EXOTIC SUPERFLUIDS: BREACHED PAIRING, MIXED PHASES AND STABILITY

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We review properties of gapless states. We construct a model where a stable breached pair (gapless) state is realized.

1. Introduction

Motivated by recent experiments in cold atoms\(^1\) and by questions in QCD at high densities\(^2\), we consider here superfluid fermion systems, in particular exotic superfluids. As suggested by Liu and Wilczek, exotic phases of matter in superfluid fermion systems involve coexistence of normal Fermi liquid and superfluid components\(^3\). Superfluid properties are described by a nonzero condensate, \(\Delta = \langle T(\psi^\dagger(x)\psi^\dagger(x')) \rangle \neq 0\), being the order parameter, while a single quasiparticle dispersion crosses the momentum axis (free Fermi surface) leading to a gapless mode; thus this phase is sometimes called a “gapless” superconductive phase. Due to nonzero Fermi condensate the ground state is a superfluid in a classical sense (i.e. it has zero viscosity). On a microscopic scale, one can envision a momentum separation in exotic superfluids. For species with noticeably different Fermi momenta, Cooper pairing takes place around the Fermi surfaces, but there is no pairing in the momentum region between surfaces (the breach). Thus in this work we use the term “breached pair” (BP) superfluidity.

To obtain exotic superfluids we consider pairing between two different fermion species whose Fermi surfaces do not match. This possibility arises in several situations: (1) Spin-up spin-down electrons in an ordinary superconductor placed in a uniform magnetic field undergo Zeeman splitting, leading to a mismatch in Fermi momentum. As found by Sarma in the

\(^1\)This work was performed in collaboration with M. Forbes, W. V. Liu and F. Wilczek.
1960’s, an exotic superconducting ground state should arise at large momentum mismatch when $\mu_B H > \Delta$. Before that, however, the first-order phase transition from the superconducting state to the normal state takes place at $\mu_B H = \Delta/\sqrt{2}$. Placing a superconductor in a spatially varying magnetic field or adding paramagnetic impurities with a strong spin-flip electron-impurity scattering amplitude stabilizes the gapless superconductor. (2) Recent experiments in cold atomic fermion gases trapped in an optical lattice and operating near Feshbach resonance deal with a mixture of two hyperfine spin components of alkali atoms. By changing the scattering length one can go from the regime of Bose-Einstein condensation to BCS superfluidity, which is of interest in this work. Laser lattice involves counterpropagating laser beams, that together generate a standing light wave leading to different AC Stark shifts for the spin-up and spin-down components. Using methods of ’engineering’ various lattice systems and by tuning effective masses one can produce exotic phases. (3) In strongly interacting quark matter at high baryon densities and low temperatures, different flavors of quarks pair and form color superconductors. Here a mismatch in Fermi momenta arises due to a nonzero strange quark mass, $m_s \neq 0$, and is triggered by imposing a charge neutrality condition. At intermediate density (2–3 nuclear densities), an exotic state may arise which links the CFL and nuclear matter phases.

Plan: First, we give a general analysis of gapless states. Then, we show how to realize a stable BP superfluid state.

### 2. Gapless state and its stability

We consider the mean-field analysis of a model with two species of fermions $A$, and $B$ of differing masses $m_A < m_B$ and in the absence of interaction with different Fermi momenta $p_A < p_B$. The Hamiltonian is

$$H = \int \frac{d^3p}{(2\pi)^3} \left( \varepsilon_p^A \psi_{Ap}^\dagger \psi_{Ap} + \varepsilon_p^B \psi_{Bp}^\dagger \psi_{Bp} \right) + H_I$$

with attractive interaction $H_I = -g \int \frac{d^3p}{(2\pi)^3} \int \frac{d^3q}{(2\pi)^3} \psi_{Ap}^\dagger \psi_{Bq}^\dagger \psi_{Bq} \psi_{Ap} - \Delta - g > 0$ and $\varepsilon_p^A = p^2/2m_A - \mu_A$, $\varepsilon_p^B = p^2/2m_B - \mu_B$, where $p_A = \sqrt{2m_A} \mu_A$, $p_B = \sqrt{2m_B} \mu_B$. At the mean-field level, the condensate $\Delta = \int \frac{d^3p}{(2\pi)^3} (\psi_{Bp}^\dagger \psi_{Ap})$ is a c-number, which permits to diagonalize the Hamiltonian. As a result, the quasiparticle excitations are $E_p^\pm = \varepsilon_p \pm \sqrt{\varepsilon_p^2 + \Delta^2}$ with $\varepsilon_p^\pm = (\varepsilon_p^A \pm \varepsilon_p^B)/2$; they contain mixture of A-particle and B-hole excitations. We minimize the thermodynamic potential $\Omega = H - \mu_A n_A - \mu_B n_B$, i.e. $\partial \Omega / \partial \Delta = 0$, to find the gap parameter $\Delta$. There are two non-trivial solutions. First one with larger gap corresponds to a fully gapped BCS state, where $E_p^\pm$ have oppo-
site signes for all momenta. Though $n_A \neq n_B$ at $g = 0$, in the presence of interaction particles redistribute so that the occupation numbers become equal $\tilde{n}_A = \tilde{n}_B = n$ with $n = 1/2 \left( 1 - \varepsilon_p^+ / \sqrt{\varepsilon_p^+ + \Delta^2} \right)$; the BCS condensation energy is the largest for $\tilde{p}_A = \tilde{p}_B$. Second solution, obtained by Sarma in 60’s, has smaller gap. Interaction is not strong enough to pull both Fermi surfaces together, leaving a breach region with single occupancy by B-particles. Pairing and superfluidity takes place primarily around the smaller Fermi surface, while there is normal component for momenta in the breach region where $E_p^\pm$ have the same signs (separation in momentum). Points where $E_p^- = 0$, i.e., $p_{1,2}^2 = (p_A^2 + p_B^2)/2 \pm 1/2 \sqrt{(p_B^2 - p_A^2)^2 - 16m_Am_B\Delta^2}$, give the free Fermi surfaces leading to gapless modes and to free fermion liquid component. Occupation numbers are $\tilde{n}_A = \tilde{n}_B = n$ for $0 \leq p \leq p_1$ and $p_2 \leq p$, and $\tilde{n}_B = 1 \tilde{n}_A = 0$ for $p_1 \leq p \leq p_2$, Figure 1.

We consider nontrivial superfluid solutions of the gap equation in grand canonical and canonical ensembles.

I. Grand canonical ensemble.

We fix the chemical potentials $\mu_A, \mu_B$ and minimize the thermodynamic potential $\Omega_S = H - \mu_A n_A - \mu_B n_B$ over all ground state superconducting wave functions, $\min \langle \Psi_S | \Omega | \Psi_S \rangle$. We obtain, apart from the trivial solution $\Delta = 0$, two nontrivial solutions: the fully gapped BCS and the gapless

Figure 1. Dispersion relations and occupation numbers for the BCS (left) and Sarma (right) states.
Figure 2. Solutions of the gap equation and corresponding pressures as functions of the Fermi momenta mismatch at fixed chemical potentials (left), and gap equation solutions and energies at fixed particle numbers (right).

Sarma solution. As a function of the Fermi momenta mismatch $\delta p = (p_A^2 - p_B^2)/4\sqrt{mamA_B}$, they are $\Delta_{BCS} = \Delta_0$ and $\Delta_{Sarma}/\Delta_0 = \sqrt{2x - 1}$ with $x = \delta p/\Delta_0$, Figure 2. Since at a given $\delta p$ and $g$, $\Delta_{BCS} > \Delta_{Sarma}$, BCS state wins, i.e., system prefers the BCS ground state over the Sarma state. At fixed Fermi momenta $p_A, p_B$, the thermodynamic potential as a function of the gap has two minima – normal (N) and the BCS (absolute min) states, and one maximum – the Sarma state. Hence the Sarma state is metastable.

We consider the pressure versus the Fermi momentum mismatch for states which are solutions of the gap equation, where normalized pressure is defined through the condensation free energy as $P_S = -\langle \langle \Omega_S \rangle - \langle \Omega_0 \rangle \rangle / \langle \langle \Omega_{BCS} \rangle - \langle \Omega_0 \rangle \rangle$ where $\langle \Omega_0 \rangle$ is the free energy of the normal state at $\delta p = 0$ and the BCS condensation energy is $\langle \Omega_{BCS} \rangle - \langle \Omega_0 \rangle = -N(0)\Delta_0^2/2$, $\Delta_0$ is the BCS gap and $N(0)$ is the density of states at the Fermi surface. In the leading order $\Delta \sim \delta p \ll p_A, p_B$ (i.e., when all quantities are written as expansions near the Fermi surface), pressures are $P_{BCS} = 1$ for $0 \leq x \leq 1$, $P_{Sarma} = 2x^2 - (1 - 2x)^2$ for $1/2 \leq x \leq 1$ and $P_N = 2x^2$ for $x \geq 0$ with $x = \delta p/\Delta_0$, Figure 2. State with maximum pressure (or minimum condensate energy) wins. For $p_A = p_B$ there is the BCS state. As we add B-particles, we create stress, and pressure of the BCS state $P_{BCS}$
drops relatively to the pressure of the normal unpaired state $P_N \sim \delta_p^2$. When $P_{BCS} - P_N \leq 0$, there is a first order phase transition at $\Delta \neq 0$ from the superconducting to the normal state. The BCS superconductor is destroyed when win from condensation energy becomes less than loss in energy needed to pull the Fermi surfaces together to create the BCS. Sarma state is tangent to normal state at $\Delta = 0$, and has always lower pressure than normal state; hence Sarma state is unstable.

II. Canonical ensemble.

We fix the particle numbers $n_A, n_B$, allowing the chemical potentials to change, and minimize the Helmholtz energy over all possible superconducting ground states with constraint of fixed $n_i i = A, B$, $\min \langle \Psi_S | H | \Psi_S \rangle_{n_i = \text{const}}$. At a single point when $n_A = n_B$ there is the BCS state with $\Delta_{BCS} = \Delta_0$. At $n_A \neq n_B$, there is Sarma state with decreasing gap as $\delta n = n_B - n_A$ increases, $\Delta_{Sarma}/\Delta_0 = \sqrt{1 - 2x}$ where $x = \delta p/\Delta_0 \sim \delta n$. The energy of Sarma state is lower than the energy of the normal state, $E_{Sarma} < E_N$; thus Sarma state is stable. Imposing neutrality condition a stable gapless superconducting state was obtained in the QCD context by Shovkovy et. al.\textsuperscript{9}.

We came to different conclusions about stability of Sarma state using grand canonical and canonical ensembles. This difference in stability analysis can be resolved by considering mixed phase, which is a mixture in space of two (or more) homogeneous states. Bedaque et. al.\textsuperscript{10} suggested to consider a mixture of the BCS and normal states, separated in $x$-space. They found that the energy of the mixed state is lower than the energy of Sarma state, $E_{mixed} < E_{Sarma}$; thus Sarma state is unstable with respect to decay into a mixed state. We confirm their findings. We define the normalized condensation energy as $E_S = (\langle H_S \rangle - \langle H_N \rangle)/\langle (H_{BCS} - \langle H_N \rangle) \rangle$ where $\langle H_{BCS} \rangle - \langle H_N \rangle = -N(0)\Delta_0^2/2$. In the leading order $\Delta \sim \delta p \ll p_A, p_B$, condensation energies are $E_{BCS} = -1$ for $x = 0$, $E_{Sarma} = -(1 - 2x)^2$ for $0 \leq x \leq 1/2$, $E_{mixed} = -(1 - \sqrt{2}x)^2$ and $E_N = 0$, $x = \delta p/\Delta_0$ and $\Delta_0$ is the BCS gap, Figure 2.

Allowing mixed states in our ansatz of trial ground state wave functions $\Psi_S$, metastable Sarma state decays (rolls down) into a mixture of the BCS and normal states, Figure 3. (In a mixed state, pressures and chemical potentials of composite states are equal, hence there are two equal minima $\Omega_{BCS} = \Omega_N$). Generally, it is difficult to include mixed states in variational ansatz in the grand canonical ensemble.

We conclude, that in grand canonical and canonical ensembles Sarma branch of the gapless solutions is unstable.
Figure 3. Thermodynamic potential as a function of the gap, and positions of each state. Sarma state decays into a mixed BCS and Normal states depicted on the right panel.

In \cite{11}, we show that conclusion about stability of a state is the same in any ensemble used. In particular, there is one-to-one correspondence between a state at fixed particle number(s) and the state that minimizes the thermodynamic potential $\Omega$ in grand canonical ensemble. Thus, there is always a stable state in the grand canonical ensemble that satisfies the constraint. Imposing constraint (over particle numbers) cannot stabilize the system. Practical guide is to look for a stable (gapless) solution in the grand canonical ensemble.

3. Breached paired superfluid state for a finite-range interaction

Our goal is to construct a stable breached paired state which has coexisting superfluid and gapless components. We use an idea that existence of the gapless modes depend on the momentum structure of the gap $\Delta(p)$. There should be two distinct regions in momentum space: first one where $\Delta_p$ is large enough to support the superfluid, and second one where $\Delta_p$ is small enough that pairing does not appreciably affect the normal free-fermion behavior. To guarantee stability, the phase must also have higher pressure than the normal state. We realize such BP states in two examples \cite{11}.

I. Cut-off interaction.

We impose a cut-off interaction such that it supports the BCS-like pairing for $p < p_A$ and it allows free dispersion relations for $p > p_A$, accommodating the excess of B-particles and leading to gapless modes. We construct this state by minimizing the thermodynamic potential, 

$$\min \langle \Psi_S | H - \mu_A n_A - \mu_B n_B | \Psi_S \rangle,$$

where $H$ includes the cut-off interaction $-g \int \frac{d^3p}{(2\pi)^3} \frac{d^3q}{(2\pi)^3} f(p) f(q) \psi^\dagger_{Ap} \psi^\dagger_{B-p} \psi_{B-q} \psi_{Aq}$ with $f(p) = 1$ for $p > p_A$ and $f(p) = 0$ for $p \leq p_A$. The gap parameter, defined
as \( \Delta = g \int \frac{d^3p}{(2\pi)^3} f(p) \langle \psi_{Bp} \psi_{A-p} \rangle \), satisfies the gap equation \( \Delta = 1/2g \int \frac{d^3q}{(2\pi)^3} \Delta f(q)/\sqrt{\varepsilon_p^2 + \Delta^2} \) where momentum integration is performed outside the breach region. The occupation numbers \( n_A, n_B \) show the evidence that it is a breached paired state, Figure 4. This state is an absolute minimum of the thermodynamic potential, hence we obtained a stable BP state.

II. Spherically symmetric static two-body potential.

With attractive potential \( V(x - x') \), interaction is \( H_I = \int d^3p/(2\pi)^3 d^3q/(2\pi)^3 V(p - q) \psi_A^\dagger p \psi_B^\dagger q \psi_{Bq} \psi_{A-p} \) and the gap parameter acquires a momentum dependence, \( \Delta_p = \int d^3q/(2\pi)^3 V(p - q) \langle \psi_{Bq} \psi_{A-q} \rangle \).

The gap equation is written \( \Delta_p = 1/2 \int d^3q/(2\pi)^3 v(p - q) \Delta q/\sqrt{\varepsilon_q^2 + \Delta_q^2} \), and quasiparticle dispersion relations are \( E_p^\pm = \varepsilon_p^\pm \pm \sqrt{\varepsilon_p^2 + \Delta_p^2} \). We take a gaussian potential for numerical simulations. Due to the BCS instability,
$\Delta_p$ picks at the effective Fermi surface given by the pole of the gap equation at $\Delta = 0$, $\varepsilon_{p_0}^+ = 0$. Therefore $\Delta_p$ supports the BCS-like pairing around $p_0$, and allows free dispersion relations, and hence free Fermi surfaces, outside the breached region, Figure 4. It is, however, difficult to verify that this state is an absolute minimum of the thermodynamic potential since instead of a number, $\Delta$, we have a function $\Delta_p$ in the variational ansatz.

We performed minimization of the thermodynamic potential numerically using different potentials. Generally, there is a central strip of fully gapped BCS phase about $p_A = p_B$ with normal unpaired phase outside. Depending on the parameters of interaction, these phases may be separated by a region of gapless BP superfluid phase, Figure 5.

Conditions to have BP phase are as follows. At $p_A = p_B$ there is standard BCS, which is a stable fully gapped solution. By adjusting the chemical potentials so as to increase the Fermi surface $p_B$, we stress the system and lower the pressure relative to the normal phase. Eventually, either before or after a transition to a BP state, the pressure becomes negative and there is a first order phase transition to the normal phase. At the point just before transition: if $\Delta_{p_B}$ is sufficiently large, the state is fully gapped (BCS) and no BP state will occur; if $\Delta_{p_B}$ is small, then it will not appreciably affect the dispersions and one finds a gapless Fermi surface coexisting with the superfluid phase. As long as $\Delta_p$ falls off sufficiently quickly, one can choose large ratio $m_B/m_A \gg 1$ so that the transition will occur with $\Delta_{p_B}$ small enough to support the BP phase. States shown at Figure 5 have $m_B/m_A = 10$. For a wider in q-space interaction, larger mass ratio is needed.

Figure 5. Phase diagram of possible homogeneous phases in coordinates of the Fermi momenta $(p_A, p_B)$ for the cut-off (left) and two-body potential (right) interactions.
4. Conclusion

We considered a Fermi system with weak attractive interaction between species A and B, where the BCS state forms at equal number densities, \( n_A = n_B \). What is the ground state of this system when \( n_A \neq n_B \)? There is a range of parameters, where a breached pair phase exist. Breached pair state is a homogeneous phase where superfluid and normal components coexist. This state is stable and can be found provided there is a momentum structure of interaction and large enough mass ratio of two species.

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