MODALITY FOR FREE: NOTES ON ADDING THE TARSKIAN MÖGLICHKEIT TO SUBSTRUCTURAL LOGICS

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ABSTRACT. We briefly examine the modal formulae that can be derived in Multiplicative Additive Linear Logic (MALL) and some extensions by using Tarski’s extensional modal operators. We also briefly compare this with a substructural form of the modal logic K.

1. INTRODUCTION

The Tarskian möglichkeit (literally, “possibility” in German) is a modal operator that was introduced by Łukasiewicz (and attributed to Tarski) in [13, §7]. This modal operator is unusual in that it is an extensional one, defined in terms of other connectives in Łukasiewicz’s many-valued logics:

\[ \Diamond A = \text{def} \, \neg \Diamond \neg A \rightarrow A \]  

(1)

The modal logic that results from this definition is unusual, in part because of the theorems such as:

\[ (\Diamond A \land \Diamond B) \rightarrow \Diamond (A \land B) \]  

(2)

In the case where \( B = \neg A \), theorem (2) appears to be paradoxical, if not absurd, and largely because of this, the Tarskian möglichkeit has been a footnote in the history of modal logic. Most of the analyses that we are aware of has been for the 3-valued logic, in [12], [8] (but omitted from [9]), [5], and [3], and it is generally critical. An application of the \( m \geq 3 \)-valued logics to describing \( m \)-state systems was suggested in [20], and an application of the infinite-valued logic applied to modelling degrees of believability was suggested by the current author in [17].

However, the infinite-valued logic can be seen as an extension of Affine Logic [4], and many of the modal formulae derivable in the infinite-valued logic are derivable in weaker substructural logics. We give an overview of some of the formal properties below, by noting modal rules and formulae in the corresponding logics. We make no claims about the applications of the Tarskian möglichkeit.

2. MULTIPlicative ADDITIVE LINEar LOGic

The sequent rules for GMALL, a calculus for Multiplicative Additive Linear Logic (MALL) [6] are given in Figure 1, using notation similar to [19]—in particular, we use \( \oplus \) for \( \text{par} \) (multiplicative disjunction) and \( \lor \) for \( \text{plus} \) (additive disjunction).

The modal rules (Figure 2) are derived in a straightforward manner. (The corresponding box operator is defined as the dual of diamond operator \( \sqbox A = \text{def} \, \neg \Diamond \neg A \).)

\[ Date: January 20, 2013. \]
Proof. Note that the rules for $\land A$ for additive implication are omitted but can be derived similarly.

Figure 1: Rules for GMALL.

Figure 2: Derived rules for Tarskian modalities in GMALL.

Remark 1. Because of the symmetries that occur in many of the proofs given in this paper, the following non-branching forms of the modal rules will be used for brevity:

Proposition 1. The following equivalences hold in MALL:

$$\diamond A \equiv A \oplus A$$

$$\Box A \equiv A \otimes A$$

Proof. Straightforward. $\square$
Remark 2. The equivalences in Proposition 1 may be used as alternative definitions of the Tarskian modalities.

Proposition 2. The following are derivable in GMALL:

\[ \neg \Box \bot \] (5)
\[ \diamond \top \] (6)
\[ \neg \Box (A \land \neg A) \] (7)
\[ \diamond (A \lor \neg A) \] (8)

where (6) corresponds to the intuitionistic modal axiom D [18].

Proof. Straightforward. □

Proposition 3. The K\Box rule and its dual

\[ \frac{\Gamma \rightarrow A}{\Box \Gamma \rightarrow \Box A} \quad \text{K\Box} \quad \frac{A \rightarrow \Delta}{\diamond A \rightarrow \diamond \Delta} \quad \text{K\Diamond} \]

are derivable in GMALL.

Proof. Straightforward. □

Proposition 4 (Distribution Theorems). The following are derivable in GMALL:

\[ \Box (A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B) \] (9)
\[ \Box (A \land B) \rightarrow (\Box A \land \Box B) \] (10)
\[ \diamond (A \land B) \rightarrow (\Box A \land \Diamond B) \] (11)
\[ (\Box A \lor \Box B) \rightarrow \Box (A \lor B) \] (12)
\[ (\Diamond A \lor \Diamond B) \rightarrow \Diamond (A \lor B) \] (13)

where (9) corresponds to the K axiom.

Proof. Straightforward. □

3. A Comparison of MALL with Substructural-K (KMALL)

Definition 1 (Substructural-K). Let Substructural-K (KMALL) be MALL augmented by extending the language of formulae with \Box A. We obtain a calculus GKMALL for KMALL by adding the following rule to GMALL:

\[ \frac{\Gamma \rightarrow A}{\Box \Gamma \rightarrow \Box A} \quad \text{K\Box} \]

which corresponds to adding to MALL the necessity rule

\[ \vdash A \quad \vdash \Box A \quad \text{N} \]

and the K axiom, \Box (A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B).

Remark 3. Cut elimination for GKMALL is shown in Appendix A.

Theorem 5. If GKMALL \vdash \Gamma \rightarrow \Delta, then GMALL \vdash [\Box/\Diamond] \Gamma \rightarrow [\Box/\Diamond] \Delta.

Proof. By induction on the derivation height. Note that \Box-formulae are be introduced into a KMALL derivation either by axioms L\bot and R\top, or by the K\Box rule. Instances of K\Box are replaced by instances of K\Diamond. □
Remark 4. \textbf{GKMALL} \not\equiv \neg \Box \bot, \Diamond \top, \neg \Box (A \land \neg A) \text{ and } \Diamond (A \lor \neg A). \ A \ form \ of \ the \ \textbf{K} \Box \ rule 
that \ allows \ for \ empty \ succedents, \ e.g. 
\[
\frac{\Gamma \Rightarrow \Delta}{\Box \Gamma \Rightarrow \Box \Delta} \text{ K\Box'}
\]
where \(|\Delta| \leq 1\), would \ allow \ for \ the \ derivation \ of \ \neg \Box \bot (5) \ and \ \Diamond \top (6) \ but \ not \ \neg \Box (A \land \neg A) (7) \ and \ \Diamond (A \lor \neg A) (8).

Remark 5. \ The converse of (10), \(\Box A \land \Box B \rightarrow \Box (A \land B)\) \ is \ not \ derivable \ in \ either \ \textbf{GMALL} \ or \ \textbf{GKMALL}.

4. \textbf{Multiplicative Additive Linear Logic with Mingle}

Linear Logics with mingle are discussed in [10, 11]. \ The mingle (also called “merge” or “mix”) rule is:
\[
\frac{\Gamma \Rightarrow \Delta, \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'} \ \text{ M}
\]

**Proposition 6.** \ The \ following \ are \ derivable \ in \ \textbf{GMALL} \ + \ M:
\[
\Box A \rightarrow \Diamond A \tag{14}
\]
\[
(\Diamond A \rightarrow \Diamond B) \rightarrow \Diamond (A \rightarrow B) \tag{15}
\]
where (14) \ corresponds \ to \ a \ form \ of \ the \ \textbf{D} \ axiom.

Remark 6. \ We \ note \ that \ the \ formulae \ (14) \ and \ (15) \ can \ be \ derived \ using \ anti-contraction \ (duplication) \ rules \ as \ well:
\[
\frac{A, \Gamma \Rightarrow \Delta}{A, A, \Gamma \Rightarrow \Delta} \ \text{ LC}^{-1} \quad \frac{\Gamma \Rightarrow \Delta, A}{\Gamma, A \Rightarrow \Delta, A} \ \text{ RC}^{-1}
\]

5. \textbf{Affine Logic}

Affine Logic [7], also called \textbf{Affine Multiplicative Additive Linear Logic (AMALL)}, \ is \ 
\textbf{MALL} \ augmented \ with \ the \ weakening \ axiom, \ (A \rightarrow 1) \land (0 \rightarrow A). \ The \ corresponding \ calculus, \ \textbf{GAMALL} \ is \ obtained \ by \ adding \ weakening \ rules \ to \ \textbf{GMALL}:
\[
\frac{\Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta} \ \text{ LW} \quad \frac{\Gamma \Rightarrow \Delta}{\Gamma, A \Rightarrow \Delta, A} \ \text{ RW}
\]

Remark 7. \ In \ Affine \ Logic, \ 0 \equiv \bot \text{ and } 1 \equiv \top.

**Proposition 7.** \ If \ \textbf{GMALL} + M \vdash A, \ then \ \textbf{GAMALL} \vdash A.

\textit{Proof.} \ M \ is \ admissible \ in \ \textbf{GAMALL}. \ \Box

Remark 8. \ Hence, \ the \ formulae \ in \ Proposition \ 6 \ are \ derivable \ in \ \textbf{GAMALL}.

**Proposition 8 (Additive Modal Rules).** \ \textbf{The \ following \ rules \ can \ be \ derived \ in \ \textbf{GAMALL}:}
\[
\frac{\Gamma \Rightarrow \Delta, A}{\Gamma \Rightarrow \Delta, \Diamond A} \ \text{ R\Diamond'} \quad \frac{\Gamma, A \Rightarrow \Delta}{\Gamma, \Box A \Rightarrow \Delta} \ \text{ Lo'}
\]

\textit{Proof.} \ Straightforward. \ \Box

**Proposition 9.** \ \textbf{The \ following \ are \ derivable \ in \ \textbf{GAMALL}:}
\[
\Box A \rightarrow A \tag{16}
\]
\[
A \rightarrow \Diamond A \tag{17}
\]
where (16) \ corresponds \ to \ the \ \textbf{T} \ axiom.
Proof. Straightforward. □

Proposition 10. The following rules are derivable in GAMALL:

\[
\begin{align*}
A \Rightarrow B \\
\Rightarrow \bigodot (A \rightarrow \Box B) & \quad \Rightarrow \bigodot (\bigodot A \rightarrow B) \\
\end{align*}
\]

Proof. Straightforward. □

Proposition 11. The following are derivable in GAMALL:

\[
\begin{align*}
\bigodot (\Box A \rightarrow \Box \Box A) & \quad (18) \\
\bigodot (\bigodot A \rightarrow \bigodot \bigodot A) & \quad (19) \\
\bigodot (A \rightarrow \bigodot \bigodot A) & \quad (20) \\
\bigodot (\bigodot A \rightarrow A) & \quad (21) \\
\bigodot (A \rightarrow \bigodot A) & \quad (22) \\
\bigodot (A \rightarrow \bigodot B) \rightarrow \bigodot (\bigodot A \rightarrow B) & \quad (23)
\end{align*}
\]

where (18), (19) and (20) are “\bigodot -forms” of the S4, S5 and B axioms, respectively.

Proof. Straightforward. □

6. Strict Logic

A calculus GSLL for Strict Linear Logic (SLL) is obtained by adding to GMALL the contraction rules:

\[
\begin{align*}
A, A, \Gamma \Rightarrow \Delta & \quad \text{LC} \\
\Gamma \Rightarrow \Delta, A, \Gamma & \quad \text{RC}
\end{align*}
\]

Proposition 12. The following can be derived in GSLL:

\[
\begin{align*}
A \rightarrow \Box A & \quad (24) \\
\bigodot A \rightarrow A & \quad (25) \\
\Box A \rightarrow \Box \Box A & \quad (26) \\
\bigodot \bigodot A \rightarrow A \bigodot & \quad (27) \\
\bigodot A \rightarrow \bigodot \bigodot A & \quad (28) \\
A \rightarrow \bigodot \bigodot A & \quad (29)
\end{align*}
\]

where (26), (28) and (29) correspond to the S4, B and S5 axioms, respectively.

Proof. Straightforward. □

Proposition 13. The following can be derived in GSLL:

\[
\begin{align*}
(\Box A \land \Box B) \rightarrow \Box (A \land B) & \quad (30) \\
(\bigodot A \land \bigodot B) \rightarrow \bigodot (A \land B) & \quad (31) \\
\Box (A \lor B) \rightarrow (\Box A \lor \Box B) & \quad (32) \\
\bigodot (A \lor B) \rightarrow (\bigodot A \lor \bigodot B) & \quad (33)
\end{align*}
\]

Proof. Straightforward. □

Remark 9. These are the converse of formulae (30) through (33). Note that (31) is the same formulae as (2) mentioned in the introduction. Indeed (30) may also be considered paradoxical.
7. Involutive Uninorm Logic

Involutive Uninorm Logic (GIUL) [15] is a substructural fuzzy logic, and has a hypersequent calculus (G) (Figure 3) based on a hyperextension of (G) [1] and the communication (Com) rule.

\[
\begin{array}{c}
\frac{0 \Rightarrow \bot}{\bot, \Gamma \Rightarrow \Delta} & \frac{\mathcal{H} \mid \Gamma \Rightarrow \Delta}{\mathcal{H} \mid \Gamma \Rightarrow \Delta, 0} R^0 & \frac{\mathcal{H} \mid \Gamma \Rightarrow \Delta}{\mathcal{H} \mid \Gamma \Rightarrow \Delta, 1} R^1 \\
\frac{\bot, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \top} L^\bot & \frac{\Gamma \Rightarrow \Delta, T}{\Gamma \Rightarrow \Delta, 1} R^T
\end{array}
\]

\[
\begin{array}{c}
\frac{\mathcal{H} \mid \Gamma, A, B \Rightarrow \Delta}{\mathcal{H} \mid A \otimes B, \Gamma \Rightarrow \Delta} L^\otimes & \frac{\mathcal{H} \mid \Gamma \Rightarrow \Delta, A}{\mathcal{H} \mid \Gamma \Rightarrow \Delta, A' \otimes B} R^\otimes \\
\frac{\mathcal{H} \mid \Gamma, A, B, \Gamma' \Rightarrow \Delta'}{\mathcal{H} \mid \Gamma \Rightarrow \Delta, A' \otimes B} L^\otimes & \frac{\mathcal{H} \mid \Gamma \Rightarrow \Delta, A, B}{\mathcal{H} \mid \Gamma \Rightarrow \Delta, A' \otimes B} R^\otimes \\
\frac{\mathcal{H} \mid A, \Gamma \Rightarrow \Delta}{\mathcal{H} \mid A \wedge B, \Gamma \Rightarrow \Delta} L^\land & \frac{\mathcal{H} \mid B, \Gamma \Rightarrow \Delta}{\mathcal{H} \mid A \wedge B, \Gamma \Rightarrow \Delta} L^\land \\
\frac{\mathcal{H} \mid A, \Gamma \Rightarrow \Delta}{\mathcal{H} \mid A \vee B, \Gamma \Rightarrow \Delta} L^\lor & \frac{\mathcal{H} \mid \Gamma \Rightarrow \Delta, A}{\mathcal{H} \mid \Gamma \Rightarrow \Delta, A' \vee B} R^\lor_1 & \frac{\mathcal{H} \mid \Gamma \Rightarrow \Delta, B}{\mathcal{H} \mid \Gamma \Rightarrow \Delta, A' \vee B} R^\lor_2 \\
\frac{\mathcal{H} \mid \Gamma \Rightarrow \Delta, A}{\mathcal{H} \mid \sim A, \Gamma \Rightarrow \Delta} L^\sim & \frac{\mathcal{H} \mid A, \Gamma \Rightarrow \Delta}{\mathcal{H} \mid \Gamma \Rightarrow \Delta, \sim A} R^\sim \\
\frac{\mathcal{H} \mid \Gamma \Rightarrow \Delta, A}{\mathcal{H} \mid A \rightarrow B, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'} L^\rightarrow & \frac{\mathcal{H} \mid A, \Gamma \Rightarrow \Delta, B}{\mathcal{H} \mid \Gamma \Rightarrow \Delta, A \rightarrow B} R^\rightarrow \\
\frac{\mathcal{H} \mid \Gamma \Rightarrow \Delta}{\mathcal{H} \mid \Gamma \Rightarrow \Delta} EW & \frac{\mathcal{H} \mid \Gamma \Rightarrow \Delta, \Gamma \Rightarrow \Delta}{\mathcal{H} \mid \Gamma \Rightarrow \Delta} EC \\
\end{array}
\]

Figure 3: Rules for GIUL.

**Proposition 14.** The following rules are derivable in GIUL:

\[
\begin{array}{c}
\frac{\mathcal{H} \mid A, \Gamma \Rightarrow \Delta \mid B, \Gamma \Rightarrow \Delta}{\mathcal{H} \mid A \wedge B, \Gamma \Rightarrow \Delta} L^\land' & \frac{\mathcal{H} \mid \Gamma \Rightarrow \Delta, A \mid \Gamma \Rightarrow \Delta, B}{\mathcal{H} \mid \Gamma \Rightarrow \Delta, A \vee B} R^\lor'
\end{array}
\]

**Proof.** Straightforward, using EC. \qed
Proposition 15. Formulae (30) through (33) are derivable in GIUL.

Proof. Straightforward, using rules from Proposition 14 and Com. A proof of (30):

\[
\frac{A \Rightarrow A}{A \Rightarrow A} \quad \frac{A \Rightarrow A | B \Rightarrow A \wedge B}{A \Rightarrow A \wedge B} \quad \frac{B \Rightarrow B}{B \Rightarrow B} \quad \frac{A \Rightarrow B | B \Rightarrow A}{A \Rightarrow B \wedge A} \quad \frac{A \Rightarrow B | B \Rightarrow B}{A \Rightarrow B \wedge B} \quad \frac{A \Rightarrow A \wedge B | B \Rightarrow A \wedge B}{R \wedge} \\
\frac{A \Rightarrow A \wedge B | B \Rightarrow A \wedge B}{R \wedge} \quad \frac{A, A \Rightarrow \Box (A \wedge B) | B, B \Rightarrow \Box (A \wedge B)}{L \wedge^2} \quad \frac{\Box A \Rightarrow \Box (A \wedge B) | \Box B \Rightarrow \Box (A \wedge B)}{L \wedge'} \quad \frac{\Box A \wedge \Box B \Rightarrow \Box (A \wedge B)}{R \rightarrow} \\
\Rightarrow (\Box A \wedge \Box B) \rightarrow \Box (A \wedge B) \quad R \rightarrow
\]

Proofs of (31) through (33) are similar.

8. Discussion and Future Work

Much of the content in this paper is straightforward. However, the formal properties of the Tarskian möglichkeit are of interest.

Theorem 5 is noteworthy, in that all of the derivable modal sequents derivable in GKMALL correspond to derivable modal sequents in GMALL using the Tarskian möglichkeit.

A semantic characterisation of the Tarskian modalities with respect to various logics is in process.

A deeper comparison of substructural logics with Tarskian modalities and their counter-part extensions to KMALL is an area of future investigation.

Acknowledgements

We’d like to thank those attending a talk at an LFCS Lab Lunch about this topic for their comments and suggestions.

Appendix A. Substructural-K (KMALL)

Lemma 16 (Cut Admissibility). GKMALL admits cut

\[
\frac{\Gamma \Rightarrow A, A \Rightarrow A \wedge B, \Gamma \Rightarrow A \wedge B}{\Gamma, \Gamma \Rightarrow \Delta, \Delta'} \quad \text{cut}
\]

Proof. Note that GMALL admits cut [19]. Adding K\Box to GKMALL also admits cut, by induction on the derivation height, with the following cases:

(1) If the cut formula is not of the form \Box A, then permute cut upwards.

(2) If the cut formula is not the principal formula on either premiss, permute the cut upward on that premiss.

(3) If the cut formula is the principal formula of either an instance of \Box \perp or \Box \top, then so is the conclusion of the cut.

(4) The remaining case is that both premisses of the cut are the conclusions of instances of K\Box. The cut is then permuted to the premisses of both K\Box instances, and K\Box is applied to the conclusion of the cut.

□
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