SOME IDENTITIES OF $q$-EULER POLYNOMIALS ARISING FROM $q$-UMBRAL CALCULUS

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ABSTRACT. Recently, Araci-Acikgoz-Sen derived some interesting identities on weighted $q$-Euler polynomials and higher-order $q$-Euler polynomials from the applications of umbral calculus (See [1]). In this paper, we develop the new method of $q$-umbral calculus due to Roman and we study new $q$-extension of Euler numbers and polynomials which are derived from $q$-umbral calculus. Finally, we give some interesting identities on our $q$-Euler polynomials related to the $q$-Bernoulli numbers and polynomials of Hegazi and Mansour.

1. Introduction

Throughout this paper we will assume $q$ to be a fixed real number between 0 and 1. We define the $q$-shifted factorials by
\[
(a : q)_0 = 1, (a : q)_n = \prod_{i=0}^{n-1} (1 - aq^i), (a : q)_\infty = \prod_{i=0}^{\infty} (1 - aq^i). \tag{1.1}
\]

If $x$ is a classical object, such as a complex number, its $q$-version is defined as $[x]_q = \frac{1 - q^x}{1 - q}$. We now introduce the $q$-extension of exponential function as follows:
\[
e_q(z) = \sum_{n=0}^{\infty} \frac{z^n}{[n]_q!} = \frac{1}{((1 - q)z : q)_\infty}, \quad (see [3, 6, 7, 8]), \tag{1.2}
\]
where $z \in \mathbb{C}$ with $|z| < 1$.

The Jackson definite $q$-integral of the function $f$ is defined by
\[
\int_0^x f(t) dq t = (1 - q) \sum_{a=0}^{\infty} f(q^a x) x q^a, \quad see [3, 6, 9]). \tag{1.3}
\]

The $q$-difference operator $D_q$ is defined by
\[
D_q f(x) = \frac{d_q f(x)}{d_q x} = \begin{cases} \frac{f(x) - f(qx)}{(1-q)x} & \text{if } x \neq 0 \\ \frac{df(x)}{dx} & \text{if } x = 0, \end{cases} \tag{1.4}
\]
where
\[
\lim_{q \to 1} D_q f(x) = \frac{df(x)}{dx}, \quad (see [3, 6, 8, 10]).
\]

By using exponential function $e_q(x)$, Hegazi and Mansour defined $q$-Bernoulli polynomials by means of
\[
\sum_{n=0}^{\infty} B_{n,q}(x) \frac{t^n}{[n]_q!} = \frac{t}{e_q(t) - 1} e_q(x), \quad (\text{see [3,6,8,12]}).
\] (1.5)

In the special case, \( x = 0 \), \( B_{n,q}(0) = B_n \) are called the \( n \)-th \( q \)-Bernoulli numbers.

From (1.5), we can easily derive the following equation:
\[
B_{n,q}(x) = \sum_{l=0}^{n} \binom{n}{l}_q B_{n-l,q} x^l = \sum_{l=0}^{n} \binom{n}{l}_q B_{l,q} x^{n-l}, \quad (1.6)
\]

where
\[
\binom{n}{l}_q = \frac{[n]_q!}{[n-l]_q! [l]_q!} = \frac{[n]_q [n-1]_q \cdots [n-l+1]_q}{[l]_q!}, \quad (\text{see [6,12]}).
\]

In the next section, we will consider new \( q \)-extensions of Euler numbers and polynomials by using the method of Hegazi and Mansour. More than five decades ago, Carlitz [2] defined a \( q \)-extension of Euler polynomials. In a recent paper (see [7]), B. A. Kupershmidt constructed reflection symmetries of \( q \)-Bernoulli polynomials which differ from Carlitz's \( q \)-Bernoulli numbers and polynomials. By using the method of B. A. Kupershmidt, Hegazi and Mansour also introduced new \( q \)-extension of Bernoulli numbers and polynomials (see [3,7,8]). From the \( q \)-exponential function, Kurt and Cenkci derived some interesting new formulae of \( q \)-extension of Genocchi polynomials. Recently, several authors have studied various \( q \)-extension of Bernoulli and Euler polynomials (see [1-10]). Let \( \mathbb{C} \) be the complex number field and let \( \mathcal{F} \) be the set of all formal power series in variable \( t \) over \( \mathbb{C} \) with
\[
\mathcal{F} = \left\{ f(t) = \sum_{k=0}^{\infty} \frac{a_k}{[k]_q!} t^k | a_k \in \mathbb{C} \right\}. \quad (1.7)
\]

Let \( \mathbb{P} = \mathbb{C}[t] \) and let \( \mathbb{P}^* \) be the vector space of all linear functionals on \( \mathbb{P} \). \( \langle L|p(x) \rangle \) denotes the action of linear functional \( L \) on the polynomial \( p(x) \), and it is well known that the vector space operations on \( \mathbb{P}^* \) are defined by
\[
\langle L + M|p(x) \rangle = \langle L|p(x) \rangle + \langle M|p(x) \rangle, \quad \langle cL|p(x) \rangle = c\langle L|p(x) \rangle,
\]
where \( c \) is complex constant (see[1,5,12]).

For \( f(t) = \sum_{k=0}^{\infty} \frac{a_k}{[k]_q!} t^k \in \mathcal{F} \), we define the linear functional on \( \mathbb{P} \) by setting
\[
\langle f(t)|x^n \rangle = a_n \text{ for all } n \geq 0. \quad (1.8)
\]

From (1.7) and (1.8), we note that
\[
\langle t^k|x^n \rangle = [n]_q! \delta_{n,k}, \quad (n,k \geq 0), \quad (1.9)
\]
where \( \delta_{n,k} \) is the Kronecker's symbol.

Let us assume that \( f_L(t) = \sum_{k=0}^{\infty} \langle L|x^n \rangle \frac{t^k}{[k]_q!} \). Then by (1.9), we easily see that
\[
\langle f_L(t)|x^n \rangle = \langle L|x^n \rangle. \quad \text{That is, } f_L(t) = L. \quad \text{Additionally, the map } L \mapsto f_L(t) \text{ is a vector space isomorphism from } \mathbb{P}^* \text{ onto } \mathcal{F}. \quad \text{Henceforth } \mathcal{F} \text{ denotes both the algebra of formal power series in } t \text{ and the vector space of all linear functionals on } \mathbb{P}, \text{ and so an element } f(t) \text{ of } \mathcal{F} \text{ will be thought as a formal power series and a linear functional. We call it the } q \text{-umbral algebra. The } q \text{-umbral calculus is the study of}
q-unbral algebra. By (1.2) and (1.3), we easily see that \( \langle e_q(yt)|x^n \rangle = y^n \) and so \( \langle e_q(yt)|p(x) \rangle = y^p(x) \) for \( p(x) \in \mathbb{P} \). The order of \( f(t) \) of the power series \( f(t) \neq 0 \) is the smallest integer for which \( a_k \) does not vanish. If \( o(f(t)) = 0 \), then \( f(t) \) is called an invertible series. If \( o(f(t)) = 1 \), then \( f(t) \) is called a delta series (see [1,5,11,12]). For \( f(t), g(t) \in \mathcal{F} \), we have \( \langle f(t)g(t)|p(x) \rangle = \langle f(t)|g(t)p(x) \rangle = \langle g(t)|f(t)p(x) \rangle \). Let \( f(t) \in \mathcal{F} \) and \( p(x) \in \mathbb{P} \). Then we have

\[
f(t) = \sum_{k=0}^{\infty} \langle f(t)|x^k \rangle \frac{t^k}{[k]_q!}, \quad p(x) = \sum_{k=0}^{\infty} \langle t^k|p(x) \rangle \frac{x^k}{[k]_q!} \quad (\text{see [11]}) \tag{1.10}
\]

From (1.10), we have

\[
p^{(k)}(x) = D_q^k p(x) = \sum_{l=k}^{\infty} \frac{\langle l^l|p(x) \rangle}{[l]_q!} [l]_q \cdots [l-k+1]_q x^{l-k}. \tag{1.11}
\]

By (1.11), we get

\[
p^{(k)}(0) = \langle t^k|p(x) \rangle \text{ and } \langle 1|p^{(k)}(x) \rangle = p^{(k)}(0). \tag{1.12}
\]

Thus from (1.12), we note that

\[
t^k p(x) = p^{(k)}(x) = D_q^k p(x). \tag{1.13}
\]

Let \( f(t), g(t) \in \mathcal{F} \) with \( o(f(t)) = 1 \) and \( o(g(t)) = 0 \). Then there exists a unique sequence \( S_n(x)(\deg S_n(x) = n) \) of polynomials such that \( \langle g(t)f(t)^k|S_n(x) \rangle = [n]_q! \delta_{n,k} \), \( (n,k \geq 0) \). The sequence \( S_n(x) \) is called the q-sheffer sequence for \( (g(t), f(t)) \) which is denoted by \( S_n(x) \sim (g(t), f(t)) \). Let \( S_n(x) \sim (g(t), f(t)) \). For \( h(t) \in \mathcal{F} \) and \( p(x) \in \mathbb{P} \), we have

\[
h(t) = \sum_{k=0}^{\infty} \frac{\langle h(t)|S_k(x) \rangle}{[k]_q!} g(t) f(t)^k, \quad p(x) = \sum_{k=0}^{\infty} \frac{\langle g(t)f(t)^k|p(x) \rangle}{[k]_q!} S_k(x), \tag{1.14}
\]

and

\[
\frac{1}{g(f(t))} e_q(y \tilde{f}(t)) = \sum_{k=0}^{\infty} \frac{S_k(y)}{[k]_q!} t^k, \quad \text{for all } y \in \mathbb{C}, \tag{1.15}
\]

where \( \tilde{f}(t) \) is the compositional inverse of \( f(t) \) (see [11,12]).

Recently, Araci–Acikgoz–Sen derived some new interesting properties on the new family of \( q \)-Euler numbers and polynomials from some applications of umbral algebra (see[1]). The properties of \( q \)-Euler and \( q \)-Bernoulli polynomials seem to be of interest and worthwhile in the areas of both number theory and mathematical physics. In this paper, we develop the new method of \( q \)-umbral calculus due to Roman and study new \( q \)-extension of Euler numbers and polynomials which are derived from \( q \)-umbral calculus. Finally, we give new explicit formulas on \( q \)-Euler polynomials relate to Hegazi–Mansour’s \( q \)-Bernoulli polynomials.

2. \( q \)-Euler Numbers and Polynomials

We consider the new \( q \)-extension of Euler polynomials which are generated by the generating function to be
\begin{equation}
\frac{2}{e_q(t) + 1} e_q(xt) = \sum_{n=0}^{\infty} E_{n,q}(x) \frac{t^n}{[n]_q!}. \tag{2.1}
\end{equation}

In the special case, \(x = 0\), \(E_{n,q}(0) = E_{n,q}\) are called the \(n\)-th \(q\)-Euler numbers. From (2.1), we note that

\begin{equation}
E_{n,q}(x) = \sum_{l=0}^{n} \binom{n}{l}_q E_{l,q} x^{n-l} = \sum_{l=0}^{n} \binom{n}{l}_q E_{n-l,q} x^l. \tag{2.2}
\end{equation}

By (2.1), we easily get

\begin{equation}
E_{0,q} = 1, E_{n,q}(1) + E_{n,q} = 2\delta_{0,n}. \tag{2.3}
\end{equation}

For example, \(E_{0,q} = 1, E_{1,q} = -\frac{1}{2}, E_{2,q} = \frac{q-1}{4}, E_{3,q} = \frac{q+1}{4}, \ldots\).

From (1.15) and (2.1), we have

\begin{equation}
te^{e_q(t) + 1} x^n = E_{n,q}(x), \quad (n \geq 0). \tag{2.5}
\end{equation}

Thus, by (1.13) and (2.5), we get

\begin{equation}
te_{n,q}(x) = \frac{2}{e_q(t) + 1} x^n = [n]_q \frac{2}{e_q(t) + 1} x^{n-1} = [n]_q E_{n-1,q}(x), \quad (n \geq 0). \tag{2.6}
\end{equation}

Indeed, by (1.9), we get

\begin{equation}
\langle \left(\frac{e_q(t) + 1}{2} t \right)^k | E_{n,q}(x) \rangle = \frac{[k]_q!}{2} \binom{n}{k}_q \langle \left(\frac{e_q(t) + 1}{2} \right)^k | E_{n-k,q}(x) \rangle = \frac{[k]_q!}{2} \binom{n}{k}_q \langle E_{n-k,q}(1) + E_{n-k,q} \rangle. \tag{2.7}
\end{equation}

From (2.4), we have

\begin{equation}
\langle \left(\frac{e_q(t) + 1}{2} \right)^k | E_{n,q}(x) \rangle = [n]_q! \delta_{n,k} \tag{2.8}
\end{equation}

Thus, by (2.7) and (2.8), we get

\begin{equation}
0 = E_{n-k,q}(1) + E_{n-k,q} = \sum_{l=0}^{n-k} \binom{n-k}{l}_q E_{l,q} + E_{n-k,q}, \quad (n, k \in \mathbb{Z}_{\geq 0} \text{ with } n > k). \tag{2.9}
\end{equation}

This is equivalent to

\begin{equation}
-2E_{n-k,q} = \sum_{l=0}^{n-k-1} \binom{n-k}{l}_q E_{l,q}, \quad \text{where } n, k \in \mathbb{Z}_{\geq 0} \text{ with } n > k. \tag{2.10}
\end{equation}

Therefore, by (2.10), we obtain the following lemma.
Lemma 2.1. For $n \geq 1$, we have

$$-2E_{n,q} = \sum_{l=0}^{n-1} \binom{n}{l}_q E_{l,q}.$$ 

From (2.2) we have

$$\int_x^{x+y} E_{n,q}(u)d_qu = \sum_{l=0}^{n} \binom{n}{l}_q E_{n-l,q} \left\{ \frac{1}{[l+1]_q} \left( (x+y)^{l+1} - x^{l+1} \right) \right\}$$

$$= \frac{1}{[n+1]_q} \sum_{l=0}^{n+1} \binom{n+1}{l+1}_q E_{n-l,q} \left\{ (x+y)^{l+1} - x^{l+1} \right\}$$

$$= \frac{1}{[n+1]_q} \sum_{l=0}^{n+1} \binom{n+1}{l}_q E_{n+1-l,q} \left\{ (x+y)^{l} - x^{l} \right\}$$

$$= \frac{1}{[n+1]_q} \left\{ E_{n+1,q}(x+y) - E_{n+1,q}(x) \right\}.$$ 

Thus, by (2.11), we get

$$\langle \frac{e_q(t) - 1}{t} | E_{n,q}(x) \rangle = \frac{1}{[n+1]_q} \left( \frac{e_q(t) - 1}{t} \right) | t E_{n+1,q}(x) \rangle$$

$$= \frac{1}{[n+1]_q} (e_q(t) - 1 | E_{n+1,q}(x))$$

$$= \frac{1}{[n+1]_q} \left\{ E_{n+1,q}(1) - E_{n+1,q} \right\}$$

$$= \int_0^1 E_{n,q}(u)d_qu.$$ 

Therefore, by (2.12), we obtain the following theorem.

Theorem 2.2. For $n \geq 0$, we have

$$\langle \frac{e_q(t) - 1}{t} | E_{n,q}(x) \rangle = \int_0^1 E_{n,q}(u)d_qu.$$ 

Let

$$P_n = \{ p(x) \in \mathbb{C}[x] | \deg p(x) \leq n \}.$$ 

For $p(x) \in P_n$, let us assume that

$$p(x) = \sum_{k=0}^{n} b_{k,q} E_{k,q}(x).$$ 

Then, by (2.4), we get

$$\left\langle \left( \frac{e_q(t) + 1}{2} \right)^k | E_{n,q}(x) \right\rangle = [n]_q! \delta_{n,k}.$$
From (2.14) and (2.15), we can derive the following equation (2.16):

\[
\langle \left( \frac{e_q(t) + 1}{2} \right)^k | p(x) \rangle = \sum_{l=0}^{n} b_{l,q} \langle \left( \frac{e_q(t) + 1}{2} \right)^k | E_{l,q}(x) \rangle = \sum_{l=0}^{n} b_{l,q}[l]_q! \delta_{l,k} = [k]_q! b_{k,q}.
\]

(2.16)

Thus, by (2.16), we get

\[
b_{k,q} = \frac{1}{[k]_q!} \langle \left( \frac{e_q(t) + 1}{2} \right)^k | p(x) \rangle = \frac{1}{2[k]_q!} \langle \langle e_q(t) + 1 \rangle^k | p(x) \rangle = \frac{1}{2[k]_q!} \{ p^{(k)}(1) + p^{(k)}(0) \},
\]

(2.17)

where \( p^{(k)}(x) = D_k^n p(x) \).

Therefore, by (2.14) and (2.17), we obtain the following theorem.

**Theorem 2.3.** For \( p(x) \in \mathbb{P}_n \), let \( p(x) = \sum_{k=0}^{n} b_{k,q} E_{k,q}(x) \). Then we have

\[
b_{k,q} = \frac{1}{2[k]_q!} \langle \langle e_q(t) + 1 \rangle^k | p(x) \rangle = \frac{1}{2[k]_q!} \{ p^{(k)}(1) + p^{(k)}(0) \},
\]

where \( p^{(k)}(x) = D_k^n p(x) \).

From (1.5), we note that \( B_{n,q}(x) \sim \left( \frac{e_q(t) - 1}{t} \right)^n, (n \geq 0). \)

(2.18)

Let us take \( p(x) = B_{n,q}(x) \in \mathbb{P}_n \). Then \( B_{n,q}(x) \) can be represented as a linear combination of \( \{ E_{0,q}(x), E_{1,q}(x), \ldots, E_{n,q}(x) \} \) as follows:

\[
B_{n,q}(x) = p(x) = \sum_{k=0}^{n} b_{k,q} E_{k,q}(x), (n \geq 0),
\]

(2.19)

where

\[
b_{k,q} = \frac{1}{2[k]_q!} \langle \langle e_q(t) + 1 \rangle^k | B_{n,q}(x) \rangle = \frac{[n]_q [n-1]_q \cdots [n-k+1]_q}{2[k]_q!} \langle e_q(t) + 1 | B_{n-k,q}(x) \rangle = \frac{1}{2} \binom{n}{k} \langle e_q(t) + 1 | B_{n-k,q}(x) \rangle = \frac{1}{2} \binom{n}{k} \{ B_{n-k,q}(1) + B_{n-k,q} \}.
\]

(2.20)

From (1.5), we can derive the following recurrence relation for the \( q \)-Bernoulli numbers:
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\[ t = \left( \sum_{l=0}^{\infty} B_{l,q} \frac{t^l}{[l]_q!} \right) (e_q(t) - 1) \]

\[ = \sum_{n=0}^{\infty} \left( \sum_{l=0}^{n} \binom{n}{l} B_{l,q} \frac{t^n}{[n]_q!} \right) - \sum_{n=0}^{\infty} B_{n,q} \frac{t^n}{[n]_q!} \]

\[ = \sum_{n=0}^{\infty} (B_{n,q}(1) - B_{n,q}) \frac{t^n}{[n]_q!}. \tag{2.21} \]

Thus, by (2.21), we get

\[ B_{0,q} = 1, B_{n,q}(1) - B_{n,q} = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } n > 1. \end{cases} \tag{2.22} \]

For example, $B_{0,q} = 1, B_{1,q} = -\frac{1}{[2]_q}, B_{2,q} = \frac{q^2}{[3]_q[4]_q}, \ldots$.

By (2.19), (2.20), and (2.22), we get

\[ B_{n,q}(x) = b_{n,q}E_{n,q}(x) + b_{n-1,q}E_{n-1,q}(x) + \sum_{k=0}^{n-2} b_{k,q}E_{k,q}(x) \]

\[ = E_{n,q}(x) + \frac{[n]_q}{2} \left( 1 - \frac{2}{[2]_q} \right) E_{n-1,q}(x) + \sum_{k=0}^{n-2} \binom{n}{k}_q B_{n-k,q}E_{k,q}(x) \tag{2.23} \]

\[ = E_{n,q}(x) - \frac{[n]_q(1-q)}{2[2]_q} E_{n-1,q}(x) + \sum_{k=0}^{n-2} \binom{n}{k}_q B_{n-k,q}E_{k,q}(x) \]

Therefore, by (2.23), we obtain the following theorem.

**Theorem 2.4.** For $n \geq 2$, we have

\[ B_{n,q}(x) = E_{n,q}(x) + \frac{[n]_q(q-1)}{2[2]_q} E_{n-1,q}(x) + \sum_{k=0}^{n-2} \binom{n}{k}_q B_{n-k,q}E_{k,q}(x). \]

For $r \in \mathbb{Z}_{\geq 0}$, the $q$-Euler polynomials, $E_{n,q}^{(r)}(x)$, of order $r$ are defined by the generating function to be

\[ \left( \frac{2}{e_q(t) + 1} \right)^r e_q(xt) = \left( \frac{2}{e_q(t) + 1} \right) \times \cdots \times \left( \frac{2}{e_q(t) + 1} \right) e_q(xt) \]

\[ = \sum_{n=0}^{\infty} E_{n,q}^{(r)}(x) \frac{t^n}{[n]_q!}. \tag{2.24} \]

In the special case, $x = 0, E_{n,q}^{(r)}(0) = E_{n,q}^{(r)}$ are called the $n$-th $q$-Euler numbers of order $r$.

Let

\[ g^r(t) = \left( \frac{e_q(t) + 1}{2} \right)^r, \ (r \in \mathbb{Z}_{\geq 0}). \tag{2.25} \]
Then $g^*(t)$ is an invertible series. From (2.24) and (2.25), we have

$$
\sum_{n=0}^{\infty} E_{n,q}^{(r)}(x) \frac{t^n}{[n]_q!} = \frac{1}{g^*(t)} e_q(x t) = \sum_{n=0}^{\infty} \frac{1}{g^*(t)} x^n \frac{t^n}{[n]_q!}.
$$

(2.26)

By (2.26), we get

$$
E_{n,q}^{(r)}(x) = \frac{1}{g^*(t)} x^n,
$$

(2.27)

and

$$
t E_{n,q}^{(r)}(x) = \frac{1}{g^*(t)} t x^n = [n]_q \frac{1}{g^*(t)} x^{n-1} = [n]_q E_{n-1,q}^{(r)}(x).
$$

(2.28)

Thus, by (2.26), (2.27) and (2.28), we see that

$$
E_{n,q}^{(r)}(x) \sim \left( \frac{e_q(t)+1}{2} \right)^r, t
$$

(2.29)

By (1.9) and (2.24), we get

$$
\langle \left( \frac{2}{e_q(t)+1} \right)^r e_q(y t) | x^n \rangle = E_{n,q}^{(r)}(y) = \sum_{l=0}^{n} \binom{n}{l} E_{n-l,q}^{(r)} y^l.
$$

(2.30)

Thus, we have

$$
\langle \left( \frac{2}{e_q(t)+1} \right)^r | x^n \rangle = \sum_{m=0}^{\infty} \left( \sum_{i_1+\cdots+i_r=m} \frac{E_{i_1,q} \cdots E_{i_r,q}}{[i_1]_q! \cdots [i_r]_q!} \right) (t^m | x^n)
$$

$$
= \sum_{i_1+\cdots+i_r=n} \frac{[n]_q!}{[i_1]_q! \cdots [i_r]_q!} E_{i_1,q} \cdots E_{i_r,q}
$$

$$
= \sum_{i_1+\cdots+i_r=n} \binom{n}{i_1, \ldots, i_r} E_{i_1,q} \cdots E_{i_r,q},
$$

(2.31)

where \( \binom{n}{i_1, \ldots, i_r} = \frac{[n]_q!}{[i_1]_q! \cdots [i_r]_q!} \).

By (2.30), we easily get

$$
\langle \left( \frac{2}{e_q(t)+1} \right)^r | x^n \rangle = E_{n,q}^{(r)}.
$$

(2.32)

Therefore, by (2.31) and (2.32), we obtain the following theorem.

**Theorem 2.5.** For $n \geq 0$, we have

$$
E_{n,q}^{(r)} = \sum_{i_1+\cdots+i_r=n} \left( \binom{n}{i_1, \ldots, i_r} E_{i_1,q} \cdots E_{i_r,q},
$$

where \( \binom{n}{i_1, \ldots, i_r} = \frac{[n]_q!}{[i_1]_q! \cdots [i_r]_q!} \).

Let us take \( p(x) = E_{n,q}^{(r)}(x) \in \mathbb{P}_n \). Then, by Theorem 2.5, we get

$$
E_{n,q}^{(r)}(x) = p(x) = \sum_{k=0}^{n} b_{k,q} E_{k,q}(x),
$$

(2.33)
Theorem 2.6. For

\[ b_{k,q} = \frac{1}{2[k]_q}((e_q(t) + 1)t^k)p(x) = \frac{1}{2[k]_q}((e_q(t) + 1)t^k)p(x) \]

\[ = \frac{(n)_q}{2}((e_q(t) + 1)|E_{n-k,q}^{(r)}(x)) = \frac{(n)_q}{2}\{E_{n-k,q}^{(r)}(1) + E_{n-k,q}^{(r)}\}. \]  

From (2.24), we have

\[ \sum_{k=0}^{\infty} \{E_{n,q}^{(r)}(1) + E_{n,q}^{(r)}\} \frac{t^n}{[n]_q!} = \left(\frac{2}{e_q(t) + 1}\right)^r (e_q(t) + 1) \]

\[ = 2 \left(\frac{2}{e_q(t) + 1}\right)^{r-1} = 2 \sum_{n=0}^{\infty} E_{n,q}^{(r-1)} \frac{t^n}{[n]_q!}. \]  

By comparing the coefficients on the both sides of (2.35), we get

\[ E_{n,q}^{(r)}(1) + E_{n,q}^{(r)} = 2E_{n,q}^{(r-1)}, \quad (n \geq 0). \]  

Therefore, by (2.33), (2.34) and (2.36), we obtain the following theorem.

Theorem 2.6. For \( n \in \mathbb{Z}_{\geq 0}, \ r \in \mathbb{Z}_{>0} \), we have

\[ E_{n,q}^{(r)}(x) = \sum_{k=0}^{\infty} \binom{n}{k} q^{k} E_{n-k,q}^{(r-1)} E_{k,q}(x). \]  

Let us assume that

\[ p(x) = \sum_{k=0}^{n} b_{k,q}^{r} E_{k,q}^{(r)}(x) \in \mathcal{P}_n. \]  

By (2.26) and (2.27), we get

\[ (\frac{e_q(t) + 1}{2})^r t^k p(x) = \sum_{l=0}^{n} b_{l,q}^{r} \left(\frac{e_q(t) + 1}{2}\right)^r t^k E_{l,q}^{(r)}(x) \]

\[ = \sum_{l=0}^{n} b_{l,q}^{r} [l]_q! b_{l,k} = [k]_q! b_{k,q}^{r}. \]  

From (2.38), we have

\[ b_{k,q}^{r} = \frac{1}{[k]_q!} \left(\frac{e_q(t) + 1}{2}\right)^r t^k p(x) = \frac{1}{2[r][k]_q!}((e_q(t) + 1)^r |t^k p(x)) \]

\[ = \frac{1}{2[r][k]_q!} \sum_{l=0}^{r} \binom{r}{l} \sum_{m \geq 0} \binom{m}{i_1, \cdots, i_l} q^{i_1, \cdots, i_l} \frac{1}{[m]_q!} (1|t^{m+k} p(x)) \]

\[ = \frac{1}{2[r][k]_q!} \sum_{l=0}^{r} \binom{r}{l} \sum_{m \geq 0} \binom{m}{i_1, \cdots, i_l} q^{i_1, \cdots, i_l} \frac{1}{[m]_q!} \rho^{(m+k)}(0). \]  

Therefore by (2.37) and (2.39), we obtain the following theorem.
Theorem 2.7. For \( n \geq 0 \), let \( p(x) = \sum_{k=0}^{n} b_{k,q}^{r} E_{k,q}^{(r)}(x) \in \mathbb{P}_n \).
Then we have

\[
\begin{align*}
b_{k,q}^{r} &= \frac{1}{2^r[k]_q!} \langle (e_q(t) + 1)^r \rangle^{k} p(x) \\
&= \frac{1}{2^r[k]_q!} \sum_{m \geq 0} \sum_{l=0}^{r} \binom{r}{l} \sum_{i_1 + \cdots + i_l = m} \binom{m}{i_1, \ldots, i_l}_q \frac{1}{[m]_q!} p^{m+k}(0),
\end{align*}
\]

where \( p^{(k)}(x) = D_k^r p(x) \).
Let us take \( p(x) = E_{n,q}(x) \in \mathbb{P}_n \). Then, by Theorem 2.7, we get

\[
E_{n,q}(x) = p(x) = \sum_{k=0}^{n} b_{k,q}^{r} E_{k,q}^{(r)}(x),
\]

(2.40)

where

\[
\begin{align*}
b_{k,q} &= \frac{1}{2^r[k]_q!} \sum_{m=0}^{n-k} \sum_{l=0}^{r} \binom{r}{l} \sum_{i_1 + \cdots + i_l = m} \binom{m}{i_1, \ldots, i_l}_q \\
&\times \frac{1}{[m]_q!} [n]_q \cdots [n - m - k + 1]_q E_{n-m-k,q} \\
&= \frac{1}{2^r} \sum_{m=0}^{n-k} \sum_{l=0}^{r} \binom{r}{l} \sum_{i_1 + \cdots + i_l = m} \binom{m}{i_1, \ldots, i_l}_q \\
&\times \frac{[m + k]_q! [n]_q \cdots [n - m - k + 1]_q}{[m + k]_q!} E_{n-m-k,q} \\
&= \frac{1}{2^r} \sum_{m=0}^{n-k} \sum_{l=0}^{r} \binom{r}{l} \binom{m}{i_1, \ldots, i_l}_q \binom{m+k}{m}_q \binom{n}{m+k}_q E_{n-m-k,q} \\
&\times E_{n-m-k,q} E_{k,q}^{(r)}(x),
\end{align*}
\]

(2.41)

Therefore, by (2.40) and (2.41), we obtain the following theorem.

Theorem 2.8. For \( n, r \geq 0 \), we have

\[
E_{n,q}(x) = \frac{1}{2^r} \sum_{k=0}^{n} \left\{ \sum_{m=0}^{n-k} \sum_{l=0}^{r} \binom{r}{l} \binom{m}{i_1, \ldots, i_l}_q \binom{m+k}{m}_q \binom{n}{m+k}_q E_{n-m-k,q} \right\} E_{k,q}^{(r)}(x).
\]

For \( r \in \mathbb{Z}_{\geq 0} \), let us consider \( q \)-Bernoulli polynomials of order \( r \) which are defined by the generating function to be

\[
\left( \frac{t}{e_q(t) - 1} \right)^r e_q(x) = \left( \frac{t}{e_q(t) - 1} \right) \times \cdots \times \left( \frac{t}{e_q(t) - 1} \right) \times e_q(x)
\]

(2.42)

\[
= \sum_{n=0}^{\infty} B_{n,q}^{(r)}(x) \frac{t^n}{[n]_q},
\]
In the special case, \( x = 0 \), \( B_{n,q}^{(r)}(0) = B_{n,q}^{(r)} \) are called the \( n \)-th \( q \)-Bernoulli numbers of order \( r \). By (2.42) we easily get

\[
B_{n,q}^{(r)}(x) = \sum_{l=0}^{n} \binom{n}{l}_q B_{l,q}^{(r)} x^{n-l} \in \mathbb{P}_n. \tag{2.43}
\]

Let us take \( p(x) = B_{n,q}^{(r)}(x) \in \mathbb{P}_n \). Then, by Theorem 2.7, we get

\[
B_{n,q}^{(r)}(x) = p(x) = \sum_{k=0}^{n} b_{k,q}^{(r)} E_{k,q}^{(r)}(x), \tag{2.44}
\]

where

\[
b_{k,q}^{(r)} = \frac{1}{2^r [k]_q!} ((e_q(t) + 1)^r t^k | B_{n,q}^{(r)}(x)))
\]

\[
= \frac{1}{2^r [k]_q!} \sum_{m=0}^{n-k} \sum_{l=0}^{r} \binom{r}{l} \sum_{i_1+\cdots+i_l=m} \binom{m}{i_1,\cdots,i_l}_q \times \frac{n! \cdots [n-m-k+1]_q}{[m]_q!} B_{n-m-k,q}^{(r)}
\]

\[
= \frac{1}{2^r} \sum_{m=0}^{n-k} \sum_{l=0}^{r} \sum_{i_1+\cdots+i_l=m} \binom{r}{l} \binom{m}{i_1,\cdots,i_l}_q \binom{m+k}{m}_q \binom{n}{m+k}_q B_{n-m-k,q}^{(r)}.
\tag{2.45}
\]

Therefore, by (2.44) and (2.45), we obtain the following theorem.

**Theorem 2.9.** For \( n, r \geq 0 \), we have

\[
B_{n,q}^{(r)}(x) = \frac{1}{2^r} \sum_{k=0}^{n} \left( \sum_{m=0}^{n-k} \sum_{l=0}^{r} \sum_{i_1+\cdots+i_l=m} \binom{r}{l} \binom{m}{i_1,\cdots,i_l}_q \binom{m+k}{m}_q \binom{n}{m+k}_q \times B_{n-m-k,q}^{(r)} \right) E_{k,q}^{(r)}(x).
\]

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