} I_i^{-\frac{1}{2}} + I_q^{\frac{1}{2}} I_p^{-\frac{1}{2}} I_i^{-\frac{1}{2}} - I_p^{-\frac{1}{2}} I_q^{-\frac{1}{2}} I_i^{\frac{1}{2}} \right],$$

for any $\{i, p, q\} = \{1, 2, 3\}$. Here we have used the homogeneity of the metric $g$ for the calculation of the derivatives $\frac{dg_{ii}}{dr}$, which can be found in [33].

We choose $(i, p, q) = (1, 2, 3)$ and $(2, 3, 1)$. Now by (3.26), we have

$$(\phi_1 - 1 - \phi_1 \phi_2) y_2' - 2 \phi_1 \phi_2 y_3' = O(1) \varepsilon (1 - x^2),$$

$$2 y_2' + (\phi_1 \phi_2 - \phi_1 + 1) y_3' = O(1) \varepsilon (1 - x^2),$$

for $x \in [\frac{3}{4}, 1]$, with $|O(1)| \leq \frac{32}{9} \phi_1^\frac{1}{2}(0)$. Therefore, using the initial values and the monotonicity of $\phi_i$, we have

$$y_2'(x) = O(1) \varepsilon (1 - x^2),$$

$$y_3'(x) = O(1) \varepsilon (1 - x^2),$$

for $\frac{3}{4} \leq x \leq 1$ with $O(1)$ uniformly bounded, depending only on the lower bound $\delta_0$ of the initial data. We then integrate these two inequalities on the interval $(x, 1)$, and use the monotonicity of $y_i$ and the fact $y_i(1) = 0$ to have

$$y_2(x) = O(1) \varepsilon (1 - x^2)^2,$$

$$y_3(x) = O(1) \varepsilon (1 - x^2)^2,$$
for \( \frac{3}{4} \leq x \leq 1 \) with \( O(1) \) uniformly bounded, depending on the lower bound \( \delta_0 \). For \( y'_1 \), we multiply \( x^{-1}(1-x^2)^2 K^\frac{1}{6} \) on both sides of (2.4) and do integration on \( (x,1) \) for \( \frac{3}{4} \leq x \leq 1 \),

\[
(x^{-1}(1-x^2)^2 K^\frac{1}{6} y'_1) + \frac{1}{3} x^{-1}(1-x^2)^2 K^\frac{1}{6} [(y_2')^2 + y'_2 y'_3 + (y_3')^2]] = 0, \quad \text{and}
\]

\[
y'_1(x) = \frac{1}{3} K(x)^{-\frac{1}{6}} x (1-x^2)^{-2} \int_x^1 s^{-1}(1-s^2)^2 K^\frac{1}{6} (s)[(y'_2)^2 + y'_2 y'_3 + (y'_3)^2]ds
\]

\[
\leq C \varepsilon^2 (1-x^2)^3,
\]

with \( C = C(\delta_0) > 0 \), where for the last inequality we have used monotonicity of \( y_i \) and the lower bound of \( K \) given by (3.13) and (3.14). Integrating the inequality on \((x,1)\), we have

\[
|y'_1(x)| \leq C \varepsilon^2 (1-x^2)^4,
\]

with some constant \( C = C(\delta_0) > 0 \), for \( \frac{3}{4} \leq x \leq 1 \).

Substituting these estimates back to (2.4)–(2.7), we have

\[
|y''_1(x)| \leq C \varepsilon^2 (1-x^2)^2,
\]

\[
|y''_1(x)| \leq C \varepsilon,
\]

for \( i = 1, 2 \) and \( \frac{3}{4} \leq x \leq 1 \), with some constant \( C = C(\delta_0) > 0 \).

Now we turn to the uniqueness discussion.

**Proof** By [33], the uniqueness of the metric is equivalent to the uniqueness of the solution to the boundary value problem (2.4)–(2.8).

Assume that we have two solutions \((y_{11}, y_{12}, y_{13})\) and \((y_{21}, y_{22}, y_{23})\) to the boundary value problem (2.4)–(2.8), with \( y_{ij} = \log(\phi_{ij}(\cdot - 1)) \) for \( i = 1, 2 \) and \( j = 2, 3 \) and \( y_{1i} = \log(K_i) \) for \( i = 1, 2 \). Denote \( z_i = y_{1i} - y_{2i} \) correspondingly for \( 1 \leq i \leq 3 \).

If these two solutions have the same conformal infinity \([\hat{g}]\) and the same non-local term \( g^{(3)} \) in the expansion, then they coincide by [8]. For \( 1 \leq i \leq 3 \), by the Einstein equation, the zeroes of \( z_i \) are discrete unless \( z_i \) is identically zero.

Let \( \varepsilon = T = 2\sqrt{6} \) and \( \delta_0 \in (0,1) \) be a given constant. By the non-positivity of the sectional curvature of \( g \) and the Einstein equation (2.1), it holds that \( |W|_{\hat{g}} \leq T \) pointwisely on \( M \). Assume that \( \phi_1(0), \phi_2(0) \geq \delta_0 \) and by assumption, \( \phi_1(0), \phi_2(0) \) satisfy the condition in Lemma 3.1. We first consider the cases \( i = 2, 3 \). For \( i = 2, 3 \), on the domain

\[
D_i^- = \{ x \in (0,1] | z_i(x) \leq 0 \},
\]

let \( b_{i1}^- < \cdots < b_{im_i}^- \) be the set of local minimum points of \( z_i \) on \( D_i^- \), and we pick up the (maximal) non-increasing intervals of \( z_i \) on \( D_i^- \) (the closure of \( D_i^- \)):

\[
[a_{i1}^-, b_{i1}^-] \cup [a_{i2}^-, b_{i2}^-] \cup \cdots \cup [a_{im_i}^-, b_{im_i}^-],
\]

such that \( a_{i1}^- < b_{i1}^- < a_{i2}^- < \cdots < a_{im_i}^- < b_{im_i}^- \) with \( m_i \) some integer; while on the domain

\[
D_i^+ = \{ x \in (0,1] | z_i(x) \geq 0 \},
\]

let \( b_{i1}^+ < \cdots < b_{in_i}^+ \) be the set of local maximum points of \( z_i \) on \( D_i^+ \) with \( n_i \) some integer, and we pick up all the (maximal) non-decreasing intervals of \( z_i \) on \( D_i^+ \) (the closure of \( D_i^+ \)):

\[
[a_{i1}^+, b_{i1}^+] \cup [a_{i2}^+, b_{i2}^+] \cup \cdots \cup [a_{in_i}^+, b_{in_i}^+].
\]
such that \( a_{i1}^+ < b_{i1}^- < a_{i2}^+ < \ldots < a_{im_i}^+ < b_{in_i}^- \). Since \( z_i \in C^\infty([0, 1]) \) and \( z_i' \) has finitely many zeroes, it is clear that \( z_i \) is of bounded variation on \( x \in [0, 1] \). For an interval \([a, b] \subseteq [0, 1] \), we denote \( V_{ai}(z_i) \) the total variation of \( z_i \) on \( x \in [a, b] \), and we denote \( V(z_i) \) the total variation of \( z_i \) on \( x \in [0, 1] \).

Recall that \( z_i(0) = z_i(1) = 0 \) for \( i = 2, 3 \). By the mean value theorem, there exist zeroes of \( z_i' \) on \( x \in (0, 1) \). And also for \( 1 \leq j \leq m_i \), either \( z_i(a_{ij}^-) = 0 \), or \( z_i(a_{ij}^+) \leq 0 \) with \( a_{ij}^- \) a local maximum of \( z_i \) on \([0, 1]\) and \( z_i'(a_{ij}^-) = 0 \). Similarly, for \( 1 \leq j \leq n_i \), either \( z_i(a_{ij}^-) = 0 \), or \( z_i(a_{ij}^+) \geq 0 \) with \( a_{ij}^+ \) a local minimum of \( z_i \) on \([0, 1]\) and \( z_i'(a_{ij}^+) = 0 \). It is clear that for \( i = 2, 3 \),

\[
\frac{1}{2} V(z_i) = \sum_{j=1}^{m_i} V_{a_{ij}}^-(z_i) + \sum_{j=1}^{n_i} V_{a_{ij}}^+(z_i)
= \sum_{j=1}^{m_i} |z_i(b_{ij}^-) - z_i(a_{ij}^-)| + \sum_{j=1}^{n_i} |z_i(b_{ij}^+) - z_i(a_{ij}^+)|.
\]

We substitute the two solutions to (3.23), and take difference to have

\[
0 = ((x - x^3)z_2' - (3 + x^2)z_2' + \frac{1}{2}(x - x^3)(y_{11}z_2' + z_1'y_2')
+ 32x(1 - x^2)^{-1}K_1^{-\frac{3}{2}}\phi_{11}^{-\frac{1}{2}}\phi_{12}^{-\frac{3}{2}}(1 + \phi_{11} - \phi_{11}\phi_{12})(\phi_{21} - \phi_{11})
+ 32x(1 - x^2)^{-1}[K_1^{-\frac{3}{2}}\phi_{11}^{-\frac{1}{2}}\phi_{12}^{-\frac{3}{2}}(1 + \phi_{11} - \phi_{11}\phi_{12})
- K_2^{-\frac{1}{2}}\phi_{21}^{-\frac{1}{2}}\phi_{22}^{-\frac{3}{2}}(1 + \phi_{21} - \phi_{21}\phi_{22})](1 - \phi_{21}),
\]

where the term

\[
|32x(1 - x^2)^{-1}[K_1^{-\frac{3}{2}}\phi_{11}^{-\frac{1}{2}}\phi_{12}^{-\frac{3}{2}}(1 + \phi_{11} - \phi_{11}\phi_{12})
- K_2^{-\frac{1}{2}}\phi_{21}^{-\frac{1}{2}}\phi_{22}^{-\frac{3}{2}}(1 + \phi_{21} - \phi_{21}\phi_{22})](1 - \phi_{21})|
\leq \frac{32}{3}x(1 - x^2)^{-1}|1 - \phi_{21}| \times (K(0)^{-\frac{3}{2}}\phi_{11}^{-\frac{1}{2}}\phi_{12}^{-\frac{3}{2}}(0) + K(0)^{-\frac{1}{2}}\phi_{21}^{-\frac{1}{2}}\phi_{22}^{-\frac{3}{2}}(0) \times \left[ \phi_{11}(0)\phi_{22}(0) + K_{12}(0)\phi_{11}(0)\phi_{22}(0) \right])
\leq Cx(1 - x^2)^{-1}|1 - \phi_{21}| \times (|z_1| + |z_2| + |z_3|),
\]

with some constant \( C > 0 \) depending on the lower bound \( \delta_0 \) of \( \phi_1(0) \) and here \( K(0) \) we mean the lower bound of \( K_{ij}(0) \) \((i = 1, 2)\) obtained in (3.11). Pick up \( 1 \leq j \leq m_2 \) (resp. \( n_2 \)). Now we do integration of the equation on \([a_{2j}, b_{2j}] \) to have

\[
(x - x^3)z_2'(x)|_{x=a_{2j}} + 3(z_2(b_{2j}^+) - z_2(a_{2j})) + \int_{a_{2j}}^{b_{2j}} x^2z_2'(x)dx
= \int_{a_{2j}}^{b_{2j}} \left[ \frac{1}{2}(x - x^3)(y_{11}z_2' + z_1'y_2') + O(1)x(1 - x^2)^{-1}|1 - \phi_{21}| \times (|z_1| + |z_2| + |z_3|)
+ 32x(1 - x^2)^{-1}K_1^{-\frac{3}{2}}\phi_{11}^{-\frac{1}{2}}\phi_{12}^{-\frac{3}{2}}(1 + \phi_{11} - \phi_{11}\phi_{12})(\phi_{21} - \phi_{11}) \right]dx,
\]

with \( O(1) \) uniformly bounded, depending only on \( \delta_0 \). It is clear that the three terms on the left hand side of the equation have the same sign. On the right hand side the third term can
not be controlled by the left hand side in general on the interval $x \in [0, 1]$, while $(\phi_{21} - \phi_{11})$ has a different sign from the left hand side on the interval $[a_{2j}^+, b_{2j}^+]$. Therefore,

$$3|z_2(b_{2j}^+) - z_2(a_{2j}^+)| \leq \int_{a_{2j}^+}^{b_{2j}^+} \left[ \frac{1}{2} (x - x^3)(|y''_1 z'_2 + |z'_1 y''_2|) + C_{21} x(1 - x^2)^{-1}|1 - \phi_{21}| \times (|z_1| + |z_2| + |z_3|) \right] dx,$$

(3.27)

for some constant $C_{21} > 0$ uniformly bounded depending only on $\delta_0$. By (3.18) we have

$$0 < \frac{1}{2} (x - x^3) y'_1(x) \leq \frac{1}{2} C_2 (|1 - \phi_1(0)| + |1 - \phi_2(0)|) x^2 (1 - x^2),$$

(3.28)

for $x \in (0, \frac{3}{4}]$; while by (3.24), we have

$$0 \leq \frac{1}{2} (x - x^3) y'_1(x) \leq \frac{1}{2} C_3 \varepsilon^2 x(1-x^2)^4,$$

(3.29)

for $x \in [\frac{3}{4}, 1]$, with the constants $C_2, C_3 > 0$ depending on $\delta_0$. Also, by (3.19) and the monotonicity of $\phi_{i1}$ we have

$$|y''_1(x)| \leq C_{24} |1 - \phi_1(0)| x,$$

(3.30)

$$x (1 - x^2)^{-1} |1 - \phi_{i1}(x)| \leq \frac{12}{7} |1 - \phi_1(0)|,$$

(3.31)

for $x \in (0, \frac{3}{4})$ and $i = 1, 2$, with some constant $C_{24} = C_{24} (\delta_0) > 0$; while by (3.25),

$$|y''_2(x)| \leq C_{25} \varepsilon (1 - x^2),$$

(3.32)

$$x (1 - x^2)^{-1} |1 - \phi_{i1}(x)| \leq C_{25} \varepsilon (1 - x^2),$$

(3.33)

for $x \in [\frac{3}{4}, 1]$ and $i = 1, 2$, with $C_{25} = C_{25} (\delta_0) > 0$. Now we let $\varepsilon_0 \in (0, 1)$ be a constant satisfying that for $\varepsilon_0 \leq x \leq 1$,

$$\frac{1}{2} \varepsilon^2 (1 - x^2)^4 C_3 \leq \frac{1}{2}, \quad \frac{1}{2} (1 - x^2)^2 C_{25} \varepsilon \leq \frac{1}{4},$$

(3.34)

$$C_{21} C_{25} (1 - x^2) \varepsilon \leq \frac{1}{4}.$$  

(3.35)

For that we let $\varepsilon_0 \in (0, 1)$ satisfy that

$$(1 - \varepsilon_0^2) \leq \min\left\{ \frac{1}{(24C_3)^{\frac{1}{2}}}, \frac{1}{(4\sqrt{6}C_{25})^{\frac{1}{2}}}, \frac{1}{8\sqrt{6}} \frac{1}{21} \frac{1}{25} \right\}.$$ 

(3.36)

Similarly, now we consider $z_3$. Multiplying $(x - x^3)$ on both sides of (2.7), we have

$$((x - x^3)y'_3)' - (3 + x^2)y'_3 + \frac{1}{2} (x - x^3) y'_3 y'_3 + 32 x (1 - x^2)^{-1} K^{-\frac{1}{2}} \phi_1^{-\frac{1}{2}} \phi_2^{-\frac{1}{2}} (1 - \phi_2)(-1 + \phi_1 + \phi_1 \phi_2) = 0.$$  

(3.37)
We substitute the two solutions to (3.37), and take difference of the two equations obtained to have

\[ 0 = (x - x^3)z_3' - (3 + x^2)z_3 + \frac{1}{2}(x - x^3)(y_1'z_3' + z_1'y_2') + 32x(1 - x^2)^{-1}K_1^{-\frac{1}{3}}\phi_1^{-\frac{1}{3}}\phi_2^{-\frac{1}{3}}(-1 + \phi_1 + \phi_1\phi_1\phi_2)(\phi_22 - \phi_12) + 32x(1 - x^2)^{-1}[K_1^{-\frac{1}{3}}\phi_1^{-\frac{1}{3}}\phi_2^{-\frac{1}{3}}(-1 + \phi_1 + \phi_1\phi_1\phi_2) - K_2^{-\frac{1}{3}}\phi_21^{-\frac{1}{3}}\phi_22^{-\frac{1}{2}}(-1 + \phi_21 + \phi_21\phi_22)](1 - \phi_22), \]

where the term

\[ |32x(1 - x^2)^{-1}[K_1^{-\frac{1}{3}}\phi_1^{-\frac{1}{3}}\phi_2^{-\frac{1}{3}}(-1 + \phi_1 + \phi_1\phi_1\phi_2) - K_2^{-\frac{1}{3}}\phi_21^{-\frac{1}{3}}\phi_22^{-\frac{1}{2}}(-1 + \phi_21 + \phi_21\phi_22)](1 - \phi_22)| \]

\[ \leq \frac{32}{3}x(1 - x^2)^{-1}|1 - \phi_22| \times K(0)^{-\frac{1}{3}}\phi_1^{-\frac{1}{3}}\phi_2^{-\frac{1}{3}}(0) \times [\phi_1(0)\phi_2(0)|K_1 - K_2| + K(0)|\phi_2(0)(1 + \phi_2(0))|\phi_1 - \phi_21 + K(0)|\phi_2(0)(2 + \phi_2(0))|\phi_21 - \phi_22|] \]

\[ \leq Cx(1 - x^2)^{-1}|1 - \phi_22| \times (|z_1| + |z_2| + |z_3|), \]

with some constant \( C = C(\delta_0) > 0 \) and here for \( K(0) \) we mean the lower bound of \( K_i(0) \) \((i = 1, 2)\) obtained in (3.11). Pick up \( 1 \leq j \leq m_3 \) (resp. \( n_3 \)). Now we do integration of the equation on \([a_{3j}^\pm, b_{3j}^\pm]\) to have

\[ (x - x^3)z_3' \bigg|_{x = a_{3j}^\pm} + 3(z_3(b_{3j}^\pm) - z_3(a_{3j}^\pm)) + \int_{a_{3j}^\pm}^{b_{3j}^\pm} x^2z_3'(x)dx \]

\[ = \int_{a_{3j}^\pm}^{b_{3j}^\pm} \left[ \frac{1}{2}(x - x^3)(y_1'z_3' + z_1'y_2') + O(1)x(1 - x^2)^{-1}|1 - \phi_22| \times (|z_1| + |z_2| + |z_3|) + 32x(1 - x^2)^{-1}K_1^{-\frac{1}{3}}\phi_1^{-\frac{1}{3}}\phi_2^{-\frac{1}{3}}(-1 + \phi_1 + \phi_1\phi_1\phi_2)(\phi_22 - \phi_12) \right] dx, \]

with \( O(1) \) uniformly bounded, depending only on \( \delta_0 \). It is clear that the three terms on the left hand side of the equation have the same sign. Similarly as the case of \( z_2 \), the third term on the right hand side has a different sign from the left hand side on the interval \([a_{3j}^\pm, b_{3j}^\pm]\). Therefore,

\[ 3|z_3(b_{3j}^\pm) - z_3(a_{3j}^\pm)| \leq \int_{a_{3j}^\pm}^{b_{3j}^\pm} \left[ \frac{1}{2}(x - x^3)(|y_1'z_3'| + |z_1'y_2'|) + C_{31}x(1 - x^2)^{-1}|1 - \phi_22| \times (|z_1| + |z_2| + |z_3|) \right] dx, \]

(3.38)

for some constant \( C_{31} = C_{31}(\delta_0) > 0 \) uniformly bounded. Also, by (3.20) we have

\[ |y_1'z_3'(x)| \leq C_{34}|1 - \phi_2(0)|x, \]

(3.39)

\[ x(1 - x^2)^{-1}|1 - \phi_22(x)| \leq \frac{12}{7}|1 - \phi_2(0)|, \]

(3.40)

for \( x \in (0, \frac{3}{2}) \) and \( i = 1, 2 \), with some constant \( C_{34} = C_{34}(\delta_0) > 0 \); while by (3.25),

\[ |y_2'z_3(x)| \leq C_{35} \varepsilon(1 - x^2), \]

(3.41)

\[ x(1 - x^2)^{-1}|1 - \phi_22(x)| \leq C_{35} \varepsilon(1 - x^2), \]

(3.42)
for \( x \in \left[ \frac{3}{4}, 1 \right] \) and \( i = 1, 2 \), with \( C_{35} = C_{35}(\delta_0) > 0 \). Now we let \( \varepsilon_0 \in (0, 1) \) be a constant satisfying that for \( \varepsilon_0 \leq x \leq 1 \),

\[
\frac{1}{2}(1 - x^2)^2C_{35} \varepsilon \leq \frac{1}{4}, \quad (3.43)
\]

\[
C_{31}C_{35} \varepsilon (1 - x^2) \leq \frac{1}{4}, \quad (3.44)
\]

For that we let \( \varepsilon_0 \in (0, 1) \) satisfy that

\[
(1 - \varepsilon_0^2) \leq \min\left\{ \frac{1}{(4\sqrt{6}C_{35})^2}, \frac{1}{8\sqrt{6}C_{31}C_{35}} \right\}. \quad (3.45)
\]

We then turn to the estimate of \( z_1 \).

Without loss of generality, we assume that \( z_1(0) \geq 0 \). If \( z_1(0) > 0 \), by (2.5), we have \( z_1''(0) > 0 \), and hence \( z_1 > 0 \) and \( z_1' > 0 \) on the first interval of monotonicity \((0, b_1)\).

Otherwise, if \( z_1(0) = 0 \), in the expansion of \( g_1 \), denote the non-local term

\[ g^{(3)} = \frac{1}{4} \tilde{h}^{(3)} = \begin{bmatrix} -\phi_1^{-1}(0)(a_1 + \phi_2^{-1}(0)b_2) & a_1 \\ a_2 & a_2 \end{bmatrix}. \]

It has been shown that if the two solutions admit the same initial data \( \phi_1(0), \phi_2(0) \) and the nonlocal term \( g^{(3)} \), then the two solutions coincide on \( x \in [0, 1] \); see [8]. In the expansion \( z_1(x) = \sum_{k=1}^\infty z_1^{(k)}(0)x^k \) at \( x = 0 \), the coefficients \( z_1^{(k)}(0) \) can be calculated explicitly and inductively using the Einstein equation and expressed by the data \( \phi_1(0), \phi_2(0) \) and \( g^{(3)} \).

In the expansion of \( z_1 = y_{11} - y_{21} \) at \( x = 0 \) by the equation (2.5), one has that the nonlocal term is involved starting from the term \( z_1^{(5)}(0) \) and \( z_1^{(k)}(0) = 0 \) for \( k \leq 4 \). Assume that these two solutions have different non-local terms \( g^{(3)} \), then there exists \( k \geq 5 \) so that \( z_1^{(k)}(0) \neq 0 \).

Without loss of generality, when \( z_1(0) = 0 \), we assume \( z_1' > 0 \) on the first interval of monotonicity \((0, x_1)\) of \( z_1 \).

Let \( 0 = x_0 < x_1 < \ldots < x_{k_1} < x_{k_1+1} = 1 \) be all the local maximum points and local minimum points of \( z_1 \) on \( x \in [0, 1] \), with \( k_1 \) some integer. Therefore, for any \( 0 \leq j \leq k_1 \), we have that \( z_1' \) keeps the same sign on \( x \in (x_j, x_{j+1}) \) with possibly finitely many zeroes on the interval.

Multiplying \( x(1 - x^2) \) on both sides of (2.4), we have

\[
(x(1 - x^2)y_1') - 2y_1' + \frac{1}{6}x(1 - x^2)(y_1')^2 + \frac{1}{3}x(1 - x^2)((y_1')^2 + y_2'y_3' + (y_3')^2) = 0. \quad (3.46)
\]

Substituting the two solutions into (3.46) and take difference of the two equations obtained, we have

\[
0 = (x(1 - x^2)z_1') - 2z_1' + \frac{1}{6}x(1 - x^2)(y_1' + y_2')z_1'
+ \frac{1}{3}x(1 - x^2)[(y_1')^2 + y_1'y_2' + (y_2')^2 - (y_2'y_1')^2 - y_2'y_3' - (y_3')^2]. \quad (3.47)
\]
For each $0 \leq j \leq k_1$, we do integration of (3.47) on the interval $x \in [x_j, x_{j+1}]$,

$$2(z_1(x_{j+1}) - z_1(x_j)) - \frac{1}{6} \int_{x_j}^{x_{j+1}} x(1 - x^2)(y_1'' + y_2')z_1' dx$$

$$= \frac{1}{3} \int_{x_j}^{x_{j+1}} x(1 - x^2)[(y_1')^2 + y_1'y_1'y_1' + (y_1')^2 - (y_2')^2 - y_2'y_2' - (y_2')^2] dx,$$

$$= \frac{1}{3} \int_{x_j}^{x_{j+1}} x(1 - x^2)[(y_1' + y_2')z_1' + (y_1' + y_2')z_2' + (y_1' + y_2')z_3'] dx.$$

(3.48)

By (3.28) and (3.29), we have

$$0 < x(1 - x^2)(y_1'' + y_2') \leq 2C_2(1 - \phi_1(0) + |1 - \phi_2(0)|)x^2(1 - x^2),$$

(3.49)

for $x \in (0, \frac{3}{4}]$, and

$$0 \leq x(1 - x^2)(y_1'' + y_2') \leq 2C_3\varepsilon x(1 - x^2)^4,$$

(3.50)

for $x \in [\frac{3}{4}, 1]$, with the constants $C_2 = C_2(\delta_0) > 0$ and $C_3 = C_3(\delta_0) > 0$. Also, by (3.30) and (3.39), we have

$$x(1 - x^2)|y_1'' + y_2' + y_1'| \leq (2C_{24}1 - \phi_1(0)) + C_{34}|1 - \phi_2(0)|x^2(1 - x^2),$$

(3.51)

$$x(1 - x^2)|y_1'' + y_2' + y_1'| \leq (C_{24}|1 - \phi_1(0)| + 2C_{34}|1 - \phi_2(0)|x^2(1 - x^2),$$

(3.52)

for $x \in (0, \frac{3}{4}]$, with some constants $C_{24} = C_{24}(\delta_0)$ and $C_{34} = C_{34}(\delta_0)$ independent of the solution and the initial data; while by (3.32) and (3.41), we have

$$x(1 - x^2)|y_1'' + y_2' + y_1'| \leq (2C_{25} + C_{35})\varepsilon x(1 - x^2)^2,$$

(3.53)

$$x(1 - x^2)|y_1'' + y_2' + y_1'| \leq (C_{25} + 2C_{35})\varepsilon x(1 - x^2)^2,$$

(3.54)

for $x \in [\frac{3}{4}, 1]$, with some constants $C_{25} = C_{25}(\delta_0) > 0$ and $C_{35} = C_{35}(\delta_0) > 0$.

Now we let $\varepsilon_0 \in (0, 1)$ be a constant satisfying that for $\varepsilon_0 \leq x \leq 1$,

$$\frac{1}{3} x(1 - x^2)^4 C_3\varepsilon^2 \leq 1,$$

(3.55)

$$\varepsilon^2 \leq \min\{ (\varepsilon (2C_{25} + C_{35}))^{-\frac{1}{2}}, (\varepsilon (C_{25} + 2C_{35}))^{-\frac{1}{2}} \}.$$

(3.56)

For that we let $\varepsilon_0 \in (0, 1)$ satisfy that

$$(1 - \varepsilon_0^2) \leq \min\{ (8C_3)^{-\frac{1}{2}}, (2\sqrt{6}(2C_{25} + C_{35}))^{-\frac{1}{2}}, (2\sqrt{6}(C_{25} + 2C_{35}))^{-\frac{1}{2}} \}.$$

(3.57)

We now choose $\varepsilon_0 \in (0, 1)$ to be the smallest number which satisfies (3.36), (3.45) and (3.57). By the monotonicity of $y_i$ on $x \in [0, 1]$ and the interior estimates of the second order elliptic equations (2.4)–(2.7) on $x \in [\frac{3}{4}, \varepsilon_0]$, there exists some constant $C_4 = C_4(\delta_0, \varepsilon_0) > 0$ independent of the initial data and the solution so that

$$|y_1^{(k)}(x)| \leq C_4(|1 - \phi_1(0)| + |1 - \phi_2(0)|),$$

and

$$|y_1^{(k)}(x)| \leq C_4(|1 - \phi_{i-1}(0)|),$$

for $2 \leq i \leq 4, 0 \leq k \leq 4$ and $x \in [\frac{3}{4}, \varepsilon_0]$. 
Now we assume that $\phi_1(0)$ and $\phi_2(0)$ satisfy that
\[
\frac{1}{8} C_2 (|1 - \phi_1(0)| + |1 - \phi_2(0)|) \leq \frac{1}{2}, \quad \frac{1}{8} C_{24} |1 - \phi_1(0)| \leq \frac{1}{4},
\] (3.58)
\[
\frac{\varepsilon_0}{1 - \varepsilon_0^2} |1 - \phi_1(0)| C_{21} \leq \frac{1}{4}, \quad \frac{1}{2} \times \frac{21}{64} C_4 (|1 - \phi_1(0)| + |1 - \phi_2(0)|) \leq \frac{1}{2}.
\] (3.59)

Combining the conditions (3.58)--(3.59) on $\phi_1(0)$ and $\phi_2(0)$, using the control (3.60)--(3.61), and by the choice of $\varepsilon_0 \in (0, 1)$, we apply the inequality (3.37) to have
\[
3 |z_2(b_j^\pm) - z_2(a_j^\pm)| \leq \frac{1}{2} |z_3(b_j^\pm) - z_3(a_j^\pm)| + \frac{1}{4} V_{a_j^\pm} V_{b_j^\pm} (z_1) + \frac{1}{4} (b_j^\pm - a_j^\pm) \sup_{x \in [0, 1]} (|z_1| + |z_2| + |z_3|),
\]
\[
\frac{5}{2} |z_2(b_j^\pm) - z_2(a_j^\pm)| \leq \frac{1}{4} V_{a_j^\pm} V_{b_j^\pm} (z_1) + \frac{1}{4} (b_j^\pm - a_j^\pm) (V_1 + V_2 + V_3).
\]
Summarizing this inequality for all $j$, we have
\[
\frac{5}{4} V(z_2) \leq \frac{1}{4} V(z_1) + \frac{1}{4} (V(z_1) + V_2 + V_3),
\]
and hence
\[
V(z_2) \leq \frac{1}{2} V(z_1) + \frac{1}{4} V(z_3).
\] (3.60)

If in addition, we assume $\phi_2(0)$ satisfies that
\[
\frac{1}{8} C_{34} |1 - \phi_2(0)| \leq \frac{1}{4}, \quad \frac{\varepsilon_0}{1 - \varepsilon_0^2} |1 - \phi_2(0)| \leq \frac{1}{4},
\] (3.61)
then by the choice of $\varepsilon_0 > 0$ and the control (3.39)--(3.42), substituting the conditions (3.58)--(3.59) and (3.61) on $\phi_1(0)$ and $\phi_2(0)$ to the inequality 3.38), we have
\[
3 |z_3(b_j^\pm) - z_3(a_j^\pm)| \leq \frac{1}{2} |z_3(b_j^\pm) - z_3(a_j^\pm)| + \frac{1}{4} V_{a_j^\pm} V_{b_j^\pm} (z_1) + \frac{1}{4} (b_j^\pm - a_j^\pm) \sup_{x \in [0, 1]} (|z_1| + |z_2| + |z_3|),
\]
\[
\frac{5}{2} |z_3(b_j^\pm) - z_3(a_j^\pm)| \leq \frac{1}{4} V_{a_j^\pm} V_{b_j^\pm} (z_1) + \frac{1}{4} (b_j^\pm - a_j^\pm) (V_1 + V_2 + V_3).
\]
Summarizing this inequality for all $j$, we have
\[
\frac{5}{4} V(z_3) \leq \frac{1}{4} V(z_1) + \frac{1}{4} (V(z_1) + V_2 + V_3),
\]
and hence
\[
V(z_3) \leq \frac{1}{2} V(z_1) + \frac{1}{4} V(z_2).
\] (3.62)

Now for the estimate of the total variation of $z_1$ on $x \in [0, 1]$, we give a third condition on the initial data $\phi_1(0)$ and $\phi_2(0):
\[
\frac{1}{3} \times \frac{21}{64} C_4 (|1 - \phi_1(0)| + |1 - \phi_2(0)|) \leq 1, \quad \frac{21}{64} C_4 (|1 - \phi_1(0)| + |1 - \phi_2(0)|) \leq 1,
\] (3.63)
\[
\frac{21}{64} C_4 (|1 - \phi_1(0)| + 2|1 - \phi_2(0)|) \leq 1, \quad \frac{1}{4} (2C_{24} |1 - \phi_1(0)| + C_{34} |1 - \phi_2(0)|) \leq 1,
\] (3.64)
\[
\frac{1}{4} (C_{24} |1 - \phi_1(0)| + 2C_{34} |1 - \phi_2(0)|) \leq 1.
\] (3.65)
Then by (3.48), we have
\[ |z_1(x_j+1) - z_1(x_j)| \leq \frac{1}{3}(V^{x_{j+1}}(z_2) + V^{x_{j+1}}(z_3)). \]  
(3.66)

Summarizing this inequality for all \( j \), one has
\[ V(z_1) \leq \frac{1}{3}(V(z_2) + V(z_3)). \]  
(3.67)

We summarize the above argument: Let \( \varepsilon_0 \in (0, 1) \) be a constant defined below (3.57). Assume that the initial data \( \phi_1(0) \) and \( \phi_2(0) \) satisfy the assumptions (3.58)–(3.59), (3.61) and (3.63)–(3.65), then by (3.60), (3.62) and (3.67), and the non-negativity of \( V(z_i) \) (\( 1 \leq i \leq 3 \)), one has
\[ V(z_i) = 0, \]
for \( 1 \leq i \leq 3 \). Therefore, \( z_i = 0 \) for \( 1 \leq i \leq 3 \). The uniqueness of the solution is proved. In particular, when we take \( \delta_0 = \frac{63}{64} \), we can choose \( (1 - \varepsilon_0^2) = 0.000025, C_4 = 3 \times 10^7 \), and \( \eta_0 = 1 - 3 \times 10^{-8} \) (this is not the optimal constant in the calculation and it could be smaller). If one could estimate \( |W|_g \) to be small, then by the same approach of the above estimate, \( \eta_0 \in (0, 1) \) in the theorem could be much smaller. This completes the proof of Theorem 1.1.
\[ \square \]

4 Uniqueness and existence of non-positively curved conformally compact Einstein metrics with high dimensional homogeneous sphere conformal infinity

In the first subsection, we will use symmetric extension to deform the prescribed conformal infinity problem of the Einstein equation to a two-point boundary value problem of an ODE system, provided that the conformal infinity \((\mathbb{S}^n, [\hat{g}])\) satisfies that \( \hat{g} \) is SU\((k+1)\) invariant on \( \mathbb{S}^{2k+1} \) or Sp\((k+1)\) invariant on \( \mathbb{S}^{4k+3} \) and the CCE manifold is an Hadamard manifold. In the second subsection, we will prove Theorem 1.3, and in the third subsection, we will prove Theorem 1.4.

4.1 The boundary value problem

A Riemannian manifold \((N, g)\) is called a homogeneous Riemannian manifold, if there exists a group \( G \) of isometries acting on \((N, g)\) transitively, and hence \( g \) is a \( G \)-invariant metric. Notice that we have the diffeomorphism \( N \cong G/H \), with \( H \) the isotropy group of some point on \( N \). Any \( G \)-invariant metric \( g \) on \( N \) has the structure described as follows (see for instance [17,48]). Let \( g, \mathfrak{h} \) be the Lie algebra of \( G \) and \( H \). \( g \) is equivalent to the Lie algebra of the Killing vector fields of the metric \( g \) on \( N \). Moreover, if \( H \) is compact, \( g \) has an ad\( \mathfrak{h} \) invariant splitting \( g = \mathfrak{h} \oplus \mathfrak{p} \) such that \([\mathfrak{h}, \mathfrak{p}] \subseteq \mathfrak{p} \). \( H \) acts on \( \mathfrak{p} \) by the adjoint map which induces a splitting:
\[ \mathfrak{p} = \mathfrak{p}_0 \oplus \mathfrak{p}_1 \oplus \ldots \oplus \mathfrak{p}_p, \]  
(4.1)

where \( \mathfrak{p}_i \) is an irreducible subspace for \( 1 \leq i \leq p \), and \( H \) acts on \( \mathfrak{p}_0 \) trivially. Let \( B \) be a bi-invariant metric on \( G \). Any \( G \)-invariant metric \( g \) on \( N \cong G/H \) is determined by its value on \( \mathfrak{p} \), which has a splitting...
\[ g = h \big|_{p_0} + \sum_{i=1}^{p} \alpha_i B \big|_{p_i} \]

with \( h \big|_{p_0} \) an arbitrary metric on \( p_0 \), and any \( \alpha_i > 0 \).

In this paper, for a homogeneous space we always mean a homogeneous Riemannian space.

The homogeneous metrics on spheres are \( G \)-invariant metrics under the transitive action of some Lie group \( G \) on the spheres. It is well known that Lie Groups acting effectively and transitively on spheres have been classified by D. Montgomery and H. Samelson ([38]), and A. Borel ([9,10]); see [5] (p. 179) and also [6,48] for instance.

Up to a scaling factor and isometry, homogeneous metrics on spheres are in one of the three classes: a one parameter family of \( \text{U}(k+1) \) -invariant metrics (equivalently, \( \text{SU}(k+1) \)-invariant) on \( S^{2k+1} \cong \text{SU}(k+1)/\text{SU}(k) \) \( (k \geq 1) \), a three parameter family of \( \text{Sp}(k+1) \)-invariant metrics on \( S^{4k+3} \cong \text{Sp}(k+1)/\text{Sp}(k) \) (containing the \( \text{SU}(2k+2) \)-invariant metrics as a subset in these dimensions) with \( k \geq 0 \), and a one parameter family of \( \text{Spin}(9) \)-invariant metrics on \( S^{15} \cong \text{Spin}(9)/\text{Spin}(7) \). For more details, see [48]. The Killing vector fields of the homogeneous metrics determine the symmetry structure of the \( G \)-invariant metrics, and generate the group action of \( G \) on the spheres.

Recall that \( p = p_0 \oplus p_1 \) for the first two classes, where the adjoint action of \( H \) on \( p_0 \) is trivial and \( p_1 \) is an invariant subspace which is irreducible under the adjoint action of \( H \). Here \( \dim(p_0) = 1 \) for \( S^{2k+1} \cong \text{SU}(k+1)/\text{SU}(k) \) and the action of \( \text{SU}(k) \) on \( p_1 \) is the usual action of \( \text{SU}(k) \) on \( \mathbb{C}^k \), while \( \dim(p_0) = 3 \) for \( S^{4k+3} \cong \text{Sp}(k+1)/\text{Sp}(k) \) and the action of \( \text{Sp}(k) \) on \( p_1 \) is the usual action of \( \text{Sp}(k) \) on \( \mathbb{H}^k \). For the third case, \( p = p_1 \oplus p_2 \) with \( \dim(p_1) = 7 \) and \( \dim(p_2) = 8 \), where the isotropy representations on \( p_1 \) and \( p_2 \) are the unique irreducible representations of \( \text{Spin}(7) \) in that dimension.

Let \( (M^{n+1}, g) \) be a CCE manifold which is Hadamard with its conformal infinity \( (S^n, \widehat{g}) \) so that \( (S^n, \widehat{g}) \) is a homogeneous space. Let \( p_0 \in M \) be the center of gravity, \( r \) be the distance function to \( p_0 \) on \( (M, g) \) and \( x = e^{-r} \) the geodesic defining function about \( 
abla \widehat{g} \) with some constant \( C > 0 \) as discussed in Sect. 2. Now we pick up a point \( q \in \partial M = S^n \). Let \( \gamma \) be the geodesic connecting \( q \) and \( p_0 \). To choose the polar coordinate \( (r, \theta) \) centered at \( p_0 \) near \( \gamma \), we turn to the choice of the local coordinates \( \theta = (\theta_1, ..., \theta^n) \) on \( S^n \) near the point \( q \) based on the above decomposition of the Lie algebra. For a given transitive \( G \)-action on \( S^n \), we can choose a basis of \( G \)-invariant action fields \( Y_1, ..., Y_n \in p \) and a coordinate \( (\theta^1, ..., \theta^n) \) near \( q \) so that \( d\theta^i = \sigma_i \) at \( q \) with \( \sigma_i \) the 1-form corresponding to \( Y_i \) such that under the coordinate \( (\theta^1, ..., \theta^n) \) each \( G \)-invariant metric \( h \) has the diagonal form

\[
\begin{bmatrix}
I_1 & & \\
& I_2 & \\
& & \ddots \\
& & & I_2
\end{bmatrix}
\]

at \( q \), with \( I_1, I_2 \) some positive numbers, for \( S^n \cong \text{SU}(k+1)/\text{SU}(k) \) with \( G = \text{SU}(k+1) \) and \( n = 2k + 1 \); while

\( \circledast \) Springer
On uniqueness and existence of conformally compact Einstein

\[ h = \begin{bmatrix} I_1 & & & \\ & I_2 & & \\ & & I_3 & \\ & & & I_4 \\ & \cdots & & \\ & & & I_4 \end{bmatrix} \]

with \( I_1, \ldots, I_4 \) some positive numbers, up to an SO(3) rotation in the subspace \( p_0 \) generated by \( Y_1, Y_2 \) and \( Y_3 \) for \( \mathbb{S}^n \cong \text{Sp}(k+1)/\text{Sp}(k) \) with \( G = \text{Sp}(k+1) \) and \( n = 4k + 3 \); see [48]. Indeed, we consider the natural embedding \( \mathbb{S}^n \subseteq \mathbb{R}^{n+1} \) with \( \mathbb{S}^n \) the unit sphere on the Euclidean space \( \mathbb{R}^{n+1} \), under the coordinates \((x_1, y_1; x_2, y_2; \ldots; x_{n+1}, y_{n+1})\). Without loss of generality, we assume that \( q = (1, 0, \ldots, 0) \), and choose \((\theta^1, \ldots, \theta^n) = (y_1, x_2, y_2, \ldots, x_{n+1}, y_{n+1})\) near \( q \). Now assume the SU\((k+1)\)-invariant metric \( \hat{g} \) on \( \mathbb{S}^n \) with \( n = 2k + 1 \) has the standard diagonal form

\[ \hat{g} = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_2 \end{bmatrix} \tag{4.2} \]

at \( q \) which is \( \theta_0 = (0, \ldots, 0) \) under the coordinate \( \theta = (\theta^1, \ldots, \theta^{2k+1}) \), with \( \lambda_1, \lambda_2 > 0 \); while the Sp\((k+1)\)-invariant metric \( \hat{g} \) on \( \mathbb{S}^n \) with \( n = 4k + 3 \) has the standard diagonal form

\[ \hat{g} = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \lambda_3 & \\ & & & \lambda_4 \\ & \cdots & & \\ & & & \lambda_4 \end{bmatrix} \tag{4.3} \]

at \( q \) which is \( \theta_0 = (0, \ldots, 0) \) under the coordinate \( \theta = (\theta^1, \ldots, \theta^{4k+3}) \), with \( \lambda_i > 0, 1 \leq i \leq 4 \). Then by the extension of the Killing vector fields and the smoothness result in Theorem 2.3, under the polar coordinate \((r, \theta)\), the SU\((k+1)\)-invariant metric \( g_r \) on \( \partial B_r(p_0) \) for \( r \geq 0 \) has the form

\[ g_r = \sinh^2(r) \begin{bmatrix} I_1(x) & & \\ & I_2(x) & \\ & & \cdots \end{bmatrix} \tag{4.4} \]

at the points along the geodesic \( \gamma = \{(r, \theta_0)| r \geq 0\} = \{(r, 0, \ldots, 0)| r \geq 0\} \), for some positive functions \( I_1(x), I_2(x) \in C^\infty([0, 1]) \), and the Ricci curvature of \( g_r \) has the expression

\[ \text{Ric}(g_r) = \begin{bmatrix} (n-1)I_1^2(x)I_2^{-2}(x) & & \\ & (n+1) - 2I_1(x)I_2^{-1}(x) & \\ & & \cdots \end{bmatrix} \tag{4.5} \]
along $\gamma$, with respect to the conformal infinity $(\mathbb{S}^n, [\hat{g}])$ for $n = 2k + 1$; while the Sp$(k + 1)$-invariant metric $g_r$ on $\partial B_r(p_0)$ for $r \geq 0$ has the form

$$g_r = \sinh^2(r) \begin{bmatrix} I_1(x) & I_2(x) & I_3(x) & I_4(x) \\ I_2(x) & I_3(x) & \vdots & I_4(x) \\ I_3(x) & \vdots & \ddots & I_4(x) \\ I_4(x) & I_4(x) \end{bmatrix}$$

(4.6)

along $\gamma = \{(r, \theta_0) | r \geq 0\}$, for some positive functions $I_i(x) \in C^\infty([0, 1])$ ($1 \leq i \leq 4$), and the Ricci curvature of $g_r$ has the expression (see [48])

$$\text{Ric}(g_r) = \begin{bmatrix} (4nt^2 + \frac{2(t^2 - (t_2 - t_3)^2)}{\eta_{13}}) & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ (4nt^2 + \frac{2(t^2 - (t_1 - t_3)^2)}{\eta_{12}}) & \cdots & \cdots \\ (4nt^2 + \frac{2(t^2 - (t_1 - t_2)^2)}{\eta_{12}}) & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ (4nt^2 + \frac{2(t^2 - (t_1 - t_3)^2)}{\eta_{12}}) & \cdots & \cdots \\ (4nt^2 + \frac{2(t^2 - (t_1 - t_3)^2)}{\eta_{12}}) & \cdots & \cdots \\ 4n + 8 - 2(t_1 + t_2 + t_3) & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ 4n + 8 - 2(t_1 + t_2 + t_3) & \cdots & \cdots \\ \cdots & \cdots & \cdots \end{bmatrix}$$

(4.7)

along $\gamma$, where $t_i = \frac{I_i(x)}{I_4(x)}$ ($1 \leq i \leq 3$), with respect to the conformal infinity $(\mathbb{S}^n, [\hat{g}])$ for $n = 4k + 3$.

Now we give a way to use the symmetry extension to derive (4.4) and (4.5) from the initial data (4.2) and the Einstein equation as in [33]. Notice that the extended Killing vector fields $X_j$ in $(M, g)$ with respect to $Y_j$ satisfy $X_j = \sum_{m=1}^{n} X_j^m \frac{\partial}{\partial \theta^m} = \sum_{m=1}^{n} Y_j^m \frac{\partial}{\partial \theta^m}$ near $\gamma$ on $(M, g)$, with the components $X_j^m$ independent of $r$ for $j = 1, \ldots, n$, under the polar coordinate $(r, \theta) = (r, \theta^1, \ldots, \theta^n)$. And also

$$g = dr^2 + g_r = dr^2 + \sum_{i,j=1}^{n} g_{ij} d\theta^i d\theta^j.$$

Therefore

$$\frac{\partial}{\partial \theta^i} (X_j^m g_{mj}) + \frac{\partial}{\partial \theta^i} (X_j^m g_{mi}) - 2 \Gamma^p_{ij}(g) X_j^m g_{mp} = 0,$$

which is

$$\frac{\partial}{\partial \theta^i} X_j^p g_{pj} + \frac{\partial}{\partial \theta^i} X_j^p g_{pi} + X_j^m \frac{\partial}{\partial \theta^m} g_{ij} = 0.$$

Define the inverse of the matrix with elements $X_j^i$ for $1 \leq i, j \leq n$ as

$$\left( Z_i^j \right) = \left( X_i^j \right)^{-1}.$$

We denote

$$C_{ij}^p = Z_i^q \frac{\partial}{\partial \theta^j} X_q^p,$$
and

$$T^p_{ij} = -T^p_{ji} = C^p_{ij} - C^p_{ji} = Z^q_j Z^m_j [X_m, X_q]^p,$$  \hfill (4.8)

with \([X_m, X_q]\) the Lie bracket of \(X_m\) and \(X_q\). Notice that \(C^p_{ij}\) and \(T^p_{ij}\) are independent of \(r\) and the metric. Then

$$\frac{\partial}{\partial \theta^q} g_{ij} = -C^m_{q1} g_{mj} - C^m_{qj} g_{mi},$$  \hfill (4.9)

$$\Gamma^p_{ij}(g_r) = \frac{1}{2} \left[ -(C^p_{ij} + C^p_{ji}) + g^{pq} (C^m_{iq} g_{mj} - C^m_{jq} g_{mi} + C^m_{qi} g_{mj} + C^m_{qj} g_{mi}) \right].$$  \hfill (4.10)

The Ricci curvature of \(g_r\) on the geodesic spheres has the expression (see [33])

$$R_{ij}(g_r) = \frac{1}{2} \left( \frac{\partial}{\partial \theta^p} C^p_{ij} + C^p_{jq} T^q_{ip} + C^p_{iq} T^q_{jp} + C^p_{qp} T^q_{ij} \right) - \frac{1}{2} g^{pq} \left( \frac{\partial}{\partial \theta^p} T^q_{ij} + C^s_{ip} T^q_{sj} + C^s_{jq} T^q_{is} - C^s_{ps} T^q_{sj} \right) g_{mj}$$

$$- \frac{1}{2} g^{pq} \left( \frac{\partial}{\partial \theta^p} T^q_{jq} + C^s_{jp} T^q_{is} + C^s_{pq} T^q_{js} - C^s_{ps} T^q_{ji} \right) g_{mi} + \frac{1}{2} T^p_{j} T^p_{i} - \frac{1}{4} g^{pq} T^p_{i} T^p_{j} g_{mn}$$

$$- \frac{1}{8} g^{pq} T^p_{j} T^p_{i} g_{mn} - \frac{1}{2} g^{pq} T^p_{j} T^p_{i} g_{mn} - \frac{1}{8} g^{pq} T^p_{j} T^p_{i} g_{mn} + \frac{1}{4} g^{pq} T^p_{i} T^p_{j} g_{mn} + \frac{1}{4} g^{pq} T^p_{i} T^p_{j} g_{mn}.$$

Let

$$g_r = \sinh^2 (r) \tilde{h} = \frac{x^{-2} (1 - x^2)^2}{4} \tilde{h}_{ij} d\theta^i d\theta^j.$$

Therefore, at the points along the line \(\gamma = \{(r, \theta_0) | r > 0\}\), the Einstein equation is equivalent to the equations (see [33])

$$\frac{d}{dx} (x(1 - x^2)^2 \tilde{h}_{pq} \frac{d}{dx} \tilde{h}_{pq}) + \frac{1}{2} x(1 - x^2)^2 \tilde{h}_{ps} \frac{d}{dx} \tilde{h}_{sm} \tilde{h}_{mq} \frac{d}{dx} \tilde{h}_{pq} - 2 \tilde{h}_{pq} \frac{d}{dx} \tilde{h}_{pq} = 0,$$  \hfill (4.12)

$$C^p_{pq} \tilde{h}_{qs} \frac{d}{dx} \tilde{h}_{si} + C^m_{iq} \tilde{h}_{pq} \frac{d}{dx} \tilde{h}_{mp} - C^p_{qp} \tilde{h}_{qs} \frac{d}{dx} \tilde{h}_{si} - C^p_{pi} \tilde{h}_{pq} \frac{d}{dx} \tilde{h}_{pq} = 0,$$  \hfill (4.13)

$$- \frac{1}{8} x(1 - x^2)^2 \frac{d^2}{dx^2} \tilde{h}_{ij} + \frac{1}{8} \left[ \frac{(n - 1) + (n + 1) x^2}{8} \left( 1 - x^2 \right) \frac{d}{dx} \tilde{h}_{ij} + \frac{x(1 - x^2)^2}{8} \tilde{h}_{pq} \frac{d}{dx} \tilde{h}_{pq} \frac{d}{dx} \tilde{h}_{pq} \frac{d}{dx} \tilde{h}_{pq} \frac{d}{dx} \tilde{h}_{ij} + (n - 1) x \tilde{h}_{ij} + x R_{ij} (\tilde{h}) = 0,$$  \hfill (4.14)

for \(1 \leq i, j \leq n\), with \(C^p_{ij}(\theta_0)\) independent of \(x\), and \(R_{ij}(\tilde{h}) = R_{ij}(g_r)\). By the regularity result in Theorem 2.3 (see also [15]), \(\tilde{h} \in C^\infty ([0, 1])\). This is a system of ordinary differential equations for \(\tilde{h}_{ij}\) on \(x \in [0, 1]\), with the boundary conditions

$$\tilde{h}_{ij}(0) = [\tilde{g}_i(\theta_0)], \quad \tilde{h}_{ij}(1) = g^0_{ij}, \quad \frac{d}{dx} \tilde{h}_{ij}(0) = 0, \quad \frac{d}{dx} \tilde{h}_{ij}(1) = 0,$$  \hfill (4.15)

with \(g^0\) the round metric on \(S^n\).

Now we calculate \(C^p_{ij}\) and \(\frac{\partial}{\partial \theta^p} T^p_{ij}\) at \(\theta = \theta_0\) for \(S^5 = SU(3)/SU(2)\). Consider the Euclidean space \(\mathbb{R}^6 = \mathbb{C}^3\) as the three dimensional complex space with the coordinate \((z_1, z_2, z_3)\), where \(z_j = x_j + i y_j\) \((j = 1, 2, 3)\). At \(q = (1, 0, 0)\), we choose a basis of the
Lie algebra \( su(3) \) (see for example [29])

\[
v_1 = \begin{pmatrix} 2i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & -i \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},
\]

\[
v_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad v_5 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad v_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -i \end{pmatrix},
\]

\[
v_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad v_8 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{pmatrix}.
\]

It is clear that the subspace generated by \( \{v_6, v_7, v_8\} \) is tangent to \( \frac{1}{SU(2)} \subset SU(3) \) and is isomorphic to \( su(2) \). Notice that \( v_1 \in p_0 \) is \( Ad_{SU(2)} \)-invariant, and so is the subspace \( p_2 \) spanned by \( \{v_2, v_3, v_4, v_5\} \). The corresponding \( SU(3) \)-invariant vector field \( Y_j \) with respect to \( v_j \) has the expression \( Y_j = (z_1 z_2 z_3) v_j^T \) at the point \((z_1, z_2, z_3) \in S^5, 1 \leq j \leq 8\). Therefore,

\[
Y_1 = \begin{pmatrix} 2iz_1 \\ -iz_2 \\ -iz_3 \end{pmatrix}^T, \quad Y_2 = \begin{pmatrix} z_2 \\ -z_1 \\ 0 \end{pmatrix}^T, \quad Y_3 = \begin{pmatrix} iz_2 \\ iz_1 \\ 0 \end{pmatrix}^T, \quad Y_4 = \begin{pmatrix} z_3 \\ 0 \\ -z_1 \end{pmatrix}^T,
\]

\[
Y_5 = \begin{pmatrix} iz_3 \\ 0 \\ iz_1 \end{pmatrix}^T, \quad Y_6 = \begin{pmatrix} 0 \\ iz_2 \\ -iz_3 \end{pmatrix}^T, \quad Y_7 = \begin{pmatrix} 0 \\ z_3 \\ -z_2 \end{pmatrix}^T, \quad Y_8 = \begin{pmatrix} 0 \\ iz_3 \\ iz_2 \end{pmatrix}^T.
\]

at the point \((z_1, z_2, z_3) \in S^5\). Direct calculations show that

\[
[Y_1, Y_2] = -3Y_3, \quad [Y_1, Y_3] = 3Y_2, \quad [Y_1, Y_4] = -3Y_5, \quad [Y_1, Y_5] = 3Y_4,
\]

\[
[Y_2, Y_3] = -Y_1 + Y_6, \quad [Y_2, Y_4] = Y_7, \quad [Y_2, Y_5] = Y_8, \quad [Y_3, Y_4] = -Y_8,
\]

\[
[Y_3, Y_5] = Y_7, \quad [Y_4, Y_5] = -Y_1 - Y_6.
\] (4.16)

We choose \( Y_1, ..., Y_5 \) as a basis of the Killing vector field at \( q \). Under the coordinate \((\theta^1, ..., \theta^5) = (y_1, x_2, y_2, x_3, y_3) \) near \( q \), we have the expression of \( X_j = Y_j \) \((1 \leq j \leq 8)\),

\[
X_1 = \left( 2\sqrt{1 - \sum_{k=1}^{5}(\theta^k)^2}, \theta^3, -\theta^2, \theta^5, -\theta^4 \right), \quad X_2 = \left( \theta^3, -\sqrt{1 - \sum_{k=1}^{5}(\theta^k)^2}, -\theta^1, 0, 0 \right),
\]

\[
X_3 = \left( \theta^2, -\theta^1, \sqrt{1 - \sum_{k=1}^{5}(\theta^k)^2}, 0, 0 \right), \quad X_4 = \left( \theta^5, 0, 0, -\sqrt{1 - \sum_{k=1}^{5}(\theta^k)^2}, -\theta^1 \right),
\]

\[
X_5 = \left( \theta^4, 0, 0, -\theta^1, \sqrt{1 - \sum_{k=1}^{5}(\theta^k)^2} \right), \quad X_6 = \left( 0, -\theta^3, \theta^2, \theta^5, -\theta^4 \right),
\]

\[
X_7 = \left( 0, \theta^4, \theta^5, -\theta^2, -\theta^3 \right), \quad X_8 = \left( 0, -\theta^5, \theta^4, -\theta^3, \theta^2 \right).
\]

Hence at \( \theta = \theta_0 \),

\[
C_{12}^3 = -C_{13}^2 = C_{14}^5 = -C_{15}^4 = -\frac{1}{2},
\]

\[
C_{21}^3 = -C_{23} = -C_{31}^2 = C_{32}^5 = C_{41}^5 = -C_{51}^4 = C_{54} = -C_{45}^1 = 1,
\]

\( \text{Springer} \)
and $C_{ij}^p = 0$ otherwise. Substituting to (4.13), we have

$$\frac{d}{dx} \tilde{h}_{ij} = 0,$$

for $i \neq j$ and $0 \leq x \leq 1$ and moreover

$$\frac{d}{dx} \tilde{h}_{ii} = \frac{d}{dx} \tilde{h}_{jj},$$

for $i, j > 1$. Using the initial data (4.15) and (4.2), we have that $\tilde{h}_{ij} = 0$ for $i \neq j$ and hence denote $\tilde{h}$ as

$$\tilde{h} = \begin{bmatrix} I_1(x) \\ I_2(x) \\ \vdots \\ I_n(x) \end{bmatrix}$$

for $0 \leq x \leq 1$ at $\theta = \theta_0$, where $I_i(x) \in C^\infty([0, 1])$, by Theorem 2.3. By (4.8) and (4.16), we have

$$\frac{\partial}{\partial \theta} T^p_{ij} = -\alpha_{cba}(-Z_i^d \frac{\partial}{\partial \theta} X^q_d Z_j^b Z_j^a X^p_a - Z_i^d Z_j^d \frac{\partial}{\partial \theta} X^q_d Z_j^b X^p_a + Z_i^d Z_j^b \frac{\partial}{\partial \theta} X^p_a)$$

= $-\alpha_{cba}(-C_{im}^q Z_j^b X^p_a - Z_j^c C_{jm}^q Z_j^b X^p_a + Z_j^c Z_j^b X^p_a C_{qm})$,

with $\alpha_{cba}$ the structure constants in (4.16). Therefore, for $0 \leq x \leq 1$ and $\theta = \theta_0$, the expression of $Ric(\tilde{h}) = Ric(g_r)$ in (4.5) holds for $k = 2$. Based on the choice of the basis of the SU$(k+1)$-invariant vector fields on $S^{2k+1}$ for $k = 2$ in [29] and $k = 3$ in [43], by induction one can easily get the general formula of the choice of the basis of the SU$(k+1)$-invariant vector fields on $S^{2k+1}$ for general $k \geq 2$, and do the above calculations to get the expression (4.4) of $g_r$ and (4.5) for $Ric(g_r)$. For $S^{4k+3} \cong Sp(k+1)/Sp(k)$, similar calculations can be done and the basis of Lie algebra $sp(k+1)$ is chosen in [48]. To deal with the possible SO(3) rotation in $p_0$, one has to view (4.13) as a system of 1-order linear homogeneous ODEs of $\tilde{h}_{ij}$ ($1 \leq i < j \leq 3$) and solve the initial value problem of (4.13) with homogeneous initial data as the generalized Berger metric case, for details see Lemma 4.2 in [33], and hence (4.6) holds and the calculations of (4.7) can be found in [48].

On $(M^{n+1}, g)$ for $n = 2k + 1$ with its conformal infinity $(S^{2k+1}, [\tilde{g}])$ where $\tilde{g}$ is an SU$(k+1)$-invariant metric, let $(r, \theta)$ be the polar coordinate chosen above. Assume $\tilde{g}$ has the form (4.2) at $q$ under the local coordinate $\theta = (\theta^1, ..., \theta^n)$. We substitute (4.4) and (4.5) to the Einstein equation (4.12)-(4.14) to have

$$\frac{d}{dx} [x(1-x^2)(I^{-1}_1 \frac{\partial}{\partial x} I_1 + (n-1)I^{-1}_2 \frac{\partial}{\partial x} I_2)] + \frac{1}{2} x(1-x^2)((I^{-1}_1 \frac{\partial}{\partial x} I_1)^2 + (n-1)(I^{-1}_2 \frac{\partial}{\partial x} I_2)^2)$

$$- 2(I^{-1}_1 \frac{\partial}{\partial x} I_1 + (n-1)I^{-1}_2 \frac{\partial}{\partial x} I_2) = 0,$$

$$- \frac{1}{8} x(1-x^2)^2 \frac{d^2}{dx^2} I_1 + \frac{1}{8} [(n-1) + (1+n)x^2] (1-x^2) \frac{d}{dx} I_1 + \frac{x(1-x^2)^2}{8} I^{-1}_1 \frac{d}{dx} I_1$,

$$+ \frac{1}{8} (1+x^2)(1-x^2)(I^{-1}_1 \frac{\partial}{\partial x} I_1 + (n-1)I^{-1}_2 \frac{\partial}{\partial x} I_2) I_1$,

$$- \frac{1}{16} x(1-x^2)^2 (I^{-1}_1 \frac{\partial}{\partial x} I_1 + (n-1)I^{-1}_2 \frac{\partial}{\partial x} I_2) \frac{d}{dx} I_1 + (1-n) x I_1 + (n-1)x I_1 I^{-1}_2 I_2^2 = 0,$$

$$- \frac{1}{8} x(1-x^2)^2 \frac{d^2}{dx^2} I_2 + \frac{1}{8} [(n-1) + (1+n)x^2] (1-x^2) \frac{d}{dx} I_2 + \frac{x(1-x^2)^2}{8} I^{-1}_2 \frac{d}{dx} I_2$$.  

Springer
\[ + \frac{1}{8} (1 + x^2)(1 - x^2)(I_1^{-1} \frac{d}{dx} I_1 + (n - 1)I_2^{-1} \frac{d}{dx} I_2)I_2 \]
\[ - \frac{1}{16} x(1 - x^2)^2 (I_1^{-1} \frac{d}{dx} I_1 + (n - 1)I_2^{-1} \frac{d}{dx} I_2) \frac{d}{dx} I_2 + (1 - n)x I_2 + x(n + 1 - 2I_1 I_2^{-1}) = 0, \]
on \( x \in [0, 1] \) where \( I_i' = \frac{d}{dx} I_i \), with the boundary condition
\[ \frac{I_1(0)}{I_2(0)} = \frac{\lambda_1}{\lambda_2}, \quad I_1(1) = I_2(1) = 1, \quad I_1'(0) = I_2'(0) = I_1'(1) = I_2'(1) = 0. \] (4.17)

Denote \( K = I_1 I_2^{n-1} \) and \( \phi = \frac{I_2}{I_1}, y_1 = \log(K) \) and \( y_2 = \log(\phi) \) so that
\[ I_1 = (K\phi^{1-n})^{\frac{1}{n}}, \quad I_2 = (K\phi)^{\frac{1}{n}}. \] (4.18)

Therefore, the boundary value problem of the Einstein metrics becomes
\[ y''_1 + \frac{1}{2n} [(y_1')^2 + (n - 1)(y_2')^2] - x^{-1}(1 + 3x^2)(1 - x^2)^{-1}y'_1 = 0, \] (4.19)
\[ y''_1 - [2(n - 1) + (1 + 2n)x^2] x^{-1}(1 - x^2)^{-1}y'_1 + \frac{1}{2} (y'_1)^2 \]
\[ + 8(n - 1)(1 - x^2)^{-2} [n - (n + 1)K^{-\frac{1}{n}} \phi^{-\frac{1}{n}} + K^{-\frac{1}{n}} \phi^{-\frac{n+1}{n}}] = 0, \] (4.20)
\[ y''_2 - [(n - 1) + (1 + n)x^2] x^{-1}(1 - x^2)^{-1}y'_2 + \frac{1}{2} y'_2 y'_2 + 8(n + 1)(1 - x^2)^{-2} K^{-\frac{1}{n}} \phi^{-\frac{1}{n}} (\phi^{-1} - 1) = 0, \] (4.21)
for \( y_1(x), y_2(x) \in C^\infty([0, 1]) \) with the boundary condition
\[ \phi(0) = \frac{\lambda_2}{\lambda_1}, \quad K(1) = \phi(1) = 1, \quad y_1'(0) = y_2'(0) = y_1'(1) = y_2'(1) = 0. \] (4.22)

Combining (4.19) and (4.20), we have
\[ (y_1')^2 - (y_2')^2 - 4nx^{-1}(1 + x^2)(1 - x^2)^{-1}y'_1 + 16n(1 - x^2)^{-2}(n - (n + 1)(K\phi)^{-\frac{1}{n}} \]
\[ + K^{-\frac{1}{n}} \phi^{-\frac{n+1}{n}}) = 0. \] (4.23)

By (2.12), we have the expansion of \( y_1 \) and \( y_2 \) at \( x = 0 \), which can also be done directly using the system (4.19)–(4.21) and the boundary data (4.22). Let \( \Phi(x) \) be the function on the left hand side of the equation (4.23). Take derivative of \( \Phi \) and use the equations (4.20) and (4.21) we have
\[ \Phi' + (y_1' - 2x^{-1}(n - 1 + (n + 1)x^2)(1 - x^2)^{-1})\Phi = 0. \] (4.24)

Consider \( y_1' \) as a given function. Using the expansion (2.12), we can derive that (4.24) has a unique solution \( \Phi = 0 \), which is (4.23). Therefore, (4.20) and (4.21) combining with the expansion of the Einstein metric imply (4.23). Similarly, any two of the equations (4.19)–(4.21) and (4.23) combining with the boundary expansion of the Einstein metric give the other two equations. Notice that the coefficients of the expansion of the metric can be solved inductively by the equations (4.20)–(4.21) and the initial data (4.22) before the order \( x^n \).

On \( (M^{n+1}, g) \) for \( n = 4k + 3 \) with its conformal infinity \((S^{4k+3}, [\hat{g}])\) where \( \hat{g} \) is an \( \text{Sp}(k + 1) \)-invariant metric, let \( (r, \theta) \) be the polar coordinate chosen above. Assume \( \hat{g} \) satisfies (4.3) at \( g \) under the local coordinate \( \theta = (\theta_1, \ldots, \theta^n) \). Denote \( K = I_1 I_2 I_3 I_4^{n-3}, \quad t_i = \frac{I_i}{I_4} \) for \( 1 \leq i \leq 3 \). Let \( y_1 = \log(K) \) and \( y_{i+1} = \log(t_i) \) for \( 1 \leq i \leq 3 \). Substituting (4.6) and (4.7)
to the Einstein equation (4.12)–(4.14), we have

\[
y''_1 - x^{-1}(1 + 3x^2)(1 - x^2)^{-1}y'_1 + \frac{1}{2n^2}[n(y'_1)^2 + ((n - 1)y'_2 - y'_3 - y'_4)^2 \\
+ (-y'_2 + (n - 1)y'_3 - y'_4)^2 + (-y'_2 - y'_3 + (n - 1)y'_4)^2 + (n - 3)(y'_2 + y'_3 + y'_4)^2] = 0,
\]

(4.25)

\[
y''_2 - x^{-1}(2n - 1 + (2n + 1)x^2)(1 - x^2)^{-1}y'_1 + \frac{1}{2}(y'_1)^2 \\
+ 8(1 - x^2)^{-2}[n(n - 1) - (K^{-1}t_1t_2t_3)^{\frac{1}{2}} ((n - 3)(n + 5) - (n - 3)(t_1 + t_2 + t_3) \\
+ \frac{2(2t_1t_2 + 2t_1t_3 + 2t_2t_3 - t_1^2 - t_2^2 - t_3^2)}{t_1t_2t_3} )] = 0,
\]

(4.26)

\[
y''_3 - x^{-1}(n - 1 + (n + 1)x^2)(1 - x^2)^{-1}y'_2 + \frac{1}{2}y'_1y'_2 \\
- 8(1 - x^2)^{-2}(K^{-1}t_1t_2t_3)^{\frac{1}{2}}[(n - 1)t_1 + 2t_2 + 2t_3 - n - 5 + \frac{2(t_1^2 - (t_2 - t_3)^2)}{t_1t_2t_3}] = 0,
\]

(4.27)

\[
y''_4 - x^{-1}(n - 1 + (n + 1)x^2)(1 - x^2)^{-1}y'_3 + \frac{1}{2}y'_1y'_3 \\
- 8(1 - x^2)^{-2}(K^{-1}t_1t_2t_3)^{\frac{1}{2}}[(n - 1)t_2 + 2t_1 + 2t_3 - n - 5 + \frac{2(t_2^2 - (t_1 - t_3)^2)}{t_1t_2t_3}] = 0,
\]

(4.28)

\[
y''_4 - x^{-1}(n - 1 + (n + 1)x^2)(1 - x^2)^{-1}y'_4 + \frac{1}{2}y'_1y'_4 \\
- 8(1 - x^2)^{-2}(K^{-1}t_1t_2t_3)^{\frac{1}{2}}[(n - 1)t_3 + 2t_1 + 2t_2 - n - 5 + \frac{2(t_3^2 - (t_1 - t_2)^2)}{t_1t_2t_3}] = 0,
\]

(4.29)

for \(y_i(x) \in C^\infty([0, 1]) \ (1 \leq i \leq 4)\) with the boundary condition

\[t_i(0) = \frac{\lambda_i}{\lambda_4}, \quad K(1) = t_i(1) = 1, \quad y'_i(0) = y''_i(1) = 0,\]

(4.30)

for \(1 \leq i \leq 3\) and \(1 \leq j \leq 4\). Combining (4.25) and (4.26), we have

\[
(y'_1)^2 - 4nx^{-1}(1 + x^2)(1 - x^2)^{-1}y'_1 - \frac{1}{n(n - 1)}[((n - 1)y'_2 - y'_3 - y'_4)^2 + (-y'_2 + (n - 1)y'_3 - y'_4)^2 \\
+ (-y'_2 - y'_3 + (n - 1)y'_4)^2 + (n - 3)(y'_2 + y'_3 + y'_4)^2] + \frac{16n}{n - 1}(1 - x^2)^{-2} \left[ n(n - 1) - (K^{-1}t_1t_2t_3)^{\frac{1}{2}} \left( (n - 3)(n + 5) - (n - 3)(t_1 + t_2 + t_3) + \frac{2(2t_1t_2 + 2t_1t_3 + 2t_2t_3 - t_1^2 - t_2^2 - t_3^2)}{t_1t_2t_3} \right) \right] = 0.
\]

(4.31)

Let \(\Phi(x)\) be the function on the left hand side of the equation (4.31). Take derivative of \(\Phi\) and use the equations (4.25) and (4.27)–(4.29), we have

\[
\Phi' + \left( \frac{1}{n} y'_1 - 4x(1 - x^2)^{-1} \right) \Phi = 0.
\]

(4.32)

Consider \(y'_1\) as a given function. By the expansion of the metric at \(x = 0\) and the initial data, similar as the case of the SU\((k + 1)\)-invariant metrics, the equation has a unique solution \(\Phi =\)
0. Therefore, (4.25) and (4.27)–(4.29), combining with the expansion of the Einstein metric at \( x = 0 \), imply (4.31). Similarly, any four equations in the system of the six equations (4.25)–(4.29) and (4.31) containing at least two of (4.27)–(4.29), combining with the expansion of the Einstein metric at \( x = 0 \), imply the other two equations.

4.2 Uniqueness of the solution to the boundary value problem (4.19)–(4.22)

The approach to the proof of uniqueness of the solution to the boundary value problem (4.19)–(4.22) for \( n = 3 \) (which is the Berger sphere case), used in [33], can be generalized directly to all odd dimensions. We only list the statement of the lemmas required and the conclusion. For details of the proof, one is referred to [33]. By similar argument as in Sect. 3, using the volume comparison and the result in [35], we have

\[
\left( \frac{Y(S^n, [\hat{g}])}{Y(S^n, [\hat{g}])} \right)^2 \leq K(0) = \lim_{x \to 0} \frac{\det(\tilde{h})}{\det(\tilde{h}^{\tilde{h}^{n+1}}(x))} = \lim_{r \to +\infty} \frac{\det(g_r)}{\det(g_r^{\tilde{h}^{n+1}}(r))} < 1, \tag{4.33}
\]

with \( Y(S^n, [\hat{g}]) \) the Yamabe constant of \((S^n, [\hat{g}])\) and \( g^{\tilde{h}} \) the round sphere metric, where

\[
g^{\tilde{h}^{n+1}} = dr^2 + g_r^{\tilde{h}^{n+1}}(r) = x^{-2} \left( dx^2 + \frac{1 - x^2}{4} h^{\tilde{h}^{n+1}} \right)
\]

is the hyperbolic metric, \( g_r^{\tilde{h}^{n+1}}(r) = \sinh(r)^2 h^{\tilde{h}^{n+1}} \) and \( h^{\tilde{h}^{n+1}} \) is the round metric on \( S^n \).

**Lemma 4.1** For the initial data \( \phi(0) \neq 1 \), we have \( y'_2(x) > 0 \) for \( x \in (0, 1) \). Also, it holds that \( y'_2(x) > 0 \) and \( \phi(0) < \phi(x) < 1 \) for \( x \in (0, 1) \) if \( \phi(0) < 1 \); while \( y'_2(x) < 0 \) and \( 1 < \phi(x) < \phi(0) \) for \( x \in (0, 1) \) if \( \phi(0) > 1 \). That is to say, \( K \) and \( \phi \) are monotonic on \( (0, 1) \).

For the proof of Lemma 4.1, see Lemma 5.1 in [33]. As in Sect. 3, by (4.23) and the initial data (4.22), we have

\[
y'_1 = 2nx^{-1}(1 - x^2)^{-1}[1 + x^2 - \sqrt{(1 + x^2)^2 + \frac{1}{4n^2}(1 - x^2)^2y'^2_2 - \frac{4}{n}y^2(1 + x^2) + \frac{1}{n}(\phi K)^{-\frac{1}{n}} + K^{-1} \phi^{(\frac{n+1}{n})}}}], \tag{4.34}
\]

and

\[
n - (n + 1)(\phi K)^{-\frac{1}{n}} + K^{-\frac{1}{n}} \phi^{(\frac{n+1}{n})} > 0, \tag{4.35}
\]

for \( x \in [0, 1) \). This gives a lower bound of \( K(0) \) under the assumption \( \phi(0) > \frac{1}{(n+1)} \), and hence,

\[
y'_1 < 2nx^{-1}(1 - x^2)^{-1}[1 + x^2 - \sqrt{(1 - x^2)^2} = 4nx(1 + x^2)^{-1}. \tag{4.36}
\]

We assume that the boundary value problem (4.19)–(4.22) admits two solutions \((y_{11}, y_{12})\) and \((y_{21}, y_{22})\) with \( y_{11} = \log(K_1), y_{12} = \log(\phi_1), y_{21} = \log(K_2) \) and \( y_{22} = \log(\phi_2) \). Let \( z_i = y_{1i} - y_{2i}, i = 1, 2 \). By the same argument in Lemma 5.3 in [33], we have

**Lemma 4.2** Assume that \( \frac{1}{n+1} \leq \phi(0) \neq 1 \). For any two zeroes \( 0 < x_1 < x_2 \leq 1 \) of \( z_1 \) so that there is no zero of \( z'_1 \) on the interval \( x \in (x_1, x_2) \), there exists a point \( x_3 \in (x_1, x_2) \) so that

\[
(y'_{12} + y'_{22})z'_1z'_2|_{x=x_3} < 0. \tag{4.37}
\]
Also, for any zero $0 < x_2 < 1$ of $z'_1$, there exists $\epsilon > 0$ so that for any $x_2 - \epsilon < x < x_2$, we have

$$ (y'_i(x) + y''_i(x))z'_i(x)z''_i(x) > 0. \quad (4.38) $$

Based on Lemma 4.2, by the same proof of Theorem 5.4 in [33], we obtain the uniqueness of the solution to the boundary value problem (4.19)–(4.22).

**Theorem 4.3** The solution to the boundary value problem (4.19)–(4.22) for $\frac{1}{n} < \phi(0) < 1$ and $1 < \phi(0) < n$ must be unique if it exists.

The main idea of the proof Theorem 4.3 is arguing by contradiction like this: Let $(y_{i1}, y_{i2})$ with $i = 1, 2$ be two different solutions to the problem and $z_i = y_{1i} - y_{2i}$ for $i = 1, 2$ be defined as above. Let $x_1$ be the largest zero of $z'_1$ on $[0, 1)$ (When $z_1(0) = 0$, by the mean value theorem, there exists a zero of $z'_1$ on $(0, 1)$ and hence $x_1 > 0$). Then as the proof of Theorem 5.4 in [33], we can show that there exists a zero of $z'_2$ on $(x_1, 1)$, and we take $x_2$ to be the largest zero of $z'_2$ on $(x_1, 1)$. We multiply $x^{1-n}(1 - x^2)^n$ on both sides of (4.21) for these two solutions, take difference, and do integration on $[x_2, 1]$ to have

$$ 8(n + 1) \int_{x_2}^{1} x^{1-n}(1 - x^2)^{n-2} [K_1^{-\frac{1}{n}} - K_2^{-\frac{1}{n}}] \phi_1^{-\frac{1}{n}} (\phi_1^{-1} - 1) + K_2^{-\frac{1}{n}} \phi_2^{-\frac{1}{n}} (\phi_2^{-1} - 1) \frac{1}{2} x^{1-n}(1 - x^2)^n (z'_1y_{12} + y'_2z'_2)dx = 0. $$

Substituting the condition $\phi(0) < n + 1$ to this equation, we obtain a contradiction by Lemma 4.1 and Lemma 4.2. (Notice that Theorem 5.2 in [33] is not needed in the proof of Theorem 5.4 in [33]. In particular, for the case $K_1(0) = K_2(0)$, by the mean value theorem, there exists a zero of $z'_1$ in $x \in (0, 1)$, and Theorem 5.4 in [33] covers this case.)

Now we are ready to prove the uniqueness result Theorem 1.3.

**Proof of Theorem 1.3** For the case $\phi(0) = 1$ i.e., $\lambda_1 = \lambda_2$ so that the conformal infinity is the round sphere, the theorem has been proved in [4,18,35,42].

Now we assume that $\lambda_1 \neq \lambda_2$. Pick up a point $q \in \partial M = \mathbb{S}^n$. Let $x$ be the geodesic defining function about $C\hat{g}$ with $C > 0$ some constant so that $x = e^{-r}$ with $r$ the distance function on $(M, g)$ to the center of gravity $p_0 \in M$; see Theorem 3.6 in [33]. Under the polar coordinate $(x, \theta)$ with $0 \leq x \leq 1$ and $\theta = 0$ along the geodesic $\gamma$ connecting $q$ and $p_0$, by the discussion above we have that the Einstein equation with prescribed conformal infinity $(\mathbb{S}^n, [\hat{g}])$ with $\hat{g}$ the homogeneous metric in (1.2), is equivalent to the boundary value problem (4.19)–(4.22) along the geodesic $\gamma$ provided that the solution has non-positive sectional curvature. Then by Theorem 4.3, up to isometries, the CCE metric is unique.

### 4.3 The existence result

We now give some estimates on the solution $(y_1, y_2)$ with $y_1 = \log(K)$ and $y_2 = \log(\phi)$ to the boundary value problem (4.19)–(4.22) based on the monotonicity lemma and (4.35). The estimates are similar as that in Sect. 3.

**Lemma 4.4** Let $\epsilon > 0$ be any small number. For the initial data $\frac{1}{(n+1)} + \epsilon < \phi(0) < (n+1) - \epsilon$, there exists a constant $C = C(\epsilon) > 0$ independent of the solution and the initial
data $\phi(0)$ such that

$$|y_1^{(k)}(x)| \leq C, \quad (4.39)$$

$$|y_2^{(k)}(x)| \leq C, \quad (4.40)$$

with $y_i^{(k)}$ the $k$–th order derivative of $x$, for $1 \leq k \leq 3$, $i = 1, 2$ and $x \in (0, \frac{3}{4}]$.

**Proof** By the monotonicity of $y_i$ for $x \in [0, 1]$ and (4.35), we have

$$\left(\frac{(n+1)\phi(0) - 1}{n\phi} \right)^{\frac{n+1}{n}} < K(0) \leq K(x) \leq 1,$$

and $\phi(x)$ lies on the interval between 1 and $\phi(0)$ for $x \in [0, 1]$. By the interior estimates of the elliptic equations (4.20) and (4.21), there exists a constant $C = C(\epsilon) > 0$ such that

$$|y_i^{(k)}(x)| \leq C |1 - \phi(0)|,$$

for $\frac{1}{2} \leq x \leq \frac{3}{4}$ and $i = 1, 2$. Now we multiply $K^{\frac{1}{2}}x^{1-n}(1 - x^2)^n$ on both sides of (4.21) to have

$$(K^{\frac{1}{2}}x^{1-n}(1 - x^2)^n y_2')(x) + 8(n+1)x^{1-n}(1 - x^2)^n - 2 K^{\frac{1}{2}}x^{1-n} \phi^{-\frac{(k+1)n}{n}} - 1 - \phi(x) = 0.$$ 

For $x \in (0, \frac{1}{2})$, we integrate the equation on the interval $[x, \frac{1}{2}]$ to have

$$K^{\frac{1}{2}}x^{1-n}(1 - x^2)^n y_2'(x) = K^{\frac{1}{2}}x^{\frac{1}{2}} 2^{n-1} 4^n \left(\frac{3}{4}\right)^n y_2'(\frac{1}{2})$$

$$+ 8(n+1) \int_x^{\frac{1}{2}} s^{1-n}(1 - s^2)^n K^{\frac{1}{2}}x^{1-n} \phi^{-\frac{(k+1)n}{n}} (s)(1 - \phi(s)) ds. \quad (4.41)$$

Therefore, there exists a constant $C = C(\epsilon) > 0$ independent of the solution and $\phi(0)$, such that

$$|y_2'(x)| \leq C |1 - \phi(0)| x.$$ 

By (4.36), for $x \in [0, \frac{1}{2}]$,

$$|y_1'(x)| \leq \frac{4n}{3} x.$$

Substituting these two estimates to (4.20) and (4.21), we have that there exists a constant $C = C(\epsilon) > 0$ independent of the solution and the initial data such that

$$|y_1''(x)| \leq C,$$

for $x \in [0, \frac{1}{2}]$ and $i = 1, 2$. We take derivative of (4.19) with respect to $x$ and obtain

$$y_1''' = -\frac{1}{n} \{ y_1'y_1'' + (n - 1)y_2'y_2'' \} + \frac{1 + 3x^2}{x(1 - x^2)} \left( -x^{-1}y_1' + y_1'' \right) + x^{-1} \frac{d}{dx} \left[ (1 + 3x^2)(1 - x^2)^{-1} y_1' \right]$$

$$= -\frac{1}{n} \{ y_1'y_1'' + (n - 1)y_2'y_2'' \} + \frac{1 + 3x^2}{x(1 - x^2)} \left[ -\frac{1}{2n} (y_1')^2 + (n - 1)(y_2')^2 + 4x(1 - x^2)^{-1} y_1' \right]$$

$$+ x^{-1} \frac{d}{dx} \left[ (1 + 3x^2)(1 - x^2)^{-1} y_1' \right]$$
where for the second identity we have used the equation (4.19) again. Therefore, by the above estimates, there exists a constant $C = C(\epsilon) > 0$ independent of the solution and $\phi(0)$ such that

$$|y''_1(x)| \leq Cx,$$

for $0 \leq x \leq \frac{1}{2}$. We take derivative of (4.21) with respect to $x$ and obtain

$$y''_2 = [(n - 1) + (1 + n)x^2]x^{-1}(1 - x^2)^{-1}(-x^{-1}y_2' + y''_2) + x^{-1}y_2' \frac{d}{dx} \left( \frac{(n - 1) + (1 + n)x^2}{1 - x^2} \right) - \frac{1}{2}(y''_2 + y_2' y''_2) - 8(n + 1) \frac{d}{dx} [(1 - x^2)^{-2} K^{-\frac{1}{2}} \phi^{-\frac{1}{2}} (\phi^{-1} - 1)],$$

and therefore, by the above estimates on $y_i$ up to second order derivatives, there exists a constant $C = C(\epsilon) > 0$ independent of the solution and $\phi(0)$ such that

$$|y''_2(x)| \leq C(x^{-1} - x^{-1}y_2' + y''_2 + x), \quad (4.42)$$

for $0 \leq x \leq \frac{1}{2}$. Now we turn to the estimate of the term $x^{-1} - x^{-1}y_2' + y''_2$. For $x \in [0, \frac{1}{2}]$, by (4.41), we have

$$y'_2(x) = (K(\frac{1}{2}))^{-\frac{3}{4}} \frac{1}{2} x^{-1} (1 - x^2)^{-n} - \frac{1}{2} x^{-1} (1 - x^2)^{-n} - 8(n + 1)(K(\frac{1}{2}))^{-\frac{1}{2}} x^{-1} (1 - x^2)^{-n}$$

$$\int_{x}^{1} s^{1-n}(1 - s^2)^n - (\frac{1}{2})(\phi(0))^{-\frac{1}{2}} (1 - \phi(0)) ds + 8(n + 1)(K(\frac{1}{2}))^{-\frac{1}{2}} x^{-1} (1 - x^2)^{-n}$$

$$\int_{x}^{1} s^{2-n}(1 - s^2)^{n-2} \times O(1) ds = O(1)x^2 - 8(n + 1)(K(\frac{1}{2}))^{-\frac{1}{2}} (\phi(0))^{-\frac{1}{2}} (1 - \phi(0)) (K(\frac{1}{2}))^{-\frac{1}{2}} x^{-1} (1 - x^2)^{-n}$$

$$\int_{x}^{1} s^{1-n}(1 - s^2)^{n-2} ds = O(1)x^2 + \frac{8(n + 1)}{n - 2} (K(\frac{1}{2}))^{-\frac{1}{2}} (\phi(0))^{-\frac{1}{2}} (1 - \phi(0)) (K(\frac{1}{2}))^{-\frac{1}{2}} x^{-1} (1 - x^2)^{-n} \quad (4.43)$$

where there exists a constant $C = C(\epsilon) > 0$ independent of the solution and $\phi(0)$ so that $|O(1)| \leq C$ for all terms $O(1)$ in the formula, by the above estimates on $y_i$ and $y'_i$ for $i = 1, 2$. Substituting (4.43) back to (4.21), we have

$$y''_2 = \frac{8(n + 1)(n - 1)}{n - 2} (K(\frac{1}{2}))^{-\frac{1}{2}} (\phi(0))^{-\frac{1}{2}} (1 - \phi(0)) + O(1)x$$

$$- 8(n + 1)(K(\frac{1}{2}))^{-\frac{1}{2}} (\phi(0))^{-\frac{1}{2}} (1 - \phi(0))$$

$$= \frac{8(n + 1)}{n - 2} (K(\frac{1}{2}))^{-\frac{1}{2}} (\phi(0))^{-\frac{1}{2}} (1 - \phi(0)) + O(1)x \quad (4.44)$$

where there exists a constant $C > 0$ independent of the solution and $\phi(0)$ so that $|O(1)| \leq C$ for all terms $O(1)$ in the formula, by the above estimates on $y_i$ and $y'_i$ for $i = 1, 2$. Therefore, by (4.43) and (4.44), there exists a constant $C > 0$ independent of the solution and $\phi(0)$ such that

$$x^{-1} - x^{-1}y_2' + y''_2 \leq C,$$

for $x \in [0, \frac{1}{2}]$, and hence by (4.42) there exists a constant $C = C(\epsilon) > 0$ independent of the solution and $\phi(0)$ such that for $x \in [0, \frac{1}{2}]$,

$$|y''_2(x)| \leq C.$$

This completes the proof of the lemma. □
Recall that in [22], Graham and Lee used a gauge fixing method and the Fredholm theory on certain weighted functional spaces to show that for the hyperbolic space \((M^{n+1}, g)\) with its conformal infinity \((\partial M, [\hat{g}])\) (where \((\partial M, \hat{g})\) is the round sphere), there exists a CCE metric on \(M\) for any given conformal infinity \((\partial M, [\hat{g}_1])\) where \(\hat{g}_1\) is a small perturbation of \(\hat{g}\) in \(C^2, \alpha\) \((0 < \alpha < 1)\) sense. Later Lee ([32]) generalized this perturbation result to a more general class of CCE manifolds \((M^{n+1}, g)\) with a corresponding conformal infinity \((\partial M, [\hat{g}_1])\). In particular, the perturbation result holds for \(g\) with non-positive sectional curvature, and also for \(g\) which has sectional curvature bounded above by \(\frac{\eta^2 - 8\eta}{8\eta - 8}\) with the Yamabe constant of \(\hat{g}\) non-negative.

Now we are ready to prove Theorem 1.4, the existence theorem of the boundary value problem.

**Proof of Theorem 1.4.** Notice that this existence result is not a perturbation result in nature. We use the continuity method. When \(\lambda = 1\), \(\hat{g}^1\) is the round metric on the sphere \(S^n\), and the CCE filling \(g^1\) is the hyperbolic metric. For openness, let \(\lambda_0 \in \left(\frac{1}{n+1}, 1\right)\) (resp. \(\lambda_0 \in [1, n+1)\)) such that \(g^{\lambda_0}\) is a CCE metric on \(B_1\) which is non-positively curved with \((S^n, [\hat{g}^{\lambda_0}])\) as its conformal infinity. Then by [32], there exists \(\epsilon > 0\) such that for \(\lambda \in (\lambda_0 - \epsilon, \lambda_0 + \epsilon)\), there exists a CCE metric \(g^\lambda\) on \(B_1\) with \((S^n, [\hat{g}^\lambda])\) as its conformal infinity and the sectional curvature of \(g^\lambda\) is sufficiently close to that of \(g^{\lambda_0}\) at the corresponding points on \(B_1\) for \(\epsilon > 0\) small enough, and moreover, by Theorem 2.3, \(g^\lambda\) has the smooth expansion (2.12).

By [33] and the argument at the beginning of this section, if a CCE metric \(g^\lambda\) is non-positively curved, then the Killing vector fields on \(M\) with \(g^\lambda\) have the same center of gravity as defined at the beginning of this section. Then by the above argument, it has the form (2.2), (4.4) and (4.18), which satisfies the boundary value problem (4.19)–(4.22).

For compactness, let \(\{\lambda_j\}_{j>0}\) be a sequence of points converging to \(\lambda_0 \in \left(\frac{1}{n+1}, (n+1)\right)\), with \(g^{\lambda_j}\) a non-positively curved CCE metric with \((S^n, [\hat{g}^{\lambda_j}])\) as its conformal infinity. Here \(\{\hat{g}^{\lambda_j}\}\) are invariant under the same SU(2k+1)-group action on \(S^n\). Let \(p^j_0\) be the center of gravity of \((B_1, g^{\lambda_j})\) for \(j > 0\). For any \((B_1, g^{\lambda_j})\), there exists a natural diffeomorphism \(F_j: U_{p^j_0} \rightarrow \partial B_1\) between the unit tangent sphere at \(p^j_0\) and \(\partial B_1\), induced by the exponential map at \(p^j_0\). Now for \(j > 1\), we define a map \(H_j : \overline{B_1} \rightarrow \overline{B_1}\), with \(H_j|_{\partial B_1} = \text{Id}\) and

\[
H_j(\text{Exp}_{p^j_0}(t F_1^{-1}(p))) = \text{Exp}_{p^j_0}(t F_j^{-1}(p)),
\]

for \(p \in \partial B_1\) and \(t \geq 0\), where \(\text{Exp}_{p^j_0}\) is the exponential map at \(p^j_0\) on \((B_1, g^{\lambda_j})\). It is easy to check that \(H_j\) is a diffeomorphism. Let \(g^*_j = H_j^* g_j\), which is still denoted as \(g_j\), for \(j \geq 2\). Now \((B_1, g_j)\) has the same center of gravity \(p_0 \in B_1\) for all \(j \geq 2\). Let \(r\) be the distance function to \(p_0\), and hence \(x = e^{-r}\) is the geodesic defining function of each \(j > 0\). Then for a given point \(q \in S^n\) on the boundary, let \(\gamma\) be the geodesic connecting \(q\) and \(p_0\) and \((r, \theta)\) be the polar coordinate near \(\gamma\) defined as above. Then, under the polar coordinates \((r, \theta)\),

\[
g^{\lambda_j} = dr^2 + g^j_r = dr^2 + \sinh^2(r) \tilde{h}^j = x^{-2}(dx^2 + \left(\frac{1-x^2}{2}\right)^2 \tilde{h}^j),
\]
and moreover these metrics have the form (2.2), (4.4) and (4.18) with \((g_r, \tilde{h}, I_l(x), \phi(x), K(x))\) replaced by \((g^j_r, \tilde{h}^j, I^j_l(x), \phi^j(x), K^j(x))\) and \(y_j(x)\) replaced by \(y^j_l(x)\) with \(i = 1, 2\) correspondingly for any \(j \geq 1\), and also these metrics satisfy the boundary value problem (4.19)–(4.22) for \(x \in [0, 1]\). By Lemma 4.4, there exists \(C > 0\) independent of \(j\) such that

\[
|(|y^j_l|^{(k)}(x)| \leq C,
\]
on \(x \in [0, \frac{1}{2}]\) for \(i = 1, 2, 0 \leq k \leq 3\) and \(j \geq 1\), with \((y^j_l)^{(k)}\) the \(k\)-th order derivative of \(y^j_l\).

Therefore, up to a subsequence, \(y^j_l\) converges in \(C^{2, \alpha}\) norm to some function \(y_l\) on \(x \in [0, \frac{1}{2}]\) for any \(\alpha \in (0, 1)\). Moreover, by the non-positivity of the sectional curvatures of \(g^{x_j}\) and the Einstein equation, the norm of the Weyl tensor has the bound \(|W|_{g^{x_j}} \leq \sqrt{(n^2 - 1)n}\). The same argument in Lemma 3.3 yields that there exists a constant \(C > 0\) independent of \(j\) such that

\[
|(|y^j_l|^{(k)}(x)| \leq C,
\]
on \(x \in [\frac{1}{2}, 1]\) for \(i = 1, 2, 0 \leq k \leq 2\) and \(j \geq 1\). Moreover, by the uniform bound of \(|W|_{g^{x_j}}\) and the Einstein equation, the sequence \(g^{x_j}\) are uniformly bounded in \(x \in [\frac{1}{2}, 1]\) under any \(C^k (k \geq 1)\) norm. Therefore, up to a subsequence, \(g^{x_j}\) converges to an Einstein metric on the domain \(x \in [\frac{1}{2}, 1]\) which is \(r \leq \ln(2)\), in \(C^k\) norm for any \(k > 0\) with the same Killing vector fields as \(g^{x_j}\). Therefore, up to a subsequence, \((B_1, p_0, g^{x_0})\) converges to a CCE manifold \((B_1, p_0, g^{x_0})\) in \(C^{k, \alpha}\) for any \(k \geq 1\) and \(\alpha \in (0, 1)\) in pointed Cheeger-Gromov sense, which is non-positively curved. Also, with \(x\) the same geodesic defining function of \(g^{x_0}\) and \(g^{x_j}\), \(x^2 g^{x_j} \rightarrow x^2 g^{x_0}\) in \(x \in [0, \frac{1}{2}]\) in \(C^{2, \alpha}\) and \((\mathbb{S}^n, [\hat{g}^{x_0}])\) is the conformal infinity of \((B_1, g^{x_0})\). Moreover, By Theorem 2.3, the metric \(g^{x_0}\) is smooth and has a smooth expansion at \(x = 0\). This proves the compactness.

Here we want to emphasize that in the above convergence argument, the volume is locally non-collapsed. Indeed, under the condition \(\lambda = \phi(0) > \frac{1}{n+1}\), the inequality (4.35) gives a positive lower bound of \(K(0) = \lim_{r \to \infty} \frac{|\partial B_r(p_0)|}{|\partial B_r^{(n+1)}|}\). Then we use Gromov’s argument to conclude: for any \(p \in M\), by the Bishop-Gromov relative volume comparison theorem,

\[
\frac{|B_r(p)|}{|B_r^{(n+1)}|} \geq \frac{|B_{r+s}(p)|}{|B_{r+s}^{(n+1)}|} \geq \frac{|B_{r+2s}(p_0)|}{|B_{r+2s}^{(n+1)}|} = \frac{|B_{r+2s}(p_0)|}{|B_{r+2s}^{(n+1)}|} \frac{|B_{r+2s}^{(n+1)}|}{|B_{r+2s+2}^{(n+1)}|} \frac{|B_{r+2s+2}^{(n+1)}|}{|B_{r+2s+2+2}^{(n+1)}|} \frac{|B_{r+2s+2+2}^{(n+1)}|}{|B_{r+2s+2+2+2}^{(n+1)}|} \cdots 
\]

\[
\geq \frac{|\partial B_{r+2s}(p_0)|}{|\partial B_{r+2s}^{(n+1)}|} \frac{|B_{r+2s}^{(n+1)}|}{|B_{r+2s+2}^{(n+1)}|} \frac{|B_{r+2s+2}^{(n+1)}|}{|B_{r+2s+2+2}^{(n+1)}|} \cdots \geq K(0) \frac{|B_{r+2s+2}^{(n+1)}|}{|B_{r+2s+2+2}^{(n+1)}|} \frac{|B_{r+2s+2+2}^{(n+1)}|}{|B_{r+2s+2+2+2}^{(n+1)}|} \cdots
\]

for \(r > 0\), where \(s\) is the distance between \(p\) and \(p_0\).

Then a direct argument of continuity method starting from \(\lambda = 1\) concludes the theorem.

\[\square\]

**Acknowledgements** The author would like to thank Professor Jie Qing and Professor Yuguang Shi for helpful discussion and constant support. The author is grateful to Professor Fuquan Fang and Xiaoyang Chen for helpful discussion on homogeneous spaces. He is grateful to Professor Matthew Gursky, Professor S.-Y. A. Chang and Professor Robin Graham for their interests and encouragement. Thanks also due to Wei Yuan for pointing him the treatise [7].

By Springer
References

1. Anderson, M.: Boundary regularity, uniqueness and non-uniqueness for AH Einstein metrics on 4-manifolds. Adv. Math. 179, 205–249 (2003)
2. Anderson, M.: Einstein metrics with prescribed conformal infinity on 4-manifolds. Geom. Funct. Anal. 18, 305–366 (2008)
3. Anderson, M., Chruściel, P. T., Delay, E.: Non-trivial, static, geodesically complete, vacuum space-times with a negative cosmological constant II (n > 4), AdS-CFT correspondence: Einstein metrics and their conformal boundaries, IRMA Lectures in Mathematics and Theoretical Physics, 8, 2005, 165–205
4. Andersson, L., Dahl, M.: Scalar curvature rigidity for asymptotically hyperbolic manifolds. Ann. Global Anal. Geo. 16, 1–27 (1998)
5. Besse, A.: Einstein Manifolds. Springer, Berlin (1987)
6. Bettiol, R., Paolo, P.: Bifurcation and local rigidity of homogeneous solutions to the Yamabe problem on spheres. Calc. Var. Part. Differ. Equ. 47(3–4), 789–807 (2013)
7. Biquard, O.: Métriques d’Einstein asymptotiquement symétriques, Astérisque No. 265 (2000)
8. Biquard, O.: Continuation unique `a partir de l’infini conforme pour les métriques d’Einstein. Math. Res. Lett. 15(6), 1091–1099 (2008)
9. Borel, A.: Some remarks about Lie groups transitive on spheres and tori. Bull. A.M.S. 55, 580–587 (1949)
10. Buttsworth, T.: The Dirichlet problem for Einstein metrics on cohomogeneity one manifolds. Ann. Global Anal. Geom. 54(1), 155–171 (2018)
11. Cai, M., Galloway, G.: Boundaries of zero scalar curvature in the AdS/CFT Correspondence. Adv. Theor. Math. Phys. 69, 1769–1783 (2000)
12. Chang, S.Y.A., Ge, Y., Qing, J.: Compactness of conformally compact Einstein 4-manifolds II. Adv. Math. 373, 107325 (2020)
13. Chen, X., Lai, M., Wang, F.: Escobar-Yamabe compactifications for Poincaré-Einstein manifolds and rigidity theorems. Adv. Math. 343, 16–35 (2019)
14. Chruściel, P.T., Delay, E., Lee, J.M., Skinner, D.N.: Boundary regularity of conformally compact Einstein metrics. J. Differ. Geom. 69(1), 111–136 (2005)
15. Cortés, V., Saha, A.: Quarter-pinched einstein metrics interpolating between real and complex hyperbolic metrics. Math. Z. 290(1–2), 155–166 (2018)
16. Deng, S.: Homogeneous Finsler Spaces, Springer Monographs in Mathematics., p. xiv+240. Springer, New York (2012)
17. Dutta, S., Javaheri, M.: Rigidity of conformally compact manifolds with the round sphere as conformal infinity. Adv. Math. 224, 525–538 (2010)
18. Eberlein, P.: Geodesic flows in manifolds of nonpositive curvature. Proc. Symp. Pure Math. 69, 525–571 (2001)
19. Fefferman, C., Graham, C.R.: Conformal invariants, in Elie Cartan et les Mathematiques d’aujourd’hui, Asterisque (1985), 95–116
20. Graham, R.: Volume and area renormalizations for conformally compact Einstein metrics. Rend. Circ. Mat. Palermo. Ser. II(Suppl. 63), 31–42 (2000). arXiv:math/9909042v1
21. Graham, R., Lee, J.: Einstein metrics with prescribed conformal infinity on the ball. Adv. Math. 87(2), 186–225 (1991)
22. Grove, K., Ziller, W.: Comomogeneity one manifolds with positive Ricci curvature. Invent. Math. 149(3), 619–646 (2002)
23. Gursky, M., Han, Q.: Non-existence of Poincaré-Einstein manifolds with prescribed conformal infinity. Geom. Funct. Anal. 27(4), 863–879 (2017)
24. Gursky, M., Han, Q., Stolz, S.: An invariant related to the existence of conformally compact Einstein fillings, arXiv: 1801.04474v1
25. Gursky, M., Székelyhidi, G.: A local existence result for Poincaré-Einstein metrics. Adv. Math. 361, 106912 (2020)
26. Hawking, S.W., Page, D.N.: Thermodynamics of black holes in Anti-de Sitter space. Comm. Math. Phys. 87(2), 577–588 (1983)
27. Hitchin, N.J.: Twistor spaces, Einstein metrics and isomonodromic deformations. J. Differ. Geom. 42(1), 30–112 (1995)
28. Kerin, M., Wraith, D.: Homogeneous metrics on spheres. Irish Math. Soc. Bull. 51, 59–71 (2003)
29. Kichenassamy, S.: On a conjecture of Fefferman and Graham. Adv. Math. 184(2), 268–288 (2004)
30. LeBrun, C.: $\mathcal{H}$-space with a cosmological constant. Proc. Roy. Soc. London Ser. A 380, 171–185 (1982)
32. Lee, J.M.: Fredholm operators and Einstein metrics on conformally compact manifolds. Mem. Amer. Math. Soc. 183(864), vi+83 (2006)
33. Li, G.: On Uniqueness of conformally compact Einstein metrics with homogeneous conformal infinity. Adv. Math. 340, 983–1011 (2018)
34. Li, G.: On Uniqueness And Existence of Conformally Compact Einstein Metrics with Homogeneous Conformal Infinity. II, preprint, arXiv:1801.07969
35. Li, G., Qing, J., Shi, Y.: Gap phenomena and curvature estimates for conformally compact Einstein manifolds. Trans. Amer. Math. Soc. 369(6), 4385–4413 (2017)
36. Matsumoto, Y.: A construction of Poincaré-Einstein metrics of cohomogeneity one on the ball. Proc. Amer. Math. Soc. 147(9), 3983–3993 (2019)
37. Mazzeo, R.: Elliptic theory of differential edge operators I. Comm. Partial Differ. Equ. 16(10), 1615–1664 (1991)
38. Montgomery, D., Samelson, H.: Transformation groups on spheres. Ann. of Math. 44, 454–470 (1943)
39. Obata, M.: The conjectures on conformal transformations of Riemannian manifolds. J. Differ. Geom. 6, 247–258 (1971)
40. Pedersen, H.: Einstein metrics, Spinning top motions and monopoles. Math. Ann. 274, 35–59 (1986)
41. Petersen, P.: Riemannian Geometry, Graduate Texts in Mathematics 171. Springer, New York (1998)
42. Qing, J.: On the rigidity for conformally compact Einstein manifolds. Int. Math. Res. Not. 2003(21), 1141–1153
43. Sbaih, M.A.A., Srour, M.K.H., Hamada, M.S., Fayad, H.M.: Lie algebra and representation of SU(4). Electron. J. Theor. Phys. 10, 28 (2013)
44. Shi, Y., Tian, G.: Rigidity of asymptotically hyperbolic manifolds. Commun. Math. Phys 259, 545–559 (2005)
45. Wang, X.: On conformally compact Einstein manifolds. Math. Res. Lett. 8, 671–688 (2001)
46. Witten, E., Yau, S.T.: Connectedness of the boundary in the ADS/CFT Correspondence. Adv. Theor. Math. Phys. 3(6), 1635–1655 (1999)
47. Yano, K.: The theory of Lie derivatives and its applications. North-Holland, Amsterdam (1957)
48. Ziller, W.: Homogeneous Einstein metrics on spheres and projective spaces. Math. Ann. 259, 351–358 (1982)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.