Optimal state discrimination with a fixed rate of inconclusive results: Analytical solutions and relation to state discrimination with a fixed error rate

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We study an optimum measurement for quantum state discrimination, which maximizes the probability of correct results when the probability of inconclusive results is fixed at a given value. The measurement describes minimum-error discrimination if this value is zero, while under certain conditions it corresponds to optimized maximum-confidence discrimination, or to optimum unambiguous discrimination, respectively, when the fixed value reaches a definite minimum. Using operator conditions that determine the optimum measurement, we derive analytical solutions for the discrimination of two mixed qubit states, including the case of two pure states occurring with arbitrary prior probabilities, and for the discrimination of N symmetric states, both pure and mixed. We also consider a case where the given density operators resolve the identity operator, and we specify the optimality conditions for partially symmetric states. Moreover, we show that from the complete solution for arbitrary values of the fixed rate of inconclusive results one can always obtain the optimum measurement in another strategy where the error rate is fixed, and vice versa.

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I. INTRODUCTION

In quantum state discrimination we want to determine the actual state of a quantum system that is known to be in a certain state belonging to a given set of states. This is an essential problem for many tasks in quantum communication and quantum cryptography. Since nonorthogonal quantum states cannot be distinguished perfectly, various optimized discrimination strategies have been developed. The best known of these are minimum-error discrimination [1, 2] and optimum unambiguous discrimination, where the latter strategy was originally derived for pure states [3–6] and later also considered for mixed states [7,8]. In unambiguous discrimination errors do not occur, that is, the total error rate $P_e$ is required to vanish, $P_e = 0$. This can be achieved at the expense of admitting inconclusive results, where the measurement fails to give a definite answer. Optimum unambiguous discrimination minimizes this failure probability, or failure rate, $Q$, yielding the minimum value $Q = Q_{\text{min}}$.

Unambiguous state discrimination is not possible for pure states that are linearly dependent [13], or for mixed states described by density operators with identical supports [7,8], where the support is the Hilbert space spanned by the eigenvectors with non-zero eigenvalues. When unambiguous discrimination is impossible, related measurements can be applied, which discriminate the states with maximum possible confidence for each conclusive outcome [10, 11]. Optimized maximum-confidence discrimination is achieved by the particular one of these measurements that minimizes the failure probability [18–22]. This measurement corresponds to optimum unambiguous discrimination when for each of the different conclusive results the maximum confidence is equal to unity.

While the confidence is defined separately for each conclusive outcome, in minimum-error discrimination the overall error rate $P_e$, averaged over all outcomes, is minimized. In this strategy inconclusive results are not allowed, that is, $Q = 0$. Chefles and Barnett [23] introduced a more general strategy that minimizes $P_e$ when a certain fixed rate $Q$ of inconclusive results is admitted, thus reducing the minimum achievable value of $P_e$. For discriminating two equiprobable nonorthogonal pure states these authors obtained the optimum solution, which interpolates between minimum-error discrimination and optimum unambiguous discrimination when the fixed rate $Q$ grows from $Q = 0$ to $Q = Q_{\text{min}}$. Their research triggered further investigations [24–29]. In particular, optimum state discrimination with a fixed rate $Q$ of inconclusive results was extended to mixed states [25, 26], and the relation to the strategy of maximum-confidence discrimination was briefly discussed in our previous papers [18, 22]. Clearly, in a measurement where $Q$ is fixed, minimizing $P_e$ corresponds to maximizing the overall rate of correct results, $P_c$. Fiurášek and Ježek [23] derived general operator conditions that have to be fulfilled in the optimum measurement with an arbitrary fixed value $Q$, holding for the discrimination of an arbitrary number $N$ of mixed states. For $N = 2$ they solved the optimization problem in the special case of two mixed qubit that have the same purity and occur with equal prior probabilities [25], which includes the solution for two equiprobable pure states [23]. Other solutions were not obtained.

Recently optimum state discrimination has been also investigated for a measurement where a certain fixed error rate $P_e$ is admitted [27, 28]. With a fixed rate of $P_e$, maximizing the rate of correct results $P_c$ corresponds to minimizing the failure rate $Q$. The optimum measurement in this strategy was studied for the discrimination of two states, $N = 2$, where the solution was derived for two pure states occurring with arbitrary prior probabilities [28, 29]. However, it was not recognized that there exists a relation between optimum state discrimination with a fixed value of $Q$, on the one hand, and with a fixed...
value of $P_c$, on the other hand. This relation implies that by completely solving the optimization problem for one of these two discrimination strategies one can also obtain the solution for the other strategy, as will be shown in the present paper.

The purpose of our paper is twofold. First, for optimum state discrimination with a fixed rate of $Q$ we derive analytical solutions going beyond the solutions obtained so far \cite{22,24}, and we also outline the relation to maximum-confidence discrimination, extending our previous discussions \cite{18,22}. Second, we show that when the maximum rate of correct results $P_c$ is known as a function of the fixed failure probability $P_f$, then from this function one can also obtain the maximum rate of correct results in dependence of a fixed error rate $P_e$, and vice versa, due to a general relation between the solutions of the two optimized strategies that holds for the discrimination of an arbitrary number $N$ of mixed states.

The paper is organized as follows: We begin in Sec. II with an alternative derivation of the optimality conditions for state discrimination with fixed $Q$, and with establishing the relation to maximum-confidence discrimination. In Sec. III together with Appendix A we apply these conditions to the discrimination of two mixed qubit states occurring with arbitrary prior probabilities, while Sec. IV together with Appendix B is devoted to the discrimination of $N$ symmetric states. In Sec. V we consider a case where the density operators resolve the identity operator, and we also specify the optimality conditions for a case of partially symmetric states. The general relation between the two optimization strategies where either $P_c$ or $Q$ has a fixed value is presented in Sec. VI, together with an example. Sec. VII provides a summary of results and concludes the paper.

II. OPTIMUM MEASUREMENT

A. Conditions for optimality

We suppose that a quantum system is prepared with the prior probability $\eta_j$ in one of $N$ given states described by the density operators $\rho_j$ ($j = 1, \ldots, N$), where $\sum_{j=1}^{N} \eta_j = 1$. The general task is to perform a discrimination measurement in order to infer in which of the $N$ possible states the system was prepared. The measurement is described by $N + 1$ positive detection operators $\Pi_1, \ldots, \Pi_N$ and $\Pi_0$, where $\text{Tr} (\rho_j \Pi_j)$ is the conditional probability that a system is inferred to be in the state $\rho_j$ given it had been prepared in the state $\rho_k$, while $\text{Tr} (\rho_0 \Pi_0)$ is the conditional probability that in this case an inconclusive result is obtained and the measurement fails to discriminate the states. The detection operators fulfill the completeness relation $\sum_{j=0}^{N} \Pi_j = I$, where $I$ is the identity operator in the $d$-dimensional Hilbert space $\mathcal{H}_d$ spanned by the eigenstates of the operators $\rho_1, \ldots, \rho_N$ that belong to nonzero eigenvalues. Unless each of the detection operators is a projector, the measurement is a generalized one. Once the detection operators are known, implementations of the generalized measurement as a projective measurement in an enlarged Hilbert space can be obtained using standard methods \cite{30,31}.

We are interested in the specific discrimination measurement where the overall failure probability

$$Q = \sum_{j=1}^{N} \eta_j \text{Tr} (\rho_j \Pi_0) = \text{Tr} (\rho \Pi_0) \quad \text{with} \quad \rho = \sum_{j=1}^{N} \eta_j \rho_j$$

has a given fixed value $Q < 1$, while the overall probability of getting correct results, $P_c$, is as large as possible. Here we defined the total density operator $\rho$, which has its support in the full Hilbert space $\mathcal{H}_d$ and is thus an operator of rank $d$. Our optimization problem can be expressed as follows:

$$\text{maximize} \quad P_c = \sum_{j=1}^{N} \eta_j \text{Tr} (\rho_j \Pi_j) = 1 - Q - P_e, \quad (2.2)$$

subject to

$$Q = \text{Tr} (\rho \Pi_0) = \text{const}, \quad \sum_{j=0}^{N} \Pi_j = I. \quad (2.3)$$

In Eq. (2.2) we introduced the overall error probability $P_e$. Since $Q$ is fixed, the maximum of the absolute rate of correct results, $P_c$, determines also the maximum of the relative rate of correct results at the same value of $Q$, defined as $R_c = P_c/(1 - Q)$ \cite{25}. We thus get

$$P_c^{\max} |_Q = R_c^{\max} |_Q (1 - Q). \quad (2.4)$$

In the special case where $R_c^{\max} |_Q = 1$ errors do not occur and the states are unambiguously discriminated.

As will be shown in Sec. II B, if $Q$ exceeds a certain minimum, denoted by $Q’$, the maximum relative rate of correct results, $R_c^{\max} |_Q$, stays constant with growing $Q$. The optimum measurement resulting from Eqs. (2.2) and (2.3) with $Q > Q’$ then yields a maximum of $P_c$ that is smaller than $P_c^{\max} |_Q$ and is therefore without practical relevance. One could modify the optimization problem and ask for the maximum of $P_c$ under the constraint that $Q$ does not exceed a certain fixed margin, $Q_M$. The maximum of $P_c$ under this modified constraint then follows from the solution of the original maximization problem, Eqs. (2.2) and (2.3), and is given by $P_c^{\max} |_{Q_M}$ if $Q_M < Q’$ and by $P_c^{\max} |_Q$ if $Q_M > Q’$.

In order to derive analytical solutions of the optimization problem posed by Eqs. (2.2) and (2.3) we use the operator conditions \cite{22,24} that have to be fulfilled in the optimum measurement. Let us begin by re-deriving these optimality conditions. For this purpose we introduce a Hermitian operator $Z$ and a scalar real amplifier $a$. Due to the two constraints in Eq. (2.3) the equation

$$\text{Tr} Z - aQ - P_c = \text{Tr} [(Z - a \rho) \Pi_0] + \sum_{j=1}^{N} \text{Tr} [(Z - \eta_j \rho_j) \Pi_j] \quad (2.5)$$
is identically fulfilled for any operator $Z$ and any multiplier $a$. Since the detection operators are positive it follows that the positivity conditions

$$Z - a\rho \geq 0, \quad Z - \eta_j \rho_j \geq 0 \quad (2.6)$$

($j = 1,\ldots, N$) imply that Tr $Z - aQ - P_c \geq 0$. Hence when the positivity conditions in Eq. (2.6) are fulfilled, the minimum of Tr $Z - aQ$ establishes an upper bound for $P_c$. In the optimum measurement, where $P_c$ is equal to this bound, both sides of Eq. (2.5) vanish for the optimum multiplier $a$ and the optimality conditions results by multiplying Eqs. (2.6) and (2.7) from the left and right by the Hermitian of the optimality conditions. This representation and positive detection operators $\bar{\Pi}$ of the optimality conditions it is sometimes advantageous to introduce transformed operators $\bar{\Pi}_0$ and $\bar{\Pi}_j$ ($j = 1,\ldots, N$). Equations (2.6) and (2.7) together therefore establish sufficient optimality conditions, first derived by Fiurášek and Ježek [22] with the help of Lagrangian multipliers. Using methods of semidefinite programming, the optimality conditions have been shown to be not only sufficient, but also necessary [26]. For solving the optimization problem it is sometimes advantageous to use

$$\bar{\Pi}_0 = \rho^{1/2} \Pi_0 \rho^{-1/2}, \quad \Gamma = \rho^{1/2} Z \rho^{-1/2}, \quad (2.8)$$

$$\bar{\Pi}_j = \rho^{1/2} \Pi_j \rho^{-1/2}, \quad \hat{\rho}_j = \rho^{-1/2} \rho_j \rho^{1/2} \quad (2.9)$$

for $j = 1,\ldots, N$, Eqs. (2.6) and (2.7) yield the optimality conditions

$$\Gamma - \hat{\rho}_j \geq 0, \quad (\Gamma - \hat{\rho}_j) \bar{\Pi}_j = 0 \quad (j = 1,\ldots, N), \quad (2.10)$$

$$\Gamma - aI \geq 0, \quad (\Gamma - aI) \bar{\Pi}_0 = 0, \quad (2.11)$$

where $\bar{\Pi}_0 + \sum_{j=1}^N \bar{\Pi}_j = \rho$, due to the completeness relation of the detection operators. This representation of the optimality conditions results by multiplying Eqs. (2.6) and (2.7) from the left and right by the Hermitian operator $\rho^{-1/2}$, taking into account that for any operator $A$ in $\mathcal{H}$ the relation $A \rho^{-1/2} = 0$ can only hold when $A = 0$, since the support of $\rho$ is the full Hilbert space $\mathcal{H}$.

Provided that an operator $Z$, a scalar multiplier $a$ and positive detection operators $\Pi_1,\ldots,\Pi_N$ with $\Pi_0 = I - \sum_{j=1}^N \Pi_j \geq 0$ satisfy Eqs. (2.6) and (2.7), or Eqs. (2.8) - (2.11), respectively, then the detection operators determine the optimum measurement, which maximizes $P_c$ with the fixed value $Q = \text{Tr}(\rho \bar{\Pi}_0) = \text{Tr} \bar{\Pi}_0$ and yields

$$P_c^{\max} = \sum_{j=1}^N \eta_j \text{Tr}(\rho \bar{\Pi}_j) = \sum_{j=1}^N \text{Tr}(\hat{\rho}_j \bar{\Pi}_j). \quad (2.12)$$

By taking the trace in both equalities in Eq. (2.7) and summing over all states in the second equality we arrive at the expressions

$$\text{Tr}(Z \Pi_0) = aQ, \quad P_c^{\max} = \text{Tr}[Z(I - \Pi_0)] = \text{Tr} Z - aQ. \quad (2.13)$$

If $Q = 0$, that is, if inconclusive results are not allowed, the optimum measurement corresponds to minimum-error discrimination, described by the well-known optimality conditions [25] arising from Eqs. (2.6) and (2.7) with $\Pi_0 = 0$ and $a = 0$ [1, 2].

## B. Limiting case of sufficiently large $Q$ and relation to maximum-confidence discrimination

When the fixed failure probability $Q$ is getting larger and larger, the operator $\bar{I}_0$, defined in Eq. (2.8), will turn into an operator of rank $d$ as soon as $Q$ reaches a certain minimum value $Q'$ [25], due to the fact that for $Q = 1$ we must have $\Pi_0 = I$ and thus $\Pi_0 = \rho$ with rank($\rho$) = $d$. This means that for $Q \geq Q'$ the operator $\bar{I}_0$ has its support in the full Hilbert space $\mathcal{H}_d$ and the equality in Eq. (2.11) can only hold when $\Gamma - aI = 0$.

For $Q > Q'$ the optimality conditions therefore reduce to

$$aI - \hat{\rho}_j \geq 0, \quad (aI - \hat{\rho}_j) \bar{\Pi}_j = 0 \quad (j = 1,\ldots, N) \quad (2.14)$$

[25]. If $Q < 1$ at least one of the operators $\bar{I}_j$ has to be different from zero. The equality condition in Eq. (2.14) implies that for any $j$ where $\bar{I}_j \neq 0$ the eigenstates of $aI - \hat{\rho}_j$ belonging to nonzero eigenvalues cannot span the full Hilbert space $\mathcal{H}_d$. Therefore when $\bar{I}_j \neq 0$ at least one of the eigenvalues of $aI - \hat{\rho}_j$ is equal to zero in the optimum measurement with $Q \geq Q'$. Together with the positivity condition in Eq. (2.11) this requires that

$$a = \max\{C_1,\ldots, C_N\}, \quad C_j = \max\{\text{eig}(\hat{\rho}_j)\}. \quad (2.15)$$

that is, where $C_j$ is the largest eigenvalue of $\hat{\rho}_j$. For any state $j$ with $C_j < a$ the operator $aI - \hat{\rho}_j$ has its support in the full Hilbert space and the equality in Eq. (2.11) can only hold when $\Pi_j = 0$. For those states $j$ where $\bar{I}_j \neq 0$, the support of the operator $\bar{I}_j$ is the eigenspace of $\hat{\rho}_j$ belonging to its largest eigenvalue, since this guarantees that $\bar{I}_j$ is orthogonal to $aI - \hat{\rho}_j$. By taking the trace in the equality in Eq. (2.11), summing over all states and inserting the value of $a$ we arrive at

$$P_c^{\max} = \max\{C_j\}(1 - Q) \quad \text{if} \quad Q \geq Q'. \quad (2.16)$$

In order to determine $Q'$, we have to minimize $Q = 1 - \sum_j \text{Tr} \bar{I}_j$ on the conditions that the operators $\bar{I}_j$ have the required supports, as described after Eq. (2.15), and that $\Pi_0 = \rho - \sum_{j=1}^N \bar{I}_j \geq 0$.

The eigenvalues $C_j$ introduced in Eq. (2.15) have a definite meaning in state discrimination. They determine the maximum confidence that can be achieved for the individual measurement outcome $j$ or, equivalently, the maximum achievable ratio between all instances where the outcome $j$ is correct and all instances where the outcome $j$ occurs. In fact, with Eq. (2.15) it follows that

$$\max_{n_j} \left\{ \frac{\eta_j \text{Tr}(\rho \bar{\Pi}_j)}{\text{Tr}(\rho \bar{\Pi}_j)} \right\} = \max_{n_j} \left\{ \frac{\text{Tr}(\hat{\rho}_j \bar{\Pi}_j)}{\text{Tr} \bar{\Pi}_j} \right\} = C_j, \quad (2.17)$$
where for each individual \( j \) the maximization is performed with respect to all choices for the detection operator \( \Pi_j \), or \( \Pi_{j'} \), respectively. Clearly, in a measurement with \( C_j = 1 \) the state \( j \) is unambiguously discriminated. From Eq. \( (2.17) \) it becomes obvious that the maximum confidence for the outcome \( j \), equal to \( C_j \), is obtained when \( \Pi_j \) has its support in the eigenspace of \( \tilde{\rho}_j \) belonging to the eigenvalue \( C_j \). \(^{16} \) When this condition holds for each of the \( N \) states, that is, when for each outcome \( j \) the confidence is maximal, the measurement is called a maximum-confidence measurement. \(^{16} \)

Optimized maximum-confidence discrimination is achieved by the specific maximum-confidence measurement where the probability of inconclusive results takes its smallest possible value, \( Q_{\text{MC}}^{\text{MC}} \). \(^{18} \) \(^{22} \) A comparison with the constant \( Q' \) characterized after Eq. \( (2.16) \) reveals that

\[
Q' = Q_{\text{MC}}^{\text{MC}} \quad \text{if} \quad C_1 = \cdots = C_N = C. \quad (2.18)
\]

Hence in all cases where the maximum confidence is the same for each conclusive outcome, the measurement maximizing \( P_c \) when \( Q \) is fixed at the value \( Q' \) is equal to the measurement for optimized maximum-confidence discrimination. When \( C = 1 \), the latter measurement corresponds to optimum unambiguous discrimination.

### III. TWO MIXED QUBIT STATES

#### A. Method for applying the optimality conditions

In this paper we want to determine the optimum measurement for an arbitrary value of the fixed failure probability \( Q \), restricting ourselves to cases that allow a simple analytical solution. Let us start with the discrimination of two mixed qubit states in a joint two-dimensional Hilbert space, where we use the optimality conditions in the form of Eqs. \( (2.10) \) and \( (2.11) \). For \( N = 2 \) the transformed density operators \( \tilde{\rho}_1 \) and \( \tilde{\rho}_2 \) defined in Eq. \( (2.9) \) have identical systems of eigensates, due to the relation \( \tilde{\rho}_1 + \tilde{\rho}_2 = I \). Their spectral representations therefore can be written as

\[
\tilde{\rho}_1 = C_1 |\nu_1\rangle \langle \nu_1| + (1 - C_2) |\nu_2\rangle \langle \nu_2|, \quad (3.1)
\]

\[
\tilde{\rho}_2 = (1 - C_1) |\nu_1\rangle \langle \nu_1| + C_2 |\nu_2\rangle \langle \nu_2|, \quad (3.2)
\]

where \( C_1 > 1/2 \) and \( C_2 > 1/2 \), since the case \( C_1 = C_2 = 1/2 \) would imply that \( \rho_1 = \rho_2 \). The constants \( C_1 \) and \( C_2 \) have the meaning of the maximum achievable confidence for the two respective outcomes, cf. Eqs. \( (2.15) \) and \( (2.17) \). It is convenient to use the orthonormal eigenstates \( |\nu_1\rangle \) and \( |\nu_2\rangle \) as the basis states for solving the optimization problem. For this purpose we define the matrix elements

\[
|\nu_j\rangle |\nu_j\rangle = \rho_{jj}, \quad \langle \nu_1| \rho |\nu_2\rangle = \rho_{12} = |\rho_{12}| e^{i\theta}, \quad (3.3)
\]

\[
|\nu_j\rangle |\Gamma\rangle |\nu_j\rangle = \Gamma_{jj}, \quad \langle \nu_1| \Gamma |\nu_2\rangle = \Gamma_{12} = |\Gamma_{12}| e^{i\phi} \quad (3.4)
\]

with \( j = 1, 2 \), where \( \rho_{11} + \rho_{22} = 1 \). In the special case when the two given states are pure, \( \rho_j = |\psi_j\rangle \langle \psi_j| \), we get from Eq. \( (2.9) \)

\[
\tilde{\rho}_j = |\nu_j\rangle \langle \nu_j| \text{ with } \rho^{1/2} |\nu_j\rangle = \sqrt{\eta_j} |\psi_j\rangle,
\]

which yields

\[
\rho_{12} = \sqrt{\eta_1 \eta_2} \langle \psi_1 | \psi_2 \rangle, \quad \rho_{jj} = \eta_j, \quad C_j = 1. \quad (3.5)
\]

The relation \( C_1 = C_2 = 1 \) reflects the fact that each of the two pure states can be unambiguously discriminated. Using Eq. \( (2.9) \) we obtain \( \text{Tr}(\rho \tilde{\rho}_j) = \eta_j \), leading to

\[
C_1 \rho_{11} + (1 - C_2) \rho_{22} = \eta_1, \quad (1 - C_1) \rho_{11} + C_2 \rho_{22} = \eta_2. \quad (3.6)
\]

Now we are prepared to apply the optimality conditions. We start with the assumption that for the given value of \( Q \) these conditions are satisfied by a solution where \( \Gamma - aI > 0 \) and where all three detection operators are different from zero. Equation \( (2.11) \) then requires that the operators \( \Gamma - aI \) and \( \Pi_j \) have both the rank \( 1 \) and are mutually orthogonal. Likewise we conclude from Eq. \( (2.10) \) that the operators \( \Gamma - \tilde{\rho}_j \) and \( \Pi_j \) are mutually orthogonal rank-one operators for \( j = 1, 2 \). In fact, if for instance \( \Pi_1 \) would be an operator of rank two, the equality condition in Eq. \( (2.10) \) could be only satisfied if \( \rho = \tilde{\rho}_1 \), which would violate the positivity constraint \( \Gamma - \tilde{\rho}_2 \geq 0 \). We are thus led to the ansatz

\[
\Gamma - aI = |\tilde{\mu}_0\rangle \langle \tilde{\mu}_0|, \quad \Pi_0 = Q |\tilde{\pi}_0\rangle \langle \tilde{\pi}_0|, \quad \langle \tilde{\mu}_0 | \tilde{\pi}_0\rangle = 0, \quad (3.7)
\]

\[
\Gamma - \tilde{\rho}_j = |\tilde{\mu}_j\rangle \langle \tilde{\mu}_j|, \quad \Pi_j = |\tilde{\beta}_j\rangle \langle \tilde{\beta}_j|, \quad \langle \tilde{\mu}_j | \tilde{\beta}_j\rangle = 0 \quad (3.8)
\]

\((j = 1, 2)\), where we introduced positive constants \( \beta_1 \) and \( \beta_2 \). Here the state vectors characterized by a tilde are non-normalized, while \( |\tilde{\pi}_0\rangle \) is normalized to unity in order to yield \( \text{Tr}(\Pi_0) = Q \). Eqs. \( (3.7) \) and \( (3.8) \) imply that for each of the three positive operators \( \Gamma - aI, \Gamma - \tilde{\rho}_1 \) and \( \Gamma - \tilde{\rho}_2 \) one of the two eigenvalues is equal to zero. Using the orthonormal basis \( \{|\nu_1\rangle, |\nu_2\rangle\} \), this requires that

\[
|\Gamma_{12}|^2 = (\Gamma_{11} - a)(\Gamma_{22} - a) = (\Gamma_{11} - C_1)(\Gamma_{22} - 1 + C_2) = (\Gamma_{11} - 1 + C_1)(\Gamma_{22} - C_2), \quad (3.9)
\]

due to the fact that the determinants of the three operators have to vanish. With the help of Eq. \( (3.9) \) we obtain for each of the three operators the respective eigenstate that belongs to its zero eigenvalue. Taking into account that these eigenstates are orthogonal to the states \( |\tilde{\mu}_j\rangle \) \((j = 0, 1, 2)\) and are therefore proportional to \( |\tilde{\pi}_0\rangle, |\tilde{\pi}_1\rangle \) and \( |\tilde{\pi}_2\rangle \), respectively, we find that

\[
|\tilde{\pi}_0\rangle = \frac{\Gamma_{22} - a |\nu_1\rangle - e^{-i\delta} \sqrt{\Gamma_{11} - a}|\nu_2\rangle}{\sqrt{\Gamma_{11} + \Gamma_{22} - 2a}}, \quad (3.10)
\]

\[
|\tilde{\pi}_1\rangle = \frac{\sqrt{\Gamma_{22} - 1 + C_2} |\nu_1\rangle - e^{-i\delta} \sqrt{\Gamma_{11} - C_1}|\nu_2\rangle}{\sqrt{\Gamma_{11} - 1 + C_1}}, \quad (3.11)
\]

\[
|\tilde{\pi}_2\rangle = \frac{\sqrt{\Gamma_{22} - C_2} |\nu_1\rangle - e^{-i\delta} \sqrt{\Gamma_{11} - 1 + C_1}|\nu_2\rangle}{\sqrt{\Gamma_{11} - 1 + C_1}}, \quad (3.12)
\]

When the expressions under the square-root signs are positive and Eq. \( (3.9) \) holds, the positivity conditions in Eqs. \( (2.10) \) and \( (2.11) \) are satisfied.

Next we invoke the completeness relation \( \sum_{j=1}^2 \Pi_j = I \). Due to Eq. \( (2.8) \) the latter relation takes the form
optimality conditions are satisfied when the considerations that led to Eq. (2.16), in this case the for and (3.8) to the matrix representation

Because of Eq. (3.9) we can express \( \Gamma \) with this case.

Let us apply the general treatment of the limiting case \( \Gamma - a I \geq 0 \), that is, we have to consider the limiting problem in dependence of \( Q \). In order to determine its range of validity, we again have to use the positivity constraints. For the values of \( Q \) where a solution fulfilling these conditions does not exist, we have to search for the optimum measurement anew. Assuming that one of the two operators \( \Pi_1 \) or \( \Pi_2 \) vanishes and again supposing that \( \Gamma - a I > 0 \), we obtain another solution of the optimization problem in dependence of \( Q \).

\begin{align}
\Pi_j &= \alpha_j |\nu_j\rangle \langle \nu_j| \quad \text{with} \quad (\rho_{11} - \alpha_1) (\rho_{22} - \alpha_2) \geq |\rho_{12}|^2.
\end{align}

Here the two constants \( \alpha_1 \) and \( \alpha_2 \) both have to be non-negative, with \( \alpha_1 + \alpha_2 = \text{Tr} (\Pi_1 + \Pi_2) = 1 - Q \). The following cases have to be distinguished:

\begin{enumerate}
\item If \( |\rho_{12}| \leq \min \{ \rho_{11}, \rho_{22} \} \), the inequality in Eq. (3.19) holds true whenever \( 0 \leq \alpha_j \leq |\nu_j| - |\nu_{2j}| \) (\( j = 1, 2 \)), that is whenever the given failure probability falls in the range \( 2|\rho_{12}| \leq Q \leq 1 \). When \( Q = 2|\rho_{12}| \) all three detection operators have the rank 1.
\item If \( |\rho_{12}| \geq \min \{ \rho_{11}, \rho_{22} \} \) we first assume that \( \rho_{11} < \rho_{22} \) and therefore \( |\rho_{12}| \geq \rho_{11} \). Putting \( \alpha_1 = 0 \), the constraint expressed in Eq. (3.19) takes the form we discussed before Eq. (3.17), which is satisfied if \( Q \geq Q_1 \), where the measurement is projective for \( Q = Q_1 \). Similar considerations hold for \( \rho_{22} \leq \rho_{11} \). In summary, for \( C_1 = C_2 = C \) we obtain
\end{enumerate}

\begin{align}
P_{c_{\text{max}}} &= \left\{ \begin{array}{l}
C_{2}(1 - Q) \quad \text{for} \quad Q \geq Q' = Q_{\text{MC}} \min \quad (3.20) \\
Q_1 \quad \text{if} \quad |\rho_{12}| \geq \rho_{11}, \\
Q_2 \quad \text{if} \quad |\rho_{12}| \geq \rho_{22}.
\end{array} \right.
\end{align}

In Eq. (3.20) we took into account that for \( C_1 = C_2 \) the value of \( Q' \) determines the minimum failure probability \( Q_{\text{MC}} \min \) necessary for maximum-confidence discrimination, see Eq. (2.18). We remark that in our earlier paper we calculated \( Q_{\text{MC}} \min \) for two mixed qubit states and found that the latter is given by the expressions in Eq. (3.21) for arbitrary values of \( C_1 \) and \( C_2 \).

As outlined in Sec. II after Eq. (2.4), a discrimination measurement maximizing \( P_{c} \) when \( Q \) is fixed at a value \( Q > Q' \) is without practical importance. In the present case such a measurement would be a generalized measurement realizable in an extended Hilbert space, where projections onto two orthogonal directions would indicate an inconclusive result since \( \Pi_0 \) has the rank 2, which implies that also \( \Pi_0 \) is a rank-two operator.
C. Complete solution for the case $C_1 = C_2 \equiv C$

In order to treat the case $\Gamma - aI > 0$ we use the method developed in Sec. III A. It turns out that for $C_1 \neq C_2$ this leads to a fourth order polynomial equation in the variable $a(1 - a)$. To obtain a simple analytical solution we therefore assume that $C_1 = C_2 \equiv C$, restricting ourselves to cases where the maximum achievable confidence is equal for the two possible outcomes. Here we present ourselves to cases where the maximum achievable confidence is given by

$$P_{c_{\text{max}}} = \begin{cases} P_c^{(0)}(Q) & \text{if } 0 \leq Q \leq 2|\rho_{12}|, \\ C(1 - Q) & \text{if } 2|\rho_{12}| \leq Q \leq 1, \end{cases} \quad (3.22)$$

where

$$P_c^{(0)} = \frac{1 - Q}{2} + \frac{2C - 1}{2} \sqrt{(1 - 2|\rho_{12}|)(1 + 2|\rho_{12}| - 2Q)}.$$ \hspace{1cm} (3.23)

(ii) If $|\rho_{12}| \geq \min\{\rho_{11}, \rho_{22}\}$ we assume without lack of generality that $\rho_{11} < \rho_{22}$, which means that $|\rho_{12}| \geq \rho_{11}$. Then we get

$$P_{c_{\text{max}}} = \begin{cases} P_c^{(0)}(Q) & \text{if } 0 \leq Q \leq Q_{cr}, \\ P_c^{(1)}(Q) & \text{if } Q_{cr} < Q \leq Q_1, \\ C(1 - Q) & \text{if } Q_1 \leq Q \leq 1, \end{cases} \quad (3.24)$$

where $Q_1$ is defined in Eq. (3.17). $Q_{cr}$ is given by

$$Q_{cr} = \frac{2\Delta}{1 - 2|\rho_{12}|} \quad \text{with} \quad \Delta = \rho_{11}\rho_{22} - |\rho_{12}|^2.$$ \hspace{1cm} (3.25)

and denotes a critical failure probability that separates the regions where the optimum measurement is a generalized measurement and where it is projective, see below. Note that $\Delta > 0$ since $\rho = \eta_1 \rho_1 + \eta_2 \rho_2$ is a mixed state. In Eq. (3.24) we introduced

$$P_c^{(1)} = C(1 - Q) - \frac{2C - 1}{1 - 4\Delta} \left[\Delta(1 - 2\rho_{11}) - (2\Delta - \rho_{11})(1 - Q) - 2|\rho_{12}|\sqrt{\Delta(Q - Q^2 - \Delta)}\right].$$ \hspace{1cm} (3.26)

The corresponding result for $\rho_{22} < \rho_{11}$ is obtained when in Eqs. (3.24) and (3.26) $Q_1$ and $\rho_{11}$ are replaced by $Q_2$ and $\rho_{22}$, respectively.

When the conditions in the upper lines of Eqs. (3.24) or (3.26), respectively, are fulfilled, the optimum detection operators are given by Eq. (A8) together with Eq. (A4) and Eqs. (A9) – (A11), see Appendix A. They describe a generalized measurement if $Q \neq 0$ and $Q \neq Q_{cr}$. For $Q = 0$ the optimum measurement is equal to the projective measurement for minimum-error discrimination, yielding the maximum probability of correct results $P_{c_{\text{max}}} = P_{c_{ME}} = \frac{1}{2}(1 + \text{Tr} |\eta_2 \rho_2 - \eta_1 \rho_1|)$. \hspace{1cm} [1, 2]

When the middle line of Eq. (3.24) applies, the optimum detection operators follow from Eq. (A14) together with Eqs. (A15) and (A19). In this case the optimum measurement is a projective measurement, where only one of the states, here the state $\rho_2$, is conclusively discriminated with a certain probability, while in the presence of the other state an inconclusive result is always obtained. The lower lines of Eqs. (3.24) or (3.26), respectively, correspond to the limiting case $Q > Q'$ discussed in the previous paragraph.

In the special case when $C = 1$, $\rho_{11} = \eta_1$, $\rho_{22} = \eta_2$, and $\rho_{12} = \sqrt{\eta_1 \eta_2} |\psi_1 \rangle |\psi_2 \rangle$ our solution refers to the discrimination of two pure states, see Eq. (A6). Two arbitrary qubit states that are mixed can be represented as

$$\rho_1 = \rho_1 |\psi_1 \rangle \langle \psi_1 | + \frac{1 - \rho_1}{2} I, \quad \rho_2 = \rho_2 |\psi_2 \rangle \langle \psi_2 | + \frac{1 - \rho_2}{2} I,$$ \hspace{1cm} (3.27)

where $|\psi_1 \rangle$ and $|\psi_2 \rangle$ are normalized pure states with the overlap $|\langle \psi_1 | \psi_2 \rangle| \leq S$ and where $Tr I = 2$. The parameters $\rho_1$ and $\rho_2$ with $0 \leq \rho_1, \rho_2 \leq 1$ are related to the purities of the two states. The condition $C_1 = C_2 = C$ with $C < 1$ is fulfilled provided that $(\eta_1^{-1} - 1)^2 = \frac{1 - \rho_1^2}{1 - \rho_2^2}$, see Appendix A. Clearly, for any two mixed qubit states $\rho_1$ and $\rho_2$ there exist particular prior probabilities of occurrence for which our solution holds, and for any given prior probability $\eta_1$ we can find a whole class of state pairs where the solution applies. In Figs. 1 and 2 the ratio $R_{c_{\text{max}}} = P_{c_{\text{max}}} |Q/(1 - Q)$ is plotted for the discrimination of pure states ($C = 1$) and of mixed states ($C < 1$). The smallest value of $Q$ where $R_{c_{\text{max}}} |Q = C$ is equal to the value $Q'$ given by Eq. (3.24) and corresponds to the minimum failure probability necessary for unambiguous discrimination if $C = 1$ [18], or necessary for maximum-confidence discrimination if $C < 1$ [18].

We emphasize that for $Q = 0$ and sufficiently large values of $Q$ the analytical solution of our optimization problem is known for two arbitrary qubit states, occurring...
ring with arbitrary prior probabilities. The first case cor-
responds to minimum-error discrimination \[1, 2\], where
for two states the measurement is projective and can be
easily determined analytically when the Hilbert space is
two-dimensional. The second case was treated in Sec.
III B and applies when \( Q \geq Q' \), where \( Q' \) is given by
Eqs. (3.18) or (3.21), respectively. For \( 0 < Q < Q' \), how-
ever, we obtained a simple analytical solution only for
those states and prior probabilities where the maximum
confidence is equal for the two outcomes, \( C_1 = C_2 \equiv C \).

IV. \( N \) SYMMETRIC STATES

A. Optimality conditions for equiprobable states and the limiting case of sufficiently large \( Q \)

Another class of analytically solvable cases refers to
discriminating \( N \) states that are symmetric, which means
that for \( j = 1, \ldots, N \)

\[
\rho_j = V^{(j-1)} \rho V^{(j-1)}^\dagger \quad \text{with} \quad V^\dagger V = V N = I
\]  

\[ (4.1) \]

where without lack of generality an arbitrary state of
the given set of states can be chosen as the reference
state \( \rho_1 \).

Assuming that each state is prepared with the same
prior probability, \( \eta_j = 1/N \), we find that \( \rho = \frac{1}{N} \sum_{j=1}^{N} \rho_j = V \rho V^\dagger \) and thus \( [V, \rho] = 0 \), which implies
that the Hermitian operator \( \rho \) and the unitary operator
\( V \) can be diagonalized in the same orthonormal basis \[32\].
Introducing the orthonormal basis states \( |r_i\rangle \), we get the spectral representations

\[
\rho = \frac{1}{N} \sum_{j=1}^{N} \rho_j = \sum_{l=1}^{d} r_l |r_l\rangle \langle r_l|, \quad V = \sum_{l=1}^{d} v_l |r_l\rangle \langle r_l|,
\]

\[ (4.2) \]

where \( \sum_{j=1}^{N} (v_l v_l^*)^j = N \delta_{l' l} \) and \( |v_l|^2 = v_l^N = 1 \), see also \[22\]. We now focus on the optimum measurement, where
the optimality conditions, given by Eqs. (2.6) and (2.7),
are satisfied. Let us suppose that \( \Pi_1 \) is an element of the
set of optimum detection operators. We introduce

\[
\Pi_j = V^{(j-1)} \Pi_1 V^{(j-1)}^\dagger
\]

\[ (4.3) \]

\( (j = 1, \ldots, N) \), where due to the symmetry the operator
\( \sum_{j=1}^{N} \Pi_j = I - \Pi_0 \) commutes with \( V \). Using \( [V, \rho] = 0 \),
this implies that

\[
[\Pi_0, V] = [\Pi_0, \rho] = [\Pi_0, Z] = [\rho, Z] = [V, Z] = 0.
\]

\[ (4.4) \]

Here the second equality sign follows from the fact that in
the optimum measurement \( (Z - a \rho) \Pi_0 = (Z - a \rho) \) is zero,
which leads to \( [\Pi_0, Z] = a [\Pi_0, \rho] \). Upon inserting Eq.
(4.3) into Eqs. (2.6) and (2.7) with \( \eta_j = 1/N \), taking
Eq. (4.1) into account, it becomes obvious that if \( \Pi_1 \) ful-
fills the optimality conditions, then these conditions are
also fulfilled by each of the operators \( \Pi_j \) defined in Eq.
(4.3), in analogy to the derivation of the optimality con-
ditions for maximum-confidence discrimination of sym-
metric mixed states in our previous paper \[22\]. Hence
the detection operators for the optimum measurement
can always be chosen in the form of Eq. (4.3). The opti-
mality conditions thus reduce to the conditions

\[
Z - a \rho \geq 0, \quad (Z - a \rho) \Pi_0 = 0,
\]

\[ (4.5) \]

\[
Z - \rho \frac{1}{N} \geq 0, \quad (Z - \rho \frac{1}{N}) \Pi_1 = 0,
\]

\[ (4.6) \]

which were first derived by applying group-theoretical
methods \[26\]. Using Eq. (4.3) and the properties of \( V \)
we find that

\[
\Pi_0 = \sum_{l=1}^{d} \left( 1 - N \langle r_l | \Pi_1 | r_l \rangle \right) |r_l\rangle \langle r_l|, \quad Z = \sum_{l=1}^{d} z_l |r_l\rangle \langle r_l|,
\]

\[ (4.7) \]

where we introduced the spectral representation of \( Z \),
taking into account that \( \Pi_0 \) and \( Z \) commute when the
optimality conditions are fulfilled. With the help of Eqs.
(2.12) and (4.3) we can represent the maximum proba-
bility of correct results achievable with the fixed value
\( Q = \text{Tr} (\rho \Pi_0) \) as

\[
P_{c, \text{max}} |Q = \text{Tr} (\rho_1 \Pi_1) \quad \text{where} \quad Q = 1 - N \text{Tr} (\rho \Pi_1).
\]

(4.8)

Let us discuss the limiting case of sufficiently large \( Q \),
which we treated in general in Sec. II B. According to
Eq. (2.11) in this limit the optimality conditions, Eqs.
(4.5) and (4.6), reduce to the conditions \( a I - \rho_1 \geq 0 \) and
\( (a I - \rho_1) \Pi_1 = 0 \), where we used the transformed operators
introduced in Eqs. (2.8) and (2.9). The spectral repre-
sentation of \( \rho_1 \) can be written as

\[
\tilde{\rho}_1 = \frac{1}{N} \rho^{-1/2} \rho_1 \rho^{-1/2} = C \sum_{k=1}^{k_1} |v_k\rangle \langle v_k| + \sum_{k=k_1+1} \nu_k |v_k\rangle \langle v_k|,
\]

\[ (4.9) \]
where $C$ denotes the largest eigenvalue and $k_1$ is its degree of degeneracy. The optimality conditions are fulfilled when $a = C$ and when the support of $\Pi_1 = \rho^{1/2}\Pi_1\rho^{1/2}$ is the Hilbert space spanned by the eigenstates $|\nu_1 \rangle \ldots |\nu_{k_1} \rangle$. We then obtain $P_{c}^{\max}|_{Q} = CNTr \Pi_1$, which yields

$$P_{c}^{\max}|_{Q} = C(1 - Q) \text{ if } Q \geq Q' = Q_{\min}^{MC}.$$  \hfill (4.10)

Here we took into account that the maximum confidence is the same for discriminating each of the equiprobable symmetric states $\rho_j = V^{(j-1)}\rho_1 V^{(j-1)}$ since the largest eigenvalues of the operators $\tilde{\rho}_j$ are identical, that is,

$$C_1 = \ldots = C_N \equiv C.$$  \hfill (4.11)

$Q'$ therefore corresponds to the failure probability $Q_{\min}^{MC}$ in optimized maximum-confidence discrimination, or, if $C = 1$, in optimum unambiguous discrimination, respectively, see Eq. (2.15). In order to determine $Q'$ we have to find the minimum of $Q = 1 - NTr \Pi_1$ on the condition that with the given support of $\Pi_1$ the positivity constraint $\Pi_0 \geq 0$ is satisfied, which because of Eq. (4.17) is equivalent to the constraint

$$\sum_{l=1}^{d} |r_l - N(r_l|\Pi_1 |r_l\rangle \langle r_l| |r_l\rangle |r_l\rangle \langle r_l| \geq 0,$$  \hfill (22)

In general, this is a nontrivial task. In the special case when the largest eigenvalue of $\rho_1$ is nondegenerate, $\Pi_1$ is proportional to $|\nu_1 \rangle \langle \nu_1|$. In agreement with our previous paper [22] we then obtain $Q' = Q_{\min}^{MC} = 1 - \min_{i} \{ |\langle r_l|\nu_1 \rangle |^2 \}$ for $k_1 = 1$.

**B. $N$ equiprobable symmetric pure qudit states**

1. **Solution for a special class of states**

First we consider the discrimination of $N$ symmetric equiprobable pure states spanning a $d$-dimensional Hilbert space $\mathcal{H}_d$. This means that $d \leq N$ and implies that all expansion coefficients $c_i$ of the states with respect to the $d$-dimensional eigenbasis of the symmetry operator $\mathcal{V}$ are different from zero [22]. In general, $N$ symmetric pure qudit states are given by

$$|\psi_j \rangle = \mathcal{V}^{(j-1)} |\psi_1 \rangle \text{ with } |\psi_1 \rangle = \sum_{i=1}^{d} c_i |r_i \rangle \text{ (} j = 1, \ldots, N \text{)}.$$  \hfill (4.12)

In order to obtain an analytical solution we restrict ourselves to special states where only two different values of the expansion coefficients $c_i$ occur. We assume that

$$|\psi_1 \rangle = c_1 \sum_{l=1}^{m} |r_l \rangle + c_2 \sum_{l=m+1}^{d} |r_l \rangle, \text{ } m|c_1|^2 + (d-m)|c_2|^2 = 1,$$  \hfill (4.13)

where $m \leq d/2$ and where we distinguish the cases $|c_1| \geq |c_2|$ and $|c_1| \leq |c_2|$ with $c_1, c_2 \neq 0$. In Appendix B we derive the maximum probability of correct results $P_{c}^{\max}|_{Q}$ with the fixed probability $Q$ of inconclusive results, making use of the optimality conditions, Eqs. (4.15) and (4.16). Using Eq. (B10) with $p = 1$ and Eq. (B13) we obtain

$$P_{c}^{\max}|_{Q} = \begin{cases} \frac{P^{(0)}(Q)}{d} \text{ if } Q \leq Q' = 1 - d \min \{|c_1|^2, |c_2|^2\} \\ \frac{d}{d} \frac{1}{Q} \text{ if } Q \geq Q' \end{cases}$$  \hfill (4.14)

with

$$P^{(0)}(Q) = \begin{cases} \frac{1}{d} \left[ \sqrt{d} \rho \mathcal{V}^{(j-1)} |\psi_j \rangle \langle \mathcal{V}^{(j-1)} |\psi_j \rangle \right] \text{ if } |c_1| \geq |c_2| \\ \frac{1}{d} \left[ \sqrt{d} \rho \mathcal{V}^{(j-1)} |\psi_j \rangle \langle \mathcal{V}^{(j-1)} |\psi_j \rangle \right] \text{ if } |c_2| \leq |c_1| \leq |c_2|. \end{cases}$$  \hfill (4.15)

The lower line of Eq. (4.14) follows from Eq. (4.10) and from the fact that $P_{c}^{(0)}(Q')/(1 - Q') = C = \frac{d}{d}$. The optimum detection operators are given by Eqs. (B3) and (B7).

When $Q = 0$ the optimum measurement corresponds to minimum-error discrimination. For this case the optimum detection operators discriminating the equiprobable states given by Eq. (4.12) are known to be $\Pi_{j}^{ME} = \frac{1}{d} \rho^{1/2} |\psi_j \rangle \langle \mathcal{V}^{(j-1)} |\psi_j \rangle \rho^{1/2}$, yielding for an arbitrary reference state $|\psi_1 \rangle$ the maximum probability of correct results

$$P_{c}^{\max}|_{Q=0} = \frac{d}{d} \left( \sum_{i=1}^{d} |c_i|^2 \right)^{2}.$$  \hfill (4.16)

On the other hand, when $Q = Q' = Q_{\min}^{MC}$ the optimum measurement corresponds to optimized maximum-confidence discrimination. For $N$ equiprobable symmetric pure qudit states with an arbitrary reference state $|\psi_1 \rangle$, it has been shown that $Q_{\min}^{MC} = 1 - d \min \{|c_1|^2, |c_2|^2\}$ [21, 22], including the special case of unambiguous discrimination when $d = 33$. The results obtained from Eq. (4.14) for the maximum confidence $C$, for $Q' = Q_{\min}^{MC}$, and for $P_{c}^{\max}|_{Q=0}$ agree with these previous results. Note that for $|c_1| = |c_2| = \frac{1}{d}$ we obtain $Q' = 0$, which means that in this case minimum-error discrimination and optimized maximum-confidence discrimination are equivalent, cf. also Sec. V A.

We emphasize that while in the limiting cases $Q = 0$ and $Q = Q'$ the optimum measurement with a fixed value of $Q$ is known for $N$ equiprobable symmetric pure qudit states which are arbitrary, our complete analytical solution, Eq. (4.14), that interpolates between these limiting cases, is restricted to the special class of those symmetric states where the reference state is given by Eq. (4.13).

2. **Application to $N$ linearly independent symmetric states**

When the number of pure states is equal to the dimension of the Hilbert space spanned by them, $N = d$, the states are linearly independent. This means that without lack of generality the eigenvalues of the symmetry operator $\mathcal{V}$ in Eq. (4.12) can be written as $v_l = \exp\left(2\pi i \frac{l}{N} \right)$ for $l = 1, \ldots, N$ [33]. With the help of Eqs. (4.12) and (4.13) we obtain for $d = N$ the mutual overlaps...
\[\langle \psi_j | \psi_k \rangle = \sum_{i=1}^{N} v_i^j v_i^k |\langle \psi_1 | r_i \rangle|^2, \text{ yielding with } j \neq k\]

\[\langle \psi_j | \psi_k \rangle = (|c_1|^2 - |c_2|^2) \sum_{i=1}^{m} \exp \left[ 2\pi i \frac{(-1)^{j+k}}{N}(k-j) \right]. \quad (4.16)\]

Clearly, when \( m = 1 \) all mutual overlaps are equal and real. If \( d = N = 3 \) this is always the case, because of the requirement \( m \leq d/2 \) in Eq. (4.13). For \( d = N \geq 4 \) and \( m \geq 2 \), however, different values of the overlap occur for the linearly independent symmetric states specified by Eq. (4.13).

We now specialize to \( N \) linearly independent symmetric pure states with equal mutual overlaps \( S \), where

\[\langle \psi_j | \psi_k \rangle = S = |c_1|^2 - |c_2|^2 = 1 - N|c_2|^2 \quad (4.17)\]

according to Eq. (4.16) with \( m = 1 \). Using the normalization condition given in Eq. (4.13), we get the expressions \( |c_1|^2 = \frac{1}{N}(N-1)S \) and \( |c_2|^2 = \frac{1}{N}S \), from which we conclude that \(-N^{-1} \leq S \leq 1 \) since neither \( |c_1|^2 \) nor \( |c_2|^2 \) can be negative. Upon inserting these expressions into Eq. (4.15) we obtain for \( |c_1| \geq |c_2| \), that is for \( S \geq 0 \),

\[P^c_c(0) = \frac{1}{N} \left[ \sqrt{\frac{1}{N} - 1} \sqrt{\frac{(N-1)(1-S)}{N}} - Q + \frac{1}{N} \sqrt{\frac{1}{N} - 1} \right]^2 \]

if \( 0 \leq S < 1 \), for \( Q \leq Q' = S \). (4.18)

while for negative overlaps we arrive at

\[P^c_c(0) = \frac{1}{N} \left[ \sqrt{\frac{1}{N} - 1} \sqrt{\frac{(N-1)(1-S)}{N}} - Q + \frac{1}{N} \sqrt{\frac{1}{N} - 1} \right]^2 \]

if \( -\frac{1}{N-1} < S \leq 0 \), for \( Q \leq Q' = -(N-1)|S| \). (4.19)

If \( Q \geq Q' \) Eq. (4.14) yields \( P^c_{c \max}(Q) = 1 - Q \), which corresponds to unambiguous discrimination since the probability of errors vanishes in this case, \( P_c = 1 - Q - P_e = 0 \). The minimum failure probability necessary for unambiguous discrimination is thus given by \( Q' \). When \( S = 0 \) Eqs. (4.18) and (4.19) apply only for \( Q = 0 \) and we get \( P^c_{c \max}(Q=0) = 1 \), as expected for mutually orthogonal states. On the other hand, in the limit where \( S \) approaches \( -\frac{1}{N-1} \) and \( c_1 \) therefore approaches zero, the states get linearly dependent and span a Hilbert space of dimension \( N - 1 \) since \( m = 1 \). For example, if \( N = 3 \) and \( \langle \psi_j | \psi_k \rangle = -\frac{1}{2} = \cos \frac{2\pi}{3} \) \( (j \neq k) \) we arrive at the trine states that can be represented by three real symmetric state vectors spanning a two dimensional Hilbert space. We note that for \( N = 2 \) Eqs. (4.18) and (4.19) are identical and reproduce the result obtained for the discrimination of two equiprobable pure states [3]. For \( N \geq 3 \), however, the explicit expressions for \( P^c_{c \max} \) and \( Q' \) depend on the sign of the overlap between the states, see Fig. 3.

C. \( N \) equiprobable symmetric mixed qubit states

The density operator of any mixed qubit state can be written as \( \rho = \rho |\psi\rangle \langle \psi| + \frac{1}{2} I \) with a certain normalized pure state \( |\psi\rangle \) and with \( 0 < \rho < 1 \). Hence the most general representation for \( N \) symmetric mixed states in a two-dimensional joint Hilbert space with the identity operator \( I \) is given by the density operators \( \rho_j = V^j V^j \rho_{j=1} V^{j-1} \) with \( j = 1, \ldots, N \) and with \( V^1 V = V^N = I \), where

\[\rho_j = p \rho \rho_j + \frac{1}{2} I, \quad \rho_j = c_1 |r_1\rangle + c_2 |r_2\rangle. \quad (4.20)\]

Without loss of generality we assume that \( |c_1| \geq |c_2| \). Using Eqs. (4.18), (4.19) and (4.14) (see Appendix B) we find that the maximum probability of correct results with the fixed failure probability \( Q \) is given by

\[P^c_{c \max}(Q) = \begin{cases} P^c_{c}(Q) \quad & \text{if } Q \leq Q' = p(1 - 2|c_2|^2) \quad (4.21) \end{cases}\]

with

\[P^c_{c}(0) = \frac{1}{N} \left[ |c_1| \sqrt{1 - \frac{2Q}{1 + p(1 - 2|c_2|^2)}} + |c_2| \right]^2 + \frac{1}{N} \left[ 1 - \frac{Q}{1 + p(1 - 2|c_2|^2)} \right]. \quad (4.22)\]
The ratio $P_{c}^{\text{max}}|Q/(1-Q)$ is plotted in Fig. 4. By calculating $C = P_{c}^{(0)}(Q')/(1-Q')$ we arrive at the maximum confidence $C = \frac{1}{N} \left( 1 + \frac{2p|c_{2}|}{\sqrt{1-p(1-2|c_{2}|)^{2}}} \right)$, in accordance with the result derived for $C$ in our previous paper [22]. $Q'$ coincides with the smallest failure probability necessary to achieve maximum-confidence discrimination [22]. On the other hand, for $Q = 0$ we get the result

$$P_{c}^{\text{max}}|_{Q=0} = P_{c}^{\text{ME}} = \frac{1}{N} \left( 1 + p|c_{1}|c_{2} \right),$$

(4.23)

which means that using our general solution for $P_{c}^{\text{max}}|Q$, we determined the maximum probability of correct results in minimum-error discrimination of the $N$ mixed qubit states.

The special case $|c_{1}|^{2} = |c_{2}|^{2} = 0.5$ is worth mentioning, where $Q' = 0$ and therefore $P_{c}^{\text{max}}/(1-Q) = C = \frac{1}{1+2p}$ for any value of $Q$. The measurements for minimum-error discrimination and for optimized maximum-confidence discrimination are then the same, cf. Sec. V A. When $N = 3$ this applies for the depolarized trine states, described by Eq. (4.20) with $\langle \psi_{1} \rangle = \frac{1}{\sqrt{2}} \left[ \exp (i \frac{\pi}{4}) |r_{1} \rangle + \exp (-i \frac{\pi}{4}) |r_{2} \rangle \right]$.

### D. $N$ special symmetric mixed states of rank $D$ spanning a joint Hilbert space of dimension $ND$

In the following we treat a special case where the optimum measurement for discriminating $N$ symmetric mixed states with a fixed probability $Q$ of inconclusive results can be obtained by applying the pure-state solution. We consider $N$ mixed states of rank $D$, occurring with the prior probabilities $\eta_{j}$ ($j = 1, \ldots, N$) and being described by the special density operators

$$\rho_{j} = \sum_{l=1}^{D} s_{l} |s_{l}^{j} \rangle \langle s_{l}^{j} | \quad \text{with} \quad \langle s_{l}^{j} | s_{l'}^{j} \rangle = \delta_{ll'},$$

(4.24)

where we assume that the overlaps of basis states belonging to different density operators obey the special relation

$$\langle s_{l}^{j} | s_{l'}^{j'} \rangle = S_{ll'} \quad \text{with} \quad -\frac{1}{N} < S < 1 \quad (i \neq j)$$

(4.25)

and with $l, l' = 1, \ldots, D$. For each value of $l$ the states $|s_{l}^{1} \rangle, \ldots, |s_{l}^{N} \rangle$ form a set of $N$ linearly independent symmetric states with equal mutual overlaps, as becomes obvious from Eq. (4.17). The spectral representations of the density operators can be written as

$$\rho_{j} = \sum_{k=1}^{D} \lambda_{k}^{(k)} \rho_{j}^{(k)} \quad \text{with} \quad \rho_{j}^{(k)} = |\psi_{j}^{k} \rangle \langle \psi_{j}^{k} |, \quad \sum_{k=1}^{D} \lambda_{k}^{(k)} = 1,$$

(4.26)

where $\langle \psi_{j}^{k} | \psi_{j'}^{k} \rangle = \delta_{kk'}$ and $\langle \psi_{j}^{k} | \psi_{j'}^{k'} \rangle = S_{kk'} \delta_{kk'}$ for $i \neq j$. The latter equation follows from Eq. (4.25) after expanding the normalized eigenstates as $|\psi_{j}^{k} \rangle = \sum u_{kl} |s_{l}^{k} \rangle$, taking into account that the expansion coefficients $u_{kl} = \langle \psi_{j}^{k} | s_{l}^{k} \rangle$ and the eigenvalues $\lambda_{k}$ are identical for the different states $j$ since the matrix elements $\langle s_{l}^{k} | \rho_{j} | s_{l'}^{k} \rangle$ do not depend on $j$. Hence for each index $k$ also the $N$ eigenstates $|\psi_{j}^{k} \rangle, \ldots, |\psi_{N}^{k} \rangle$, belonging each to a different density operator, represent a set of linearly independent symmetric states.

Let us introduce the $N$-dimensional subspace $H_{N}^{k}$ spanned by the eigenstates $|\psi_{j}^{k} \rangle, \ldots, |\psi_{N}^{k} \rangle$ and let $I^{(k)}$ be the identity operator in this subspace. Introducing the symmetry operator $V^{(k)}$ with $|\psi_{j}^{k} \rangle = V^{(k)}|\psi_{j-1}^{k} \rangle$, it follows that $\rho_{j} = V \rho_{j-1} V^\dagger$, where $V = V^{(1)} \otimes \cdots \otimes V^{(D)}$. Clearly, the $N$ mixed states given by Eq. (4.24) or, equivalently, by Eq. (4.26) are symmetric, but in the following we do not necessarily assume that they occur with equal prior probabilities $\eta_{j}$.

The optimum measurement, discriminating the mixed states with a maximum probability of correct results for a fixed value of the failure probability $Q$, is determined by the optimality conditions, Eqs. (2.6) and (2.7), where because of Eq. (4.26) $\rho = \sum_{k=1}^{D} \lambda_{k}^{(k)} \rho_{j}^{(k)}$ with $\rho_{j} = \sum_{j=1}^{N} \eta_{j} \rho_{j}^{(k)}$. Using the orthogonality of the different subspaces $H_{N}^{k}$ it follows that the optimality conditions are satisfied for a certain value of the real multiplier $a$ and for certain operators $Z$ and $\Pi_{i}$ ($i = 0, 1, \ldots, N$) when

$$Z = \sum_{k=1}^{D} \lambda_{k}^{(k)} Z^{(k)}, \quad \Pi_{i} = \sum_{k=1}^{D} \Pi_{i}^{(k)} \quad \text{with} \quad \sum_{i=0}^{N} \Pi_{i}^{(k)} = I^{(k)},$$

(4.27)

provided that $Z^{(k)}$ and $\Pi_{i}^{(k)}$ ($i = 0, 1, \ldots, N$) satisfy the corresponding optimality conditions in their respective subspaces $H_{N}^{k}$ for the same value of $a$. The latter requirement can indeed be fulfilled in our problem, since in each of the different subspaces $H_{N}^{k}$ the structure of the resulting optimality conditions is identical, due to the fact that $\langle \psi_{j}^{k} | \psi_{j}^{k} \rangle$ does not depend on $k$. In particular, this means that in the optimum measurement $\text{Tr} (\rho_{j}^{(k)} \Pi_{i}^{(k)}) = \text{Tr} (\rho_{j}^{(1)} \Pi_{i}^{(1)})$ for $k = 2, \ldots, D$. Consequently, using again the orthogonality of the different subspaces $H_{N}^{k}$, we find that the detection operators given in Eq. (4.27) maximize $P_{c}$ at the fixed failure probability $Q = \text{Tr} (\rho P_{0})$ that can be written as $Q = \sum_{k=1}^{D} \lambda_{k} \text{Tr} (\rho_{j}^{(k)} \Pi_{i}^{(k)}) = \text{Tr} (\rho_{j}^{(1)} \Pi_{i}^{(1)})$. According to Eq. (2.12) they yield the maximum probability of correct results

$$P_{c}^{\text{max}}|_{Q} = \sum_{k=1}^{D} \lambda_{k} \sum_{j=1}^{N} \eta_{j} \text{Tr} (\rho_{j}^{(k)} \Pi_{i}^{(k)}) = \sum_{j=1}^{N} \eta_{j} \text{Tr} (\rho_{j}^{(1)} \Pi_{i}^{(1)}).$$

(4.28)
The dependence of $P_{c}^{\text{max}}|_{Q}$ on $Q$ is thus exactly the same as the respective dependence that arises from the corresponding discrimination problem for the states $|\psi_{i}^{k}\rangle, \ldots, |\psi_{N}^{k}\rangle$ in any one of the orthogonal subspaces $\mathcal{H}_{k}^{N}$ $(k = 1, \ldots, D)$. Without lack of generality in Eq. $4.29$ we referred to the subspace $\mathcal{H}_{N}^{N}$. In other words, provided that the pure-state optimization problem can be solved for the set of linearly independent symmetric states $\{|\psi_{j}\rangle\}$ occurring with the prior probabilities $\eta_{j}$ $(j = 1, \ldots, N)$, that is when the optimum detection operators in the subspace $\mathcal{H}_{N}^{N}$ can be determined, we know the complete solution.

We emphasize that $P_{c}^{\text{max}}|_{Q}$ does not depend on the rank $D$ of the mixed states $\rho_{j}$ nor on their matrix elements $s_{ij}$, or their eigenvalues $\lambda_{k}$, respectively. Using the pure-state results derived in this paper, we obtain analytical solutions in two cases:

(i) For $N$ mixed states that are defined by Eq. $4.24$ or, equivalently, by $4.26$ and occur with equal prior probabilities $\eta_{j} = 1/N$, the final result $P_{c}^{\text{max}}|_{Q}$ is represented by Eq. $4.14$ with $d = N$, together with Eqs. $4.18$ and $4.19$. The failure probability $Q$ given by $Q' = S$ if $S \geq 0$ and $Q' = (N - 1)|S|$ if $- (N - 1) < S \leq 0$, now corresponds to the minimum failure probability necessary for unambiguously discriminating the $N$ mixed states.

(ii) When $N = 2$ and $\eta_{1} \leq \eta_{2}$ the final result $P_{c}^{\text{max}}|_{Q}$ for optimally discriminating the two mixed states described by Eqs. $4.21$ or $4.26$, respectively, is obtained by substituting the expressions $\rho_{11} = \eta_{1}, \rho_{22} = \eta_{2}, \rho_{12} = \sqrt{\eta_{1}\eta_{2}}$ and $C = 1$ into Eqs. $5.22$ - $5.29$, thus using the corresponding solution for the optimum discrimination of two pure states occurring with arbitrary prior probabilities. We note that two pure states can be always written as symmetric states with respect to a suitable basis. Our general result for $P_{c}^{\text{max}}|_{Q}$ reveals that the minimum failure probability $Q$ required for unambiguously discriminating the two mixed states is given by Eq. $5.24$, in agreement with earlier results for the optimum unambiguous discrimination of these two special mixed states [11, 12].

V. FURTHER APPLICATIONS

A. $N$ equiprobable mixed qudit states resolving the identity operator

When the dimension $d$ of the joint Hilbert space is larger than two and the states are genuinely mixed, that is when the discrimination problem cannot be reduced to the problem of discriminating pure states, it is in general hard to obtain analytical solutions. However, there is an exceptional case. We consider $N$ states that occur with equal prior probabilities, $\eta_{j} = 1/N$, and are described by the special density operators $\rho_{j}$ $(j = 1, \ldots, N)$, where

$$\rho_{j} = p|\psi_{j}\rangle\langle\psi_{j}| + \frac{1 - p}{d} I \quad \text{with} \quad p = \frac{1}{N} \sum_{j=1}^{N} \rho_{j} = \frac{I}{d}. \quad \text{(5.1)}$$

Here $I$ is the identity operator in $\mathcal{H}_{d}$, and $0 \leq p \leq 1$. Eq. $5.1$ means that the identity operator can be resolved as a weighted sum over the density operators $\rho_{j}$. The largest eigenvalue of any one of the operators $\tilde{\rho}_{j} = \rho^{-1/2}\rho_{j}\rho^{-1/2} = \frac{1}{N}\rho_{j}$, determining the maximum confidence $\tilde{C}_{j}$ of the outcome $j$, is given by $(pd+1-p)/N$, and the corresponding eigenstate is $|\psi_{j}\rangle$. Taking into account that $\rho$ is proportional to $I$, it follows from the considerations in Sec. II B that for maximum-confidence discrimination both the operators $\Pi_{j}$ and $\Pi_{j} = \rho^{-1/2}\Pi_{j}\rho^{-1/2}$ are proportional to $|\psi_{j}\rangle\langle\psi_{j}|$. Since Eq. $5.1$ implies that

$$\Pi_{0} = 0, \quad \Pi_{j} = \frac{d}{N}|\psi_{j}\rangle\langle\psi_{j}| \quad (j = 1, \ldots, N) \quad \text{(5.2)}$$

fulfill the completeness relation. Therefore maximum-confidence discrimination is possible without inconclusive results, and we get with the help of Eq. $2.18$

$$C_{1} = \cdots = C_{N} \equiv C = \frac{1 + p(d-1)}{N}, \quad Q' = Q_{\text{MC}} = 0. \quad \text{(5.3)}$$

From Eq. $2.10$ we then obtain

$$P_{c}^{\text{max}}|_{Q} = C(1 - Q) \quad \text{if} \quad Q \geq 0, \quad \text{(5.4)}$$

which in particular means that $P_{c}^{\text{max}}|_{Q=0} = \frac{1+p(d-1)}{N}$. The detection operators given by Eq. $5.2$ describe optimized maximum-confidence discrimination since they yield the smallest possible failure probability, $Q = 0$. On the other hand, they also describe minimum-error discrimination, since they maximize the probability of correct results when $Q$ is fixed at the value $Q = 0$. Hence both measurements coincide and the relative rate of correct results cannot be increased by admitting inconclusive results.

Eqs. $5.2$ - $5.4$ agree with our previous results [22] obtained by studying the optimized maximum-confidence discrimination of symmetric states obeying Eq. $5.1$. However, they show that these results also hold in a more general case. While the derivation in [22] supposes symmetric states $|\psi_{j}\rangle$ $(j = 1, \ldots, N)$ with equal expansion coefficients $c_{j}$ with respect to the basis of the symmetry operator, $c_{j} = 1/\sqrt{d}$ for $l = 1, \ldots, d$ (see also Appendix B), we emphasize that in Eq. $5.1$ the $N$ states $\rho_{j}$, or $|\psi_{j}\rangle$, respectively, need not necessarily be symmetric.

Moreover, the expression for $P_{c}^{\text{max}}|_{Q=0}$ following from Eqs. $5.3$ and $5.4$ generalizes the known result for minimum-error discrimination of pure states resolving the identity operator [24, 32] to a special class of mixed states. We mention that the corresponding pure-state measurement has been experimentally realized for a set
of qubit states \((d = 2)\) with \(N = 3\), given by the symmetric trine states, and also for a set with \(N = 4\), given by the tetrad states \([34]\). The latter are defined as

\[
|\psi_j\rangle = \frac{\sqrt{2} \exp(\frac{2\pi i j}{3})|1\rangle - |0\rangle}{\sqrt{3}} \quad (j = 1, 2, 3), \quad |\psi_4\rangle = |0\rangle. \tag{5.5}
\]

Note that the tetrad states do not belong to the class of fully symmetric states considered in the previous section, but possess only a partial symmetry. It is easy to check that they fulfill the requirement \(\frac{1}{3} \sum_{j=1}^{N} |\psi_j\rangle\langle\psi_j| = |\frac{1}{2}\rangle\). When \(p < 1\), the end points of the Bloch vectors belonging to the four depolarized tetrad states resulting from Eq. (5.1) together with Eq. (5.5) form a regular tetrahedron within the Bloch sphere, with radius \(p\sqrt{8/3}\). In the special case \(N = 4\) and \(d = 2\) we get \(P_e^{\max}|_{Q=0} = (1 + p)/4\), in accordance with the result following from a recent solution \([36]\) for the discrimination of mixed qubit states with Bloch vectors forming a regular polyhedron, where an approach to study minimum-error discrimination was applied that is based on a geometrical method using Helstrom families of ensembles in convex optimization \([36, 37]\).

B. Partially symmetric states

The methods developed in this paper can be extended to the problem of discriminating between states that possess a certain partial symmetry. We assume that the given set of states consists of two sets of a equiprobable symmetric states and that the total density operators resulting from each set commute. More precisely, we refer to the discrimination of \(N\) states \((N > M \geq 1)\), where the states \(\rho_1, \ldots, \rho_N\) are symmetric as described in Sec. IV, occurring with equal prior probabilities \(\eta_j/M\), and where the remaining states \(\rho_{M+1}, \ldots, \rho_N\) with equal prior probabilities \((1 - \eta_j)/(N - M)\) are also symmetric. In addition, we assume that the operators \(\sum_{j=1}^{M} \rho_j\) and \(\sum_{j=M+1}^{N} \rho_j\) have the same eigenbasis, which implies that the symmetry operators \(V\) and \(U\), referring to the two symmetric sets, are both diagonal in this eigenbasis. In analogy to the derivation of Eqs. (4.5) and (4.6) it follows that in the optimum measurement the detection operators can be supposed to obey the same symmetry as the density operators, that is, \(\Pi_j = V^{(j-1)}\Pi_0 V^{(j-1)}\) for \(j = 1, \ldots, M\) and \(\Pi_{M+j} = U^{(j-1)}\Pi_M U^{(j-1)}\) for \(j = 1, \ldots, N - M\). Eqs. (2.8) and (2.7) then reduce to the optimality conditions

\[
Z - ap \geq 0, \quad (Z - ap) \Pi_0 = 0, \tag{5.6}
\]

\[
Z - \frac{\eta_j}{M} \rho_1 \geq 0, \quad (Z - \frac{\eta_j}{M} \rho_1) \Pi_1 = 0, \tag{5.7}
\]

\[
Z - \frac{\eta_j \rho_{M+1}}{N - M} \geq 0, \quad (Z - \frac{\eta_j \rho_{M+1}}{N - M}) \Pi_{M+1} = 0, \tag{5.8}
\]

where \(\eta_1 + \eta_2 = 1\). These conditions contain only \(\Pi_0\) and the two detection operators belonging to the reference states \(\rho_1\) and \(\rho_{M+1}\) of the two symmetric sets.

VI. RELATION TO STATE DISCRIMINATION WITH A FIXED ERROR PROBABILITY

A. General considerations

So far we considered the measurement strategy that maximizes the overall probability of getting a correct result, \(P_e\), with a fixed value of the failure probability \(Q\). The general relation \(P_e + P_e + Q = 1\) implies that

\[
P_e^{\max}|_{P_e} = 1 - Q - P_e(Q), \tag{6.1}
\]

where \(P_e(Q)\) is the minimum overall error probability that can be obtained at the same fixed value of \(Q\). Another discrimination strategy maximizes \(P_e\) under the constraint that the overall error probability \(P_e\) has a fixed value \([28, 29]\). We then get

\[
P_e^{\max}|_{P_e} = 1 - P_e - Q(P_e), \tag{6.2}
\]

where \(Q(P_e)\) is the minimum failure probability necessary to achieve the same fixed error rate \(P_e\).

Let us investigate the relation between the optimization problems posed by these two strategies. For this purpose we suppose that \(P_e(Q)\), introduced in Eq. (6.1), is a monotonously decreasing function of \(Q\) in a certain interval around a value \(Q = Q_\alpha\). Then from the assumption \(P_e(Q_\alpha) = P_e(1)\) it follows that \(Q(P_e(\alpha)) = Q(\alpha)\) (2), where \(Q(P_e)\) is introduced in Eq. (6.2). In order to verify this intuitive statement, we use an indirect proof. Suppose that \(Q(P_e(\alpha)) = Q(\beta)\), where \(Q(\beta) < Q(\alpha)\). This means that the failure probability \(Q(\alpha)\) is not large enough to achieve the value \(P_e(\alpha)\), or, in other words, when \(Q\) is fixed at \(Q_\alpha\), we get a minimum overall error probability \(P_e\) that is larger than \(P_e(\alpha)\), in contradiction to the assumption (1). Now suppose that \(Q(P_e(\alpha)) = Q(\beta)\), where \(Q(\beta) > Q(\alpha)\). This means that the value \(P_e(\alpha)\) can be already reached at a value of \(Q\) that is smaller than \(Q_\alpha\) which together with the assumption (1) is a contradiction to the fact that \(P_e(Q)\) is monotonously decreasing. Hence the conclusion (2) indeed follows from the assumption (1). In an analogous way the equation \(P_e(Q_\alpha) = P_e(\alpha)\) can be shown to follow from \(Q(P_e(\alpha)) = Q_\alpha\). These findings are summarized as

\[
P_e(Q_\alpha) = P_e(\alpha) \Leftrightarrow Q(P_e(\alpha)) = Q_\alpha. \tag{6.3}
\]

Taking Eqs. (6.1) and (6.2) into account, we thus obtain the relation \(P_e^{\max}|_{Q_\alpha} = P_e^{\max}|_{P_e}\). Hence we conclude that the detection operators maximizing \(P_e\) at the fixed failure probability \(Q_\alpha\) (or minimizing \(P_e\) at this value \(Q_\alpha\), respectively), are the same as the detection operators maximizing \(P_e\) at the fixed error probability \(P_e^{\alpha} = 1 - Q_\alpha - P_e^{\max}|_{Q_\alpha}\) (or minimizing \(Q\) at this value \(P_e^{\alpha}\), respectively).
The latter conclusion can be also obtained in more formal terms. Let us assume that the detection operators $\Pi_0, \Pi_1, \ldots, \Pi_N$, the operator $Z$ and the scalar multiplier $\alpha$ fulfill the optimality conditions, Eqs. (2.10) and (2.27). Then the detection operators determine the measure-ment that maximizes $P_e$ at the fixed failure probability $Q = \alpha^{-1}\text{Tr}(Z \Pi_0)$, as becomes obvious from Eq. (2.13). This yields the probability of correct results $P_e^\text{max}|_Q = 1 - P_e - Q = \text{Tr} Z - aQ$, see Eq. (2.13). From the latter equality it follows that in the optimum measurement $P_e = 1 - \text{Tr} Z - (1 - a)Q$. Taking again Eq. (2.13) into account, we conclude that the same optim-imum detection operators also characterize a measure-ment which maximizes $P_e$ at the fixed overall error prob-ability $P_e = 1 - \text{Tr} Z - (\alpha^{-1} - 1)\text{Tr}(Z \Pi_0)$.

Due to the connection between the optimization prob-lems in the two strategies we can directly determine the solution $P_e^\text{max}|_P$ from the solution $P_e^\text{max}|_Q$ and vice versa, using Eqs. (6.1) and (6.2) and taking into ac-count that according to Eq. (6.3) the function $Q(P_e)$ is the inverse of the function $P_e(Q)$. We note that the determination of $P_e^\text{max}|_P$ is only of practical interest if

$$P_e \leq P_E = 1 - P_e^\text{max}|_Q = 0 \quad (6.4)$$

with $P_E$ denoting the minimum error probability ob-tainable in the strategy of minimum-error discrimina-tion $[1,2]$, where inconclusive results do not occur. Admitting a larger value of $P_e$ does not yield any advantage since for $P_e > P_E$ we get $P_e^\text{max}|_P \leq 1 - P_e < 1 - P_E$, where Eq. (6.2) has been used.

### Example

In the following we present an example where we derive the maximum probability of correct results with a fixed error rate, $P_e^\text{max}|_P$, with the help of the result for $P_e^\text{max}|_Q$. We consider the discrimination of two mixed qubit states with $C_1 = C_2 = C$, see Sec. III C, assuming that $\rho_{11} \leq \rho_{22}$. From Eq. (6.1) together with Eqs. (6.2) and (6.24) we find that

$$P_e(Q) = \frac{1 - Q}{2} - \frac{2C - 1}{2} \sqrt{(1 - 2|\rho_{12}|)(1 + 2|\rho_{12}| - 2Q)},$$

provided that $Q$ is restricted to a certain interval. Inserting the boundaries of this interval into Eq. (6.5) leads to the restriction $P_e' \leq P_e(Q) \leq P_e$, where $P_e$ refers to minimum-error discrimination, see Eq. (6.3), and

$$P_e' = \begin{cases} P_e(2|\rho_{12}|) = (1 - C)(1 - 2|\rho_{12}|) & \text{if } |\rho_{12}| \leq \rho_{11}, \\ P_e(Q_{cr}) & \text{if } |\rho_{12}| \geq \rho_{11} \end{cases}$$

with $P_e(Q_{cr}) = \frac{(\rho_{11} - |\rho_{12}|)^2}{1 - 2|\rho_{12}|} + (1 - C)(\rho_{22} - \rho_{11})$. From the function $P_e(Q)$ we can derive the inverse function $Q(P_e)$. Making use of Eq. (6.2) we obtain

$$P_e^\text{max}|_P = P_e + (2C - 1)^2(1 - 2|\rho_{12}|)$$

$$+ 2(2C - 1)\sqrt{(1 - 2|\rho_{12}|)(1 - C)(1 - 2|\rho_{12}|)},$$

if $P_e' \leq P_e \leq P_E$, which reduces to $P_e^\text{max}|_P = (\sqrt{P_e} + \sqrt{1 - 2|\rho_{12}|})^2$ when $C = 1$, that is when the states are pure, in accordance with Eq. (6.7). Next we use Eq. (6.11) together with Eq. (6.20) and get $P_e(Q) = (1 - C)(1 - Q)$ for $Q' \leq Q \leq 1$, where $Q'$ is given by Eq. (6.21). This restricts $P_e(Q)$ to the interval $0 \leq P_e(Q) \leq P_e''$ with

$$P_e'' = \begin{cases} (1 - C)(1 - 2|\rho_{12}|) = P_e' & \text{if } |\rho_{12}| \leq \rho_{11}, \\ (1 - C)(1 - Q_1) & \text{if } |\rho_{12}| \geq \rho_{11}. \end{cases}$$

For $C \neq 1$ we can determine the inverse function $Q(P_e)$ and insert it into Eq. (6.2), arriving at

$$P_e^\text{max}|_P = P_e \frac{C}{1 - C}$$

if $0 \leq P_e \leq P_e''$ ($C \neq 1$).

Equation (6.9) reflects the fact that unambiguous discrimina-tion, where $P_e = 0$, is impossible when $C \neq 1$, since then $P_e^\text{max} = 0$ which means that $Q = 1$ and the measurement always fails.

If $|\rho_{12}| \leq \rho_{11}$ Eqs. (6.7) - (6.9) determine the complete solution since in this case $P_e' = P_e''$. For $C = 1$ the solution is in agreement with the result obtained for pure states by directly performing the optimization when $P_e$ is fixed.

If $|\rho_{12}| \geq \rho_{11}$, that is if $P_e' \neq P_e''$, we still have to consider the interval $P_e'' = P_e(Q_1) \leq P_e \leq P_e(Q_{cr}) = P_e'$. From Eqs. (6.1) and (A20) we find that

$$P_e(Q) = (1 - C)(1 - Q) + (2C - 1)\gamma_2(Q)$$

(6.10)

if $Q_{cr} \leq Q \leq Q_1$, where $\gamma_2(Q_1) = 0$. Using Eq. (A19) it is in principle possible to invert Eq. (6.10), that is, to determine the function $Q(P_e)$, which yields $P_e^\text{max}|_P$ according to Eq. (6.2). Since for $C \neq 1$ the calculations are rather involved, we specialize to the case $C = 1$, where $P_e'' = 0$ and $P_e(Q) = \gamma_2(Q)$, which due to Eq. (6.2) yields $P_e^\text{max}|_P = 1 - \gamma_2 - \gamma_2(Q)$. The right-hand side of the latter equation is equivalent to $\gamma_2$, as becomes obvious from Eq. (A19). Substituting $\gamma_2 = P_e$ into the equation for $\gamma_2$ given by Eq. (A19), we obtain for $|\rho_{12}| \geq \rho_{11}$

$$P_e^\text{max}|_P = \frac{1}{\rho_{11}} \left(\rho_{12}\sqrt{P_e} + \sqrt{\Delta} \sqrt{\rho_{11} - P_e}\right)^2$$

(6.11)

if $C = 1$ and $0 \leq P_e \leq (\rho_{12} - |\rho_{12}|)^2 = P_e'$

where $\Delta = \rho_{11}\rho_{22} - |\rho_{12}|^2$. Equation (6.11) together with Eq. (6.7) for $C = 1$ determines the complete solution when the states are pure with $|\rho_{12}| \geq \rho_{11}$. Using Eq. (6.5) we find that this solution is in agreement with the results obtained for two pure states by directly performing the optimization for a fixed value of $P_e$. [29]
VII. SUMMARY AND CONCLUSIONS

In the main part of the paper we considered a measurement for state discrimination that minimizes the error probability $P_e$, or maximizes the probability $P_c$ of correct results, respectively, when a certain fixed probability $Q$ of inconclusive results is admitted. For a number of problems not treated before we derived analytical solutions for the optimum measurement:

(i) We investigated the discrimination of two arbitrary mixed qubit states that occur with arbitrary prior probabilities. For the case that the two conclusive outcomes can be discriminated with the same maximum confidence we obtained the complete solution, see Eqs. (3.22) - (3.26). This solution includes the discrimination of two pure states occurring with arbitrary prior probabilities.

(ii) We studied the discrimination of $N$ symmetric states spanning a $d$-dimensional Hilbert space. For a certain class of symmetric equiprobable pure qubit states ($d \leq N$) we derived the solution, given by Eqs. (4.11) and (4.15). As a special case, this solution contains the discrimination of $N$ symmetric linearly independent pure states with equal mutual overlaps, see Eqs. (4.18) and (4.19). Moreover, we also obtained the solution for the discrimination of $N$ symmetric equiprobable mixed qubit states, given by Eqs. (4.21) and (4.22), and of $N$ special symmetric mixed states of rank $D$ spanning a joint Hilbert space of dimension $ND$, see Eq. (4.25).

(iii) We solved the optimization problem for a case of mixed qubit states that are complete in the sense that a weighted sum of their density operators is equal to the identity operator, and we also specified the general optimality conditions for a certain kind of partially symmetric states. The treatment of the optimization problem resulting for the latter case is left for further investigations.

In the final part of the paper we showed that there exists a general relation between the solutions for optimum state discrimination in the two different discrimination strategies where either the rate $Q$ of inconclusive results, or the overall error rate $P_e$, has a fixed value. This relation, expressed by Eqs. (5.1) - (5.3), holds for an arbitrary number $N$ of mixed states. It implies that by solving the optimization problem in one of the two strategies, one can also obtain the solution in the other strategy. As an illustration we presented an example where for two mixed qubit states the maximum rate of correct results with a fixed error rate $P_e$ is derived from the solution for optimum state discrimination with a fixed rate $Q$ of inconclusive results.

In order to solve the optimization problem for a fixed probability $Q$ of inconclusive results, we applied the operator conditions [25] determining the optimum measurement. As discussed in our paper, these optimality conditions, Eqs. (2.6) and (2.7), provide a very general approach for treating various optimized state discrimination measurements, as far as only overall probabilities, averaged over all outcomes, are considered. In the appropriate limiting cases, they describe minimum-error discrimination, on the one hand, while on the other hand they refer to optimized maximum-confidence discrimination provided that the maximum confidence is the same for each conclusive outcome, or to optimum unambiguous discrimination, respectively, in the special case when the maximum confidence is equal to unity.

We note that related work has been done independently by E. Bagan and R. Muñoz-Tapia (Barcelona) and G. A. Olivares-Rentería and J. A. Bergou (New York) [8].

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Appendix A

Here we present the derivation of the optimum measurement that discriminates two mixed qubit states when $C_1 = C_2 = C$. Eq. (3.3), valid for $\Gamma - aI > 0$, yields

$$\Gamma_{22} = \Gamma_{11} = \frac{a^2 - C(1 - C)}{2a - 1}, \quad |\Gamma_{12}| = \Gamma_{11} - a, \quad (A1)$$

where we have to require that $\Gamma_{11} > a$ and $\Gamma_{11} > C > 1/2$ to ensure the positivity of the expressions under the square-root signs in Eqs. (3.10) - (3.12). From Eqs. (3.14) and (3.16) we obtain the relations

$$e^{i\phi} = -e^{i\phi}, \quad \beta_1 + \beta_2 = \frac{1 - Q}{2\Gamma_{11} - 1} = \frac{2|\rho_{12}| - Q}{|\Gamma_{12}|}. \quad (A2)$$

Using Eq. (A1) in the last equality of Eq. (A2) we get

$$a = \frac{1}{2} + \frac{2C - 1}{2} \sqrt{\frac{1 - 2|\rho_{12}|}{1 + 2|\rho_{12}| - 2Q}}. \quad (A3)$$

After calculating the matrix elements $\Gamma_{11}$ and $\Gamma_{22}$ that result for this value of $a$, we obtain from Eqs. (3.14) and (3.15) the solutions

$$\beta_{1/2} = \frac{\sqrt{(1 - 2|\rho_{12}|)(1 + 2|\rho_{12}| - 2Q)} \pm (\rho_{11} - \rho_{22})}{2(2C - 1)}. \quad (A4)$$

Due to Eqs. (A2) and (A3) a solution with positive constants $\beta_1$ and $\beta_2$ can only result when the two conditions

$$Q \leq 2|\rho_{12}|, \quad Q \leq \frac{\rho_{11}\rho_{22} - |\rho_{12}|^2}{\frac{1}{2} - |\rho_{12}|} \equiv Q_{cr} \quad (A5)$$

are simultaneously fulfilled. In the second condition we used the relation $(\rho_{11} - \rho_{22})^2 = 1 - 4\rho_{11}\rho_{22}$ and introduced the critical value $Q_{cr}$ of the failure probability. For
later purposes we note that
\begin{align}
2|\rho_{12}| & \leq Q_{cr} \quad \text{if} \quad |\rho_{12}| \leq \min\{\rho_{11}, \rho_{22}\}, \\
2|\rho_{12}| & \geq Q_{cr} \quad \text{if} \quad |\rho_{12}| \geq \min\{\rho_{11}, \rho_{22}\}.
\end{align}

(A6) To see this, let us assume that $|\rho_{12}| \leq \rho_{11} \leq \frac{1}{2}$, which leads to $(\frac{1}{2} - |\rho_{12}|)^2 \geq (\frac{1}{2} - \rho_{11})^2 = 1 - \rho_{11}\rho_{22}$. Hence it follows that $\rho_{11}\rho_{22} - |\rho_{12}|^2 \geq |\rho_{12}| - |\rho_{12}|^2$ and consequently $2|\rho_{12}| \leq Q_{cr}$. Analogous considerations hold for the other possible cases.

When the two conditions in Eq. (A5) are met, we obtain from Eq. (3.8) the optimum detection operators
\[ \Pi_i = \beta_i \rho_i^{-1/2}|\tilde{\pi}_i\rangle\langle \tilde{\pi}_i|\rho_i^{-1/2} \quad (i = 1, 2), \]
with
\[ |\tilde{\pi}_1\rangle = \sqrt{\frac{2C_1 - 1}{2}} \left( |\nu_1\rangle + e^{-i\phi} |\nu_2\rangle \right) \]
\[ |\tilde{\pi}_2\rangle = \sqrt{\frac{2C_1 - 1}{2}} \left( |\nu_1\rangle + e^{-i\phi} |\nu_2\rangle \right) \]
with $b = -\frac{1}{\sqrt{(1 - 2|\rho_{12}|)(1 + 2|\rho_{12}| - 2Q)}}.$

(A9) For completeness, we also give an explicit expression for the failure operator. Because of Eqs. (3.11) and (3.12), the latter can be written as
\[ \Pi_0 = Q \rho_0^{-1/2}|\tilde{\pi}_0\rangle\langle \tilde{\pi}_0|\rho_0^{-1/2} = \frac{Q}{Q_{cr}} |\pi_0\rangle\langle \pi_0|, \]
where $|\tilde{\pi}_0\rangle = \sqrt{Q_{cr}}\rho_0^{-1/2}|\tilde{\pi}_0\rangle$ with $\langle \pi_0|\pi_0\rangle = 1$. Here we took into account that
\[ |\tilde{\pi}_0\rangle = \frac{|\nu_1\rangle + e^{-i\phi} |\nu_2\rangle}{\sqrt{2}}, \]
and
\[ \langle \pi_0|\rho_0^{-1/2}|\tilde{\pi}_0\rangle = \frac{1}{Q_{cr}}. \]
where in the second equality use has been made of the matrix elements of $\rho^{-1}$ with respect to the basis $\{|\nu_1\rangle, |\nu_2\rangle\}.$

If $|\rho_{12}| \leq \min\{\rho_{11}, \rho_{22}\}$ and therefore $2|\rho_{12}| \leq Q_{cr}$, the condition $Q \leq 2|\rho_{12}|$ is sufficient to guarantee that both inequalities in Eq. (A5) hold, which means that the detection operators given by Eqs. (A8) - (A12) describe the optimum measurement. Using Eq. (2.12) together with the expressions for $\rho_1$ and $\rho_2$ that ensue from Eqs. (3.11) and (3.12) when $C_1 = C_2 = C$, we obtain with the help of Eqs. (A8) the maximum probability of correct results $P_c^{(0)}$ given in Eq. (3.20). After combining this result with Eq. (3.20) we arrive at Eq. (3.22).

We now focus on the case that $|\rho_{12}| \geq \min\{\rho_{11}, \rho_{22}\}$, where due to Eqs. (A5) and (A7) the detection operators given by Eqs. (A8) - (A12) describe the optimum measurement only as long as $Q \leq Q_{cr}$. In order to derive the solution for $Q \geq Q_{cr}$, we assume without loss of generality that $|\rho_{12}| \geq \rho_{11}$, which implies that $\rho_{11} < \rho_{22}$ and that $P_c^{max} = C(1 - Q)$ if $Q \geq Q_1$, see Eqs. (3.11) and (3.20). Equation (A4) reveals that in this case the parameter $\beta_1$ vanishes for $Q = Q_{cr}$, which means that $\Pi_1 = 0$. Therefore if $Q_{cr} \leq Q \leq Q_1$ we have to search for a projective measurement with
\[ \Pi_1 = 0, \quad \Pi_2 = |\pi_2\rangle\langle \pi_2|, \quad \Pi_0 = I - \Pi_2 \]
that maximizes $P_c$ at the fixed value $Q$ and yields $P_c^{max}(Q_1) = C(1 - Q_1)$. Instead of using again the optimality conditions, we can solve the optimization problem directly, due to the simple structure of the detection operators. For this purpose we make the convenient general ansatz
\[ |\pi_2\rangle = \sqrt{\gamma_1 \rho^{-1/2}} |\nu_1\rangle + e^{i\chi} \sqrt{\gamma_2} \rho^{-1/2} |\nu_2\rangle, \]
where the phase $\chi$ and the nonnegative parameters $\gamma_1$ and $\gamma_2$ have to be determined. The normalization condition $|\pi_2|\langle \pi_2| = 1$ leads to the constraint
\[ \gamma_1 \rho_{22} + \gamma_2 \rho_{11} - 2\sqrt{\gamma_1 \gamma_2} \rho_{12} |\cos(\chi + \phi)| = \rho_{11} \rho_{22} - |\rho_{12}|^2 \geq \Delta, \]
(A16) where we used the matrix elements of $\rho^{-1}$ with respect to the basis $\{|\nu_1\rangle, |\nu_2\rangle\}$ and took into account that $\rho_{12} = |\rho_{12}| e^{i\phi}$. The fixed probability $Q$ of inconclusive results can be expressed as
\[ Q = 1 - Tr(\rho \Pi_2) = 1 - (\gamma_1 + \gamma_2). \]
From $P_c = \eta_2 Tr(\rho \Pi_2) = Tr(\rho \rho_2^{1/2} \Pi_2 \rho_1^{1/2})$ we obtain the probability of correct results,
\[ P_c = (1 - C) \gamma_1 + C \gamma_2, \]
where Eq. (3.22) has been used. In order to maximize $P_c$ we need to allow for the largest possible values of $\gamma_1$ and $\gamma_2$ consistent with Eq. (A16), that is we have to put $\cos(\chi + \phi) = 1$, which means that $\chi = -\phi$. Using Eqs. (A16) and (A17) we then find that
\[ \gamma_2 = \frac{(|\rho_{12}| \sqrt{\gamma_1} + \sqrt{\Delta \sqrt{\rho_{11} - \gamma_1}})^2}{\rho_{11}} = 1 - Q - \gamma_1. \]
(A19) The equation arising from the second equality sign in Eq. (A19) can be solved to yield an explicit expression for $\gamma_1(Q)$. Making use of Eq. (A18) together with Eq. (A17) we get the solution
\[ P_c^{max} = P_c^{(1)} = C(1 - Q) - (2C - 1) \gamma_1(Q), \]
which is explicitly given in Eq. (3.20). Clearly, our derivation requires that $\gamma_1(Q) \geq 0$. According to Eq. (A19) the boundary case $\gamma_1 = 0$ implies that $\Delta / \rho_{11} = 1 - Q$ and therefore $Q = Q_{1}$, see Eq. (3.17). Hence Eq. (A20) is valid for $Q_{cr} \leq Q \leq Q_1$, yielding $P_c^{max}(Q_1) = C(1 - Q_1)$, as required. Taking Eq. (3.20) into account we arrive at the final result, Eq. (3.21). A direct calculation shows that indeed $P_c^{(0)}(Q_{cr}) = P_c^{(1)}(Q_{cr})$.

We still consider the question under which condition the relation $C_1 = C_2 = C$ holds true when the mixed
qubit states are represented with the help of the general ansatz \( \rho_{1/2} = \rho_{1/2} |\psi_{1/2}\rangle \langle \psi_{1/2}| + \frac{\rho_{1/2}}{2} I \), see Eq. (3.27). After calculating the spectral representations of \( \rho_j = \rho^{-1/2} \eta_j \rho \rho^{-1/2} \) \((j = 1, 2)\) and comparing the result with Eqs. (3.21) and (3.22), we arrive at the condition \((\eta_1 p_1)^2 - (\eta_2 p_2)^2 = \eta_1 - \eta_2\), leading to the explicit expression \((2C - 1)^2 = 1 - \frac{2(1 - p_1^2)(1 - p_2^2)}{1 + \sqrt{(1 - p_1^2)(1 - p_2^2) + p_1 p_2(1 - 2|S|^2)}} \) (A21) with \( S = |\psi_1\rangle \langle \psi_2| \). Taking into account that \( \text{Tr}(\rho_1 \rho_2) = \text{Tr}(\rho_1 \rho_2)/\langle \eta_1 \eta_2 \rangle \) we find the help of Eqs. (3.27), (3.21) and (3.22) that \[ 1 - p_1 p_2(1 - 2|S|^2)/2 = C(1 - C)(p_1^2 + p_2^2 - 2|p_1 p_2|^2)\] \( \eta_1(1 - \eta_1) \) (A22)

Together with Eq. (3.6) and with the relation \((\eta_1 - 1)^2 = \frac{1 - 2 - \rho}{p_2^2}\), following from the condition for \( C_1 = C_2 = C\), Eqs. (A21) and (A22) form a system of four equations for determining the values of \( \eta_1, p_1, p_2 \) and \( |S| \) when \( C \) and the matrix elements \( \rho_{ij} \) are given.

**Appendix B**

We consider the discrimination of \( N \) symmetric qudit states \( \rho_j \) \((j = 1, \ldots, N)\) described by

\[ \rho_j = p \langle \psi_j | \psi_j \rangle + \frac{1 - p}{d} I \quad \text{with} \quad |\psi_j\rangle = V^{(j - 1)}|\psi_1\rangle, \]  

(B1)

where \( 0 \leq p \leq 1 \) and \( |\psi_1\rangle = c_1 \sum_{i=1}^{m} |r_i\rangle + c_2 \sum_{i=m+1}^{d} |r_i\rangle \), see Eq. (4.13). Here \( I \) is the identity operator in the \( d \)-dimensional Hilbert space \( \mathcal{H}_d \) spanned by the \( N \) pure states \( |\psi_j\rangle \). Using the properties of the symmetry operator \( V \), given after Eq. (4.22), we can determine the spectral representation of \( \rho = \frac{1}{N} \sum_{j=1}^{N} \rho_j \), arriving at

\[ \rho = r_1 \sum_{i=1}^{m} \langle r_i | r_i \rangle + r_2 \sum_{i=m+1}^{d} \langle r_i | r_i \rangle, \quad r_i = p |c_i|^2 + \frac{1 - p}{d} \]  

(B2)

with \( i = 1, 2 \). The structure of \( \rho_1 \) suggests the ansatz

\[ \Pi_1 = |\tilde{v}\rangle \langle \tilde{v}| \quad \text{with} \quad |\tilde{v}\rangle = b_1 \sum_{i=1}^{m} |r_i\rangle + b_2 \sum_{i=m+1}^{d} |r_i\rangle. \]  

(B3)

To be specific, we assume that \( 0 < |b_1| \leq |b_2| \). We shall use the optimality conditions, Eqs. (4.10) and (4.11), in order to investigate whether this ansatz yields the optimum solution for the case that \( Z - ap > 0 \).

The equality in Eq. (4.5) implies that \( \Pi_0 \) cannot span the whole \( d \)-dimensional Hilbert space if \( Z - ap > 0 \). Using Eqs. (4.7) and (B3) this leads to the requirement

\[ N|b_2|^2 = 1, \quad \Pi_0 = (1 - N|b_1|^2) \sum_{i=1}^{m} |r_i\rangle \langle r_i|, \]  

(B4)

which arises from the positivity constraint \( \Pi_0 \geq 0 \) with \( |b_2| \geq |b_1| \). The resulting failure probability is given by

\[ Q = \text{Tr}(\rho \Pi_0) = (1 - N|b_1|^2)mr_1. \]  

(B5)

Taking into account that \( [Z, \rho] = 0 \) it becomes obvious from Eqs. (B2) and (B4) that the equality in the first optimality condition, Eq. (4.6), holds true if

\[ Z = z_1 \sum_{l=1}^{m} |r_l\rangle \langle r_l| + z_2 \sum_{l=m+1}^{d} |r_1\rangle \langle r_1| \quad \text{and} \quad a = \frac{z_1}{mr_1} \]  

(B6)

Now we turn to the equality in the second optimality condition, Eq. (4.6), leading to \((NZ - \rho_1)|\tilde{v}\rangle \langle \tilde{v}| = 0\), which implies that \( N(r_1 Z|\tilde{v}\rangle = (r_1|\tilde{r}|) \) for \( l = 1, \ldots, d \). After expressing \( \rho_1 \) with the help of Eqs. (B1) and (B3) we arrive at the condition

\[ z_1 = \frac{p}{N} \frac{c_1}{b_1} \left[ m c_1^2 b_1 + (d - m) c_2^2 b_2 \right] + \frac{1 - p}{dN} \]  

(i = 1, 2).

(B7)

Since \( Z \) is Hermitian we choose the phase of \( b_1 \) to coincide with the phase of \( c_1 \) \((i = 1, 2)\). Equations (B4) and (B5) then lead to

\[ b_2 = \frac{1}{\sqrt{N}} \frac{c_2}{|c_2|}, \quad b_1 = \frac{w}{\sqrt{N}} \frac{c_1}{|c_1|} \quad \text{with} \quad w = \sqrt{\frac{mr_1 - Q}{mr_1}} \]  

(B8)

From Eq. (B7) we obtain the relation

\[ z_1 - \frac{1 - p}{dN} = \frac{p}{N} \frac{c_1}{c_2} \frac{m}{|c_1|} \left( \frac{d - m}{w} \right) = \frac{|c_1|^2}{w|c_2|} \left( z_2 - \frac{1 - p}{dN} \right), \]  

(B9)

yielding explicit expressions for \( z_1 \) and \( z_2 \). Using Eq. (4.8) we get the maximum probability of correct results,

\[ p_{E \text{max}} |Q = (\tilde{v})|\rho_1 \rangle \langle \tilde{v}| = p_{E (0)}, \]  

where because of Eq. (B8)

\[ p_{E (0)} = \frac{p}{N} \left[ |m| |c_1| w + (d - m) |c_2| \right]^2 + \frac{1 - p}{dN} (mw^2 + d - m). \]  

(B10)

In order to decide whether this is indeed the optimum solution we have to check whether the positivity constraints in Eqs. (4.10) and (4.11) are fulfilled.

We start with the positivity constraint in Eq. (4.10), which takes the form \( Z - \frac{p}{N} |\psi_1\rangle \langle \psi_1| - \frac{1 - p}{d} I \geq 0 \). Clearly, the constraint is satisfied provided that \( N d|\psi_1| Z|\psi_1| \geq 1 - p + pd \). After inserting the explicit representation of the operator \( Z \), this inequality can be transformed into

\[ m^2 |c_1|^4 + m(d - m)|c_1 c_2|^2 \left( \frac{|c_1|^2}{w^2} + \frac{w^2}{|c_1|^2} \right) + (d - m)^2 |c_2|^4 \geq 1. \]  

(B11)

Due to the normalization condition in Eq. (4.13) and to the fact that \( x + x^{-1} \geq 2 \) for any \( x \) it is obvious that Eq. (B11) always holds true, which implies that the positivity constraint in Eq. (4.10) is satisfied.

The positivity constraint given in Eq. (4.5), on the other hand, which results from the fact that \( Q \) is fixed
and different from zero, imposes the relevant restriction. Because of Eq. (B10), this constraint takes the form
\[ z_2 - a r_2 = z_2 - z_1 \frac{r_2}{r_1} \geq 0, \]  
(B12)
where \( z_1 \) and \( z_2 \) follow from Eq. (B9). It turns out that the constraint is satisfied in the following general cases:

(i) When \( p = 1 \) and therefore \( r_i = |c_i|^2 \) (\( i = 1, 2 \), Eq. (B12) is fulfilled if \( w \geq |c_2|/|c_1| \), which with the help of Eqs. (B8) and (B13) leads to the condition
\[ Q \leq m(|c_1|^2 - |c_2|^2) = 1 - d|c_2|^2 = Q', \]  
(B13)
(ii) When \( d = 2 \), and consequently \( m = d - m = 1 \), a straightforward calculation shows that Eq. (B12) holds true provided that \( w^2 \geq r_2/r_1 \). Because of Eq. (B8), this yields the requirement \( Q \leq r_1 - r_2 \), or, using Eq. (B12),
\[ Q \leq p(|c_1|^2 - |c_2|^2) = p(1 - 2|c_2|^2) = Q'. \]  
(B14)
(iii) When \( |c_1|^2 = |c_2|^2 = \frac{1}{d} \) for arbitrary values of \( p \) and \( d \), implying that also \( r_1 = r_2 = \frac{1}{d} \). Eq. (B12) is fulfilled if \( w = 1 \), that is, if \( Q = 0 \). This example belongs to the special case where the density operators resolve the identity operator, as discussed in Sec. V A.

The preceding calculations have been performed assuming that \( |b_2| \geq |b_1| \). Since \( Q \) has to be positive, we conclude from Eqs. (B13) and (B14) that this assumption is justified if \( |c_2| \leq |c_1| \). In the opposite case, \( |c_2| \geq |c_1| \), the derivation could be performed in a completely analogous way, where the results arise from the previous ones when the indexes 1 and 2 are interchanged and when \( m \) is replaced by \( d - m \), and vice versa.

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