Logarithmic Hölder continuous mappings and Beltrami equation

Evgeny Sevost’yanov1,2 · Sergei Skvortsov1

Received: 22 January 2021 / Revised: 27 June 2021 / Accepted: 28 June 2021
Published online: 9 July 2021
© The Author(s), under exclusive licence to Springer Nature Switzerland AG 2021

Abstract
The paper is devoted to the study of mappings satisfying the inverse Poletsky inequality. We study the local behavior of these mappings. We are most interested in the case when the corresponding majorant is integrable on some set of spheres of positive linear measure. Our main result is a logarithmic Hölder continuity of such mappings at inner points. As a corollary, we have established the existence of a continuous $AC L$-solution of the Beltrami equation, which is logarithmic Hölder continuous.

Mathematics Subject Classification Primary 30C65; Secondary 31A15 · 31B25

1 Introduction

As known, mappings $f : D \to \mathbb{R}^n$, $n \geq 2$, with bounded distortion satisfy the inequality

$$M(\Gamma) \leq N(f, A) \cdot K \cdot M(f(\Gamma)), \quad (1.1)$$

where $M$ is the modulus of a family of paths $\Gamma$ in $A$, $N(y, f, A) = \text{card} \{x \in A : f(x) = y\}$, $N(f, A) = \sup_{y \in \mathbb{R}^n} N(y, f, A)$, $A$ is a Borel set in a domain $D$, and $K \geq 1$ is some constant defined as $K = \text{ess sup} K_O(x, f)$. Here we use the common notation $K_O(x, f) = \|f'(x)\|^n / J(x, f)$ for $J(x, f) \neq 0$; $K_O(x, f) = 1$ for $f'(x) = 0$, and $K_O(x, f) = \infty$ for $f'(x) \neq 0$ and $J(x, f) = 0$. In addition, regarding relation (1.1), we could point to the following papers, see e.g., [14, Theorem 3.2] or

---

1 Zhytomyr Ivan Franko State University, 40 Bol’shaya Berdichevskaya Str., 10 008 Zhytomyr, Ukraine
2 Institute of Applied Mathematics and Mechanics of NAS of Ukraine, 1 Dobrovol’skogo Str., 84 100 Slavyansk, Ukraine
Theorem 1.1, Lemma 4.2, Theorem 4.1]). Some similar condition applies to mappings that have unbounded outer dilatations. In particular, the relation

\[ M(\Gamma) \leq \int_{f(E)} K_I \left( y, f^{-1}, E \right) \cdot \rho^n_\ast(y) \, dm(y), \quad (1.2) \]

holds for the so-called mappings \( f : D \to \mathbb{R}^n, n \geq 2 \), with a finite length distortion, where \( E \) is an arbitrary measurable subset of \( D \), \( \Gamma \) is a family of paths in \( E \) and \( \rho_\ast(y) \in \text{adm} \, f(\Gamma) \) (see, e.g., [18, Theorem 8.5]). In this manuscript, the main object of the study are mappings which satisfy some more general inequality than (1.2). The words “more general” should be understood in the sense that all mappings satisfying condition (1.2) satisfy some other condition for special families of paths and densities, but a more abstract Lebesgue measurable function \( Q \) in place of \( K_I \), however, this condition, generally speaking, does not imply the inequality (1.2). More precisely, in relation (1.2), we will assume that there is a more abstract Lebesgue measurable function \( Q \), and we will try to connect to the behaviour of this function all the problems related to Hölder continuity which are studied in this article. For estimates of Hölder type for many well-known classes of mappings, such as mappings with bounded distortion and quasiconformal mappings, see, e.g., [6, Theorem 1.1], [7, Theorem 5], [12, Theorem 3.2.II], [26, Theorem 1.1.2], [35, Theorem 18.2, Remark 18.4] and [15, Theorem 3.2]. Regarding the more general Hölder logarithmic continuity, we point out articles and monographs [34, Theorems 1.1.V and 2.1.V], [18, Theorem 7.4], [19, Theorem 3.1] and [24, Theorem 5.11], cf. [3, Theorems 4 and 5]. We emphasize that, under some very special restrictions, such mappings are even Lipschitz, but this fact is very rare for mappings of such a general nature (see, e.g., [23, Theorem 1.1, Lemma 4.2, Theorem 4.1]).

In what follows, \( D \) is a domain in \( \mathbb{R}^n, n \geq 2 \), \( M \) denotes the \( n \)-modulus of a family of paths, see [35], and the element \( dm(x) \) corresponds to a Lebesgue measure in \( \mathbb{R}^n \). For the sets \( A, B \subset \mathbb{R}^n \) we set, as usual,

\[ \text{diam} \, A = \sup_{x, y \in A} |x - y|, \quad \text{dist} \, (A, B) = \inf_{x \in A, y \in B} |x - y|. \]

Sometimes, instead of \( \text{dist} \, (A, B) \), we also write \( d(A, B) \), if a misunderstanding is impossible. Given sets \( E \) and \( F \) and a given domain \( D \) in \( \mathbb{R}^n = \mathbb{R}^n \cup \{ \infty \} \), we denote by \( \Gamma(E, F, D) \) the family of all paths \( \gamma : [0, 1] \to \mathbb{R}^n \) joining \( E \) and \( F \) in \( D \), that is, \( \gamma(0) \in E, \gamma(1) \in F \) and \( \gamma(t) \in D \) for all \( t \in (0, 1) \). Everywhere below, unless otherwise stated, the boundary and the closure of a set are understood in the sense of the extended Euclidean space \( \overline{\mathbb{R}}^n \). Let \( x_0 \in \overline{D}, x_0 \neq \infty \).

\[
\begin{align*}
B(x_0, r) &= \{ x \in \mathbb{R}^n : |x - x_0| < r \}, \quad \mathbb{B}^n = B(0, 1), \\
S(x_0, r) &= \{ x \in \mathbb{R}^n : |x - x_0| = r \}, S_i = S(x_0, r_i), \quad i = 1, 2, \\
A &= A(x_0, r_1, r_2) = \{ x \in \mathbb{R}^n : r_1 < |x - x_0| < r_2 \}.
\end{align*}
\quad (1.3)
\]
Let $f : D \to \mathbb{R}^n$, $n \geq 2$, and let $Q : \mathbb{R}^n \to [0, \infty)$ be a Lebesgue measurable function such that $Q(y) \equiv 0$ for $y \in \mathbb{R}^n \setminus f(D)$. Let $A = A(y_0, r_1, r_2)$. Let $\Gamma_f(y_0, r_1, r_2)$ denotes the family of all paths $\gamma : [a, b] \to D$ such that $f(\gamma) \in \Gamma(S(y_0, r_1), S(y_0, r_2), A(y_0, r_1, r_2))$, i.e., $f(\gamma(a)) \in S(y_0, r_1)$, $f(\gamma(b)) \in S(y_0, r_2)$, and $f(\gamma(t)) \in A(y_0, r_1, r_2)$ for any $a < t < b$. We say that $f$ satisfies the inverse Poletsky inequality at $y_0 \in f(D)$ if the relation

$$M(\Gamma_f(y_0, r_1, r_2)) \leq \int_A Q(y) \cdot \eta^n(|y - y_0|) \, dm(y)$$

holds for any $0 < r_1 < r_2 < r_0 := \sup_{y \in D'} |y - y_0|$ and any Lebesgue measurable function $\eta : (r_1, r_2) \to [0, \infty]$ such that

$$\int_{r_1}^{r_2} \eta(r) \, dr \geq 1. \quad (1.5)$$

Note that the first author established the openness and discreteness of mappings in (1.4) under certain conditions on the function $Q$, see, e.g., [31]. In a more general case, the validity of these properties is not guaranteed. Note also that the equicontinuity of homeomorphisms with condition (1.4) with somewhat less general constraints on the domain and the corresponding mappings is studied in detail in [32]. In this manuscript, we focus on mappings with branching.

We now formulate the main results of this article. To this end, we recall a few more definitions. A mapping $f : D \to \mathbb{R}^n$ is called discrete if the preimage $\{f^{-1}(y)\}$ of each point $y \in \mathbb{R}^n$ consist of isolated points, and open if the image of any open set $U \subset D$ is an open set in $\mathbb{R}^n$. In the extended Euclidean space $\overline{\mathbb{R}}^n = \mathbb{R}^n \cup \{\infty\}$, we use the chordal metric $h$ defined by the equalities

$$h(x, y) = \frac{|x - y|}{\sqrt{1 + |x|^2} \sqrt{1 + |y|^2}}, \quad x \neq \infty \neq y, \quad h(x, \infty) = \frac{1}{\sqrt{1 + |x|^2}}. \quad (1.6)$$

For a given set $E \subset \overline{\mathbb{R}}^n$, we set

$$h(E) := \sup_{x, y \in E} h(x, y). \quad (1.7)$$

The quantity $h(E)$ in (1.7) is called the chordal diameter of the set $E$.

For given sets $A, B \subset \overline{\mathbb{R}}^n$, we put $h(A, B) = \inf_{x \in A, y \in B} h(x, y)$, where $h$ is a chordal metric defined in (1.6). Note that the article considers three different classes of mappings satisfying the inverse Poletsky inequality: $\mathcal{R}_Q^*(D, D')$, $\mathcal{R}_Q(D, D')$ and $\mathcal{F}_Q(D)$. Accordingly, $\mathcal{R}_Q^*(D, D')$ are homeomorphisms of a domain $D'$ onto a domain $D$, $\mathcal{R}_Q(D, D')$ are open discrete mappings with between fixed domains $D$ and $D'$, and $\mathcal{F}_Q(D)$ are open discrete mappings of some domain $D$, whose image can change. The following text provides a detailed definition of each of these classes.
For domains $D, D' \subset \mathbb{R}^n, n \geq 2$, and a Lebesgue measurable function $Q : \mathbb{R}^n \to [0, \infty]$ equal to zero outside the domain $D'$, we define by $\mathcal{R}_Q(D, D')$ the family of all open discrete mappings $f : D \to D'$ such that relation (1.4) holds for each point $y_0 \in D'$. Note that the definition of class $\mathcal{R}_Q(D, D')$ does not require that the domain $D$ be mapped onto the domain $D'$ surjectively under the mapping $f \in \mathcal{R}_Q(D, D')$. In what follows, $\mathcal{H}^{n-1}$ denotes $(n - 1)$-dimensional Hausdorff measure.

**Theorem 1.1** Let $D$ and $D'$ be domains in $\mathbb{R}^n, n \geq 2$, and let $D'$ be a bounded domain. Suppose that, for each point $y_0 \in D'$ and for every $0 < r_1 < r_2 < r_0 := \sup_{y \in D'} |y - y_0|$ there is a set $E \subset [r_1, r_2]$ of a positive linear Lebesgue measure such that the function $Q$ is integrable with respect to $\mathcal{H}^{n-1}$ over the spheres $S(y_0, r)$ for every $r \in E$. Then the family of mappings $\mathcal{R}_Q(D, D')$ is equicontinuous at each point $x_0 \in D$.

Theorem 1.1 generalizes the result of [32, Theorem 1.5] to the case when mappings may have branch points, and the corresponding function $Q$ may turn out to be non-integrable in the considered domain $D$. In particular, Theorem 1.1 implies the following obvious.

**Corollary 1.1** Assume that under the conditions of Theorem 1.1

$$\int_{S(y_0, r)} Q(y) d\mathcal{H}^{n-1}(y) < \infty$$

for almost all $r \in (0, r_0)$ and any $y_0 \in D'$. Then the family of mappings $\mathcal{R}_Q(D, D')$ is equicontinuous at each point $x_0 \in D$.

For a domain $D \subset \mathbb{R}^n, n \geq 2$, and a Lebesgue measurable function $Q : \mathbb{R}^n \to [0, \infty], Q(y) \equiv 0$ for $y \in \mathbb{R}^n \setminus f(D)$, we denote by $\mathcal{F}_Q(D)$ the family of all open discrete mappings $f : D \to \mathbb{R}^n$ such that relation (1.4) holds for each point $y_0 \in f(D)$. In the case where the function $Q$ behaves somewhat better, and the domain $D$ is the unit ball, the following result holds.

**Theorem 1.2** Let $n \geq 2$, and let $Q \in L^1(\mathbb{R}^n)$. Suppose that $K$ is a compact set in $\mathbb{B}^n$, where $\mathbb{B}^n$ is defined by (1.3). Then the inequality

$$|f(x) - f(y)| \leq \frac{C_n \cdot \|Q\|_1^{1/n}}{\log^{1/n} \left(1 + \frac{r_0}{2|x-y|}\right)}$$

holds for any $x, y \in K$ and $f \in \mathcal{F}_Q(\mathbb{B}^n)$, where $\|Q\|_1$ denotes the $L^1$-norm of $Q$ in $\mathbb{R}^n$, $C_n > 0$ is some constant depending only on $n$, and $r_0 = d(K, \partial \mathbb{B}^n)$.

A separate case of Theorems 1.1 and 1.2 and also the Corollary 1.1 is a situation when $f$ is a homeomorphism in $D$. In this case, let us denote $g := f^{-1}$ and remark that

$$g(\Gamma(S(y_0, r_1), S(y_0, r_2), f(D))) = \Gamma_f(y_0, r_1, r_2).$$
In fact, if \( \gamma \in g(\Gamma(S(y_0, r_1), S(y_0, r_2), f(D))) \), then \( \gamma : [a, b] \to \mathbb{R}^n \), and \( \gamma = g \circ \alpha \), \( \alpha : [a, b] \to \mathbb{R}^n \) and \( \alpha(a) \in S(y_0, r_1), \alpha(b) \in S(y_0, r_2), \alpha(t) \in f(D) \) for \( a \leq t \leq b \). Now, \( \gamma(t) \in D \) for \( a \leq t \leq b \) and \( f(\gamma) = \alpha \in \Gamma(S(y_0, r_1), S(y_0, r_2), f(D)) \), i.e., \( \gamma \in \Gamma_f(y_0, r_1, r_2) \). Thus, \( g(\Gamma(S(y_0, r_1), S(y_0, r_2), f(D))) \subset \Gamma_f(y_0, r_1, r_2) \). The inverse inclusion is proved similarly.

For domains \( D, D' \subset \mathbb{R}^n, n \geq 2 \), and a Lebesgue measurable function \( Q : \mathbb{R}^n \to [0, \infty] \) equal to zero outside the domain \( D \), we denote by \( \mathcal{R}_Q^*(D, D') \) the family of all homeomorphisms \( g \) of \( D' \) onto \( D \) such that relation

\[
M(f(\Gamma(S(x_0, r_1), S(x_0, r_2), D))) \leq \int_A Q(x) \cdot \eta^n(|x - x_0|) \, dm(x) \tag{1.10}
\]

holds for \( f = g^{-1} \), each \( x_0 \in D \), any \( 0 < r_1 < r_2 < d_0 = \sup_{x \in D} |x - x_0| \) and any Lebesgue measurable function \( \eta : (r_1, r_2) \to [0, \infty] \) with the condition (1.5). Taking into account Theorems 1.1, 1.2 and Corollary 1.1 as well as relation (1.9), we obtain the following statements, which were also obtained for the first time in this paper.

**Corollary 1.2** Let \( D \) and \( D' \) be domains in \( \mathbb{R}^n, n \geq 2 \), and let \( D \) be a bounded domain. Suppose that \( Q : \mathbb{R}^n \to [0, \infty] \) is a Lebesgue measurable function and for each point \( x_0 \in D \) and for every \( 0 < r_1 < r_2 < d_0 := \sup_{x \in D} |x - x_0| \) there is a set \( E \subset (r_1, r_2) \) of positive linear Lebesgue measure such that the function \( Q \) is integrable over the spheres \( S(x_0, r) \) for every \( r \in E \). Then the family of mappings \( \mathcal{R}_Q^*(D, D') \) is equicontinuous at each point \( y_0 \) of \( D' \).

**Corollary 1.3** Let \( n \geq 2 \) and let \( Q \in L^1(\mathbb{R}^n) \). Suppose that \( K \) is a compact set in \( \mathbb{R}^n \). Then the inequality

\[
|g(x) - g(y)| \leq \frac{C_n \cdot (\|Q\|_1)^{1/n}}{\log^{1/n} \left(1 + \frac{r_0}{2|x - y|}\right)}
\]

holds for every \( x, y \in K \) and every \( g \in \mathcal{R}_Q^*(D, \mathbb{R}^n) \), where \( \|Q\|_1 \) denotes \( L^1 \)-norm of \( Q \) in \( \mathbb{R}^n \), \( C_n > 0 \) is some constant depending only on \( n \), and \( r_0 = d(K, \partial \mathbb{R}^n) \).

## 2 The case when a majorant is integrable over some set of spheres

Let us formulate, first of all, a simple but very important topological statement, which will be used repeatedly in what follows (see, e.g., [11; Theorem 1.15.46]).

**Proposition 2.1** Let \( A \) be a set in a topological space \( X \). If the set \( C \) is connected and \( C \cap A \neq \emptyset \neq C \setminus A \), then \( C \cap \partial A \neq \emptyset \).

Let \( D \subset \mathbb{R}^n, f : D \to \mathbb{R}^n \) be a discrete open mapping, \( \beta : [a, b) \to \mathbb{R}^n \) be a path, and \( x \in f^{-1}(\beta(a)) \). A path \( \alpha : [a, c) \to D \) is called a maximal \( f \)-lifting of \( \beta \) starting at \( x \), if (1) \( \alpha(a) = x \); (2) \( f \circ \alpha = \beta|_{[a, c)} \); (3) for \( c < c' \leq b \), there is no a path \( \alpha' : [a, c') \to D \) such that \( \alpha = \alpha'|_{[a, c)} \) and \( f \circ \alpha' = \beta|_{[a, c')} \). Similarly, we
may define a maximal \( f \)-lifting \( \alpha : (c, b] \to D \) of a path \( \beta : (a, b] \to \mathbb{R}^n \) ending at \( x \in f^{-1}(\beta(b)) \). The following assertion holds (see [16, Lemma 3.12]).

**Proposition 2.2** Let \( f : D \to \mathbb{R}^n, n \geq 2 \), be an open discrete mapping, let \( x_0 \in D \), and let \( \beta : (a, b) \to \mathbb{R}^n \) be a path such that \( \beta(a) = f(x_0) \) and such that either \( \lim_{t \to b} \beta(t) \) exists, or \( \beta(t) \to \partial f(D) \) as \( t \to b \). Then \( \beta \) has a maximal \( f \)-lifting \( \alpha : [a, c) \to D \) starting at \( x_0 \). If \( \alpha(t) \to x_1 \in D \) as \( t \to c \), then \( c = b \) and \( f(x_1) = \lim_{t \to b} \beta(t) \). Otherwise \( \alpha(t) \to \partial D \) as \( t \to c \).

**Proof of Theorem 1.1** We prove Theorem 1.1 by contradiction. Suppose that the conclusion of this theorem does not hold, that is, the family of maps \( \mathcal{R}_Q(D, D') \) is not equicontinuous at some point \( x_0 \in D \). Then there exists \( \varepsilon_0 > 0 \) such that for each \( m \in \mathbb{N} \) there are \( x_m \in D \) and a mapping \( f_m \in \mathcal{R}_Q(D, D') \), such that \( |x_m - x_0| < 1/m \), however,

\[
|f_m(x_m) - f_m(x_0)| \geq \varepsilon_0. \tag{2.1}
\]

Consider a straight line \( r = r_m(t) = f_m(x_0) + (f_m(x_m) - f_m(x_0))t, -\infty < t < \infty \), passing through points \( f_m(x_m) \) and \( f_m(x_0) \), see Fig. 1. Since the domain \( D' \) is bounded, there are points \( x_1^m \) and \( x_2^m \) in the intersection of the line \( r_m \) with the boundary of the domain \( D' \), see Proposition 2.1. We may assume that \( x_1^m \) and \( x_2^m \) are located on opposite sides of the segment \( f_m(x_0) f_m(x_m) \). Let \( \gamma_m^1 : [1, c_m) \to D, 1 < c_m \leq \infty \), be a maximal \( f_m \)-lifting of \( r = r_m(t), t \geq 1 \), starting at \( x_m \), existing by Proposition 2.2.

Let us to prove that the situation when \( \gamma_m^1 \to x_1 \in D \) as \( t \to c_m \) is impossible. Indeed, in this case, by Proposition 2.2 \( c_m = \infty \) and \( f_m(x_1) = \lim_{t \to +\infty} r_m(t) \). Then, on the one hand, \( f_m(x_1) \in f_m(D) \) due to the openness of \( f_m \), and on the other hand, \( f_m(x_1) = \infty \) due to the definition of \( r = r_m(t) \). Since \( \infty \notin f_m(D) \), we obtain a contradiction. Thus, the case \( \gamma_m^1(t) \to x_1 \in D \) as \( t \to c_m \) is impossible, as required.

By Proposition 2.2

\[
h(\gamma_m^1(t), \partial D) \to 0 \tag{2.2}
\]

as \( t \to c_m - 0 \). Similarly, let \( \gamma_m^2 : (d_m, 0] \to D, -\infty \leq d_m < 0 \) be a maximal \( f_m \)-lifting of \( r = r_m(t), t \leq 0 \), ending at \( x_0 \), which exists by Proposition 2.2. As in (2.2), we have that

\[
h(\gamma_m^2(t), \partial D) \to 0 \tag{2.3}
\]

as \( t \to d_m + 0 \). By (2.2) and (2.3), there exist \( 1 \leq t_{1m} < c_m \) and \( d_m \leq t_{2m}^2 < 0 \) such that

\[
h(z_1^m, \partial D) < 1/m \quad \text{and} \quad h(z_2^m, \partial D) < 1/m \quad \text{where} \quad \gamma_m(t_{1m}^1) = z_1^m \quad \text{and} \quad \gamma_m(t_{2m}^2) = z_2^m.
\]

Set \( P_m = \gamma_m^1_{[1, t_{1m}^1]}, Q_m = \gamma_m^2_{[t_{2m}^2, 0]} \) and \( \Gamma_m := \Gamma(P_m, Q_m, D) \). Since the inner points of any domain are weakly flat (see, e.g., [32, Lemma 2.2]), we obtain that

\[
M(\Gamma_m) = M(\Gamma(P_m, Q_m, D)) \to \infty, \quad m \to \infty. \tag{2.4}
\]

We show that condition (2.4) leads to a contradiction with the definition of the class of mappings \( \mathcal{R}_Q(D, D') \) in (1.4). We denote

\[
a_m := |f_m(z_1^m) - f_m(x_m)|, \quad b_m = |f_m(z_1^m) - f_m(x_0)|.
\]
Obviously, by construction

\[ \begin{align*}
  f_m(\mathcal{P}_m) &\subset B(f_m(z_1^m), a_m), \\
  f_m(\mathcal{Q}_m) &\subset \mathbb{R}^n \setminus B(f_m(z_1^m), b_m).
\end{align*} \tag{2.5} \]

Since the domain \( D' \) is bounded, we may assume that all three sequences \( f_m(z_1^m) \), \( f_m(x_0) \) and \( f_m(x_m) \) converge to some elements \( \widetilde{x}, \widetilde{x}_1 \) and \( \widetilde{x}_2 \) at \( m \to \infty \), respectively. Now \( a_m \to a_0 \) and \( b_m \to b_0 \) as \( m \to \infty \), where \( a_0 = |\widetilde{x} - \widetilde{x}_2| \) and \( b_0 = |\widetilde{x} - \widetilde{x}_1| \).

Fix \( 0 < \varepsilon < \varepsilon_0/2 \). \tag{2.6}

Let \( x^* \in D' \) be such that \(|x^* - \widetilde{x}| < \varepsilon/3\). Additionally, let \( m_0 \in \mathbb{N} \) be such that

\[ |f_m(z_1^m) - \widetilde{x}| < \varepsilon/3, \quad a_0 - \varepsilon/3 < a_m < a_0 + \varepsilon/3 \quad \forall \ m > m_0. \tag{2.7} \]

Let \( x \in B(f_m(z_1^m), a_m) \). Now, by (2.7) and the triangle inequality we obtain that

\[ |x - x^*| \leq |x - f_m(z_1^m)| + |f_m(z_1^m) - \widetilde{x}| + |\widetilde{x} - x^*| \]

\[ < a_m + \varepsilon/3 + \varepsilon/3 < a_0 + \varepsilon, \quad m > m_0. \tag{2.8} \]

It follows from (2.8) that

\[ B(f_m(z_1^m), a_m) \subset B(x^*, a_0 + \varepsilon), \quad m > m_0. \tag{2.9} \]

Let \( R_0 \) be such that

\[ a_0 + \varepsilon < R_0 < a_0 + \varepsilon_0 - \varepsilon. \tag{2.10} \]

Note that the choice of the number \( R_0 \) in the formula (2.10) is possible by the definition of the number \( \varepsilon \) in (2.6). Let \( y \in B(x^*, R_0) \). Again, by the triangle inequality, and equations (2.7), (2.10) we obtain that

\[ |y - f_m(z_1^m)| \leq |y - x^*| + |x^* - \widetilde{x}| + |\widetilde{x} - f_m(z_1^m)| \]

\[ < R_0 + 2\varepsilon/3 < a_0 + \varepsilon_0 - \varepsilon + 2\varepsilon/3 \]

\[ = a_0 + \varepsilon_0 - \varepsilon/3 < a_m + \varepsilon_0 \leq b_m. \tag{2.11} \]

It follows from (2.11) that

\[ B(x^*, R_0) \subset B(f_m(z_1^m), b_m), \quad m > m_0. \tag{2.12} \]

Recall that, a family of paths \( \Gamma_1 \) in \( \mathbb{R}^n \) is said to be minorized by a family of paths \( \Gamma_2 \) in \( \mathbb{R}^n \), abbr. \( \Gamma_1 > \Gamma_2 \), if, for every path \( \gamma_1 \in \Gamma_1 \), there is a path \( \gamma_2 \in \Gamma_2 \) such that \( \gamma_2 \) is a restriction of \( \gamma_1 \). In this case,

\[ \Gamma_1 > \Gamma_2 \quad \Rightarrow \quad M(\Gamma_1) \leq M(\Gamma_2) \tag{2.13} \]
(see [5, Theorem 1]). Taking into account relations (2.5), (2.9) and (2.12), as well as Proposition 2.1, we obtain that
\[
\Gamma_m > \Gamma_{f_m}(x^*, a_0 + \varepsilon, R_0), \quad m > m_0.
\] (2.14)
In turn, from relation (2.14), as well as from the definition of the class \(R_Q(D, D')\), it follows that
\[
M(\Gamma_m) \leq M(\Gamma_{f_m}(x^*, a_0 + \varepsilon, R_0)) \leq \int_A Q(y) \cdot \eta^n(|y - x^*|) \, dm(y), \quad m > m_0,
\] (2.15)
where \(A = A(x^*, a_0 + \varepsilon, R_0)\), and \(\eta\) is an arbitrary non-negative Lebesgue measurable function satisfying condition (1.5) for \(r_1 = a_0 + \varepsilon\) and \(r_2 = R_0\). Below we use the standard conventions \(a/\infty = 0\) for \(a \neq \infty\) and \(a/0 = \infty\) if \(a > 0\) and \(0 \cdot \infty = 0\) (see e.g. [30, 3.1]). Put now
\[
I = \int_{a_0 + \varepsilon}^{R_0} \frac{dt}{t q_{x^*}^{1/(n-1)}(t)},
\] (2.16)
where
\[
q_{y_0}(r) = \frac{1}{\omega_{n-1} r^{n-1}} \int_{S(y_0, r)} Q(y) \, dH^{n-1}(y),
\] (2.17)
\(\omega_{n-1}\) is the area of the unit sphere \(S^{n-1}\) in \(\mathbb{R}^n\), and \(q_{x^*}(t)\) is defined in (2.17) for \(y_0 := x^*\). By the hypothesis of the theorem, there is a set \(E \subset [a_0 + \varepsilon, R_0]\) of positive linear measure such that the function \(q_{x^*}(t)\) is finite for \(t \in E\). Thus, \(I \neq 0\) in (2.16). In this case, the function \(\eta_0(t) = \frac{1}{I q_{x^*}^{1/(n-1)}(t)}\) satisfies (1.5) for \(r_1 = a_0 + \varepsilon\) and \(r_2 = R_0\). Substituting this function on the right side of the inequality (2.15) and applying Fubini’s theorem, we obtain
\[
M(\Gamma_m) \leq \frac{\omega_{n-1}}{In^{-1}} < \infty, \quad m > m_0,
\] (2.18)
where \(\omega_{n-1}\) is the area of the unit sphere \(S^{n-1}\) in \(\mathbb{R}^n\). However, relation (2.18) contradicts (2.4). The obtained contradiction refutes the assumption made in (2.1). \(\Box\)

To illustrate Theorem 1.1, we consider the following examples.

**Example 1** This example is devoted to the study of the case when the result of Theorem 1.1 (Corollary 1.2) can be applied to some family of mappings, although the corresponding function \(Q\) is not integrable in the domain under consideration. In this case, however, the so-called Lehto type integral diverges for the function \(Q\) (see e.g. [18, (7.50)]). We consider the following function \(\varphi : (0, 1] \rightarrow \mathbb{R}\), defined as follows:
\[
\varphi(t) = \begin{cases} 
1, & t \in \left(\frac{1}{2k+1}, \frac{1}{2k}\right), k = 1, 2, \ldots, \\
\frac{1}{t^n}, & t \in \left[\frac{1}{2k}, \frac{1}{2k-1}\right], k = 1, 2, \ldots,
\end{cases}
\]
As usual, put
\[ q_0(t) = \frac{1}{\omega_{n-1} t^{n-1}} \int_{S(0,t)} Q(x) \, d\mathcal{H}^{n-1}(x). \]

Using the Fubini theorem, as well as the countable additivity of the Lebesgue integral, we will have:
\[
\int_{\mathbb{B}^n} Q(x) \, dm(x) = \int_0^1 \int_{S(0,r)} Q(x) \, d\mathcal{H}^{n-1}(x) \, dr
\]
\[
= \omega_{n-1} \int_0^1 r^{n-1} \varphi(r) \, dr \geq \omega_{n-1} \sum_{k=1}^{\infty} \int_{1/(2k)}^{1/(2k-1)} \frac{dr}{r} = \omega_{n-1} \sum_{k=1}^{\infty} \ln \left( \frac{2k}{2k-1} \right). \]

(2.20)

Note that the series on the right-hand side of (2.20) diverges. Indeed, by virtue of the Lagrange’s mean value theorem \( \ln \left( \frac{2k}{2k-1} \right) = \ln(2k) - \ln(2k-1) = \frac{1}{\theta(k)} \geq \frac{1}{2k} \), where \( \theta(k) \in [2k-1, 2k] \). Since \( \sum_{k=1}^{\infty} \frac{1}{2k} = \infty \), we obtain that \( \sum_{k=1}^{\infty} \ln \left( \frac{2k}{2k-1} \right) = \infty \) and consequently,
\[ \int_{\mathbb{B}^n} Q(x) \, dm(x) = \infty. \]

On the other hand,
\[ \int_0^1 \frac{dt}{t q_0^{1/(n-1)}(t)} = \sum_{k=1}^{\infty} \frac{1/(2k)}{r^k} = \sum_{k=1}^{\infty} \ln \frac{2k+1}{2k} = \infty. \tag{2.21} \]

Define a sequence of mappings \( f_m : \mathbb{B}^n \to \mathbb{R}^n \) by
\[ f_m(x) = \frac{x}{|x|} \rho_m(|x|), \quad f_m(0) := 0, \]
where
\[
\rho_m(r) = \exp \left\{ -\frac{1}{r} \int_{tq_0^{1/(n-1)}(t)} dt \right\}, \quad q_{0,m}(r) := \frac{1}{\omega_{n-1} r^{n-1}} \int_{|x|=r} Q_m(x) \, dS,
\]

\[
Q_m(x) = \begin{cases} 
Q(x), & |x| > 1/m, \\
1, & |x| \leq 1/m.
\end{cases}
\]

Note that each of the mappings \( f_m \) is a homeomorphism of the unit ball \( \mathbb{B}^n \) onto itself. Now we show that every \( f_m, m = 1, 2, \ldots, \) satisfies the relation (1.10) where the function \( Q \) defined by (2.19). First of all, we note that each mapping \( f_m \) belongs to the class \( ACL \); moreover, their norm and Jacobian are calculated by the relations
\[
\| f_m'(x) \| = \frac{\exp \left\{ -\frac{1}{|x|} \int_{tq_0^{1/(n-1)}(t)} dt \right\}}{|x|}, \quad |J(x, f_m)| = \frac{\exp \left\{ -n \frac{1}{|x|^n} t q_0^{1/(n-1)}(t) \right\}}{|x|^n \omega_1^{1/(n-1)}(|x|)},
\]
see [10, Proof of Theorem 5.2]. Thus, \( f_m \in W^{1,n}_{loc}(\mathbb{B}^n \setminus 0) \). Set
\[
K_I(x, f) = \begin{cases} 
\frac{|J(x, f)|}{l(f'(x))}, & J(x, f) \neq 0, \\
1, & f'(x) = 0, \\
\infty, & \text{otherwise}
\end{cases}
\]
where \( l(f'(x)) = \inf_{|h|=1} |f'(x)h| \). Observe that the inner dilatation \( K_I(x, f_m) \) of \( f_m \) at the point \( x \) is calculated as \( K_I(x, f_m) = q_{0,m}(|x|) \leq q_0(|x|) \), where
\[
q_{0,m}(r) = \frac{1}{\omega_{n-1} r^{n-1}} \int_{S(0,r)} Q_m(x) \, d\mathcal{H}^{n-1}(x),
\]
see [10, Proof of Theorem 5.2]. In this case, $f_m$ satisfy relation (1.10) with $Q = K_I(x, f) = q_0(|x|)$ (see, for example, [18, Corollary 8.5 and Theorem 8.6]).

Note that the function $Q$, extended by zero outside the unit ball, is integrable over almost all spheres centered at each point $x_0 \in \mathbb{B}^n$, since this function is locally bounded in $\mathbb{B}^n \setminus \{0\}$. Thus, the maps $g_m = f_m^{-1}, m = 1, 2, \ldots$ satisfy all the conditions of Corollary 1.2 (more generally, Theorem 1.1), and by this Corollary the family of mappings $\{g_m\}_{m=1}^{\infty}$ is equicontinuous in the unit ball. Moreover, the function $Q$ is not integrable in the unit ball due to relation (2.20), however, the family $\{f_m\}_{m=1}^{\infty}$ is equicontinuous in $\mathbb{B}^n$ due to condition (2.21) and [18, Theorem 7.6]. However, the equicontinuity of the family $\{f_m\}_{m=1}^{\infty}$ can be verified directly; moreover, the equicontinuity of the inverse family $\{g_m\}_{m=1}^{\infty}$ may also be obtained from the fact that $f_m$ converges to some homeomorphism $f$ as $m \to \infty$ locally uniformly. In this case, $g_m$ also converges locally uniformly to some homeomorphism $g = f^{-1}$ (see [22, Lemma 3.1]).

**Example 2** Now consider the sequence of functions

$$Q_m(x) = \begin{cases} \frac{1}{|x|}, & |x| > 1/m, \\ 1, & |x| \leq 1/m. \end{cases}$$

In the same way as above, we put

$$f_m(x) = \frac{x}{|x|} \rho_m(|x|), \quad f_m(0) := 0,$$

where

$$\rho_m(r) = \exp \left\{- \int_r^1 \frac{dt}{1/(n-1)}(t) \right\}, \quad q_{0,m}(r) := \frac{1}{\omega_{n-1}r^{n-1}} \int_{|x|=r} Q_m(x) \, dS,$$

Note that the mappings $f_m$, as well as inverse mappings $g_m := f_m^{-1}$ can be written in explicit form, namely,

$$f_m(x) = \begin{cases} mx \cdot \exp\left(\frac{n-1}{n} \cdot (1/m)^{n/(n-1)} - 1\right), & |x| \leq 1/m, \\ \frac{x}{|x|} \exp\left(\frac{n-1}{n} \cdot (|x|^{(n/(n-1))-1}) - 1\right), & |x| > 1/m, \end{cases}$$

$$g_m(y) = \begin{cases} \frac{y}{m} \cdot \exp\left(\frac{1-n}{n} ((1/m)^{n/(n-1)} - 1)\right), & |y| \leq \exp\left(\frac{n-1}{n} \cdot (1/m)^{n/(n-1)} - 1\right), \\ \frac{y}{|y|} \cdot \frac{(n \ln |y| + (n-1)(y/(n-1))^{(n-1)/n})}{(n-1)/(n)} , & |y| > \exp\left(\frac{n-1}{n} \cdot (1/m)^{n/(n-1)} - 1\right). \end{cases}$$

Arguing as in Example 1, it can be shown that the maps $f_m$ satisfy relation (1.10) for $D = \mathbb{B}^n$ and $Q(x) = \frac{1}{|x|^{n}}$. Note that $\int_{\mathbb{B}^n} Q(x) \, dm(x) = \infty$, although in this case we have $\int_0^1 \frac{dt}{1/(n-1)}(t) < \infty$. Note that, in contrast to the mappings from Example 1, we have a locally uniform convergence of $g_m$ to $g$ as $m \to \infty$, where $g$ is some continuous mapping. Moreover, the ”direct” sequence of mappings of $f_m, m = 1, 2, \ldots$, is neither
convergent, nor equicontinuous, in $\mathbb{B}^n$. This fact about the equicontinuity of the maps $g_m$ is also the result of Corollary 1.2 (Theorem 1.1), since the function $Q$ has finite mean values over almost all spheres.

3 Logarithmic Hölder continuity of mappings of the unit ball

Note that the original (local) version of this result was published by us in [33, Theorem 1.1] [cf. (3.1)–(3.11) vs. [33, (2.1)–(2.12)], using $B(x_0, 2r_0) \setminus B(x_0, r_0)$ vs. using $B(0, 1 - \frac{r_0}{2}) \setminus B(0, 1 - r_0) \subset B(0, 1 - \frac{r_0}{2}) \setminus K$. By the words “local version” we mean that earlier we established a similar result in some ball of a fixed point $x_0$. Below we give a proof of a similar statement concerning an arbitrary compact set $K$ of the unit ball.

Proof of Theorem 1.2 We fix $x, y \in K \subset \mathbb{B}^n$ and $f \in \mathcal{F}_Q(\mathbb{B}^n)$. Put

$$|f(x) - f(y)| := \varepsilon_0. \quad (3.1)$$

If $\varepsilon_0 = 0$, there is nothing to prove. Let $\varepsilon_0 > 0$. Draw through the points $f(x)$ and $f(y)$ a straight line $r = r(t) = f(x) + (f(x) - f(y))t$, $-\infty < t < \infty$ (see Fig. 2).

Let $\gamma^1 : [1, c) \to \mathbb{B}^n$, $1 < c \leq \infty$ be a maximal $f$-lifting of the ray $r = r(t), t \geq 1$, starting at $x$, which exists by Proposition 2.2. Let us to prove that the situation when $\gamma^1(t) \to x_1 \in \mathbb{B}^n$ as $t \to c$ is impossible. Indeed, in this case, by Proposition 2.2 $c = \infty$ and $f(x_1) = \lim_{t \to +\infty} r(t)$. Then, on the one hand, $f(x_1) \in f(\mathbb{B}^n)$ due to the openness of $f$, and on the other hand, $f(x_1) = \infty$ due to the definition of $r = r(t)$. Since $\infty \notin f(\mathbb{B}^n)$, we obtain a contradiction. Thus, the case $\gamma^1(t) \to x_1 \in \mathbb{B}^n$ as $t \to c$ is impossible, as required. By Proposition 2.2

$$h(\gamma^1(t), \partial \mathbb{B}^n) \to 0 \quad (3.2)$$

as $t \to c - 0$. Similarly, let $\gamma^2 : (d, 0] \to \mathbb{B}^n$, $-\infty \leq d < 0$ be a maximal $f$-lifting of the ray $r = r(t), t \leq 0$, with end at a point $y$, which exists by Proposition 2.2. Just like in (3.2) we have that

$$h(\gamma^2(t), \partial \mathbb{B}^n) \to 0 \quad (3.3)$$

as $t \to d + 0$. Let $z = \gamma^1(t_1)$ be some point on the path $\gamma^1$ located at the distance $r_0/2$ from the boundary of the unit ball, where $r_0 := d(K, \partial \mathbb{B}^n)$. Put $\gamma^* := \gamma^1|_{[1, t_1]}$. Now, by the triangle inequality, $\text{diam } (|\gamma^*|) \geq r_0/2$. Let $\Gamma := \Gamma(|\gamma^*|, |\gamma^2|, \mathbb{B}^n)$. By [37, Lemma 4.3]

$$M(\Gamma) \geq (1/2) \cdot M(\Gamma(|\gamma^*|, |\gamma^2|, \mathbb{R}^n)), \quad (3.3)$$

and on the other hand, by [36, Lemma 7.38]

$$M(\Gamma(|\gamma^*|, |\gamma^2|, \mathbb{R}^n)) \geq c_n \cdot \log \left(1 + \frac{1}{m}\right), \quad (3.4)$$
where $c_n > 0$ is some constant depending only on $n$,

$$m = \frac{\text{dist}(|\gamma^*|, |\gamma^2|)}{\min\{\text{diam}(|\gamma^*|), \text{diam}(|\gamma^2|)\}}.$$ 

Note that $\text{diam}(|\gamma^i|) = \sup_{\omega, w \in |\gamma^i|} |\omega - w| \geq r_0/2$, $i = 1, 2$. Then, by combining (3.3) and (3.4) and taking into account that $\text{dist}(|\gamma^*|, |\gamma^2|) \leq |x - y|$, we obtain that

$$M(\Gamma) \geq \tilde{c}_n \cdot \log \left(1 + \frac{r_0}{2\text{dist}(|\gamma^*|, |\gamma^2|)}\right) \geq \tilde{c}_n \cdot \log \left(1 + \frac{r_0}{2|x - y|}\right), \quad (3.5)$$

where $\tilde{c}_n > 0$ is some constant depending only on $n$.

Let $z_1 := f(z), \varepsilon^1 := |f(x) - z^1|$ and $\varepsilon^2 := |f(y) - z^1|$. Observe that

$$|f(x) - f(y)| + \varepsilon^1 = |f(y) - f(x)| + |f(x) - z^1| = |z^1 - f(y)| = \varepsilon^2, \quad (3.6)$$

and, thus, $\varepsilon^1 < \varepsilon^2$. Now let us prove the upper bound for $M(\Gamma)$. Let $P = |f(\gamma^*)|$ and $Q = |f(\gamma^2)|$. Put

$$A := A(z^1, \varepsilon^1, \varepsilon^2) = \{x \in \mathbb{R}^n : \varepsilon^1 < |x - z^1| < \varepsilon^2\}.$$ 

We show that

$$f(\Gamma) > \Gamma(S(z^1, \varepsilon^1), S(z^1, \varepsilon^2), A). \quad (3.7)$$

Indeed, let $\gamma \in \Gamma$. Then $f(\gamma) \in f(\Gamma), f \circ \gamma : [0, 1] \rightarrow \mathbb{R}^n$, $f(\gamma(0)) \in P$, $f(\gamma(1)) \in Q$ and $f(\gamma(s)) \in f(\mathbb{R}^n)$ for $0 < s < 1$. Let $q > 1$ be a number such that

$$z^1 = f(y) + (f(x) - f(y))q.$$ 

Since $f(\gamma(0)) \in P$, there is $1 \leq t \leq q$ such that $f(\gamma(0)) = f(y) + (f(x) - f(y))t$. So,

$$|f(\gamma(0)) - z^1| = |(f(x) - f(y))(q - t)|$$
$$\leq |(f(x) - f(y))(q - 1)| = |(f(x) - f(y))q + f(y) - f(x)|$$
$$= |f(x) - z^1| = \varepsilon^1. \quad (3.8)$$

On the other hand, since $f(\gamma(1)) \in Q$, there is a $p \leq 0$ such that

$$f(\gamma(1)) = f(y) + (f(x) - f(y))p.$$ 

In this case, we obtain that
\[ |f(y(1)) - z^1| = |(f(x) - f(y))(q - p)| \]
\[ \geq |(f(x) - f(y))q| = |(f(x) - f(y))q + f(y) - f(y)| \]
\[ = |f(y) - z^1| = \varepsilon^2. \] (3.9)

Since \( \varepsilon^1 < \varepsilon^2 \), due to (3.9) we obtain that
\[ |f(y(1)) - z^1| > \varepsilon^1. \] (3.10)

It follows from (3.8) and (3.10) that \( |f(y)| \cap B(z^1, \varepsilon^1) \neq \emptyset \neq (f(\mathbb{R}^n) \setminus B(z^1, \varepsilon^1)) \cap |f(y)| \). In this case, by Proposition 2.1 there is \( t_1 \in (0, 1) \) such that \( f(y(t_1)) \in S(z^1, \varepsilon^1) \). We may assume that \( f(y(t)) \notin B(z^1, \varepsilon^1) \) for \( t \in (t_1, 1) \). Put \( \alpha^1 := f(y)|_{[t_1, 1]} \).

On the other hand, since \( \varepsilon^1 < \varepsilon^2 \) and \( f(y(t_1)) \in S(z^1, \varepsilon^1) \), we obtain that \( |\alpha^1| \cap B(z^1, \varepsilon^2) \neq \emptyset \). By (3.9) we obtain that \( (f(\mathbb{R}^n) \setminus B(z^1, \varepsilon^2)) \cap |\alpha^1| \neq \emptyset \). Thus, again by Proposition 2.1 there is \( t_2 \in (t_1, 1) \) such that \( \alpha^1(t_2) \in S(z^1, \varepsilon^2) \). We may assume that \( f(y(t)) \in B(z^1, \varepsilon^2) \) for \( t \in (t_1, t_2) \). Set \( \alpha^2 := \alpha^1|_{[t_1, t_2]} \). Now, \( f(y) > \alpha^2 \) and \( \alpha^2 \in \Gamma(S(z^1, \varepsilon^1), S(z^1, \varepsilon^2), A) \). Thus, (3.7) is proved.

It follows from (3.7) that \( \Gamma > \Gamma f(z^1, \varepsilon^1, \varepsilon^2) \). Now, we set
\[ \eta(t) = \begin{cases} \frac{1}{\varepsilon_0}, & t \in [\varepsilon^1, \varepsilon^2], \\ 0, & t \notin [\varepsilon^1, \varepsilon^2], \end{cases} \]
where \( \varepsilon_0 \) is a number from (3.1). Note that \( \eta \) satisfies the relation (1.5) for \( r_1 = \varepsilon^1 \) and \( r_2 = \varepsilon^2 \). Indeed, it follows from (3.1) and (3.6) that
\[ r_1 - r_2 = \varepsilon^2 - \varepsilon^1 = |f(y) - z^1| - |f(x) - z^1| = |f(x) - f(y)| = \varepsilon_0. \]

Then \( \int_{\varepsilon^1}^{\varepsilon^2} \eta(t) \, dt = (1/\varepsilon_0) \cdot (\varepsilon^2 - \varepsilon^1) \geq 1 \). By the inequality (3.7) and the relation (1.4) applied at the point \( z^1 \), we obtain that
\[ M(\Gamma) \leq M(\Gamma f(z^1, \varepsilon^1, \varepsilon^2)) \leq \frac{1}{\varepsilon_0} \int_{\mathbb{R}^n} Q(z) \, dm(z) = \frac{\|Q\|_1}{|f(x) - f(y)|^{n'}}. \] (3.11)

By (3.5) and (3.11), we obtain that
\[ \tilde{c}_n \cdot \log \left( 1 + \frac{r_0}{2|x - y|} \right) \leq \frac{\|Q\|_1}{|f(x) - f(y)|^{n'}}. \]

From the latter ratio, the desired inequality (1.8) follows, while \( C_n := \tilde{c}_n^{-1/n} \). \( \square \)
4 On logarithmic Hölder type maps in arbitrary domains

As we indicated above, the problem of Hölder continuity for mappings of arbitrary domains has not yet been resolved. Nevertheless, under certain not too rough conditions, a result of this kind can be obtained from the main theorem of the previous section. The following result holds.

**Theorem 4.1** Let $n \geq 2$, and let $Q \in L^1(\mathbb{R}^n)$. Suppose that $K$ is a compact set in $D$, and beside that, $D'$ is bounded. Now, the inequality

$$|f(x) - f(y)| \leq \frac{C}{\log^{1/n} \left( 1 + \frac{r_0}{2|x-y|} \right)}$$  \quad (4.1)

holds for any $x, y \in K$ and all $f \in \mathcal{R}_Q(D, D')$, where $C = C(n, K, \|Q\|_1, D, D') > 0$ is some constant depending only on $n$, $K$ and $\|Q\|_1$. $\|Q\|_1$ denotes $L^1$-norm of $Q$ in $\mathbb{R}^n$, and $r_0 = d(K, \partial D)$.

**Proof** It suffices to bound the expression

$$|f(x) - f(y)| \cdot \log^{1/n} \left( 1 + \frac{r_0}{2|x-y|} \right)$$  \quad (4.2)

for all $x, y \in K$ and $f \in \mathcal{R}_Q(D, D')$.

Fix $x, y \in K$ and $f \in \mathcal{R}_Q(D, D')$. If $|x - y| \geq r_0/2$, then the expression in (4.2) is trivially bounded. Indeed, by the triangle inequality,

$$|f(x) - f(y)| \leq |f(x)| + |f(y)| \leq 2M_0,$$  \quad (4.3)
where $M_0 = \sup_{z \in D'} |z|$. Observe that $M_0 < \infty$ because $D'$ is bounded. By (4.3) we obtain that

$$|f(x) - f(y)| \cdot \log^{1/n} \left( 1 + \frac{r_0}{2|x - y|} \right) \leq M_0 \cdot \log^{1/n} 2,$$

(4.4)
as required.

Let $|x - y| < r_0/2$. In this case, $y \in B(x, r_0)$. Let $\psi$ be the conformal mapping of the unit ball $B^n$ onto the ball $B(x, r_0)$, namely, $\psi(z) = zr_0 + x$, $z \in B^n$. In particular, $\psi^{-1}(B(x, r_0/2)) = B(0, 1/2)$. Applying the restriction of the mapping $\tilde{f} := f|_{B(x, r_0)}$ and considering the auxiliary map $F := \tilde{f} \circ \psi, F : B^n \rightarrow D'$, we conclude that it also satisfies condition (1.4) with the same $Q$. Now, by Theorem 1.2

$$|F(\psi^{-1}(x)) - F(\psi^{-1}(y))| \leq \frac{C_2 \cdot (\|Q\|_1)^{1/n}}{\log^{1/n} \left( 1 + \frac{1}{4|\psi^{-1}(x) - \psi^{-1}(y)|} \right)}.$$  

(4.5)

Since $F(\psi^{-1}(x)) = f(x)$ and $F(\psi^{-1}(y)) = f(y)$, we may rewrite (4.5) as

$$|f(x) - f(y)| \leq \frac{C_2 \cdot (\|Q\|_1)^{1/n}}{\log^{1/n} \left( 1 + \frac{1}{4|\psi^{-1}(x) - \psi^{-1}(y)|} \right)}.$$  

(4.6)

Note that the maps $\psi^{-1}(y)$ are Lipschitz with the Lipschitz constant $\frac{1}{r_0}$. In this case, we obtain from relation (4.6) that

$$|f(x) - f(y)| \leq \frac{C_2 \cdot (\|Q\|_1)^{1/n}}{\log^{1/n} \left( 1 + \frac{r_0}{4|x - y|} \right)}.$$  

(4.7)

Finally, we note that $\log^{1/n} \left( 1 + \frac{1}{nt} \right) \sim \log^{1/n} \left( 1 + \frac{1}{kt} \right)$ as $t \rightarrow +0$ for various fixed $k, n > 0$, which may be verified using the L’Hôpital rule. It follows that

$$\frac{C_2 \cdot (\|Q\|_1)^{1/n}}{\log^{1/n} \left( 1 + \frac{r_0}{4|x - y|} \right)} \leq \frac{C_1 \cdot (\|Q\|_1)^{1/n}}{\log^{1/n} \left( 1 + \frac{r_0}{2|x - y|} \right)}$$

for some constant $C_1 = C_1(r_0) > 0$. In this case, it follows from (4.7) that

$$|f(x) - f(y)| \leq \frac{C_1 \cdot (\|Q\|_1)^{1/n}}{\log^{1/n} \left( 1 + \frac{r_0}{2|x - y|} \right)}.$$  

(4.8)

It follows from (4.4) and (4.8) that the inequality (4.1) holds for

$$C := \max\{C_1 \cdot (\|Q\|_1)^{1/n}, M_0 \cdot \log^{1/n} 2\}.$$
5 Applications to the Beltrami equation

We note that the theory of the local behavior of mappings with the inverse Poletsky inequality developed by us may be applied to the problem of the existence of solutions to Beltrami equations. More specifically, we will demonstrate how the result of Theorems 1.1 and 1.2 may be applied to obtain results of this kind (see Theorem 5.1 and Corollary 5.1 below).

Recently, the topic related to the existence of solutions of degenerate Beltrami differential equations has been actively developed (see, e.g., [8,9,27,28]). The main results on this topic are compiled in a relatively recent monograph [9], with links to publications by these and other authors. One of the problems posed in the study of Beltrami equations is to find the conditions for a complex coefficient that ensure that their solutions exist. Finding solutions usually holds in the class of $ACL$-homeomorphisms, although it is quite correct to consider just continuous $ACL$-solutions. Recall that a (continuous) mapping $f : D \to \mathbb{R}^n$, $n \geq 2$, is absolutely continuous on lines, abbr. $f \in ACL$, if, for every closed parallelepiped $P$ in $D$ whose sides are perpendicular to the coordinate axes, each coordinate function of $f|P$ is absolutely continuous on almost every line segment in $P$ which is parallel to the coordinate axes, see e.g. [35, section 26]. In this section, we obtain another result on the existence of solutions of degenerate Beltrami equation, which is based on the transition to inverse mappings. Compared to Ryazanov et al. [27,28] and Gutlyanskii et al. [8], we are somewhat weakening the conditions on the complex coefficient. The obtained solution of the equation may not be homeomorphic, but relative to the previous results, the degree of its smoothness is $W^{1,p}_{loc}$, $p > 1$, and therefore somewhat higher.

We turn now to the definitions. Let $D$ be a domain in $\mathbb{C}$. In what follows, a mapping $f : D \to \mathbb{C}$ is assumed to be sense-preserving, moreover, we assume that $f$ has partial derivatives almost everywhere. Put $f_{\bar{z}} = (f_x + if_y)/2$ and $f_z = (f_x - if_y)/2$. The complex dilatation of $f$ at $z \in D$ is defined as follows: $\mu(z) = f_{\bar{z}}/f_z$ for $f_{\bar{z}} \neq 0$ and $\mu(z) = 0$ otherwise. The maximal dilatation of $f$ at $z$ is the following function:

$$K_\mu(z) = K_{\mu,f}(z) = \frac{1 + |\mu(z)|}{1 - |\mu(z)|}. \tag{5.1}$$

Note that the Jacobian of $f$ at $z \in D$ may be calculated according to the relation

$$J(z, f) = |f_{\bar{z}}|^2 - |f_z|^2.$$

Since we assume that the map $f$ is sense preserving, the Jacobian of this map is non-negative at all points where $f$ is differentiable. Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, and let $\mu : D \to \mathbb{D}$ be a Lebesgue measurable function. We define the maximal dilatation corresponding to a complex dilatation $\mu$ by (5.1). It is easy to see that

$$K_{\mu,f}(z) = \frac{|f_z| + |f_{\bar{z}}|}{|f_{\bar{z}}| - |f_z|}.$$
whenever partial derivatives of $f$ exist at $z \in D$ and, in addition, $J(z, f) \neq 0$. We also define the inner dilatation of the order $p \geq 1$ of the map $f$ by the relation

$$K_{I,p}(z, f) = \frac{|f_{z}|^2 - |f_{\bar{z}}|^2}{(|f_{z}| - |f_{\bar{z}}|)^p}$$

whenever $J(z, f) \neq 0$. In addition, we set $K_{I,p}(z) = 1$ provided that $|f_{z}| + |f_{\bar{z}}| = 0$, and $K_{I,p}(z) = \infty$ when $J(z, f) = 0$, but $|f_{z}| + |f_{\bar{z}}| \neq 0$. Observe that $K_{I,2}(z) = K_{\mu}(z)$. Set $\|f'(z)\| = |f_{z}| + |f_{\bar{z}}|$. Recall that a homeomorphism $f : D \to \mathbb{C}$ is said to be quasiconformal if $f \in W^{1,2}_{\text{loc}}(D)$ and, in addition, $\|f'(z)\|^2 \leq K \cdot |J(z, f)|$ for some constant $K \geq 1$ almost everywhere.

A Beltrami equation is a differential equation of the form

$$f_{\bar{z}} = \mu(z) \cdot f_z,$$  
(5.3)

where $\mu = \mu(z)$ is a given measurable function with $|\mu(z)| < 1$ a.a. Given $k \geq 1$ we set

$$\mu_k(z) = \begin{cases} \mu(z), & K_{\mu}(z) \leq k, \\ 0, & K_{\mu}(z) > k. \end{cases}$$  
(5.4)

Let $f_k$ be a homeomorphic ACL-solution of the equation $f_{\bar{z}} = \mu_k(z) \cdot f_z$, which maps the unit disk onto itself and satisfies the normalization conditions $f_k(0) = 0$, $f_k(1) = 1$. This solution exists by Ahlfors [1, Theorem 3.B.V] or Bojarski [2, Theorem 8.2]. Note that the inverse mapping $g_k = f_k^{-1}$ is quasiconformal; in particular, it is differentiable almost everywhere. Let $g_k$ be an inverse mapping to $f_k$, then its complex dilation $\mu_{g_k}$ is calculated according to the relation $\mu_{g_k}(w) = -\mu_k(g_k(w)) = -\mu_k(f_k^{-1}(w))$, see e.g., [1,(4).C.I]. In this case, the maximal dilation of $g_k$ is calculated by the relation

$$K_{\mu_{g_k}}(w) = \frac{1 + |\mu_k(f_k^{-1}(w))|}{1 - |\mu_k(f_k^{-1}(w))|}.  \quad (5.5)$$

Accordingly, the inner dilatation of the order $p$ of the map $g_k$ can be calculated according to relation (5.2), namely,

$$K_{I,p}(w, g_k) = \frac{|(g_k)_w|^2 - |(g_k)_{\bar{w}}|^2}{|(g_k)_w| - |(g_k)_{\bar{w}}|)^p.  \quad (5.6)$$

The following statement holds.

**Theorem 5.1** Let $\mu : \mathbb{D} \to \mathbb{D}$ be a Lebesgue measurable function, and let $\mu_k$, $k = 1, 2, \ldots$, be a function defined by the relation (5.4). Assume that $f_k$ is a homeomorphic ACL-solution of the equation $f_{\bar{z}} = \mu_k(z) \cdot f_z$, that maps the unit disk onto itself and satisfies the normalization conditions $f_k(0) = 0$, $f_k(1) = 1$. Let $g_k = f_k^{-1}$ and let $Q : \mathbb{D} \to [1, \infty)$ be a Lebesgue measurable function. Assume that the following conditions hold:
(1) for each \(0 < r_1 < r_2 < 1\) and \(y_0 \in \mathbb{D}\) there is a set \(E \subset [r_1, r_2]\) of positive linear Lebesgue measure such that the function \(Q\) is integrable over the circles \(S(y_0, r)\) for any \(r \in E\);

(2) there exist a number \(1 < p \leq 2\) and a constant \(M > 0\) such that
\[
\int_{\mathbb{D}} K_{I,p}(w, g_k)\, dm(w) \leq M \tag{5.7}
\]
for all \(k = 1, 2, \ldots,\) where \(K_{I,p}(w, g_k)\) is defined in (5.6);

(3) the inequality
\[
K_{\mu g_k}(w) \leq Q(w) \tag{5.8}
\]
holds for a.e. \(w \in \mathbb{D}\), where \(K_{\mu g_k}\) is defined in (5.5).

Then the Eq. (5.3) has a continuous \(W^{1,p}_{\text{loc}}(\mathbb{D})\)-solution \(f\) in \(\mathbb{D}\).

**Corollary 5.1** In particular, the conclusion of Theorem 5.1 holds if, in this theorem, we abandon condition (1), accept condition (3), and replace condition (2) with the requirement \(Q \in L^1(\mathbb{D})\). In this case, the solution \(f\) of Eq. (5.3) can be chosen such that
\[
|f(x) - f(y)| \leq C \cdot (\|Q\|_1)^{1/2} \log^{1/2} \left(1 + \frac{r_0}{2|x - y|}\right) \tag{5.9}
\]
for any compact set \(K \subset \mathbb{D}\) and \(x, y \in K\), where \(\|Q\|_1\) denotes \(L^1\)-norm of \(Q\) in \(\mathbb{D}\), \(C > 0\) is some constant, and \(r_0 = d(K, \partial \mathbb{D})\).

Let \(D\) be a domain in \(\mathbb{C}\). Suppose that a function \(\varphi : D \to \mathbb{R}\) is locally integrable in some neighborhood of a point \(z_0 \in D\). We say that \(\varphi\) has a finite mean oscillation at \(z_0 \in D\), and we write \(\varphi \in FMO(z_0)\), if the relation
\[
\limsup_{\varepsilon \to 0} \frac{1}{\pi \varepsilon^2} \int_{B(z_0, \varepsilon)} |\varphi(z) - \overline{\varphi}_\varepsilon|\, dm(z) < \infty
\]
holds, where \(\overline{\varphi}_\varepsilon = \frac{1}{\pi \varepsilon^2} \int_{B(z_0, \varepsilon)} \varphi(z)\, dm(z)\) (see, e.g., [28, section 2]). We say that a function \(\varphi\) has a finite mean oscillation in \(D\), and we write \(\varphi \in FMO(D)\), if \(\varphi \in FMO(z_0)\) for any \(z_0 \in D\). The following statement holds.

**Corollary 5.2** If, under the conditions of Corollary 5.1, we require in addition that either \(Q(z) \in FMO(\mathbb{D})\), or
\[
\int_0^{\delta(w_0)} \frac{dt}{t q_{w_0}(t)} = \infty \tag{5.10}
\]
for any \(w_0 \in \mathbb{D}\) and some \(\delta(w_0) > 0\), \(q_{w_0}(r) = \frac{1}{2\pi} \int_0^{2\pi} Q(w_0 + re^{i\theta}) \, d\theta\), then \(f\) can be chosen as a homeomorphism in \(\mathbb{D}\).
Before proceeding to the proof of the main results of the section and their corollaries, we present the following fundamental statement, see, for example, [26, Lemma III.3.5]; cf. [26, Theorem I.2.4] and [29, Lemma 2.1].

**Proposition 5.1** Let \( f_\nu : U \to \mathbb{R}^n \) be a sequence of functions of class \( W^{1,p}(U) \), where \( U \) is an open set in \( \mathbb{R}^n \), and \( p > 1 \). Assume that the norm sequence \( \| f_\nu \|_{1,p,U} \), \( \nu = 1, 2, \ldots \), is bounded and the functions \( f_\nu \) converge in \( L_1 \) to a function \( f_0 : U \to \mathbb{R} \) as \( \nu \to \infty \). Then \( f_0 \in W^{1,p}(U) \) and the functions \( f'_\nu \) converge weakly in \( L^p(U) \) to \( f'_0 \).

The proof of Theorem 5.1, as well as Corollaries 5.1 and 5.2, will be given later in the text. Before this, we formulate and prove the following convergence lemma. On this occasion, see also similar statements related to the study of equations and mappings with some another conditions, e.g. [9, Ch. 2] and [29, Theorem 3.1].

**Lemma 5.1** Let \( 1 < p \leq 2 \), and let \( \mu : D \to \mathbb{D} \) be a Lebesgue measurable function. Suppose that \( f_k, k = 1, 2, \ldots \) is a sequence of sense-preserving \( W^{1,2}_{\text{loc}}(D) \)-homeomorphisms of \( D \) onto itself with complex coefficients \( \mu f_k(z) \). Suppose that \( f_k \) converges locally uniformly in \( D \) to some mapping \( f : D \to \mathbb{C} \) as \( k \to \infty \), and the sequence \( \mu f_k(z) \) converges to \( \mu \) as \( k \to \infty \) for almost all \( z \in D \). Suppose also that the inverse mappings \( g_k := f_k^{-1} \) belong to the class \( W^{1,2}_{\text{loc}}(D) \) and, in addition,

\[
\int_D K_{1,p}(w, g_k) \, dm(w) \leq M
\]

for some \( M > 0 \), each \( k = 1, 2, \ldots \). Now \( f \in W^{1,p}_{\text{loc}}(D) \) and, moreover, here \( \mu \) is a complex dilatation of the map \( f \), in other words, \( fz = \mu(z) \cdot f_k(z) \) for almost all \( z \in D \).

**Proof** In general, we will follow the scheme described in the proof of [29, Theorem 3.1], cf. [9, Theorem 2.1] and [26, Lemma III.3.5]. We denote \( \partial f = f_z \) and \( \overline{\partial} f = f_{\overline{z}} \). Let \( C \) be an arbitrary compact set in \( D \). Since, by assumption, the maps \( g_k = f_k^{-1} \) belong to \( W^{1,2}_{\text{loc}} \), then \( g_k \) possess the Luzin \( N \)-property (see, for example, [25, Theorem 3], cf. [13, Corollary B]). Now, the Jacobian \( J(z, f_k) \) is almost everywhere nonzero, see, for example, [20, Theorem 1]. Since \( f_k \in W^{1,2}_{\text{loc}} \), a change of variables in the integral is true (see e.g. [4, Theorem 3.2.5]). In this case, we have that

\[
\int_C \| f'_k(z) \|^p \, dm(z) = \int_C \frac{\| f'_k(z) \|^p}{J(z, f_k)} \cdot J(z, f_k) \, dm(z) = \int_{f_k(C)} \int_K_{1,p}(w, g_k) \, dm(w) \leq M < \infty.
\]

It follows from (5.11) that \( f \in W^{1,p}_{\text{loc}} \), \( \partial f_k \) and \( \overline{\partial} f_k \) weakly converge in \( L^1_{\text{loc}}(D) \) to \( \partial f \) and \( \overline{\partial} f \), respectively (see Proposition 5.1).
It remains to show that the map \( f \) is a solution of the Beltrami equation \( f \frac{\partial}{\partial z} = \mu(z) \cdot \frac{\partial}{\partial \bar{z}} \). Put \( \zeta(z) = \bar{\partial} f(z) - \mu(z) \partial f(z) \) and show that \( \zeta(z) = 0 \) almost everywhere. Let \( B \) be an arbitrary disk lying with its closure in \( D \). By the triangle inequality

\[
\left| \int_B \zeta(z) \, dm(z) \right| \leq I_1(k) + I_2(k), \quad k \in \mathbb{N},
\]

where

\[
I_1(k) = \left| \int_B (\bar{\partial} f(z) - \bar{\partial} f_k(z)) \, dm(z) \right|
\]

and

\[
I_2(k) = \left| \int_B (\mu(z) \partial f(z) - \mu f_k(z) \partial f_k(z)) \, dm(z) \right|.
\]

We have proved above that \( I_1(k) \to 0 \) as \( k \to \infty \). It remains to deal with the expression \( I_2(k) \). To do this, note that \( I_2(k) \leq I_2'(k) + I_2''(k) \), where

\[
I_2'(k) = \left| \int_B \mu(z)(\partial f(z) - \partial f_k(z)) \, dm(z) \right|
\]

and

\[
I_2''(k) = \left| \int_B (\mu(z) - \mu f_k(z)) \partial f_k(z) \, dm(z) \right|.
\]

Due to the weak convergence of \( \partial f_k \to \partial f \) in \( L^1_{\text{loc}}(D) \) as \( k \to \infty \), we obtain that \( I_2'(k) \to 0 \) for \( k \to \infty \), since \( \mu \in L^\infty(D) \). Moreover, for a given \( \varepsilon > 0 \) there is \( \delta = \delta(\varepsilon) > 0 \) such that

\[
\int_E |\partial f_k(z)| \, dm(z) \leq \int_E |\partial f_k(z) - \partial f(z)| \, dm(z) + \int_E |\partial f(z)| \, dm(z) < \varepsilon,
\]

whenever \( m(E) < \delta \), \( E \subset B \), and \( k \) is sufficiently large.

Finally, by Egorov’s theorem (see [30, Theorem III.6.12]) for each \( \delta > 0 \) there exists a set \( S \subset B \) such that \( m(B \setminus S) < \delta \) and \( \mu f_k(z) \to \mu(z) \) uniformly on \( S \). Then \( |\mu f_k(z) - \mu(z)| < \varepsilon \) for all \( k \geq k_0 \), some \( k_0 = k_0(\varepsilon) \in \mathbb{N} \) and all \( z \in S \). Now, by (5.15) and (5.11), as well as by Hölder’s inequality,

\[
I_2''(k) \leq \varepsilon \int_S |\partial f_k(z)| \, dm(z) + 2 \int_{B \setminus S} |\partial f_k(z)| \, dm(z)
\]
\[
\begin{align*}
&\leq \varepsilon \cdot \left\{ \left( \int_{D} K_{1,p}(w, g_k) \, dm(w) \right)^{1/p} \cdot (m(B))^{(p-1)/p + 2} \right\} \\
&\leq \varepsilon \cdot \left( M^{1/p} \cdot (m(B))^{(p-1)/p + 2} \right). \tag{5.16}
\end{align*}
\]

for \( k \geq k_0 \). From (5.12), (5.13), (5.14) and (5.16) it follows that \( \int_{B} \zeta(z) \, dm(z) = 0 \) for all disks \( B \), compactly embedded in \( D \). Based on the Lebesgue theorem on differentiation of an indefinite integral (see [30, IV(6.3)]), it follows that \( \zeta(z) = 0 \) almost everywhere in \( D \). The lemma is proved. \( \Box \)

**Proof of Theorem 5.1** Consider a sequence of complex-valued functions

\[
\mu_k(z) = \begin{cases} 
\mu(z), & K_\mu(z) \leq k, \\
0, & K_\mu(z) > k,
\end{cases} \tag{5.17}
\]

where \( K_\mu(z) \) is defined by (5.1). Note that \( \mu_k(z) \leq \frac{k+1}{k+1} < 1 \), therefore, the equation (5.3), in which instead of \( \mu \) on the right side we take \( \mu := \mu_k \), and \( \mu_k \) is defined by the relation (5.17), has a homeomorphic \( W^{1,2}_{\text{loc}}(D) \)-solution \( f_k : D \to D \) with normalizations \( f_k(0) = 0, f_k(1) = 1 \), which is \( k \)-quasiconformal in \( D \) (see [1, Theorem 3.B.V] or [2, Theorem 8.2]). By the same theorem, \( f_k \) maps the unit disk onto itself; moreover, \( g_k = f_k^{-1} \) are also quasiconformal, in particular, they belong to the class \( W^{1,2}_{\text{loc}}(D) \). By [17, Theorem 6.10] and by (5.8)

\[
M(g_k(\Gamma)) \leq \int_{D} K_{\mu_k} \cdot \rho_\ast^2 \, dm(w) \leq \int_{D} Q \cdot \rho_\ast^2 \, dm(w)
\]

for any \( k \in \mathbb{N} \) and any path family \( \Gamma \) in \( D \), and each function \( \rho_\ast \in \text{adm} \Gamma \). By Theorem 1.1 the family \( \{f_k\}^\infty_{k=1} \) is equicontinuous in \( D \). Thus, by the Arzela–Ascoli theorem \( f_k \) is a normal family of mappings (see e.g. [35, Theorem 20.4]), in other words, there is a subsequence \( f_{k_l} \) of \( f_k \), converging locally uniformly in \( D \) to some map \( f : D \to \overline{D} \). Note also that \( \mu_k(z) \to \mu(z) \) as \( k \to \infty \) for almost all \( z \in D \), because \( |\mu(z)| < 1 \) a.e. and, therefore, \( K_\mu(z) \) in (5.1) is finite for almost all \( z \in D \). Then by (5.7) and Lemma 5.1 the map \( f \) belongs to the class \( W^{1,p}_{\text{loc}}(D) \) and, in addition, \( f \) is a solution of (5.3). \( \Box \)

**Proof of Corollary 5.1** It follows from Theorem 5.1. Indeed, by the Fubini’s theorem the condition \( Q \in L^1(D) \) implies that the integrals \( \int_{S(x_0,r) \cap D} Q(x) \, d\mathcal{H}^1(x) \) are measurable functions by \( r \) and finite for a.e. \( 0 < r < \infty \) (see e.g. [30, Theorem 8.1.III]). In this case, conditions (5.7) and (5.8) are simultaneously satisfied, where \( p = 2 \). Thus, the existence of a solution of the equation (5.3) and its belonging to the class \( W^{1,2}_{\text{loc}}(D) \) follow directly from Theorem 5.1.
Moreover, by Theorem 1.2

\[ |f_k(x) - f_k(y)| \leq \frac{C \cdot (\|Q\|_1)^{1/2}}{\log^{1/2} \left( 1 + \frac{r_0}{2|x-y|} \right)} \quad \forall \ x, y \in K \]

where $\|Q\|_1$ is the $L^1$-norm of $Q$ in $\mathbb{D}$, $C$ is some constant and $r_0 := \text{dist}(K, \partial \mathbb{D})$. Passing here to the limit as $k \to \infty$, we obtain relation (5.9). Corollary 5.1 is proved. \hfill \Box

**Proof of Corollary 5.2** Suppose that $Q \in \mathcal{FMO}(\mathbb{D})$, or that relation (5.10) holds. Then the sequence $g_k$ forms an equicontinuous family of mappings (see [24, Theorems 6.1 and 6.5]). Therefore, by the Arzela–Ascoli theorem $g_k$ is a normal family (see e.g. [35, Theorem 20.4]), in other words, there is a subsequence $g_{kl}$ of $g_k$ converging locally uniformly in $D$ to some map $g: D \to D$. By the normalization conditions, $g_{kl}(0) = 0$ and $g_{kl}(1) = 1$ for all $l = 1, 2, \ldots$. Then, by virtue of Ryazanov et al. [22, Theorem 4.1] the mapping $g$ is a homeomorphism in $D$. Furthermore, by Ryazanov et al. [22, Lemma 3.1] we also have that $f_{kl} \to f = g^{-1}$ as $l \to \infty$ locally uniformly in $D$. Next, we apply arguments similar to those used in the case of an integrable function $Q$. Since $\mu_{kl}(z) \to \mu(z)$ as $k \to \infty$ and for almost all $z \in D$, by Lemma 5.1 the map $f$ belongs to the class $W_{1,2}^{\text{loc}}(\mathbb{D})$ and moreover, $f$ is a solution of (5.3). Inequality (5.9) follows from Corollary 5.1. \hfill \Box

**Example 3** Let $p = 2$, let $q \geq 1$ be an arbitrary number and let $0 < \alpha < 2/q$. As usual, we use the notation $z = re^{i\theta}$, $r \geq 0$, $\theta \in [0, 2\pi)$. Put

\[ \mu(z) = \begin{cases} e^{2i\theta} \frac{2r-\alpha(2r-1)}{2r+\alpha(2r-1)}, & 1/2 < |z| < 1, \\ 0, & |z| \leq 1/2. \end{cases} \tag{5.18} \]

Using the ratio

\[ \mu_f(z) = \frac{\overline{\partial} f}{\partial f} = e^{2i\theta} \frac{rf_r + if_\theta}{rf_r - if_\theta}, \]

see (11.129) in [18], we obtain that the mapping

\[ f(z) = \begin{cases} \frac{z}{|z|} (2|z| - 1)^{1/\alpha}, & 1/2 < |z| < 1, \\ 0, & |z| \leq 1/2 \end{cases} \tag{5.19} \]

is a solution of the equation $f_{\mu} = \mu(z) \cdot f_{\overline{z}}$, where $\mu$ is defined by (5.18). Note that the existence of a solution of this equation is ensured by Corollary 5.1 (for this, we verify that all conditions of this Corollary are satisfied). Note that for $\mu$ in (5.18), the corresponding maximal dilatation $K_{\mu}$ is the function

\[ K_{\mu}(z) = \begin{cases} \frac{2|z|}{\alpha(2|z| - 1)}, & 1/2 < |z| < 1, \\ 1, & |z| \leq 1/2 \end{cases} \tag{5.20} \]
Let \( k > 1/\alpha \). Observe that \( K_{\mu}(z) \leq k \) for \( |z| \geq \frac{1}{2} \cdot \frac{ka}{ka-1} \) and \( K_{\mu}(z) > k \) otherwise. As above, we set
\[
\mu_k(z) = \begin{cases} 
\mu(z), & K_{\mu}(z) \leq k, \\
0, & K_{\mu}(z) > k.
\end{cases}
\]
Observe that the mappings
\[
f_k(z) = \begin{cases} 
\frac{z}{|z|^2}(2|z| - 1)^{1/\alpha}, & \frac{1}{2} \cdot \frac{ka}{ka-1} < |z| < 1, \\
\frac{z}{(\frac{ka}{ka-1})^{1/\alpha}}, & |z| \leq \frac{ka}{ka-1}
\end{cases}
\]
are solutions of the equation \( f_\mu = \mu_k(z) \cdot f_\mu \). Besides that, the inverse mappings \( g_k(y) = f_k^{-1}(y) \) are calculated by the relations
\[
g_k(y) = \begin{cases} 
y(\frac{|y|^\alpha + 1}{|y|^\alpha}), & \left(\frac{ka}{ka-1} - 1\right)^{1/\alpha} < |y| < 1, \\
y \cdot \frac{ka}{ka-1}, & |y| \leq \left(\frac{ka}{ka-1} - 1\right)^{1/\alpha}
\end{cases}
\]
(5.21)
It follows from (5.20) that
\[
K_{\mu_k}(z) = \begin{cases} 
\frac{4|z|}{2a(2|z|-1)}, & \frac{1}{2} \cdot \frac{ka}{ka-1} < |z| < 1, \\
1, & |z| \leq \frac{1}{2} \cdot \frac{ka}{ka-1}
\end{cases}
\]
(5.22)
We should check that relation (5.7) holds for some function \( Q \) that is integrable in \( \mathbb{D} \). For this purpose, we substitute the maps \( g_k \) from (5.21) into the maximal dilatation \( K_{\mu_k} \) defined by the equality (5.22). Then
\[
K_{\mu_k}(y) = \begin{cases} 
\frac{|y|^\alpha + 1}{|y|^\alpha}, & \left(\frac{ka}{ka-1} - 1\right)^{1/\alpha} < |y| < 1, \\
1, & |y| \leq \left(\frac{ka}{ka-1} - 1\right)^{1/\alpha}
\end{cases}
\]
Note that \( K_{\mu_{g_k}}(y) \leq Q(y) := \frac{|y|^\alpha + 1}{|y|^{\alpha}} \) for all \( y \in \mathbb{D} \). Moreover, the function \( Q \) is integrable in \( \mathbb{D} \) even in the degree \( q \), and not only in the degree 1 (see the arguments used in considering [18, Proposition 6.3]). By the construction \( f_k(0) = 0 \) and \( f_k(1) = 1 \). Therefore, all the conditions of Corollary 5.1 are satisfied with \( p = 2 \), and the map \( f = f(z) \) in (5.19) may be considered as the desired solution of the equation \( f_\mu = \mu(z) \cdot f_\mu \). Moreover, it follows from the proof of this Corollary that the map \( f \) is exactly the solution of the equation indicated there, since it is a locally uniform limit of the sequence \( f_k \). Note that the map \( f \) is not a homeomorphic solution, it is also not open and discrete.

We show that for a function \( \mu \) in (5.18) there is no homeomorphic \( W_{\text{loc}}^{1,2}(\mathbb{D}) \)-solution of the Beltrami equation (5.3). Indeed, let \( g : \mathbb{D} \to \mathbb{D} \) be such a solution. Due to the Riemann mapping theorem, we may assume that \( g \) maps the unit disk onto itself.
Note that \( f \) and \( g \) are locally quasiconformal in \( \{ 1/2 < |z| < 1 \} \), therefore, due to the uniqueness theorem (see [9, Proposition 5.5]), \( g = \varphi \circ f \), where \( \varphi \) is some conformal mapping. Observe that \( \varphi \) is defined in the punctured ball \( \mathbb{D} \setminus \{0\} \), because \( f(\{ 1/2 < |z| < 1 \}) = \mathbb{D} \setminus \{0\} \). Thus, \( g \circ f^{-1} = \varphi \) and, since \( \varphi \) is conformal in \( \mathbb{D} \setminus \{0\} \), \( \varphi \) has a continuous extension to the origin. The last condition cannot be fulfilled, since \( f^{-1}(y) = \frac{y}{|y|^{p+1}} \), and \( g \) is some automorphism of the unit disk. This contradiction disproves the assumption that there exists a homeomorphic \( W^{1,2}_{\text{loc}}(\mathbb{D}) \)-solution \( g \) of (5.3).

Example 4 In conclusion, we also give an example of the Beltrami equation in which the existence of an \( W^{1,p}_{\text{loc}}(\mathbb{D}) \)-solution is ensured by Theorem 5.1 for some \( 1 < p < 2 \), although, at the same time, Corollaries 5.1 and 5.2 are not applicable. For this purpose, we use the already existing construction of family of mappings from Example 2.

As usual, we use the notation \( z = re^{i\theta} \), \( r \geq 0 \) and \( \theta \in [0, 2\pi) \).

Using the ratio

\[
\mu_f(z) = \frac{\partial f}{r \partial f} = e^{2i\theta} \frac{r f_r + i f_\theta}{r f_r - i f_\theta},
\]

see (11.129) in [18], we obtain that the mapping

\[
f(z) = \begin{cases} \frac{z}{|z|} (2 \ln |z| + 1)^{1/2}, & e^{-1/2} < |z| < 1, \\ 0, & |z| \leq e^{-1/2} \end{cases}
\]

is a solution of the equation \( f_z = \mu(z) \cdot f_z \), where \( \mu \) is defined by (5.23). Note that the existence of a solution of this equation is ensured by Theorem 5.1 (for this, we verify that all conditions of this Theorem are satisfied). Note that for \( \mu \) in (5.23), the corresponding maximal dilatation \( K_\mu \) is the function

\[
K_\mu(z) = \begin{cases} \frac{1}{1+2 \ln r}, & e^{-1/2} < |z| < 1, \\ 1, & |z| \leq e^{-1/2} \end{cases}
\]

Observe that \( K_\mu(z) \leq k \) for \( |z| \geq e^{\frac{1-k}{2k}} \) and \( K_\mu(z) > k \) otherwise. As above, we set

\[
\mu_k(z) = \begin{cases} \mu(z), & K_\mu(z) \leq k, \\ 0, & K_\mu(z) > k. \end{cases}
\]

Observe that the mappings

\[
f_k(z) = \begin{cases} \frac{z}{|z|} (2 \ln |z| + 1)^{1/2}, & e^{\frac{1-k}{2k}} < |z| < 1, \\ ze^{\frac{k-1}{2k}} \cdot k^{-1/2}, & |z| \leq e^{\frac{1-k}{2k}} \end{cases}
\]
are solutions of the equation \( f_z = \mu_k(z) \cdot f_z \). Besides that, the inverse mappings \( g_k(y) = f_k^{-1}(y) \) are calculated by the relations

\[
g_k(y) = \begin{cases} \frac{y}{|y|} \cdot e^{\frac{|y|^2 - 1}{2}}, & k^{-1/2} < |y| < 1, \\ ye^{\frac{1-k}{2}} \cdot k^{1/2}, & |y| \leq k^{-1/2} \end{cases}.
\] (5.25)

It follows from (5.24) that

\[
K_{\mu_k}(z) = \begin{cases} \frac{1}{1+2\ln|z|}, & e^{\frac{1-k}{2|x|}} < |z| < 1, \\ 1, & |z| \leq e^{\frac{1-k}{2k}}. \end{cases}
\] (5.26)

Now, we substitute the maps \( g_k \) from (5.25) into the maximal dilatation \( K_{\mu_k} \) defined by the equality (5.26). We obtain that

\[
K_{\mu_gk}(y) = \begin{cases} \frac{1}{|y|^2}, & k^{-1/2} < |y| < 1, \\ 1, & |y| \leq k^{-1/2} \end{cases}.
\]

We observe that \( K_{\mu_gk}(y) \) converges pointwise to \( Q(y) = \frac{1}{|y|^2} \). Moreover, by direct calculation we may verify that the function \( Q \) is not integrable in the unit disk \( \mathbb{D} \). It also follows from this that there is no other function \( \varphi \) integrable in the unit disk and such that \( K_{\mu_gk}(y) \leq \varphi(y) \) a.e. Indeed, if such a function existed, then passing here to the limit as \( k \to \infty \) we obtain that \( Q(y) \leq \varphi(y) \), which contradicts the condition of non-integrability of \( Q(y) = \frac{1}{|y|^2} \) in \( \mathbb{D} \). On the other hand, the function \( Q \) (extended by zero outside the unit ball) is integrable over almost all circles \( S(x_0, r) \) for any \( x_0 \in \mathbb{D} \), namely, those that do not pass through the origin.

To complete the consideration of the example, we still need to calculate \( K_{I,p} \) in (5.6). For this purpose, we may use the approach taken when considering [18, Proposition 6.3]. In the notation of this Proposition, we obtain this for the mapping \( g_k \),

\[
\delta_r = e^{\frac{|y|^2 - 1}{2}} \cdot |y|, \quad \delta_r = |y| \cdot e^{\frac{|y|^2 - 1}{2}}, \quad k^{-1/2} < |y| < 1.
\]

Thus, \( \delta_r \geq \delta_r \) and, consequently,

\[
K_{I,p}(y, g_k) = e^{\left(\frac{|y|^2 - 1}{2}\right)(2-p)} \frac{|y|^p}{|y|^p}, \quad k^{-1/2} < |y| < 1.
\] (5.27)

Similarly,

\[
K_{I,p}(y, g_k) = e^{\frac{1-k}{2k} \cdot k^{1/2}} \cdot |y| \leq k^{-1/2}.
\] (5.28)
It follows from (5.27) and (5.28) that

$$\int_{D} K_{I,p}(y, g_k) \, dm(y)$$

$$= \int_{|y| \leq k^{-1/2}} K_{I,p}(y, g_k) \, dm(y) + \int_{k^{-1/2} < |y| < 1} K_{I,p}(y, g_k) \, dm(y)$$

$$= \pi \cdot k^{-1} \cdot \left( e^{\frac{k-1}{2p}} \right)^{2-p} + \int_{k^{-1/2} < |y| < 1} \frac{e^{(\frac{|y|^2 - 1}{2})}(2-p)}{|y|^p} \, dm(y)$$

$$\leq \pi \cdot k^{-1 + \frac{2-p}{2}} + \int_{0 < |y| < 1} \frac{dm(y)}{|y|^p} = \pi k^{-\frac{p}{2}}$$

$$+ 2\pi \int_{0}^{1} r^{1-p} \, dr \leq \pi + 2\pi (2 - p)^{-1} < \infty.$$

Thus, (5.7) is fulfilled with $M = \pi + 2\pi (2 - p)^{-1}$.

**Acknowledgements** The datasets generated and/or analysed during the current study are available from the corresponding author on reasonable request.

**Declarations**

**Conflict of interest** The authors have no relevant financial or non-financial interests to disclose.

**References**

1. Ahlfors, L.: Lectures on Quasiconformal Mappings. Van Nostrand, New York (1966)
2. Bojarski, B.: Generalized solutions of a system of differential equations of the first order of the elliptic type with discontinuous coefficients. Mat. Sb. 43(85), 451–503 (1957)
3. Cristea, M.: Open discrete mappings having local $ACL^p$ inverses. Complex Var. Ellipt. Equ. 55(1–3), 61–90 (2010)
4. Federer, H.: Geometric Measure Theory. Springer, Berlin (1969)
5. Fuglede, B.: Extremal length and functional completion. Acta Math. 98, 171–219 (1957)
6. Gol’dshtein, V., Gurov, L., Romanov, A.: Homeomorphisms that induce monomorphisms of Sobolev spaces. Israel J. Math. 91, 31–60 (1995)
7. Gol’dshtein, V., Ukhlov, A.: About homeomorphisms that induce composition operators on Sobolev spaces. Compl. Vari. Ellipt. Equ. 55(8–10), 833–845 (2010)
8. Gutlyanskii, V., Ryazanov, V., Yakubov, E.: The Beltrami equations and prime ends. J. Math. Sci. 210(1), 22–51 (2015)
9. Gutlyanskii, V.Y., Ryazanov, V.I., Srebro, U., Yakubov, E.: The Beltrami Equation: A Geometric Approach. Springer, New York (2012)
10. Il’yutko, D.P., Sevost’yanov, E.A.: Boundary behaviour of open discrete mappings on Riemannian manifolds. Sborn. Math. 209(5), 605–651 (2018)
11. Kuratowski, K.: Topology, vol. 2. Academic Press, New York, London (1968)
12. Lehto, O., Virtanen, K.: Quasiconformal Mappings in the Plane. Springer, New York (1973)
13. Maly, J., Martio, O.: Lusin’s condition $N$ and mappings of the class $W^{1,n}_{\text{loc}}$. J. Reine Angew. Math. 458, 19–36 (1995)

14. Martio, O., Rickman, S., Väisälä, J.: Definitions for quasiregular mappings. Ann. Acad. Sci. Fenn. Ser. A1(448), 1–40 (1969)

15. Martio, O., Rickman, S., Väisälä, J.: Distortion and singularities of quasiregular mappings. Ann. Acad. Sci. Fenn. Ser. A1(465), 1–13 (1970)

16. Martio, O., Rickman, S., Väisälä, J.: Topological and metric properties of quasiregular mappings. Ann. Acad. Sci. Fenn. Ser. A1(488), 1–31 (1971)

17. Martio, O., Ryazanov, V., Srebro, U., Yakubov, E.: Mappings with finite length distortion. J. d’Anal. Math. 93, 215–236 (2004)

18. Martio, O., Ryazanov, V., Srebro, U., Yakubov, E.: Moduli in Modern Mapping Theory. Springer, LLC, New York (2009)

19. Martio, O., Ryazanov, V., Srebro, U., Yakubov, E.: On $Q$-homeomorphisms. Ann. Acad. Sci. Fenn. Math. 30(1), 49–69 (2005)

20. Ponomarev, S.P.: The $N^{-1}$-property of mappings, and Lusin’s $(N)$ condition. Math. Notes 58(3), 960–965 (1995)

21. Rickman, S.: Quasiregular Mappings. Springer, Berlin (1993)

22. Ryazanov, V., Salimov, R., Sevost’yanov, E.: On convergence analysis of space homeomorphisms. Siber. Adv. Math. 23(4), 263–293 (2013)

23. Ryazanov, V., Salimov, R., Sevost’yanov, E.: On the Hölder property of mappings in domains and on boundaries. J. Math. Sci. 246(1), 60–74 (2020)

24. Ryazanov, V., Sevost’yanov, E.: Toward the theory of ring $Q$-homeomorphisms. Israel J. Math. 168, 101–118 (2008)

25. Reshetnyak, Y.G.: Some geometric properties of functions and mappings with generalized derivatives. Siber. Math. J. 7, 704–732 (1967)

26. Reshetnyak, Y.G.: Space mappings with bounded distortion. In: Transl. Math. Monographs, vol. 73. AMS (1989)

27. Ryazanov, V., Srebro, U., Yakubov, E.: On ring solutions of Beltrami equations. J. d’Anal. Math. 96, 117–150 (2005)

28. Ryazanov, V., Srebro, U., Yakubov, E.: Finite mean oscillation and the Beltrami equation. Israel Math. J. 153, 247–266 (2006)

29. Ryazanov, V., Srebro, U., Yakubov, E.: On convergence theory for Beltrami equations. Ukr. Mat. Visnyk 5(4), 524–535 2008. transl. in Ukr. Math. Bull. 5(4), 517–528 (2008)

30. Saks, S.: Theory of the Integral. Dover, New York (1964)

31. Sevost’yanov, E.A.: On open and discrete mappings with a modulus condition. Ann. Acad. Sci. Fenn. 41, 41–50 (2016)

32. Sevost’yanov, E.A., Skvortsov, S.A.: On mappings whose inverse satisfy the Poletsky inequality. Ann. Acad. Sci. Fenn. Math. 45, 257–259 (2020)

33. Sevost’yanov, E.A., Skvortsov, S.A., Dovhopiatyi, O.P.: On nonhomeomorphic mappings with the inverse Poletsky inequality. J. Math. Sci. 252(4), 541–557 (2021)

34. Suvorov, G.D.: Generalized Principle of Length and Area in Mapping Theory. Naukova Dumka, Kiev (1985)

35. Väisälä, J.: Lectures on $n$-dimensional quasiconformal mappings. In: Lecture Notes in Math., vol. 229. Springer, Berlin (1971)

36. Vuorinen, M.: Conformal geometry and quasiregular mappings. In: Lecture Notes in Math., vol. 1319. Springer, Berlin (1988)

37. Vuorinen, M.: On the existence of angular limits of $n$-dimensional quasiconformal mappings. Ark. Math. 18, 157–180 (1980)

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.