Marcinkiewicz-Zygmund and ordinary strong laws for empirical distribution functions and plug-in estimators

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Abstract

Both Marcinkiewicz-Zygmund strong laws of large numbers (MZ-SLLNs) and ordinary strong laws of large numbers (SLLNs) for plug-in estimators of general statistical functionals are derived. It is used that if a statistical functional is “sufficiently regular”, then a (MZ-) SLLN for the estimator of the unknown distribution function yields a (MZ-) SLLN for the corresponding plug-in estimator. It is in particular shown that many L-, V- and risk functionals are “sufficiently regular”, and that known results on the strong convergence of the empirical process of α-mixing random variables can be improved. The presented approach does not only cover some known results but also provides some new strong laws for plug-in estimators of particular statistical functionals.

Keywords: statistical functional, plug-in estimator, Marcinkiewicz-Zygmund strong law of large numbers, ordinary strong law of large numbers, empirical process, α-mixing, function bracket, L-statistic, law-invariant risk measure, V-statistics
1 Introduction

Let \( F \) be a class of distribution functions on the real line, and \( T : F \to V' \) be a statistical functional, where \( (V', \| \cdot \|_V') \) is a normed vector space. Let \( (X_i)_{i \in \mathbb{N}} \) be a sequence of identically distributed real random variables on some probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \) with distribution function \( F \in \mathcal{F} \). If \( \hat{F}_n \) denotes a reasonable estimator for \( F \) based on the first \( n \) observations \( X_1, \ldots, X_n \), then \( T(\hat{F}_n) \) can provide a reasonable estimator for \( T(F) \).

In the context of nonparametric statistics, a central question concerns the rate of almost sure convergence of the plug-in estimator \( T(\hat{F}_n) \) to \( T(F) \). That is, one wonders for which exponents \( r' \geq 0 \) the convergence

\[
n'^r \| T(\hat{F}_n) - T(F) \|_V \to 0 \quad \mathbb{P}\text{-a.s.} \tag{1}\]

holds, where it is assumed that the left-hand side is \( \mathcal{F} \)-measurable for every \( n \in \mathbb{N} \).

This article is concerned with the convergence in (1) for both \( r' > 0 \) and \( r' = 0 \) and general statistical functionals \( T \). In the case \( r' > 0 \) the convergence in (1) can be seen as a Marcinkiewicz-Zygmund strong law of large numbers (MZ-SLLNs), and in the case \( r' = 0 \) it can be seen as an ordinary strong law of large numbers (SLLNs).

Let \( (V, \| \cdot \|_V) \) be a normed vector space with \( V \) a class of real functions on \( \mathbb{R} \), and assume that the difference \( F_1 - F_2 \) of every two distribution functions \( F_1, F_2 \in \mathcal{F} \) are elements of \( V \). So \( \| \cdot \|_V \) can in particular be seen as a metric on \( \mathcal{F} \). Assume that \( \hat{F}_n \) is a \( \mathcal{F} \)-valued estimator for \( F \) based on \( X_1, \ldots, X_n \), that \( \| \hat{F}_n - F \|_V \) is \( \mathcal{F} \)-measurable for every \( n \in \mathbb{N} \), and that

\[
n^r \| \hat{F}_n - F \|_V \to 0 \quad \mathbb{P}\text{-a.s.} \tag{2}\]

for some \( r \geq 0 \). Finally, let \( \hat{\mathbb{F}}_n = \{ \hat{F}_n(\omega) : \omega \in \Omega \} \) be the range of \( \hat{F}_n \), and \( \hat{\mathbb{F}} \) be the union of the \( \hat{F}_n, n \in \mathbb{N} \). Then, if for every sequence \( (F_n) \subset \hat{\mathbb{F}} \) with \( \| F_n - F \|_V \to 0 \) we have that

\[
\| T(F_n) - T(F) \|_V = O(\| F_n - F \|_V) \tag{3}\]

for some fixed \( \beta > 0 \), we obtain by choosing \( F_n := \hat{F}_n (\omega\text{-wise}) \) that (1) holds for \( r' = r\beta \).

If for every sequence \( (F_n) \subset \hat{\mathbb{F}} \) with \( \| F_n - F \|_V \to 0 \) we only have that

\[
\| T(F_n) - T(F) \|_V = o(1), \tag{4}\]

then we obtain that (1) holds at least for \( r' = 0 \); again choose \( F_n := \hat{F}_n (\omega\text{-wise}) \). That is, in order to obtain a MZ-SLLN for \( T(\hat{F}_n) \) it suffices to have a MZ-SLLN for \( \hat{F}_n \) and to verify (2), and in order to obtain a SLLN for \( T(\hat{F}_n) \) it suffices to have a SLLN for \( \hat{F}_n \) and to verify (3). We refer to (3) as Hölder-\( \beta \) continuity of \( T(\hat{F}_n) \) with \( \mathcal{F} \)-w.r.t. \( (\| \cdot \|_V, \| \cdot \|_{V'}) \) and to (4) as continuity of \( T(\hat{F}_n) \) with \( \mathcal{F} \)-w.r.t. \( (\| \cdot \|_V, \| \cdot \|_{V'}) \) and \( \hat{\mathbb{F}} \).

Concerning \( \hat{F}_n \) we will restrict ourselves to the empirical distribution function. That is, from now on we assume that \( \hat{F}_n = \frac{1}{n} \sum_{i=1}^n 1_{\{X_i, \infty\}} \). In particular, \( \hat{\mathbb{F}} \) will always be...
contained in the class of all empirical distribution functions
\( \frac{1}{n} \sum_{i=1}^{n} 1_{[x_i, \infty)} \) with \( n \in \mathbb{N} \) and \( x_1, \ldots, x_n \in \mathbb{R} \). The rest of the article is organized as follows. In Section 2 we will first present some results that illustrate (2) for uniform and nonuniform sup-norms. Thereafter, in Section 3 we will show that several statistical functionals are \((\text{Hölder}-\beta)\) continuous w.r.t. uniform or nonuniform sup-norms. The proofs of the results of Section 2 will be given in Sections 4–6.

2 Strong laws for \( \hat{F}_n \)

An intrinsic example for \((\mathcal{V}, \| \cdot \|_\mathcal{V})\) is the normed vector space \((\mathcal{D}_\phi, \| \cdot \|_\phi)\) of all càdlàg functions \( \psi \) with \( \| \psi \|_\phi < \infty \), where \( \| \psi \|_\phi := \| \psi \phi \|_\infty \) refers to the nonuniform sup-norm based on some weight function \( \phi \). By weight function we mean any continuous function \( \phi: \mathbb{R} \to \mathbb{R}^+ \) which is bounded away from zero, i.e. \( \phi(\cdot) \geq \varepsilon \) for some \( \varepsilon > 0 \), and u-shaped, i.e. nonincreasing on \((-\infty, x_\phi]\) and nondecreasing on \([x_\phi, \infty)\) for some \( x_\phi \in \mathbb{R} \). In Section 3 we will see that many statistical functionals are \((\text{Hölder}-\beta)\) continuous w.r.t. \((\| \cdot \|_\phi, | \cdot |)\) and \( \hat{F} \). Here we will first present some results that illustrate (2) for \( \| \cdot \|_\mathcal{V} = \| \cdot \|_\phi \).

We begin with the case of independent observations. The following result strongly relies on [2, Theorem 7.3]. The proof can be found in Section 4.

**Theorem 2.1** Let \((X_i)\) be an i.i.d. sequence of random variables with distribution function \( F \). Let \( \phi \) be a weight function and \( r \in [0, \frac{1}{2}) \). If

\[
\int_{-\infty}^{\infty} \phi(x)^{1/(1-r)} dF(x) < \infty,
\]

then

\[
n^r \| \hat{F}_n - F \|_\phi \longrightarrow 0 \quad \mathbb{P}\text{-a.s.}
\]

Let us now turn to the case of weakly dependent data. We will assume that the sequence \((X_i)\) is \( \alpha \)-mixing in the sense of [26], i.e. that the mixing coefficient \( \alpha(n) := \sup_{k \geq 1} \sup_{A,B} |\mathbb{P}[A \cap B] - \mathbb{P}[A]\mathbb{P}[B]| \) converges to zero as \( n \to \infty \), where the second supremum ranges over all \( A \in \sigma(X_1, \ldots, X_k) \) and \( B \in \sigma(X_{k+n}, X_{k+n+1}, \ldots) \). For an overview on mixing conditions see [12, 15].

**Theorem 2.2** Let \((X_i)\) be a sequence of identically distributed random variables with distribution function \( F \). Suppose that \((X_i)\) is \( \alpha \)-mixing with mixing coefficients \( \alpha(n) \). Let \( r \in [0, \frac{1}{2}) \) and assume that \( \alpha(n) \leq Kn^{-\theta} \) for all \( n \in \mathbb{N} \) and some constants \( K > 0 \) and \( \theta > 2r \). Then

\[
n^r \| \hat{F}_n - F \|_\infty \longrightarrow 0 \quad \mathbb{P}\text{-a.s.}
\]

For the proof of Theorem 2.2 which can be found in Section 5 we will combine arguments of [23] and [25]. Under the stronger mixing conditions \( \alpha(n) \leq Kn^{-3} \) and \( \alpha(n) \leq Kn^{-(3+\varepsilon)} \), \( \varepsilon > 0 \), the convergence in (5) is already known from [7, 23] and [37], respectively. If in (5) almost sure converges is replaced by convergence in probability, then the result is known from [38]. The more recent article [6] contains a version of
Theorem 2.2 for empirical processes of so-called S-mixing sequences. The concept of S-mixing seems to be less restrictive than the concept of \( \alpha \)-mixing, but the two concepts are not directly comparable.

To compare Theorem 2.2 above with Theorem 1 in [6] anyway, let \( X_t := \sum_{s=0}^{\infty} a_s Z_{t-s}, \) \( t \in \mathbb{N} \), be a linear process with \( (Z_s)_{s \in \mathbb{Z}} \) a sequence of i.i.d. random variables with expectation zero, a finite absolute \( p \)th moment for some \( p \geq 2 \), and a Lebesgue density \( f \) satisfying
\[
|f(x) - f(y)| dx \leq M|x - y| \text{ for all } x, y \in \mathbb{R} \text{ and some finite constant } M > 0.
\]
For instance, these conditions are fulfilled when \( Z_0 \) is centered normal. If \( a_s = s^{-\gamma} \) for some \( \gamma > (2 + p)/p \), then results in [19] show that \( (X_t) \) is \( \alpha \)-mixing with \( \alpha(n) \leq Kn^{-\vartheta} \) for \( \vartheta = (p(\gamma - 1) - 2)/(1 + p) \). So, if we choose \( \gamma = (3 + 2p)/p \), then we have \( \vartheta = 1 \) and therefore Theorem 2.2 yields
\[
n^r \| \hat{F}_n - F \|_\infty \rightarrow 0 \quad \mathbb{P}\text{-a.s.}, \quad \forall r \in [0, 1/2).
\]
On the other hand, in order to obtain (6) with the help of Theorem 1 and the considerations in Section 3.1 of [6], one has to choose \( \gamma = (A + (A + 1)p)/p \) for some \( A > 4 \). Since \( (A + (A + 1)p)/p > (3 + 2p)/p \) for every \( A > 4 \), Theorem 2.2 above appears to be less restrictive in the \( \alpha \)-mixing case than Theorem 1 in [6]. On the other hand, Theorem 1 in [6] covers even the two-parameter empirical process.

It seems to be hard to modify the arguments of the proof of Theorem 2.2 in such a way that they can be applied to the case of a nonconstant weight function \( \phi \). To the best of the author’s knowledge, there is no respective results in the literature so far. Results of [13] cover the case where in [5] the sup-norm is replaced by the \( L^p \)-norm w.r.t. a \( \sigma \)-finite measure for \( p > 1 \). However, as the case \( p = 1 \) is excluded, the results do not cover the \( L^1 \)-Wasserstein distance. Notice that several statistical functionals can be shown to be continuous w.r.t. the \( L^1 \)-Wasserstein distance.

If one is content with \( r = 0 \), i.e. with an ordinary SLLN, then the following Theorem 2.3 gives a respective result for nonconstant weight functions \( \phi \) and \( \alpha \)-mixing data. The proof of Theorem 2.3 can be found in Section 6. To the best of the author’s knowledge, Theorem 2.3 provides the first result on the strong convergence of the empirical distribution function \( \hat{F}_n \) of \( \alpha \)-mixing random variables to the underlying distribution function \( F \) w.r.t. a nonuniform sup-norm. For any nonincreasing function \( h : \mathbb{R}_+ \rightarrow [0, 1] \), we let \( h^{-}\) := \( \sup\{x \in \mathbb{R}_+ : h(x) > y\}, y \in [0, 1] \), be its right-continuous inverse, with the convention \( \sup\emptyset := 0 \).

**Theorem 2.3** Let \( (X_i) \) be a sequence of identically distributed random variables with distribution function \( F \). Let \( \phi \) be a weight function, and suppose that \( \int_{-\infty}^{\infty} \phi dF < \infty \). Suppose that \( (X_i) \) is \( \alpha \)-mixing with mixing coefficients \( \alpha(n) \), let \( \alpha(t) := \alpha([t]) \) be the càdlàg extension of \( \alpha(\cdot) \) from \( \mathbb{N} \) to \( \mathbb{R}_+ \), and assume that
\[
\int_{0}^{1} \log (1 + \alpha^{-}(s/2)) \overline{G}^{-}(s) ds < \infty
\]
where
\[
G(s) := \int_{0}^{s} \alpha^{-}(t) dt.
\]
for $\overline{G} := 1 - G$, where $G$ denotes the distribution function of $\phi(X_1)$. Then
\[
\|\hat{F}_n - F\|_\phi \longrightarrow 0 \quad \mathbb{P}\text{-a.s.}
\]

(8)

Remark 2.4 Notice that (7) holds in particular if $\mathbb{E}[\phi(X_1) \log^+ \phi(X_1)] < \infty$ and $\alpha(n) = \mathcal{O}(n^{-\vartheta})$ for some arbitrarily small $\vartheta > 0$; cf. [24, Application 5, p. 924].

3 Strong laws for $T(\hat{F}_n)$ for particular functionals $T$

In this section we will show that several statistical functionals $T$ are continuous w.r.t. nonuniform sup-norms $\| \cdot \|_\phi$ or w.r.t. the uniform sup-norm $\| \cdot \|_{\infty}$. As a consequence we will obtain MZ-SLLNs and SLLNs for $T(\hat{F}_n)$, cf. the discussion in the Introduction.

3.1 L-functionals

Let $K$ be the distribution function of a probability measure on $(0, 1)$, and $\mathbb{F}_K$ be the class of all distribution function $F$ on the real line for which
\[
\int_{-\infty}^{\infty} |x| dK(F(x)) < \infty.
\]
The functional $\mathcal{L}$, defined by
\[
\mathcal{L}(F) := \mathcal{L}_K(F) := \int_{-\infty}^{\infty} x dK(F(x)), \quad F \in \mathbb{F}_K,
\]
is called L-functional associated with $K$. It was shown in [8] that if $K$ is continuous and piecewise differentiable, the (piecewise) derivative $K'$ is bounded above and $F \in \mathbb{F}_K$ takes the value $d \in (0, 1)$ at most once if $K$ is not differentiable at $d$, then for every $\lambda > 1$ the functional $\mathcal{L} : \mathbb{F}_K \to \mathbb{R}$ is quasi-Hadamard differentiable at $F$ tangentially to $\mathbb{D}_{\phi_\lambda}$, where $\phi_\lambda(x) := (1 + |x|)^\lambda$. This implies in particular that $\mathcal{L}$ is also Hölder-1 continuous at $F$ w.r.t. $(\| \cdot \|_{\phi_\lambda}, | \cdot |)$ and $\hat{F}$. The assumption that $K'$ be bounded can be relaxed at the cost of a more sophisticated choice of the weight function $\phi$; cf. the following Lemma 3.1. In the lemma we will assume without loss of generality that $F(x) \in (0, 1)$ for all $x \in \mathbb{R}$. If $F$ reaches 0 or 1, then the weight function $\phi_{\gamma,F}$, defined in (10) below, can be modified in the obvious way.

Lemma 3.1 Let $F \in \mathbb{F}_K$, $\overline{F} := 1 - F$, $0 \leq \beta' < \gamma \\leq 1$, and assume that

(a) $K$ is locally Lipschitz continuous at $x$ with local Lipschitz constant $L(x) > 0$ for all $x \in (0, 1)$, and $L(x) \leq C' x^{-\beta'} (1 - x)^{-\beta'}$ for all $x \in (0, 1)$ and some constant $C' > 0$.

(b) $\int_{-\infty}^{0} F(x)^{\gamma - \beta'} dx + \int_{0}^{\infty} \overline{F}(x)^{\gamma - \beta'} dx < \infty$. 

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Assume $F(x) \in (0, 1)$ for all $x \in \mathbb{R}$, and define the weight function

$$
\phi_{\gamma,F}(x) := F(x)^{-1}1_{[-\infty, 0)}(x) + F(x)^{-1}1_{[0, \infty)}(x), \quad x \in \mathbb{R}.
$$

Then the functional $\mathcal{L} : \mathbb{F}_K \to \mathbb{R}$ is Hölder-1 continuous at $F$ w.r.t. $(\| \cdot \|_{\phi_{\gamma,F}}, \cdot | \cdot)$ and $\hat{F}$.

**Proof** Since $\mathcal{L}(F)$ can be written as $\mathcal{L}(F) = - \int_{-\infty}^{0} K(F(x)) \, dx + \int_{0}^{\infty} (1 - K(F(x))) \, dx$, we obtain by assumption (a)

$$
|\mathcal{L}(F_n) - \mathcal{L}(F)| \leq \int_{-\infty}^{\infty} |K(F_n(x)) - K(F(x))| \, dx \\
\leq \int_{-\infty}^{\infty} L(F(x)) |(F_n - F)(x)| \, dx \\
\leq \left( C' \int_{-\infty}^{\infty} F(x)^{-\beta} \overline{F}(x)^{-\beta'} \phi_{\gamma,F}(x)^{-1} \, dx \right) \|F_n - F\|_{\phi_{\gamma,F}}
$$

for every sequence $(F_n) \subset \hat{F}$; notice that $\|F_n - F\|_{\phi_{\gamma,F}}$ is finite because of $\gamma \leq 1$. Since the latter integral is finite by assumption (b), we obtain $|\mathcal{L}(F_n) - \mathcal{L}(F)| = \mathcal{O}(\|F_n - F\|_{\phi_{\gamma,F}})$ when $\|F_n - F\|_{\phi_{\gamma,F}} \to 0$. □

**Remark 3.2** Assumption (a) in Lemma 3.1 is fulfilled for every continuous convex distribution function $K$ on the unit interval satisfying $1 - K(x) \leq C(1 - x)^{\beta}$ (for all $x \in [0, 1]$ and some $C > 0$) with $\beta = 1 - \beta'$ and $0 \leq \beta' < 1$. In this case we can choose $L(x) = C(1 - x)^{-\beta'}$ and $C' = C$. For instance, $K(x) := 1 - (1 - x)^{\beta}$ provides such a distribution function when $0 < \beta \leq 1$.

**Remark 3.3** Lemma 3.1 shows that the functional $\mathcal{L}$ is Hölder-1 continuous at $F$ when $K$ is locally Lipschitz continuous on $(0, 1)$ and at least Hölder continuous (of a certain order) at 0 and 1. If the kernel $K$ is only piecewise Hölder-\(\hat{\beta}\) continuous on $[0, 1]$ for some $\beta \in (0, 1)$, and $F \in \mathbb{F}_K$ satisfies $\|F - 1_{[0, \infty)}\|_{\phi_{\gamma}} < \infty$ for some $\gamma > 1/\beta$, then one can derive at least Hölder-\(\hat{\beta}\) continuity of $\mathcal{L}$ at $F$ w.r.t. $(\| \cdot \|_{\phi_{\gamma}}, \cdot | \cdot)$ and $\hat{F}$; cf. [39] Theorem 2. □

**Theorem 3.4** Let $X_1, X_2, \ldots$ be identically distributed random variables with distribution function $F \in \mathbb{F}_K$. Let $0 \leq \beta' < \gamma \leq 1$, and assume that conditions (a)–(b) of Lemma 3.1 hold.

(i) If the $X_i$ are independent and $F$ satisfies the assumptions of Theorem 2.7 for $r \in [0, \frac{1}{2})$ and $\phi = \phi_{\gamma,F}$ defined in (10), then we have $n^r |\mathcal{L}(\hat{F}_n) - \mathcal{L}(F)| \to 0$ $\mathbb{P}$-a.s.

(ii) If the sequence $(X_i)$ is $\alpha$-mixing and satisfies the assumptions of Theorem 2.8 for $\phi = \phi_{\gamma,F}$ defined in (10), then we have at least $|\mathcal{L}(\hat{F}_n) - \mathcal{L}(F)| \to 0$ $\mathbb{P}$-a.s.
In view of Lemma 3.1 and the discussion in the Introduction, assertions (i) and (ii) in Theorem 3.4 are immediate consequences of Theorems 2.1 and 2.3, respectively. Example 3.5 below sheds light on the assumptions of Theorem 3.4. Part (i) of Theorem 3.4 recovers results from [11, 20, 35, 39]. Ordinary SLLNs for L-statistics in the fashion of part (ii) of Theorem 3.4 can be found in [33] for i.i.d. data, in [5] for $\phi$-mixing data, and in [1, 5, 17] for ergodic stationary data. In the case of $\alpha$-mixing data the conditions in [5, 17] are comparable to those of part (ii) in Theorem 3.4. That is, the statements of Theorem 3.4 are basically already known. Nevertheless our approach leads to simple proofs once Theorems 2.1 and 2.3 are established. In the context of general law-invariant risk measures, in Section 3.2 below, we will also take advantage of the method of proof of Theorem 3.4.

**Example 3.5** Let $0 \leq \beta' < \gamma \leq 1$, and assume that condition (a) in Lemma 3.1 holds. Further assume that $F(x) = c_1|x|^{-\alpha}$ for all $x \leq -x_0$, and $F(x) = c_2x^{-\alpha}$ for all $x \geq x_0$, for some constants $\alpha, x_0, c_1, c_2 > 0$. In this case, assumption (b) in Lemma 3.1 and the integrability condition in Theorem 2.1 (with $\phi = \phi_{\gamma, F}$) read as

$$\int_{-\infty}^{-1} |x|^{-\alpha(\gamma - \beta')} \, dx + \int_{1}^{\infty} x^{-\alpha(\gamma - \beta')} \, dx < \infty$$

and

$$\int_{-\infty}^{-1} x^{2r-1} \, dx + \int_{1}^{\infty} x^{2r-1} \, dx < \infty,$$

respectively. Condition (11) holds if and only if $\gamma > \beta' + \frac{1}{\alpha}$, and condition (12) holds if and only if $\gamma < 1 - r$. So, if we assume $0 \leq \beta' + \frac{1}{\alpha} < 1 - r$ and $0 \leq r < \frac{1}{2}$, then the assumptions on $K$ and $F$ imposed in the setting of part (i) of Theorem 3.4 are fulfilled (with any $\gamma \in (\beta' + \frac{1}{\alpha}, 1 - r)$). In particular, if we assume $0 \leq \beta' + \frac{1}{\alpha} < 1$, then the assumptions on $K$ and $F$ imposed in the setting of part (ii) of Theorem 3.4 are fulfilled (with any $\gamma \in (\beta' + \frac{1}{\alpha}, 1)$). ✤

In the following theorem we restrict ourselves to empirical quantile estimators based on $\alpha$-mixing data. However, it can easily be extended to plug-in estimators of more general L-functionals $L_K$ with $dK$ having compact support strictly within $(0, 1)$. Under the stronger mixing conditions $\alpha(n) \leq K e^{-\varepsilon n}$, $\varepsilon > 0$, and $\alpha(n) \leq K n^{-\vartheta}$ the result of Theorem 3.6 is basically already known from [4] and [36], respectively. We let $H^+(x) := \inf\{y \in \mathbb{R} : H(y) \geq x\}$, $x \in \mathbb{R}$, denote the left-continuous inverse of any nondecreasing function $H : \mathbb{R} \to \mathbb{R}$, with the convention $\inf\emptyset := \infty$.

**Theorem 3.6** Let $(X_i)$ be an $\alpha$-mixing sequence of identically distributed random variables with distribution function $F$. Let $r \in \left[0, \frac{1}{2}\right)$, and assume that the mixing coefficients satisfy $\alpha(n) \leq K n^{-\vartheta}$ for all $n \in \mathbb{N}$ and some constants $K > 0$, $\vartheta > 2r$. Let
y ∈ (0, 1), and assume that F is differentiable at F−(y) with F′(F−(y)) > 0. Then, 
n^r|\hat{F}^{-}_{n}(y) - F^{-}(y)| → 0 \mathbb{P}\text{-a.s.}

**Proof** Since F−(y) = L_{K_y}(F) with K_y = 1_{[y,1]}, the proof of Theorem 2 in [39] shows that, under the above assumptions on F, \mathbb{P}\text{-a.s. there is some constant } C > 0 such that 

|\hat{F}^{-}_{n}(y) - F^{-}(y)| ≤ C \|\hat{F}^{-}_{n} - F\|_{\infty} \text{ for all } n \in \mathbb{N}.

Now the claim follows directly from Theorem 3.2. □

### 3.2 Law-invariant coherent risk measures

Let \( \rho \) be a law-invariant coherent risk measure on \( \mathcal{X} := L^p(\Omega, \mathcal{F}, \mathbb{P}) \) for some \( p \in [1, \infty] \), i.e. \( \rho \) be a mapping from \( \mathcal{X} \) to \( \mathbb{R} \) being

- monotone: \( \rho(X) \leq \rho(Y) \) for all \( X,Y \in \mathcal{X} \) with \( X \leq Y \) \( \mathbb{P} \)-almost surely,
- translation-equivariant: \( \rho(X + m) = \rho(X) + m \) for all \( X \in \mathcal{X} \) and \( m \in \mathbb{R} \),
- subadditive: \( \rho(X + Y) \leq \rho(X) + \rho(Y) \) for all \( X,Y \in \mathcal{X} \),
- positively homogenous: \( \rho(\lambda X) = \lambda \rho(X) \) for all \( X \in \mathcal{X} \) and \( \lambda \geq 0 \).

Since \( \rho \) is law-invariant, we may regard it as a functional \( \mathcal{R} \) on the set \( \mathcal{F}^p \) of all distribution functions of random variables in \( L^p(\Omega, \mathcal{F}, \mathbb{P}) \). If the underlying probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \) is rich enough to support a random variable with continuous distribution (which is equivalent to \( (\Omega, \mathcal{F}, \mathbb{P}) \) being atomless in the sense of [16, Definition A.26]), then the functional \( \mathcal{R} \) admits the representation

\[
\mathcal{R}(F) = \sup_{K \in \mathcal{K}_R} \mathcal{L}_K(F) \quad \forall F \in \mathcal{F}^p,
\]

where \( \mathcal{L}_K \) is the L-functional associated with kernel \( K \) (cf. (9)) and \( \mathcal{K}_R \) is a suitable set of continuous convex distribution functions on the unit interval. This was shown in [16, Corollary 4.72] for \( p = \infty \), and in [22] for the general case. Notice that in [22] the role of \( K \) is played by \( \hat{g} \).

If condition (a) in Lemma 3.1 holds for every \( K \in \mathcal{K}_R \) with the same \( L(x), \beta', C' \), and \( F \in \mathcal{F}^p \) satisfies condition (b) in Lemma 3.1, then, in view of

\[
|\mathcal{R}(F_n) - \mathcal{R}(F)| = \left| \sup_{K \in \mathcal{K}_R} \mathcal{L}_K(F_n) - \sup_{K \in \mathcal{K}_R} \mathcal{L}_K(F) \right| \leq \sup_{K \in \mathcal{K}_R} |\mathcal{L}_K(F_n) - \mathcal{L}_K(F)|
\]

the proof of Lemma 3.1 shows that the functional \( \mathcal{R} : \mathcal{F}^p \rightarrow \mathbb{R} \) is Hölder-1 continuous at \( F \) w.r.t. \((\| \cdot \|_{\phi_{\infty,F}}, \| \cdot \|)\) and \( \hat{F} \). So, in this case assertions (i)–(ii) in Theorem 3.4 also hold for \( \mathcal{R} \) in place of \( \mathcal{L} \). This seems to be the first general respective result in the context of law-invariant coherent risk measures.

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Example 3.7 It is easy to show that
\[ \rho_{p,a}(X) := \mathbb{E}[X] + a \mathbb{E}[(X - \mathbb{E}[X])^+]^{1/p} \]
provides a law-invariant coherent risk measure (called risk measure based on one-sided moments) on \( L^p(\Omega, \mathcal{F}, \mathbb{P}) \) for every \( p \in [1, \infty) \) and \( a \in [0,1] \). It was shown in [22, Lemma A.5] that the associated functional \( \mathcal{R}_{p,a} : \mathbb{P} \to \mathbb{R} \) is not a L-functional when \( a > 0 \). But according to our preceding discussion \( \mathcal{R}_{p,a} \) can be represented as in (13). We clearly have
\[ 1 - K(x) = \mathcal{L}_K(F_{B_{1-x}}) \leq \mathcal{R}_{p,a}(F_{B_{1-x}}) = (1 - x) + a((1 - x)x^p)^{1/p} \leq (1 + a)(1 - x)^{1/p} \]
(where \( F_{B_{1-x}} \) is the Bernoulli distribution function with expectation \( 1 - x \)) for all \( x \in (0,1) \) and \( K \in \mathcal{K}_{p,a} \). Thus Remark 3.2 and the preceding discussion show that the risk functional \( \mathcal{R}_{p,a} \) is Hölder-1 continuous at \( F \) w.r.t. \( (\| \cdot \|_{\phi^*}, | \cdot |) \) and \( \hat{F} \), provided \( F \) satisfies condition (b) in Lemma 3.1 with \( \beta' = 1 - \frac{1}{p} \).

3.3 V-functionals

Let \( g : \mathbb{R}^2 \to \mathbb{R} \) be a measurable function, and \( \mathcal{F}_g \) be the class of all distribution functions \( F \) on the real line for which \( \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |g(x_1, x_2)|dF(x_1)dF(x_2) < \infty \). The functional \( \mathcal{V} \), defined by
\[ \mathcal{V}(F) := \mathcal{V}_g(F) := \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x_1, x_2) dF(x_1)dF(x_2), \quad F \in \mathcal{F}_g, \]
is called von Mises-functional (or simply V-functional) associated with \( g \). It was shown in [10] that under fairly weak assumptions on \( g \) and \( F \in \mathcal{F}_g \) the functional \( \mathcal{V} \) is Hölder-1 continuous at \( F \) w.r.t. \( (\| \cdot \|_{\phi^*}, | \cdot |) \) and \( \hat{F} \). Thus, from Theorems 2.1–2.3 one can easily derive MZ-SLLNs and SLLNs for \( \mathcal{V}(\hat{F}_n) \); see also [10].

MZ-SLLNs for i.i.d. data that can be obtained with the help of Theorem 2.1 are already known from [18, 27, 30]. Related ordinary SLLNs can be found in [21] for i.i.d. data, in [34] for \( \phi^s \)-mixing data, in [3] for \( \beta \)-mixing data, and in [1] for ergodic stationary data. The proofs in [34] contain gaps as was revealed in [3, p.14]. The conditions on \( g, F \) and the mixing coefficients in [3, Theorem 1] are comparable to those under which Theorem 2.3 and Remark 2.4 above yield ordinary SLLNs, but in our setting we can consider even \( \alpha \)-mixing. The assumptions on the kernel \( g \) in [1] are more restrictive than the conditions we would have to impose in our setting. On the other hand, ergodicity is a weaker assumption than \( \alpha \)-mixing.
To the best of the author’s knowledge, so far MZ-SLLNs for weakly dependent data can be found only in [14]. In [14] the data are assumed to be \( \beta \)-mixing. In the case of a bounded kernel \( g \), Theorem 1 in [14] assumes that the mixing coefficients satisfy
\[
\sum_{n=1}^{\infty} n\beta(n) < \infty
\]
in order to obtain a MZ-SLLN for any \( r \in (0,1/2) \). With the help of Theorem 2.2 above this condition can be relaxed to \( \alpha(n) = \mathcal{O}(n^{-1}) \), even in the less restrictive case of \( \alpha \)-mixing. On the other hand, Theorem 2 in [14] covers also the case of unbounded kernels \( g \).

It was shown in [10] that V-functionals that are degenerate w.r.t. \((g,F)\) are typically even Hölder-2 continuous at \( F \) w.r.t. \((\| \cdot \|_\phi, | \cdot |)\) and \( \hat{F} \). So the rate of convergence of degenerate V-statistics is typically twice the rate of convergence of non-degenerate V-statistics; for details see again [10].

4 Proof of Theorem 2.1

By the usual quantile transformation [29, p.103], we may and do choose a sequence of i.i.d. \( U[0,1] \)-random variables, possibly on an extension \((\Omega, \mathcal{F}, \mathbb{P})\) of the original probability space \((\Omega, \mathcal{F}, \mathbb{P})\), such that the corresponding empirical distribution function \( \hat{G}_n \) satisfies \( \hat{F}_n = \hat{G}_n(F) \) \( \mathbb{P} \)-a.s. Then
\[
n^r \| \hat{F}_n - F \|_\phi = n^r \sup_{x \in \mathbb{R}} |\hat{G}_n(F(x)) - F(x)| \phi(x)
\leq n^r \sup_{s \in (0,1)} |\hat{G}_n(s) - s| w(s)
\]
with \( w(s) := \phi(\max\{F^{-}\!(s); F^{+}\!(s)\}) \), where \( F^{-}\! \) and \( F^{+}\! \) denote the left- and the right-continuous inverse of \( F \), respectively. According to Theorem 7.3 (3) in [2], the latter bound converges \( \mathbb{P} \)-a.s. to 0 as \( n \to \infty \) if and only if \( \int_{(0,1)} w(s)^{1/(1-r)} ds < \infty \). Since
\[
\int_{(0,1)} w(s)^{1/(1-r)} ds = \int_{\mathbb{R}} \phi(x)^{1/(1-r)} dF(x)
\]
by a change-of-variable (and the fact that \( F^{-}\! = F^{+}\! \) \( ds \)-almost everywhere), and since this integral is finite by assumption, we thus obtain \( n^r \| F_n - F \|_\phi \to 0 \) \( \mathbb{P} \)-a.s.

5 Proof of Theorem 2.2

In this section, we will prove Theorem 2.2. By the usual quantile transformation [29, p.103] (which works also for mixing data) it suffices to prove the result in the special case of \( U[0,1] \)-distributed random variables. Let \( (U_i) = (U_i)_{i \in \mathbb{N}} \) be an \( \alpha \)-mixing sequence of identically \( U[0,1] \)-distributed random variables on some probability space \((\Omega, \mathcal{F}, \mathbb{P})\). Let \( I \) be the identity on \([0,1]\), and \( \hat{G}_n := \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}_{[U_i,1]} \) be the empirical distribution function of \( U_1, \ldots, U_n \).
Theorem 5.1 Let \( r \in [0, 1/2) \), \( C > 0 \) and \( \vartheta > 2r \). Suppose that the mixing coefficients \((\alpha(n))\) of the sequence \((U_i)\) satisfy \( \alpha(n) \leq Cn^{-\vartheta} \) for all \( n \in \mathbb{N} \). Then
\[
n^r \| \hat{G}_n - I \|_{\infty} \to 0 \quad \mathbb{P}\text{-a.s.} \tag{14}
\]

The proof of Theorem 5.1 will be carried out in three steps (Sections 5.1–5.3). For every \( p \in \mathbb{N}_0 \), \( q \in \mathbb{N} \) and \( t \in [0, 1] \), define
\[
Z_{p,q}(t) := \left| \sum_{i=p+1}^{p+q} (1_{[0,t]}(U_i) - t) \right|.
\]
Thus, in order to verify (14), we have to show
\[
\frac{1}{n^{1-r}} \sup_{t \in (0,1)} Z_{0,n}(t) \to 0 \quad \mathbb{P}\text{-a.s.} \tag{15}
\]

In Sections 5.1 we will collect some elementary properties of \( Z_{p,q}(t) \). In Section 5.2 we will prove some nontrivial properties of \( Z_{p,q}(t) \). Finally, in Section 5.3 we will prove (15).

5.1 Auxiliary results, Part I

Of course, for every \( p \in \mathbb{N}_0 \) and \( q, u \in \mathbb{N} \) with \( q < u \), and every \( t \in (0, 1) \), the elementary inequality
\[
Z_{p,u}(t) \leq Z_{p,q}(t) + Z_{p+q,u-q}(t) \tag{16}
\]
holds; see also [23, p. 333]. Let
\[
N_n := \left\lfloor \frac{\log n}{\log 2} \right\rfloor \tag{17}
\]
be the largest \( N \in \mathbb{N}_0 \) with \( 2^N \leq n \). Then \( n \) can be represented as
\[
n = 2^{N_n} + \sum_{j=1}^{N_n} h_j(n) 2^{j-1} \tag{18}
\]
for suitable \( h_j(n) \in \{0, 1\}, \ j = 1, \ldots, N_n \). Equation (18) and a repeated application of (16) yield that for every \( n \in \mathbb{N} \) and \( t \in (0, 1) \)
\[
Z_{0,n}(t) \leq Z_{0,2^{N_n}}(t) + \sum_{j=1}^{N_n} Z_{2^{N_n} + b_j(n)2^j,2^j-1}(t) \tag{19}
\]
holds for suitable integers \( b_j(n) \in \{0, \ldots, 2^{N_n-j} - 1\}, \ j \in \{1, \ldots, N_n\} \).
5.2 Auxiliary results, Part II

Lemma 5.3 below will be crucial for the central part of the proof of Theorem 5.1 (cf. Section 5.3). For the proof of Lemma 5.3 we will need the following lemma, which in turn is an immediate consequence of Proposition 7.1 in [25] and Markov’s inequality.

**Lemma 5.2** For all $p \in \mathbb{N}_0$, $q \in \mathbb{N}$ and $x > 0$,

$$
\mathbb{P}\left[q^{-1/2} \sup_{t \in (0,1)} Z_{p,q}(t) \geq x \right] \leq \frac{1}{2^q} \left(1 + 4 \sum_{i=0}^{q-1} \alpha(i)\right)(2 + \log q)^2.
$$

(20)

Now, let $R > r$ be sufficiently close to $r$ (to be concretized later on) and $\beta > 0$ be sufficiently close to zero (to be concretized later on). For every $N \in \mathbb{N}$, define the event

$$
F_N := \left\{ \sup_{t \in (0,1)} Z_{0,2^N}(t) \geq 2^{N(1-R)} \right\}.
$$

For every $N \in \mathbb{N}$, $j \in \{1, \ldots, N\}$ and $b \in \{0, \ldots, 2^{N-j} - 1\}$ define the event

$$
H_N(j,b) := \left\{ \sup_{t \in (0,1)} Z_{2^N+b2^j,2^j-1}(t) \geq 2^{N(1-R)} 2^{-\beta(N-j)} \right\}.
$$

Moreover, for every $N \in \mathbb{N}$ define the event

$$
H_N := \bigcup_{j=1}^N \bigcup_{b=0}^{2^{N-j}-1} H_N(j,b).
$$

**Lemma 5.3** $\mathbb{P}[\limsup_{N \to \infty} F_N] = \mathbb{P}[\limsup_{N \to \infty} H_N] = 0$. In particular, $\mathbb{P}$-a.s. there are some constants $K_1, K_2 > 0$ such that

$$
\sup_{t \in (0,1)} Z_{0,2^N}(t) \leq K_1 2^{N(1-R)}
$$

for all $N \in \mathbb{N}$, and

$$
\sup_{t \in (0,1)} Z_{2^N+b2^j,2^j-1}(t) \leq K_2 2^{N(1-R)} 2^{-\beta(N-j)}
$$

for all $N \in \mathbb{N}$, $j \in \{1, \ldots, N\}$ and $b \in \{0, \ldots, 2^{N-j} - 1\}$.

**Proof** By Lemma 5.2 and the assumption $\alpha(i) \leq C i^{-\vartheta}$,

$$
\mathbb{P}[F_N] = \mathbb{P}\left[2^{-N/2} \sup_{t \in (0,1)} Z_{0,2^N}(t) \geq 2^{N(1/2-R)} \right]
\leq \frac{1}{2^{N(1-2R)}} \left(1 + 4 \sum_{i=0}^{2^N-1} C i^{-\vartheta}\right)(2 + \log 2^N)^2
\leq K 2^{N(2R-\vartheta)N^2}
$$
for some finite constant \( K > 0 \), where we assumed without loss of generality that \( \vartheta \in (0, 1) \). Choosing \( R \) sufficiently close to \( r \), and taking the assumption \( \vartheta > 2r \) into account, we obtain \( \sum_{N=1}^{\infty} \mathbb{P}[F_N] < \infty \). Now the Borel-Cantelli lemma yields \( \mathbb{P}[\lim \sup_{N \to \infty} F_N] = 0 \).

Again by Lemma 5.2 and the assumption \( \alpha(i) \leq C i^{-\vartheta} \),

\[
\mathbb{P}[H_N(j, b)] = \mathbb{P}\left[2^{-(j-1)/2} \sup_{t \in (0, 1)} Z_{2N+2j, 2^{j-1}}(t) \geq 2^{-(j-1)/2} 2^{N(1-R)} 2^{-\beta(N-j)}\right] \\
\leq \frac{1}{2^{(j-1)2N(1-R)} 2^{-\beta(N-j)}} \left(1 + 4 \sum_{i=0}^{2^{-1}} C i^{-\vartheta} \right) (2 + \log 2^{j-1})^2 \\
= \ K 2^{j(2-2\beta-\vartheta)} 2^{-N(2-2\beta-\vartheta)} j^2
\]

for some finite constant \( K > 0 \), where we again assumed without loss of generality that \( \vartheta \in (0, 1) \). Therefore,

\[
\mathbb{P}[H_N] \leq K 2^{-N(2-2\beta-\vartheta)} \sum_{j=1}^{N} \sum_{b=0}^{2^{N-j-1}} 2^{j(2-2\beta-\vartheta)} j^2 \\
\leq K 2^{-N(1-2\beta-\vartheta)} \sum_{j=1}^{N} j^2 2^{j(2-2\beta-\vartheta)} j^2 \\
\leq K' 2^{-N(1-2\beta-\vartheta)} 2^{N(1-\beta-\vartheta)} \\
= K' 2^{-N(\vartheta-2\beta-\vartheta)}
\]

for some finite constant \( K' > 0 \). Choosing \( R \) sufficiently close to \( r \), choosing \( \beta \) sufficiently close to zero, and taking the assumption \( \vartheta > 2r \) into account, we obtain \( \sum_{N=1}^{\infty} \mathbb{P}[H_N] < \infty \). Now the Borel-Cantelli lemma yields \( \mathbb{P}[\lim \sup_{N \to \infty} H_N] = 0 \).

### 5.3 Completion of the proof of Theorem 5.1

We now prove (15). By (19) and the definition of \( N_n \) as the largest \( N \in \mathbb{N}_0 \) with \( 2^{N} \leq n \) (cf. (17)), we have

\[
\frac{1}{n^{1-r}} \sup_{t \in (0, 1)} Z_{0,n}(t) \leq \frac{1}{n^{1-r}} \sup_{t \in (0, 1)} Z_{0,2N_n}(t) + \frac{1}{n^{1-r}} \sum_{j=1}^{N_n} \sup_{t \in (0, 1)} Z_{2^{N_n}+b_j(n) 2^{j}, 2^{j-1}}(t) \\
=: I_{n,1} + I_{n,2}
\]

for suitable \( b_j(n) \in \{0, \ldots, 2^{N_n-j} - 1 \} \). In the sequel we will show that \( I_{n,1} \) and \( I_{n,2} \) converge to zero \( \mathbb{P} \)-a.s. This will complete the proof of Theorem 5.1.

As for \( I_{n,1} \), we observe that by Lemma 5.3 there is \( \mathbb{P} \)-a.s. a constant \( K_1 > 0 \) such that \( I_{n,1} \leq n^{r-1} K_1 2^{N_n(1-R)} = K_1 n^{r-R} \) for all \( n \in \mathbb{N} \). Since \( R > r \), the summand \( I_{n,1} \) thus converges to zero \( \mathbb{P} \)-a.s.
As for $I_{n,2}$, we observe that by Lemma 5.3 there is $P$-a.s. a constant $K_2 > 0$ such that

$$I_{n,2} \leq \frac{1}{2^{N_n(1-R)}} \sum_{j=1}^{N_n} K_2 2^{N_n(1-R)} 2^{-\beta(N_n-j)}$$

$$\leq K_2 2^{-N_n(R-r)} \sum_{j=0}^{N_n-1} 2^{-\beta j}$$

holds for all $n \in \mathbb{N}$. Since $R > r$, the summand $I_{n,2}$ thus converges to zero $P$-a.s. This completes the proof of Theorem 2.2.

6 Proof of Theorem 2.3

Without loss of generality we assume $x_\phi = 0$. So $\phi$ can be seen as a nonincreasing function on $[-\infty, 0]$. We will only show that

$$\sup_{x \in (-\infty, 0]} |\tilde{F}_n(x) - F(x)| \phi(x) \to 0 \quad P\text{-a.s.}$$

The analogous result for the positive real line can be shown in the same way. We will proceed in three steps, where we will combine arguments of [31-32] (Steps 1–2) with Rio’s SLLN for $\alpha$-mixing data (Step 3). The latter can be found in [24, Theorem 1 (ii)] and will be recalled in the following theorem. As before, the rightcontinuous inverse $h^+$ of any nonincreasing function $h : \mathbb{R}_{+} \to [0, 1]$ will be defined by $h^+(y) := \sup\{x \in \mathbb{R}_{+} : h(x) > y\}$, $y \in [0, 1]$, with the convention $\sup \emptyset := 0$.

**Theorem 6.1** (Rio) Let $\xi_1, \xi_2, \ldots$ be identically distributed random variables on some probability space $(\Omega, \mathcal{F}, P)$ with $\mathbb{E}[|\xi_1|] < \infty$. Suppose that $(\xi_i)$ is $\alpha$-mixing with mixing coefficients $\alpha(n)$, and let $\alpha(y) := \alpha(|y|)$ be the càdlàg extension of $\alpha(\cdot)$ from $\mathbb{N}$ to $\mathbb{R}_{+}$. Let $G$ be the distribution function of $|\xi_1|$, and set $G^\alpha := 1 - G$. If

$$\int_0^1 \log \left(1 + \alpha^\alpha(y/2)\right) G^\alpha(y) \, dy < \infty,$$

then $\frac{1}{n} \sum_{i=1}^{n} (\xi_i - \mathbb{E}[\xi_i]) \to 0$ $P$-a.s.

**Step 1.** Let $L^1(d\xi)$ be the space of all Lebesgue integrable functions on $[0, 1]$, and $[l, u] := \{f \in L^1(d\xi) : l \leq f \leq u\}$ be the bracket of two functions $l, u \in L^1(d\xi)$ with $l \leq u$ pointwise. For any $\varepsilon > 0$, a bracket $[l, u]$ is called $\varepsilon$-bracket if $\int_0^1 (u - l) \, d\xi < \varepsilon$. Set

$$w(t) := \phi(F^+(t)) \mathbb{1}_{[0, F(0)]}(t), \quad t \in [0, 1].$$

Since our assumption $\int_{-\infty}^\infty \phi \, dF < \infty$ implies $\int_0^1 w \, d\xi < \infty$, we can find as in [31] Example 19.12] a finite partition $0 = t_0^\alpha < t_1^\alpha < \cdots < t_{m}^\alpha = 1$ of $[0, 1]$ such that $[t_i^\alpha, u_i^\alpha]$ with

$$t_i^\alpha(\cdot) := w(t_i^\alpha) \mathbb{1}_{[t_i^\alpha, u_i^\alpha]}(\cdot)$$

$$u_i^\alpha(\cdot) := w(t_i^\alpha - 1) \mathbb{1}_{[t_i^\alpha - 1, u_i^\alpha]}(\cdot) + w(\cdot) \mathbb{1}_{[t_i^\alpha - 1, t_i^\alpha]}(\cdot)$$
(i = 1, . . . , mε) are ε-brackets in L1(Ⅰ) covering the class Ew := {wμ : μ ∈ [0, 1]} of functions

\[ w_s(·) := w(s)1_{[0,s]}(·). \]

**Step 2.** By the usual quantile transformation, we can find a sequence of U[0, 1]-random variables U1, U2, . . . (possibly on an extension (Ω, F, P) of the original probability space (Ω, F, P)) such that the sequence (Ui) has the same mixing coefficients (under P) as the sequence (X1) under P and such that the corresponding empirical distribution function \( \hat{G}_n \) satisfies \( \hat{F}_n = \hat{G}_n \circ F \) P-a.s. Here we will show as in the proof of Theorem 2.4.1 in [32] that

\[ \sup_{x \leq 0} |\hat{F}_n(x) - F(x)| \phi(x) \leq \max_{i=1,\ldots,m} \max \left\{ \int_0^1 u^ε_i d(\hat{G}_n - 1) ; \int_0^1 l^ε_i d(1-\hat{G}_n) \right\} + ε \]  

(22)

for every ε > 0. Since

\[ \sup_{x \leq 0} |\hat{F}_n(x) - F(x)| \phi(x) = \sup_{x \leq 0} |\hat{G}_n(F(x)) - F(x)| \phi(x) \leq \sup_{s \in (0,1)} |\hat{G}_n(s) - s| w(s) \]

\[ = \sup_{s \in (0,1)} \left| \int_0^1 w_s d\hat{G}_n - \int_0^1 w_s dl \right|. \]

for (22) it suffices to show that

\[ \sup_{s \in (0,1)} \left| \int_0^1 w_s d\hat{G}_n - \int_0^1 w_s dl \right| \leq \max_{i=1,\ldots,m} \max \left\{ \int_0^1 u^ε_i d(\hat{G}_n - 1) ; \int_0^1 l^ε_i d(1-\hat{G}_n) \right\} + ε. \]  

(23)

To prove (23), we note that for every \( s \in [0, 1] \) there is some \( i_s \in \{1, \ldots, mε\} \) such that \( w_s \in [l^ε_{i_s}, u^ε_{i_s}] \); cf. Step 1. Therefore, since \( [l^ε_{i_s}, u^ε_{i_s}] \) is an ε-bracket,

\[ \int_0^1 w_s d\hat{G}_n - \int_0^1 w_s dl \leq \int_0^1 u^ε_{i_s} d\hat{G}_n - \int_0^1 w_s dl \]

\[ = \int_0^1 u^ε_{i_s} d(\hat{G}_n - 1) + \int_0^1 (u^ε_{i_s} - w_s) dl \]

\[ \leq \int_0^1 u^ε_{i_s} d(\hat{G}_n - 1) + \int_0^1 (u^ε_{i_s} - l^ε_{i_s}) dl \]

\[ \leq \max_{i=1,\ldots,m} \int_0^1 u^ε_{i} d(\hat{G}_n - 1) + ε. \]

Analogously we obtain

\[ \int_0^1 w_s d\hat{G}_n - \int_0^1 w_s dl \geq -\left(\max_{i=1,\ldots,m} \int_0^1 l^ε_{i} d(1-\hat{G}_n) + ε\right). \]
That is, (22) holds true.

Step 3. Because of (22), for (8) to be true it suffices to show that both \( \dot{\beta}_1^0 l_{\epsilon i} d(G_n - \bar{G}) \) and \( \dot{\beta}_1^0 u_{\epsilon i} d(G_n - \bar{G}) \) converge \( P \)-a.s. to zero for every \( i = 1, \ldots, m_e \). The second convergence follows from the representation

\[
\int_0^1 u_{\epsilon i} d(G_n - \bar{G}) = \frac{1}{n} \sum_{j=1}^n \left( w(t_{\epsilon i-1}) \mathbb{1}_{[0,t_{\epsilon i-1}]}(U_j) - \mathbb{E}_P \left[ w(t_{\epsilon i-1}) \mathbb{1}_{[0,t_{\epsilon i-1}]}(U_1) \right] \right) + \frac{1}{n} \sum_{j=1}^n \left( w(U_j) \mathbb{1}_{(t_{\epsilon i-1},t_{\epsilon i}]}(U_j) - \mathbb{E}_P \left[ w(U_1) \mathbb{1}_{(t_{\epsilon i-1},t_{\epsilon i}]}(U_1) \right] \right)
\]

and Theorem 6.1, noting that (7) implies (21) for both \( \xi_j := w(t_{\epsilon i-1}) \mathbb{1}_{[0,t_{\epsilon i-1}]}(U_j) \) and \( \xi_j := w(U_j) \mathbb{1}_{(t_{\epsilon i-1},t_{\epsilon i}]}(U_j) \). The verification of the first convergence is even easier. This completes the proof of Theorem 2.3.

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