A note on large rainbow matchings in edge-coloured graphs

Allan Lo∗ Ta Sheng Tan†

May 2, 2014

Submitted to Graphs and Combinatorics on April 19, 2012.

Abstract

A rainbow subgraph in an edge-coloured graph is a subgraph such that its edges have distinct colours. The minimum colour degree of a graph is the smallest number of distinct colours on the edges incident with a vertex over all vertices. Kostochka, Pfender, and Yancey showed that every edge-coloured graph on \( n \) vertices with minimum colour degree at least \( k \) contains a rainbow matching of size at least \( k \), provided \( n \geq \frac{17}{4}k^2 \). In this paper, we show that \( n \geq 4k - 4 \) is sufficient for \( k \geq 4 \).

1 Introduction

Let \( G \) be a simple graph, that is, no loops or multiple edges. We write \( V(G) \) for the vertex set of \( G \) and \( \delta(G) \) for the minimum degree of \( G \). An edge-coloured graph is a graph in which each edge is assigned a colour. We say such an edge-coloured \( G \) is proper if no two adjacent edges have the same colour. A subgraph \( H \) of \( G \) is rainbow if all its edges have distinct colours. Rainbow subgraphs are also called totally multicoloured, polychromatic, or heterochromatic subgraphs.

For a vertex \( v \) of an edge-coloured graph \( G \), the colour degree of \( v \) is the number of distinct colours on the edges incident with \( v \). The smallest colour degree of all vertices in \( G \) is the minimum colour degree of \( G \) and is denoted by \( \delta^c(G) \). Note that a properly edge-coloured graph \( G \) with \( \delta(G) \geq k \) has \( \delta^c(G) \geq k \).

In this paper, we are interested in rainbow matchings in edge-coloured graphs. The study of rainbow matchings began with a conjecture of Ryser [11], which states that every Latin square of odd order contains a Latin transversal. Equivalently, for \( n \) odd, every properly \( n \)-edge-colouring

∗School of Mathematics, University of Birmingham, B15 2TT, United Kingdom. Email: s.a.lo@bham.ac.uk. Allan Lo was supported by the ERC, grant no. 258345.
†Department of Pure Mathematics and Mathematical Statistics, Centre for Mathematical Sciences, University of Cambridge, Wilberforce Road, Cambridge CB3 0WB, United Kingdom. Email: T.S.Tan@dpmms.cam.ac.uk.
of $K_{n,n}$, the complete bipartite graph with $n$ vertices on each part, contains a rainbow copy of perfect matching. In a more general setting, given a graph $H$, we wish to know if an edge-coloured graph $G$ contains a rainbow copy of $H$. A survey on rainbow matchings and other rainbow subgraphs in edge-coloured subgraph can be found in [4]. From now onwards, we often refer to $G$ for an edge-coloured graph $G$ (not necessarily proper) of order $n$.

Li and Wang [3] showed that if $\delta'(G) = k$, then $G$ contains a rainbow matching of size $\lceil \frac{5k-3}{12} \rceil$. They further conjectured that if $k \geq 4$, then $G$ contains a rainbow matching of size $\left\lceil \frac{k}{2} \right\rceil$. This bound is tight for properly edge-coloured complete graphs. LeSaulnier et al. [8] proved that if $\delta'(G) = k$, then $G$ contains a rainbow matching of size $\left\lceil \frac{k}{2} \right\rceil$. Furthermore, if $G$ is properly edge-coloured with $G \neq K_1$ or $|V(G)| \neq \delta(G) + 2$, then there is a rainbow matching of size $\left\lceil \frac{k}{2} \right\rceil$. The conjecture was later proved in full by Kostochka and Yancey [7].

What happens if we have a larger graph? Wang [12] proved that every properly edge-coloured graph $G$ with $\delta(G) = k$ and $|V(G)| \geq \frac{8k}{3}$ contains a rainbow matching of size at least $\left\lceil \frac{2k}{3} \right\rceil$. He then asked if there is a function, $f(k)$, such that every properly edge-coloured graph $G$ with $\delta(G) \geq k$ and $|V(G)| \geq f(k)$ contains a rainbow matching of size $k$. The bound on the size of rainbow matching is sharp, as shown for example by any $k$-edge-coloured $k$-regular graph. If $f(k)$ exists, then we trivially have $f(k) \geq 2k$. In fact, $f(k) > 2k$ for even $k$ as there exists $k \times k$ Latin square without any Latin transversal (see [1, 13]). Diemunsch et al. [2] gave an affirmative answer to Wang’s question and showed that $f(k) \leq \frac{12k}{5}$. The bound was then improved to $f(k) \leq \frac{9k}{5}$ in [10], and shortly thereafter, to $f(k) \leq \frac{9k}{5}$ in [3].

Kostochka, Pfender and Yancey [6] considered a similar problem with $\delta'(G)$ instead of properly edge-coloured graphs. They showed that if $G$ is such that $\delta'(G) \geq k$ and $n > \frac{4k}{3}k^2$, then $G$ contains a rainbow matching of size $k$. Kostochka [5] then asked: can $n$ be improved to a linear bound in $k$? In this paper, we show that $n \geq 4k - 4$ is sufficient for $k \geq 4$. Furthermore, this implies that $f(k) \leq 4k - 4$ for $k \geq 4$.

**Theorem 1.1.** If $G$ is an edge-coloured graph on $n$ vertices with $\delta'(G) \geq k$, then $G$ contains a rainbow matching of size $k$, provided $n \geq 4k - 4$ for $k \geq 4$ and $n \geq 4k - 3$ for $k \leq 3$.

## 2 Main Result

We write $[k]$ for $\{1, 2, \ldots, k\}$. For an edge $uv$ in $G$, we denote by $c(uv)$ the colour of $uv$ and let the set of colours be $\mathbb{N}$, the set of natural numbers.

The idea of the proof is as follows. By induction, $G$ contains a rainbow matching $M$ of size $k - 1$. Suppose that $G$ does not contain a rainbow matching of size $k$. We are going to show that there exists another rainbow matching $M'$ of size $k - 1$ in $V(G) \setminus V(M)$. Clearly, the colours of $M$ equal to the colours of $M'$. If $n \geq 4k - 3$, then there exists a vertex $z$ not in $M \cup M'$. Since $\delta'(G) \geq k$, $z$ has a neighbour $w$ such that $zw$ does not use any colour of $M$. Hence, it is easy to deduce that $G$ contains a rainbow matching of size $k$. 

2
Proof of Theorem 1.7. We proceed by induction on $k$. The theorem is trivially true for $k = 1$. So fix $k > 1$ and assume that the theorem is true for $k - 1$. Let $G$ be an edge-coloured graph with $\delta^c(G) \geq k$ and $n = |V(G)| \geq 4k - 4$ if $k \geq 4$ and $n \geq 4k - 3$ otherwise. Suppose that the theorem is false and so $G$ does not contain a rainbow matching of size $k$.

By induction, there exists a rainbow matching $M = \{x_iy_i : i \in [k-1]\}$ in $G$, say with $c(x_iy_i) = i$ for each $i \in [k-1]$. Let $M'$ be another rainbow matching of size $s$ (which could be empty) in $G$ vertex-disjoint from $M$. Clearly $s \leq k - 1$ and the colours on $M'$ is a subset of $[k-1]$, as otherwise $G$ contains a rainbow matching of size $k$. Without loss of generality, we may assume that $M' = \{z_iw_i : i \in [s]\}$ with $c(z_iw_i) = i$ for each $i \in [s]$. We further assume that $M$ and $M'$ are chosen such that $s$ is maximal. Now, let $W = V(G) \setminus V(M \cup M')$ and $S = \{x_i, y_i, z_i, w_i : i \in [s]\}$. Clearly, if there is an edge in $W$, it must have colour in $[s]$, otherwise $G$ contains a rainbow matching of size $k$, or $s$ is not maximal.

Fact A If $uw$ is an edge in $W$, then $c(uw) \in [s]$.

Furthermore, if $uw$ is an edge with $u \in S$ and $v \in W$, then $c(uv) \in [k-1]$, otherwise $G$ contains a rainbow matching of size $k$. First, we are going to show that $s = k - 1$. Suppose the contrary, $s < k - 1$. We then claim the following.

Claim By relabeling the indices of $i$ (in the interval $\{s + 1, s + 2, \ldots, k - 1\}$) and swapping the roles of $x_i$ and $y_i$ if necessary, there exist distinct vertices $z_{k-1}, z_{k-2}, \ldots, z_s$ in $W$ such that for $s + 1 \leq i \leq k - 1$ the following holds for $s + 1 \leq i \leq k - 1$:

(a) $y_i z_i$ is an edge and $c(y_i z_i) \notin [i]$.

(b) Let $T_i$ be the vertex set $\{x_j, y_j, z_j : i \leq j \leq k - 1\}$. For any colour $j$, there exists a rainbow matching of size $k - i$ on $T_i$ which does not use any colour in $[i-1] \cup \{j\}$.

(c) Let $W_i = W \setminus \{z_i, z_{i+1}, \ldots, z_{k-1}\}$. If $x_iw$ is an edge with $w \in W_i$, then $c(x_iw) \in [s]$.

(d) If $uw$ is an edge with $u \in S$ and $w \in W_i$, then $c(uw) \in [i - 1]$.

(e) If $uw$ is an edge with $u \in S \cup T_i \cup W$ and $w \in W_i$, then $c(uw) \in [i - 1]$ or $u \in \{y_i, \ldots, y_{k-1}\}$.

Proof of Claim. Let $W_k = W$ and $T_k = \emptyset$. Observe that part (d) and (e) of the claim hold for $i = k$. For each $i = k - 1, k - 2, \ldots, s + 1$ in terms, we are going to find $z_i$ satisfying (a) – (e). Suppose that we have already found $z_{k-1}, z_{k-2}, \ldots, z_{i+1}$.

Note that $|W_{i+1}| \geq n - 2(k - 1) - 2s - (k - i - 1) \geq 1$, so $W_{i+1} \neq \emptyset$. Let $z$ be a vertex in $W_{i+1}$. By the colour degree condition, $z$ must incident with at least $k$ edges of distinct colours, and in particular, at least $k - i$ distinct coloured edges not using colours in $[i]$. Then, there exists a vertex $u \in \{x_j, y_j : s + 1 \leq j \leq i\}$ such that $uz$ is an edge with $c(uz) \notin [i]$ by part (e) of the claim for the case $i + 1$. Without loss of generality, $u = y_i$ and we set $z_i = z$.

Part (b) of the claim is true for colour $j \neq i$, simply by taking the edge $x_iy_i$ together with a rainbow matching of size $k - i - 1$ on $T_{i+1}$ which does not use any colour in $[i] \cup \{j\}$. For colour
Let \( s \) is not maximal.

As \( \delta (G) \geq k \geq 4 \), \( x_1 \) must have a neighbour \( v \) such that \( c(v) \neq k-1 \).

Since \( \delta^c (G) \geq k-1 \), any vertex \( u \in \{ x_1, y_1, z_1, w_1 \} \) must have a neighbour \( v \) such that \( c(v) \neq k-1 \).

If \( v \notin \{ x_1, y_1, z_1, w_1 \} \), then \( G \) contains a rainbow matching of size \( k \). So, without loss of generality, \( x_1z_1 \) and \( y_1w_1 \) are edges in \( G \) with \( c(x_1z_1), c(y_1w_1) \neq k-1 \). By symmetry, we may assume that for each \( i \in [k-1] \), \( x_iz_i \) and \( y_iw_i \) are edges in \( G \) with \( c(x_iz_i), c(y_iw_i) \neq k-1 \).

As \( \delta^c (G) \geq k \geq 4 \), \( x_1 \) must have a neighbour \( v \notin \{ y_1, z_1, w_1 \} \) with \( c(x_1v) \neq 1 \). Without loss of generality, we may assume \( v = z_j \) for some \( j \) and \( c(x_1z_j) = 2 \). Now, \( \{ x_1z_j, z_1w_1, y_2w_2, \} \cup \{ x_iy_i : i \in \{ 3, 4, \ldots, k-1 \} \} \) is a rainbow matching of size \( k \) in \( G \), which again is a contradiction. This completes the proof of the theorem.

### 3 Remarks

In Theorem 1.1, the bound on \( n \), the number of vertices, is sharp for \( k = 2, 3 \) (and trivially for \( k = 1 \)), as shown by properly 3-edge-coloured \( K_4 \) for \( k = 2 \) and by properly 3-edge-coloured two disjoint copies of \( K_4 \) for \( k = 3 \). However, we do not know if the bound is sharp for \( k \geq 4 \).

**Question.** Given \( k \), what is the minimum \( n \) such that every edge-coloured graph \( G \) of order \( n \) with \( \delta^c (G) = k \) contains a rainbow matching of size \( k \)?
Acknowledgment

The authors thank Alexandr Kostochka for suggesting the problem during ‘Probabilistic Methods in Graph Theory’ workshop at University of Birmingham, United Kingdom. We would also like to thank Daniela Kühn, Richard Mycroft and Deryk Osthus for organizing this nice event.

References

[1] R. A. Brualdi and H. J. Ryser, Combinatorics matrix theory, Encyclopedia of Mathematics and its Applications, vol. 39, Cambridge University Press, 1991.

[2] J. Diemunsch, M. Ferrara, C. Moffatt, F. Pfender, and P. S. Wenger, Rainbow matching of size $\delta(G)$ in properly-colored graphs, [arXiv:1108.2521](arXiv:1108.2521).

[3] J. Diemunsch, M. Ferrara, A. Lo, C. Moffatt, F. Pfender, and P. S. Wenger, Rainbow matching of size $\delta(G)$ in properly-colored graphs, submitted.

[4] M. Kano and X. Li, Monochromatic and heterochromatic subgraphs in edge-colored graphs – a survey, Graphs Combin. 24 (2008), 237-263.

[5] A. Kostochka, personal communication.

[6] A. Kostochka and M. Yancey, Large rainbow matchings in edge-coloured graphs, Combinatorics, Probability and Computing 21 (2012), 255-263.

[7] A. Kostochka, F. Pfender and M. Yancey, Large rainbow matchings in large graphs, [arXiv:1204.3193](arXiv:1204.3193).

[8] T. D. LeSaulnier, C. Stocker, P. S.Wenger, and D. B. West, Rainbow matching in edge-colored graphs, Electron. J. Combin. 17 (2010), #N26.

[9] H. Li and G. Wang, Heterochromatic matchings in edge-colored graphs, Electron. J. Combin. 15 (2008), #R138.

[10] A. Lo, A note on rainbow matchings in properly edge-coloured graphs, [arXiv:1108.5273](arXiv:1108.5273).

[11] H. J. Ryser, Neuere probleme der kombinatorik, Vortrage über Kombinatorik Oberwolfach, Mathematisches Forschungsinstitut Oberwolfach (1967), 24-29.

[12] G. Wang, Rainbow matchings in properly edge colored graphs, Electron. J. Combin. 18 (2011), #P162.

[13] I. M. Wanless, Transversals in latin squares: a survey, Surveys in Combinatorics 2011, London Math. Soc., 2011.