Shortcut To Adiabaticity for an Anisotropic 2D Bose Gas Containing Defects

D.J. Papoular and S. Stringari
INO-CNR BEC Center and Dipartimento di Fisica, Università di Trento, 38123 Povo, Italy
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We present a Shortcut To Adiabaticity (STA) protocol applicable to two-dimensional weakly–interacting Bose gases containing defects such as vortices or solitons. Our protocol relies on a new class of exact scaling solutions to the Gross–Pitaevskii equation applied to the 2D Bose gas in the presence of anisotropic time–dependent harmonic traps. It connects stationary states corresponding to initial and final traps having the same frequency ratio. The duration of the STA can be made very short so as to realize a quantum quench from one stationary state to another. We show that the STA conserves the shape of the defects present inside the anisotropic cloud, which sharply constrasts with the strong distortion of an anisotropic vortex during the free expansion of the cloud.

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Solitons and quantized vortices are fundamental excitations of non–linear media and play a key role in superfluid dynamics [1]. In anisotropic geometries, the energetically favored defects are solitonic vortices, characterized by a non–circular velocity field which nevertheless presents non–zero circulation [2, 3]. The investigation of their dynamics has been initiated by recent experiments where they were created either deterministically by phase imprinting [4], or spontaneously in a system which is quickly driven through a phase transition [5]. These defects exhibit intricate dynamics and decay mechanisms. For example, the snake instability [6], which affects solitons in 2D and in 3D, can lead to the creation of a solitonic vortex, a process which is likely to have played a role in [7]. The size of these defects is set by the healing length [8, chap. 5], which is too small to allow for in–situ observation. Up to now they have always been observed after the free expansion of the cloud, which increases the core dimensions. However, in the anisotropic case, the free expansion strongly distorts the cloud, and its use in the presence of a solitonic vortex leads to a very short duration of the STA [10].

An alternative to the free expansion of an ultracold cloud is provided by the recently–introduced Shortcut To Adiabaticity (STA) protocols [10, 11]. These offer a reversible way to evolve a many–body system from one state to another, achieving the same target state as an adiabatic transformation within a much shorter time over which decoherence and losses are minimal. They allow for the manipulation of the momentum spread of a wavepacket without the time constraints associated with delta kick cooling [12, 13]. Besides their practical usefulness, the investigation of these protocols has led to the discovery of novel solutions to seasoned models such as the Boltzmann equation [14]. They can be formulated in terms of counterdiabatic driving [15, 16]. They are being discussed in the context of the preparation of many–body states [17, 18], and have also motivated an experimental exploration of the quantum speed limit [20] as well as encouraged further reflection on the third law of thermo-dynamics [21].

The construction of an STA relies on the existence of a scaling solution to the equation describing the many–body dynamics of the system in the presence of a time–dependent confinement. Such a solution exists for the ideal gas in a harmonic trap [22], and it has been used to construct an STA solution [10] which has been demonstrated experimentally on a Bose cloud above T_c. [23]. Allowing for interactions, a scaling solution exists for the hydrodynamic equations describing the condensate dynamics in the Thomas–Fermi approximation [24, 25], and the corresponding STA has been implemented experimentally [26]. Scaling solutions also exist for a class of many–body systems [27] including interacting quasi–1D Bose gases, where an STA trajectory has been studied experimentally [28], and for box potentials [29].

Up to now, STAs have been demonstrated either in the ideal–gas regime or the Thomas–Fermi limit. Their application to gases containing defects requires going beyond these approximations because the existence and the dynamics of these defects result from the interplay between quantum pressure and interactions, even if the latter are strong enough for the bulk of the gas to be in the hydrodynamic regime [30, chap. 5]. In this context, harmonically–trapped 2D Bose gases are specially promising. In the isotropic case, they allow for an exact scaling solution to the many–body dynamics [24], which holds for all interaction strengths spanning from the ideal gas limit to the hydrodynamic regime. This 2D scaling requires no time–dependent manipulation of the interaction strength, unlike in 3D and 1D geometries [30]. It is due to a hidden symmetry of the harmonically–trapped system holding at all temperatures [31, 32]. The absence of damping of the breathing mode, which follows from this symmetry, has been confirmed experimentally [33].

In this Letter, we introduce a novel STA protocol applicable to anisotropic 2D Bose gases containing phase defects. This protocol links two stationary states corresponding to initial and final traps with the same anisotropy. It allows for a reversible and arbitrarily fast
FIG. 2. (A) Thick lines: Squared trapping frequencies which achieve an anisotropic STA linking the stationary states in two π the maximum density, and phases are normalized to free expansion of the same initial cloud during the same time (ω01t = 1.8). All six plots result from a numerical solution of Eq. (4) in imaginary time (left) and real time (center, right). Lengths are in units of (ℏ/mω01)^1/2, densities are normalized to the maximum density, and phases are normalized to π.

compression or expansion of the cloud which conserves the aspect ratio and acts as a homothetic transformation on the defects. This sharply contrasts with the free expansion of the anisotropic cloud, which leads to a time–dependent aspect ratio and an inversion of the cloud anisotropy, and which dramatically affects the density and phase profiles of vortices [9] (see Fig. 1 right). Our STA protocol can be used to quench the ultracold cloud [34] from one anisotropic stationary state to another. It relies on a new and exact scaling solution for the dynamics of a 2D Bose gas in a time–dependent anisotropic harmonic trap. Exact solutions to the time–dependent non–linear Schrödinger equation are rare, and we believe our result to be the first analytical solution to the Gross–Pitaevskii (GP) equation in the presence of a time–dependent anisotropic trap. This solution is applicable if the spatial aspect ratio of the cloud remains constant in time, though the ratio of the trapping frequencies need not be constant.

**Scaling solution for anisotropic time–dependent traps.** We consider a Bose gas which is tightly confined in the axial direction z, and trapped in the harmonic potential \( V(r, t) = m(\omega_1^2(t)x^2 + \omega_2^2(t)y^2)/2 \), with \( m \) being the atomic mass, in the directions \( x \) and \( y \). Note that \( V(r, t) \) is both time–dependent and anisotropic. We assume that the oscillator length \( l_z \) associated with the axial confinement is both larger than the scattering length \( a_{3D} \) characterizing 3D collisions and smaller than the healing length. The first assumption ensures that the scattering amplitude is momentum–independent, thus avoiding the quantum anomaly affecting the scale invariance of 2D systems [35], while the second means that the atomic density does not spill beyond the axial ground state [36]. These conditions are experimentally realistic [37], and they allow for a description of the \( T = 0 \) dynamics of the system in terms of the 2D Gross–Pitaevskii equation [8] chap. 17:

\[
\frac{i\hbar}{2m} \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \Psi + V(r, t)\Psi + g_{2D}|\Psi|^2\Psi ,
\]

where \( V(r, t) \) is the time–dependent anisotropic trap defined above, \( \Psi(r, t) \) is the wavefunction, \( \Delta = \partial_x^2 + \partial_y^2 \) is the Laplacian, and the coupling constant \( g_{2D} \) satisfies \( g_{2D}m/\hbar^2 = \sqrt{8\pi a_{3D}/l_z} \) [38]. We assume that, for \( t \leq 0 \), \( \Psi \) is a stationary solution of Eq. (1) with the constant trap \( V_0(r) = m(\omega_{01}^2x^2 + \omega_{02}^2y^2)/2 \). The wavefunction \( \Psi \) need not represent the lowest–energy stationary state.
Following the approach of Refs. [24, 39], we seek an exact solution to Eq. (1) of the form:

$$\Psi(r, t) = \frac{1}{b(t)} \chi_0[\rho, \tau(t)] \exp[i\Phi(r, t)],$$  \hspace{1cm} (2)

with $b(t)$ being the only scaling parameter. In Eq. (2), we have introduced the new coordinates $\rho = r/b$, the reduced time $\tau(t) = \int^t dt'/b^2(t')$, and the phase $\Phi(r, t) = (m/2\hbar)[r^2/b^2]$. The positive function $b(t)$ satisfies the boundary conditions $b(0) = 1$, $b(0) = 0$, and $b(0) = 0$, where the dots denote derivation with respect to the time $t$. The insertion of Eq. (2) into Eq. (1) yields an evolution equation for $\chi_0$ where the terms proportional to $\nabla\chi_0$ cancel [39]. We find that, if the trapping frequencies satisfy:

$$\omega^2_j(t) = \frac{\omega^2_j}{b^2} - \frac{\dot{b}}{b} \quad \text{and} \quad \omega^2_2(t) = \frac{\omega^2_2}{b^2} - \frac{\dot{b}}{b},$$  \hspace{1cm} (3)

then the function $\chi_0(\rho, \tau)$ is a solution of the following Gross–Pitaevskii equation:

$$i\hbar \frac{\partial \chi_0}{\partial \tau} = -\frac{\hbar^2}{2m} \Delta_\rho \chi_0 + \frac{m}{2} \sum_j \omega^2_j \rho^2_j \chi_0 + g_{2D}|\chi_0|^2 \chi_0,$$  \hspace{1cm} (4)

with $\Delta_\rho = \partial^2_{\rho_1} + \partial^2_{\rho_2}$. The trapping potential appearing in Eq. (4) is constant equal to $V_0(\rho)$, which coincides with the potential of Eq. (1) for $t \leq 0$, so that $\chi_0$ is a stationary solution of Eq. (4) whose evolution in time simply leads to a phase which is linear in $\tau$. Thus, the joint rescaling of the space and time coordinates, and the gauge transform defined by $\Phi(r, t)$, allow us to describe the dynamics of the system exactly in terms of the coupled differential equations of Eq. (4). In particular, Eq. (2) shows that the mean–squared radii of the gas satisfy $\langle x^2(t) \rangle/\langle x^2 \rangle_0 = \langle y^2(t) \rangle/\langle y^2 \rangle_0 = b^2(t)$, where $\langle x^2 \rangle_0$ and $\langle y^2 \rangle_0$ are the mean–squared radii corresponding to the stationary configuration for $t \leq 0$. Hence, the spatial aspect ratio $R = \sqrt{\langle y^2 \rangle/\langle x^2 \rangle}$ of the cloud remains constant in time, even though the frequency ratio $\omega_2/\omega_1$ determined from Eqs. (3) does depend on time.

Shortcut to adiabaticity in $2D$. — We now use the exact scaling solution described above to construct an STA protocol which is applicable to anisotropic classical clouds containing defects. This protocol transforms the initial stationary state into another stationary state corresponding to a trap with the same anisotropy, i.e. $\omega_{f2}/\omega_{f1} = \omega_{2}/\omega_{1}$, where $\omega_{f1}$ and $\omega_{f2}$ characterize the final state, reached at the time $t_f$ which can be chosen at will.

Our scheme is reminiscent of a recently–proposed STA protocol applicable to isotropic classical gases [14], where the role of our scaling parameter $b(t)$ is played by $T^{-1/2}$, with $T$ being the temperature. We first choose an appropriate time dependence for $b(t)$, and then determine the two time–dependent trapping frequencies $\omega_1(t)$ and $\omega_2(t)$ which achieve the sought transformation. We impose $\omega_j(t \geq t_f) = \omega_{fj}$ for $j = 1, 2$, and we require that $\Psi(r, t)$ be stationary for $t \geq t_f$. In terms of the scaling parameter, the stationary condition means that $b(t_f) = 0$ and $\dot{b}(t_f) = 0$. Combined with Eq. (3), the latter requirement defines the final value $b_f$ of the scaling parameter in terms of the initial and final trapping frequencies:

$$\omega_{f1} \omega_{2} = \omega_{f2} = \frac{1}{b^2(t_f)}.$$  \hspace{1cm} (5)

Provided that Eq. (5) holds, the scaling parameter $b(t)$ must satisfy 3 boundary conditions for $t = 0$ and 3 others for $t = t_f$ [40]. Numerous choices are possible for $b(t)$, the simplest being the following fifth–order polynomial:

$$b(t) = 1 + (b_f - 1) \left[10 \left(\frac{t}{t_f}\right)^3 - 15 \left(\frac{t}{t_f}\right)^4 + 6 \left(\frac{t}{t_f}\right)^5\right].$$  \hspace{1cm} (6)

Once $b(t)$ is known, the frequencies $\omega_1(t)$ and $\omega_2(t)$ which achieve the STA trajectory are determined using Eq. (3).

The evolution in time of these frequencies depends on the initial values $\omega_{0j}$ and on the duration $t_f$, but not on the interaction strength $g_{2D}$ or on the number of atoms $N$. The stationarity condition implies that, for $t \geq t_f$, the time evolution of $\Psi(r, t)$ simply leads to a phase which is linear in $(t - t_f)$.

For given values of $m$ and $g_{2D}$, the shape of the STA trajectory $\{\omega^2_1(t), \omega^2_2(t)\}$ is dictated by three key parameters. First, the scaled duration $(\omega_{01}\omega_{02})^{1/2}t_f$ determines the attractive or repulsive nature of the required trapping frequencies $\omega^2_j(t)$. In analogy with the isotropic scheme of Ref. [14], shorter durations require potentials that are repulsive (i.e. $\omega^2_2(t) < 0$) for longer fractions of the total time. Second, the ratio $\omega_{01}/\omega_{f1}$ sets the final dimensions of the cloud through Eq. (5). Third, the ratio $\omega_{2}/\omega_{02}$, which encodes the initial anisotropy of the gas, sets the time–independent value of the spatial aspect ratio $R$. Our exact solution is applicable for all coupling strengths compatible with the mean–field Eq. (1), spanning from the ideal–gas limit $(R = \sqrt{\omega_{01}/\omega_{02}})$ to the hydrodynamic regime $(R = \omega_{01}/\omega_{02})$.

We now demonstrate this new STA protocol on the expansion of an anisotropic cloud containing a single vortex (see Figs. 1 and 2). The single–vortex state is a stationary state of Eq. (4), albeit not the one with the lowest energy. However, it is the lowest–energy state satisfying the symmetry conditions $\Psi(-x, -y, t) = -\Psi(x, y, t)$ and $\Psi(-x, y, t) = \chi_\ast(x, y, t)$ (up to an arbitrary constant in the definition of the phase). Imposing these conditions on the initial wavefunction allows for its calculation using imaginary–time evolution despite the absence of any gauge or rotation term in Eq. (1). We start from the single–vortex stationary state with $\omega_{02} = 10\omega_{01}$ and $mgN\hbar^2 = 3100$, whose density and phase profiles are represented on Fig. 1 (left, top and bottom). Over the short time $\omega_{01}t_f = 1/2$, the STA trajectory reaches a new stationary state where both trapping frequencies are four
times as small as their initial values. Figure 2A shows the time dependence of $\omega_x^2$ and $\omega_y^2$ from $t = 0$ to $t_f$. The exact prediction for the scaling parameter $b(t)$ given by Eq. (6) is compared on Fig. 2B to the calculated values for the ratio $\Delta x/\Delta x_0 = \Delta y/\Delta y_0$ characterizing the quadratic mean radii of the cloud, obtained through a numerical solution of the Gross–Pitaevskii Eq. (1) for times up to $t = 6t_f = 3/\omega_{01}$. The density and phase profiles of the cloud at $t = 3.6t_f$, shown on Fig. 1 (center), are those of a stationary anisotropic vortex in the expanded trap. The small residual oscillations seen on Fig. 2B for $t > t_f$, which start before $t_f$ and survive for longer times, are an artefact of the numerical simulation, and their amplitude decreases with increasing resolutions of the spatial grid.

The above results illustrate three important properties of our STA protocol. First, the squared frequencies determined by Eq. (3) can be transiently negative, corresponding to an expulsive potential, as already noted in Ref. [14]. Second, despite the initial and final trap anisotropies being the same, the ratio $\omega_x^2(t)/\omega_x^2(t)$ is not constant in time. Indeed, Fig. 2A shows that $\omega_x^2(t)$ is positive at all times whereas $\omega_y^2(t)$ is negative for intermediate times, which is a dramatic consequence of anisotropy. Third, the trapping frequencies are continuous at $t = 0$ and $t = t_f$, but they need not go smoothly to the initial and final values, as long as the scaling parameter $b(t)$ satisfies the correct boundary conditions at $t = 0$ and $t = t_f$. By contrast, even a very smooth non–STA trajectory which links the initial and final states within the same time $t_f$ results in large–amplitude oscillations of the mean–square radii for $t > t_f$ (see Fig. 2C). The discontinuity in the derivatives of the STA trapping frequencies at $t = 0$ and $t_f$ is a consequence of our choice for $b(t)$ (Eq. 6). It can be avoided by choosing a higher–order polynomial, in accordance with the results of Ref. [14].

The stationary behavior of the cloud following an STA (Fig. 2B) sharply contrasts with its behavior during free expansion, which leads to a time–dependent aspect ratio and an inversion of the anisotropy of the cloud for all interaction strengths (see Fig. 3). Comparing the density and phase distributions obtained, at the same instant, from the same initial cloud undergoing an STA (Fig. 1 center) or a free expansion (Fig. 1 right) further confirms that the STA acts as a homogeneous dilation on the vortex, whereas the free expansion leads to the twisted nodal line characteristic of soliton vortices [9].

The free expansion of a gas is practically irreversible. On the contrary, the time reversal of an STA trajectory leads to another STA trajectory which performs the inverse transformation. More precisely, if $\{\omega_1(t), \omega_2(t)\}$ is an STA trajectory satisfying Eq. (3) with the scaling parameter $b(t)$, then $\{\omega_1(t_f - t), \omega_2(t_f - t)\}$ is an STA trajectory associated with the scaling function $\bar{b}(t) = b(t_f - t)/b_f$. Thus, an STA which compresses the cloud can be obtained by time–reversing the trajectory of Fig. 2A.

The duration $t_f$ of the STA trajectory can be chosen to be arbitrarily small. Therefore, the STA does not have any associated quantum speed limit [41]. It allows for an ultrafast expansion of the cloud, leading to a final state in a very weakly–confining trap. Equation (5) shows that, if $\omega_{f1}/\omega_{01}$ is sufficiently small, $b_f$ can be made large. Hence, the cloud can be expanded to an arbitrarily large radius within an arbitrarily small time, thus realizing a quantum quench from one stationary state to another, with the caveat that the part of the trajectory involving expulsive potentials ($\omega_y^2 < 0$) becomes more pronounced.

We have presented our scaling solution and the corresponding STA trajectories at $T = 0$, in which regime Eq. (1) provides a convenient description of the cloud dynamics. However, our results are more general. Under the assumptions justifying the use of Eq. (1), the dynamical symmetry our results rely on applies to all eigenstates of the complete quantum Hamiltonian [32], and we expect the STA trajectories to be applicable for all temperatures at which $s$–wave scattering is dominant.

In conclusion, our exact scaling solution for the dynamics of an anisotropic 2D Bose cloud, applicable if the aspect ratio is constant in time, is an important step towards the analytical description of the solutions of the 2D Gross–Pitaevskii equation. Quasi–2D gases are readily created and manipulated experimentally [37]. The anisotropic (trapping or expulsive) quadratic potentials required for the implementation of our STA protocol can be obtained by combining red– and blue–detuned optical potentials [42]. The new generation of manipulation methods to which STA protocols belong will allow for a detailed experimental investigation of the dynamics and decay mechanisms of defects in anisotropic clouds such as those observed in [41,17]. For instance, the atomic cloud can be cooled and condensed in a strongly–confining trap where evaporative cooling is efficient, and then quickly expanded using the STA, without any distortion, into a weakly–confining trap where defects can more easily be
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