THE $\gamma$-POSITIVITY OF BASIC EULERIAN POLYNOMIALS
VIA GROUP ACTIONS

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Abstract. We prove the $\gamma$-positivity of the basic Eulerian polynomials that enumerate permutations by the excedance statistic and the major index as well as the corresponding results for derangements. These refine the classical $\gamma$-positivity results for the Eulerian polynomials and the derangement polynomials. Our main tools are the Modified Foata–Strehl action on permutations and the recent triple statistic (des, rix, aid) equidistributed with (exc, fix, maj).

1. Introduction

A permutation $\sigma$ of $[n] := \{1, 2, \ldots, n\}$ is a bijection from $[n]$ to $[n]$, which will be identified with the word $\sigma = \sigma_1 \cdots \sigma_n$, where $\sigma_i = \sigma(i)$. We denote by $\mathcal{S}_n$ the set of all permutations of $[n]$. Given a $\sigma \in \mathcal{S}_n$, an integer $i \in [n]$ is an excedance (resp. fixed point) of $\sigma$ if $\sigma_i > i$ (resp. $\sigma_i = i$). Denote by $\text{exc}(\sigma)$ and $\text{fix}(\sigma)$ the number of excedances and fixed points of $\sigma$, respectively. Let $\text{maj}(\sigma) := \sum_{\sigma_i > \sigma_{i+1}} i$ be the major index of $\sigma$.

Shareshian and Wachs [13, 14] introduced and studied the basic Eulerian polynomials

$$A_n(t, r, q) := \sum_{\sigma \in \mathcal{S}_n} t^{\text{exc}(\sigma)} r^{\text{fix}(\sigma)} q^{\text{maj}(\sigma) - \text{exc}(\sigma)}.$$ 

In [14, Remark 5.5] they noticed that using an unpublished result of Gessel they could prove, among other things, that

1. $A_n(t, 1, q)$ has coefficients in $\mathbb{N}[q]$ when expanded in the basis

$$\left\{ t^k(1 + t)^{n-2k} \right\}_{k=0}^{[n/2]}.$$

2. $A_n(t, 0, q)$ has coefficients in $\mathbb{N}[q]$ when expanded in the basis

$$\left\{ t^k(1 + t)^{n-1-2k} \right\}_{k=0}^{[(n-1)/2]}.$$

It is then natural to ask whether there are any combinatorial (see [2, 5]) or geometric meaning (see [11, 12]) of these coefficients. Our main results, Theorems 4 and 5 provide combinatorial interpretations for these coefficients. We first review some partial and related recent results on this topic.

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For each $\sigma \in \mathfrak{S}_n$ we call $\sigma_i$ ($1 \leq i \leq n$) a double descent (resp. double ascent, peak, valley) of $\sigma$ if $\sigma_{i-1} > \sigma_i > \sigma_{i+1}$ (resp. $\sigma_{i-1} < \sigma_i < \sigma_{i+1}$, $\sigma_{i-1} > \sigma_i < \sigma_{i+1}$, $\sigma_{i-1} > \sigma_i < \sigma_{i+1}$), where we use the convention $\sigma_0 = \sigma_{n+1} = +\infty$. Denote by $dd(\sigma)$ (resp. $da(\sigma)$, $peak(\sigma)$, $valley(\sigma)$) the number of double descents (resp. double ascent, peak, valley) of $\sigma$. For each $\sigma \in \mathfrak{S}_n$ let $cda(\sigma) := |\{i : \sigma^{-1}(i) < i < \sigma(i)\}|$ be the number of double excedances or cyclic double ascents of $\sigma$ and $cyc(\sigma)$ denote the number of cycles of $\sigma$. For $1 \leq k \leq n$ define the sets

$\mathcal{D}_{n,k} := \{\sigma \in \mathfrak{S}_n : dd(\sigma) = 0, \text{des}(\sigma) = k\}$, (1.1)

$\tilde{\mathcal{D}}_{n,k} := \{\sigma \in \mathcal{D}_{n,k-1} : \sigma_{n-1} < \sigma_n\}$, (1.2)

$\mathcal{E}_{n,k} := \{\sigma \in \mathfrak{S}_n : \text{fix}(\sigma) = 0, cda(\sigma) = 0, \text{exc}(\sigma) = k\}$. (1.3)

A classical result due to Foata and Schützenberger [5] is that the Eulerian polynomials have the following expansion, which implies both the symmetry and unimodality (see for instance [14] for definitions) of the sequence of Eulerian numbers.

**Theorem 1 ([5]).** One has

$$A_n(t, 1, 1) = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} |\mathcal{D}_{n,k}| t^k (1 + t)^{n-1-2k}. \quad (1.4)$$

The following result was proved by Shin and Zeng [15, Theorem 11] via continued fractions.

**Theorem 2 ([15]).** One has

$$\sum_{\sigma \in \mathfrak{S}_n \atop \text{fix}(\sigma) = 0} \beta^{\text{cyc}(\sigma)} t^{\text{exc}(\sigma)} = \sum_{k=1}^{\lfloor n/2 \rfloor} \left( \sum_{\sigma \in \mathcal{E}_{n,k}} \beta^{\text{cyc}(\sigma)} \right) t^k (1 + t)^{n-2k}. \quad (1.5)$$

It follows that

$$A_n(t, 0, 1) = \sum_{k=1}^{\lfloor n/2 \rfloor} |\mathcal{E}_{n,k}| t^k (1 + t)^{n-2k}. \quad (1.6)$$

A permutation $\sigma \in \mathfrak{S}_n$ is called alternating if $\sigma_1 < \sigma_2 > \sigma_3 < \sigma_4 > \cdots$. An inversion of a permutation $\sigma \in \mathfrak{S}_n$ is a pair $(\sigma_i, \sigma_j)$ such that $1 \leq i < j \leq n$ and $\sigma_i > \sigma_j$. Let $\text{inv}(\sigma)$ be the number of inversions of $\sigma$. The set of alternating permutations of order $n$ is denoted by $\mathfrak{A}_n$. Foata and Han [4, Theorem 1] proved the following result:

**Theorem 3 ([4]).** One has $A_{2n}(-1, 1, q) = A_{2n-1}(-1, 0, q) = 0$ ($n \geq 1$) and

$$A_{2n+1}(-1, 1, q) = (-1)^n \sum_{\sigma \in \mathfrak{A}_{2n+1}} q^{\text{inv}(\sigma)} \quad (n \geq 0); \quad (1.7)$$

$$A_{2n}(-1, 0, q) = (-1)^n \sum_{\sigma \in \mathfrak{A}_{2n}} q^{\text{inv}(\sigma)} \quad (n \geq 0). \quad (1.8)$$
Several $q$-analogs of Eulerian polynomials with combinatorial meanings have been studied in the literature (see [13] and the references therein) and various extensions of (1.4) and (1.6) have already been obtained in [2, 8, 15]. The following two theorems are the main results of this paper.

**Theorem 4.** We have

$$A_n(t, 1, q) = \sum_{k=0}^{\lfloor(n-1)/2\rfloor} \left( \sum_{\sigma \in \mathcal{D}_{n,k}} q^{\text{inv}(\sigma)} \right) t^k (1 + t)^{n-1-2k}. \quad (1.9)$$

For example, for $n = 4$, we have

$$\mathcal{D}_{4,0} = \{1234\} \quad \text{and} \quad \mathcal{D}_{4,1} = \{1324, 1423, 2314, 2413, 3412, 1243, 1342, 2341\}.$$  

Hence

$$A_4(t, 1, q) = 1 + (3 + 2q + 3q^2 + 2q^3 + q^4)t + (3 + 2q + 3q^2 + 2q^3 + q^4)t^2 + t^3$$

$$= (1 + t)^3 + (2q + 3q^2 + 2q^3 + q^4)t(1 + t).$$

**Theorem 5.** We have

$$A_n(t, 0, q) = \sum_{k=1}^{\lfloor n/2 \rfloor} \left( \sum_{\sigma \in \tilde{\mathcal{D}}_{n,k}} q^{\text{inv}(\sigma)} \right) t^k (1 + t)^{n-2k}. \quad (1.10)$$

For $n = 4$ we have

$$\tilde{\mathcal{D}}_{4,1} = \{1234\} \quad \text{and} \quad \tilde{\mathcal{D}}_{4,2} = \{1324, 1423, 2314, 2413, 3412\}.$$  

Hence

$$A_4(t, 0, q) = t + (2 + q + 2q^2 + q^3 + q^4)t^2 + t^3$$

$$= t(1 + t)^2 + (q + 2q^2 + q^3 + q^4)t^2.$$  

**Remark 1.** Obviously Theorem 4 is a $q$-analogue of Theorem 1. To show that (1.10) reduces to (1.6), we have to prove that $|\tilde{\mathcal{D}}_{n,k}| = |\mathcal{E}_{n,k}|$. A bijection between $\tilde{\mathcal{D}}_{n,k}$ and $\mathcal{E}_{n,k}$ will be given in Remark 4. Since $\mathcal{A}_{2n+1} = \mathcal{D}_{2n+1,n}$ and $\mathcal{A}_{2n} = \tilde{\mathcal{D}}_{2n,n}$, Theorems 4 and 5 reduce immediately to Theorem 3 with $t = -1$.

**Remark 2.** For any $\sigma \in S_n$ let $\text{DES}(\sigma) = \{i \in [n-1] : \sigma_i > \sigma_{i+1}\}$ be its descent set. According to a result of Foata and Schützenberger [7], for any $S \subseteq [n-1],$

$$\sum_{\sigma \in S_n} q^{\text{inv}(\sigma)} = \sum_{\sigma \in \mathcal{S}_n} q^{\text{imaj}(\sigma)} \quad (\sigma \in S_n \quad \text{and} \quad \text{DES}(\sigma) = S),$$

where $\text{imaj}(\sigma) = \text{maj}(\sigma^{-1})$. Hence we can replace the statistic inv by imaj in Theorems 4 and 5.

We will prove Theorems 4 and 5 by making use of Modified Foata–Strehl action and an alternative interpretation of the basic Eulerian polynomials involving the descent statistic.
Definition 1. Let \( w = w_1w_2 \cdots w_k \) be a word of length \( k \) with distinct letters from \([n]\). An admissible inversion of \( w \) is a pair \((w_i, w_j)\) such that \( 1 \leq i < j \leq k \) and \( w_i > w_j \) and satisfies either of the following conditions:

- \( 1 < i \) and \( w_{i-1} < w_i \) or
- there is some \( l \) such that \( i < l < j \) and \( w_i < w_l \).

Let \( ai(w) \) be the number of admissible inversions of \( w \). The statistic \( rix \) is defined recursively as follows:

\[
rix(w) := \begin{cases} 
0, & \text{if } i = 1 < k; \\
1 + rix(w_1w_2 \cdots w_{k-1}), & \text{if } i = k; \\
rix(w_{i+1}w_{i+2} \cdots w_k), & \text{if } 1 < i < k.
\end{cases}
\]

For example, if \( w = 291753468 \), then there are 14 inversions, except (5, 3) and (5, 4), all others are admissible, hence \( ai(w) = 12 \); for the computation of \( rix(w) \), we have successively \( rix(w) = rix(1753468) = 1 + rix(175346) = 1 + rix(5346) = 2 + rix(534) = 2 \).

Recall the following interpretation of the basic Eulerian polynomials [9, Theorem 8] (see also [3] for an equivalent version), of which the special \( r = 1 \) case was first proved in [10] using Rees product of posets.

Lemma 6 ([3][9][14]). One has

\[
A_n(t, r, q) = \sum_{\sigma \in S_n} t^{\text{des}(\sigma)} t^{rix(\sigma)} q^{ai(\sigma)}.
\] (1.11)

The rest of this paper is organized as follows. In Section 2, after recalling the Modified Foata–Strehl (MFS) action, we prove Theorem 4. In Section 3, we introduce an alternative algorithm to compute the statistic \( rix \), which connects our model of \( R_0^{n,k} \) to \( E_{n,k} \). Proof of Theorem 5 will be given in Section 4. We derive the recurrence relations and the generating functions of the \( \gamma \)-coefficients in (1.9) and (1.10) in Section 5. The paper is concluded with some further remarks in Section 6.

2. Proof of Theorem 4

Let \( \sigma \in S_n \), for any \( x \in [n] \), the \( x \)-factorization of \( \sigma \) reads \( \sigma = w_1w_2xw_3w_4 \), where \( w_2 \) (resp. \( w_3 \)) is the maximal contiguous subword immediately to the left (resp. right) of \( x \) whose letters are all smaller than \( x \). Following Foata and Strehl [7] we define the action \( \varphi_x \) by

\[
\varphi_x(\sigma) = w_1w_3xw_2w_4.
\]

For instance, if \( x = 4 \) and \( \sigma = 2743156 \in S_7 \), then \( w_1 = 27, w_2 = \emptyset, w_3 = 31 \) and \( w_4 = 56 \). Thus \( \varphi_x(\sigma) = 2731456 \). Clearly, \( \varphi_x \) is an involution acting on \( S_n \) and it is not hard to
see that $\varphi_x$ and $\varphi_y$ commute for all $x, y \in [n]$. Brändén [2] modified $\varphi_x$ to be

$$
\varphi'_x(\sigma) := \begin{cases} 
\varphi_x(\sigma), & \text{if } x \text{ is a double ascent or double descent of } \sigma; \\
\sigma, & \text{if } x \text{ is a valley or a peak of } \sigma.
\end{cases}
$$

Again it is clear that $\varphi'_x$'s are involutions and commute. For any subset $S \subseteq [n]$ we can then define the function $\varphi'_S : \mathfrak{S}_n \to \mathfrak{S}_n$ by

$$
\varphi'_S(\sigma) = \prod_{x \in S} \varphi'_x(\sigma).
$$

Hence the group $\mathbb{Z}_2^n$ acts on $\mathfrak{S}_n$ via the functions $\varphi'_S$, $S \subseteq [n]$. This action will be called the Modified Foata–Strehl action ($\text{MFS-action for short}$) as depicted in Fig. 1.

We first show that the statistic $a_i$ is invariant under the MFS action.

**Lemma 7.** Let $\sigma \in \mathfrak{S}_n$. For each $x \in [n]$, we have $a_i(\sigma) = a_i(\varphi'_x(\sigma))$.

**Proof.** If $x$ is a peak or a valley of $\sigma$, then $\varphi'_x(\sigma) = \sigma$ and the result is true. If $x$ is a double descent of $\sigma$, then $w_3$ is empty in the $x$-factorization of $\sigma = w_1xw_3w_4$ and there is no admissible inversions of $\sigma$ between $x$ and the word $w_3$. As $\varphi'_x(\sigma) = w_1w_3xw_4$, there is no inversions of $\varphi'_x(\sigma)$ between the word $w_3$ and $x$. Let $(\sigma_i, \sigma_j) \notin \{(x, y) : y \text{ is a letter in } w_3\}$ be a pair of $\sigma$ such that $i < j$. We claim that $(\sigma_i, \sigma_j)$ is an admissible inversion of $\sigma$ if and only if it is an admissible inversion of $\varphi'_x(\sigma)$, from which the result follows.

For a word $w$, we write $a \in w$ if $a$ is a letter in $w$. To check the claim, there are 6 cases to be considered: (1) $\sigma_i \in w_1$ and $\sigma_j \in w_1$; (2) $\sigma_i \in w_1$ and $\sigma_j \in xw_3$; (3) $\sigma_i \in w_1$ and $\sigma_j \in w_4$; (4) $\sigma_i \in w_3$ and $\sigma_j \in w_3$; (5) $\sigma_i \in xw_3$ and $\sigma_j \in w_4$; (6) $\sigma_i \in w_4$ and $\sigma_j \in w_4$. We will only show case (2), other cases are similar. If $(\sigma_i, \sigma_j)$ is an admissible inversion of $\sigma$, then $\sigma_{i-1} < \sigma_i > \sigma_j$ or $\sigma_j < \sigma_i < \sigma_k$ for some $i < k < j$. Clearly, $(\sigma_i, \sigma_j)$ is an admissible of $\varphi'_x(\sigma)$ if $\sigma_k \neq x$. Otherwise if $\sigma_k = x$, then we denote $x'$ the last letter of $w_1$ and consider the triple $(\sigma_i, x', \sigma_j)$. This indicates that $(\sigma_i, \sigma_j)$ is an admissible inversion of $\varphi'_x(\sigma)$, since $x' > x > \sigma_i$. To show that, if $(\sigma_i, \sigma_j)$ is an admissible inversion of $\varphi'(\sigma)$ then $(\sigma_i, \sigma_j)$ is an admissible inversion of $\sigma$, is similar and we omit. This finishes the proof of our claim in case (2). \hfill \Box

For any permutation $\sigma \in \mathfrak{S}_n$, let $\text{Orb}(\sigma) = \{g(\sigma) : g \in \mathbb{Z}_2^n\}$ be the orbit of $\sigma$ under the MFS-action. The MFS-action divides the set $\mathfrak{S}_n$ into disjoint orbits. Moreover, for
σ ∈ \mathcal{S}_n$, if $x$ is a double descent of $σ$, then $x$ is a double ascent of $\varphi'_x(σ)$. Hence, there is a unique permutation in each orbit which has no double descent. Now, let $\bar{σ}$ be such a unique element in $\text{Orb}(σ)$, then $\text{da}(\bar{σ}) = n - \text{peak}(\bar{σ}) - \text{valley}(\bar{σ})$ and $\text{des}(\bar{σ}) = \text{peak}(\bar{σ}) = \text{valley}(\bar{σ}) - 1$. Thus
\[
\sum_{\pi \in \text{Orb}(σ)} q^{\text{ai}(\pi)} t^{\text{des}(\pi)} = q^{\text{ai}(σ)} t^{\text{des}(σ)} (1 + t)^{\text{da}(σ)} = q^{\text{ai}(σ)} t^{\text{des}(σ)} (1 + t)^{n - 1 - 2\text{des}(σ)}.
\]
Therefore, by (1.11), we have
\[
A_n(t, 1, q) = \sum_{σ ∈ \mathcal{S}_n} t^{\text{des}σ} q^{\text{ai}σ} = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \left( \sum_{σ ∈ \mathcal{D}_{n,k}} p^{\text{ai}(σ)} \right) t^{k(1 + t)^{n - 1 - 2k}}.
\]
Thus Theorem IV is a consequence of the following result.

**Lemma 8.** For each $σ ∈ \mathcal{D}_{n,k}$, we have $\text{ai}(σ) = \text{inv}(σ)$.

**Proof.** Let $σ ∈ \mathcal{S}_n$ be a permutation without double descents. Let $(σ_i, σ_j)$ be an inversion of $σ$. If $i = 1$, then $σ_1 < σ_2$ and so $(σ_i, σ_j)$ is an admissible inversion of $σ$. If $i ≥ 2$, then there are two cases to be considered: $σ_{i-1} < σ_i$ or $σ_{i-1} > σ_i$. In the first case, $(σ_i, σ_j)$ is an admissible inversion of $σ$. In the second case, we must have $σ_{i+1} > σ_i$, otherwise $σ_i$ will be a double descent of $σ$. Thus in this case, $(σ_i, σ_j)$ is also an admissible inversion of $σ$. This shows that $\text{ai}(σ) ≥ \text{inv}(σ)$ and therefore the desired result. \(\square\)

### 3. Rix-factorization

We first give a slightly different (less recursive) description of the “rix” statistic. Let $w = w_1 \cdots w_k$ be a word with distinct letters from $[n]$. We say that $w_i$ is a descent top of $w$ if $w_i > w_{i+1}$. If the greatest letter of $w$ is $w_k$ (resp. $w_1$ and $k ≥ 2$), then $w$ is called a quasi-hook (resp. inverse quasi-hook).

**Definition 2** (Rix factorization). We factorize each permutation $σ ∈ \mathcal{S}_n$ as
\[
σ = α_1α_2 \cdots α_iβ,
\] where each $α_i$ (if any) is a quasi-hook and $β$ is a quasi-hook or an inverse quasi-hook, by applying the following algorithm:

(i) $w ← σ$; $i ← 0$;
(ii) if $w$ is an increasing word, let $β = w$ and we get (3.1); otherwise, $i ← i + 1$, let $x$ be the greatest descent top of $w$ and write $w = w'xw''$ for some subwords $w', w''$;
(iii) if $w' ≠ ∅$, then $α_i = w'x$, $w ← w''$ and go to (ii); otherwise, let $β = w$ and we get (3.1).

The factorization (3.1) will be called the rix-factorization of $σ$. Denote by $β_1(σ)$ the first letter of $β$. Then a letter in the maximal increasing suffix of $β$ that is not smaller than $β_1(σ)$ will be called a rixed point of $σ$. Denote by $\text{RIX}(σ)$ the set of all rixed points of $σ$. 
For example, the algorithm gives

\[
\begin{align*}
\sigma &= 21879354610 = 2187935|\beta, \quad \text{with } \beta = 4610 \text{ (quasi-hook)} \\
\tau &= 61108497253 = 6110849|\beta \quad \text{with } \beta = 7253 \text{ (inverse quasi-hook)} \\
\pi &= 11047625389 = 110476253|\beta \quad \text{with } \beta = 625389 \text{ (quasi-hook)}.
\end{align*}
\]

We have

\[
\begin{align*}
\beta_1(\sigma) &= 4, \quad \text{RIX}(\sigma) = \{4, 6, 10\}; \\
\beta_1(\tau) &= 7, \quad \text{RIX}(\tau) = \emptyset; \\
\beta_1(\pi) &= 6, \quad \text{RIX}(\pi) = \{8, 9\}.
\end{align*}
\]

**Proposition 9.** Let \( \sigma = \alpha_1\alpha_2\cdots\alpha_i\beta \) be the rix-factorization of \( \sigma \in \mathfrak{S}_n \). Let \( x_k \) be the last letter of \( \alpha_k \) for \( k = 1, \ldots, i \). Then

\[
x_1 > x_2 > \cdots > x_i > \beta_1(\sigma).
\]

**Proof.** Since \( x_i \) is a descent top of \( \alpha_i\beta \), we have \( x_i > \beta_1(\sigma) \). For \( k = 1, \ldots, i-1 \), since \( x_k \) is the greatest descent top of \( \alpha_k\cdots\alpha_i\beta \), it is obvious that \( x_k > x_{k+1} \). \( \square \)

**Proposition 10.** For each \( \sigma \in \mathfrak{S}_n \) we have

\[
\text{rix}(\sigma) = |\text{RIX}(\sigma)|.
\]

In particular, \( \text{rix}(\sigma) = 0 \) if and only if \( \beta \) is an inverse quasi-hook in the rix-factorization of \( \sigma \).

**Proof.** It follows from the observation that the letters to the left of any descent top do not give contribution to the statistic “rix”. \( \square \)

For each permutation \( \sigma \in \mathfrak{S}_n \), we define its standard cycle form (SCF) as the product of ordered cycles satisfying the following conditions:

- each cycle \((a \sigma(a) \cdots \sigma^j(a))\), with \( \sigma^{j+1}(a) = a \), is written with its largest element first \( a \geq \sigma^k(a) \) for all \( k \geq 1 \);
- the cycles of length \( \geq 2 \) are arranged in decreasing order of their largest element;
- the cycles of length 1 are arranged in increasing order of their elements and after the cycles of length \( \geq 2 \).

For example, the SCF of \( \sigma = 129753468 \) is \( \text{SCF}(\sigma) = (9863)(74)(1)(2)(5) \).

Let us introduce the set

\[
\mathcal{R}_{n,k}^0 := \{ \sigma \in \mathfrak{S}_n : \text{rix}(\sigma) = 0, \text{dd}(\sigma) = 1, \text{des}(\sigma) = k \}.
\]

Let \( \text{FIX}(\sigma) \) be the set of all fixed points of \( \sigma \in \mathfrak{S}_n \).

**Proposition 11.** There is a bijection \( \Phi : \mathfrak{S}_n \to \mathfrak{S}_n \) satisfying

\[
\text{des}(\sigma) = \text{exc}(\Phi(\sigma)) \quad \text{and} \quad \text{RIX}(\sigma) = \text{FIX}(\Phi(\sigma))
\]

for each \( \sigma \in \mathfrak{S}_n \). Moreover, the restriction of \( \Phi \) on \( \mathcal{R}_{n,k}^0 \) is a bijection from \( \mathcal{R}_{n,k}^0 \) to \( \mathcal{E}_{n,k} \).
Proof. Starting from $\sigma \in S_n$ with rix-factorization $\alpha_1\alpha_2\cdots\alpha_i\beta$. Deleting all the rixed points of $\sigma$ we get

$$\sigma' = \alpha_1\alpha_2\cdots\alpha_i\beta',$$

where $\beta'$ is the word (may be empty) obtained from $\beta$ by removing all the rixed points. Clearly, $\beta'$ is an inverse quasi-hook. To any quasi-hook or inverse quasi-hook $\alpha = a_1a_2\cdots a_l$, we associate the cycle

$$\tilde{\alpha} = \begin{cases} (a_l, a_{l-1}, \ldots, a_1), & \text{if } \alpha \text{ is a quasi-hook;} \\ (a_1, a_l, a_{l-1}, \ldots, a_2), & \text{if } \alpha \text{ is an inverse quasi-hook.} \end{cases}$$

Now, we define the SCF of $\Phi(\sigma)$ by

$$\Phi(\sigma) := \tilde{\alpha}_1\tilde{\alpha}_2\cdots\tilde{\alpha}_i\tilde{\beta}'(\sigma_k)(\sigma_{k+1})\cdots(\sigma_n),$$

where $\text{RIX}(\sigma) = \{\sigma_k, \sigma_{k+1}, \ldots, \sigma_n\}$. By Proposition 9, the mapping $\Phi$ is well-defined. For example, if $\sigma = 7\ 6\ 9\ 1\ 8\ 4\ 2\ 3\ 5\ 10 = 7\ 6\ 9\ |\ 1\ 8\ |\ 4\ 2\ 3\ 5\ 10$, then $\sigma' = 7\ 6\ 9\ 1\ 8\ = 7\ 6\ 9\ |\ 1\ 8$ and thus

$$\Phi(\sigma) = (9, 6, 7)(8, 1)(4, 3, 2)(5)(10) = 8\ 4\ 2\ 3\ 5\ 7\ 9\ 1\ 6\ 10.$$ 

Clearly, $\text{des}(\sigma) = 4 = \text{exc}(\Phi(\sigma))$ and $\text{RIX}(\sigma) = \{5, 10\} = \text{FIX}(\Phi(\sigma))$.

To show that $\Phi$ is a bijection, we define explicitly its inverse. Given a permutation $\sigma \in S_n$, we write $\sigma$ in cycle form as

$$\sigma = C_1C_2\cdots C_iO_1O_2\cdots O_j,$$

where $C_i$'s are cycles of length $\geq 2$ and $O_k = (o_k)$ is one point cycle for $k = 1, 2, \ldots, j$. For each cycle $C = (c_1, c_2, \ldots, c_l)$ of length $l \geq 2$, we define the two words (quasi-hook or inverse quasi-hook)

$$\tilde{C} := c_lc_{l-1}\cdots c_1 \quad \text{and} \quad C' := c_1c_{l-1}\cdots c_2.$$ 

Then $\Phi^{-1}(\sigma)$ (viewed as a word) is defined as

$$\Phi^{-1}(\sigma) = \begin{cases} \tilde{C}_1\tilde{C}_2\cdots C'_1o_1\cdots o_j, & \text{if } o_1 > \text{the largest element of } C_i; \\ \tilde{C}_1\tilde{C}_2\cdots \tilde{C}_io_1\cdots o_j, & \text{otherwise}. \end{cases}$$

For example, $\Phi^{-1}((9, 6, 7)(8, 1)(4, 3, 2)(5)(10)) = 7\ 6\ 9\ 1\ 8\ 4\ 2\ 3\ 5\ 10$.

It is straightforward to check the desired properties of $\Phi$. \hfill \Box

Remark 3. In view of $(111)$, the two triples $(\text{exc, fix, maj})$ and $(\text{des, rix, aid})$ are equidistributed on $S_n$, where $\text{aid}(\sigma) = \text{ai}(\sigma) + \text{des}(\sigma)$. Note that $(\text{FIX, maj})$ and $(\text{RIX, aid})$ are not equidistributed on $S_3$. 

It is easy to construct a bijection between $\mathcal{R}_{n,k}^0$ and $\tilde{\mathcal{D}}_{n,k}$. Define the mapping
\[ f : \mathcal{R}_{n,k}^0 \to \tilde{\mathcal{D}}_{n,k} \] by
\[ f(\sigma) = \varphi_x'(\sigma), \quad (3.2) \]
where $\sigma$ is any permutation in $\mathcal{R}_{n,k}^0$ and $x = \beta_1(\sigma)$.

**Corollary 12.** The mapping $f$ is a bijection between $\mathcal{R}_{n,k}^0$ and $\tilde{\mathcal{D}}_{n,k}$.

**Proof.** For each $\sigma \in \mathcal{R}_{n,k}^0$, by Propositions 10 and 9 the letter $\beta_1(\sigma)$ is the only double descent of $\sigma$, which becomes a double ascent of $f(\sigma)$. As $\text{rix}(\sigma) = 0$, $\beta_1(\sigma)$ is the last letter of $f(\sigma)$. Thus $f(\sigma)$ is a permutation in $\tilde{\mathcal{D}}_{n,k}$ and $f$ is well-defined. To finish the proof that $f$ is a bijection we define its inverse $f^{-1}$ as
\[ f^{-1}(\sigma) = \varphi_y'(\sigma), \]
where $\sigma$ is any permutation in $\tilde{\mathcal{D}}_{n,k}$ and $y$ is the last letter of $\sigma$. All we need to show is that now $f^{-1}(\sigma)$ is a permutation in $\mathcal{R}_{n,k}^0$. Clearly, $y$ becomes the only double descent in $f^{-1}(\sigma)$ and $\text{des}(f^{-1}(\sigma)) = k$. Moreover, $y = \beta_1(f^{-1}(\sigma))$ by the algorithm of the rix-factorization, and so $\text{rix}(f^{-1}(\sigma)) = 0$. Therefore, $f^{-1}(\sigma)$ is a permutation in $\mathcal{R}_{n,k}^0$. □

**Remark 4.** In view of Proposition 11, the composition $\Phi \circ f^{-1}$ is a bijection between $\tilde{\mathcal{D}}_{n,k}$ and $\mathcal{E}_{n,k}$. This bijection is illustrated in Table 1.

| $\tilde{\mathcal{D}}_{4,2}$ | 1324 | 1423 | 2314 | 2413 | 3412 |
|-----------------------------|------|------|------|------|------|
| $\mathcal{R}_{4,2}^0$      | 4132 | 1432 | 4213 | 2431 | 3421 |
| $\mathcal{E}_{4,2}$        | 4312 | 4321 | 2413 | 3412 | 2143 |

Table 1. The bijection $\Phi \circ f^{-1} : \tilde{\mathcal{D}}_{4,2} \xrightarrow{f^{-1}} \mathcal{R}_{4,2} \xrightarrow{\Phi} \mathcal{E}_{4,2}$.

**4. Proof of Theorem 5**

For any $x \in [n]$, we introduce a new action $\varphi''_x : \mathcal{S}_n \to \mathcal{S}_n$ as
\[ \varphi''_x(\sigma) = \begin{cases} \sigma, & \text{if } x \in \{ \beta_1(\sigma) \} \cup \text{RIX}(\sigma); \\ \varphi'_x(\sigma), & \text{otherwise}. \end{cases} \]

Since $\varphi'_x$'s are involutions and that they commute, so do $\varphi''_x$'s. Thus, for any subset $S \subseteq [n]$ we can define the mapping $\varphi''_S : \mathcal{S}_n \to \mathcal{S}_n$ by
\[ \varphi''_S(\sigma) = \prod_{x \in S} \varphi''_x(\sigma). \]
This is a new $\mathbb{Z}_2^n$-action on $\mathfrak{S}_n$ via the functions $\varphi''_x$, $S \subseteq [n]$, that we called \textit{restricted MFS-action}. An important property of the action $\varphi''_x$ is that it preserves the rix-factorization type of permutations, namely,

\textbf{Lemma 13.} Let $\sigma \in \mathfrak{S}_n$ be a permutation with rix-factorization $\alpha_1\alpha_2\cdots\alpha_i\alpha_{i+1}$ with $\alpha_{i+1} = \beta$. For any $x \in [n]$, if $x$ is a letter of $\alpha_k$ ($1 \leq k \leq i+1$), then the rix-factorization of $\varphi''_x(\sigma)$ is $\alpha_1\cdots\alpha_{i'}\cdots\alpha_i\alpha_{i+1}$, where $\alpha'_{i'}$ is a rearrangement of $\alpha_k$. Moreover,

$$\beta_1(\varphi''_x(\sigma)) = \beta_1(\sigma) \quad \text{and} \quad \text{RIX}(\varphi''_x(\sigma)) = \text{RIX}(\sigma). \quad (4.1)$$

\textit{Proof.} For $x \in [n]$ we apply $\varphi''_x$ to $\sigma$ and distinguish two cases.

a) $x$ is a letter of $\alpha_k$ ($1 \leq k \leq i$). Let $x_k$ be the last letter of $\alpha_k$, which is also the greatest letter in $\alpha_k$. If $x = x_k$, then $x$ is a peak of $\sigma$ by Proposition 9, so $\varphi''_x(\sigma) = \sigma$. If $x \neq x_k$, then $x < x_k < x_{k-1}$ by Proposition 9, where $x_0 = +\infty$. Therefore, $\varphi''_x(\sigma) = \alpha_1\cdots\alpha_{k-1}\alpha_k\alpha_{k+1}\cdots\alpha_i\beta$ with $x_k$ as the last letter of $\varphi''_x(\alpha_k)$.

b) $x$ is a letter of $\alpha_{i+1} = \beta$. If $x \in \{\beta_1(\sigma)\} \cup \text{RIX}(\sigma)$, then $\varphi''_x(\sigma) = \sigma$; otherwise, $x < \beta_1(\sigma)$. Therefore, $\varphi''_x(\sigma) = \alpha_1\cdots\alpha_i\varphi''_x(\beta)$ with $\beta_1(\sigma)$ as the first letter of $\varphi''_x(\beta)$.

In both cases, the result follows from the rix-factorization. \hfill \Box

Let $\mathcal{R}_n$ be the set of permutations in $\mathfrak{S}_n$ without rixed points.

\textbf{Lemma 14.} We have

$$\sum_{\sigma \in \mathcal{R}_n} q^{ai(\sigma)}t^{des(\sigma)} = \sum_{k=1}^{\lfloor n/2 \rfloor} \left( \sum_{\sigma' \in \mathcal{R}_{n,k}} q^{ai(\sigma')} \right) t^k (1 + t)^{n-2k}.$$

\textit{Proof.} First we note that the restricted MFS-action is stable on $\mathcal{R}_n$, that is, $\sigma \in \mathcal{R}_n$ implies $\varphi''_x(\sigma) \in \mathcal{R}_n$ for any $x \in [n]$. This follows from the property of $\varphi''_x$ in (1.1).

For a permutation $\sigma \in \mathcal{R}_n$ let $\text{Orb}(\sigma) = \{g(\sigma) : g \in \mathbb{Z}_2^n\}$ be the orbit of $\sigma$ in $\mathcal{R}_n$ under the restricted MFS-action. By Propositions 9 and 10, the letter $\beta_1(\sigma)$ is a double descent of $\sigma$. If $x \in [n]$ is a double descent of $\sigma$ different from $\beta_1(\sigma)$, then $x$ becomes a double ascent of $\varphi''_x(\sigma)$ and $\text{des}(\sigma) = \text{des}(\varphi''_x(\sigma)) + 1$. Therefore, there is a unique element in $\text{Orb}(\sigma)$, denoted $\tilde{\sigma}$, which has only one double descent $\beta_1(\tilde{\sigma})$ and

$$\sum_{\pi \in \text{Orb}(\sigma)} q^{ai(\pi)}t^{des(\pi)} = q^{ai(\tilde{\sigma})}t^{des(\tilde{\sigma})} (1 + t)^{\text{da}(\tilde{\sigma})}.$$

For any $\sigma \in \mathfrak{S}_n$ we have $\text{valley}(\sigma) = \text{peak}(\sigma) + 1$. Thus

$$\text{dd}(\sigma) + \text{da}(\sigma) + 2\text{peak}(\sigma) = n - 1. \quad (4.2)$$

The result then follows from (4.2) and the observation that $\text{da}(\tilde{\sigma}) = 1$ and $\text{peak}(\tilde{\sigma}) = \text{des}(\tilde{\sigma}) - 1$. \hfill \Box
It follows from Eq. (1.11) and Lemmas 7 and 14 that

\[ A_n(t, 0, q) = \sum_{\sigma \in \mathcal{R}_n} t^{\text{des}(\sigma)} q^{\text{ai}(\sigma)} = \sum_{k=1}^{\lfloor n/2 \rfloor} \left( \sum_{\sigma \in \mathcal{R}_{n,k}} q^{\text{ai}(\sigma)} \right) t^k (1 + t)^{n-2k}. \]

Applying the bijection \( f : \mathcal{R}_{n,k}^0 \to \tilde{\mathcal{D}}_{n,k} \) defined in (3.2), we get

\[ A_n(t, 0, q) = \sum_{k=1}^{\lfloor n/2 \rfloor} \left( \sum_{\sigma \in \tilde{\mathcal{D}}_{n,k}} q^{\text{ai}(\sigma)} \right) t^k (1 + t)^{n-2k}, \]

which is equivalent to expansion (1.10) in view of Lemma 8.

5. Generating functions of \( \gamma \)-coefficients

In this section, we study the recurrences and generating functions of the two \( \gamma \)-coefficients in (1.9) and (1.10), namely,

\[ \gamma_{n,k}(q) := \sum_{\sigma \in \mathcal{D}_{n,k}} q^{\text{inv}(\sigma)} \quad \text{and} \quad \tilde{\gamma}_{n,k}(q) := \sum_{\sigma \in \tilde{\mathcal{D}}_{n,k}} q^{\text{inv}(\sigma)}. \]

Let \( \mathcal{D}_n = \{ \sigma \in S_n : \text{dd}(\sigma) = 0 \} \) and \( \tilde{\mathcal{D}}_n = \{ \sigma \in \mathcal{D}_n : \sigma_{n-1} < \sigma_n \} \). Then

\[ \Gamma_n(y, q) := \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \gamma_{n,k}(q)y^k = \sum_{\sigma \in \mathcal{D}_n} y^{\text{des}(\sigma)} q^{\text{inv}(\sigma)}; \]

\[ \tilde{\Gamma}_n(y, q) := \sum_{k=1}^{\lfloor n/2 \rfloor} \tilde{\gamma}_{n,k}(q)y^k = \sum_{\sigma \in \tilde{\mathcal{D}}_n} y^{\text{des}(\sigma)+1} q^{\text{inv}(\sigma)}. \]

The first values of \( \Gamma_n(y, q) \) and \( \tilde{\Gamma}_n(y, q) \) for \( 1 \leq n \leq 5 \) are:

\[ \Gamma_1(y, q) = \Gamma_2(y, q) = 1, \quad \Gamma_3(y, q) = 1 + y(q + q^2), \]
\[ \Gamma_4(y, q) = 1 + y(q + q^2)(2 + q + q^2), \]
\[ \Gamma_5(y, q) = 1 + y(3q + 5q^2 + 5q^3 + 5q^4 + 2q^5 + 2q^6) \]
\[ + y^2(q + q^3)(1 + q + q^2 + q^3)(q + q^2); \]

and

\[ \tilde{\Gamma}_1(y, q) = 0, \quad \tilde{\Gamma}_2(y, q) = \tilde{\Gamma}_3(y, q) = y, \]
\[ \tilde{\Gamma}_4(y, q) = y + y^2(q + 2q^2 + q^3 + q^4), \]
\[ \tilde{\Gamma}_5(y, q) = y + y^2(2q + 4q^2 + 4q^3 + 4q^4 + 2q^5 + 2q^6). \]
Proposition 15. The exponential generating functions for $A_n$ with initial condition $A_0 = 1$ and the initial conditions $A_1 = (1 + t)^n \Gamma_n(y, q)$, $A_2 = (1 + t)^n \tilde{\Gamma}_n(y, q)$, where $\Gamma_0(y, q) = \tilde{\Gamma}_0(y, q) = 1$ and $t = y = t/(1 + t)^2$. 

Proof. Clearly Theorems 4 and 5 are equivalent to, for $n \geq 1$,

$$A_n(t, 1, q) = (1 + t)^n \Gamma_n(y, q),$$

$$A_n(t, 0, q) = (1 + t)^n \tilde{\Gamma}_n(y, q),$$

where $\Gamma_0(y, q) = \tilde{\Gamma}_0(y, q) = 1$ and $y = t/(1 + t)^2$. Substituting (5.7) and (5.8) into (5.4) yields (5.5) and (5.6).

It is proved in [9] by applying the $q$-differential operator to both sides of (5.4) that the basic Eulerian polynomials $A_n(t, r, q)$ ($n \geq 1$) satisfy the following recurrence:

$$A_{n+1}(t, r, q) = (r + t)A_n(t, r, q) + t \sum_{j=1}^{n-1} \binom{n}{j} q^j A_j(t, r, q) A_{n-j}(t, 1, q)$$

with initial condition $A_1(t, r, q) = r$. Substituting (5.7) into (5.9) we obtain the recurrences for $\gamma$-coefficients.

Proposition 16. The polynomials $\Gamma_n(y, q)$ and $\tilde{\Gamma}_n(y, q)$ satisfy the following recurrences:

$$\Gamma_{n+1}(y, q) = \Gamma_n(y, q) + y \sum_{i=1}^{n-1} q^i \binom{n}{i} \Gamma_i(y, q) \Gamma_{n-i}(y, q);$$

$$\tilde{\Gamma}_{n+1}(y, q) = \Gamma_n(y, q) + y \sum_{i=2}^{n-1} q^i \binom{n}{i} \tilde{\Gamma}_i(y, q) \Gamma_{n-i}(y, q);$$

for $n \geq 1$ and the initial conditions $\Gamma_0(t, q) = \tilde{\Gamma}_0(t, q) = 1$ and $\tilde{\Gamma}_1(t, q) = 0$. 

For $n \geq 1$ let $(q; q)_n := \prod_{i=1}^{n}(1 - q^i)$ be the shifted $q$-factorial and $(q; q)_0 = 1$. The $q$-exponential function and $q$-binomial coefficients are then defined by

$$e(z; q) := \sum_{n \geq 0} z^n / (q; q)_n$$

and

$$\binom{n}{k}_q := (q; q)_n / (q; q)_{n-k}.$$
In fact, we can give a direct combinatorial proof of Proposition 16. Recall that the $q$-binomial coefficient has the combinatorial interpretation \[16, Proposition 1.3.17\]

\[{n \choose k}_q = \sum_{(A,B)} q^{\text{inv}(A,B)}, \quad (5.12)\]

where the sum is over all ordered set partitions $(A, B)$ of $[n]$ such that $|A| = k$ and

\[\text{inv}(A, B) := \{(i, j) \in A \times B : i > j\}.\]

**Combinatorial proof of Proposition 16.** For $2 \leq j \leq n$ let $\mathcal{D}_n^{(j)} := \{\sigma \in \mathfrak{S}_n : \sigma_j = n\}$ and define

\[\Gamma_n^{(j)}(y, q) := \sum_{\sigma \in \mathcal{D}_n^{(j)}} y^{\text{des}(\sigma)} q^{\text{inv}(\sigma)}.\]

Clearly, for $n \geq 1$

\[\Gamma_{n+1}(y, q) = \sum_{j=2}^{n+1} \Gamma_{n+1}^{(j)}(y, q) = \Gamma_n(y, q) + \sum_{j=2}^{n} \Gamma_n^{(j)}(y, q). \quad (5.13)\]

For a set $X$ let $\binom{X}{m}$ denote the set of the $m$-element subsets of $X$ and $\mathfrak{S}_X$ the set of permutations of $X$ without double descents. Also let $\mathcal{T}(n, j)$ be the set of all triples $(S, \sigma_L, \sigma_R)$ such that $S \in \binom{[n]}{j}$ and $\sigma_L \in \mathfrak{S}_S$, $\sigma_R \in \mathfrak{S}_{[n] \setminus S}$. For $2 \leq j \leq n - 1$, define the mapping $\sigma \mapsto (S, \sigma_L, \sigma_R)$ by

- $S = \{\sigma_i : 1 \leq i \leq j - 1\}$;
- $\sigma_L = \sigma_1 \sigma_2 \cdots \sigma_{j-1}$ and $\sigma_R = \sigma_{j+1} \sigma_{j+2} \cdots \sigma_n$.

It is not hard to see that this mapping is a bijection between $\mathcal{D}_n^{(j)}$ and $\mathcal{T}(n-1, j-1)$ and satisfies

\[\text{des}(\sigma) = \text{des}(\sigma_L) + \text{des}(\sigma_R) + 1\]

and

\[\text{inv}(\sigma) = \text{inv}(\sigma_L) + \text{inv}(\sigma_R) + \text{inv}(S, [n-1] \setminus S) + n - j.\]

Thus, for $2 \leq j \leq n$ we have

\[\Gamma_{n+1}^{(j)}(y, q) = \sum_{\sigma \in \mathcal{D}^{(j)}_{n+1}} y^{\text{des}(\sigma)} q^{\text{inv}(\sigma)}\]

\[= y q^{n+1-j} \sum_{(S, \sigma_L, \sigma_R) \in \mathcal{T}(n-1, j-1)} q^{\text{inv}(S, [n] \setminus S)} q^{\text{inv}(\sigma_L)} y^{\text{des}(\sigma_L)} q^{\text{inv}(\sigma_R)} y^{\text{des}(\sigma_R)}\]

\[= y q^{n+1-j} \sum_{S \in \binom{[n]}{j-1}} q^{\text{inv}(S, [n] \setminus S)} \sum_{\sigma_L \in \mathfrak{S}_S} q^{\text{inv}(\sigma_L)} y^{\text{des}(\sigma_L)} \sum_{\sigma_R \in \mathfrak{S}_{[n] \setminus S}} q^{\text{inv}(\sigma_R)} y^{\text{des}(\sigma_R)}\]

\[= y q^{n+1-j} \left[ \sum_{j-1}^{n} \binom{n}{j-1} \Gamma_{j-1}(y, q) \Gamma_{n+1-j}(y, q) \right], \quad (5.14)\]
where we apply (5.12) to the last equality. Substituting (5.14) into (5.13) we obtain (5.10). Similarly we can prove (5.11), the details are left to the interested reader. □

6. Concluding remarks

Using the combinatorics developed in Section 3, we can also give a combinatorial proof of Theorem 2, of which a similar proof for the special $\beta = 1$ case was given in [1, 17].

Proof of Theorem 2 Define the statistic “lyc” by $\text{lyc}(\sigma) := \text{cyc}(\Phi(\sigma))$ for $\sigma \in S_n$. Since the function $\varphi' : S_n \to S_n$ preserves the rix-factorization type of permutations for any $x \in [n]$, the statistic “lyc” is invariant under the restricted MFS-action. Thus by Lemma 14, we have

$$\sum_{\sigma \in R_n} \beta \text{lyc}(\sigma) t \text{des}(\sigma) = \frac{n}{2} \sum_{j=1} \left( \sum_{\sigma \in R_{n,j}} \beta \text{lyc}(\sigma) \right) t^j (1 + t)^{n-2j}. $$

By Proposition 11, applying the bijection $\Phi : S_n \to S_n$ to both sides of the above expansion we get (1.5). □

We can further extend (1.5) and (1.10) to permutations with a given number of fixed points. Indeed, it is known [14, Eq.(4.3)] and easy to deduce from (5.4) that

$$\sum_{\sigma \in S_n} t^{\text{exc}(\sigma)} q^{\text{maj}(\sigma) - \text{exc}(\sigma)} = \begin{bmatrix} n \end{bmatrix}_j A_{n-j}(t, 0, q). \tag{6.1}$$

Also, it is easy to see that

$$\sum_{\sigma \in S_n} \beta^{\text{cyc}(\sigma)} t^{\text{exc}(\sigma)} = \begin{bmatrix} n \end{bmatrix}_k \beta^j \sum_{\sigma \in S_{n-k, j}} \beta^{\text{cyc}(\sigma)} t^{\text{exc}(\sigma)}. \tag{6.2}$$

We derive from Theorem 2 and Theorem 5 the following corollaries.

Corollary 17. For $1 \leq k \leq n$ we have

$$\sum_{\sigma \in S_n} \beta^{\text{cyc}(\sigma)} t^{\text{exc}(\sigma)} = \begin{bmatrix} n/2 \end{bmatrix}_k \left( \sum_{\sigma \in S_{n-k, j}} \beta^{\text{cyc}(\sigma) + j} \right) t^j (1 + t)^{n-2j}, \tag{6.3}$$

and

$$\sum_{\sigma \in S_n} t^{\text{exc}(\sigma)} q^{\text{maj}(\sigma) - \text{exc}(\sigma)} = \sum_{j=1} \begin{bmatrix} (n-k)/2 \end{bmatrix}_k q^{\text{inv}(\sigma)} t^j (1 + t)^{n-k-2j}. \tag{6.4}$$
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