The Vlasov-Poisson and Vlasov-Poisson-Fokker-Planck systems in stochastic electromagnetic fields: local well-posedness

Jacob Bedrossian* Stavros Papathanasiou†

November 8, 2022

Abstract

In this paper, we construct unique, local-in-time strong solutions to the Vlasov-Poisson (VP) and Vlasov-Poisson-Fokker-Planck (VPFP) systems subjected to external, spatially regular, white-in-time electromagnetic fields in $\mathbb{T}^d \times \mathbb{R}^d$. Initial conditions are taken $H^\sigma$ with $\sigma > d/2 + 1$ (in addition to polynomial velocity weights). We additionally show that solutions to the VPFP are instantly $C^\infty_{x,v}$ due to hypoelliptic regularization if the external force fields are smooth. The external forcing arises in the kinetic equation as a stochastic transport in velocity, which means, together with the anisotropy between $x$ and $v$ in the nonlinearity, that the local theory is a little more complicated than comparable fluid mechanics equations subjected to either additive stochastic forcing or stochastic transport. Although stochastic electromagnetic fields are often discussed in the plasma physics literature, to our knowledge, this is the first mathematical study of strong solutions to nonlinear stochastic kinetic equations.

Contents

1 Introduction 1
2 Outline 5
3 Very smooth solutions and pathwise uniqueness 10
4 Proof of main theorem 29
5 Hypoelliptic regularization for Vlasov-Poisson-Fokker-Planck 34

1 Introduction

In this paper we prove the local-in-time existence and uniqueness of (probabilistically strong) solutions of the Vlasov and Vlasov-Fokker-Planck equations for a distribution of charged particles subjected to a stochastic external electric field

\[
\begin{aligned}
\frac{df}{dt} + v \cdot \nabla_x f dt + E \cdot \nabla_v f dt + \nabla_v f \otimes dW_t &= \nu \nabla_v \cdot (\nabla_v f + f v) dt \\
\rho &= \int f dv \\
E &= \nabla_x (-\Delta_x)^{-1}(\rho - 1) \\
\rho(t,x) &= \int_{\mathbb{R}^d} f(t,x,v) dv,
\end{aligned}
\]  

\tag{1.1}

*Department of Mathematics, University of Maryland, College Park, MD 20742, USA jacob@math.umd.edu. Both J.B. and S.P. were supported by National Science Foundation Award DMS-2108633.

†Department of Mathematics, University of Maryland, College Park, MD 20742, USA stavrosp@math.umd.edu
where \( \nu \geq 0 \) is the collision frequency – we treat both the case \( \nu > 0 \) and \( \nu = 0 \) (i.e. the Vlasov–Poisson equations). Below we denote the Fokker-Planck operator

\[
\mathcal{L} f := \Delta_v f + \nabla_v \cdot (v f),
\]

which is a commonly used simplification for collisions of charged particles against a background (see e.g. [7]).

Here, we consider the problem in the periodic box \( (x,v) \in \mathbb{T}^d \times \mathbb{R}^d \), although the case \( (x,v) \in \mathbb{R}^d \times \mathbb{R}^d \) could be approached with similar arguments. The process \( W_t \) is a white-in-time, colored-in-space, vector-valued Gaussian process which plays the role of an external fluctuating electric field which we describe in more detail below. Our analysis works for general \( 1 \leq d \leq 3 \) and also applies to external magnetic fields. For simplicity, we will take initial conditions in the velocity-weighted Sobolev space \( H^m_\sigma \) defined by the norm:

\[
\| f \|_{H^m_\sigma}^2 := \sum_{|\alpha| + |\beta| \leq \sigma} \int_{\mathbb{T}^d \times \mathbb{R}^d} |\partial^\alpha_x \partial^\beta_v f(x,v)|^2 (1 + |v|^2)^{m/2} \, dv \, dx.
\]

Stochastic and randomly fluctuating electromagnetic fields are a classical topic in the plasma physics literature where they are used as a model for studying various dynamics in “turbulent”-like situations in both confined fusion and astrophysical applications. Much of the work is on studying the motion of charged particles (i.e. Lagrangian trajectories or passive scalars) subjected to stochastic electromagnetic fields of various kinds; see e.g. [2, 31–33, 38, 39, 55, 59, 60] and the references therein for a tiny fraction of the existing work on the topic. Another line of work regards subjecting gyrokinetic equations or other macroscopic models to randomly fluctuating external force fields of this type for the purpose of studying plasma turbulence; see e.g. [47, 53, 54] and the references therein. The purpose of this work is to begin laying down some rigorous mathematical theory for studying nonlinear kinetic theory models of plasmas in these kinds of settings.

It is sometimes useful to make a concrete representation of \( W_t \) and for simplicity we will show how to do this in \( d = 3 \); the extension to other dimensions is straightforward and is omitted. To make this concrete representation of \( W_t \), we define a real Fourier basis on \( L^2(\mathbb{T}^3; \mathbb{R}^3) \) by defining for each \( k = (\ell, i) \in \mathbb{K} := \mathbb{Z}_0^d \times \{1, 2, 3\} \)

\[
e_k(x) = \begin{cases} c_d \gamma^i_\ell \sin(\ell \cdot x), & \ell \in \mathbb{Z}_+^d \\
c_d \gamma^i_\ell \cos(\ell \cdot x), & \ell \in \mathbb{Z}_-^d,
\end{cases}
\]

where \( \mathbb{Z}_0^d := \mathbb{Z}^d \setminus \{0, \ldots, 0\} \), \( \mathbb{Z}_+^d = \{ \ell \in \mathbb{Z}_0^d : \ell^{(d)} > 0 \} \cup \{ \ell \in \mathbb{Z}_0^d : \ell^{(1)} > 0, \ell^{(d)} = 0 \} \) and \( \mathbb{Z}_-^d = -\mathbb{Z}_+^d \), and for each \( \ell \in \mathbb{Z}_0^d \), \( \{\gamma^i_\ell\}^3_{i=1} \) is a set of three orthonormal vectors with \( \{\gamma^1_\ell, \gamma^2_\ell, \gamma^3_\ell\} \) spanning the plane perpendicular to \( \ell \in \mathbb{R}^3 \) with the property that \( \gamma^1_{-\ell} = -\gamma^1_\ell \) and \( \gamma^2_{\ell} \) parallel to \( \ell \). The constant \( c_d = \sqrt{2/(2\pi)}^{-d/2} \) is a normalization factor so that \( e_k(x) \) are a complete orthonormal basis on \( L^2 \). With this, we define our external electric field as

\[
W_t(x) = \sum_{k \in \mathbb{K}} \sigma_k e_k(x) W_t^{(k)},
\]

with \( \{W_t^{(k)}\}_{k \in \mathbb{K}} \) is a family of independent standard Wiener processes defined on a given stochastic basis \((\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})\). The \( \sigma_k \) are coloring coefficients satisfying at least \( \sum_{k \in \mathbb{K}} |\sigma_k|^2 < \infty \), however, more stringent regularity requirements will be assumed below (here we make the natural definition \( |k| = |\ell| \) for \( k = (\ell, i) \in \mathbb{K} \)). We can also treat the case of fluctuating magnetic fields; see Remark 1.4 below.

Local well-posedness of strong solutions for the deterministic problem is classical; see e.g. [35, 36] for the Vlasov equations and [48, 56] for the Vlasov-Fokker-Planck equation. Global existence for the
In this case, we say such pathwise solutions are **unique** and for a stopping time and $f$ subelliptic, and so stochastic transport enjoys a special structure that makes it possible to develop a strong evolution, as the Stratonovich-Itô correction term is of second order. However, this correction term is so it is clear that stochastic transport cannot be treated perturbatively with respect to the deterministic well-posedness theory, and in fact, it is sometimes possible to produce a better well-posedness theory for stochastic transport than for deterministic transport [28]. Due to this special structure and the many physical applications, there have been a great number of works studying stochastic transport equations recently; see for example [4, 13, 24, 25, 46] and also [20, 26] in the kinetic case.

There have been many works on fluid equations subjected to multiplicative or transport-type stochastic forcing. For the Navier-Stokes equations see for example [8, 10, 12, 44, 45]. The 2D Euler equations in vorticity form subjected to transport noise was studied in, for example [9, 16], where strong solutions with bounded vorticity were constructed (see also [15]). The aforementioned papers [15, 16] belong to the so-called theory of “Stochastic Advection by Lie Transport” (SALT), see the foundational paper [34], as well as [1, 17]. The work [11] studies the 3D primitive equations with transport noise.

The works [21, 22] provide a fairly general framework to study a wide class of dissipative fluid equations forced with multiplicative noise, such as the Navier-Stokes equations or the primitive equations. For the 2D Euler equations with various types of general multiplicative noise, see [29] and the references therein.

In comparison to stochastic fluid dynamics, the work on nonlinear, stochastically forced kinetic equations is significantly thinner. The paper [50] constructs global-in-time renormalized martingale (probabilistically weak) solutions of the Boltzmann equations with external stochastic forcing similar to that used in (1.1). In work with a clear relationship with our own, [23] constructs global solutions of interacting point charges (i.e. Vlasov–Poisson with solutions given by a finite number of Dirac masses) subjected to stochastic external electric fields; see also [14].

In this paper we continue the study of stochastic kinetic theory by proving local existence and pathwise uniqueness of strong solutions. Let us recall some standard notions for probabilistically strong solutions of SPDEs that may experience finite-time blow up (we follow the presentation used in [22, 29]), which are nothing more than the natural stochastic analogues of the deterministic notions of local-in-time existence, uniqueness, and maximally-extended solutions.

**Definition 1.1.** A local pathwise solution of (1.1) is a pair $(f, \tau)$ with $\tau$ an almost-surely strictly positive stopping time and $f$ an adapted stochastic process satisfying the regularity

$$f(\cdot \land \tau) \in C([0, \infty); H^σ_m)$$

and for $t \geq 0$ satisfies,

$$f(t \land \tau) - f(0) + \int_0^{t \land \tau} (v \cdot \nabla_x f(s) + E(s) \cdot \nabla_v f - \nu \mathcal{L} f(s)) \, d\tau = \int_0^{t \land \tau} \nabla_v f(s) \circ dW_s.$$

Moreover, we say such pathwise solutions are **unique** if for any pair $(f_1, \tau_1), (f_2, \tau_2)$ we have

$$\mathbb{P}(f_1(t) - f_2(t) = 0 \ \forall 0 \leq t < \tau_1 \land \tau_2 | f_1(0) = f_2(0)) = 1.$$

In this case, $f_1$ and $f_2$ are called **indistinguishable**.
The following definition of maximal pathwise solution provides a continuation criterion for strong solutions. For this we will use $H^{d/2+1+}$, sharper continuation criteria will be considered in future work. That is, we show that local $H^m$ solutions can be uniquely extended provided some $H^m_0$ norm remains finite for $s_0 > 1 + d/2$ and $m_0 > d$ fixed and arbitrary.

**Definition 1.2.** Fix $s_0 > d/2 + 1$ an integer and $m_0 > d$. We call a maximal pathwise solution a triple of a solution $f$, an increasing sequence of almost-surely positive stopping times $\{\tau_n\}_{n \geq 0}$, and a limiting stopping time $\xi$ such that each pair $(f, \tau_n)$ is a local pathwise solution, $\lim_{n \to \infty} \tau_n = \xi$, and

$$\sup_{0 \leq t \leq \tau_n} \|f(t)\|_{H^{s_0}_0} \geq n$$
onumber

on the set $\{\xi < \infty\}$.

In this paper, we prove the following local existence and uniqueness theorem. Global existence of these strong solutions will be considered in a future work.

**Theorem 1.3.** Let $1 \leq d \leq 3$. Let $\sigma > s_0$ and $m > m_0$ be fixed integers and assume that

$$\sum_{k \in \mathbb{K}} |k|^{2\sigma'} |\sigma_k|^2 < \infty$$

for some $\sigma' > \sigma + 4$ (integer). Suppose that the initial condition $f_0$ is an $\mathcal{F}_0$-measurable random variable such that $f_0 \in H^m_0$ almost-surely. Then, there exists a unique, maximal pathwise solution to (1.1) for any $\nu \geq 0$.

**Remark 1.4.** Our proof also applies when there is a stochastic magnetic field. We may similarly treat the case of independent electric and magnetic fields as the following, for example (denoting $c > 0$ the speed of light),

$$W_t(x) \mapsto \sum_{k \in \mathbb{K}} \sigma_k^{(E)} e_k(x) W_t^{k;E} + \frac{1}{c} \nu \sum_{\ell \in \mathbb{Z}^3_0} \sigma_{\ell}^{(B;1)} e_{(\ell,1)}(x) W_t^{(\ell,1);B} + \frac{1}{c^2} \nu \sum_{\ell \in \mathbb{Z}^3_0} \sigma_{\ell}^{(B;2)} e_{(\ell,2)}(x) W_t^{(\ell,2);B},$$

with

$$\sum_{k=(\ell,j) \in \mathbb{K}} |k|^{2\sigma'} \left( |\sigma_k^{(E)}|^2 + |\sigma_{\ell}^{(B;1)}|^2 + |\sigma_{\ell}^{(B;2)}|^2 \right) < \infty,$$

or when electromagnetic fields are correlated, for example one could use forcing of the following potentially natural form

$$W_t \mapsto \sum_{\ell \in \mathbb{Z}^3_0} \sigma_{\ell}^{(1)} e_{(\ell,1)}(x) + \frac{v \times e_{(\ell,2)}(x)}{c} W_t^{(\ell,1)} + \sum_{\ell \in \mathbb{Z}^3_0} \sigma_{\ell}^{(2)} e_{(\ell,2)}(x) + \frac{v \times e_{(\ell,1)}(x)}{c} W_t^{(\ell,2)}.$$

Sufficiently regular-in-space deterministic external electromagnetic fields or random fields that are smoother in time than white noise (for example, Ornstein-Uhlenbeck processes and variations thereof as in the Langevin antenna forcing used in the plasma physics literature \[53\]) can also be easily included in the analysis without any significant changes. For simplicity of presentation, we will mainly focus on the case of external electric fields and simply make comments about what changes when considering an additional magnetic field.
Remark 1.5. The methods of this paper can also deal with with more general mean-field interactions, replacing the self-consistent electric field with:

\[ E = \nabla_x K * (\rho - 1), \]

for any kernel \( K \) such that \( \left\| \nabla_x^{j+1} E \right\|_{L^p} \lesssim \left\| \langle \nabla_x \rangle^j \rho \right\|_{L^p} \) for all \( 1 < p < \infty \).

Remark 1.6. It should be straightforward to extend to \( d \geq 4 \). It should also be possible to treat non-integer \( \sigma, s_0, \) and \( \sigma' \), however, this would require more delicate (anisotropic) commutator estimates.

Remark 1.7. We believe our methods could be extended to the Landau collision operators for initial data \( f_0 \) sufficiently close to a global Maxwellian to prove local-in-time existence and uniqueness of strong solutions to e.g. the Vlasov–Poisson–Landau equations with stochastic external electromagnetic fields. This extension may be considered in future work.

Remark 1.8. In light of the classical deterministic theory of bounded solutions of the Vlasov equations (see e.g. [41, 42]), it is natural to expect an analogue of [9] in kinetic theory. Similarly, we expect local (and global) existence and uniqueness of the Vlasov-Fokker-Planck equations using only e.g. \( f_0 \in L^2_{m_0} \). These extensions may be considered in future work.

Finally, in Section 5 we present a proof of the following hypoelliptic regularization result. This is proved using a time-dependent hypoellipticity norm in the spirit of [58].

**Theorem 1.9.** Let \( f \) be a maximal pathwise solution to (1.1) as in Theorem 1.3. Suppose that for all \( N \) there holds

\[ |\sigma_k| \lesssim_N |k|^{-N}. \]

Then if \( \nu > 0 \), then \( f(t) \in C^\infty_{x,v} \) for all \( t \in (0, \xi) \).

2 Outline

Let us outline the general idea of how to prove Theorem 1.3 and then provide the details in the main body of the text. See Section 5 for how to prove Theorem 1.9.

As in e.g. [22, 29], we first construct solutions to (1.1) with smoother initial data \( H^\sigma_{m_0} \) with \( \sigma' > \sigma + 4 \) and \( m' > m + 3 \) (both integers) with trajectories in \( L^{\infty}_{t,loc} H^\sigma_{m'} \cap C_{t,loc} H^\sigma_m \). This procedure is done in Section 3. Then we regularize the initial condition and pass to the limit to obtain solutions with initial data in \( H^\sigma_m \) that take values in \( C_{t,loc} H^\sigma_m \). In addition to obtaining solutions with lower regularity, what is more important for many purposes, is that this constructs solutions which take values continuously in the highest regularity available. This latter procedure is done in Section 4.

To construct maximal pathwise solutions to (1.1) we first introduce a standard trick for regularizing the nonlinearity in a way which allows to close necessary probabilistic moment estimates. Consider a smooth nonnegative and nonincreasing function \( \theta : [0, \infty) \to \mathbb{R} \) such that:

\[ \theta(x) = \begin{cases} 
1 & \text{if } x \leq 1, \\
0 & \text{if } x \geq 2,
\end{cases} \]

and define:

\[ \theta_R(x) = \theta \left( \frac{x}{R} \right). \]

Then we define the regularized SPDE

\[ df + v \cdot \nabla_x f dt + \theta_R(\|f\|_{H^\sigma_{m_0}}) E \cdot \nabla_v f dt + \nabla_v f \odot dW_t = \nu L f dt, \quad (2.1) \]
We show in Section 3 that this SPDE admits global-in-time, unique, pathwise solutions (i.e. \( \xi = \infty \) with probability 1 in the definition of maximal solutions) starting from \( H_m^{\sigma_m'} \) initial conditions. Specifically, we prove the following.

**Lemma 2.1.** Let \( f_0 \) be a \( \mathcal{F}_0 \)-measurable random variable such that \( \forall p \geq 2 \),

\[
E \|f_0\|_{H_m^{\sigma_m'}}^p < \infty.
\]

Then, there exists an \( f \in C([0, \infty); H_m^{\sigma_m'-4}) \) \( \cap \mathcal{L}_{t,loc}([0, \infty); H_m^{\sigma_m'-1}) \) \( \mathbb{P} \)-a.s. which is a solution to (2.1) in the sense that

\[
f(t) = f_0 + \int_0^t \left( -v \cdot \nabla_x f(s) - \theta_R(\|f\|_{H_m^{\sigma_m'}}) E(s) \cdot \nabla_v f(s) + \nu \mathcal{L} f(s) \right) ds - \int_0^t \nabla_v f(s) \circ dW_s. \quad \mathbb{P} \text{–a.s.,}
\]

where the equality holds in \( C([0, \infty); H_m^{\sigma_m'-4}) \). Moreover, if \( \tilde{f} \) is any other solution in the above sense, then \( f = \tilde{f} \) almost surely in the sense that

\[
\mathbb{P} \left( f(t) - \tilde{f}(t) = 0 \quad \forall 0 \leq t < \infty \right) = 1.
\]

It is clear that solutions of (2.1) are also solutions to (1.1) for as long as \( \|f\|_{H_m^{\sigma_m}} < R \), and so by considering the increasing sequence of \( R = n \) and defining the stopping times

\[
\tau_n = \inf \left\{ t \geq 0 : \|f(t)\|_{H_m^{\sigma_m}} > n \right\},
\]

we may use (2.1) to construct local pathwise solutions to (1.1). A standard cutting procedure (described below) also shows how to remove the moment requirement on the initial data.

**Lemma 2.2.** Let \( f_0 \) be an \( \mathcal{F}_0 \)-measurable random variable with \( f_0 \in H_m^{\sigma_m'} \) almost surely. Then, Lemma 2.1 implies the pathwise existence and uniqueness of a maximal solution \((f, \tau)\) to (1.1) with initial data \( f_0 \) with trajectories \( f \) satisfying

\[
f(\cdot \wedge \tau) \in \mathcal{L}_{t,loc}([0, \infty); H_m^{\sigma_m'-1}) \cap \mathcal{C}_{t,loc}([0, \infty); H_m^{\sigma_m'-4}).
\]

**Proof.** First consider the case that \( \|f_0\|_{H_m^{\sigma_m'}} < M \) almost-surely. Then, we choose \( R = M + 1 \) in (2.1), and define the stopping time:

\[
\tau = \inf\{ t \geq 0 : \|f(t)\|_{H_m^{\sigma_m}} > R \},
\]

where \( f \) is the solution to (2.1) with initial data \( f_0 \), guaranteed to exist and be unique from Lemma 2.1. Note that up to time \( \tau \), the process \( f \) also solves (1.1), since for \( t < \tau \) we have \( \|f(t)\|_{H_m^{\sigma_m}} \leq \|f_0\|_{H_m^{\sigma_m}} \leq R \) and therefore \( \theta_R(\|f(t)\|_{H_m^{\sigma_m}}) = 1 \). Clearly \( \tau > 0 \) almost surely since \( H_m^{\sigma_m'-2} \subset H_m^\sigma \) and \( f \) takes values continuously in \( H_m^{\sigma_m'-2} \). The pair \((f, \tau)\) is thus a local solution of (1.1) within the higher regularity framework of this lemma, which is unique by Lemma 3.13 below. Now we will extend \( f \) to a maximal solution.

Let \( \mathcal{T} \) be the collection of all stopping times corresponding to a local solution and define \( \xi = \sup \mathcal{T} \). Define also:

\[
\tau_n := \inf\{ t \geq 0 : \|f(t)\|_{H_m^{\sigma_m}} > n \}.
\]

Fix \( T > 0 \) finite but arbitrary and assume \( \mathbb{P}(\xi = \tau_n \wedge T) > 0 \) for some \( n \). This implies that

\[
\sup_{t \leq \xi} \|f(t)\|_{H_m^{\sigma_m}} \leq n \text{ on the set } \{ \xi = \tau_n \wedge T \},
\]

We denote this maximal solution by \( \xi = \tau_n \wedge T \) and define a new initial condition \( f_0 \) such that \( \|f_0\|_{H_m^{\sigma_m'}} < \frac{M}{2} \). We can then repeat the above procedure to construct a maximal solution with initial data \( f_0 \) almost surely in \( C_{t,loc}([0, \infty); H_m^{\sigma_m'-4}) \).
and thus $f$ can be continued to a solution of (2.1) with $R = n + 1$ and thus of (1.1) up to a stopping time past $\xi$ - contradicting $\xi$'s maximality. Since $T$ was arbitrary, we either have $\xi = \infty$, or $\tau_n < \xi < \infty$ for all $n$. In the latter case, we also get $\sup_{t < \xi} \|f(t)\|_{H^\alpha_{m_0}} \geq n$ for all $n$ and thus $\sup_{t < \xi} \|f(t)\|_{H^\alpha_{m_0}} = \infty$.

Now we drop the almost-sure uniform boundedness requirement. If $\|f_0\|_{H^\alpha_m} < \infty$ almost surely, we decompose $f_0 = \sum_{k=0}^{\infty} f_{0,k}$, where $f_{0,k} := 1_{\{k \leq \|f_0\|_{H^\alpha_m} < k+1\}} f_0$. Now each $f_{0,k}$ generates a maximal solution $(f_k, \tau_k)$ where $\tau_k$ is the corresponding maximal existence time, and we define the “total” maximal solution (in high regularity) of (1.1) as $(\tilde{f}, \tilde{\tau})$ with:

$$
\tilde{f} = \sum_{k=0}^{\infty} 1_{\{k \leq \|f_0\|_{H^\alpha_m} < k+1\}}(\omega) f_k,
$$

$$
\tilde{\tau} = \sum_{k=0}^{\infty} 1_{\{k \leq \|f_0\|_{H^\alpha_m} < k+1\}}(\omega) \tau_k.
$$

\[\square\]

Solutions to (2.1) are constructed using a two-step procedure. First, we regularize the nonlinearity again and use an iteration procedure to construct a solution to the regularized SPDE and then second, we pass to the limit in the additional regularization parameter. Let $\varphi \in C^\infty_c(B(0,2))$ with \(\int_{\mathbb{R}^d} \varphi \, dx = 1\) and define $\varphi_\epsilon = e^{-d} \varphi(\epsilon^{-1} \cdot)$. Specifically, we seek a solution to the following regularized SPDE (here the convolution in $x$ has been periodized),

$$
d\tilde{f} + v \cdot \nabla_x \tilde{f} \, dt + \nabla_v \tilde{f} \circ dW_t + \theta_R\left( \|\tilde{f}\|_{H^\alpha_{m_0}} \right)(\varphi_\epsilon \ast \tilde{E}) \cdot \nabla_v \tilde{f} = \nu \mathcal{L} \tilde{f} \, dt
$$

(2.2)

$$
\tilde{E} = \nabla_x (-\Delta_x)^{-1} \left( \int_{\mathbb{R}^d} \tilde{f}(t, \cdot, v) \, dv - 1 \right).
$$

This is done by an iteration method, specifically the following

$$
f^0 = f_0
$$

$$
df^0 + v \cdot \nabla_x f^0 \, dt + \nabla_v f^0 \circ dW_t = \nu \mathcal{L} f^0 \, dt
$$

(2.3)

$$
df^{j+1} + v \cdot \nabla_x f^{j+1} \, dt + \theta_R\left( \|f^j\|_{H^\alpha_{m_0}} \right)(\varphi_\epsilon \ast E[f^j]) \cdot \nabla_v f^{j+1} \, dt + \nabla_v f^{j+1} \circ dW_t = \nu \mathcal{L} f^{j+1} \, dt
$$

(2.4)

where we denote

$$
E[f] := \nabla_x (-\Delta_x)^{-1} \left( \int_{\mathbb{R}^d} f(t, \cdot, v) \, dv - 1 \right).
$$

The solutions to (2.3) - (2.4) are constructed by the method of characteristics. Indeed, (2.4) is the forward Kolmogorov equation associated to the SDE

$$
\begin{cases}
    dX_t = V_t \, dt \\
    dV_t = -\nu V_t \, dt + \theta_R(\|f^j\|_{H^\alpha_{m_0}})(\varphi_\epsilon \ast E[f^j])(t, X_t) \, dt + \sqrt{2\nu} d\tilde{W}_t + \sum_k \sigma_k e_k(X_t) \circ dW_t^{(k)},
\end{cases}
$$

(2.5)

where $(\tilde{W}_t)$ is a $d$-dimensional Brownian motion defined in a new stochastic basis $(\Omega', \mathcal{F}', P')$ (independent of the original basis). That is, (2.5) are the stochastic characteristics corresponding to (2.4), which generates a global stochastic flow of volume-preserving diffeomorphisms $\phi_t$ on $\mathbb{T}^d \times \mathbb{R}^d$, defined on the product space

$$(\Omega \times \Omega', \mathcal{F} \otimes \mathcal{F}', P \times P'),$$

7
which map $H_{m'}^{α'}$ back to itself for all finite times almost-surely. The multiplicative (linear!) SPDE (2.4) is then solved by a “partial Feynman-Kac” formula with respect to $P'$:

$$f^{j+1} = E_{P'} f_0 \circ \phi_t^{-1}.$$ 

See [40] for more details.

**Remark 2.3.** Note that this type of regularization procedure has the added benefit of retaining non-negativity of $f$ as well as the preservation of the Casimir conservation laws, e.g. if $ν = 0$ then $\|f^j\|_{L^p} = \|f_0\|_{L^p}$ and for $ν > 0$ one at least has $\|f^j\|_{L^p} \leq e^{d_{out}} \|f_0\|_{L^p}$. However, these properties do not play an important role here.

Next, we need uniform a priori estimates to enable passing $j \to \infty$. These are obtained via Eulerian energy methods and come out as $\forall p < \infty$, $α \in (0, 1/2)$, and $T < \infty$,

$$\begin{align*}
\sup_{j \geq 1} E \|f^j\|^p_{L^\infty(0,T;H_{m'}^{α'})} & \lesssim_{p,T,ε} 1 \\
\sup_{j \geq 1} E \|f^j\|^p_{W^{α,p}(0,T;H_{m'^2-1}^{α'})} & \lesssim_{p,T,ε,α} 1.
\end{align*}$$

See Lemma 3.1 for the proof of these estimates. Several previous works, for example [11, 22, 29] have used compactness to pass to similar limits, extract martingale solutions (i.e. probabilistically weak) using the Skorokhod embedding theorem, and then subsequently upgrade these solutions using a Gyöngy-Krylov lemma [30] argument and pathwise uniqueness. However, this technique seems not to apply in a clear manner to the Lagrangian iteration (2.4). Instead, we prove directly that there is a stopping time $ξ$ which is almost-surely greater than 1 such that $\{f^j\}_{j \geq 0}$ forms a Cauchy sequence in $L^2(Ω; C_t([0, ξ); H_{m0}^{α'}))$, at which point it is not hard to pass to the limit, iterate in $t$, and construct global solutions to (2.2) in the desired regularity classes. This is proved in Lemma 3.6, where, in analogy with a classical Picard iteration, we show that $f^{j+1} - f^j$ is nearly comparable in size to the $j$-th term of a power series in powers of $\sqrt{T}$ of the solution. This procedure finally yields

**Lemma 2.4.** Let $f_0$ be an $F_0$-measurable random variable such that $\forall p \geq 2$,

$$E \|f_0\|^p_{H_{m'}^{α'}} < \infty.$$ 

Then, there exists an $f ∈ C([0, ∞); H_{m'-2}^{α'-3}) \cap L_t,loc([0, ∞); H_{m'}^{α'})$ $P$–a.s. which is a solution to (2.2) in the sense that, $P$–a.s.:

$$f(t) = f_0 + \int_0^t \left( -v \cdot \nabla_x f(s) - θ_R(\|f\|_{H_{m0}^{α'}}) \varphi_ε \ast E(s) \cdot \nabla_v f(s) + νL f(s) \right) ds - \int_0^t \nabla_v f(s) \circ dW_s,$$

where the equality holds in $C([0, ∞); H_{m'-2}^{α'-3})$. Moreover, if $\tilde{f}$ is any other solution in the above sense, then $f = \tilde{f}$ almost surely in the sense that

$$P \left( f(t) - \tilde{f}(t) = 0 \quad \forall 0 ≤ t < ∞ \mid f(0) = \tilde{f}(0) \right) = 1.$$ 

The next step in the proof of Lemma 2.1 is to remove the superfluous mollifier $\varphi_ε$, which begins with obtaining $ε$-independent estimates (now indexing solutions to (2.2) by $ε$),

$$\sup_{ε \in (0,1)} E \|f_ε\|^p_{L^\infty(0,T;H_{m'}^{α'})} \lesssim_{p,T,R} 1.$$
\[
\sup_{\epsilon \in (0,1)} \mathbb{E} \left\| f_\epsilon \right\|_{W^{\alpha, p}(0,T;H^{\sigma'-2}_m)}^p \lesssim_{p,T,R,\alpha} 1.
\]

See Lemma 3.10 for the proof of these estimates. These estimates can be considered the probabilistic analogue of the common deterministic method of sharpening a continuation criterion a posteriori, specifically, the thrust of the estimates is to show that the \(H^{s_m}_m\) norm controls all \(H^{\sigma_m'}_m\) norms for \(m' > m_0\) and \(\sigma' > m_0\). At this step, it does not seem straightforward to prove that \(\{f_\epsilon\}_{\epsilon \in (0,1)}\) is Cauchy, and so we follow the martingale approach. Specifically, we use these uniform bounds to apply the Skorokhod embedding theorem to produce probabilistically weak solutions to (2.1) (see Proposition 3.11 below). These solutions are subsequently upgraded to probabilistically strong solutions by proving pathwise uniqueness (Lemma 3.13) and an application of the Gyöngy-Krylov lemma (from [30]; see Lemma 3.12 below). This general procedure is rather standard at this point; see for example [11, 22, 29]. This step completes the proof of Lemma 2.1.

The final step in the proof of Theorem 1.3 is to pass to a suitable limit in order to construct solutions in \(C_tH^\sigma_m\) from \(H^\sigma_m\) initial data, which is done in Section 4. We perform a regularization procedure on the initial data by defining a sequence of initial conditions

\[
f_{0;n} = \mathcal{R}^n f_0 = \theta_n(\cdot) m^{2d} \eta \left( \frac{\cdot}{n^2} \right) \ast f_0,
\]

where \(\eta \in C_c^\infty(\mathbb{R}^{2d})\) and satisfies \(\eta \geq 0\) and \(\int_{\mathbb{R}^{2d}} \eta \, dx \, dv = 1\). Note these have been both mollified and cut-off in velocity (to improve both regularity and localization). For all \(f_0 \in H^\sigma_m \cap L^1_{\mathrm{loc}}\), we hence have \(f_{0;n} \in H^\sigma_{m'} \cap L^1_{\mathrm{loc}}\) for all \(\sigma', m'\). Subsequently, there are unique local pathwise solutions to (1.1) \((f_n, \tau_n)\) with

\[
f_n(\cdot \land \tau_n) \in L^\infty_{\mathrm{loc}}(0, \infty; H^\sigma_{m'}) \cap C([0, \infty); H^\sigma_m).
\]

By obtaining suitable uniform-in-\(n\) upper bounds on the \(C_tH^\sigma_m\) norm, we may pass to the limit \(n \to \infty\) and hence extract local pathwise solutions to the original problem in \(C_tH^\sigma_m\); see Lemmas 4.2 and 4.3 for details.

**Notation and conventions**

For technical reasons, it is sometimes necessary (particularly when passing to the limit in the proofs of Lemmas 3.11 and 2.1) to view the fluctuating field as coming from a cylindrical Wiener process. Specifically, let \(\mathfrak{U}\) be a separable Hilbert space, with an orthonormal basis \((g_k)_{k \in \mathbb{K}}\). We can formally define a cylindrical Wiener process \(\mathcal{W}_t\) on \(\mathfrak{U}\) by the formula

\[
\mathcal{W}_t := \sum_{k \in \mathbb{K}} g_k W^{(k)}_t.
\]

Since this sum is divergent on \(\mathfrak{U}\), one frequently employs the larger Hilbert space:

\[
\mathfrak{U}_0 := \left\{ \sum_{k} \alpha_k g_k : \sum_{k} k^{-2} \alpha_k^2 < \infty \right\},
\]

\[
\left\| \sum_{k} \alpha_k g_k \right\|_{\mathfrak{U}_0}^2 = \sum_{k} k^{-2} \alpha_k^2,
\]

where it can be shown that the formal sum for \(\mathcal{W}_t\) converges and defines a process whose paths are almost surely in \(C([0,T];\mathfrak{U}_0)\). Moreover, the embedding \(\mathfrak{U} \subset \mathfrak{U}_0\) is Hilbert–Schmidt. For any separable Hilbert space \(X\), we denote the space of all Hilbert–Schmidt operators from \(\mathfrak{U}\) to \(X\) by \(L_2(\mathfrak{U}; X)\); the definition of this space is:

\[
L_2(\mathfrak{U}; X) := \left\{ T \in L(\mathfrak{U}; X) : \sum_{k} \| T g_k \|_{X}^2 < \infty \right\},
\]
\[ \|T\|^2_{L^2(\Omega; X)} = \sum_k \|Tg_k\|^2_X. \]

For more details on cylindrical Wiener processes and the relevant functional analytic setting, see [19]. At various points, we use the notation \( f \in L^p \) to signify that \( f \) is in any \( L^q \) space for \( q < p \).

We often employ the common notation:
\[ A(f) \lesssim_{p_1, p_2, \ldots} B(f) \]
which means that there exists a constant \( C > 0 \) depending only on the parameters \( p_1, p_2, \ldots \) but not on the argument \( f \), such that \( A(f) \leq CB(f) \) for all relevant \( f \). We omit the parameters if they are unimportant or clear from the context.

For the velocity-weighted \( L^2 \) norms and inner products, we set:
\[ \langle f, g \rangle_m := \int_{T^d \times \mathbb{R}^d} f(x, v) g(x, v) \langle v \rangle^m \, dv \, dx \]
\[ \|f\|_{L^2_m}^2 := \langle f, f \rangle_m. \]

Finally, at various points we use the mixed weighted norms:
\[ \|f\|_{L^2,n,H^2} := \left( \int_{\mathbb{R}^d} \left[ \int_{T^d} |f(x, v)|^p \, dv \right]^{2/p} \langle v \rangle^n \, dx \right)^{1/2}, \]
\[ \|f\|_{L^2,n,P^p} := \left( \int_{\mathbb{R}^d} \left[ \int_{T^d} |\nabla_v f(x, v)|^p \, dv \right]^{2/p} \langle v \rangle^n \, dx \right)^{1/2}. \]

### 3 Very smooth solutions and pathwise uniqueness

#### 3.1 Proof of Lemma 2.4

As discussed in Section 2, a key step in proving Lemma 2.4 is constructing a convergent sequence of approximate solutions derived from a Lagrangian iteration scheme for (2.2). In particular, consider a sequence \( f^j \) defined inductively as:
\[
\begin{cases}
  df^0 + \nu \cdot \nabla_x f^0 dt + \nabla_v f^0 \circ dW_t = \nu L f^0 dt, \\
  df^{j+1} + \nu \cdot \nabla_x f^{j+1} dt + \nabla_v f^{j+1} \circ dW_t = \nu L f^{j+1} dt - \theta_R(\|f_j\|_{H^{m_0}}) (\varphi \ast E^j) \cdot \nabla_v f^{j+1} dt,
\end{cases}
\tag{3.1}
\]

As discussed in Section 2, for a given \( f^j \in f \in C([0, \infty); H^{\sigma'; -3}_{m'-2} \cap L^\infty_{t,loc}([0, \infty), H^{\sigma'}_{m'}) \), the solution \( f^{j+1} \) is constructed via the method of stochastic characteristics. First, we provide \( j \)-independent estimates in order to pass to the limit \( j \to \infty \), for which we need appropriate compactness estimates for the iterates \( f^j \) defined above. We remark that studying the stochastic flow of diffeomorphisms could show that \( f^j \in f \in C([0, \infty); H^{\sigma'; -3}_{m'-2} \cap L^\infty_{t,loc}([0, \infty), H^{\sigma'}_{m'}) \), however, providing \( j \)-independent (and eventually \( \epsilon \)-independent) bounds seems to be significantly more complicated than an Eulerian energy method approach, which is hence the approach we take. The main ingredient is provided by the following lemma:

**Lemma 3.1.** Let \( (f^j)_{j \geq 1} \) be a sequence of global solutions to the iterative scheme (3.1) with \( E \|f_0\|^p_{H^{\alpha'}} < M_p < \infty \) for all \( p \geq 2 \). For \( \alpha \in (0, \frac{1}{2}) \), \( p \geq 2 \), we have the uniform estimates:
\[
\sup_{j \geq 1} E \sup_{t \leq T} \|f^j(t)\|^p_{H^{\alpha'}} \leq C_{T,R,\epsilon,M}.
\tag{3.2}
\]
Proof of Lemma 3.2. For any \( m > d \) there exists a constant \( C_{m,d} \) such that

\[
\sup_{j \geq 1} \mathbb{E} \left\| f_j \right\|_{W^{m-2,p}([0,T];H^{m-2}_x)}^p \leq C_{T,R,\varepsilon,M}.
\] (3.3)

Before we begin, let us begin by recalling a few standard estimates. The first shows how to estimate the density in terms of the distribution function using sufficiently many velocity moments.

**Lemma 3.2.** For any \( m > d \) there exists a constant \( C_{m,d} \) such that

\[
\left\| \int f(x,v) dv \right\|_{L^2_x} \leq C \| f \|_{L^2_m}
\]

Next, we recall the following Gagliardo-Nirenberg-Sobolev estimate: for all integers \( 0 \leq i \leq \sigma \) and functions \( f \in H^\sigma \) (in \( \mathbb{T}^d \) or \( \mathbb{R}^n \)) there holds

\[
\sum_{|\alpha| = i} \| \partial^n_x \|_{L^2_x} \leq C \| f \|_{L^2} \left( \sum_{|\alpha| = \sigma} \| \partial^n_x \|_{L^2} \right)^{\frac{1}{\sigma}}.
\] (3.4)

The next estimate recalls how to adapt Sobolev space product rules to the anisotropic nonlinearity. We give a proof for the readers’ convenience.

**Lemma 3.3.** Let \( g \in H^{n_0-1}_x, \ f \in H^{n}_x \) for some \( n_0 > d, \ s > \frac{d}{2} + 1, \) and \( n \geq 0 \) arbitrary. Denote \( E^g := \nabla_x \Delta_x^{-1} \left( \int g dv - 1 \right) \). Then, for a constant \( C \) that does not depend on \( g \) or \( f \):

\[
\| E^g \cdot \nabla_v f \|_{H^s_x} \leq C \| g \|_{H^{n_0-1}_x} \| \nabla_v f \|_{H^s_x}
\] (3.5)

and

\[
\sum_{|\alpha| + |\beta| \leq s} \left( \partial^n_x \partial^n_v (E^g \cdot \nabla_v f), \partial^n_x \partial^n_v f \right)_{H^n_x} \leq C (\| E^g \|_{W^{1,\infty}} + \| g \|_{H^{n_0-1}_x}) \| f \|_{H^n_x}^2.
\] (3.6)

**Proof of Lemma 3.3.** In what follows, \( \eta \in (0, \frac{1}{2}) \) will be fixed, which implies \( H^{d+\eta}_x \subset L^\infty_x \). Denote the multivariate binomial coefficients by

\[
\binom{\alpha}{\alpha'} := \frac{\Pi_{j=1}^d \alpha_j!}{\Pi_{j=1}^d (\alpha'_j)! \Pi_{j=1}^d (\alpha_j - \alpha'_j)!}.
\]

We begin the proof of (3.5) by using the product rule and the triangle inequality:

\[
\| E^g \cdot \nabla_v f \|_{H^s_x} \leq \sum_{|\alpha| + |\beta| \leq s} \left( \frac{\alpha}{\alpha_1} \right) \left\| \partial^n_x \partial^n_v f \right\|_{L^2_x}.
\]

Now, we split into four separate cases:

**Case 1 \( \alpha_1 = \alpha \):** In this case, we use Hölder’s inequality, the embedding \( H^{d+\eta}_x \subset L^\infty_x \), and Lemma 3.2

\[
\| E^g \cdot \nabla_v \partial^n_x \partial^n_v f \|_{L^2_x} \leq \| E^g \|_{L^\infty} \| \nabla_v \partial^n_x \partial^n_v f \|_{L^2_x} \leq C \| g \|_{H^{d-2+\eta}_x} \| \nabla_v f \|_{H^s_x}.
\] (3.7)

In the following cases, \( \alpha_1 < \alpha \).
Case 2 $|\alpha_1| + |\beta| = |\alpha| + |\beta| - 1$ or $|\alpha| + |\beta| = 1$: Here, we have $|\alpha - \alpha_1| = 1$, and hence similarly to the previous case

$$||\partial_{x}^{\alpha_1} E^g \cdot \nabla_v \partial_{x}^{\alpha_1} \partial_v f||_{L^2_{x}} \leq ||\nabla_x E^g||_{L^\infty} ||\nabla_v f||_{H^s_{v}} \leq C ||g||_{H^{\alpha_1 + \eta}_{x,v}} ||\nabla_v f||_{H^s_{v}}.$$  

Case 3 $|\alpha_1 - \alpha| = |\alpha| + |\beta| \geq 2$: Here, we necessarily have $\alpha_1 = \beta = 0$, and hence (using the Sobolev embedding now on $\nabla_v f$),

$$||\partial_{x}^{\alpha_1} E^g \cdot \nabla_v f||_{L^2_{x}} \leq ||\partial_{x}^{\alpha_1} E^g||_{L^2_{x}} ||\nabla_v f||_{L^\infty_{x,v}} \leq C ||g||_{H^{2+\eta}_{x,v}} ||\nabla_v f||_{H^{2+\eta}_{x,v}}.$$  

Case 4 $|\alpha_1| + |\beta| \leq |\alpha| + |\beta| - 2$: Here $|\alpha - \alpha_1| \geq 2$, so:

$$||\partial_{x}^{\alpha_1} E^g \cdot \nabla_v \partial_{x}^{\alpha_1} \partial_v f||_{L^2_{x}} \leq ||\partial_{x}^{\alpha_1} E^g||_{L^2_{x}} ||\nabla_v \partial_{x}^{\alpha_1} \partial_v f||_{H^{\alpha_1}_{x,v}} \leq C ||g||_{H^{\alpha_1}_{x,v}} ||\nabla_v \partial_{x}^{\alpha_1} \partial_v f||_{H^{\alpha_1}_{x,v}}.$$  

Summing over the various cases, (3.6) follows.

The proof of (3.6) is similar but just slightly more subtle after using the cancellation that occurs when all of the derivatives land on $\nabla_v f$. First, distribute the derivatives with Leibniz’s rule.

$$\sum_{|\alpha| + |\beta| \leq s} \left< \partial_{x}^{\alpha_1} \partial_v^\beta (E^g \cdot \nabla_v f), \partial_{x}^{\alpha_1} \partial_v f \right>_{n} \leq \sum_{|\alpha| + |\beta| \leq s} \sum_{\alpha_1 \leq \alpha} \left< \partial_{x}^{\alpha_1} E^g \cdot \nabla_v \partial_{x}^{\alpha_1} \partial_v^\beta f, \partial_{x}^{\alpha_1} \partial_v f \right>_{n}.$$  

We distinguish the following cases:

Case 1 $\alpha_1 = \alpha$: by integrating by parts the $\nabla_v$ onto the weight, we have

$$\left< E^g \cdot \nabla_v \partial_{x}^{\alpha_1} \partial_v^\beta f, \partial_{x}^{\alpha_1} \partial_v^\beta f \right>_{n} = \frac{1}{2} \int_{T^d \times [0,1]} E^g \cdot \nabla_v \partial_{x}^{\alpha_1} \partial_v^\beta f^{2} (f^\alpha)^{n} dv dx \leq C ||E^g||_{L^\infty} ||\partial_{x}^{\alpha_1} \partial_v^\beta f||_{L^2_{x}}^{2}.$$  

Case 2 $|\alpha - \alpha_1| = 1$: Cauchy-Schwarz gives

$$\left< \partial_{x}^{\alpha_1} E^g \cdot \nabla_v \partial_{x}^{\alpha_1} \partial_v^\beta f, \partial_{x}^{\alpha_1} \partial_v f \right>_{n} \leq ||\partial_{x}^{\alpha_1} E^g||_{L^\infty} ||f||_{H^s_{x}} \leq C ||\nabla_x E^g||_{L^\infty} ||f||_{H^s_{x}}.$$  

Case 3 $|\alpha - \alpha_1| \geq 2$: We have:

$$\left< \partial_{x}^{\alpha_1} E^g \cdot \nabla_v \partial_{x}^{\alpha_1} \partial_v^\beta f, \partial_{x}^{\alpha_1} \partial_v f \right>_{n} \leq ||\partial_{x}^{\alpha_1} E^g||_{L^2_{x}} ||\nabla_v \partial_{x}^{\alpha_1} \partial_v^\beta f||_{L^2_{x}} ||\partial_{x}^{\alpha_1} \partial_v f||_{L^2_{x}}.$$  

12
Similarly, for the dissipative term, integrating by parts gives:

\[ \mathcal{D}_{\alpha,\beta}(f^j) = -2\|\nabla_v \partial_x^\alpha \partial_v^\beta f^j\|_{L^2_{m'}}^2 \, dt + \int_{\mathbb{T}^d \times \mathbb{R}^d} |\partial_x^\alpha \partial_v^\beta f^j|^2 \Delta_v (v)^m \, dv \, dx \, dt \leq -2\|\nabla_v \partial_x^\alpha \partial_v^\beta f^j\|_{L^2_{m'}}^2 \, dt + C\|f^j\|^2_{H^{[\alpha']+[\beta]}} \, dt \]

Similarly, for the dissipative term, integrating by parts gives:

\[ \mathcal{D}_{\alpha,\beta}(f^j) = -2\|\nabla_v \partial_x^\alpha \partial_v^\beta f^j\|_{L^2_{m'}}^2 \, dt + \int_{\mathbb{T}^d \times \mathbb{R}^d} |\partial_x^\alpha \partial_v^\beta f^j|^2 \Delta_v (v)^m \, dv \, dx \, dt \leq -2\|\nabla_v \partial_x^\alpha \partial_v^\beta f^j\|_{L^2_{m'}}^2 \, dt + C\|f^j\|^2_{H^{[\alpha']+[\beta]}} \, dt \]

Next, we prove Lemma 3.1.

**Proof of Lemma 3.1.** We begin with an estimate for \( \|f\|^2_{H^{[\alpha']}_{m'}} \), which we then use to derive an estimate for \( \|f\|^p_{H^{[\alpha']}_{m'}} \), for \( p > 2 \). Applying Itô’s formula to \( \|\partial_x^\alpha \partial_v^\beta f^j\|^2_{L^2_{m'}} \), we have:

\[ \begin{align*}
d\|\partial_x^\alpha \partial_v^\beta f^j\|^2_{L^2_{m'}} &= -2 \left\langle \partial_x^\alpha \partial_v^\beta (v \cdot \nabla_x f^j), \partial_x^\alpha \partial_v^\beta f^j \right\rangle_{m'} \, dt \\
&\quad + 2 \left\langle \Delta_v \partial_x^\alpha \partial_v^\beta f^j, \partial_x^\alpha \partial_v^\beta f^j \right\rangle_{m'} \, dt \\
&\quad + 2 \left\langle \partial_x^\alpha \partial_v^\beta (\text{div}_v (f^j v)), \partial_x^\alpha \partial_v^\beta f^j \right\rangle_{m'} \, dt \\
&\quad - 2 \left\langle \theta_R \partial_x^\alpha \partial_v^\beta (\varphi_t \cdot E^{j-1} \cdot \nabla v f^j), \partial_x^\alpha \partial_v^\beta f^j \right\rangle_{m'} \, dt \\
&\quad - 2 \left\langle \partial_x^\alpha \partial_v^\beta (\nabla_v f^j \cdot dW_t), \partial_x^\alpha \partial_v^\beta f^j \right\rangle_{m'} \, dt \\
&\quad + \sum_k \left\langle \partial_x^\alpha \partial_v^\beta [(\sigma_k e_k \cdot \nabla v)^2 f^j], \partial_x^\alpha \partial_v^\beta f^j \right\rangle_{m'} \, dt \\
&\quad + \sum_k \|\partial_x^\alpha \partial_v^\beta (\sigma_k e_k \cdot \nabla_v f^j)\|_{L^2_{m'}}^2 \, dt \\
&= : \mathcal{T}_{\alpha,\beta}(f^j) + \mathcal{D}_{\alpha,\beta}(f^j) + \mathcal{F}_{\alpha,\beta}(f^j) + \mathcal{N}_{\alpha,\beta}(f^j) + \mathcal{M}_{\alpha,\beta}(f^j) + \mathcal{C}_{\alpha,\beta}(f^j) \tag{3.9} \end{align*} \]

where

\[ \mathcal{C}_{\alpha,\beta}(f^j) = \sum_k \left\langle \partial_x^\alpha \partial_v^\beta [(\sigma_k e_k \cdot \nabla v)^2 f^j], \partial_x^\alpha \partial_v^\beta f^j \right\rangle_{m'} \, dt + \sum_k \|\partial_x^\alpha \partial_v^\beta (\sigma_k e_k \cdot \nabla_v f^j)\|_{L^2_{m'}}^2 \, dt. \]

Here, the \( \mathcal{T}, \mathcal{D}, \mathcal{F}, \mathcal{N}, \mathcal{M}, \mathcal{C} \) terms abbreviate transport, dissipation, friction, nonlinear electric field, martingale, and correction contributions, respectively. We begin by observing that by integration by parts,

\[ \mathcal{T}_{\alpha,\beta}(f^j) + \mathcal{F}_{\alpha,\beta}(f^j) \leq C\|f^j\|^2_{H^{[\alpha']+[\beta]}} \, dt. \tag{3.10} \]

Similarly, for the dissipative term, integrating by parts gives:
Next, turn to the Itô correction terms, which need to be treated carefully in order to not lose derivatives. Distributing derivatives gives

\[ C_{\alpha,\beta}(f^j) = \sum_k \left( \| \partial_x^\alpha \partial_v^\beta (\sigma_k e_k \cdot \nabla_v) f^j \|_{L^2_{m'}}^2 + \left( \partial_x^\alpha \partial_v^\beta (\sigma_k e_k \cdot \nabla_v)^2 f^j, \partial_x^\alpha \partial_v^\beta f^j \right)_{m'} \right) \, dt \]

\[ = \sum_k \sum_{\alpha' < \alpha} \frac{\alpha' \alpha}{\alpha} \left( \partial_x^{\alpha'-\alpha} (e_k \otimes e_k) : \nabla_v^2 \partial_x^\alpha \partial_v^\beta f^j, \partial_x^\alpha \partial_v^\beta f^j \right)_{m'} \, dt \quad (3.11) \]

\[ + \sum_k \| \partial_x^\alpha \partial_v^\beta (\sigma_k e_k \cdot \nabla_v) f^j \|_{L^2_{m'}}^2 \, dt. \quad (3.12) \]

Now, (3.12) provides a term of highest order that cancels the Itô correction \( \| \partial_x^\alpha \partial_v^\beta (\sigma_k e_k \cdot \nabla_v) f^j \|_{L^2_{m'}}^2 \), and terms of lower order that can either be readily controlled by \( \| f^j \|^2_{H_a^{m'}} \) or cancel out with a corresponding term in (3.11). Integrating by parts in (3.12) we have,

\[ (3.12) = \sum_k \left( \partial_x^\alpha \partial_v^\beta f^j, \partial_x^\alpha \partial_v^\beta f^j \right)_{m'} - \left( \sum_k \sigma_k e_k \cdot \nabla_v \partial_x^\alpha \partial_v^\beta f^j, \partial_x^\alpha \partial_v^\beta f^j \right)_{m'} \]

\[ - \int \int_{T^d \times \mathbb{R}^d} \partial_x^\alpha \partial_v^\beta f^j (\sigma_k e_k \cdot \nabla_v \partial_x^\alpha \partial_v^\beta f^j) \sigma_k e_k \cdot \nabla_v (\langle v \rangle^m) \, dv \, dx \]

\[ = \left( \partial_x^\alpha \partial_v^\beta f^j, \sigma_k e_k \cdot \nabla_v \partial_x^\alpha \partial_v^\beta f^j \right)_{m'} - \left( \sum_k \sigma_k e_k \cdot \nabla_v \partial_x^\alpha \partial_v^\beta f^j, \sigma_k e_k \cdot \nabla_v \partial_x^\alpha \partial_v^\beta f^j \right)_{m'} \]

\[ + \frac{1}{2} \left( \partial_x^\alpha \partial_v^\beta f^j, \partial_x^\alpha \partial_v^\beta f^j \right)_{m'} (\langle v \rangle^m) \, dv \, dx \]

\[ = \left( \partial_x^\alpha \partial_v^\beta f^j, \sigma_k e_k \cdot \nabla_v \partial_x^\alpha \partial_v^\beta f^j \right)_{m'} - \left( \sum_k \sigma_k e_k \cdot \nabla_v \partial_x^\alpha \partial_v^\beta f^j, \sigma_k e_k \cdot \nabla_v \partial_x^\alpha \partial_v^\beta f^j \right)_{m'} \]

\[ + \frac{1}{2} \left( \partial_x^\alpha \partial_v^\beta f^j, \partial_x^\alpha \partial_v^\beta f^j \right)_{m'} (\langle v \rangle^m) \, dv \, dx. \quad (3.14) \]

Thus, we observe that (3.14) cancels the Itô correction (3.13), (3.17) is bounded above by \( C \| \partial_x^\alpha \partial_v^\beta f^j \|_{L^2_{m'}}^2 \), and (3.16) only contains derivatives of order lower than \(|\alpha| + |\beta|\) and is thus bounded above by \( C \| f^j \|^2_{H_a^{m'}} \).

Next, we turn to (3.15). This term contains (A) terms from the commutator where the total number of derivatives on \( f^j \) is strictly less than \(|\alpha| + |\beta|\), which can be treated by integration by parts of the \( \sigma_k e_k \cdot \nabla_v \) and are thus bounded above by \( C \| f^j \|^2_{H_a^{m'}} \); and (B) a highest order term which we deal with as follows:

\[ 2 \sum_{\alpha' < \alpha \atop |\alpha'| = 1} \rho^\alpha \partial_v^\alpha (e_k \cdot \nabla_v \partial_x^\alpha \partial_v^\alpha \partial_v^\beta f^j) \cdot \partial_x^\alpha \partial_v^\alpha f^j \right)_{m'} \]
\[ -2 \sum_{\alpha' < \alpha} \sigma_k^2 \left\langle \left( \partial_{\alpha'} e_k \cdot \nabla_v (e_k \cdot \nabla_v) \partial_{\alpha' - \alpha} \partial_\nu^3 f_j, \partial_\nu^3 \partial_\nu^3 f_j \right) \right\rangle_m \\
\] 
\[ - \sum_{\alpha' < \alpha} \sigma_k^2 \int_{T^d \times \mathbb{R}^d} \partial_\nu^2 \partial_\nu^3 f_j e_k \cdot \nabla_v \partial_{\alpha' - \alpha} \partial_\nu^3 f_j e_k \cdot \nabla_v (\langle v \rangle^m') d\nu dx \\
\] 
\[ = - \sum_{\alpha' < \alpha} \sigma_k^2 \left\langle \partial_{\alpha'} (e_k \otimes e_k) : \nabla_v^2 \partial_{\alpha' - \alpha} \partial_\nu^3 f_j, \partial_\nu^3 \partial_\nu^3 f_j \right\rangle_{m'} \quad (3.18) \]
\[ - \sum_{\alpha' < \alpha} \sigma_k^2 \int_{T^d \times \mathbb{R}^d} \partial_\nu^2 \partial_\nu^3 f_j e_k \cdot \nabla_v \partial_{\alpha' - \alpha} \partial_\nu^3 f_j e_k \cdot \nabla_v (\langle v \rangle^m') d\nu dx. \quad (3.19) \]

Notice that to obtain the prefactor 1 in (3.18) we used the symmetry of the tensor \( e_k \otimes e_k \). Now, (3.19) can be directly bounded by \( C \| f_j \|^2_{W^\sigma_{m'}} \), while (3.18) cancels out the highest order term in (3.11). Therefore, we finally conclude using (3.11)–(3.19) that we have
\[ C_{\alpha,\beta}(f_j) \leq C \| f_j \|^2_{W^\sigma_{m'}} dt. \]

Next we treat the contribution of the electric field term. It follows from (3.7)–(3.8) in the proof of (3.6) that:
\[ \mathcal{N}_{\alpha,\beta}(f_j) \leq C\theta_R(\| f_j \|^1_{H^\sigma_{m'}})(\| \varphi * E^{j-1} \|_{W^{1,\infty}} + \| \varphi * E^{j-1} \|_{H^{\sigma-1}_{m'}}) \| f_j \|^2_{W^\sigma_{m'}} dt \]
\[ \lesssim \epsilon, R \| f_j \|^2_{W^\sigma_{m'}} dt. \]

Finally, the martingale contribution is given by
\[ \mathcal{M}_{\alpha,\beta}(f_j) = \int_{T^d \times \mathbb{R}^d} |\partial_\nu^2 \partial_\nu^3 f_j|^2 dW_t \cdot \nabla_v (\langle v \rangle^m') d\nu dx - 2 \left\langle \left[ \partial_\nu^2 \partial_\nu^3, dW_t \right] \nabla_v f_j, \partial_\nu^3 \partial_\nu^3 f_j \right\rangle_{m'}. \quad (3.20) \]

We sum (3.10)–(3.20) over \( |\alpha| + |\beta| \leq \sigma' \) and obtain
\[ \| f_j \|^2_{H^\sigma_{m'}} \leq C \| f_j \|^2_{W^\sigma_{m'}} dt + 2\nu \| \nabla_v f_j \|^2_{H^\sigma_{m'}} dt + \sum_{0 \leq |\alpha| + |\beta| \leq \sigma} \mathcal{M}_{\alpha,\beta}(f_j), \]
so integrating in time and using the Burkholder–Davis–Gundy inequality (see e.g. [18]) (hereinafter abbreviated as BDG) we obtain:
\[ \mathbb{E} \sup_{t \leq T} \| f_j(t) \|^2_{H^\sigma_{m'}} \leq \mathbb{E} \| f_0 \|^2_{H^\sigma_{m'}} + C \int_0^T \mathbb{E} \| f_j(t) \|^2_{H^\sigma_{m'}} dt \\
\] 
\[ + C \mathbb{E} \left( \int_0^T \| f_j(t) \|^4_{H^\sigma_{m'}} dt \right)^{\frac{1}{2}} \leq \| f_0 \|^2_{H^\sigma_{m'}} + C \int_0^T \mathbb{E} \| f_j(t) \|^2_{H^\sigma_{m'}} dt \\
\] 
\[ + \frac{1}{2} \mathbb{E} \sup_{t \leq T} \| f_j(t) \|^2_{H^\sigma_{m'}}, \]
where the second line followed from Hölder’s inequality. After rearranging and applying Grünwall’s inequality we obtain the uniform-in-\( j \) estimate:
\[ \mathbb{E} \sup_{t \leq T} \| f_j(t) \|^2_{H^\sigma_{m'}} \leq C \mathbb{E} \| f_0 \|^2_{H^\sigma_{m'}}, \]
where the constant \( C \) depends on \( \epsilon, R, T, m', \sigma' \), but not \( f_0 \) or \( j \). Thus we have obtained (3.2) for \( p = 2 \).

Now, we use Itô’s formula again, this time for \( \| f^j \|_{H^\sigma(m')}^p \), with \( p > 2 \):

\[
\begin{align*}
\frac{d}{dt} \| f^j \|_{H^\sigma(m')}^p &= \frac{p}{2} \| f^j \|_{H^\sigma(m')}^p - \frac{p-2}{2} \| f^j \|_{H^\sigma(m')}^2 dt + \frac{p-2}{2} \| f^j \|_{H^\sigma(m')}^{p-2} dB^j \sum_{|\alpha|+|\beta| \leq \sigma} \left( \partial^\alpha_x \partial^\beta_v (\sigma_k \epsilon_k \cdot \nabla_v f^j), \partial^\alpha_x \partial^\beta_v f^j \right)_{m'} dt.
\end{align*}
\]

The latter term is treated by a straightforward commutator estimate, and together with the above estimates on \( d \| f^j \|_{H^\sigma(m')}^2 \), we obtain

\[
\begin{align*}
d\| f^j \|_{H^\sigma(m')}^p &\leq C \| f^j \|_{H^\sigma(m')}^p dt - \| f^j \|_{H^\sigma(m')}^{p-2} \sum_{|\alpha|+|\beta| \leq \sigma} \left( \partial^\alpha_x \partial^\beta_v (\nabla_v f^j \cdot dW_t), \partial^\alpha_x \partial^\beta_v f^j \right)_{m'} dt.
\end{align*}
\]

After integrating in time, using the BDG inequality, and applying Hölder’s inequality, we have

\[
\begin{align*}
\mathbb{E} \sup_{t \leq T} \| f^j \|_{H^\sigma(m')}^p &\leq \mathbb{E} \| f^j \|_{H^\sigma(m')}^p + C \int_0^T \mathbb{E} \| f^j \|_{H^\sigma(m')}^p ds + \frac{1}{2} \mathbb{E} \sup_{t \leq T} \| f^j \|_{H^\sigma(m')}^p.
\end{align*}
\]

By rearranging and using Grönwall’s lemma, we obtain (3.2).

We now turn to the proof of (3.3). We have:

\[
\begin{align*}
\mathbb{E} \| f^j \|_{W^{s,p}(0,T;H^\sigma(m')_{m'-1})}^p &\leq C \| f_0 \|_{H^\sigma(m')_{m'-1}}^p + C \mathbb{E} \left( \left\| \int_0^T -v \cdot \nabla_x f^j + \Delta_v f^j + \text{div}_v(f^j(v)) \right\|_{W^{1,p}(0,T;H^\sigma(m')_{m'-1})}^p \right) \\
&+ C \mathbb{E} \left( \left\| \int_0^T \theta R \varphi \ast E^{j-1} \cdot \nabla_v f^j \right\|_{W^{1,p}(0,T;H^\sigma(m')_{m'-1})}^p \right) \\
&+ C \mathbb{E} \left( \left\| \frac{1}{2} \int_0^T \sum_k (\sigma_k \epsilon_k \cdot \nabla_v) f^j ds \right\|_{W^{1,p}(0,T;H^\sigma(m')_{m'-1})}^p \right) \\
&+ C \mathbb{E} \left( \left\| \int_0^T \nabla_v f^j \cdot dW_t \right\|_{W^{0,p}(0,T;H^\sigma(m')_{m'-1})}^p \right).
\end{align*}
\]

The terms that are regular in time are estimated in a straightforward manner using the available regularity:
\[
E \left\| \int_0^t \sum_k (\sigma_k e_k \cdot \nabla v)^2 f^j \right\|^p_{W^{1,p}([0,T];H^{\sigma'_m-2})} \leq C E \sup_{t' \leq T} \| f^j(t') \|^p_{H^{\sigma'_m-1}}.
\]
\[
E \left\| \int_0^t \nabla v f^j \cdot dW_t \right\|^p_{W^{1,p}([0,T];H^{\sigma'_m-2})} \leq C E \left( \sup_{t' \leq T} \| f^j(t') \|^p_{H^{\sigma'_m-1}} \right) \cdot \left( \int_0^T \| \nabla f^j(s) \|^p_{H^{\sigma'_m-2}} \, ds \right).
\]

The time-regularity is only limited by the stochastic integral, which is estimated by a variant of the BDG inequality adapted to fractional regularity estimates in time (see e.g. [Lemma 2.1; [27]] for a proof), namely
\[
E \left\| \int_0^t \nabla v f^j \cdot dW_t \right\|^p_{W^{1,p}([0,T];H^{\sigma'_m-2})} \leq C E \sup_{t' \leq T} \| f^j(t') \|^p_{H^{\sigma'_m-1}}.
\]

Therefore, using that \( W^{1,p}([0,T];H^{\sigma'_m-2}) \subset W^{\alpha,p}([0,T];H^{\sigma'_m-2}) \) continuously and (3.2), we obtain:
\[
E \left\| f^j \right\|^p_{W^{\alpha,p}([0,T];H^{\sigma'_m-2})} \leq C_{R,T} E \left\| f_0 \right\|^p_{H^{\sigma'_m}}
\]
uniformly in \( j \), which implies (3.3), completing the proof of Lemma 3.1.

**Remark 3.4.** By examining the proof above, one can see that one can also treat magnetic fields, due to the special structure of the Lorentz force \( v \times B(x) \), which ensures both \( \nabla v \cdot (v \times B) = 0 \) and, despite the power of \( v \), the estimates do not lose any moments in \( v \) as \( v \times B \) is orthogonal to \( v \) (nor does the \( v \) dependence create any issues controlling higher regularity).

We continue the proof of Lemma (2.4). The approximation procedure mixes \( f^j \) and \( f^{j+1} \) in a way that makes it difficult to apply the usual method of using tightness of the laws in pathspace and applying the Skorohod embedding theorem to construct probabilistically weak solutions which are subsequently upgraded to strong solutions (see e.g. [11, 21, 22, 29]). Instead we will prove that \( \{f^j\}_{j=1}^\infty \) is Cauchy in a suitable topology. For this we first need the following consequence of Lemma 3.1 and the Borel-Cantelli lemma.

**Lemma 3.5.** For all \( \delta > 0 \), \( \exists \) a \( F_1 \)-measurable, almost-surely finite, random constant \( C_0 \) such that for all \( j \geq 0 \) there holds
\[
\sup_{s \leq 1} \| f^j(s) \|_{H^{\sigma'_m}} < C_0(\omega, \delta) \langle j \rangle^\delta.
\]
Moreover, \( \forall \alpha, n \) there holds,
\[
P(C_0 > n) \leq \delta, \alpha n^{-\alpha}.
\]

**Proof.** Recall the uniform in \( j \) bound (3.2) for the iterates for \( T = 1 \):
\[
\sup_{j \geq 1} E \sup_{s \leq 1} \| f^j(s) \|_{H^{\sigma'_m}} \leq C_{p,R,\epsilon,M} < \infty.
\]
This estimate implies:
\[
E \sum_{j=0}^\infty \sup_{s \leq 1} \| f^j(s) \|_{H^{\sigma'_m}} \langle j \rangle^p \leq C_{p,R,\epsilon,M},
\]
for $p > \frac{1}{\delta}$. Denote by $A_j$ the sets:

$$A_j := \{ \omega \in \Omega : \sup_{s \leq 1} \| f_j^s(s) \|_{H^{m_0'}} > \langle j \rangle^\delta \},$$

and note that by Chebyshev’s inequality:

$$\sum_{j=0}^{\infty} \mathbb{P}(A_j) \leq \sum_{j=0}^{\infty} \mathbb{E} \sup_{s \leq 1} \| f_j^s(s) \|^p_{H^{m_0'}} \langle j \rangle^{\delta p} < \infty.$$

It then follows by the Borel–Cantelli lemma that

$$\mathbb{P}(\limsup_{j \to \infty} A_j) = 0,$$

implying that $\mathbb{P}$-a.s., $\sup_{s \leq 1} \| f_j^s(s) \|_{H^{m_0'}} > \langle j \rangle^\delta$ at most for a finite number of $j$’s. Denote the largest such $j$ by $j_0(\omega)$. We then see that there is a random constant $C_0(\omega, \delta)$ such that

$$\sup_{j \geq 0} \left( \langle j \rangle^{-\delta} \sup_{s \leq 1} \| f_j^s(s) \|_{H^{m_0'}} \right) < C_0(\omega, \delta)$$

$\mathbb{P}$–almost surely. In particular, we can take:

$$C_0(\omega, \delta) := \inf \left\{ n \in \mathbb{N} : \sup_{j \leq j_0(\omega)} \left( \langle j \rangle^{-\delta} \sup_{s \leq 1} \| f_j^s(s) \|_{H^{m_0'}} \right) < n \right\}.$$ 

To bound the probability that $C_0$ is large, we observe:

$$\mathbb{P}(C_0 > n) \leq \mathbb{P} \left( \sup_{j \geq 0} \left( \langle j \rangle^{-\delta} \sup_{s \leq 1} \| f_j^s(s) \|_{H^{m_0'}} > n \right) \right)$$

$$\leq \sum_{j=0}^{\infty} \langle j \rangle^{-\delta p} \mathbb{E} \sup_{s \leq 1} \| f_j^s(s) \|^p_{H^{m_0'}} n^{-p}$$

$$\lesssim n^{-p}.$$ 

This completes the proof of the lemma. \qed

The next lemma is the crucial convergence estimate.

**Lemma 3.6.** There exists an increasing sequence of stopping times $\tau_n$ such that $\{ f_j^s \}_{j=1}^{\infty}$ is Cauchy in $L_\omega^2 C([0, \tau_n]; H^{m_0}_{m_0})$ and the stopping time

$$\lim_{n \to \infty} \tau_n = \xi,$$

is almost-surely greater than 1.

**Proof.** Define the increasing sequence of stopping times

$$\tau_n = \inf \left\{ t : \exists j : \| f_j^s(t) \|_{H^{m_0+1}} > n \langle j \rangle^\delta \right\}.$$
Note that by Lemma 3.5 there holds

\[ P(\tau_n \geq 1) = P \left( \sup_{j \geq 0} \langle j \rangle^{-\delta} \sup_{t < 1} \| f^j(t) \|_{H^{s_0+\delta}} < n \right) \]
\[ \geq P(C_0 < n) \]
\[ \geq 1 - P(C_0 > n) \]
\[ \geq 1 - O(n^{-\alpha}). \]

Therefore, \( \lim_{n \to \infty} P(\tau_n > 1) = 1 \) and so if we define

\[ \xi = \lim_{n \to \infty} \tau_n, \]

then \( \xi \) is almost-surely greater than or equal to 1.

Let \( \delta \in (0, 1/6) \) be fixed arbitrary. We will show by induction that \( \exists K_0 > 0 \) (deterministic constant depending on \( \delta \)) such that for all \( j \geq 1 \), there holds

\[ E \sup_{s < t \wedge \tau_n} \| f^j - f^{j-1} \|^2_{H^{s_0}_m} \leq \frac{(K_0 n^4 t)^j j^{4+j}}{j!}. \quad (3.21) \]

First consider the case \( j = 1 \). The calculation of \( d \left\| \partial^\alpha \partial^\beta_v (f^1 - f^0) \right\|_{L^2_{m_0}}^2 \) is the same in Lemma 3.1 except for the nonlinear terms. That is, for \( |\alpha| + |\beta| \leq s_0 \) we have for some constant \( C > 0 \)

\[ d \left\| \partial^\alpha \partial^\beta_v (f^1 - f^0) \right\|_{L^2_{m_0}}^2 \leq C \left\| f^1 - f^0 \right\|^2_{H^{s_0}_m} dt \]
\[ - \frac{\theta_R \left( \left\| f^0 \right\|_{H^{s_0}_m} \right) \partial^\alpha \partial^\beta_v (\varphi * E^0 \cdot \nabla_v f^1), \partial^\alpha \partial^\beta_v (f^1 - f^0) \right\|_{m_0} dt \]
\[ - \frac{\partial^\alpha \partial^\beta_v (\nabla_v (f^1 - f^0) \cdot dW_t), \partial^\alpha \partial^\beta_v (f^1 - f^0) \right\|_{m_0}}{m_0}. \]

For the nonlinear term we note that by (3.6) we have, recalling the definition of \( \tau_n \)

\[ \left\langle \theta_R \left( \left\| f^0 \right\|_{H^{s_0}_m} \right) \partial^\alpha \partial^\beta_v (\varphi * E^0 \cdot \nabla_v f^1), \partial^\alpha \partial^\beta_v (f^1 - f^0) \right\|_{m_0} \right\rangle \leq \theta_R \left( \left\| f^0 \right\|_{H^{s_0}_m} \right) \partial^\alpha \partial^\beta_v (\varphi * E^0 \cdot \nabla_v f^1), \partial^\alpha \partial^\beta_v (f^1 - f^0) \right\|_{m_0} \right\rangle \leq C \left( \left\| f^1 - f^0 \right\|_{H^{s_0}_m} \right) \]

Integrating in time and using the BDG inequality as above, we obtain (note that \( f^1 \) and \( f^0 \) have the same initial data),

\[ E \sup_{s < t \wedge \tau_n} \| f^1(s) - f^0(s) \|_{H^{s_0}_m}^2 \leq C \int_0^{t \wedge \tau_n} E \| f^1(s) - f^0(s) \|_{H^{s_0}_m}^2 ds \]
\[ + C n^2 t + C E \left( \int_0^{t \wedge \tau_n} \| f^1(s) - f^0(s) \|_{H^{s_0}_m}^4 ds \right)^{1/2} \]
\[ \leq C n^2 t + C \int_0^{t \wedge \tau_n} E \| f^1(s) - f^0(s) \|_{H^{s_0}_m}^2 ds \]
\[ + \frac{1}{2} E \sup_{s \leq t \wedge \tau_n} \| f^1(s) - f^0(s) \|_{H^{s_0}_m}^2. \]

Therefore, Grönwall’s inequality verifies (3.21) for \( j = 1 \) and some large \( K_0 \).
Next consider the inductive step. Hence, suppose that (3.21) holds for $j$ and we wish to verify that it holds for $j + 1$. As above, for some constant $C > 0$

\[
d\|\partial_x^\alpha \partial_v^\beta (f^{j+1} - f^j)\|_{L^2_{\tau_n}}^2 \leq C \|f^{j+1} - f^j\|_{H^{\alpha_0}_{\tau_n}}^2 \ dt
\]

\[
- 2 \left( \theta_R(\|f^j\|_{H^{\alpha_0}_{\tau_n}}) \partial_x^\alpha \partial_v^\beta (\varphi_\epsilon \ast E^j \cdot \nabla_v f^{j+1}), \partial_x^\alpha \partial_v^\beta (f^{j+1} - f^j) \right)_{m_0} \ dt
\]

\[
+ 2 \left( \theta_R(\|f^{j-1}\|_{H^{\alpha_0}_{\tau_n}}) \partial_x^\alpha \partial_v^\beta (\varphi_\epsilon \ast E^{j-1} \cdot \nabla_v f^j), \partial_x^\alpha \partial_v^\beta (f^{j+1} - f^j) \right)_{m_0} \ dt
\]

\[
- 2 \left( \partial_x^\alpha \partial_v^\beta (\nabla_v (f^{j+1} - f^j)) \cdot dW_t, \partial_x^\alpha \partial_v^\beta (f^{j+1} - f^j) \right)_{m_0}.
\]

The nonlinearity separates into several natural terms, namely

\[
(*) = -2 \left( \theta_R(\|f^j\|_{H^{\alpha_0}_{\tau_n}}) \partial_x^\alpha \partial_v^\beta (\varphi_\epsilon \ast E^j \cdot (\nabla_v f^{j+1} - \nabla_v f^j)), \partial_x^\alpha \partial_v^\beta (f^{j+1} - f^j) \right)_{m_0} \ dt
\]

\[
- 2 \left( \theta_R(\|f^{j-1}\|_{H^{\alpha_0}_{\tau_n}}) \partial_x^\alpha \partial_v^\beta (\varphi_\epsilon \ast E^{j-1} \cdot \nabla_v f^j), \partial_x^\alpha \partial_v^\beta (f^{j+1} - f^j) \right)_{m_0} \ dt
\]

\[
- 2 \left( \theta_R(\|f^{j-1}\|_{H^{\alpha_0}_{\tau_n}}) \partial_x^\alpha \partial_v^\beta ((\varphi_\epsilon \ast E^j - \varphi_\epsilon \ast E^{j-1}) \cdot \nabla_v f^j), \partial_x^\alpha \partial_v^\beta (f^{j+1} - f^j) \right)_{m_0} \ dt
\]

\[
= N_1 + N_2 + N_3.
\]

The term $N_1$ is treated via (3.6) in the same manner as in Lemma 3.1, giving

\[
N_1 \lesssim R \|f^{j+1} - f^j\|_{H^{\alpha_0}_{\tau_n}}^2.
\]

The terms $N_2, N_3$ however are different. The term $N_3$ is estimated via the following for $t < \tau_n$:

\[
N_3 \lesssim R \|f^j\|_{H^{\alpha_0}_{\tau_n}+1} \|f^j - f^{j-1}\|_{H^{\alpha_0}_{\tau_n}} \|f^{j+1} - f^j\|_{H^{\alpha_0}_{\tau_n}}
\]

\[
\lesssim n^j \|f^j - f^{j-1}\|_{H^{\alpha_0}_{\tau_n}} \|f^{j+1} - f^j\|_{H^{\alpha_0}_{\tau_n}}.
\]

The term $N_2$ requires a control on the difference $\theta_R(\|f^j\|_{H^{\alpha_0}_{\tau_n}}) - \theta_R(\|f^{j-1}\|_{H^{\alpha_0}_{\tau_n}})$:

\[
\theta_R(\|f^j\|_{H^{\alpha_0}_{\tau_n}}) - \theta_R(\|f^{j-1}\|_{H^{\alpha_0}_{\tau_n}})
\]

\[
= \int_0^1 \theta'_R(z) \|f^j\|_{H^{\alpha_0}_{\tau_n}} + (1 - z) \|f^{j-1}\|_{H^{\alpha_0}_{\tau_n}} (\|f^j\|_{H^{\alpha_0}_{\tau_n}} - \|f^{j-1}\|_{H^{\alpha_0}_{\tau_n}}) \, dz.
\]

Therefore, for $t < \tau_n$

\[
N_2 \lesssim \|f^j\|_{H^{\alpha_0}_{\tau_n}} - \|f^{j-1}\|_{H^{\alpha_0}_{\tau_n}} \|f^j\|_{L^2_{\tau_n}} \|f^{j+1} - f^j\|_{H^{\alpha_0}_{\tau_n}}
\]

\[
\lesssim \|f^j - f^{j-1}\|_{H^{\alpha_0}_{\tau_n}} \|f^j\|_{L^2_{\tau_n}} \|f^{j+1} - f^j\|_{H^{\alpha_0}_{\tau_n}}
\]

\[
\lesssim n^j \|f^j - f^{j-1}\|_{H^{\alpha_0}_{\tau_n}} \|f^{j+1} - f^j\|_{H^{\alpha_0}_{\tau_n}}.
\]

Integrating in time and using the BDG inequality as above, we obtain (noting that $f^{j+1}$ and $f^j$ have the same initial data) for $t < \tau_n$:

\[
E \sup_{s \leq t} \|f^{j+1}(s) - f^j(s)\|_{H^{\alpha_0}_{\tau_n}}^2 \leq C \int_0^t E \|f^{j+1}(s) - f^j(s)\|_{H^{\alpha_0}_{\tau_n}}^2 \, ds.
\]
\[ + C_0 n^4 j^{4\delta} E \int_0^t \| f^j(s) - f^{j-1}(s) \|_{H_{m0}^{\delta}}^2 ds \]
\[ + \frac{1}{2} E \sup_{s \leq t} \| f^{j+1}(s) - f^j(s) \|_{H_{m0}^{\delta}}^2 \]

By the inductive hypothesis

\[ C n^4 j^{4\delta} \int_0^t \| f^j(s) - f^{j-1}(s) \|_{H_{m0}^{\delta}}^2 ds \leq C_0 n^4 j^{4\delta} \int_0^t \frac{(C_0 n^4 s)^j j^{4\delta j}}{j!} ds \]
\[ \leq \frac{(C_0 n^4 t)^j j^{4\delta(j+1)}}{(j+1)!}, \]

and so we have verified (3.21).

Finally, we show that (3.21) implies that \{f^j\} is Cauchy in \(L_\omega^2 L_t^2([0, \tau_n]; H_{m0}^{\delta})\). Indeed, let \(k < \ell\) and

\[ E \sup_{s < t \wedge \tau_n} \| f^\ell - f^k \|_{H_{m0}^{\delta}}^2 \leq \sum_{j=k}^{\ell} \frac{(K_0 n^4 t)^j j^{4\delta j}}{j!}. \]

Hence, if we choose \(k > (2C_0 n^4 t)^{1/\delta}\), then

\[ E \sup_{s < t \wedge \tau_n} \| f^\ell - f^k \|_{H_{m0}^{\delta}}^2 \leq \sum_{j=k}^{\ell} \frac{1}{2^{-\delta j}} \frac{k^{\delta j} j^{4\delta j}}{j!} \leq \sum_{j=k}^{\ell} \frac{1}{2^{-\delta j}} \frac{j^{5\delta j}}{j!}. \]

By Stirling’s formula we have the following uniformly in \(j\) (using \(5\delta < 1\)),

\[ \frac{j^{5\delta j}}{j!} \lesssim 1, \]

therefore

\[ E \sup_{s < t \wedge \tau_n} \| f^\ell - f^k \|_{H_{m0}^{\delta}}^2 \lesssim \sum_{j=k}^{\ell} \frac{1}{2^{-\delta j}} \lesssim \frac{1}{2^{-\delta k}}. \]

We conclude that the sequence is Cauchy as claimed in the lemma.

\[ \square \]

**Lemma 3.7.** For each \(n\), the iterates \(\{f^j\}_{j=1}^\infty\) converge uniformly in \(H_{m0}^{\delta}\) on compact subintervals of \([0, \tau_n]\) to a strong pathwise solution of the SPDE (2.2) on the set \(\{\tau_n > 0\} \subset \Omega\).

**Proof.** Consider only \(\omega \in \{\tau_n > 0\} \subset \Omega\). Let \(f\) be the limiting process of the \(f^j\) in \(L_\omega^2 C([0, \tau_n]; H_{m0}^{\delta})\) - whose existence is guaranteed by Lemma 3.6. We will show that each term in (3.1) converges to the corresponding term in (2.2). The convergence of the linear terms is straightforward:

\[ E \sup_{t \leq T \wedge \tau_n} \left\| \int_0^t v \cdot \nabla_x (f^{j+1} - f) ds \right\|_{H_{m0-1}^{\delta}}^2 \lesssim_{m0} T^2 E \sup_{t \leq T \wedge \tau_n} \| f^{j+1} - f \|_{H_{m0}^{\delta}}^2 \to 0, \]
\[ E \sup_{t \leq T \wedge \tau_n} \left\| \int_0^t \Delta_x (f^{j+1} - f) ds \right\|_{H_{m0-2}^{\delta}}^2 \lesssim_{m0} T^2 E \sup_{t \leq T \wedge \tau_n} \| f^{j+1} - f \|_{H_{m0}^{\delta}}^2 \to 0, \]

21
Combining the above, we see that
\[ f \text{ is a solution of } (2.2). \]

**Corollary 3.8.** There exists a global, strong pathwise solution of the SPDE (2.2) such that \( \forall p \in [2, \infty), f \in L_p^p C_t, \text{loc} H^{\delta' - 3}_{m'} \cap L_p^p L_{t, \text{loc}}^\infty H^{\delta'}_{m'} \).
Proof. By sending $n \to \infty$ and using that $\tau_n$ is a non-decreasing sequence such that $\lim_{n \to \infty} \mathbb{P}(\tau_n > 1) = 1$ we see that almost-surely, $\{ f_j \}_{j=1}^{\infty}$ converges uniformly in $H^{s_0}_{m_0}$ on compact subintervals of $[0, 1]$ to a limiting function $f \in C([0, 1); H^{s_0}_{m_0})$. By Sobolev interpolation, and uniform boundedness in $L^\infty_{t, \text{loc}} H^{s'}_{m'}$, we obtain similar uniform convergence in $H^{s''}_{m''}$ for all $0 \leq s'' < s'$ and $m'' < m'$. At the same time, the uniform bounds on $\{ f_j \}$ from Lemma 3.1 imply that $\forall p \in [2, \infty)$, $f \in L^p_\omega C([0, 1); H^{s_0}_{m_0})$ by the lower semicontinuity of weak convergence. By Lemma 3.7, the limiting function $\tilde{f}$ is also a solution of (2.2). Now, we simply iterate the construction starting at $t = 1/2, 3/2, \ldots$ to obtain the existence of a global solution satisfying the desired bounds. □

The following lemma proves uniqueness of solutions to (2.2), thus completing the proof of Lemma 2.4.

Lemma 3.9. Let $f, f'$ be two global pathwise solutions to (2.2) on the same stochastic basis with $f(0) = f'(0) = f_0$ for some $\mathcal{F}_0$-measurable $f_0$ with $\mathbb{E}[|f_0|^p_{H^{s'}_{m'}}] < \infty$ for some $p > 2$ and such that for all $\epsilon > 0$, $f, f' \in L^{p-\epsilon}_\omega C([0, 1); H^{s_0}_{m_0}) \cap L^p_\omega L^\infty_{t, \text{loc}} H^{s'}_{m'}$. Then $f, f'$ are indistinguishable, that is:

$$\mathbb{P}\left( f(t) = f'(t) \text{ for all } 0 \leq t \right) = 1.$$

Proof. This is proved by an energy estimate on $\|f - f'\|_{H^{s'}_{m'}(\omega)}^2$. Similarly to the proof of Lemma 3.1, for $|\alpha| + |\beta| \leq \sigma' - 1$ we have:

$$d\|\partial^\alpha_x \partial^\beta_v (f - f')\|_{H^{s'}_{m'}(\omega)}^2 \leq C\|f - f\|_{H^{s'}_{m'}(\omega)}^2 dt - 2\left\langle \partial^\alpha_x \partial^\beta_v (\theta_R \varphi_* E \cdot \nabla_v f - \theta_R' \varphi_* E' \cdot \nabla_v f'), \partial^\alpha_x \partial^\beta_v (f - f') \right\rangle_{m'} dt$$

$$- 2 \int_0^t \left\langle \partial^\alpha_x \partial^\beta_v (\nabla_v (f - f') \cdot dW_t), \partial^\alpha_x \partial^\beta_v (f - f') \right\rangle_{m'} dt. \tag{3.22}$$

We split the electric field contributions as:

$$\left\langle \partial^\alpha_x \partial^\beta_v (\theta_R \varphi_* E \cdot \nabla_v f - \theta_R' \varphi_* E' \cdot \nabla_v f'), \partial^\alpha_x \partial^\beta_v (f - f') \right\rangle_{m'} = N_1 + N_2 + N_3,$$

where:

$$N_1 := \left\langle (\theta_R - \theta_R') \partial^\alpha_x \partial^\beta_v (\varphi_* E \cdot \nabla_v f), \partial^\alpha_x \partial^\beta_v (f - f') \right\rangle_{m'},$$

$$N_2 := \left\langle \theta_R' \partial^\alpha_x \partial^\beta_v (\varphi_* (E - E') \cdot \nabla_v f), \partial^\alpha_x \partial^\beta_v (f - f') \right\rangle_{m'},$$

$$N_3 := \left\langle \theta_R' \partial^\alpha_x \partial^\beta_v (\varphi_* E' \cdot \nabla_v (f - f')), \partial^\alpha_x \partial^\beta_v (f - f') \right\rangle_{m'}.$$

These are estimated as follows:

$$|N_1| \lesssim_R \|f\|_{H^{s'}_{m'}(\omega)} \|f - f'\|_{H^{s'}_{m'}(\omega)}^2, \tag{3.23}$$

$$|N_2| \lesssim \|f\|_{H^{s'}_{m'}(\omega)} \|f - f'\|_{H^{s'}_{m'}(\omega)}^2, \tag{3.24}$$

$$|N_3| \lesssim_R \|f - f'\|_{H^{s'}_{m'}(\omega)}^2, \tag{3.25}$$

where in (3.23) we used the mean value theorem for $\theta_R$ and (3.5), in (3.24) we used (3.5), and in (3.25) we used (3.6) - in addition to Lemma 3.2 for each electric field.
Now, fix $K > 0$. Since $f, f' \in L^p_\omega L^\infty_{t,loc} H^s_{m'}$, the stopping time:

$$\xi_K = \inf \{ t \geq 0 : \sup_{s \leq t} \| f(s) \|_{H^s_{m'}} + \sup_{s \leq t} \| f'(s) \|_{H^s_{m'}} > K \} \wedge \tau \wedge \tau'$$

is almost surely finite. Even though it is not clear that $\xi_K$ is almost surely positive in general, for almost every $\omega \in \Omega$ there exists $K > 0$ such that $\xi_K > 0$, and in addition $\xi_K \rightarrow \tau \wedge \tau' \text{ P–a.s. as } K \rightarrow \infty$. With this in mind, we fix $T > 0$ and use (3.23)-(3.25) and the BDG inequality in (3.22), to obtain:

$$\mathbb{E} \sup_{s \leq t \wedge \xi_K} \| f(s) - f'(s) \|^2_{H^s_{m'-1}} \leq \mathbb{E} \int_0^t \sup_{s \leq s' \wedge \xi_K} \| f(s) - f'(s) \|^2_{H^{s'-1}_{m'-1}} ds'$$

$$+ C \mathbb{E} \int_0^t \sup_{s \leq s' \wedge \xi_K} \| f(s) - f'(s) \|^2_{H^{s'-1}_{m'-1}} ds'$$

$$\leq C \mathbb{E} \int_0^t \sup_{s \leq s' \wedge \xi_K} \| f(s) - f'(s) \|^2_{H^{s'-1}_{m'-1}} ds'$$

for all $t \leq T$, whereby the usual rearrangement and Grönwall’s lemma give:

$$\mathbb{E} \sup_{s \leq T \wedge \xi_K} \| f(s) - f'(s) \|^2_{H^s_{m'-1}} = 0.$$  

Taking $K \rightarrow \infty$ and then $T \rightarrow \infty$, the conclusion follows. ☐

### 3.2 Proof of Lemma 2.1

Next, we want to pass to the limit $\epsilon \rightarrow 0$, for which we need uniform-in-$\epsilon$ estimates similar to those of Lemma 3.1, but this time for a family $\{ f_\epsilon \}_{\epsilon > 0}$ of solutions to (2.2). Note that since $f_\epsilon \in L^\infty_{t,loc} H^s_{m'} \cap C_t H^{s'-\beta}_{m'-2}$, the highest norm in which we know $f_\epsilon$ is continuous is $C_t H^{s'-1}_{m'-1}$ and thus we use this as the base for our estimates.

**Lemma 3.10.** Let $f$ be a solution of (2.2). For $\alpha \in (0, \frac{1}{2})$, $p \geq 2$, we have the uniform in $\epsilon$ estimates:

$$\mathbb{E} \sup_{t \leq T} \| f(t) \|^p_{H^s_{m'-1}} \lesssim_{p,T,R,f_0} 1$$  \hspace{1cm} (3.26)

and

$$\mathbb{E} \| f \|^p_{W^{\alpha,p([0,T]; H^{s'-3}_{m'-2})}} \lesssim_{p,T,R,f_0} 1.$$  \hspace{1cm} (3.27)

**Proof.** The proof proceeds by induction in the number of derivatives\(^1\). The inductive hypothesis is that for $s > d/2$ derivatives on a solution $f$ of (2.2), we have:

$$\mathbb{E} \sup_{t' \leq T} \| f \|^p_{H^s_{m'-1}} \lesssim_{p,R,T,f_0} 1.$$  \hspace{1cm} (3.28)

We show that this implies the same estimate for $s + 1$. Begin by using Itô’s formula on $\| \partial^\alpha_x \partial^\beta_v f \|^2_{L^2_{m'-1}}$ for $|\alpha| + |\beta| = s + 1$, where similarly to (3.9) we obtain:

$$d \| \partial^\alpha_x \partial^\beta_v f \|^2_{L^2_{m'-1}} = -2 \left( \partial^\alpha_x \partial^\beta_v (v \cdot \nabla_x f), \partial^\alpha_x \partial^\beta_v f \right)_{m'-1} \, dt$$

\(^1\)see for instance [43] for similar inductive energy estimates for the relativistic Vlasov–Maxwell system.
The linear terms are treated as in the proof of Lemma 3.1, and the only term that requires new attention is $N_{\alpha,\beta}(f)$. By the classical Gagliardo-Nirenberg inequality (see e.g. [Proposition A.3 [52]]) we have:

$$
\left| \left\langle \partial_x^\alpha \partial_t^\beta (\varphi \ast E \cdot \nabla_v f), \partial_x^\alpha \partial_t^\beta f \right\rangle \right|_{m'-1} \leq C \left\| \varphi \ast E \right\|_{W^{1,\infty}} \left\| f \right\|_{L^{p+1}}^{2} dt
$$

$$
+ \sum_{|\alpha-\gamma| \leq 2} \left\| \partial_x^{-\gamma} \varphi \ast E \right\|_{L^2} \left\| \nabla_v \partial_x^\alpha \partial_t^\beta f \right\|_{L^2} \left\| \partial_x^\alpha \partial_t^\beta f \right\|_{L^2} \left\| f \right\|_{L^2}^{2} dt
$$

$$
\leq C \left\| \varphi \ast E \right\|_{W^{1,\infty}} \left\| f \right\|_{L^{p+1}}^{2} dt
$$

$$
+ C \sum_{\substack{\gamma < \alpha \\
|\alpha-\gamma| \geq 2}} \left\| \partial_x^{-\gamma} \varphi \ast E \right\|_{L^2} \left\| \varphi \ast E \right\|_{H^{1+1}} \left\| \nabla_v \partial_x^\alpha \partial_t^\beta f \right\|_{L^2} \left\| f \right\|_{L^2}^{2} dt, \quad (3.29)
$$

where for fixed $\gamma$, the interpolation parameter $\eta$ is given by:

$$
\eta = \frac{|\gamma|}{|\alpha|-1} + \frac{d}{|\alpha|-1} \left( \frac{1}{2} - \frac{|\beta| + |\gamma| + 1}{2s} \right)
$$

and thus $\eta < 1$ provided $s > \frac{d}{2}$. By Young's inequality and (3.29) it follows that

$$
N_{\alpha,\beta} \leq C \left( 1 + \left\| f \right\|_{H^{m'+1}_{m'-1}}^p \right) dt + CR \left\| f \right\|_{H^{m'+1}_{m'-1}}^{2} dt.
$$

Plugging this back into (3.28), and using the same procedure as in the proof of Lemma 3.1, we obtain:

$$
d\left\| f \right\|_{H^{m'+1}_{m'-1}} \leq C(1 + R) \left\| f \right\|_{H^{m'+1}_{m'-1}}^{2} dt + \sum_{|\alpha|+|\beta|=s+1} \mathcal{M}_{\alpha,\beta}(f) + \mathcal{M}_{0,0}(f)
$$

$$
+ C(1 + \left\| f \right\|_{H^{m'+1}_{m'-1}}^p) dt.
$$

We again integrate in time and apply the BDG inequality as in the proof of (3.2) for $p = 2$, where the only difference is the term $\| f \|_{H^{m'+1}_{m'-1}}^p$, which is now controlled by the inductive hypothesis, and we get:

$$
\mathbb{E} \sup_{t \leq T} \| f \|_{H^{m'+1}_{m'-1}}^2 \lesssim R, p, T, f_0. 1.
$$
With this in hand, we can directly transfer the proof of (3.2) for \( p > 2 \) and obtain (3.26). Then the same argument as the proof of (3.3) (i.e. using the variant of BDG from [27, Lemma 2.1]) gives (3.27).

The last thing that remains is to demonstrate the inductive base of the preceding scheme. Here this is done by first estimating the \( H^2_{m'} \) norm of \( f \). This is sufficient to start the inductive scheme above in \( 1 \leq d \leq 3 \) as \( 2 > d/2 \). As the linear terms are always controlled in the same way, we only focus on the electric field contributions. As always, we have:

\[
\left| \left\langle \partial_x^d \partial_v^\beta (E \cdot \nabla_v f), \partial_x^d \partial_v^\beta f \right\rangle_{m'} \right| \leq C \|E\|_{W^{1,\infty}} \|f\|_{H^{2}_{m'}}^2 + \sum_{\gamma < \alpha} \sum_{|\alpha-\gamma| \geq 2} \left| \left\langle \partial_x^{\alpha-\gamma} E \cdot \nabla_v \partial_v^\beta f, \partial_x^\alpha \partial_v^\beta f \right\rangle_{m'} \right|,
\]

but since only two derivatives are acting on \( f \) at this point, the terms in the summation are only present when \( |\alpha| = 2 \) and \( |\beta| = |\gamma| = 0 \). Let \( q = 4 \) in \( d = 1, 2 \), and for \( d \geq 3 \) let \( q \) be arbitrary such that \( 2 < q < \frac{2d}{d-2} \). Then by Hölder’s inequality and Sobolev embeddings we have

\[
\left| \left\langle \partial_x^d E \cdot \nabla_v f, \partial_x^d f \right\rangle_{m'} \right| \leq \|\partial_x^d E\|_{L^{\frac{q}{2} - 1}} \|\nabla_v f\|_{L^{q}_{v,m'} L^2_{x}} \|f\|_{H^{2}_{m'}},
\]

\[
\leq \|\nabla_x^d E\|_{H^{\frac{q(a-q)}{2q}}_{m'}} \|f\|_{H^{2}_{m'}}^2, \quad \text{(3.30)}
\]

where we have used that \( s_0 > \frac{d}{2} + 1 > \frac{d(q-2)}{2q} \) and the embedding \( H^1_x \subset L^q_x \) which holds for all \( d \geq 1 \) due to our choice of \( q \). From this point on the procedure is the same as in the inductive step. We plug (3.30) into (3.28) for \( \alpha, \beta \) with \( |\alpha| + |\beta| = 2 \), sum over all such \( \alpha, \beta \) as well as the case when \( \alpha = \beta = 0 \), integrate in time, apply the BDG inequality and Grönnwall’s lemma and obtain:

\[
\mathbb{E} \sup_{t \leq T} \|f(t)\|_{H^{2}_{m'}}^p \lesssim_{T,R,f_0} 1.
\]

Then applying the same argument as in the proof of (3.2) for \( p > 2 \), we also obtain for \( p > 2 \):

\[
\mathbb{E} \sup_{t \leq T} \|f(t)\|_{H^{2}_{m'}}^p \lesssim_{p,T,R,f_0} 1.
\]

This provides the inductive base and therefore the proof of the lemma is complete for \( 1 \leq d \leq 3 \). \( \square \)

For solutions to (2.2), it is unclear how to prove \( \{f^\epsilon\}_{\epsilon > 0} \) forms a Cauchy sequence as \( \epsilon \to 0 \). Instead, we employ a standard procedure based on the Skorokhod embedding theorem (see e.g. [37]) to produce probabilistically weak (called martingale) solutions on a new stochastic basis, and then upgrade them to probabilistically strong using the Győngy–Krylov lemma from [30] (see Lemma 3.12 below). We let \( (\epsilon_n)_{n=1}^\infty \) be a decreasing sequence of positive numbers with \( \epsilon_n \to 0 \) as \( n \to \infty \) and define the corresponding sequence \( f_n := f^{\epsilon_n} \) of solutions to (2.2), which we have shown satisfy the uniform bounds (3.26) and (3.27). For \( \alpha \in (0, \frac{1}{2}) \) and \( p > 2 \) such that \( \alpha p > 1 \), we define the pathspace

\[
X := W^\alpha_{\text{loc}}([0, \infty); H^{\alpha-1}_{m'-1}) \cap L^\infty_{\text{loc}}([0, \infty); H^{\alpha-1}_{m'-1}).
\]

Recall that since \( \alpha p > 1 \) and \( H^{\alpha-3}_{m'-2} \subset H^{\alpha-4}_{m'-3} \) compactly, from [27, Theorem 2.2], we have:

\[
W^\alpha_{\text{loc}}([0, \infty); H^{\alpha-3}_{m'-2}) \subset C_t H^{\alpha-4}_{m'-3}.
\]

By the uniform estimates (3.26) and (3.27), the laws \( \nu_n := \mathcal{L}(f_n) \) are bounded in probability in \( X \), and thus they are tight in the smaller pathspace

\[
X_c := C([0, \infty); H^{\alpha-4}_{m'-3}) \cap L^\infty_{\text{loc}}([0, \infty); H^{\alpha-1}_{m'-1}).
\]
Note that the tightness in $L^\infty_{loc}([0, \infty); \mathcal{H}_{m'}^{\sigma'-1})$ is in the weak-$*$ topology. We now use this to obtain a martingale solution to (2.1) in high regularity.

**Proposition 3.11.** Let $\mu_0$ be a probability measure on $\mathcal{H}_{m'}^{\sigma'}$ so that $\int_{\mathcal{H}_{m'}^{\sigma'}} \|f\|_{\mathcal{H}_{m'}^{\sigma'}}^p \, d\mu_0(f)$ for some $p > 2$. Then there exists a stochastic basis $\tilde{S} = (\tilde{\Omega}, \tilde{\mathcal{F}}, \{\tilde{\mathcal{F}}_t\}, \tilde{P})$ and a predictable process

$$\tilde{f} \in L^p_\omega C_t H_{m'}^{\sigma'-4} \cap L^p_\omega L^\infty_{t, loc} H_{m'}^{\sigma'-1}$$

such that $\mathcal{L}(\tilde{f}(0)) = \mu_0$ and $\tilde{f}$ solves (2.1) in the sense that, there is a sequence of i.i.d Brownian motions $\{\tilde{W}_t(k)\}$ such that the following equality holds in $C([0, \infty); \mathcal{H}_{m'}^{\sigma'-3})$

$$\tilde{f}(t) = f_0 + \int_0^t \left( -v \cdot \nabla_x \tilde{f}(s) - \theta R(\|f\|_{H_{m_0}^{\sigma'}}) \tilde{E}(s) \cdot \nabla_v \tilde{f}(s) + \nu \mathcal{L}(\tilde{f}(s)) \right) \, ds - \int_0^t \nabla_v \tilde{f}(s) \, d\tilde{W}_s \text{ P-a.s.,}$$

with

$$\tilde{E} = \nabla_x (-\Delta_x)^{-1} \left( \int_{\mathbb{R}^d} \tilde{f}(t, \cdot, v) \, dv - 1 \right).$$

**Proof.** Let $\mu^n = \mathcal{L}(f_n, \mathcal{W})$ in $X_\times C([0, \infty); \mathcal{U}_0)$. The sequence $(\mu^n)_{n=1}^\infty$ is tight by the uniform estimates (3.26) and (3.27) combined with the fact that its projection onto $C([0, \infty); \mathcal{U}_0)$ is the same for each $n$. By Prokhorov’s theorem, $(\mu^n)_{n=1}^\infty$ has a weakly convergent subsequence - reindexed to $\mu^n$. By Skorokhod’s embedding theorem, there exists a new probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ and on it random elements $(\tilde{f}_n, \tilde{W}_n)$ with laws $\mu^n$ which converge $\tilde{P}$-a.s. to some limit $(\tilde{f}, \tilde{\mathcal{W}})$ in the product topology of $X_\times C([0, \infty); \mathcal{U}_0)$. Then by a variation of the mollification technique employed in the proof of [5, Equation 4.17] the random elements $(\tilde{f}_n, \tilde{W}_n)$ satisfy (2.2) just like $(f_n, \mathcal{W})$, but in the new probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$:

$$\tilde{f}_n(t) - \tilde{f}_n(0) + \int_0^t \left( v \cdot \nabla_x \tilde{f}_n + \tilde{E}_n \cdot \nabla_v \tilde{f}_n - \frac{1}{2} \sum_k (\sigma_k e_k \cdot \nabla_v)^2 \tilde{f}_n - \nu \mathcal{L}(\tilde{f}_n) \right) \, ds + \int_0^t \nabla_v \tilde{f}_n \cdot d\tilde{W}_t^n = 0,$$

where we have denoted by $\tilde{W}_t^n$ the external electric field corresponding to $\tilde{W}_n$; that is to say, if:

$$\tilde{W}_n = \sum_k g_k \tilde{W}_k^n,$$

then $\tilde{W}_n$ is simply given by:

$$\tilde{W}_n = \sum_k \sigma_k e_k \tilde{W}_k^n.$$

The passage to the limit $n \to \infty$ in the SPDEs (2.2) satisfied by $(\tilde{f}_n, \tilde{W}_n)$ to obtain that the limit $(\tilde{f}, \tilde{\mathcal{W}})$ solves (2.1) can be carried out by combining the convergences $\tilde{f}_n \to \tilde{f}$ and $\tilde{W}_n \to \tilde{\mathcal{W}}$ with [21, Lemma 2.1], so we omit the proof for technicalities. We simply note that the presence of the transport noise does not cause any additional difficulties in our setting.  

As our goal is to construct solutions in $\mathcal{S}$, we need to “upgrade” the martingale solutions of the preceding lemma to probabilistically strong solutions. With the above in mind, we now state the Gyöngy–Krylov lemma from [30], which will allow us to combine the tightness of $(f_n)_{n=1}^\infty$ with the pathwise uniqueness of the limit (Lemma 3.13 below) to show that in fact (2.1) has (unique) global solutions on the original stochastic basis $\mathcal{S}$.

\(^2\)See also [11, Proposition 3.2, (iii)] for an application to the primitive equations where the noise is present as a stochastic transport term.
**Lemma 3.12** (Gyöngy–Krylov). Let \( (Y_n)_{n=1}^\infty \) be a sequence of \( X \)-valued random variables, where \( X \) is a complete separable metric space. Then \( (Y_n)_n \) converges in probability if and only if for every two subsequences \( Y_{n_k}, Y_{l_k} \) the joint sequence \( (Y_{n_k}, Y_{l_k}) \) has a subsequence \( (Y_{n_k'}, Y_{l_k'}) \) whose laws converge weakly to a probability measure \( \nu \) supported on the diagonal of \( X \times X \):

\[
\nu \left( \{(x,y) \in X \times X : x = y\} \right) = 1.
\]

With this lemma at hand, we now set to prove pathwise uniqueness of solutions to (2.1), which is the content of the following:

**Lemma 3.13.** Let \( f, f' \) be global solutions to (2.1) on the same stochastic basis with \( f(0) = f'(0) = f_0 \) almost surely, where \( \mathbb{E}[f_0]_{H^{\sigma'}} < \infty \) for some \( p > 2 \) and such that \( f, f' \in L^p_{t, \text{loc}}(\mathbb{R}^{m'} \cap L^\infty_{t, \text{loc}} H^{\sigma'-1}_m) \). Then \( f, f' \) are indistinguishable, that is:

\[
P \left( f(t) = f'(t) \text{ for all } t \geq 0 \right) = 1. \tag{3.31}
\]

**Proof.** First of all, notice that since \( f, f' \in L^\infty_{t, \text{loc}} H^{\sigma'-1}_m \), for \( K > 0 \) the stopping times

\[
\xi_K := \inf \{ t > 0 : \| f \|^2_{H^{\sigma'-1}_m} + \| f' \|^2_{H^{\sigma'-1}_m} \geq K \}
\]

are well defined and satisfy \( \xi_K \to \infty \) as \( K \to \infty \), \( \mathbb{P} \)-almost surely. We now perform an energy estimate on \( H^{m}_m \). We use Itô’s formula on the quantity \( \| \partial_x^\alpha \partial_v^\beta (f - f') \|_{L^2_m}^2 \), for \( |\alpha| + |\beta| = s_0 \):

\[
d\| \partial_x^\alpha \partial_v^\beta (f - f') \|_{L^2_m}^2 = 2 \left( \partial_x^\alpha \partial_v^\beta (v \cdot \nabla_x (f - f')), \partial_x^\alpha \partial_v^\beta (f - f') \right)_{m_0} dt \\
+ 2 \left( \Delta_v \partial_x^\alpha \partial_v^\beta (f - f'), \partial_x^\alpha \partial_v^\beta (f - f') \right)_{m_0} dt \\
+ 2 \left( \partial_x^\alpha \partial_v^\beta \text{div}_v ((f - f') v), (f - f') \right)_{m_0} dt \\
- 2 \left( \partial_x^\alpha \partial_v^\beta (\theta_R E^f \cdot \nabla_v f - \theta_R' E^{f'} \cdot \nabla_v f'), \partial_x^\alpha \partial_v^\beta (f - f') \right)_{m_0} dt \\
- 2 \left( \partial_x^\alpha \partial_v^\beta (\nabla_v (f - f') \cdot dW_t), \partial_x^\alpha \partial_v^\beta (f - f') \right)_{m_0} dt \\
+ \sum_k \left( \partial_x^\alpha \partial_v^\beta ((\sigma_k e_k \cdot \nabla_v)^2 (f - f')), \partial_x^\alpha \partial_v^\beta (f - f') \right)_{m_0} dt \\
- \sum_k \| \partial_x^\alpha \partial_v^\beta (\sigma_k e_k \cdot \nabla_v (f - f')) \|_{L^2_m}^2 dt.
\]

Clearly, all terms except those involving the electric fields can be estimated as in the proof of (3.2), so we only examine the electric field term:

\[
\left| \left( \partial_x^\alpha \partial_v^\beta (\theta_R E^f \cdot \nabla_v f - \theta_R' E^{f'} \cdot \nabla_v f'), \partial_x^\alpha \partial_v^\beta (f - f') \right)_{m_0} \right| \\
\leq \left| (\theta_R - \theta_R') \left( \partial_x^\alpha \partial_v^\beta (E^f \cdot \nabla_v f), \partial_x^\alpha \partial_v^\beta (f - f') \right)_{m_0} \right| \\
+ \left| \theta_R \left( \partial_x^\alpha \partial_v^\beta ((E^f - E^{f'}) \cdot \nabla_v f'), \partial_x^\alpha \partial_v^\beta (f - f') \right)_{m_0} \right| \\
+ \left| \theta_R' \left( \partial_x^\alpha \partial_v^\beta (E^{f'} \cdot \nabla_v (f - f')), \partial_x^\alpha \partial_v^\beta (f - f') \right)_{m_0} \right|
\]

28
\[ \leq C_R \|f\|_{H_{\kappa_0}^{\alpha}} - \|f'\|_{H_{\kappa_0}^{\alpha}} \cdot \|E_f\|_{H_{\kappa_0}^{\alpha}} \cdot \|f - f'\|_{H_{\kappa_0}^{\alpha}} \\
+ C\|E_f - E_f'\|_{H_{\kappa_0}^{\alpha}} \|\nabla f\|_{H_{\kappa_0}^{\alpha}} \|f - f'\|_{H_{\kappa_0}^{\alpha}} \\
+ CR\|f - f'\|_{H_{\kappa_0}^{\alpha}}^2 \leq R.K\|f - f'\|_{H_{\kappa_0}^{\alpha}}^2 \]

for \( t \leq \xi_K \), where we have used the mean value theorem on \( \theta_R \) and Lemma 3.3. Thus, arguing similarly to the proof of Lemma 3.9 (i.e. by the BDG and Grönwall inequalities), we have:

\[ \mathbb{E}\sup_{t' \leq t \wedge \xi_K \wedge T} \|f - f'\|_{H_{\kappa_0}^{\alpha}}^2 \leq C_{R,K,T}\|f(0) - f'(0)\|_{H_{\kappa_0}^{\alpha}}^2 = 0. \]

Since \( \xi_K \to \infty \) as \( K \to \infty \), the monotone convergence theorem implies that for all \( T > 0 \) we have:

\[ \mathbb{E}\sup_{t \leq T} \|f(t) - f'(t)\|_{H_{\kappa_0}^{\alpha}}^2 = 0, \]

which implies (3.31) since \( T \) is arbitrary.

We now have everything we need to (subsequentially) pass to the limit \( \epsilon \to 0 \) in the original stochastic basis.

**Proof of Lemma 2.1.** We define the joint laws \( \mu_{\epsilon,l} = \mathcal{L}(f_n, f_l, \mathcal{W}) \). Similarly to the discussion in Lemma 3.11, for any sequence \( \mu_{\epsilon,l}^{n,k} \) with \( n_k, l_k \to \infty \) as \( k \to \infty \), by Prokhorov’s theorem the estimates (3.26) and (3.27) (and the fact that \{\mathcal{W}\} is a singleton) provide a weakly convergent subsequence of probability measures in \( X_c \times X_c \times C([0,\infty); \Omega_0) \), which we still denote (after relabelling) by \( \mu_{\epsilon,l}^{n,k} \), and we denote its limit by \( \mu \). By the Skorokhod embedding theorem, we can construct a new stochastic basis again denoted by \( \hat{\mathcal{S}} = (\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathcal{P}}) \) and on it a sequence of random elements \((\hat{f}_n, \hat{f}_l, \hat{\mathcal{W}}^k)\) and \((\hat{f}, \hat{f}_l, \hat{\mathcal{W}})\) such that \( \mathcal{L}(\hat{f}_n, \hat{f}_l, \hat{\mathcal{W}}^k) = \mu_{\epsilon,l}^{n,k} \), \( \mathcal{L}(\hat{f}, \hat{f}_l, \hat{\mathcal{W}}) \) and:

\[ (\hat{f}_n, \hat{f}_l, \hat{\mathcal{W}}^k) \to (\hat{f}, \hat{f}_l, \hat{\mathcal{W}}) \quad \text{in} \quad X_c \times X_c \times C([0,\infty); \Omega_0), \quad \hat{\mathcal{P}}-\text{a.s..} \]

As in Lemma 3.11, \((\hat{f}_n, \hat{\mathcal{W}}^k)\) and \((\hat{f}_l, \hat{\mathcal{W}}^k)\) satisfy the SPDE (2.2) in the new stochastic basis (by the method of [5, Section 4.3.4]), so we can pass to the limit \( k \to \infty \) in all the terms of (2.2) for \((\hat{f}_n, \hat{\mathcal{W}}^k)\) and \((\hat{f}_l, \hat{\mathcal{W}}^k)\) (using [21, Lemma 2.1] for the stochastic integrals) to show that \((\hat{f}, \hat{\mathcal{W}})\) is a solution to (2.1) on the new stochastic basis. Since \( \mathcal{P}(f_n(0) = f_l(0)) = 1 \), we also have \( \hat{\mathcal{P}}(\hat{f}_n(0) = \hat{f}_l(0)) = 1 \), and thus in the limit \( k \to \infty \) we obtain \( \hat{\mathcal{P}}(\hat{f}(0) = \hat{f}(0)) = 1 \). Therefore, by Lemma 3.13, \( \hat{f} \) and \( \hat{f} \) are indistinguishable. This means that the measure \( \hat{\mu} \), defined as the projection of \( \mu \) onto the first two components \( X_c \times X_c \), is in fact supported on the diagonal of \( X_c \times X_c \). Thus, by Lemma 3.12, a subsequence of \((f_n)\) converges in probability in the original stochastic basis \( S \) in the topology of \( X_c' \) to a limiting process \( f \) which solves (2.1). This concludes the proof of Lemma 2.1.

**Boxed Claim 2.2.**

**4 Proof of main theorem**

In this section, with the results of Section 3 at hand, we prove the main result of the paper, Theorem 1.3. At first, we consider initial data \( f_0 \) satisfying \( \|f_0\|_{H_{\kappa_0}^{\alpha}} \leq M < \infty \) almost surely, for a fixed deterministic \( M \). This assumption can be removed at the end by a cutting argument similar to that of Lemma 2.2. We treat the initial data \( f_0 \) with a sequence of regularization and velocity cutoff operators \( R^n \), obtaining a sequence of regularized data

\[ f^n_0 := R^n f_0, \]
defined as
\[ R_n f := \theta_n(v)n^{2d} \eta \left( \frac{x}{n} \right) \ast_{x,v} f, \]
where \( \eta \in C_c^\infty(\mathbb{R}^{2d}) \) satisfies \( \eta(x,v) \geq 0 \) and \( \int \int \eta(x,v) dv dx = 1 \). We note the following properties of this regularization. The proofs are standard and are omitted for brevity.

**Lemma 4.1.** Let \( s' \geq s \geq 0 \) be integers and \( m' \geq m \geq 0 \) be arbitrary. Then,

(i) The regularization operators \( R_n \) are uniformly bounded on \( H^{\sigma}_m \)
\[ \sup_{n \geq 1} \| R_n f \|_{H^\sigma_m} \lesssim m, \sigma \| f \|_{H^\sigma_m}. \]

(ii) The regularization operators satisfy: for \( f \in H^\sigma_m \)
\[ \| R_n f \|_{H^{\sigma'}_{m'}} \lesssim n^{\sigma' - \sigma} n^{(m' - m)/2} \| f \|_{H^\sigma_m}. \]

(iii) The regularization operators converge in the following senses: for \( f \in H^\sigma_m \) there holds
\[ \lim_{n \to \infty} \| R_n f - f \|_{H^\sigma_m} = 0 \]
\[ \lim_{n \to \infty} n \| R_n f - f \|_{H^{\sigma}-1} = 0. \tag{4.1} \]

In the previous section, we showed that each of the \( f_n^0 \) generates a maximal solution \( f^n \) of (1.1) in \( C([0, \tau_n]; H^\sigma_m) \cap L^\infty([0, \tau_n]; H^{\sigma'-1}_{m'-1}) \), where \( \tau_n \) is the maximal time of existence of \( f^n \). We now show that the sequence \( (f^n)_{n=1}^\infty \) of approximate solutions has a strongly convergent subsequence.

We start by defining the stopping times:
\[ \tau^n_T := \inf \{ t \geq 0 : \| f^n(t) \|_{H^\sigma_m} > \| f^0_n \|_{H^\sigma_m} + 2 \} \]
\[ \tau^n_{T,k} = \tau^n_T \land \tau^n_k \]

The following is similar to [44, Lemma 37] or [29, Lemma 7.1]:

**Lemma 4.2.** Let \( \tau_n \) be a sequence of stopping times and suppose that a sequence of predictable processes \( f^n \in C([0, \tau_n]; H^\sigma_m) \) satisfy:
\[ \lim_{n \to \infty} \sup_{t' \leq \tau^n_{T,k}} \| f^n - f^l \|_{H^\sigma_m} = 0, \tag{4.2} \]
\[ \lim_{n \to \infty} \sup_{t' \leq \tau^n_{T,k}} \| f^n \|_{H^\sigma_m} = 0. \tag{4.3} \]

Then, there exists a stopping time \( \tau \) with \( \mathbb{P}(0 < \tau \leq T) = 1 \), a predictable process \( f \in C([0, \tau]; H^\sigma_m) \), and a subsequence \( (f^{n_j})_{j=1}^\infty \) of \( (f^n)_{n=1}^\infty \) such that
\[ \sup_{t \leq \tau} \| f^{n_j}(t) - f(t) \|_{H^\sigma_m} \to 0 \text{ as } j \to \infty, \quad \mathbb{P}-a.s. \]
and
\[ \sup_{t \leq \tau} \| f(t) \|_{H^\sigma_m} \leq 2 + \sup_{n} \| f^0_n \|_{H^\sigma_m}, \quad \mathbb{P}-a.s. \]

We now verify that the regularized solutions \( \{ f^n \}_{n \geq 1} \) satisfy the conditions of Lemma 4.2.
Lemma 4.3. The solutions \((f^n)_{n=1}^{\infty}\) generated by the regularized data \((f^n_0)_{n=1}^{\infty}\) satisfy (4.2) and (4.3).

Proof. We begin with proving that the Cauchy property (4.2) holds for the sequence \(f^n\) of solutions to (2.1) with initial data \(f^n_0\). This is done via an energy estimate with some similarities with the uniqueness and convergence proofs in Section 3 with one significant difference. For \(t \leq \tau_{n,t}\), we have:

\[
\begin{aligned}
&d\|\partial_x \partial_v^\beta (f^n - f^l)\|_{L^2_m}^2 = -2 \left(\partial_x \partial_v^\beta (v \cdot \nabla_x (f^n - f^l)), \partial_x \partial_v^\beta (f^n - f^l)\right)_m \ dt
+ 2 \left(\Delta_v (\partial_x \partial_v^\beta (f^n - f^l)), \partial_x \partial_v^\beta (f^n - f^l)\right)_m \ dt
+ 2 \left(\partial_x \partial_v^\beta \text{div}_v ((f^n - f^l)v), \partial_x \partial_v^\beta (f^n - f^l)\right)_m \ dt
- 2 \left(\partial_x \partial_v^\beta (E^n \cdot \nabla_v f^n - E^l \cdot \nabla_v f^l), \partial_x \partial_v^\beta (f^n - f^l)\right)_m \ dt
- 2 \left(\partial_x \partial_v^\beta (f^n - f^l) \cdot \text{d}W_t), \partial_x \partial_v^\beta (f^n - f^l)\right)_m \ dt
+ \sum_k \left(\partial_x \partial_v^\beta (\sigma_k e_k \cdot \nabla_v (f^n - f^l))^2, \partial_x \partial_v^\beta (f^n - f^l)\right)_m \ dt
+ \sum_k \|\partial_x \partial_v^\beta (\sigma_k e_k \cdot \nabla_v (f^n - f^l))\|_{L^2_m}^2 \ dt.
\end{aligned}
\]

We will control the above for \(|\alpha| + |\beta| = \sigma\). Of course, the linear terms are treated in the same way as in the estimates of Section 3. We now explain how the electric field terms are to be estimated. For \(t \leq \tau_{n,t}\), we have:

\[
\begin{aligned}
&\left|\left(\partial_x \partial_v^\beta (E^n \cdot \nabla_v f^n - E^l \cdot \nabla_v f^l), \partial_x \partial_v^\beta (f^n - f^l)\right)_m\right|
\leq \left|\left(\partial_x \partial_v^\beta ((E^n - E^l) \cdot \nabla_v f^n), \partial_x \partial_v^\beta (f^n - f^l)\right)_m\right|
+ \left|\left(\partial_x \partial_v^\beta ((E^l \cdot \nabla_v (f^n - f^l)), \partial_x \partial_v^\beta (f^n - f^l)\right)_m\right|
\leq C \left\|f^n - f^l\right\|_{H^{m-1}} \left\|f^n\right\|_{H^{m+1}} + C \left\|f^l\right\|_{H^{m-1}} \left\|f^n\right\|_{H^{m+1}}^2
\leq C \left\|f^n - f^l\right\|_{H^{m}}^2 + \left\|f^n\right\|_{H^{m-1}}^2 \left\|f^n\right\|_{H^{m+1}}^2.
\end{aligned}
\]

(4.4)

where we have used Lemma 3.3. Combining our estimates from the previous section with (4.4) and the fact that we are taking \(t \leq \tau_{j,k}\), we have:

\[
\begin{aligned}
&d \left(\left\|f^n - f^l\right\|_{H^{m}}^2\right) \leq C \left\|f^n - f^l\right\|_{H^{m}}^2 \ dt + C \left\|f^n - f^l\right\|_{H^{m-1}}^2 \left\|f^n\right\|_{H^{m+1}} \ dt
- 2 \sum_{|\alpha|+|\beta|=\sigma} \left(\partial_x \partial_v^\beta (\nabla_v (f^n - f^l) ), \partial_x \partial_v^\beta (f^n - f^l)\right)_m
- 2 \left(\nabla_v (f^n - f^l) \cdot \text{d}W_t, f^n - f^l\right)_m.
\end{aligned}
\]

(4.5)

In what follows we denote

\[
\begin{aligned}
&\mathcal{M}_\sigma (f^n - f^l) := -2 \sum_{|\alpha|+|\beta|=\sigma} \left(\partial_x \partial_v^\beta (\nabla_v (f^n - f^l)), \partial_x \partial_v^\beta (f^n - f^l)\right)_m
- 2 \left(\nabla_v (f^n - f^l) \cdot \text{d}W_t, f^n - f^l\right)_m.
\end{aligned}
\]

(4.6)

The estimate (4.5) would close similarly to before (i.e. by using BDG and Grönnwall’s inequalities), save for the fact that we do not a priori know that the term \(\left\|f^n - f^l\right\|_{H^{m-1}}^2 \left\|f^n\right\|_{H^{m+1}}^2\) is in \(L^{\infty}_\omega L^1_t\), so we now estimate
it separately\(^3\) (compare to [29, Lemma 7.2]). The stochastic product rule gives:

\[
d(\|f^n - f^l\|_{H^\sigma_{m-1}}^2 d\|f^n\|_{H^\sigma_{m+1}}^2) = \|f^n\|_{H^\sigma_{m+1}}^2 d(\|f^n\|_{H^\sigma_{m-1}}^2 + \|f^n - f^l\|_{H^\sigma_{m-1}}^2 d\|f^n\|_{H^\sigma_{m+1}}^2) + d(\|f^n - f^l\|_{H^\sigma_{m-1}}^2) d(\|f^n\|_{H^\sigma_{m+1}}^2).
\]

(4.7)

The correction term in (4.7) is:

\[
(d\|f^n - f^l\|_{H^\sigma_{m-1}}^2)(d\|f^n\|_{H^\sigma_{m+1}}^2) = \sum_k B_{k,\sigma-1}(f^n - f^l) B_{k,\sigma+1}(f^n)
\]

where:

\[B_{k,\alpha}(h) := \langle \sigma_k e_k \cdot \nabla_v(h), h \rangle_m + \sum_{|\alpha|+|\beta|=s} \langle \partial^\alpha_x \partial^\beta_v (\sigma_k e_k \cdot \nabla_v(h)), \partial^\alpha_x \partial^\beta_v h \rangle_m.\]

Note that \(B_{k,\alpha}(h) \leq C \sigma_k \|e_k\|_{W^{s,\infty}} \|h\|_{H^\sigma}^2,\) so:

\[
d(\|f^n - f^l\|_{H^\sigma_{m-1}}^2) d(\|f^n\|_{H^\sigma_{m+1}}^2) \leq C \|f^n - f^l\|_{H^\sigma_{m-1}}^2 ||f^n\|_{H^\sigma_{m+1}}^2 dt.
\]

(4.8)

For the main terms of (4.7) we have similar estimates as before. For the difference \(f^n - f^l\) and for \(t \leq \tau_{n,l}\) we have (recalling the definition (4.6)):

\[
d\|f^n - f^l\|_{H^\sigma_{m-1}}^2 dt + C \|f^n - f^l\|_{H^\sigma_{m-1}}^2 \|f^n\|_{H^\sigma_{m+1}}^2 dt
\]

\[
- 2 \sum_{|\alpha|+|\beta|=\sigma-1} \langle \partial^\alpha_x \partial^\beta_v (\nabla_v(f^n - f^l)) \cdot \nabla_v(f^n - f^l) \rangle_m
\]

\[
- 2 \langle \nabla_v(f^n - f^l) \cdot dW_t, f^n - f^l \rangle_m
\]

\[
\leq C \|f^n - f^l\|_{H^\sigma_{m-1}}^2 dt + \mathcal{M}_{\sigma-1}(f^n - f^l),
\]

(4.9)

where we used (4.4) for \(\sigma - 1\) instead of \(\sigma\) derivatives and the definition of the stopping time \(\tau^{T}_{n}\). Similarly, for the norm of \(f^n\) and for \(t \leq \tau_{n,l}\) we have:

\[
d\|f^n\|_{H^\sigma_{m+1}}^2 dt \leq C \|f^n\|_{H^\sigma_{m+1}}^2 dt
\]

\[
+ C \|f^n\|_{H^\sigma_{m+1}}^2 \|f^n\|_{H^\sigma_{m+1}}^2 dt
\]

\[- 2 \sum_{|\alpha|+|\beta|=\sigma+1} \langle \partial^\alpha_x \partial^\beta_v (\nabla_v f^n) \cdot dW_t, \partial^\alpha_x \partial^\beta_v f^n \rangle_m
\]

\[- 2 \langle \nabla_v f^n \cdot dW_t, f^n \rangle_m
\]

\[
\leq C \|f^n\|_{H^\sigma_{m+1}}^2 dt + \mathcal{M}_{\sigma+1}(f^n),
\]

(4.10)

where we again used the definition of the stopping time \(\tau^{T}_{n}\). Now, plugging (4.8), (4.9) (4.10) into (4.7), we obtain:

\[
d(\|f^n - f^l\|_{H^\sigma_{m-1}}^2 \|f^n\|_{H^\sigma_{m+1}}^2) \leq C \|f^n - f^l\|_{H^\sigma_{m-1}}^2 \|f^n\|_{H^\sigma_{m+1}}^2 dt
\]

\[+ \mathcal{M}_{\sigma+1}(f^n) \|f^n - f^l\|_{H^\sigma_{m-1}}^2 + \mathcal{M}_{\sigma-1}(f^n - f^l) \|f^n\|_{H^\sigma_{m+1}}^2,
\]

(4.11)

---

\(^3\)This loss of probabilistic moments was addressed by the cutoff in the approximation scheme of Section 3.
Integrating (4.11) in time and using the BDG inequality, we obtain:

\[
\begin{align*}
\mathbb{E} \sup_{t' \leq t \land \tau_{n,l}} \left( \| f^n - f^l \|_{H^m_{\sigma-1}}^2 \right) & \leq \mathbb{E} \left( \| f^n_0 - f^l_0 \|_{H^m_{\sigma-1}}^2 \right) \\
& + C \mathbb{E} \int_0^t \sup_{t' \leq s \land \tau_{n,l}} \| f^n - f^l \|_{H^m_{\sigma-1}}^2 ds \\
& + C \mathbb{E} \left( \int_0^t \| f^n - f^l \|_{H^m_{\sigma-1}}^2 \right)^{1/2}
\end{align*}
\]

so after rearranging and using Grönwall’s inequality as done previously in e.g. the proof of Lemma 3.1, we get:

\[
\mathbb{E} \sup_{t' \leq t \land \tau_{n,l}} \left( \| f^n - f^l \|_{H^m_{\sigma-1}}^2 \right) \leq C \mathbb{E} \left( \| f^n_0 - f^l_0 \|_{H^m_{\sigma-1}}^2 \right).
\] (4.12)

Now returning to (4.5), integrating in time, using the BDG inequality, plugging in (4.12), we obtain:

\[
\begin{align*}
\mathbb{E} \sup_{t' \leq t \land \tau_{n,l}} \| f^n - f^l \|_{H^m_{\sigma-1}}^2 & \leq \mathbb{E} \| f^n_0 - f^l_0 \|_{H^m_{\sigma-1}}^2 \\
& + C \int_0^t \mathbb{E} \sup_{t' \leq s \land \tau_{n,l}} \| f^n - f^l \|_{H^m_{\sigma-1}}^2 ds \\
& + C \mathbb{E} \left( \int_0^t \| f^n - f^l \|_{H^m_{\sigma-1}}^2 ds \right)^{1/2} \\
& + C \mathbb{E} \left( \| f^n_0 - f^l_0 \|_{H^m_{\sigma-1}}^2 \right)^{1/2}
\end{align*}
\]

Therefore we have

\[
\begin{align*}
\mathbb{E} \sup_{t' \leq t_n} \| f^n - f^l \|_{H^m_{\sigma-1}}^2 & \leq C \mathbb{E} \| f^n_0 - f^l_0 \|_{H^m_{\sigma-1}}^2 \\
& + C \mathbb{E} \left( \| f^n_0 - f^l_0 \|_{H^m_{\sigma-1}}^2 \right). 
\end{align*}
\] (4.13)

Then (4.2) follows from (4.13) and Lemma 4.1 (in particular, note (4.1)).

Next, we move to the proof of (4.3). By Itô’s formula, we have:

\[
\begin{align*}
\frac{d}{dt} \| f^n \|_{H^m_{\sigma}}^2 & \leq C \| f^n \|_{H^m_{\sigma}}^2 dt + \| E^n \|_{H^\sigma}^2 dt \\
& - 2 \sum_{|\alpha|+|\beta|=\sigma} \left\langle \partial_x^\alpha \partial_v^\beta (\nabla_v f^n) \cdot dW_t, \partial_x^\alpha \partial_v^\beta f^n \right\rangle_m \\
& - 2 \left\langle \nabla_v f^n \cdot dW_t, f^n \right\rangle_m. 
\end{align*}
\] (4.14)

Let us denote

\[
\mathcal{M}_\sigma(f^n) := 2 \sum_{|\alpha|+|\beta|=\sigma} \left\langle \partial_x^\alpha \partial_v^\beta (\nabla_v f^n) \cdot dW_t, \partial_x^\alpha \partial_v^\beta f^n \right\rangle_m + 2 \left\langle \nabla_v f^n \cdot dW_t, f^n \right\rangle_m
\]

so that for \( t \leq t_n \land \epsilon \), after integrating in time, (4.14) gives:

\[
\| f^n(t) \|_{H^m_{\sigma}}^2 \leq \| f^n_0 \|_{H^m_{\sigma}}^2 + C \int_0^t \| f^n(s) \|_{H^m_{\sigma}}^2 ds + \int_0^t \mathcal{M}_\sigma(s).
\]
Therefore, by Chebyshev’s inequality for the usual deterministic integral and Doob’s inequality for the martingale, we have

\[
P(\sup_{t' \leq \tau_n^{T} \wedge \epsilon} \| f^n \|_{\dot{H}_m^\sigma}^2 > \| f_0 \|_{\dot{H}_m^\sigma}^2 + 1) \leq P \left( C \int_0^{\tau_n^{T} \wedge \epsilon} \| f^n(s) \|_{\dot{H}_m^\sigma}^2 \, ds > \frac{1}{2} \right) + \mathbb{P} \left( \sup_{t' \leq \tau_n^{T} \wedge \epsilon} \left| \int_0^{t'} \mathcal{M}^n(s) \right| > \frac{1}{2} \right) \]

\[
\leq C \mathbb{E} \int_0^{\tau_n^{T} \wedge \epsilon} \| f^n(s) \|_{\dot{H}_m^\sigma}^2 \, ds + CE \| f^n \|_{\dot{H}_m^\sigma}^2 \mathbb{E} \int_0^{\tau_n^{T} \wedge \epsilon} \| f^n(s) \|_{\dot{H}_m^\sigma}^2 \, ds \leq C \epsilon \mathbb{E} \| f_0 \|_{\dot{H}_m^\sigma}^2 \| f_0 \|_{\dot{H}_m^\sigma} \leq CM \epsilon; \]

note we also used that \( t \leq \tau_n^T \) implies \( \| f^n(t) \|_{\dot{H}_m^\sigma} \leq C \) for a constant \( C > 0 \) that depends on the size of the initial data \( f_0 \) uniformly in \( n \), since \( \| f_0 \|_{\dot{H}_m^\sigma} \leq C \| f_0 \|_{\dot{H}_m^\sigma} \) independently of \( n \). Taking \( \epsilon \to 0 \), we obtain (4.3).

Combining Lemmas 4.2 and 4.3, we obtain the existence of a local strong solutions to (1.1) when \( \| f_0 \|_{\dot{H}_m^\sigma} \leq M < \infty \) almost surely. A splitting of the general random initial condition similar to the one in Lemma 2.2 can now provide a local solution whenever \( f_0 \) is \( \mathcal{F}_0 \)-measurable with \( \| f_0 \|_{\dot{H}_m^\sigma} < \infty \) \( \mathbb{P} \)-a.s.. Specifically, since \( f_0 = \sum_{M=0}^{\infty} 1_{M \leq \| f_0 \|_{\dot{H}_m^\sigma} < M+1} f_0 \), each component \( f_{0,M} := 1_{M \leq \| f_0 \|_{\dot{H}_m^\sigma} < M+1} f_0 \) generates a local strong solution \( (f_M, \tau_M) \) to (1.1) and we re-construct the full \( f \) and \( \tau \) using

\[
f = \sum_{M=0}^{\infty} 1_{M \leq \| f_0 \|_{\dot{H}_m^\sigma} < M+1} f_M, \]

and

\[
\tau = \sum_{M=0}^{\infty} 1_{M \leq \| f_0 \|_{\dot{H}_m^\sigma} < M+1} \tau_M. \]

This completes the proof of Theorem 1.3.

5 Hypoelliptic regularization for Vlasov-Poisson-Fokker-Planck

Theorem 1.9 follows by a priori regularization estimates of (2.1), specifically, it suffices to prove that solutions to (2.1) are almost-surely \( C_{x,v}^\infty \) for \( t > 0 \).

We first prove that if \( f_0 \in \dot{H}_m^\sigma \), then the solution lies in \( f(t) \in H_m^{\sigma+1} \) for \( t > 0 \) (with size depending only on the \( H^\sigma \) norm of the initial condition). As mentioned in Section 1, this hypoelliptic regularization is proved using a time-weighted variation of the classical hypocoercive energy functional for the kinetic Fokker-Planck equation (see [58]). For the linear case, a related hypoelliptic regularization estimate can be found in [20]. Taking the standard energy from [58] and scaling derivatives with the powers of \( t \) expected from known hypoelliptic regularization estimates (alternatively, one can deduce them from scaling arguments; see e.g. [6]) we have

\[
\mathcal{E}_1[t,f] = \| f(t) \|_{L^2_m}^2 + at \| \nabla_v f(t) \|_{L^2_m}^2 + bt^2 \langle \nabla_v f(t), \nabla_v f(t) \rangle_{m} + ct^3 \| \nabla_x f(t) \|_{L^2_m}^2. \]
For $H_\sigma^m$ estimates we hence define

$$E_\sigma[t, f] := \sum_{0 \leq q \leq \sigma} E_{1,m}[t, \nabla_{x}^{\sigma-q} \nabla_v^q f].$$

The constants are chosen (indepedent of $\sigma$) such that $0 < c < b < a$ and $b^2 < \sqrt{ac}$ so that

$$E_\sigma \approx \|f\|_{H_\sigma^m} + t \|\nabla_v f\|_{H_\sigma^m}^2 + t^3 \|\nabla_x f\|_{H_\sigma^m}^2.$$ 

The parameters $a, b, c$ are chosen more specifically to satisfy: for some sufficiently small $\varepsilon \ll 1$ we require

$$\begin{cases} 
1 \geq \frac{a}{\varepsilon} \geq \frac{b}{\varepsilon^2} \geq \frac{c}{\varepsilon^3} \\
 a \leq \varepsilon \sqrt{1 - b}, \quad b \leq \varepsilon \sqrt{a - c}.
\end{cases} \quad (5.1)$$

We recall the proof that such $a, b, c$ exist in Lemma 5.2 below. Note that these conditions imply $bt^2 \leq \varepsilon at + \varepsilon ct^3$, a fact we use repeatedly below.

Hence, an estimate on $E_\sigma$ in terms of $\|f_0\|_{H_\sigma^m}$ implies the desired regularization estimates (along with some more quantitative information that we will not directly use here). The main result of this Section is the following.

**Proposition 5.1.** Let $f_0$ be a $\mathcal{F}_0$-measurable initial data and suppose that for all $p < \infty$, $\exists M_p > 0$ such that for some $\sigma > \frac{4}{2} + 1$ we have:

$$\mathbb{E} \|f_0\|_{H_\sigma^m}^p < M_p. \quad (5.2)$$

Let $R < \infty$, let $f$ be the unique pathwise solution of (2.1).

Then $\exists T > 0$ depending only on $\sigma$ such that there holds for all $p < \infty$,

$$\mathbb{E} \left( \sup_{0 < t < T} E_\sigma[t, f(t)] \right)^p \leq C(R, p, M_2, M_3, \ldots).$$

Therefore, almost surely $f(t) \in H_\sigma^{\sigma + 1}$ for all $0 < t < T$.

Proposition 5.1 implies a corresponding instantaneous regularization for the maximal pathwise solution of (1.1). Once the above proposition is proved, one may simply iterate it, observing that for all $\delta > 0$, $f(\delta)$ is an $\mathcal{F}_\delta$-measurable random variable with

$$\mathbb{E} \|f(\delta)\|_{H_\sigma^{\sigma + 1}}^p < \infty.$$ 

Therefore, we may apply Proposition 5.1 to the initial data $f(\delta)$ with $\sigma \mapsto \sigma + 1$. Finally, similar to the proof of Lemma 2.2, a simple cutting procedure can be applied to remove the moment constraint on the initial condition. Hence, to prove Theorem 1.9, it suffices to prove Proposition 5.1.

**Proof.** For notational simplicity, we will take $\nu = 1$ but the same arguments (up to a suitable rescaling of the coefficients $a, b, c$) apply for any $\nu > 0$. Define the dissipation rate:

$$\mathbb{D}_\sigma(t, f(t)) := \sum_{\alpha + \beta \leq \sigma} \left( \left\| \nabla_v \partial_x^\alpha \partial_v^\beta f(t) \right\|_{L_2^n}^2 + at \left\| \nabla_v \partial_x^\alpha \partial_v^\beta f(t) \right\|_{L_2^n}^2 \right) + \frac{b}{2} \left( \left\| \nabla_x \partial_x^\alpha \partial_v^\beta f(t) \right\|_{L_2^n}^2 + ct^3 \left\| \nabla_v \nabla_x \partial_x^\alpha \partial_v^\beta f(t) \right\|_{L_2^n}^2 \right),$$

35
which we show arises from $\text{d} \mathcal{E}_\sigma$. Note that this is almost the same as the contribution from $\text{d} \mathcal{E}_\sigma$ that arises when the time derivative lands on the powers of $t$ in front of the higher-order terms. In order to reduce some of the notation in the ensuing calculation, we use $B(h, g)$ to denote an $L^2_m$-bounded bilinear form, the exact form of which is irrelevant, i.e, a form which is linear in both arguments and such that for any $h, g \in L^2_m$

$$
\|B(h, g)\|_{L^2_m} \lesssim \|h\|_{L^2_m} \|g\|_{L^2_m}.
$$

The main step of the proof is to calculate the following

$$
\text{d} \mathcal{E}_\sigma = \sum_{|\alpha| + |\beta| \leq \sigma} \text{d} \left\| \partial_x^{\alpha} \partial_{v}^{\beta} f(t) \right\|^2_{L^2_m} + \text{d} \left( a t \left\| \nabla_{v} \partial_x^{\alpha} \partial_{v}^{\beta} f(t) \right\|^2_{L^2_m} \right)

+ \sum_{|\alpha| + |\beta| \leq \sigma} \text{d} \left( b t^2 \left( \nabla_{v} \partial_x^{\alpha} \partial_{v}^{\beta} f(t), \nabla_{x} \partial_x^{\alpha} \partial_{v}^{\beta} f(t) \right) \right) + \text{d} \left( c t^3 \left\| \nabla_x \partial_x^{\alpha} \partial_{v}^{\beta} f(t) \right\|^2_{L^2_m} \right).
$$

As in the proof of the various bounds in Sections 3–4, we have:

$$
\text{d} \left\| \partial_x^{\alpha} \partial_{v}^{\beta} f \right\|^2_{L^2_m} = -2 \left( \partial_x^{\alpha} \partial_{v}^{\beta} (v \cdot \nabla_x f), \partial_x^{\alpha} \partial_{v}^{\beta} f \right) \, dt

+ 2 \left( \Delta_v \partial_x^{\alpha} \partial_{v}^{\beta} f, \partial_x^{\alpha} \partial_{v}^{\beta} f \right) \, dt

+ 2 \left( \partial_x^{\alpha} \partial_{v}^{\beta} \text{div}_v(f v), \partial_x^{\alpha} \partial_{v}^{\beta} f \right) \, dt

- 2 \left( \partial_x^{\alpha} \partial_{v}^{\beta} (\theta_R(\|f\|_{H^s_{\text{m}}}) E \cdot \nabla_v f), \partial_x^{\alpha} \partial_{v}^{\beta} f \right) \, dt

- 2 \left( \partial_x^{\alpha} \partial_{v}^{\beta} (\nabla_v f \cdot d W_t), \partial_x^{\alpha} \partial_{v}^{\beta} f \right) \, dt

+ \sum_k \left( \partial_x^{\alpha} \partial_{v}^{\beta} ((\sigma_k e_k \cdot \nabla_v)^2 f), \partial_x^{\alpha} \partial_{v}^{\beta} f \right) \, dt

- \sum_k \left\| \partial_x^{\alpha} \partial_{v}^{\beta} (\sigma_k e_k \cdot \nabla_v f) \right\|^2_{L^2_m} \, dt.
$$

(5.3)

This formula, and its straightforward variations, are then used to expand most of the terms of $\text{d} \mathcal{E}_\sigma$, with the exception of the cross-terms (i.e. those multiplied by $b$). For the cross-terms we instead have

$$
\text{d} \left( \partial_x^{\alpha} \partial_{v}^{\beta+\epsilon_j} f, \partial_x^{\alpha+\epsilon_j} \partial_{v}^{\beta} f \right) \, dt

= - \left( \left( \partial_x^{\alpha} \partial_{v}^{\beta+\epsilon_j} (v \cdot \nabla_x f), \partial_x^{\alpha+\epsilon_j} \partial_{v}^{\beta} f \right) \right) \, dt

- \left( \theta_R \partial_x^{\alpha} \partial_{v}^{\beta+\epsilon_j} (E \cdot \nabla_v f), \partial_x^{\alpha+\epsilon_j} \partial_{v}^{\beta} f \right) \, dt

+ \left( \partial_x^{\alpha} \partial_{v}^{\beta+\epsilon_j} (\nabla_v f \cdot d W_t), \partial_x^{\alpha+\epsilon_j} \partial_{v}^{\beta} f \right) \, dt

- \left( \partial_x^{\alpha} \partial_{v}^{\beta+\epsilon_j} (\sigma_k e_k \cdot \nabla_v)^2 f, \partial_x^{\alpha+\epsilon_j} \partial_{v}^{\beta} f \right) \, dt

+ \frac{1}{2} \sum_k \left( \partial_x^{\alpha} \partial_{v}^{\beta+\epsilon_j} (\sigma_k e_k \cdot \nabla_v)^2 f, \partial_x^{\alpha+\epsilon_j} \partial_{v}^{\beta} f \right) \, dt

+ \frac{1}{2} \sum_k \left( \partial_x^{\alpha} \partial_{v}^{\beta+\epsilon_j} f, \partial_x^{\alpha+\epsilon_j} \partial_{v}^{\beta} (\sigma_k e_k \cdot \nabla_v)^2 f \right) \, dt
$$

36
+ \sum \left\langle \partial^2_x \sigma_{\alpha} \sigma_{\beta} (\sigma_k e_k \cdot \nabla_v f), \sigma_{x}^{\alpha+e_j} \sigma_{v}^{\beta} (\sigma_k e_k \cdot \nabla_v f) \right\rangle_m \, dt
\]

\[= \mathcal{T}_{c,\alpha,\beta,j}(f) + N_{c,\alpha,\beta,j}(f) + D_{c,\alpha,\beta,j}(f) + F_{c,\alpha,\beta,j}(f) + M_{c,\alpha,\beta,j}(f) + C_{c,\alpha,\beta,j}(f),\]

where we abbreviated \( \theta_R := \theta_R(\|f\|_{H^3_{\sigma}}) \) and \( \mathcal{T}_{c,\alpha,\beta,j}, N_{c,\alpha,\beta,j}, D_{c,\alpha,\beta,j}, F_{c,\alpha,\beta,j}, M_{c,\alpha,\beta,j}, C_{c,\alpha,\beta,j} \) indicate transport, nonlinear (electric field), dissipation, friction, martingale, and (Itô) correction contributions (which incorporate the last three terms) to the cross-terms, respectively.

**Linear, deterministic contributions:**

First, we collect the contributions of the linear terms, namely those that arise from the \( v \cdot \nabla_x \) free transport and the Fokker–Planck operator. The main effect of these terms is to introduce the dissipation \( \mathbb{D}_\sigma \). The calculation is standard (see [58]) and so we omit most of the details. We define the total contribution of the linear terms of the SPDE for \( f \) to \( d\mathcal{E} \) by:

\[
\text{Lin}(\alpha, \beta) := a \left\| \nabla_v \partial^\alpha_x \partial^\beta_v f \right\|^2_m + 2bt \left\langle \nabla_v \partial_x^\alpha \nabla_v^\beta f, \nabla_v \partial_x^\alpha \partial_v^\beta f \right\rangle_m + 3ct^2 \left\| \nabla_x \partial^\alpha_x \partial_v^\beta f \right\|^2_{L^2_m}
\]

\[- 2 \left\langle \partial^\alpha_x \partial_v^\beta (v \cdot \nabla_x f), \partial^\alpha_x \partial_v^\beta f \right\rangle_m + 2 \left\langle \Delta_v \partial^\alpha_x \partial_v^\beta f, \partial^\alpha_x \partial_v^\beta f \right\rangle_m
\]

\[+ 2 \left\langle \partial^\alpha_x \partial_v^\beta \partial_v \partial_v(f v), \partial^\alpha_x \partial_v^\beta f \right\rangle_m
\]

\[- 2at \left\langle \nabla_v \partial^\alpha_x \partial_v^\beta (v \cdot \nabla_x f), \nabla_v \partial^\alpha_x \partial_v^\beta f \right\rangle_m + 2at \left\langle \nabla_v \partial^\alpha_x \partial_v^\beta \Delta_v f, \nabla_v \partial^\alpha_x \partial_v^\beta f \right\rangle_m
\]

\[+ 2at \left\langle \partial^\alpha_x \partial_v^\beta \partial_v \nabla_v(f v), \nabla_v \partial^\alpha_x \partial_v^\beta f \right\rangle_m
\]

\[- bl^2 \left( \sum_j \mathcal{T}_{c,\alpha,\beta,j} + \mathcal{D}_{c,\alpha,\beta,j} + \mathcal{F}_{c,\alpha,\beta,j} \right)
\]

\[- 2ct^3 \left\langle \nabla_x \partial^\alpha_x \partial_v^\beta (v \cdot \nabla_x f), \nabla_v \partial^\alpha_x \partial_v^\beta f \right\rangle_m
\]

\[+ 2ct^3 \left\langle \Delta \partial^\alpha_x \partial_v^\beta \partial_v \partial_v(f v), \partial^\alpha_x \partial_v^\beta \partial_v \partial_v f \right\rangle_m + 2ct^3 \left\langle \partial^\alpha_x \partial_v^\beta (\partial_v \partial_v(f v)), \partial^\alpha_x \partial_v^\beta \partial_v \partial_v f \right\rangle_m \tag{5.4}
\]

For the rest of this proof, we denote \( p := |\alpha|, q := |\beta| \). By integration by parts we may write

\[\mathcal{T}_{c,\alpha,\beta,j}(f) = - \left\langle \partial^\alpha_x \partial_v^\beta \partial_v (v \cdot \nabla_x f), \partial^\alpha_x \partial_v^\beta (v \cdot \nabla_x f) \right\rangle_m \, dt - \left\langle \partial^\alpha_x \partial_v^\beta \partial_v (v \cdot \nabla_x f), \partial^\alpha_x \partial_v^\beta (v \cdot \nabla_x f) \right\rangle_m \, dt
\]

\[= \left\| \partial^\alpha_x \partial_v^\beta f \right\|^2_{L^2_m} \, dt
\]

\[- \left\langle \nabla_v (\partial^\alpha_x \partial_v^\beta f), \partial^\alpha_x \partial_v^\beta f \right\rangle_m \, dt - \left\langle \nabla_v (\partial^\alpha_x \partial_v^\beta f), \partial^\alpha_x \partial_v^\beta f \right\rangle_m \, dt
\]

\[+ \sum_{\beta' < \beta} \left( \begin{array}{c} \beta' \\ \alpha \end{array} \right) \left( \left\langle \partial^\alpha_x \partial_v^{\beta' + \beta} \partial_v (v \cdot \nabla_x f), \partial^\alpha_x \partial_v^{\beta' + \beta} \partial_v (v \cdot \nabla_x f) \right\rangle_m + \left\langle \partial^\alpha_x \partial_v^{\beta' + \beta} \partial_v (v \cdot \nabla_x f), \partial^\alpha_x \partial_v^{\beta' + \beta} \partial_v (v \cdot \nabla_x f) \right\rangle_m \right) \, dt
\]

\[= \left\| \partial^\alpha_x \partial_v^\beta f \right\|^2_{L^2_m} \, dt, \quad \text{if } \beta = 0
\]

\[= \left\| \partial^\alpha_x \partial_v^\beta f \right\|^2_{L^2_m} \, dt + \sum_{q=0}^{q-1} B(\nabla_{x}^{p+1} \nabla_{v}^{q} f, \nabla_{x}^{p+1} \nabla_{v}^{q} f) \, dt \quad \text{if } |\beta| > 0, \tag{5.5}
\]

where recall from above that \( B(\cdot, \cdot) \) denotes an \( L^2_m \) bounded bilinear form, the exact form of which is not relevant. The dissipation term is more easily treated, yielding

\[
\mathcal{D}_{c,\alpha,\beta}(f) = \left\langle \partial^\alpha_x \partial_v^{\beta+e_j} \Delta_v f, \partial^\alpha_x \partial_v^{\beta+e_j} \partial_v \partial_v f \right\rangle_m \, dt + \left\langle \partial^\alpha_x \partial_v^{\beta+e_j} f, \partial^\alpha_x \partial_v^{\beta+e_j} \Delta_v f \right\rangle_m \, dt
\]

37
\[
= -2 \left( \partial_x^a \partial_v^b \nabla_v f, \partial_x^{a+e_j} \partial_v^{b+e_j} \nabla_v f \right)_m dt \\
+ B(\nabla_x^{p+q+1} f, \nabla_x^{p+1} f) dt + B(\nabla_x^{q+1} f, \nabla_x^{p+1} f) dt.
\]

(5.6)

The friction term can be re-arranged as follows

\[
\mathcal{F}_{c, \alpha, \beta}(f) = \left( \partial_x^a \partial_v^b (\nabla_x \nabla_v f), \partial_x^{a+e_j} \partial_v^{b+e_j} f \right)_m dt + \left( \partial_x^a \partial_v^b \nabla_v f, \partial_x^{a+e_j} \partial_v^{b+e_j} (\div \nabla_v f) \right)_m dt \\
+ \sum_{\beta' < \beta | \beta'| = 1} \left( \partial_x^{a+e_j} \partial_v^{b+\beta'} f, \partial_x^a \partial_v^{b+e_j} f \right)_m dt + \sum_{\beta' < \beta | \beta'| = 1} \left( \partial_x^{a+e_j} \partial_v^{b+\beta'} f, \partial_x^a \partial_v^{b+e_j} f \right)_m dt
\]

(5.7)

The fundamental structure of the hypocoercive norm \( E \) is that the \( T \) term gives rise to the \( \nabla_x \) dissipation term that would otherwise be missing from the dissipation of a kinetic equation. That is, from (5.5), we obtain:

\[
b t^2 T_{c, \alpha, \beta, j}(f) + b t^2 \left\| \partial_x^{a+e_j} \partial_v^{b+e_j} f \right\|_{L^2}^2 dt \leq b t^2 B \left( \nabla_v \left( \nabla_x^{p+1} \nabla_v^{q+1} f \right), \nabla_v \left( \nabla_x^{q+1} \nabla_v^{p+1} f \right) \right) dt \\
+ b t^2 B \left( \nabla_x^{q+1} \nabla_v^{p+1} f, \nabla_v \left( \nabla_x^{q+1} \nabla_v^{p+1} f \right) \right) dt \\
\lesssim \frac{a}{t} \| \mathcal{D}_\sigma \| dt + b t^2 \mathcal{E}_\sigma dt \\
\lesssim \frac{b}{t} \| \mathcal{D}_\sigma \| dt + b t^2 \mathcal{E}_\sigma dt.
\]

(5.8)

and similarly, from (5.6) and (5.7):

\[
b t^2 D_{c, \alpha, \beta, j} + b t^2 \mathcal{F}_{c, \alpha, \beta, j} \lesssim \left( \frac{b}{a} + \frac{bq}{c} \right) \| \mathcal{D}_\sigma \| dt + \mathcal{E}_\sigma dt \\
\lesssim \| \mathcal{D}_\sigma \| dt + \mathcal{E}_\sigma dt.
\]

(5.9)

Putting together the negative definite terms that arise from \( T_{c, \alpha, \beta, j} \) and those in (5.4), we obtain for short \( t \) and for some \( C' > 0 \):

\[
Lin(\alpha, \beta) \leq - (2 - Ct) \| \mathcal{D}_\sigma \| + C \mathcal{E}_\sigma.
\]

In fact this is somewhat sub-optimal, as the second term on the right-hand side above can be taken in weaker norms. However, such refinements will be irrelevant here as we are only interested in short time regularization.

**Nonlinear contributions:**

Next, we collect the contributions of the nonlinear electric field. Namely,

\[
NL(\alpha, \beta) := -2 \left( \partial_x^a \partial_v^b \theta_R(\| f \|_{H_{w_{00}}}^m) E \cdot \nabla_v f, \partial_x^{a+e_j} \partial_v^{b+e_j} f \right)_m \\
- 2at \left( \partial_x^a \partial_v^b \nabla_v (\theta_R(\| f \|_{H_{w_{00}}}^m) E \cdot \nabla_v f), \partial_x^{a+e_j} \partial_v^{b+e_j} \nabla_v f \right)_m
\]

38
Note that in the above, \( \eta < 1 \) since \( \frac{d}{2(\sigma-1)} < 1 \). Therefore,

\[
\| E \|_{L^\infty W^{1,1}} \| E \|_{H^\sigma} \left\| \nabla_v \partial_v^\beta f \right\|_{L^2_m} \| f \|_{H^\sigma_m} \lesssim \| f \|_{H^\sigma_m}^2 + \left( \| E \|_{L^\infty W^{1,1}} \| E \|_{H^\sigma_m} \right)^{\frac{2-2\eta}{4-2\eta}} \| f \|_{H^\sigma_m}^{\frac{4-2\eta}{4-2\eta}}.
\]

Note that in this term, it was not necessary to make use of the dissipation, as the first term in the final inequality above is controlled by at most \( H^\sigma_m \) and the second term, which is derived from the factor \( \| E \|_{L^\infty W^{1,1}} \| E \|_{H^\sigma_m} \| \nabla_v \partial_v^\beta f \|_{L^2_m} \), contains at most \( \sigma - 1 \) derivatives (since \( |\beta| \leq \sigma - 2 \) whenever \( |\alpha - \gamma| \geq 2 \)).

In a similar manner, the corresponding “second term” in the electric field contributions to \( \partial_x^\alpha \partial_v^\beta f \) and \( \partial_x^\alpha \partial_v^{\beta+\epsilon_j} f \) terms of the energy will contain at most \( \sigma \) derivatives. This means that all in all, for short \( t \) we can bound the electric field contributions to (5.3), as well as those to the higher order terms in the definition of \( E_\sigma \), from (5.10):

\[
\left\| \left\langle \nabla_v \partial_v^\beta (E \cdot \nabla_v f) \bigg| \nabla_v \partial_v^\beta f \right\rangle \right\|_{m} \lesssim \| f \|_{H^\sigma_{m-1}}^p + \mathcal{E}_\sigma \quad \text{at} \quad \left\| \left\langle \nabla_v \partial_v^{\beta+\epsilon_j} (E \cdot \nabla_v f) \bigg| \nabla_v \partial_v^{\beta+\epsilon_j} f \right\rangle \right\|_{m} \lesssim \| f \|_{H^\sigma_m}^p + \mathcal{E}_\sigma
\]

for some \( p > 2 \) fixed. Note that the high power of \( \| f \|_{H^\sigma_m} \) is a priori controlled (in \( L^1_t L^\infty_{x,loc} \)) by the finite \( p \)-th moment assumptions (5.2).
We now move to estimating the electric field contribution to the cross term. First, we integrate by parts for convenience, in order to “symmetrize” $N_{c,\alpha,\beta,j}$ up to a lower order term:

$$N_{c,\alpha,\beta,j} = \left\langle \partial_x^\alpha \partial_y^{\alpha+\varepsilon_j} (E \cdot \nabla_v f), \partial_x^\alpha \partial_y^{\alpha+\varepsilon_j} (E \cdot \nabla_v f) \right\rangle_m + \left\langle \partial_x^\alpha \partial_y^{\alpha+\varepsilon_j} (E \cdot \nabla_v f), \partial_x^\alpha \partial_y^{\alpha+\varepsilon_j} (E \cdot \nabla_v f) \right\rangle_m$$

$$= - \left\langle \partial_x^\alpha \partial_y^{\alpha+\varepsilon_j} (E \cdot \nabla_v f), \partial_x^\alpha \partial_y^{\alpha+\varepsilon_j} (E \cdot \nabla_v f) \right\rangle_m + \left\langle \partial_x^\alpha \partial_y^{\alpha+\varepsilon_j} (E \cdot \nabla_v f), \partial_x^\alpha \partial_y^{\alpha+\varepsilon_j} (E \cdot \nabla_v f) \right\rangle_m$$

$$= 2 \left\langle \partial_x^\alpha \partial_y^{\alpha+\varepsilon_j} (E \cdot \nabla_v f), \partial_x^\alpha \partial_y^{\alpha+\varepsilon_j} (E \cdot \nabla_v f) \right\rangle_m$$

$$+ \iint \partial_x^\alpha \partial_y^{\alpha+\varepsilon_j} (E \cdot \nabla_v f) \partial_x^\alpha \partial_y^{\alpha+\varepsilon_j} f \partial_v (|v|^m) \, dv \, dx.$$  \hspace{1cm} (5.13)

We analyze the first term in the final equality above:

$$\left\langle \partial_x^\alpha \partial_y^{\alpha+\varepsilon_j} (E \cdot \nabla_v f), \partial_x^\alpha \partial_y^{\alpha+\varepsilon_j} f \right\rangle_m \lesssim \sum_{\gamma \leq \alpha} \left| \left\langle \partial_x^\alpha \partial_y^{\alpha+\varepsilon_j} (E \cdot \nabla_v f), \partial_x^\alpha \partial_y^{\alpha+\varepsilon_j} f \right\rangle_m \right|$$

$$\leq \|E\|_{L_{x\gamma}^\infty} \|\nabla_v^{q+2} \nabla_{x\gamma}^p f\|_{L_m^2} \|\nabla_v^{p+\varepsilon_j+1} f\|_{L_m^2}$$

$$+ \|\nabla_x E\|_{L_{x\gamma}^\infty} \|\nabla_v^{q+2} \nabla_{x\gamma}^{p-1} f\|_{L_m^2} \|\nabla_x^{p+1} \nabla_v^{q+1} f\|_{L_m^2}$$

$$+ \sum_{\gamma < \alpha} \left\| \nabla_x^{\alpha-\varepsilon} E \right\|_{L_{x,\gamma}^{2,\sigma}} \left\| \nabla_v^{q+2} \nabla_{x\gamma}^\beta f \right\|_{L_m^{\sigma}} \left\| \nabla_v^{p+1} \nabla_{x\gamma}^\beta f \right\|_{L_m^{\sigma}}.$$  \hspace{1cm} (5.14)

To estimate each term in the summation above, we again interpolate as in the proof of Lemma 3.10:

$$\left\| \nabla_x^{\alpha-\varepsilon} E \right\|_{L_{x,\gamma}^{2,\sigma}} \lesssim \|\nabla_x E\|_{L_{x,\gamma}^{2,\sigma}}^{1-\frac{|\alpha-\varepsilon|}{2\sigma}} \|\nabla_x^{\alpha-\varepsilon} E\|_{L_{x,\gamma}^{2,\sigma}}^{\frac{|\alpha-\varepsilon|}{2\sigma}}.$$  \hspace{1cm} (5.15)

$$\left\| \nabla_v^{q+2} \nabla_{x\gamma}^\beta f \right\|_{L_m^{\sigma}} \lesssim \left\| \nabla_v^{q+2} f \right\|_{L_m^{\sigma}}^{1-\eta} \left\| \nabla_v^{q+2} \nabla_{x\gamma}^\beta f \right\|_{L_m^{\sigma}}^\eta.$$  \hspace{1cm} (5.16)

where

$$\eta = \frac{\beta}{\alpha} + \frac{d}{2} \left( \frac{1}{2} - \frac{\epsilon + \gamma}{2\sigma} \right),$$

which again satisfies $\eta < 1$ since $\sigma > d/2$. With these exponents, plugging (5.15)-(5.16) into (5.14) we get:

$$\left\langle \partial_x^\alpha \partial_y^{\alpha+\varepsilon_j} (E \cdot \nabla_v f), \partial_x^\alpha \partial_y^{\alpha+\varepsilon_j} f \right\rangle_m \lesssim \|E\|_{L_{x\gamma}^\infty} \|\nabla_v^{q+2} \nabla_{x\gamma}^p f\|_{L_m^2} \|\nabla_v^{q+2} \nabla_{x\gamma}^{p+1} f\|_{L_m^2}$$

$$+ \|\nabla_x E\|_{L_{x\gamma}^\infty} \|\nabla_v^{q+2} \nabla_{x\gamma}^{p-1} f\|_{L_m^2} \|\nabla_x^{p+1} \nabla_v^{q+1} f\|_{L_m^2}$$

$$+ \sum_{\gamma < \alpha} \|\nabla_x^{\alpha-\varepsilon} E\|_{L_{x,\gamma}^{2,\sigma}} \|\nabla_v^{q+2} \nabla_{x\gamma}^\beta f\|_{L_m^{\sigma}} \|\nabla_v^{p+1} \nabla_{x\gamma}^\beta f\|_{L_m^{\sigma}}.$$  \hspace{1cm} (5.17)

The lower order term in (5.13) produces a less significant contribution - as it contains a smaller number of derivatives - and hence we omit the treatment for brevity. Therefore, since $|\alpha - \gamma| \geq 2$ implies $q + 2 \leq \sigma$, we obtain from (5.17) that $\exists C(\varepsilon) > 0$ such that

$$bt^2 N_{c,\alpha,\beta,j} \leq \frac{\varepsilon}{10} \mathbb{D}_{\sigma} \varepsilon + C(\varepsilon) E \varepsilon dt + C(\varepsilon) \|f(t)\|_{H_m^n} dt.$$  \hspace{1cm} (5.18)

This completes the required estimates on the electric field.
Itô corrections:

Next, let us analyze the contributions of the corrections to the cross term. This equals:

\[
C_{c, \alpha, \beta, j}(f) = \frac{1}{2} \sum_k \left( \langle \partial_x^\alpha \partial_v^{\beta+\epsilon_j}(\sigma_k e_k \cdot \nabla_v)^2 f, \partial_x^{\alpha+\epsilon_j} \partial_v^{\beta} f \rangle_m + \langle \partial_x^\alpha \partial_v^{\beta+\epsilon_j} f, \sigma_k \partial_x^{\alpha+\epsilon_j} \partial_v^{\beta} (\sigma_k e_k \cdot \nabla_v)^2 f \rangle_m \right) + \sum_k \left( \langle \partial_x^\alpha \partial_v^{\beta+\epsilon_j}(\sigma_k e_k \cdot \nabla_v f), \partial_x^{\alpha+\epsilon_j} \partial_v^{\beta} (\sigma_k e_k \cdot \nabla_v f) \rangle_m \right) dt
\]

\[
= \sum_{p' \leq p} \left( B(\nabla_v^{p+1} \nabla_v^{q+1} f, \nabla_v^{p'+1} \nabla_v^{q+1} f) + B(\nabla_v^{p+1} \nabla_v^{q+1} f, \nabla_v^{p'} \nabla_v^{q+1} f) \right) dt
\]

As mentioned above, here \(B\) denotes a bilinear form which is bounded on \(L^2_m \times L^2_m\), the exact form of which is irrelevant. It follows that

\[
bt^2 C_{c, \alpha, \beta, j} \lesssim (\varepsilon + t) \mathbb{D}_\sigma + \mathcal{E}_\sigma.
\]

As in (5.3), from similar calculations to those in the proof of (3.2) in Lemma 3.1 for \(p = 2\), we have

\[
\sum_k \langle \partial_x^\alpha \partial_v^{\beta}(\sigma_k e_k \cdot \nabla_v)^2 f, \partial_x^{\alpha+\epsilon_j} \partial_v^{\beta} f \rangle_m - \sum_k \left\| \partial_x^\alpha \partial_v^{\beta}(\sigma_k e_k \cdot \nabla_v f) \right\|^2_{L^2_m} \lesssim \mathcal{E}_\sigma
\]

\[
at \sum_k \left\{ \langle \nabla_v \partial_x^\alpha \partial_v^{\beta}(\sigma_k e_k \cdot \nabla_v)^2 f, \nabla_v \partial_x^{\alpha+\epsilon_j} \partial_v^{\beta} f \rangle_m - \sum_k \left\| \nabla_v \partial_x^\alpha \partial_v^{\beta}(\sigma_k e_k \cdot \nabla_v f) \right\|^2_{L^2_m} \right\} \lesssim \mathcal{E}_\sigma
\]

\[
ct^2 \sum_k \left\{ \langle \nabla_x \partial_x^\alpha \partial_v^{\beta}(\sigma_k e_k \cdot \nabla_v)^2 f, \nabla_x \partial_x^{\alpha+\epsilon_j} \partial_v^{\beta} f \rangle_m - \sum_k \left\| \nabla_x \partial_x^\alpha \partial_v^{\beta}(\sigma_k e_k \cdot \nabla_v f) \right\|^2_{L^2_m} \right\} \lesssim \mathcal{E}_\sigma.
\]

This completes the necessary estimates on the Itô correction terms.

Final estimate:

For \(t\) sufficiently small, combining the estimates on the linear terms of (5.3) from the above arguments with (5.8), (5.9), (5.18) and (5.11)-(5.12), we ultimately obtain

\[
d\mathcal{E}_\sigma \leq C \mathcal{E}_\sigma dt - (2 - C\varepsilon) \mathbb{D}_\sigma dt + \mathcal{M}_\sigma + C \left\| f \right\|_{H^2_m}^2 dt,
\]

where \(\mathcal{M}_\sigma\) denotes all of the martingale terms:

\[
\mathcal{M}_\sigma = -2 \left\langle \partial_x^\alpha \partial_v^{\beta}(\nabla_v f \cdot dW_t), \partial_x^{\alpha+\epsilon_j} \partial_v^{\beta} f \right\>_m
\]

\[
- 2at \sum_{j=1}^d \left\langle \partial_x^\alpha \partial_v^{\beta+\epsilon_j}(\nabla_v f \cdot dW_t), \partial_x^\alpha \partial_v^{\beta+\epsilon_j} f \right\>_m
\]

\[
- bt^2 \sum_{j=1}^d \left\langle \partial_x^\alpha \partial_v^{\beta+\epsilon_j}(\nabla_v f \cdot dW_t), \partial_x^{\alpha+\epsilon_j} \partial_v^{\beta} f \right\>_m
\]

\[
41
\]
\[-bt^2 \sum_{j=1}^{d} \left\langle \partial_x^2 \partial_v^{\alpha+\epsilon_j} f, \partial_x^{\alpha+\epsilon_j} \partial_v^\beta (\nabla_v f \cdot dW_t) \right\rangle_m \]
\[-2ct^3 \sum_{j=1}^{d} \left\langle \partial_x^{\alpha+\epsilon_j} \partial_v^\beta (\nabla_v f \cdot dW_t), \partial_x^{\alpha+\epsilon_j} \partial_v^\beta f \right\rangle_m.\]

Therefore, integrating in time and using the BDG inequality, we obtain
\[
E \sup_{t' \leq \tau} E_\sigma(t') + (2 - C\varepsilon) E \int_0^\tau D_\sigma(s)ds \leq E \|f_0\|_{H^m}^2 + CE \int_0^\tau E_\sigma(s)ds + CE \int_0^\tau \|f(s)\|_{H^m}^p ds
\]
\[
+ CE \left( \int_0^\tau (E_\sigma(s) + \varepsilon \sqrt{E_\sigma(s)} D_\sigma(s))^2 ds \right)^{\frac{1}{2}}
\]
\[
\leq E \|f_0\|_{H^m}^2 + CE \int_0^\tau E_\sigma(s)ds + CE \int_0^\tau \|f(s)\|_{H^m}^p ds
\]
\[
+ CE \sup_{t' \leq \tau} E_\sigma^\frac{1}{2}(t') \left( \int_0^\tau (E_\sigma(s)^{\frac{1}{2}} + \varepsilon D_\sigma(s)^{\frac{1}{2}})^2 ds \right)^{\frac{1}{2}}
\]
\[
\leq E \|f_0\|_{H^m}^2 + CE \int_0^\tau E_\sigma(s)ds + CE \int_0^\tau \|f(s)\|_{H^m}^p ds
\]
\[
+ \frac{1}{2} E \sup_{t' \leq \tau} E_\sigma(t') + C\varepsilon^2 E \int_0^\tau D_\sigma(s)ds.
\]

Rearranging terms and applying Grönwall’s inequality, we obtain
\[
E \sup_{t' \leq \tau} E_\sigma(t') \lesssim 1.
\]

As is standard (and as in the proofs of the $p > 2$ estimates (3.2) and (3.26) in Section 3), a straightforward variation of the above argument extends to prove
\[
E \left( \sup_{t' \leq \tau} E_\sigma(t') \right)^p \lesssim 1,
\]
completing the desired estimates. \qed

For the reader’s convenience, we include the elementary proof that such constants $a, b, c$ as prescribed in (5.1) do indeed exist. A more general situation is treated in [57, Lemma A.16].

**Lemma 5.2.** Let $\varepsilon > 0$. Then there exist constants $a, b, c$ such that (5.1) holds:

\[
\begin{cases}
1 \geq \frac{a}{\varepsilon} \geq \frac{b}{\varepsilon} \geq \frac{c}{\varepsilon} \\
a \leq \varepsilon \sqrt{1 \cdot b}, \quad b \leq \varepsilon \sqrt{a \cdot c}
\end{cases}
\]

**Proof.** Let $\vartheta > 0$, and pick $m_1, m_2, m_3 > 0$ such that
\[
m_1 = 1, \quad m_2 \in (1, 2), \quad m_3 \in (m_2, 2m_2 - 1)
\]
and set:
\[
a = \vartheta^{m_1} = \vartheta, \quad b = \vartheta^{m_2}, \quad c = \vartheta^{m_3}.
\]

The proof of the lemma is concluded by picking $\vartheta$ sufficiently small. \qed
References

[1] D. Alonso-Orán and A. Bethencourt de León, *On the well-posedness of stochastic Boussinesq equations with transport noise*, Journal of Nonlinear Science 30 (2020), no. 1, 175–224.

[2] R. Balest, H.-D. Wang, and J. Misguich, *Langevin equation versus kinetic equation: Subdiffusive behavior of charged particles in a stochastic magnetic field*, Physics of Plasmas 1 (1994), no. 12, 3826–3842.

[3] J. Batt and G. Rein, *Global classical solutions of the periodic Vlasov-Poisson system in three dimensions*, Comptes rendus de l'Académie des sciences. Série 1, Mathématique 313 (1991), no. 6, 411–416.

[4] L. Beck, F. Flandoli, M. Gubinelli, and M. Maurelli, *Stochastic ODEs and stochastic linear PDEs with critical drift: regularity, duality and uniqueness*, Electronic Journal of Probability 24 (2019), 1–72.

[5] A. Bensoussan, *Stochastic Navier–Stokes equations*, Acta Applicandae Mathematica 38 (1995), no. 3, 267–304.

[6] F. Bouchut, *Existence and uniqueness of a global smooth solution for the Vlasov-Poisson-Fokker-Planck system in three dimensions*, Journal of Nonlinear Science 30 (2020), no. 1, 175–224.

[7] T. M Boyd, T. Boyd, and J. Sanderson, *The physics of plasmas*, Cambridge university press, 2003.

[8] Z Brzeźniak, M Capiński, and F. Flandoli, *Stochastic Navier-Stokes equations with multiplicative noise*, Stochastic Analysis and Applications 10 (1992), no. 5, 523–532.

[9] Z. Brzeźniak, F. Flandoli, and M. Maurelli, *Existence and uniqueness for stochastic 2D Euler flows with bounded vorticity*, Archive for Rational Mechanics and Analysis 221 (2016), no. 1, 107–142.

[10] Z. Brzeźniak and E. Motyl, *Existence of a martingale solution of the stochastic Navier–Stokes equations in unbounded 2D and 3D domains*, Journal of Differential Equations 254 (2013), no. 4, 1627–1685.

[11] Z. Brzeźniak and J. Slavík, *Well-posedness of the 3D stochastic primitive equations with transport noise*, arXiv preprint arXiv:2008.00274 (2020).

[12] M Capinski and N. Cutland, *Navier–Stokes equations with multiplicative noise*, Nonlinearity 6 (1993), no. 1, 71.

[13] N. Champagnat and P.-E. Jabin, *Strong solutions to stochastic differential equations with rough coefficients*, The Annals of Probability 46 (2018), no. 3, 1498–1541.

[14] M. Coghi and F. Flandoli, *Propagation of chaos for interacting particles subject to environmental noise*, The Annals of Applied Probability 26 (2016), no. 3, 1407–1442.

[15] D. Crisan, F. Flandoli, and D. D Holm, *Solution properties of a 3D stochastic Euler fluid equation*, Journal of Nonlinear Science 29 (2019), no. 3, 813–870.

[16] D. Crisan and O. Lang, *Well-posedness for a stochastic 2D Euler equation with transport noise*, arXiv preprint arXiv:1907.00451 (2019).

[17] D. Crisan and O. Lang, *Local well-posedness for the great lake equation with transport noise*, arXiv preprint arXiv:2003.03357 (2020).

[18] G. Da Prato and J. Zabczyk, *Ergodicity for infinite dimensional systems*, Vol. 229, Cambridge university press, 1996.

[19] G. Da Prato and J. Zabczyk, *Stochastic equations in infinite dimensions*, Cambridge university press, 2014.

[20] S. de Moor, L. M. M. Rodrigues, and J. Vovelle, *Invariant measures for a stochastic Fokker-Planck equation*, Kinetic and Related Models 11 (2018), no. 2, 357–395.

[21] A. Debussche, N. Glatt-Holtz, and R. Temam, *Local martingale and pathwise solutions for an abstract fluids model*, Physica D: Nonlinear Phenomena 240 (2011), no. 14–15, 1123–1144.

[22] A. Debussche, N. Glatt-Holtz, R. Temam, and M. Ziane, *Global existence and regularity for the 3D stochastic primitive equations of the ocean and atmosphere with multiplicative white noise*, Nonlinearity 25 (2012), no. 7, 2093.

[23] F. Delarue, F. Flandoli, and D. Vincenzi, *Noise prevents collapse of Vlasov-Poisson point charges*, Communications on Pure and Applied Mathematics 67 (2014), no. 10, 1700–1736.

[24] E. Fedrizzi and F. Flandoli, *Pathwise uniqueness and continuous dependence for SDEs with non-regular drift*, Stochastics: An International Journal of Probability and Stochastic Processes 83 (2011), no. 03, 241–257.

[25] E. Fedrizzi and F. Flandoli, *Noise prevents singularities in linear transport equations*, Journal of Functional Analysis 264 (2013), no. 6, 1329–1354.

[26] E. Fedrizzi, F. Flandoli, E. Priola, and J. Vovelle, *Regularity of stochastic kinetic equations*, Electronic Journal of Probability 22 (2017), 1–42.

[27] F. Flandoli and D. Gatarek, *Martingale and stationary solutions for stochastic Navier-Stokes equations*, Probability Theory and Related Fields 102 (1995), no. 3, 367–391.
[28] F. Flandoli, M. Gubinelli, and E. Priola, *Well-posedness of the transport equation by stochastic perturbation*, Inventiones mathematicae **180** (2010), no. 1, 1–53.

[29] N. E Glatt-Holtz and V. C Vicol, *Local and global existence of smooth solutions for the stochastic Euler equations with multiplicative noise*, The Annals of Probability **42** (2014), no. 1, 80–145.

[30] I. Gyöngy and N. Krylov, *Existence of strong solutions for Itô’s stochastic equations via approximations*, Probability theory and related fields **105** (1996), no. 2, 143–158.

[31] D. Hall and P. Sturrock, *Diffusion, scattering, and acceleration of particles by stochastic electromagnetic fields*, The Physics of Fluids **10** (1967), no. 12, 2620–2628.

[32] D. E Hall, *Particle dynamics in stochastic electromagnetic fields*, Plasma Instabilities in Astrophysics (1969), 253.

[33] K. Hasselmann and G Wibberenz, *Scattering of charged particles by random electromagnetic fields*, Zeitschrift für Geophysik **34** (1968), 353–388.

[34] D. D Holm, *Variational principles for stochastic fluid dynamics*, Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences **471** (2015), no. 2176, 2014963.

[35] E Horst, *Global strong solutions of Vlasov’s equation–necessary and sufficient conditions for their existence*, Banach Center Publications **1** (1987), no. 19, 143–153.

[36] E. Horst and H Neunzert, *On the classical solutions of the initial value problem for the unmodified non-linear Vlasov equation I general theory*, Mathematical Methods in the Applied Sciences **3** (1981), no. 1, 229–248.

[37] N. Ikeda and S. Watanabe, *Stochastic differential equations and diffusion processes*, Elsevier, 2014.

[38] U Jaekel and R Schlickeiser, *The Fokker-Planck coefficients of cosmic ray transport in random electromagnetic fields*, Zeitschrift für Geophysik **34** (1968), 353–388.

[39] J. A Krommes, C. Oberman, and R. G Kleva, *Plasma transport in stochastic magnetic fields. Part 3. Kinetics of test particle diffusion*, Journal of plasma physics **30** (1983), no. 1, 11–56.

[40] H. Kunita, *Stochastic flows and stochastic differential equations*, Vol. 24, Cambridge university press, 1997.

[41] P.-L. Lions and B. Perthame, *Propagation of moments and regularity for the 3-dimensional Vlasov-Poisson system*, Inventiones mathematicae **105** (1991), no. 1, 415–430.

[42] G. Loeper, *Uniqueness of the solution to the Vlasov–Poisson system with bounded density*, Journal de mathématiques pures et appliquées **86** (2006), no. 1, 68–79.

[43] J. Luk and R. M Strain, *Strichartz estimates and moment bounds for the relativistic Vlasov–Maxwell system*, Archive for Rational Mechanics and Analysis **219** (2016), no. 1, 445–552.

[44] R. Mikulevicius and B. L Rozovskii, *Stochastic Navier–Stokes equations for turbulent flows*, SIAM Journal on Mathematical Analysis **35** (2004), no. 5, 1250–1310.

[45] R. Mikulevicius and B. L Rozovskii, *Global L2-solutions of stochastic Navier–Stokes equations*, The Annals of Probability **33** (2005), no. 1, 137–176.

[46] S.-E. A Mohammed, T. K Nilssen, and F. N Proske, *Sobolev differentiable stochastic flows for sdes with singular coefficients: Applications to the transport equation*, The Annals of Probability **43** (2015), no. 3, 1535–1576.

[47] A. B. Navarro, B. Teaca, D. Told, D. Groselj, P. Crandall, and F. Jenko, *Structure of plasma heating in gyrokinetic Alfvénic turbulence*, Physical review letters **117** (2016), no. 24, 245101.

[48] H. Neunzert, M. Pulvirenti, and L. Triolo, *On the Vlasov-Fokker-Planck equation*, Mathematical methods in the applied sciences **6** (1984), no. 1, 527–538.

[49] K. Pfaffelmoser, *Global classical solutions of the Vlasov-Poisson system in three dimensions for general initial data*, Journal of Differential Equations **95** (1992), no. 2, 281–303.

[50] S. Punshon-Smith and S. Smith, *On the Boltzmann equation with stochastic kinetic transport: global existence of renormalized martingale solutions*, Archive for Rational Mechanics and Analysis **229** (2018), no. 2, 627–708.

[51] J. Schaeffer, *Global existence of smooth solutions to the Vlasov Poisson system in three dimensions*, Communications in partial differential equations **16** (1991), no. 8–9, 1313–1335.

[52] T. Tao, *Nonlinear dispersive equations: local and global analysis*, American Mathematical Soc., 2006.

[53] J. TenBarge, G. G Howes, W. Dorland, and G. W Hammett, *An oscillating Langevin antenna for driving plasma turbulence simulations*, Computer Physics Communications **185** (2014), no. 2, 578–589.

[54] D. Told, F. Jenko, J. M TenBarge, G. G Howes, and G. W Hammett, *Multiscale nature of the dissipation range in gyrokinetic simulations of Alfvénic turbulence*, Physical review letters **115** (2015), no. 2, 025003.
[55] E. Vanden Eijnden and R. Balescu, *Statistical description and transport in stochastic magnetic fields*, Physics of plasmas 3 (1996), no. 3, 874–888.

[56] H. D. Victory Jr and B. P O’Dwyer, *On classical solutions of Vlasov-Poisson Fokker-Planck systems*, Indiana University mathematics journal (1990), 105–156.

[57] C. Villani, *A review of mathematical topics in collisional kinetic theory*, Handbook of mathematical fluid dynamics 1 (2002), no. 71-305, 3–8.

[58] C. Villani, *Hypocoercivity*, American Mathematical Soc., 2009.

[59] H.-D. Wang, M. Vlad, E V. Eijnden, F Spineanu, J. Misguich, and R. Balescu, *Diffusive processes in a stochastic magnetic field*, Physical Review E 51 (1995), no. 5, 4844.

[60] A Wingen, S. Abdullaev, K. Finken, M Jakubowski, and K. Spatschek, *Influence of stochastic magnetic fields on relativistic electrons*, Nuclear fusion 46 (2006), no. 11, 941.