Minimizing the expected search time of finding the hidden object by maximizing the discount effort reward search

Mohamed Abd Allah El-Hadidy and Ajab A. Alfreedi

ABSTRACT
A new search technique is developed to locate the hidden target (object) in one of the N-disjoint regions that are not identical. The lost object follows a bivariate distribution. Minimizing the search effort with discount reward has been applied instead of reducing the expected search time. Moreover, the minimum number of searchers is determined in order to minimize the total expected cost. Assuming the object’s position has a Circular Normal distribution, the Kuhn–Tucker necessary conditions are implemented to get the optimum search plan.

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1. Introduction
The detection of explosives in places where they are likely to be present in the shortest possible time will prevent many innocent victims. Minimizing the time and the effort to detect one of the explosive needs to increase the number of searchers and the coordination process among them. The case of the randomly located object with a known distribution on the real line has been discussed by Reyniers [1,2]. She considered two unit speeds searchers aim to find this object in the shortest possible time. In addition, Mohamed et al. [3,4] studied this problem for the hidden object which has a known distribution in an open area.

On the other hand, for a stochastically moving object, El-Hadidy and Abou-Gabal [5] and El-Hadidy and Alzuulaibani [6,7] used a linear coordinated search technique to present a finite search plan which minimizes the first collision time expected value between one of the searchers and the stochastically moving object. Moreover, this technique is applied by using the Bayesian approach (see, e.g. [8–11]). Earlier, many interesting methods to track the stochastically moving object have been presented by Dai et al. [12] and Deilami et al. [13]. Besides that, Mohamed et al. [14,15], Mohamed and El-Hadidy [16], Mohamed and El-Hadidy [17], Beltagy and El-Hadidy [18], Abou-Gabal and El-Hadidy [19], Mohamed et al. [20], Kassem and El-Hadidy [21], El-Hadidy and El-Bagoury [22], El-Hadidy [23–31], El-Hadidy and Alzuulaibani [6,7], and El-Hadidy et al. [32] provided many different mathematical treatments of this issue in both cases stochastically moving and hidden objects.

This paper aims to coordinate the search technique that allows the M-searchers $S_j, j = 1, 2, \ldots, M$ (where $M$ is an even number), start together and searching for a hidden object from the centre of each region $R_\ell, \ell = 1, 2, \ldots, N$ (a point $(0, 0)$), as shown in Figure 1 (the search path in the region $R_\ell$ which has 4-unit speed searchers). The purpose is to find the minimum expected value of the detection time by achieving the optimal search plan after applying the discount effort reward function which has been applied before in [25].

In this work, Section 2 explains the problem and presents the expected reward cost of detection. In Section 3, the optimal search policy that minimizes the expected reward cost of detection is presented after considering the Circular Normal distribution of the object’s position. A discussion of the results and future works is presented in the conclusion part.

2. Problem formulation
The mathematical model of this problem is formulated by considering the discounted search effort reward. This model gives the expected value of the discounted effort reward for detection in one of N-disjoint and not identical regions $R_\ell, \ell = 1, 2, \ldots, N$. 

CONTACT Mohamed Abd Allah El-Hadidy melhadidi@science.tanta.edu.eg

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The search space: The space is divided into $N$ disjoint and not identical regions $R_\ell$, $\ell = 1, 2, \ldots, N$. Each region has two roads intersected at the centre of this region. These roads divide the region into four identical parts. The two roads are considered as $x$ and $y$ axes (see Figure 1). Here, the report for the object position is given at $(0, 0)$.

The object: The object is randomly located in one of these regions with symmetric distribution about $(0, 0)$.

The means of search: Let $M$ searchers start the searching process for the object from the $(0, 0)$ in each region. Each region has an even number of searchers. The searchers go along the two axes (positive and negative parts) with equal speeds. The sectors and their tracks are searched with regular speed. The searchers return to $(0, 0)$ and still searching until the object detected.

2.1. The searching process

Let the position of the object in $R_\ell$ be defined by the two independent random variables ($X, Y$). The surface of $R_\ell$ is a “Standard Euclidean 2-space $E$”, with points $(x, y)$. In $R_\ell$, we have four searchers, $S_{k\ell}, k = 1, 2, 3, 4$, each of them is always searching one part from the four parts (see Figure 1). We will divide the region into many sectors, as shown in Figure 1.

2.2. 4-Coordinated search technique in the region $R_\ell$

2.2.1. The Searching path

To find the object, the searchers $S_{k\ell}, k = 1, 2, 3, 4$, follow $\delta_{k\ell}, \lambda_{k\ell}, \Xi_{k\ell}$ and $\Omega_{k\ell}$ (the search paths), respectively. The first search path $\delta_{1\ell}$ of $S_{1\ell}$ is defined as follows:

(i) Begin at the point $(0, 0)$ and go along the right part of the $y$-axis (positive part) as far as $a_{1\ell}$.

(ii) Look for the object in $g_{1\ell}$ and its track.

(iii) When $S_{1\ell}$ reaches the point $(-a_{1\ell}, 0), S_{1\ell}$ will return to the origin through $-ve$ part of the $x$-axis.

In addition, one can define the second search path $\delta_{2\ell}$ of $S_{2\ell}$ as in the above steps (ii), (ii) and (iii), where $S_{2\ell}$ goes a distance $a_{2\ell}$ to search the sector $g_{2\ell}$ and its track, etc. Thus, $\delta_{k\ell}$ of $S_{k\ell}$ is completely defined by a sequence $[\delta_{k\ell}, i \geq 0]$. The first search path $\lambda_{1\ell}$ of $S_{2\ell}$ is defined as follows:

(a) Begin at $(0, 0)$ and go along the left part of the $y$-axis ($-ve$ part) as far as $a_{1\ell}$.

(b) Search the sector $h_{1\ell}$ and its track.

(c) When $S_{2\ell}$ reaches the point $(0, a_{1\ell})$, $S_{2\ell}$ will return to the origin through $-ve$ part of the $x$-axis.

And, $\lambda_{2\ell}$ (the second search path) of $S_{2\ell}$ is defined as in the above steps from (a) to (c), where $S_{2\ell}$ goes a distance $a_{2\ell}$ to search $h_{2\ell}$ and its track, etc. Thus, $\lambda_{k\ell}$ of $S_{2\ell}$ is completely defined by the sequence $[\lambda_{k\ell}, i \geq 0]$.

Also, by considering the searchers’ movement on the sectors and tracks done in anticlockwise, then the search paths of $S_{3\ell}, S_{4\ell}$ are $[\Omega_{3\ell}, i \geq 0]$ and $[\Xi_{3\ell}, i \geq 0]$, respectively, where $S_{3\ell}, S_{4\ell}$ search the parts III and IV.

Each searcher goes along the $x$-axis with speed $v = 1$ and searches the circles, the tracks with regular speed $\beta_{\ell}$. The time that the searcher takes it through going on the $x$-axis will add to the time of the searching process (sectors and tracks searching time).

Let the surface of $R_\ell$ be a “Standard Euclidean 2-space $E$” and the object position has the probability density function $f(x, y)$. Also, let $t_{k\ell}, i = 1, 2, \ldots, k = 1, 2, 3, 4$ be the time that the searchers $S_{k\ell}, k = 1, 2, 3, 4$ take it in $[\delta_{k\ell}, i \geq 0], [\lambda_{k\ell}, i \geq 0], [\Omega_{k\ell}, i \geq 0]$ and $[\Xi_{k\ell}, i \geq 0]$ in the four parts to $(0, 0)$, where any track $i$ has a width $a_{i\ell} - a_{(i-1)\ell}$. They go on the $y$-axis and $x$-axis from the origin before searching the sectors. In addition, they return after finishing the searching process with equal speeds $v = 1$ to $(0, 0)$. Then, the time of going through the $y$-axis is equal to the distances which done. They are searching $g_{1\ell}, h_{1\ell}, w_{1\ell}, z_{1\ell}, i = 1, 2, \ldots$ (sectors and tracks) with $\beta_{\ell}$. Then, the searching time is equal to $t_{1\ell} = 2\pi/\omega_{1\ell}$, where $\tau_{1\ell}$ is the time league and $\omega_{1\ell}$ is called angular velocity. The searching time $\tau_{1\ell}$ depends on $\omega_{1\ell}$ which depends on the radius $a_{1\ell}$ and the time of detection $t(\phi_{\ell})$.

We choose the discounted effort function as an exponential function $D_{i}(i) = d_{i}^{\alpha}, 0 < d_{i} < 1$, that will reduce the possible rewards at the revolution number $i$ (see [25]). The adjust parameter $d_{i}$ gives permission to make the decision indirectly, and this helps the searcher to take appropriate actions in the future.

**Theorem 2.1**: the expected reward cost of detection for the lost object is given by
\[
E(t|\phi_t) = \sum_{i=1}^{\infty} \left( 8a_{i\ell} + \frac{2\pi}{\omega_{1\ell}} \right) \cdot 4d_{i\ell}
\]

\[
\cdot \left[ \sum_{i=0}^{\infty} \left( \sum_{j=0}^{n} \int_{\theta_{i-1\ell}}^{\theta_{i\ell}} \int_{\theta_{j-1\ell}}^{\theta_{j\ell}} g_{(r, \theta_{i\ell})} (r, \theta) r_{i\ell} d\theta d\theta_{i\ell} \right) \right].
\]

(1)

**Proof:** The object may be in one of the four parts in \( R_t \). Thus, more appropriate formulas for the expected time are available. This leads to the following:

If the object is located at any point on the track of \( g_{1\ell} \), then \( t_{2\ell} = 2a_{1\ell} + \pi/2\omega_{1\ell} \).

If the object is located at any point on the track of \( g_{2\ell} \), then \( t_{2\ell} = 2(a_{1\ell} + a_2 + a_3 + \pi/2)(1/\omega_{1\ell} + 1/\omega_{2\ell}) \).

If the object is located at any point on the track of \( g_{3\ell} \), then \( t_{2\ell} = 2(a_{1\ell} + a_2 + a_3 + \pi/2)(1/\omega_{1\ell} + 1/\omega_{2\ell} + 1/\omega_{3\ell}) \), etc.

If the object is located at any point on the track of \( h_{1\ell} \), then \( t_{1\ell} = a_{1\ell} + \frac{1}{2}(2\pi/\omega_{1\ell}) + a_2 = 2a_{1\ell} + \pi/2\omega_{1\ell} \).

If the object is located at any point on the track of \( h_{2\ell} \), then \( t_{1\ell} = 2(a_{1\ell} + a_2 + \pi/2)(1/\omega_{1\ell} + 1/\omega_{2\ell}) \).

If the object is located at any point on the track of \( h_{3\ell} \), then \( t_{1\ell} = 2(a_{1\ell} + a_2 + a_3 + \pi/2)(1/\omega_{1\ell} + 1/\omega_{2\ell} + 1/\omega_{3\ell}) \), etc.

If the object is located at any point on the track of \( w_{1\ell} \), then \( t_{1\ell} = a_{1\ell} + \frac{1}{2}(2\pi/\omega_{1\ell}) + a_2 = 2a_{1\ell} + \pi/2\omega_{1\ell} \).

If the object is located at any point on the track of \( w_{2\ell} \), then \( t_{1\ell} = 2(a_{1\ell} + a_2 + \pi/2)(1/\omega_{1\ell} + 1/\omega_{2\ell}) \).

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If the object is located at any point on the track of \( z_{1\ell} \), then \( t_{2\ell} = 2a_{1\ell} + \pi/2\omega_{1\ell} \).

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If the object is located at any point on the track of \( z_{3\ell} \), then \( t_{2\ell} = 2(a_{1\ell} + a_2 + a_3 + \pi/2)(1/\omega_{1\ell} + 1/\omega_{2\ell} + 1/\omega_{3\ell}) \), etc.

Each sector is divided into equal small sectors \( I_{u\ell} \), \( u = 1, 2, \ldots, n \), where these sectors make a set of equal cones. As in Figure 2, these cones have the same vertex (0, 0). Thus, each searcher can cover a track with width \( a_{i\ell} - a_{(i-1)\ell} \) which has equal small areas of cones in the track number \( i \). These cones are determined by a set of lines with equations \( x = m_{u\ell} y = \tan \theta_{u\ell} \), y, where \( \theta_{u\ell} = \theta_{(u-1)\ell} + \pi/2\omega_{1\ell} \), \( u = 1, 2, \ldots, n \). These equations give a range of equal small spaces to equalize the searching process. Applying the polar coordinates with \( x = r \cos \theta \) and \( y = r \sin \theta \), \( r \leq a_{i\ell} \), \( i = 1, 2, 3, \ldots \), and \( \theta : \theta_{i\ell} = \theta_{(i-1)\ell} \rightarrow \theta_{i\ell} \), \( u = 1, 2, 3, \ldots, n \), where \( a_{i\ell} = r_{i\ell} = 0 \) and \( \theta_{i\ell} = 0 \) to evaluate the expected searching time to detect the object. The searching process is performed in the anticlockwise direction.

By using our assumptions where the object has symmetric distribution and applying the discounted effort

**Figure 2.** The small search area in \( R_t \) which is made by small sectors \( I_{u\ell} \), \( u = 1, 2, \ldots, n \), which made by the sectors inside the circles with radiusses \( a_{i\ell} \), \( i = 1, 2, 3, \ldots \) 

reward function in each revolution, we have

\[
E(t|\phi_t) = \left( 2a_{1\ell} + \frac{\pi}{2\omega_{1\ell}} \right) \cdot d_{1\ell}^{1}
\]

\[
\cdot \left( \int_{0}^{a_{1\ell}} \int_{0}^{\theta_{1\ell}} g_{(r, \theta)}(r, \theta) r_{1\ell} d\theta d\theta_{1\ell} + \cdots \right)
\]

\[
+ \int_{0}^{a_{1\ell}} \int_{0}^{\theta_{1\ell}} g_{(r, \theta)}(r, \theta) r_{1\ell} d\theta d\theta_{1\ell}
\]

\[
+ \left( 2(a_{1\ell} + a_2) + \frac{\pi}{2} \left( \frac{1}{\omega_{1\ell}} + \frac{1}{\omega_{2\ell}} \right) \right) \cdot (d_{1\ell}^{1} + d_{2\ell}^{1})
\]

\[
\times \left( \int_{0}^{a_{2\ell}} \int_{0}^{\theta_{2\ell}} g_{(r, \theta)}(r, \theta) r_{2\ell} d\theta d\theta_{2\ell} + \cdots \right)
\]

\[
+ \int_{0}^{a_{2\ell}} \int_{0}^{\theta_{2\ell}} g_{(r, \theta)}(r, \theta) r_{2\ell} d\theta d\theta_{2\ell}
\]

\[
+ \left( 2(a_{1\ell} + a_2 + a_3) + \frac{\pi}{2} \right) \cdot \left( \frac{1}{\omega_{1\ell}} + \frac{1}{\omega_{2\ell}} + \frac{1}{\omega_{3\ell}} \right) \cdot (d_{1\ell}^{1} + d_{2\ell}^{1} + d_{3\ell}^{1})
\]

\[
\times \left( \int_{0}^{a_{3\ell}} \int_{0}^{\theta_{3\ell}} g_{(r, \theta)}(r, \theta) r_{3\ell} d\theta d\theta_{3\ell} + \cdots \right)
\]

\[
+ \left( 2a_{1\ell} + \frac{\pi}{2\omega_{1\ell}} \right) \cdot d_{1\ell}^{1}
\]

\[
\times \left( \int_{0}^{a_{1\ell}} \int_{0}^{\theta_{1\ell}} g_{(r, \theta)}(r, \theta) r_{1\ell} d\theta d\theta_{1\ell} + \cdots \right)
\]

\[
+ \int_{0}^{a_{1\ell}} \int_{0}^{\theta_{1\ell}} g_{(r, \theta)}(r, \theta) r_{1\ell} d\theta d\theta_{1\ell}
\]
\[
\begin{align*}
&M. A. A. EL-HADIDY AND A. A. ALFREEDI
\end{align*}
\]
we can notice that the expected value of the time in the
other searches the sectors in the right part of the y-axis and the
return to the expected reward cost of detection for the two searchers to
the object has been detected as in (2) after the object has been detected in (2) will
become

\[ E(t(\phi_k)) = \sum_{i=1}^{\infty} \left( 4a_{i\ell} + \frac{\pi}{\omega_{i\ell}} \right) \cdot 2d_i \]

which is less than the expected value of the time in (2). Also, we can notice that the expected value of the time in the case of \( q_i \) searchers, where \( q_i \) is even number, is smaller than the expected value of \( (q_i - 2) \) searchers; this leads to the following:

**Theorem 2.3:** For any even number \( q_i \) of searchers in one of the non-identical \( N \)-regions, where \( \sum_{i=1}^{N} q_i = M \), the total expected reward cost of detection is given by

\[ E(t(\phi)) = \sum_{i=1}^{\infty} \left( \sum_{\ell=1}^{N} \left( 2q_i a_{i\ell} + \frac{q_i \pi}{2 \omega_{i\ell}} \right) \cdot q_i d_i \right) \cdot \int g(\theta, r) r dr d\theta \]

**Proof:** By the same method that prove Theorem 2.1 we can prove this theorem.

**Corollary 2.4:** If the width is fixed (i.e. \( a_{i\ell} = a_{(i-1)\ell} = a \)), then \( a_1 = a, a_2 = 2a, a_3 = 3a, \ldots \), and if \( q_i = q_i \theta_{i\ell} - \theta_{(i-1)\ell} = \theta \) then in (4) we have

\[ E(t(\phi)) = \sum_{i=1}^{\infty} \left( 2a + \frac{\pi}{2 \omega_i} \right) \cdot N_i q_i^2 d_i \cdot \int g(\theta, r) r dr d\theta \]

The above result shows that this technique is more suitable to detect an important object (like a bomb or a person in a wilderness area) by using \( q_i \) searchers.

3. Optimal search plan

Since the main contribution of this technique is to minimize the expected cost of detection, then we need to get the optimal search plan \( \phi^* \in \Phi \), which gives

\[ E(t(\phi^*)) = \min \{ E(t(\phi)), \phi \in \Phi \} \]

**Proof:** Let us assume, from now on, that \( a_{i\ell} = a_{(i-1)\ell} = a \), and \( \beta^* = \inf \{ \beta \} \).

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The above result shows that this technique is more suitable to detect an important object (like a bomb or a person in a wilderness area) by using \( q_i \) searchers.
It is clear that, if $\hat{Q} \subseteq Q$ (the class of all possible search plans) for which there is only one element and if $a^*_{t,\ell}$ is an optimal search path on the $x$-axis, then all the optimal search paths will be $\delta^*, \lambda^*, \Sigma^*$ and $\Omega^*$ which belongs to $\hat{Q}$. Consequently, besides the condition $A(F) < \infty$, we can assume that for the necessary condition on the known object’s distribution, there exists a search path $\delta^* = (\delta^*_i, i \geq 0, \ell = 1, 2, \ldots, N)$, $\lambda^* = (\lambda^*_i, i \geq 0, \ell = 1, 2, \ldots, N)$, $\Sigma^* = (\Sigma^*_i, i \geq 0, \ell = 1, 2, \ldots, N)$ and $\Omega^* = (\Omega^*_i, i \geq 0, \ell = 1, 2, \ldots, N)$ from class $Q$ such that $D(\delta^*, \lambda^*, \Sigma^*, \Omega^*, D^*; F) = m$.

**Theorem 3.1:** If $A(F) < \infty$ and $\mathcal{D}(i) = d^*_t, 0 < d^*_\ell < 1$ in each region $R_{t,\ell}, \ell = 1, 2, \ldots, N$, then there exists a search path from class $Q$ with finite expected reward cost, which can also lead to $m < \infty$ [33].

After proving that objective function (4) is convex, we will use the Kuhn–Tucker conditions to obtain these optimal values which minimizes $E(t(\phi))$. Hence, we will obtain the following non-linear programming problem (NLP(1)):

**NLP(1):**

\[
\begin{align*}
\text{min} & \quad \max_{a_{t,\ell}, d^*_t} E(t(\phi)) \\
\text{subject to} & \quad 0 \leq \sum_{i=1}^{\infty} \sum_{\ell=0}^{N} \sum_{i=1}^{n} \int_{a_{t-1} \ell}^{a_{t} \ell} g_{\ell} \left( r_t, \theta_{\ell} \right) dr_t d\theta_{\ell} = 1, \\
& \quad a_{t,\ell} - a_{t-1,\ell} > 0, \quad q_{\ell} > 0, \\
& \quad 0 < d^*_\ell < 1, \quad i = 2, 3, \ldots, \infty,
\end{align*}
\]

where

\[
E(t(\phi)) = \sum_{t=1}^{N} \left( \sum_{i=1}^{n} \int_{a_{t-1} \ell}^{a_{t} \ell} \int_{\theta_{t-1} \ell}^{\theta_{t} \ell} g_{\ell} \left( r_t, \theta_{\ell} \right) dr_t d\theta_{\ell} \right).
\]

This is equivalent to the following:

**NLP(2):**

\[
\begin{align*}
\min & \quad \sum_{t=1}^{N} \left( \sum_{i=1}^{n} \int_{a_{t-1} \ell}^{a_{t} \ell} \int_{\theta_{t-1} \ell}^{\theta_{t} \ell} g_{\ell} \left( r_t, \theta_{\ell} \right) dr_t d\theta_{\ell} - d^*_\ell \right) \\
\text{subject to} & \quad \sum_{i=1}^{n} \int_{a_{t-1} \ell}^{a_{t} \ell} \int_{\theta_{t-1} \ell}^{\theta_{t} \ell} g_{\ell} \left( r_t, \theta_{\ell} \right) dr_t d\theta_{\ell} \leq 1, \\
& \quad a_{t,\ell} - a_{t-1,\ell} > 0, \quad q_{\ell} > 0, \\
& \quad 0 < d^*_\ell < 1, \quad i = 2, 3, \ldots, \infty.
\end{align*}
\]

From the Kuhn–Tucker conditions, we obtain

\[
\begin{align*}
\frac{\partial E(t(\phi))}{\partial a_{t,\ell}} &= \sum_{x=1}^{4} U_x \frac{\partial G_i(a_{t,\ell}, q_{\ell}, d^*_t)}{\partial a_{t,\ell}} = 0, \\
\frac{\partial E(t(\phi))}{\partial q_{\ell}} &= \sum_{x=1}^{4} U_x \frac{\partial G_i(a_{t,\ell}, q_{\ell}, d^*_t)}{\partial q_{\ell}} = 0, \\
\frac{\partial E(t(\phi))}{\partial d^*_t} &= \sum_{x=1}^{4} U_x \frac{\partial G_i(a_{t,\ell}, q_{\ell}, d^*_t)}{\partial d^*_t} = 0,
\end{align*}
\]

\[
\begin{align*}
\frac{\partial G_i(a_{t,\ell}, q_{\ell}, d^*_t)}{\partial a_{t,\ell}} &\leq 0, \\
\frac{\partial G_i(a_{t,\ell}, q_{\ell}, d^*_t)}{\partial q_{\ell}} &\leq 0, \\
\frac{\partial G_i(a_{t,\ell}, q_{\ell}, d^*_t)}{\partial d^*_t} &\leq 0, \\
U_x \frac{\partial G_i(a_{t,\ell}, q_{\ell}, d^*_t)}{\partial d^*_t} &\geq U_x, \quad U_x \geq 0.
\end{align*}
\]

Since $\omega_{t,\ell} = \beta / a_{t,\ell}$, we have

\[
\begin{align*}
\sum_{t=1}^{N} \sum_{i=1}^{n} \left( \frac{2q_{\ell} - 2\omega_{t,\ell} q_{\ell} \pi}{4\omega_{t,\ell}} \right) \cdot q_{\ell} \left( 1 - d^*_\ell \right) \\
+ \sum_{t=1}^{N} \sum_{i=1}^{n} \int_{a_{t-1} \ell}^{a_{t} \ell} \int_{\theta_{t-1} \ell}^{\theta_{t} \ell} g_{\ell} \left( r_t, \theta_{\ell} \right) r_t dr_t d\theta_{\ell} \\
+ \sum_{t=1}^{N} \sum_{i=1}^{n} \left( \frac{2q_{\ell} a_{t,\ell} + q_{\ell} \pi}{2\omega_{t,\ell}} \right) \cdot q_{\ell} \left( 1 - d^*_\ell \right) \\
+ \sum_{t=1}^{N} \sum_{i=1}^{n} \int_{a_{t-1} \ell}^{a_{t} \ell} \int_{\theta_{t-1} \ell}^{\theta_{t} \ell} g_{\ell} \left( r_t, \theta_{\ell} \right) r_t dr_t d\theta_{\ell} \\
+ \sum_{t=1}^{N} \sum_{i=1}^{n} \left( \frac{2q_{\ell} a_{t,\ell} + q_{\ell} \pi}{2\omega_{t,\ell}} \right) \cdot q_{\ell} \left( 1 - d^*_\ell \right) \\
+ \sum_{t=1}^{N} \sum_{i=1}^{n} \int_{a_{t-1} \ell}^{a_{t} \ell} \int_{\theta_{t-1} \ell}^{\theta_{t} \ell} g_{\ell} \left( r_t, \theta_{\ell} \right) r_t dr_t d\theta_{\ell} \\
+ \sum_{t=1}^{N} \sum_{i=1}^{n} \left( \frac{2q_{\ell} a_{t,\ell} + q_{\ell} \pi}{2\omega_{t,\ell}} \right) \cdot q_{\ell} \left( 1 - d^*_\ell \right) \\
+ \sum_{t=1}^{N} \sum_{i=1}^{n} \int_{a_{t-1} \ell}^{a_{t} \ell} \int_{\theta_{t-1} \ell}^{\theta_{t} \ell} g_{\ell} \left( r_t, \theta_{\ell} \right) r_t dr_t d\theta_{\ell} \\
+ \sum_{t=1}^{N} \sum_{i=1}^{n} \left( \frac{2q_{\ell} a_{t,\ell} + q_{\ell} \pi}{2\omega_{t,\ell}} \right) \cdot q_{\ell} \left( 1 - d^*_\ell \right) \\
+ \sum_{t=1}^{N} \sum_{i=1}^{n} \int_{a_{t-1} \ell}^{a_{t} \ell} \int_{\theta_{t-1} \ell}^{\theta_{t} \ell} g_{\ell} \left( r_t, \theta_{\ell} \right) r_t dr_t d\theta_{\ell} + U_x \left( -1 \right) = 0.
\end{align*}
\]
Many cases have been found to solve Equations (5)–(11) as follows:

\( U_{ij} = 0 \), \( \chi = 1, 2, 3, 4; \)  
\( U_{ij} > 0 \), \( \chi = 1, 2, 3, 4; \)  
\( N \)  
\( \sum \)

3.1 The case of the initial position given by a Circular Normal distribution

In all search strategies, we consider that the debris diffusion of an aeroplane crash over the oceans and seas have a Circular Normal distribution, why? By considering the disaster of Air Force Flight 447, Stone et al. [34] answered this question. They proved that all impact points of debris are found within a 20 nautical mile radius circle from the crashed point. After they analysed the data about these impact points of debris, they found that the distribution of these points is a Circular Normal distribution with centre at the last known position. Thus, we let \( X, Y \) are two independent random variables that represent the last position of the object (black box), and they have a Circular Normal distribution with joint probability density function which considered in [27]:

\[
f(x, y) = \frac{1}{2\pi \sigma^2} \exp \left[ -\frac{(x^2 + y^2)}{2\sigma^2} \right],
\]

\[-\infty \leq x, y \leq \infty.\]  

(15)
It is noticed from (16) to (18) that \( a_2 \) is a function of \( a_1 \) and \( a_{i+1} \) is a function of \( a_i \). Let the set of the critical search paths is not empty, then we can address ourselves to solve (16)–(18) to obtain the optimal values of searchers in one of the non-identical \( N \)-regions. By assuming the Circular Normal distribution of the object position, we obtain the optimal values of \( a_{it}^*, d_{it}^* \), \( q_{it}^* \), \( i = 1, 2, \ldots, \infty, \ell = 1, 2, \ldots, N \), that give the optimal search plan after solving a difficult optimization problem.

In the future, this proposed model will be generalized to find multiple hidden objects by using a group of searchers.

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**ORCID**

Mohamed Abd Allah El-Hadidy [http://orcid.org/0000-0002-9407-9586](http://orcid.org/0000-0002-9407-9586)

Ajab A. Alfreedi [http://orcid.org/0000-0003-0365-4555](http://orcid.org/0000-0003-0365-4555)

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