Cosmic Ray Origins in Supernova Blast Waves

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ABSTRACT

We extend the self-similar solution derived by Chevalier (1983a) for a Sedov blast wave accelerating cosmic rays (CR) to show that the Galactic CR population can be divided into: (A) CR with energies above $\sim 200\text{GeV}$ released upstream during CR acceleration by supernova remnants (SNR), (B) CR advected into the interior of the SNR during expansion and then released from the SNR at the end of its life to provide the Galactic CR component below $\sim 200\text{GeV}$. The intersection between the two populations may correspond to a measured change in the Galactic CR spectral index at this energy (Adriani et al 2011).

Key words: cosmic rays, acceleration of particles, shock waves, magnetic field, ISM: supernova remnants

1 INTRODUCTION

Supernova remnants (SNR) are the most probable source of Galactic cosmic rays (CR) at energies up to a few PeV. CR gain energy at the outer shocks of supernova blast waves by first order Fermi diffusive shock acceleration (Krymsky 1977, Axford Leer & Skadron 1977, Bell 1978, Blandford & Ostriker 1978), although second order Fermi processes may also contribute (Ostrowski 1999). CR may also be accelerated by shocks associated with star formation, the large scale Galactic wind, or activity at the centre of the Galaxy.

Diffusive shock acceleration (DSA) efficiently produces a $T^{-2}$ CR energy spectrum where $T$ is the CR energy in eV. The predicted maximum CR energy produced by SNR shocks is close to a PeV, although it appears that the historical supernova remnants (SNR) are unable to reach this energy since their shocks are already significantly decelerated (Zirakashvili & Ptuskin 2008, Bell et al 2013). In order to explain the Galactic CR population it is essential not only that CR protons should be accelerated to a few PeV but also that the CR should be able to escape the SNR without large energy loss. Bell et al (2013) showed that the highest energy CR escape upstream from the shock into the interstellar medium. However most of the shock-accelerated CR, by energy content as well as number, are carried downstream into the interior of the SNR. In this paper we examine the fate of these lower energy CR as they are advected into the SNR, by energy content as well as number, are carried downstream into the interior of the SNR. An individual CR accelerated early in the Sedov phase has a much reduced energy by the time it is released into the Galaxy. This is often perceived as a difficulty in explaining CR origins. However, the CR energy lost by adiabatic expansion is in fact re-used to drive the blast wave and accelerate a new generation of CR at a later time. Chevalier (1983a) derived a self-similar Sedov blast-wave solution that includes CR pressure. He showed that the CR pressure dominates the thermal plasma pressure at the centre of the remnant even if only a relatively small fraction of the available energy is given to CR by the shock. Because CR have a smaller ratio of specific heats ($\gamma = 4/3$) than thermal particles ($\gamma = 5/3$), CR lose less energy during adiabatic expansion. Thermal particles preferentially lose energy as they drive the blast wave and accelerate more CR, whereas CR preferentially keep their energy for release into the ISM at the end of the SNR’s life.

In this paper we extend Chevalier’s self-similar model to derive the CR energy spectrum and the maximum CR energy inside a blast wave. We show that CR produced by SNR can be divided into two populations: (A) CR with energies above $\sim 200\text{GeV}$ that escape ahead of the shock during SNR expansion (B) CR advected into the interior of the SNR during expansion and then released from the SNR at the end of its life to provide the Galactic CR component below $\sim 200\text{GeV}$. Instead of limiting the efficiency of Galactic CR production, adiabatic losses during SNR expansion increase the efficiency by filtering energy from the thermal plasma into CR. The underlying principles of the calculation apply to any blast wave, possibly including any launched from the centre of the Galaxy or from star forming regions.

Using the formulation developed by Bell et al (2013) we derive energy spectra and energy densities of CR within the SNR and the total energy of CR released into the surround-
ing medium. Bell et al (2013) showed that the maximum CR energy is determined by the growth rate of the instability amplifying the magnetic field needed to confine CR in the shock environment during acceleration. The results derived using Bell et al (2013) differ from those derived on an assumption that the energy density of the amplified magnetic field is proportional to the kinetic energy density $\rho_0 u_0^2$ of plasma with density $\rho_0$ overtaken by a shock with velocity $u_0$ (e.g Berezko & Völk 2004, 2007). CR produced by SNR can be divided into populations A and B as defined above. The overlap of the two populations at an energy of about 200 GeV may be related to the break in the CR energy spectrum measured by PAMELA (Adriani et al, 2011) and other experiments (Ahn 2010, Tomassetti 2012).

Throughout this paper we consider only proton acceleration. Wherever CR are mentioned we refer to protons unless otherwise stated.

2 SEDOV SELF-SIMILARITY

In this section we derive the Sedov self-similar solution including the CR pressure as well as the thermal pressure. Chevalier (1983a) has previously derived this self-similar solution but we present the derivation in a form that facilitates calculation of the self-similar CR energy distribution inside the blast wave. A detailed time-dependent numerical study of the effect of efficient CR acceleration on SNR dynamics in the Sedov phase can be found in Castro et al (2011).

The essential feature of the Sedov solution for an expanding blast wave is that the total energy is conserved. At any time during self-similar expansion into a uniform medium measured by PAMELA (Adriani et al, 2011) and other experiments but we present the derivation in a form that facilitates calculation of the self-similar CR energy distribution inside the blast wave. A detailed time-dependent numerical study of the effect of efficient CR acceleration on SNR dynamics in the Sedov phase can be found in Castro et al (2011).

The essential feature of the Sedov solution for an expanding blast wave is that the total energy is conserved. At any time during self-similar expansion into a uniform medium with density $\rho_0$ the energy in the blast wave is proportional to $\rho_0 r^3 u_0^2$ since the energy density at any point inside the blast wave is proportional to $\rho_0 u_0^2$ (assuming that the shock Mach number is high) and the volume is proportional to $r^3$, where $r$ is the radius of outer shock. From energy conservation $r^3 u_0^2$ is constant, so $r_s \propto t^{2/5}$ and $u_s \propto t^{-3/5}$.

In reality, and as part of this model, some energy is lost from the blast wave due to CR escaping upstream as estimated below in equation 19. If the energy loss due to CR escaping upstream is $0.03 \rho_0 u_0^3$ per unit shock area (Bell et al 2013) then $dE/ds \approx -0.13E/r_s$ for $E \approx 3\rho_0 u_0^2 r_s^3$, which gives $\beta \approx 0.05$, $r_s \propto t^{0.34}$ and $u_s \propto t^{-0.61}$ instead of $r_s \propto t^{0.4}$ and $u_s \propto t^{-0.6}$. This will produce a very slight flattening in the CR spectrum since it reduces the energy given to low energy CR later in the life of the SNR. Because the effect is small we neglect the effect of energy loss to CR and proceed on the assumption that $r_s \propto t^{0.4}$ and $u_s \propto t^{-0.6}$.

Self-similarity is independent of the ratio of specific heats $\gamma$. It also holds for a mixture of CR and thermal gases with different $\gamma$ provided the CR acceleration efficiency is constant in time. We consider the case in which the immediately post-shock CR pressure $P_{CR}$ is a fraction $\epsilon$ of the total post-shock pressure $P_s$ with the thermal pressure $P_t$ providing the balance of the post-shock pressure:

$$P_{CR}(r_s) = \epsilon P_s \quad P_t(r_s) = (1 - \epsilon) P_s$$ (1)

In reality, $\epsilon$ probably varies as the shock speed changes, but for simplicity, and because it is unclear whether $\epsilon$ increases or decreases, we assume that it remains constant throughout the Sedov phase. For convenience and usefulness in later sections of this paper we introduce $R(r)$ as the radius of the shock when the fluid element presently at position $r$ was overtaken by the shock. Since the mass presently inside the radius $r$ is equal to the mass inside the shock when the shock was at radius $R_r$,

$$\int_0^{r} 4\pi \rho(r)r^2dr = \frac{4\pi}{3} \rho_0 R^3$$ (2)

where $\rho$ is the present density profile. $R(r)$ is a function of the present radius $r$. Since $u_s \propto r_s^{3/2}$ the post-shock pressure was $(R/r_s)^{-3} P_t$ when the fluid element now at radius $r$ passed through the shock. Hence the CR pressure at radius $r$ is reduced by adiabatic expansion to $\epsilon P_s (R/r_s)^{-3}(\rho/\rho_s)^{4/3}$ where $\rho_s$ is the post-shock density. Similarly the thermal pressure is $(1 - \epsilon) P_s (R/r_s)^{-3}(\rho/\rho_s)^{5/3}$ so the total pressure at radius $r$ is

$$P = \frac{P_s}{R^3} \left[ (1 - \epsilon) \left( \frac{P_t}{\rho_s} \right)^{5/3} + \epsilon \left( \frac{P_t}{\rho_s} \right)^{4/3} \right]$$ (3)

where all quantities are defined at the present time. This equation assumes that all CR remain relativistic even as they cool adiabatically. In practice mildly relativistic CR become non-relativistic as they cool adiabatically and their $\gamma$ changes from 4/3 to 5/3. We neglect this effect under the assumption that most of the CR energy resides in CR that remain relativistic. For example, if a $T^{-2}$ spectrum extends to 1 PeV, CR with a Lorentz factor less than two account for only 6% of the total CR energy and CR with a Lorentz factor less than ten account for 15% of the total. See Chevalier (1983a,b) for a discussion of this issue when the shock-accelerated spectrum is steeper than $T^{-2}$.

We also assume that CR diffusion can be neglected and that CR remain localised to the same fluid element after passing through the shock. This is a good assumption for most CR, since CR are spatially localised by their small Larmor radius: the Larmor radius of a CR with energy $\gamma E$ in GeV in a 10 $\mu$G magnetic field is only $10^{-7} T_{10}^{1/2}$ parsec. Furthermore it is part of the theory of diffusive shock acceleration that all except the very highest energy CR exit the acceleration process by being advected away downstream with the thermal plasma. Hence advection dominates diffusion over most of the CR energy range, and diffusion can be neglected for bulk properties of CR such as the integrated energy density of all CR from the lowest to the highest energy.

From self-similarity, $\rho = \rho/r_s^{3}$, the fluid velocity inside the SNR takes the form $u = t^{3/5} f(r/r_s)$, and pressure takes the form $P = t^{3/5} g(r/r_s)$ where $f$ and $g$ represent the shape of the velocity and pressure profiles. The self-similar equation for mass conservation is then

$$\frac{u_s r_s^3 \partial \rho}{r_s \partial r} = \frac{1}{r^2} \frac{\partial (r^2 \rho u)}{\partial r}$$ (4)

where the left hand side of the equation is the self-similar
The resulting profiles are given in figure 1 for various shock
derived from the asymptotic solution given in equation 7
\[ \rho \text{ is lost close to } \tau \text{ is derived in appendix A:} \]

The asymptotic solution close to the centre of the blast wave
is derived in appendix A:

\[ \frac{u}{u_s} = \left\{ -\frac{1}{5(1-\epsilon)} \left[ \left( 16\epsilon^2 - 10(1-\epsilon)u_P r^3 \right)^{1/2} - 4\epsilon \right] \right\}^{-\frac{3}{2}} \]

where \( P = P_c \) and \( \partial P/\partial r = 0 \) at zero radius. For \( \epsilon \to 0 \) (negligible CR pressure) \( \rho \propto r^{3/2} \). For non-zero \( \epsilon \) (CR dominant at the centre) \( \rho \propto r^3 \) as \( r \to 0 \). The asymptotic forms of \( R \) and \( u \) can be derived from equations 3, 6 and 7. Boundary conditions are imposed at the shock where

\[ \rho_s = (4 + 3\epsilon)\rho_0 \; ; 
\quad P_s = \frac{3 + 3\epsilon}{4 + 3\epsilon} \rho_0 u_s^2 \; ; 
\quad R = r_s \]

We solve the equations numerically by integrating towards the centre from the shock radius \( r_s \). The numerical accuracy is lost close to \( r = 0 \) due to the density becoming very small (\( \rho \propto r^9 \) for small \( r \)). The profiles close to \( r = 0 \) are derived from the asymptotic solution given in equation 7 and fitted to the numerical solution by suitable choice of \( P_c \).

The resulting profiles are given in figure 1 for various shock acceleration efficiencies (see also Tables 1 to 5 of Chevalier (1983a)). We define \( \phi \) as the ratio of the CR energy density \( U_{CR} \) to sum of the thermal \( U_t \) and CR energy densities: \( \phi = U_{CR}/(U_t + U_{CR}) \). The subscript \( s \) denotes the value at the shock. \( \epsilon \) and \( \phi_s \) are related by

\[ \phi_s = \frac{2\epsilon}{1+\epsilon} \; ; 
\quad \epsilon = \frac{\phi_s}{2 - \phi_s} \]

and the post-shock thermal and CR energy densities are

\[ U_{ts} = \frac{18(1 - \phi_s)\rho_0 u_s^2}{(8 - \phi_s)(2 - \phi_s)} \; ; 
\quad U_{CR,s} = \frac{18\phi_s \rho_0 u_s^2}{(8 - \phi_s)(2 - \phi_s)} \]

In figure 1 we see that the thermal energy density always decreases towards the centre of the blast wave. In contrast, for all cases plotted in figure 1 the CR energy density is greater at the centre than immediately downstream of the shock. The central part of the blast wave can be characterised as a CR bubble with low thermal energy density and low mass density. The radius of the CR bubble decreases as the CR fraction \( \phi_s \) decreases, but even when only 5 percent of the post-shock energy density is given to CR (\( \phi_s = 0.05 \)) the CR bubble extends out to 10-20 percent of the shock radius. Adiabatic expansion inside the blast wave acts as a filter which transfers thermal energy to CR energy.

Figure 2 plots the CR, thermal and kinetic total energies as a function of \( \phi_s \) (see also Table 6 of Chevalier (1983a)). It shows that as much as 70-80% of the total energy in the blast wave can be given to CR if CR acceleration at the shock is highly efficient. If equal energies are given to CR and thermal particles at the shock, CR contribute 53% of the total blast wave energy (thermal+CR+kinetic). Even if only 10% of the total CR plus thermal energy at the shock is given to CR, the CR energy in the blast wave is still 14% of the total. In the limit of small \( \phi_s \), the total energy of CR is \( E_{CR} = 1.5\phi_s E_0 = 3\epsilon E_0 \) where \( E_0 \) is the total energy of the blast wave. Far from reducing the efficiency of CR production, the hydrodynamics of the blast wave gives...
a proportion of the total blast wave energy to CR which is
greater than the fraction \( \phi_s \) of energy given to CR at the
shock. Instead of being a problem for CR production,
adiabatic expansion works to increase the fraction of the
supernova energy given to CR. When a SNR finally disperses
the CR energy released into the ISM may be a large fraction
of the energy of the original explosion.

\section{Approach to Self-Similarity}

\( R(r) \) is the radius of the shock front at the time when the
fluid element now at position \( r \) was overtaken by the shock.
As seen in figure 1, a fluid element presently located about
half way between the centre of the blast wave and the present
shock radius was overtaken by the shock when it was only
\( \sim 10\% \) of its present radius. Consequently the early non-
Sedov evolution of the blast wave affects a large part of
its interior. Self-similarity cannot be naively assumed even
when the blast wave has expanded to \( 10 \times \) or even \( 100 \times \)
the radius \( r_f \) at which it completed the ejecta-dominated phase
(sometimes known as the free expansion phase) and entered
the Sedov phase. We characterise \( r_f \) as the radius at which the
swept-up mass \( 4\pi \rho_0 r_f^3/3 \) is equal to the ejected mass
\( M_{ej} \). Figure 3 provides insight into the late-time effect of
the early pre-Sedov history of the blast wave. It plots \( R(r) \)
and the shock velocity \( u_s(r) \) defined as the velocity of the shock
at the time when the fluid element now at \( r \) was overtaken by
the shock. Curves are plotted for different CR acceleration
efficiencies: \( \phi_s = 0.1 \) and \( \phi_s = 0.5 \). \( u_s \) is plotted relative
to its present value at \( r = r_s \). For both values of \( \phi_s \) the figure
shows that a fluid element now at radius \( r_s / 2 \) passed through
the shock when the shock velocity was \( \sim 30 \times \) its present
value. For example if the present shock velocity is 100km s\(^{-1}\)
a fluid element at \( r_s / 2 \) would have passed through the
shock when its velocity was \( \sim 30000 \) km s\(^{-1}\). Fluid elements
close to the centre of the blast wave would have been shocked
at unrealistically high velocities. This casts doubt on the
realism of the Sedov solution for the inner parts of the blast
wave.

We examine the approach to Sedov self-similarity by
time-dependent Lagrangian hydrodynamic calculation of a
blast wave driven by a thin spherical shell with mass \( M_{ej} \)
initially expanding into a uniform medium of density \( \rho_0 \)
with velocity \( u_{ej} \). In reality the hydrodynamic structure of
the early ejecta-dominated phase is much more complicated
(Chevalier 1982, Truelove & McKee 1999). Ejecta are
launched with a range of velocities rather than a single vel-
ocity \( u_{ej} \) but the thin shell model provides guidance on the
validity of the Sedov model that is our concern here. The
solution converges to the self-similar Sedov solution when
the shock radius \( r_s \) is much greater than the radius \( r_f \).
The comparison is shown in figure 4 for \( \phi_s = 0.25 \) where the
profiles of the mass density \( \rho \), the CR energy density \( U_{cr} \),
and the thermal energy density \( U_t \) are plotted when the blast
wave has expanded to 10, 100 and 1000 times the radius
\( r_f \). The density profile is nearly unaffected by the pre-Sedov
history. The energy densities are nearly unaffected when the
blast wave has expanded by a factor of 1000 in radius, but
strongly affected when the blast wave has expanded by a
factor of 10. However, for all values of \( r_s / r_f \) in figure 4, the
CR energy density exceeds the thermal energy density in the
inner half (by radius) of the blast wave. Hence the conclu-
sion of section 2 still stands that adiabatic expansion acts to
filter energy into CR and the inner parts of the blast wave
are dominated by CR pressure. The total pressure (CR plus
thermal) at the centre of the blast wave is approximately in-
dependent of \( r_s / r_f \) in figure 4 since it is determined by the
need to drive the blast wave into the surrounding medium.

In passing we note that the agreement between the
curve for \( r_s / r_f = \infty \) and \( r_s / r_f = 1000 \) in figure 4 for all
except small radius where they would be expected to differ
evidence that both the self-similar and the thin shell calcu-
lations are reliable since the the curves were calculated with
different computer codes using different numerical methods.
4 THE MAXIMUM CR ENERGY INSIDE THE BLAST WAVE

In this section we derive the maximum CR energy as a function of radius. We will assume self-similarity in this section and then examine effects arising from the pre-Sedov history in section 5. A fluid element presently at radius $r$ passed through the shock when its radius was $R$. We assume that the CR accelerated by the shock followed a $T^{-2}$ energy spectrum up to a maximum CR energy $T_s(R)$ in eV. After adiabatic expansion, the CR spectrum is still proportional to $T^{-2}$, but the maximum CR energy at radius $r$ is reduced to

$$T_{\text{max}}(r) = T_s(R) \left(\frac{\rho}{\rho_s}\right)^{1/3}$$  \hspace{1cm} (11)

$T_s(R)$ is determined by the microphysics of CR acceleration and the CR-driven amplification of magnetic field in the shock precursor. We consider three different models (A, B & C) for $T_s(R)$ as follows.

Equation 7 depends on the assumption that CR diffusion is small. While diffusion has negligible effect on bulk CR properties such as the CR energy density, as discussed in section 2, it could be more important for CR with energy $T_{\text{max}}$ which have a relatively large Larmor radius. As discussed below the maximum CR energy in the centre of a SNR at the end of its life is about 10 TeV and these have a Larmor radius in a 10 $\mu$G magnetic field of 0.001 parsec which is very much less than the SNR radius during the Sedov phase. Consequently, CR diffusion inside old SNR can only be important if the interior magnetic field is very small, and even then CR would be unable to escape through the larger compressed interstellar magnetic field closer to the shock. At early times during the Sedov phase $T_{\text{max}}$ is larger but the magnetic field is also larger due to field amplification. Neglect of diffusion therefore seems reasonable, but the validity of the assumption might be tested with more complete calculations.

The dependence of $T_{\text{max}}(r)$ on $\rho(r)$ as given in equation 11 determines the maximum CR energy inside the blast wave for a given maximum CR energy $T_s(R)$ at the shock. Bell et al. (2013) showed that $T_s(R)$ in the early evolution of an SNR is determined by the growth rate of the instability that amplifies the magnetic field. Model C below is based on this understanding, but firstly for comparison we consider two other models, A and B, based on simpler ways of estimating $T_s(R)$ at the shock. Model A neglects magnetic field amplification during acceleration and assumes Bohm diffusion. Magnetic field amplification is well attested by observation as well as theory so Model B includes magnetic field amplification but still assumes Bohm diffusion. Model C both includes magnetic field amplification and avoids assuming Bohm diffusion.

**Model A:** Firstly we consider the option that $T_s = u_s B_0 R_s / 8$ which is derived from Lagage & Cesarsky (1983a,b) where $B_0$ is the upstream magnetic field (ie no magnetic field amplification ahead of the shock). This expression for $T_s$ is based on Bohm diffusion (defined here as $D_{\text{Bohm}} = r_\psi c$ where $r_\psi$ is the CR Larmor radius) in a magnetic field $B_0$ during shock acceleration. The factor 1/8 assumes that CR spend equal times upstream and downstream during acceleration (Bell 2013). Apart from the factor 1/8 this is also the maximum CR energy derived by Hillas (1984) for generalised CR acceleration. For self-similar expansion $u_s \propto r_s^{-1/2}$. Here and throughout the rest of the paper, for a SNR approaching the end of its life, we assume the following standard values:

$$B_0 = 5 \mu G \quad u_s = 30 \text{ km s}^{-1} \quad r_s = 100 \text{ pc}$$

The maximum CR energy at a radius $r$ inside the blast wave is then

$$T_{\text{max}}(r) = 5 \left(\frac{R}{r_s}\right)^{-1/2} \left(\frac{\rho}{\rho_s}\right)^{1/3} \text{ TeV}$$  \hspace{1cm} (12)

as plotted in figure 5, where $R$ and $\rho$ are functions of $r$. The curves labelled ‘model A’ in figure 5 show that the maximum CR energy falls away slowly inside the blast wave, but remains greater than 1 TeV until very close to the centre. With this recipe for the magnetic field, CR released into the ISM when a SNR reaches the end of its life can only replenish the Galactic CR population up to energies of a few TeV.

**Model B:** As pointed out by Lagage & Cesarsky (1983a,b) CR cannot be accelerated to PeV energies if the magnetic field at the shock is limited to interstellar values of a few $\mu$G. Magnetic field amplification (Bell 2004) facilitates CR acceleration to PeV energies. Option C will apply the latest theories of magnetic field amplification, but before that we consider the case in which $T_s = u_s B_0 R_s / 8$ and the pre-shock magnetic field is amplified such that the magnetic energy density at the shock is a fixed fraction of the available energy, $B^2 / 2 \mu_0 = \xi \rho v^2_s$. Völk et al (2005) suggest from observations that the downstream magnetic energy density is $\sim 3\%$ of $\rho v^2_s$ implying $\xi \sim 0.003$ (depending on the magnetic field orientation) when allowance is made for magnetic field compression at the shock (increasing $B^2$ by $\sim 10$) when estimating the upstream magnetic field. In this case, the upstream magnetic field is the maximum of the amplified field and a typical ISM field of 5 $\mu$G

$$\frac{B}{\mu G} = \max \left[5, \ 1.2 \left(\frac{\xi}{0.003}\right)^{1/2} \left(\frac{n_e}{0.01 \text{ cm}^{-3}}\right)^{1/2} \left(\frac{u_s}{30 \text{ km s}^{-1}}\right)\right]$$  \hspace{1cm} (13)
forces exerted by the CR current on the thermal plasma exceed the magnetic force \(-B \times (\nabla \times B)/\mu_0\) acting within the thermal plasma. At shock velocities less than \(\sim 1,000\ \text{km s}^{-1}\) the NRH instability is inactive (Schure \& Bell 2013) and magnetic fluctuations are excited by the resonant Alfvén instability (Lerche 1967, Kulsrud \& Pearce 1969, Wentzel 1974) that generates Alfvén waves with a wavelength \(2\pi/k\) matching the CR Larmor radius \(r_s\). The Alfvén instability operates differently from the NRH instability and dominates in a different regime but its maximum growth rate is a numerical factor times \(0.5j_{CR}\sqrt{\mu_0/\rho}\). The numerical factor is close to one as noted by Zirakashvili \& Ptuskin (2008) but depends upon the form of the CR energy distribution (see Appendix B). Hence the argument based on the NRH instability (Bell et al 2013) for high velocity shocks also applies to the Alfvén instability at low velocity shocks, and equation 15 can be applied to SNR throughout the Sedov phase. The corresponding profiles of \(T_{\text{max}}\) inside the blast wave are plotted as the curves labelled ‘model C’ in figure 5. \(T_{\text{max}}\) at a radius \(r\) is calculated from equation 15 with \(R_\text{pc}\) and \(u_7\) set to the shock radius and shock velocity when the fluid element at \(r\) was overtaken by the shock.

The results obtained with models A, B \& C are discussed further in the next two sections.

5 \(T_{\text{MAX}}\) NEAR THE CENTRE

According to figure 5 the maximum CR energy \(T_{\text{max}}\) is unbounded at zero radius in models B and C. This is an artifact due to the projection of self-similar Sedov expansion back to zero SNR radius. In the pre-Sedov ejecta-dominated phase, the shock velocity is much lower than that given by the Sedov model in which the expansion velocity is infinite at \(t = 0\).

In section 3 (see figure 4) the effects of initial ejecta-dominated were estimated using a time dependent model in which the shock was driven by a thin shell representing the ejected mass \(M_{ej}\). The same thin-shell model can be used to estimate \(T_{\text{max}}\) near the centre of the blast wave where the history of the ejecta-dominated phase is important. The shock velocity is nearly constant during the ejecta-dominated pre-Sedov phase and the radius is small initially so \(T_{\text{max}}\) turns over on approaching the centre of the blast wave as plotted in figure 6 in accord with equation 15. Nevertheless, \(T_{\text{max}}\) at the centre of the blast wave can be \(\sim 100\)–\(1,000\) times larger than \(T_{\text{max}}\) at the shock. In old SNR \((r_s/r_f = 100 \sim 1,000)\) CR energies may reach \(\sim 10 \sim 100\) TeV in the centre of an SNR even though CR are currently accelerated only to \(\sim 100\) GeV at the shock. Early in the Sedov phase \((r_s/r_f \sim 10)\), \(T_{\text{max}}\) at the centre of the blast wave is only \(\sim 10\) times larger than \(T_{\text{max}}\) at the shock.

6 THE LIMITATIONS OF MODELS A AND B

Models A and B predict larger CR energies than model C in the outer parts of the blast wave because they incorrectly assume Bohm diffusion in old SNR when the Alfvén instability is weakly driven. Bohm diffusion occurs when CR trajectories are scattered with a mean free path equal to the CR Larmor radius. This is only possible if rapidly growing
plasma instabilities produce large fluctuations in the field on the scale of a Larmor radius. If the magnetic field remains essentially uniform on the Larmor scale the CR are unscattered and diffusive shock acceleration is too slow for CR to reach the energy \( E > T_{\text{rms}} \), which is the maximum value of \( T_{\text{rms}} \) at the shock. The Sedov self-similar result is given by \( r_s/r_f = \infty \).

Model B also overestimates the maximum CR energy in the centre of the blast wave. Model B assumes that Bohm diffusion applies and that the Bohm diffusion coefficient should be calculated from the total magnetic field. In reality, fluctuations in the magnetic field grow on a wide range of scales from the Larmor radius of GeV protons to the Larmor radius of the highest energy CR. Bohm diffusion depends on a match between the Larmor radius of the scattered CR with the scalelength of the magnetic field. Only components of the magnetic field structured on the scale of the CR Larmor radius are effective in scattering a particular CR. The magnetic field derived from x-ray synchrotron emission at the shock (Berezhko et al 2003, Vink & Lamming 2005) is the total magnetic field. The component of the magnetic field on the Larmor radius of a particular CR is smaller. Model B uses the observed magnetic field as calculated by Völk et al (2005) to predict \( T_{\text{max}} \) and therefore model B overestimates the maximum CR energy. Compensation for this effect would probably reduce \( T_{\text{max}} \) in agreement with model C.

### 7 THE CR ENERGY SPECTRUM

The maximum CR energy \( T_{\text{max}} \) is plotted in figures 5 and 6 for different models as a function of radius \( r \). Working on the basis that the energy spectrum at any radius follows a \( T^{-2} \) power up to the local maximum CR energy \( T_{\text{max}}(r) \) we integrate in radius to calculate the differential energy spectrum of the total CR population inside the blast wave. The full lines in figure 7 present the CR spectrum calculated for model C for two different CR acceleration efficiencies, \( \phi_r = 0.1 \) and 0.5. The CR energy \( T \) is normalised to \( T_{\text{rms}} \), which is the current value of \( T_{\text{rms}} \) at the shock. The spectrum is proportional to \( T^{-2} \) for \( T < T_{\text{rms}} \) since this power law applies at all points inside the blast wave in this energy range. The local maximum CR energy \( T_{\text{max}} \) increases towards the centre of the remnant so CR reach the highest energies only in a small volume close to the centre. Consequently the spectrum is steeper for \( T > T_{\text{rms}} \) but still follows a power law. The spectral index of 2.6 for \( T > T_{\text{rms}} \) is close to that of Galactic CR up to the knee, but this must be coincidental since the spectrum of CR arriving at the Earth is expected to be steepened by energy-dependent losses during propagation from the source. The shape of the spectrum is nearly independent of \( \phi_r \), but slightly flatter for \( \phi_r = 0.1 \).

The self-similar spectrum calculated for model C extends without limit towards infinite CR energy, representing CR acceleration by an infinitely fast Sedov blast wave expanding from a central singularity. The dashed curves in figure 7 plot the spectrum calculated with the time-dependent code that models the pre-Sedov phase as described in sections 3 and 5. This more realistic thin-shell model of early expansion causes the CR spectrum to terminate instead of extend to infinite energy. The radius of a SNR expands by about 50 during the Sedov phase (\( r_s/r_f \sim 50 \)) in which case the spectrum terminates at \( \sim 200T_{\text{rms}}^4 \) at the end of the Sedov phase. In other words, towards the end of the Sedov phase, CR near the centre of the SNR reach energies which are about 200 times larger than the maximum CR energy at the shock.

The total CR spectrum inside the SNR has two important energies: (i) the energy \( T_{\text{rms}} \) which is the maximum CR energy at the shock and at which the spectral index steepens from 2.0 to 2.6, (ii) \( T_{\text{max}} \), which is the maximum CR energy anywhere in the SNR and the energy at which the spectrum terminates.

From equation 15, \( T_{\text{max},s} = 230\rho_e^{1/2}u_2^{-3/2}R_p \), where \( R_p \) and \( u_2 \) are the shock radius and the shock velocity. The total energy of a Sedov blast wave is \( E = 3\rho_0u_2^2R_p^3 \) for \( \gamma = 5/3 \), which is the case of negligible CR pressure, so this formula can be re-cast as \( T_{\text{max},s} = 400\rho_0^{1/6}B_{5}^{3/4}E_{44}^{-1/2} \). This is the blast wave energy in units of 10^{44}J. CR are only confined at the shock if the shock velocity is greater than the Alfvén speed \( v_A \), since the resonant Alfvén instability is only excited by CR drifting faster than the Alfvén speed. 

\[
T_{\text{max},s} = 400\rho_0^{1/6}B_{5}^{3/4}E_{44}^{-1/2} \]

Under these assumptions, and with our standard parameters, CR are released into the ISM from the interior of the SNR at the end of its life with a power law spectrum \( T^{-2} \) at energies less than 200 GeV. At energies above 200 GeV the spectrum is steeper inside the blast wave and proportional to \( T^{-2.6} \). The estimate of 200 GeV as the maximum energy to which CR are accelerated at the end of the SNR lifetime will be reduced if collisional damping in a dense partially ionised plasma inhibits the growth of CR-driven Alfvén waves as may be the case for the middle-aged SNR
adiabatic processes increase the overall efficiency of CR blast wave and accelerate a new generation of CR. In fact, losses due to adiabatic expansion are stronger for the thermal plasma ($\gamma = 5/3$) than for CR ($\gamma = 4/3$). As shown in figures 1 & 2, most of the energy in the blast wave can be given to CR. The blast wave acts as a filter to accumulate CR which are then released into the ISM as the SNR eventually dissipates.

Adiabatic expansion operates to increase the total SNR energy passed to CR but it works against the production of CR with high energies reaching the knee in the spectrum. As estimated in section 7, the maximum CR energy $T_{\text{max},t}$ inside the SNR at the end of of its life is of the order of 20 TeV. SNR in the late Sedov phase may efficiently produce the Galactic CR population up to the maximum energy $T_{\text{max},s}$ of CR being accelerated by the shock at the end of the SNR’s life. As shown in figure 7, the CR spectrum inside the SNR steepens at this point before terminating at $T_{\text{max},t}$. It was shown by Bell et al (2013) and Schure & Bell (2013) that CR above 200 GeV can instead be produced efficiently by young SNR, but these are released into the Galaxy by escaping upstream without passing into the interior of the SNR. They are the highest energy CR being accelerated by the shock at any time by the expanding SNR. They have long scattering mean free paths and carry the electrical current needed to excite instabilities upstream of the shock.

CR accelerated by SNR can therefore be divided into two populations. A high energy population (population A), extending from $\sim 200$ GeV to $\sim 1$ PeV, escapes upstream with a $T^{-2}$ energy spectrum when averaged over the Sedov phase. A low energy population (population B), with a $T^{-2.6}$ energy spectrum below $\sim 200$ GeV and $T^{-2.8}$ between $\sim 200$ GeV and $\sim 20$ TeV, is released into the ISM by old SNR after residing inside the remnant between acceleration and release. Although both populations contribute Galactic CR between $\sim 200$ GeV and $\sim 20$ TeV, population A increasingly dominates toward the higher end of this range because of its flatter spectrum.

The production of the two CR populations is strongly related and they both have the same spectral index under the assumption that shock acceleration produces a $T^{-2}$ spectrum. However their history between acceleration and release into the ISM is different so they may not connect seamlessly at the cross-over energy at $\sim 200$ GeV. We assess the connectivity of the two populations by comparing the energy released into the ISM in each population.

Initially we compare the energy of each population in the limit of low acceleration efficiency in which $\epsilon$ is small. From section 2 and figure 2 the energy released in low energy CR, population B, is

$$E_B \approx 3 \epsilon E_0$$

for small $\epsilon$ where $E_0$ is the total blast wave energy.

The energy released into the ISM as population A can be estimated from equations 2-4 from Bell et al (2013) in which CR escape ahead of the shock at energy $T_{\text{max}}$ with electric current $j_{CR}$ and consequent energy flux $j_{CR} T_{\text{max}}$. From these equations, the rate of CR energy escape from unit surface area of the shock is $0.75 P_{CR} u_s / \log(e T_{\text{max}}/m_p c^2)$ where $P_{CR}$ is the CR pressure at the shock, $u_s$ is the shock velocity and $e T_{\text{max}}/m_p c^2$ is the Lorentz factor of escaping CR protons. The total energy $E_A$ released into the ISM with population A can be estimated by integrating over CR released as the SNR expands from the

8 GALACTIC CR

In this paper we have shown that adiabatic losses do not reduce the total CR energy released into the ISM. Any energy lost by CR due to adiabatic expansion is used to drive the blast wave and accelerate a new generation of CR. In fact, adiabatic processes increase the overall efficiency of CR production.
radius $R_s$ at the beginning of the Sedov phase to a radius $R_f$ when CR are released into the ISM, giving

$$E_A = \int_{R_f}^{R_s} \frac{0.75\rho_{CR}u_s}{4\pi R^2 \log(eT_{max}/m_p c^2)} dt$$

$$\approx \frac{3\pi}{4} \frac{\log(R_s/R_f)}{\log(eT_{max}/m_p c^2)} \epsilon E_0$$  \hspace{0.5cm} (18)$$

$eT_{max}/m_p c^2 = 10^6$ for acceleration to 1PeV at the beginning of the Sedov phase, and $R_s = 50R_f$ for deceleration from 10,000 km s$^{-1}$ to 30 km s$^{-1}$ during the Sedov phase in which $R_s \propto u_s^{2/3}$, giving

$$E_A \approx 0.7\epsilon E_0$$  \hspace{0.5cm} (19)$$

where $E_0 = 3\rho_0u_s^2R_s^3$. These estimates (equations 17 & 19) gloss over a number of complicating factors, but they are sufficient to suggest that the energies $E_A$ and $E_B$ in each population are comparable except that the energy in the higher energy population A is probably $\sim 0.25$ times that in population B. Hence the connection at around 200GeV can be expected to be reasonably smooth. If we take our estimates of $E_A$ and $E_B$ at face value, the spectrum at source below 200GeV is proportional to $T^{-2}$. At energies a little above 200GeV the spectrum steepens to $T^{-2.6}$ as seen in figure 7 before flattening again to $T^{-2}$ as population A begins to dominate. Of course, the spectrum of CR arriving at the Earth is steepened due to energy losses during propagation. Also, the spectrum at source may deviate from a $T^{-2}$ spectrum as discussed for example by Bell et al (2011).

The formula $E_B = 3\epsilon E_0$ is correct for small $\epsilon$ and $\phi_A$. If CR acceleration is more efficient and $\phi_A = 0.5$ then the formula overestimates $E_B$ by a factor of 2 (see figure 2). $E_B$ and $E_A$ are then closer in value but still $E_B > E_A$ and the overall picture of the Galactic CR spectrum is more or less unchanged.

Adriani et al (2011) find evidence in PAMELA data for a flattening in the Galactic CR spectral index above 200GeV. The detailed spectral structure observed at 200GeV might be open to question in the light of AMS data (Ting et al 2013), but the change in index is supported by other data (Ahn et al 2010, Tomassetti 2012). Our model suggests that the structure at 200GeV might be due to the joining of population A and population B. If anything we predict a local steepening above 200GeV rather than a flattening of the spectrum, and our prediction of the join occurring at 200GeV is uncertain easily by a factor of 2. However, it appears very likely that Galactic CR at PeV and GeV energies must be done at very different stages of SNR evolution and escaped into the interstellar medium by different routes at different times. More detailed modelling and observation is needed to establish whether the measured structure in the Galactic CR spectrum can be explained by our model or whether the answer lies in energy-dependent CR propagation from the SNR to the Earth as proposed for example by Blasi et al (2012) or Tomassetti (2012) or in spectral concavity due to non-linear effects as proposed by Ptuskin et al (2013).

9 OBSERVATIONAL CONSEQUENCES

Finally we briefly note some observational consequences for SNR and other blast waves. Our analysis predicts the existence of CR bubbles at the centres of older SNR as shown by Chevalier (1983a). These bubbles may extend 10s of parsec and extend half-way to the outer edge of the SNR. Inside the bubble, the CR energy density exceeds the thermal energy density, and the maximum CR energy $T_{max}$ exceeds that of CR close to the shock. Despite the large CR energy density in the interior, CR protons are not strong emitters of $\gamma$-rays because of the low interior mass density and the consequent lack of thermal protons as targets for proton-proton interactions (see figure 1). CR electrons may be more detectable in the interior especially if they interact with a uniform photon density to emit inverse Compton radiation. The radio synchrotron luminosity in the interior is uncertain since it depends on the unknown magnitude of the magnetic field. The interior magnetic field is strongly reduced by adiabatic expansion but magnetic field amplification at the shock prior to expansion may compensate for this. Given the low predicted $\gamma$-ray emission by protons and the uncertainties in the radio emission, inverse Compton emission from CR electrons appears to be the most accessible signature of the presence of a CR bubble inside a blast wave.

The discussion presented in this paper may be applied to blast waves launched by any rapid energy release such as may occur in the centre of the Galaxy or any other galaxy, leading possibly to the formation of the Galactic Fermi bubbles (Carretti et al 2013) or the SNR-like shocks observed in Centaurus A (Croston et al 2009).

10 CONCLUSIONS

Our principal conclusions are that:

- SNR in the Sedov phase contain a CR bubble at their centre that extends to a quarter or a half of the SNR radius as previously shown by Chevalier (1983a).
- Adiabatic expansion serves to increase rather than decrease the efficiency of Sedov-phase SNR as producers of Galactic CR.
- Galactic CR can be divided into two populations: (A) CR at higher energies that escape upstream of the shock into the ISM as part of the acceleration process as discussed by Bell et al (2013), (B) CR with energies up to about 200GeV constituting CR bubbles that are released into the Galaxy at the end of the SNR’s life.
- The intersection of the two populations may tentatively be identified with the change in spectral index detected by Adriani et al (2011) at $\sim 200GeV$.
- The CR electrons in the bubble may be detected by inverse Compton $\gamma$-rays but CR protons may be relatively undetectable due to the low mass density in the centre of blast wave. Synchrotron radio emission depends upon the magnetic energy density at the centre of a blast wave.
- The above discussion may be applicable to blast waves originating from the centres of our Galaxy or other galaxies.
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APPENDIX A: THE SOLUTION PROFILES CLOSE TO ZERO RADIUS

In this appendix we derive the asymptotic profiles with respect to radius. Multiplying equation 3 by $R^3/r^3$ and differentiation with respect to radius gives

$$R^3 \frac{\partial P}{\partial r} + \frac{3P}{P_0} = \left( \frac{5}{3} (1 - \epsilon) \left( \frac{\rho}{\rho_s} \right)^{3/5} + \frac{4}{3} \left( \frac{\rho}{\rho_s} \right)^{1/3} \right) \frac{1}{\rho_s} \frac{\partial \rho}{\partial r} \quad (A1)$$

After rearrangement,

$$\left( 3 \frac{P_0}{P} \right) \frac{P}{\rho^2} \frac{\partial P}{\partial r} \frac{dr}{r^3} = \frac{P_0}{\rho_0} \left( \frac{5}{3} (1 - \epsilon) \left( \frac{\rho}{\rho_s} \right)^{-1/3} + \frac{4}{3} \left( \frac{\rho}{\rho_s} \right)^{-2/3} \right) \frac{\partial \rho}{\partial r} \quad (A2)$$

From equation 5 $\partial P/\partial r \to 0$ as $r \to 0$ since $u \to 0$ as $r \to 0$ and $\partial u/\partial r$ is finite. The pressure must be non-zero at $r = 0$ since the motions are sub-sonic at the centre of the blast wave ($u \to 0$). Hence the term including $\partial P/\partial r$ can be neglected in equation A2. We define $P_c$ as the pressure at $r = 0$ and integrate equation A2 with respect to $r$ to obtain

$$\frac{5}{2} (1 - \epsilon) \left( \frac{\rho}{\rho_s} \right)^{2/3} + 4 \epsilon \left( \frac{\rho}{\rho_s} \right)^{1/3} - \rho_0 \rho \frac{r^3}{P_0 r^3} = 0 \quad (A3)$$

This quadratic in $\rho/\rho_s$ can be solved to obtain an expression for $\rho$ which is reproduced in equation 7.

$$\frac{\rho}{\rho_s} = \left( \frac{1}{5(1 - \epsilon)} \left[ \left( 16 \epsilon^2 + 10(1 - \epsilon) \rho_0 P_0 r^3 \right)^{1/2} - 4 \epsilon \right] \right)^3 \quad (A4)$$

At the centre ($r \to 0$) $\rho \propto r^9$ unless $\epsilon = 0$ in which case $\rho \propto r^{9/2}$. The strong dependence of $\rho$ on $r$ strengthens the assertion above from equation 5 that $\partial P/\partial r$ can be neglected in equation A2.

APPENDIX B: THE MAXIMUM CR ENERGY AT A SHOCK

Equation 15 for the maximum CR energy at a shock was derived by Bell et al (2013) on the basis of a sufficient electric current must escape upstream of the shock to amplify the magnetic field through the growth of the NRH instability by about 5 e-foldings at its maximum growth rate. This determines the energy of the escaping CR since for a given CR energy flux set to a fixed fraction of $\nu_0^2$ the electric current is too small if the energy of CR carrying the current is very large. Conversely, if the energy of escaping CR is too low the CR current is large and the instability grows so rapidly that the magnetic field is strongly amplified and

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the CR are unable to escape upstream. For more details of the model see Bell et al (2013).

The argument of Bell et al (2013) and the derivation of equation 15 for the maximum CR energy were based on the assumption that the NRH instability is active and dominant. This is true for young SNR with high shock velocities, but the NRH instability is inactive for SNR in the late Sedov phase. When the CR current drops below a characteristic value $j_c = B/\langle \mu_0 \rho \rangle$ the $j_{CR} \times B$ force is too weak to overcome the tension in the magnetic field and the NRH instability ceases to operate. For a magnetic field of $5\mu G$ and an electron density of $1\text{cm}^{-3}$, $j_{CR}$ drops below the crossover value $j_c$ when the shock velocity falls below $1,000\text{km s}^{-1}$ At shock velocities below this the Alfven instability (Lerche 1967, Kulsrud & Pearce 1969, Wentzel 1974) driven by CR streaming dominates. The Alfven instability causes the growth of Alfven waves in spatial resonance with the CR Larmor radius. Because Alfven waves are natural modes of the system they are undamped by tension in the magnetic field and can grow even if the growth rate drops below the natural frequency of the wave. The NRH and Alfven instabilities drive modes with opposite circular polarisations. In this appendix we set out the derivation of the maximum growth rates of both the Alfven and NRH instabilities for monoenergetic streaming CR using the formalism of Bell (2004), showing that the maximum growth rates for each instability are given by very similar expressions, differing only by 10 percent. The similarity of the two growth rates was previously noted by Zirakashvili & Ptuskin (2008).

Because of the similar growth rates the estimate of the maximum CR energy based on instability growth rates derived by Bell et al (2013) for the NRH instability at high shock velocities also applies to the Alfven instability at low shock velocities. Crucially for this paper, equation 15 can be applied to SNR throughout the Sedov phase.

The dispersion relation for CR-driven instability can be found in equation 7 of Bell (2004) (see also Achterberg 1983). It includes both the Alfven and NRH instabilities. The dispersion relation is

$$\omega^2 = k^2 v_A^2 + (1 - \sigma) \frac{k j_{CR} B_{||}}{\rho} \quad (B1)$$

where $k$, $j_{CR}$ (named $j_c$ in Bell (2004)) and $B_{||}$ are the wavenumber, CR electric current and magnetic field respectively, each aligned parallel to the shock normal. $v_A$ is the Alfven speed. A small term in $\omega/k\nu_A$ has been omitted from equation 7 of Bell (2004) as justified therein. The function $\sigma$ describes the response of the streaming CR to perturbations in the magnetic field. For monoenergetic CR with a Larmor radius $r_g$ propagating diffusively relative to the background plasma, and $\lambda = 1/k r_g$,

$$\sigma = \frac{3}{4} \lambda(1 - \lambda^2) \left[ \ln \left( \frac{1+\lambda}{1-\lambda} \right) + \frac{3}{2} \lambda^2 \right] \quad (B2)$$

for long wavelengths, $k < r^{-1}_g$, $\lambda > 1$, and

$$\sigma = \frac{3}{4} \lambda(1 - \lambda^2) \left[ \ln \left( \frac{1+\lambda}{1-\lambda} \right) + i\pi \right] + \frac{3}{2} \lambda^2 \quad (B3)$$

for short wavelengths, $k > r^{-1}_g$, $\lambda < 1$. The imaginary term $(i\pi)$ at short wavelengths results from the spatial resonance with the CR Larmor radius. No such resonance occurs at wavelengths longer than the CR Larmor radius, which accounts for the absence of the imaginary term for $kr_g < 1$.

When the CR current $j_{CR}$ is small and magnetic perturbations grow by the Alfven instability, $k^2 v_A^2$ dominates the real part of the right hand side of equation B1. In this limit, and with $kr_g > 1$,

$$\omega^2 = k^2 v_A^2 - \frac{3\pi i}{4} \left( \frac{k^2 r_g^2 - 1}{k^2 r_g^2} \right) \frac{k j_{CR} B_{||}}{\rho} \quad (B4)$$

In the limit of small $j_{CR}$, the maximum growth rate is

$$\gamma_{\text{max}} = \frac{\pi}{4} \sqrt{\frac{\mu_0}{3\rho}} j_{CR} = 0.45 \sqrt{\frac{\mu_0}{\rho}} j_{CR} \quad (B5)$$

which occurs when $kr_g = \sqrt{3}$.

In contrast, when the CR current $j_{CR}$ is large and magnetic perturbations grow by the NRH instability, the maximum growth rate occurs at wavelengths much shorter than the CR Larmor radius ($\lambda \ll 1$). Both the real and imaginary parts of $\sigma \ll 1$ can then be neglected giving

$$\omega^2 = k^2 v_A^2 + \frac{k j_{CR} B_{||}}{\rho} \quad (B6)$$

In the appropriate polarisation, $k j_{CR} B_{||} < 0$,

$$\omega = \pm i \left( \frac{|k j_{CR} B_{||}|}{\rho} - k^2 v_A^2 \right)^{1/2} \quad (B7)$$

and the maximum growth rate is

$$\gamma_{\text{max}} = 0.5 \sqrt{\frac{\mu_0}{\rho}} j_{CR} \quad (B8)$$

which occurs when $|k| = 0.5 \mu_0 j_{CR} B_{||}$.

Although the NRH and Alfven instabilities operate in different ways and in different polarisations, equations B5 and B8 show that the maximum growth is very similar in the low $j_{CR}$ Alfven limit and the high $j_{CR}$ NRH limit. The maximum growth rate for $j_{CR}$ across the range from the
Alfven to the NRH limit is plotted in figure 8. To a good approximation the maximum growth rate can be assumed to be $0.5\sqrt{\mu_0/\rho} j_{CR}$ across the whole range of $j_{CR}$. Consequently, equation 15 provides a good estimate of the maximum CR energy at a shock at all times during the Sedov phase of SNR expansion.

The discussion in this appendix has treated the CR distribution as monoenergetic. This is reasonable for escaping CR which have to reach a certain energy before they escape and are not accelerated beyond this energy. Bell (2004) derives the dispersion relation for a $T^{-2}$ CR distribution and similar results can be obtained from the plots of the real and imaginary parts of $\sigma$ in figure 1 of that paper.