Duality, Hidden Symmetry and Dynamic Isomerism in 2D Hinge Structures

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Recently, a new type of duality was reported in some deformable mechanical networks which exhibit Kramers-like degeneracy in phononic spectrum at the self-dual point. In this work, we clarify the origin of this duality and propose a design principle of 2D self-dual structures with arbitrary complexity. We find that this duality originates from the partial central inversion (PCI) symmetry of the hinge, which belongs to a more general end-fixed scaling transformation. This symmetry gives the structure an extra degree of freedom without modifying its dynamics. This results in dynamic isomers, i.e., dissimilar 2D mechanical structures, either periodic or aperiodic, having identical dynamic modes, based on which we demonstrate a new type of wave-guide without reflection or loss. Moreover, the PCI symmetry allows us to design various 2D periodic isostatic networks with hinge duality. At last, by further studying a 2D non-mechanical magnonic system, we show that the duality and the associated hidden symmetry should exist in a broad range of Hamiltonian systems.

Introduction

Space group symmetries are cornerstones of condensed matter physics [1–4]. Nevertheless, physical systems can also have hidden symmetries which cannot be captured by space group theory [5–12]. Some of these hidden symmetries were previously reported at some specific points in the Brillouin zone (BZ) [6, 7, 12]. Recently, a hidden symmetry in the full BZ induced by self-duality was discovered in 2D mechanical isostatic networks [13, 14]. These structures have a low coordination number, which creates many local hinges with unconstrained degrees of freedom [15–19]. By tuning the open angle of hinges $\vartheta$, one can change these structures continuously from an open to a folded state [20–24]. Interestingly, these structures with small $\vartheta$ are dual to the structures with large $\vartheta$, thus there exists a critical $\vartheta^*$, at which the dual counterpart of the structure is itself. At this self-dual point, hidden symmetry emerges, resulting in Kramers-like double degeneracy [24], and many other interesting phenomena, like mechanical non-abelian spintronics [13], the degeneracy of elastic modulus [26], the critically-tilted Dirac cone [14], the symmetric boundary effect [27], topological corner states and Maxwell modes [28, 29] etc. Nevertheless, so far, only a few meticulous structures are found to be self-dual. The origin of the duality and the resulting hidden symmetry remain mysterious [14, 30].

In this work, we illuminate the origin of this duality and propose a design principle of 2D structures with such duality. We find that this duality, which we rephrase as ‘hinge duality’, originates from a special partial central inversion (PCI) symmetry of the hinge, which gives the structure an extra degree of freedom without changing the Hamiltonian. When multiple hinges are connected into hinge chains or networks, the combination of local PCI generates dynamic isomers, i.e., different configurations having exactly the same dynamic modes. Based on the PCI and resulting hinge duality, we also propose design rules to generate arbitrarily complicated 2D periodic structures with self-duality. Furthermore, we show that PCI belongs to a more general end-fixed scaling transformation, which can further broaden the range of dynamic isomers. At last, we also demonstrate the existence of hinge duality in a non-mechanical magnonic system, which suggests this duality is a generic property of hinge structures, existing in a broad range of Hamiltonian systems.

FIG. 1: (a-c): Dual transformation for a single hinge, which is composed by PCI (a $\rightarrow$ b) and a global 90° rotation (b $\rightarrow$ c), with respect to the hinge point. The arrows indicate the vibrational degrees of freedom. (d-g) Dynamic isomers of hinge chain with different hinge sequences.
Duality of a single hinge  We first consider a simple hinge composed by two structurally identical arms a and b, which can freely rotate around the hinge point in dimension $d=2$ [31][32]. Each arm is modelled as a mechanical spring network with $n+1$ nodes. The total degrees of freedom in the system thus is $(2n+1)d$, with the hinge node shared by two arms. Under the harmonic approximation, all nodes do small vibrations around their equilibrium positions, and the time-dependent position of node $i$ in arm a, b can be written as

$$r_i^a(t) = R_i^a + u_i^a(t); \quad r_i^b(t) = R_i^b + u_i^b(t),$$  \hspace{1cm} (1)$$

respectively, where $R_i^a$ and $R_i^b$ are the equilibrium positions, $u_i^a(t)$ and $u_i^b(t)$ are the time-dependent vibrational displacements for nodes $i \in [0,1,2,\ldots,n]$ in arm a and b, respectively. The position of the hinge node is $r_0(t) = r_0^a = r_0^b = R_0 + u_0(t)$. Thus $R_0^a = R_0^b = R_0$ and $u_0^a = u_0^b = u_0$. Since two arms are structurally identical, they are connected by the rotational operation $\hat{R}(\vartheta)$ with $\vartheta$ the open angle between two arms. The Hamiltonian of the system is

$$H = H_k + \sum_{i,j} \frac{\lambda_{ij}}{2} \left\{ \left[ (u_i^a - u_j^a) \cdot e_{ij}^a \right]^2 + \left[ (u_i^b - u_j^b) \cdot e_{ij}^b \right]^2 \right\}$$  \hspace{1cm} (2)$$

where $H_k$ represents the kinetic energy, $\lambda_{ij}$ is the spring constant for corresponding bond pair in two arms. $e_{ij}^k = (R_k^i - R_k^j) / ||R_k^i - R_k^j||$ ($k = a,b$) is the normalized bond vector, with $i,j$ running over all pairwise spring (bond) connections in each arm. The vibrational displacement vector $X = \{ u_0^a, u_0^b, u_1^a, \ldots, u_n^a, u_n^b \}$ satisfies the dynamic equation $\mathcal{M} \cdot \partial_t^2 X = D \cdot X$ with D the dynamic matrix and $\mathcal{M}$ the mass matrix [33][35].

Based on general symmetry consideration, when an isolated mechanical system under the central inversion with respect to arbitrary fixed point, the Hamiltonian is invariant. However, this is usually not true for a part of the system. Nevertheless, one can prove that Hamiltonian Eq. (2) is invariant when central inversion is conducted only for one arm with respect to the hinge point, e.g.,

$$R_i^{b'} = 2R_0 - R_i^b, \hspace{1cm} i \in [0,1,2,\ldots,n].$$  \hspace{1cm} (3)$$

We call this transformation partial central inversion (PCI), as depicted in Fig. 1a→b. This transformation leaves the vibrational degrees of freedom $u_i^b$ intact. In addition, rotation symmetry guarantees the invariance of the system in a 90° rotation around the hinge point (Fig. 1b→c). Therefore, by defining these two consecutive transformations as $V_0$, we have the commutation relationship $[V_0, H] = 0$ or $[V_0, D] = 0$. Equivalently, $V_0$ can also be interpreted as the combination of operator $K$ that changes the open angle of the hinge from $\vartheta$ to $\vartheta^* = \pi - \vartheta$, i.e., $\hat{K}D(\vartheta)\hat{K}^{-1} = D(\vartheta^*)$, and operator $\hat{U}_0$ that switches the corresponding nodes in two arms ($r_i^a = r_i^b$) and rotates all vibrational degrees of freedom by 90° at the same time, namely, $V_0 = K\hat{U}_0$. In the case of $n = 2$, which is shown in Fig. 1, $\hat{U}_0$ can be written as [29]

$$\hat{U}_0 = \begin{pmatrix}
\hat{r} & 0 & 0 & 0 & 0 \\
0 & 0 & \hat{r} & 0 & 0 \\
0 & \hat{r} & 0 & 0 & 0 \\
0 & 0 & 0 & \hat{r} & 0 \\
0 & 0 & 0 & 0 & \hat{r}
\end{pmatrix}$$  \hspace{1cm} (4)$$

where $\hat{r} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ is the 90° rotation operator. As can be seen, $\hat{U}_0$ performs the node switching $(a,1) \rightarrow (b,1)$ and $(a,2) \rightarrow (b,2)$, while leaves the position of hinge node unchanged. Since $K$ commutes with $\hat{U}_0$, from $[V_0, D] = 0$ one has

$$\hat{U}_0 D(\vartheta)\hat{U}_0^{-1} = D(\vartheta^*).$$  \hspace{1cm} (5)$$

Eq. (5) expresses the dual relationship between two hinge configurations at open angles $\vartheta$ and $\vartheta^*$, which we call “hinge duality”. Especially, $\vartheta^* = \vartheta$ corresponds to the self-dual point at which the hinge structure remains intact under $K$, but the dynamic modes are transformed by $U_0$. One can generally prove that the vibrational modes before and after this transformation are orthogonal to each other, i.e., $X \cdot (\hat{U}_0 \cdot X) = 0$. This guarantees that at the self-dual point, an arbitrary vibration energy (frequency) level of the system is at least double degenerated. One can also prove that changing the mass of the corresponding node pair, or the spring constant $\lambda_{ij}$, does not affect the hinge duality. Moreover, this hinge duality is independent of the number of nodes in the arm. This means that even in the continuous limit of the arm $(n \rightarrow \infty)$, the hinge duality is still preserved.

Dynamic isomerism  From the above analysis, one can see that the PCI gives the hinge an extra degree of freedom which preserves the dynamic modes of the system. When multiple hinges are inter-connected to form a hinge chain, these extra degrees of freedom are additive, i.e., a hinge chain with $N$ hinges has $2^N$ dissimilar configurations whose dynamic modes are exactly the same. We call these configurations dynamic isomers, which can be labelled by binary sequences like ”1011011...” of length $N$. Here, 1 and 0 indicate two dual states with open angle $\vartheta$ and $\vartheta^*$ for a single hinge. In Fig. 1f–g, we show several dynamically isomeric chains with different sequences, where sequence ”0000000...” or ”1111111...” represents two simplest periodic hinge chains (Fig. 1f, f). We also show an intermediate configuration between these two states in Fig. 1h, and a more disordered chain configuration in Fig. 1i. It’s surprising that such a disordered chain has the same vibrational eigenmodes as that of the periodic chains. It should be noticed that mode propagation direction in Fig. 1 is rotated by 90° compared with
of vibrational modes and shifts the node \((l, 1)\) one period distance \(\mathbf{a}_1\). Thus, this transformation for periodic hinge chain should be written as \(\hat{V}_1 = \hat{K}\hat{U}_1\) with

\[
\hat{U}_1 = \begin{pmatrix}
\mathbf{r} & 0 & 0 & 0 \\
0 & \mathbf{r} & 0 & 0 \\
0 & 0 & \mathbf{r} & 0 \\
0 & 0 & 0 & \hat{T}_{a_1} \mathbf{r}
\end{pmatrix} \mathcal{I}
\]

Here, the switching between \((l, 1)\) and \((l+1, 1)\) is expressed as the combination of shifting operator \(\hat{T}_{a_1} = e^{-i\mathbf{q} \cdot \mathbf{a}_1}\) and complex conjugation \(\mathcal{I}\) which reverses the sign of wave vector \(\mathbf{q}\). Therefore, \(\hat{U}_1\) is an anti-unitary matrix satisfying \(\hat{U}_1^2 = -1\). Since \(\hat{V}_1\) commutes with \(\mathcal{D}\), we have the dual transformation relationship for the periodic chain

\[
\hat{U}_1 \mathcal{D}(\vartheta, \mathbf{q}) \hat{U}_1^{-1} = \mathcal{D}(\vartheta^*, \mathbf{q}).
\]

Similar to that in single hinge, the above dual relationship is also independent of the number of nodes in the arms. In Fig. 2b, we show the phononic spectrum for the periodic hinge chains in Fig. 1a. We find the identical spectrum for systems at \(\vartheta = 60^\circ\) and \(\vartheta = 120^\circ\), as well as the double degeneracy in the whole BZ zone at \(\vartheta = 90^\circ\), a hallmark of self-dual point [13].

**Duality in 2D periodic hinge networks** From the above analysis, one can see that the existence of a hinge and corresponding dual transformation in the unit cell are the requisites for periodic chains to achieve hinge duality. This inspires us to propose design rules of 2D periodic networks with hinge duality: i) constructing a single hinge in which each arm of the hinge has more than three nodes; ii) making the corresponding nodes in two arms as a node pair; iii) choosing two node pairs and constructing two vectors that connect the nodes in each pair. These two vectors define the two lattice vectors \(\mathbf{a}_1\) and \(\mathbf{a}_2\) of 2D periodic networks. Following the above procedure, one can generate arbitrarily complex 2D periodic structures with hinge duality by designing the hinge arm. In Fig. 3a, we show the minimal 2D periodic networks in which the hinge arm is composed by three nodes and two bonds. When the hinge arm is a three-node triangle, one obtains a generalized twisted Kagome structure [35] (Fig. 3b). All these structures have the same dual transformation \(\hat{V}_2 = \hat{K}\hat{U}_2\) with

\[
\hat{U}_2 = \begin{pmatrix}
\mathbf{r} & 0 & 0 \\
0 & \hat{T}_{a_2} \mathbf{r} & 0 \\
0 & 0 & \hat{T}_{a_1} \mathbf{r}
\end{pmatrix} \mathcal{I},
\]

where the nodes switchings \((l_1, l_2, 1) \rightarrow (l_1 + 1, l_2, 1)\) and \((l_1, l_2, 2) \rightarrow (l_1, l_2 + 1, 2)\) in unit cell \((l_1, l_2)\) are related with the shifting operator \(\hat{T}_{a_1}\) and \(\hat{T}_{a_2}\) along \(\mathbf{a}_1\) and \(\mathbf{a}_2\), respectively. If the hinge arm is a hourglass made up by two equilateral triangles, one has a lattice with p2g space group symmetry (Fig. 3c), which can be obtained...
FIG. 3: By constructing different hinges in the unit cell, one can obtain various 2D periodic structures with hinge duality. (a) A minimal 2D hinge network, (b) generalized twisted Kagome, (c) dynamic isomer of twisted kagome with two hinge arms having different size, (d) p2g twisted kagome, (e) snub square lattice, (f) pmg lattice [14]. The corresponding bond pairs are drawn in the same color in each figure, which have the same spring constant.

by twisting the standard Kagome lattice [22]. When the hinge arm is a perfect square, a snub square lattice with p4g symmetry is obtained (Fig. 3e). The phononic spectrums for these structures (Fig. S2-S6) and a general proof of the dual relationship in 2D periodic hinge structures is also provided in [39].

The above design principles can also have more complicated variation. In Fig. 2f, we show the network with pmg symmetry reported in [14]. The unit cell of this structure is a hinge made up by two rhombus arms with a ‘dangling’ bond on each arm. Nevertheless, the two ‘dangling’ bonds are arranged in an opposite direction, making two arms unable to map to each other by rotation. In fact, the dual transformation in this structure involves additional PCI transformation for the dangling bond (see the dashed blue bond in Fig. 3f). This example suggests that there exist more complicated design rules for hinge-dual structures based on multiple PCIs. Moreover, we find that PIC in Eq. (3) belongs to a more general scaling transformation, with scaling ratio $\xi = -1$, i.e.,

$$R_i^{\xi'} = R_0 - \xi (R_0 - R_i^k), \quad i \in [0, 1, 2, \cdots, n]$$  \hspace{1cm} (9)

and $k = a, b$. In this transformation, the hinge point is fixed. One can prove that Eq. (9) with arbitrary $\xi$ also preserves the Hamiltonian as long as $\lambda_{ij}$ remains unchanged during the transformation, which can be realized easily in mechanical systems. Under this condition, we can have 2D periodic dynamic isomers composed by hinge arms that have different sizes (Fig. 3c). In these dynamic isomers, the propagation directions of vibrational waves are also modified [39].

It should be mentioned that structures in Fig. 3 are all deformable networks, some of which are Maxwell (isostatic) structures. In fact, the existence of free hinge in the unit cell guarantees the deformability of these structures, thus there must be an corresponding Guest-Hutchinson mode [23, 37]. Nevertheless, as discussed in [39], structures with hinge duality are not necessary to be deformable. Hinge structures with either node pinning or bond orientation constraint can also have hinge duality as shown in Fig. S7-8 [39]. This broadens the functionality and potential application scenarios of these structures.

At last, we also consider a non-mechanical magnon system, whose Hamiltonian can be written as

$$H = -2J \sum_{i,j} s_i \cdot s_j$$  \hspace{1cm} (10)

where $(i,j)$ indicates the nearest spin pairs. Magnons are the collective magnetic excitations (spin waves) associated with the precession of the spin moments [38]. In
the weak perturbation regime, $s_x, s_y \ll s_z \simeq s$, magnons obey the first-order dynamic equation rather than the second-order one for phonons, i.e.,

$$\partial_t S_\perp = \frac{2J_s}{\hbar} D_m \cdot \mathbf{S}_\perp$$

with $S_\perp = (s_1^x, s_1^y, s_1^z, s_2^y, \cdots)$ the spin moments in $xy$ plane and $S_\perp = (s_1^x, -s_1^y, s_1^z, -s_2^y, \cdots)$ and $D_m$ the dynamic matrix (see [39] for details). In Fig.4, we show the magnonic spectrums for the p31m twisted Kagome network under different open angles. Similar to the phononic systems, we observe the identical magnonic spectrum at two dual open angles $\vartheta = 60^\circ$ and $\vartheta = 120^\circ$, as well as the Kramers-like degeneracy at the self-dual point $\vartheta = 90^\circ$. These results suggest that the hinge duality should be a generic property of hinge structures, insensitive to specific Hamiltonian or dynamics of the system.

**Discussion and Conclusion** In conclusion, we unveil the origin of hinge duality and the resulting hidden symmetry in 2D Hamiltonian hinge systems. We find the hinge structure has a unique PCI symmetry related with the end-fixed scaling, based on which the hinge can have a Hamiltonian-invariant dual transformation between two different configurations. This intriguing property leads to 2D periodic or aperiodic dynamic isomers of hinge chains/networks, which can be utilized to build flexible wave-guides or networks with applications in phononic circuits [36]. It is also interesting to explore the dynamic isomerism at the molecular scale [40, 41]. Furthermore, we propose simple rules to design self-dual 2D periodic structures with arbitrary complexity, which provides a guideline to fabricate self-dual metamaterials with protected new topological phases or other unconventional mechanical/phononic properties [12–14, 26–29]. At last, we show that the hinge duality also exists in non-mechanical magnonic system. Thus we expect it a generic property of a broad range of Hamiltonian or dynamic systems. In fact, during the preparing of this work, we noticed Ref. [30] which discussed this topic from a different point of view. We expect further breakthrough in discovering other kinds of duality-induced hidden symmetry, especially in aperiodic 2D [42] and periodic 3D isostatic systems [18].

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