Unimodular Gauge in Perturbative Gravity and Supergravity

Laurent Baulieu *

LPTHE, Sorbonne Université, CNRS
4 Place Jussieu, 75005 Paris, France

Abstract

This paper explains the Unimodular gauge fixing of gravity and supergravity in the framework of a perturbative BRST construction. The unphysical sector contains additional BRST-exact quartets to suppress possible ambiguities and impose both the Unimodular gauge fixing condition on the metric and a gauge condition for the reparametrization symmetry of the unimodular part of the metric. The Unimodular gauge choice of the metric must be completed by a γ-Traceless gauge condition for the Rarita–Schwinger field in the case of supergravity. This gives an interesting new class of gauges for gravity and supergravity.
1 Introduction

Albert Einstein recognised as early as in 1916 that there is a preferred gauge in classical gravity. He recommended the choice of a system of coordinates such that the determinant of the space-time metric $g_{\mu\nu}$ is locally unimodular. This means the gauge choice $- \det(g_{\mu\nu}) = 1$ for solving the Einstein equations of motion [1]. Since this epoch, there has been some interesting activities about the concept of the Unimodular gauge. The word “Unimodular gravity” has actually become quite common. The non-exhaustive series of papers [2][3][4][5][6][7][8][9][10] and references therein address interesting questions related to this domain. A priori, the so-called Unimodular gravity has a different physical content than the standard Einstein theory. When looking at the literature, there are mainly two formulations: one that imposes $g \equiv - \det(g_{\mu\nu}) = 1$ as a gauge choice and another one that imposes the constraint $g = 1$. Some confusion is spread around these formulations, although their difference is actually quite clear.

Working in the “Unimodular gauge” $g = 1$ for the Einstein theory is nothing but a possible choice, maybe unfamiliar and difficult to enforce, but formally equivalent to any other gauge choice. The BRST methodology to enforce this gauge choice is the subject of this paper. It exhibits interesting non trivialities that we find worth being published. The end of this introduction sketches physical motivations for using this gauge. Maybe the most striking one is that the gravity observables can be represented as functionals of the unimodular part of the metric, because of the physical redundancy between metrics related by a Weyl transformation. This last property was underlined in a different way in the classical theory in [12].

In contrast, the “Unimodular gravity” means that one changes the theory by varying classically the 'Einstein-Hilbert action by only considering variation of metrics with $g = 1$. One motivation of the “Unimodular gravity” is that the cosmological constant is introduced as a constant of integration that can be chosen at will, while the “Unimodular gauge” fixes the cosmological constant as a parameter of the Lagrangian from the beginning.

Imposing $g = 1$ is a well-defined classical local gauge condition for gravity made possible by the reparametrization invariance of the theory. Thus, as a matter of principle, there should be no ambiguity to define a perturbative quantum field theory of gravity in this gauge. A solution must exist at least perturbatively for quantizing gravity by imposing the unimodular condition $g = 1$ on $g_{\mu\nu}$ and gauge fixing afterwards the residual reparametrization invariance of its unimodular part $\tilde{g}_{\mu\nu}$. If supergravity is involved, the Unimodular condition on the metric implies a $\gamma$-traceless condition on the spin 3/2 Rarita–Schwinger field $\Psi_{\mu}$ and the residual local supersymmetry of its pure spin 3/2 part $\tilde{\Psi}_{\mu}$ must be further gauge fixed.

This paper is thus aimed at building a local quantum Lagrangian that defines the Unimodular gravity and supergravity unambiguously, at least for defining a consistent perturbative BRST invariant quantum field theory. A BRST exact gauge fixing action will be build that enforces consistently the Unimodular gauge condition, to be added to the Einstein action (and the Rarita–Schwinger action). We don’t fear a possible anomaly for this process in the $d = 4$ case, because consistent 4d gravitational anomalies cannot possibly exist due to the structure of the $SO(3,1)$ Lie algebra.

Getting a BRST symmetry invariant gauge fixing is necessary to possibly enforce all relevant Ward identities that define the theory eventually. Gravity is non renormalisable by power counting but, presumably, the Unimodular gauge fixing procedure can be made stable under radiative corrections by introducing the needed counterterms that are compatible with the Ward identities, order by order in perturbation theory.

The Unimodular gauge quantum Lagrangian built in this work makes explicit the particularities of diffeomorphisms with a divergence-less vector parameter. One of its subtleties is that a formal Faddeev–Popov gauge fixing of Unimodular metrics provides a singular determinant with a ghost of ghost phenomenon. To control this phenomenon, techniques analogous as those used to currently define TQFT’s with a gauge invariance are needed. This provides a localisation of fields around their Unimodular components with a remaining gauge invariance to be further gauge fixed in a BRST invariant way. The standard BRST field content of perturbative gravity must be therefore enlarged with additional BRST trivial quartets to define the Unimodular gauge.

Although this work is self contained, it has a hidden motivation that is the stochastic quantization of gravity. The latter remedies the absence of a well-defined Lorentz time evolution in quantum gravity by the stochastic quantization time as the variable that orders the non-perturbative quantum gravity phenomena. In this framework, [13] indicates that the conformal factor of the metric behaves as a spectator, while the non trivial aspects of the quantum gravity dynamics are carried by the Unimodular part of the metric. But [13] also predicts that, at the perturbative level, the limit at infinite stochastic time of the stochastically quantized
gravity is the well-defined (modulo UV questions) standard 4d perturbative quantum theory, for which the Lorentz time can be defined. It thus appeared necessary to dispose of a precise construction of semi-classical gravity in the Unimodular gauge, the subject of this paper.

Interestingly, having a well-defined perturbative quantization of gravity in the Unimodular gauge makes contact with the work of York [12], who showed that what the classical Einstein equations truly propagate are the equivalence classes of metrics defined modulo Weyl transformations. Solving the Einstein equations is a Cauchy problem. York pointed out that, taking as initial conditions two metrics related by a Weyl transformation, their evolution at any given future time provides two metrics that are also related by a Weyl transformation. This fact holds true although the gravity equations of motions are not Weyl invariant [12]. This makes the principle of gauge invariance and the definition of observables more subtle in gravity than in Yang-Mills and $p$-form gauge invariant theories. Showing that one can gauge fix the metric to be unimodular in a BRST invariant way is a way to generalise at the quantum level the classical arguments of York, since the set of the Weyl classes of metrics can be represented by the set of unimodular metrics.

Our work suggests that the (super)gravity observables should be defined as the functionals of unimodular metric (and gravitino $\gamma$-Traceless components) although the (super)gravity equations are not Weyl invariant. This property is very natural when one works in the Unimodular gauge. Since the BRST invariance ensures that the same physics can be computed with any other (well-defined) choice of gauge, the same conclusion must be true in other gauges. The expression of observables may then occur with more complicated expressions.

## 2 Pure Gravity

### 2.1 Improved BRST symmetry for the Unimodular gauge

The current method to perturbatively gauge fix gravity in Lagrangian formalism is by introducing a BRST symmetry operation $s$ transforming the metric field $g_{\mu\nu}(x)$ and the vector ghost field $\xi^\mu(x)$ of the reparametrization symmetry. The covariance of the BRST trivial pair made of a reparametrization antighost and a Lagrange multiplier depends on the gauge condition one wishes to use. Choosing the gauge function $\partial_\nu g^{\mu\nu}$, the anticommuting antighost and commuting Lagrange multiplier are both vectors $\xi^\mu(x)$ and $b^\mu$. The BRST symmetry is then defined by the following action of the graded differential operator $s$ on the gravity BRST multiplet fields

\[
\begin{align*}
sg_{\mu\nu} &= \text{Lie}_s g_{\mu\nu} \\
sc_\mu &= \xi^\nu \partial_\nu \xi^\mu \\
sb^\mu &= b^\mu \\
sb^\mu &= 0.
\end{align*}
\]

One has $[s, \partial_\mu] = 0$ and the nilpotency $s^2 = 0$. t’Hooft and Veltman defined the perturbation expansion of quantum gravity in the de-Donder gauge adding the invariant s-exact term $s(\xi^\mu \partial_\nu g_{\mu\nu})$ to the Einstein action [14]

\[
L_{\text{Einstein}} \rightarrow L_{\text{Einstein}} + 8(\xi^\mu \partial_\nu g_{\mu\nu}) = L_{\text{Einstein}} + b^\mu \partial_\nu g_{\mu\nu} - \xi^\mu \text{Lie}_s \partial_\nu g_{\mu\nu}.
\]

They used the Feynman rules for the metric and the ghosts and antighosts that stem from the local action [2]. Their gravity Ward identities are implied by the symmetry [1].

The use of a unimodular metric gauge choice seems impossible with only the standard Fadeev–Popov fields. The condition $\partial_\nu g^{\mu\nu}$ in [2] exhausts the 4 gauge freedoms allowed by the Lagrange multiplier components $b^\mu$.

Something more refined than the standard Fadeev–Popov construction must be done to define the gauge fixing of $g_{\mu\nu}$ to its unimodular part $\hat{g}_{\mu\nu}$ ($g = 1$) with a further gauge fixing of the reparametrization symmetry of $\hat{g}_{\mu\nu}$.

$\hat{g}_{\mu\nu}$ and $g$ can be considered as independent field variables. This generalizes for $d > 2$ the decomposition of a 2d metrics in its conformal form and its Beltrami parameter [15]. The off-shell decomposition of $g_{\mu\nu}$ in $\hat{g}_{\mu\nu}$ and $g$ is justified because the variations of $g_{\mu\nu}$ are not irreducible Lorentz tensors and split into trace and traceless components. In fact, an off-shell decomposition of any given Lorentz tensor fields in irreducible representations should be done systematically for spin values larger than 1. For spin 3/2, the Rarita–Schwinger field $\Psi_\mu$ must be split in its $\gamma$-Trace and $\gamma$-Traceless irreducible components, and so on.
Thus, one must be quite precise when attempting to gauge fix the gravity field $g_{\mu\nu}$ with separate gauge functions for $\sqrt{g}$ and its unimodular part (here $d = 4$)

$$\hat{g}_{\mu\nu} \equiv g_{\mu\nu}/\sqrt{g}^2. \quad (3)$$

The gauge fixing problem of gravity draws us deeper in the BRST symmetry formalism than the Yang–Mills theory and the theory of $p$-form fields whose field variations belong to irreducible Lorentz representations.

If one formally applies the Faddeev–Popov method to impose both gauge conditions $\partial_\nu \hat{g}^{\mu\nu} = 0$ and $\sqrt{g} \equiv \sqrt{-\text{det} g_{\mu\nu}} = 1$, (which make sense classically), the (formal) Faddeev–Popov determinant for the gauge fixing functional $\partial_\nu \hat{g}^{\mu\nu}$ gets zero modes, meaning degenerate equations of motion for $\xi^\mu$ and $\xi^\mu$ in the corresponding Faddeev–Popov action, as well as for $b_\mu$. These zero modes are configurations of $\xi^\mu$ and $\xi^\mu$ with $\partial_\mu \xi^\mu = 0$ and $\partial_\mu \xi^\mu = 0$. The reason for the trouble is that any given variation $\delta \hat{g}_{\mu\nu}$ of an unimodular metric $\hat{g}_{\mu\nu}$ is traceless

$$\det \hat{g}_{\mu\nu} = -1 \Rightarrow \hat{g}^{\mu\nu} \delta \hat{g}_{\mu\nu} = 0. \quad (4)$$

In fact, since $s\sqrt{g} = \partial_\mu (\xi^\mu \sqrt{g})$, the constraint $\sqrt{g} = 1$ is left invariant by the diffeomorphism transformations $sg_{\mu\nu} = \text{Lie}_\xi g_{\mu\nu}$ only under the condition

$$s\sqrt{g} \big|_{g=1} = \partial_\mu \xi^\mu = 0. \quad (5)$$

This constraint explains why the Faddeev–Popov determinant associated to the gauge function $\partial_\nu \hat{g}^{\mu\nu}$ is singular and that its formal (and inconsistent) representation by a path integral over ghost and antighost fields would give a Lagrangian with the above mentioned ghost and antighost longitudinal zero modes. The BRST transformation of $b^\mu = s\xi^\mu$ has also longitudinal zero modes. Analogously, the separate gauge fixing of $\sqrt{g}$ leads to a ghost action with transverse zero modes because $s\sqrt{g} = \partial_\mu (\sqrt{g} \xi^\mu)$.

One must therefore find a way to define the path integral while separating in a BRST invariant way the phase space of diffeomorphism ghost and antighost fields into its longitudinal and transverse sectors. The trouble of having ghost and/or antighost fields that are defined modulo a longitudinal part can be cured by introducing ghosts of ghosts. The longitudinal and transverse components of the auxiliary field $b_\mu$ must be also separated. Its longitudinal component must undergo a BRST invariant gauge fixing. Ghosts of ghosts correct the wrong statement that the 5 conditions $\partial_\nu \hat{g}^{\mu\nu} = 0, g = 1$ might imply an over-gauge fixing.

To clarify those points, the current understanding of topological quantum field theories with gauge symmetries, which involve systematically ghosts of ghosts and avoid in this way the danger over-gauge fixings, can be used as a road map. To solve all issues at once, an extended BRST symmetry involving new fields organised under the form of BRST-exact quartets must be introduced. They count altogether for zero degrees of freedom, of which some play the role of Lagrange multipliers. They allow a BRST invariant gauge fixing of the unimodular part of the metric and of the resulting degeneracy of the diffeomorphism ghosts. Eventually, the construction of a BRST invariant path integral will emerge with a functional measure using $\hat{g}_{\mu\nu}$ and $\sqrt{g}$ as fundamental fields.

One thus completes the ordinary BRST system in Eq. (11) by addition of the trivial BRST quartet

$$L^{(00)}, \eta^{(10)}, \eta^{(01)}, b^{(11)}. \quad (6)$$

The scalar bosonic fields $L, b$ and fermionic fields $\eta, \bar{\eta}$ count altogether for zero=1+1-1-1 degrees of freedoms in unitary relations provided their dynamics is governed by an s-exact action defining invertible propagators.

Having available this extra set of unphysical fields is exactly what one needs to get a Lagrangian with invertible propagators in the Unimodular gauge, with a BRST invariant gauge fixing of zero modes that otherwise would spoil the definition of gravity by a path integral the in Unimodular gauge. Eventually, a path integral with a measure depending only on the unimodular part of the metric will be obtained as a generalization of the classical prescription of Einstein [1].

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The following diagram displays suggestively all necessary ghosts, antighosts and Lagrange multipliers:

\[
\begin{array}{c|c|c}
\Phi & \xi^{(10)} & \eta^{(10)} \\
\hline
\xi^{(10)} & \xi^{(01)} & \eta^{(01)} \\
\hline
\eta^{(10)} & b^{(11)} & b^{(11)} \\
\hline
\end{array}
\]

The numbers \(-1, 0, 1\) in the bottom line indicate the net ghost number of fields that are aligned vertically above each number. The BRST transformations that generalises [1] are

\[
s g_{\mu \nu} = \text{Lie}_\xi g_{\mu \nu} \\
s \xi^\mu = \text{Lie}_\xi \xi^\mu \\
s \bar{\xi}^\mu = b^\mu \\
s L = \eta \\
s s = b.
\]

One has still \([s, \partial_{\mu}] = 0\), \([s, d] = 0\) and \(s^2 = 0\) on this extended set of fields. In fact \(d, s, i_\xi, \text{Lie}_\xi = [i_\xi, d]\) build a system of nilpotent graded differentials operators\[2\] The last two lines in Eqs. (8) identify \(L, \eta, \bar{\eta}, b\) as the elements of a BRST exact quartet. The commuting scalar \(b\) is an additional scalar Lagrange multiplier with ghost number 0. Both anticommuting scalar \(\eta, \bar{\eta}\) are odd Lagrange multipliers with ghost numbers \(-1\) and 1.

2.2 The BRST invariant quantum Lagrangian for the Unimodular gauge

We now define a class of BRST invariant gauge fixing actions with the 4 gauge functions \(\partial_{\mu} \hat{g}^{\mu \nu}\) and the gauge condition \(\sqrt{g} = 1\). Using Eqs. (8), we can complete \(\int dx \sqrt{g} R(g_{\mu \nu})\) by addition of an \(s\)-exact term. We define

\[
\int dx \mathcal{L}_{\text{gauge fixed}}^{\text{BRST inv.}} = \int dx \left( \sqrt{g} R(g_{\mu \nu}) + s \left[ \xi^\mu (\hat{g}_{\mu \nu} \partial_{\rho} \hat{g}^{\rho \nu} + \gamma \partial_{\mu} L + \frac{\alpha}{2} b^\mu), + \bar{\eta}(\sqrt{g} - 1) \right] \right)
\]

The class of Unimodular gauges is characterized by the gauge parameters \(\gamma, \alpha\). Observables are the elements of the cohomology of \(s\). Their expectation values is expected to be independent on the choice of \(\alpha, \gamma\). These parameters will be chosen as \(\alpha = 0\) and \(\gamma = 1\) in what follows. By expanding the \(s\)-exact term one gets

\[
\int dx \mathcal{L}_{\text{gauge fixed}}^{\text{BRST inv.}}(\hat{g}_{\mu \nu}, L, b_{\mu, \xi^\mu, \bar{\xi}^\mu, \eta, \bar{\eta}, b) = \int dx \left( \sqrt{g} R(\hat{g}_{\mu \nu}) + b^\mu (\hat{g}_{\mu \nu} \partial_{\rho} \hat{g}^{\rho \nu} + \partial_{\mu} L) + b(\sqrt{g} - 1)
\]

\[\hat{\xi}^\mu (\hat{g}_{\mu \nu} \partial_{\rho} \text{Lie}_\xi \hat{g}^{\rho \nu} + (\text{Lie}_\xi \hat{g}_{\mu \nu}) \partial_{\rho} \hat{g}^{\rho \nu} - \hat{\xi}^\nu \partial_{\rho} \eta + \bar{\eta} \nabla_{\rho} \hat{\xi}^\nu) \right].
\]

The elimination of the scalar Lagrange multiplier \(b\) by its equation of motion implies \(g = 1\) and thus \(g_{\mu \nu} = \hat{g}_{\mu \nu}\). The action \([10]\) is the BRST invariant proposal for gravity in the Unimodular gauge. It can be written as

\[
\mathcal{I}_{\text{Unimodular}}^{\text{BRST}}(\hat{g}_{\mu \nu}, L, b_{\mu, \xi^\mu, \bar{\xi}^\mu, \eta, \bar{\eta}) = \int dx \left( R(\hat{g}_{\mu \nu}) + b^\mu (\hat{g}_{\mu \nu} \partial_{\rho} \hat{g}^{\rho \nu} + \partial_{\mu} L) \right) - \hat{\xi}^\mu (\hat{g}_{\mu \nu} \partial_{\rho} \text{Lie}_\xi \hat{g}^{\rho \nu} + (\text{Lie}_\xi \hat{g}_{\mu \nu}) \partial_{\rho} \hat{g}^{\rho \nu} ) - \hat{\xi}^\nu \partial_{\rho} \eta + \bar{\eta} \nabla_{\rho} \hat{\xi}^\nu) \right]
\]

Beware that \(R(\hat{g}_{\mu \nu}) \equiv R(g_{\mu \nu})\) at \(g = 1\). This construction of the BRST invariant Unimodular gauge action is both consistent and direct.

Eq. ([10]) shows that the field \(L\) is a Lagrange multiplier for the longitudinal part of \(b_{\mu}\). The latter would be undetermined in the absence of \(L\) or with \(\gamma = 0\) because the quadratic form \(b_{\mu} \partial_{\nu} \hat{g}^{\mu \nu}\) is degenerate when
\( b_\mu \to b_\mu + \partial_\mu \epsilon \) due to the unimodularity of \( \hat{g}^{\mu \nu} \). Likewise, \( \eta \) and \( \eta \) are fermionic Lagrange multipliers that gauge fix the longitudinal zero modes of \( \xi^\mu \) and \( \xi^\nu \) stemming from the unimodularity of \( \hat{g}^{\mu \nu} \) in \( \xi^\mu s(\hat{g}_{\mu \nu} \partial_\mu \hat{g}^{\nu \rho}) \).

Eventually the BRST invariant action \( I_{\text{Unimodular}}^{\text{BRST}} \) in (11) only depends on \( \hat{g}^{\mu \nu} \) with the gauge condition \( \partial_\mu \hat{g}^{\mu \nu} + \partial_\nu L = 0 \). The unimodular components \( \hat{g}_{\mu \nu} \) of \( g_{\mu \nu} \) circulate in Feynman diagrams loops while \( \sqrt{g} \) remains a spectator field with some compensations due to bosonic loops of \( L \) and fermionic loops of \( \eta \) and \( \eta \). All propagators between the bosons \( \hat{g}_{\mu \nu}, b_\mu, b, L \) and the fermions \( \xi^\mu, \xi^\nu, \eta, \eta \) are invertible. Thus, the action (11) is well suited for a quantum description of gravity with the Unimodular gauge choice \( g = 1 \), giving some concrete sense to the visionary classical prescription of Einstein [11] when it is interpreted as a gauge fixing prescription. For perturbations around non-trivial classical backgrounds, the latter must be expressed in the Unimodular gauge.

### 2.3 Gravity observables

Mean values of observables are defined as

\[
\langle \mathcal{O}(\hat{g}_{\mu \nu}) \rangle \equiv \int [d\hat{g}_{\mu \nu}] [d\xi^\mu] [d\xi^\nu] [dL] [d\eta] [d\bar{\eta}] \mathcal{O}(\hat{g}_{\mu \nu}) \exp - \frac{1}{\hbar} \hat{I}_{\text{Unimodular}}^{\text{BRST}}(\hat{g}_{\mu \nu}, L, b_\mu, \xi^\mu, \xi^\nu, \eta, \bar{\eta}).
\]

If matter is coupled, the gauge fixing \( g_{\mu \nu} = \hat{g}_{\mu \nu} \) also affects its energy momentum tensor, which then depends on \( g_{\mu \nu} \) only through \( \hat{g}_{\mu \nu} \). Because gravitational anomalies cannot exits in \( d = 4 \), we expect one can enforce the Unimodular gauge order by order at any finite order of perturbation theory, modulo the necessity of adding more and more invariant counterterms. The Ward identities should guarantee the stability of the gauge \( \sqrt{g} = 1 \).

### 3 Unimodular supergravity

We consider the supergravity \( N = 1, d = 4 \). We choose the new minimal system of auxiliary fields of Søninius and West [17] with the notations of [18]. Auxiliary fields are often necessary for the nilpotency of the BRST symmetry operator in supergravity. Their role is secondary in this paper.

We use a Lorentz signature. The flat metric \( g_{\mu \nu} \) has signature \((- , +, +, +)\). The Dirac matrices \( \gamma^\mu \) are real and \( \gamma^5 \equiv \gamma^0 \gamma^1 \gamma^2 \gamma^3 = \frac{1}{4} \epsilon_{\mu \nu \rho \sigma} \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \). One has \((\gamma^5)^2 = -1, \gamma^5 = \gamma^5 \) and \( \gamma^\mu = \gamma^0 \gamma^\mu \gamma^0 \). The Dirac conjugate of a spinor \( X \) is \( X^\star \equiv X^\dagger \gamma^0 \). One chooses the charge conjugation matrix \( C \) to be \( \gamma^0 \) with \( X^C \equiv (C X^\star)^T \). Majorana spinors have 4 real components since by definition \( X^C = X \).

With these conventions, the Rarita–Schwinger Lagrangian of the spin 3/2 Majorana gravitino \( \Psi_{\mu} \) is

\[
L_{\text{RS}} = -\frac{1}{2} e^{\mu \nu \rho \sigma} \Psi^* \gamma^5 \gamma_\nu D_\rho \Psi_\sigma.
\]

(13)

To make more transparent the gravitino gauge fixing, one has advantage to use the 3/2 spin projection operators as in [13] and [18] (that satisfy all relevant orthogonality conditions)

\[
P_{\mu \nu} = \frac{1}{3} \delta_{\mu \nu} - \frac{1}{3} \gamma_{\mu} \gamma_{\nu}
\]

\[
(P_{11}^\dagger)_{\mu \nu} = \frac{1}{3} \gamma_{\mu} \gamma_{\nu} \quad (P_{12}^\dagger)_{\mu \nu} = \frac{1}{\sqrt{3}} \gamma_{\mu} \omega_{\nu}
\]

\[
(P_{21}^\dagger)_{\mu \nu} = \frac{1}{\sqrt{3}} \gamma_{\nu} \omega_{\mu} \quad (P_{22}^\dagger)_{\mu \nu} = \frac{1}{3} \omega_{\mu} \omega_{\nu}
\]

\[
\omega_{\mu} \equiv \frac{\partial_\mu}{\gamma^5 \cdot \partial} \quad \gamma^5 = \gamma^\mu - \omega^\mu = \eta_{\mu \nu} - \omega_{\mu \nu}. \quad (14)
\]

\[1\] There is a mixed propagator between \( \hat{g}_{\mu \nu} \) and \( L \) as a consequence of the the choice \( \alpha = 0 \). For \( \alpha \neq 0 \), one has a Feynman type propagator for \( \hat{g}_{\mu \nu} \) and a Klein–Gordon propagator for \( L \) after the algebraic elimination of \( b^\mu \).

\[2\] We refer to [15] as well as to [18] for properties of the covariant derivative \( D_\mu = \partial_\mu + \omega_{\alpha \beta} \frac{1}{2} \gamma_{\alpha \beta} A + B \) in the new minimal set of \( N = 1, d = 4 \) supergravity. Here we have omitted the dependence in the auxiliary fields \( A \) and \( B \) without loss of generality. To incorporate the auxiliary fields in the the Unimodular gauge fixing of supergravity, one can consistently use [18], although it is devoted to the different subject of \( N = 1, d = 4 \) supergravity superHiggs mechanism.
The free part of the Rarita–Schwinger Lagrangian, invariant under the transformation $\Psi_\mu \to \Psi_\mu + \partial_\mu \epsilon$, is

$$L_{\text{RS}}^{\text{free}} = -\frac{1}{2} i \epsilon^{\mu\nu\rho\sigma} \Psi_\mu \gamma_\nu \partial_\rho \Psi_\sigma \equiv \Psi_\mu (\partial_\mu - P_{11}^{\mu} - \frac{1}{2}) \Psi_\nu. \quad (15)$$

An interesting observation is that $L_{\text{RS}}^{\text{free}} = \Psi^\dagger_\mu (g_{\mu\nu} - \partial_\mu \partial_\nu) \Psi_\nu + (\ldots) \Psi$. Suppose now that one has the following algebraic constraint on $\Psi_\mu$

$$\Psi \equiv \gamma^\mu \Psi_\mu = 0. \quad (16)$$

It can be enforced by using a fermionic spin 1/2 Lagrange multiplier $\bar{\tau}$ and adding the term $\bar{\tau} \Psi$ to $L_{\text{RS}}^{\text{free}}$

$$L_{\text{RS}}^{\text{free}} + \bar{\tau} \Psi = \Psi^\dagger_\mu (g_{\mu\nu} - \partial_\mu \partial_\nu) \Psi_\nu + (\bar{\tau} + \ldots) \Psi. \quad (17)$$

The non-locality seemingly presents in $\partial_\mu \partial_\nu \bar{\tau} \Psi$ is spurious as it will be shown shortly. Eq. (17) expresses the naturalness of the off-shell gauge condition $\Sigma$. One defines the following off-shell decomposition of $\Psi_\mu$ that will be convenient to complete the condition $\sqrt{\Sigma} = 1$ on the metric by a $\gamma$-Traceless gauge condition on $\Psi_\mu$

$$\Psi_\mu = \hat{\Psi}_\mu + \gamma^\mu \hat{\Psi} \quad \text{where} \quad \hat{\Psi}_\mu \equiv \frac{1}{d} \Psi_\mu - d \gamma_\mu \Psi. \quad (18)$$

### 3.1 Fields for $\gamma$ traceless gauge in supergravity

We wish to separately gauge fix in a BRST invariant way both Lorentz irreducible spin 1/2 and spin 3/2 spinors $\hat{\Psi}$ and $\hat{\Psi}_\mu$ in Eq. (15) with the gauge functions $\hat{\Psi}$ and $\partial^\mu \hat{\Psi}_\mu$. This choice of spinorial gauge functions is quite different than the conventional one in supergravity. The latter amounts to add to the Rarita–Schwinger Lagrangian a gauge fixing term $\Psi^\dagger \partial \Psi \Psi$ with additional subtleties in the massive case. However, this commonly used gauge fixing term vanishes for $\Psi = 0$. It is thus inconsistent with the off-shell $\gamma$-Traceless condition of $\Psi_\mu$ that will be used shortly to build the Unimodular gauge supergravity. We must advance with caution because a ghost of ghost phenomenon for the local supersymmetry ghosts is expected if we impose $\Psi = 0$, analogous to that occurring for the reparametrization ghosts in the gravity gauge $g = 1$.

Call $\chi$ the commuting supersymmetry Majorana spinor ghost of supergravity. $\bar{\tau}$ is the commuting antighost and $d = s \tau$ is the anticommuting spinor auxiliary field (often known as the Nielsen–Kallosh ghost). $d$ is nothing but the fermionic Lagrange multiplier needed to possibly enforce a spin 1/2 gauge condition on $\hat{\Psi}_\mu$. $d$ is the analogous of $b^\mu$ for the a vector gauge condition on $\hat{g}_{\mu\nu}$. The standard BRST symmetry transformations of the $N = 1$, $d = 4$ supergravity are

$$sg_{\mu\nu} = L \epsilon \xi_{\mu\nu} + i \Psi^\dagger_\mu (\gamma_\nu) \chi$$
$$s\Psi_\mu = L \epsilon \xi_\mu + D_\mu \chi$$
$$s\xi^\mu = L \epsilon \xi^\mu + i \chi^\vee \gamma^\mu \chi = \xi^\nu \partial_\nu \xi^\mu + i \chi^\vee \gamma^\mu \chi$$
$$s\chi = L \epsilon \xi \chi = \xi^\nu \partial_\nu \xi^\mu - \frac{1}{2} \chi \partial_\nu \xi^\mu$$
$$s\bar{\tau} = d$$
$$sd = 0.$$

The novel feature is of using $d$ to only gauge fix the irreducible component $\hat{\Psi}_\mu$ of $\Psi_\mu$, instead of the full $\Psi_\mu$. This makes the situation completely different than that of the gauge choices used eg in [16] and [18]. Here comes the point. The Unimodularity constraint $g = 1$ implies for consistency that the supersymmetry variation of $\sqrt{\Sigma}$ vanishes. Thus, Eq. (15) must be generalized as

$$0 = s \sqrt{g}_{|g = 1} = \partial_\mu (\sqrt{\Sigma} \xi^\mu) + ig_{\mu\nu} \chi^\vee \gamma_\mu \Psi_\nu = \partial_\mu \xi^\mu + i \chi^\vee \Psi. \quad (19)$$
The supergravity path integral measure must therefore involve BRST invariant “transverse” and “longitudinal” off-shell field separations, according to the following Unimodularity gauge relevant “longitudinality” definitions
\[ \partial_{\mu} \xi^\mu = 0 \quad \text{and} \quad \Psi = 0. \]  

(20)

In the pure gravity case, the BRST trivial quartet \( L, \eta, \overline{\eta}, b \) allows one to possibly enforce the Unimodularity gauge, with \( \tilde{g}_{\mu\nu} \) and \( \sqrt{g} \) considered as independent field variables.

For the Rarita–Schwinger action, one wishes to consider \( \hat{\Psi} \) and \( \tilde{\Psi} \) as independent classical components of the Rarita–Schwinger field, to be gauge fixed separately. One thus completes the standard supergravity BRST fields in Eq. (19) by addition of a spinorial trivial quartet \( \lambda, a, \overline{\lambda}, \overline{a} \), obviously associated to \( L, \eta, \overline{\eta}, b \).

The BRST gravity fields in (7) get therefore the following Rarita–Schwinger partners
\[ \Psi_{\mu} = (\hat{\Psi}_{\mu}, \tilde{\Psi}, a^{(00)}), \]
\[ \chi^{(10)}, \lambda^{(01)}, d^{(11)}, \pi^{(11)} \]

(21)

The BRST transformations, which complete Eqs. (19) and express that \( \lambda, a, \overline{\lambda}, \overline{a} \) is a trivial BRST quartet, are
\[ s a = \lambda \quad s \lambda = 0 \]
\[ s \overline{\lambda} = \overline{a} \quad s \overline{a} = 0. \]  

(22)

### 3.2 BRST exact-terms for the Unimodular gauge in supergravity

To impose the \( \gamma \)-Traceless condition \( \Psi = 0 \) on the Rarita-Schwinger field, one defines
\[ L_{gf}^{\Psi} = s(\overline{\chi} \Psi) = \overline{\chi} \Psi + \overline{\chi} \delta \chi + \overline{\chi} (s e^\mu_a) \gamma^a \Psi_{\mu}. \]  

(23)

The equation of motion of \( \Psi \) enforces \( \Psi = 0 \) for the spin 3/2 field, analogously as that of \( b \) enforces \( g = 1 \).

To impose the longitudinal gauge function \( \partial \cdot \Psi \) on the \( \gamma \)-Traceless spin 3/2 field \( \hat{\Psi}_{\mu} \), one defines
\[ L_{gf}^{\partial \Psi} = s(\overline{\chi} (\partial \cdot \hat{\Psi} + \beta \delta a + \frac{\delta}{2} \delta d)) = \delta a^\mu \frac{\delta}{2} d + d^\ast (\partial \cdot \hat{\Psi} + \beta \delta a) + \overline{\chi} (\partial \cdot D\chi + \beta \delta \lambda) \]
+ ghost interaction terms proportional to \( s g_{\mu\nu} \).

\[ \beta \text{ and } \delta \text{ are parameters. The field } a \text{ is the fermionic analogous of the boson } L \text{ in Eq (10).} \]

The proposed gauge fixed action of the massless Rarita–Schwinger field in the \( \gamma \)-traceless gauge is therefore
\[ \int dx L_{BRST}^{RS}(\Psi_{\mu}, \chi, d, \lambda, \overline{\lambda}, a, \overline{a}) \equiv \int dx (L_{BRST}^{RS} + L_{gf}^{\Psi} + L_{gf}^{\partial \Psi}). \]  

(25)

The \( \gamma \)-traceless condition (10) \( \Psi = 0 \) holds everywhere after the elimination of \( \overline{a} \) by its algebraic equation of motion from \( L_{BRST}^{RS} \). In particular, the free quadratic part of \( \int dx L_{BRST}^{RS} \) is gauge fixed to (17). One must check that all fields in (21) have invertible propagators stemming from the action (25).

The term \( s(\overline{\chi} \delta a) \) in \( L_{gf}^{\partial \Psi} \) enforces the propagation of the fields \( \lambda \) and \( a \). In order its coefficient doesn’t vanish, one has the following condition, analogous to \( \gamma \neq 0 \) in (10),
\[ \beta \neq 0. \]  

(26)

\( L_{gf}^{\partial \Psi} \) determines a mixed propagator between \( \partial^\mu \hat{\Psi}_{\mu} \) and the spin 1/2 field \( d \). The second order propagation term \( \overline{\chi} \partial^2 \chi \) between the supersymmetry ghosts \( \chi \) and \( \overline{\chi} \) is a mere consequence of the choice of a gauge fixing function \( \partial \cdot \Psi \) to gauge fix the remaining of the local supersymmetry invariance after having imposed \( \Psi = 0 \).
### 3.3 Free quadratic approximation of the BRST invariant Lagrangian

To verify that the gauge fixing is complete, let us compute the quadratic approximation of the Lagrangian

\[
\int dx (L_{\text{free}}^R + L_{\text{g}}^F + L_{\text{g}}^A) \equiv \int dx (L_{\text{free}}^F(\Psi, a, \pi) + L_{\text{free}}^B(\chi, \pi, \lambda, \overline{\lambda}).
\] (27)

The fermionic part is

\[
L_{\text{free}}^F = (\pi^+ + \ldots) \Psi + \eta^{\mu\nu} \Psi^\mu \partial \Psi^\nu - \partial \cdot \Psi \frac{1}{\partial} \partial \cdot \Psi^* + \delta d^a \frac{\partial}{\partial d} + d^* (\partial \cdot \Psi + \beta \partial a)
\]

\[
\sim \eta^{\mu\nu} \hat{\Psi}^\mu \partial \hat{\Psi}^\nu - \partial \cdot \hat{\Psi}^* \frac{1}{\partial} \partial \cdot \hat{\Psi}^* + \delta d^a \frac{\partial}{\partial d} + d^* (\partial \cdot \hat{\Psi} + \beta \partial a)
\]

\[
\sim \eta^{\mu\nu} \hat{\Psi}^\mu \partial \hat{\Psi}^\nu - \partial \cdot \Psi^* \frac{1}{\partial} \partial \cdot \Psi^* - (\partial \cdot \Psi^* + \beta \partial a) \frac{\delta}{\partial} (\partial \cdot \hat{\Psi} + \beta \partial a).
\] (28)

In the last two lines, the Rarita–Schwinger field dependance is only through its spin 3/2 \(\gamma\)-traceless component \(\hat{\Psi}_\mu\) after the elimination of \(\pi\) by its equation of motion. Taking \(\delta = -1\), the terms \(\partial \cdot \Psi^* \frac{1}{\partial} \partial \cdot \Psi^*\) cancel and

\[
L_{\text{free}, \delta = -1}^F \sim \eta^{\mu\nu} \hat{\Psi}^\mu \partial \hat{\Psi}^\nu + \beta^2 a^* \partial a + 2 \beta a \partial \cdot \hat{\Psi}.
\] (29)

This local Lagrangian defines the fermionic free propagators of \(\hat{\Psi}_\mu\) and \(a\) for \(\beta \neq 0\).

The bosonic quadratic approximation of the gauge fixed free action is

\[
L_{\text{free}, \delta = -1}^B = (\pi^* \pi^\ast) \left( \begin{array}{cc} \partial^2 & \beta \partial \\ \beta \partial & 0 \end{array} \right) \left( \begin{array}{c} \chi \\ \lambda \end{array} \right).
\] (30)

It gives the following matrix of free bosonic propagators

\[
(\pi^* \pi^\ast) \left( \begin{array}{cc} 0 & \frac{1}{\beta} \\ \beta & -\frac{1}{\beta} \end{array} \right) \left( \begin{array}{c} \chi \\ \lambda \end{array} \right).
\] (31)

This expression shows the relevance of having \(\beta \neq 0\) also in the bosonic sector.

All the propagators have the standard dimensions and are suitable for a perturbative expansion.

The constraint \(\Psi = 0\) holds when computing the Feynman rules of interactions. The spin 1/2 component \(\Psi\) of the gravitino doesn’t circulate within loops, a phenomenon that is compensated by a circulation of appropriate ghosts. The complete decoupling of \(\Psi\) is analogous to that the conformal factor \(g\) in the Unimodular gauge.

### 3.4 The complete supergravity action

In the Unimodular and \(\gamma\)-Traceless gauges for the graviton and gravitino, the previous results show that the BRST invariant gauge fixing of the supergravity action \(\int dx \sqrt{\gamma} R(g_{\mu\nu}) - \frac{i}{2} \epsilon^{\mu
u\rho\sigma} \Psi^*_{\mu} \gamma_{\nu} \gamma_{\rho} D_{\sigma} \Psi\) determines an action of the form:

\[
I_{\text{sugra}}[g_{\mu\nu}, \hat{\Psi}_\mu, \text{ghosts}] = \int dx \left( R(g_{\mu\nu}) - \frac{1}{2} \epsilon^{\mu
u\rho\sigma} \hat{\Psi}^*_{\mu} \gamma_{\nu} D_{\rho} \Psi_{\sigma} \right) + \text{ghost terms}.
\] (32)

The fields \(b\) and \(\overline{\lambda}\) have been eliminated by their algebraic equations of motion to get this BRST invariant action, giving a dependance on the metric and Rarita–Schwinger fields only through their unimodular and \(\gamma\)-Traceless gauge components.

Such a simplified metric and gravitino dependence implies technical simplifications. In particular, when one writes the Ward identities, all variations of the \(\hat{g}_{\mu\nu}\) are traceless and all interactions between the spin 1/2 and 3/2 components of \(\Psi_\mu\) disappear because \(\Psi = 0\).
4 Conclusion

The completion of the ordinary BRST field content of supergravity ($g_{\mu\nu}, \xi^\mu, \bar{\xi}^\mu, b^\mu$) and ($\Psi^\mu, \chi, \bar{\chi}, d$) by the pair of both BRST trivial quartets counting for zero degrees of freedom

$$(L, \eta, \bar{\eta}, b) \quad \text{and} \quad (\lambda, a, \bar{\lambda}, \bar{\pi})$$

allows an off-shell BRST invariant gauge fixing of the metric and Rarita–Schwinger fields into their unimodular part $\hat{g}_{\mu\nu}$ and $\gamma$-Traceless part $\hat{\Psi}_\mu$.

This Unimodular choice of gauge, with $g = 1$ for the metric and the $\gamma$-Traceless condition $\Psi = 0$ for the gravitino can be further completed by the gauge functions $\partial^\mu \hat{g}_{\mu\nu}$ and $\partial^\mu \hat{\Psi}_\mu$. One gets an off-shell decoupling of the conformal factor of the metric and of the $\gamma$-Trace of the gravitino. This new class of gauges gives a different and maybe quite interesting perturbative theory of gravity and supergravity.

The BRST technology involved in this paper is analogous to that one often uses in topological quantum field theories with a gauge invariance.

A virtue of our extended BRST analysis is to provide a clearer approach to the definition of observables in gravity and supergravity. In view of Eq. (12) (and its extension to supergravity), observables can be defined as functionals of $\hat{g}_{\mu\nu}$ and $\hat{\Psi}_\mu$ in the cohomology of $s$. The set of physical S-matrix elements one should compute in this gauge are those with external legs made of unimodular components of the graviton and $\gamma$-traceless gravitino. Getting the conformal factor and the spin $1/2$ component of the gravitino as spectators extends at the quantum level, at least semi-perturbatively, the early classical intuition of Einstein [1] and the work of [12].

Interestingly, this paper extends to all dimensions $d > 2$ the possibility of formulating gravity (and supergravity) with the reduced fields $\hat{g}_{\mu\nu}$ (and $\hat{\Psi}_\mu$) as fundamental fields, modulo some ghosts that are not in the physical spectrum. [15] observed for instance that a 2d worldsheet is best described in terms of the Beltrami parametrisation of 2d metric (and 2d gravitino). In the 2d case, this gives a good mathematical understanding of the factorisation properties of (super)strings, the decoupling of the conformal factor of the worldsheets, the nature of 2d (super)conformal anomalies, the definition of (super)string observables. The 2d (super)conformal factor fully disappears from the path integral measure and the Liouville fields couple only to the (super) Beltrami components of the 2d metric (and 2d gravitino).

The perspectives of using the Unimodular gauge for $d > 2$ are not yet obvious. It might be for instance interesting to revisit the Velo–Zwanziger phenomenon [19] in the Unimodular gauge. It could be also quite illuminating to reconsider the BRST superHiggs effect analysis of [18] in this different gauge for the Rarita–Schwinger field.

Finally, given that any perturbation around an unimodular background is purely traceless, as a classical graviton is, one may consider the Unimodular gauge as a kind of physical gauge for gravity. Reformulating known general relativity solutions in this gauge could be quite instructive.

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