Cohomology of vector bundles and non-pluriharmonic loci

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Abstract
In this paper, we study cohomology groups of vector bundles on neighborhoods of a non-pluriharmonic locus in Stein manifolds and in projective manifolds. By using our results, we show variants of the Lefschetz hyperplane theorem.

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1 Introduction
Let \( X \) be a Stein manifold. Let \( \varphi \) be an exhaustive plurisubharmonic function on \( X \). The support of \( i\partial \bar{\partial} \varphi \) has some interesting properties. In this paper, we study the cohomology of holomorphic vector bundles on open neighborhoods of \( \text{supp} \: i\partial \bar{\partial} \varphi \). Here we denote by \( \text{supp} \: T \) the support of a current \( T \). Let \( F \) be a holomorphic vector bundle over \( X \). Let \( A \subset X \) be a closed set. For any open neighborhoods \( V \subset U \) of \( A \), the inclusion map induces \( \mathcal{H}_q(U, F) \to \mathcal{H}_q(V, F) \). We define the direct limit \( \lim_{\to} \bigcap_{j=1}^m \text{supp} \: i\partial \bar{\partial} \varphi_j \subset V \) where \( V \) runs through all open neighborhoods of \( A \). Our first main result is the following:

**Theorem 1** Let \( X \) be a Stein manifold of dimension \( n \) (\( n \geq 3 \)). Let \( m \) be a positive integer which satisfies \( 1 \leq m \leq n - 2 \). Let \( \varphi_1, \ldots, \varphi_m \) be non-constant plurisubharmonic functions on \( X \) such that for every \( r < \sup_X \varphi_j \) the sublevel set \( \{ z \in X \mid \varphi_j(z) \leq r \} \) is compact (\( 1 \leq j \leq m \)). Let \( F \) be a holomorphic vector bundle over \( X \). Then the natural map

\[
H^0(X, F) \to \lim_{\to} \bigcap_{j=1}^m \text{supp} \: i\partial \bar{\partial} \varphi_j \subset V \quad \mathcal{H}^q(V, F),
\]

is an isomorphism and

\[
\lim_{\to} \bigcap_{j=1}^m \text{supp} \: i\partial \bar{\partial} \varphi_j \subset V \quad \mathcal{H}^q(V, F) = 0
\]

for \( 0 < q < n - m - 1 \).
Let \( X \) be a projective manifold. We have the Hodge decomposition \( H^2(X, \mathbb{C}) = H^{2,0}(X, \mathbb{C}) \oplus H^{1,1}(X, \mathbb{C}) \oplus H^{0,2}(X, \mathbb{C}) \). Define \( H^{1,1}(X, \mathbb{C}) = H^{1,1}(X, \mathbb{C}) \cap H^2(X, \mathbb{R}) \). Let \( NS(X) \otimes_{\mathbb{Z}} \mathbb{R} \subset H^{1,1}(X, \mathbb{R}) \) be the Neron-Severi group of \( X \) tensored with the real numbers. Let \( K_{NS} \subset NS(X) \otimes \mathbb{R} \) be the open cone generated by classes of ample divisors (see Section 6 of [4]). Our second main result is the following:

**Theorem 2** Let \( X \) be a projective manifold of dimension \( n \) \((n \geq 3)\). Let \( m \) be a positive integer which satisfies \( 1 \leq m \leq n - 2 \). Let \( T_1, \ldots, T_m \) be closed positive currents of type \((1, 1)\) on \( X \) whose cohomology classes belong to \( K_{NS} \). Let \( F \) be a holomorphic vector bundle over \( X \). Then the natural map

\[
H^q(X, F) \to \lim_{\cap_j \supp T_j \subset V} H^q(V, F)
\]

is an isomorphism for \( 0 \leq q < n - m - 1 \) and is injective for \( q = n - m - 1 \).

Let \( t \in H^0(X, L) \) be a non-zero holomorphic section of the ample line bundle \( L \) over \( X \). We define the hypersurface \( Y = \{ z \in X \mid t(z) = 0 \} \). Let \( h_0 \) be a smooth Hermitian metric of \( L \) such that \( \omega_0 := 2\pi c_1(L, h_0) \) is Kähler form. Here \( c_1(L, h_0) \) is the Chern form of \( L \) associated to \( h_0 \). Then \( Y = \supp(\omega_0 + i \partial \overline{\partial} \log |t|_{h_0}^2) \). Because of the vanishing theorem of cohomology groups with compact supports in the Stein manifold \( X \setminus Y \), we have that the natural map

\[
H^q(X, F) \to \lim_{Y \subset V} H^q(V, F)
\]

is an isomorphism for \( q < n - 1 \) and is injective for \( q = n - 1 \). Theorem 2 is a counterpart of this result. Unfortunately, we do not know whether Theorems 1 and 2 hold in the case when the degree is \( n - m \).

Let \( T^*_X \) be the holomorphic cotangent bundle over \( X \). If we take \( F = \bigwedge^p T^*_X \), our main results imply the following variants of the Lefschetz hyperplane theorem (see Lemma 1 of [10]).

**Corollary 1** Let \( X, \varphi_1, \ldots, \varphi_m \) be as in Theorem 1. Then the natural map

\[
H^q(X, \mathbb{C}) \to \lim_{\cap_j \supp (i \partial \overline{\partial} \varphi_j) \subset V} H^q(V, \mathbb{C})
\]

is an isomorphism for \( q < n - m - 1 \) and is injective for \( q = n - m - 1 \).

**Corollary 2** Let \( X, T_1, \ldots, T_m \) be as in Theorem 2. Then the natural map

\[
H^q(X, \mathbb{C}) \to \lim_{\cap_j \supp T_j \subset V} H^q(V, \mathbb{C})
\]

is an isomorphism for \( q < n - m - 1 \) and is injective for \( q = n - m - 1 \).

Corollary 1 generalizes the main theorem of [10]. The degree which appears in [10] is \( \max\{n - 4, 1\} \). On the other hand, those which appear in our main results are \( n - 2 \) when \( m = 1 \). The improvement of the degree is due to the method of Lee and Nagata ([8]) and the estimate of the Sobolev norm.

We prove the case \( m = 1 \) of Theorems 1 and 2 in Sect. 3, 4 and 5. By using Mayer–Vietoris sequence, we prove the general case in Sect. 6.
2 Preliminaries

Let $X$ be a Kähler manifold and let $\omega$ be a Kähler metric on $X$. We assume that $X$ is weakly pseudoconvex, that is, there exists a smooth plurisubharmonic exhaustion function on $X$. Let $F$ be a holomorphic vector bundle over $X$ and let $H$ be a smooth Hermitian metric of $F$. We denote by $L^{(p,q)}(X, F, H, \omega)$ the Hilbert space of $F$-valued $(p, q)$-forms $u$ which satisfy

$$\|u\|^2_{H, \omega} = \int_X |u|^2_{H, \omega} dV_\omega < +\infty.$$  

Here $dV_\omega = \omega^n/\pi^n$. Let $i\Theta(F, H)$ be the Chern curvature tensor of $(F, H)$ and let $\Lambda$ be the adjoint of multiplication of $\omega$. Suppose that the operator $[i\Theta(F, H), \Lambda]$ acting on $(n, q)$-forms with values in $F$ is positive definite on $X$ ($q \geq 1$). Then, for any $\bar{\partial}$-closed form $u \in L^{(n, q)}(X, F, H, \omega)$ which satisfies $\int_X ([i\Theta(F, H), \Lambda]^{-1}u, u)_{H, \omega} dV_\omega < +\infty$, there exists $v \in L^{(n, q-1)}(X, F, H, \omega)$ such that $\bar{\partial}v = u$ and that

$$\|v\|^2_{H, \omega} \leq \int_X ([i\Theta(F, H), \Lambda]^{-1}u, u)_{H, \omega} dV_\omega$$

(cf. [3]). We note that $\omega$ is possibly non complete.

3 $L^2$-estimate

In [10], the surjectivity between the cohomology groups was proved by the Donnelly–Fefferman–Berndtsson type $L^2$-estimates for $(0, q)$-forms ([1,5]). In [8], Lee and Nagata showed that $L^2$-Serre duality and $L^2$-estimates for not $(0, q)$ but $(n, q)$-forms improve the main result of [9]. We apply their method to the proof of Proposition 1 below.

Let $X$ be a Stein manifold of dimension $n$ and let $D$ be a relatively compact subdomain in $X$. Assume that there exist $\phi, \eta \in C^\infty(D)$ which are negative and plurisubharmonic on $D$. We assume that $\max\{\phi(z), \eta(z)\} \rightarrow 0$ when $z \rightarrow \partial D$ and that $\inf_D \eta < -1$. Define $\phi = -\log(-\phi)$ and $\rho = \max\{-\log(-\phi), 0\}$. Here $\epsilon > 0$ is a small positive number and $\max_\epsilon$ is a regularized max function (see Chapter I, Section 5 of [3]). Let $F$ be a holomorphic vector bundle over $X$ and $H$ be a smooth Hermitian metric of $F$. We define $F^*, H^*$ to be the dual of $F, H$. Let $\psi \in C^\infty(D)$ be a strictly plurisubharmonic function. We take a large positive integer $N$ such that the Hermitian vector bundle $(F^*, H^* e^{-(N-1)\psi})$ is Nakano positive on $D$. Let $\delta > 0$ be a positive number. Put $\omega = i\partial\bar{\partial}(\psi/\delta + \rho + \phi)$. Then $\omega$ is a complete Kähler metric on $D$. Let $\kappa \in C^\infty(\mathbb{R})$ such that $\kappa'(t) \geq 1, \kappa''(t) \geq 0$ for $t \geq 0$. Put $\xi = N\psi + \kappa \circ \rho - \delta \phi$. The proof of the following lemma is completely similar to that of Lemma 3.1 of [8].

**Lemma 1** Let $f \in L^{(n, q)}(D, F^*, H^* e^{-\xi}, \omega)$ such that $\bar{\partial}f = 0$. Assume that $\delta < q$. Then there exists $u \in L^{(n, q-1)}(D, F^*, H^* e^{-\xi}, \omega)$ such that $\bar{\partial}u = f$ and that

$$\int_D |u|^2_{H^*, \omega} e^{-\xi} dV_\omega \leq C_{q, \delta} \int_D |f|^2_{H^*, \omega} e^{-\xi} dV_\omega.$$  

Here $C_{q, \delta}$ depends only on $q$ and $\delta$.

**Proof** There exist relatively compact weakly pseudoconvex subdomains $D_1 \subset D_2 \subset \cdots \subset D$ which exhaust $D$. Because of the Nakano positivity of $(F^*, H^* e^{-(N-1)\psi})$, there exists $u_k$...
in $L^{(n,q-1)}(D_k, F^*, H^*e^{-N\psi - \kappa \rho})$ such that $\overline{\partial}u_k = f$ and that
\[
\int_{D_k} |u_k|^2_{H^{\kappa}, \omega} e^{-N\psi - \kappa \rho} dV_\omega \leq \int_{D_k} \langle [i \Theta(F^*, H^*e^{-N\psi - \kappa \rho}), \Lambda ]^{-1} f, f \rangle_{H^\kappa, \omega} e^{-N\psi - \kappa \rho} dV_\omega.
\]
Since $\phi$ and $\overline{\partial}\phi$ are bounded in $D_k$, we have that $u_k e^{\delta \phi}$ is the minimal solution of $\overline{\partial}(u_k e^{\delta \phi}) = (f + \delta \overline{\partial}\phi \wedge u_k) e^{\delta \phi}$ in $L^{(n,q-1)}(D_k, F^*, H^*e^{-N\psi - \kappa \rho - \delta \phi}, \omega)$. Then
\[
\int_{D_k} |u_k|^2_{H^\kappa, \omega} e^{-\delta \phi} dV_\omega = \int_{D_k} |u_k e^{\delta \phi}|^2_{H^\kappa, \omega} e^{-N\psi - \kappa \rho - \delta \phi} dV_\omega \\
\leq \int_{D_k} \langle [i \Theta(F^*, H^*e^{-N\psi - \kappa \rho - \delta \phi}), \Lambda ]^{-1} (f + \delta \overline{\partial}\phi \wedge u_k), f + \delta \overline{\partial}\phi \wedge u_k \rangle_{H^\kappa, \omega} e^{-\delta \phi} dV_\omega \\
\leq (1 + \frac{1}{t}) \int_{D_k} \langle [i \Theta(F^*, H^*e^{-N\psi - \kappa \rho - \delta \phi}), \Lambda ]^{-1} f, f \rangle_{H^\kappa, \omega} e^{-\delta \phi} dV_\omega \\
+ (1 + t)\frac{\delta^2}{q} \int_{D_k} \langle [i \Theta(F^*, H^*e^{-N\psi - \kappa \rho - \delta \phi}), \Lambda ]^{-1} \overline{\partial}\phi \wedge u_k, \overline{\partial}\phi \wedge u_k \rangle_{H^\kappa, \omega} e^{-\delta \phi} dV_\omega
\]
for every $t > 0$. We have that $\langle [i \Theta(F^*, H^*e^{-N\psi - \kappa \rho - \delta \phi}), \Lambda ]v, v \rangle_{H^\kappa, \omega} \geq q \delta |v|^2_{H^\kappa, \omega}$ for any $F$-valued $(n,q)$-form $v$ since $i \delta \overline{\partial}(\psi + \kappa \circ \rho + \delta \phi) \geq \delta \omega$. Because $|\overline{\partial}\phi|_{\omega} \leq 1$, we have that
\[
(1 + t)\frac{\delta^2}{q} \int_{D_k} \langle [i \Theta(F^*, H^*e^{-N\psi - \kappa \rho - \delta \phi}), \Lambda ]^{-1} \overline{\partial}\phi \wedge u_k, \overline{\partial}\phi \wedge u_k \rangle_{H^\kappa, \omega} \leq (1 + t)\frac{\delta}{q} |u_k|^2_{H^\kappa, \omega}.
\]
By taking $t$ sufficiently small, we have that $1 - (1 + t)\frac{\delta}{q} < 0$ and
\[
\int_{D_k} |u_k|^2_{H^\kappa, \omega} e^{-\delta \phi} dV_\omega \leq C_{q,\delta} \int_{D_k} |f|^2_{H^\kappa, \omega} e^{-\delta \phi} dV_\omega \leq C_{q,\delta} \int_{D} |f|^2_{H^\kappa, \omega} e^{-\delta \phi} dV_\omega.
\]
Here $C_{q,\delta} = \frac{(1 + t)^{-1}}{1 - (1 + t)\frac{\delta}{q}}$. Hence we may choose a subsequence of $\{u_k\}_{k \in \mathbb{N}}$ converging weakly in $L^{(n,q-1)}(D, F^*, H^*e^{-\kappa \rho}, \omega)$ to $u$. Then $u$ is the $F$-valued $(n,q)$-form we are looking for. $\square$

**Lemma 2** Let $\alpha \in L^{(0,q)}(D, F, He^\kappa, \omega)$ such that $\overline{\partial}\alpha = 0$. Assume that $q \geq 1$ and that $\delta < n - q$. Then there exists $\beta \in L^{(0,q-1)}(D, F, He^\kappa, \omega)$ such that $\overline{\partial}\beta = \alpha$ and that
\[
\int_{D} |\beta|^2_{H^\kappa, \omega} e^{\kappa \rho} dV_\omega \leq C_{q,\delta} \int_{D} |\alpha|^2_{H^\kappa, \omega} e^{\kappa \rho} dV_\omega.
\]

**Proof** Let $\star_F$ be the Hodge-star operator $L^{(0,q)}(D, F, He^\kappa, \omega) \rightarrow L^{(n,n-q)}(D, F^*, H^*e^{-\kappa \rho}, \omega)$ as in [2]. Let $\overline{\partial}^*_F$ be the Hilbert space adjoint to $\overline{\partial} : L^{(n,n-q)}(D, F^*, H^*e^{-\kappa \rho}, \omega) \rightarrow L^{(n,n-q)}(D, F^*, H^*e^{-\kappa \rho}, \omega)$. We note that the formal adjoint of $\overline{\partial}$ is equal to $-\star_F \overline{\partial} \star_F$. Since $\omega$ is complete, we have that $\star_F \alpha \in L^{(n,n-q)}(D, F^*, H^*e^{-\kappa \rho}, \omega)$ is contained in the domain of $\overline{\partial}_F$ and $\overline{\partial}^*_F * F \alpha = 0$. Lemma 1 shows that there exists $u \in L^{(n,n-q+1)}(D, F^*, H^*e^{-\kappa \rho}, \omega)$ such that $\overline{\partial}_F u = \star_F \alpha$ and that $\|u\|^2_{H^{n-q+1}, \omega} \leq C_{q,\delta} \|\star_F \alpha\|^2_{H^{n-q+1}, \omega} = C_{q,\delta} \|\alpha\|^2_{H^\kappa, \omega}$. By the completeness of $\omega$ again, we have that $\beta = \star_F u \in L^{(0,q-1)}(D, F, He^\kappa, \omega)$ is contained in the domain of $\overline{\partial}$ and $\overline{\partial}\beta = \alpha$. We have that $\|\beta\|^2_{H^\kappa, \omega} \leq C_{q,\delta} \|\alpha\|^2_{H^\kappa, \omega}$ and this completes the proof. $\square$
We denote by $\Omega^{(0,q)}(D, F)$ (resp. $\Omega^{(0,q)}(\overline{D}, F)$) the space of $F$-valued smooth $(0, q)$-forms on $D$ (resp. on a neighborhood of $\overline{D}$). Let $\partial^c D = \{ z \in \partial D \mid \varphi(z) = 0 \}$.

**Proposition 1** Let $1 \leq q \leq n - 2$. Assume that $\varphi$ is pluriharmonic on a neighborhood of $\partial^c D$ and $d\varphi \not\equiv 0$ on $\partial^c D$. Let $\alpha \in \Omega^{(0,q)}(\overline{D}, F)$ such that $\overline{\partial}\alpha = 0$ in $D$ and that $\text{supp} \alpha \cap D \subset \{ z \in D \mid \rho = 0 \}$. Then there exists $\beta \in \Omega^{(0,q-1)}(D, F)$ such that $\overline{\partial}\beta = \alpha$ and that $\text{supp} \beta \subset \{ z \in D \mid \rho \leq 1 \}$.

**Proof** Let $1 < \delta < 2$. If $a \in \{ z \in \partial D \mid \eta(z) \neq 0 \}$, then $a \in \partial^c D$. There exists a small neighborhood $U \subset X$ of $a$ such that $\varphi$ is pluriharmonic and $i\partial\bar{\partial}\varphi = \frac{i\partial\varphi \wedge \bar{\partial}\varphi}{\varphi^2}$ on $U \cap D$. Since $\eta(a) \neq 0$, we may assume that $i\partial\bar{\partial}\rho$ and $\kappa \circ \rho$ are bounded. Then $e^\xi dV_\omega \leq C|\varphi|^{-2}(i\partial\bar{\partial}\varphi)^n$ on $U \cap D$. Hence

$$\int_{U \cap D} |\alpha|^2_{H \omega} e^\xi dV_\omega \leq C \int_{U \cap D} |\alpha|^2_{H \omega} |\varphi|^{-2}(i\partial\bar{\partial}\varphi)^n < +\infty.$$ 

If $b \in \{ z \in \partial D \mid \eta(z) = 0 \}$, there exists a small open neighborhood $U \subset X$ of $b$ such that $U \cap \text{supp} \alpha = \emptyset$. Hence we have that $\alpha \in L^{(0,q)}(D, F, He^\xi, \omega)$.

Let $\kappa_j \in C^\infty(\mathbb{R})$ $(j = 1, 2, \ldots)$ be functions which satisfies the following conditions:

(i) $\kappa_j(t) \leq \kappa_{j+1}(t)$ for any $j \in \mathbb{N}$ and $t \geq 0$,

(ii) $\kappa_j'(t) \geq 1$ and $\kappa_j''(t) \geq 0$ for any $j \in \mathbb{N}$ and $t \geq 0$,

(iii) there exists a positive constant $C$ such that $\kappa_j(t) \leq C$ for any $j \in \mathbb{N}$ and $0 \leq t \leq 1/2$,

(iv) $\lim_{j \to \infty} \kappa_j(t) = +\infty$ for any $t \geq 1$.

Let $\xi_j = N\psi + \kappa_j \circ \rho - \delta \phi$. We have that $\alpha \in L^{(0,q)}(D, F, He^{\xi_j}, \omega)$. Since $q \leq n - 2$, there exist $\beta_j \in L^{(0,q-1)}(D, F, He^{\xi_j}, \omega)$ such that $\overline{\partial}\beta_j = \alpha$ and $\|\beta_j\|_{H^q, \omega}^2 \leq C_q, \delta \|\alpha\|^2_{H^q, \omega}$ by Lemma 2. Because $\kappa_j \circ \rho \leq C$ on $\text{supp} \alpha$, we have that $\|\alpha\|^2_{H^q, \omega}$ does not depend on $j$.

Take a weakly convergent subsequence $\{ \beta_{j_i} \}_{i \in \mathbb{N}}$ in $L^{(0,q-1)}(D, F, He^{N\psi - \delta \phi}, \omega)$ and there exists the weak limit $\beta \in L^{(0,q-1)}(D, F, He^{N\psi - \delta \phi}, \omega)$ such that $\overline{\partial}\beta = \alpha$ and $\text{supp} \beta \subset \{ z \in D \mid \rho(z) \leq 1 \}$. The regularity of $\beta$ will be discussed in Sect. 4. □

### 4 Estimate of the Sobolev norm

It is enough to consider the case $q \geq 2$. In the proof of Proposition 1, we take $\kappa_j \in C^\infty(\mathbb{R})$ $(j \in \mathbb{N})$ which satisfy four conditions. We add the following condition to $\{ \kappa_j \}_{j \in \mathbb{N}}$:

(v) For any non-negative integer $k$, there exists positive constant $C_k$ which does not depend on $j$ and satisfies

$$\frac{\kappa_j^{(k)}(t)}{(\kappa_j(t))^{k+1}} \leq C_k$$

for any $j \in \mathbb{N}$ and $t \geq 0$.

**Lemma 3** There exist functions $\kappa_j \in C^\infty(\mathbb{R})$ $(j \in \mathbb{N})$ which satisfy the above five conditions.

**Proof** We define $\kappa_j(t) = \sum_{l=0}^{j} t^l$. It is easy to see that $\{ \kappa_j \}_{j \in \mathbb{N}}$ satisfies the conditions (i), (ii), (iii), (iv). For $t \geq 0$, we have that

$$\left(\kappa_j(t)\right)^{k+1} \geq \sum_{l=0}^{j} \left( l + k \right) t^l \geq \frac{1}{k!} \sum_{l=0}^{j} \frac{(l + k)!}{l!} t^l \geq \frac{1}{k!} \kappa_j^{(k)}(t).$$
Hence $\{\kappa_j\}_{j \in \mathbb{N}}$ satisfies (v). \hfill \Box

Assume $\{\kappa_j\}_{j \in \mathbb{N}}$ satisfies the above five conditions. Take $\{\beta_j\}_{j \in \mathbb{N}}$ and $\beta$ as in the proof of Proposition 1. We may assume that $\beta_j$ is orthogonal to the kernel of $\overline{\partial} : L^{(0,q-1)}(D,F,H e^{\kappa_j},\omega) \to L^{(0,q)}(D,F,H e^{\kappa_j},\omega)$ and $\beta_j$ is smooth (cf. [7]). Let $a \in D$ and let $\chi \in C^\infty(D)$ be a non-negative function such that $\chi = 1$ on a neighborhood of $a$ and $\text{supp } \chi$ is sufficiently small. Denote by $H^j = H e^{N \psi - \delta \phi}$ the Hermitian metric of $F$. Then $H^j e^{\kappa_j \rho} = H e^{\kappa_j}$. We may assume that $\text{supp } \chi$ is contained in a complex chart $U$ and that $F$ is trivialized there. Let $(x_1,\ldots, x_{2n})$ be a local coordinates on $U$. Let $K = (k_1, k_2, \ldots, k_{2n})$ be the multi-index and let $|K| = k_1 + \cdots + k_{2n}$. Define $D^K : \Omega^{(0,q-1)}(U,F) \to \Omega^{(0,q-1)}(U,F)$ by $D^K = \left( \frac{\partial}{\partial x_1} \right)^{k_1} \cdots \left( \frac{\partial}{\partial x_{2n}} \right)^{k_{2n}}$.

**Lemma 4** Let $k$ be a non-negative integer. Then
\[
\sum_{|K|=k} \int_U |D^K(\chi \beta_j)|^2_{H^j,\omega} e^{\kappa_j \rho}/2^{k+1} \omega < C_k
\]
for any $j \in \mathbb{N}$ where $C_k$ is a constant which does not depend on $j$.

Let $W^k_{(0,q-1)}(U,F)$ be the Sobolev space of $F$-valued $(0,q-1)$-forms whose derivatives up to order $k$ are in $L^{(0,q-1)}(U,F,H^j,\omega)$. Lemma 4 implies that the subset $\{\chi \beta_j\}_{j \in \mathbb{N}}$ is bounded in $W^k_{(0,q-1)}(U,F)$. By taking weakly convergent subsequence of $\{\chi \beta_j\}_j$, we have that $\beta \in W^k_{(0,q-1)}(U',F)$ where $U'$ is a sufficiently small open neighborhood of $a$ contained in $U$. By the Sobolev lemma (cf. [3]), it follows that $\beta$ is smooth in $D$.

The proof of Lemma 4 proceeds by induction on $k$. We have seen that $\|\beta_j\|_{H^j,\omega}^2$ does not depend on $j$, and the case $k = 0$ holds. Let $\nabla_j$ be the Hermitian connection on $F$ which is compatible with $H^j e^{\kappa_j \rho}$. By the trivialization of $F$ in $U$, we have that $\nabla_j = d + \Gamma H^j + \partial \kappa_j \circ \rho \text{ Id}_F$ where $\Gamma H^j$ is the connection form defined by $H^j$. Let $* \partial$ be the Hodge-star operator from $F$-valued $(p,q)$-form to $F$-valued $(n-q,n-p)$-form. Define $\overline{\partial} = * \partial * : \Omega^{(0,q-1)}(U,F) \to \Omega^{(0,q-2)}(U,F)$. Then $\overline{\partial} \beta$ is the formal adjoint of $\overline{\partial} : L^{(0,q-1)}(U,F,H Id_d,\omega) \to L^{(0,q)}(U,F,H Id_d,\omega)$. Here $H Id_d$ is the flat metric of the trivial vector bundle $F$ over $U$.

**Lemma 5** Under the hypothesis that Lemma 4 holds up to $k$, we have that
\[
\sum_{|K|=k} \int_U |\partial D^K(\chi \beta_j)|^2_{H^j,\omega} e^{\kappa_j \rho}/2^{k+1} \omega < C_{k+1}
\]
for any $j \in \mathbb{N}$ where $C_{k+1}$ is a constant which does not depend on $j$.

**Proof** We denote by $\overline{\partial} * \beta_j$ the Hilbert space adjoint to $\overline{\partial} : L^{(0,q-2)}(D,F,H e^{\kappa_j \rho},\omega) \to L^{(0,q-1)}(D,F,H e^{\kappa_j \rho},\omega)$. The formal adjoint of $\overline{\partial}$ is given by $-*(\partial + \Gamma H^j + \partial \kappa_j \circ \rho \text{ Id})*$. Since $\beta_j$ is orthogonal to the kernel of $\overline{\partial}$ in $L^{(0,q-1)}(D,F,H e^{\kappa_j \rho},\omega)$, we have that $\overline{\partial} * \beta_j = 0$. Then $* \partial * \beta_j = -* \Gamma H^j * \beta_j - * \partial \kappa_j \circ \rho \wedge * \beta_j$. Hence
\[
* \partial * (\chi \beta_j) = * (\partial \chi \wedge * \beta_j) - * \Gamma H^j * (\chi \beta_j) - * \partial \kappa_j \circ \rho \wedge (\chi \beta_j).
\]
Let $K = (k_1,\ldots,k_{2n})$ such that $k = k_1 + \cdots + k_{2n}$. It follows that the order of the differential operator $* D^K - D^K *$ is $k - 1$. Then
\[
\partial D^K(\chi \beta_j) = * \partial * D^K(\chi \beta_j) = D^K * \partial (\chi \beta_j) + \partial \beta_j,
\]
where $L$ is a differential operator of order $k$. The term $D^K \ast \partial \ast (\chi \beta_j)$ is written as

$$\sum_{k' \leq k, l \leq k+1} \sum_{|K'| = k'} t_{K', l} \kappa_j^{(l)} \circ \rho D^{K'} \beta_j$$

where $t_{K', l}$ is a function which does not depend on $j$. It follows that

$$\int_U |\partial D^K (\chi \beta_j)|^2_{H', \omega} e^{\kappa_j \circ \rho / 2^{k+1}} dV_\omega \leq C \sum_{k' \leq k, l \leq k+1} \sum_{|K'| = k'} \int_U (|t_{K', l} \kappa_j^{(l)} \circ \rho|^2 |D^{K'} \beta_j|^2_{H', \omega} + |\beta_j|^2_{H', \omega}) e^{\kappa_j \circ \rho / 2^{k+1}} dV_\omega$$

and

$$e^{-\kappa_j \circ \rho / 2^{k+1}} dV_\omega,$$

where $C$ does not depend on $j$. Since $\sup_{r \geq 0} r^{2(l+1)} e^{-r/2^{k+1}} < +\infty$, the condition (v) of $\{\kappa_j\}_{j \in \mathbb{N}}$ shows that the last term of the above inequality is bounded from above by a constant which does not depend on $j$. \qed

A standard computation shows that Lemma 5 implies Lemma 4 (cf. Chapter 5 of [7]).

**Proof of Lemma 4** We may assume that there exists an orthonormal frame $(\theta_1, \ldots, \theta_n)$ of $T_X^*$ on $U$. For any $f \in C^\infty(U)$, we define $\partial f / \partial \theta_1$ and $\partial f / \partial \bar{\theta}_1$ by $df = \sum_{i=1}^n \partial f / \partial \theta_i \theta_i + \partial f / \partial \bar{\theta}_1 \bar{\theta}_1$. Let $I = (i_1, \ldots, i_q-1)$ be a multi-index with $i_1 < \cdots < i_q-1$ and $\bar{\theta}_I = \bar{\theta}_{i_1} \wedge \cdots \wedge \bar{\theta}_{i_{q-1}}$. Let $g = \sum_{|I| = q-1} g_I \theta_I$ be a smooth $(0, q-1)$-form on $U$ with compact support.

Define $A_g = \sum_{|I| = q-1} \sum_{|J| = 1} g_I \kappa_j \circ \rho / 2^{k+1} \bar{\theta}_I$ and $B_g = \sum_{|I| = q-2} \sum_{|J| = 1} g_I \kappa_j \circ \rho / 2^{k+1} \bar{\theta}_I$. Then $\bar{\partial} g - A_g$ and $-\bar{\partial} g - B_g$ have no term where $g_I$ is differentiated. Hence we have that

$$\int_U (|A_g|^2_{g_\omega} + |B_g|^2_{g_\omega}) e^{\kappa_j \circ \rho / 2^{k+1}} dV_\omega \leq \int_U (2|\bar{\partial} g|^2_{g_\omega} + 2|\bar{\partial} g|^2_{g_\omega} + C|g_\omega|^2) e^{\kappa_j \circ \rho / 2^{k+1}} dV_\omega. \quad (1)$$

The left-hand side of the above inequality is equal to

$$\sum_{|I| = q-1} \sum_{l=1}^n \int_U \left| \frac{\partial g_I}{\partial \theta_l} \right|^2 e^{\kappa_j \circ \rho / 2^{k+1}} dV_\omega - \sum_{|J| = q-2} \sum_{l,t} \int_U \left( \frac{\partial^2 g_{IJ}}{\partial \theta_l \partial \theta_t} - \frac{\partial^2 g_{IJ}}{\partial \bar{\theta}_l \partial \bar{\theta}_t} \right) e^{\kappa_j \circ \rho / 2^{k+1}} dV_\omega.$$ 

By integrating by parts, the second integral of the above is equal to

$$\int_U \left( \frac{\partial^2 g_{IJ}}{\partial \theta_l \partial \theta_t} - \frac{\partial^2 g_{IJ}}{\partial \bar{\theta}_l \partial \bar{\theta}_t} \right) \bar{g}_{IJ} e^{\kappa_j \circ \rho / 2^{k+1}} dV_\omega + \int_U R e^{\kappa_j \circ \rho / 2^{k+1}} dV_\omega.$$

Here $R$ is written as the sum of $s_1 g_{IJ} \kappa_j^{(l)} \circ \rho \bar{\partial} g_{IJ} / \partial \bar{\theta}_1$ and $s_2 g_{IJ} g_{IJ} \kappa_j^{(l)} \circ \rho (i = 0, 1, 2)$ where $s_1$ and $s_2$ are the smooth functions which depend on neither $g$ nor $j$. The order of the
differential operator $\frac{\partial^2}{\partial \theta_i \partial \tilde{\theta}_i} - \frac{\partial^2}{\partial \tilde{\theta}_i \partial \theta_i}$ is one. The condition (v) of \{$$\kappa_j$$\} shows that
\[
\int_U |g_I \frac{\partial g_{I_2}}{\partial \theta_i}|^2 e^{\kappa_j \rho / 2^{k+1}} dV_\omega \\
\leq \frac{1}{\varepsilon} \int_U |g_I \kappa_j \rho / 2^{k+1}|^2 e^{\kappa_j \rho / 2^{k+1}} dV_\omega + \varepsilon \int_U |\frac{\partial g_{I_2}}{\partial \theta_i}|^2 e^{\kappa_j \rho / 2^{k+1}} dV_\omega \\
\leq C \varepsilon \int_U |g_I|^2 e^{\kappa_j \rho / 2^k} dV_\omega + \varepsilon \int_U |\frac{\partial g_{I_2}}{\partial \theta_i}|^2 e^{\kappa_j \rho / 2^{k+1}} dV_\omega
\]
for any $\varepsilon > 0$. Hence the left-hand side of (1) is bounded below by
\[
\frac{1}{2} \sum_{|l|=q-1} \sum_{l=1}^n \int_U |\frac{\partial g_I}{\partial \theta_i}|^2 e^{\kappa_j \rho / 2^{k+1}} dV_\omega - C \int_U |g_I|^2 e^{\kappa_j \rho / 2^k} dV_\omega,
\]
where $C$ does not depend on $j$. Because the order of the differential operator $\frac{\partial^2}{\partial \theta_i \partial \tilde{\theta}_i} - \frac{\partial^2}{\partial \tilde{\theta}_i \partial \theta_i}$ is one, we have that
\[
\int_U |\frac{\partial g_I}{\partial \theta_i}|^2 e^{\kappa_j \rho / 2^{k+1}} dV_\omega - \int_U |\frac{\partial g_I}{\partial \tilde{\theta}_i}|^2 e^{\kappa_j \rho / 2^{k+1}} dV_\omega \\
\leq \varepsilon \int_U \left( |\frac{\partial g_I}{\partial \theta_i}|^2 + |\frac{\partial g_I}{\partial \tilde{\theta}_i}|^2 \right) e^{\kappa_j \rho / 2^{k+1}} dV_\omega + C \varepsilon \int_U |g_I|^2 e^{\kappa_j \rho / 2^k} dV_\omega
\]
for any $\varepsilon > 0$. Since $F$ is trivialized in $U$ and $H'$ is equivalent to the flat metric $H_{id}$, we can replace $g$ by $D^K(\chi \beta_j)$ in the above calculus, and we obtain
\[
\sum_{l=1}^n \int_U \left( |\frac{\partial D^K(\chi \beta_j)}{\partial \theta_i}|^2 H',\omega + |\frac{\partial D^K(\chi \beta_j)}{\partial \tilde{\theta}_i}|^2 H',\omega \right) e^{\kappa_j \rho / 2^{k+1}} dV_\omega \\
\leq C \int_U (D^K\overline{\chi}(\beta_j))^2_{H',\omega} \\
\quad + |\theta D^K(\chi \beta_j)^2_{H',\omega} e^{\kappa_j \rho / 2^{k+1}} dV_\omega + C \int_U |D^K(\chi \beta_j)|^2_{H',\omega} e^{\kappa_j \rho / 2^k} dV_\omega.
\]
Note that $\overline{\chi}(\beta_j) = \chi \alpha + \overline{\partial} \chi \wedge \beta_j$. The induction hypothesis and Lemma 5 show that the last term of the above inequality does not depend on $j$. \hfill \square

5 Extension of closed $F$-valued forms in manifolds

In this section, we extend closed $F$-valued forms as in [10].

Let $X$ be a Stein manifold and let $\varphi$ be a non-constant plurisubharmonic function on $X$ such that for every $r < \sup_X \varphi$ the sublevel set \{$z \in X \mid \varphi(z) < r$\} is compact. Let $F$ be a holomorphic vector bundle over $X$. Let $\psi \in C^\infty(X)$ be an exhaustive strictly plurisubharmonic function. We define $D_\psi(r) = \{z \in X \mid \varphi(z) < r\}$.

**Lemma 6** Let $U \subset X$ be an open neighborhood of $\text{supp } i \partial \overline{\partial} \varphi$ and let $u \in \Omega^{0,q-1}(U, F)$ ($1 \leq q \leq n - 2$) such that $\overline{\partial}u = 0$. Let $r < \sup \varphi$ such that $d\varphi \neq 0$ on $\partial D_\psi(r) \setminus \text{supp } i \partial \overline{\partial} \varphi$. (Note that $\varphi$ is smooth on $X \setminus \text{supp } i \partial \overline{\partial} \varphi$). Then there exists $v \in \Omega^{0,q-1}(D_\psi(r), F)$ such that $u = v$ on a neighborhood of $\text{supp } i \partial \overline{\partial} \varphi \cap D_\psi(r)$. \hfill \cotton
**Proof** We take $\chi \in C^\infty(X)$ such that $0 \leq \chi \leq 1$, $\chi = 1$ on a neighborhood of $\text{supp } i\partial\overline{\partial}\varphi$ and that $\text{supp } \chi \subset U$. Then $\alpha := \overline{\partial}(\chi u)$ is $\overline{\partial}$-closed. By Lemma 7 below, there exists a smooth plurisubharmonic function $\varphi\varepsilon$ on a neighborhood of $D_\varphi(r)$ such that $\varphi\varepsilon \geq \varphi$ on $D_\varphi(r)$, $\varphi\varepsilon > \varphi$ on $\text{supp } i\partial\overline{\partial}\varphi \cap D_\varphi(r)$, and that $\varphi\varepsilon = \varphi$ on $\text{supp } \alpha$.

Let $\eta = \varphi\varepsilon - \varphi$. It follows that $\eta$ is plurisubharmonic on $D_\varphi(r) \setminus \text{supp } i\partial\overline{\partial}\varphi$ and that $\eta = 0$ on $\text{supp } \alpha$. Since $\eta$ is lower semi-continuous on $D_\varphi(r)$, there exists $c > 0$ such that $\eta > c$ on $D_\varphi(r) \cap \text{supp } i\partial\overline{\partial}\varphi$. Let $D' = \{z \in D \mid \varphi(z) - r < 0, \eta(z) - c/2 < 0\}$. Note $\eta \in C^\infty(D \setminus \text{supp } i\partial\overline{\partial}\varphi)$ since $\varphi \in C^\infty(D \setminus \text{supp } i\partial\overline{\partial}\varphi)$. By Proposition 1, there exist $c' < \frac{c}{\varepsilon}$ and $\beta \in \Omega^{(0,q-1)}(D', F)$ such that $\overline{\partial}\beta = \alpha$ and that $\text{supp } \beta \subset \{z \in D' \mid \eta(z) \leq c'\}$. (If $D'$ is a disjoint union of bounded pseudoconvex domains, we apply Proposition 1 to each component.) We extend $\beta$ by zero on $D_\varphi(r) \setminus D'$, and we consider $\beta$ as an element of $\Omega^{(0,q-1)}(D_\varphi(r), F)$. We have that $\overline{\partial}\beta = \alpha$ on $D_\varphi(r)$. Then $\nu = \chi u - \beta$ is the form we are looking for.

**Lemma 7** Let $V \subset X$ be an open neighborhood of $\text{supp } i\partial\overline{\partial}\varphi$. Let $t < \text{sup } \varphi$. There exists a smooth plurisubharmonic function $\varphi\varepsilon(t)$ on $D_\varphi(t)$ such that $\varphi\varepsilon \geq \varphi$ on $D_\varphi(t)$, $\varphi\varepsilon > \varphi$ on $\text{supp } i\partial\overline{\partial}\varphi(t)$, and that $\varphi\varepsilon = \varphi$ on $D_\varphi(t) \setminus V$.

**Proof** Since $X$ is Stein, we may assume that $X$ is a submanifold of $\mathbb{C}^m$. By the theorem of Dcquier and Grauert, there exists an open neighborhood $W \subset \mathbb{C}^m$ of $X$ and a holomorphic retraction $\mu : W \to X$ (cf. Chapter VIII of [6]). Let $h : \mathbb{C}^m \to \mathbb{R}^+$ be a smooth function depending only on $|z|$ ($z \in \mathbb{C}^m$) whose support is contained in the unit ball and whose integral is equal to one. Define $h_\varepsilon(z) = (1/\varepsilon^m)h(z/\varepsilon)$ for $\varepsilon > 0$. Let $W' \subset W$ be a relatively compact open neighborhood of $D_\varphi(t)$. If $\varepsilon > 0$ is sufficiently small, we can define $(\varphi \circ \mu)_\varepsilon = (\varphi \circ \mu)(z)h_\varepsilon$ on $W'$. Put $\varphi\varepsilon = (\varphi \circ \mu)_\varepsilon$ on $D_\varphi(t)$. Then we have $\varphi\varepsilon \geq \varphi$ on $D_\varphi(t)$ and $\varphi\varepsilon > \varphi$ on $\text{supp } i\partial\overline{\partial}\varphi$. Since $\mu^{-1}(\text{supp } i\partial\overline{\partial}\varphi) \cap (W' \setminus \text{supp } \varphi) = \emptyset$, there exists small $\varepsilon > 0$ such that the ball of radius $\varepsilon$ centered at every point of $W' \setminus \text{supp } \varphi$ does not intersect $\mu^{-1}(\text{supp } i\partial\overline{\partial}\varphi)$. Then $(\varphi \circ \mu)_\varepsilon = \varphi \circ \mu$ on $W' \setminus \text{supp } \varphi$.

**Lemma 8** Let $\varphi$ be a plurisubharmonic function on $X$ as in Theorem 1. Let $r < s < t < \text{sup } X \varphi$ such that $D_\varphi(r) \subset D_\varphi(s) \subset D_\varphi(t)$. Then there exists a plurisubharmonic function $\phi \varepsilon$ on $X$ which satisfies the following conditions:

(a) $\varphi = \phi \varepsilon$ on $\{z \in D \mid \varphi(z) \geq t\}$.
(b) $\text{supp } i\partial\overline{\partial}\phi \varepsilon \cup D_\varphi(r) \subset \text{supp } i\partial\overline{\partial}\phi$.

**Proof** Let $\psi$ be an exhaustive strongly plurisubharmonic function on $X$. We may assume that $\varphi > 0$ on $X$. Let $M = \sup_{D_\varphi(t)} \psi$ and let $\tau(z) = r + \frac{(s-r)}{M+1}$. Then $r < \tau < s$ on the closure of $D_\varphi(t)$. We define

$$\phi(z) = \begin{cases} 
\varphi(z) & \text{if } z \in \{x \in X \mid \varphi(x) \geq t\} \\
\max(\varphi(z), \tau(z)) & \text{if } z \in D_\varphi(t)
\end{cases}$$

Then $\phi$ is a plurisubharmonic function on $X$. It is easy to see that $\phi$ satisfies the conditions of the lemma.

**Lemma 9** Let $X$ and $\varphi$ be as in Theorem 1. Let $U \subset X$ be an open neighborhood of $\text{supp } i\partial\overline{\partial}\varphi$ and let $u \in \Omega^{(0,q-1)}(U, F)$ ($1 \leq q \leq n - 2$) such that $\overline{\partial}u = 0$. Let $s_1 < s_2 < \cdots < \text{sup } X \varphi$ such that $\lim_{j \to \infty} s_j = \sup D_\varphi$ and that

$$D_\varphi(s_1) \subset D_\varphi(s_2) \subset \cdots \subset X.$$ 

Then there exist $V_j \subset X$ and $v_j \in \Omega^{(0,q-1)}(V_j, F)$ ($j = 3, 4, \ldots$) which satisfy the following:
(i) \( V_j \) is an open neighborhood of \( \text{supp} \ i \partial \overline{\partial} \phi \cap D_\phi(s_j) \).

(ii) \( \overline{\partial} v_j = 0 \) on \( V_j \).

(iii) \( v_{j+1} = v_j \) on \( D_\phi(s_{j-2}) \).

(iv) \( v_j = u \) on a neighborhood of \( \text{supp} \ i \partial \overline{\partial} \phi \).

**Proof** By Lemma 6, there exists \( \nu \in \Omega^{(0,q-1)}(D_\phi(s_5), F) \) such that \( \overline{\partial} \nu = 0 \) and that \( \nu = u \) on a neighborhood of \( \text{supp} \ i \partial \overline{\partial} \phi \cap D_\phi(s_5) \). We may assume that \( U \) is sufficiently small. Then \( \nu \) on \( D_\phi(s_4) \) and \( u \) on \( U \) can be glued together to give the form \( v_3 \) on \( V_3 := U \cup D_\phi(s_4) \).

Assume that there exist \( V_j \) and \( v_j \) which satisfy the condition of the lemma \((j \geq 3)\). By Lemma 8, there exists plurisubharmonic function \( \phi \) on \( X \) such that \( \phi = \phi \) on \([z \in D | \phi(z) \geq s_j]\) and that \( \text{supp} \ i \partial \overline{\partial} \phi \cup D_\phi(s_{j-2}) \subset \text{supp} \ i \partial \overline{\partial} \phi \). Then \( v_j \) is defined on a neighborhood of \( \text{supp} \ i \partial \overline{\partial} \phi \). By Lemma 6, there exists \( \tilde{\nu} \in \Omega^{(0,q-1)}(D_\phi(s_{j+3}), F) \) such that \( \overline{\partial} \tilde{\nu} = 0 \) and that \( \tilde{\nu} = v_j \) on an open neighborhood of \( D_\phi(s_{j+3}) \cap \text{supp} \ i \partial \overline{\partial} \phi \). As in the case of \( v_3 \), we can glue \( \tilde{\nu} \) and \( v_j \) together, and we obtain \( v_{j+1} \) and \( V_{j+1} \) which satisfy the conditions of the lemma. Hence we obtain \( V_k \) and \( v_k \) \((k = 3, 4, \ldots)\) inductively. \( \square \)

Now we prove that the natural map

\[ H^0(X, F) \to \lim_{\sup \, \text{supp} \ i \partial \overline{\partial} \phi \subset V} H^0(V, F). \]

is an isomorphism and

\[ \lim_{\sup \, \text{supp} \ i \partial \overline{\partial} \phi \subset V} H^q(V, F) = 0. \]

for \( 0 < q < n - 2 \).

**Proof** Let \( U \) be an open neighborhood of \( \text{supp} \ i \partial \overline{\partial} \phi \) in \( X \). Let \( u \in \Omega^{(0,q-1)}(U, F) \) \((1 \leq q \leq n-2)\) such that \( \overline{\partial} u = 0 \). Take \( \{s_j\} \) and \( \{v_j\} \) as in Lemma 5. We can define \( \nu \in \Omega^{(0,q-1)}(X, F) \) by \( \nu(z) = v_{j+2}(z) \) when \( z \in D_\phi(s_j) \). Then \( \overline{\partial} \nu = 0 \) and \( \nu \) is equal to \( u \) on a neighborhood of \( \text{supp} \ i \partial \overline{\partial} \phi \). Hence the natural map \( H^{q-1}(X, F) \to \lim_{\sup \, \text{supp} \ i \partial \overline{\partial} \phi \subset V} H^{q-1}(V, F) \) is surjective.

Since \( H^{q-1}(X, F) = 0 \) for \( q \geq 2 \), this completes the proof. \( \square \)

Let \( X \) be a projective manifold and let \( T \) be a closed positive current of type \((1, 1)\) on \( X \) such that the cohomology class \( \{T\} \) belongs to \( K_{NS} \). There exist very ample line bundles \( L_1, \ldots, L_p \) and positive numbers \( a_1, \ldots, a_p \) such that \( \{T\} = a_1 c_1(L_1) + \cdots + a_p c_1(L_p) \) where \( c_1(L) \) is the first Chern class of \( L \). Let \( \omega_j \) be a smooth closed positive form such that \( \omega_j \in c_1(L_j) \). Put \( \omega = a_1 \omega_1 + \cdots + a_p \omega_p \). Then there exists an almost plurisubharmonic function \( \phi \) on \( X \) such that \( T = \omega + i \partial \overline{\partial} \phi \) (see Section 14 of [4]). Here we say that a function \( \phi \) is almost plurisubharmonic if, for any \( x \in X \), there exists a smooth function \( \psi \) on a neighborhood of \( x \) such that \( \phi + \psi \) is a plurisubharmonic function. Then we have that points where \( \phi \) is not continuous belong to \( \text{supp} \ T \). (We note that this claim needs not hold if \( \phi \) is not almost plurisubharmonic, even when \( \phi \) is a locally integrable upper-semicontinuous function.)

First we assume that \( \phi \) is bounded on \( X \). Take non-zero holomorphic sections \( s_1 \in \Gamma(X, L_1), \ldots, s_p \in \Gamma(X, L_p) \). Define \( s = s_1 \otimes \cdots \otimes s_p \in \Gamma'(X, L_1 \otimes \cdots \otimes L_p) \).

**Lemma 10** Let \( U \subset X \) be an open neighborhood of \( \text{supp} \ T = \text{supp} \, (\omega + i \partial \overline{\partial} \phi) \). Here \( \phi \) is a bounded almost plurisubharmonic function. Let \( u \in \Omega^{(0,q-1)}(U, F) \) \((1 \leq q \leq n-2)\) such that \( \overline{\partial} u = 0 \). Let \( K \subset X \setminus \{z \in X | s(z) = 0\} \) be a compact set. Then there exist

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an open neighborhood $U_1 \subset X$ of $K \cup \text{supp} \ (\omega + i \delta \partial \varphi)$, a $\overline{\partial}$-closed $F$-valued form $u_1 \in \Omega^{(0,q-1)}(U_1, F)$ and a bounded almost plurisubharmonic function $\varphi_1$ on $X$ which satisfy the following conditions:

(a) $u = u_1$ on an open neighborhood of $\text{supp} \ (\omega + i \partial \varphi)$,
(b) $\omega + i \delta \partial \varphi \geq 0$,
(c) $K \cup \text{supp} \ (\omega + i \partial \varphi) \subset \text{supp} \ (\omega + i \delta \partial \varphi) \subset U_1$.

**Proof** Put $Y = \{ z \in X \mid s(z) = 0 \}$. Then $X \setminus Y$ is a Stein manifold. Let $\| \cdot \|_j$ be a smooth Hermitian metric of $L_j$ whose Chern curvature is $\omega_j$. Let $\tilde{\varphi} = \varphi - \sum_{j=1}^{p} \frac{1}{2\pi} \log \|s_j\|^{2a_j}$ on $X \setminus Y$. Then $\tilde{\varphi}$ is an exhaustive plurisubharmonic function and $\text{supp} \ i \partial \delta \tilde{\varphi} = (X \setminus Y) \setminus \text{supp} \ (\omega + i \partial \varphi)$. Let $D_{\tilde{\varphi}}(r) = \{ z \in X \setminus Y \mid \tilde{\varphi}(z) < r \}$. Take $s < t$ such that $K \subset D_{\tilde{\varphi}}(s) \subset D_{\tilde{\varphi}}(t)$. There exists $v \in \Omega^{(0,q-1)}(D_{\tilde{\varphi}}(t), F)$ such that $\overline{\partial} v = 0$ and that $u = v$ on an open neighborhood of $\overline{\partial} \varphi$. As in Lemma 9, we can glue $v$ and $u$ together, and we obtain an open neighborhood $U_1 \subset X$ of $K \cup \text{supp} \ (\omega + i \partial \varphi)$ and $u_1 \in \Omega^{(0,q-1)}(U_1, F)$ which satisfy (a). By Lemma 8, there exists bounded from below, exhaustive plurisubharmonic function $\tilde{\varphi}_1$ such that $K \cup \text{supp} \ i \partial \delta \tilde{\varphi} \subset \text{supp} \ i \partial \delta \tilde{\varphi}_1 \subset U_1$ and that $\tilde{\varphi}_1 = \tilde{\varphi}$ on $X \setminus (U_1 \cup D_{\tilde{\varphi}}(t))$. Put $\varphi_1 = \tilde{\varphi}_1 + \sum_{j=1}^{p} \frac{1}{2\pi} \log \|s\|^{2a_j}$. Since $\varphi_1 = \varphi$ on a neighborhood of $\overline{\partial} \varphi$, we can consider $\varphi_1$ as a bounded almost plurisubharmonic function on $X$. Then $\varphi_1$ is a function we are looking for.

Now we prove that the natural map

$$H^q(X, F) \to \lim_{\supp (\omega + i \partial \varphi) \subset V} H^q(V, F)$$

is an isomorphism for $0 \leq q < n - 2$ and is injective for $q = n - 2$.

**Proof of the case where $\varphi$ is bounded** We first show that $H^q(X, F) \to \lim_{\supp (\omega + i \partial \varphi) \subset V} H^q(V, F)$ is surjective for $0 \leq q \leq n - 3$. Let $u \in \Omega^{(0,q)}(U, F)$ be a $\overline{\partial}$-closed $F$-valued form where $U$ is an open neighborhood of $\text{supp} \ (\omega + i \partial \varphi)$. We can take compact sets $K_i \subset X$ and $s_{i,j} \in \Gamma(X, L_j)$ $(1 \leq l \leq N, 1 \leq j \leq p)$ such that $\bigcup_{l=1}^{N} K_l = X$ and that $K_l \cap \bigcup_{j=1}^{p} \{ z \in X \mid s_{i,j}(z) = 0 \} = \emptyset$ for any $l$. Put $u_{\varphi_0} = \varphi$, $u_{0,0} = u$. By using Lemma 10 repeatedly, there exist $U_1 \subset X, u_1 \in \Omega^{(0,q)}(U_1, F)$ $(1 \leq l \leq N)$ and $\varphi_l (1 \leq l \leq N - 1)$ such that $\bigcup_{l=1}^{N} K_l \cup \text{supp} \ (\omega + i \delta \partial \varphi) \subset \text{supp} \ (\omega + i \partial \delta \varphi_1) \cup U_1$ and that $u_l = u_{l-1}$ on a neighborhood of $\text{supp} \ (\omega + i \delta \partial \varphi_{l-1})$. Hence $u_N \in \Omega^{(0,q)}(X, F)$ is $\overline{\partial}$-closed $F$-valued form such that $u_N = u$ on a neighborhood of $\text{supp} \ (\omega + i \partial \varphi)$. This proves the surjectivity.

Next, we show that $H^q(X, F) \to \lim_{\supp (\omega + i \partial \varphi) \subset V} H^q(V, F)$ is injective for $0 \leq q \leq n - 2$.

Let $\alpha \in \Omega^{(0,q)}(X, F)$ such that $\overline{\partial} \alpha = 0$. Suppose that there exists an open neighborhood $U \subset X$ of $\text{supp} \ (\omega + i \delta \partial \varphi)$ and $v \in \Omega^{(0,q-1)}(U, F)$ such that $\alpha = \overline{\partial} v$ on $U$. Let $\chi \in C^\infty(X)$ be a function such that $\text{supp} \ \chi \subset U$ and that $\chi = 1$ on a neighborhood of $\text{supp} \ (\omega + i \partial \varphi)$. Then $\alpha = \overline{\partial} (\chi v)$ is $\overline{\partial}$-closed $F$-valued form which vanishes on a neighborhood of $\text{supp} \ (\omega + i \delta \partial \varphi)$.

Take $K_1, \ldots, K_N$ as in the proof of the surjectivity. As in the proof of Lemma 6, there exists $F$-valued $(0, q - 1)$-form $v'_1$ which is defined in an open neighborhood of $K_1$ such that $\overline{\partial} v'_1 = \alpha = \overline{\partial} (\chi v)$ and that $v'_1 = 0$ on a neighborhood of $\text{supp} \ (\omega + i \delta \partial \varphi)$. By the trivial extension, we may assume that $v'_1$ is defined on a neighborhood $U_1$ of $K_1 \cup \text{supp} \ (\omega + i \delta \partial \varphi)$. Define $v_1 = \chi v + v'_1$ on $U_1$. We have that $\overline{\partial} v_1 = \alpha$ and that $v_1 = v$ on a neighborhood of $\text{supp} \ (\omega + i \delta \partial \varphi)$. As in Lemma 10, there exists a bounded function $\varphi_1$ such that $\omega + i \delta \partial \varphi_1 \geq 0$.
and that $K_1 \cup \text{supp} (\omega + i \partial \bar{\partial} \varphi) \subset \text{supp} (\omega + i \partial \bar{\partial} \varphi_1) \subset U_1$. If we replace $K_1, v, U, \varphi$ by $K_2, v_1, U_1, \varphi_1$ respectively, we obtain $v_2, U_2, \varphi_2$ which satisfy the suitable conditions. By repeating this process, we obtain $v_N \in \Omega^{(0, q-1)} (X, F)$ such that $\partial v_N = \alpha$. \hfill \Box

**proof of the case where $\varphi$ is unbounded** Define $\varphi_c = \max \{ \varphi, c \}$ for $c \in \mathbb{R}$ and $T_c = \omega + i \partial \bar{\partial} \varphi_c$. Then $\varphi_c$ is bounded and $T_c \geq 0$ since $\omega > 0$. It is easy to see that $\text{supp} T = \text{supp} (\omega + i \partial \bar{\partial} \varphi) \subset \text{supp} T_c$. Let $U$ be any open neighborhood of $\text{supp} T$. We show that $\text{supp} T_c \subset U$ for sufficiently small $c$. Assume that there exist points $x_k \in X \setminus U$ such that $x_k \in \text{supp} T_{-k}$ for any $k \in \mathbb{N}$. Let $\{x_{k(j)}\}_{j \in \mathbb{N}}$ be a convergent subsequence and $x = \lim_{j \to \infty} x_{k(j)} \in X \setminus U$. Then almost plurisubharmonic function $\varphi$ is unbounded on a neighborhood of $x$, and is not continuous at $x$. We have that $x \in \text{supp} T$, which gives a contradiction. Now we can reduce the proof to the case where $\varphi$ is bounded. \hfill \Box

6 Proof of main results

The proof of Theorem 1 is similar to that of Theorem 2. Hence we only prove Theorem 2 by induction on $m$. The case $m = 1$ holds by the arguments in Sect. 5. Assume now that $m \geq 2$ and that the case $m - 1$ has already been proved. We first note that the cohomology class of $T_j + T_k$ is in $\mathcal{K}_{NS}$ and

$$\text{supp} T_j \cup \text{supp} T_k = \text{supp} (T_j + T_k)$$

for any $1 \leq j, k \leq m$. Put $A = \bigcap_{j=1}^{m-1} \text{supp} T_j$ and $B = \text{supp} T_m$. Then Mayer–Vietoris exact sequence yields the commutative diagram

$$\begin{array}{ccc}
\cdots & \longrightarrow & H^q (X, F) \\
\downarrow & \downarrow & \downarrow \\
\lim_{A \subseteq V_1} H^q (V_1 \cup V_2, F) & \longrightarrow & H^q (V_1, F) \oplus H^q (V_2, F) \\
\downarrow & \downarrow & \downarrow \\
\lim_{B \subseteq V_2} H^q (V_1 \cap V_2, F) & \longrightarrow & H^q (V_1 \cap V_2, F) \\
\cdots & \longrightarrow & \cdots
\end{array}$$

where the rows are exact sequences. We have that

$$\lim_{A \subseteq V_1} H^q (V_1 \cup V_2, F) \simeq \lim_{\bigcap_{j=1}^{m-1} (\text{supp} T_j + T_m) \subseteq V} H^q (V, F)$$

and

$$\lim_{A \subseteq V_1} H^q (V_1 \cap V_2, F) \simeq \lim_{\bigcap_{j=1}^{m} \text{supp} T_j \subseteq V} H^q (V, F).$$

Then we complete the proof by the induction hypothesis and a diagram-chasing argument. \hfill \Box

Let $X = \mathbb{P}^n$ be the complex projective space of dimension $n \geq 3$. Then any non-zero closed positive current of type $(1, 1)$ belongs to $\mathcal{K}_{NS}$ and we can apply Theorem 2 to this case.

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