Many-body localization in one dimension as a dynamical renormalization group fixed point

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We formulate a dynamical real space renormalization group approach to describe the time evolution of a random spin-1/2 chain, or interacting fermions, initialized in a state with fixed particle positions. Within this approach we identify a many-body localized state of the chain as a dynamical infinite randomness fixed point. Near this fixed point our method becomes asymptotically exact, allowing analytic calculation of time dependent quantities. In particular we explain the striking universal features in the growth of the entanglement seen in recent numerical simulations: unbounded logarithmic growth delayed by a time inversely proportional to the interaction strength. The particle number fluctuations by contrast exhibit much slower growth as log log t indicating blocked particle transport. Lack of true thermalization in the long time limit is attributed to an infinite set of approximate integrals of motion revealed in the course of the RG flow, which become asymptotically exact conservation laws at the fixed point. Hence we identify the many-body localized state with an emergent generalized Gibbs ensemble.

What is the effect of interactions on Anderson localization? One common wisdom is that any amount of interaction will give rise to collective excitations that could assist transport at non-vanishing temperature even if single particle states are all localized. But the belief, that there are no strict many-body insulators at $T > 0$, has been challenged by theoretical arguments, dating as far back as Anderson’s original paper, which suggest a many-body localization transition marking a critical point in the transport properties of a closed quantum system[1, 2]. The idea has recently gained support from numerical studies[3–6]. Furthermore, simulations of one dimensional systems have revealed remarkably universal behavior of the dynamics in the putative many-body localized state[7–9]. For example the time evolution following a quench from a state with fixed particle positions shows blocked particle transport accompanied by unbounded logarithmic growth of the entanglement entropy.

In this paper we formulate a real space renormalization group (RG) scheme that describes the time evolution of a random spin chain on multiple time scales. For certain initial conditions we can establish a many-body localized state at an infinite randomness fixed point of the dynamics near which the RG scheme is asymptotically exact. The results of this theory exhibit many of the universal features observed in the numerical simulations[7–9], although they are derived for a somewhat different model. As our starting point we consider the Hamiltonian of the random spin-1/2 XXZ chain without local Zeeman fields:

$$H = \sum_i \frac{J_i}{2} \left( S_i^+ S_{i+1}^- + S_i^- S_{i+1}^+ + 2\Delta_i S_i^z S_{i+1}^z \right).$$

The couplings $J_i$ and anisotropy parameters $\Delta_i$ on sites $i$ are random variables drawn from uncorrelated probability distributions. The couplings may be positive or negative and we assume $|\Delta_i| < 1$. The spins in (1) can be mapped using a Jordan-Wigner transformation to spinless fermions with nearest neighbor interactions, subject to bond disorder.

To study the propagation of information through the chain we investigate the time evolution of the system starting from a non entangled initial state, which for simplicity we take as an antiferromagnetic Neél state with spins pointing along the z-axis. We shall see that this choice of initial state greatly simplifies the scheme and allows us to obtain well controlled results for the dynamics at long times.

**RG scheme** – Before proceeding with the details, let us outline the idea underlying the RG solution of the time evolution. As in the standard strong disorder RG scheme[10–12], we take advantage of the large local separation of energy scales induced by the randomness to gradually eliminate degrees of freedom. However, instead of targeting the ground state of the system, our aim is the time evolution of the chain starting from a specified initial state. The dynamics at the shortest time scales are oscillations of frequency $\Omega$ performed by the most strongly coupled pairs of spins on the chain, which are effectively decoupled on these time scales from their typically much slower neighbors. On time scales longer than $\Omega^{-1}$ the rapid oscillations performed by the strong bonds can be eliminated using time dependent perturbation theory, which leads to renormalization of the coupling between the slow spins. In this way we gain the essential information about the dynamics of the chain at all scales. If the distribution of coupling constants flows to a wide distribution at the dynamical fixed point, then the perturbative RG approach becomes increasingly well controlled, or even asymptotically exact if the system flows to infinite randomness[12]. Similar ideas have been applied to solve classical dynamics in certain disordered[13, 14] as well as clean[15] systems, but to our knowledge not to quantum dynamics.

We now apply this scheme to the Model (1) with a staggered (Neél) initial state. The strong bond, with ex-
change coupling denoted by $\Omega$, then always connects a pair of anti-aligned spins. At time scales of order $\Omega^{-1}$ the strong pair is effectively decoupled from the neighboring spins that are presumably connected to it by much weaker couplings. The only significant dynamics in this neighborhood of the chain is the rapid oscillation of the strong pair with frequency $\Omega$ between the $|\uparrow\downarrow\rangle$ and $|\downarrow\uparrow\rangle$ states.

The effective Hamiltonian for the dynamics at time scales much larger than $\Omega^{-1}$ is derived using time-dependent perturbation theory in the coupling of the strong pair to the rest of the chain. Specifically, we move to the interaction picture with respect to the Hamiltonian of the strong pair and compute the evolution of the density matrix $\rho(t) = U(t)^{\dagger} \rho_0 U(t)$, where $|\psi_0\rangle \langle \psi_0|$ is the initial state of the strong pair, up to second order, while averaging over rapid oscillations of frequency $\Omega$. The resulting time evolution can be matched term by term to an effective time evolution $\exp(-iH_{\text{eff}}(t)) |\psi_0\rangle \langle \psi_0| \exp(iH_{\text{eff}} t)$. Therefore this procedure amounts to derivation of an effective Hamiltonian.

At this order of perturbation theory the $|\uparrow\downarrow\rangle$ and $|\downarrow\uparrow\rangle$ are not populated and therefore truncated from the Hilbert space. The retained states $|\pm\rangle = 2^{-1/2}(|\uparrow\downarrow\rangle \pm |\downarrow\uparrow\rangle)$ of the strong pair can be taken as the $\uparrow/\downarrow$ states of a new pseudo spin variable $\hat{S}_n$, which initially points along the positive or negative $x$-axis. The effective Hamiltonian derived in this procedure for our model is then given by:

$$H_{\text{eff}} = H_{\text{chain}} + h_n S_n^z + \frac{J_L J_R}{2\Omega(1 - \Delta_S^2)} \left( S_L^+ S_R^- + S_L^- S_R^+ \right) + \frac{\Delta_S J_L J_R}{2\Omega} \left[ \frac{1}{1 - \Delta_S^2} \left( S_L^+ S_R^- + S_L^- S_R^+ \right) - \frac{\Delta_L \Delta_R}{\Delta_S} S_L^z S_R^z \right] S_n^z$$

Because $[H_{\text{eff}}, S_n^z] = 0$ the time evolution can be computed separately for each eigenvalue $\pm \frac{1}{2}$ of $S_n^z$, using $H_{\text{eff}}^{\pm}$ that does not depend on the operator $S_n^z$. The different evolution under $H_{\text{eff}}^\pm$ together with the fact that the new spin starts in a superposition of $\uparrow$ and $\downarrow$ leads to entanglement between the effective-spin on the strong bond and its two neighboring spins. Full entanglement is generated after a time $t_{\text{ent}} = 2\Omega/(J_L J_R \Delta_S)$, set by the difference in the exchange constant in $H_{\text{eff}}^\pm$. This process will be important later on for computing the evolution of the entanglement entropy.

Apart from generating entanglement, the difference between the evolution given $\uparrow_n$ or $\downarrow_n$ is not crucial for the subsequent dynamics in the sense that they both lead to the same recipe for renormalization of coupling constants. $H_{\text{eff}}^+$ and $H_{\text{eff}}^-$ have the same form as the original Hamiltonian and we can directly read off the coupling generated between $\hat{S}_L$ and $\hat{S}_R$, neighboring the strong bond to the left and right, upon decimation of that bond: $J = J_L J_R/\Omega$ and $|\Delta| = |\Delta_L| |\Delta_R|/4$, where we neglected the linear $\Delta$ correction to $J$. This approximation will be justified a posteriori by the fact that $\Delta$ flows to zero. The renormalization of the exchange coupling is then identical to that found in the random Heisenberg chain at $T = 0$ and leads to the Random singlet phase[10–12]. Note also that we keep only the absolute value of the anisotropy. The sign will randomize in the course of the RG flow because it depends on the state of $S_n^z$. These RG steps are now iterated as we gradually approach lower frequency scales and longer times.

The RG steps are iterated to produce a flow of the probability distributions with decreasing frequency cutoff $\Omega$ starting from the microscopic cutoff $\Omega_0$. Using the scaling variables $\zeta = \ln \frac{\Omega}{\Omega_0}$ and $\beta = -\ln |\Delta|$, and $\Gamma = \ln(\Omega_0/\Omega) = \ln(\Omega_0 t)$ we obtain the following equation for the joint probability distribution $P(\zeta, \beta; \Gamma)$:

$$\frac{\partial P}{\partial \Gamma} = \frac{\partial P}{\partial \zeta} + \rho(0; \Gamma) \int_0^\infty d\beta \lambda_\delta(\zeta - \zeta_L - \zeta_R) \delta(\beta - \beta_L - \beta_R - \ln 4) P(\zeta_L, \beta_L; \Gamma) P(\zeta_R, \beta_R; \Gamma).$$

(3)

By integrating over $\beta$ we obtain an equation for $\rho(\zeta; \Gamma)$,

$$\frac{\partial \rho}{\partial \Gamma} = \frac{\partial \rho}{\partial \zeta} + \rho(0; \Gamma) \int_0^\infty d\zeta_L d\zeta_R \delta(\zeta - \zeta_L - \zeta_R) \rho(\zeta_L) \rho(\zeta_R).$$

(4)

This equation is identical to the flow leading to the random-singlet ground state and it is solved by the same ansatz[12], $\rho(\zeta; \Gamma) = a(\Gamma) e^{-a(\Gamma) \zeta}$ with

$$a(\Gamma) = \frac{1}{\Gamma + 1/\alpha_0}.$$

(5)

Of course the above solution includes only partial information on the fixed point of the dynamics. Important information for calculation of physical quantities is held in the conditional average of the interaction variable $\beta$ given a value of $\zeta$ on the same bond, $\bar{\beta}(\zeta, \Gamma) \equiv \int_0^\infty d\beta P(\zeta, \beta; \Gamma)/\rho(\zeta; \Gamma)$. Proceeding as in Ref. [12], we derive the equation for this moment by multiplying Eq.
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determined by the initial condition.

Another important ingredient for calculation of phys-
ical properties is the distance between remaining spins
(i.e. length of decimated clusters) at time $t$. An il-
ustration of such clusters is shown in Fig. 1. Since
the RG flow is the same as in Ref. [12] we similarly
obtain $L_{\Gamma} = (a_0 \Gamma + 1)^2 = [a_0 \ln(\Omega_0 t) + 1]^2$. Which
anes as $\ln^2(\Omega_0 t)$ at long times. From this result we
an immediately infer the decay of the staggered moment
which is simply the fraction of remaining spins at time $t$: $m_s = 1/L(t) = 1/(a_0 \ln(\Omega_0 t) + 1)^2$. It is interesting
to contrast this behavior with the decay of the staggered
moment in the analogous quench of a clean XXZ model.
In that case Ref. [16] found oscillatory decay of the stag-
gered moment (for $\Delta < 1$) with an envelope that decays
exponentially in time.

We note that in the non-interacting case $\Delta = 0$ the
RG flow generalizes in a straight forward way to an ar-itrary Ising initial states rather than a Ne\'el state. If
the strongest exchange coupling happens to be between
spins with the same orientations, this pair will freeze in
their state, while quantum fluctuations lead to an effec-
tive coupling between the neighbors that is the same in
absolute value as that obtained for anti-aligned spins.
Hence the distribution of the absolute values of the ex-
change couplings is governed by the same RG equations.
In the interacting case, such aligned pairs lead to further
complications that are left for later work.

Results – We now use the solutions of the flow equa-
tions to compute the time evolution of important phys-
ical quantities. Specifically to gain information on par-
ticle transport and on thermalization of the system we
consider the growth of the total particle number fluctu-
ation and of the entanglement entropy in a sub-system
Corresponding to half of the chain.

To compute the particle number fluctuations we note that
the total particle number (or $S^z_{\text{tot}}$) within a deci-
mated pair of sites is a conserved quantity in the RG
scheme. Therefore the only contribution to the particle
number fluctuation in one half of the system comes from
decimated pairs that reside on different sides of the in-
terface. Each oscillating pair that cuts the interface adds
$1/8$ to the number fluctuation on time average. Compu-
ting the total particle number fluctuation then amounts
to counting the number of decimated bonds that cuts the
interface. Proceeding exactly as in the random sin-
glet phase[17] we have $N_p \approx \int_0^\infty \! d\nu \alpha(\nu) = \ln (\Gamma + 1/a_0)$.
Hence the particle number fluctuation grows extremely
slowly as $(\delta N^2) = (1/24) \ln(\ln(\Omega_0 t))$ at long times. In-
terestingly this result is independent of the interaction
strength $\Delta$.

We now turn to the growth of the entanglement en-
tropy, between the two halves ($A$ and $B$) of the system,
$S \equiv -\text{tr} \rho_A \log_2 \rho_A$. Consider first the simpler (non-
interacting) case $\Delta = 0$, where no entanglement is gen-
erated between a decimated pair and the rest of the sys-

Then, the only source of entanglement between the
two halves of the system are decimated pairs that cut
the interface, each contributes a time average of
$S_p = 2 - 1/\ln 2 \approx 0.557$. The growth of the entropy
is therefore similar to that of the particle number fluctu-
ation

$$S_0(t) \approx S_p \left( \frac{1}{3} \ln (\ln(\Omega_0 t) + 1/a_0) \right)$$

We can generalize this result (for $\Delta = 0$) to a quench
from an arbitrary Ising state with a fraction $q$ of anti-
aligned neighbors. Because $q$ is an invariant of the RG
and aligned pairs do not contribute to the entropy the
prefactor in (8) changes to $qS_p$.

We turn to our main focus, which is the interacting
system with $\Delta \neq 0$. In this case a pair decimated at
time $t_1$ will eventually get entangled with the neighbor-

spins according to Eq. (2) after a characteristic time
t_{\text{ent}}(t_1) = 2\Gamma_1/(J_0^2 \Delta_1).$ In particular from the time of the
quench, entanglement will be generated by the in-
teraction only after a delay time $t_{\text{delay}} = 2\Omega_0/(J_0^2 \Delta_0) =
2(\Omega_0/J_0)(1/J_0^2)$, where $J_0 \equiv J_0 \Delta_0$ is the typical value of the
bare interaction energy.

The interaction-generated entanglement entropy found
at time $t$ originates from entanglement of pairs elimi-
nated at an earlier time $t_1 = t - t_{\text{ent}}$, which corresponds
to the scaling parameter $\Gamma_1 = \ln \Omega_0 t_1$. To estimate this
contribution to the entropy we recall that spins on the
renormalized chain at time $t_1$ are separated by clusters
of length $L(\Gamma_1)$ of spins decimated at even earlier times
and are oscillating at higher frequencies. By the time $t$
that a pair of spins decimated at $t_1$ entangles with their
neighbors, the spins inside the decimated clusters must
also be entangled with each other. So, it is safe to assume
that by the observation time $t$ entanglement propagates
to a distance $L(\Gamma_1)$ giving rise to entanglement entropy
$S \approx 0.5 L(\Gamma_1) \approx 0.5 (a_0 \Gamma_1 + 1)^2$. The factor 0.5 stems

FIG. 1. Schematic illustration of remaining spins and clusters of decimated pairs in the renormalized chain at time $t$. 

(3) by $\beta$ and then integrating over $\beta$

$$\partial_t \ln (\beta(\zeta)) = \partial_{\zeta} \ln (\beta(\zeta)) + \frac{2a(\Gamma)}{\beta(\zeta)} \int_0^\zeta d\zeta' \beta(\zeta')$$

Note that we have neglected the $\ln 4$ in (3). This is jus-
itified near the fixed point since the typical $\beta$ flows to $\infty$.
After substituting the solution for $\rho(\zeta, \Gamma)$ in Eq. (6) we
find the solution

$$\beta(\zeta) = \frac{1}{b_0} (a_0 \Gamma + 1)^{\phi} \left( 1 + \frac{\zeta \phi}{\Gamma + 1/a_0} \right)$$

where $\phi = (1 + \sqrt{5})/2$ is the golden ratio and $b_0$ is deter-
determined by the initial condition.
from the number of available degrees of freedom: the two states with aligned spins in each decimated pair remain unpopulated and therefore do not contribute to the entropy. To write this as a function of the time $t$ we use the relation between $t$ and $t_1$,

$$t = t_1 + t_{\text{sat}} = t_1 \left(1 + \frac{2\Omega_0^2}{J_LJ_R\Delta_1}\right) \approx t_1 \frac{2\Omega_0^2}{J_LJ_R\Delta_1}. \quad (9)$$

We now take the logarithm of both sides and replace the scaling variables by their appropriate average values $\zeta \rightarrow 1/\alpha(\Gamma_1)$ and $\beta \rightarrow \bar{\beta}(\zeta = 0; \Gamma_1)$. Here the correlations between the random variables came in to play. We needed the average of $\beta$ on the bonds with strongest $J$ ($\zeta = 0$), which is different than the global average of $\beta$.

Using the solutions (5) and (7) for the typical values we find $\Gamma = 3\Gamma_1 + 1/\delta(a_0\Gamma_1 + 1)^{\phi} + 2/a_0 + \ln 2$. This equation should be inverted to express $\Gamma_1$ as a function of $\Gamma$ and used to obtain $S(\Gamma) = 0.5L(\Gamma_1, \Gamma)$. In general, the inversion cannot be done analytically, but we can easily find the limiting regimes. At sufficiently long times the term $\Gamma_1^\phi$ dominates the right hand side and then we have $a_0\Gamma_1 = [b_0(\Gamma - 2/a_0 - \ln 2)]^{\phi} - 1$. On the other hand at shorter times, when the linear term dominates, we get $\Gamma_1 = 1/2(\Gamma - 1/a_0 - 1/b_0 - \ln 2)$.

The crossover time $t^*$ separating the two regimes depends on the initial conditions through the coefficients of the terms $\Gamma_1$ and $\Gamma_1^\phi$. If $b_0 \gg a_0$, that is for stronger disorder in hopping than in the interactions, we have $t^* = t_{\text{delay}} \exp\left[6(3b_0/a_0)^{\phi}/a_0\right]$. In the opposite regime $b_0 \ll a_0$ the term $\Gamma_1^\phi$ dominates from the outset and $t^* = t_{\text{delay}}$. We can now write an expression for the growth of the entanglement entropy valid in the limiting regimes

$$S(t) \approx \frac{1}{2} \left(\frac{\ln(t/t_{\text{delay}})}{\ln(\Omega_0/J_0)} + 1\right)^2 \theta(t - t_{\text{delay}}) \theta(t^* - t)$$

$$+ \frac{1}{2} \left(\frac{\ln(t/t_{\text{delay}})}{\ln(1/\Delta_0)} + 1\right)^{2\phi} \theta(t - t^*) - \frac{1}{2}. \quad (10)$$

Interestingly, Eq. (10) gives the unbounded logarithmic growth of the entanglement entropy seen in the numerical simulations and even the delay of this interaction induced growth by a time that scales as the inverse interaction strength $\Omega_0$. It is important to note however that the numerically simulated model differs from ours by having disordered Zeeman coupling, which may ultimately drive it to a different fixed point [18]. Nevertheless we can attempt a comparison even of the more subtle form of the logarithmic growth by crudely accounting for this difference. Through second order perturbation theory the Zeeman strong disorder generates predominantly strong randomness in the hopping, but not in the interactions. Hence the natural regime to compare the numerics to our model is $a_0 \ll b_0$ where the first line of (10) holds at any reasonable time-scales (i.e. below the exponentially long time $t^*$). Even within this regime Eq. (10) predicts a crossover between growth of the entanglement entropy as $\ln t$ at times $t < (\Omega_0/J_0)^2T_{\text{delay}}$ to growth as $\ln^2 t$ at longer times. The slope of the logarithmic growth $1/\ln(\Omega_0/J_0)$ is completely independent of the interaction strength $\Delta$, consistent with the universality identified in Ref. [9].

To complete the comparison, we note that the growth of particle number fluctuations which was computed above is much slower ($\sim \ln \ln t$) than the growth of the entanglement entropy. Again this is consistent with the numerical results, which implies that particle transport is essentially blocked. Indeed the particle number fluctuations are in general only a lower bound of the entanglement entropy [19].

Having found that the entanglement entropy increases without bound it is natural to ask if this leads to thermalization. To address this issue let us consider the saturation of the entanglement entropy in a finite system, or in a finite sub-system of length $L_s$. Eq. (10) implies that the entropy will approach its maximal value $S_{\text{sat}}$ after a time $t_{\text{sat}} \approx t_{\text{delay}} \exp\left[-\ln(\Delta_0) L_0^{\phi/2}\right]$. Does the saturation value $S_{\text{sat}}$ correspond to a state in thermal equilibrium?

Provided we start from a symmetric distribution of $\Delta_i$ such that $(\Delta_i) = 0$, then the initial Neel state has zero mean energy, exactly in the middle of the many-body energy spectrum. If this state thermalized following the quench, the entanglement entropy would have to saturate to its infinite temperature value of $L$. But as we have pointed out above, the RG flow implies a saturation entropy that is at most half of the infinite temperature value because half of the degrees of freedom remain frozen in the dynamics. This fact is embodied in the infinite set of emergent integrals of motion $I_P = (S_1^2 + S_2^2)_{\rho}$, which account for the fact that a pair of decimated spins, never flip their relative orientation within the perturbative RG scheme. Note that this remains true even for generalized initial states allowing for aligned neighboring spins. In the case of an initial Neel state we also have the particle number on oscillating pairs ($S_1^2 + S_2^2$) as additional integrals of motion. These emergent conservation laws become asymptotically exact for well separated pairs of spins decimated at long times as the perturbative RG scheme becomes asymptotically exact near the infinite randomness fixed point. We conclude that the long time steady state of the chain with non vanishing interaction is characterized by the generalized Gibbs ensemble (GGE) [20], which describes thermalization within a subspace constrained by the values of the emergent integrals of motion $I_P$.

It is interesting to note that the long time steady state attained by the non interacting state $\Delta_i = 0$ is markedly different. The extremely slow increase of the entanglement entropy as $\ln t$ given by Eq. (8) together with the relation between length and time scales
ln Ω_0t = Γ ∼ √L imply saturation of the entanglement entropy to S_∞ ≈ \frac{2}{\phi} ln L. This result, as well as the ln ln t growth of the entropy matches with numerical results obtained for the random transverse field Ising chain [21] that can be similarly described by a model of noninteracting fermions.

So far we have used the RG approach to establish and characterize a many-body localized state. It is interesting to examine the criterion for validity of the RG scheme, which may indicate a transition to a different, possibly thermalizing and delocalized, state. Such a criterion can be obtained from the first term in Eq. (2). The perturbation theory at the base of the renormalization step is valid while \langle J_L J_R / \Omega \rangle \ll \langle J \rangle (1 - Δ_0^2). This can be expressed in terms of the average values of parameters at the beginning of the flow. Using \langle J_0 / Ω_0 \rangle = a_0 / (a_0 + 1), we obtain the criterion a_0 < Δ_0^2 - 1. Recall that increasing a_0 corresponds to decreasing disorder. It is tempting to speculate that for weaker disorder a_0 > Δ^2 - 1 the dynamics will give rise to a normal thermalizing state. However the above requirement may be too strict. Even if the criterion a_0 < Δ_0^2 - 1 is not satisfied initially, this may be corrected in later stages of the flow as Δ independently flows to smaller values. It is therefore also possible that the transition to a different state occurs only for \langle Δ^2 \rangle ≈ 1 independent of the disorder strength.

Conclusion – Using a real space RG scheme formulated in real time, we gave a dynamical description of a many-body localized state in a random spin chain, equivalent to interacting fermions with random hopping. Within this approach the localized state is characterized by a flow to an infinite randomness fixed point. Solution of the flow equations allows us to characterize this state in a rather detailed way. The results are found to be in excellent agreement with recent numerical simulations done on a similar, albeit not identical, model [9].

Particle localization is manifest in the extremely slow growth \sim ln ln t of the particle number fluctuations in half the system that is seen both in the interacting and non-interacting systems. The entanglement entropy S reveals a dramatic difference between the Anderson localized state of noninteracting fermions and the many-body localized state established when interactions are present. In the non-interacting system S grows together with the particle number fluctuation as \langle S(t) \rangle \sim log log t and saturates to a non-extensive value \sim ln L in a finite system. Interactions lead to much faster growth of \langle S(t) \rangle as log^{2/φ} t at long times, but they take effect only after a delay time that scales as the inverse of the interaction strength t_{delay} \sim 1/J^2. Furthermore, the log^{2/φ} t behavior seen in the long time limit is preceded by log t growth up to an intermediate time scale t_{in} \sim t_{delay}(Ω_0 / J_0)^2 ≫ t_{delay}. It is interesting to note that the growth of entanglement as ln^{2/φ} t exceeds the upper bound \sim ln t proved for non-interacting Anderson localized chains [22].

The RG flow toward the infinite randomness fixed point has direct consequences on the equilibration in this system. In a sub-system of length L the entanglement entropy saturates to an extensive value S_∞ \sim L, which is however smaller than it would reach had the system attained true thermal equilibrium. We attribute the lack of thermalization to an infinite set of emergent integrals of motion, which become asymptotically exact conservation laws near the infinite randomness dynamical fixed point. The dynamics of the system can therefore be viewed as thermalization within a GGE characterized by the emergent set of conserved quantities, a possibility suggested in Ref. [23, 24]. Here we demonstrated that such a GGE emerges in a non integrable random system as a dynamical fixed point of the renormalization group and captures the essence of a many-body localized state. The nature of the critical point marking the transition to the normal thermalizing state remains an interesting question for future study as are generalizations of our scheme to more generic disorder models and initial states.

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