Introduction

As predicted by Casimir [1], two parallel uncharged ideal metal planes separated by a distance \( a \) at zero temperature should attract each other by the force

\[
F(a) = -\frac{\pi^2 c}{240} \frac{\hbar}{a^4},
\]

which depends on the Planck constant \( \hbar \) and the speed of light \( c \). According to Casimir, this force is caused by the zero-point oscillations of quantized electromagnetic field whose spectrum is altered by the presence of ideal metal planes. Within the formalism of quantum electrodynamics, in the absence of planes the zero-point energy is given by an integral over a continuous wave vector \( \mathbf{k} \). In the presence of planes, however, the tangential component of electric field vanishes on the plane surfaces and the component \( k_z \) in direction perpendicular to the planes becomes discrete. The modified zero-point energy of the electromagnetic field is an integral over \( k_x \) and \( k_y \), but a discrete sum in \( k_z \). Although both zero-point energies in the absence and in the presence of planes are infinitely large, their difference is finite. As a result, the negative derivative of this difference with respect to \( a \) is equal to the Casimir force \([1]\).

More recently, it was understood that the Casimir force, as well as the more familiar van der Waals force, belongs to a wide class of physical phenomena determined by the zero-point and thermal fluctuations of the electromagnetic field. Lifshitz \([2]\) created the general theory of forces of this kind acting between two parallel material plates (semispaces) kept at any temperature in thermal equilibrium with the environment. In the framework of the Lifshitz theory, the ideal metal boundary conditions are replaced by the electrodynamic continuity conditions which take into account real material properties by means of the frequency-dependent dielectric permittivity and (for magnetic plates) magnetic permeability.

There are two main approaches to a derivation of basic expressions of the Lifshitz theory for the force of fluctuation origin. One of them is based on the fluctuation-dissipation theorem of statistical physics \([3, 4]\) and another one, which goes back to Casimir, on quantum field theory with appropriate boundary conditions \([5, 6]\). By the frequently used present-day terminology, the name van der Waals force refers to the fluctuation-induced forces at separations of a few nanometers which do not depend on \( c \). The term Casimir force is used in all remaining cases, i.e., when the interaction of fluctuation origin depends on \( \hbar, c, \) material properties and temperature. By now the Lifshitz theory is generalized for the case of boundary surfaces of arbitrary geometric shape \([5, 12]\). A systematic investigation of the thermal Casimir force between parallel plates made of real metals traces back to 2000. In Ref. \([13]\) it was shown that if the low-frequency dielectric response of metals is described by the realistic Drude model taking a proper account of the relaxation properties of conduction electrons, the Lifshitz theory predicts large thermal correction which arises at short separations of a few hundred nanometers at room temperature and decreases the force magnitude. The thermal
correction of this type does not arise if the low-frequency dielectric response of metals is described by the plasma model which disregards the relaxation properties of conduction electrons \[14\]. It should be remembered, however, that the plasma model is in fact applicable only at high frequencies in the range of infrared optics where the relaxation properties do not play any role. The surprising thing is that the Casimir entropy calculated with the Lifshitz theory employing the Drude model violates the third law of thermodynamics (the Nernst heat theorem) for metals with perfect crystal lattices \[15\] \[21\] (the Nernst heat theorem is followed for metals with the defects of structure \[22\] \[24\], but this does not solve the problem because perfect crystal lattice is a system with the nondegenerate ground state, so that it should satisfy the third law of thermodynamics). If, however, the Casimir entropy is found using the plasma model dielectric response, the Nernst heat theorem is satisfied without trouble \[15\] \[21\].

Of even greater surprise is that the theoretical predictions of the Lifshitz theory using the Drude dielectric response at low frequencies have been excluded by the measurement data of many experiments performed with both nonmagnetic (Au) and magnetic (Ni) metallic test bodies by the two different experimental groups \[25\] \[37\]. The same measurement data were found to be in a very good agreement with the predictions of the Lifshitz theory employing the plasma dielectric response \[23\] \[37\]. In the most striking experiment of Ref. \[33\] using the differential measurement scheme, a difference between the excluded and confirmed theoretical predictions was by up to a factor of 1000. Thus, the Drude mode, which provides an adequate description of numerous optical and electrical physical phenomena, does not work in application to the Casimir force.

Note that the measurement data of one experiment performed at separations of a few micrometers were found to be in better agreement with theoretical predictions using the Drude model \[38\]. To obtain this conclusion, the Casimir force calculated using the Drude or the plasma model was subtracted from by the order of magnitude larger measured force. The obtained differences were fitted to the theoretical electric force originating from the surface patches. The results of Ref. \[38\] were shown to be, however, uncertain because in this experiment the surface imperfections, which are unavoidably present on a surface of the used spherical lens of centimeter-size radius, were ignored \[39\]. Recent Casimir experiment performed in the micrometer separation range, where the patch potentials were directly measured by means of Kelvin probe microscopy, demonstrated an agreement with theoretical predictions using the plasma model and excluded those using the Drude model \[37\].

An apparent disagreement of theoretical predictions of the fundamental Lifshitz theory using the Drude model, which was fully validated in the area of electromagnetic and optical phenomena other than the Casimir effect, with the measurement data, as well as with the basic principles of thermodynamics, is puzzling and calls for a satisfactory explanation. This subject was hotly debated in the literature starting from 2000 and many attempts have been made directed to reaching an agreement with thermodynamics, looking for some unaccounted systematic effects in measurements of the Casimir force, or developing the more exact theory for a sphere-plate geometry used in experiments. Considerable advances have been made in this way (see Refs. \[40\] \[45\] for a review), but the ultimate resolution of the Casimir puzzle still remains to be found.

One possible approach to understanding of this problem is that all models of the electromagnetic response of macroscopic bodies are in some sense phenomenological, either relying on assumptions about bulk properties or upon very simplified models of the response of individual atoms to an applied field. This means that no model derived in any macroscopic framework using, e.g., the Boltzmann transport equations or Kubo theory can be expected to work under all circumstances, especially, in the very extreme conditions in Casimir force experiments. From this standpoint, it is not at all surprising that some physical situations give results in contrast with the predictions of commonly used models. There are, however, some limitations following from fundamental physics that exacerbate the situation. Thus, according to Maxwell equations, the dielectric response of metals to electromagnetic field in the quasistatic limit is inverse proportional to the frequency. The Drude model satisfies this demand and describes the relaxation properties of conduction electrons whereas the dielectric permittivity of the plasma model is inverse proportional to the second power of frequency and does not describe the relaxation properties.

At one time it was hoped that a resolution of this problem may come from the investigation of graphene which is a two-dimensional sheet of carbon atoms packed in the hexagonal crystal lattice. At energies below a few eV, which are characteristic for the Casimir effect at not too short separations, graphene is well described by the Dirac model as a set of massless or very light electronic quasi-particles satisfying the Dirac equation where \(c\) is replaced with the Fermi velocity \(v_F\) \[46\] \[47\]. The spatially nonlocal dielectric response of graphene to electromagnetic field was found precisely on the basis of first principles of quantum electrodynamics using the polarization tensor \[48\] \[51\]. The Lifshitz theory employing this dielectric response turned out to be in agreement with experiments measuring the Casimir force in graphene systems \[52\] \[55\] and with the Nernst heat theorem \[56\] \[60\]. The question arises of whether the spatially nonlocal dielectric response could be helpful for a resolution of the Casimir puzzle.

Unfortunately, an application of the conventional spa-
tially nonlocal dielectric permittivities derived in the literature for theoretical description of the anomalous skin effect, screening effects etc. [61–67], although leaves room for a resolution of thermodynamic problems, does not remedy a contradiction between the Lifshitz theory and the measurement data [68–72]. In this situation, some phenomenological models are worth consideration.

With this approach, Ref. [73] proposed the spatially nonlocal dielectric permittivities which describe nearly the same response, as the Drude model, to electromagnetic waves on the mass shell (i.e., to the propagating waves), but quite a different response, than the Drude model, to the off-the-mass-shell waves, which are also called evanescent. These permittivities depend only on the magnitude of the wave vector projection on the plane of Casimir plates $k_\perp$. It was shown that they satisfy the Kramers-Kronig relations [73], and the respective Casimir entropy goes to zero with vanishing temperature in agreement with the Nernst heat theorem [74]. With the aim of solving the above problems, spatially nonlocal permittivities were also introduced in Ref. [73].

The most important thing is that the proposed nonlocal permittivities bring the Lifshitz theory in agreement with the measurement data of experiments performed with two nonmagnetic (Au) [73] and two magnetic (Ni) [74] test bodies. To perform the theory-experiment comparison, the expressions for the reflection coefficients entering the Lifshitz formula via the surface impedances were used. The latter have been found in Refs. [62, 63] for nonmagnetic metals in the approximation of specular reflection of electrons on the boundary surfaces as some functionals of nonlocal dielectric permittivities. For the case of magnetic metals, similar surface impedances were recently derived in Ref. [75].

In this paper, we suggest the spatially nonlocal response functions which, similar to Ref. [73], lead to the same results, as the standard Drude model, for electromagnetic waves on the mass shell, but to alternative results for the off-the-mass-shell fields. However, unlike Ref. [73], the response functions considered below depend on all components of the wave vector $k$ what is fully consistent with the used formalism of surface impedances in the approximation of specular reflection developed in Refs. [62, 63] (see Sec. II for more detail).

We calculate the surface impedances and reflection coefficients for two independent polarizations of the electromagnetic field in the case of Au and Ni surfaces. The obtained reflection coefficients for Au surfaces are used to perform computations of the effective Casimir pressure between two Au-coated plates [27] and the Casimir force between an Au-coated sphere and an Au-coated plate [37] in the experiments performed by means of a micromechanical torsional oscillator. Using the same reflection coefficients, we also compute the gradient of the Casimir force between an Au-coated sphere and an Au-coated plate in the experiments performed in different separation regions by means of an atomic force microscope [24, 36]. With the help of reflection coefficients on both Au and Ni surfaces, we perform computations of the gradient of the Casimir force between an Au-coated sphere and a Ni-coated plate measured in the experiment [39]. Finally, the gradient of the Casimir force between a Ni-coated sphere and a Ni-coated plate is computed which was measured in Refs. [31, 32].

According to our results, in all cases the predictions of the Lifshitz theory using the suggested nonlocal response functions, which take the proper account of the relaxation properties of conduction electrons in the region of propagating waves, are in a very good agreement with the measurement data. Future applications of these results are discussed.

The paper is organized as follows. In Sec. II, we present the formalism of the Lifshitz theory in the approximation of specular reflection. In Sec. III, the spatially nonlocal response functions depending on all components of the wave vector are introduced and their properties are investigated. Section IV is devoted to computations of the effective Casimir pressure and Casimir force in different experiments using a micromechanical torsional oscillator. In Secs. V and VI computations of the gradient of the Casimir force are performed between nonmagnetic and with magnetic test bodies, respectively, measured by means of an atomic force microscope. In Sec. VII, the reader will find our conclusions and a discussion.

**FORMALISM OF THE LIFSHITZ THEORY IN THE APPROXIMATION OF SPECULAR REFLECTION**

The Casimir free energy per unit area and pressure in the configuration of two thick parallel plates (semispheres) spaced at separation $a$ at temperature $T$ in thermal equilibrium with the environment are given by the famous Lifshitz formulas [2, 3] (see also [40, 41] for modern notations in terms of the reflection coefficients used below)

$$ F(a, T) = \frac{k_B T}{2\pi} \sum_{l=0}^{\infty} \int_0^\infty k_\perp dk_\perp \times \sum_\alpha \ln \left[ 1 - r_A^{(1)}(i\xi_l, k_\perp) r_A^{(2)}(i\xi_l, k_\perp) e^{-2aq_l} \right], $$

$$ P(a, T) = -\frac{k_B T}{\pi} \sum_{l=0}^{\infty} q_l k_\perp dk_\perp \times \sum_\alpha \left[ r_A^{(1)}(i\xi_l, k_\perp) r_A^{(2)}(i\xi_l, k_\perp) e^{2aq_l} \right]^{-1}. \quad (2) $$

Here, $k_B$ is the Boltzmann constant, $k_\perp = (k_x^2 + k_y^2)^{1/2}$ is the magnitude of the wave vector projection on the
plane of plates, $\xi_l = 2\pi k_B T l/h$ are the Matsubara frequencies, $q_l = (k_{l\perp}^2 + \xi_l^2/c^2)^{1/2}$, and the prime on the summation sign divides by 2 the term of the first sums with $l = 0$. The reflection coefficients of electromagnetic waves with the transverse magnetic ($\alpha = \text{TM}$) and transverse electric ($\alpha = \text{TE}$) polarizations on the first and second plates are $r_{\text{TM}}^{(1)}(i\xi_l, k_{l\perp})$ and $r_{\text{TE}}^{(2)}(i\xi_l, k_{l\perp})$, respectively.

In the original version of the Lifshitz theory [2, 3], it is assumed that the plate materials possess only the temporal dispersion, i.e., their dielectric permittivities, $\varepsilon_n(\omega)$, and magnetic permeability, $\mu_n(\omega)$, where $n = 1, 2$ for the first and second plates, depend on the frequency $\omega$ (a dependence on $T$, e.g., for metals is also allowed). In this case the reflection coefficients are given by the familiar Fresnel formulas considered at $\omega = i\xi_l$

$$r_{\text{TM}}^{(n)}(i\xi_l, k_{l\perp}) = \frac{\varepsilon_n(i\xi_l)q_l - k_n(i\xi_l, k_{l\perp})}{\varepsilon_n(i\xi_l)q_l + k_n(i\xi_l, k_{l\perp})},$$

$$r_{\text{TE}}^{(n)}(i\xi_l, k_{l\perp}) = \frac{\mu_n(i\xi_l)q_l - k_n(i\xi_l, k_{l\perp})}{\mu_n(i\xi_l)q_l + k_n(i\xi_l, k_{l\perp})},$$

where

$$k_n(i\xi_l, k_{l\perp}) = \left[k_{l\perp}^2 + \varepsilon_n(i\xi_l)\mu_n(i\xi_l)\xi_l^2/c^2\right]^{1/2}. \quad (4)$$

The reflection coefficients in Eq. (2) can also be expressed in terms of the exact surface impedances $Z_{\text{TM}}^{(n)}(i\xi_l, k_{l\perp})$ and $Z_{\text{TE}}^{(n)}(i\xi_l, k_{l\perp})$

$$r_{\text{TM}}^{(n)}(i\xi_l, k_{l\perp}) = \frac{c q_l - i\xi_l Z_{\text{TM}}^{(n)}(i\xi_l, k_{l\perp})}{c q_l + i\xi_l Z_{\text{TM}}^{(n)}(i\xi_l, k_{l\perp})},$$

$$r_{\text{TE}}^{(n)}(i\xi_l, k_{l\perp}) = \frac{c q_l Z_{\text{TE}}^{(n)}(i\xi_l, k_{l\perp}) - i\xi_l}{c q_l Z_{\text{TE}}^{(n)}(i\xi_l, k_{l\perp}) + i\xi_l},$$

where for materials possessing only the temporal dispersion the impedances are connected with the dielectric permittivities and magnetic permeabilities as $\varepsilon_n(\omega)$ and $\mu_n(\omega)$

$$Z_{\text{TM}}^{(n)}(i\xi_l, k_{l\perp}) = \frac{\varepsilon_n(i\xi_l)k_{l\perp}}{\xi_l \varepsilon_n(i\xi_l)},$$

$$Z_{\text{TE}}^{(n)}(i\xi_l, k_{l\perp}) = \frac{\xi_l \mu_n(i\xi_l)}{c k_n(i\xi_l, k_{l\perp})}.$$ \quad (6)

It is evident that the substitution of Eq. (6) in Eq. (5) returns us back to the reflection coefficients (3).

According to generalizations of the Lifshitz theory in the framework of the scattering approach [11, 12], Eq. (2) with appropriately defined reflection coefficients remains valid for any planar structures. If materials of the plates, besides temporal, possess the spatial dispersion, derivation of the exact expressions for the reflection coefficients runs into problems. The point is that the response of spatially dispersive material filling in the entire 3-dimensional space to electric fields parallel and perpendicular to the wave vector $\mathbf{k} = (k_x, k_y, k_z)$ is described by the longitudinal, $\varepsilon^{\text{L}}(\omega, \mathbf{k})$, and transverse, $\varepsilon^{\text{Tr}}(\omega, \mathbf{k})$, dielectric permittivities [66, 77].

In the strict sense, these permittivities can be introduced only under a condition of translational invariance which is violated by the presence of Casimir plates separated by a vacuum gap [78–80]. The standard Lifshitz theory deals with plate materials possessing only the temporal dispersion. Therefore the dielectric permittivities depend only on $\omega$ and the violation of translational invariance makes no problem. For the plate materials with spatial dispersion, this violation, however, makes impossible an employment of the standard continuity boundary conditions and derivation of the Fresnel reflection coefficients [3].

This difficulty can be circumvented as follows. Since the Casimir force is determined by the dielectric properties in the bulk and appropriate boundary conditions, it is possible to preserve the translational invariance in fictitious homogeneous medium by assuming the specular reflection of charge carriers (electrons) on the boundary surfaces of Casimir plates. In doing so an electron reflected on an interface between the plate and the vacuum gap is indistinguishable from an electron coming freely from the source side of a fictitious medium. Then one can introduce the longitudinal and transverse dielectric permittivities $\varepsilon^{\text{L}}$ and $\varepsilon^{\text{Tr}}$ which depend on $\omega$ and all components of the wave vector $\mathbf{k}$ because the fictitious medium is translationally invariant in all directions, and not only in the plane of Casimir plates [62–63]. Physically there is no sufficient reason to exclude one of the wave vector components. This would be justified for graphene and other two-dimensional materials but not for a bulky matter possessing the spatial dispersion. Note that the response functions depending on all components of $\mathbf{k}$ are fully consistent with the Lifshitz theory where the reflection coefficients depend only on $k_{l\perp}$. As a result, these coefficients preserve the form of Eq. (5) whereas the surface impedances are obtained from the nonlocal bulk permittivities depending on $\omega$ and $\mathbf{k}$ by the integration with respect to $k_{l\perp}$ as:

$$Z_{\text{TM}}^{(n)}(i\xi_l, k_{l\perp}) = \frac{c q_l \mu_n(i\xi_l)}{\pi} \int_{-\infty}^{\infty} \frac{dk_{l\perp}}{k_{l\perp}^2} \left[\frac{k_{l\perp}^2}{\varepsilon_n^2(i\xi_l, k_{l\perp})\xi_l^2} + \frac{k_{l\perp}^2}{\varepsilon_n(i\xi_l, k_{l\perp})\mu_n(i\xi_l)\xi_l^2} + c^2 k_{l\perp}^2\right],$$

$$Z_{\text{TE}}^{(n)}(i\xi_l, k_{l\perp}) = \frac{c q_l \mu_n(i\xi_l)}{\pi} \int_{-\infty}^{\infty} \frac{dk_{l\perp}}{\varepsilon_n^2(i\xi_l, k_{l\perp})\mu_n(i\xi_l)\xi_l^2}.$$ \quad (7)

For nonmagnetic metals Eq. (7) was derived in Refs. 62, 63 (see also Ref. 81 for a review) and generalized for the case of magnetic ores in Ref. 70. There is also a generalization of Eq. (4) for the case of diffuse reflection [67, 82]. Note, however, that for sufficiently smooth boundary surfaces used in experiments on measuring the Casimir force the approximation of specular reflection is
well applicable.

As mentioned in Sec. I, the surface impedances with \( \mu_n(i\xi_l) = 1 \) and the dielectric permittivities \( \varepsilon^{(n)}(\omega, k) \) and \( \varepsilon^{(l)}(\omega, k) \) describing the anomalous skin effect were used to calculate the Casimir force between Au surfaces in Refs. [73, 71]. It was found, however, that corrections to the force due to spatial nonlocality in the region of the anomalous skin effect are too small and incapable to explain a disagreement between the measurement data and theoretical predictions.

**SPATIALLY NONLOCAL DIELECTRIC FUNCTIONS PROVIDING AN ALTERNATIVE RESPONSE TO THE OFF-THE-MASS-SHELL FIELDS**

The phenomenological nonlocal dielectric functions depending on \( k_\perp \), which bring the Lifshitz theory in agreement with experiments on measuring the Casimir force between two similar plates and with thermodynamics do not disregarding the dissipation of conduction electrons, were introduced in Refs. [72, 74, 76]. Here, we propose the nonlocal dielectric permittivities depending on all components of the wave vector

\[
\varepsilon^{(n)}_{tr}(\omega, k) = \varepsilon^{(n)}(\omega) - \frac{\omega^2 p,n}{\omega(\omega + i\gamma_n)} \left( 1 + i\frac{v^{(n)}_{tr} k}{\omega} \right),
\]

\[
\varepsilon^{(l)}_{tr}(\omega, k) = \varepsilon^{(l)}(\omega) - \frac{\omega^2 p,n}{\omega(\omega + i\gamma_n)} \left( 1 + i\frac{v^{(l)}_{tr} k}{\omega} \right)^{-1},
\]

where \( \varepsilon^{(n)}(\omega) \) is the contribution determined by the core electrons, \( \omega_{p,n} \) is the plasma frequency, \( \gamma_n \) is the relaxation parameter, \( k = |k| \) is the magnitude of the wave vector, and \( v^{(n)}_{tr}, v^{(l)}_{tr} \) are constants of the order of Fermi velocity \( v_F,n \) (as before, \( n = 1, 2 \) for materials of the first and second plates).

The response functions introduced in Refs. [73, 74, 76] are obtained from Eq. (8) if the magnitude of the wave vector \( k \) is replaced with the magnitude of its projection on the plane of the Casimir plates \( k_\perp \). It was shown [45, 73] that the Lifshitz theory using the resulting dielectric functions is in good agreement with the measurement data of experiments [27, 30] measuring the Casimir interaction between two similar Au test bodies if \( v^{(n)}_{tr} = v^{(l)}_{tr} = 7v_F,Au \). With the same numerical coefficient, \( v^{(n)}_{tr} = v^{(l)}_{tr} = 7v_F,Ni \), the Lifshitz theory using these permittivities was found in agreement [76] with measurements of the Casimir interaction between two similar magnetic (Ni) test bodies [31, 32]. However, as discussed in previous section, the nonlocal permittivities depending only on \( k_\perp \) are not fully consistent with the usual formalism of surface impedances in the approximation of specular reflection and represent only some kind of a simplified particular case.

We return to the permittivities which depend on all components of the wave vector. For the electromagnetic waves on the mass shell (the propagating waves) they describe approximately the same response as the standard Drude model supplemented by the oscillator terms \( \varepsilon^{(n)}_{osc}(\omega) \) taking into account the interband transitions [83]. This is because the additions to unity depending on \( k \) in the parantheses in Eq. (8) under a condition \( ck \lesssim \omega \) become negligibly small

\[
\frac{v^{(n)}_{tr} k}{\omega} \sim \frac{v_F,n}{c} \frac{ck}{\omega} \ll 1.
\]

In doing so, the response functions take into account dissipation of conduction electrons by means of the relaxation parameter \( \gamma_n \) as does the Drude model. If, however, \( ck > \omega \), as it holds for the evanescent waves which are off the mass shell in free space, the permittivities can lead to an electromagnetic response differing from that of the Drude model.

Below we demonstrate that the response functions take into account dissipation of conduction electrons by means of the relaxation parameter \( \gamma_n \) as does the Drude model. If, however, \( ck > \omega \), as it holds for the evanescent waves which are off the mass shell in free space, the permittivities can lead to an electromagnetic response differing from that of the Drude model.

We begin with calculation of the reflection coefficients at zero Matsubara frequency \( \xi_l = 0 \) which plays the major role in the problems of Casimir physics discussed in Sec. I. Substituting \( N = i\xi_l \) in Eq. (8), one obtains the expressions for nonlocal permittivities at the pure imaginary Matsubara frequencies

\[
\varepsilon^{(n)}_{tr}(i\xi_l, k) = \varepsilon^{(n)}(i\xi_l) + \frac{\omega^2 p,n}{\xi_l(\xi_l + i\gamma_n)} \left( 1 + i\frac{v^{(n)}_{tr} k}{\xi_l} \right),
\]

\[
\varepsilon^{(l)}_{tr}(i\xi_l, k) = \varepsilon^{(l)}(i\xi_l) + \frac{\omega^2 p,n}{\xi_l(\xi_l + i\gamma_n)} \left( 1 + i\frac{v^{(l)}_{tr} k}{\xi_l} \right)^{-1}.
\]
Substituting Eq. (10) to the first expression in Eq. (7) and considering vanishing \( \xi_0 \), we find the asymptotic behavior of \( Z_{TM}^{(n)} \) in this case

\[
Z_{TM}^{(n)}(i\xi_0, k_\perp) = \frac{2\varepsilon k^2}{\pi \varepsilon_0} \int_0^\infty \frac{dk_x}{\sqrt{k_x^2 + b_n^2 + k_\perp^2}},
\]

where \( b_n = \omega_{n}{^2}/(\gamma_n v_n^L) \). Performing here the change of integration variable from \( k_x \) to \( k = (k_x^2 + k_\perp^2)^{1/2} \), we rewrite Eq. (11) in the form

\[
Z_{TM}^{(n)}(i\xi_0, k_\perp) = \frac{2\varepsilon k^2}{\pi \varepsilon_0} \int_{k_\perp}^\infty \frac{dk}{\sqrt{k^2 - k_\perp^2}}(k + b_n).
\]

This integral can be calculated using 1.2.45.3 and 1.2.45.5 in Ref. 84 with the result

\[
Z_{TM}^{(n)}(i\xi_0, k_\perp) = \frac{2\varepsilon k^2}{\pi \varepsilon_0} \left\{ \begin{array}{ll}
\ln \frac{b_n + \sqrt{b_n^2 - k_\perp^2}}{k_\perp}, & k_\perp < b_n, \\
\arccos \frac{b_n}{k_\perp}, & k_\perp > b_n.
\end{array} \right.
\]

Substituting this result in the first expression of Eq. (5) and putting \( \xi_0 = 0 \), we finally obtain

\[
r_{TM}^{(n)}(0, k_\perp) = \frac{\pi}{\sqrt{b_n^2 - k_\perp^2}} \ln \frac{b_n + \sqrt{b_n^2 - k_\perp^2}}{k_\perp} \quad \text{for} \quad k_\perp < b_n
\]

and

\[
r_{TM}^{(n)}(0, k_\perp) = \frac{\pi}{\sqrt{b_n^2 - k_\perp^2}} \arccos \frac{b_n}{k_\perp} \quad \text{for} \quad k_\perp > b_n
\]

for \( k_\perp < b_n \) and \( k_\perp > b_n \). For \( k_\perp = b_n \), both Eqs. (14) and (15) lead to the result

\[
r_{TM}^{(n)}(0, k_\perp) = \frac{\pi - 2}{\pi + 2}.
\]

Now we consider the value of the TE reflection coefficient at zero Matsubara frequency. For this purpose, we substitute the first expression in Eq. (10) into the second formula in Eq. (7) and find the following asymptotic expression in the case of vanishing \( \xi_0 \):

\[
Z_{TE}^{(n)}(i\xi_0, k_\perp) = \frac{2\varepsilon_0 \mu_n(i\xi_0)}{\pi c} \int_0^\infty \frac{dk_x}{B_n \sqrt{k_x^2 + k_\perp^2 + k_\parallel^2 + k_{\parallel}^2}},
\]

where \( B_n = \mu_n(i\xi_0)\omega_n^2 v_n^L/\gamma_n c^2 \).

The integral in Eq. (17) has the same form as in Eq. (11). Calculating it in the same way as above, one obtains

\[
Z_{TE}^{(n)}(i\xi_0, k_\perp) = \frac{2\varepsilon_0 \mu_n(i\xi_0)}{\pi c} \left\{ \begin{array}{ll}
\ln \frac{B_n + \sqrt{B_n^2 - k_\perp^2}}{k_\perp}, & k_\perp < B_n, \\
\arccos \frac{B_n}{k_\perp}, & k_\perp > B_n.
\end{array} \right.
\]

Substituting this equation in the second expression of Eq. (9) and putting \( \xi_0 = 0 \), we find the following results:

\[
r_{TE}^{(n)}(0, k_\perp) = \frac{2\mu_n(0)k_\perp \ln \frac{B_n + \sqrt{B_n^2 - k_\perp^2}}{k_\perp} - \pi \sqrt{B_n^2 - k_\perp^2}}{2\mu_n(0)k_\perp + \pi \sqrt{B_n^2 - k_\perp^2}}
\]

for \( k_\perp < B_n \) and

\[
r_{TE}^{(n)}(0, k_\perp) = \frac{2\mu_n(0)k_\perp \arccos \frac{B_n}{k_\perp} - \pi \sqrt{B_n^2 - B_n^2}}{2\mu_n(0)k_\perp + \pi \sqrt{B_n^2 - B_n^2}}
\]

for \( k_\perp > B_n \). For \( k_\perp = B_n \), Eqs. (19) and (20) lead to

\[
r_{TE}^{(n)}(0, k_\perp) = \frac{2\mu_n(0) - \pi}{2\mu_n(0) + \pi}.
\]

From Eqs. (14), (15) and (19), (20) one can see that at zero Matsubara frequency the magnetic properties make an impact only on the TE reflection coefficient in the Lifshitz formula (2).

The values of impedances and reflection coefficients at all Matsubara frequencies with \( l \geq 1 \) are more complicated. We again substitute Eq. (10) in the first expression in Eq. (7) and introduce in the obtained integrals the following dimensionless integration variable and projection magnitude of the wave vector on the plane of plates:

\[
x = \frac{k_x c}{\xi_l}, \quad p_l = \frac{k_{\parallel} c}{\xi_l}.
\]

Then the impedance \( Z_{TM} \) takes the form

\[
Z_{TM}^{(n)}(i\xi_l, k_\perp) = \frac{2\varepsilon k^2}{\pi \xi_l^2} \int_0^\infty \frac{(c + v_n^L \sqrt{p_l^2 + x^2}) dx}{(p_l^2 + x^2)[\mu_n(i\xi_l)(A_l^{(n)} + D_l^{(n)} \sqrt{p_l^2 + x^2}) + p_l^2 + x^2]}.
\]

(23)
where
\[ A_l^{(n)} = \varepsilon_c^{(n)}(i\xi_l) + \frac{\omega_p^2}{\xi_l(\xi_l + \gamma_n)}, \]
\[ D_l^{(n)} = \frac{\omega_p^2 v_{n}^{Tr}}{\xi_l(\xi_l + \gamma_n)}. \]

Equations (24) and (25) are convenient for numerical computations performed in the next sections.

In the end of this section, we note that the spatially nonlocal dielectric permittivities (8) introduced above satisfy the Kramers-Kronig relations as it should be in accordance with the condition of causality. This can be proven similar to Ref. [73] if to take into account that the dielectric permittivity of core electrons in Eq. (8) takes the form (41, 83)

\[ \varepsilon^{(n)}_c(\omega) = 1 + \frac{1}{\pi} \sum_{j=1}^{K_n} \frac{g_{n,j}}{\omega^2 - \omega_n^2 - i\gamma_n\omega}, \]

where \(\omega_n, j \neq 0\) are the oscillator frequencies, \(g_{n,j}\) are the oscillator strengths, \(\gamma_n, j\) are the relaxation parameters and \(K_n\) are the numbers of oscillators for the first (\(n = 1\)) and second (\(n = 2\)) Casimir plates, respectively. The permittivity (24) satisfies the standard Kramers-Kronig relations (66, 77).

In this case the derivation presented in Ref. [73] is repeated with the only replacement of \(k_\perp\) with \(k = (k_\perp^2 + k_z^2)^{1/2}\). As a result, for \(\varepsilon^{(n)}_c\) one obtains the following Kramers-Kronig relations

\[
\begin{align*}
\text{Re} \varepsilon^{Tr}_n(\omega, k) &= 1 + \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{dx}{\omega - \omega} \left[ \text{Im} \varepsilon^{Tr}_n(x, k) + \frac{\omega_p^2 v_{n}^{Tr} k}{x^2 \gamma_n} \right], \\
\text{Im} \varepsilon^{Tr}_n(\omega, k) &= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{dx}{\omega - \omega} \left[ \text{Re} \varepsilon^{Tr}_n(x, k) + \frac{\omega_p^2 v_{n}^{Tr} k}{x^2 \gamma_n} \right] \\
&+ \frac{4\pi \text{Re} \sigma^{Tr}_{n,0}(0)}{\omega},
\end{align*}
\]  

where the integrals are understood as the principal values and the real part of the static transverse conductivity is given by (73)

\[ \text{Re} \sigma^{Tr}_{n,0}(0) = \frac{\omega_p^2 (\gamma_n - v_{n}^{Tr} k)}{4\pi \gamma_n^2}. \]

It should be mentioned that the last term on the right-hand side of the first equality in Eq. (27) originates from the second-order pole of the dielectric permittivity \(\varepsilon^{Tr}_n(\omega)\) at \(\omega = 0\). The last term on the right-hand side of the second equality in Eq. (27) is caused by the first-order pole so that in the local limit, \(k \to 0\), Eq. (28) represents the static conductivity of the standard Drude model (77).

The dielectric permittivity \(\varepsilon^{Tr}_n(\omega)\) defined in Eq. (8) has no poles at \(\omega = 0\). For this reason, the Kramers-Kronig relations for this permittivity take the same simplest form as for the permittivity of core electrons (66, 77)

\[ \begin{align*}
\text{Re} \varepsilon^{Tr}_n(\omega, k) &= 1 + \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{dx}{\omega - \omega} \text{Im} \varepsilon^{Tr}_n(x, k), \\
\text{Im} \varepsilon^{Tr}_n(\omega, k) &= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{dx}{\omega - \omega} \text{Re} \varepsilon^{Tr}_n(x, k).
\end{align*} \]  

From Eqs. (27) and (29), it is easy to find the expressions for nonlocal dielectric permittivities along the imaginary frequency axis (73, 77)

\[ \begin{align*}
\varepsilon^{Tr}_n(i\xi, k) &= 1 + \frac{2}{\pi} \int_{0}^{\infty} \frac{dx}{x^2 + \xi^2} \left[ \text{Re} \varepsilon^{Tr}_n(x, k) + \frac{\omega_p^2 v_{n}^{Tr} k}{x^2 \gamma_n} \right], \\
\varepsilon^{Tr}_n(i\xi, k) &= 1 + \frac{2}{\pi} \int_{0}^{\infty} \frac{dx}{x^2 + \xi^2},
\end{align*} \]  

which are useful for computations by means of the Lifshitz formula (2).

**COMPARISON WITH THEORY FOR MEASUREMENTS BETWEEN NONMAGNETIC TEST BODIES BY MEANS OF A MICROMECHANICAL TORSIONAL OSCILLATOR**

In the series of dynamic experiments performed in high vacuum in the configuration of an Au-coated sphere of radius \(R\) and an Au-coated plate by means of a micromechanical torsional oscillator at room temperature (23, 28), the measurement data for the gradient of the Casimir force \(F'_{sp}(a, T)\) was represented in terms of the effective Casimir pressure between two Au plates

\[ P^{expt}(a, T) = -\frac{1}{2\pi R} F'_{sp}(a, T). \]  

This was done by using the proximity force approximation (40, 41) which leads to a relative error of less than
It was found that the theoretical predictions of the Lifshitz theory computed by Eq. (12) with taken into account surface roughness are excluded by the measurement data if the dielectric response of Au is described by the Drude model supplemented by the permittivity of core electrons. This response function is described by the Drude model supplemented by the relaxation properties of conduction electrons, i.e., the plasma-like model which disregards the relaxation properties of free electrons. The respective theoretical bands cannot be distinguished from the upper (red) theoretical bands in Figs. 1(a), 1(d) over four separation regions from 162.03 to 745.98 nm. The experimental data are shown as crosses. The horizontal and vertical arms of these crosses indicate the total measurement errors found at the 95% confidence level as a combination of systematic and random errors. The lower (black) theoretical bands in Figs. 1 indicate the theoretical predictions of the Lifshitz theory obtained by Eqs. (2) and (3) using the standard local Drude response function. The width of each theoretical band is determined by the errors in all the above theoretical parameters used in computations.

As is seen in Fig. 1, the measurement data are in a very good agreement with theoretical predictions of the Lifshitz theory made using the spatially nonlocal response functions which take into account the dissipation properties of conduction electrons. Almost the same theoretical predictions in agreement with the measurement data were made by the Lifshitz theory using the plasma response function given by Eq. (32) with \( \gamma_n = 0 \), i.e., with the relaxation properties of free electrons disregarded. The respective theoretical bands cannot be distinguished from the upper (red) bands shown in Fig. 1. As to the lower (black) theoretical bands in Fig. 1, they are excluded by the measurement data over the entire range of separations.

We are coming now to the recently performed experiment on measuring the differential Casimir force between an Au-coated sphere of \( R = 149.7 \text{ \mu m} \) radius and top and bottom of the Au-coated rectangular trenches. As most of precise measurements of the Casimir interaction, this one was made at room temperature in high vacuum. Thanks to the differential character of this measurement, it has been made possible to obtain the meaningful data up to separation distances of a few micrometers using the same setup of a micromechanical torsional oscillator. Due to the sufficiently deep trenches used, the effectively measured Casimir force was that acting between a sphere and a plate which served as the trench top.

In the framework of the proximity force approximation, the Casimir force acting between a sphere and a plate is given by

\[
F_{sp}(a, T) = 2\pi R F (a, T),
\]

where \( F (a, T) \) is the single fitting parameter of our phenomenological model. The chosen value of this parameter leads to the best agreement between experiment and theory. We recall that for the nonlocal permittivities depending only on \( k_i \) this parameter is equal to 7 resulting in larger (but as yet sufficiently small) deviations from the standard Drude model in optical experiments.

The computational results for the magnitude of the effective Casimir pressure were shown by the upper (red) theoretical bands in Figs. 1(a), 1(d) over four separation regions from 162.03 to 745.98 nm. The experimental data are shown as crosses. The horizontal and vertical arms of these crosses indicate the total measurement errors found at the 95% confidence level as a combination of systematic and random errors. The lower (black) theoretical bands in Fig. 1 indicate the theoretical predictions of the Lifshitz theory obtained by Eqs. (2) and (3) using the standard local Drude response function. The width of each theoretical band is determined by the errors in all the above theoretical parameters used in computations.

As is seen in Fig. 1, the measurement data are in a very good agreement with theoretical predictions of the Lifshitz theory made using the spatially nonlocal response functions which take into account the dissipation properties of conduction electrons. Almost the same theoretical predictions in agreement with the measurement data were made by the Lifshitz theory using the plasma response function given by Eq. (32) with \( \gamma_n = 0 \), i.e., with the relaxation properties of free electrons disregarded. The respective theoretical bands cannot be distinguished from the upper (red) bands shown in Fig. 1. As to the lower (black) theoretical bands in Fig. 1, they are excluded by the measurement data over the entire range of separations.
where the Casimir free energy in the configuration of two parallel plates is presented in Eq. (2). In Ref. [37], the force $F_{sp}$ was computed both approximately using Eqs. (2), (3), and (34) and precisely on the basis of first principles of quantum electrodynamics at nonzero temperature using the scattering approach [89–92] and the gradient expansion [93–96]. It was shown [37] that all differences between the approximate and exact results are well below the measurement errors within the separation region from 0.2 to 8 $\mu$m, irrespective of whether the Drude or plasma response function is used in computations.

The obtained theoretical results employing the plasma model given by Eq. (32) with $\gamma_n = 0$ were found to be in a good agreement with the measurement data over the entire range of separations. The results computed similarly, but using the Drude model, were excluded by the data over the separation region from 0.2 to 4.8 $\mu$m. In so doing the background electric force due to patch potentials was investigated with the help of Kelvin probe microscopy [97] and included in the total experimental error of the Casimir force determined at the 95% confidence level.

Here, we compute the Casimir force in the configuration of the experiment [37] by the first equality in Eq. (2) and Eq. (32) using the proposed spatially nonlocal dielectric functions (10). For the reflection coefficients with $l = 0$, Eqs. (14), (15) and (20), (21) have been used, and for $l \geq 1$ numerical computations were performed by Eqs. (23) and (25) as described above with the following
values of all parameters of Au [37]: $\hbar \omega_{p, Au} = 9.0$ eV, $\hbar \gamma_{Au} = 35.0$ meV, $v_{F, Au} = 1.78 \times 10^6$ m/s. The obtained results were multiplied by a correction factor accounting for an inaccuracy of the proximity force approximation which was calculated in Ref. [37].

The obtained computational results in the range of separations from 1 to 4.9 $\mu$m are shown by the upper (red) band in Fig. 2. In the same figure, the lower (black) band is computed by Eqs. (2), (3) and (34) using the Drude response function (32). The measurement data are indicated as crosses. As is seen in Fig. 2, the upper and lower vertical arms of the crosses differ from one another. This is because the attractive electric force due to patch potentials is included as part of the error in measuring the Casimir force. According to Fig. 2, the theoretical predictions of the Lifshitz theory using the proposed nonlocal response functions (8), (10) are in agreement with the measurement data. Good agreement also holds over the ranges of separations from 0.2 to 1 $\mu$m and from 4.8 to 8 $\mu$m which are not shown in Fig. 2. The predictions of the same theory using the Drude model are excluded over the range of separations from 1 to 4.8 $\mu$m (in Ref. [37], it is shown that they are also excluded in the measurement range from 0.2 to 1 $\mu$m but here we are more interested in the region of large separations exceeding 1 $\mu$m).

The Casimir forces computed by Eqs. (2), (3) and (34) using the plasma model given by Eq. (42) with $\gamma_n = 0$ are almost the same as the ones computed above with the spatially nonlocal response functions. Thus, at separations of 1, 3, 5, and 7 $\mu$m the pairs of force magnitudes (in nN) computed using the nonlocal response functions and the plasma model are (372.98, 374.62), (21.44, 21.67), (7.24, 7.30), and (3.69, 3.71), i.e., only 0.44%, 1.06%, 0.8%, and 0.54% relative differences, well below the respective experimental errors. Although these results are not experimentally distinguishable, that ones obtained using the nonlocal response functions should be considered as preferable as they are obtained with taken into account relaxation properties of conduction electrons.

MEASUREMENTS BETWEEN NONMAGNETIC TEST BODIES BY MEANS OF AN ATOMIC FORCE MICROSCOPE

Another experimental setup for measuring the Casimir interaction is an atomic force microscope whose sharp tip is replaced with a sphere of sufficiently large radius [96]. Here, we compare the measurement data of three most precise experiments on measuring the gradient of the Casimir force between an Au-coated sphere and an Au-coated plate obtained by means of a dynamic atomic force microscope [29, 37] with theoretical predictions using the nonlocal dielectric functions (10). All measurements were performed in high vacuum at room temperature. In interpretation of all these experiments the same parameters of Au, i.e., the values of $\omega_{p, Au}$, $\gamma_{Au}$, and $\varepsilon_c(Au)$, have been used as already listed in Sec. IV when describing measurements of the differential Casimir force between a sphere and a plate with rectangular trenches. We start with the experiment of Ref. [29] which employed the sphere of $R = 41.3$ $\mu$m radius. The theoretical force gradients were computed using the Lifshitz theory and the proximity force approximation with taken into account correction for its inaccuracy [29]

$$F'_{sp}(a, T) = -2\pi RP(a, T) \left[1 + \theta(a, T) \frac{a}{R}\right],$$

where the Casimir pressure $P$ is given by the second expression in Eq. (2) with the Fresnel reflection coefficients and at separations below 1 $\mu$m the coefficient $\theta$ is negative and does not exceed unity (see Refs. [93, 95] and the more complete results for different dielectric functions in Refs. [92, 96]). The effect of surface roughness was taken into account perturbatively and shown to be negligibly small.

According to the results of Ref. [29], the theoretical predictions obtained using the Drude model [42] are excluded by the measurement data within the range of separations from 235 to 420 nm. The same data turned out to be in a good agreement with theoretical predictions found by using the plasma model which does not take into account the relaxation properties of free electrons, i.e., put $\gamma_{Au} = 0$. Thus, the results obtained earlier in Refs. [25, 28] by means a micromechanical torsional oscillator were confirmed independently by using quite different laboratory setup.

We have computed the gradients of the Casimir force [35] in the experimental configuration of Ref. [29] using
the spatially nonlocal response functions \((10)\) and reflection coefficients \((5)\) expressed via the surface impedances as described in Secs. III and IV. The same parameters of Au, as in Ref. \([29]\), have been used and \(v^\text{Tr,L}_{\text{Au}} = v^\text{Tr,L}_{\text{Au}} = 3v_{\text{F,Au}}/2\) as indicated above. The computational results are shown by the upper (red) bands in Fig. 3 where the experimental data are presented as crosses whose arms indicate the total measurement errors determined at the 67% confidence level. The lower (black) bands in Fig. 3 reproduce the computational results of Ref. \([29]\) obtained using the Drude model \((32)\). In fact, our computational results using the nonlocal response functions are almost coinciding with the results of Ref. \([29]\) using the dissipationless plasma model. In doing so, our results are in a good agreement with the measurement data over the entire range of separations which exclude the theoretical predictions using the Drude model over the separation range from 235 to 420 nm.

An upgraded setup employing the atomic force microscope with increased sensitivity of the cantilever through a decrease of its spring constant was used in Refs. \([35, 36]\) in more precise measurements of the Casimir force gradients up to larger separation distances. The important property of an upgraded setup was an employment of the two-step cleaning procedure of the vacuum chamber and test body surfaces by means of UV light followed by Argon bombardment. The radius of the sphere used was \(R = 43.47\ \mu\text{m}\).

The comparison between experiment and theory in Ref. \([36]\) was made as described above in this section using Eq. \((33)\) and the dielectric response \((32)\) with \(\gamma_{\text{Au}} \neq 0\) (the Drude model) and \(\gamma_{\text{Au}} = 0\) (the plasma model). In the measurements with smaller oscillation amplitude of the cantilever equal to 10 nm, the theoretical predictions using the Drude model were excluded over the range of separations from 250 to 820 nm whereas those using the plasma model were found to be in agreement with the measurement data over the entire measurement range.

We have computed the gradient of the Casimir force in the experimental configuration of Refs. \([32, 36]\) using the proposed dielectric functions \((10)\) with the same parameters of Au, as in these references, and \(v^\text{Tr,L}_{\text{Au}} = 3v_{\text{F,Au}}/2\) as above. The computational results are shown by the upper (red) bands in Fig. 4 as a function of separation. The measurement data with their errors determined at the 67% confidence level are shown as crosses. The theoretical predictions obtained using the Drude model \([36]\) are presented by the lower (black) bands. As is seen in Fig. 4 our results, which take into account the dissipation of free electrons, are in agreement with the data over the entire separation region from 250 to 950 nm.

Another set of measurements was performed in

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**FIG. 3:** The gradients of the Casimir force between an Au-coated sphere and an Au-coated plate computed using the spatially nonlocal response functions and the Drude model are shown as functions of separation by the upper (red) and lower (black) bands, respectively. The experimental data \([29]\) are shown as crosses whose arms indicate the total errors determined at the 67% confidence level.

**FIG. 4:** The gradients of the Casimir force between an Au-coated sphere and an Au-coated plate computed using the spatially nonlocal response functions and the Drude model are shown as functions of separation by the upper (red) and lower (black) bands, respectively. The experimental data \([36]\) obtained with smaller oscillation amplitude of the cantilever are shown as crosses whose arms indicate the total errors determined at the 67% confidence level.
Ref. [36] with a larger oscillation amplitude of cantilever equal to 20 nm. This made it possible to get the meaningful measurement data at larger separation distances and exclude theoretical predictions using the Drude model up to 1.1 μm.

Our computational results for the gradient of the Casimir force obtained with the nonlinear dielectric functions [10] are presented by the upper (red) bands in Fig. 5 over the range of separations from 0.6 to 1.3 μm of the data set in Ref. [36], measured with a larger oscillation amplitude. They are in a good agreement with the measurement data indicated as crosses. Our results accounting for the dissipation properties of free electrons are almost coinciding with those obtained in Ref. [36] using the dissipationless plasma model, but deviate significantly from those obtained by means of the Drude model. The latter are shown by the lower (black) bands.

One can conclude that the Lifshitz theory employing the proposed nonlinear dielectric permittivity is in equally good agreement with the measurement data of precise experiments performed by two different experimental groups by means of a micromechanical torsional oscillator and an atomic force microscope using test bodies made of a nonmagnetic metal.

![Figure 5](image-url)

**FIG. 5:** The gradients of the Casimir force between an Au-coated sphere and an Au-coated plate computed using the spatially nonlinear response functions and the Drude model are shown as functions of separation by the upper (red) and lower (black) bands, respectively. The experimental data [36] obtained with larger oscillation amplitude of the cantilever are shown as crosses whose arms indicate the total errors determined at the 67% confidence level.

**THEORY-EXPERIMENT COMPARISON WITH MAGNETIC TEST BODIES**

The action of magnetic properties of the plate materials on the Casimir force has attracted considerable attention in the literature. Thus, although in Refs. [2] and [3] the Lifshitz formulas were derived for nonmagnetic test bodies, they were rewritten with account of magnetic properties in Ref. [99]. The Casimir force acting between an ideal metal plate and an infinitely permeable sphere and an Au-coated plate computed using the Drude and plasma models are almost coinciding and are in a good agreement with the measurement data of Au and Ni, respectively, as described in Refs. [29, 30, 32, 40, 41]. The predictions obtained by means of the plasma model with magnetic properties of boundary plates make an impact on the Casimir free energy and pressure entirely through the zero-frequency terms of the Lifshitz formulas (2). This is caused by the fact that the frequency-dependent magnetic permeability becomes equal to unity at frequencies which are much smaller than the first Matsubara frequency at not too low temperature.

As was emphasized in Ref. [104], the magnetic properties of boundary plates make an impact on the Casimir force between one magnetic and one nonmagnetic plates, as well as between two magnetic plates, was considered by many authors (see, e.g., Refs. [12, 78, 101–103]). This problem attracted special attention in connection with the possibility of repulsive Casimir forces [42].

We begin with an experiment of Ref. [30] where the gradient of the Casimir force was measured between an Au-coated sphere of R = 64.1 μm radius and a plate coated with a magnetic metal Ni by means of an atomic force microscope. Note that in measuring of the Casimir force using magnetic metals they are not magnetized and do not give rise to a gradient of any additional force of magnetic origin [32]. In Ref. [30] the theoretical force gradients were computed by Eq. (35) with the Fresnel reflection coefficients [3] and spatially local dielectric permittivities [82] with γ_{Au} ≠ 0 (the Drude model) and γ_{Ni} = 0 (the plasma model) at room temperature. The following values of parameters for Au (n = 1) and Ni (n = 2) have been used: \( \hbar \omega_{Au,1} = \hbar \omega_{p,Au} = 9.0 \) eV, \( h \gamma_{1} = h \gamma_{Au} = 35.0 \) meV [40, 11, 88] and \( \mu_{1}(i \xi) = \mu_{Au}(i \xi) = 1 \); \( h \omega_{p,Ni} = 4.89 \) eV, \( h \gamma_{2} = h \gamma_{Ni} = 43.6 \) meV [88, 107] and \( \mu_{2}(0) = \mu_{Ni}(0) = 110, \mu_{2}(i \xi) = \mu_{Ni}(i \xi) = 1 \) for \( l \geq 1 \). The values of \( \varepsilon_{c}^{(1)}(i \xi) = \varepsilon_{c}^{(Au)}(i \xi) \) and \( \varepsilon_{c}^{(2)}(i \xi) = \varepsilon_{c}^{(Ni)}(i \xi) \) were obtained from the optical data of Au and Ni, respectively, as described in Refs. [29, 30, 32, 40, 41].

According to the results of Ref. [30], within the measurement range from 220 to 500 nm the theoretical predictions of the Lifshitz theory using the Drude and the plasma models are almost the same and are in a good agreement with the measurement data. (Note that at larger separations of a few micrometers the gradients of the Casimir force between a nonmagnetic-metal sphere and a magnetic-metal plate computed using the Drude and plasma models are different [104].)
properties of the Ni plate disregarded \([\mu_{Ni}(i\xi_l) = 1\) for all \(l \geq 0\)\] are excluded by the measurement data over the range of separations from 220 to 420 nm. If the Drude model is used in computations, the obtained results do not depend on whether the magnetic properties of Ni plate are included or omitted.

Here, we compute the gradient of the Casimir force in the experimental configuration of Ref. \([31, 32]\) using Eq. \((35)\) and the spatially nonlocal dielectric permittivities \([10]\). When computing the Casimir pressure, we have used the impedance reflection coefficients \([9]\) for Au and Ni leading to Eqs. \((14), (15)\) and \((19), (20)\) for \(l = 0\) and Eqs. \((22), (25)\) for \(l \geq 1\). The same parameters for Au and Ni, as indicated above have been used and the Fermi velocity for Ni was found from Eq. \((33)\) under an assumption of the spherical Fermi surface, \(v_{F, Ni} = 1.31 \times 10^6 \text{ m/s}\) (for Au we used \(v_{F, Au} = 1.78 \times 10^6 \text{ m/s}\) employed in Sec. IV and V). As in all previously considered experiments, the best agreement between the measurement data and theoretical predictions is reached for \(v_{Tr}^{Au} = v_{F, Au}^L = 3v_{F, Au}/2\) and \(v_{Tr}^{Ni} = v_{F, Ni}^L = 3v_{F, Ni}/2\).

Computational results obtained using the spatially nonlocal dielectric permittivities \([10]\) are shown by the solid (red) bands in Fig. 6 where the experimental data with their total errors determined at the 67\% confidence level are presented as crosses. As is seen in Fig. 6 the theoretical predictions of the Lifshitz theory employing the proposed nonlocal dielectric functions are in a very good agreement with the measurement data over the entire range of experimental separations. For the configuration of Au-Ni test bodies in the range from 220 to 500 nm almost the same theoretical predictions in equally good agreement with the measurement data are obtained when the conduction electrons are described by the spatially local Drude or plasma model.

Next, we consider the experiment of Refs. \([31, 32]\) on measuring the gradient of the Casimir force between a sphere of \(R = 61.71 \mu\text{m}\) radius and a plate both coated with a magnetic metal Ni performed by means of an atomic force microscope. In these references computations of the force gradient using the standard Lifshitz theory were performed as in Sec. V but with the parameters \(\omega_{p, 1} = \omega_{p, Ni}, \gamma_1 = \gamma_2 = \gamma_{Ni}\), and \(\varepsilon_c^{(1)}(i\xi_l) = \varepsilon_c^{(2)}(i\xi_l) = \varepsilon_c^{(Ni)}(i\xi_l)\) presented above. An important result found in Ref. \([31]\) is that for two magnetic metals the gradients of the Casimir force calculated using the Drude model are larger than those found by means of the plasma model. This is different from the case of two Au plates (see Figs. 3–5). It was shown \([31, 32]\) that the theoretical predictions obtained with the Drude model are excluded by the data over the separation region from 223 to 420 nm, whereas similar predictions made with the help of a plasma model are in a very good agreement with the measurement results. Similar to the case of test bodies coated with Au films, this result is puzzling because at low frequencies the relaxation properties of conduction electrons are well studied in many physical phenomena other than the Casimir effect.

Here, we computed the gradient of the Casimir force between the Ni-coated surfaces of a sphere and a plate by Eq. \((35)\). The Casimir pressure in this equation was found from the second line in Eq. \((2)\), reflection coefficients \([5]\) and impedance functions \([7]\), as described above, using the spatially nonlocal dielectric permittivities \([10]\) with the same parameters of Ni as above. The computational results are shown by the lower (red) bands in Fig. 7 as a function of separation. They are in a very good agreement with the measurement data indicated as crosses. As in all other experiments using an atomic force microscope, the total experimental errors are determined at the 67\% confidence level. The upper (black) bands in Fig. 7 reproduce the results of Refs. \([31, 32]\) computed by the Lifshitz formula and the spatially local Drude model \([42]\). It is seen that these results are excluded by the measurement data over the separation region from 223 to 420 nm in accordance with the conclusion made in Refs. \([31, 32]\). The point is that at short separations up to a few hundred nanometers the force gradients computed using the local plasma model, which disregards the relaxation of free electrons, and the spatially nonlocal dielectric functions \([10]\), which take relaxation into account, are almost coinciding. In this situation, a failure of the Drude model may be explained by an inadequate
description of the dielectric response to electromagnetic waves off the mass shell.

CONCLUSIONS AND DISCUSSION

In this paper, we have proposed the phenomenological spatially nonlocal dielectric functions which provide nearly the same response to electromagnetic waves on the mass shell, as does the standard Drude model, but respond differently to the off-the-mass-shell fields. Unlike the previously suggested response functions of this kind [73, 74, 76], the permittivities presented here depend on all the three components of the wave vector which is a more general case in the approximation of specular reflection used.

As discussed in Sec. I, the problem of disagreement between theoretical predictions of the fundamental Lifshitz theory and the experimental data for metallic test bodies remains unresolved for almost 20 years. Many attempts of its resolution have been undertaken (including a consideration of the frequency-dependent relaxation parameter [108], i.e., the so-called Gurzhi model), but the problem is as yet unresolved. Similar problem arises for dielectric materials [43, 109]. All this makes it warranted to consider some phenomenological approaches suggested by analogy with graphene which, due to its simplicity in comparison with metallic materials, allows the fundamental calculation of its spatially nonlocal dielectric response based on the first principles of quantum electrodynamics at nonzero temperature.

Using this line of reasoning, the surface impedances and reflection coefficients determined by the proposed nonlocal dielectric functions have been found in the approximation of specular reflection. This made it possible to calculate the effective Casimir pressure between two parallel metallic plates, the Casimir force between a sphere and a plate, and its gradient predicted by the suggested approach in configurations of several precise experiments performed during the last 15 years by two different experimental groups by means of micromechanical torsional oscillator and atomic force microscope. In doing so the experiments with both nonmagnetic and magnetic test bodies were considered.

It was shown that the suggested spatially nonlocal dielectric functions taking into account the dissipation properties of conduction electrons bring the Lifshitz theory to equally good agreement with the measurement data of all the performed experiments as does the plasma model which disregards the dissipation of conduction electrons. Good agreement with seven considered experiments, which were performed between both nonmagnetic and magnetic metallic surfaces (Au-Au, Au-Ni, and Ni-Ni), was reached with the velocity parameters common to the transverse and longitudinal permittivities $v_n^T = v_n^L = v_n = 3v_F/n / 2$, where the Fermi velocities $v_F, n$ are determined in the approximation of a spherical Fermi surface. In doing so, the theoretical predictions are rather sensitive to the value of $v_n^T$ but are almost independent on $v_n^L$ in the region from 0 to $10v_F/n$. In the regions of experimental separations these predictions differ from the previously made experimentally consistent predictions using the dissipationless plasma models only in the limits of measurement errors.

The single precise experiment which was not compared with theoretical predictions of the suggested nonlocal approach to calculation of the Casimir force is the measurement of differential Casimir force between the Ni-Ni surfaces of a sphere and a plate performed by means of a micromechanical torsional oscillator [33]. Taking into account, however, that experiments by means of both an atomic force microscope and micromechanical torsional oscillator are in a good agreement with this calculation approach for Au surfaces, the results of Sec. VI for two magnetic surfaces can be safely extended to the experiment of Ref. [33].

To conclude, the suggested spatially nonlocal dielectric functions include the dissipation of conduction electrons leading to only negligibly small deviations from the standard Drude dielectric response in the area of propagating waves on the mass shell, satisfy the Kramers-Kronig relations, and bring the theoretical predictions of the Lif-
shitz theory in agreement with the measurement data of all precise experiments on measuring the Casimir force. In the future it is desirable to provide some grounding in theory to these permittivities based, e.g., on the polarization tensor in (3+1)-dimensions, or quantum field theoretical approach to correlation functions and Boltzmann kinetic theory.

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