\[ \pi \text{-FORMULAS FROM DUAL SERIES OF THE DOUGALL THEOREM} \]

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By means of the extended Gould–Hsu inverse series relations, we show that the dual relation of Dougall’s summation theorem for a well-poised \( _7F_6 \)-series can be used to construct numerous interesting Ramanujan-like infinite-series expressions for \( \pi^{-1} \) and \( \pi^{-2} \), including an elegant formula for \( \pi^{-2} \) due to Guillera.

1. Introduction and Motivation

In 1973, Gould and Hsu \cite{27} discovered a useful pair of inverse series relations, which can be equivalently reproduced in what follows. Let \( \{a_i, b_i\} \) be any two complex sequences such that the \( \varphi \)-polynomials defined by

\[ \varphi(x; 0) \equiv 1 \quad \text{and} \quad \varphi(x; n) = \prod_{k=0}^{n-1} (a_k + xb_k) \quad \text{for} \quad n \in \mathbb{N} \]

differ from zero for \( x, n \in \mathbb{N}_0 \). Then the following inverse-series relations hold:

\[ f(n) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \varphi(k; n) g(k), \]
\[ g(n) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} (a_k + kb_k) \varphi(n; k+1) f(k). \]

This inverse pair has wide applications to terminating hypergeometric-series identities \cite{9–12, 15, 24}. The duplicate form with applications can be found in \cite{17, 18, 20}. We can also mention the \( q \)-analogs due to Carlitz \cite{6}, which have applications to \( q \)-series identities \cite{13, 14, 16, 19, 25, 26}.

The Gould–Hsu inversions have the following extended form (cf. \cite{4, 9, 15}):

\[ f(n) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \varphi(\lambda + k; n)\varphi(-k; n) \frac{\lambda + 2k}{\lambda + n}_{k+1} g(k), \quad (1a) \]
\[ g(n) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} (a_k + \lambda b_k + kb_k)(a_k - kb_k) \varphi(\lambda + k; n + k+1)\varphi(-n; k+1) (\lambda + k)_n f(k), \quad (1b) \]
where the shifted factorials are defined by

\[(x)_0 = 1 \quad \text{and} \quad (x)_n = \frac{\Gamma(x + n)}{\Gamma(x)} = x(x + 1)\ldots(x + n - 1) \quad \text{for} \quad n \in \mathbb{N}.
\]

There exist numerous hypergeometric-series identities (see, e.g., [5], Chapter 8, and [7–12, 15, 23, 24]). One of the well-known summation theorems is originally due to Dougall [22] and deals with terminating well-poised \(\text{\,}_7F_6\)-series. Examining its dual formulas by using (1a) and (1b), we conclude that their limit relations unexpectedly result in \(\pi\)-related infinite-series expressions, including the following elegant formula discovered by Guillera [28–30]:

\[
\frac{32}{\pi^2} = \sum_{k=0}^{\infty} \left[ \frac{1}{2^k} \cdot \frac{1}{2^k} \cdot \frac{1}{4^k} \cdot \frac{3}{4^k} \right] \frac{3 + 34k + 120k^2}{16^k}.
\]

By means of the duplicate forms of (1a) and (1b), in the next section, we develop, in detail, the dual formulas of the Dougall summation theorem. Then the applications are presented in Section 3, where several \(\pi\)-related infinite Ramanujan-like series [32] with a convergence rate of \(\frac{1}{16}\) are illustrated as examples.

Recall that the \(\Gamma\)-function (see, e.g., [31], § 8) is defined by the beta integral

\[
\Gamma(x) = \int_0^\infty u^{x-1} e^{-u} \, du \quad \text{for} \quad \Re(x) > 0,
\]

which admits Euler’s reflection property

\[
\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x} \quad \text{with} \quad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.
\]

The asymptotic formula

\[
\Gamma(x + n) \approx n^x (n - 1)! \quad \text{as} \quad n \to \infty
\]

is useful for the evaluation of the limits of \(\Gamma\)-function quotients.

For the sake of brevity, the product and quotient of shifted factorials are, respectively, abbreviated to

\[
[\alpha, \beta, \ldots, \gamma]_n = (\alpha)_n (\beta)_n \ldots (\gamma)_n,
\]

\[
\begin{bmatrix} \alpha, \beta, \ldots, \gamma \\ A, B, \ldots, C \end{bmatrix}_n = \frac{(\alpha)_n (\beta)_n \ldots (\gamma)_n}{(A)_n (B)_n \ldots (C)_n}.
\]

Similar notation is used for the \(\Gamma\)-function quotient

\[
\Gamma \begin{bmatrix} \alpha, \beta, \ldots, \gamma \\ A, B, \ldots, C \end{bmatrix} = \frac{\Gamma(\alpha)\Gamma(\beta)\ldots\Gamma(\gamma)}{\Gamma(A)\Gamma(B)\ldots\Gamma(C)}.
\]
2. Main Theorems from the Duplicate Inversions

The fundamental identity discovered by Dougall [22] (see also [3], § 4.3) for very well poised terminating \( \gamma F_0 \)-series can be represented as follows:

\[
\Omega_n(a; b, c, d) := \binom{1 + a, 1 + a - b - c, 1 + a - b - d, 1 + a - c - d}{1 + a - b, 1 + a - c, 1 + a - d, 1 + a - b - c - d}_n
\]

\[
= \sum_{k=0}^{n} \frac{a + 2k}{a} \binom{a, b, c, d, e, -n}{1 + a - b, 1 + a - c, 1 + a - d, 1 + a - e, 1 + a + n}_k,
\]

where the series is 2-balanced because \( 1 + 2a + n = b + c + d + e \).

For all \( n \in \mathbb{N}_0 \), it is well known that

\[
n = \left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{1 + n}{2} \right\rfloor,
\]

where \( \lfloor x \rfloor \) denotes the greatest integer that does not exceed \( x \). Thus, it is not difficult to check that the Dougall formula (3) is equivalent to the following formula:

\[
\Omega_n \left( a; b + \left\lfloor \frac{n}{2} \right\rfloor, c, d + \left\lfloor \frac{1 + n}{2} \right\rfloor \right) = \binom{1 + a - c - d, b + c - a}{1 + a - d, b - a}_n \binom{1 + a, b + d - a}{1 + a - c, b + c + d - a}_n
\]

\[
\times \binom{1 + a - b - c, c + d - a}{1 + a - b, d - a}_\left\lfloor \frac{1 + n}{2} \right\rfloor
\]

whose parameters are subject to \( 1 + 2a = b + c + d + e \). We reformulate the above equality as a binomial sum

\[
\sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{b + k, b - a - k}{\left\lfloor \frac{n}{2} \right\rfloor, d + k, d - a - k}_n \binom{1 + a - c - d, b + c - a}{1 + a - d, b - a}_n \binom{1 + a, b + d - a}{1 + a - c, b + c + d - a}_n
\]

\[
\times \frac{a + 2k}{(a + n)k + 1} \binom{a, b, c, d, 1 + 2a - b - c - d}{1 + a - b, 1 + a - c, 1 + a - d, b + c + d - a}_k
\]

\[
= \binom{b, 1 + a - c - d, b + c - a}{1 + a - d}_n \binom{d, 1 + a - b - c, c + d - a}{1 + a - b}_n \binom{1 + a - c - d, b + c - a}{1 + a - d}_n
\]

\[
\times \binom{a, b + d - a}{1 + a - c, b + c + d - a}_n.
\]
This equality exactly matches relation (1a) under the conditions that \( \lambda \to a \) and

\[
\varphi(x; n) = (b - a + x)\left[\frac{n}{2}\right](d - a + x)\left[\frac{1+n}{2}\right],
\]

as well as

\[
f(n) = \left[\begin{array}{c}
1 + a - c - d, b, b + c - a \\
1 + a - d
\end{array}\right]_{\frac{n}{2}} \left[\begin{array}{c}
a, b + d - a \\
1 + a - c, b + c + d - a
\end{array}\right]_{n}
\]

\[
\times \left[\begin{array}{c}
1 + a - b - c, d, c + d - a \\
1 + a - b
\end{array}\right]_{\frac{n+1}{2}},
\]

\[
g(k) = \left[\begin{array}{c}
a, b, c, d, 1 + 2a - b - c - d \\
1 + a - b, 1 + a - c, 1 + a - d, b + c + d - a
\end{array}\right]_{k}
\]

The dual relation corresponding to (1b) can be explicitly formulated, according to the parity of \( k \) and \((a)_k(a + k)_n = (a)_n(a + n)_k\), as follows:

\[
\left[\begin{array}{c}
b, c, d, 1 + 2a - b - c - d \\
1 + a - b, 1 + a - c, 1 + a - d, b + c + d - a
\end{array}\right]_{n}
\]

\[
= \sum_{k \geq 0} \binom{n}{2k} \frac{(d + 3k)(d - a - k)(a + n)_{2k}}{(b + n, b - a - n)_{k+1} (d + n, d - a - n)_{k+1}}
\]

\[
\times \left[\begin{array}{c}
1 + a - c - d, b, b + c - a \\
1 + a - d
\end{array}\right]_{k} \left[\begin{array}{c}
1 + a - b - c, d, c + d - a \\
1 + a - b
\end{array}\right]_{k}
\]

\[
\times \left[\begin{array}{c}
b + d - a \\
1 + a - c, b + c + d - a
\end{array}\right]_{2k}
\]

\[
- \sum_{k \geq 0} \binom{n}{2k+1} \frac{(b + 3k + 1)(b - a - k - 1)(a + n)_{2k+1}}{(b + n, b - a - n)_{k+1} (d + n, d - a - n)_{k+1}}
\]

\[
\times \left[\begin{array}{c}
1 + a - c - d, b, b + c - a \\
1 + a - d
\end{array}\right]_{k+1} \left[\begin{array}{c}
1 + a - b - c, d, c + d - a \\
1 + a - b
\end{array}\right]_{k+1}
\]

\[
\times \left[\begin{array}{c}
b + d - a \\
1 + a - c, b + c + d - a
\end{array}\right]_{2k+1}.
\]
Further, multiplying by \(n^2\) across this binomial relation and then letting \(n \to \infty\), we evaluate the limits of the left term by using (2) and of the corresponding right term by the Weierstrass \(M\)-test on uniform convergence of a series (cf. [33], §3.106). After certain routine simplification, the resulting limit relation can be explicitly expressed by the following lemma:

**Lemma 1** (infinite series identity).

\[
\Gamma \left[ \begin{array}{c}
1 + a - b, \ 1 + a - c, \ 1 + a - d, \ b + c + d - a \\
 b, \ c, \ d, \ 1 + 2a - b - c - d
\end{array} \right]
\]

\[
= \sum_{k \geq 0} \frac{(d + 3k)(a - d)}{(2k)!} \left[ \begin{array}{c}
b + d - a \\
1 + a - c, \ b + c + d - a
\end{array} \right]_{2k}
\]

\[
\times \left[ \begin{array}{c}
1 + a - c - d, \ b, \ b + c - a \\
a - d
\end{array} \right]_k \left[ \begin{array}{c}
1 + a - b - c, \ d, \ c + d - a \\
1 + a - b
\end{array} \right]_k
\]

\[
+ \sum_{k \geq 0} \frac{(b + 3k + 1)(a - b)}{(2k + 1)!} \left[ \begin{array}{c}
b + d - a \\
1 + a - c, \ b + c + d - a
\end{array} \right]_{2k+1}
\]

\[
\times \left[ \begin{array}{c}
1 + a - c - d, \ b, b + c - a \\
1 + a - d
\end{array} \right]_k \left[ \begin{array}{c}
1 + a - b - c, \ d, \ c + d - a \\
a - b
\end{array} \right]_{k+1}.
\]

By using this lemma, we now establish two main theorems used in the next section to deduce infinite-series expressions for \(\pi^{\pm 1}\) and \(\pi^{\pm 2}\).

For the equality in Lemma 1, multiplying both sides by \((1 + a - c)(b + c + d - a)\) and then unifying the two sums, we obtain the following infinite-series identity:

**Theorem 1** (infinite-series identity).

\[
\Gamma \left[ \begin{array}{c}
1 + a - b, \ 2 + a - c, \ 1 + a - d, \ 1 - a + b + c + d \\
 b, \ c, \ d, \ 1 + 2a - b - c - d
\end{array} \right]
\]

\[
= \sum_{k = 0}^{\infty} \mathcal{P}(k) \frac{[b, \ d, \ 1 + a - b - c, \ 1 + a - c - d, \ b + c - a, \ c + d - a]_k (b + d - a)_{2k}}{(2k + 1)! \ [1 + a - b, \ 1 + a - d]_k [2 + a - c, \ 1 - a + b + c + d]_{2k}}
\]

where \(\mathcal{P}(k)\) is a polynomial given by

\[
\mathcal{P}(k) = (1 + a - b - c + k)(d + k)(c + d - a + k)(b + d - a + 2k)(1 + b + 3k)
\]

\[
+ (1 + 2k)(a - d + k)(1 + a - c + 2k)(b + c + d - a + 2k)(d + 3k).
\]

Alternatively, shifting backward \(k \to k - 1\) in the second sum and then unifying it with the first sum, we get, in a similar way, another infinite-series identity from Lemma 1.
Theorem 2 (infinite-series identity).

\[
\Gamma \left[ 1 + a - b, 1 + a - c, 1 + a - d, b + c + d - a \right] \\
\quad \begin{array}{c}
\quad b, \\
\quad c, \\
\quad d, \\
\quad 1 + 2a - b - c - d
\end{array}
\]

\[
= \sum_{k=0}^{\infty} Q(k) \frac{[b, d, 1 + a - b - c, 1 + a - c - d, b + c - a, c + d - a]_k}{(2k)! [1 + a - b, 1 + a - d]_k [1 + a - c, b + c + d - a]_2 k},
\]

where \( Q(k) \) is a rational function given by

\[
Q(k) = (a - d + k)(d + 3k) \\
\quad \times \left\{ 1 + \frac{(2k)(a - b + k)(a - c + 2k)(b + c + d - a - 1 + 2k)(b - 2 + 3k)}{(a - c - d + k)(b - 1 + k)(b + c - a - 1 + k)(b + d - a - 1 + 2k)(d + 3k)} \right\}.
\]

3. Infinite Series for \( \pi^{\pm 1} \) and \( \pi^{\pm 2} \)

By using Theorems 1 and 2, we can deduce numerous infinite-series identities. They are presented in what follows and grouped in seven classes whose weight polynomial degrees are not greater than 3. In all examples, the parameter settings \([a, b, c, d] \) and eventual references are highlighted in the headers. In order to ensure the accuracy, all summation formulas in this section are experimentally verified by appropriately devised Mathematica commands.

3.1. Series for \( \pi^{-2} \).

Example 1 Guillera [28–30]:

\[
\begin{array}{c}
\frac{1}{2}, \\
\frac{1}{2}, \\
\frac{1}{2}, \\
\frac{1}{2}
\end{array}
\]

in Theorem 1.

\[
\frac{32}{\pi^2} = \sum_{k=0}^{\infty} \frac{[1, 1, 1, 1, 3]}{1, 1, 1, 1, 1} \frac{120k^2 + 34k + 3}{16^k}.
\]

Example 2 Chu and Zhang [21]:

\[
\begin{array}{c}
\frac{3}{2}, \\
\frac{3}{2}, \\
\frac{3}{2}, \\
\frac{3}{2}
\end{array}
\]

in Theorem 1.

\[
\frac{128}{\pi^2} = \sum_{k=0}^{\infty} \frac{[1, 1, 1, 1, 1]}{1, 1, 1, 2, 2} \frac{120k^2 + 118k + 13}{16^k}.
\]

Example 3

\[
\begin{array}{c}
\frac{3}{2}, \\
\frac{3}{2}, \\
\frac{3}{2}, \\
\frac{3}{2}
\end{array}
\]

in Theorem 1.

\[
\frac{256}{3\pi^2} = \sum_{k=0}^{\infty} \frac{[1, 1, 1, 3, 3]}{1, 1, 1, 2, 2} \frac{80k^3 + 148k^2 + 80k + 9}{16^k}.
\]
Example 4 \[ \frac{3 \ 3 \ 1 \ 3}{2 \ 2 \ 2 \ 2} \] in Theorem 1.

\[
\frac{512}{\pi^2} = \sum_{k=0}^{\infty} \left[ \frac{1 \ 1 \ 3 \ 3 \ 5}{2 \ 2 \ 2 \ 2 \ 4} \right]_{k} \frac{240k^3 + 532k^2 + 336k + 45}{16^k}.
\]

Example 5 \[ \frac{1 \ 1 \ 1 \ 1}{2 \ 2 \ 2 \ 2} \] in Theorem 1.

\[
\frac{32}{\pi^2} = \sum_{k=0}^{\infty} \left[ \frac{3 \ 1 \ 1 \ 1 \ 1}{2 \ 2 \ 2 \ 2 \ 4} \right]_{k} \frac{3 - 10k - 40k^2}{16^k}.
\]

Example 6 \[ \frac{3 \ 1 \ 3 \ 1}{2 \ 2 \ 2 \ 2} \] in Theorem 1.

\[
\frac{256}{3\pi^2} = \sum_{k=0}^{\infty} \left[ \frac{3 \ 1 \ 1 \ 1 \ 1 \ 3 \ 7}{2 \ 2 \ 2 \ 2 \ 4 \ 4} \right]_{k} \frac{9 - 38k - 40k^2}{16^k}.
\]

Example 7 \[ \frac{3 \ 3 \ 1 \ 3}{2 \ 2 \ 2 \ 2} \] in Theorem 2.

\[
\frac{8}{\pi^2} = \sum_{k=1}^{\infty} \left[ \frac{3 \ 1 \ 1 \ 1 \ 3 \ 7}{2 \ 2 \ 2 \ 2 \ 4} \right]_{k} \frac{k(3 - 18k + 40k^2)}{16^k}.
\]

Example 8 \[ \frac{3 \ 1 \ 3 \ 1}{2 \ 2 \ 2 \ 2} \] in Theorem 2.

\[
\frac{24}{\pi^2} = \sum_{k=0}^{\infty} \left[ \frac{1 \ 1 \ 1 \ 1 \ 3 \ 5 \ 5}{2 \ 2 \ 2 \ 2 \ 4 \ 4} \right]_{k} \frac{3 + 8k + 20k^2}{16^k}.
\]

Example 9 \[ \frac{3 \ 3 \ 1 \ 1}{2 \ 2 \ 2 \ 2} \] in Theorem 1.

\[
\frac{256}{9\pi^2} = \sum_{k=0}^{\infty} \left[ \frac{3 \ 5 \ 1 \ 1 \ 3 \ 1 \ 3}{2 \ 2 \ 2 \ 2 \ 4} \right]_{k} \frac{5 + 12k - 68k^2 - 80k^3}{16^k}.
\]
3.2. Series for $\pi^2$.

**Example 10** Chu and Zhang [21]: $\frac{3}{2}, 1, 1, 1$ in Theorem 1.

$$\frac{9\pi^2}{8} = \sum_{k=0}^{\infty} \left[ \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{3}{4} \right] \frac{11 + 64k + 111k^2 + 60k^3}{16^k}.$$ 

**Example 11** $\frac{5}{2}, 2, 1, 2$ in Theorem 1.

$$\frac{225\pi^2}{32} = \sum_{k=0}^{\infty} \left[ \frac{2}{3} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{3}{4} \right] \frac{68 + 206k + 197k^2 + 60k^3}{16^k}.$$ 

**Example 12** $\frac{5}{2}, 1, 2, 2$ in Theorem 1.

$$\frac{135\pi^2}{64} = \sum_{k=0}^{\infty} \left[ \frac{2}{3} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{3}{4} \right] \frac{21 + 93k + 110k^2 + 40k^3}{16^k}.$$ 

**Example 13** $\frac{5}{2}, 1, 2, 1$ in Theorem 2.

$$\frac{3\pi^2}{32} = 1 + \sum_{k=1}^{\infty} \left[ \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{3}{4} \right] \frac{3 + 3k - 22k^2 - 40k^3}{16^k}.$$ 

**Example 14** $\frac{7}{2}, 1, 2, 1$ in Theorem 2.

$$\frac{15\pi^2}{256} = \frac{1}{3} + \sum_{k=1}^{\infty} \left[ \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{3}{4} \right] \frac{1 - 3k + 2k^2 + 8k^3}{16^k}.$$
Example 15 $\left\lfloor \frac{7}{2} \right\rfloor, 2, 2, 2$ in Theorem 2.

$$\frac{27\pi^2}{128} = \sum_{k=0}^{\infty} \left[ \frac{2 - 21k - 66k^2 - 40k^3}{2^k} \right].$$

Example 16 $\left\lfloor \frac{7}{2} \right\rfloor, 1, 2, 2$ in Theorem 2.

$$\frac{405\pi^2}{256} = 18 + \sum_{k=1}^{\infty} \left[ \frac{48 - 59k - 194k^2 - 120k^3}{2^k} \right].$$

3.3. Series for $\pi^2/\Gamma^3$.

Example 17 $\left\lfloor \frac{1}{2} \right\rfloor, 1, 1, 1, 2, 3, 3, -\frac{2}{3}$ in Theorem 1.

$$\frac{98\pi^2}{3\Gamma \left( \frac{2}{3} \right)^3} = \sum_{k=0}^{\infty} \left[ \frac{118 + 45k - 1098k^2 - 1080k^3}{2^k} \right].$$

Example 18 $\left\lfloor \frac{3}{2} \right\rfloor, 1, 1, 4, 2, 3, 3, 3, 3$ in Theorem 1.

$$\frac{637\pi^2}{16\Gamma \left( \frac{2}{3} \right)^3} = \sum_{k=0}^{\infty} \left[ \frac{1080k^3 + 2286k^2 + 1395k + 161}{2^k} \right].$$

Example 19 $\left\lfloor \frac{2}{3} \right\rfloor, -\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}$ in Theorem 1.

$$\frac{275\pi^2}{\Gamma \left( \frac{1}{3} \right)^3} = \sum_{k=0}^{\infty} \left[ \frac{125 - 351k - 1602k^2 - 1080k^3}{2^k} \right].$$
Example 20 \[
\begin{array}{c}
\frac{3 \cdot 2 \cdot 5 \cdot 1}{2 \cdot 3 \cdot 3 \cdot 3} - \frac{1}{3}
\end{array}
\] in Theorem 1.

\[
\frac{825\pi^2}{8\Gamma\left(\frac{1}{3}\right)^3} = \sum_{k=0}^{\infty} \left[ \begin{array}{cccccc}
2 & 1 & 1 & 7 & 11 & -12 \\
\frac{3}{6} & \frac{6}{6} & \frac{6}{9} & \frac{6}{12} & \frac{7}{12}
\end{array} \right] k
\]

\[
\frac{53 - 315k - 1278k^2 - 1080k^3}{16^k}
\]

Example 21 \[
\begin{array}{c}
\frac{3 \cdot 2 \cdot 5 \cdot 1}{2 \cdot 3 \cdot 3 \cdot 3} - \frac{1}{3}
\end{array}
\] in Theorem 1.

\[
\frac{2805\pi^2}{4\Gamma\left(\frac{1}{3}\right)^3} = \sum_{k=0}^{\infty} \left[ \begin{array}{cccccc}
2 & 5 & 7 & 7 & 13 & 14 & 5 & 11 \\
\frac{3}{3} & \frac{3}{6} & \frac{6}{6} & \frac{6}{9} & \frac{6}{12} & \frac{12}{12}
\end{array} \right] k
\]

\[
\frac{1080k^3 + 2790k^2 + 2151k + 478}{16^k}
\]

Example 22 \[
\begin{array}{c}
-\frac{1}{2} - \frac{2}{3} - \frac{2}{3} - \frac{2}{3}
\end{array}
\] in Theorem 2.

\[
\frac{3872\pi^2}{243\Gamma\left(\frac{2}{3}\right)^3} = \sum_{k=0}^{\infty} \left[ \begin{array}{cccccc}
2 & 5 & 5 & 11 & 11 & 4 & 5 & 11 \\
\frac{3}{3} & \frac{3}{6} & \frac{6}{6} & \frac{6}{9} & \frac{6}{12} & \frac{12}{12}
\end{array} \right] k
\]

\[
\frac{1080k^3 - 954k^2 - 585k + 242}{16^k}
\]

Example 23 \[
\begin{array}{c}
\frac{1}{2} - \frac{2}{3} - \frac{4}{3} - \frac{2}{3}
\end{array}
\] in Theorem 2.

\[
\frac{2380\pi^2}{27\Gamma\left(\frac{2}{3}\right)^3} = \sum_{k=0}^{\infty} \left[ \begin{array}{cccccc}
2 & 5 & 1 & 5 & 5 & 4 & 11 & 17 \\
\frac{3}{3} & \frac{3}{6} & \frac{6}{6} & \frac{6}{9} & \frac{6}{12} & \frac{12}{12}
\end{array} \right] k
\]

\[
\frac{1080k^3 - 1278k^2 + 99k + 170}{16^k}
\]

Example 24 \[
\begin{array}{c}
\frac{1}{2} - \frac{2}{3} - \frac{2}{3}
\end{array}
\] in Theorem 2.

\[
\frac{770\pi^2}{27\Gamma\left(\frac{2}{3}\right)^3} = \sum_{k=0}^{\infty} \left[ \begin{array}{cccccc}
1 & 2 & 5 & 11 & 11 & 7 & 5 & 1 \\
\frac{3}{3} & \frac{3}{6} & \frac{6}{6} & \frac{6}{9} & \frac{6}{12} & \frac{12}{12}
\end{array} \right] k
\]

\[
\frac{55 + 441k - 234k^2 - 1080k^3}{16^k}
\]
Example 25 \[ \frac{3 \cdot 4 \cdot 2 \cdot 4}{2 \cdot 3 \cdot 3 \cdot 3} \] in Theorem 2.

\[
\frac{-1001 \pi^2}{18 \Gamma \left( \frac{2}{3} \right)^3} = \sum_{k=0}^{\infty} \left[ \frac{1}{3^k} \frac{4}{6^k} \frac{5}{6^k} \frac{11}{6^k} \frac{10}{6^k} \frac{1}{12^k} \frac{7}{12^k} \right] \frac{1080 k^3 + 1422 k^2 + 351 k + 44}{16^k}.
\]

Example 26 \[ \frac{3 \cdot 4 \cdot 4 \cdot 2}{2 \cdot 3 \cdot 3 \cdot 3} \] in Theorem 2.

\[
\frac{385 \pi^2}{36 \Gamma \left( \frac{2}{3} \right)^3} = \sum_{k=0}^{\infty} \left[ \frac{1}{3^k} \frac{2}{6^k} \frac{1}{6^k} \frac{5}{6^k} \frac{7}{6^k} \frac{5}{12^k} \frac{11}{12^k} \right] \frac{55 + 189 k + 90 k^2 - 1080 k^3}{16^k}.
\]

Example 27 \[ \frac{2}{3} \frac{-1}{2} \frac{1}{3} \frac{2}{3} \] in Theorem 2.

\[
\frac{910 \pi^2}{9 \Gamma \left( \frac{1}{3} \right)^3} = \sum_{k=0}^{\infty} \left[ \frac{1}{3^k} \frac{1}{6^k} \frac{1}{6^k} \frac{1}{6^k} \frac{1}{6^k} \frac{11}{6^k} \frac{4}{12^k} \frac{5}{12^k} \right] \frac{1080 k^3 - 774 k^2 - 225 k + 91}{16^k}.
\]

Example 28 \[ \frac{3 \cdot 2 \cdot 2 \cdot 2}{2 \cdot 3 \cdot 3 \cdot 3} \] in Theorem 2.

\[
\frac{1225 \pi^2}{6 \Gamma \left( \frac{1}{3} \right)^3} = \sum_{k=0}^{\infty} \left[ \frac{2}{3^k} \frac{1}{6^k} \frac{1}{6^k} \frac{1}{6^k} \frac{8}{6^k} \frac{1}{12^k} \frac{7}{12^k} \right] \frac{98 + 153 k - 414 k^2 - 1080 k^3}{16^k}.
\]

3.4. Series for \( \frac{\Gamma^3}{\pi^2} \).

Example 29 \[ \frac{-1}{2} \frac{-5}{6} \frac{1}{3} \frac{1}{6} \] in Theorem 1.

\[
\frac{180 \Gamma \left( \frac{2}{3} \right)^3}{\pi^2} = \sum_{k=0}^{\infty} \left[ \frac{1}{6^k} \frac{1}{6^k} \frac{5}{6^k} \frac{7}{6^k} \frac{1}{12^k} \frac{12^k} {12^k} \frac{19}{18} \right] \frac{35 + 228 k - 540 k^2 - 2160 k^3}{16^k}.
\]
Example 30 \[
\begin{align*}
\frac{8748 \Gamma \left( \frac{2}{3} \right)^3}{7\pi^2} &= \sum_{k=0}^{\infty} \left[ \begin{array}{cccccccc}
1 & 5 & 5 & 7 & 11 & 11 & 17 & 25 \\
6 & 6 & 6 & 6 & 6 & 12 & 12 & 18 \\
1 & 2 & 3 & 3 & 1 & 2 & 2 & 7 \\
\end{array} \right] \frac{593 + 1344k - 1404k^2 - 2160k^3}{16^k}.
\end{align*}
\]

Example 31 \[
\begin{align*}
\frac{960 \Gamma \left( \frac{2}{3} \right)^3}{7\pi^2} &= \sum_{k=0}^{\infty} \left[ \begin{array}{cccccccc}
1 & 1 & 7 & 13 & 5 & 11 & 25 \\
6 & 6 & 6 & 6 & 6 & 12 & 12 & 18 \\
1 & 1 & 1 & 3 & 1 & 4 & 5 & 7 \\
\end{array} \right] \frac{65 + 372k - 756k^2 - 2160k^3}{16^k}.
\end{align*}
\]

Example 32 \[
\begin{align*}
\frac{960 \Gamma \left( \frac{2}{3} \right)^3}{\pi^2} &= \sum_{k=0}^{\infty} \left[ \begin{array}{cccccccc}
1 & 1 & 5 & 5 & 7 & 11 & 11 & 25 \\
6 & 6 & 6 & 6 & 6 & 12 & 12 & 18 \\
1 & 2 & 3 & 2 & 4 & 2 & 1 & 7 \\
\end{array} \right] \frac{245 - 204k - 2052k^2 - 2160k^3}{16^k}.
\end{align*}
\]

Example 33 \[
\begin{align*}
\frac{7776 \Gamma \left( \frac{2}{3} \right)^3}{7\pi^2} &= \sum_{k=0}^{\infty} \left[ \begin{array}{cccccccc}
1 & 5 & 5 & 5 & 7 & 11 & 5 & 12 \\
6 & 6 & 6 & 6 & 6 & 6 & 12 & 18 \\
1 & 2 & 3 & 3 & 1 & 2 & 4 & 1 \\
\end{array} \right] \frac{360k^2 + 546k + 191}{16^k}.
\end{align*}
\]

Example 34 \[
\begin{align*}
\frac{1024 \Gamma \left( \frac{2}{3} \right)^3}{21\pi^2} &= \sum_{k=0}^{\infty} \left[ \begin{array}{cccccccc}
1 & 1 & 7 & 13 & 1 & 5 & 25 \\
6 & 6 & 6 & 6 & 6 & 12 & 12 & 18 \\
1 & 1 & 1 & 3 & 4 & 5 & 7 & 7 \\
\end{array} \right] \frac{2160k^3 + 2268k^2 + 60k + 13}{16^k}.
\end{align*}
\]
Example 35 \(-\frac{1}{2}, -\frac{1}{6}, \frac{5}{6}, \frac{5}{6}\) in Theorem 1.

\[
\frac{2916 \Gamma \left( \frac{1}{3} \right)^3}{5 \pi^2} = \sum_{k=0}^\infty \left[ \begin{array}{cccccccc}
\frac{1}{6} & \frac{5}{6} & \frac{7}{6} & \frac{7}{6} & 13 & \frac{7}{6} & \frac{13}{12} & \frac{23}{18} \\
1, & 3, & 3, & 1, & 1, & 2, & 5, & \frac{5}{18}
\end{array} \right] \frac{1}{16^k} \frac{697 + 1056k - 1836k^2 - 2160k^3}{16^k}.
\]

Example 36 \(-\frac{1}{2}, -\frac{1}{6}, \frac{11}{6}, \frac{5}{6}\) in Theorem 1.

\[
\frac{2592 \Gamma \left( \frac{1}{3} \right)^3}{25 \pi^2} = \sum_{k=0}^\infty \left[ \begin{array}{cccccccc}
\frac{1}{6} & \frac{5}{6} & \frac{7}{6} & \frac{7}{6} & 13 & 1 & \frac{7}{6} & \frac{23}{18} \\
1, & 3, & 3, & 1, & 2, & 5, & 5, & \frac{5}{18}
\end{array} \right] \frac{1}{16^k} \frac{223 - 888k - 3348k^2 - 2160k^3}{16^k}.
\]

Example 37 \(-\frac{5}{2}, -\frac{1}{6}, -\frac{1}{6}\) in Theorem 1.

\[
\frac{32 \Gamma \left( \frac{1}{3} \right)^3}{5 \pi^2} = \sum_{k=0}^\infty \left[ \begin{array}{cccccccc}
\frac{1}{6} & \frac{1}{6} & \frac{5}{6} & \frac{5}{6} & 11 & \frac{1}{6} & \frac{7}{12} & \frac{23}{18} \\
1, & 1, & 1, & 3, & 2, & 4, & 5, & \frac{5}{18}
\end{array} \right] \frac{1}{16^k} \frac{2160k^3 + 1188k^2 - 84k + 11}{16^k}.
\]

Example 38 \(-\frac{5}{2}, -\frac{1}{6}, -\frac{11}{6}\) in Theorem 1.

\[
\frac{2592 \Gamma \left( \frac{1}{3} \right)^3}{55 \pi^2} = \sum_{k=0}^\infty \left[ \begin{array}{cccccccc}
\frac{1}{6} & \frac{5}{6} & \frac{7}{6} & \frac{11}{6} & 13 & \frac{19}{12} & 12 & \frac{29}{18} \\
1, & 2, & 3, & 3, & 2, & 4, & 1, & \frac{11}{18}
\end{array} \right] \frac{1}{16^k} \frac{151 + 264k - 2052k^2 - 2160k^3}{16^k}.
\]

Example 39 \(-\frac{3}{2}, -\frac{5}{6}, -\frac{1}{6}\) in Theorem 1.

\[
\frac{256 \Gamma \left( \frac{1}{3} \right)^3}{9 \pi^2} = \sum_{k=0}^\infty \left[ \begin{array}{cccccccc}
\frac{1}{6} & \frac{1}{6} & \frac{5}{6} & \frac{5}{6} & 11 & \frac{1}{6} & \frac{5}{12} & \frac{23}{18} \\
1, & 1, & 1, & 3, & 4, & 5, & 8, & \frac{5}{18}
\end{array} \right] \frac{1}{16^k} \frac{55 - 348k - 2700k^2 - 2160k^3}{16^k}.
\]
Example 40 \[
\begin{bmatrix}
3 & 5 & 5 & 11 \\
2 & 6 & 6 & 6
\end{bmatrix}
\] in Theorem 1.

\[
\frac{6912 \Gamma \left( \frac{1}{3} \right)^3}{55 \pi^2} = \sum_{k=0}^{\infty} \begin{bmatrix}
1 & 1 & 5 & 7 & 11 & 7 & 13 & 29 \\
6 & 6 & 6 & 6 & 6 & 12 & 12 & 18
\end{bmatrix} \frac{2160k^3 + 3564k^2 + 1680k + 251}{16^k}.
\]

Example 41 \[
\begin{bmatrix}
-1 & 1 & 7 & 1 \\
-2 & 6 & 6 & 6
\end{bmatrix}
\] in Theorem 2.

\[
\frac{2673 \Gamma \left( \frac{2}{3} \right)^3}{16 \pi^2} = \sum_{k=0}^{\infty} \begin{bmatrix}
1 & 5 & 5 & 11 & -11 & -5 & 13 \\
6 & 6 & 6 & 6 & 6 & 12 & 12 & 18
\end{bmatrix} \frac{11 + 1380k + 1188k^2 - 2160k^3}{16^k}.
\]

Example 42 \[
\begin{bmatrix}
-1 & 7 & -5 & 7 \\
-2 & 6 & -6 & 6
\end{bmatrix}
\] in Theorem 2.

\[
\frac{13365 \Gamma \left( \frac{2}{3} \right)^3}{16 \pi^2} = \sum_{k=0}^{\infty} \begin{bmatrix}
1 & 5 & 5 & 7 & 11 & 17 & 19 \\
6 & 6 & 6 & 6 & 6 & 12 & 12 & 18
\end{bmatrix} \frac{2160k^3 - 2484k^2 - 1092k + 385}{16^k}.
\]

Example 43 \[
\begin{bmatrix}
1 & 7 & 1 \\
2 & 6 & 6
\end{bmatrix}
\] in Theorem 2.

\[
\frac{675 \Gamma \left( \frac{2}{3} \right)^3}{2 \pi^2} = \sum_{k=0}^{\infty} \begin{bmatrix}
1 & 5 & 5 & 7 & 5 & 11 & 19 \\
6 & 6 & 6 & 6 & 6 & 12 & 12 & 18
\end{bmatrix} \frac{2160k^3 - 972k^2 - 660k + 175}{16^k}.
\]

Example 44 \[
\begin{bmatrix}
3 & 7 & 7 \\
2 & 6 & 6
\end{bmatrix}
\] in Theorem 2.

\[
\frac{180 \Gamma \left( \frac{2}{3} \right)^3}{\pi^2} = \sum_{k=0}^{\infty} \begin{bmatrix}
1 & 5 & 5 & 7 & 1 & 5 & 19 \\
6 & 6 & 6 & 6 & 6 & 12 & 12 & 18
\end{bmatrix} \frac{35 + 228k - 540k^2 - 2160k^3}{16^k}.
\]
Example 45 \[ \frac{-\frac{1}{2} - \frac{1}{6} + \frac{1}{6} - \frac{1}{6}}{1} \text{ in Theorem 2.} \]

\[
\frac{1053\left(\frac{1}{3}\right)^3}{32\pi^2} = \sum_{k=0}^{\infty} \left\{ \begin{array}{c}
\frac{1}{6} - \frac{1}{6} - \frac{1}{6} - \frac{1}{6}, \\
\frac{1}{6} - \frac{1}{6} - \frac{1}{6} - \frac{1}{6}, \\
\frac{1}{6} - \frac{1}{6} - \frac{1}{6} - \frac{1}{6}
\end{array} \right\} \frac{2160k^3 - 756k^2 - 1668k + 65}{16^k}.
\]

Example 46 \[ \frac{-\frac{1}{2} - \frac{1}{6} + \frac{1}{6} - \frac{1}{6}}{1} \text{ in Theorem 2.} \]

\[
\frac{3969\left(\frac{1}{3}\right)^3}{32\pi^2} = \sum_{k=0}^{\infty} \left\{ \begin{array}{c}
\frac{1}{6} - \frac{1}{6} - \frac{1}{6} - \frac{1}{6}, \\
\frac{1}{6} - \frac{1}{6} - \frac{1}{6} - \frac{1}{6}, \\
\frac{1}{6} - \frac{1}{6} - \frac{1}{6} - \frac{1}{6}
\end{array} \right\} \frac{2160k^3 - 2052k^2 - 1092k + 245}{16^k}.
\]

Example 47 \[ \frac{\frac{1}{2} - \frac{1}{6} - \frac{1}{6} - \frac{1}{6}}{1} \text{ in Theorem 2.} \]

\[
\frac{63\left(\frac{1}{3}\right)^3}{10\pi^2} = \sum_{k=0}^{\infty} \left\{ \begin{array}{c}
\frac{1}{6} - \frac{1}{6} - \frac{1}{6} - \frac{1}{6}, \\
\frac{1}{6} - \frac{1}{6} - \frac{1}{6} - \frac{1}{6}, \\
\frac{1}{6} - \frac{1}{6} - \frac{1}{6} - \frac{1}{6}
\end{array} \right\} \frac{432k^3 - 108k^2 - 132k + 7}{16^k}.
\]

Example 48 \[ \frac{\frac{3}{2} - \frac{1}{6} - \frac{1}{6} - \frac{1}{6}}{1} \text{ in Theorem 2.} \]

\[
\frac{84\left(\frac{1}{3}\right)^3}{5\pi^2} = \sum_{k=0}^{\infty} \left\{ \begin{array}{c}
\frac{1}{6} - \frac{1}{6} - \frac{1}{6} - \frac{1}{6}, \\
\frac{1}{6} - \frac{1}{6} - \frac{1}{6} - \frac{1}{6}, \\
\frac{1}{6} - \frac{1}{6} - \frac{1}{6} - \frac{1}{6}
\end{array} \right\} \frac{2160k^3 - 324k^2 - 516k + 77}{16^k}.
\]

Example 49 \[ \frac{\frac{3}{2} - \frac{1}{6} - \frac{1}{6} - \frac{1}{6}}{1} \text{ in Theorem 2.} \]

\[
\frac{84\left(\frac{1}{3}\right)^3}{\pi^2} = \sum_{k=0}^{\infty} \left\{ \begin{array}{c}
\frac{1}{6} - \frac{1}{6} - \frac{1}{6} - \frac{1}{6}, \\
\frac{1}{6} - \frac{1}{6} - \frac{1}{6} - \frac{1}{6}, \\
\frac{1}{6} - \frac{1}{6} - \frac{1}{6} - \frac{1}{6}
\end{array} \right\} \frac{175 + 228k - 972k^2 - 2160k^3}{16^k}.
\]
3.5. Series for $\pi^{-1}$.

**Example 50** Chu and Zhang [21]: \[
\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{3} \text{ in Theorem 1.}
\]
\[
\frac{15\sqrt{3}}{\pi} = \sum_{k=0}^{\infty} \left[ \frac{1}{2} \cdot \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{2}{3} \right]_{k} 135k^2 + 75k + 8 \cdot \frac{16^k}{1}. \]

**Example 51** \[
\frac{21\sqrt{3}}{\pi} = \sum_{k=0}^{\infty} \left[ \frac{1}{2} \cdot \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{2}{3} \right]_{k} 810k^3 + 684k^2 + 141k + 10 \cdot \frac{16^k}{1}. \]

**Example 52** \[
\frac{48}{\pi} = \sum_{k=0}^{\infty} \left[ \frac{1}{2} \cdot \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{3}{8} \right]_{k} 480k^2 + 212k + 15 \cdot \frac{16^k}{1}. \]

**Example 53** \[
\frac{80}{3\pi} = \sum_{k=0}^{\infty} \left[ \frac{1}{2} \cdot \frac{1}{3} \cdot \frac{3}{8} \cdot \frac{1}{3} \cdot \frac{7}{8} \right]_{k} 640k^3 + 560k^2 + 112k + 7 \cdot \frac{16^k}{1}. \]

**Example 54** \[
\frac{256}{3\pi \sqrt{3}} = \sum_{k=0}^{\infty} \left[ \frac{1}{2} \cdot \frac{1}{3} \cdot \frac{5}{3} \cdot \frac{5}{3} \cdot \frac{4}{3} \right]_{k} 720k^3 + 804k^2 + 236k + 15 \cdot \frac{16^k}{1}. \]
Example 55 \[
\frac{192}{\pi \sqrt{3}} = \sum_{k=0}^{\infty} \begin{bmatrix}
\frac{1}{2}, \frac{1}{6}, \frac{1}{6}, 7 \\
\frac{1}{2}, \frac{1}{6}, \frac{1}{12}, \frac{12}{12} \\
1, 1, 1, \frac{4}{3}, \frac{4}{3} \\
\end{bmatrix}_k \frac{6480k^3 + 4284k^2 + 840k + 35}{16^k}.
\]

Example 56 Chu and Zhang [21]: \[
\frac{384}{\pi \sqrt{3}} = \sum_{k=0}^{\infty} \begin{bmatrix}
\frac{1}{2}, \frac{1}{6}, \frac{1}{6}, \frac{5}{12} \\
\frac{1}{2}, \frac{1}{6}, \frac{12}{12}, \frac{12}{12} \\
1, 1, 1, \frac{2}{3}, \frac{5}{3} \\
\end{bmatrix}_k \frac{1080k^2 + 798k + 55}{16^k}.
\]

Example 57 \[
\frac{105 \sqrt{5} - 2 \sqrt{5}}{\pi} = \sum_{k=0}^{\infty} \begin{bmatrix}
\frac{1}{2}, \frac{1}{5}, \frac{3}{5}, \frac{4}{5}, \frac{1}{10} \\
\frac{1}{2}, \frac{1}{5}, \frac{5}{5}, \frac{5}{5}, \frac{10}{10} \\
1, 1, 1, \frac{13}{10}, \frac{17}{20}, \frac{27}{20} \\
\end{bmatrix}_k \frac{3750k^3 + 2525k^2 + 505k + 24}{16^k}.
\]

Example 58 \[
\frac{45 \sqrt{5} + 2 \sqrt{5}}{\pi} = \sum_{k=0}^{\infty} \begin{bmatrix}
\frac{1}{2}, \frac{1}{5}, \frac{2}{5}, \frac{2}{5}, \frac{3}{5}, \frac{7}{10} \\
\frac{1}{2}, \frac{1}{5}, \frac{5}{5}, \frac{5}{5}, \frac{7}{10}, \frac{10}{10} \\
1, 1, 1, \frac{11}{10}, \frac{19}{20}, \frac{29}{20} \\
\end{bmatrix}_k \frac{3750k^3 + 2800k^2 + 595k + 42}{16^k}.
\]

Example 59 \[
\frac{55 \sqrt{5} + 2 \sqrt{5}}{3\pi} = \sum_{k=0}^{\infty} \begin{bmatrix}
\frac{1}{2}, \frac{3}{5}, \frac{3}{5}, \frac{3}{5}, \frac{3}{5}, \frac{3}{10} \\
\frac{1}{2}, \frac{3}{5}, \frac{5}{5}, \frac{5}{5}, \frac{5}{5}, \frac{5}{10}, \frac{10}{10} \\
1, 1, 1, \frac{9}{10}, \frac{21}{20}, \frac{31}{20} \\
\end{bmatrix}_k \frac{1250k^3 + 1025k^2 + 215k + 16}{16^k}.
\]
Example 60 \[ \frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{5} \] in Theorem 1.

\[
\frac{195\sqrt{5} - 2\sqrt{3}}{\pi} = \sum_{k=0}^{\infty} \left[ \frac{1}{3} \frac{1}{5} \frac{4}{5} \frac{2}{5} \frac{4}{5} \frac{9}{10} \right] \frac{3750k^3 + 3350k^2 + 655k + 36}{16^k}.
\]

Example 61 \[ \frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{8} \] in Theorem 1.

\[
\frac{480}{\pi(\sqrt{2} + 1)} = \sum_{k=0}^{\infty} \left[ \frac{1}{2} \frac{1}{3} \frac{7}{8} \frac{1}{16} \frac{11}{16} \frac{21}{16} \right] \frac{15360k^3 + 9920k^2 + 1888k + 63}{16^k}.
\]

Example 62 \[ \frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{3}{8} \] in Theorem 1.

\[
\frac{224}{3\pi(\sqrt{2} - 1)} = \sum_{k=0}^{\infty} \left[ \frac{1}{2} \frac{3}{8} \frac{3}{8} \frac{5}{16} \frac{3}{16} \frac{11}{16} \right] \frac{5120k^3 + 3776k^2 + 800k + 55}{16^k}.
\]

Example 63 \[ \frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{5}{8} \] in Theorem 1.

\[
\frac{288}{\pi(\sqrt{2} - 1)} = \sum_{k=0}^{\infty} \left[ \frac{1}{2} \frac{5}{8} \frac{5}{8} \frac{5}{16} \frac{13}{16} \right] \frac{15360k^3 + 12736k^2 + 2656k + 195}{16^k}.
\]

Example 64 \[ \frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{7}{8} \] in Theorem 1.

\[
\frac{1056}{\pi(\sqrt{2} + 1)} = \sum_{k=0}^{\infty} \left[ \frac{1}{2} \frac{1}{8} \frac{7}{8} \frac{7}{16} \frac{15}{16} \right] \frac{15360k^3 + 14144k^2 + 2656k + 105}{16^k}.
\]
Example 65 \[ \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{6} \] in Theorem 2.

\[
10 \sqrt{3} \pi = \sum_{k=0}^{\infty} \left[ \frac{1}{2} \cdot \frac{1}{6} \cdot \frac{1}{12} \cdot \frac{5}{12} \right]_{k} \frac{2160k^3 - 372k^2 + 68k + 5}{16k}.
\]

Example 66 \[ \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{3} \] in Theorem 2.

\[
6 \sqrt{3} \pi = \sum_{k=0}^{\infty} \left[ \frac{1}{2} \cdot \frac{3}{3} \cdot \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{2}{3} \right]_{k} \frac{135k^3 - 48k^2 - 7k + 2}{16k}.
\]

Example 67 \[ \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{4} \] in Theorem 2.

\[
20 \pi = \sum_{k=0}^{\infty} \left[ \frac{1}{2} \cdot \frac{1}{4} \cdot \frac{1}{4} \cdot \frac{1}{8} \cdot \frac{5}{8} \right]_{k} \frac{960k^3 - 232k^2 - 38k + 5}{16k}.
\]

Example 68 \[ \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{6} \cdot \frac{1}{2} \] in Theorem 2.

\[
10 \sqrt{3} \cdot 9 \pi = \sum_{k=0}^{\infty} \left[ \frac{1}{2} \cdot \frac{1}{4} \cdot \frac{1}{4} \cdot \frac{5}{6} \cdot \frac{5}{6} \right]_{k} \frac{k(120k^2 - 26k + 5)}{16k}.
\]

Example 69 \[ \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{6} \cdot \frac{1}{2} \] in Theorem 2.

\[
6 \sqrt{3} \pi = \sum_{k=0}^{\infty} \left[ \frac{1}{2} \cdot \frac{1}{4} \cdot \frac{1}{4} \cdot \frac{5}{6} \cdot \frac{5}{6} \right]_{k} \frac{720k^3 - 300k^2 - 4k + 3}{16k}.
\]
Example 70 \[ \frac{1}{2} \frac{1}{2} - \frac{1}{2} \frac{7}{6} \] in Theorem 2.

\[ \frac{27 \sqrt{3}}{\pi} = \sum_{k=0}^{\infty} \left[ \frac{3}{2} \frac{1}{6} \frac{1}{12} \frac{7}{12} \right] \left( \frac{1}{1}, \frac{1}{1}, \frac{1}{3}, -\frac{2}{3} \right)_k \frac{21 + 292k - 420k^2 - 2160k^3}{16^k}. \]

Example 71 \[ \frac{1}{2} \frac{1}{2} - \frac{1}{2} \frac{1}{4} \] in Theorem 2.

\[ \frac{96}{\pi} = \sum_{k=0}^{\infty} \left[ \frac{3}{2} \frac{1}{4} \frac{1}{8} \frac{3}{8} \right] \left( \frac{1}{1}, \frac{1}{1}, \frac{3}{8}, -\frac{1}{8} \right)_k \frac{9 + 102k - 424k^2 - 960k^3}{16^k}. \]

Example 72 \[ \frac{1}{2} \frac{1}{2} - \frac{1}{2} \frac{1}{4} \] in Theorem 2.

\[ \frac{160}{\pi} = \sum_{k=0}^{\infty} \left[ \frac{3}{2} \frac{5}{4} \frac{1}{8} \frac{5}{8} \right] \left( \frac{1}{1}, \frac{1}{1}, \frac{1}{8}, -\frac{3}{8} \right)_k \frac{960k^3 - 232k^2 - 710k + 75}{16^k}. \]

Example 73 \[ \frac{1}{2} \frac{1}{2} - \frac{1}{6} \frac{1}{2} \] in Theorem 2.

\[ \frac{28}{3 \sqrt{3} \pi} = \sum_{k=0}^{\infty} \left[ \frac{1}{2} \frac{1}{4} \frac{1}{6} \frac{1}{12} \frac{7}{12} \frac{13}{12} \right] \left( \frac{1}{1}, \frac{1}{1}, \frac{2}{3}, \frac{4}{12} \right)_k \frac{k(60k^2 - 8k - 7)}{16^k}. \]

Example 74 \[ \frac{1}{2} \frac{1}{2} - \frac{1}{2} \frac{1}{3} \] in Theorem 2.

\[ \frac{162 \sqrt{3}}{5 \pi} = \sum_{k=0}^{\infty} \left[ \frac{3}{2} \frac{1}{3} \frac{2}{3} \frac{4}{3} \frac{4}{3} \right] \left( \frac{1}{1}, \frac{1}{1}, \frac{1}{12} \frac{5}{12} \right)_k \frac{135k^3 - 48k^2 - 106k + 24}{16^k}. \]
3.6. Series for $\pi$.

**Example 75** $\frac{5}{2}, 2, 2, \frac{3}{4}$ in Theorem 2.

\[
\frac{5\pi}{16} = \sum_{k=0}^{\infty} \left[ \begin{array}{cccc}
-\frac{1}{2} & -\frac{1}{4} & -\frac{1}{8} & -\frac{3}{8} \\
1 & 3 & 9 & 13 \\
2 & 4 & 8 & 8 \\
\end{array} \right]_k \left( 1 - 7k + 40k^2 \right) \frac{1}{16^k}.
\]

**Example 76** $\frac{5}{2}, 2, 2, \frac{1}{4}$ in Theorem 2.

\[
\frac{25\pi}{16} = \sum_{k=0}^{\infty} \left[ \begin{array}{cccc}
-\frac{1}{2} & -\frac{1}{4} & -\frac{1}{8} & -\frac{5}{8} \\
1 & 9 & 7 & 11 \\
2 & 4 & 8 & 8 \\
\end{array} \right]_k \left( 120k^2 + 77k + 5 \right) \frac{1}{16^k}.
\]

**Example 77** $\frac{3}{2}, 2, 1, \frac{1}{4}$ in Theorem 2.

\[
\frac{3\pi}{8} = \sum_{k=0}^{\infty} \left[ \begin{array}{cccc}
-\frac{1}{2} & -\frac{1}{4} & -\frac{1}{8} & -\frac{3}{8} \\
1 & 5 & 7 & 11 \\
2 & 4 & 8 & 8 \\
\end{array} \right]_k \frac{1 + 11k + 106k^2 + 240k^3}{16^k}.
\]

**Example 78** $\frac{3}{2}, 1, 1, \frac{5}{6}$ in Theorem 1.

\[
\frac{36\pi}{5\sqrt{3}} = \sum_{k=0}^{\infty} \left[ \begin{array}{cccc}
1 & 1 & 2 & 1 \\
2 & 3 & 6 & 9 \\
3 & 5 & 7 & 9 \\
2 & 3 & 4 & 6 \\
\end{array} \right]_k \frac{60k^2 + 64k + 13}{16^k}.
\]

**Example 79** Chu and Zhang [21]: $\frac{3}{2}, 1, \frac{5}{6}, \frac{1}{6}$ in Theorem 1.

\[
\frac{20\pi}{9\sqrt{3}} = \sum_{k=0}^{\infty} \left[ \begin{array}{cccc}
1 & 1 & 2 & 1 \\
3 & 3 & 3 & 7 \\
2 & 2 & 5 & 6 \\
\end{array} \right]_k \frac{12k^2 + 15k + 4}{16^k}.
\]
Example 80 \(\frac{1}{2}, 1, \frac{1}{6}, 1\) in Theorem 1.

\[
\frac{20\pi}{27\sqrt{3}} = \sum_{k=0}^{\infty} \begin{bmatrix} 1 & \frac{1}{3} & \frac{2}{3} & 3 \vspace{1mm} \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{5}{3} \vspace{1mm} \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{5}{3} \vspace{1mm} \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{5}{3} \vspace{1mm} \\
\end{bmatrix}_k \frac{4 - 11k - 69k^2 - 60k^3}{16k}.
\]

Example 81 \(\frac{3}{2}, 1, \frac{5}{6}, 2\) in Theorem 1.

\[
\frac{140\pi}{27\sqrt{3}} = \sum_{k=0}^{\infty} \begin{bmatrix} 2 & \frac{4}{3} & -\frac{1}{3} & \frac{5}{4} \vspace{1mm} \\
1 & 3 & 3 & 11 \vspace{1mm} \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{5}{3} \vspace{1mm} \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{5}{3} \vspace{1mm} \\
\end{bmatrix}_k \frac{60k^3 + 133k^2 + 85k + 13}{16k}.
\]

Example 82 \(\frac{1}{2}, 1, -\frac{1}{6}, 2\) in Theorem 1.

\[
\frac{700\pi}{243\sqrt{3}} = \sum_{k=0}^{\infty} \begin{bmatrix} 2 & -\frac{1}{3} & \frac{4}{3} & \frac{5}{4} \vspace{1mm} \\
1 & 3 & 3 & 11 \vspace{1mm} \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{5}{3} \vspace{1mm} \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{5}{3} \vspace{1mm} \\
\end{bmatrix}_k \frac{25 - 3k - 91k^2 - 60k^3}{16k}.
\]

Example 83 \(\frac{3}{2}, 1, \frac{5}{6}, 1\) in Theorem 2.

\[
\frac{4\pi}{9\sqrt{3}} = \frac{2}{3} + \sum_{k=1}^{\infty} \begin{bmatrix} 1 & \frac{2}{3} & -\frac{2}{3} & \frac{1}{4} \vspace{1mm} \\
1 & 3 & \frac{1}{2} & \frac{7}{6} \vspace{1mm} \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{6}{6} \vspace{1mm} \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{6}{6} \vspace{1mm} \\
\end{bmatrix}_k \frac{20k^2 + 7k + 2}{16k}.
\]

Example 84 \(\frac{1}{2}, 1, \frac{1}{6}, 1\) in Theorem 2.

\[
\frac{4\pi}{81\sqrt{3}} = \frac{2}{3} + \sum_{k=1}^{\infty} \begin{bmatrix} 1 & \frac{2}{3} & -\frac{2}{3} & \frac{1}{4} \vspace{1mm} \\
1 & 3 & \frac{1}{2} & \frac{7}{6} \vspace{1mm} \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{6}{6} \vspace{1mm} \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{6}{6} \vspace{1mm} \\
\end{bmatrix}_k \frac{2 + 7k - 20k^2}{16k}.
\]

Example 85 \(\frac{5}{2}, 1, \frac{5}{6}, 2\) in Theorem 2.

\[
\frac{10\pi}{7\sqrt{3}} = 2 + \sum_{k=1}^{\infty} \begin{bmatrix} 2 & \frac{5}{3} & -\frac{5}{3} & \frac{1}{4} \vspace{1mm} \\
1 & 5 & 7 & 11 \vspace{1mm} \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{6}{6} \vspace{1mm} \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{6}{6} \vspace{1mm} \\
\end{bmatrix}_k \frac{12k^2 + 17k + 8}{16k}.
\]
Example 86 \( \frac{3^2 \cdot 1 \cdot 2 \cdot 5}{6} \) in Theorem 2.

\[
\frac{16\pi}{3\sqrt{3}} = 20 + \sum_{k=1}^{\infty} \left[ \frac{1^4 \cdot 4^4 \cdot 1^4 \cdot 5^5}{2^3 \cdot 3^3 \cdot 6^3 \cdot 6} \right] \frac{56 - 33k - 270k^2}{16^k}.
\]

Example 87 \( \frac{1}{2}, 1, \frac{1}{6}, 2 \) in Theorem 2.

\[
\frac{350\pi}{243\sqrt{3}} = 30 + \sum_{k=1}^{\infty} \left[ \frac{2^2 \cdot 5^5 \cdot 3^3 \cdot 5^5}{2^3 \cdot 3^3 \cdot 4^3 \cdot 4} \right] \frac{40 - k - 60k^2}{16^k}.
\]

Example 88 \( \frac{3}{2}, 1, 1, 2 \) in Theorem 2.

\[
\frac{32\pi}{27\sqrt{3}} = -8 + \sum_{k=1}^{\infty} \left[ \frac{2^2 \cdot 3^2 \cdot 4^1 \cdot 3^3}{2^3 \cdot 3^3 \cdot 4^3 \cdot 4} \right] \frac{k(15k + 2)(1 - 12k)}{16^k}.
\]

Example 89 \( \frac{5}{2}, 1, 2, \frac{5}{6} \) in Theorem 1.

\[
\frac{270\pi}{7\sqrt{3}} = \sum_{k=0}^{\infty} \left[ \frac{1^1 \cdot 1^1 \cdot 1^1 \cdot 5^5}{2^3 \cdot 3^3 \cdot 6^3 \cdot 6} \right] \frac{70 + 409k + 627k^2 + 270k^3}{16^k}.
\]

Example 90 \( \frac{5}{2}, 1, 2, \frac{1}{6} \) in Theorem 1.

\[
\frac{756\pi}{275\sqrt{3}} = \sum_{k=0}^{\infty} \left[ \frac{1^2 \cdot 1^3 \cdot 2^1 \cdot 1^1 \cdot 1^1}{2^3 \cdot 3^3 \cdot 6^3 \cdot 6} \right] \frac{5 + 92k + 258k^2 + 135k^3}{16^k}.
\]

3.7. BBP-Series. In 1995, S. Plouffe discovered the following amazing BBP-formula (named after Bailey–Borwein–Plouffe [2] (Theorem 1)):

\[
\pi = \sum_{k=0}^{\infty} \left( \frac{1}{16} \right)^k \left\{ \frac{4}{8k+1} - \frac{2}{8k+4} - \frac{1}{8k+5} - \frac{1}{8k+6} \right\}
\]
providing a digit-extraction algorithm for \( \pi \) in base 10. By decomposing the factorial fraction of the term into partial fractions, we can show that the next five series are all equivalent to the presented BBP-formula.

**Example 91** \( \left[ \frac{3}{2}, 1, 1, \frac{3}{4} \right] \) in Theorem 1.

\[
15\pi = \sum_{k=0}^{\infty} \left[ \begin{array}{rrr} 1 & 3 & 1 \\ 2' & 4' & 8' \\ 3 & 7 & 9 & 13 \\ 2' & 4' & 8' & 8 \end{array} \right] k \frac{120k^2 + 151k + 47}{16^k}.
\]

**Example 92** \( \left[ \frac{5}{2}, 1, 2, \frac{3}{4} \right] \) in Theorem 1.

\[
\frac{63\pi}{2} = \sum_{k=0}^{\infty} \left[ \begin{array}{rrr} 1 & 3 & 1 \\ 2' & 4' & 8' \\ 5 & 11 & 9 & 13 \\ 2' & 4' & 8' & 8 \end{array} \right] k \frac{120k^2 + 235k + 99}{16^k}.
\]

**Example 93** \( \left[ \frac{3}{2}, 1, 2, -\frac{1}{4} \right] \) in Theorem 2.

\[
\frac{21\pi}{8} = 7 + \sum_{k=1}^{\infty} \left[ \begin{array}{rrr} -1 & 1 & 1 \\ 2' & 4' & 8' \\ 1 & 7 & 5 & 9 \\ 2' & 4' & 8' & 8 \end{array} \right] k \frac{480k^2 - 172k - 9}{16^k}.
\]

**Example 94** \( \left[ \frac{5}{2}, 1, 2, \frac{3}{4} \right] \) in Theorem 2.

\[
\frac{21\pi}{10} = 7 + \sum_{k=1}^{\infty} \left[ \begin{array}{rrr} -1 & 1 & 1 \\ 2' & 4' & 8' \\ 5 & 3 & 5 & 9 \\ 2' & 4' & 8' & 8 \end{array} \right] k \frac{23 + 10k - 240k^2}{16^k}.
\]

**Example 95** \( \left[ \frac{3}{2}, 1, 2, -\frac{5}{4} \right] \) in Theorem 2.

\[
\frac{77\pi}{8} = -\frac{55}{3} + \sum_{k=1}^{\infty} \left[ \begin{array}{rrr} 1 & 5 & 7 \\ 2' & 4' & 8' \\ 1 & 11 & 1 & 5 \\ 2' & 4' & 8' & 8 \end{array} \right] k \frac{160k^2 - 36k - 13}{16^k}.
\]

There is another BBP-formula proposed in the article by Adamchik–Wagon [1]

\[
2\pi = \sum_{k=0}^{\infty} \left( \frac{1}{16} \right)^k \left\{ \frac{8}{8k + 2} + \frac{4}{8k + 3} + \frac{4}{8k + 4} - \frac{1}{8k + 7} \right\}.
\]
Then the same approach of partial fractions can show that it has the following different infinite-series representations:

**Example 96** \[ \frac{3}{2}, 1, 1, \frac{1}{4} \] in Theorem 2.

\[
\frac{5\pi}{9} = 5 + \sum_{k=1}^{\infty} \left[ \begin{array}{cccc}
-1 & 1 & 1 & 5 \\
2 & 4 & 8 & 16 \\
3 & 5 & 3 & 7 \\
2 & 4 & 8 & 8 \\
\end{array} \right] \frac{7 - 6k - 80k^2}{16^k}.
\]

**Example 97** \[ \frac{5}{2}, 1, 2, \frac{1}{4} \] in Theorem 2.

\[
\frac{15\pi}{14} = 3 + \sum_{k=1}^{\infty} \left[ \begin{array}{cccc}
-1 & 1 & 5 & 9 \\
2 & 4 & 8 & 16 \\
5 & 9 & 3 & 7 \\
2 & 4 & 8 & 8 \\
\end{array} \right] \frac{19 - 62k - 80k^2}{16^k}.
\]

**Example 98** \[ \frac{3}{2}, 1, 2, \frac{1}{4} \] in Theorem 2.

\[
\frac{15\pi}{8} = 5 + \sum_{k=1}^{\infty} \left[ \begin{array}{cccc}
-1 & 3 & 1 & 5 \\
2 & 4 & 8 & 16 \\
1 & 7 & 11 & 16 \\
2 & 4 & 8 & 8 \\
\end{array} \right] \frac{160k^2 - 108k + 21}{16^k}.
\]

**Example 99** \[ \frac{3}{2}, 1, 2, -\frac{3}{4} \] in Theorem 2.

\[
\frac{45\pi}{8} = 16 + \sum_{k=0}^{\infty} \left[ \begin{array}{cccc}
-1 & 3 & 5 & 9 \\
2 & 4 & 8 & 16 \\
1 & 3 & 7 & 16 \\
2 & 4 & 8 & 8 \\
\end{array} \right] \frac{11 + 260k - 480k^2}{16^k}.
\]

**Example 100** \[ \frac{3}{2}, 1, 1, \frac{5}{4} \] in Theorem 1.

\[
21\pi = \sum_{k=0}^{\infty} \left[ \begin{array}{cccc}
1 & 1 & 3 & 7 \\
2 & 4 & 8 & 8 \\
3 & 5 & 11 & 15 \\
2 & 4 & 8 & 8 \\
\end{array} \right] \frac{65 + 413k + 812k^2 + 480k^3}{16^k}.
\]
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