GLOBAL EXISTENCE AND OPTIMAL DECAY RATE OF SOLUTIONS TO HYPERBOLIC CHEMOTAXIS SYSTEM IN BESOV SPACES

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Abstract. In this paper, we study the qualitative behavior of hyperbolic system arising from chemotaxis models. Firstly, by establishing a new product estimates in multi-dimensional Besov space $\dot{B}^{\frac{d}{2}}_{2, r}(\mathbb{R}^d)(1 \leq r \leq \infty)$, we establish the global small solutions in multi-dimensional Besov space $\dot{B}^{\frac{d}{2} - 1}_{2, r}(\mathbb{R}^d)$ by the method of energy estimates. Then we study the asymptotic behavior and obtain the optimal decay rate of the global solutions if the initial data are small in $B^{\frac{d}{2} - 1}_{2,1}(\mathbb{R}^d) \cap \dot{B}^{0}_{1,\infty}(\mathbb{R}^d)$.

1. Introduction. In this paper, we consider the Cauchy problem of a hyperbolic system arising from chemotaxis models

$$\begin{cases}
\partial_t a - \text{div}(a q) = D \Delta a, & x \in \mathbb{R}^d, t \geq 0, \\
\partial_t q - \nabla a - \epsilon \Delta q = \epsilon \nabla(|q|^2), & x \in \mathbb{R}^d, t \geq 0, \\
(a, q)|_{t=0} = (a_0, q_0), & x \in \mathbb{R}^d.
\end{cases} \tag{1}$$

Chemotaxis is a biological phenomenon describing the oriented movements of cells and micro-organisms population density in response to an external chemical stimulus that spreads in their environment. The mathematical prototype of chemotaxis models, dating to Keller and Segel [8, 9] was proposed to describe the aggregation of cellular slime molds Dictyostelium discoideum. The extensively studied Keller-Segel chemotaxis model takes the following formulation

$$\begin{cases}
\partial_t u = \text{div}(D(u,c)\nabla u - \chi(u,c)\nabla(\phi(c))), \\
\partial_t c = \epsilon(u,c)\Delta c + g(u,c),
\end{cases} \tag{2}$$

where $x = (x_1, x_2, \cdots, x_d) \in \mathbb{R}^d$, $u = u(x,t)$ and $c = c(x,t) > 0$ are called the cell density and the chemical concentration, respectively. $D > 0$ and $\epsilon > 0$ denote the diffusion rate of the cells(bacteria) and chemical substance, respectively. The function $\chi(u,c)$ represents the sensitivity with respect to chemotaxis and $\chi > 0(< 0)$ corresponds to attractive(repulsive) chemotaxis. The potential function $\phi(c)$, also called chemotactic sensitivity function, describes the signal detection mechanism, and $g(u,c)$ characterizes the chemical growth and degradation. With different

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choices of φ(c) and g(u, c), various results have been established, see [7, 13, 11, 2] and the references therein.

If we assume D and ε are positive constants and choose χ(u, c) = χ₀u, φ(c) = ln c and g(u, c) = uc − µc, then the Keller-Segel system (2) takes the following form

\[
\begin{align*}
\partial_t u &= \text{div}(D\nabla u - χ₀u\nabla \ln c), \\
\partial_t c &= \epsilon \Delta c + uc - µc.
\end{align*}
\]

We consider the repulsive (χ₀ < 0) case for (3), then system (1) is derived from (3) under Hopf-Cole transformation

\[
q = \frac{\nabla c'}{c}, \quad \tilde{a} = u
\]

where c' = e^{μt}c, together with scalings

\[
\tilde{x} = x\sqrt{-\frac{1}{χ₀}}, \quad \tilde{q} = q\sqrt{-χ₀}, \quad \tilde{D} = \frac{D}{-χ₀}, \quad \tilde{c} = \frac{c}{-χ₀}
\]

where tilde has been dropped.

Now, let us briefly review some results related to our problem. When the diffusion of chemical substance is so small that it is negligible, i.e. ε = 0, (1) becomes to the hyperbolic-parabolic chemotaxis model and there are many results concerning the qualitative behavior of solutions. Guo et al.[5], Zhang et al.[22] investigated the large-amplitude classical solutions on R without any smallness assumption on the initial data. The long-time behavior of one-dimensional small/large-amplitude classical solutions were established in [22] and [12], respectively. Related results on finite interval in one-dimensional case, we refer the readers to [4, 23, 17] and the references therein. For multi-dimensional case, Li et al. [11] investigated the local/global existence in H^s(\mathbb{R}^d) with s > 1 + \frac{d}{2}. Moreover, blow up criterion and quantitative decay of perturbations of classical solutions were also shown. Hao [6] obtained the global small-amplitude strong solution for initial data close to a constant equilibrium state in critical hybrid Besov spaces B^{\frac{d}{2} - 2}(\mathbb{R}^d) \times B^{\frac{d}{4} - \frac{d}{2} - 1}(\mathbb{R}^d). Fan and Zhao [3] extended the local well-posedness of large-amplitude classical solutions in [11] to H^s(\mathbb{R}^d) with s > \frac{d}{2} and gave several blow up criteria of local strong solutions.

Compared with the wide study on (1) with ϵ = 0, the results on the chemically diffusible mode (i.e. (1) with ϵ > 0) are relatively less. Recently, the authors in [14, 15, 16, 10] considered the traveling wave solution of (1) and its nonlinear stability were established on \mathbb{R}. The global well-posedness, long-time behavior of one-dimensional classical solutions for large data on finite intervals with Neumann-Dirichlet boundary conditions and Dirichlet-Dirichlet boundary conditions were established in [20, 17], respectively. Wang et al. [21] established the global existence, asymptotic decay rates and diffusion convergence rate of small solutions in H^k(\mathbb{R}^d) with d = 2, 3 and some integer k ≥ 2. Martinez et al. [18] obtained the global asymptotic stability of constant ground states, and showed the explicit decay rate of solutions under very mild conditions on initial data in H^2(\mathbb{R}). The purpose of this paper is to establish the asymptotic behavior (global existence and time decay rate) of solutions of (1) in Besov spaces which is a larger important space containing Sobolev space and L^p space. Using the Besov space technique, by establishing a new product estimates in multi-dimensional Besov space \dot{B}^{\frac{d}{2}}_{2,r}(\mathbb{R}^d)(1 ≤ r ≤ \infty), combining with the energy estimates, we establish the global small solutions in multi-dimensional Besov space \dot{B}^{\frac{d}{2} - 1}_{2,r}(\mathbb{R}^d).
Our first result of this paper is stated as follows.

**Theorem 1.1.** Let \( d \geq 2 \) and \( 1 \leq r \leq \infty \). Assume that \((a_0 - \bar{a}, q_0) \in \dot{B}^{\frac{d}{2} - 1}_{2,r}\) for some constant background state \( \bar{a} > 0 \). There exists a constant \( \sigma > 0 \) such that if
\[
\| (a_0 - \bar{a}, q_0) \|_{\dot{B}^{\frac{d}{2} - 1}_{2,r}} \leq \sigma,
\]
then (1) has a unique global-in-time solution \((a - \bar{a}, q) \in \dot{L}^{\infty}_t(\dot{B}^{\frac{d}{2} - 1}_{2,r}) \cap \dot{L}^1_t(\dot{B}^{\frac{d}{2} + 1}_{2,r}).\)
Moreover, there exists a constant \( C_0 > 0 \), and we have
\[
\| (a - \bar{a}, q) \|_{L^\infty_t(\dot{B}^{\frac{d}{2} - 1}_{2,r}) \cap L^1_t(\dot{B}^{\frac{d}{2} + 1}_{2,r})} \leq C_0 \| (a_0 - \bar{a}, q_0) \|_{\dot{B}^{\frac{d}{2} - 1}_{2,r}}
\]
for all \( T \geq 0 \).

Our second result concerns the asymptotic behavior of solutions. The classical Littlewood-Paley decomposition implies that \( f \in \dot{H}^{-s}(\mathbb{R}^d) \) for any \( s \in (0, \frac{d}{2}) \) if \( f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d) \). Wang et al. showed in [21] that if the initial data further satisfied \((a_0 - \bar{a}, q_0) \in \dot{H}^{-s}(\mathbb{R}^d) \) for some \( s \in (0, \frac{d}{2}) \) and some constant background state \( \bar{a} > 0 \), then
\[
\| \nabla^l (a - \bar{a}) \|_{L^2(\mathbb{R}^d)} + \| \nabla^l q \|_{L^2(\mathbb{R}^d)} \leq C (1 + t)^{-\frac{s}{2}} , \quad l = 0, 1, ..., k - 1 ,
\]
\[
\| \nabla^k (a - \bar{a}) \|_{L^2(\mathbb{R}^d)} + \| \nabla^k q \|_{L^2(\mathbb{R}^d)} \leq C (1 + t)^{-\frac{k}{2}}
\]
for any \( k \). The classical Littlewood-Paley decomposition also yields that the low frequency part of \( f \) in \( \dot{H}^{-s}(\mathbb{R}^d) \) lies in \( \dot{B}^{1}_{1,\infty}(\mathbb{R}^d) \) and \( L^1(\mathbb{R}^d) \subset \dot{B}^{0}_{1,\infty}(\mathbb{R}^d) \), thus we see that \( \dot{B}^0_{1,\infty}(\mathbb{R}^d) \) is a natural function space for system (1). Oktia [19] established the perturbations decay for global solutions to the compressible Navier-Stokes equations in critical Besov spaces, if the initial perturbations of density and velocity are small in \( \dot{B}^{\frac{d}{2}}_{2,1}(\mathbb{R}^d) \cap \dot{B}^0_{1,\infty}(\mathbb{R}^d) \) and \( \dot{B}^{\frac{d}{2} - 1}_{2,1}(\mathbb{R}^d) \cap \dot{B}^0_{1,\infty}(\mathbb{R}^d), \) respectively. Inspired by Okita’s work, our second purpose in this paper is to study the asymptotic behavior for the obtained solutions. Precisely, we have the following decay estimates.

**Theorem 1.2.** Let \( d \geq 2 \). Assume that \((a_0 - \bar{a}, q_0) \in \dot{B}^{\frac{d}{2} - 1}_{2,1} \cap \dot{B}^{0}_{1,\infty}.\) There exist two positive constants \( \sigma \) and \( \bar{C}_0 \) such that if
\[
\| (a_0 - \bar{a}, q_0) \|_{\dot{B}^{\frac{d}{2} - 1}_{2,1} \cap \dot{B}^{0}_{1,\infty}} \leq \sigma,
\]
then (1) has a unique global-in-time solution \((a - \bar{a}, q) \in L^\infty([0, \infty); \dot{B}^{\frac{d}{2} - 1}_{2,1})\) satisfying
\[
\| (a - \bar{a}, q)(t) \|_{\dot{B}^{\frac{d}{2} - 1}_{2,1}} \leq \bar{C}_0 (1 + t)^{-\frac{d}{4}}
\]
for all \( t \geq 0 \).

**Remark 1.** We note that the decay rate of solutions obtained in Theorem 1.2 is optimal in the sense that it attains the decay rate of solutions to the linearized system (see Section 3 and Lemma 4.1). Indeed, by \( \dot{B}^{\frac{d}{2} - 1}_{2,1} \subset L^2 \), the decay rate of solutions in Theorem 1.2 has the same decay rate as the solutions of heat equation.

The rest of this paper is structured as follows. In Section 2 we present some notions and basic tools. In Section 3 we establish a priori estimates for the linearized equations of system (1) which will be crucial in the proof of Theorem 1.1. In Section 4 we prove Theorem 1.1 and Theorem 1.2 in detail.
2. Littlewood-Paley theory and Besov spaces. Let $S(\mathbb{R}^d)$ be the Schwartz class of rapidly decreasing functions. Given $f(x) \in S(\mathbb{R}^d)$, the Fourier transform $\mathcal{F}f(\xi) = \hat{f}(\xi)$ of $f(x)$ is defined by

$$\hat{f}(\xi) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{-ix\cdot\xi} f(x) \, dx.$$ 

The inverse of $\mathcal{F}$ is denoted by $\mathcal{F}^{-1}$, which is defined by

$$\mathcal{F}^{-1}f(x) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{ix\cdot\xi} f(\xi) \, d\xi.$$ 

Now we provide a characterization of Besov space based on the Littlewood-Paley decomposition [1].

We start with the dyadic partition of unity. Choose two nonnegative radial functions $\chi, \varphi \in S(\mathbb{R}^d)$, supported in the ball $B = \{ \xi \in \mathbb{R}^d, |\xi| \leq \frac{4}{3} \}$ and in the ring $C = \{ \xi \in \mathbb{R}^d, \frac{3}{4} \leq |\xi| \leq \frac{8}{3} \}$ respectively, such that

$$\chi(\xi) + \sum_{j \geq 0} \varphi(2^{-j}\xi) = 1, \quad \xi \in \mathbb{R}^d.$$ 

$$\sum_{j \in \mathbb{Z}} \varphi(2^{-j}\xi) = 1, \quad \xi \in \mathbb{R}^d \setminus \{0\}.$$ 

For every $f \in S'(\mathbb{R}^d)$, the homogeneous (or nonhomogeneous) dyadic blocks $\dot{\Delta}_j$ (or $\Delta_j$) and homogeneous (or nonhomogeneous) low-frequency cut-off operator $\dot{S}_j$ (or $S_j$) are defined as follows

$$\forall j \in \mathbb{Z}, \quad \dot{\Delta}_j f = \varphi(2^{-j}D)f \quad \text{and} \quad \dot{S}_j f = \chi(2^{-j}D)f;$$

$$\Delta_j f = \begin{cases} 
0, & j \leq -2; \\
\chi(D)f, & j = -1; \\
\varphi(2^{-j}D)f, & j \geq 0;
\end{cases}$$

and

$$S_j f = \sum_{q=-1}^{j-1} \Delta_q f.$$ 

Then we have the formal Littlewood-Paley decomposition in the nonhomogeneous case

$$f = \sum_{j \geq -1} \Delta_j f, \quad \forall f \in S'(\mathbb{R}^d). \quad (6)$$ 

Unfortunately, for the homogeneous case, the Littlewood-Paley decomposition is invalid. We need a new space to modify it, namely,

$$S'_h = \left\{ f \in S'(\mathbb{R}^d) : \lim_{j \to -\infty} ||\chi(2^{-j}D)f||_{L^\infty} = 0 \right\}.$$ 

Thus we have the formal Littlewood-Paley decomposition in the homogeneous case

$$f = \sum_{j \in \mathbb{Z}} \Delta_j f, \quad \forall f \in S'_h(\mathbb{R}^d). \quad (7)$$ 

With suitable choice of $\chi$ and $\varphi$, one can easily verify that the Littlewood-Paley decomposition satisfies the property of almost orthogonality:

$$\dot{\Delta}_j \dot{\Delta}_k f \equiv 0 \quad \text{if} \quad |j - k| \geq 2 \quad \text{and} \quad \Delta_{-1} \Delta_k f \equiv 0 \quad \text{if} \quad k \geq 1. \quad (8)$$
Next we recall Bony’s decomposition from [1]:

\[ uv = \hat{T}_u v + \hat{T}_v u + \hat{R}(u, v), \]  

with

\[ \hat{T}_u v = \sum_{j \in \mathbb{Z}} \hat{S}_{j-1} u \Delta_j v, \quad \hat{R}(u, v) = \sum_{j \in \mathbb{Z}} \hat{\Delta}_j u \Delta_j v, \quad \Delta_j v = \sum_{|j'| \leq 1} \Delta_{j'} v. \]

The operators \( \hat{\Delta}_j \) and \( \Delta_j \) help us recall the definition of the homogenous Besov spaces and the inhomogenous Besov spaces (see [1]).

**Definition 2.1.** Let \( s \in \mathbb{R}, T > 0 \) and \( 1 \leq p, q \leq \infty \). The homogeneous Besov space \( B^s_{p, q} \) is the set of tempered distribution \( f \in \mathcal{S}'_k \) satisfying

\[ \| f \|_{B^s_{p, q}} = \left\| \left( 2^{js} \| \hat{\Delta}_j f \|_{L^p} \right) \right\|_r < \infty. \]

The time-space homogeneous Besov spaces is the set of tempered distribution \( f \) satisfying

\[ \lim_{j \to -\infty} \| \hat{S}_j f \|_{L^\infty} = 0 \]

and

\[ \| f \|_{\dot{B}^s_{p, q}} = \left\| \left( 2^{js} \| \Delta_j f \|_{L^p} \right) \right\|_r < +\infty. \]

The nonhomogeneous Besov space \( B^s_{p, r} \) is the set of tempered distribution \( f \) satisfying

\[ \| f \|_{B^s_{p, r}} = \left\| \left( 2^{js} \| \Delta_j f \|_{L^p} \right) \right\|_r < \infty. \]

Now we present some useful lemmas which will play an important role in the proof of our main results.

**Lemma 2.2.** ([1]) Let \( 1 \leq p \leq q \leq \infty \) and \( B \) be a ball and \( C \) a ring of \( \mathbb{R}^d \). Assume that \( f \in L^p \), then for any \( \alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d \), there exists a constant \( C \) independent of \( f, j \) such that

\[ \text{Supp} \hat{f} \subset \lambda B \Rightarrow \sup_{|\alpha|=k} \| \partial^\alpha f \|_{L^p} \leq C^{k+1} \lambda^{d(1 - \frac{k+1}{q})} \| f \|_{L^p}, \]

\[ \text{Supp} \hat{f} \subset \lambda C \Rightarrow C^{-k-1} \lambda^k \| f \|_{L^p} \leq \sup_{|\alpha|=k} \| \partial^\alpha f \|_{L^p} \leq C^{k+1} \lambda^k \| f \|_{L^p}. \]

**Lemma 2.3.** Let \( 1 \leq r \leq \infty \) and \( f, g \in \dot{B}^s_{2, r} \). Then \( f, g \in \dot{B}^s_{2, r} \) and

\[ \| f g \|_{\dot{B}^s_{2, r}} \leq C(\| f \|_{\dot{B}^s_{2, r}}, \| g \|_{\dot{B}^s_{2, r}}, \| f \|_{\dot{B}^s_{2, s+1}}, \| g \|_{\dot{B}^s_{2, s+1}}). \]

**Proof.** Using the Bony paraproduct decomposition (9) and the property of quasiorthogonality (8), we have

\[ \hat{\Delta}_k(fg) = \sum_{|j-k| \leq 4} \hat{\Delta}_k(\hat{S}_{j-1} f \hat{\Delta}_j g) + \sum_{|j-k| \leq 4} \hat{\Delta}_k(\hat{S}_{j-1} g \hat{\Delta}_j f) \]

\[ + \sum_{j \geq k-3, |\alpha| \leq 1} \hat{\Delta}_k(\hat{\Delta}_j f \hat{\Delta}_{j-\alpha} g), \]
which means that
\[
\|fg\|_{\dot{B}_{2,r}^{\frac{d}{r}}} = \|\langle 2^{j} k \hat{\Delta}_k (fg) \rangle_{L^2} \|_{L^r} \\
\leq \|\langle 2^{j} k \hat{\Delta}_k (\hat{S}_{j-1} f \hat{\Delta}_j g) \rangle_{L^2} \|_{L^r} \\
+ \|\langle 2^{j} k \hat{\Delta}_k (\hat{S}_{j-1} g \hat{\Delta}_j f) \rangle_{L^2} \|_{L^r} \\
+ \|\langle 2^{j} k \hat{\Delta}_k (\hat{\Delta}_j f \hat{\Delta}_j g) \rangle_{L^2} \|_{L^r}
\]
\[
= I_1 + I_2 + I_3. \tag{11}
\]

We shall estimate the above three terms separately. Using the Young inequality and the Hölder inequality, we get
\[
I_1 \leq C \|\langle 2^{j} k \hat{\Delta}_k (\hat{S}_{j-1} f \hat{\Delta}_j g) \rangle_{L^2} \|_{L^r} \\
\leq C \|\langle 2^{j} k \hat{\Delta}_k f \rangle_{L^\infty} \| \hat{\Delta}_j g \|_{L^2} \|_{L^r} \\
\leq C \|\langle 2^{j} k \hat{\Delta}_k f \rangle_{L^\infty} \| \hat{\Delta}_j g \|_{L^2} \|_{L^r} \\
\leq C \|\langle 2^{j} k \hat{\Delta}_k f \rangle_{L^\infty} \| \hat{\Delta}_j g \|_{L^2} \|_{L^r} \tag{12}
\]

\[I_2\text{ can be estimated in a similar way, and we have}
\]
\[
I_2 \leq C \|\langle g \rangle_{\dot{B}_{\infty,\infty}^{-1}} \| f \|_{\dot{B}_{2,r}^{\frac{d}{r}+1}}. \tag{13}
\]

For the term \(I_3\), the Hölder inequality together with the Young inequality yield
\[
I_3 \leq C \|\langle 2^{j} k \hat{\Delta}_j f \rangle_{L^2} \| \hat{\Delta}_j g \|_{L^\infty} \|_{L^r} \\
\leq C \|\langle 2^{j} k \hat{\Delta}_j f \rangle_{L^2} \| \hat{\Delta}_j g \|_{L^\infty} \|_{L^r} \\
\leq C \|\langle 2^{j} k \hat{\Delta}_j f \rangle_{L^2} \| \hat{\Delta}_j g \|_{L^\infty} \|_{L^r} \tag{14}
\]

Taking (12)-(14) into (11) and using the Besov embedding \(\dot{B}_{2,r}^{\frac{d}{r}-1} \hookrightarrow \dot{B}_{\infty,\infty}^{-1}\), we obtain the desired inequality (10).

\[\]

**Lemma 2.4.** (\cite{19}) (i) Let \(a, b > 0\) and \(\max \{a, b\} > 1\). Then
\[
\int_0^t (1+s)^{-a}(1+t-s)^{-b} ds < C(1+t)^{-\min \{a, b\}}, \quad t > 0.
\]

(ii) Let \(a, b > 0\) and \(f \in L^1(0, \infty)\). Then
\[
\int_0^t (1+s)^{-a}(1+t-s)^{-b} f ds < C(1+t)^{-\min \{a, b\}} \int_0^t |f| ds, \quad t > 0.
\]
3. Reformulation and a priori estimate. Due to the physical significance of $\epsilon$ and $D$ as diffusive parameters, without loss of generality, we assume $\epsilon = D = 1$ in what follows. Let $p = a - \bar{a}$, then system (1) can be rewritten as

$$\begin{align*}
\begin{cases}
\partial_t p - \Delta p - \bar{a} \text{div} q = \text{div}(pq), \\
\partial_t q - \Delta q - \nabla p = \nabla(|q|^2), \\
(p, q)|_{t=0} = (p_0, q_0).
\end{cases}
\end{align*}$$

(15)

We first consider the following linearized equations of system (15) with a general function $(f, g)$:

$$\begin{align*}
\begin{cases}
\partial_t p - \Delta p - \bar{a} \text{div} q = f, \\
\partial_t q - \Delta q - \nabla p = g, \\
(p, q)|_{t=0} = (p_0, q_0).
\end{cases}
\end{align*}$$

(16)

Let $U(t) = (p(t), q(t))^T$, $F = (f, g)^T$. Set

$$A = \begin{pmatrix} \Delta & \bar{a} \text{div} \\ \nabla & \Delta \end{pmatrix},$$

by using operator $A$, problem (16) is written as

$$\partial_t U - AU = F, \quad U|_{t=0} = U_0.$$ 

(17)

We introduce a semigroup associated with $A$. We set

$$E(t)u := \mathcal{F}^{-1}(e^{\hat{A}(\xi)t}\hat{u}) \text{ for } u \in L^2,$$

where

$$\hat{A}(\xi) = \begin{pmatrix} -|\xi|^2 & \bar{a}i \xi \\ i \xi^T & -|\xi|^2 E_d \end{pmatrix},$$

here $E_d$ denotes the identity matrix of $d$ dimension.

We have the following lemma.

**Lemma 3.1.** Let $(p, q)$ be a solution of system (16) on $[0, T)$, then there holds

$$|||p, q|||_{L^\infty(B_2^{-\frac{d}{2}-1}) \cap L^1(B_2^{rac{d}{2}+1})} \leq C(|||p_0, q_0|||_{B_2^{-\frac{d}{2}-1}} + |||f, g|||_{L^1(B_2^{-\frac{d}{2}+1})}).$$

**Proof.** Applying the operator $\hat{A}_j$ to Equations (16)$_1$ and (16)$_2$, respectively, yields that

$$\begin{align*}
\begin{cases}
\partial_t p_j - \Delta p_j - \bar{a} \text{div} q_j = f_j, \\
\partial_t q_j - \Delta q_j - \nabla p_j = g_j,
\end{cases}
\end{align*}$$

(18)

here and in the sequel, we always denote $\phi_j = \hat{A}_j \phi$.

Taking $L^2$ inner product of Equations (18)$_1$ and (18)$_2$ with $p_j$ and $\bar{a}q_j$, respectively, then integrating by parts, we get

$$\begin{align*}
\begin{cases}
\frac{1}{2} \frac{d}{dt}||p_j||_{L^2}^2 + ||\nabla p_j||_{L^2}^2 - (\bar{a} \text{div} q_j, p_j) = (f_j, p_j), \\
\frac{1}{2} \frac{d}{dt}||q_j||_{L^2}^2 + \bar{a}||\nabla q_j||_{L^2}^2 - (\nabla p_j, \bar{a} q_j) = (g_j, \bar{a} q_j).
\end{cases}
\end{align*}$$

(19)

Notice the fact that $(\bar{a} \text{div} q_j, a_j) = - (\nabla a_j, \bar{a} q_j)$ and use Lemma 2.2, then we get from (19) that

$$\frac{d}{dt}||p_j, q_j||_{L^2}^2 + c_0 2^2 ||(p_j, q_j)||_{L^2}^2 \leq C||(f_j, g_j)||_{L^2} ||(p_j, q_j)||_{L^2},$$

where $c_0 = \frac{1}{2} (\frac{1}{2^{\frac{d}{2}+1}} - \frac{1}{2^{\frac{d}{2}-1}})$.
which leads to
\[ \frac{d}{dt} \|(p_j, q_j)\|_{L^2} + c_0 2^{2j} \|(p_j, q_j)\|_{L^2} \leq C \|(f_j, g_j)\|_{L^2}. \]  
(20)

Multiplying both sides of (20) by \(2^l \|2^{-1}\) and taking \(l\) norm over \(j \in \mathbb{Z}\), we infer that
\[ \|(p, q)\|_{L_t^\infty(B_{2^l_r}^{\frac{d}{2}+1})} \leq C \left( \|(p_0, q_0)\|_{B_{2^l_r}^{\frac{d}{2}+1}} + \|(f, g)\|_{L_t^1(B_{2^l_r}^{\frac{d}{2}+1})} \right). \]  
(21)

Hence, we complete the proof of lemma 3.1.

\[ \square \]

4. Proof of Theorem 1.1 and Theorem 1.2.

4.1. Proof of Theorem 1.1. Before proving our result, we point out that the local solution can be established by the classical Friedrichs’ regularization method \([1, 6]\). Now, it remains to establish a global priori estimate under condition (4).

Firstly, using the Bernstein inequality in Lemma 2.2 and the product estimates in Lemma 2.3 together with the Hölder inequality, we obtain
\[ \|\text{div}(pq)\|_{L_t^1(B_{2^l_r}^{\frac{d}{2}+1})} \leq C \|pq\|_{L_t^1(B_{2^l_r}^{\frac{d}{2}})} \leq C \left( \|q\|_{L_t^\infty(B_{2^l_r}^{\frac{d}{2}+1})} \|p\|_{L_t^1(B_{2^l_r}^{\frac{d}{2}+1})} + \|p\|_{L_t^\infty(B_{2^l_r}^{\frac{d}{2}+1})} \|q\|_{L_t^1(B_{2^l_r}^{\frac{d}{2}+1})} \right) \leq C \|p, q\|_{L_t^1(B_{2^l_r}^{\frac{d}{2}+1})} \|p, q\|_{L_t^1(B_{2^l_r}^{\frac{d}{2}+1})}. \]  
(22)

Similarly, we have
\[ \|\nabla(|q|^2)\|_{L_t^1(B_{2^l_r}^{\frac{d}{2}+1})} \leq C \|q\|_{L_t^\infty(B_{2^l_r}^{\frac{d}{2}+1})} \|q\|_{L_t^1(B_{2^l_r}^{\frac{d}{2}+1})}. \]  
(23)

Then, according to Lemma 3.1, we conclude that
\[ \|(p, q)\|_{L_t^\infty(B_{2^l_r}^{\frac{d}{2}+1})} + \|(p, q)\|_{L_t^1(B_{2^l_r}^{\frac{d}{2}+1})} \leq C \left( \|(p_0, q_0)\|_{B_{2^l_r}^{\frac{d}{2}+1}} + \|p, q\|_{L_t^1(B_{2^l_r}^{\frac{d}{2}+1})} \|p, q\|_{L_t^1(B_{2^l_r}^{\frac{d}{2}+1})} \right), \]  
which together with condition (4) imply the result in Theorem 1.1 by using a standard continuity argument provided \(\sigma\) is suitable small.

4.2. Proof of Theorem 1.2. Let \(d \geq 2\) and \((p_0, q_0) \in B_{2^l_1}^{\frac{d}{2}-1}\). Under the assumption of Theorem 1.2, according to Theorem 1.1, we have \((p, q) \in L^\infty([0, \infty); B_{2^l_1}^{\frac{d}{2}-1})\).

Next, we are devoted to establish the decay estimates for \((p, q)\) in \(B_{2^l_1}^{\frac{d}{2}+1}\). Let \(U(t) = (p(t), q(t))^T\) be a solution of (16) with \((f, g) = (\text{div}(pq), \nabla(|q|^2))\), then \(U(t)\) can be written as the following integral form:
\[ U(t) = E(t)U_0 + \int_0^t E(t - \tau)F(U(\tau))d\tau, \]
where \(U_0 = (p_0, q_0)^T\) and \(F(U(t)) = (f, g)^T\).

To simplify the notation, we set
\[ M_l(t) = \sup_{0 \leq \tau \leq t} (1 + \tau)^{\frac{d}{2}} \|\Delta_{-1} U(\tau)\|_{L^2}, \]
\[ M_h(t) = \sup_{0 \leq \tau \leq t} (1 + \tau)^{\frac{d}{2}} \sum_{j \geq 0} 2^{(\frac{d}{2}+1)j} \|\Delta_j U(\tau)\|_{L^2}, \]
which implies
\[ ||U(\tau)||_{B^{\frac{d}{2}-1}_{2,1}} \leq (1 + \tau)^{-\frac{d}{4}} M(t) \quad \text{for} \quad \tau \in [0,t], \]
where \( M(t) = M_e(t) + M_h(t) \). Then, we have
\[
M(t) \leq \sup_{0 \leq \tau \leq t} (1 + \tau)^{\frac{d}{4}} ||E(\tau)\Delta_{-1}U_0||_{L^2} \\
+ \sup_{0 \leq \tau \leq t} (1 + \tau)^{\frac{d}{4}} \sum_{j \geq 0} 2^{j(\frac{d}{4} - 1)} ||E(\tau)\hat{\Delta}_jU_0||_{L^2} \\
+ \sup_{0 \leq \tau \leq t} (1 + \tau)^{\frac{d}{4}} \int_0^\tau ||E(\tau - s)\Delta_{-1}F(U(s))||_{L^2} ds \\
+ \sup_{0 \leq \tau \leq t} (1 + \tau)^{\frac{d}{4}} \sum_{j \geq 0} 2^{j(\frac{d}{4} - 1)} \int_0^\tau ||E(\tau - s)\hat{\Delta}_jF(U(s))||_{L^2} ds,
\]
since \( \Delta_j \) coincides with \( \hat{\Delta}_j \) for \( j \geq 0 \).

To obtain the decay estimate of Theorem 1.2, we state the estimate of the semi-group operator \( E(t) \) associated with \( A \).

**Lemma 4.1.** Let \( U_0 = (p_0, q_0)^T \). Then the operator \( E(t) \) satisfies the following estimates
\[
||E(t)\Delta_{-1}U_0||_{L^2} \leq C(1 + t)^{-\frac{d}{4}} ||U_0||_{B^0_{1,\infty}},
\]
\[
\sum_{j \geq 0} 2^{j(\frac{d}{4} - 1)} ||E(t)\hat{\Delta}_jU_0||_{L^2} \leq C(1 + t)^{-\frac{d}{4}} ||U_0||_{B^{\frac{d}{2}-1}_{2,1}}.
\]

**Proof.** According to the action of semi-group of the heat equation over spectrally supported funtions [1], it can be checked that
\[
||\mathcal{F}(E(t)\hat{\Delta}_jU_0)(\xi)||_{L^2} \leq C e^{-\sigma_0 t^2} ||\hat{\Delta}_jU_0||_{L^2}, \quad \text{for} \quad j \in \mathbb{Z}.
\]
Due to the fact that [1]: for any \( \sigma > 0 \) there exists a constant \( C_\sigma \) such that
\[
\sup_{t \geq 0} \sum_{k \in \mathbb{Z}} t^{\frac{d}{4}} 2^{k\sigma} e^{-\frac{t^2}{4} 2^k} \leq C_\sigma.
\]

Firstly using the homogeneous Littlewood-Paley decomposition (7) and then the property of almost orthogonality (8), we arrive at
\[
t^{\frac{d}{4}} ||E(t)\Delta_{-1}U_0||_{L^2} \leq t^{\frac{d}{4}} \sum_{k \leq 0} ||E(t)\hat{\Delta}_k\Delta_{-1}U_0||_{L^2} \\
\leq Ct^{\frac{d}{4}} \sum_{k \leq 0} ||E(t)\hat{\Delta}_kU_0||_{L^2} + ||\Delta_{-1}U_0||_{L^1} \\
\leq Ct^{\frac{d}{4}} \sum_{k \leq 0} 2^{k\frac{d}{2}} e^{-\mu t^2} ||\hat{\Delta}_kU_0||_{L^1} \\
\leq C ||U_0||_{B^0_{1,\infty}} \sum_{k \leq 0} t^{\frac{d}{4}} 2^{k\frac{d}{2}} e^{-\mu t^2} \\
\leq C ||U_0||_{B^0_{1,\infty}}.
We also find that
\[ \|E(t)\Delta_{-1}U_0\|_{L^2} \leq \sum_{k \leq 0} \|E(t)\Delta_k\Delta_{-1}U_0\|_{L^2} \]
\[ \leq C \sum_{k \leq 0} \|E(t)\Delta_kU_0\|_{L^2} \]
\[ \leq C \sum_{k \leq 0} 2^k \frac{2}{\mu t} e^{-\mu t^2} \|\Delta_kU_0\|_{L^1} \]
\[ \leq C \|U_0\|_{L^1_{\infty}} \sum_{k \leq 0} 2^k \frac{2}{\mu t} \]
\[ \leq C \|U_0\| \cdot b_{\infty}^{1} \cdot E_{\infty}^1. \]
This implies the first estimate of our result.

To obtain the second estimate of this lemma, we find that
\[ t^2 \sum_{j \geq 0} 2^{j(\frac{1}{2} - 1)} \|E(t)\Delta_jU_0\|_{L^2} \leq Ct^2 \sum_{j \geq 0} 2^{j(\frac{1}{2} - 1)} e^{-\mu t^2} \|\Delta_jU_0\|_{L^2} \]
\[ \leq C \sum_{j \geq 0} 2^{j(\frac{1}{2} - 1)} \|\Delta_jU_0\|_{L^2} \]
\[ \leq C \|U_0\| \cdot b_{\infty}^{-1}. \]
and
\[ \sum_{j \geq 0} 2^{j(\frac{1}{2} - 1)} \|E(t)\Delta_jU_0\|_{L^2} \leq C \sum_{j \geq 0} 2^{j(\frac{1}{2} - 1)} e^{-\mu t^2} \|\Delta_jU_0\|_{L^2} \]
\[ \leq C \sum_{j \geq 0} 2^{j(\frac{1}{2} - 1)} \|\Delta_jU_0\|_{L^2} \]
\[ \leq C \|U_0\| \cdot b_{\infty}^{-1}. \]
This completes the proof of Lemma 4.1. \( \square \)

Now, we will give a detailed proof of Theorem 1.2. Firstly, by the Hölder inequality, we get
\[ \|F(U)\|_{L^1} \leq \|\text{div}(pq)\|_{L^2} + \|\nabla(|q|^2)\|_{L^1} \]
\[ \leq C(||q||_{L^2}||\nabla p||_{L^2} + ||p||_{L^2}||\text{div} q||_{L^2} + ||q||_{L^2}||\nabla |q||_{L^2}). \]
Making full use of the embedding \( B_{2,1}^s \hookrightarrow L^2 (s \geq 0) \), nonhomogeneous Littlewood-Paley decomposition (6) and the property of almost orthogonality (8), we conclude that
\[ ||q||_{L^2} ||\nabla p||_{L^2} \leq ||q||_{B_{2,1}^s} \left( ||\Delta_{-1}\nabla p||_{L^2} + \sum_{j \geq 0} \Delta_j \nabla p||_{B_{2,1}^s} \right) \]
\[ \leq C||q||_{B_{2,1}^s} \left( ||p||_{B_{2,1}^s} + ||p||_{B_{2,1}^{s+1}} \right) \]
\[ \leq C||U||_{B_{2,1}^s} \left( ||U||_{B_{2,1}^s} + ||U||_{B_{2,1}^{s+1}} \right). \]
The other terms can be estimated in a similar way. Hence we have
\[ ||F(U)||_{L^1} \leq C||U||_{B_{2,1}^s} \left( ||U||_{B_{2,1}^s} + ||U||_{B_{2,1}^{s+1}} \right). \]
Then by Lemma 4.1 and Lemma 2.4, for \( \tau \in [0, t] \), we have
\[
\int_0^\tau \|\Delta_{-1}E(\tau - s)F(U(s))\|_{L^2} \, ds \\
\leq \int_0^\tau (1 + \tau - s)^{-\frac{q}{4}} \|F(U(s))\|_{B^1_{q, \infty}} \, ds \\
\leq \int_0^\tau (1 + \tau - s)^{-\frac{q}{4}} \|F(U(s))\|_{L^1} \, ds \\
\leq \int_0^\tau (1 + \tau - s)^{-\frac{q}{4}} (1 + s)^{-\frac{q}{4}} M(\tau) \left(1 + s\right)^{-\frac{q}{4}} M(\tau) + \|U\|_{B^{\frac{q}{4}+1}_{2,1}} \right) \, ds \\
\leq C(1 + \tau)^{-\frac{q}{4}} M(\tau)\|U\|_{L^1_t(B^{\frac{q}{4}+1}_{2,1})} + C(1 + \tau)^{-\frac{q}{4}} M^2(\tau).
\]
(24)

Using (22)-(23) with \( r = 1 \) and by \( B^{\frac{q}{4}+1}_{2,1} \), we have
\[
\|F(U)\|_{B^{\frac{q}{4}+1}_{2,1}} \\
\leq \left(\|\text{div}(aq)\|_{B^{\frac{q}{2}+1}_{2,1}} + \|\nabla q\|^2\|_{B^{\frac{q}{4}+1}_{2,1}}\right) \\
\leq C\|U\|_{B^0_{1,\infty}}\|U\|_{B^{\frac{q}{4}+1}_{2,1}}.
\]

According to the second estimate in Lemma 4.1, then by Lemma 2.4, for \( \tau \in [0, t] \), we obtain
\[
\sum_{j \geq 0} 2^{j(\frac{q}{4}+1)} \int_0^\tau \|E(\tau - s)\hat{\Delta}_j F(U(s))\|_{L^2} \, ds \\
\leq \int_0^\tau (1 + \tau - s)^{-\frac{q}{4}} \|F(U(s))\|_{B^{\frac{q}{4}+1}_{2,1}} \, ds \\
\leq \int_0^\tau (1 + \tau - s)^{-\frac{q}{4}} (1 + s)^{-\frac{q}{4}} M(\tau)\|U\|_{B^{\frac{q}{4}+1}_{2,1}} \, ds \\
\leq C(1 + \tau)^{-\frac{q}{4}} M(\tau)\|U\|_{L^1_t(B^{\frac{q}{4}+1}_{2,1})}.
\]
(25)

By Lemma 4.1, we have
\[
M(t) \leq C\left(\|U_0\|_{B^0_{1,\infty} \cap B^{\frac{q}{4}+1}_{2,1}} + \sup_{0 \leq \tau \leq t} (1 + \tau)^{\frac{q}{4}} \int_0^\tau \|E(\tau - s)\Delta_{-1} F(U(s))\|_{L^2} \, ds \\
+ \sup_{0 \leq \tau \leq t} (1 + \tau)^{\frac{q}{4}} \sum_{j \geq 0} 2^{j(\frac{q}{4}+1)} \int_0^\tau \|E(\tau - s)\hat{\Delta}_j F(U(s))\|_{L^2} \, ds\right).
\]
(26)

Noticing that \( \|U\|_{L^1_t(B^{\frac{q}{4}+1}_{2,1})} \leq C\|U_0\|_{B^{\frac{q}{4}+1}_{2,1}} \) and taking (24)-(25) into (26), we conclude that
\[
M(t) \leq C\left(\|U_0\|_{B^0_{1,\infty} \cap B^{\frac{q}{4}+1}_{2,1}} + M(t)\|U_0\|_{B^0_{1,\infty} \cap B^{\frac{q}{4}+1}_{2,1}} + M^2(t)\right).
\]

If \( \|U_0\|_{B^0_{1,\infty} \cap B^{\frac{q}{4}+1}_{2,1}} \) is sufficiently small enough, the standard continuous method shows that for all \( t \geq 0 \),
\[
M(t) \leq \tilde{C}_0\|U_0\|_{B^0_{1,\infty} \cap B^{\frac{q}{4}+1}_{2,1}},
\]
for some positive constant \( \tilde{C}_0 \). Thus, we complete the proof of Theorem 1.2.
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