SOME TOPICS RELATED TO BERGMAN KERNEL

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Actually we will discuss some topics related to Bergman kernel on Cartan-Hartogs domain. Cartan-Hartogs domain is introduced by Guy Roos and Weiping Yin in 1998, which is built on the Cartan domains (classical domains). The four big types of Cartan domains can be written as [1]:

\[ R_I(m, n) := \{ Z \in \mathbb{C}^{mn} : I - ZZ^t > 0, \} , \]

\[ R_{II}(p) := \{ Z \in \mathbb{C}^{\frac{p(p+1)}{2}} : I - ZZ^t > 0, \} , \]

\[ R_{III}(q) := \{ Z \in \mathbb{C}^{\frac{q(q-1)}{2}} : I - ZZ^t > 0, \} , \]

\[ R_{IV}(n) := \{ Z \in \mathbb{C}^{n} : 1 + |ZZ^t| - 2ZZ^t > 0, \}
\]

\[ 1 - |ZZ^t|^2 > 0 \} . \]

where \( Z \) is \( m \times n \) matrix, \( p \) degree symmetric matrix, \( q \) degree skew symmetric matrix and \( n \) dimensional complex vector respectively. Then the Cartan-Hartogs domains can be introduced as follows:

\( Y_I := Y_I(N, m, n; K) := \{ W \in \mathbb{C}^N, Z \in R_I(m, n) : |W|^{2K} < \det(I - ZZ^t), K > 0 \} , \)

\( Y_{II} := Y_{II}(N, p; K) := \{ W \in \mathbb{C}^N, Z \in R_{II}(p) : |W|^{2K} < \det(I - ZZ^t), K > 0 \} , \)

\( Y_{III} := Y_{III}(N, q; K) := \{ W \in \mathbb{C}^N, Z \in R_{III}(q) : |W|^{2K} < \det(I - ZZ^t), K > 0 \} , \)

\( Y_{IV} := Y_{IV}(N, n; K) := \{ W \in \mathbb{C}^N, Z \in R_{IV}(n) : |W|^{2K} < 1 - 2ZZ^t + |ZZ^t|^2, K > 0 \} , \)

where \( \det \) indicates the determinant, \( N, m, n, p, q \) are natural numbers. These domains are also called super-Cartan domains.

If the right hand of above inequalities are denoted by the \( N_j := N_j(Z, Z) \), \( j = I, II, III, IV \) respectively, then the definition of Cartan-Hartogs domain can be also written as

\( Y_j = \{ W \in \mathbb{C}^N, Z \in R_j : |W|^{2K} < N_j(Z, Z) \} , j = I, II, III, IV. \)

The following topics will be discussed:

I. The zeroes of Bergman kernel of Cartan-Hartogs domain;

II. The classical (Cannonical) metrics (Bergman metric, Caratheodory metric, Kaehler-Einstein metric, Kobayashi metric) on Cartan-Hartogs domain, which contains Bergman metric equivalent to Kaehler-Einstein metric, Lu Qikeng constant, Bergman Kaehler-Einstein metric and some good new metrics.

III. Generalized Cartan-Hartogs domain;
IV. The centre of representative domain and applications;
V. The solution of Dirichlet’s problem of complex Monge-Ampère equation on Cartan-Hartogs domain and Kaehler-Einstein metric with explicit formula.

I. The zeroes of Bergman kernel on Cartan-Hartogs domain

The Cartan-Hartogs domain of the first type is defined by

\[ Y_I(N, m, n; K) = \{ W \in \mathbb{C}^N, Z \in R_I(m, n) : |W|^{2K} < \det(I - ZZ^*) \}, \ K > 0 \].

And

\[ Y_I(1, 1, n; K) = \{ W \in \mathbb{C}, Z \in \mathbb{C}^n : |W|^{2K} + |z_1|^2 + |z_2|^2 + \cdots + |z_n|^2 < 1 \}. \]

Then the Bergman kernel of \( Y \) is

\[ K_I((W, Z); (\zeta, \xi)) = K^{-n\pi^{-(n+1)}}F(Y)det(I - \overline{Z\xi})^{-(1+n+1/K)}. \]

Where

\[ F(Y) = \sum_{i=0}^{n+1} b_i \Gamma(i + 1)Y^{i+1}, Y = (1 - X)^{-1}, X = W\zeta[\det(I - \overline{Z\xi})]^{-1/K}. \]

And \( b_0 = 0 \), let

\[ P(x) = (x + 1)[(x + 1 + Kn)(x + 1 + K(n - 1)) \cdots (x + 1 + K)]. \]

Then the others \( b_i \) can be got by

\[ b_i = [P(-i - 1) - \sum_{k=0}^{i-1} b_k(-1)^k\Gamma(i + 1)/\Gamma(i - k + 1)][(-1)^i\Gamma(i + 1)]^{-1}. \] (1.1)

Recently, Liyou Zhang prove that above formula can be rewritten as

\[ b_i = \sum_{j=1}^{i} \frac{(-1)^j P(-j - 1)}{\Gamma(j + 1)\Gamma(i - j + 1)}. \]

It is well known that for the first type of Cartan-Hartogs domains there exists the holomorphic automorphism \((W^*, Z^*) = F(W, Z)\) such that \( F(W, Z_0) = (W^*, 0) \) if \((W, Z_0) \in Y_I\). Due to the transformation rule of Bergman kernel, one has

\[ K_I((W, Z); (\zeta, \xi)) = (\det J_F(W, Z))|_{Z_0=Z} K_I([W^*, 0], (\zeta^*, \xi^*)) (\det J_{\overline{F}}(\zeta, \xi)). \]

Therefore the zeroes of \( K_I((W, Z); (\zeta, \xi)) \) are same as the zeroes of \( K_I([W^*, 0], (\zeta^*, \xi^*)) = K^{-n\pi^{-(n+1)}}F(Y) \). Let \( W^* \) be the \( W \), and \( \zeta^* \) be the \( \zeta \), then we have

\[ K^{-n\pi^{-(n+1)}}F(Y) = K^{-n\pi^{-(n+1)}}F(y), y = (1 - \overline{W\xi})^{-1}. \] (1.2)

Where

\[ F(y) = \sum_{i=0}^{n+1} b_i \Gamma(i + 1)y^{i+1}. \] (1.3)

If \((W, 0), (\zeta, 0), (W^*, 0)\) and \((\zeta^*, 0)\) belong to \( Y_I \), then their norms \(|W|, |\zeta|, |W^*|, |\zeta^*|\) are less than 1.

1.1. Let \( t = \overline{W\xi} \), then \(|t| < 1 \), and \( F(y) = (1 - t)^{-(n+2)}G(t) \), where \( G(t) = \sum_{i=0}^{n+1} b_i \Gamma(i + 1)(1 - t)^{n+1-i} \). Therefore to discuss the the presence or absence of zeroes of the Bergman kernel function of \( Y_I(1, 1, n; K) \) can be reduced to discuss the zeroes of polynomial with real coefficients in the unit disk in \( \mathbb{C}[3] \).
Because $y = (1 - W^2)^{-1} = \frac{1}{1 - y^2}$, which maps the unit disk in $t$-plane onto the half-plane in $y$-plane $Re_y > 1/2$. Therefore to discuss the presence or absence of zeroes of the Bergman kernel function of $Y_I(1, 1, n; K)$ can be reduced to discuss the zeroes of polynomial with real coefficients in the right half-plane $Re_y > 1/2$.

Above two statements are true not only for the $Y_I(1, 1, n; K)$ but also for all of the Cartan-Hartogs domains (and Hua domains).

1.2. In the low dimension case, it is very easy to answer the Lu Qi-Keng problem for the Cartan-Hartogs domain. For example, we can say that $Y_I(1, 1, 1; K)$ is Lu Qi-Keng domain.

At this time $Y_I(1, 1, 1; K) = \{ W \in \mathbb{C}, Z \in \mathbb{C} : |W|^{2K} + |Z|^2 < 1 \}$, and the zeroes of its Bergman kernel function $K_I[(W, Z), (\zeta, \xi)]$ are same as the zeroes of $K_I[(W^*, 0), (\zeta^*, 0)]$. But

$$K_I[(W, 0), (\zeta, 0)] = K^{-n}(-1)^2 F(y), F(y) = \sum_{i=0}^{2} b_i \Gamma(i + 1)y^{i+1}$$

$y = (1-t)^{-1}$.

Where $b_1 = K - 1$, $b_2 = 1$, $b_0 = 0$, therefore

$$F(y) = (K - 1)y^2 + 2y^3 = y^3[(K - 1)(1 - t) + 2].$$

But the zeroes of $F(y)$ are equal to $t = (K + 1)/(K - 1)$, its norm $|t| > 1$, it is impossible.

Therefore the Bergman kernel function of $Y_I(1, 1, 1; K)$ is zero-free, that is the $Y_I(1, 1, 1; K)$ is Lu Qi-Keng domain. Therefore we also prove that:

If $D \subset C^2$ is a bounded pseudoconvex domain with real analytic boundary and its holomorphic automorphism group is noncompact, then $D$ is the Lu Qi-Keng domain due to the E. Bedford and S. I. Pinchuk’s following theorem [4].

**Theorem 1.1:** If $D$ is a bounded pseudoconvex domain with real analytic boundary and its holomorphic automorphism group is noncompact, then $D$ is biholomorphically equivalent to a domain of the form

$$E_m = \{(z_1, z_2) \in C^2 : |z_1|^{2m} + |z_2|^2 < 1\}$$

for some positive integer $m$.

**II. THE CLASSICAL (CANNONICAL) METRICS ON CARTAN-HARTOGS DOMAIN**

Let $D$ be the bounded domain in $C^M$, $B_D, C_D, K_D$ denote the Bergman metric, Carathéodory metric, Kaehler-Einstein metric, Kobayashi metric respectively, then we have $C_D \leq B_D, C_D \leq K_D$ [5], and if $D$ is also the convex domain then $C_D = K_D$ [5]. On the other hand, there is no clear relationship between the $B_D$ and $K_D$.

2.1. But we have that $B_D \leq cK_D$ for the Cartan-Hartogs domain where $c$ is the constant [6-9].

2.2. We proved the Bergman metric is equivalent to Kaehler-Einstein metric [10], that is $B_D \sim K_D$ on Cartan-Hartogs domains. For example, we consider the Cartan-Hartogs of the first type $Y_I = Y_I(N, m, n; K)$. Let

$$G_\lambda = G_\lambda(Z, W) = Y^\lambda[det(I - Z\bar{Z}^t)]^{-(m+n+\delta)}, \lambda > 0,$$
\[ T_{\lambda}(Z,W;\overline{Z},\overline{W}) = (g_{i\overline{j}}) = \left( \frac{\partial^2 \log G_{\lambda}}{\partial z_i \partial \overline{z}_j} \right), \]

where

\[ Y = (1 - X)^{-1}, \quad X = |W|^2[\det(I - ZZ^\top)]^{-\frac{1}{n}}. \]

Then \( G_{\lambda} \) induces a metric

\[ Y(I\lambda) := \left( \frac{\langle dw, dz \rangle_{T_{\lambda}(Z,W;\overline{Z},\overline{W})}}{\langle dw, dz \rangle} \right)^{1/2}. \]

Firstly, by the direct computations one can prove that \( B_{Y_I} \sim Y(I\lambda) \).

The \( Y(I\lambda) \) has good properties: Its holomorphic sectional curvature and Ricci curvature are bounded from above and below by the Negative constants. Then based on above good properties and using the Yau’s Schwarz lemma[11] one can prove \( KE_{Y_I} \sim Y(I\lambda) \). Therefore \( KE_{Y_I} \sim B_{Y_I} \), and the metric \( Y(I\lambda) \) may be useful for us.

2.3. Definition: A complex manifold \( M^n \) is called holomorphic homogeneous regular if there are positive constants \( r < R \) such that for each point \( p \in M \), there is a one to one holomorphic map \( f : M \rightarrow \mathbb{C}^n \) such that

i) \( f(p) = 0 \);

ii) \( B_r \subset f(M) \subset B_R \), where \( B_r \) and \( B_R \) are balls with radius \( r \) and \( R \) respectively .

Theorem 2.1(Liu-Sun-Yau)[11,12]: For holomorphic homogeneous regular manifolds, the Bergman metric, the Kobayashi metric and the Caratheodory metric are equivalent.

Therefore if Cartan-Hartogs domains are the holomorphic homogeneous regular manifolds, then the Bergman metric, the Kobayashi metric, the Caratheodory metric and Kaehler-Einstein metric are equivalent. But whether the Cartan-Hartogs domains are the holomorphic homogeneous regular manifolds? This problem remains open.

2.4. From an immediate consequence of an inequality due to Lu Qikeng’s paper [13], we have the following

Theorem 2.2(Lu Qikeng): Let \( D \) be a bounded domain in \( \mathbb{C}^n \). Then for each tangent \( v \in T_z(D) = \mathbb{C}^n \) at \( z \in D \), \( B_D(z,v) \geq C_D(z,v) \). Where \( B_D(z,v) \) equals the length of \( v \) w.r.t. the Bergman metric \( B_D \), and \( C_D(z,v) \) equals the length of \( v \) w.r.t. the differential Caratheodory metric \( C_D \).

Therefore Cheung and Wong introduce the definition of Lu constant \( L(D) \) of a bounded domain \( D \) in \( \mathbb{C}^n \) as follows[5].

Definition:

\[ L(D) = \sup_{z \in D} \sup_{v \neq 0 \in T_z(D)} \left( \frac{C_D(z,v)}{B_D(z,v)} \right). \]

Lu’s theorem says that \( L(D) \leq 1 \). \( L(D) = (1/(n + 1))^{1/2} \) when \( D \) is the unit ball in \( \mathbb{C}^n \). One can try to determine the Lu’s constants of all Cartan domains and all Cartan-Hartogs domains.

2.5. Some years ago S.T.Yau proposed an intricate problem to look for a characterization of the bounded pseudoconvex domains on which the Bergman metrics are complete Kaehler-Einstein metric[5].

The following Lu’s theorem can be viewed as a particular case of Yau’s problem of which the Bergman metric is of constant negative holomorphic sectional curvature:

Theorem 2.3(Lu Qikeng)[14]: Let \( D \) be a bounded domain in \( \mathbb{C}^n \) with a complete Bergman metric \( B_D \). If the holomorphic sectional curvature is equal to
a negative constant \(-c^2\), then \(D\) is biholomorphic to the Euclidean ball \(B_n\) and \(c^2 = \frac{2}{n+1}\).

S.Y. Cheng conjectures that a \textit{strange pseudoconvex domain} whose Bergman metric is Kaehler-Einstein must be biholomorphic to the Euclidean ball[5]. We can also prove that if the Bergman metric of Cartan-Hartogs domain is Kaehler-Einstein, then this Cartan-Hartogs domain must be homogeneous(See below).

III. Generalized Cartan-Hartogs domain

Some years ago we generate the Cartan-Hartogs domain to the Hua domain as follows[3]:

\[
\{ W_j \in \mathbb{C}^N, Z \in R_s : \sum_{j=1}^r \left| W_j \right|^2_{[N_s(Z,\overline{Z})]K_j} < 1, \quad p_j > 0, K_j > 0, j = 1, \ldots, r \}, s = I, II, III, IV.
\]

Right now we will generate the Cartan-Hartogs domain from another way.

Let \(\Omega\) be a domain in \(\mathbb{C}^n\), \(\rho\) a positive continuous function on \(\Omega\), and let \(D\) be a (fixed) irreducible bounded symmetric domain in \(\mathbb{C}^d\). Then Roos, Englš and Zhang define a new domain in \(\mathbb{C}^{n+d}\) as follows[15,16]:

\[
\Omega^D := \{ (w, z) \in \mathbb{C}^d \times \Omega : \frac{w}{\rho(z)} \in D \}. \tag{3.1}
\]

Let \(B(0,1)\) be the unit ball in \(\mathbb{C}^d\), and let \(D = B(0,1), \rho(z) = N_j(z, z)^{1/(2K)}\), then one has

\[
\Omega^{B(0,1)} := \{ (w, z) \in \mathbb{C}^d \times \Omega : \frac{w}{N_j(z, z)^{1/(2K)}} \in B(0,1) \}. \tag{3.2}
\]

The \(\frac{w}{N_j(z, z)^{1/(2K)}} \in B(0,1)\) can be denoted by

\[
\left( \frac{w}{N_j(z, z)^{1/(2K)}} \right) \left( \frac{w}{N_j(z, z)^{1/(2K)}} \right) < 1.
\]

That is

\[
|w|^{2K} < N_j(z, z).
\]

Therefore

\[
\Omega^{B(0,1)} = \{ (w, z) \in \mathbb{C}^d \times \Omega : |w|^{2K} < N_j(z, z) \}. \tag{3.3}
\]

Above (3.3) is the definition of Cartan-Hartogs domain.

Let

\[
D = R_j, \rho(z) = N_i(z, z)^{1/(2K)}, d = \text{dim}D, \Omega = R_i,
\]

then we get the following new domain, which generalizes the Cartan-Hartogs, and is called generalized Cartan-Hartogs domain:

\[
R_i^R_j = \{ (w, z) \in \mathbb{C}^d \times R_i : \frac{w}{N_i(z, z)^{1/(2K)}} \in R_j \}. \tag{3.4}
\]

where \(i, j = I, II, III, IV\). Therefore we get 16 types of generalized Cartan-Hartogs domain as follows:

\[
Y(I, I) = R_i^R_i = \{ (w, z) \in \mathbb{C}^{mn} \times R_i : \frac{w}{\text{det}(I - zz^T)^{1/(K)}} \}.
\]
Y(I, II) := R_{I}^{R_{II}} = \{(w, z) \in C^{p(n+1)/2} \times R_{I} : w\overline{w} < \det(I - z\overline{z})^{1/(K)} f^{(p(n+1)/2)}\}.

Y(I, III) := R_{I}^{R_{III}} = \{(w, z) \in C^{q(n-1)/2} \times R_{I} : w\overline{w} < \det(I - z\overline{z})^{1/(K)} f^{(q(n-1)/2)}\}.

Y(I, IV) := R_{I}^{R_{IV}} = \{(w, z) \in C^{N} \times R_{I} : 2det(I - z\overline{z})^{1/(K)} w\overline{w} - |ww'|^2 < \det(I - z\overline{z})^{2/(K)},
|ww'| < \det(I - z\overline{z})^{1/(K)}\}.

Y(II, I) := R_{II}^{R_{I}} = \{(w, z) \in C^{m} \times R_{I} : w\overline{w} < \det(I - z\overline{z})^{1/(K)} f^{(m)}\}.

Y(II, II) := R_{II}^{R_{II}} = \{(w, z) \in C^{p(n+1)/2} \times R_{II} : w\overline{w} < \det(I - z\overline{z})^{1/(K)} f^{(p)}\}.

Y(II, III) := R_{II}^{R_{III}} = \{(w, z) \in C^{q(n-1)/2} \times R_{II} : w\overline{w} < \det(I - z\overline{z})^{1/(K)} f^{(q)}\}.

Y(II, IV) := R_{II}^{R_{IV}} = \{(w, z) \in C^{N} \times R_{II} : 2det(I - z\overline{z})^{1/(K)} w\overline{w} - |ww'|^2 < \det(I - z\overline{z})^{2/(K)},
|ww'| < \det(I - z\overline{z})^{1/(K)}\}.

Y(III, I) := R_{III}^{R_{I}} = \{(w, z) \in C^{m} \times R_{I} : w\overline{w} < \det(I - z\overline{z})^{1/(K)} f^{(m)}\}.

Y(III, II) := R_{III}^{R_{II}} = \{(w, z) \in C^{p(n+1)/2} \times R_{III} : w\overline{w} < \det(I - z\overline{z})^{1/(K)} f^{(p)}\}.

Y(III, III) := R_{III}^{R_{III}} = \{(w, z) \in C^{q(n-1)/2} \times R_{III} : w\overline{w} < \det(I - z\overline{z})^{1/(K)} f^{(q)}\}.

Y(III, IV) := R_{III}^{R_{IV}} = \{(w, z) \in C^{N} \times R_{III} : 2det(I - z\overline{z})^{1/(K)} w\overline{w} - |ww'|^2 < \det(I - z\overline{z})^{2/(K)},
|ww'| < \det(I - z\overline{z})^{1/(K)}\}.

Y(IV, I) := R_{IV}^{R_{I}} = \{(w, z) \in C^{m} \times R_{IV} : w\overline{w} < (1 + |z\overline{z}'|^2 - 2z\overline{z}')^{1/(K)} f^{(m)}\}.

Y(IV, II) := R_{IV}^{R_{II}} = \{(w, z) \in C^{p(n+1)/2} \times R_{IV} : w\overline{w} < (1 + |z\overline{z}'|^2 - 2z\overline{z}')^{1/(K)} f^{(p)}\}.

Y(IV, III) := R_{IV}^{R_{III}} = \{(w, z) \in C^{q(n-1)/2} \times R_{IV} : w\overline{w} < (1 + |z\overline{z}'|^2 - 2z\overline{z}')^{1/(K)} f^{(q)}\}.

Y(IV, IV) := R_{IV}^{R_{IV}} = \{(w, z) \in C^{N} \times R_{IV} : 2(1 + |z\overline{z}'|^2 - 2z\overline{z}')^{1/(K)} w\overline{w} - |ww'|^2 < (1 + |z\overline{z}'|^2 - 2z\overline{z}')^{2/(K)},
|ww'| < (1 + |z\overline{z}'|^2 - 2z\overline{z}')^{1/(K)}\}.

These are the new research fields, one can seeks the Bergman kernel, Szegö and consider other topics.

IV. THE CENTRE OF REPRESENTATIVE DOMAIN AND APPLICATIONS

The Riemann mapping theorem characterizes the planar domains that are biholomorphically equivalent to the unit disk. In the higher dimensions, there is no Riemann mapping theorem, and the following problem arises:

Are there canonical representatives of biholomorphic equivalence classes of domains?

In the dimension one, if $K(z, w)$ is the Bergman kernel function of simply connected domain $D \neq C$, it is well known that the biholomorphic mapping

$$F(z) = \frac{1}{K(t, t)} \frac{\partial}{\partial\overline{w}} \log K(z, w)|_{w=t}$$

maps the $D$ onto unit disk.

In the higher dimensions, Stefan Bergman introduced the notion of a "representative domain" to which a given domain may be mapped by "representative coordinates". If $D$ is a bounded domain in $C^n$, $K(Z, W)$ is the Bergman kernel function of $D$, let

$$T(Z, Z) = (g_{ij}) = \left( \frac{\partial^2 \log K(Z, W)}{\partial z_i \partial \overline{z}_j} \right)$$
and its converse is $T^{-1}(Z, W) = (g_j^{-1})$. Then the local representative coordinates based at the point $t$ is

$$f_i(Z) = \sum_{j=1}^n g_j^{-1} \frac{\partial}{\partial W_j} \log \frac{K(Z, W)}{K(W, W)} \bigg|_{W=t}, i = 1, \ldots, n.$$  

Or

$$F(Z) = (f_1, \ldots, f_n) = \frac{\partial}{\partial W} \log \frac{K(Z, W)}{K(W, W)} \bigg|_{W=t} T^{-1}(t, t),$$

where

$$\frac{\partial}{\partial W} = \left( \frac{\partial}{\partial W_1}, \frac{\partial}{\partial W_2}, \ldots, \frac{\partial}{\partial W_n} \right).$$

These coordinates take $t$ to 0 and have complex Jacobian matrix at $t$ is equal to the identity. **The $F(D)$ is called the representative domain of $D$. If $D$ is biholomorphic equivalent to $D_1$, then $D$ and $D_1$ have same representative domain.**

Zeroes of the Bergman kernel function $K(Z, W)$ evidently pose an obstruction to the global definition of Bergman representative coordinates. This observation was Lu Qi-Keng’s motivation for asking which domains have zero-free Bergman kernel functions. This problem is called Lu Qi-Keng conjecture by M.Skwarczynski in 1969 in his paper [17]. **If the Bergman kernel function of $D$ is zero-free, that means the Lu Qi-Keng conjecture has a positive answer, then the domain $D$ is called the Lu Qi-Keng domain.**

4.1. In 1981 Lu Qikeng introduces an another definition of "representative domain"[18].

**Definition:** A bounded domain in $\mathbb{C}^n$ is called a representative domain, if there is a point $t \in D$ such that the matrix of the Bergman metric tensor $T(z, t)$ is independent of $z \in D$. The point $t$ is called the centre of the representative domain.

If $D$ is representative domain in the sense of Lu, and $D_1$ is the representative domain of $D$ in the sense of Bergman, then $D$ is same as the $D_1$ under an affine transformation.

4.2. In 1981, Lu Qikeng[18] proved the following

**Theorem 4.1:** Let $D$ be a bounded domain and $D_1$ be a representative domain of $D_1$ in $\mathbb{C}^n$ with centre $s_0$. If $f : D \rightarrow D_1$ is a biholomorphic mapping, then $f$ is of the form

$$f(z) = s_0 + \frac{\partial}{\partial t} \log \frac{K(z, \tilde{t})}{K(t, t)} |_{t=t_0} T^{-1}(t_0, \tilde{t}) A.$$  

Moreover $K(z, \tilde{t}_0)$ is zero free when $z \in D$. Where $s_0 = f(t_0)$, $A = (\frac{\partial f}{\partial z})_{z=t_0}$, and $\frac{\partial}{\partial t} = (\frac{\partial}{\partial t_1}, \ldots, \frac{\partial}{\partial t_n})$.

**Corollary 1:** Let $D$ be a bounded domain and $D_1$ be a representative domain of $D_1$ in $\mathbb{C}^n$ with centre $s_0$, if $D$ is biholomorphic equivalent $D_1$, the Bergman kernel of $D_1$ is $K_1(w, \tilde{w})$, then $K_1(w, \tilde{w})$ is zero free when $w \in D_1$.

**Corollary 2:** If $s_0 = 0$, $D = D_1$, then $A = I$, and the holomorphic automorphism $f(z)$ of $D$ has the following form:

$$f(z) = [\frac{\partial}{\partial t} \log \frac{K(z, \tilde{t})}{K(t, t)}]_{t=t_0} T^{-1}(t_0, \tilde{t}).$$
Where \( 0 = f(t_0) \).

4.3. If the full group of holomorphic automorphism is denoted by \( Aut(D) \), and let \( S = \{ z : f(z) = 0, f \in Aut(D) \} \). Then the \( Aut(D) \) is constituted by

\[
f(z) = \left[ \frac{\partial^{\gamma} \log K(z, \overline{t})}{K(t, \overline{t})} \right]_{t = t_0} T^{-1}(t_0, \overline{t}) .
\]

Where \( t_0 \) spreads all over \( S \).

Therefore if \( 0 \) is the centre of representative domain \( D \), and the set \( S \) is got explicitly, then the \( Aut(D) \) can be got explicitly as above. From this, we can get the \( Aut(D) \) with explicit form if \( D \) is the Cartan-Hartogs domain.

V. THE SOLUTION OF DIRICHLET’S PROBLEM OF COMPLEX MONGE-AMPERÉ EQUATION ON CARTAN-HARTOGS DOMAIN AND KAHLER-EINSTEIN METRIC WITH EXPlicit FORMULA

Complex Monge-Ampère equation is the nonlinear equation with high degree, therefore to get its solution is very difficult.

S.Y. Cheng, N.M. Mok, S.T. Yau consider the following Dirichlet’s problem of the complex Monge-Ampère equation:

\[
\begin{aligned}
&\det \left( \frac{\partial^2 g}{\partial z_i \partial \overline{z}_j} \right) = e^{(n+1)g} \quad z \in D, \\
g = \infty &\quad z \in \partial D,
\end{aligned}
\]

And they proved that the above problem exists unique solution[19,20], where the \( g \) can induce the Kaehler-Einstein metric as follows:

\[
(KE_D)^2 = dz \left( \frac{\partial^2 g}{\partial z_i \partial \overline{z}_j} \right)_{\overline{i}, i}. 
\]

We consider the explicit solution of Dirichlet’s problem of complex Monge-Ampère equation on \( Y_1 \):

\[
\begin{aligned}
&\det \left( \frac{\partial^2 g}{\partial z_i \partial \overline{z}_j} \right)_{1 \leq i, j \leq M} = e^{(M+1)g} \quad z \in Y_1, \\
g = \infty &\quad z \in \partial Y_1,
\end{aligned}
\] (5.1)

where \( M = N + mn \) is the complex dimension of \( Y_1 \).

Because \( Y_1 \) is pseudoconvex domain. Therefore the solution of problem (5.1) is existent and unique.

5.1. We prove that the solution of problem (5.1) can be got in semi-explicit formula, and the explicit solution is obtained in special case. That is the following theorem is proved[21]:

**Theorem 5.1:** If \( G(X) \) is the solution of the following problem

\[
\begin{aligned}
&(M + 1)^{-1} \left[ X G' + (m + n + \frac{N}{K}) G' + \left( GG'' - (G')^2 \right) X \right]_{\overline{G}^N_{m+1}} = G, \\
&G(0) = K^{-m}; \lim_{X \to -1} G(X) = \infty, \\
&G(X) = \infty, \\
&(G(X))^{m+1} = G,
\end{aligned}
\] (5.2)

then

\[
g = (M + 1)^{-1} \log[G(X) \det(I - Z \bar{Z}^{m-(m+n+N/K)}].
\]
is the solution of the problem (5.1); if $K = \frac{mn+1}{m+n}$, and

$$G(X) = \left(\frac{m+n}{mn+1}\right)^{mn}(1-X)^{-(M+1)},$$

then the following $g$ is the special solution of the problem (5.1):

$$g = (M+1)^{-1}\log[(\frac{m+n}{mn+1})^{mn}(1-X)^{-(M+1)}\det(I - Z \overline{Z})^{-(m+n+N/K)}]$$

$$= \log[(1-X)^{-1}\det(I - Z \overline{Z})^{-(m+n)/(mn+1)}], \quad (5.3)$$

where

$$X = X(Z, W) = |W|^2[\det(I - Z \overline{Z})]^{-1/K}, G' = \frac{dG(X)}{dX}, G'' = \frac{d^2G(X)}{dX^2}.$$

**Remark 1:** The complex Monge-Ampère equation is the nonlinear equation, hence to get its explicit solution is very difficult. Therefore mathematicians hope to get the solutions for the problem (5.1) by using the numerical method. Due to the above results the numerical method of the problem (5.1) is reduced to the numerical method of the problem (5.2). Which reduce the complexity of the numerical method of problem (5.1) consumedly. Next, if the numerical method of problem (5.1) or the numerical method of problem (5.2) is appeared in the future, then the special solution $g$ (see (5.3)) can be used to check these numerical methods. And if one reduces the complex Monge-Ampère equation in (5.1) by the linearization method, then the $g$ of (5.3) can be also to check the precision and the rationality for the linearization method.

**Remark 2:** Although the problem (5.1) have not been got the explicit solution in general case, but its semi-explicit solution has the form

$$g = (M+1)^{-1}\log[G(X)\det(I - Z \overline{Z})^{-(m+n+N/K)}],$$

where $G(X)$ satisfies the (5.2).

**Remark 3:** The Bergman kernel function of $Y_I(N, m, n; K)$ is

$$K_I(W, Z; W', Z') = K^{-(mn\pi)^{-(m+n+N)}G(X)\det(I - Z \overline{Z})^{-(m+n+N/K)}}.$$

Where

$$G(X) = \sum_{i=0}^{mn+1} b_i \Gamma(N + i)(1-X)^{-(N+i)}, X = X(W, Z) = |W|^2[\det(I - Z \overline{Z})]^{-1/K},$$

$$|W|^2 = \sum_{j=1}^{N} |W_j|^2,$$

and $b_i$ are constants.

If $Y_I$ is homogeneous domain, then its Bergman metric is equal to its Kaehler-Einstein metric, that is $G(X) = \sum_{i=0}^{mn+1} b_i \Gamma(N + i)(1-X)^{-(N+i)}$ must satisfies the equation (5.2). If $G(X)$ is not satisfies (5.2), then $Y_I$ is not homogeneous.

If $G(X)$ is satisfies (5.2), then the Bergman metric of $Y_I$ is equal to the Kaehler-Einstein metric.

By computations, we prove that the $G(X)$ satisfies the equation (5.2) if and only if $m = 1$. That is the $Y_I$ is the unit ball(homogeneous domain).
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