Local properties of Schubert Varieties in the Symplectic Grassmannian via a bounded RSK correspondence

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Abstract

In a paper by Ghorpade and Raghavan, they provide an explicit combinatorial description of the Hilbert function of the tangent cone at any point on a Schubert variety in the symplectic grassmannian, by giving a certain “degree-preserving” bijection between a set of monomials defined by an initial ideal and a “standard monomial basis”. We prove here that this bijection is in fact a bounded RSK correspondence.

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1 Introduction

In [1], Kreiman gives an explicit gröbner basis for the ideal of the tangent cone at any $T$-fixed point of a Richardson variety in the ordinary grassmannian, where
The organization of this paper is as follows. In §2.4 of this paper, we show how our bijection given in [2] is the same as the map BRSK of [1]. In §2.4, we show how our problem after stating all necessary definitions and notation. In §2.4, we provide a proof of this fact here.

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The symplectic grassmannian and a Schubert variety therein are defined in [2.1] of this paper. In this paper, we consider the bijection given by proposition 4.1 of [3] for Schubert varieties in the symplectic grassmannian. We prove here that this bijection is a bounded RSK correspondence. The proof of this fact actually reduces to proving that the bijection \( \tilde{\pi} \) given in [2] is the same as the map BRSK of [1], the reduction being shown in §2.4 of this paper. It will be nice if one can prove that the map BRSK of [1] is the only natural bijection between the two given combinatorially defined sets.

The organization of this paper is as follows. In §2.4, firstly we state our problem after stating all necessary definitions and notation. In §2.4, we show how our initial problem is reduced to another problem (namely, the problem of proving that the map \( \tilde{\pi} \) of [2] is the same as the map BRSK of [1]). In §2.4 we state the main theorem (namely, theorem 3.0.1), its corollary, and then we prove some important results needed to prove the main theorem. In §4 we provide a proof of the main theorem.

2 Stating the problem

2.1 Some necessary definitions and notation

The following definitions and notation are written in the same way as given in the papers [3] and [2].

Given any positive integer \( n \), we denote by \([n]\) the set \( \{1, 2, \ldots, n\} \). Given positive integers \( r \) and \( n \) with \( r \leq n \), we denote by \( I(r, n) \) the set of all \( r \)-element subsets of \([n]\) . Let \( \alpha = (\alpha_1, \ldots, \alpha_r) \in I(r, n) \) where \( 1 \leq \alpha_1 < \ldots < \alpha_r \leq n \). If \( \beta = (\beta_1, \ldots, \beta_r) \in I(r, n) \) be such that \( 1 \leq \beta_1 < \ldots < \beta_r \leq n \), then we say that \( \alpha \leq \beta \) if \( \alpha_i \leq \beta_i \forall i = 1, \ldots, r \). Clearly, \( \leq \) defines a partial order on \( I(r, n) \).

A positive integer \( d \) will be kept fixed throughout this paper. For \( j \in [2d] \), set \( j^* := 2d + 1 - j \). Let \( I(d) \) denote the set of all \( d \)-element subsets \( v \) of \([2d]\) with the property that exactly one of \( j, j^* \) belongs to \( v \) for every \( j \in [d] \). Clearly \( I(d) \subseteq I(d, 2d) \). In particular, we have the partial order \( \leq \) on \( I(d) \) induced from \( I(d, 2d) \).

We denote by \( \epsilon \) the element \((1, \ldots, d)\) of \( I(d) \). The \( \epsilon \)-degree of an element \( x \) of \( I(d) \) is the cardinality of \( x \setminus [d] \) or equivalently that of \( d \setminus [x] \). More generally, given any \( v \in I(d) \), the \( v \)-degree of an element \( x \) of \( I(d) \) is the cardinality of \( x \setminus v \) or equivalently that of \( v \setminus x \). An ordered pair \( w = (x, y) \) of elements of \( I(d) \) is called an admissible pair if \( x \geq y \) and the \( \epsilon \)-degrees of \( x \) and \( y \) are...
equal. Sometimes \(x\) and \(y\) are referred as the top and bottom of \(w\) and written as \(\text{top}(w)\) for \(x\) and \(\text{bot}(w)\) for \(y\). Given an admissible pair \(w = (x, y)\), we define the \(v\)-degree of \(w\) by \(v\)-degree\((w)\) := \(\frac{1}{2}(|x \setminus v| + |y \setminus v|)\).

Given any two admissible pairs \(w = (x, y)\) and \(w' = (x', y')\), we say that \(w \geq w'\) if \(y \geq x'\), that is, if \(x \geq y \geq x' \geq y'\). An ordered sequence \((\omega_1, \ldots, \omega_t)\) of admissible pairs is called a standard tableau if \(\omega_i \geq \omega_{i+1}\) for \(1 \leq i < t\). Sometimes \(\omega_1 \geq \ldots \geq \omega_t\) is written to denote a standard tableau \((\omega_1, \ldots, \omega_t)\).

Given any \(w \in I(d)\), we say that a standard tableau \(\omega_1 \geq \ldots \geq \omega_t\) is \(w\)-dominated if \(w \geq \text{top}(\omega_1)\). Given any \(v \in I(d)\), we say that the standard tableau \(\omega_1 \geq \ldots \geq \omega_t\) is \(v\)-compatible if for each \(\omega_i\), either \(v \geq \text{top}(\omega_i)\) or \(\text{bot}(\omega_i) \geq v\), and \(\omega_i \neq (v, v)\). Given \(v\) and \(w\) in \(I(d)\), we denote by \(SM_v^w\) the set of all \(w\)-dominated \(v\)-compatible standard tableaux. For any positive integer \(m\), let \(SM_v^w(m)\) denote the set of all \(w\)-dominated \(v\)-compatible standard tableaux of degree \(m\), where the degree of a standard tableau \(\omega_1 \geq \ldots \geq \omega_t\) is defined to be the sum of the \(v\)-degrees of \(\omega_1, \ldots, \omega_t\). Let \(SM_v^w\) denote the set of all \(v\)-compatible standard tableaux that are anti-dominated by \(v\): a standard tableau \(\omega_1 \geq \ldots \geq \omega_t\) is called anti-dominated by \(v\) if \(\text{bot}(\omega_t) \geq v\).

Fix a vector space \(V\) of dimension \(2d\) over an algebraically closed field of arbitrary characteristic. Fix a non-degenerate skew-symmetric bilinear form \(\langle , \rangle\) on \(V\). Fix a basis \(e_1, \ldots, e_{2d}\) of \(V\) such that

\[
\langle e_i, e_j \rangle = \begin{cases} 
  1 & \text{if } i = j^* \text{ and } i < j \\
  -1 & \text{if } i = j^* \text{ and } i > j \\
  0 & \text{otherwise.}
\end{cases}
\]

A linear subspace \(W\) of \(V\) is said to be isotropic if the form \(\langle , \rangle\) vanishes identically on it. Denote by \(\mathfrak{Sp}_d(V)\) the grassmannian of all \(d\)-dimensional subspaces of \(V\) and by \(\mathfrak{M}_d(V)\) the set of all maximal isotropic subspaces of \(V\). Then \(\mathfrak{M}_d(V)\) is a closed subvariety of \(\mathfrak{Sp}_d(V)\) and is called the symplectic grassmannian.

The group \(\mathfrak{Sp}(V)\) of linear automorphisms of \(V\) preserving \(\langle , \rangle\) acts transitively on \(\mathfrak{M}_d(V)\) – this follows from Witt’s theorem that an isometry between subspaces can be lifted to one of the whole vector space. The elements of \(\mathfrak{Sp}(V)\) that are diagonal with respect to the basis \(e_1, \ldots, e_{2d}\) form a maximal torus \(T\) of \(\mathfrak{Sp}(V)\). Similarly the elements of \(\mathfrak{Sp}(V)\) that are upper triangular with respect to \(e_1, \ldots, e_{2d}\) form a Borel subgroup \(B\) of \(\mathfrak{Sp}(V)\).

The \(T\)-fixed points of \(\mathfrak{M}_d(V)\) are parametrized by \(I(d)\): for \(v = (v_1, \ldots, v_d)\) in \(I(d)\), the corresponding \(T\)-fixed point, denoted by \(e^v\), is the span of \(e_{v_1}, \ldots, e_{v_d}\). These points lie in different \(B\)-orbits, and the union of their \(B\)-orbits is all of \(\mathfrak{M}_d(V)\). A Schubert variety \(X_w\) in \(\mathfrak{M}_d(V)\) is by definition the closure of such a \(B\)-orbits with the reduced scheme structure. Schubert varieties are thus indexed by the \(T\)-fixed points and so in turn by \(I(d)\). Given \(w\) in \(I(d)\), we denote by \(X_w\), the closure of the \(B\)-orbit of the \(T\)-fixed point \(e^w\).

Fix elements \(v, w \in I(d)\) with \(v \leq w\). Define \(R^v := \{(r, c) \in [2d] \setminus v \times v : r \leq c^v\}\) and \(R^w := \{(r, c) \in [2d] \setminus v \times v : r > c\}\). Let \(S^v\) denote the set of all monomials in \(R^v\) and \(T^v\) the set of all monomials in \(R^v\). Given any \(\beta_1 = (r_1, c_1), \beta_2 = (r_2, c_2)\) in \(R^v\), we say that \(\beta_1 \succ \beta_2\) if \(r_1 > r_2\) and \(c_1 < c_2\). A sequence \(\beta_1 > \ldots > \beta_t\) of elements of \(R^v\) is called a
v-chain. Given a v-chain \( \beta_1 = (r_1, c_1) > \ldots > \beta_t = (r_t, c_t) \), we define

\[
s_{\beta_1, \ldots, s_{\beta_t}} := (\{v_1, \ldots, v_d\} \setminus \{c_1, \ldots, c_t\}) \cup \{r_1, \ldots, r_t\}.
\]

We say that \( w \) dominates the v-chain \( \beta_1 > \ldots > \beta_t \) if \( w \geq s_{\beta_1, \ldots, s_{\beta_t}} \). Let \( \mathcal{S} \) be a monomial in \( \mathcal{R}^v \). By a v-chain in \( \mathcal{S} \), we mean a sequence \( \beta_1 > \ldots > \beta_t \) of elements of \( \mathcal{S} \cap \mathcal{R}^v \). We say that \( w \) dominates every v-chain in \( \mathcal{S} \). Let \( S^w_m \) denote the set of all \( w \)-dominated monomials in \( \mathcal{R}^v \), and (for any positive integer \( m \)) let \( S^w_m(m) \) denote the set of such monomials of degree \( m \).

2.2 The result of Ghorpade and Raghavan

Theorem 2.2.1. Let \( v, w \) be elements of \( I(d) \) with \( v \leq w \). Let \( X_w \) be the Schubert variety corresponding to \( w \), \( e^v \) the \( T \)-fixed point in \( X_w \) corresponding to \( v \), and \( R \) be the coordinate ring of the tangent cone to \( X_w \) at the point \( e^v \). Then the dimension as a vector space of the \( m^{th} \) graded piece \( R(m) \) of \( R \) equals the cardinality of \( S^w_m(m) \).

2.3 Our problem

The proof of theorem 2.2.1 (as given in [3]) relies on a bijection between the two combinatorially defined sets \( SM^w_v(m) \) and \( S^w_v(m) \). And this bijection in turn, relies upon a bijection between \( SM^w_v \) and \( T^v \), which is stated in Proposition 4.1 of [3]. Our problem is to prove that the bijection between \( SM^w_v \) and \( T^v \) (as mentioned in Proposition 4.1 of [3]) is a bounded RSK correspondence.

2.4 Reduction of our problem to another problem

Consider \( v \) as an element of \( I(d, 2d) \). A standard monomial in \( I(d, 2d) \) is a totally ordered sequence \( \theta_1 \geq \ldots \geq \theta_r \) of elements of \( I(d, 2d) \). Such a monomial is called \( v \)-compatible if each \( \theta_j \) is comparable to \( v \) but no \( \theta_j \) equals \( v \); it is anti-dominated by \( v \) if \( \theta_j \geq v \).

Let \( \tilde{SM}^v \) denote the set of all \( v \)-compatible standard monomials in \( I(d, 2d) \) anti dominated by \( v \). Let \( \tilde{R}^v \) denote the set of all ordered pairs \( (r, c) \) such that \( r \in [2d] \setminus v \) and \( c \in v \). Let \( \tilde{R}^v \) denote the subset of \( \mathcal{R}^v \) consisting of those \( (r, c) \) with \( r > c \). Let \( \tilde{T}^v \) denote the set of all monomials in \( \tilde{R}^v \).

There is a natural injection \( f: SM^w_v \to SM^w_v \) given by

\[
f(w_1 \geq \ldots \geq w_t) := \text{top}(w_1) \geq \text{bot}(w_2) \geq \ldots \geq \text{top}(w_t) \geq \text{bot}(w_t).
\]

Composing this map \( f \) with the bijection \( \tilde{\phi} \) (of §4, [2]) from \( \tilde{SM}^v \to \tilde{T}^v \), we get an injection of \( SM^v \) into \( T^v \). It then follows from lemma 4.5 of [3] that under this composition, the image of \( SM^v \) in \( \tilde{T}^v \) is the set \( \mathcal{E} \) of all special monomials, where the definition of a special monomial of \( T^v \) is given in definition 4.4 of [3]. On the other hand, there is a bijective map (call it \( g \)) from the set \( \mathcal{E} \) of all special monomials to \( T^v \) as given in §4.1 of [3]: Given any \( \mathcal{S} \) in \( \mathcal{E} \), to get \( g(\mathcal{S}) \), replace those \( (r, c) \) of \( \mathcal{S} \) with \( r > c^* \) by \( (c^*, r^*) \) and then take the (positive) square root. The composition \( \eta := g \circ \tilde{\phi} \circ f \) is the required bijection from \( SM^v \) to \( T^v \).
Therefore, to prove that the composition map $\eta$ is a bounded RSK correspondence, it suffices to show that the map $\phi$ (of \cite{1}, \cite{2}) is a bounded RSK correspondence. But recall the map $\tilde{\pi}$ from \cite{2}, and also the fact that the maps $\tilde{\pi}$ and $\phi$ (of \cite{2}) are inverses of each other. Hence, it now suffices to show that the map $\tilde{\pi}$ of \cite{2} is equal to the map $\text{BRSK}$ of \cite{1}. This fact will be proved in corollary \textbf{3.0.1} below. Corollary \textbf{3.0.1} follows immediately from theorem \textbf{3.0.1}

Hence, our goal is now to prove theorem \textbf{3.0.1} below.

### 3 The main theorem

**Theorem 3.0.1.** Let $U$ be a finite monomial in $\tilde{\mathcal{W}}$. Recall the map $\pi$ from \cite{2}. Let $\pi(U) = (w_0, U^{(1)}), \pi(U^{(1)}) = (w_1, U^{(2)}).....$ and so on till $\pi(U^{(m)}) = (w_n, \phi)$, where $\phi$ is the empty monomial. Then for each $r \in \{0, 1, \ldots, m\}$, the following holds:

(i) All the row numbers of the distinguished subset $S_{w_r}$ (corresponding to $w_r$) consist of the $(m+1) - r$ th row entries of the left hand notched tableau of $\text{BRSK}(U)$.

(ii) All the column numbers of $S_{w_r}$ comprise of the $(m+1) - r$ th row entries of the right hand notched tableau of $\text{BRSK}(U)$.

**Corollary 3.0.1.** For any finite monomial $U$ in $\tilde{\mathcal{W}}$, $\tilde{\pi}(U) = \text{BRSK}(U)$.

### 3.1 Some results needed to prove theorem \textbf{3.0.1}

We will prove theorem \textbf{3.0.1} by induction on the cardinality $n$ of the monomial $U$. If $n = 1$, then the theorem is obvious. Let us henceforth assume that theorem \textbf{3.0.1} holds true for all finite monomials in $\tilde{\mathcal{W}}$ of cardinality $\leq n - 1$.

To prove theorem \textbf{3.0.1} above, we need some notation, lemmas and definitions. We will mention those first. In the rest of this paper, we are going to assume the induction hypothesis, that is, theorem \textbf{3.0.1} holds true for all finite monomials in $\tilde{\mathcal{W}}$ of cardinality $\leq n - 1$. Also, in the rest of this paper, (unless otherwise mentioned) we are going to use the same terminology and notation as in the papers \cite{2} and \cite{1}.

**Notation:** Let $\iota$ be the involution map, which was defined in \cite{1}. Let $U$ be a monomial in $\tilde{\mathcal{W}}$ of degree $n$. Arrange $\iota(U)$ in lexicographic order, say, $\iota(U) = \{(a_1, b_1), (a_2, b_2), \ldots, (a_n, b_n)\}$.

Let $F = \{(b_1, a_1), (b_2, a_2), \ldots, (b_{n-1}, a_{n-1})\}$.

That is, $\iota(F) = \{(a_1, b_1), (a_2, b_2), \ldots, (a_{n-1}, b_{n-1})\}$.

Then $b_n \leq b_{n-1}$ and if $b_n = b_{n-1}$, we have $a_{n-1} \geq a_n$.

The element $(b_n, a_n)$ enters into $F$ to make it $U$.

Let $\text{BRSK}(\iota(F)) = (P^{(n-1)}, Q^{(n-1)})$.

Let $p_{ij}$ denote the entry in the $i$-th row and $j$-th column of $P^{(n-1)}$. Similarly, let $q_{ij}$ denote the entry in the $i$-th row and $j$-th column of $Q^{(n-1)}$.

Clearly, $\text{BRSK}(\iota(U)) = (P^{(n-1)}, Q^{(n-1)})$ if $(a_n, b_n)$, the entries of $P^{(n-1)}$ are $\{a_1, \ldots, a_n\}$ and the entries of $Q^{(n-1)}$ are $\{b_1, \ldots, b_n\}$. Let $(P^{(n)}, Q^{(n)})$ denote $(P^{(n-1)}, Q^{(n-1)})$ if $(a_n, b_n)$.

Let $k$ be a positive integer such that elements of depth $k$ exist in the monomial $F$. Let $\{(r_1, c_1), \ldots, (r_p, c_p)\}$ denote the topmost block of $F$ of depth $k$, where the elements of the block are written in non-decreasing order of both row and
column numbers. It then follows from the induction hypothesis that \( c_1 \) is an entry in the first row of \( P^{(n-1)} \).

**Lemma 3.1.1.** The entries in the first row of \( P^{(n-1)} \) which are strictly less than \( c_1 \), are the column numbers of the topmost elements of some blocks of \( F \) of depth \( < k \).

**Proof:** Suppose not. Say, some \( p_{1j'} \) is the column number of the topmost element of some block \( B \) of \( F \) of depth \( \geq k \). Say, the first element of the block \( B \) is \( (R, p_{1j'}) \).

Case (i) : \( R > r_1 \)
In this case, \((r_1, c_1)\) will have depth \( > k \) in \( F \), a contradiction.

Case (ii) : \( R = r_1 \)
In this case, \((r_1, c_1)\) will either have depth \( > k \) (a contradiction) or both \((r_1, c_1)\) and \((R, p_{1j'})\) have depth equal to \( k \). But in the later situation, \((r_1, c_1)\) cannot be the first element in the topmost block of \( F \) of depth \( k \), again a contradiction.

Case (iii) : \( R < r_1 \)
If \((R, p_{1j'})\) is of depth \( k \), then clearly \((r_1, c_1)\) cannot be the first element of the topmost block of \( F \) of depth \( k \).
If \((R, p_{1j'})\) has depth \( s > k \), then \( \exists \) a \( v \)-chain of length \( s \) having tail \((R, p_{1j'})\).
Say, the \( v \)-chain is \((e_1, f_1) \rightarrow (e_2, f_2) \rightarrow \ldots \rightarrow (e_s, f_s) = (R, p_{1j'})\). Then \((e_k, f_k)\) will have depth \( k \) in \( F \).
If \( e_k \leq r_1 \), then actually \((r_1, c_1)\) is not the topmost element of the topmost block of \( F \) of depth \( k \), a contradiction.
If \( e_k > r_1 \), then \((e_k, f_k) \rightarrow (r_1, c_1)\), a contradiction to the depth of \((r_1, c_1)\) in \( F \).
\( \square \)

**Lemma 3.1.2.** Let \((b_n, a_n)\) enter into the monomial \( F \) to make it \( U \) in such a way that the singleton set \( \{(b_n, a_n)\} \) is the topmost block of \( U \) of depth \( k \), and the next block of \( U \) of depth \( k \) from the top being \( \{(r_1, c_1), (r_2, c_2), \ldots, (r_p, c_p)\} \). Then the entries in the first row of \( P^{(n-1)} \) which are strictly less than \( b_n \) are all strictly less than \( a_n \).

**Proof:** It follows from the hypothesis of this lemma that \( b_n < c_1, b_n < r_1 \leq r_2 \leq \cdots \leq r_p \) and \( a_n < c_1 \leq c_2 \leq \cdots \leq c_p \). It also follows from the induction hypothesis of theorem 3.1.1 that the first row of \( P^{(n-1)} \) contains the smallest column numbers of each block of \( F \). In particular, it contains the entry \( c_1 \).

We will prove this lemma by method of contradiction. Suppose the conclusion of this lemma does not hold. Say, some entry \( p_{1j_0} \) of the first row of \( P^{(n-1)} \) which is strictly less than \( b_n \) is \( \geq a_n \). Then we have \( a_n \leq p_{1j_0} < b_n < c_1 \). By induction hypothesis we know that \( p_{1j_0} \) is the smallest column number of some block of \( F \), say block \( D \). Since \( p_{1j_0} < c_1 \), therefore (by lemma 3.1.1), the block \( D \) has depth \( \leq k \). Say, \( D \) is a block of depth \( s(\leq k) \).
Now, since \((b_n, a_n)\) is of depth \( k \) in \( U \), therefore there exists an element \((R, C)\) in \( U \) of depth \( s(\leq k) \) such that \((R, C)\) and \((b_n, a_n)\) form a \( v \)-chain. That is , \( b_n < R \) and \( C < a_n \).
Say, \((R, C)\) lies in the block \( D \) of \( U \). Clearly then , \( D \neq D \) (because the smallest column number of the block \( D \) is \( p_{1j_0} \), which is \( \geq a_n \)).
Let \((\hat{R}, \hat{C})\) be the bottom-most element of the block \( D \). Then \( R \leq \hat{R} \) and
\[ C \leq \hat{C}. \]

Hence, we have \( \hat{R} \geq R > b_n > P_{ij_n} \Rightarrow \hat{R} > P_{ij_n} \). That is, \( \mathcal{B} \) and \( \mathcal{D} \) are not two different blocks of depth \( s \), a contradiction. \( \square \)

**Definition 3.1.1.** Let \( \{(R_1, C_1), (R_2, C_2), \ldots, (R_p, C_p)\} \) be a block \( \mathcal{B} \) of some finite monomial \( \mathcal{G} \) of \( \mathbb{R}^n \). Let \( b_0 \leq R_1 \) be such that \( b_0 \in [2d] \setminus v \) and \( b_0 > C_1 \). Let \( a_0 \in v \) be such that \( a_0 \leq C_1 \). Then we say that, \( \{(a_0, a_0), (R_1, C_1), \ldots, (R_p, C_p)\} \) is a left concatenation of \( \mathcal{B} \) by \( (b_0, a_0) \).

**Lemma 3.1.3.** Let \( \{(r_1, c_1), \ldots, (r_p, c_p)\} \) be the topmost block of \( F \) of depth \( k \). Let \( (b_n, a_n) \) enter into the monomial \( F \) to make it \( U \) in such a way that \( \{(b_n, a_n), (r_1, c_1), \ldots, (r_p, c_p)\} \) becomes the topmost block of \( U \) of depth \( k \). Let \( m \) be a positive integer as given in the statement of theorem 3.1. Then \( \exists \) an integer \( k' \) where \( 0 \leq k' \leq m - 1 \) such that:

(i) For each \( t \in \{0, 1, \ldots, k'\} \), all the blocks of \( U^{(t)} \) except one are the same as the blocks of \( F^{(t)} \). The one block of \( U^{(t)} \) that is different, is in fact, a left-concatenation of a block of \( F^{(t)} \) by \( (b_n, \ast) \), where \( \ast \) is some entry of \( v \) which is \( \geq a_n \).

(ii) The set of all blocks of \( U^{(k'+1)} \) is equal to the set of all blocks of \( F^{(k'+1)} \) union one more block, which is of the form \( \{(b_n, \ast)\} \), where \( \ast \) is some entry of \( v \) which is \( \geq a_n \).

(iii) For each \( t \in \{k'+2, \ldots, m\} \), the set of all blocks of \( U^{(t)} \) is the same as the set of all blocks of \( F^{(t)} \).

**Proof:** Clearly all blocks of \( U \) except one (namely, \( \{(b_n, a_n), (r_1, c_1), \ldots, (r_p, c_p)\} \)) are the same as all blocks of \( F \). And the block \( \{(b_n, a_n), (r_1, c_1), \ldots, (r_p, c_p)\} \) is a left concatenation of the block \( \{(r_1, c_1), \ldots, (r_p, c_p)\} \) of \( F \) by \( (b_n, a_n) \). Let \( \mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_{a_n} \) denote the all other blocks of \( U \) or \( F \) (they are the same!). \( U^{(1)} \) contains the elements \( (b_n, c_1), (r_1, c_2), \ldots, (r_{p-1}, c_p) \), and all other elements of \( U^{(1)} \) are given by \( \mathcal{B}_1', \ldots, \mathcal{B}_{a_n}' \). If \( \{(b_n, c_1)\} \) is a single block in \( U^{(1)} \), then \( k' = 0 \). Now suppose \( \{(b_n, c_1)\} \) is not a single block in \( U^{(1)} \), then \( U^{(1)} \) and \( F^{(1)} \) differ only by one element, namely \( (b_n, c_1) \) is the extra element that \( U^{(1)} \) contains. Moreover, \( (b_n, c_1) \) is the only element on \( U^{(1)} \) having the lowest possible row number \( b_n \). So, the blocks of \( U^{(1)} \) and \( F^{(1)} \) are essentially the same except one block, which is a left-concatenation of a block of \( F^{(1)} \) by \( (b_n, c_1) \). Let that block of \( U^{(1)} \) be \( \{(b_n, c_1), (r_1, c_2), \ldots, (r_{k'-1}, c_i)\} = \mathcal{C} \).

Then \( \mathcal{C}' = \{(b_n, \tilde{c}_2), (r_1, \tilde{c}_3), \ldots, (r_{k'-2}, \tilde{c}_i)\} \), where \( c_1 \leq \tilde{c}_2 \).

Now, \( U^{(2)} \) contains the elements of \( \mathcal{C}' \). If \( \{(b_n, \tilde{c}_2)\} \) is a single block in \( U^{(2)} \), take \( k' = 1 \). Otherwise proceed similarly.

This process will stop at a stage \( k' \leq m - 1 \), because \( U \) is a finite monomial, and at some stage, we will surely get a block consisting of a single element \( \{(b_n, \ast)\} \), where \( \ast \) is some entry of \( v \) which is \( \geq a_n \).

And after the \( (k'+1) \)-th stage, again the blocks of \( U^{(t)} \) and \( F^{(t)} \) will remain the same. \( \square \)

**Remark 3.1.0.1.** Lemma 3.1.3 above simply means that in the process \((P^{(n-1)}, Q^{(n-1)}) \leftarrow (a_n, b_n)\), there is bumping in \( P^{(n-1)} \) upto the \( k' \)-th stage.
At the \((k^1 + 1)\)-th stage, the bumping stops and \(b_n\) is placed in a new box in some row of \(Q^{(n-1)}\). The remaining entries of \(P^{(n-1)}\) and \(Q^{(n-1)}\) will remain unchanged.

**Lemma 3.1.4.** Let \(b_n < b_i\) for all \(i \in \{1, \ldots, n-1\}\). Let \(\{(r_1, c_1), \ldots, (r_p, c_p)\}\) be the topmost block of \(F\) of depth \(k\). Let \((b_n, a_n)\) enter into the monomial \(F\) to make it \(U\) in such a way that \(\{(b_n, a_n), (r_1, c_1), \ldots, (r_p, c_p)\}\) becomes the topmost block of \(U\) of depth \(k\). Let \((P^{(n)}, Q^{(n)}) = BRSK (U) = (P^{(n-1)}, Q^{(n-1)}) \leftarrow (a_n, b_n)\).

Then \(a_n\) is an entry in the first row of \(P^{(n)}\) and \(r_p\) is an entry of the first row of \(Q^{(n)}\). \(a_n\) bumps \(c_1\) from the first row of \(P^{(n-1)}\), \(c_1\) is placed in the second row of \(P^{(n-1)}\). It either bumps an entry from the 2nd row of \(P^{(n-1)}\) or it does not bump anything. And the bumping by \(c_1\) happens iff \(\exists \) a block in \(U^{(1)}\) having \(c_1\) as the least column number (and having at least two elements). This process of bumping continues up to a finite stage until \(\{(b_n, \star)\}\) is a single block, where \(\star\) is some entry of \(v\) which is \(\geq a_n\). At this stage, \(\{(b_n, \star)\}\) is a single block, \(\star\) is placed in a new box in some row of \(P^{(n-1)}\) and \(b_n\) is placed in a new box in the corresponding row of \(Q^{(n-1)}\). All other entries of \(P^{(n-1)}\) and \(Q^{(n-1)}\) remain unchanged in the same rows.

**Proof:** By induction hypothesis, the first row of \(P^{(n-1)}\) contains the smallest column number of each block of \(F\). In particular, it contains the entry \(c_1\).

And the first row of \(Q^{(n-1)}\) contains the entry \(r_p\). In the process \(P^{(n-1)} \leftarrow (a_n, b_n)\) of bounded insertion, firstly all entries of \(P^{(n-1)}\) which are \(\geq b_n\) are removed. Since \(c_1 < b_n\) therefore \(c_1\) is not removed.

**Claim:** \(a_n\) bumps \(c_1\) from the first row of \(P^{(n-1)}\).

**Proof of the claim:** Clearly \(a_n \leq c_1\). It suffices to show that all the entries in the first row of \(P^{(n-1)}\) which are \(\leq c_1\) are also strictly less than \(a_n\).

Suppose not. Say, \(p_{1j_0}\) is an entry in the first row of \(P^{(n-1)}\) such that \(a_n \leq p_{1j_0} < c_1\).

By **Lemma 3.1.1** it follows that \(p_{1j_0}\) is the column number of the topmost element of some block \(C\) of \(F\) of depth \(s < k\).

Say, the first (or the topmost) element of the block \(C\) is \((R, p_{1j_0})\). Now, since \((b_n, a_n)\) is an element of depth \(k\) in \(U\) and \(s < k\), therefore \(\exists \) an element \((e_s, f_s)\) in \(U\) of depth \(s\) such that \((e_s, f_s) \geq (a_n, b_n)\).

Clearly then \(e_s > b_n\) and \(f_s < a_n \leq p_{1j_0}\). Also \(e_s \leq R\), because if \(e_s > R\), then \((e_s, f_s) > (R, p_{1j_0})\), a contradiction to the depth of \((R, p_{1j_0})\) in \(U\).

Hence we have \(b_n < e_s \leq R\) and \(f_s < p_{1j_0}\). Now \((e_s, f_s)\) and \((R, p_{1j_0})\) are two elements in \(U\) of depth \(s\). Also \(e_s > b_n > c_1 > p_{1j_0}\). Which implies \(e_s > p_{1j_0}\), that is, \((e_s, f_s)\) belongs to the same block \(C\) as \((R, p_{1j_0})\).

Since \(e_s \leq R\) and \(f_s < p_{1j_0}\), we get a contradiction to the fact that \((R, p_{1j_0})\) is the topmost element of the block \(C\).

Hence the claim.

Now, since \(a_n\) bumps \(c_1\) from the first row of \(P^{(n-1)}\), therefore \(a_n\) is placed at the position of \(c_1\) in the first row of \(P^{(n)}\).

Also \(c_1\) gets inserted in the 2nd row of \(P^{(n-1)}\).

Again, it is clear that \(\exists\) a block of \(U^{(1)}\) whose smallest column number is \(c_1\), because \(b_n < b_i\) \(\forall i \in \{1, \ldots, n-1\}\) and \((b_n, c_1)\) belongs to \(U^{(1)}\).

If \(c_1\) does not bump anything from the 2nd row of \(P^{(n-1)}\), that means that \(\{(b_n, c_1)\}\) is itself a block of \(U^{(1)}\). In this case, \(b_n\) is placed in a new box in the
Let $BRSK(\mathcal{U})$ be the topmost block of $\mathcal{U}$. Then by lemma 3.1.2, we know that the entries in the first row of $\mathcal{U}$ are in lexicographic order, say, $(a_1, b_1, \ldots, a_n, b_n)$. We will prove the theorem by induction on $n$. Suppose the theorem is true for all monomials in $\mathcal{R}^w$ of degree $\leq n-1$. Arrange $\iota(\mathcal{U})$ in lexicographic order, say, 

$\iota(\mathcal{U}) = \{(a_1, b_1), (a_2, b_2), \ldots, (a_n, b_n)\}$.

Let $F = \{(b_1, a_1), (b_2, a_2), \ldots, (b_{n-1}, a_{n-1})\}$. That is $\iota(F) = \{(a_1, b_1), (a_2, b_2), \ldots, (a_{n-1}, b_{n-1})\}$. Then $b_n \leq b_{n-1}$ and if $b_n = b_{n-1}$, we have $a_{n-1} \geq a_n$.

The following cases arise:

(i) $b_n < b_{n-1}$ and $a_{n-1} > a_n$.

(ii) $b_n < b_{n-1}$ and $a_{n-1} = a_n$.

(iii) $b_n < b_{n-1}$ and $a_{n-1} < a_n$.

(iv) $b_n = b_{n-1}$ and $a_{n-1} = a_n$.

(v) $b_n = b_{n-1}$ and $a_{n-1} > a_n$.

(b) enters into $F$ to make it $\mathcal{U}$.

**Case (i)** $b_n < b_{n-1}$ and $a_{n-1} > a_n$:

Since $b_n < b_{n-1}$, so $(b_n, a_n)$ cannot change the depth of any element of $F$. Again since $a_{n-1} > a_n$ and $(a_1, b_1, \ldots, (a_n, b_n))$ are in lexicographic order, therefore $(b_n, a_n)$ cannot make a new block of higher depth (that is, if the maximum possible depth of any element of $F$ is $t$, then it is not possible that the maximum possible depth of any element in $\mathcal{U}$ becomes strictly bigger than $t$). So the only thing that can happen is that the element $(b_n, a_n)$ gets added to $F$ as the topmost element of some block. Hence after $(b_n, a_n)$ enters into $F$, only two things can happen: Either $(b_n, a_n)$ becomes the topmost block of $\mathcal{U}$ of depth $k$ containing a single element or $(b_n, a_n, (r_1, c_1), \ldots, (r_p, c_p))$ becomes the topmost block of $\mathcal{U}$ of depth $k$, where $(r_1, c_1, \ldots, (r_p, c_p))$ is the topmost block of $\mathcal{U}$ of depth $k$.

**Subcase (a):** Suppose the singleton set $(b_n, a_n)$ becomes the topmost block of $\mathcal{U}$ of depth $k$:

$$(b_n, a_n)$$ is the topmost block of $\mathcal{U}$ of depth $k$, the next block of $\mathcal{U}$ of depth $k$ from the top being $\{(r_1, c_1), (r_2, c_2), \ldots, (r_p, c_p)\}$. Then $b_n < c_1$, $b_n < r_1 \leq r_2 \leq \ldots \leq r_p$ and $a_n < c_1 \leq c_2 \leq \ldots \leq c_p$. Let $BRSK(\iota(F)) = (P^{(n-1)}, Q^{(n-1)})$. Let $p_{ij}$ denote the entry in the $i$-th row and $j$-th column of $P^{(n-1)}$. Similarly $q_{ij}$.

Clearly, $BRSK(\iota(\mathcal{U})) = (P^{w-1}, Q^{w-1}) \leftarrow (a_n, b_n)$.

By induction hypothesis, it follows that the first row of $P^{(n-1)}$ contains the smallest column numbers of each block of $F$. In particular, it contains the entry $c_1$. Then by lemma 3.1.2, we know that the entries in the first row of $P^{(n-1)}$ which are strictly less than $b_n$ are all strictly less than $a_n$. Hence we are done.

**Subcase (b):** $(b_n, a_n, (r_1, c_1), \ldots, (r_p, c_p))$ becomes the topmost block of $\mathcal{U}$ of depth $k$.
Then we have $b_n > c_1$, $b_n < r_1 \leq r_2 \leq \ldots \leq r_p$ and $a_n \leq c_1 \leq c_2 \leq \ldots \leq c_p$.

Let $(P^{(n)}, Q^{(n)}) = BRSK_d(U) = (P^{(n-1)}, Q^{(n-1)}) \leftarrow (a_n, b_n)$.

By lemma \[3.1.4\] we are done.

**Case (ii) $b_n < b_{n−1}$ and $a_{n−1} = a_n$:**

Since $(b_1, a_1), \ldots, (b_n, a_n)$ are in lexicographic order, therefore $(b_{n−1}, a_{n−1})$ is the topmost element of the topmost block of $F$ of some depth, say $k$. In this case, since $(b_1, a_1), \ldots, (b_n, a_n)$ are in lexicographic order, $b_n < b_{n−1}$ and $a_{n−1} = a_n$, therefore $(b_n, a_n)$ can only get added to the block of $F$ whose topmost element is $(b_{n−1}, a_{n−1})$. Also since $b_n < b_{n−1}$ and $a_{n−1} = a_n$, therefore $(b_n, a_n)$ will be the topmost element of that block.

Now if $(b_{n−1}, a_{n−1})$ had been a single block of $F$ of depth $k$, then $(b_n, a_n)$, $(b_{n−1}, a_{n−1})$ will be the first block of $U$ of depth $k$. Then the proof of the theorem is similar to the proof of subcase (b), case(i).

And if $(b_{n−1}, a_{n−1}), (r_1, c_1), \ldots, (r_p, c_p)$ is the topmost block of $F$ of depth $k$, then $(b_n, a_n), (b_{n−1}, a_{n−1}), (r_1, c_1), \ldots, (r_p, c_p)$ will be the topmost block of $U$ of depth $k$. Then again the proof of the theorem is similar to the proof of subcase (b), case(i).

**Case (iii) $b_n < b_{n−1}$ and $a_{n−1} < a_n$:**

Since $b_n < b_i \forall i \in \{1, \ldots, n−1\}$, therefore $(b_n, a_n)$ cannot change the depth of any element of $F$. So only two things can happen, which are given in the subcases (a) and (b) below:

**Subcase (a):** $(b_n, a_n)$ gets added to a block of $F$ for some depth $k$ as the topmost element.

**Proof:** Same as in case (i).

**Subcase (b):** The singleton set $(b_n, a_n)$ becomes the topmost block of depth $(k + 1)$ in $U$, where $k$ is the maximum possible depth of any element of $F$.

**Proof:** In this subcase, we will have to show that $a_n$ is strictly greater than all entries in the first row of $P^{(n−1)}$ which are $\leq b_n$.

Suppose not.

Then $\exists$ an entry (call it $p_{1j_0}$) in the first row of $P^{(n−1)}$ which is such that

\[a_n \leq p_{1j_0} < b_n\]

Clearly then, $p_{1j_0}$ is the smallest column number of some block $\mathcal{B}$ of $F$, say of depth $s$, where $s \leq k$.

Let $(R, p_{1j_0})$ denote the topmost element of the block $\mathcal{B}$.

**Claim 1:** The block $\mathcal{B}$ cannot be the topmost block of depth $s$ in $F$.

**Proof of the claim 1:** Suppose not. Say $\mathcal{B}$ is the topmost block of depth $s$ in $F$. Since $(b_n, a_n)$ has depth $k + 1$ in $U$ and $s < k + 1$, therefore $\exists$ an element $(\beta, \alpha)$ in $U(n−1)$ such that $(\beta, \alpha)$ has depth $s$ in $F$ and $(\beta, \alpha) > (b_n, a_n)$ is a $v$-chain. Then $b_n < \beta$ and $a_n > \alpha$. The element $(\beta, \alpha)$ of $F$ lies in some block of $F$ of depth $s$. Since $\mathcal{B}$ is the topmost block of depth $s$ in $F$ and $p_{1j_0}$ is the topmost column number of $\mathcal{B}$, therefore we must have $p_{1j_0} \leq \alpha$. But then we get $p_{1j_0} \leq \alpha < a_n$, which contradicts $(\ast)$.

Hence Claim 1 is proved.

**Claim 2:** $\exists$ an element $(e_s, f_s)$ in the topmost block of depth $s$ in $F$ such that $(e_s, f_s) > (b_n, a_n)$ is a $v$-chain.

**Proof of claim 2:** Since $(b_n, a_n)$ is of depth $k + 1$ in $U$ and $s < k + 1$, therefore $\exists$ an element $(e_s, f_s)$ of depth $s$ in $F$ such that $(e_s, f_s) > (b_n, a_n)$ is a $v$-chain. It suffices to show that $(e_s, f_s)$ belongs to the topmost block of depth $s$ in $F$. 

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Suppose not. Let $\mathcal{B}$ denote the block of $F$ of depth $s$ where $(e_s, f_s)$ lies. Let $\mathcal{C}$ denote the topmost block of $F$ of depth $s$. Let $(b, a)$ denote the bottommost element of $\mathcal{C}$. Then since $\mathcal{B} \neq \mathcal{C}$, we have $b \leq f_s$. Now since $(e_s, f_s) > (b_n, a_n)$, we have $f_s < a_n$. So we get $b \leq f_s < a_n$, which implies $(b, a_n) \notin \mathcal{R}^v$. But $b_n < b$ (since $b_n < b_i$ for all $i \in \{1, \ldots, n-1\}$). So, we get $b_n < a_n$, a contradiction to the fact that $(b_n, a_n) \in \mathcal{R}^v$.

Hence Claim 2 is proved.

**Assuming claim 1 and claim 2:** It follows from claim 2 that $\exists (e_s, f_s)$ in the topmost block of depth $s$ in $F$ such that $(e_s, f_s) > (b_n, a_n)$ is a $v$-chain. $\Rightarrow b_n < e_s$. Now, $\star$ says that $a_n \leq p_{1j_n} < b_n$. This together with $b_n < e_s$ imply that $p_{1j_n} < e_s$. But this implies the two elements $(e_s, f_s)$ and $(R, p_{1j_n})$ of depth $s$ lie in the same block, which is a contradiction to claim 1.

**Case (iv) $b_n = b_{n-1}$ and $a_n = a_{n-1}$:**

In this case $(b_n, a_n)$ will be added in the block of $(b_{n-1}, a_{n-1})$ and it will be the first element of that block. The rest follows similarly as in subcase (b) of case (i).

**Case (v) $b_n = b_{n-1}$ and $a_{n-1} > a_n$:**

Since $b_n = b_{n-1}$ and $a_{n-1} > a_n$, therefore $(b_n, a_n)$ cannot make a new block of higher depth. Again since $b_n \leq b_i \ \forall i \in \{1, \ldots, n-1\}$, therefore $(b_n, a_n)$ cannot change the depth of any element of $F$. So the only thing that can happen is that the element $(b_n, a_n)$ gets added to $F$ as the topmost element of some block (of some depth $k$ say). In this case also, the proof is similar to the proof of case (i). □

**References**

[1] Kreiman, V.: *Local properties of Richardson varieties in the Grassmannian via a bounded Robinson-Schensted-Kunth correspondence*, J Algebra Comb (2008) 27: 351–382.

[2] Kodiyalam, V., Raghavan, K.N.: *Hilbert functions of points on Schubert varieties in Grassmannians*, J. Algebra 270(1), 28–54 (2003).

[3] Ghorpade, S., Raghavan, K.N.: *Hilbert functions of points on Schubert varieties in the symplectic Grassmannians*, Trans. Am. Math. Soc. 358(12), 5401–5423, 2006.

[4] Upadhyay, S.: *Initial ideals of tangent cones to the Richardson varieties in the Orthogonal Grassmannian*, International Journal of Combinatorics, Hindawi Publishing Corporation, Volume 2013, Article ID 392437, 19 pages.