A NEW DERIVATION OF THE INNER PRODUCT FORMULA FOR THE MACDONALD SYMMETRIC POLYNOMIALS

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Abstract. We give a short proof of the inner product conjecture for the symmetric Macdonald polynomials of type $A_{n-1}$. As a special case, the corresponding constant term conjecture is also proved.

1. Introduction

Macdonald’s inner product formula, conjectured in [4], was recently proved for arbitrary root systems by Cherednik [1], using the double affine Hecke algebras. In addition to Cherednik’s proof, a combinatorial proof by Macdonald [4] and representation-theoretic proof by Etingof and Kirillov Jr. [2] have been given for the $A_{n-1}$ case. The aim of the present note is to give a short proof for the $A_{n-1}$ case by means of asymptotic analysis with $q$-Selberg type integrals. One of our motivations is to clarify the argument on the integral representation of solutions of eigenvalue problems of the Macdonald type [7]. In that case, choice of cycles associated with the integral corresponds to the choice of different solutions. Such study on the cycles leads to the present argument, another proof of the inner product conjecture for the Macdonald symmetric polynomials of type $A_{n-1}$. Our argument includes a new proof of the corresponding constant term conjecture as a special case (see also [5]).

Throughout this note, we consider $q$ as a real number satisfying $0 < q < 1$ and $t = q^k$, where $k \in \mathbb{N}$.

2. Inner Product Formula

We begin recalling some fundamental facts. For a basic reference, we refer the reader to [3].

A partition $\lambda$ is a sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ of non-negative integers in decreasing order; $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \geq 0$. The number of non-zero elements $\lambda_i$ is called the length of $\lambda$, denoted by $l(\lambda)$. The sum of the $\lambda_i$ is the weight of $\lambda$, denoted by $|\lambda|$. Given a partition
\( \lambda \), we define the conjugate partition \( \lambda' = (\lambda'_1, \lambda'_2, \ldots, \lambda'_n) \) by \( \lambda'_i = \text{Card}\{ j; \lambda_j \geq i \} \).

On partitions, the dominance (or natural) ordering is defined by
\[
\lambda \geq \mu \iff |\lambda| = |\mu| \text{ and } \lambda_1 + \cdots + \lambda_i \geq \mu_1 + \cdots + \mu_i \text{ for all } i \geq 1.
\]

We consider the ring \( \mathbb{C}[x] = \mathbb{C}[x_1, \ldots, x_n] \) of polynomials in \( n \) variables \( x = (x_1, \ldots, x_n) \). The subring of all symmetric polynomials is denoted by \( \mathbb{C}[x]^{S_n} \).

For \( f = \sum \beta \beta x^{\beta} \in \mathbb{C}[x] \), we define
\[
\bar{f} = \sum \beta \beta x^{-\beta}
\]
and let \([f]_1\) denote the constant term of \( f \).

The inner product is defined by
\[
\langle f, g \rangle = \frac{1}{n!}[f \bar{g} \Delta]_1
\]
for \( f, g \in \mathbb{C}[x] \), with
\[
\Delta = \Delta(x) = \prod_{1 \leq i < j \leq n} (x_i/x_j; q)_{\infty} \prod_{1 \leq i < j \leq n} (tx_i/x_j; q)_{\infty} = \prod_{1 \leq i < j \leq n} (x_i/x_j; q)_k,
\]
where \((a; q)_{\infty} = \prod_{i \geq 0} (1 - aq^i)\) and \((a; q)_n = (a; q)_{\infty}/(q^n a; q)_{\infty}\).

Then there is a unique family of symmetric polynomials \( P_\lambda(x) = P_\lambda(x; q, t) \in \mathbb{C}[x]^{S_n} \) indexed by the partition \( \lambda = (\lambda_1, \ldots, \lambda_n) \) such that
\begin{enumerate}
  \item \( P_\lambda = m_\lambda + \sum_{\mu < \lambda} c_{\lambda\mu} m_\mu \),
  \item \( \langle P_\lambda, P_\mu \rangle = 0 \quad \text{if} \quad \lambda \neq \mu \),
\end{enumerate}
where each \( m_\mu \) expresses the monomial symmetric polynomial indexed by \( \mu \). The polynomials \( P_\lambda \) are called Macdonald symmetric polynomials (associated with the root system of type \( A_{n-1} \)).

Our aim is to prove the following:

**Theorem.** We have
\[
\langle P_\lambda, P_\lambda \rangle = \prod_{1 \leq i < j \leq n} \prod_{r=1}^{k-1} \left( 1 - q^{\lambda_i - \lambda_j + r(j-i)} \right) / \left( 1 - q^{\lambda_i - \lambda_j - r(j-i)} \right).
\]

When \( \lambda = 0 \) (so that \( P_\lambda = 1 \)), the formula gives the constant term of \( \Delta(x) \). This is the constant term conjecture of type \( A_{n-1} \) (see [3]).
3. Proof of Theorem

Lemma. If \( m \geq n \), for a polynomial \( \psi(x) = \psi(x_1, \ldots, x_n) \), we have

\[
\left( \frac{1}{2\pi \sqrt{-1}} \right)^n \int_{T^n} \prod_{1 \leq i \leq m} \prod_{1 \leq j \leq n} \frac{1}{(y_i/x_j; q)_k} \Delta(x) \psi(x) \frac{dx_1 \cdots dx_n}{x_1 \cdots x_n} = \sum_{\{i_1, \ldots, i_n\} \subset \{1, \ldots, n\}} \sum_{0 \leq l_1, \ldots, l_n \leq k-1} \text{Res}_{x = (y_1, q^{l_1}, \ldots, y_n, q^{l_n})} \left\{ \prod_{1 \leq i \leq m} \prod_{1 \leq j \leq n} \frac{1}{(y_i/x_j; q)_k} \Delta(x) \psi(x) \frac{dx_1 \cdots dx_n}{x_1 \cdots x_n} \right\},
\]

where \( i_1, \ldots, i_n \) are distinct, and \( T^n = \{(t_1, \ldots, t_n) \in \mathbb{C}^n; |t_i| = 1 (1 \leq i \leq n)\} \).

Proof. For a polynomial \( \psi(x_1, x_2) \) and \( 0 \leq l \leq k - 1 \), we have the equality

\[
\text{Res}_{x = (yq^l/x_2; q)_k} \frac{(x_1/x_2; q)_k(x_2/x_1; q)_k}{(y/x_1; q)_k(y/x_2; q)_k} \psi(x_1, x_2) \frac{dx_1 dx_2}{x_1 x_2} = \frac{(yq^l/x_2; q)_k(x_2q^{-l}/y; q)_k}{(q^{-l}; q)_l(q; q)_{k-l}(y/x_2; q)_k} \psi(yq^l, x_2) \frac{dx_2}{x_2}. \tag{3.1}
\]

Because \( (y/x_2; q)_k \) divides \( (yq^l/x_2; q)_k(x_2q^{-l}/y; q)_k \), the 1-form (3.1) has no poles on the \( x_2 \)-plane. This shows that the set of poles of

\[
\prod_{1 \leq i \leq m} \prod_{1 \leq j \leq n} \frac{1}{(y_i/x_j; q)_k} \Delta(x_1, x_2) \psi(x_1, x_2) \frac{dx_1 dx_2}{x_1 x_2}
\]

is the union of \( (x_1, x_2) = (y_{i_1} q^{l_1}, y_{i_2} q^{l_2}) \) for \( 1 \leq i_1 \neq i_2 \leq m \) and \( 0 \leq l \leq k - 1 \), which implies the assertion of the above Lemma in the \( n = 2 \) case. Repeating this procedure, we have the desired result in case of general \( n \).

It is known ((3.11) in [4]) that

\[
\sum_{\lambda} b_{\lambda} P_{\lambda}(y) P_{\lambda}(x) = \prod_{1 \leq i \leq m} (ty_i x_j; q)_\infty \prod_{1 \leq i \leq m} \frac{1}{(y_i x_j; q)_k} \tag{3.2}
\]

with

\[
b_{\lambda} = b_{\lambda}(q, t) = \prod_{s \in \lambda} \frac{1 - q^{a(s) t^{l(s)} + 1}}{1 - q^{a(s) + 1 t^{l(s)}}}.
\]
Here the sum is taken over all partitions $\lambda$ such that $l(\lambda) \leq \min\{m, n\}$, and the arm-length $a(s)$ (resp. the leg-length $l(s)$) is defined by $a(s) = \lambda_i - j$ (resp. $l(s) = \lambda'_j - i$) for a square $s = (i, j)$ in the diagram $\lambda$.

The formula (3.2) in the $m = n$ case with the orthogonality relation gives

$$b_\lambda P_\lambda(1) P_\lambda = \frac{1}{n!} \left( \frac{1}{2\pi \sqrt{-1}} \right)^n \int_{T^n} \prod_{1 \leq i, j \leq n} \frac{1}{(y_i/x_j; q)_k} P_\lambda(x) \Delta(x) \frac{dx_1 \cdots dx_n}{x_1 \cdots x_n} \text{ (3.3)}$$

$$= \frac{1}{n!} \sum_{\sigma \in S_n} \sum_{0 \leq l_1, \ldots, l_n \leq k-1} \text{Res}_{x=(y_{\sigma(1)}q^{l_1}, \ldots, y_{\sigma(n)}q^{l_n})} \left\{ \prod_{1 \leq i, j \leq n} \frac{1}{(y_i/x_j; q)_k} P_\lambda(x) \Delta(x) \frac{dx_1 \cdots dx_n}{x_1 \cdots x_n} \right\}$$

$$= \sum_{0 \leq l_1, \ldots, l_n \leq k-1} \text{Res}_{x=(y_1q^{l_1}, \ldots, y_nq^{l_n})} \left\{ \prod_{1 \leq i, j \leq n} \frac{1}{(y_i/x_j; q)_k} P_\lambda(x) \Delta(x) \frac{dx_1 \cdots dx_n}{x_1 \cdots x_n} \right\}.$$
which tends to

\[
\sum_{0 \leq l_1, \ldots, l_n \leq k-1} \text{Res}_{x=(q^{l_1}, \ldots, q^{l_n})} \left\{ \frac{x^n (x_1 x_2)^k \cdots (x_1 \cdots x_{n-1})^k}{\prod_{i=1}^{n} (1/x_i)^k} \right\} \{ (y_1 x_1)^{\lambda_1} \cdots (y_n x_n)^{\lambda_n} + \text{lower order terms} \} \frac{dx_1 \cdots dx_n}{x_1 \cdots x_n}
\]

\[
= y^\lambda \sum_{0 \leq l_1, \ldots, l_n \leq k-1} \text{Res}_{x=(q^{l_1}, \ldots, q^{l_n})} \left\{ \prod_{i=1}^{n} \frac{(x_i)^{\lambda_i+(n-i)k}}{(1/x_i; q)_k} dx_1 \cdots dx_n \right\}
\]

+ lower order terms

\[
= y^\lambda \prod_{i=1}^{n} \frac{(q^{\lambda_i+(n-i)k+1}; q)_{k-1}}{(q; q)_{k-1}} + \text{lower order terms},
\]

if

\[
1 > |y_1| \gg |y_2| \gg \cdots \gg |y_n|.
\]

Comparing the coefficients of \(y^\lambda\) of (3.3) in the region (3.4) leads to

\[
b_\lambda \langle P_\lambda, P_\lambda \rangle = \prod_{i=1}^{n} \frac{(q^{\lambda_i+(n-i)k+1}; q)_{k-1}}{(q; q)_{k-1}},
\]

which is equivalent to

\[
\langle P_\lambda, P_\lambda \rangle = \prod_{1 \leq i < j \leq n} \frac{(q^{\lambda_i+\lambda_j+1+(j-1)k}; q)_{k-1}}{(q^{\lambda_i+\lambda_j+1+(j-1)k}; q)_{k-1}}.
\]

Here we used the equality

\[
b_\lambda = \prod_{1 \leq i < j \leq n} \frac{(q^{\lambda_i+\lambda_j+1+(j-1)k}; q)_{k-1}}{(q^{\lambda_i+\lambda_j+1+(j-1)k}; q)_{k-1}} \prod_{i=1}^{n} \frac{(q^{\lambda_i+1+k(n-i)}; q)_{k-1}}{(q; q)_{k-1}}.
\]

This completes the proof of our Theorem.

**Remark.** When we would like to consider the \(q = 1\) case directly, we need only modify the proof of Lemma and the calculation of the residue at the final step.

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