SOME FORMULAS OF SANTALÓ TYPE IN FINSLER GEOMETRY AND ITS APPLICATIONS

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Abstract. In this paper, we establish two Santaló type formulas for general Finsler manifolds. As applications, we derive a universal lower bound for the first eigenvalue of the nonlinear Laplacian, two Croke type isoperimetric inequalities, and a Yamaguch type finiteness theorem in Finsler geometry.

1. Introduction

In [16, 17], Santaló considered the kinematic measure and established a formula which describes the Liouville measure on the unit sphere bundle of a Riemannian manifold in terms of the geodesic flow and the measure of a hypersurface. This formula plays an important role in global Riemannian geometry. Some of its applications are universal bounds for the first eigenvalue [5], Croke’s isoperimetric inequality [10] and a generalization of Berger’s theorem [8]. Moreover, with Santaló’s formula, Croke in [7] solved a famous isoperimetric problem in dimension 4. See [5, 7, 8, 9, 10, 11, 16, 17] for more details.

A Finsler manifold is a differentiable manifold, on which every tangent space is endowed a Minkowski norm instead of a Euclidean norm. There is only one reasonable notion of the measure for Riemannian manifolds (cf. [4]). However, the measure on a Finsler manifold can be defined in various ways and essentially different results may be obtained, e.g., [1, 2, 18]. Hence, it is interesting to ask whether an analogue of Santaló’s formula still holds for Finsler manifolds.

Let $\m (M, F)$ be a Finsler manifold. Denote by $\pi : SM \to M$ the unit sphere bundle. If $F(y) = F(-y)$ for any $y \in SM$, then $F$ is reversible. In a reversible Finsler manifold, the reverse of a geodesic is still a geodesic (see [3, 18]). In [23], Shen and Zhao considered the problem above and established a Santaló type formula for reversible Finsler manifolds.

There are infinitely many nonreversible Finsler metrics. For example, a Randers metric in the form $F = \alpha + \beta$ is non-reversible, where $\alpha$ is a Riemannian metric and $\beta$ is a 1-form. The reverse of a geodesic in a non-reversible Finsler manifold is in general not a geodesic. Moreover, in a non-reversible Finsler manifold, the measure of a hypersurface induced by the inward normal vector field may be different from the one induced by the outward normal vector filed (see Example 1 in Section 5 below). The purpose of this paper is to establish some Santaló type formulas for general Finsler manifolds.

Let $(M, \partial M, F, d\mu)$ be a compact Finsler manifold with smooth boundary, where $F$ is possibly non-reversible and $d\mu$ is a measure on $M$. Denote by $\mathbf{n}_+$ and $\mathbf{n}_-$ the 2010 Mathematics Subject Classification. Primary 53B40, Secondary 47J10, 28A75.

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unit inward and outward normal vector fields along \( \partial M \), respectively. The measures on \( \partial M \) induced by \( n_{\pm} \) are defined by \( dA_{\pm} := i^* (n_{\pm} | d\mu) \). Let \( S^+ \partial M \) and \( S^- \partial M \) be the bundles of inwardly and outwardly pointing unit vectors along \( \partial M \), i.e., \( S^\pm \partial M = \{ y \in SM |_{\partial M} : g_{n_{\pm}} (n_{\pm}, y) > 0 \} \). The measures on \( S^\pm \partial M \) are the product measures \( d\chi_{\pm} (y) := dk_{\pi(y)} (y) d\Lambda_{\pm} (\pi(y)) \), where \( dk_{\pi(y)} (y) \) is the Riemannian measure on \( S_{\pi} M := \pi^{-1} (y) \) induced by \( F \). For each \( y \in S^+ \partial M \), set \( t(y) := \sup \{ t > 0 : \gamma_y (s) \in M, 0 < s < t \} \) and \( \iota(y) := \min \{ \iota(y), t(y) \} \), where \( \iota(y) \) is the cut value of \( y \).

Since \( F \) may be non-reversible, to investigate the asymmetry of the Finsler manifold, we introduce the reverse of \( F \), which is defined by \( \bar{F}(y) := F(-y) \). Clearly, \( \bar{F} \) is a Finsler metric as well. Let \( \iota(-\cdot) \), \( \bar{\iota}(-\cdot) \) and \( \tilde{\iota}(-\cdot) \) be defined as above on \( (M, \partial M, \bar{F}) \). Then we have the following Santaló type formulas.

**Theorem 1.1.** For all integral function \( f \) on \( SM \), we have

\[
\begin{align*}
(i) & \int_{V_M^+} f dV_M = \int_{y \in S^+ \partial M} e^\tau(y) g_{n_+} (n_+, y) d\chi_+(y) \int_0^{\iota(y)} f(\varphi(y)) dy, \\
(ii) & \int_{V_M^-} f dV_M = \int_{y \in S^- \partial M} e^\tau(y) g_{n_-} (n_-, y) d\chi_-(y) \int_0^{\iota(y)} f(\varphi(y)) dy,
\end{align*}
\]

where \( dV_M \) is the canonical Riemannian measure on \( SM \), \( \tau \) is the distortion of \( d\mu \), \( \varphi(y) \) is the geodesic flow of \( F \), \( V_M^+ := \{ y \in SM : \iota(y) \leq \tilde{\iota}(y) \} \) and \( V_M^- := \{ y \in SM : \iota(y) \leq \tilde{\iota}(y) \} \).

One can easily see that Theorem 1.1 implies the Santaló type formulas for reversible Finsler manifolds \[10\] and for Riemannian manifolds \[16, 17\]. It is remarkable that, in a non-reversible Finsler manifold, \( \Lambda_- (\partial M) \neq \Lambda_+ (\partial M) \) and the formulas \[10\] and \[17\] contain information about \( \Lambda_+ (\partial M) \) and \( \Lambda_- (\partial M) \), respectively.

Before giving some applications of Theorem 1.1 we shall recall some notions and basic facts of the first eigenvalue in the Finsler setting. The first eigenvalue \( \lambda_1 (M, d\mu) \) in \( (M, F, d\mu) \) is defined as the smallest positive eigenvalue of the nonlinear Laplacian \( \Delta_{d\mu} \) introduced by Shen (cf. \[14, 18, 19\]). It should be noted that both \( \Delta_{d\mu} \) and \( \lambda_1 (M, d\mu) \) are dependent on the choice of the measure \( d\mu \). Theorem 1.1 now yields the following.

**Theorem 1.2.** Let \( (M, \partial M, F) \) be a compact Finsler \( n \)-manifold with smooth boundary such that every geodesic ray in \( (M, F) \) minimizes distance up to the point that it intersects \( \partial M \). Then

\[
\lambda_1 (M, d\mu) \geq \begin{cases} \\
\frac{\lambda_D (S_D^+)}{\Lambda_F}, & d\mu \text{ is the Busemann-Hausdorff measure,} \\
\frac{\lambda_D (S_D^+)}{\Lambda_F}, & d\mu \text{ is the Holmes-Thompson measure,}
\end{cases}
\]

where \( D := \text{diam}(M) \), \( \Lambda_F \) is the uniform constant of \( F \), and \( S_D^+ \) denotes the \( n \)-dimensional Riemannian hemisphere of the constant sectional curvature space having diameter equal to \( D \). The equality holds if and only if \( (M, F) \) is isometric to \( S_D^+ \).

Note that a Finsler metric \( F \) is Riemannian if and only if \( \Lambda_F = 1 \). Hence, Theorem 1.2 implies Croke’s sharp universal lower bound for the first eigenvalue \[5, 10\].
Let \((M, \partial M, F)\) be as before. For each \(x \in M\), set
\[
\omega := \inf_{x \in M} \min\{\omega^+_x, \omega^-_x\},
\]
where \(\omega^\pm_x := c_n^{-1} \int_{U^\pm_x} e^\tau(y) d\nu_x(y), \ U^\pm_x := \pi_{V^\pm_x}^{-1}(x)\) and \(c_n = \text{Vol}(\mathbb{S}^{n-1})\). Then Theorem 1.1 furnishes the following inequalities.

**Theorem 1.3.** Let \((M, \partial M, F, d\mu)\) be a compact Finsler \(n\)-manifold with smooth boundary, where \(d\mu\) is either the Busemann-Hausdorff measure or the Holmes-Thompson measure. Then

1. \[
\frac{A_\pm(\partial M)}{\mu(M)} \geq \frac{(n-1)c_{n-1}\omega}{c_{n-2} D^2 n^{2n+\frac{1}{2}}},
\]
where \(D := \text{diam}(M)\).
2. \[
\frac{A_\pm(\partial M)}{\mu(M)^{1-\frac{1}{n}}} \geq \frac{c_{n-1}\omega^{1+\frac{1}{n}}}{(c_n/2)^{1-\frac{1}{n}} D^2 n^{2n+\frac{1}{2}}},
\]
with equality if and only if \((M, F)\) is a Riemannian hemisphere of a constant sectional curvature sphere.

If \(F\) is reversible, then \(\omega_+ = \omega_-\) and \(A_+(\partial M) = A_-(\partial M)\). Hence, Theorem 1.3 implies Croke type isoperimetric inequalities for reversible Finsler manifolds [23, Theorem 1.6] and for Riemannian manifolds [10].

As an application of Theorem 1.3, we obtain a Finslerian version of Yamaguchi’s finiteness theorem.

**Theorem 1.4.** For any \(n\) and positive numbers \(i, V, \delta\), the class of closed Finsler \(n\)-manifolds \((M, F)\) with injectivity radius \(i_M \geq i\), \(\Lambda_F \leq \delta\) and \(\mu(M) \leq V\), contains at most finitely many homotopy types. Here, \(\mu(M)\) is either the Busemann-Hausdorff volume or the Holmes-Thompson volume of \(M\).

2. **Preliminaries**

In this section, we recall some definitions and properties about Finsler manifolds. See [3, 18] for more details.

Let \((M, F)\) be a (connected) Finsler \(n\)-manifold with Finsler metric \(F : TM \to [0, \infty)\). Let \((x, y) = (x^i, y^i)\) be local coordinates on \(TM\). Define
\[
g_{ij}(x, y) := \frac{1}{2} \frac{\partial^2 F^2(x, y)}{\partial y^i \partial y^j}, \quad G^i(y) := \frac{1}{4} g^{ij}(y) \left\{ 2 \frac{\partial g_{jl}}{\partial x^k}(y) - \frac{\partial g_{jk}}{\partial x^l}(y) \right\} y^j y^k,
\]
where \(G^i\) are the geodesic coefficients. A smooth curve \(\gamma(t)\) in \(M\) is called a (constant speed) geodesic if it satisfies
\[
\frac{d^2 \gamma^i}{dt^2} + 2G^i \left( \frac{d\gamma}{dt} \right) = 0.
\]
We always use \(\gamma_y(t)\) to denote the geodesic with \(\gamma_y(0) = y\).

The Ricci curvature is defined by \(\text{Ric}(y) := \sum_{i=1}^n R^i_i(y)\), where
\[
R^i_k(y) := 2\frac{\partial G^i}{\partial x^k} - y^l \frac{\partial^2 G^i}{\partial x^l \partial y^k} + 2G^j \frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^j} \frac{\partial G^j}{\partial y^k}.
\]
Let $\pi: SM \to M$ be the unit sphere bundle, i.e., $S_x M := \{y \in T_x M : F(x, y) = 1\}$ and $SM := \cup_{x \in M} S_x M$. The measure on $SM$ is defined by

$$dV_{SM}(x,y) = \sqrt{\det g_{ij}(x,y)}dx^1 \wedge \cdots \wedge dx^n \wedge d\nu_x(y)$$

$$= e^{\tau(y)} \pi^*(d\mu(x)) \wedge d\nu_x(y).$$

where

$$d\nu_x(y) := \sqrt{\det g_{ij}(x,y)} \left( \sum_{i=1}^n (-1)^{i-1} y^i dy^1 \wedge \cdots \wedge \hat{dy^i} \cdots \wedge dy^n \right).$$

is the Riemannian measure on $S_x M$ induced by $F$.

The reversibility $\lambda_F$ and the uniformity constant $\Lambda_F$ of $(M, F)$ are defined by $\lambda_F := \sup_{x \in M} \lambda_F(x)$ and $\Lambda_F := \sup_{x \in M} \Lambda_F(x)$, where

$$\lambda_F(x) := \sup_{y \in S_x M} F(x, -y), \quad \Lambda_F(x) := \sup_{X,Y \in S_x M} g_X(Y,Y).$$

Clearly, $\Lambda_F \geq \lambda_F^2 \geq 1$. $\lambda_F = 1$ if and only if $F$ is reversible, while $\Lambda_F = 1$ if and only if $F$ is Riemannian.

The dual Finsler metric $F^*$ on $M$ is defined by

$$F^*(\eta) := \sup_{x \in T_x M \setminus 0} \frac{\eta(X)}{F(X)}, \quad \forall \eta \in T^*_x M.$$ 

The Legendre transformation $\mathcal{L}: TM \to T^* M$ is defined as

$$\mathcal{L}(X) := \begin{cases} g_X(X, \cdot) & X \neq 0, \\ 0 & X = 0. \end{cases}$$

In particular, $F^*(\mathcal{L}(X)) = F(X)$. Now let $f: M \to \mathbb{R}$ be a smooth function on $M$. The gradient of $f$ is defined by $\nabla f = \mathcal{L}^{-1}(df)$. Thus, $df(X) = g_{\nabla f}(\nabla f, X)$.

Let $d\mu$ be a measure on $M$. In a local coordinate system $(x^i)$, express $d\mu = \sigma(x)dx^1 \wedge \cdots \wedge dx^n$. In particular, the Busemann-Hausdorff measure $d\mu_{BH}$ and the Holmes-Thompson measure $d\mu_{HT}$ are defined by

$$d\mu_{BH} = \sigma_{BH}(x)dx := \frac{\text{Vol}(\mathbb{B}^n)}{\text{Vol}(\{y \in T_x M : F(x, y) < 1\})}dx^1 \wedge \cdots \wedge dx^n,$$

$$d\mu_{HT} = \sigma_{HT}(x)dx := \left( \frac{1}{e_n - 1} \int_{S_x M} \sqrt{\det g_{ij}(x,y)d\nu_x(y)} \right) dx^1 \wedge \cdots \wedge dx^n.$$

For $y \in T_x M \setminus 0$, define the distortion of $(M, F, d\mu)$ as

$$\tau(y) := \log \frac{\sqrt{\det g_{ij}(x,y)}}{\sigma(x)}.$$

By the same argument as in [21], one can show the following lemma.

**Lemma 2.1.** Let $(M, F)$ be a Finsler $n$-manifold with finite uniform constant $\Lambda_F$. Let $d\mu$ denote either the Busemann-Hausdorff measure or the Holmes-Thompson measure on $M$. Then the distortion $\tau$ of $d\mu$ satisfy $\Lambda_F^n \leq e^{\tau(y)} \leq \Lambda_F^n$, for all $y \in SM$.

The reverse of a Finsler metric $F$ is defined by $\tilde{F}(y) := F(-y)$. It is not hard to see that $\tilde{G}(y) = G^i(-y)$ and $\tilde{d}\mu = d\mu$, where $\tilde{G}$ (resp. $G^i$) are the geodesic coefficients of $\tilde{F}$ (resp. $F$), and $d\tilde{\mu}$ (resp. $d\mu$) denotes the Busemann-Hausdorff
measure or the Holmes-Thompson measure of $\tilde{F}$ (resp. $F$). In particular, if $\gamma$ is a geodesic of $F$, then the reverse of $\gamma$ is a geodesic of $\tilde{F}$.

3. Santaló Type Formulas

Let $(M, \partial M, F)$ be compact Finsler manifold with smooth boundary. Denote by $n_+$ (resp. $n_-$) the unit inward (resp. outward) normal vector field along $\partial M$. Define $N_+ := \{k \cdot n_+(x) : x \in \partial M, k \in \mathbb{R}\}$. The exponential map $\text{Exp}_+ : N_+ \to M$, $k \cdot n_+(x) \mapsto \exp_x(kn_+(x))$.

We always identify $\partial M$ with the zero section of $N_+$. The same arguments as in 
[23] Lemma 5.1, Remark 5.1 show the following lemma.

Lemma 3.1. $\text{Exp}_+$ maps a neighborhood of $\partial M \subset N_+$ $C^1$-diffeomorphically onto a neighborhood of $\partial M \subset \overline{M}$. Hence, there exists a small $\delta > 0$ such that $\text{Exp}_+ : M_\delta \to \text{Exp}_+(M_\delta)$ is $C^1$-diffeomorphic, where $M_\delta := \{k \cdot n_+(x) : x \in \partial M, 0 \leq k < \delta\}$.

Define $\rho : \overline{M} \to \mathbb{R}_+$ by $\rho(x) = d(\partial M, x)$. Lemma 3.1 together with the proofs of 
[23] Lemma 5.2-5.3, Corollary 5.1 and [18] Lemma 3.2.3 yields

Lemma 3.2. Let $\sigma(t), 0 \leq t < \epsilon$, be a $C^1$-curve with $\sigma(0) \in \partial M$ and $\sigma((0, \epsilon)) \subset M$. Then

$$0 \leq \frac{d}{dt} \bigg|_{t=0^+} \rho \circ \sigma(t) = g_{n_+}(n_+, \sigma(0)).$$

Hence, $\nabla \rho_+(x) = n_+(x)$, for any $x \in \partial M$.

Set $S^\pm \partial M := \{y \in SM|_{\partial M} : g_{n_+}(n_+, y) > 0\}$. By the Legendre transformations, one can show that $S^\pm \partial M$ are two submanifolds of $\overline{M}$. 

Remark 1. In general, $n_+ \neq -n_-$. However, it follows from the Legendre transformations that $S^\pm \partial M = \{y \in SM|_{\partial M} : g_{n_+}(n_+, y) < 0\}$.

Set $Z := \{y \in \partial M : \exists t > 0$ such that $\gamma_{y}(0, t) \subset M\}$. Define a function $t : SM \cup S^+ \partial M \cup Z \to \mathbb{R}_+$ by $t(y) := \sup\{t > 0 : \gamma_{y}(s) \in M, 0 < s < t\}$, which is called the $t$-function. By the same argument as in 
[23] Lemma 5.4, one can show that $t$-function is low semi-continuous on $SM \cup S^+ \partial M$.

Since $(M, \partial M, F)$ is compact, we can define a map

$$\Psi : \{(t, y) : y \in S^+ \partial M, 0 \leq t \leq t(y)\} \to SM, (t, y) \mapsto \varphi_t(y),$$

where $\varphi_t$ is the geodesic flow of $F$. Let $\widetilde{t}$ (resp. $\widetilde{t}$) denote the $t$-function (resp. the cut value function) defined on $(M, \partial M, \widetilde{F})$, where $\widetilde{F}(y) := F(-y)$. Set

$$U_{\widetilde{M}} := \{y \in SM : \widetilde{t}(-y) < i(-y)\}.$$ 

Since $y \in SM$ implies that $\widetilde{F}(-y) = 1$, $U_{\widetilde{M}}$ is well-defined. In particular, we have the following

Lemma 3.3. $\Psi|_{\mathcal{N}_+} : \mathcal{N}_+ \to U_{\widetilde{M}} \setminus U_{\overline{Z}}$ is a one-one map. Here, $\mathcal{N}_+ := \{(t, y) : y \in S^+ \partial M, t \in (0, l(y))\}$, $U_{\overline{Z}} := \{\varphi_t(y) : y \in Z, t \in (0, l(y))\}$, and $l(y) := \min\{i(y), t(y)\}$. 

Proof. Since $\overline{M}$ is compact, for each $y \in U_{\overline{M}}^-$, $0 < \hat{t}(y) < \hat{i}(y) < \infty$. Clearly, $\hat{\gamma}_{-y}(t), 0 \leq t \leq \hat{t}(y)$ is a unit speed minimal geodesic in $(\overline{M}, \overline{F})$. Set $Y := -\hat{\gamma}_{-y}(\hat{t}(y))$. Thus,

$$F(Y) = \overline{F}(-Y) = \overline{F}(\tilde{\gamma}_{-y}(\hat{t}(y))) = 1.$$  

It follows from Lemma 3.2 that $g_{n_+}(n_+, Y) = 0$. Hence, $Y \in S^+\partial M \cup \mathbb{Z}$.

Let $d$ (resp. $\tilde{d}$) denote the distance function induced by $F$ (resp. $\overline{F}$). Let $p := \pi(y)$ and $q := \pi(Y)$. Then $L_F(\gamma_Y([0, \hat{t}(y)])) = \hat{t}(y) = d(p, q) = d(q, p)$, which implies that $i(Y) \geq \hat{t}(y)$. We claim that $i(Y) > \hat{t}(y)$. If not, then $p$ is the cut point of $q$ along $\gamma_Y$. If $p$ is also a conjugate point of $q$, then there exists a non-vanishing Jacobi field $J(t)$ along $\gamma_Y(t)$ such that $J(0) = 0$ and $J(\hat{t}(y)) = 0$. It is easy to check that $J(t) := J(\hat{t}(y) - t)$ is a Jacobi field along $\tilde{\gamma}_{-y}$ in $(\overline{M}, \overline{F})$. Hence, $q$ is a conjugate point of $p$ along $\tilde{\gamma}_{-y}$ in $(\overline{M}, \overline{F})$, which contradicts $\hat{t}(y) < \hat{i}(y)$. Since $p$ is not a conjugate point of $q$, by the proof of [3, Proposition 8.2.1], one can show that there exists another minimal geodesic from $q$ to $p$ in $(\overline{M}, F)$. Thus, there exist two distinct minimal geodesics from $p$ to $q$ with the length $\hat{t}(y)$ in $(\overline{M}, \overline{F})$, which also contradicts $\hat{t}(y) < \hat{i}(y)$. Hence, the claim is true, which implies that $\hat{t}(y) < \min\{t(Y), i(Y)\} = l(Y)$.

From above, we show that for each $y \in U_{\overline{M}}^-$, there exist $Y \in S^+\partial M \cup \mathbb{Z}$ and $t := \hat{t}(y) < l(Y)$ such that $y = \Psi(t, Y)$. Let $N_\mathbb{Z} := \{(t, y) : y \in \mathbb{Z}, t \in (0, l(y))\}$. Then $\Psi|_{n_+ \cup N_\mathbb{Z}} : n_+ \cup N_\mathbb{Z} \to U_{\overline{M}}^-$ is subjective. Since $\Psi$ is injective, we are done by $\Psi(N_\mathbb{Z}) = U_{\overline{M}}^-$. \hfill $\Box$

Given any measure $d\mu$ on $M$, the induced volume forms on $\partial M$ by $n_\pm$ are defined by $d\Lambda_\pm := \hat{i}^*(n_\pm) d\mu$, where $\hat{i} : \partial M \hookrightarrow M$ is the inclusion map (cf. [15]). Now we have the following Santaló type formulas.

**Theorem 3.4.** Let $(M, \partial M, F, d\mu)$ be a compact Finsler manifold with smooth boundary. Thus, for all integral function $f$ on $SM$, we have

\begin{equation}
\int_{V_M^-} f dV_{SM} = \int_{y \in S^+\partial M} e^{\varphi(y)} g_{n_+}(n_+, y) d\chi_+(y) \int_0^{\hat{t}(y)} f(\varphi_t(y)) dt, \tag{1}
\end{equation}

\begin{equation}
\int_{V_M^+} f dV_{SM} = \int_{y \in S^-\partial M} e^{\varphi(y)} g_{n_-}(n_-, y) d\chi_-(y) \int_0^{\hat{i}(y)} f(\varphi_{-t}(y)) dt, \tag{2}
\end{equation}

where $V_M^\pm := \{y \in SM : \hat{t}(y) \leq \hat{i}(y)\}$, $V_M^+ := \{y \in SM : \hat{t}(y) \leq \hat{i}(y)\}$ and $d\chi_\pm(y) = d\Lambda_\pm(\pi(y)) \wedge d\nu_{\pi(y)}(y)$.

**Proof.** (1). Given any $y \in S^+\partial M$. We identify $T_y(S^+\partial M)$ with its image in $T_{(0,y)}(\mathbb{R} \times S^+\partial M)$. Since $\Psi_*((0,y))(X) = X, \forall X \in T_y(S^+\partial M)$, we have

\begin{equation}
\Psi^*(d\chi_+(y)) \equiv d\chi_+|_{(0,y)} \mod dt. \tag{3.1}
\end{equation}

We claim that $[\Psi^* \pi^* d\rho]|_{(0,y)} = 0 \mod dt$. In fact, for each $X \in T_y(S^+\partial M)$, there exists a curve $\xi : [0, +\varepsilon] \to S^+\partial M$ with $\xi(0) = y$ and $\xi(0) = X$. Thus,

$$\langle X, \Psi^* \pi^* d\rho \rangle|_{(0,y)} = \langle \pi_* (\Psi_*((0,y))X), d\rho \rangle = \langle \pi_* X, d\rho \rangle = \frac{d}{ds}|_{s=0} \rho(\pi(\xi(s))) = 0.$$
The claim is true. Lemma \[3.2\] now yields
\[
[\Psi^* \pi_1^* d\rho] |_{(0,y)} = \left\langle \frac{\partial}{\partial t}, \Psi^* \pi_1^* d\rho \right\rangle_{(0,y)} dt
\]
(3.2)
\[
= \left. \left( \frac{d}{dt} \right) \rho \circ \gamma_y(t) \right|_{t=0^+} dt = g_{n_+}(n_+, y) dt.
\]

Define a function \( \eta \in C^\infty(\mathbb{R} \times S^+ \partial M) \) by \( \Psi^* (dV_{SM}) = \eta \cdot \beta \), where \( \beta \mid_{(t,y)} = dt \wedge d\chi_+(y) \) is a \((2n-1)\) form on \( \mathbb{R} \times S^+ \partial M \). It is easy to check that \( \eta(t,y) = \eta(0,y) \) (cf. [23, Lemma 5.6]). By the co-area formula (see [18, Theorem 3.3.1]), \( (3.1) \) and \( (3.2) \), we have
\[
[\eta dt \wedge d\chi_+] |_{(0,y)} = \Psi^* (dV_{SM}(y)) = \Psi^* \left( e^{\tau(y)} \pi^* (d\mu(y) \wedge d\nu_{\tau(y)}(y)) \right)
\]
\[
= \Psi^* \left( e^{\tau(y)} \pi^* (d\rho(y) \wedge dA_+(y)) \wedge d\nu_{\tau(y)}(y) \right)
\]
\[
= \left[ e^{\tau(y)} g_{n_+}(n_+, y) dt \wedge d\chi_+ \right](0,y),
\]
that is, \( \eta(0,y) = e^{\tau(y)} g_{n_+}(n_+, y) \). It follows from the definition of \( \eta \) that
\[
(3.3) \quad \Psi^* (dV_{SM}(\varphi_1(y))) = e^{\tau(y)} g_{n_+}(n_+, y) dt \wedge d\chi,
\]
which implies that \( \Psi \) is of maximal rank. Hence, Lemma \[3.3\] yields that \( \Psi \mid_{\partial \mathcal{N}} \) is a diffeomorphism.

Let \( \mathcal{N} := \{ y \in SM : \tilde{l}(-y) = \tilde{l}(y) \} \). Thus, \( V_{SM}^- = U_{SM}^- \cup \mathcal{N} \). By an argument similar to the proof of Lemma \[3.3\] one has \( \mathcal{N} \subset \{ \varphi_{\tilde{l}}(y) : y \in S^+ \partial M \cup Z, l(t) = \tilde{l}(y) \} \), which implies that \( \mathcal{N} \) has measure zero with respect to \( dV_{SM} \). Also note that \( V_{SM}(U_{SM}^\circ \Psi(\partial \mathcal{N})) = V_{SM}(U_Z) = 0 \). Hence, by \( (3.3) \), we have
\[
\int_{V_{SM}^-} f dV_{SM} = \int_{U_{SM}^-} f dV_{SM}
\]
\[
= \int_{\Psi(\partial \mathcal{N})} f dV_{SM} = \int_{\partial \mathcal{N}^+} \Psi^* (f dV_{SM})
\]
\[
= \int_{S^+ \partial M} e^{\tau(y)} g_{n_+}(n_+, y) d\chi(y) \int_0^{\tilde{l}(y)} f(\varphi_1(y)) dt.
\]
(2) By considering \((M, \partial M, \tilde{F})\) and using the formula \( (1) \), we have
\[
\int_{\tilde{V}_{SM}^-} f(-y) d\tilde{V}_{SM}(y) = \int_{y \in S^+ \partial M} e^{\tau(y)} g_{\tilde{n}_+}(\tilde{n}_+, y) d\tilde{\chi}_+(y) \int_0^{\tilde{l}(y)} f(-\varphi_{\tilde{t}}(y)) dt,
\]
where the quantities \( \tilde{\cdot} \) denote the quantities \( \cdot \) defined by \( \tilde{F} \). Note that \( \tilde{n}_+ = -n_- \) and \( -\varphi_{\tilde{t}}(y) = \varphi_{-\tilde{t}}(-y), 0 \leq t \leq \tilde{l}(y) \). The formula \( (2) \) now follows from the transformation \( y \mapsto -y \).

4. A universal lower bound for the first eigenvalue of the nonlinear Laplacian

**Definition 4.1** ([14] [19]). Let \((M, F, d\mu)\) be a compact Finsler manifold. Denote \( \mathcal{H}_0(M, d\mu) \) by
\[
\mathcal{H}_0(M, d\mu) := \left\{ f \in W^1_2(M) : \int_M f d\mu = 0, \partial M = \emptyset, \right\}
\]
\[
\left\{ f \in W^1_2(M) : f \mid_{\partial M} = 0, \partial M \neq \emptyset \right\}.
\]
Define the canonical energy functional $E_{d\mu}$ on $\mathcal{H}_0(M, d\mu) - \{0\}$ by
\[
E_{d\mu}(u) := \frac{\int_M F^*(du)^2 d\mu}{\int_M u^2 d\mu}.
\]

$\lambda$ is an eigenvalue if there is a function $u \in \mathcal{H}_0(M, d\mu) - \{0\}$ such that $d_u E_{d\mu} = 0$ with $\lambda = E_{d\mu}(u)$. In this case, $u$ is called an eigenfunction corresponding to $\lambda$. The first eigenvalue $\lambda_1(M, d\mu)$ is defined by
\[
\lambda_1(M, d\mu) := \inf_{u \in \mathcal{H}_0(M, d\mu) - \{0\}} E_{d\mu}(u),
\]
which is the smallest positive critical value of $E_{d\mu}$.

**Remark 2.** $u$ is an eigenfunction corresponding to $\lambda$ if and only if
\[
\Delta_{d\mu} u + \lambda u = 0 \text{ (in the weak sense)},
\]
where $\Delta_{d\mu}$ is the nonlinear Laplacian introduced by Shen [14, 13, 19]. It should be noted that $\Delta_{d\mu}$ is dependent on the choice of $d\mu$.

**Proposition 4.2.** Let $(M, F)$ be a Finsler $n$-manifold. Then for any $p \in M$ and $f \in C^\infty(M)$, we have
\[
F^*(df|_p)^2 \geq \frac{n}{e_{n-1} \Lambda_F^{n+1}(p)} \int_{S_p M} \langle y, df \rangle^2 d\nu_p(y),
\]
with equality if and only if $F(p, \cdot)$ is a Euclidean norm.

**Proof.** Without loss of generality, we may suppose $df|_p \neq 0$. Set $B_p M := \{y \in T_p M : F(p, y) < 1\}$. By [21], one can choose a $g_F$-orthonormal basis $\{e_i\}$ of $T_p M$ such that $e_n = \nabla f / F(\nabla f)$ and $\deg g_{ij}(p, y) \leq \Lambda_F^i(p)$. Let $\{y^i\}$ denote the corresponding coordinates. By Stokes’ formula, we have
\[
\int_{S_p M} \langle y, df \rangle^2 d\nu_p(y) \\
\leq \Lambda_F^2(p) F^2(\nabla f) \int_{S_p M} (y^n)^2 \sum_{k=1}^n (-1)^{k-1} y^k dy^1 \wedge \cdots \wedge \hat{dy}^k \wedge \cdots \wedge dy^n \\
= (n + 2) \Lambda_F^2(p) F^2(\nabla f) \int_{B_p M} (y^n)^2 dy^1 \wedge \cdots \wedge dy^n \\
\leq (n + 2) \Lambda_F^2(p) F^2(\nabla f) \int_{B_n(M(\sqrt{\Lambda_F(p)})} (y^n)^2 dy^1 \wedge \cdots \wedge dy^n \\
= \frac{c_{n-1}}{n} \Lambda_F^{n+1}(p) F^2(\nabla f)
\]
If equality holds in (4.1), then it follows from (4.2) that $B_p M = \mathbb{B}_n(\sqrt{\Lambda_F(p)})$. Namely, $F(y) = 1$ if and only if $g_F(y, y) = \Lambda_F(p)$. In particular, $1 = F(e_n) = g_F(e_n, e_n) = \Lambda_F(p)$, which implies that $F(p, \cdot)$ is a Euclidean norm. \qed

**Theorem 4.3.** Let $(M, \partial M, F)$ be a compact Finsler $n$-manifold with smooth boundary such that every geodesic ray in $(M, F)$ minimizes distance up to the point that it intersects $\partial M$. Then
\[
\lambda_1(M, d\mu) \geq \left\{ \begin{array}{ll}
\frac{\lambda_1(M, d\mu)}{\Lambda_F^{n+1}}, & d\mu = d\mu_{BH}, \\
\frac{\lambda_1(M, d\mu)}{\Lambda_F^{n+1}}, & d\mu = d\mu_{HT},
\end{array} \right.
\]

where $D := \text{diam}(M)$ and $\mathbb{S}^+_D$ denotes the $n$-dimensional Riemannian hemisphere of the constant sectional curvature sphere having diameter equal to $D$. The equality holds if and only if $(M, F)$ is isometric to $\mathbb{S}^+_D$.

Proof. Lemma 2.1 yields that

\[
\int_{S^+_pM} e^{\tau(y)} d\mu_p(y) = c_{n-1} \frac{\sigma_{HT}(p)}{\sigma(p)} \geq \begin{cases} \frac{c_{n-1}}{\Lambda_F}, & d\mu = d\mu_{BH}, \\ c_{n-1}, & d\mu = d\mu_{HT}. \end{cases}
\]

Since $V^+_M = SM$, Theorem 3.4 together with Proposition 4.2 and (4.4) then yields

\[
\int_M F^{*2}(df)d\mu 
\]

\[(4.5) \geq \frac{n}{c_{n-1} \Lambda_F^{n+2}} \int_M d\mu(p) \int_{S^+_pM} \langle y, df \rangle^2 d\nu_p(y) 
= \frac{n}{c_{n-1} \Lambda_F^{n+2}} \int_{SM} e^{-\tau(y)}(y, df)^2 dV_p(y) 
= \frac{n}{c_{n-1} \Lambda_F^{n+2}} \int_{y \in S^* \partial M} e^{\tau(y)} g_{n-}(n, y) d\chi(y) \int_{\tilde{l}(y)}^0 e^{-\tau(\tilde{l}(y))} \langle \tilde{l}(y), df \rangle^2 dt 
\geq \frac{n}{c_{n-1} \Lambda_F^{2n+1}} \int_{y \in S^* \partial M} e^{\tau(y)} g_{n-}(n, y) d\chi(y) \int_{\tilde{l}(y)}^0 \left( \frac{d}{dt} f(\gamma(t)) \right)^2 dt 
\geq \frac{n}{c_{n-1} \Lambda_F^{2n+1}} \left( \frac{\pi}{D} \right)^2 \int_{SM} f^2(\pi(y)) dV_p(y) 
\geq \begin{cases} \frac{\lambda_1(\mathbb{S}^+_D)}{\Lambda_F}, & d\mu = d\mu_{BH}, \\ \frac{\lambda_1(\mathbb{S}^+_D)}{\Lambda_F}, & d\mu = d\mu_{HT}. \end{cases}
\]

If we have equality in (4.3), then (4.5) together with Proposition 4.2 implies $\Lambda_F = 1$. Hence, $(M, F)$ is a Riemannian manifold and $\lambda_1(M) = \lambda_1(\mathbb{S}^+_D)$. By the standard argument (see [5] p.131 or [10]), one can show that $(M, F)$ is isometric to $\mathbb{S}^+_D$. \hfill $\square$

In [19], Shen shows that the first eigenvalue of a forward metric ball is bounded from above by a constant depending only on the dimension and lower bounds on the Ricci curvature and the S-curvature. From Theorem 4.3 we obtain a lower bound for the first eigenvalue of a forward metric ball.

**Corollary 4.4.** Let $(M, F, d\mu)$ be a forward complete Finsler $n$-manifold of injectivity radius $i_M$. For any $0 < r < i_M/(1 + \sqrt{\Lambda_F})$ and any $p \in M$, we have

\[
\lambda_1(B^+_F(r)) \geq \begin{cases} \frac{\lambda_1(\mathbb{S}^+_D)}{\Lambda_F^{n+2}}, & d\mu = d\mu_{BH}, \\ \frac{\lambda_1(\mathbb{S}^+_D)}{\Lambda_F^{n+2}}, & d\mu = d\mu_{HT}. \end{cases}
\]
with equality if and only if $B^+_F(r)$ is isometric to $S^+_{x,r}$.

5. Croke Type Isoperimetric Inequalities

In this section, we shall establish Theorem 5.3 and give some applications.

**Lemma 5.1.** For each $x \in \partial M$, we have
\[
\int_{S^+_F \partial M} g_{n_x}(\mathbf{n}_x, y)e^{\tau}(y) \, d\nu_x(y) \leq \frac{c_{n-2}}{n-1} \Lambda_F^{2n+4}(x),
\]
with equality if and only if $F(x, \cdot)$ is a Euclidean norm. Here, "+" denotes either "\(+\)" or "\(-\)", and $S^+_F \partial M := \{y \in S_x M : g_{n_x}(\mathbf{n}_x, y) > 0\}$.

**Proof.** Suppose $\tau = +$. By [21], one can choose a $g_{n_x}$-orthnormal basis $\{e_1\}$ of $T_x M$ such that $e_n = \mathbf{n}_x$ and $\det g_{n_x}(y) \leq \Lambda_F(y)$. Let $\{y^i\}$ be the corresponding coordinates. Set $\|\cdot\| := \sqrt{g_{n_x}(\cdot, \cdot)}$. Define
\[
B^+_x := \{y \in T_x M : F(y) < 1, \ y^n > 0\}, \quad B^+_{x,r} := \{y \in T_x M : F(y) = 1, \ y^n = r\}
\]
\[
\mathbb{B}_{x,r}(s) := \{y \in T_x M : y^n = r, \ \|y^n e_n\| < s\}, \quad \omega := g_{n_x}(\mathbf{n}_+ \cdot y)e^{\tau}(y) \, d\nu_F(y).
\]
For each $y \in B^+_x$, $y^n = g_{n_x}(\mathbf{n}_+ \cdot y) \leq F(\mathbf{n}_+)F(y) \leq 1$. Stokes’ formula together with Lemma 2.1 then yields
\[
\int_{S^+_F \partial M} \omega \leq \Lambda_F^{3n/2}(x) \int_{S^+_F \partial M} y^n \sum_{k=1}^{n} (-1)^{k-1} y^k \, dy^1 \wedge \cdots \wedge \widehat{dy^k} \wedge \cdots \wedge dy^n
\]
\[
= (n+1)\Lambda_F^{3n}(x) \int_{B^+_x} y^n \, dy^1 \wedge \cdots \wedge dy^n
\]
\[
= (n+1)\Lambda_F^{3n}(x) \int_0^1 \text{Vol}(B^+_{x,y^n}) y^n \, dy^n
\]
\[
\leq (n+1)\Lambda_F^{3n}(x) \int_0^{\sqrt{\Lambda_F(x)}} \text{Vol}(\mathbb{B}_{x,y^n}(\sqrt{\Lambda_F(x)} - (y^n)^2)) y^n \, dy^n
\]
\[
= \frac{c_{n-2}}{n-1} \Lambda_F^{2n+4}(x),
\]
with equality if and only if $\Lambda_F(x) = 1$, i.e., $F(x, \cdot)$ is a Euclidean norm.

Suppose $\tau = -$. Note that $\Lambda_F(x) = \Lambda_F(x)$. Using the same method as in Theorem 5.3, one can get the formula. \hfill \square

Given any point $x \in M$, let $(r, y)$ denote the polar coordinates about $x$. Set $\mathcal{F}(r, y) = e^{\tau(\gamma_u(r))}\bar{\sigma}_x(r, y)$, where $d\mu_{(r, y)} := \bar{\sigma}_x(r, y) \, dr \wedge d\nu_F(y)$. Then we have the following inequality of Berger-Kazdan type [23 Theorem 1.3]

**Lemma 5.2** (23). Let $(M, F)$ be a compact Finsler $n$-manifold. For each $y \in SM$ and $0 < t \leq l \leq i_y$, we have
\[
\int_0^l \int_0^{l-r} \mathcal{F}(t, \varphi_r(y)) \, dt \geq \frac{\pi c_n}{2c_{n-1}} \left( \frac{1}{\pi} \right)^{n+1},
\]
with equality if and only if
\[
R_{\gamma_u(t)}(\cdot, \gamma_y(t))\gamma_y(t) = \left( \frac{\pi}{l} \right)^2 \text{id}, \quad 0 \leq t \leq l,
\]
where $R$ is the (Riemannian) curvature tensor acting on $\gamma_y(t)$.\hfill \square
Now we have the following theorem.

**Theorem 5.3.** Let \((M, \partial M, F, d\mu)\) be a compact Finsler \(n\)-manifold with smooth boundary, where \(d\mu\) is either the Basuann-Hausdorff measure or the Holmes-Thompson measure. Set

\[
\omega := \inf_{x \in M} \min\{\omega_x^+, \omega_x^-\} = \min \left\{ \inf_{x \in M} \omega_x^+, \inf_{x \in M} \omega_x^- \right\},
\]

where \(\omega_x^\pm := c_{n-1}^{-1} \int_{U_x^\pm} e^r(y) d\nu_x(y)\) and \(U_x^\pm := \pi[\nu_x^\pm](x)\). Then

\[
(1) \quad \frac{A_\pm(\partial M)}{\mu(M)} \geq \frac{(n-1)c_{n-1}\omega}{c_{n-2} D \Lambda_F^{2n+\frac{1}{2}}},
\]

where \(D := \text{diam}(M)\).

\[
(2) \quad \frac{A_\pm(\partial M)}{\mu(M)^{1-\frac{1}{n}}} \geq \frac{c_{n-1}\omega^{1+\frac{1}{n}}}{(c_{n}/2)^{1-\frac{1}{n}} \Lambda_F^{2n+\frac{2}{n}}},
\]

with equality if and only if \((M, F)\) is a Riemannian hemisphere of a constant sectional curvature.

**Proof.** (1) Theorem 1.1 together with Lemma 5.1 furnishes

\[
c_{n-1}\omega^\pm(M) \leq c_{n-1} \int_M \omega_x^\pm d\mu(x) = \int_{x \in M} d\mu(x) \int_{U_x^\pm} e^r(y) d\nu_x(y) = V_{SM}(\nu_x^\pm) \leq D \int_{S^+\partial M} e^r(y) g_{n\pm}(n\pm, y) d\chi_{\pm}(y) \leq D A_\pm(\partial M) \frac{c_{n-2}}{n-1} \Lambda_F^{2n+\frac{1}{2}}.
\]

(2) For each \(y \in S^+\partial M, l(\varphi_t(y)) \geq l(y) - t\), for any \(0 \leq t \leq l(y)\). By Theorem 2.1, Theorem 5.2 and H"older’s inequality, we have

\[
\mu^2(M) = \int_M d\mu(x) \int_{S^1 M} d\nu_x(y) \int_0^{l(y)} \hat{\sigma}_x(r, y) dr = \int_{S^1 M} d\nu_x(y) \int_0^{l(y)} e^{-r(y)} \hat{\sigma}_x(r, y) dr \\
\geq \int_{S^1 M} d\nu_x(y) \int_0^{l(y)} e^{-r(y)} \hat{\sigma}_x(r, y) dr \\
= \int_{S^+\partial M} e^{r(y)} g_{n\pm}(n\pm, y) d\lambda(y) \int_0^{l(y)} dt \int_0^{l(y)} e^{-t(y) \hat{\sigma}_x(r, \varphi_t(y))} dr \\
\geq \Lambda_F^{-2n} \int_{S^+\partial M} e^{r(y)} g_{n\pm}(n\pm, y) d\lambda(y) \int_0^{l(y)-t} dt \int_0^{l(y)} e^{-t(y) \hat{\sigma}_x(r, \varphi_t(y))} dr \\
\geq \frac{c_n}{2c_{n-1} \pi^n \Lambda_F^{2n}} \int_{S^+\partial M} l(y)^{n+1} e^{r(y)} g_{n\pm}(n\pm, y) d\lambda(y) \\
\geq \frac{c_n}{2c_{n-1} \pi^n \Lambda_F^{2n}} \left( \int_{S^+\partial M} l(y)^{n+1} e^{r(y)} g_{n\pm}(n\pm, y) d\lambda(y) \right)^{n+1} \left( \int_{S^+\partial M} e^{r(y)} g_{n\pm}(n\pm, y) d\lambda(y) \right)^{-n}.
\]
\[
\begin{align*}
(5.2) \quad & \geq \frac{c_n}{2c_{n-1}\pi^n A_F^{2n}} V_{SM}(V_M^\gamma)^{n+1} \left( \frac{n-1}{c_{n-2} A_+(\partial M) A_F^{2n+\frac{\pi}{2}}} \right)^n \\
& \geq \frac{(c_{n-1})^n \mu(M)^n+1}{(c_n/2)^{n-1} A_0^n(\partial M) A_F^{2n+\frac{\pi}{2}}}. \\
\end{align*}
\]

That is,

\[
(5.3) \quad \frac{A_+(\partial M)}{\mu(M)^{1-\frac{\pi}{2n}}} \geq \frac{c_{n-1} \omega^{1+\frac{1}{2}}}{(c_n/2)^{1-\frac{\pi}{2}} A_F^{2n+\frac{\pi}{2}}}. \\
\]

Let \( \tilde{A}_\pm \) and \( \tilde{\omega} \) be defined as before on \((M, \tilde{F})\). It is easy to check that \( \tilde{A}_\pm(\partial M) = A_\pm(\partial M) \) and \( \tilde{\omega} = \omega \). From above, we obtain

\[
(5.4) \quad \frac{A_-(\partial M)}{\mu(M)^{1-\frac{\pi}{2n}}} = \frac{\tilde{A}_+(\partial M)}{\tilde{\mu}(M)^{1-\frac{\pi}{2n}}} \geq \frac{c_{n-1} \omega^{1+\frac{1}{2}}}{(c_n/2)^{1-\frac{\pi}{2}} A_F^{2n+\frac{\pi}{2}}}. \\
\]

(5.3) together with (5.4) yields (5.1).

Suppose that equality holds in (5.1). Then we have equality in (5.3) or (5.4). It follows from (5.2) and Lemma 5.1 that \( 1 = \Lambda_F = \Lambda_{\tilde{F}} \). Hence, \( F \) is an Riemannian metric and (5.1) becomes

\[
\frac{A(\partial M)}{\mu(M)^{1-\frac{\pi}{2n}}} = \frac{c_{n-1} \omega^{1+\frac{1}{2}}}{(c_n/2)^{1-\frac{\pi}{2}} A_F^{2n+\frac{\pi}{2}}}. \\
\]

Since \( V_M = SM, t(y) \leq i_y \), for all \( y \in SM \). Hölder's inequality implies \( t(y) \) is constant, say, equal to \( t \), on all of \( S^+ \partial M \). Hence, \( t(y) = l \), for all \( y \in S^+ \partial M \). And Theorem 5.2 yields \( M \) has constant sectional curvature equal to \((\pi/l)^2\), i.e., \( M \) is a hemisphere. \( \square \)

From above, it is easy to see that Theorem 5.3 becomes Croke’s isoperimetric inequality \(10\) if \( \Lambda_F = 1 \). In the Finslerian case, the upper bound on \( \Lambda_F \) in Theorem 5.3 is very important as the following example shows.

**Example 1.** Let \( B^n \) be the unit open ball in \( \mathbb{R}^n \) equipped with a Funk metric \( F \), that is,

\[
F(x, y) = \frac{\sqrt{1 - |x|^2}|y|^2 + (x \cdot y)^2 + x \cdot y}{1 - |x|^2},
\]

where "|·|" (resp. "·") denotes the Euclidean norm (resp. inner product). For \( r \in (0, 1) \), set \( \Omega_r := \{ x \in B^n : |x| < r \} \). Then \( (\Omega_r, \partial \Omega_r, F|_{\Omega_r}) \) is a compact Finsler manifold with smooth boundary. By directly computing, we have \( \mu_{BH}(\Omega_r) = \frac{c_{n-1}}{\pi} r^n \) and \( A_{\pm}(\partial \Omega_r) = c_{n-1}(1 \pm r)^{n-1} \), where \( d A_{\pm} \) are induced by \( d \mu_{BH} \). Clearly,

\[
\lim_{r \to 1} \frac{A_+(\partial \Omega_r)}{A_-(\partial \Omega_r)} = +\infty. \\
\]

Note that

\[
\Lambda_F(\partial \Omega_r) = \left( \frac{1 + r}{1 - r} \right)^2, \quad \text{diam}(\Omega_r) = \log \left( \frac{1 + r}{1 - r} \right). \\
\]

For any \( x \in \Omega_r \),

\[
\omega_x^\pm = \frac{1}{(1 - |x|^2)^{\frac{n+\frac{\pi}{2}}{2}}} \geq 1, \text{ i.e., } \omega = 1. \\
\]
Hence, we have
\[
\frac{A_+(\partial \Omega_r)}{\mu_{BH}^\prime(\Omega_r)} > \frac{(n-1)c_{n-1}\omega}{c_{n-2}\text{diam}(\Omega_r)A_F^{2n+\frac{2}{2}}\mu_{BH}^\prime(\Omega_r)^{1-\frac{1}{n}}} > \frac{c_{n-1}\omega_{1+\frac{1}{n}}}{(c_n/2)^{1-\frac{1}{n}}A_F^{2n+\frac{2}{2}}}.
\]

In particular,
\[
\lim_{r \to 1} A_F^\prime = +\infty, \quad \lim_{r \to 1} \frac{A_-(\partial \Omega_r)}{\mu_{BH}^\prime(\Omega_r)} = \lim_{r \to 1} \frac{A_-(\partial \Omega_r)}{\mu_{BH}^\prime(\Omega_r)^{1-\frac{1}{n}}} = 0.
\]

Before giving some applications of Theorem 5.3, we introduce the definitions of the Sobolev constant, Cheeger’s constant and the isoperimetric constant of a closed Finsler manifold.

**Definition 5.4.** Let \((M, F, d\mu)\) be a closed Finsler manifold. The Sobolev constant \(S(M, d\mu)\) is defined as
\[
S(M, d\mu) := \inf_{f \in C^\infty(M)} \left\{ \int_M F^*(df)d\mu \right\}^{\frac{1}{n}}.
\]
Cheeger’s constant \(h(M, d\mu)\) and the isoperimetric constant \(I(M, d\mu)\) are defined by
\[
h(M, d\mu) := \inf_{\Gamma} \frac{\min\{A_+(\Gamma)\}}{\min\{\mu(M_1), \mu(M_2)\}}, \quad I(M, d\mu) := \inf_{\Gamma} \frac{\min\{A_+(\Gamma)\}^n}{\min\{\mu(M_1), \mu(M_2)\}^{n-1}},
\]
where \(\Gamma\) varies over compact \((n-1)\)-dimensional submanifolds of \(M\) which divide \(M\) into disjoint open submanifolds \(M_1, M_2\) of \(M\) with common boundary \(\partial M_1 = \partial M_2 = \Gamma\).

**Remark 3.** By using the co-area formula (cf. [18, Theorem 3.3.1]) and the same argument as in [13], one can obtain a Cheeger type inequality
\[
\lambda_1(M, d\mu) \geq \frac{h^2(M, d\mu)}{4\lambda_F^2}.
\]
And we also have a Federer-Fleming type inequality (see Proposition 6.1 below), i.e.,
\[
I(M, d\mu) \leq S(M, d\mu) \leq 2I(M, d\mu).
\]

**Corollary 5.5.** Let \((M, F, d\mu)\) be a closed Finsler \(n\)-manifold with \(\text{Ric} \geq (n-1)k\), where \(d\mu\) denotes either the Busemann-Hausdorff measure or the Holmes-Thompson measure. Then
\[
\lambda_1(M, d\mu) \geq \left[ \frac{(n-1)\mu(M)}{4c_{n-2}^nA_F^{4n+1}\text{diam}(M)\int_0^{\text{diam}(M)} s^{n-1}(t)dt} \right]^2,
\]
\[
S(M, d\mu) \geq \frac{\mu(M)^{n+1}}{4c_{n-1}^n(c_n)^{n-1}A_F^{4n^2+2n}\left(\int_0^{\text{diam}(M)} s^{n-1}(t)dt\right)^{n+1}}.
\]
Hence, both \(\lambda_1(M, d\mu)\) and \(S(M, d\mu)\) can be bounded from below in terms of the diameter, volume, uniform constant and a lower bound for the Ricci curvature.
Proof. **Step 1.** Let \( \Gamma \) be any \((n - 1)\)-dimensional compact submanifold of \( M \) dividing \( M \) into two open submanifolds \( M_1 \) and \( M_2 \), such that \( \partial M_1 = \partial M_2 = \Gamma \). Given \( x \in M_1 \), let
\[
O_x := \{ q \in M : \exists y \in U_x^- \text{ such that } q = \tilde{\gamma}_y(t), t \in (0, \tilde{t}(y)) \},
\]
where \( \tilde{\gamma}_y(t) \) is the geodesic in \( (M, \tilde{F}) \) with \( \tilde{\gamma}_y(0) = -y \).

For any \( q \in M_2 \), there exists a minimal unit speed geodesic, say \( \gamma_X(t) \), from \( x \) to \( q \). Clearly, \( \gamma_X(t) \) must hit the boundary and therefore, \( \tilde{t}(X) \leq \tilde{t}(X) \). Since \( F(-X) = \tilde{F}(X) = 1 \), \( q \in O_x \) which implies that \( M_2 \subset O_x \).

Note that \( \text{Ric} \geq (n - 1)k \), \( \Lambda_F = \Lambda_F \) and \( d\tilde{\mu} = d\mu \). Hence, by Lemma 2.1 and the volume comparison theorem (cf. [23, Theorem 3.1]), we have
\[
\mu(M_2) = \tilde{\mu}(O_x) = \int_{y \in U_x^-} d\tilde{\mu}_x(-y) \int_0^{\tilde{t}_r} \sigma_x(r, \gamma_y) dr
\leq \Lambda_F^n \int_{y \in U_x^-} d\tilde{\mu}_x(-y) \int_0^{\tilde{t}_r} \sigma_k^n(r) dr
\leq c_{n-1} \Lambda_F^{2n} \omega_1(x) \int_0^{\text{diam}(M)} \sigma_k^n(r) dr.
\]
That is,
\[
\omega_i^- \geq \frac{\mu(M_j)}{c_{n-1} \Lambda_F^{2n} \int_0^{\text{diam}(M)} \sigma_k^n(t) dt}, \quad i \neq j.
\]

Set \( O'_x := \{ q \in M : \exists y \in U_x^+ \text{ such that } q = \gamma_y(t), t \in (0, i(y)) \}. \) It is easy to see that \( M_2 \subset O'_x \). By the similar argument, one can show that
\[
\omega_i^+ \geq \frac{\mu(M_j)}{c_{n-1} \Lambda_F^{2n} \int_0^{\text{diam}(M)} \sigma_k^n(t) dt}, \quad i \neq j.
\]

**Step 2.** The inequalities above together with Theorem 5.3 yield
\[
\mathcal{H}(M, d\mu) \geq \frac{(n-1)\mu(M)}{2c_{n-2} \Lambda_F^{4n+4} \int_0^{\text{diam}(M)} \sigma_k^{n-1}(t) dt},
\]
\[
\mathcal{II}(M, d\mu) \geq \frac{\mu(M)^{n+1}}{4c_{n-1} (c_n)^{n-1} \Lambda_F^{4n+2} \left( \int_0^{\text{diam}(M)} \sigma_k^{n-1}(t) dt \right)^{n+1}}.
\]

Corollary now follows from Remark 3 \( \Box \)

**Corollary 5.6.** Let \((M, F, d\mu)\) be a closed Finsler \( n \)-manifold, where \( d\mu \) is either the Busemann-Hausdorff measure or the Holmes-Thompson measure. Then for any \( x \in M \) and \( 0 < r < i_M/(1 + \sqrt{\Lambda_F}) \), we have
\[
\mu(B^+_x(r)) \geq \frac{C(n, \Lambda_F)}{n^n} r^n, \quad A_\pm (S^+_x(r)) \geq \frac{C(n, \Lambda_F)}{n^{n-1}} r^{n-1}.
\]

Proof. The similar argument as in Lemma 5.2 shows \( i_M = \tilde{i}_M \), where \( \tilde{i}_M \) is the injectivity radius of \((M, \tilde{F})\). Hence, \( U_x^\pm = S_xM \) for all \( x \in B^+_x(r) \). By Theorem 5.3 and 4.3, we have
\[
\frac{\tilde{\mu}(B^+_x(r))}{\mu(B^+_x(r))^{1-\frac{n}{k}}} = \frac{\Lambda_i(S^+_x(r))}{\mu(B^+_x(r))^{1-\frac{n}{k}}} \geq C(n, \Lambda_F),
\]

which implies that
\[
\mu(B^+_x(r)) \geq \frac{C(n, \Lambda_F)}{n^n} r^n.
\]

Theorem 5.3 together with (5.5) yields
\[
\Lambda_\pm(S^+_x(r)) \geq \frac{C(n, \Lambda_F)}{n^{n-1}} r^{n-1}.
\]

\[\square\]

In order to establish Theorem 1.5, let us recall some definitions and properties of general LGC spaces first. Refer to [20, 24] for more details.

**Definition 5.7** ([20, 24]). A general metric space is a pair \((X, d)\), where \(X\) is a set and \(d : X \times X \to \mathbb{R}^+ \cup \{\infty\}\), called a metric, is a function, satisfying the following two conditions: (a) \(d(x, y) \geq 0\), with equality \(\Leftrightarrow x = y\); (b) \(d(x, y) + d(y, z) \geq d(x, z)\). The reversibility \(\lambda_X\) of a general metric space \((X, d)\) is defined by
\[
\lambda_X := \sup_{x \neq y} \frac{d(x, y)}{d(y, x)}.
\]

A contractibility function \(\rho : [0, r) \to [0, +\infty)\) is a function satisfying: (a) \(\rho(0) = 0\), (b) \(\rho(\epsilon) \geq \epsilon\), (c) \(\rho(\epsilon) \to 0\), as \(\epsilon \to 0\), (d) \(\rho\) is nondecreasing. A general metric space \(X\) is LGC(\(\rho\)) for some contractibility function \(\rho\), if for every \(\epsilon \in [0, r)\) and \(x \in X\), the forward ball \(B^+_x(\epsilon)\) is contractible inside \(B^+_x(\rho(\epsilon))\).

**Lemma 5.8** ([24]). Fix a function \(N : (0, \alpha) \to (0, \infty)\) with
\[
\lim_{\epsilon \to 0^+} \sup_n N(\epsilon) < \infty
\]
and a contractibility function \(\rho : [0, r) \to [0, +\infty)\). The class
\[
\mathcal{C}(N, \rho) := \{X \in \mathcal{M}^\delta : X \text{ is LGC(}\rho\text{)}, \text{ Cov}(X, \epsilon) \leq N(\epsilon) \text{ for all } \epsilon \in (0, \alpha)\}
\]
contains only finitely many homotopy types. Here, \(\mathcal{M}^\delta\) denotes the collection of compact general metric spaces with reversibility \(\leq \delta\) and Cov\((X; \epsilon)\) denotes the minimum number of forward \(\epsilon\)-balls it takes to cover \(X\).

Corollary 5.6 together with Lemma 5.8 yields the following

**Theorem 5.9.** For any \(n\) and positive numbers \(i, V, \delta\), the class of closed Finsler \(n\)-manifolds \((M, F)\) with injectivity radius \(i_M \geq i\), \(\Lambda_F \leq \delta\) and \(\mu(M) \leq V\), contains at most finitely many homotopy types. Here, \(\mu(M)\) is either the Busemann-Hausdorff volume or the Holmes-Thompson volume of \(M\).

**Proof.** Let \(c_M\) denote the contractibility radius of \((M, F)\) (cf. [24]). Since \(c_M \geq i_M \geq i\), \((M, F)\) is LGC(\(\rho\)), where \(\rho\) is the identity map of \([0, i)\). Corollary 5.6 implies that \(\mu(B^+_x(\epsilon)) \geq C(n, \delta) \epsilon^n\) for all \(p \in M\) and \(\epsilon < i/(1 + \sqrt{\delta})\). It follows from [20] Proposition 3.11 that
\[
\text{Cov}(M, \epsilon) \leq \frac{\mu(M)}{C(n, \delta)(\epsilon/(2\sqrt{\delta}))^n} = C'(n, \delta, V) \epsilon^{-n}.
\]

Define the covering function \(N(\epsilon) := C'(n, \delta, V) \epsilon^{-n}, \epsilon \in (0, i/(1 + \sqrt{\delta})\). The conclusion now follows from Lemma 5.8 \(\square\)

One can easily see that Theorem 5.9 implies Yamaguchi’s finiteness theorem [22] and [24, Theorem 1.3].
6. Appendix

**Proposition 6.1.** Let \((M, F, d\mu)\) be a closed Finsler manifold. Then
\[
\| (M, d\mu) \| \leq S(M, d\mu) \leq 2\| (M, d\mu) \|.
\]

**Proof.** Fix \(\Gamma\) with \(\mu(M_1) \leq \mu(M_2)\). Define a Lipschitz function \(f_+^\epsilon\) by
\[
f_+^\epsilon(x) := \begin{cases} 1, & x \in M_1, \ d(\Gamma, x) \geq \epsilon, \\ \frac{1}{\epsilon}d(\Gamma, x), & x \in M_1, \ d(\Gamma, x) < \epsilon, \\ 0, & x \in M_2. \end{cases}
\]

By letting \(\epsilon \to 0^+\), we obtain that
\[
\inf_{\alpha \in \mathbb{R}} \left( \int_M |f_+^\epsilon - \alpha|^{\frac{n}{n-1}}d\mu \right)^{\frac{n-1}{n}} \geq \inf_{\alpha \in \mathbb{R}} \left\{ \int (1 - \alpha|\nabla \alpha|)\mu(M_1) + |\alpha|\frac{n}{n-1}\mu(M_2) \right\}^{\frac{n-1}{n}}
\]
\[
\geq \mu(M_1)^{n-1}\inf_{\alpha \in \mathbb{R}} \left\{ (1 - \alpha|\nabla \alpha|) + |\alpha|\frac{n}{n-1} \right\}^{\frac{n-1}{n}}
\]
\[
\geq \mu(M_1)^{n-1}/2.
\]

Set \(\rho_+(x) = d(\Gamma, x), \ x \in \overline{M_1}\). Lemma 3.2 yields that \(\nabla \rho_+|_\Gamma = \mathbf{n}_+\), where \(\mathbf{n}_+\) denotes the unit inward normal vector field along \(\partial M_1 = \Gamma\). By the co-area formula (cf. [18 Theorem 3.3.1]), we see that
\[
\int_M F^\epsilon (df_+^\epsilon)d\mu = \frac{1}{\epsilon} \int_0^\epsilon dt \int_{\rho_+^\epsilon(t)} d\Lambda_+ \to \Lambda_+ (\Gamma).
\]

Hence, \(2\Lambda_+ (\Gamma)^n \geq S(M, d\mu) \cdot \mu(M_1)^{n-1}\). Similarly, define a Lipschitz function \(f_-^\epsilon\) by
\[
f_-^\epsilon(x) := \begin{cases} 0, & x \in M_2, \ d(\Gamma, x) > \epsilon \\ \frac{1}{\epsilon}d(\Gamma, x) - 1, & x \in M_2, \ d(\Gamma, x) \leq \epsilon, \\ -1, & x \in M_1. \end{cases}
\]

Then one can show \(2\Lambda_- (\Gamma)^n \geq S(M, d\mu) \cdot \mu(M_1)^{n-1}\). Therefore, \(S(M, d\mu) \leq 2\| (M, d\mu) \|\).

Given \(f \in C^\infty\), let \(\alpha_0\) be a median of \(f\), i.e.,
\[
\mu(\{x : f(x) \geq \alpha_0\}) \geq \frac{1}{2}\mu(M), \ \mu(\{x : f(x) \leq \alpha_0\}) \geq \frac{1}{2}\mu(M).
\]

Set \(M_1 := \{x : f(x) < \alpha_0\}\) and \(M_2 := \{x : f(x) > \alpha_0\}\). Then \(\mu(M_i) \leq \mu(M)/2\), for \(i = 1, 2\). Let \(h := f - \alpha_0\) and \(h_i := h|_{M_i} \in C^\infty(M_i), \ i = 1, 2\).

Let \(M_i := \{x : h_2(x) > t\}\). Since \(\mu(M_i)\) is decreasing, we have
\[
\frac{d}{ds} \left( \int_0^s \mu(M_t) \frac{d}{dt} dt \right)^{\frac{n}{n-1}} = \frac{n}{n-1} \left( \int_0^s \mu(M_t) \frac{d}{dt} dt \right)^{\frac{n}{n-1}} \mu(M_s) \frac{n}{n-1} \geq \frac{n}{n-1} s^n \mu(M_s),
\]
which implies
\[
\left( \int_0^s \mu(M_t) \frac{d}{dt} dt \right)^{\frac{n}{n-1}} \geq \int_0^s \mu(M_t) dt \frac{n}{n-1}.
\]
Note that $\nabla h_2$ is the inward normal vector field along $\partial M_i$. Thus,
\[
\int_{M_2} F^*(dh_2)d\mu = \int_0^\infty A_+(\partial M_i)dt \geq I(M_2)^\frac{1}{2} \int_0^\infty \mu(M_i)^{\frac{n-1}{n}} dt \\
\geq I(M_2)^\frac{1}{2} \left( \int_0^\infty \mu(M_i) dt \right)^{\frac{n-1}{n}} = I(M_2)^\frac{1}{2} \left( \int_0^\infty t^\frac{n-1}{n} d\mu(M_i) \right)^{\frac{n-1}{n}} \\
= I(M_2)^\frac{1}{2} \left( \int_0^\infty i^\frac{n-1}{n} dt \int_{\partial M_i} dA h_2 \right)^{\frac{n-1}{n}} = I(M_2)^\frac{1}{2} \left( \int_M h_2^\frac{n-1}{n} d\mu \right)^{\frac{n-1}{n}}.
\]
Here, $I(M_i)$ is defined by
\[
\inf_\Omega \frac{\min \{ A_+(\partial \Omega) \}}{\mu(\Omega)^{n-1}},
\]
where $\Omega$ range over all open submanifolds of $M_i$ with compact closures in $M_i$ and smooth boundary. Clearly, $I(M_i) \neq 0$.

Likewise, one can show that $\int_M F^*(dh_1)d\mu \geq I(M_1)^\frac{1}{2} \|h_1\|_{n/(n-1)}$. Since $\mu(M_i) \leq \mu(M)/2$, $I(M_i) \geq I(M, d\mu)$. Let $\chi_i$ be the characteristic function of $M_i$, $i = 1, 2$. Thus,
\[
\int_M F^*(df)d\mu = \int_M F^*(dh_1)d\mu = \sum_j \int_{M_j} F^*(dh_j)d\mu \\
\geq I(M, d\mu)^\frac{1}{2} \sum_j \left\{ \int_M \chi_j |f - \alpha_0|^{\frac{n}{n-1}} \right\}^{\frac{n-1}{n}} \\
\geq I(M, d\mu)^\frac{1}{2} \|f - \alpha_0\|_{n/(n-1)} \geq I(M, d\mu)^\frac{1}{2} \inf_{\alpha \in \mathbb{R}} \|f - \alpha\|_{n/(n-1)},
\]
which implies that $S(M, d\mu) \geq I(M, d\mu)$. \hfill \Box

REFERENCES

[1] J. Alvarez-Paiva and G. Berck, What is wrong with the Hausdorff measure in Finsler spaces, Adv. in Math., 204(2006), 647-663.
[2] J. Alvarez-Paiva and A.C. Thompson, Volumes in normed and Finsler spaces, A Sampler of Riemann-Finsler geometry (Cambridge) (D. Bao, R. Bryant, S.S. Chern, and Z. Shen, eds.), Cambridge University Press, 2004, pp. 1-49.
[3] D. Bao, S. S. Chern and Z. Shen, An introduction to Riemannian-Finsler geometry, GTM 200, Springer-Verlag, 2000.
[4] D. Burago, Y. Burago and S. Ivanov, A course in metric geometry, A.M.S, 2001.
[5] I. Chavel, Eigenvalues in Riemannian geometry, Academic Press, New York, 1984.
[6] B. Chen,Some geometric and analysis problems in Finsler geometry, Doctoral thesis, Zhejiang University, 2010.
[7] C. Croke, A sharp four dimensional isoperimetric inequality, Comment. Math. Helv. 59(1984), 187-192.
[8] C. Croke, Curvature Free Volume Estimates, Inventiones Mathematicae, 76(1984), 515-521.
[9] C. Croke, N. Dairbekov, Lengths and volumes in Riemannian manifolds, Duke Math. J., 125 (2004), 1-14.
[10] C. Croke, Some isoperimetric inequalities and eigenvalue estimates, Ann. Sci. Ec. Norm. Super, Ser. 13(1980), 419-435.
[11] C. Croke, and M. Katz, Universal volume bounds in Riemannian manifolds, Surveys in Differential Geometry VIII, Lectures on Geometry and Topology held in honor of Calabi, Lawson, Siu, and Uhlenbeek at Harvard University, May 3-5, 2002, edited by S.T. Yau (Somerville, MA: International Press, 2003.) pp. 109-137.
[12] D. Egloff, Uniform Finsler Hadamard manifolds, Ann. Inst. Henri Poincaré, 66(1997), 323-357.
[13] M. Ledoux, *A simple analytic proof of an inequality by P. Buser*, Proc. Amer. Math. Soc. **121** (1994), 951-959.

[14] Y. Ge and Z. Shen, *Eigenvalues and eigenfunctions of metric measure manifolds*, Proc. London Math. Soc., **82** (2001), 725-746.

[15] H. Rademacher, *Nonreversible Finsler metrics of positive ag curvature*, A sampler of Riemann-Finsler geometry, Cambridge Univ. Press, Cambridge, 2004, pp. 261-302.

[16] L. Santaló, *Integral Geometry and Geometric Probability*, Encyclopedia Math. Appl., Addison-Wesley, Reading, MA, 1976.

[17] L. Santaló, *Measure of sets of geodesics in a Riemannian space and applications to integral formulas in elliptic and hyperbolic spaces*, Summa Brasil. Math., **3** (1952), 1-11.

[18] Z. Shen, *Lectures on Finsler geometry*, World Sci., Singapore, 2001.

[19] Z. Shen, *The non-linear Laplacian for Finsler manifolds*, The theory of Finslerian Laplacians and applications, vol. 459 of Math. Appl., Kluwer Acad. Publ., Dordrecht, 1998, pp. 187-198.

[20] Y. Shen and W. Zhao, *Gromov pre-compactness theorems for nonreversible Finsler manifolds*, Diff. Geom. Appl., **28** (2010), 565-581.

[21] B. Wu, *Volume form and its applications in Finsler geometry*, Publ. Math. Debrecen, **78** (2011), 723-741.

[22] T. Yamaguchi, *Homotopy type finiteness theorems for certain precompact families of Riemannian manifolds*, Proc. Am. Math. Soc. **102** (1988), 660-666.

[23] W. Zhao and Y. Shen, *A Universal Volume Comparison Theorem for Finsler Manifolds and Related Results*, Can. J. Math., **65** (2013), 1401-1435.

[24] W. Zhao, *Homotopy finiteness theorems for Finsler manifolds*, Publ. Math. Debrecen, **83** (2013), 329-358.

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