Existence of mild solutions for the Hamilton-Jacobi equation with critical fractional viscosity in the Besov spaces

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Abstract
We consider the Cauchy problem for the Hamilton-Jacobi equation with critical dissipation,
\begin{align*}
\partial_t u + (-\Delta)^{1/2} u = |\nabla u|^p, \quad x \in \mathbb{R}^N, \quad t > 0, \quad u(x, 0) = u_0(x), \quad x \in \mathbb{R}^N,
\end{align*}
where \( p > 1 \) and \( u_0 \in B^{1,1}_r(\mathbb{R}^N) \cap B^{1,1}_{\infty,1}(\mathbb{R}^N) \) with \( r \in [1, \infty] \). We show that for sufficiently small \( u_0 \in \dot{B}^{1,1}_{\infty,1}(\mathbb{R}^N) \), there exists a global-in-time mild solution. Furthermore, we prove that the solution behaves asymptotically like suitable multiplies of the Poisson kernel.

1 Introduction

We consider the Hamilton-Jacobi equation with fractional viscosity,
\begin{align*}
\left\{ \begin{array}{l}
\partial_t u + (-\Delta)^{\alpha/2} u = |\nabla u|^p, \quad x \in \mathbb{R}^N, \quad t > 0, \\
u(x, 0) = u_0(x), \quad x \in \mathbb{R}^N,
\end{array} \right.
\end{align*}
(1.1)
where \( N \geq 1 \), \( \partial_t = \partial/\partial t \), \( \nabla = (\partial_{x_1}, \cdots, \partial_{x_N}) \), \( \partial_{x_j} = \partial/\partial x_j \) \( (j = 1, \cdots, N) \), \( \alpha \in (0, 2] \), \( p > 1 \) and \( u_0 \) is a nontrivial measurable function in \( \mathbb{R}^N \). Here the operator \((-\Delta)^{\alpha/2}\), which called the Lévy operator, is defined by the Fourier transform \( \mathcal{F} \) such that
\begin{align*}
(-\Delta)^{\alpha/2} f := \mathcal{F}^{-1} \left[ |\xi|^\alpha \mathcal{F}[f] \right].
\end{align*}
In this paper we study the existence of global-in-time solutions to the problem (1.1) with \( \alpha = 1 \), and investigate the asymptotics of solutions.

The problem (1.1) with \( \alpha = 2 \) is the well-known viscous Hamilton-Jacobi (VHJ) equation. The VHJ equation possesses both mathematical and physical interest. Indeed, in
mathematical points of view, it is the simplest example of a parabolic PDE with a nonlinearity depending only on the first order spatial derivatives of \( u \), and it describes a model for growing random interfaces, which is known as the Kardar-Parisi-Zhang equation (see [20, 23]). On the other hand, the problem (1.1) with \( \alpha \in (0,2) \) often appears in the context of mathematical finance as Bellman equations of optimal control of jump diffusion processes (see, for example, [9, 11, 17, 18, 28]).

The VHJ equation has been studied in many papers about various topics. For the existence and uniqueness of solutions, it is well known that, for any \( u_0 \in W^{1,\infty}(\mathbb{R}^N) \), the problem (1.1) with \( \alpha = 2 \) has a unique global-in-time mild solution, i.e., a solution of the integral equation

\[
u(t) = e^{t\Delta} u_0 + \int_0^t e^{(t-\tau)\Delta} |\nabla u(\tau)|^p \, d\tau, \quad t > 0,
\]

where \( e^{t\Delta} \) denotes the convolution operator with the heat kernel (see, for example, [2, 4, 6, 10]). Furthermore, this solution is classical for positive time, and by the maximum principle, we see that, if \( u_0 \geq 0 \), then \( u \geq 0 \), and if \( u_0 \leq 0 \), then \( u \leq 0 \). From this property, the nonlinearity \( |\nabla u|^p \) behaves like a source term for nonnegative initial data and an absorption term for nonpositive initial data. Similarly to the case of the semilinear heat equation \( \partial_t u - \Delta u = \lambda |u|^{p-1} u \) with \( \lambda = \pm 1 \), the asymptotics of solutions to this equation is determined by the balance of effects from the diffusion term \( \Delta u \) and the one from the nonlinearity \( |\nabla u|^p \), and there are many results on the asymptotic behavior of solutions. See, for example, [2]–[6], [10, 19, 24] and the references therein. Among others, in [3], Benachour, Karch and Lancrencot proved that, for the case \( u_0 \in L^1(\mathbb{R}^N) \cap W^{1,\infty}(\mathbb{R}^N) \) with \( u_0 \not\equiv 0 \), the following hold.

(i) Assume that \( u_0 \geq 0 \).

(a) For the case \( p \geq 2 \), there exists a limit

\[
C_\ast := \lim_{t \to \infty} \int_{\mathbb{R}^N} u(x,t) \, dx = \int_{\mathbb{R}^N} u_0(x) \, dx + \int_0^\infty \int_{\mathbb{R}^N} |\nabla u(x,t)|^p \, dx \, dt \quad (1.2)
\]

such that

\[
\lim_{t \to \infty} t^{\frac{N}{p} - \frac{1}{2} + \frac{1}{q}} \|\nabla^j [u(t) - C_\ast G(t)]\|_{L^q(\mathbb{R}^N)} = 0, \quad q \in [1, \infty], \quad j = 0, 1, \quad (1.3)
\]

where \( G(x,t) \) is the heat kernel.

(b) For the case \( p \in (p_c, 2) \) with \( p_c := (N + 2)/(N + 1) \), there exists a positive constant \( \varepsilon = \varepsilon(N,p) \) such that, if

\[
\|u_0\|_{L^1(\mathbb{R}^N)} \|\nabla u_0\|_{L^\infty(\mathbb{R}^N)}^{(N+1)p-(N+2)} < \varepsilon,
\]

then (1.3) holds true.

(ii) Assume that \( u_0 \leq 0 \). For any \( p > p_c \), (1.3) holds true.
For the case $\alpha \in (1, 2)$, Karch and Woyczyński [19] studied similar topics. They showed that, for any $u_0 \in W^{1, \infty}(\mathbb{R}^N)$, the problem (1.1) with $\alpha \in (1, 2)$ has a unique global-in-time mild solution. Furthermore, for the case $p > (N+\alpha)/(N+1)$, they proved that there exists a (mild) solution which behaves asymptotically like suitable multiples of the kernel of the integral equation. For notions of another weak solutions, Droniou and Imbert [9] constructed a unique global-in-time viscosity solution in $W^{1, \infty}(\mathbb{R}^N)$ for the case $\alpha \in (0, 2)$ (see also [11, 27]).

On the other hand, the case $\alpha = 1$ is completely different from the case $\alpha \in (1, 2)$. In fact, for the case $\alpha \in (0, 2]$, the semigroup $e^{-t(-\Delta)^{\alpha/2}}$ satisfies the following decay estimates
\[
\|\nabla^j e^{-t(-\Delta)^{\alpha/2}} f\|_{L^q(\mathbb{R}^N)} \leq C t^{-\frac{N}{2}(1-\frac{j}{\alpha})-\frac{j}{\alpha}} \|f\|_{L^1(\mathbb{R}^N)}, \quad q \in [1, \infty], \quad j = 0, 1,
\]
for all $t > 0$ (see, for example, [13]). For the case $\alpha \in (1, 2]$, since $t^{-1/\alpha}$ is integrable locally, we can easily prove the existence of local-in-time mild solutions in $W^{1, \infty}(\mathbb{R}^N)$ (see [19, Proposition 3.1]). However, for the case $\alpha = 1$, since $t^{-1}$ is not integrable, we need to impose the regularity of one order derivative on the solution. In this sense the value $\alpha = 1$ is critical. Similar situation appears in the fractional Burgers equation,
\[
\partial_t u + u \partial_x u + (-\partial_{xx})^{\alpha/2} u = 0, \quad x \in \mathbb{R}, \quad t > 0. \tag{1.4}
\]
For (1.4), the value $\alpha = 1$ is a threshold for the occurrence of singularity in finite time or the global regularity (see [1, 7, 8, 21]). In [16], the first author of this paper studied (1.4) with $\alpha = 1$, and constructed a small global-in-time mild solution in the Besov space $\dot{B}^{0}_{\infty, 1}(\mathbb{R})$ which is the critical space under the scaling invariance (see also [26]). Furthermore, he proved that, for small initial data in $L^1(\mathbb{R}) \cap \dot{B}^{0}_{\infty, 1}(\mathbb{R})$, the corresponding solution behaves like the Poisson kernel as $t \to \infty$.

In this paper, modifying the argument in [16], we show that there exists a global-in-time mild solution of the problem (1.1) with $\alpha = 1$ in the critical Besov space. Furthermore, we prove that global-in-time solutions with some suitable decay estimates behave asymptotically like suitable multiples of the Poisson kernel.

We introduce some notations. Throughout this paper we put $\mathcal{L} := -(-\Delta)^{1/2}$ for simplicity. Let $P_t$ be the Poisson kernel, that is,
\[
P_t(x) := t^{-N} P(x/t), \quad x \in \mathbb{R}^N, \quad t > 0,
\]
where $P$ is defined by
\[
P(x) := \mathcal{F}^{-1}[e^{-|\xi|^2}](x) = c_N (1 + |x|^2)^{-(N+1)/2}, \quad x \in \mathbb{R}^N,
\]
and $c_N$ is a constant chosen so that
\[
\int_{\mathbb{R}^N} P(x) \, dx = 1. \tag{1.5}
\]
Then, for all $t > 0$, $e^{t\mathcal{L}}$ denotes the convolution operator with $P_t$, that is,
\[
[e^{t\mathcal{L}} f](x) := \int_{\mathbb{R}^N} P_t(x-y) f(y) \, dy, \quad x \in \mathbb{R}^N, \quad t > 0. \tag{1.6}
\]
and $f$ is a measurable function. For $q \in [1, \infty]$, we denote by $\| \cdot \|_{L^q}$ the usual norm of $L^q := L^q(\mathbb{R}^N)$. Furthermore, for $s \in \mathbb{R}$, $q \in [1, \infty]$ and $\sigma \in (0, \infty]$, we denote by $\| \cdot \|_{B^s_{q,\sigma}}$ and $\| \cdot \|_{\dot{B}^s_{q,\sigma}}$ the usual norm of inhomogeneous and homogeneous Besov spaces $B^s_{q,\sigma} := B^s_{q,\sigma}(\mathbb{R}^N)$ and $\dot{B}^s_{q,\sigma} := \dot{B}^s_{q,\sigma}(\mathbb{R}^N)$, respectively. (See Section 2 for more precise details.)

Now we are ready to state the main result of this paper. We consider the integral equation corresponding to (1.1) with $\alpha = 1$, that is,

$$u(t) = e^{t\mathcal{L}}u_0 + \int_0^t e^{(t-\tau)\mathcal{L}}|\nabla u(\tau)|^p \, d\tau, \quad t \geq 0,$$

and obtain the following result.

**Theorem 1.1** Let $N \geq 1$, $p > 1$ and $r \in [1, \infty]$. Assume $u_0 \in B^1_{q,1}$ for all $q \in [r, \infty]$. Then the following hold.

(i) There exists a positive constant $\delta = \delta(N, p)$ such that, if

$$\|u_0\|_{\dot{B}^1_{\infty,1}} \leq \delta,$$

then there exists a unique global-in-time solution $u$ of (1.1) satisfying

$$u \in C([0, \infty), B^1_{q,1}) \cap L^1(0, \infty; \dot{B}^2_{q,1}),$$

$$\sup_{t \geq 0} (1 + t)^N(\frac{1}{r} - \frac{1}{q})^+ \|\nabla^j u(t)\|_{L^q} < \infty,$$

$$\int_0^\infty t^{N(\frac{1}{r} - \frac{1}{q})^+} \|u(t)\|_{\dot{B}^2_{q,1}} \, dt < \infty,$$

for all $q \in [r, \infty]$ and $j = 0, 1$.

(ii) Let $v$ be a global-in-time solution of (1.1) satisfying (1.9) and (1.10). Then for any $j \in \{0, 1\}$, the following hold.

(a) If $1 < r < \infty$, then

$$t^{N(\frac{1}{r} - \frac{1}{q})^+} \|\nabla^j [v(t) - e^{t\mathcal{L}}u_0]\|_{L^q} = \begin{cases}
O(t^{-N(r-1)\frac{N}{r}}) & \text{if } p \geq r, \\
O(t^{-N(r-1)\frac{N}{r}}) & \text{if } p < r,
\end{cases}$$

as $t \to \infty$, for any $q \in [r, \infty]$.

(b) If $r = 1$, then the limit $C_*$ given in (1.2) exists and

$$\lim_{t \to \infty} t^{N(1-\frac{1}{r})^+} \|\nabla^j [v(t) - C_*P_{t+1}]\|_{L^q} = 0, \quad 1 \leq q \leq \infty.$$
Remark 1.1 (i) Let \( u \) be a mild solution \( u \) of (1.1) with \( \alpha = 1 \), i.e., solution of (1.7). For any \( \lambda > 0 \), put

\[
u_\lambda(x,t) := \lambda^{-1} u(\lambda x, \lambda t), \quad u_{0,\lambda}(x) := \lambda^{-1} u_0(\lambda x).
\]

Then the function \( u_\lambda \) is also a solution of (1.1) with \( \alpha = 1 \) and the initial function \( u_{0,\lambda} \) satisfying

\[C^{-1}\|u_0\|_{B_{1,1}^1} \leq \|u_{0,\lambda}\|_{B_{1,1}^1} \leq C\|u_0\|_{B_{1,1}^1},\]

where \( C \) is a positive constant independent of \( \lambda \). This means that the condition (1.8) is invariant with respect to the similarity transformation (1.13). This is the reason why we say that \( B_{1,1}^1 \) is the critical Besov space with respect to (1.1).

(ii) In the assertion (ii) of Theorem 1.1, if we only consider the case \( j = 0 \), then we can remove the assumption that the solution \( u \) satisfies (1.10). See Section 5.

(iii) As is seen from our proof, it is possible to replace (1.10) with

\[
\int_0^\infty t^{N(\frac{1}{p} - \frac{1}{q}) + \beta} \|u(t)\|_{B_{q,1}^1} dt < \infty,
\]

where \( 1/p < \beta < 1 \). We also note focusing on the linear part that for \( \beta = 1 \), the maximal regularity estimate and the embedding implies that

\[
\int_0^\infty t^{N(\frac{1}{p} - \frac{1}{q}) + 1} \|e^{t\mathcal{L}} u_0\|_{B_{q,1}^1} dt \leq C\|u_0\|_{B_{q,1}^1}, \quad B_{1,1}^1 \subset B_{r,1}^0,
\]

and one cannot expect the time decay with \( \beta = 1 \) for initial data in \( B_{1,1}^1 \). Therefore, the expected maximal decay order is given as the case \( \beta < 1 \) expect for \( \beta = 1 \), and the case \( \beta = 1/p \) is a sufficient decay to prove the asymptotic behavior.

(iv) By the embedding \( B_{1,1}^1 \subset C^1 \) and \((-\Delta)^{1/2} f \in C(\mathbb{R}^N) \) for \( f \in B_{1,1}^1 \), the solution \( u \) in Theorem 1.1(i) satisfies the problem (1.1) in the classical sense. We also see that \( u(t) \) is in the class \( C^2 \) for almost every \( t \) since \( u \in L^1(0, \infty; B_{1,1}^2) \). Compared with the results [9, 11, 27], our framework in the Besov spaces is the one with higher regularity than theirs, since their initial data are in \( W^{1,\infty} \) and solutions are considered in the sense of viscosity solutions and \( B_{1,1}^1 \subset W^{1,\infty} \).

(v) In Theorem 1.1(ii)-(a), it is possible to prove that

\[
\lim_{t \to \infty} t^{N(\frac{1}{p} - \frac{1}{q}) + j} \|\nabla^j u(t)\|_{L^p} = 0
\]

since one can show \( t^{N(\frac{1}{p} - \frac{1}{q}) + j} \|\nabla^j e^{t\mathcal{L}} u_0\|_{L^p} = o(1) \) as \( t \to \infty \) for any \( u_0 \in L^p \) by the density argument due to \( C_0^\infty \subset L^p \).

This paper is organized as follows. In Section 2, we give the definition of the Besov spaces, its properties and estimates for the nonlinearity \( |\nabla u|^p \). We also introduce the linear estimates for \( e^{t\mathcal{L}} f \) in the Lebesgue spaces and the Besov spaces. Sections 3 and 4 are devoted to the proof of the assertions (i) and (ii) in Theorem 1.1, respectively.
2 Preliminary

In this section we prove some estimates in the Besov spaces and recall some preliminary results on $e^{t\mathcal{L}}f$. In what follows, for any two nonnegative functions $f_1$ and $f_2$ on a subset $D$ of $[0, \infty)$, we say

$$f_1(t) \preceq f_2(t), \quad t \in D$$

if there exists a positive constant $C$ such that $f_1(t) \leq Cf_2(t)$ for all $t \in D$. In addition, we say

$$f_1(t) \asymp f_2(t), \quad t \in D$$

if $f_1(t) \preceq f_2(t)$ and $f_2(t) \preceq f_1(t)$ for all $t \in D$. We denote the function spaces of rapidly decreasing functions by $S(\mathbb{R}^N)$ and tempered distributions by $S'(\mathbb{R}^N)$. We define $\mathcal{Z}(\mathbb{R}^N)$ by

$$\mathcal{Z}(\mathbb{R}^N) := \left\{ f \in S(\mathbb{R}^N) \mid \int_{\mathbb{R}^N} x^\alpha f(x) dx = 0 \text{ for all } \alpha \in \mathbb{N}^N \right\}$$

with the topology of $S(\mathbb{R}^N)$, and $\mathcal{Z}'(\mathbb{R}^N)$ by the topological dual of $\mathcal{Z}(\mathbb{R}^N)$. We first give the definition of the inhomogeneous and homogeneous Besov spaces (see Triebel [29]).

**Definition 2.1** Let $\phi \in S(\mathbb{R}^N)$ satisfy

$$\text{supp} \mathcal{F}[\phi] \subset \{ \xi \in \mathbb{R}^N \mid 2^{-1} \leq |\xi| \leq 2 \}, \quad \sum_{j \in \mathbb{Z}} \mathcal{F}[\phi](2^{-j} \xi) = 1 \text{ for any } \xi \in \mathbb{R}^N \setminus \{0\},$$

Let $\{\phi_j\}_{j \in \mathbb{Z}}$ and $\psi$ be defined by

$$\phi_j(x) := 2^{Nj}\phi(2^j x), \quad \psi(x) = \mathcal{F}^{-1} \left[ 1 - \sum_{j \geq 1} \mathcal{F}[\phi_j] \right](x)$$

For $s \in \mathbb{R}$, $q \in [1, \infty]$ and $\sigma \in (0, \infty]$, we define the following.

(i) The inhomogeneous Besov space $B^s_{q, \sigma}$ is defined by

$$B^s_{q, \sigma} := \left\{ u \in \mathcal{S}'(\mathbb{R}^N) \mid \|u\|_{B^s_{q, \sigma}} < \infty \right\},$$

where

$$\|u\|_{B^s_{q, \sigma}} := \begin{cases} \|\psi * u\|_{L^q} + \left\{ \sum_{j \geq 1} \left( 2^{js}\|\phi_j * u\|_{L^q} \right)^\sigma \right\}^{1/\sigma} & \text{if } 0 < \sigma < \infty, \\ \|\psi * u\|_{L^q} + \sup_{j \geq 1} 2^{js}\|\phi_j * u\|_{L^q} & \text{if } \sigma = \infty. \end{cases}$$

(ii) The homogeneous Besov space $\dot{B}^s_{q, \sigma}$ is defined by

$$\dot{B}^s_{q, \sigma} := \left\{ u \in \mathcal{Z}'(\mathbb{R}^N) \mid \|u\|_{\dot{B}^s_{q, \sigma}} < \infty \right\},$$

where

$$\|u\|_{\dot{B}^s_{q, \sigma}} := \begin{cases} \left\{ \sum_{j \in \mathbb{Z}} \left( 2^{js}\|\phi_j * u\|_{L^q} \right)^\sigma \right\}^{1/\sigma} & \text{if } 0 < \sigma < \infty, \\ \sup_{j \in \mathbb{Z}} 2^{js}\|\phi_j * u\|_{L^q} & \text{if } \sigma = \infty. \end{cases}$$
Remark 2.1 It is known that \( Z(\mathbb{R}^N) \subset S(\mathbb{R}^N) \subset S'(\mathbb{R}^N) \subset Z'(\mathbb{R}^N) \) and \( Z'(\mathbb{R}^N) \simeq S'(\mathbb{R}^N)/P(\mathbb{R}^N) \), where \( P(\mathbb{R}^N) \) is the set of all polynomials, and the homogeneous Besov spaces can also be considered as subspaces of the quotient space \( S'(\mathbb{R}^N)/P(\mathbb{R}^N) \). Then we use the following equivalence, which is due to the argument by e.g. Kozono and Yama- 

zaki [22], for the nonlinear term in (1.1) to construct solutions in the homogeneous spaces with \( u(t) \in S'(\mathbb{R}^N) \). If \( s < n/q \) or \( (s, \sigma) = (n/q, 1) \), then the homogeneous Besov space \( \dot{B}^s_{q, \sigma} \) is regarded as \( \left\{ u \in S'(\mathbb{R}^N) \left| \| u \|_{\dot{B}^s_{q, \sigma}} < \infty, u = \sum_{j \in \mathbb{Z}} \phi_j * u \text{ in } S'(\mathbb{R}^N) \right. \right\} \).

Hence, \( u \in \dot{B}^s_{q, \sigma} \) can be regarded as an element of \( S'(\mathbb{R}^N) \). We also see from the analogous argument to theirs that \( \nabla u \) can be regarded as an element of \( S'(\mathbb{R}^N) \) if \( u \in \dot{B}^s_{\infty, 1} \) with \( s \leq 1 \). This is used for the nonlinear term \( |\nabla u|^p \) when we construct global solutions.

Next we give some interpolation inequalities in the Besov spaces.

Lemma 2.1 Let \( s \in \mathbb{R}, \alpha, \beta > 0, q \in [1, \infty] \) and \( \sigma \in (0, \infty] \). Then it holds that

\[
\| f \|_{\dot{B}^s_{q, \sigma}} \leq \| f \|_{\dot{B}^s_{q, \infty}}^{\alpha} \| f \|_{\dot{B}^s_{\infty, \infty}}^{\beta} \| f \|_{\dot{B}^s_{\infty, \infty}}^{\alpha - \beta} \tag{2.1}
\]

for all \( f \in \dot{B}^{s+\alpha}_{q, \infty} \cap \dot{B}^{s-\beta}_{q, \infty} \).

Proof. The estimate (2.1) is known for the case \( 1 \leq \sigma \leq \infty \) by the result of Machihara-Ozawa [25]. The case \( 0 < \sigma < 1 \) follows from the analogous argument to their proof, thus the proof is left to readers. \( \square \)

The following proposition is on the equivalence between the norm of the Besov spaces defined as above and that by differences (see Triebel [29]).

Proposition 2.1 Let \( s > 0, q \in [1, \infty] \) and \( \sigma \in (0, \infty] \). If \( M \in \mathbb{N} \) satisfies \( M > s \), then there holds that

\[
\| f \|_{\dot{B}^s_{q, \sigma}} \asymp \left\{ \int_{\mathbb{R}^N} \left( \| \nabla^M f \|_{L^q} \right)^{\sigma} d\eta \right( \frac{d\eta}{|\eta|^N} \right)^{1/\sigma} \tag{2.2}
\]

for all \( f \in \dot{B}^s_{q, \sigma} \), where \( \nabla_y f(x) := f(x + y) - f(x) \) and \( \nabla^M_y f := (\nabla_y)^M f \).

By using Proposition 2.1 we have the following.

Lemma 2.2 Let \( p, s \) and \( \varepsilon \) satisfy \( p > 1, 0 < s < \min\{2, p\} \) and \( 0 < \varepsilon < \min\{1, p - 1\} \).
Then, for any $q \in [1, \infty]$, 
\[ \|f^p\|_{\dot{B}^q_{\infty,1}} \leq \|f\|_{\dot{B}^0_{\infty,1}}^{p-1} \|f\|_{\dot{B}^q_{\infty,1}}, \] 
\[ \|f^p - |g|^p\|_{\dot{B}^q_{\infty,1}} \leq (\|f\|_{\dot{B}^0_{\infty,1}}^{p-1} + \|g\|_{\dot{B}^0_{\infty,1}}^{p-1}) \|f - g\|_{\dot{B}^q_{\infty,1}}, \] 
\[ \begin{cases} 
\left( \|f\|_{\dot{B}^0_{\infty,1}}^{p-1} \|f\|_{\dot{B}^1_{\infty,1}} + \|g\|_{\dot{B}^0_{\infty,1}}^{p-1} \|g\|_{\dot{B}^1_{\infty,1}} \right) \|f - g\|_{\dot{B}^q_{\infty,1}}, & \text{if } 1 < p < 2, \\
\left( \|f\|_{\dot{B}^0_{\infty,1}}^{p-2} + \|g\|_{\dot{B}^0_{\infty,1}}^{p-2} \right) \left( \|f\|_{\dot{B}^1_{\infty,1}} + \|g\|_{\dot{B}^1_{\infty,1}} \right) \|f - g\|_{\dot{B}^q_{\infty,1}}, & \text{if } p \geq 2,
\end{cases} \] 
for all $f, g \in \dot{B}^0_{\infty,1} \cap \dot{B}^1_{\infty,1} \cap \dot{B}^0_{q,1} \cap \dot{B}^0_{q,1}$. 

In order to prove this lemma, we use the following fundamental inequality.

**Lemma 2.3** Let $p > 1$. Then, for any $A, B, C, D \in \mathbb{R}$, 
\[ \|A^p - B^p - (|C|^p - |D|^p)\| \]
\[ \leq (\|C\|^{p-1} + |D|^{p-1})|A - B - (C - D)| \]
\[ + \begin{cases} 
\|A - C\|^{p-1} |A - B|, & \text{if } 1 < p < 2, \\
\|A - C\|^{p-2} |A - B|, & \text{if } p \geq 2.
\end{cases} \] 

**Proof.** Let $p > 1$ and $A, B, C, D \in \mathbb{R}$. It follows from the fundamental theorem of calculus that 
\[ |A|^p - |B|^p - (|C|^p - |D|^p) \]
\[ = \int_0^1 \left\{ \partial_\theta |\theta A + (1 - \theta)B|^p - \partial_\theta |\theta C + (1 - \theta)D|^p \right\} d\theta \]
\[ = \int_0^1 \left\{ |\theta A + (1 - \theta)B|^{p-2} (\theta A + (1 - \theta)B) (A - B) - |\theta C + (1 - \theta)D|^{p-2} (\theta C + (1 - \theta)D) (C - D) \right\} d\theta \] 
\[ = \int_0^1 \left\{ [|\theta A + (1 - \theta)B|^{p-2} (\theta A + (1 - \theta)B) - |\theta C + (1 - \theta)D|^{p-2} (\theta C + (1 - \theta)D)] (A - B) + |\theta C + (1 - \theta)D|^{p-2} (\theta C + (1 - \theta)D) (A - B - (C - D)) \right\} d\theta. \]

Furthermore, we have 
\[ \|E^{p-2}E - |F|^{p-2}F\| \leq C \begin{cases} 
\|E - F|^{p-1}, & \text{if } 1 < p < 2, \\
(\|E|^{p-2} + |F|^{p-2})|E - F|, & \text{if } p \geq 2.
\end{cases} \]
for any $E, F \in \mathbb{R}$. This together with (2.6) yields (2.5). Thus Lemma 2.3 follows. □

**Proof of Lemma 2.2.** For the proof of (2.3), we utilize the equivalent norm (2.2) of the Besov spaces $\dot{B}^{s}_{q,1}$ by differences, and it suffices to estimate the following

$$\int_{\mathbb{R}^N} \left( |y|^{-s} \sup_{|y| \leq |\eta|} \left\| \triangle_y^2 f \right\|_{L^p} \right) \frac{d\eta}{|\eta|^N}.$$

In order to estimate $\triangle_y^2 f^p$, we apply Lemma 2.2. Put

$$A = f(x + 2y), \quad B = C = f(x + y), \quad D = f(x),$$

and we note that

$$A - B = (\triangle_y f)(x + y), \quad C - D = (\triangle_y f)(x), \quad A - B - (C - D) = (\triangle_y^2 f)(x).$$

In the case $1 < p < 2$, by (2.5) and the Hölder inequality we have

$$\left\| \triangle_y^2 f \right\|_{L^p} \leq \left\| f \right\|_{L^{p-1}} \left\| \triangle_y f \right\|_{L^q} + \left( \left\| \triangle_y f \right\|_{L^q} \right)^{p-1} \left\|^p \right\|_{L^{p-1}} \left\| \triangle_y f \right\|_{L^{pq}} \quad (2.7)$$

On the estimate of $I_1$, we get

$$\int_{\mathbb{R}^N} \left( |y|^{-s} \sup_{|y| \leq |\eta|} I_1(y) \right) \frac{d\eta}{|\eta|^N} \leq \left\| f \right\|_{L^{p-1}} \int_{\mathbb{R}^N} \left( |y|^{-s} \sup_{|y| \leq |\eta|} \left\| \triangle_y f \right\|_{L^q} \right) \frac{d\eta}{|\eta|^N} \leq \left\| f \right\|_{B^{0}_{q,1}} \left\| f \right\|_{B^{s}_{q,1}}. \quad (2.8)$$

On the estimate of $I_2$, applying the Hölder inequality and the embedding $\dot{B}^{0}_{q,1} \hookrightarrow \dot{B}^{0}_{\infty,\infty}$, we have

$$\int_{\mathbb{R}^N} \left( |y|^{-s} \sup_{|y| \leq |\eta|} I_2(y) \right) \frac{d\eta}{|\eta|^N} = \int_{\mathbb{R}^N} \left( |y|^{-\frac{p}{q'}} \sup_{|y| \leq |\eta|} \left\| \triangle_y f \right\|_{L^p} \right)^p \frac{d\eta}{|\eta|^N} \leq \left\| f \right\|_{B^{p q, p}_{q,1}} \leq \left\| f \right\|_{B^{0}_{q,1}} \left\| f \right\|_{B^{s}_{q,1}} \quad (2.9)$$

In the case $p \geq 2$, by (2.5) and the Hölder inequality again we obtain

$$\left\| \triangle_y^2 f \right\|_{L^p} \leq \left\| f \right\|_{L^{p-1}} \left\| \triangle_y f \right\|_{L^q} + \left\| f \right\|_{L^q} \left( \left\| \triangle_y f \right\|_{L^{p-1}} \right)^{p-2} \left\|^2 \right\|_{L^{q}} \times (\left\| \triangle_y f \right\|_{L^{q}} + \left\| \triangle_y f \right\|_{L^{q}}) \leq \left\| f \right\|_{L^{p-1}} \left\| \triangle_y f \right\|_{L^q} + \left\| f \right\|_{L^{q}} \left\| \triangle_y f \right\|_{L^{q}} \quad (2.10)$$

$$= I_1(y) + I_3(y).$$
For the estimate of $I_3$, it follows from the same estimate as (2.9) with taking $p = 2$ for $\|\triangle_y f\|_{L^q}$ that

$$\int_{\mathbb{R}^N} |\eta|^{-\varepsilon} \sup_{|y| \leq |\eta|} J_3(y) \frac{d\eta}{|\eta|^N} \leq \|f\|_{B^{p-1}_{\infty,1}}^{p-2} \left( \|f\|_{B^s_{q,1}} \|f\|_{B^p_{q,1}} \right) = \|f\|_{B^{p-1}_{\infty,1}} \|f\|_{B^s_{q,1}}. \quad (2.11)$$

Combining (2.7), (2.8), (2.9), (2.10) and (2.11), the estimate (2.3) holds.

For the proof of (2.4), we also utilize the equivalent norm (2.2) of the Besov space $\dot{B}^s_{\infty,1}$ by differences, and it suffices to estimate the following

$$\int_{\mathbb{R}^N} \left( |\eta|^{-\varepsilon} \sup_{|y| \leq |\eta|} \|\triangle_y(|f|^p - |g|^p)\|_{L^q} \right) \frac{d\eta}{|\eta|^N}.$$ 

In order to estimate $\triangle_y(|f|^p - |g|^p)$, we also apply Lemma (2.2) Put

$$A = f(x + y), \quad B = g(x + y), \quad C = f(x), \quad D = g(x),$$

we note that

$$A - C = (\triangle_y f)(x), \quad B - D = (\triangle_y g)(x), \quad A - B - (C - D) = \triangle_y(f - g).$$

In the case $1 < p < 2$, by (2.5) and the Hölder inequality we have

$$\|\triangle_y(|f|^p - |g|^p)\|_{L^q} \leq (\|f\|_{L^\infty}^{p-1} + \|g\|_{L^\infty}^{p-1}) \|\triangle_y(f - g)\|_{L^q} + (\|\triangle_y f\|_{L^\infty}^{p-1} + \|\triangle_y g\|_{L^\infty}^{p-1}) \|f - g\|_{L^q} \quad (2.12)$$

$$=: J_1(y) + J_2(y).$$

On the estimate of $J_1$, we get

$$\int_{\mathbb{R}^N} |\eta|^{-\varepsilon} \sup_{|y| \leq |\eta|} J_1(y) \frac{d\eta}{|\eta|^N} \leq \left( \|f\|_{L^\infty}^{p-1} + \|g\|_{L^\infty}^{p-1} \right) \int_{\mathbb{R}^N} \left( |\eta|^{-\varepsilon} \sup_{|y| \leq |\eta|} \|\triangle_y(f - g)\|_{L^q} \right) \frac{d\eta}{|\eta|^N}$$

$$\leq (\|f\|_{B^{p-1}_{\infty,1}} + \|g\|_{B^{p-1}_{\infty,1}}) \|f - g\|_{\dot{B}^p_{q,1}}. \quad (2.13)$$

On the estimate of $J_2$, by (2.1) and the embeddings $\dot{B}^s_{q,1} \hookrightarrow \dot{B}^s_{q,\infty}$ ($s = 0, 1$) and $\dot{B}^0_{q,1} \hookrightarrow L^q$ we have

$$\int_{\mathbb{R}^N} |\eta|^{-\varepsilon} \sup_{|y| \leq |\eta|} J_2(y) \frac{d\eta}{|\eta|^N}$$

$$\leq \int_{\mathbb{R}^N} \left( |\eta|^{-\varepsilon} \sup_{|y| \leq |\eta|} \left( \|\triangle_y f\|_{L^\infty} + \|\triangle_y g\|_{L^\infty} \right) \right)^{p-1} \frac{d\eta}{|\eta|^N} \|f - g\|_{L^q}$$

$$\leq \left\{ \|f\|_{B^{p-1}_{\infty,\infty}} \|f\|_{B^p_{\infty,\infty}}^{1-\varepsilon} + \left( \|g\|_{B^{p-1}_{\infty,\infty}} \|g\|_{B^p_{\infty,\infty}} \right)^{p-1} \right\} \|f - g\|_{L^q}$$

$$\leq \left( \|f\|_{B^{p-1}_{\infty,1}}^{1-\varepsilon} \|f\|_{B^p_{\infty,1}} + \|g\|_{B^{p-1}_{\infty,1}}^{1-\varepsilon} \|g\|_{B^p_{\infty,1}} \right) \|f - g\|_{B^0_{q,1}}. \quad (2.14)$$
In the case $p \geq 2$, by (2.5) and the Hölder inequality again we obtain

$$
\|\triangle_y(|f|^p - |g|^p)\|_{L^q} \leq (\|f\|_{L^\infty}^{p-1} + \|g\|_{L^\infty}^{p-1})\|\triangle_y(f - g)\|_{L^q} + (\|f\|_{L^\infty}^{p-2} + \|g\|_{L^\infty}^{p-2})\|\triangle_y f\|_{L^\infty} + \|\triangle_y g\|_{L^\infty})\|f - g\|_{L^q}
$$

(2.15)

For the estimate of $J_3$, it follows from (2.1) and the embeddings $\dot{B}^s_q \hookrightarrow \dot{B}^s_{q,\infty}$ $(s = 0, 1)$ and $\dot{B}^0_{q,1} \hookrightarrow L^q$ that

$$
\int_{\mathbb{R}^N} |\eta|^{-\varepsilon} \sup_{|y| \leq |\eta|} J_3(y) \frac{d\eta}{|\eta|^N} \leq (\|f\|_{L^\infty}^{p-2} + \|g\|_{L^\infty}^{p-2}) \left(\|\triangle_y f\|_{L^\infty} + \|\triangle_y g\|_{L^\infty}\right) \frac{d\eta}{|\eta|^N} \|f - g\|_{L^q}
$$

(2.16)

Therefore, (2.4) is obtained by (2.12), (2.13), (2.14), (2.15) and (2.16).

The end of this section we recall some results on $e^{t\mathcal{L}} f$.

**Lemma 2.4** [13, 16] Let $s \in \mathbb{R}$ and $q \in [1, \infty]$.  
(i) For $j = 0, 1$, $\alpha \geq 0$, $r \in [1, q]$ and $\sigma \in [1, \infty]$, it holds that

$$
\|\nabla^j P_{t+1} f\|_{L^q} \leq (1 + t)^{-N(1-\frac{1}{q})-j},
$$

(2.17)

$$
\|\nabla^j e^{t\mathcal{L}} f\|_{L^q} \leq t^{-N(1-\frac{1}{q})-j} \|f\|_{L^r},
$$

(2.18)

$$
\|e^{t\mathcal{L}} f\|_{\dot{B}^s_{q,\sigma}} \leq t^{-N(1-\frac{1}{q})-\alpha} \|f\|_{\dot{B}^s_{r,\sigma}}.
$$

(2.19)

(ii) It holds that

$$
\|e^{t\mathcal{L}} f\|_{L^1_t(0,\infty;\dot{B}^s_{q,1})} \leq \|f\|_{\dot{B}^s_{q,1}}.
$$

(2.20)

(iii) It holds that

$$
\left\| \int_0^t e^{(t-\tau)\mathcal{L}} f(\tau) \, d\tau \right\|_{L^1_t(0,\infty;\dot{B}^s_{q,1})} \leq \|f\|_{L^1(0,\infty;\dot{B}^s_{q,1})}.
$$

(2.21)

**Lemma 2.5** ([Proposition 3.1] [14]). For any $f \in L^1$, if

$$
\int_{\mathbb{R}^N} f(x) \, dx = 0,
$$

then

$$
\lim_{t \to \infty} \|e^{t\mathcal{L}} f\|_{L^1} = 0.
$$
3 Existence of global-in-time solutions and Decay estimates

In this section we prove the assertion (i) of Theorem 1.1. We apply the contraction mapping principle in a suitable complete metric space. Let \( \Psi(u) \) be defined by

\[
\Psi(u)(t) := e^{tL}u_0 + \int_0^t e^{(t-\tau)L} |\nabla u(\tau)|^p \, d\tau,
\]

and we define the following norms

\[
\|u\|_{\dot{X}_q^s} := \sup_{t>0} \|u(t)\|_{\dot{B}^{s+1}_{q,1}} + \int_0^\infty \|u(t)\|_{\dot{B}^{s+1}_{q,1}} \, dt,
\]

\[
\|u\|_{\dot{Y}_q^s} := \|u\|_{\dot{X}_q^s} + \sup_{t>0} t^{N(\frac{1}{r} - \frac{1}{q} + 1 - s)} \|u(t)\|_{\dot{B}^{1}_{r,1}} + \int_0^\infty t^{N(\frac{1}{r} - \frac{1}{q} + 1 - s)} \|u(t)\|_{\dot{B}^{2}_{r,1}} \, dt.
\]

\( \dot{X}_q^s \) with the norm \( \|u\|_{\dot{X}_q^s} \) is defined by the space of all functions \( u \) such that

\[
u \in L^\infty(0, \infty; \dot{B}^{s+1}_{q,1}) \cap L^1(0, \infty; \dot{B}^{s+1}_{q,1}) \quad \text{and} \quad \|u\|_{\dot{X}_q^s} < \infty,
\]

and \( \dot{Y}_q^s \) with the norm \( \|\cdot\|_{\dot{Y}_q^s} \) is also defined by the space of all functions \( u \) such that

\[
u \in \dot{X}_q^s \quad \text{and} \quad \|u\|_{\dot{Y}_q^s} < \infty.
\]

Here, let \( \varepsilon \) and \( \lambda \) be fixed constants satisfying

\[
0 < \varepsilon < \min\{1, p-1\} \quad \text{and} \quad 0 < \lambda < 1,
\]

and we introduce the following metric space \( \mathcal{X} \)

\[
\mathcal{X} := \{ u \in \dot{X}_r^\varepsilon \cap \dot{X}_\infty^\varepsilon \cap \dot{X}_r^1 \cap \dot{X}_\infty^1 \mid \|u\|_{\dot{X}_q^s} \leq 2C_0\|u_0\|_{\dot{B}^{1}_{q,1}} \text{ for any } q \in [r, \infty],
\]

\[
\|u\|_{\dot{Y}_r^\lambda \cap \dot{Y}_\infty^\lambda} \leq 2C_0\|u_0\|_{\dot{B}^{\lambda}_{r,1} \cap \dot{B}^{\lambda}_{\infty,1}} \},
\]

with the metric

\[
d(u, v) := \|u - v\|_{\dot{X}_r^\varepsilon \cap \dot{X}_\infty^\varepsilon},
\]

where \( C_0 \) will be taken later. We first show that \( \mathcal{X} \) is a complete metric space.

**Lemma 3.1** \( \mathcal{X} \) is a complete metric space.

**Proof.** It is easy to see that \( \mathcal{X} \) is a metric space, then we show the completeness only. Let \( \{u_n\} \) be a Cauchy sequence in \( \mathcal{X} \). Since \( \dot{X}_r^\varepsilon \cap \dot{X}_\infty^\varepsilon \) is complete, there exists \( u \in \dot{X}_r^\varepsilon \cap \dot{X}_\infty^\varepsilon \) such that \( u_n \) converges to \( u \) in \( \dot{X}_r^\varepsilon \cap \dot{X}_\infty^\varepsilon \) as \( n \to \infty \). Then we also have

\[
\lim_{n \to \infty} \|\phi_j * (u_n(t) - u(t))\|_{L^r \cap L^\infty} = 0 \quad \text{for almost every } t \text{ and } j \in \mathbb{Z},
\]

\[
\lim_{n \to \infty} \int_0^L \|\phi_j * (u_n(t) - u(t))\|_{L^r \cap L^\infty} \, dt = 0 \quad \text{for any } L > 0 \text{ and } j \in \mathbb{Z}.
\]
There imply that, for any \( L > 0 \),
\[
\lim_{n \to \infty} \sum_{|j| \leq L} 2^j \| \phi_j * u_n(t) \|_{L^\infty} = \sum_{|j| \leq L} 2^j \| \phi_j * u(t) \|_{L^\infty},
\]
\[
\lim_{n \to \infty} t^{N(\frac{1}{r} - \frac{1}{q})+1-\lambda} \sum_{|j| \leq L} 2^j \| \phi_j * u_n(t) \|_{L^q} = t^{N(\frac{1}{r} - \frac{1}{q})+1-\lambda} \sum_{|j| \leq L} 2^j \| \phi_j * u_n(t) \|_{L^q}
\]
for almost every \( t \), and
\[
\lim_{n \to \infty} \int_0^L \sum_{|j| \leq L} 2^j \| \phi_j * u_n(t) \|_{L^\infty} dt = \int_0^L \sum_{|j| \leq L} 2^j \| \phi_j * u(t) \|_{L^\infty} dt,
\]
\[
\lim_{n \to \infty} \int_0^L t^{N(\frac{1}{r} - \frac{1}{q})+1-\lambda} \sum_{|j| \leq L} 2^j \| \phi_j * u_n(t) \|_{L^\infty} dt
\]
\[
= \int_0^L t^{N(\frac{1}{r} - \frac{1}{q})+1-\lambda} \sum_{|j| \leq L} 2^j \| \phi_j * u(t) \|_{L^\infty} dt.
\]

The terms in right hand side of the above four equalities are bounded uniformly with respect to \( L > 0 \) since \( \{u_n\} \subset X \), and they are monotone increasing, so that, they converges as \( L \to \infty \). Then we deduce that \( u \) satisfies
\[
\|u\|_{\dot{X}^{\lambda}} \leq 2C_0 \|u_0\|_{\dot{B}^{\lambda}_{q,1}} \quad \text{for any } q \in [r, \infty) \quad \text{and} \quad \|u\|_{Y^{\lambda} \cap \dot{Y}^{\lambda}} \leq 2C_0 \|u_0\|_{\dot{B}^{\lambda}_{q,1} \cap \dot{B}^{\lambda}_{q,1}},
\]
hence, \( u \in X \). Therefore the completeness of \( X \) follows. \( \square \)

In order to estimate the terms in (3.1), we prepare the following proposition.

**Proposition 3.1** Let \( p, q, r, \varepsilon \) and \( \lambda \) satisfy \( p > 1, 1 \leq r \leq q \leq \infty \) and (3.2). Then there holds that
\[
\|e^{\varepsilon \mathcal{L}} u_0\|_{\dot{X}^{\lambda}_{r}} \leq \|u_0\|_{\dot{B}^{\lambda}_{q,1}},
\]
\[
\|e^{\varepsilon \mathcal{L}} u_0\|_{Y^{\lambda}} \leq \|u_0\|_{\dot{B}^{\lambda}_{q,1}} + \|u_0\|_{\dot{B}^{\lambda}_{q,1}},
\]
\[
\left\| \int_0^t e^{(t-\tau)\mathcal{L}} |\nabla u(\tau)|^p d\tau \right\|_{L^q} \leq t \|u\|^{p-1}_{\dot{X}^{\lambda}_{\infty}} \|u\|_{\dot{X}^{\lambda}_{r}},
\]
\[
\left\| \int_0^t e^{(t-\tau)\mathcal{L}} |\nabla u(\tau)|^p d\tau \right\|_{\dot{X}^{\lambda}_{q}} \leq \|u\|^{p-1}_{\dot{X}^{\lambda}_{\infty}} \|u\|_{\dot{X}^{\lambda}_{r}},
\]
\[
\left\| \int_0^t e^{(t-\tau)\mathcal{L}} |\nabla u(\tau)|^p d\tau \right\|_{Y^{\lambda}_{q}} \leq \|u\|^{p-1}_{\dot{X}^{\lambda}_{\infty}} (\|u\|_{Y^{\lambda}} + \|u\|_{Y^{\lambda}}),
\]
\[
\left\| \int_0^t e^{(t-\tau)\mathcal{L}} (|\nabla u(\tau)|^p - |\nabla v(\tau)|^p) d\tau \right\|_{\dot{X}^{\lambda}_{q}} \leq (\|u\|^{p-1}_{\dot{X}^{\lambda}_{\infty}} + \|v\|^{p-1}_{\dot{X}^{\lambda}_{\infty}}) \|u - v\|_{\dot{X}^{\lambda}_{r}}.
\]

**Remark 3.1** We should note that the nonlinear term \( |\nabla u(\tau)|^p \) is in \( \mathcal{S}'(\mathbb{R}^N) \) if \( u \in \dot{X}^{\lambda}_{\infty} \). Although \( \dot{B}^{\lambda}_{q,1}(\mathbb{R}^N) \) is considered as a subspace of \( \mathcal{Z}(\mathbb{R}^N) \), \( \nabla u(\tau) \) is determined independently of the choice of representative elements in \( \dot{B}^{\lambda}_{q,1} \) by \( \nabla u(\tau) \in \dot{B}^{0}_{q,1}(\mathbb{R}^N) \) and Remark 2.1 hence, \( \nabla u(\tau) \in \mathcal{S}'(\mathbb{R}^N) \). We also see \( \nabla u(\tau)|^p \in L^\infty(\mathbb{R}^N) \subset \mathcal{S}'(\mathbb{R}^N) \) by \( \dot{B}^{0}_{q,1} \subset L^\infty \). In addition the estimate (3.4) implies that the term in the left hand side is in \( L^q \) if \( u \in X \).
Proof. The linear estimate (3.3) is verified by the use of (2.19) and (2.20). In fact, the estimates of \( \| e^{t\mathcal{L}}u_0 \|_{X_q^s} \) \((s = 1, \lambda)\) is obtained by the boundedness of \( e^{t\mathcal{L}} \), (2.19) and (2.20), and for the second and third terms in the definition of \( \| \cdot \|_{q,s} \) we also apply (2.19) and (2.20) to get

\[
t^N(\frac{1}{q} - \frac{1}{\ell}) + 1 - \lambda \| e^{t\mathcal{L}}u_0 \|_{B_{r,1}^1} \lesssim \| u_0 \|_{B_{r,1}^1},
\]

\[
\int_0^\infty t^N(\frac{1}{q} - \frac{1}{\ell}) + 1 - \lambda \| e^{t\mathcal{L}}u_0 \|_{B_{r,1}^2} dt \leq \int_0^\infty \| e^{\frac{1}{2}t\mathcal{L}}u_0 \|_{B_{r,1}^{1+\lambda}} dt \lesssim \| u_0 \|_{B_{r,1}^{1+\lambda}}.
\]

Then (3.3) is obtained.

The estimate (3.4) is obtained by applying the boundedness of \( e^{(t-\tau)\mathcal{L}} \) from \( L^q \) to itself, the Hölder inequality and the embedding \( B_{q,1}^0 \hookrightarrow L^q \). Thus we omit the detail.

To prove the nonlinear estimates (3.5), (3.6) and (3.7), we prepare the following nonlinear estimates that, for \( s = 1, \lambda, \)

\[
\| \nabla u \|_{B_{q,1}^s} \lesssim \| \nabla u \|_{B_{q,1}^{s-1}} \| \nabla u \|_{B_{q,1}^s} \lesssim \| u \|_{B_{q,1}^{s-1}} \| u \|_{B_{q,1}^s},
\]

(3.8)

\[
\| |\nabla u|^p - |\nabla v|^p \|_{B_{q,1}^s} \lesssim \| \nabla u \|_{B_{q,1}^{s-1}} \| \nabla u \|_{B_{q,1}^s} \lesssim \| u \|_{B_{q,1}^{s-1}} \| u \|_{B_{q,1}^s},
\]

(3.9)

which are obtained by (2.3), (2.4) and the interpolation inequality in the Besov spaces, that is,

\[
\| f \|_{B_{q,1}^s} = \sum_{j \in \mathbb{Z}} 2^j \| \phi_j * f \|_{L^q} = \sum_{j \in \mathbb{Z}} (2^j \| \phi_j \|_{L^1} \| f \|_{L^q}) \leq \| f \|_{B_{q,1}^s} \lesssim \| f \|_{B_{q,1}^s} \| f \|_{B_{q,1}^{1+\lambda}}.
\]

On the estimate of (3.5), by the boundedness of \( e^{(t-\tau)\mathcal{L}} \), (2.21), (3.8) with \( s = 1 \) and the Hölder inequality we have

\[
\left\| \int_0^t e^{(t-\tau)\mathcal{L}}|\nabla u(\tau)|^p d\tau \right\|_{X_q^s} \lesssim \int_0^\infty \| \nabla u(\tau) \|_{B_{q,1}^s} \| u \|_{L^1(0,\tau; B_{q,1}^s)} \lesssim \| u \|_{X_q^s} \| u \|_{X_q^s}.
\]

Then (3.5) is obtained.
On the estimate of (3.6), the first norm \( \| \cdot \|_{\dot{X}^{\lambda}} \) in the definition of \( \| \cdot \|_{\dot{Y}^{\lambda}} \) can be treated in the same way as the proof of (3.5) with (3.8) \((s = \lambda)\), thus we omit the estimate on \( \| \cdot \|_{\dot{X}^{\lambda}} \) to consider the second and third terms only. We put

\[
K_1(x, t) := \int_0^{t/2} e^{(t-\tau)\mathcal{L}} |\nabla u(\tau)|^p \, d\tau, \quad K_2(x, t) := \int_{t/2}^t e^{(t-\tau)\mathcal{L}} |\nabla u(\tau)|^p \, d\tau, \tag{3.10}
\]

for all \( x \in \mathbb{R}^N \) and \( t > 0 \). On the second term, by (2.19) and (3.8) we have

\[
t^N(\frac{1}{r} - \frac{1}{q}) + 1 - \lambda \|K_1(t)\|_{\dot{B}^{1,q}_1,0} \leq \int_0^{t/2} \|u(\tau)\|_{\dot{B}^{1,q}_1,1} \|u(\tau)\|_{\dot{B}^{\lambda+1}_q,1} \, d\tau \leq \int_0^{t/2} \|u(\tau)\|_{\dot{B}^{1,q}_1,1} \|u(\tau)\|_{\dot{B}^{\lambda+1}_q,1} \, d\tau \leq \|u\|_{\dot{X}^{\lambda} \infty} \|u\|_{\dot{Y}^{\lambda}}, \quad t > 0. \tag{3.11}
\]

Furthermore, by the boundedness of \( e^{(t-\tau)\mathcal{L}} \) and (3.8) we obtain

\[
t^N(\frac{1}{r} - \frac{1}{q}) + 1 - \lambda \|K_2(t)\|_{\dot{B}^{1,q}_1,0} \leq \int_0^t \|\nabla u(\tau)\|_{\dot{B}^{1,q}_1} \, d\tau \leq \|u\|_{L^\infty(0,\infty;\dot{B}^{1,q}_1,1)} \int_0^t \|u(\tau)\|_{\dot{B}^{\lambda+1}_q,1} \, d\tau \leq \|u\|_{\dot{X}^{\lambda} \infty} \|u\|_{\dot{Y}^{\lambda}}, \quad t > 0. \tag{3.12}
\]

On the third norm, by (2.19), (2.20) and (3.8) we have

\[
\int_0^\infty \int_0^{\infty} t^N(\frac{1}{r} - \frac{1}{q}) + 1 - \lambda \|K_1(t)\|_{\dot{B}^{2,q}_1} \, dt \, d\tau \leq \int_0^\infty \int_0^{\infty} e^{\frac{t}{2}\mathcal{L}} |\nabla u(\tau)|^p \|\nabla u(\tau)\|_{\dot{B}^{\lambda+1}_q,1} \, d\tau \, dt \leq \int_0^\infty \|\nabla u(\tau)\|_{\dot{B}^{\lambda+1}_q,1} \, d\tau \leq \|u\|_{\dot{X}^{\lambda} \infty} \|u\|_{\dot{Y}^{\lambda}}. \tag{3.13}
\]
Furthermore, by (2.21) and (3.8) we obtain

\[ \int_0^\infty t^{N\left(\frac{1}{2} - \frac{1}{q}\right) + 1 - \lambda} \| K_2(t) \|_{\dot{B}^1_{q,1}} \, dt \]

\[ \leq \int_0^\infty t^{N\left(\frac{1}{2} - \frac{1}{q}\right) + 1 - \lambda} \int_\mathbb{T} \| e^{(t-\tau)L} |\nabla u(\tau)|^p \|_{\dot{B}^1_{q,1}} \, d\tau \, dt \]

\[ \leq \int_0^\infty t^{N\left(\frac{1}{2} - \frac{1}{q}\right) + 1 - \lambda} \| |\nabla u(\tau)|^p \|_{\dot{B}^1_{q,1}} \, d\tau \]

\[ \leq \| u \|_{L^\infty(0,\infty; \dot{B}^1_{q,1})} \int_0^\infty t^{N\left(\frac{1}{2} - \frac{1}{q}\right) + 1 - \lambda} \| u(\tau) \|_{\dot{B}^1_{q,1}} \, d\tau \]

\[ \leq \| u \|_{L^\infty(0,\infty; \dot{B}^1_{q,1})} \| u \|_{\dot{Y}^1_q}. \]  

(3.14)

Then, (3.6) is obtained by (3.10), (3.11), (3.12), (3.13) and (3.14).

For the proof of (3.7), it follows from the boundedness of $e^{(t-\tau)L}$, (2.21), (3.8) and the Hölder inequality that, if $1 < p < 2$, then

\[ \left\| \int_0^t e^{(t-\tau)L} \left( |\nabla u(\tau)|^p - |\nabla v(\tau)|^p \right) \, d\tau \right\|_{X^1_q} \]

\[ \leq \int_0^\infty \| |\nabla u(\tau)|^p - |\nabla v(\tau)|^p \|_{\dot{B}^1_{q,1}} \, d\tau \]

\[ \leq \left( \| u \|_{L^\infty(0,\infty; \dot{B}^1_{q,1})} + \| v \|_{L^\infty(0,\infty; \dot{B}^1_{q,1})} \right) \| u - v \|_{L^1(0,\infty; \dot{B}^{1+\varepsilon}_{q,1})} \]

\[ + \| u - v \|_{L^\infty(0,\infty; \dot{B}^1_{q,1})} \| u - v \|_{L^1(0,\infty; \dot{B}^{1+\varepsilon}_{q,1})} \times \]

\[ \times \left( \| u \|_{L^\infty(0,\infty; \dot{B}^1_{q,1})} \| v \|_{L^\infty(0,\infty; \dot{B}^2_{q,1})} \right) \]

\[ \leq \left( \| u \|_{L^1(0,\infty; \dot{B}^1_{q,1})} + \| v \|_{L^1(0,\infty; \dot{B}^1_{q,1})} \right) \| u - v \|_{X^1_q}, \]

and, if $p \geq 2$, then

\[ \left\| \int_0^t e^{(t-\tau)L} \left( |\nabla u(\tau)|^p - |\nabla v(\tau)|^p \right) \, d\tau \right\|_{X^1_q} \]

\[ \leq \left( \| u \|_{L^\infty(0,\infty; \dot{B}^1_{q,1})} + \| v \|_{L^\infty(0,\infty; \dot{B}^1_{q,1})} \right) \| u - v \|_{L^1(0,\infty; \dot{B}^{1+\varepsilon}_{q,1})} \]

\[ + \| u - v \|_{L^\infty(0,\infty; \dot{B}^1_{q,1})} \| u - v \|_{L^1(0,\infty; \dot{B}^{1+\varepsilon}_{q,1})} \times \]

\[ \times \left( \| u \|_{L^\infty(0,\infty; \dot{B}^1_{q,1})} \| v \|_{L^\infty(0,\infty; \dot{B}^2_{q,1})} \right) \]

\[ \leq \left( \| u \|_{L^1(0,\infty; \dot{B}^1_{q,1})} + \| v \|_{L^1(0,\infty; \dot{B}^1_{q,1})} \right) \| u - v \|_{X^1_q}, \]

Therefore, (3.7) is obtained and the proof of all estimates is completed. \( \square \)

In what follows, we prove that the solution exists globally in time by applying the contraction mapping principle in \( X \) for initial data \( u_0 \) in \( B^1_{r,1} \cap B^1_{\infty,1} \) and small in \( B^1_{\infty,1} \), and that the solutions satisfy the decay estimates (1.9) and (1.10).
Proof of existence of global-in-time solutions in $\mathfrak{X} \cap C([0, \infty), B^{1}_{r,1} \cap B^{1}_{\infty,1})$. Let the constant $C_0$ in the definition of $\mathfrak{X}$ be a constant which satisfy the all estimates in Proposition 3.1 and we assume the initial data satisfies

$$u_0 \in B^{1}_{r,1} \cap B^{1}_{\infty,1} \text{ and } \|u_0\|_{\dot{B}^{1}_{r,1}} \leq (2^{p+1}C_0^p)^{-\frac{1}{p}}.$$ (3.15)

For any $u, v \in \mathfrak{X}$, it follows from Proposition 3.1 that

$$\|\Psi(u)\|_{X^s_\varepsilon} \leq C_0\|u_0\|_{\dot{B}^{1}_{r,1}} + C_0\|u\|_{X^s_{\infty}} \leq C_0\|u_0\|_{\dot{B}^{1}_{r,1}} + C_0(2C_0\|u_0\|_{\dot{B}^{1}_{r,1}})^{p-1} \cdot 2C_0\|u_0\|_{\dot{B}^{1}_{r,1}}$$

$$\leq 2C_0\|u_0\|_{\dot{B}^{1}_{r,1}}^p.$$ (3.16)

for any $s = \varepsilon, 1$. $\Psi$ is a contraction map from $\mathfrak{X}$ to itself and the global solution for small initial data is obtained in $\mathfrak{X}$. Then $u(t) = \Psi(u)(t)$ in $\mathcal{Z}'(\mathbb{R}^{N})$ for all almost every $t$, and we have to find a fixed point such that the equality $u(t) = \Psi(u)(t)$ holds in $\mathcal{S}'(\mathbb{R}^{N})$. For this purpose, we take a sequence $\{u_n\}$ such that

$$u_1 := e^{t\mathcal{L}}u_0, \quad u_n := \Psi(u_{n-1}), \quad n \geq 2.$$ (3.14)

The previous contraction argument implies that $u_n$ converges to $u$ in $X^\varepsilon_{t} \cap X^\varepsilon_{\infty}$. Here, we see that $\Psi(u_{n-1})$ tends to $\Psi(u)$ in $L^\infty$ as $n \to \infty$ for each $t$ since we have from (3.14)

$$\|\Psi(u_{n-1})(t)\|_{L^\infty} \leq \|u_0\|_{L^\infty} + t\|u_{n-1}\|_{X^1_{\infty}}^p,$$ $\|\Psi(u)(t)\|_{L^\infty} \leq \|u_0\|_{L^\infty} + t\|u\|_{X^\infty_1}^p,$

$$\| \int_0^t e^{(t-\tau)\mathcal{L}}(\nabla u_{n-1}(\tau))^p - \nabla u(\tau))^p d\tau \|_{L^\infty} \leq \int_0^t (\|\nabla u_{n-1}(\tau)\|_{L^\infty}^{p-1} + \|\nabla u(\tau)\|_{L^\infty}^{p-1}) \|\nabla u_{n-1}(\tau) - \nabla u(\tau)\|_{L^\infty} \, d\tau$$

$$\leq \left(\|u_{n-1}\|_{X^\infty_1}^{p-1} + \|u\|_{X^\infty_1}^{p-1}\right) \int_0^t \|u_{n-1}(\tau) - u(\tau)\|_{\dot{B}^{1}_{r,1}} \, d\tau$$

$$\leq \|u_0\|_{\dot{B}^{1}_{\infty,1}}^{p-1} \int_0^t \|u_{n-1}(\tau) - u(\tau)\|_{X^\varepsilon_{t}}^{\varepsilon_1} \|u_{n-1}(\tau) - u(\tau)\|_{\dot{B}^{1}_{r,1}}^{1-\varepsilon} \, d\tau$$

$$\leq \|u_0\|_{\dot{B}^{1}_{\infty,1}}^{p-1} \int_0^t \|u_{n-1}(\tau) - u(\tau)\|_{L^\infty(0,\infty;\dot{B}^{1}_{\infty,1})} \|u_{n-1}(\tau) - u(\tau)\|_{L^1(0,\infty;\dot{B}^{1+\varepsilon}_{r,1})} \, d\tau$$

$$\leq \|u_0\|_{\dot{B}^{1}_{\infty,1}}^{p-1} \int_0^t \|u_{n-1}(\tau) - u(\tau)\|_{X^\varepsilon_{\infty}} \to 0 \quad \text{as } n \to \infty.$$ 17
Therefore, \( u_n(t) = \Psi(u_{n-1})(t) \) is a Cauchy sequence in \( L^\infty \), so that, there exists \( v(t) \in L^\infty \) such that \( u_n(t) \) converges to \( v(t) \) in \( L^\infty \) as \( n \to \infty \). It follows from \( L^\infty \subset S'(\mathbb{R}^N) \subset Z'(\mathbb{R}^N) \) and the uniqueness of the limit in \( Z'(\mathbb{R}^N) \) that \( u_n(t) \) also converges to \( v(t) \) in \( Z'(\mathbb{R}^N) \) as \( n \to \infty \) and \( v(t) = u(t) \) in \( Z'(\mathbb{R}^N) \). Since \( u(t) \in B_{\infty,1}^1 \) and \( \nabla u(t) \in S'(\mathbb{R}^N) \) by Remark 2.7, it holds that \( \nabla v(t) = \nabla u(t) \) and \( \Psi(v)(t) = \Psi(u)(t) \) in \( S'(\mathbb{R}^N) \) for all \( t \). Then taking the limit in the topology of \( L^\infty \) on the equation \( u_n(t) = \Psi(u_{n-1})(t) \) for each \( t \), we obtain

\[
v(t) = \Psi(u)(t) = \Psi(v)(t) \quad \text{in} \quad L^\infty .
\]

By \( \|e^{t\mathcal{L}}u_0\|_{L^q} \leq \|u_0\|_{L^q} \) and (3.4), \( v \) satisfies \( v(t) \in L^r \cap B_{r,1}^1 \cap L^\infty \cap B_{\infty,1}^1 = B_{r,1}^1 \cap B_{\infty,1}^1 \). Hence, the fixed point \( v \) is a solution in \( X \cap C([0,\infty), B_{r,1}^1 \cap B_{\infty,1}^1) \).

It remains to show the uniqueness. Let \( u, v \in X \) satisfies \( u, v \in C([0,\infty), B_{r,1}^1 \cap B_{\infty,1}^1) \), \( u = \Psi(u) \) and \( v = \Psi(v) \), and we show that \( u(t) = v(t) \) in \( S'(\mathbb{R}^N) \) for all \( t \). The contraction property (3.16) implies that \( u(t) = v(t) \) in \( B_{\infty,1}^1 \subset Z'(\mathbb{R}^N) \). Since \( 0 < \varepsilon < 1 \), there exists a constant \( c(t) \) independent of \( x \in \mathbb{R}^N \) such that \( u(t) = v(t) + c(t) \) in \( S'(\mathbb{R}^N) \). It follows from \( \nabla u(t) = \nabla v(t) \) in \( S'(\mathbb{R}^N) \) that

\[
u(t) = e^{t\mathcal{L}}u_0 + \int_0^t e^{(t-\tau)\mathcal{L}}|\nabla u(\tau)|^p d\tau = e^{t\mathcal{L}}u_0 + \int_0^t e^{(t-\tau)\mathcal{L}}|\nabla v(\tau)|^p d\tau = v(t) \quad \text{in} \quad S'(\mathbb{R}^N).
\]

Therefore, \( c(t) \equiv 0 \) in \( S'(\mathbb{R}^N) \), and the uniqueness follows. \( \Box \)

**Proof of the decay estimates (1.9) and (1.10).** According to the above proof of global existence, let

\[
\lambda = (p - 1)/p, \quad (3.17)
\]

and it is sufficient to show only (1.9) for the solution \( u \) satisfying

\[
\|u\|_{X^1_{\lambda} \cap X^\infty_{\lambda} \cap Y^1_{\lambda} \cap Y^\infty_{\lambda}} < \infty, \quad (3.18)
\]

since (1.10) is obtained by \( \|u\|_{X^1_{\lambda} \cap Y^\infty_{\lambda}} < \infty \). In the case \( 0 \leq t \leq 1 \), the boundedness in time on the norms \( \|\nabla^j u(t)\|_{L^q} \) \( (j = 0, 1) \) is obtained by the Hölder inequality, the inequalities \( \|u(t)\|_{L^q} \leq \|u(t)\|_{B^0_{p,1}}, \|\nabla u(t)\|_{L^q} \leq \|u(t)\|_{B^1_{p,1}} \), and (3.18), so that it suffices to consider the case \( t > 1 \). We show the estimate with derivative:

\[
\|\nabla u(t)\|_{L^q} \leq (1 + t)^{-N(\frac{1}{p} - \frac{1}{q}) - 1}, \quad t > 0. \quad (3.19)
\]

Once (3.19) is proved, it is possible to show the decay estimate of \( \|u(t)\|_{L^r} \). In fact, by (2.18) and (3.19) we see that

\[
\|u(t)\|_{L^q} \\
\leq t^{-N(\frac{1}{p} - \frac{1}{q})} \|u_0\|_{L^p} + \int_0^{t/2} (t - \tau)^{-N(\frac{1}{p} - \frac{1}{q})} \|\nabla u(\tau)\|^p_{L^r} d\tau + \int_{t/2}^t \|\nabla u(\tau)\|^p_{L^q} d\tau \\
\leq t^{-N(\frac{1}{p} - \frac{1}{q})} \int_0^{t/2} \|\nabla u(\tau)\|^p_{L^{pr}} d\tau + \int_{t/2}^t \|\nabla u(\tau)\|^p_{L^q} d\tau \\
\leq t^{-N(\frac{1}{p} - \frac{1}{q})} \int_0^{t/2} \left\{ (1 + \tau)^{-N(\frac{1}{p} - \frac{1}{q})} \right\} d\tau + \int_{t/2}^t \left\{ \tau^{-N(\frac{1}{p} - \frac{1}{q})} \right\} d\tau \\
\leq t^{-N(\frac{1}{p} - \frac{1}{q})}, \quad t \geq 1,
\]

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so that, the decay estimate in $L^q(\mathbb{R}^n)$ is obtained.

We show (3.19). It follows from (2.18) that
\[
\|\nabla u(t)\|_{L^q} \leq t^{-N\left(\frac{1}{p} - \frac{1}{q}\right) - 1} \|u_0\|_{L^p} + \int_0^{t/2} \|\nabla e^{(t-\tau)\mathcal{L}} \nabla u(\tau)\|_{L^q} d\tau. \tag{3.20}
\]

We first consider the case $r < \infty$. By (2.18), (3.17) and (3.18) we have
\[
\int_0^{t/2} \|\nabla e^{(t-\tau)\mathcal{L}} \nabla u(\tau)\|_{L^q} d\tau \leq \int_0^{t/2} (t-\tau)^{-N\left(\frac{1}{p} - \frac{1}{q}\right) - 1} \|\nabla u(\tau)\|_{L^p} d\tau \leq t^{-N\left(\frac{1}{p} - \frac{1}{q}\right) - 1} \int_0^{t/2} \|\nabla u(\tau)\|_{L^p}^p d\tau \leq t^{-N\left(\frac{1}{p} - \frac{1}{q}\right) - 1} \int_0^{t/2} \{1 + (1 + \tau)^{-N\left(\frac{1}{p} - \frac{1}{q}\right) - 1 + \lambda}\}^p d\tau \leq t^{-N\left(\frac{1}{p} - \frac{1}{q}\right) - 1}. \tag{3.21}
\]

On the other hand, by (3.17), (3.18), the boundedness of $e^{(t-s)\mathcal{L}}$ in $L^q$ and the H"older inequality we obtain
\[
\int_0^{t} \|\nabla e^{(t-\tau)\mathcal{L}} \nabla u(\tau)\|_{L^q}^p d\tau \leq \int_0^{t} \|\nabla |\nabla u(\tau)|\|_{L^q}^p d\tau \leq \int_0^{t} \|u(\tau)\|_{B_{\infty,1}^q}^{-1} \|u(\tau)\|_{B_{q,1}^2}^p d\tau \leq \|u\|_{B_{\infty,1}^q}^{-1} \int_0^{t} (t-\tau)^{-N\left(\frac{1}{p} - \frac{1}{q}\right) - 1 + \lambda} \|u(\tau)\|_{B_{q,1}^2}^p d\tau \leq \int_0^{t} \{1 + \tau\}^{-N\left(\frac{1}{p} - \frac{1}{q}\right) - 1 + \lambda} \|u\|_{B_{\infty,1}^q}^{-1} \|u\|_{B_{q,1}^2}^p d\tau \leq t^{-N\left(\frac{1}{p} - \frac{1}{q}\right) - 1}. \tag{3.22}
\]

This together with (3.20) and (3.21) yields (3.19) for the case $r < \infty$.

Next we consider the case $r = \infty$. In this case, the problem is that the integral in the third line of (3.21) diverges as $t \to \infty$. Then corresponding estimate to (3.21) is the following with taking $r = q = \infty$
\[
\int_0^{t/2} \|\nabla e^{(t-\tau)\mathcal{L}} \nabla u(\tau)\|_{L^\infty}^p d\tau \leq t^{-1} \int_0^{t/2} \{1 + \tau\}^{-1 + \lambda} \|u\|_{Y_{\infty,1}^p} d\tau \leq t^{-1} \log(1 + t),
\]
and the same estimate as (3.22) holds. This implies that
\[
\|\nabla u(t)\|_{L^\infty} \leq t^{-1} \log(1 + t), \quad t > 1.
\]

By this decay estimate, we can improve the corresponding one to (3.21) as
\[
\int_0^{t/2} \|\nabla e^{(t-\tau)\mathcal{L}} \nabla u(\tau)\|_{L^\infty}^p d\tau \leq t^{-1} \int_0^{t/2} \{1 + \tau\}^{-1} \log(2 + t)\}^p d\tau \leq t^{-1}.
\]
Therefore, we also have the estimate (3.19) for the case \( r = \infty \), and the proof of (1.9) is completed. □

4 Asymptotic behavior

In this section we prove the assertion (ii) of Theorem 1.1. The proof is based on the arguments in [14, Theorem 1.2] and [15, Theorem 1.1] (see, also, [12]). Throughout this section we assume that \( u \) is a global-in-time solution of (1.7) satisfying (1.9) and (1.10).

Proof of Theorem 1.1 (ii)-(a). Let \( r \in (1, \infty) \) and \( q \in [r, \infty] \). By (1.7), for any \( j \in \{0, 1\} \), we have

\[
\|\nabla^j[v(t) - e^{tL}u_0]\|_{L^q} \leq \left\| \nabla^j \int_{t/2}^t e^{(t-\tau)L} |\nabla v(\tau)|^p \, d\tau \right\|_{L^p} + \left\| \nabla^j \int_0^{t/2} e^{(t-\tau)L} |\nabla v(\tau)|^p \, d\tau \right\|_{L^p}
\]

(4.1)

for all \( t > 0 \). We first estimate \( I_{1,j}(t) \). By (1.9) and (2.18) we obtain

\[
I_{1,0}(t) \leq \int_{t/2}^t \|e^{(t-\tau)L} |\nabla v(\tau)|^p\|_{L^q} \, d\tau \\
\leq \int_{t/2}^t \|\nabla v(\tau)\|_{L^\infty}^{p-1} \|\nabla v(\tau)\|_{L^q} \, d\tau \\
\leq \int_{t/2}^t \tau^{-(N_1(p-1)+1)} \tau^{-N(1/2 - 1/q)} \, d\tau \\
\leq t^{-N(1/2 - 1/q)} \cdot (N_1(p-1)+1), \quad t \geq 1.
\]

(4.2)

Furthermore, applying the argument similar to (3.22) with (1.9) and (1.10), we see that

\[
I_{1,1}(t) \leq \int_{t/2}^t \|\nabla e^{(t-\tau)L} |\nabla^2 v(\tau)|^p\|_{L^q} \, d\tau \\
\leq \int_{t/2}^t \|\nabla v(\tau)\|_{L^\infty} \|\nabla^2 v(\tau)\|_{L^q} \, d\tau \\
\leq \int_{t/2}^t \tau^{-(N_1(p-1)+1)+1} \tau^{-N(1/2 - 1/q)} \|\nabla^2 v(\tau)\|_{L^q} \, d\tau \\
\leq t^{-N(1/2 - 1/q)} \cdot (N_1(p-1)+1) \int_{0}^{\infty} \tau^{N(1/2 - 1/q) + 1} \|v(\tau)\|_{B^2_{2,q}} \, d\tau \\
\leq t^{-N(1/2 - 1/q)} \cdot (N_1(p-1)+1), \quad t \geq 1.
\]

(4.3)

Since \( (p-1)p + 1 > p \) for all \( p > 1 \), it follows from (4.2) and (4.3) that

\[
t^{N(1/2 - 1/q)} I_{1,j}(t) = O(t^{-N/(p-1)})
\]

(4.4)
as $t \to \infty$, for any $j \in \{0, 1\}$.

Next we estimate $I_{2,j}$. For the case $1 < r \leq p$, by (1.9) and (2.18) we obtain

\[
I_{2,j}(t) \leq \int_0^{t/2} \|\nabla^j e^{(t-\tau)\mathcal{L}}|\nabla v(\tau)|^p\|_{L^q} d\tau \\
\leq \int_0^{t/2} (t-\tau)^{-N(1-\gamma)/2-j} \|\nabla v(\tau)|^p\|_{L^q} d\tau \\
\leq t^{-N(1-\gamma)/2-j} \int_0^{t/2} \|\nabla v(\tau)|^p\|_{L^q} d\tau \\
\leq t^{-N(1-\gamma)/2-j} \int_0^{t/2} (1 + \tau)^{-N(p-1)/p} d\tau \\
\leq t^{-N(1-\gamma)/2-j} \int_0^{t/2} (1 + \tau)^{-N(p-1)/p} d\tau \\
\leq t^{-N(1-\gamma)/2-j} \int_0^{t/2} (1 + \tau)^{-N(p-1)/p} d\tau, \quad t \geq 1.
\]

(4.5)

For the case $r > p$, by (1.9) and (2.18) again we see that

\[
I_{2,j}(t) \leq \int_0^{t/2} \|\nabla^j e^{(t-\tau)\mathcal{L}}|\nabla v(\tau)|^p\|_{L^q} d\tau \\
\leq \int_0^{t/2} (t-\tau)^{-N(p-1)/q-j} \|\nabla v(\tau)|^p\|_{L^r} d\tau \\
\leq t^{-N(p-1)/q-j} \int_0^{t/2} \|\nabla v(\tau)|^p\|_{L^r} d\tau \\
\leq t^{-N(p-1)/q-j} \int_0^{t/2} (1 + \tau)^{-N(p-1)/r} d\tau \leq t^{-N(p-1)/r-j} \int_0^{t/2} (1 + \tau)^{-N(p-1)/r} d\tau, \quad t \geq 1.
\]

This together with (4.5) yields

\[
t^{-N(1-\gamma)/2-j} I_{2,j}(t) = \begin{cases} 
O(t^{-N(p-1)/r}) & \text{if } p \geq r, \\
O(t^{-N(p-1)/q}) & \text{if } p < r,
\end{cases}
\]

(4.6)

as $t \to \infty$, for any $j \in \{0, 1\}$. Therefore, substituting (4.4) and (4.6) into (4.1), we have (1.11), and the assertion (ii)-(a) of Theorem 1.1 follows. ∎

**Proof of Theorem 1.1 (ii)-(b).** Let $r = 1$. Put

\[
c(t) := M(u_0) + \int_0^t M(|\nabla v(\tau)|^p) d\tau,
\]

(4.7)

where

\[
M(f) = \int_{\mathbb{R}^N} f(x) dx.
\]

(4.8)

Then, by (1.9) we have

\[
|c(t_2) - c(t_1)| = \int_{t_1}^{t_2} M(|\nabla v(\tau)|^p) d\tau = \int_{t_1}^{t_2} \|\nabla v(\tau)|^p\|_{L^p} d\tau \leq \int_{t_1}^{t_2} (1 + \tau)^{-N(p-1)/p} d\tau
\]

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for all $t_2 \geq t_1 \geq 0$. This implies that there exists the limit $C_*$ given by (1.2) such that
\[
\left| \int_{\mathbb{R}^N} v(x,t) \, dx - C_* \right| = |c(t) - C_*| = O(t^{-(N+1)(p-1)})
\] (4.9)
as $t \to \infty$. Furthermore, (2.17) and (4.9) yield
\[
\lim_{t \to \infty} t^{N(1-\frac{1}{q})+j} \|\nabla_j [c(t)P_{t+1} - C_*P_{t+1}]\|_{L^q} = \lim_{t \to \infty} |c(t) - C_*| = 0
\] (4.10)
for all $q \in [1, \infty]$ and $j = 0, 1$. Put
\[
w(x,t) := [e^{t\mathcal{L}}u_0](x) - M(u_0)P_{t+1}(x)
\] (4.11)
for all $(x,t) \in \mathbb{R}^N \times (0, \infty)$. Since it follows from the semigroup property of $P_t$ that
\[
[e^{t\mathcal{L}}P_1](x) = P_{t+1}(x),
\] (4.12)
we have
\[
w(x,t) = [e^{t\mathcal{L}}w(0)](x).
\]
On the other hand, by (1.5), (4.8) and (4.11) we obtain
\[
\int_{\mathbb{R}^N} F(x,t) \, dx = 0,
\] (4.15)
for all $q \in [1, \infty]$ and $j = 0, 1$.

Let
\[
F(x,t) := |\nabla v(x,t)|^p - M(|\nabla v(t)|^p)P_{t+1}(x).
\] (4.14)
Then, by (1.5) and (4.8) we have
\[
\int_{\mathbb{R}^N} F(x,t) \, dx = 0, \quad t \geq 0.
\] (4.15)
Since it follows from (4.12) and (4.14) that
\[
\int_0^t e^{(t-\tau)\mathcal{L}}|\nabla v(\tau)|^p \, d\tau - \int_0^t M(|\nabla v(\tau)|^p) \, d\tau P_{t+1}(x)
\]
\[
= \int_0^t e^{(t-\tau)\mathcal{L}} \left\{ |\nabla v(\tau)|^p - M(|\nabla v(\tau)|^p)P_{t+1} \right\} \, d\tau = \int_0^t e^{(t-\tau)\mathcal{L}} F(\tau) \, d\tau,
\]
by (1.7), (4.7) and (4.11) we see that
\[
v(x,t) - c(t)P_{t+1}(x)
\]
\[
= e^{t\mathcal{L}}u_0 + \int_0^t e^{(t-\tau)\mathcal{L}} |\nabla v(\tau)|^p \, d\tau - \left[ M(u_0) + \int_0^t M(|\nabla v(\tau)|^p) \, d\tau \right] P_{t+1}(x)
\]
\[
= w(x,t) + \int_0^t e^{(t-\tau)\mathcal{L}} F(\tau) \, d\tau.
\]
This together with (4.10) and (4.13) implies that
\[
\lim_{t \to \infty} t^{N(1 - \frac{1}{q}) + j} \| \nabla^j [v(t) - C_* P_{t+1}] \|_{L^q} \\
= \lim_{t \to \infty} t^{N(1 - \frac{1}{q}) + j} \| \nabla^j [v(t) - c(t) P_{t+1}] \|_{L^q} + \lim_{t \to \infty} t^{N(1 - \frac{1}{q}) + j} \| \nabla^j [c(t) P_{t+1} - C_* P_{t+1}] \|_{L^q} \\
= \lim_{t \to \infty} t^{N(1 - \frac{1}{q}) + j} \left\| \nabla^j \int_0^t e^{(t - \tau) \mathcal{L}} F(\tau) d\tau \right\|_{L^q}.
\]

Therefore, in order to obtain (1.12), it suffices to prove
\[
\lim_{t \to \infty} t^{N(1 - \frac{1}{q}) + j} \left\| \nabla^j \int_0^t e^{(t - \tau) \mathcal{L}} F(\tau) d\tau \right\|_{L^q} = 0. \tag{4.16}
\]

For any \( j \in \{0, 1\} \), put
\[
J_{1,j}(t) := \int_{t/2}^t \nabla^j e^{(t - \tau) \mathcal{L}} F(\tau) d\tau, \\
J_{2,j}(t) := \int_L^{t/2} \nabla^j e^{(t - \tau) \mathcal{L}} F(\tau) d\tau, \\
J_{3,j}(t) := \int_0^L \nabla^j e^{(t - \tau) \mathcal{L}} F(\tau) d\tau,
\]
for \( t \geq 2L \), where \( L \geq 1 \). Since it follows from (1.10), (2.17) and (4.14) that
\[
\sup_{t > 0} (1 + t)^{N(1 - \frac{1}{q}) + N(p-1) + p} \| F(t) \|_{L^q} < \infty, \tag{4.17}
\]
by (2.18) we have
\[
t^{N(1 - \frac{1}{q})} \| J_{1,0}(t) \|_q \leq t^{N(1 - \frac{1}{q})} \int_{t/2}^t \| F(\tau) \|_{L^q} d\tau \leq t^{-N-1} (p-1) = o(1) \tag{4.18}
\]
as \( t \to \infty \). Furthermore, by (1.9), (2.17) and (4.4) we obtain
\[
t^{N(1 - \frac{1}{q}) + 1} \| J_{1,1}(t) \|_{L^q} \leq t^{N(1 - \frac{1}{q}) + 1} \left[ I_{1,1}(t) + \int_{t/2}^t \| \nabla v(\tau) \|_{L^q} \| \nabla P_{\tau+1} \|_{L^q} d\tau \right] \\
\leq t^{N(1 - \frac{1}{q}) + 1} I_{1,1}(t) + \int_{t/2}^t \tau^{-N(p-1)-p} d\tau \\
\leq t^{N(1 - \frac{1}{q}) + 1} I_{1,1}(t) + t^{-N+1} (p-1) = o(1) \tag{4.19}
\]
as \( t \to \infty \). Moreover, by (2.18) and (4.17) we have
\[
t^{N(1 - \frac{1}{q}) + j} \| J_{2,j}(t) \|_{L^q} \leq t^{N(1 - \frac{1}{q}) + j} \int_L^{t/2} (t - \tau)^{-N(1 - \frac{1}{q}) - j} \| F(\tau) \|_{L^1} d\tau \\
\leq \int_L^{t/2} \| F(\tau) \|_{L^1} d\tau \leq \int_L^{t/2} \tau^{-N(p-1) - p} d\tau \leq L^{-N+1} (p-1) \tag{4.20}
\]

as \( t \to \infty \).
for all sufficiently large $t$. Similarly, we see that
\[ t^{N(1-\frac{1}{q})+j} \| J_{3,j}(t) \|_{L^q} \leq t^{N(1-\frac{1}{q})+j} \int_0^L \left\| \nabla^j e^{\frac{t-\tau}{N}} e^{\frac{(t-\tau)\mathcal{L}}{2}} F(\tau) \right\|_{L^q} d\tau \]
\[ \leq \int_0^L \left\| e^{\frac{(t-\tau)\mathcal{L}}{2}} F(\tau) \right\|_{L^1} d\tau \]
for all $t \geq 2L$. On the other hand, for any $L > 0$, it follows from Lemma 2.6 with (4.15) that
\[ \lim_{t \to \infty} \left\| e^{\frac{(t-\tau)\mathcal{L}}{2}} F(\tau) \right\|_{L^1} = 0 \]
for all $s \in (0, L)$. Furthermore, by (2.18) we have
\[ \sup_{t \geq 2L} \left\| e^{\frac{(t-\tau)\mathcal{L}}{2}} F(\tau) \right\|_{L^1} \leq \| F(\tau) \|_{L^1}. \]
Then, applying the Lebesgue dominated convergence theorem with (4.22) and (4.23) to (4.21), we obtain
\[ \lim_{t \to \infty} t^{N(1-\frac{1}{q})+j} \| J_{3,j}(t) \|_{L^q} = 0. \]
Therefore, by (4.18), (4.19), (4.20) and (4.24) we see that
\[ \limsup_{t \to \infty} t^{N(1-\frac{1}{q})+j} \left\| \nabla^j \int_0^t e^{(t-\tau)\mathcal{L}} F(\tau) d\tau \right\|_{L^q} \leq C_1 L^{-(N+1)(p-1)} \]
for some constant $C_1$ independent of $L$. Therefore, since $L$ is arbitrary, by $p > 1$ we have (4.16), and the proof of the assertion (ii)-(b) of Theorem 1.1 is complete. □

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References

[1] N. Alibaud, C. Imbert and G. Karch, Asymptotic properties of entropy solutions to fractal Burgers equation, SIAM J. Math. Anal., 42 (2010), 354–376.

[2] L. Amour and M. Ben-Artzi, Global existence and decay for viscous Hamilton-Jacobi equations, Nonlinear Anal. 31 (1998), 621–628.

[3] S. Benachour, G. Karch and Ph. Laurençot, Asymptotic profiles of solutions to viscous Hamilton-Jacobi equations, J. Math. Pures Appl. 83 (2004), 1275–1308.

[4] S. Benachour and Ph. Laurençot, Global solutions to viscous Hamilton-Jacobi equations with irregular initial data, Comm. Partial Differential Equations 24 (1999) 1999–2021.
[5] M. Ben-Artzi and H. Koch, Decay of mass for a semilinear parabolic equation, Comm. Partial Differential Equations 24 (1999), 869–881.

[6] M. Ben-Artzi, Ph. Souplet and F.B. Weissler, The local theory for viscous Hamilton-Jacobi equations in Lebesgue spaces, J. Math. Pures Appl. 81 (2002), 343–378.

[7] H. Dong, D. Du and D. Li, Finite time singularities and global well-posedness for fractal Burgers equations, Indiana Univ. Math. J., 58 (2009), 807–822.

[8] J. Droniou, T. Gallouët and J. Vovelle, Global solution and smoothing effect for a non-local regularization of a hyperbolic equation, J. Evol. Equ., 4 (2003), 479–499.

[9] J. Droniou and C. Imbert, Fractal first order partial differential equations, Arch. Rational Mech. Anal. 182 (2006), 299–331.

[10] B. Gilding, M. Guedda and R. Kersner, The Cauchy problem for $u_t = \Delta u + |\nabla u|^q$, J. Math. Anal. Appl. 284 (2003), 733–755.

[11] C. Imbert, A non-local regularization of first order Hamilton-Jacobi equations, J. Differential Equations 211 (2005), 218–246.

[12] K. Ishige and T. Kawakami, Refined asymptotic profiles a semilinear heat equation, Math. Ann., 353 (2012), 161–192.

[13] K. Ishige, T. Kawakami and K. Kobayashi, Global solutions for a nonlinear integral equation with a generalized heat kernel, Discrete Contin. Dyn. Syst. Ser. S. 7 (2014), 767–783.

[14] K. Ishige, T. Kawakami and K. Kobayashi, Asymptotics for a nonlinear integral equation with a generalized heat kernel, J. Evol. Equ. 14 (2014), 749–777.

[15] K. Ishige and K. Kobayashi, Convection-diffusion equation with absorption and non-decaying initial data, J. Differential Equations 254 (2013), 1247–1268.

[16] T. Iwabuchi, Global solutions for the critical Burgers equation in the Besov spaces and the large time behavior, Ann. Inst. H. Poincaré Anal. Non Linéaire 32 (2015), 687–713.

[17] E. R. Jakobsen and K. H. Karlsen, Continuous dependence estimates for viscosity solutions of integro-PDEs, J. Differential Equations 212 (2005), 278–318.

[18] E. R. Jakobsen and K. H. Karlsen, A ”maximum principle for semicontinuous functions” applicable to integro-partial differential equations, NoDEA Nonlinear Differential Equations Appl. 13 (2006), 137–165.

[19] G. Karch and W. A. Woyczyński, Fractal Hamilton-Jacobi-KPZ equations, Trans. Am. Math. Soc. 360 (2008), 2423–2442.

[20] M. Kardar, G. Parisi and Y. C. Zhang, Dynamic scaling of growing interfaces, Phys. Rev. Lett. 56 (1986), 889–892.
[21] A. Kiselev, F. Nazarov and R. Shterenberg, Blow up and regularity for fractal Burgers equation, Dyn. Partial Differential Equations, 5 (2008), 211–240.

[22] H. Kozono, M. Yamazaki, *Semilinear heat equations and the Navier-Stokes equation with distributions in new function spaces as initial data*, Comm. Partial Differential Equations, 19 (1994), no. 5-6, 959–1014.

[23] J. Krug and H. Spohn, Universality classes for deterministic surface growth, Phys. Rev. A. 38 (1988), 4271–4283.

[24] Ph. Laurençot and Ph. Souplet, On the growth of mass for a viscous Hamilton-Jacobi equation, J. Anal. Math. 89 (2003), 367–383.

[25] S. Machihara, T. Ozawa, *Interpolation inequalities in Besov spaces*, Proc. Amer. Math. Soc. 131 (2003), no. 5, 1553–1556.

[26] C. Miao and G. Wu, Global well-posedness of the critical Burgers equation in critical Besov spaces, J. Differential Equations, 247 (2009), 1673–1693.

[27] L. Silvestre, On the differentiability of the solution to the Hamilton-Jacobi equation with critical fractional diffusion, Advances in Mathematics 226 (2011), 2020–2039.

[28] H. M. Soner, Optimal control with state-space constraint. II, SIAM J. Control Optim. 24 (1986), 1110–1122.

[29] H. Triebel, “Theory of Function Spaces,” Birkhäuser-Verlag, Basel, 1983.