De-phasing of a flux-qubit coupled to a harmonic oscillator

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Abstract

Decoherence in superconducting qubits is known to arise because of a variety of environmental degrees of freedom. In this article, we focus on the influence of thermal fluctuations in a weakly damped circuit resonance coupled to the qubit. Because of the coupling, the qubit frequency is shifted by an amount $n\delta\nu_0$ if the resonator contains $n$ energy quanta. Thermal fluctuations induce temporal variations $n(t)$ and thus dephasing. We give an approximate formula for the qubit dephasing time as a function of $\delta\nu_0$. We discuss the specific case of a flux-qubit coupled to the plasma mode of its DC-SQUID detector. We first derive a plasma mode-qubit interaction hamiltonian which, in addition to the usual Jaynes-Cummings term, has a coupling term quadratic in the oscillator variables coming from the flux-dependence of the SQUID Josephson inductance. Our model predicts that $\delta\nu_0$ cancels in certain non-trivial bias conditions for which dephasing due to thermal fluctuations should be suppressed.

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I. INTRODUCTION

A series of recent experiments have made it clear that it is possible to manipulate the quantum state of macroscopic electrical circuits based on Josephson junctions [1, 2, 3]. This breakthrough opens the way to the realization of fundamental tests of quantum mechanics, up to now confined to atomic physics and quantum optics, in a solid-state physics context. An additional interest comes from the eventual possibility of using these circuits as building blocks for a quantum computer [4]. In view of this latest application, it is highly desirable to understand better how they become dephased by environmental noise.

To estimate the dephasing rates, the Bloch-Redfield theory assumes that the qubit is weakly coupled to a bath at temperature $T$ with a memory short compared to all relevant timescales (white noise). In that limit, it is well known that the dephasing rate is proportional only to the low-frequency part of the environment spectral density. It becomes increasingly clear however that this description is inadequate in a number of cases. For example, the Bloch-Redfield assumptions are obviously unjustified when dephasing is due to the fluctuations of slow environmental degrees of freedom which typically have a $1/f$ spectrum [5]. This is also the case when a resonance of large quality factor ($Q \gg 1$) occurs in the environment at a frequency comparable to the qubit, since the memory of the environment can not be neglected then [6]. Both processes are relevant in our experiments [3, 7]. We study the quantum coherence of a circuit called the flux-qubit, measured by a DC-SQUID. The flux-qubit is sensitive to a number of microscopic degrees of freedom: motion of nearby vortices trapped in superconducting thin-films, fluctuations of the junctions critical current, and charge noise. In addition, it is strongly coupled to the harmonic oscillator (called HO in the remaining of this work) constituted by the underdamped DC-SQUID and a shunt capacitor to which it is connected to improve its resolution as a detector. In recent experiments we observed clear signatures of the strong coupling between the two systems, manifested by the appearance of sideband resonances in the spectrum [7]. In the present article we investigate theoretically the effects of this coupling on the qubit decoherence. Thermal fluctuations of the photon number $n$ stored in the oscillator shift the qubit frequency by an amount $n\delta \nu_0$ and lead to dephasing. We note that a similar effect has been recently observed in the case of a Cooper-pair box coupled to a waveguide resonator [8, 9]. When the resonator was driven to perform the measurement, the qubit line was shifted and broadened due to ac-Stark shift.
and photon shot noise.

In the first part of this article, we propose a simple analytical formula giving the pure dephasing time as a function of the system parameters. In the remaining we apply this model to the specific case of our circuit. We start by deriving the coupling hamiltonian between a superconducting flux-qubit and the HO. In addition to the linear coupling term \[6, 10\], we find it necessary to consider the next order term which is quadratic in the oscillator variables. We finally investigate the dependence of the shift per photon \(\delta \nu_0\) on the system bias parameters, namely the magnetic flux enclosed by the qubit loop \(\Phi_x\) and the SQUID bias current \(I_b\). In particular, we find that it is possible to cancel the dephasing per photon \(\delta \nu_0\) for specific bias conditions, so that the influence of thermal fluctuations on the qubit should be suppressed.

II. DERIVATION OF AN APPROXIMATE FORMULA FOR THE DEPHASING TIME

Let us consider the situation where a qubit of frequency \(\nu_q\) is linearly coupled to an underdamped HO of frequency \(\nu_p\) with a coupling strength \(g\). The qubit is supposed to be an ideal undamped two-level system, whereas the HO is coupled to a bath at temperature \(T\) which damps its dynamics with a rate \(\kappa\). We are interested in the limit \(\kappa << \nu_p\) where the oscillator is underdamped. We can write the total hamiltonian as \(H = h[-(1/2)\nu_q \sigma_z + \nu_p a^\dagger a + g\sigma_x(a + a^\dagger)]\), where we introduced the Pauli matrices \(\sigma_x, \sigma_y, \sigma_z\) in the qubit Hilbert space and the usual annihilation (creation) operator \(a (a^\dagger)\) for the HO. This is the well-studied Jaynes-Cummings hamiltonian \[11, 12\]. Let us recall a few results useful in the following. In the limit where \(|\delta| \equiv |\nu_q - \nu_p| >> g\) (called the dispersive regime), the energy eigenstates of the coupled system can be written as a function of the uncoupled energy states \(|i, n\rangle\), where \(i = 0, 1\) refers to the qubit state and \(n\) to the photon state of the HO, as \[14\]

\[
|+n\rangle \approx |1, n\rangle + \frac{g\sqrt{n+1}}{\delta}|0, n+1\rangle \\
|-n\rangle \approx -\frac{g\sqrt{n+1}}{\delta}|1, n\rangle + |0, n+1\rangle
\]

(1)
their energies being

\[ E_{\pm n} = (n + 1)\hbar \nu_p \pm \frac{\hbar}{2} \delta \pm \frac{\hbar g^2(n + 1)}{\delta} \]  

One sees that because of the coupling, the energy eigenstates are shifted by a quantity \( \pm \hbar g^2(n + 1) / \delta \). In the presence of \( n \) photons, the dressed qubit excited state is \( |+, n \rangle \) and the ground state \(|-, n - 1 \rangle \) so the qubit transition is \( E_{+, n} - E_{-, n-1} = \hbar (\nu_q + 2g^2(n + 1) / \delta) \). This means that the qubit frequency is shifted by an amount \( \nu_{q,n} - \nu_{q,0} = n(2g^2 / \delta) = n\delta\nu_0 \). Thus, any temporal fluctuation of the photon number will lead to dephasing. Let us introduce the mean photon number in the HO assumed to be at thermal equilibrium \( \bar{n} = 1/(\exp(\hbar \nu_p / kT) - 1) \). The stationary photon number distribution is given by a Boltzmann law

\[ p(n) = \frac{1}{(\bar{n} + 1)(\bar{n}/(\bar{n} + 1))^n} \]

The temporal fluctuations are characterized by the two-time correlation function \( C(\tau) = \langle n(0) n(\tau) \rangle \). It is possible to estimate \( C(\tau) \) using a quantum Langevin equation approach as in [13]. We show in this way in the annex A that

\[ C(\tau) = \bar{n}(\bar{n} + 1) \exp(-\kappa |\tau|) + \bar{n}^2 \]  

In order to quantify the effect of these fluctuations on the qubit coherence, we follow the analysis of Blais et al. [14]. The total phase accumulated by the qubit during a free evolution is \( \phi(t) = 2\pi \int_0^t \nu_q(t') dt' = \phi(t) + \delta\phi(t) \) where we isolated the deterministic quantity \( \phi(t) = 2\pi(\nu_{q,0} + \bar{n}\delta\nu_0)t \) from the fluctuating \( \delta\phi(t) = 2\pi\delta\nu_0 \int_0^t (n(t') - \bar{n}) dt' \). Dephasing is described by the quantity \( f_\phi(t) = \langle \exp(i\delta\phi(t)) \rangle \) called the dephasing factor. In the limit where \( t >> \kappa^{-1} \), the variable \( \delta\phi(t) \) should have gaussian statistics so that

\[ f_\phi(t) = \exp(-\langle \delta\phi(t)^2 \rangle / 2) = \exp[-(2\pi\delta\nu_0)^2/2 \int_0^t \int_0^t < (n(t') - \bar{n})(n(t'') - \bar{n}) > dt'dt''] \]

Combining equations 3 and 4 we obtain that

\[ f_\phi(t) = \exp[-((2\pi\delta\nu_0)^2\bar{n}(\bar{n} + 1)/2) \int_0^t \int_0^t \exp(-\kappa|t' - t''|) dt'dt''] \]

Since \( \int_0^t \int_0^t \exp(-\kappa|t' - t''|) = (2/\kappa)t + (2/\kappa^2)[\exp(-\kappa t) - 1] \), we find that the long-time decay of the dephasing factor is given by

\[ < \exp(i\delta\phi(t)) > = \exp[-\frac{1}{2}(2\pi\delta\nu_0)^2\bar{n}(\bar{n} + 1) \frac{2}{\kappa} t] \]
This describes an exponential decay of time constant

\[ \tau_\phi = \frac{\kappa}{(2\pi \delta \nu_0)^2 \bar{n}(\bar{n} + 1)} \]  

(7)

It is interesting to compare this formula with the one derived in [14] for the case when the HO is driven by a coherent field of mean photon \(\bar{n}\). The photon-photon correlator is then \(C(\tau) = \bar{n} \exp (-\kappa |t|/2)\). Compared to equation (3) we notice a factor \(1/2\) in the exponent which is due to the presence of the external drive, and a replacement of the \(\bar{n}(\bar{n} + 1)\) by \(\bar{n}\).

This reflects the fact that a coherent field has a poissonian distribution of photon numbers of variance \(\bar{n}\) whereas a thermal field has a superpoissonian distribution of variance \(\bar{n}(\bar{n} + 1)\).

We will now show that in the limit where formula (7) applies (namely \(\tau_\phi >> \kappa^{-1}\)) the same formula also gives the decay of the spin-echo time \(\tau_E\). The reason is that the fluctuations of the photon number occur on a much shorter timescale than dephasing so that they can not be compensated by a refocusing pulse. More quantitatively, the phase accumulated during the echo sequence is

\[ \delta \phi_E(t) = 2\pi \delta \nu_0 \left[ \int_0^{t/2} (n(t') - \bar{n})dt' - \int_{t/2}^{t} (n(t') - \bar{n})dt' \right] \]  

(8)

so that the fluctuations are

\[ <\delta \phi_E(t)^2> = (2\pi \delta \nu_0)^2 \left[ \int_0^{t/2} \int_0^{t/2} (n(t') - \bar{n})(n(t'') - \bar{n})dt'dt'' + \int_{t/2}^{t} \int_{t/2}^{t} (n(t') - \bar{n})(n(t'') - \bar{n})dt'dt'' - 2 \int_0^{t/2} \int_{t/2}^{t} (n(t') - \bar{n})(n(t'') - \bar{n})dt'dt'' \right] \]  

(9)

Obviously we need to calculate only the last term. Using the expression for the correlation function given earlier we find that

\[ \int_0^{t/2} \int_{t/2}^{t} (n(t') - \bar{n})(n(t'') - \bar{n})dt'dt'' = \frac{1}{\kappa^2} \bar{n}(\bar{n} + 1)(1 - \exp(-\kappa t/2))^2 \]

Combining with previous results we obtain

\[ <\delta \phi_E(t)^2> = (2\pi \delta \nu_0)^2 \bar{n}(\bar{n} + 1) \left[ \frac{2}{\kappa} t + \frac{4}{\kappa^2} (\exp(-\kappa t/2) - 1) - \frac{2}{\kappa^2} (1 - \exp(-\kappa t/2))^2 \right] \]  

(10)
In the limit where \( t >> \kappa^{-1} \), the long-time decay is still dominated by the term linear in \( t \) and we obtain \( \tau_E = \tau_\phi \). If on the other hand we were in the opposite limit in which the dephasing time is shorter than the photon correlation time, the decay of the Ramsey signal would be gaussian and the echo would decay with a \( \exp(-\kappa t^3/12) \) law much slower than the Ramsey decay. We note that this crossover between a lorentzian and a gaussian lineshape when the dephasing time becomes shorter than \( \kappa^{-1} \) has been observed in \cite{9}.

Whereas in this reasoning we only considered the case where the qubit-HO coupling is linear, it can also be applied evidently for a more complex interaction hamiltonian whenever the qubit frequency shift is proportional to the photon number. As we will see in the next paragraph, this is the case for our circuit in which a flux-qubit is coupled to the plasma mode of its measuring DC-SQUID.

### III. QUBIT-PLASMA MODE COUPLING HAMILTONIAN

#### A. Description of the system

The flux-qubit is a superconducting loop containing three Josephson junctions threaded by an external flux \( \Phi_x \equiv f(\Phi_0/2\pi) \). It is coupled to a DC-SQUID detector shunted by an external capacitor \( C_{sh} \) whose role is to limit phase fluctuations across the SQUID as well as to filter high-frequency noise from the dissipative impedance. The SQUID is threaded by a flux \( \Phi_{Sq} \equiv f'(\Phi_0/2\pi) \). The circuit diagram is shown in figure 1a. There, the flux-qubit is the loop in red containing the three junctions of phases \( \phi_i \) and capacitances \( C_i \) \( (i = 1, 2, 3) \). It also includes an inductance \( L_1 \) which models the branch inductance and eventually the inductance of a fourth larger junction \cite{17}. The two inductances \( K_1 \) and \( K_2 \) model the kinetic inductance of the line shared by the SQUID and the qubit. The SQUID is the larger loop in blue. The junction phases are called \( \phi_4 \) and \( \phi_5 \) and their capacitances \( C_4 \) and \( C_5 \). The critical current of the circuit junctions is written \( I_{C_i} \) \( (i = 1 \text{ to } 5) \). The SQUID loop also contains two inductances \( K_3 \) and \( L_2 \) which model its self-inductance. The SQUID is connected to the capacitor \( C_{sh} \) through superconducting lines of parasitic inductance \( L_s \). The phase across the stray inductance and the SQUID is denoted \( \phi_A \). The whole circuit is biased by a current source \( I_b \) in parallel with a dissipative admittance \( Y(\omega) \). Since our goal is primarily to determine the qubit-plasma mode coupling hamiltonian, we will neglect the
FIG. 1: (a) qubit biased by $\Phi_x$ and SQUID biased by current $I_b$. (b) Simplified electrical scheme: the SQUID-qubit system is seen as an inductance $L_J$ connected to the shunt capacitor $C_{sh}$ through inductance $L_{sh}$. $\Phi_a$ is the flux across the two inductances $L_J$ and $L_{sh}$ in series.

We start writing the total Hamiltonian of the circuit shown in figure using the circuit theory presented in [18]. We first choose a spanning tree containing all the capacitors as
shown in figure 1a. We then write the loop submatrices

$$\mathbf{F}_{CL} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & -1 \end{pmatrix}, \quad \mathbf{F}_{CB} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$$

(11)

and

$$\mathbf{F}_{KL} = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}, \quad \mathbf{F}_{KB} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

(12)

We note $M$ the mutual inductance between the qubit and SQUID loops. In the notations of [18] the inductance matrices are

$$\mathbf{L} = \begin{pmatrix} L_1 & M & 0 \\ M & L_2 & 0 \\ 0 & 0 & L_3 \end{pmatrix}, \quad \mathbf{L}_K = \begin{pmatrix} K_1 & 0 & 0 \\ 0 & K_2 & 0 \\ 0 & 0 & K_3 \end{pmatrix}, \quad \mathbf{L}_{LL} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

(13)

and

$$\mathbf{L}_{LL} = \begin{pmatrix} L_1 + K_1 + K_2 & M - K_1 - K_2 & K_1 \\ M - K_1 - K_2 & L_2 + K_1 + K_2 & -K_1 - K_3 \\ K_1 & -K_1 - K_2 & K_1 + K_3 + L_3 \end{pmatrix}$$

(14)

We note $l_{ij}^{-1}$ the matrix elements of $L_{LL}^{-1}$ whose expressions can be easily computed. We finally obtain the total hamiltonian as

$$H_S = H_{kin} + (\Phi_0/2\pi)^2 U(\phi)$$

(15)

where

$$H_{kin} = \left(\frac{\Phi_0}{2\pi}\right)^2 \left(\frac{1}{2} \sum_{i=1}^{5} Q_i^2/C_i + Q_A^2/C_{sh}\right)$$

$$U(\phi) = -\sum_{i=1}^{5} \frac{1}{L_{ij}} \cos \phi_i + \frac{1}{2l_{21}} \left(\sum_{i=1}^{3} \phi_i - f\right)^2$$

$$+ \left(\sum_{i=1}^{3} \phi_i - f\right) \left[l_{21}^{-1}(\phi_4 + \phi_5 - f') - l_{31}^{-1}(\phi_5 + \phi_A)\right] + u(\phi_4, \phi_5, \phi_A)$$

(16)
The \( Q_i \) (\( Q_A \)) are the charges stored on the capacitors \( C_i \) (\( C_{sh} \)) and \( u(\phi_4, \phi_5, \phi_A) \) is defined by

\[
u = +\frac{1}{2l_{12}}(\phi_4 + \phi_5 - f')^2 - \frac{1}{2l_{12}}[(\phi_5 + \phi_A - f')^2] + (\phi_4 + \phi_5)^2 - (\phi_4 - \phi_A)^2 \]

\[
+ \frac{1}{2l_{12}}(\phi_5 + \phi_A)^2 + \frac{2\pi}{\Phi_0} I_b \phi_A
\]

(17)

Our first goal will be here to simplify this hamiltonian so that the coupling of the relevant degrees of freedom is made clear. We will consider here that they are only two: the qubit, in the two-level approximation, and the plasma mode considered, if uncoupled to the qubit, as a harmonic oscillator. In particular, we will neglect the SQUID junctions capacitance which bring additional resonances at higher frequencies, and only consider the shunt capacitance \( C_{sh} \). Our approach is justified by the fact that only the plasma mode and the qubit have comparable energy scales, that the plasma mode is strongly coupled to the environment and therefore relevant for studying dephasing and relaxation, and that it undergoes thermal fluctuations because of its relatively low frequency. We also observed experimental evidence for the qubit-plasma mode strong coupling [7]. These results are a clear indication that a quantum-mechanical description of the coupled “qubit-plasma mode” is indeed needed. We will start by doing the two-level approximation on the qubit variables.

**B. Qubit hamiltonian and two-level approximation**

The hamiltonian for the qubit alone is

\[
H_q(f, I_b) = H_{kin} - (\Phi_0/2\pi)^2 \left[ \sum_i \frac{1}{L_{J,i}} \cos \phi_i + \frac{1}{2l_{11}} (\sum \phi_i - f)^2 \right]
\]

(18)

We first write this hamiltonian in a two-level approximation in terms of the Pauli matrices, which is valid here around \( \Phi_0/2 \) because of the specific properties of the circuit eigenstates. We define the states \( |0\rangle \) and \( |1\rangle \) as the eigenstates of \( H_q(\pi, 0) \equiv H_q^{(0)} \). Then when restraining ourselves to the 0, 1 states we have by definition \( H_q^{(0)} = -(h\Delta/2)\sigma_z \). We define \( H_q^{(1)}(f, I_b) = H_q(f, I_b) - H_q^{(0)} \). We have

\[
H_q^{(1)}(f, I_b) = (\Phi_0/2\pi)^2[(1/2l_{11})(\sum \phi_i - f)^2 - (\sum \phi_i - \pi)^2] \]

\[
= (\Phi_0/2\pi)^2[-(1/2l_{11})(2\sum \phi_i - 2\pi - (f - \pi))(f - \pi)]
\]

\[
= -(\Phi_0/2\pi)^2(\sum \phi_i - \pi) (f - \pi)/l_{11}
\]

(19)
forgetting constant terms in the last equation. We now want to decompose $H_q^{(1)}(f, I_b)$ on the $|0\rangle, |1\rangle$ subspace. We start by writing that $H_q^{(1)}(f, I_b) = (1/2)[(h_{00} + h_{11})I + (h_{00} - h_{11})\sigma_z + (h_{01} + h_{10})\sigma_x + (ih_{01} - ih_{10})\sigma_y]$, where $h_{ij} = \langle i|H_q^{(1)}|j\rangle$. For symmetry reasons $\langle 0|(\sum \phi_i - \pi)|0\rangle = \langle 1|(\sum \phi_i - \pi)|1\rangle = 0$ so that $h_{00} = h_{11} = 0$. Indeed, the hamiltonian $H_q^{(0)}$ is invariant under the transformation $T \phi_1 \to -\phi_1, \phi_2 \to -\phi_2$ and $\phi_3 \to 2\pi - \phi_3$. This means that the eigenstates of $H_q^{(0)}$ have to also be eigenstates of $T$. Since $T^2 = I$, these eigenstates should be either 1 or $-1$ so that $\psi_i(-\phi_1, -\phi_2, 2\pi - \phi_3) = \pm \psi_i(\phi_1, \phi_2, \phi_3)$ and $|\psi_i(-\phi_1, -\phi_2, 2\pi - \phi_3)|^2 = |\psi_i(\phi_1, \phi_2, \phi_3)|^2$. This implies that the matrix elements $h_{00} = h_{11} = 0$.

Since we can always chose the global phases of $|0\rangle$ and $|1\rangle$ so that $h_{01}$ and $h_{10}$ are real, we obtain that $H_q^{(1)}(f, I_b) = h_{01}\sigma_x$ where $h_{01} = (\Phi_0/2\pi)^2\langle 0|\sum \phi_i|1\rangle[-(f - \pi)/l_{11}]$. We next define $I_p \equiv (\Phi_0/2\pi)(1/l_{11}) < 0|\sum \phi_i|1\rangle$, $\epsilon = 2I_p(\Phi_x - \Phi_0/2)/\hbar$. In the end we can write the total hamiltonian as

$$H_q(f) = -\frac{\hbar}{2}(\Delta \sigma_z + \epsilon \sigma_x) \quad (20)$$

We note that this derivation generalizes the analysis of [19] which was made under the assumption that the qubit loop self-inductance is negligible. Here we retrieve the result of [20] which showed numerically that the form of the qubit hamiltonian was little affected by taking into account this inductance.

FIG. 2: Qubit frequency $\nu_q$ as a function of the bias $\epsilon$ for $\Delta = 5.5 GHz$ (minimum frequency in the figure). The dashed line indicates the phase-noise insensitive bias point $\epsilon = 0$ where $d\nu_q/d\epsilon = 0$.
The Hamiltonian \( H \) yields qubit transition frequency \( \nu = \sqrt{\Delta^2 + \epsilon^2} \). The corresponding dependence is plotted in figure 2 for realistic parameters. An interesting property is that when the qubit is biased at \( \epsilon = 0 \) (dashed line in figure 2), it is insensitive to first order to noise in the bias variable \( \epsilon \).

C. Plasma mode Hamiltonian

Next we turn to the “SQUID+shunt capacitor” variables \( \phi_4, \phi_5, \phi_A \). As already explained, we will here make a crude approximation and completely neglect the SQUID junction capacitance. This is justified by the fact that at the bias current that we use the modes to which they correspond are at frequencies much higher than the qubit and plasma mode. Moreover, we will keep only the terms of second order in the SQUID potential, which is equivalent to considering the SQUID as one inductance \( L_J(I_b, f') \). On the other hand we will keep in the analysis the dependence on flux of \( L_J \) which has important effects.

In this approximation, the only dynamical variable in the system is \( \phi_A \). Its Hamiltonian is very simply given by the Hamiltonian of a harmonic oscillator of capacitance \( C_{sh} \) and inductance \( L = L_s + L_J(I_b, f') \)

\[
H_p = \frac{Q_A^2}{2C_{sh} + (\Phi_0/2\pi)^2(\phi_A - \overline{\phi_A})^2/2L}
\]

(21)

where \( \overline{\phi_A} = (2\pi/\Phi_0)LJb \) is the mean value of \( \phi_A \). We call \( a \) and \( a^\dagger \) the creation and annihilation operators corresponding to this harmonic oscillator:

\[
a = \frac{\Phi_0}{2\pi} \sqrt{\frac{\pi C_{sh} \nu_0}{\hbar}} (\phi_A - \overline{\phi_A}) + \frac{i}{\sqrt{2 C_{sh} \hbar \nu_0}} Q_A
\]

(22)

and \( a^\dagger \) is the conjugate operator. Then the Hamiltonian is simply \( H_p = \hbar \omega_0(a^\dagger a + 1/2) \), and the phase \( \phi_A \) can be written : \( \phi_A = \overline{\phi_A} + \delta \phi_0 (a + a^\dagger) \) \( (\delta \phi_0 \) is the rms amplitude of the vacuum fluctuations of \( \phi_A \).

In our model where the SQUID junctions have no capacitance, for a given value of \( \phi_A \) and \( f' \) all the phases of the SQUID are well-defined functions \( \phi_{4,5}(\phi_A, f') \). So the quantum fluctuations of \( \phi_A \) translate directly into fluctuations of \( \phi_4 \) and \( \phi_5 \) as follows:

\[
\phi_{4,5} = \overline{\phi_{4,5}} + (d\phi_{4,5}/d\phi_A)\delta \phi_0(a + a^\dagger) + (1/2)(d^2 \phi_{4,5}/d\phi_A^2)(\delta \phi_0)^2(a + a^\dagger)^2
\]

(23)
where $\phi_{4,5} = \phi_{4,5}(\phi_A)$. We develop the functions to second order in $a$ and $a^\dagger$ for consistency. Again, the sensitivity coefficients $d\phi_{4,5}/d\phi_A$ and $d^2\phi_{4,5}/d\phi_A^2$ depend on $I_b$ and $f'$ and can be easily calculated. We also note that $d\phi_A = (2\pi/\Phi_0)LdI_b$ so that

$$\phi_{4,5} = \phi_{4,5} + \frac{\Phi_0}{2\pi L} \frac{d\phi_4}{dI_b} \delta \phi_0(a + a^\dagger) + \frac{1}{2} \left( \frac{\Phi_0}{2\pi L} \right)^2 \frac{d^2\phi_{4,5}}{dI_b^2} (\delta \phi_0)^2 (a + a^\dagger)^2$$

(24)

We will finally show how to evaluate the sensitivity coefficients for any SQUID parameters. Following [22, 23] we introduce the parameters $x = (\phi_4 + \phi_5)/2$, $y = (\phi_4 - \phi_5)/2$, $s = I_b/(I_{C4} + I_{C5})$, $b = \Phi_0/\pi L_{SQ}(I_{C4} + I_{C5})$, $U_0 = \Phi_0(I_{C4} + I_{C5})/2\pi$, $\alpha = (I_{C4} - I_{C5})/(I_{C4} + I_{C5})$, and $\delta = (K_3 + K_1 - L_2 - K_2)/L_{SQ}$. The stationary solutions for the SQUID phases are obtained by minimizing the 2-dimensional potential

$$U(x, y) = U_0[-s x - \cos x \cos y - \alpha \sin x \sin y - \delta s y + b(y - f'/2)^2]$$

(25)

that is to solve the equations

$$\partial U/\partial x = U_0[-s + \sin x \cos y - \alpha \cos x \sin y] = 0$$

$$\partial U/\partial y = U_0[\cos x \sin y - \alpha \sin x \cos y - \delta s + 2b(y - f'/2)] = 0$$

(26)

From this it is possible to obtain numerically the functions $\Phi_{4,5}(\phi_A, f')$ and thus all the sensitivity coefficients needed in the model.

D. Qubit-plasma mode coupling hamiltonian

The coupling hamiltonian is due to two contributions. First, the explicit coupling term in the equation 36. But we also need to rewrite the plasma mode hamiltonian since the parameters of this hamiltonian (notably the SQUID Josephson inductance) now depends on the qubit state. We therefore need to reconsider the following variables in the plasma mode hamiltonian:

$$f' \rightarrow f' + (2\pi/\Phi_0)MI_p\sigma_x$$

$$L \rightarrow L + \delta L\sigma_x$$

$$\bar{\phi}_A \rightarrow \bar{\phi}_A + (2\pi/\Phi_0)\delta LI_b\sigma_x$$

(27)
where $\delta L \equiv (2\pi/\Phi_0)(\partial L_I/\partial f')MI_p$. The SQUID-qubit coupling term writes

$$V = (\Phi_0/2\pi)^2 \sum_{i=1}^{3} (\phi_i - f)[l_{21}^{-1}(\phi_4 + \phi_5 - f') - l_{31}^{-1}(\phi_5 + \phi_A)]$$  \hspace{1cm} (28)

Since $\sigma_x = (\sum \phi_i - \pi)/ <0|\sum \phi_i|1>$, we can rewrite $V$ in the form

$$V = (\Phi_0/2\pi)l_{11}I_p\sigma_x[l_{21}^{-1}(\phi_4 + \phi_5 - f') - l_{31}^{-1}(\phi_5 + \phi_A)]$$ \hspace{1cm} (29)

Keeping only the terms which contain explicit couplings we obtain that

$$V = (\Phi_0/2\pi)l_{11}I_p\sigma_x[(\Phi_0/2\pi L)(l_{21}^{-1}(d\phi_4/dI_b) + (l_{21}^{-1} - l_{31}^{-1})(d\phi_5/dI_b))\delta\phi_0(a + a^\dagger) + (1/2)(\Phi_0/2\pi L)^2(l_{21}^{-1}(d^2\phi_4/dI_b^2) + (l_{21}^{-1} - l_{31}^{-1})(d^2\phi_5/dI_b^2))\delta\phi_0^2(a + a^\dagger)^2]$$ \hspace{1cm} (30)

On the other hand, the plasma mode hamiltonian now writes

$$H_{pl-q} = Q_A^2/2C_{sh} + (\Phi_0/2\pi)^2[\phi_A - \bar{\phi}_A - (2\pi/\Phi_0)\delta LI_b\sigma_x]^2/2(L + \delta L\sigma_x)$$

$$= Q_A^2/2C_{sh} + (\Phi_0/2\pi)^2 \left[ \frac{(\phi_A - \bar{\phi}_A)^2}{2L} - \frac{2\pi\delta LI_b}{\Phi_0L}(\phi_A - \bar{\phi}_A)\sigma_x - \frac{\delta L}{2L^2}(\phi_A - \bar{\phi}_A)^2\sigma_x \right]$$ \hspace{1cm} (31)

so that

$$H_{pl-q} = h\nu_0(a^\dagger a + 1/2) - \frac{\Phi_0\delta LI_b}{2\pi L} \delta\phi_0(a + a^\dagger) - \left( \frac{\Phi_0}{2\pi} \right)^2 \frac{\delta\phi_0^2}{2L^2} (\delta L/2L^2)^2(a + a^\dagger)^2\sigma_x$$ \hspace{1cm} (32)

Finally the total interaction hamiltonian $H_I = V + H_{pl-q}$ can be written as

$$H_I = h[g_1(a + a^\dagger) + g_2(a + a^\dagger)^2]\sigma_x$$ \hspace{1cm} (33)

The coupling constants $g_1$ and $g_2$ could be deduced from the above expressions. Nevertheless we propose a way to determine them experimentally. We first note that this coupling hamiltonian can be rewritten

$$H_I = h[\lambda_1\delta\phi_A + \lambda_2\delta\phi_A^2]\sigma_x$$ \hspace{1cm} (34)

where $\delta\phi_A = \phi_A - \bar{\phi}_A$. This gives us a very direct way of evaluating the coupling constants $g_1$ and $g_2$: indeed varying the bias current $I_b$ through the SQUID by an amount $\delta I_b$ is equivalent to a variation $\delta\phi_A = 2\pi L\delta I_b/\Phi_0$. Since we can experimentally measure the
dependence of the qubit frequency on the bias current \( \epsilon(I_b) \) and thus measure \( \partial \epsilon / \partial I_b \) and \( \partial^2 \epsilon / \partial I_b^2 \), we can obtain the coupling constants from the experiment by the following expressions:

\[

g_1 = (1/2)(\partial \epsilon / \partial I_b)(\Phi_0/2\pi L)\delta \phi_0 \\
g_2 = (1/4)(\partial^2 \epsilon / \partial I_b^2)(\Phi_0/2\pi L)^2\delta \phi_0^2
\]

(35)

Finally the total qubit-plasma mode hamiltonian can be written as

\[
H/h = (1/2)(-\Delta \sigma_z - \epsilon \sigma_x) + \nu_0(a^\dagger a + 1/2) + [g_1(I_b)(a + a^\dagger) + g_2(I_b)(a + a^\dagger)^2] \sigma_x
\]

(36)

We note that a more rigorous derivation for the coupling between the plasma mode and the qubit after elimination of the internal dynamics of the SQUID thanks to the Feynman-Vernon influence functional has been performed in [25] and gives ultimately the same form if we develop their interaction hamiltonian to the second order in the oscillator variables. It is also instructive to compare this hamiltonian to the simpler situation studied in [8]. There a Cooper-pair box is capacitively coupled to a coplanar waveguide resonator. The interaction hamiltonian contains only one term, linear in the oscillator variables, and with a fixed coupling constant depending on the geometry of the circuit. In our case the coupling is mediated by the SQUID flux-dependent and current-dependent inductance; therefore the coupling constants are tunable and higher-order terms are of importance. This made possible to induce transitions in which both the HO and the qubit state are modified [7].

E. Coupling constants

We now want to give analytical formulae for the coupling constants \( g_1 \) and \( g_2 \) in the simplest case where a number of assumptions are made: 1) the SQUID-qubit coupling is supposed to be symmetric \( (K_1 = K_2) \) so that the qubit bias is only coupled to the current \( J \) circulating in the SQUID loop \( \epsilon = (2I_p/h)(\Phi_x + MJ(I_b)) \) 2) the SQUID loop self-inductance and the stray inductance \( L_s \) are negligible so that the total inductance of the plasma mode is the SQUID Josephson inductance \( L = L_J(f, I_b) \). Within these assumptions the equations are easily solved and yield that
\[
x = \arcsin \left( \frac{I_b}{2I_C \cos \left( \frac{f'}{2} \right)} \right)
\]

\[
y = \frac{f'}{2}
\]

which implies that the current circulating in the SQUID loop is

\[
J \equiv (I_1 - I_2)/2 = I_C \sqrt{1 - \left( \frac{I_b}{2I_C \cos \left( \frac{f'}{2} \right)} \right)^2} \sin \left( \frac{f'}{2} \right)
\]

The rms phase fluctuations of the plasma mode are \( \delta \phi_0 = \frac{(2\pi/\Phi_0)}{\sqrt{h\nu_pL/2}} \) so that we obtain

\[
g_1 = -\frac{MI_p}{h} \frac{\sin \left( \frac{f'}{2} \right)}{4I_C \cos^2 \left( \frac{f'}{2} \right)} \sqrt{\frac{h\nu_p}{2L}} \left[ 1 - \left( \frac{I_b}{2I_C \cos \left( \frac{f'}{2} \right)} \right)^2 \right]^{-1/2}
\]

\[
g_2 = -\frac{MI_p}{h} \frac{\sin \left( \frac{f'}{2} \right)}{8I_C \cos^2 \left( \frac{f'}{2} \right)} \sqrt{\frac{h\nu_p}{2L}} \left[ -\left( \frac{I_b}{2I_C \cos \left( \frac{f'}{2} \right)} \right)^2 \left( 1 - \left( \frac{I_b}{2I_C \cos \left( \frac{f'}{2} \right)} \right)^2 \right)^{-3/2} + \left( 1 - \left( \frac{I_b}{2I_C \cos \left( \frac{f'}{2} \right)} \right)^2 \right)^{-1/2} \right]
\]

Around \( I_b = 0 \) these expressions can be simplified by keeping only the lowest order in \( I_b \):

\[
g_1 \simeq -\frac{MI_p}{h} \frac{\sin \left( \frac{f'}{2} \right)}{4I_C \cos^2 \left( \frac{f'}{2} \right)} \sqrt{\frac{h\nu_p}{2L}} \frac{I_b}{L}
\]

\[
g_2 \simeq -\frac{1}{16} \frac{\sin \left( \frac{f'}{2} \right)}{\cos^2 \left( \frac{f'}{2} \right)} \frac{MI_p}{L I_C} \nu_p
\]

We will now discuss quantitatively the behaviour of \( g_1 \) and \( g_2 \) for actual sample parameters: \( I_C = 3.4 \mu A, M = 6.5 pH, I_p = 240 nA, \Delta = 5.5 GHz, \nu_p = 3.1 GHZ, L_J = 300 pH, \)

\( f'/2 = 1.45 \pi \). We will restrict ourselves to a range of bias conditions relevant for our conditions, supposing that \( I_b \) varies between \( \pm 300 nA \) and that \( f'/2 \) varies by \( df' = \pm 4 \cdot 10^{-3} \pi \) around \( 1.45 \pi \). We chose such an interval for \( f' \) because it corresponds to changing the qubit bias point \( \epsilon \) by \( \pm 2 GHz \) around 0. The constants \( g_1 \) and \( g_2 \) are plotted in figure 3 as a function of \( I_b \) for two different values of \( f' \) (\( g_1 \) is shown as a full line, \( g_2 \) as a dashed line, and the two different values of \( f' \) are symbolized by gray for \( df' = -2\pi \cdot 10^{-3} \) and black
FIG. 3: Coupling constants $g_1$ (solid line) and $g_2$ (dashed line) as a function of the bias current, for two values of the reduced SQUID flux bias $f'$ differing by $df' = 4 \cdot 10^{-3}$ (gray and black lines).

for $df' = 0$). It can be seen that the coupling constants only weakly depend on the value of the flux in this range, so that we will neglect this dependence in the following and consider that $g_1$ and $g_2$ only depend on the bias current $I_b$. Moreover we see from figure 3 that the approximations made in equation (40) are justified in this range of parameters since $g_1$ is closely linear in $I_b$ and $g_2$ nearly constant. We also note that $g_1 = 0$ for $I_b = 0$. This fact can be generalized to the case where the SQUID-qubit coupling is not symmetric and the junctions critical current are dissimilar: in certain conditions these asymmetries can be compensated for by applying a bias current $I_b^*$ for which $g_1(I_b^*) = 0$. At the current $I_b^*$, the qubit is effectively decoupled from the measuring circuit fluctuations to first order.

IV. ENERGY LEVELS AND DEPHASING

In this paragraph we investigate the discuss the energy levels of the hamiltonian (36) and we estimate the frequency shift per photon $\delta \nu_0$. We will only consider the case where the qubit is detuned from the plasma mode $\nu_q - \nu_p \gg g_1$ and also $|\nu_q - 2\nu_p| \gg g_2$.

A. Energy levels

If the qubit and the plasma mode were uncoupled (case $g_1 = g_2 = 0$), the system energy eigenstates would be $|i\rangle \otimes |n\rangle$, where $i$ refers to the qubit state and can be either 0 or
1, and \( n \) to the plasma mode occupation number. The energy levels would simply be 
\[ E^{(u)}_{i,n} = h(i\sqrt{\Delta^2 + \epsilon^2 + n\nu_0}). \]
When either \( g_1 \) or \( g_2 \) are non zero, these eigenstates are modified, but for convenience we will still label them thanks to the uncoupled state from which they are the closest \( |i, n\rangle \). However, the energies are now modified: 
\[ E_{i,n} = E^{(u)}_{i,n} + \delta\nu_{i,n}. \]
The aim of this paragraph is to estimate the quantity \( \delta\nu_{i,n} \).

We first rewrite the hamiltonian in a more convenient way for our purpose, by introducing the rotated axes \( X \) and \( Z \) defined as 
\[ \sigma_Z = (\Delta\sigma_z + \epsilon\sigma_x)/\sqrt{\Delta^2 + \epsilon^2} \quad \text{and} \quad \sigma_X = (-\epsilon\sigma_z + \Delta\sigma_x)/\sqrt{\Delta^2 + \epsilon^2}. \]
The angle \( \theta \) is defined so that \( \cos \theta \equiv \epsilon/\sqrt{\Delta^2 + \epsilon^2} \) and \( \sin \theta \equiv \Delta/\sqrt{\Delta^2 + \epsilon^2} \). The system hamiltonian now writes
\[
H/h = -(1/2)\sqrt{\Delta^2 + \epsilon^2}\sigma_Z + \nu_p(a^\dagger a + 1/2) + [g_1(a + a^\dagger) + g_2(a + a^\dagger)^2](\cos \theta \sigma_Z + \sin \theta \sigma_X) \quad (41)
\]

1. **Linear term**

Let us first suppose that \( g_2 = 0 \). Then the coupling is linear in the oscillator variables, with a longitudinal component proportional to \( \cos \theta \) and a transverse component proportional to \( \sin \theta \).

We first notice that the longitudinal component has no effect on the energy states. Indeed, the term \( \sigma_Z(a + a^\dagger) \) does not shift the energy levels to first order of the perturbation theory since \( \langle i, n | \sigma_Z(a + a^\dagger) | i, n \rangle = 0 \) for all states. To second order, all energy levels are shifted by the same quantity \( (g_1 \cos \theta)^2/\nu_p \) which implies that all the transition frequencies stay unchanged. This conclusion stays true to all orders of perturbation theory.

On the other hand, the transverse coupling term \( g_1 \sin \theta \sigma_X(a + a^\dagger) \) produces the well-known dispersive shift in cavity QED \[24\] to second-order in perturbation theory, as calculated in the first section of this article. In the rotating wave approximation, we recall that \( \delta\nu^1_{i,n} = i(g_1 \sin \theta)^2/\delta + (2i-1)n(g_1 \sin \theta)^2/\delta \) where \( \delta = \sqrt{\Delta^2 + \epsilon^2} - \nu_0 \) is the qubit-plasma mode detuning. However, it is necessary here to go beyond the rotating wave approximation since \( \delta \) is of the same order of magnitude as the qubit and the oscillator frequency. It is easily seen that to second order of perturbation theory, one obtains
\[
\delta\nu^1_{i,n} = 2i(g_1 \sin \theta)^2 \frac{\sqrt{\Delta^2 + \epsilon^2}}{\Delta^2 + \epsilon^2 - \nu_0^2} + (2i-1)2n(g_1 \sin \theta)^2 \frac{\sqrt{\Delta^2 + \epsilon^2}}{\Delta^2 + \epsilon^2 - \nu_0^2} \quad (42)
\]
The first term of this equation describes the Lamb shift, which simply renormalizes the bare qubit frequency and has no influence on dephasing. We will thus neglect it in the following. The frequency shift per photon is

$$\delta\nu_0^1 = 4(g_1 \sin \theta)^2 \frac{\sqrt{\Delta^2 + \epsilon^2}}{\Delta^2 + \epsilon^2 - \nu_p^2}$$  \hspace{1cm} (43)$$

From the previous expression it is clear that the sign of $\delta\nu_0^1$ is fully determined by the sign of $\delta$. In particular, in our experiments where $\nu_q > \nu_p$, $\delta\nu_0^1 > 0$.

2. Quadratic term

Next, we consider the case $g_1 = 0$ but $g_2 > 0$ (which is the case notably at the decoupled current $I_b = I_b^* \text{[10]}$). The quadratic coupling term produces effects which are sensibly different from the cavity QED case. Indeed, it generates a frequency shift to first order in perturbation theory via the term $2g_2 \cos \theta \sigma_Z a^\dagger a$. Considering that the $g_2$ coupling is already second order, we only keep the first order of perturbation theory. We therefore obtain that $\delta\nu_0^2 = -(2i - 1)2g_2n \cos \theta$ so that the shift per photon is

$$\delta\nu_0^2 = -4g_2 \cos \theta$$  \hspace{1cm} (44)$$

Contrary to the shift produced by the linear coupling term, the sign of this frequency shift now depends on $\epsilon$. Since $g_2$ is negative (see figure 3), $\delta\nu_0^2$ actually has the same sign as $\epsilon$. We also note that the quadratic term has no effect on the qubit when $\epsilon = 0$, since at that point the average flux generated by both qubit states $|0\rangle$ and $|1\rangle$ averages out to zero so that the SQUID Josephson inductance is unchanged.

3. Total frequency shift and dependence on the bias parameters

The total frequency shift per photon is the sum of the two contributions identified above:

$$\delta\nu_0(\epsilon, I_b) \equiv \delta\nu_0^1 + \delta\nu_0^2$$

$$= 4 \left[ (g_1 \sin \theta)^2 \frac{\sqrt{\Delta^2 + \epsilon^2}}{\Delta^2 + \epsilon^2 - \nu_p^2} - g_2 \cos \theta \right]$$  \hspace{1cm} (45)$$
FIG. 4: Frequency shift per photon $\delta \nu_0$ as a function of $I_b$ and $\epsilon$. The white regions correspond to $-15 \text{MHz}$ and the black to $+35 \text{MHz}$. The solid line $\epsilon_m(I_b)$ indicates the bias conditions for which $\delta \nu_0 = 0$. The dashed line indicates the phase noise insensitive point $\epsilon = 0$; the dotted line indicates the decoupling current $I_b = I^*_b$.

Because of the different dependence on $\epsilon$ of the two contributions discussed above, we expect a cancellation of the AC-Zeeman term (due to $g_1$) by the quadratic term (due to $g_2$) for some bias parameters corresponding to a negative value of $\epsilon$. This is shown in figure 4 where we plotted $\delta \nu_0(\epsilon, I_b)$ as calculated with the formula above for the sample parameters considered in the previous paragraph. The curved full line corresponds to the points $\epsilon_m(I_b)$ for which $\delta \nu_0 = 0$. For these bias conditions, it is expected that the qubit is insensitive to the thermal fluctuations of the plasma mode (see formula 7). Therefore we predict an increase of the dephasing time whenever $\epsilon = \epsilon_m(I_b)$.

We stress that these biasing conditions are non-trivial in the sense that they do not satisfy an obvious symmetry in the circuit. This point is emphasized in figure 4 where we plotted as a dashed line the bias conditions $\epsilon = 0$ for which the qubit is insensitive to phase noise (due to flux or bias current noise); and as a dotted line the decoupling current conditions $I_b = I^*_b$ for which the qubit is effectively decoupled from its measuring circuit. The $\epsilon_m(I_b)$ line shares only one point with these two curves: the point $(I^*_b, \epsilon)$ which is optimal with respect to flux, bias current, and photon noise. For the rest, the three lines are obviously
distinct. This makes it possible to experimentally discriminate between the various noise sources limiting the qubit coherence by studying the dependence of $\tau_\phi$ on bias parameters.

V. CONCLUSION

Superconducting qubits are often measured by circuits behaving as underdamped oscillators to prevent energy relaxation of the qubit. If these oscillators have a frequency comparable to $kT$, their photon number undergoes thermal fluctuations. This induces frequency dispersive frequency shifts of the qubit frequency $n\delta\nu_0$ and leads to dephasing. In this article we derive a simple formula to account for this process. We apply our model to the specific case of a flux-qubit coupled to the plasma mode of its DC-SQUID. Because of the SQUID internal degrees of freedom (circulating current), the interaction Hamiltonian contains two terms, one linear in the oscillator variables which describes an effective inductive coupling between the two circuit, but also a quadratic term due to the flux-dependence of the SQUID Josephson inductance. Moreover, the coupling constants can be tuned over a wide range by changing the SQUID bias current. We study the qubit frequency shift per photon $\delta\nu_0$ and find that $\delta\nu_0 = 0$ for non-trivial biasing conditions. When they are fulfilled, the effect of thermal fluctuations on the qubit should be suppressed.

VI. ANNEX A

Here we show how we evaluate the correlation function $C(t) = \langle a^\dagger(0)a(0)a^\dagger(t)a(t) \rangle$. In order to do so, we follow [13]. We model the damping of the HO by a linear coupling to a bath of harmonic oscillators

\[
H = H_{\text{HO}} + H_{\text{bath}} + H_{\text{int}} = \hbar\nu_p(a^\dagger a + \frac{1}{2}) + \sum_\omega \hbar\omega(A^\dagger(\omega)A(\omega) + \frac{1}{2}) + \sum_\omega \hbar[g(\omega)a^\dagger A(\omega) + g^*(\omega)A^\dagger(\omega)a]
\]  

(46)

Under the assumption that the bath has a short memory, it can be shown that the evolution of the HO variables in the Heisenberg representation is given by

\[
\dot{a} = (-i2\pi\nu_p - \kappa/2)a(t) - F(t)
\]  

(47)
that reasoning holds of course to show that a at thermal equilibrium, calculate the last term: N \int F(t') \exp (i2\pi\nu_p + \kappa/2)t'dt' \quad (48)

This allows us to calculate C(t) = < a^\dagger(0)a(0)a^\dagger(t)a(t) >

C(t) = < a^\dagger(0)a(0)a^\dagger(t)a(t) >
= < n(0)^2 > \exp(-\kappa t)
- \exp(-\kappa t) \int_0^t < a^\dagger(0)a(0)F^\dagger(t')a(0) > \exp(-i2\pi\nu_p + (\kappa/2))t'
+ \exp(-\kappa t) < a^\dagger(0)a(0)a^\dagger(0)F(t') > \exp(i2\pi\nu_p + (\kappa/2))t'
+ \exp(-\kappa t) \int_0^t \int_0^t < a^\dagger(0)a(0)F^\dagger(t')F(t'') > \exp[-i2\pi\nu_p(t' - t'')] \exp[(\kappa/2)(t' + t'')] dt'dt'' \quad (49)

In this equation, the second and third term vanish. Indeed,

< a^\dagger(0)a(0)F^\dagger(t')a(0) > = i \sum_\omega g(\omega) \exp^{-i\omega t} < a^\dagger(0)a(0)a^\dagger(0)A(\omega, 0) >
= i \sum_\omega g(\omega) \exp^{-i\omega t} < a^\dagger(0)a(0)a^\dagger(0) > < A(\omega, 0) > \quad (50)

since it is assumed that at time t = 0 the bath and the HO are uncorrelated. For a bath at thermal equilibrium, < A(\omega, 0) >= 0 so that < a^\dagger(0)a(0)F^\dagger(t')a(0) > = 0. The same reasoning holds of course to show that < a^\dagger(0)a(0)a^\dagger(0)F(t') > = 0 as well. Using the fact that < F^\dagger(t)F(t'') > = \kappa N(2\pi\nu_p)\delta(t' - t'') \quad (13), where N(\omega) = < A^\dagger(\omega, 0)A(\omega, 0) >, we can calculate the last term:

\exp(-\kappa t) \int_0^t \int_0^t < a^\dagger(0)a(0)F^\dagger(t')F(t'') > \exp[-i2\pi\nu_p(t' - t'')] \exp[(\kappa/2)(t' + t'')] dt'dt''
= < n(0) > \kappa N(2\pi\nu_p) \exp(-\kappa t) \int_0^t \int_0^t \delta(t' - t'') \exp[-i2\pi\nu_p(t' - t'')] \exp[\kappa(t' + t'')/2] dt'dt''
= < n(0) > \kappa N(2\pi\nu_p) \exp(-\kappa t) \int_0^t \exp(\kappa t') dt' = N(2\pi\nu_p) < n(0) > (1 - \exp(-\kappa t)) \quad (51)

Since we assume that the HO and the bath are permanently in thermal equilibrium, N(2\pi\nu_p) = < n(0) > = \bar{n}, whereas < n^2(0) > - < n(0) >^2 = \bar{n}(\bar{n} + 1) (non-poissonian photon statistics of a thermal field). Therefore we obtain
\[ C(\tau) = [<n^2(0)> - <n(0)^2>] \exp(-\kappa\tau) + <n(0)^2> \\
= \bar{n}(\bar{n} + 1) \exp(-\kappa\tau) + \bar{n}^2 \] (52)

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