Linear Strain Tensors on Hyperbolic Surfaces and
Asymptotic Theories for Thin Shells

Peng-Fei YAO

Key Laboratory of Systems and Control
Institute of Systems Science, Academy of Mathematics and Systems Science
Chinese Academy of Sciences, Beijing 100190, P. R. China
School of Mathematical Sciences
University of Chinese Academy of Sciences, Beijing 100049, China
e-mail: pfyao@iss.ac.cn

Abstract We perform a detailed analysis of the solvability of linear strain equations
on hyperbolic surfaces. We prove that if the surface is a smooth noncharacteristic
region, any first order infinitesimal isometry can be matched to an infinitesimal isom-
etry of an arbitrarily high order. The implications of this result for the elasticity of
thin hyperbolic shells are discussed.

Keywords hyperbolic surface, shell, nonlinear elasticity, Riemannian geometry

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1 Introduction

Let $M \subset \mathbb{R}^3$ be a surface and let the middle surface of a shell be an open set $\Omega \subset M$. The linear strain tensor of a displacement $y \in W^{1,2}(\Omega, \mathbb{R}^3)$ of the shell takes the form

$$\text{sym } \nabla y = U \quad \text{for } x \in \Omega,$$

where $2 \text{sym } \nabla y = \nabla y + \nabla^T y$. This equation plays a fundamental role in the theory of thin shells, see [5, 6, 7, 8]. When $U = 0$, a solution $y$ to (1.1) is referred to as an infinitesimal isometry. In a hierarchy of shell models (introduced in the setting of plates and justified in [2], and in the setting of shells in [7] through $\Gamma$-convergence) under the assumption

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that the elastic energy of deformations scales like $h^\beta$, $h$ being the thickness of a shell, for $\beta > 2$ the deformations of the mid-surface, modulo a rigid motion, look like $\text{id} + \varepsilon V$ up to its first order of expansion where $V$ is an infinitesimal isometry [7]. In addition, the construction of the recovery sequence necessary for establishing the upper bound in the context of $\Gamma$-convergence is closely related to the solvability of (1.1) ([5, 6, 7, 8]).

Equation (1.1) can be translated into a scalar second order partial differential equation [4]. Thus the solvability of (1.1) is equivalent to that of this scalar equation, which is subject to the geometry of the middle surface $\Omega$. The main observation is that the scalar equation on $\Omega$ is, respectively, elliptic, hyperbolic or degenerate if $\Omega$ is of positive, negative or zero Gaussian curvature. The work [8] is specific to elliptic surfaces, where the matching property of infinitesimal isometries is established and the recovery sequences are constructed for the scaling regime $2 < \beta < 4$. In [5] the authors establish a similar property for smooth developable surfaces with no flat part. A survey on this topic is presented in [6].

We state our main results for the hyperbolic surfaces as follows.

A region $\Omega \subset M$ is said to be hyperbolic if its Gaussian curvature $\kappa$ is strictly negative. We assume throughout this paper that

$$\kappa(x) < 0 \quad \text{for} \quad x \in \Omega.$$

In this case equation (1.1) is hyperbolic and a solution to it behaves like a wave. The well-posedness of (1.1) obeys the shape and the (partial) boundary data of the mid-surface $\Omega$ due to the property of the wave propagation. We introduce the notion of a noncharacteristic region below, subject to the second fundamental form $\Pi$ of the surface $M$.

**Definition 1.1** A region $\Omega \subset M$ is said to be noncharacteristic if

$$\Omega = \{ \alpha(t,s) \mid (t,s) \in (0,a) \times (0,b) \},$$

where $\alpha : [0,a] \times [0,b] \to M$ is an imbedding map which is a family of regular curves with two parameters $t, s$ such that

$$\Pi(\alpha_t(t,s),\alpha_t(t,s)) \neq 0, \quad \text{for all} \quad (t,s) \in [0,a] \times [0,b],$$

$$\Pi(\alpha_s(0,s),\alpha_s(0,s)) \neq 0, \quad \Pi(\alpha_s(a,s),\alpha_s(a,s)) \neq 0, \quad \text{for all} \quad s \in [0,b],$$

$$\Pi(\alpha_t(0,s),\alpha_s(0,s)) = \Pi(\alpha_t(a,s),\alpha_s(a,s)) = 0, \quad \text{for all} \quad s \in [0,b].$$

Consider a surface given by the graph of a function $h : \mathbb{R}^2 \to \mathbb{R}$,

$$M = \{ (x, h(x)) \mid x = (x_1, x_2) \in \mathbb{R}^2 \}.$$
Under the coordinate system $\psi(p) = x$ for $p = (x, h(x)) \in M$, \[ \partial x_1 = (1, 0, h_x(x)), \quad \partial x_2 = (0, 1, h_y(x)), \quad \vec n = \frac{1}{\sqrt{1 + |\nabla h|^2}}(-\nabla h, 1), \]
\[ \Pi = -\frac{1}{\sqrt{1 + |\nabla h|^2}}\nabla^2 h, \quad \kappa = \frac{h_{x1x1}h_{x2x2} - h_{x1x2}^2}{(1 + |\nabla h|^2)^2}. \]

(i) Let $h(x) = h_2(x_1) + h_2(x_2)$ where $h_i : \mathbb{R} \to \mathbb{R}$ are $C^2$ functions with $h''(x_1)h''(x_2) < 0$. Let $\sigma_i \in \mathbb{R}$ for $1 \leq i \leq 4$ with $\sigma_1 < \sigma_2$ and $\sigma_3 < \sigma_4$. Then \[ \Omega = \{ (x, h(x)) | \sigma_1 < x_1 < \sigma_2, \sigma_3 < x_2 < \sigma_4 \} \]
is noncharacteristic.

(ii) Let $h(x) = x_1^3 - 3x_1x_2^2$. Then \[ \kappa(p) < 0 \text{ for } p = (x, h(x)), \quad x \in \mathbb{R}^2, \quad |x| > 0. \]

For $\varepsilon > 0$ and $\sigma_1 < \sigma_2$ given \[ \Omega = \{ (x, h(x)) | \varepsilon < x_1 < \frac{1}{\varepsilon} \cdot \sigma_1, \sigma_3 < x_2 < \sigma_4 \} \]
is noncharacteristic.

We say that a noncharacteristic region $\Omega \subset M$ is of class $C^{m,1}$ for some integer $m \geq 0$ if the surface $M$ is of class $C^{m,1}$ and all the curves $\alpha(0, \cdot)$, $\alpha(a, \cdot)$, and $\alpha(\cdot, s)$ for each $s \in [0, b]$ are of class $C^{m,1}$. The points $\alpha(0, 0)$, $\alpha(a, 0)$, $\alpha(0, b)$, and $\alpha(a, b)$ are angular points of $\Omega$ even if $\Omega$ is smooth. Define:

\[ \mathcal{V}(\Omega, \mathbb{R}^3) = \{ V \in W^{2,2}(\Omega, \mathbb{R}^3) \mid \text{sym} \nabla V = 0 \}. \]

We have the following.

**Theorem 1.1** Let $\Omega$ be a noncharacteristic region of class $C^{m+2,1}$ for some integer $m \geq 0$. Then, for every $V \in \mathcal{V}(\Omega, \mathbb{R}^3)$ there exists a sequence $\{ V_k \} \subset \mathcal{V}(\Omega, \mathbb{R}^3) \cap C^{m,1}(\Omega, \mathbb{R}^3)$ such that
\[ \lim_{k \to \infty} \| V - V_k \|_{W^{2,2}(\Omega, \mathbb{R}^3)} = 0. \tag{1.2} \]

A one parameter family $\{ u_\varepsilon \}_{\varepsilon > 0} \subset C^{0,1}(\Omega, \mathbb{R}^3)$ is said to be a (generalized) $m$th order infinitesimal isometry if the change of metric induced by $u_\varepsilon$ is of order $\varepsilon^{m+1}$, that
\[ \| \nabla^T u_\varepsilon \nabla u_\varepsilon - g \|_{L^{\infty}(\Omega, \mathbb{R}^3)} = O(\varepsilon^{m+1}) \text{ as } \varepsilon \to 0, \]
where $g$ is the induced metric of $M$ from $\mathbb{R}^3$, see [5]. If $V \in \mathcal{V}(\Omega, \mathbb{R}^3) \cap C^{0,1}(\Omega, \mathbb{R}^3)$, then $w_\varepsilon = \text{id} + \varepsilon V$ is a first order isometry.
Theorem 1.2 Let \( \Omega \) be a noncharacteristic region of class \( C^{2m+1,1} \). Given \( V \in V(\Omega, \mathbb{R}^3) \cap C^{2m-1,1}(\Omega, \mathbb{R}^3) \), there exists a family \( \{ w_\varepsilon \}_{\varepsilon > 0} \subset C^{1,1}(\Sigma, \mathbb{R}^3) \), equibounded in \( C^{1,1}(\Omega, \mathbb{R}^3) \), such that for all small \( \varepsilon > 0 \) the family:

\[
  u_\varepsilon = \text{id} + \varepsilon V + \varepsilon^2 w_\varepsilon
\]
is a \( m \)th order isometry of class \( C^{1,1} \).

For \( V \in V(\Omega, \mathbb{R}^3) \) given, there exists a unique \( A \in W^{1,2}(\Omega, T^2) \) such that

\[
  \nabla_\alpha V = A(x) \alpha \quad \text{for} \quad \alpha \in M_x, \quad A(x) = -A^T(x), \quad x \in \Omega. \quad (1.3)
\]

The finite strain space is the following closed subspace of \( L^2(\Omega, T^2_{\text{sym}}) \):

\[
  \mathcal{B}(\Omega, T^2_{\text{sym}}) = \left\{ \lim_{h \to 0} \text{sym} \nabla w_h \mid w_h \in W^{1,2}(\Omega, R^3) \right\}
\]

where limits are taken in \( L^2(\Omega, T^2_{\text{sym}}) \), see [3, 9]. A region \( \Omega \subset M \) is said to be approximately robust if

\[
  (A^2)_\tan \in \mathcal{B}(\Omega, T^2_{\text{sym}}) \quad \text{for} \quad V \in V(\Omega, \mathbb{R}^3),
\]

see [7].

Theorem 1.3 Let \( \Omega \subset M \) be a noncharacteristic region of class \( C^{2,1} \). Then \( \Omega \) is approximately robust.

Let \( \bar{n} \) be the normal field of surface \( M \). Consider a family \( \{ \Omega_h \}_{h > 0} \) of thin shells of thickness \( h \) around \( \Omega \),

\[
  \Omega_h = \left\{ x + t\bar{n}(x) \mid x \in \Omega, \ |t| < h/2 \right\}, \quad 0 < h < h_0,
\]

where \( h_0 \) is small enough so that the projection map \( \pi : \Omega_h \to \Omega, \pi(x + t\bar{n}) = x \) is well defined. For a \( W^{1,2} \) deformation \( u_h : \Omega_h \to \mathbb{R}^3 \), we assume that its elastic energy (scaled per unit thickness) is given by the nonlinear functional:

\[
  E_h(u_h) = \frac{1}{h} \int_{\Omega_h} W(\nabla u_h) dz.
\]

The stored-energy density function \( W : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R} \) is \( C^2 \) in an open neighborhood of \( \text{SO}(3) \), and it is assumed to satisfy the conditions of normalization, frame indifference and quadratic growth: For all \( F \in \mathbb{R}^3 \times \mathbb{R}^3, R \in \text{SO}(3) \),

\[
  W(R) = 0, \quad W(RF) = W(F), \quad W(F) \geq C \text{dist}^2(F, \text{SO}(3)),
\]

with a uniform constant \( C > 0 \). The potential \( W \) induces the quadratic forms ([1])

\[
  Q_3(F) = D^2 W(Id)(F, F), \quad Q_2(x, F_\tan) = \min\{ Q_3(\hat{F}) \mid \hat{F} = F_\tan \}.
\]
We shall consider a sequence $e_h > 0$ such that:

$$0 < \lim_{h \to 0} e_h/h^\beta < \infty \quad \text{for some } 2 < \beta \leq 4. \quad (1.4)$$

Let

$$\beta_m = 2 + 2/m.$$  

Recall the following result.

**Theorem 1.4** [7] Let $\Omega$ be a surface embedded in $\mathbb{R}^3$, which is compact, connected, oriented, of class $C^{1,1}$, and whose boundary $\partial \Omega$ is the union of finitely many Lipschitz curves. Let $u_h \in W^{1,2}(\Omega_h, \mathbb{R}^3)$ be a sequence of deformations whose scaled energies $E_h(u_h)/e_h$ are uniformly bounded. Then there exist a sequence $Q_h \in SO(3)$ and $c_h \in \mathbb{R}^3$ such that for the normalized rescaled deformations

$$y_h(z) = Q_h u_h(x + \frac{h}{h_0^2} t\tilde{n}(x)) - c_h, \quad z = x + t\tilde{n}(x) \in \Omega_h,$$

the following holds.

(i) $y_h$ converges to $\pi$ in $W^{1,2}(\Omega_h, \mathbb{R}^3)$.

(ii) The scaled average displacements

$$V_h(x) = \frac{h}{h_0 \sqrt{e_h}} \int_{-h_0/2}^{h_0/2} [y_h(x + t\tilde{n}) - x] dt$$

converges to some $V \in \mathcal{V}(\Omega, \mathbb{R}^3)$.

(iii) $\liminf_{h \to 0} E_h(u_h)/e_h \geq I(V)$, where

$$I(V) = \frac{1}{24} \int_\Omega Q_2 \left( x, (\nabla (A\tilde{n}) - A\nabla \tilde{n})_{\text{tan}} \right) dg, \quad (1.5)$$

where $A$ is given in (1.3).

The above result proves the lower bound for the $\Gamma$-convergence. We now state the upper bound in the $\Gamma$-convergence result for a smooth noncharacteristic region.

**Theorem 1.5** Let $\Omega \subset M$ be a noncharacteristic region of class $C^{2,1}$. Assume that (1.4) holds for $\beta = 4$. Then for every $V \in \mathcal{V}(\Omega, \mathbb{R}^3)$ there exists a sequence of deformations $\{u_h\} \subset W^{1,2}(\Omega, \mathbb{R}^3)$ such that (i) and (ii) of Theorem 1.4 hold. Moreover,

$$\lim_{h \to 0} \frac{1}{e_h} E_h(u_h) = I(V),$$

where $I(V)$ is given in (1.5).

**Theorem 1.6** Let $\Omega \subset M$ be a noncharacteristic region of class $C^{2m+1,1}$, where $m \geq 2$ is given such that

$$e_h = o(h^{\beta m}).$$

Then the results in Theorem 1.5 hold.
2 Linear Strain Equations

We reformulate some expressions from [4] to reduce (1.1) to a coordinate free, scalar equation which can be solved by selecting special charts.

Let \( k \geq 1 \) be an integer. Let \( T \in T^k(M) \) be a \( k \)th order tensor field and let \( X \in \mathcal{X}(M) \) be a vector field. We define a \( k-1 \)th order tensor field by

\[
l_X T(X_1, \cdots, X_{k-1}) = T(X, X_1, \cdots, X_{k-1}) \quad \text{for} \quad X_1, \cdots, X_{k-1} \in \mathcal{X}(M),
\]

which is called an inner product of \( T \) with \( X \). For any \( T \in T^2(M) \) and \( \alpha \in M_x \),

\[
\text{tr} \ l_{\alpha} DT
\]

is a linear functional on \( M_x \). Thus there is a vector, denoted by \( \Lambda(T) \), such that

\[
\langle \Lambda(T), \alpha \rangle = 1_{\alpha} DT \quad \text{for} \quad \alpha \in M_x, \ x \in M. \tag{2.1}
\]

Clearly, the above formula defines a vector field \( \Lambda(T) \in \mathcal{X}(M) \).

We also need another linear operator \( Q \) as follows. Let \( M \) be oriented and \( \mathcal{E} \) be a volume element of \( M \) with the positive orientation. Let \( x \in M \) be given and let \( e_1, e_2 \) be an orthonormal basis of \( M_x \) with the positive orientation, that is,

\[
\mathcal{E}(e_1, e_2) = 1 \quad \text{at} \quad x.
\]

We define \( Q : M_x \to M_x \) by

\[
Q\alpha = \langle \alpha, e_2 \rangle e_1 - \langle \alpha, e_1 \rangle e_2 \quad \text{for all} \quad \alpha \in M_x. \tag{2.2}
\]

\( Q \) is well defined in the following sense: Let \( \hat{e}_1, \hat{e}_2 \) be a different orthonormal basis of \( M_x \) with the positive orientation,

\[
\mathcal{E}(\hat{e}_1, \hat{e}_2) = 1.
\]

Let

\[
\hat{e}_i = \sum_{j=1}^{2} \alpha_{ij} e_j \quad \text{for} \quad i = 1, 2.
\]

Then

\[
1 = \mathcal{E}(\hat{e}_1, \hat{e}_2) = \alpha_{11} \alpha_{22} - \alpha_{12} \alpha_{21}.
\]

Using the above formula, a simple computation yields

\[
\langle \alpha, \hat{e}_2 \rangle \hat{e}_1 - \langle \alpha, \hat{e}_1 \rangle \hat{e}_2 = \langle \alpha, e_2 \rangle e_1 - \langle \alpha, e_1 \rangle e_2.
\]

Clearly, \( Q : M_x \to M_x \) is an isometry and

\[
Q^T = -Q, \quad Q^2 = -\text{Id}.
\]
Remark 2.1 $Q: M_x \to M_x$ is the rotation by $\pi/2$ along the clockwise direction.

The operator, defined above, defines an operator, still denoted by $Q: T(M) \to T(M)$, by

$$(QX)(x) = QX(x), \quad x \in M, \quad X \in T(M).$$

For each $k \geq 2$, the operator $Q$ further induces an operator, denoted by $Q^*: T^k(M) \to T^k(M)$ by

$$(Q^*T)(X_1, \cdots, X_k) = T(QX_1, \cdots, QX_k), \quad X, \cdots, X_k \in T(M), \quad T \in T^k(M). \quad (2.3)$$

Notice that orientability of $M$ is necessary to operators $Q$ or $Q^*$.

Let $x \in \Omega$ be given and let $y \in W^{1,1}(\Omega, \mathbb{R}^3)$. Set

$$p(y)(x) = \frac{1}{2}[\nabla y(e_2, e_1) - \nabla y(e_1, e_2)] \quad \text{for} \quad x \in \Omega, \quad (2.4)$$

where $\nabla y(\alpha, \beta) = \langle \nabla_\beta y, \alpha \rangle$ for $\alpha, \beta \in M_x$, $\nabla$ is the differential in the Euclidean space $\mathbb{R}^3$, and $e_1, e_2$ is an orthonormal basis of $M_x$ with the positive orientation. It is easy to check that the value of the right hand side of (2.4) is independent of choice of a positively orientated orthonormal basis. Thus

$$p: \quad W^{1,1}(\Omega, \mathbb{R}^3) \to L^2(\Omega)$$

is a linear operator.

For $U \in T_{sym}^2(M)$ given, consider problem

$$\text{sym} \nabla y(\alpha, \beta) = U(\alpha, \beta) \quad \text{for} \quad \alpha, \beta \in M_x, \quad x \in M, \quad (2.5)$$

where $y \in W^{1,2}(\Omega, \mathbb{R}^3)$.

Let $x \in \Omega$ be given. To simplify computation we use many times the following special frame field: Let $E_1, E_2$ be a positively orientated frame field normal at $x$ with following properties

$$\langle E_i, E_j \rangle = \delta_{ij} \quad \text{in some neighbourhood of} \quad x,$$

$$D_{E_i}E_j = 0, \quad \nabla_{E_i}\vec{n} = \lambda_iE_i \quad \text{at} \quad x \quad \text{for} \quad 1 \leq i, j \leq 2, \quad (2.6)$$

where $\nabla$ is the connection of the Euclidean space $\mathbb{R}^3$, $D$ is the connection of $M$ in the induced metric, $\vec{n}$ is the normal field of $M$, and $\lambda_1\lambda_2 = \kappa$ is the Gaussian curvature. It follows from (2.6) that

$$\Pi(E_i, E_j) = \lambda_i\delta_{ij}, \quad \nabla_{E_i}E_j = -\lambda_i\delta_{ij}\vec{n} \quad \text{at} \quad x \quad \text{for} \quad 1 \leq i, j \leq 2, \quad (2.7)$$

where $\Pi(\alpha, \beta) = \langle \nabla_{\alpha}\vec{n}, \beta \rangle$ is the second fundamental form of $M$.  

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Let \( y \in W^{1,1}(\Omega, \mathbb{R}^3) \) be a solution to problem (2.5). Then (2.5) reads
\[
\begin{aligned}
\nabla y(E_1, E_1) &= U(E_1, E_1), \\
\nabla y(E_2, E_1) + \nabla y(E_1, E_2) &= 2U(E_1, E_2), \quad \text{in some neighbourhood of } x. \\
\nabla y(E_2, E_2) &= U(E_2, E_2),
\end{aligned}
\]

Let
\[
v = p(y)
\]
and define
\[
u = \nabla y(\vec{n}, E_1)E_1 + \nabla y(\vec{n}, E_2)E_2.
\]
We can check easily that \( u \) is a globally defined vector field on \( \Omega \). Moreover, \( v \) satisfies
\[
v + U(E_2, E_1) = \nabla y(E_2, E_1), \quad v - U(E_1, E_2) = -\nabla y(E_1, E_2)
\]
in some neighbourhood of \( x \). Therefore, \( \{v, u\} \) determines \( \nabla \alpha y \) for \( \alpha \in M_x \), that is,
\[
\begin{aligned}
\nabla_{E_1} y &= U(E_1 E_1)E_1 + [v + U(E_1, E_2)]E_2 + \langle u, E_1 \rangle \vec{n}, \\
\nabla_{E_2} y &= [-v + U(E_1, E_2)]E_1 + U(E_2, E_2)E_2 + \langle u, E_2 \rangle \vec{n}.
\end{aligned}
\]
The relation (2.11) can be rewritten as in a form of coordinate free
\[
\nabla \alpha y = 1_a U - vQ\alpha + \langle u, \alpha \rangle \vec{n} \quad \text{for} \quad \alpha \in M_x, \ x \in \Omega.
\]
The function \( v \) and the vector field \( u \) are the new dependent variables and we proceed to find the differential equations they satisfy.

Differentiating the first equation in (2.10) with respect to \( E_2 \) and using the relations (2.6) and (2.7), we have
\[
\begin{aligned}
E_2(v) + DU(E_2, E_1, E_2) &= \nabla^2 y(E_2, E_1, E_2) + \nabla y(\nabla_{E_2} E_2, E_1) \\
&= E_1[\nabla y(E_2, E_2)] - \lambda_2 \nabla y(\vec{n}, E_1) \\
&= DU(E_2, E_2, E_1) - \lambda_2 \langle u, E_1 \rangle \quad \text{at } x,
\end{aligned}
\]
where the following formula has been used
\[
\nabla^2 y(E_2, E_1, E_2) = \nabla^2 y(E_2, E_2, E_1) \quad \text{at } x.
\]
Similarly, we obtain
\[
E_1(v) - DU(E_1, E_2, E_1) = -DU(E_1, E_1, E_2) + \lambda_1 \langle u, E_2 \rangle \quad \text{at } x.
\]
Combining (2.12), (2.13) and (2.6) yields

\[
Dv = [DU(E_1, E_2, E_1) - DU(E_1, E_1, E_2)]E_1 + \lambda_1 (u, E_2)E_1 \\
+ [DU(E_2, E_2, E_1) - DU(E_2, E_1, E_2)]E_2 - \lambda_2 (u, E_1)E_2
\]

\[
= Q[[DU(E_2, E_1, E_1) + DU(E_2, E_2, E_2)]E_2 - [DU(E_2, E_2, E_2) + DU(E_1, E_1, E_2)]E_1 \\
+ [DU(E_1, E_2, E_2) + DU(E_1, E_1, E_1)]E_1 - [DU(E_1, E_1, E_1) + DU(E_2, E_2, E_1)]E_1}
\]

\[
+ \nabla \bar{n}Qu
\]

\[
= Q[\Lambda(U) - D(tr U)] + \nabla \bar{n}Qu \quad \text{for} \quad x \in \Omega,
\]  \hspace{1cm} (2.14)

where the operator \( Q : M_x \rightarrow M_x \) is defined in (2.2), \( \Lambda(U) \in \mathcal{X}(\Omega) \) is given in (2.1), and \( \nabla \bar{n} : M_x \rightarrow M_x \) is the shape operator, defined by

\[
\nabla \bar{n} \alpha = \nabla_{\alpha} \bar{n} \quad \text{for} \quad \alpha \in M_x, \ x \in M.
\]

Now we proceed to derive the differential equations for which the function \( v \) satisfies. Since

\[
\kappa = \Pi(E_1, E_1)\Pi(E_2, E_2) - \Pi^2(E_1, E_2) \quad \text{in a neighbourhood of} \ x,
\]

from (2.6) and (2.7) we compute

\[
D\kappa = [D\Pi(E_1, E_1, E_1)\lambda_2 + \lambda_1 D\Pi(E_2, E_2, E_1)]E_1 \\
+ [D\Pi(E_1, E_1, E_2)\lambda_2 + \lambda_1 D\Pi(E_2, E_2, E_2)]E_2 \quad \text{at} \ x.
\]

Using (2.14), (2.6) and (2.7), we have

\[
D(\nabla \bar{n}Qu)(E_1, E_1) = E_1 \langle \nabla \bar{n}Qu, E_1 \rangle = E_1 \langle u, Q^T \nabla_E \bar{n} \rangle
\]

\[
= Du(Q^T \nabla_E \bar{n}, E_1) + \langle u, D_{E_1}(Q^T \nabla_E \bar{n}) \rangle
\]

\[
= \lambda_1 Du(E_2, E_1) + D\Pi(E_1, E_1, E_1) \langle u, E_2 \rangle - D\Pi(E_1, E_1, E_2) \langle u, E_1 \rangle \quad \text{at} \ x, \quad (2.16)
\]

where the symmetry of \( D\Pi \) is used. A similar computation yields

\[
D(\nabla \bar{n}Qu)(E_2, E_2) = -\lambda_2 Du(E_1, E_2)
\]

\[
+ D\Pi(E_1, E_2, E_2) \langle u, E_2 \rangle - D\Pi(E_2, E_2, E_2) \langle u, E_1 \rangle \quad \text{at} \ x. \quad (2.17)
\]

Multiplying (2.16) by \( \lambda_2 \) and (2.17) by \( \lambda_1 \), respectively, summing them, and using (2.15), we obtain

\[
\langle D(\nabla \bar{n}Qu), Q^\star \Pi \rangle = \kappa [Du(E_2, E_1) - Du(E_1, E_2)] + \langle Qu, D\kappa \rangle. \quad (2.18)
\]

Note that the function \( Du(E_2, E_1) - Du(E_1, E_2) \) is globally defined on \( \Omega \) which is independent of choice of a positively orientated orthonormal basis when the vector field \( u \) is given. From (2.14) and (2.18), we obtain

\[
\langle D^2v, Q^\star \Pi \rangle = \langle D \{ Q[\Lambda(U) - D(tr U)] \}, Q^\star \Pi \rangle + \kappa [Du(E_2, E_1) - Du(E_1, E_2)] + \langle Qu, D\kappa \rangle.
\]
Next, let us consider the compatibility conditions which ensure that a $y$ to satisfy (2.11) exists when the function $v$ and the vector field $u$ are given to satisfy (2.14). We define $B : M_x \to M_x$ for $x \in \Omega$ by

$$B\alpha = l_\alpha U - vQ\alpha + \langle u, \alpha \rangle \vec{n} \quad \text{for} \quad \alpha \in M_x. \quad (2.19)$$

It is easy to check that there is a $y : \Omega \to \mathbb{R}^3$ such that

$$\nabla_\alpha y = B\alpha \quad \text{for} \quad \alpha \in M_x, \ x \in \Omega$$

if and only if the operator $B$ satisfies

$$\nabla_X(BY) = \nabla_Y(BX) + B[X, Y] \quad \text{for} \quad X, Y \in \mathcal{X}(\Omega). \quad (2.20)$$

Using (2.6), (2.7), (2.13), and (2.19), we have

$$\nabla_{E_1}(BE_2) = \left[DU(E_2, E_1, E_1) - E_1(v) + \lambda_1 \langle u, E_2 \rangle \right]E_1 + DU(E_2, E_2, E_1)E_2$$

$$+ \left[Du(E_2, E_1) - \lambda_1 U(E_2, E_1) + v\lambda_1 \right] \vec{n}$$

$$= DU(E_1, E_1, E_2)E_1 + DU(E_2, E_2, E_1)E_2$$

$$+ \left[Du(E_2, E_1) - \lambda_1 U(E_2, E_1) + v\lambda_1 \right] \vec{n} \quad \text{at} \quad x. \quad (2.21)$$

Similarly, we obtain

$$\nabla_{E_2}(BE_1) = DU(E_1, E_1, E_2)E_1 + DU(E_2, E_2, E_1)E_2$$

$$+ \left[Du(E_2, E_1) - \lambda_2 U(E_1, E_2) - v\lambda_2 \right] \vec{n} \quad \text{at} \quad x. \quad (2.22)$$

It follows from (2.21) and (2.22) that the relation (2.20) holds if and only if

$$Du(E_2, E_1) - Du(E_1, E_2) + \text{tr} U(Q\nabla \vec{n}, \cdot) + v \text{tr} \Pi = 0 \quad \text{for} \quad x \in \Omega.$$

Moreover, we assume that

$$\kappa(x) \neq 0 \quad \text{for all} \quad x \in \bar{\Omega}. \quad (2.23)$$

From (2.14), we obtain

$$u = Q(\nabla \vec{n})^{-1}Q[\Lambda(U) - D(\text{tr} U)] - Q(\nabla \vec{n})^{-1}Dv \quad \text{for} \quad x \in \Omega. \quad (2.24)$$

The above derivation yields the following.

**Theorem 2.1** ([4]) Suppose that (2.23) holds. Let $v$ be a solution to problem

$$\langle D^2 v, Q^* \Pi \rangle = P(U) - v \kappa \text{tr} \Pi + X(v) \quad \text{for} \quad x \in \Omega, \quad (2.25)$$
where
\[
P(U) = \langle D\{Q[\Lambda(U) - D(\text{tr}U)]\}, Q^*\Pi \rangle - \langle Q[\Lambda(U) - D(\text{tr}U)], (\nabla\bar{n})^{-1}D\kappa \rangle
\]
\[= -\kappa \text{tr} U(Q\nabla\bar{n}, \cdot),
\]
\[X = (\nabla\bar{n})^{-1}D\kappa. \quad (2.26)
\]

Let \( u \) be given by (2.24). Then there is a \( y \) to satisfy (2.5) such that (2.11) holds. Moreover,
\[
|\nabla y|^2(x) = |U|^2(x) + 2v^2(x) + |u(x)|^2 \quad \text{for} \quad x \in \Omega.
\]

If, in addition, \( y = W + w\bar{n}, w = (y, \bar{n}) \), then
\[
u = Dw - lW\Pi,
\]
\[Dw = lW\Pi - Q(\nabla\bar{n})^{-1}Dv + Q(\nabla\bar{n})^{-1}Q[\Lambda(U) - D(\text{tr}U)].
\]

Remark 2.2 A solution \( y \), modulo a constant vector, in Theorem 2.1 is unique when a solution \( v \) to (2.25) is given.

Remark 2.3 If \( \Omega \) is elliptic and \( \Pi \gg 0 \), then \( \hat{g} = \Pi \) is another metric on \( \Omega \). From [11] we have
\[
\langle D^2v, Q^*\Pi \rangle = \kappa\Delta_{\hat{g}}v + \frac{1}{2\kappa}\Pi(QD\kappa, QDv) \quad \text{for} \quad x \in \Omega,
\]
where \( \Delta_{\hat{g}} \) is the Laplacian of the metric \( \hat{g} \). Thus, in this case equation (2.25) becomes
\[
\Delta_{\hat{g}}v = \frac{1}{\kappa}P(U) - v\text{tr}\Pi + \frac{1}{2\kappa}X(v) \quad \text{for} \quad x \in \Omega.
\]

3 Solvability Regions for Normal Equations

We make preparations to solve problem (2.25). We will show that for hyperbolic surfaces equation (2.25) takes the form of a normal equation locally in an asymptotic coordinate system, see Proposition 4.1 in Section 4 later. So we discuss the solvability of the normal equation in this section.

We consider the following normal equation
\[
\eta_{x_1x_2}(x) = \eta(f, w) \quad \text{for} \quad x = (x_1, x_2) \in \mathbb{R}^2 \quad (3.1)
\]
where
\[
\eta(f, w) = f + f_0(x)w(x) + X(w),
\]
f_0 is a function, and \( X = (X_1, X_2) \) is a vector field on \( \mathbb{R}^2 \).

We shall work out some basic regions in which problem (3.1) is uniquely solvable when \( h \) and some data on part of boundary are given. Let \( k \geq 0 \) be an integer. Let \( f_0 \) and \( X \)
be of class $C^{k-1,1}$, where $C^{-1,1} = L^\infty$. A curve $\gamma(t) = (\gamma_1(t), \gamma_2(t)) : [a, b] \to \mathbb{R}^2$ is said to be noncharacteristic if

$$\gamma'(t)\gamma_2'(t) \neq 0 \quad \text{for} \quad t \in [a, b].$$

We define a linear operator $F : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$Fx = (x_2, -x_1) \quad \text{for} \quad x = (x_1, x_2) \in \mathbb{R}^2. \quad (3.2)$$

Let $\gamma(t) = (\gamma_1(t), \gamma_2(t)) : [0, t_0] \to \mathbb{R}^2$ be a noncharacteristic curve with $\gamma_1'(0)\gamma_2'(0) < 0$. We assume that

$$\gamma_1'(t) > 0, \quad \gamma_2'(t) < 0 \quad \text{for} \quad t \in [0, t_0]. \quad (3.3)$$

Otherwise, we consider the curve $z(t) = \gamma(-t + t_0)$. Set

$$E(\gamma) = \{ (x_1, x_2) \in \mathbb{R}^2 \mid \gamma_1 \circ \gamma_2^{-1}(x_2) < x_1 < \gamma_1(t_0), \gamma_2(t_0) < x_2 < \gamma_2(0) \}. \quad (3.4)$$

Consider the boundary data

$$w \circ \gamma(t) = q_0(t), \quad \langle \nabla w, F\gamma \rangle \circ \gamma(t) = q_1(t) \quad \text{for} \quad t \in (0, t_0). \quad (3.5)$$

Let $f$ be a function with its domain $E$. For simplicity, we denote $\|f\|_{C^{k,1}} = \|f\|_{C^{k,1}(E)}$, $\|f\|_{W^{k,2}} = \|f\|_{W^{k,2}(E)}$, and so on.

**Proposition 3.1** Let $q_0$ be of $C^{k,1}$ and $q_1$, $f$ be of $C^{k-1,1}$, respectively. Then problem (3.1) admits a unique solution $w \in C^{k,1}(\overline{E(\gamma)})$ with the data (3.5). Moreover, there is a $C > 0$, independent of solutions $w$, such that

$$\|w\|_{C^{k,1}} \leq C(\|q_0\|_{C^{k,1}} + \|q_1\|_{C^{k-1,1}} + \|f\|_{C^{k-1,1}}). \quad (3.6)$$

For $\sigma \in (0, t_0)$, let

$$\beta_\sigma(t) = \gamma(\sigma) - tF\gamma(\sigma) \quad \text{for} \quad t \in (0, t_\sigma),$$

where $t_\sigma > 0$ is such that $\beta_\sigma(t_\sigma) \in \partial E(\gamma)$.

**Proposition 3.2** Let $f_0$ and $X$ be of class $C^{0,1}$. Let $q_0$ be of class $W^{2,2}$ and $q_1$, $f$ be of class $W^{1.2}$, respectively. Then problem (3.1) admits a unique solution $w \in W^{2,2}(E(\gamma))$ with the data (3.5). Moreover, there is a $C > 0$, independent of solutions $w$, such that

$$\|w\|_{W^{2,2}}^2 + \|w_{x_2} \circ \beta_\sigma\|_{W^{1,2}}^2 \leq C(\|q_0\|_{W^{2,2}}^2 + \|q_1\|_{W^{1,2}}^2 + \|f\|_{W^{1,2}}). \quad (3.7)$$

Let

$$\Gamma(\gamma, w) = \sum_{j=0}^1 \|\nabla^j w \circ \gamma\|_{L^2(0,t_0)}^2 + \int_0^{t_0} [w_{x_1x_1} \circ \gamma(t)]^2 t + [w_{x_2x_2} \circ \gamma(t)]^2 (t_0 - t)]dt. \quad (3.8)$$
Proposition 3.3 Let $f_0$ and $X$ be of class $C^{0,1}$. Then there are $0 < c_1 < c_2$ such that for all solutions $w \in W^{2,2}(E(\gamma))$ to problem (3.1)

$$c_1 \Gamma(\gamma, w) \leq \|f\|_{W^{1,2}}^2 + \|w\|_{W^{2,2}}^2 \leq c_2 [\|f\|_{W^{1,2}}^2 + \Gamma(\gamma, w)],$$

(3.9)

$$\|w(\cdot, \gamma_2(0))\|_{W^{1,2}}^2 + \int_{\gamma_1(0)}^{\gamma_1(t_0)} |w_{x_1x_1}(t_1, \gamma_2(0))|^2 (x_1 - \gamma_1(0)) dx_1 \leq c_2 [\|f\|_{W^{1,2}}^2 + \Gamma(\gamma, w)],$$

$$\|w(\gamma_1(t_0), \cdot)\|_{W^{1,2}}^2 + \int_{\gamma_2(t_0)}^{\gamma_2(0)} |w_{x_2x_2}(\gamma_1(t_0), x_2)|^2 (x_2 - \gamma_2(t_0)) dx_1 \leq c_2 [\|f\|_{W^{1,2}}^2 + \Gamma(\gamma, w)].$$

For $z = (z_1, z_2) \in \mathbb{R}^2$, $a > 0$, and $b > 0$ given, let

$$R(z, a, b) = (z_1, z_1 + a) \times (z_2, z_2 + b).$$

Consider the boundary data

$$w(x_1, z_2) = p_1(x_1), \quad w(z_1, x_2) = p_2(x_2)$$

(3.10)

for $x_1 \in [z_1, z_1 + a]$ and $x_2 \in [z_2, z_2 + b]$, respectively.

Proposition 3.4 Let $p_1$ and $p_2$ be of class $C^{k,1}$ with $p_1(z_1) = p_2(z_2)$. Let $f$ be of class $C^{k-1,1}$. Then there is a unique solution $w \in C^{k,1}(R(z, a, b))$ to (3.1) with the data (3.10). Moreover, there is a $C > 0$ such that

$$\|w\|_{C^{k,1}} \leq C(\|p_1\|_{C^{k,1}} + \|p_2\|_{C^{k,1}} + \|f\|_{C^{k-1,1}}).$$

Proposition 3.5 Let $f_0$ and $X$ be of class $C^{0,1}$. Let $p_1$ and $p_2$ be of class $W^{2,2}$ with $p_1(z_1) = p_2(z_2)$. Let $f$ be of class $W^{1,2}$. Then there is a unique solution $w \in W^{2,2}(R(z, a, b))$ to (3.1) with the data (3.10). Moreover, the estimates

$$\|w\|_{W^{2,2}} \leq C(\|p_1\|_{W^{2,2}}^2 + \|p_2\|_{W^{2,2}}^2 + \|f\|_{W^{1,2}}^2)$$

hold.

Proposition 3.6 Let $f_0$ and $X$ be of class $C^{0,1}$. Then there are $0 < c_1 < c_2$ such that for all solutions $w \in W^{2,2}(R(z, a, b))$ to problem (3.1)

$$c_1 \|w(\cdot, z_2)\|_{W^{2,2}}^2 + \|w(z_1, \cdot)\|_{W^{2,2}}^2 \leq \|w\|_{W^{2,2}}^2 + \|f\|_{W^{1,2}}^2 \leq c_2 [\|w(\cdot, z_2)\|_{W^{2,2}}^2 + \|w(z_1, \cdot)\|_{W^{2,2}}^2 + \|f\|_{W^{1,2}}^2].$$

Let $\beta = (\beta_1, \beta_2) : [0, t_0] \to \mathbb{R}^2$ be a noncharacteristic curve with $\beta_1'(0) \beta_2'(0) > 0$. We assume

$$\beta_i'(t) > 0 \quad \text{for} \quad t \in [0, t_0], \quad i = 1, 2.$$  

(3.11)
Otherwise, we consider the curve \( z(t) = \beta(-t + t_0) \). Set
\[
\mathcal{P}_1(\beta) = \{ (x_1, x_2) \mid \beta_1 \circ \beta_2^{-1}(x_2) < x_1 < \beta_1(t_0), \beta_2(0) < x_2 < \beta_2(t_0) \},
\]
and consider the boundary data
\[
w_{x_2} \circ \beta(t) = p(t), \quad t \in (0, t_0); \quad w(x_1, \beta_2(0)) = p_1(x_1), \quad x_1 \in (\beta_1(0), \beta_1(t_0)). \tag{3.12}
\]
Set
\[
P_2(\beta) = \{ (x_1, x_2) \mid b_1(0) < x_1 < \beta_1 \circ \beta_2^{-1}(x_2), \ b_2(0) < x_2 < \beta_2(t_0) \},
\]
and consider the boundary data
\[
w_{x_1} \circ \beta(t) = p(t), \quad t \in (0, t_0); \quad w(\beta_1(0), x_2) = p_2(x_2), \quad x_1 \in (b_1(0), \beta_1(t_0)). \tag{3.13}
\]

**Proposition 3.7** Let the curve \( \beta \) be of class \( C^{k-1,1} \). Let \( p_1 \) (or \( p_2 \)) be of class \( C^{k,1} \) and let \( p, f \) be of class \( C^{k-1,1} \). Then problem (3.1) admits a unique solution \( w \in C^{k,1}(\overline{P_1(\beta)}) \) (or \( C^{k,1}(\overline{P_2(\beta)}) \)) with the data (3.12) (or (3.13)). Furthermore, the similar estimates as in Proposition 3.4 hold.

**Proposition 3.8** Let the curve \( \beta \) be of class \( C^1 \). Let \( f_0 \) and \( X \) be of class \( C^{0,1} \). Let \( p_1 \) (or \( p_2 \)) be of class \( W^{2,2} \) and let \( p, f \) be of class \( W^{1,2} \). Then problem (3.1) admits a unique solution \( w \in W^{2,2}(\overline{P_1(\beta)}) \) (or \( W^{2,2}(\overline{P_2(\beta)}) \)) with the data (3.12) (or (3.13)). Furthermore, the similar estimates as in Proposition 3.5 hold.

Let
\[
\Gamma(P_1, w) = \int_0^{t_0} |p'(t)|^2(t_0 - t)dt + \|p_1\|_{W^{1,2}}^2 + \int_{\beta_1(0)}^{\beta_1(t_0)} |p''_1(x_1)|^2(x_1 - z_i)|dx_i, \quad i = 1, 2.
\]

**Proposition 3.9** Let the curve \( \beta \) be of class \( C^1 \). Let \( f_0 \) and \( X \) be of class \( C^{0,1} \). Then there are \( 0 < c_1 < c_2 \) such that for all solutions \( w \in W^{2,2}(\overline{P_1(\beta)}) \) to problem (3.1)
\[
c_1 \Gamma(P_1, w) \leq \|w\|_{W^{2,2}}^2 + \|f\|_{W^{1,2}}^2 \leq c_2[\Gamma(P_1, w) + \|f\|_{W^{1,2}}^2].
\]
\[
c_1 \|w(\beta_1(t_0), \cdot)\|_{W^{2,2}}^2 \leq \int_0^{t_0} |p'(t)|^2 dt + \int_{\beta_1(0)}^{\beta_1(t_0)} |p''_1(x_1)|^2(x_1 - b_1(0))|dx_1 + \|p_1\|_{W^{1,2}}^2 + \|f\|_{W^{1,2}}^2
\]
\[
\leq c_2(\|w(\beta_1(t_0), \cdot)\|_{W^{2,2}}^2 + \int_{\beta_1(0)}^{\beta_1(t_0)} |p''_1(x_1)|^2(x_1 - b_1(0))|dx_1 + \|p_1\|_{W^{1,2}}^2 + \|f\|_{W^{1,2}}^2). \tag{3.14}
\]
Moreover, the corresponding estimates for solutions \( w \in W^{2,2}(\overline{P_2(\beta)}) \) hold.

**Remark 3.1** (3.14) implies that \( p \in W^{1,2} \) if and only if \( w(\beta_1(t_0), \cdot) \in W^{2,2} \). However, the case of \( p \notin W^{1,2} \) may happen.
Let $\gamma : [0, t_1] \to \mathbb{R}^2$ and $\beta : [0, t_0] \to \mathbb{R}^2$ be two noncharacteristic curves with $\gamma(0) = \beta(0)$ such that (3.3) and (3.11) hold, respectively. We assume that

$$\sup_{t \in [0, t_1]} [\gamma_1(t) + \gamma_2(t)] < \beta_1(t_0) + \beta_2(t_0), \quad \beta_1(t_0) + \beta_2(t_0) - \beta_2(0) \leq \gamma_1(t_1).$$

Set

$$\Xi_1(\beta, \gamma) = \{ (x_1, x_2) \in \mathbb{R}^2 \mid \beta_1(0) < x_1 \leq \beta_1(t_0), \quad \gamma_2 \circ \gamma_1^{-1}(x_1) < x_2 < \beta_2 \circ \beta_1^{-1}(x_1) \}$$

$$\cup \{ (x_1, x_2) \in \mathbb{R}^2 \mid \beta_1(t_0) \leq x_1 < \gamma_1(t_1), \quad \gamma_2 \circ \gamma_1^{-1}(x_1) < x_2 < -x_1 + \beta_1(t_0) + \beta_2(t_0) \}. \quad (3.15)$$

Consider the boundary data

$$w_{x_2} \circ \beta(t) = p(t) \quad \text{for} \quad t \in [0, t_0], \quad (3.16)$$

$$w \circ \gamma(t) = q_0(t), \quad \langle \nabla w, \mathcal{F} \gamma \rangle \circ \gamma(t) = q_1(t) \quad \text{for} \quad t \in (0, t_1), \quad (3.17)$$

where $\mathcal{F}$ is given by (3.2).

To have a $C^{k,1}$ solution on $\Xi_1(\beta, \gamma)$, we need some kind of compatibility conditions at the point $z$. At this moment, we assume that Propositions 3.1 and 3.7 hold in advance. Their proofs will be given later. From Proposition 3.1, problem (3.1) admits a unique solution $u \in C^{k,1}(\overline{E(\gamma)})$ with the data (3.17). From Proposition 3.7, there is a unique solution $v \in C^{k,1}(P_1(\beta))$ to problem (3.1) with the data

$$v_{x_2} \circ \beta(t) = p(t), \quad t \in (0, t_0), \quad v(x_1, \beta_2(0)) = u(x_1, \beta_2(0)), \quad x_1 \in [\beta_1(0), \beta_1(t_0)]. \quad (3.18)$$

In terms of the uniqueness, if problem (3.1) has a unique solution $w \in C^{k,1}(\overline{E(\gamma)})$ with the data (3.16) and (3.17) together, then

$$w(x) = \begin{cases} v(x) & \text{for} \quad x \in P_1(\beta), \\ u(x) & \text{for} \quad x \in E(\gamma). \end{cases} \quad (3.19)$$

Conversely, if we define $w$ by the formula (3.19), then whether it is a $C^{k,1}$ solution to (3.1) on $\Xi_1(\beta, \gamma)$ depends on the $C^{k,1}$ regularity of $w$ at the point $\beta(0)$. Thus, compatibility conditions are something which can guarantee that $w$ is $C^{k,1}$ at $\beta(0)$, that is

$$\nabla^j u \circ \gamma(0) = \nabla^j v \circ \beta(0) = \nabla^j u \circ \gamma(0) \quad \text{for} \quad 0 \leq j \leq k. \quad (3.20)$$

The solution $u$ with the data (3.17) yields

$$\nabla u \circ \gamma(t) = \frac{1}{|\gamma'(t)|^2} (\gamma_1'(t)q_0(t) + \gamma_2'(t)q_1(t), \quad \gamma_2'(t)q_0(t) - \gamma_1'(t)q_1(t)) \quad (3.21)$$

for $t \in [0, t_1]$. Using (3.1) and (3.21), we have

$$u_{x_2x_1} \circ \gamma(t) = f \circ \gamma(t) + \frac{1}{|\gamma'(t)|^2} [\gamma_2'(t)X_1 \circ \gamma(t) - \gamma_1'(t)X_2 \circ \gamma(t)] q_1(t) + f_0 \circ \gamma(t) q_0(t) + \frac{1}{|\gamma'(t)|^2} [\gamma_1'(t)X_1 \circ \gamma(t) + \gamma_2'(t)X_2 \circ \gamma(t)] q_0(t) \quad (3.22)$$
for $t \in (0, a)$. Next, differentiating the second component in (3.21) with respect to variable $t$ and using (3.22), we obtain

$$u_{x_2x_2} \circ \gamma(t) = -\frac{\gamma'_1}{\gamma_2} f \circ \gamma(t) - \frac{\gamma'_1}{\gamma_2} f_0 \circ \gamma q_0$$

$$-\left[ \frac{2(\gamma''_1, \gamma'_1)}{|\gamma'|^4} + \frac{\gamma'_1}{|\gamma'|^2} \gamma'_2 (\gamma'_1 X_1 \circ \gamma + \gamma'_2 X_2 \circ \gamma) - \frac{\gamma''_2}{|\gamma'|^2 \gamma_2} q'_0 + \frac{1}{|\gamma'|^2} q''_0 \right]$$

$$+ \left[ \frac{2(\gamma''_1, \gamma'_1) \gamma'_1}{|\gamma'|^4 \gamma_2^2} - \frac{\gamma'_1}{|\gamma'|^2 \gamma_2^2} (\gamma'_2 X_1 \circ \gamma - \gamma'_1 X_2 \circ \gamma) - \frac{\gamma''_1}{|\gamma'|^2 \gamma_2} q_1 - \frac{\gamma'_1}{|\gamma'|^2 \gamma_2} q'_1 \right]. \quad (3.23)$$

By repeating the above procedure, we have shown that, for $1 \leq j \leq k - 1$, there are $j$ order tensor fields $A_{\alpha\beta}(t), A^1_\alpha(t)$, and $A^0_\alpha(t)$ such that

$$\nabla^j u_{x_2} \circ \gamma(t) = \sum_{\alpha + \beta \leq j - 1} \partial_{x_1}^\alpha \partial_{x_2}^\beta f \circ \gamma(t) A_{\alpha\beta}(t) + \sum_{\alpha \leq j} q_1^{(a)}(t) A^1_\alpha(t)$$

$$+ \sum_{\alpha \leq j + 1} q_0^{(j)}(t) A^0_\alpha(t) \quad \text{for} \quad t \in [\beta_1(0), \beta_1(t_0)]. \quad (3.24)$$

Let $v \in C^{k,1}(\partial_1(\beta))$ be the solution to (3.1) with the data (3.18). Then

$$p'(t) = \langle \nabla u_{x_2}(\beta(t)), \dot{\beta}(t) \rangle, \quad p''(t) = \langle \nabla^2 u_{x_2}(\beta(t)), \dot{\beta}(t) \otimes \ddot{\beta}(t) \rangle + \langle \nabla u_{x_2}(\beta(t)), \dddot{\beta}(t) \rangle$$

for $t \in [0, t_0]$. Some computations show that

$$p^{(l)}(t) = \langle \nabla^l u_{x_2}(\beta(t)), \dddot{\beta}(t) \rangle$$

$$+ \sum_{j_1 + \cdots + j_l = l, 1 \leq i \leq l - 1} a_{j_1 \cdots j_l} \langle \nabla^{j_1} u_{x_2}(\beta(t)), \dddot{\beta}(j_1)(t) \otimes \dddot{\beta}(j_l)(t) \rangle \quad (3.25)$$

for $t \in [0, t_0]$, and $1 \leq l \leq k$, where $a_{j_1 \cdots j_l}$ are positive integers. Then assumption (3.20) is stated as the following.

**Definition 3.1** Let the curves $\beta$ and $\gamma$ be of class $C^{k,1}$. Let $q_0$ be of class $C^{k,1}$ and $p, q_1, f$ of class $C^{k-1,1}$, respectively. It is said that the $k$th order compatibility conditions hold at $z$ if $|\gamma'(0)|^2 p(0) = \gamma'_2(0)q'_0(0) - \gamma'_1(0)q_1(0)$ and

$$p^{(l)}(0) = \langle \nabla^l u_{x_2} \circ \gamma(0), \dot{\beta}(0) \otimes \cdots \otimes \dddot{\beta}(0) \rangle$$

$$+ \sum_{j_1 + \cdots + j_l = l, 1 \leq i \leq l - 1} a_{j_1 \cdots j_l} \langle \nabla^{j_1} u_{x_2} \circ \gamma(0), \dddot{\beta}(j_1)(0) \otimes \cdots \otimes \dddot{\beta}(j_l)(0) \rangle \quad (3.26)$$

for $1 \leq l \leq k - 1$, where $\nabla^{j_1} u_{x_2} \circ \gamma(0)$ and $a_{j_1 \cdots j_l}$ are given in (3.24) and (3.25), respectively.

**Proposition 3.10** Let the curves $\beta$ and $\gamma$ be of class $C^{k,1}$. Let $q_0$ be of class $C^{k,1}$ and $p, q_1, f$ of class $C^{k-1,1}$, respectively. If $k \geq 1$, we assume that the $k$th order compatibility conditions hold at $\gamma(0)$. Then problem (3.1) admits a unique solution $w \in C^{k,1}(\partial_1(\beta, \gamma))$ with the data (3.16) and (3.17). Moreover, the following estimates hold

$$\|w\|_{C^{k,1}} \leq C(\|p\|_{C^{k-1,1}} + \|q_0\|_{C^{k,1}} + \|q_1\|_{C^{k-1,1}} + \|f\|_{C^{k-1,1}}).$$
Proposition 3.11 Let the curves $\beta$ and $\gamma$ be of class $C^1$. Let $f_0$ and $X$ be of class $C^0,1$. Let $q_0$ be of class $W^{2,2}$ and $p, q_1, f$ of class $W^{1,2}$, respectively, such that the 1st order compatibility conditions hold true at $\gamma(0)$. Then problem (3.1) admits a unique solution $w \in W^{2,2}(\Xi_1(\beta, \gamma))$ with the data (3.16) and (3.17). Moreover, the following estimates hold

$$
\|w\|_{W^{2,2}} \leq C(\|p\|_{W^{1,2}} + \|q_0\|_{W^{2,2}} + \|q_1\|_{W^{1,2}} + \|f\|_{W^{1,2}}).
$$

Let

$$
\Gamma_i(\beta, w) = \int_0^{t_0} |p'(s)|^2(t_0 - s)ds, \quad p(s) = w_{x_i} \circ \beta(s) \quad \text{for} \quad s \in (0, t_0), \quad i = 1, 2. \quad (3.27)
$$

Proposition 3.12 Let the curves $\beta$ and $\gamma$ be of class $C^1$. Let $f_0$ and $X$ be of class $C^0,1$. Then there are $0 < c_1 < c_2$ such that for all solutions $w \in W^{2,2}(\Xi_1(\beta, \gamma))$ to problem (3.1)

$$
c_1[\Gamma(\gamma, w) + \Gamma(\beta, w)] \leq \|w\|^2_{W^{2,2}} + \|f\|^2_{W^{1,2}} \leq c_2[\Gamma(\gamma, w) + \Gamma(\beta, w)] + \|f\|^2_{W^{1,2}},
$$

where $\Gamma(\gamma, w)$ is given in (3.8).

Let $\gamma : [0, t_1] \to \mathbb{R}^2$ and $\beta : [0, t_0] \to \mathbb{R}^2$ be two noncharacteristic curves with $\gamma(0) = \beta(0)$ such that (3.11) holds. We here assume that

$$
\gamma_2(t_1) > \beta_2(t_0), \quad \gamma_1'(t) < 0, \quad \gamma_2'(t) > 0 \quad \text{for} \quad t \in [0, t_1].
$$

Set

$$
\Xi_2(\beta, \gamma) = \{(x_1, x_2) \in \mathbb{R}^2 \mid \beta_1(0) < x_1 < \beta_1(t_0), \quad \beta_2 \circ \beta_1^{-1}(x_1) \leq x_2 < -x_1 + \beta_1(t_0) + \beta_2(t_0) \}
$$

$$
\cup \{(x_1, x_2) \in \mathbb{R}^2 \mid \gamma_1(t_1) < x_1 < \gamma_1(0), \quad \gamma_2 \circ \gamma_1^{-1}(x_1) < x_2 < -x_1 + \gamma_1(t_0) + \beta_2(t_0) \}.
$$

Consider the data

$$
w_{x_1} \circ \beta(t) = p(t) \quad \text{for} \quad t \in [0, t_0], \quad (3.28)
$$

$$
w \circ \gamma(t) = q_0(t), \quad \langle \nabla w, \mathcal{F} \gamma \rangle \circ \gamma(t) = q_1(t) \quad \text{for} \quad t \in (0, t_1), \quad (3.29)
$$

where $\mathcal{F}$ is given by (3.2).

Proposition 3.13 The corresponding results as in Propositions 3.10, 3.11, and 3.12 hold where $\Xi_1(\beta, \gamma)$ and $\Gamma_2(\beta, w)$ are replaced with $\Xi_2(\beta, \gamma)$ and $\Gamma_1(\beta, w)$, respectively.

Let $\beta$ and $\gamma$ be noncharacteristic curves with $\beta(0) = \gamma(0)$ such that (3.3) and (3.11) hold. Let $\hat{\beta} = (\hat{\beta}_1, \hat{\beta}_2) : [0, t_0] \to \mathbb{R}^2$ be noncharacteristic such that

$$
\beta_1(t_0) + \beta_2(t_0) = \hat{\beta}_1(t_0) + \hat{\beta}_2(t_0),
$$

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Consider the boundary data

\[ \beta_1(t_0) + \beta_2(t_0) \leq \gamma_1(t_1) + \gamma_2(0), \quad \gamma(t_1) = \hat{\beta}(0), \quad \hat{\beta}_1'(t) > 0, \quad \hat{\beta}_2'(t) > 0 \quad \text{for} \quad t \in [0, t_0]. \]

Set

\[
\Phi(\beta, \hat{\beta}) = \{ (x_1, x_2) | \beta_1(0) < x_1 < \beta_1(t_0), \quad \gamma_2 \circ \gamma_1^{-1}(x_1) \leq x_2 < \beta_2 \circ \beta_1^{-1}(x_1) \} 
\cup \{ (x_1, x_2) | \beta_1(t_0) \leq x_1 < \gamma_1(t_1), \quad \gamma_2 \circ \gamma_1^{-1}(x_1) \leq x_2 < -x_1 + \beta_1(t_0) + \beta_2(t_0) \} 
\cup \{ (x_1, x_2) | \gamma_1(t_1) \leq x_1 < \hat{\beta}_1(t_0), \quad \hat{\beta}_2 \circ \hat{\beta}_1^{-1}(x_1) < x_2 < -x_1 + \hat{\beta}_1(t_0) + \beta_2(t_0) \}.
\]

Consider the boundary data

\[ w_{x_2} \circ \beta(t) = p_1(t), \quad t \in [0, t_0], \quad w_{x_1} \circ \hat{\beta}(t) = p_2(t), \quad t \in (0, \hat{t}_0), \quad (3.30) \]

\[ w \circ \gamma(t) = q_0(t), \quad \langle \nabla w, \mathcal{F} \rangle \circ \gamma(t) = q_1(t) \quad \text{for} \quad t \in (0, t_1). \quad (3.31) \]

We have the following.

**Theorem 3.1** Let the curves \( \beta, \gamma, \) and \( \hat{\beta} \) be of class \( C^{k,1} \). Let \( q_0 \) be of class \( C^{k,1} \), and \( p_1, p_2, q_1, f \) of class \( C^{k-1,1} \) such that the \( k \)-th order compatibility conditions hold true at \( \gamma(0) \) and \( \gamma(t_1) \), respectively. Then problem (3.1) admits a unique solution \( w \in C^{k,1}(\Phi(\beta, \hat{\beta})) \) with the data (3.30) and (3.31). Moreover, the following estimates hold

\[ \|w\|_{C^{k,1}}^2 \leq C(\|p_1\|_{C^{k-1,1}}^2 + \|p_2\|_{C^{k-1,1}}^2 + \|q_0\|_{C^{k,1}}^2 + \|q_1\|_{C^{k-1,1}}^2 + \|f\|_{C^{k-1,1}}^2). \]

**Theorem 3.2** Let the curves \( \beta, \gamma, \) and \( \hat{\beta} \) be of class \( C^{1} \). Let \( f_0 \) and \( X \) be of class \( C^{1} \). Let \( q_0 \) be of class \( W^{2,2} \), and \( p_1, p_2, q_1, f \) of class \( W^{1,2} \), such that the 1-st order compatibility conditions hold true at \( \gamma(0) \) and \( \gamma(t_1) \), respectively. Then problem (3.1) admits a unique solution \( w \in W^{2,2}(\Phi(\beta, \hat{\beta})) \) with the data (3.30) and (3.31). Moreover, the following estimates hold

\[ \|w\|_{W^{2,2}}^2 \leq C(\|p_1\|_{W^{1,2}}^2 + \|p_2\|_{W^{1,2}}^2 + \|q_0\|_{W^{2,2}}^2 + \|q_1\|_{W^{1,2}}^2 + \|f\|_{W^{1,2}}^2). \]

**Theorem 3.3** Let the curves \( \beta, \gamma, \) and \( \hat{\beta} \) be of class \( C^{1} \). Let \( f_0 \) and \( X \) be of class \( C^{0,1} \). Then there are \( 0 < c_1 < c_2 \) such that for all solutions \( w \in W^{2,2}(\Phi(\beta, \gamma)) \) to problem (3.1)

\[ c_1[\Gamma(\gamma, w) + \Gamma_1(\hat{\beta}, w) + \Gamma_2(\beta, w)] \leq \|w\|_{W^{2,2}}^2 + \|f\|_{W^{1,2}}^2 \leq c_2[\Gamma(\gamma, w) + \Gamma_1(\hat{\beta}, w) + \Gamma_2(\beta, w) + \|f\|_{W^{1,2}}^2], \]

where \( \Gamma(\gamma, w), \Gamma_1(\hat{\beta}, w), \) and \( \Gamma_2(\beta, w) \) are given in (3.8) and (3.27), respectively.

The remaining of this section is devoted to proofs of Propositions 3.1-3.13 and Theorems 3.1, 3.2, and 3.3.

**Lemma 3.1** Let \( T > 0 \) be given. There is a \( \varepsilon_T > 0 \) such that if \( |\gamma(0)| \leq T \) and \( \max\{\gamma_1(t_0) - \gamma_1(0), \gamma_2(t_0) - \gamma_2(0)\} < \varepsilon_T \), Proposition 3.1 holds true.
Proof. The proof is broken into several steps as follows.

Step 1. Let \( k = 0 \) and let \( w \in C^{0,1}(E(\gamma)) \) be a solution to (3.1) with the data (3.5). It follows from (3.5) that

\[
w_{x_1} \circ \gamma(t) = \frac{1}{|\gamma'(t)|^2} [\gamma_1'(t)q_0(t) + \gamma_2'(t)q_1(t)], \quad w_{x_2} \circ \gamma(t) = \frac{1}{|\gamma'(t)|^2} [\gamma_2'(t)q_0(t) - \gamma_1'(t)q_1(t)].
\]

Let \( x = (x_1, x_2) \in E(\gamma) \) be given. We integrate (3.1) with respect to the first variable \( \zeta_1 \) over \( (\gamma_1 \circ \gamma_2^{-1}(\zeta_2), x_1) \) for \( \zeta_2 \in (\gamma_2 \circ \gamma_1^{-1}(x_1), x_2) \) to have

\[
w_{x_2}(x_1, \zeta_2) = w_{x_2} \circ \gamma(\gamma_2^{-1}(\zeta_2)) + \int_{\gamma_1 \circ \gamma_2^{-1}(\zeta_2)}^{x_1} \eta(f, w)(\zeta_1, \zeta_2) d\zeta_1. \tag{3.32}
\]

Then integrating the above identity over \( (\gamma_2 \circ \gamma_1^{-1}(x_1), x_2) \) with respect to the second variable \( \zeta_2 \) yields

\[
w(x_1, x_2) = \mathcal{B} (q_0, q_1) + \int_{E(x)} \eta(f, w) d\zeta,
\]

where

\[
\mathcal{B} (q_0, q_1) = q_0 \circ \gamma_1^{-1}(x_1) + \int_{\gamma_1^{-1}(x_1)}^{x_2} \frac{\gamma_2'(t)}{|\gamma'(t)|^2} [\gamma_2'(t)q_0(t) - \gamma_1'(t)q_1(t)] dt, \tag{3.33}
\]

\[
E(x) = \{ (\zeta_1, \zeta_2) | \gamma_1 \circ \gamma_2(\zeta_2) < \zeta_1 < x_1, \quad \gamma_2 \circ \gamma_1^{-1}(x_1) < \zeta_2 < x_2 \}. \tag{3.34}
\]

Step 2. We define an operator \( I : C^{0,1}(E(\gamma)) \to C^{0,1}(E(\gamma)) \) by

\[
I(w) = \mathcal{B} (q_0, q_1) + \int_{E(x)} \eta(f, w) d\zeta \quad \text{for} \quad w \in C^{0,1}(E(\gamma)). \tag{3.35}
\]

It is easy to check that \( w \in C^{0,1}(E(\gamma)) \) solves (3.1) with the data (3.5) if and only if \( I(w) = w \).

Next, we shall prove that there is a \( 0 < \varepsilon_T \leq 1 \) such that when \( |\gamma(0)| \leq T \) and \( 0 < \max\{\gamma_1(t_0) - \gamma_1(0), \gamma_2(t_0) - \gamma_2(0)\} < \varepsilon_T \), the map \( I : C^{0,1}(E(\gamma)) \to C^{0,1}(E(\gamma)) \) is contractible. Thus the existence and uniqueness of solutions in the case \( k = 0 \) follows from Banach’s fixed point theorem.

A simple computation shows that for \( w \in C^{0,1}(E(\gamma)) \)

\[
[I(w)]_{x_1} = \frac{1}{|\gamma'(t)|^2} [\gamma_1'(t)q_0(t) + \gamma_2'(t)q_1(t)]_{t=\gamma_1^{-1}(x_1)} + \int_{\gamma_2 \circ \gamma_1^{-1}(x_1)}^{x_2} \eta(f, w)(x_1, \zeta_2) d\zeta_2,
\]

\[
[I(w)]_{x_2} = \frac{1}{|\gamma'(t)|^2} [\gamma_2'(t)q_0(t) - \gamma_1'(t)q_1(t)]_{t=\gamma_2^{-1}(x_2)} + \int_{\gamma_1 \circ \gamma_2^{-1}(x_2)}^{x_1} \eta(f, w)(\zeta_1, x_2) d\zeta_1.
\]

The above formulas yield for \( w_1, w_2 \in C^{0,1}(E(\gamma)) \),

\[
\|I(w_1) - I(w_2)\|_{C^{0,1}(E(\gamma))} \leq C_T \max\{\lambda, \lambda^2\} \|w_1 - w_2\|_{C^{0,1}(E(\gamma))},
\]

where \( C_T \) is a constant depending on \( T \) and the function \( \gamma \).
where
\[ \lambda = \max\{\gamma_1(t_0) - \gamma_1(0), \gamma_2(t_0) - \gamma_2(0)\}, \quad C_T = \|f_0\|_{L^\infty(|x| \leq 2T)} + \|X\|_{L^\infty(|x| \leq 2T)}. \]

Thus, the map \( I : C^{0,1}(\overline{E(\gamma)}) \to C^{0,1}(\overline{E(\gamma)}) \) is contractible if \( \lambda > 0 \) is small.

**Step 3.** Consider the case \( k = 1 \). Let \( q_0 \in C^{1,1}[0, t_0] \), \( q_1 \in C^{0,1}[0, t_0] \), and \( f \in C^{0,1}(\overline{E(\gamma)}) \) be given. By Step 2, there is a \( \varepsilon_T > 0 \) such that if \( |\gamma(0)| \leq T \) and \( 0 < \lambda < \varepsilon_T \), problem (3.1) has a unique solution \( w \in C^{0,1}(\overline{E(\gamma)}) \) with the data (3.5). A formal computation shows that \( u = w_{x_1} \) solves problem
\[ w_{x_1} = \eta(f, u) \quad \text{for} \quad x \in E(\gamma), \tag{3.36} \]
with the data
\[ u \circ \gamma(t) = \hat{q}_0(t), \quad \langle \nabla u, F\hat{\gamma} \rangle \circ \gamma(t) = \hat{q}_1(t) \quad \text{for} \quad t \in (0, t_0), \tag{3.37} \]
where
\[ \hat{f} = f_{x_1} + f_{0x_1}w + \nabla_{0x_1}X(w), \quad \hat{q}_0(t) = \frac{1}{|\gamma(t)|^2} [\gamma_2'(t)q_0(t) - \gamma_1'(t)q_1(t)], \]
\[ \hat{q}_1(t) = \gamma_2'(t)w_{x_1} \circ \gamma(t) - \gamma_1'(t)w_{x_1} \circ \gamma(t). \]

In the above formula, \( w_{x_1} \circ \gamma(t) \) and \( w_{x_1} \circ \gamma(t) \) are given by (3.22) and (3.23), respectively. We apply Step 2 to problem (3.36) and (3.37) to obtain \( u = w_{x_1} \in C^{0,1}(\overline{E(\gamma)}) \) when \( 0 < \lambda < \varepsilon_T \). A similar argument yields \( w_{x_2} \in C^{0,1}(\overline{E(\gamma)}) \). Thus \( w \in C^{1,1}(\overline{E(\gamma)}) \).

By repeating the above procedure, the existence and uniqueness of the solutions in the cases \( k \geq 2 \) are obtained.

**Step 4.** Let map \( I : C^{k,1}(\overline{E(\gamma)}) \to C^{k,1}(\overline{E(\gamma)}) \) be defined in Step 2 and let \( w \in C^{k,1}(\overline{E(\gamma)}) \) be the solution to problem (3.1) with the data (3.5). Then
\[ \|w\|_{C^{k,1}} = \|I(w)\|_{C^{k,1}} \leq \|I(0)\|_{C^{k,1}} + \|I(w) - I(0)\|_{C^{k,1}} \]
\[ \leq C(\|q_0\|_{C^{k,1}} + \|q_1\|_{C^{k-1,1}} + \|f\|_{C^{k-1,1}}) + C_T \max\{\lambda, \lambda^2\} \|w\|_{C^{k,1}}. \]

Thus, the estimate (3.6) follows if \( \lambda > 0 \) is small. \( \square \)

**Lemma 3.2** Let \( T > 0 \) be given. There is \( \varepsilon_T > 0 \) such that if \( |\gamma(0)| \leq T \) and \( \max\{\gamma_1(t_0) - \gamma_1(0), \gamma_2(0) - \gamma_2(t_0)\} < \varepsilon_T \), Proposition 3.2 holds true.

**Proof** Let \( I(w) \) be defined by the formula (3.35) this time for \( w \in W^{2,2}(E(\gamma)) \). The similar arguments as in the proof of Lemma 3.1 show that the operator \( I : W^{2,2}(E(\gamma)) \to W^{2,2}(E(\gamma)) \) is contractible when \( \max\{\gamma_1(t_0) - \gamma_1(0), \gamma_2(0) - \gamma_2(t_0)\} \) is small, thus, a unique solution \( w \in W^{2,2}(E(\gamma)) \) with the data (3.5) exists, and the estimate
\[ \|w\|_{W^{2,2}}^2 \leq C(\|q_0\|_{W^{2,2}}^2 + \|q_1\|_{W^{1,2}}^2 + \|f\|_{W^{1,2}}) \tag{3.38} \]
follows.

Let $\beta_\sigma(t) = (\beta_{\sigma_1}(t), \beta_{\sigma_2}(t))$. Using equation (3.1), we have

$$w_{x_2x_2} \circ \beta_\sigma(t) = w_{x_2x_2} \circ \gamma \circ \gamma_2^{-1} \circ \beta_{\sigma_2}(t) + \int_{\gamma_1 \circ \gamma_2^{-1} \circ \beta_{\sigma_2}(t)}^{\beta_{\sigma_1}(t)} [\eta(f, w)]_{x_2}(\zeta_1, \beta_{\sigma_2}(t)) d\zeta_1,$$

which yields

$$|w_{x_2x_2} \circ \beta_\sigma(t)|^2 \leq 2|w_{x_2x_2} \circ \gamma \circ \gamma_2^{-1} \circ \beta_{\sigma_2}(t)|^2 + 2[\gamma_1(t_0) - \gamma_1(0)] C_T \int_{\gamma_1 \circ \gamma_2^{-1} \circ \beta_{\sigma_2}(t)}^{\beta_{\sigma_1}(t)} ([|f|^2 + |\nabla f|^2 + |w|^2 + |\nabla w|^2 + |\nabla^2 w|^2](\zeta_1, \beta_{\sigma_2}(t)) d\zeta_1.$$

Integrating the above inequality over $(0, t_\sigma)$ with respect to $t$ and using (3.23), we obtain

$$\|w_{x_2x_2} \circ \beta_\sigma\|^2_{L^2} \leq C(\|f\|^2_{W^{1,2}} + \|q_0\|^2_{W^{2,2}} + \|q_1\|^2_{W^{1,2}} + \|w\|^2_{W^{2,2}}).$$

A similar computation shows that $\|\nabla w_{x_1} \circ \beta_\sigma\|^2_{L^2}$, $\|\nabla w \circ \beta_\sigma\|^2_{L^2}$, and $\|w \circ \beta\|^2_{L^2}$ can be bounded also by the right hand side of the above inequality. Thus estimate (3.7) follows from (3.8).

**Lemma 3.3** Let $T > 0$ be given. There is $\varepsilon_T > 0$ such that if $|\gamma(0)| \leq T$ and $\max\{\gamma_1(t_0) - \gamma_1(0), \gamma_2(0) - \gamma_2(t_0)\} < \varepsilon_T$, Proposition 3.3 holds.

**Proof** Step 1 Using (3.1) we have

$$w_{x_1x_1}(x) = w_{x_1x_1} \circ \gamma \circ \gamma_1^{-1}(x_1) + \int_{\gamma_2 \circ \gamma_1^{-1}(x_1)}^{x_2} [\eta(f, w)]_{x_1}(x_1, \zeta_2) d\zeta_2,$$

which yields

$$|w_{x_1x_1}(x)|^2 \leq 2|w_{x_1x_1} \circ \gamma \circ \gamma_1^{-1}(x_1)|^2 + 2[x_2 - \gamma_2 \circ \gamma_1^{-1}(x_1)] \int_{\gamma_2 \circ \gamma_1^{-1}(x_1)}^{\gamma_2(0)} [\eta(f, w)]_{x_1}(x_1, \zeta_2)^2 d\zeta_2,$$

and

$$|w_{x_1x_1} \circ \gamma \circ \gamma_1^{-1}(x_1)|^2 \leq 2|w_{x_1x_1}(x)|^2 + 2[x_2 - \gamma_2 \circ \gamma_1^{-1}(x_1)] \int_{\gamma_2 \circ \gamma_1^{-1}(x_1)}^{\gamma_2(0)} [\eta(f, w)]_{x_1}(x_1, \zeta_2)^2 d\zeta_2,$$

respectively. Integrating thee above two inequalities, respectively, first with respect to $x_2$ over $(\gamma_2 \circ \gamma_1^{-1}(x_1), \gamma_2(t_0))$ and then with respect to $x_1$ over $(\gamma_1(0), \gamma_1(t_0))$ respectively, we obtain

$$\|w_{x_1x_1}\|^2_{L^2} \leq 2\sigma_{12} \int_0^{t_0} |w_{x_1x_1} \circ \gamma(t)|^2 dt + C_T \lambda^2 (\|f\|^2_{W^{1,2}} + \|w\|^2_{W^{2,2}})$$

and

$$\sigma_{11} \int_0^{t_0} |w_{x_1x_1} \circ \gamma(t)|^2 dt \leq 2\|w_{x_1x_1}\|^2_{L^2} + C_T \lambda^2 (\|f\|^2_{W^{1,2}} + \|w\|^2_{W^{2,2}}),$$

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where
\[ \sigma_{11} = \inf_{t \in (0, t_0)} \left[ \gamma_2(0) - \gamma_2(t) \right] \gamma'_1(t) / t, \quad \sigma_{12} = \sup_{t \in (0, t_0)} \left[ \gamma_2(0) - \gamma_2(t) \right] \gamma'_1(t) / t, \]
\[ \lambda = \sup_{|x| \leq 2T} (1 + f_0^2 + |\nabla f_0|^2 + |X|^2 + |\nabla X|^2). \]

By similar arguments, we establish the following
\[ \|w_{x_2} x_2\|^2 \leq 2\sigma_{22} \int_0^{t_0} \|w_{x_2} x_2 \circ \gamma(t)\|^2 dt + C_T \lambda^2 (\|f\|_{W, 1, 2}^2 + \|w\|_{W, 2, 2}^2), \]
\[ \sigma_{21} \int_0^{t_0} \|w_{x_2} x_2 \circ \gamma(t)\|^2 dt \leq 2\|w_{x_2} x_2\|^2 + C_T \lambda^2 (\|f\|_{W, 1, 2}^2 + \|w\|_{W, 2, 2}^2), \]
where
\[ \sigma_{21} = \inf_{t \in (0, t_0)} \left[ \gamma_1(0) - \gamma_1(t) \right] \left[ -\gamma'_2(t) \right]/(t_0 - t), \quad \sigma_{22} = \sup_{t \in (0, t_0)} \left[ \gamma_1(0) - \gamma_1(t) \right] \left[ -\gamma'_2(t) \right]/(t_0 - t). \]

**Step 2** As in Step 1, we have
\[ \|w_{x_1} x_1\|^2 \leq 2\sigma_{12} \int_0^{t_0} \|w_{x_1} \circ \gamma(t)\|^2 dt + C_T \lambda^2 (\|f\|_{L, 2}^2 + \|w\|_{W, 1, 2}^2) \]
\[ \leq 2\sigma_{12} t_0 \|w_{x_1} \circ \gamma\|^2_{L, 2(0, t_0)} + C_T \lambda^2 (\|f\|_{L, 2}^2 + \|w\|_{W, 1, 2}^2), \]
\[ \sigma_{11} \int_0^{t_0} \|w_{x_1} \circ \gamma(t)\|^2 dt \leq 2\|w_{x_1}\|^2_{L, 2} + C_T \lambda^2 (\|f\|_{L, 2}^2 + \|w\|_{W, 1, 2}^2). \quad (3.39) \]

In addition, since
\[ w_{x_1} \circ \gamma \circ \gamma^{-1}_2 (x_2) = w_{x_1} (x) - \int_{\gamma_1 \circ \gamma^{-1}_2 (x_2)} w_{x_1 x_1} (\zeta_1, x_2) d\zeta_1 \quad \text{for} \quad x_2 \in (\gamma_2(t_0), \gamma_2(0)), \]

it follows that
\[ \sigma_{21} \int_0^{t_0} \|w_{x_1} \circ \gamma(t)\|^2 (t_0 - t) dt \leq 2\|w_{x_1}\|^2_{L, 2} + \lambda^2 \|w_{x_1 x_1}\|^2_{L, 2}. \quad (3.40) \]

Combing (3.39) and (3.40), we have
\[ \min\{\sigma_{11}, \sigma_{21}\} \|w_{x_1} \circ \gamma\|^2_{L, 2(0, t_0)} \leq \frac{2}{t_0} (\sigma_{21} \int_0^{t_0/2} \|w_{x_1} \circ \gamma(t)\|^2 dt + \sigma_{11} \int_{t_0/2}^{t_0} \|w_{x_1} \circ \gamma(t)\|^2 dt) \]
\[ \leq \frac{2}{t_0} [4 + \lambda^2 (C_T + 1)] (\|f\|_{L, 2}^2 + \|w\|_{W, 2, 2}^2). \]

By a similar computation, we obtain
\[ \|w_{x_2} x_2\|^2 \leq 2\sigma_{12} t_0 \|w_{x_2} \circ \gamma\|^2_{L, 2(0, t_0)} + C_T \lambda^2 (\|f\|_{L, 2}^2 + \|w\|_{W, 1, 2}^2), \]
\[ \min\{\sigma_{11}, \sigma_{21}\} \|w_{x_2} \circ \gamma\|^2_{L, 2(0, t_0)} \leq \frac{2}{t_0} [4 + \lambda^2 (C_T + 1)] (\|f\|_{L, 2}^2 + \|w\|_{W, 2, 2}^2), \]

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\[ \|w\|_{L^2}^2 \leq 2 \sigma_{12} t_0 \|w \circ \gamma\|_{L^2(0, a)}^2 + \lambda^2 \|w_{x_2}\|_{L^2}^2, \]
\[ \min\{\sigma_{11}, \sigma_{21}\} \|w \circ \gamma\|_{L^2(0, t_0)}^2 \leq \frac{2}{t_0} [4 + \lambda^2 (C_T + 1)] \|w\|_{W^{1,2}}^2. \]

**Step 3** From Steps 1 and 2, we obtain
\[ [1 - (4C_T + 1) \lambda^2] \|w\|_{W^{2,2}}^2 \leq 2(\sigma_{12}(1 + t_0) + \sigma_{22}) \Gamma(\gamma, w) + (4C_T + 1) \lambda^2 \|f\|_{W^{1,2}}^2, \]
when \( \lambda \) is small, and
\[ \min\{\sigma_{11}, \sigma_{21}\} \Gamma(\gamma, w) \leq 2\{2 + C_T \lambda^2 + \frac{3}{t_0} [4 + (C_T + 1) \lambda^2]\}(\|f\|_{W^{1,2}}^2 + \|w\|_{W^{2,2}}^2), \]
respectively.

**Step 4** We have
\[ w_{x_1 x_1}(x_1, \gamma_2(0)) = w_{x_1 x_1} \circ \gamma \circ \gamma_1^{-1}(x_1) + \int_{\gamma_2 \circ \gamma_1^{-1}(x_1)} \eta(f, w)_{x_1}(x_1, \xi_2) d\xi_2, \]
which gives, by (3.9),
\[ \int_{\gamma_1(t_0)} \gamma_1(t) \int_{\gamma_1(t_0)} |w_{x_1 x_1}(x_1, \gamma_2(0))|^2(x_1 - \gamma_1(0)) dx_1 \leq 2 \int_0^{t_0} |w_{x_1 x_1} \circ \gamma(t)|^2[\gamma_1(t) - \gamma_1(0)] dt \]
\[ + C(\|f\|_{W^{1,2}}^2 + \|w\|_{W^{2,2}}^2) \leq C(\|f\|_{W^{1,2}}^2 + \Gamma(\gamma, w)). \]
A similar argument completes the proof.

A similar argument as in Step 4 of the proof of Lemma 3.3 yields

**Lemma 3.4** Let \( T > 0 \) be given. There is \( \varepsilon_T > 0 \) such that if \( |\gamma(0)| \leq T \) and \( \min\{\gamma_1(t_0) - \gamma_1(0), \gamma_2(0) - \gamma_2(t_0)\} < \varepsilon_T \), then for all solutions \( w \in W^{2,2}(E(\gamma)) \) to problem (3.1) satisfy
\[ c_1 \|w(\cdot, \gamma_2(0))\|_{W^{2,2}}^2 \leq \sum_{j=0}^{2} \|w_{x_1} \circ \gamma\|_{L^2}^2 + \|f\|_{W^{1,2}}^2 + \|w\|_{W^{2,2}}^2 \leq c_2(\|w(\cdot, \gamma_2(0))\|_{W^{2,2}}^2 + \|f\|_{W^{1,2}}^2 + \|w\|_{W^{2,2}}^2), \]
\[ c_1 \|w(\gamma_1(t_0), \cdot)\|_{W^{2,2}}^2 \leq \sum_{j=0}^{2} \|w_{x_2} \circ \gamma\|_{L^2}^2 + \|f\|_{W^{1,2}}^2 + \|w\|_{W^{2,2}}^2 \leq c_2(\|w(\gamma_1(t_0), \cdot)\|_{W^{2,2}}^2 + \|f\|_{W^{1,2}}^2 + \|w\|_{W^{2,2}}^2), \]
where \( 0 < c_1 < c_2 \) are some constants.

By a similar argument as for Lemma 3.1, we obtain the following lemmas 3.5 and 3.6.

**Lemma 3.5** Let \( T > 0 \) be given. There is \( \varepsilon_T > 0 \) such that if \( |z| \leq T \) and \( 0 < \max\{a, b\} < \varepsilon_T \), Propositions 3.4 and 3.5 hold.

**Lemma 3.6** Let \( T > 0 \) be given. There is \( \varepsilon_T > 0 \) such that if \( |\beta(0)| \leq T \) and \( 0 < \max\{\beta_1(t_0) - \beta_1(0), \beta_2(t_0) - \beta_2(0)\} < \varepsilon_T \), Propositions 3.7 and 3.8 hold.
By a similar argument as for Lemma 3.3, we have the following lemmas.

**Lemma 3.7** Let $T > 0$ be given. There is $\varepsilon_T > 0$ such that if $|z| \leq T$ and $0 < \max\{a, b\} < \varepsilon_T$, Proposition 3.6 holds.

**Lemma 3.8** Let $T > 0$ be given. There is $\varepsilon_T > 0$ such that if $|\beta(0)| \leq T$ and $\max\{\beta_1(t_0) - \beta_1(0), \beta_2(t_0) - \beta_2(0)\} < \varepsilon_T$, Proposition 3.9 holds.

Let $\Xi_1(\beta, \gamma)$ be given by (3.15). For $\varepsilon > 0$, set

$$ \Xi_1(\varepsilon) = \{ x \in \Xi_1(\beta, \gamma) | \gamma_1(0) < x_1 < \gamma_1(0) + \varepsilon \}. \quad (3.41) $$

**Lemma 3.9** Let all the assumptions in Proposition 3.10 hold. Let $T > 0$ be given. Then there is a $\varepsilon_T > 0$ such that if $|\gamma(0)| \leq T$ and $0 < \varepsilon < \varepsilon_T$, problem (3.1) admits a unique solution $w \in C^{k,1}(\Xi_1(\varepsilon))$ with the data (3.16) for $t \in [0, \beta^{-1}_1(\gamma_1(0) + \varepsilon)]$ and (3.17) for $t \in [0, \gamma^{-1}_1(\gamma_1(0) + \varepsilon)]$, respectively. Moreover, a similar estimate as in Proposition 3.10 holds.

**Proof** We define an operator $I$ over $C^{0,1}(\Xi_1(\varepsilon))$ by letting, for $w \in C^{0,1}(\Xi_1(\varepsilon))$ and $x \in \Xi_1(\varepsilon)$,

$$ I(w) = b(p, q_0, q_1)(x) + \int_{\mathcal{P}(x)} \eta(f, w)(\zeta) d\zeta, \quad (3.42) $$

where

$$ b(p, q_0, q_1)(x) = \begin{cases} \int_0^{\beta^{-1}_2(x_2)} p(t)\beta'_2(t) dt + \mathcal{B}(q_0, q_1)(x_1, \beta_2(0)), & \text{if } \beta_2(0) \leq x_2 < \beta_2 \circ \beta^{-1}_1(\gamma_1(0) + \varepsilon), \\ \mathcal{B}(q_0, q_1)(x), & \text{if } \gamma_2 \circ \gamma^{-1}_1(\gamma_1(0) + \varepsilon) < x_2 < \gamma_2(0), \end{cases} $$

$$ \mathcal{P}(x) = \begin{cases} \{ \zeta_1, \zeta_2 \} | \beta_1 \circ \beta^{-1}_2(\zeta_2) < \zeta_1 < x_1, \beta_2(0) < \zeta_2 < x_2 \} & \text{if } \beta_2(0) \leq x_2 < \beta_2 \circ \beta^{-1}_1(\gamma_1(0) + \varepsilon), \\ \cup E(x_1, \beta_2(0)) & \text{for } \beta_2(0) \leq x_2 < \beta_2 \circ \beta^{-1}_1(\gamma_1(0) + \varepsilon), \\ E(x) & \text{for } \gamma_2 \circ \gamma^{-1}_1(\gamma_1(0) + \varepsilon) < x_2 < \gamma_2(0), \end{cases} $$

where $\mathcal{B}(q_0, q_1)$ and $E(x)$ are given in (3.33) and (3.34), respectively.

Clearly, the regularity of $p \in C^{-1,1}$, $q_0 \in C^{0,1}$, and $q_1 \in C^{-1,1}$ assures that $b(p, q_0, q_1) \in C^{0,1}(\Xi_1(\varepsilon))$ and thus $I(w) \in C^{0,1}(\Xi_1(\varepsilon))$ when $w \in C^{0,1}(\Xi_1(\varepsilon))$.

Let us assume that $p \in C^{0,1}$, $q_0 \in C^{1,1}$, and $q_1 \in C^{0,1}$. It is easy to check that the 1th order order compatibility conditions,

$$ |\gamma'(0)|^2 p(0) = \gamma'_2(0) q_0'(0) - \gamma'_1(0) q_1(0), $$

imply that $b(p, q_0, q_1) \in C^{1,1}$. Thus $I(w) \in C^{1,1}(\Xi_1(\varepsilon))$ if $w \in C^{1,1}(\Xi_1(\varepsilon))$.

Thus, the map $I$ maps $C^{k,1}(\Xi_1(\varepsilon))$ into themselves for $k = 0$, and 1, respectively. As in the proof of lemma 3.1, there is a $\varepsilon_T > 0$ such that when $0 < \varepsilon < \varepsilon_T$, $I$ is contractive. Thus, the lemma holds for $k = 0, 1$. 

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Step 2. We proceed by induction in $k \geq 1$. Let $w \in C^{k,1}(\overline{\Omega}(\varepsilon))$ be the solution to (3.1) with the data (3.16) and (3.17). From Lemmas 3.1, 3.5, and 3.6, the formula (3.19) holds. Thus, we have

$$
\partial_{x_1}^i \partial_{x_2}^j v(x_1, \beta_2(0)) = \partial_{x_1}^i \partial_{x_2}^j u(x_1, \beta_2(0)) \quad \text{for } \beta_1(0) \leq x_1 \leq \beta_1(0) + \varepsilon,
$$

(3.43)

for $0 \leq i + j \leq k$.

To complete the proof, we need to show that if the $k+1$th order compatibility conditions hold, then (3.43) are true with $i + j = k + 1$. Since $v(x_1, \beta_2(0)) = u(x_1, \beta_2(0))$ for all $x_1 \in [\beta_1(0), \beta_1() + \varepsilon]$, it follows that

$$
\partial_{x_1}^{k+1} v(x_1, \beta_2(0)) = \partial_{x_1}^{k+1} u(x_1, \beta_2(0)) \quad \text{for all } x_1 \in [\beta_1(0), \beta_1(0) + \varepsilon].
$$

Let $i + j = k + 1$ with $j \geq 1$. If $i \geq 1$, then $j = k + 1 - i \leq k$ and, by the induction assumptions,

$$
\partial_{x_2}^j v(x_1, \beta_2(0)) = \partial_{x_2}^j u(x_1, \beta_2(0)) \quad \text{for all } x_1 \in [\beta_1(0), \beta_1(0) + \varepsilon],
$$

which yield

$$
\partial_{x_1}^i \partial_{x_2}^j v(x_1, \beta_2(0)) = \partial_{x_1}^i \partial_{x_2}^j u(x_1, \beta_2(0)) \quad \text{for all } x_1 \in [\beta_1(0), \beta_1(0) + \varepsilon].
$$

(3.44)

To complete the induction, it remains to check the case of $i = 0$ and $j = k + 1$.

Using (3.1), we have

$$
\left( \partial_{x_2}^{k+1} v(x_1, \beta_2(0)) \right)_{x_1} = \partial_{x_2}^k (v_{x_1x_2})(x_1, \beta_2(0)) = \partial_{x_2}^k [f + f_0 v + X_1 v_{x_1} + X_2 v_{x_2}](x_1, \beta_2(0)) = X_2(x_1, \beta_2(0)) \partial_{x_2}^k v(x_1, \beta_2(0)) + \partial_{x_2}^k [f + f_0 v + X_1 v_{x_1}](x_1, \beta_2(0))
$$

$$
+ \sum_{i=1}^{k} C_i^j \partial_{x_2}^i X_2 \partial_{x_2}^{k-i+1} v(x_1, \beta_2(0)).
$$

(3.45)

Let

$$
\rho(x_1) = \partial_{x_2}^k [f + f_0 v + X_1 v_{x_1}](x_1, \beta_2(0)) + \sum_{i=1}^{k} C_i^j \partial_{x_2}^i X_2 \partial_{x_2}^{k-i+1} v(x_1, \beta_2(0))
$$

for $x_1 \in [\beta_1(0), \beta_1(0) + \varepsilon]$. Thus $\tau(x_1) = \partial_{x_2}^{k+1} v(x_1, \beta_2(0))$ is the solution to problem

$$
\begin{cases}
\tau'(x_1) = X_2(x_1, \beta_2(0)) \tau(x_1) + \rho(x_1) \quad \text{for } x_1 \in (\beta_1(0), \beta_1(0) + \varepsilon), \\
\tau(\beta_1(0)) = \partial_{x_2}^{k+1} v(\beta(0)).
\end{cases}
$$

(3.46)

Moreover, the induction assumptions, $w \in C^{k,1}(\overline{\Omega}(\beta, \gamma))$, yield

$$
\langle \nabla^i v_{x_2}(z), \beta^{(j_1)}(0) \otimes \cdots \beta^{(j_l)}(0) \rangle = \langle \nabla^i u_{x_2}(z), \beta^{(j_1)}(0) \otimes \cdots \beta^{(j_l)}(0) \rangle
$$

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for $j_1 + \cdots + j_i = l$, $1 \leq i \leq l - 1$, and $1 \leq l \leq k$. Then the $k + 1$th order compatibility conditions imply
\[
\langle \nabla^k v_{x_2} \circ \beta(0), \dot{\beta}(0) \otimes \cdots \otimes \dot{\beta}(0) \rangle = \langle \nabla^k u_{x_2} \circ \gamma(0), \dot{\beta}(0) \otimes \cdots \otimes \dot{\beta}(0) \rangle.
\]
Using (3.44) and $\beta'_2(0) > 0$, we obtain
\[
\partial^{k+1}_{x_2} u \circ \gamma(0) = \partial^{k+1}_{x_2} v \circ \beta(0).
\]
In addition, it follows from the induction assumptions and (3.44) that
\[
\rho(x_1) = \partial^k_{x_2} [f + f_0 u + X_1 u_{x_1}](x_1, \beta_2(0)) + \sum_{i=1}^k C_i^k \partial^i_{x_2} X_2 \partial^{k-i+1}_{x_2} u(x_1, \beta_2(0)),
\]
for $x_1 \in [\beta_1(0), \beta_1(0) + \varepsilon]$. By a similar computation as in (3.45), $\partial^{k+1}_{x_2} u(x_1, \beta_2(0))$ is also a solution to problem (3.46). The uniqueness of solutions of problem (3.46) yields
\[
\partial^{k+1}_{x_2} v(x_1, \beta_2(0)) = \partial^{k+1}_{x_2} u(x_1, \beta_2(0)), \quad x_1 \in [\beta_1(0), \beta_1(0) + \varepsilon].
\]
The induction is complete.

The similar estimates in as Proposition 3.10 follow from an argument as in the proof of Lemma 3.1.

**Lemma 3.10** Let $\Xi_1(\varepsilon)$ be given in (3.41). Let $T \geq |\beta(0)|$ be given. Then there exist a $\varepsilon_T > 0$ and $0 < c_1 < c_2$ such that for all solutions $w \in W^{2,2}(\Xi_1(\varepsilon))$ to problem (3.1) and $0 < \varepsilon < \varepsilon_T$,
\[
c_1 [\Gamma(\gamma|_{0^\varepsilon}, w) + \Gamma_{I}(\beta|_{0^\varepsilon}, w)] \leq \|w\|_{W^{2,2}}^2 + \|f\|_{W^{1,2}}^2 \leq c_2 [\Gamma(\gamma|_{0^\varepsilon}, w) + \Gamma_{I}(\beta|_{0^\varepsilon}, w) + \|f\|_{W^{1,2}}^2],
\]
where $\Gamma(\gamma, w)$ and $\Gamma_{I}(\beta, w)$ are given (3.8) and (3.27), respectively, and
\[
t_1 \varepsilon = \gamma_1^{-1}(\gamma_1(0) + \varepsilon), \quad t_0 \varepsilon = \beta_1^{-1}(\beta_1(0) + \varepsilon).
\]

**Proof Step 1** Let
\[
\Xi_1(\varepsilon) = P(\varepsilon) \cup E(\varepsilon),
\]
where
\[
P(\varepsilon) = \{ x \mid \beta_1 \circ \beta_2^{-1}(x_2) < x_1 < \beta_1(0) + \varepsilon, \beta_2(0) \leq x_2 < \beta_2(t_0 \varepsilon) \},
\]
\[
E(\varepsilon) = \{ x \mid \gamma_1 \circ \gamma_2^{-1}(x_2) < x_1 < \gamma_1(0) + \varepsilon, \gamma_2(t_1 \varepsilon) < x_2 < \gamma_2(0) \}.
\]

It follows from Lemma 3.3 that
\[
c_1 \Gamma(\gamma|_{0^\varepsilon}, w) \leq \|w\|_{W^{2,2}(E(\varepsilon))}^2 + \|f\|_{W^{1,2}(E(\varepsilon))}^2 \leq c_2 [\Gamma(\gamma|_{0^\varepsilon}, w) + \|f\|_{W^{1,2}(E(\varepsilon))}^2].
\]
A similar argument as in the proof of Lemma 3.3 also yields
\[ c_1 \Gamma(\beta_0^\ell, w) \leq \|w\|^2_{W^{2,2}(P(\xi))} + \|f\|^2_{W^{1,2}(P(\xi))} \leq c_2[\Gamma(\beta_0^\ell, w) + \|f\|^2_{W^{1,2}(P(\xi))}]. \]

**Step 2** Using (3.1), we have
\[ w_{x_1x_1} \circ \beta \circ \beta_1^{-1}(x_1) = w_{x_1x_1} \circ \gamma \circ \gamma_1^{-1}(x_1) + \int_{\gamma_2 \circ \gamma_1^{-1}(x_1)} [\eta(f, w)]_{x_1}(x_1, \zeta_2) d\zeta_2, \]
from which we obtain a \( \sigma > 0 \) such that
\[ \int_0^{T_{\max}} |w_{x_1x_1} \circ \beta(t)|^2 dt \leq \sigma \int_0^{T_{\max}} |w_{x_1x_1} \circ \gamma(t)|^2 dt + C_T \varepsilon^2 (\|w\|^2_{W^{2,2}} + \|f\|^2_{W^{1,2}}) \]
and
\[ \int_0^{T_{\max}} |w_{x_1x_1} \circ \gamma(t)|^2 dt \leq \sigma \int_0^{T_{\max}} |w_{x_1x_1} \circ \beta(t)|^2 dt + C_T \varepsilon^2 (\|w\|^2_{W^{2,2}} + \|f\|^2_{W^{1,2}}). \]
Similarly, we have
\[ \sum_{j=0}^1 \|\nabla w \circ \beta\|_{L^2(0,t_{\max})}^2 \leq \sigma \sum_{j=0}^1 \|\nabla w \circ \gamma\|_{L^2(0,t_{\max})}^2 + C_T \varepsilon \|w\|^2_{W^{2,2}}, \]
\[ \sum_{j=0}^1 \|\nabla w \circ \gamma\|_{L^2(0,t_{\max})}^2 \leq \sigma \sum_{j=0}^1 \|\nabla w \circ \beta\|_{L^2(0,t_{\max})}^2 + C_T \varepsilon \|w\|^2_{W^{2,2}}. \]

Finally, using those estimates in Step 1, we obtain the estimates of the lemma. We omit the details. \( \square \)

**Proof of Proposition 3.1.** We shall show that the assumptions \( |\gamma(0)| \leq T \) and \( \max\{\gamma_1(t_0) - \gamma_1(0), \gamma_2(t_0) - \gamma_2(0)\} < \varepsilon_T \) in Lemma 3.1 are unnecessary. Let \( T > 0 \) be given such that
\[ E(\gamma) \subset \{ x \in \mathbb{R}^2 \mid |x| \leq T \}. \]
Let \( \varepsilon_T > 0 \) be given such that Lemmas 3.1 and 3.5 hold. We divide the curve \( \gamma \) into \( m \) parts with the points \( \tau_0 = 0, \tau_0 < \tau_1 < \cdots < \tau_m = t_0 \) such that
\[ |\gamma(\tau_{i+1}) - \gamma(\tau_i)| = \frac{\varepsilon_T}{2}, \quad 0 \leq i \leq m - 2, \quad |\gamma(t_0) - \gamma(\tau_{m-1})| \leq \frac{\varepsilon_T}{2}. \]
For simplicity, we assume that \( m = 3 \). The other cases can be treated by a similar argument.

In the case of \( m = 3 \), we have
\[ \overline{E(\gamma)} = (\bigcup_{i=0}^2 \overline{E}_i) \cup (\bigcup_{i=1}^3 \overline{R}_i) \]
(3.47)
where
\[ E_i = \{ x \in E(\gamma) \mid \gamma_1(\tau_i) \leq x \leq \gamma_1(\tau_{i+1}), \gamma_2(\tau_{i+1}) \leq x \leq \gamma_2(\tau_i) \} \quad i = 0, 1, 2, \]

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\[ R_1 = [\gamma_1(\tau_1), \gamma_1(\tau_2)] \times [\gamma_2(\tau_1), \gamma_2(0)], \quad R_2 = [\gamma_1(\tau_2), \gamma_1(t_0)] \times [\gamma_2(\tau_2), \gamma_2(\tau_1)], \]
\[ R_3 = [\gamma_1(\tau_2), \gamma_1(t_0)] \times [\gamma_2(\tau_1), \gamma_2(0)]. \]

From Lemma 3.1, problem \((3.1)\) admits a unique solution \(w_i \in C^{k,1}(\overline{E_i})\) for each \(i = 0, 1, 2, 3\), respectively, with the corresponding data and the corresponding estimates. We define \(w \in C^{k,1}(\bigcup_{i=0}^{2} \overline{E_i})\) by
\[
w(x) = w_i(x) \quad \text{for} \quad x \in \overline{E_i} \quad \text{for} \quad i = 0, 1, 2.
\]

We extend the domain of \(w\) from \(\bigcup_{i=0}^{2} \overline{E_i}\) to \(\overline{E(\gamma)}\) by the following way. By Lemma 3.5, we define \(w \in C^{k,1}(\overline{R_i})\) to be the solution \(u_i \in C^{k,1}(\overline{R_i})\) to problem \((3.1)\) with the data
\[
u_i(\gamma_1(\tau_1), x_2) = w_{i-1}(\gamma_1(\tau_1), x_2) \quad \text{for} \quad x_2 \in [\gamma_2(\tau_1), \gamma_2(\tau_{i-1})],
\]
\[
u_i(x_1, \gamma_2(\tau_i)) = w_i(x_1, \gamma_2(\tau_i)) \quad \text{for} \quad x_1 \in [\gamma_1(\tau_i), \gamma_1(\tau_{i+1})],
\]
for \(i = 1, 2\), respectively. Then we extend \(w\) on \(C^{k,1}(\overline{R_3})\) to be the solution \(u_3\) of \((3.1)\) with the data
\[
u_3(\gamma_1(\tau_2), x_2) = u_1(\gamma_1(\tau_2), x_2) \quad \text{for} \quad x_2 \in [\gamma_2(\tau_1), \gamma_2(0)],
\]
\[
u_3(x_1, \gamma_2(\tau_2)) = u_2(x_1, \gamma_2(\tau_2)) \quad \text{for} \quad x_1 \in [\gamma_1(\tau_2), \gamma_1(t_0)].
\]

To complete the proof, we have to show that \(w\) is a \(C^{k,1}\) solution on all the connection segments between any two subregions above. Consider the subregion
\[
\overline{E} = \overline{E_0} \cup \overline{E_1} \cup \overline{R_1}.
\]
Since \(|\gamma(\tau_2) - \gamma(0)| \leq \varepsilon_T\), Lemma 3.1 insures that problem \((3.1)\) admits a unique solution \(\tilde{w} \in C^{k,1}(\overline{E})\) with the corresponding data. Then the uniqueness implies that \(w(x) = \tilde{w}(x)\) for \(x \in \overline{E}\). In particular, \(w\) is \(C^{k,1}\) on the segments \(\{(\gamma_1(\tau_1), x_2) \mid x_2 \in [\gamma_2(\tau_1), \gamma_2(0)]\}\) and \(\{(x_1, \gamma_2(\tau_1)) \mid x_1 \in [\gamma_1(\tau_1), \gamma_1(\tau_2)]\}\), respectively. By a similar argument, we show that \(w\) is also \(C^{k,1}\) on all the other segments.

The estimates in \((3.6)\) follow from the ones in Lemmas 3.1 and 3.2. \(\square\)

**Proofs of Propositions 3.3, 3.6, 3.9, 3.12, and 3.13** We cut the region \(E(\gamma)\) in subregions in the same way as for Proposition 3.1 to apply Lemmas 3.3 and 3.4 to those subregions to obtain the estimates \((3.9)\). Similarly, we have Propositions 3.6, 3.9, 3.12, and 3.13 from Lemmas 3.1–3.10, respectively. \(\square\)

**Proofs of Theorems 3.1, 3.2, and 3.3** Consider a decomposition of
\[
\Phi(\beta, \beta) = \Xi_1(\beta, \gamma) \cup E \cup P_2(\beta),
\]

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where
\[ \Xi_1(\beta, \gamma) = \{ x \mid \beta_1(0) < x_1 \leq \beta_1(t_0), \gamma_2 \circ \gamma_1^{-1}(x_1) < x_2 < \beta_2 \circ \beta_1^{-1}(x_1) \} \]
\[ \cup \{ x \mid \beta_1(t_0) < x_1 < -\gamma_2(0) + \beta_1(t_0) + \beta_2(t_0), \gamma_2 \circ \gamma_1^{-1}(x_1) < x_2 < \beta_2 \circ \beta_1^{-1}(x_1) \}, \]
\[ E = \{ x \mid \gamma_0(t_1) \leq x_1 < -\beta_1(t_0) + \beta_2(t_0), \hat{\beta}_2(t_0) \leq x_2 < -\gamma_0(t_1) + \beta_1(t_0) + \beta_2(t_0) \}, \]
\[ P_2(\hat{\beta}) = \{ x \mid \gamma_0(t_1) \leq x_1 < \hat{\beta}_1 \circ \hat{\beta}_2^{-1}(x_2), \gamma_2(t_1) < x_2 < \hat{\beta}_2(t_0) \}. \]

We apply Propositions 3.10 and 3.7 to \( \Xi_1(\beta, \gamma) \) and \( P_1(\hat{\beta}) \), respectively, to obtain the solution \( \Xi_1(\beta, \gamma) \cup P_2(\hat{\beta}) \) with corresponding data and then extend the domain of it to \( \Phi(\beta, \hat{\beta}) \) as in the proof of Proposition 3.1.

A similar argument proves Theorems 3.2 and 3.3. \( \square \)

4 Solvability for Hyperbolic Surfaces

Let \( M \subset \mathbb{R}^3 \) be a hyperbolic surface with the normal field \( \vec{n} \) and let \( \Omega \subset M \) be a non-characteristic region, where
\[ \Omega = \{ \alpha(t, s) \mid (t, s) \in (0, a) \times (0, b) \}. \]
We consider solvability of problem under appropriate part boundary data
\[ \langle D^2w, Q^*\Pi \rangle = f + f_0w + X(w) \quad \text{for} \quad x \in \Omega, \quad (4.1) \]
where \( f_0 \) is a function on \( M \) and \( X \in X(M) \) is a vector field on \( M \). Clearly, equation (2.25) takes the form of (4.1).

Let the linear operator \( Q : M_x \to M_x \) be given in (2.2) for \( x \in M \). Recall that the shape operator \( \nabla \vec{n} : M_x \to M_x \) is defined by \( \nabla \vec{n}X = \nabla_X \vec{n}(x) \) for \( X \in M_x \). Now we define operators \( T_i : M_x \to M_x \) by
\[ T_iX = \frac{1}{2} \left[ X + (-1)^i \varrho(X)Q \nabla \vec{n}X \right] \quad \text{for} \quad X \in M_x, \quad i = 1, 2, \quad (4.2) \]
where
\[ \varrho(X) = \frac{1}{\sqrt{-K}} \text{sign} \Pi(X, X) \quad (4.3) \]
and \( \text{sign} \) is the sign function.

We shall consider the part boundary data
\[ \langle Dw, T_1\alpha_s \rangle \circ \alpha(0, s) = p_1(s), \quad \langle Dw, T_2\alpha_s \rangle \circ \alpha(a, s) = p_2(s) \quad \text{for} \quad s \in (0, b), \quad (4.4) \]
\[ w \circ \alpha(t, 0) = q_0(t), \quad \frac{1}{\sqrt{2}} \langle Dw, (T_2 - T_1)\alpha_t \rangle \circ \alpha(t, 0) = q_1(t) \quad \text{for} \quad t \in (0, a). \quad (4.5) \]

To have a smooth solution, we need some kind of compatibility conditions as follows.
Let $A$ and $B$ be $k$th order and $m$th order tensor fields on $M$, respectively, with $k \geq m$. We define $A(1-)B$ to be a $(k-m)$th order tensor field by
\[ A(1-)B(X_1, \ldots, X_{k-m}) = \langle 1_{X_{k-m}} \cdots 1_{X_1} A, B \rangle(x) \text{ for } x \in M, \]
where $X_1, \ldots, X_{k-m}$ are vector fields on $M$.

For convenience, we assume that
\[ |\alpha_t(t, 0)| = 1 \text{ for } t \in [0, a]. \]
Then $Q\alpha_t$, $\alpha_t$ is an orthonormal basis of $M_{\alpha(t, 0)}$ with the positive orientation for all $t \in [0, a]$ and
\[ Q\nabla n\alpha_t = \Pi(\alpha_t, \alpha_t)Q\alpha_t - \Pi(\alpha_t, Q\alpha_t)\alpha_t \text{ for } t \in [0, a]. \]

Let $k \geq 1$ be an integer. First, we assume that $w$ is a $C^{k,1}$ solution to (4.1) in a neighborhood of the curve $\alpha(t, 0)$ with the data (4.5). Then
\[ Dw(\alpha(t, 0)) = B_1(t)q_0(t) + C_0(t)q_1(t) \text{ for } t \in [0, a], \]
where
\[ B_1(t) = [\alpha_t + \frac{\Pi(\alpha_t, Q\alpha_t)}{\Pi(\alpha_t, \alpha_t)} Q\alpha_t], \quad C_0(t) = \frac{1}{g(\alpha_t)\Pi(\alpha_t, \alpha_t)} Q\alpha_t, \]
are vector fields along the curve $\alpha(t, 0)$, from which we obtain
\[ D_\alpha Dw(\alpha(t, 0)) = D_\alpha B_1 q_0(t) + B_1(t)q_0''(t) + D_\alpha C_0 q_1(t) + C_0(t)q_1'(t). \]

Using (4.1) and the above formula, we compute along the curve $\alpha(t, 0)$ to have
\[ D^2w(Q\alpha_t, Q\alpha_t)\Pi(\alpha_t, \alpha_t) = f + f_0 w + \langle Dw, X \rangle - \langle D_\alpha Dw, \alpha_t \rangle \Pi(Q\alpha_t, Q\alpha_t) \]
\[ + 2\langle D_\alpha Dw, Q\alpha_t \rangle \Pi(Q\alpha_t, \alpha_t) \]
\[ = f + f_0 q_0(t) + [(X, B_1(t)) + \langle D_\alpha B_1, Z(t) \rangle]q_0'(t) + \langle B_1(t), Z(t) \rangle q_0''(t) \]
\[ + [(X, C_0(t)) + \langle D_\alpha C_0, Z(t) \rangle]q_1(t) + \langle C_0(t), Z(t) \rangle q_1'(t) \text{ for } t \in [0, a], \]
where
\[ Z(t) = 2\Pi(Q\alpha_t, \alpha_t)Q\alpha_t - \Pi(Q\alpha_t, Q\alpha_t)\alpha_t. \]

Since $\Pi(\alpha_t, \alpha_t) \neq 0$ for all $t \in [0, a]$, we have obtained two order tensor fields, $A^2(t)$, $B^2_i(t)$, and $C^2_i(t)$, that are given by $f_0$, $X$, $\Pi$, $Q\alpha_t$, $\alpha_t$, and their differentials, such that
\[ D^2w(\alpha(t, 0)) = A^2(t) f + \sum_{i=0}^2 B^2_i(t)q_0^{(i)}(t) + \sum_{i=0}^1 C^2_i(t)q_1^{(i)}(t) \text{ for } t \in [0, a]. \]
By repeating the above procedure, we obtain \((k + i)\)th order tensors fields \(A_i^{k+i}(t)\), and \(k\)th order tensor fields \(B_i^k(t), C_i^k(t)\), such that

\[
D^k w(\alpha(t, 0)) = Q_k(q_0, q_1, f)(t) \quad \text{for} \quad t \in [0, a],
\]

where

\[
Q_k(q_0, q_1, f)(t) = \sum_{i=0}^{k-2} A_i^{k+i}(t)(1-)D^i f(\alpha(t, 0)) + \sum_{i=0}^{k} B_i^k(t)q_0^{(i)}(t) + \sum_{i=0}^{k-1} C_i^k(t)q_1^{(i)}(t) \tag{4.8}
\]

for \(t \in [0, a]\) and \(k \geq 2\), where \(“(1-)”\) is defined in (4.6).

**Definition 4.1** Let \(q_0\) be of class \(C^{k,1}\), and \(p_1, p_2, q_1, f\) of class \(C^{k-1,1}\) to be said to satisfy the \(k\)th order compatibility conditions at \(\alpha(0, 0)\) and \(\alpha(a, 0)\) if

\[
p_j(t_j) = \langle B_1(t_j), T_1 \alpha_s \rangle q_0^{(i)}(t_j) + \langle C_0(t_j), T_1 \alpha_s \rangle q_1(t_j), \tag{4.9}
\]

\[
p_j^{(i)}(0) = \langle Q_i(q_0, q_1, f)(t_j), \dot{\gamma}_j(0) \otimes \cdots \otimes \dot{\gamma}_j(0) \rangle + \sum_{j_1+\cdots+j_l=t, 1 \leq l \leq k-1} a_{j_1\cdots j_l} \langle Q_i(q_0, q_1, f)(t_j), \dot{\gamma}_j^{(j_1)}(0) \otimes \cdots \otimes \dot{\gamma}_j^{(j_l)}(0) \rangle \tag{4.10}
\]

for \(1 \leq l \leq k-1\), where \(a_{j_1\cdots j_l}\) are positive integers given in (3.25), \(j = 1, 2\), \(\gamma_1(s) = \alpha(0, s)\), \(\gamma_2(s) = \alpha(a, s)\), \(t_1 = 0\), and \(t_2 = a\).

Our main task in this section is to establish the following.

**Theorem 4.1** Let \(\Omega\) be a noncharacteristic region of class \(C^{m+2,1}\) and let \(f_0\) and \(X\) be of class \(C^{m-1,1}\). Let \(q_0\) be of class \(C^{m,1}\), and \(p_1, p_2, q_1, f\) be of \(C^{m-1,1}\), respectively. If \(m \geq 1\), we assume that the \(m\)th compatibility conditions holds. Then there is a unique solution \(w \in C^{m,1}(\Omega)\) to problem (4.1) with the data (4.4) and (4.5). Moreover, there is \(C > 0\), independent of \(w\), such that

\[
\|w\|_{C^{m,1}(\Omega)} \leq C(\|q_1\|_{C^{m-1,1}[0,a]} + \|q_0\|_{C^{m,1}[0,a]} + \|p_1\|_{C^{m-1,1}[0,b]} + \|p_2\|_{C^{m-1,1}[0,b]} + \|f\|_{C^{m-1,1}(\Omega)}). \tag{4.11}
\]

**Remark 4.1** If \(p_1, p_2 \in C^{m-1,1}_0(0, b)\), \(q_0 \in C^{m,1}_0(0, a)\), \(q_1 \in C^{m-1,1}_0(0, a)\), and \(f \in C^{m-1,1}_0(\Omega)\) for an integer \(m \geq 0\), then the \(m\)th order compatibility conditions are clearly true.

**Theorem 4.2** Let \(\Omega\) be a noncharacteristic region of class \(C^{2,1}\) and let \(f_0\) and \(X\) be of class \(C^{0,1}\). Let \(q_0\) be of class \(W^{2,2}\), and \(p_1, p_2, q_1, f\) of class \(W^{1,2}\) to satisfy the 1th order compatibility conditions. Then there is a unique solution \(w \in W^{2,2}(\Omega)\) to problem
with the data \((4.4)\) and \((4.5)\). Moreover, there is \(C > 0\), in dependent of solution \(w\), such that

\[
\|w\|_{W^{2,2}(\Omega)}^2 \leq C (\|q_0\|_{W^{2,2}(0,a)}^2 + \|q_1\|_{W^{1,2}(0,a)}^2 + \|p_1\|_{W^{1,2}(0,b)}^2 + \|p_2\|_{W^{1,2}(0,b)}^2 + \|f\|_{W^{1,2}(\Omega)}^2).
\]  

(4.12)

We define

\[
\Gamma(\Omega, w) = \int_{a}^{b} (|p'_1(s)|^2 + |p'_2(s)|^2)(b - s)ds + \Gamma(\alpha(\cdot, 0), w),
\]

where \(p_1, p_2\) are given in \((4.4)\), and

\[
\Gamma(\alpha(\cdot, 0), w) = \sum_{j=0}^{1} \|\nabla^j w \circ \alpha(\cdot, 0)\|_{L^2(0,a)}^2 + \int_{a}^{b} \left[D^2w(T_1^t_1, T_2^t_1)\right]^2t + \left[D^2w(T_2^t_2, T_2^t_2)\right]^2(a - t)|dt.
\]

(4.13)

**Theorem 4.3** Let \(\Omega\) be a noncharacteristic region of class \(C^{2,1}\) and let \(f_0\) and \(X\) be of class \(C^{0,1}\). Then there are \(0 < c_1 < c_2\) such that for all solutions \(w \in W^{2,2}(\Omega)\) to problem \(4.1\)

\[
c_1 \Gamma(\Omega, w) \leq \|w\|_{W^{2,2}(\Omega)}^2 + \|f\|_{W^{1,2}(\Omega)}^2 \leq c_2 (\|f\|_{W^{1,2}(\Omega)}^2 + \Gamma(\Omega, w)).
\]

(4.14)

Next, we assume that \(f = 0\) to consider problem

\[
\langle D^2w, Q^*\Pi \rangle = f_0w + X(w) \quad \text{for} \quad x \in \Omega.
\]

(4.15)

Denote by \(\Upsilon(\Omega)\) all the solutions \(w \in W^{2,2}(\Omega)\) to problem \(4.15\). For \(w \in \Upsilon(\Omega)\), we let

\[
\Gamma(w) = \int_{0}^{b} (|p'_1(s)|^2 + |p'_2(s)|^2)ds + \|q_0\|_{W^{2,2}(0,a)}^2 + \|q_1\|_{W^{1,2}(0,a)}^2,
\]

where \(p_1, p_2, q_0,\) and \(q_1\) are given in \((4.4)\) and \((4.5)\), respectively. We define

\[
\mathcal{H}(\Omega) = \{ w \in \Upsilon(\Omega) \text{ with the 1th order compatibility conditions } | \Gamma(w) < \infty \ \}.
\]

**Theorem 4.4** Let \(\Omega\) be a noncharacteristic region of class \(C^{2,1}\) and \(X\) of class \(C^{0,1}\). For each \(w \in \Upsilon(\Omega)\), there exists a sequence \(w_n \in \mathcal{H}(\Omega)\) such that

\[
\lim_{n \to \infty} \|w_n - w\|_{W^{2,2}(\Omega)} = 0.
\]

The remains of this section is devoted to the proofs of Theorems 4.1-4.4.
We shall solve (4.1) locally in asymptotic coordinate systems and then paste the local solutions together. A chart \( \psi(p) = (x_1, x_2) \) on \( M \) is said to be an *asymptotic coordinate system* if

\[
\Pi(\partial x_1, \partial x_1) = \Pi(\partial x_2, \partial x_2) = 0.
\]  

(4.16)

If \( M \) is hyperbolic, an asymptotic coordinate system exists locally([10]).

In an asymptotic coordinate system, equation (4.1) takes a normal form. We have the following.

**Proposition 4.1** Let \( M \) be a hyperbolic orientated surface and let \( \psi(p) = (x_1, x_2) : U(\subset M) \to \mathbb{R}^2 \) be an asymptotic coordinate system on \( M \) with the positive orientation.

Then

\[
\langle D^2 w, Q^* \Pi \rangle = \pm 2 \sqrt{-\kappa} \det G_{w,x_1x_2}(x) + \text{the first order terms},
\]

(4.17)

where \( w(x) = w \circ \psi^{-1}(x) \) and the sign takes \(-\) if \( \Pi(\partial x_1, \partial x_2) > 0 \) and \(+\) if \( \Pi(\partial x_1, \partial x_2) < 0 \), respectively, and

\[
G(p) = \begin{pmatrix}
|\partial x_1|^2 & \langle \partial x_1, \partial x_2 \rangle \\
\langle \partial x_1, \partial x_2 \rangle & |\partial x_2|^2
\end{pmatrix}
\]

for \( p \in U \).

**Proof** Let \( p \in U \) be fixed. Let \( \alpha_i = (\alpha_{i1}, \alpha_{i2})^T \in \mathbb{R}^2 \) be such that

\[
(\alpha_1, \alpha_2) \in \text{SO}(2), \quad \det(\alpha_1, \alpha_2) = 1, \quad G(p)\alpha_i = \eta_i \alpha_i \quad \text{for} \quad i = 1, 2,
\]

where \( \eta_i > 0 \) are the eigenvalues of the matrix \( G(p) \). Set

\[
E_i = \alpha_{i1} \partial x_1 + \alpha_{i2} \partial x_2 \quad \text{for} \quad i = 1, 2.
\]

(4.18)

Since

\[
\langle E_i, E_j \rangle = \alpha_i^T G(p) \alpha_j = \eta_i \delta_{ij} \quad \text{for} \quad 1 \leq i, j \leq 2,
\]

\[
\frac{E_1}{\sqrt{\eta_1}}, \frac{E_2}{\sqrt{\eta_2}}
\]

forms an orthonormal basis of \( M_p \). Moreover, \( \frac{E_1}{\sqrt{\eta_1}}, \frac{E_2}{\sqrt{\eta_2}} \) is of the positive orientation due to

\[
\det\left(\frac{E_1}{\sqrt{\eta_1}}, \frac{E_2}{\sqrt{\eta_2}}, \vec{n}\right) = \det\left(\langle \partial x_1, \partial x_2, \vec{n}\rangle \begin{pmatrix}
\alpha_{11} & \alpha_{12} & 0 \\
\alpha_{21} & \alpha_{22} & 0 \\
0 & 0 & 1
\end{pmatrix} \right) = \det(\partial x_1, \partial x_2, \vec{n}) \frac{1}{\sqrt{\eta_1 \eta_2}} = 1.
\]

It follows from (2.2) that

\[
Q \frac{E_1}{\sqrt{\eta_1}} = -\frac{E_2}{\sqrt{\eta_2}}, \quad Q \frac{E_2}{\sqrt{\eta_2}} = \frac{E_1}{\sqrt{\eta_1}}.
\]
Using the above relations and the formulas (4.16), we have at $p$
\[
\eta_1\eta_2(D^2w, Q^\Pi) = D^2w(E_1, E_1)\Pi(E_2, E_2) - 2D^2w(E_1, E_2)\Pi(E_1, E_2)
+ D^2w(E_2, E_2)\Pi(E_1, E_1)
= 2(\alpha_{21}\alpha_{22}D^2w(E_1, E_1) - (\alpha_{11}\alpha_{22} + \alpha_{12}\alpha_{21})D^2w(E_1, E_2)
+ \alpha_{11}\alpha_{12}D^2w(E_2, E_2))\Pi(\partial x_1, \partial x_2)
= 2D^2w(\alpha_{21}E_1 - \alpha_{11}E_2, \alpha_{22}E_1 - \alpha_{12}E_2)\Pi(\partial x_1, \partial x_2)
= -2(\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21})^2D^2w(\partial x_2, \partial x_2)\Pi(\partial x_1, \partial x_2)
= -2[w_{x_1x_2} - D_{\partial x_1, \partial x_2}(w)]\Pi(\partial x_1, \partial x_2),
\] (4.19)
where the formula
\[
\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21} = \det(\alpha_1, \alpha_2) = 1,
\]
has been used.

(4.17) follows from (4.19) since $\kappa = -\frac{\Pi^2(\partial x_1, \partial x_2)}{\eta_1\eta_2}$. \qed

**Lemma 4.1** There is a $\sigma_0 > 0$ such that, for all $p \in \overline{M}$, there exist asymptotic coordinate systems $\psi : B(p, \sigma_0) \to \mathbb{R}^2$ with $\psi(p) = (0, 0)$, where $B(p, \sigma_0)$ is the geodesic plate in $M$ centered at $p$ with radius $\sigma_0$.

**Proof.** For $p \in \overline{M}$, let $\sigma(p)$ denote the least upper bound of the radii $\sigma$ for which an asymptotic systems $\psi = x : B(p, \sigma) \to \mathbb{R}^2$ with $\psi(p) = (0, 0)$ exists. From the existence of local asymptotic coordinate systems, $\sigma(p) > 0$ for all $p \in \overline{M}$. Let $p, q \in \overline{M}$, and $q \in B(p, \sigma(p))$. Let
\[
\sigma_1(q) = \inf_{z \in M, d(z, p) = \sigma(p)} d(q, z),
\]
where $d(\cdot, \cdot)$ is the distance function on $M \times M$ in the induced metric. Then $\sigma_1(q) > 0$ and $B(q, \sigma_1(q)) \subset B(p, \sigma(p))$, since $q \in B(p, \sigma(p))$.

For any $0 < \tilde{\sigma} < \sigma_1(q)$, $B(q, \tilde{\sigma}) \subset B(p, \sigma(p))$. Thus, there is a $0 < \sigma < \sigma(p)$ such that $B(q, \tilde{\sigma}) \subset B(p, \sigma)$. Let $\psi = x : B(p, \sigma) \to \mathbb{R}^2$ be an asymptotic system with $\psi(p) = (0, 0)$. Set $\hat{\psi}(z) = \psi(z) - \psi(q)$ for $z \in B(q, \tilde{\sigma})$. Then $\hat{\psi} : B(q, \tilde{\sigma}) \to \mathbb{R}^2$ is an asymptotic coordinate system with $\hat{\psi}(q) = (0, 0)$, that is,
\[
\sigma(q) \geq \sigma_1(q) \quad \text{for} \quad q \in B(p, \sigma(p)).
\]
Thus, $\sigma(p)$ is lower semi-continuous in $\overline{M}$ and $\min_{p \in \overline{M}} \sigma(p) > 0$ since $\overline{M}$ is compact. \qed

**Lemma 4.2** Let $\gamma : [0, a] \to M$ be a regular curve without self intersection points. Then there is a $\sigma_0 > 0$ such that, for all $p \in \{ \gamma(t) \mid t \in (0, a) \}$, $S(p, \sigma_0)$ has at most two intersection points with $\{ \gamma(t) \mid t \in [0, a] \}$, where $S(p, \sigma_0)$ is the geodesic circle centered at
If \( p = \gamma(0) \), or \( \gamma(a) \), then \( S(p, \sigma_0) \) has at most one intersection point with \( \{ \gamma(t) \mid t \in [0, a] \} \).

**Proof.** By contradiction. Let the claim in the lemma be not true. For each integer \( k \geq 1 \), there exists \( t_k < t_k^1 < t_k^2 \) (or \( t_k > t_k^1 > t_k^2 \)) in \([0, a]\) such that

\[
d(\gamma(t_k), \gamma(t_k^1)) = d(\gamma(t_k), \gamma(t_k^2)) = \frac{1}{k} \quad \text{for} \quad k \geq 1.
\]

(4.20)

We may assume that

\[
t_k \to t^0, \quad t_k^1 \to t^1, \quad t_k^2 \to t^2 \quad \text{as} \quad k \to \infty,
\]

for certain points \( t^0, t^1, t^2 \in [0, a] \). Then \( 0 \leq t^0 \leq t^1 \leq t^2 \leq a \) and

\[
\gamma(t^0) = \gamma(t^1) = \gamma(t^2).
\]

The assumption that the curve \( \gamma \) has no self intersection point implies that

\[
t^0 = t^1 = t^2.
\]

For \( k \geq 1 \), let

\[
f_k(t) = \frac{1}{2} \rho_k^2(\gamma(t)), \quad \text{for} \quad t \in [0, a],
\]

where \( \rho_k(p) = d(\gamma(t_k), p) \) for \( p \in M \). It follows from (4.20) that there is a \( \zeta_k \) with \( t_k < \zeta_k < t_k^2 \) such that

\[
f_k'(\zeta_k) = 0.
\]

On the other hand, the formula

\[
f_k'(t) = \rho_k(\gamma(t)) \langle D\rho_k(\gamma(t)), \dot{\gamma}(t) \rangle
\]

implies that \( f_k'(t_k) = 0 \). Thus, we obtain \( \eta_k \in (t_k, \zeta_k) \) such that

\[
f_k''(\eta_k) = 0 \quad \text{for} \quad k \geq 1.
\]

Since

\[
f_k''(t) = D(\rho_k D\rho_k)(\dot{\gamma}(t), \dot{\gamma}(t)) + \rho_k(\gamma(t)) \langle D\rho_k(\gamma(t)), D\dot{\gamma}(t) \dot{\gamma} \rangle,
\]

we have

\[
|\dot{\gamma}(t^0)|^2 = f_0''(t^0) = \lim_{k \to \infty} f_k''(\eta_k) = 0,
\]

which contradicts the regularity of the curve \( \gamma \), where

\[
f_0(t) = \frac{1}{2} d^2(\gamma(t^0), \gamma(t)) \quad \text{for} \quad t \in [0, a].
\]

\( \Box \)

We need the following.

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Proposition 4.2 \( (i) \) \( \det (Q \nabla \tilde{n} X, X, \tilde{n}(x)) = \Pi(X, X)(x) \) for \( X \in M_x, x \in M \).

\( (ii) \) \( \Pi(Q \nabla \tilde{n} X, Q \nabla \tilde{n} X) = \kappa \Pi(X, X) \) for \( X \in M_x, x \in M \).

**Proof** Let \( x \in M \) be given. Let \( e_1, e_2 \) be an orthonormal basis of \( M_x \) with the positive orientation such that

\[
\Pi(e_i, e_j)(x) = \lambda_i \delta_{ij} \quad \text{for} \quad 1 \leq i, j \leq 2. \quad (4.21)
\]

Then

\[
\det (Q \nabla \tilde{n} X, X, \tilde{n}) = \det (e_1, e_2, \tilde{n}) \left( \begin{array}{cc}
\lambda_2 \langle X, e_2 \rangle & \langle X, e_1 \rangle \langle X, e_2 \rangle \langle X, e_2 \rangle \\
-\lambda_1 \langle X, e_1 \rangle & \langle X, e_1 \rangle \langle X, e_1 \rangle \langle X, e_2 \rangle \\
0 & 0
\end{array} \right) = \Pi(X, X). \quad (4.22)
\]

In addition, using (4.21), we have

\[
\Pi(Q \nabla \tilde{n} X, Q \nabla \tilde{n} X) = \Pi\left( -\lambda_1 \langle X, e_1 \rangle e_2 + \lambda_2 \langle X, e_2 \rangle e_1, -\lambda_1 \langle X, e_1 \rangle e_2 + \lambda_2 \langle X, e_2 \rangle e_1 \right)
= \lambda_1^2 \lambda_2 \langle X, e_1 \rangle^2 + \lambda_2^2 \lambda_1 \langle X, e_2 \rangle^2 = \kappa \Pi(X, X).
\]

\( \Box \)

**Lemma 4.3** Let \( p_0 \in M \) and let \( B(p_0, \sigma) \) be the geodesic ball centered at \( p_0 \) with radius \( \sigma > 0 \). Let \( \gamma : [-a, a] \to B(p_0, \sigma) \) and \( \beta : [-b, b] \to B(p_0, \sigma) \) be two noncharacteristic curves of class \( C^1 \), respectively, with

\[
\gamma(0) = \beta(0) = p_0, \quad \Pi(\dot{\gamma}(0), \dot{\beta}(0)) = 0.
\]

Let \( \hat{\psi} : B(p_0, \sigma) \to \mathbb{R}^2 \) be an asymptotic coordinate system. Then there exists an asymptotic coordinate system \( \psi : B(p_0, \sigma) \to \mathbb{R}^2 \) with \( \psi(p_0) = (0, 0) \) such that

\[
\psi(\gamma(t)) = (t, -t) \quad \text{for} \quad t \in [-a, a], \quad (4.23)
\]

\[
\beta'_1(s) > 0, \quad \beta'_2(s) > 0 \quad \text{for} \quad s \in [-b, b], \quad (4.24)
\]

where \( \psi(\beta(s)) = (\beta_1(s), \beta_2(s)) \). Moreover, for \( X = X_1 \partial x_1 + X_2 \partial x_2 \) with \( \Pi(X, X) \neq 0 \), we have

\[
\varrho(X) Q \nabla \tilde{n} X = \begin{cases}
X_1 \partial x_1 - X_2 \partial x_2, & X_1 X_2 > 0, \\
-X_1 \partial x_1 + X_2 \partial x_2, & X_1 X_2 < 0,
\end{cases} \quad (4.25)
\]

where \( \varrho(X) \) is given in (4.3).

**Proof.** Let \( \hat{\psi}(p_0) = (0, 0) \) and

\[
\hat{\psi}(\gamma(t)) = (\gamma_1(t), \gamma_2(t)) \quad \text{for} \quad t \in [-a, a].
\]
Since $\gamma$ is noncharacteristic,

$$\Pi(\dot{\gamma}(t), \dot{\gamma}(t)) = \gamma'_1(t)\gamma'_2(t)\Pi(\partial x_1, \partial x_2) \neq 0 \quad \text{for} \quad t \in [-a, a].$$

Without loss of generality, we assume that

$$\gamma'_1(t) > 0, \quad \gamma'_2(t) < 0 \quad \text{for} \quad t \in [-a, a]. \quad (4.26)$$

We extend the domain $[-a, a]$ of $\gamma(t)$ to $\mathbb{R}$ such that

$$\lim_{t \to \pm \infty} \gamma_1(t) = \pm \infty, \quad \lim_{t \to \pm \infty} \gamma_2(t) = \mp \infty,$$

and the relations (4.26) hold for all $t \in \mathbb{R}$. Consider a diffeomorphism $\varphi(x) = y : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$\varphi(x) = (\gamma_1^{-1}(x_1), -\gamma_2^{-1}(x_2)) \quad \text{for} \quad x = (x_1, x_2) \in \mathbb{R}^2. \quad (4.27)$$

Then $\varphi \circ \hat{\psi} : B(p_0, \sigma) \to \mathbb{R}^2$ is an asymptotic coordinate system such that

$$\varphi \circ \hat{\psi}(\gamma(t)) = (t, -t) \quad \text{for} \quad t \in [-a, a]. \quad (4.28)$$

Let $\varphi \circ \hat{\psi}(\beta(s)) = (\beta_1(s), \beta_2(s))$. Since $\beta$ is noncharacteristic,

$$\beta'_1(s)\beta'_2(s) \neq 0 \quad \text{for} \quad s \in [-b, b].$$

In addition, the assumption $\Pi(\dot{\gamma}(0), \dot{\beta}(0)) = 0$ and the relation (4.28) imply that

$$0 = \Pi(\dot{\gamma}(0), \dot{\beta}(0)) = \Pi(\partial x_1 - \partial x_2, \beta'_1(0)\partial x_1 + \beta'_2(0)\partial x_2) = [\beta'_2(0) - \beta'_1(0)]\Pi(\partial x_1, \partial x_2),$$

that is, $\beta'_1(0) = \beta'_2(0)$. If $\beta'_1(0) > 0$, we let $\psi(p) = \varphi \circ \hat{\psi}(p)$ to have (4.24). If $\beta'_1(0) < 0$, we define instead of (4.27)

$$\varphi(x) = (\gamma_2^{-1}(x_2), -\gamma_1^{-1}(x_1)) \quad \text{for} \quad x = (x_1, x_2) \in \mathbb{R}^2.$$}

Thus (4.24) follows again.

Next, we prove (4.25). Let $Q\nabla \bar{n}X = Y_1\partial x_1 + Y_2\partial x_2$. Since $(Y_1 X_2 + Y_2 X_1)\Pi(\partial x_1, \partial x_1) = \Pi(Q\nabla \bar{n}X, X) = \langle Q\nabla \bar{n}X, \nabla \bar{n}X \rangle = 0$, we have

$$Q\nabla \bar{n}X = \sigma(X_1\partial x_1 - X_2\partial x_2),$$

where $\sigma$ is a function. Using Proposition 4.2 (ii), we obtain

$$\sigma^2 = -\kappa.$$ 

Thus (4.25) follows from Proposition 4.2 (i). \qed

Denote

$$\Omega(0, s_0) = \{ \alpha(t, s) | t \in (0, a), \ s \in (0, s_0) \} \quad \text{for} \quad s_0 \in [0, b]. \quad (4.29)$$

Then $\Omega = \Omega(0, b)$. 37
Lemma 4.4 Let the assumptions in Theorem 4.1 hold. Then there is a $0 < \omega \leq b$ such that problem (4.1) admits a unique solution $w \in C^{m,1}(\Omega(0,\omega))$ with the data (4.4) where $s \in [0,\omega]$, and (4.5) to satisfy
\[
\|w\|_{C^{m,1}(\Omega(0,\omega))} \leq C(\|p_1\|_{C^{m-1,1}[0,b]} + \|p_2\|_{C^{m-1,1}[0,b]} + \|q_0\|_{C^{m-1}[0,a]}
+ \|q_1\|_{C^{m-1,1}[0,a]} + \|f\|_{C^{m-1,1}(\Omega)}) \tag{4.30}
\]

Proof. Let $\sigma_0 > 0$ be given small such that the claims in Lemmas 4.1 and 4.2 hold, where $\gamma(t) = \alpha(t,0)$ in Lemma 4.2. We divide the curve $\alpha(t,0)$ into $m$ parts with the points $\lambda_i = \alpha(t_i,0)$ such that
\[
\lambda_0 = \alpha(0,0), \quad \lambda_m = \alpha(a,0), \quad d(\lambda_i, \lambda_{i+1}) = \frac{\sigma_0}{3}, \quad 0 \leq i \leq m - 2, \quad d(\lambda_{m-1}, \lambda_m) \leq \frac{\sigma_0}{3},
\]
where $t_0 = 0$, $t_1 > 0$, $t_2 > t_1$, ⋯, and $t_m = a > t_{m-1}$. For simplicity, we assume that $m = 3$. The other cases can be treated by a similar argument.

We shall construct a local solution in a neighborhood of $\alpha(t,0)$ by the following steps.

Step 1. Let $\delta_0 > 0$ be small such that
\[
\alpha(0,s) \in B(\lambda_0, \sigma_0) \quad \text{for} \quad s \in [0, \delta_0].
\]
From Lemma 4.3, there is asymptotic coordinate system $\psi_0(p) = x : B(\lambda_0, \sigma_0) \to \mathbb{R}^2$ with $\psi_0(\lambda_0) = (0,0)$ such that
\[
\psi_0(\alpha(t,0)) = (t, -t) \quad \text{for} \quad t \in [0, t_2], \tag{4.31}
\]
\[
\beta_0'(s) > 0, \quad \beta_0'(s) > 0 \quad \text{for all} \quad s \in [0, \delta_0],
\]
where $\psi_0(\beta_0(s)) = (\beta_0(s), \beta_0(s))$ and $\beta_0(s) = \alpha(0,s)$. Moreover, it follows from (4.25) that
\[
T_1\dot{\gamma}(t) = \partial x_1, \quad T_2\dot{\gamma}(t) = -\partial x_2 \quad \text{for} \quad t \in (0, t_2),
\]
\[
T_1\dot{\beta}_0(s) = \beta_0'(s)\partial x_2, \quad T_2\dot{\beta}_0(s) = \beta_0'(s)\partial x_1 \quad \text{for} \quad s \in (0, \delta_0).
\]
Let $0 < s_0 \leq \delta_0$ be given such that $\beta_0(s_0) + \beta_0(s_0) \leq t_2$. Let $\gamma_0(t) = \psi_0(\alpha(t,0))$. Set
\[
\Xi_1(\beta_0, \gamma_0) = \{ x \in \mathbb{R}^2 \mid 0 < x_1 \leq \beta_0(s_0), -x_1 \leq x_2 < \beta_0(\gamma_0^{-1}(x_1)) \}
\]
\[
\cup \{ x \in \mathbb{R}^2 \mid \beta_0(s_0) < x_1 < t_2, -x_1 < x_2 < -x_1 + \beta_0(s_0) + \beta_0(s_0) \}. \tag{4.32}
\]
From Proposition 4.1, solvability of problem (4.1) on $\Omega \cap \psi_0^{-1}(\Xi_1(\beta_0, \gamma_0))$ is equivalent to that of problem (3.1) over the region $\Xi_1(\beta_0, \gamma_0)$. Next, we consider the transfer of the boundary data under the chart $\psi_0$. The corresponding part data are
\[
w_{x_2} \circ \beta_0(s) = (Dw, T_1\dot{\beta}_0(s)) \circ \alpha(0,s) / \beta_0'(s) = p_1(s) / \beta_0'(s) \quad \text{for} \quad s \in [0, s_0],
\]
\[
w(t, -t) = w \circ \psi_0^{-1}(t, -t) = w(\alpha(t,0)) = q_0(t) \quad \text{for} \quad t \in [0, t_2].
\]

\[
\frac{\partial}{\partial \nu} w(t, -t) = \frac{1}{\sqrt{2}} \langle Dw, (T_2 - T_1) \alpha_t \circ \alpha(t, 0) \rangle = q_1(t) \quad \text{for} \quad t \in [0, a],
\]

where
\[
w(x) = w \circ \psi^{-1}(x).
\]

It is easy to check that \(p_1/\beta_0, q_0, q_1\), and \(f\) are \(m\)th order compatible at \(\alpha(0, 0)\) in the sense of Definition 4.1 is equivalent to that \(p_1/\beta_0, q_0, q_1\), and \(\hat{f}\) do in the sense of Definition 3.1, where
\[
\hat{f} = \frac{f \circ \psi^{-1}(x)}{2} \sqrt{\frac{\det G(x)}{-k \circ \psi^{-1}(x)}} \quad \text{for} \quad x \in \psi_0(B(\lambda_0, \sigma_0)) \subset \mathbb{R}^2,
\]

where \(G(x) = \det((\partial x_i, \partial x_j))\).

From Proposition 3.10, problem (3.1) admits a unique solution \(w \in C^{m-1}(\Xi_1(\beta_0, \gamma_0))\) with the corresponding boundary data. Thus, we have obtained a solution, denoted by \(w_0 \in C^{m-1}(\Omega_0)\), to problem (4.1) with the data
\[
\langle Dw_0, T_1 \alpha_s \rangle \circ \alpha(0, s) = p_1(s) \quad \text{for} \quad s \in [0, s_0],
\]
\[
w_0 \circ \alpha(t, 0) = q_0(t), \quad \frac{1}{\sqrt{2}} \langle Dw_0, (T_2 - T_1) \alpha_t \circ \alpha(t, 0) \rangle = q_1(t) \quad \text{for} \quad t \in [0, t_2],
\]

where
\[
\Omega_0 = \Omega \cap \psi_0^{-1}[\Xi_1(\beta_0, \gamma_0)].
\]

We define a curve on \(\Omega_0\) by
\[
\beta_1(s) = \psi_0^{-1} \circ \gamma_{t_1}(s) \quad \text{for} \quad s \in [0, s_{t_1}], \quad (4.33)
\]

where
\[
\gamma_{t_1}(s) = (s + t_1, s - t_1), \quad s_{t_1} = \begin{cases} 
  t_1 & \text{if} \quad t_1 \in (0, \frac{t_2}{2}], \\
  t_2 - t_1 & \text{if} \quad t_1 \in (\frac{t_2}{2}, t_2). 
\end{cases}
\]

Then \(\beta_1(s)\) is noncharacteristic and
\[
\Pi(\beta_1(0), \alpha_t(t_1, 0)) = \Pi(\partial x_1 + \partial x_2, \partial x_1 - \partial x_2) = 0. \quad (4.34)
\]

Let
\[
\hat{\Omega}_0 = \Omega \cap \psi_0^{-1}[\Xi_1(\beta_0, \gamma_0)] \cap \{ x \in \mathbb{R}^2 \mid x_1 - x_2 \leq 2t_1 \}.
\]

It follows from the estimate in Proposition 3.10 that
\[
\|w_0\|_{C^{m-1}(\Omega_0)} \leq C \Gamma_{mC}(p_1, p_2, q_0, q_1, f), \quad (4.35)
\]

where
\[
\Gamma_{mC}(p_1, p_2, q_0, q_1, f) = \|p_1\|_{C^{m-1,1}(0, a)} + \|p_2\|_{C^{m-1,1}(0, a)} + \|q_0\|_{C^{m,1}(0, a)} + \|q_1\|_{C^{m-1,1}(0, a)} + \|f\|_{C^{m,1}(\hat{\Omega})}.
\]
Step 2. Let the curve $\beta_1$ be given in (4.33). Let $\hat{s}_1 > 0$ be small such that

$$\beta_1(s) \in B(\lambda_1, \sigma_0) \quad \text{for} \quad s \in [0, \hat{s}_1].$$

From the noncharacteristicness of $\beta_1(s)$ and the relation (4.34) and Lemma 4.3 again, there exists an asymptotic coordinate system $\psi_1(p) = x : B(\lambda_1, \sigma_0) \to \mathbb{R}^2$ with $\psi_1(\lambda_1) = (0, 0)$ and

$$\psi_1(\alpha(t + t_1, 0)) = (t, -t) \quad \text{for} \quad t \in [0, t_3 - t_1],$$

$$\beta_{11}'(s) > 0, \ \beta_{12}'(s) > 0 \quad \text{for} \quad s \in [0, s_1],$$

where $\psi_1(\beta_1(s)) = (\beta_{11}(s), \beta_{12}(s))$. Since $\psi_1(\beta_1(0)) = (0, 0)$, we take $0 < s_1 \leq \hat{s}_1$ such that

$$\beta_{11}(s_1) + \beta_{12}(s_1) \leq t_3 - t_1.$$

By some similar arguments in Step 1, we obtain a unique solution $w_1 \in C^{m,1}(\Omega_1')$ to problem (4.1) with the data

$$\langle Dw_1, T_1 \hat{\beta}_1 \rangle \circ \beta_1(s) = \langle Dw_0, T_1 \hat{\beta}_1(s) \rangle \circ \beta_1(s) \quad \text{for} \quad s \in [0, s_1],$$

$$w_1(\alpha(t, 0)) = q_0(t), \ \frac{1}{\sqrt{2}}(Dw_1, (T_2 - T_1)\alpha_t) \circ \alpha(t, 0) = q_1(t) \quad \text{for} \quad t \in [t_1, t_3],$$

where

$$\Omega_1 = \Omega \cap \psi_1^{-1}[\Xi_1(\beta_1, \gamma_1)], \ \gamma_1(t) = \psi_1(\alpha(t + t_1, 0)),$$

$w_0$ is the solution of (4.1) on $\Omega_0$, given in Step 1, and

$$\Xi_1(\beta_1, \gamma_1) = \{ x \in \mathbb{R}^2 \mid 0 < x_1 \leq \beta_{11}(s_1), \ -x_1 \leq x_2 < \beta_{12} \circ \beta_{11}^{-1}(x_1) \}$$

$$\cup \{ x \in \mathbb{R}^2 \mid \beta_{11}(s_1) < x_1 < t_3 - t_1, \ -x_1 < x_2 < -x_1 + \beta_{11}(s_1) + \beta_{12}(s_1) \}.$$

As in Step 1, we define a curve on $\Omega_1$ by

$$\beta_2(s) = \psi_1^{-1}(s + t_2 - t_1, s + t_1 - t_2) \quad \text{for} \quad s \in [0, s_{t_2}],$$

where

$$s_{t_2} = t_2 - t_1 \quad \text{if} \quad t_2 - t_1 \leq \frac{t_3 - t_1}{2}; \quad s_{t_2} = t_3 - t_2 \quad \text{if} \quad t_2 - t_1 > \frac{t_3 - t_1}{2}.$$

Then $\beta_2(s)$ is noncharacteristic and

$$\Pi(\beta_2(0), \alpha_t(t_2, 0)) = \Pi(\partial x_1 + \partial x_2, \partial x_1 - \partial x_2) = 0. \quad (4.36)$$

Let

$$\hat{\Omega}_1 = \Omega \cap \psi_1^{-1}[\Xi_1(\beta_1, \gamma_1)] \cap \{ x \in \mathbb{R}^2 \mid x_1 - x_2 \leq 2(t_2 - t_1) \}.$$
As in Step 1, using the estimates in Proposition 3.10 and (4.35), we have

\[
\|w_1\|_{C^{m,1}(\Omega_1)} \leq C(\|Dw_0\|_{\mathcal{T}_1 T_1} + \beta_1 \|Q_1\|_{C^{m-1,1}[0,a]} + \|q_0\|_{C^{m,1}[0,a]} + \|q_1\|_{C^{m-1,1}[0,a]} + \|f\|_{C^{m-1,1}(\Omega_1)}) \leq CT_{m,C}(p, q_0, q_1, h). \tag{4.37}
\]

**Step 3.** Let \( \hat{s}_2 > 0 \) be small such that

\[
\beta_2(s), \quad \alpha(a, s) \in B(\lambda_2, \sigma_0) \quad \text{for} \quad s \in [0, \hat{s}_2].
\]

Let \( \psi_2(p) = x : B(\lambda_2, \sigma_0) \to \mathbb{R}^2 \) be an asymptotic coordinate system with \( \psi_2(\lambda_2) = (0, 0) \),

\[
\psi_2(\alpha(t + t_2, 0)) = (t, -t) \quad \text{for} \quad t \in [0, a - t_2], \tag{4.38}
\]

and

\[
\beta'_2(1)(s) > 0, \quad \beta'_2(2)(s) > 0 \quad \text{for} \quad s \in [0, \hat{s}_2],
\]

where \( \psi_2(\beta_2(s)) = (\beta(1)(s), \beta(2)(s)) \). Next, we prove that

\[
\beta'_2(1)(s) > 0, \quad \beta'_2(2)(s) > 0 \quad \text{for} \quad s \in [0, \hat{s}_2], \tag{4.39}
\]

where \( \psi_2(\beta_3(s)) = (\beta(1)(s), \beta(2)(s)) \) and \( \beta_3(s) = \alpha(a, s) \), by contradiction. Since \( \beta_3(s) \) is noncharacteristic, using (4.38) and the assumption \( \Pi(\alpha_1(a, 0), \beta'_3(0)) = 0 \), we have

\[
\beta'_3(0) = \beta'_3(0); \quad \text{thus} \quad \beta'_2(1)(s) \beta'_2(2)(s) > 0 \quad \text{for} \quad s \in [0, \hat{s}_2].
\]

Let

\[
p(t, s) = \alpha_1(t, s) + \alpha_2(t, s), \quad \psi_2(\alpha(t + t_2, s)) = (\alpha_1(t, s), \alpha_2(t, s)).
\]

Let (4.39) be not true, that is, \( \beta'_2(1)(s) < 0, \beta'_2(2)(s) < 0 \) for \( s \in [0, \hat{s}_2] \). Thus

\[
p(0, s) = \beta_2(1)(s) + \beta_2(2)(s) > \beta_2(1)(0) + \beta_2(2)(0) = 0 \quad \text{for} \quad s \in (0, \hat{s}_2],
\]

\[
p(a, s) = \beta_2(1)(s) + \beta_2(2)(s) < \beta_2(1)(0) + \beta_2(2)(0) = 0 \quad \text{for} \quad s \in (0, \hat{s}_2].
\]

Let \( t(s) \in (0, a - t_2) \) be such that

\[
\alpha_1(t(s), s) + \alpha_2(t(s), s) = 0 \quad \text{for} \quad s \in (0, \hat{s}). \tag{4.40}
\]

Since \( \alpha_1(t(0), 0) = 1 \) and \( \alpha(t + t_2, s) \) are noncharacteristic for all \( s \in [0, \hat{s}] \), we have \( \alpha_1(t, s) > 0 \) and

\[
0 < \alpha_1(0, s) < \alpha_1(t(s), s) < \alpha_1(a - t_2, s) = \beta_3(s) < \beta_3(0) = a - t_2.
\]

Thus, equality (4.40) means that \( \alpha_1(t(s), s), 0) = \alpha(t(s), s) \), which is a contradiction since \( \alpha : [0, a] \times [a, b] \to M \) is an imbedding map.

On the other hand, from (4.25), we have

\[
\mathcal{T}_1 \beta_3(s) = \beta_3(2)(s) \partial x_2, \quad \mathcal{T}_2 \beta_3(s) = \beta_3(1)(s) \partial x_1.
\]

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Let $0 < s_2 \leq \hat{s}_2$ be such that
\[ \beta_{21}(s_2) + \beta_{22}(s_2) \leq a - t_2. \]

Then take $0 < s_3 < b$ such that
\[ \beta_{21}(s_2) + \beta_{22}(s_2) = \beta_{31}(s_3) + \beta_{32}(s_3). \]

Applying Theorem 3.1, problem (4.1) admits a unique solution $w_2 \in C^{m,1}((\Omega_2))$ with the data
\[ \langle Dw_2, T_1 \beta_2 \rangle \circ \beta_2(s) = \langle Dw_1, T_1 \beta_2 \rangle \circ \beta_2(s), \quad \langle Dw_2, T_1 \alpha_2 \rangle \circ \alpha(a, s) = p_2(s) \quad \text{for} \quad s \in [0, s_2], \]
\[ w_2(\alpha(t, 0)) = q_0(t), \quad \frac{1}{\sqrt{2}}\langle Dw_2, (T_2 - T_1)\alpha \rangle \circ \alpha(t, 0) = q_1(t) \quad \text{for} \quad t \in [t_2, a], \]
where
\[ \hat{\Omega}_2 = \Omega_2 = \Omega \cap \psi_2^{-1}(\Phi(\beta_2, \beta_3)), \]
\[ \Phi(\beta_2, \beta_3) = \{ x \in \mathbb{R}^2 \mid 0 < x_1 < \beta_{21}(s_2), -x_1 < x_2 < \beta_{22} \circ \beta_{21}^{-1}(x_1) \} \]
\[ \cup \{ x \in \mathbb{R}^2 \mid \beta_{21}(s_2) \leq x_1 < a - t_2, -x_1 \leq x_2 < -x_1 + \beta_{21}(s_2) + \beta_{22}(s_2) \} \]
\[ \cup \{ x \in \mathbb{R}^2 \mid a - t_2 \leq x_1 < \beta_{31}(s_3), \beta_{32} \circ \beta_{31}^{-1}(x_1) < x_2 < -x_1 + \beta_{31}(s_3) + \beta_{32}(s_3) \}. \]

Using the estimates in Theorem 3.1 and (4.37), we obtain
\[ \| w_2 \|_{C^{m,1}((\Omega_2))} \leq C(\| Dw_2, T_1 \beta_2 \|_{C^{m-1,1}[0, s_2]} + \| p_2 \|_{C^{m-1,1}[0, a]} + \| q_0 \|_{C^{m,1}[0, a]} \]
\[ + \| q_1 \|_{C^{m-1,1}[0, a]} + \| f \|_{C^{m-1,1}(\Omega_2)}) \leq CT_{mC} (p_1, p_2, q_0, q_1, f). \quad (4.41) \]

**Step 4.** We define
\[ w = w_i \quad \text{for} \quad p \in \hat{\Omega}_i \quad \text{for} \quad i = 0, 1, 2. \]
Let $\omega > 0$ be small such that
\[ \alpha(t, s) \in \hat{\Omega}_0 \cup \hat{\Omega}_1 \cup \hat{\Omega}_2 \quad \text{for} \quad (t, s) \in (0, a) \times (0, \omega). \]

Then $w \in C^{m,1}(\Omega(0, \omega))$ will be a solution to (4.1) with the corresponding data if we show that
\[ w_0(p) = w_1(p) \quad \text{for} \quad p \in \Omega_0 \cap \Omega_1; \quad w_1(p) = w_2(p) \quad \text{for} \quad p \in \Omega_1 \cap \Omega_2. \quad (4.42) \]

Since
\[ w_{1s_2} \circ \beta_1(s) = w_{0s_2} \circ \beta_1(s) \quad \text{for} \quad s \in [0, s_1], \]
\[ w_0(t, -t) = w_1(t, -t), \quad \frac{\partial w_0}{\partial \nu}(t, -t) = \frac{\partial w_1}{\partial \nu}(t, -t) \quad \text{for} \quad t \in [t_1, t_2], \]

\[ \cdots \]
from the uniqueness in Proposition 3.10, we have
\[ w_0(x) = w_1(x) \quad \text{for} \quad x \in \Xi_1(\beta_0, \gamma_0) \cap \Xi_1(\beta_1, \gamma_1), \]
which yields the first identity in (4.42). A similar argument shows that the second identity in (4.42) is true.

Finally, the estimate (4.30) follows from (4.35), (4.37), and (4.41).

From a similar argument as for the proof of Lemma 4.4, we obtain the following.

**Lemma 4.5** Let the assumptions in Theorem 4.2 hold. Then there is a \( 0 < \omega \leq b \) such that problem (4.1) admits a unique solution \( w \in W^{2,2}(\Omega(0,\omega)) \) with the data (4.4) where \( s \in (0,\omega) \), and (4.5) to satisfy
\[
\|w\|^2_{W^{2,2}(\Omega(0,\omega))} \leq C(\|q_0\|^2_{W^{2,2}(0,a)} + \|q_1\|^2_{W^{1,2}(0,a)} + \|p_1\|^2_{W^{2,2}(0,b)} + \|p_2\|^2_{W^{2,2}(0,b)} + \|f\|_{W^{1,2}(\Omega)}). \tag{4.43}
\]

**Lemma 4.6** Let the assumptions in Theorem 4.3 hold. Then there are \( 0 < \omega \leq b \) and \( C > 0 \) such that for all solutions \( w \in W^{2,2}(\Omega) \) to problem (4.1)
\[
\|w\|^2_{W^{2,2}(\Omega(0,\omega))} \leq C[\|f\|^2_{W^{1,2}(\Omega)} + \Gamma(\Omega, w)], \tag{4.44}
\]
where \( \Omega(0,\omega) \) is given in (4.29).

**Proof** We keep all the notion in the proof of Lemma 4.4. Let \( \omega > 0 \) be given in Step 4. Then
\[ w_0(x) = w \circ \psi_0^{-1}(x) \]
is a solution to problem (3.1) on the region \( \Xi_1(\beta_0, \gamma_0) \), where \( \Xi_1(\beta_0, \gamma_0) \) is given in (4.32) and
\[ \beta_0(s) = \psi_0(\alpha(0, s)) = (\beta_{01}(s), \beta_{02}(s)) \quad \text{for} \quad s \in (0, s_0), \quad \gamma_0(t) = (t, -t) \quad \text{for} \quad t \in (0, t_2). \]

From the formula (4.25), we have
\[
p_1(s) = \langle Dw, T_1 \alpha \rangle + \alpha(0, s) = w_{0x_2} \circ \beta_0(s) \beta'_2(s),
\]
\[
D^2 w(T_1 \gamma_0(t), T_1 \gamma_0(t)) = w_{0x_1x_1} \circ \gamma_0(t) - D_{0x_1} \partial x_1(w).
\]
It follows that
\[
\left| D^2 w(T_1 \gamma_0(t), T_1 \gamma_0(t)) - |w_{0x_1x_1} \circ \gamma_0(t)| \right| \leq C|\nabla w_0 \circ \gamma_0(t)|.
\]
Similarly, we have
\[
\left| D^2 w(T_2 \gamma_0(t), T_2 \gamma_0(t)) - |w_{0x_2x_2} \circ \gamma_0(t)| \right| \leq C|\nabla w_0 \circ \gamma_0(t)|.
\]
Using the above relations, we obtain

\[
\Gamma(\gamma_0, w_0) + \Gamma_2(\beta_0, w_0) \leq C \Gamma(\Omega, w), \tag{4.45}
\]

where \(\Gamma(\gamma_0, w_0)\) and \(\Gamma_2(\beta_0, w_0)\) are given in (3.8) and (3.27), respectively.

Applying Proposition 3.12 to \(\Xi_1(\beta_0, \gamma_0)\) and using (4.45), we have

\[
\|w\|^2_{W^{2,2}(\Omega_0)} \leq C \|w_0\|^2_{W^{2,2}(\Xi_1(\beta_0, \gamma_0))} \leq C(\|f\|^2_{W^{1,2}} + \Gamma(\Omega, w)).
\]

Using (3.7) by a similar argument as for the above estimates, we obtain

\[
\|w\|^2_{W^{2,2}(\Omega_i)} \leq C(\|f\|^2_{W^{1,2}} + \Gamma(\Omega, w)), \quad \text{for} \quad i = 1, 2.
\]

Thus the estimate (4.44) follows. \(\square\)

**Lemma 4.7** Let the assumptions in Theorem 4.3 hold. Then there is \(C > 0\) such that for all solutions \(w \in W^{2,2}(\Omega)\) to problem (4.1)

\[
\Gamma(\Omega, w) \leq C(\|w\|^2_{W^{2,2}(\Omega)} + \|f\|^2_{W^{1,2}(\Omega)}).
\]  

**Proof Step 1** We claim that for each \(\varepsilon > 0\), there is \(C_{\varepsilon} > 0\) such that

\[
\sum_{j=0}^{2} \int_{a-\varepsilon}^{a-\varepsilon} |D^j w \circ \alpha(t, 0)|^2 dt \leq C_{\varepsilon}(\|w\|^2_{W^{2,2}(\Omega)} + \|f\|^2_{W^{1,2}(\Omega)}). \tag{4.47}
\]

Let \(t_0 \in (0, a)\) be fixed and let \(p_0 = \alpha(t_0, 0)\). From Lemma 4.3, there are \(0 < \sigma_0 < \min\{t_0, a - t_0\}\) and an an asymptotic coordinate system \(\psi : B(p_0, \sigma_0) \to \mathbb{R}^2\) with \(\psi(p_0) = (0, 0)\) such that \(\psi(B(p_0, \sigma_0) \cap \Omega) \subset \{ x \mid x \in \mathbb{R}^2, \ x_1 + x_2 > 0 \}\) and

\[
\psi(\alpha(t + t_0, 0)) = (t, -t) \quad \text{for} \quad t \in (-\sigma_0, \sigma_0),
\]

\[
\mathcal{T}_1 \alpha(t + t_0, 0) = \partial x_1, \quad \mathcal{T}_2 \alpha(t + t_0, 0) = \partial x_2 \quad \text{for} \quad t \in (-\sigma_0, \sigma_0),
\]

where the operators \(\mathcal{T}_i\) are given in (4.2). In addition, we take \(0 < \sigma_1 < \sigma_0\) small enough such that

\[
E(\gamma) \subset \psi(B(p_0, \sigma_0) \cap \Omega),
\]

where

\[
\gamma(t) = (t, -t), \quad E(\gamma) = \{ x \mid -x_2 < x_1 < \sigma_1, \ -\sigma_2 < x_2 < \sigma_1 \}.
\]

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Observe that \( w_0(x) = w \circ \psi^{-1}(x) \) is a solution to problem (3.1) on the region \( E(\gamma) \). Applying Proposition 3.3, we have

\[
\sum_{j=0}^{2} \int_{-\sigma_1/2}^{\sigma_1/2} |D^j w \circ \alpha(t + t_0, s)|^2 \, dt \leq C \sum_{j=0}^{2} \int_{-\sigma_1/2}^{\sigma_1/2} |\nabla^j w_0(t, -t)|^2 \, dt
\]

\[
\leq C(\|w_0\|_{W^{2,2}(E(\gamma))}^2 + \|f \circ \psi^{-1}\|_{W^{1,2}(E(\gamma))}^2) \leq C(\|w\|_{W^{2,2}}^2 + \|f\|_{W^{1,2}}^2).
\]

Thus the estimates (4.47) follows from the finitely covering theorem. By a similar argument, we have

\[
\sum_{j=0}^{2} \int_{b-\varepsilon}^{b} |D^j w \circ \alpha(t_k, s)|^2 \, ds \leq C_{\varepsilon}(\|w\|_{W^{2,2}(\Omega)}^2 + \|f\|_{W^{1,2}(\Omega)}^2), \quad k = 1, 2, \quad (4.48)
\]

where \( t_1 = 0 \) and \( t_2 = a \).

**Step 2** Let \( \varepsilon > 0 \) be given small. From Lemma 4.3, there is an asymptotic coordinate system \( \psi : B(\alpha(0, b), 3\varepsilon) \rightarrow \mathbb{R}^2 \) with \( \psi(\alpha(0, b)) = (0, 0) \) such that

\[
\psi(\alpha(t, b)) = (t, -t) \quad \text{for} \quad t \in [0, t_1], \quad d(\alpha(0, b), \alpha(t_1, b)) = 2\varepsilon,
\]

\[
\beta'_1(s) > 0, \quad \beta'_2(s) > 0, \quad T_1 \alpha_s(0, b + s - \varepsilon) = \beta'_2(s) \partial x_2 \quad \text{for} \quad s \in [0, \varepsilon],
\]

where \( \psi(\alpha(0, b + s - \varepsilon)) = (\beta_1(s), \beta_2(s)) \). For \( 0 < \varepsilon_1 \leq \varepsilon \), let

\[
\beta(s) = \psi(\alpha(0, b + s - \varepsilon_1)), \quad \gamma(t) = \psi(\alpha(t, b - \varepsilon_1)) = (\gamma_1(t), \gamma_2(t)).
\]

We fix \( 0 < \varepsilon_1 \leq \varepsilon \) such that

\[
\beta_1(\varepsilon_1) + \beta_2(\varepsilon_1) - \beta_2(0) = -\beta_2(0) \leq t_1 = \gamma_1(t_1), \quad \text{for some} \ t_2 > 0 \quad \gamma_1(t_2) = t_1.
\]

Set

\[
\Xi_1(\beta, \gamma) = \{ x \mid \beta_1(0) < x_1 \leq 0, \ \gamma_2 \circ \gamma_1^{-1}(x_1) < x_2 < \beta_2 \circ \beta_1^{-1}(x_1) \}
\]

\[
\cup \{ x \mid 0 \leq x_1 < t_1, \ \gamma_2 \circ \gamma_1^{-1}(x_1) < x_2 < -x_1 \}.
\]

It follows from Proposition 3.12 that

\[
\int_{b-\varepsilon_1}^{b} |p_1'(s)|^2 (b - s) \, ds \leq C(\|f \circ \psi^{-1}\|_{W^{1,2}(\Xi_1(\beta, \gamma))}^2 + \|w \circ \psi^{-1}\|_{W^{1,2}(\Xi_1(\beta, \gamma))}^2)
\]

\[
\leq C(\|f\|_{W^{1,2}}^2 + \|w\|_{W^{2,2}}^2).
\]

By a similar argument, we can show that the terms,

\[
\int_{b-\varepsilon}^{b} |p_2'(s)|^2 (b - s) \, ds,
\]

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\[
\int_0^\varepsilon |p'_1(s)|^2(b - s) ds + \sum_{j=0}^1 \|\nabla^j w \circ \alpha(\cdot, 0)\|_{L^2(0, \varepsilon)}^2 \\
+ \int_0^\varepsilon \|[D^2 w(T_+ \alpha_t, T_+ \alpha_t)]^2 t + [D^2 w(T_- \alpha_t, T_- \alpha_t)]^2(a - t)\|dt,
\]
and
\[
\int_0^\varepsilon |p'_2(s)|^2(b - s) ds + \sum_{j=0}^1 \|\nabla^j w \circ \alpha(\cdot, 0)\|_{L^2(\alpha - \varepsilon, \alpha)}^2 \\
+ \int_{\alpha-\varepsilon}^\alpha \|[D^2 w(T_+ \alpha_t, T_+ \alpha_t)]^2 t + [D^2 w(T_- \alpha_t, T_- \alpha_t)]^2(a - t)\|dt,
\]
are all bounded by \(C(\|f\|_{W^{1,2}}^2 + \|w\|_{W^{2,2}}^2)\).

Thus estimate (4.46) follows by combing the above estimates with those in Step 1. \(\square\)

**Proof of Theorem 4.1** Let \(\mathbb{N}\) be the set of all \(0 < \omega \leq b\) such that the claims in Lemma 4.4 hold. We shall prove

\[b \in \mathbb{N}.
\]

Let \(\omega_0 = \sup_{\omega \in \mathbb{N}} \omega\). Then \(0 < \omega_0 \leq b\). Thus there is a unique solution \(w \in C^{m,1}(\Omega(0, \omega_0))\) to (4.1) with the data (4.4), where \(s \in [0, \omega_0]\), and (4.5).

Let \(\sigma_0 > 0\) be given small such that the claims in Lemmas 4.1 and 4.2 hold, where \(\gamma(t) = \alpha(t, \omega_0)\) in Lemma 4.2. As in the proof of Lemma 4.4, we divide the curve \(\alpha(t, \omega_0)\) into \(m\) parts with the points \(\lambda_i = \alpha(t_i, \omega_0)\) such that

\[
\lambda_0 = \alpha(0, \omega_0), \quad \lambda_m = \alpha(a, \omega_0), \quad d(\lambda_i, \lambda_{i+1}) = \frac{\sigma_0}{3}, \quad 0 \leq i \leq m - 2, \quad d(\lambda_{m-1}, \lambda_m) \leq \frac{\sigma_0}{3},
\]

where \(t_0 = 0, t_1 > 0, t_2 > t_1, \cdots, \) and \(t_m = a > t_m\). Again, we shall treat the case of \(m = 3\) for simplicity.

**Step 1.** Let \(\psi_0(p) = x : B(\lambda_0, \sigma_0) \to \mathbb{R}^2\) be an asymptotic coordinate system with \(\psi_0(\lambda_0) = (0, 0)\) such that

\[
\psi_0(\alpha(t, \omega_0)) = (t, -t) \quad \text{for} \quad t \in [0, t_2],
\]

\[
\zeta_1'(s) > 0, \quad \zeta_2'(s) > 0 \quad \text{for} \quad s \in [\omega_0 - \varepsilon_0, \omega_0],
\]

\[
T_1 \alpha_s(0, s) = \zeta_2'(s) \partial x_2, \quad T_2 \alpha_s(0, s) = \zeta_1'(s) \partial x_1 \quad \text{for} \quad s \in [\omega_0 - \varepsilon_0, \omega_0],
\]

where

\[
\psi_0(\alpha(0, s)) = (\zeta_1(s), \zeta_2(s)) \quad \text{for} \quad s \in [\omega_0 - \varepsilon_0, \omega_0].
\]

For \(\varepsilon_0 > 0\), let

\[
\beta_0(s) = \psi_0(\alpha(0, s + \omega_0 - \varepsilon_0)) = (\beta_{01}(s), \beta_{02}(s)), \quad \gamma_0(t) = \psi(\alpha(t, \omega_0 - \varepsilon_0)) = (\gamma_{01}(t), \gamma_{02}(t)),
\]
where \( \beta_0(s) = \zeta_i(s + \omega_0 - \varepsilon_0) \) for \( i = 1, 2 \). Let \( \varepsilon_0 > 0 \) be given small such that there is \( t_2^0 > 0 \) satisfying
\[
\gamma_0( t_2^0 ) = t_2, \quad \beta_0(0) + t_2 \geq 0.
\]

Set
\[
\Xi(\beta_0, \gamma_0) = \{ x \mid \beta_0(0) < x_1 \leq 0, \gamma_0 \circ \gamma_0^{-1}(x_1) < x_2 < \beta_0 \circ \beta_0^{-1}(x_1) \}
\]
\[
\cup \{ x \mid 0 < x_1 < t_2, \gamma_0 \circ \gamma_0^{-1}(x_1) < x_2 < -x_1 \},
\]
\[
\Omega_0 = \psi_0^{-1}(\Xi(\beta_0, \gamma_0)) \cap \{ x \mid x_1 - x_2 \leq 2t_1 \}.
\]

We apply the estimates in Proposition 3.10 to \( \Xi(\beta_0, \gamma_0) \) to have
\[
\| w \|_{C^{m,1}(\overline{\Omega_0})} \leq C\| w \circ \psi_0^{-1} \|_{C^{m,1}(\Xi(\beta_0, \gamma_0))}
\]
\[
\leq C\| p_1 \|_{C^{m-1,1}} + \| w \circ \gamma_0 \|^2_{C^{m,1}} + \| \frac{\partial w}{\partial \nu} \c_{m-1,1} + \| f \circ \psi_0^{-1} \|_{C^{m,1}}
\]
\[
\leq C\| p_1 \|_{C^{m-1,1}} + \| w \circ \alpha(\cdot, \omega_0 - \varepsilon) \|^2_{C^{m,1}} + \| Dw, (T_2 - T_1)\alpha_t \r_o \alpha(\cdot, \omega_0 - \varepsilon) \|_{C^{m-1,1}} + \| f \|_{C^{m-1,1}}
\]
\[
\leq C\| p_1 \|_{C^{m-1,1}} + \| p_2 \|_{C^{m-1,1}} + \| q_0 \|_{C^{m,1}} + \| q_0 \|_{C^{m-1,1}} + \| f \|_{C^{m-1,1}},
\]
where \( \nu \) denotes the outside normal of \( \Xi(\beta_0, \gamma_0) \) along \( \gamma_0 \).

We define a curve on \( M \) by
\[
\hat{\beta}_1(s) = \psi_0^{-1}(s + t_1, s - t_1) \quad \text{for} \quad s \in [-s_{t_1}, 0],
\]
where \( s_{t_1} > 0 \) is the number such that \( \hat{\beta}(-s_{t_1}) = \gamma_0(t_1^0) \) for some \( t_1^0 > 0 \). It is easy to check that \( \hat{\beta}_1(s) \) is noncharacteristic and
\[
\Pi(\hat{\beta}_1(0), \alpha_t(t_1, \omega_0)) = 0.
\]  \( 4.49 \)

**Step 2.** By a similar argument as in Steps 2 and 3 in the proof of Lemma 4.4, we proceed to obtain a \( \varepsilon > 0 \) small such that \( \| w \|_{C^{m,1}(\overline{\Omega(\omega_0 - \varepsilon, \omega_0)})} \) is bounded by the right hand side of (4.30). Thus (4.30) is true when \( \omega \) is replaced with \( \omega_0 \) in the left hand side of (4.30). Then \( \omega_0 \in \mathbb{N} \).

**Step 3.** We now claim \( b = \omega_0 \) by contradiction. If \( \omega_0 < b \), then Lemma 4.3 allows us to have \( \omega_0 < \omega_1 \leq b \) and \( \omega_1 \in \mathbb{N} \).

**Proof of Theorem 4.2** A similar argument as in the proof of Theorem 4.1 completes the proof.

**Proof of Theorem 4.3** Let \( R \) be the set of all \( 0 < \omega \leq b \) such that estimate (4.44) is true. Set \( \omega_0 = \sup_{\omega \in R} \omega \). By Lemmas 4.6 and 4.7, it is sufficient to prove
\[
\omega_0 \in R.
\]
By following the proof of Theorem 4.2, we obtain a \( \varepsilon > 0 \) small such that
\[
\|w\|_{W^{2,2}(\Omega(\omega_0 - \varepsilon, \omega_0))} \leq C\left[\int_{\omega_0 - \varepsilon}^{\omega_0} (|p'_1(s)|^2 + |p'_2(s)|^2)(\omega_0 - s)ds + \Gamma(\alpha(\cdot, \omega_0 - \varepsilon), w) + \|f\|_{W^{1,2}}^2, \right]
\]
where \( \Gamma(\alpha(\cdot, \omega_0 - \varepsilon), w) \) is given in (4.13). On the other hand, we fix \( 0 < \varepsilon_1 < \varepsilon \) and apply Lemma 4.7 to the region \( \Omega(\omega_0 - \varepsilon, \omega_0 - \varepsilon_1) \) to obtain
\[
\Gamma(\alpha(\cdot, \omega_0 - \varepsilon), w) \leq C[\|w\|_{W^{2,2}(\Omega(\omega_0 - \varepsilon, \omega_0 - \varepsilon_1))}^2 + \|f\|_{W^{2,2}(\Omega(\omega_0 - \varepsilon, \omega_0 - \varepsilon_1))}^2]
\]
\[
\leq C[\|w\|_{W^{2,2}(\Omega(\omega_0 - \varepsilon, \omega_0 - \varepsilon_1))}^2 + \|f\|_{W^{2,2}(\Omega(\omega_0 - \varepsilon, \omega_0 - \varepsilon_1))}^2] (\text{by (4.44)})
\]
\[
\leq C[\|f\|_{W^{1,2}(\Omega)} + \Gamma(\Omega, w)].
\]
By Lemma 4.6, we have \( \omega_0 \in \mathcal{R} \). By Lemma 4.7, we obtain \( \omega_0 = b \).

**Proof of Theorem 4.4** Let \( w \in \mathcal{Y}(\Omega) \) and \( \varepsilon > 0 \) be given. We shall find a \( \hat{w} \in \mathcal{H}_2(\Omega) \) such that
\[
\|w - \hat{w}\|_{W^{2,2}(\Omega)} < \varepsilon.
\]

Let \( p_1, p_2, \) and \( q_0, q_1 \) be given in (4.4) and (4.5), respectively. Towards approximating \( w \) by \( \mathcal{H}(\Omega) \) functions, we first approximate its traces \( q_0, q_1, p_1, \) and \( p_2 \). From Theorem 4.3, those traces are regular except for points \( \alpha(0, 0), \alpha(0, b), \alpha(a, 0), \) and \( \alpha(a, b) \). Next, we change their values near those points to make them regular and to let the 1th order compatibility conditions hold at \( \alpha(0, 0) \) and \( \alpha(a, 0) \).

**Step 1** Consider the point \( \alpha(0, 0) \). Let \( \sigma > 0 \) be given small by Lemma 4.3 such that there is an asymptotic coordinate system \( \psi : B(\alpha(0, 0), \sigma) \to \mathbb{R}^2 \) with \( \psi(\alpha(0, 0)) = (0, 0) \) such that
\[
\psi(\alpha(t, 0)) = (t, -t) \quad \text{for} \quad t \in [0, t_0),
\]
\[
\beta'_1(0) = \beta'_2(0), \quad \beta'_1(s) > 0, \quad \beta'_2(s) > 0 \quad \text{for} \quad s \in [0, t_0),
\]
for some \( 0 < t_0 < \min\{a, b\}/4 \) small, where \( \psi(\alpha(0, s)) = (\beta_1(s), \beta_2(s)) \).

From Lemma 4.3, we have
\[
p_1(s) = w_{x_2} \circ \beta(s)\beta'_2(s) \quad \text{for} \quad s \in [0, t_0],
\]
\[
q'_0(t) = w_{x_1}(t, -t) - w_{x_2}(t, -t), \quad -\sqrt{2}q_1(t) = w_{x_1}(t, -t) + w_{x_2}(t, -t) \quad \text{for} \quad t \in (0, t_0),
\]
where \( w(x) = w \circ \psi^{-1}(x) \). Moreover, we have
\[
D^2w(T_1\alpha_t, T_1\alpha_t) = D^2w(\partial x_1, \partial x_1) = w_{x_1x_1}(t, -t) - D_{\partial x_1} \partial x_1(w)(t, -t) = \varphi_{11} + \phi_1,
\]
and
\[
D^2w(T_2\alpha_t, T_2\alpha_t) = \varphi_{22} + \phi_2,
\]
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where, by Proposition 4.1,
\[
\varphi_{11} = \frac{1}{2} [q''_0(t) - \sqrt{2} q'_1(t)], \quad \varphi_{22} = \frac{1}{2} [q''_0(t) + \sqrt{2} q'_1(t)] \quad \text{for} \quad t \in (0, t_0),
\]
\[
\phi_1 = \pm \frac{1}{2} \sqrt{\frac{\det G}{-\kappa}} [f_0 w + X(w)] - D\partial_1 x_1(w), \quad \phi_2 = \pm \frac{1}{2} \sqrt{\frac{\det G}{-\kappa}} [f_0 w + X(w)] - D\partial_2 x_2(w).
\]
By Theorem 4.3
\[
p_1 \in W^{1,2}(0, t_0), \quad q_0 \in W^{1,2}(0, t_0), \quad q_1, \varphi_{11}^{1/2}, \varphi_{22}, \phi_1, \phi_2 \in L^2(0, t_0).
\]
Thus
\[
q_0^{t^{1/2}} = (\varphi_{11} + \varphi_{22})^{t^{1/2}} \in L^2(0, t_0), \quad q_1^{t^{1/2}} = \frac{1}{\sqrt{2}} (\varphi_{22} - \varphi_{11})^{t^{1/2}} \in L^2(0, t_0).
\]
We also need the following.

**Lemma 4.8** Let
\[
z(t) = \frac{1}{2} [q'_0(t) + \sqrt{2} q_1(t)] \quad \text{for} \quad t \in (0, t_0).
\]
Then \(z \in C [0, t_0]\) and
\[
p(0) + z(0) \beta_2'(0) = 0. \quad (4.50)
\]

**Proof of Lemma 4.8** It follows from \(z' = \varphi_{22} \in L^2(0, t_0)\) that \(z \in C [0, t_0]\).

We have
\[
w_{x_2} \circ \beta \circ \beta_1^{-1}(t) - w_{x_2}(t, -t) = \int_{-t}^{\beta_2 \circ \beta_1(t)} w_{x_2}(t, s) ds,
\]
from which we obtain
\[
|w_{x_2} \circ \beta \circ \beta_1^{-1}(t) - w_{x_2}(t, -t)|^2 \leq [\beta_2 \circ \beta_1^{-1}(t) + t] \int_{-t}^{\beta_2 \circ \beta_1(t)} |w_{x_2}(t, s)|^2 ds.
\]
For \(\varepsilon > 0\) given, let \(\vartheta \in [\varepsilon/2, \varepsilon]\) be fixed such that
\[
|w_{x_2} \circ \beta \circ \beta_1^{-1}(\vartheta) - w_{x_2}(\vartheta, -\vartheta)|^2 = \inf_{t \in [\varepsilon/2, \varepsilon]} |w_{x_2} \circ \beta \circ \beta_1^{-1}(t) - w_{x_2}(t, -t)|^2.
\]
Then
\[
|w_{x_2} \circ \beta \circ \beta_1^{-1}(\vartheta) - w_{x_2}(\vartheta, -\vartheta)|^2 \leq \frac{2}{\varepsilon} [\beta_2 \circ \beta_1^{-1}(\varepsilon) + \varepsilon] \int_{\varepsilon/2}^{\varepsilon} \int_{-t}^{\beta_2 \circ \beta_1(t)} |w_{x_2}(t, s)|^2 ds
\]
\[
\leq \sigma \varepsilon \int_{0}^{\varepsilon} \int_{-t}^{\beta_2 \circ \beta_1(t)} |w_{x_2}(t, s)|^2 ds.
\]
Thus,
\[
w_{x_2} \circ \beta(0) - w_{x_2}(0, 0) = 0,
\]
that is, (4.50).

Let \( 0 < \varepsilon < t_0 \) given small. We shall construct \( \hat{q}_0 \) and \( \hat{q}_1 \) to satisfy the following.

1. \( \hat{q}_0(t) = q_0(t), \hat{q}_1(t) = q_1(t) \) for \( t \in [\varepsilon, a] \);
2. \( \hat{q}_0 \in W^{2,2}(0, a) \) and \( \hat{q}_1 \in W^{1,2}(0, a) \);
3. The following 1th order compatibility conditions hold at the point \( \alpha(0,0) \),

\[
2p(0) + [\hat{q}_0'(0) + \sqrt{2}\hat{q}_1(0)]\beta'_2(0) = 0;
\]

4. If \( \hat{w} \in \mathcal{Y}(\Omega) \) is such that

\[
\hat{w} \circ \alpha(t,0) = \hat{q}_0(t), \quad \frac{1}{2}\langle D\hat{w}, (T_2 - T_1)\alpha_t \rangle \circ \alpha(t,0) = \hat{q}_1(t) \quad \text{for} \quad t \in (0,a),
\]

\[
\langle D\hat{w}, T_1 \alpha_s \rangle \circ \alpha(0,s) = p_1(s), \quad \langle D\hat{w}, T_2 \alpha_s \rangle \circ \alpha(0,s) = p_2(s) \quad \text{for} \quad s \in (0,b),
\]

then

\[
\Gamma(\Omega, \hat{w} - w) \to 0 \quad \text{as} \quad \varepsilon \to 0.
\]

For the above purposes, we define

\[
\hat{q}_0(t) = \begin{cases} 
\sigma_0(\varepsilon) + [q_0(t_0) - \int_0^{t_0} \varphi_{11} ds - \int_0^{t_0} \varphi_{22} ds]t + \int_0^t (t - s)\varphi_{22}(s) ds, & t \in [0, \varepsilon), \\
q_0(t), & t \in [\varepsilon, a],
\end{cases}
\]

and

\[
\hat{q}_1(t) = \begin{cases} 
q_1(t_0) + \frac{1}{\sqrt{2}} \int_0^{t_0} \varphi_{11} ds - \frac{1}{\sqrt{2}} \int_t^{t_0} \varphi_{22} ds, & t \in (0, \varepsilon), \\
q_1(t), & t \in [\varepsilon, a],
\end{cases}
\]

where

\[
\sigma_0(\varepsilon) = q_0(\varepsilon) - q_0'(\varepsilon)\varepsilon + \int_0^\varepsilon s\varphi_{22}(s) ds.
\]

Clearly, (1) and (2) hold for the above \( \hat{q}_0 \) and \( \hat{q}_1 \). Since

\[
q_0'(t_0) - \int_\varepsilon^{t_0} \varphi_{11} ds - \int_0^{t_0} \varphi_{22} ds = q_0'(\varepsilon) - \int_0^\varepsilon \varphi_{22}(s) ds = q_0(\varepsilon) - z(\varepsilon) + z(0), \quad \text{for} \quad t \in (0, \varepsilon),
\]

\[
\frac{1}{\sqrt{2}} \int_\varepsilon^{t_0} \varphi_{11} ds - \frac{1}{\sqrt{2}} \int_t^{t_0} \varphi_{22} ds = q_1(\varepsilon) - q_1(t_0) + \frac{1}{\sqrt{2}} [z(t) - z(\varepsilon)],
\]

using (4.50), we have

\[
2p(0) + [\hat{q}_0'(0) + \sqrt{2}\hat{q}_1(0)]\beta'_2(0) = q_0'(\varepsilon) + \sqrt{2}q_1(\varepsilon) - 2z(\varepsilon) = 0.
\]

Next, we check (4). It follows that

\[
|q_0(t) - \hat{q}_0(t)|^2 = \left| \int_\varepsilon^t \int_s^\varepsilon q_0''(\tau) d\tau ds + \int_t^{(t-s)} \varphi_{22}(s) ds \right|^2
\]

\[
\leq 2(\varepsilon - t + t \ln \frac{t}{\varepsilon}) \int_\varepsilon^\varepsilon |q_0''(\tau)|^2 d\tau + \frac{2}{3} \varepsilon^3 \int_\varepsilon^\varepsilon |\varphi_{22}(s)|^2 ds \quad \text{for} \quad t \in (0, \varepsilon).
\]
In addition,
\[ |q_0(t) - \hat{q}_0(t)|^2 = | \int_t^\varepsilon \varphi_{11}(s)ds |^2 \leq (\ln \frac{\varepsilon}{t}) \int_0^\varepsilon |\varphi_{11}(s)|^2 ds \quad \text{for} \quad t \in (0, \varepsilon). \]
Similarly, we have
\[ |q_1(t) - \hat{q}_1(t)|^2 \leq (\ln \frac{\varepsilon}{t}) \int_0^\varepsilon |\varphi_{11}(s)|^2 ds \quad \text{for} \quad t \in (0, \varepsilon). \]
Using (4.13) and the above estimates, we have
\[
\Gamma (\alpha, 0, w - \hat{w}) = \sum_{j=0}^1 \| D(w - \hat{w}) \circ \alpha (\cdot, 0) \|_{L^2(0, \varepsilon)}^2 + \int_0^\varepsilon \| D^2 (w - \hat{w})(T_1 \alpha_t, T_1 \alpha_t) \| dt \\
+ \| D^2 (w - \hat{w})(T_1 \alpha_t, T_1 \alpha_t) \| (a - t) dt \\
\leq C \int_0^\varepsilon [|q_0(s) - \hat{q}_0(s)]^2 + |q_0'(t) - \hat{q}_0'(t)|^2 + |q_1(t) - \hat{q}_1(t)|^2 \\
+ |\varphi_{11} - \hat{\varphi}_{11}|^2 dt + |\varphi_{22} - \hat{\varphi}_{22}|^2 dt \\
\leq C \int_0^\varepsilon \left[ (|q_0''(t)|^2 + |\varphi_{11}(t)|^2) t + |\varphi_{22}(t)|^2 \right] dt, \quad (4.51)
\]
where
\[ \hat{\varphi}_{11} = \frac{1}{2} q_0''(t) - \sqrt{2} \hat{q}_0'(t) = 0, \quad \hat{\varphi}_{22} = \frac{1}{2} q_0''(t) + \sqrt{2} \hat{q}_0'(t) = \varphi_{22}. \]
Thus (4) follows.

Step 2 As in Step 1, we change the values of \( q_0 \) and \( q_1 \) near the point \( \alpha(0, 0) \) to get \( \hat{q}_0 \) and \( \hat{q}_1 \) in \( W^{2,2}(0, a) \) and in \( W^{1,2}(0, a) \), respectively, such that the 1th order compatibility conditions hold to approximate \( q_0 \) and \( q_1 \). Then we change the values of \( p_1 \) and \( p_2 \) near the points \( \alpha(0, b) \) and \( \alpha(a, b) \), respectively, such that \( \hat{p}_1, \hat{p}_2 \in W^{1,2}(0, b) \) approximate \( p_1 \) and \( p_2 \), respectively. Thus the proof completes from Theorem 4.2.

5 Proofs of Main Results in Section 1

Let \( \Omega \subset M \) be a noncharacteristic region. For \( U \in L^2(\Omega, T_{sym}^2) \) given, we consider problem
\[
sym \nabla y = U \quad \text{on} \quad \Omega. \quad (5.1)
\]
We have the following.

Theorem 5.1 Let \( \Omega \) be a noncharacteristic region of class \( C^{2,1} \). For \( U \in C^{1,1}(\Omega, T_{sym}^2) \), there exists a solution \( y = W + w\tilde{n} \) such that
\[
sym DW + w\Pi = U \quad \text{for} \quad x \in \Omega \quad (5.2)
\]
satisfying the bounds
\[
\| W \|_{C^{1,1}(\Omega, T)} + \| w \|_{C^{0,1}(\Omega)} \leq C \| U \|_{C^{1,1}(\Omega, T_{sym}^2)}. \quad (5.3)
\]
If, in addition, \( \Omega \in C^{m+2,1} \), \( U \in C^{m+1,1}(\Omega, T^{2}_{sym}) \) for some \( m \geq 1 \), then

\[
\|W\|_{C^{m+1,1}(\Omega,T)} + \|w\|_{C^{m,1}(\Omega)} \leq C\|U\|_{C^{m+1,1}(\Omega, T^{2}_{sym})},
\] (5.4)

**Proof** (1) Consider problem

\[
\langle D^{2}v, Q^{*} \Pi \rangle = P(U) - 2v \kappa \text{tr} \Pi + X(v) \quad \text{for} \quad x \in \Omega,
\] (5.5)

where \( P(U) \) and \( X \) are given in (2.26) and (2.27), respectively, with the boundary data

\[
\langle Dv, \mathcal{T}_{1}\alpha_{s} \rangle \circ \alpha(0, s) = \langle Dv, \mathcal{T}_{2}\alpha_{s} \rangle \circ \alpha(a, s) = 0 \quad \text{for} \quad s \in (0, b),
\] (5.6)

\[
v \circ \alpha(t, 0) = \frac{1}{\sqrt{2}}(Dv, (\mathcal{T}_{2} - \mathcal{T}_{1})\alpha_{t}) \circ \alpha(t, 0) = 0 \quad \text{for} \quad t \in (0, a),
\] (5.7)

where \( \mathcal{T}_{1} \) and \( \mathcal{T}_{2} \) are given in (4.2).

Since

\( P(U) \in L^{\infty}(\Omega), \quad X \in L^{\infty}(\Omega), \)

it follows from Theorem 4.1 that problem (5.5) with the data (5.6) and (5.7) has a unique solution \( v \in C^{0,1}(\Omega) \) with the bounds

\[
\|v\|_{C^{0,1}(\Omega)} \leq C\|U\|_{C^{1,1}(\Omega, T^{2}_{sym})}.
\] (5.8)

From Theorem 2.1, there is a solution \( y \in C^{0,1}(\Omega, \mathbb{R}^{3}) \) to (5.1). Let

\[
w = \langle y, \vec{n} \rangle, \quad W = y - w\vec{n}.
\]

Then \( w \in C^{0,1}(\Omega) \). It follows from [12, lemma 4.3] that \( W \in C^{1,1}(\Omega, T) \) and (5.3) holds.

(2) Let \( \Omega \in C^{m+2,1} \) and \( U \in C^{m+1,1}(\Omega, T^{2}_{sym}) \) be given for some \( m \geq 1 \). Let

\[
q_{0}(t) = q_{1}(t) = 0 \quad \text{for} \quad t \in [0, a].
\]

Let \( Q_{k}\left(0, 0, P(U)\right)(t) \) be given in the formula (4.8) for \( t \in [0, a] \) and \( 1 \leq k \leq m - 1 \). We define

\[
\phi_{j}(s) = \begin{cases} 0, & m = 1, \\
\sum_{l=1}^{m-1} \frac{p_{j}^{(l)}(t_{j})}{n_{j}^{(l)}} s^{l}, & m \geq 2,
\end{cases} \quad \text{for} \quad s \in [0, b], \quad j = 1, 2,
\] (5.9)

where \( p_{j}^{(l)}(t_{j}) \) are given by the right hand sides of (4.10) for \( 1 \leq l \leq m - 1 \) and \( 1 \leq j \leq 2 \), where \( q_{0} = q_{1} = 0 \) and \( f = P(U) \). Clearly, the \( m \)th compatibility conditions hold true for the above \( q_{0}, q_{1}, \phi_{1}, \phi_{2}, \) and \( P(U) \). From Theorem 4.1, there is a solution \( v \in C^{m,1}(\Omega) \) to problem (5.5) with the data

\[
\langle Dv, \mathcal{T}_{1}\alpha_{s} \rangle \circ \alpha(0, s) = \phi_{1}(s), \quad \langle Dv, \mathcal{T}_{2}\alpha_{s} \rangle \circ \alpha(a, s) = \phi_{2}(s) \quad \text{for} \quad s \in (0, b),
\]

\[
v \circ \alpha(t, 0) = \frac{1}{\sqrt{2}}(Dv, (\mathcal{T}_{2} - \mathcal{T}_{1})\alpha_{t}) \circ \alpha(t, 0) = 0 \quad \text{for} \quad t \in (0, a).
\]
Moreover, it follows from (4.11) and (2.26) that
\[ \|v\|_{C^{m,1}(\bar{\Omega})} \leq C\|U\|_{C^{m+1,1}(\Omega,T^2)}, \]
which implies the estimate (5.4) is true. \qed

**Proof of Theorem 1.1** Let
\[ V = W + w\vec{n}, \quad w = \langle V, \vec{n} \rangle. \]
The regularity of
\[ \text{sym} DW = -w\Pi \in W^{2,2}(\Omega, \mathbb{R}^3) \]
implies
\[ W \in W^{3,2}(\Omega, T). \]

Let \( E_1, E_2 \) be a frame field on \( \Omega \) with the positive orientation and let
\[ v = \frac{1}{2} [\nabla V(E_2, E_1) - \nabla V(E_1, E_2)]. \]
From Theorem 2.1 \( v \) is a solution to problem
\[ \langle D^2 v, Q^*\Pi \rangle = -2v\kappa \text{tr} \Pi + X(v) \quad \text{for} \quad x \in \Omega, \quad (5.10) \]
where \( \kappa \text{tr} \Pi \in C^{m,1}(\bar{\Omega}) \) and \( X = (\nabla \vec{n})^{-1}D\kappa \in C^{m-1,1}(\bar{\Omega}, T) \), where \( C^{m-1,1}(\bar{\Omega}, T) = L^\infty(\Omega, T) \).

It is easy to check that
\[ \nabla_{E_i} V = D_{E_i} W + w\nabla_{E_i} \vec{n} + [E_i(w) - \Pi(W, E_i)]\vec{n} \quad \text{for} \quad i = 1, 2. \]
Thus
\[ v = DW(E_2, E_1) - DW(E_1, E_2) \in W^{2,2}(\Omega). \]
From Theorems 4.4, 4.1, and 4.2, there are solutions \( v_n \in C^{m,1}(\bar{\Omega}) \) to problem (5.10) such that
\[ \lim_{n \to \infty} \|v_n - v\|_{W^{2,2}(\Omega)} = 0. \]
Let
\[ u_n = -Q(\nabla \vec{n})^{-1}Dv_n, \quad u = -Q(\nabla \vec{n})^{-1}Dv. \]
Then \( u_n \in C^{m-1,1}(\bar{\Omega}) \).

From Theorem 2.1, there exist \( \hat{V}_n \in C^{m,1}(\bar{\Omega}, \mathbb{R}^3) \) such that
\[ \left\{ \begin{array}{l} \nabla_{E_1} \hat{V}_n = v_n E_2 + \langle u_n, E_1 \rangle \vec{n}, \\ \nabla_{E_2} \hat{V}_n = -v_n E_1 + \langle u_n, E_2 \rangle \vec{n}, \end{array} \right. \quad \text{for} \quad n = 1, 2, \ldots. \quad (5.11) \]
Define
\[ V_n(\alpha(t, s)) = \hat{V}_n(\alpha(0, s)) - \hat{V}_n(\alpha(0, 0)) + V(\alpha(0, 0)) + \int_0^t \nabla_{\alpha^t} \hat{V}_n dt \quad \text{for} \quad n = 1, 2, \ldots. \]
Thus \( V_n \in \mathcal{V}(\Omega, IR^3) \cap C^{m, 1}(\Omega, IR^3) \) satisfy (1.2).

**Proof of Theorem 1.2** As in [5] we conduct in \( 2 \leq i \leq m \). Let
\[ u_{\varepsilon} = \sum_{j=0}^{i-1} \varepsilon^j w_j \]
be an \((i-1)\)th order isometry of class \( C^{2(m-i+1)+1,1}(\overline{\Omega}, IR^3) \), where \( w_0 = \text{id} \) and \( w_1 = V \) for some \( i \geq 2 \). Then
\[ \sum_{j=0}^k \nabla^T w_j \nabla w_{k-j} = 0 \quad \text{for} \quad 0 \leq k \leq i-1. \]

Next, we shall find out \( w_i \in C^{2(m-i)+1,1}(\overline{\Omega}, IR^3) \) such that
\[ \phi_{\varepsilon} = u_{\varepsilon} + \varepsilon^i w_i \]
is an \( i \)th order isometry. From Theorem 5.1 there exists a solution \( w_i \in C^{2(m-i)+1,1}(\overline{\Omega}, IR^3) \) to problem
\[ \text{sym} \nabla w_i = -\frac{1}{2} \text{sym} \sum_{j=1}^{i-1} \text{sym} \nabla^T w_j \nabla w_{i-j} \]
which satisfies
\[ \| w_i \|_{C^{2(m-i)+1,1}(\overline{\Omega}, IR^3)} \leq C \| \sum_{j=1}^{i-1} \text{sym} \nabla^T w_j \nabla w_{i-j} \|_{C^{2(m-i+1)+1,1}(\overline{\Omega}, IR^3)} \]
\[ \leq C \sum_{j=1}^{i-1} \| w_j \|_{C^{2(m-i+1)+1,1}(\overline{\Omega}, IR^3)} \| w_{i-j} \|_{C^{2(m-i+1)+1,1}(\overline{\Omega}, IR^3)}. \]
The conduction completes. \( \square \)

**Proof of Theorem 1.3** will follow from the proposition below.

**Proposition 5.1** Let \( \Omega \subset M \) be a noncharacteristic region of class \( C^{2,1} \). Then for \( U \in W^{2,2}(\Omega, T^2_{\text{sym}}) \) there exits a solution \( w \in W^{1,2}(\Omega, IR^3) \) to problem
\[ \text{sym} \nabla w = U. \]
Proof Consider problem (5.5) with the data (5.6) and (5.7). By (4.9) the first order compatibility conditions hold. Since $P(U) \in L^2(\Omega)$, the proposition follows from Theorems 4.2 and 2.1.

Proof of Theorem 1.5 follows from [7, Theorem 2.3] and Theorem 1.3.

Proof of Theorem 1.6 Using Theorems 1.1, 1.2, and following the proof of [5, Theorem 6.2], we completes the proof. The details are omitted here.

References

[1] G. Friesecke, R. James, S. Muller, A theorem on geometric rigidity and the derivation of nonlinear plate theory from three dimensional elasticity. Commun. Pure Appl. Math. 55, 1461-1506 (2002).

[2] G. Friesecke, R. James, S. Muller, A hierarchy of plate models derived from nonlinear elasticity by gamma-convergence. Arch. Ration. Mech. Anal. 180(2), 183-236 (2006).

[3] G. Geymonat, E. Sanchez-Palencia, On the rigidity of certain surfaces with folds and applications to shell theory. Arch. Ration. Mech. Anal. 129(1), 11-45 (1995)

[4] Q. Han, J.-X. Hong, Isometric embedding of Riemannian Manifolds in Euclidean Spaces, Mathematical Surveys and Monographs, 130 American Mathematical Society, Providence, RI, 2006

[5] P. Hornung, M. Lewicka, M. R. Pakzad, Infinitesimal isometries on developable surfaces and asymptotic theories for thin developable shells. J. Elasticity 111 (2013), no. 1, 1-19.

[6] M. Lewicka, M. R. Pakzad, The infinite hierarchy of elastic shell models: some recent results and a conjecture. Infinite dimensional dynamical systems, 407-420, Fields Inst. Commun., 64, Springer, New York, 2013.

[7] M. Lewicka, M. G. Mora, M. R. Pakzad, Shell theories arising as low energy -limit of 3d nonlinear elasticity. Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 9 (2010), no. 2, 253-295.

[8] —, The matching property of infinitesimal isometries on elliptic surfaces and elasticity of thin shells. Arch. Ration. Mech. Anal. 200 (2011), no. 3, 1023-1050.

[9] E. Sanchez-Palencia, Statique et dynamique des coques minces. II. Cas de flexion pure inhibe. Approximation membranaire. C. R. Acad. Sci. Paris Sr. I Math. 309(7), 531-537 (1989)
[10] M. Spivak, A comprehensive introduction to differential geometry. Vol. III. Second edition. Publish or Perish, Inc., Wilmington, Del., 1979. xii+466 pp. ISBN: 0-914098-83-7.

[11] P. F. Yao, Space of Infinitesimal Isometries and Bending of Shells, 2012.

[12] —, Modeling and control in vibrational and structural dynamics. A differential geometric approach. Chapman & Hall/CRC Applied Mathematics and Nonlinear Science Series. CRC Press, Boca Raton, FL, 2011.