Abstract

We formulate the $O(3)$ $\sigma-$ model on fuzzy sphere and construct the Hopf term. We show that the field can be expanded in terms of the ladder operators of Holstein-Primakoff realisation of $SU(2)$ algebra and the corresponding basis set can be classified into different topological sectors by the magnetic quantum numbers. We obtain topological charge $Q$ and show that $-2j \leq Q \leq 2j$. We also construct BPS solitons. Using the covariantly conserved current, we construct the Hopf term and show that its value is $Q^2$ as in the commutative case. We also point out the interesting relation of physical space to deformed $SU(2)$ algebra.

Keywords: Fuzzy sphere, $O(3)$ $\sigma-$ model, Hopf term, BPS solution,
1 Introduction

Field theories on non-commutative spaces [1] have been extensively studied in last few years. These studies have emanated from attempts to understand the renormalisation program at a deeper level or string theories [2], quantum gravity and matrix models [3]. These theories present many interesting features and some of these features are not shared by their commutative counterparts. Aspects such as UV/IR mixing [4], soliton and instanton solutions [5, 6], monopoles [7], gauge invariance [8], renormalisability [9] etc have been widely investigated (for a recent review, see [10]).

Study of field theories on compact, non-commutative spaces are of interest as they provide an alternative way of regularising field theories with finite degrees of freedom [11, 12]. Fuzzy sphere ($S_F^2$) whose co-ordinates are non-commuting was introduced and study of fuzzy field theory models was initiated in [13]. Topological solutions in non-linear sigma model on fuzzy sphere were obtained and their commutative limit were studied in [14]. Chiral anomaly and the Fermion doubling problem in this setting were investigated by Balachandran et al [15, 16]. Gauge theories on fuzzy sphere have been introduced and studied in [17]. UV/IR mixing in the case of $\Phi^4$ theory on fuzzy sphere has been investigated in detail in [18]. Topological properties of $S_F^2$ has been studied in [19, 20, 21]. Non-commutative $CP^n$ model and its supersymmetric version have been studied recently [22, 23] and BPS solutions were obtained.

In this paper we study, fuzzy $CP^1$ model with Hopf term. $CP^n$ model [24] which shares many interesting properties of gauge theories like asymptotic freedom has been a testing ground for many ideas [25] underlying non-perturbative physics. $CP^1$ model (or equivalently $O(3)$ non-linear sigma model) has topological solitons which acquire fractional statistics when coupled to Hopf term [26]. These models with Hopf term have been analysed in canonical framework and the existence of fractional charge was verified [27]. $O(3)$ $\sigma-$ model has also been studied in relation with many condensed matter systems [28]. It has been studied in path integral framework and interesting interplay of topological and Noether charges has been shown. The equivalence of $CP^1$ model with Hopf term to Fermionic theory has also been shown [29, 30]. It is interesting to study the $CP^1$ model with Hopf term in the fuzzy sphere setting.

On the fuzzy sphere ($S_F^2$) the derivations are obtained by the adjoint action of the angular momentum operators $\mathcal{L}_i$. We take $\mathcal{L}_i$ in the Holstein-
Primakoff [31] realisation of $SU(2)$ algebra which acts on a finite dimensional Hilbert space. The corresponding state vectors can be classified into different topological sectors using $L_3$ eigenvalues which are integers. We have shown that the soliton charge $Q$ which is expressed in terms of a covariantly conserved current is integer and equal to the $L_3$ eigenvalue. We also obtain the BPS soliton solution. Using the covariantly conserved current, we construct the Fuzzy Hopf term and demonstrate that its value is proportional to $Q^2$ as in the commutative case.

Fuzzy sphere $S^2_F$ is obtained by quantising the co-ordinates of $S^2$ [13]. Thus the fuzzy sphere $(S^2_F)$ is defined by the co-ordinates obeying

$$[x_i, x_j] = i\lambda\epsilon_{ijk}x_k, \quad x_i x_i = R^2. \quad (1)$$

where $R$ is the radius of the sphere. The fields on $S^2_F$ are functions of the non-commutative co-ordinates and hence when one expands the fields in terms of spherical harmonics $Y_{lm}$, there is a cut off for the allowed values of $l$ ($l \leq 2j = N$). These fields are realised as operators acting on a $(N+1)^2$ dimensional Hilbert space. The fields on $S^2_F$ are now complex $(N+1) \times (N+1)$ matrices.

In the next section, we present the fuzzy sphere as a deformed Hopf fibration and define the finite dimensional Hilbert space associated with it [20]. Here we also show how the fields are defined on $S^2_F$. In section 3, first we present the $CP^1$ model on $R \otimes S^2_F$. We then get the BPS bound and define topological charge. We also show the BPS solutions for different topological sectors. In section 4, we present the construction Hopf term and evaluate it. In section 5, we point out the interesting connection between $S^2_F$ and $SU_q(2)$ operators, details of which will be taken up later. Finally in section 6, we present the concluding remarks.

## 2 Fuzzy Sphere

The coordinates of $S^2_F$ obey Eqn. (1) and hence they are $(2j+1) \times (2j+1)$ dimensional matrices and can be realised by the generators of the $2j+1$ dimensional irreducible representation of $SU(2)$. In the above $\lambda = \frac{R}{\sqrt{j(j+1)}}$,

where $R$ is the radius of the sphere. From Eqn. (1) we see that in the limit $\lambda \to 0$ (i.e., $j \to \infty$), we recover the commutative sphere. To get the non-commutative plane we should take $j \to \infty$, $R \to \infty$ such that $\frac{R}{j} = \lambda$ is a constant. Continuum is obtained by taking $\lambda \to 0$. 

3
We define the $S_F^2$ using the non-commutative analogue of the Hopf fibration $S^3 \to S^2$. For this purpose we start with the creation and annihilation operators of two uncoupled harmonic oscillators obeying
\[
[A_\alpha, A^\dagger_\beta] = \delta_{\alpha \beta}, \quad \alpha = 1, 2; \quad A_1 = a, A_2 = b
\]
with $a^\dagger a = N_1$ and $b^\dagger b = N_2$. The operators $a, \ a^\dagger, \ b, \ b^\dagger$ act on the infinite dimensional number states
\[
|n\rangle_{1, 2} = \frac{(A^\dagger_\alpha)^n}{\sqrt{n!}} |0\rangle_{1, 2}, \quad n = 0, 1, 2, ..
\]
with $A_\alpha|0\rangle_{1, 2} = 0$. Using these one can define two sets of operators
\[
J_\alpha^{(\alpha)} = A^\dagger_\alpha \sqrt{N - N_\alpha}, \quad J_\alpha^{(-\alpha)} = \sqrt{N - N_\alpha} A_\alpha, \quad J_3^{(\alpha)} = (N_\alpha - N^2) / 2.
\]
Each of this (Holstein-Primakoff realisation) forms a spin $N / 2 = j$ representation of a $SU(2)$ for any given value of $N$. For maintaining the unitarity of the representation of $SU(2)$, these ladder operators in Eqn. (4) should act only on a finite dimensional subspaces $n, m \leq N$. Here, we note that in the case of $SU_q(2)$, action of ladder operators can be restricted on a finite dimensional space in a natural way by taking $q$ to be an $(N + 1)th$ root of unity (see section 5).

Using the operators $a, \ a^\dagger, \ b, \ b^\dagger$, we define
\[
\mathcal{L}_+ = a^\dagger \sqrt{N - a^\dagger a} + b^\dagger \sqrt{N - b^\dagger b},
\]
\[
\mathcal{L}_- = \sqrt{N - a^\dagger a} a + \sqrt{N - b^\dagger b} b,
\]
\[
\mathcal{L}_3 = (a^\dagger a + b^\dagger b - N).
\]
These operators obey
\[
[\mathcal{L}_+, \mathcal{L}_-] = 2\mathcal{L}_3, \quad [\mathcal{L}_3, \mathcal{L}_\pm] = \pm \mathcal{L}_\pm.
\]
\[ |N, N\rangle \] \[ m = N \]
\[
\ldots
\]
\[ |N, 1\rangle |N - 1, 2\rangle \ldots |2, N - 1\rangle |1, N\rangle \] \[ m = 1 \]
\[ |N, 0\rangle |N - 1, 1\rangle \ldots |1, N - 1\rangle |0, N\rangle \] \[ m = 0 \]
\[ |N - 1, 0\rangle |N - 2, 1\rangle \ldots |1, N - 1\rangle |0, N\rangle \] \[ m = -1 \]
\[
\ldots
\]
\[ |0, 0\rangle \] \[ m = -N \]

where \( m = 0, \pm 1, \ldots, \pm N, N = 2j \) are the corresponding \( L_3 \) integer eigenvalues which classify these states. For a given state, \( m = n_1 + n_2 - N \) where \( n_1 \) and \( n_2 \) are the eigenvalues of \( N_1 = a\dagger a \) and \( N_2 = b\dagger b \) respectively. For any given \( j \), there are \((2j + 1)^2\) number of states which are classified into \( 2N + 1 \) sectors by the corresponding value of \( m \). Notice that for any given value of \( j \) \((= \frac{N}{2})\), \( m \) is always integer. Note that there is a symmetry between the number of states for any given value \( n \) and \(-n \) of \( m \). Any row in the above with a given \( m \) value can be thought as a representation of \( SU(2) \) with spin \( j = \frac{N}{2}, \frac{N-1}{2}, \frac{N-2}{2}, \ldots \). Ladder operators of this \( SU(2) \) are:

(a) for \( m = -N \) to 0,

\[ j_+ = a\dagger b, \quad j_- = b\dagger a \quad \text{and} \quad j_3 = \frac{1}{2}(a\dagger a - b\dagger b). \]

(b) for \( m = 1 \) to \( N \)

\[ J_+ = a\dagger \sqrt{\frac{(N - N_1)(N - N_2)}{(N_1 + 1)(N_2 + 1)}} b, \quad J_- = J_+^\dagger, \quad j_3 = \frac{1}{2}(a\dagger a - b\dagger b). \]

A generic scalar field \( \Phi \) on the fuzzy sphere is defined as

\[ \Phi = \sum_{k,l} a_{kl}(t) \mathcal{L}_+^k \mathcal{L}_-^l, \quad k, l \leq N. \]  \hspace{1cm} (8)

From this we find

\[ [\mathcal{L}_3, \Phi] = \sum_{kl} (k - l)a_{kl}(t) \mathcal{L}_+^k \mathcal{L}_-^l, \]  \hspace{1cm} (9)

which for given values of \( k - l \) becomes

\[ [\mathcal{L}_3, \Phi] = m\Phi, \quad k - l \equiv m = 0, \pm 1, \ldots \pm N. \]  \hspace{1cm} (10)
We notice from Eqn. (8) that under
\[
\mathcal{L}_\pm \rightarrow e^{\pm i\psi} \mathcal{L}_\pm, \\
\Phi \rightarrow \Phi e^{im\psi}.
\]  
(11)

We will see in the next section that this gauge transformation is a symmetry of \(CP^1\) model.

Here we see that the number of independent functions \(a_{kl}\) for any given \(N\) is same as obtained if one expands the scalar field in terms of the spherical harmonics, i.e., \(\phi = \sum_{lm} A_{lm} Y_{lm}\).

In [22] Holstein-Primakoff realisation of \(SU(2)\) generators was used to study the \(CP^n\) model. Here the fields \(\Phi\) are expanded in terms of the creation and annihilation operators of the uncoupled harmonic oscillators, \(A_\alpha^\dagger, A_\alpha, \alpha = 1, 2\). In contrast, here we expand the fields \(\Phi\) in terms of \(\mathcal{L}_\pm\).

In the present case, the basis set is given by \(|N, 0\), \(|N - 1, 1\), \(|1, N - 1\), \(|0, N\) and all other states are generated from these by the action of \(\mathcal{L}_\pm\). As pointed out earlier, number of states in topological sectors with \(\mathcal{L}_3\) eigenvalues \(\pm m\) are same.

Differentiation on fuzzy sphere is realised by the adjoint action of \(\mathcal{L}_i\). We work with \(\mathcal{L}_i\) in anti-hermitian representation so that \([\mathcal{L}_i, \mathcal{L}_j] = \epsilon_{ijk} \mathcal{L}_k\). Thus we have

\[
\mathcal{L}_i^\dagger = -\mathcal{L}_i, \quad \mathcal{L}_i \Phi = [\mathcal{L}_i, \Phi], \quad (\mathcal{L}_i \Phi)^\dagger = [\mathcal{L}_i, \Phi^\dagger] = \mathcal{L}_i \Phi^\dagger.
\]  
(12)

3 \(CP^1\) model on \(R \otimes S_F^2\)

In this section, we present the action for \(CP^1\) model on \(R \otimes S_F^2\). Here the time coordinate which is \(R\) commutes with coordinates of fuzzy sphere. This, in the planar non-commutative limit, will lead to \(CP^1\) model on \(R \otimes R_\theta^2\) and in commutative limit gives \(CP^1\) model in \(2 + 1\) dimensions.

The action for \(CP^1\) model on \(R \otimes S_F^2\) is given by

\[
S = \frac{2}{2j + 1} tr \int t \partial_0 \Phi^\dagger \partial_0 \Phi + \Phi^\dagger \partial_0 \Phi \Phi^\dagger \partial_0 \Phi - [\mathcal{L}_i, \phi]^\dagger [\mathcal{L}_i, \phi] - \Phi^\dagger \mathcal{L}_i \Phi \Phi^\dagger \mathcal{L}_i \Phi. \tag{13}
\]

Here \(\Phi\) is a non-zero complex doublet i.e. \(\Phi^\dagger = (\Phi_1^*, \Phi_2^*)\) which obeys the
condition $\Phi^\dagger \Phi = 1$. We rewrite the above action as

$$S = \text{Tr} \left( \partial_0 \Phi^\dagger \partial_0 \Phi + A_0^2 + 2i A_0 \Phi^\dagger \partial_0 \Phi - (s L_i, \phi)^\dagger [L_i, \phi] + A_i^2 + 2i A_i \Phi^\dagger L_i \Phi \right).$$  \hspace{1cm} (14)

Here $\text{Tr} = \frac{2}{2j+1} \text{tr} f_i$. By eliminating $A_0$ and $A_i$ using their equations of motion from (14), we get back (13). Above action along with the constraint $\Phi^\dagger \Phi = 1$ can be expressed as

$$S = \text{Tr} \left( |D_0 \Phi|^2 - |D_i \Phi|^2 + \lambda (\Phi^\dagger \Phi - 1) \right).$$  \hspace{1cm} (15)

Here,

$$D_0 \Phi = \partial_0 \phi - i \Phi A_0, \quad D_i \Phi = L_i \Phi - i \Phi A_i.$$  \hspace{1cm} (16)

It is easy to see that the equations of motion for fields $\Phi$ as well as $\Phi^\dagger$ have equal powers of $L_\pm$. Thus the fields in a given topological sector stay in the same sector under time evolution. The action Eqn. (15) is invariant under the right action of local $U(1)$ group,

$$\Phi \to \Phi G, \quad \quad A_\mu \to G^\dagger A_\mu G + id_\mu G^\dagger G,$$  \hspace{1cm} (17, 18)

where $G \in U$ and $d_0 \equiv \partial_0$, and $d_i \equiv L_i$. This invariance allows us to remove the extra degree of freedom (that of the phase of $\Phi$). From Eqn. (11), we identify $G = \exp L_3 \Phi$. Equations of motion of $A_\mu$ following from Eqn. (15) gives

$$A_\mu = -i \Phi^\dagger d_\mu \Phi.$$  \hspace{1cm} (19)

We can re-write using (19),

$$D_\mu \Phi = \mathcal{P} d_\mu \Phi, \quad (D_\mu \Phi)^\dagger = (d_\mu \Phi)^\dagger \mathcal{P}$$  \hspace{1cm} (20)

where $\mathcal{P} = 1 - \Phi \Phi^\dagger$, and $\mathcal{P}^2 = \mathcal{P}$. Here we note that for any $\Phi$ obeying $[L_3, \Phi] = im \Phi$,

$$A_3 = -i \Phi^\dagger L_3 \Phi = -i (\Phi_1^* L_3 \Phi_1 + \Phi_2^* L_3 \Phi_2) = (m_1 |\Phi_1|^2 + m_2 |\Phi_2|^2) = m \Phi^\dagger \Phi = mI,$$  \hspace{1cm} (21)

\footnote{In the limit $N \to \infty$, $L_i \to i L_i$, we get the action on $R \otimes S^2$ and by taking $\frac{\Phi^\dagger}{R} \to p_i$, we get commuting plane $-R^3$.}
where we have taken $m_1 = m_2 = m$ (i.e., both $\Phi_1$ and $\Phi_2$ have same eigenvalue for $L_3$.) We have

$$\Phi^\dagger D_\mu \Phi = 0,$$  \hspace{1cm} (22)

and with Eqn. (21) and $[\mathcal{L}_i, \Phi] = i m \Phi$ we get

$$D_3 \Phi = L_3 \Phi - i \Phi A_3 = 0.$$ \hspace{1cm} (23)

This Eqn. (23) is a consequence of the gauge invariance under Eqn. (17,18).

Using these, we define

$$\Phi^\dagger [D_\mu, D_\nu] \Phi = -i F_{\mu\nu} = -i [d_\mu A_\nu - d_\nu A_\mu + i[A_\mu, A_\nu] - \epsilon_{\mu\nu\alpha} A_\alpha],$$ \hspace{1cm} (24)

where $F_{\mu\nu}$, $\mu, \nu = 0, 1, 2$ is the field strength on $R \otimes S_F^2$. Notice that the last term in (24), viz $-\epsilon_{0\mu\nu} A_3$ is present only in the fuzzy case and it vanishes in the continuum limit.

Under the gauge transformation the field strength transform as

$$F_{\mu\nu} \rightarrow G^\dagger F_{\mu\nu} G,$$ \hspace{1cm} (25)

The Bianchi identity following from Eqn. (24), is

$$\epsilon_{\mu\nu\lambda} D_\mu (\Phi F_{\nu\lambda}) = 0,$$ \hspace{1cm} (26)

which, after using Eqn. (22) leads to

$$\epsilon_{\mu\nu\lambda} D_\mu F_{\nu\lambda} = 0$$ \hspace{1cm} (27)

$$= \epsilon_{\mu\nu\lambda} D_\mu (d_\nu A_\lambda - d_\lambda A_\nu + i[A_\nu, A_\lambda]),$$ \hspace{1cm} (28)

where we have used the fact that $D_\mu A_3 = 0$.

### 3.1 Topological Charge

We now construct lower bound on energy for the $CP^1$ model defined by the action Eqn. (13) in a topological sector and show that is is proportional to the topological charge. In static case, with the gauge choice $A_0 = 0$, we get

$$E = \frac{1}{2j + 1} tr |D_i \Phi|^2.$$ \hspace{1cm} (29)

Using the identity

$$\frac{1}{2} tr \left[ (D_i \Phi \pm i \epsilon_{ij} D_j \Phi)^\dagger (D_i \Phi \pm i \epsilon_{ij} D_j \Phi) \right] + \frac{1}{2} tr (D_3 \Phi)^\dagger (D_3 \Phi) \geq 0,$$ \hspace{1cm} (30)
where $i = 1, 2$. We see

\[
\frac{1}{2j+1} tr |D_i \Phi|^2 + |D_3 \Phi|^2 \geq \mp \frac{i}{2j+1} tr \epsilon_{ij} (D_i \Phi)^\dagger D_j \Phi = Q, \tag{31}
\]

i.e., $E \geq |Q|$ \tag{32}

which is re-expressed using Eqn. (24) as

\[
Q = -\frac{1}{2j+1} tr F_{12} = -\frac{1}{2j+1} tr J_0, \tag{33}
\]

where $J_0$ is the zeroeth component of

\[
 J_\mu = \epsilon_{\mu\nu\lambda} F_{\nu\lambda}, \tag{34}
\]

which is covariantly conserved due to Eqn. (27), i.e, $D_\mu J_\mu = 0$.

From Eqn. (31) we have

\[
 Q = \frac{i}{2j+1} tr \left[ (D_1 \Phi)^\dagger (D_2 \Phi) - (D_2 \Phi)^\dagger (D_1 \Phi) \right] \tag{35}
\]

\[
= -\frac{i}{2j+1} tr \Phi^\dagger [L_3, \Phi] = m, \tag{36}
\]

where $\Phi^\dagger [L_3, \Phi] = \Phi_1^* [L_3, \Phi_1] + \Phi_2^* [L_3, \Phi_2] = i(m_1 |\Phi_1|^2 + m_2 |\Phi_2|^2 = im|\Phi|^2 = imI)$. We can also use Eqns. (33, 24) and Eqn. (21) and get $Q = m$.

### 3.2 BPS Solution

The BPS equations following from Eqn. (30) are

\[
 D_i \Phi \pm i \epsilon_{ij} D_j \Phi = 0, \tag{37}
\]

\[
 D_3 \Phi = 0. \tag{38}
\]

$\Phi$ which satisfy the above, saturates the bound and has topological charge $Q = m$. We recall from Eqn. (21) and Eqn. (23) that for any $\Phi$ which is an eigenvector of $L_3$, Eqn. (38) is identically satisfied.

Eqn. (37) can be re-expressed as

\[
 P L_\pm \Phi = 0 \tag{39}
\]
where $\mathcal{P} = 1 - \Phi \Phi^\dagger$ and we have used Eqn. (6) and Eqn. (19). Thus any configuration satisfying
\[ [\mathcal{L}_\pm, \Phi] = 0 \] (40)
will be a BPS (anti-) soliton solution. Thus the generic BPS solutions for (anti-) soliton with $Q = \pm m$ are
\[ \Phi_{\text{soliton}} = a_m \mathcal{L}_+^m, \quad \Phi_{\text{anti-soliton}} = a_0 \mathcal{L}_-^m. \] (41)

With the parameterisation
\[ \Phi = \mathcal{F} \frac{1}{\sqrt{\mathcal{F}^\dagger \mathcal{F}}}, \] (42)
BPS equation (39) becomes
\[ \mathcal{P} \left[ \mathcal{L}_\pm, \mathcal{F} \right] \frac{1}{\sqrt{\mathcal{F}^\dagger \mathcal{F}}} = 0, \] (43)
where $\mathcal{P} = 1 - \mathcal{F} \frac{1}{\sqrt{\mathcal{F}^\dagger \mathcal{F}}} \mathcal{F}^\dagger$. It is easy to see that both $\Phi$ and $\mathcal{F}$ have same $\mathcal{L}_3$ eigenvalue. Any $\mathcal{F}$ satisfying
\[ [\mathcal{L}_\pm, \mathcal{F}] = 0 \] (44)
will give a BPS (anti)-soliton solution configuration.

4 Hopf term

Using the covariantly conserved current $J_\mu$ given in Eqn. (34), we construct the Hopf invariant term as
\[ H = \frac{1}{2\pi} Tr \left[ \epsilon_{\mu\nu\lambda} \left( A_\mu F_{\nu\lambda} - \frac{2i}{3} A_\mu A_\nu A_\lambda + \epsilon_{\nu\lambda\rho} A_\mu A_\rho \right) \right] \] (45)
Under the transformation $\delta A_\mu$ of $A_\mu$,
\[ \delta H = \frac{1}{\pi} Tr \epsilon_{\mu\nu\lambda} \delta A_\mu F_{\nu\lambda}, \] (46)
which is total derivative (after using Eqn. (28)). This sets $H = \text{integer}$. Using Eqn. (24) and Eqn. (19), we express $H$ as
\[ H = - \frac{1}{\pi} \left[ Tr \epsilon_{\mu\nu\lambda} \Phi^\dagger d_\mu \Phi d_\nu \Phi^\dagger d_\lambda \Phi + Tr \Phi^\dagger \partial_0 \Phi \Phi^\dagger [j_3, \Phi] \right] \] (47)
Hopf term given above, in the planar limit reduces to

\[ H = \frac{1}{2\pi} \epsilon_{\mu\nu\lambda} \Phi^\dagger \partial_\mu \Phi \partial_\nu \Phi \partial_\lambda \Phi \]  

(48)

with \( m = 0 \).

For evaluating the Hopf term, we rotate static soliton configuration through \( \theta = 2n\pi \) by

\[ e^{i\theta} L_3 \Phi = e^{i\theta} \Phi, \quad \Phi^\dagger e^{i\theta} L_3 = \Phi^\dagger e^{i\theta} \]

(49)

Under this rotation

\[ \epsilon_{0ij} L_i \Phi^\dagger L_j \Phi \rightarrow \epsilon_{0ij} L_i \Phi^\dagger L_j \Phi \]

\[ \Phi^\dagger \partial_0 \Phi \rightarrow i \partial_0 \theta m. \]

(50)

Using the fact that \( \int dt \partial_0 \theta = \theta(t) - \theta(0) = 2\pi \) (or integer multiple) we get the Hopf term in Eqn. (47) to be

\[ H = m^2. \]

(51)

From Eqn. (36) and Eqn. (51), we get the relation \( H = Q^2 \) which is exactly same as that in the commutative plane.

5 Fuzzy Sphere and \( SU_q(2) \)

The coordinates of the fuzzy sphere, which is obtained by quantising \( S^2 \) obeys \( SU(2) \) algebra. In section 2 we have seen that these coordinates can be realised by the generators of \( SU(2) \). These generators \( L_i \) can be constructed from the creation and annihilation operators of two uncoupled oscillators, either by Schwinger or Holstein-Primakoff realisation of \( SU(2) \). Since these creation and annihilation operators acts on infinite dimensional oscillator space, one has to restrict the action of \( L_i \) to a finite dimensional subspace. In section 2, we have done this by imposing the condition \( n \leq N, \ m \leq N \) (see Eqn. (3) and discussion after Eqn.(4)). We show here that in the case of \( SU_q(2) \) such a restriction can be introduced in a natural way and hence the generators of \( SU_q(2) \) [32] can be used as realisation of the coordinates of fuzzy sphere. In this section, we briefly discuss this aspect. First, for the completeness and also to fix the notations we present the \( SU_q(2) \) construction through q-oscillator [33]
The q-creation and annihilation operators obey

\[ a_q a_q^\dagger - q^{\frac{N}{2}} a_q^\dagger a_q = q^{-\frac{N}{2}} \]  

(52)

where the number operator satisfy

\[ [N, a_q] = -a_q, \quad [N, a_q^\dagger] = a_q^\dagger. \]  

(53)

These operators act on the q-oscillator states obeying

\[ |n\rangle_q = \frac{(a_q^\dagger)^n |0\rangle_q}{\sqrt{|n|!}}, \quad N|n\rangle_q = n|n\rangle_q, \]  

(54)

\[ a_q |0\rangle_q = 0, \quad \text{where} \quad [n] = \frac{q^{\frac{n}{2}} - q^{-\frac{n}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}} \equiv [n]_q. \]

Here by taking \( q \) to be \( (n + 1)th \) root of unity, i.e., \( q = \exp \left( \frac{2\pi i}{n+1} \right) \), we get \( [n + 1]_q = 0. \) With this \( q \) we get only a finite dimensional space with basis \( |0\rangle_q, ..., |n\rangle_q \). Thus by fixing the value of \( n \) in the definition for \( q \), we are guaranteed to have a finite dimensional Hilbert space.

With two uncoupled q-oscillator operators

\[ a_q a_q^\dagger - q^{\frac{N}{2}} a_q^\dagger a_q = q^{-\frac{N}{2}}, \quad b_q b_q^\dagger - q^{\frac{N}{2}} b_q^\dagger b_q = q^{-\frac{N}{2}} \]  

(55)

the Schwinger realisation of \( SU_q(2) \) generators [33] are given by

\[ J_+ = a_q^\dagger b_q, \quad J_- = b_q^\dagger a_q, \quad J_3 = \frac{1}{2} (N_1 - N_2), \]  

(56)

obeying

\[ [J_3, J_\pm] = \pm J_\pm, \quad [J_+, J_-] = [2J_3]_q. \]  

(57)

These generators now acts on a finite dimensional states. The q- analogue of Holstein-Primakoff realisation of \( SU_q(2) \) generators [34] are

\[ \mathcal{L}_\pm^{(q)} = L_\pm^{(1)} \otimes q^{\frac{L^{(2)}}{2}} + q^{-\frac{L^{(1)}}{2}} \otimes L_\pm^{(2)} \]  

\[ \mathcal{L}_3^{(q)} = L_3^{(1)} \otimes 1 + 1 \otimes L_3^{(2)}, \]  

(58)

obeying

\[ [\mathcal{L}_+^{(q)}, \mathcal{L}_-^{(q)}] = [2L_3^{(q)}]_q \quad \text{and} \quad [L_3^{(q)}, \mathcal{L}_\pm^{(q)}] = \pm \mathcal{L}_\pm^{(q)}, \]  

(60)
The above $q$–operators $L_{\pm}$ (Eqn.58) acts exactly on the $q$-analogue of the states on which the ladder operators in Eqn. (5) acts. Note here that we do not have to fix any condition on $n$ and $m$ unlike in the case of $SU(2)$.

In terms of the $SU_q(2)$ generators Eqn. (58) and Eqn. (59) we can define the generic scalar field

$$\Phi = \sum_{k,l} a_{kl}(L_{\pm}^{(q)})^k (L_{\pm}^{(q)})^l,$$  \hspace{1cm} (62)

obeying

$$[L_3^{(q)}, \Phi] = \sum_{k,l} (k - l) a_{kl}(L_{\pm}^{(q)})^k (L_{\pm}^{(q)})^l. \hspace{1cm} (63)$$

Thus as in the section 2, with given $k$ and $l$, we get

$$[L_3^{(q)}, \Phi] = m\Phi, \hspace{0.5cm} k - l \equiv m. \hspace{1cm} (64)$$

Study of $CP^1$ model with Hopf term in this setting will be presented elsewhere.

6 Conclusion

In this paper we have constructed the $CP^1$ model with Hopf term on $R \otimes S^2_F$. We have shown that the fields on fuzzy sphere can be expressed in terms of the ladder operators $L_{\pm}$ of the Holstein-Primakoff realisation of $SU(2)$ algebra and they are classified into different sectors by the $L_3$ eigenvalues. The finite dimensional Hilbert space of these operators (for a given $N = 2j$) can be classified into different sectors by $m$, $-2N \leq m \leq 2N$ which is the eigenvalue of $L_3$. Moreover, number of states with a given $L_3$ eigenvalue $m = \pm n$ are the same. Using the fields defined in terms of $L_{\pm}$, we have formulated the $CP^1$ model on $R \otimes S^2_F$ and obtained BPS bound. The local gauge invariance sets the $L_3$ eigenvalues of both $CP^1$ fields $\Phi_1$ and $\Phi_2$ to be same. The current associated with the topological charge $Q$ of the (anti-)

$$L_{\pm}^{(1)} = a_q^\dagger \sqrt{N - N_1}, \quad L_{\pm}^{(1)} = (L_{\pm}^{(1)})^\dagger, \quad L_3^{(1)} = N_1 - \frac{N}{2} \hspace{1cm} (61)$$

$$L_{\pm}^{(2)} = b_q^\dagger \sqrt{N - N_2}, \quad L_{\pm}^{(2)} = (L_{\pm}^{(2)})^\dagger, \quad L_3^{(2)} = N_2 - \frac{N}{2}$$
solition is conserved covariantly. This charge is shown to be the $L_3$ eigenvalue of the $CP^1$ fields. We have also obtained the general solutions for the BPS equations. For any given $j$, there are $2N + 1$ distinct soliton-anti-soliton solutions. Using the covariantly conserved current, we have then constructed the Hopf term for $R \otimes S^2$ and showed that in the commutative limit, it goes to the known form. We have also evaluated the Hopf term and showed that it is equal to $Q^2$ as in the commutative case. We have argued that the generators of $SU_q(2)$ allows a natural realisation of fuzzy sphere since the dimension of the Hilbert space corresponding to $SU_q(2)$ can be tuned to given $n$ by taking $q$ to be $n + 1$th root of unity.

It has been shown earlier that the $O(3)$ $\sigma-$ model with Hopf term with a coefficient $\pi$ is equivalent to spin $\frac{1}{2}$ theory with four Fermi interactions [30]. Having developed the $O(3)$ $\sigma-$ model on fuzzy sphere, one can now analyse this equivalence in this non-perturbative formulation. It has been shown that the $CP^1$ model with Hopf term is equivalent to spin $s$ theory. Here the spin $s$ is related to the coefficient of the Hopf term $\theta$ and is given by $\theta = \frac{\pi}{2s}, \ s = \frac{1}{2}, 1, \frac{3}{2}, ...$ [30]. Studies to see whether these equivalences go through in the present case are in progress.

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References

[1] A. Connes, Noncommutative geometry, Academic Press, London, (1994).

[2] N. Seiberg and E. Witten, JHEP 09 (1999) 032.

[3] A. Connes, M. R. Douglas and A. Schwarz, JHEP 9802 (1998) 033.

[4] S. Minwalla, M. Van Raamsdonk and N. Seiberg, JHEP 0002 (2000) 020; A. Matusis, L. Susskind, N. Toumbas, JHEP 0012 (2000) 002; B. P. Dolan, D. O’Conner and P. Presnajder, JHEP 0203 (2002) 013.

[5] N. A. Nekrasov, ’Trieste lectures on solitons in noncommutative gauge theories’, hep-th/0011095; J. A. Harvey, ’Komaba lectures on non-
commutative solutions and D-branes’, hep-th/0102076, R. Gopakumar, S. Minwalla and A. Strominger, JHEP 0005 (2000) 020.

[6] N. Nekrasov and A. Schwarz, Comm. Math. Phys. 198 (1998) 689; K. Furuuchi, Prog. Theor. Phys. 103 (2000) 1043; D. H. Correa, G. Lozano, E. F. Moreno and F. A. Schaposnik, Phys. Lett. B515 (2001) 206; C. Chu, V. V. Khoze and G. Travaglini, Nucl. Phys. B621 (2002) 101.

[7] S. Moriyama, Phys. Lett. B485 (2000) 278; D. J. Gross and N. A. Nekrasov, JHEP 0007 (2000) 034.

[8] J. A. Harvey, hep-th/0105242; C. Sochichiu, hep-th/0202014.

[9] S. Gubser and S. L. Sondhi, Nucl. Phys. B605 (2001) 395; E. T. Akhmedov, P. De Boer, G. W. Semenoff, Phys. Rev. D64 (2001) 065005; S. Sarkar and B. Sathiapalan, JHEP 0105, 049 (2001) 049; S. Sarkar, JHEP 0206 (2002) 003; L. Griguolo and M. Pietroni, JHEP 0105 (2001) 032.

[10] M. R. Douglas and N. A. Nekrasov, Rev. Mod. Phys. 73 (2001) 977.

[11] H. Grosse, C. Klimcik and P. Presnajder, Int. J. Theor. Phys. 35, (1996) 231; H. Grosse and A. Strohmaier, Lett. Math. Phys. 48 (1999) 163.

[12] P. Presnajder, J. Math. Phys. 41 (2000) 2789, H. Aoki, S. Iso, K. Nagao, hep-th/0209137.

[13] J. Madore, Class. Quant. Grav. 9 (1992) 69.

[14] H. Grosse, C. Klimcik and P. Presnajder, Commun. Math. Phys. 1788 (1996) 507; S. Baez, A. P. Balachandran, S. Vaidya and B. Ydri, Commun. Math. Phys. 208 (2000) 787; A. P. Balachandran, X. Martin and D. O’Connor, Int. J. Mod. Phys. A16 (2001) 2577; S. Vaidya, JHEP 0201 (2002) 011.

[15] A. P. Balachandran and S. Vaidya, Int. J. Mod. Phys. A16 (2001) 17.

[16] A. P. Balachandran, T. R. Govindarajan and B. Ydri, Mod. Phys. Lett. A15 (2000) 1279; A. P. Balachandran, T. R. Govindarajan and B. Ydri, hep-th/0006216; B. Ydri, Fuzzy Physics, PhD thesis (Syracuse Uni), hep-th/0110006.
[17] C. Klimcik, Commun. Math. Phys. 199 (1998) 257; U. C-Watamura and S. Watamura, Commun. Math. Phys. 212 (2000) 395; S. iso, Y. Kimura, K. Tanaka and K. Wakatsuki, Nucl. Phys. B604, (2001) 121.

[18] S. Vaidya, Phys. Lett. B512 (2001) 403; C-S Chu, J. Madore and H. Steinacker, JHEP 0108 (2001) 038.

[19] H. Grosse, C. Klimcik and P. Presnajder, Commun. Math. Phys. 178 (1996) 507.

[20] C-T Chan, C-M Chen, H. S. Yang, hep-th/0106269.

[21] H. Grosse and C. W. Rupp, math-ph/0103003; H. Grosse, C. W. Rupp and A. Strohmaier, J. Geom. Phys. 42 (2002) 54.

[22] B-H. Lee, K. Lee and H. S. Yang, Phys. Lett. B498 (2001) 277; C-T. Chan, C-M. Chen, F-L. Lin and H. S. Yang, Nucl. Phys. B625 (2002) 327.

[23] H. O. Girotti, M. Gomes, V. O. Rivelles and A. J. da Silva, Int. J. Mod. Phys. A17 (2002) 1503; H. O. Girotti, M. Gomes, A. Yu. Petrov, V. O. Rivelles and A. J. da Silva, Phys. Lett. B521 (2001) 119.

[24] A. A. Belavin and A. M. Polyakov, JETP Lett. 22 (1975) 245.

[25] H. Eichenherr, Nucl. Phys. B146 (1978) 215; V. Golo and A. M. Perelemov, Phys. Lett. 79B (1978) 112; E. Cremmer and J. Scherck, Phys. Lett. 74B (1978) 341; D’Adda, M. Lüscher and P. Di Vecchia, Nucl. Phys. B146 (1978) 63, ibid B152 (1979) 125.

[26] F. Wilczek and A. Zee, Phys. Rev. Lett. 50 (1983) 2250; Y-S Wu and A. Zee, Phys. Lett. 147B (1984) 325.

[27] M. J. Bowick, D. Karabali and L. C. R. Wijewardhana, Nucl. Phys. B271 (1986) 417; A. Kovner, Phys. Lett. 224B (1989) 299.

[28] E. Fradkin, ‘Field theories of Condensed Matter Systems’, Addison-Wesley (1991).

[29] A. M. Polyakov, Mod. Phys. Lett. A3 (1988) 325; A. Yu. Alekseev and S. L. Shatashvili, Mod. Phys. Lett. A3 (1998) 1551.
[30] R. Shankar and M. Sivakumar, Mod. Phys. Lett. A6 (1991) 2379; T. R. Govindarajan, N. Shaji, R. Shankar and M. Sivakumar, Phys. Rev. Lett. 69 (1992) 721; ibid, Int. J. Mod. Phys. A8 (1993) 3965; S. K. Paul, R. Shankar and M. Sivakumar, Mod. Phys. Lett. A6 (1991) 553.

[31] T. Holstein and H. Primakoff, Phys. Rev. 58 (1940) 1098.

[32] E. K. Sklyanin, Funct. Anal. Appl. 16 (1982) 262; P. P. Kulish and N. Y. Reshetikhin, J. Sov. Math. 23 (1983) 2435.

[33] L. C. Biedenharn, J. Phys. A: Math. Gen. 22, (1989) L879; A. J. Macfarlane, J. Phys. A: Math. Gen. 22 (1989) 4581.

[34] T. Curtright, D. Fairlie and C. Zachos (Ed), Quantum Groups, (Proceedings of the Argonne Workshop, World Scientific), (1991); Also see, Yu Manin, Quantum Groups and Non-Commutative Geometry, Montreal: Les Publications CRM (1988).

[35] A. P. Balachandran and G. Immirzi, Fuzzy Nambu-Goldstone Physics, hep-th/0212133.