Economic order quantity model with deterioration factor and all-units discount

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Abstract. Inventory related with the illiquid assets for a company and its amount must be carefully calculated. Inventory cost usually consists of purchasing, ordering, holding and shortage cost; however, when there is a possibility that the goods are damage, deteriorated or exceed its expiration date, the so-called deteriorated cost must be considered in the total inventory cost. In this paper we consider an economic order quantity (EOQ) inventory model by considering the possibility of goods to be deteriorated or damage or exceed its expiration date. We also consider an all-units discount scheme offered by the supplier that affect the inventory cost formation in this model. Numerical examples are given to illustrate the model and sensitivity analysis of the shortage cost is also provided. Higher shortage cost will make the total inventory cost higher and the optimal order quantity is obtained by taking the cheapest price the supplier has offered.

Keywords: inventory model, EOQ, deterioration, expiration date, all-units discount.

1. Introduction
Inventory is related with the storage of raw material or finished goods with the aim to fulfil demand. Inventory management is a crucial factor in a company especially for a retailer. Having too many inventories in hands will cause holding costs such as maintenance cost, storage or insurance cost higher. However, on the other hand, having less inventory will incur more ordering cost and shortage cost due to the unavailability to satisfy customers’ demands. Therefore, one of the solutions to overcome this problem is to find the satisfactory middle point between those two policies. [8]

For the food retailer or chemical industry, expiration date becomes one of the substantial factor to consider that affect the total inventory cost. For items that can deteriorate or damage before their expiration date, this problem will make inventory management more complicated. When items are close to their expiration date, their value decrease or even have no value when they exceed the expiration date. Having more goods will incur more deterioration cost for the company, however having less will also incur more ordering cost and perhaps shortage cost. Basically, deterioration or expiration date or damages can occur for almost every goods and this will be one consideration in building the mathematical model in this paper. There are a lot of literature discussed inventory models including the basic one, Economic Order Quantity (EOQ) model (see [3], [7], [8], [9]). EOQ model with expiration date has been discussed by [1]. The extension of the model by adding all-units discount is studied in [5] although there was an ambiguity in the model formulation and the holding cost fraction that was quite big enough. Another variant with stochastic deterioration can be found in [4], with the review on deteriorating inventory study in [2] and [6]. This paper tries to tackle the problem from different point of view especially in the model development that is quite different compared to [5]. We still focusing on an EOQ model with deterioration factor and all-units discount scheme offered by the supplier.
The organization of this paper is as follows. An economic order quantity model will be presented in Section 2 and in Section 3 our model is proposed by considering deterioration and all-units discount. Numerical examples are provided in Section 4, while conclusions and further research are put in the last section.

2. An Economic Order Quantity Model

According to [3], [7], [8] and [9], total inventory cost usually consists of purchasing cost, ordering/setup cost, holding cost and shortage/stock out cost. However, there are possibilities that other costs are involved in the calculation of total inventory cost, but usually these costs are negligible since the effect to the total inventory cost is quite small. Also, involving too many costs that are not substantial in the total inventory cost will make our model more complicated to be mathematically analysed. Figure 1 gives an illustration of the four cost components in the total inventory cost.

![Figure 1. Four cost components in the total inventory cost](image)

Therefore, mathematically the total inventory cost can be expressed as follows.

\[
\text{Total inventory cost} = \text{Purchasing cost} + \text{Ordering cost} + \text{Holding cost} + \text{Shortage cost}
\]

An EOQ inventory model is the simplest inventory model along with its assumptions such as: (1). Demand is known and constant. (2). The model only applied for single item and no interaction with other items. (3). Replenishment is occurred immediately after an order is released or in other words there is no lead time. If there is lead time, then lead time is known and constant from time to time. (4). Shortage is not allowed. (5). Order quantity is always the same for every replenishment during a period. (6). Purchasing cost is proportional to the ordered amount. (7). Holding cost depends on the averaged of stored units. [4]

The notations used in the EOQ model are:
- \( R \) = Demand per year.
- \( C \) = Ordering cost per order.
- \( H \) = Holding cost per unit per year.
- \( P \) = Purchasing cost per unit.
- \( Q \) = Optimal order quantity.
- \( B \) = Reorder point.
- \( TC \) = Total inventory cost.
Suppose demand of an item is constant with rate of \( R \) units per year, ordering cost of \( C \) per order and holding cost per unit per year of \( H \), purchasing cost of \( P \) per unit and the highest inventory level is \( Q \) when a replenishment occurs.

Since it is assumed that there is no shortage in the EOQ inventory model, then equation (1) becomes

\[
\text{Total inventory cost} = \text{Purchasing cost} + \text{Ordering cost} + \text{Holding cost}
\]

(2)

We also assume that one planning period is one year. As a consequence, the total annual inventory cost for the EOQ inventory model is given by

\[
TC(Q) = RP + \frac{CR}{Q} + \frac{HQ}{2}
\]

(3)

To find the optimal value of \( Q \) that minimizes the total annual inventory cost, we need a necessary condition of \( \frac{dTC}{dQ} = 0 \) that leads to the optimal order quantity of \( Q = \sqrt{\frac{2CR}{H}} \). From equation (3), we have \( TC(Q) = RP + \frac{CR}{Q} + \frac{HQ}{2} \). The second derivative of the total annual inventory cost with respect to \( Q \) is

\[
\frac{d^2TC}{dQ^2} = \frac{2CR}{Q^3}
\]

and this value is positive due to the positive values of \( C \), \( R \), and \( Q \). Therefore, the value of \( Q = \sqrt{\frac{2CR}{H}} \) will minimize \( TC(Q) = RP + \frac{CR}{Q} + \frac{HQ}{2} \).

3. An EOQ Model with Deterioration and All-units Discount

The basic concept of the model in this section is coming from the EOQ model. The optimal order quantity and when to order are crucial in determining the total inventory cost. In this model the order quantity will also has influence in the amount of deteriorated goods. The assumptions of this model are:

1. Demand is known and constant.
2. The model is only for single item and no interaction with other items.
3. Lead time is known and constant.
4. The order quantity is always the same in every replenishment.
5. Shortage will occur when items are (almost) deteriorated or damage.
6. Deterioration time is known with certainty.
7. All the (almost) deteriorated items will be sold at cheaper price, and therefore there is no holding cost for deteriorated items.

The notations in this model are:

\[
\begin{align*}
Q & \quad \text{Optimum order quantity.} \\
Q_k & \quad \text{The number of deteriorated goods.} \\
P_i & \quad \text{Purchasing cost per unit.} \\
D & \quad \text{Demand in one planning period.} \\
S & \quad \text{Ordering cost per order.} \\
h & \quad \text{Holding cost fraction per unit per planning period.} \\
C_s & \quad \text{Shortage cost per unit.} \\
J & \quad \text{Selling price per unit for deteriorated items.} \\
P_p & \quad \text{Purchasing cost per one planning period.} \\
P_o & \quad \text{Ordering cost per one planning period.} \\
P_S & \quad \text{Holding cost per one planning period.}
\end{align*}
\]
\( C_{So} \) = Shoetage cost per one planning period.
\( C_{kd} \) = Deteriorated cost per one planning period.
\( T \) = One planning period.
\( B \) = Reorder point.
\( L \) = Lead time.
\( t \) = Small cycle period.
\( t_1 \) = Holding period before deterioration.
\( t_2 \) = Period when shortage occurs.
\( TC \) = Total inventory cost.
\( U \) = Order quantity limit with different purchasing cost per unit.

Figure 2. An EOQ model with deterioration [1]

Figure 2 shows that the maximum inventory level is \( Q \) units, with the number of deteriorated items of \( Q_k \) units that occurs at the end of \( t_1 \). \( L \) denotes lead time, \( t_2 \) denotes the duration of shortage, \( T \) denotes one planning period and the company must reorder when the inventory level reaches \( B \) units.

The four cost components that affect total inventory cost still to be considered in this model plus another component of deterioration cost. Also in this model, there is all-units discount scheme offered by the supplier that also affect the purchasing cost. Thus, mathematically the total inventory cost can be expressed as

\[
\text{Total inventory cost} = \text{Purchasing cost} + \text{Ordering cost} + \text{Holding cost} + \text{Shortage cost} + \text{Deterioration cost}
\]

1. Purchasing cost occurs when company purchases goods to fulfil customers’ demands. Since in this model, supplier offers all-units discount scheme, the purchasing cost per unit is defined as follows.
\begin{equation}
\begin{aligned}
P_i = \begin{cases} 
P_0 & \text{for } V_0 \leq Q < V_1 \\
P_1 & \text{for } V_1 \leq Q < V_2 \\
\vdots \\
P_j & \text{for } V_j \leq Q < V_{j+1}
\end{cases}
\end{aligned}
\end{equation}

where \( P_j > P_{j+1}, j = 0, 1, 2, \ldots \) for each unit.

If in a year demands are \( D \) units, then the purchasing cost in a year is \( C_p = PD \).

2. Ordering cost occurs every time an order is released. If every order need an ordering cost of \( S \), the amount of ordering cost in a year is \( C_o = \frac{SD}{Q} \).

3. Holding cost usually for maintenance, rent or insurance for the stored items. If the holding cost per unit is expressed as a fraction of purchasing cost per unit, that is \( P_i h \), then the holding cost in a year is \( C_h = \frac{P_i h (Q^2 - Q_s^2)}{2Q} \).

4. Shortage cost (penalty cost) occurs when there is shortage due to deteriorated items. Shortage occurs during period of \( t_2 \). If the shortage cost per unit per time unit is \( C_k \), then the amount of shortage cost in a year is \( C_{so} = \frac{C_k Q_s^2}{2Q} \).

5. Deterioration cost occurs due to the sales of deteriorated items at cheaper price and ordering the same amount at a higher price. The sales occurs at \( t_1 \) and the company will buy a new items at higher price. This formulation of deteriorated cost that make this model different from the model [3].

If deteriorated items are sold at time \( t_1 \) with price \( J \), then the amount of deterioration cost in a year is \( C_{ld} = \frac{Q_k (P_i - J)D}{Q} \).

The total inventory cost for this model is given by

\begin{equation}
TAC(Q, Q_s) = P_D + \frac{SD}{Q} + \frac{P_i h (Q^2 - Q_s^2)}{2Q} + \frac{C_k Q_s^2}{2Q} + \frac{Q_k (P_i - J)D}{Q}
\end{equation}

To find the minimum total inventory cost, two necessary conditions must be satisfied, viz. \( \frac{\partial TAC}{\partial Q} = 0 \) and \( \frac{\partial TAC}{\partial Q_s} = 0 \). From \( \frac{\partial TAC}{\partial Q} = 0 \), we have

\begin{equation}
Q = \sqrt{2SD + \frac{Q_s^2 (C_k - P_i h) + 2Q_s (P_i - J)D}{P_i h}},
\end{equation}

and for \( \frac{\partial TAC}{\partial Q_s} = 0 \), we have

\begin{equation}
Q_s = \frac{(P_i - J)D}{P_i h - C_k}.
\end{equation}
By substituting equation (6) to equation (5), it is obtained

\[ Q = \sqrt{\frac{2SD}{P_i h} + \frac{(P_i - J)^2 D^2}{P_i h(P_i h - C_i)}} \]  

(7)

The procedure to find the optimal order quantity to minimize the total inventory cost can be performed using the well-known all-units discount algorithm ([3], [9]):

1. Calculate \( Q \) for each purchasing unit in the all-units discount scheme.
2. Compare \( Q \) with \( V \). If \( Q \) lies within interval \( V(V_j \leq Q < V_{j+1}) \), then \( Q \) is valid and continue to step 4.
3. If \( Q \) is not valid (lies outside interval \( V_j \leq Q < V_{j+1} \)), then
   (i). If \( Q \leq V_j \), set \( Q = V_j \).
   (ii). If \( Q > V_{j+1} \), set \( Q = V_{j+1} \).
4. Calculate the number of deteriorated items \( (Q_k) \).
5. Calculate \( TC \) for each valid \( Q \) and all possible \( V \) from step 3.
6. Compare the values of \( TC \) for valid \( Q \) with \( TC \) for all possible \( V \).
7. Choose the order quantity \( (Q) \) that gives the minimum \( TC \).

4. Numerical Results
A company in the food procurement industry needs 500 units of goods in a year with ordering cost of Rp. 200,000 per order and holding cost fraction of 0.8 from the purchasing cost per unit. When the goods are (almost) deteriorated or damage, they can be sold for Rp. 11,500 per unit. As the impact of these deterioration or damages, there is shortage cost of Rp. 50 per unit. The supplier offers all-units discount with the following scheme ([3], with some modifications):

| Order quantity (units) | Price/Unit |
|------------------------|------------|
| \( \leq 160 \)         | Rp. 15.000 |
| 161–180                | Rp. 14.000 |
| 181–200                | Rp. 13.000 |
| > 200                  | Rp. 12.000 |

We need to determine the optimal order quantity to minimize the total inventory cost. From the above information, we have \( D = 500 \) units, \( S = Rp. 200,000 \), \( h = 0.8 \), \( J = Rp. 11.500 \) and \( C_k = Rp. 50 \). We employ the all-units discount algorithm to find the optimal order quantity that minimize the total inventory cost.

1. Using equation (7) we will find the value of \( Q \) and the value of valid \( Q \) for each purchasing cost per unit and the results are depicted in Table 1 below.

| Order quantity (units) | Price/Unit | \( Q \) (units) | Valid \( Q \) (units) |
|------------------------|------------|----------------|---------------------|
| \( \leq 160 \)         | Rp. 15.000 | 212            | 160                 |
| 161–180                | Rp. 14.000 | 190            | 180                 |
| 181–200                | Rp. 13.000 | 167            | 181                 |
| > 200                  | Rp. 12.000 | 154            | 201                 |
2. Using equation (6), the number of deteriorated goods for each purchasing cost per unit can be calculated and found in Table 2 below.

| Order quantity (unit) | Price/Unit | $Q_i$ (units) |
|-----------------------|------------|---------------|
| $\leq 160$            | Rp. 15.000 | 147           |
| 161–180               | Rp. 14.000 | 113           |
| 181–200               | Rp. 13.000 | 73            |
| $> 200$               | Rp. 12.000 | 27            |

Table 3. Calculation of TC

| Order quantity (unit) | Price/Unit | $Q$ (units) | $Q_i$ (units) | TC          |
|-----------------------|------------|-------------|---------------|-------------|
| $\leq 160$            | Rp. 15.000 | 160         | 147           | Rp. 9,885,851 |
| 161–180               | Rp. 14.000 | 180         | 113           | Rp. 8,952,793 |
| 181–200               | Rp. 13.000 | 181         | 73            | Rp. 8,143,810 |
| $> 200$               | Rp. 12.000 | 201         | 27            | Rp. 7,478,576 |

Therefore, the optimal order quantity is 201 units with 27 units of deteriorated goods and total inventory cost of Rp. 7,478,576.

We also give sensitivity analysis of the shortage cost to the optimal solution in Table 4. We vary the shortage cost from Rp. 100 per unit to Rp. 500 per unit. It is found that as the shortage cost increases, the total inventory cost also increases and consistently the minimum total inventory cost occurs when ordering of 201 units of goods with deteriorated goods around 27 units. The minimum total inventory costs for different values of shortage cost per unit are between Rp. 7,478,667 to Rp. 7,479,391.

5. Conclusions and Further Research

From the numerical examples in the previous section, we found that this model is quite good given that the value of $(P_c - J)$ is relatively small (less than 10%), or $(P_c - J) < (P_i h - C_i)$. This means that in the model, the company is better to sell the (almost) deteriorated goods rather than facing up the shortage that causes shortage cost. This model is also suitable in the condition where the difference between the purchase cost and the selling price of the (almost) deteriorated goods is quite small and the shortage cost per unit due to deteriorated cost is also quite small. Below, we give an illustration of the output for the case with different values of shortage cost per unit (see Table 4).

From Table 4, we can see that as the shortage cost per unit is bigger, then the number of deteriorated goods will increase (from equation (6)) and as a consequence the total inventory cost will also increase (from equation (4)). The optimal order quantity for each case in Table 4 is 201 units.

We have developed an inventory model to determine the optimal order quantity by considering deterioration factor and all-units discount. Other aspects that are interested to pursue as further research are developing other discount schemes such as incremental discount or a continuous quantity discount function, developing multi-item case or probabilistic inventory models with discount, deterioration and other relevant factors.

| $C_i$ = 100 | $C_i$ = 150 | $C_i$ = 250 | $C_i$ = 500 |
|-------------|-------------|-------------|-------------|
| $Q$         | $Q_i$       | TC          | $Q$         | $Q_i$       | TC          | $Q$         | $Q_i$       | TC          |

Table 4. Illustration of the model for different values of $C_i$
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Acknowledgments
Funding from the Directorate of Research and Community Services, Indonesian Ministry of Research, Technology and Higher Education is highly acknowledged. We would also like to appreciate the comments from reviewers to the earlier version of this paper.