The slice rank of a direct sum

Abstract

We show that the slice rank of the direct sum of two tensors is equal to the sum of their slice ranks. This result generalizes the fact, shown by Tao, that the slice rank of a diagonal tensor is equal to the number of non-zero entries of that tensor. The proof uses the duality method of Sawin and Tao in a straightforward way.

1 Introduction

By a $d$-tensor over a field $\mathbb{F}$, we shall mean a function of the form $T : X_1 \times \cdots \times X_d \to \mathbb{F}$, where $X_1, \ldots, X_d$ are finite sets. When $d = 2$, we can think of $T$ as an $|X_1| \times |X_2|$ matrix, and an important invariant associated with it is its rank. It is natural to try to generalize the notion of rank to higher-order tensors, but it turns out that there are several competing generalizations, each with different advantages and disadvantages, more than one of which is genuinely useful.

If $u_i : X_i \to \mathbb{F}$ for $i = 1, \ldots, d$, write $u_1 \otimes \cdots \otimes u_d$ for the tensor $T$ given by

$$T(x_1, \ldots, x_d) = u_1(x_1)u_2(x_2)\cdots u_d(x_d).$$

Tensors of this form are said to have tensor rank equal to 1. Then the tensor rank of $T$ is the smallest $r$ such $T$ is a sum of $r$ tensors of tensor rank 1. Note that when $d = 2$ this definition is one of the standard ways of defining the rank of a matrix.

A second definition of rank can be obtained by changing what we count as a rank-1 tensor. Let us say that a tensor has partition rank 1 if there is a partition of $\{1, \ldots, d\}$ into non-empty sets $S_1$ and $S_2$ and $T$ splits up as a product $T = T_1T_2$, where each $T_i$ depends only on the variables $x_j$ such that $j \in S_i$. Note that for $d \geq 2$ a tensor of tensor rank 1 has partition rank 1 and that any partition of $\{1, \ldots, d\}$ into two disjoint sets can be used. In general, the partition rank of a tensor $T$ is the smallest $r$ such that $T$ is a sum of $r$ tensors of partition rank 1.

An intermediate definition is that of slice rank. Here, the tensors of rank 1 are defined as for partition rank except that we insist that $S_1$ is a singleton. So for instance if $d = 4$, then a tensor of the form $u(x_1, x_2)v(x_3, x_4)$ has partition rank 1 but does not necessarily have slice rank 1, whereas a tensor of the form $u(x_3)v(x_1, x_2, x_4)$ has slice rank 1 and partition rank 1. As one would expect, the slice rank of a tensor $T$ is the smallest $r$ such that $T$ is a sum of $r$ tensors of slice rank 1.

Since a tensor of tensor rank 1 has slice rank 1 and a tensor of slice rank 1 has partition rank 1, we find that the tensor rank is at least as big as the slice rank, which is at least as big as the partition rank.
In a remarkable and very quick sequence of developments in 2016, Croot, Lev and Pach proved that subsets of $\mathbb{Z}_2^n$ that do not contain an arithmetic progression of length 3 have exponentially small density \[2\], and then Ellenberg and Gijswijt proved the same for subsets of $\mathbb{F}_3^n$, thereby solving the famous cap-set problem in additive combinatorics \[3\]. Soon after that, Tao gave a more conceptual reformulation of the argument \[9\], in which the following lemma (in the case $d = 3$) played a crucial role.

**Lemma 1** (Tao). Let $T : X^d \to \mathbb{F}$ be a $d$-tensor and suppose that $T(x_1, \ldots, x_d) = 0$ except if $x_1 = x_2 = \cdots = x_d$. Then the slice rank of $T$ is equal to the number of non-zero entries of $T$.

We briefly sketch his proof in the case $d = 3$. Suppose that one has a decomposition

$$T(x, y, z) = \sum_{i=1}^{r} a_i(x)b_i(y, z) + \sum_{j=1}^{s} c_j(y)d_j(x, z) + \sum_{k=1}^{t} e_k(z)f_k(x, y).$$

Then a simple linear algebra argument shows that there is a function $h : X \to \mathbb{F}$ such that $\sum_x h(x)a_i(x) = 0$ for $i = 1, \ldots, r$ and such that $h(x) = 0$ for at most $r$ values of $x$. Take such an $h$ and consider the matrix $M(y, z) = \sum_x h(x)T(x, y, z)$. Then $M$ is diagonal, and $M(y, y) = h(y)T(y, y, y)$. If the number of non-zero entries of $T$ is $m$, then the number of non-zero entries of $M$ is at least $m - r$, so $M$ has rank at least $m - r$.

On the other hand, $M$ has a decomposition

$$M(y, z) = \sum_{j=1}^{s} c_j(y)u_j(z) + \sum_{k=1}^{t} v_k(y)e_k(z),$$

where $u_j(z) = \sum_x h(x)d_j(x, z)$ and $v_k(y) = \sum_x h(x)f_k(x, y)$ for each $j, k$. It follows that $M$ has rank at most $s + t$.

Putting these two estimates together, we deduce that $m - r \leq s + t$. Since the initial decomposition of $T$ was arbitrary, this proves that the slice rank of $T$ is at least $m$, as we wanted.

In this paper, we shall prove the following result. Suppose we have finite sets $X_1, \ldots, X_d$ and for each $i$ let $X_i = X_i^1 \cup X_i^2$, where this is a disjoint union. Given two tensors $T_i : X_i^1 \times \cdots \times X_i^1 \to \mathbb{F}$, $i = 1, 2$, their direct sum $T_1 \oplus T_2$ is the tensor that takes the value $T_1(x_1, \ldots, x_d)$ if $x_i \in X_i^1$ for each $i$, $T_2(x_1, \ldots, x_d)$ if $x_i \in X_i^2$ for each $i$, and 0 otherwise.

Let us write $\sigma(T)$ for the slice rank of $T$.

**Theorem 2.** For any two tensors, we have $\sigma(T_1 \oplus T_2) = \sigma(T_1) + \sigma(T_2)$.

Note that this immediately implies that $\sigma(T_1 \oplus \cdots \oplus T_m) = \sigma(T_1) + \cdots + \sigma(T_m)$ (where the definition of $T_1 \oplus \cdots \oplus T_m$ is obvious), and hence Tao’s lemma, which is the special case where each $T_i$ is a $1 \times \cdots \times 1$ tensor.

To prove the theorem, it is tempting to try to modify Tao’s argument, but the following example, with $d = 3$, seems to indicate that that cannot be done straightforwardly.

**Example.** Let $\epsilon$ be the $3 \times 3 \times 3$ Levi-Civita symbol. That is, it is defined on $\{1, 2, 3\}^3$, and $\epsilon(x, y, z) = 0$ if any two of $x, y, z$ are equal, and otherwise $\epsilon(x, y, z) = 1$ if $(x, y, z)$ is an even permutation of $(1, 2, 3)$ and $-1$ if it is an
odd permutation. (It would more normally be written $\epsilon_{ijk}$, but we write it $\epsilon(x,y,z)$ for consistency with our earlier notation.) This tensor is supported on an antichain, meaning that if $x \leq x', y \leq y', z \leq z'$, and both $(x, y, z)$ and $(x', y', z')$ belong to the support, then $(x, y, z) = (x', y', z')$. If we define a slice to be a subset of $\{1, 2, 3\}^3$ defined by holding one of the coordinates constant, then the number of slices needed to cover the support of $\epsilon$ is 3, since each slice contains two points of the support. A result of Sawin and Tao [7] states that if a tensor is supported on an antichain, then its slice rank is equal to the number of slices needed to cover the support, which implies that $\epsilon$ has slice rank 3.

If, however, $h$ is any function from $\{1, 2, 3\}$ to $\mathbb{F}$, then the $3 \times 3$ matrix

$$M(y, z) = \sum_x h(x)\epsilon(x, y, z)$$

is antisymmetric, and therefore has rank at most 2.

To see why this is a problem, let $T = \epsilon \oplus \cdots \oplus \epsilon$, where we take $m$ copies, and suppose we have a decomposition

$$T(x, y, z) = \sum_{i=1}^r a_i(x)b_i(y, z) + \sum_{j=1}^s c_j(y)d_j(x, z) + \sum_{k=1}^t e_k(z)f_k(x, y).$$

We can find $h$ with at most $r$ zeros such that $\sum_x h(x)a_i(x) = 0$ for $i = 1, 2, \ldots, r$, and the matrix

$$M(y, z) = \sum_x h(x)T(x, y, z)$$

has rank at most $s + t$.

However, in the other direction all we know is that the rank of $M$ is twice the number of copies of $\epsilon$ that are not projected to zero – that is, twice the number of $q$ such that at least one of $h(3q - 2), h(3q - 1)$ and $h(3q)$ is non-zero. The number of such $q$ is at least $m - \lfloor r/3 \rfloor$, but can in principle be that low. For example, if for each $i \leq r$, $a_i$ is the $i$th standard basis vector, then $h(i)$ is forced to be zero for $i = 1, \ldots, r$, so for $q \leq r/3$ we have that $h(3q - 2), h(3q - 1)$ and $h(3q)$ are all zero. So the best lower bound we can obtain in general is that $2\lceil r/3 \rceil + s + t \geq 2m$. By symmetry we obtain similar estimates with the role of $r$ played by $s$ and $t$. But if $r, s$ and $t$ are all equal and are multiples of 3, then we find that $8(r + s + t)/9 \geq 2m$, from which we can conclude only that $r + s + t \geq 9m/4$.

Note that the result of Sawin and Tao that shows that $\sigma(\epsilon) = 3$ also shows that $\sigma(\epsilon \oplus \cdots \oplus \epsilon) = 3m$ (where there are still $m$ copies of $\epsilon$), but there are tensors for which their method does not give optimal estimates, so this argument will only work for special cases of the problem.

Remark. The example just presented relied on a “non-trivial” space of low-rank matrices, namely the $3 \times 3$ antisymmetric matrices. We regard a space $Z$ of matrices of rank at most $r$ as trivial if there are spaces $U$ and $V$ of dimensions $s$ and $t$ with $s + t \leq r$ such that $Z$ is the sum of the space of matrices with rows in $U$ and the space of matrices with columns in $V$. It is not a straightforward problem to understand spaces of low-rank matrices in general. See for example a paper of Eisenbud and Harris [3], which was what led us to think of the example above, and which can probably be used to construct other examples of a similar type.
An earlier version of this note contained a more complicated argument. I would like to thank Thomas Karam for pointing out that certain parts of that argument were imprecise to the point of not being obviously correct. Although it turned out that the argument could be rescued in the case \( d = 3 \) (and probably also for general \( d \) but that is trickier), during subsequent conversations with Thomas Karam a simpler proof emerged, after which it became clear that the result could in fact be proved using a simple modification of the argument of Sawin and Tao just mentioned, a possibility that I had previously considered but, as a result of an incorrect heuristic argument, discounted. While this makes the result not interesting enough to publish formally, it still seems worth keeping it as an arXiv preprint, since at some point it may save somebody some time if it can be readily found online. As this document is not intended for publication, we include the modified old argument for the case \( d = 3 \), just in case elements of the proof are of use to anyone.

## 2 Proof of Theorem 2

For the convenience of the reader, we begin by recalling one or two facts from a blog post of Sawin and Tao [7]. The first is that we can think of tensors in two different ways – either in “matrix form” as functions \( T : X_1 \times \cdots \times X_d \to \mathbb{F} \) or as elements of a tensor product \( V_1 \otimes \cdots \otimes V_d \). Given a function \( T : X_1 \otimes \cdots \otimes X_d \to \mathbb{F} \), the corresponding element of the tensor product \( \mathbb{F}^{X_1} \otimes \cdots \otimes \mathbb{F}^{X_d} \) is the sum

\[
\sum_{x_1, \ldots, x_d} T(x_1, \ldots, x_d) e_{x_1} \otimes \cdots \otimes e_{x_d},
\]

where, given \( x_i \in X_i \), the vector \( e_{x_i} \) is the standard basis vector in \( \mathbb{F}^{X_i} \) that takes the value 1 at \( x_i \) and 0 everywhere else. In the other direction, given an element \( \tau \) of a tensor product \( V_1 \otimes \cdots \otimes V_d \) of finite-dimensional vector spaces, take a basis \( \{ e_{i1}, \ldots, e_{ir_i} \} \) of each \( V_i \), write \( \tau \) in the unique way possible as

\[
\tau = \sum_{j_1, \ldots, j_d} \lambda(j_1, \ldots, j_d) e_{1j_1} \otimes \cdots \otimes e_{dj_d},
\]

let \( X_j = \{ 1, 2, \ldots, r_i \} \), and set \( T = \lambda \).

In the tensor-product formulation, the slice rank of a tensor \( T \in V_1 \otimes \cdots \otimes V_d \) is the smallest \( r \) such that it is possible to write \( T \) in the form

\[
\sum_{i=1}^{d} \sum_{j=1}^{r_i} u_{ij} \otimes v_{ij},
\]

with \( r_1 + \cdots + r_d = r \), where for each \( i \), \( u_{ij} \in V_i \) and \( v_{ij} \in V_1 \otimes \cdots \otimes V_{i-1} \otimes V_{i+1} \otimes \cdots \otimes V_d \). (This is a slight abuse of notation because what we are really doing is “inserting” \( u_{i,j} \) into \( v_{i,j} \). More precisely, if \( v \) is a pure tensor \( w_1 \otimes \cdots \otimes w_{i-1} \otimes w_{i+1} \otimes \cdots \otimes w_d \) and \( u \in V_i \), then by \( u \otimes v \) we mean the tensor \( w_1 \otimes \cdots \otimes w_{i-1} \otimes u \otimes w_{i+1} \otimes \cdots \otimes w_d \), and then this map can be extended linearly.)

It is simple to check that this tensor-product definition of slice rank agrees with the definition given earlier. We shall therefore pass freely between the two, using whichever formulation is more convenient at any one moment.

**Lemma 3.** Let \( V_1, \ldots, V_d \) be finite-dimensional vector spaces over a field \( \mathbb{F} \) and let \( T \in V_1 \otimes \cdots \otimes V_d \). Then \( T \) has slice rank at most \( r \) if and only if there
exist subspaces $U_i \subset V_i^*$ with $\sum_i \text{codim}(U_i) \leq r$ such that $\langle T, u \rangle = 0$ for every $u \in U_1 \otimes \cdots \otimes U_d$.

Proof. Suppose first that $T$ has slice rank at most $r$. Then we can write $T$ as a sum $\sum_{i=1}^d \sum_{j=1}^{r_i} v_{ij} \otimes w_{ij}$, where for each $i$ and $j$, $v_{ij} \in V_i$ and $w_{ij} \in V_1 \otimes \cdots \otimes V_{i-1} \otimes V_{i+1} \otimes \cdots \otimes V_d$, and $\sum_{i=1}^d r_i \leq r$.

For each $i$ let $U_i$ be the set of all $u \in V_i^*$ such that $\langle v_{ij}, u \rangle = 0$ for $j = 1, \ldots, r_i$. Then $U_i$ is a subspace of codimension at most $r_i$. Moreover, if $u_i \in U_i$ for $i = 1, \ldots, d$, then $\langle \sum_{i=1}^d \sum_{j=1}^{r_i} v_{ij} \otimes w_{ij}, u_1 \otimes \cdots \otimes u_d \rangle = 0$, since for each $i, j$ we have that $\langle v_{ij}, u_i \rangle = 0$. Extending linearly we find that $\langle T, u \rangle = 0$ for every $u \in U_1 \otimes \cdots \otimes U_d$, and we also have that $\sum_i \text{codim}(U_i) \leq r$.

In the reverse direction, suppose that such subspaces $U_i$ exist. For each $i$ choose a basis of $U_i$ and extend it to a basis of $V_i^*$. By considering the expansion of $T$ with respect to the dual bases of these bases, we see that $T$ must be contained in the subspace $\sum_{i=1}^d V_1 \otimes \cdots \otimes V_{i-1} \otimes U_i^\perp \otimes V_{i+1} \otimes \cdots \otimes V_d$ of $V_1 \otimes \cdots \otimes V_d$. Since $U_i^\perp$ has a basis of size $r_i$, this yields a decomposition of $T$ of the required form.

Proof of Theorem 2. Let $V_1, \ldots, V_d$ be finite-dimensional vector spaces with $V_i = V_i^1 \oplus V_i^2$, and let $T = T_1 + T_2$, where $T_1 \in V_1^1 \otimes \cdots \otimes V_d^1$ and $T_2 \in V_1^2 \otimes \cdots \otimes V_d^2$. We would like to show that $\sigma(T_1) + \sigma(T_2) \leq \sigma(T)$, the reverse inequality being trivial.

Let $r = \sigma(T)$ and choose subspaces $U_i \subset V_i^*$ with $\text{codim}(U_i) = r_i$ and $\sum_i r_i = r$ such that $\langle T, u \rangle = 0$ for every $u \in U_1 \otimes \cdots \otimes U_d$.

For each $i$, choose a basis $v_{1i}, \ldots, v_{imi}$ of $V_i$ that starts with a basis of $V_i^1$ and ends with a basis of $V_i^2$. Let $v_{1i}^*, \ldots, v_{imi}^*$ be the dual basis, and let $w_{1i}, \ldots, w_{imi}$ be a basis of $U_i$, where $m_i = n_i - r_i$. Each $w_{ij}$ can be expanded in terms of the dual basis. Let us write $u_{ij}(h)$ for the $h$th coefficient of $u_{ij}$ with respect to this basis: that is,

$$u_{ij} = \sum_{h=1}^{n_i} u_{ij}(h)v_{ih}.$$  

By applying Gaussian elimination, we may assume for any given $i$ that the first $h$ for which $u_{ij}(h)$ is non-zero is a strictly increasing function of $j$. Alternatively, we may assume for any given $i$ that the last $h$ for which $u_{ij}(h)$ is non-zero is a strictly decreasing function of $i$. (That is, for each $i$ we may assume one or the other of these two statements: we do not claim that both can be assumed at once.)

Suppose that for a particular $i$ we have chosen the first option: that is, the first $h$ for which $u_{ij}(h) \neq 0$ is strictly increasing with $j$. If $\dim(V_i^1) = s_i$, then for every $j$ such that the first such $h$ is greater than $s_i$, we have that $u_{ij}$ vanishes on $V_i^1$. We now define a sequence $w_{1i}, \ldots, w_{imi}$ as follows. For every $j$ such that the first $h$ is less than $s_i$, we let $w_{ij}$ be the projection of $u_{ij}$ on to the first $s_i$ coordinates, and note that $w_{ij}$ and $u_{ij}$ agree on $V_i^1$. Let the number of such $j$ be $k_i$. For every $j > k_i$, we let $w_{ij} = u_{ij}$, and as just mentioned we have that $w_{ij}$ vanishes on $V_i^1$.

Similarly, if we have chosen the second option, then we can define a sequence $w_{1i}, \ldots, w_{imi}$ and $k_i$ such that for $j \leq k_i$ we have that $w_{ij}$ vanishes on $V_i^2$ and for $j > k_i$ we have that $w_{ij}$ agrees with $u_{ij}$ on $V_i^2$.

In both cases we start with the vectors $u_{1i}, \ldots, u_{imi}$ and obtain a sequence $w_{1i}, \ldots, w_{imi}$, and some $k_i$ such that $w_{1i}, \ldots, w_{ik_i} \in (V_i^1)^k$ and $w_{i,k_i+1}, \ldots, w_{imi} \in (V_i^2)^{m_i - k_i}$. 

5
\((V^2)\). For each \(i\) let \(U^1_i\) be the span of \(w_{i1}, \ldots, w_{ik_i}\) and let \(U^2_i\) be the span of \(w_{i,k_i+1}, \ldots, w_{im_i}\). Since \(\dim(U^1_i) + \dim(U^2_i) = m_i\), we have that \(\text{codim}(U^1_i) + \text{codim}(U^2_i) = n_i - m_i = r_i\). (Here by the codimension of \(U^1_i\) we mean its codimension as a subspace of \((V^1)^*_i\), and similarly for \(U^2_i\).)

Assume now that there exists \(i_0\) such that the second option is chosen. We claim that if \(u \in U^1_i \otimes \cdots \otimes U^1_d\), then \(\langle T^1, u \rangle = 0\). It is enough to prove this when \(u = u_{i1} \otimes \cdots \otimes u_d\) with \(u_i \in U^1_i\). Furthermore, it is enough to prove it when each \(u_i\) is equal to \(w_{ij}\) for some \(j \leq k_i\).

If \(i\) is such that the first option is chosen, and \(j \leq k_i\), then \(w_{ij}\) agrees with \(u_{ij}\) on \(V^1_i\). If \(i\) is such that the second option is chosen, and \(j \leq k_i\), then \(w_{ij} = u_{ij}\) and therefore also agrees with \(u_{ij}\) on \(V^1_i\). Since \(T^1 \in V^1_i \otimes \cdots \otimes V^2_d\), it follows that \(\langle T^1, u \rangle\) does not change if we replace each \(w_{ij}\) by \(u_{ij}\). But if each \(u_i\) is one of the vectors \(w_{ij}\), then \(\langle T, u \rangle = 0\), by hypothesis. Also, since the second option is chosen for \(i_0\), \(u_{i_0}\) vanishes on \(V^2_{i_0}\). It follows that \(\langle T^2, u \rangle = 0\), and therefore that \(\langle T^1, u \rangle = 0\).

Similarly, if \(u \in U^2_i \otimes \cdots \otimes U^2_d\) and we choose the first option for at least one \(i\), then \(\langle T^2, u \rangle = 0\).

Since \(d \geq 2\), we can choose the first option for at least one \(i\) and the second option for at least one \(i\), so the result is proved.

An examination of the above argument shows that it can be used to prove stronger statements as well. Suppose, for instance, that \(T\) is of the form \(T^1 + T^2\) where \(T^1\), as before, belongs to \(V^1_1 \otimes \cdots \otimes V^1_d\), but all we assume about \(T^2\) is that it belongs to \(V^1 \otimes \cdots \otimes V^d_{d-1} \otimes V^2_d\). We now run the proof, choosing the first option for \(i = 1, 2, \ldots, d - 1\) and the second option for \(i = d\).

Suppose that \(w = w_{11} \otimes \cdots \otimes w_{id}\), where each \(w_i\) is equal to \(w_{ij}\) for some \(j \leq k_i\). For \(i = 1, 2, 3, \ldots, d - 1\) let us replace \(w_i\) by some \(u_i \in U^1_i\) that agrees with \(w_{ij}\) on \(V^1_i\). Letting \(u = u_1 \otimes \cdots \otimes u_{d-1} \otimes w_d\), we then have that \(\langle T^1, w \rangle = \langle T^1, u \rangle\).

Because we chose the second option for \(i = d\), \(w_d\) vanishes on \(V^2_d\), and therefore \(\langle T^2, w \rangle = 0\). It follows that \(\langle T^1, u \rangle = \langle T, u \rangle = 0\), where the last equality holds by hypothesis.

Now suppose that \(w = w_{11} \otimes \cdots \otimes w_{id}\), where this time each \(w_i\) is equal to \(w_{ij}\) for some \(j > k_i\). Then there exists \(u_d \in U^2_d\) that agrees with \(w_{ij}\) on \(V^2_d\), while for \(i = 1, 2, \ldots, d - 1\) we have that \(w_i\) vanishes on \(V^1_i\). Let \(u = w_{11} \otimes \cdots \otimes w_{d-1} \otimes u_d \in U^2_1 \otimes \cdots \otimes U^2_d\) and note that \(\langle T^1, u \rangle = 0\).

Given \(\alpha \in \{1, 2\}^d\), let \(T^\alpha\) stand for the projection of \(T\) to \(V^\alpha_1 \otimes \cdots \otimes V^\alpha_d\). (To be more explicit, given \(v_i \in V_i\) we can write it uniquely as \(v^1_i + v^2_i\) with \(v^1_i \in V^1_i\) and \(v^2_i \in V^2_i\). This allows us to decompose \(v^1_1 \otimes \cdots \otimes v^1_d \in V^1 \otimes \cdots \otimes V^d\) into \(2^d\) parts \(v^\alpha_1 \otimes \cdots \otimes v^\alpha_d\), one for each \(\alpha\).) Then \(T = \sum T^\alpha\).

If any of \(\alpha_1, \ldots, \alpha_{d-1}\) is equal to 1, then because \(w_i\) vanishes on \(V^1_i\) for \(i \leq d - 1\), we have that \(\langle T^\alpha, w \rangle = 0\). Also, when \(\alpha = (2, 2, \ldots, 2, 1)\), we have that \(T^\alpha = 0\). It follows that

\[
\langle T^{22\ldots2}, w \rangle = \langle T^2, w \rangle = \langle T^2, u \rangle = \langle T, u \rangle = 0,
\]

where again the last equality holds by hypothesis.

This proves the following statement.

**Theorem 4.** Let \(V_1, \ldots, V_d\) be finite-dimensional vector spaces with \(V_i = V^1_i \oplus V^2_i\) for each \(i\). Let \(T \in V_1 \otimes \cdots \otimes V_d\) and suppose that the component \(T^\alpha\) (see just above for the definition) is zero unless either \(\alpha_d = 2\) or \(\alpha_1 = \cdots = \alpha_d = 1\).

Then

\[
\sigma(T) \geq \sigma(T^{11\ldots1}) + \sigma(T^{22\ldots2}).
\]
Note that the conditions of this theorem are satisfied in particular if $T^\alpha$ is non-zero only for increasing sequences $\alpha$. This gives us a simple corollary about “block upper triangular” tensors. Here we let $T \in V_1 \otimes \cdots \otimes V_d$ as before, but this time $V_i = V_i^1 \oplus \cdots \oplus V_i^k$ for some $k$. We call a tensor block upper triangular (with respect to the given decompositions) if the component $T^\alpha$ (defined in the obvious way for each $\alpha \in [k]^d$) is non-zero only for increasing sequences $\alpha$.

**Corollary 5.** Let $V_1, \ldots, V_d$ be as above and let $T \in V_1 \otimes \cdots \otimes V_d$ be upper triangular. Then

$$\sigma(T) \geq \sigma(T^{11\ldots1}) + \cdots + \sigma(T^{kk\ldots k}).$$

**Proof.** For each $i$ let $W_i^1 = V_i^1 \oplus \cdots \oplus V_i^{k-1}$ and let $W_i^2 = V_i^k$. Then $V_i = W_i^1 \oplus W_i^2$. For $\alpha \in \{1, 2\}^d$ let $S^\alpha$ be the component of $T$ in $W_1^{\alpha_1} \otimes \cdots \otimes W_d^{\alpha_d}$. Then $T$ is block upper triangular with respect to the decompositions $V_i = W_i^1 \oplus W_i^2$, from which it follows, using the theorem just proved, that $\sigma(T) \geq \sigma(S^{11\ldots1}) + \sigma(S^{22\ldots2})$.

But $S^{22\ldots2} = T^{22\ldots2}$, and $S^{11\ldots1} \in W_1^1 \otimes \cdots \otimes W_d^1$ is block upper triangular with respect to the decompositions $W_i^1 = V_i^1 \oplus \cdots \oplus V_i^{k-1}$. By induction on $k$ we have that

$$\sigma(S^{11\ldots1}) \geq \sigma(T^{11\ldots1}) + \cdots + \sigma(T^{k-1,k-1\ldots,k-1}),$$

and the proof is complete.

### 3 An alternative proof of Theorem 2 for 3-tensors

There seems no harm in including the argument mentioned earlier that works when $d = 3$, even though it is a little more complicated, as the lemmas along the way may be of some interest. However, the reader just interested in obtaining some proof of Theorem 2 can safely skip this section.

For this proof we shall use the more “matrix-like” conception of tensors.

**Lemma 6.** Let $V$ and $W$ be two vector spaces with $V \cap W = \{0\}$, let $v_1, \ldots, v_n \in V$ and $w_1, \ldots, w_n \in W$ be two sequences of vectors, and let $U \subset V + W$ be the subspace generated by the vectors $v_i + w_i$. Then there exists a sequence $v''_1 + w''_1, \ldots, v''_n + w''_n$ that generates $U$ with each $v''_i$ in $V$ and each $w''_i$ in $W$, such that the non-zero $v''_i$ are linearly independent and the non-zero $w''_i$ are linearly independent.

**Proof.** Without loss of generality $v_1, \ldots, v_m$ is a maximal linearly independent subset of $v_1, \ldots, v_n$. Then for each $j > m$ we can write

$$v_j = \sum_{i=1}^m \lambda_{ji} v_i.$$

For $j > m$ let $w'_j = w_j - \sum_{i=1}^m \lambda_{ji} w_i$ and let $v'_j = 0$, and observe that the $v'_j + w'_i$ generate the same subspace as the $v_i + w_i$. (We let $v'_j = v_i$ and $w'_j = w_i$ when $i \leq m$.) We also have that the non-zero $v_i$ are linearly independent.
Now let us choose \( v'_1, \ldots, v'_n \) and \( w''_1, w''_2, \ldots, w''_n \) as follows, with the aim of ensuring that for every \( s \) we have that
\[
\langle w''_s, w''_{s+1}, \ldots, w''_n \rangle = \langle w'_s, w'_{s+1}, \ldots, w'_n \rangle.
\]

We start by setting \( w''_n = w'_n \). Once we have chosen \( w''_{s+1}, \ldots, w''_n \) with the desired property, if
\[
w'_s = \sum_{s+1}^n \mu_i w''_i,
\]
then we set \( w''_s = 0 \) and \( v''_s = v'_s - \sum_{s+1}^n \mu_i v''_i \). Otherwise – that is, if \( w'_s \) is not a linear combination of \( w''_{s+1}, \ldots, w''_n \) – we set \( w''_s = w'_s \) and \( v''_s = v'_s \).

Since \( v'_{m+1} = \cdots = v'_n = 0 \), we find that \( v''_{m+1} = \cdots = v''_n = 0 \) as well. Also, the non-zero \( v''_i \) are linearly independent, as are the vectors \( v''_1, \ldots, v''_n \), and the vectors \( v''_i + w''_i \) generate the same subspace as the vectors \( v_i + w_i \). \( \square \)

In the next lemma, we write \( a \otimes b \) for the function that takes the value \( a(x)b(y,z) \) at \( (x, y, z) \). Note that the lemma is really about matrices – the fact that the \( b_i \) are functions of two variables is irrelevant, but it is the case we shall use when we apply the lemma.

**Lemma 7.** If \( a_1, \ldots, a_r \) and \( a'_1, \ldots, a'_r \) generate the same subspace, then any tensor \( \sum_i a_i(x)b_i(y,z) \) is equal to some tensor \( \sum_j a'_j(x)b'_j(y,z) \).

**Proof.** Let \( a_i = \sum_{j=1}^r \theta_{ij} a'_j \) for each \( i \), which we can do because the \( a'_j \) contain the \( a_i \) in their linear span. Then
\[
\sum_i a_i \otimes b_i = \sum_{i,j} \theta_{ij} a'_j \otimes b_i = \sum_j a'_j \otimes (\sum_i \theta_{ij} b_i),
\]
so we can take \( b'_j = \sum_i \theta_{ij} b_i \) for each \( j \). \( \square \)

**Remark.** The lemma just proved highlights the main difference, for this question, between slice rank and tensor rank, and indeed various other kinds of rank. Each \( b'_j \) is a linear combination of the \( b_i \), and is therefore a function of the same type. But if we were considering tensor rank, then each \( b_i \) would be a rank-1 matrix, and we would not be able to conclude that each \( b'_j \) was a rank-1 matrix. Thus, there is a flexibility associated with slice-rank decompositions that we do not have with tensor-rank decompositions.

We now take three finite sets \( X, Y, \) and \( Z \), each partitioned into two subsets, so \( X = X^1 \cup X^2, Y = Y^1 \cup Y^2 \) and \( Z = Z^1 \cup Z^2 \). (We shall use superscripts to denote elements of the set \( \{1,2\} \) and subscripts to index the functions we use in decompositions.) Given a function \( a : X \to F \), we define \( a^\alpha \) to be the projection of \( a \) to \( X^\alpha \); that is, \( a^\alpha(x) = a(x) \) if \( x \in X^\alpha \) and \( a^\alpha(x) = 0 \) otherwise. We do the same for functions defined on \( Y \) and \( Z \). Similarly, if \( b : Y \times Z \to F \), then \( b^{\beta\gamma} \) is the projection of \( b \) to \( Y^{\beta} \times Z^\gamma \), and so on. In particular, if \( T : X \times Y \times Z \to F \) is a tensor, then \( T^{\alpha\beta\gamma} \) is the projection of \( T \) to \( X^\alpha \times Y^{\beta} \times Z^\gamma \).

We shall also sometimes use this notation to refer to restrictions rather than projections. For example, if we say that \( T = T^{111} \oplus T^{222} \), we mean that \( T^{\alpha\beta\gamma} = 0 \) except if \( \alpha = \beta = \gamma \). In other words, it is sometimes convenient to regard \( T^{\alpha\beta\gamma} \) as defined on \( X^\alpha \times Y^{\beta} \times Z^\gamma \), and it is sometimes convenient to regard it as defined on all of \( X \times Y \times Z \) but supported on \( X^\alpha \times Y^{\beta} \times Z^\gamma \), and similarly for functions of fewer variables. We hope that no confusion will arise.
Corollary 8. Let \( X = X^1 \cup X^2 \), \( Y = Y^1 \cup Y^2 \) and \( Z = Z^1 \cup Z^2 \) be three finite sets each partitioned into two subsets, and let \( T : X \times Y \times Z \rightarrow \mathbb{F} \) be a tensor. Suppose that \( T \) has a decomposition

\[
T(x, y, z) = \sum_{i=1}^{r} a_i(x)b_i(y, z) + \sum_{j=1}^{s} c_j(y)d_j(x, z) + \sum_{k=1}^{t} e_k(z)f_k(x, y). \tag{1}
\]

Then \( T \) has such a decomposition with the additional property that for all \( \alpha, \beta, \gamma \in \{1, 2\} \) the non-zero \( a_i^\alpha \) are linearly independent, the non-zero \( c_j^\beta \) are linearly independent, and the non-zero \( e_k^\gamma \) are linearly independent.

Proof. Applying Lemma \( 5 \) with \( V = \mathbb{F}^{X^1} \), \( W = \mathbb{F}^{X^2} \), \( v_i = a_i^1 \) and \( w_i = a_i^2 \) for each \( i \), we obtain a sequence \( a_1', \ldots, a_r' \) with the same linear span as \( a_1, \ldots, a_r \), such that the non-zero vectors \( (a_j')^1 \) are linearly independent and the non-zero vectors \( (a_j')^2 \) are linearly independent. By Lemma \( 4 \) we can find functions \( b_1', \ldots, b_r' : Y \times Z \rightarrow \mathbb{F} \) such that \( \sum_i a_i(x)b_i(y, z) = \sum_j a_j'(x)b_j'(y, z) \) for every \( x, y, z \). By symmetry we can rewrite the other two terms in a similar way, and the result is proved.

We need one further linear algebra lemma.

Lemma 9. Let \( U, V, W \) be vector spaces and let \( W' \) be a subspace of \( W \). Let \( u_1, \ldots, u_r \in U \) be linearly independent and let \( v_1, \ldots, v_s \in V \) be linearly independent. Suppose that we have a linear combination \( \sum_{i=1}^{r} \sum_{j=1}^{s} u_i \otimes v_j \otimes w_{ij} \) that belongs to the subspace \( U \otimes V \otimes W' \). Then all the vectors \( w_{ij} \) belong to the subspace \( W' \).

Proof. Suppose not, and let \( \phi : W \rightarrow \mathbb{F} \) be a linear functional that vanishes on \( W' \) but not on every vector \( w_{ij} \). Define \( \psi(u \otimes v \otimes w) = \phi(w)u \otimes v \) and extend this to a linear map \( \psi : U \otimes V \otimes W \rightarrow U \otimes V \). Then \( \psi \) vanishes on \( U \otimes V \otimes W' \). However, the image of \( \sum_{i=1}^{r} \sum_{j=1}^{s} u_i \otimes v_j \otimes w_{ij} \) is a non-zero linear combination of the \( u_i \otimes v_j \), which are linearly independent, so it is non-zero. This is a contradiction.

Now let us adopt our main hypothesis, namely that we have a tensor \( T \) as in Corollary \( 8 \) and that \( T = T^{111} \oplus T^{222} \). Suppose also that \( T \) has a decomposition as in (1) above, and that the conclusion of Corollary \( 8 \) holds for this decomposition. Our hypothesis is equivalent to the statement that \( T^{\alpha \beta \gamma} = 0 \) except if \( \alpha = \beta = \gamma \).

For \( \alpha, \beta, \gamma \in \{1, 2\} \) let \( A^\alpha = \{ i : a_i^\alpha \neq 0 \} \), let \( C^\beta = \{ j : c_j^\beta \neq 0 \} \), and let \( E^\gamma = \{ k : e_k^\gamma \neq 0 \} \). Then for each \( \alpha, \beta, \gamma, x, y, z \), we have that

\[
T^{\alpha \beta \gamma}(x, y, z) = \sum_{i \in A^\alpha} a_i^\alpha(x)b_i^\beta(y, z) + \sum_{j \in C^\beta} c_j^\beta(y)d_j^\gamma(x, z) + \sum_{k \in E^\gamma} e_k^\gamma(z)f_k^{\alpha \beta}(x, y).
\]

In the next lemma, we shall use bracketed superscripts to denote dependencies and non-bracketed superscripts to denote the parts that a function applies to. So for example, in the statement, the function \( p_{ij}^{(\alpha \gamma)} \) is defined on \( Z^\gamma \) and depends on \( \alpha \) (because it will be made out of the functions \( d_j^{\beta \gamma} \), which are defined on \( X^\alpha \times Z^\gamma \)).
Lemma 10. Let \( \alpha, \beta, \gamma \) be not all equal and let \( i \in A^\alpha \). Then there exist functions \( p_{ij}^{(\alpha)\gamma} : Z^\gamma \to F \) and \( q_{ik}^{(\alpha)\beta} : Y^\beta \to F \) such that

\[
b_{ij}^{\alpha\beta\gamma} = \sum_{j \in C^\beta} c_j^{\beta} \otimes p_{ij}^{(\alpha)\gamma} + \sum_{k \in E^\gamma} q_{ik}^{(\alpha)\beta} \otimes e_k^\gamma,
\]

with similar decompositions for \( d_{ij}^{\alpha\beta\gamma} \) and \( f_k^{\alpha\beta} \).

Proof. Since the \( a_i^\alpha \) with \( i \in A^\alpha \) are linearly independent, the matrix \( (a_i^\alpha(x)) \), where \( i \) ranges over \( A^\alpha \) and \( x \) over \( X \), has rank \( |A^\alpha| \). It follows that we can find for each \( i \) a function \( h_i^{(\alpha)} : X \to F \) such that \( \sum_x h_i^{(\alpha)}(x)a_i^\alpha(x) = \delta_{il} \) for every \( l \in A^\alpha \). Then since \( T^{\alpha\beta\gamma} = 0 \), we have that

\[
0 = \sum_x h_i^{(\alpha)}(x)T^{\alpha\beta\gamma}(x,y,z) = b_{ij}^{\alpha\beta\gamma}(y,z) - \sum_{j \in C^\beta} c_j^{\beta}(y)p_{ij}^{(\alpha)\gamma}(z) - \sum_{k \in E^\gamma} q_{ik}^{(\alpha)\beta}(y)e_k^\gamma(z),
\]

where

\[
p_{ij}^{(\alpha)\gamma}(z) = -\sum_x h_i^{(\alpha)}(x)d_{ij}^{\alpha\beta\gamma}(x,z)
\]

and

\[
q_{ik}^{(\alpha)\beta}(y) = -\sum_x h_i^{(\alpha)}(x)f_k^{\alpha\beta}(x,y).
\]

The corresponding results for the functions \( d_{ij}^{\alpha\beta\gamma} \) and \( f_k^{\alpha\beta} \) are proved in the same way. \( \square \)

Using Lemma 10 we can rewrite the decomposition of \( T^{\alpha\beta\gamma} \) above in the form

\[
\sum_{i \in A^\alpha} \sum_{j \in C^\beta} a_i^\alpha \otimes c_j^{\beta} \otimes p_{ij}^{(\alpha)\gamma} + \sum_{i \in A^\alpha} \sum_{k \in E^\gamma} a_i^\alpha \otimes q_{ik}^{(\alpha)\beta} \otimes e_k^\gamma + \sum_{i \in A^\alpha} \sum_{j \in C^\beta} a_i^\alpha \otimes c_j^{\beta} \otimes g_{ij}^{(\beta)\gamma} + \sum_{j \in C^\beta} \sum_{k \in E^\gamma} h_{jk}^{(\beta)} \otimes c_j^{\beta} \otimes e_k^\gamma + \sum_{i \in A^\alpha} \sum_{k \in E^\gamma} a_i^\alpha \otimes u_{ik}^{(\gamma)} \otimes e_k^\gamma + \sum_{j \in C^\beta} \sum_{k \in E^\gamma} v_{jk}^{(\gamma)} \otimes c_j^{\beta} \otimes e_k^\gamma.
\]

We are still assuming here that \( \alpha, \beta \) and \( \gamma \) are not all equal.

Since \( T^{\alpha\beta\gamma} \) is also equal to 0 under this assumption, it follows from Lemma 9 that \( p_{ij}^{(\alpha)\gamma} + g_{ij}^{(\beta)\gamma} \) is a linear combination of the \( e_k^\gamma \) with \( k \in E^\gamma \), with similar statements for \( q_{ik}^{(\alpha)\beta} + v_{ik}^{(\gamma)} \) and for \( h_{jk}^{(\beta)} + u_{jk}^{(\gamma)} \).

We now show that the result is true in the extreme case that \( A^1 = A^2 = B^1 = B^2 = C^1 = C^2 = \{t\} \).

Corollary 11. Suppose that \( A^1 = A^2 = \{r\} \), \( B^1 = B^2 = \{s\} \) and \( C^1 = C^2 = \{t\} \). Then the slice ranks of \( T_{111} \) and \( T_{222} \) are both at most \( \min\{r,s,t\} \).

Proof. For this proof, let us adopt the convention that summing over \( i \) means summing over \( i \in A^\alpha = A^3 \), and similarly for \( j \) and \( k \).

From what we have just proved, with \( (\alpha, \beta, \gamma) = (2,2,1) \), we have for all \( i,j \) that \( p_{ij}^{(\alpha=2)\gamma} + g_{ij}^{(\beta=2)\gamma} \) is a linear combination of the \( e_k^1 \), and we have similar conclusions for \( q_{ik}^{(\alpha=2)\beta} + u_{ik}^{(\gamma=2)} \) and \( h_{jk}^{(\beta=2)} + v_{jk}^{(\gamma=2)} \). Here we are writing
\( p_{ij}^{(\alpha=2)1} \) to denote the function \( p_{ij}^{(\alpha)1} \) in the case \( \alpha = 1 \), and so on. (It would be nice to be able to write the simpler \( p_{ij}^{(2)1} \), but then it would not be clear that 2 was the value taken by \( \alpha \).)

Now recall that for all \( \alpha, \beta, \gamma \), we have that

\[
T^{\alpha\beta\gamma}(x, y, z) = \sum_i a_i^\alpha(x)b_i^\beta(y, z) + \sum_j c_j^\beta(y)d_j^\gamma(x, z) + \sum_k e_k^\gamma(z)f_k^{\alpha\beta}(x, y).
\]

Substituting the formulae obtained in Lemma 10 for \( b_{ij}^{11}, d_{ij}^{11} \) and \( f_{jk}^{11} \) by taking \( (\alpha, \beta, \gamma) = (2, 1, 1), (1, 2, 1) \) and \( (1, 1, 2) \), respectively, we obtain the formula

\[
T^{111} = \sum_{i,j} a_i^1 \otimes c_j^1 \otimes p_{ij}^{(\alpha=2)1} + \sum_{i,k} a_i^1 \otimes q_{ik}^{(\alpha=2)1} \otimes e_k^1
+ \sum_{i,j} a_i^1 \otimes e_j^1 \otimes g_{ij}^{(\beta=2)1} + \sum_{j,k} h_{jk}^{(\beta=2)1} \otimes c_j^1 \otimes e_k^1
+ \sum_{i,k} a_i^1 \otimes u_{ik}^{(\gamma=2)} \otimes e_k^1 + \sum_{j,k} v_{jk}^{(\gamma=2)} \otimes c_j^1 \otimes e_k^1.
\]

The observations in the second paragraph of this proof imply that the right hand side belongs to the linear span of the functions \( a_i^1 \otimes c_j^1 \otimes e_k^1 \). From this the result for \( T^{111} \) follows. The proof for \( T^{222} \) is similar.

Since \( 2 \min\{r, s, t\} \leq r + s + t \), we are done in this case.

To do the general case, we reduce to the case covered by Corollary 11 using an inductive argument.

**Proof of Theorem 2 for 3-tensors.** Suppose now that the hypothesis of Corollary 11 does not hold. Then without loss of generality \( a_1^2 = 0 \). Let \( P \) be the matrix of a projection to the one-dimensional subspace of \( \mathbb{F}^X \) generated by \( a_1 \) such that \( P \) vanishes on all functions supported in \( X^2 \), and let \( Q = I - P \). Then

\[
T(x, y, z) = \sum_{x'} P(x, x')T(x', y, z) + \sum_{x'} Q(x, x')T(x', y, z).
\]

For every \( y, z \), the sum in the first term is a function of \( x \), and that function is a multiple of \( a_1^1 \). Therefore, it can be written in the form \( a_1^1(x)b(y, z) \). Also, if \( (y, z) \notin Y^1 \times Z^1 \), then \( T(x', y, z) = 0 \) for every \( x' \in X^1 \), and therefore the first term vanishes, by the condition that \( P \) vanishes on functions supported in \( X^2 \). It follows that \( b \) is supported on \( Y^1 \times Z^1 \).

As for the second term, writing \( Qg(x, u_1, \ldots, u_m) \) as shorthand for the sum \( \sum_{x'} Q(x, x')g(x', u_1, \ldots, u_m) \), it is equal to

\[
\sum_{i=1}^r Qa_i(x)b_i(y, z) + \sum_{j=1}^s c_j(y)Qd_j(x, z) + \sum_{k=1}^t e_k(z)Qf_k(x, y).
\]

But \( Qa_1 = 0 \), so this is a decomposition of \( QT \) into \( (r - 1) + s + t \) pieces. Furthermore, since \( PT \) is supported in \( X^1 \times Y^1 \times Z^1 \), it follows that \( QT \) is also a direct sum. Therefore, by induction on \( r + s + t \), \( \sigma((QT)^{111}) + \sigma((QT)^{222}) \leq r - 1 + s + t \). Since \( (PT)^{111}(x, y, z) = a_1^1(x)b(y, z) \) and \( (PT)^{222} = 0 \), it follows that \( \sigma(T^{111}) + \sigma(T^{222}) \leq r + s + t \).

\[
11
\]
4 Further remarks and questions

There are other basic statements about matrix rank that do not generalize to slice rank for higher-degree tensors. For instance, it is not true in general that $\sigma(S \otimes T) = \sigma(S)\sigma(T)$. Indeed, if one takes three reasonably generic $n \times n \times n$ slice-rank-1 tensors with slices in different directions – that is, of the kind $a(x)b(y, z)$, $c(y)d(x, z)$, and $e(z)f(x, y)$ – then their tensor product will tend to have large slice rank. For instance, if $a, e$ and $f$ are all equal to the standard basis vector $e_1$ and $b, d$ and $f$ are all equal to the identity matrix, then the tensor product of the three tensors is equivalent to the so-called matrix multiplication tensor, which has rank $n^2$ (see [1, Remark 4.9]). And for an example in the other direction, if $T : \mathbb{F}_3^n \to \mathbb{F}_3$ is the characteristic function of the set $\{(x, y, z) \in \mathbb{F}_3^3 : x + y + z = 0\}$, then it has slice rank 3. (To see this, observe that if not, then it has a decomposition into two functions of slice rank 1, so without loss of generality there is no function of type $e(z)f(x, y)$ involved in the decomposition. But if we then fix $z$, we obtain a matrix of rank 2, but it is also a permutation matrix so it has rank 3, a contradiction.) However, the $n$th tensor power of $T$ can be thought of as the characteristic function of the set $\{(x, y, z) \in (\mathbb{F}_3^n)^3 : x + y + z = 0\}$, which, as the polynomial method shows, has slice rank exponentially smaller than $3^n$.

A special case of Theorem 2 is that $\sigma(S \otimes T) = \sigma(S)\sigma(T)$ when $S$ is a diagonal tensor, so we obtain equality for this case, but we know in advance that the argument cannot be simple enough to generalize to all tensor products.

Another related question is a long-standing conjecture of Strassen that tensor rank was additive for direct sums, which, despite being true in a number of special cases, was eventually disproved by Shitov in 2017, who found a highly non-obvious counterexample [8].

We conclude with three questions. The first is whether there is a simultaneous generalization of the main theorem of this paper and of the result of Sawin and Tao mentioned earlier. To make this question more precise, suppose that $X_i$ is partitioned into sets $X_{i1}, \ldots, X_{ir_i}$ for each $i$. Define the block support of a tensor $T : X_1 \times \cdots \times X_d$ to be the set of $(j_1, \ldots, j_d)$ such that $T$ restricted to the block $X_{1j_1} \times \cdots \times X_{dj_d}$ is not identically zero. Define a block slice of $T$ to be the restriction of $T$ to a set of the form

$$X_1 \times \cdots \times X_{h-1} \times X_{hj} \times X_{h+1} \times \cdots \times X_d.$$

Call a block $X_{1j_1} \times \cdots \times X_{dj_d}$ maximal if $(j_1, \ldots, j_d)$ is a maximal element of the block support.

If the non-zero blocks of $T$ are covered by some set of block slices, it is trivial that the slice rank of $T$ is at most the sum of the slice ranks of those block slices. However, sometimes we can improve on this bound. For instance, suppose that the block support of a 3-tensor $T$ is contained in three planes, and contains the intersection of those three planes. Suppose also that the block corresponding to that intersection has high slice rank $r$, and that if that block is removed, then the three block slices have small slice rank $s$. With a suitable example like this, one can arrange that the sum of the slice ranks of block slices that cover the non-zero blocks is minimized in the obvious way, which gives an upper bound of at least $3r$. But one can obtain a better upper bound of $r + 3s$ by first decomposing the block at the intersection and then decomposing the rest of the slices.

With that example in mind, let us define a partial block slice to be the restriction of $T$ to a union of blocks that forms a subset of a block slice.
Question. Let $T$ be a $d$-tensor as above and let $S$ be its block support. Does it follow that the slice rank of $T$ is at least the minimum of the sum of the slice ranks of a set of partial block slices that cover all the maximal blocks of $T$?

A positive answer to that question may be too much to hope for, in which case a much weaker preliminary question one might ask is whether if all non-zero blocks have slice rank at least $r$, and if $m$ block slices are needed to cover the maximal blocks, then the slice rank of $T$ is at least $mr$.

Another obvious question is the following.

Question. Is partition rank additive for direct sums?

It seems reasonable to guess that the answer is no, since the proof just given for slice rank appears to fail quite badly. But that is a pure guess, and it might not be a simple matter to find a counterexample. Naslund showed that if an appropriate extra step is added to Tao’s proof of Lemma 1 then it can be made to yield the stronger result that the partition rank of a diagonal tensor is also equal to the number of non-zero entries, so diagonal tensors do not give counterexamples.

Finally, we ask a more open-ended question.

Question. Does Theorem 2 have any interesting combinatorial applications?

The answer to this is not obvious, given that up to now combinatorial applications have tended to be of the result for diagonal tensors (that is, of Lemma 1).

We do not have a promising suggestion for how to apply the result, but can at least point out one constraint on what a genuine application would need to look like. Suppose that $T_1, \ldots, T_m$ are tensors and that the result of Sawin and Tao can be used to show that $\sigma(T_i) \geq r_i$. It then follows easily that $\sigma(T_1 \oplus \cdots \oplus T_m) \geq r_1 + \cdots + r_m$. (We observed this in the introduction in the special case where $T_1 = \cdots = T_m = \epsilon$.) Therefore, an application of the main result of this paper would have to be to tensors $T_1, \ldots, T_m$ to which the approach of Sawin and Tao does not apply, which in practice, given the current state of knowledge, means tensors for which we probably do not know how to calculate their slice rank.

That refers to applications that use direct sums of specific tensors. Another possibility might be an argument in which tensors $T_1, \ldots, T_m$ are defined in terms of some unknown objects (such as subsets of a finite group) that satisfy certain hypotheses that are used to derive lower bounds for the slice ranks $\sigma(T_i)$. However, for the result of this paper to be used in an essential way, there would still be constraints on the nature of the derivation.

Just before this result was posted, an interesting preprint appeared by Sauermann, who for the first time proved a combinatorial result using a lower bound for the slice rank of a non-diagonal tensor: to obtain the lower bound she relied on the approach of Sawin and Tao. That at least suggests that there is value in extending the known methods for calculating slice rank.

References

[1] J. Blasiak, T. Church, H. Cohn, J. Grochow, E. Naslund, W. Sawin and C. Umans, On cap sets and the group-theoretic approach to matrix multiplication
Discrete Analysis 2017:3, 27 pp.
[2] E. Croot, V. Lev and P. Pach, *Progression-free sets in $\mathbb{Z}_n^4$ are exponentially small*, Ann. of Math. 185 (2017), 331-337.

[3] D. Eisenbud and J. Harris, *Vector spaces of matrices of low rank*, Adv. Math. 70 (1988), 135-155.

[4] J. Ellenberg and D. Gijswijt, *On large subsets of $\mathbb{F}_q^n$ with no three-term arithmetic progression* Ann. of Math. 185 (2017), 339-343.

[5] E. Naslund, *The partition rank of a tensor and $k$-right corners in $\mathbb{F}_q^n$*, Jour. Combin. Th. A 174 (2020), 105190.

[6] L. Sauermann, *Finding solutions with distinct variables to systems of linear equations over $\mathbb{F}_p$*, arXiv:2105.06863.

[7] W. Sawin and T. Tao, *Notes on the “slice rank” of tensors*, https://terrytao.wordpress.com/2016/08/24/notes-on-the-slice-rank-of-tensors/

[8] Y. Shitov, *Counterexamples to Strassen’s direct sum conjecture*, Acta Math. 222 (2019), 363-379.

[9] T. Tao, *A symmetric formulation of the Croot-Lev-Pach- Ellenberg-Gijswijt capset bound*, https://terrytao.wordpress.com/2016/05/18/a-symmetric-formulation-of-the-croot-lev