Actions for Gravity, with Generalizations: A Review

Peter Peldán

Institute of Theoretical Physics
Chalmers University of Technology
and University of Göteborg
S-412 96 Göteborg, Sweden

Abstract

The search for a theory of quantum gravity has for a long time been almost fruitless. A few years ago, however, Ashtekar found a reformulation of Hamiltonian gravity, which thereafter has given rise to a new promising quantization project; the canonical Dirac quantization of Einstein gravity in terms of Ashtekar’s new variables. This project has already given interesting results, although many important ingredients are still missing before we can say that the quantization has been successful.

Related to the classical Ashtekar Hamiltonian, there have been discoveries regarding new classical actions for gravity in (2+1)- and (3+1)-dimensions, and also generalizations of Einstein’s theory of gravity. In the first type of generalization, one introduces infinitely many new parameters, similar to the conventional Einstein cosmological constant, into the theory. These generalizations are called “neighbours of Einstein’s theory” or “cosmological constants generalizations”, and the theory has the same number of degrees of freedom, per point in spacetime, as the conventional Einstein theory. The second type is a gauge group generalization of Ashtekar’s Hamiltonian, and this theory has the correct number of degrees of freedom to function as a theory for a unification of gravity and Yang-Mills theory. In both types of generalizations, there are still important problems that are unresolved: e.g. the reality conditions, the metric-signature condition, the interpretation, etc.

In this review, I will try to clarify the relations between the new and old actions for gravity, and also give a short introduction to the new generalizations. The new results/treatments in this review are: 1. A more detailed constraint analysis of the Hamiltonian formulation of the Hilbert-Palatini Lagrangian in (3+1)-dimensions. 2. The canonical transformation relating the Ashtekar- and the ADM-Hamiltonian in (2+1)-dimensions is given. 3. There is a discussion regarding the possibility of finding a higher dimensional Ashtekar formulation.

There are also two clarifying figures (in the beginning of chapter 2 and 3, respectively) showing the relations between different action-formulations for Einstein gravity in (2+1)- and (3+1)-dimensions.

1 Email address: tfepp@fy.chalmers.se
Contents

1 Introduction 1

2 Actions in (3+1)-dimensions 4
   2.1 The Einstein-Hilbert Lagrangian 4
   2.2 The Hilbert-Palatini Lagrangian 5
   2.3 The ADM-Hamiltonian 7
   2.4 Hamiltonian formulation of the H-P Lagrangian 11
   2.5 Self dual Hilbert-Palatini Lagrangian 21
   2.6 The Ashtekar Hamiltonian 26
   2.7 The CDJ-Lagrangian 29
   2.8 The Plebanski Lagrangian 31

3 Actions in (2+1)-dimensions 35
   3.1 The E-H and the H-P Lagrangian, and the ADM-Hamiltonian 36
   3.2 The self dual H-P and the Plebanski Lagrangian 36
   3.3 Hamiltonian formulation of the H-P Lagrangian, and the Ashtekar Hamiltonian 36
   3.4 The CDJ-Lagrangian 39

4 Generalizations 41
   4.1 The cosmological constants 42
   4.2 Gauge group generalization 44
   4.3 Higher dimensions 46

5 Outlook 48

A Conventions and Notation 51

B Definitions of Connections and Curvature 53

References 58
Chapter 1

Introduction

The greatest challenge in theoretical physics today is to find the theory of quantum gravity. That is, the union of the theory for microscopic particles, quantum mechanics, and the theory for "cosmological" objects, the general theory of relativity. Despite the fact that a lot of physicists have been attacking the problem of quantum gravity, using a variety of different methods during a period of at least 40 years, there has been no real progress in this quest. One reason for this failure could be that the tools and methods used just are not adequate for this task. Most of the quantization attempts, so far, have treated gravity as just another particle field theory. There are, however, great differences between the theory of gravity and theories describing the other fundamental forces and particles in nature. One of the most striking one is that in a conventional particle theory one assumes that the fields are propagating on a non-dynamical background spacetime, while in the theory of gravity it is exactly the "background" spacetime that is the dynamical field; "the stage is participating in the play".

Perhaps we "only" need to invent new methods specially constructed for quantization of diffeomorphism invariant theories, like gravity, without any need of really changing quantum mechanics or the theory of relativity. It could, however, also be true that already quantum mechanics and/or the theory of relativity are wrong, so that any attempt to unite these two theories will be bound to fail.

Without any experimental guidelines (as the blackbody radiation was for quantum mechanics, and perhaps the Michelson-Morley experiment was to the special theory of relativity) of how to modify either our methods or our theories, we either have to keep on struggling with the quantization of the standard formulations, or we could make a "theoretical excursion", leaving the experimentally confirmed way, and try to guess what kind of modifications our theories need, possibly guided by theoretical beauty or other temptations.

String theory is a kind of "theoretical excursion", without any experimental support. It has, for the last 10 years, been regarded as the strongest candidate for a theory of quantum gravity, or even for a theory of everything. However, the optimism regarding string theory seems to have decreased slightly, recently, due to technical difficulties and absence of progress. The basic idea behind string theory is that the fundamental constituents of matter are string-like, extended, one-dimensional objects, instead of pointlike which is otherwise believed. In a way, string theory can be seen as a synthesis of many earlier attempts of quantizing gravity: supergravity, Kaluza-Klein theories, higher spin
In the latest years, another way of tackling quantum gravity has received a lot of attention. When Ashtekar \cite{2} in 1986 managed to reformulate the Einstein theory of gravity in terms of new variables, it soon became clear that this new formulation had some appealing features that made it suitable for quantization à la Dirac. Since then, physicists have been working on this project, and perhaps for the first time in the history of quantum gravity there has really been some progress regarding solutions to the constraints of quantum gravity \cite{3}, \cite{4}. (These solutions are the physical wavefunctionals that are annihilated by the operator valued constraints, in the canonical formulation.) However, some people might say that this is not really worth anything until we also have an inner product and observables so that we can calculate a physical quantity. Of course, it could be that this critique is correct, and that these quantization attempts will never lead to the desired result without the introduction of modifications somewhere. It is, however, anyhow very valuable to try the conventional quantization of the conventional theory first, in order to let the theory itself indicate in what direction we should search for modifications.

Besides these two main routes towards a theory of quantum gravity, there also exist attempts using path integral quantization and numerical calculations in simplicial quantum gravity.

As mentioned above, the new promising quantization scheme is based on the new Ashtekar reformulation of Hamiltonian gravity. This Ashtekar reformulation can be seen as a shift of emphasis from the metric to the connection as the fundamental field for gravity. Later, Capovilla, Dell and Jacobson (CDJ) \cite{5} managed to go even further and found an action written (almost) purely in terms of the connection. This discovery soon led to the finding of two different types of generalizations of both the CDJ-Lagrangian as well as the Ashtekar Hamiltonian: the cosmological constants in refs.\cite{6}, \cite{7} and \cite{8}, and also the gauge group generalization in \cite{9}.

The purpose of this review is to describe most of the known actions for classical gravity in (2+1)- and (3+1)-dimensions, and also to show how these different actions are related. I also want to briefly give the basic ideas behind the generalizations mentioned above.

In chapter 2, I describe the actions for (3+1)-dimensional gravity, and do most of the calculations in great detail. Chapter 3 contains the actions for (2+1)-dimensions, and chapter 4 presents the generalized Ashtekar Hamiltonians.

Throughout this review I neglect surface terms that appear in partial integrations in the actions. That is, I assume compact spacetime or fast enough fall-off behavior at infinity for the fields. Note, however, that surface terms normally need a careful treatment \cite{10}, \cite{11}.

My notation and conventions are partly given in each section and partly collected in two appendices. Definitions and notation introduced in one section are, however, only valid inside that section. This is specially true for the covariant derivatives of which, in this review, there exist at least five different types of, but only three different symbols for $D_a$, $D_a$ and $\nabla_a$. 
The Hilbert-Palatini Lagrangian $L(e_α^I, \omega_β^{(+)}J^K)$

The Plebanski Lagrangian $L(\Sigma^i_{\alpha\beta}, \omega_β^{(+)}i^j, \Psi^{ij}, \eta)$

### Fig. 1 Actions for gravity in (3+1)-dimensions
Chapter 2

Actions in (3+1)-dimensions

In fig. 1, I have tried to collect most of the known classical actions for gravity in (3+1)-dimensions, and their connecting relations. Among these actions there is one for which the explicit form is not known, today. That is "the pure $SO(1,3)$ spin-connection Lagrangian", and it can be found through an elimination of the tetrad field from the Hilbert-Palatini Lagrangian or from a Legendre transform from the Hamiltonian formulation of the H-P Lagrangian. The reason why this Lagrangian is not known explicitly is due to technical difficulties in the above mentioned calculations. (There is, however, an "affine connection form" of this Lagrangian, that is explicitly known; the Schrödinger Lagrangian $[12]$. This Lagrangian can be found through an elimination of the metric from the Hilbert-Palatini Lagrangian, in metric and affine connection form, and the Lagrangian equals the square-root of the determinant of the Ricci-tensor.) There is also one known action that is missing in fig.1; in ref. $[13]$ 'tHooft presented an $SL(3)$ and diffeomorphism invariant action, and it was shown that the equations of motion, following from this action, are the Einstein’s equations. Furthermore, if this $SL(3)$-invariance is gauge-fixed to $SU(2)$, this action reduces to the CDJ-action. I do not treat this action here, mainly due to the fact that this formulation differs from all the actions in fig.1 in that it is constructed with the use of an enlarged gauge-invariance.

2.1 The Einstein-Hilbert Lagrangian

In an attempt to find a Lagrangian that gives, as equations of motion, the Einstein equations for gravity, the obvious candidate is the simplest scalar function of the metric and its derivatives; the curvature scalar. And in order to get an action which is generally coordinate invariant one must densitize the curvature scalar with the help of the determinant of the metric. The resulting Lagrangian is called the Einstein-Hilbert Lagrangian for pure gravity. In this section I will analyze this second order Einstein-Hilbert Lagrangian for pure gravity with a cosmological constant and show that the equations of motion following from the variation of the action are Einstein’s equations. The Lagrangian is

$$\mathcal{L}_{EH} = e \left( e_i^\alpha e_j^\beta R_{\alpha \beta}^{IJ}(\omega(e)) + 2\lambda \right)$$

(2.1)

where $e_i^\alpha$ is the tetrad field and $R_{\alpha \beta}^{IJ}(\omega(e))$ is the curvature of the unique torsion-free spin-connection $\omega_{\alpha}^{IJ}$, compatible with $e_i^\alpha$. $e$ is the determinant of the inverse tetrad $e_{\alpha I}$. 
To find the change in $S_{EH}$ under a variation of $e_{aI}$, I need to calculate the variation of $e, e^I, R_{\alpha\beta}{}^{IJ}$ first:

$$e = \frac{1}{24} \epsilon^{\alpha\beta\gamma\delta} \epsilon^{IJKLM} e_{\alpha I} e_{\beta J} e_{\gamma K} e_{\delta L}$$  \hspace{1cm} (2.2)

$$\Rightarrow \delta e = \frac{1}{6} \epsilon^{\alpha\beta\gamma\delta} \epsilon^{IJKLM} (\delta e_{\alpha I}) e_{\beta J} e_{\gamma K} e_{\delta L} = ee^I \delta e_{\alpha I}$$  \hspace{1cm} (2.3)

$$\delta (e^I_{\alpha} e_{\alpha J}) = (\delta e^I_{\alpha}) e_{\alpha J} + e^I_{\alpha} \delta e_{\alpha J} = \delta \eta_{IJ} = 0$$ \hspace{1cm} (2.4)

$$\Rightarrow \delta e^I_{\alpha} = -e^I_{\beta} (\delta e_{\beta J}) e_{\alpha J}$$ \hspace{1cm} (2.5)

The variation of $R_{\alpha\beta}{}^{IJ}$ is given in Appendix B.

$$\delta R_{\alpha\beta}{}^{IJ} = \frac{1}{2} \left( e^{\gamma[I} D_{\alpha} D_{\beta]} \delta e_{\gamma]} + e^{\gamma[I} e^{\mu J]} e_{\mu K} D_{\alpha} D_{\beta} \delta e_{K]} \right)$$ \hspace{1cm} (2.6)

Using (2.3), (2.5), (2.6) and the fact that the covariant derivative annihilates $e_{aI}$, the variation of $S_{EH}$ becomes:

$$\delta S_{EH} = \int_M d^4 x \ e \left( -2e^\alpha_K e^\gamma_I e^\beta_J R_{\gamma\beta}{}^{KJ} + (2\lambda + e^\gamma_K e^\beta_J R_{\gamma\beta}{}^{KJ}) e^\alpha_I \right) \delta e^I_{\alpha}$$

$$+ \int_M d^4 x \ \partial_{\alpha} \left( e (g^{\gamma[I} e^{\beta_J} D_{\beta]} \delta e_{\gamma]} + g^{\gamma[I} g^{\beta_J} e_{K\beta} D_{\beta} \delta e_{K]} \right)$$ \hspace{1cm} (2.7)

Neglecting the surface term, the requirement that the action should be stationary under general variations of $e_{aI}$, implies Einstein’s equation:

$$e^K_I e^\gamma_I e^\beta_J R_{\gamma\beta}{}^{KJ} - (\lambda + \frac{1}{2} e^K_I e^\gamma_I e^\beta_J R_{\gamma\beta}{}^{KJ}) e^\alpha_I = 0$$ \hspace{1cm} (2.8)

This shows that the Einstein-Hilbert Lagrangian is a good Lagrangian for gravity, in the sense that Einstein’s equation follows from its variation.

### 2.2 The Hilbert-Palatini Lagrangian

The Hilbert-Palatini Lagrangian is a first order Lagrangian for gravity which one gets from the Einstein-Hilbert Lagrangian simply by letting the spin-connection, in the argument of the Riemann-tensor, become an independent field. Normally when one lets a field become independent like this one needs to add a Lagrange multiplier term to the Lagrangian, implying the original relation, in order to get the same equations of motion. The reason why this is not needed here, is that the variation of the action with respect to the spin-connection will itself imply the correct relation. (This is really a rather remarkable feature of this particular Lagrangian.)

Therefore, I consider:
Chapter 2  Actions in (3+1)-dimensions

\[ \mathcal{L}_{HP} = e \left( e_\gamma^\alpha e_\beta^J R_{\alpha\beta}^{IJ} (\omega_\gamma^{KL}) + 2\lambda \right) \]  

(2.9)

The only difference between \( \mathcal{L}_{EH} \) and \( \mathcal{L}_{HP} \) is that in (2.9) the spin-connection \( \omega_\gamma^{KL} \) is an independent field, while in (2.1) it is a given function of the tetrad field. When varying the action with respect to general variations \( \delta e_\alpha^I \) and \( \delta \omega_\alpha^{IJ} \), one needs to know the variation of \( e_\alpha^I \), \( e_\alpha^I \) and \( R_{\alpha\beta}^{IJ} \). The two first variations are given in (2.3) and (2.5), and the variation of \( R_{\alpha\beta}^{IJ} \) is given in Appendix B.

\[ \delta e_\alpha^I = e e_\alpha^I \delta e_\alpha^I \]

\[ \delta e_\alpha^I = -e_\alpha^I (\delta e_\beta^J) e_\beta^J \]  

(2.10)

\[ \delta R_{\alpha\beta}^{IJ} = \mathcal{D}_{[\alpha} \omega_{\beta]}^{IJ} \]

Using this together with the identity

\[ e e_\alpha^I e_\beta^J = \frac{1}{2} \epsilon^{\alpha\beta\gamma\delta} e_{IJKLMNOP} \epsilon^K e^L \]  

(2.11)

gives

\[ \delta S_{HP} = \int_M d^4x e \left( -2e_\alpha^I e_\beta^J R_{\alpha\beta}^{IJ} (\omega_\gamma^{KL}) + (2\lambda + e_\gamma^\alpha e_\beta^J R_{\alpha\beta}^{IJ}) e_\alpha^I \right) \delta e_\alpha^I \]

\[ - \int_M d^4x \frac{1}{2} \mathcal{D}_{\alpha} \left( \epsilon^{\alpha\beta\gamma\delta} e_{IJKLMNOP} \epsilon^K e^L \right) \delta \omega_\beta^{IJ} - \partial_\alpha (ee_\gamma^\alpha e_\beta^J \delta \omega_\beta^{IJ}) \]  

(2.12)

Neglecting the surface term again, the equations of motion are:

\[ e_\alpha^I e_\beta^J R_{\alpha\beta}^{IJ} (\lambda + \frac{1}{2} e_\gamma^\alpha e_\beta^J R_{\alpha\beta}^{IJ}) e_\alpha^I = 0 \]  

(2.13)

\[ \mathcal{D}_{[\alpha} e^K_{\gamma]} = 0 \]  

(2.14)

But (2.14) is just the zero-torsion condition, which can be solved to get the unique torsion-free spin-connection compatible with \( e_\alpha^I \). Inserting this solution to (2.14) into (2.13) one gets Einstein’s equation again.

This shows that the Hilbert-Palatini action also is a good action for gravity, again in the sense that its equations of motion are the Einstein equations. There are, however, cases when the Einstein-Hilbert Lagrangian and the Hilbert-Palatini Lagrangian gives different theories. The first example is in a path-integral approach to quantum gravity. In a path-integral one is supposed to vary all independent fields freely over all not gauge-equivalent field-configurations. And since the two Lagrangians are only equivalent ”on-shell”, and not for general field-configurations, the path-integrals will probably differ. The second example is gravity-matter couplings where the spin-connection couples directly to some matter field. This is the case for fermionic matter. In that case, the variation of the Hilbert-Palatini action with respect to the spin-connection will not in general yield the torsion-free condition (2.14). Instead the theory will have torsion, which, however, can be avoided by adding an extra term to the Lagrangian. See, for instance, [14] for a recent discussion.
2.3 The ADM-Hamiltonian

Here, I will derive the well-known ADM-Hamiltonian for triad gravity, starting from the Einstein-Hilbert Lagrangian (2.1). The derivation will closely follow [15].

The Einstein-Hilbert Lagrangian:

\[ L_{EH} = e \left( R(e^I_\alpha) + 2\lambda \right) \] (2.15)

To be able to find the Hamiltonian formulation of (2.15), I will assume spacetime \( M \) to be topologically \( \Sigma \times \mathbb{R} \), where \( \Sigma \) is some space-like submanifold of \( M \), and \( \mathbb{R} \) stands for the time direction. I will also partly break the manifest spacetime covariance by choosing the \( x^0 \) coordinate to be my time-coordinate.

Before defining the momenta and doing the Legendre transform, it is preferable to slightly rewrite the Lagrangian. The reason is that (2.15) contains second derivatives of the tetrad, which can be partially integrated away, leaving a dependence solely on \( e^I_\alpha \) and its first derivatives.

First, I define two covariant derivatives: \( D_\alpha \) and \( \nabla_\alpha \). \( D_\alpha \) is covariant with respect to both general coordinate transformations in spacetime as well as local Lorentz transformations on the flat index, while \( \nabla_\alpha \) is only covariant under general coordinate transformations. Or in other words: \( D_\alpha \) ”knows how to act” on both spacetime indices as well as Lorentz indices, while \( \nabla_\alpha \) only ”knows how to act” on spacetime indices.

\[ D_\alpha \lambda^\beta I := \partial_\alpha \lambda^\beta I + \Gamma^\beta_\alpha^\gamma I \lambda^\gamma J + \omega^I_{\alpha J} \lambda^\beta J \] (2.16)

\[ \nabla_\alpha \lambda^\beta I := \partial_\alpha \lambda^\beta I + \Gamma^\beta_\alpha^\gamma I \lambda^\gamma J \] (2.17)

where \( \Gamma^\beta_\alpha^\gamma \) in (2.16) and (2.17) are the same connection. Then I require that these covariant derivatives are compatible with the tetrad and the metric:

\[ D_\alpha e^\beta I = 0 \] (2.18)

\[ \nabla_\alpha g^\beta_\gamma = 0 \] (2.19)

Using (2.18) and requiring \( \Gamma^\gamma_\alpha^\beta \) = 0 (no torsion) it is possible to uniquely determine \( \Gamma^\gamma_\alpha^\beta \) and \( \omega^I_{\alpha J} \) as functions of the tetrad. See Appendix B for details. The Riemann-tensor is then defined:

\[ R_{\alpha\beta I J} \lambda_J := D_{[\alpha} D_{\beta]} \lambda_I \] (2.20)

\[ R_{\alpha\beta\mu}^I e_\mu := \nabla_{[\alpha} \nabla_{\beta]} \lambda_I \] (2.21)

\[ \Rightarrow D_{[\alpha} D_{\beta]} \lambda_I \mu J = R_{\alpha\beta I J} \lambda_I + R_{\alpha\beta I}^J \lambda_\mu J \] (2.22)

Using vectors like \( \lambda_K := e^\mu_K \lambda_\alpha \) in (2.20) and (2.21) it is straightforward to show that \( R_{\alpha\beta I J}^\delta = R_{\alpha\beta I}^J e_\gamma J e^\delta_\gamma \), and with the help of definition (2.21) the Einstein-Hilbert Lagrangian can be rewritten:

\[ R(e^I_\alpha) := g^\alpha^\gamma R_{\alpha\beta I}^\gamma \beta = e^\beta J \nabla_{[\alpha} \nabla_{\beta]} e^\gamma J \] (2.23)
Then, using (2.18), it follows that
\[ \nabla_\alpha e^\beta J = -\omega^J_{\alpha K} e^K_\beta \] (2.24)
which together with (2.23) gives
\[ R(e_\alpha) = \nabla_\alpha (e^J_\beta \nabla_\beta e^\alpha J) - e^\beta J_{|\alpha} J_{\beta|J} e^{\alpha M} \] (2.25)
Now, the solution to (2.18) for \( \omega^J_{\alpha I} \) is
\[ \omega^I_{\alpha J} = \frac{1}{2} e_{\alpha K} (\Omega^K_{IJ} + \Omega^J_{IK} - \Omega^I_{JK}) \] (2.26)
where I have defined the anholonomy
\[ \Omega^I_{KJ} := e^\beta K e^\alpha I \partial_{[\alpha} e^J_{\beta]} \] (2.27)
Using (2.25), (2.26) and neglecting the surface term the Lagrangian (2.15) becomes:
\[ L = e (\Omega^L_{KJ} \Omega^I_{IL} - \frac{1}{2} \Omega^I_{KL} \Omega^J_{LK} - \frac{1}{4} \Omega^I_{LK} \Omega^J_{LI} + 2\lambda) \] (2.28)
Note that \( \nabla_\alpha e = 0 \). Now the Lagrangian has a form that makes it rather straightforward to do the Legendre transform. The Lagrangian should first be (3+1)-decomposed, and that is achieved by splitting the tetrad.
\[ e_{0I} = N N_I + N^a V_{aI}, \quad e_{aI} = V_{aI}; \quad N^I V_{aI} = 0; \quad N^I N_I = -1 \] (2.29)
This is a completely general decomposition and puts no restriction on the tetrad. The inverse tetrad becomes:
\[ e^{0I} = -\frac{N^I}{N}, \quad e^{aI} = V^{aI} + \frac{N^a N^I}{N}; \quad V^{aI} V_{bI} = \delta^a_b; \quad V^{aI} N_I = 0 \] (2.30)
And the metric gets the standard ADM-form:
\[ g_{\alpha\beta} = \begin{pmatrix} -N^2 + N^a N_a & N_a \\ N_a & V_{aI} V^I_b \end{pmatrix} \] (2.31)
\[ g^{\alpha\beta} = \begin{pmatrix} -\frac{N^2}{N^2} & \frac{N^a}{N^2} \\ \frac{N^a}{N^2} & V^{aI} V^b_I - \frac{N^a N^b}{N^2} \end{pmatrix} \] (2.32)
\[ g := det(g_{\alpha\beta}) = -N^2 det(V_{aI} V^I_b) \] (2.33)
\[ e := det(e_{aI}) = N \sqrt{det(V_{aI} V^I_b)} \] (2.34)
where \( N_a := V_{aI} V^I_b N^b \). \( N \) and \( N^a \) are normally called the lapse-function and the shift-vector, respectively.
From here, it is rather straightforward, although tedious, to perform the Legendre transform using the (3+1)-decomposition (2.29). There is, however, a simple way of
reducing this calculation dramatically. By partly breaking the manifest Lorentz invariance
with the gauge-choice $N^I = (1, 0, 0, 0)$, the Legendre transform simplifies a lot. And
since the only difference at the Hamiltonian level should be that the Lorentz invariance
is reduced to $SO(3)$ invariance, I use this gauge-choice here. With this choice for $N^I$
there is no inconsistency in notation with $V^{ai}$ called $e^{ai}$, so I change notation to: $e^{ai} = V^{ai}$, $e_{ai} = V_{ai}$.

Now, I use the $(3+1)$-decomposition of $e^{ai}$ for decomposing $\Omega^{ijk}$ also.

$$\Omega^{0jk} = \frac{1}{N}e^{bj}\partial_0 e^{k}_j - \frac{N^a}{N}e^{bj}\partial_0 e^{k}_a$$

$$\Omega^{0j0} = \frac{1}{N}e^{bj}\partial b N$$

$$\Omega^{ij0} = 0$$

$$\Omega^{ijk} = e^{ai}e^{bj}\partial_0 e^{k}_a$$

The Lagrangian (2.28) should also be $(3+1)$-decomposed:

$$L = N^{(3)}e \left( -\frac{1}{2}\Omega^{0(kl)}\Omega_{0kl} + \Omega^{0}_k\Omega_{0j}^j + 2\Omega^{i0}_0\Omega_{ij}^j \\
-\frac{1}{4}\Omega^{ijk}\Omega_{ijk} - \frac{1}{2}\Omega^{ijk}\Omega_{ikj} + \Omega^{i0}_k\Omega_{ij}^j + 2\lambda \right)$$

where $(3)e = det(e_{ai})$. By using (2.25), but now for the three-dimensional covariant
derivative $\nabla_a$ on $\Sigma$, defined to annihilate $e_{ai}e^{a}_b$, one can easily show that:

$$N^{(3)}e^{(3)}R = N^{(3)}e \left( \Omega^{ik}_k\Omega_{ij}^j - \frac{1}{4}\Omega^{ijk}\Omega_{ijk} - \frac{1}{2}\Omega^{ijk}\Omega_{ikj} \right) + \nabla_a \left( N^{(3)}e^{bk}_a e^{a}_b \Omega_{ij}^j \right) - 2^{(3)e}e^{b}_i\partial_b e^{a}_c g^{ca}\partial_a N$$

Using this in (2.36) and again neglecting surface terms gives

$$L = N^{(3)}e \left( -\frac{1}{2}\Omega^{0(kl)}\Omega_{0kl} + \Omega^{0}_k\Omega_{0j}^j + (3)R + 2\lambda \right)$$

where I have used (2.33) to eliminate the last term in (2.37). Now it is time to define the
momenta

$$\pi^{ai} := \frac{\delta L}{\delta \dot{e}_{ai}} = (3)e(\Omega^{0(ki)}e^{a}_k - 2\Omega^{0}_k e^{ai})$$

which means that there exists a primary constraint:

$$L^i := \epsilon^{ijk}\pi^j e_{ak} \approx 0$$

It will be shown later that this constraint is the generator of $SO(3)$ rotations. Inversion
of (2.37) yields:

$$\Omega^{0(ki)} = \frac{1}{(3)e}(\pi^{ai}e^{a}_k - \frac{1}{2}(\pi^{bij}e_{bj})\delta^{ik})$$
Altogether, the total Hamiltonian becomes:

\[
\mathcal{H} = \pi^{ai}\dot{e}_{ai} - \mathcal{L} + \Lambda_i L^i = \frac{N}{2\pi^c} \left( \pi^{ij}\pi_{ij} - \frac{1}{2}(\pi^i)^2 - 2(\pi^i)^2 R - 4\pi^i\lambda \right) - N^c e_{ci} \left( ^{(3)}D_a \pi^{ai} + \Lambda_i L^i \right)
\]  

where \( \pi^{ij} := \pi^{ai} e_{ai} \), \( ^{(3)}D_a \) is the covariant derivative defined to annihilate \( e_{ai} \), and \( \Lambda_i \) is an arbitrary Lagrange multiplier. The fundamental Poisson bracket is

\[
\{ e_{ai}(x), \pi^{bj}(y) \} = \delta_i^b \delta^j \delta(x - y)
\]  

The geometrical interpretation of these phase space variables is that \( e_{ai} \) is the triad field on the spatial hypersurface, and \( \pi^{bj} \) is closely related to the extrinsic curvature of the hypersurface. Since \( N \) and \( N^a \) have vanishing momenta the variation of them implies the secondary constraints:

\[
\mathcal{H} := \frac{1}{2} \left( ^{(3)}e \right) \left( \pi^{ij}\pi_{ij} - \frac{1}{2}(\pi^i)^2 - 2(\pi^i)^2 R - 4\pi^i\lambda \right) \approx 0
\]  

\[
\mathcal{H}_a := -e_{ai} \left( ^{(3)}D_b \pi^{bi} \right) \approx 0
\]  

normally called the Hamiltonian- and the vector-constraint.

Then, in order to check if the Hamiltonian \( (2.42) \) is the complete Hamiltonian for this theory one needs to calculate the time evolution of the constraints. The requirement is that the time evolution of the constraint must be weakly zero. For a total Hamiltonian which is just a linear combination of constraints, this corresponds to requiring the constraints to be first-class. To calculate the constraint algebra, I will first show that the transformations generated by \( L^i \) are \( SO(3) \) rotations, and the ones generated by \( \mathcal{H}_a \) are spatial diffeomorphisms modulo \( SO(3) \) rotations. The transformations generated by \( L^i \):

\[
\delta^{L^i} e_{ai} = \{ e_{ai}, L^i[\Lambda_i] \} = -\epsilon_{ijk} \Lambda^j e^k_a
\]  

\[
\delta^{L^i} \pi^{ai} = \{ \pi^{ai}, L^i[\Lambda_i] \} = -\epsilon_{ijk} \Lambda^j \pi^{ka}
\]  

where \( L^i[\Lambda_i] := \int_2 d^3x \, L^i(x) \Lambda_i(x) \). This shows that \( L^i \) is the generator of \( SO(3) \) rotations. Then, I will define the new constraint \( \tilde{\mathcal{H}}_a \) as a linear combination of \( L^i \) and \( \mathcal{H}_a \), and show that the transformations generated by this new constraint are spatial diffeomorphisms:

\[
\tilde{\mathcal{H}}_a := \mathcal{H}_a + M_{ai}(\epsilon_{bj}) L^i
\]  

where

\[
M_{ai}(\epsilon_{bj}) := \frac{1}{2} \epsilon_{ijk} \left( ^{(3)}e \right) \omega_c^{jk} = \frac{1}{2} \epsilon_{ijk} \epsilon^{aj} \epsilon^{bk} (e_{ai} \partial_a e_{bj}^l + e_{aj} \partial_a e_{bi}^l).
\]  

The transformations of the fundamental fields are:

\[
\delta^{\tilde{\mathcal{H}}_a} e_{ai} = \{ e_{ai}, \tilde{\mathcal{H}}_b[N^b] \} = N^b \partial_b e_{ai} + e_{bi} \partial_a N^b = \mathcal{L}_{N^b} e_{ai}
\]  

\[
\delta^{\tilde{\mathcal{H}}_a} \pi^{ai} = \{ \pi^{ai}, \tilde{\mathcal{H}}_b[N^b] \} = N^b \partial_b \pi^{ai} - \pi^{ai} \partial_b N^b + \pi^{ai} \partial_b N^b = \mathcal{L}_{N^b} \pi^{ai}
\]
where $\mathcal{L}_{N^b}$ denotes the Lie-derivative in the direction $N^b$. The last term in (2.51) is needed since $\pi^{ai}$ is a tensor density of weight plus one. With the information that $L^i$ generates $SO(3)$ rotations and $\mathcal{H}_a$ generates spatial diffeomorphisms, it is really simple to calculate all Poisson brackets containing these two constraints. All one needs to know is how the constraints transform under these transformations. Under $SO(3)$ rotations, $L^i$ is a vector and $\mathcal{H}$ is a scalar. Under spatial diffeomorphisms, $L^i$ and $\mathcal{H}$ are scalar densities, and $\mathcal{H}_a$ is a covariant vector density. (Note that $\mathcal{H}_a$ is not an $SO(3)$ scalar since $K_{ai}$ does not transform covariantly under $SO(3)$ transformations. Note also that due to the anti-symmetry in the derivatives in $K_{ai}$, $K_{ai}$ and $\mathcal{H}_a$ do, however, transform covariantly under spatial diffeomorphisms.) This gives the algebra:

\[
\begin{align*}
\{L^i[\Lambda_i], L^j[\gamma_j]\} &= L^k[\epsilon_{kij} \gamma^i \Lambda^j] \\
\{L^i[\Lambda_i], \mathcal{H}_a[N^a]\} &= L^i[-\mathcal{L}_{N^a} \Lambda_i] \\
\{L^i[\Lambda_i], \mathcal{H}[N]\} &= 0 \quad (2.52) \\
\{\mathcal{H}_a[N^a], \mathcal{H}_b[M^b]\} &= \mathcal{H}_a[\mathcal{L}_{N^b} M^a] \\
\{\mathcal{H}[N], \mathcal{H}_a[N^a]\} &= \mathcal{H}[-\mathcal{L}_{N^a} N]
\end{align*}
\]

The only Poisson bracket left to calculate is the one including two Hamiltonian constraints: $\{\mathcal{H}[N], \mathcal{H}[M]\}$. This calculation is a bit trickier, but since the result must be anti-symmetric in $N$ and $M$, one can neglect all terms not containing derivatives on these fields. Then, one needs the variation of $^{(3)}R$ with respect to $e_{ai}$:

\[
\frac{\delta^{(3)}R(e_{bij}(y))}{\delta e_{ai}(x)} = \mathcal{D}_b \left( g^{ab}(e^{dj}_{[i} \mathcal{D}_d \delta^3(x - y) + g^{de} e^j_{d} \mathcal{D}_e \delta^3(x - y)) \right) \quad (2.53)
\]

A straightforward calculation gives

\[
\{\mathcal{H}[N], \mathcal{H}[M]\} = \int_{\Sigma} d^3z \int_{\Sigma} d^3x \left( -N(x) \frac{\delta^{(3)}R(x)}{\delta e_{ai}(z)} M(z) - \pi^{ij}(z) e_{ai}(z) - \frac{1}{2} \pi^{kij}(z) e_{a}(z) \right) - (N \leftrightarrow M)\]

\[
\quad \quad = \cdots = \mathcal{H}_a[g^{ab}(N \partial_b M - M \partial_b N)] \quad (2.54)
\]

where I have neglected terms proportional to $L^i$. This shows that the set of constraints $\mathcal{H}, \mathcal{H}_a$ and $L^i$ really forms a first class set, meaning that the total Hamiltonian (2.42) is complete and consistent.

The Hamiltonian (2.42) is the well-known ADM-Hamiltonian for gravity, [14]. In attempts at canonical quantization of this Hamiltonian, the complicated non-polynomial form of the Hamiltonian constraint has always been a huge obstacle. No one has yet found an explicit solution to the quantum version of $\mathcal{H} = 0$.

### 2.4 Hamiltonian formulation of the H-P Lagrangian

In this section, I will perform the Legendre transform from the Hilbert-Palatini Lagrangian. The basic canonical coordinate will here be the $SO(1,3)$ spin-connection, and
the metric will only be a secondary object expressible in terms of the canonical fields. The straightforward Hamiltonian analysis of this Lagrangian reveals that there are ten first class constraints, generating the local symmetries of the Lagrangian (diffeomorphisms and Lorentz transformations), and also twelve second class constraints. The second class constraints are needed since the \( SO(1,3) \) connection has too many independent components in comparison with the tetrad field.

After deriving the complete and consistent Hamiltonian for this system, I will go on and solve the second class constraints, and show that the resulting Hamiltonian is the \( SO(1,3) \) ADM-Hamiltonian. Due to the appearance of tertiary second-class constraints, in this analysis, the calculations in this section will be rather lengthy and tedious.

This Legendre transform has already been done in \([17]\) and also in \([18]\). The differences in my approach are that I give the constraint algebra in more detail, and also that I solve the second class constraints without breaking the manifest \( SO(1,3) \) invariance.

The first order Lagrangian density is

\[
L = e(\epsilon^\alpha_I e^\beta_J R_{\alpha\beta}^{IJ} (\omega^K_L) + 2\lambda) \tag{2.55}
\]

where \( e^\alpha_I \) is the tetrad field, \( R_{\alpha\beta}^{IJ} (\omega^K_L) \) is the curvature of the \( SO(1,3) \) connection \( \omega^K_L \), here treated as an independent field. The \( (3+1) \)-decomposition gives

\[
L = e(2e^0_1 e^a_J R_{aJ} + e^a_1 e^b_J R_{ab}^{IJ} + 2\lambda) \tag{2.56}
\]

I define the momenta conjugate to \( \omega^a_{IJ} \):

\[
\pi^a_{IJ} := \frac{\delta L}{\delta \dot{\omega}^a_{IJ}} = ee^0_I e^a_J \tag{2.57}
\]

Since \( \pi^a_{IJ} \) has 18 components, while the right hand side of \((2.57)\) has only 12 independent components, there are six primary constraints:

\[
\Phi^{ab} := \frac{1}{2} \pi^{a}_{IJ} \pi^{b}_{KL} \epsilon^{IJKL} \approx 0 \tag{2.58}
\]

The Lorentz-signature condition on the metric also imposes the non-holonomic constraint: \( det(\pi^{a}_{IJ} \pi^{bl}_{IJ}) < 0 \). For a discussion of the equivalence of \((2.57)\) and \((2.58)\) together with the non-holonomic constraint, see \([18]\).

Now, I want to express the Lagrangian in terms of the coordinate \( \omega^a_{IJ} \), the momenta \( \pi^a_{IJ} \), and possible Lagrange multiplier fields, and to do that I make use of the general decomposition of the tetrad given in \((2.30)\),

\[
e^0_I = -\frac{N^I}{N}, \quad e^a_I = V^a_I + \frac{N^a N^I}{N}; \quad N^I V^a_I = 0; \quad N^I N_I = -1 \tag{2.59}
\]

Using this decomposition it is possible to show that

\[
\eta^{IJ} = -N^I N^J + V^a_I V^a_J := -N^I N^J + \tilde{\eta}^{IJ} \tag{2.60}
\]

where \( V^a_I := V_{ab} V^{bl} \) and \( V_{ab} V^{bl} \tilde{V}^c_I = \delta^c_a \). This formula can then be used to project out components parallel and normal to \( N^I \). With the use of this, and the fact that

\[
\Phi^{ab} = 0 \Rightarrow \pi^a_{IJ} \tilde{\eta}^{IK} \tilde{\eta}^{JL} = 0 \tag{2.61}
\]
it is straightforward to rewrite the terms in the Lagrangian as:

\[
e e^a e^b R_{ab}^{IJ} = -\frac{N^2}{e} Tr(\pi^a \pi^b R_{ab}) + N^a Tr(\pi^b R_{ab}) - \lambda_{ab} \Phi^{ab}
\]

\[
e = \det(e_{ai}) = N \sqrt{\det(V_{ab})} = \frac{N^2}{e} \sqrt{\det(Tr(\pi^a \pi^b))}
\]

where I have included all terms containing \(\pi^I \tilde{\eta}^{JK} \tilde{\eta}^{IL}\) into the Lagrange multiplier term \(\lambda_{ab} \Phi^{ab}\). Redefining the lapse function \(\tilde{N} := \frac{N^2}{e}\), the total Lagrangian becomes

\[
\mathcal{L} = -Tr(\pi^a \dot{\omega}_a) - Tr(\omega_0 G) - \tilde{N} \mathcal{H} - N^a \mathcal{H}_a - \lambda_{ab} \Phi^{ab}
\]

where

\[
G^{IJ} := D_a \pi^{aIJ} = \partial_a \pi^{aIJ} + [\omega_a, \pi^{aIJ}] \approx 0
\]

\[
\mathcal{H} := Tr(\pi^a \pi^b R_{ab}) - 2\lambda \sqrt{\det(Tr(\pi^a \pi^b))} \approx 0
\]

\[
\mathcal{H}_a := -Tr(\pi^b R_{ab}) \approx 0
\]

\[
\Phi^{ab} := \frac{1}{2} \pi^{aIJ} \pi^{bKL} \epsilon_{IJKL} \approx 0
\]

and the fundamental Poisson bracket is

\[
\{\omega_{aIJ}(x), \pi^{bKL}(y)\} = \frac{1}{2} \delta^b_a \delta^{[K}_i \delta^{L]}_j \delta^a(x - y)
\]

The SO(1, 3) trace is defined as:

\[
Tr(ABC) := A^I_J B^J_K C^K_I
\]

\[
Tr(AB) := A^I_J B^I_J
\]

The preliminary Hamiltonian can now be read off in (2.64), using: \(\mathcal{H}_{tot} = -Tr(\pi^a \dot{\omega}_a) - \mathcal{L}\). The Lagrange multiplier fields are \(\tilde{N}, N^a, \omega_{IJ}, \lambda_{ab}\), and their variation imposes the constraints in (2.65). For this Hamiltonian to be a complete and consistent one, one needs to check if the time evolution of all the constraints vanishes weakly. That is, a field-configuration that initially satisfies all the constraints must stay on the constraint surface under the time evolution. In our case, the total Hamiltonian is just a linear combination of constraints, which means that the above requirement corresponds to demanding that the constraint algebra should close. If it is not closed, one has two options; either one fixes some of the Lagrange multipliers so that the time evolution is consistent, or one introduces secondary constraint.

To simplify the constraint analysis, I prefer to change to vector notation in SO(1, 3) indices.

\[
\pi^{aIJ} := \pi^{ai} T_{i}^{IJ}; \quad \omega^{ij} := \omega^{ij} T_{i}^{IJ}
\]

where \(T_{i}^{IJ}\) are the generators of the \(so(1, 3)\) Lie-algebra. \(N.B\), the indices; \(i, j k\) take values 1, 2, ..., 6 here since \(so(1, 3)\) is a six-dimensional Lie-algebra, while in other sections \(i, j, k\) denote \(SO(3)\) indices. The following definitions and identities are also useful:
Chapter 2  Actions in (3+1)-dimensions

\[ q_{ij} := -Tr(T_i T_j) \quad q^{ij} q_{kj} = \delta_j^i \]
\[ q_{ij}^* := -Tr(T_i^* T_j^*) := -\frac{1}{2} \epsilon^{IJKL} T_i^K T_j^L \]
\[ \pi^a_i := \pi^{a j} q_{ij} \; ; \; \pi_i^a := \pi^{a j} q_{ij}^* \quad (2.68) \]
\[ q^{* ij} := q^{ik} q_{jl} q^{* kl} \Rightarrow q^{* ij} q^{* jk} = -\delta_i^j \]
\[ [T_i, T_j] = f_{ij}^k T_k; \quad f_{ijk} := f_{ij}^l q_{kl} = -2 Tr(T_i T_j T_k) \]
\[ f_{ijk} f_{lm}^k = \frac{1}{2} (q_{ij} [q_{kl}] - q_{ik} [q_{mj}] ) \]

\[ f_{ijk} \] has all the index-symmetries (anti-symmetries) that \( f_{ijk} \) has. The *-operation really corresponds to the duality-operation in \( so(1,3) \) indices. Using this notation, the Hamiltonian becomes

\[
\mathcal{H}_{tot} = \hat{N} \mathcal{H} + N^a \mathcal{H}_a + \lambda_{ab} \Phi^{ab} - \omega_{ai} G^i 
\]
\[ \mathcal{G}^i = D_a \pi^{ai} = \partial_a \pi^{ai} + f_{ijk} \omega_{aj} \pi^{ak} \approx 0 \]
\[ \mathcal{H} = -\frac{1}{2} f_{ijk} \pi^{ai} \pi^{bj} R_{ab}^k - 2 \lambda \sqrt{-det(\pi^{ai} \pi_i^a)} \approx 0 \]
\[ \mathcal{H}_a = \pi^{bi} R_{abi} \approx 0 \]
\[ \Phi^{ab} = \pi^{ai} \pi_i^b \approx 0 \]

where " \( \approx \) " denotes weak equality, which means equality on the constraint-surface, or equality modulo the constraints.

The fundamental Poisson bracket is

\[
\{ \omega_{ai}(x), \pi^{bj}(y) \} = \delta^b_i \delta^j_a \delta^3(x - y) \quad (2.70)
\]

To calculate the constraint algebra I will again use the shortcut described in section 2.3. First, the generator of spatial diffeomorphisms and the generator of Lorentz transformations are identified. Then, using their fundamental transformations all Poisson brackets containing these generators are easily calculated.

The transformations generated by \( \mathcal{G}^i \) are:

\[
\delta^{\mathcal{G}^i} \pi^{ai} := \{ \pi^{ai}, \mathcal{G}^j [\Lambda_j] \} = f_{ijk}^{\; \; \; \; \; \; \; \; l} \Lambda^j_l \pi^{al} \quad (2.71)
\]
\[
\delta^{\mathcal{G}^i} \omega^i_a := \{ \omega^i_a, \mathcal{G}^j [\Lambda_j] \} = -D_a \Lambda^i \quad (2.72)
\]

which shows that \( \mathcal{G}^i \) is the generator of \( SO(1,3) \) transformations. Then I define

\[
\tilde{\mathcal{H}}_a := \mathcal{H}_a - \omega_{ai} \mathcal{G}^i 
\]

and calculate the transformations it generates:
This leaves only three Poisson brackets left to calculate:

\[ \delta \tilde{H}_a \pi^{ai} := \{ \pi^{ai}, \tilde{H}_b[N^b] \} = N^b \partial_b \pi^{ai} - \pi^{bi} \partial_b N^a + \pi^{ai} \partial_b N^b = \mathcal{L}_{N^b} \pi^{ai} \]

(2.74)

\[ \delta \tilde{H}_a \omega^i_a := \{ \omega^i_a, \tilde{H}_b[N^b] \} = N^b \partial_b \omega^i_a + \omega^i_a \partial_a N^b = \mathcal{L}_{N^b} \omega^i_a \]

(2.75)

which shows that \( \tilde{H}_a \) is the generator of spatial diffeomorphisms. \( (\mathcal{L}_{N^b} \) is the Lie-derivative along the vector field \( N^b \).) This makes it easy to calculate all Poisson brackets containing \( G^i \) or \( \tilde{H}_a \). See section 2.3 for details.

\[ \{ G^i[\Lambda_i], G^j[\gamma_j] \} = G_i[\Lambda^k \gamma^j f_{kj}^i] \]

(2.76)

\[ \{ H[N], G^i[\gamma_j] \} = 0 \]

(2.77)

\[ \{ \Phi^{ab}[\lambda_{ab}], G^j[\gamma_j] \} = 0 \]

(2.78)

\[ \{ G^i[\Lambda_i], \tilde{H}_a[N^a] \} = G_i[-\mathcal{L}_{N^a} \Lambda^i] \]

(2.79)

\[ \{ H_b[M^b], \tilde{H}_a[N^a] \} = \tilde{H}_b[-\mathcal{L}_{N^a} M^b] \]

(2.80)

\[ \{ H[N], \tilde{H}_a[N^a] \} = H[-\mathcal{L}_{N^a} N] \]

(2.81)

\[ \{ \Phi^{ab}[\lambda_{ab}], \tilde{H}_a[N^a] \} = \Phi^{ab}[-\mathcal{L}_{N^a} \lambda_{ab}] \]

(2.82)

This leaves only three Poisson brackets left to calculate: \( \{ H[N], H[M] \}, \{ H[N], \Phi^{ab}[\lambda_{ab}] \} \) and \( \{ \Phi^{ab}[\lambda_{ab}], \Phi^{cd}[\gamma_{cd}] \} \). They are rather straightforward to calculate, but it simplifies to notice that \( \{ H[N], H[M] \} \) is anti-symmetric in \( N \) and \( M \) meaning that it is only terms containing derivatives on these fields that survive. One needs also the structure-constant identity in (2.68).

\[ \{ H[N], H[M] \} = H_b[\frac{1}{2} \pi^{bj} \pi^c_j (M \partial_c N - N \partial_c M)] + \Phi^{bc}[\pi^{sam} R_{bam} (N \partial_c M - M \partial_c N)] \]

(2.83)

\[ \{ \Phi^{ab}[\lambda_{ab}], \Phi^{cd}[\gamma_{cd}] \} = 0 \]

(2.84)

\[ \{ \Phi^{ab}[\lambda_{ab}], H[N] \} = \int \Delta^3 x \ N \lambda_{ab} \pi^{smc} f_{min} \pi^{bi} D_c \pi^{na} := \Psi^{ab}[N \lambda_{ab}] \]

(2.85)

The constraint algebra fails to close due to (2.85). Notice that, if we should simply remove the constraint \( \Phi^{ab} \approx 0 \) from the theory, the constraint algebra would still fail to close, this time due to (2.83). There is, however, an alternative strategy that really gives a closed constraint algebra: forget about \( \Phi^{ab} \), and define instead two new constraints \( \mathcal{H}^*: = \frac{1}{2} f_{ijk} \pi^{ai} \pi^{bj} R_{ab}^k \approx 0 \) and \( \mathcal{H}_a^*: = \pi^{bi} R_{abi} \approx 0 \). This will make the constraints \( G^i, H, \mathcal{H}, \mathcal{H}_a, \mathcal{H}_a^* \) form a first class set. The theory will of course not be ordinary Einstein gravity, more like twice the theory of Einstein gravity. (The number of degrees of freedom is four per spacetime point, and if one splits the Euclidean theory into self dual and anti-self dual parts, the action will really just be two copies of the pure gravity action. The Lorentzian case is more complicated due to the reality conditions.)

Now, returning to the theory that really follows from the H-P Lagrangian we notice that since the constraint algebra fails to close, the time evolution of the constraints \( H \) and \( \Phi^{ab} \) will not automatically be consistent. That is, the time evolution will bring the theory.
out of the solution-space to these constraints. This must be taken care of in order to get a fully consistent theory. And as mentioned earlier, there are two different strategies available; either solve for some Lagrange multiplier, or introduce secondary constraints. The first method is often the preferred one since that solution maximizes the number of degrees of freedom. I try this method first. The equations that should be solved are

\[
\dot{\Phi}^{ab}[K_{ab}] = \{\Phi^{ab}[K_{ab}], H_{tot}\} \approx \Psi^{ab}[\tilde{N}K_{ab}] \approx 0 \quad (2.86)
\]

\[
\dot{H}[M] = \{H[M], H_{tot}\} \approx \Psi^{ab}[-M\lambda_{ab}] \approx 0 \quad (2.87)
\]

Here, \(K_{ab}\) and \(M\) are arbitrary test functions, and \(\tilde{N}\) and \(\lambda_{ab}\) are the Lagrange multiplier fields that sit in the total Hamiltonian (2.69). These equations must be solved for the Lagrange multipliers so that they are satisfied for all test-functions \(M\) and \(K_{ab}\). Assuming \(\Psi^{ab}\) to be a non-degenerate matrix, the minimal solution is

\[
\tilde{N} = 0 \quad (2.88)
\]

\[
\lambda_{ab} = \tilde{\lambda}_{ab} - \frac{1}{3}\Psi_{ab}\Psi^{cd}\tilde{\lambda}_{cd} \quad (2.89)
\]

where \(\Psi_{ab}\) is the inverse to \(\Psi^{ab}\), and \(\tilde{\lambda}_{cd}\) is an arbitrary Lagrange multiplier. In a generic field theory, this would probably be a good choice to get a consistent Hamiltonian formulation. In our case, however, we do know that the Lagrange multiplier field \(\tilde{N}\) has an important physical interpretation, it is the lapse-function, and putting that to zero would mean that the spacetime metric always would be degenerate. Before leaving this ”unphysical” solution let us calculate the physical degrees of freedom: half the phase space coordinates (18) minus the number of first class constraints (14) minus half the number of second class constraints (1), gives three degrees of freedom per point. Thus, it looks like the ”degenerate metric theory” has more freedom than the conventional non-degenerate one. (If this ”degenerate theory” is liberated from second-class constraints, and if also all first-class constraints except \(\mathcal{H}_a\) and the rotational part of \(\mathcal{Q}^i\) are solved and properly gauge-fixed, then we would possibly have an \(SO(3)\) and \(diff(\Sigma)\)-invariant theory on \(SO(3)\) Yang-Mills phase space: the Husain-Kuchař-theory \([13]\). Although, to really be sure that this ”degenerate theory” is the Husain-Kuchař theory, one must show that this reduction can be done so that the remaining phase space coordinate is an \(SO(3)\) connection\([1]\).

Excluding this solution, we are forced to include secondary constraints. The new constraint, which should be added to the total Hamiltonian, is

\[
\Psi^{ab} := \frac{1}{2}\pi^{smc}f_{min}(bD_c\pi^a)n \approx 0 \quad (2.90)
\]

meaning that the total Hamiltonian now becomes

\[
\mathcal{H}'_{tot} = \mathcal{H}_{tot} + \gamma_{ab}\Psi^{ab} \quad (2.91)
\]

With this new Hamiltonian, one needs to check again if the time evolution of all constraints vanishes weakly, and the new ingredients are all Poisson brackets containing the constraint

\[1\]I thank Ingemar Bengtsson for suggesting this.
\(\Psi^{ab}\). The Poisson brackets containing \(\tilde{\mathcal{H}}_a\) and \(G^i\) are again easy to calculate. Note that, due to the fact that \(D_a\) is only covariant with respect to \(SO(1,3)\) transformations, \(\Psi^{ab}\) is not manifestly covariant under spatial diffeomorphisms. It is, however, covariant, which can be seen by adding a fiducial affine connection to \(D_a\) and note that it drops out from \(\Psi^{ab}\). (This is of course also true for \(G^i\).) Thus, since \(\Psi^{ab}\) transforms covariantly under both \(SO(1,3)\) transformations as well as spatial diffeomorphisms, the Poisson brackets are

\[
\{ \Psi^{ab}[\gamma_{ab}], G^i[\Lambda_i] \} = 0
\]

\[
\{ \Psi^{ab}[\gamma_{ab}], \tilde{\mathcal{H}}_c[N^c] \} = \Psi^{cd}[\mathcal{L}_{N^c}\gamma_{cd}]
\]

(2.92)

(2.93)

The Poisson bracket between \(\Psi^{ab}\) and \(\Phi^{ab}\) is also easily calculated:

\[
\{ \Phi^{ab}[\rho_{ab}], \Psi^{cd}[M_{cd}] \} = \int d^3x \left( -\frac{1}{2}\rho_{ab}\pi^{bc}M_{cd}\pi^{da} + \frac{1}{2}\rho_{ab}\pi^{ab}\gamma_{cd}\pi^{cd} \right) \approx 0
\]

(2.94)

where \(\pi^{ab} := \pi^{ai}\pi^{ib}\). This shows that \(\Phi^{ab}\) and \(\Psi^{ab}\) are second-class constraints. (The rank of the "constraint-matrix" in (2.94) is, for generic field-configurations, maximal, i.e. rank twelve.) Now, two more Poisson brackets should be calculated, \(\{ \Psi^{ab}, \Psi^{cd} \}\) and \(\{ \mathcal{H}, \Psi^{ab} \}\). But these calculations are really horrible, and since I do not need the exact result in order to show that the Hamiltonian now can be put in a consistent form, I will not write out these Poisson brackets here. Instead, I start with the time evolution of \(\Phi^{ab}\):

\[
\dot{\Phi}^{ab}[\rho_{ab}] = \{ \Phi^{ab}[\rho_{ab}], H'_\text{tot} \} \approx \int d^3x \left( -\frac{1}{2}\rho_{ab}\pi^{bc}\gamma_{cd}\pi^{da} + \frac{1}{2}\rho_{ab}\pi^{ab}\gamma_{cd}\pi^{cd} \right) \dot{\gamma}_{ab} \approx 0
\]

(2.95)

And since this equation is required to be satisfied for all test functions \(\rho_{ab}\), the only solution is

\[
\gamma_{ab} = 0
\]

(2.96)

meaning that the constraint \(\Psi^{ab}\) drops out from the total Hamiltonian. With this solution for \(\gamma_{ab}\), the time evolution for \(\mathcal{H}\) is automatically okay and it is only \(\Psi^{ab}\)’s time evolution left to check. First, I define

\[
\{ \Psi^{ab}[K_{ab}], \mathcal{H}[\tilde{N}] \} := \Sigma^{ab}[K_{ab}]
\]

(2.97)

Whatever complicated result the above Poisson bracket gives, I partially integrate it so that the test function \(K_{ab}\) is free from derivatives, and then I call the result \(\Sigma^{ab}\). With this definition, the time evolution of \(\Psi^{ab}\) becomes

\[
\dot{\Psi}^{ab}[K_{ab}] = \{ \Psi^{ab}[K_{ab}], H_{\text{tot}} \} \approx \int d^3x \left( \frac{1}{2}K_{ab}\pi^{bc}\lambda_{cd}\pi^{da} - \frac{1}{2}K_{ab}\pi^{ab}\lambda_{cd}\pi^{cd} \right) + \Sigma^{ab}[K_{ab}] \approx 0
\]

(2.98)

This equation can always be solved for \(\lambda_{ab}\) so that the equation is satisfied for all test-functions \(K_{ab}\).
\begin{equation}
\lambda_{ab} = 2 \pi_{ae} \Sigma^{ce} \pi_{cb} - \pi_{ab} \pi_{ce} \Sigma^{ce}
\end{equation}

Here, \(\pi_{ab}\) is the inverse to \(\pi_{ab} := \pi_{ai} \pi_{ib}\). So, regardless of the complicated form of \(\Sigma^{ab}\) one can always solve for \(\lambda_{ab}\). Note that it can happen that \(\Sigma^{ab}\) contains derivatives of the lapse-function \(\tilde{N}\), but in the action these can always be partially integrated away to get a multiplicative Lagrange multiplier.

To summarize the Hamiltonian analysis; the total and consistent Hamiltonian is

\begin{equation}
H_{\text{tot}} = \tilde{N} \mathcal{H} + N^a \mathcal{H}_a - \omega_0 G^i + \lambda_{ab}(\tilde{N}, \pi, \omega) \Phi^{ab}
\end{equation}

where \(\lambda_{ab}\) is the specific function of \(\tilde{N}, \pi_{ai}\) and \(\omega_{ai}\), given in (2.97) and (2.99). There are twelve second-class constraints: \(\Phi^{ab}\) and \(\Psi^{ab}\), and ten first class constraints: \(G^i, H_a\) and \(\tilde{H} := \mathcal{H} + \frac{1}{\tilde{N}} \lambda_{ab}(\tilde{N}, \pi, \omega) \Phi^{ab}\). The number of degrees of freedom are: half the number of phase space coordinates (18) minus the number of first-class constraints (10) minus half the number of second-class constraints (6), which gives two degrees of freedom per point.

The next task in this section is to eliminate the second-class constraints from the theory, leaving a Hamiltonian formulation with first-class constraints only. In doing this elimination, I find it convenient to switch back to the original \(SO(1,3)\) notation. First, to solve \(\Phi^{ab} \approx 0\), I use the origin of that constraint; namely (2.57). So, I fix a trio of non-degenerate space-like \(SO(1,3)\) vectors \(E^a_I\), and also the (up to a sign) unique time-like unit vector, orthogonal to \(E^a_I\), \(N^I\).

\begin{align*}
det(E^a_I E^b_I) & > 0 \\
E^a_I N^I & = 0 \\
N^I N^I & = -1
\end{align*}

\(N^I\) can be explicitly defined as

\begin{equation}
N^I = -\frac{\epsilon^{IJKL} E^a_J E^b_K E^c_L \epsilon_{abc}}{6 \sqrt{\det(E^a_I E^b_I)}}
\end{equation}

Note the similarity between the pair \((N^I, E^a_J)\) and \((N^I, V^a_J)\) in (2.59). It is also useful to define the projection operator

\begin{equation}
\tilde{\eta}^{IJ} := E^a_I E^a_J = \eta^{IJ} + N^I N^J
\end{equation}

where \(E^a_I := E_{ab} E^{bI}\), and \(E_{ab}\) is the inverse to \(E^a_I E^b_I\). Using this projection operator, one can separate all \(SO(1,3)\) indices into "boost" and "rotational" parts. The solution to \(\Phi^{ab} \approx 0\) can now be written as

\begin{equation}
\pi^{aIJ} = N^{[I} E^{aJ]}
\end{equation}

where the twelve physical degrees of freedom in \(\pi^{aIJ}\) now is captured in \(E^a_I\). Now it is possible to solve also \(\Psi^{ab}\) for \(\omega_{aIJ}\), but instead of doing so, I will take a shortcut via the symplectic form \(\pi^{aIJ} \omega_{aIJ}\). I use the solution to \(\Phi^{ab} \approx 0\), (2.104) in the symplectic form, and see what components of \(\omega_{aIJ}\), \(E^a_I\) will project out. To do that, I first split \(\omega_{aIJ}\) into two separate pieces.
\[ \omega_{aIJ} = M_{aIJ} + \Gamma_{aIJ} \]  

(2.105)

where \( \Gamma_{aIJ} \) is the unique torsion-free connection compatible with \( E^a_I \):

\[ D_a E^b_I = \partial_a E^b_I + \Gamma^b_{ca} E^c_I + \Gamma^I_{IJ} E^b_J - \Gamma^c_{ca} E^b_I = 0 \]  

(2.106)

See Appendix B for a treatment of this type of “hybrid” spin-connection. Thus, (2.105) can be seen as a definition of \( M_{aIJ} \). Then, in order to rewrite the symplectic form I need some formulas relating \( \dot{N}^I \) and \( \dot{E}^a_I \). From (2.101) it follows that

\[ \dot{N}^I N^I = 0 \]

\[ \dot{N}^I E^a_I = -\dot{E}^a_I N^I \]  

(2.107)

Now, the symplectic form becomes

\[ \pi^{aIJ} \omega_{aIJ} = 2 N^I E^a_I (\dot{\Gamma}_{aIJ} + \dot{M}_{aIJ}) = 2 N^I \left( D_a \dot{E}^a_I - D_a \dot{E}^a_I + E^a_I \dot{M}_{aIJ} \right) \]

\[ = -2 \partial_a (N^I \dot{E}^a_I) + 2 N^I E^a_I \dot{M}_{aIJ} \]

\[ = -2 \partial_a (N^I \dot{E}^a_I) + 2 (N^I E^a_I M_{aIJ}) + \dot{E}^a_I (2 N^I E^a_I E^b_K M_{bIK} - 2 N^I M_{aIJ}) \]  

(2.108)

Neglecting the surface terms, one can now read off the momenta conjugate to \( E^a_I \),

\[ \{ E^a_I (x), K_{bJ}(y) \} = \delta^a_b \delta^I_J \delta^3(x-y) \]

\[ K_{aJ} := 2 N^I E^a_I E^b_K M_{bJK} - 2 N^I M_{aIJ} \]  

(2.109)

This means that the physical components \( E^a_I \) of \( \pi^{aIJ} \), project out twelve components \( K_{aI} \) of \( \omega_{aIJ} \). The question is then; are these twelve components \( K_{aI} \) of \( \omega_{aIJ} \), the twelve components that survives the constraint \( \Psi^{ab} \)? To answer this question, I first invert the relation (2.109).

\[ M_{aIJ} = -\frac{1}{2} E^{bK} K_{aK} N_{[J} E_{bI]} - \frac{1}{4} N^K K_{eK} E^{e[I} E_{aJ]} \]

\[ + T_{a}^{bc} E_{bI} E_{cJ} \]  

(2.110)

where \( T_{a}^{bc} = 0 \) and \( T_{a}^{ab} = 0 \). \( T_{a}^{bc} \) are the six components of \( \omega_{aIJ} \) that is orthogonal to \( E^a_I \) in the symplectic form. Now, putting this into the other second-class constraint \( \Psi^{ab} \), one gets

\[ \Psi^{ab} = -\frac{1}{2} \pi^{cIJ} \epsilon_{IJ} K_{LMN} (bLM) K_{M}^{a} = \cdots \sim \epsilon^{cd(a} T^{b)g} E_{gd} \]  

(2.111)

And requiring that \( \Psi^{ab} = 0 \) imposes six conditions on the six components in \( T_{a}^{bc} \), which can easily be solved to get: \( T_{a}^{bc} = 0 \). This means that the solution to \( \Phi^{ab} = 0 \) really
projects out the solution to the other second-class constraint \( \Psi^{ab} \), and that the physical degrees of freedom surviving both \( \Phi^{ab} \) and \( \Psi^{ab} \) are captured in \( E^{aI} \) and \( K_{aI} \). To summarize the solution of the second-class constraints, we have

\[
\begin{align*}
\pi^{aIJ} &= N^I E^{aJ} \\
\omega_{aIJ} &= \Gamma_{aIJ} - \frac{1}{2} E^{bK} K_{aK} N_{[J} E_{bI]} - \frac{1}{4} N^K K_{eK} E^e_{[J} E^{a]}_{I]}
\end{align*}
\]

\[
\{E^{aI}(x), K_{bj}(y)\} = \delta^a_b \delta^I_j \delta^3(x-y) \tag{2.112}
\]

With this solution of the second-class constraints at hand, it is time to rewrite the total Hamiltonian in terms of the physical fields \( E^{aI} \) and \( K_{aI} \). A straightforward calculation gives

\[
\begin{align*}
G_{IJ} &= D_a \pi^{aIJ} = \ldots = -\frac{1}{2} E^a_{[I} K_{aJ]} \approx 0 \\
H_a &= \pi^{bIJ} R_{abIJ} = \ldots = -D_{(a} (E^{bM} K_{b|M}) \approx 0 \\
H &= \pi^{aIJ} \tilde{R}_{abIJ} - 2\lambda \sqrt{\det(\pi^{aIJ} \pi_{IJ})} = \ldots \\
&= -E^{aI} E^{bJ} \tilde{R}_{abIJ} - 2\lambda \sqrt{\det(E^{aI} E^{bJ})} - \frac{1}{4} E^{aI} E^{bJ} K_{[aI} K_{bJ]} \approx 0 \tag{2.115}
\end{align*}
\]

where \( D_a \) is the \( E^{aI} \) compatible torsion-free covariant derivative, and \( \tilde{R}_{abIJ} \) is its curvature: \( \tilde{R}_{abIJ} := \partial_{[a} \Gamma_{b]IJ} + [\Gamma_{a}, \Gamma_{b}]_{IJ} \). In equations (2.114) and (2.115), I have neglected terms proportional to Gauss’ law \( G_{IJ} \).

But this Hamiltonian is easily identified as the well-known ADM-Hamiltonian with the full \( SO(1,3) \) invariance unbroken. If one wants to compare this to section 2.3 one first needs to gauge-fix \( E^{a0} = 0 \), and then solve the corresponding ”boost-part” of Gauss’ law, \( G^{aI} = 0 \) \((K_{a0} = 0)\). One should also de-densitize the coordinate \( E^{ai} \). I will not do this comparison in detail, I just end this section by a short summary of what has been done here.

The straightforward Hamiltonian formulation of the first-order H-P Lagrangian gave a Hamiltonian system with twelve second-class constraints. The reason why the second-class constraints appear can be traced to the mismatch between the number of algebraically independent components of the spin-connection and the tetrad field. (In (2+1)-dimensions or in (3+1)-dimensions with only a self dual spin-connection, the mismatch disappears and there are no second-class constraints.) Then, when the second-class constraints are solved to get a Hamiltonian containing only first-class constraints, one ends at the ADM-Hamiltonian.

In doing this Hamiltonian analysis of the full Hilbert-Palatini Lagrangian, it is striking how complicated the analysis is, compared to the analysis of the self dual H-P Lagrangian, in next section. And the reason why the constraint analysis, in this section, is so complicated is of course the appearance of second-class constraints. We will see that a reduction of the number of algebraically independent components in the spin-connection will significantly simplify the Hamiltonian analysis. This reduction is accomplished by only using the self dual part of the spin-connection in the Lagrangian.
2.5 Self dual Hilbert-Palatini Lagrangian

In this section, the H-P Lagrangian containing only the self dual part of the curvature will be examined. I will first show that the equations of motion following from it, are the same as those from the full H-P Lagrangian. This is due to the fact that the two Lagrangians give the same equation of motion from the variation of the spin-connection, and once this equation is solved, the two Lagrangians differs only by a term that vanishes due to the Bianchi-identity.

After proving that this action is a good action for gravity, the Legendre transform will be performed, and the theory will be brought into the Ashtekar Hamiltonian formulation. In this Hamiltonian formulation, it will become clear that all the second-class constraints from section 2.4 now have disappeared. And the reason why they do not appear, is that with only the self dual spin-connection present, the tetrad has enough independent components to function as momenta to the spin-connection, without restrictions.

The analysis of the self dual H-P Lagrangian was first performed in [20]. Later, this Lagrangian has been examined in [17], [18], [21] and [22].

Before I start working on the action, I will give a few basic features of self duality. Consider an \( so(1,3) \) Lie-algebra valued field: \( A^{IJ} \). The dual of \( A^{IJ} \) is defined as

\[
A^{*IJ} := \frac{1}{2} \epsilon^{IJ} K L A^{KL}
\]

and the dual of the dual becomes

\[
A^{**IJ} = \frac{1}{4} \epsilon^{IJK} \epsilon^{LMN} A^{MN} = -A^{IJ}
\]

The minus-sign follows from the Lorentz-signature of the Minkowski metric. (For Euclidean signature, \( so(4) \), there is a plus-sign there instead.) Now, since the duality-operation imposed twice has the eigen-value minus one for any \( A^{IJ} \), it is possible to diagonalize \( A^{IJ} \) as follows

\[
A^{IJ} = A^{(+)}IJ + A^{(-)}IJ
\]

where

\[
A^{*IJ} = +iA^{(+)}IJ - iA^{(-)}IJ
\]

\( A^{(+)}IJ \) and \( A^{(-)}IJ \) are called the self dual and the anti-self dual part of \( A^{IJ} \). They can be explicitly defined as

\[
A^{\pm IJ} := \frac{1}{2} (A^{IJ} \mp i A^{*IJ}) = \frac{1}{2} (A^{IJ} \mp i \epsilon^{IJ} K L A^{KL})
\]

The self dual and anti-self dual parts can in some sense be seen as orthogonal components, and the following relations are easily proven with (2.120) and the \( \epsilon - \delta \) identity.

\[
A^{(+)IJ} B^{(-)}_{IJ} = 0
\]

\[
A^{(+)IJ} B^{(+)}_{IJ} = A^{(+)IJ} B^{(+)I J}
\]

\[
[A, B]^{IJ} = [A^{(+)}, B^{(+)I J} + A^{(-)}, B^{(-)}I J]
\]
Equation (2.121) is just the orthogonality relation, and (2.123) shows that the complexification of the Lorentz-algebra splits into its self dual and anti-self dual sub-algebras: $so(1,3;C) \sim so(3) \times so(3)$. Equation (2.123) also shows that the self dual curvature is the curvature of the self dual spin-connection:

$$R^{IJ}_{\alpha \beta} (\omega^{(+)}_{\alpha \beta}) = R^{(+)IJ}_{\alpha \beta} (\omega^{(+)}_{\alpha \beta}) = R^{(+)}_{\alpha \beta} (\omega_{\alpha \beta})^{KL}$$

(2.124)

Now, since the original $A^{IJ}$ has 6 algebraically independent components, while $A^{(+)}_{IJ}$ and $A^{(-)}_{IJ}$ has only three algebraically independent components, there exist relations between different components of $A^{(+)}_{IJ}$ and $A^{(-)}_{IJ}$. For instance

$$A_{kl}^{(\pm)} = \pm i A_{0i}^{(\pm)} \epsilon_{ikl}$$

(2.125)

$$A_{0i}^{(\pm)} = \mp i \epsilon_{ikl} A_{kl}^{(\pm)}$$

(2.126)

where $\epsilon_{ijl} := \epsilon_{0lj}^{ij}$ and $\epsilon_{ijkl} = \epsilon_{ijk}$. This split can also be made Lorentz-covariant. (Note that, if $A^{(+)}_{IJ}$ is a Lorentz-covariant object, it transforms only under the self dual Lorentz-transformations, while a self dual Lorentz-connection transforms under the full Lorentz-group.)

To make the split (2.125) and (2.126) Lorentz covariant, I define a trio of non-degenerate space-like $SO(1,3)$ vector-fields $V_{aI}$, and also the (up to a sign) unique time-like unit vector-field $N_I$, orthogonal to $V_{aI}$.

$$\det(V_{aI}V_{bI}) > 0, \quad N_I V_{aI} = 0, \quad N_I^N_I = -1$$

(2.127)

An explicit definition of $N_I$ is

$$N_I := -\frac{\epsilon_{ijkl} V^a_I V^b_J V^c_K V^d_M \epsilon_{abc}}{6 \sqrt{\det(V_{aI}V_{bI})}}$$

(2.128)

I also define the projection operator

$$\tilde{\eta}^{IJ} := V_{aI} V^a_J = \eta^{IJ} + N_I^N_J$$

(2.129)

where $V_{aI} := V_{ab} V^{bI}$, and $V_{ab}$ is the inverse to $V_{ab} := V_{aI} V^b_I$. Using this projection operator, all $so(1,3)$ indices can be decomposed into parts parallel and orthogonal to $N_I$.

$$K^{I'} := -N_I^N_J K^J$$

(2.130)

$$K^I := \tilde{\eta}^{IJ} K^J$$

(2.131)

$$\Rightarrow K^I = K^{I'} + K^{\tilde{I}}$$

(2.132)

Note that

$$N_I = N_I^{I'}, \quad V_{aI} = V_{a\tilde{I}}, \quad \tilde{\eta}^{IJ} = \tilde{\eta}^{\tilde{I}\tilde{J}}$$

(2.133)

I also define
2.5 Self dual Hilbert-Palatini Lagrangian

\[ \epsilon^{JKLM} := N_I \epsilon^{IJKL} \Rightarrow \epsilon^{JKLM} = -N_I \epsilon^{JKLM} \]  \hspace{1cm} (2.134)

Then, using the \( \epsilon - \delta \) identity for the full \( \epsilon^{IJKL} \), it is possible to show that

\[ \epsilon^{JKLM} \epsilon^{LMNP} = \delta^{[JKLM]}_{MNP} \]  \hspace{1cm} (2.135)

Now, with all this machinery, it is straightforward to make the split (2.125), (2.126) covariant. First

\[ A^{(+)}^{I'}J = \frac{1}{2} (A^{I'}J + \frac{i}{2} N_I \epsilon^{JKLM} A_KL) \]  \hspace{1cm} (2.136)
\[ A^{(\pm)}_{I\bar{J}} = \frac{1}{2} (A^{I\bar{J}} + iN^I \epsilon^{I\bar{J}L} A_{K\bar{L}}) \]  \hspace{1cm} (2.137)

which give the relations:

\[ A^{(\pm)}_{I\bar{J}} = \pm i N^K \epsilon^{I\bar{J}L} A^{(\pm)}_{K\bar{L}} \]  \hspace{1cm} (2.138)
\[ A^{(\pm)}_{I\bar{J}} = \pm \frac{i}{2} N^K \epsilon^{I\bar{J}L} A^{(\pm)}_{K\bar{L}} \]  \hspace{1cm} (2.139)

To compare (2.125), (2.126) with (2.138), (2.139), one can make the choice:

\[ N^I = (-1, 0, 0, 0) \Rightarrow V^{a0} = 0. \]

Now, it is time to introduce the self dual Hilbert-Palatini Lagrangian.

\[ \mathcal{L}_{HP}^{(+)} := e \left( \Sigma^{(+)}_{IJ} \epsilon^{(\gamma)}_{K} R^{(+)}_{IJ} (\omega^{(+)}_{KL}) + \lambda \right) \]  \hspace{1cm} (2.140)

where \( \Sigma^{\alpha\beta}_{IJ} (\epsilon^{(\gamma)}_{K}) := \frac{1}{2} \epsilon^{\alpha\beta}_{IJ} \epsilon^{(\gamma)}_{K} \), and \( \Sigma^{(+)}_{IJ} (\epsilon^{(\gamma)}_{K}) \) is its self dual part. \( R^{(+)}_{IJ} \) is the self dual part of the curvature of the spin-connection, and as mentioned earlier, the self dual part of the curvature of the full spin-connection equals the full curvature of the self dual part of the spin-connection. See (2.124).

First, when I want to show that the equations of motion following from (2.140) are the same as the ones following from the full H-P Lagrangian, I will regard \( R^{(+)}_{IJ} \) as the self dual part of the full curvature. That is

\[ R^{(+)}_{\alpha\beta} (\omega^{(+)}_{IJ}) = \frac{1}{2} \left( R^{IJ}_{\alpha\beta} (\omega) - \frac{i}{2} \epsilon^{I\bar{J}L} (R^{K\bar{L}}_{\alpha\beta} (\omega)) \right) \]  \hspace{1cm} (2.141)

The equations of motion following from the variation of \( \omega^{IJ}_{\alpha} \) are

\[ \frac{\delta S_{HP}^{(+)}_{\alpha}}{\delta \omega^{IJ}_{\alpha}} = -2D_{\alpha} (e \Sigma^{(+)}_{IJ} \epsilon^{\gamma}_{K}) = 0 \]  \hspace{1cm} (2.142)

But, since both \( \epsilon^{\gamma}_{K} \) and \( \omega^{IJ}_{\alpha} \) are real, the real part of (2.142) is just the normal zero-torsion condition. And the imaginary part of (2.142) is the dual of the real part, and therefore contains the same information. Then, if one solves (2.142) for \( \omega^{IJ}_{\alpha} \), and uses that solution in the Lagrangian (2.140), the imaginary part of the Lagrangian vanishes due to the Bianchi-identity. And the real part of the Lagrangian is just the conventional
Chapter 2  Actions in (3+1)-dimensions

Hilbert-Palatini Lagrangian (or Einstein-Hilbert Lagrangian, when the solution to (2.142) is used). So, altogether this means that the variation of $\omega^{I\alpha}_{\beta}$ implies the normal zero-torsion condition, and when using the solution to that equation, the variation of $e^\alpha_I$ implies Einstein’s equations. Note that the reality condition on the spin-connection really is superfluous. It is enough to require the tetrad to be real, then equation (2.142) will take care of the reality of the spin-connection. It is this fact that Ashtekar uses when he only imposes reality conditions on the metric-variables, and not on the connection. Note, however, that this is only true for non-degenerate metrics. With a degenerate metric, (2.142) cannot be solved to get a unique spin-connection.

Before doing the (3+1)-decomposition and Legendre transformation of (2.140), I will eliminate the non-independent parts of $\omega^{(+)I\alpha}_{\beta}$ and $\omega^{(+)I\alpha}_{\beta}$. Using (2.138) and (2.139), it follows that

$$A^{(+)I\beta}B^{(+)I\beta} = 2A^{(+)I\beta}B^{(+)I\beta} + A^{(+)I\beta}B^{(+)I\beta} = 4A^{(+)I\beta}B^{(+)I\beta}$$

(2.143)

This means that the (3+1)-decomposed Lagrangian becomes

$$\mathcal{L}^{(+)HP} = e\left(8\Sigma^{(+)0b}_{P\beta} R^{(+)0\beta} + 4\Sigma^{(+)ab}_{P\beta} R^{(+)0\beta} + \lambda\right)$$

(2.144)

Then, I define the momenta conjugate to $\omega^{(+)I\alpha}_{\beta}$:

$$\pi^{b}_{P\alpha} := \frac{\delta \mathcal{L}}{\delta \dot{\omega}^{(+)I\alpha}_{\beta}} = 8e\Sigma^{(+)0b}_{P\beta}$$

(2.145)

Remember that $\Sigma^{a\beta}_{I\beta}$ is not an independent field, it is just the anti-symmetric product of two tetrads. The next step in the Legendre transform is to rewrite the Lagrangian in terms of the phase space variables, but before doing so, I will introduce the ADM-like tetrad decomposition. I decompose the tetrad as follows

$$e^{0I} = -\frac{N^I}{N} \Rightarrow e^{0I}e^{0I} = -\frac{1}{N^2}$$

$$e^{aI} = V^{aI} + \frac{N^a N^I}{N}; \quad V^{aI}N_I = 0$$

(2.146)

This decomposition is not a restriction on the tetrad, the tetrad is completely general. Instead, it is a choice for the previously introduced vector fields: $N^I$, $V^{aI}$. Note that this decomposition gives the standard ADM-form of the metric:

$$g^{\alpha\beta} = \left(\begin{array}{cc}
-\frac{1}{N^2} & V^{aI}V^b_I - \frac{N^a N^b}{N^2} \\
\frac{N^a}{N^2} & \frac{N^a N^b}{N^2}
\end{array}\right)$$

(2.147)

Using this decomposition in (2.145), gives

$$\pi^{b}_{P\alpha} = 2e(e^{0I}_{I\beta} - i\epsilon_{P\alpha} \bar{K}L e^{0I}_{K\beta} e^{b}_{L}) = -\frac{2e}{N}N_I V^b_I$$

(2.148)

Note that the imaginary part of $\pi^{b}_{P\alpha}$ automatically vanishes with this choice of unit time-like vector-field. Here, I want to emphasize that it is not a gauge-choice involved here, the tetrad is completely general, the choice lies instead in what twelve components of $\omega^{(+)I\alpha}_{\beta}$ that should be regarded as independent, and the clever choice is to choose $\omega^{(+)I\alpha}_{\beta}$’s
projection along $e^{0t}$. This choice is clever, since it makes the momenta real and simple. Note, however, that there is nothing that says that the momenta must be real. A priori, we only know that the tetrad is real, and that can be achieved by the weaker requirement $\text{Im}(\pi^a_{I,J} \pi^{b'I,J}) = 0$.

Now, returning to the Legendre transform, relation (2.148) should be inverted,

$$V^b_J = \frac{N}{2e} N^I \pi^b_J$$

and the two last terms in (2.144) should be rewritten in terms of the momenta.

$$4e \Sigma^{(+)}_{I,J} R^{(+)}_{ab} = -N^a \pi^b_{I,J} R^{(+)}_{ab} - \frac{iN^2}{4e} \pi^a_{I,J} \pi^{b'I,J} N_{ab} N^I \pi^{(+)}_{ab} N^I \pi^{(+)}_{ab}$$

$$e = \frac{N^2}{8e} \det \pi^a_{I,J} = \frac{N^2}{8e} 6 \epsilon_{abc} \epsilon^{JKL} \pi^a_{I,J} \pi^b_{I,J} \pi^c_{I,J} N_{ab} N_{ab} N_{ab}$$

(2.150)

Before I write out the total phase space Lagrangian, I introduce the Ashtekar variables:

$$A_{ai} := -2N^J \omega^{(+)}_{ai,J}, \quad F_{ab} := -2N^J R^{(+)}_{ab,J} = \partial_a A_{b|i} + i\epsilon_{ijk} A_{a} A_{b}$$

$$E^{ai} := \frac{1}{2} N_{I,J} \pi^{a,I,J}, \quad \tilde{N} := \frac{N^2}{e}$$

(2.151)

where I used the fact that, since the "tilded" indices are orthogonal to a time-like direction, they are really $SO(3)$ indices. ($i, j, k ...$ are $SO(3)$ indices.) In terms of these variables, the total Lagrangian becomes

$$\mathcal{L}_{HP}^{(+)} = E^{ai} \dot{A}_{ai} - N^a \mathcal{H}_a - \tilde{N} \mathcal{H} + A_{ai} \mathcal{G}^i$$

(2.152)

where

$$\mathcal{H}_a := E^{b|i} F_{abi} \approx 0$$

$$\mathcal{H} := \frac{i}{2} E^{ai} E^{bj} F_{ab} k \epsilon_{ijk} - \lambda \text{det}(E^{ai}) \approx 0$$

$$\mathcal{G}^i := \mathfrak{D}_a E^{ai} = \partial_a E^{ai} + i\epsilon_{ijk} A_{aj} E^{aq} \approx 0$$

are the constraints that follow from the variation of the Lagrange multiplier fields $N^a$, $\tilde{N}$, and $A_{0i}$. The fundamental Poisson bracket is \{ $A_{ai}(x), E^{bj}(y)$ \} = $\delta^a_b \delta^i_j \delta^3(x - y)$.

This is the famous Ashtekar formulation, and to complete the analysis of it, a constraint analysis must be performed. This will be done in the next section. But, before I leave this section, I will give the metric formula in terms of Ashtekar’s variables. Using (2.146) and (2.151), the densitized spacetime metric is

$$\tilde{g}^{\alpha\beta} = \sqrt{-g} g^{\alpha\beta} = \left( -\frac{1}{N} N^a \tilde{N} E^{ai} E^b_i - \frac{N^a N^b}{N} \right)$$

(2.153)

The most appealing feature of the Ashtekar formulation is perhaps the simple and polynomial form of all the constraints. It is this feature that has enkindled the old
attempts of non-perturbative canonical quantization of gravity. Another fact that is new here in the Ashtekar formulation, compared to the other Hamiltonians in section 2.3 and 2.4, is that the phase space coordinate here is a (gauge-)connection. In the ADM-Hamiltonian, the phase space variables are all gauge-covariant objects, and in the Hamiltonian formulation of the full H-P Lagrangian, the spin-connection can only be used as the phase space coordinate if one is prepared to pay the price of keeping second-class constraints in the theory. Once these constraints are eliminated, the coordinate is again a gauge-covariant object. This fact, that the phase space coordinate is a gauge-connection has made it feasible to import techniques and methods from the more well-studied analysis of Yang-Mills theories.

2.6 The Ashtekar Hamiltonian

Here, I will do the constraint analysis for the Ashtekar Hamiltonian given in section (2.5). Then, the reality conditions will be examined in more detail, and finally the canonical transformation, relating this formulation to the ADM-Hamiltonian, will be given.

The Ashtekar Hamiltonian was originally found through the above mentioned canonical transformation [2], and it soon became clear that this Hamiltonian could be very useful in attempts to quantize gravity canonically. Prior to the existence of this Hamiltonian, the attempts to quantize gravity canonically had all started from the complicated ADM-Hamiltonian. The ADM-Hamiltonian is complicated mostly due to the non-polynomial and inhomogeneous form of the Hamiltonian constraint. This complicated form has made it practically impossible to find any quantum solution, in this formulation. Ashtekar’s Hamiltonian, however, has a simple polynomial and homogeneous Hamiltonian constraint, and quantum solutions to this formulation were soon found [3], [4].

The Ashtekar Hamiltonian formulation can be summarized as:

\[
H_{tot} := \tilde{N}H + N^aH_a - A_0G^i
\]

\[
H_a := E^{bi}F_{ab} \approx 0
\]

\[
H := \frac{i}{2}E^{ai}E^{bj}F_{ab}^k\epsilon_{ijk} - \lambda det(E^{ai}) \approx 0
\]

\[
G^i := D_aE^{ai} = \partial_aE^{ai} + i\epsilon_{ijk}A_{aj}E^{ak} \approx 0
\]

The fundamental Poisson bracket is \(\{A_{ai}(x), E^{bj}(y)\} = \delta^b_a\delta^j_i\delta^3(x - y)\), and the other fields are Lagrange multiplier fields, whose variation implies the constraints given in (2.154). The densitized spacetime metric can be constructed from the phase space fields, in a solution, as

\[
\tilde{g}_{\alpha\beta} = \sqrt{-g}g^{\alpha\beta} = \left(\frac{-\frac{1}{N}}{N^a}, \frac{N^a}{\tilde{N}E^{ai}E^{bj} - \frac{N^aN^b}{N}}\right)
\]

And the reality conditions will be given later. Next, I will show that this is a consistent Hamiltonian formulation in the sense that a field-configuration that initially satisfies the constraints will stay on the constraint surface under time evolution. This is the same as requiring the time evolution of the constraints to be weakly vanishing. To show that, I will calculate the constraint algebra. First, the transformations generated by \(G^i\) are given:
\[ \delta^G E^{ai} := \{E^{ai}, G^j[\Lambda_j]\} = \epsilon^i_{jk} \Lambda^j E^{ak} \]  
\[ \delta^G A^i_a := \{A^i_a, G^j[\Lambda_j]\} = -D_a \Lambda^i \]

which shows that \( G^i \) is the generator of \( SO(3) \) transformations. (Note that \( SO(3) \) here is the self-dual part of the Lorentz-group, not the rotation part.) Then, I define the generator of spatial diffeomorphisms:

\[ \tilde{H}^a := H^a - A^a_i G_i \]

and calculate the transformations it generates:

\[ \delta \tilde{H}^a E^{ai} := \{E^{ai}, \tilde{H}^b[N^b]\} = N^b \partial_b E^{ai} - E^{bi} \partial_b N^a + E^{ai} \partial_b N^b = \mathcal{L}_{N^b} E^{ai} \]

\[ \delta \tilde{H}^a A^i_a := \{A^i_a, \tilde{H}^b[N^b]\} = N^b \partial_b A^i_a + A^i_a \partial_b N^b = \mathcal{L}_{N^b} A^i_a \]

Now, one needs to know how the constraints transform under \( SO(3) \) rotations and spatial diffeomorphisms. Under \( SO(3) \) transformations; \( G^i \) is a vector, \( \tilde{H}_a \) is non-covariant, and \( H \) is a scalar. Under spatial diffeomorphisms; \( G^i \) is a scalar density of weight plus one, \( \tilde{H}_a \) is a covariant vector density of weight plus one, and \( H \) is a scalar density of weight plus two. Note that, although \( \tilde{H}_a \) is not covariant under \( SO(3) \) rotations, the Poisson bracket between \( G^i \) and \( \tilde{H}_a \) can yet easily be calculated since \( G^i \) transforms covariantly under spatial diffeomorphisms.

\[ \{G^i[\Lambda_i], G^j[\gamma_j]\} = G^i[\Lambda^k \gamma^j \epsilon_{kj}] \]  
\[ \{H[N], G^j[\gamma_j]\} = 0 \]  
\[ \{G^i[\Lambda_i], H_a[N^a]\} = G^i[-\mathcal{L}_{N^a} \Lambda^i] \]  
\[ \{H_b[M^b], H_a[N^a]\} = H_b[-\mathcal{L}_{N^a} M^b] \]  
\[ \{H[N], \tilde{H}_a[N^a]\} = \tilde{H}_b[-\mathcal{L}_{N^a} N^b] \]

The only Poisson bracket left to calculate is \( \{H, H\} \), and again this calculation simplifies by noting that only terms with derivatives on either \( N \) or \( M \) contribute:

\[ \{H[N], H[M]\} = \tilde{H}_a[E^{ai} E^b_i (N \partial_b M - M \partial_b N)] \]

This shows that all the constraints are first-class, and since the total Hamiltonian is a linear combination of these constraints, the time evolution of the constraints are automatically consistent.

In section 2.5 it was shown that it is sufficient to require the tetrad to be real in order to get real general relativity from the self dual H-P Lagrangian. With a real tetrad, the spin-connection will automatically be real in a solution to the equations of motion. How can this be translated into the Hamiltonian formulation, where one normally wants to impose all restrictions on the phase space variables? The equation that took care of the reality condition for the spin-connection was: \( \frac{\delta S^I}{\delta \omega^I} = 0 \), and here this becomes:
\[ \frac{\delta S_{HP}^{(+)}}{\delta A_{bi}} = \mathcal{G}^i = 0 \] (2.167)

\[ \frac{\delta S_{HP}^{(+)}}{\delta A_{ai}} = \dot{E}^{ai} + \frac{\delta H_{tot}}{\delta A_{ai}} = 0 \] (2.168)

In section 2.5 it was enough to require the tetrad to be real, and the solution to (2.167), (2.168) gave automatically a real spin-connection. (Note, however, that \(A_{ai}\), which in a solution is the self dual part of this real spin-connection, is not real.) Here, in the Hamiltonian formulation, this is a bit unconvenient since the velocity \(\dot{E}^{ai}\) appears in (2.168), and requiring that \(\dot{E}^{ai}\) is real will lead to conditions on the Lagrange multiplier fields \(\tilde{N}\) and \(A_{0i}\). Also, requiring the momenta \(E^{ai}\) to be real will forbid complex \(SO(3)\) gauge-transformations, which is a bit unnatural since \(SO(3)\) here really stands for the self dual part of \(SO(1, 3)\), which is a complex Lie-algebra.

The way out of this is to concentrate the reality requirements on the metric instead of the tetrad.

\[ Im(\tilde{N}) = 0, \quad Im(N^a) = 0, \quad Im(E^{ai}E^b_i) = 0 \] (2.169)

Then, imposing that \((E^{ai}E^b_i)\) should be real, will, through its equation of motion, lead to:

\[ Im(E^{cj}E^{(ai)D_cE^b_k}\epsilon_{ijk}) = 0 \] (2.170)

which is a good condition in the sense that it does not involve any Lagrange multipliers. Using the reality conditions (2.169) and (2.170) it is then straightforward to show that the metric will stay real under time evolution, provided the metric is invertible.

Now, I want to show the relation between this Ashtekar Hamiltonian and the conventional ADM-Hamiltonian in triad form. They are related through a complex canonical transformation, and to show that, I first define the new field \(K_{ai}\):

\[ K_{ai} := A_{ai} + i\Gamma_{ai}(E) \] (2.171)

where \(\Gamma_{ai}\) is the unique torsion-free spin-connection defined to annihilate \(E^{ai}\):

\[ D_aE^{bi} := \partial_aE^{bi} + \Gamma^{bi}_{ac}E^c_i - \Gamma^{c}_{ac}E^{bi} + \epsilon^{ijk}\Gamma_{aj}E^b_k = 0 \] (2.172)

Now, using the definition (2.171) to rewrite all the constraints in terms of \(K_{ai}\) and \(E^{ai}\), one gets

\[ \mathcal{G}^i = \mathcal{D}_aE^{ai} = D_aE^{ai} + i\epsilon^{ijk}K_{aj}E^a_k = i\epsilon^{ijk}K_{aj}E^a_k \] (2.173)

\[ \mathcal{H}_a = E^{bj}(-iR_{abi} + D_{[a}K_{bj]} + i\epsilon_{ijk}K_{aj}K_{bk}) \approx E^{bj}D_{[a}K_{bj]} \] (2.174)

\[ \mathcal{H} = \frac{1}{2}E^{ai}E^{bj}R_{ab}^k\epsilon_{ijk} - \lambda\text{det}(E^{ai}) + \frac{i}{2}E^{ai}E^{bj}D_{[a}K_{bj]}^k - \frac{1}{2}E^{ai}E^{bj}K_{ai}K_{bj} \approx \frac{1}{2}E^{ai}E^{bj}R^{k}_{ab}\epsilon_{ijk} - \lambda\text{det}(E^{ai}) - \frac{1}{2}E^{ai}E^{bj}K_{ai}K_{bj} \] (2.175)
where \( R^k_{ab} := \partial [a \Gamma^k_{b}^i] + \epsilon^{kln} \Gamma_{ai} \Gamma_{bn} \), and I have used the Bianchi-identity in (2.174), and neglected terms proportional to \( G^i \) in both (2.174) and (2.175). Comparing these constraints with the ADM-constraints in section 2.3, one notices that these constraints have exactly the same structure. To get exact agreement, one needs to change \( E^a_i \) into its non-densitized inverse \( e^a_i \), which is the coordinate used in section 2.3. I will not do this canonical transformation here, instead I will now show that the transformation (2.171) really is a canonical one. To show that, I first define the undensized \( E^a_i \) and its inverse

\[
e^a_i := \frac{1}{\sqrt{\text{det}(E^b_j)}} E^a_i, \quad e^a_i e^a_i = \delta^a_i, \quad (^3e) := \text{det} e^a_i = \sqrt{\text{det} E^a_i} \tag{2.176}
\]

Then, the following is true

\[
D[a e^a_i] = \partial[a e^a_i] + \epsilon^{ijk} \Gamma^j_{[a} \Gamma^k_{i]} = 0 \tag{2.177}
\]

and the time-derivative of (2.177) must also vanish.

\[
D[a \dot{e}^a_i] + \epsilon^{ijk} \dot{\Gamma}^j_{[a} \dot{e}^k_{i]} = 0 \tag{2.178}
\]

Solving (2.178) for \( \dot{\Gamma}^i_{a} \), and contracting its indices with an \( E^a_i \) then gives:

\[
\dot{\Gamma}^i_{a} E^a_i = -\frac{1}{2} (^3e) e^{aj} e^{bk} \epsilon_{ijk} D_a \dot{e}^c_i = -\frac{1}{2} \partial_a ( (^3e) e^{aj} e^{bk} \epsilon_{ijk} \dot{e}^c_i) \tag{2.179}
\]

This means that \( E^a_i \dot{A}^i_{ai} = E^a_i \dot{K}^i_{ai} \) up to the surface term (2.179), which shows that the fundamental Poisson bracket now is

\[
\{ K^i_{ai}(x), E^b_j(y) \} = \delta^b_j \delta^i_3 (x - y) \tag{2.180}
\]

and that the transformation really is a canonical one. Note that a comparison between (2.173) and the ADM-Hamiltonian in section 2.3 shows that \( K^i_{ai} E^a_i \) has the interpretation as the (densitized) extrinsic curvature. Further details about this canonical transformation can be found in e.g [2] and [23].

### 2.7 The CDJ-Lagrangian

From section 2.1 to 2.6 the emphasis has gradually shifted from the metric towards the connection as the fundamental field for gravity. Here in this section, the shift will reach almost completion, when the metric is eliminated from the Ashtekar Hamiltonian (except for the conformal factor), leaving the connection as the fundamental field.

The pure spin-connection formulation of gravity was first found [5] through a field-elimination from the Plebanski action in section (2.8). It was, however, also clear that the Hamiltonian formulation of the pure spin-connection Lagrangian equals the Ashtekar Hamiltonian. So, the Ashtekar Hamiltonian and the CDJ-Lagrangian are related via a Legendre transform.

In this section, I want to explicitly perform the Legendre transform, starting from the Ashtekar Hamiltonian, and in order not to complicate things more than necessary, I will
put $\lambda = 0$ here. The Legendre transform for $\lambda \neq 0$ is much more complicated to perform, and details can be found in [24].

In doing the Legendre transform, I will also assume that the vector constraint, in Ashtekar’s Hamiltonian, can be found as a primary constraint to the resulting Lagrangian. This means that the equations that should be solved are

$$\dot{A}_{ai} = \frac{i\tilde{N}}{2} \epsilon_{abc} \epsilon_{ijk} E^{bj} B^{ck} + \frac{1}{2} N^b \epsilon_{bac} B^c_i + D_a A_{0i} \quad (2.181)$$

$$\epsilon_{abc} E^{bi} B^c_i = 0 \quad (2.182)$$

where $B^{ai}$ is the magnetic field: $B^{ai} := \epsilon^{abc} F_{bc}^i$. Now, doing the Legendre transform means solving the equation of motion for the momentum, to get the momenta as a function of the coordinate and its velocity. Here, this corresponds to solving (2.181) for $E^{ai}$. Equation (2.182) should then also be solved for $E^{ai}$ in order to find $\mathcal{H}_a$ as a primary constraint.

To be able to solve these two equations, I must require the magnetic field to be non-degenerate, i.e

$$\text{det}(B^{ai}) := \frac{1}{6} \epsilon_{abc} \epsilon_{ijk} B^{ai} B^{bj} B^{ck} \neq 0 \quad (2.183)$$

This restriction of the magnetic field will exclude some field-configurations. In [5], these excluded field-configurations are given in terms of Petrov-classes. Note that flat Minkowski-space is excluded. Now, if $B^{ai}$ is non-degenerate, it can serve as a basis for all spatial vector-fields (or $SO(3)$ vector-fields). For instance

$$E^{ai} = \Psi^{ij} B^a_j \quad (2.184)$$

is then always true for some $SO(3)$ matrix $\Psi^{ij}$. Using this relation in (2.182), implies

$$\Psi^{[ij]} = 0 \quad (2.185)$$

Then, multiplying (2.181) by $B^{aj}$ and symmetrizing in the $SO(3)$ indices, yields

$$\Omega^{ij} = \frac{1}{\eta} (\Psi^{ij} - \delta^{ij} Tr \Psi) \quad (2.186)$$

where $\Omega^{ij} := \epsilon^{\alpha\beta\gamma\delta} F_{\alpha\beta}^i F_{\gamma\delta}^j = 2 B^{a(i} F_{j)a}$ and $\eta := -\frac{1}{2iN \text{det}(B^{ai})}$. This equation is easily solved

$$\Psi^{ij} = \eta (\Omega^{ij} - \frac{1}{2} \delta^{ij} Tr \Omega) \quad (2.187)$$

and the Legendre transform is done. Putting this solution into the Lagrangian $\mathcal{L} = E^{ai} \dot{A}_{ai} - \mathcal{H}^{Ash}$, one gets

$$\mathcal{L}_{CDJ} = \frac{\eta}{8} (Tr \Omega^2 - \frac{1}{2} (Tr \Omega)^2) \quad (2.188)$$

which is the sought for, pure spin-connection Lagrangian for pure gravity. Note that $\mathcal{L}_{CDJ}$ is not a totally metric-free Lagrangian since $\eta$ is related to the metric via the lapse.
function. With a cosmological constant and/or matter fields it is, however, in principle possible to eliminate $\eta$ also. See [24]. To get a formula for the metric here in the CDJ-formulation, one can return to the metric-formula in the Ashtekar formulation, and then follow the fields through the Legendre transform. The result is

$$\tilde{g}^{\alpha\beta} = \sqrt{-g}g^{\alpha\beta} = -\frac{2i}{3}\eta\epsilon_{ijk}\epsilon^{\gamma\delta\epsilon}F_{\gamma\delta}^iF_{\epsilon\kappa}^jF_{\mu\nu}^k\epsilon^\mu\nu\kappa$$  \hspace{1cm} (2.189)$$

which is called the Urbantke formula. Before the CDJ-formulation was known, Urbantke [25] gave the metric-formula (2.189) as the solution to the problem: with respect to what metric is a given $SO(3)$ field strength, self dual? Since this question is insensitive to the conformal factor of the metric, Urbantke could only give the metric up to an arbitrary conformal factor, and the result was (2.189). Further details about the CDJ-formulation can be found in: the original discovery [5] and [26], relation to the Ashtekar Hamiltonian [27], [28], [24] and [8], with new cosmological constants [6], [7], [29] and [8], generalization to other gauge groups, [9].

### 2.8 The Plebanski Lagrangian

In section 2.5, we learned that it is enough to have only the self dual part of the H-P Lagrangian in order to get full general relativity. This means that the only combination of the tetrad fields that is needed is the self dual part of the anti-symmetric product of two tetrads. Plebanski made use of that fact, and promoted this combination of tetrads into an independent field. Then, to still get the same physical content in the action, he had to add a Lagrange multiplier term, imposing the original relation.

Here, I want to briefly go through the steps that take us from the self dual H-P Lagrangian to the Plebanski Lagrangian. Then, I will also derive the CDJ-Lagrangian by eliminating fields from the Plebanski Lagrangian. Details about the Plebanski action can be found in [30], [8].

The self dual H-P Lagrangian:

$$\mathcal{L}_{HP}^{(+)} := e\left(\Sigma_{IJ}^{(+)}\epsilon_{IJ}[e^\gamma]R_{\alpha\beta}(e_{K})\omega_{\alpha\beta}(\omega_{KL}) + \lambda\right)$$  \hspace{1cm} (2.190)$$

where $\Sigma_{IJ}(e_{K}) := \frac{1}{2}\epsilon_{[I}^\alpha\epsilon_{J]}^\beta$, and $\Sigma_{IJ}^{(+)}(e_{K})$ is its self dual part. Now, using the identity

$$\frac{1}{2}\epsilon_{[I}^\alpha\epsilon_{J]}^\beta = \frac{1}{4}\epsilon^{\alpha\beta\gamma\delta}\epsilon_{\gamma}^K\epsilon_{\delta}^L\epsilon_{IJKL}$$  \hspace{1cm} (2.191)$$

and defining

$$\Sigma_{IJ} := \frac{1}{2}\epsilon_{[I}^\alpha\epsilon_{J]}^\beta$$  \hspace{1cm} (2.192)$$

it is straightforward to show that

$$\epsilon\Sigma_{IJ}^{(+)} = \frac{i}{2}\epsilon_{[I}^\alpha\epsilon_{J]}^\beta\Sigma_{\alpha\beta}^{(+)}$$  \hspace{1cm} (2.193)$$

The determinant of $e_{\alpha}^I$ can also be written solely in terms of $\Sigma$'s:
Chapter 2  Actions in (3+1)-dimensions

\[ e := \text{det} e^i_a = \frac{i}{6} \epsilon_{\alpha\beta\gamma\delta} \sum_{\alpha\beta}^{(+)I} \sum_{\gamma\delta}^{(+)} \]  \hspace{1cm} (2.194)

With all this, the Lagrangian now becomes

\[ L^{(+)}_{HP} = \frac{i}{2} \epsilon_{\alpha\beta\gamma\delta} \sum_{\alpha\beta}^{(+)I} \left( R^{(+)\gamma\delta} + \frac{\lambda}{3} \sum_{\gamma\delta}^{(+)} \right) (2.195) \]

And, since both \( \Sigma \) and \( R \) are self dual, they have only 18 algebraically independent components. To make this more manifest, I do as in section 2.5 and split \( \Sigma \) and \( R \) into theirs boost and rotation parts:

\[ A^{IJ} = A^{I'}_{J'} + A^{I\tilde{J}} \]  \hspace{1cm} (2.196)

where \( A^{IJ} \) is a general \( so(1, 3) \) valued field, and the primed index stands for the projection along a time-like unit vector-field \( N^I \), while the tilded index stands for the projection into the orthogonal space-like surface orthogonal to \( N^I \). See section 2.5 for details. For a self dual field \( A^{IJ} \) it is then true that

\[ A^{IJ} B_{IJ} = 2 A^{I'}_{J'} B_{I'J'} + A^{I\tilde{J}} B_{I\tilde{J}} = 4 A^{I'}_{J'} B_{I'J'} \]  \hspace{1cm} (2.197)

Then, I introduce the \( SO(3) \) notation:

\[ \Sigma_{\alpha\beta} := \Sigma^{(+)I'}_\alpha N^I \]
\[ F_{\alpha\beta i} := -2 R^{(+)I'}_{\alpha\beta} N^I \]
\[ A_{ai} := -2 \omega^{(+)I'}_{\alpha} N^I \]  \hspace{1cm} (2.198)

where the factor of two is introduced just to get agreement with the Ashtekar curvature in section 2.3. Now, the Lagrangian is

\[ L^{(+)}_{HP} = i \epsilon_{\alpha\beta\gamma\delta} \sum_{\alpha\beta}^{i} \left( F_{\gamma\delta i} - \frac{2 \lambda}{3} \sum_{\gamma\delta}^{i} \right) \]  \hspace{1cm} (2.199)

Remember that \( \Sigma_{\alpha\beta} \) is still just a given function of the tetrad-field. The idea is now to let \( \Sigma_{\alpha\beta} \) become an independent field, and at the same time add a Lagrange multiplier term, imposing the original definition of it. A constraint that does the job, is

\[ \tilde{M}^{i\bar{j}} := M^{i\bar{j}} - \frac{1}{3} \delta^{i\bar{j}} M^k_k = 0 \]
\[ M^{i\bar{j}} := \epsilon^{\alpha\beta\gamma\delta} \sum_{\alpha\beta}^{i} \sum_{\gamma\delta}^{i} \]  \hspace{1cm} (2.200)

To show that this is a necessary condition on \( \Sigma \), one can just put in the original definition of \( \Sigma_{\alpha\beta} \) in \( M^{i\bar{j}} \):

\[ M^{IJKL} := \epsilon^{\alpha\beta\gamma\delta} \sum_{\alpha\beta}^{(+)IJ} \sum_{\gamma\delta}^{(+)KL} = \frac{1}{2} \epsilon^{IJKL} + \frac{e}{2} \eta^{I}[K\eta^L]^J \]  \hspace{1cm} (2.201)

which means that
2.8 The Plebanski Lagrangian

\[ M^{ij} := N_N N_K M^{NJK} = -i \epsilon^{\gamma \delta \epsilon} \widehat{\eta}^{ij} = \frac{1}{3} \epsilon^{\gamma \delta \epsilon} N_N N_K \tilde{\eta}_{LM} M^{NLKM} = \frac{1}{3} \delta^{ij} M_k^k \quad (2.202) \]

To get an indication that \( \tilde{M}^{ij} = 0 \) also is a sufficient condition, one may count the number of degrees of freedom in \( \Sigma \) constrained by \( \tilde{M}^{ij} \), and compare with the degrees of freedom in \( \epsilon^I_\alpha \): \( \Sigma \) has 18 independent components, and is constrained by the traceless symmetric three-by-three matrix \( \tilde{M}^{ij} \), leaving 13 degrees of freedom. \( \epsilon^I_\alpha \) has 16 free components but the self dual anti-symmetric product is invariant under anti-self dual Lorentz transformations, leaving again 13 degrees of freedom. An explicit proof of the adequacy of \( \tilde{M}^{ij} = 0 \) can be found in [5]. Now, when adding the constraint \( \tilde{M}^{ij} = 0 \) to the Lagrangian with a Lagrange multiplier, the tracelessness of \( \tilde{M}^{ij} \) can be shifted over to the Lagrange multiplier, and then I let that condition also be implied by a Lagrange multiplier term,

\[ \mathcal{L} = i \epsilon^{\alpha \beta \gamma \delta} \Sigma^{ij} \left( F_{\gamma \delta i} - \frac{2\lambda}{3} \Sigma_{\gamma \delta i} + \Psi^{ij} \Sigma_{\gamma \delta} \right) + \mu \Psi^k \quad (2.203) \]

where the independent fields now are \( \Sigma^{ij}, A^i_\alpha, \Psi^{ij} \) and \( \mu \). Note that \( \Psi^{ij} \) is an arbitrary symmetric \( SO(3) \) matrix-field, its tracelessness is imposed by the variation of \( \mu \). This is the Plebanski Lagrangian, and in order to extract real general relativity from it, one must impose reality conditions on \( \Sigma \) that turns out to be rather awkward for Lorentzian spacetime. I will not go into more details concerning this action, instead I want to show how to reach the CDJ-Lagrangian from here.

The idea is the following; since the Lagrangian now has an inhomogeneous dependence on \( \Sigma \), it is possible to eliminate it from the action, through its equation of motion. And once this is done, the Lagrangian instead gets an inhomogeneous dependence on \( \Psi \) making it possible to eliminate also this field, leaving only the curvature \( F^i_{\alpha \beta} \) and the Lagrange multiplier \( \mu \). For simplicity, I will put \( \lambda = 0 \). For a treatment of the \( \lambda \neq 0 \) case, see [26]. See also [31].

\[ \frac{\delta S}{\delta \Sigma^{ij}_{\alpha \beta}} = i \epsilon^{\alpha \beta \gamma \delta} \left( F_{\gamma \delta i} + 2 \Psi^{ij} \Sigma^{\gamma \delta} \right) = 0 \quad (2.204) \]

Assuming \( \Psi^{ij} \) to be invertible, the solution is

\[ \Sigma^{ij}_{\alpha \beta} = -\frac{1}{2} \Psi^{-1ij} F_{\alpha \beta j} \quad (2.205) \]

With this solution, the Lagrangian becomes

\[ \mathcal{L} = -\frac{i}{4} Tr(\Psi^{-1} \Omega) + \mu Tr \Psi \quad (2.206) \]

where I have introduced a convenient matrix notation, and defined: \( \Omega^{ij} := \epsilon^{\alpha \beta \gamma \delta} F^i_{\alpha \beta} F^j_{\gamma \delta} \).

Now, using the characteristic equation for three-by-three matrices:

\[ A^3 - A^2 Tr A + \frac{1}{2} A \left( (Tr A)^2 - Tr A^2 \right) - 1 det A = 0 \quad (2.207) \]
it follows that
\[ Tr \Psi = \frac{1}{2} det \Psi \left( (Tr \Psi^{-1})^2 - Tr \Psi^{-2} \right) \] (2.208)
Putting this into the Lagrangian, and varying the action with respect to \( \Psi^{-1} \) one gets:
\[ \frac{\delta S}{\delta \Psi^{-1}_{ij}} = i \left( -\frac{1}{4} \Omega_{ij} + \rho (Tr \Psi^{-1} \delta_{ij} - \Psi^{-1}_{ij}) \right) = 0 \] (2.209)
where I have redefined the Lagrange multiplier: \( \rho := -i \ \mu \det \Psi \). This equation is now easily solved for \( \Psi^{-1} \):
\[ \Psi^{-1}_{ij} = -\frac{1}{4\rho} \left( \Omega_{ij} - \frac{1}{2} \delta_{ij} Tr \Omega \right) \] (2.210)
And finally, putting this solution into the Lagrangian, and redefining the Lagrange multiplier again: \( \eta := \frac{1}{4\rho} \), the result is the CDJ-Lagrangian
\[ \mathcal{L} = \frac{\eta}{8} \left( Tr \Omega^2 - \frac{1}{2} (Tr \Omega)^2 \right) \] (2.211)
Chapter 3

Actions in (2+1)-dimensions

In this section, I intend to briefly go through the (2+1)-dimensional counterparts of the various actions dealt with in section 2. Here, I will not do all calculations in the same detail as for (3+1)-dimensions.

The reason why (2+1)-dimensions here is treated specially, is that both the Ashtekar Hamiltonian as well as the CDJ-Lagrangian are known to exist only in (2+1)- and (3+1)-dimensions. See, however, section 4.3 regarding speculations about higher dimensional formulations. The (2+1)-dimensional version of fig.1 is given in fig.2, where we see that the number of different actions now has decreased significantly. Two actions are absent, due to the non-existence of self dual two-forms in (2+1)-dimensions, and four of the actions from fig.1 have merged into two; the pure $SO(1,2)$ spin-connection Lagrangian

---

Fig.2 Actions for gravity in (2+1)-dimensions.
equals here the CDJ-Lagrangian, and the Hamiltonian formulation of the H-P Lagrangian equals the Ashtekar Hamiltonian.

There are at least one action missing in fig.2; the Chern-Simons Lagrangian \[32\], \[33\]. The reason why I do not treat the Chern-Simons Lagrangian here is that it is a purely \((2+1)\)-dimensional formulation, which does not exist in other dimensions, and I am mainly interested in formulations that exist in \((3+1)\)-dimensions, as well.

### 3.1 The E-H and the H-P Lagrangian, and the ADM-Hamiltonian

All these three action formulations of gravity work perfectly all right in arbitrary spacetime dimensions (dimensions greater than \((1+1)\), in \((1+1)\)-dimensions the E-H and the H-P Lagrangian are just total divergencies). All calculations that were made in sections \((2.1)-(2.3)\), except one, are trivially generalized to other dimensions. The calculation that needs a slight modification is the variation of the H-P Lagrangian with respect to the spin-connection. In arbitrary spacetime dimensions, the variation yields

\[
\frac{\delta S_{HP}}{\delta \omega^I_{\alpha \beta}} = \mathcal{D}_\beta \left( e e^\alpha_{[I} e^\beta_{J]} \right) = 0 \tag{3.1}
\]

And in dimensions higher than \((3+1)\) it is a bit tricky to show that \((3.1)\) implies the zero-torsion condition (It is, however, true that it does):

\[
\mathcal{D}_{[\alpha} e^I_{\beta]} = 0 \tag{3.2}
\]

In \((2+1)\)-dimensions, however, \((3.2)\) follows directly from \((3.1)\), if one knows the "inverse-formula":

\[
ee^{\alpha}_{[I} e^\beta_{J]} = \epsilon^{\alpha \beta \gamma} \epsilon_{IJK} e^K_\gamma \tag{3.3}
\]

### 3.2 The self dual H-P and the Plebanski Lagrangian

These two Lagrangians are unique for \((3+1)\)-dimensions, since the construction of them relies heavily on the use of self dual two-forms, which only exist in four dimensional spacetimes. One could of course construct a Plebanski-like Lagrangian without relation to self duality; let the combination \(e^I_{[\alpha} e^J_{\beta]}\) become an independent field, and add a Lagrange multiplier term imposing the original definition.

### 3.3 Hamiltonian formulation of the H-P Lagrangian, and the Ashtekar Hamiltonian

Here is one of the formulations where \((2+1)\)-dimensions is special. It will be shown here that it is exactly in \((2+1)\)-dimensions that the Hamiltonian formulation of the H-P Lagrangian does not give rise to second class constraints. In fact, here in \((2+1)\)-dimensions, the Hamiltonian formulation equals the Ashtekar formulation. The reason
why no second-class constraints appear here, as they did in \((3+1)\)-dimensions, can be found from a counting of degrees of freedom: as we saw in section (2.4), the second class constraints were needed since the tetrad did not have enough independent components to function as an unrestricted momenta for the spin-connection. In an arbitrary spacetime with dimension \((D+1)\), the spatial restriction of the spin-connection has \(\frac{D(D+1)D}{2}\) number of algebraically independent components, while the “tetrad” has \((D+1)^2\) components. Then, \((D+1)\) of the “tetrad”’s components are needed as Lagrange multipliers, imposing the diffeomorphism constraints. This means that, in order not to get restrictions on the momenta, we must have:

\[
\frac{D(D+1)D}{2} \leq (D+1)^2 - (D+1) \tag{3.4}
\]

and the solution is \(D \leq 2\), singling out \((2+1)\)-dimensions as the unique spacetime where the H-P Lagrangian will not produce second class constraints. (The reason why I say that the additional constraints are second-class, is that a first-class constraint always corresponds to a local symmetry. And the H-P Lagrangian has only gauge and diffeomorphism symmetries.)

The standard H-P Lagrangian is

\[
\mathcal{L}_{HP} = \left( ee_I^\alpha e_J^\beta R_{\alpha\beta}^{IJ}(\omega) + \lambda e \right) \tag{3.5}
\]

where \(e_I^\alpha\) is the triad field, \(e\) the determinant of its inverse, and \(\omega^{IJ}\) an independent \(SO(1,2)\) connection. The Hamiltonian formulation of (3.5) has previously been studied in [32], [21], and in [34]. See also [35]. Now, I split the triad

\[
e^0_I = -\frac{N_I}{N}, \quad e^I = V_I^a + \frac{N^a N_I}{N}; \quad N > 0, \quad N^I N_I = -1, \quad V_I^a N^I = 0 \tag{3.6}
\]

Putting this in (3.5) and defining: \(A^I_\alpha := e^{JK} \omega_{\alpha J K}, F_{\alpha\beta}^I := e^{JK} R_{\alpha\beta J K}\), gives

\[
\mathcal{L}_{HP} = -\frac{e}{N} N_I V_I^a e^{JK} F_{0aK} + \frac{1}{2} e V_I^a V_J^b e^{JK} F_{abK} + \frac{e}{N} N^a N_I V_I^b e^{JK} F_{abK} + \lambda e \tag{3.7}
\]

The momenta conjugate to \(A^I_\alpha\) is

\[
\pi^{aI} := \frac{\delta \mathcal{L}}{\delta \dot{A}^I_a} = -\frac{e}{N} N_K V_K^J e^{JKI} \tag{3.8}
\]

Now, using the epsilon-delta identity \(\epsilon^{ab} \epsilon_{cd} = \delta_c^a \delta^d_b\), and the orthogonality between \(N_I\) and \(V^aI\) it follows that

\[
\pi^{aI} = -\frac{e}{N} \epsilon^{ab} V_b^I \left( \frac{1}{2} N_J V_K^c V_M^d \epsilon^{JKM} \epsilon_{cd} \right) \tag{3.9}
\]

where \(V^a_I\) is the inverse to \(V^aI\): \(V_{al} V^{bl} = \delta^b_a\), and \(V_{al} N^l = 0\). Eq. (3.6) also implies:

\[
e := \left( \frac{1}{6} \epsilon_{\alpha\beta\delta} e^{JKI} e^{aI} e^{bJ} e^{cK} \right)^{-1} = \left( -\frac{1}{2N} N^a_N V_K^c V_M^d \epsilon^{JKM} \epsilon_{cd} \right)^{-1} \tag{3.10}
\]
which makes it easy to invert (3.9):

$$V_a^I = \epsilon_{ba} \pi_{bI}$$  

(3.11)

Putting this into the Lagrangian, and rescaling the lapse function: \( \tilde{N} := N^2 / \epsilon \), gives

$$\mathcal{L} = \pi^{ai} \dot{A}_a^i - \tilde{N} \mathcal{H} - N^o \mathcal{H}_a + A_0 \mathcal{G}^I$$  

(3.12)

$$\mathcal{H} := \frac{1}{2} \pi^{ai} \pi^{bj} F_{ab}^K \epsilon_{IJK} - \lambda \det(\pi^{ai} \pi^b_j) \approx 0$$

$$\mathcal{H}_a := \pi^b_a F_{ab}^I \approx 0$$

$$\mathcal{G}^I := D_a \pi^a I - \partial_a \pi^a I + \epsilon^{IJK} A_{aj} \pi^a_K \approx 0$$

which is the (2+1)-dimensional Ashtekar Hamiltonian. Compare with (2.152). Note that, for non-degenerate metrics, there is a unique solution to the constraints \( \mathcal{H} \) and \( \mathcal{H}_a \):

$$e^{ab} F_{ab}^I = - \lambda \epsilon^{IJK} \pi^a_j \pi^b_K \epsilon_{ab}. \text{ So, instead of the constraints } \mathcal{H} \text{ and } \mathcal{H}_a \text{ given in (3.12), one can use the constraint: } \Psi^I := \epsilon^{ab} F_{ab}^I + \lambda \epsilon^{IJK} \pi^a_j \pi^b_K \epsilon_{ab} \approx 0.$$  

The constraint algebra for the Hamiltonian in (3.12) is exactly the same as for the (3+1)-dimensional case, and there are no reality conditions here in (2+1)-dimensions. Following the fields through the Legendre transform above, it is easy to write down the metric formula for the Ashtekar variables:

$$\tilde{g}^{\alpha \beta} = \sqrt{-g} g^{\alpha \beta} = \left( \begin{array}{cc} -\frac{1}{N} & \frac{N^a}{N} \\ \frac{N^a}{N} & \frac{N^a N^b}{N} \end{array} \right)$$  

(3.13)

Then, it is only the relation between the Ashtekar Hamiltonian and the ADM-Hamiltonian left here to show. These two Hamiltonians are related by a canonical transformation of similar type as in the (3+1)-dimensional case. First I define the field \( K_{ai} \):

$$K_{ai} := A_{ai} - \Gamma_{ai}$$  

(3.14)

where \( \Gamma_{ai} \) is the unique torsion-free spin-connection compatible with \( \pi^{ai} \):

$$D_a \pi^{bl} = \partial_a \pi^{bl} + \Gamma^{b}_{ac} \pi^{al} - \Gamma^{c}_{ac} \pi^{bl} + \epsilon^{IJK} \Gamma_{aj} \pi^{a} _K = 0$$  

(3.15)

Despite the fact that \( \pi^{ai} \) does not have an inverse “in the SO(1, 2) indices”, (3.13) can be uniquely solved for \( \Gamma_{ai} \) as a function of \( \pi^{ai} \). This can be understood from a counting of degrees of freedom in (3.15): equation (3.15) represents 12 equations, and \( \Gamma_{ab} \) and \( \Gamma_{ai} \) are 6 + 6 unknown, meaning that there is enough information in (3.15) to solve for all components. See Appendix B for details. (Note, however, that the alternative requirement: \( D_{[a} \pi^{i]} = 0 \), is not enough to uniquely specify \( \Gamma_{ai} \), since it gives only 3 equations for 6 unknown.)

Now, using (3.14) and (3.13), the Ashtekar constraints in (3.12) becomes:

$$\mathcal{G}^I = D_a \pi^{ai} + \epsilon^{IJK} K_{aj} \pi^a_K = \epsilon^{IJK} K_{aj} \pi^a_K$$  

(3.16)

$$\mathcal{H}_a := \pi^{bl} R_{ab} \Gamma + \pi^{bl} D_{[a} K_{bj]} + \pi^{bl} \epsilon_{IJK} K_{bj} \epsilon_{ab} \approx \pi^{bl} D_{[a} K_{bj]}$$  

(3.17)

$$\mathcal{H} = \frac{1}{2} \pi^{ai} \pi^{bj} R_{ab} \epsilon_{IJK} - \lambda \det(\pi^{ai} \pi^b_j) + \pi^{ai} \pi^{bj} (D_a K^b_j) \epsilon_{IJK} + \frac{1}{2} \pi^{ai} \pi^{bj} K_{[a} K_{b]}$$
3.4 The CDJ-Lagrangian

\[ \frac{1}{2} R(\Gamma) - \lambda \det (\pi^a I \pi^b I) + \frac{1}{2} \pi^a I \pi^b J K_{[a I} K_{b J]} \] (3.18)

where \( R_{ab}^I := \partial_{[a I} \Gamma_{b]}^I + \epsilon^{IJK} \Gamma_{aJ} \Gamma_{bK} \) and \( R := \epsilon_{IJK} \pi^a I \pi^b J R_{ab}^K \), and I have neglected terms proportional to \( G^I \) in both (3.17) and (3.18). In (3.17) the Bianchi identity: \( \pi^{bl} R_{ab}^I = 0 \) was also used. Thus, the transform (3.14) really gives the wanted ADM-Hamiltonian, and what is left to prove is that the transformation really is a canonical one. To do that, I first define the undensitized \( \pi^a I \) and its inverse:

\[ e^a I := \frac{1}{\sqrt{\det (\pi^a I \pi^b I)}} \pi^a I \]

\[ e_a I e_b I = \delta_a^b, \quad \epsilon^{IJK} e_a I e_J e_K = 0 \] (3.19)

Then (3.13) gives

\[ D_{[a I} e_{b I]} = \partial_{[a I} e_{b I]} + \epsilon^{IJK} \Gamma_{aJ} e_{b K} = 0 \] (3.20)

or

\[ \epsilon^{ab} \left( D_a \dot{e}_b^I + \epsilon^{IJK} \dot{\Gamma}_{a J} e_{b K} \right) = 0 \] (3.21)

Then, projecting (3.21) along the vector field \( \epsilon^{IJK} e^a J e^K e^b \), gives

\[ e^a J \dot{\Gamma}_{a J} = 2 \epsilon^{IJK} e^a J e^K D_a \dot{e}_b l \] (3.22)

which means that

\[ \pi^a I \dot{A}_{a I} = \pi^a I \dot{K}_{a I} + \partial_a \left( \frac{2}{\sqrt{\det (e^a I e_b I)}} \epsilon^{IJK} e^a K e^K e_b l \right) \] (3.23)

showing that, up to the surface term, \( \pi^a I \) and \( K_{a I} \) are conjugate variables. So, assuming compact spacetime or fast enough fall-off behavior at infinity, we have a canonical transformation.

3.4 The CDJ-Lagrangian

The CDJ-Lagrangian in (2+1)-dimensions can be found either by a Legendre transform from the Ashtekar Hamiltonian, or from an elimination of the triad field from the H-P Lagrangian. The latter method is the simpler one.

\[ \mathcal{L}_{HP} = \epsilon^{a \beta \gamma} e_{\gamma I} F_{a \beta}^I + \lambda e \] (3.24)

where \( F_{a \beta}^I := \partial_{[a} A_{\beta]}^I + \epsilon^{IJK} A_{a \alpha} A_{\beta K} \), and \( A_{a \alpha}^I := \epsilon^{IJK} \omega_{a \alpha} e {K} \) and \( \omega_{a \alpha} e {K} \) is an \( SO(1,2) \) connection. The equation of motion following from variation of \( e_{a I} \) is

\[ \frac{\delta S}{\delta e_{a I}} = F^{* a I} + \lambda e e_{a I} = 0 \] (3.25)
Chapter 3  Actions in (2+1)-dimensions

where $F^{*\alpha I} := e^{\alpha \beta \gamma} F_{\beta \gamma}^I$ is the dual of $F_{\alpha \beta I}$. The idea is now to solve (3.25) for $e^{\alpha I}$, and then put the solution back into the Lagrangian, yielding a totally metric-free formulation.

Taking the determinant of (3.25), gives

$$\chi^3 e^2 = -\det(F^{*\alpha I})$$

which is solved by

$$e = \pm \sqrt{-\frac{1}{\chi^3}} \det(F^{*\alpha I})$$

Then, (3.27) and (3.25) give the complete solution to $\frac{\delta S}{\delta e^{\alpha I}} = 0$:

$$e^{\alpha I} = \mp \frac{1}{\chi \sqrt{-\frac{1}{\chi^3} \det(F^{*\alpha I})}} F^{*\alpha I}$$

and putting this solution back into the Lagrangian, one gets

$$\mathcal{L} = \mp 2 \, \text{sign}(\chi) \sqrt{-\frac{1}{\chi} \det(F^{*\alpha I})}$$

which is the wanted pure spin-connection Lagrangian for (2+1)-dimensional gravity with a cosmological constant. Note that this Lagrangian is totally metric-free, the only independent field is the spin-connection. For a treatment of the CDJ-Lagrangian without a cosmological constant, and with a coupling to a scalar field, see [34]. See also [38].

The metric formula in this formulation, follows directly from (3.28).

$$g^{\alpha \beta} = e^{\alpha I} e^{\beta I} = \frac{-\chi}{\det(F^{*\alpha I})} F^{*\alpha I} F^{*\beta I}$$

Note that the Lorentz-signature condition $\det g_{\alpha \beta} = -e^2 < 0$ corresponds to $\frac{1}{\chi} \det(F^{*\alpha I}) < 0$. See for instance (3.26).
Chapter 4

Generalizations

In this section, I want to present two recent generalizations of the Ashtekar Hamiltonian and the CDJ Lagrangian. The first kind of generalization was first discovered in [1] and in [8]. See also [4] and [29]. It is a cosmological constant type of generalization in the sense that it does not increase the number of degrees of freedom. At the canonical level (the Ashtekar formulation), the generalization is achieved by adding terms to the Hamiltonian constraint, and at the Lagrangian level (the CDJ Lagrangian) one has to add terms to the Lagrangian. This generalized theory still has an interpretation in terms of Riemannian geometry, and it is possible to extract the spacetime metric out of the constraint algebra. Explicit spherically symmetric solutions to the generalized theory have been studied in [8] and [36], in which it is shown that the generalized theory really is physically different from the conventional Einstein theory. Two remaining problems with this generalization are the reality condition and the metric-signature requirement. No one has yet given a reality condition that works for any of the new cosmological constants. (The problem arises for Lorentzian spacetimes. For Euclidean signature, where all fields are real, it is possible to show that there exist ranges for some of the new cosmological constants that will ensure positive definiteness of the metric [37].) In [34], it was also shown that this generalization of Ashtekar variables does not have any direct counterpart in (2+1)-dimensions. In (2+1)-dimensions the ordinary cosmological constant seems to be the unique generalization of the pure gravity theory, that does not increase the number of degrees of freedom.

The other type of generalization that I will present here is a generalization of the theory to other gauge groups. That is, in (3+1)-dimensions, I consider a canonical formulation of a gauge and diffeomorphism invariant theory that reduces to the Ashtekar formulation if the gauge group is chosen to be $SO(3)$. See [9]. In (2+1)-dimensions, this type of generalization is also possible, but there the pure gravity gauge group is $SO(1,2)$. The difference between (3+1)- and (2+1)-dimensions is that in (3+1)-dimensions there exist an infinite number of different theories that has the above mentioned behavior, while in (2+1)-dimensions there is only one theory of this type. This is closely related to the existence of the infinite number of new cosmological constants in (3+1)-dimensions. This gauge group generalization does increase the number of degrees of freedom, and in fact has the correct number of degrees of freedom to be a candidate theory for Einstein gravity coupled to Yang-Mills theory. In (2+1)-dimensions, it was been shown in [38] that this generalization gives the conventional Einstein-Yang-Mills theory to lowest order in
Yang-Mills fields.

In (2+1)-dimensions, there exists an extension of the pure gravity formulation, which has one degree of freedom per point in spacetime; topologically massive gravity \[39\]. See also \[34\] for the relation to the Ashtekar formulation. I do not treat this extension here, mainly due to the fact that this is a purely (2+1)-dimensional formulation which has no counterpart in other spacetime dimensions.

4.1 The cosmological constants

In a canonical formulation of a diffeomorphism invariant theory, there exists a set of first class constraints generating the diffeomorphism symmetry. If these generators are split into parallel and orthogonal parts, \( \tilde{H}_a \) and \( H \), where \( \tilde{H}_a \) generates spatial diffeomorphisms on the spatial hypersurface and \( H \) generates diffeomorphisms off the hypersurface, these first class constraints always obey the following constraint algebra:

\[
\{ \tilde{H}_a[N^a], \tilde{H}_b[M^b] \} = \tilde{H}_a[\mathcal{L}_{N^a} M^a] 
\]

(4.1)

\[
\{ \tilde{H}_a[N^a], H[N] \} = H[\mathcal{L}_{N^a} N] 
\]

(4.2)

\[
\{ H[N], H[M] \} = \tilde{H}_a[q^{ab}(N \partial_b M - M \partial_b N)] 
\]

(4.3)

where \( \mathcal{L}_{N^a} \) denotes the Lie-derivative along the vector field \( N^a \), and \( q^{ab} \) is the spatial metric on the hypersurface. Just by requiring path-independence of deformations of the hypersurface, it was shown in \[10\] that any any canonical formulation of a diffeomorphism invariant theory, with a metric, must give a representation of this algebra. Thus, this algebra can be used in two different ways; both as a test if a given theory is diffeomorphism invariant, and also as a definition of what the spatial metric is in terms of the phase space variables. (in fact, it is possible to extract the entire spacetime metric from the constraint algebra and the expression for the time-evolution of the spatial metric. See \[10\] for details.)

Now, given a particular set of phase space variables, and the fact that \( \tilde{H}_a \) should generate spatial diffeomorphisms, it is in general easy to find the unique realization of \( \tilde{H}_a \) in terms of the phase space variables. More specifically, if \( \tilde{H}_a \) is the generator of spatial diffeomorphisms, it has the action of the Lie-derivative on the fundamental fields:

\[
\delta \tilde{H}_a q := \{ q, \tilde{H}_a[N^a] \} = \mathcal{L}_{N^a} q 
\]

(4.4)

\[
\delta \tilde{H}_a p := \{ p, \tilde{H}_a[N^a] \} = \mathcal{L}_{N^a} p 
\]

(4.5)

where \((q,p)\) denotes the phase space variables. This normally gives a uniquely solvable system of equations, from which it is easy to solve for \( \tilde{H}_a(q,p) \). As an example, study the \( SO(3) \) Yang-Mills phase space, where the fundamental fields are \( A_{ai} \) and \( E_{ai} \), an \( SO(3) \) connection and its conjugate momenta. Writing out (4.4) and (4.5) in details for these phase space variables gives an easily solvable system of equations, with the unique solution

\[
\tilde{H}_a = H_a - A_{ai} G^i = E_{bi} F_{abi} - A_{ai} D_b E_{bi} 
\]

(4.6)
4.1 The cosmological constants

All this means that in an attempt to generalize a diffeomorphism invariant theory, the only freedom one has lies in the Hamiltonian constraint, $\mathcal{H}$. That is, since the theory should be diffeomorphism invariant, it must obey the algebra \((4.1)-(4.3)\), meaning that $\mathcal{H}_a$ is the generator of spatial diffeomorphisms, which is unique. And furthermore, since $\mathcal{H}_a$ is the generator of spatial diffeomorphisms, \((4.1)\) and \((4.2)\) is automatically satisfied if $\mathcal{H}_a$ and $\mathcal{H}$ are diffeomorphism covariant objects. This means that in order to generalize a diffeomorphism invariant theory, without breaking this invariance, the generalization must reside in $\mathcal{H}$, and the requirement it has to fulfill is that it should be a diffeomorphism covariant object, satisfying \((4.3)\).

In Ashtekar variables, there is also the $SO(3)$ symmetry, which is generated by $\mathcal{G}$. In the following, I will neglect this part of the algebra and just make sure that the generalized $\mathcal{H}$ is an $SO(3)$ scalar, in order not to break this part of the constraint algebra. Thus, the first basic requirements a candidate $\mathcal{H}$ has to satisfy is that it should be invariant under $SO(3)$ rotations and covariant under spatial diffeomorphisms. The fact that it should be invariant under $SO(3)$ transformations means that it should be constructed out of gauge covariant objects, like $E^{ai}$ and $B^{ai} := \epsilon^{abc} F^{i}_{bc}$, with all indices properly contracted. And covariance under spatial diffeomorphisms just means that all spatial indices should be properly contracted as well. This immediately singles out four good candidates as basic building blocks: $A := \epsilon_{abc} \epsilon_{ijk} E^{ai} E^{bj} E^{ck}$, $B := \epsilon_{abc} \epsilon_{ijk} E^{ai} E^{bj} B^{ck}$, $C := \epsilon_{abc} \epsilon_{ijk} B^{ai} B^{bj} B^{ck}$, $D := \epsilon_{abc} \epsilon_{ijk} B^{ai} B^{bj} B^{ck}$. These four scalar densities all satisfy the above mentioned requirements, and therefore any function of them will also do so; an expression such as

$$\mathcal{H} = f(A, B, C, D)$$

should therefore be a rather general Ansatz for the Hamiltonian constraint. Or, introducing parameters that serve as new cosmological constants, one could consider functions like:

$$f(A, B, C, D) = \alpha_1 A + \alpha_2 B + \alpha_3 C + \alpha_4 D + \alpha_5 \frac{AB}{D} + \cdots$$

What is left to check, to ensure that the theory is diffeomorphism invariant, is \((4.3)\). I will not give the detailed calculation here, but it is shown in \(7\) that it is in general true that \((4.3)\) is satisfied with this type of Ansatz.

( In (2+1)-dimensions there exist only two different basic $SO(1, 2)$ vector fields: $\Psi^I := \epsilon^{ab} F^I_{ab}$ and $B_I := \epsilon_{IJK} \pi^{aI} \pi^{bK} \epsilon_{ab}$, meaning that the basic building blocks for $\mathcal{H}$ should be: $A := \Psi^I \Psi_I$, $B := \Psi^I B_I$ and $C := B^I B_I$. But $B + \lambda C$ is just the conventional Ashtekar Hamiltonian for pure gravity with a cosmological constant, and $A$ is proportional to $B$, when $\mathcal{H}_a = 0$ is satisfied. Therefore, there exists no generalization of this type of the pure gravity theory in (2+1)-dimensions)

At the Lagrangian level (the CDJ Lagrangian), it is even easier to discover the generalization. The conventional pure gravity Lagrangian is

$$\mathcal{L}_{CDJ} = \frac{\eta}{8} \left( Tr \Omega^2 - \frac{1}{2} (Tr \Omega)^2 \right)$$

where $\Omega^{ij} := \epsilon^{\alpha \beta \gamma \delta} F^i_{\alpha \beta} F^j_{\gamma \delta}$ and $\eta$ is a scalar density of weight minus one. Now, since $\Omega^{ij}$ is a three by three matrix, there exist three independent traces: $Tr \Omega$, $Tr \Omega^2$ and $Tr \Omega^3$. 
All other scalars constructed from $\Omega^{ij}$ can be written as functions of these three traces. This follows from the characteristic equation for three by three matrices. Thus, the most general $SO(3)$ invariant and diffeomorphism covariant Lagrangian density constructed out of these building blocks, are

$$\mathcal{L} = f(\eta, \text{Tr}\Omega, \text{Tr}\Omega^2, \text{Tr}\Omega^3)$$  \hspace{1cm} (4.10)

where one just has to make sure that the Lagrangian density is a scalar density of weight plus one. Or, again introducing ”the cosmological constants”:

$$\mathcal{L} = \frac{\beta_1}{\eta} + \beta_2 \text{Tr}\Omega + \beta_3 \eta \text{Tr}\Omega^2 + \beta_4 \eta (\text{Tr}\Omega)^2 + \beta_5 \eta^2 \text{Tr}\Omega^3 + \cdots$$  \hspace{1cm} (4.11)

These two Lagrangians are both $SO(3)$ invariant and diffeomorphism covariant, and will therefore in general give a Hamiltonian formulation that has the required constraint algebra (4.1)-(4.3). (For special values of the cosmological constants it can, however, happen that additional second class constraints appear. This is what happens if one in (4.9) changes the factor $\frac{1}{2}$ into $\frac{1}{3}$.)

A rather remarkable fact in this cosmological constants generalization is that the Urbantke formula (2.189) still holds as an expression for the spacetime metric. This is shown by identifying the metric from the constraint algebra, and then following the fields through a Legendre transform. See [41].

### 4.2 Gauge group generalization

Here, I just want to briefly give the basic ideas of how to generalize the Ashtekar formulation to other gauge groups. For further details, see [9] and VIII.

As mentioned in section (1.1), in a canonical formulation of a diffeomorphism invariant theory it is always possible to find first-class constraints obeying the algebra (4.1)-(4.3). And the problem with the Ashtekar Hamiltonian for other gauge groups, is that the crucial part of the algebra (4.3) fails to close. The ordinary Hamiltonian constraint for pure gravity, in Ashtekar’s variables, is

$$\mathcal{H} = \frac{i}{4} \epsilon_{ijk} E^{ai} E^{bj} B^{ck}$$  \hspace{1cm} (4.12)

And, in order to generalize this expression to higher dimensional gauge groups, the $\epsilon_{ijk}$ must be changed into some other gauge covariant object. ($\epsilon_{ijk}$ is only well defined for three dimensional gauge groups.) The obvious candidate is $f_{ijk}$, the structure constant for the Lie-algebra. ($f_{ijk} = \epsilon_{ijk}$ for $SO(3)$.) This will however not work, since the Poisson bracket $\{\mathcal{H}, \mathcal{H}\}$ will fail to close. The reason why it closes for $SO(3)$ is that there exist a structure constant identity (the $\epsilon - \delta$-identity) for that Lie-algebra, while there for an arbitrary gauge group does not exist any such identity. The idea is then to eliminate the structure constant (or $\epsilon_{ijk}$) from $\mathcal{H}$ without changing the content of it. And to do that, one can use the above mentioned structure constant identity. For instance:

$$\mathcal{H} = \frac{\text{det}B^{ai}}{\text{det}B^{ai}} \mathcal{H} = \frac{1}{24} \epsilon_{def} \epsilon_{lmn} B^{a d} B^{e m} B^{f n} i \epsilon_{ijk} E^{ai} E^{bj} B^{ck}$$
4.2 Gauge group generalization

\[ \frac{i}{4} \epsilon_{abc} \epsilon_{def} (E^{ai} B^d_i) (E^{bj} B^e_j) (B^{ck} B^f_k) \sqrt{\text{det} B^{ai} B^b_i} \]  

(4.13)

is a Hamiltonian constraint that works also for higher dimensional gauge groups, and that reduces to the Ashtekar Hamiltonian constraint for three dimensional gauge groups. A perhaps even simpler way of finding the generalized Hamiltonian constraint is to use the Ansatz

\[ E^{ai} = \Psi^{ij} B^a_j \]  

(4.14)

in the ordinary Ashtekar Hamiltonian constraint,

\[ \mathcal{H} = \frac{i}{4} \epsilon_{abc} \epsilon_{ijk} E^{ai} E^{bj} B^{ck} = \frac{i}{4} \text{det} B^{ai} \left( (\text{Tr} \Psi)^2 - \text{Tr} \Psi^2 \right) \]  

(4.15)

and then try to construct the same constraint with the building blocks: \( E^{ai} E^b_i \), \( E^{ai} B^b_i \) and \( B^{ai} B^b_i \).

\[ E^{ai} E^b_i b_{ab} = \text{Tr} \Psi^2 \]  

(4.16)

\[ E^{ai} B^b_i b_{bc} E^{cj} B^d_j b_{da} = \text{Tr} \Psi^2 \]  

(4.17)

\[ E^{ai} B^b_i b_{ab} = \text{Tr} \Psi \]  

(4.18)

where I have used the fact that \( \mathcal{H}_a = 0 \) means that \( \Psi^{ij} = 0 \), and \( b_{ab} \) is the inverse to \( B^{ai} B^b_i \). This means that a suitable \( \mathcal{H} \) is

\[ \mathcal{H} = -\frac{i}{4} \text{det} B^{ai} \left( \alpha E^{ai} E^b_i b_{ab} + (1 - \alpha) E^{ai} B^b_i b_{bc} E^{cj} B^d_j b_{da} - (E^{ai} B^b_i b_{ab})^2 \right) \]  

(4.19)

In checking the constraint algebra however, it becomes clear that it is only two values of \( \alpha \) that give a closed algebra: \( \alpha = 2 \) and \( \alpha = 0 \). Thus, it seems like the ”arbitrary gauge group Hamiltonian” is more restrictive than the \( SO(3) \) Hamiltonian, in which arbitrary functions of the basic building blocks give a closed algebra.

A third way of finding the generalized theory is to start from the CDJ Lagrangian which is trivially generalized to other gauge groups, and then perform the Legendre transform to the Hamiltonian formulation without using any \( SO(3) \) identities. This procedure should always give a closed algebra, for the generators of diffeomorphisms, since the starting point is a manifestly diffeomorphism covariant object.

One of the still unresolved problems with this gauge group generalization, is that the metric, which can be read off from the constraint algebra, looks rather awkward in terms of the phase space variables, and therefore the reality conditions seems to be very hard to find. (This is also true for the cosmological constant generalization, in section (4.1).)

The idea behind these type of generalization is that it may be possible to find a unified theory of gravity and Yang-Mills theory in this way. But, before the reality conditions are found, no such interpretation can be given. Another problem here in (3+1)-dimensions, is that there exists an infinite number of different generalizations that all reduce to the pure gravity Ashtekar formulation when \( SO(3) \) is chosen, and it is not an easy task to select the correct one, if there exists one. An optimistic expectation regarding this, is that
once the correct generalization is found, the reality condition problem and the "metric
problem" could possibly be naturally solved.

In (2+1)-dimensions however, the corresponding generalized theory is unique, and
there the theory really has an interpretation as gravity coupled to Yang-Mills theory.
To lowest order in Yang-Mills fields, the generalized theory exactly coincide with the
conventional Einstein-Yang-Mills coupling.

4.3 Higher dimensions

Here, I want to discuss the possibility of finding higher dimensional generalizations of the
Ashtekar Hamiltonian and the CDJ-Lagrangian. I have no real results to present here.
When the Ashtekar Hamiltonian first was found, it seemed clear that it was a purely
(3+1)-dimensional formulation, which relied on the use of self dual two-forms, which only
exist in four dimensional spacetimes. Later, it was shown \[21\] that the Ashtekar formula-
ation also exists in (2+1)-dimensions. The natural question is then: is it also possible to
find an Ashtekar Hamiltonian in dimensions higher than (3+1)? To try to answer that
question, I will define two different meanings of "higher dimensional Ashtekar formula-
tion".

First, the perhaps obvious definition is: a canonical formulation of Einstein grav-
ity on the Yang-Mills phase space. If we assume that this theory has only gauge and
diffeomorphism symmetries, we can calculate the number of degrees of freedom in a gen-
teral theory of this type, and compare with the ADM-formulation of Einstein gravity:
let the spacetime have dimension \(D+1\), and the Yang-Mills gauge group have dimen-
sion \(N\). Then, the number of degrees of freedom per point in spacetime is; for the
ADM-formulation: \(\frac{D(D+1)}{2} - (D + 1) = \frac{(D-2)(D+1)}{2}\), and for the "Ashtekar formulation":
\(D \times N - N - (D + 1) = N(D - 1) - (D + 1)\), where I have subtracted the number of first
class constraints from half the number of phase space variables. (In the ADM-formulation,
the phase space coordinate is the \(D\)-dimensional symmetric spatial metric, and the only
local symmetries are the diffeomorphism symmetries. In the "Ashtekar formulation", the
phase space coordinate is the spatial restriction of the gauge connection, and the symme-
tries are \(N\) gauge symmetries and \((D+1)\) diffeomorphism symmetries.) In order to have
a realization of Einstein gravity on Yang-Mills phase space, with the above mentioned
symmetries, the number of degrees of freedom must coincide, meaning that

\[ N = \frac{D(D + 1)}{2(D - 1)} \]  

(4.20)

Checking the known results for (2+1)- and (3+1)-dimensions, gives \(N = 3\) in both cases,
which is the dimensionality for \(SO(1, 2)\) and \(SO(3)\). Now, if it should be possible to find
this generalization, the dimensionality \(N\) of the gauge group must be an integer value.
However, it is easy to see from the equation above that there are no \(D > 3\), that will give
an integer \(N\). If \(N\) is an integer, then it must be possible to factorize out the factor \((D - 1)\)
from the numerator. But the numerator consists of the product of the two subsequent
integers greater than \((D - 1)\), and, although I do not give a proof of this statement, it
is obvious that this is not possible for \(D > 3\). This means that it is impossible to find
a realization of Einstein gravity on Yang-Mills phase space, with only gauge and diffeomorphism symmetries. If one adds further symmetries to the theory, it may of course be possible to find this realization. Altogether this gives the conclusion that the obvious higher dimensional Ashtekar formulation does not exist.

Another definition of a higher dimensional Ashtekar formulation is: a gauge and diffeomorphism invariant canonical formulation on the Yang-Mills phase space. Thus, one just relaxes the requirement that the theory must equal the Einstein theory of gravity, and instead allows the theory to have an arbitrary number of degrees of freedom. With this definition, I believe it is possible to find a higher dimensional Ashtekar formulation. (In fact, the conventional (3+1)-dimensional Ashtekar Hamiltonian with gauge group $SO(3)$, is trivially generalized to higher dimensions. However, in dimensions higher than (3+1), the spatial metric will then always be degenerate.) In order to find the Ashtekar Hamiltonian one must again concentrate on finding a suitable Hamiltonian constraint. Gauss’ law and the vector constraint are trivially generalized to arbitrary spacetime dimensions and gauge groups. And again, the requirements the Hamiltonian constraint must satisfy are that it should be gauge and diffeomorphism covariant, and obey (4.3). I will not try to guess or derive any Hamiltonian constraints here, instead I will describe another way of finding this formulation. If we start at the CDJ-level, which is easier generalized to higher dimensions, the Legendre transform will take us to the wanted Hamiltonian. To construct the CDJ-Lagrangian in arbitrary spacetime dimensions, one may use the building blocks: the scalar density field $\eta$, two epsilon tensor densities, $\epsilon^{\alpha\beta\gamma\cdots}$ and (D+1) field strengths, $F^{I}_{\alpha\beta}$. Then, one needs some gauge covariant objects like the killing-metric and/or the structure constants from the Lie-algebra, to properly contract the gauge-indices. This, in general non-unique, Lagrangian is both gauge and diffeomorphism covariant, and the Hamiltonian formulation will therefore satisfy the required algebra (4.3)-(4.3). (It could of course happen that additional second class constraints appear, as well.) However, the major problem with this type of construction is that the Legendre transform generally is rather problematic to perform. (Nobody has yet showed how this should be done, in dimensions higher than (3+1).)

So, to summarize the speculations; it seems that the obvious generalization of Ashtekar’s variables to higher dimensions, is not possible to find, without the introduction of additional symmetries. However, if one just seeks a gauge and diffeomorphism invariant formulation on Yang-Mills phase space, it ought to be possible to find this formulation from a Legendre transform from the higher dimensional CDJ-Lagrangian.
Chapter 5

Outlook

When it became clear that conventional Einstein gravity is perturbatively non-renormalizable, and that the attempts of improving this behavior, such as higher derivative theories and supergravity, also had failed, many physicists took this as a sign that gravity and/or quantum mechanics need a drastic modification before it will be possible to unite them. The alternative conclusion from this failure is that it is the methods that are wrong, and, in fact, knowing the failure of perturbative quantization of gravity, it is easy to find reasons why a perturbative approach to quantum gravity is bound to fail; see e.g. [4], [18], [42] and [43] for enlightening discussions of perturbative versus non-perturbative treatments of quantum gravity.

Without the adequate skill of handling non-perturbative path integrals, we are left with canonical quantization à la Dirac, in order to be able to handle the quantization of gravity, non-perturbatively. Earlier attempts of quantizing gravity canonically, have all been based on the ADM-Hamiltonian, and these attempts have failed due to both technical as well as conceptual problems. The technical problems are mostly concentrated in the complicated Hamiltonian constraint; the Wheeler-DeWitt equation. Since the Hamiltonian constraint, in the ADM-formulation, has a complicated non-polynomial dependence on the basic phase space coordinate, the spatial metric, no one has yet been able to find an explicit, and well-defined, quantum solution to this constraint. The conceptual problems are the problems that any attempt of quantizing gravity eventually will have to face; e.g. the problem of time, and the problem of finding/defining physical observables in a theory of quantum gravity [18], [42] [44], [45].

When Ashtekar [2] reformulated the Hamiltonian for gravity to be a formulation on $SO(3)$ Yang-Mills phase space, with rather simple polynomial and homogeneous constraints, the above mentioned technical problems were reduced significantly. Instead, the Ashtekar Hamiltonian has another new feature that was absent in the ADM-Hamiltonian; the theory is complex, and one needs reality conditions in order to be able to extract real general relativity. This could be a serious drawback for the Ashtekar formulation, or as an optimist would say: the reality condition may be something positive in that they will help us to select the correct inner product for the theory.

Anyway, since the discovery of the Ashtekar Hamiltonian, the quantization program, using Dirac quantization of the Ashtekar Hamiltonian, has grown to be an active field of research. There already exist two excellent books covering this subject [18], [42], a number of different reviews [16], [17], [48], [49], and more than 300 publications [50], all...
related to Ashtekar’s variables.

The first attempts, in this program, used the so-called connection representation [3], where the wave functionals are holomorphic functionals of the self-dual Ashtekar connection, and the trace of the holonomy of this connection around a closed loop can be shown to satisfy both Gauss’ law and the Hamiltonian constraint. The vector constraint is not solved, however. This soon led to the loop-representation, where the wave functionals are complex functionals of loops on the spatial hypersurface [4], [51], [52], [53]. In this representation, the algebra that is quantized is a non-canonical graded $T$-algebra, where $T$ stands for traces of holonomies with momentum fields inserted along the loop. This algebra is automatically $SU(2)$ ($SO(3)$) invariant, which means that Gauss’ law is already taken care of at the classical level (reduced phase space quantization). The vector constraint, which classically generates spatial diffeomorphisms, is solved a la Dirac by only considering wave functionals of knot and link classes of loops. (One really considers generalized knot and link classes, for details see e.g. [54].) It was also soon clear that the Hamiltonian constraint is solved by using only non-intersecting, smooth loops [4]. Later, several other solutions to all the quantum constraints have been found [52], [53]. However, there are still two extremely important ingredients missing; the inner product and observables. Without these, one cannot calculate any physical quantities, and the solutions do not really give any information about the theory. Another important ingredient is the general solution to all the constraints. Perhaps that is too much to ask for, but what is really needed is an understanding of the importance of the known solutions. For all we know, the solutions that have been found so far could all belong to a ”degenerate set of measure zero”. (That is, they could be totally unimportant.)

The quantization program also includes work on linearized gravity [55], [56], (2+1)-dimensional gravity [57], Maxwell fields [58], [59], (1+1)-dimensional QED [60], etc.

Finally, I want to give my opinion and expectations of how the CDJ-formulation as well as the generalizations of Einstein gravity can contribute to a better understanding of classical and quantum gravity. So far, nothing really important has come out of the discovery of the CDJ-Lagrangian. (When the CDJ-Lagrangian was found [5], the general solution to the classical diffeomorphism constraints, modulo Gauss’ law, were also found, but this could have been found without the knowledge of the CDJ-Lagrangian.) There have been attempts to make progress with path integral quantization, and discretized approximations using the CDJ-Lagrangian [71], [72], but so far there are no real results coming out of this. Related to the CDJ-Lagrangian there have also been some new discoveries concerning gravitational instantons [54], [64], [65]. In my opinion, the most interesting outcome of the CDJ-Lagrangian is, so far, the gauge group generalization. This generalization could also have been found without the knowledge of the CDJ-formulation, but it is really techniques and ideas coming from this pure connection formulation that have enabled the finding of this generalization.

Why is the generalizations, presented in section 4, so interesting? So far, none of these generalizations has a satisfactory treatment of the reality conditions, or the metric signature condition. However, if these, and possibly other, problems can be solved, the new cosmological constants are on the same footing as the conventional Einstein cosmological constant, and we will need a theoretical explanation of why they are all zero, or keep them in the theory. Perhaps, the introduction of all these new cosmological constants
will eventually help us understand why it is the pure gravity Einstein equation without any cosmological constants that seems to be the correct equation for describing "classical nature". A positive expectation regarding the gauge group generalization of Ashtekar’s variables, is that this generalization may be a suitable theory for the loop-representation quantization of coupled gravity-Yang-Mills theory. This is otherwise a problem in the Ashtekar formulation; the coupling of gravity to Yang-Mills theory has a Hamiltonian constraint that is seemingly not suited for the loop-representation quantization.
Appendix A

Conventions and Notation

Indices: $\alpha, \beta, \gamma, \ldots$ denote spacetime indices, $I, J, K, \ldots$ denote $SO(1, 3)$ indices (or $SO(1, 2)$ in $(2+1)$-dimensions), $a, b, c, \ldots$ denote spatial indices.

Symmetrization and Antisymmetrization: Symmetrization and antisymmetrization of indices are denoted by brackets, according to

\[
A^{[ab]} := \frac{1}{2} (A^{ab} + A^{ba}) \\
A^{(ab)} := \frac{1}{2} (A^{ab} - A^{ba}) \\
A^{(abc)} := A^{(ab)c} + A^{(ca)b} + A^{(bc)a} \\
A^{[abc]} := A^{[ab]c} + A^{[ca]b} + A^{[bc]a} \\
A^{(a|b|c)} := A^{abc} + A^{cba}
\]

The Minkowski metric: The Minkowski metric is chosen to be $\eta^{IJ} := \text{Diag}(-1, 1, 1, 1)$ in $(3+1)$-dimensions, and $\eta^{IJ} := \text{Diag}(-1, 1, 1)$ in $(2+1)$-dimensions.

$(3+1)$-dimensional $\epsilon$-symbol: $\epsilon^{\alpha\beta\gamma\delta}$ and $\epsilon_{\alpha\beta\gamma\delta}$ are totally antisymmetric, and $\epsilon^{0123} = \epsilon_{0123} = 1$ in every coordinate system, implying that $\epsilon^{\alpha\beta\gamma\delta}$ is a tensor density of weight plus one, and $\epsilon_{\alpha\beta\gamma\delta}$ has weight minus one. For any non-degenerate tensor $K_{\alpha\beta}$, the following is true

\[
\epsilon_{\alpha\beta\gamma\delta} = \frac{1}{K} K_{\alpha\epsilon} K_{\beta\sigma} K_{\gamma\rho} K_{\delta\kappa} \epsilon^{\epsilon\sigma\rho\kappa}
\]

where $K := \det K_{\alpha\beta} = \frac{1}{24} \epsilon^{\alpha\beta\gamma\delta} \epsilon^{\epsilon\sigma\rho\kappa} K_{\alpha\epsilon} K_{\beta\sigma} K_{\gamma\rho} K_{\delta\kappa}$.

Spatial restriction of $\epsilon^{\alpha\beta\gamma\delta}$: $\epsilon^{abc} := \epsilon^{0abc}$, $\epsilon_{abc} := \epsilon_{0abc}$, meaning that $\epsilon^{abc}$ satisfies the following identities

\[
\epsilon^{abc} \epsilon_{def} = \delta^{[a}_{[d} \delta^{b}_{e} \delta^{c]}_{f}] \\
\epsilon^{abc} \epsilon_{ade} = \delta^{[b}_{[d} \delta^{c]}_{e} \\
\epsilon^{abc} \epsilon_{abc} = 2\delta^{c}_{c}
\]
SO(1, 3) $\epsilon$ -symbol: $\epsilon^{IJKL}$ is totally antisymmetric, and $\epsilon^{0123} = 1$.

$\epsilon_{IJKL} := \eta_{IM} \eta_{JN} \eta_{KP} \eta_{LQ} \epsilon^{MNPQ}$, meaning that $\epsilon_{0123} = -1$, and the following identities are valid:

\[
\begin{align*}
\epsilon^{IJKL} \epsilon_{MNPQ} &= -\delta^{[I}_{M}[\delta^{J}_{N} \delta^{K}_{P} \delta^{L]}_{Q} \quad (A.10) \\
\epsilon^{IJKL} \epsilon_{1NPQ} &= -\delta^{[I}_{N} \delta^{J}_{P} \delta^{K}_{L} \delta^{0}_{Q} \quad (A.11) \\
\epsilon^{IJKL} \epsilon_{1JPQ} &= -2\delta^{[K}_{P} \delta^{L}_{Q} \quad (A.12) \\
\epsilon^{IJKL} \epsilon_{1JKQ} &= -6\delta^{L}_{Q} \quad (A.13)
\end{align*}
\]

$(2+1)$-dimensional $\epsilon$ -symbol: $\epsilon^{\alpha\beta\gamma}$ and $\epsilon_{\alpha\beta\gamma}$ are totally antisymmetric, and $\epsilon^{012} = \epsilon_{012} = 1$ in every coordinate system, implying that $\epsilon^{\alpha\beta\gamma}$ is a tensor density of weight plus one, and $\epsilon_{\alpha\beta\gamma}$ has weight minus one. For any non-degenerate tensor $K_{\alpha\beta}$, the following is true

\[
\epsilon_{\alpha\beta\gamma} = \frac{1}{K} K_{\alpha\epsilon} K_{\beta\delta} K_{\gamma\rho} \epsilon^{\epsilon\sigma\rho} \quad (A.14)
\]

where $K := \det K_{\alpha\beta} = \frac{1}{6} \epsilon^{\alpha\beta\gamma} \epsilon_{\epsilon\sigma\rho} K_{\alpha\epsilon} K_{\beta\delta} K_{\gamma\rho}$.

Spatial restriction of $\epsilon^{\alpha\beta\gamma}$: $\epsilon^{ab} := \delta^{ab}$, $\epsilon_{ab} := \epsilon_{0ab}$, meaning that $\epsilon^{ab}$ satisfies the following identities

\[
\begin{align*}
\epsilon^{ab} \epsilon_{de} &= \delta^{[a}_{d} \delta^{b]}_{e} \quad (A.15) \\
\epsilon^{ab} \epsilon_{ae} &= \delta^{b}_{e} \quad (A.16)
\end{align*}
\]

SO(1, 2) $\epsilon$ -symbol: $\epsilon^{IJK}$ is totally antisymmetric, and $\epsilon^{012} = 1$.

$\epsilon_{IJK} := \eta_{IM} \eta_{JN} \eta_{KP} \epsilon^{MNP}$, meaning that $\epsilon_{012} = -1$, and the following identities are valid:

\[
\begin{align*}
\epsilon^{IJK} \epsilon_{MNP} &= -\delta^{[I}_{M}[\delta^{J}_{N} \delta^{K}_{P}] \quad (A.17) \\
\epsilon^{IJK} \epsilon_{1NP} &= -\delta^{[J}_{N} \delta^{K}_{P} \delta^{I]}_{P} \quad (A.18) \\
\epsilon^{IJK} \epsilon_{1JP} &= -2\delta^{K}_{P} \quad (A.19)
\end{align*}
\]
Appendix B

Definitions of Connections and Curvature

Definition of metric compatible affine connection and spin-connection. The affine connection $\Gamma^\gamma_{\alpha\beta}$ and the spin-connection $\omega^{IJ}_\alpha$ are here defined to annihilate the tetrad $e_\alpha^I$:

$$D_\alpha e_\beta^I := \partial_\alpha e_\beta^I - \Gamma^\gamma_{\alpha\beta} e_\gamma^I + \omega^{IJ}_\alpha e_\beta^J = 0 \quad \text{(B.1)}$$

(Zero-torsion is assumed, and therefore $\Gamma^\gamma_{[\alpha\beta]} = 0$.) This means that

$$D_\alpha (e_\beta^I e_\gamma^J) = D_\alpha (e_\beta^I) = \partial_\alpha g^{\gamma\gamma} - \Gamma^\gamma_{\alpha\beta} g_{\gamma\gamma} - \Gamma^\gamma_{\gamma\alpha} g_{\alpha\beta} = 0 \quad \text{(B.2)}$$

or, $\omega^{IJ}_\alpha$ is antisymmetric, and is therefore so(1,3) Lie-algebra valued. Now, to solve for $\Gamma^\gamma_{\alpha\beta}$ and $\omega^{IJ}_\alpha$ from (B.1), it is convenient to first solve for $\Gamma^\gamma_{\alpha\beta}$ from the equation:

$$D_\alpha g_{\beta\gamma} = D_\alpha (e_\beta^I e_\gamma^J) = \partial_\alpha g_{\beta\gamma} - \Gamma^\epsilon_{\alpha\beta} g_{\epsilon\gamma} - \Gamma^\epsilon_{\epsilon\gamma} g_{\alpha\beta} = 0 \quad \text{(B.3)}$$

To solve for $\Gamma^\gamma_{\alpha\beta}$, one does a cyclic permutation of the three indices $\alpha$, $\beta$ and $\gamma$ two times. The two resulting equations are then added together, and finally equation (B.3) is subtracted from this sum. Due to the symmetry of $\Gamma^\gamma_{\alpha\beta}$, the result is

$$\partial_\alpha g_{\beta\gamma} + \partial_\gamma g_{\alpha\beta} - \partial_\beta g_{\gamma\alpha} - 2\Gamma^\epsilon_{\gamma\alpha} g_{\epsilon\beta} = 0 \quad \text{(B.4)}$$

which is easily solved.

$$\Gamma^\epsilon_{\alpha\beta} = \frac{1}{2} g^{\gamma\gamma} (\partial_\alpha g_{\epsilon\beta} + \partial_\beta g_{\epsilon\alpha} - \partial_\epsilon g_{\alpha\beta}) \quad \text{(B.5)}$$

Then, putting (B.5) into (B.1), the solution for $\omega^{IK}_\alpha$ becomes:

$$\omega^{IK}_\alpha = -e^K_\beta \partial_\alpha e_\beta^I + e^K_\beta \Gamma^\gamma_{\alpha\beta} e_\gamma^I = \frac{1}{2} e^I [\partial_\alpha e^K_\epsilon + e^K_\epsilon e_\alpha \partial_\beta e_\epsilon] \quad \text{(B.6)}$$

Thus, (B.5) and (B.6) are the unique solution to (B.1), and an indication to why it is possible to uniquely solve for these connections, can be found by counting degrees of freedom: In $(D+1)$-dimensions, equation (B.1) has $(D + 1)^3$ components, while $\Gamma^\gamma_{\alpha\beta}$ and $\omega^{IJ}_\alpha$ has $(D+1)(D+2)/(2D+2)(D+1)$ and $(D+1)(D+2)/(2D+2)(D+1)$ algebraically independent components, showing that the number of unknowns are the same as the number of equations. So, as long as
the system of equations (B.1) is non-degenerate, there will always exist a unique solution. (The non-degeneracy of (B.1) is closely connected to the non-degeneracy of the tetrad field.)

Another derivation of the spin-connection. Instead of (B.1), the spin-connection can be defined as follows:

\[ \mathcal{D}_{[\alpha e_{\beta}I]} := \partial_{[\alpha e_{\beta}I} + \omega_{[\alpha I}^J e_{\beta]J} = 0 \]  

(B.7)

Note that (B.1) implies (B.7) but the converse is not true. (Γ_{\alpha \beta}^I is not even defined in (B.7).) Do (B.1) and (B.7) then have the same solution for ω_{IJ}^I? Yes, they have, and that can be understood by dimensional counting again: In (D+1)-dimensions, (B.7) represents \( \frac{(D+1)D}{2} (D + 1) \) equations, and the spin-connection has the same number of algebraically independent components, meaning that (B.7) has a unique solution. Then, since both (B.1) and (B.7) have unique solutions, and (B.1) implies (B.7), the solution must be the same.

To directly solve (B.7) for ω_{IJ}^I, one can use the same trick that was used to solve for Γ_{\alpha \beta}^I. First, convert all free indices in (B.7) into flat SO(1, 3) indices:

\[ e_{J}^\alpha e_{K}^\beta \left( \partial_{[\alpha e_{\beta}I} + \omega_{[\alpha I}^L e_{\beta]L} \right) = 0 \]  

(B.8)

Then, do a cyclic permutation of the free indices I, J and K two times, and add and subtract the three resulting equations:

\[ \Omega_{JKI} + \Omega_{IKJ} - \Omega_{KIJ} + 2 e_{J}^\alpha \omega_{\alpha IK} = 0 \]  

(B.9)

where I have defined \( \Omega_{JKI} := e_{J}^\alpha e_{K}^\beta \partial_{[\alpha e_{\beta}I}. \) The solution for the spin-connection is

\[ \omega_{\alpha KI} = \frac{1}{2} e_{J}^\alpha (\Omega_{JKI} + \Omega_{IKJ} - \Omega_{KIJ}) \]  

(B.10)

Comparing (B.10) to (B.6) gives exact agreement.

Definition of the curvature field. The Riemann tensor is defined as follows:

\[ \mathcal{D}_{[\alpha \mathcal{D}_\beta}] \lambda_\epsilon = R_{\alpha \beta \epsilon}^\mu \lambda_\mu \]  

(B.11)

\[ \mathcal{D}_{[\alpha \mathcal{D}_\beta]} \lambda_I = R_{\alpha \beta I}^J \lambda_J \]  

(B.12)

By using these definitions for a vector field of the form \( \lambda_\epsilon = e^I_\epsilon \lambda_I, \) one can show that

\[ e^I_\epsilon \mathcal{D}_{[\alpha \mathcal{D}_\beta]} \lambda_\epsilon = e^I_\epsilon R_{\alpha \beta \epsilon}^\mu e^J_\mu \lambda_J = R_{\alpha \beta I}^J \lambda_J \]  

(B.13)

And, since this is valid for all vectors \( \lambda_I, \) the following relation must hold:

\[ R_{\alpha \beta I}^J = R_{\alpha \beta \epsilon}^\mu e^I_\epsilon e^J_\mu \]  

(B.14)

Now, using the definitions (B.11) and (B.12) together with the definition of the covariant derivative (B.1), it is straightforward to derive the explicit form of the Riemann tensor:
Definitions of Connections and Curvature

\[ R_{\alpha \beta} = \partial_{[\alpha} \Gamma_{\beta]}^{\prime} + \Gamma_{[\alpha}^\rho \Gamma_{\beta] \mu}^{\rho} \]  
(B.15)

\[ R_{\alpha \beta}^{IJ} = \partial_{[\alpha} \omega_{\beta]}^{IJ} + \omega_{[\alpha}^{I K} \omega_{\beta] K}^{J} \]  
(B.16)

Einstein’s equation. Einstein’s equation for pure gravity with a cosmological constant, in the metric formulation, is

\[ R_{\mu \nu} - \frac{1}{2} g_{\mu \nu} R - \lambda g_{\mu \nu} = 0 \]  
(B.17)

Using the relation (B.14), the definition of the Ricci tensor

\[ R_{\mu \nu} := R_{\mu \alpha \nu \alpha} \]  
and the definition of the curvature scalar

\[ R := g_{\mu \nu} R_{\mu \nu} \]  
Einstein’s equation can be rewritten in terms of the tetrad.

\[ R_{\nu \alpha I J} e^{I} e^{J} - \frac{1}{2} e^{I} e^{\nu K} R^{\gamma I} R_{\gamma K}^{J} + 2 \lambda = 0 \]  
(B.18)

Or, multiplying with \( e^{\mu I} \) to get the form that follows from the Einstein-Hilbert Lagrangian:

\[ R_{\nu \alpha I J} e^{I} e^{J} e^{\nu} - \frac{1}{2} e^{I} e^{\nu K} R^{\gamma I} R_{\gamma K}^{J} + 2 \lambda = 0 \]  
(B.19)

"Hybrid connections." Here, I want to show that it is possible to define a unique spin-connection compatible with a sort of "hybrid" vielbein variable \( e^{I}_{a} \), where the dimensionality of the spatial index is one unit lower than the dimensionality of the internal index.

Consider a \((D + 1)\)-dimensional spacetime, and a "vielbein" field \( e_{aI} \) where \( a \) take values \( 1, 2, \ldots, D \), and the \( I \)-index is a flat Lorentz index, taking values \( 0, 1, \ldots, D \). In a Hamiltonian formulation where one has to partly break the manifest spacetime covariance, the "vielbein" is split into two parts, \( e_{0I} \) and \( e_{aI} \), where \( a \) is a spatial index taking values \( 1, 2, \ldots, D \). Then, I define a spin-connection \( \omega^{I J}_{a} \) compatible with this "hybrid" field \( e^{I}_{a} \):

\[ D_{a} e_{bI} := \partial_{a} e_{bI} - \Gamma^{c}_{ab} e_{cI} + \omega^{I J}_{a} e_{bJ} = 0 \]  
(B.20)

At first sight, it seems to be impossible to find a unique solution for \( \omega^{I J}_{a} \), since there exists no inverse to \( e_{aI} \), such that \( e^{aI} e_{aI} = \delta^{I}_{J} \). It is however possible to uniquely solve for both \( \Gamma^{c}_{ab} \) and \( \omega^{I J}_{a} \), and that can again be understood by counting degrees of freedom: Equation (B.20) represents \( D^{2}(D + 1) \) equations, and \( \Gamma^{c}_{ab} \) and \( \omega^{I J}_{a} \) have \( \frac{D(D+1)}{2} D \) and \( \frac{D(D+1)}{2} D \) number of algebraically independent components, giving exact agreement between the number of unknown and the number of equations. Another way of understanding this is to note that since \( \omega^{I J}_{a} \) is antisymmetric, one does not lose any information in \( \omega^{I J}_{a} \) by projecting it on \( e_{aI} \) as long as \( e_{aI} \) represents \( D \) linearly independent \( SO(1, D) \) vectors.

To explicitly solve (B.20) for \( \omega^{I J}_{a} \), it is convenient to introduce the unit, time-like vector field, \( N^{I} \), orthogonal to \( e^{I}_{a} \):
\[ N^I e_{aI} = 0, \quad N^I N_I = -1, \quad \Rightarrow N^I = \pm \frac{e^{IJK} \cdots P e_{aI} e_{bJ} e_{cK} \cdots e_e \delta^{abc \cdots g}}{D! \sqrt{\det(e_{aI} e_{bJ})}} \] (B.21)

Then, I define the projection operator:

\[ \tilde{\eta}^{IJ} = e^a_I e^a_J = \eta^{IJ} + N^I N^J \] (B.22)

where \( e^a_I := q^{ab} e^b_I \) and \( q^{ab} \) is the inverse to \( q_{ab} := e^a_I e^b_I \). I use this projection operator to project out time-like (primed) and space-like (tilded) indices.

\[ \lambda^I = \lambda^I' + \lambda^I \] (B.23)

Using this, \( \omega^{IJ}_a \) can be written as

\[ \omega^{IJ}_a = \omega^{[I'}_{aJ'] + \omega^{IJ}_a } \] (B.24)

Now, it is straightforward to solve (B.20) for \( \Gamma^c_{ab} \), \( \omega^{[I'}_{aJ'] } \) and \( \omega^{IJ}_a \). First, \( \Gamma^c_{ab} \) can be solved for as usual:

\[ \Gamma^c_{ab} = \frac{1}{2} e^a_{Ib} \left( \partial_a q_{eb} + \partial_b q_{ea} - \partial_e q_{ab} \right) \] (B.25)

and then the spin-connection is easily given:

\[ \omega^{IJ}_a = e^b_J N_I N^K \partial_a e_{bK} \] (B.26)
\[ \omega^{IJ}_a = -\tilde{\eta}^{bJ} e^b_J \partial_a e_{bK} + e^b_J \Gamma^c_{ab} e_{cI} \] (B.27)

Thus, (B.24), (B.25), (B.26) and (B.27) give the unique solution to (B.20).

Note however that the alternative definition

\[ D[a e_b]_I = \partial_a [e_b]_I + \omega_{[aI} e_{bJ]} = 0 \] (B.28)

does not have a unique solution for \( \omega_{aI}^J \), which can be seen by dimensional counting: The number of equations are \( \frac{D(D-1)}{2}(D+1) \) while the number of unknowns are \( \frac{D(D+1)}{2}D \), showing that there are to few equations to uniquely specify the spin-connection.

Variations of \( \omega^I_J \) and \( R_{\alpha\beta}^{IJ} \). From (B.16) it follows that

\[ \delta R_{\alpha\beta}^{IJ} = \partial_{[\alpha} \delta \omega_{\beta]}^{IJ} + \delta \omega_{[\alpha}^{IK} \omega_{\beta]K}^{J} + \omega_{[\alpha}^{IK} \delta \omega_{\beta]K}^{J} = D_{[\alpha} \delta \omega_{\beta]}^{IJ} \] (B.29)

Note that \( \delta \omega_{IJ} \) is a Lorentz covariant object since it is the difference between two Lorentz connections. To find the variation of \( \omega^I_J \) with respect to variations of \( e_{aI} \), one can use the defining equation:

\[ D_{[\alpha} e_{\beta]} = \partial_{[\alpha} e_{\beta]} + \omega_{[\alpha} e_{\beta]}^{J} e_{\beta]} = 0 \] (B.30)

Varying this equation, gives
Definitions of Connections and Curvature

\[ D_{[\alpha} \delta \varepsilon_{\beta]} + \delta \omega_{[\alpha I} \varepsilon_{\beta]J} = 0 \]  \hspace{1cm} (B.31)

But, (B.31) is the same equation as (B.7), with \( \omega^{I J} \rightarrow \delta \omega^{I J} \) and \( \partial_{[\alpha} \varepsilon_{\beta]} \rightarrow D_{[\alpha} \delta \varepsilon_{\beta]}, \)
meaning that the solution to (B.31) is

\[ \delta \omega^{I J} = \frac{1}{2} e^I \left( D_{[\alpha} \delta \varepsilon^J + e^{\beta J}_\varepsilon \varepsilon^K D_{\beta} \delta \varepsilon^K \right) \]  \hspace{1cm} (B.32)

Together, (B.29) and (B.32) then give the variation of \( R_{\alpha \beta I J} \) with respect to \( e_{\alpha I}. \)
Bibliography

[1] Green, M., Schwarz, J. H. and Witten, E., "Superstring theory", (Cambridge Univ. Press., 1987).

[2] Ashtekar, A., Phys. Rev. Lett. 57 (1986) 2244, Phys. Rev. D36 (1987) 1587.

[3] Jacobson, T. and Smolin, L., Nucl. Phys. B299 (1988) 295.

[4] Rovelli, C. and Smolin, L., Nucl. Phys. B331 (1990) 80.

[5] Capovilla, R., Dell, J. and Jacobson, T., Phys. Rev. Lett. 63 (1989) 2325
    Capovilla, R., Dell, J., Jacobson, T. and Mason, L., Class. Quant. Grav. 8 (1991) 41.

[6] Capovilla, R., Nucl. Phys. B373 (1992) 233.

[7] Bengtsson, I., Phys. Lett. B254 (1991) 55.

[8] Bengtsson, I. and Peldán, P., Int. J. Mod. Phys. A7 (1992) 1287.

[9] Peldán, P., Phys. Rev. D46 (1992) R2279.

[10] Regge, T. and Teitelboim, C., Ann. Phys. 88 (1974) 286.

[11] Soloviev, V. O., Phys. Lett. 292 (1992) 30.

[12] Schrödinger, E., "Space-time structure", (Cambridge U. Press 1950).

[13] ’tHooft, G., Nucl. Phys. B357 (1991) 211.

[14] Ashtekar, A., Romano, J. D. and Tate, R. S., Phys. Rev. D40 (1989) 2572.

[15] Nicolai, H. and Matschull, H. J., Aspects of Canonical Gravity and Supergravity, DESY preprint 92-099.

[16] Arnowitt, R., Deser, S. and Misner, C. W., "Gravitation: An introduction to current research." (L. Witten, Ed.), (Wiley, New York, 1962).

[17] Ashtekar, A., Balachandran, A. P. and Jo, S. G., Int. J. Mod. Phys. A4 (1989) 1493.

[18] Ashtekar, A. "Lectures on non-perturbative canonical gravity." (World Scientific, Singapore, 1991).
[19] Husain, V. and Kuchař, K., Phys. Rev. D42 (1990) 4070.

[20] Samuel, J., Pramana-J Phys. 28 (1987) L429
    Jacobson, T. and Smolin, L., Class. Quant. Grav. 5 (1988) 583.

[21] Bengtsson, I., Int. J. Mod. Phys. A4 (1989) 5527.

[22] Wallner, R. P., Phys.Rev. D46 (1992) 4263.

[23] Henneaux, M., Nelson, J. E. and Schomblond, C., Phys. Rev. D39 (1989) 434.

[24] Peldán, P., Class. Quant. Grav. 8 (1991) 1765.

[25] Urbantke, H., J. Math. Phys. 25 (1984) 2321.

[26] Capovilla, R. and Jacobson, T., Mod. Phys. Lett. 7 (1992) 1871.

[27] Bengtsson, I. and Peldán, P., Phys. Lett. B244 (1990) 261.

[28] Peldán, P., Phys. Lett. B248 (1990) 62.

[29] Bengtsson, I., ”Ashtekar’s variables and the cosmological constants”, Göteborg preprint ITP 91-33.

[30] Plebanski, J. F., J. Math. Phys. 18 (1977) 2511.

[31] Koshti, S., Class. Quant. Grav. 9 (1992) 1937.

[32] Witten, E., Nucl. Phys. B311 (1988) 46.

[33] Romano, J. D., Geometrodynamics vs. Connection Dynamics (in the context of (2+1)- and (3+1)-gravity), Ph.D. Thesis, Syracuse University 1991.

[34] Peldán, P., Class. Quant. Grav. 9 (1992) 2079.

[35] Fülöp, G., Mod. Phys. Lett. A7 (1992) 3495.

[36] Bengtsson, I. and Boström, O., Class. Quant. Grav. 9 (1992) L47.
    Boström, O., ”Cosmological Constants Galore”, Chalmers University of Technology Licentiate Thesis (1992).

[37] private communication from I. Bengtsson.

[38] Peldán, P., Nucl. Phys. B395 (1993) 239.

[39] Deser, S., Jackiw, R. and Templeton, S., Ann. Phys. (NY) 140 (1982) 372.
    Deser, S. and Xiang, X., Phys. Lett. B263 (1991) 39.

[40] Hojman, S., Kuchař, K. and Teitelboim, C., Ann. Phys., NY, 96 (1976) 88.

[41] Bengtsson, I., J. Math. Phys. 32 (1991) 3158.

[42] Ashtekar, A. ”New Perspectives in Canonical Gravity”, (Bibliopolis, Napoli, 1988).
[43] Isham, C., "Conceptual Problems in Quantum Gravity", Imperial preprint TP/90-91/4.

[44] Proceeding of the Osgood Hill Conference on Conceptual Problems in Quantum Gravity, eds. Ashtekar, A. and Stachel, J. (Birkhäuser, Boston 1991).

[45] Isham, C., Canonical Quantum Gravity and the Problem of Time, Imperial/TP/91-92/25.

[46] Rovelli, C., Class. Quant. Grav. 8 (1991) 1613.

[47] Kodama, H. "Quantum gravity by the complex canonical formulation", gr-qc/9211024, Int. J. Mod. Phys. To appear.

[48] Pullin, J., "Knot theory and quantum gravity: A primer.", University of Utah preprint UU-REL-93/1/9, hep-th/9301028.

[49] Smolin, L., "What can we learn from the study of non-perturbative quantum general relativity?", gr-qc/9211019.

[50] Brügmann, B., "Bibliography of Publications related to Classical and Quantum Gravity in terms of the Ashtekar’s Variables", gr-qc/9303013.

[51] Blencowe, M. P., Nucl. Phys. 341 (1990) 213.

[52] Brügmann, B., Gambini, R. and Pullin, J., Phys. Rev. Lett. 68 (1992) 431.

[53] Brügmann, B., Gambini, R. and Pullin, J., Nucl. Phys. B385 (1992) 581.

[54] Smolin, L., Recent Developments in nonperturbative quantum gravity, To appear in "Proceedings of the XXII Gift International Seminar on Theoretical Physics, Quantum Gravity and Cosmology", June 1991, Catalonia, Spain (World Scientific, Singapore 1992).

[55] Ashtekar, A., Rovelli, C. and Smolin, L., Phys. Rev. D44 (1991) 1740, Phys. Rev. Lett. 69 (1992) 237.

[56] Zegward, J., Nucl. Phys. B378 (1992) 288.

[57] Ashtekar, A., Husain, V., Rovelli, C., Samuel, J. and Smolin, L., Class. Quant. Grav. 6 (1989) L185.

Ashtekar, A., "Lessons from 2+1 dimensional quantum gravity", In Strings 90 eds. R. Arnowitt et al, (World Scientific, Singapore, 1990).

Marolf, D., "Loop representations for 2+1 gravity on a torus", Syracuse preprint SU-GP-93/3-1.

[58] Ashtekar, A. and Rovelli, C., Class. Quant. Grav. 9 (1992) 1121.

[59] Gambini, R. and Pullin, J., "Quantum Einstein-Maxwell fields: a unified viewpoint from the loop-representation." hep-th@9210110.

[60] Hallin, J., "$QED_{1+1}$ by Dirac Quantization", Göteborg preprint ITP 93-8.
[61] Smolin, L., Class. Quant. Grav. 9 (1992) 883.

[62] Miller, M. and Smolin, L., ”A new discretization of classical and quantum general relativity”, gr-qc/9304005.

[63] Samuel, J., Class. Quant. Grav. 5 (1988) L123.

[64] Capovilla, R., Dell, J. and Jacobson, T., Class. Quant. Grav. 7 (1990) L1.

[65] Torre, C. G., Phys. Rev. D41 (1990) 3620.