SHARP CONSTANTS OF APPROXIMATION THEORY. V. AN ASYMPTOTIC EQUALITY RELATED TO POLYNOMIALS WITH GIVEN NEWTON POLYHEDRA

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Abstract. Let \( V \subset \mathbb{R}^m \) be a convex body, symmetric about all coordinate hyperplanes, and let \( \mathcal{P}_aV, a \geq 0, \) be a set of all algebraic polynomials whose Newton polyhedra are subsets of \( aV. \) We prove a limit equality as \( a \to \infty \) between the sharp constant in the multivariate Markov-Bernstein-Nikolskii type inequalities for polynomials from \( \mathcal{P}_aV \) and the corresponding constant for entire functions of exponential type with the spectrum in \( V. \)

1. Introduction

We continue the study of the sharp constants in multivariate inequalities of approximation theory that began in [14, 15, 16, 17]. In this paper we prove an asymptotic equality between the sharp constants in the multivariate Markov-Bernstein-Nikolskii type inequalities for entire functions of exponential type and algebraic polynomials whose Newton polyhedra are subsets of the given convex body.

Notation. Let \( \mathbb{R}^m \) be the Euclidean \( m \)-dimensional space with elements \( x = (x_1, \ldots, x_m), y = (y_1, \ldots, y_m), t = (t_1, \ldots, t_m), u = (u_1, \ldots, u_m), \) the inner product \( t \cdot x := \sum_{j=1}^{m} t_j x_j, \) and the norm \( |x| := \sqrt{x \cdot x}. \) Next, \( \mathbb{C}^m := \mathbb{R}^m + i\mathbb{R}^m \) is the \( m \)-dimensional complex space with elements \( z = (z_1, \ldots, z_m) = x + iy \) and the norm \( |z| := \sqrt{|x|^2 + |y|^2}; \) \( \mathbb{Z}^m \) denotes the set of all integral lattice points in \( \mathbb{R}^m; \) and \( \mathbb{Z}^m_+ \) is a subset of \( \mathbb{Z}^m \) of all points with nonnegative coordinates. We also use multi-indices \( s = (s_1, \ldots, s_m) \in \mathbb{Z}^m_+, \beta = (\beta_1, \ldots, \beta_m) \in \mathbb{Z}^m_+, \) and \( \alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{Z}^m_+ \) with

\[
|s| := \sum_{j=1}^{m} s_j, \quad |\beta| := \sum_{j=1}^{m} \beta_j, \quad |\alpha| := \sum_{j=1}^{m} \alpha_j, \quad y^\beta := y_1^{\beta_1} \cdots y_m^{\beta_m}, \quad D^\alpha := \frac{\partial^{\alpha_1}}{\partial y_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_m}}{\partial y_m^{\alpha_m}}.
\]

Given \( \sigma \in \mathbb{R}^m, \sigma_j \neq 0, 1 \leq j \leq m, \) and \( M > 0, \) let \( \Pi^m(\sigma) := \{ t \in \mathbb{R}^m : |t_j| \leq |\sigma_j|, 1 \leq j \leq m \}, \) \( Q^m(M) := \{ t \in \mathbb{R}^m : |t_j| \leq M, 1 \leq j \leq m \}, \) \( \mathcal{B}^m(M) := \{ t \in \mathbb{R}^m : |t_j| \leq M \}, \) and \( O^m(M) := \{ t \in \mathbb{R}^m : \sum_{j=1}^{m} |t_j| \leq M \} \) be the \( m \)-dimensional parallelepiped, cube, ball, and octahedron.
respectively. In addition, $|\Omega|_k$ denotes the $k$-dimensional Lebesgue measure of a measurable set $\Omega \subseteq \mathbb{R}^m$, $1 \leq k \leq m$. We also use the floor function $\lfloor a \rfloor$.

Let $L_r(\Omega)$ be the space of all measurable complex-valued functions $F$ on a measurable set $\Omega \subseteq \mathbb{R}^m$ with the finite quasinorm

$$\|F\|_{L_r(\Omega)} := \begin{cases} \left(\int_{\Omega} |F(x)|^r \, dx\right)^{1/r}, & 0 < r < \infty, \\ \text{ess sup}_{x \in \Omega} |F(x)|, & r = \infty. \end{cases}$$

This quasinorm allows the following ”triangle” inequality:

$$\left\| \sum_{j=1}^l F_j \right\|_{L_r(\Omega)} \leq \sum_{j=1}^l \|F_j\|_{L_r(\Omega)}, \quad F_j \in L_r(\Omega), \quad 1 \leq j \leq l, \quad (1.1)$$

where $l \in \mathbb{N} := \{1, 2, \ldots\}$ and $\hat{r} := \min\{1, r\}$ for $r \in (0, \infty]$.

In this paper we will need certain definitions and properties of convex bodies in $\mathbb{R}^m$. Throughout the paper $V$ is a centrally symmetric (with respect to the origin) closed convex body in $\mathbb{R}^m$ and $V^* := \{y \in \mathbb{R}^m : \forall t \in V, |t \cdot y| \leq 1\}$ is the polar of $V$. It is well known that $V^*$ is a centrally symmetric (with respect to the origin) closed convex body in $\mathbb{R}^m$ and $V^{**} = V$ (see, e.g., [30] Sect. 14]). The set $V$ generates the following dual norm on $\mathbb{C}^m$ by

$$\|z\|_V^* := \sup_{t \in V} \left| \sum_{j=1}^m t_j z_j \right|, \quad z \in \mathbb{C}^m.$$

Throughout the paper we assume that the body $V \subseteq \mathbb{R}^m$ satisfies the parallelepiped condition (\Pi-condition), that is, for every vector $t \in V$ with nonzero coordinates, the parallelepiped $\Pi^m(t)$ is a subset of $V$. It is easy to verify that $V$ satisfies the \Pi-condition if and only if $V$ is symmetric about all coordinate hyperplanes, that is, for every $t \in V$ the vectors $(\pm |t_1|, \ldots, \pm |t_m|)$ belong to $V$. In particular, given $\lambda \in [1, \infty]$ and $\sigma \in \mathbb{R}^m$, $\sigma_j > 0$, $1 \leq j \leq m$, the set $V_{\lambda, \sigma} := \left\{ t \in \mathbb{R}^m : \left( \sum_{j=1}^m |t_j/\sigma_j|^1/\lambda \right)^{1/\lambda} \leq 1 \right\}$, satisfies the \Pi-condition. Therefore, the sets $\Pi^m(\sigma)$ (for $\lambda = \infty$), $Q^m(M)$ (for $\lambda = \infty$ and $\sigma = (M, \ldots, M)$), $\mathfrak{B}^m(M)$ (for $\lambda = 2$ and $\sigma = (M, \ldots, M)$), and $O^m(M)$ (for $\lambda = 1$ and $\sigma = (M, \ldots, M)$) satisfy the \Pi-condition as well.

Given $\alpha \geq 0$, the set of all trigonometric polynomials $T(x) = \sum_{\theta \in aV \cap \mathbb{Z}^m} c_\theta \exp[i(\theta \cdot x)]$ with complex coefficients is denoted by $T_{aV}$.

**Definition 1.1.** We say that an entire function $f : \mathbb{C}^m \to \mathbb{C}^1$ has exponential type $V$ if for any $\varepsilon > 0$ there exists a constant $C_0(\varepsilon, f) > 0$ such that for all $z \in \mathbb{C}^m$, $|f(z)| \leq C_0(\varepsilon, f) \exp((1 + \varepsilon)\|z\|_V^*)$.

The class of all entire function of exponential type $V$ is denoted by $B_V$. In the univariate case we use the notation $B_\lambda := B_{[-\lambda, \lambda]}$, $\lambda > 0$. Throughout the paper, if no confusion may occur, the
same notation is applied to \( f \in B_V \) and its restriction to \( \mathbb{R}^m \) (e.g., in the form \( f \in B_V \cap L_p(\mathbb{R}^m) \)). The class \( B_V \) was defined by Stein and Weiss \[33\] Sect. 3.4. For \( V = \Pi^m(\sigma) \), \( V = Q^m(M) \), and \( V = \mathfrak{B}^m(M) \), similar classes were defined by Bernstein \[6\] and Nikolskii \[29\] Sects. 3.1, 3.2.6], see also \[9\] Definition 5.1. Properties of functions from \( B_V \) have been investigated in numerous publications (see, e.g., \[6\] \[29\] \[33\] \[34\] \[35\] \[36\] \[37\] \[38\] \[39\] \[40\] and references therein). Some of these properties are presented in Lemma \[22\].

Given \( a \geq 0 \), let \( \mathcal{P}_{aV} \) be a set of all polynomials \( P(x) = \sum_{\beta \in aV \cap \mathbb{Z}_m^m} c_\beta x^\beta \) in \( m \) variables with complex coefficients whose Newton polyhedra are subsets of \( aV \). In the univariate case we use the notation \( \mathcal{P}_a = \mathcal{P}_{[a]} := \mathcal{P}_{a[-1,1]} \). In the case of \( V = O^m(1) \), \( \mathcal{P}_{aV} = \mathcal{P}_{O^m(n)} \) coincides with the set of all polynomials in \( m \) variables of total degree at most \( n \), \( n \in \mathbb{N} \). It is easy to verify that if \( V_1 \subseteq V_2 \), then \( B_{V_1} \subseteq B_{V_2} \) and \( \mathcal{P}_{aV_1} \subseteq \mathcal{P}_{aV_2} \).

Throughout the paper \( C, C_1, C_2, \ldots, C_{25} \) denote positive constants independent of essential parameters. Occasionally we indicate dependence on certain occurrences. The same symbol \( C \) does not necessarily denote the same constant in different occurrences, while \( C_k, 1 \leq k \leq 25 \), denotes the same constant in different occurrences.

**Markov-Bernstein-Nikolskii Type Inequalities.** Let \( D_N := \sum_{|\alpha|=N} b_\alpha D^\alpha \) be a linear differential operator with constant coefficients \( b_\alpha \in \mathbb{C}^1, |\alpha| = N, N \in \mathbb{Z}_+^1 \). We assume that \( D_0 \) is the corresponding imbedding or identity operator.

Next, we define sharp constants in multivariate Markov-Bernstein-Nikolskii type inequalities for algebraic and trigonometric polynomials and entire functions of exponential type. Let

\[
M_{p,D_N,n,m,V} := n^{-N-m/p} \sup_{P \in \mathcal{P}_{O^m(n)} \setminus \{0\}} \frac{|D_N(P)(0)|}{\|P\|_{L_p(V^*)}}, \tag{1.2}
\]

\[
\tilde{M}_{p,D_N,n,m,V} := a^{-N-m/p} \sup_{P \in \mathcal{P}_{aV} \setminus \{0\}} \frac{|D_N(P)(0)|}{\|P\|_{L_p(Q^m(1))}}, \tag{1.3}
\]

\[
P_{p,D_N,n,m,V} := a^{-N-m/p} \sup_{T \in \mathcal{T}_{aV} \setminus \{0\}} \frac{\|D_N(T)\|_{L_\infty(Q^m(\pi))}}{\|T\|_{L_p(Q^m(\pi))}} \tag{1.4}
\]

\[
E_{p,D_N,n,m,V} := \sup_{f \in (B_V \cap L_p(\mathbb{R}^m)) \setminus \{0\}} \frac{\|D_N(f)\|_{L_\infty(\mathbb{R}^m)}}{\|f\|_{L_p(\mathbb{R}^m)}}.
\]

Here, \( a > 0, N \in \mathbb{Z}_+^1, n \in \mathbb{N}, V \subseteq \mathbb{R}^m, \) and \( p \in [0, \infty) \). In a sense, \( M_{p,D_N,n,m,V} \) and \( \tilde{M}_{p,D_N,n,m,V} \), \( n \in \mathbb{N} \), are dual sharp constants since the domain of integration \( V^* \) in \( \text{(1.2)} \) is the polar of the polynomial "degree" \( V \) in \( \text{(1.3)} \), and the domain of integration \( Q^m(1) = (O^m(1))^* \) in \( \text{(1.3)} \) is the polar of the polynomial "degree" \( O^m(1) \) in \( \text{(1.2)} \). In particular, \( M_{p,D_N,n,m,O^m(1)} = \tilde{M}_{p,D_N,n,m,O^m(1)}, n \in \mathbb{N} \).
We show in this paper that the equality can be asymptotically extended to any $V$, satisfying the II-condition.

Newton polyhedra and polynomial classes $\mathcal{P}_{aV}$ associated with Newton polyhedra play an important role in algebra, geometry, and analysis (see, e.g., a survey [3, Sect. 3]). However, the only sharp estimate for polynomials from $\mathcal{P}_{aV}$ we know in multivariate approximation theory is a sharp V. A. Markov-type inequality for polynomial coefficients with $a \in \mathbb{N}$ and $V = \Pi^m(\sigma)$, $\sigma_j \in \mathbb{N}$, $1 \leq j \leq m$, proved by Bernstein [5, Theorem 1] (see (1.8) below). The purpose of this paper is to prove a limit relation between $E_{p,DN,m,V}$ and $\tilde{M}_{p,DN,a,m,V}$ as $a \to \infty$ for $V$, satisfying the II-condition.

The following limit relation for multivariate trigonometric polynomials

$$\lim_{a \to \infty} P_{p,DN,a,m,V} = E_{p,DN,m,V}, \quad p \in (0, \infty], \quad (1.5)$$

was proved by the author [13, Theorem 1.3]. In the univariate case of $V = [-1, 1]$, $D_N = d^N/dx^N$, and $a \in \mathbb{N}$, (1.5) was proved by the author and Tikhonov [18]. In earlier publications [23, 24], Levin and Lubinsky established versions of (1.5) on the unit circle for $N = 0$. Quantitative estimates of the remainder in asymptotic equalities of the Levin-Lubinsky type were found by Gorbachev and Martyanov [19]. Certain extensions of the Levin-Lubinsky’s results to the $m$-dimensional unit sphere in $\mathbb{R}^{m+1}$ were recently proved by Dai, Gorbachev, and Tikhonov [8].

The first sharp constant in the inequality for polynomial coefficients was found by V. A. Markov [25] (see also [26, Eqs. (5.1.4.1)]) in the form ($n \in \mathbb{N}$)

$$M_{\infty,d^N/dx^N,n,1,[-1,1]} = \tilde{M}_{\infty,d^N/dx^N,n,1,[-1,1]}$$

$$= \mu_n^N := n^{-N} \left\{ \begin{array}{ll}
T_n^{(N)}(0), & n - N \text{ is odd,} \\
T_n^{(N)}(0), & n - N \text{ is even} 
\end{array} \right.$$

$$= 1 + o(1) = (1 + o(1))E_{\infty,d^N/dx^N,1,[-1,1]}, \quad (1.6)$$

as $n \to \infty$, where $T_n \in \mathcal{P}_n$ is the Chebyshev polynomial of the first kind. For $p = 2$ Labelle [22] proved the equalities ($n \in \mathbb{N}$, $N \leq n$)

$$M_{2,d^N/dx^N,n,1,[-1,1]} = \tilde{M}_{2,d^N/dx^N,n,1,[-1,1]}$$

$$= \frac{(2N)!}{2^N N!} \sqrt{N + 1/2} \, n^{-(N+1/2)} \left( \frac{[(n - N)/2] + N + 1/2}{N + 1/2} \right) = \frac{1 + o(1)}{\sqrt{\pi(2N + 1)}}$$

$$= (1 + o(1))E_{2,d^N/dx^N,1,[-1,1]}, \quad (1.7)$$

as $n \to \infty$. 
The following sharp constant in the multivariate inequality for polynomial coefficients was found in [5, Theorem 1]:

\[ \tilde{M}_{\infty,D^a,a,m,\Pi^m(\sigma)} = a^{-|\alpha|} \prod_{j=1}^{m} [a\sigma_j]^{|\alpha_j|} \mu_{\langle a\sigma_j \rangle}^{\alpha_j} = (1 + o(1)) \prod_{j=1}^{m} \sigma_j^{\alpha_j} = (1 + o(1)) E_{\infty,D^a,m,\Pi^m(\sigma)}, \]  

(1.8)
as \( a \to \infty \), where \( \mu_{\langle a\sigma_j \rangle}^{\alpha_j} \) is defined in (1.6) and \( \sigma_j > 0, 1 \leq j \leq m \). Note that

\[ \tilde{M}_{\infty,D^a,a,m,\Pi^m(\sigma)} \leq \prod_{j=1}^{m} \sigma_j^{\alpha_j}, \]  

(1.9)
which follows from the left equality in (1.8) and the corresponding univariate version of (1.9) \( \mu^N_{\eta} \leq 1 \) (see [34, Eq. 2.6(9)] with its proof in [16, Lemma 2.5]).

A crude estimate

\[ |c_{\beta}| \leq \left( \prod_{j=1}^{m} \beta_j! \right)^{-1} (A(V)a/M)^{|\beta|} \|P\|_{L_\infty(Q^m(M))}, \quad \beta \in aV \cap \mathbb{Z}_+^m, \]  

(1.10)
for coefficients of a polynomial \( P(x) = \sum_{\beta \in aV \cap \mathbb{Z}_+^m} c_{\beta} x^\beta \) from \( \mathcal{P}_aV \) follows immediately from (1.9) if we choose a cube \( Q^m(A), A = A(V) \), such that \( V \subseteq Q^m(A) \) and use (1.9) for \( \Pi^m(\sigma) = Q^m(A) \).

The author [16, Theorem 1.2] extended (1.6) and (1.7) to a general asymptotic relation for a multivariate \( L_p \)-version of the V. A. Markov constant for polynomial coefficients in the following form \( (n \in \mathbb{N}, p \in (0, \infty)) \):

\[ \lim_{n \to \infty} M_{p,D_N,n,m,V} = E_{p,D_N,m,V}. \]  

(1.11)
For \( m = 1, D_N = d^N/dx^N \), and \( V = [-1, 1] \) this equality was proved by the author in [13, Theorem 1.1]. A special case of (1.11) for an even \( N \in \mathbb{Z}_+^1, p \in [1, \infty] \), the unit ball \( V = \mathbb{B}^m(1) \), and the operator \( D_N = \Delta^{N/2} \), where \( \Delta \) is the Laplace operator, was obtained by the author in [15, Corollary 4.4].

Note that relations (1.5) and (1.11) are valid for any centrally symmetric \( V \) (see [14, 16]). Note also that certain properties of the sharp constants in univariate weighted spaces are discussed by Arestov and Deikalova [2]. In addition, note that the Bernstein-Nikolskii sharp constants \( E_{p,D_N,m,V} \) can be easily found only for \( p = 2 \) (see [14, Eq. (1.6)]).

Despite the fact that the constants \( M_{p,D_N,n,m,V} \) and \( \tilde{M}_{p,D_N,a,m,V} \) for \( m > 1 \) are defined differently by (1.2) and (1.3), it turns out that they are asymptotically equal. In this paper we extend (1.6), (1.7), and (1.8) to a general asymptotic relation for \( \tilde{M}_{p,D_N,a,m,V} \), which is similar to (1.11).

**Main Results and Remarks.** Recall that \( V \) is a closed convex body in \( \mathbb{R}^m \), satisfying the \( \Pi \)-condition. In particular, \( V \) is centrally symmetric (with respect to the origin).
Theorem 1.2. If $N \in \mathbb{Z}_+^1$, $V \subset \mathbb{R}^m$, and $p \in (0, \infty]$, then \( \lim_{a \to \infty} \tilde{M}_{p,D_N,a,m,V} \) exists and
\[
\lim_{a \to \infty} \tilde{M}_{p,D_N,a,m,V} = E_{p,D_N,m,V}.
\] (1.12)

In addition, there exists a nontrivial function $f_0 \in B_V \cap L_p(\mathbb{R}^m)$ such that
\[
\lim_{a \to \infty} \tilde{M}_{p,D_N,a,m,V} = \|D_N(f_0)\|_{L_\infty(\mathbb{R}^m)}/\|f_0\|_{L_p(\mathbb{R}^m)}.
\] (1.13)

The following corollary is a direct consequence of relations (1.5), (1.11), and (1.12).

Corollary 1.3. If $n \in \mathbb{N}$, $N \in \mathbb{Z}_+^1$, $V \subset \mathbb{R}^m$, and $p \in (0, \infty]$, then
\[
\lim_{n \to \infty} M_{p,D_N,n,m,V} = \lim_{a \to \infty} \tilde{M}_{p,D_N,a,m,V} = \lim_{a \to \infty} P_{p,D_N,a,m,V} = E_{p,D_N,m,V}.
\]

Remark 1.4. Relations (1.12) and (1.13) show that the function $f_0 \in B_V \cap L_p(\mathbb{R}^m)$ from Theorem 1.2 is an extremal function for $E_{p,D_N,m,V}$.

Remark 1.5. In definitions (1.3) and (1.4) of the sharp constants we discuss only complex-valued functions $P$ and $f$. We can define similarly the "real" sharp constants if the suprema in (1.3) and (1.4) are taken over all real-valued functions on $\mathbb{R}^m$ from $\mathcal{P}_a V \setminus \{0\}$ and $(B_V \cap L_p(\mathbb{R}^m)) \setminus \{0\}$, respectively. It turns out that the "complex" and "real" sharp constants coincide. For $m = 1$ this fact was proved in [13, Sect. 1] (cf. [18, Theorem 1.1] and [16, Remark 1.5]), and the case of $m > 1$ can be proved similarly.

Remark 1.6. Answering a referee’s question, we announced in [16, Remark 1.6] relation (1.12) for $V = Q^m(M)$ and $a \in \mathbb{N}$ with a typo ($a^{-N-m/p}$ was missing).

Remark 1.7. Now and then we call $M_{p,D_N,n,m,V}$ and $\tilde{M}_{p,D_N,a,m,V}$ the V. A. Markov constants for polynomial coefficients because of relations (1.6). However, there are different constants $\mathcal{M}_{p,D_N,n,m,V}$ and $\tilde{\mathcal{M}}_{p,D_N,a,m,V}$, defined by (1.2) and (1.3), respectively, with $n^{-N-m/p}|D_N(P)(0)|$ replaced by the corresponding $L_\infty$-norm. They are associated with the name of V. A. Markov as well because he [25] found the sharp constant for $m = 1$, $p = \infty$, $D_N = d^N/dx^N$, and $V = [-1,1]$. A brief survey on $\mathcal{M}_{p,d^N/dx^N,n,1,[-1,1]} = \tilde{\mathcal{M}}_{p,d^N/dx^N,n,1,[-1,1]}$ and its asymptotic behaviour were presented in [13] (see also [15, Corollary 4.6]). Certain estimates of $\mathcal{M}_{p,D_0,n,m,V}$ were surveyed in [16, Remark 1.8].

The proof of Theorem 1.2 is presented in Section 3. It follows general ideas developed in [17, Corollary 7.1]. Section 2 contains certain properties of functions from $B_V$ and $\mathcal{P}_a V$. 
2. Properties of Entire Functions and Polynomials

In this section we discuss certain properties of entire functions of exponential type and polynomials that are needed for the proof of Theorem 1.2. We start with three standard properties of multivariate entire functions of exponential type.

**Lemma 2.1.** (a) If \( f \in B_V \), then there exists \( M = M(V) > 0 \) such that \( f \in B_{Q^m(M)} \).

(b) The following crude Bernstein and Nikolskii type inequalities hold true:

\[
\|D^\alpha(f)\|_{L_\infty(\mathbb{R}^m)} \leq C \|f\|_{L_\infty(\mathbb{R}^m)}, \quad f \in B_V \cap L_\infty(\mathbb{R}^m), \quad \alpha \in \mathbb{Z}_+^m,
\]

\[
\|f\|_{L_\infty(\mathbb{R}^m)} \leq C \|f\|_{L_p(\mathbb{R}^m)}, \quad f \in B_V \cap L_p(\mathbb{R}^m), \quad p \in (0, \infty),
\]

where \( C \) is independent of \( f \).

(c) For any sequence \( \{f_n\}_{n=1}^\infty \), \( f_n \in B_V \cap L_\infty(\mathbb{R}^m) \), \( n \in \mathbb{N} \), with \( \sup_{n \in \mathbb{N}} \|f_n\|_{L_\infty(\mathbb{R}^m)} = C \), there exist a subsequence \( \{f_{n_d}\}_{d=1}^\infty \) and a function \( f_0 \in B_V \cap L_\infty(\mathbb{R}^m) \) such that for every \( \alpha \in \mathbb{Z}_+^m \),

\[
\lim_{d \to \infty} D^\alpha f_{n_d} = D^\alpha f_0
\]

uniformly on any compact set in \( \mathbb{C}^m \).

**Proof.** Statement (a) follows from the obvious inclusion \( V \subseteq Q^m(M) \) for a certain \( M = M(V) > 0 \) (cf. [16, Lemma 2.1 (a)]). Inequality (2.1) for \( V = Q^m(M) \), \( M > 0 \), is well known (see, e.g., [29, Eq. 3.2.2(8)]), while for any \( V \), (2.1) follows from statement (a) (cf. [16, Lemma 2.1(c)]). Inequality (2.2) was established in [28, Theorem 5.7]. Statement (e) was proved in [14, Lemma 2.3]. □

Given \( a \geq 0 \), \( \gamma > 0 \), and a univariate continuous function \( f \in L_\infty(\mathbb{R}^1) \), let

\[
E(f, \mathcal{P}_a, L_\infty([-\gamma, \gamma])) := \inf_{R \in \mathcal{P}_a} \|f - R\|_{L_\infty([-\gamma, \gamma])} = \|f - R_a\|_{L_\infty([-\gamma, \gamma])}
\]

be the error of best approximation of \( f \) by polynomials from \( \mathcal{P}_a \) in the norm of \( L_\infty([-\gamma, \gamma]) \). Here, \( R_a(\cdot) = R_a(f, \gamma, \cdot) \in \mathcal{P}_a \) is the polynomial of best uniform approximation to \( f \). Some elementary properties of \( R_a \) are discussed in the next lemma.

**Lemma 2.2.** (a) The following inequality holds true:

\[
\|R_a\|_{L_\infty([-\gamma, \gamma])} \leq 2\|f\|_{L_\infty(\mathbb{R}^1)}.
\]
(b) If \( f_\mu(v) := f(\mu v), \mu \neq 0, \) then \( R_a(f_\mu, \gamma/|\mu|, v) = R_a(f, \gamma, \mu v), \) \( v \in [-\gamma/|\mu|, \gamma/|\mu|]. \)

(c) For \( a_j \geq 0, \gamma_j > 0, \) and \( t \in \mathbb{R}^d \) with \( t_j \neq 0, 1 \leq j \leq d, \) the following inequality holds true:

\[
\max_{|x_j| \leq \gamma_j/|t_j|, 1 \leq j \leq d} \left| \prod_{j=1}^{d} f(t_j x_j) - \prod_{j=1}^{d} R_{a_j}(f, \gamma_j, t_j x_j) \right| \leq \|f\|_{L_\infty([\gamma, \gamma])} \left( \sum_{j=1}^{d} 2^{j-1} E(f(t_j \cdot), P_{a_j}, L_\infty([-\gamma_j/|t_j|, \gamma_j/|t_j|])) \right). 
\] (2.6)

**Proof.** Statement (a) follows from the inequalities

\[
\|R_a\|_{L_\infty([-\gamma, \gamma])} \leq \|f\|_{L_\infty([-\gamma, \gamma])} + E(f, P_a, L_\infty([-\gamma, \gamma])) \leq 2 \|f\|_{L_\infty([-\gamma, \gamma])},
\]

while statement (b) is an immediate consequence of the Kolmogorov characterization of an element of best approximation to a complex-valued function \( [21] \) (see also \( [32, \text{Theorem 1.9} ] \) and \( [1, \text{Sect. 47}] \)).

To prove statement (c), we note that for \( |x_j| \leq \gamma_j/|t_j|, 1 \leq j \leq d, \) the following relations hold true by (2.5):

\[
\left| \prod_{j=1}^{d} f(t_j x_j) - \prod_{j=1}^{d} R_{a_j}(f, \gamma_j, t_j x_j) \right| = \sum_{j=1}^{d} \left| f(t_j x_j) - R_{a_j}(f, \gamma_j, t_j x_j) \right| \prod_{k=j+1}^{d} f(t_k x_k) \prod_{k=1}^{j-1} R_{a_k}(f, \gamma_k, t_k x_k) \leq \|f\|_{L_\infty([\gamma, \gamma])} \sum_{j=1}^{d} 2^{j-1} \left\| f(t_j \cdot) - R_{a_j}(f, \gamma_j, t_j) \right\|_{L_\infty([-\gamma_j/|t_j|, \gamma_j/|t_j|])},
\] (2.7)

where \( \prod_{k=q}^{l} := 1 \) for \( q < l. \) Note that the proof of identity (2.7) is simple and left as an exercise to the reader. Then (2.6) follows from (2.8) since \( R_{a_j}(f, \gamma_j, t_j x_j) = R_{a_j}(f, \gamma_j, t_j/|t_j|) \) by statement (b) for \( a = a_j, \mu = t_j \neq 0, \) and \( v = x_j, 1 \leq j \leq d. \) \( \square \)

**Remark 2.3.** Concerning Lemma 2.2 (b), we note that for every fixed \( v \in \mathbb{R}^1 \) the polynomial

\[
\begin{cases} 
R_a(f, \gamma, \mu v), \mu \neq 0, \\
f(0), \mu = 0,
\end{cases}
\]

is obviously a continuous function of \( \mu \in \mathbb{R}^1 \setminus \{0\}, \) but it can be discontinuous at \( \mu = 0 \) since \( R_a(f, \gamma, 0) \) is not necessarily equal to \( f(0). \)

In the next four lemmas we discuss estimates of the error of polynomial approximation for functions from \( B_V. \)
Lemma 2.4. Let \( g \in B_\lambda \cap L_\infty(\mathbb{R}^1) \) be a univariate entire function of exponential type at most \( \lambda > 0 \). Given \( a \geq 1 \) and \( \tau \in (0, 1) \), the following inequality holds true:

\[
E(g, P_a, L_\infty([-a\tau/\lambda, a\tau/\lambda])) \leq C_1(\tau) \exp[-C_2(\tau) a] \|g\|_{L_\infty(\mathbb{R}^1)},
\]

where

\[
C_1(\tau) := 2 \left( 1 + 1/\sqrt{1 - \tau^2} \right), \quad C_2(\tau) := \log \left( 1 + \sqrt{1 - \tau^2} \right) - \log \tau - \sqrt{1 - \tau^2} > 0.
\]

Proof. It is known (see, e.g., [34, Sect. 5.4.4]) that for any \( \lambda > 0 \),

\[
\text{Step 1.}
\]

\[
\text{Step 2.}
\]

\[
\text{Step 3.}
\]

Therefore,

\[
E(g, P_a, L_\infty([-a\tau/\lambda, a\tau/\lambda])) \leq \frac{2 \exp[a\tau\delta]}{\delta \left( \delta + \sqrt{1 + \delta^2} \right)^2} \|g\|_{L_\infty(\mathbb{R}^1)}.
\]

Setting \( \delta = \sqrt{1 - \tau^2}/\tau \) in (2.11), we arrive at (2.9) and (2.10).

In case of \( a \in \mathbb{N} \), versions of Lemma 2.4 were proved by the author [10, Lemma 4.1] and Bernstein [4, Theorem VI] (see also [34, Sect. 5.4.4] and [1, Appendix, Sect. 83]). More general and more precise inequalities were obtained in [10] and [11].

Lemma 2.5. For given \( a \geq 1 \) and \( \tau \in (0, 1) \) and for every \( t \in V \), there exists a polynomial \( P_t(x) = P_{t,a,V,\tau}(x) = \sum_{\beta \in aV \cap \mathbb{Z}_+^m} c_\beta(t) x^\beta \) from \( P_aV \) such that \( c_\beta = c_{\beta,a,V,\tau} \in L_\infty(V) \), \( \beta \in aV \cap \mathbb{Z}_+^m \), and the following inequality holds true:

\[
\text{ess sup}_{t \in V} \max_{x \in Q^m(\alpha t)} |\exp[i(t \cdot x)] - P_t(x)| \leq C_3(\tau, m) \exp[-C_4(\tau, V) a], \quad t \in V.
\]

Proof. We prove the lemma in three steps.

Step 1. We first obtain the univariate inequality (\( \lambda \neq 0 \))

\[
E(\exp[i\lambda \cdot], P_a, L_\infty([-a\tau/|\lambda|, a\tau/|\lambda|])) \leq C_1(\tau) \exp[-C_2(\tau) a]
\]

by using Lemma 2.4 for \( g(\cdot) = \exp[i\lambda \cdot] \in B_\lambda \cap L_\infty(\mathbb{R}^1) \).

Step 2. Next, we prove (2.12) for a parallelepiped \( V = \Pi^m(u) \), where \( u \in \mathbb{R}^m \), \( u_j \neq 0 \), \( 1 \leq j \leq m \), in the following form:

\[
\text{ess sup}_{t \in \Pi^m(u)} \max_{x \in Q^m(\alpha t)} |\exp[i(t \cdot x)] - P_{t,a,\Pi^m(u),\tau}(x)| \leq C_3(\tau, m) \exp[-C_4(\tau, \Pi^m(u)) a].
\]

(2.13)
Here,
\[ C_3(\tau, m) = m^{2m-1} C_1(\tau), \quad C_4(\tau, \Pi^m(u)) = \min_{1 \leq j \leq m} |u_j| C_2(\tau), \quad (2.15) \]
and the constants \( C_1 \) and \( C_2 \) in (2.13) and (2.15) are defined by (2.10). To prove (2.14), for any \( t \in \Pi^m(u) \) we define a polynomial
\[ P_t(x) = P_{t,a,\Pi^m(u),\tau}(x) := \begin{cases} \prod_{t_j \neq 0, 1 \leq j \leq m} R_{a|u_j|}(\exp[i\cdot], a\tau|u_j|, t_j x_j), & t \neq 0, \\ 1, & t = 0, \end{cases} \quad (2.16) \]
from the class \( P_{a,\Pi^m(u)} = P_{\Pi^m(u)} \). We recall that \( R_a = R_a(f, \gamma, \cdot) \) is defined by (2.4). Since \( |t_j| \leq |u_j|, 1 \leq j \leq m \), we obtain from (2.16), (2.6), and (2.13)
\[
\max_{x \in Q^m(a\tau)} |\exp[i(t \cdot x)] - P_t(x)| \\
\leq \max_{t_j \neq 0, |x_j| \leq a\tau|u_j|/|t_j|, 1 \leq j \leq m} |\exp[i(t \cdot x)] - P_t(x)| \\
\leq 2^{m-1} \sum_{t_j \neq 0, 1 \leq j \leq m} E(\exp[i t_j \cdot], P_{a|u_j|}, L_\infty([-a\tau|u_j|/|t_j|, a\tau|u_j|/|t_j|])) \\
\leq m 2^{m-1} C_1(\tau) \exp \left[ - \min_{1 \leq j \leq m} |u_j| C_2(\tau) a \right].
\]
This proves (2.14) and (2.15).

Note that by formula (2.16), all coefficients \( c_\beta(t), \beta \in \Pi^m(u) \cap \mathbb{Z}^m_+ \), of the polynomial \( P_t \) are continuous in \( t \in \mathbb{R}^m \setminus \bigcup_{j=1}^m H_j \), where \( H_j \) is the \( j \)-th \((m-1)\)-dimensional coordinate hyperplane in \( \mathbb{R}^m \), \( 1 \leq j \leq m \). We also note that the coefficients can be discontinuous on \( H := \bigcup_{j=1}^m H_j \) (see Remark 2.3). However, \( c_\beta = c_{\beta, a, \Pi^m(u), \tau} \in L_\infty(\mathbb{R}^m), \beta \in \Pi^m(u) \cap \mathbb{Z}^m_+ \). Indeed, using relations (2.16) and (2.5), we obtain the inequality
\[ \max_{x \in Q^m(a\tau)} |P_t(x)| \leq 2^m \]
for every \( t \in \mathbb{R}^m \). Therefore, for coefficients of \( P_t \) we have the estimate \( \sup_{t \in \mathbb{R}^m} |c_\beta(t)| < \infty, \beta \in \Pi^m(u) \cap \mathbb{Z}^m_+ \), by (1.10) and (2.17). Then \( c_\beta \in L_\infty(\mathbb{R}^m), \beta \in \Pi^m(u) \cap \mathbb{Z}^m_+ \), since \( |H|_m = 0 \).

Step 3. Finally, let \( V \) be a convex body, satisfying the \( \Pi \)-condition.

Step 3a. First of all, given \( \delta \in (1, \infty) \), we construct a finite family of parallelepipeds \( \{\Pi^m(u^{(k)})\}_{k=1}^K \) such that
\[ V \subseteq \bigcup_{k=1}^K \Pi^m(u^{(k)}) \subseteq \delta V, \quad \min_{1 \leq j \leq m, 1 \leq k \leq K} |u_j^{(k)}| \geq C_5(\delta, V), \quad (2.18) \]
where \( K = K(\delta, m, V) \).

To construct the family, we first consider the following parallelepipeds
\[ \Pi_l := \{ x \in \mathbb{R}^m : |x_i| \leq C_6(\delta, V), |x_j| \leq C_7(\delta, V), j \neq l \}, \]
where $C_6(\delta, V) := \min_{1 \leq l \leq m} \sqrt{(1 + \delta)/2} |O_1 \cap V|$ (where $O_1$ is the $l$th coordinate axis, $1 \leq l \leq m$), and $C_7(\delta, V)$ is chosen such that

$$\Pi_l \subseteq \sqrt{\delta} V, \quad 1 \leq l \leq m, \quad \inf_{x \in V} \min_{1 \leq j \leq m} |x_j| \geq C_7(\delta, V). \tag{2.19}$$

Since $(1 + \delta)/2 < \sqrt{\delta}$, there is a small enough $C_7(\delta, V) < C_6(\delta, V)$ such that (2.19) holds true.

Next, let $x \in V \setminus \bigcup_{l=1}^{m} \Pi_l$. Then $|x_j| > 0$, $1 \leq j \leq m$, by the second relation of (2.19), and $x$ is an interior point of $\Pi^m(\delta x)$. In addition, since $V$ satisfies the $\Pi$-condition, we see that $\Pi^m(\delta x) \subseteq \delta V$.

Furthermore, setting

$$\Pi_x := \begin{cases} \sqrt{\delta} \Pi_l, & x \in \Pi_l \cap V, \quad 1 \leq l \leq m, \\ \Pi^m(\delta x), & x \in V \setminus \bigcup_{l=1}^{m} \Pi_l, \end{cases} \tag{2.20}$$

for every $x \in V$, we see by the construction of $\Pi_x = \Pi^m(u)$, $u = u(x) \in \delta V$, and by relations (2.19) and (2.20) that

$$V \subseteq \bigcup_{x \in V} \Pi_x \subseteq \delta V, \quad \min_{x \in V} \min_{1 \leq j \leq m} |u_j(x)| \geq \sqrt{\delta} C_7(\delta, V). \tag{2.21}$$

To construct the family $\{\Pi^m(u^{(k)})\}_{k=1}^{K}$ with $K = K(\delta, m, V)$, we need the following special case of Morse’s theorem [27] (see also [20, Remark 1.4]):

**Lemma 2.6.** Let for every $x \in V$ there exist a parallelepiped $\tilde{\Pi}_x$ (not necessarily centered at the origin), satisfying the condition: there exists a fixed constant $C \geq 1$ independent of $x$, and there exist two balls $x + B^m(r(x))$ and $x + B^m(Cr(x))$ centered at $x$ of radii $r(x)$ and $Cr(x)$, respectively, such that $x + B^m(r(x)) \subseteq \tilde{\Pi}_x \subseteq x + B^m(Cr(x))$. Then a family $\{\tilde{\Pi}_x\}_{x \in V}$ contains a subfamily $\pi := \{\tilde{\Pi}_x(d)\}_{d=1}^{\infty}$ with the following properties:

(a) $V \subseteq \bigcup_{d=1}^{\infty} \tilde{\Pi}_x(d)$;

(b) there exist subfamilies $\pi_k$, $1 \leq k \leq K_1(m, C)$, of mutually disjoint parallelepipeds such that $\pi = \bigcup_{k=1}^{K_1} \pi_k$.

Then the family $\{\tilde{\Pi}_x\}_{x \in V} = \{\Pi_x\}_{x \in V}$ defined by (2.20) satisfies the condition of Lemma 2.6. Indeed, by the construction of $\Pi_x$, the condition of Lemma 2.6 is satisfied for

$$r(x) = C_8(\delta, V) := \left(\sqrt{\delta} - 1\right) C_7, \quad x \in V; \quad C = \delta D(V)/C_8,$$

where $D(V)$ is the diameter of $V$. In addition, note that any two parallelepipeds defined by (2.20) have nonempty intersection. Hence subfamilies $\pi_k$, $1 \leq k \leq K_1(m, C)$, from property (b) of Lemma 2.6 contain no more than one parallelepiped. Therefore, by Lemma 2.6, there exists a finite subfamily of parallelepipeds $\{\tilde{\Pi}_x(d)\}_{d=1}^{K_1} = \{\Pi^m(u^{(k)})\}_{k=1}^{K}$ with $K(\delta, m, V) := K_1(m, C)$ and $u^{(k)} \in \delta V$, $1 \leq k \leq K$, such that (2.18) holds true for $C_5 = \sqrt{\delta} C_7$ by (2.21).
Step 3b). Furthermore, given \( \tau \in (0, 1) \), let us set \( \delta = 1/\tau \), and let \( \{\Pi^m(u^{(k)})\}_{k=1}^{K} \) be a finite family of parallelepipeds, where \( K = K(\delta, m, V) \) and \( u^{(k)} \in \delta V, 1 \leq k \leq K \), such that (2.18) holds true. Let us define \( P_t(x) = P_{t,a,V,\tau}(x), x \in Q^m(\alpha r), t \in V \), by the formula

\[
P_{t,a,V,\tau}(x) := P_{t,a/\delta,\Pi^m(u^{(k)}),\tau}(x), \quad t \in V \cap \left( \cap_{l=1}^{k-1} \Pi^m(u^{(l)}) \cup \Pi^m(u^{(l)}) \right), \quad 1 \leq k \leq K. \quad (2.22)
\]

Recall that the polynomial \( P_{t,a,\Pi^m(u^{(k)}),\tau}(x) \) is defined by (2.16), and its coefficients belong to \( L_\infty(\Pi^m(u^{(k)})), 1 \leq k \leq K \). Since \( V \subseteq \bigcup_{k=1}^{K} \Pi^m(u^{(k)}) \) by (2.18), we see from (2.22) that the coefficients of \( P_{t,a,V,\tau} \) belong to \( L_\infty(V) \). Next, since \( \bigcup_{k=1}^{K} \Pi^m(u^{(k)}) \subseteq \delta V \) by (2.18), \( P_{t,a,V,\tau} \in \mathcal{P}_{(a/\delta)(\delta V)} = \mathcal{P}_a \) for each fixed \( t \in V \).

Furthermore, we obtain from (2.22), (2.14), (2.15), and (2.18)

\[
\begin{align*}
\text{ess sup}_{t \in V} \max_{x \in Q^m(\alpha r)} |\exp[i(t \cdot x)] - P_{t,a,V,\tau}(x)| \\
\leq \max_{1 \leq k \leq K} \text{ess sup}_{t \in \Pi^m(u^{(k)}), x \in Q^m(\alpha r)} |\exp[i(t \cdot x)] - P_{t,a,V,\Pi^m(u^{(k)}),\tau}(x)| \\
\leq C_3(\tau, m) \exp \left[ -\min_{1 \leq k \leq K} \min_{1 \leq j \leq m} |u_j^{(k)}| C_2(\tau) a \right] \\
\leq C_3(\tau, m) \exp \left[ -C_5(1/\tau, V) C_2(\tau) a \right] \\
= C_3(\tau, m) \exp \left[ -C_4(\tau, V) a \right].
\end{align*}
\]

This completes the proof of Lemma 2.5. \( \square \)

**Lemma 2.7.** For any \( f \in \mathcal{B}_V \cap L_\infty(\mathbb{R}^m), \tau \in (0, 1), \) and \( a \geq 1 \), there is a polynomial \( P_a = P_{a,V,\tau,f} \in \mathcal{P}_a \) such that for each \( \alpha \in \mathbb{Z}_+^m \) and \( r \in (0, \infty] \),

\[
\lim_{a \to \infty} \|D^\alpha(f) - D^\alpha(P_a)\|_{L_r(Q^m(\alpha r))} = 0. \quad (2.23)
\]

**Proof.** We prove the lemma in three steps.

**Step 1.** We first assume that \( f \in \mathcal{B}_V \cap L_2(\mathbb{R}^m) \). By the Paley-Wiener type theorem (33) Theorem 4.9, there exists \( \varphi \in L_2(V) \) such that \( f(x) = (2\pi)^{-m/2} \int_V \varphi(t) \exp[i(t \cdot x)] dt, x \in \mathbb{R}^m \).

Let \( P_t(x) \) be a polynomial from Lemma 2.5. Then for \( a \geq 1 \) the integral \( (x \in \mathbb{R}^m) \)

\[
P_a(x) = P_a(f, V, \tau, x) := (2\pi)^{-m/2} \int_V \varphi(t) P_t(x) dt = (2\pi)^{-m/2} \sum_{\beta \in aV \cap \mathbb{Z}_+^m} \int_V \varphi(t) c_\beta(t) dt x^\beta
\]
exists since \(c_{\beta} = c_{\beta,a,V,\tau} \in L_\infty(V)\), \(\beta \in aV \cap \mathbb{Z}_+^m\). Therefore, \(P^*_a \in \mathcal{P}_{aV}\). Next, it follows from (2.12) that given \(\tau \in (0,1)\),

\[
\|f - P^*_a\|_{L_\infty(Q^m(\tau v))} \leq (2\pi)^{-m/2} \int_V |\varphi(t)| dt \sup_{t \in V} \max_{x \in Q^m(\tau v)} |\exp[i(t \cdot x)] - P_t(x)| \\
\leq |V|^{1/2}C_3(\tau, m) \exp[-C_4(\tau, V) a] \|f\|_{L_2(\mathbb{R}^m)}, \tag{2.24}
\]

where \(C_3\) and \(C_4\) are the constants from Lemma 2.3.

**Step 2.** Next, let \(f \in B_V \cap L_\infty(\mathbb{R}^m)\). Then given \(\tau \in (0,1)\) and \(\varepsilon \in (0, (1 - \tau)/(2\tau C_9)]\), where \(C_9(m,V) := (m + 1) \sup_{z \in \mathbb{C}^m} |z|/\|z\|_V\), the function

\[
f_1(z) := f(z) \left( \frac{\sin \left( \frac{\varepsilon}{(\sum_{j=1}^m z_j^2)^{1/2}} \right)}{\varepsilon \left( \sum_{j=1}^m z_j^2 \right)^{1/2}} \right)^{m+1}
\]

belongs to \(B_{(1 + \varepsilon C_9) V} \cap L_2(\mathbb{R}^m)\) and \(\|f_1\|_{L_2(\mathbb{R}^m)} \leq C_{10}(m) \varepsilon^{-m/2} \|f\|_{L_\infty(\mathbb{R}^m)}\). Replacing now \(a\) with \(a/(1 + \varepsilon C_9)\) and \(\tau\) with \((1 + \varepsilon C_9) \leq (1 + \tau)/2\) in (2.24), we see from (2.24) that there exists a polynomial \(P_a(\cdot) = P_{a,V,\tau,f,\varepsilon}(\cdot) := P^*_{a/(1 + \varepsilon C_9)}(f_1, (1 + \varepsilon C_9) V, \cdot) \in \mathcal{P}_{aV}\), where \(\varepsilon\) will be chosen later, such that

\[
\|f_1 - P_a\|_{L_\infty(Q^m(\tau v))} \leq \|f_1 - P_a\|_{L_\infty(Q^m((a/(1 + \varepsilon C_9))((1 + \tau)/2))} \\
\leq C_{11}(\tau, m, V) \varepsilon^{-m/2} \exp[-C_4((1 + \tau)/2, V) 2a\tau/(1 + \tau)] \|f\|_{L_\infty(\mathbb{R}^m)} \\
= C_{11}(\tau, m, V) \varepsilon^{-m/2} \exp[-C_{12}(\tau, V) a] \|f\|_{L_\infty(\mathbb{R}^m)}. \tag{2.25}
\]

Furthermore, using an elementary inequality \(v - \sin v \leq v^3/6, v \geq 0\), we have

\[
\|f - f_1\|_{L_\infty(Q^m(\tau v))} \leq (m + 1) \sup_{x \in Q^m(\tau v)} \left| 1 - \frac{\sin(\varepsilon |x|)}{\varepsilon |x|} \right| \|f\|_{L_\infty(\mathbb{R}^m)} \\
\leq (1/6)(m + 1)m \varepsilon^2 a^2 \|f\|_{L_\infty(\mathbb{R}^m)}. \tag{2.26}
\]

Combining (2.25) and (2.26), we obtain

\[
\|f - P_a\|_{L_\infty(Q^m(\tau v))} \leq C_{13}(\tau, m, V) \left( \varepsilon^2 a^2 + \varepsilon^{-m/2} \exp[-C_{12}(\tau, V) a] \right) \|f\|_{L_\infty(\mathbb{R}^m)}. \tag{2.27}
\]

Finally minimizing the right-hand side of (2.27) over all \(\varepsilon \in (0, (1 - \tau)/(2\tau C_9)]\), we arrive at the following inequality:

\[
\|f - P_a\|_{L_\infty(Q^m(\tau v))} \leq C_{14}(\tau, m, V) a^{2m+4} \exp[-C_{15}(\tau, m, V) a] \|f\|_{L_\infty(\mathbb{R}^m)}, \tag{2.28}
\]

where \(C_{15} = 4C_{12}/(m + 4)\). Note that if the minimum occurs at \(\varepsilon = \varepsilon_0\), then \(P_a = P_{a,V,\tau,f,\varepsilon_0}\) in (2.28).
Step 3. First of all, for \( P_b \in \mathcal{P}_{bV}, b \geq 1, M > 0, \) and \( \alpha \in \mathbb{Z}^m_+ \), we need the following crude Markov-type inequality:

\[
\| D^\alpha (P_b) \|_{L_\infty(Q^m(M))} \leq C_{16}(m, V, |\alpha|)(b^2/M)^{|\alpha|} \| P_b \|_{L_\infty(Q^m(M))}.
\] (2.29)

To prove (2.29), we note that there exists a constant \( C_{17}(V) \) such that \( P_b \) is a polynomial of total degree at most \( n = [C_{17}b] \in \mathbb{N} \) (that is, \( P_b \in \mathcal{P}_{Q^m(n)} \)). Then inequality (2.29) easily follows from a multivariate A. A. Markov-type inequality proved by Wilhelmsen [35, Theorem 3.1].

Next, let \( \{P_{a+k}\}_{k=0}^\infty \) be the sequence of polynomials, satisfying inequality (2.28) with \( a \) replaced by \( a + k, k = 0, 1, \ldots \). Then the series

\[
\sum_{k=0}^\infty (P_{a+k+1} - P_{a+k}) = \lim_{L \to \infty} (P_{a+L+1} - P_a) = \lim_{L \to \infty} (P_{a+L+1} - f + f - P_a)
\]

converges to \( f - P_a \) in the metric of \( L_\infty(Q^m(a\tau)) \) by (2.28). In addition, for any \( \alpha \in \mathbb{Z}^m_+ \) we obtain by (2.29) for \( M = a\tau \) and by (2.28)

\[
\sum_{k=0}^\infty \| D^\alpha (P_{a+k+1} - P_{a+k}) \|_{L_\infty(Q^m(a\tau))}
\]

\[
\leq C_{16}(a\tau)^{-|\alpha|} \sum_{k=0}^\infty (a + k + 1)^2|\alpha| \| P_{a+k+1} - P_{a+k} \|_{L_\infty(Q^m(a\tau))}
\]

\[
\leq C_{16}(a\tau)^{-|\alpha|} \sum_{k=0}^\infty (a + k + 1)^2|\alpha| \left( \| f - P_{a+k+1} \|_{L_\infty(Q^m((a+k+1)\tau))} + \| f - P_{a+k} \|_{L_\infty(Q^m((a+k)\tau))} \right)
\]

\[
\leq 2C_{14}C_{16}(a\tau)^{-|\alpha|} \exp[-C_{15}a] \sum_{k=0}^\infty (a + k + 1)^{2|\alpha|+2m/(m+4)} \exp[-C_{15}a] \| f \|_{L_\infty(\mathbb{R}^m)}
\]

\[
\leq C_{18}(\tau, m, V, |\alpha|, r) a^{|\alpha|+2} \exp[-C_{15}a] \| f \|_{L_\infty(\mathbb{R}^m)}.
\] (2.30)

Hence the series \( \sum_{k=0}^\infty D^\alpha (P_{a+k+1} - P_{a+k}) \) is uniformly convergent on \( Q^m(a\tau) \) by the Weierstrass M-test, and this series converges to \( D^\alpha (f - P_a) \) in the metric of \( L_\infty(Q^m(a\tau)) \) by the Differentiation Theorem from multivariate calculus. It remains to take account of the following inequalities:

\[
\| D^\alpha (f) - D^\alpha (P_a) \|_{L_\tau(Q^m(\alpha\tau))} \leq (2a\tau)^m/r \| D^\alpha (f - P_a) \|_{L_\infty(Q^m(\alpha\tau))}
\]

\[
\leq (2a\tau)^m/r \sum_{k=0}^\infty \| D^\alpha (P_{a+k+1} - P_{a+k}) \|_{L_\infty(Q^m(\alpha\tau))}.
\] (2.31)

Thus (2.23) follows from (2.31) and (2.30), and the proof of the lemma is completed.

Remark 2.8. Note that limit relation (2.23) holds true for the same polynomial \( P_a \) and any \( \alpha \in \mathbb{Z}^m_+ \) and \( r \in (0, \infty] \). The proof of this fact in Lemma 2.7 is based on the exponential approximation rate in (2.28).
A certain polynomial estimate is discussed in the following lemma.

**Lemma 2.9.** Given $a \geq 1$, $M > 0$, $p \in (0, \infty)$, $\tau \in (0,1)$, and $P \in \mathcal{P}_aV$, the following inequality holds true:

$$\|P\|_{L^\infty(Q^m(\tau M))} \leq C_{19}(\tau, m, V, p)(a/M)^{m/p}\|P\|_{L^p(Q^m(M))}. \quad (2.32)$$

**Proof.** Inequality (2.32) for $V = Q^m(1)$ and $a \in \mathbb{N}$ follows from a more general inequality proved in [16, Lemma 2.7 (b)]. To prove (2.32) for any $V$, we note that there exists a constant $C_{20}(V)$ such that $P$ is a polynomial of degree at most $n = \lfloor C_{20}a \rfloor \in \mathbb{N}$ in each variable (that is, $P \in \mathcal{P}_{Q^m(n)}$). Then (2.32) follows from [16, Lemma 2.7 (b)].

In the next lemma we discuss special properties of polynomials from $\mathcal{P}_aV$.

**Lemma 2.10.** Given $a \geq 1$, $b \geq 1$, and $P(x) = \sum_{\beta \in aV \cap \mathbb{Z}^m_n} c\beta x^\beta \in \mathcal{P}_aV$, let

$$R_{a,b}(t) := P(b \sin(t_1/b), \ldots, b \sin(t_l/b)), \quad t \in \mathbb{R}^m,$$

be a trigonometric polynomial. Then the following statements are valid.

(a) $R_{a,b} \in B_{(a/b)V}$.

(b) For $\alpha \in \mathbb{Z}_+^m$ the following estimate holds true:

$$|D^\alpha (R_{a,b})(0) - D^\alpha (P)(0)| \leq C_{21}(m, \alpha) \max_{0 \leq s_j \leq \alpha_j, 1 \leq j \leq m, s \neq \alpha} |D^s (P)(0)| / b. \quad (2.33)$$

**Proof.** (a) We see that

$$R_{a,b}(bt) = \sum_{\beta \in aV \cap \mathbb{Z}^m_n} b^{\beta} c\beta \prod_{j=1}^m \sin^{b_j} t_j = \sum_{\beta \in aV \cap \mathbb{Z}^m_n} b^{\beta} c\beta \prod_{j=1}^m \sum_{\beta \in \mathbb{Z}^m_n, 0 \leq |\beta| \leq \beta_j, 1 \leq j \leq m} d_{\theta, \beta} \exp[i(\theta \cdot t)].$$

Then $R_{a,b}(bt) \in T_{aV}$, since $V$ satisfies the $\Pi$-condition, and therefore, $R_{a,b}(\cdot) \in B_{(a/b)V}$.

(b) To prove this statement, we need the identity

$$D^\alpha (R_{a,b})(0) = \sum_{s_1=1}^{\alpha_1} \cdots \sum_{s_m=1}^{\alpha_m} b^{s_1-|\alpha_1|} D^s (P)(0) \prod_{j=1}^m c(s_j, \alpha_j), \quad (2.34)$$

where

$$c(l, k) := \sum_{p_1 + \ldots + p_3 = l} k! p_1(1)! p_3(3)! \ldots, \quad (2.35)$$

and the sum in (2.35) is taken over all nonnegative integers $p_1, p_3, \ldots$, such that $1p_1 + 3p_3 + \ldots = k$ and $p_1 + p_3 + \ldots = l, 0 \leq l \leq k$.

Identity (2.34) for $m = 1$ follows from Faà di Bruno’s formula for derivatives of the composite function $\psi(b \sin(\cdot/b))$ (see for example [31] or [7]). For $m > 1$, (2.34) can be proved by induction in $m$.

Since $c(k, k) = 1, k \in \mathbb{N}$, by (2.35), estimate (2.33) follows immediately from (2.34). \qed
3. Proof of Theorem 1.2

Throughout the section we use the notation \( \tilde{p} = \min\{1, p\} \), \( p \in (0, \infty] \), introduced in Section 1.

Proof of Theorem 1.2. We first prove the inequality

\[
E_{p, DN, m, V} \leq \liminf_{a \to \infty} \tilde{M}_{p, DN, a, m, V}, \quad p \in (0, \infty]. \tag{3.1}
\]

Let \( f \) be any function from \( BV \cap L_p(\mathbb{R}^m) \), \( p \in (0, \infty] \). Then \( f \in B_{Q^m(M)} \), \( M = M(V) > 0 \), by Lemma 2.7 (a); hence \( D_N(f) \in B_{Q^m(M)} \) by [29 Sect. 3.1] (see also [14 Lemma 2.1 (d)]). In addition, \( f \in L_\infty(\mathbb{R}^m) \) by Nikolskii’s inequality (2.2) and \( D_N(f) \in L_p(\mathbb{R}^m) \) by Bernstein’s and Nikolskii’s inequalities (2.1) and (2.2) and by the “triangle” inequality (1.1). Therefore,

\[
\lim_{|x| \to \infty} D_N(f)(x) = 0, \quad p \in (0, \infty). \tag{3.2}
\]

Indeed, since \( D_N(f) \in B_{Q^m(M)} \cap L_p(\mathbb{R}^m) \), (3.2) is known for \( p \in [1, \infty) \) (see, e.g., [29 Theorem 3.2.5]), and for \( p \in (0, 1) \) it follows from (2.2), since if \( D_N(f) \in L_p(\mathbb{R}^m) \), \( p \in (0, 1) \), then \( D_N(f) \in L_1(\mathbb{R}^m) \).

Let us first prove (3.1) for \( p \in (0, \infty) \). Then by (3.2), there exists \( x_0 \in \mathbb{R}^m \) such that \( \|D_N(f)\|_{L_\infty(\mathbb{R}^m)} = |D_N(f)(x_0)| \). Without loss of generality we can assume that \( x_0 = 0 \). Let \( \tau \in (0, 1) \) be a fixed number. Then using polynomials \( P_a \in \mathcal{P}_{aV}, a \geq 1 \), from Lemma 2.7 we obtain for \( r = \infty \) by (2.28) and (1.1),

\[
\|D_N(f)\|_{L_\infty(\mathbb{R}^m)} = |D_N(f)(0)|
\leq \lim_{a \to \infty} |D_N(f)(0) - D_N(P_a)(0)| + \liminf_{a \to \infty} |D_N(P_a)(0)|
= \lim_{a \to \infty} |D_N(P_a)(0)| \leq \tau^{-(N+m/p)} \liminf_{a \to \infty} \left( \tilde{M}_{p, DN, a, m, V} \|P_a\|_{L_p(Q^m(a\tau))} \right). \tag{3.3}
\]

Using again Lemma 2.7 (for \( \alpha = 0 \) and \( r = p \), we have from (1.1)

\[
\limsup_{a \to \infty} \|P_a\|_{L_p(Q^m(a\tau))} \leq \lim_{a \to \infty} \left( \|f - P_a\|_{L_p(Q^m(a\tau))}^p + \|f\|_{L_p(Q^m(a\tau))}^{p^\tilde{p}} \right)^{1/p} = \|f\|_{L_p(\mathbb{R}^m)}. \tag{3.4}
\]

Combining (3.3) with (3.4), and letting \( \tau \to 1- \), we arrive at (3.1) for \( p \in (0, \infty) \).

In the case \( p = \infty \), for any \( \varepsilon > 0 \) there exists \( x_0 \in \mathbb{R}^m \) such that \( \|D_N(f)\|_{L_\infty(\mathbb{R}^m)} < (1 + \varepsilon) |D_N(f)(x_0)| \). Without loss of generality we can assume that \( x_0 = 0 \). Then similarly to (3.3) and (3.4) we can obtain the inequality

\[
\|D_N(f)\|_{L_\infty(\mathbb{R}^m)} < (1 + \varepsilon) \tau^{-N} \liminf_{a \to \infty} \tilde{M}_{\infty, DN, a, m, V} \|f\|_{L_\infty(\mathbb{R}^m)}. \tag{3.5}
\]

Finally letting \( \tau \to 1- \) and \( \varepsilon \to 0+ \) in (3.5), we arrive at (3.1) for \( p = \infty \). This completes the proof of (3.1).
Furthermore, we will prove the inequality

$$\lim \sup_{a \to \infty} \tilde{M}_{p,D,N,a,m,V} \leq E_{p,D,N,m,V}, \quad p \in (0, \infty],$$

(3.6)

by constructing a nontrivial function $f_0 \in B_V \cap L_p(\mathbb{R}^m)$, such that

$$\lim \sup_{a \to \infty} \tilde{M}_{p,D,N,a,m,V} \leq \|D_N(f_0)(0)\|_{L_\infty(\mathbb{R}^m)}/\|f_0\|_{L_p(\mathbb{R}^m)} \leq E_{p,D,N,m,V}. \quad (3.7)$$

Then inequalities (3.1) and (3.6) imply (1.12). In addition, $f_0$ is an extremal function in (1.12), that is, (1.13) is valid.

It remains to construct a nontrivial function $f_0$, satisfying (3.7). We first note that

$$\inf_{a \geq 1} \tilde{M}_{p,D,N,a,m,V} \geq C_{22}(p,N,D_N,m,V). \quad (3.8)$$

This inequality follows immediately from (3.1). Let $U_a \in P_{aV}$ be a polynomial, satisfying the equality

$$\tilde{M}_{p,D,N,a,m,V} = a^{-N-m/p} |D_N(U_a)(0)| / \|U_a\|_{L_p(Q^m(a))}, \quad a \geq 1. \quad (3.9)$$

The existence of an extremal polynomial $U_a$ in (3.9) can be proved by the standard compactness argument (see, e.g., [18, Proof of Theorem 1.5] and [14, Proof of Theorem 1.3]). Next, setting

$$P_a(x) := U_a(x/a),$$

we have from (3.9) that

$$\tilde{M}_{p,D,N,a,m,V} = |D_N(P_a)(0)| / \|P_a\|_{L_p(Q^m(a))} = 1/\|P_a\|_{L_p(Q^m(a))}, \quad (3.10)$$

since we can assume that

$$|D_N(P_a)(0)| = 1. \quad (3.11)$$

Then it follows from (3.10), (3.11), and (3.8) that

$$\|P_a\|_{L_p(Q^m(a))} = 1/\tilde{M}_{p,D,N,a,m,V} \leq 1/C_{22}(p,N,D_N,m,V).$$

Hence using Lemma 2.9 for $M = a$ and $\tau \in (0, 1)$, we obtain the estimate

$$\sup_{a \geq 1} \|P_a\|_{L_\infty(Q^m(a))} \leq C_{19}/C_{22} = C_{23}(\tau, p, N, D_N, m, V). \quad (3.12)$$

In addition, combining estimates (1.10) for $M = a\tau$ and (3.12), we have for any $s \in \mathbb{Z}_m$:

$$|D^s(P_a)(0)| \leq (A(V)/\tau)^{|s|}C_{23} = C_{24}(\tau, p, N, D_N, m, V, s). \quad (3.13)$$

Furthermore, we define a trigonometric polynomial

$$R_{a,\tau}(t) := P_a(a\tau \sin(t_1/(a\tau)), \ldots, a\tau \sin(t_m/(a\tau))), \quad t \in \mathbb{R}^m.$$  

Then $R_{a,\tau}$ satisfies the following properties:

(P1) $R_{a,\tau} \in B_{(1/\tau)V}$. 

(P2) The following relations hold true:
\[
\sup_{a \geq 1} \| R_{a, \alpha} \|_{L^\infty(Q^m(\alpha \pi/2))} = \sup_{a \geq 1} \| R_{a, \alpha} \|_{L^\infty(\mathbb{R}^m)} \leq C_{23}. \quad (3.14)
\]

(P3) For \( \alpha \in \mathbb{Z}_+^m \) and \( \alpha \geq 1 \),
\[
|D^\alpha(R_{a, \alpha})(0) - D^\alpha(P_a)(0)| \leq C_{21} \max_{0 \leq s_j \leq \alpha_j, 1 \leq j \leq m} C_{24}(\tau, p, N, D_N, m, V, s)/(\alpha)
= C_{25}(\tau, p, N, D_N, m, V, \alpha)/a. \quad (3.15)
\]

(P4) For \( a \geq 1, p \in (0, \infty) \), and \( m \in (a \tau/\sqrt{m}] \),
\[
\| P_a \|_{L_p(Q^m(a))} \geq (1 - m M^2 (\alpha \tau)^{-2})^{1/p} \| R_{a, \alpha} \|_{L_p(Q^m(M))}. \quad (3.16)
\]

Indeed, property (P1) follows from Lemma 2.10 (a), while (P2) is an immediate consequence of (3.12). Next, property (P3) follows from Lemma 2.10 (b) and relations (3.13). To prove (P4), we note first that for \( p = \infty \) inequality (3.16) is trivial. Next, setting \( f(\cdot) = \cos(\cdot) \) and replacing \( R_{a_j}, 1 \leq j \leq m, \) with 1 in identity (2.7), we obtain
\[
1 - \prod_{j=1}^m \cos(t_j x_j) = \sum_{j=1}^m (1 - \cos(t_j x_j)) \prod_{k=j+1}^m \cos(t_k x_k) \leq (1/2) \sum_{j=1}^m (t_j x_j)^2. \quad (3.17)
\]

Furthermore, for \( a \geq 1, p \in (0, \infty) \), and \( M \in (0, a \tau/\sqrt{m}] \),
\[
\| P_a \|_{L_p(Q^m(a))}^p = \| P_a \|_{L_p(Q^m(\alpha \tau))}^p \int_{Q^m(\alpha \pi/2)} \left| R_{a, \alpha \tau}(t) \right|^p \prod_{j=1}^m \cos(t_j/(\alpha \tau)) dt
\]
\[
\geq \| R_{a, \alpha \tau} \|_{L_p(Q^m(M))}^p - \int_{Q^m(M)} \left| R_{a, \alpha \tau}(t) \right|^p \left( 1 - \prod_{j=1}^m \cos(t_j/(\alpha \tau)) \right) dt. \quad (3.18)
\]

Finally using estimate (3.17) for \( x_j = 1/(\alpha \tau), 1 \leq j \leq m \), we obtain (3.16) from (3.18).

Let \( \{a_n\}_{n=1}^\infty \) be an increasing sequence of numbers such that \( \inf_{n \in \mathbb{N}} a_n \geq 1 \), \( \lim_{n \to \infty} a_n = \infty \), and
\[
\lim_{a \to \infty} \| \tilde{M}_{p, D_N, a, m, V} \| = \lim_{n \to \infty} \| \tilde{M}_{p, D_N, a_n, m, V} \|. \quad (3.19)
\]

Property (P1) and relation (3.14) of property (P2) show that the sequence of trigonometric polynomials \( \{R_{a_n, a_n \tau}\}_{n=1}^\infty = \{f_n\}_{n=1}^\infty \) satisfies the conditions of Lemma 2.1 (c) with \( B_V \) replaced by \( B(1/\tau)V \). Therefore, there exist a subsequence \( \{R_{a_n, a_n \tau}\}_{d=1}^\infty \) and a function \( f_0, \tau \in B(1/\tau)V \) such that
\[
\lim_{d \to \infty} R_{a_n, a_n \tau} = f_0, \tau, \quad \lim_{d \to \infty} D_N \left( R_{a_n, a_n \tau} \right) = D_N(f_0, \tau), \quad (3.20)
\]
uniformly on any cube \( Q^m(M), M > 0 \).
Moreover, by \((3.11), (3.15),\) and \((3.20),\)

\[
|D_N(f_0, \tau)(0)| = \lim_{d \to \infty} |D_N\left( R_{a_{n_d}} \right)(0)| = \lim_{d \to \infty} |D_N\left( P_{a_{n_d}} \right)(0)| = 1. \tag{3.21}
\]

In addition, using \((1.1), (3.22), (3.16), (3.10),\) and \((3.19),\) we obtain for any cube \(Q^m(M), M > 0,\)

\[
\|f_0, \tau\|_{L_p(Q^m(M))} \leq \lim_{d \to \infty} \left( \|f_0, \tau - R_{a_{n_d}}\|_{L_p(Q^m(M))} + \|R_{a_{n_d}}\|_{L_p(Q^m(M))} \right)^{1/p}
\]

\[
= \lim_{d \to \infty} \left| R_{a_{n_d}} \right|_{L_p(Q^m(M))} \leq \lim_{d \to \infty} \|P_{a_{n_d}}\|_{L_p(Q^m(M))} = 1/ \lim_{d \to \infty} \tilde{M}_{p, D_N, a_{n_d}, m, V}. \tag{3.22}
\]

Next using \((3.22)\) and \((3.8),\) we see that

\[
\|f_0, \tau\|_{L_p(\mathbb{R}^m)} \leq 1/C_{22}(p, N, D_N, m, V). \tag{3.23}
\]

Therefore, \(f_0, \tau\) is a nontrivial function from \(B_{(1/\tau)} \cap L_p(\mathbb{R}^m),\) by \((3.23)\) and \((3.24).\) Thus for any cube \(Q^m(M), M > 0,\) we obtain from \((3.19), (3.10), (3.16), (3.20),\) and \((3.21),\)

\[
\limsup_{a \to \infty} \tilde{M}_{p, D_N, a_{n_d}, V} = \lim_{d \to \infty} \left( \|P_{a_{n_d}}\|_{L_p(Q^m(M))} \right)^{-1}
\]

\[
\leq \lim_{d \to \infty} \left( \left| R_{a_{n_d}} \right|_{L_p(Q^m(M))} \right)^{-1}
\]

\[
= |D_N(f_0, \tau)(0)| / \|f_0, \tau\|_{L_p(Q^m(M))}. \tag{3.24}
\]

It follows from \((3.24)\) that

\[
\limsup_{a \to \infty} \tilde{M}_{p, D_N, a_{n_d}, V} \leq E_{p, D_N, m, (1/\tau)V} = \tau^{-N - m/p} E_{p, D_N, m, V}. \tag{3.25}
\]

Then letting \(\tau \to 1^-\) in \((3.25),\) we arrive at \((3.6).\) However, we need to prove stronger relations \((3.5).\)

To construct \(f_0,\) note first that \(f_0, \tau(\tau) \in B_V\) and by \((3.23)\) and \((2.2),\)

\[
\sup_{\tau \in (1/2, 1)} \|f_0, \tau\|_{L_\infty(\mathbb{R}^m)} = \sup_{\tau \in (1/2, 1)} \|f_0, \tau(\tau)\|_{L_\infty(\mathbb{R}^m)} \leq C \sup_{\tau \in (1/2, 1)} \tau^{-m/p} \|f_0, \tau\|_{L_p(\mathbb{R}^m)} < \infty.
\]

Therefore, by Lemma \((2.1)\) (c) applied to a sequence \(\{f_{0, \tau_n}\}_{n=1}^\infty,\) where \(\tau_n \in (1/2, 1), n \in \mathbb{N},\) and \(\lim_{n \to \infty} \tau_n = 1,\) there exist a subsequence \(\{f_{0, \tau_{n_d}}\}_{d=1}^\infty\) and a function \(f_0 \in B_V \cap L_\infty(\mathbb{R}^m) = \bigcap_{d=1}^\infty \left( B_{(1/\tau_{n_d})} \cap L_\infty(\mathbb{R}^m) \right)\) such that for every \(\alpha \in \mathbb{Z}_+^m,\lim_{d \to \infty} D^\alpha \left( f_{0, \tau_{n_d}} \right) = D^\alpha(f_0)\) uniformly on any compact set in \(\mathbb{C}^m.\)
Note that by (3.21) and (3.23), \( f_0 \) is a nontrivial function from \( B_V \cap L_p(\mathbb{R}^m) \). Then using (3.24), we obtain

\[
\limsup_{a \to \infty} M_{p,D_{N,a,m,V}} \leq \lim_{M \to \infty} \lim_{n \to \infty} |D_N(f_0,\tau_n)(0)| / \|f_0,\tau_n\|_{L_p(Q^m(M))} \\
= |D_N(f_0)(0)| / \|f_0\|_{L_p(\mathbb{R}^m)} \\
\leq \|D_N(f_0)\|_{L_{\infty}(\mathbb{R}^m)} / \|f_0\|_{L_p(\mathbb{R}^m)} \\
\leq E_{p,D_{N,m,V}}.
\]

Thus (3.7) holds true, and this completes the proof of the theorem. \( \square \)

**Remark 3.1.** The proof of (3.1) is all but identical to the proof of the inequality \( E_{p,D_{N,m,V}} \leq \liminf_{n \to \infty} M_{p,D_{N,n,m,V}} \) from [16] though these proofs are based on different lemmas. However, the proof of (3.6) is different compared with the proof of the inequality \( \limsup_{n \to \infty} M_{p,D_{N,n,m,V}} \leq E_{p,D_{N,N,m,V}} \) from [16]. The latter proof is based on V. A. Markov-type inequalities for polynomials from \( P_{O^m(n)} \). We do not know if there are analogues of these inequalities for polynomials from \( P_{a \Gamma} \) but the case of \( V = \Pi^m(\sigma) \), see (1.8). That is why, (3.6) is reduced to certain relations for trigonometric polynomials (cf. [14]).

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