Wavefunctions for highly anisotropic homogeneous cosmologies

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Abstract

The canonical quantization of homogeneous cosmologies is considered in the high anisotropic limit. Exact wavefunctions are found in this limit when the momentum constraints are reduced at the classical level. Lorentzian solutions that represent tunnelling from classically forbidden regimes are identified. Solutions to the modified Wheeler-DeWitt equation are also found for the vacuum Bianchi IX model when a quantum reduction of the momentum constraints is considered.

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Quantum cosmology applies the concepts of quantum mechanics to the Universe as a whole. The majority of studies in this field invoke the Dirac quantization procedure \cite{1}. In this approach the wavefunction of the Universe is annihilated by the operator versions of the Hamiltonian and momentum constraints \cite{2}. However, it is not known how to solve these constraints in full generality. Indeed, the configuration space of the wavefunction is infinite-dimensional. In practice, therefore, one invokes the ‘minisuperspace’ approximation and applies the procedure only to those cosmologies that represent homogeneous solutions to Einstein’s field equations \cite{2,3}. These are the Bianchi models and the Kantowski-Sachs Universe. Hence, the problem is reduced to a quantum mechanical system with a finite number of degrees of freedom.

It is not clear whether the results derived from this minisuperspace quantization represent a valid approximation to the full theory of quantum gravity \cite{4,5}. However, the main justification for this approach is that it provides a solvable framework in which the problems associated with quantum cosmology may be addressed. For example, there are difficulties in extracting physical predictions from the wavefunction and there are also problems with defining a Hilbert space of states. (For a review see, e.g., Ref \cite{6}).

One may adopt one of two approaches when performing a canonical quantization of homogeneous Universes. The simplest method is to reduce the momentum constraints at the classical level before quantization and most investigations to date have adopted this view. The Wheeler-DeWitt equation represents the quantum Hamiltonian constraint and governs the evolution of the wavefunction on minisuperspace \cite{2}. However, a more rigorous and consistent method is to reduce the momentum constraints at the quantum level. In this way all constraints are treated in a similar fashion. These two approaches are not necessarily equivalent and can lead to different results \cite{7,8}.

In either case, however, very few exact wavefunctions have been found to date. The purpose of this letter is to derive new families of solutions to the quantum constraints of the Bianchi class A models. We consider wavefunctions on a region of minisuperspace corresponding to the high anisotropic limit of these models.

We first investigate the approach whereby the momentum constraints are reduced prior to quantization. We show that the resulting Wheeler-DeWitt equation may be written in the form of the unit-mass Klein-Gordon equation. We then solve the Wheeler-DeWitt equation for the vacuum Bianchi IX model when a full quantum reduction of the momentum constraints is performed.

The homogeneous Bianchi space-times have a topology $R \times G_3$, where $G_3$ is a three-dimensional Lie group of isometries transitive on space-like three-dimensional orbits \cite{9}. The world-interval of the space-time is

$$ds^2 = -dt^2 + h_{ab} \omega^a \omega^b, \quad a, b = 1, 2, 3,$$

where the metric $h_{ab}$ on the surfaces of homogeneity is a function of $t$ alone and $\omega^a$ are one-forms. The isometry of the three-surface is determined by the structure constants
Table 1: The diagonal components of $m^{ab}$ for each Bianchi type in the class A.

| Type | I   | II  | VI₀ | VII₀ | VIII | IX  |
|------|-----|-----|-----|------|------|-----|
| $m^{ab}$ | (0, 0, 0) | (1, 0, 0) | (1, −1, 0) | (1, 1, 0) | (1, 1, −1) | (1, 1, 1) |

$C^{abc}$ of the Lie algebra of $G_3$. These may be decomposed as $C^{abc} = m^{ad} \varepsilon_{dbc} + \delta^{a}{}_{[b} a_{c]}$, where $a_{c} \equiv C^{a}{}_{ac}$ and $m^{ab}$ is symmetric \cite{10}. The Jacobi identity $C^{a}{}_{b[c} C^{b}{}_{de]} = 0$ is satisfied if and only if $a_{b}$ is transverse to $m^{ab}$, i.e., $m^{ab} a_{b} = 0$. The Lie algebra belongs to the Bianchi class A if $a_{b} = 0$ and to the class B if $a_{b} \neq 0$ \cite{10}. It is well known that the evolution of class B models cannot be described in terms of a standard Hamiltonian treatment. This is related to the fact that these models cannot admit a spatially compact topology \cite{7}. Consequently, we restrict our attention to the class A. This consists of Bianchi types I, II, VI₀, VII₀, VIII and IX. The Lie algebra of each type is uniquely determined up to isomorphisms by the rank and signature of $m^{ab}$.

We will assume that the matter source is a single, massless, minimally coupled scalar field $\phi$. The classical dynamics of these cosmologies is determined by the time-space and time-time components of the Einstein field equations. These may be expressed as the momentum and Hamiltonian constraints, respectively. The former is given by

$$P_{d} \equiv \pi^{a}{}_{c} m^{cb} \varepsilon^{dab} = 0,$$

where the conjugate momentum variable $\pi^{ab}$ represents the integral of the momentum density over the spatial hypersurfaces. The Hamiltonian constraint, on the other hand, takes the form \cite{11}

$$\left(\pi^{ab} \pi_{ab} - \frac{1}{2} \pi^{2}\right) + \mathcal{V}^{2} \left(m^{ab} m_{ab} - \frac{1}{2} m^{2}\right) + \frac{1}{24} \pi^{2}_{\phi} = 0,$$

where $\pi_{\phi}$ is the momentum conjugate to the scalar field, $\mathcal{V}$ represents the volume of space and indices are raised and lowered with $h^{ab}$ and $h_{ab}$, respectively.

At the classical level, the momentum constraints \cite{11} imply that $\pi^{ab}$, $m^{ab}$ and $h_{ab}$ may be simultaneously diagonalized on a given spatial hypersurface of constant $t$. It then follows directly from the field equations that these quantities are diagonal on all other hypersurfaces. Hence, one may diagonalize the variables at the classical level without loss of generality and the momentum constraints therefore become trivial.

In this case one may choose a basis where each diagonal component of $m^{ab}$ is $\pm 1$ or 0. The six cases that constitute the Bianchi class A are shown in Table 1. The
three-metric is then written as  

\[ h_{ab}(t) = e^{2\alpha(t)} \left( e^{2\beta(t)} \right)_{ab} \]  

(4)

where the matrix \( \beta_{ab} \equiv \text{diag} \left[ \beta_+ + \sqrt{3} \beta_-, \beta_+ - \sqrt{3} \beta_-, -2\beta_+ \right] \) represents the degree of anisotropy in these models. The parameter \( e^\alpha \) may be viewed as an averaged scale factor of the Universe. The Hamiltonian constraint (3) then reduces to  

\[ -\pi_\alpha^2 + \pi_{\beta_+}^2 + \pi_{\beta_-}^2 + \pi_\phi^2 + U = 0 \]  

(5)

where \( \pi_j \ (j = \alpha, \beta_\pm, \phi) \) are the conjugate momenta,

\[ U = \frac{1}{3} \left[ (m_{11} h_{11})^2 + (m_{22} h_{22})^2 + (m_{33} h_{33})^2 - 2m_{11} m_{22} h_{11} h_{22} - 2m_{11} m_{33} h_{11} h_{33} - 2m_{22} m_{33} h_{22} h_{33} \right] \]  

(6)

represents the ‘superpotential’ and \( m^{ab} = \text{diag} [m_{11}, m_{22}, m_{33}] \). \( \text{(8)} \). (The direct dependence of the superpotential on \( V \) is eliminated by performing a linear translation on \( \alpha \)). The superpotential for the Bianchi type I cosmology vanishes identically and we shall not consider this model further.

Quantization of these cosmologies follows by imposing the algebra \( [j, \pi_j]_- = i \). The Hamiltonian constraint (5) is then promoted to an operator that annihilates the state vector \( \tilde{\Psi} \) of the Universe. The result is the Wheeler-DeWitt equation on minisuperspace:

\[
\left[ \frac{\partial^2}{\partial \alpha^2} - \frac{\partial^2}{\partial \beta_+^2} - \frac{\partial^2}{\partial \beta_-^2} + 2p \frac{\partial}{\partial \alpha} - \frac{\partial^2}{\partial \phi^2} + U \right] \tilde{\Psi} = 0,
\]  

(7)

where the constant \( p \) accounts for ambiguities that arise in the operator ordering \( \text{[12]} \). It is convenient to rescale the wavefunction by \( \tilde{\Psi} = \Psi e^{i q \phi - p \alpha} \), where \( q \) is an arbitrary, complex constant. Eq. (7) then simplifies to

\[
\left[ \frac{\partial^2}{\partial \alpha^2} - \frac{\partial^2}{\partial \beta_+^2} - \frac{\partial^2}{\partial \beta_-^2} + q^2 - p^2 + U \right] \Psi = 0.
\]  

(8)

A unified exact solution to this equation can be found for all Bianchi types if we choose the factor ordering \( p^2 = 9 + q^2 \). It can be verified by direct substitution that the wavefunction has the form

\[
\tilde{\Psi} = e^{(3-p)\alpha + i q \phi} e^{-I},
\]  

(9)

where

\[
I = \pm \frac{1}{6} m^{ab} h_{ab}
\]  

(10)

and summation over indices is implied. This solution generalizes the exact solution found previously for the type IX model \( \text{[13]} \). In the vacuum case \( (q = 0) \), the function \( I \)
is itself a solution to the Euclidean Hamilton-Jacobi equation. It therefore represents a Euclidean action for these cosmologies.

In general, an oscillating solution to the Wheeler-DeWitt equation may be interpreted as a classically allowed, Lorentzian geometry. A cosmological singularity then arises when the wavefunction undergoes an infinite number of oscillations. On the other hand, a classically forbidden Euclidean geometry corresponds to a non-oscillatory, exponential wavefunction. In particular, if the wavefunction is suitably damped at large three-geometries and remains regular when the spatial metric degenerates, it may be interpreted as a quantum wormhole. Solution (9) is an example of this latter type of wavefunction and therefore represents an anisotropic quantum wormhole.

Further unified solutions to Eq. (8) can be found in the limit where \( \beta \gg 1 \). This region of minisuperspace is interesting because quantum fluctuations of the gravitational field in the early Universe may have introduced considerable anisotropy into the initial conditions. However, observations indicate that the present Universe is highly isotropic. It is possible, therefore, that a better understanding of this high anisotropic regime may provide insight into how the Universe evolved into its current state.

It follows that \( h_{22}/h_{11} \ll 1 \) and \( h_{33}/h_{11} \ll 1 \) when \( \beta \gg 1 \). The first term in Eq. (8) therefore dominates the superpotential. Formally, this is equivalent to choosing \( m_{11} = 1 \) and \( m_{22} = m_{33} = 0 \). Hence, this particular limit of the Bianchi class A may be investigated by considering the type II model. The Wheeler-DeWitt equation (8) simplifies to

\[
\left[ \frac{\partial^2}{\partial \alpha^2} - \frac{\partial^2}{\partial \beta^2} + q^2 - p^2 + \frac{1}{3} e^{4\alpha+4\beta+4\sqrt{3}\beta} \right] \Psi = 0
\]  

and may be solved by defining a new wavefunction

\[
\Phi(\alpha, \beta) = \Psi(\alpha, \beta)e^{-r(\sqrt{3}\beta+\beta)/2},
\]  

where \( r^2 \equiv q^2 - p^2 \). Substitution of this ansatz into Eq. (11) implies that

\[
\left[ \frac{\partial^2}{\partial \alpha^2} - \frac{\partial^2}{\partial \beta^2} - \sqrt{3}r \frac{\partial}{\partial \beta} + r \frac{\partial}{\partial \beta} + \frac{1}{3} e^{4\alpha+4\beta+4\sqrt{3}\beta} \right] \Phi = 0.
\]  

We now perform a change of variables to \( \{u, v, w\} \), where

\[
u \equiv \frac{1}{12} e^{6\alpha+3\beta+3\sqrt{3}\beta}
\]  

and \( w = w(\alpha, \beta) \) is an arbitrary, twice continuously differentiable function of the minisuperspace coordinates. We then search for wavefunctions that are independent
of this variable. In this case, Eq. (13) simplifies to the canonical, unit-mass Klein-Gordon equation

\[
\left[ \frac{\partial^2}{\partial u \partial v} - 1 \right] \Phi = 0.
\]  
(15)

Hence, the \{u, v\} variables may be viewed as null coordinates over a region of (1 + 1)-dimensional Minkowski space-time spanned by the space-like coordinate \( X = u + v \) and time-like coordinate \( T = u - v \). These null coordinates are restricted to lie in the range \((u, v) \in (0, +\infty)\), so \( X \geq |T| \). We may therefore view the wavefunction \( \Phi \) as a classical particle of unit mass that is confined within the Rindler wedge of (1 + 1)-dimensional Minkowski space-time.

The semi-classical limit of the vacuum models may also be analyzed by introducing the variables (14). In the WKB approximation, one substitutes solutions of the form \( \Phi_{\text{WKB}} \approx e^{-iS/\hbar} \) into the Wheeler-DeWitt equation and considers the limit \( \hbar \to 0 \). Applying this substitution in Eq. (15) implies that the Hamilton-Jacobi equation takes the form \( S_u S_v = -1 \), where a subscript denotes partial differentiation. One solution to this equation is \( S = -i(cu + c^{-1}v) \), where \( c \) is an arbitrary, complex constant. However, the family of wavefunctions \( \Phi_c = e^{-cu - v/c} \) are also exact solutions to the full Wheeler-DeWitt equation (15) [16]. Hence, the WKB approximation is exact in this case.

\(|\Phi_c|\) is bounded everywhere when \( \text{Re } c \geq 0 \) and the wavefunction is exponentially damped for large \( \alpha \) if \( \text{Im } c = 0 \). Unfortunately, however, it does not satisfy the quantum wormhole boundary conditions because it decays too rapidly [15]. On the other hand, more general solutions to Eq. (15) may be generated in terms of linear superpositions of this family. The class of wavefunctions \( \Phi_c \) may therefore be physically relevant. In general, the superpositions have the form

\[
\Phi = \int_C dcM(c)e^{-cu - v/c},
\]
(16)

where \( M(c) \) is an arbitrary function of the parameter \( c \) and \( C \) represents the contour of integration in the complex plane [16].

Different solutions to Eq. (15) correspond to different choices for \( M(c) \) and \( C \). For example, if we specify \( M(c) = \frac{1}{2}c^{i(\epsilon-3)/3} \), where \( \epsilon \) is an arbitrary, real constant, and perform the integration over the positive half of the real axis, we find that

\[
\Phi_\epsilon = \frac{1}{2} \int_0^\infty dc c^{i(\epsilon-3)/3} e^{-cu - v/c}
= \left( \frac{u}{\epsilon} \right)^{i\epsilon/6} K_{i\epsilon/3} \left( 2\sqrt{uv} \right)
= K_{i\epsilon/3} \left[ \frac{1}{6} e^{2\alpha + 2\beta_+ + 2\sqrt{3}\beta_-} \right] \exp \left[ -\frac{i\epsilon}{6} (8\alpha + 2\beta_+ + 2\sqrt{3}\beta_-) \right],
\]
(17)

where \( K \) is the modified Bessel function of order \( i\epsilon/3 \) [16].
The argument of this function is given by

$$2\sqrt{uv} = |I| = \frac{1}{6}e^{2\alpha+2\beta_i+2\sqrt{3}i}$$

and corresponds to the high anisotropic limit of the Euclidean action (10). It is also directly proportional to the square root of the superpotential (3). The modified Bessel function is exponentially damped for sufficiently large arguments and the wavefunction takes the form $$\Phi_\epsilon \propto e^{-|I|}$$ for $$\sqrt{uv} > |\epsilon|/6$$. In this region of minisuperspace, therefore, the wavefunction reduces to the Euclidean form of Eq. (10).

The nature of solution (17) is different for smaller values of the scale factor. Indeed, it oscillates for $$0 < |I| < |\epsilon|/3$$, so the boundary $$I = |\epsilon|/3$$ represents a point of maximum expansion. The wavefunction has the asymptotic form $$\Phi_\epsilon \propto u^{-i\epsilon/3}$$ as $$\alpha \to -\infty$$ and therefore represents plane waves in the minisuperspace variables when $$\epsilon \neq 0$$. It oscillates an infinite number of times as the spatial metric degenerates and this behaviour is interpreted as a cosmological singularity.

Solution (16) may provide insight into the quantum nature of cosmological singularities in these highly anisotropic Universes. The singularity arose in this case because a particular superposition of Euclidean solutions was considered. In view of this, it is natural to investigate whether the singularity may be eliminated by considering alternative superpositions.

In order to pursue this possibility further, we introduce the new coordinate pair

$$\mu \equiv \frac{v}{2} + \sqrt{2}u, \quad \nu \equiv \frac{v}{2} - \sqrt{2}u.$$  

(19)

This change of variables transforms Eq. (15) into

$$\left[ \frac{\partial^2}{\partial \mu^2} - \frac{\partial^2}{\partial \nu^2} - \mu + \nu \right] \Phi = 0$$

(20)

and the general, separable solution to this equation is given by

$$\Phi_m = [c_1\text{Ai}(m + \mu) + c_2\text{Bi}(m + \mu)] [c_3\text{Ai}(m + \nu) + c_4\text{Bi}(m + \nu)]$$  

(21)

where Ai(x) and Bi(x) are Airy functions and $$\{m, c_j\}$$ are arbitrary constants.

This solution may be expressed in the form of Eq. (16) when $$c_j$$ satisfy appropriate conditions. If we specify

$$M(c) = \frac{\sqrt{2i}}{c^{3/2}} \exp \left[ \frac{2}{3c^3} - \frac{2m}{c} \right].$$

(22)

Eq. (16) takes the form

$$\Phi_m = \int_C \frac{dc}{c^{3/2}} \exp \left[ -\frac{c^3}{12} + (\mu + \nu + 2m)\frac{c}{2} + \frac{1}{4c}(\mu - \nu)^2 \right].$$

(23)
where \( \tilde{c} \equiv -2/c \). Halliwell and Louko have shown how integrals of this form may be evaluated [17]. Different choices for the contour \( C \) result in different products of Airy functions. Since we are interested in superimposing bounded wavefunctions, we assume that \( \text{Re } c > 0 \). We therefore choose the contour of integration to lie to the left of the origin. The result of the integration is

\[
\Phi_m = \text{Ai}(\mu + m)\text{Ai}(\nu + m)
\]

and, modulo a constant of proportionality, this is equivalent to Eq. (21) with \( c_2 = c_4 = 0 \).

This solution has an interesting feature. The Airy function \( \text{Ai}(x) \) is exponentially damped for large positive arguments, but exhibits oscillatory behaviour if \( x < 0 \) [18]. However, the variable \((\mu + m)\) is positive-definite for all values of the scale factor if \( m > 0 \). In this case, it follows that the wavefunction will represent classically forbidden geometries when \((\nu + m) > 0\), but will correspond to Lorentzian solutions when \((\nu + m) < 0\). Now, when the spatial volume of the Universe becomes vanishingly small \((\alpha \to -\infty)\), \( u \to 0 \) and \( v \to +\infty \), so \( \mu \to \nu \to +\infty \). Hence, the wavefunction vanishes, but does not oscillate, as the spatial metric degenerates. Consequently, there is no singular behaviour at the origin. As the scale factor grows, however, \((\nu + m)\) decreases and eventually becomes negative. At this point the wavefunction begins to exhibit oscillatory behaviour. In effect, the Universe tunnels from a classically forbidden regime into a Lorentzian domain when \( \nu \approx -m \).

The above solutions were derived after the momentum constraints had been reduced at the classical level. However, it is more accurate to view these constraints as operators that annihilate the wavefunction of the Universe. Consequently, it is better not to impose the assumption of diagonality before quantization. Indeed, Ashtekar and Samuel have argued that this restriction may result in a loss of generality at the quantum level [7]. In the remainder of this work, therefore, we shall consider the quantization procedure when the assumption of diagonality is dropped. In particular, we will investigate the vacuum Bianchi IX model.

The quantum constraints for this model were recently derived by Higuchi and Wald [8]. We briefly summarize their main results here. They parametrized each point in minisuperspace in terms of an orthogonal matrix \( \mathbf{O} \) and the eigenvalues \( \{\lambda_1, \lambda_2, \lambda_3\} \) of the metric \( \mathbf{h} \). These eigenvalues satisfy the conditions \( \lambda_1 \geq \lambda_2 \geq \lambda_3 > 0 \) and \( \lambda_1 \lambda_2 \lambda_3 = 1 \). The matrix \( \mathbf{O} \) rotates the eigenvectors of the metric into a new orthonormal basis. It can be shown that the quantum momentum constraints are satisfied if the wavefunction is invariant under the action of \( \mathbf{O} \mathbf{h} \mathbf{O}^{-1} \). It must therefore be symmetric under permutations in \( \lambda_i \).

If one chooses \( \lambda_1 \equiv e^{2(\beta_+ + \sqrt{3}\beta_-)} \), \( \lambda_2 \equiv e^{2(\beta_- - \sqrt{3}\beta_+)} \) and \( \lambda_3 \equiv e^{-4\beta_+} \), the Wheeler-DeWitt equation may be written as

\[
\left[ \frac{\partial^2}{\partial \alpha^2} - \frac{1}{C(\beta_\pm)} \sum_{j=\pm} \frac{\partial}{\partial \beta_j} C(\beta_\pm) \frac{\partial}{\partial \beta_j} + U - 90 \xi \right] \Psi = 0, \tag{25}
\]
where
\[ C(\beta_{\pm}) = 8 \left| \sinh \left( 2\sqrt{3}\beta_{-} \right) \sinh \left( 3\beta_{+} - \sqrt{3}\beta_{-} \right) \sinh \left( 3\beta_{+} + \sqrt{3}\beta_{-} \right) \right| \]
(26)
and \( \xi \) is a numerical constant. The non-trivial contribution from \( C(\beta_{\pm}) \) arises because the volume element on superspace is affected by a term that depends on the eigenvalues of the metric.

In this representation, the momentum constraints are satisfied if the wavefunction is invariant under 120-degree rotations in the \((\beta_{+}, \beta_{-})\) plane and reflections in \(\beta_{-}\). An appropriate linear combination of wavefunctions satisfying these symmetry conditions can always be constructed. Formally, one may write the full wavefunction as \( \Psi = \Phi + R\Phi + R^{-1}\Phi + \Phi(\beta_{-} \to -\beta_{-}) \), where \( \Phi \) represents a particular solution to Eq. (25) and \( R \) is the 120-degree rotation matrix in the \((\beta_{+}, \beta_{-})\) plane [19]. On a practical level, therefore, one need only find solutions to the modified Wheeler-DeWitt equation (25).

However, it is not clear at present how one might proceed to solve this equation in full generality. In view of this, we shall consider the two limiting cases where \( \beta_{\pm} \gg 1 \) and \( \beta_{+} \gg 1, |\beta_{-}| \ll 1 \).

In the former case \( C(\beta_{\pm}) \approx e^{6\beta_{+} + 2\sqrt{3}\beta_{-}} \) and \( U = 48uv \). Eq. (25) therefore simplifies to
\[
\left[ \frac{\partial^2}{\partial \alpha^2} - \frac{\partial^2}{\partial \beta_{+}^2} - \frac{\partial^2}{\partial \beta_{-}^2} - 6 \frac{\partial}{\partial \beta_{+}} - 2\sqrt{3} \frac{\partial}{\partial \beta_{-}} + U - 90\xi \right] \Psi = 0.
\]
(27)

A quantum wormhole solution to this equation may be found by rescaling the wavefunction such that \( \varphi = \Psi e^{3\beta_{+} + \sqrt{3}\beta_{-}} \). This wavefunction satisfies an equation that is formally equivalent to Eq. (8), where \( q = 0 \) and \( p^2 = 90\xi - 12 \). Hence, one solution is given by \( \varphi = e^{-2\sqrt{uv}} \) for the special case \( \xi = 21/90 \).

Further solutions may be found by defining a new wavefunction \( \Phi \):
\[
\Phi(\alpha, \beta_{\pm}) = \Psi(\alpha, \beta_{\pm}) e^{-c_{+}\beta_{+} + c_{-}\beta_{-}},
\]
(28)
where
\[
c_{+} \equiv \frac{\sqrt{3}}{2} \sqrt{12 - 90\xi} - 3,
\]
\[
c_{-} \equiv \frac{1}{2} \sqrt{12 - 90\xi + \sqrt{3}}.
\]
(29)
The Wheeler-DeWitt equation (27) transforms to
\[
\left[ \frac{\partial^2}{\partial \alpha^2} - \frac{\partial^2}{\partial \beta_{+}^2} - \frac{\partial^2}{\partial \beta_{-}^2} - \sqrt{3} \left( 12 - 90\xi \right)^{1/2} \frac{\partial}{\partial \beta_{+}} + \left( 12 - 90\xi \right)^{1/2} \frac{\partial}{\partial \beta_{-}} + \frac{1}{3} e^{4\alpha + 4\beta_{+} + 4\sqrt{3}\beta_{-}} \right] \Phi = 0
\]
(30)

\(^1\)Note that we are assuming implicitly that \( \beta_{+} \geq \beta_{-} \) in this analysis. This condition ensures that \( C(\beta_{\pm}) \) has an approximately exponential form.
and this equation is formally equivalent to Eq. (13) with \( r = (12 - 90 \xi)^{1/2} \). It therefore transforms into the unit-mass Klein-Gordon equation (15) when the null variables (14) are introduced. Hence, the above analysis and solutions will also be relevant in this more general quantization procedure if the model is sufficiently anisotropic. The range of \( \beta_\pm \) must be such that \( C(\beta_\pm) \) can be viewed as a purely exponential function.

We will conclude by considering the case where \( \beta_+ \gg 1 \) and \( |\beta_-| \ll 1 \). It follows that \( C(\beta_\pm) \approx 4\sqrt{3}|\beta_-|e^{6\beta_+} \) in this limit and the Wheeler-DeWitt equation (25) therefore takes the approximate form

\[
\left[ \frac{\partial^2}{\partial \alpha^2} - \frac{\partial^2}{\partial \beta_+^2} - \frac{\partial^2}{\partial \beta_-^2} - \frac{1}{\beta_-} \frac{\partial}{\partial \beta_-} + (9 - 90 \xi) + e^{4\alpha + 4\beta_+} \beta_-^2 \right] \Theta = 0, \tag{31}
\]

where \( \Theta \equiv e^{3\beta_+} \Psi \). Without loss of generality, we have eliminated the numerical constant in front of the superpotential via a linear shift in \( \alpha \).

To proceed, we change variables to the null coordinates

\[
\rho \equiv \alpha + \beta_+, \quad \eta \equiv \alpha - \beta_+. \tag{32}
\]

In this case, Eq. (31) transforms to

\[
\left[ \frac{\partial^2}{\partial \rho \partial \eta} - \frac{\partial^2}{\partial \beta_+^2} - \frac{\partial^2}{\partial \beta_-^2} - \frac{1}{\beta_-} \frac{\partial}{\partial \beta_-} + (9 - 90 \xi) + \beta_-^2 e^{4\rho} \right] \Theta = 0 \tag{33}
\]

and it follows that the wavefunction \( \Theta \) is an eigenstate of \( \eta \). In order to solve this equation, we assume that \( \Theta \) has the generic form \[4\]

\[
\Theta = \exp \left[ -iE\eta - B(\rho) - \lambda A(\rho)\beta_-^2 \right], \tag{34}
\]

where \( A \) and \( B \) are arbitrary functions of the null variable \( \rho \) and \( \{\lambda, E\} \) are arbitrary constants. After substitution of this ansatz, Eq. (33) separates into two, first-order ordinary differential equations:

\[
4iEB' + 4\lambda A + 9 - 90 \xi = 0 \tag{35}
\]

\[
4iE\lambda A' - 4\lambda^2 A^2 + e^{4\rho} = 0, \tag{36}
\]

where a prime denotes differentiation with respect to \( \rho \).

If we introduce a new function \( D(\rho) \), where \( A(\rho) \equiv d\ln D/d\rho \), and choose \( \lambda = -iE \), Eq. (36) simplifies to

\[
D'' + \frac{1}{4E^2} e^{4\rho} D = 0. \tag{37}
\]

The general solution to this equation is given by

\[
D = Z_0 \left[ \frac{1}{4E^2} e^{2(\alpha + \beta_+)} \right], \tag{38}
\]
where \( Z_0 \) is an arbitrary, linear combination of ordinary, zero-order Bessel functions. Finally, Eq. (35) may now be solved by direct integration after substitution of Eq. (38). We conclude, therefore, that one wavefunction satisfying Eq. (31) is

\[
\Psi = \left[ Z_0 \left( e^{2\rho/4E} \right) \right]^{-1} \exp \left[ iE (A\beta^2 - \eta) - \frac{i}{E} \left( \frac{9 - 90\xi}{4} \right) \rho - 3\beta_+ \right]. \quad (39)
\]

To summarize, we have considered the quantization of the homogeneous Bianchi class A cosmologies in the limit of high anisotropy. A number of exact solutions to the quantum constraints were found. We considered first the approach whereby the momentum constraints are reduced at the classical level prior to quantization. Wavefunctions that represent tunnelling from a classically forbidden state were presented. These solutions may be expressed as a linear superposition of purely Euclidean wavefunctions. We then considered a quantum reduction of the momentum constraints for the vacuum Bianchi IX cosmology. The modified Wheeler-DeWitt equation may be solved if appropriate conditions are satisfied. These solutions should prove useful for investigating some of the fundamental questions that arise in quantum cosmology.

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