Minimal and maximal lengths of quantum gravity from non-Hermitian position-dependent noncommutativity

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Abstract

A minimum length scale of the order of Planck length is a feature of many models of quantum gravity that seek to unify quantum mechanics and gravitation. Recently, Perivolaropoulos in his seminal work [Phys. Rev.D 95, 103523 (2017)] predicted the simultaneous existence of minimal and maximal length measurements of quantum gravity. More recently, we have shown that both measurable lengths can be obtained from position-dependent noncommutativity [J. Phys. A: Math. Theor. 53, 115303 (2020)]. In this paper, we present an alternative derivation of these lengths from non-Hermitian position-dependent noncommutativity. We show that a simultaneous measurement of both lengths form a family of discrete spaces. In one hand, we show the similarities between the maximal uncertainty measurement and the classical properties of gravity. On the other hand, the connection between the minimal uncertainties and the non-Hermicity quantum mechanic scenarios. The existence of minimal uncertainties are the consequences of non-Hermicities of some operators that are generators of this noncommutativity. With an appropriate Dyson map, we demonstrate by a similarity transformation that the physically meaningfulness of dynamical quantum systems is generated by a hidden Hermitian position-dependent noncommutativity. This transformation preserves the properties of quantum gravity but removes the fuzziness induced by minimal uncertainty measurements at this scale. Finally, we study the eigenvalue problem of a free particle in a square-well potential in these new Hermitian variables.

Keywords: Position-deformed algebra; General uncertainty principle, Non-Hermitian Hamiltonian; Hidden Hermiticity.

1 Introduction

The idea of noncommutativity of space-time might provide deep indications about the quantum nature of space-time at a very small distance, where a full theory of quantum gravity must be invoked, has its root in string theory [9]. In fact, the noncommutativity of space-time is one of the promising candidate theories to the unification of quantum theory and General Relativity (GR). All the other candidate theories of unification such
as string theory \cite{4}, black hole theory \cite{5}, loop quantum gravity \cite{6} predicted the existence of minimal measurement of quantum gravity at the Planck scale. To theoretically realize this minimal length scale in quantum mechanics, one has introduced a simple model, the so-called Generalized Uncertainty Principle (GUP) \cite{7,8,9} which is a gravitational correction to quantum mechanics. Mostly, these theories of quantum gravity are restricted to the case where there is a nonzero minimal uncertainty in the position. Only Doubly Special Relativity (DSR) theories \cite{10,11,12} suggest an addition to the minimal length, the existence of a maximal momentum. Recently, Perivolaropoulos proposed a consistent algebra that induces for a simultaneous measurement, a maximal length and a minimal momentum \cite{1}. In this approach, the maximal length of quantum gravity is naturally arisen in cosmology due to the presence of particle horizons. Perivolaropoulos also predicted the simultaneous existence of maximal and minimal position uncertainties. More recently, we have shown that both position uncertainties can simultaneously be obtained from position-dependent noncommutativity and the minimal momentum is provided by the position-dependent deformed Heisenberg algebra \cite{2}. In continuation of this work, we show that both lengths can also be derived from non-Hermitian position-dependent noncommutativity. The simultaneous presence of both lengths at this scale form a lattice system in which each site represented by the minimal length is spaced by the maximal length. At each minimal length point results of the unification of magnetic and gravitational fields. As has been recently shown \cite{13}, the maximal length allows probing quantum gravitational effects with low energies and manifests properties similar to the classical ones of General Relativity (GR).

It is well known that the existence of minimal uncertainties in quantum mechanics induces among other consequences \cite{14,15,16,17,18} a non Hermicity of some operators that generate the corresponding Hilbert space \cite{6,19,20,21}. In the present case, the minimal length in the X-direction and the minimal momentum in the \(P_y\)-direction lead to the non-Hermiticity of operators \(\hat{X}\) and \(\hat{P}_y\) that generate the noncommutative space \cite{1}. Consequently, Hamiltonians \(\hat{H}\) of systems involving these operators will in general also not be Hermitian. The corresponding eigenstates no longer form an orthogonal basis and the Hilbert space structure will be modified. In order to map these operators into their Hermitian counterparts. We introduce a positive-definite Dyson map \(\eta\) \cite{22} and its associated metric operator \(\rho\) which generate a hidden Hermitian position-dependent noncommutativity by means of similarity transformation of the non-Hermitian one i.e \(\eta(\hat{X}, \hat{Y}, \hat{P}_x, \hat{P}_y, \hat{H}) = (\hat{x}, \hat{y}, \hat{p}_x, \hat{p}_y, \hat{h})\). Doing so, we tie a connection between the quantum mechanic noncommutativity with GUP \cite{23,24,25,26,27,28,29,30,31,32} and the non-Hermiticity quantum mechanic scenarios \cite{33,34,35,36,37,38,39,40,41,42,43}. Furthermore, this transformation preserves the uncertainty measurements at this scale but removes the fuzziness induced by the minimal uncertainty measurements. Finally, within this hidden Hermitian space, we present the eigensystems of a free particle in a box. We show that the existence of maximal length induces strong quantum gravitational effects in this box. These effects are manifested by the deformations of quantum energy and these deformations are more pronounced as one increases the quantum levels, allowing the particle to jump from one
state to another with low energies and with high probability densities \[13\]. These properties are similar to the classical gravity of General relativity where the gravitational field becomes stronger for heavy systems that curve the space, enabling the surrounding light systems to fall down with low energies. The resulting time inside of this space runs out more slowly as the gravitational effects increase.

In what follows, we explore in section 2, the similarities between our recent deformed noncommutativity with GUP and the pseudo-Hermiticity quantum mechanic scenarios. We show that these deformations lead to a non Hermiticity of the position operator $\hat{X}$ and the momentum operator $\hat{P}_y$. By constructing a Dyson map \[22\] we provide their corresponding set of Hermitian counterparts. As a consequence of these deformations, we derive in section 3, the uncertainty measurements resulting from these deformations. In section 4, we study in terms of our new set of variables, the model of particles in a 2D box. We present our conclusion in section 5.

2 Non-Hermitian position dependent noncommutativity

Given a set operators of $\hat{X}, \hat{Y}, \hat{P}_x, \hat{P}_y$ defined on the 2D Hilbert space and satisfy the following commutation relations and all possible permutations of the Jacobi identities \[1\]

\[
\begin{align*}
[\hat{X}, \hat{Y}] &= i\theta(1 - \tau \hat{Y} + \tau^2 \hat{Y}^2), \\
[\hat{Y}, \hat{P}_y] &= i\hbar(1 - \tau \hat{Y} + \tau^2 \hat{Y}^2), \\
[\hat{X}, \hat{P}_x] &= i\hbar(1 - \tau \hat{Y} + \tau^2 \hat{Y}^2), \\
[\hat{X}, \hat{P}_y] &= i\theta \tau (2\tau \hat{Y} \hat{X} - \hat{X}) + i\theta \tau (2\tau \hat{Y} \hat{P}_y - \hat{P}_y) \\
[\hat{P}_x, \hat{P}_y] &= 0, \\
[\hat{Y}, \hat{P}_x] &= 0.
\end{align*}
\]

(1)

Using the asymmetrical Bopp-shift \[42\], we can relate the noncommutative operators \[2\] to the ordinary commutations ones as follows:

\[
\begin{align*}
\hat{x}_0, \hat{y}_0 &= i\theta, \\
\hat{x}_0, \hat{p}_{x_0} &= i\hbar, \\
\hat{y}_0, \hat{p}_{y_0} &= i\hbar, \\
\hat{p}_{x_0}, \hat{p}_{y_0} &= 0, \\
\hat{x}_0, \hat{p}_{y_0} &= 0, \\
\hat{y}_0, \hat{p}_{x_0} &= 0.
\end{align*}
\]

(2)

Using the asymmetrical Bopp-shift \[22\], we can relate the noncommutative operators \[2\] to the ordinary commutations ones as follows:

\[\hat{x}_0 = \hat{x}_s - \frac{\theta}{2\hbar} \hat{p}_{y_s}, \quad \hat{y}_0 = \hat{y}_s,\]

(3)

where the Hermitian operators $\hat{x}_s, \hat{y}_s, \hat{p}_{x_s}, \hat{p}_{y_s}$ satisfy the ordinary 2D Heisenberg algebra

\[
\begin{align*}
[\hat{x}_s, \hat{y}_s] &= 0, \\
[\hat{x}_s, \hat{p}_{x_s}] &= i\hbar, \\
[\hat{y}_s, \hat{p}_{y_s}] &= i\hbar.
\end{align*}
\]
In terms of the standard flat-Hermitian noncommutative operators \( \{p_x, p_y\} = 0 \), \( \{\hat{x}, \hat{p}_y\} = 0 \), \( \{\hat{y}, \hat{p}_x\} = 0 \).

In the non-Hermitian version of the theory, we may now represent the algebra (1) as follows
\[
\hat{X} = (1 - \tau \hat{y}_0 + \tau^2 \hat{y}_0^2)\hat{x}_0, \quad \hat{Y} = \hat{y}_0, \quad \hat{P}_x = \hat{p}_{x_0}, \quad \hat{P}_y = (1 - \tau \hat{y}_0 + \tau^2 \hat{y}_0^2)\hat{p}_{y_0}.
\]
From this representation follows immediately that some of the operators involved are no longer Hermitian. We observe
\[
\hat{X}^\dagger = \hat{X} - i\theta\tau(1 - 2\tau\hat{Y}), \quad \hat{Y}^\dagger = \hat{Y}, \quad \hat{P}^\dagger_x = \hat{P}_x, \quad \hat{P}^\dagger_y = \hat{P}_y + i\hbar\tau(1 - 2\tau\hat{Y}).
\]
As is apparent, the operators \( \hat{X} \) and \( \hat{P}_y \) are not Hermitian. As an immediate consequence, the Hamiltonian of the system involving these operators will in general also not be Hermitian i.e. \( \hat{H}(\hat{X}, \hat{Y}, \hat{P}_x, \hat{P}_y) \neq \hat{H}(\hat{X}, \hat{Y}, \hat{P}_x, \hat{P}_y) \). In order to map this operator into Hermitian ones, some synonymously used concepts are introduced in the literature such as the PT-symmetry [33, 34, 35, 36], the quasi-Hermiticity [37, 38], the pseudo-Hermiticity [39, 40, 41] or the cryptoHermiticity [42, 43, 44]. It has been clarified in [34] that a non-Hermitian operator \( \mathcal{O} \) having all eigenvalues real is connected to its Hermitian conjugate \( \mathcal{O}^\dagger \) through a linear, Hermitian, invertible, and bounded metric operator \( \rho \) such as \( \rho\mathcal{O}\rho^{-1} = \mathcal{O}^\dagger \). Factorizing this operator into a product of a Dyson map can be taken to be
\[
\eta = (1 - \tau\hat{Y} + \tau^2\hat{Y}^2)^{-1/2},
\]
so the hidden Hermitian variables \( \hat{x}, \hat{y}, \hat{p}_x, \hat{p}_y \) can be stated in terms of \( \theta \)-deformed space operators as follows
\[
\begin{align*}
\hat{x} &= \eta\hat{X}\eta^{-1} = (1 - \tau\hat{y}_0 + \tau^2\hat{y}_0^2)^{1/2}\hat{x}_0(1 - \tau\hat{y}_0 + \tau^2\hat{y}_0^2)^{1/2} = \hat{x}^\dagger, \\
\hat{p}_x &= \eta\hat{P}_x\eta^{-1} = \hat{p}_{x_0} = \hat{p}_x, \\
\hat{y} &= \eta\hat{Y}\eta^{-1} = \hat{y}_0 = \hat{y}^\dagger, \\
\hat{p}_y &= \eta\hat{P}_y\eta^{-1} = (1 - \tau\hat{y}_0 + \tau^2\hat{y}_0^2)^{1/2}\hat{p}_{y_0}(1 - \tau\hat{y}_0 + \tau^2\hat{y}_0^2)^{1/2} = \hat{p}_y^\dagger.
\end{align*}
\]
As is well established in [34], a consequence of the non-Hermiticity of an operator $O$, its eigenstates no longer form an orthonormal basis and the Hilbert space representation has to be modified. This is achieved by utilizing the operator $\rho$ as a metric to define a new inner product $\langle \cdot | \cdot \rangle_\rho$ in terms of the standard inner product $\langle \cdot | \cdot \rangle$ defined as

$$\langle \Phi | \Psi \rangle_\rho := \langle \Phi | \rho | \Psi \rangle,$$  \hfill (14)

for arbitrary states $\langle \Phi \rangle$ and $| \Psi \rangle$. The observables $O$ are then Hermitian with respect to this new metric

$$\langle \Phi | O | \Psi \rangle_\rho = \langle O | \Phi | \rho | \Psi \rangle.$$  \hfill (15)

An important physical consequence resulting from the algebra (1), is the loss of the Hermiticity of certain operators which deformed the structure of the Hilbert space (14) as were predicted by the theory of Kempf et al [7]. In the next section, let us study Heisenberg’s uncertainty principle applied to a simultaneous measurement of operators of this algebra.

### 3 Minimal and maximal measurements

For the system of operators satisfying the commutation relations in (1), the generalized uncertainty principle is defined as follows

$$\Delta A \Delta B \geq \frac{1}{2} |\langle [\hat{A}, \hat{B}]_\rho \rangle| \quad \text{for} \quad \hat{A}, \hat{B} \in \{\hat{X}, \hat{Y}, \hat{P}_x, \hat{P}_y\},$$  \hfill (16)

where $\Delta A = \sqrt{\langle (\hat{A} - \langle \hat{A} \rangle_\rho)^2 \rangle_\rho}$ and for $\hat{B}$. An interesting features can be observed through the following uncertainty relations:

$$\Delta X \Delta Y \geq \frac{\hbar}{2} \left(1 - \tau \langle \hat{Y} \rangle_\rho + \tau^2 \langle \hat{Y}^2 \rangle_\rho\right),$$  \hfill (17)

$$\Delta Y \Delta P_y \geq \frac{\hbar}{2} \left(1 - \tau \langle \hat{Y} \rangle_\rho + \tau^2 \langle \hat{Y}^2 \rangle_\rho\right).$$  \hfill (18)

i) For the uncertainty relation (17), using $\langle \hat{Y}^2 \rangle_\rho = \Delta Y^2 + \langle \hat{Y} \rangle_\rho^2$, the inequality (17) is reduced to

$$\Delta X \Delta Y \geq \frac{\theta}{2} \left(1 + \tau^2 \Delta Y^2\right) \quad \text{for} \quad \langle \hat{Y} \rangle_\rho = 0.$$  \hfill (19)

This equation (19) can be rewritten as a second order equation of $\Delta Y$ and the solutions are given by

$$\Delta X = \frac{\Delta Y}{\theta \tau^2} \pm \sqrt{\left(\frac{\Delta Y}{\theta \tau^2}\right)^2 - \frac{1}{\tau^2}}.$$  \hfill (20)
These solutions lead to the absolute minimal uncertainty $\Delta X_{\text{min}}$ in $X$-direction and to the absolute maximal uncertainty $\Delta Y_{\text{max}}$ in $Y$-direction as predicted by Perivolaropoulos \cite{1}

\begin{align*}
\Delta X_{\text{min}} &= \theta \tau = \frac{\tau}{B} = l_{\text{min}}, \quad (21) \\
\Delta Y_{\text{max}} &= \frac{1}{\tau} = l_{\text{max}}. \quad (22)
\end{align*}

Different versions of minimal length uncertainties have been introduced in the literature \cite{60,61,62,63} which significantly improve the one proposed by Kempf et al \cite{7}. It is well known that these minimal length uncertainties induce a singularity of position representation at the Planck scale i.e they are inevitably bounded by minimal quantities beyond which any further localization of particles is not possible. Conversely to these results, here the obtained minimal length $\Delta X_{\text{min}} = \frac{\tau}{B}$ induces a broken singularity at the Planck scale due to the external magnetic fields $B$. In fact this scenario can be regarded as the Landau problem where the Planck scale bounded by the weak quantum gravitational field $\tau$ is orthogonally subjected to the superstrong magnetic field $B$ of the parallel universe causing its bouncing at this minimal point. This broken singularity manifested by a big bang unifies the weak quantum gravitational field and the superstrong magnetic field as minimal length (see Figure 1). A simultaneous measurement of

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{diagram.png}
\caption{Representation of minimal length scale $\Delta X_{\text{min}}$.}
\end{figure}

the minimal length $\Delta X_{\text{min}}$ and the maximal length $\Delta Y_{\text{max}}$ generates the inverse of the magnetic field as follows

$$\Delta X_{\text{min}} \Delta Y_{\text{max}} = \frac{1}{B} = l_{\text{min}} l_{\text{max}}. \quad (23)$$
If we consider $n$-dimensional sets of the algebra \[(1),\] based on the equation \[(23)\] we obtain a sequence of minimal lengths alternated by maximal lengths

\[
...l_{\text{min}}l_{\text{max}}l_{\text{min}}l_{\text{max}}... = \frac{1}{B^n}. \tag{24}
\]

This sequence forms a sort of discreteness of the Planck space and can be compared to a lattice system in which each site represented by $l_{\text{min}}$ is spaced by $l_{\text{max}}$ (see Figure 2). Note that two of the strongest competing candidates for a theory of quantum gravity, String Theory (ST) and Loop Quantum Gravity (LQG) have been thought of as describing different regimes of quantum gravity. ST fundamentally attempts to unify forces while LQG attempts to quantize spacetime at the Planck scale. Based on the proposal \[(23)\] and (Figure 2), we can argue that ST and LQG are in fact not fundamentally different from each other, they are just two different methods to approach the same problem.

ii) Repeating the same calculation and argumentation in the situation of uncertainty relation \[(22)\] for simultaneous $\hat{Y}, \hat{P}_y$-measurement, we find the absolute maximal uncertainty $\Delta Y_{\text{max}}$ \[(22)\] and an absolute minimal uncertainty momentum $\Delta P_{y_{\text{min}}}$ for $\langle \hat{Y} \rangle_{\rho} = 0$

\[
\Delta Y_{\text{max}} = \frac{1}{\tau} = l_{\text{max}}, \quad \Delta P_{y_{\text{min}}} = h\tau = p_{\text{min}}. \tag{25}
\]

These results \[(25)\] are consistent with the ones obtained by Perivolaropoulos \[1\]. From these results, it is interesting to observe that the GUP is reduced into $\Delta Y_{\text{max}}\Delta P_{y_{\text{min}}} = \hbar$.

It is well known from the Heisenberg principle that the latter relation can be cast into

\[
\Delta Y_{\text{max}}\Delta E = \hbar c \implies \Delta E = \frac{\hbar c}{\Delta Y_{\text{max}}}. \tag{26}
\]

where $\Delta E = \Delta P_{y_{\text{min}}}c$. Unlike the results obtained in the minimal length scenarios \[7\ \[10\ \[55\ \[57\ \[58\ \[59\ \[60\ \[61\ \[62\], here the required uncertainty energy is weak since the
dimension of length $\Delta Y_{\text{max}}$ is very large. This indicates that a maximal localization of quantum gravity induces weak energies for its measurement. Let us now consider the equation $\Delta Y_{\text{max}} = \Delta t_c$. Inserting this equation in (26), one obtains

$$\Delta t = \frac{\hbar}{\Delta E}. \quad (27)$$

Since, the uncertainty energy is low in this space due to the maximal measurement of quantum gravity, then its time $\Delta t$ strongly increases i.e the time runs more slowly in this space. In comparison with the minimal length theories, the concept of a maximal length of quantum gravity developed in this paper admits a close analogy with the properties of gravity in GR in the sense that the gravitational field becomes stronger for heavy systems that curve the space, allowing the surrounding light systems to fall down with low energies. The resulting time inside of this space is dilated and length contraction takes effect. As will be shown in the next section, the increase of the quantum gravitational parameter $\tau$ in an infinite square well potential curves the quantum levels to enable the particles to jump from one state to another with low energy and with high probability densities. The wavefunction compresses and contracts inward as one increases the effect of quantum gravitational effects.

iii) Finally, simultaneous measurements of operators $(\hat{X}, \hat{P}_y)$, $(\hat{X}, \hat{P}_x)$ and $(\hat{X}, \hat{P}_y)$ are spatial isotropy since their measurements do not present any minimal/maximal length or minimal momentum.

Moreover, repeating the GUP calculations with the hidden position-dependent noncommutativity (13), one generates the same uncertainty measurements

$$\Delta x_{\text{min}} = \theta \tau, \quad \Delta x_{\text{max}} = \frac{1}{\tau}, \quad \Delta p_{\text{min}} = \hbar \tau. \quad (28)$$

This indicates that the Dyson map does not remove the characteristics of quantum gravity at this scale i.e it only removes the fuzziness induced by the singular points by shedding light on the hidden Hermitian space. Consequently particles can be localized in precise ways in this new space. This situation could be compared to the gravitational holographic principle where the real information inside the Black hole is virtually projected at its event horizon. Taking into account the latter, the formal simultaneous representation of minimal length $\Delta X_{\text{min}}$ and maximal length $\Delta X_{\text{max}}$ (Figure 2) can be illustrated as follows (Figure 3).

4 Hidden-Hermitian free particle Hamiltonian in a box

The Hamiltonian of free particle in 2D non-Hermitian position-dependent noncommutative space reads as follows

$$\hat{H}_F = \frac{1}{2m_0} \left( \hat{P}_x^2 + \hat{P}_y^2 \right). \quad (29)$$
As mentioned, any Hamiltonian depending on the operators $\hat{X}$ or $\hat{P}_y$ will obviously no longer be Hermitian. Thus, using the relations (5), we can transform the Hamiltonian (29) into the standard $\theta$-deformed operator (2) as follows

$$\hat{H}_F = \frac{1}{2m_0} \left[ \hat{p}_x^2 + (1 - \tau \hat{y}_0 + \tau^2 \hat{y}_0^2)^{1/2} \hat{p}_{y_0} (1 - \tau \hat{y}_0 + \tau^2 \hat{y}_0^2) \hat{p}_{y_0} (1 - \tau \hat{y}_0 + \tau^2 \hat{y}_0^2)^{1/2} \right].$$  \hspace{1cm} (30)

Evidently this Hamiltonian is non-Hermitian $\hat{H} \neq \hat{H}^\dagger$. Thus, the Hermicity requirement of this operator is achieved by means of a similarity transformation. Since all our variables are converted into Hermitian ones by the same Dyson map, this will also hold for any function in these variables, as for instance the Hamiltonian. Thus, the Hermitian counterpart Hamiltonian becomes

$$\hat{h}_F = \eta \hat{H}_F \eta^{-1} = \frac{1}{2m_0} (\hat{p}_x^2 + \hat{p}_y^2).$$  \hspace{1cm} (31)

Using the relation (10,12), we rewrite the Hamiltonian in terms of the $\theta$-noncommutative operators

$$\hat{h}_F = \frac{1}{2m_0} \left[ \hat{p}_{\tau y_0}^2 + (1 - \tau \hat{y}_0 + \tau^2 \hat{y}_0^2)^{1/2} \hat{p}_{\tau y_0} (1 - \tau \hat{y}_0 + \tau^2 \hat{y}_0^2) \hat{p}_{\tau y_0} (1 - \tau \hat{y}_0 + \tau^2 \hat{y}_0^2)^{1/2} \right].$$  \hspace{1cm} (32)

Appealing to the nonsymmetric Bopp-shift (3), we may rewrite, the above Hamiltonian as follows

$$\hat{h}_F = \frac{1}{2m_0} \left[ \hat{p}_{\tau y_0}^2 + (1 - \tau \hat{y}_0 + \tau^2 \hat{y}_0^2)^{1/2} \hat{p}_{\tau y_0} (1 - \tau \hat{y}_0 + \tau^2 \hat{y}_0^2) \hat{p}_{\tau y_0} (1 - \tau \hat{y}_0 + \tau^2 \hat{y}_0^2)^{1/2} \right].$$  \hspace{1cm} (33)
The time-independent Schrödinger equation is given by
\[
\hat{h}_F \psi(x_s, y_s) = E \psi(x_s, y_s),
\]
\[
(\hat{h}_F^x + \hat{h}_F^y) \psi(x_s, y_s) = E \psi(x_s, y_s).
\] (34)

As it is clearly seen, the system is decoupled and the solution to the eigenvalue equation (34) is given by
\[
\psi(x_s, y_s) = \psi(x_s) \psi(y_s), \quad E = E_x + E_y
\] (35)

where \(\psi(x_s)\) is the wave function in the \(x_s\)-direction and \(\psi(y_s)\) the wave function in the \(y_s\)-direction. Since the particle is free in the \(x_s\)-direction, the wave function is given by
\[
\psi_k(x_s) = \int_{-\infty}^{+\infty} dk g(k) e^{ikx_s}, \quad (36)
\]

where \(g(k)\) determines the shape of the wave packet and the energy spectrum is continuous \([26]\)
\[
E_x = E_k = \frac{\hbar^2 k^2}{2m_0}. \quad (37)
\]

In \(y_s\)-direction, we have to solve the following equation
\[
\frac{1}{2m_0} \left(1 - \tau \hat{y}_s + \tau^2 \hat{y}_s^2 \right)^{1/2} \hat{p}_{y_s} \left(1 - \tau \hat{y}_s + \tau^2 \hat{y}_s^2 \right) \hat{p}_{y_s} \left(1 - \tau \hat{y}_s + \tau^2 \hat{y}_s^2 \right)^{1/2} \psi(y_s) = E_y \psi(y_s). \quad (38)
\]

This equation is an agreement with the one introduced by von Roos \([64]\) for systems with a position-dependent mass (PDM) operator and it can be rewritten as \([68]\)
\[
\left( -\frac{\hbar^2}{2m_0} \left( \frac{m_0}{m(y_s)} \frac{\partial}{\partial y_s} \right)^2 \right) \psi(y_s) = E_y \psi(y_s), \quad (39)
\]

where
\[
m(\hat{y}_s) = \frac{m_0}{\left(1 - \tau \hat{y}_s + \tau^2 \hat{y}_s^2 \right)^2}, \quad (40)
\]

being the PDM of the system strongly perturbed by quantum gravity \([13]\). Figure 4 illustrates the PDM as a function of the position \(y_s\) \((0 < y_s < 0.3)\). One observes that the effective mass \(m(\hat{y}_s)\) in this description increases with \(\tau\). This indicates that quantum gravitational fields increase with \(m(\hat{y}_s)\). In otherwise, by increasing experimentally the PDM, one can make the quantum gravitational effects stronger for a measurement through the variation of PDM of the system. Furthermore, the increase of PDM with the quantum gravitational effect will be consequence of the deformation of the quantum energy levels allowing, the particle to jump from one state to another with low energies \([6]\). This observation is in perfect analogy with the theory of GR where massive objects induce strong gravitational fields and curve the space enabling the surrounding light systems to fall down with low energies.
The equation (39) can be conveniently rewritten by means of the transformation
\[ \psi(y_s) = \sqrt[4]{m(y_s)/m_0} \phi(y_s) \]
as in [68]
\[ -\frac{\hbar^2}{2m_0} \left( \frac{m_0}{m(y_s)} \frac{\partial}{\partial y_s} \right)^2 \phi(y_s) = E_y \phi(y_s), \quad \text{with} \quad E_y > 0, \quad (41) \]
or
\[ -\frac{\hbar^2}{2m_0} \left[ (1 - \tau y_s + \tau^2 y_s^2)^2 \frac{\partial^2}{\partial y_s^2} - \tau(1 - 2\tau y_s)(1 - \tau y_s + \tau^2 y_s^2) \frac{\partial}{\partial y_s} \right] \phi(y_s) = E_y \phi(y_s), \quad (42) \]
The solution of this equation (42) is given by [13]
\[ \phi_\lambda(y_s) = A \exp \left( \frac{2\lambda}{\tau \sqrt{3}} \left[ \arctan \left( \frac{2\tau y_s - 1}{\sqrt{3}} \right) + \frac{\pi}{6} \right] \right), \quad (43) \]
\[ \psi_\lambda(y_s) = \frac{A}{\sqrt{1 - \tau y_s + \tau^2 y_s^2}} \exp \left( \frac{2\lambda}{\tau \sqrt{3}} \left[ \arctan \left( \frac{2\tau y_s - 1}{\sqrt{3}} \right) + \frac{\pi}{6} \right] \right), \quad (44) \]
where \( \lambda = \sqrt{2m_0 E_y/\hbar} \) and A is the normalization constant. We notice that if the standard wave-function \( \psi_\lambda(y_s) \) is normalized, then \( \phi_\lambda(y_s) \) is normalized under a \( \tau \)-deformed
integral. Indeed, we have
\[
\int_{-\infty}^{+\infty} dy_s \psi_\lambda^*(y_s) \psi_\lambda(y_s) = \int_{-\infty}^{+\infty} dy_s \frac{1}{1 - \tau y_s + \tau^2 y_s^2} \phi_\lambda^*(y_s) \phi_\lambda(y_s) = 1. \tag{45}
\]
Based on this equation (45), the normalized constant $A$ is determined as follows
\[
1 = \int_{-\infty}^{+\infty} dy_s \frac{1}{1 - \tau y_s + \tau^2 y_s^2} \phi_\lambda^*(y_s) \phi_\lambda(y_s) \tag{46}
\]
and
\[
A^2 \int_{-\infty}^{+\infty} dy_s \frac{1}{1 - \tau y_s + \tau^2 y_s^2}, \tag{47}
\]
so, we find
\[
A = \sqrt{\frac{\tau \sqrt{3}}{2\pi}}. \tag{48}
\]
The next important point concerns the quantization of the energy spectrum; we will show below that this property comes directly from the orthogonality of these solutions. Since the operator $\hat{h}_y$ is Hermitian, then the corresponding eigenfunctions $\psi_\lambda(y_s)$ are orthogonal. This property can be shown by considering the integral
\[
\int_{-\infty}^{+\infty} dy_s \hat{h}_y \psi_\lambda^*(y_s) \psi_\lambda(y_s) = \int_{-\infty}^{+\infty} dy_s \hat{\psi}_\lambda^*(y_s) \hat{h}_y \hat{\psi}_\lambda(y_s), \tag{49}
\]
which becomes, after an integration by parts,
\[
E_{\lambda'} \int_{-\infty}^{+\infty} dy_s \hat{\psi}_\lambda^*(y_s) \hat{h}_y \hat{\psi}_\lambda(y_s) = E_\lambda \int_{-\infty}^{+\infty} dy_s \hat{\psi}_\lambda^*(y_s) \hat{h}_y \hat{\psi}_\lambda(y_s). \tag{50}
\]
Since these two integrals are equal, one has
\[
(E_{\lambda'} - E_\lambda) \int_{-\infty}^{+\infty} dy_s \psi_\lambda^*(y_s) \psi_\lambda(y_s) = 0, \tag{51}
\]
and
\[
(E_{\lambda'} - E_\lambda) \int_{-\infty}^{+\infty} dy_s \frac{A^2}{1 - \tau y_s + \tau^2 y_s^2} e^{i \frac{2(\lambda - \lambda')}{\tau \sqrt{3}}} \left[ \arctan \left( \frac{2\tau y_s}{\sqrt{3}} \right) + \frac{\pi}{2} \right] = 0, \tag{52}
\]
and
\[
\sin \left( \frac{\lambda - \lambda'}{\tau \sqrt{3}} \pi \right) = 0. \tag{53}
\]
The quantization follows from the equation (53) and leads to the equation
\[
\frac{\lambda - \lambda'}{\tau \sqrt{3}} \pi = n\pi \quad \text{and} \quad \lambda - \lambda' = \lambda_n = \tau n \sqrt{3}, \quad n \in \mathbb{N}, \tag{54}
\]
where one notices the case \( n = 0 \) i.e. \( \lambda = \Lambda \), corresponding to the normalization condition considered in (49). Then, the energy spectrum of the particle is written as

\[
E_y = E_n = \frac{3\tau^2\hbar^2}{2m_0} n^2.
\]  

As it is clearly obtained, the presence of this deformed parameter \( \tau \) in \( y_s \)-direction quantized the energy of a free particle. This fact comes to confirm the fundamental property of gravity which consists of contracting and discretizing the matter.

Then, the total eigensystem is given by

\[
E = \begin{cases} 
E_x = \frac{\hbar^2 k^2}{2m_0}, \\
E_y = \frac{3\tau^2\hbar^2}{2m_0} n^2.
\end{cases}
\]  

and

\[
\psi(x_s, y_s) = \begin{cases} 
\psi_k(x_s) = \int_{-\infty}^{+\infty} dk g(k)e^{ikx_s}, \\
\psi_n(y_s) = A \frac{\hbar}{\sqrt{1-\tau y_s + \tau^2 y_s^2}} \exp\left(i2n \left[ \arctan\left(\frac{2\tau y_s - 1}{\sqrt{3}}\right) + \frac{\pi}{6}\right]\right).
\end{cases}
\]  

Now, we consider the above free particle of mass \( m_0 \) captured in a two-dimensional box of length \( 0 \leq x_s \leq a \) and height \( 0 \leq y_s \leq a \). The boundaries of the box are located.

We impose the wave functions \( \psi(0) = 0 = \psi(a) \). The eigensystems in \( x_s \)-direction are given by

\[
\psi_n(x_s) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi}{a} x_s\right), \quad E_n = \frac{n^2 \hbar^2}{2m_0a^2}, \quad E_1 = \frac{\pi^2 \hbar^2}{2m_0a^2}.
\]  

Taking the results (58) as a witness, we study what follows the influence of the deformed parameter \( \tau \) on the system. In \( y_s \)-direction, the solution is given by

\[
\psi_k(y_s) = \frac{B}{\sqrt{1-\tau y_s + \tau^2 y_s^2}} \exp\left(i \frac{2k}{\tau \sqrt{3}} \left[ \arctan\left(\frac{2\tau y_s - 1}{\sqrt{3}}\right) + \frac{\pi}{6}\right]\right),
\]  

where \( k = \frac{\sqrt{2m_0E'}}{\hbar} \). Then by normalization, \( \langle \psi_k|\psi_k \rangle = 1 \), we have

\[
1 = B^2 \int_0^a \frac{dy_s}{1-\tau y_s + \tau^2 y_s^2},
\]  

so we find

\[
B = \sqrt{\frac{\tau \sqrt{3}}{2}} \left[ \arctan\left(\frac{2\tau a - 1}{\sqrt{3}}\right) + \frac{\pi}{6}\right]^{-1/2}.
\]  

Based on the reference [7], the scalar product of the formal eigenstates is given by

\[
\langle \psi_{k'}|\psi_k \rangle = \frac{\tau \sqrt{3}}{2(k-k')} \left[ \arctan\left(\frac{2\tau a - 1}{\sqrt{3}}\right)\right] \sin\left(\frac{2(k-k') \left[ \arctan\left(\frac{2\tau a - 1}{\sqrt{3}}\right)\right]}{\tau \sqrt{3}}\right).
\]  

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This relation shows that, the normalized eigenstates (59) are no longer orthogonal. However, if one tends \((k - k') \to \infty\), these states become orthogonal

\[
\lim_{(k-k') \to \infty} \langle \psi_{k'} | \psi_k \rangle = 0. \tag{63}
\]

These properties show that, the states \(|\psi_k\rangle\) are essentially Gaussians centered at \((k - k') \to 0\) (see Figure 5). This observation indicates primordial fluctuations at this scale and these fluctuations increase with the quantum gravitational effects. As will be shown in the forthcoming development, the increase of fluctuations is manifested by high probability densities of particles. The states \(|\psi_k\rangle\) can be compared to the coherent states of harmonic oscillator [68, 66, 67] which are known as states that mediate a smooth transition between the quantum and classical worlds. This transition is manifested by the saturation of the Heisenberg uncertainty principle \(\Delta_x \Delta_p = \frac{\hbar}{2}\). In comparison with coherent states of harmonic oscillator, the states \(|\psi_k\rangle\) strongly saturate the GUP \((\Delta_{\psi_k} X \Delta_{\psi_k} P = \hbar)\) at the Planck scale and could be used to describe the transition states between the quantum world and unknown world for which the physical descriptions are out of reach.

![Figure 5: Variation of \(\langle \psi_{k'} | \psi_k \rangle\) versus \(k - k'\) with \(a = 1\).](image)

We suppose that, the wave function satisfies the Dirichlet condition i.e it vanishes at the boundaries \(\psi_k(0) = 0 = \psi_k(a)\). Thus, using especially the boundary condition
ψ_k(0) = 0, the above wavefunctions (59) becomes
\[ \psi_k(y_s) = \frac{B}{\sqrt{1 - \tau y_s + \tau^2 y_s^2}} \sin \left( \frac{2k}{\tau \sqrt{\frac{3}{\tau}} \arctan \left( \frac{2\tau y_s - 1}{\sqrt{3}} \right) + \frac{\pi}{6}} \right). \] (64)

The quantization follows from the boundary condition ψ_k(a) = 0 and leads to the equation
\[ \frac{2k_n}{\tau \sqrt{\frac{3}{\tau}}} \left[ \arctan \left( \frac{2\tau a - 1}{\sqrt{3}} \right) + \frac{\pi}{6} \right] = n\pi \quad \text{with} \quad n \in \mathbb{N}^*, \] (65)
\[ k_n = \frac{\pi \tau \sqrt{\frac{3}{\tau}}}{2 \left[ \arctan \left( \frac{2\tau a - 1}{\sqrt{3}} \right) + \frac{\pi}{6} \right]}. \] (66)

Then, the energy spectrum of the particle is written as
\[ E'_n = \frac{3\pi^2 \tau^2 \hbar^2 n^2}{8m_0 \left[ \arctan \left( \frac{2\tau a - 1}{\sqrt{3}} \right) + \frac{\pi}{6} \right]^2}. \] (67)

At the limit τ → 0, we have
\[ \lim_{\tau \to 0} E'_n = E_n = \frac{\pi^2 \hbar^2 n^2}{2m_0 a^2}. \] (68)
Thus, the energy levels can be rewritten as
\[ E_n = \frac{3}{4} \left[ \frac{\tau L}{\arctan \left( \frac{2\tau a - 1}{\sqrt{3}} \right) + \frac{\pi}{6}} \right]^2 E_n \leq E_n. \] (69)

The effects of the parameter τ in y_s direction induce deformations of quantum levels, which consequently lead to a decrease in the amplitude of the energy levels. The corresponding wave functions to the energies (67) are given by
\[ \psi_n(x) = \frac{B}{\sqrt{1 - \tau y_s + \tau^2 y_s^2}} \sin \left( \frac{n\pi}{\left[ \arctan \left( \frac{2\tau a - 1}{\sqrt{3}} \right) + \frac{\pi}{6} \right]^2} \left[ \arctan \left( \frac{2\tau y_s - 1}{\sqrt{3}} \right) + \frac{\pi}{6} \right] \right). \] (70)

The total eigenvalues of the system are given by
\[ E'_n = n^2 \frac{\pi^2 \hbar^2}{2m_0 a^2} + \frac{3\pi^2 \tau^2 \hbar^2 n^2}{8m_0 \left[ \arctan \left( \frac{2\tau a - 1}{\sqrt{3}} \right) + \frac{\pi}{6} \right]^2}, \]
\[ = \left( 1 + \frac{3\tau^2 a}{\left[ \arctan \left( \frac{2\tau a - 1}{\sqrt{3}} \right) + \frac{\pi}{6} \right]^2} \right) E_n. \] (71)
and
\[
\lim_{\tau \to 0} E_n^t = 2E_n, \quad (72)
\]

The wave function in \(x, y\)-directions are given by
\[
\psi(x, y) = \frac{B\sqrt{\frac{2}{\pi}}}{\sqrt{1 - \tau y + \tau^2 y^2}} \sin \left( \frac{n\pi}{a} x \right)
\times \sin \left[ \frac{n\pi}{\arctan \left( \frac{2\tau a - 1}{\sqrt{3}} \right) + \frac{\pi}{6} } \right]^2 \left[ \arctan \left( \frac{2\tau y - 1}{\sqrt{3}} \right) + \frac{\pi}{6} \right]. \quad (73)
\]

At the limit \(\tau \to 0\), we have
\[
\lim_{\tau \to 0} \psi(x, y) = \frac{2}{a} \sin \left( \frac{n\pi}{a} x \right) \sin \left( \frac{n\pi}{a} y \right). \quad (74)
\]

The corresponding probability density is given by
\[
\rho(x, y) = \frac{2B}{a(1 - \tau y + \tau^2 y^2)} \sin^2 \left( \frac{n\pi}{a} x \right)
\times \sin \left[ \frac{n\pi}{\arctan \left( \frac{2\tau a - 1}{\sqrt{3}} \right) + \frac{\pi}{6} } \right]^2 \left[ \arctan \left( \frac{2\tau y - 1}{\sqrt{3}} \right) + \frac{\pi}{6} \right]. \quad (75)
\]

Figure 6 illustrates the energy levels of the particle as functions of the quantum number \(n\) and the quantum gravitational parameter \(\tau\). Figure (a) shows how the increase of \(\tau\) gradually curves the energy levels as one increases the quantum number \(n\) from the fundamental level. Figure (b) shows energy levels versus the quantum number \(n\) for fixed values of \(\tau\). Conversely to the graph obtained in [68, 69, 70, 71], one observes that, the amplitudes of energy levels \(E_n^t/E_1\) decrease when \(\tau\) increases. In fact, increasing the quantum gravitational effects lead to the enhancement of binding quantum levels allowing particles to jump from one state to another with low energies [13].

Figure 7 illustrates a comparison between eigenfunctions \(\psi_n(x)\) and \(\psi_n(y)\) for fixed values of \(n\) and \(\tau\). The wave function in \(x\)-direction is taken as a witness with respect to that of the \(y\)-direction where the effects of quantum gravity are strongly applied. For \(n \in \{1; 5; 15; 20\}\), \(\psi(y)\) compresses and contracts inward as one increases \(\tau\). This fact comes to confirm the fundamental property of gravity which is length contraction.

Unlike the figure reported in [69, 70], here the figure 8 shows the plots of the probability density for the three lower states \(n = 1; n = 5; n = 15; n = 30\) for a fixed value of \(\tau\) (\(\tau = 0.1\)), it can be seen that the probability to find a particle is practically the same everywhere in the square well and this probability strongly increases with the quantum number. This indicates that the deformations allow particles to jump from one state to another with low energies and with high probability densities.
Figure 6: The energy $E_n/E_1$ of the particle in 2D box of length $a = 1$ with mass $m_0 = 1$ and $\hbar = 1$. 
Figure 7: Comparison graph of $\psi_n(x_s)$ and $\psi_n(y_s)$ for a particle confined in an infinite square well of length $a = 1$ deformed by the gravity parameter $\tau$ in $y_s$ direction.
Figure 8: The probability density of a two-dimensional infinite square well for $\tau = 0.1$. 
5 Concluding remarks

In this paper, we revisited our recent concept of minimal and maximal lengths [2] (previously predicted by Perivolaropoulos [1]) in the context of non-Hermitian position-dependent noncommutativity. We have shown that the existence of both lengths has an interesting and significant properties of quantum gravity at the Planck scale. We have shown that a simultaneous measurement of both lengths form a lattice system in which each site represented by a minimal length $l_{\text{min}}$ is spaced by a maximal length $l_{\text{max}}$. At each singular point $l_{\text{min}}$ results from the unification of strong magnetic and weak quantum gravitational fields. Furthermore, we have demonstrated that the maximal length of quantum gravity at this scale manifests properties similar to classical gravity and allows probing quantum gravitational effects with low energies. Moreover the existence of minimal uncertainties lead almost unavoidably to non-Hermicity of some operators that generate the noncommutative algebra [1]. Consequently, Hamiltonians of systems involving these operators will not be Hermitian and the corresponding Hilbert space structure is modified. In order to map these operators into Hermitian ones, we have introduced an appropriate Dyson map and by means of a similarity transformation, we have generated a hidden Hermitian position-dependent noncommutativity [13]. This transformation preserves properties of quantum gravity at this but remove the fuzziness induced by the minimal uncertainties. Finally, to find the representation of a free particle in this new space, we have solved a non-linear Schrödinger equation. To do so, we have transformed this equation into von Roos equation [64], then by an appropriate change of variable, we reduced this equation into a simple and solvable non-linear Schrödinger equation. We observed that the increase of quantum gravitational effects $\tau$ in this region curves the quantum energy levels. These curvatures are more pronounced as one increases the quantum levels, allowing the particle to jump from one state to another with low energies and with high probability densities. Furthermore, they contract and compress the wave function in $y_s$-direction. However, one can wonder about what happens in the case of a harmonic oscillator? In this way, the Hamiltonian of the system is given by

$$\hat{H}_{ho} = \frac{1}{2m_0} \left( \hat{P}_x^2 + \hat{P}_y^2 \right) + \frac{1}{2} m_0 \omega^2 (\hat{X}^2 + \hat{Y}^2).$$

(76)

In terms of the $\theta$-deformed variables, this Hamiltonian can also be re-written as follows

$$\hat{H}_{ho} = \frac{1}{2m_0} \left[ \hat{P}_{x_0}^2 + \left(1 - \tau \hat{y}_0 + \tau^2 \hat{y}_0^2 \right)^2 \hat{P}_{y_0}^2 - i\hbar \tau (1 + 2\tau \hat{y}_0)(1 - \tau \hat{y}_0 + \tau^2 \hat{y}_0^2) \hat{P}_{y_0} \right] + \frac{1}{2} m_0 \omega^2 \left[ \left(1 - \tau \hat{y}_0 + \tau^2 \hat{y}_0^2 \right)^2 \hat{x}_0^2 - i\theta (1 - 2\tau \hat{y}_0) (1 - \tau \hat{y}_0 + \tau^2 \hat{y}_0^2) x_0 + y_0^2 \right].$$

(77)

Since this Hamiltonian $\hat{H}_{ho}$ is evidently non-Hermitian, we have to employ a Dyson map to convert it into Hermitian one as in the previous example

$$\hat{h}_{ho} = \eta \hat{H}_{ho} \eta^{-1} = \frac{1}{2m_0} \left( \hat{P}_x^2 + \hat{P}_y^2 \right) + \frac{1}{2} m_0 \omega^2 (\hat{x}^2 + \hat{y}^2).$$

(78)
This Hamiltonian may be also re-expressed as follows using the representation (9,10,11,12)

\[ \hat{h}_{ho} = \frac{1}{2m_0} \left[ \hat{p}_{x_0}^2 + (1 - \tau \hat{y}_0 + \tau^2 \hat{y}_0^2)^{1/2} \hat{p}_{y_0} (1 - \tau \hat{y}_0 + \tau^2 \hat{y}_0^2)^{1/2} \right] + \frac{1}{2} m_0 \omega^2 \left[ \hat{y}_0^2 + (1 - \tau \hat{y}_0 + \tau^2 \hat{y}_0^2)^{1/2} \hat{x}_0 (1 - \tau \hat{y}_0 + \tau^2 \hat{y}_0^2)^{1/2} \right] ^{79} \]

The eigensystems of the equation (79) are far more complicated to obtain with the same method as in the previous models as the system viewed as a differential equation no longer decouples in $x_0$ and $y_0$. We leave the construction of solutions for this model by alternative means for future work.

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