Fermat Theorems — Simple Proofs

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Abstract

The Modular Group provides simple proofs of Fermat’s representations:

\[ X^2 + Y^2 \] for primes congruent to 1 (mod 4) and by

\[ X^2 + 3Y^2 \] for primes congruent to 1 (mod 3)

1. INTRODUCTION. Among the observations by Fermat [1,4] in the mid seventeenth century about representations of primes as sums of squares were the following:

Theorem A: Primes of the form \( p = 4N + 1 \) can be represented as \( p = X^2 + Y^2 \).

Theorem B: Primes of the form \( p = 3N + 1 \) can be represented as \( p = X^2 + 3Y^2 \).

Since the original proofs by Euler in the following century, many proofs (especially of the first) have been given using a variety of techniques. However, using the modular group, the arguments that follow provide proofs that are certainly the simplest. In addition, they provide a striking example of the close relation between number theory and complex analysis. We recall in section 2 some standard facts from algebra and complex analysis and in sections 3 and 4 we present two (essentially equivalent) proofs of each of the theorems; one from the point of view of complex analysis and one in the context of arithmetic groups. Preliminary results were obtained in [2] and [3]. Proofs of both theorems start, as usual, with preliminary divisibility observations for the relevant prime; for the first that \( p = 4N + 1 \) is a factor of \( m^2 + 1 \) for some \( m \), and for the second that \( p = 3N + 1 \) is a factor of \( m^2 + m + 1 \) for some \( m \). Following Euler, these facts follow directly from Fermat’s Little Theorem by factoring the equation \( a^{p-1} - 1 \equiv 0 \, (mod \, p) \) and setting \( m = a^N \).

2. MODULAR GROUP. The modular group (projective special linear group) \( PSL(2,\mathbb{Z}) \) consists of 2x2 unimodular matrices with integer entries, where matrices \( A \) and \( -A \) are identified. An element of this group acts on the upper half complex plane \( \mathcal{H} \) as a fractional linear transformation \( T(z) = (az + b)(cz + d)^{-1} \) (with determinant \( ad - bc = 1 \)); the group \( \Gamma \) of transformations also being referred to as the modular group. It is well known (and easy to verify) that the set \( \{ z \in \mathcal{H} : |z| > 1, |Re(z)| < 1/2 \} \), together with that part of its boundary with \( Re(z) \geq 0 \) is a fundamental set \( \mathcal{F} \); every point in \( \mathcal{H} \) is \( \Gamma \)-equivalent to a point in \( \mathcal{F} \) and no two points of \( \mathcal{F} \) are \( \Gamma \)-equivalent.
Among the elements of the modular group are the elliptic elements of order two 
\((E_2^2) = \text{identity}\). They have zero trace and, with \(r = (1 + m^2)s^{-1}\), the form
\[
E_{(2)} = \begin{pmatrix}
m & -r \\
s & -m
\end{pmatrix}
\]  

Elements of the two conjugacy classes of elliptic elements of order three
\((E_3^3) = \text{identity}\) have trace one and the forms (with \(r = (1 + m + m^2)s^{-1}\))
\[
E_{(3)} = \begin{pmatrix}
1 + m & -r \\
s & -m
\end{pmatrix}
\]  
and
\[
E_{(3)}^{-1} = \begin{pmatrix}
-m & -r \\
s & 1 + m
\end{pmatrix}
\]
Note that, if \(s = 1\) or a prime of the form \(p = 4N + 1\) in the first case \((3N + 1\) in the second or third), then \(r\) is an integer and the matrix is in \(PSL(2, Z)\). As elements of \(\Gamma\), \(E_{(2)}\) has the “elliptic” fixed point \((m + i)/p\) while both \(E_{(3)}\) and \(E_{(3)}^{-1}\) have the fixed point \([2m + 1 + i\sqrt{3}]/2p\).

3.1 PROOF OF THEOREM A: The fixed point \((m + i)/p\) of the transformation corresponding to \(E_{(2)}\) (with \(s\) the prime \(p = 4N + 1\)) must be \(\Gamma\)-equivalent, by a transformation \(T\) in \(\Gamma\), to the fixed point \(z = i\) of the transformation \(z \rightarrow -1/z\), the unique order two elliptic fixed point in the fundamental set \(\mathcal{F}\) of \(\Gamma\). Comparing the imaginary parts of \(T(i) = (a + ib)/(c + id) = (bd - ac + i)/(c^2 + d^2)\) with that of \((m + i)/p\) gives immediately \(p = c^2 + d^2\).

3.2 PROOF OF THEOREM B. With \(s\) the prime \(p = 3N + 1\), both \(E_{(3)}^{-1}\) and \(E_{(3)}\) have the fixed point \([2m + 1 + i\sqrt{3}]/2p\) which must be equivalent to the unique fixed point \([1 + i\sqrt{3}]/2\) of order 3 in \(\mathcal{F}\). Comparing imaginary parts of \([2m + 1 + i\sqrt{3}]/2p\) with \(T([1 + i\sqrt{3}]/2)\) one finds, at first, that \(p = d^2 + dc + c^2\). If either \(c\) or \(d\) are even, we can write \(p = (d + c/2)^2 + 3(c/2)^2\) or \(p = (c + d/2)^2 + 3(d/2)^2\) while if both \(c\) and \(d\) are odd, \(p = ((d - c)/2)^2 + 3((d + c)/2)^2\). In any event, as claimed in Fermat Theorem B, \(p\) is representable by the form \(X^2 + 3Y^2\).
4: VARIATION OF PROOFS. The above simple arguments can be translated into equally simple ones about the arithmetic group $\text{PSL}(2, \mathbb{Z})$. In this group there is only one conjugacy class of order two elliptic elements so the matrix $E_2$ with $p = 4N + 1$ (and corresponding $m$) is conjugate to the matrix with $s = 1$ and $m = 0$. Performing the conjugation by a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ establishes that $p = c^2 + d^2$ proving Fermat’s Theorem A.

For a given $p$ the matrices $E_3$ and $E_3^{-1}$ of section 2 are conjugate respectively to the corresponding matrices with $s = 1$ and $m = 0$. Performing the conjugations and comparing the result, one obtains in either case that $c^2 + dc + d^2$. As before, one sees that this gives the representation $p = X^2 + 3Y^2$ of Theorem B.

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