Does Excess Quantum Noise Exist in Spontaneous Processes?

A. Aiello, M. P. van Exter, G. Nienhuis, J. P. Woerdman
Huygens Laboratory, Leiden University, P.O. Box 9504, Leiden, The Netherlands

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Abstract

We investigate the role of excess quantum noise in type-II degenerate parametric down conversion in a cavity with non-orthogonal polarization eigenmodes. Since only two modes are involved we are able to derive an analytical expression for the twin-photon generation rate measured outside the cavity as a function of the degree of mode nonorthogonality. Contrary to recent claims we conclude that there is no evidence of excess quantum noise for a parametric amplifier working so far below threshold that spontaneous processes dominate.

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The ultimate noise limit of optical devices is set by the quantum noise. A textbook example is formed by the laser, where phase and intensity fluctuations have a quantum origin. These fluctuations can be seen as the consequence of having "one noise photon in the laser mode" [Ref. 1, p. 72]. For small lasers, in particular semiconductor lasers, this quantum noise limit is easily reached in practical cases. More recently, there has been a large body of work pointing at the fact that the quantum noise may be enhanced by the so called excess noise factor or Petermann $K$ factor. This excess noise may occur when the laser cavity has nonorthogonal eigenmodes; it is then as if there are $K$ noise photons in the laser mode. There is no doubt regarding the physical reality of excess quantum noise; it has been verified in many experiments on lasers with nonorthogonal longitudinal, transverse and polarization modes.

However, there is a continuing debate whether the noise enhancement arises from an amplification of spontaneously emitted photons by the gain medium or from a cavity-enhanced single-atom decay rate. If the latter interpretation applied, excess noise would also be a valid concept (far) below the oscillation threshold of the device under consideration. In this case excess noise could be very useful; for instance it has been claimed that it could lead to an enhanced generation of twin photons in Spontaneous Parametric Down Conversion (SPDC), by placing the nonlinear crystal in an unstable cavity (which has nonorthogonal transverse eigenmodes).

The reason that there is room for uncertainty is that all theoretical approaches of this problem are necessarily model dependent. The most common experimental realization of mode nonorthogonality concerns the transverse modes of an unstable cavity. However, this case is intrinsically difficult to treat: one deals with an infinite manifold of transverse modes which cannot be truncated since there is no sharp distinction between system modes (= cavity modes) and reservoir modes (= free space modes). This has motivated us to study the effect of excess noise on cavity-enhanced SPDC, for a case where one can construct an exactly solvable quantum theory of mode nonorthogonality. In fact, we use a cavity with nonorthogonal polarization modes (instead of transverse modes). Our model comprises (and reduces to those as particular subcases) two theoretical models both of which have been experimentally verified. For the type-II degenerate parametric amplifier we use the Gardiner and Savage model [16] whose validity has been also recently verified by Lu and Ou [14]. For the cavity with two non-orthogonal polarization modes we adopt the model of Bretenaker and coworkers [6] where a large polarization $K$-factor has been demonstrated. The restriction to the polarization case does not limit the validity
of our conclusions: in earlier works it has been shown experimentally and theoretically that the polarization $K$ factor shows all the features of a generic $K$ factor. The main message of our Letter is that there is no spontaneous emission enhancement in parametric down conversion.

SPDC constitutes a natural framework in which to study polarization excess noise. In a type-II SPDC process, two orthogonally polarized photons are generated. Because of crystal anisotropy, for a fixed angular frequency only a restricted set of spatial directions is allowed for the emitted photons. In the degenerate case one can achieve a single allowed direction for a collinear emission, thus, besides imperfect phase-matching effects, single transverse mode operation can be realized. An optical cavity also allows for several resonant longitudinal modes but the double resonance condition for SPDC restricts this number. It can be shown that because of crystal birefringence, for a type-II process the double resonance condition can only be satisfied at degenerate frequency so that the number of allowed longitudinal modes is reduced to one.

We consider a cavity having one perfectly reflecting mirror at position $z = -L$, and a partially reflecting mirror at $z = 0$, as shown in Fig. 1. We decompose the electric field inside the cavity into left (subscript $L$) and right (subscript $R$) propagating waves. In degenerate type-II SPDC two orthogonally polarized fields are generated. Thus, we have only two modes $a$ and $b$, with the same angular frequency $\omega$ and orthogonal polarization directions, parallel to the $x$ and $y$ axis, respectively. Two other modes, $f_L$ and $f_R$, having the same polarization as mode $a$, are introduced in order to assure the unitarity of the model. Mode non-orthogonality is achieved by inserting in the cavity a phase anisotropy due to circular birefringence (polarization rotator) and a loss anisotropy generated by linear dichroism (polarization dependent absorber), following the scheme given in.

The optical elements inside the cavity are: an absorber modeled as a beam-splitter acting only on the $x$ polarization, a crystal with nonlinear gain $G$ and a rotator which rotates the polarization axes by an angle $\phi$ along $z$ axis. A delay line has been put in front of the left mirror to account for the mode propagation inside the
cavity. We assume that the optical elements are infinitesimally thin and we put the operator phases at \( z = 0 \) equal to zero. It should be noticed that because phase matching conditions can be fulfilled only by left-travelling or right-travelling waves (we assume left), the light is amplified only during the first half round trip whereas it is freely propagating during the second half round trip. We characterize both lossless passive and active optical elements as unitary devices with two input ports and two output ports related by a scattering matrix.

On the right mirror the output annihilation operators belonging to mode \( a \), are related to the input operators on the same mode by the transformation

\[
\begin{align*}
\hat{a}_{\text{out}} &= \mathcal{T} \hat{a}_{1R} + \mathcal{R} \hat{a}_{\text{in}}, \\
\hat{a}_{1L} &= \mathcal{R} \hat{a}_{1R} + \mathcal{T} \hat{a}_{\text{in}},
\end{align*}
\]

where \( \mathcal{R} = -\sqrt{\mathcal{R}} \), \( \mathcal{T} = i \sqrt{1 - \mathcal{R}} \) and \( 0 \leq R < 1 \). For mode \( b \) the above relations hold if we make everywhere the substitution \( a \rightarrow b \). The effect of the rotator on left-travelling mode operators can be represented as:

\[
\begin{align*}
\hat{a}_{2L} &= \cos \phi \hat{a}_{1L} + \sin \phi \hat{b}_{1L}, \\
\hat{b}_{2L} &= -\sin \phi \hat{a}_{1L} + \cos \phi \hat{b}_{1L}.
\end{align*}
\]

Note that we have chosen as a rotator a device antisymmetric with respect to temporal inversion (e.g., a Faraday rotator). Then after a round trip the total rotation angle is equal to \( 2 \phi \). The corresponding matrix for right-travelling modes is obtained by substituting in the above formula \( 1 \leftrightarrow 2 \) and \( L \rightarrow R \). The scattering matrix for the parametric crystal is given by:

\[
\begin{align*}
\hat{a}_{3L} &= \mathcal{G} \hat{a}_{2L} + (G^2 - 1)^{1/2} \hat{b}_{2L}^\dagger, \\
\hat{b}_{3L}^\dagger &= (G^2 - 1)^{1/2} \hat{a}_{2L} + \mathcal{G} \hat{b}_{2L}^\dagger,
\end{align*}
\]

where the real gain \( G \) satisfies \( G > 1 \). The corresponding transformation for the right-travelling modes follows after the substitutions \( 3 \leftrightarrow 2 \), \( L \rightarrow R \) and \( G = 1 \) (transparent medium). Since Eqs. (3) preserve bosonic commutation rules, for a parametric amplifier with a classical non-depleted pump it is not necessary to add noise from an external bath to account for pump fluctuations. In our model only the down-converted field is confined by the cavity, not the pump field, therefore the cavity mode structure cannot affect the pump beam fluctuations. For the absorber, which introduces losses only for mode \( a \) we have:

\[
\begin{align*}
\hat{a}_{4L} &= t \hat{a}_{3L} + r \hat{f}_{\text{Lin}}, \\
\hat{b}_{4L} &= \hat{b}_{3L}, \\
\hat{f}_{\text{out}} &= r \hat{a}_{3L} + t \hat{f}_{\text{Lin}},
\end{align*}
\]

where \( r = i \sqrt{1 - t^2} \) and the parameter \( t \) (0 \leq t \leq 1) represents the ratio between field amplitudes along \( x \) and \( y \) polarization directions. For right-travelling modes \( 4 \leftrightarrow 3 \) and \( L \rightarrow R \). For the delay line we have \( \hat{a}_{5L} = \exp(i \theta) \hat{a}_{4L} \), where \( \theta = \omega L/c \). For right-travelling modes \( 5 \leftrightarrow 4 \) and \( L \rightarrow R \). The same relation holds for mode \( b \). Finally, on the left mirror \( \hat{a}_{5R} = -\hat{a}_{5L} \) and similarly for mode \( b \).

The equations given above can be solved to express right-travelling mode operators in terms of left-travelling mode operators:

\[
\begin{align*}
\begin{pmatrix}
\hat{a}_{1R} \\
\hat{b}_{1R}
\end{pmatrix} &= -\mathcal{G} \begin{pmatrix}
C_{12}^+ & S_1 \\
-S_1 & C_{12}^-
\end{pmatrix} \begin{pmatrix}
\hat{a}_{1L} \\
\hat{b}_{1L}
\end{pmatrix} + \mathcal{G} \begin{pmatrix}
S_2 & -C_{21}^+ \\
C_{21}^- & S_2
\end{pmatrix} \begin{pmatrix}
\hat{a}_{1L}^\dagger \\
\hat{b}_{1L}^\dagger
\end{pmatrix} + \begin{pmatrix}
\hat{f}_a \\
\hat{f}_b
\end{pmatrix},
\end{align*}
\]

where \( \mathcal{G} \equiv (G^2 - 1)^{1/2} \), \( \gamma_j = \exp(2i \theta)(t^2 - (-1)^j)/2 \), \( S_j = \gamma_j \sin(2\phi) \), \( C_{ij}^\pm = \gamma_i \cos(2\phi) \pm \gamma_j \), \( \pm (i, j = 1, 2) \) and

\[
\begin{align*}
\begin{pmatrix}
\hat{f}_a \\
\hat{f}_b
\end{pmatrix} &= \mathcal{R} \begin{pmatrix}
(t \hat{f}_{\text{Lin}} + \hat{f}_{\text{Rin}}) \cos \phi \\
-(t \hat{f}_{\text{Lin}} + \hat{f}_{\text{Rin}}) \sin \phi
\end{pmatrix}.
\end{align*}
\]

3
Figure 2: Plot of the total average photon number $\bar{N} \equiv \bar{n}_a + \bar{n}_b$ of the sub-threshold OPO, calculated at resonance, as function of the absorber transmission $t$ and of the rotator angle $\phi$. The values of the other parameters are: $G = 1.01$, $R = 0.2$. For $t = 0$ photons in mode $a$ are fully absorbed and the residual value of $\bar{N}$ is due to contribution of only mode $b$.

For a non-zero rotation angle $\phi$, the quantum noise operators $\hat{f}_a$ and $\hat{f}_b$ do not commute:

$$\begin{align*}
[\hat{f}_a, \hat{f}_a^\dagger] &= (1 - t^4) \cos^2 \phi, \\
[\hat{f}_b, \hat{f}_b^\dagger] &= (1 - t^4) \sin^2 \phi, \\
[\hat{f}_a, \hat{f}_b^\dagger] &= -(1 - t^4) \sin \phi \cos \phi.
\end{align*}$$

This noise correlation disappears when the eigenmodes of the classical cold cavity round-trip matrix $M$ become orthogonal ($\phi = 0$ and/or $t = 1$). The matrix $M$ can be obtained from Eq. (5) by putting $G = 1$ (cold cavity) and disregarding the quantum noise term (classical). In fact $M$ coincides with the matrix in the first row of Eq. (5). Diagonalizing the non-unitary matrix $M$ we find for the Petermann $K$ factor

$$K_\triangleleft = \frac{(1 - t^2)^2}{(1 - t^2)^2 - (1 + t^2)^2 \sin^2 2\phi},$$

when $t < t_c$ (locked regime), and

$$K_\triangleright = \frac{(1 + t^2)^2 \sin^2 2\phi}{(1 + t^2)^2 \sin^2 2\phi - (1 - t^2)^2},$$

for $t > t_c$ (unlocked regime), where $t_c(\phi) = \sqrt{(1 - |\sin 2\phi|)/(1 + |\sin 2\phi|)}$. Apart from notation these results agree with earlier works. We conclude that our model correctly accounts for non-orthogonal polarization modes.

The next step is to calculate the SPDC rate and to study how it depends on the "non-orthogonality parameters" $t$ and $\phi$. Equations (10) together with Eqs. (5) can be straightforwardly solved to express "out" operators in terms of "in" operators. The resulting expressions are very cumbersome and it is not useful to write them explicitly. Their general form is

$$\hat{a}_{\text{out}} = \sum_{\alpha = a, b} \left( M_{1\alpha} \hat{a}_{\text{in}} + M_{2\alpha} \hat{a}_{\text{in}}^\dagger + M_{3\alpha} \hat{f}_\alpha + M_{4\alpha} \hat{f}_\alpha^\dagger \right),$$

(10)
Figure 3: Dotted-dashed line: "geometrical" Petermann $K$ factor, given by Eqs. (8-9) for a cavity without crystal, calculated for $\phi = \pi/8$, as a function of the absorber transmission $t$. The value of $K$ diverges for $t \to t_c$ ($\phi = \pi/8$) $\approx 0.41$. Dashed line: total average photon number $\bar{N}$ calculated at resonance and $\phi = \pi/8$. The values of the other parameters are $G = 1.01$, $R = 0.2$, corresponding to a sub-threshold OPO. The grey band represents all possible values of $\bar{N}$ for non-orthogonal modes. Note that $\bar{N}$ is not enhanced for $t = t_c$.

and similarly for mode $b$, where $M_{\alpha\beta}$ are complicated functions of $t, \phi, G, R$ and $L$. From the above formula we calculate the average photon number emitted on modes $a$ and $b$:

$$\bar{n}_\alpha = \langle \hat{a}_\alpha^\dagger \hat{a}_\alpha \rangle_{\text{vac}}, \quad (\alpha = a, b),$$

where the subscript "vac" indicates that the quantum expectation value is calculated for incoming vacuum field. When both absorber and rotator are switched off (orthogonal modes case) $\bar{n}_a = \bar{n}_b = \bar{n}$, where

$$\bar{n} = (G^2 - 1) \left[ \frac{1 - R}{1 - 2G\sqrt{R} \cos(2\omega L/c) + R} \right]^2.$$

The term inside the square brackets, when calculated for $G = 1$, coincides with the spontaneous emission modification factor $F$ but in our case it is quadratic because of nonlinearity. At resonance ($L = m\pi c/\omega$, with $m$ integer), a divergence appears for $\bar{n}$ when $G = (1 + R)/(2\sqrt{R}) > 1$, corresponding to the threshold of oscillation. However, we are interested only in the sub-threshold case where a privileged lasing mode is not selected. We report in Fig. 2 the total average photon number $\bar{N} = \bar{n}_b + \bar{n}_a$, evaluated at resonance, as function of the absorber transmission coefficient $t$ and of the rotation angle $\phi$ due to the rotator. The nonlinear gain $G$ and the output mirror reflectivity $R$ have been chosen as $G = 1.01$, $R = 0.2$, so that sub-threshold operation is achieved.

From Fig. 2 it is clear that the local maxima of $\bar{N}$, for $t$ variable, are located on the curve $\phi = 0$ which corresponds to a cavity with orthogonal modes. This curve constitutes the upper boundary of the grey band shown in Fig. 3. The other points in the grey band represent all possible values of $\bar{N}$, calculated with the same parameters as in Fig. 2, for cavities with non-orthogonal modes. All these points are below the curve corresponding to orthogonal modes; so we do not find any twin-photon enhancement under these conditions.

This may be compared with the behavior of the "geometrical" $K$ factor, as given by Eqs. (8-9), that is the $K$ factor which appears in a "cold cavity" (i.e., without gain) as simple consequence of mode nonorthogonality. Fig. 3 shows the behavior of this $K$ factor with respect to $\bar{N}$, as function of the absorber transmission $t$. Both $K$ and $\bar{N}$ are evaluated for $\phi = \pi/8$. The values of the other parameters for $\bar{N}$ are the same as in Fig. 2. From a geometrical point of view, when $t = t_c$ the cavity eigenmodes become parallel and the corresponding $K$ factor diverges. In Fig. 3 this resonant behavior of $K$, when $t$ approaches $t_c$, is evident, but at the same time there is
no signature of a critical behavior of $\bar{N}$. Therefore we conclude that for a sub-threshold OPO, the total average photon number $\bar{N}$ does not depend on $K$.

In conclusion we have shown that there is no excess quantum noise enhancement in type-II SPDC. On the contrary, the use of a cavity with nonorthogonal (instead of orthogonal) eigenmodes leads to a reduction of the twin photon generation rate. Excess quantum noise must therefore be exclusively ascribed to amplification of spontaneously emitted photons; the spontaneous emission process itself is not affected. Excess quantum noise becomes effective only very close to threshold when one of the cavity eigenmodes is "selected" as the lasing mode which dominates over the other modes.

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