Mass generation without phase coherence in the Chiral Gross-Neveu Model at finite temperature and small $N$ in 2+1 dimensions.

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The chiral Gross-Neveu model is one of the most popular toy models for QCD. In the past, it has been studied in detail in the large-N limit. In this paper we study its small-N behavior at finite temperature in 2+1 dimensions. We show that at small $N$ the phase diagram of this model is principally different from its behavior at $N \rightarrow \infty$. We show that for a small number $N$ of fermions the model possesses two characteristic temperatures $T_{KT}$ and $T^*$. That is, at small $N$, along with a quasordered phase $0 < T < T_{KT}$ the system possesses a very large region of precursor fluctuations $T_{KT} < T < T^*$ which disappear only at a temperature $T^*$, substantially higher than the temperature $T_{KT}$ of Kosterlitz-Thouless transition.

In this letter we discuss small-N behavior of the Gross-Neveu (GN) model with U(1)-symmetry in 2+1 dimensions at finite temperature. The Gross-Neveu model is a field theoretic model of zero-mass fermions with quartic interaction, which provides us with considerable insight into the mechanisms of spontaneous symmetry breakdown and is considered to be an illuminating toy model for QCD. Our small-N study is motivated by recent results in the theory of superconductivity in the regimes when BCS mean-field theory is not valid.

Recently a remarkable progress was made in the theory of superconductivity in understanding mechanisms of symmetry breakdown in the regimes of strong interaction and low carrier density. It was observed that, in general, a Fermi system with attraction possess two distinct characteristic temperatures corresponding to pair formation and pair condensation. That is, in a strong-coupling superconductor Cooper pairs are formed at a certain temperature $T^*$ although there is no macroscopic occupation of zero momentum level and thus no phase coherence and no symmetry breakdown. The temperature should be lowered to $T_c(< T^*)$ in order to make these pairs condense and establish phase coherence. The large region $T_c < T < T^*$ where there exist Cooper pairs but no continual symmetry is broken is called pseudogap phase. Thus the symmetry breakdown in a strong-coupling superconductor resembles onset of long range order in $^4$He where, formally one also can introduce a characteristic temperature of thermal decomposition of a $^4$He atom, however it does not mean that this temperature is connected in any respect with temperature of the onset of phase coherence. In fact, the BCS scenario when superconductive phase transition can approximately be described as a pair-formation transition (i.e. when there is only one characteristic temperature $T_c$) is a very exceptional case. That is, the BCS scenario holds true only at infinitesimally weak coupling strength or very high carrier density.

Similarly as the BCS theory became a source of inspiration in particle physics (in particular it had direct influence on appearance of the Nambu–Jona-Lasinio and Gross-Neveu models), recently the pseudogap concept started to penetrate from the theory of superconductivity to the high-energy physics, sparking many intensive discussions. The pseudogap concept was first introduced to particle physics in $^3P_2$

Let us now return to the Gross-Neveu model. At finite temperatures in the limit of large $N$ its behavior closely resembles a BCS superconductor $^3P_2$. Below, we show that a very rich physical structure, similar to the phase diagram of a strong-coupling superconductor, emerges in the small-N limit in the chiral GN model.

Existence of the pseudogap phase in the chiral GN model at small $N$ provides us with an example when generation of fermion mass happens without spontaneous breakdown of continual symmetry and suggests possibility of importance of this concept in particle physics.

The chiral GN model $^3P_2$ has the following Lagrange density $^3P_2$

$$\mathcal{L} = \bar{\psi}_a i \not{\partial} \psi_a + \frac{g_0}{2N} \left[ (\bar{\psi}_a \psi_a)^2 + (\bar{\psi}_a i \gamma_5 \psi_a)^2 \right],$$

where the index $a$ runs from 1 to $N$. The fields $\psi(x)$ can be integrated out yielding collective field action:

$$\mathcal{A}_{col} [\sigma] = N \left\{ -\frac{\sigma^2 + \pi^2}{2g_0} - i \text{Tr} \log [i \not{\sigma} - \sigma(x) - i \gamma_5 \pi] \right\}.$$ 

This model is invariant under the set of chiral $O(2)$ transformations which rotate $\sigma$ and $\pi$ fields into each other. In large-N limit, the model has a second order phase transition at which fermions acquire mass. At zero temperature in 2+1 dimensions it is accompanied by an appearance of a massless composite Goldstone boson (for details see e.g. the review $^3P_2$). In the symmetry-broken phase the model is characterized by a typical “mexican

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hat" effective potential. The propagator of the massive \( \sigma \) fluctuations can be readily extracted and it coincides with the \( \sigma \)-propagator of the ordinary GN model \( \hat{G} \):
\[
G_{\sigma'\sigma} = -\frac{i}{N} \left[ g_0^{-1} - i \frac{1}{2\pi^2} \frac{1}{|q|^2 + \Delta^2} \right]^{-1}
\]

Where \( M \) is the mass dynamically acquired by fermions. At finite temperature according to standard dimensional reduction arguments the system is effectively two dimensional and thus the Coleman theorem forbids the spontaneous breakdown of the \( U(1) \)-symmetry. However, as it was shown by Witten, this does not preclude the system from generating fermion mass. That is, as was shown in \( \hat{G} \), employing "modulus-phase" variables
\[
\sigma + i\pi = |\Delta| e^{i\theta}
\] one can see that the system generates fermion mass \( M = |\Delta| \) that coincides with the modulus of the complex order parameter, but its phase remains incoherent and the correlators of the complex order parameter have algebraic decay. Existence of the local gap modulus \( \Delta \) does not contradict Coleman theorem since \( \Delta \) is neutral under \( U(1) \) transformations. Thus at low temperature in \( 2+1 \) dimensions there appears an "almost" Goldstone boson that becomes a real Goldstone boson at exactly zero temperature.

Let us start our study with an inspection of the effective potential of the model at finite temperature in the leading order approximation and then take into account the next-to-leading order corrections. Following \( \hat{G} \), the fermion mass at finite temperature is given by the gap equation which coincides with the gap equation for ordinary GN model with discrete symmetry (for detailed calculations see \( \hat{G} \)):
\[
1 = g_0 \text{tr} \left( \int \frac{d^2k}{(2\pi)^2} \frac{1}{2E} \tanh \left( \frac{E}{2T} \right) \right),
\]

where \( E \) stands for \( \sqrt{k^2 + \Delta^2} \). In the leading order mean-field approximation we have a situation similar to BCS superconductor: there is a gap that disappears at certain temperature which, in what follows, we denote by \( T^* \). It can be expressed via the gap function at zero temperature:
\[
T^* = \frac{\Delta(0)}{2 \log 2}.
\]

Near \( T^* \) the gap function has, in the mean-field approximation, the following behavior:
\[
\Delta(T) = T^* 4 \sqrt{\log 2} \sqrt{1 - \frac{T}{T^*}}.
\]

On the other hand at low temperatures the gap function receives exponentially small temperature correction:
\[
\Delta(T) = \Delta(0) - 2T \exp \left[ -\frac{\Delta(0)}{T} \right]
\] (5)

Lets us stress once more: a straightforward calculation of next-to-leading order corrections gives that the gap should be exactly zero at any finite temperature in \( 2+1 \) dimensions. However as it was shown in the paper mentioned above \( \hat{G} \), such a direct calculation of corrections misses essential physics of a two-dimensional problem. The fluctuations can be made arbitrarily weak by decreasing temperature in \( 2+1 \) dimensions and at \( T = 0 \) a local gap becomes a real gap. Most interesting effect happens however when temperature is increased. It was anticipated before that there is only one characteristic temperature in a such system - namely the temperature of the Kosterlitz-Thouless (KT) transition which (as was anticipated) coincides with temperature of the formation of the local gap. We show below that this scenario holds true only at very large number of field components. In general, as we show, the model has two characteristic temperatures like in the case of a superconductor with a pseudogap. In order to show it we have to go beyond mean-field approximation.

Let us make an expansion around a saddle point solution and derive the propagator of the imaginary part of the field \( \Delta \) that has a pole at \( q^2 = 0 \). The procedure is standard and will not be reproduced here, see for details \( \hat{G} \).

\[
G_{\Delta^{i\pi} \Delta^{i\pi}} = \frac{1}{N} \left[ \frac{1}{8\pi\Delta(T)} \tanh \left( \frac{\Delta(T)}{2T} \right) \right]^{-1} \frac{i}{q^2}
\] (6)

The propagator \( \hat{G} \) characterizes stiffness of the phase fluctuations in the degenerate minimum of the effective potential. It gives the following expression for the kinetic term of phase fluctuations for the chiral GN model:
\[
E_{\text{kin}} = \int d^2x \frac{N}{8\pi} \Delta(T) \tanh \left( \frac{\Delta(T)}{2T} \right) |\nabla \theta|^2
\] (7)

Now we have all the tools to find the position of the KT transition in the chiral GN model. It is well-known, that the KT transition can not be found by straightforward perturbative methods. In order to find a position of the KT transition one should first observe that in variables
In two dimensions such a system possesses excitations of the form of vortices and antivortices that are attracted to each other by a Coulomb potential. At low temperatures, the vortices and antivortices form bound pairs. In order to study the phase disorder transition in the chiral GN model we should solve a set of equations: equation for \( T_{KT}(\Delta, N) \) that follows from the kinetic term and equation \( \mathcal{H} \) for the gap modulus \( \Delta(T_{KT}, N) \) that follows from effective potential. I.e. in our case the phase transition is a competition of two processes: the thermal excitation of directional fluctuations in the degenerate valley of the effective potential and the thermal depletion of the stiffness coefficient.

Let us first consider the case of small \( N \). From expressions (3), (4), (5) we see that in the regime of small \( N \) the vortex pairs break up which is the Kosterlitz-Thouless phase-disorder transition. The key feature of the GN models, namely number of field components and \( \Delta \) [see eq. (7)].

The temperature of the Kosterlitz-Thouless phase transition in the system (8) is given by (3):

\[
T_{KT} = \frac{\pi}{2} \beta.
\]

In order to study the phase disorder transition in the chiral GN model we should solve a set of equations: equation for \( T_{KT}(\Delta, N) \) that follows from the kinetic term and equation \( \mathcal{H} \) for the gap modulus \( \Delta(T_{KT}, N) \) that follows from effective potential. I.e. in our case the phase transition is a competition of two processes: the thermal excitation of directional fluctuations in the degenerate valley of the effective potential and the thermal depletion of the stiffness coefficient.

Thus the Kosterlitz-Thouless transition will take place at the temperature:

\[
T_{KT} = \frac{N}{8} \Delta(0)
\]

which at small \( N \) is significantly lower than the temperature (8) at which the gap modulus disappears. For the ratio \( T_{KT}/T^* \) at small \( N \) we obtain:

\[
\frac{T_{KT}}{T^*} = \frac{N \log(2)}{4}.
\]

So with decreasing \( N \) separation of \( T_{KT} \) and \( T^* \) increases. Let us now turn to the regime when \( N \) is no longer small.

Then from the equations (1), (2), (3), (4) we see that in this regime \( T_{KT} \) tends from below to \( T^* \). The Hamiltonian (2) in this limit reads near \( T^* \):

\[
H_{\text{kin}} = \int dx \frac{N}{16\pi} \frac{\Delta(T)^2}{T} |\nabla \theta|^2.
\]

From the eqs. (1), (3), (4), (5) we find the following expression for behavior of \( T_{KT} \) at large \( N \):

\[
T_{KT} \simeq T^* \left( 1 - \frac{1}{N \log(2)} \right).
\]

This equation explicitly shows merger of the temperatures \( T_{KT} \) and \( T^* \) in the limit of large \( N \). This can be interpreted as the restoration of the “BCS-like” scenario for the quasi-condensation in the limit \( N \to \infty \). The ratio \( T_{KT}/T^* \) is displayed on Fig. 1.

![FIG. 1. Recovery of a “BCS-like” scenario for quasicondensation at large N in the chiral GN model. The solid curve is the ratio of the temperatures of the KT transition (\( T_{KT} \)) and the characteristic temperature of the formation of the effective potential (\( T^* \)). As it is shown on the picture this ratio tends from below to unity (the horizontal dashed line) as N is increasing and the region of precursor fluctuations shrinks.](image)

Thus the phase diagram of the model at small \( N \) consists of the following phases at non-zero temperature: (i) \( 0 < T < T_{KT} \) - the low temperature quasi-ordered phase with bound vortex-antivortex pairs. (ii) \( T_{KT} < T < T^* \) - the phase analogous to pseudogap phase of superconductors: i.e. the chirally symmetric phase with unbounded vortex-antivortex pairs that exhibit violent precursor fluctuations and a nonzero local modulus of the order parameter. (iii) \( T > T^* \) - high temperature “normal” chirally symmetric phase.

The mechanism of the phase separation is very transparent with the key feature being the fact that the stiffness is proportional to \( N \) [see eq. (1)]. At large \( N \) the directional fluctuations are energetically extremely expensive and thus, the phase transition is controlled basically by the modulus of the order parameter. On the other hand at small \( N \), the stiffness is low, and the thermal excitation of the fluctuations in the degenerate valley of the
effective potential starts governing the phase transition in the system.

Now let us briefly discuss the physical meaning of $T^*$ and what one can expect to happen when the system reaches it at small $N$. At first, as we can conclude from simple physical reasoning in analogy with superconductivity, the appearance of the second characteristic temperature is very natural. Besides the fact that the phase analogous to the intermediate phase $T_{KT} < T < T^*$ it is the dominating region on a phase diagram of strong-coupling and low carrier density superconductors, the similar effects are known in a large variety of condensed matter systems such as exitonic condensates, Josephson junction arrays and many other systems. One of the most illuminating examples of the appearance of the pseudogap phase is the chiral GN model in $2 + \epsilon$ dimensions at zero temperature where this phenomenon is governed by quantum dynamical fluctuations at small $N$ \[5\]. In $D=2+\epsilon$ the presence of two small parameters in the problem has allowed to prove existence and different physical origin of two characteristic values of renormalized coupling constant and the formation of an intermediate pseudogap phase \[3\]. We can also observe that the mean-field approximation gives a second order phase transition at $T^*$ which is certainly an artifact since much above $T_{KT}$ there are violent thermal phase fluctuations. These fluctuations should wash out the phase transition at $T^*$ which should degenerate to a smeared crossover as it happens in superconductors. Apparently, this crossover can not be studied adequately in the framework of $1/N$-expansion. The most insight into the properties of the system in the region $T_{KT} < T < T^*$ can be obtained by numerical simulations. Although the KT transition is very hard to observe in simulations, the hint for the phase separation would be a gradual degradation of the transition at $T^*$ with decreasing $N$.

Important circumstance is that the pseudogap phase as a precursor of the chiral phase transition is not a phenomenon common only for low dimensional systems but also should occur at small $N$ in higher dimensions and also in systems with other symmetries. One should however observe that as it was shown recently by us, unfortunately no direct generalization of the discussed here nonlinear sigma model approach is possible to NJL model in $3 + 1$ dimensions \[7\], at least in a closed form.

Unfortunately the letter format of this paper does not allow us to discuss in detail the similarities and differences of these phenomena in the chiral GN model and in the models of superconductivity with precursor formation of Cooper pairs. A detailed discussion of these aspects is currently in preparation and will be presented elsewhere.

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