Penalized Sieve GEL for Weighted Average Derivatives of Nonparametric Quantile IV Regressions

Xiaohong Chen†, Demian Pouzo‡, and James L. Powell §

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Abstract

This paper considers estimation and inference for a weighted average derivative (WAD) of a nonparametric quantile instrumental variables regression (NPQIV). NPQIV is a non-separable and nonlinear ill-posed inverse problem, which might be why there is no published work on the asymptotic properties of any estimator of its WAD. We first characterize the semiparametric efficiency bound for a WAD of a NPQIV, which, unfortunately, depends on an unknown conditional derivative operator and hence an unknown degree of ill-posedness, making it difficult to know if the information bound is singular or not. In either case, we propose a penalized sieve generalized empirical likelihood (GEL) estimation and inference procedure, which is based on the unconditional WAD moment restriction and an increasing number of unconditional moments that are implied by the conditional NPQIV restriction, where the unknown quantile function is approximated by a penalized sieve. Under some regularity conditions, we show that the self-normalized penalized sieve GEL estimator of the WAD of a NPQIV is asymptotically standard normal. We also show that the quasi likelihood ratio statistic based on the penalized sieve GEL criterion is asymptotically chi-square distributed regardless of whether or not the information bound is singular.

JEL Classification: C14; C22

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†Tel.: +1 203 432 5852; Email: xiaohong.chen@yale.edu.
‡Corresponding author. Tel.: +1 510 642 6709. Email: dpouzo@econ.berkeley.edu.
§Tel.: +1 510 643 0709. Email: powell@econ.berkeley.edu.
1 Introduction

Since the seminal paper by Koenker and Bassett [1978], quantile regressions and functionals of quantile regressions have been the subjects of ever-expanding theoretical research and applications in economics, statistics, biostatistics, finance, and many other science and social science disciplines. See Koenker [2005] and the forthcoming Handbook of Quantile Regression (2017) for the latest theoretical advances and empirical applications.

The presence of endogenous regressors is common in many empirical applications of structural models in economics and other social sciences. The Nonparametric Quantile Instrumental Variable (NPQIV) regression, \( E\{1\{Y \leq h_0(W)\} - \tau | X\} = 0 \), was, to our knowledge, first proposed in Chernozhukov and Hansen [2005] and Chernozhukov et al. [2007]. This model is a leading important example of nonlinear and non-separable ill-posed inverse problems in econometrics, which has been an active research topic following the Nonparametric (mean) Instrumental Variables (NPIV) regression, \( E[Y - h_0(W)|X] = 0 \), studied by Newey and Powell [2003], Hall and Horowitz [2005], Blundell et al. [2007], Carrasco et al. [2007], Darolles et al. [2011] and others. See, for example, Horowitz and Lee [2007], Chen and Pouzo [2009, 2012, 2015], Gagliardini and Scaillet [2012], Chernozhukov and Hansen [2013], Chen et al. [2014] and others for recent work on the NPQIV and its various extensions.

In this paper, we consider estimation and inference for a Weighted Average Derivative (WAD) functional of a NPQIV. For models without nonparametric endogeneity, WAD functionals of nonparametric (conditional) mean regression, \( E[Y - h_0(X)|X] = 0 \), and of quantile regression, \( E\{1\{Y \leq h_0(X)\} - \tau | X\} = 0 \), have been extensively studied in both statistics and econometrics. In particular, under some mild regularity conditions, plug-in estimators for WADs of any nonparametric mean and quantile regressions can be shown to be semiparametrically efficient and root-\( n \) asymptotically normal (where \( n \) is the sample size). See, for example, Newey and Stoker [1993], Newey [1994], Newey and Powell [1999], Ackerberg et al. [2014] and the references therein. Although unknown functions of endogenous regressors occur frequently in empirical work, due to the ill-posed nature of NPIV and NPQIV, there is not much research on WAD functionals of NPIV and NPQIV yet. In fact, even for the simpler NPIV model that is a linear and separable ill-posed inverse problem, it is
still a difficult question whether a linear functional of a NPIV could be estimated at the root-$n$ rate; see, e.g., Severini and Tripathi [2012] and Davezies [2016]. Although Ai and Chen [2007] provide low-level sufficient conditions for a root-$n$ consistent and asymptotically normal estimator of the WAD of the NPIV model, and Ai and Chen [2012] provide a semiparametric efficient estimator of WAD for that model, to our knowledge, there is no published work on semiparametric efficient estimation of the WAD for the NPQIV model yet.

We first characterize the semiparametric efficiency bound for the WAD functional of a NPQIV model. Unfortunately, the bound depends on an unknown conditional derivative operator and hence an unknown degree of ill-posedness. Therefore, it is difficult to know if the semiparametric information bound is singular or not. Further, even if a researcher assumes that the information bound is non-singular and the WAD is root-$n$ consistently estimable, the results in Ai and Chen [2012] and Chen and Santos [2018] show that a simple plug-in estimator of a WAD might not be semiparametrically efficient. This is in contrast to the results of Newey and Stoker [1993] and Ackerberg et al. [2014] who show that plug-in estimators of a WAD of a nonparametric mean and quantile regression are semiparametrically efficient.

We then propose penalized sieve Generalized Empirical Likelihood (GEL) estimation of the WAD for the NPQIV model, which is based on the unconditional WAD moment restriction and an increasing number of unconditional moments implied by the conditional moment restriction of the NPQIV model, where the unknown quantile function is approximated by a flexible penalized sieve. Under some regularity conditions, we show that the self-normalized penalized sieve GEL estimator of the WAD of a NPQIV is asymptotically standard normal. We also show that the Quasi Likelihood Ratio (QLR) statistic based on the penalized sieve GEL criterion is asymptotically chi-squared distributed regardless of whether the information bound is singular or not; this can be used to construct confidence sets for the WAD of NPQIV without the need to estimate the variance nor the need to know the precise convergence rates of the WAD estimator.

Our estimation procedure builds upon Donald et al. [2003], who approximate a conditional moment restriction $E[\rho(Y, \theta_0)|X] = 0$ by an increasing sequence of unconditional moment restrictions, and then consider estimation of the Euclidean parameter $\theta_0$ (of fixed and finite dimension) and specification tests based on GEL (and related) procedures. For the same model $E[\rho(Y, \theta_0)|X] = 0,$
Kitamura et al. [2004] directly estimate the conditional moment restriction via kernel and then apply a kernel-based conditional empirical likelihood (EL) to estimate $\theta_0$. However, the model considered in these papers does not contain any unknown functions (say $h()$) and the residuals $\rho(., \theta)$ are assumed to be twice continuously differentiable with respect to $\theta$ at $\theta_0$. For the semiparametric conditional moment restriction $E[\rho(Y, \theta_0, h_0(\cdot))|X] = 0$ when the unknown function $h(\cdot)$ could depend on an endogenous variable, Otsu [2011] and Tao [2013] consider a sieve conditional EL extension of Kitamura et al. [2004], and Sueishi [2017] provides a sieve unconditional GEL extension of Donald et al. [2003], where the unknown function $h(\cdot)$ is approximated by a finite dimensional linear sieve (series) as in Ai and Chen [2003]. However, like Ai and Chen [2003], all these papers assume twice continuously differentiable residuals $\rho(., \theta, h(\cdot))$ with respect to $(\theta_0, h_0(\cdot))$, and hence rule out the NPQIV model.

Parente and Smith [2011] study GEL properties for non-smooth residuals $g(., .)$ in the unconditional moment models $E[g(Y, \theta_0)] = 0$, but require the dimensions of both $g(., .)$ and $\theta_0$ to be fixed and finite. Finally, Horowitz and Lee [2007], Gagliardini and Scaillet [2012], Chen and Pouzo [2009, 2012, 2015], and Chernozhukov et al. [2015] do include the NPQIV model, but none of these papers addresses the issues of estimation and inference for the WAD of the NPQIV.

The rest of the paper is organized as follows. Section 2 introduces notation and the model. Section 3 characterizes the semiparametric efficiency bound for the WAD of the NPQIV model. Section 4 introduces a flexible penalized sieve GEL procedure. Section 5 derives the consistency and the convergence rates of the penalized sieve GEL estimator for the NPQIV model. Section 6 establishes the asymptotic distributions of the WAD estimator and of the QLR statistic based on penalized sieve GEL for the WAD of a NPQIV. Section 7 concludes with a discussion of extensions.

2 Preliminaries and Notation

Let $Z \equiv (Y, W, X)$ be the observable data vector, where $Y$ is the outcome variable, $W$ is the endogenous variable and $X$ is the instrumental variable (IV); we assume the observable data, $Z$, is distributed according to a probability distribution $P$. In order to simplify the exposition, we restrict attention to real-valued continuous random variables, i.e., we assume $P$ has a density $p$
with support given by $Z \equiv Y \times W \times X \subseteq \mathbb{R}^3$; extending our results to vector-valued endogenous and instrumental variables would be straightforward but cumbersome in terms of notation.

**Notation.** For any subset, $Z$, of an Euclidean space let $\mathcal{P}(Z)$ be the class of Borel probability measures over $Z$. For any $P \in \mathcal{P}(Z)$, we use $p$ to denote its probability density function (pdf) (with respect to Lebesgue (Leb) measure) and $supp(P)$ to denote its support. We also use $P_X (p_X)$ to denote the marginal probability (pdf) of a random variable $X$; and $P_{Y|X} (p_{Y|X})$ to denote the conditional probability (pdf) of $Y$ given $X$. For expectation, we write $E_Q[.]$ to be explicit about the fact that $Q$ is the measure of integration; throughout we sometimes use $E[.] \equiv EP[.]$ when $P$ is the true probability of the data. The term “wpa1” stands for “with probability approaching one (under $P$)”; for any two real-valued sequences $(x_n, y_n)_n$ $x_n \preceq y_n$ denotes $x_n \leq C y_n$ for some $C$ finite and universal; $\preceq$ is defined analogously. For any $q \geq 1$, we use $L^q(Q) \equiv L^q(Z, Q)$ to denote the class of measurable functions $f : Z \mapsto \mathbb{R}$ such that $\|f\|_{L^q(Q)} = \left(\int_{z \in Z} |f(z)|^q Q(dz)\right)^{1/q} < \infty$; as usual $L^\infty(Leb)$ denotes the class of essentially bounded real-valued functions. We use $\|,\|_e$ to denote the Euclidean norm, $\mathbb{R}_+ = [0, \infty)$ and $\mathbb{R}_{++} = (0, \infty)$.

For any subset $S$ of a vector space $(\mathbb{S}, \|,\|_S)$, $lin\{S\}$ denotes the smallest linear space containing $S$; for any subspace $A \subseteq \mathbb{S}$, $A^\perp$ denotes its orthogonal complement in $(\mathbb{S}, \|,\|_S)$. For any linear operator, $M : (\mathbb{S}_1, \|,\|_1) \to (\mathbb{S}_2, \|,\|_2)$, let $Kernel(M) \equiv \{x \in \mathbb{S}_1 : M[x] = 0\}$ and $Range(M) \equiv \{y \in \mathbb{S}_2 : \exists x \in \mathbb{S}_1, M[x] = y\}$; it is bounded if and only if $\sup_{x \in \mathbb{S}_1: \|x\|_1 = 1} \|M[x]\|_2 < \infty$. For any linear bounded operator $M$, $M^+$ denotes its generalized inverse; see, e.g., Engl et al. [1996].

### 2.1 The WAD of the NPQIV model

Let $\mathbb{A} \equiv \mathbb{R} \times H$, where $H = \{h \in L^2(Leb) : h' \text{ exists and } \|h\|_{L^2(Leb)} < \infty\}$, i.e., $H$ is a Sobolev space of order 1, here $h'$ should be viewed as a weak derivative of $h$ (see Brezis [2010]). We note that $H$ is a Hilbert space under the norm $\|h\|_H \equiv \|h\|_{L^2(Leb)} + \|h'\|_{L^2(Leb)}$, and $A$ is a Hilbert space under the norm $\|((\theta, h))\|_A \equiv \|\theta\|_e + \|h\|_H$. In this paper we measure convergence in $A$ using another norm $\|((\theta, h))\|_A \equiv \|\theta\|_e + \|h\|$ for $\|h\| \leq \|h\|_H$ (such as $\|h\| = \|h\|_{L^2(Leb)}$). The parameter set is given by $\mathcal{A} \equiv \Theta \times \mathcal{H} \subseteq A$, where $\Theta$ is bounded and convex and $\mathcal{H}$ is a set that contains additional restrictions on $h \in H$ which will be specified below. We assume that $P$ is such that there exists a
parameter $\alpha_0 \equiv (\theta_0, h_0) \in \mathcal{A}$ that satisfies

$$0 = EP[\{Y \leq h_0(W)\} - \tau | X]$$  \hspace{1cm} (1)

$$\theta_0 = EP[\mu(W)h'_0(W)]$$  \hspace{1cm} (2)

for $\tau \in (0, 1)$, where $\mu$ is a nonnegative, continuously differentiable scalar function in $L^\infty(Leb) \cap \mathbb{R}$ and should be viewed as the weighting function of the average derivative, $\theta_0$, of $h_0$.

The following assumption ensures that the conditions above uniquely identify $\alpha_0$; it will be maintained throughout the paper and will not be explicitly referenced in the results below.

**Assumption 1.** There is a unique $\alpha_0 \in int(\mathcal{A})$ that satisfies model (1)-(2).

The interior assumption is needed only for the asymptotic distribution results in Section 6. In cases where $\mathcal{H}$ has an empty interior, one can use the concept of relative interior of $\mathcal{H}$. This assumption is clearly high level. The goal of this paper is to characterize the asymptotic behavior of a modified GEL estimator of $\alpha$, taking as given the identification part; for a discussion of primitive conditions for Assumption 1, we refer the reader to Chen et al. [2014] and references therein.

The following assumption imposes additional restrictions over the primitives: $\mu$, $P$ and $\alpha_0$.

**Assumption 2.** (i) $P$ has a continuously differentiable pdf, $p$, such that: the marginal density $p_W$ of $W$ is uniformly bounded, zero at the boundary of the support and $p'_W \in L^2(Leb)$; the marginal density $p_X$ of $X$ is uniformly bounded away from 0 on its support; $\sup_{y,w,x \in \mathbb{Z}} p_{Y|WX}(y | w, x) < \infty$, $\sup_{y,w,x \in \mathbb{Z}} \left| \frac{dp_{Y|WX}(y | w, x)}{dy} \right| < \infty$; (ii) $\mathcal{H}$ is convex and such that for all $h \in \mathcal{H}$, $\sup_{w \in W} |\mu(w)h(w)| < \infty$; (iii) $Var_P(\mu(W)h'(W)) > 0$ for all $h \in \mathcal{H}$ in a $|| \cdot ||$-neighborhood of $h_0$.

Part (i) of this condition imposes differentiability and boundedness restrictions on different elements of $p$; part (ii) ensures that $\lim_{w \to \pm \infty} p_W(w)\mu(w)h(w) = 0$ which allows for an alternative representation for $\theta_0$ using integration by parts (see expression 3 below); part (iii) is a high level assumption and essentially implies $Var_P(\mu(W)h'_0(W)) > 0$ as well as continuity of $h \mapsto Var_P(\mu(W)h'(W))$. 


3 Efficiency Bound for $\theta_0$

By definition of $\mathbb{H}$, Assumption 2 and integration by parts, it follows that

$$\theta_0 = E[\mu(W)h_0'(W)] = -\int \ell(w)h_0(w)dw \quad (3)$$

where

$$w \mapsto \ell(w) \equiv \mu'(w)p_W(w) + \mu(w)p'_W(w).$$

For the derivations of the efficiency bound, it is important to recall that $\ell$ depends on $p_W$, so we sometimes use $\ell p$ to denote $\ell$. Finally, observe that under our assumptions over $\mu$ and $p_W$, $\ell \in L^2(\text{Leb})$.

The formal definition of the efficiency bound for the unknown parameter $\theta_0$ is given at the beginning of Appendix A. Loosely speaking, the efficiency bound is a lower bound for the asymptotic variance of all locally regular and asymptotically linear estimators of $\theta_0$; see Bickel et al. [1998] for details and formal definitions. If it is infinite, then the parameter $\theta_0$ cannot be estimated at root-$n$ rate by these estimators. We now derive this bound. For this, we introduce some useful notation.

For any $(y, w, \alpha) \in \mathbb{Y} \times \mathbb{W} \times \mathbb{A}$, let

$$\rho(y, w, \alpha) \equiv (\rho_1(y, w, \alpha), \rho_2(y, w, h))^T \equiv (\theta - \mu(w)h'(w), 1\{y \leq h(w)\} - \tau)^T.$$  

Let $T : \mathbb{H} \rightarrow L^2(P_X)$ be given by

$$T[g](x) = \int p_{Y\mid W,X}(h_0(w)\mid w, x)g(w)p_{W\mid X}(w\mid x)dw$$

for all $x \in X$ and $g \in \mathbb{H}$. The fact that $T$ maps into $L^2(P_X)$ follows from Jensen inequality and the fact that $\sup_{w, x} p_{Y\mid W,X}(h_0(w)\mid w, x) < \infty$ (see Assumption 2). Its adjoint operator is denoted as $T^* : L^2(P_X) \rightarrow L^2(P_W)$. Finally, let

$$x \mapsto \Gamma(x) \equiv E[\rho_1(Y, W, \alpha_0)\rho_2(Y, W, h_0)\mid X = x]/(\tau(1 - \tau))$$

and $z \mapsto \epsilon(z) \equiv \rho_1(y, w, \alpha_0) - \Gamma(x)\rho_2(y, w, h_0)$. Then $E[\epsilon(Z)\rho_2(Y, W, h_0)\mid X] = 0$ and $E[\epsilon(Z)] = 0$.

**Theorem 3.1.** Suppose Assumptions 1 and 2 hold and $\ell \in \text{Kernel}(T)$ \perp. Then

1. The efficiency bound of $\theta_0$ is finite iff $\ell \in \text{Range}(T^*)$.
2. If it is finite, its efficient variance $V_0$ is given by

$$V_0 = ||\epsilon(\cdot)||_{L^2(P)}^2 + ||T(T^*T)^+[\ell - T^*[\Gamma]]||_{L^2(P)}^2.$$ 

Proof. See Appendix A.

The first result in Theorem 3.1 is obtained following the approach of Bickel et al. [1998]. The condition $\ell \in Kernel(T)\perp$ ensures that only the “identified part” of $h_0$ — that is, the part of $h_0$ that is orthogonal to the kernel of $T$ — matters for computing the weighted average derivative; we refer the reader to Appendix A and the paper by Severini and Tripathi [2012] for further discussion.

Severini and Tripathi [2012] provides an analogous result to Theorem 3.1(1) for linear functionals in a nonparametric linear IV regression model. Our condition $\ell \in Range(T^*)$, is analogous to theirs, but with a subtle yet important difference. In Severini and Tripathi [2012], the object that plays the role of $\ell$ does not depend on $P$, whereas in our case it does. This observation changes the nature of our condition vis-a-vis theirs, because, in our setup, $\ell \in Range(T^*)$ implies a restriction on $P$ since both quantities, $\ell$ and $T$ depend on it.\footnote{It is worth pointing out that this restriction was not imposed as one of the conditions that defined the model used to construct the tangent space; see Appendix A for a definition.} It is also important to note that, if $T$ is compact, then the range of $T^*$ is a strict subset of $L^2(P_W)$ so that $\ell \in Range(T^*)$ may not hold. Hence, in this case the weighted average derivative may not be root-n estimable, and, moreover, the condition that determines the finiteness of the efficiency bound depends on unknown quantities.

This observation highlights a difference with the no-endogeneity case, where the efficiency bound is always finite, provided that $\ell \in L^2(P_W)$ (see Newey and Stoker [1993]).

Another discrepancy between the no-endogeneity case and ours is that in the former case the “plug in” is always efficient (see Newey and Stoker [1993], Newey [1994]) due to the fact that the tangent space is the whole of $\{f \in L^2(P): E[f] = 0\}$. On the other hand, for NPQIV Chen and Santos [2018] show that the closure of the tangent space is the whole space iff the $Range(T)$ is dense in $L^2(P_X)$, which in turn is equivalent to $Kernel(T^*) = \{0\}$. This last condition is comparable to a completeness condition on the conditional distribution of the exogenous variable given the endogenous ones, which may or may not hold for a particular $P$.\footnote{In the NPIV setting, $Kernel(T^*) = \{0\}$ is equivalent to the pdf of $X$ given $W$ satisfying a completeness condition.}
The second result in Theorem 3.1 follows from projecting the influence function onto the closure of the tangent space (see Bickel et al. [1998] and Van der Vaart [2000] and references therein). So as to shed some light on the expression for the efficiency bound, we point out that it corresponds to the efficiency bound of the semiparametric sequential conditional moment model via the “orthogonalized moments” approach in Ai and Chen [2012]. In their notation, let \( \varepsilon_2(z, \alpha) \equiv \rho_2(y, w, h) \) and \( \varepsilon_1(z, \alpha) \equiv \rho_1(y, w, \alpha) - \Gamma(x)\rho_2(y, w, h) \). Note that \( E[\varepsilon_1(Z, \alpha)\varepsilon_2(Z, \alpha) \mid X] = 0 \) (and \( \varepsilon_1(z, \alpha_0) = \varepsilon(z) \)). The model (1)-(2) becomes equivalent to their orthogonalized moment model:

\[
E[\varepsilon_2(Z, \alpha) \mid X] = 0, \quad E[\varepsilon_1(Z, \alpha_0)] = 0. \tag{4}
\]

The expression in our Theorem 3.1(2) coincides with their theorem 2.3 semiparametric efficient variance bound for \( \theta_0 \) of the model (4). Also see proposition 3.3 in Ai and Chen [2012] for the semiparametric efficient variance bound for the WAD of a NPIV model.

4 The Penalized-Sieve-GEL Estimator

In this section we introduce our estimator for \( \alpha_0 \in \mathcal{A} \equiv \Theta \times \mathcal{H} \subseteq \mathbb{S} \equiv \Theta \times \mathbb{H} \). In order to do this, it will be useful to define some quantities. Given the i.i.d. sample \((Z_i)_{i=1}^n\), let \( P_n \) be the corresponding empirical probability. Let \((q_k)_{k \in \mathbb{N}}\) be a complete basis in \( L^2(\mathbb{X}, \mathbb{L}e_b) \). For any \( J \in \mathbb{N} \), let \( q^J(x) = (q_1(x), ..., q_J(x))^T \) be \( J \times 1 \) vector-valued function of \( x \), and for any \((z, \alpha) \in \mathbb{Z} \times \mathcal{A}\), let

\[
g_J(z, \alpha) \equiv (\rho_1(y, w, \alpha), \rho_2(y, w, \alpha)q^J(x)^T) = (\theta - \mu(w)h'(w), [1\{y \leq h(w)\} - \tau]q^J(x)^T)^T.
\]

Let \( \mathcal{S} \subseteq \mathbb{R} \) be an open interval that contains 0. For any \( P \in \mathcal{P}(\mathbb{Z}) \), any \( \alpha \in \mathcal{A} \) and any \( J \in \mathbb{N} \), denote \( \Lambda_J(\alpha, P) \equiv \cap_{z \in \mathrm{supp}(P)} \{ \lambda \in \mathbb{R}^{J+1} : \lambda^T g_J(z, \alpha) \in \mathcal{S} \} \), and \( \hat{\Lambda}_J(\alpha) \equiv \Lambda_J(\alpha, P_n) \).

Let \( s : \mathcal{S} \to \mathbb{R} \) be strictly concave, twice-continuously differentiable with Lipschitz continuous second derivative; and \( s'(0) = s''(0) = -1 \); see, e.g., Smith [1997] and Donald et al. [2003] for examples of such \( s(\cdot) \) functions. For any \( \lambda \in \Lambda_J(\alpha, P) \), let

\[
S_J(\alpha, \lambda, P) \equiv E_P[s(\lambda^T g_J(Z, \alpha))] - s(0), \quad \hat{S}_J(\alpha, \lambda) \equiv S_J(\alpha, \lambda, P_n).
\]

If \( \mathcal{A} \) were a finite-dimensional compact set with \( \dim(\mathcal{A}) \leq J + 1 \), then \( \alpha_0 \) could be estimated by the GEL procedure: \( \arg \min_{\alpha \in \mathcal{A}} \sup_{\lambda \in \hat{\Lambda}_J(\alpha)} \hat{S}_J(\alpha, \lambda) \) (see, e.g., Donald et al. [2003]).
Due to the presence of the infinite-dimensional nuisance parameter $h_0 \in \mathcal{H}$ in the NPQIV model (1), the parameter space $\mathcal{A} \equiv \Theta \times \mathcal{H}$ is an infinite-dimensional function space that is typically non-compact subset in $(\mathcal{A}, ||.||)$ and hence the identifiable uniqueness condition needed for consistency in $||.||$-norm might fail; see, e.g., Newey and Powell [2003] and Chen [2007]. The above GEL procedure needs to be regularized to regain consistency and/or to speed up rate of convergence in $||.||$-norm. To this end, we introduce a regularizing structure, which, jointly with $(q_k)_{k \in \mathbb{N}}$, consists of a sequence of sieve spaces $(\mathcal{A}_k \equiv \Theta \times \mathcal{H}_k)_{k \in \mathbb{N}}$ in $(\mathcal{A}, ||.||)$, and a sequence of penalties $(\gamma_k \times Pen(\cdot))_{k \in \mathbb{N}}$ with tuning parameters $\gamma_k \downarrow 0$ and a penalty function $Pen : \mathcal{A} \to \mathbb{R}_+$. The Penalized-Sieve-GEL (PSGEL) estimator is defined as

$$\hat{\alpha}_{L,n} \in \arg \min_{\alpha \in \mathcal{A}_k} \left[ \sup_{\lambda \in \Lambda, J(\alpha)} \hat{S}_J(\alpha, \lambda) + \gamma_K Pen(\alpha) \right],$$

for any $(L = (J, K), n) \in \mathbb{N}^3$. If the “arg min” in the previous expression is empty, one can replace it by an approximate minimizer.

The following assumption imposes restrictions over the regularizing structure $\{(q_k, \mathcal{H}_k, \gamma_k Pen)_{k \in \mathbb{N}}\}$. Let $(\varphi_k)_{k \in \mathbb{N}}$ be a basis functions in $\mathbb{H}$, and $\nabla \varphi^K = (\varphi'_1, \ldots, \varphi'_K)^T$. Assumption 3. (i) $(q_k)_{k \in \mathbb{N}}$ is a basis in $L^2(\mathbf{P}_X)$, and $E[q^T(X)q^T(X)^T] = I$ for each finite $J$;

(ii) For all $K$, $\mathcal{H}_K \subseteq \text{lin}\{\varphi_1, \ldots, \varphi_K\}$ is closed and convex, and $\bigcup_k \mathcal{H}_k \supseteq \mathcal{H}$, i.e., for any $\alpha \in \mathcal{A} \equiv \Theta \times \mathcal{H}$ there is an $\Pi_K \alpha \in \mathcal{A}_K \equiv \Theta \times \mathcal{H}_K$ such that $||\Pi_K \alpha - \alpha|| = o(1)$; and for some finite $C \geq 1$, $C^{-1}I \leq E \left[ (\varphi^K(W)) (\varphi^K(W))^T + (\nabla \varphi^K(W)) (\nabla \varphi^K(W))^T \right] \leq CI$;

(iii) (a) $Pen : \mathcal{A} \to \mathbb{R}_+$ is lower semi-compact (in $||.||$), $|Pen(\Pi_K \alpha_0) - Pen(\alpha_0)| = O(1)$, $Pen(\alpha_0) < \infty$, and $\gamma_k \downarrow 0$, and (b) there exists an $M < \infty$ such that for any $m \geq M$, any $K$ and any $\alpha \in \mathcal{A}_K$, if $Pen(\alpha) \leq m$ then $\sup_{w \in W} |\mu(w)h'(w)| \leq m$.

Condition (i) is mild (see Donald et al. [2003] (DIN) and the discussion therein). Condition (ii) essentially defines the sieve space. Part (a) of Condition (iii) is standard in ill-posed problems (see Chen and Pouzo [2012]); Part (b) is not. If $\mathcal{H}_K$ is $\| \cdot \|_{L^\infty(\mathbb{W}, \mu)}$ bounded, then the condition is vacuous. If this is not the case, then the condition requires $Pen$ to be “stronger” than the $\| \cdot \|_{L^\infty(\mathbb{W}, \mu)}$ norm. The need to bound $\|h'||_{L^\infty(\mathbb{W}, \mu)}$ arises from the fact that, in many instances, in the proofs we need to control $\rho(y, w, \alpha)$ uniformly on $(y, w)$ (e.g., see Lemma SM.II.3 in the
Additionally, in our setup, is useful to link $\text{Pen}$ to $\| \cdot \|_{L^\infty(W,\mu)}$ because the structure of the problem implies a natural bound for $\text{Pen}(\cdot)$ — and thus, through Assumption 3(iii), a bound for $\| \cdot \|_{L^\infty(W,\mu)}$, as shown in the following lemma.

**Lemma 4.1.** For any $L = (J, K) \in \mathbb{N}^2$ and any $\alpha \in A_K$,

$$
\gamma_K \text{Pen}(\hat{\alpha}_{L,n}) \leq \sup_{\lambda \in \hat{\Lambda}(\alpha)} \hat{S}_J(\alpha, \lambda) + \gamma_K \text{Pen}(\alpha) \text{ wpa1.}
$$

Proof. See Appendix B. \hfill \Box

The bound, however, may depend on $(J, K, n)$ and thus may affect the convergence rate. Below, we will set $\alpha$ in the right-hand-side (RHS) to a particular value in $A_K$ and use the resulting bound to construct what we call an “effective sieve space”.

## 5 Consistency and Convergence Rates of the PSGEL Estimator

This section establishes the consistency and the rates of convergence of the PSGEL estimator $\hat{\alpha}_{L,n}$ to the true parameter $\alpha_0$ under a given norm $\| \cdot \|$ over $A$. In this and the next section, we note that the implicit constants inside the $O_P$ do not depend on $(J, K, n)$.

### 5.1 Effective sieve space

Throughout the paper we use the following notation. Let $\bar{\theta} \equiv \sup_{\theta \in \Theta} |\theta| < \infty$; and $b_{p,J} \equiv (E[|q^J(X)|^p])^{1/p}$ for any $p > 0$. For any $L = (J, K) \in \mathbb{N}^2$, let

$$
\Gamma_{L,n} \equiv \left\{ \frac{\hat{g}^2_{L,0}}{n} + \|E[g_J(Z, \Pi_K\alpha_0)]\|^2 + \gamma_K \text{Pen}(\Pi_K\alpha_0) \right\}, \quad \bar{\Theta}^2 \equiv \bar{\theta} + \|\mu(\Pi_Kh_0)\|^2_{L^2(\mathcal{P})} + b_{2,J}^2.
$$

Let $(l_n)_n$ be a slowly diverging positive sequence, e.g., $l_n = \log \log n$, which is introduced solely to avoid keeping track of constants. Finally we let

$$
\bar{\mathcal{A}}_{L,n} \equiv \{ \alpha \in A_K : \text{Pen}(\alpha) \leq \bar{U}_{L,n} \}, \quad \text{where} \quad \bar{U}_{L,n} \equiv l_n \gamma_K^{-1} \Gamma_{L,n}. \quad \text{(11)}
$$

The sequence of sets, $(\bar{\mathcal{A}}_{L,n})_{L,n}$, can be viewed as the sequence of “effective” sieve spaces, because, as the following lemma shows, wpa1 the estimator (and, trivially, the sieve approximator $\Pi_K\alpha_0 \in A_K$) both belong to it.
Assumption 4. (i) \( b_{4,i}^4/n = o(1) \); (ii) \( \delta_n = o(1) \), \( \delta_n \times \bar{U}_{L,n} = o(1) \), \( b^\varrho_{\varrho,n} n \delta_n^\varrho = o(1) \) for some \( \varrho > 0 \); (iii) \( \sqrt{\frac{\varrho^2 a}{n}} + \| E[g_J(Z, \Pi_K \alpha_0)] \|_2^2 = o(\delta_n) \).

Lemma 5.1. Let Assumptions 1, 2, 3 and 4 hold. Then, for any \( L \in \mathbb{N}^2 \), \( \hat{\alpha}_{L,n} \in \bar{A}_{L,n} \) wpa1.

Proof. See Appendix D. \( \square \)

The proof of this Lemma follows from Lemma 4.1 with \( \alpha = \Pi_K \alpha_0 \) and Lemma D.1 with \( \alpha = \Pi_K \alpha_0 \) and \( P = P_n \) in Appendix D. The latter lemma provides a bound for \( \sup_{\lambda \in \hat{\Lambda}(\Pi_K \alpha_0)} \hat{S}_J(\Pi_K \alpha_0, \lambda) \) in terms of \( \| \sum_{i=1}^n g_J(Z_i, \Pi_K \alpha_0) \|^2_2 \) and \( \gamma_K \text{Pen}(\Pi_K \alpha_0) \). With this in mind, the components of \( \bar{U}_{L,n} \) are intuitive: \( \bar{g}^2_{L,0}/n \) is related to the “variance” of \( \sum_{i=1}^n g_J(Z_i, \Pi_K \alpha_0) \), where \( \bar{g}^2_{L,0} \) is a bound for \( \| g_J(., \Pi_K \alpha_0) \|_e^2 \). The term \( \| E[ g_J(Z, \Pi_K \alpha_0) ] \|_e \) is related to the “bias” and reflects the fact that \( \Pi_K \alpha_0 \in \mathcal{A}_K \) is a sieve approximate to \( \alpha_0 \).

Remark 5.1. As explained above, Lemma 5.1 and Assumption 3(iii) are used to ensure that \( \| h' \|_{L^\infty(\mathbb{W}, \mu)} \) is bounded. If the construction of \( \mathcal{H}_K \) directly implies \( \| h' \|_{L^\infty(\mathbb{W}, \mu)} \leq \bar{U} \) for some fixed constant \( \bar{U} < \infty \), then \( \bar{U} \) should replace \( \bar{U}_{L,n} \) in the definition of \( \bar{A}_{L,n} \). This is applicable every time \( \bar{U}_{L,n} \) appears below. \( \triangle \)

5.2 Relation to Penalized Sieve GMM

As expected, the asymptotic properties of the PSGEL estimator are closely related to an approximate minimizer of a GMM criterion associated to the following expression: for any \( J \in \mathbb{N} \) and any \( P \in \mathcal{P}(\mathbb{Z}) \), let

\[
\alpha \mapsto Q_J(\alpha, P) \equiv E_P[g_J(Z, \alpha)]^T H_J(\alpha_0, P)^{-1} E_P[g_J(Z, \alpha)]
\]

where \( (\alpha, P) \mapsto H_J(\alpha, P) \equiv E_P[g_J(Z, \alpha)g_J(Z, \alpha)^T] \). That is, \( Q_J(., P) \) is the optimally weighted (population) GMM criterion function associated with the vector of moments \( E[g_J(Z, .)] \).

For what follows, it will be useful to define the following intermediate quantity which can be viewed as a (sequence) of pseudo-true parameters. For each \( L \equiv (J, K) \in \mathbb{N}^2 \), let

\[
\alpha_{L,0} \equiv \arg \min_{\alpha \in \bar{A}_{L,n}} Q_J(\alpha, P).
\]
We note that $\alpha_0 \in \arg\min_{\alpha \in A} Q_J(\alpha, P)$ for any $J$; but as we restrict to the effective sieve space $\tilde{A}_{L,n}$, it could be that $\alpha_{L,0} \neq \alpha_0$ for any $L \in \mathbb{N}^2$. The following lemma guarantees that $\alpha_{L,0}$ is in fact non-empty.

**Lemma 5.2.** Let Assumptions 2 and 3 hold. Then, for each $L = (J, K) \in \mathbb{N}^2$, $\alpha_{L,0}$ is non-empty.

**Proof.** See Appendix D. \qed

While this lemma shows that $\alpha_{L,0}$ is non-empty, it may not be a singleton. Nevertheless, for model (1)-(2), it is easy to choose some finite-dimensional linear sieve $\mathcal{H}_K$ and some strict convex penalty $\text{Pen}$ such that $\alpha_{L,0}$ is in fact a singleton. Therefore the next assumption is effectively a way to suggest choices of a regularizing structure:

**Assumption 5.** For any $L \in \mathbb{N}^2$, $\alpha_{L,0}$ is single-valued.

Let $m_2(X, \alpha) \equiv \mathbb{E}_P[\rho_2(Y, W, \alpha) | X]$. For each $J \in \mathbb{N}$, the $L^2(P)$ projection of $m_2(\cdot, \alpha)$ onto the linear span of $q^J(X)$ is denoted as $\text{Proj}_J[m_2(\cdot, \alpha)](X)$, where

$$
\text{Proj}_J[m_2(\cdot, \alpha)](X) = \mathbb{E}_P \left[ m_2(X, \alpha)q^J(X)^T \right] \left( \mathbb{E}_P[q^J(X)q^J(X)^T] \right)^{-1} q^J(X)
$$

where $\left( \mathbb{E}_P[q^J(X)q^J(X)^T] \right)^{-1} = I$ by Assumption 3.

The next lemma provides sufficient conditions that ensure convergence of $\alpha_{L,0}$ to the true parameter $\alpha_0$.

**Lemma 5.3.** Let Assumptions 1, 2, 3 and 5 hold. Suppose $\lim_{n \to \infty} \sup_{\alpha \in \tilde{A}_{L,n}} ||\text{Proj}_{J_n}[m_2(\cdot, \alpha)] - m_2(\cdot, \alpha)||_{L^2(P)} = 0$. Then: $||\alpha_{L,n,0} - \alpha_0|| = o(1)$.

**Proof.** See Appendix D. \qed

### 5.3 Convergence rates

A crucial part of establishing the convergence rate of $\hat{\alpha}_{L,n}$ is to bound the rate of $||\hat{\alpha}_{L,n} - \alpha_{L,0}||$. For this it is important to quantify how well the population sieve GMM criterion function $Q_J$ separates
points in \((\tilde{A}_{L,n}, ||||)\) around \(\alpha_{L,0}\). To do this, we define, for each, \((L, n) \in \mathbb{N}^3\), \(\varpi_{L,n} : \mathbb{R}_+ \to \mathbb{R}_+\) as

\[
t \mapsto \varpi_{L,n}(t) \equiv \inf_{\alpha \in \tilde{A}_{L,n} : ||\alpha - \alpha_{L,0}|| \geq t} Q_f(\alpha, \mathbf{P}) - Q_f(\alpha_{L,0}, \mathbf{P}).
\]  

(5)

The function \(\varpi_{L,n}\) is analogous to the one used in the standard identifiable uniqueness condition (see White and Wooldridge [1991], Newey and McFadden [1994]). Within the ill-posed inverse literature this function is akin to the notion of sieve measure of ill-posedness used in Blundell et al. [2007] and Chen and Pouzo [2012, 2015]. The following lemma establishes some useful properties.

**Lemma 5.4.** Let Assumptions 2, 3 and 5 hold. Then: for each \((L, n) \in \mathbb{N}^3\), \(\varpi_{L,n}(t) = 0\) iff \(t = 0\) and \(\varpi_{L,n}\) is continuous and non-decreasing in \(t\).

**Proof.** See Appendix D. \(\square\)

It is worth noting that even though \(\varpi_{L,n}(t) > 0\) for all \(t > 0\), it could happen that \(\varpi_{L,n}(t) \to 0\) as \(L\) diverges. This behavior reflects the ill-posed nature of the problem.

We now present some high-level assumptions used to establish the convergence rate of the PSGEL estimator. The first of these assumptions introduces, and imposes restrictions on, a positive real-valued sequence \((\delta_n)_{n \in \mathbb{N}}\) that is common in the GEL literature (see the Appendix in Donald et al. [2003]). It ensures that the ball \(\{\lambda \in \mathbb{R}^{J+1} : ||\lambda||_e \leq \delta_n\}\) belongs to \(\hat{A}_J(\alpha)\) for any \(\alpha \in \tilde{A}_{L,n}\) (see Lemma SM.II.3 in the Supplemental Material SM.II). The assumption also restricts the rates of \((b_{\rho,J})_{\rho \in \mathbb{R}, J \in \mathbb{N}}\) and the rate at which \(L = (J, K) \in \mathbb{N}^2\) diverges relative to \(n\):

**Assumption 6.** (i) Assumption 4 holds; (ii) \(\delta_n l_n = o(1)\), \(b_{3,J}^2 \delta_n^q = o(1)\) for some \(q > 0\), and \((\bar{\delta}_{L,n})^4 b_{4,J}^4 / n = o(1)\).

Recall that the sequence \((l_n)_n\) diverges arbitrary slowly like \(\log \log n\), and the bound \(\bar{\delta}_{L,n}\) is allowed to grow (slowly) at the rate of \(l_n\). Assumption 6 slightly strengthens Assumption 4.

The following assumption is a high-level condition that controls the supremum of the process \(f \mapsto G_n[f] \equiv n^{-1/2} \sum_{i=1}^n \{f(Z_i) - E[f(Z_i)]\}\) over the classes \(\tilde{A}_{L,n}\) and \(G_L \equiv \{(y, w) \mapsto \rho_2(y, w, \alpha) : \alpha \in \tilde{A}_{L,n}\}\).

**Assumption 7.** There exists a positive real-valued sequence, \((\Delta_{L,n})_{L,n \in \mathbb{N}^3}\), such that, for any \(L \in \mathbb{N}^2\), \(\sup_{(\theta, h) \in \tilde{A}_{L,n}} |G_n[\mu \cdot h']| = O_P(\Delta_{L,n})\) and for all \(1 \leq j \leq J\), \(\sup_{g \in G_L} |G_n[g \cdot q_j]| = O_P(\Delta_{L,n})\).
For instance, if \( \{ \mu \cdot h': h \in \mathcal{H} \} \) and \( \{(y, w) \mapsto 1\{y \leq h(w)\}: h \in \mathcal{H} \} \) are P-Donsker, then \((\Delta_{L,n})_{L,n \in \mathbb{N}^3}\) is uniformly bounded.\(^3\) But if this is not the case, then \((\Delta_{L,n})_{L,n \in \mathbb{N}^3}\) may diverge as \(L\) (or \(n\)) grows.

The next theorem establishes the convergence rate of the PSGEL estimator; in particular it establishes the rate for the estimator of the infinite dimensional component \(h_0 \in \mathcal{H}\).

**Theorem 5.1.** Suppose Assumptions 1, 2, 3, 5 and 7 hold. For any \((\delta_n, l_n)\) satisfying Assumption 6, there exists a finite constant \(M > 0\) such that

\[
||\hat{\alpha}_{L,n} - \alpha_0|| = O_P \left( \bar{w}^{-1}_{L,n} \left( M(\delta_{1,L,n} + \delta_{2,L,n}) \right) \right) + ||\alpha_{L,0} - \alpha_0||,
\]

where

\[
\delta_{1,L,n} \equiv \sqrt{J_n} \times \Delta_{L,n} \times (\vartheta + \tilde{D}_{L,n} + b_{2,J}), \quad \delta_{2,L,n} \equiv \frac{1}{J_n} \{ \delta_n + \delta_n^{-1}\Gamma_{L,n} \}.
\]

**Proof.** See Appendix C. \(\square\)

The rate of convergence of the PSGEL estimator is composed of two standard terms reflecting the “approximation error” \(||\alpha_{L,0} - \alpha_0||\) and the “sampling error” \(\bar{w}^{-1}_{L,n} (M(\delta_{1,L,n} + \delta_{2,L,n}))\). The component \(\bar{w}^{-1}_{L,n}(\cdot)\), reflects the ill-posed nature of the estimation problem. As noted previously, even though, for a fixed \(L\), \(\bar{w}_{L,n}(t) > 0\) for \(t > 0\), this relationship can deteriorate as \(L\) diverges, which implies that \(\bar{w}^{-1}_{L,n}(t)\) may diverge as \(L\) diverges.

Below, we present an heuristic description of the proof that sheds light on the role of the sequences \((\delta_{1,L,n}, \delta_{2,L,n})_{L,n}\) and of \(\bar{w}_{L,n}\).

### 5.4 Heuristics

By the triangle inequality it suffices to bound the rate of \(||\hat{\alpha}_{L,n} - \alpha_{L,0}||\). We do this by linking the PSGEL estimator to the population sieve GMM problem defined by \(Q(J, P)\). The first step to do this is to show that the PSGEL is an approximate minimizer of the sample sieve GMM criterion \(Q(J, P_n)\) with the rate given by \(\delta_{2,L,n}\).

---

\(^3\)Restrictions on the “complexity” of these classes are implicit restrictions on the “complexity” of \(\mathcal{H}\); see Chen et al. [2003] and Van der Vaart [2000].
Lemma 5.5. Let Assumptions 2 and 3 hold. For any $(\delta_n, l_n)_{n \in \mathbb{N}}$ satisfying Assumption 6, we have:

$$Q_J(\hat{\alpha}_{L,n}, P_n) = O_P(\delta_{2,L,n}) , \quad \text{with} \quad \delta_{2,L,n} = \mathcal{U}_{\delta_{2,L,n}}^2 \{ \delta_n + \delta_n^{-1} \Gamma_{L,n} \} .$$

Proof. See Appendix D. □

The Lemma illustrates not only the role of $\delta_{2,L,n}$ but its nature. The two terms inside the curly brackets are completely analogous to those appearing in Donald et al. [2003]. The scaling by $\mathcal{U}_{\delta_{2,L,n}}^2$ is not present in Donald et al. [2003] and its appearance here is due to the fact that the bound of $\rho(\cdot, \alpha_{L,0})$ may depend, in principle, on $n$ and $L = (J, K)$. In Donald et al. [2003], on the other hand, the upper bound $\mathcal{U}_{\delta_{2,L,n}}$ can be taken to be a fixed constant due to their Assumption 6.

Lemma 5.5 implies that, for some finite $M$, the event $Q_J(\hat{\alpha}_{L,n}, P_n) > Q_J(\alpha_{L,0}, P_n) \leq M \delta_{2,L,n}$ occurs wpa1. The next step is to link the empirical GMM criterion function, $Q_J(\cdot, P_n)$, to its population analog, $Q_J(\cdot, P)$ for which we can quantify its behavior (around $\alpha_{L,0}$) using $\mathcal{W}_{L,n}$. The next lemma provides such a link by showing that $Q_J(\cdot, P_n)$ converges to its population analog.

Lemma 5.6. Let Assumptions 2, 3 and 7 hold. Then: for any $L \equiv (J, K) \in \mathbb{N}^2$,

$$\sup_{\alpha \in A_{L,n}} |Q_J(\alpha, P_n) - Q_J(\alpha, P)| = O_P(\delta_{1,L,n}) , \quad \text{with} \quad \delta_{1,L,n} = \sqrt{J} \times \Delta_{L,n} \times (\mathcal{W} + \mathcal{U}_{L,n} + b_{2,J}).$$

Proof. See Appendix D. □

The rate $(\delta_{1,L,n})_{L,n}$ has several components. The component $\sqrt{J}$ reflects the pointwise convergence rate of $\|n^{-1} \sum_{i=1}^n g_j(Z_i, \alpha) - E[g_j(Z, \alpha)]\|_e$, while the factor of $\Delta_{L,n}$ reflects the fact that we need uniform convergence of that term. Finally, the term $(\mathcal{W} + \mathcal{U}_{L,n} + b_{2,J})$ is essentially the (uniform) bound for $\alpha \mapsto \|n^{-1} \sum_{i=1}^n g_j(Z_i, \alpha)\|_e$ and $\alpha \mapsto \|E[g_j(Z, \alpha)]\|_e$ over $A_{L,n}$.

With this result at hand and simple algebra, one can show that for some finite $M$ the set $A \equiv \{Q_J(\hat{\alpha}_{L,n}, P) - Q_J(\alpha_{L,0}, P) \leq M(\delta_{1,L,n} + \delta_{2,L,n})\}$ occurs wpa1. Therefore, by standard laws of probabilities, it follows that the probability of the set $||\hat{\alpha}_{L,n} - \alpha_0|| \geq M'\mathcal{W}_{L,n}^{-1}(M(\delta_{1,L,n} + \delta_{2,L,n}))$ (for any $M'$) is — up to a vanishing term — less or equal than the probability of the intersection of the same set with $A$. Therefore, it only remains to show that the latter probability is naught for sufficiently large $M'$. This follows because this latter probability is in turn bounded above by the probability of $\mathcal{W}_{L,n}(M'\mathcal{W}_{L,n}^{-1}(M(\delta_{1,L,n} + \delta_{2,L,n}))) \leq M(\delta_{1,L,n} + \delta_{2,L,n})$. By the fact that $\mathcal{W}_{L,n}$
is non-decreasing (see Lemma 5.4), this probability is naught by sufficiently large $M'$, proving the result of Theorem 5.1.

5.5 Discussion of the elements in the Convergence Rate

We now present some observations regarding the main components of the convergence rate in Theorem 5.1, namely, the rates $(\delta_{1,n}, \delta_{2,n})_{L,n}$ and $\omega_{L,n}$ defined in expression 5. Regarding the latter, we first need to specify the norm $||.||$. We start by taking $(\varphi_k)_{k \in \mathbb{N}}$ to be an orthogonal basis with respect to the Lebesgue measure over $\mathbb{H}$. Thus, for any $\alpha = (\theta, h) \in \mathcal{A}$, there exists a real-valued sequence, $(\pi_l)_{l=0}^{\infty}$, such that $\alpha = (\theta, h) = (\pi_0 \overline{\varphi}_0, \sum_{l=1}^{\infty} \pi_l \varphi_l)$ where $\overline{\varphi}_0 = 1$ and $\pi_0 = \theta$, and, for any $k \geq 1$, $\overline{\varphi}_k = \varphi_k$ and $\pi_k$ is the “Fourier” coefficient of $h$ with respect the basis $(\varphi_k)_{k \in \mathbb{N}}$. This representation gives rise to the following norm over $\mathcal{A}$, $\alpha \mapsto \sqrt{\sum_{l=0}^{\infty} \pi_l^2} = \sqrt{\theta^2 + ||h||^2_{L^2(\text{Leb})}}$. The aforementioned norm presents itself as a “natural” norm under which we can establish convergence rate and thus we set $||.||$ as this norm; our result can be extended to norms other than this by specifying how the desired norm relates to $\alpha \mapsto \sqrt{\theta^2 + ||h||^2_{L^2(\text{Leb})}}$.

We now shed light on the behavior of $\omega_{L,n}$ under our choice of $||.||$. In particular, we will illustrate how this function is linked to the curvature of the criterion function $\alpha \mapsto \bar{Q}_J(\alpha, P)$. To do this, it is convenient to use local approximations, so we take, for each $L = (K, L) \in \mathbb{N}^2$, $\mathcal{A}_K$ to be convex, $\alpha_{L,0}$ to be such that $\text{ARC}_K(\alpha_{L,0}) \equiv \{\alpha_{L,0} + \zeta : \zeta \in \mathcal{A}_K \setminus \{\alpha_{L,0}\} \text{ and } t \in [0, 1]\} \subseteq \mathcal{A}_K$, and require that $\text{Pen}$ to be convex and twice continuously differentiable. By the mean value theorem and the fact that $\alpha_{L,0}$ is a minimizer — and thus satisfies that $\frac{d \bar{Q}_J(\alpha_{L,0}, P)}{d \alpha}[\cdot] = 0$ —, it follows that for any $\alpha \in \bar{A}_{L,n},$

$$\bar{Q}_J(\alpha, P) - \bar{Q}_J(\alpha_{L,0}, P) \geq \frac{1}{2} \inf_{\eta \in \text{ARC}_K(\alpha_{L,0})} \frac{d^2 \bar{Q}_J(\eta, P)}{d \alpha^2}[\alpha - \alpha_{L,0}, \alpha - \alpha_{L,0}].$$

By the sieve representation discussed above, the RHS in this expression can be cast as

$$\bar{Q}_J(\alpha, P) - \bar{Q}_J(\alpha_{L,0}, P) \geq (\pi^{K+1} - \pi^{K+1}_{L,0})^T I_L (\pi^{K+1} - \pi^{K+1}_{L,0})^T,$$

where $\pi^{K+1}$ denotes the first $K + 1$ coefficients of the representation of $\alpha$; $\pi^{K+1}_{L,0}$ is the same but
for \( \alpha_{L,0} \), and \( \mathcal{I}_L \) is a \((K + 1) \times (K + 1)\) matrix where the \((i, j)\)-th component is given by

\[
\mathcal{I}_L[i, j] = \frac{1}{2} \inf_{\eta \in A_{RC_K(\alpha_{L,0})}} \frac{d^2 \tilde{Q}_J(\eta, \mathbf{P})}{d \alpha^2}[\hat{\varphi}_i, \hat{\varphi}_j].
\]

This result implies that \( \bar{\omega}_{L, n}(t) \geq t^2 e_{\min}(\mathcal{I}_L) \) \( (e_{\min}(A) \) is the minimal eigenvalue of the matrix \( A)\). If \( e_{\min}(\mathcal{I}_L) > 0 \), then

\[
||\hat{\alpha}_{L,n} - \alpha_0|| = O_p \left( \left( e_{\min}(\mathcal{I}_L) \right)^{-1/2} \sqrt{\delta_{1,L,n} + \delta_{2,L,n} + ||\alpha_{L,0} - \alpha_0||} \right).
\]

The scaling factor \( (e_{\min}(\mathcal{I}_L))^{-1/2} \) summarizes the ill-posed nature of the problem, because, even though we require \( e_{\min}(\mathcal{I}_L) > 0 \) for each \( L \), we do not impose this restriction uniformly on \( L \), i.e., we allow that \( e_{\min}(\mathcal{I}_L) \to 0 \) as \( L \to \infty \).\(^4\) The speed at which this occurs depends on the local curvature of \( \tilde{Q}_J(\cdot, \mathbf{P}) \) \( (\alpha_{L,0}) \) and the growth of \( \mathcal{A}_K \); see Blundell et al. [2007] and Chen and Pouzo [2012] for a more thorough discussion.

We next discuss the rate components \((\delta_{1,L,n}, \delta_{2,L,n})_{L,n}\). As mentioned in Remark 5.1 above, if \( \sup_{h \in \mathcal{H}} ||h'||_{L^\infty(\mathcal{W}, \mu)} \) is finite, then \( \delta_{L,n} \) can be replaced by a fixed constant \( \overline{\delta} \); this fact and some algebra implies that \( \delta_{2,L,n} = O(\delta_n + \delta_n^{-1}(b_{2,J}^2/n + ||E[g_J(Z, \Pi_K \alpha_0)]||_2^2 + \gamma_K Pen(\Pi_K \alpha_0))) \), where \( \Pi_K \alpha_0 \) is the projection of \( \alpha_0 \) onto \( \mathcal{A}_K \) (see Lemma SM.II.1 in the Supplemental Material). By taking \( \delta_n \) to balance both terms, it follows \( \delta_{2,L,n} = O \left( \sqrt{b_{2,J}^2/n + ||E[g_J(Z, \Pi_K \alpha_0)]||_2^2 + \gamma_K Pen(\Pi_K \alpha_0)} \right) \).

Ignoring the term inside the curly brackets, this is the same rate than the one obtained by DIN in Lemma A.14 (note that in their setup \( b_{2,J} \leq J \)); the additional term inside the curly brackets stems from the fact that our estimation problem needs to be regularized and consequently \( \alpha_{L,0} \) is not the true parameter that nullifies the moments.

The sequence \((\delta_{1,L,n})_{L,n}\) is somewhat more standard within the semi-/non-parametric literature, e.g. Chen [2007], and its components essentially impose restrictions on the “complexity” of \( \mathcal{H} \). For instance, if \( \mathcal{A} = \Theta \times \mathcal{H} \) is such that the classes \( \{\rho_2(\cdot, \cdot, \alpha) : \alpha \in \mathcal{A}\} \) and \( \{\mu h' : h \in \mathcal{H}\} \) are \( P \)-Donsker, then \( \Delta_{L,n} = O(1) \) and \( \delta_{1,L,n} = O \left( \sqrt{\frac{J}{n} \times b_{2,J}} \right) \).

To further simplify the expression, suppose \( b_{\rho, J}^2 \leq J^{p/2} \); cf. Assumption 2 in Donald et al. [2003] (see that paper for details and further references). Thus, under these conditions, the result

\(^4\)The condition \( e_{\min}(\mathcal{I}_L) > 0 \) for each \( L \) essentially ensures that the “identifiable uniqueness” condition 5 holds for each \( L \); this requirement is common in the ill-posed inverse literature (e.g., Chen [2007]).
in Theorem 5.1 simplifies to

\[ \|\hat{\alpha}_{L,n} - \alpha_0\| = O_P \left( (e_{\min}(I_L))^{-1/2} \left( \frac{J}{\sqrt{n}} + \|E[g_J(Z, \Pi_0)]\|_\infty + \sqrt{\gamma K \text{Pen}(\Pi_0)} \right)^{1/2} + \|\alpha_{L,0} - \alpha_0\| \right); \]

a rate governed by the degree of ill-posedness, the number \( J \) of moment functions, the number \( K \) of series terms and the bias arising from \( \alpha_{L,0} \).

6 Asymptotic Distribution Theory

We now define the LR-type test statistic for the null hypothesis \( \theta_0 = \nu \). For any \( \nu \in \Theta \) and any \((L, n) \equiv (J, K, n) \in \mathbb{N}^3\), let

\[ \hat{\mathcal{L}}_{L,n}(\nu) \equiv 2 \left\{ \inf_{\{\alpha \in A_K : \theta = \nu\}} \left[ \sup_{\lambda \in \hat{\Lambda}_J(\alpha)} \hat{S}_J(\alpha, \lambda) + \gamma_K \text{Pen}(\alpha) \right] - \inf_{\alpha \in A_K} \left[ \sup_{\lambda \in \hat{\Lambda}_J(\alpha)} \hat{S}_J(\alpha, \lambda) + \gamma_K \text{Pen}(\alpha) \right] \right\}. \]

The goal of this section is to show that this statistic is asymptotically chi-square distributed with one degree of freedom. The proof of this result relies on a local quadratic approximation of the criterion function \( \hat{S}_J \) and a representation for the parameter of interest. To derive these results, we define the following quantities: For any \( \alpha \in A_K \) and any \((\theta, \zeta) \in \Lambda\), let

\[ G(\alpha)((\theta, \zeta)) = \frac{dE[g_J(Z, \alpha)]}{d\theta} \theta + \frac{dE[g_J(Z, \alpha)]}{dh} \zeta = \begin{bmatrix} \theta \\ 0 \end{bmatrix} + \begin{bmatrix} E[\ell(W)\zeta(W)] \\ E[p_Y|W|X(h(W)|W, X)\zeta(W)q^I(X)] \end{bmatrix} \]

where \( 0 \) is a \( J \times 1 \) vector of zeros. By assumption 2 these quantities are well-defined.

For any \( L \in \mathbb{N}^2 \) and for any \((\theta, \zeta) \in \Lambda\), we define another norm over \( \Lambda \) as,

\[ \|(\theta, \zeta)\|_w^2 \equiv (G(\alpha_{L,0})((\theta, \zeta)))^T H_L^{-1}(G(\alpha_{L,0})((\theta, \zeta))), \]

where \( H_L \equiv H_J(\alpha_{L,0}, \mathbf{P}) \). This norm acts as the so-called "weak norm" in Ai and Chen [2003, 2007].

6.1 Alternative Representation for the Weighted Average Derivative

Lemma F.1 in Appendix F shows that, over \( \text{lin}\{A_K\} \) for any \( L = (J, K) \in \mathbb{N}^2 \), \( \|\alpha\|_w = 0 \) iff \( \alpha = 0 \). This fact implies that linear functionals are always bounded in the space \( \text{lin}\{A_K\}, \|\cdot\|_w \). Since \( \theta \) can be interpreted as a linear functional of \( \alpha \), the following representation for \( \theta \) holds: For all
L = (J, K) ∈ N^2, there exists a v^*_L,n ∈ A_K such that for any α = (θ, h) ∈ A_K,

\[ \theta = \langle v^*_L,n, \alpha \rangle_w, \text{ and } ||v^*_L,n||_w = \sup_{a=(\theta,h) \in \text{lin}\{A_K\}, a \neq 0} \frac{||\theta||}{||a||_w}. \]

We note that, even though for each fixed L ∈ N^2, ||v^*_L,n||_w < \infty, this quantity may diverge as L diverges if \( \theta \) is not root-n estimable. Hence, we scale \( v^*_L,n \) by its norm, and define \( u^*_L,n \equiv \frac{v^*_L,n}{||v^*_L,n||_w} \). Then

\[ \hat{\theta}_L,n - \theta_{L,0} / ||v^*_L,n||_w = \langle u^*_L,n, \hat{\alpha}_L,n - \alpha_{L,0} \rangle_w. \]

**Remark 6.1** (On the relationship between the Riesz representer and the Efficiency bound). The weak norm of the Riesz representer, \( ||v^*_L,n||_w \), is the efficiency bound of \( \theta_0 \) in a model with \( J + 1 \) unconditional moments functions, \( E_P[g_j(Z, \cdot)] \), and \( K + 1 \) parameters (which define \( \alpha \in A_K \)). For a suitably chosen sequence \( L \equiv L(n) \) that increases as \( n \) does — since \( (q_j)_j \) is dense in \( L^2(\mathbb{R}, \text{Leb}) \) — one expects the sequence of unconditional moment functions to approximate the moments (1)-(2) defining the model. Thus, by the results in Chamberlain [1987] (see also lemma 3.3, lemma 4.1 and appendix A.1 in Chen and Pouzo [2015]) one expects \( (||v^*_L(n),n||_w)_n \) to converge to the efficiency bound presented in Theorem 3.1 provided it is finite. If the efficiency bound is infinite, the sequence \( (||v^*_L(n),n||_w)_n \) will diverge; this fact reflects the non root-n estimability of the weighted average derivative within the original model (1)-(2). △

### 6.2 The Asymptotic distributions of \( \hat{\theta}_L,n \) and LR statistic

For any positive real-valued sequences \( (\eta_{L,n}, \eta_{w,L,n})_{L,n} \in \mathbb{N}^3 \) (they will be restricted below) and any \( (L, n) \in \mathbb{N}^3 \), let \( N_{L,n} \equiv \{ \alpha \in \tilde{A}_{L,n} : ||\alpha - \alpha_{L,0}|| \leq \eta_{L,n} \text{ and } ||\alpha - \alpha_{L,0}||_w \leq \eta_{w,L,n} \} \).

In what follows, for any \( (L, n) \in \mathbb{N}^3 \), let \( \hat{\alpha}^\nu_{L,n} \) be the argument that minimizes the restricted criterion function, i.e., \( \hat{\alpha}^\nu_{L,n} \in \arg \min_{\alpha \in A_K : \theta = \nu} \sup_{\lambda \in \tilde{\Lambda}_j(\alpha)} \hat{S}_j(\alpha, \lambda) \). We impose the following assumption that restricts the convergence rate of the unrestricted and restricted PSGEL estimators.

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5 Ai and Chen [2003, 2012] established this claim for a richer model with conditional moments and infinite dimensional parameters.
Assumption 8. For any $L \in \mathbb{N}^2$ and $\alpha \in \{\hat{\alpha}_{L,n}, \hat{\alpha}^*_L,n\}$, if $\nu = \theta_0$: (i) $\alpha \in \text{int}(\mathcal{N}_{L,n})$; (ii) $\gamma_K \sup_{|t| \leq t_{n,\alpha}^{1/2}} |\text{Pen}(\alpha) - \text{Pen}(\alpha + tu^*_L,n)| = o_p(n^{-1})$; (iii) There exists a $C < \infty$ such that for any $h \in \mathbb{H}$, $||h||_{L^2(\text{Leb})} \leq C||(0,h)||$.

Part (i) of this assumption ensure that both estimators — the restricted and unrestricted ones — converge to $\alpha_{L,0}$ faster than $\eta_{L,n}$ and $\eta_{w,L,n}$ in the respective norms. One can use the results in Section 5 to verify this assumption. Part (ii) ensures that the penalty term is negligible (see also Chen and Pouzo [2015]). Finally part (iii) states a relationship between the norm $h \mapsto ||(0,h)||$ — used in Section 5 — and the $L^2(\text{Leb})$ norm over $\mathbb{H}$.

In the following assumption we let $\bar{G}_{L,n} \equiv \{f(\cdot, \alpha) = \rho_2(\cdot, \cdot, \alpha) - \rho_2(\cdot, \cdot, \alpha_{L,0}) : \alpha \in \mathcal{N}_{L,n}\}$.

Assumption 9. There exists positive sequence, $(\Delta_{2,L,n})_{L,n \in \mathbb{N}^2}$, such that, for any $L = (J, K) \in \mathbb{N}^2$, $\sup_{\alpha = (\theta, h) \in \mathcal{N}_{L,n}} \mathbb{E}_n[\mu \cdot (h' - h'_{L,0})] = O_P(\Delta_{2,L,n})$ and for all $1 \leq j \leq J$, $\sup_{f \in \bar{G}_{L,n}} \mathbb{E}_n[f \cdot q_j] = O_P(\Delta_{2,L,n})$.

This is a high-level assumption that controls one of the terms in the remainder of the quadratic approximation in Lemma 6.1 below. As $\mathcal{N}_{L,n}$ is shrinking, one would expect $\Delta_{2,L,n} = o(1)$; the exact rate, however, depends on the complexity of $\bar{A}_{L,n}$.

Assumption 10. There exists a positive real-valued sequence, $(\Xi_{L,n})_{L,n \in \mathbb{N}^2}$, such that, for any $L = (J, K) \in \mathbb{N}^2$, $\sup_{\alpha \in \mathcal{N}_{L,n}} \|H_J(\alpha, P_n) - H_J(\alpha_{L,0}, P_n) - \{H_J(\alpha, P) - H_J(\alpha_{L,0}, P)\}\|_e = O_P(\Xi_{L,n})$.

This high-level assumption implies stochastic equi-continuity of the process $H_J(\cdot, P_n)$, and it is used to control one of the terms in the remainder of the quadratic approximation in Lemma 6.1 below.

The final two assumptions impose additional restrictions on $(\eta_{L,n}, \eta_{w,L,n})_{L,n \in \mathbb{N}^2}$, $(b_{\rho,J})_{\rho \in \mathbb{R}, J \in \mathbb{N}}$, $(\delta_n)_{n \in \mathbb{N}}$ and the rate at which $L = (J, K)$ diverges relative to $n$.

Assumption 11. (i) $\frac{\sqrt{n}}{||\eta_{L,n}||_w} \|E[g_J(Z, \alpha_{L,0})]\|_e = o(1)$; (ii) $\frac{\sqrt{n}}{||\eta_{L,n}||_w} |\theta_{L,0} - \theta_0| = o(1)$.

This assumption implies that the “bias” terms arising from working with $\alpha_{L,0}$, as opposed to $\alpha_0$, are small relative to the rate we are using to scale the leading term of the asymptotic expansions.

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6 The results in Section 5 apply to the restricted estimator, under the null, with minimal changes.
below $\frac{\sqrt{n}}{||T_{L,n}||_w}$. Similar assumptions have been imposed in the literature, e.g. Chen and Pouzo [2015] and reference therein.

**Assumption 12.** (i) $n\delta_n^3(\Omega_{L,n}+b_{3,J})^3 = o(1)$, $n\delta_n^2\left(\frac{\theta_{L,0}^2 + ||h_{L,0}^t||_{L^\infty(W,\mu)}}{\sqrt{\frac{b_{3,J}}{n}} + \Omega_{L,n}\eta_{L,n} + \Xi_{L,n}}\right) = o(1)$ and $n\delta_n\left(\sqrt{\frac{n}{\delta}} \Delta_{2,L,n} + \eta_{L,n}^{2}b_{2,J}\right) = o(1)$; (ii) $\sqrt{\frac{\eta_{L,n}^{2}b_{2,J}}{n}} + ||E[g_{J}(Z,\alpha_{L,0})]||_e^2 + \eta_{w,L,n} = o(\delta_n)$ and $\left(\sqrt{\frac{n}{\delta}} \Delta_{2,L,n} + \eta_{L,n}^{2}b_{2,J}\right) = o(\delta_n)$; (iii) There exists a $\varrho > 0$ such that $||h_{L,0}^t||_{L^\infty(W,\mu)}^{2+\varrho}/n^{2+\varrho} = o(1)$ and $b_{2,J}^{2+\varrho}/n^{2+\varrho} = o(1)$; (iv) $A_{L,0} \equiv E[\mathbf{p}_Y|W,X(h_{L,0}(W) \mid W,X)\check{q}^T(X)\check{v}^K(W)^T]$ has full rank $K$ and $n^{-1/2}e_{\min}(A_{L,0}^TA_{L,0})^{-1} = o(\eta_{L,n})$; (v) $||h_{L,0}^t||_{L^\infty(W,\mu)}^2 b_{2,J}^{2+\varrho}/\sqrt{n} = o(1)$.

Part (i) ensures that the remainder term for the asymptotic quadratic representation of $\hat{S}_{IJ}$ is negligible (see Lemma 6.1). The sequence $(\delta_n)_n$ in part (ii) was discussed after Assumption 6. Part (iii) is used to show asymptotic normality of the leading term in Lemma 6.2 by means of a Lyapounov condition. Finally, part (iv) ensures that the weak norm is proportional to the strong norm over $\mathcal{A}_K$ (even though the constant of proportionality may vanish as $L$ diverges) and that deviations of the form $\alpha + b_n n^{-1/2}u_{L,n}^*$ stay in $\mathcal{N}_{L,n}$ (see Lemma F.4 in Appendix F). These deviations play a crucial role in the proof of Lemma 6.2.

**Remark 6.2** (The rate restrictions of Assumption 12). While parts (iii)-(v) are fairly easy to check and interpret, parts (i)-(ii) are not as easy. The goal of this remark is to illustrate the restrictions imposed by these parts on the different rates $(\delta_n, \eta_{L,n}, \eta_{w,L,n}, \Delta_{2,L,n}, \Xi_{L,n})_n$ where $(L_n)_n$ is a diverging sequence in $\mathbb{N}^2$. To do this, we take as the point of departure the setting described in Section 5.5, which allows us to simplify some expressions. Under this setup, part (i) imposes $\delta_n = o\left(n^{-1/3}J_n^{-1/2}\right)$. Given this, the restrictions in parts (i)-(ii) imply that $\eta_{L,n} = O(\min\{n^{-1/2}(\delta_n)^{-1}J_n^{1/2}, n^{-1/6}J_n^{-3/4}\})$ and $\eta_{w,L,n} = o(n^{-1/3}J_n^{-1/2})$; we note that by imposing a polynomial rate of decay, this condition rules out the so-called severely ill-posed case wherein the rate of for $(\eta_{L,n})_n$ decays slower than polynomial order (see Chen and Pouzo [2012] and references therein). Parts (i)-(ii) also imply that $\Delta_{2,L,n} = o(\sqrt{n}J_n^{-1/2})$ and $\Xi_{L,n} = o(n^{-1/2})$; for the “worst case” where $\delta_n = \left(n^{-1/3}J_n^{-1/2}\right)/l_n$, it follows that $\Delta_{2,L,n} = O(J_n n^{-1/6})$ and $\Xi_{L,n} = O(n^{-1/3}J_n)$, but the restriction can be relaxed if $(\delta_n)_n$ decays faster. Finally, parts (i)-(ii) impose restrictions on the growth of $(L_n)_n$: $J_n = O(n^{-1/6})$ and $\sqrt{\frac{1}{n}||E[g_{J_n}(Z,\alpha_{L,0})]||_e^2} = o(n^{-1/3})$. \triangle
The following result characterizes the asymptotic distribution of the LR test statistic under the null. This characterization holds regardless of whether the parameter \( \theta_0 \) is root-\( n \) estimable or not.

**Theorem 6.1.** Let Assumptions 1-5 and 8-12 hold. Then, under the null \( \theta_0 = \nu \),

\[
\hat{L}_{L,n}(\theta_0) \Rightarrow \chi^2_1.
\]

**Proof.** See Appendix E. \( \square \)

This result extends those in Parente and Smith [2011] to a non-parametric setup where the GEL is constructed using an increasing number of moment conditions, and wherein the parameter of interest may not be root-\( n \) estimable. Using a related estimator — an EL-based on conditional moments a la Kitamura et al. [2004] — Tao [2013] derived an analogous result but her assumptions rule out non-smooth residuals, relevant for the quantile IV model considered here.

As a by-product of the derivations used to prove Theorem 6.1, an asymptotic linear representation for the estimator of the WAD is obtained.

**Theorem 6.2.** Let Assumptions 1-5, 8 (for \( \hat{\alpha}_{L,n} \)), 9, 10 and 12 hold. Then

\[
\frac{\hat{\theta}_{L,n} - \theta_{L,0}}{||v_{L,n}^*||_w} = n^{-1} \sum_{i=1}^{n} (G(\alpha_{L,0})[u_{L,n}^*])^T H^{-1}_L g_J(Z_i, \alpha_{L,0}) + o_P(n^{-1/2}).
\]

Further, under Assumption 11, we have

\[
\frac{\sqrt{n}(\hat{\theta}_{L,n} - \theta_0)}{||v_{L,n}^*||_w} \Rightarrow N(0, 1).
\]

The proof is the same as that Lemma F.7 in Appendix F so it is omitted. This result illustrates the role of \( ||v_{L,n}^*||_w \) as the appropriate scaling of our estimator. If the sequence \( (||v_{L,n}^*||_w)_n \) is uniformly bounded, then this theorem implies that \( \hat{\theta}_{L,n} \) is \( \sqrt{n} \) asymptotically Gaussian. On the other hand, if the sequence diverges, Gaussianity is still preserve but the rate is slower and given by \( \sqrt{n}/||v_{L,n}^*||_w \). 

### 6.3 Heuristics

The idea is to show that, asymptotically, \( \hat{L}_{L,n} \) is a quadratic form of Gaussian random variables. The first step is to provide a quadratic approximation for the criterion function \( \hat{S}_J(\alpha, \cdot) \) as a function of \( \lambda \), as shown in the following lemma.
Lemma 6.1. Let Assumptions 1-5, 9, 10 and 12(v) hold. Then uniformly over \((\alpha, \lambda) \in \mathcal{N}_{L,n} \times \{ \lambda \in \mathbb{R}^{J+1} : \|\lambda\|_e \leq \delta_n \}\), for any \(L = (J, K) \in \mathbb{N}^2\),

\[
\hat{S}_J(\alpha, \lambda) = -\lambda^T \Delta(\alpha) - \frac{1}{2} \lambda^T H_L \lambda \\
+ O_\mathbb{P}\left( \delta_n^3 (\overline{\theta} + l_n \gamma_j^{-1} \Gamma_{L,n} + b_{3,j})^3 \right) \\
+ O_\mathbb{P}\left( \delta_n^2 \left( (\overline{\theta} + \|h_{L,0}^j\|_{L_{\infty}(\mathbb{W}, \mu)})^2 \sqrt{n^3 J^5 / n + 3 \delta_L \eta_{L,n} + \Xi_{L,n}} \right) \right) \\
+ O_\mathbb{P}\left( \delta_n \left( \sqrt{\frac{J}{n}} \Delta_{2,L,n} + \eta_{L,n}^2 b_{2,j} \right) \right).
\]

where \(\Delta(\alpha) \equiv n^{-1} \sum_{i=1}^n g_J(Z_i, \alpha_{L,0}) + G(\alpha_{L,0})[\alpha - \alpha_{L,0}]\).

Proof. See Appendix F. \(\Box\)

The “remainder” terms in the RHS (the \(O_\mathbb{P}(.)\) terms) are fairly intuitive: the order \(\delta_n^3\)-term requires boundedness of the third derivative of \(S_J(\alpha, \cdot)\); the \(\delta_n^2\)-term arises because the expansion yields a quadratic term with \(H_J(\alpha, P_n)\) as opposed to \(H_L\); and the \(\delta_n\)-term is the error of approximating \(n^{-1} \sum_{i=1}^n g_J(Z_i, \alpha) \) with \(\Delta(\alpha)\). This last part handles the non-smooth nature of the residuals \(\rho_2\) by using \(E[g_J(Z, \cdot)]\), which is a smooth function. Assumption 12(i) ensures that these ‘remainder’ terms are in fact \(o_\mathbb{P}(n^{-1})\). This fact, and the fact that \(\hat{\Lambda}_J(\alpha)\) contains a \(\delta_n\)-ball (see Lemma SM.II.3 in the Supplemental Material SM.II), imply that the expression in the Lemma provides an asymptotic characterization for \(\sup_{\lambda \in \hat{\Lambda}_J(\alpha)} \hat{S}_J(\alpha, \lambda)\) in terms of \((\Delta(\alpha))^TH_L^{-1}(\Delta(\alpha))\), which is a quadratic form in \(\alpha\).

With this result at hand and Assumption 8, one can obtain lower and upper bounds for \(\hat{L}_{L,n}(\theta_0)\) of the form,

\[
\hat{L}_{L,n}(\theta_0) \geq (\Delta(\hat{\alpha}_{L,n}))^TH_L^{-1}(\Delta(\hat{\alpha}_{L,n})) - (\Delta(\hat{\alpha}_{L,n}) + tu_{L,n}^*)^TH_L^{-1}(\Delta(\hat{\alpha}_{L,n} + tu_{L,n}^*)) + o_\mathbb{P}(1),
\]

for appropriately chosen \(t \in \mathbb{R}\), and

\[
\hat{L}_{L,n}(\theta_0) \leq (\Delta(\hat{\alpha}_{L,n}^\theta + tu_{L,n}^*))^TH_L^{-1}(\Delta(\hat{\alpha}_{L,n}^\theta + tu_{L,n}^*)) - (\Delta(\hat{\alpha}_{L,n}^\theta) + tu_{L,n}^*)^TH_L^{-1}(\Delta(\hat{\alpha}_{L,n}^\theta) + tu_{L,n}^*) + o_\mathbb{P}(1),
\]

for appropriately chosen \(t \in \mathbb{R}\). Since \(\alpha \mapsto \Delta(\alpha)\) is an affine function, the RHS in the previous expression is fairly easy to characterize. The following lemma formalizes these steps (its proof presents the explicitly choice for \(t\) in the previous two displays).
Lemma 6.2. Let Assumptions 1-5, 8-12 hold. Then, under the null $\nu = \theta_0$,

$$
\hat{L}_{L,n}(\theta_0) - \left( n^{-1/2} \sum_{i=1}^{n} (G(\alpha_{L,0})[u_{L,n}^*])^T H_{L}^{-1} g_{J}(Z_i, \alpha_{L,0}) \right)^2
\geq 2\sqrt{n} \frac{\hat{\Sigma}^{1/2}}{||v_{L,n}^*||_w} \left( n^{-1/2} \sum_{i=1}^{n} (G(\alpha_{L,0})[u_{L,n}^*])^T H_{L}^{-1} g_{J}(Z_i, \alpha_{L,0}) \right) + o_p(1).
$$

and

$$
\hat{L}_{L,n}(\theta_0) - \left( n^{-1/2} \sum_{i=1}^{n} (G(\alpha_{L,0})[u_{L,n}^*])^T H_{L}^{-1} g_{J}(Z_i, \alpha_{L,0}) \right)^2
\leq 2\sqrt{n} \frac{\hat{\Sigma}^{1/2}}{||v_{L,n}^*||_w} \left( n^{-1/2} \sum_{i=1}^{n} (G(\alpha_{L,0})[u_{L,n}^*])^T H_{L}^{-1} g_{J}(Z_i, \alpha_{L,0}) \right) + \sqrt{n} \frac{\hat{\Sigma}^{1/2}}{||v_{L,n}^*||_w} + o_p(1).
$$

Proof. See Appendix F. \hfill \Box

This lemma shows the reason for Assumption 11 in our analysis, as this assumption ensures that

$$
\hat{L}_{L,n}(\theta_0) = \left( n^{-1/2} \sum_{i=1}^{n} (G(\alpha_{L,0})[u_{L,n}^*])^T H_{L}^{-1} g_{J}(Z_i, \alpha_{L,0}) \right)^2 + o_p(1).
$$

Under mild assumptions and Assumption 11, the object inside the parenthesis is asymptotically Normal with mean 0 and variance 1. Here we see the importance of the “optimal weight”, $H_{L}^{-1}$. If $H_{L}$ differed from $E[g_{J}(Z, \alpha_{L,0})g_{J}(Z, \alpha_{L,0})^T]$, then the variance of the term inside the parenthesis will not be equal to 1, and the test statistic will only be proportional to a $\chi^2_1$ in the limit; see Chen and Pouzo [2015] for a more thorough discussion and results for this case.

7 Conclusion

Since the seminal work by Koenker and Bassett about 40 years ago (Koenker and Bassett [1978]), quantile regression models have become ubiquitous in econometrics and statistics; see Koenker [2017] for a recent survey. The original linear quantile regression model has been extended in several directions; in particular to the general non-parametric IV framework that allows for “flexible functional forms” and endogeneity of the regressors. This type of model, while very general, presents
technical challenges arising from the non-smooth nature of the criterion function as well as its ill-posedness. One goal of this paper is to shed some light on how the nonlinear ill-posedness of the non-parametric quantile IV (NPQIV) model affects not only the speed of convergence to the conditional quantile function but also the accuracy for estimating even simple linear functionals. For this, we derive the semiparametric efficiency bound for a particular linear functional of the NPQIV — the weighted average derivative (WAD).

To estimate the parameters of interest — the NPQIV function and its WAD — we propose a general penalized sieve GEL procedure based on the unconditional WAD moment restriction and an increasing number of unconditional moments that are asymptotically equivalent to the conditional moment defining the NPQIV model (1). We show that the QLR statistic based on the penalized sieve GEL is asymptotically chi-square distributed regardless of whether or not the information bound of the WAD is singular. This result can be used to construct confidence sets for the WAD without the need to estimate the variance of the estimator of the WAD. We hope these results extend even further the scope of quantile regression models.

The penalized sieve GEL procedure is more generally applicable to any semi/nonparametric conditional moment restrictions and unconditional moment restrictions, say of the following form:

\begin{align}
E[\rho_2(Y, W; \theta_{02}, h_{01}(\cdot), ..., h_{0q}(\cdot)) | X] &= 0, \quad \text{a.s.-} X, \quad (6) \\
E[\rho_1(Y, W; \theta_{01}, \theta_{02}, h_{01}(\cdot), ..., h_{0q}(\cdot))] &= 0. \quad (7)
\end{align}

Here \( Y \) denotes dependent (or endogenous) variables, \( X \) denotes conditioning (or instrumental) variables and \( W \) could be either endogenous or subset of \( X \), \( \theta = (\theta'_1, \theta'_2)' \) denotes a vector of finite dimensional parameters, and \( h(\cdot) = (h_1(\cdot), ..., h_q(\cdot)) \) a \( q \times 1 \) vector of real-valued measurable functions of \( Y, W, X \) and other unknown parameters. The residual functions \( \rho_j(y, w; \theta, h(\cdot)), \ j = 1, 2 \), could be nonlinear, pointwise non-smooth with respect to \( (\theta, h) \). And some of the \( \theta \) could have singular information bound. This is a valuable alternative to classical semiparametric two-step GMM when the second step finite dimensional parameter \( \theta \) might not be root-\( n \) estimable.

In an old unpublished draft, Chen and Pouzo [2010] study the asymptotic properties of another estimation procedure, optimally weighted penalized Sieve Minimum Distance (SMD) based on orthogonalized residuals for model (6)-(7). Under a set of regularity conditions, including the
assumption that the WAD of a NPQIV has a positive information bound, Chen and Pouzo [2010] establish that their optimally weighted penalized SMD estimator of the WAD is root-\( n \) asymptotically normal and semiparametrically efficient. It would be interesting to compare this paper’s estimator against theirs, and we leave this to future work.

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Appendix

A Proof of Theorem 3.1

To show Theorem 3.1 we need some more detailed notation and definitions. Let $\mathcal{M}$ be the set of Borel probability measures over $\mathbb{Z}$ such that for each $P \in \mathcal{M}$: (1) there exists a $(\theta(P), h(P)) \in A$ for which equations (1)-(2) hold for $P$; (2) the conditions of the Theorem are satisfied for $P$.

Given a $Q \in \mathcal{M}$, we use $(\theta(Q), h(Q))$ to denote the parameters that satisfy equation 1 and $\theta(Q) = -E_Q[\ell_Q(W)h(Q)(W)]$. For the true $P$, we simply use $(\theta_0, h_0) = (\theta(P), h(P))$.

Henceforth, let $L^2_0(P) \equiv \{g \in L^2(P): E_P[g(Z)] = 0\}$. A curve in $\mathcal{M}$ at $P$ is a mapping $[0,1] \ni t \mapsto P[t] \in \mathcal{M}$ such that there exists a $g \in L^2_0(P)$ such that

$$\lim_{t \to 0} \int \left( \frac{\sqrt{P[t](dz)}}{t} - \sqrt{P(dz)} - 0.5g(z)\sqrt{P(dz)} \right)^2 = 0.$$ 

We call $g$ the tangent of the curve; we typically use $t \mapsto P[t](g)$ to denote a curve with tangent $g$. The set of tangents for all curves in $\mathcal{M}$ at $P$ is called the tangent set; the linear span of the set is called the tangent space of $\mathcal{M}$ at $P \in \mathcal{M}$, and we denote it as $T$.

The efficiency bound of $\theta_0$ is defined as (e.g., see Bickel et al. [1998])

$$\mathcal{E}(P) \equiv \sup_{g \in T} \frac{\|\hat{\theta}(P)[g]\|}{\|g\|_{L^2(P)}}$$

where $\hat{\theta}(P)$ is the G-derivative of $P \mapsto \theta(P)$ at $P$, i.e.,

$$g \mapsto \hat{\theta}(P)[g] = \lim_{t \to 0} \frac{\theta(P[t](g)) - \theta(P)}{t}.$$ 

Henceforth, we use $T_P$ to denote the operator $T$ under the probability measure $P$; the notation $T$ is reserved for $T_P$.

(1) From the expression for $\mathcal{E}(P)$, it follows that finiteness of the efficiency bound is equivalent to boundedness of the linear functional $\hat{\theta}(P)$. In order to show this, we note that, since $\ell_P \in$
Kernel(T_P)\perp$, it follows that, for any $P \in \mathcal{M}$,

$$\theta(P) = -\int \ell_P(w) h_{id}(P)(w) dw$$

where $h_{id}(P)$ is the “identified part” of $h(P)$ under $T_P$, i.e., $h_{id}(P)$ is such that $h(P) = h_{id}(P) + \nu$ where $h_{id}(P) \in Kernel(T_P)\perp$ and $\nu \in Kernel(T_P)$. Thus, it is enough to characterize the G-derivative of the RHS, and we do it in the following lemma; for this, let $A_P : T \to L^2(P_X)$ be defined as

$$g \mapsto A_P[g](\cdot) \equiv \int \rho_2(y, w, h(P)) g(y, w, x) P_{Y \mid X}(dy, dw | \cdot).$$

Lemma A.1. For any $g \in T$,

$$\dot{\theta}(P)[g] = \theta(g \cdot P) - E_P \left[ \ell_P(W) h_{id}(P)[g](W) \right]$$

$$= \langle \mu_h'(P), g \rangle_{L^2(P)} - \langle \ell_P, \dot{h}_{id}(P)[g] \rangle_{L^2(P)}$$

and $\dot{h}_{id}(P)[g] = (T^*T)^+ T^* A_P[g]$, where $(T^*T)^+$ be the generalized inverse of $T^*T$; for a definition see Engl et al. [1996] Ch 2.

Proof. See Section SM.I.

\[\square\]

Remark A.1. This lemma illustrates the role that the condition $\ell_P \in Kernel(T_P)\perp$ plays in our proof. The previous lemma uses the conditional moment 1 to characterize the G-derivative of $P \mapsto h(P)$, and this only allow us to characterize the G-derivative of $h_{id}$, since the part of $h$ in the Kernel of $T$ vanishes. Under condition, $\ell_P \in Kernel(T_P)\perp$, however, this is enough for characterizing the G-derivative of $P \mapsto \theta(P)$. \(\triangle\)

Therefore,

$$\mathcal{E}(P) = \sup_{g \in T} \frac{\| \langle \mu_h'(P), g \rangle_{L^2(P)} - \langle \ell_P, \dot{h}(P)[g] \rangle_{L^2(P)} \|_{L^2(P)}}{\|g\|_{L^2(P)}}$$

and

$$\sup_{g \in T} \frac{\| \langle \ell_P, \dot{h}(P)[g] \rangle_{L^2(P)} \|_{L^2(P)}}{\|g\|_{L^2(P)}} = \sup_{g \in T} \frac{\| \langle \ell_P, (T^*T)^+ T^* A_P[g] \rangle_{L^2(P)} \|_{L^2(P)}}{\|g\|_{L^2(P)}}.$$

We now show that, if $\ell_P \in Range(T)$, then $\mathcal{E}(P) < \infty$. By the triangle inequality, it suffices to show that $\frac{\| \langle \mu_h'(P), g \rangle_{L^2(P)} \|_{L^2(P)}}{\|g\|_{L^2(P)}} < \infty$ and $\sup_{g \in T} \frac{\| \langle \ell_P, (T^*T)^+ T^* A_P[g] \rangle_{L^2(P)} \|_{L^2(P)}}{\|g\|_{L^2(P)}} < \infty$. The former follows
because \( \mu \) is uniformly bounded and \( h'(\mathbf{P}) \in L^2(\mathbf{P}) \). We now show that the latter holds. As \( \ell_{\mathbf{P}} \in \text{Range}(\mathbf{T}^\ast) \), then \((\mathbf{T}^\ast \mathbf{T})^\ast[\ell_{\mathbf{P}}]\) is well-defined. And thus
\[
(\ell_{\mathbf{P}}, (\mathbf{T}^\ast \mathbf{T})^\ast \mathbf{T}^\ast \mathbf{A}_g[g]_{L^2(\mathbf{P})} = \int \mathbf{T}(\mathbf{T}^\ast \mathbf{T})^\ast[\ell_{\mathbf{P}}](x)\rho_2(y, w, x_0)g(y, w, x)\mathbf{P}(dy, dw, dx)
\]
\[
= (\mathbf{T}(\mathbf{T}^\ast \mathbf{T})^\ast[\ell_{\mathbf{P}}] \cdot \rho_2, g)_{L^2(\mathbf{P})}.
\]
Also, \( \|\mathbf{T}(\mathbf{T}^\ast \mathbf{T})^\ast[\ell_{\mathbf{P}}] \cdot \rho_2\|_{L^2(\mathbf{P})} \leq 2\|\mathbf{T}(\mathbf{T}^\ast \mathbf{T})^\ast[\ell_{\mathbf{P}}]\|_{L^2(\mathbf{P})} < \infty \) because \( \mathbf{T}(\mathbf{T}^\ast \mathbf{T})^\ast \) is bounded and \( \|\ell_{\mathbf{P}}\|_{L^2(\mathbf{P})} \lesssim \|\ell_{\mathbf{P}}\|_{L^2(\mathbf{Leb})} < \infty \) under Assumption 2. Therefore \( \sup_{g \in \mathcal{T}} \|\langle \ell_{\mathbf{P}}, (\mathbf{T}^\ast \mathbf{T})^\ast \mathbf{T}^\ast \mathbf{A}_g[g] \rangle_{L^2(\mathbf{P})}\| = \infty \) when \( \ell_{\mathbf{P}} \in \text{Range}(\mathbf{T}^\ast) \), as desired.

(2) To prove part (2), we assume that \( \ell_{\mathbf{P}} \notin \text{Range}(\mathbf{T}^\ast) \). Let \( \hat{\mathbf{h}}_{id}^\ast(\mathbf{P}) : L^2(\mathbf{P}) \to T^\ast \) be the adjoint of \( \hat{\mathbf{h}}_{id}(\mathbf{P}) \) and is given by
\[
g \mapsto \hat{\mathbf{h}}_{id}^\ast(\mathbf{P})[g](y, w, x) = \mathbf{T}(\mathbf{T}^\ast \mathbf{T})^\ast[g](x)\rho_2(y, w, x)
\]
for any \((y, w, x) \in \mathbb{Z}\).

It is well known (Van der Vaart [2000] p. 363) that the efficiency bound (when it exists) is the variance of the projection of the influence function onto the tangent space, i.e.,
\[
\mathcal{E}(\mathbf{P}) = \|\text{Proj}_T \left[ \mu h'(\mathbf{P}) - \hat{\mathbf{h}}_{id}^\ast(\mathbf{P})[\ell_{\mathbf{P}}] \right] \|_{L^2(\mathbf{P})}
\]
\[
= \|\text{Proj}_T \left[ \mu h'(\mathbf{P}) - \mathbf{T}(\mathbf{T}^\ast \mathbf{T})^\ast[\ell_{\mathbf{P}}] \cdot \rho_2 \right] \|_{L^2(\mathbf{P})}
\]
where \( \text{Proj}_T : L^2(\mathbf{P}) \to T \) is the projection operator onto the closure of the tangent space. This operator is characterized in the following lemma.
Lemma A.2. For any \( f \in L^2(\mathcal{P}) \),

\[
(y, w, x) \mapsto \text{Proj}_T[f](y, w, x) = f - \text{Proj}_{L^2(\mathcal{P}, x)}[f](x) - \frac{\rho_2(y, w, \alpha(P))}{\gamma(1 - \gamma)} \cdot (I - T(T^*T)^+T^*)\text{Proj}_{L^2(\mathcal{P}, x)}[\rho_2 \cdot f](x).
\]

Proof. See Section SM.I. \( \square \)

Let \( M \equiv (I - T(T^*T)^+T^*)/(\tau(1 - \tau)) \) and \( \Gamma(x) = E[\rho_1(Y, W, \alpha(P))\rho_2(Y, W, \alpha(P)) \mid X = x]/(\tau(1 - \tau)) \), then we can write

\[
\text{Proj}_T[\mu h'(P)](y, w, x) = (\theta(P) - \mu(w)h'(P)(w)) - E[\theta(P) - \mu(W)h'(P)(W) \mid X = x] - \rho_2(y, w, \alpha(P))M\left[E[\mu(W)h'(P)(W)\rho_2(Y, W, \alpha(P)) \mid \cdot]\right](x) =\rho_1(y, w, \alpha(P)) - E[\rho_1(Y, W, \alpha(P)) \mid X = x] - \rho_2(y, w, \alpha(P))\Gamma(x)
\]

where the second line follows from the fact that \( E[\rho_2(Y, W, \alpha(P))\rho_1(Y, W, \alpha(P)) \mid X] \) equals \( E[\mu(W)h'(P)(W)\rho_1(Y, W, \alpha(P)) \mid X] \).

In addition,

\[
\text{Proj}_T[T(T^*T)^+[\ell_P] \cdot \rho_2](y, w, x) = T(T^*T)^+[\ell_P](x)\rho_2(y, w, \alpha(P)) - \rho_2(y, w, \alpha(P))M\left[T(T^*T)^+[\ell_P](\cdot)\rho_2^2 \mid \cdot\right](x) = T(T^*T)^+[\ell_P](x)\rho_2(y, w, \alpha(P)) - \rho_2(y, w, \alpha(P))MT(T^*T)^+[\ell_P](x)\tau(1 - \tau) + \rho_2(y, w, \alpha(P))T(T^*T)^+[\ell_P](x)
\]

where the last line follows from properties of the generalized inverse, namely, \((T^*T)^+T^*T(T^*T)^+ = (T^*T)^+\); see Engl et al. [1996] Proposition 2.3.
Therefore, for all \((y, w, x) \in \mathbb{Z}\),

\[
Proj_{\mathcal{T}} \left[ \mu h'(P) - \hat{h}_{\id}'(P)[\ell_P] \right](y, w, x) = \epsilon(y, w, x) - E[\epsilon(Y, W, x) \mid X = x] + \rho_2(y, w, \alpha(P))T(\mathbf{T}^\ast \mathbf{T}) \uparrow [\ell_P - \mathbf{T}^\ast \Gamma](x).
\]

where \((y, w, x) \mapsto \epsilon(y, w, x) \equiv \rho_1(y, w, \alpha(P)) - \rho_2(y, w, h(P))\Gamma(x)\). Note that

\[
E[\epsilon(Y, W, x) \mid X = x] = E[\rho_1(Y, W, \alpha(P)) \mid X = x].
\]

The proof concludes by showing that \(\epsilon(Y, W, X) - E[\epsilon(Y, W, X) \mid X]\) is orthogonal to

\[
\rho_2(Y, W, \alpha(P))T(\mathbf{T}^\ast \mathbf{T}) \uparrow [\ell_P - \mathbf{T}^\ast \Gamma](X).
\]

This follows because, conditional on \(X\), \(\epsilon(Y, W, X) - E[\epsilon(Y, W, X) \mid X]\) is orthogonal to \(\rho_2(Y, W, \alpha(P))\) by construction. □

### B Appendix for Section 4

This Appendix contains the proofs of all the Lemmas presented in Section 4.

**Proof of Lemma 4.1.** Note that for any \(\alpha \in \mathcal{A}, \hat{\Lambda}_J(\alpha) \ni 0\) wpa1, hence \(\sup_{\lambda \in \hat{\Lambda}_J(\alpha)} \hat{S}_J(\alpha, \lambda) \geq 0\) wpa1; in particular this applies to \(\alpha = \hat{\alpha}_{L,n}\). Therefore, for any \(\alpha \in \mathcal{A}_K\),

\[
\gamma K Pen(\hat{\alpha}_{L,n}) \leq \sup_{\lambda \in \hat{\Lambda}_J(\hat{\alpha}_{L,n})} \hat{S}_J(\hat{\alpha}_{L,n}, \lambda) + \gamma K Pen(\hat{\alpha}_{L,n}) \leq \sup_{\lambda \in \hat{\Lambda}_J(\alpha)} \hat{S}_J(\alpha, \lambda) + \gamma K Pen(\alpha),
\]

wpa1, where the second inequality is due to the definition of the minimizer \(\hat{\alpha}_{L,n} \in \mathcal{A}_K\). □

### C Proof for Theorem 5.1

Consider an \(L\) that satisfies the Assumptions of the theorem. By the triangle inequality, it suffices to show that for any \(\epsilon > 0\), there exists constants \(M_1, M > 0\) and \(N \in \mathbb{N}\) such that

\[
P \left( \|\hat{\alpha}_{L,n} - \alpha_{L,0}\| \geq M \varpi^{-1}_{L,n}(M_1 \delta_{L,n}) \right) \leq \epsilon
\]

for all \(n \geq N\), where \(\delta_{L,n} \equiv \delta_{1,n} + \delta_{2,n}\). Henceforth, let \(A_n(M_1, M) \equiv \{ \|\hat{\alpha}_{L,n} - \alpha_{L,0}\| \geq M \varpi^{-1}_{L,n}(M_1 \delta_{L,n}) \}\).
From the proof of Lemma 5.1, $\gamma_k \alpha_n = O_P(\Gamma_k)$ and the fact that $Q_J(\alpha, P_n) \geq 0$, imply that there exists an $M_0$ and an $N_0$ such that

$$P(Q_J(\hat{\alpha}_{L,n}, P_n) - Q_J(\alpha_{L,0}, P_n) \geq M_0 \delta_{2,L,n}) \leq \epsilon$$

for all $n \geq N_0$. This result and Lemma 5.6 in turn imply that

$$P(Q_J(\hat{\alpha}_{L,n}, P) - Q_J(\alpha_{L,0}, P) \geq 2M_0 \delta_{1,L,n} + \delta_{2,L,n}) \leq \epsilon$$

for all $n \geq N_0$. Let $B_n \equiv \{Q_J(\hat{\alpha}_{L,n}, P) - Q_J(\alpha_{L,0}, P) \leq 2M_0 \delta_{1,L,n} + \delta_{2,L,n}\}$.

The previous display implies that for any $n \geq N_0$, $P(A_n(M_1, M)) \leq P(A_n(M_1, M) \cap B_n) + \epsilon$.

By definition of $\omega_{L,n}$, for any history of data $(Z_i)_i$ in $A_n(M_1, M) \cap B_n$ it follows that

$$\omega_{L,n}(M_{\omega_{L,n}^{-1}}(M_1 \delta_{L,n})) \leq 2M_0 \delta_{1,L,n} + \delta_{2,L,n} \Leftrightarrow M_{\omega_{L,n}^{-1}}(M_1 \delta_{L,n}) \leq \omega_{L,n}^{-1}(2M_0 \delta_{1,L,n} + \delta_{2,L,n})$$

where the equivalence follows from the fact that $t \mapsto \omega_{L,n}(t)$ is non-decreasing (see Lemma 5.4). By setting $M_1 = 2M_0$ and $M > 1$ this display implies that $P(A_n(2M_0, M) \cap B_n) = 0$ thus proving the desired result. □

D Appendix for Section 5

This Appendix contains the proofs of all the Lemmas presented in Section 5.

The following lemmas are used to prove the Lemmas in Section 5; the proofs are relegated to the Supplementary Material SM.II.

**Lemma D.1.** Let Assumption 3 hold. Suppose $(\alpha, P) \in \Lambda \times \mathcal{P}(\mathcal{Z})$ and $\delta > 0$ are such that: There exists finite $C > 0$ such that (1) $\sup_{\lambda \in \Lambda, \delta} \sup_{z \in \supp(P)} s''(\lambda^T g_J(z, \alpha)) \leq -\sqrt{C}$, (2) $e_{\min}(H_1(\alpha, P)) \geq \sqrt{C}$, (3) $2C^{-1} ||E_P[g_J(Z, \alpha)]||_e < \delta$, and (4) the hypothesis of Lemma SM.II.3 are satisfied for $\epsilon > 0$.

Then, with probability higher than $1 - \epsilon$,

1. $\arg\max_{\lambda \in \Lambda_J(\alpha, P)} S_J(\alpha, \lambda, P) = \arg\max_{\lambda \in B(\delta)} S_J(\alpha, \lambda, P) = \{\lambda_J(\alpha, P)\}$, $\frac{dS_J(\alpha, \lambda_J(\alpha, P), P)}{d\lambda} = 0.$
2. $\sup_{\lambda \in \Lambda(\alpha, P)} S(\alpha, \lambda, P) \leq 2C^{-1} ||E_P[g_J(Z, \alpha)]||_e^2.$
3. $||\lambda_J(\alpha, P)||_e \leq 2C^{-1} ||E_P[g_J(Z, \alpha)]||_e.$
Lemma D.2. Let Assumption 2 hold. Then, for any (non-random) $\alpha \in \mathcal{A}$ and any $J \in \mathbb{N}$,
\[
\|E_P[g_J(Z, \alpha)]\|_e = O_P \left( \sqrt{\bar{b} + \|h'\|_{L^2(P)}^2 + \frac{b^2_{2,J}}{n}} + \|E_P[g_J(Z, \alpha)]\|_e^2 \right).
\]
(the constant implicit in the $O_P$ does not depend on $J$).

Lemma D.3. Suppose Assumptions 2 and 3 hold. Then for any $\alpha$ in a $\|\cdot\|$-neighborhood of $\alpha_0$,
1. $\|H_J(\alpha, P_n) - H_J(\alpha, P)\|_e = O_P(\{\theta^2 + \|h'\|_{L^\infty(\mathcal{W}, \mu)}^2\} \sqrt{b^4_{4,J}/n})$.\(^7\)
2. If $\{\theta^2 + \|h'\|_{L^\infty(\mathcal{W}, \mu)}^2\} \sqrt{b^4_{4,J}/n} = o(1)$, there exists a $C < \infty$ such that wpa1,
\[
1/C \leq e_{\min}(H_J(\alpha, P)) \leq e_{\max}(H_J(\alpha, P)) \leq C
\]
for $P \in \{P_n, P\}$.

Lemma D.4. Suppose Assumption 2 and 3 hold. For any $(L = (J, K), n)$, and any positive real-valued sequence $(\delta_n)_n$ satisfying Assumption 6(i)(iii)(iv)(v) and $\delta_n = o(1)$, it follows that
\[
\|n^{-1} \sum_{i=1}^n g_J(Z_i, \hat{\alpha}_{L,n})\|_e \lesssim C_{L,n} \{\delta_n + \delta^{-1}_n \{\|E_P[g_J(Z, \alpha_{L,0})]\|_e^2 + \gamma_K \text{Pen}(\alpha_{L,0})\} \}.
\]

Lemma D.5. Suppose Assumption 3(iii). Then for $P \in \{P_n, P\}$,
\[
\sup_{\alpha \in A_{L,n}} \|E_P[g_J(Z, \alpha)]\|_e \lesssim \bar{b} + l_n \gamma_K^{-1} \Gamma_{L,n} + E_P[\|q^J(X)\|_e]
\]
wpa1.

Proof of Lemma 5.1. By Lemma 4.1 and the fact that $\Pi_K \alpha_0 \in \mathcal{A}_K$, we have
\[
\text{Pen}(\hat{\alpha}_{L,n}) \leq \gamma_K^{-1} \sup_{\lambda \in A_{J}(\Pi_K \alpha_0, P_n)} S(\Pi_K \alpha_0, \lambda, P_n) + \text{Pen}(\Pi_K \alpha_0).
\]

By Lemma D.1 applied to $(\Pi_K \alpha_0, P_n)$ (by Lemma SM.II.4 in the Supplementary Material SM.II), the conditions of the Lemma D.1 hold wpa1 it follows that
\[
\text{Pen}(\hat{\alpha}_{L,n}) \lesssim \gamma_K^{-1} \|E_P[n g_J(Z, \Pi_K \alpha_0)]\|_e^2 + \text{Pen}(\Pi_K \alpha_0) \quad \text{wpa1}.
\]

By Lemma D.2, wpa1, $\|E_P[n g_J(Z, \Pi_K \alpha_0)]\|_e^2 \leq l_n (\hat{g}_{L,0}/n + \|E_P[g_J(Z, \Pi_K \alpha_0)]\|_e^2)$, and thus the result follows. \(\square\)

\(^7\)For matrices, $\|\cdot\|_e$ is the operator norm induced by the Euclidean norm.
Proof of Lemma 5.2. Throughout, fix $L = (J, K)$. We show that the “argmin” is non-empty by invoking the Weierstrass Theorem (see Zeidler [1985]). For this, note that $\alpha \mapsto Q_J(\alpha, P)$ is a continuous transformation of

$$\alpha \mapsto E[g_J(Z, \alpha)]^T = (\theta + E[\ell(W)h(W)], E[E[(F_Y|W,X)(h(W) \mid W, X) - \tau)|X]q^j(X)^T])$$

(here $F_Y|W,X$ is the conditional cdf of $Y$ given $W, X$ associated to $P$). Since $\ell \in L^2(Leb)$ and $p_W$ is bounded (see Assumption 2), then $h \mapsto E[\ell(W)h(W)]$ is continuous with respect to $\|\cdot\|_{L^2(Leb)}$. Also under Assumption 2,

$$|E[F_Y|W,X(h_1(W) \mid W, x) - F_Y|W,X(h_2(W) \mid W, x)|x]| \lesssim \int |h_1(w) - h_2(w)|p_W(w)dw$$

$$\lesssim \inf_{x} 1/p_X(x) \int |h_1(w) - h_2(w)|p_W(w)dw$$

so $h \mapsto E[(F_Y|W,X(h(W) \mid W, X) - \tau)|X]q^j(X)^T]$ is also continuous with respect to $\|\cdot\|_{L^2(Leb)}$. Under assumption 3, $\alpha \mapsto Q_J(\alpha, P) + \gamma K \text{Pen}(\alpha)$ is lower semi-compact, and $\mathcal{A}_K = \Theta \times \mathcal{H}_K$ is a finite dimensional and closed set. So all the assumptions of the Weierstrass Theorem hold. \hfill \Box

Proof of Lemma 5.3. In the proof we simply use $L$ instead of $L_n$. We first present some intermediate results.

From the definition of $\text{Proj}_J$ it follows that

$$E_P[g_J(Z, \alpha)] = E_P[M_J(X)\rho(Y, W, \alpha)] = (E_P[\rho_1(Y, W, \alpha)], E_P[\rho_2(Y, W, \alpha)q^j(X)^T])^T.$$  

This in turn implies that

$$(E_P[g_J(Z, \alpha)])^T M_J(X) = (E_P[\rho_1(Y, W, \alpha)], E_P[\rho_2(Y, W, \alpha)q^j(X)^T])M_J(X) = (E_P[\rho_1(Y, W, \alpha)], \text{Proj}_J[m_2(\cdot, \alpha)](X)).$$

Since $E_P[M_J(X)M_J(X)^T] = I_{J+1}$ under Assumption 3 the previous result implies that

$$(E_P[\rho_1(Y, W, \alpha)])^2 + \|\text{Proj}_J[m_2(\cdot, \alpha)]\|_{L^2(P)}^2 = E_P[(E_P[g_J(Z, \alpha)])^T M_J(X)M_J(X)^T (E_P[g_J(Z, \alpha)])] = \|E_P[g_J(Z, \alpha)]\|_c^2.$$  

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This observation and the proof of Lemma D.3 (applied to $\alpha_0$) implies that
\[
Q_J(\alpha, P) \geq c^{-1} E_P[g_J(Z, \alpha)^T] (E_P[M_J(X)M_J(X)^T])^{-1} E_P[g_J(Z, \alpha)]
\]
\[
= c^{-1} ||E_P[g_J(Z, \alpha)]||_c^2
\]
\[
= c^{-1} \left\{ (E_P[\rho_1(Y, W, \alpha)])^2 + ||\text{Proj}_J[m_2(:, \alpha)]||_{L^2(P)}^2 \right\}
\]
and
\[
Q_J(\alpha, P) \leq c \left\{ (E_P[\rho_1(Y, W, \alpha)])^2 + ||\text{Proj}_J[m_2(:, \alpha)]||_{L^2(P)}^2 \right\}
\]
for some $c > 1$.

By the conditions in the lemma, uniformly over $\alpha \in \bar{A}_{L,n}$,
\[
\liminf_{J \to \infty} Q_J(\alpha, P) \geq c^{-1} \left\{ (E_P[\rho_1(Y, W, \alpha)])^2 + ||m_2(:, \alpha)||_{L^2(P)}^2 \right\}
\]
and
\[
\limsup_{J \to \infty} Q_J(\alpha, P) \leq c \left\{ (E_P[\rho_1(Y, W, \alpha)])^2 + ||m_2(:, \alpha)||_{L^2(P)}^2 \right\}.
\]

We now turn to the proof of the claim in the Lemma. We show this by contradiction. That is, suppose that there exists a $c > 0$ such that $||\alpha_{L,0} - \Pi_K \alpha_0|| \geq c$ i.o. Therefore, for any $L$ for which the expression holds, it follows that
\[
\inf_{\alpha \in \bar{A}_{L,n} : ||\alpha - \Pi_K \alpha_0|| \geq c} Q_J(\alpha, P) \leq Q_J(\Pi_K \alpha_0, P).
\]

Under Assumption 3, $\{\alpha \in \bar{A}_{L,n} : ||\alpha - \Pi_K \alpha_0|| \geq c\}$ is compact under $||.||$ and by the proof of Lemma 5.2, $\alpha \mapsto Q_J(\alpha, P)$ is continuous under $||.||$. Thus, there exists a $\alpha_L \in \bar{A}_{L,n}$ such that (i) $||\alpha_L - \Pi_K \alpha_0|| \geq c$ and (ii) $Q_J(\alpha_L, P) \leq Q_J(\Pi_K \alpha_0, P)$.

By the first part of the proof and our conditions, $Q_J(\Pi_K \alpha_0, P) = o(1)$ so this implies that $Q_J(\alpha_L, P) = o(1)$. By the first part of the proof, this result implies that
\[
(E_P[\rho_1(Y, W, \alpha_L)])^2 + ||m_2(:, \alpha_L)||_{L^2(P)}^2 = o(1).
\]

Since, under our conditions, $U_{L,n} \leq K < \infty$, it follows that the sequence $(\alpha_L)_L$ belongs to $\{\alpha : \text{Pen}(\alpha) \leq K\}$ which is compact under $||.||$ by Assumption 3. Thus, there exists a convergent subsequence with limit $\alpha^*$. On the one hand, $||\alpha_L - \Pi_K \alpha_0|| \geq c$, it follows that $\alpha^* \neq \alpha_0$. But, on
the other hand, continuity of \( \alpha \mapsto (E_P[\rho_1(Y,W,\alpha)])^2 + \|m_2(\cdot,\alpha)\|_{L^2(P)}^2 \) and the previous display imply that
\[
(E_P[\rho_1(Y,W,\alpha^*)])^2 + \|m_2(\cdot,\alpha^*)\|_{L^2(P)}^2 = 0
\]
but this contradicts the identification condition in Assumption 1.

**Proof of Lemma 5.4.** The proof of Lemma 5.2 implies that \( \alpha \mapsto Q_J(\alpha,P) \) is continuous and that \( t \mapsto \{\alpha \in \bar{A}_{L,n}: \|\alpha - \alpha_{L,0}\| \geq t\} \) is a compact-valued correspondence which is also continuous. Thus by the Theorem of the Maximum, \( \varpi_{L,n} \) is continuous. This also implies that the “inf” is achieved; this and the definition of \( \alpha_{L,0} \) and assumption 5 imply that \( \varpi_{L,n}(t) = 0 \) iff \( t = 0 \). The fact that it is non-decreasing is trivial to show.

**Proof of Lemma 5.5.** Observe that
\[
Q_J(\hat{\alpha}_{L,n}, P_n) \leq \frac{1}{e_{\min}(H_J(\alpha_0, P))} \|E_P^n[g_J(Z, \hat{\alpha}_{L,n})]\|_e^2.
\]
By Lemma D.3(2), it follows that
\[
Q_J(\hat{\alpha}_{L,n}, P_n) \leq C^{-1} \|E_P^n[g_J(Z, \hat{\alpha}_{L,n})]\|_e^2.
\]
By Lemma D.4
\[
Q_J(\hat{\alpha}_{L,n}, P_n) \asymp C_{L,n} \{\delta_n + \delta_n^{-1} \{\|E_P^n[g_J(Z, \alpha_{L,0})]\|_e^2 + \gamma K P\text{en}(\alpha_{L,0})\}\}.
\]
By Lemma D.2 and definition of \( \delta_{2,L,n} \),
\[
Q_J(\hat{\alpha}_{L,n}, P_n) = O_P(\delta_{2,L,n}).
\]

**Proof of Lemma 5.6.** Note that
\[
|Q_J(\alpha, P_n) - Q_J(\alpha, P)| \leq \|H_J(\alpha_0, P)^{-1/2}(E_P^n[g_J(Z, \alpha)] - E_P[g_J(Z, \alpha)])\|_e \times \|H_J(\alpha_0, P)^{-1/2}(E_P^n[g_J(Z, \alpha)] + E_P[g_J(Z, \alpha)])\|_e
\]
This bound and Lemma D.3(2) (applied to $\alpha_0$) imply that it suffices to show that

$$\sup_{\alpha \in \bar{A}_{L,n}} \text{Term}_{2, J}(\alpha, P_n) = O_p \left( \sqrt{J n} \Delta_{L,n} \right),$$

and $$\sup_{\alpha \in \bar{A}_{L,n}} \text{Term}_{1, J}(\alpha, P_n) = O_p \left( \sqrt{J n} \Delta_{L,n} \right)$$

where

$$\text{Term}_{1, J}(\alpha, P_n) \equiv ||E_{P_n}[g_J(Z, \alpha)] - E_P[g_J(Z, \alpha)]||_e$$

$$\text{Term}_{2, J}(\alpha, P_n) \equiv ||E_{P_n}[g_J(Z, \alpha)] + E_P[g_J(Z, \alpha)]||_e.$$

By Lemma D.5,

$$\sup_{\alpha \in \bar{A}_{L,n}} ||E_{P_n}[g_J(Z, \alpha)]||_e \lesssim \sqrt{J n} + l_n \gamma_K^{-1} \Gamma_{L,n} + E_{P_n}[||q^J(X)||_e]$$

wpa; and similarly for $\sup_{\alpha \in \bar{A}_{L,n}} ||E_P[g_J(Z, \alpha)]||_e$. Hence,

$$\sup_{\alpha \in \bar{A}_{L,n}} \text{Term}_{2, J}(\alpha, P_n) = O_p \left( \sqrt{J n} \Delta_{L,n} \right).$$

Regarding $\text{Term}_{1, J}(\alpha, P_n)$, by definition of $g_J$ and simple algebra,

$$\sup_{\alpha \in \bar{A}_{L,n}} \text{Term}_{1, J}(\alpha, P_n) \leq n^{-1/2} \sup_{(\theta, h) \in \bar{A}_{L,n}} ||G_n[\mu \cdot h']||$$

$$+ \sup_{\alpha \in \bar{A}_{L,n}} ||n^{-1} \sum_{i=1}^n \rho_2(Y_i, W_i, \alpha) q^J(X_i) - E_P[\rho_2(Y, W, \alpha) q^J(X)]||_e.$$ 

By Assumption 7, the first term in the RHS is $O_p(n^{-1/2} \Delta_{L,n})$; the second term in the RHS can be bounded above by $\sqrt{J n} \max_{1 \leq j \leq J} \sup_{g \in G_K} ||G_n[g \cdot q_j]||$, which by Assumption 7 is $O_p \left( \sqrt{J n} \Delta_{L,n} \right)$.

\[ \square \]

### E Proof of Theorem 6.1

By Assumption 11(i) and Lemma F.8 in Appendix F, $n^{-1/2} \sum_{i=1}^n (G(\alpha_{L,0})[u_{L,n}^*])^T H_{L,n}^{-1} g_J(Z_i, \alpha_{L,0}) \Rightarrow N(0, 1)$. This fact, Lemma 6.2 and Assumption 11(ii), imply the desired result. \[ \square \]
F Appendix for Section 6

For any \( L = (J, K) \in \mathbb{N}^2 \), let \( M_L \in \mathbb{R}^{(1+J) \times (1+K)} \) be defined as

\[
M_L = \begin{bmatrix}
1 & E[\ell(W)\varphi^K(W)^T] \\
0 & E[p_Y|W, X]q^J(X)\varphi^K(W)^T
\end{bmatrix}
\]

where \( 0 \) is a \( J \times 1 \) vector of zeros. The following eight lemmas are proved in the Section SM.III in the Supplemental Material.

**Lemma F.1.** Let Assumptions 1-5 and 12(iv)(v) hold. Then, there exists a \( c > 0 \) such that for any \( L = (J, K) \in \mathbb{N}^2 \) and any \( \alpha = (\theta, \pi^T \varphi^K) \) some \( \pi \in \mathbb{R}^K \) (i.e., \( \alpha \in \text{lin}\{A_K\} \)), it follows that

\[
||\alpha||_e \geq c \times \sqrt{e_{\min}(M_L^T M_L)}||\theta, \pi||_e
\]

where \( e_{\min}(M_L^T M_L) > 0 \).

**Lemma F.2.** Let Assumptions 1-5, 10 and 12(v) hold. Then

\[
\sup_{\alpha \in \mathcal{N}_{L,n}} ||H_J(\alpha, P_n) - H_J(\alpha_{L0}, P)||_e = O_p(\Xi_{1,L,n})
\]

with

\[
\Xi_{1,L,n} = O_p \left( \left( \bar{\gamma} + ||h_{L0}'||_{L^\infty(\mathcal{H}, \mu)} \right)^2 \sqrt{b_{4,J}/n + \mathcal{U}_{L,n} \eta_{L,n} + \Xi_{L,n}} \right)
\]

**Lemma F.3.** For any \( L = (J, K) \in \mathbb{N}^2 \),

\[
\sup_{\alpha \in \mathcal{N}_{L,n}} \left| n^{-1} \sum_{i=1}^n g_J(Z_i, \alpha) - g_J(Z_i, \alpha_{L0}) - E[g_J(Z, \alpha) - g_J(Z, \alpha_{L0})] \right|_e 
\leq \sqrt{\frac{J}{n}} \left( \sup_{(\theta, h) \in \mathcal{N}_{L,n}} \mathbb{G}_n[\mu \cdot h'] + \max_{1 \leq j \leq J} \sup_{h \in \mathcal{G}_{L,n}} \mathbb{G}_n[g \cdot q_j] \right).
\]

**Lemma F.4.** Let Assumptions 1-5, 8(ii)(iii) and 12(iv)(v) hold. If \( \alpha \in \text{int}(\mathcal{N}_{L,n}) \), then \( \alpha + tu_{L,n}^* \in \mathcal{N}_{L,n} \) for all \( |t| \leq l_n n^{-1/2} \).

**Lemma F.5.** Let Assumptions 1-5, 8, 9 and 12 hold. Then all the conditions of Lemma D.1 hold for \((\hat{\alpha}_{L,n}, P_n)\) and \((\hat{\alpha}_{L,n}', P_n)\) \(\omega p 1\).
Lemma F.6. Let Assumptions 1-5 and 12(v) hold. For any $L = (J, K) \in \mathbb{N}^2$, any $\gamma > 0$ and any $\alpha \in \tilde{A}_{L,n}$ such that $||\alpha - \alpha_{L,0}||_w \leq \gamma$, it follows that

$$||H_L^{-1} \Delta(\alpha)||_e = O_P \left( \sqrt{\frac{\bar{g}_{L,0}^2}{n}} + ||E[g_J(Z, \alpha_{L,0})]||_e^2 + \gamma \right).$$

Lemma F.7. Let Assumptions 1-5, and 12 hold. Then

$$\langle u_{L,n}^*, \hat{\alpha}_{L,n} - \alpha_{L,0} \rangle_w = n^{-1} \sum_{i=1}^{n} (G(\alpha_{L,0})[u_{L,n}^*])^T H_L^{-1} g_J(Z_i, \alpha_{L,0}) + o_P(n^{-1/2}).$$

Lemma F.8. Let Assumptions 1, 5 and 12 hold. Then,

$$n^{-1/2} \sum_{i=1}^{n} (G(\alpha_{L,0})[u_{L,n}^*])^T H_L^{-1} \{g_J(Z_i, \alpha_{L,0}) - E[g_J(Z, \alpha_{L,0})]\} \Rightarrow N(0,1).$$

Proof of Lemma 6.1. By the calculations in the proof of Lemma D.1, for all $(\alpha, \lambda) \in \mathcal{N}_{L,n} \times B(\delta_n)$,

$$\hat{S}_J(\alpha, \lambda) \geq -\lambda^T E_P_n[g_J(Z, \alpha)] - \frac{1}{2} \lambda^T H_J(\alpha, P_n)\lambda + O(\delta^3 E_P_n[||g_J(Z, \alpha)||_e^3]);$$

the reverse inequality also can be shown in similar fashion.

By Lemma F.2, $\sup_{\alpha \in \mathcal{N}_{L,n}} ||H_J(\alpha, P_n) - H_J(\alpha_{L,0}, P)\|_e = O_P(\Xi_{1,L,n})$. Hence

$$\hat{S}_J(\alpha, \lambda) = -\lambda^T E_P_n[g_J(Z, \alpha)] - \frac{1}{2} \lambda^T H_J(\alpha_{L,0}, P)\lambda + O(\delta^3 E_P_n[||g_J(Z, \alpha)||_e^3] + \delta^2 \Xi_{1,L,n}).$$

By Lemma F.3 and Assumption 9, it follows that

$$\sup_{\alpha \in \mathcal{N}_{L,n}} \left\| n^{-1} \sum_{i=1}^{n} g_J(Z_i, \alpha) - g_J(Z_i, \alpha_{L,0}) - E[g_J(Z, \alpha) - g_J(Z, \alpha_{L,0})] \right\|_e = O_P \left( \sqrt{\frac{J}{n}} \Delta_{2,L,n} \right).$$

Hence

$$\hat{S}_J(\alpha, \lambda) = -\lambda^T \{E_P_n[g_J(Z, \alpha_{L,0})] + E[g_J(Z, \alpha) - g_J(Z, \alpha_{L,0})]\} - \frac{1}{2} \lambda^T H_J(\alpha_{L,0}, P)\lambda$$

$$+ O \left( \delta^3 E_P_n[||g_J(Z, \alpha)||_e^3] + \delta^2 \Xi_{1,L,n} + \delta \sqrt{\frac{J}{n}} \Delta_{2,L,n} \right).$$

By the mean value theorem,

$$E[g_J(Z, \alpha) - g_J(Z, \alpha_{L,0})] = G(\alpha_{L,0})[\alpha - \alpha_{L,0}]$$

$$+ \int_{0}^{1} \{G(\alpha_{L,0} + t(\alpha - \alpha_{L,0}))[\alpha - \alpha_{L,0}] - G(\alpha_{L,0})[\alpha - \alpha_{L,0}]\} dt. \quad (8)$$
Note that, for any $\alpha$ and $\alpha_1$,

$$G(\alpha)[\alpha_1] - G(\alpha_{L,0})[\alpha_1] = E [ \{ p_Y | W, X (h(W)|W, X) - p_Y | W, X (h_{L,0}(W)|W, X) \} h_1(W)q^J(X)] .$$

By Assumption 2, $|d_p |x|p_{y(x)}| \leq C$ some finite $C$ and thus, for each $j$,

$$|E [ \{ p_Y | W, X (h(W)|W, X) - p_Y | W, X (h_{L,0}(W)|W, X) \} h_1(W)q_j(X)] | \leq C E [ |h(W) - h_{L,0}(W)| \times |h_1(W)||X = x| \times E [ |q_j(X)|] .$$

Therefore,

$$|G(\alpha)[\alpha_1] - G(\alpha_{L,0})[\alpha_1]| \leq \sup_x E [ |h(W) - h_{L,0}(W)| \times |h_1(W)||X = x| \times E [ |q^J(X)|] .$$

Applying these observations to the last term in the RHS of expression (8), it follows that

$$|E [ g_J(Z, \alpha) - g_J(Z, \alpha_{L,0}) - G(\alpha_{L,0})[\alpha - \alpha_{L,0}]] | e = O (\sup_x E [ |h(W) - h_{L,0}(W)|^2 |X = x| \times E [ |q^J(X)|] )$$

where the last line follows by Assumption 2. Therefore,

$$\hat{S}_J(\alpha, \lambda) = - \lambda^T \{ E_{P_\alpha} [ g_J(Z, \alpha_{L,0})] + G(\alpha_{L,0})[\alpha - \alpha_{L,0}] \} - \frac{1}{2} \lambda^T H_J(\alpha_{L,0}, P) \lambda$$

$$+ O \left( \delta^3 E_{P_\alpha} [ |g_J(Z, \alpha)|^3] + \delta^2 \Xi_{1,L,n} + \delta \left( \sqrt{J} \Delta_{2,L,n} + ||h - h_{L,0}||_{L^2(\mathbb{L}b_{2,1})}^2 \right) \right) .$$

Since $\hat{\alpha}_{L,n} \in \hat{A}_{L,n}$ wpa 1 (Lemma 5.1), it follows by the proof of Lemma SM.II.3 that $|g_J(Z, \hat{\alpha}_{L,n})|^3 \leq (\overline{\sigma} + n^{-1})^2 \Gamma_{L,n} + ||q^J(X)||_e^3$. So by the Markov inequality

$$E_{P_\alpha} [ |g_J(Z, \hat{\alpha}_{L,n})|^3] = O_P \left( \overline{\sigma} + n^{-1} \Gamma_{L,n} + b_{3,J}^3 \right) .$$

Proof of Lemma 6.2. Step 1. We show that, for $\alpha \in \{ \hat{\alpha}_{L,n}, \hat{\alpha}_{L,n}' \}$,

$$\sup_{\lambda \in \hat{A}_{J}(\alpha)} \hat{S}_J(\alpha, \lambda) = \frac{1}{2} \Delta(\alpha)^T H_{L}^{-1}(\Delta) + O_P (n^{-1}) .$$

We first note that, for any $\alpha \in \{ \hat{\alpha}_{L,n}, \hat{\alpha}_{L,n}' \}$, by Lemma F.5 and Lemmas SM.II.3 $\{ B(\delta_n) \}$
\(\hat{\Lambda}_J(\alpha)\) wpa1. Also, by Assumption 8, under the null, \(\{\hat{\alpha}_{L,n}, \hat{\alpha}_{L,n}'\} \subseteq \mathcal{N}_n\). Hence, under Assumption 12, Lemma 6.1 implies that

\[
\sup_{\lambda \in \Lambda_J(\alpha)} \hat{S}_J(\alpha, \lambda) \leq \sup_{\lambda \in B(\delta_n)} -\lambda^T \Delta(\alpha) - 0.5\lambda^T H^{-1}_L \lambda + o_P(n^{-1})
\]

\[
\leq \frac{1}{2} \Delta(\alpha)^T H^{-1}_L \Delta(\alpha) + o_P(n^{-1})
\]

where the first inequality is valid, because \(\sup_{\lambda \in \Lambda_J(\alpha)} \hat{S}_J(\alpha, \lambda) = \sup_{\lambda \in B(\delta_n)} \hat{S}_J(\alpha, \lambda)\) by Lemmas F.5 and Lemma D.1; the last inequality follows because the RHS is obtained by maximizing over the whole \(\mathbb{R}^{J+1}\) not only \(\hat{\Lambda}_J(\alpha)\).

By Lemma F.5 and Lemma D.1,

\[
\sup_{\lambda \in \Lambda_J(\alpha)} \hat{S}_J(\alpha, \lambda) \geq -\lambda^T \Delta(\alpha) - 0.5\lambda^T H^{-1}_L \lambda + o_P(n^{-1}),
\]

for all \(\lambda \in B(\delta_n)\). By Lemma F.6 and Assumption 8, the maximizer of the RHS, \(\lambda^*\), is such that \(||\lambda^*||_e = O_P \left(\sqrt{\frac{g_{L,n}}{n}} + ||E[g_J(Z, \alpha_{L,0})||_e^2 + \eta_{w,L,n}\right)\). Hence, by Assumption 12, \(\lambda^* \in B(\delta_n)\).

Therefore,

\[
\sup_{\lambda \in \hat{\Lambda}_J(\alpha)} \hat{S}_J(\alpha, \lambda) \geq \frac{1}{2} \Delta(\alpha)^T H^{-1}_L \Delta(\alpha) + o_P(n^{-1}).
\]

**Step 2.** We now show that

\[
\hat{\mathcal{L}}_{L,n}(\nu) \geq o_P(n^{-1}) + \left(n^{-1} \sum_{i=1}^{n} (G(\alpha_{L,0})[u_{L,n}^*])^T H^{-1}_L g_J(Z_i, \alpha_{L,0})\right)^2
\]

\[
+ 2(n^{-1} \sum_{i=1}^{n} (G(\alpha_{L,0})[u_{L,n}^*])^T H^{-1}_L g_J(Z_i, \alpha_{L,0}) \right) .
\]

Using the results in step 1,

\[
\hat{\mathcal{L}}_{L,n}(\nu) \geq \{ \Delta(\hat{\alpha}_{L,n}^\nu)^T H^{-1}_L \Delta(\hat{\alpha}_{L,n}^\nu) - \Delta(\alpha)^T H^{-1}_L \Delta(\alpha) \} + \gamma_K \{\text{Pen}(\hat{\alpha}_{L,n}^\nu) - \text{Pen}(\alpha)\}
\]

\[
+ o_P(n^{-1}).
\]

For any \(\alpha \in A_K\). In particular \(\alpha = \hat{\alpha}_{L,n}^\nu - tu_{L,n}^*\) with \(t = (G(\alpha_{L,0})[u_{L,n}^*])^T H^{-1}_L E_P[g_J(Z, \alpha_{L,0})]\).

By Lemma F.8 and Assumption 11, \(|t| = O_P(n^{-1/2})\), so by Lemma F.4, under Assumption 8, this choice of \(\alpha\) belongs to \(\mathcal{N}_{L,n}\). Moreover, by Assumption 8, \(\gamma_K \{\text{Pen}(\hat{\alpha}_{L,n}^\nu) - \text{Pen}(\alpha)\} = o_P(n^{-1}).\)
Hence, after some simple calculations,

$$\hat{L}_{L,n}(\nu) \geq o(n^{-1}) - \left\{ t^2\|u_{L,n}^*\|_w^2 - 2t(G(\alpha_{L,0})[u_{L,n}^*])^TH_{L}^{-1}\Delta(\hat{\alpha}_{L,n}) \right\}. $$

Note that $\Delta(\hat{\alpha}_{L,n}) = E_{P_n}[g_J(Z, \alpha_{L,0})] + G(\alpha_{L,0})[\hat{\alpha}_{L,n}^\nu - \alpha_{L,0}]$, so

$$2(G(\alpha_{L,0})[u_{L,n}^*])^TH_{L}^{-1}\Delta(\hat{\alpha}_{L,n}) = 2(G(\alpha_{L,0})[u_{L,n}^*])^TH_{L}^{-1}E_{P_n}[g_J(Z, \alpha_{L,0})]$$

$$+ 2(G(\alpha_{L,0})[u_{L,n}^*])^TH_{L}^{-1}(G(\alpha_{L,0})[\hat{\alpha}_{L,n}^\nu - \alpha_{L,0}])$$

$$= 2(G(\alpha_{L,0})[u_{L,n}^*])^TH_{L}^{-1}E_{P_n}[g_J(Z, \alpha_{L,0})]$$

$$+ 2(\nu - \theta_{L,0})/\|v_{L,n}^*\|_w,$$

where the last line follows because $(G(\alpha_{L,0})[u_{L,n}^*])^TH_{L}^{-1}(G(\alpha_{L,0})[\hat{\alpha}_{L,n}^\nu - \alpha_{L,0}]) = \langle u_{L,n}^*, \hat{\alpha}_{L,n}^\nu - \alpha_{L,0} \rangle_w = (\bar{\theta}_{L,0} - \theta_{L,0})/\|v_{L,n}^*\|_w$.

This observation and the fact that $\|u_{L,n}^*\|_w = 1$, imply

$$\hat{L}_{L,n}(\nu) \geq o(n^{-1}) - \left\{ t^2 - 2t(G(\alpha_{L,0})[u_{L,n}^*])^TH_{L}^{-1}E_{P_n}[g_J(Z, \alpha_{L,0})] + 2t(\bar{\theta}_{L,0} - \nu) \right\}/\|v_{L,n}^*\|_w.$$ 

By our choice of $t$, it follows that

$$\hat{L}_{L,n}(\nu) \geq o(n^{-1}) + \left\{ n^{-1}\sum_{i=1}^{n}(G(\alpha_{L,0})[u_{L,n}^*])^TH_{L}^{-1}g_J(Z_i, \alpha_{L,0}) \right\}^2$$

$$+ 2(\nu - \theta_{L,0}) /\|v_{L,n}^*\|_w \left\{ n^{-1}\sum_{i=1}^{n}(G(\alpha_{L,0})[u_{L,n}^*])^TH_{L}^{-1}g_J(Z_i, \alpha_{L,0}) \right\}.$$

**Step 3.** We now show that

$$\hat{L}_{L,n}(\nu) \leq \left( \frac{\theta_{L,0} - \bar{\theta}_0}{\|v_{L,n}^*\|_w} - \left(n^{-1}\sum_{i=1}^{n}(G(\alpha_{L,0})[u_{L,n}^*])^TH_{L}^{-1}g_J(Z_i, \alpha_{L,0}) \right) \right)^2 + o(n^{-1}).$$

To do this we proceed as in Step 2. Using the results in step 1,

$$\hat{L}_{L,n}(\nu) \leq \left\{ \Delta(\alpha)^TH_{L}^{-1}\Delta(\alpha) - \Delta(\hat{\alpha}_{L,n})^TH_{L}^{-1}\Delta(\hat{\alpha}_{L,n}) \right\}$$

$$+ \gamma_K\{Pen(\alpha) - Pen(\hat{\alpha}_{L,n})\} + o(n^{-1}),$$

for any $\alpha \in \mathcal{A}_K$ such that $\theta = \nu$. Let $\alpha = \hat{\alpha}_{L,n} - tu_{L,n}^*$, where $t = \langle u_{L,n}^*, \hat{\alpha}_{L,n} - \alpha_{L,0} \rangle_w + \frac{\theta_{L,0} - \bar{\theta}_0}{\|v_{L,n}^*\|_w}$. Note that

$$\langle v_{L,n}^*, \alpha \rangle_w = \bar{\theta}_{L,n} - t\|v_{L,n}^*\|_w = \bar{\theta}_{L,n} - (\bar{\theta}_{L,n} - \theta_0) = \theta_0.$$
and under the null, $\theta_0 = \nu$. Thus $\alpha$ satisfies with the restriction $\nu = \theta$. Moreover, by Lemma F.7 and Assumption 11, it follows that $t = O_P(n^{-1/2})$. Thus, by Lemma F.4 — under Assumption 8 and the null —, $\alpha \in \mathcal{N}_{L,n}$ wp1.

This observation, Assumption 8, and analogous calculations to those in Step 1 imply

$$\hat{L}_{L,n}(\nu) \leq \left\{ t^2 - 2t(G(\alpha_{L,0})[u_{L,n}^*])^T H_L^{-1} \Delta(\hat{\alpha}_{L,n}) \right\} + o_P(n^{-1}),$$

and

$$(G(\alpha_{L,0})[u_{L,n}^*])^T H_L^{-1} \Delta(\hat{\alpha}_{L,n}) = \langle u_{L,n}^*, \hat{\alpha}_{L,n} - \alpha_{L,0} \rangle_w + n^{-1} \sum_{i=1}^n (G(\alpha_{L,0})[u_{L,n}^*])^T H_L^{-1} g_i(Z_i, \alpha_{L,0})$$

$$\equiv \frac{\hat{\theta}_{L,n} - \theta_0}{\|v_{L,n}^*\|_w} - \frac{\theta_{L,0} - \theta_0}{\|v_{L,n}^*\|_w} + F_n$$

$$= t - \frac{\theta_{L,0} - \theta_0}{\|v_{L,n}^*\|_w} + F_n.$$

Therefore,

$$\hat{L}_{L,n}(\nu) \leq \left\{ t^2 - 2t\left( \frac{\theta_{L,0} - \theta_0}{\|v_{L,n}^*\|_w} + F_n \right) \right\} + o_P(n^{-1})$$

$$= \left\{ -t^2 + 2t\left( \frac{\theta_{L,0} - \theta_0}{\|v_{L,n}^*\|_w} - F_n \right) \right\} + o_P(n^{-1})$$

$$= \left\{ -t^2 + 2t \left( \frac{\theta_{L,0} - \theta_0}{\|v_{L,n}^*\|_w} - F_n \right) - \left( \frac{\theta_{L,0} - \theta_0}{\|v_{L,n}^*\|_w} - F_n \right)^2 \right\} + \left( \frac{\theta_{L,0} - \theta_0}{\|v_{L,n}^*\|_w} - F_n \right)^2 + o_P(n^{-1})$$

$$\leq \left( \frac{\theta_{L,0} - \theta_0}{\|v_{L,n}^*\|_w} - F_n \right)^2 + o_P(n^{-1}).$$

**Step 4.** The desired result follows from imposing the null hypothesis in Step 2. \(\square\)
Supplemental Material

SM.I Supplemental Material for Appendix A

The next lemma presents some properties of the tangent space.

Lemma SM.I.1. The tangent space of $\mathcal{M}$ at $P \in \mathcal{M}$ is included in the class of all $g \in L_0^2(P)$ such that

$$\int \rho_2(y, w, h(P))g(y, w, \cdot)P_{YW|X}(dy, dw | \cdot) \in \text{Range}(T_P).$$

Proof. By our definition of $\mathcal{M}$ any curve, $t \mapsto P[t]$, must satisfy

$$\int \rho_2(y, w, h(P[t]))p[t]_{YW|X}(y, w | X)dydw = 0.$$

Since the curve is in $\mathcal{M}$ it has a pdf, which we denote as $p[t]$; also we write $\rho_2$ only as a function of $h$ and not $\alpha$ to stress the fact that $\theta$ is not present. Since $\theta(P) \in \text{int}(\Theta)$ the equation $\theta(P[t]) = -E_{P[t]}[\ell_{P[t]}(W)h(P[t])(W)]$ does not impose — locally — any restrictions; so it can be ignored for the computation of the tangent space.

Since $p_X > 0$ for any $P$ in the model, the previous display implies that

$$\int \rho_2(y, w, h(P[t]))p[t]_{YW|X}[t](y, w | X)dydw = \int (F_{Y|W|X}[t](h(P[t])(w) | w, x) - \tau)p_{W|X}[t](w, x)dw,$$

where $F_{Y|W|X}[t]$ is the conditional cdf of $Y$, given $w, x$ associated to $P[t]$. By our assumption 2, taking derivative with respect to $t$ implies

$$\int \rho_2(y, w, h(P))g(y, w, \cdot)p(y, w | x)dydw = \mathbf{T}_P \left[ \dot{h}(P)[g] \right](x)$$

for all $x \in \mathcal{X}$. Hence, for any $g$ in the tangent space, it must hold that the LHS function belongs to $\text{Range}(\mathbf{T}_P)$.

Proof of Lemma A.1. Part 1. By laws of differentiation, if the derivative exists, it has to satisfy

$$\frac{d}{dt} \int \ell_{P[t]}(w)h_{id}(P[t](g))(w)dw \bigg|_{t=0} = \frac{d}{dt} \int \ell_{P}(w)h_{id}(P[t](g))(w)dw \bigg|_{t=0} + \frac{d}{dt} \int \ell_{P[t]}(w)h_{id}(P)(w)dw \bigg|_{t=0}.$$
Since for all $P \in \mathcal{M}$,

$$
\ell_P(w) = \mu'(w)p_W(w) + \mu(w)p'_W(w) = \mu'(w) \int p(y, w, x)dydx + \mu(w) \frac{d \int p(y, w, x)dydx}{dw}
$$

it follows that

$$
\frac{d \int \ell_P[t](g)h_{id}(P)(w)dw}{dt} \bigg|_{t=0} = \frac{d \int \mu'(w)h_{id}(P)(w)p[t](g)(w)dw}{dt} \bigg|_{t=0} + \frac{d \int \mu(w)h_{id}(P)(w)p'[t](g)(w)dw}{dt} \bigg|_{t=0}.
$$

Since $p[t](g)$ belongs to the model, $p'[t](g)$ is continuous. We now show that

$$
\lim_{w \to \mp \infty} \mu(w)h(P)(w)p[t](g)(w)dw = 0.
$$

Since $p[t](g)$ is a density, it is enough to show that $||\mu h(P)||_{L^\infty(p_W)} < \infty$, but this follows by the fact that $\mu$ and $p_W$ are uniformly bounded and by definition of $\mathbb{H}$. By this result and integration by parts, it follows that

$$
\frac{d \int \ell_P[t](g)h_{id}(P)(w)dw}{dt} \bigg|_{t=0} = \frac{d \int \mu'(w)h_{id}(P)(w)p[t](g)(w)dw}{dt} \bigg|_{t=0} - \frac{d \int \{\mu'(w)h_{id}(P)(w) + \mu(w)h'_{id}(P)(w)\}p[t](g)(w)dw}{dt} \bigg|_{t=0} = - \frac{d \int \mu(w)h'_{id}(P)(w)p[t](g)(w)dw}{dt} \bigg|_{t=0}.
$$

By definition of derivative of the curve $t \mapsto p[t](g)$ and the fact that $||\mu h'(P)||_{L^2(Leb)} \lesssim ||h'_{id}(P)||_{L^2(Leb)} < \infty$, it follows by Dominated Convergence Theorem that

$$
\frac{d \int \ell_P[t](g)h_{id}(P)(w)dw}{dt} \bigg|_{t=0} = \frac{d \int \mu(w)h'_{id}(P)(w)p[t](g)(w)dw}{dt} \bigg|_{t=0} = - \int \mu(w)h'_{id}(P)(w) \int g(y, w, x)p(y, w, x)dydxdw.
$$

Since $\ell_P \in L^2(\mathcal{P}W)$, $g \mapsto \int \ell_P(w)g(w)dw$ is a bounded linear functional; this, part 2 below and the Dominated Convergence Theorem, imply

$$
\frac{d \int \ell_P(w)h(P[t](g))(w)dw}{dt} \bigg|_{t=0} = \int \ell_P(w)h_{id}(P)(g)(w)dw.
$$

Hence,

$$
\hat{\theta}(P)[g] = - \int \ell_P(w)h_{id}(P)[g](w)dw + \int \mu(w)h'_{id}(P)(w) \int g(y, w, x)p(y, w, x)dydxdw.
$$
Note that the last term in the RHS is, by definition, \( \theta(g \cdot P) \). Hence
\[
\dot{\theta}(P)[g] = - \int \ell_P(w) \dot{h}_{id}(P)[g](w) dw + \theta(g \cdot P).
\]

**Part 2.** This part of the proof follows from applying the implicit function theorem to \( (h, P) \mapsto G(h, P) \equiv \int (F_{Y|W,X}(h(w) \mid w, x) - \tau)p_{W|X}(w \mid x) dw, \) where \( F_{Y|W,X} \) is the cdf associated to the probability measure \( P \). The derivatives of the mapping \( G \) at \((h(P), P)\), with direction \((\zeta, Q)\) where \( \zeta \in \mathbb{H} \) and \( Q \) is a measure over \( \mathbb{Z} \), are given by
\[
\frac{dG(h(P), P)}{dP}[Q](x) = \int \rho_2(y, w, h(P))Q_{Y|W,X}(dy, dw \mid x) dy dw \forall x,
\]
and
\[
\frac{dG(h(P), P)}{dh}[\zeta](x) = T[\zeta](x) = T[\zeta_{id}](x) = T_{id}[\zeta_{id}](x), \forall x,
\]
where the second equality follows because any \( \zeta \in \mathbb{H} \) can be decomposed as the sum of \( \zeta_{id} \in Kernel(T)^\perp \) and an element in \( Kernel(T) \); in the third expression \( T_{id} \) is defined as the restriction of \( T \) to \( Kernel(T)^\perp \). We observe that \( T_{id} \) is invertible (in the sense that the inverse is a linear functional); its extension to the whole of \( L^2(Leb) \) is the generalized inverse and we denote it as \( T^+ \) (see Engl et al. [1996] p. 33).

Taking \( Q \) such that \( Q_{Y|W,X} = g \cdot P_{Y|W,X} \) for \( g \in \mathcal{T} \), it follows that \( \frac{dG(h(P), P)}{dP}[Q] = A_P[g] \). Moreover, by Lemma **SM.I.1**, any \( g \in \mathcal{T} \) is such that \( A_P[g] \in Range(T_{id}) = Range(T) \). Thus, it follows that by the Implicit Function Theorem that
\[
\dot{h}_{id}(P)[g] = (T)^+[A_P[g]] = (T^*T)^+[A_P[g]],
\]
where the second equality follows from the results by Engl et al. [1996] p. 35.

**Proof of Lemma A.2.** The calculations are analogous to those in Severini and Tripathi [2012] so they are omitted.
Let $\Pi_K\alpha_0$ be the projection of $\alpha_0$ onto $A_K = \Theta \times H_K$; since $H_K$ is closed and convex (Assumption 3), it is well-defined.

**Lemma SM.II.1.** Suppose Assumption 3 holds. Then

$$||E[g_J(Z, \alpha_L, 0)]||_e^2 + \gamma_K Pen(\alpha_L, 0) \lesssim ||E[g_J(Z, \Pi_K\alpha_0)]||_e^2 + \gamma_K Pen(\Pi_K\alpha_0),$$

and $||h_{L,0}'||_{L^\infty(W)} \leq \gamma_K^{-1}||E[g_J(Z, \Pi_K\alpha_0)]||_e + Pen(\Pi_K\alpha_0)$.

**Proof.** By definition of the $\alpha_L, 0$ and the fact that $\Pi_K\alpha_0 \in A_K$, $\bar{Q}_J(\alpha_L, 0, P) \leq \bar{Q}_J(\Pi_K\alpha_0, P)$. By Lemma D.3, $C^{-1}I \leq H_J^{-1}(\alpha_0, P) \leq CI$ some $C \geq 1$, thus

$$C^{-1}||E[g_J(Z, \alpha_L, 0)]||_e^2 + \gamma_K Pen(\alpha_L, 0) \leq C||E[g_J(Z, \Pi_K\alpha_0)]||_e^2 + \gamma_K Pen(\Pi_K\alpha_0)$$

and the first result follows.

The second result follows from the first result and Assumption 3(iii).

**SM.II.1 Supplementary Lemmas**

We present and prove a sequence of lemmas used in the proofs of the Lemmas of Section 5.

**Lemma SM.II.2.** For any $\alpha \in A$, any $A \equiv A_y \times A_w \times A_x \subseteq Z$ Borel, and any $L = (J, K) \in \mathbb{N}^2$,

$$\sup_{z \in A} ||g_J(z, \alpha)||_e \leq \bar{g}_L(\alpha, A) \equiv \bar{\theta} + ||h'||_{L^\infty(A_w, \mu)} + \sup_{x \in A_x} ||q^J(x)||_e.$$

**Proof.** The result follows from the fact that,

$$\sup_{z \in A} ||g_J(z, \alpha)||_e \leq \sup_{z \in A} |\theta - \mu(w)h'(w)| + \sup_{x \in A_x} ||q^J(x)||_e$$

$$\leq \bar{\theta} + \sup_w |\mu(w)h'(w)| + \sup_x ||q^J(x)||_e.$$

Throughout, for any $\delta > 0$, let $B(\delta) \equiv \{\lambda \in \mathbb{R}^{J+1}: ||\lambda||_e \leq \delta\}$. 


Lemma SM.II.3. Suppose Assumption 3 holds. Then, there exists a $\eta > 0$ such that for any $\epsilon > 0$, any $(L = (J, K), n)$ and any $\delta > 0$ for which

$$\delta \theta < \eta/3, \text{ and } b_{e,J}^0 n (3\delta)\theta / (\eta)\theta < \epsilon/2, \text{ and } \delta \times \mathcal{U}_{L,n} < \eta/3, \text{ some } \varrho > 0,$$

it follows that $P \left( B(\delta) \subseteq \Lambda(\alpha, P_n) \forall \alpha \in \bar{A}_{L,n} \right) \geq 1 - \epsilon$.

Proof. Since $S \ni 0$, there exists a $\eta > 0$ for which: For any $(Z_i)_{i=1}^{\infty}$, if

$$\sup_{\alpha \in \bar{A}_{L,n}} \sup_{z \in \text{supp}(P_n)} \max_{\lambda \in B(\delta)} |\lambda^T g_J(z, \alpha)| \leq \eta,$$

then $\lambda \in \Lambda(\alpha, P_n)$, for all $\alpha \in \bar{A}_{L,n}$.

By the Cauchy-Schwarz inequality and Lemma SM.II.2,

$$\sup_{\alpha \in \bar{A}_L} \sup_{z \in \text{supp}(P_n)} ||\lambda|| e ||g_J(z, \alpha)|| e \leq \delta \times \sup_{\alpha \in \bar{A}_{L,n}} g_L(\alpha, \text{supp}(P_n)).$$

Thus, it suffices to show that $P(\delta \times \sup_{\alpha \in \bar{A}_{L,n}} g_L(\alpha, \text{supp}(P_n)) \leq \eta) \geq 1 - \epsilon$ (for the $\epsilon$ in the statement of the lemma).

To show this we bound each term of $\sup_{\alpha \in \bar{A}_{L,n}} g_L(\alpha, \text{supp}(P_n))$. Note that $\delta \theta < \eta/3$; also note that by the Markov inequality

$$P \left( \delta \max_{1 \leq i \leq n} ||q^J(X_i)|| e \geq \eta/3 \right) \leq nE_P \left[ ||q^J(X)|| e \theta \right] / (\eta)\theta \times (3\delta)\theta \leq b_{e,J}^0 n (3\delta)\theta / (\eta)\theta,$$

it suffices that the RHS is less than $\epsilon/2$, which it does by assumption.

Finally, by definition of $\bar{A}_{L,n}$ and Assumption 3, $||h'||_{L^\infty(\mathbb{W}, \mu)} \leq \mathcal{U}_{L,n}$, hence

$$\sup_{\alpha \in \bar{A}_{L,n}} ||h'||_{L^\infty(\text{supp}(P_n), \mu)} \leq l_n \gamma_K^{-1} \Gamma_{L,n} = \mathcal{U}_{L,n}, \ a.s.$$ 

Since by assumption $\delta \times l_n \gamma_K^{-1} (\Gamma_{L,n}) < \eta/3$,

$$\square$$

Lemma SM.II.4. Suppose Assumptions 2 and 3 hold. For any $(L = (J, K), n)$ and any positive real-valued sequence, $(\delta_n)_n$, such that

1. $b_{e,J}^4 / n = o(1)$.

2. $\delta_n = o(1)$ and $b_{e,J}^0 n \delta_n^0 = o(1)$ and $\delta_n \mathcal{U}_{L,n} = o(1)$ for some $\varrho > 0$. 

5
3. \( \sqrt{\frac{g_{L,0}}{n}} + \|E[g_J(Z, \alpha_{L,0})]\|_e^2 = o(\delta_n) \).

it follows that the conditions of Lemma D.1 hold for \((\alpha_{L,0}, P_n)\) wpa1.

Proof. To show condition (1) in Lemma D.1 it suffices to show that

\[
\sup_{\lambda \in B(\delta_n)} \sup_{z \in \mathbb{Z}} \lambda^T g_J(z, \alpha_{L,0})
\]

is bounded. But this follows from the proof of Lemma SM.II.3, the fact that \(\alpha_{L,0} \in \bar{A}_{L,n}\) and the fact that all the conditions in that Lemma are satisfied.

Condition (2) in Lemma D.1 holds wpa1 for some \(C\) by Lemma D.3 applied to \(\alpha = \alpha_{L,0}\) and \(P\), provided that \(b_{4,J}^4/n = o(1)\) (note that \(|\theta| + \|h_{L,0}'\|_{L^\infty(W,\mu)}\) is bounded).

By Lemma D.2, for condition (3) in Lemma D.1 it is enough that

\[
\sqrt{\frac{g_{L,0}}{n} + \|E[g_J(Z, \alpha_{L,0})]\|_e^2} = \sqrt{\frac{\bar{g}_{L,0}^2}{n} + \|E[g_J(Z, \alpha_{L,0})]\|_e^2} = o(\delta_n)
\]

which holds by assumption.

Finally, \(\delta_n = o(1)\) and \(b_{\rho,J}^0 n \delta_n^\rho = o(1)\) and \(\delta_n l_n \gamma_K^{-1} \Gamma_{L,n} = \delta_n \bar{\Omega}_{L,n} = o(1)\), so condition (4) in Lemma D.1 holds for any \(\epsilon > 0\). \(\square\)

Remark SM.II.1. We note that the requirement of \(\delta_n l_n \gamma_K^{-1} \Gamma_{L,n} = \delta_n \bar{\Omega}_{L,n} = o(1)\) (condition 2),

\(\frac{\sqrt{g_{L,0}^2}}{n} + \|E[g_J(Z, \alpha_{L,0})]\|_e^2\)

does not contradict condition 3. Even though \(\Gamma_{L,n}\) contains \(\sqrt{\frac{g_{L,0}^2}{n} + \|E[g_J(Z, \alpha_{L,0})]\|_e^2}\) in condition 2, there is the \(l_n \gamma_K^{-1}\) term which is large, at least for sufficiently large \(K\). \(\triangle\)

Lemma SM.II.5. Suppose Assumption 3 holds. Then, for any \((L = (J, K), n)\) such that

\[
(1 + \bar{\theta} + \bar{\Omega}_{L,n}) \sqrt{\frac{1 + b_{4,J}^4}{n}} = o(1), \quad (9)
\]

it follows that

\[
P \left( e_{\max} (H_J(\hat{\alpha}_{L,n}, P_n)) \leq C_{L,n} \right) \to 1
\]

where \(C_{L,n} \equiv (1 + \bar{\theta} + \bar{\Omega}_{L,n})^2\).

Proof. Step 1. We show that

\[
n^{-1} \sum_{i=1}^n g_J(Z_i, \alpha) g_J(Z_i, \alpha)^T \leq n^{-1} \sum_{i=1}^n \|\rho(Y_i, W_i, \alpha)\|_e^2 M_J(X_i) M_J(X_i)^T,
\]
where

\[
M_J(x) = \begin{bmatrix}
1 & 0 \\
0 & q^J(x)
\end{bmatrix}
\]

\(0\) is a \(J \times 1\) vector of zeros.

To do this, note that for each \(z \in Z\),

\[
g_J(z, \alpha)g_J(z, \alpha)^T = \begin{bmatrix}
(\rho_1(y, w, \alpha))^2 & \rho_1(y, w, \alpha)\rho_2(y, w, \alpha)q^J(x)^T \\
\rho_1(y, w, \alpha)\rho_2(y, w, \alpha)q^J(x) & (\rho_2(y, w, \alpha))^2q^J(x)q^J(x)^T
\end{bmatrix}
\]

\[\equiv M_J(x)\rho(y, w, \alpha)\rho(y, w, \alpha)^T M_J(x)^T.\]

It follows that \(\rho(y, w, \alpha)\rho(y, w, \alpha)^T \leq \|\rho(y, w, \alpha)\|_2^2I\). Therefore

\[
n^{-1}\sum_{i=1}^n g_J(Z_i, \alpha)g_J(Z_i, \alpha)^T \leq n^{-1}\sum_{i=1}^n \|\rho(Y_i, W_i, \alpha)\|_2^2M_J(X_i)M_J(X_i)^T.
\]

**Step 2.** Let \((y, w) \mapsto R(y, w) \equiv \sup_{\alpha \in \bar{A}_{L,n}} \|\rho(y, w, \alpha)\|_e^2\). We now show that

\[
\left\|n^{-1}\sum_{i=1}^n R(Y_i, W_i)M_J(X_i)M_J(X_i)^T - E [R(Y, W)M_J(X)M_J(X)^T]\right\|_e = O_p\left(\frac{\sqrt{E[E[(R(Y, W))^2]X^2]}(1 + \|q^J(x)\|_e^2)}{n}\right).
\]

By the Markov inequality it is enough to bound

\[
n^{-1/2}\sqrt{E \left[\|R(Y, W)M_J(X)M_J(X)^T\|_e^2\right]} \leq n^{-1/2}\sqrt{E [(R(Y, W))^2(\text{trace}\{M_J(X)M_J(X)^T\})^2]}.
\]

Since

\[
M_J(x)M_J(x)^T = \begin{bmatrix}
1 & 0^T \\
0 & q^J(x)q^J(x)^T
\end{bmatrix},
\]

the previous display implies that

\[
n^{-1/2}\sqrt{E \left[\|R(Y, W)M_J(X)M_J(X)^T\|_e^2\right]} \leq n^{-1/2}\sqrt{E [E[(R(Y, W))^2]X^2]}(1 + \|q^J(x)\|_e^2)^2].
\]
Step 3. By Steps 1-2 and Assumption 3

\[ n^{-1} \sum_{i=1}^{n} g_J(Z_i, \alpha)T \leq \sup_{x} E[R(Y, W) \mid X = x] \times I \]

\[ + O_P \left( \sqrt{\sup_{x} E[(R(Y, W))^2 \mid X = x]} \right) \left( 1 + E \left[ \left[ \frac{\|q'(X)\|^4}{n} \right] \right] \right) \]

Also,

\[ R(y, w) \leq \sup_{\alpha \in \bar{A}_{L,n}} |\theta - \mu(w)h'(w)|^2 + 1 \leq 1 + \bar{\sigma}^2 + (l_n \gamma^{-1}_K \Gamma_{L,n})^2 \]

where the last inequality follows from Assumption 3 and definition of \( \bar{A}_{L,n} \). So, to the extent that \((1 + \theta + l_n \gamma^{-1}_K \Gamma_{L,n})^4 = o(1)\), it follows that

\[ \sup_{\alpha \in \bar{A}_{L,n}} e_{max} \left( n^{-1} \sum_{i=1}^{n} g_J(Z_i, \alpha)T \right) \leq (1 + \bar{\sigma} + l_n \gamma^{-1}_K \Gamma_{L,n})^2 \]

wpa1. Since \( \hat{\alpha}_{L,n} \in \bar{A}_{L,n} \) wpa1, this implies the result.

\[ \square \]

SM.II.2 Proofs of the Lemmas stated in Appendix D

We now present the proofs of the Lemmas stated in Appendix D.

Proof of Lemma D.1. By the mean value theorem,

\[ S_J(\alpha, \lambda, P) = s'(0)\lambda^T E_P[g_J(Z, \alpha)] + \frac{1}{2} \lambda^T \left\{ \int_0^1 E_P \left[ s''(t\lambda^T g_J(Z, \alpha)) \ g_J(Z, \alpha)g_J(Z, \alpha)^T \right] dt \right\} \lambda \]

\[ = - \lambda^T E_P[g_J(Z, \alpha)] + \frac{1}{2} \lambda^T \left\{ \int_0^1 E_P \left[ s''(t\lambda^T g_J(Z, \alpha)) \ g_J(Z, \alpha)g_J(Z, \alpha)^T \right] dt \right\} \lambda. \]

Note that

\[ \int_0^1 E_P \left[ s''(t\lambda^T g_J(Z, \alpha)) \ g_J(Z, \alpha)g_J(Z, \alpha)^T \right] dt \leq \int_0^1 \sup_z s''(t\lambda^T g_J(z, \alpha)) dt E_P \left[ g_J(Z, \alpha)g_J(Z, \alpha)^T \right]. \]

Under our assumptions, \( \sup_z s''(t\lambda^T g_J(z, \alpha)) \leq -\sqrt{C} \) for all \( t \in [0, 1] \). This and the fact that

\[ e_{min} \left( E_P \left[ g_J(Z, \alpha)g_J(Z, \alpha)^T \right] \right) \geq \sqrt{C} \]
imply that
\[ S_J(\alpha, \lambda, P) \leq -\lambda^T \mathbb{E}_P[g_J(Z, \alpha)] - \frac{C}{2}||\lambda||_e^2 \leq ||\lambda||_e ||\mathbb{E}_P[g_J(Z, \alpha)]||_e - \frac{C}{2}||\lambda||_e^2. \]

Hence, since \( S_J(\alpha, 0, P) = 0 \), by evaluating the previous expression in \( \lambda_1 \in \text{arg max}_{\lambda \in B(\delta)} S_J(\alpha, \lambda, P) \) (it exists by continuity of \( S_J(\alpha, ., P) \) and compactness of the set), we obtain
\[
||\lambda_1||_e \leq 2C^{-1}||\mathbb{E}_P[g_J(Z, \alpha)]||_e.
\]

Since \( 2C^{-1}||\mathbb{E}_P[g_J(Z, \alpha)]||_e < \delta \) by assumption, \( \lambda_1 \) is in the interior of \( B(\delta) \). Since, by Lemma SM.II.3 — all the hypothesis of the lemma are satisfied — \( \lambda_1 \in \Lambda_J(\alpha, P) \) with probability higher than \( 1 - \epsilon \), it holds that \( \lambda_1 \in \text{arg max}_{\lambda \in \Lambda_J(\alpha, P)} S_J(\alpha, \lambda, P) \) with probability higher than \( 1 - \epsilon \).

Finally, this implies that
\[
\sup_{\lambda \in \Lambda_J(\alpha, P)} S_J(\alpha, \lambda, P) = S_J(\alpha, \lambda_1, P) \leq 2C^{-1}||\mathbb{E}_P[g_J(Z, \alpha)]||_e^2,
\]
with probability higher than \( 1 - \epsilon \). \( \square \)

**Proof of Lemma D.2.** It suffices to show that
\[
||n^{-1} \sum_{i=1}^n g_J(Z_i, \alpha) - E[g_J(Z, \alpha)]||_e^2 = O_P \left( \frac{\overline{\theta} + ||\mu h'||_{L^2(P)}^2 + E[||q^f(X)||_e^2]}{n} \right).
\]

By the Markov inequality, we can study
\[
E[||n^{-1} \sum_{i=1}^n g_J(Z_i, \alpha) - E[g_J(Z, \alpha)]||_e^2] \leq n^{-1} E[||g_J(Z, \alpha)||_e^2],
\]
and it follows by Lemma D.5, \( E[||g_J(Z, \alpha)||_e^2] \leq \overline{\theta} + E[||\mu(W) h'(W)||^2] + E[||q^f(X)||_e^2] \). \( \square \)

**Proof of Lemma D.3.** Part 1. By analogous calculations to those in the proof of Lemma SM.II.5 and Lemma A.6 in DIN, \( ||H_J(\alpha, P_n) - H_J(\alpha, P)||_e = O_P(\sqrt{\sup_{\lambda} E[||\rho(Y, W, \alpha)||_e^4|X = x]} b_{\lambda, i}^4 / n) \).

Note that,
\[
E[||\rho(Y, W, \alpha)||_e^4|X = x] \leq E \left[ (1 + |\theta - \mu(W) h'(W)|^2)^2 |X = x \right] \lesssim \theta^4 + ||\mu h'||_{L^4(P)}^4 \lesssim \theta^4 + ||h'||_{L^\infty(W, \mu)}^4
\]
where the last inequality follows from Assumption 2 and some trivial algebra.

Part 2. From Part 1 and the fact that \( A \mapsto e_{\min}(A) \) is Lipschitz, is suffices to show the result
for $P = \mathbf{P}$. Note that

$$E_{\mathbf{P}}[g_J(Z, \alpha)g_J(Z, \alpha)^T] = E[M_J(X)E[\rho(Y, W, \alpha)\rho(Y, W, \alpha)^T|X]M_J(X)^T]$$

where $M_J$ is defined in the proof of Lemma SM.II.5.

We now argue that $C^{-1}I \leq E[\rho(Y, W, \alpha)\rho(Y, W, \alpha)^T|X] \leq CI$ for any $\alpha$ in some neighborhood of $\alpha_0$. First note that, under Assumption 2, $E[(\rho_1(Y, W, \alpha))^2 | X] = \int(\lambda_0 - \mu(W))h'(w))^2p_W|X(w|x)dw \geq CV_{ar}\mu(W)h'(W) > 0$ some $C > 0$. Hence the trace of $E[\rho(Y, W, \alpha)\rho(Y, W, \alpha)^T|X]$ is positive uniformly in $x$. Since $\rho_1(, \alpha)$ and $\rho_2(, \alpha)$ are not linearly dependent, by the Cauchy-Schwarz (strict) inequality the determinant is also positive; thus $C^{-1}I \leq E[\rho(Y, W, \alpha)\rho(Y, W, \alpha)^T|X]$ uniformly on $X$. The reverse inequality is obtained in a similar fashion.

Hence under Assumption 3 the desired result follows. 

\begin{proof}[Proof of Lemma D.4] By the mean value theorem, for any $\alpha \in \mathcal{A}$, $P$ and $\lambda$ in a neighborhood of 0,

$$S_J(\alpha, \lambda, P) = s'(0)\lambda^TE_{\mathbf{P}}[g_J(Z, \alpha)] + \frac{1}{2}\lambda^T \left\{ \int_0^1 E_{\mathbf{P}} [s''(t\lambda^T g_J(Z, \alpha))g_J(Z, \alpha)g_J(Z, \alpha)^T]dt \right\} \lambda$$

$$= -\lambda^TE_{\mathbf{P}}[g_J(Z, \alpha)] + \frac{1}{2}\lambda^T \left\{ \int_0^1 E_{\mathbf{P}} [s''(t\lambda^T g_J(Z, \alpha))g_J(Z, \alpha)g_J(Z, \alpha)^T]dt \right\} \lambda.$$

By our assumptions over $s$ (in particular, that it has Hölder continuous second derivative), it follows

$$|s''(\lambda^T g_J(z, \alpha)) - s''(0)| \leq C \times |\lambda^T g_J(z, \alpha)|$$

for some $C > 0$. Hence

$$\left| \lambda^T \left\{ \int_0^1 E_{\mathbf{P}} [s''(t\lambda^T g_J(Z, \alpha))g_J(Z, \alpha)g_J(Z, \alpha)^T]dt \right\} \lambda \right|$$

$$\leq \int_0^1 E_{\mathbf{P}} \left[(s''(t\lambda^T g_J(Z, \alpha)) - s''(0)) \lambda^T g_J(Z, \alpha)g_J(Z, \alpha)^T \lambda \right] dt$$

$$\leq C \times E_{\mathbf{P}} \left[ |\lambda^T g_J(Z, \alpha)| \lambda^T g_J(Z, \alpha)g_J(Z, \alpha)^T \lambda \right]$$

$$\leq C \times E_{\mathbf{P}} \left[ ||g_J(Z, \alpha)||^2 \right].$$

\end{proof}
Therefore, for \( \alpha = \hat{\alpha}_{L,n} \) and \( P = P_n \),

\[
S_J(\alpha, \lambda, P) \geq -\lambda^T E_P [g_J(Z, \alpha)] - \frac{1}{2} \lambda^T H_J(\alpha, P) \lambda - C||\lambda||_e^3 \times E_P \left[ ||g_J(Z, \alpha)||_e^3 \right] \\
\geq -\lambda^T E_P [g_J(Z, \alpha)] - \frac{C \times C_{L,n}}{2} ||\lambda||_e^2 - C||\lambda||_e^3 \times E_P \left[ ||g_J(Z, \alpha)||_e^3 \right]
\]

where the second inequality follows from Lemma SM.II.5.

Let \( \tilde{\lambda} \equiv -\delta_n E_P_n [g_J(Z, \hat{\alpha}_{L,n})]/||E_P_n [g_J(Z, \hat{\alpha}_{L,n})]||_e \) with \( \delta_n \) satisfying the assumptions in the statement of the Lemma. Since \( ||\tilde{\lambda}||_e = \delta_n \) and \( \hat{\alpha}_{L,n} \in \tilde{\mathcal{A}}_{L,n} \) wpa1 (Lemma 5.1), by Lemma SM.II.3, this choice of \( \delta_n \) ensures that \( \tilde{\lambda} \in \Lambda_f(\hat{\alpha}_{L,n}, P_n) \) wpa1.

The previous expression for \( S_J(\alpha, \lambda, P) \) yields

\[
S_J(\hat{\alpha}_{L,n}, \tilde{\lambda}, P_n) \geq \delta_n ||E_P_n [g_J(Z, \hat{\alpha}_{L,n})]||_e - \frac{C \times C_{L,n}}{2} \delta_n^2 \times E_P_n \left[ ||g_J(Z, \hat{\alpha}_{L,n})||_e^3 \right].
\]

By definition of \( \hat{\alpha}_{L,n} \),

\[
S_J(\hat{\alpha}_{L,n}, \tilde{\lambda}, P_n) + \gamma_K Pen(\hat{\alpha}_{L,n}) \leq \sup_{\lambda \in \Lambda_f(\alpha_{L,0}, P_n)} S_J(\alpha_{L,0}, \lambda, P_n) + \gamma_K Pen(\alpha_{L,0}) \quad \text{wpa1}.
\]

By Lemma SM.II.4, all the conditions in Lemma D.1 hold for \( (\alpha_{L,0}, P_n) \) wpa1. In this case, the previous inequality implies that \( S_J(\hat{\alpha}_{L,n}, \tilde{\lambda}, P_n) + \gamma_K Pen(\hat{\alpha}_{L,n}) \leq ||E_P_n [g_J(Z, \alpha_{L,0})]||_e^2 + \gamma_K Pen(\alpha_{L,0}) \) wpa1. Therefore

\[
\delta_n ||E_P_n [g_J(Z, \hat{\alpha}_{L,n})]||_e - \frac{C \times C_{L,n}}{2} \delta_n^2 \times E_P_n \left[ ||g_J(Z, \hat{\alpha}_{L,n})||_e^3 \right] \\
\leq ||E_P_n [g_J(Z, \alpha_{L,0})]||_e^2 + \gamma_K Pen(\alpha_{L,0}) \quad \text{wpa1}.
\]

Since \( \hat{\alpha}_{L,n} \in \tilde{\mathcal{A}}_{L,n} \) wpa1 (Lemma 5.1), it follows by the proof of Lemma SM.II.3 that \( ||g_J(Z, \hat{\alpha}_{L,n})||_e^3 \leq (\bar{\theta} + l_n \gamma^{-1}_K \Gamma_{L,n} + ||q^f(X)||_e)^3 \). We now show that \( \delta_n^3 \left\{ (\bar{\theta} + l_n \gamma^{-1}_K \Gamma_{L,n})^3 + E_P_n [||q^f(X)||_e^3] \right\} = o_p(\delta_n^2 C_{L,n}) \), so that the term \( C\delta_n^3 \times E_P_n \left[ ||g_J(Z, \hat{\alpha}_{L,n})||_e^3 \right] \) can be ignored.

To show this, note by definition of \( C_{L,n} \), it suffices to show \( \delta_n^3 \left\{ (C_{L,n})^2 + E_P_n [||q^f(X)||_e^3] \right\} = o_p(\delta_n^2 C_{L,n}) \). By the conditions in the lemma \( \delta_n C_{L,n} = o(1) \) and taking \( C_{L,n} \geq 1 \) (if this is not the case, the solution is trivial), and thus \( \delta_n^3 (C_{L,n})^2 = o(\delta_n^2 C_{L,n}) \). By the Markov inequality, it remains to show that \( \delta_n b_{3,j}^3 = o(C_{L,n}) \). As we take \( C_{L,n} \geq 1 \) and \( \delta_n b_{3,j}^3 = o(1) \), the desired equality holds.
Hence

$$||E_P[g_J(Z, \hat{\alpha}_{L,n})]||_e \lesssim C_{L,n} \delta_n + \delta_n^{-1} \{||E_P[g_J(Z, \alpha_{L,0})]||_e^2 + \gamma_K Pen(\alpha_{L,0})\}, \text{ wpa}.\]$$

Proof of Lemma D.5. By definition of $g_J$ and simple algebra, it follows that

$$||E_P[g_J(Z, \alpha)]||_e \leq \bar{\theta} + \sup_w |h'(w)\mu(w)| + E_P[||q^J(X)||_e].$$

By the fact that $\alpha \in \bar{\mathcal{A}}_{L,n}$, $Pen(\alpha) \leq \mathcal{U}_{L,n} = l_n \gamma_K^{-1} \Gamma_{L,n}$. So, by Assumption 3(iii) it follows that $\sup_w |h'(w)\mu(w)| \leq \mathcal{U}_{L,n} = l_n \gamma_K^{-1} \Gamma_{L,n}$, uniformly on $h$. \qed

SM.III Supplemental Material for Appendix F

Proof of Lemma F.1. Recall that $|| \cdot ||_w^2 \equiv (G(\alpha_{L,0})[\cdot] )^T H_L^{-1}(G(\alpha_{L,0})[\cdot])$. Since $\alpha \in \text{lin}\{\mathcal{A}_K\}$, we can cast $\alpha = (\theta, \varphi^K(\cdot)^T \pi)$ some $\pi \in \mathbb{R}^K$. Hence

$$G(\alpha_{L,0})[\alpha] = \begin{bmatrix} \theta - E[\ell(W)\varphi^K(W)^T \pi] \\ E[p_Y|W X(h_{L,0}(W)|W, X)q^J(X)\varphi^K(W)^T \pi] \end{bmatrix} = M_L \times (\theta, \pi)^T.$$ 

Hence

$$||\alpha||_w^2 = (\theta, \pi) M_L^T H_L^{-1} M_L (\theta, \pi)^T.$$ 

Thus, it is sufficient to show that $M_L^T H_L^{-1} M_L$ is positive definite. By Lemma D.3 — since $\theta_{L,0}^2 + \|\mu h_{L,0}(w)^2\|_{\infty} = o(1)$ — it follows that $e_{\min}(H_L^{-1}) \geq \epsilon^2 > 0$, Hence,

$$||\alpha||_w^2 \geq \epsilon^2 e_{\min}(M_L^T M_L) ||\theta, \pi||_e^2.$$ 

Note that the eigenvalues of $M_L^T M_L$ are those of

$$(E[p_Y|W X(h_{L,0}(W)|W, X)q^J(X)\varphi^K(W)^T])^T (E[p_Y|W X(h_{L,0}(W)|W, X)q^J(X)\varphi^K(W)^T])$$
and 1. By Assumption 12(iv) $E[pY|W,X(h_{L,0}(W)|W,X)q'(X)\varphi^K(W)^T]$ has full rank and thus the eigenvalues are positive.

**Proof of Lemma F.2.** Note that

$$
\|H_J(\alpha, P_n) - H_J\|_e \leq \|H_J(\alpha, P_n) - H_J(\alpha_{L,0}, P_n) - \{H_J(\alpha, P) - H_J(\alpha_{L,0}, P)\}\|_e \\
+ \|H_J(\alpha_{L,0}, P_n) - H_J(\alpha_{L,0}, P) + H_J(\alpha, P) - H_J\|_e \\
\leq \|H_J(\alpha, P_n) - H_J(\alpha_{L,0}, P_n) - \{H_J(\alpha, P) - H_J(\alpha_{L,0}, P)\}\|_e \\
+ \|H_J(\alpha_{L,0}, P_n) - H_J(\alpha_{L,0}, P)\|_e \\
+ \|H_J(\alpha, P) - H_J\|_e \\
\equiv \text{Term}_{1,n} + \text{Term}_{2,n} + \text{Term}_{3,n}.
$$

The term $\text{Term}_{1,n}$ is controlled by Assumption 10. By Lemma D.3 applied to $\alpha_{L,0}$, $\text{Term}_{2,n} = O_P\left(\{\theta_{L,0} + ||h'_{L,0}||_{L^\infty(W,\mu)}\}^2 \sqrt{b_4/n}\right)$.

Note that

$$
H_J(\alpha, P) - H_J = E_P\left[M_J(X)\{\rho(Y,W,\alpha)\rho(Y,W,\alpha)^T - \rho(Y,W,\alpha_{L,0})\rho(Y,W,\alpha_{L,0})^T\}M_J(X)^T\right] \\
= E_P\left[M_J(X)E[\{\rho(Y,W,\alpha)\rho(Y,W,\alpha)^T - \rho(Y,W,\alpha_{L,0})\rho(Y,W,\alpha_{L,0})^T\}|X]M_J(X)^T\right] \\
\leq \sup_x |\text{trace}\{E_P[\rho(Y,W,\alpha)\rho(Y,W,\alpha)^T - \rho(Y,W,\alpha_{L,0})\rho(Y,W,\alpha_{L,0})^T]|X = x]\}| \\
\times E_P\left[M_J(X)M_J(X)^T\right].
$$

Moreover,

$$
|\text{trace}\{E_P[\rho(Y,W,\alpha)\rho(Y,W,\alpha)^T - \rho(Y,W,\alpha_{L,0})\rho(Y,W,\alpha_{L,0})^T]|X = x]\}| \\
= |E_P[\rho_1(Y,W,\alpha)^2 - \rho_1(Y,W,\alpha_{L,0})^2]|X = x]\| \\
+ |E_P[\rho_2(Y,W,\alpha)^2 - \rho_2(Y,W,\alpha_{L,0})^2]|X = x]\| \\
= |E_P[(\theta - \mu(W)h'(W))^2 - (\theta_{L,0} - \mu(W)h'_{L,0}(W))^2]|X = x]\| \\
+ |(1 - 2\tau)|E_P[(1\{Y \leq h(W)\} - 1\{Y \leq h_{L,0}(W)\})]|X = x]\| \\
= |E_P[\theta^2 - \theta_{L,0}^2 - 2\theta\mu(W)h'(W) + (\mu(W)h'(W))^2 - (\mu(W)h'_{L,0}(W))^2 + 2\theta_{L,0}\mu(W)h'_{L,0}(W)]|X = x]\| \\
+ |(1 - 2\tau)|E_P[(F_{Y|W,X}(h(W)|W,X) - F_{Y|W,X}(h_{L,0}(W)|W,X))]|X = x]\|. 
$$
By assumption 2, $$|E_{\mathcal{P}}[(F_{Y|W,X}(h(W)|W,X) - F_{Y|W,X}(h_{L,0}(W)|W,X))|X = x]| \lesssim ||h - h_{L,0}||_{L^2(W,\mathcal{E})}.$$ Since $$E[M_f(X)M_f(X)^T] = I$$ by assumption 3, it follows that
$$||H_f(\alpha, \mathcal{P}) - H_f||_e = O(\bar{\delta}_{L,n} \times ||\alpha - \alpha_{L,0}||).$$

Proof of Lemma F.3. Observe that
$$\left\| \frac{1}{n} \sum_{i=1}^{n} g_f(Z_i, \alpha) - g_f(Z_i, \alpha_{L,0}) - E[g_f(Z, \alpha) - g_f(Z, \alpha_{L,0})] \right\|_e \leq \left\| \frac{1}{n} \sum_{i=1}^{n} \mu(W_i) \{h'(W_i) - h'_{L,0}(W_i)\} - E[\mu(W)\{h'(W) - h'_{L,0}(W)\}] \right\|_e + \left\| \frac{1}{n} \sum_{i=1}^{n} (1\{Y_i \leq h(W_i)\} - 1\{Y_i \leq h_{L,0}(W_i)\})q^f(X_i) - E[(1\{Y \leq h(W)\} - 1\{Y \leq h_{L,0}(W)\})q^f(X)] \right\|_e \leq \sqrt{\frac{J}{n}} \left( \sup_{(\theta, h) \in \mathcal{N}_{L,n}} \mathbb{E}_{n}[\mu \cdot (h' - h'_{L,0})] + \max_{1 \leq j \leq J} \sup_{g \in \mathcal{G}_{L,n}} \mathbb{E}_{n}[g \cdot q_j] \right).$$

Proof of Lemma F.4. By the triangle inequality and definition of $$u^*_L,n$$, it suffices to check that $$||\alpha - \alpha_{L,0}||_w + t \leq \eta_{w,L,n}$$, which holds since $$\eta_{w,L,n} \gtrsim l_n n^{-1/2}$$. Regarding the latter term, by the triangle inequality it suffices to check that $$||\alpha - \alpha_{L,0}|| + t||u^*_L,n|| \leq \eta_{L,n}$$. By Lemma F.1 and Assumption 3, $$||u^*_L,n|| \lesssim (e_{\min}(M^T_L M_L))^{-1}$$. By Assumption 8(iii), $$l_n n^{-1/2}(e_{\min}(M^T_L M_L))^{-1} = o(\eta_{L,n}).$$

It follows that by Assumption 8, $$\text{Pen}(\alpha + tu^*_L,n) \leq \text{Pen}(\alpha) + n^{-1}/\gamma_K$$. Since $$\alpha \in \bar{A}_{L,n}$$ and $$\Gamma_{L,n} \gtrsim n^{-1}$$, it follows that $$\alpha + tu^*_L,n \in \bar{A}_{L,n}.$$ 

Proof of Lemma F.5. Throughout the proof, let $$\alpha \in \{\hat{\alpha}_{L,n}, \hat{\alpha}''_{L,n}\}$$.

By the calculations in the proof of Lemma SM.II.3 and the fact that $$\alpha \in \bar{A}_{L,n}$$ wpa1 (for $$\alpha = \hat{\alpha}''_{L,n}$$; this follows from Assumption 8), $$||\lambda||_e \times ||E_{\mathcal{P}}[g_f(Z, \alpha)]||_e \lesssim (||\lambda||_e \bar{\delta} + ||\lambda||_e \bar{\delta}_{L,n} + b^\theta_{e,j} n ||\lambda||_e^2)$$. Since $$\lambda \in B(\delta_n)$$, under Assumption 12 (with $$\theta = 3$$), $$|\lambda^T g_f(z, \alpha)| = o(1)$$, since $$s''$$ is continuous, this implies condition 1.
Regarding Condition 3, note that

\[
||E_{P_n}[g_J(Z, \alpha)]||_e \leq ||E_{P_n}[g_J(Z, \alpha) - g_J(Z, \alpha_{L,0})] - E_P[g_J(Z, \alpha) - g_J(Z, \alpha_{L,0})]||_e \\
+ ||E_{P_n}[g_J(Z, \alpha_{L,0})]||_e + ||E_P[g_J(Z, \alpha) - g_J(Z, \alpha_{L,0})]||_e.
\]

By Lemma F.3 and Assumption 9, the first term in the RHS is of order $\sqrt{J/n}\Delta_{2,J,n}$. By the proof of Lemma 6.1, the third term is of order $\eta_{L,n}^2 b_{2,J} + \eta_w L,n$. Finally, by Lemma D.2, the second term if of order $\sqrt{\eta_{L,0}^2 / n} + ||E_P[g_J(Z, \alpha_{L,0})]||_e^2$. These results and assumption 12 imply the condition.

By the triangle inequality,

\[
||E_{P_n}[g_J(Z, \alpha)]||_e \leq ||E_{P_n}[g_J(Z, \alpha) - E_{P_n}[g_J(Z, \alpha_{L,0})] - \{E_P[g_J(Z, \alpha) - E_P[g_J(Z, \alpha_{L,0})]\}||_e \\
+ ||E_{P_n}[g_J(Z, \alpha_{L,0})]||_e + ||E_P[g_J(Z, \alpha) - g_J(Z, \alpha_{L,0})]||_e.
\]

The first term in the RHS is of order $O_P(\Delta_{2,L,n})$ by Assumption 9. By Lemma D.2, the second term is of order $O_P(\sqrt{\eta_{L,0}^2 / n} + ||E_P[g_J(Z, \alpha_{L,0})]||_e^2)$. Finally, by the proof of Lemma 6.1 the third term is of order $O_P(||\alpha - \alpha_{L,0}||_w + ||\alpha - \alpha_{L,0}||^2_{b_{2,J}})$.

Thus, since $\alpha \in \mathcal{N}_n$ wp1 (by Lemma F.4),

\[
||E_{P_n}[g_J(Z, \alpha)]||_e = O_P \left( \Delta_{2,L,n} + \sqrt{\frac{\theta + ||\mu h_{L,0}'||_{L^2(P)}^2 + b_{2,J}^2}{n}} + ||E_P[g_J(Z, \alpha_{L,0})]||_e^2 + \eta_w L,n + \eta_{L,n}^2 b_{2,J} \right)
\]

By Assumption 12(ii), this result implies Condition 4. Condition 5 holds by assumption.

\[\square\]

**Proof of Lemma F.6.** By Lemma D.3(i) (applied to $\alpha = \alpha_{L,0}$) it suffices to bound $||\Delta(\alpha)||_e$. By Lemma D.2, $||E_{P_n}[g_J(Z, \alpha_{L,0})]||_e = O_P \left( \sqrt{\frac{\eta_{L,0}^2}{n}} + ||E[g_J(Z, \alpha_{L,0})]||_e^2 \right)$.

Also, by Lemma D.3 (applied to $\alpha = \alpha_{L,0}$),

\[
||G(\alpha_{L,0})[\alpha - \alpha_{L,0}]||_e \preceq ||\alpha - \alpha_{L,0}||_w.
\]

\[\square\]

**Proof of Lemma F.7.** By Lemma F.4, $\tilde{\alpha}_{L,n} - tu_{L,n}^* \in \mathcal{N}_{L,n}$ for all $t = O(l_n^{-1/2})$. By the same
calculations of Step 1 in the proof of lemma 6.2 and definition of \( \hat{\alpha}_{L,n} \), for any \( t = O(l_n n^{-1/2}) \).

\[
0 \leq \left\{ (\Delta(\hat{\alpha}_{L,n} + tu_{L,n}^*))^TH^{-1}_L(\Delta(\hat{\alpha}_{L,n} + tu_{L,n}^*)) \right\} - \left\{ (\Delta(\hat{\alpha}_{L,n}))^TH^{-1}_L(\Delta(\hat{\alpha}_{L,n})) \right\} + \text{rem}_n
\]

\[
= \{ \bar{t}^2 - 2t(\Delta(\hat{\alpha}_{L,n})^TH^{-1}_L(G(\alpha_{L,0})[u_{L,n}^*])) \} + \text{rem}_n,
\]

with \( \text{rem}_n = o_P(n^{-1}) \). Taking \( t = \pm \sqrt{\text{rem}_n} \), it follows that

\[
\left| (\Delta(\hat{\alpha}_{L,n})^TH^{-1}_L(G(\alpha_{L,0})[u_{L,n}^*])) \right| \lesssim \sqrt{\text{rem}_n}.
\]

Since \( \Delta(\hat{\alpha}_{L,n})^TH^{-1}_L(G(\alpha_{L,0})[u_{L,n}^*]) = \langle u_{L,n}^*, \hat{\alpha}_{L,n} - \alpha_{L,0} \rangle w + n^{-1} \sum_{i=1}^n (G(\alpha_{L,0})[u_{L,n}^*])^TH^{-1}_Lg_J(Z_i, \alpha_{L,0}) \), the desired result follows.

\( \square \)

**Proof of Lemma F.8.** Let \( \zeta_{L,n} \equiv (G(\alpha_{L,0})[u_{L,n}^*])^TH^{-1}_L\{g_J(Z_i, \alpha_{L,0}) - E[g_J(Z, \alpha_{L,0})] \} \). It is clear that \( E[\zeta_{L,n}] = 0 \) and \( \text{Var}(\zeta_{L,n}) = 1 \), so to show asymptotic normality it suffices to show that the Lyapounov condition holds.

By Cauchy-Swarchz inequality and definition of \( u_{L,n}^* \), for any \( \varrho > 0 \),

\[
E_P \left[ |\zeta_{L,n}|^{2+\varrho} \right] \leq E_P \left[ ||u_{n}^*||_{w}^{2+\varrho} ||g_J(Z, \alpha_{L,0}) - E[g_J(Z, \alpha_{L,0})]||_{e}^{2+\varrho} \right]
\]

\[
= E_P \left[ ||g_J(Z, \alpha_{L,0}) - E[g_J(Z, \alpha_{L,0})]||_{e}^{2+\varrho} \right]
\]

\[
\lesssim E_P \left[ ||g_J(Z, \alpha_{L,0})||_{e}^{2+\varrho} \right].
\]

By the calculations in the proof of Lemmas SM.II.2 and SM.II.3, \( ||g_J(Z, \alpha_{L,0})||_{e} \leq \overline{\vartheta} + ||h_{L,0}'||_{L_{\infty}(\mathcal{W}, \mu)} + ||q'(X)||_{e}. \) Thus

\[
E_P \left[ |\zeta_{L,n}|^{2+\varrho} \right] \lesssim (\overline{\vartheta} + ||h_{L,0}'||_{L_{\infty}(\mathcal{W}, \mu)})^{2+\varrho} + b_{2+\varrho, J}'.
\]

By Assumption 12, \( b_{2+\varrho, J}'/n = o(1) \) and \( (\overline{\vartheta} + ||h_{L,0}'||_{L_{\infty}(\mathcal{W}, \mu)})^{2+\varrho}/n^{2+\varrho} = o(1) \) for some \( \varrho > 0 \). Thus, the Lyapounov condition holds.

\( \square \)