A Riemannian Genuine Measure of Entanglement for Pure States

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While several measures exist for entanglement of multipartite pure states, a true entanglement measure for mixed states still eludes us. A deeper study of the geometry of quantum states may be the way to address this issue, in which context we present a measure for multipartite pure state entanglement (GBR) based on a geodesic distance in the space of quantum states. Our measure satisfies all the desirable properties of a “Genuine Measure of Entanglement” (GME), that exhibits some advantages over other GMEs.

**Keywords:** Pure state entanglement measure, Geometry of quantum states, genuine measure of entanglement

I. INTRODUCTION

Entanglement is a crucial property of quantum systems that has come to the forefront of research in recent times as a resource in quantum computing and communication. Quantification entanglement in different contexts for pure, mixed, bipartite and multipartite states is therefore a subject of intense research in current times.

Historically several measures of pure state entanglement have been formulated from different approaches, exploring different aspects of what appears to be a multifaceted property of multiparticle quantum systems. Algebraic measures starting from concurrence, such as negativity [1] and reshuffling negativity [2] are straightforward and have been proven to be computationally inexpensive. These measures are not sufficiently discriminant. There are operational measures such as entanglement cost [3] and distillable entanglement based on the amount of resources required to generate entanglement. Among different approaches there have been measures of entanglement based on operator algebra [4] and more recently, coherence of an entangled state in the Schmidt basis [5].

To relate entanglement more closely to the nature of the states, an approach using the differential geometry of quantum state space [6] is interesting. It was the work pioneered by Amari and Nagaoka [7] that showed that the application of differential geometry to information theory would give meaningful insights. The Riemannian structure of quantum state space was studied in great detail by Morozova and Çentsov [8]. Later along with the contribution of Petz [9], a concrete definition of all possible monotone Riemannian metrics on quantum state space was formulated.

The basic definition of a geometric entanglement measure is the distance between the state under consideration and the closest separable state. Calculation of such measures involving minimisation over the (infinite) set of all states is naturally computationally hard, especially for higher dimensional state spaces. Avenues to simplify the procedure for calculating such a measure are well worth pursuing.

Several authors [10–12] have attempted to provide geometric entanglement measures using different metrics on quantum state space to find the closest separable state. These approaches largely involve minimization procedures. We need more efficient and computationally less expensive measures.

There are complications involved in defining entanglement measures for multipartite systems, not least due to the existence of different kinds of entanglement depending on which parties one selects. One can choose to focus on Genuine Measures of Entanglement (GME), a concept that characterizes entanglement as a resource for several quantum protocols, both in the sense of information theory and for understanding physics of many-body systems.

An n-qubit entanglement monotone is a Genuine Measure of Entanglement (GME) then it must satisfy the following properties ([13, 14]):

**P1** GME= 0 for all product states and for all biseparable states.

**P2** GME≠ 0 for all non-biseparable states.

**P3** GME ranks the GHZ state (Equation 11) as more entangled than the W state (Equation 12).

Existing GMEs seem to fall short in some way or other. For example, the multi-qubit entanglement measure given by Meyer et al [15] and interpreted by Brennen [16] is non-zero for biseparable states. Later Love et al [17] gave a correction to this measure which qualified it as a GME. In works by Li et al [18], Xi et al [19], Sabín et al [20] and Markiewicz et al [21], bipartite measures have been utilised to construct a GME. The measures given by Pan et al [22] and by Carvalho et al [23] use addition of bipartite measures, which in general are non-zero for biseparable states and thus do not satisfy **P1**. Cai et al [24] have defined a GME based on von Neumann entropy. The well-known measure 3-tangle, which is an algebraic extension of concurrence given by Coffman et al [25] and later extended by Miyake [26], does not satisfy **P2** and is always zero for W-class entangled three
qubit states. Bounds on geometric measures of entanglement using z-spectral radius for some classes of pure states were given by Xiong et al [27]. Haddadi et al [28] give an overview of different geometric measures of multi-qubit entanglement.

In this work we aim to construct a GME using the geometry of the space of reduced density operators, for a subsystem of the of the full entangled system. The space of reduced density operators was earlier described by Calabrese et al [29]. We demonstrate a straightforward way to construct a measure of two-qubit entanglement based on a Riemannian metric, that does not rely on minimization procedures. We generalize this approach to bipartite systems of higher dimensions, that is then extended to pure multipartite systems of higher dimension to define a genuine measure of entanglement.

II. GEOMETRY OF QUANTUM STATE SPACE

We first discuss the geometry of the space of states of $d$-level quantum systems. States are represented by density operators on the finite-dimensional Hilbert space $\mathcal{H}^d$, which are $d \times d$ non-negative Hermitian operators $\rho$ of unit trace.

A very useful representation of such operators is affine parametrization using so-called generalized Pauli operators $\sigma_i$, which are $d^2 - 1$ traceless, Hermitian, mutually orthogonal operators satisfying $\text{Tr}(\sigma_i \sigma_j) = \delta_{ij} d$. Any $d$-dimensional Hermitian matrix can be represented in this parametrization, and a Hermitian trace 1 operator is given by

$$\hat{\rho}(x) = \frac{1}{d} \mathbb{1} + \frac{\sqrt{d - 1}}{d} \sum_{i=1}^{d^2 - 1} x_i \sigma_i,$$

with real coefficients $x_i$. The choice of the coefficients in this expression ensure normalization of $\hat{\rho}$. It represents a valid quantum state density operator by imposing further conditions to ensure non-negativity [31–33]. A necessary (but not sufficient) condition for this is that the norm of the vector formed by the $x_i$ is less than unity

$$\left(\sum_{i=1}^{d^2 - 1} x_i^2\right)^\frac{1}{2} = r \leq 1.$$  \hspace{1cm} (2)

The space spanned by the $x_i$ is thus a $d^2 - 1$-dimensional hyperspherical ball of radius $r = |\vec{x}|$. We will call this the parameter space $\mathcal{B}(\mathcal{H}^d) = \mathcal{B}_n$ of unit trace, Hermitian operators on $\mathcal{H}^d$. For pure states, we have the condition

$$\text{Tr} \rho^2 = 1 \implies r = 1.$$  \hspace{1cm} (3)

For the smallest non-trivial quantum system, namely a single qubit, Equation (2) is also a sufficient condition for non-negativity. Thus $\mathcal{B}_2$ is the 2-d Bloch ball, with pure states on the surface and mixed states in the interior, the maximally mixed state at the center $r = 0$.

However, for $d > 2$, the condition Equation (2) is not sufficient to ensure non-negativity. Further restrictions ([33]) select a subset $\mathcal{Q}_d \subset \mathcal{B}_d$. While all pure states lie on the boundary $\partial \mathcal{B}_d$, there are points on the surface $r = 1$ that do not correspond to states in $\mathcal{Q}_d$. Pure states are extreme points of $\mathcal{B}_d$, not just boundary points. Although many points inside $\mathcal{B}_d$ will not correspond to quantum states, the center of the hypersphere, $r = 0$ represents the maximally mixed state for all $d$. All pure states are equidistant from this point.

Riemannian monotone distances on this state space can be defined using spherical polar representation of $x_i$. The Morozova-\reflectbox{C}entsov-Petz theorem [34] classifies all possible metrics, which generally split into a radial part and an angular part, the latter having variations depending on the choice of a multiplicative factor depending on the Morozova-\reflectbox{C}entsov function. We will use the explicit form for such a metric on the single-qubit space, which also exhibits rotational invariance.

III. ENTANGLEMENT MEASURE FOR TWO-QUBIT PURE STATES

Given a distance measure $D(.,.)$ on the state space, a geometric entanglement measure for a pure state $\psi$ is defined as the distance to the closest separable state $\phi[35]$. If $\mathcal{S}$ is the set of all pure separable states,

$$E(\psi_{AB}) = \min_{\phi \in \mathcal{S}} D(\psi, \phi).$$ \hspace{1cm} (4)

For a generic state $\rho$ that could be mixed, a measure of entanglement is a positive real function $E(\rho)$ that is required to satisfy [36]: (E1) Monotonicity under stochastic local operations and classical communications (SLOCC), (E2) Discriminance: $E(\rho) = 0$ iff $\rho$ is separable and (E3) Convexity, among other desirable properties such as asymptotic continuity, normalizability and computability.

If we are to build an entanglement measure using distances in the space of density operators, the problem of minimization is hard. One way to circumvent this is to consider a corresponding distance in the reduced state space of a subsystem that is part of the entanglement.

Consider a state $\psi_{AB} \in \mathcal{Q}_{AB}$ of a bipartite system. The closest separable state $\phi_{AB} = \phi_A \otimes \phi_B$ maps to two pure states, one each on the surfaces of the reduced state spaces $\mathcal{Q}_A$ and $\mathcal{Q}_B$. In the reduced space $\mathcal{Q}_A$ of the subsystem $A$, we need to find the distance from the reduced state $\rho_A$ to the closest point on the surface. Any Riemannian metric can be used to prove that this point is radially outward on the surface, as illustrated in Figure 1. We construct an entanglement measure for $\psi_{AB}$ in terms of a suitably normalized Riemannian measure of this distance.

In the simplest case of a two-qubit pure state $\psi_{AB}$, partial trace over $B$ maps the state of the qubit $A$ to a point $a$ in the single qubit Bloch ball. The closest
connecting two states

The geodesic distance along a curve parametrized by \( \lambda \) corresponding to the space of pure single-qubit states. This reduces to the Fubini-Study metric on the surface of the maximally mixed state and diverges near pure states.

The non-Euclidean nature of this metric, and brings out the characteristic equation, which is computationally less efficient than finding the REM.

We test the performance of the REM by comparing with two popular measures namely the concurrence \( C(\psi) = \sqrt{2 - 2 \text{Tr} \rho_A^2} \) and the entanglement entropy \( E(\psi_{AB}) = -\text{Tr}(\rho_A \log \rho_A) \) for some classes of states that occur in the literature for their usefulness.

**Example 1.** Figure 3 shows comparative plots for superpositions of symmetric and antisymmetric states [38]:

\[
\left| \psi_1(\theta) \right> = \frac{1}{\sqrt{3}} \cos \theta \left( \left| \phi_0 \right> + \left| \phi_1 \right> + \left| \phi_2 \right> \right) + \sin \theta \left| \phi_3 \right>,
\]

where

\[
\left| \phi_0 \right> = \left| 00 \right>, \quad \left| \phi_1 \right> = \frac{1}{\sqrt{2}} \left( \left| 01 \right> + \left| 10 \right> \right), \quad \left| \phi_2 \right> = \left| 11 \right>, \quad \left| \phi_3 \right> = \frac{1}{\sqrt{2}} \left( \left| 01 \right> - \left| 10 \right> \right)
\]

symmetric, antisymmetric.

**Example 2:** In Figure 4, comparisons are made for entanglement measures on the “unconditionally entangled states” [39] that are superpositions of a Bell state \( \left| \beta_{01} \right> = \frac{1}{\sqrt{2}} \left( \left| 00 \right> - \left| 11 \right> \right) \) and an orthogonal product state \( |++ \rangle \) where \( |+ \rangle = \frac{1}{\sqrt{2}} \left( \left| 00 \right> + \left| 11 \right> \right) \):

\[
\left| \psi_2(\theta) \right> = \cos \theta \left| \beta_{01} \right> + \sin \theta |+ \rangle |+ \rangle.
\]

Both sets of examples show that the behaviour of REM is consistent with the behaviour of the other measures. This encourages us to extend our measure to higher number of qubits.
These classes are also generalized to which only one qubit in the state $|\psi_1(\theta)\rangle$ of Equation (9). Several works have discussed multipartite entanglement measures but few satisfy all the criteria for a GME. Now it was pointed out in by Love et al[17] and Yu et al[42] that any bipartite measure can be used to define a GME. We therefore define a GME using a geometric measure for bipartite entanglement.

### A. Bipartite measure of n-qubit entanglement

Along the same lines as our two qubit entanglement measure, we construct an REM for non-biseparability of tripartite states. A three-qubit system in a pure state $\psi_{ABC}$ allows three possible bipartitions: $(AB)C$, $A(BC)$ and $AB(C)$. Let us consider the bipartition $(AB)C$ for which the measure of entanglement (between the subsystems $AB$ and $C$) is the distance between the state $\psi_{ABC}$ and the closest biseparable state $\phi \in S_C$, the set of all biseparable states in the $(AB)C$ bipartition.

$$E(\psi_{(AB)C}) = \min_{\phi \in S_C} d(\psi, \phi).$$

Calculating this by minimization is computationally hard. Generalizing the REM of the previous section will reduce this complexity.

The reduced density operator $\rho_{AB}$ lives in the parameter space $B_2$. The closest biseparable state $\rho_p$ is radially outward on the surface of this hypersphere. We measure this distance using the Bures measure $D_B(\ldots)$ [43, 44], and normalize it to define a bipartite Riemannian entanglement measure for $(AB)C$ as

$$bREM(\psi_{(AB)C}) = \frac{D_B(\rho_{AB}, \rho_p)}{N_2},$$

$N_2$ being the normalization constant.

Normalized $bREM$. The 3-qubit GHZ state Equation (15) the absolute maximally entangled state, with respect to which we normalize our measure. While we might expect the maximum value to be 1, we find that the distance measure is typically less than 1. This is mainly because the rank of the 3-qubit state is less than the dimensionality of the $AB$ Hilbert space. This can be seen using a generalised Schmidt decomposition to explicitly cast the 3-qubit state into bipartite form[45]. The $AB$ subspace of the 3-qubit entangled $ABC$ system is four dimensional, whereas the generalised Schmidt form of the GHZ state has only two components:

$$|GHZ\rangle_{ABC} = \frac{1}{\sqrt{2}} (|000\rangle + |111\rangle)$$

with redefined basis vectors for $AB$ subsystem:

$$\{ |00\rangle, |11\rangle, |01\rangle, |10\rangle \} \equiv \{ |0'\rangle, |1'\rangle, |2'\rangle, |3'\rangle \}.$$

Equation (16) resembles the two qubit Bell state in the generalised Schmidt basis so that we might expect the
entanglement of \((AB)C\) bipartition to be maximal, i.e. 1. This ought to be reflected in the maximal randomness of measurement outcomes in the \(AB\) subsystem, equivalent to those of a Bell state. However this is true only for the projections \(\{|0\rangle\langle 0|, \ |1\rangle\langle 1|\}\). Projections along the other two basis vectors \(\{|2\rangle, \ |3\rangle\}\) are always definite (i.e. they are always zero!). Thus although the \((AB)C\) entanglement of the \((GHZ)\) state is maximal, it cannot measure to 1 since there is some residual determinacy in the measurement outcomes. This is borne out by the distance \(D_B(\rho_{AB|GHZ}, \rho_B) = 1 - (2 - \sqrt{2})^{1/2} < 1\). We use this as the value of \(N_2\) to normalize the measure \(bREM\).

It turns out to be computationally easier to evaluate this distance as the complement of the distance from the state at \(r = 0\) to \(\rho_{AB}\). The center of the Bloch ball \(r = 0\) represents the maximally mixed state \(I/4\). The Bures distance between two states can be expressed in terms of their density operators:

\[
D^2(\rho_1, \rho_2) = 2 - 2 \text{Tr}(\sqrt{\rho_1 \rho_2}).
\]

Using this one can obtain an algebraic expression for \(bREM\):

\[
bREM(\psi_{(AB)C}) = 1 - \frac{D(\rho_{AB|GHZ}, \rho_B)}{N_2} = 1 - \left(\frac{1}{2} - \text{Tr}\sqrt{\rho_B}\right)^{1/2}.
\]

We can similarly calculate the \(bREM\)'s of the other two bipartitions.

B. GME based on \(bREM\)

We define a GME as the geometric mean of \(bREM\)s of all possible bipartitions. We call this the GBR (Geometric mean of Bipartite Riemannian measures of entanglement):

\[
GBR(\psi_{ABC}) = (b_1 b_2 b_3)^{1/3}
\]

where \(b_i\) is the \(bREM\) for the \(i\)th bipartition.

This measure is readily generalised to the \(n\)-qubit case. The number \(m\) of unique possible bipartitions is given by[18],

\[
m = \begin{cases} \frac{(n-1)!}{2^{(n-1)/2}} \binom{n}{i} & \text{for odd } n, \\ \frac{(n-2)!}{2^{(n-2)/2}} \binom{n}{i} + \frac{1}{2} \binom{n/2}{n} & \text{for even } n. \end{cases}
\]

For each bipartition, we calculate \(b_i\) using the Bures distance from the reduced density operator of the larger partition, to its closest pure state. We choose the subsystem with larger \(d\) to calculate \(b_i\) since this leads to better smoothness. We then take the geometric mean:

\[
GBR(\psi_n) = \prod_{i=1}^{m} (b_i)^{1/m}.
\]

Properties of GBR. The multiqubit entanglement measure of Equation (20) satisfies all the desirable properties of a GME. In particular, it is invariant under local unitary transformations (LU-invariant) and monotonic under SLOCC. It is easy to show these properties by examining the same for the constituent \(bREM\)s.

The normalized \(bREM\) in terms of the reduced density operator \(\rho_i\) of a partition can be expressed algebraically as

\[
b_i = 1 - k \left(\sqrt{d} - \text{Tr}\sqrt{\rho_i}\right)^{1/2}, \text{ where } k = \frac{\sqrt{2}}{d^{1/4}}.
\]

Since trace is cyclically invariant, application of local unitaries on the system will leave this measure unchanged. Due to majorization [46], it is SLOCC monotonic. Another way to see that SLOCC monotonicity holds is that the Bures distance to the radially outward state is a positive function of the radial coordinate of the reduced density operator, which is monotonic under SLOCC[3].

It is easily verified that GBR also satisfies the properties (P1, P2, P3) required of a GME. For instance, (P1) and (P2) follow since GBR is defined using the product of bipartite measures. Calculations show that GBR\((GHZ) = 1\), while GBR\((W) = 0.94\), satisfying P3. Some useful features of this measure are discussed next.

V. DISCUSSION

We have derived a new pure-state entanglement measure, the GBR, whose main advantage is avoiding cumbersome minimization procedures to find the closest separable state. An interesting feature of our measure is that it takes into account the curvature of the space of states. This will distinguish it from measures that use Euclidean distances in the space of states.

We compare this measure with some GMEs recently proposed in the literature such as Generalised Geometric Mean (GGM)[47, 48], Geniunely Multipartite Concurrence (GMC)[49], Concurrence Fill (F)[50] and Geometric Mean of Bipartite Concurrence (GBC)[18].

Concurrence Fill is an elegant and visualisable measure for 3-qubit systems, but is hard to generalize to higher number of qubits. Though there are beautiful entanglement polygon inequalities[51], there is no reason in general to expect the GME based on bipartite concurrences to obey volume or area laws for the \(n\)-qubit case[52].

Discrimination is the effectiveness of different measures in distinguishing states that belong to LU-inequivalent classes. For 3-qubit states, there are only two SLOCC inequivalent classes, namely the GHZ-type and W-type[41]. All valid GME’s including GMC and GGM can differentiate the degree of entanglement between these two classes.
However, there are six sub-classes\cite{20, 53} based on the entanglement between different subsystems. Among these are four classes of genuinely entangled states, which form LU-equivalence classes. They can be identified by the non-zero coefficients in their generalised Schmidt decomposition (GSD). The GSD of a 3-qubit pure state takes the canonical form

\[ |\psi\rangle = \lambda_0 |000\rangle + \lambda_1 |100\rangle + \lambda_2 |101\rangle + \lambda_3 |110\rangle + \lambda_4 |111\rangle. \]

The four classes each have non-zero \(\lambda_0\) and \(\lambda_4\), and are differentiated by the the non-vanishing of the other \(\lambda_s\):

- **C1**: all of \(\lambda_1, \lambda_2, \lambda_3\) vanish, (GHZ-class);
- **C2**: any two of \(\lambda_1, \lambda_2, \lambda_3\) vanish;
- **C3**: any one of \(\lambda_1, \lambda_2, \lambda_3\) vanishes
- **C4**: none of \(\lambda_1, \lambda_2, \lambda_3\) vanish. (W-class).

It turns out that GGM and GMC, both of which use the minimum entanglement among all bipartitions, are relatively poor at discriminating certain states that belong to these LU-inequivalent classes. Xie et al\cite{50} show by example that F successfully differentiates the entanglement content of two states belonging to two different sub-classes where GMC and GGM fail. We demonstrate that GBR too is successful in discriminating these classes, using two example families of states:

\begin{align*}
\text{C1: } & |\chi_1(\theta)\rangle = \cos \frac{\theta}{2} |000\rangle + \sin \frac{\theta}{2} |111\rangle, \quad (21a) \\
\text{C2: } & |\chi_2(\theta)\rangle = \frac{1}{\sqrt{2}} (\sin \theta |000\rangle + \cos \theta |110\rangle + |111\rangle), \quad (21b)
\end{align*}

\[ 0 < \theta \leq \frac{\pi}{2} \]

\(|\chi_1(\theta)\rangle\) and \(|\chi_2(\theta)\rangle\) belong to classes C1 and C2 respec-
FIG. 6. Comparison of GBR with other GME’s for the state of class $\chi_3(\theta)$ to show better smoothness of GBR. The boxed region is zoomed in below to highlight the difference between the curves at higher resolution.

As shown in Figure 6, GGM and GMC show cusps in their variation with $\theta$. This is a limitation of measures involving non-analytic functions such as the minimum. In contrast, we see that GBC and GBR are smooth. Figure 6(b) is a close-up to highlight the difference between GBC and GBR, that appear to overlap in Figure 6(a).

VI. CONCLUSION

In this work we give a measure for genuine pure state entanglement using the Riemannian structure of quantum state space. The two-qubit measure REM is a computationally efficient method of calculating the distance to the closest separable state. This inspires an extension to multi-party systems through bipartitions. We construct a function, GBR, of the $n$-qubit state, that satisfies all the requisite properties of a Genuine Measure of Entanglement.

It is noteworthy that this measure uses entanglement information from all bipartitions. Therefore it is better at discriminating different LU-invariant sub-classes of three qubit states than some of the existing GME’s in literature that are based on minimum entanglement among bipartitions.

GBR explicitly uses the Riemannian structure of state space, and consequently varies smoothly with state parameters. Entropic measures for instance, are functions of the Euclidean distance, and do not pick up the curvature of the state space.

While measures like concurrence fill do not easily generalise to $n$ qubits, GBR by its definition can be constructed for all finite dimensional pure states.

As for mixed states, this measure can be readily extended using convex roof construction. A detailed analysis of computational cost reduction is work in progress.

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