Reconfiguring Directed Trees in a Digraph*

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Abstract

In this paper, we investigate the computational complexity of subgraph reconfiguration problems in directed graphs. More specifically, we focus on the problem of determining whether, given two directed trees in a digraph, there is a (reconfiguration) sequence of directed trees such that for every pair of two consecutive trees in the sequence, one of them is obtained from the other by removing an arc and then adding another arc. We show that this problem can be solved in polynomial time, whereas the problem is PSPACE-complete when we restrict directed trees in a reconfiguration sequence to form directed paths. We also show that there is a polynomial-time algorithm for finding a shortest reconfiguration sequence between two directed spanning trees.

1 Introduction

Reconfiguration problems ask, given two feasible solutions, to determine whether there is a step-by-step transformation between the two solutions whose intermediate solutions are all feasible. There are numerous results regarding reconfiguration problems from several perspectives of Theoretical Computer Science (see [12, 9]). In particular, reconfiguration problems on graphs have received special attention in the literature.

Let \( S(G) \) be a set of subgraphs of a graph \( G \). Given two subgraphs \( H \) and \( H' \) in \( S(G) \), the problem of deciding whether there is a sequence of subgraphs \( \langle H = H_0, H_1, \ldots, H_\ell = H' \rangle \) such that all subgraphs \( H_i \) belong to \( S(G) \) and \( |E(H_i) \setminus E(H_{i+1})| = |E(H_{i+1}) \setminus E(H_i)| = 1 \) for \( 0 \leq i < \ell \) is one of the central problems in the field of reconfiguration problems. This formulation includes several known tractable and intractable reconfiguration problems. Ito et al. [6] showed that \textsc{Spanning Tree Reconfiguration} and \textsc{Matching Reconfiguration} can be solved in polynomial time, that is, \( S(G) \) is considered as the set of all spanning trees or the set of all matchings with the same cardinality in \( G \), respectively. \textsc{Matching Reconfiguration} can be extended to degree-constrained subgraphs, which is also tractable shown by Mühlenberth [7]. Hanaka et al. [5] studied reconfiguration problems for several subgraphs, including trees and paths. In particular, they showed that if \( S(G) \) consists of all trees in \( G \), every instance of the corresponding reconfiguration problem is a yes-instance unless the two input trees have different numbers of edges. Motivated by applications in motion planning, Biasi and Ophelders [1], Demaine et al. [3], and Gupta et al. [4] studied some variants of reconfiguring undirected paths and showed that these problems are PSPACE-complete, while they are fixed-parameter tractable (FPT) when parameterized by the length of input paths.

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Figure 1: There is no reconfiguration sequence between the red tree and the blue tree.

In contrast to these reconfiguration problems on undirected graphs, however, results of reconfiguration problems on directed graphs would be scarce. In this paper, we investigate the complexity of several reconfiguration problems on directed graphs. The main focus of this paper is the following problem: Given a directed graph $G = (V, A)$ and two directed trees $T^s$ and $T^t$ in $G$, the problem asks to decide whether there is a sequence $⟨T_0, T_1, \ldots, T_\ell⟩$ of directed trees in $G$ with $T_0 = T^s$ and $T_\ell = T^t$ such that $|A(T_i) \setminus A(T_{i+1})| = |A(T_{i+1}) \setminus A(T_i)| = 1$ for $0 \leq i < \ell$. We refer to this problem as Directed Tree Reconfiguration, which is a directed counterpart of Undirected Tree Reconfiguration studied in [5]. The precise definition of directed trees is given in Section 2.

The main result of this paper is as follows.

**Theorem 1.** Directed Tree Reconfiguration can be solved in time $O(|V||A|)$. Moreover, if the answer is affirmative, we can construct a reconfiguration sequence between $T^s$ and $T^t$ with length $O(|V|^2)$ within the same running time bound.

As a variant of Directed Tree Reconfiguration, we consider the following problem, which we call Shortest Directed Tree Reconfiguration: Given an integer $\ell$ in addition to the input of Directed Tree Reconfiguration, the problem asks whether there is a reconfiguration sequence of length at most $\ell$ between $T^s$ and $T^t$. For Shortest Directed Tree Reconfiguration, we develop a polynomial-time algorithm, provided that the input trees are directed spanning trees. More precisely, we show that there always exists a reconfiguration sequence between two directed spanning trees $T^s$ and $T^t$ with length $|A(T^s) \setminus A(T^t)|$.

**Theorem 2.** Shortest Directed Tree Reconfiguration can be solved in polynomial time, provided that the input trees $T^s$ and $T^t$ are directed spanning trees of the input graph $G$.

Directed Tree Reconfiguration for directed spanning trees can be viewed as a special case of Matroid Intersection Reconfiguration for a graphic matroid and (a truncation of) a partition matroid. Here Matroid Intersection Reconfiguration asks, given two matroids and their two common bases $B^s$ and $B^t$, to determine if there is a reconfiguration sequence of common bases between $B^s$ and $B^t$; see [10] for matroids. While it is known [6] that Maximum Bipartite Matching Reconfiguration, which can be viewed as Matroid Intersection Reconfiguration for two (truncations of) partition matroids, is polynomially solvable, the complexity of Matroid Intersection Reconfiguration is still open. Theorem 2 provides a new tractable class of Matroid Intersection Reconfiguration, particularly, its “shortest” version.

We also discuss other variants of Directed Tree Reconfiguration. When the two input and all intermediate directed trees are restricted to have a fixed vertex $r$ as their root, we show that these two trees are always reconfigurable as opposed to the fact that Directed Tree Reconfiguration has no-instances (see Fig. 1). As another variant of Directed Tree Reconfiguration, we consider Directed Path Reconfiguration and Directed Path Sliding (see Section 5 for their precise definitions), where the two input and all intermediate directed trees are restricted to directed paths. In contrast to the tractability of Directed Tree Reconfiguration, both problems are PSPACE-complete. Finally, we discuss the complexity of reconfiguring directed acyclic graphs, which is also PSPACE-complete. Our results are summarized as in Table 1.

1 There are several results on motion planning in directed graphs [2, 8, 13].
Table 1: The table summarizes our results in this paper: Tree, r-Tree, Span. Tree, r-Span. Tree, Path, and DAG indicate directed trees, r-directed trees, directed spanning trees, r-directed spanning trees, directed paths, and directed acyclic graphs, respectively. The cells “Yes” mean the corresponding problems only have yes-instances.

|                | Tree | r-Tree | Span. Tree | r-Span. Tree | Path | DAG               |
|----------------|------|--------|------------|--------------|------|------------------|
| Reachability   | P    | Yes    | Yes        | Yes          | P    | PSPACE-comp.     |
| Shortest       | Open | Open   | P          | P            | -    | -                |

2 Preliminaries

Let $G = (V,A)$ be a directed graph. We denote by $V(G)$ and $A(G)$ the vertex and arc sets of $G$, respectively. We may abuse $G$ to denote the arc set of $G$ when no confusions arise, and then we use $|G|$, called the size of $G$, to denote the number of arcs in $G$. Let $e = (u,v)$ be an arc of $G$. We say that $e$ is directed from $u$ or directed to $v$. The vertex $u$ (resp. $v$) is called the tail (resp. head) of $e$. For $v \in V$, we denote by $N_G^{-}(v)$ the set of out-neighbors of $v$ in $G$, i.e., $N_G^{-}(v) = \{ w \in V : (v,w) \in A \}$. The in-degree (resp. out-degree) of $v$ is the number of arcs directed to $v$ (resp. directed from $v$) in $G$. For $X \subseteq V$, the subgraph of $G$ induced by $X$ is denoted as $G[X]$. For an arc $(u,v) \in G$ and a subgraph $H$ of $G$, we denote by $H + (u,v)$ and $H - (u,v)$ the directed graphs obtained from $H$ by adding $(u,v)$ and by removing $(u,v)$, respectively.

A directed tree is a directed graph $T$ whose underlying undirected graph forms a tree such that every vertex except for a vertex $r \in V(T)$ has in-degree exactly 1. The unique vertex $r$ of in-degree 0 is called the root of $T$, and $T$ is called an r-directed tree. An arc in a directed tree $T$ is called a leaf arc if the out-degree of its head is 0 in $T$. A directed graph consisting of a disjoint union of directed trees is called a directed forest or an R-directed forest, where $R$ is the set of roots of its (weakly) connected components. A directed path is a directed tree that has at most one leaf arc.

Let $H^s$ and $H^t$ be two subgraphs of $G$ that has the same size. A sequence of subgraphs $\langle H_0, H_1, \ldots, H_\ell \rangle$ of $G$ is called a reconfiguration sequence between $H^s$ and $H^t$ if $H^s = H_0$, $H^t = H_\ell$, and $|A(H_i) \setminus A(H_{i+1})| = |A(H_{i+1}) \setminus A(H_i)| = 1$ for all $0 \leq i < \ell$. In other words, $H_{i+1}$ is obtained by removing an arc from $H_i$ and then adding another arc to it for each $0 \leq i < \ell$. We call $\ell$ the length of the reconfiguration sequence. If there is a reconfiguration sequence (with respect to some graph property) between $H^s$ and $H^t$, we say that $H^t$ is reconfigurable from $H^s$. Note that reconfiguration sequences are symmetric: $H^t$ is reconfigurable from $H^s$ if and only if $H^s$ is reconfigurable from $H^t$.

3 Variants of Directed Tree Reconfiguration

3.1 Directed forests

Let $S \subseteq 2^U$ be a collection of subsets of a finite set $U$. Suppose that every set in $S$ has the same cardinality. We say that $S$ satisfies the weak exchange property if for $S, S' \in S$ with $S \neq S'$, there exist $e \in S \setminus S'$ and $e' \in S' \setminus S$ such that $S \setminus \{e\} \cup \{e'\} \in S$. This property is closely related to the (simultaneous) exchange property of bases of matroids: Recall that if $\mathcal{B}$ is the collection of bases of a matroid, then for $B, B' \in \mathcal{B}$ with $B \neq B'$ and for $e \in B \setminus B'$, there is $e' \in B' \setminus B$ such that $B \setminus \{e\} \cup \{e'\} \in \mathcal{B}$. The weak exchange property is not only a weaker version of the exchange property but also gives an important consequence for reconfiguration problems.

In reconfiguration problems, we can consider $S$ as the set of feasible solutions. Given this, we say that $S$ is reconfigurable if for $S, S' \in S$, there is a reconfiguration sequence $\langle S = S_0, S_1, \ldots, S_\ell = S' \rangle$ between $S$ and $S'$ such that $S_i \in S$ for $0 \leq i \leq \ell$ and $|S_i \setminus S_{i+1}| = |S_{i+1} \setminus S_i| = 1$ for $0 \leq i < \ell$. We also say that $S$ is tightly reconfigurable if there is a reconfiguration sequence of length $|S' \setminus S'|$ between $S$ and $S'$. Since every reconfiguration sequence has length at least $|S \setminus S'|$, this is a shortest reconfiguration sequence for $S$. It is easy to observe that $S$ is tightly reconfigurable if and only if it satisfies the weak exchange property. As we observed, the collection of bases of a matroid is tightly reconfigurable which is also observed in [6]. In this paper, we show that the collection
of directed spanning trees in a directed graph satisfies the weak exchange property, implying that Shortest Directed Tree Reconfiguration can be solved in polynomial time, provided that $T^*$ and $T^\circ$ are directed spanning trees of $G$.

**Theorem 3.** The set of all directed spanning trees in $G$ satisfies the weak exchange property.

**Proof.** Let $T$ and $T'$ be arbitrary directed spanning trees in $G$ with $T \neq T'$. Suppose first that $T$ and $T'$ have a common root $r$. Let $e' = (u, v)$ be an arc in $T' \setminus T$ such that the path from $r$ to $u$ in $T'$ is contained in $T$. Clearly, we have $v \neq r$. Let $e$ be the unique arc directed to $v$ in $T$. From the definition of $e$ and $e'$, we have $e \neq e'$. Let $R = T + e' - e$. Now in $T + e'$, the vertex $v$ is the only vertex that has two arcs ($e$ and $e'$) directed to it. Thus, in $R$, no vertex has in-degree 2 or more. Moreover, all vertices in $R$ are reachable from $r$: the paths in $T'$ that use $e$ are rerouted to use $e'$ in $R$, and all other paths in $T$ still exist in $R$. Since $|T| = |R|$, $R$ is a directed spanning tree in $G$.

Suppose next that $T$ and $T'$ have different roots $r$ and $r'$, respectively. Let $e'$ be the unique arc in $T'$ directed to $r$, that is, $e' = (u, r)$ for some $u \in V$. Let $P$ be the path from $r$ to $u$ in $T$. Since $P + e'$ is a directed cycle, there is an arc $e = (v, w) \in P$ that does not belong to $T'$. Let $R = T + e' - e$. Observe that no vertex in $R$ has in-degree 2 or more since it holds already in $P + e'$. Observe also that all vertices in $R$ are reachable from $w$: for the descendants of $w$ in $T'$, $R$ contains the same path from $w$; and for the other vertices, we first follow the path from $w$ to $u$ in $T$, use the arc $e' = (u, r)$, and then follow the path in $T$ from $r$. Since $|T| = |R|$, $R$ is a directed spanning tree (rooted at $w$) in $G$. □

This proves Theorem 2. Note that the above theorem shows that the set of $r$-directed spanning trees also satisfies the weak exchange property.

**Corollary 1.** The set of all $r$-directed spanning trees in $G$ satisfies the weak exchange property.

We also prove that the set of all directed forests with the same size satisfies the weak exchange property, which implies that Shortest Directed Forest Reconfiguration is also polynomial-time solvable.

**Theorem 4.** The set of all directed forests having the same size in $G$ satisfies the weak exchange property.

**Proof.** Let $F$ and $F'$ be distinct directed forests in $G$ with $|F| = |F'|$. We first consider the case where there is some arc $e' \in F' \setminus F$ such that the endpoints of $e'$ do not belong to the same (weakly) connected component of $F$, that is, either $e'$ connects two connected components of $F$ or at least one of the endpoints of $e'$ does not belong to $F$. Now, we show that there is an arc $e \in F' \setminus F$ such that $F + e' - e$ is a directed forest of $G$. If $F + e'$ is a directed forest, then we can select any arc in $F \setminus F'$ as $e$. Assume that $F + e'$ is not a directed forest. By the assumption in this case, the underlying undirected graph of $F + e'$ contains no (undirected) cycle. Thus there is a vertex of in-degree at least 2 in $F + e'$. Since $F$ is a directed forest, only the head of $e'$, say $v$, can be such a vertex, and its in-degree is exactly 2. As $e$, we select the other arc in $F + e'$ that has $v$ as its head. Since $e' \in F' \setminus F$, this arc $e$ does not belong to $F'$. Since $F + e' - e$ does not contain any cycle in the underlying graph nor any vertex of in-degree 2 or more, it is a directed forest in $G$.

Next we consider the case where every arc $e' \in F' \setminus F$ has both endpoints in the same connected component of $F$. Let $F_1, \ldots, F_c \subseteq F$ be the connected components of $F$, and let $F'_1, \ldots, F'_c \subseteq F'$ be the subsets of $F'$ such that $F'_i = \{ e' \in F' \mid e' \text{ has both endpoints in } F_i \}$. We claim that $|F_i| = |F'_i|$. To see this, observe that if $|F_i| < |F'_i|$ for some $i$, then $F'_i$ is not a directed tree since $V(F'_i) \subseteq V(F_i)$ and $F_i$ is a directed spanning tree of the subgraph of $G$ induced by $V(F_i)$. This proves the claim as $|F| = |F'|$. Since both endpoints of every arc in $F'_i$ belong to $F_i$, we also have $V(F'_i) = V(F_i)$ for all $1 \leq i \leq c$. As $F \neq F_i$, there is a connected component $F_i$ in $F$ with $F_i \neq F'_i$ and by Theorem 3 the theorem follows. □

As mentioned in Introduction, Directed Tree Reconfiguration for spanning directed trees (and Directed Forest Reconfiguration) form subclasses of Matroid Intersection Reconfiguration. Theorems 3 and 4 would give a new insight on matroid intersection in terms of the weak exchange property.
3.2 Directed forests with fixed roots

In this subsection, we prove that every instance of Directed Tree Reconfiguration is a yes-instance, provided the input directed trees have the same size and a common root r. More precisely, we prove the following theorem.

**Theorem 5.** For every pair of r-directed trees T and T' in G with |T| = |T'| = k, there is a reconfiguration sequence \( (T = T_0, T_1, \ldots, T_k = T') \) such that all intermediate directed trees have the same root r. Moreover, the length l of the reconfiguration sequence is at most k.

**Proof.** We say that an arc e in a directed tree T'' is fixed (with respect to T') if the directed path from r to the head of e in T'' appears in T'. An arc is unfixed if it is not fixed. Let h be the number of unfixed arcs in T. We prove that there is a reconfiguration sequence between T and T' of length at most h by induction on h. If h = 0, then we have T = T'. In the following, we assume that h ≥ 1 and that for every r-directed tree T'' that has k arcs and contains fewer than h unfixed arcs with respect to T', there is a reconfiguration sequence from T'' to T' of length at most h − 1.

Let e = (u, v) be an arc in T' such that e is not included in T' but all other arcs in the path P from r to the tail of e in T' are included in T. Such an arc exists since T ≠ T' and they share the root r. Note that all arcs in P are fixed.

Assume for now that there is an unfixed arc f in T such that T''' := T + e − f is a directed tree in G. Note that T''' is still rooted at r since e is an arc of a directed tree rooted at r. Observe that e is fixed in T''' as both T''' and T' contain the path P and that the fixed arcs of T remain fixed in T''' since we only removed the unfixed arc f. Thus T''' has fewer than h unfixed arcs. By the induction hypothesis, there is a reconfiguration sequence from T''' to T' of length at most h − 1, and thus T' is reconfigurable from T as |T \ T''| = |T'' \ T| = 1. Therefore, it suffices to find such an arc f.

If the head v of e is included in T, then we set f to the arc directed to v in T. Then f is unfixed since T' cannot contain it and T + e − f is a directed tree obtained from T by changing the parent of v to u. Otherwise, v is not included in T, then we set f to an unfixed leaf arc of T, which exists since h ≥ 1. Since T + e is a directed tree and f is a leaf arc of T + e as well, T + e − f is a directed tree.

This result can be extended to R-directed forests.

**Theorem 6.** For every pair of R-directed forests F and F' in G with |F| = |F'| = k, there is a reconfiguration sequence \( (F = F_0, F_1, \ldots, F_k = F') \) such that all intermediates F_i are R-directed forests in G. Moreover, the length l of the reconfiguration sequence is at most k.

**Proof.** Since every arc between vertices in R cannot belong to any R-directed forest, we assume that there are no arcs between them. Let G' be a directed multigraph obtained from G by identifying vertices in R into a single vertex r. Observe that a set X ⊆ E forms an R-directed forest in G if and only if X is an r-directed tree of G'. By Theorem 5, the theorem follows.

4 An algorithm for Directed Tree Reconfiguration

This section is devoted to proving our main result, namely a polynomial-time algorithm for Directed Tree Reconfiguration. The idea of the algorithm is as follows. Let G = (V, A) be a directed graph and let k be a positive integer. For v ∈ V, we let T(v) be the collections of all v-directed trees T in G with |T| = k. By Theorem 5 there is a reconfiguration sequence between any pair of v-directed trees in T(v) such that all internal directed trees in the sequence belong to T(v). This enables us to “compress” all directed trees in T(v) into a single representative for each v ∈ V, and it suffices to seek the reachability in the “compressed” solution space. In the rest of this section, when we refer to reconfiguration sequences, every subgraph in these sequences are directed trees with k arcs.

Let u and v be distinct vertices in G and let T ∈ T(u) and T' ∈ T(v).

**Lemma 1.** Suppose that G has an arc (u, v) or (v, u). Then, there is a reconfiguration sequence between T and T'.


Proof. Assume without loss of generality that $G$ has an arc $(u, v)$. Since $v$ is the root of $T'$, we have $(u, v) \notin T'$. If $u \notin V(T')$, the subgraph $T''$ obtained from $T'' + (u, v)$ by removing arbitrary one of the leaf arcs is a directed tree in $T(u)$. Thus, by Theorem 5, there is a reconfiguration sequence between $T$ and $T''$ and then we are done in this case. Otherwise, $T + (u, v)$ has a directed cycle passing through $(u, v)$. Then, the graph obtained from $T + (u, v)$ by removing the arc directed to $u$ in the cycle is a directed tree in $T(u)$. Again, by Theorem 5, the lemma follows.

By inductively applying Theorem 5 and this lemma, we have the following corollary.  

**Corollary 2.** Suppose that $G$ has a directed path from $u$ to $v$ or from $v$ to $u$. Then, there is a reconfiguration sequence between $T$ and $T'$.  

**Lemma 2.** If there is a vertex $w \in N^+(u) \cap N^-(v)$ such that $G[V \setminus \{u, v\}]$ has a $u$-directed tree of size $k - 1$, then there is a reconfiguration sequence between $T$ and $T'$.  

**Proof.** Let $T''$ be a directed tree in $G[V \setminus \{u, v\}]$ that has $k - 1$ arcs and root $w \in N^+(u) \cap N^-(v)$. Since $T'' + (u, w)$ and $T'' + (v, w)$ are directed trees that belong to $T(u)$ and $T(v)$, respectively, by Lemma 5, there are reconfiguration sequences between $T$ and $T'' + (u, w)$ and between $T'' + (v, w)$ and $T'$. As $T'' + (v, w)$ is reconfigurable from $T'' + (u, w)$, concatenating these sequences yields a reconfiguration sequence between $T$ and $T'$.  

The above corollary and lemma give sufficient conditions for finding a reconfiguration sequence between $T$ and $T'$. The following lemma ensures that these conditions are also necessary conditions for a “single step”.  

**Lemma 3.** Suppose that $|T \setminus T'| = |T' \setminus T| = 1$. Then, at least one of the following conditions hold: (1) $G$ has a directed path from $u$ to $v$ or from $v$ to $u$ or (2) there is $w \in N^+(u) \cap N^-(v)$ such that $G[V \setminus \{u, v\}]$ has a $u$-directed tree of size $k - 1$.  

**Proof.** Suppose that $v \in V(T)$. Then, there is a directed path $P$ from $u$ to $v$ in $T$ and hence we are done. Symmetrically, the lemma follows when $u \in V(T')$. Thus, we assume that $v \notin V(T)$ and $u \notin V(T')$. This assumption implies that there is a unique arc $e$ directed from $u$ in $T$ as otherwise we have $|T \setminus T'| \geq 2$. Also, there is a unique arc $e'$ directed from $v$ in $T'$. By the fact that $|T \setminus T'| = |T' \setminus T| = 1$, $T - e = (T' - e')$ must be a directed tree with root $w \in N^+(u) \cap N^-(v)$ that has $k - 1$ arcs in $G[V \setminus \{u, v\}]$.  

To find a reconfiguration sequence between $T^s$ and $T^r$, we construct an auxiliary graph $G$ as follows. We assume that $G$ is (weakly) connected. For each $v \in V$, $G$ contains a vertex $v$ if $G$ has a $v$-directed tree of size $k$. For each pair of distinct $u$ and $v$ in $V(G)$, we add an (undirected) edge between them if (1) $G$ has a directed path from $u$ to $v$ or from $v$ to $u$; or (2) there is a vertex $w \in N^+(u) \cap N^-(v)$ such that $G[V \setminus \{u, v\}]$ has a $w$-directed tree of size $k - 1$. The graph $G$ can be constructed in $O(|V||A|)$ time. Our algorithm simply finds a path in $G$ between the two roots of given directed trees $T^s$ and $T^r$. The correctness of the algorithm immediately follows from the following lemma, which also proves the first part of Theorem 1.  

**Lemma 4.** Let $T^s$ and $T^r$ be directed trees in $G$ with $|T^s| = |T^r| = k$ whose roots are $r^s$ and $r^r$, respectively. Then, there is a path between $r^s$ and $r^r$ in $G$ if and only if there is a reconfiguration sequence between $T^s$ and $T^r$.  

**Proof.** We first show the forward implication. Suppose that there is a path $P$ between $r^s$ and $r^r$ in $G$. By Corollary 2 and Lemma 2, there is a reconfiguration sequence between $T^s$ and $T^r$ that can be constructed along the path $P$.  

For the converse implication, suppose that there is a reconfiguration sequence between $T^s$ and $T^r$. Let $T$ and $T'$ be two directed trees that appear consecutively in the sequence. We claim that either $T$ and $T'$ have a common root or the roots of $T$ and $T'$ are adjacent in $G$. If $T$ and $T'$ have a common root, the claim obviously holds. Suppose otherwise. Let $u$ and $v$ be the roots of $T$ and $T'$, respectively. By Lemma 3 at least one of the conditions (1) and (2) holds, implying that $u$ and $v$ are adjacent in $G$.  

It is easy to check that our algorithm turns into the one that finds an actual reconfiguration sequence of length $O(|V|^2)$ if the answer is affirmative, and hence Theorem 1 follows.
5 Intractable cases

Directed Path Reconfiguration is a variant of Directed Tree Reconfiguration, where the two input trees $T^a$, $T^b$ and intermediate trees are all directed paths in $G$. Here, we use $(P_0, P_1, \ldots, P_t)$ with $P_0 = P^a$ and $P_t = P^b$ to denote a reconfiguration sequence between two directed paths $P^a$ and $P^b$. Directed Path Sliding consists of the same instance of Directed Path Reconfiguration and we are allowed the following adjacency relation in a valid reconfiguration sequence: for every pair of consecutive directed paths $P = (v_1, v_2, \ldots, v_k)$ and $P' = (v'_1, v'_2, \ldots, v'_k)$, either $v_i = v'_{i+1}$ holds for all $1 \leq i < k$ or $v_i = v'_{i-1}$ holds for all $1 < i < k$. Since $P'$ is obtained by “sliding” in a forward or backward direction, we call the problem Directed Path Sliding. In this section, we show that Directed Path Reconfiguration and Directed Path Sliding are both PSPACE-complete.

To this end, we first show that both problems are equivalent with respect to polynomial-time many-one reductions. Let $G$ be a directed graph and let $P = (v_1, v_2, \ldots, v_k)$ be a directed path in $G$ with arc $e_i = (v_i, v_{i+1})$ for $1 \leq i < k$. We denote by $t(P)$ the tail $v_1$ of $P$ and by $h(P)$ the head $v_k$ of $P$. Observe that for a directed path $P'$ in $G$ with $|P \setminus P'| = |P' \setminus P| = 1$, at least one of the following conditions holds:

- **sliding**: $P' = (v_2, v_3, \ldots, v_k, v)$ or $P' = (v, v_1, v_2, \ldots, v_{k-1})$ for some $v \in V \setminus V(P)$;
- **turning**: $P' = (v_1, v_2, \ldots, v_k, v_{k-1})$ or $P' = (v, v_2, v_3, \ldots, v_k)$ for some $v \in V \setminus V(P)$;
- **shifting**: $P' = (v_1, v_{i+1}, \ldots, v_k, v_{i-1})$ for some $1 < i < k$. This can be done when $P + (v_k, v_1)$ forms a directed cycle.

See Fig. 2 for an illustration.

We can regard these conditions as operations to obtain $P'$ from $P$. Since shifting can be simulated by $i-1$ sliding operations along the directed cycle $P + (v_k, v_1)$, the essential difference between directed path reconfiguration and directed path sliding is turning operation in order to solve these problems. Now, we perform polynomial-time reductions between these problems in both directions.

Let $(G = (V, A), P^a, P^b)$ be an instance of Directed Path Reconfiguration. For each vertex $v$ in $G$, we add two vertices $v^\text{in}, v^\text{out}$ and two arcs $(v^\text{in}, v), (v, v^\text{out})$. These two vertices are called pendant vertices. We let $G'$ be the graph obtained in this way. Then, we show the following lemma.

**Lemma 5.** $(G, P^a, P^b)$ is a yes-instance of Directed Path Reconfiguration if and only if $(G', P^a, P^b)$ is a yes-instance of Directed Path Sliding.

**Proof.** Let $(P_0, P_1, \ldots, P_t)$ be a reconfiguration sequence between $P^a = P_0$ and $P^b = P_t$ of Directed Path Reconfiguration. By the above argument, we can assume that $P_{i+1}$ is obtained from $P_i$ by applying either sliding or turning. Let $P_i = (v_1, v_2, \ldots, v_k)$. We replace the subsequence $(P_i, P_{i+1})$ with $(P_i, P', P_{i+1})$, where $P' = (v_i^\text{in}, v_1, v_2, \ldots, v_{k-1})$ if $t(P_i) = t(P_{i+1})$ and $P' = (v_2, v_3, \ldots, v_k, v_1^\text{out})$ otherwise. Clearly, $P'$ and $P_{i+1}$ are obtained from $P_i$ and $P'$ by applying sliding operations, respectively. By replacing each subsequence for $0 \leq i < t$, we have a reconfiguration sequence of Directed Path Sliding in $G'$.

Conversely, let $(P_0, P_1, \ldots, P_t)$ be a reconfiguration sequence $P^a = P_0$ and $P^b = P_t$ of Directed Path Sliding. Similarly to the other direction, we construct a reconfiguration sequence of Directed Path Reconfiguration. Assume that $P^a \neq P^b$ as otherwise we are done. Observe that each path $P_i = (v_1, v_2, \ldots, v_k)$ contains at most one pendant vertex. This follows from the
also gives a reconfiguration sequence of $Q$. Specifically, given two directed acyclic subgraphs $P^s$ and $P^t$, the problem is equivalent to reconfiguring directed feedback arc sets in directed graphs. More formally, let $(Q_0, Q_1, \ldots, Q_k)$ be a reconfiguration sequence of $P^s$ and $P^t$, respectively. Since $P^s$ can be obtained from $P^t$ by turning, we can construct a reconfiguration sequence of Directed Path Reconfiguration by omitting paths having pendant vertices.

For the converse direction, we let $(G, P^s, P^t)$ be an instance of Directed Path Sliding. Let $G'$ be the directed graph obtained from $G$ by subdividing each arc $e = (u, v)$ with a new vertex $v_e$, that is, we replace $e$ with $v_e$ and add two arcs $(u, v_e)$ and $(v_e, v)$. Let $Q^s$ and $Q^t$ be defined accordingly from $P^s$ and $P^t$, respectively. In $G'$, we say that a path $P'$ is a standard path if $h(P)$ and $t(P)$ belong to $V$ and it is a nonstandard path otherwise.

**Lemma 6.** $(G, P^s, P^t)$ is a yes-instance of Directed Path Sliding if and only if $(G', Q^s, Q^t)$ is a yes-instance of Directed Path Reconfiguration.

**Proof.** It is easy to transform any reconfiguration sequence of $(G, P^s, P^t)$ for Directed Path Sliding to that of $(G', Q^s, Q^t)$ for Directed Path Reconfiguration. Conversely, let $(Q_0, Q_1, \ldots, Q_k)$ be a reconfiguration sequence of Directed Path Reconfiguration between $Q^s = Q_0$ and $Q^t = Q_k$ in $G'$. Observe that turning is allowed only for nonstandard paths. This means that for any two standard paths $Q_i$ and $Q_j$ in a reconfiguration sequence such that $Q_k$ is nonstandard for $i < k < j$, $Q_j$ is obtained from $Q_i$ by two sliding operations. Thus, by replacing each subsequence $(Q_i, Q_{i+1}, \ldots, Q_j)$ in this way, we obtain that $(G', Q^s, Q^t)$ for Directed Path Sliding, which also gives a reconfiguration sequence of $(G, P^s, P^t)$ for Directed Path Sliding as well.

Now, we show that the PSPACE-completeness of Directed Path Sliding.

**Theorem 7.** Directed Path Sliding is PSPACE-complete.

**Proof.** By a standard argument in reconfiguration problems, the problem belongs to PSPACE. By non-deterministically guessing the “next solution” in a reconfiguration sequence, the problem can be solved in non-deterministic polynomial space, while by Savitch’s theorem $[11]$, we can solve the problem in deterministic polynomial space as well.

It is easy to observe that the undirected version of Directed Path Sliding can be reduced to Directed Path Sliding by simply replacing each (undirected) edge of an input graph with two arcs with opposite directions. As the undirected version is known to be PSPACE-complete $[3]$, the directed version is also PSPACE-complete.

By Lemma 6, we immediately have the following corollary.

**Corollary 3.** Directed Path Reconfiguration is PSPACE-complete.

Suppose that subgraphs in a reconfiguration sequence are relaxed to be acyclic. Observe that the problem is equivalent to reconfiguring directed feedback arc sets in directed graphs. More specifically, given two directed acyclic subgraphs $H^s$ and $H^t$ in a directed graph $G = (V, A)$, the problem asks to determine whether there is a reconfiguration sequence of directed acyclic subgraphs $(H^s = H_0, H_1, \ldots, H_\ell = H^t)$ such that $|A(H_i) \setminus A(H_{i+1})| = |A(H_{i+1}) \setminus A(H_i)| = 1$ for all $0 \leq i < \ell$. Seeing this problem from the complement, the problem is equivalent to finding a reconfiguration sequence $(A_1, A_2, \ldots, A_\ell)$ of subsets of $A$ such that $H_i = G - A_i$ is acyclic for all $0 \leq i \leq \ell$. Since each $A_i$ is a feedback arc set of $G$, we call this problem Directed Feedback Arc Set Reconfiguration. There is another variant of this problem, called Directed Feedback Vertex Set Reconfiguration, in which we are asked to determine given two subsets $V^s$ and $V^t$ of $V$, there is a sequence of vertex subsets $(V^s = V_0, V_1, \ldots, V_\ell = V^t)$ of $V$ such that $G[V \setminus V_i]$ is acyclic and $|V_i \setminus V_{i+1}| = |V_{i+1} \setminus V_i| = 1$ for all $0 \leq i < \ell$.

**Theorem 8.** Directed Feedback Arc Set Reconfiguration and Directed Feedback Vertex Set Reconfiguration are PSPACE-complete.
Proof. By an analogous argument in Theorem 4 these problems belong to PSPACE.

It is easy to observe that Directed Feedback Vertex Set Reconfiguration is PSPACE-hard. To see this, consider an undirected graph $G = (V, E)$ and the directed graph $D = (V, A)$ obtained from $G$ by replacing all undirected edge $\{u, v\}$ with two arcs $(u, v)$ and $(v, u)$. Observe that every vertex cover of $G$ is also a directed feedback vertex set of $D$ and vice versa. By the PSPACE-hardness of reconfiguring independent sets [9], Directed Feedback Vertex Set Reconfiguration is PSPACE-hard.

To prove the PSPACE-hardness of Directed Feedback Arc Set Reconfiguration, we perform a standard polynomial-time reduction from Directed Feedback Vertex Set Reconfiguration.

Let $G = (V, A)$ be a directed graph. We construct a directed multigraph $G' = (V', A')$ as follows. We first add a pair of copies $\{v^{\text{in}}, v^{\text{out}}\}$ for each $v \in V$ and add an arc $(v^{\text{in}}, v^{\text{out}})$ to $G'$. We call this arc an internal arc of $v$. The vertex set of $G'$ is defined as $V' = \bigcup_{v \in V} \{v^{\text{in}}, v^{\text{out}}\}$. For $(u, v) \in A$, add $|V| + 1$ parallel arcs $(u^{\text{out}}, v^{\text{in}})$ to $G'$. For two (directed) feedback vertex sets $X^\#$ and $X^\ell$ in $G$ with $|X^\#| = |X^\ell| = k$, $Y^\#$ and $Y^\ell$ defined as the sets of internal arcs corresponding to $X^\#$ and $X^\ell$, respectively. Now, we show that $G$ contains a reconfiguration sequence of feedback vertex sets between $X^\#$ and $X^\ell$ in $G$ if and only if there is a reconfiguration sequence of (directed) feedback arc sets between $Y^\#$ and $Y^\ell$ in $G'$.

Since $X$ is a feedback vertex set of $G$, the corresponding internal arc set $Y$ is a feedback arc set of $G'$. Thus, the forward implication is straightforward. Conversely, suppose that there is a reconfiguration sequence $(Y_0, Y_1, \ldots, Y_t)$ between $Y_1 = Y^\#$ and $Y_t = Y^\ell$ such that all the intermediate sets $Y_i$ is a feedback arc set of $G'$. For $0 \leq i \leq t$, let $Y_i^\#$ be the set of internal arcs in $Y_i$ and $X_i^\ell$ is the set of vertices in $G$, each of which corresponds to an (internal) arc in $Y_i^\#$. To prove the backward implication, it suffices to show that $X_i^\ell$ is a feedback vertex set of $G'$. To see this, suppose that there is a directed cycle $C \subseteq G[V \setminus X_i^\ell]$. For every arc $(u, v)$ in $C$, there is at least one arc from $v^{\text{out}}$ to $u^{\text{in}}$ in $G' - Y_i$ as there are $|V| - 1$ copies there. Thus, the cycle also induces a directed cycle in $G' - Y_i$, contradicting the fact that $Y_i$ is a feedback arc set of $G'$. □

6 Concluding remarks

There are several possible open questions related to our results. Shortest Directed Tree Reconfiguration would be a notable open question arising in our work. Contrary to the cases of directed spanning trees and directed $r$-spanning trees, the sets of directed trees and $r$-directed trees with $k < |V| - 1$ arcs do not satisfy the weak exchange property, which makes Shortest Directed Tree Reconfiguration highly nontrivial. It would be also interesting to know whether Directed Path Reconfiguration and Directed Path Sliding are fixed-parameter tractable (FPT) parameterized by the length of input paths. Although the undirected counterparts are known to be FPT [8][4], it would be difficult to apply their techniques directly to our cases.

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