Universal Hypermultiplet Metrics

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Abstract

Some instanton corrections to the universal hypermultiplet moduli space metric of the type-IIA string theory compactified on a Calabi-Yau threefold arise due to multiple wrapping of BPS membranes and fivebranes around certain cycles of Calabi-Yau. The classical universal hypermultiplet metric is locally equivalent to the Bergmann metric of the symmetric quaternionic space $SU(2,1)/U(2)$, whereas its generic quaternionic deformations are governed by the integrable $SU(\infty)$ Toda equation. We calculate the exact (non-perturbative) UH metrics in the special cases of (i) the D-instantons (the wrapped D2-branes) in the absence of fivebranes, and (ii) the fivebrane instantons with vanishing charges, in the absence of D-instantons. The solutions of the first type preserve the $U(1) \times U(1)$ classical symmetry, while they can be interpreted as the gravitational dressing of the hyper-Kähler D-instanton solutions. The solutions of the second type preserve the non-abelian $SU(2)$ classical symmetry, while they can be interpreted as the gradient flows in the universal hypermultiplet moduli space.

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1 Introduction

Non-perturbative (instanton) quantum corrections in compactified M-theory/strings are believed to be crucial for solving the fundamental problems of vacuum degeneracy and supersymmetry breaking [1]. Some instanton corrections to various physical quantities in the effective lower-dimensional supergravity theory, originating from the string/M-theory compactification on a Calabi-Yau (CY) threefold \( \mathcal{Y} \), can be understood in terms of the Euclidean five-branes wrapped about the entire CY space and the Euclidean membranes (two-branes) wrapped about special (supersymmetric) three-cycles \( \mathcal{C} \) of \( \mathcal{Y} \) [2, 3]. Being solitonic (BPS) classical solutions to the higher dimensional (Euclidean) equations of motion with non-vanishing topological charges, these wrapped branes are localized in the uncompactified dimensions and thus can be identified with the instantons. The instanton actions are essentially given by the volumes of the cycles on which the branes are wrapped.

For instance, the compactification of the type-IIA superstring theory on \( \mathcal{Y} \) gives rise to the four-dimensional (4d) \( N=2 \) superstrings whose Low-Energy Effective Action (LEEA) is given by the 4d, \( N=2 \) supergravity coupled to \( N=2 \) vector supermultiplets and hypermultiplets. Any other \( N=2 \) matter multiplet (of physical spin \( \leq 1 \) or \( \leq 1/2 \)) can be dualized to an \( N=2 \) vector multiplet or a hypermultiplet, respectively. The numbers of the \( N=2 \) matter vector multiplets and hypermultiplets are dictated by the topological data about \( \mathcal{Y} \), namely, the Hodge numbers \( h_{1,1} \) and \( h_{1,2} + 1 \), respectively, where \( h_{p,q} \) are the dimensions of the Dolbeaux cohomology groups of \( \mathcal{Y} \) [1]. The hypermultiplet LEEA is most naturally described by the Non-Linear Sigma-Model (NLSM), whose scalar fields parametrize the quaternionic target space \( \mathcal{M}_H \) [3]. The instanton corrections to the LEEA due to the wrapped fivebranes and membranes can be easily identified and distinguished from each other in the semi-classical limit, since the fivebrane instanton corrections are organized by powers of \( e^{-1/g_{\text{string}}^2} \), whereas the membrane instanton corrections are given by powers of \( e^{-1/g_{\text{string}}} \), where \( g_{\text{string}} \) is the type-IIA superstring coupling constant [6].

From the M-theory perspective, the compactified field theory (LEEA) is given by the five-dimensional supergravity with the same number of unbroken supercharges. The effective supergravities in four and five dimensions are related via the compactification of the latter on a circle, which does not affect the NLSM target space \( \mathcal{M}_H \).

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2 The supersymmetric 3-cycle \( \mathcal{C} \) is a three-dimensional submanifold of the CY space, which obeys \( J|_\mathcal{C} = \text{Im}(\Omega)|_\mathcal{C} = 0 \), where \( J \) is the Kähler form and \( \Omega \) is the holomorphic (3,0) form in \( \mathcal{Y} \). The supersymmetric cycles minimize volume in their homology class [4], while the corresponding wrapped brane configurations lead to the BPS states.
in the hypermultiplet sector. The vacuum expectation value of the four-dimensional
dilaton field $\langle \phi \rangle$ in the compactified type-IIA superstring is simply related to the
CY volume $V_{\text{CY}}$ in M-theory, $V_{\text{CY}} = e^{-2\langle \phi \rangle}$, so that the type-IIA superstring loop
expansion amounts to the derivative expansion of the M-theory action [7].

The unbroken local supersymmetry with eight supercharges (e.g., N=2 in 4d)
put severe constraints on the LEEA of matter supermultiplets or, equivalently, on
their moduli space. In fact, it requires the whole moduli space $\mathcal{M}$ be the local
product, $\mathcal{M} = \mathcal{M}_V \times \mathcal{M}_H$, where $\mathcal{M}_V$ is the special Kähler manifold associated
with the N=2 vector multiplets [8], and $\mathcal{M}_H$ is the quaternionic manifold associated
with the hypermultiplets [5]. Any CY compactification has the co-called Universal
Hypermultiplet (UH) containing a dilaton, an axion, a complex RR-type pseudo-scalar
and a Dirac dilatino [9]. The constraints imposed by 4d, N=2 local supersymmetry on
the four-dimensional UH moduli space are even stronger then in higher dimensions:
the target space of the universal hypermultiplet NLSM has to be an Einstein space
with the (anti)self-dual Weyl tensor (these spaces are called the self-dual Einstein
spaces or simply the Einstein-Weyl spaces in the mathematical literature).

The N=2 vector multiplet moduli space $\mathcal{M}_V$ is relatively well understood, whereas
much less is known about the hypermultiplet moduli space $\mathcal{M}_H$ [10]. The conjectured
duality between the type-IIA superstring compactification on CY and the heterotic
string compactification on $K3 \times T^2$, which is supposed to exchange $\mathcal{M}_V$ and $\mathcal{M}_H$
in the same moduli space $\mathcal{M}$, was used in the past to get information about $\mathcal{M}_H$
on the type-II string side from the knowledge about $\mathcal{M}_V$ on the heterotic string
side, although with only partial success [11]. Since the UH transforms into the N=2
supergravity multiplet under type-II mirror symmetry, any quantum corrections to
the classical UH metric are essentially gravitational in nature [7].

The standard instanton calculus gives us another technical device after taking
into account the partial supersymmetry breaking induced by the BPS branes [4]. Be-
ing applied to the bosonic instanton background, broken supersymmetries generate
fermionic (Goldstino) zero modes that are to be absorbed by extra terms in the ef-
fective field theory. These instanton-induced interactions are quartic in the fermionic
fields (by index theorems) and thus contribute to the curvature tensor of the sup-
persymmetric NLSM. By supersymmetry, the instanton corrections to the NLSM
curvature imply the corresponding deformations in the NLSM metric. To calculate
these non-perturbative corrections by using the standard instanton technology, one
may integrate over the fermionic zero modes and then compute the fluctuation deter-

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3See e.g., ref. [10] for a review.
In this paper we restrict ourselves to a calculation of the instanton corrections to the universal hypermultiplet NLSM metric, by analyzing generic quaternionic deformations of the classical UH metric. We apply the most direct procedure based on the fact that the (anti)self-dual Weyl tensor already implies the integrable system of partial differential equations on the components of the UH moduli space metric. Additional simplifications arise due to the Einstein condition and the physically motivated isometries. The UH metric in question is supposed to be regular and complete.

The local data about the underlying CY threefold \( \mathcal{Y} \) is described in terms of the \( h_{p,q} \) complex moduli. In the compactified theory, the \( h_{1,2} \) moduli are promoted to the non-universal matter hypermultiplets. The universal hypermultiplet is ‘universal’ in the sense that its own moduli space does not depend upon the CY moduli, or, equivalently, the UH is merely sensitive to the global information about the CY space (like topology, volume, charges and symmetries).

Some instanton solutions to the effective Euclidean (UH + supergravity) equations of motion and the corresponding instanton actions were found in refs. [3, 13, 14]. The instanton solutions carry charges descending from the wrapped brane charges, and they preserve a part (half) of type-IIA supersymmetry. The instanton actions are also dependent upon the charges of the instanton and the complex structure (or central charge) at the moduli space infinity. Unfortunately, no exact quaternionic metrics in the UH moduli space beyond string perturbation theory were constructed, which would amount to a calculation of the exact LEEA of the universal hypermultiplet. In this paper we report about the results of our investigation initiated in ref. [15].

The paper is organized as follows. In sect. 2 we review the four-dimensional classical LEEA (or NLSM) of the universal hypermultiplet in the background of 4d, \( N=2 \) supergravity [16], and discuss its perturbative deformations [7, 17] and the origin of non-perturbative corrections due to the wrapped BPS branes [2, 3, 13, 14]. The symmetry structure of the classical UH moduli space is given in sect. 3. The D-instanton corrections due to the wrapped membranes (D2-branes) in the hyper-Kähler limit [18] are presented in sect. 4, whereas their quaternionic generalizations are described in sect. 5. In sect. 6 we demonstrate that the zero-charge fivebrane instanton corrections to the UH metric are described by the Tod-Hitchin metric. Its physical interpretation as the gradient flow in the UH moduli space is also given in sect. 6. Generic UH metrics are briefly discussed in sect. 7. In Appendix A we collect basic facts about the quaternionic symmetric space \( SU(2,1)/U(2) \). Appendix B is devoted to basic definitions of theta functions and some useful identities between them.
2 M-theory, type-IIA string perturbation theory, and the UH classical moduli space metric

The eleven-dimensional (11d) M-theory is supposed to substitute the ten-dimensional (10d) type-IIA superstring theory at strong string coupling, while the M-theory LEEA is described by 11d supergravity [20]. The action of 11d supergravity is unique, and its bosonic part reads [21]

\[
S_{11} = \frac{1}{2\kappa_{11}^2} \int d^{11}x \sqrt{-\hat{g}} \left( \mathcal{R} - \frac{1}{48} \hat{F}^2 \right) + \frac{1}{12\kappa_{11}^2} \int \hat{A}_3 \wedge \hat{F}_4 \wedge \hat{F}_4 ,
\]

(2.1)

where \( \kappa_{11} \) is the 11d gravitational constant, \( \hat{F}_4 = d\hat{A}_3 \), and all quantities with ‘hats’ refer to 11d. Though the 11d supergravity action is only the leading (LEEA) part of the full (unknown) M-theory action, it is already enough to study the BPS states in M-theory. The latter are given by the solitonic classical solutions (called M-branes) to 11d supergravity, which preserve some part (say, a half) of 11d supersymmetry. The BPS condition amounts to the existence of a Majorana-Killing 11d spinor \( \varepsilon_\alpha(x) \) obeying the first-order linear differential equation

\[
\bar{D}_M \varepsilon \equiv \left[ (\partial_M + \frac{1}{4} \omega^{BC}_M \Gamma_{BC}) - \frac{1}{288} \left( \Gamma^M_{NPQS} - 8\delta^N_M \Gamma^{PQS} \right) F_{NPQS} \right] \varepsilon = 0 ,
\]

(2.2)

where \( M = 0, 1, \ldots, 10 \), \( \omega^{BC}_M \) is the 11d supergravity spin connection, and \( \Gamma \)'s are the antisymmetric products (with unit weight) of 11d Dirac matrices. The M-theory supersymmetry algebra is just the most general super-Poincaré algebra in 11d [22],

\[
\{Q_\alpha, Q_\beta\} = (C \Gamma^M)_{\alpha\beta} P_M + \frac{1}{27} (C \Gamma^{MN})_{\alpha\beta} Z_{MN} + \frac{1}{141} (C \Gamma^{NPQS})_{\alpha\beta} Y_{MNPQS} ,
\]

(2.3)

where \( Q_\alpha, \alpha = 1, \ldots, 32 \), are the 11d supersymmetry charges, \( P_M \) are 11d translations, \( C \) is the 11d charge conjugation matrix, whereas \( Z_{MN} \) and \( Y_{MNPQS} \) stand for the ‘electric’ and ‘magnetic’ charges, respectively,

\[
Z_{MN} = \int_{M2\text{-brane}} dx^M \wedge dx^N ,
\]

\[
Y_{MNPQS} = \int_{M5\text{-brane}} dx^M \wedge dx^N \wedge dx^P \wedge dx^Q \wedge dx^S .
\]

(2.4)

Equation (2.3) is just the 11d Fierz rearrangement formula for \( \{Q_\alpha, Q_\beta\} \in 528 \) of \( Spin(32) \). The M-theory solitons are thus given by membranes [23] and fivebranes [24], which are in fact dual to each other under the ‘electric-magnetic’ duality in 11d.

4The 11d signature is \((-,-,+,...,+\).
The physical and topological significance of the abelian (‘electric’ and ‘magnetic’) charges in eq. (2.3) become clear from the 11d equations of motion, which follow from the action (2.1). In particular, one has
\[ d^*_{11}F_4 + \frac{1}{2} \hat{F}_4^2 = d \left( *_{11} \hat{F}_4 + \frac{1}{2} \hat{A}_3 \wedge \hat{F}_4 \right) = 0 , \] (2.5)
where \( *_{11} \hat{F}_4 \) is the 11d dual to \( \hat{F}_4 \). Hence, the ‘electric’ charge of an M2-brane [25],
\[ Q_e = \frac{1}{\kappa_{11} \sqrt{2}} \int_{S^7} \left( *_{11} \hat{F}_4 + \frac{1}{2} \hat{A}_3 \wedge \hat{F}_4 \right) , \] (2.6)
is conserved, where \( S^7 \) is the asymptotic seven-sphere surrounding the M2-brane (cf. Gauss law in Maxwell electrodynamics). Similarly, the ‘magnetic’ charge of the M5-brane [25],
\[ Q_m = \frac{1}{\kappa_{11} \sqrt{2}} \int_{S^4} \hat{F}_4 , \quad d\hat{F}_4 = 0 , \] (2.7)
is also conserved, where \( S^4 \) is the asymptotic four-sphere surrounding the M5-brane (cf. Dirac monopole charge). The Dirac quantization condition implies
\[ Q_e Q_m = 2\pi (hc)Z . \] (2.8)

Demanding the membrane and fivebrane (worldvolume) actions be well defined in quantum theory gives rise to a quantization of their electric and magnetic charges or, equivalently, the M-brane tension quantization [20, 21],
\[ 2\pi T_5 = T_2^2 \quad \text{and} \quad \kappa_{11}^2 T_2 T_5 = \pi Z . \] (2.8)

A connection to type-IIA superstrings is obtained by compactifying M-theory/11d supergravity on a circle of radius \( r = \sqrt{\alpha'} \) with the Kaluza-Klein (KK) Ansatz [22]
\[ ds^2_{11} = e^{-2\phi/3} \tilde{s}^2_{10} + e^{4\phi/3} (dx^{11} - A_m dx^m)^2 , \] (2.9)
where \( \tilde{s}^2_{10} = g_{mn} dx^m dx^n \) is the 10d spacetime metric (in the string frame) with \( m, n = 0, 1, \ldots, 9 \), \( A_m \) is the KK (type-IIA) vector field, and \( \phi \) is the 10d dilaton. The gravitational constants in 11d and 10d are related by \( \kappa_{10}^2 = \kappa_{11}^2 / (2\pi \sqrt{\alpha'}) \). Performing the dimensional reduction of the action (2.1) on the circle with the help of eq. (2.9) gives rise to the bosonic part of the type-IIA supergravity action in the string frame. The latter is related to the canonical form of the type-IIA supergravity action (in the Einstein frame) via the Weyl rescaling \( \tilde{g}_{mn} = e^{\phi/2} g_{mn} \). This yields the standard bosonic part of the 10d type-IIA supergravity action,
\[ S_{10} = \frac{1}{2\kappa_{10}^2} \int d^{10}x \sqrt{-g} \left[ \mathcal{R} - \frac{1}{2} (\partial_m \phi)^2 \right] - \frac{1}{4\kappa_{10}^2} \int \left[ e^{\phi/2} F_4 \wedge *_{10} F_4 + e^{-\phi} H_3 \wedge *_{10} H_3 \right] \\
+ \frac{1}{4\kappa_{10}^2} \int B_2 \wedge F_4 \wedge F_4 , \] (2.10)
where $H_3 = dB_2$ and $F_4 = dA_3$ originate from the three-form $\hat{A}_3$ and its field strength $\hat{F}_4$ in 11d, e.g., $\hat{A}_3 = A_3 + B_2 \wedge dx^{11}$, etc.

The 10d type-IIA supergravity is known to be the LEEA of 10d type-IIA strings in the (string) tree approximation \cite{1}. The double dimensional reduction of the 11d (M-theory) membrane yields a (type-IIA) fundamental string in 10d. The fundamental string in 10d is dual to a solitonic (NS-NS) five-brane. In M-theory compactified on the circle $S^1$ the 11d fivebrane can be wrapped about $S^1$, which gives rise to the (RR-charged) type-IIA fourbrane called D4-brane \cite{22}. In 10d this D4-brane is dual to the 10d (RR-charged) membrane called D2-brane \cite{22}.

The non-perturbative (instanton) corrections to the four-dimensional UH originate from the 10d type-IIA membranes (D2-branes) wrapped about certain (supersymmetric or special Lagrangian) 3-cycles $C$ of CY, and the 10d type-IIA NS-NS fivebranes wrapped about the entire CY space \cite{2, 3, 13, 14}. From the 11d M-theory perspective, the dilaton expectation value is related to the volume of the CY space (sect. 1), whereas the expectation value of the RR scalar $C$ is related to the CY period $\int C \Omega$.

The universal sector (UH) of the CY compactification of the 10d type-IIA supergravity in four dimensions is most easily obtained by using the following Ansatz for the 10d metric \cite{3}:

\begin{equation}
    ds_{10}^2 = g_{mn} dx^m dx^n = e^{-\phi/2} ds_{CY}^2 + e^{3\phi/2} g_{\mu \nu} dx^\mu dx^\nu ,
\end{equation}

while keeping only $SU(3)$ singlets in the internal CY indices \cite{19} and ignoring all CY complex moduli. In eq. (2.11) $\phi(x)$ stands for the 4d dilaton, $g_{\mu \nu}(x)$ is the spacetime metric in four uncompactified dimensions, $\mu, \nu = 0, 1, 2, 3$, and $ds_{CY}^2$ is the (Kähler and Ricci-flat) metric of the internal CY threefold $\mathcal{Y}$ in complex coordinates,

\begin{equation}
    ds_{CY}^2 = g_{ij}(y, \bar{y}) dy^i d\bar{y}^j ,
\end{equation}

where $i, j = 1, 2, 3$. By definition, the CY threefold $\mathcal{Y}$ possesses the $(1, 1)$ Kähler form $J$ and the holomorphic $(3, 0)$ form $\Omega$. The universal hypermultiplet (UH) unites the dilaton $\phi$, the axion $D$ coming from dualizing the three-form field strength $H_3 = dB_2$ of the NS-NS two-form $B_2$ in 4d, and the complex scalar $C$ representing the RR three-form $A_3$ with $A_{ijk}(x, y) = \sqrt{2} C(x) \Omega_{ijk}(y)$. When using a flat (or rigid) CY (cf. a six-torus of equal radii $r = \sqrt{\alpha'}$), with

\begin{equation}
    g_{ij} = \delta_{ij} \quad \text{and} \quad \Omega_{ijk} = \varepsilon_{ijk} ,
\end{equation}

the reduction of eq. (2.10) down to four dimensions with the help of eqs. (2.12) and
\( (2.13) \) yields the so-called Ferrara-Sabharwal action \[ S_4 = \frac{1}{2\kappa_4^2} \int d^4x \sqrt{-g} \left[ R - 2(\partial_m \phi)^2 - 2e^{2\phi} |\partial_\mu C|^2 \right] \]

\( - \frac{1}{4\kappa_4^2} \int \left[ e^{-4\phi} H_3 \wedge *H_3 - 2iH_3 \wedge \bar{C}^\dagger \bar{d}C - 4H_3 \wedge dD \right] , \]

where the real 4d Lagrange multiplier \( D \) has been introduced to enforce the Bianchi relation \( dH_3 = 0 \). Removing \( H_3 \) via its equations of motion from eq. (2.14) results in the dual action \[ S_4 = -\frac{1}{\kappa_4^2} \int d^4x \sqrt{-g} \left[ -\frac{1}{2} R + (\partial_m \phi)^2 + e^{2\phi} |\partial_\mu C|^2 + e^{4\phi} \left( \partial_\mu D + \frac{i}{2} \bar{C}^\dagger \bar{d}C \right)^2 \right] . \]

The equivalent action in the string frame reads \[ S_4, \text{string} = \frac{1}{\kappa_4^2} \int d^4x \sqrt{-g} e^{-2\phi} \left\{ \frac{1}{2} \bar{R} + 2(\partial_m \phi)^2 - \frac{1}{6} H_{\mu\nu\lambda}^2 \right\} \]

\( - \frac{1}{\kappa_4^2} \int d^4x \sqrt{-g} \left[ \partial_\mu C \partial^\mu \bar{C} + \frac{i}{2} H^\mu \left( C \partial_\mu \bar{C} - \bar{C} \partial_\mu C \right) \right] , \]

where \( H^\rho = \frac{1}{\sqrt{-g}} \epsilon^{\rho\mu\nu\lambda} H_{\mu\nu\lambda} \) is the Hodge dual of the field strength \( H_3 = dB_2 \). The N=2 supergravity background in eqs. (2.14), (2.15) and (2.16) is merely represented by the 4d spacetime metric. The NLSM action of the UH thus has the following target space metric \[ -\kappa_4^2 ds_{\text{classical}}^2 = d\phi^2 + e^{2\phi} |dC|^2 + e^{4\phi} \left( dD + \frac{i}{2} \bar{C}^\dagger \bar{d}C \right)^2 , \]

whose right-hand-side is regular and positively definite. This metric is of purely gravitational origin, with \( \kappa_4^2 \) being the only coupling constant.

The perturbative (one-loop) string corrections to the UH metric (2.17) originate from the \((\text{Riemann})^4\) terms \[ \text{in M-theory compactified on a CY threefold } Y \, [4]. \] These quantum corrections are proportional to the CY Euler number \( \chi = 2(h_{1,1} - h_{1,2}) \, [23]. \) The one-loop corrected action of UH coupled to gravity in the string frame reads \[ S_4, \text{one-loop} = \frac{1}{\kappa_4^2} \int d^4x \sqrt{-g} \left\{ (e^{-2\phi} + \chi) \left( \frac{1}{2} \bar{R} - \frac{1}{6} H_{\mu\nu\lambda}^2 + \frac{2e^{-4\phi}}{e^{2\phi} + \chi} (\partial_\mu \phi)^2 \right) \right\} \]

\( - \frac{1}{\kappa_4^2} \int d^4x \sqrt{-g} \left[ \partial_\mu C \partial^\mu \bar{C} + \frac{i}{2} H^\mu \left( C \partial_\mu \bar{C} - \bar{C} \partial_\mu C \right) \right] , \]

\[ \text{The 4d spacetime signature is } (-, +, +, +). \] The 4d gravitational constant \( \kappa_4 \) is related to the 10d gravitational constant \( \kappa_{10} \) via the equation \( \kappa_4^2 = \kappa_{10}^2/(2\pi r^6) = \kappa_{10}^2/(2\pi \sqrt{\alpha'})^6. \)
where $\chi$ is proportional to $\chi$. The superstring (loop) perturbation theory is controlled by the powers of $e^{-2\phi}$, whereas the second line of eq. (2.18) cannot be multiplied by any power of $e^{-2\phi}$ without breaking the perturbative Peccei-Quinn type symmetry $C \to C + \text{const}$. This non-renormalization argument does not, however, exclude the non-perturbative (gravitational-type) corrections due to the wrapped branes, which break the Peccei-Quinn type symmetries (sects. 4 and 5).

The string one-loop corrected NLSM metric dictated by eq. (2.18) is related to the classical metric of eq. (2.17) by a local field redefinition, $e^{-2\phi} \to e^{-2\phi} + \chi$, so that the local UH geometry is not affected by perturbative string corrections \cite{7}. String duality, however, implies some discrete (global) identifications in the UH moduli space, which are the consequence of the brane charge (and tension) quantization in M-theory \cite{3} (see the next sect. 3).

3 Classical metric and perturbative deformations

The NLSM of UH with the target space metric (2.17) can be rewritten to the form \cite{2, 4}

$$L_{\text{NLSM}} = -\frac{1}{\kappa^2} \left( K_{,SS} \partial_\mu S \partial^\mu \bar{S} + K_{,SC} \partial_\mu S \partial^\mu \bar{C} + K_{,CS} \partial_\mu C \partial^\mu \bar{S} + K_{,CC} \partial_\mu C \partial^\mu \bar{C} \right) \quad (3.1)$$

in terms of two complex variables, $C$ and $S$,

$$S = e^{-2\phi} + 2iD + C\bar{C} \quad , \quad (3.2)$$

where commas in subscripts denote partial derivatives, and the Kähler potential reads

$$K = -\log \left( S + \bar{S} - 2C\bar{C} \right) = -\log \left( 2e^{-2\phi} \right) . \quad (3.3)$$

Equation (3.1) makes manifest the Kähler nature of the classical NLSM metric,

$$ds^2_{\text{NLSM}} = e^{2K} \left[ dS d\bar{S} - 2CdS d\bar{C} - 2\bar{C} d\bar{S} dC + 2(S + \bar{S}) dC d\bar{C} \right] . \quad (3.4)$$

Other useful parametrizations are given in Appendix A. In particular, after the change of variables (A.9) or its inverse,

$$S = \frac{1 - z_1}{1 + z_1} , \quad C = \frac{z_2}{1 + z_1} , \quad (3.5)$$

and the Kähler gauge transformation (A.4) with $f(z) = \log \left[ \frac{1}{2}(1 + z_1) \right]$, the Kähler potential (3.3) takes the form (A.3) that is associated with the standard (Bergmann)

\footnote{The relative coefficient varies in the literature, see e.g., ref. \cite{7}.}
metric (A.1) of the non-compact symmetric space $SU(2,1)/SU(2) \times U(1)$. When using the new coordinates (A.5), one arrives at the classical UH metric in the form (A.6), with the $SU(2)$ isometry of the metric being manifest.

The classical quaternionic space $Q = G/H \equiv SU(2,1)/SU(2) \times U(1)$ has the eight-dimensional non-abelian isometry group $G = SU(2,1)$ whose left action on $Q$ after the compensating right action of the ‘gauge’ subgroup $H = SU(2) \times U(1)$ leaves the metric (A.1) of $Q$ intact.

As far as quantum corrections within the type-IIA string (loop) perturbation theory are concerned, the UH metric is supposed to be invariant under the so-called Peccei-Quinn type symmetries [2, 7]. These symmetries are given by constant shifts of the NS-NS axion $D$,

$$S \rightarrow S + i\eta , \quad (3.6)$$

and constant shifts of the R-R field $C$ [8],

$$C \rightarrow C + \gamma - i\beta , \quad S \rightarrow S + 2(\gamma + i\beta)C + \gamma^2 + \beta^2 , \quad (3.7)$$

where $\eta$, $\beta$ and $\gamma$ are the real parameters. The Peccei-Quinn type transformations (3.6) and (3.7) form the non-abelian Heisenberg group [3]. It is usually assumed [2, 3, 7] that the Peccei-Quinn type symmetries do not change their form in string perturbation theory. Since our considerations are purely local, we ignore the global group structure and refer to the symmetry (3.6) as $U_D(1)$.

The instantons originating from the fivebranes wrapped over the entire CY space break both symmetries (3.6) and (3.7) (cf. the breaking of the translational invariance of the $\theta$-parameter in QCD by instantons), whereas the D-instantons (wrapped membranes) break the symmetry (3.7) but keep the symmetry (3.6) [8].

The $U(1)$ subgroup of the $SU(2) \subset H$ symmetry is given by the duality rotations $U_C(1)$ of the complex R-R pseudo-scalar $C$,

$$C \rightarrow e^{i\alpha}C , \quad S \rightarrow S , \quad (3.8)$$

with the real parameter $\alpha$. The duality rotations (3.8) are believed to be exact in quantum theory, even when all quantum (or instanton) corrections are taken into account [3, 8].

The remaining four classical symmetries of the coset $SU(2,1)/U(2)$ are given by scale transformations,

$$S \rightarrow S + \lambda S , \quad C \rightarrow C + \frac{1}{2}\lambda C , \quad (3.9)$$
with the real parameter $\lambda$, and

$$S \rightarrow S + \frac{i}{4} \epsilon_1 S^2 + \frac{1}{2} (\epsilon_2 + i \epsilon_3) CS ,$$

$$C \rightarrow C + \frac{i}{4} \epsilon_1 CS + \frac{1}{2} \epsilon_2 (C^2 - \frac{1}{2} S) + \frac{i}{2} \epsilon_3 (C^2 + \frac{1}{2} S) ,$$

with the real parameters $\epsilon_1$, $\epsilon_2$ and $\epsilon_3$. The symmetries (3.9) and (3.10) are always broken by instanton corrections.

The conserved Noether charges associated with the isometries (3.6) and (3.7) were calculated in ref. [3],

$$J_\eta = \frac{i}{4} \kappa e^{2\kappa} \left( dS - d\bar{S} + 2 \epsilon_2 \frac{\epsilon_2}{\epsilon_2} dC \right) ,$$

$$J_\beta = - \frac{2i}{4} \kappa e^{\kappa} \left( dC - d\bar{C} \right) + 2 \left( C + \bar{C} \right) J_\eta ,$$

$$J_\gamma = - \frac{2}{4} \kappa e^{\kappa} \left( dC + d\bar{C} \right) - 2i \left( C - \bar{C} \right) J_\eta .$$

The Noether charge (over a supersymmetric three-cycle $C$)

$$Q_\eta = \int_C \ast J_\eta$$

descends from the fivebrane charges (2.4), whereas the other two Noether charges,

$$Q_\beta = \int_{\Sigma_3} \ast J_\beta \quad \text{and} \quad Q_\gamma = \int_{\Sigma_3} \ast J_\gamma ,$$

descend from the D2-brane (RR) charges. The existence of two charges is related to the existence of two homology classes of a three-cycle $C$ [3]. The BPS brane charge quantization (sect. 2) implies that only a discrete subgroup of the continuous Peccei-Quinn type symmetries (3.6) and (3.7) is going to survive in full quantum theory after taking into account both membrane and fivebrane instanton corrections [3]. The discrete identifications of the UH scalars can be read off from eqs. (3.6) and (3.7),

$$S \sim S + in_\eta + 2(n_\gamma + in_\beta)C + n_\eta^2 + n_\beta^2 ,$$

$$C \sim C + n_\gamma - in_\beta ,$$

where now all $n_\eta, n_\gamma, n_\beta$ are integers [3]. The transformations (3.14) define a discrete non-abelian group $Z$. The global (topological) structure of the UH moduli space $\mathcal{M}$ is thus given by $Q/Z$ [3].

Since the type-IIA string loop corrections are invariant under the perturbative Peccei-Quinn type symmetries (3.6) and (3.7), it is natural to rewrite the UH metric in terms of the new quantities $(u, v, \phi)$ defined by eq. (A.14), which are all invariant.
under (3.6) and (3.7). When using the notation (A.14), it is not difficult to convince oneself that the unique quaternionic deformation of the classical UH action within the string perturbation theory is described by eq. (2.18) indeed [7]. Hence, the local UH metric (2.17) does not receive any perturbative quaternionic corrections modulo the UH field redefinitions (sect. 2).

4 Hyper-Kähler versus quaternionic geometry

The D-instanton corrections to the UH moduli space metric due to the wrapped D2-branes were explicitly calculated in the hyper-Kähler limit by Ooguri and Vafa [18]. In this limit the 4d, N=2 supergravity decouples [5], and the c-map applies [9].

In the absence of fivebrane corrections, the unbroken symmetry of the UH metric is given by $U_D(1) \times U_C(1)$ (sect. 3). In adapted coordinates with respect to the $U_D(1)$ isometry the UH metric can be written down in the LeBrun form [31],

$$ds_K^2 \equiv g_{ab}d\phi^a d\phi^b = W^{-1}(dt + \Theta_1)^2 + W \left[ e^u(dx^2 + dy^2) + d\omega^2 \right],$$

(4.1)

where two potentials $W$ and $u$, and a one-form $\Theta_1$ have been introduced. The metric Ansatz (4.1) is valid for \textit{any} Kähler metric $g_{ab}$ in four real dimensions, $a, b = 1, 2, 3, 4$, with a Killing vector $K^a$ that preserves the Kähler structure. We use the adapted coordinates $\phi^a = (t, x, y, \omega)$ where $t$ is the coordinate along the trajectories of the Killing vector associated with $U_D(1)$, whereas $(x, y, \omega)$ are the coordinates in the space of trajectories, $W^{-1} = g_{ab}K^aK^b \neq 0$. Accordingly, no metric components are dependent upon $t$ in eq. (4.1).

The Kähler condition on the metric (4.1) also implies a linear equation on $\Theta_1$ [31],

$$d\Theta_1 = W_x dy \wedge d\omega + W_y d\omega \wedge dx + (W e^u)_{,\omega} dx \wedge dy .$$

(4.2)

In turn, this gives rise to the following integrability condition on $W$ [31]:

$$W_{xx} + W_{yy} + (W e^u)_{,\omega \omega} = 0 .$$

(4.3)

The hyper-Kähler geometry implies, by definition, the existence of \textit{three} linearly independent Kähler structures $(J_k)_a^\, b$, $k = 1, 2, 3$, which are covariantly constant, $\nabla_c (J_k)_a^\, b = 0$, and obey a quaternionic algebra. Moreover, in four real dimensions, a hyper-Kähler metric necessarily has the \textit{Anti-Self-Dual} (ASD) Riemann curvature, and vice versa [32]. As regards four-dimensional hyper-Kähler metrics, they are just Kähler and Ricci-flat, and vice versa [33].
If the Killing vector $K^\a$ is triholomorphic (i.e. it is consistent with N=2 supersymmetry), one may further restrict the metric (4.1) by taking $u = 0$. The Riemann ASD condition then amounts to a linear system \[34\],

\[ \Delta W = 0 \quad \text{and} \quad \bar{\nabla} W + \bar{\nabla} \times \Theta = 0 , \tag{4.4} \]

where $\Delta$ is the Laplace operator in three flat dimensions, $\Delta = 4g_{\text{string}}^2 \partial_z \bar{\partial}_z + \partial_w^2$, and $z = g_{\text{string}}(x + iy)$. The complex coordinate $z$ represents the RR-type complex scalar, so that the unbroken $U_C(1)$ symmetry (3.8) implies that a solution to eq. (4.4) may depend upon $z$ only via its absolute value $|z|$. 

One may now think of $W$ as the electro-static potential for a collection of electric charges distributed in three dimensions near the axis $z = 0$ with unit density in $\omega$. The unique regular (outside the positions of charges) solution to this problem in the limit $g_{\text{string}} \to 0$, while keeping $|z|/g_{\text{string}}$ finite, reads \[35\]

\[ W = \frac{1}{4\pi} \log \left( \frac{\mu^2}{zz} \right) + \sum_{m=1}^{\infty} \frac{\cos(2\pi m\omega)}{\pi} K_0 \left( \frac{2\pi |mz|}{g_{\text{string}}} \right) , \tag{4.5} \]

where $K_0$ is the modified Bessel function. The solution (4.5) can be trusted for large $|z|$, where it amounts to the infinite D-instanton/anti-instanton sum \[18\],

\[ W = \frac{1}{4\pi} \log \left( \frac{\mu^2}{zz} \right) + \sum_{m=1}^{\infty} \exp \left( - \frac{2\pi |mz|}{g_{\text{string}}} \right) \cos(m\omega) \]

\[ \times \sum_{n=0}^{\infty} \frac{\Gamma(n + \frac{1}{2})}{\sqrt{\pi} n! \Gamma(-n + \frac{1}{2})} \left( \frac{g_{\text{string}}}{4\pi |mz|} \right)^{n + \frac{1}{2}} . \tag{4.6} \]

The exp $(-1/g_{\text{string}})$ type dependence of the solution (4.6) apparently agrees with the general expectations \[3\] that it describes the D-instantons indeed.

If merely the vanishing scalar curvature of the Kähler metric (4.1) or a non-triholomorphic isometry of the hyper-Kähler metric were required, we would end up with the non-linear equation \[31, 34\]

\[ u_{xx} + u_{yy} + (e^u)_{\omega \omega} = 0 . \tag{4.7} \]

The equation (4.7) is known as the $SU(\infty)$ Toda field equation \[36\] since it appears in the large-$N$ limit of the standard (two-dimensional) Toda system for $SU(N)$. See, e.g., ref. \[37\] for more about the Toda equation (4.7), and ref. \[38\] for more about the four-dimensional hyper-Kähler NLSM. The Einstein-Kähler deformations of the Bergmann metric of $SU(2,1)/U(2)$ in eq. (A.1) were investigated in refs. \[39, 40\].
5 D-instantons and quaternionic UH metric

A quaternionic manifold admits three independent almost complex structures \((\tilde{J}_k)^a_b\), which are, however, not covariantly constant but satisfy \(\nabla_a(\tilde{J}_k)^b_c = (T_a)_k^n(\tilde{J}_n)^b_c\), where \((T_a)_k^n\) is the NLSM torsion [32]. This torsion is induced by 4d, N=2 supergravity because the quaternionic condition on the hypermultiplet NLSM target space metric is the direct consequence of local N=2 supersymmetry in four spacetime dimensions [3]. As regards four-dimensional quaternionic manifolds (relevant for UH), they all have Einstein-Weyl geometry of negative scalar curvature [5, 32], i.e.

\[ W_{abcd} = 0 \quad , \quad R_{ab} = \frac{\Lambda}{2} g_{ab} \quad , \quad (5.1) \]

where \(W_{abcd}\) is the Weyl tensor and \(R_{ab}\) is the Ricci tensor for the metric \(g_{ab}\). The overall coupling constant of the 4d NLSM has the same dimension as \(\kappa^2\), while in the N=2 locally supersymmetric NLSM these coupling constants are proportional to each other with the dimensionless coefficient \(\Lambda < 0\) [5]. We take \(\kappa^2 = 1\) for simplicity.

Since the quaternionic and hyper-Kähler conditions are not compatible, the canonical form (4.1) should be revised. Nevertheless, the exact quaternionic metric is governed by the same three-dimensional Toda equation (4.7) [13]. Indeed, when using another (Tod) Ansatz [11]

\[ ds^2_Q = \frac{P}{\omega^2} \left[ e^u(dx^2 + dy^2) + d\omega^2 \right] + \frac{1}{P\omega^2}(dt + \Theta_1)^2 \quad (5.2) \]

for a quaternionic metric with an abelian isometry, it is straightforward to prove that the restrictions (5.1) on the metric (5.2) precisely amount to eq. (4.7) on the potential \(u = u(x,y,\omega)\), while \(P\) is given by [11]

\[ P = \frac{1}{2\Lambda} (\omega u_\omega - 2) \quad , \quad (5.3) \]

whereas the one-form \(\Theta_1\) obeys the linear equation [11]

\[ d\Theta_1 = -P_x \, dy \wedge d\omega - P_y \, d\omega \wedge dx - e^n(P_\omega + \frac{2}{\omega}P + \frac{2\Lambda}{\omega^2}P^2)dx \wedge dy \quad . \quad (5.4) \]

The limit \(\Lambda \to 0\), where 4d, N=2 supergravity decouples, should be taken with care. After rescaling \(u \to \Lambda u\) in eq. (4.7) we get

\[ u_{xx} + u_{yy} + \frac{1}{\Lambda} (e^{\Lambda u})_{\omega\omega} = 0 \quad . \quad (5.5) \]

\[ ^7\text{A generic Einstein-Weyl manifold does not have a Kähler structure.} \]
This equation gives rise to the 3d Laplace (linear!) equation when $\Lambda \to 0$, as expected. We can, therefore, conclude that the non-linear Toda equation (4.7) substitutes the linear Laplace equation (4.4) in the presence of 4d, N=2 supergravity. The ‘cosmological’ constant $\Lambda$ can be considered as the most relevant parameter of the deformation that converts a given UH hyper-Kähler metric into the ‘gravitationally dressed’ quaternionic UH metric via eq. (5.2), based on the same solution to the Toda equation (4.7) \[15\].

In terms of the complex coordinate $\zeta = x + iy$, the 3d Toda equation (4.7) takes the form

$$4u\zeta\bar{\zeta} + (e^u)\omega\bar{\omega} = 0 \quad (5.6)$$

It is not difficult to check that this equation is invariant under holomorphic transformations of $\zeta$,

$$\zeta \to \hat{\zeta} = f(\zeta) \quad (5.7)$$

with an arbitrary function $f(\zeta)$, provided that it is accompanied by the shift of the Toda potential,

$$u \to \hat{u} = u - \log(f') - \log(\bar{f}') \quad (5.8)$$

where the prime means differentiation with respect to $\zeta$ or $\bar{\zeta}$, respectively. The transformations (5.7) can be interpreted as the residual diffeomorphisms in the NLSM target space of the universal hypermultiplet, which keep invariant the quaternionic Ansatz (5.2) under the compensating ‘Toda gauge transformations’ (5.8).

To make contact with particular four-dimensional hyper-Kähler geometries (with isometries) \[38\], it is natural to search for separable exact solutions to the Toda equation, having the form

$$u(\zeta, \bar{\zeta}, \omega) = F(\zeta, \bar{\zeta}) + G(\omega) \quad (5.9)$$

Equation (4.7) now reduces to two separate equations,

$$F\zeta\bar{\zeta} + \frac{c^2}{2}e^F = 0 \quad (5.10)$$

and

$$\partial_\omega e^G = 2c^2 \quad (5.11)$$

where $c^2$ is a separation constant. After taking into account the positivity of $e^G$, the general solution to eq. (5.11) reads

$$e^G = c^2(\omega^2 + 2\omega b \cos \alpha + b^2) \quad (5.12)$$

where $b$ and $\alpha$ are arbitrary real integration constants.
Equation (5.10) is the 2d Liouville equation that is well known in 2d quantum gravity [42]. Its general solution reads
\[ e^F = \frac{4 |f'|^2}{(1 + c^2 |f|^2)^2} \]  
(5.13)
in terms of arbitrary holomorphic function \( f(\zeta) \). The ambiguity associated with this function is, however, precisely compensated by the Toda gauge transformation (5.8), so that we have the right to choose \( f(\zeta) = \zeta \) in eq. (5.13). This yields the following regular exact solution to the 3d Toda equation:
\[ e^u = \frac{4c^2(\omega^2 + 2\omega b \cos \alpha + b^2)}{(1 + c^2 |\zeta|^2)^2} \]  
(5.14)
It is obvious now that the constant \( c^2 \) is positive indeed. It also follows from eqs. (5.2), (5.3) and (5.14) that any separable exact solution to the quaternionic UH metric possesses the rigid \( U_C(1) \) duality symmetry with respect to the duality rotations \( \zeta \to e^{i\alpha} \zeta \) of the complex RR-field \( \zeta \).

Though the quaternionic NLSM metric defined by eqs. (5.2), (5.3), (5.4) and (5.14) is apparently different from the classical UH metric (2.17), these metrics are nevertheless equivalent in the classical region of the UH moduli space where all quantum corrections are suppressed. The classical approximation corresponds to the conformal limit \( \omega \to \infty \) and \( |\zeta| \to \infty \), while keeping the ratio \( |\zeta|^2 / \omega \) finite. Then one easily finds that \( P \to -\Lambda^{-1} = \text{const.} > 0 \), whereas the metric (5.2) takes the form
\[ ds^2 = \frac{1}{\lambda^2} \left( |dC|^2 + d\lambda^2 \right) + \frac{1}{\lambda^4} (dD + \Theta)^2 \]  
(5.15)
in terms of the new variables \( C = 1/\zeta \) and \( \lambda^2 = \omega \), after a few rescalings. The metric (5.15) reduces to that of eq. (2.17) when using \( \lambda^{-2} = e^{2\phi} \). Another interesting limit is \( \omega \to 0 \) and \( |\zeta| \to \infty \), where one gets a conformally flat metric \( AdS_4 \).

Based on the fact that both hyper-Kähler and quaternionic metrics under consideration are governed by the same Toda equation, a natural mechanism of generating the quaternionic metrics from known hyper-Kähler metrics in the same (four) dimensions arises: first, one deduces a solution to the Toda equation (4.7) from a given four-dimensional hyper-Kähler metric having a non-triholomorphic or rotational isometry, by rewriting it to the form (4.1), and then one inserts the obtained exact solution into the quaternionic Ansatz (5.2) to deduce the corresponding quaternionic metric with the same isometry. Being applied to the D-instantons, this mechanism results in their dressing with respect to 4d, N=2 supergravity background [15].

\[ \text{The different mechanism, which associates the quaternionic metric in } 4(n+1) \text{ real dimensions to a given (special) hyper-Kähler metric in } 4n \text{ real dimensions, was proposed in ref. [14].} \]
The $SU(\infty)$ Toda equation is known to be notoriously difficult to solve, while a very few its exact solutions are known [37]. Nevertheless, the proposed connection to the hyper-K"{a}hler metrics can be used as the powerful vehicle for generating exact solutions to eq. (4.7). It is worth mentioning that eq. (4.3) follows from eq. (4.7) after a substitution

$$W = \partial_u \omega,$$

while eq. (5.2) is solved by

$$\Theta_1 = \mp \partial_y u(dx) \pm \partial_x u(dy).$$

This is known as the Toda frame for a hyper-K"{a}hler metric [34, 44, 45]. It is not difficult to verify that the separable solution (5.14) is generated from the Eguchi-Hanson (hyper-K"{a}hler) metric along these lines [44, 38]. Another highly non-trivial solution to the Toda equation (4.17) follows from the Atiyah-Hitchin (hyper-K"{a}hler) metric [33, 45]. The transform to the Toda frame for the Atiyah-Hitchin metric is known [46],

$$y + ix = K(k)\sqrt{1 + k'^2 \sinh^2 \nu} \left( \cos \vartheta + \frac{\tanh \nu}{K(k)} \int_0^{\pi/2} d\gamma \frac{\sqrt{1 - k^2 \sin^2 \gamma}}{1 - k^2 \tanh^2 \nu \sin^2 \gamma} \right),$$

where $(\vartheta, \psi, \varphi; k)$ are the new coordinates (in four dimensions). The parameter $k$ plays the role of modulus here, $0 < k < 1$, while $k' = \sqrt{1 - k^2}$ is called the complementary modulus. The remaining definitions are

$$\nu \equiv \log \left( \tan \frac{\theta}{2} \right) + i \psi , \quad \tau = 2 \left( \varphi + \arg(1 + k'^2 \sin^2 \nu) \right),$$

in terms of the standard complete elliptic integrals of the first and second kind, $K(k)$ and $E(k)$, respectively. The solution to the Toda equation (4.7) now reads [44, 38]

$$e^u = \frac{1}{16} K^2(k) \sin^2 \vartheta \left| 1 + k'^2 \sin^2 \nu \right| .$$

The physical significance of the related quaternionic metric solution is apparent in the perturbative region $k \to 1$, where [47]

$$k' \propto e^{-S_{\text{inst}}}, \quad \text{and} \quad S_{\text{inst.}} \to +\infty .$$

In this limit the Atiyah-Hitchin metric is exponentially close to the Taub-NUT metric [33], while the exponentially small corrections can be interpreted as the (mixed) D
instantons and anti-instantons. The D-instanton action $S_{\text{inst}}$ represents here the volume of the corresponding supersymmetric three-cycle $C$, on which the D-brane is wrapped [13]. It is worth mentioning that the same exact solution also describes the hypermultiplet moduli space metric in the 3d, N=4 supersymmetric Yang-Mills theory with the $SU(2)$ gauge group, which was obtained via the c-map in ref. [48].

6 Fivebrane instantons and Tod-Hitchin metric

As was demonstrated in ref. [13], the BPS condition on the fivebrane instanton solution with the vanishing charge $Q_\eta$ of eq. (3.12) defines a gradient flow in the hypermultiplet moduli space. The flow implies the $SU(2)$ isometry of the UH metric since the non-degenerate action of this isometry in the four-dimensional UH moduli space gives rise to the well defined three-dimensional orbits which can be parametrized by the ‘radial’ coordinate to be identified with the flow parameter. In the case of the classical UH moduli space $SU(2,1)/SU(2) \times U(1)$ the radial coordinate ($r$) is defined by eq. (A.5), while its relation to the complex ($S,C$) coordinates is given by

$$r^2 = \frac{|1 - S|^2 + 4|C|^2}{|1 + S|^2},$$

(6.1)

where we have used eqs. (3.5) and (A.5). It is worth mentioning that the non-abelian $SU(2)$ symmetry includes the abelian duality rotations (3.8). However, it does not imply the preservation of the rest of the $SU(2,1)/SU(2) \times U(1)$ symmetries (other than $SU(2)$) including the Pecei-Quinn type symmetries (3.6) and (3.7). Hence, the $SU(2)$-invariant deformations of the classical UH metric, subject to the quaternionic constraints (5.1), should describe the zero-charge fivebrane instanton corrections.

The action of a fivebrane instanton with the vanishing charge may merely depend upon the complex structure modulus at infinity [13]. Therefore, we expect the corresponding UH moduli space metric be merely dependent upon the complex structure on the boundary of the coset $SU(2,1)/SU(2) \times U(1)$. The conformal boundary metric for the coset $SU(2,1)/SU(2) \times U(1)$ is known to be degenerate, i.e. it possess a zero eigenvalue. This happens because the conformal structure, associated with the Bergmann metric in the form (A.6) inside the unit ball in $C^2$, does not extend across the boundary since the coefficient at $\sigma_2^2$ in eq. (A.6) decays faster than the coefficients at $\sigma_1^2$ and $\sigma_3^2$. However, the conformal structure survives in the two-dimensional (2d) subspace annihilated by $\sigma_2$. The 2d complex structure has a single real parameter – the central charge [12] – which should appear in the UH metric. The exact metric solutions, given below in this section, confirm these expectations.
Let’s consider a generic $SU(2)$-invariant metric in four Euclidean dimensions. In the Bianchi IX formalism, where the $SU(2)$ symmetry is manifest, the general Ansatz for such metrics reads (see Appendix A for our notation)

$$ds^2 = w_1 w_2 w_3 dt^2 + \frac{w_3 w_1}{w_1} \sigma_1^2 + \frac{w_2}{w_2} \sigma_2^2 + \frac{w_1 w_2}{w_3} \sigma_3^2,$$  \hfill (6.2)

in terms of the (left)-invariant one-forms $\sigma_i$ defined by eq. (A.7), and the radial coordinate $t$. The metric (6.2) is dependent upon three functions $w_i(t)$, $i = 1, 2, 3$. The most general $SU(2)$-invariant metric is given by a quadratic form with respect to $\sigma_i$. However, it can always be chosen in the diagonal form, as in eq. (6.2), without loss of generality.

The quaternionic constraints on the metric amount to the ASD-Weyl equation and the Einstein equation — see eq. (5.1). The exact $SU(2)$-invariant solutions to these equations were found by Tod [49] and Hitchin [50]. The main results of refs. [49, 50] are briefly described below.

Being applied to the metric (6.2), the ASD Weyl condition gives rise to a system of Ordinary Differential Equations (ODE) [49],

$$\begin{align*}
\dot{A}_1 &= - A_2 A_3 + A_1 (A_2 + A_3), \\
\dot{A}_2 &= - A_3 A_1 + A_2 (A_3 + A_1), \\
\dot{A}_3 &= - A_1 A_2 + A_3 (A_1 + A_2),
\end{align*}$$  \hfill (6.3)

where the dots denote differentiation with respect to $t$, and the functions $A_i(t)$ are defined by the auxiliary ODE system,

$$\begin{align*}
\dot{w}_1 &= - w_2 w_3 + w_1 (A_2 + A_3), \\
\dot{w}_2 &= - w_3 w_1 + w_2 (A_3 + A_1), \\
\dot{w}_3 &= - w_1 w_2 + w_3 (A_1 + A_2).
\end{align*}$$  \hfill (6.4)

The ODE system (6.3) is known in the mathematical literature as the classical Halphen system [51]. The Bergmann metric (A.6) is the simplest solution to eqs. (6.3) and (6.4) with $A_i = 0$ for all $i = 1, 2, 3$. This follows from comparison of eqs. (6.3), (6.4) and (A.6). Note that despite of the fact that all $A_i = 0$, the Bergmann metric (A.6) is, nevertheless, non-trivial (or non-flat) since eq. (A.6) is still a non-trivial solution to eq. (6.4).

Given a metric solution to the ASD Weyl equations, the Einstein equation of eq. (5.1) can be easily satisfied after proper Weyl rescaling of the ASD Weyl metric,
because any local Weyl transformation does not affect the vanishing Weyl tensor. Having obtained an explicit solution to the Halphen system (6.3), it may be substituted into the ODE system (6.4). To solve eq. (6.4), it is convenient to change variables as \[49\]

\[
\begin{align*}
  w_1 &= \frac{\Omega_1 \dot{x}}{\sqrt{x(1-x)}}, \\
  w_2 &= \frac{\Omega_2 \dot{x}}{\sqrt{x^2(1-x)}}, \\
  w_3 &= \frac{\Omega_3 \dot{x}}{\sqrt{x(1-x)^2}},
\end{align*}
\]

(6.5)

where the new variables $\Omega_i(x), i = 1, 2, 3$, are constrained by an algebraic condition,

\[
\Omega_2^2 + \Omega_3^2 - \Omega_1^2 = \frac{1}{4}.
\]

(6.6)

The algebraic relation (6.6) reduces the number of the newly introduced functions in eq. (6.5) from four to three, as it should. In fact, eq. (6.6) is also dictated by the quaternionic nature of the metric \[49, 50\].

In terms of the new variables (6.5), the ODE system (6.4) takes the form \[49, 50\]

\[
\begin{align*}
  \Omega_1' &= -\frac{\Omega_2\Omega_3}{x(1-x)}, \\
  \Omega_2' &= -\frac{\Omega_3\Omega_1}{x}, \\
  \Omega_3' &= -\frac{\Omega_1\Omega_2}{1-x},
\end{align*}
\]

(6.7)

where the primes denote differentiation with respect to $x$. It is not difficult to verify that the algebraic constraint (6.6) is preserved under the flow (6.7), so that the highly non-linear transformation (6.5) is fully consistent. In terms of the new variables $(x, \Omega_i)$, the Einstein condition of eq. (5.1) on the metric (6.2) in the form

\[
ds^2 = e^{2u} \left[ \frac{dx^2}{x(1-x)} + \frac{\sigma_1^2}{\Omega_1^2} + \frac{(1-x)\sigma_2^2}{\Omega_2^2} + \frac{x\sigma_3^2}{\Omega_3^2} \right]
\]

(6.8)

amounts to the algebraic relation \[49\]

\[
96\kappa^2 e^{2u} = \frac{8x\Omega_1^2\Omega_2^2\Omega_3^2 + 2\Omega_1\Omega_2\Omega_3(x(\Omega_1^2 + \Omega_2^2) - (1 - 4\Omega_3^2)(\Omega_2^2 - (1-x)\Omega_1^2))}{(x\Omega_1\Omega_2 + 2\Omega_3(\Omega_2^2 - (1-x)\Omega_1^2))^2}
\]

(6.9)

which yields the Weyl factor $u(x)$ in terms of the functions $\Omega_i(x)$.

The Halphen system (6.3) has a long history \[52\]. Perhaps, its most natural (manifestly integrable) derivation is provided via a reduction of the $SL(2, \mathbb{C})$ anti-self-dual
Yang-Mills equations from four Euclidean dimensions to one \[53\]. A classification of all possible reductions is known in terms of the so-called Painlevé groups that give rise to six different types of integrable Painlevé equations \[53\]. It remains to identify those of them that lay behind the ASD-Weyl (or quaternionic-Kähler) geometry with the $SU(2)$ symmetry. There are only two natural (or nilpotent, in the terminology of ref. \[53\]) types (III and VI) that give rise to a single non-linear integrable equation. In the geometrical terms, it is the Painlevé III equation that lays behind the four-dimensional Kähler spaces with vanishing scalar curvature \[54\], whereas the Painlevé VI equation is known to be behind the ASD-Weyl geometries having the $SU(2)$ symmetry \[49, 50, 55\]. A generic Painlevé VI equation has four real parameters \[53\], but they are all fixed by the quaternionic property \[49, 50\], as in eq. (6.6). This means that the quaternionic metrics with the $SU(2)$ symmetry are all governed by the particular Painlevé VI equation:

$$y'' = \frac{1}{2} \left( \frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-x} \right) (y')^2 - \left( \frac{1}{x} + \frac{1}{x-1} + \frac{1}{y-x} \right) y' + \frac{y(y-1)(y-x)}{x^2(x-1)^2} \left[ \frac{1}{8} - \frac{x}{8y^2} + \frac{x-1}{8(y-1)^2} + \frac{3x(x-1)}{8(y-x)^2} \right] ,$$

(6.10)

where $y = y(x)$, and the primes denote differentiation with respect to $x$.

The equivalence between eqs. (6.3) and (6.10) via eqs. (6.4) and (6.5) is known to mathematicians \[49, 50, 53\]. Explicitly, in the Einstein case, it is given by the relations

$$\Omega_1^2 = \frac{(y-x)^2y(y-1)}{x(1-x)} \left( v - \frac{1}{2(y-1)} \right) \left( v - \frac{1}{2} \right) ,$$

$$\Omega_2^2 = \frac{(y-x)y^2(y-1)}{x} \left( v - \frac{1}{2(y-x)} \right) \left( v - \frac{1}{2(y-1)} \right) ,$$

$$\Omega_3^2 = \frac{(y-x)y(y-1)^2}{(1-x)} \left( v - \frac{1}{2y} \right) \left( v - \frac{1}{2} \right) ,$$

(6.11)

where the auxiliary variable $v$ is defined by the equation

$$y' = \frac{y(y-1)(y-x)}{x(x-1)} \left( 2v - \frac{1}{2y} - \frac{1}{2(y-1)} + \frac{1}{2(y-x)} \right) .$$

(6.12)

An exact solution to the Painlevé VI equation (6.10), which leads to a regular (and complete) quaternionic metric (6.2), is unique \[50\]. The Hitchin solution \[50\] can be expressed in terms of the standard theta-functions $\vartheta_\alpha(z|\tau)$, where $\alpha = 1, 2, 3, 4$.  

\footnote{We use the standard definitions and notation for the theta functions \[10\] — see Appendix B.}
In order to write down the Hitchin solution to eq. (6.10), the theta-function arguments should be related, 
\[ z = \frac{1}{2}(\tau - k), \]
where \( k \) is an arbitrary (real and positive) parameter. The variable \( \tau \) is related to the variable \( x \) of eq. (6.10) via
\[ x = \vartheta_3^4(0)/\vartheta_4^4(0), \quad (6.13) \]
where the value of the variable \( z \) is explicitly indicated, as usual. One finds \[ y(x) = \frac{\vartheta_1'''(0)}{3\pi^2\vartheta_4^4(0)\vartheta_1'(0)} + \frac{1}{3} \left[ 1 + \frac{\vartheta_3^4(0)}{\vartheta_4^4(0)} \right] \]
\[ + \frac{\vartheta_1'''(z)\vartheta_1'(z) - 2\vartheta_1''(z)\vartheta_1'(z) + 2\pi i(\vartheta_1''(z)\vartheta_1'(z) - \vartheta_1'^2(z))}{2\pi^2\vartheta_4^4(0)\vartheta_1'(z)(\vartheta_1'(z) + \pi i\vartheta_1(z))}, \quad (6.14) \]

The parameter \( k > 0 \) describes the monodromy of the solution (6.14) around its essential singularities (branch points) \( x = 0,1,\infty \). This (non-abelian) monodromy is generated by the matrices (with the purely imaginary eigenvalues \( \pm i \)) \[ M_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0 & i^{1-k} \\ i^{1+k} & 0 \end{pmatrix}, \quad M_3 = \begin{pmatrix} 0 & i^{-k} \\ -i^k & 0 \end{pmatrix}. \quad (6.15) \]

Another explicit (equivalent) form of the Hitchin exact solution to the metric coefficients \( w_i \) in eqs. (6.2) and (6.4) was derived in ref. [58], in terms of the theta functions with characteristics, by the use of the fundamental Schlesinger system and the isomonodromic deformation techniques.

The function (6.14) is meromorphic outside its essential singularities at \( x = 0,1,\infty \), while also has simple poles at \( \bar{x}_1, \bar{x}_2, \ldots \), where \( \bar{x}_n \in (x_n, x_{n+1}) \) and \( x_n = x(ik/(2n - 1)) \) for each positive integer \( n \). Accordingly, the metric is well-defined (complete) for \( x \in (\bar{x}_n, x_{n+1}] \), i.e. in the unit ball with the origin at \( x = x_{n+1} \) and the boundary at \( x = \bar{x}_n \). Near the boundary the Tod-Hitchin metric (6.2) has the asymptotical behaviour
\[ ds^2 = \frac{dx^2}{(1-x)^2} + \frac{4}{(1-x)^2\cosh^2(\pi k/2)}\sigma_1^2 + \frac{16}{(1-x)^2\sinh^2(\pi k/2)\cosh^2(\pi k/2)}\sigma_2^2 \]
\[ + \frac{4}{(1-x)^2\sinh^2(\pi k/2)}\sigma_3^2 + \text{regular terms}. \quad (6.16) \]

As is clear from eq. (6.16), the coefficient at \( \sigma_2^2 \) vanishes faster than the coefficients at \( \sigma_1^2 \) and \( \sigma_3^2 \) when approaching the boundary, \( x \to 1^- \), similarly to eq. (A.6). On the two-dimensional boundary annihilated by \( \sigma_2 \) one has the natural conformal structure
\[ \sinh^2(\pi k/2)\sigma_1^2 + \cosh^2(\pi k/2)\sigma_3^2. \quad (6.17) \]
The only relevant parameter $\tanh^2(\pi k/2)$ in eq. (6.17) represents the central charge (or the conformal anomaly) on the boundary. This result is apparently consistent with (i) the fact that the type-IIA fivebranes can be described in terms of an exact (super)conformal field theory [59], (ii) the Zamolodchikov $c$-theorem [60], and (iii) the holographic principle [61].

The constraints (5.1) do not seem to imply any quantization condition on the monodromy parameter $k$ since the regular metric solutions exist for any $k > 0$. The central charge (or the critical exponent $k$) is quantized in solvable 2d, N=2 superconformal field theories (the minimal N=2 superconformal models) which are associated with compact (simply-laced) Lie groups [42]. The absence of central charge quantization in our case may be related to the negative scalar curvature of the metrics. The Einstein-Weyl metrics of the positive scalar curvature take the similar form given by eqs. (6.2) or (6.8), while they are known to be related to the so-called Poncelet $n$-polygons that give rise to the quantization condition $k = 2/n$, where $n \in \mathbb{Z}$ [62]. It is thus the non-compact nature of the Lie group $SU(2,1)$ that is responsible for the absence of the central charge quantization in the boundary conformal field theory.

### 7 Conclusion

The universal hypermultiplet gives us the unique opportunity to learn more about the non-perturbative quantum corrections in string/M-theory via better understanding of the exact quaternionic geometry governing the hypermultiplet LEA (or NLSM) in the 4d, N=2 supergravity background.

The D-instanton corrections to the classical UH moduli space metric are calculable due to the residual $U_D(1) \times U_C(1)$ symmetry (sects. 4 and 5). No such corrections arise when all the other fields (except UH) are turned off. The fivebrane instanton corrections are calculable at vanishing fivebrane charges (sect. 6). In a generic case with non-vanishing charges, there may be also contributions from the membranes ending on fivebranes wrapped on a CY space [13]. Then only $U_C(1)$ isometry may be left. However, the quaternionic Ansatz (5.2) still applies, this time with respect to the $U_C(1)$ isometry. The absence of any other symmetry means that generic instanton corrections are described by non-separable exact solutions to the three-dimensional Toda equation (4.7). This observation is consistent with another observation that the instanton action is not a sum of membrane and fivebrane contributions [13].

More general (than Atiyah-Hitchin) regular hyper-Kähler four-manifolds with a rotational isometry are known [13], though in the rather implicit form (as the algebraic
curves). It is also known that the Toda equation (4.7) can be reduced in a highly non-trivial way to the Painlevé equations [64]. It is not clear to us how to extract the specific information from this knowledge, which would be relevant for the problem mentioned in the title.

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Appendix A: coset space $SU(2,1)/SU(2) \times U(1)$

The homogeneous symmetric space $SU(2,1)/SU(2) \times U(1)$ is topologically equivalent to the open ball in $\mathbb{C}^2$ with the Bergmann metric [32]

$$ds^2 = \frac{dz_1 d\bar{z}_1 + dz_2 d\bar{z}_2}{1 - |z_1|^2 - |z_2|^2} + \frac{(\bar{z}_1 dz_1 + \bar{z}_2 dz_2)(z_1 d\bar{z}_1 + z_2 d\bar{z}_2)}{(1 - |z_1|^2 - |z_2|^2)^2}, \quad (A.1)$$

where $|z_1|^2 + |z_2|^2 < 1$ and $|z|^2 = z\bar{z}$. The metric (A.1) is Kähler,

$$ds^2 = (\partial_a \bar{\partial}_b K) dz_a d\bar{z}_b = e^K dz_a d\bar{z}_a + e^{2K}(\bar{z}_a dz_a)(z_b d\bar{z}_b), \quad a, b = 1, 2, \quad (A.2)$$

with the Kähler potential

$$K = -\log(1 - |z_1|^2 - |z_2|^2). \quad (A.3)$$

The Kähler potential is defined modulo the Kähler gauge transformations,

$$K(z, \bar{z}) \rightarrow \tilde{K}(z, \bar{z}) = K(z, \bar{z}) + f(z) + \bar{f}(\bar{z}), \quad (A.4)$$

with an arbitrary holomorphic function $f(z_1, z_2)$.

Equation (A.3) clearly shows that the Bergmann metric is dual to the Fubini-Study metric on the compact complex projective space $\mathbb{CP}^2 = SU(3)/SU(2) \times U(1)$ [65]. The homogeneous space $\mathbb{CP}^2$ is symmetric, while it is also an Einstein space of
positive scalar curvature with the (anti)self-dual Weyl tensor.\footnote{The Weyl tensor is the traceless part of the Riemann curvature.} The non-compact coset space $SU(2, 1)/U(2)$ is, therefore, also an Einstein space (though of negative scalar curvature), with the (anti)self-dual Weyl tensor. In other words, the coset space $SU(2, 1)/U(2)$ is an Einstein-Weyl (or a self-dual Einstein) space. In four dimensions, the Einstein-Weyl spaces are called quaternionic by definition\footnote{[32]}. After the coordinate change
\begin{equation}
    z_1 = r \cos \frac{\theta}{2} e^{i(\varphi + \psi)/2}, \quad z_2 = r \sin \frac{\theta}{2} e^{-i(\varphi - \psi)/2}, \tag{A.5}
\end{equation}
the metric \((A.1)\) can be rewritten to the diagonal form in the Bianchi IX formalism with manifest \(SU(2)\) symmetry,
\begin{equation}
    ds^2 = \frac{dr^2}{(1 - r^2)^2} + \frac{r^2 \sigma_2^2}{(1 - r^2)^2} + \frac{r^2}{(1 - r^2)^2} (\sigma_1^2 + \sigma_3^2), \tag{A.6}
\end{equation}
where we have introduced the \(su(2)\) (left)-invariant one-forms
\begin{align*}
    \sigma_1 &= -\frac{1}{2} (\sin \psi \sin \theta d\varphi + \cos \psi d\theta), \\
    \sigma_2 &= \frac{1}{2} (d\psi + \cos \theta d\varphi), \\
    \sigma_3 &= \frac{1}{2} (\sin \psi d\theta - \cos \psi \sin \theta d\varphi),
\end{align*}
in terms of four real coordinates $0 \leq r < 1$, $0 \leq \theta < \pi$, $0 \leq \varphi < 2\pi$ and $0 \leq \psi < 4\pi$. The one-forms \((A.6)\) obey the relations
\begin{equation}
    \sigma_i \wedge \sigma_j = \frac{1}{2} \varepsilon_{ijk} d\sigma_k, \quad i,j,k = 1,2,3. \tag{A.8}
\end{equation}

Another useful parametrization of the coset $SU(2, 1)/U(2)$ arises after the following change of variables:
\begin{equation}
    z_1 = \frac{1 - S}{1 + S}, \quad z_2 = \frac{2C}{1 + S}. \tag{A.9}
\end{equation}
The Kähler potential in terms of the new complex variables $(S, C)$ reads
\begin{equation}
    \mathcal{K} = -\log \left( S + \bar{S} - 2C\bar{C} \right), \tag{A.10}
\end{equation}
with the Kähler metric
\begin{equation}
    ds^2 = e^{2\mathcal{K}} \left( dSd\bar{S} - 2CdSdC - 2CdCdC + 2(S + \bar{S})dCd\bar{C} \right). \tag{A.11}
\end{equation}
This form of the metric was used by Ferrara and Sabharwal\footnote{[16]} in their analysis of the type-II superstring vacua on Calabi-Yau spaces, in terms of the complex RR-type scalar $C$ and the complex scalar
\begin{equation}
    S = e^{-2\phi} + 2iD + C\bar{C}, \tag{A.12}
\end{equation}
where $\phi$ stands for the dilaton field and $D$ stands for the axion field in four spacetime dimensions. The metric (A.11) can also be rewritten to the form

$$ds^2 = |u|^2 + |v|^2,$$

(A.13)

where

$$u \equiv e^{\phi}dC \quad \text{and} \quad v \equiv e^{2\phi}\left(\frac{1}{2}dS - \bar{C}dC\right),$$

(A.14)

by using the relations

$$\phi = -\frac{1}{2}\ln\left[(S + \bar{S} - 2C\bar{C})/2\right]$$

(A.15)

and

$$e^K = \frac{1}{2}e^{2\phi},$$

(A.16)

in accordance with eqs. (A.10) and (A.12). The notation (A.13) and (A.14) was used by Strominger [7] in his investigation of the one-loop string corrections to the universal hypermultiplet. Of course, the metric (A.13) is not flat, since the one-forms $u$ and $v$ of eq. (A.14) are not exact. Here are some useful identities [7]

$$d\phi = -\frac{1}{2}(v + \bar{v}),$$

(A.17)

AppendixB: Basic facts about theta-functions

The first theta-function $\vartheta_1(z|\tau)$ is defined by the series [56]

$$\vartheta_1(z) \equiv \vartheta_1(z|\tau) = -i \sum_{n=-\infty}^{+\infty} (-1)^n \exp\left\{\left(n + \frac{1}{2}\right)^2 \pi \tau + (2n + 1)z\right\}$$

(B.1)

$$= 2 \sum_{n=0}^{+\infty} (-1)^n q^{(n+1/2)^2} \sin(2n + 1)z, \quad q = e^{i\pi \tau},$$

where $\tau$ is regarded as the fundamental complex parameter, whose imaginary part must be positive, $q$ is called the nome of the theta-function, $|q| < 1$, and $z$ is the complex variable. The other theta-functions are defined by [56]

$$\vartheta_2(z|\tau) = \vartheta_1(z + \frac{1}{2} \tau)|\tau) = \sum_{n=-\infty}^{+\infty} q^{(n+1/2)^2} e^{i(2n+1)z}$$

(B.2)

$$= 2 \sum_{n=0}^{+\infty} q^{(n+1/2)^2} \cos(2n + 1)z,$$
\[ \vartheta_3(z|\tau) = \vartheta_4(z + \frac{1}{2}\pi)|\tau) = \sum_{n=-\infty}^{+\infty} q^n e^{2inz} \]

\[ = 1 + 2 \sum_{n=1}^{+\infty} q^n \cos 2nz , \]

and

\[ \vartheta_4(z|\tau) = \sum_{n=-\infty}^{+\infty} (-1)^n q^n e^{2inz} = 1 + 2 \sum_{n=1}^{+\infty} (-1)^n q^n \cos 2nz . \]  

The identities \([56]\)

\[ \vartheta_3^4(0) = \vartheta_2^4(0) + \vartheta_4^4(0) , \]  

\[ \vartheta_3'(0) = \vartheta_2(0)\vartheta_3(0)\vartheta_4(0) , \]  

and

\[ \frac{\vartheta_4''(0)}{\vartheta_4'(0)} = \frac{\vartheta_2''(0)}{\vartheta_2(0)} + \frac{\vartheta_3''(0)}{\vartheta_3(0)} + \frac{\vartheta_4''(0)}{\vartheta_4(0)} , \]  

where the primes denote differentiation with respect to \(z\), may be used to rewrite eq. (6.4) to other equivalent forms (cf. \([50, 57, 58]\)).

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