I. INTRODUCTION

Tunneling problems of quantum mechanics are distinct from classical physics. The problems are normally discussed for a quantum particle passing through a potential barrier when the process is forbidden in classical mechanics. For this problem, the most useful approximation would be the Wentzel - Kramers - Brillouin (WKB or WKBJ) approximation, which is a method of finding approximate solutions to linear differential equations. For a particle with an energy $E$ and the rest mass $m$ moving in a one-dimensional potential $V(x)$, where $E < V(x)$ for a range $a \geq x \geq b$, the tunneling amplitude is described by

$$e^{-\frac{i}{\hbar} \int_a^b \sqrt{2m(V(x)-E)} \, dx} = e^{-\frac{i}{\hbar} \int_a^b p(x) \, dx},$$

where $p(x)$ denotes the canonical momentum.

The above description of the WKB approximation is very elementary. In this formalism, one will normally consider the continuity of the waves at each turning point $(x = a, b)$ to get connection formulae between left and right-moving solutions. If $V(x)$ is given by an inverted quadratic potential, one can find the exact solutions described by the special functions (e.g., the Weber function). It is also known that local analysis near a turning point is well described by the Airy function. In more general situations, the connection formulae, which are explained by the Stokes phenomenon of the ordinary differential equation, are described by using the Exact WKB(Exact WKB) formalism. Initially, the formalism was valid only for the second-order equations, but later by the discovery of new stokes lines and virtual turning points, it was found to be valid for higher-order differential equations. See Refs. 4, 11 for mathematical details of the EWKB formalism. After the Borel resummation, one will find finite and exact results from the divergent power series of the WKB expansion.

Besides the conventional tunneling described above in Eq.(1) for $E < V(x)$, one can see a similar quantum effect for $E > V(x)$. In this case, a reflection of the particle is forbidden in classical mechanics, while reflection is allowed in quantum mechanics. One might be tempted to describe the phenomena using Eq.(1), but in this case, the situation is less trivial. First, the turning points, defined by $p(x) = 0$, are not on the real axis of $x$. For the simplest case of an inverted quadratic potential, a pair of turning points will appear on the imaginary axis of $x$. The Stokes lines of the EWKB analysis show that the pair forms a Merged pair of Turning Points (MTP), whose exact solutions are given by the Weber function. We will describe the situation in Sec.I A. For an inverted quartic potential $V(x) \propto -x^4$, two pairs of turning points will appear on the complex $x$ plane away from the origin, showing that local expansion around the top of the potential is not always giving a good approximation. In this case, from the Stokes lines, one can understand that each pair forms the MTP structure, whose local approximate solutions are given by the Weber function. If a differential equation has a pole in addition to the conventional turning points, the situation becomes less easy, but still, it can be well described by the EWKB formalism. One can see that the Stokes lines of Hawking radiation have a loop structure around a pole, and the Stokes lines seem to be absent near the horizon. See also the calculation given in Ref. 13.

One could suspect that the EWKB formalism is nothing but the classical method of the steepest descent path since turning points and the Stokes lines of the EWKB analysis might seem to be corresponding to saddle points and steepest descent paths, respectively. To avoid confusion, later in this section, we will explicitly calculate the connection formulae using the method of steepest descent path. One can compare it with the exact WKB formalism described in Sec.I A.

The above statements are about quantum processes of tunnelings and reflections, but Hawking radiation is particle production from the vacuum. To make the relationship between these ideas clearer, we will describe explicitly the correspondence between the connection formulae of the quantum problem and the Bogoliubov coefficients. Calculation of the Bogoliubov coefficients normally re-
quires more than Eq. [1]. What is “more” will be shown in Eq. (10) and (11). Note however the EWKB uses another approach to the problem.

Before mentioning the details of the EWKB formalism, we will first review the method of the steepest descent path for calculating the Bogoliubov coefficients. Intuitively, the calculation describes what happens in Eq. (1) when \( E > V(x) \). What we should calculate is the quantum reflection, not the quantum tunneling. The method of steepest descent path was first applied to cosmological particle production in Ref. [14] and extended for higher polynomials in Ref. [15, 16]. Consider an equation of motion given by

\[
\ddot{\chi}_k + \left[k^2 + m_0^2 - V(t)\right] \chi_k = 0, \tag{2}
\]

where we have considered \( V(t) = -\frac{g^2}{M^2(1 - t)^2} \) in Ref. [13, 16]. For \( n = 1 \), this equation coincides with the original preheating scenario of Ref. [17] and is similar to the scattering problem by an inverted quadratic potential. One can write down the (conventional) WKB solution given by

\[
\chi_k = \frac{\alpha_k}{\sqrt{2\omega_k}} e^{-i \int_{-\infty}^{t'} dt' \omega_k(t')} + \frac{\beta_k}{\sqrt{2\omega_k}} e^{i \int_{-\infty}^{t'} dt' \omega_k(t')}, \tag{3}
\]

where \( \omega_k \equiv \sqrt{k^2 + m_0^2 - V(t)} \tag{4} \) and \( \alpha_k, \beta_k \) have to satisfy

\[
|\alpha_k|^2 - |\beta_k|^2 = 1. \tag{5}
\]

Since the coefficients of the WKB solution are time-dependent, one has to consider the additional constraint

\[
0 = \frac{\dot{\alpha}_k}{\sqrt{2\omega_k}} e^{-i \int_{-\infty}^{t'} dt' \omega_k(t')} + \frac{\dot{\beta}_k}{\sqrt{2\omega_k}} e^{i \int_{-\infty}^{t'} dt' \omega_k(t')}. \tag{6}
\]

Substituting the representation (4) into (2) with the above constraint, one obtains

\[
\dot{\alpha}_k = \frac{\dot{\omega}_k}{2\omega_k} e^{2i \int_{-\infty}^{t'} dt' \omega_k(t')}, \tag{7}
\]

\[
\dot{\beta}_k = \frac{\dot{\omega}_k}{2\omega_k} e^{-2i \int_{-\infty}^{t'} dt' \omega_k(t')}. \tag{8}
\]

Supposing that the initial state has no particle, we have

\[
\alpha_k(-\infty) = 1, \quad \beta_k(-\infty) = 0. \tag{9}
\]

One can estimate the Bogoliubov coefficients for \( \beta_k \ll 1 \) by the integration given by

\[
\beta_k(+\infty) = \int_{-\infty}^{+\infty} dt \frac{\dot{\omega}_k(t)}{2\omega_k(t)} \exp \left[ -2i \int_{-\infty}^{t} dt' \omega_k(t') \right]. \tag{10}
\]

In the above calculation, poles at \( \omega_k(t) = 0 \), which are called “turning points” are essential.\(^1\) One can evaluate the integral of Eq. (10) using the steepest descent method on the complex \( t \) plane. Details of the calculation can be found in Ref. [13, 16].

Naively, we thought that the above calculation has to be valid for the WKB tunneling method of Hawking radiation, but we found the situation is not so trivial. The above calculation requires the Stokes phenomena between “in” and “out” asymptotic states, but they (i.e., the Stokes phenomena) are not very clear in the previous calculations. The WKB calculations of static black holes are normally used to derive the connection formulae of the modes beyond the horizon, not directly calculating the Bogoliubov coefficients, at least in the sense we have described above for Eq. (10). One way that might be used to avoid this situation is a collapsing shell model [18], which is not static.

WKB tunneling methods of calculating the Unruh and Hawking effects are expected to explain the heuristic picture of Hawking described in Ref. [12]. Hawking explains the effect as a tunneling outward of positive energy modes and a tunneling inward of negative energy modes. This heuristic picture has been explored later in Ref. [13, 18, 20–23], in which the action of a particle picks up an imaginary contribution when it crosses the horizon (the integration path is taken on the complex plane), and the imaginary contribution was interpreted as tunneling probability. Viewing the situation of the static black hole using the Kruskal coordinates, which are valid on both sides of the horizon, the factor calculated by the “tunneling”\(^2\) is nothing but the factor required for the smooth connection of the WKB solutions between inside and outside the horizon [23], where the Stokes phenomena do not appear in the calculation, at least explicitly.

In that case, the Bogoliubov transformation can be calculated using [18, 20–23]

\[
\alpha_{kk'} = \frac{1}{2\pi u_k(r)} \int_{-\infty}^{\infty} dt e^{i\omega_k t} v_k(t, r),
\]

\[
\beta_{kk'} = -\frac{1}{2\pi u_k(r)} \int_{-\infty}^{\infty} dt e^{i\omega_k t} \bar{v}_k(t, r), \tag{11}
\]

which are essentially Fourier transforms of the modes. Note that the difference between Eq. (10) and Eq. (11) is not trivial. Here the modes are decomposed as

\[
\phi(t, r) = \int dk \left[ a_k u_k(r) e^{-i\omega_k t} + a_k^* u_k^*(r) e^{i\omega_k t} \right], \tag{12}
\]

for an observer at infinity (an asymptotic observer) and

\[
\phi(t, r) = \int dk \left[ b_k v_k(t, r) + b_k^* v_k^*(t, r) \right], \tag{13}
\]

\(^1\) Note that there are no classical turning points on the real axis in this model. All turning points are on the complex plane.

\(^2\) Note however that normally the modes have \( E > V(r) \) near the horizon.
for an infalling observer. Eq. (10) calculates the connection matrix of the ± solutions of a differential equation, while Eq. (11) compares modes of different vacua. The discrepancy between these calculations is not a trivial issue.

To be more specific, $v_k(t, r)$ of the collapsing shell model of Ref. [18] is described as inertial vacuum. Although distinguishable, in which an accelerating observer is just seeing the different vacua, but it is not calculating the evolution phenomena. In fact, $v_k$ has the saddle point at $r = 2(M + \omega_k)$ which gives $M' \simeq M + \omega_k \frac{e^{\frac{i}{2 \pi r} - \frac{i}{2 \pi r}}}{1 + e^{\frac{i}{2 \pi r} - \frac{i}{2 \pi r}}}$. (15)

Then, the integral

$$\int_{-\infty}^{\infty} dt e^{i\omega t} v_k^*$$

has the saddle point at

$$e^{\frac{i}{2 \pi r} - \frac{i}{2 \pi r}} = \frac{i}{2}$$, (17)

which gives $t = 4i\pi M + \text{real}$ at zeroth order in $\omega_k$. The calculation has been extended in Ref. [19] to give a simple formula.

$$ds^2 = -[N_t(r, M + H)dt]^2 + [dr + N_r(r, M + H)dt]^2,$$ (18)

they showed that the coefficients can be calculated by the “tunneling” given by

$$|\beta_{kk'}| = e^{-i\int_{-\infty}^{\infty} p_-(r)dr}$$

$$p_-(r) = \int_{0}^{\infty} \frac{dH'}{N_t(r, M + H') - N_r(r, M + H')}.$$ (19)

The calculation is not using explicitly the Stokes phenomena. In fact, $v_k$ used to calculate the Bogoliubov coefficients in Ref. [18] has no Stokes phenomena (mixing of the ± solutions) near the horizon. The most important part of the calculation is Eq. (11), which is comparing different vacua, but it is not calculating the evolution (the Stokes phenomena) of the solutions. Of course, Eq. (11) is quite natural if it is applied to the Unruh effect, in which an accelerating observer is just seeing the inertial vacuum. Although $\beta_{kk'}$ is very similar to Eq. (11), Hawking’s tunneling picture requires more. Therefore, at this moment, we have to conclude that the explicit relation between Hawking’s tunneling picture and the Stokes phenomena is vague in the conventional WKB tunneling calculations.

To avoid confusion, we stress that we are not criticizing the previous calculations. The Stokes phenomena could be hidden somewhere in the calculation, but unfortunately, we could not find where they are. Indeed, Giovanazzi [27] found that a microscopic description of Hawking radiation in sonic black holes clearly explains Hawking’s original picture of tunneling. Giovanazzi’s model explains how the Stokes phenomena can be responsible for Hawking radiation in sonic black holes. These analog systems are considered expected to gain a better understanding of Hawking radiation avoiding the (theoretical) problems of the original black hole radiation. For our calculation in this paper, we consider the microscopic description of Hawking radiation in sonic black holes advocated by Giovanazzi in Ref. [27]. The merit of using this model is the affinity for the Exact WKB analysis and the Stokes phenomena. We will show that the model is very useful for our calculation.

### A. Introduction to the Exact WKB analysis

The WKB approximation is a method for finding approximate solutions to linear differential equations. It is well known that the WKB expansion gives divergent power series, but their first term is normally giving an excellent approximation (e.g. the WKB approximation of a harmonic oscillator). The Exact WKB analysis considers the Borel resummation to explain why and how the WKB approximation can be justified in the context of exact calculation.

The mathematical formulation of the EWKB uses $\eta \equiv h^{-1} \gg 1$ for the expansion, instead of using the Planck (Dirac) constant $\hbar$. Following Ref. [4, 9], our starting point is the “Schrödinger equation” in quantum mechanics described as

$$\left[-\frac{d^2}{dx^2} + \eta^2 Q(x)\right] \psi(x, \eta) = 0.$$ (21)

Introducing the “potential” $V(x)$ and the “energy” $E$, we define

$$Q(x) = V(x) - E.$$ (22)

Writing the solution as $\psi(x, \eta) = e^{-iE\eta/\hbar}$, we have

$$\psi = e^{\int_{x_0}^{x} S(x, \eta)dx}.$$ (23)

---

3 We stick to the Stokes phenomena near the horizon because of Hawking’s tunneling picture.
for \( S(x, \eta) \equiv \partial R/\partial x \). For \( S \), we have

\[
- \left( S^2 + \frac{\partial S}{\partial x} \right) + \eta^2 Q = 0. \tag{24}
\]

If one expands \( S \) as \( S(x, \eta) = \sum_{n=-\infty}^{n=\infty} \eta^n S_n \), one will find

\[
S = \eta S_{-1}(x) + S_0(x) + \eta^{-1} S_1(x) + \ldots, \tag{25}
\]

which leads

\[
S_{-1}^2 = Q \tag{26}
\]

\[
2S_{-1} S_j = - \left[ \sum_{k+l=j-1, k \geq 0, l \geq 0} S_k S_l + \frac{dS_{j-1}}{dx} \right] (j \geq 0). \tag{27}
\]

Using the relation between the odd and the even series, one will have

\[
\psi = \frac{1}{\sqrt{S_{\text{odd}}}} e^{\int_{x_0}^x S_{\text{odd}} dx} \tag{28}
\]

\[
S_{\text{odd}} \equiv \sum_{j=0}^{\infty} \eta^{1-2j} S_{2j-1}. \tag{29}
\]

Depending on the sign of the first \( S_{-1} = \pm \sqrt{Q(x)} \), there are two solutions \( \psi_\pm \), which are given by

\[
\psi_\pm = \frac{1}{\sqrt{S_{\text{odd}}}} \exp \left( \pm \int_{x_0}^x S_{\text{odd}} dx \right) = e^{\pm \eta \int \sqrt{Q} dx} \sum_{n=0}^{\infty} \eta^{-n-1/2} \psi_{\pm,n}(x). \tag{30}
\]

The above WKB expansion is usually divergent but is Borel-summable. The Borel transform is taken as

\[
\psi_\pm^B = \sum_{n=0}^{\infty} \frac{\psi_{\pm,n}(x)}{\Gamma(n + \frac{1}{2})} (y \pm s(x))^{n - \frac{1}{2}}. \tag{31}
\]

What is important here is the shift of the integral of the inverse-Laplace integration (Borel sum). The shift is determined by \( S_{-1} \) as

\[
\psi_\pm \rightarrow \Psi_\pm \equiv \int_{\mp s(x)}^{\infty} e^{-y\eta} \psi_\pm^B (x, y) dy, \tag{32}
\]

\[
s(x) \equiv \int_{x_0}^{x} S_{-1}(x) dx, \tag{33}
\]

where the \( y \)-integral is parallel to the real axis. Note also that the original Laplace-like transformation, which is used to define \( \psi_\pm^B \), was done for \( \eta \), not for \( x \).

Let us see what the Stokes phenomena look like in the EWKB formalism. Just for simplicity, we first consider a familiar case, the Airy function for \( Q(x) = x \). In this case, one can find that \( \psi_\pm^B \) is given explicitly by the hypergeometric function. On the complex \( x \)-plane, three Stokes lines are going out of a turning point. The Stokes lines are the solutions of \( \text{Im}[s(x)] = 0 \). Since the two paths on the complex \( y \)-plane of the Borel resummation (starting from \( \mp s(x) \), which correspond to \( \pm \) solutions) overlap when \( x \) is on the Stokes line, one of them will develop additional contributions as \( x \) goes across the Stokes line. Then, one can develop the following connection formulae, which are of course consistent with the exact solutions described by the Airy function.

- Crossing the \( \psi_+ \)-dominant Stokes line\(^5\) with anti-clockwise rotation (seen from the turning point)

\[
\Psi_+ \rightarrow \Psi_+ + i \Psi_- \tag{34}
\]

\[
\Psi_- \rightarrow \Psi_- \tag{35}
\]

- Crossing the \( \psi_- \)-dominant Stokes line with anti-clockwise rotation (seen from the turning point)

\[
\Psi_- \rightarrow \Psi_- + i \Psi_+ \tag{36}
\]

\[
\Psi_+ \rightarrow \Psi_+ \tag{37}
\]

- Inverse rotation gives a minus sign in front of \( i \).

In some typical cases, the Stokes lines have degenerated and a gap appears for complex \( \eta \), between \( \text{Im} \eta > 0 \) and \( \text{Im} \eta < 0 \). The gap can be solved by a normalization factor (called Voros coefficient\(^6\)), whose analytic calculation is given in Ref.\(^6\) for MTP (Merged pair of simple Turning Points) and a loop structure of a Bessel-like equation\(^28\). Therefore, these gaps are already solved analytically. More easily, one can calculate the normalization factor using simple consistency relations\(^1\), although the phase cannot be determined in that way.

To show explicitly the calculation of the connection matrix by using the EWKB, we start with

\[
\frac{d^2 \psi}{dt^2} + \eta^2 \left[ E + \frac{y_0^2 v^2 t^2}{2} \right] \psi = 0. \tag{38}
\]

For \( E < 0 \), turning points will appear on the real axis. This is a conventional tunneling problem of quantum mechanics. The Stokes lines are shown in Fig.\(^\text{1}\) and the connection matrix starting from \( t = -\infty \) to \( t = +\infty \) can be calculated using the EWKB or the Weber function.

For \( E > 0 \), turning points will appear on the imaginary \( t \)-axis. In Fig.\(^2\), we show the Stokes lines for \( V(x) = -1 - t^2 \) (\( E = 0 \)).

For \( E = k^2 > 0 \), the connection matrix starting from \( t = -\infty \) to \( t = +\infty \) can be calculated using the EWKB, which gives (See Ref.\(^10\) for calculational details)

\[
\left( \begin{array}{cc} N^2 \left( e^{-K_{ud}} + e^{K_{ud}} \right) & -ie^{-K_{ud}} \\ ie^{-K_{ud}} & e^{K_{ud}} \end{array} \right), \tag{39}
\]

\(^5\) On the Stokes line, where \( \text{Im}[s(x)] = 0 \), both \( \pm s(x) \) are on the real axis of the \( y \)-plane. In Ref.\(^28\), one can find a picture describing the situation.

\(^6\) This equation appears in the cosmological preheating scenario of Ref.\(^17\).
FIG. 1: The potential $V(x) = 1 - t^2$ ($E = 0$) and the Stokes lines for $\text{Im}\eta^2 = 0$, $\text{Im}\eta^2 > 0$ and $\text{Im}\eta^2 < 0$ are shown. One can see that the Stokes lines have degenerated for $\text{Im}\eta^2 = 0$ and there is a gap between $\text{Im}\eta^2 > 0$ and $\text{Im}\eta^2 < 0$. The structure is called Merged pair of Tunneling Points (MTP). The classical turning points are appearing at $x = \pm 1$.

where we defined

$$K_{ud} \equiv \int_{t_*}^{t_*'} S_{odd}dt$$  \hspace{1cm} (40)$$

for $t_*' \equiv \pm i\frac{k}{2\pi}$. To make the diagonal elements consistent, one may put a condition given by

$$|N|^{-2} = \sqrt{1 + e^{2K_{ud}}}. \hspace{1cm} (41)$$

Finally, the connection matrix becomes

$$\begin{pmatrix}
\sqrt{1 + e^{-2K_{ud}}} & -ie^{K_{ud}} \\
-ie^{-K_{ud}} & \sqrt{1 + e^{-2K_{ud}}}
\end{pmatrix}, \hspace{1cm} (42)$$

where the phase of $N$ has been neglected because we have used the consistency condition instead of the analytic calculation. The result is consistent with the exact result obtained from the Weber function\[10].

One can also discuss the Schwinger effect using the EWKB\[11, 26, 32–35]. Consider a complex charged scalar field interacting with an electromagnetic field with the vector potential (constant electric field $E_z = E$)

$$A_\mu(t) = (0, 0, 0, -Et), \hspace{1cm} (43)$$

one will have

$$\hbar^2 \frac{d^2\phi_k}{dt^2} + \left[ m^2 + (p - qA(t))^2 \right] \phi_k = 0, \hspace{1cm} (44)$$

which gives for the EWKB,

$$Q(t) = -m^2 - (p - qA(t))^2. \hspace{1cm} (45)$$

Using the EWKB, one can easily draw the Stokes lines and calculate the connection matrix for the solutions. In the above formula, $t = 0$ might seem to be special, but considering the gauge symmetry, the time should be arbitrary for a constant electric field. See also Ref.\[35] for the Stokes phenomena and Schwinger pair production in time-dependent laser pulses.

Let us see shortly what the Stokes lines look like in the Rindler space. We consider the two-dimensional Rindler metric\[26]

$$ds^2 = -(1 + ax_R)^2 dt_R^2 + dx_R^2. \hspace{1cm} (46)$$

Corresponding EWKB formulation will have\[25]

$$Q(t) = -\frac{\omega^2 - m^2(1 + ax_R)^2}{(1 + ax_R)^2}, \hspace{1cm} (47)$$

where one will find two turning points at $x_R = (\pm \omega/m - 1)/a$ and a pole at the midpoint $x_R = -1/a$ (at the center of the loop structure of the Stokes lines in Fig.3). In the $m \to 0$ limit, these turning points will flow to infinity. Since there is no Stokes line within the loop structure, one cannot expect the Stokes phenomena in this case. The “potential” and the Stokes lines are shown in Fig.3.

To find the Stokes phenomena of an accelerating observer, we write a pair of $e^{\pm it / \omega dt}$ of an inertial observer's
FIG. 3: The upper panel is the “potential” of $Q(t) = -\frac{\omega^2 - m^2}{(1 + ax)^2}$ for $a = 1, \omega = 0.3, m = 0.2$, and the lower shows the Stokes lines. A pole is shown at the center of the loop.

FIG. 4: The Stokes lines for $Q(\tau) = -(\cosh^2 2\tau + 0.1 + 0.1i)(1 + 0.1i)$ are shown on the complex $\tau$ plane. The double turning points and the degenerated Stokes lines are both split by introducing small parameters.

Consider a local expansion of $Q(\tau)$ around $\tau = 0$ as $Q(\tau) = -\omega^2 \cosh^2 (a\tau)$, we find $\frac{d}{d\tau} = e^{\pm i \omega t} \cosh (a\tau) d\tau$, which gives for the EWKB formalism:

$$Q(\tau) = -\omega^2 \cosh^2 (a\tau).$$

Considering a one-dimensional Fermi-degenerate liquid squeezed by a smooth barrier forming a transonic flow, Giovanazzi\[27\] presented a microscopic description of Hawking radiation in sonic black holes. The fundamental equations of fluid dynamics for an inviscid fluid in one dimension are the equation of continuity and Euler’s equation given by

$$\partial_t \rho + \partial_x (\rho v) = 0,$$

$$\rho \partial_t v + \rho v \partial_x v = -\partial_x p - \frac{\rho}{m} \partial_x V(x),$$

where $\rho(t, x), v(t, x), p(t, x), m$ and $V(x)$ are the density, velocity of flow, pressure, the mass of a particle and external potential of the particle, respectively. Then expand $\rho$ and $v$ as

$$\rho = \rho_0 + \epsilon \rho_1 + \epsilon^2 \rho_2 + ...,\]

$$v = \rho_0 + \epsilon v_1 + \epsilon^2 v_2 + ...,\]

where $\rho_0$ and $v_0$ denote the background flow, and $\rho_i$ and $v_i$ are giving the sound waves. Introducing the velocity potential $\phi$ such that $v_1 = -\partial_x \phi$, one will find

$$\partial_{\mu} g^{\mu \nu} \partial_\nu \phi = 0,$$

where $\mu, \nu = t, x$. Here the “acoustic metric” is given by

$$g_{\mu \nu} = \frac{\epsilon^2}{\rho_0^2} \begin{pmatrix} \epsilon^2 - v_0^2 & v_0 \\ v_0 & -1 \end{pmatrix},$$

where $\epsilon$ in this calculation, we are expecting that the plus and the minus sign solutions of an inertial observer’s vacuum will be mixed if they are described by the accelerating observer’s time.

Here we have skipped the formal WKB expansion.
where

\[ c_s \equiv \sqrt{\frac{\partial p}{\partial \rho}}. \]  

(54)

Choosing an appropriate potential \( V(x) \), one will find that Hawking radiation is realized at the sound horizon which divides regions of \( v_0 < c_s \) and \( v_0 > c_s \). The temperature of the radiation at the horizon is

\[ T = \frac{\hbar c_s}{4\pi k_B} \frac{\partial}{\partial x} \left( 1 - \frac{v_0^2}{c_s^2} \right) \]  

(55)

Giovanazzi presented a microscopic description of Hawking radiation of sonic black holes in Ref. [27]. A one-dimensional Fermi-degenerate liquid squeezed by a smooth barrier \( V = -\frac{1}{2} \lambda x^2 \) forms a transonic flow. Using \( p = \hbar^2 \pi^2 \rho^3 / 3m^4 \) for the Fermi-degenerate liquid and classical solutions

\[
\begin{align*}
\rho_0 &= \frac{m}{2\pi \hbar} \left( \sqrt{2m(E_F + \lambda x^2 / 2)} - \sqrt{m \lambda x} \right) \\
v_0 &= \frac{1}{2m} \left( \sqrt{2m(E_F + \lambda x^2 / 2)} + \sqrt{m \lambda x} \right),
\end{align*}
\]

(56)

where \( E_F \) is the energy of a particle on the Fermi surface, one will see that the sound horizon appears at \( x = 0 \). Here the sound velocity is given by \( c_s = \hbar \pi \rho_0 / m^2 \). After quantization, one will find that the temperature of Hawking radiation is given by

\[ T = \frac{\hbar}{2\pi k_B} \sqrt{\frac{\lambda}{m}}. \]  

(57)

By choosing \( \lambda = m \omega_x^2 \), we have

\[ T = \frac{\hbar \omega_x}{2\pi k_B}. \]  

(58)

Giovanazzi’s microscopic treatment of the quantum process established a close relationship between sonic Hawking radiation and quantum tunneling (and quantum scattering) through the barrier. Indeed, quantum scattering of a particle with \( E > 0 \) will radiate positive energy back outward the horizon, while a particle with \( E < 0 \) can tunnel the horizon barrier into the black hole. These quantum processes are beautifully explaining the original Hawking’s picture of “Hawking radiation as tunneling”.

Following Ref. [27], we consider the EWKB with a potential \( V(x) = -\frac{1}{2} m a^2 x^2 \), which gives the temperature of the sonic black hole

\[ T = \frac{\hbar}{2\pi k_B} a. \]  

(59)

To compare the sonic black hole and the conventional Hawking radiation, we have relations

\[
\begin{align*}
a &\rightarrow \frac{c^3}{4GM} \\
\dot{M} &\equiv \frac{\hbar c^4}{15360\pi G^2 M^2} \frac{1}{2}\sqrt{E + a_0^2 x^2 / 2}
\end{align*}
\]

(60)

where we set \( G \) and \( M \) are the gravitational constant and the black hole mass, respectively. We also have \( \dot{a} \propto M / M^2 \propto M^{-4} \propto a^4 \).

For a real black hole, the radiation continuously takes away the mass of the black hole. Then, what an observer sees as the black hole mass depends on the distance from the horizon. An inner observer sees a lighter black hole compared to an outer observer. To realize the same phenomena in the sonic black hole, we consider \( a(x) \approx a_0 + \alpha x / 2 \), where \( a_0 \) is a constant defined at the horizon \( (x = 0) \).

Using the EWKB with \( a(x) \approx a_0 + \alpha x / 2 \), the problem becomes scattering by a potential

\[ V(x) = - \frac{m}{2} a_0^2 x^2 - \frac{\alpha a_0}{2} x^3 - \frac{\alpha^2}{8} x^4. \]  

(61)

For simplicity, consider

\[ Q(x) = -E - \frac{m}{2} a_0^2 x^2 - \frac{\alpha a_0}{2} x^3 - \frac{\alpha^2}{8} x^4. \]  

(62)

What we have to do is (1) Draw the Stokes lines and (2) Calculate the \( \alpha \)-correction of the radiation. The Stokes lines are given in Fig. 5 which shows that the structure near the origin is the well-known MTP structure of the Weber function. Such structure typically appears in scattering by an inverted quadratic potential [11].

![Stokes lines diagram](image)

FIG. 5: The Stokes lines of \( Q(x) = -E - \frac{m}{2} a_0^2 x^2 - \frac{\alpha a_0}{2} x^3 - \frac{\alpha^2}{8} x^4 \) for \( E = 1, a_0 = 1, \alpha = -0.6 \) is shown. The turning points (MTP structure) on the left float away to \( -\infty \) as \( \alpha \rightarrow 0 \).

From the Stokes lines, we find that the differences from the static \( (a(x) = a_0) \) system will appear in (1) the small displacement of the pair of turning points and (2) the shift of \( S_{-1} \). Defining the original \( S_{-1} \) as \( S^o_{-1} \equiv S_{-1\alpha=0} \), we have

\[ S_{-1} \approx \sqrt{-E - \frac{m}{2} a_0^2 x^2 - \frac{\alpha a_0}{2} x^3 - \frac{\alpha^2}{8} x^4} \approx S^o_{-1} + \frac{\alpha a_0 x^3 / 2 + \alpha^2 x^4 / 8}{2\sqrt{E + a_0^2 x^2 / 2}}. \]  

(63)
Then, integration between the pair of turning points can be written as

\[
\int_{i\sqrt{2E/a_0}+\varepsilon}^{i\sqrt{2E/a_0}} \left[ S^{-1}_0 + i a_0 x^3/2 + \frac{a^2 x^4}{8} \right] dx \\
\approx \int_{-i\sqrt{2E/a_0}}^{i\sqrt{2E/a_0}} S^{-1}_0 dx + \varepsilon S^{-1}_0(ia_0^{-1}) \\
- \varepsilon S^{-1}_0(-ia_0^{-1}) + i \int_{-i\sqrt{2E/a_0}}^{i\sqrt{2E/a_0}} \frac{a_0 x^3/2 + a^2 x^4/8}{2\sqrt{E + a_0^2 x^2/2}} dx
\]

where \( S^{-1}_0(\pm ia_0^{-1}) = 0 \) has been used. The integration of the term proportional to \( x^3 \) will vanish. One can evaluate the integral to give

\[
i \int_{-i\sqrt{2E/a_0}}^{i\sqrt{2E/a_0}} \frac{a^2 x^4/8}{2\sqrt{E + a_0^2 x^2/2}} dx \\
= -\frac{3\pi a^2 E^2}{16\sqrt{2a_0^3}}
\]  

What is important here is how the relativistic effect of the above deviation evolves with time. Since the standard scenario of black hole radiation gives \( \alpha \propto \dot{a} \propto a^4 \), one can see that the relative contribution from the second term (deviation) increases as \( a^4 \). Therefore, for Hawking radiation of the sonic black hole, our result suggests that non-thermal contribution will grow as \( a^3 \propto 1/M^3 \) and the ratio between non-thermal and thermal contributions increases as \( a^4 \propto 1/M^4 \).

III. CONCLUSIONS AND DISCUSSIONS

In this paper, using the EWKB analysis, we have calculated the distortion of Hawking-like radiation from an unstable sonic black hole. The Stokes phenomena are playing a very important role in our calculation.

One thing that is still not very clear is the relation between Giovannazzi’s microscopic picture of Hawking-like radiation in sonic black holes and Hawking’s tunneling picture of real Hawking radiation. At this moment, at least from the viewpoint of “particle production as the Stokes phenomena”, we could not find a clear relationship between them. Nevertheless, we are expecting that our viewpoint will contribute to a better understanding of the issue.

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