Numerical evidence for the Maldacena conjecture in two-dimensional $\mathcal{N} = (8, 8)$ super Yang–Mills theory

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The $\mathcal{N} = (8, 8)$ super Yang–Mills theory in 1 + 1 dimensions is solved at strong coupling to directly confirm the predictions of supergravity at weak coupling. The calculations are done in the large-$N_c$ approximation using Supersymmetric Discrete Light-Cone Quantization. The stress-energy correlator is obtained as a function of the separation $r$; for intermediate values of $r$, the correlator behaves in a manner consistent with the $1/r^5$ behavior predicted by weak-coupling supergravity.

1. INTRODUCTION

The conjectured correspondences [1,2] between certain string theories and supersymmetric Yang–Mills (SYM) theories at large-$N_c$ can be tested directly if one is able to solve an SYM theory at strong coupling. The solution can then be compared to a small-curvature supergravity approximation to the corresponding string theory. This approach requires a nonperturbative technique for SYM theories, and one has been developed over the past several years as a supersymmetric form [3,4] of discrete light-cone quantization [5,6], known as SDLCQ.

The SDLCQ approach is a Hamiltonian formulation in a Fock basis using light-cone coordinates [7]. The choice of coordinates allows for a simple vacuum and a consistent Fock expansion. The momentum-space wave functions in the different Fock sectors then satisfy coupled integral equations which can be discretized to obtain a matrix representation of the bound-state eigenvalue problem. The discretization is actually done at the level of second-quantized operators expanded in momentum modes; however, this ultimately yields the same matrix eigenvalue problem.

The quantity which we compute in order to test a correspondence is a two-point correlator of the stress-energy tensor. The correlator is computed in SDLCQ by inserting a sum over the eigenstates obtained in the matrix diagonalization. To keep the calculation manageable, we consider the case of $\mathcal{N} = (8, 8)$ SYM theory in two dimensions which corresponds to a particular type IIB string theory [2]. The supergravity approximation to the correlator is discussed in [8]. The potential for an SDLCQ check of this correspondence was explored there and in subsequent work [9], but the numerical resolution available in this earlier work was insufficient for a true test. We have now reached a resolution sufficient to provide evidence of consistency in the correspondence [10].

An outline of the remainder of the paper is as follows. Section 2 describes the SDLCQ method. The particular SYM theory is described in Sec. 3 and the calculation of the correlator is formulated in Sec. 4. For comparison purposes, we consider both $\mathcal{N} = (8, 8)$ and $\mathcal{N} = (2, 2)$ theories. The results are presented and discussed in Sec. 5. Some additional discussion is provided in Sec. 6.

2. SUPERSYMMETRIC DISCRETE LIGHT-CONE QUANTIZATION

We use light-cone coordinates as suggested by Dirac [7]. The time coordinate is $x^+ = (t+z)/\sqrt{2}$, and the space coordinate is $x^- \equiv (t-z)/\sqrt{2}$. The conjugate variables are the light-cone energy $p^- = (E-p_z)/\sqrt{2}$ and momentum $p^+ \equiv (E+p_z)/\sqrt{2}$. The mass-shell condition $p^2 = m^2$ then
The discretization proposed by Pauli and Brodsky, known as discrete light-cone quantization (DLCQ), is based on the imposition of periodic boundary conditions on a light-cone box \(-L < x^- < L\). This yields a discrete grid in momentum space where individual momenta are specified by \(p_i^+ = \frac{2\pi}{L} n_i\), with \(n_i\) a positive integer. For fixed total momentum \(P^+\), the limit \(L \to \infty\) is exchanged for a limit in terms of the integer resolution \(K \equiv \frac{2\pi}{L} P^+\). Since the individual momenta are strictly positive, \(K\) is the upper limit on the number of particles allowed by the discretization. Integrals are replaced by discrete sums

\[
\int dp^+ f(p^+) \approx \frac{\pi}{L} \sum_n f(n\pi/L),
\]
and Dirac delta functions become Kronecker delta functions

\[
\delta(p^+ - p'^+) \to \frac{L}{\pi} \delta_{nn'}.
\]

Supersymmetric DLCQ (SDLCQ) is constructed to maintain the supersymmetry algebra

\[
\{Q^+, Q^-\} = 2\sqrt{2}P^+,
\]
\[
\{Q^-, Q^-\} = 2\sqrt{2}P^-,
\]
\[
\{Q^+, Q^-\} = -4P_+.
\]

This algebra is satisfied explicitly by first discretizing the supercharge \(Q^-\) and computing

\[
P_{SDLCQ} = \frac{1}{2\sqrt{2}} \{Q^-, Q^-\}.
\]

In ordinary DLCQ, the \(P^-\) operator is discretized directly and the supersymmetry algebra is not satisfied, except in the limit of infinite resolution. By preserving the supersymmetry algebra, SDLCQ preserves supersymmetry in the spectrum, even at finite resolution.

3. SUPER YANG–MILLS THEORIES

The \(N=(8,8)\) SYM theory is obtained by reducing \(N = 1\) SYM theory from ten to two dimensions. The action in light-cone gauge (\(A_\perp = 0\)) is

\[
S_{1+1}^{LC} = \int dx^+ dx^- \text{tr} \left[ \partial_\perp X_I \partial_- X_I + i\theta_\perp^R \partial_\perp \theta_R + i\theta_\perp^L \partial_- \theta_L + \frac{1}{2}(\partial_- A_\perp)^2 + gA_\perp J^+ + \sqrt{2}g\theta_\perp^R \beta_I [X_I, \theta_R] + g^2 \left[ X_I, X_J \right]^2 \right].
\]

Here the \(X_I\), with \(I = 1, \ldots, 8\), are the scalar remnants of the transverse components of the ten-dimensional gauge field \(A_\mu\). The two-component spinor fields \(\theta_R\) and \(\theta_L\) are remnants of the right-moving and left-moving projections of the original sixteen-component spinor. We also have the current \(J^+ = i[X_I, \partial_- X_I] + 2\theta_\perp^R \theta_R\) and two matrices \(\beta_1 \equiv \sigma_1, \beta_2 \equiv \sigma_3\). The supercharges for this theory can be obtained by dimensionally reducing the ten-dimensional superscurrent, which yields

\[
Q_\alpha^- = \int dx^- \text{tr} \left[ -\frac{3}{4}J^+ \frac{1}{\partial_-} u_\alpha + 2^{-1/4} i[X_I, X_J](\beta_I \beta_J)_{\alpha \eta} u_\eta \right],
\]
where \(\alpha, \eta = 1, \ldots, 8\) and the \(u_\alpha\) are the components of \(\theta_R\). The mode expansions of the dynamical fields are

\[
X_{Ipq}(x^-) = \frac{1}{\sqrt{2\pi}} \int_0^\infty \frac{dk^+}{\sqrt{2k^+}} \times \left[ A_{Ipq}(k^+) e^{-ik^+ x^-} + A_{Iqiq}(k^+) e^{ik^+ x^-} \right],
\]
and

\[
u_{\alpha pq}(x^-) = \frac{1}{\sqrt{2\pi}} \int_0^\infty \frac{dk^+}{\sqrt{2}} \times \left[ B_{\alpha pq}(k^+) e^{-ik^+ x^-} + B_{\alpha qpi}(k^+) e^{ik^+ x^-} \right],
\]
where \(p, q = 1, 2, \ldots, N_c\).

The operators \(A\) and \(B\) satisfy the (anti)commutation relations

\[
[A_{Ipq}(k^+), A_{Jrs}^\dagger(k^+) ] = \delta_{IJ} \delta_{pr} \delta_{qs} \delta(k^+ - k'^+),
\]

\[
\{B_{\alpha pq}(k^+), B_{\beta rs}^\dagger(k^+) \} = \delta_{\alpha \beta} \delta_{pr} \delta_{qs} \delta(k^+ - k'^+).
\]
In the discrete approximation, we rescale the annihilation operators as
\[
\sqrt{\frac{L}{\pi}}a(k) = A(k^+ = \frac{\pi k}{L}),
\]
\[
\sqrt{\frac{L}{\pi}}b(k) = B(k^+ = \frac{\pi k}{L}).
\]
We then have
\[
[a_{tpq}(k), a^\dagger_{tpq}(k')] = \delta_{lq} \delta_{pr} \delta_{qs} \delta_{kk'},
\]
\[
\{b_{tpq}(k), b^\dagger_{tpq}(k')\} = \delta_{lq} \delta_{pr} \delta_{qs} \delta_{kk'},
\]
and
\[
X_{tpq}(x^-) = \frac{1}{\sqrt{2\pi}} \sum_{k=1}^{\infty} \frac{1}{\sqrt{2k}} [a_{tpq}(k)e^{-i\frac{2\pi}{L}kx^-} + a^\dagger_{tpq}(k)e^{i\frac{2\pi}{L}kx^-}] \tag{13}
\]
\[
u_{tpq}(x^-) = \frac{1}{\sqrt{2L}} \sum_{k=1}^{\infty} \frac{1}{\sqrt{2}} [b_{tpq}(k)e^{-i\frac{2\pi}{L}kx} + b^\dagger_{tpq}(k)e^{i\frac{2\pi}{L}kx}].
\]

The presence of extended supersymmetry means that there are eight different Hamiltonians \(P_{\alpha} = \{Q_{\alpha}, Q_{\beta}\} / 2\sqrt{2}\), any one of which can be diagonalized. There exist unitary transformations between them to guarantee that the spectrum is the same for all. We will work with \(P^\alpha\).

This Hamiltonian can be block diagonalized by taking advantage of various symmetries. The generators of these symmetries are given in Table I. The block diagonalization significantly reduces the computational load for diagonalization.

The \(N=2\) SYM theory is obtained through reduction of the \(N=1\) SYM theory from four to two dimensions. The action is the same as for \(N=8\), except that the indices run from 1 to 2 instead of 1 to 8. Just as there are fewer dynamical fields, there are also fewer symmetries. The smaller number of fields allows calculations at higher resolution; we have reached \(K = 14\) for the \((2, 2)\) theory vs \(K = 11\) for the \((8, 8)\) theory. However, there is no conjecture of correspondence or any separate estimate of the correlator’s behavior for the \((2, 2)\) theory.

4. STRESS-ENERGY CORRELATOR

The stress-energy correlation function that we compute is
\[
F(x^-, x^+) \equiv \langle T^{++}(x)T^{++}(0)\rangle.
\]
For the string theory corresponding to two-dimensional \(N=8\) SYM theory, \(F\) can be calculated on the string-theory side in a weak-coupling super-gravity approximation. Its behavior for intermediate separations \(r \equiv \sqrt{2x^+x^-}\) is
\[
F(x^-, x^+) = \frac{N^2_c}{g_{\text{SYM}}^4}. \tag{15}
\]
We will compute \(F\) in SYM theory and compare, considering both \(N=8\) and \(2, 2\) theories.

We fix the total momentum \(P^+ = P^-\) and compute the Fourier transform, which can be expressed in a spectral decomposed form as
\[
\tilde{F}(P^-, x^+)
\]
\[
= \frac{1}{2L} \langle T^{++}(P^-, x^+)T^{++}(-P^-, 0)\rangle
\]
\[
= \sum_i \frac{1}{2L} \langle 0|T^{++}(P^-, 0)|i\rangle e^{-iP^+x^+} \times \langle i|T^{++}(-P^-, 0)|0\rangle.
\]
The position-space form is recovered by the inverse transform, with respect to \(P_0 = K\pi/L\). The continuation to Euclidean space is made by taking \(r\) to be real. This yields
\[
F(x^-, x^+) = \sum_i \frac{1}{\pi} \langle 0|T^{++}(K)|i\rangle^2 \tag{17}
\]
\[
\times \left(\frac{x^+}{x^-}\right)^2 \frac{M_4 K_4 (M_4 \sqrt{2x^+x^-})}{8\pi^2 K^3}.
\]

The stress-energy operator \(T^{++}\) is given by
\[
T^{++}(x^-, x^+) = \text{tr} \left[ (\partial_- X^I)^2 \right. \tag{18}
\]
\[
+ \frac{1}{2} (iu^\alpha \partial_- u^\alpha - i(\partial_- u^\alpha)u^\alpha) \right].
\]
In terms of the discretized creation operators, we find
\[
T^{++}(-K)|0\rangle \tag{19}
\]
Table 1
Generators of symmetries for the Hamiltonian $P^-_8$.

|   | $a_1$ | $a_2$ | $a_3$ | $a_4$ | $a_5$ | $a_6$ | $a_7$ | $a_8$ | $b_1$ | $b_2$ | $b_3$ | $b_4$ | $b_5$ | $b_6$ | $b_7$ |
|---|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| 1 | $a_1$ | $a_8$ | $-a_5$ | $-a_4$ | $-a_3$ | $a_6$ | $-a_7$ | $a_2$ | $b_1$ | $b_1$ | $-b_3$ | $b_2$ | $b_7$ | $-b_5$ | $b_2$ |
| 2 | $a_2$ | $a_1$ | $-a_5$ | $-a_4$ | $-a_3$ | $-a_2$ | $a_6$ | $-a_7$ | $b_1$ | $b_3$ | $b_2$ | $b_1$ | $b_3$ | $-b_6$ | $-b_7$ |
| 3 | $a_2$ | $a_1$ | $-a_6$ | $a_7$ | $-a_3$ | $a_5$ | $a_4$ | $b_1$ | $-b_2$ | $b_6$ | $b_5$ | $b_4$ | $b_3$ | $b_6$ | $b_7$ |
| 4 | $-a_1$ | $-a_2$ | $-a_3$ | $-a_4$ | $-a_5$ | $-a_6$ | $a_7$ | $a_8$ | $b_1$ | $b_2$ | $b_3$ | $b_4$ | $b_5$ | $b_6$ | $b_7$ |
| 5 | $a_1$ | $a_2$ | $a_3$ | $-a_4$ | $-a_5$ | $a_6$ | $-a_7$ | $a_8$ | $b_1$ | $b_2$ | $b_3$ | $b_4$ | $b_5$ | $b_6$ | $b_7$ |
| 6 | $-a_1$ | $a_2$ | $a_3$ | $-a_4$ | $-a_5$ | $-a_6$ | $-a_7$ | $a_8$ | $b_1$ | $-b_2$ | $b_3$ | $-b_4$ | $-b_5$ | $b_6$ | $b_7$ |
| 7 | $a_1$ | $-a_2$ | $a_3$ | $-a_4$ | $-a_5$ | $-a_6$ | $-a_7$ | $a_8$ | $-b_1$ | $b_2$ | $b_3$ | $-b_4$ | $-b_5$ | $b_6$ | $-b_7$ |

\[
\begin{align*}
= \pi \frac{\pi}{2L} \sum_{k=1}^{K-1} \left[ -\sqrt{k(K-k)}a_{ij}^\dagger(K-k)a_{ij}(k) \\
+ \frac{K}{2 - k} b_{\alpha ij}^\dagger(K-k)b_{\alpha ij}(k) \right] |0\rangle.
\end{align*}
\]

Thus $(L/\pi)(0|T^{++}(K)|i)$ is independent of $L$. Also, only one symmetry sector contributes.

The correlator behaves like $1/r^4$ at small $r$, as can be seen by taking the limit to obtain

\[
\left(\frac{x^-}{x^+}\right)^2 F(x^-,x^+) \sim \frac{N^2_c(2n_b+n_f)}{4\pi^2 r^4}(1 - \frac{1}{K}).
\]

To simplify the appearance of this behavior, we rescale $F$ by defining

\[
f \equiv \langle T^{++}(x)T^{++}(0) \rangle \left(\frac{x^-}{x^+}\right)^2 \times \frac{4\pi^2 r^4}{N^2_c(2n_b+n_f)}.
\]

Then $f$ is just $(1 - 1/K)$ for small $r$.

We compute $f$ numerically by obtaining the entire spectrum for small matrices and by using Lanczos iterations for large matrices. The Lanczos technique \[\footnote{The Lanczos technique is a powerful method for obtaining eigenvalues and eigenvectors of large matrices.} \] generates an approximate tridiagonal representation of the Hamiltonian which captures the important contributions after only a few iterations and which is easily diagonalized to compute the sum over eigenstates.

5. RESULTS

The log-log derivative of the rescaled correlator $f$ is plotted in Fig. 4 for both SYM theories and for a range of resolution values. At small $r$, the graphs for different $K$ match the expected $(1 - 1/K)$ behavior. At large $r$, the behavior is different between odd and even $K$, although in the intermediate region, the difference gets smaller as $K$ gets bigger. The difference in behavior at large $r$ is due to the absence of an exactly massless state among states that contribute for $K$ odd. For each even $K$ there is a contributing massless state, which allows the correlator to return to the proper $1/r^4$ behavior at large $r$. For odd $K$, the lowest contributing state becomes massless only in the large-$K$ limit. In the $\mathcal{N} = (8,8)$ theory, the corresponding supergravity solution for intermediate $r$ implies that the log-log derivative of the rescaled correlator should be equal to $-1$.

For small and intermediate $r$ we can extrapolate the values of $f$ to infinite resolution. Typical extrapolations for the $(8,8)$ theory are given in Fig. 4. The range of values obtained with fits of different orders to odd and even $K$ provides an estimate of the interval within which the actual value should lie. We display these intervals in Fig. 4. The two theories clearly differ in their behavior for intermediate $r$ and only the $(8,8)$ theory is consistent with the $1/r^5$ behavior predicted for it by the supergravity approximation to the dual string theory.

6. CONCLUSIONS

The calculations that we have done succeed in distinguishing between theories that differ in the amount of extended supersymmetry. Only the result for the $\mathcal{N} = (8,8)$ theory is consistent with a $1/r^5$ behavior and thus with the Maldacena conjecture for this theory \[\footnote{The Maldacena conjecture relates AdS/CFT duality to the AdS supergravity approximation to the dual string theory.} \]. The errors in the extrapolations remain too large to support a claim of computing the expected $1/r^5$ behavior, but the numerical evidence is consistent. Additional cal-
Figure 1. Plots of the log-log derivative of the rescaled correlator $f$ for the (a) $\mathcal{N} = (2,2)$ and (b) $(8,8)$ SYM theories. Each curve corresponds to a different resolution $K$, with $K$ ranging from 3 to 14 in (a) and from 3 to 11 in (b). For odd $K$ the curves are solid, and for even $K$ they are dashed. The separation $r$ is measured in units of $\sqrt{\pi/g^2 N_c}$.

Figure 2. Sample extrapolations for the log-log derivative of the rescaled correlator $f$ in the $\mathcal{N} = (8,8)$ theory for (a) $\log_{10}(r) = 0.2$ and (b) $\log_{10}(r) = 0.5$. The lines show quadratic (solid) and cubic (dashed) fits to the computed points, with the fits done separately for odd and even $K$. 
Figure 3. Summary of extrapolations to infinite resolution for the (a) $\mathcal{N} = (2, 2)$ and (b) $(8, 8)$ SYM theories. The vertical segments represent the intervals obtained by various choices of extrapolations, such as those shown in Fig. 2.

Calculations at higher resolution may soon be possible, and these may well provide explicit confirmation of the conjecture.

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