Stability of Anisotropic Stellar Filaments

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Abstract

The study of perturbation of self-gravitating celestial cylindrical object have been carried out in this paper. We have designed a framework to construct the collapse equation by formulating the modified field equations with the background of \( f(R,T) \) theory as well as dynamical equations from the contracted form of Bianchi identities with anisotropic matter configuration. We have encapsulated the radial perturbations on metric and material variables of the geometry with some known static profile at Newtonian and post-Newtonian regimes. We examined a strong dependence of unstable regions on stiffness parameter which measures the rigidity of the fluid. Also, the static profile and matter variables with \( f(R,T) \) dark source terms control the instability of compact cylindrical system.

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1 Introduction

One of the most important and remarkable research outcomes, over the past few decades, is that our cosmos is expanding with an accelerating rate. The roots of this discovery laid from the observations of high red shift supernova Ia [1], which was then reinforced by the cross comparison with the large scale structure [2] and cosmic microwave background radiations [3]. To explore such cosmic puzzle, one requires to introduce a constituent in the cosmic matter distribution equipped with a huge negative pressure gradient in most

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conventional gravitational theory, i.e., general relativity (GR). Scientists dubbed this form as dark energy (DE). Further, cosmological indications point flat configurations of our universe with its approximate ratio of constituents as $\frac{2}{3}$ dark energy and $\frac{1}{3}$ dark matter. The need of the hour is to study the unknown and ambiguous nature of DE as well as dark matter (DM). Due to this, many radical various cosmic models have been suggested, like, a tiny no negative and non zero cosmological constant, phantoms, quintessence, Chaplygin gas, brane worlds DE and many more [See the review articles [4–7] and references therein].

Besides this, there is another challenging issue in relativistic astrophysics, i.e., deep analysis of structure and formation of compact stellar objects, like black holes, white dwarfs and neutron stars. It is well-known that, on scales much smaller than the horizon size, the DE fluctuations are insignificant [8], their influences on the relativistic fluid over densities evolution are remarkable [9]. This made the researchers to investigate how DE alters the alluring phenomenon of stellar gravitational collapse. Generally, it is believed that DE produces non-attractive force on its environment, and thus it is expected that DE may produce hindrances in the star collapse. Further, the burning question is how DE mollify the dynamics of already established black holes. It has been analyzed, recently, that black hole mass decreases in the presence phantom energy accretion and approaches to zero on reaching the Big Rip phase [10]. Qadir et al. [11] discussed various aspects of modified relativistic dynamics and proposed that GR may need to be modified to resolve various cosmological issues, like quantum gravity and the dark matter problem.

The effects of some well-consistent $f(R)$ corrections on the formation of celestial structures lead to various phenomenological consequences. In particular, gravity induced by extreme curvature terms could yield significant deviations from that of GR. The first consistent accelerating universe model from $f(R)$ gravity was suggested in [12]. In this scenario, number of results have been found (see Ref. [13] for a review), including restricted axially symmetric astronomical solutions [14], cylindrical voids relevant models [15], Raychaudhuri [16] as well modified-Ellis [17] equations, existence of charged stellar bodies [18], post-inflationary reheating phenomena [19] or the current accelerated cosmic expansion [20]. The $f(R,T)$ gravity theory [21] is one of the extensions of Einstein’s theory of gravity (GR) based on the coupling of relativistic geometrical configurations with its matter content. In this theory, the Lagrangian for EH action includes the extra degrees of freedom along with trace of stress energy tensor.

It is worthy to stress that there exists wide range of papers that considered the problem of Jeans analysis and referred subjects like collapsing stellar systems and black holes in modified gravities. For instance, the Jeans analysis for some celestial bodies has been performed in $f(R)$ gravity by different astrophysicist [22]. Clifton et al. [23] discussed gravitational collapse of relativistic spherical distributions and analyzed some of their dynamical properties in the context of $f(R)$ gravity. However, Sanders [24] and Malekjani et al. [25] addressed the issue of
stellar collapse in modified Newtonian dynamics. Martino et al. [26] studied the evolution of spherical relativistic model model and found that modified gravity corrections substantially affect evolutionary phases of huge cosmic relativistic structures. Recently, Moraes et al. [56] presented some theoretical predictions of f(R, T) gravity by analyzing the existence and evolution static wormholes models.

Santos [28] investigated the existence of spherical compact stellar bodies in extended version of f(R) gravity and found widely different structure of relativistic neutron stars from that found in GR. Ghosh and Maharaj [29] examined collapse of relativistic non-interacting particles in f(R) gravity and determined relatively less unstable matter distribution in the mysterious dark cosmos. Cembranos et al. [30] also performed such analysis and established that gravity predicted f(R) gravity supports more compact stellar system configurations. Alavirad and Weller [31] studied the effects of finite logarithmic f(R) terms in the structure formation of stellar interior and observed some attractive observational outcomes differing from the dynamics induced by GR. Sebastiani et al. [32] found both stable as well as unstable black holes configurations in the presence of gravity induced by some f(R) models. various dynamical features of wormholes [33,34], black holes [35,36] and curvature singularity [37,38] in the relativistic celestial bodies have been explored in f(R) gravity.

Sharif with his research fellows [39] examined various stability islands for some relativistic compact matter distributions. Further, they also found role of some matter variables in this context [40]. Farinelli et al. [41] numerically solved the extended versions of Lane-Emden equation for relativistic systems and determined that f(R) gravity is probably to host massive stable compact objects. Bhatti and Yousaf [42] checked the existence of various interesting configurations of stellar objects and encountered wide variety of huge celestial homogeneous objects in f(R) gravity. Recently, Yousaf et al. [43] explored that how modified gravity affects the initial regular cosmic environment for the collapse of compact celestial systems.

The analysis of stability issue has been motivated from some gravitational tests obtained from pulsar-timing experiments. The pioneer work in the development of dynamical instability was done by Chandrasekhar [44] by giving a relation about the stability of spherical star with the help of adiabatic index, Γ1. The interesting feature of Γ1 is that it provides enough information about the stiffness of the relativistic collapsing interiors. The non-adiabatic phenomenon of spherical gravitational collapse has been explored by [45] and confronted that dissipation caused by heat radiations boost up the unstable phases of the evolving stars at both Newtonian (N) and post-Newtonian (pN) epochs. Chan et al. [46] investigated that tiny value of anisotropy in the pressure configurations could dramatically disturb the stable phases of the self-gravitating spherical interior. Herrera and Santos [47] presented a detailed review on the importance of anisotropic pressure in the modeling of collapsing stellar interiors. Sharif and his collaborators [48] investigated the relationship between fluid thickness and matter parameters for various cosmological compact models. Yousaf and Bhatti [49]
identified instability constraints for the stability of expansion-free self-gravitating compact objects in the weak-field approximations of $f(R, T)$ gravity.

Here, we study the role of stiffness parameter and $f(R)$ dark source terms to understand the unstable epochs of anisotropic relativistic interiors in $f(R, T)$ gravity. The paper is organized as follows. In section one, we provide $f(R, T)$ field equations, junction conditions as well as dynamical equations. Section two provides well-known radial configuration of perturbation. We then employ this over various fundamental equations of our dynamical system. In section three, we use Harrison-Wheeler state equation to relate perturbed pressure gradient with the systems energy density, after that we find $f(R, T)$ collapse equation. Section four explores the instability regions of cylindrical anisotropic system with N pN limits. We conclude our results in the last section.

2 $f(R, T)$ Theory and Stellar Filament

Though considered in many other astrophysical frameworks, the concept of $f(R)$ gravity received considerable attention mainly due to the reason that it could presents plausible description to the alluring phenomenon of accelerating universe expansion [50]. The main idea in $f(R, T)$ gravity is to replace the tiny value of cosmological constant in usual EH with an algebraic generic expression of $R$ and $T$, where $R$ is the Ricci scalar, while $T$ is the trace of energy momentum tensor. It is given as [21]

$$S_{f(R, T)} = \int d^4x \sqrt{-g}[f(R, T) + L_M], \quad (1)$$

where $g$ is the metric trace, while $L_M$ is indicating the presence of matter Lagrangian. It worthy to mention that the dynamical equations of motions in $f(R, T)$ gravity theory is directly related the matter contents contribution, therefore, relativistic astrophysicists can get any particular configurations of equations by choosing any form $L_M$. Here, we consider $L_M = \mu$ (where $\mu$ stands for system energy density). Now, we vary the above action with $g_{\mu\beta}$, after some manipulation, we obtain the following distributions of $f(R, T)$ field equations (for details [26])

$$G_{\mu\beta} = T_{\mu\beta}^{\text{eff}}, \quad (2)$$

where

$$T_{\mu\beta}^{\text{eff}} = \left[(1 + f_T(R, T))T^{(m)}_{\mu\beta} - \mu g_{\mu\beta} f_T(R, T) - \left(\frac{f(R, T)}{R} - f_R(R, T)\right)\frac{R}{2} + \left(\nabla_\mu \nabla_\beta + g_{\mu\beta} \Box\right) f_R(R, T)\right] \frac{1}{f_R(R, T)}.$$
Here, $G_{\mu\beta}$ and $T^{\text{eff}}_{\mu\beta}$ are Einstein and effective $f(R,T)$ energy-momentum tensors, respectively. Further, $\Box$ is the de-Alembert’s operator expressed by means of covariant derivation operator as $\nabla_\mu \nabla^\mu$ while subscripts $T$ and $R$ indicate derivations with respect to $T$ and $R$ of the corresponding quantities.

We take a cylindrical self-gravitating system filled with locally anisotropic fluid configurations. We assume that the evolution phases of this system is separated by a 3D timelike surface. This surface is symbolized with $\Sigma$. This boundary differentiates our cylindrical manifold, $V$, into couple of portions, in which the exterior one is denoted with the help of $+$ sign, while $-$ sign indicates configuration of interior manifold. The $V^-$ manifold can be mentioned through the following diagonal and non-rotating metric

$$ds^2 = A^2 dt^2 - B^2 dr^2 - C^2 d\phi^2 - D^2 dz^2,$$

where metric coefficients are the functions of $t$ and $r$. For the sake of simplicity, we are considering $D$ to be unity. Such type of assumption has been taken by many researchers [51-53]. The spacetime for $V^+$ is

$$ds_+^2 = e^{2(\gamma - \nu)}(dv^2 - d\rho^2) + e^{-2\nu}\rho^2 d\phi^2 + e^{2\nu}dz^2,$$

where $\gamma$ and $\nu$ are the functions of $\nu$ and $\rho$, while the coordinates are numbered as $x^\beta = (\nu, \rho, \phi, z)$. The corresponding vacuum field equations provide

$$\rho(v_\nu^2 + v_\rho^2) = \frac{f - Rf_R}{2f_R}e^{2(\gamma - \nu)},$$

$$2v_\nu v_\rho \rho = \gamma_\nu,$$

$$v_{\nu\nu} - \frac{\nu}{\rho} - v_{\rho\rho} - \frac{e^{2(\gamma - \nu)}}{4\rho}\left(\frac{f - Rf_R}{f_R}\right)\left\{\rho e^{-4\nu} + \frac{e^{2\gamma}}{\rho}\right\},$$

where subscripts $\rho$ and $\nu$ show partial differentiations with respect to $\rho$ and $\nu$, respectively. To calculate the $f(R,T)$ field equations, we choose the following form of the usual energy-momentum tensor [55]

$$T^-_{\mu\beta} = (\mu + P_r)V_\mu V_\beta - P_r g_{\mu\beta} + (P_z - P_r)S_\mu S_\beta + (P_\phi - P_r)K_\mu K_\beta,$$

where $P_z$, $P_\phi$ and $P_r$ are pressure gradients along $z$, $\phi$ and $r$ directions, respectively. Moreover, $V_\beta = A\delta^\beta_0$ is the four-velocity, while $S_\beta = \delta^\beta_3$, $K_\beta = C\delta^\beta_2$ are four-vectors. We now consider that our system is configuring in the comoving coordinate system. In this system, above four vectors obey

$$V^\beta V_\beta = -1, \quad K^\beta K_\beta = S^\beta S_\beta = 1, \quad V^\beta K_\beta = S^\beta K_\beta = V^\beta S_\beta = 0.$$
The expansion scalar $\Theta = V^\mu_{, \mu}$ for our cylindrically interior turns out to be

$$\Theta = \frac{1}{A} \left( \frac{\dot{B}}{B} + \frac{\dot{C}}{C} \right),$$  \hspace{1cm} (9)$$

where over dot means $\frac{\partial}{\partial t}$.

The $f(R,T)$ field equations (2) for the cylindrically symmetric spacetime (3) yield

$$G_{00} = \frac{A^2}{f_R} \left[ \mu - \left( f_R - \frac{f}{R} \right) \frac{R}{2} + \psi_{tt} \right], \quad G_{01} = \psi_{tr},$$  \hspace{1cm} (10)$$

$$G_{11} = \frac{B^2}{f_R} \left[ P_r + (P_r + \mu) f_T + \left( f_R - \frac{f}{R} \right) \frac{R}{2} + \psi_{rr} \right],$$  \hspace{1cm} (11)$$

$$G_{22} = \frac{C^2}{f_R} \left[ P_\phi + (P_\phi + \mu) f_T + \left( f_R - \frac{f}{R} \right) \frac{R}{2} + \psi_{\phi\phi} \right],$$  \hspace{1cm} (12)$$

$$G_{33} = \frac{1}{f_R} \left[ P_z + (P_z - \mu) f_T + \left( f_R - \frac{f}{R} \right) \frac{R}{2} + \psi_{zz} \right],$$  \hspace{1cm} (13)$$

where

$$\psi_{tt} = \frac{\partial_{rr} f_R}{B^2} - \left( \frac{\dot{B}}{B} + \frac{\dot{C}}{C} \right) \frac{\partial_t f_R}{A^2} + \left( \frac{C'}{C} - \frac{B'}{B} \right) \frac{\partial_r f_R}{B^2},$$  \hspace{1cm} (14)$$

$$\psi_{tr} = \frac{1}{f_R} \left( \partial_r \partial_t f_R - \frac{\dot{B}}{B} \partial_r f_T - \frac{A'}{A} \partial_t f_R \right),$$  \hspace{1cm} (15)$$

$$\psi_{rr} = \frac{\partial_{rr} f_R}{A^2} - \left( \frac{\dot{A}}{A} - \frac{\dot{C}}{C} \right) \frac{\partial_t f_R}{A^2} - \left( \frac{A'}{A} + \frac{C'}{C} \right) \frac{\partial_r f_R}{B^2},$$  \hspace{1cm} (16)$$

$$\psi_{\phi\phi} = \frac{\partial_{rr} f_R}{B^2} + \frac{\partial_t f_R}{A^2} + \left( \frac{\dot{B}}{B} - \frac{\dot{A}}{A} \right) \frac{\partial_t f_R}{A^2} + \left( \frac{B'}{B} - \frac{A'}{A} \right) \frac{\partial_r f_R}{B^2},$$  \hspace{1cm} (17)$$

$$\psi_{zz} = \frac{\partial_{tt} f_R}{A^2} - \left( \frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C} \right) \frac{\partial_t f_R}{A^2} - \frac{\partial_{rr} f_R}{B^2} + \left( \frac{B'}{B} - \frac{A'}{A} - \frac{C'}{C} \right) \frac{\partial_r f_R}{B^2},$$  \hspace{1cm} (18)$$

where prime indicates $\frac{\partial}{\partial r}$ operator. The Ricci invariant for the non-ideal cylindrical metric with $f(R,T)$ gravity are found as

$$R(t,r) = \frac{2}{B^2} \left[ \frac{A''}{A} + \frac{C''}{C} + \frac{A'}{A} \left( \frac{C'}{BC} - \frac{B'}{C} \right) - \frac{B'C'}{BC} \right]$$

$$- \frac{2}{A^2} \left[ \frac{\dot{B}}{B} + \frac{\dot{C}}{C} - \frac{\dot{A}}{A} \left( \frac{\dot{B}}{BC} + \frac{\dot{C}}{B} \right) + \frac{\dot{B'C}}{BC} \right].$$  \hspace{1cm} (19)$$
3 Dynamics

In this section, we shall perform dynamical investigations of a stellar filaments coupled with anisotropic matter content in \( f(R,T) \) gravity. In this setting, we shall compute the hydrodynamical equation for the system which is undergoing oscillations. We shall also find instability epochs of the perturbed cylindrical body.

3.1 Hydrodynamics

Now, we shall formulate dynamical equations with the help of energy-momentum tensor divergence. This will assist us to analyze dynamical phases of cylindrical relativistic stellar bodies with \( f(R,T) \) background. It is worthy to note that in this modified gravity theory, the divergence of energy momentum tensor is non-zero. In this case, we have the following configurations of the divergence \[20\]

\[ \nabla^\mu T_{\mu \beta} = \left[ (\Theta_{\mu \beta} + T_{\mu \beta}) \nabla^\mu \ln f_T + \nabla^\mu \Theta_{\mu \beta} - \frac{1}{2} g_{\mu \beta} \nabla^\mu T \right] \frac{f_T}{1 - f_T}. \]

For our locally anisotropic cylindrical system, the above equation gives

\[ 21\]

\[ \begin{align*}
\dot{\mu} \left( \frac{1 + 2f_T}{f_R(1 + f_T)} \right) - & B \frac{B^2}{A^2 f_R} (1 + f_T)(\mu + P_r) - \frac{\mu}{f_R} \partial_t f_R - \frac{C C'}{A^2 f_R} (1 + f_T)(\mu + P_\phi) \\
& + \frac{\partial_t T}{2(1 + f_T)} + \frac{2\mu}{1 + f_T} \partial_t f_T + D_0 = 0, \\
\frac{P_r}{f_R} & + \frac{A A'}{A^2 f_R} (1 + f_T)(\mu + P_r) + \frac{P_r}{f_R} \left\{ \partial_r f_T - \frac{(1 + f_T)\partial_r f_R}{f_R} \right\} + \frac{\mu'}{f_R} + \frac{\mu}{f_R} \\
& \times \left\{ \partial_r f_T - \frac{f_T \partial_r f_R}{f_R} \right\} + \frac{1}{(1 + f_T)} \left\{ \left( \frac{T'}{2} + \mu' \right) f_T - (\mu - P_r) \partial_r f_T \right\} + \frac{C'}{C B^2 f_R} \\
& \times (P_r - P_\phi)(1 + f_T) + D_1 = 0,
\end{align*} \]

where \( D_0 \) and \( D_1 \) are higher degrees of freedom offered by \( f(R,T) \) theory. These are the functions of both temporal and radial coordinates. The dark source terms are energy variations corrections along with the time and adjacent surfaces in the cylindrical self-gravitating celestial objects, respectively. These are calculated and written in Appendix A.

Thorne [57] defined the C-energy for cylindrically symmetric spacetime as follows

\[ \tilde{E}(t,r) = m(t,r) = \frac{1}{8} \left( 1 - l^{-2} \nabla^\alpha \tilde{r} \nabla_\alpha \tilde{r} \right). \]
The circumference radius, \( \rho \), specific length, \( l \), and areal radius \( \tilde{r} \) of the cylindrical geometry obey following relations
\[
\rho^2 = \xi_{(1)} \xi_{(1)}, \quad l^2 = \xi_{(2)} \xi_{(2)}, \quad \tilde{r} = \rho l.
\]
where \( \xi_{(1)} = \frac{\partial}{\partial \phi} \) and \( \xi_{(2)} = \frac{\partial}{\partial z} \) are the Killing vectors in cylindrical system. The specific energy in our cylindrical geometry is found as
\[
m(t, r) = \frac{l}{8} \left( 1 - \frac{C'^2}{B^2} + \frac{\dot{C}^2}{A^2} \right). \tag{23}
\]

Now, we define couple of well-known operators, i.e., proper and radial derivative operators. This will helps us to find the cylindrical mass function variations among its adjacent surfaces. These operators are
\[
D_J \equiv \frac{1}{A} \frac{\partial}{\partial t}, \quad D_C \equiv \frac{1}{C'} \frac{\partial}{\partial r}. \tag{24}
\]
Using Eqs.(23) and (24), we get
\[
E \equiv \frac{C'}{B} = \sqrt{1 - \frac{8m}{l} + U^2}. \tag{25}
\]
where \( U \) is the relativistic system velocity and is given as
\[
U = D_J C = \frac{\dot{C}}{A}. \tag{26}
\]
Equations (10), (11) and (23)-(26) provide
\[
D_C m = \frac{l}{4} \left[ \mu - \left( f_R - \frac{f}{R} \right) \frac{R}{2} + \psi_{tr} - \frac{\psi_{tr} U}{BAE} \right] \frac{C}{f_R}. \tag{27}
\]
This equation describes total energy variation among neighboring anisotropic fluid surfaces. It is evident that the first three mathematical terms on the right hand side in the above equation are coming due to \( f(R,T) \) system energy density corrections. The last term \( (\psi_{tr}/BA)(U/E) \) contributes to affect the evolutionary mechanism of the cylindrical system due to its non-attractive behavior. It is note worthy that \( \psi_{tr} > 0 \) and for collapsing models, \( U \) must be negative, therefore \( (\psi_{tr}/BA)(U/E) < 0 \) in the above equation. This will make \( -(\psi_{tr}/BA)(U/E) \) to be greater than zero. The solution of Eq.(27) is
\[
m = \frac{1}{4} \int_0^C \left[ \left\{ \mu - \left( f_R - \frac{f}{R} \right) \frac{R}{2} + \psi_{tr} - \frac{\psi_{tr} U}{BAE} \right\} \frac{l}{f_R} \right] dC. \tag{28}
\]
For the smooth matching of the interior and exterior geometry over the boundary surface $\Sigma$, we will use the Darmois [58] as well as Senovilla [59] junction conditions. We consider timelike hypersurface for which, we impose $r = \text{constant}$ in Eq. (3) and $\rho(\nu)$ in Eq. (4). In this context, the first fundamental form on $\Sigma$, provide

$$d\tau_\Sigma \equiv e^{2\gamma - 2\nu} \sqrt{1 - \left(\frac{d\rho}{d\nu}\right)^2} \, d\nu = A dt,$$

$$C \equiv e^\nu, \quad e^\nu = \frac{1}{\rho},$$

with $1 - \left(\frac{d\rho}{d\nu}\right)^2 > 0$. The second fundamental form gives us the following set of equations

$$e^{2\gamma - 2\nu} [\nu_{\tau\tau} \rho_\tau - \rho_{\tau\tau} \nu_\tau - \{\rho_\tau (\gamma_\nu - \nu_\nu) + \nu_\tau (\gamma_\rho - \nu_\rho)\}] \equiv \frac{-A'}{AB},$$

$$e^{2\nu} (\rho_\tau \nu_\nu + \nu_\tau \rho_\nu) \equiv \frac{CC'}{B}, \quad \rho_\tau \nu_\nu + \nu_\tau \rho_\nu \equiv \frac{V_\tau}{\rho}. \quad (32)$$

Using Eqs. (29)-(32) and field equations, we get

$$P_{\tau\Sigma} \equiv 0. \quad (33)$$

Further, we have

$$R|^+_- = 0, \quad f_{,RR}[\partial_\nu R]^+_- = 0, \quad f_{,RR} \neq 0. \quad (34)$$

The constraint mentioned in Eq. (33) is due to Darmois junction conditions. Equation (35) is required to be obeyed over the hypersurface due to modified gravity which affirms the continuity of Ricci curvature invariant over $\Sigma$ even for matter thin shells. By making use of Eqs. (33) and (35), we have

$$P_{\tau} \equiv \frac{1}{1 + f_T} \left(\frac{f - R f_R}{2}\right). \quad (35)$$

This equations shows that radial pressure on the boundary surface is non-zero and depends on the dark source terms coming from $f(R, T)$ gravity with constant $R$ and $T$ background. Thus, modified gravity corrections are directly linked with the flux of momentum of the gravitational wave emerging from the relativistic cylindrical system.

For the presentation of cosmological and theoretical well-consistent $f(R, T)$ theory, the selection of its generic function holds fundamental importance. In this study, we take specific class of $f(R, T)$ models given as follows

$$f(R, T) = f_1(R) + f_2(R) f_3(T). \quad (36)$$
The model of this type of configurations are originated from the explicit non-minimal coupling of curvature and matter. We now study $f(R,T)$ power law type model mentioned as follows

$$f(R) = R + \lambda R^2 T^2. \tag{37}$$

Such functional form meet with the Lagrangian form of $f(R,T)$ gravity observed in Eq.\((36)\). In the above equation, we have taken $f_1(R) = R$, $f_2(R) = R^2$ along with $f_3(T) = T^2$. By considering $\lambda \to 0$, all GR solutions can be maintained.

A detailed review on the viability of $f(R,T)$ models has been presented in [60]. We have assumed $f(R,T)$ model mentioned in Eq.(37), in which the value of $\lambda$ should indeed be small in order to satisfy the Solar system tests. If one takes $f(R,T) = R + \lambda T$, then one can obtain dynamics identical with that of GR. Thus, the simple case $f(R,T) = R + \lambda T$ is fully equivalent with standard GR, after rescaling $\lambda$. The value of $\lambda$ should be small enough, i.e. $\lambda \ll 1$. The $f(R,T)$ theory with this model is just general relativity with strange dynamical cosmological constant $f(T)$ (covariant!) or if one wish with strange matter coupling. Further, the gravitational dynamics induced by $R + 2\lambda T$ gravity model give results comparable with that gravitational model having an effective cosmological constant, i.e., $\Lambda_{\text{eff}} \propto H^2$ (where $H$ is a Hubble parameter). In this standpoint, the gravitational coupling would behave as an effective time dependent coupling, thereby providing $G_{\text{eff}} = G \pm \lambda$. The study of the cosmological perturbations of $f(R,T) = R + \lambda R^2 T^2$ model has not been performed yet, and this would give another set of constraints on $\lambda$.

### 3.2 Oscillations

In this section, we shall proceed our analysis by considering a well-known radial perturbation method. We shall employ this technique over $f(R,T)$ field and dynamical equations. In this scheme, the quantity $\alpha$ is known as perturbation parameter that would control the impact of fluctuations within the cylindrical anisotropic system. We would consider effects upto $O(\alpha)$. It is worthy to stress that $\alpha \in (0,1)$ in our examination. This perturbation scheme was brought in by Herrera et al. [61]. The “static” quantities of the corresponding expressions are represented with zero subscript. Here, we take perturbation with the assumption that all structural variables of the systems in $f(R,T)$ gravity are dependent upon the same time dependent function, i.e., $\eta(t)$. The perturbation technique is

$$X(t,r) = X_0(r) + \alpha \eta(t)x(r), \tag{38}$$

$$Y(t,r) = Y_0(r) + \alpha Y(t,r), \tag{39}$$

$$f(t,r) = R_0(1 + T_0^2 \lambda R_0) + \alpha \eta(t) e(r)(1 + 2R_0 \lambda T_0^2), \tag{40}$$

$$f_R(t,r) = (1 + 2R_0 \lambda T_0^2) + 2\alpha \eta(t) e(r) \lambda T_0^2, \tag{41}$$
\[ \Theta(t, r) = 0 + \alpha \bar{\Theta}(t, r), \]  

where Eq. (38) shows oscillations of metric variables while oscillations of matter variables are represented in Eq. (39). The non-static perturbed configurations of Eq. (19) is

\[
- \epsilon \eta = \left( \frac{b}{B_0} + \frac{c}{r} \right) \frac{2 \dot{\eta}}{A_0^2} + \eta \left[ \frac{4 R_0 b}{B_0^3} - \frac{2}{B_0^3} \left\{ \frac{c''}{r} - \frac{a A''_0}{A_0^2} + \frac{a''}{A_0} - \frac{A'_0}{A_0} \right\} \left( \frac{b}{B_0} \right) \right] 
- \left( \frac{c}{r} \right)' + \left( \frac{1}{r} - \frac{B'_0}{B_0} \right) \left( \frac{a}{A_0} \right)' - \left( \frac{c}{r} \right)' \frac{B'_0}{B_0} - r^{-1} \left( \frac{b}{B_0} \right)' \right],
\]

while its static form is

\[ R_0(r) = 2 \left[ \frac{A'_0}{A_0} \left( \frac{1}{r} - \frac{B'_0}{B_0} \right) + \frac{A''_0}{A_0} - \frac{B'_0}{B_0 r} \right] \frac{1}{B_0^2}. \]

The static form of the locally anisotropic cylindrical \( f(R, T) \) field equations (10)-(13), with \( C_0 = r \), are given as follows

\[
(1 + 2 \lambda R_0 T_0^2) \left( \frac{B'_0}{B_0} \right) = r B_0^2 \left[ \mu_0 - \frac{\lambda}{2} R_0^2 T_0^2 + \psi_{ut} \right], \tag{43}
\]

\[
(1 + 2 \lambda R_0 T_0^2) \left( \frac{A'_0}{A_0} \right) = r B_0^2 \left[ (P_{00} + \mu_0)(1 + 2 \lambda R_0^2 T_0) + \frac{\lambda}{2} R_0^2 T_0^2 + \psi_{rr} \right], \tag{44}
\]

\[
(1 + 2 \lambda R_0 T_0^2) \left( \frac{B'_0}{B_0} + \frac{A''_0}{B_0 A'_0} \right) A'_0 = B_0 A_0 \left[(P_{00} + \mu_0)(1 + 2 \lambda R_0^2 T_0) + \frac{\lambda}{2} R_0^2 T_0^2 + \psi_{\phi \phi} \right], \tag{45}
\]

\[
(1 + 2 \lambda R_0 T_0^2) \left\{ \left( \frac{A'_0}{A_0 r} + \frac{A''}{A_0} \right) - \frac{B'_0}{B_0} \left( \frac{1}{r} + \frac{A'_0}{A_0} \right) \right\} = B_0^2 [(P_{z0} - \mu_0)(1 + 2 \lambda R_0 T_0) + \frac{\lambda}{2} R_0^2 T_0^2 + \psi_{zz} \right], \tag{46}
\]

where overset index \((S)\) points out static form of the respective \( f(R, T) \) dark component terms. These quantities are computed as follows

\[
^{(S)} \psi_{ut} = \frac{2 T_0 \lambda}{B_0^2} \left[ 2 T'_0 R_0 + R''_0 T_0 + 2 T''_0 R_0 - \frac{2 T'^2_0}{T_0} R_0 + \left( \frac{B'_0}{B_0} - \frac{1}{r} \right) \left( R'_0 T_0 + 2 T'_0 R_0 \right) \right],
\]

\[
^{(S)} \psi_{rr} = - \frac{2 T_0 \lambda}{B_0^2} \left( R'_0 T_0 + 2 T'_0 R_0 \right) \left( \frac{1}{r} + \frac{A'_0}{A_0} \right),
\]
The non-static perturbed form of (00), (11) and (22) components of $f(R, T)$ field are

$$
\psi_{\phi \phi}^{(S)} = \frac{2\lambda T_0}{B_0^2} \left[ 2T'_0R'_0 + R''_0T_0 + 2T'_0R_0 + 2\frac{T'_0}{T_0}R_0 \right. \\
\left. - \left( \frac{A'_0}{A_0} - \frac{B'_0}{B_0} \right) (T_0R'_0 + 2T'_0R_0) \right],
$$

$$
\psi_{zz}^{(S)} = \frac{2\lambda T_0}{B_0^2} \left[ (R'_0T_0 + 2T'_0R_0) \left( \frac{B'_0}{B_0} - \frac{1}{r} - \frac{A'_0}{A_0} \right) - 2T''_0R'_0 - R''_0T_0 - 2T''_0R_0 - 2\frac{T''_0}{T_0}R_0 \right].
$$

The non-static perturbed form of (00), (11) and (22) components of $f(R, T)$ field are

$$
\bar{\mu} = \chi_2 \eta,
$$

$$
\bar{P}_r + \bar{\mu} = \eta \chi_3 - \frac{1}{(1 + 2\lambda R^2_0 T_0)} \left( \frac{P_0}{T_0^2} \right) \left( \psi_{tt} + 2\eta \left( \psi_{tt} + \frac{c}{r A_0^2} (1 + 2\lambda R^2_0 T_0) \right) \right)
$$

$$
\times \left( \frac{b}{B_0^2} + T^2_0 e^\lambda \right),
$$

$$
\bar{P}_0 + \bar{\mu} = \eta \chi_5 - \dot{\psi} \left[ \frac{b}{B_0 A_0^2} (1 + 2\lambda R^2_0 T_0^2) + \psi_{\phi \phi} \right] \left( \frac{1}{1 + 2\lambda R^2_0 T_0} \right),
$$

where $\chi_2, \chi_3$ and $\chi_5$ contain combinations of static metric parameters corresponding to $f(R, T)$ field equations, respectively. These quantities are described in Appendix A. We see that Eq. (21) is trivially obeyed under perturbation, however, Eq. (22) provides the following dynamical equation

$$
\left( \frac{P'_0}{(1 + 2\lambda R^2_0 T_0^2)} \right) + \frac{2P_0}{(1 + 2\lambda R^2_0 T_0^2)} \left[ \frac{R_0(R'_0T'_0 + 2T'_0R'_0)}{(1 + 2\lambda R^2_0 T_0^2)} \right. \\
\left. - \left( \frac{2\lambda e T^2_0}{(1 + 2\lambda R^2_0 T_0^2)} + \frac{2\mu_0}{(1 + 2\lambda R^2_0 T_0^2)} \right) \right]
$$

$$
\times \left[ \frac{a}{A_0} + \frac{b}{B_0} + \frac{2\lambda e T^2_0}{(1 + 2\lambda R^2_0 T_0^2)} \right] + \left( \frac{\mu'_0}{(1 + 2\lambda R^2_0 T_0^2)} \right) \left( \frac{2\lambda R^2_0 T_0^2}{(1 + 2\lambda R^2_0 T_0^2)} \right) \\
\times \left[ (R_0T'_0 + 2T'_0R'_0)R_0 - \frac{2T^2_0\lambda R^2_0(2R'_0T'_0 + T'_0R'_0)}{(1 + 2\lambda R^2_0 T_0^2)} \right] - \frac{2R_0\lambda}{(1 + 2\lambda R^2_0 T_0^2)} (P_r + P_\phi)
$$

$$
+ \left( \frac{\mu_0 - P_0}{(1 + 2\lambda R^2_0 T_0^2)} \right) \left( \frac{T'_0}{2} + \mu'_0 \right) + \frac{r (1 + 2\lambda R^2_0 T_0)}{B^2_0 (1 + 2\lambda R^2_0 T_0^2)} (P_r + P_\phi)
$$

$$
- \frac{A_0 A'_0}{(1 + 2\lambda R^2_0 T_0^2) B_0^2} (P_r + P_\phi)(1 + 2\lambda R^2_0 T_0) + D_{1S} = 0,
$$

where $D_{1S}$ indicates that $D_1$ is evaluated under static form of perturbation method. Its value is

$$
D_{1S} = \psi_{rr}^{(S)} - \frac{A_0 A'_0}{(1 + 2\lambda R^2_0 T_0^2) B_0^2} \left( \psi_{tt}^{(S)} + \psi_{\phi \phi}^{(S)} \right) + \left( \frac{R^2_0 T^2_0}{2} \right) \frac{\lambda}{2} + \frac{r (\psi_{rr}^{(S)} + \psi_{\phi \phi}^{(S)})}{(1 + 2\lambda R^2_0 T_0^2) B_0^2}.
$$
The hydrostatic as well as non-hydrostatic distributions of C-energy function \((23)\) is

\[
m_0 = \frac{l}{8} - \frac{l}{8B_0^2}, \quad \bar{m} = \frac{\eta}{B_0^2} \left( \frac{b}{B_0} - c \right) \frac{l}{4},
\]

The mathematical interaction between radial components \(b\) and \(B_0\) can be found from Eq.\((11)\) as

\[
\frac{b}{B_0} = \left( \psi_{tr} + \frac{c'}{r} - \frac{cA_0'}{rA_0} \right) r.
\]

Equations \((21)\) and \((22)\) after using Eqs.\((38)-(41)\) give

\[
\bar{\mu} + X_1(r)\dot{\eta} = 0, \quad \bar{\mu}_r + \frac{2\lambda\bar{\mu}}{(1 + 2R_0^2\lambda T_0^2)} \left( (R_0T_0' + 2T_0R_0')R_0 - (2R_0T_0' + T_0R_0')T_0 \right)
\]

\[
\times \left( \psi_{tr} + \frac{c'}{r} - \frac{cA_0'}{rA_0} \right) r.
\]

(52)

where \(X_1\) contains combinations of radial functions of \(f(R, T)\) unusual relativistic cylindrical energy density and \(D_3\) comprises of corrections due to extra degrees of freedom of \(f(R, T)\) gravity. These quantities are given in Appendix A. The modified versions of dynamical equations \((52)\) and \((53)\) would a source of power stimulus in the examination of stability islands of cylindrical self-gravitating interiors.

The boundary condition \((33)\) after using Eq.\((38)-(41)\) give

\[
P_{\text{eff}} r_0 \bar{\mu} \Sigma = -\lambda R_0^2 T_0^2 \left( 1 + 2\lambda R_0 T_0 \right),
\]

(54)

where

\[
\bar{\mu} \Sigma = -e\lambda R_0 T_0^2 \left( 1 + 2\lambda R_0 T_0 + \lambda R_0^3 \right).
\]

(55)
By making use of Eqs. (48), (54) and (55), we get
\[
\delta_1 \ddot{\eta} + \delta_2 \dot{\eta} + \delta_3 \eta \Sigma^{(e)} = 0, \tag{56}
\]
where
\[
\delta_1 = \left\{ \frac{P_2}{\psi_{rr}} + \frac{(1 + 2R_0 \lambda T_0^2) c}{r A_0^2} \right\} \frac{1}{(1 + 2R_0 \lambda T_0)}, \quad \delta_2 = \frac{P_1}{\psi_{11}} \frac{1}{(1 + 2R_0 \lambda T_0)}, \quad \delta_3 = \chi_4 + \chi_2 - \chi_3.
\]
Equation (56) has two independent different solutions in terms of their graphical representations. One of these solutions represents unstable while the other one describes the stable nature of the collapsing cylindrical stellar object. In this manuscript, we are concerned with the exploration of unstable phases of the compact system in \( f(R, T) \) gravity. So, we will restrict our analysis by taking an oscillating solution. As seen from the perturbation scheme that initially, our stellar model is in the phase of complete hydrostatic equilibrium. For the examination of collapsing model, it is required to limit radial perturbation parameters, i.e., \( a, b, c \) and \( e \), are all positive. Due to this, one can easily check that \( \omega_\Sigma^2 > 0 \). In this chain, the required solution of the above equation is
\[
\eta = -\exp(\omega_\Sigma t), \quad \text{where} \quad \omega_\Sigma = \frac{-\delta_2 + \sqrt{\delta_2^2 - 4\delta_1 \delta_3}}{2\delta_1}. \tag{57}
\]

### 3.3 Hydrodynamical Equation

In this section, we shall formulate \( f(R, T) \) collapse equation which would be helpful to check the instability regimes for our system. For this purpose, we shall make use of a familiar equation of state (EoS), i.e., Harrison-Wheeler type EoS. It is useful to note that this EoS peculiarly relate \( \bar{\mu} \) with \( \bar{P}_i \). This is given as follows \[62\]
\[
\bar{P}_i = \Gamma_1 \frac{P_{i0}}{P_{i0} + \mu_0} \bar{\mu} \tag{58}
\]
where \( \Gamma_1 \) is known as adiabatic index. It measures the fractional variations between pressure and energy density experienced by matter configurations following the motion. In this way, it reports how stiff a relativistic fluid is, thus suggesting its alternative name, stiffness parameter. We assume \( \Gamma_1 \) to be a constant entity throughout in our stability analysis of relativistic stellar objects. In the scenario of N regime, the instability of spherical self-gravitating systems depend purely on the mean value of stiffness parameter \[64\]. However, in GR, the stability relies not only on the average value of \( \Gamma_1 \) but also on the star radius. In the study
of relativistic self-gravitating systems in modified gravity, the situation is quite different. Chandrasekhar [44] calculated a simple numeric value, $\frac{4}{3}$, through $\Gamma_1$ for the stability of relativistic isotropic sphere. But in modified gravity, finding of such type of numeric value is impossible. The dark source terms coming from modified gravity would greatly make ones calculations cumbersome. Due to this reason, many researchers [63] have found such instability ranges through $\Gamma_1$ depending upon radial configurations of matter variables. We are also expecting such type of solution in our approach. Continuing in this way, the integration of Eq.(52) yields

$$\bar{\mu} = \chi_1 \eta, \text{ where } \chi_1 = -X_1.$$  \hspace{1cm} (59)

Using this value in Eq.(58), we get

$$\bar{\mu} - \bar{P}_r = \chi_1 \left\{ 1 - \frac{P_{r0} \Gamma_1}{(P_{r0} + \mu_0)} \right\} \eta,$$ \hspace{1cm} (60)

$$\bar{P}_r - \bar{P}_\phi = \Gamma_1 \chi_1 \left\{ \frac{P_{r0}}{P_{r0} + \mu_0} - \frac{P_{\phi0}}{P_{\phi0} + \mu_0} \right\} \eta.$$ \hspace{1cm} (61)

Using Eqs.(59)-(61) in Eq.(53), we obtain

$$\Gamma_1 \left( \frac{P_{r0} \chi_1}{P_{r0} + \mu_0} \right) \eta + \frac{\Gamma_1 P_{r0} \chi_1 \xi}{(P_{r0} + \mu_0)(1 + 2R_0^2 \lambda T_0)} \eta - \frac{A_0 A_0' X_1 \eta}{B_0^2} + \left\{ \frac{P_{r0}}{P_{r0} + \mu_0} - \frac{P_{\phi0}}{P_{\phi0} + \mu_0} \right\} \eta \frac{r \chi_1 (1 + 2R_0^2 \lambda T_0)}{B_0^2 (1 + 2R_0^2 \lambda T_0^2)} \eta + \Phi \eta = 0,$$ \hspace{1cm} (62)

where

$$\Phi = \frac{r}{(1 + 2R_0^2 \lambda T_0^2)B_0^2} \left\{ \left( \frac{P_{2}'}{(P_{2}')^2} - \psi_{rr} - \frac{(P_{1})}{(P_{1})^2} \right) \omega^2 + \frac{(P_{1})}{(P_{1})^2} \right\} + \frac{A_0 A_0'}{B_0^2 (1 + 2R_0^2 \lambda T_0^2)} \left( \omega\psi_{rr} + \omega^2 \psi_{rr} \right)$$

$$+ \left( \frac{2 \lambda R_0^2 T_0 \chi_1}{(1 + 2R_0^2 \lambda T_0^2)} \right) \eta - \left( \frac{2 \lambda \chi_1}{(1 + 2R_0^2 \lambda T_0^2)} \right) \eta,$$ \hspace{1cm} (63)

$$\xi = 2\lambda \left[ (R_0 T_0' + 2T_0 R_0') R_0 - (2R_0 T_0' + T_0 R_0') T_0 \right] \left( \frac{1 + 2R_0^2 \lambda T_0^2}{1 + 2R_0^2 \lambda T_0^2} \right) + 2R_0 \lambda (2R_0 T_0 + T_0 R_0') - \frac{A_0 A_0'}{B_0^2 (1 + 2R_0^2 \lambda T_0)}.$$ \hspace{1cm} (64)

This is the hydrodynamical equation for our considered systems which is framed in $f(R, T)$ gravity. It includes all the ingredients that are required to study the unstable phases of the cylindrical geometry which is coupled with locally anisotropic fluid relativistic stellar matter.
3.4 Newtonian Limit

In order to assess N limit constraint, we consider
\[ \mu_0 \gg P_{j0}, \quad A_0 = 1, \quad B_0 = 1. \]

Under this chain, the collapse equation (62) provides
\[ \Gamma_1 [(P_{r0}\chi_{1N})' + P_{r0}\chi_{1N}\xi_N + (P_{r0} - P_{\phi 0})\chi_{1N}\zeta]\eta = |\Phi_N(1 + 2\lambda R^2 T_0)\eta|, \tag{65} \]

where \( \zeta = \frac{1 + 2\lambda R^2 T_0}{1 + 2m_0 G r C^2} \) and subscript \( N \) symbolizes that corresponding terms are approximated within \( N \) epoch. Now, taking the value of \( \eta \) from Eq. (57), and manipulating above equation, one can achieve above results independent of temporal coordinate. The instability constraint for cylindrically symmetric celestial object in \( f(R, T) \) gravity is obtained as follows
\[ \Gamma_1 < \frac{|\Phi_N(1 + 2\lambda R^2 T_0)|}{[(P_{r0}\chi_{1N})'] + |P_{r0}\chi_{1N}\xi_N| + |(P_{r0} - P_{\phi 0})\chi_{1N}\zeta|}. \tag{66} \]

For cylindrical system coupled with isotropic pressure, we find
\[ \Gamma_1 < \frac{|\Phi_N(1 + 2\lambda R^2 T_0)|}{|P_{0}\chi_{1N}'| + |P_{0}\chi_{1N}\xi_N|}. \tag{67} \]

The system will undergo in the stable phase against collapse if modified gravitational forces induced by \( |\Phi_N(1 + 2\lambda R^2 T_0)| \) are higher than that produced by \( |(P_{r0}\chi_{1N})'| + |P_{r0}\chi_{1N}\xi_N| \), i.e., principal stresses and non-attractive gravitational forces thereby providing stability constraint \( \Gamma_1 > 1 \). For hydrostatic equilibrium, the system would need to balance forces produced by \( |\Phi_N(1 + 2\lambda R^2 T_0)| \) and \( |(P_{0}\chi_{1N})'| + |P_{0}\chi_{1N}\xi_N| \). This leads the system to reach at equilibrium condition i.e., \( \Gamma_1 = 1 \). On the other hand, if a stellar model has attained forces coming from \( |\Phi_N(1 + 2\lambda R^2 T_0)| \) lesser than that of \( |(P_{0}\chi_{1N})'| + |P_{0}\chi_{1N}\xi_N| \), then the relativistic interior will come in unstable window thus giving \( \Gamma_1 \in (0, 1) \). The quantities \( \xi \) and \( \chi_1 \) contain \( f(R, T) \) corrections controlled by a parameter \( \lambda \). Therefore, higher order curvature terms have greatly modify the range of dynamical instability of relativistic cylindrical object. It can be seen from expressions (66) and (67) that anisotropy in pressure along with dark source terms have greatly restrict the instability regions thus producing difficulty for the system to leave stable configurations against collapse.

3.5 Post Newtonian Limit

For instability conditions with pN approximation, we take
\[ \mu_0 \gg P_{j0}, \quad A_0 = 1 - \frac{m_0 G}{r C^2}, \quad B_0 = 1 + \frac{m_0 G}{r C^2}, \tag{68} \]
where \( G \) and \( C \) are gravitational constant and speed of light, respectively. It is worthy to note that we are interested to consider effects upto \( O \left( \frac{m_0}{r} \right) \) in the following calculations. Equation (46) provides
\[
\frac{A''_0}{A_0} = -\frac{A'_0 B'_0}{A_0 B_0} + \frac{B^2_0}{(1 + 2\lambda R_0 T^2_0)} \left[ (\mu_0 + P_{\phi 0})(1 + 2\lambda R^2_0 T_0) + \frac{\lambda}{2} R^2_0 T^2_0 + (S) \right].
\] (69)

By making use of Eqs. (44) and (51), we have
\[
\frac{B'_0}{B_0} = \frac{4m'_0}{(l - 8m_0)},
\] (70)
\[
\frac{A'_0}{A_0} = \frac{2r^2(\mu_0 + P_{r 0})(1 + 2\lambda R^2_0 T_0) + \lambda R^2_0 T^2_0 r^2 l - 4(l - 8m_0)\lambda T^2_0}{2r(l - 8m_0)(1 + 2\lambda R^2_0 T^2_0 + 2\lambda r T_0)} = \varphi \text{(say)}.
\] (71)

Using Eqs. (68)-(71) and (57), Eq.(62) yields
\[
\frac{\Gamma_1 \eta \Pi}{(1 + 2\lambda R^2_0 T_0)} = |\Phi_{pN} \eta| + \left| \left( 1 - \frac{4m_0}{r} \right) \varphi \chi_{1pN} \eta \right|,
\] (72)

where \( \Pi = \left[ \frac{P_{\phi 0} \chi_{1pN}}{\mu_0 + P_{r 0}} \right]' + \frac{P_{\phi 0} \chi_{1pN} \xi_{pN}}{\mu_0 + P_{r 0}} + \left\{ \frac{P_{\phi 0}}{\mu_0 + P_{r 0}} - \frac{P_{r 0}}{\mu_0 + P_{\phi 0}} \right\} \frac{\chi_{1pN} \xi_{pN}}{B^2_0} \right]. \)

For the physical applicability of the above conditions, it is worthy to constraint that all terms in the above are positive. The instability constraint for locally anisotropic cylindrical system in \( f(R, T) \) is calculated as follows
\[
\Gamma_1 < \frac{|\Phi_{pN}(1 + 2\lambda R^2_0 T_0)| + \left| \left( 1 - \frac{4m_0}{r} \right) \varphi \chi_{1pN}(1 + 2\lambda R^2_0 T_0) \right|}{\Pi}.
\] (73)

For locally isotropic cylindrical systems, we reach the same constraint as above with the difference that \( \Pi \) will has the following value
\[
\Pi = \left( \frac{P_{\phi 0} \chi_{1pN}}{\mu_0 + P_{r 0}} \right)' + \frac{P_{\phi 0} \chi_{1pN} \xi_{pN}}{\mu_0 + P_{r 0}}
\] (74)

The locally isotropic self-gravitating system will enter in the unstable window, lest it satisfies the relation (73) with \( \Pi \) given in (74). This shows that pN unstable epochs relies on combinations of principal stresses gradients, repulsive \( f(R, T) \) gravity dark source terms and adiabatic index, \( \Gamma_1 \). These terms then depends upon the static profiles of the corresponding quantities. This asserts the importance of hydrostatic equilibrium parameters. The quantities \( \chi_{pN} \) and \( \xi_{pN} \) are producing gravitational effects due to dark source terms. The extra degrees of freedom induced by \( \lambda \) terms produce complexity in the instability regions, thus delaying the collapse.
4 Summary and Discussion

The study of compact cylindrical object in modified gravity captures many realistic features of the cosmos. The energy distribution of universe also poses interesting puzzles to astrophysicists and cosmologists. Compact celestial objects are an ideal natural laboratory to look for observational signatures and possible modifications in Einstein’s gravity. There have been number of researchers who have discussed the stable as well as unstable regions of compact celestial objects within the framework of Einstein’s gravity. Some researchers have also examined these epochs with higher order corrections in Einstein’s theory. This paper explores instability constraints for cylindrically symmetric anisotropic celestial object in the background of fourth order $f(R,T)$ gravity. The $f(R,T)$ theories yield corrections to the field connected with the matter as compared to the usual GR field. Many researchers used expansion-free scenario to keep the stiffness parameter irrelevant in the study of dynamical instability. Here, we focussed our analysis by encapsulating the effects predicted by adiabatic index or stiffness parameter.

A very promising way to explain this is to keep the non-zero contribution of expansion scalar during evolution of cylindrical model. We explored couple of dynamical equations using contracted form of Bianchi identities with $f(R,T)$ effective energy momentum tensor. It is an established fact that the stars are in a state of hydrostatic equilibrium due to the balance of two conflicting effects, namely, the gravitational force and the opposite internal thermal pressure in the star’s interior provided by nuclear fusion of the elements. When any of these forces overcome the other one then stability of the star is affected and can make explosive events depending upon the size of star. In other words, any galactic model may be stable in one state and turns out to be unstable in later stage. The critical aspect is to determine the dynamical instability via linear perturbations, but one cannot find that up to what degree these techniques can demonstrate the stability problem. The instability issues have much significance as it is closely related with our current understanding of black hole physics and future evolution of the universe.

Due to the complicated framework of $f(R,T)$ theory, it would be significant to gather the instability constraints under some approximations. Therefore, we impose radial perturbation technique on dynamical as well as $f(R,T)$ field equations. The expansion as well as Ricci scalars are also perturbed using linear perturbations on metric as well as material profiles. The static configurations of the field and dynamical equations are constructed within the environment of a viable $f(R,T)$ model. A specific $f(R,T)$ model is used to investigate both qualitative and quantitative behavior of inhomogeneous and unstable system when compared with GR field. For non-zero expansion case, we use adiabatic index to construct collapse equation for the study of instability epochs in N and pN regimes. At N regime, we have a flat background metric while for pN region, we have adopted some known profile of the
The main findings from the collapse equation under both approximations are summarized as follows:

(i) With a physical interpretation and understanding of the under discussion model, we have found the significant dependence of the stiffness parameter on the static profile of the system including the geometrical and physical quantities, particularly, the extra curvature ingredients coming due to $f(R, T)$ theory of gravity. We would like to mention here that there exist some expansion-free dynamical instability analysis in the literature that explored instability ranges at both N and pN regimes independent of this adiabatic index (stiffness parameter) \cite{49,65}.

(ii) It is observed that the anisotropy due to pressure gradients makes increment in the unstable regions of cylindrical object. It indicates that the cylindrical geometry will be stable against fluctuations until it violates the inequalities obtained in (66) and (73) for both N and pN limits.

(iii) The results investigated here are well consistent with those already exist in the literature for GR \cite{39} and modified gravity \cite{63} under some specific constraints.

It is significant to mention that adiabatic index have a particular numerical value, i.e., $\frac{4}{3}$ and 1 for isotropic spherical and cylindrical celestial objects, respectively, in the framework of GR. The instability ranges depend upon the choice of $f(R, T)$ gravity models. One can have different instability regions for different models, i.e., by changing the model, the results may differ from the present one as different models correspond to different eras. Finally, we mention that our all consequences correspond to that of GR under $f(R, T) = R$ limit.

The modified gravity theories (like $f(R, T)$ gravity) provide a cosmologically viable explanation for the current expansion in our cosmos and have been the center of attention during the last few decades. The dynamical equations in this modified gravity have contribution from matter parts, consequently, one can have particular system of equations with every selection of $L_M$.

In order to provide physically realistic $f(R, T)$ gravity theory, one has to study well-consistent as well as physically acceptable models of gravity. Such models not only obeys the constraints in relativistic background set by the solar system and terrestrial experiments but also shed light over the current acceleration in our cosmos. Also, they satisfy the constraints required for their theoretical viability. Any cosmological model in modified gravity should avoid instabilities (Ostrogradski's instability, Dolgov-Kawasaki instability and tachyons) and describe the exact cosmological dynamics. The stability analysis is a cornerstone in astrophysical discussions on gravitational collapse. Haghani et al. \cite{66} and Odintsov and Sáez-Gómez \cite{67} commented that Dolgov-Kawasaki instability in $f(R, T)$ gravity requires similar sort of limitations on the arbitrary function in the Lagrangian as in $f(R)$ gravity.

The following conditions should hold for physically realistic $f(R)$ models \cite{68}:
• The positivity of $f_R(R)$ is needed to avoid the emergence of ghost state with $R > \tilde{R}$, where $\tilde{R}$ is the today value of the Ricci invariant. Ghost appears very often, while describing the modified gravity theories that indicates DE as a source behind current accelerating cosmos. This may emerge due to the mysterious repulsive force between supermassive or massive celestial objects at long distances. The condition of keeping positive the effective gravitational constant, $G_{eff} = \frac{G}{1 + 2\lambda T^2 R}$, is also of great importance for the attractive feature of gravity.

• The positivity of $f_{RR}(R)$ is developed to avoid the appearance of tachyons (which is any hypothetical particle traveling faster than speed of light) with $R > \tilde{R}$. One can consider the imaginary rest-mass so that moving mass must be real as the moving mass of these particles is imaginary.

If $f(R)$ models do not satisfy these conditions, then it would be regarded as unviable. In addition to the above mentioned limitations on the arbitrary function in the Lagrangian of $f(R)$ gravity, we require $1 + 2\lambda T R^2 > 0$ for $G_{eff} > 0$. So, for the physically acceptable $f(R, T)$ models, one needs to satisfy the following constraints

$$1 + 2\lambda T^2 R > 0, \quad 1 + 2\lambda T R^2 > 0, \quad \lambda T^2 > 0, \quad R \geq \tilde{R}.$$

On the other hand, cylindrical systems have been surprising relativists since Levi-Civita constructed its vacuum solution. They serve as a natural tool to explore physics that lies behind the two independent parameters in Levi-Civita spacetime and in particular, the one that describes the Newtonian energy per unit length looks the most elusive. The fact that there are two parameters while in its counterpart, Newtonian theory, has only one parameter looks a sufficient justification for deserving more research. Besides, there has been renewed interest in cylindrically symmetric sources in relation with different, classical and quantum, aspects of gravitation. Such sources may serve as test bed for numerical relativity, quantum gravity and for probing cosmic censorship and hoop conjecture, among other important issues, and represent a natural tool to seek the physics that lies behind the two independent parameters in Levi-Civita metric.

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Appendix A

The dark source terms \( D_0 \) and \( D_1 \) are given as follows

\[
D_0 = \frac{\psi_{01,1}}{A^2} - \frac{\psi_{01}}{A^2} \left( \frac{A'}{A} + \frac{B'}{B} + \frac{2AA'}{B^2} - \frac{CC'}{B^2} \right) - \frac{\psi_{00}}{A^2 f_R} (B \dot{B} + C \dot{C}) - \frac{\psi_{11}}{A^2 f_R} \times B \dot{B} - \frac{\psi_{22}}{A^2 f_R} C \dot{C},
\]

\[
D_1 = \frac{\psi_{01,0}}{B^2} - \frac{\psi_{01}}{B^2} \left( \frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{2B \dot{B}}{A^2} \right) - \frac{AA'}{B^2 f_R} (\psi_{00} + \psi_{11}) + \frac{CC'}{B^2 f_R} \times (\psi_{11} - \psi_{22}) - \frac{f - R f_R}{2} - \psi_{11} \right) .
\]

(A1)

The components of extra curvature terms \( \chi_i \)s are

\[
\chi_2 = \frac{1}{B_0^2 (1 + 2 \lambda R_0 T_0^2)} \left\{ \left( \frac{c}{r} \right)' \left( \frac{B_0'}{B_0} \right)' + \frac{1}{r} \left( \frac{b}{B_0} \right)' - \frac{c''}{r} \right\} - \left( \psi_{00} - e \lambda R_0 T_0^2 \right) - \left( \mu_0 - \lambda R_0^2 T_0^2 + \psi_{00} \right) \left( \frac{2b}{B_0^2} + \frac{2 \lambda T_0^2}{B_0^2} \right),
\]

\[
\chi_3 = \frac{1}{(1 + 2 \lambda R_0^2 T_0)} \left\{ (1 + 2 \lambda R_0 T_0^2) \left\{ \frac{A'_0}{A_0} \left( \frac{c}{r} \right)' + \frac{1}{r} \left( \frac{a}{A_0} \right)' \right\} - \left\{ \lambda \frac{R_0^2 T_0^2}{2} + \psi_{11} \right\} + (\mu_0 + P_{\phi_0})(1 + 2 \lambda R_0^2 T_0) \right\} \left\{ \frac{2b}{B_0^2} + 2 \lambda T_0^2 \right\} - \left[ 2 e \lambda \lambda R_0^2 (\mu_0 + P_{\phi_0}) + e \lambda R_0 T_0^2 \right] ,
\]

\[
\chi_5 = - \frac{(1 + 2 \lambda R_0 T_0^2)}{A_0 B_0^2 (1 + 2 \lambda R_0^2 T_0)} \left\{ b' A'_0 + a' B'_0 - \frac{2b}{B_0} (A'_0 B'_0 + B_0 a'' - b A''_0) \right\} - (2 R_0 P_{\phi_0})
\]

\[
+ 2 \mu_0 R_0^2 + T_0^2 + \frac{\psi_{22}}{e \lambda R_0} \left( \frac{e \lambda R_0}{(1 + 2 \lambda R_0 T_0)} - (1 + 2 \lambda R_0^2 T_0) \right) \left\{ \frac{\lambda R_0^2 T_0^2}{2} + \psi_{22} \right\}
\]

\[
+ (\mu_0 + P_{\phi_0})(1 + 2 \lambda R_0^2 T_0) \right\} \left\{ \frac{b}{B_0} + \frac{a}{A_0} + \frac{2 e \lambda T_0^2}{(1 + 2 \lambda R_0 T_0)} \right\} .
\]

The mathematical expressions mentioned in Eqs. (52) and (53) are

\[
X_1 (r) = \left[ 2 \lambda T_0 \mu_0 \frac{(e T_0 + 2 R_0 z)}{1 + 2 \lambda R_0 T_0^2} + (1 + 2 \lambda R_0^2 T_0) (\mu_0 + P_{\phi_0}) \left( \frac{c}{A_0} + \frac{b B_0}{A_0^2} \right) \right]
\]

21
\[
\begin{align*}
-4\lambda_0 R_0 \frac{(e T_0 R_0 z)}{(1 + 2\lambda R_0^2 T_0)} - \frac{\psi_0}{(A_0^2(1 + 2\lambda R_0^2 T_0))} & \left\{ \psi_{11}^{(S)} \right\} \\
+ (b B_0 + rc) & \left\{ \psi_{00}^{(S)} + rc \psi_{22}^{(S)} \right\} - \psi_0 \frac{1}{A_0^2} - \psi_0 \frac{1}{A_0^2} \left( \frac{A_0^2}{A_0} + \frac{B_0^2}{B_0} - \frac{r}{B_0} \right) \\
+ \frac{2A_0 A_0'}{B_0^2} & \left[ \frac{(1 + 2\lambda R_0^2 T_0)(1 + 2\lambda R_0^2 T_0)}{(1 + 4\lambda R_0^2 T_0)} \right], \\
D_3 = & \frac{2\lambda R_0 P_{00}}{(1 + 2\lambda R_0^2 T_0)} (2T_0 e' + 2e R_0 + R_0 z') - \frac{4\lambda^2 T_0 P_{00}}{(1 + 2\lambda R_0^2 T_0)^2} \left( P_{00} (2T_0 R_0' + R_0 T_0') - \frac{4\lambda^2 T_0 P_{00}}{(1 + 2\lambda R_0^2 T_0)^2} (T_0 R_0' + R_0 T_0') \right) \\
+ & \frac{2\lambda R_0 R_0'}{(1 + 2\lambda R_0^2 T_0)} \left\{ \frac{(1 + 2\lambda R_0^2 T_0)}{(1 + 2\lambda R_0^2 T_0)} - R_0^2 \right\} + \frac{2\lambda R_0 P_{00}}{(1 + 2\lambda R_0^2 T_0)^2} \left( T_0 e' + 2e R_0 T_0' \right) \\
+ & \frac{2R_0 R_0'}{(1 + 2\lambda R_0^2 T_0)} \left\{ (1 + 2\lambda R_0^2 T_0) - \frac{2\lambda R_0 P_{00}}{(1 + 2\lambda R_0^2 T_0)^2} \right\} \left( T_0 e' + 2e R_0 T_0' \right) \\
\times (\mu_0 + P_{00}) + & \frac{4\lambda^2 T_0 P_{00}}{(1 + 2\lambda R_0^2 T_0)^2} \left[ 2\lambda R_0^2 T_0 (T_0 R_0' + R_0 T_0') \right] - \frac{4\lambda^2 T_0^2 R_0^2 \mu_0}{(1 + 2\lambda R_0^2 T_0)^2} (2R_0 z' + T_0 e') \\
+ & \frac{2\lambda R_0 R_0'}{(1 + 2\lambda R_0^2 T_0)} \left\{ (1 + 2\lambda R_0^2 T_0) \frac{2\lambda R_0^2 T_0}{(1 + 2\lambda R_0^2 T_0)^2} \right\} \left\{ \frac{2\lambda R_0}{(1 + 2\lambda R_0^2 T_0)^2} - R_0^2 \right\} \\
+ & \frac{2\lambda R_0^2}{(1 + 2\lambda R_0^2 T_0)} \left[ \mu_0 + \frac{T_0}{2} \right] \left[ 1 - \frac{2\lambda R_0^2 T_0}{(1 + 2\lambda R_0^2 T_0)} \right] + \frac{2\lambda R_0 T_0}{(1 + 2\lambda R_0^2 T_0)} \left( \mu_0 - P_{00} \right) \left[ \frac{2\lambda R_0^2}{(1 + 2\lambda R_0^2 T_0)^2} \right] (T_0 R_0') \\
+ & \frac{R_0 T_0'}{(1 + 2\lambda R_0^2 T_0)} - 2T_0 e' - 2e R_0' - R_0 z' + r \frac{(P_{00} - P_{00})}{(1 + 2\lambda R_0^2 T_0)^2} (1 + 2\lambda R_0^2 T_0) \left\{ \frac{1}{r} \right\} \\
+ & \frac{c'}{(1 + 2\lambda R_0^2 T_0)} - \frac{2b}{B_0} \left\{ \frac{2e \lambda R_0^2}{(1 + 2\lambda R_0^2 T_0)^2} (P_{00} - P_{00}) \right\} - \left\{ \psi_{00}^{(S)} \right\} \\
+ & \psi_{11}^{(S)} \left[ \frac{A_0 A_0'}{A_0 A_0'} - \frac{2b}{B_0} \right] - \frac{2b}{B_0} - \frac{2e \lambda T_0}{(1 + 2\lambda R_0^2 T_0)^2} \right\} + \psi_{00} + \psi_{11}^{(P)} \right] \frac{A_0 A_0'}{B_0^2(1 + 2\lambda R_0^2 T_0)} \]
\[ \lambda(eR_0T - 0^2)' + \psi_{0,1}^{(P)} + \frac{r}{B_0^2(1 + 2\lambda R_0 T_0^2)} \left( \psi_{11}^{(P)} - \psi_{22}^{(P)} \right). \]  

(A4)

References

[1] A. G. Riess, et al. [Supernova Search Team Collaboration], Astron. J. 116, 1009 (1998) [arXiv:astro-ph/9805201]; S. Perlmutter et al. [Supernova Cosmology Project Collaboration], Astrophys. J. 517, 565 (1999) [arXiv:astro-ph/9812133]; A. G. Riess et al. [Supernova Search Team Collaboration], Astrophys. J. 607, 665 (2004) [arXiv:astro-ph/0402512].

[2] M. Tegmark et al. [SDSS Collaboration], Phys. Rev. D 69, 103501 (2004) [arXiv:astro-ph/0310723]; K. Abazajian et al., [arXiv:astro-ph/ 0410239]; K. Abazajian et al. [SDSS Collaboration], Astron. J. 128, 502 (2004) [arXiv:astro-ph/0403325]; K. Abazajian et al. [SDSS Collaboration], Astron. J. 126, 2081 (2003) [arXiv:astro-ph/0305492]; E. Hawkins et al., Mon. Not. R. Astron. Soc. 346, 78 (2003) [arXiv:astro-ph/0212375]; L. Verde et al., Mon. Not. Roy. Astron. Soc. 335, 432 (2002) [arXiv:astro-ph/0112161].

[3] C. L. Bennett et al., Astrophys. J. Suppl. 148, 1 (2003) [arXiv:astro-ph/0302207]; D. N. Spergel et al. [WMAP Collaboration], Astrophys. J. Suppl. 148, 175 (2003) [arXiv:astro-ph/0302209].

[4] V. Sahni and A. A. Starobinsky, Int. J. Mod. Phys. D 9, 373 (2000).

[5] S. M. Carroll, Living Rev. Rel. 4, 1 (2000) [arXiv:astro-ph/0004075].

[6] P. J. E. Peebles and B. Ratra, Rev. Mod. Phys. 75, 559 (2002) [arXiv:astro-ph/0207347].

[7] T. Padmanabhan, Phys. Rep. 380, 235 (2003); M. Sharif and Z. Yousaf, Astrophys. Space Sci. 357, 49 (2015).

[8] P. Bode, C. Ma, R. R. Caldwell and L. Wang, Astrophys. J., L1 521, (1999).

[9] P. G. Ferreira and M. Joyce, Phys. Rev. Lett. 79, 4740 (1997); D. F. Mota, C. van de Bruck, Astron. Astrophys. 421, 71 (2004).

[10] V. Dokuchaev, E. Babichev and Yu. Eroshenko, Phys. Rev. Lett. 93, 021102 (2004) [arXiv:hep-th/ 0408170].
[11] A. Qadir, H. W. Lee, K. Y. Kim, Int. J. Mod. Phys. D 26 (2017) 1741001.
[12] S. Nojiri and S. D. Odintsov, Phys. Rev. D 68, 123512 (2003).
[13] J. Páramos, Proc. QSO Astrophys., arXiv:1111.2740 [gr-qc].
[14] M. Sharif and Z. Yousaf, J. Cosmol. Astropart. Phys. 06, 019 (2014);
[15] Z. Yousaf and M. Z. Bhatti, Mon. Not. R. Astron. Soc. 458, 1785 (2016).
[16] M. Sharif and Z. Yousaf, Gen. Relativ. Gravit. 47, 48 (2015); Can. J. Phys. 93, 905 (2015); M. Z. Bhatti and Z. Yousaf, Int. J. Mod. Phys. D 26, 1750029 (2017).
[17] M. Sharif and Z. Yousaf, Eur. Phys. J. C 75, 58 (2015); Astrophys. Space Sci. 352, 321 (2014).
[18] M. Sharif and Z. Yousaf, Astrophys. Space Sci. 354, 431 (2014); M. Z. Bhatti, Z. Yousaf and S. Ashraf, Ann. Phys. 383, 439 (2017); Z. Yousaf, M. Z. Bhatti and A. Rafaqat, Astrophys. Space Sci. 68, 362 (2017).
[19] O. Bertolami, P. Frazão, and J. Páramos, Phys. Rev. D 83, 044010 (2011).
[20] S. Nojiri and S. D. Odintsov, Phys. Lett. B 657, 238 (2007); O. Bertolami, P. Frazão, and J. Páramos, Phys. Rev. D 81, 104046 (2010).
[21] T. Harko, F. S. N. Lobo, S. Nojiri and S. D. Odintsov, Phys. Rev. D 84, 024020 (2011).
[22] S. Capozziello, M. De Laurentis, S. D. Odintsov, and A. Stabile, Phys. Rev. D 83, 064004 (2011); S. Capozziello, M. De Laurentis, I. De Martino, M. Formisano, and S. D. Odintsov, Phys. Rev. D 85, 044022 (2012); E. V. Arbuzova, A. D. Dolgov, and L. Reverberi, arXiv:1406.7104.
[23] T. Clifton, P. G. Ferreira, A. Padilla, and C. Skordis, Phys. Rep. 513, 1 (2012).
[24] R. H. Sanders, Mon. Not. R. Astron. Soc. 296, 1009 (1998).
[25] M. Malekjani, S. Rahvar, and H. Haghj, Astrophys. J. 694, 1220 (2009).
[26] M. C. Martino, H. F. Stabenau and R. K. Sheth, Phys. Rev. D 79, 084013 (2009).
[27] P. H. R. S. Moraes, R. A. C. Correa and R. V. Lobato, arXiv:1701.01028 [gr-qc].
[28] E. Santos, Astrophys. Space Sci. 341, 411 (2012).
[29] S. G. Ghosh and S. D. Maharaj, Phys. Rev. D 85, 124064 (2012).

[30] J. A. R. Cembranos, A. de la Cruz-Dombriz and B. M. Núñez, J. Cosmol. Astropart. Phys. 04, 021 (2012).

[31] H. Alavirad and J. M. Weller, Phys. Rev. D 88, 124034 (2013).

[32] L. Sebastiani, D. Momeni, R. Myrzakulov and S. D. Odintsov, Phys. Rev. D 88, 104022 (2013).

[33] C. Bambi, A. Cardenas-Avendano, G. J. Olmo and D. Rubiera-Garcia, Phys. Rev. D 93, 064016 (2016).

[34] Z. Yousaf, M. Ilyas, M. Z. Bhatti, Eur. Phys. J. Plus 132 (2017) 268.

[35] G. J. Olmo and D. Rubiera-Garcia, Phys. Rev. D 84, 124059 (2011).

[36] G. J. Olmo and D. Rubiera-Garcia, Universe 1, 173 (2015).

[37] E. V. Arbuzova and A. D. Dolgov, Phys. Lett. B 700, 289 (2011).

[38] K. Bamba, S. Nojiri and S. D. Odintsov, Phys. Lett. B 698, 451 (2011).

[39] M. Sharif and M. Z. Bhatti, J. Cosmol. Astropart. Phys. 11, 014 (2013); Phys. Lett. A 378, 469 (2014); Astropart. Phys. 56, 35 (2014); Mon. Not. R. Astron. Soc. 450, 1015 (2015).

[40] M. Sharif and M. Z. Bhatti, Mod. Phys. Lett. A 29, 1450094 (2014); ibid. 1450129.

[41] R. Farinelli, M. De Laurentis, S. Capozziello and S. D. Odintsov, Mon. Not. R. Astron. Soc. 440, 2909 (2014).

[42] M. Z. Bhatti and Z. Yousaf, Eur. Phys. J. C 76, 219 (2016) [arXiv: 1604.01395 [gr-qc]]; M. Z. Bhatti, Eur. Phys. J. Plus 131, 428 (2016).

[43] Z. Yousaf, K. Bamba and M. Z. Bhatti, Phys. Rev. D 93, 064059 (2016) [arXiv:1603.03175 [gr-qc]]; ibid. Phys. Rev. D 93, 124048 (2016) [arXiv:1606.00147 [gr-qc]]; Phys. Rev. D 95, 024024 (2017) [arXiv:1701.03067 [gr-qc]]; Z. Yousaf, M. Z. Bhatti and U. Farwa, Mon. Not. R. Astron. Soc. 464, 4509 (2017); Z. Yousaf, Eur. Phys. J. Plus 132, 71 (2017); Z. Yousaf, Eur. Phys. J. Plus 132, 276 (2017).

[44] S. Chandrasekhar, Astrophys. J. 140, 417 (1964).
[45] L. Herrera, G. Le Denmat and N. O. Santos, Mon. Not. R. Astron. Soc. **237**, 257 (1989); R. Chan, S. Kichenassamy, G. Le Denmat and N. O. Santos, Mon. Not. R. Astron. Soc. **239**, 91 (1989).

[46] R. Chan, L. Herrera and N. O. Santos, Mon. Not. R. Astron. Soc. **265**, 533 (1993).

[47] L. Herrera and N. O. Santos, Phys. Rep. **286**, 53 (1997).

[48] M. Sharif and Z. Yousaf, Mon. Not. R. Astron. Soc. **440**, 3479 (2014); Eur. Phys. J. C **75**, 194 (2015) [arXiv:1504.04367 [gr-qc]]; M. Sharif and M. Z. Bhatti, Astrophys. Space Sci. **355**, 389 (2015); Z. Yousaf, M. Z. Bhatti and U. Farwa, Class. Quantum Grav. **34**, 145002 (2017); Z. Yousaf, M. Z. Bhatti and U. Farwa, Eur. Phys. J. C **77**, 359 (2017) [arXiv:1705.06975 [physics.gen-ph]].

[49] Z. Yousaf and M. Z. Bhatti, Eur. Phys. J. C **76**, 267 (2016) [arXiv:1604.06271 [physics.gen-ph]].

[50] S. M. Carroll, V. Duvvuri, M. Trodden, M. S. Turner, Phys. Rev. D **70**, 043528 (2004); S. M. Carroll, A. De Felice, V. Duvvuri, D. A. Easson, M. Trodden, et al., Phys. Rev. D **71**, 063513 (2005).

[51] J. Carot, J. M. M. Senovilla and R. Vera, Class. Quantum Grav. **16**, 3025 (1999).

[52] M. Sharif and Z. Yousaf, Can. J. Phys. **90**, 865 (2012) [arXiv:1311.2874 [gr-qc]]; M. Sharif and M. Z. Bhatti, J. Cosmol. Astropart. Phys. **10**, 056 (2013).

[53] Y. Kurita and K.-I. Nakao, Phys. Rev. D **73**, 064022 (2006).

[54] A. Einstein and N. Rosen, J. Franklin Inst. **223**, 43, (1937)

[55] A. Di Prisco, L. Herrera, M. A. H. MacCallum and N. O. Santos, Phys. Rev. D **80**, 064031 (2009).

[56] J. Barrientos and G. F. Rubilar, Phys. Rev. D **90**, 028501, (2014).

[57] K. S. Thorne, Phys. Rev. B**138**, 251 (1965).

[58] G. Darmois, *Memorial des Sciences Mathematiques* (Gauthier-Villars, Paris, 1927), Fasc. 25.

[59] J. M. M. Senovilla, Phys. Rev. D **88**, 064015 (2013).

[60] S. Nojiri and S. D. Odintsov, Phys. Rep. **505**, 59 (2011) [arXiv:1011.0544 [gr-qc]].
[61] L. Herrera, N. O. Santos and G. Le Denmat, Mon. Not. R. Astron. Soc. 237, 257 (1989).

[62] B. K. Harrison, K. S. Thorne, M. Wakano and J. A. Wheeler, *Gravitation Theory and Gravitational Collapse* (University of Chicago Press, 1965).

[63] M. Sharif and Z. Yousaf, Astrophys. Space Sci. 354, 471 (2014); H. R. Kausar, Mon. Not. R. Astron. Soc. 445, 3650 (2014); M. Sharif and Z. Yousaf, Astrophys. Space Sci. 355, 317 (2015); A. Jawad, D. Momeni, S. Rani, and R. Myrzakulov, Astrophys. Space Sci. 361, 141 (2016).

[64] S. Chandrasekhar, Rev. Mod. Phys. 56, 2 (1984).

[65] L. Herrera, G. Le Denmat and N. O. Santos, Gen. Relativ. Gravit. 44, 1143 (2012).

[66] Z. Haghani, T. Harko, F. S. N. Lobo, H. R. Sepangi and S. Shahidi, Phys. Rev. D 88, 044023 (2013).

[67] S. D. Odintsov and D. Sáez-Gómez, Phys. Lett. B 725, 437 (2013).

[68] S. Capozziello, Int. J. Mod. Phys. D 11, 483 (2002); S. Nojiri and S. D. Odintsov, Phys. Rep. 505, 59 (2011); K. Bamba et al., Astrophys. Space Sci. 342, 155 (2012).