An Incompressible PH-Pfaffian State for $\nu = 5/2$ Quantum Hall Effect

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The PH-Pfaffian state having 1/2 central charge is consistent with the thermal Hall conductance measurement of $\nu = 5/2$ fractional quantum Hall system, but lacks support from the existing numerical results. In this report we propose a new state described by the wave function $Pf(\sum_{i<j}^N (z_i - z_j)^2) \prod_{i<j} (z_i - z_j)^2$, with $Pf(A)$ being the Pfaffian of an antisymmetric matrix $A$. We call this new state the incompressible PH-Pfaffian state, as it is formed by increasing the relative angular momentum by two for each of the paired composite fermions of the PH-Pfaffian state (as seen from the numerator $(z_i - z_j)^2$ inside the Pfaffian symbol $Pf$), making the PH-Pfaffian state from possibly compressible to incompressible. We argue that while the incompressible PH-Pfaffian state is distinct, it has the same central charge as the PH-Pfaffian state, and is therefore consistent with the thermal Hall conductance measurement. In spherical geometry, the incompressible PH-Pfaffian state has the same magnetic flux number $N_\phi = 2N + 1$ as the anti-Pfaffian state, allowing a direct numerical comparison between the two states. Results of exact diagonalization of finite systems in the second Landau level show that, by increasing the short range component of the Coulomb interaction, the ground state undergoes a phase transition from the anti-Pfaffian state to the incompressible PH-Pfaffian state, thus providing the numerical support to the latter.

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After thirty five years since its discovery, the understanding of the fractional quantum Hall effect (FQHE) at $\nu = 5/2$ filling factor is still elusive and remains a great intellectual challenge. Among the three primary candidates, Pfaffian state and its PH conjugate, the anti-Pfaffian state, form two degenerate but distinct states in the absence of Landau level mixing [6][7]. When the Landau level mixing is properly taken into account, the degeneracy between the Pfaffian and anti-Pfaffian states is lifted, and the numerical studies are in favor of the PH-Pfaffian state energetically [8], making the anti-Pfaffian state a more likely a candidate for the ground state of the $\nu = 5/2$ FQHE. The third topologically different state that is Particle-hole (PH) symmetric, hence termed PH-Pfaffian state [9][10][11] has also attracted a great attention. Unfortunately it lacks any numerical support. In spherical geometry, no consistent gapped ground state is found to exist at the total flux number $N_\phi = 2N - 1$ as required by the PH symmetry even with a wide range of variations of Coulomb interactions [11]. This is consistent with the findings that the PH-Pfaffian state may fail to represent a gapped, incompressible phase [12].

While the existing numerical results seem to converge to a consensus that the anti-Pfaffian state is the most likely candidate for the ground state of the $\nu = 5/2$ FQHE, the thermal Hall conductance measurement [13] casts a great doubt on this consensus. Although the thermal Hall conductance measurement, $\kappa_{xy} = 5/2$ in units of $\frac{2e^2}{h}$, is rather encouraging and pointing to the existence of non-abelian quasiparticles, it is incompatible with the edge structure of anti-Pfaffian, but rather consistent with the PH-Pfaffian topological order. In view of the discrepancy between the numerical and experimental results, some more complicated proposals are put forward to resolve the discrepancy. Among them are disorder induced mesoscopic puddles composed of Pfaffian and anti-Pfaffian states with effective edge structure that is consistent with the thermal conductance measurement [13][13][16], and incomplete thermal equilibration on an anti-Pfaffian edge, which are all under debate [17][18].

In the spherical geometry, the Pfaffian state is formed at $N_\phi = 2N - 3$, the anti-Pfaffian at $N_\phi = 2N + 1$, and the PH-Pfaffian state at $N_\phi = 2N - 1$. In an attempt to search for numerical support for the PH-Pfaffian topological order, in [15] we asked the following question: can we form an incompressible state at $N_\phi = 2N - 3$ or $N_\phi = 2N + 1$ that is not PH symmetric yet maintains the PH-Pfaffian topological order and is energetically more favorable (or has larger overlap with the exact ground state) than the Pfaffian state or anti-Pfaffian state, at least for a certain parameter range of the Coulomb interactions? The answer was yes. The idea is to add two Laughlin type abelian quasiparticles to the PH-Pfaffian state, and make the two quasiparticles form a uniform state with the maximum avoidance from one another. The resulting state is termed as a compressed PH-Pfaffian state, as it can be viewed as the result of “compressing” the PH-Pfaffian state with two flux quanta removed. The compressed PH-Pfaffian state is not PH symmetric but possesses the PH-Pfaffian topological order. Since both the compressed PH-Pfaffian state and the Pfaffian state formed at $N_\phi = 2N - 3$, it allows for a direct numerical comparison between the two. The finite size numerical results show that the overlap of the exact ground state with the compressed PH-Pfaffian state exceeds that with the Pfaffian state when the short range component of the Coulomb interaction increases to a certain level, lending a numerical support
to the compressed PH-Pfaffian state.

While the compressed PH-Pfaffian state blows a life to the PH-Pfaffian topological order numerically, a question was raised if the finite size result would survive in the thermal dynamic limit as one would think two quasiparticles in the ground state would make no difference in the thermal dynamic limit \[20 \text{[21].} \] It is the main purpose of this brief report to address this apparent "two quasiparticles" thermal dynamic shortcoming with a more elegant resolution. To this end, we propose a new state described by the following wave function

\[
\Psi_{1PH} = P_{LLL} P \frac{1}{z_i^* - z_j^*} \prod_{i<j} (z_i - z_j)^2 \tag{1}
\]

where \(z_j = x_j + iy_j \) is the complex coordinate of the \(j\)th electron, \(z_j^* = x_j - iy_j\), \(N \) is the total number of electrons, \(P_{LLL}\) is the lowest Landau level projection operator, and \( Pf[A] \) is the Pfaffian of an antisymmetric matrix \( A \). We call this new state the incompressible PH-Pfaffian state, as compared to the following PH-Pfaffian wave function

\[
\Psi_{PH} = P_{LLL} P \frac{1}{z_i^* - z_j^*} \prod_{i<j} (z_i - z_j)^2 \tag{2}
\]

it can be viewed as the result of increasing the relative angular momentum by two for each of the paired composite fermions of the PH-Pfaffian state, as seen from the numerator \((z_i - z_j)^2\) inside the Pfaffian symbol \( Pf \) in Eq. (1). We argue that while the incompressible PH-Pfaffian state is topologically distinct, it has the same central charge as the PH-Pfaffian state, and is therefore consistent with the thermal Hall conductance measurement. Furthermore, as the increasing of the relative angular momentum by two for each of the paired composite fermions of the PH-Pfaffian state is "built in" intrinsically in the incompressible PH-Pfaffian state, we believe it is not subject to the apparent "two quasiparticles" thermal dynamic shortcoming suffered by the compressed PH-Pfaffian state as discussed above.

To make it numerically easier to deal with when projecting to the lowest Landau level, we will use an alternative form of Eq. (1):

\[
 Pf \left( \frac{\partial}{\partial z_i} - \frac{\partial}{\partial z_j} \right) \prod_{i<j} (z_i - z_j)^3 \tag{3}
\]

or written in spherical geometry

\[
 Pf \left( \frac{\partial}{\partial u_i} - \frac{\partial}{\partial v_i} \right) \prod_{i<j} (\partial_{u_i} - \partial_{v_j}) \Phi_3 \tag{4}
\]

where \( \Phi_3 = \prod_{i<j} (u_i v_j - u_j v_i)^3 \) is the Laughlin wave function, and \((u, v)\) are the spinor variables describing electron coordinates.

Since the total flux number \( N_\phi \) corresponding to the PH-Pfaffian wave function is \( N_\phi = 2N - 1 \), and the incompressible PH-Pfaffian is formed from the PH-Pfaffian state by increasing the relative angular momentum by two for each of the paired composite fermions of the PH-Pfaffian state, the relationship between the flux number \( N_\phi \) and the number of electrons \( N \) is \( N_\phi = 2N + 1 \). This is the same \( N_\phi - N \) relationship for the anti-Pfaffian state. The fact that both the incompressible PH-Pfaffian state and the anti-Pfaffian state have the same \( N_\phi - N \) relationship \( N_\phi = 2N + 1 \), allows for a direct numerical comparison between the two to determine which wave function, therefore which topological order and at what condition, represents the exact ground state.

In Fig. 1 we calculated and plotted the overlap of the exact ground state of a finite system \((N_\phi, N) = (13, 6)\) with the anti-Pfaffian state and with the incompressible PH-Pfaffian state respectively. The exact ground state is obtained in the second Landau level free of Landau level mixing, with the ratios of \( V_1 / V_1^c \) ranging from 1 to 1.5, where \( V_1^c \) is the Coulomb value of \( V_1 \) in the second Landau level. We see the ground state undergoes a phase transition from the anti-Pfaffian state to the incompressible PH-Pfaffian state as the short range component of the Coulomb interaction increases. The transition occurs at \( V_1 / V_1^c \approx 1.15 \), as at this point the overlap of the exact ground state with the incompressible PH-Pfaffian state exceeds that with the anti-Pfaffian state. Therefore, if in the real system the Coulomb interaction falls in a range such that the ground state is the incompressible PH-Pfaffian state, the system will support the edge structure that is consistent with the thermal Hall conductance measurement.

![FIG. 1: For \( N = 6 \) and \( N_\phi = 13 \). Overlap of the exact ground state with the anti-Pfaffian state (solid line) and the incompressible PH-Pfaffian state (dashed line) as the function of the pseudopotential \( V_1 \) normalized by its Coulomb value \( V_1^c \) in the second Landau level.](image-url)
As there exists a PH conjugate state of the Pfaffian state, the anti-Pfaffian, there also exists a PH conjugate state of the incompressible PH-Pfaffian state. Since the number of electrons \( N \) is related to the number of holes \( N_h \) of the PH conjugate state by \( N + N_h = N_\phi + 1 \), the relationship between the flux number and the number of holes of the anti-Pfaffian or the PH conjugate state of the incompressible PH-Pfaffian state is \( N_\phi = 2N_h - 3 \). While Pfaffian and anti-Pfaffian are two topologically distinct states, we believe the incompressible PH-Pfaffian state and its PH conjugate state have the same topological order. In the absence of PH symmetry breaking factors such as Landau level mixing, the same transition from the Pfaffian state to the PH conjugate of the incompressible PH-Pfaffian state will take place at the same short range interaction strength.

Before closing, we would like to point out that Eq.(1) can be generalized to:

\[
Pf\left(\frac{(z_i - z_j)^p}{(z_i^* - z_j^*)^q}\right) \prod_{i<j} (z_i - z_j)^m \tag{5}
\]

and

\[
Pf\left(\frac{(z_i^* - z_j^*)^p}{(z_i - z_j)^q}\right) \prod_{i<j} (z_i - z_j)^m \tag{6}
\]

where \( p, q, \) and \( m \) are positive integers, with \( m \geq q \) and \( m + p + q \) being odd for fermions and even for bosons. Among them, it is of a particular interest when \( p = 1, m = q = 2 \) as Eq.(5) becomes

\[
Pf\left(\frac{(z_i - z_j)}{(z_i^* - z_j^*)}\right) \prod_{i<j} (z_i - z_j)^2 \tag{7}
\]

describing a state at \( N_\phi = 2N + 1 \). The study of Eq.(7) will be reported in future works.

In conclusion, we have proposed an incompressible PH-Pfaffian state by increasing the relative angular momentum by two for each of the paired composite fermions of the PH-Pfaffian state, making the PH-Pfaffian state from compressible to incompressible. The incompressible PH-Pfaffian state is not particle-hole symmetric but possesses the PH-Pfaffian topological order. In the spherical geometry, results of exact diagonalization of finite systems in the second Landau level show that, by increasing the short range component of the Coulomb interaction, the ground state undergoes a phase transition from the anti-Pfaffian state to the incompressible PH-Pfaffian state before further entering into a gapless state. More works are required on a few fronts: First we need to see if the parameter range of the Coulomb interaction at which the system is in the incompressible PH-Pfaffian state matches realistic conditions. Secondly, we need to study larger size finite systems to verify if the incompressible PH-Pfaffian state can survive in the thermal dynamic limit. It will also be interesting to study Eq.(5) and Eq.(6) with various values of \( p, q, \) and \( m \).

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