Regression Discontinuity Design with Potentially Many Covariates

Yoichi Arai,† Taisuke Otsu‡ and Myung Hwan Seo§

February 20, 2024

Abstract

This paper studies the case of possibly high-dimensional covariates in the regression discontinuity design (RDD) analysis. In particular, we propose estimation and inference methods for the RDD models with covariate selection which perform stably regardless of the number of covariates. The proposed methods combine a localization approach using kernel weights with $\ell_1$-penalization to handle high-dimensional covariates. We provide theoretical and numerical results which illustrate the usefulness of the proposed methods. Theoretically, we present risk and coverage properties for our point estimation and inference methods, respectively. Under certain special cases, the proposed estimator becomes more efficient than the conventional covariate adjusted estimator at the cost of an additional sparsity condition. Numerically, our simulation experiments and empirical example show the robust behaviors of the proposed methods to the number of covariates in terms of bias and variance for point estimation and coverage probability and interval length for inference.

1 Introduction

In causal or treatment effect analysis, discontinuities in regression functions induced by an assignment variable can provide useful information to identify certain causal effects. The regression

---

*This paper is a developed version of the previous manuscript (https://sticerd.lse.ac.uk/dps/em/em601.pdf) inspired by the discussion with Matias Cattaneo. We are grateful to Matias Cattaneo and Sebastian Calonico for helpful comments and discussions. This research was supported by Grants-in-Aid for Scientific Research 20K01598 from the Japan Society for the Promotion of Science (Arai) and the ERC Consolidator Grant (SNP 615882) (Otsu). Financial support from the Center for National Competitiveness in the Institute of Economic Research of Seoul National University and the Ministry of Education of the Republic of Korea and the National Research Foundation of Korea (NRF-2018S1A5A2A01033487) is gratefully acknowledged (Seo).

†School of Social Sciences, Waseda University, 1-6-1 Nishiwaseda, Shinjuku-ku, Tokyo 169-8050, Japan. Email: yarai@waseda.jp

‡Department of Economics, London School of Economics, Houghton Street, London, WC2A 2AE, UK. Email: t.otsu@lse.ac.uk

§Department of Economics, Seoul National University, 1 Gwankro Gwanakgu, Seoul, 08826, Korea. Email: myunghseo@snu.ac.kr
discontinuity design (RDD) has been widely applied in observational studies to identify the average treatment effect at the discontinuity point. For the RDD, the causal parameters of interest are identified by some contrasts of the left and right limits of the conditional mean functions. See e.g. Imbens and Lemieux (2008), Cattaneo, Titiumik and Vazquez-Bare (2020), an edited volume by Cattaneo and Escanciano (2017), and references therein.

In the growing literature on the RDD analysis, this paper focuses on the RDDs where covariates are included in the estimation, which are extensively studied by Calonico, Cattaneo, Farrell and Titiumik (2019) (hereafter, CCFT). See also Frölich and Huber (2019) for an alternative estimation method based on kernel smoothing after localization around the cutoff. In practice, researchers often augment the regression models for the RDD analysis with various additional predetermined covariates such as demographic or socioeconomic characteristics for data units. For several RDD estimators using covariates based on local polynomial regression methods, CCFT investigated the MSE expansion, asymptotic efficiency, and data-driven bandwidth selection methods. Furthermore, CCFT developed asymptotic distributional approximations for those estimators and proposed valid inference procedures by constructing bias and variance estimators with covariate adjustment. These results may be considered as extensions of the analyses in Calonico, Cattaneo and Titiumik (2014) (hereafter, CCT) combined with robust bias correction methods in Calonico, Cattaneo and Farrell (2018, 2020) to incorporate covariates in the RDD analysis. See also Calonico, Cattaneo, Farrell and Titiumik (2017) for a statistical package on these methods.

In randomized controlled trials, regression adjustment using covariates is a common practice since it is always helpful to improve asymptotic efficiency of the causal effect estimator as far as a full set of treatment-covariate interactions is included (Lin, 2013). Also a recent paper by Lei and Ding (2021) proposed a bias correction method for the regression adjustment estimator with a diverging number of covariates. On the other hand, in the RDD analysis, which is a quasi-experiment setup, the efficiency gain by introducing covariates is not necessarily guaranteed, and CCFT provided a concrete guideline by clarifying the conditions to achieve consistency and efficiency gain for the covariate adjusted RDD estimator. Typically the efficiency improves when the projection coefficients of the covariates on the outcome are equal for both control and treatment groups. Since practitioners also commonly incorporate covariates for the RDD analysis, CCFT’s guideline has a large impact in applied research. When we use covariates, it is common to employ their transformations and interactions, and the number of these terms can be pretty large. This paper adds a further guideline for practitioners who also face a large number of covariates. To begin with, the (weighted) OLS estimation in CCFT is not applicable when
the number of covariates is larger than the sample size. Also, in the above scenario for efficiency improvement, it is beneficial to augment CCFT’s procedure with covariate selection by high-dimensional statistical methods particularly when the regression coefficients for the conditional mean function satisfy certain sparsity.

For point estimation on the causal effect parameter identified by the RDD, we consider the Lasso estimator and its post-selection estimator based on the local linear regression (i.e., eq. (2) of CCFT). The combination of localization using kernel weights and ℓ₁-penalization to deal with high-dimensional covariates is particularly relevant for the RDD analysis, where the effective sample size would be typically small due to the localization so that the effect of dimensionality of covariates becomes severer. Theoretically, we derive the ℓ₁-risk properties of our local Lasso estimator and its post-selection version. Practically, based on our simulation study, we recommend the CCFT estimator after selecting covariates by the ℓ₁-penalization even for a relatively small number of covariates, which exhibits desirable MSE properties and stability across different setups.

For inference, we propose to select covariates with the local Lasso. We show that the inference based on the selected covariates can be implemented in the same manner as in CCFT. We also show that when the effect of the additional covariates on the potential outcomes with or without treatment is invariant, our approach can lead to improved efficiency at the cost of an additional sparsity condition. This sparsity condition is trivially satisfied when the set of active covariates is unknown but fixed. Our simulation results demonstrate that our post-selection confidence interval exhibits robust performances in terms of both coverages and lengths, even for a relatively small number of covariates.

This paper also contributes to the large literature on high-dimensional methods in econometrics and statistics (see, e.g., Bühlmann and van de Geer, 2011, and Belloni et al., 2018, for an overview) by combining the kernel localization with ℓ₁-penalization to handle high-dimensional covariates. Our inference problem can be formulated as the one for low-dimensional parameters in high-dimensional models. In statistics literature, many papers investigated this issue, such as Belloni, Chernozhukov and Hansen (2014), van de Geer, et al. (2014), and Zhang and Zhang (2014). However, these approaches are not directly applicable to the RDD context because the current problem concerns the inference on a jump in a nonparametric regression model.

This paper is organized as follows. Section 2.1 introduces our basic setup and local Lasso estimator, and presents the ℓ₁-risk properties. In Section 2.2, we discuss the validity of CCFT’s

---

1A recent paper by Kreiß and Rothe (2023) investigates a similar estimator to ours, and discusses an inference method based on the approach by Armstrong and Kolesár (2018).
inference after selecting covariates by our Lasso procedure. Section 3 provides discussions on some extensions. A step-by-step procedure for implementation of our method is described in Section 4. To illustrate the proposed method, Section 5 conducts a simulation study, and Section 6 presents an empirical example based on the Head Start data.

2 Main result

2.1 Setup and local Lasso estimator for covariate selection

In this subsection, we present our basic setup and introduce the local Lasso estimator for the RDD with possibly high-dimensional covariates. For each unit \( i = 1, \ldots, n \), we observe an indicator variable \( T_i \) for a treatment (\( T_i = 1 \) if treated and \( T_i = 0 \) otherwise), and outcome \( Y_i = Y_i(0) \cdot (1 - T_i) + Y_i(1) \cdot T_i \), where \( Y_i(0) \) and \( Y_i(1) \) are potential outcomes for \( T_i = 0 \) and \( T_i = 1 \), respectively. Note that we cannot observe \( Y_i(0) \) and \( Y_i(1) \) simultaneously. Our purpose is to make inference on the causal effect of the treatment, or more specifically, some distributional aspects of the difference of the potential outcomes \( Y_i(1) - Y_i(0) \). The RDD analysis focuses on the case where the treatment assignment \( T_i \) is completely or partly determined by some observable covariate \( X_i \), called the running variable. For example, to study the effect of class size on pupils’ achievements, it is reasonable to consider the following setup: the unit \( i \) is school, \( Y_i \) is an average exam score, \( T_i \) is an indicator variable for the class size (\( T_i = 0 \) for one class and \( T_i = 1 \) for two classes), and \( X_i \) is the number of enrollments. For more examples, see e.g. Imbens and Lemieux (2008), Cattaneo, Titunik and Vazquez-Bare (2020), Cattaneo and Escanciano (2017), and references therein.

Depending on the assignment rule for \( T_i \) based on \( X_i \), we have two cases, called the sharp and fuzzy RDDs. In this section, we focus on the sharp RDD and discuss the fuzzy RDD in Section 3.1. In the sharp RDD, the treatment is deterministically assigned as \( T_i = I\{X_i \geq \bar{x}\} \), where \( I\{\cdot\} \) is the indicator function and \( \bar{x} \) is a known discontinuity (cutoff) point. Throughout the paper, we normalize \( \bar{x} = 0 \) to simplify the presentation. A parameter of interest, in this case, is the average causal effect at the discontinuity point:

\[
\tau = E[Y_i(1) - Y_i(0)|X_i = 0].
\]  

Since the difference \( Y_i(1) - Y_i(0) \) is unobservable, we need a tractable representation of \( \tau \) in terms of quantities that can be estimated by data. If the conditional mean functions \( E[Y_i(1)|X_i = x] \) and \( E[Y_i(0)|X_i = x] \) are continuous at the cutoff point \( x = 0 \), then the average causal effect \( \tau \)
can be identified as a contrast of the left and right limits of the conditional mean $\mathbb{E}[Y_i|X_i = x]$ at $x = 0$, that is

$$\tau = \lim_{x \downarrow 0} \mathbb{E}[Y_i|X_i = x] - \lim_{x \uparrow 0} \mathbb{E}[Y_i|X_i = x]. \quad (2)$$

As argued in CCFT, it is usually the case that practitioners have access to additional covariates (denoted by $Z_i \in \mathbb{R}^p$) and augment their empirical models with $Z_i$ to estimate the causal effect $\tau$ of interest. This practically relevant setup is extensively studied in CCFT for the case where $Z_i$ is low-dimensional. In this paper, we consider the case of possibly high-dimensional $Z_i$, and propose a new point estimation method for $\tau$ and an adjustment of CCFT’s inference method.

We examine the case where the additional covariates $Z_i$ are predetermined in the sense that $Z_i = Z_i(0) \cdot (1 - T_i) + Z_i(1) \cdot T_i$ but $Z_i(0) =_{d} Z_i(1)$ for the potential covariates $Z_i(0)$ and $Z_i(1)$ for $T_i = 0$ and $T_i = 1$, respectively. Motivated by CCFT’s recommended model (in their eq. (2)), we propose the local Lasso estimator $\hat{\theta} = (\hat{\alpha}, \hat{\tau}, \hat{\beta}_-, \hat{\beta}_+)$ that solves

$$\min_{\alpha, \tau, \beta_-, \beta_+, \gamma} \frac{1}{n} \sum_{i=1}^{n} K \left( \frac{X_i}{b_n} \right) (Y_i - \alpha - T_i \tau - X_i \beta_- - T_i X_i \beta_+ - Z_i' \gamma)^2 + \lambda_n |\gamma|_1, \quad (3)$$

where $|\gamma|_1 = \sum_{j=1}^{p} |\gamma_j|$ is the $\ell_1$-norm of $\gamma$, $\gamma_j$ means the $j$-th element of $\gamma$, $K(\cdot)$ is a kernel function, $b_n$ is a bandwidth, and $\lambda_n$ is a penalty level. Popular choices for $K(\cdot)$ are the uniform and triangular kernels supported on $[-b_n, b_n]$. Based on (3), our point estimator for $\tau$ is given by $\hat{\tau}$.

Our preliminary simulation results suggest that the local Lasso estimator for $\tau$ is somewhat biased in finite samples. Therefore, our recommendation for point estimation is to employ a post-selection method. Let $\hat{S} = \{j : |\hat{\gamma}_j| \geq \zeta_n\}$ for a non-negative sequence $\{\zeta_n\}$, and $Z_{\hat{S},i}$ be a subvector of $Z_i$ selected by $\hat{S}$. Then the local post-Lasso estimator $\hat{\theta} = (\hat{\alpha}, \hat{\tau}, \hat{\beta}_-, \hat{\beta}_+ \tau')$ is defined as a solution of the local least square:

$$\min_{\alpha, \tau, \beta_-, \beta_+} \frac{1}{n} \sum_{i=1}^{n} K \left( \frac{X_i}{h_n} \right) (Y_i - \alpha - T_i \tau - X_i \beta_- - T_i X_i \beta_+ - Z_{\hat{S},i}')^2, \quad (4)$$

where $h_n$ is another bandwidth, and the estimator for $\tau$ is given by $\hat{\tau}$.

Several points are worthy of remark for this estimator. First, without the $\ell_1$-penalization, our estimator reduces to the local linear-type estimator recommended by CCFT’s eq. (2). Therefore, the proposed estimator is a natural generalization of CCFT’s when the dimension of $Z_i$ is high. Second, without the kernel weights for localization, our estimator in (3) reduces to the conventional Lasso estimator. However, since our parameter of interest $\tau$ is identified as a local object.
in (2), it is crucial to introduce such localization to avoid misspecification bias of the conditional mean functions. Third, it is often the case that the kernel function $K(\cdot)$ has bounded support. In this case, the effective sample size would be typically of orders $nb_n$ and $nh_n$. Thus even if the dimension of $Z_i$ is relatively small compared to the original sample size $n$, the $\ell_1$-penalization would be useful especially for small values of $b_n$ and $h_n$. Finally, the trimming term $\zeta_n$ to obtain the set $\hat{S}$ is introduced to stabilize numerical results (see (7) below for our recommended choice based on simulation studies), and theoretically we may set as $\zeta_n = 0$.

We now present risk properties of the local Lasso estimators $\hat{\theta}$ and $\bar{\theta}$. Let $G_i = (1, T_i, X_i, T_iX_i, Z_i)'$ be the vector of regressors in (3), $G_{i,j}$ be the $j$-th element of $G_i$, and $\Theta_n = \arg \min_{\theta} E[K(X_i/b_n)(Y_i - G_i^T\theta)^2]$ be an argmin set. We impose the following assumptions.

**Assumption 1.** There exists a sequence $\theta_n^* \in \Theta_n$ that satisfies the following conditions.

1. Let $\epsilon_i = \sqrt{K(X_i/b_n)(Y_i - G_i^T\theta_n^*)}$. There exists some $C \in (0, \infty)$ such that
   \[
   E[|K(X_i/b_n)G_{i,j}\epsilon_i|^m] \leq b_n m!C^{m-2}/2,
   \]
   for all $j = 1, \ldots, p$ and $m = 2, 3, \ldots$.

2. Let $\delta_A$ be the subvector of $\delta$ for an index set $A$, $S^* = \{j : \theta_{n,j}^* \neq 0\}$, and $(S^*)_c = \{j : \theta_{n,j}^* = 0\}$, where $\theta_{n,j}^*$ is the $j$-th element of $\theta_n^*$. There exists some $\phi^* \in (0, \infty)$ such that
   \[
   \frac{s^*}{|\delta_{S^*}|_1^2} \cdot \min_{\delta' \in [S^*)_c]_{\leq 3}} \delta' \left( \frac{1}{nb_n} \sum_{i=1}^n K \left( \frac{X_i}{b_n} \right) G_i G_i' \right) \delta \geq (\phi^*)^2,
   \]
   with probability approaching one, where $s^* = |S^*|$.

**Assumption 2.** $K : \mathbb{R} \to \mathbb{R}$ is a bounded and symmetric second-order kernel function which is continuous with a compact support. The bandwidth $b_n$ is a positive sequence satisfying $b_n \to 0$ and $nb_n \to \infty$ as $n \to \infty$.

Assumption 1 defines $\theta_n^*$ as an approximate linear predictor or the linear projection on the set of included variables in the index set $S^*$ since $E[K(X_i/b_n)G_{i,j}\epsilon_i] = 0$ for all $j \in S^*$. This assumption is general enough to cover the setup in CCFT, which assumes $p$ is fixed. In RDD analyses, it is common to introduce many generated covariates, such as transformations of initial covariates like polynomials, interactions, and various basis functions, without knowing which of them are relevant a priori.\(^2\)

\(^2\)To motivate the use of generated covariates, it is insightful to note that the asymptotic variance of CCFT’s
Although it is beyond the scope of this paper, the definition of $\theta_n^*$ could be modified to be an approximate minimizer which does not belong to $\Theta_n$ but gets closer to it at some suitable rate. Since it complicates the exposition and derivation as in Kreiš and Rothe (2023) or Belloni, Chernozhukov and Hansen (2014), we maintain this exact sparsity assumption. For example, such an extension for approximate sparsity will be useful to allow the situation where the conditional mean satisfies $E[Y|X,Z] = E[Y|X,Z_S]$ for some sparse set $S$ but the conditional mean function $E[Y|X,Z_S]$ is nonlinear in $Z_S$ so that the exact sparsity assumption typically fails.

Assumption 1 (1) contains a set of moment conditions, which are introduced to verify a local-type Bernstein inequality in Lemma A.1. The extra factor $b_n$ is due to the presence of the kernel weight $K(X_i/b_n)$. Assumption 1 (2) is a localized version of the compatibility condition.

A sufficient condition for this is the so-called restricted eigenvalue condition. More specifically, $\min_{\|\beta\|_0 \leq s^*} \sum_{i=1}^n K \left( \frac{X_i}{n} \right) \frac{\beta G_i \beta_G}{\beta_0}$, where $\|\beta\|_0$ denotes the cardinality of $\beta$, provides a lower bound for the compatibility constant $(\phi^*)^2$ (see, e.g., Section 6.13 of Bühlmann and van der Geer, 2011). If CCFT’s method is feasible with each subset of covariates of dimension $2s^*$, then the restricted eigenvalue condition is indeed satisfied. Assumption 2 contains assumptions on the kernel $K$ and bandwidth $b_n$, which are standard in the literature of nonparametric methods. Note that since $\theta_n^*$ and $S^*$ depend on $b_n$, Assumption 1 should be satisfied along each sequence $\{b_n\}$. Also, the deviation bounds on the prediction and estimation errors of $\hat{\theta}$ will be given as functions of $s^*$. While a precise condition on $s^*$ is hard to specify and depends on the sampling distribution, it would be typically smaller order than $\sqrt{n b_n}$ to satisfy the compatibility condition in Assumption 1 (2).

Let $\hat{\gamma}_{S,i}$ be the subvector of $\hat{\gamma}$ selected by $\hat{S}$, $\hat{\theta} = (\hat{\alpha}, \hat{\tau}, \hat{\beta}_{-}, \hat{\beta}_{+}, \hat{\gamma}_{S,i}', Z_{S,i})'$, $Z_{S,i}$ be the subvector of $Z_i$ selected by $\hat{S}$, $G_{S,i} = (1, T_i, X_i, T_i X_i, Z_{S,i}')'$, and $m_n = \lambda_{\min} \left( \frac{1}{n b_n} \sum_{i=1}^n K(X_i/h_n) G_{S,i} G_{S,i}' \right)^{-1}$, where $\lambda_{\min}(A)$ means the minimum eigenvalue of a matrix $A$. Also let $\hat{S}_1 = \{i : 0 < |\hat{\gamma}_i| < \zeta_n\}$, $S_n = \hat{S} \cup \hat{S}_1 = \{i : \hat{\gamma}_i \neq 0\}$, $Z_{S_1,i}$ be the subvector of $Z_i$ selected by $\hat{S}_1$, $G_{S_1,i} = (1, T_i, X_i, T_i X_i, Z_{S_1,i}')'$, and $m_{1n} = \lambda_{\max} \left( \frac{1}{n b_n} \sum_{i=1}^n K(X_i/h_n) G_{S_1,i} G_{S_1,i}' \right)$, where $\lambda_{\max}(A)$ means the maximum eigenvalue of a matrix $A$. The $\ell_1$-risk properties of the local Lasso and post-Lasso estimators (for the case of $h_n = b_n$) are obtained as follows.

**Theorem 1.** Suppose $\lambda_n^{-1} \sqrt{\log p/(nb_n)} \to 0$.

RDD estimator is proportional to $\text{Var}(\{Y_i(1) - Y_i(0)\} - (Z_i(1) - Z_i(0))\gamma) \{X_i = 0\}$ for some $\gamma$, which is considered as the (conditional) variance of the linear projection error. Although CCFT considered linear projection due to the constraint on dimensionality, it is clear that the asymptotic variance is minimized by employing the conditional expectation $E[Y(1) - Y(0)|Z_i(1) - Z_i(0), X_i = 0]$ instead of the linear projection $(Z_i(1) - Z_i(0))\gamma$. Therefore, it is natural to extend CCFT’s approach to high-dimensional settings by employing generated covariates or series approximation for the conditional mean.
Under Assumptions 1-2, it holds
\[ |\hat{\theta} - \theta^*_n|_1 \leq C \frac{\lambda_n s^*}{\phi^*}, \tag{5} \]
for some \( C \in (0, \infty) \) with probability approaching one.

Under Assumptions 1-2 and \( h_n = b_n \), it holds
\[ |\bar{\theta}_{S_n} - \hat{\theta}_{S_n}|_1 \leq (m_n |S_n| \lambda_n) \lor (|\hat{S}_{1}| \zeta_n) \lor (m_{1n} |\hat{S}_{1}| \zeta_n^2 / \lambda_n). \tag{6} \]

The proof of this theorem is presented in Appendix A.1. This theorem characterizes the risk properties of the estimators \( \hat{\theta} \) and \( \bar{\theta} \) around \( \theta^*_n \). The risk bound of \( \hat{\theta} \) depends on the tuning parameter \( \lambda_n \), the number of non-zero coefficients \( s^* \), and the compatibility constant \( \phi^* \). Note that the decay rate of \( \lambda_n \) is bounded from below by \( \sqrt{\log p/(nb_n)} \). Thus, the risk bound of \( \hat{\theta} \) gets worse as the number of covariates \( p \) increases or the effective sample size \( nb_n \) due to the kernel localization decreases. The result \( (5) \) for the post-selection estimator \( \hat{\theta} \) shows that the deviation from the original Lasso estimator \( \hat{\theta} \) is small when tuning parameter \( \lambda_n \) or the number of selected covariates \( |S_n| \) is small, or the minimum eigenvalue of \( \frac{1}{nh_n} \sum_{i=1}^{n} K(X_i/b_n)G_{\hat{S},i}G_{\hat{S},i}' \) is large. If the trimming parameter \( \zeta_n \) is of similar magnitude of \( \lambda_n \), then the three terms in the bounds are of similar magnitude. If \( \zeta_n \) is smaller order of magnitude than \( \lambda_n \), then the first term will dominate. We suggest some practical choice of the trimming term \( \zeta_n \) in Section 5 based on our simulation studies.

The above theorem is on estimation of the coefficients of the best linear predictor \( \theta^*_n \) defined in Assumption 1 (1). Additionally suppose that the assumptions of Lemma 1 of CCFT hold true, and the covariates \( Z_i \) are predetermined. Then we can guarantee that the second element of \( \theta^*_n \) coincides with the average causal effect \( \tau \) in (1) so that Theorem 1 provides the conditions for the consistency and convergence rate of \( \hat{\tau} \) to \( \tau \). If \( Z_i \) are not predetermined (i.e., \( Z_i(0) \neq d Z_i(1) \)), then \( \hat{\tau} \) typically converges to \( \tau \) minus some bias component, which is obtained as a limit of CCFT’s bias term in their Lemma 1.

Our estimators and above theorem can be extended to other regression models that contain the covariates \( \{T_iZ_i, (1-T_i)Z_i\}, (Z_i - \bar{Z}), \) or \( \{T_i(Z_i - \bar{Z}), (1-T_i)(Z_i - \bar{Z})\} \) as in CCFT. However, as shown in Lemma 1 of CCFT, such estimators require more stringent conditions to guarantee the consistency for \( \tau \). Furthermore, the local Lasso regression \( (3) \) can be extended to incorporate polynomials of \( X_i \) and \( T_iX_i \) even though this paper focuses on the local linear model.

Finally, we discuss the choices of the localization bandwidths \( b_n \) and \( h_n \) and regularization parameter \( \lambda_n \). We can use the MSE-optimal bandwidth based on the suggestion by CCFT.
and the regularization parameter $\lambda_n$ using cross-validation by Friedman, Hastie and Tibshirani (2010) or a data-driven choice by Belloni, Chernozhukov and Hansen (2014) among others. See Theorem 2 in the next subsection for their justification, and Section 4 for a detail on our practical recommendation.

### 2.2 Inference

We next consider interval estimation and hypothesis testing on the average causal effect $\tau$. For finite or low-dimensional $Z_i$, we recommend to use CCFT’s bias corrected inference method. This subsection argues that we can still apply CCFT’s inference procedure for high-dimensional $Z_i$, provided that CCFT’s conditions remain valid for $S^*$ and the subvector $\theta^*_{n,S}$ of $\theta^*_n$ selected by $S^*$.

In this subsection, we specify the tuning constant $\zeta_n$ to obtain $\hat{S} = \{j : |\hat{\gamma}_j| \geq \zeta_n\}$ as

$$\zeta_n = \lambda_n g_n \sum_{j=1}^p \mathbb{E} \{\hat{\gamma}_j \neq 0\},$$

where we set $g_n = \log \log \log n$. This choice of $g_n$ is based on the simulation experiments in Section 5 and it is not shown to be optimal but works reasonably well. Based on the selected covariates by $\hat{S}$ with $\zeta_n$ in (7), we apply CCFT’s bias corrected t-ratio to conduct statistical inference on the causal effect parameter $\tau$.

Consider the local post-Lasso estimator $\hat{\tau}$ defined by (4). As shown in Appendix A.2, the dominant term of $\hat{\tau}$ can be characterized as

$$\hat{\tau} = e_2' \left( \frac{1}{nh_n} \sum_{i=1}^n K \left( \frac{X_i}{h_n} \right) G_{1i} G_{1i}' \right)^{-1} \frac{1}{nh_n} \sum_{i=1}^n K \left( \frac{X_i}{h_n} \right) G_{1i} \xi_i + o_p((nh_n)^{-1/2}),$$

where $e_2 = (0, 1, 0, 0)'$, $G_{1i} = (1, T_i, X_i, T_iX_i)'$, $\xi_i = T_i \xi_i(1) + (1 - T_i) \xi_i(0)$ with $\xi_i(t) = Y_i(t) - Z_i(t)\gamma_Y$, and

$$\gamma_Y = \arg \min_{\gamma: \gamma_j = 0 \text{ for } j \notin S^*} \mathbb{E}[(\hat{Y} - Z'\gamma)^2|X = 0],$$

with $\hat{Y} = Y(1) - Y(0) - \mathbb{E}[Y(1) - Y(0)|X = 0]$. Indeed, the asymptotic linear form in (8) is analogous to the one derived for the case of fixed dimensional $Z_i$ in CCFT (except that $\gamma_{Y,j} = 0$ for $j \notin S^*$). Therefore, the pre-asymptotic bias and variance of $\hat{\tau}$ can be analogously written as

$$B = \frac{1}{2} e_2' \Gamma^{-1} \partial_- (q' \mu_-^{(2)}) + \frac{1}{2} e_2' \Gamma^{-1} \partial_+ (q' \mu_+^{(2)}),$$

$$V = (q \otimes P' e_1)' \Sigma_- (q \otimes P' e_1) + (q \otimes P'_+ e_1)' \Sigma_+ (q \otimes P'_+ e_1),$$

where $\Sigma_- = \mathbb{E}[\partial_- (Z' \gamma)^2|X = 0]$, $\Sigma_+ = \mathbb{E}[\partial_+ (Z' \gamma)^2|X = 0]$. This suggests that $\hat{\tau}$ is approximately normally distributed with mean $\tau$ and variance $V$.
respectively, where \( e_1 = (1, 0)', \) \( R = \begin{bmatrix} 1 & \cdots & 1 \\ X_1/h_n & \cdots & X_n/h_n \end{bmatrix}', q = (1, -\gamma')', and

\[
K_- = h_n^{-1} \text{diag}([\{X_1 < 0\}K(X_1/h_n), \ldots, \{X_n < 0\}K(X_n/h_n)],
\]

\[
K_+ = h_n^{-1} \text{diag}([\{X_1 \geq 0\}K(X_1/h_n), \ldots, \{X_n \geq 0\}K(X_n/h_n)],
\]

\[
\Gamma_- = n^{-1} R' K_- R, \quad \Gamma_+ = n^{-1} R' K_+ R,
\]

\[
\vartheta_- = n^{-1} R' K_- [X^2_1/h_n^2, \ldots, X^2_n/h_n^2]', \quad \vartheta_+ = n^{-1} R' K_+ [X^2_1/h_n^2, \ldots, X^2_n/h_n^2]',
\]

\[
\mu_-^{(2)} = \left[ \frac{\partial^2 \mathbb{E}[Y_i(0)|X_i = x]}{\partial x^2} \right]_{x=0}', \quad \mu_+^{(2)} = \left[ \frac{\partial^2 \mathbb{E}[Z_i^*(0)|X_i = x]}{\partial x^2} \right]_{x=0}',
\]

\[
P_- = \sqrt{\frac{\gamma_-}{n}} R' K_-, \quad P_+ = \sqrt{\frac{\gamma_+}{n}} R' K_+,
\]

\[
\Sigma_- = \text{Var}(\text{vec}(\mathbf{Y}(0), \mathbf{Z}^*(0)|\mathbf{X} = 0)), \quad \Sigma_+ = \text{Var}(\text{vec}(\mathbf{Y}(1), \mathbf{Z}^*(1)|\mathbf{X} = 0),
\]

with \( \mathbf{Y}(0) = (Y_1(0), \ldots, Y_n(0))', \mathbf{Y}(1) = (Y_1(1), \ldots, Y_n(1))', \mathbf{X} = (X_1, \ldots, X_n)', \mathbf{Z}^*(0) = (Z_{S,1}(0), \ldots, Z_{S,n}(0))', \) and \( \mathbf{Z}^*(1) = (Z_{S,1}(1), \ldots, Z_{S,n}(1))'. \)

By estimating the unknown components, the pre-asymptotic bias and variance can be estimated as

\[
\tilde{B} = \frac{1}{2} e_1' \Gamma_-^{-1} \vartheta_- (\tilde{\mu}_-^{(2)}) + \frac{1}{2} e_1' \Gamma_+^{-1} \vartheta_+ (\tilde{\mu}_+^{(2)}),
\]

\[
\tilde{V} = (q \otimes P'_1 e_1)' \Sigma_- (q \otimes P'_1 e_1) + (q \otimes P'_1 e_1)' \Sigma_+ (q \otimes P'_1 e_1),
\]

respectively, where \( \tilde{q}' = (1, -\gamma')' \), \( \tilde{\mu}_-^{(2)} \) and \( \tilde{\mu}_+^{(2)} \) are local polynomial estimators of \( \mu_-^{(2)} \) and \( \mu_+^{(2)} \) for the elements corresponding to \( Z_{S,i} \), respectively, and \( \tilde{\Sigma}_- \) and \( \tilde{\Sigma}_+ \) are conditional variance estimators of \( \Sigma_- \) and \( \Sigma_+ \), respectively, such as the nearest neighborhood or plug-in estimator in Section 7.9 of CCFT’s supplement. Based on these estimators, the t-ratio for \( \tau \) is obtained as

\[
T_\tau = \frac{\tilde{\tau} - h_n^2 \tilde{B} - \tau}{\sqrt{(nh_n)^{-1} \tilde{V}}},
\]

which is exactly the same as the t-ratio in Theorem 2 of CCFT but using the selected covariates \( Z_{S,i} \). By extending the theoretical developments in CCFT, we obtain the following result.

**Theorem 2.** Suppose Assumptions \( \square \) hold true. Suppose for all \( x \) in a neighborhood of 0 and \( t = 0, 1 \), the density of \( X_i \) is continuous and bounded away from zero, \( \mathbb{E}[|Y_i(t)|, Z_i(t)'|X_i = x] \) is three times continuously differentiable, \( \partial^2 \mathbb{E}[Z_i(0)|X_i = x]/\partial x^2 = \partial^2 \mathbb{E}[Z_i(1)|X_i = x]/\partial x^2, \)
\[ \mathbb{E}[Z_i(t)Y_i(t)|X_i = x] \text{ is continuously differentiable, } \text{Var}((Y_i(t), Z_i(t'))|X_i = x) \text{ is continuously differentiable and invertible, and } \mathbb{E}[(Y_i(t), Z_i(t'))^4|X_i = x] \text{ is continuous. For all } j, k \text{ and positive integers } m, \text{ and some finite } C, \mathbb{E}[|K(X_i/h_n)G_{1i,j}Z_{i,k}|^m] \leq h_n m! C^{m-2}/2. \]

Finally, assume \( h_n \to 0, nh_n \to \infty, \lambda_n^{-1} \sqrt{\log p/(nb_n)} \to 0, \) and \( (\sqrt{\log p} + \sqrt{nh_n} h_n^2) (h_n^2 + b_n^2 + \lambda_n + \zeta_n + \xi_n^2/\lambda_n) s* \to 0. \) Then the conditional MSE expansion of the first term in (8) (denoted by \( \hat{\tau}_n \)) is obtained as

\[ \mathbb{E}[(\hat{\tau}_n - \tau)^2|X] = h_n^2 B^2 \{1 + o_p(1)\} + \frac{1}{nh_n} \mathcal{V}. \quad (13) \]

Furthermore, if we additionally assume \( \sqrt{nh_n^2 (B - B)} \xrightarrow{p} 0 \) and \( \sqrt{n} \xrightarrow{p} 1, \) then

\[ T_\tau \xrightarrow{d} \mathcal{N}(0, 1). \quad (14) \]

The results in (13) and (14) are analogous to CCFT’s Theorems 1 and 2, respectively. This theorem theoretically supports to employ the bias correction and bandwidth selection methods by CCFT based on the selected covariates \( Z_{S,i}. \) See Section 4 below for our practical recommendation. The assumption \( \partial^2 \mathbb{E}[Z_i(0)|X_i = x]/\partial x^2 = \partial^2 \mathbb{E}[Z_i(1)|X_i = x]/\partial x^2 \) is natural for predetermined covariates but may be relaxed by introducing additional regularity conditions (see, Kreiß and Rothe, 2023). Other assumptions except for the last one are also imposed in CCFT. The assumption \( (\sqrt{\log p} + \sqrt{nh_n} h_n^2) (h_n^2 + b_n^2 + \lambda_n + \zeta_n + \xi_n^2/\lambda_n) s* \to 0 \) is used to control the remainder term in (8), and can be considered as a sparsity assumption to restrict the growth rate of \( s* \) \(^3\) Although the conditions \( \sqrt{nh_n^2 (B - B)} \xrightarrow{p} 0 \) and \( \sqrt{n} \xrightarrow{p} 1 \) are high level, these are typically satisfied for the bias and variance estimators discussed in CCFT \(^4\) See Remark 4 below for a specific example of the variance estimator \( \mathcal{V}. \)

**Remark 1.** [Efficiency comparison] It should be noted that the asymptotic variance \( \mathcal{V} \) in (10) of the estimator \( \hat{\tau} \) (or the bias corrected version \( \hat{\tau} - h_n^2 B \)) takes the same form as the one in CCFT even when the dimension of \( Z_{S,i} \) grows as \( n \) increases. As investigated in CCFT, we can see that the relative efficiency of \( \hat{\tau} \) compared to the conventional RDD estimator without covariates (say,

\(^3\)In the standard Lasso literature, we typically impose \( s* \sqrt{\log p/n} \xrightarrow{p} 0 \) and the minimal penalty level requirement on \( \lambda_n. \) For comparison, consider the following standard setting for tuning parameters, where \( b_n \sim h_n \sim n^{-1/5}, \lambda_n = a_n \sqrt{\log p/(nb_n)} \) with a slowly diverging \( a_n \) and \( \zeta_n = O(\lambda_n). \) Then the condition on \( s* \) reduces to \( s* \sqrt{\log p/n} \xrightarrow{p} 0, \) which is analogous to the standard case.

\(^4\)Under the assumption \( \partial^2 \mathbb{E}[Z_i(0)|X_i = x]/\partial x^2 = \partial^2 \mathbb{E}[Z_i(1)|X_i = x]/\partial x^2, \) the components \( \mu_{\gamma}^{(2)} \) and \( \mu_{\gamma}^{(2)} \) in \( \mathcal{B} \) become \( \mu_{\gamma}^{(2)} = \partial^2 \mathbb{E}[Y_i(0)|X_i = x]/\partial x^2 \big|_{x=0} \) and \( \mu_{\gamma}^{(2)} = \partial^2 \mathbb{E}[Y_i(1)|X_i = x]/\partial x^2 \big|_{x=0}, \) respectively. Thus, in this case, the convergence rates of the conventional local polynomial estimators for \( \mu_{\gamma}^{(2)} \) and \( \mu_{\gamma}^{(2)} \) guarantee \( \sqrt{nh_n^2 (B - B)} \xrightarrow{p} 0 \) (see, e.g., Fan and Gijbels, 1992, and Ruppert and Wand, 1994).
\[ \frac{\mathcal{V}}{\mathcal{V}_{\text{unadjusted}}} = \frac{\sum_{t=0}^{1} \text{Var}(Y_i(t) - Z_t(t)\gamma Y|X_i = 0)}{\sum_{t=0}^{1} \text{Var}(Y_i(t)|X_i = 0)}, \]  

where \( \mathcal{V}_{\text{unadjusted}} \) is the asymptotic variance of \( \hat{\tau}_{\text{unadjusted}} \) and \( \gamma Y \) is defined in [9]. Generally there is no clear ranking for these asymptotic variances. However, letting \( \gamma Y, S^* \) be an \( s^* \)-dimensional subvector of \( \gamma Y \) selected by \( S^* \), in an important special case where

\[ \gamma Y, S^* = \text{Var}(Z_{S^*},i(t)|X_i = 0)^{-1}\mathbb{E}[(Z_{S^*},i(t) - \mathbb{E}[Z_{S^*},i(t)|X_i])Y_i(t)|X_i = 0] \quad \text{for } t = 0 \text{ and } 1, \]

the coefficient vector \( \gamma Y, S^* \) becomes the best linear approximation by \( Z_{S^*}(t) \) for each group and thus \( \hat{\tau} \) is asymptotically more efficient than \( \hat{\tau}_{\text{unadjusted}} \).

The relative efficiency in (15) is also insightful to illustrate the merit of our Lasso approach compared to CCFT. Let \( Z = [Z_{S^*} : Z_{S^*}] \) and \( Z_C \) be covariates employed to apply the CCFT estimator \( \hat{\tau}_{\text{ccft}} \) (without the Lasso covariates selection). Consider the special case in [16]. As far as \( Z_C \) contains \( Z_{S^*} \), \( \hat{\tau} \) and \( \hat{\tau}_{\text{ccft}} \) achieve the same asymptotic efficiency \( \mathcal{V} \). On the other hand, if \( Z_C \) does not contain some elements of \( Z_{S^*} \), then under (16), the CCFT estimator \( \hat{\tau}_{\text{ccft}} \) is asymptotically less efficient than the post-Lasso estimator \( \hat{\tau} \). Therefore, when researchers are less certain whether all elements of \( Z_{S^*} \) are included in \( Z_C \) typically due to too many candidates in \( Z \) or too small effective sample sizes used for estimation, our Lasso-based approach may be more attractive to achieve asymptotic efficiency \( \mathcal{V} \) in more broader situations. We emphasize that such an efficiency gain of our estimator \( \hat{\tau} \) can be achieved at the cost of the additional sparsity condition in Assumption [1] which is trivially satisfied when the set of active covariates is unknown but fixed.

**Remark 2.** [Finite sample comparison] Although there is no gain of using \( \hat{\tau} \) instead of \( \hat{\tau}_{\text{ccft}} \) in terms of asymptotic efficiency as far as \( Z_C \) contains \( Z_{S^*} \), the estimator \( \hat{\tau} \) may exhibit better finite sample performance even in such a scenario. To see this point, let \((\hat{\beta}_C', \hat{\gamma}_C')\) be the OLS estimator for the regression of \( K^{1/2}Y \) on \( K^{1/2}(1, T, X, TX) \) and \( K^{1/2}Z_C \) so that \( \hat{\tau}_{\text{ccft}} \) is the second element of \( \hat{\beta}_C \). As can be seen from CCFT’s supplement, one of the remainder terms of \( \sqrt{\frac{nh}{n}}(\hat{\tau}_{\text{ccft}} - \tau) \) involves a linear combination of the estimation error \( \hat{\gamma}_C - (\gamma^{s'}, 0)' \). Since the \( \ell_2 \)-convergence rate of \( \hat{\gamma}_C - (\gamma^{s'}, 0)' \) is typically of order \( \sqrt{\text{dim} Z_C / n} \), this remainder term is of larger order than \( \hat{\gamma} - \gamma Y \), where \((\hat{\beta}^{s'}, \hat{\gamma}^{s'})\) is the OLS estimator for the regression of \( K^{1/2}Y \) on \( K^{1/2}(1, T, X, TX) \) and \( K^{1/2}Z_{S^*} \). Even though these remainder terms do not appear in the first order asymptotic distribution, they contribute to the finite sample behaviors of \( \hat{\tau} \) and \( \hat{\tau}_{\text{ccft}} \). Another finite sample issue we encounter in our simulation study below is that as the dimension
of $Z_C$ increases, the values of the MSE-optimal bandwidth tend to be smaller (due to larger bias estimates but relatively stable variance estimates). Thus the effective sample size used for the RDD estimation tends to be smaller so that we observe larger variations in the resulting RDD estimates across simulation draws. Finally, our simulation results in Section 5 (particularly DGPs 2 and 3 with large $p$) illustrate that the covariate selection approach exhibits smaller standard deviations than CCFT in finite samples even when the asymptotic variances may be equivalent.

Remark 3. [Selection consistency] Although Theorem 2 is our main result on inference of the causal effect $\tau$, it is also possible to derive the consistency of the selection procedure (i.e., $\mathbb{P}\{\hat{S} = S^*\} \rightarrow 1$) under some additional $\beta$-min type condition (i.e., there exists some $\varepsilon > 0$ such that $|\gamma_j^*| > \lambda_n \delta_n s^*(1 + \varepsilon)$ for each $j \in S^*$). See a working paper version of this paper (Arai, Otsu and Seo, 2021) for more details on the selection consistency.

Remark 4. [Estimation of $\mathcal{V}$] An example of the estimator $\hat{V}$ for the asymptotic variance $\mathcal{V}$ is the nearest neighborhood estimator, which is employed in CCT, CCFT, and our numerical illustrations. Let

$$\bar{\varepsilon}_{V-,i} = I\{X_i < 0\} \sqrt{\frac{J}{J+1}} \left( V_i - \frac{1}{J} \sum_{j=1}^{J} V_{\ell_-,j(i)} \right),$$

$$\bar{\varepsilon}_{V+,i} = I\{X_i \geq 0\} \sqrt{\frac{J}{J+1}} \left( V_i - \frac{1}{J} \sum_{j=1}^{J} V_{\ell_+,j(i)} \right),$$

for $V \in \{Y, Z_1, \ldots, Z_p\}$ and a fixed positive integer $J$, where $\ell_-(i)$ is the index of the $j$-th closest unit to unit $i$ among $\{i : X_i < 0\}$, and $\ell_+(i)$ is the index of the $j$-th closest unit to unit $i$ among $\{i : X_i \geq 0\}$. The nearest neighborhood estimators of $\Sigma_-$ and $\Sigma_+$ in (11) are defined as

$$\Sigma_{-} = \begin{bmatrix} \Sigma_{YY} & \Sigma_{YZ_1} & \cdots \Sigma_{YZ_{|S|-}} \\ \Sigma_{Z_1Y} & \Sigma_{Z_1Z_1} & \cdots \Sigma_{Z_1Z_{|S|-}} \\ \vdots & \vdots & \ddots \vdots \\ \Sigma_{Z_{|S|-}Y} & \Sigma_{Z_{|S|-}Z_1} & \cdots \Sigma_{Z_{|S|-}Z_{|S|-}} \end{bmatrix},$$

$$\Sigma_{+} = \begin{bmatrix} \Sigma_{YY} & \Sigma_{YZ_1} & \cdots \Sigma_{YZ_{|S|}} \\ \Sigma_{Z_1Y} & \Sigma_{Z_1Z_1} & \cdots \Sigma_{Z_1Z_{|S|}} \\ \vdots & \vdots & \ddots \vdots \\ \Sigma_{Z_{|S|}Y} & \Sigma_{Z_{|S|}Z_1} & \cdots \Sigma_{Z_{|S|}Z_{|S|}} \end{bmatrix},$$

where $\Sigma_{YW_-}$ and $\Sigma_{YW_+}$ are $n \times n$ matrices whose $(i,j)$-th elements are

$$[\Sigma_{YW_-},i,j] = I\{X_i < 0\} I\{X_j < 0\} I\{i = j\} \bar{\varepsilon}_{V-,i} \bar{\varepsilon}_{W-,j},$$

$$[\Sigma_{YW_+},i,j] = I\{X_i \geq 0\} I\{X_j \geq 0\} I\{i = j\} \bar{\varepsilon}_{V+,i} \bar{\varepsilon}_{W+,j},$$

for $i, j = 1, \ldots, n$ and $V, W \in \{Y, Z_1, \ldots, Z_p\}$. Then nearest neighborhood estimator of $\mathcal{V}$ is
defined as
\[ \hat{V}_{NN} = (\bar{q} \otimes P'_1 e_1)' \Sigma_{NN}^{-1} \bar{q} \otimes P'_1 e_1 + (\bar{q} \otimes P'_+ e_1)' \Sigma_{NN}^{+1} \bar{q} \otimes P'_+ e_1. \]

By adapting the proof of Section S.2.4 in the supplement of CCT, the consistency of this variance estimator is obtained as follows.

**Proposition 1.** Suppose that the assumptions of Theorem 2 hold true. Additionally assume that Var((Y_i(t), Z_i(t)'|X_i = x) is Lipschitz continuous in a neighborhood of 0, and

\[ \varpi_n^{-1} \{(m_n|S_n|\lambda_n) \lor (|\hat{S}_1|\zeta_n) \lor (m_{1n}|\hat{S}_1|\zeta_n^2/\lambda_n)\} \xrightarrow{p} 0, \]

\[ (\varpi_n n h_n)^{-1} s^* \sqrt{\log s^*} \xrightarrow{p} 0, \quad \zeta_n^{-1} \lambda_n s^* \rightarrow 0, \]

where \( \varpi_n \) is defined in (30). Then \( \frac{\hat{V}_{NN}}{\hat{V}} \xrightarrow{p} 1. \)

3 Discussion

3.1 Fuzzy RDD

Although the discussion so far focuses on the sharp RDD analysis, it is possible to extend our approach to the fuzzy RDD analysis, where the forcing variable \( X_i \) is not informative enough to determine the treatment \( W_i \) but still affects the treatment probability. In particular, the fuzzy RDD assumes that the conditional treatment probability \( \mathbb{P}\{W_i = 1|X_i = x\} \) jumps at the cutoff point \( \bar{x} \). As in the last section, we normalize \( \bar{x} = 0 \). To define a reasonable parameter of interest for the fuzzy case, let \( W_i(x) \) be a potential treatment for unit \( i \) when the cutoff level for the treatment is set at \( x \), and assume that \( W_i(x) \) is non-increasing in \( x \) at \( x = 0 \). Using the terminology of Angrist, Imbens and Rubin (1996), unit \( i \) is called a complier if her cutoff level is \( X_i \) (i.e., \( \lim_{x \downarrow X_i} W_i(x) = 0 \) and \( \lim_{x \uparrow X_i} W_i(x) = 1 \)). A parameter of interest in the fuzzy RDD, suggested by Hahn, Todd and van der Klaauw (2001), is the average causal effect for compliers at \( x = 0 \),

\[ \tau_f = \mathbb{E}[Y_i(1) - Y_i(0)|i \text{ is complier}, X_i = 0]. \]

Hahn, Todd and van der Klaauw (2001) showed that under mild conditions the parameter \( \tau_f \) can be identified by the ratio of the jump in the conditional mean of \( Y_i \) at \( x = 0 \) to the jump in the conditional treatment probability at \( X_i = 0 \), i.e.,

\[ \tau_f = \frac{\lim_{x \downarrow 0} \mathbb{E}[Y_i|X_i = x] - \lim_{x \uparrow 0} \mathbb{E}[Y_i|X_i = x]}{\lim_{x \downarrow 0} \mathbb{P}\{W_i = 1|X_i = x\} - \lim_{x \uparrow 0} \mathbb{P}\{W_i = 1|X_i = x\}}. \]
In this case, letting \(T_i = I\{X_i \geq 0\}\), the numerator and denominator of (17) can be estimated by the local Lasso estimators \(\hat{\theta}_1 = (\hat{\alpha}_1, \hat{\tau}_1, \hat{\beta}_1, \ldots, \hat{\beta}_1, \hat{\gamma}_1')\) and \(\hat{\theta}_2 = (\hat{\alpha}_2, \hat{\tau}_2, \hat{\beta}_2, \ldots, \hat{\beta}_2, \hat{\gamma}_2')\), which solve

\[
\min_{\alpha_1, \tau_1, \beta_1, \ldots, \beta_1, \gamma_1} \frac{1}{n b_{1n}} \sum_{i=1}^{n} K \left( \frac{X_i}{b_{1n}} \right) (Y_i - \alpha_1 - T_i \tau_1 - X_i \beta_1 - T_i X_i \beta_1 - Z_i' \gamma_1)^2 + \lambda_1 n |\gamma_1|_1,
\]

\[
\min_{\alpha_2, \tau_2, \beta_2, \ldots, \beta_2, \gamma_2} \frac{1}{n b_{2n}} \sum_{i=1}^{n} K \left( \frac{X_i}{b_{2n}} \right) (W_i - \alpha_2 - T_i \tau_2 - X_i \beta_2 - T_i X_i \beta_2 - Z_i' \gamma_2)^2 + \lambda_2 n |\gamma_2|_1,
\]

where \(Z_{1i}\) and \(Z_{2i}\) are predetermined covariates used for the first and second regressions, respectively. Then, based on the selected covariates from the above local Lasso regressions (i.e., \(\tilde{S}_1 = \{j : |\gamma_{1j}| > \zeta_n\}\) with a subvector \(Z_{1\tilde{S}_1}\) of \(Z_{1i}\) selected by \(\tilde{S}_1\), and \(\tilde{S}_2 = \{j : |\gamma_{2j}| > \zeta_n\}\) with a subvector \(Z_{2\tilde{S}_2}\) of \(Z_{2i}\) selected by \(\tilde{S}_2\)), we implement the local least squares as in CCFT:

\[
\min_{\alpha_1, \tau_1, \beta_1, \ldots, \beta_1, \gamma_1} \frac{1}{n h_{1n}} \sum_{i=1}^{n} K \left( \frac{X_i}{h_{1n}} \right) (Y_i - \alpha_1 - T_i \tau_1 - X_i \beta_1 - T_i X_i \beta_1 - Z_i' \gamma_1, t_1)^2,
\]

\[
\min_{\alpha_2, \tau_2, \beta_2, \ldots, \beta_2, \gamma_2} \frac{1}{n h_{2n}} \sum_{i=1}^{n} K \left( \frac{X_i}{h_{2n}} \right) (W_i - \alpha_2 - T_i \tau_2 - X_i \beta_2 - T_i X_i \beta_2 - Z_i' \gamma_2, t_2)^2.
\]

The numerator and denominator of (17) are given by the estimated coefficients of \(\tau_1\) and \(\tau_2\) for the above minimizations, respectively. If the treatment variable \(W\) satisfies analogous conditions for Theorem 1 (by replacing \(W\) with \(Y\)), we expect that analogous results to the sharp RDD case can be established.

### 3.2 Regression kink design

Our high-dimensional method can be extended for the RKDs. For each unit \(i = 1, \ldots, n\), we observe continuous outcome and explanatory variables denoted by \(Y_i\) and \(X_i\), respectively. The RKD analysis is concerned with the following nonseparable model

\[
Y = f(Q, X, U),
\]

where \(U\) is an error term (possibly multivariate) and \(Q = q(X)\) is a continuous policy variable of interest with known \(q(\cdot)\). In general, even though we know the function \(q(\cdot)\), we are not able to identify the treatment effect by the policy variable \(Q\). However, it is often the case that the policy function \(q(\cdot)\) has some kinks (but is continuous). For instance, suppose \(Y\) is duration of unemployment and \(X\) is earnings before losing the job. We are interested in the effect of unemployment benefits \(Q = q(X)\). In many unemployment insurance systems (e.g.,
the one in Austria), \( q(\cdot) \) is specified by a piecewise linear function. In such a scenario, one may exploit changes of slopes in the conditional mean \( \mathbb{E}[Y|X = x] \) to identify a treatment effect of \( Q \). Suppose \( q(\cdot) \) is kinked at 0. Otherwise, we redefine \( X \) by subtracting the kink point \( c \) from \( X \). In particular, Card, et al. (2015) showed that a treatment on treated parameter \( \tau_k = \int \frac{\partial f(q,x,u)}{\partial q} dF_{U|Q=q,X=x}(u) \) is identified as

\[
\tau_k = \frac{\lim_{x \downarrow 0} \frac{d}{dx} \mathbb{E}[Y|X = x] - \lim_{x \uparrow 0} \frac{d}{dx} \mathbb{E}[Y|X = x]}{\lim_{x \downarrow 0} \frac{d}{dx} q(x) - \lim_{x \uparrow 0} \frac{d}{dx} q(x)}.
\]

To estimate \( \tau_k \), we propose the following local Lasso regression

\[
\min_{\alpha, \delta, \beta, \zeta, \eta, \gamma} \frac{1}{nb_n} \sum_{i=1}^{n} K \left( \frac{X_i}{b_n} \right) (Y_i - \alpha - T_i X_i \delta - X_i \beta - T_i X_i^2 \zeta - X_i^2 \eta - Z_i \gamma)^2 + \lambda_n |\gamma|_1. \tag{19}
\]

Let \( \hat{\delta} \) be the Lasso estimator of \( \delta \) by (19). Since the denominator \( q_0 = \lim_{x \downarrow c} \frac{d}{dx} q(x) - \lim_{x \uparrow c} \frac{d}{dx} q(x) \) in (18) is assumed to be known, the estimator of \( \tau_k \) is given by \( \hat{\tau}_k = \hat{\delta} / q_0 \). Under analogous conditions to Theorem 1, we expect that analogous results to the sharp RDD case can be established.

4 Recommendation for Implementation

We recommend the following steps to implement our method in R. The most important difference from CCFT is the covariate selection step (Step 2). This procedure is employed in our numerical studies in the following sections.

1. Without using data on covariates, obtain the MSE-optimal bandwidth \( b_n = b_n^{\text{CCT}} \) developed by CCT. This can be implemented by an R package, \texttt{rdrobust} (Calonico, Cattaneo and Titiunik, 2015b).

2. Using the data on covariates, implement the Lasso estimation in (3) to compute \( \hat{\theta} \) by setting \( b_n = b_n^{\text{CCT}} \) and choosing \( \lambda_n \) by a data-driven choice. This can be implemented by an R package, \texttt{hdm} (https://cran.r-project.org/web/packages/hdm/index.html) for example.

3. Based on the Lasso estimates \( \hat{\theta} \) obtained in Step 2, select the covariates \( Z_{\hat{S},i} \) by using the trimming term \( \zeta_n \) in (7). Then implement CCFT’s RDD estimation by (4) and inference procedure including the bias correction and bandwidth selection. For this step, we recommend to follow the procedures detailed in CCFT’s supplement by using \texttt{rdrobust}.

The MSE-optimal bandwidth with the full set of covariates can be too narrow when the number
of covariates is large relative to the number of observations and many of covariates are not relevant. Given that we do not know the exact identities of \( S^* \) a priori, we instead employ the MSE-optimal bandwidth by CCT in Step 1, which does not include any covariate. This yields more robust results as shown in the simulation in the next section. Furthermore, we note that CCFT’s bandwidth selection in Step 3 is derived under the assumption of fixed dimensional covariates even though the dimension of the selected covariates \( Z_{\hat{S},i} \) may grow. We recommend the above procedure because of its practicality given the available packages. Further theoretical analysis for an optimal choice of the bandwidth parameters and other tuning constants in the present setup is beyond the scope of this paper. We note that the choice described in the above procedure works well for all data generating processes considered in our numerical studies.

5 Simulation

In this section, we conduct simulation experiments to investigate finite sample properties of our covariate selection approach for estimation and inference on the sharp RDD analysis. We consider three simulation designs based on CCFT with introducing additional covariates. The additional covariates are generated based on the simulation designs in Belloni, Chernozhukov and Hansen (2014). Let \( B(a, b) \) be a beta distribution with parameters \( a \) and \( b \). The data generating process (DGP) is specified as follows

\[
Y = \mu_1(X) + \mu_2(Z) + \mu_3(W) + \varepsilon_y, \quad X \sim 2B(2, 4) - 1, \quad Z = \mu_z(X) + \varepsilon_z,
\]

\[
\mu_z(x) = \begin{cases} 
0.49 + 1.06x + 5.74x^2 + 17.14x^3 + 19.75x^5 + 7.47x^5 & \text{for } x < 0, \\
0.49 + 0.61x + 0.23x^2 - 3.46x^3 + 6.43x^4 - 3.48x^5 & \text{for } x \geq 0,
\end{cases}
\]

\[
W = (W_1, \ldots, W_p)' \sim N(0, \Sigma_W) \text{ with } \Sigma(W_I^2) = 1 \text{ and } \text{Cov}(W_I, W_{I_1}) = 0.5^{|I-I_1|}, \text{ and}
\]

\[
\begin{pmatrix} 
\varepsilon_y \\
\varepsilon_z
\end{pmatrix} \sim N(0, \Sigma), \quad \Sigma = \begin{pmatrix} 
\sigma_y^2 & \rho \sigma_y \sigma_z \\
\rho \sigma_y \sigma_z & \sigma_z^2
\end{pmatrix},
\]

with \( \sigma_y = 0.1295 \) and \( \sigma_z = 0.1353 \).

For the functions \( \mu_1, \mu_2, \) and \( \mu_3, \) we consider three cases. For DGP1, we set \( \rho = 0.2692 \), all the coefficients of \( \mu_2(z) \) and \( \mu_3(w) \) to be zero, and

\[
\mu_1(x) = \begin{cases} 
0.48 + 1.27x + 7.18x^2 + 20.21x^3 + 21.54x^4 + 7.33x^5 & \text{for } x < 0, \\
0.52 + 0.84x - 3.00x^2 + 7.99x^3 - 9.01x^4 + 3.56x^5 & \text{for } x \geq 0,
\end{cases}
\]
For DGP2, we set \( \rho = 0.2692 \).

\[
\mu_1(x) = \begin{cases} 
0.36 + 0.96x + 5.47x^2 + 15.28x^3 + 15.87x^4 + 5.14x^5 & \text{for } x < 0, \\
0.38 + 0.62x - 2.84x^2 + 8.42x^3 - 10.24x^4 + 4.31x^5 & \text{for } x \geq 0,
\end{cases}
\]

and \( \mu_3(w) = \sum_{l=0}^{p-1} \pi_l w_l \) with \( \pi_l = 0.2^l \). For DGP3, we set \( \mu_1(x) \) and \( \mu_2(z) \) as in DGP2, and \( \mu_3(w) = \sum_{l=0}^{p-1} \pi_l w_l \) with \( \pi_l = 0.5^l \). DGP1 is the baseline model considered by CCFT which contains no covariates. DGPs 2 and 3 introduce additional covariates to Model 2 of CCFT with different degrees of magnitude in the coefficients of \( z \). For DGP4, we set \( \mu_1(x) \) and \( \mu_2(z) \) as in DGP2, and \( \mu_3(w) = \sum_{l=1}^{p} \pi_l w_l \) with \( \pi_l = 0.2l^{-1} \). DGP4 is considered to see how all the approaches behave when the (approximate) sparsity assumption fails. The sample size is set as \( n = 500 \) for all cases. The number of the covariates \( p \) varies from 5 to 500. The results are based on 1,000 Monte Carlo replications.

Table 1 shows the biases and RMSEs of the following four point estimation methods. For the bandwidth \( h_n \), the first two methods use the MSE-optimal bandwidth without covariates proposed by CCT. The third method uses the MSE-optimal bandwidth with covariates proposed by CCFT. The fourth method, called “Adaptive”, is the bandwidth for our covariate selection approach which uses the MSE-optimal bandwidth without covariates for the covariate selection stage and uses that with covariates in the estimation stage. For estimation methods, the first method uses the standard RDD estimation method without covariates by CCT. The second and the third methods use the RDD estimation with covariates by CCFT. The fourth method uses the RDD estimation with the selected covariates. See Section 4 for more details on the fourth method.

Our findings are summarized as follows. First, the RMSEs of the covariate adjusted estimation get larger irrespective of the bandwidths as the number of covariates increases across all DPGs. These increases in the RMSEs are due to inflated standard errors caused by a large number of covariates. This result clearly indicates the need for covariate selection. Second, the covariate selection approach shows excellent performances for all cases. Both the biases and

---

4Strictly speaking, DGPs 2 and 3 do not satisfy the sparsity assumption (when \( p \) is large) because the coefficients \( \pi_l \) are not exactly equal to zero even for very large \( l \). However, our unreported simulation results show that the variable selection approach produces identical results to Tables 1-4 when the DGPs are modified to satisfy the exact sparsity assumption, where \( \pi_l \) are set to zero for \( l > n^{9/10} \). Because the covariate adjusted approaches are unstable, their results differ but are similar qualitatively.
RMSEs are stable for different values of $p$ for all designs. Finally, all methods work equally well for DGP1, where all the additional covariates are irrelevant. However, for DGP2 and DGP3, we find substantial efficiency loss of the standard method. The results for DGP4 show that the standard and covariate selection approaches preform stably even when the sparsity assumption is not satisfied. However, the covariate adjusted approaches by CCFT perform unstably in terms of the RMSE. Overall, we recommend the covariate selection even for relatively small $p$.

Table 2 reports the number of selected covariates for our covariate selection approach. It is quite natural that the number of selected covariates increases when the number of non-zero coefficients of covariates increases. It is interesting to note that the average number of selected covariates decreases as the number of covariates increase.

Table 3 shows the coverage probabilities and interval lengths of the robust confidence intervals for the causal effect. The nominal coverage level is 0.95. The following points are notable. First, the performances of our covariate selection approach are stable for all DGPs, although the coverage probabilities tend to be a little bit smaller than the nominal level. Second, for the covariate adjusted approaches, the coverage probabilities decrease and the interval lengths get shorter as $p$ increases. Third, the coverage of CCT is more stable and better than other methods, especially for DGP2 and DGP3. However, the average lengths of CCT are substantially longer than the other methods. Fourth, even when the sparsity assumption fails as in DGP4, the standard and covariate selection approaches preform stably but the covariate adjusted approach exhibits erratic behaviors in terms of the coverages and interval lengths. Overall, the covariate selection approach is promising for inference as well since it exhibits robust performances in both coverages and lengths for different numbers of covariates and DGPs.

Finally, Table 4 presents the properties of the MSE-optimal bandwidths. We observe that the MSE-optimal bandwidth without covariates and the adaptive one are very stable while the MSE-optimal bandwidth with covariates shrinks as $p$ increases. This is possibly the main source of the increased RMSE and under-coverages.

We note that the unstable performance of the covariate adjusted approaches when the number of covariates is large can be because their optimal bandwidths are not available for cases where the number of covariates is large. It would be more appropriate to develop optimal bandwidths for the case of high-dimensional covariates and make comparisons, but it is beyond the scope of this paper.
6 Empirical illustration: Head Start data

To illustrate our variable selection approach, we revisit the problem of the Head Start program first studied by Ludwig and Miller (2007) where they investigate the effect of the Head Start program on various outcomes related to health and schooling. The federal government provided grant-writing assistance to the 300 poorest counties based on the poverty index to apply for the Head Start program. This leads to the RDD with the poverty index as a running variable where the cut-off value is set as $\bar{x} = 59.1984$. Ludwig and Miller (2007) conducted their RDD analysis using no covariate, and CCFT examined the impact of the covariance-adjustment. CCFT employed nine pre-intervention covariates from the U.S. Census, which include total population, percentage of population, percentages of black and urban population, and levels and percentages of population in three age groups (children aged 3 to 5, children aged 14 to 17, and adults older than 25). The main finding by CCFT is that the covariate adjusted RDD inference yields shorter confidence intervals while the RDD point estimates remain stable.

An important aspect of the Head start example is that it is unclear which covariates become useful to improve efficiency mainly due to the lack of economic theories behind the problem. We conduct the empirical exercises of CCFT by applying our variable selection approach with two extensions. First, we introduce 36 interaction terms in addition to the nine original covariates. Second, we also implement those estimation and inference for subsamples to see the effect of changes in the ratio of the number of covariates ($p$) to that of observations ($n$). Hereafter, as in CCFT, we focus on child mortality among many outcome variables.

Table 5 shows the results of our empirical illustration. Four columns correspond to four estimation procedures which are the same as those used in the simulation experiments. The first panel shows the full sample results ($n = 2799$ and $p/n = 0.016$). The RDD causal effect estimates are presented in the first row. The next three rows show 95% confidence intervals, their percentage length changes relative to the one in the first column, and their associated $p$-values where these are obtained without restriction on the MSE optimal bandwidth for the local linear regression ($h$) and the pilot bandwidth ($b$). See CCT and CCFT for more details on the robust inference methods. These results are also obtained under the restriction $h/b = 1$, which are reported in the following three rows. The next two rows in the same panel present the bandwidths $(h, b)$, and effective sample sizes $(n_-, n_+)$ used for the RDD estimation. The effective sample sizes are the numbers of observations of the running variable in the intervals $[\bar{x} - h, \bar{x}]$ and $[\bar{x}, \bar{x} + h]$. We also report the selected covariates for our covariate selection approach. We use subsamples of the first 1000 and 500 observations for the second and third panels, leading to
|      | MSE-optimal bandwidths: | w/o Covariates | w/ Covariates | Adaptive |
|------|-------------------------|----------------|---------------|----------|
|      |                         | Standard       | Covariate adjusted | Covariate adjusted | Covariate selection |
|      |                         | Bias | RMSE | Bias | RMSE | Bias | RMSE | Bias | RMSE |
| DGP1 |                         | p    |      |      |      |      |      |      |      |
|      | 5                       | 0.022| 0.062| 0.023| 0.064| 0.022| 0.065| 0.022| 0.062|
|      | 10                      | 0.021| 0.065| 0.022| 0.068| 0.019| 0.070| 0.020| 0.065|
|      | 20                      | 0.020| 0.061| 0.019| 0.066| 0.013| 0.074| 0.019| 0.061|
|      | 30                      | 0.021| 0.064| 0.022| 0.074| 0.018| 0.089| 0.020| 0.064|
|      | 40                      | 0.022| 0.064| 0.022| 0.080| 0.002| 0.323| 0.022| 0.064|
|      | 50                      | 0.021| 0.065| 0.027| 0.101| 0.003| 0.316| 0.021| 0.065|
|      | 100                     | 0.019| 0.063| 0.038| 0.387| 0.004| 0.133| 0.018| 0.063|
|      | 250                     | 0.018| 0.065| 0.026| 0.082| 0.006| 0.102| 0.017| 0.065|
|      | 500                     | 0.021| 0.064| 0.028| 0.072| 0.009| 0.100| 0.020| 0.064|
| DGP2 |                         | 5    | 0.019| 0.540| 0.000| 0.058| 0.029| 0.081| 0.029| 0.090|
|      |                         | 10   | -0.007| 0.525| -0.001| 0.061| 0.026| 0.084| 0.028| 0.094|
|      |                         | 20   | 0.008| 0.543| -0.007| 0.064| 0.020| 0.094| 0.026| 0.099|
|      |                         | 30   | -0.006| 0.530| -0.006| 0.070| 0.025| 0.123| 0.027| 0.105|
|      |                         | 40   | 0.002| 0.521| -0.012| 0.080| -0.001| 1.485| 0.026| 0.102|
|      |                         | 50   | 0.043| 0.531| -0.010| 0.098| 0.002| 1.116| 0.025| 0.105|
|      |                         | 100  | 0.042| 0.544| -0.033| 0.551| -0.052| 0.874| 0.023| 0.107|
|      |                         | 250  | -0.007| 0.505| -0.059| 1.162| 0.027| 1.166| 0.019| 0.116|
|      |                         | 500  | 0.003| 0.535| -0.058| 1.636| -0.066| 1.578| 0.027| 0.118|
| DGP3 |                         | 5    | 0.013| 0.708| -0.004| 0.058| 0.029| 0.081| 0.028| 0.103|
|      |                         | 10   | -0.010| 0.681| -0.005| 0.060| 0.026| 0.084| 0.030| 0.179|
|      |                         | 20   | 0.002| 0.706| -0.011| 0.064| 0.020| 0.094| 0.005| 0.177|
|      |                         | 30   | -0.009| 0.691| -0.011| 0.070| 0.025| 0.123| 0.030| 0.186|
|      |                         | 40   | 0.000| 0.683| -0.017| 0.080| 0.021| 0.321| 0.018| 0.176|
|      |                         | 50   | 0.038| 0.686| -0.015| 0.085| 0.042| 0.784| 0.017| 0.190|
|      |                         | 100  | 0.048| 0.722| -0.036| 0.301| 0.045| 0.605| 0.013| 0.201|
|      |                         | 250  | -0.019| 0.668| -0.055| 0.347| -0.021| 0.957| 0.009| 0.216|
|      |                         | 500  | -0.004| 0.707| -0.050| 0.485| 0.015| 0.965| 0.014| 0.217|
| DGP4 |                         | 5    | 0.028| 0.180| 0.018| 0.066| 0.029| 0.081| 0.022| 0.131|
|      |                         | 10   | 0.021| 0.187| 0.015| 0.068| 0.026| 0.084| 0.019| 0.185|
|      |                         | 20   | 0.019| 0.193| 0.010| 0.071| 0.020| 0.094| 0.015| 0.192|
|      |                         | 30   | 0.028| 0.197| 0.013| 0.080| 0.025| 0.123| 0.024| 0.196|
|      |                         | 40   | 0.021| 0.192| 0.008| 0.092| 0.001| 1.480| 0.019| 0.190|
|      |                         | 50   | 0.029| 0.198| 0.011| 0.110| -0.026| 1.491| 0.022| 0.194|
|      |                         | 100  | 0.036| 0.206| -0.053| 0.853| -0.024| 0.470| 0.032| 0.203|
|      |                         | 250  | 0.017| 0.194| -0.039| 0.547| -0.072| 0.578| 0.013| 0.190|
|      |                         | 500  | 0.020| 0.196| -0.026| 0.549| -0.053| 0.627| 0.016| 0.194|
| DGP   | $p$ | Average | Min | Max |
|-------|-----|---------|-----|-----|
| DGP1  | 5   | 0.376   | 0   | 1   |
|       | 10  | 0.362   | 0   | 1   |
|       | 20  | 0.357   | 0   | 1   |
|       | 30  | 0.353   | 0   | 1   |
|       | 40  | 0.332   | 0   | 1   |
|       | 50  | 0.379   | 0   | 1   |
|       | 100 | 0.350   | 0   | 1   |
|       | 250 | 0.331   | 0   | 1   |
|       | 500 | 0.264   | 0   | 1   |
| DGP2  | 5   | 2.868   | 2   | 3   |
|       | 10  | 2.780   | 2   | 3   |
|       | 20  | 2.689   | 1   | 3   |
|       | 30  | 2.631   | 2   | 3   |
|       | 40  | 2.555   | 1   | 3   |
|       | 50  | 2.574   | 1   | 3   |
|       | 100 | 2.478   | 1   | 3   |
|       | 250 | 2.341   | 1   | 3   |
|       | 500 | 2.303   | 1   | 3   |
| DGP3  | 5   | 3.910   | 3   | 5   |
|       | 10  | 3.063   | 2   | 5   |
|       | 20  | 3.045   | 2   | 5   |
|       | 30  | 3.003   | 1   | 4   |
|       | 40  | 2.959   | 1   | 5   |
|       | 50  | 2.920   | 1   | 5   |
|       | 100 | 2.850   | 1   | 4   |
|       | 250 | 2.683   | 1   | 4   |
|       | 500 | 2.547   | 1   | 4   |
| DGP4  | 5   | 1.771   | 1   | 3   |
|       | 10  | 0.983   | 0   | 3   |
|       | 20  | 0.860   | 0   | 3   |
|       | 30  | 0.787   | 0   | 3   |
|       | 40  | 0.719   | 0   | 3   |
|       | 50  | 0.730   | 0   | 3   |
|       | 100 | 0.680   | 0   | 3   |
|       | 250 | 0.593   | 0   | 3   |
|       | 500 | 0.479   | 0   | 3   |
Table 3: Simulation: Inference

| Estimation methods | w/o Covariates | w/ Covariates | Adaptive |
|--------------------|----------------|---------------|----------|
|                   | CP Length CP Length | CP Length CP Length | CP Length CP Length |
| **DGP1**           |                |               |          |
| p=5                | 0.918 0.270    | 0.898 0.246   | 0.901 0.246   | 0.916 0.270 |
| p=10               | 0.892 0.213    | 0.855 0.211   | 0.856 0.205   | 0.891 0.213 |
| p=20               | 0.925 0.293    | 0.815 0.259   | 0.779 0.280   | 0.919 0.275 |
| p=30               | 0.914 0.260    | 0.734 0.148   | 0.658 0.156   | 0.914 0.260 |
| p=40               | 0.915 0.241    | 0.649 0.182   | 0.476 0.107   | 0.916 0.241 |
| p=50               | 0.915 0.246    | 0.540 0.152   | 0.247 0.076   | 0.917 0.246 |
| p=100              | 0.913 0.246    | 0.194 0.342   | 0.186 0.040   | 0.911 0.246 |
| p=250              | 0.900 0.252    | 0.108 0.040   | 0.174 0.048   | 0.904 0.233 |
| p=500              | 0.905 0.188    | 0.138 0.020   | 0.155 0.065   | 0.903 0.188 |
| **DGP2**           |                |               |          |
| p=5                | 0.926 2.474    | 0.661 0.232   | 0.845 0.283   | 0.867 0.331 |
| p=10               | 0.932 1.732    | 0.627 0.209   | 0.781 0.292   | 0.873 0.413 |
| p=20               | 0.922 2.261    | 0.591 0.261   | 0.677 0.252   | 0.876 0.395 |
| p=30               | 0.938 1.932    | 0.517 0.136   | 0.505 0.143   | 0.864 0.305 |
| p=40               | 0.934 2.351    | 0.472 0.170   | 0.305 0.122   | 0.879 0.344 |
| p=50               | 0.930 1.853    | 0.407 0.127   | 0.267 0.231   | 0.884 0.300 |
| p=100              | 0.918 2.063    | 0.250 0.069   | 0.566 2.065   | 0.876 0.570 |
| p=250              | 0.932 2.538    | 0.778 2.501   | 0.808 3.648   | 0.867 0.559 |
| p=500              | 0.915 2.286    | 0.881 6.466   | 0.875 5.146   | 0.876 0.494 |
| **DGP3**           |                |               |          |
| p=5                | 0.926 3.173    | 0.615 0.219   | 0.845 0.283   | 0.870 0.456 |
| p=10               | 0.932 2.243    | 0.601 0.202   | 0.781 0.292   | 0.905 0.724 |
| p=20               | 0.922 2.593    | 0.564 0.236   | 0.677 0.252   | 0.908 0.609 |
| p=30               | 0.931 2.498    | 0.495 0.151   | 0.505 0.143   | 0.881 0.680 |
| p=40               | 0.928 3.277    | 0.456 0.174   | 0.295 0.122   | 0.914 1.118 |
| p=50               | 0.927 2.462    | 0.397 0.132   | 0.189 0.082   | 0.888 0.444 |
| p=100              | 0.912 2.750    | 0.182 0.073   | 0.164 0.116   | 0.898 0.846 |
| p=250              | 0.931 3.578    | 0.119 0.139   | 0.175 0.362   | 0.895 0.745 |
| p=500              | 0.922 3.235    | 0.130 0.158   | 0.157 0.255   | 0.892 0.635 |
| **DGP4**           |                |               |          |
| p=5                | 0.910 1.083    | 0.807 0.269   | 0.845 0.283   | 0.899 0.525 |
| p=10               | 0.924 0.619    | 0.762 0.253   | 0.781 0.292   | 0.912 0.624 |
| p=20               | 0.923 0.948    | 0.689 0.277   | 0.677 0.252   | 0.904 0.854 |
| p=30               | 0.912 0.631    | 0.588 0.149   | 0.505 0.143   | 0.909 0.631 |
| p=40               | 0.932 0.933    | 0.489 0.162   | 0.296 0.122   | 0.925 0.997 |
| p=50               | 0.920 0.666    | 0.375 0.113   | 0.232 0.121   | 0.914 0.673 |
| p=100              | 0.914 0.714    | 0.306 0.289   | 0.471 0.835   | 0.914 0.714 |
| p=250              | 0.916 0.918    | 0.677 1.547   | 0.705 1.436   | 0.920 0.882 |
| p=500              | 0.910 0.821    | 0.853 1.862   | 0.791 3.522   | 0.910 0.810 |
| DGP | \(p\) | Mean  | SD   | Mean  | SD   | Mean  | SD   |
|-----|-----|-------|------|-------|------|-------|------|
| DGP1 | 5   | 0.198 | 0.046 | 0.190 | 0.043 | 0.197 | 0.046 |
|    | 10  | 0.195 | 0.046 | 0.179 | 0.041 | 0.195 | 0.046 |
|    | 20  | 0.197 | 0.043 | 0.163 | 0.035 | 0.197 | 0.044 |
|    | 30  | 0.196 | 0.044 | 0.147 | 0.031 | 0.196 | 0.045 |
|    | 40  | 0.197 | 0.044 | 0.128 | 0.026 | 0.196 | 0.043 |
|    | 50  | 0.194 | 0.045 | 0.108 | 0.024 | 0.193 | 0.045 |
|    | 100 | 0.196 | 0.047 | 0.075 | 0.024 | 0.196 | 0.047 |
|    | 250 | 0.194 | 0.045 | 0.069 | 0.019 | 0.194 | 0.045 |
|    | 500 | 0.197 | 0.043 | 0.071 | 0.020 | 0.196 | 0.043 |
| DGP2 | 5   | 0.177 | 0.025 | 0.108 | 0.011 | 0.111 | 0.013 |
|    | 10  | 0.177 | 0.025 | 0.107 | 0.011 | 0.113 | 0.014 |
|    | 20  | 0.176 | 0.026 | 0.105 | 0.011 | 0.115 | 0.015 |
|    | 30  | 0.177 | 0.025 | 0.102 | 0.011 | 0.115 | 0.015 |
|    | 40  | 0.176 | 0.025 | 0.099 | 0.012 | 0.117 | 0.015 |
|    | 50  | 0.176 | 0.026 | 0.094 | 0.014 | 0.116 | 0.015 |
|    | 100 | 0.176 | 0.025 | 0.130 | 0.026 | 0.118 | 0.016 |
|    | 250 | 0.176 | 0.025 | 0.159 | 0.029 | 0.121 | 0.016 |
|    | 500 | 0.177 | 0.025 | 0.173 | 0.031 | 0.121 | 0.016 |
| DGP3 | 5   | 0.182 | 0.028 | 0.108 | 0.011 | 0.117 | 0.014 |
|    | 10  | 0.183 | 0.029 | 0.107 | 0.011 | 0.138 | 0.015 |
|    | 20  | 0.181 | 0.029 | 0.105 | 0.011 | 0.139 | 0.017 |
|    | 30  | 0.183 | 0.028 | 0.102 | 0.011 | 0.138 | 0.015 |
|    | 40  | 0.182 | 0.028 | 0.099 | 0.012 | 0.139 | 0.016 |
|    | 50  | 0.183 | 0.029 | 0.092 | 0.013 | 0.140 | 0.015 |
|    | 100 | 0.183 | 0.029 | 0.078 | 0.021 | 0.141 | 0.016 |
|    | 250 | 0.181 | 0.028 | 0.076 | 0.022 | 0.142 | 0.018 |
|    | 500 | 0.183 | 0.029 | 0.074 | 0.021 | 0.143 | 0.017 |
| DGP4 | 5   | 0.141 | 0.015 | 0.108 | 0.011 | 0.126 | 0.016 |
|    | 10  | 0.144 | 0.015 | 0.107 | 0.011 | 0.142 | 0.015 |
|    | 20  | 0.144 | 0.016 | 0.105 | 0.011 | 0.142 | 0.016 |
|    | 30  | 0.144 | 0.015 | 0.102 | 0.011 | 0.143 | 0.015 |
|    | 40  | 0.145 | 0.015 | 0.099 | 0.012 | 0.144 | 0.016 |
|    | 50  | 0.144 | 0.016 | 0.092 | 0.013 | 0.143 | 0.016 |
|    | 100 | 0.145 | 0.015 | 0.104 | 0.021 | 0.143 | 0.016 |
|    | 250 | 0.144 | 0.016 | 0.132 | 0.025 | 0.143 | 0.016 |
|    | 500 | 0.144 | 0.015 | 0.150 | 0.025 | 0.143 | 0.016 |
\( p/n = 0.045 \) and 0.090, respectively.

For the full sample case, the covariate adjusted estimates mildly deviate from the standard one while our estimate based on the variable selection is identical to the standard one. Although the confidence intervals of the covariate adjusted approaches are shorter than the standard one, this might induce under-coverages for the case of many covariates as illustrated in the simulation experiment. As the sample size gets smaller, the observations made here are amplified. In contrast, we can see the stable performance of the variable selection approach and its mild contribution to shorten the confidence intervals. Table 6 presents the corresponding results for the case of nine covariates as considered by CCFT. The results are essentially similar to those of Table 5, although the influence of the covariates is less dramatic.
Table 5: Empirical illustration: Head Start data (45 covariates)

| MSE-optimal bandwidths: | w/o Covariates | w/ Covariates | Adaptive |
|-------------------------|----------------|--------------|----------|
| $n = 2779$              |                |              |          |
| $p/n = 0.016$           |                |              |          |
| $h/b$ unrestricted      |                |              |          |
| Robust 95% CI           | $[-5.46, -0.1]$ | $[-3.85, -0.07]$ | $[-6.05, -1.24]$ | $[-5.46, -0.1]$ |
| CI length change (%)    | 0              | 0             | 0         |
| Robust p-value          | 0.042          | 0.042         | 0.033     | 0.042     |
| $h/b = 1$               |                |              |          |
| Robust 95% CI           | $[-6.41, -1.09]$ | $[-5.44, -1.10]$ | $[-6.55, -1.14]$ | $[-6.41, -1.09]$ |
| CI length change (%)    | 0              | 0             | 0         |
| Robust p-value          | 0.006          | 0.003         | 0.005     | 0.006     |

| $n = 1000$              |                |              |          |
| $p/n = 0.045$           |                |              |          |
| $h/b$ unrestricted      |                |              |          |
| Robust 95% CI           | $[-5.45, 1.75]$ | $[-5.44, -0.29]$ | $[-6.14, -1.34]$ | $[-5.08, 1.79]$ |
| CI length change (%)    | 0.314          | 0.29         | 0.022     | 0.347     |
| Robust p-value          | 0.063          | 0.025         | 0.064     | 0.070     |

| $n = 500$               |                |              |          |
| $p/n = 0.090$           |                |              |          |
| $h/b$ unrestricted      |                |              |          |
| Robust 95% CI           | $[-7.25, 2.48]$ | $[-7.44, -2.06]$ | $[-9.64, -4.63]$ | $[-6.93, 2.25]$ |
| CI length change (%)    | 0.337          | 0.01         | 0.000     | 0.317     |
| Robust p-value          | 0.139          | 0.00         | 0.000     | 0.110     |

| $n = 500$               |                |              |          |
| $p/n = 0.090$           |                |              |          |
| $h/b = 1$               |                |              |          |
| Robust 95% CI           | $[-10.22, 1.42]$ | $[-9.03, -3.43]$ | $[-9.62, -4.29]$ | $[-10.1, 1.07]$ |
| CI length change (%)    | 0.139          | 0.00         | 0.000     | 0.110     |

| Selected covariates     | % of adult population | % of adult population |
|-------------------------|-----------------------|-----------------------|
| $n_-, n_+$              | 234, 180              | 234, 180              |

| $n = 500$               |                |              |          |
| $p/n = 0.090$           |                |              |          |
| $h/b$ unrestricted      |                |              |          |
| Robust 95% CI           | $[-10.22, 1.42]$ | $[-9.03, -3.43]$ | $[-9.62, -4.29]$ | $[-10.1, 1.07]$ |
| CI length change (%)    | 0.139          | 0.00         | 0.000     | 0.110     |

| Selected covariates     | % of adult population | % of adult population |
|-------------------------|-----------------------|-----------------------|
| $n_-, n_+$              | 234, 180              | 234, 180              |

| $n = 500$               |                |              |          |
| $p/n = 0.090$           |                |              |          |
| $h/b = 1$               |                |              |          |
| Robust 95% CI           | $[-10.22, 1.42]$ | $[-9.03, -3.43]$ | $[-9.62, -4.29]$ | $[-10.1, 1.07]$ |
| CI length change (%)    | 0.139          | 0.00         | 0.000     | 0.110     |

| Selected covariates     | % of adult population | % of adult population |
|-------------------------|-----------------------|-----------------------|
| $n_-, n_+$              | 234, 180              | 234, 180              |
Table 6: Empirical illustration: Head Start data (9 covariates)

| MSE-optimal bandwidths: | w/o Covariates | w/ Covariates | Adaptive |
|-------------------------|----------------|--------------|----------|
| n = 2779                |                |              |          |
| p/n = 0.003             |                |              |          |
| h/b unrestricted        |                |              |          |
| Robust 95% CI           | [−5.46, −0.1]  | [−5.37, −0.45] | [−5.21, −0.37] | [−5.46, −0.1] |
| CI length change (%)    | −8.24          | −9.74        | 0        |          |
| Robust p-value          | 0.042          | 0.021        | 0.024    | 0.042    |
| h/b = 1                 |                |              |          |
| Robust 95% CI           | [−6.41, −1.09] | [−6.63, −1.46] | [−6.54, −1.39] | [−6.41, −1.09] |
| CI length change (%)    | −2.87          | −3.23        | 0        |          |
| Robust p-value          | 0.006          | 0.002        | 0.003    | 0.006    |
| h, b                    | 6.81, 10.73    | 6.81, 10.73  | 6.98, 11.64 | 6.81, 10.73 |
| n−, n+                  | 234, 180       | 234, 180     | 240, 184 | 234, 180 |
| Selected covariates     | None           |              |          |          |
| n = 1000                |                |              |          |
| p/n = 0.009             |                |              |          |
| h/b unrestricted        |                |              |          |
| Robust 95% CI           | [−5.45, 1.75]  | [−6.35, −1.1] | [−7.7, −2.28] | [−5.08, 1.79] |
| CI length change (%)    | −7.12          | −20.0        | −4.49    |          |
| Robust p-value          | 0.314          | 0.141        | 0.237    | 0.347    |
| h/b = 1                 |                |              |          |
| Robust 95% CI           | [−8.26, 0.22]  | [−8.26, −0.63] | [−5.93, −0.88] | [−7.94, 0.27] |
| CI length change (%)    | −10.20         | −19.75       | −3.28    |          |
| Robust p-value          | 0.063          | 0.022        | 0.145    | 0.070    |
| h, b                    | 6.52, 10.23    | 6.52, 10.23  | 9.14, 13.93 | 9.26, 8.07 |
| n−, n+                  | 74, 77         | 79, 79       | 121, 98  | 79, 79   |
| Selected covariates     | % of adult population |          |          |
| n = 500                 |                |              |          |
| p/n = 0.018             |                |              |          |
| h/b unrestricted        |                |              |          |
| Robust 95% CI           | [−7.25, 2.48]  | [−7.25, 1.28] | [−6.75, 1.6] | [−6.93, 2.25] |
| CI length change (%)    | −12.36         | −14.24       | −5.74    |          |
| Robust p-value          | 0.337          | 0.170        | 0.227    | 0.317    |
| h/b = 1                 |                |              |          |
| Robust 95% CI           | [−10.22, 1.42] | [−9.97, −0.03] | [−9.74, 0.03] | [−10.107] |
| CI length change (%)    | −14.58         | −16.10       | −4.94    |          |
| Robust p-value          | 0.139          | 0.049        | 0.051    | 0.110    |
| h, b                    | 6.37, 9.16     | 6.37, 9.16   | 6.58, 9.88 | 4.31, 6.82 |
| n−, n+                  | 60, 56         | 61, 56       | 63, 58   | 61, 56   |
| Selected covariates     | % of adult population |          |          |          |
To better understand the proposed method in this paper, we investigate behaviors of three approaches, two covariate adjusted and variable selection approaches, through the Head start example where the approaches are the same as those considered in the simulation, and Tables 5 and 6. The approach denoted by “Cov-adjusted 1” corresponds to the covariate adjusted approach using the CCT MSE-optimal bandwidths, and the one denoted by “Cov-adjusted 2” to the one using the CCFT MSE-optimal bandwidths. We construct 50 subsamples of $k$ observations out of the Head start data where $k$ is set to 1000, and 500. We estimate the RDD treatment effects based on the subsamples using nine or forty-five covariates where nine covariates are those considered
Figure 3: Comparison of three approaches ($n = 1000, 9$ covariates) for point estimates (left panel) and CI lengths (right panel).

Figure 4: Comparison of three approaches ($n = 500, 9$ covariates) for point estimates (left panel) and CI lengths (right panel).
in CCFT and forty-five are those considered in Table 5. Figures 1-4 show boxplots of the results on estimation (left panel) and inference (right panel). The top and bottom of the boxes are the third and first quartiles, and the top- and bottom-bars show the maximum and minimum values less than (the third quartiles + 1.5 times the interquartile range) and greater than (the first quartile - 1.5 times the interquartile range), respectively. The left panels in Figures 1-4 show the differences in the RDD treatment effects between the three approaches and the standard one where the standard one is based on CCT. The right panel in each figure shows the corresponding differences in the confidence interval lengths.

First, the left panels of Figures 1 and 2 show the point estimates by the covariate adjusted approaches deviate from the standard one by a large extent when the number of covariates is large, and the differences get larger as the sample size becomes small. We observe a similar tendency for the situations with nine covariates, while differences are less dramatic. Second, we note that the point estimates based on the standard and variable selection approaches are very close, and they are stable. Third, we observe that the covariate adjusted approaches tend to produce large decreases in the interval length, which can reflect under-coverages as observed in Table 3. Fourth, the variable selection approach leads to reductions of around 5%, which are still nonnegligible. Fifth, the results for nine covariates given in Figures 3 and 4 are similar to those for forty-five covariates, although the magnitude of changes for nine covariates is not as extreme as that for forty-five covariates. These results show the usefulness of our variable selection approach not only for the situation where the number of covariates is large but also for that where the number of covariates is relatively small, when the number of observations is not so large.

We often encounter situations where a number of potentially useful covariates are available. Furthermore, it is common to employ transformations of covariates such as interaction and quadratic terms. However, we are typically uncertain about which covariates contribute to improving efficiency possibly due to the lack of economic theories. The RDD analysis is local in nature, and the number of covariates relative to the effective sample size can be pretty large. We have seen that covariate adjusted approaches can become misleading in several examples. The steady performance of the variable selection approach under various circumstances is noteworthy, and it can provide an essential second opinion for the standard and covariate adjusted approaches.

The results shown in Figures 1-3 are based on the bandwidths with \( h_n/b_n \) unrestricted, and those with the bandwidths with \( h_n/b_n = 1 \) are qualitatively similar.

The variable selection approach produced the identical result to the standard one about 20 times for all setups. The boxplots are drawn based on the non-identical results.
A Mathematical appendix

A.1 Proof of Theorem 1

We use the following modification of Bernstein’s inequality.

**Lemma A.1.** Under Assumptions 1-2, it holds

\[
\mathbb{E} \left[ \max_{1 \leq j \leq p} \left| \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left\{ K \left( \frac{X_i}{b_n} \right) G_{i,j} \epsilon_i - \mathbb{E} \left[ K \left( \frac{X_i}{b_n} \right) G_{i,j} \epsilon_i \right] \right\} \right|^m \right] \leq 2C^n b_n^{m/2} \log^{m/2} p, 
\]

for \( m \leq 1 + \log p \).

**Proof of Lemma A.1.** It follows from Bernstein’s inequality (e.g., Lemma 14.12 in Bühlmann and van de Geer, 2011) that

\[
\mathbb{E} \left[ \max_{1 \leq j \leq p} \left| \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left\{ K \left( \frac{X_i}{b_n} \right) G_{i,j} \epsilon_i - \mathbb{E} \left[ K \left( \frac{X_i}{b_n} \right) G_{i,j} \epsilon_i \right] \right\} \right|^m \right] \leq 2C^n b_n^{m/2} \log^{m/2} p, 
\]

for \( m \leq 1 + \log p \). \( \Box \)

To make the proof more accessible and comparable to the standard Lasso, we begin with \( \tilde{\theta} \) as the solution of

\[
\min_{\theta} \frac{1}{nb_n} \sum_{i=1}^{n} K \left( \frac{X_i}{b_n} \right) (Y_i - \alpha - T_i \tau - X_i \beta - T_i X_i \beta + Z_i' \gamma)^2 + \lambda_n |\theta|_1. \tag{20}
\]

A.1.1 Deviation bounds for \( \tilde{\theta} \)

Under Assumptions 1-2, Lemma A.1 implies

\[
\mathbb{E} \left[ \max_{1 \leq j \leq p} \left| \frac{1}{\sqrt{n}} \sum_{i=1}^{n} K \left( \frac{X_i}{b_n} \right) G_{i,j} \epsilon_i \right| \right] \leq 2b^{1/2} \log^{1/2} p. 
\]

Thus, we have

\[
\mathbb{P} \{ A_n \} := \mathbb{P} \left\{ \frac{4}{nb_n} \left| \sum_{i=1}^{n} K \left( \frac{X_i}{b_n} \right) G_{i,j} \epsilon_i \right| \leq \lambda_n \right\} \rightarrow 1, \tag{21}
\]

as \( n \to \infty \), provided that \( \sqrt{\log p}/(nb_n) = o(\lambda_n) \).

Let \( Y = (Y_1 K_1^{1/2}, \ldots, Y_n K_n^{1/2})', e = (\epsilon_1 K_1^{1/2}, \ldots, \epsilon_n K_n^{1/2})' \), and \( G = (G_1 K_1^{1/2}, \ldots, G_n K_n^{1/2})' \). Since \( \tilde{\theta} \) is a minimizer, we have

\[
\frac{1}{nb_n} |Y - G\tilde{\theta}|_2^2 + \lambda_n |\tilde{\theta}|_1 \leq \frac{1}{nb_n} |Y - G\theta^*|_2^2 + \lambda_n |\theta^*|_1.
\]
By plugging $Y = G\theta^* + e$ into the above, we obtain
\[
\frac{2}{nb_n} |G(\tilde{\theta} - \theta^*)|^2 \leq \frac{4}{nb_n} e'G(\tilde{\theta} - \theta^*) + 2\lambda_n |\theta^*|_1 - 2\lambda_n |\tilde{\theta}|_1
\leq \frac{4}{nb_n} |e'|_\infty |\tilde{\theta} - \theta^*|_1 + 2\lambda_n (|\theta^*|_1 - |\tilde{\theta}|_1)
\leq 3\lambda_n |\tilde{\theta}_S^* - \theta_S^*|_1 - \lambda_n |\tilde{\theta}_{S^c}^*|_1,
\] (22)
conditionally on $A_n$, where $S^*_c$ is the complement of $S^*$, the second inequality follows from the Hölder inequality, and the third inequality follows from the definition of $A_n$ and the following facts
\[
|\tilde{\theta} - \theta^*|_1 = |\tilde{\theta}_S - \theta_S^*|_1 + |\tilde{\theta}_{S^c}|_1,
\]
\[
|\theta^*|_1 - |\tilde{\theta}|_1 = |\theta_S^*|_1 - |\tilde{\theta}_S|_1 - |\tilde{\theta}_{S^c}|_1 \leq |\theta_S^* - \theta_S^*|_1 - |\tilde{\theta}_{S^c}|_1,
\]
(23)
due to the triangle inequality. Thus, (22) implies $3|\tilde{\theta}_S - \theta_S^*|_1 \geq |\tilde{\theta}_{S^c}^*|_1$ and
\[
\frac{2}{nb_n} |G(\tilde{\theta} - \theta^*)|^2 + \lambda_n |\tilde{\theta} - \theta^*|_1 \leq 4\lambda_n |\tilde{\theta}_S^* - \theta_S^*|_1,
\]
(24)
by using (23).

Now, Assumption 1 implies that
\[
4\lambda_n |\tilde{\theta}_S^* - \theta_S^*|_1 \leq 4\lambda_n \sqrt{\frac{s^*}{nb_n \phi^2}} |G(\tilde{\theta} - \theta^*)|_2 \leq \frac{1}{nb_n} |G(\tilde{\theta} - \theta^*)|^2 + 4\lambda_n^2 \frac{s^*}{\phi^2},
\]
with probability approaching one, where we note that $2ab \leq a^2 + b^2$ for the second inequality. Combining this with (24), we have
\[
\frac{1}{nb_n} |G(\tilde{\theta} - \theta^*)|^2 + \lambda_n |\tilde{\theta} - \theta^*|_1 \leq 4\lambda_n^2 \frac{s^*}{\phi^2},
\]
with probability approaching one.

Also note that this result implies that the number of nonzero elements in $\tilde{\theta}$ is $O(s^*)$ by the argument given in Bickel et al. (2009, eq. (B.3)).

A.1.2 Proof of (i)

Let $G_i = (G_{1i}', G_{2i}')'$, where $G_{2i} = K_1^{1/2} Z_i$ and $G_{1i}$ is the other in the partition of $G_i$. Define $G_1 = (G_{11}, \ldots, G_{1n})'$ and $G_2 = (G_{21}, \ldots, G_{2n})'$ and partition $\theta$ accordingly. The compatibility condition for $G_1$ is implied by the usual full column rank condition in the classical linear
regression since $|\theta_1|^2 \leq \dim(\theta_1)|\theta_1|^2$.

Note that the result in (21) still holds. Since $\hat{\theta}$ is a minimizer, we have

$$\frac{1}{\lambda_n} |Y - G\hat{\theta}|^2 + \lambda_n |\hat{\theta}_2| \leq \frac{1}{\lambda_n} |Y - G\theta^*|^2 + \lambda_n |\theta_2^*|.$$  

By plugging $Y = G\theta^* + \epsilon$ into the above, we have

$$\frac{2}{\lambda_n} |G(\hat{\theta} - \theta^*)|^2 \leq \frac{4}{\lambda_n} e'G(\hat{\theta} - \theta^*) + 2\lambda_n |\theta_2^*| - 2\lambda_n |\hat{\theta}_2|$$

$$\leq \frac{4}{\lambda_n} |e'G|\infty|\hat{\theta} - \theta^*| + 2\lambda_n(|\theta_2^*| - |\hat{\theta}_2|)$$

$$\leq \lambda_n |\hat{\theta}_1 - \theta_1^*| + 3\lambda_n |\hat{\theta}_2 - \theta_2^*| + \lambda_n |\hat{\theta}_2 - \theta_2^*|,$$

(25)

conditionally on $A_n$, where the second inequality follows from the Hölder inequality, and the third inequality follows from the definition of $A_n$ and the following facts

$$|\hat{\theta} - \theta^*| \leq |\hat{\theta}_2 - \theta_2^*| + |\hat{\theta}_2^* - \theta_2^*|,$$

(26)

$$|\theta^*| - |\hat{\theta}| \leq |\theta_2 - \theta_2^*| - |\hat{\theta}_2 - \theta_2^*| \leq |\theta_2 - \theta_2^*| + |\hat{\theta}_2 - \theta_2^*|,$$  

due to the triangle inequality. Thus, (25) implies $3|\hat{\theta}_2 - \theta_2^*| \geq |\hat{\theta}_2|$, and

$$\frac{2}{\lambda_n} |G(\hat{\theta} - \theta^*)|^2 + \lambda_n |\hat{\theta}_2 - \theta_2^*| \leq \lambda_n |\hat{\theta}_1 - \theta_1^*| + 4\lambda_n |\hat{\theta}_2 - \theta_2^*|,$$

(27)

by using (26).

Now Assumption [T] implies

$$4\lambda_n |\hat{\theta}_2 - \theta_2^*| \leq 4\lambda_n \sqrt{\frac{s^*}{\lambda_n \phi^{*2}}} |G_2(\hat{\theta}_2 - \theta_2^*)| \leq \frac{1}{\lambda_n} |G_2(\hat{\theta}_2 - \theta_2^*)|^2 + 4\lambda_n \frac{s^*}{\phi^{*2}},$$

$$\lambda_n |\hat{\theta}_1 - \theta_1^*| \leq \lambda_n \sqrt{\frac{s^*}{\lambda_n \phi^{*2}}} |G_1(\hat{\theta}_1 - \theta_1^*)| \leq \frac{1}{\lambda_n} |G_1(\hat{\theta}_1 - \theta_1^*)|^2 + 4\lambda_n \frac{s^*}{\phi^{*2}},$$

with probability approaching one. Combining these inequalities with (27), we have

$$\frac{1}{\lambda_n} |G(\hat{\theta} - \theta^*)|^2 + \lambda_n |\hat{\theta}_2 - \theta_2^*| \leq 5\lambda_n \frac{s^*}{\phi^{*2}},$$

which implies

$$|\hat{\theta}_2 - \theta_2^*| \leq 5\lambda_n \frac{s^*}{\phi^{*2}}.$$  

(28)
Turning to the finite dimensional component \( \hat{\theta}_1 \), note that

\[
\hat{\theta}_1 - \theta^*_1 = \left( \frac{1}{nb_n} G'_1 G_1 \right)^{-1} \left( \frac{1}{nb_n} G'_1 e - \frac{1}{nb_n} G'_1 G_2 (\hat{\theta}_2 - \theta^*_2) \right)
= O_p((nb_n)^{-1/2}) + O_p(|\hat{\theta}_2 - \theta^*_2|_1).
\]

Combining this with (28) yields the conclusion in (5).

A.1.3 Proof of (ii)

Note that we assume \( h_n = b_n \). Let \( G_{\hat{S}} = (G_{\hat{S},1}, \ldots, G_{\hat{S},n})' \). Since \( \hat{\theta}_{\hat{S}} \) is the OLS estimator that minimizes the sum of the squared residuals in the regression of \( Y \) on \( G_{\hat{S}} \), it holds

\[
|Y - G_{\hat{S}} \hat{\theta}_{\hat{S}}|^2 \leq |Y - G_{\hat{S}} \hat{\theta}_{\hat{S}}|^2.
\]

Let \( \hat{S}_1 = \{i : 0 < |\hat{\theta}_i| < \zeta_n \} \) and \( S_n = \hat{S} \cup \hat{S}_1 \). Note also that

\[
|Y - G_{S_n} \hat{\theta}_{S_n}|_2^2 = |Y - G_{\hat{S}} \hat{\theta}_{\hat{S}}|^2 \leq |Y - G_{\hat{S}} \hat{\theta}_{\hat{S}}|^2.
\]

Then, we get

\[
\frac{1}{nh_n} |G_{S_n} (\hat{\theta}_{S_n} - \hat{\theta}_{S_n})|^2 \leq \frac{1}{nh_n} |G_{\hat{S}_1} \hat{\theta}_{\hat{S}_1}|^2 + \frac{2}{nh_n} |\hat{e}' G_{\hat{S}_1} \hat{\theta}_{\hat{S}_1}| + \frac{2}{nh_n} |\hat{e}' G_{S_n} (\hat{\theta}_{S_n} - \hat{\theta}_{S_n})| \leq \frac{1}{nh_n} |G_{\hat{S}_1} \hat{\theta}_{\hat{S}_1}|^2 + \lambda_n |\hat{\theta}_{\hat{S}_1}|_1 + \lambda_n |\hat{\theta}_{S_n} - \hat{\theta}_{S_n}|_1,
\]

(29)
due to the Hölder inequality and the Karush–Kuhn–Tucker condition for \( \hat{e} \). On the other hand,

\[
\lambda_{\min} \left( \frac{1}{nh_n} G'_S G_S \right) |\hat{\theta}_{S} - \hat{\theta}_{S}|^2 \leq \frac{1}{nh_n} |G_S (\hat{\theta}_{S} - \hat{\theta}_{S})|^2.
\]

Also note that \( |a|_1 \leq \sqrt{s}|a|_2 \) for an \( s \)-dimensional vector \( a \). Then, we consider the three cases. First, let the last term in (29) be the biggest among the three terms. Then, the bound becomes

\[
\lambda_{\min} \left( \frac{1}{nh_n} G'_S G_S \right)^{-1} |S_n|_{\lambda_n}. \]

If the second is the biggest, then \( |\hat{S}_1|_{\zeta n} \geq |\hat{\theta}_{\hat{S}_1}|_1 \geq |\hat{\theta}_{S_n} - \hat{\theta}_{S_n}|_1 \). Finally, if the first term is the biggest, then \( \lambda_n |\hat{\theta}_{S_n} - \hat{\theta}_{S_n}|_1 \leq \frac{1}{nh_n} |G_{S_n} \hat{\theta}_{S_n}|_2 \leq \lambda_{\max} \left( \frac{1}{nh_n} G'_{S_1} G_{S_1} \right) |\hat{S}_1|_{\zeta n}^2 \). Therefore, we get the desired bound under \( h_n = b_n \) and Assumption

\[\square\]
A.2 Proof of Theorem 2

First, we show the asymptotic expansion in (8). Let \( \gamma \) be a coefficient vector of \( Z_i \), where the elements correspond to \( Z_{S,i} \) in (4), and other elements are zero. Observe that

\[
\bar{\gamma} = \gamma - \gamma^{(0)}.
\]

Thus it is sufficient for (8) to show that the second term is of order \( o_p((nh_n)^{-1/2}) \).

Note that

\[
\| \frac{1}{nh_n} \sum_{i=1}^{n} K \left( \frac{X_i}{h_n} \right) G_{1i}^2 \|_{\infty} \leq \lambda_{\min} \left( \frac{1}{nh_n} \sum_{i=1}^{n} K \left( \frac{X_i}{h_n} \right) G_{1i}^2 \right)^{-1} \| \frac{1}{\sqrt{nh_n}} \sum_{i=1}^{n} K \left( \frac{X_i}{h_n} \right) G_{1i} Z_i^t \|_{\infty} + O_p(\sqrt{nh_n h_n^2}) = O_p(\sqrt{\log p}) + O_p(\sqrt{nh_n h_n^2}),
\]

where the inequality follows from the fact that \( (\sum_{i=1}^{n} K(X_i/h_n)G_{1i}G_{1i}^t)^{-1} \sum_{i=1}^{n} K(X_i/h_n)G_{1i}Z_i^t \) is a vector of the local linear RDD estimator for the outcome \( Z_i \) and its bias is of order \( O(h_n^2) \) from Lemma SA-2 of CCFT, and the equality follows from the local Bernstein’s inequality in Lemma A.1.

Turing to bound \( \gamma - \gamma^{(1)} \), recall that \( \theta^{S*} = \arg \min_{\theta^{S*}} E[K(X_i/b_n)(Y_i - G_{1i}^t \theta^{S*})^2/b_n] \) while \( \gamma_{Y} = \arg \min_{\gamma} E[(\hat{Y} - Z_{S*}^t \gamma)^2|X = 0] \), where \( \hat{Y} = Y(1) - Y(0) - E[Y(1) - Y(0)|X = 0] \) and \( Z_{S*} = Z_{S*}(1) - Z_{S*}(0) \). Due to van der Vaart and Wellner (1996, Theorem 3.4.1), we can bound the contrast \( |\gamma_{Y} - \gamma^{*}| \) by the difference in the criterions, which is of order \( O(s^* b_n^2) \) by the standard bias calculation. For the same reasoning, we obtain \( |\gamma(b_n) - \gamma(h_n)| = O_p(s^*(h_n^2 + b_n^2)) \), where we highlight by \( \gamma(b_n) \) the dependence on the bandwidth used to compute \( \gamma \).

Since \( |\gamma(b_n) - \gamma^{(1)}| = O_p(\lambda_n s^*) \) by Theorem 1 and thus \( |\gamma_{Y} - \gamma(h_n)|_1 = O_p(s^*(h_n^2 + b_n^2) + \lambda_n s^*) \) due to the preceding derivation, Hölder’s inequality and the assumption \( (\sqrt{\log p} + \sqrt{nh_n h_n^2})(s^*(h_n^2 + b_n^2) + \lambda_n s^*) \rightarrow 0 \) guarantee (8).

Second, we note that \( \hat{\tau}_z \) is the conventional local linear RDD estimator without covariates for the outcome variable \( \xi_i \). Thus, the proof of CCFT’s Theorem 1 is directly applicable and the
MSE expansion in (13) follows. Finally, we show (14). By (8), we have

$$T_\tau = \sqrt{\frac{n}{n}} \phi \left( \hat{\tau}_\xi - h_n \mathcal{B} - \tau \right) - \sqrt{\frac{n}{n}} \phi \left( h_n \mathcal{B} - \tau \right).$$

Since $\hat{\tau}_\xi$ is the conventional local linear RDD estimator without covariates for the outcome variable $\xi_i$, Lemma SA-10 of CCFT yields $\sqrt{\frac{n}{n}} \phi (\hat{\tau}_\xi - h_n \mathcal{B} - \tau) \xrightarrow{d} N(0,1)$. Therefore, the assumptions $\sqrt{\frac{n}{n}} \phi (\mathcal{B} - \mathcal{B}) \xrightarrow{P} 0$ and $\sqrt{\frac{n}{n}} \phi \xrightarrow{P} 1$ imply the conclusion in (14).

### A.3 Proof of Proposition 1

Observe that $\phi_{\tau}^T$ can be written as

$$\phi_{\tau}^T = \phi_{\tau} - \phi_{\tau}^T \phi_{\tau} + \phi_{\tau}^T \phi_{\tau} \phi_{\tau},$$

where $\phi_{\tau} = \phi \otimes \Gamma^{-1} e_1$, $\phi_{\tau}^T = \phi \otimes \Gamma^{-1} e_1$, and

$$\phi_{\tau} = \begin{bmatrix} \phi_{\tau}^N \phi_{\tau}^W \end{bmatrix}, \quad \phi_{\tau}^N = \begin{bmatrix} \phi_{\tau}^N_{Y-} \phi_{\tau}^N_{Y+} \cdots \phi_{\tau}^N_{Z_{i,-}} \end{bmatrix}, \quad \phi_{\tau}^W = \begin{bmatrix} \phi_{\tau}^W_{Y-} \phi_{\tau}^W_{Y+} \cdots \phi_{\tau}^W_{Z_{i,-}} \end{bmatrix},$$

$$\phi_{\tau}^N = \frac{1}{n} \sum_{i=1}^n I \{X_i < 0 \} K(X_i/h_n)^2 \begin{bmatrix} 1 & X_i/h_n & (X_i/h_n)^2 \\ X_i/h_n & (X_i/h_n)^2 \end{bmatrix} \bar{\epsilon}_{\phi_{\tau}^W-\phi_{\tau}^W-i},$$

for $V, W \in \{Y, Z_1, \ldots, Z_p\}$. Similarly, the asymptotic variance $\phi$ can be written as

$$\phi = \phi_{\tau} \phi_{\tau} - \phi_{\tau}^T \phi_{\tau} + \phi_{\tau}^T \phi_{\tau} \phi_{\tau}.$$

36
where \( r_- = q \otimes \Gamma^{-1}_- e_1 \), \( r_+ = q \otimes \Gamma^{-1}_+ e_1 \), and 

\[
\Psi_- = \begin{bmatrix}
\Psi_{YY} & \Psi_{YZ_1} & \cdots & \Psi_{YZ_s} \\
\Psi_{Z_1Y} & \Psi_{Z_1Z_1} & & \\
\vdots & & \ddots & \\
\Psi_{Z_sY} & & & \Psi_{Z_sZ_s}
\end{bmatrix}, \quad \Psi_+ = \begin{bmatrix}
\Psi_{YY} & \Psi_{YZ_1} & \cdots & \Psi_{YZ_s} \\
\Psi_{Z_1Y} & \Psi_{Z_1Z_1} & & \\
\vdots & & \ddots & \\
\Psi_{Z_sY} & & & \Psi_{Z_sZ_s}
\end{bmatrix},
\]

\[
\Psi_{VV}^- = \frac{1}{n h_n} \sum_{i=1}^n \P\{X_i < 0\} K(X_i/h_n)^2 \begin{bmatrix} 1 & X_i/h_n \\ X_i/h_n & (X_i/h_n)^2 \end{bmatrix} \Cov(V_i(0), W_i(0)|X_i),
\]

\[
\Psi_{VV}^+ = \frac{1}{n h_n} \sum_{i=1}^n \P\{X_i \geq 0\} K(X_i/h_n)^2 \begin{bmatrix} 1 & X_i/h_n \\ X_i/h_n & (X_i/h_n)^2 \end{bmatrix} \Cov(V_i(1), W_i(1)|X_i),
\]

for \( V, W \in \{Y, Z_1, \ldots, Z_p\} \).

Since \( \P(\hat{S} \subset S^*) \to 1 \) due to the deviation bound in Theorem 1, the following statements are all conditional on the event \( \hat{S} \subset S^* \). Without loss of generality, suppose \( \hat{S} = \{1, \ldots, |\hat{S}|\} \) and \( S^* = \{1, \ldots, |\hat{S}|, |\hat{S}| + 1, \ldots, s^*\} \). Let

\[
\varpi_n = \min\{\lambda_{\min}(\Psi_-), \lambda_{\min}(\Psi_+))\}. \quad (30)
\]

We decompose

\[
\frac{\hat{V}^{NN}}{V} - 1 = \frac{1}{\P} \begin{bmatrix} \bar{r}_- & \Psi_{YY}^- & \cdots & \Psi_{YZ_s^-} \\ 0 & \Psi_{YY}^+ & \cdots & \Psi_{YZ_s^+} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \Psi_{YY}^+ \end{bmatrix} \begin{bmatrix} \bar{r}_- \\ 0 \\ \vdots \\ 0 \\ \bar{r}_+ \end{bmatrix} + \frac{1}{\P} \begin{bmatrix} \bar{r}_- & \mu_- & \cdots & \mu_- \\ 0 & \mu_+ & \cdots & \mu_+ \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mu_+ \end{bmatrix} \begin{bmatrix} \bar{r}_- \\ 0 \\ \vdots \\ 0 \\ \bar{r}_+ \end{bmatrix}.
\]

\[
=: T_{1-} + T_{2-} + T_{1+} + T_{2+}.
\]

For \( T_{2-} \), we have

\[
|T_{2-}| \leq \varpi_n^{-1} \begin{bmatrix} \bar{r}_- \\ 0 \end{bmatrix} + \mu_- \begin{bmatrix} \bar{r}_- \\ 0 \end{bmatrix} \leq \varpi_n^{-1} \begin{bmatrix} (m_n|S_n| \lambda_n) \vee (|\hat{S}_1| \zeta_n) \vee (m_{1n}|\hat{S}_1| \zeta_n^2 / \lambda_n) \end{bmatrix},
\]

where the inequality follows from \( V \geq \varpi_n \) and Hölder’s inequality, and the equality follows from Theorem 1(ii). A similar argument yields that \( T_{2+} \) is of the same stochastic order.
For $T_{1-}$, letting $\Psi_{11}^{W}$ be the first $|\hat{S}| \times |\hat{S}|$ block component of $\Psi_{-}$, we have

$$|T_{1-}| \leq \varepsilon_n^{-1} \max_{1 \leq j_1, j_2 \leq 2S} |\Psi_{-}^{NN} - \Psi_{-}^{11}| \mathcal{O}_p(\varepsilon_n^{-1} s^4).$$

So it remains to obtain the stochastic order of $\max_{1 \leq j_1, j_2 \leq 2S} |\Psi_{-}^{NN} - \Psi_{-}^{11}|$. From p. 36 of the supplement of CCT, we can decompose

$$\Psi_{-}^{NN} - \Psi_{-}^{11} = \eta_1^{VW} + \eta_2^{VW} + \eta_3^{VW},$$

for $V, W \in \{Y, Z_1, \ldots, Z_p\}$, where $\varepsilon_{V,i} = V_i - \mathbb{E}[V_i | X_i]$, $\varepsilon_{W,i} = W_i - \mathbb{E}[W_i | X_i]$, $\mu_{\cdot -}(x) = \mathbb{E}[V_{i}(0) | X_i = x]$, $\mu_{\cdot -}(x) = \mathbb{E}[W_{i}(0) | X_i = x]$, and

$$\eta_1^{VW} = \frac{1}{J(J+1)} \sum_{j=1}^J \eta_1^{VW,j},$$

$$\eta_{1,j}^{VW} = \frac{1}{nh_n} \sum_{i=1}^n \mathbb{I}\{X_i < 0\} \mathcal{K}(X_i/h_n)^2 \begin{bmatrix} 1 & X_i/h_n \\ X_i/h_n & (X_i/h_n)^2 \end{bmatrix} \varepsilon_{V,\ell_{+,j}(i)} \varepsilon_{W,\ell_{-,j}(i)} - \varepsilon_{V,i} \varepsilon_{W,j},$$

$$\eta_2^{VW} = \frac{2}{J(J+1)} \sum_{1 \leq j \leq J} \sum_{0 \leq k \leq J} \eta_2^{VW,j,k},$$

$$\eta_{2,j,k}^{VW} = \frac{1}{n} \sum_{i=1}^n \mathbb{I}\{X_i < 0\} \mathcal{K}(X_i/h_n)^2 \begin{bmatrix} 1 & X_i/h_n \\ X_i/h_n & (X_i/h_n)^2 \end{bmatrix} \varepsilon_{V,\ell_{-,j}(i)} \varepsilon_{W,\ell_{-,j}(i)},$$

$$\eta_3^{VW} = \frac{1}{n} \sum_{i=1}^n \mathbb{I}\{X_i < 0\} \mathcal{K}(X_i/h_n)^2 \begin{bmatrix} 1 & X_i/h_n \\ X_i/h_n & (X_i/h_n)^2 \end{bmatrix} \varepsilon_{V,\ell_{+,j}(i)} \varepsilon_{W,\ell_{+,j}(i)}$$

$$\times \{(\tilde{\sigma}_{3,i}^{VW})^2 - (\tilde{\sigma}_{4,i}^{VW})^2 - (\tilde{\sigma}_{5,i}^{VW})^2 + (\tilde{\sigma}_{6,i}^{VW})^2 + (\tilde{\sigma}_{7,i}^{VW})^2 - (\tilde{\sigma}_{8,i}^{VW})^2 - (\tilde{\sigma}_{9,i}^{VW})^2\},$$

$$(\tilde{\sigma}_{3,i}^{VW})^2 = \frac{1}{J(J+1)} \left( \sum_{j=1}^J \{\mu_{\cdot -}(X_i) - \mu_{\cdot -}(X_{\ell_{+,j}(i)})\} \right) \left( \sum_{j=1}^J \{\mu_{\cdot -}(X_i) - \mu_{\cdot -}(X_{\ell_{+,j}(i)})\} \right),$$

$$(\tilde{\sigma}_{4,i}^{VW})^2 = \varepsilon_{V,i} \frac{1}{J+1} \sum_{j=1}^J \varepsilon_{W,\ell_{-,j}(i)}, \quad (\tilde{\sigma}_{5,i}^{VW})^2 = \varepsilon_{W,i} \frac{1}{J+1} \sum_{j=1}^J \varepsilon_{V,\ell_{-,j}(i)},$$

$$(\tilde{\sigma}_{6,i}^{VW})^2 = \varepsilon_{V,i} \frac{1}{J+1} \sum_{j=1}^J \{\mu_{\cdot -}(X_i) - \mu_{\cdot -}(X_{\ell_{-,j}(i)})\},$$

$$(\tilde{\sigma}_{7,i}^{VW})^2 = \varepsilon_{W,i} \frac{1}{J+1} \sum_{j=1}^J \{\mu_{\cdot -}(X_i) - \mu_{\cdot -}(X_{\ell_{-,j}(i)})\},$$

$$(\tilde{\sigma}_{8,i}^{VW})^2 = \frac{1}{J+1} \sum_{j=1}^J \varepsilon_{V,\ell_{-,j}(i)} \sum_{k=1}^J \{\mu_{\cdot -}(X_i) - \mu_{\cdot -}(X_{\ell_{-,j}(i)})\},$$

$$(\tilde{\sigma}_{9,i}^{VW})^2 = \frac{1}{J+1} \sum_{j=1}^J \varepsilon_{W,\ell_{-,j}(i)} \sum_{k=1}^J \{\mu_{\cdot -}(X_i) - \mu_{\cdot -}(X_{\ell_{-,j}(i)})\}.$$
Since the argument is similar as the one in Section S.2.4 of CCT’s supplement, we only present the result for $\eta_{1VW}^T$. Since $J$ is fixed, take any $j$. We want to obtain the stochastic order of

$$\max_{V,W \in \{\bar{Y}, \bar{Z}_1, \ldots, \bar{Z}_{|\hat{S}|}\}} \left\{ \frac{1}{nh_n} \sum_{i=1}^{n} \mathbb{I}\{X_i < 0\} K(X_i/h_n)^2 (\varepsilon_{V,\ell-j(i)} - \varepsilon_{V,i}) \right\}.$$

Note that

$$\max_{V,W \in \{\bar{Y}, \bar{Z}_1, \ldots, \bar{Z}_{|\hat{S}|}\}} \left\{ \frac{1}{nh_n} \sum_{i=1}^{n} \mathbb{I}\{X_i < 0\} K(X_i/h_n)^2 (\varepsilon_{V,\ell-j(i)} - \varepsilon_{V,i}) \right\}$$

$$\leq \max_{V,W \in \{\bar{Y}, \bar{Z}_1, \ldots, \bar{Z}_{|\hat{S}|}\}} \left\{ \frac{1}{nh_n} \sum_{i=1}^{n} \mathbb{I}\{X_i < 0\} K(X_i/h_n)^2 \left\{ \begin{array}{l} (\varepsilon_{V,\ell-j(i)} - \varepsilon_{V,i}) \\ (\sigma_{VW}^{2}(X_{\ell-j(i)}) - \sigma_{VW}^{2}(X_{i})) \end{array} \right\} \right\}$$

$$= T_{11} + T_{12}.$$

For $T_{11}$, we can apply the localized Bernstein inequality in Lemma A.1, which implies

$$T_{11} = O_{p} \left( \sqrt{\log s^* \frac{s^*}{nh_n}} \right).$$

For $T_{12}$, note that

$$T_{12} = \max_{V,W \in \{\bar{Y}, \bar{Z}_1, \ldots, \bar{Z}_{|\hat{S}|}\}} \left\{ \frac{1}{nh_n} \sum_{i=1}^{n} \mathbb{I}\{X_i < 0\} K(X_i/h_n)^2 (\sigma_{VW}^{2}(X_{\ell-j(i)}) - \sigma_{VW}^{2}(X_{i})) \right\}$$

$$\leq \max_{1 \leq i \leq n} |X_{\ell-j(i)} - X_i| \max_{V,W \in \{\bar{Y}, \bar{Z}_1, \ldots, \bar{Z}_{|\hat{S}|}\}} |C_{VW}| \left\{ \frac{1}{nh_n} \sum_{i=1}^{n} \mathbb{I}\{X_i < 0\} K(X_i/h_n)^2 \right\}$$

$$= O_{p} \left( \max_{1 \leq i \leq n} |X_{\ell-j(i)} - X_i| \right) = O_{p}(n^{-1}),$$

where $C_{VW}$ is the Lipschitz constant for $\sigma_{VW}^{2}$ (and we assume $\max_{V,W \in \{\bar{Y}, \bar{Z}_1, \ldots, \bar{Z}_{|\hat{S}|}\}} |C_{VW}|$ is bounded), and the second equality follows from Abadie and Imbens (2006, Theorem 1). Combining these results, we have

$$T_{1-} = O_{p} \left( \frac{s^* \sqrt{\log s^*}}{\omega_n nh_n} \right).$$
References

[1] Abadie, A. and G. W. Imbens (2006) Large sample properties of matching estimators for average treatment effects, *Econometrica*, 74, 253-267.

[2] Angrist, J. D., Imbens, G. W. and D. B. Rubin (1996) Identification of causal effects using instrumental variables, *Journal of the American Statistical Association*, 91, 444-455.

[3] Arai, Y., Otsu T. and M. Seo (2021) Regression discontinuity design with potentially many covariates, [arXiv:2109.08351](https://arxiv.org/abs/2109.08351).

[4] Armstrong, T. B. and M. Kolesár (2018) Optimal inference in a class of regression models, *Econometrica*, 86, 655-683.

[5] Belloni, A., Chernozhukov, V., Chetverikov, D., Hansen, C. and K. Kato (2018) High-dimensional econometrics and regularized GMM, Working paper.

[6] Belloni, A., Chernozhukov, V. and C. Hansen (2014) Inference on treatment effects after selection amongst high-dimensional controls, *Review of Economic Studies*, 81, 608-650.

[7] Bickel, P. J., Ritov, Y. A. and A. B. Tsybakov (2009) Simultaneous analysis of Lasso and Dantzig selector, *Annals of Statistics*, 37, 1705-1732.

[8] Bühlmann, P. and S. van de Geer (2011) *Statistics for High-Dimensional Data*, Springer.

[9] Calonico, S., Cattaneo, M. D. and M. H. Farrell (2018) On the effect of bias estimation on coverage accuracy in nonparametric inference, *Journal of the American Statistical Association*, 113, 767-779.

[10] Calonico, S., Cattaneo, M. D. and M. H. Farrell (2020) Coverage error optimal confidence intervals for local polynomial regression, [arXiv:1808.01398](https://arxiv.org/abs/1808.01398).

[11] Calonico, S., Cattaneo, M. D., Farrell, M. H., and R. Titiunik (2017) rdrobust: Software for regression discontinuity designs, *Stata Journal*, 17, 372-404.

[12] Calonico, S., Cattaneo, M. D., Farrell, M. H., and R. Titiunik (2019) Regression discontinuity designs using covariates, *Review of Economics and Statistics*, 101, 442-451.

[13] Calonico, S., Cattaneo, M. D. and R. Titiunik (2014) Robust nonparametric confidence intervals for regression-discontinuity designs, *Econometrica*, 82, 2295-2326.
[14] Calonico, S., Cattaneo, M. D. and R. Titiunik (2015a) Optimal data-driven regression discontinuity plots, *Journal of the American Statistical Association*, 110, 1753-1769.

[15] Calonico, S., Cattaneo, M. D. and R. Titiunik (2015b) Rdrobust: An R package for robust inference in regression discontinuity design, *R Journal*, 7, 38-51.

[16] Card, D., Lee, D. S., Pei, Z. and A. Weber (2015) Inference on causal effects in a generalized regression kink design, *Econometrica*, 83, 2453-2483.

[17] Cattaneo, M. D. and J. C. Escanciano (2017) *Regression Discontinuity Designs: Theory and Applications*, in *Advances in Econometrics*, vol. 38, Emerald Group Publishing.

[18] Cattaneo, M. D., Titiunik, R. and G. Vazquez-Bare (2020) The regression discontinuity design, in Curini, L. and R. J. Franzese (eds.), *Handbook of Research Methods in Political Science and International Relations*, ch. 44, pp. 835-857, Sage Publications.

[19] Fan, J., and I. Gijbels (1992) Variable bandwidth and local linear regression smoothers, *Annals of Statistics*, 20, 2008-36.

[20] Friedman, J. H., Hastie, T. and R. Tibshirani (2010) Regularization paths for generalized linear models via coordinate descent, *Journal of Statistical Software*, 33, 1-22.

[21] Frölich, M. and M. Huber (2019) Including covariates in the regression discontinuity design, *Journal of Business & Economic Statistics*, 37, 736-748.

[22] Hahn, J., Todd, P. and W. van der Klaauw (2001) Identification and estimation of treatment effects with a regression-discontinuity design, *Econometrica*, 69, 201-209.

[23] Imbens, G. W. and T. Lemieux (2008) Regression discontinuity designs: a guide to practice, *Journal of Econometrics*, 142, 615-635.

[24] Kreiß, A. and C. Rothe (2023) Inference in regression discontinuity designs with high-dimensional covariates, *Econometrics Journal*, 26, 105-123.

[25] Lei, L. and P. Ding (2021) Regression adjustment in completely randomized experiments with a diverging number of covariates, *Biometrika*, 108, 815-828.

[26] Lin, W. (2013) Agnostic notes on regression adjustments to experimental data: reexamining Freedman’s critique, *Annals of Applied Statistics*, 7, 295-318.

[27] Ludwig, J. and D. L. Miller (2007) Does Head Start improve children’s life chances? Evidence from a regression discontinuity design, *Quarterly Journal of Economics*, 122, 159-208.
[28] Ruppert, D. and M. P. Wand (1994) Multivariate locally weighted least squares regression, *Annals of Statistics*, 22, 1346-1370.

[29] van de Geer, S., Bühlmann, P., Ritov, Y. and R. Dezeure (2014) On asymptotically optimal confidence regions and tests for high-dimensional models, *Annals of Statistics*, 42, 1166-1202.

[30] van der Vaart, A. W. and J. A. Wellner (1996) *Weak Convergence and Empirical Processes*, Springer.

[31] Zhang, C.-H. and S. S. Zhang (2014) Confidence intervals for low dimensional parameters in high dimensional linear models, *Journal of the Royal Statistical Society*, B, 76, 217-242.