VIRTUAL RATIONAL BETTI NUMBERS OF NILPOTENT-BY-ABELIAN GROUPS

BEHROOZ MIRZAI AND FATEMEH Y. MOKARI

ABSTRACT. In this paper we study virtual rational Betti numbers of a nilpotent-by-abelian group $G$, where the abelianization $N/N'$ of its nilpotent part $N$ satisfies certain tameness property. More precisely, we prove that if $N/N'$ is $2(c(n - 1) - 1)$-tame as a $G/N$-module, $c$ the nilpotency class of $N$, then

$$v_b_j(G) := \sup_{M \in A_G} \dim Q H_j(M, Q)$$

is finite for all $0 \leq j \leq n$, where $A_G$ is the set of all finite index subgroups of $G$.

INTRODUCTION

The virtual rational Betti numbers of a finitely generated group studies the growth of the Betti numbers of the group as one follows passage to subgroups of finite index. Following [7] and [13], we define the $n$-th virtual rational Betti number of a finitely generated group $G$ as

$$v_b_n(G) := \sup_{M \in A_G} \dim Q H_n(M, Q),$$

where $A_G$ is the set of all subgroups of finite index in $G$.

In [7] Bridson and Kochloukova introduced and studied the first virtual rational Betti number of a finitely generated group $G$ and showed that if $G$ is either a finitely presented nilpotent-by-abelian group or an abelian-by-polycyclic group of type $FP_3$, then $v_b_1(G)$ is finite. Moreover, they conjectured that this should be true for all finitely presented soluble groups. As they have shown the finiteness of the first virtual rational Betti numbers of a metabelian group $G$, with normal abelian subgroup $A$ and abelian quotient $Q$ is closely related to the 2-tameness of $A$ as a $Q$-module, an invariant of metabelian groups introduced by Bieri and Strebel [6].

In [13], Kochloukova and the second author extended these results to higher virtual rational Betti numbers of abelian-by-polycyclic groups, by replacing higher tameness with finitely generatedness of high tensor powers of abelian normal subgroups. More precisely, let $A$ be a normal abelian subgroup of $G$ such that the quotient group $Q := G/A$ is polycyclic. If $Q$ is not abelian, we assume that $G$ is of type $FP_3$. Then it is shown in [13, Theorem A] that if $\bigotimes_{Q}^{2n}(A \otimes \mathbb{Z} Q)$ is finitely generated as a $QQ$-module via the diagonal action, then $v_b_j(G)$ is finite for $0 \leq j \leq n$. Note that if $G$
is metabelian, then finitely generatedness of $\bigotimes^n_{\mathbb{Q}}(A \otimes_{\mathbb{Z}} \mathbb{Q})$ is equivalent to $2n$-tameness of $A$ as a $Q$-module (see Theorem 4.1).

Finitely generated soluble groups occurring in applications are often nilpotent -by-abelian-by-finite, that is, any such group $G$ contains subgroups $N \unlhd H \unlhd G$ such that $N$ is nilpotent, $H/N$ abelian and $G/H$ finite. In this paper, we study the virtual rational Betti numbers of nilpotent-by-abelian-by-finite groups. Since $\text{vb}_n(G) = \text{vb}_n(H)$ (Lemma 5.5), it is sufficient to study virtual rational Betti numbers of nilpotent-by-abelian groups. Here is our main theorem.

**Theorem 5.4.** Let $N \rightarrow G \rightarrow Q$ be an exact sequence of groups, where $G$ is finitely generated, $N$ is nilpotent of class $c$ and $Q$ is abelian. If $N/N'$ is $2(c(n-1)+1)$-tame, then for any $0 \leq j \leq n$, $\text{vb}_j(G)$ is finite.

As a motivation for the study of virtual rational Betti numbers, one can mention a result of Lück which says that the $L^2$-Betti numbers can be computed as a limit involving the ordinary Betti numbers of subgroups of finite index. Here we show that for these groups there is no growth, i.e. the sequences remain bounded. This result therefore confirms Lück’s formula by establishing a stronger property for this class of groups [16].

To prove our main theorem we needed to study certain aspects of homology of nilpotent groups. Nilpotent groups have a great deal of commutativity built into their structure and they are groups that are “almost abelian”. So it is natural to expect that some of the properties of homology of abelian groups, in some way, may be shared by nilpotent groups. In this article, we will study two such properties. For more similarity between homology of abelian and nilpotent groups we refer the interested reader to [9], [18], [10].

The $n$-th homology of an abelian group $A$ with rational coefficients is isomorphic to $\bigwedge^n_{\mathbb{Q}}(A \otimes_{\mathbb{Z}} \mathbb{Q})$. We prove the analogue of this result for nilpotent groups. More precisely, if $N$ is a nilpotent group of class $c$, then we show that there exists a natural filtration of $H_j(N, \mathbb{Q})$,

$$0 = E_0 \subseteq E_1 \subseteq \cdots \subseteq E_{l-1} \subseteq E_l = H_j(N, \mathbb{Q}),$$

such that for any $0 \leq k \leq l$, $E_k/E_{k-1}$ is a natural subquotient of a vector space from the set $\{\bigotimes^s_{\mathbb{Q}} V\}_{0 \leq s \leq c(j-1)+1}$, where $V := (N/N') \otimes_{\mathbb{Z}} \mathbb{Q}$. When our group is free nilpotent, we show that the above theorem is true even with integral coefficients. Although the existence of the above filtration is not a surprise and can be obtain by easy induction, but the bound $c(j-1)+1$ is new and important for our applications. Furthermore, for groups with small $c$ we show that this bound is sharp. The proofs of these results occupy Sections 1 and 2.

Let $N$ be a nilpotent normal subgroup of a group $G$. If $G$ acts nilpotently on $N/N'$, then Theorem 2.1 implies that $G$ acts nilpotently on $H_k(N, \mathbb{Q})$. But with a direct method we can prove a more general result. Let $T$ be an $RG$-module, where $R$ is a commutative ring. In Section 3, we will show that if $G$ acts nilpotently on both $N/N'$ and $T$, then $G$ acts nilpotently on
each $H_k(N, T)$ and $H^k(N, T)$. As an application, we show that if moreover $G/N$ is finite and $l$-torsion and $1/l \in R$, then the natural action of $G/N$ on $H_k(N, T)$ and $H^k(N, T)$ is trivial and therefore the natural maps
\[
\text{cor}^G_N : H_k(N, T) \to H_k(G, T), \quad \text{res}^G_N : H^k(G, T) \to H^k(N, T)
\]
are isomorphisms.

Both of these results about the homology of nilpotent groups are used in the proof of our main theorem (Theorem 5.4).

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1. Differentials of the Lyndon-Hochschild-Serre spectral sequence

Let $G$ be a group, $A$ an abelian normal subgroup of $G$ and $Q := G/A$. Let
\[
M^2_{p,q} = H_p(Q, H_q(A, M)) \Rightarrow H_{p+q}(G, M)
\]
be the Lyndon-Hochschild-Serre spectral sequence associated to the exact sequence of groups
\[
A \hookrightarrow G \twoheadrightarrow Q,
\]
where here $M$ is either $\mathbb{Z}$ or $\mathbb{Q}$ with the trivial action of $G$. In this section, we would like to give an explicit formula for the differentials
\[
d^2_{2,q} : \mathbb{Q}E^2_{2,q} \to \mathbb{Q}E^2_{0,q+1},
\]
for any $q \geq 0$, when $A$ is central, i.e. $A \subseteq Z(G)$.

Let $\phi : A \otimes_{\mathbb{Z}} H_q(A, M) \to H_{q+1}(A, M)$ be the natural product map [8, Chap. V, §5], say induced by the shuffle product on the bar resolution, and consider the following composition
\[
(1.1) \quad H^2(Q, A) \otimes_{\mathbb{Z}} H_p(Q, H_q(A, M)) \xrightarrow{- \cap -} H_{p-2}(Q, A \otimes_{\mathbb{Z}} H_q(A, M)) \xrightarrow{H_{p-2}(\text{id}_Q, \phi)} H_{p-2}(Q, H_{q+1}(A, M)),
\]
where $- \cap -$ is the cap product [8, Chap. V, §3].

Let $\rho$ be the element of $H^2(Q, A)$ associated to $A \hookrightarrow G \twoheadrightarrow Q$ [8, Chap. IV, Theorem 3.12] and set
\[
\Delta(\rho) := H_{p-2}(\text{id}_Q, \phi) \circ (\rho \cap -) : H_p(Q, H_q(A, M)) \to H_{p-2}(Q, H_{q+1}(A, M)).
\]
André has proved the following fact.
Proposition 1.1. Let an exact sequence \( A \rightarrowtail G \twoheadrightarrow Q \) be given as in above. Then
\[
d_{p,q}^2 = d_{p,q}^2 + \Delta(\rho),
\]
where \( d_{p,q}^2 \) is the differential of the Lyndon-Hochschild-Serre spectral sequence associated to the semidirect product extension \( A \rightarrow A \times Q \rightarrow Q \).

Proof. See [3, p. 2670] \( \square \)

Now let \( A \) be a central subgroup of \( G \). Then the conjugate action of \( Q \) on \( A \) is trivial and thus \( A \times Q = A \times Q \). It is well-known and easy to prove that in this case, for any \( p \) and \( q \), \( d_{p,q}^2 = 0 \) and therefore
\[
d_{p,q}^2 = \Delta(\rho).
\]

Moreover, since \( A \) is central, the action of \( Q \) on \( H_q(A, M) \) is trivial. Thus for \( M = Q \), the Universal Coefficient Theorem implies that
\[
\bigotimes Q_{p,q}^2 = H_p(Q, Z) \otimes Z H_q(A, Q) \simeq H_p(Q, Z) \otimes Z \Lambda_q^q(A \otimes Z Q).
\]
If \( p = 2 \), then (1.1) finds the following form
\[
H^2(Q, A) \otimes Z H_2(Q, Z) \otimes Z H_q(A, Q) \xrightarrow{(-\cap-) \otimes \text{id}} A \otimes Z H_q(A, Q) \xrightarrow{\Delta} H_{q+1}(A, Q),
\]
where
\[
- \cap - : H^2(Q, A) \otimes Z H_2(Q, Z) \rightarrow A
\]
is the cap product. Therefore from formula (1.2), we obtain the following explicit formula
\[
d_{2,q}^2 : \bigotimes Q_{2,q}^2 = H_2(Q, Z) \otimes Z \Lambda_q^2(A \otimes Z Q) \rightarrow \bigotimes Q_{0,q+1}^2 = \Lambda_q^q(A \otimes Z Q),
\]
\[
x \otimes (a_1 \land \cdots \land a_q) \mapsto (\rho \cap x) \land a_1 \land \cdots \land a_q.
\]
Thus we have proved the following proposition.

Proposition 1.2. Let \( G \) be a group, \( A \) a central subgroup of \( G \) and \( Q := G/A \). Let
\[
\bigotimes Q_{p,q}^2 = H_p(Q, H_q(A, Q)) \Rightarrow H_{p+q}(G, Q)
\]
be the Lyndon-Hochschild-Serre spectral sequence associated to the extension \( A \rightarrowtail G \twoheadrightarrow Q \). Then for any \( q \geq 0 \), the differential
\[
d_{2,q}^2 : \bigotimes Q_{2,q}^2 = H_2(Q, Z) \otimes Z \Lambda_q^2(A \otimes Z Q) \rightarrow \bigotimes Q_{0,q+1}^2 = \Lambda_q^q(A \otimes Z Q),
\]
is given by the formula \( x \otimes (a_1 \land \cdots \land a_q) \mapsto (\rho \cap x) \land a_1 \land \cdots \land a_q \). Here \( \rho \) is the element of \( H^2(G, A) \) associated to the above extension and the map \(- \cap - : H^2(Q, A) \otimes Z H_2(Q, Z) \rightarrow A\) is the cap product. If \( A \) is torsion free, then the same result is true for \( d_{2,q}^2 : \bigotimes Q_{2,q}^2 \rightarrow \bigotimes Q_{0,q+1}^2 \).

The following corollary will be needed in the next section.

Corollary 1.3. Let \( G, A, Q \) and \( \bigotimes Q_{p,q}^2 \) be as in Proposition 1.2. If \( A \subseteq Z(G) \cap G' \), then \( d_{2,q}^2 : \bigotimes Q_{2,q}^2 \rightarrow \bigotimes Q_{0,q+1}^2 \) is surjective for any \( q \geq 0 \) and therefore \( \bigotimes Q_{\infty,q}^2 = \bigotimes Q_{0,q}^2 \). Moreover, if \( A \) is torsion free, then the same results hold for \( d_{2,q}^2 : \bigotimes Q_{2,q}^2 \rightarrow \bigotimes Q_{0,q+1}^2 \).
But, from the above, we know that this map is given by the formula 
\[ \rho \] the action of 
\[ N/\gamma \] conjugate action of 
nilpotent groups of class 
\[ d \gamma \] From the exact sequence 

Proof. The spectral sequence \( M^2 \mathcal{E}_{p,q} \), gives us the five term exact sequence 
\[
H_2(G, M) \rightarrow H_2(Q, M) \xrightarrow{d_{2,0}} H_1(A, M)_Q \rightarrow H_1(G, M) \rightarrow H_1(Q, M) \rightarrow 0,
\]
[8, Chap. VII, Corollary 6.4]. Clearly \( H_1(G, Z) \simeq H_1(Q, Z) \simeq G/G' \). Since the action of \( Q \) on \( A \) is trivial, we have \( H_1(A, Z)_Q \simeq H_1(A, Z) = A \). Thus from the above exact sequence, we obtain the surjective map 
\[ d_{2,0}^2 : H_2(Q, Z) \rightarrow A. \]
But, from the above, we know that this map is given by the formula \( x \mapsto \rho \cap x \). Now by Proposition \( 1.2 \), \( d_{2,q}^2 \) is surjective and this immediately implies that \( \mathcal{E}_{0,q}^\infty = \mathcal{E}_{0,q}^3 = 0. \)

2. Homology of nilpotent groups

Let \( N \) be a nilpotent group of class \( c \) and consider its lower central series, 
\[ 1 = \gamma_{c+1}(N) \subset \gamma_c(N) \subset \cdots \subset \gamma_2(N) \subset \gamma_1(N) = N. \]
From the exact sequence \( \gamma_c(N) \twoheadrightarrow N \twoheadrightarrow N/\gamma_c(N) \), we obtain the Lyndon-Hochschild-Serre spectral sequence 
\[
E_{p,q}^2 = H_p(N/\gamma_c(N), H_q(\gamma_c(N), T)) \Rightarrow H_{p+q}(N, T),
\]
where \( T \) is a \( N \)-module.

Since \( \gamma_{c+1}(N) = [\gamma_c(N), N] = 1 \), it follows that \( \gamma_c(N) \subseteq Z(N) \). So the conjugate action of \( N/\gamma_c(N) \) on \( \gamma_c(N) \) is trivial. This also implies that the action of \( N/\gamma_c(N) \) on \( H_q(\gamma_c(N), T) \) is trivial, provided that the action of \( N \) on \( T \) is trivial.

Theorem 2.1. Let \( N \) be a nilpotent group of class \( c \). Then there exists a natural filtration of \( H_j(N, Q) \), 
\[
0 = E_0 \subseteq E_1 \subseteq \cdots \subseteq E_{l-1} \subseteq E_l = H_j(N, Q),
\]
such that for any \( 0 \leq k \leq l \), \( E_k/E_{k-1} \) is a natural subquotient of a vector space from the set \( \{ \bigotimes_{Q}^j V \}_{0 \leq s \leq c(j-1)+1}, \) where \( V := (N/N') \otimes_{Q} Q \).

Proof. We prove the claim by induction on \( c \). All filtrations, homomorphisms and subquotients that will be considered in this proof are natural. If \( c = 1 \), then \( N' = \gamma_2(N) = 1 \). Thus \( N \) is abelian and by [8, Theorem 6.4, Chap. V] we have 
\[
H_j(N, Q) \simeq (\bigwedge_{Z}^j N) \otimes_{Q} Q \simeq \bigwedge_{Q}^j V.
\]
Clearly \( \bigwedge_{Q}^j V \) is of the form \( (\bigotimes_{Q}^j V)/T \), for some subspace \( T \) of \( \bigotimes_{Q}^j V \). Since \( j = 1(j-1) + 1 = c(j-1) + 1 \), our claim is valid for \( c = 1 \).

Now let \( c \geq 2 \) and assume that the claim of the theorem is true for all nilpotent groups of class \( d \), \( 1 \leq d \leq c - 1 \). The spectral sequence (2.1) gives us a filtration of \( H_j(N, Q) \)
\[
0 = F_{-1}H_j \subseteq F_0H_j \subseteq \cdots \subseteq F_{j-1}H_j \subseteq F_jH_j = H_j(N, Q),
\]
such that $E_{i,j-i}^\infty \simeq F_iH_j/F_{i-1}H_j$, $0 \leq i \leq j$. By Corollary 1.3, $E_{0,j}^\infty = 0$, so $F_0H_j = F_0H_j/F_{-1}H_j \simeq E_{0,j}^\infty = 0$.

We know that $E_{i,j-i}^2$ is a subquotient of

$$E_{i,j-i}^2 \simeq H_i(N/\gamma_c(N), \mathbb{Q}) \otimes_{\mathbb{Q}} H_{j-i}(\gamma_c(N), \mathbb{Q}).$$

The group $\gamma_c(N)$ is abelian, so

$$H_{j-i}(\gamma_c(N), \mathbb{Q}) \simeq \bigwedge_{\mathbb{Q}}^{j-i}(\gamma_c(N) \otimes \mathbb{Q}).$$

There is a natural surjective map $\bigotimes_{\mathbb{Q}}\mathbb{C}(N/N') \to \gamma_c(N)$, which induces a surjective map

$$\bigwedge_{\mathbb{Q}}^{j-i}(\bigotimes_{\mathbb{Q}}^C V) \to \bigwedge_{\mathbb{Q}}^{j-i}(\gamma_c(N) \otimes \mathbb{Q})$$

and clearly from this we obtain a surjective map

$$(2.2) \quad \bigotimes_{\mathbb{Q}}^C (j-i) \to H_{j-i}(\gamma_c(N), \mathbb{Q}).$$

This implies that $F_iH_j/F_{i-1}H_j$ is a subquotient of

$$(2.3) \quad H_i(N/\gamma_c(N), \mathbb{Q}) \otimes_{\mathbb{Q}} \bigotimes_{\mathbb{Q}}^C(j-i) V.$$

On the other hand, since $N/\gamma_c(N)$ is nilpotent of class $c-1$, by the induction hypothesis, for any $1 \leq i \leq j$, we have a filtration of $H_i(N/\gamma_c(N), \mathbb{Q})$,

$$0 = G_{0,i} \subseteq G_{1,i} \subseteq \cdots \subseteq G_{k_i,i} \subseteq G_{k_i,i} = H_i(N/\gamma_c(N), \mathbb{Q})$$

such that for any $0 \leq t \leq k_i$, $G_{t,i}/G_{t-1,i}$ is a subquotient of some $\bigotimes_{\mathbb{Q}}^s V$, where $0 \leq s_{t,i} \leq (c-1)(i-1) + 1$ (Note that $(N/\gamma_c(N))/(N/\gamma_c(N))' = N/N'$). This together with (2.3) imply that $F_iH_j/F_{i-1}H_j$ is a subquotient of some $\bigotimes_{\mathbb{Q}}^s V$, where

$$0 \leq s_t \leq (c-1)(i-1) + 1 + c(j-i) = c(j-1) - i + 2 \leq c(j-1) + 1.$$

This finishes the induction step and so the proof of the theorem. □

With some restriction on $N$, one can obtain similar results for integral homology.

**Proposition 2.2.** Let $N$ be a free nilpotent group of class $c$. Then there exists a natural filtration of $H_j(N, \mathbb{Z})$,

$$0 = E_0 \subseteq E_1 \subseteq \cdots \subseteq E_{l-1} \subseteq E_l = H_j(N, \mathbb{Z}),$$

such that for any $0 \leq k \leq l$, $E_k/E_{k-1}$ is a natural subquotient of a $\mathbb{Z}$-module from the set $\{\bigotimes_{\mathbb{Q}}^V V\}_{0 \leq s \leq c(j-1)+1}$, where $V := N/N'$.

**Proof.** Since $N$ is a free nilpotent group, $\gamma_c(N)$ is torsion free. Thus

$$H_n(\gamma_c(N), \mathbb{Z}) \simeq \bigwedge_{\mathbb{Z}}^n \gamma_c(N)$$

(see [8, Theorem 6.4, Chap. V]) and so it is torsion free. This implies that

$$E_{i,j-i}^2 \simeq H_i(N/\gamma_c(N), \mathbb{Z}) \otimes_{\mathbb{Z}} H_{j-i}(\gamma_c(N), \mathbb{Z}).$$

Now the proof is similar to the proof of Theorem 2.1. □
Remark 2.3. We believe that \( c(j - 1) + 1 \) is a sharp bound for the existence of a filtration with the above property for \( H_j(N, \mathbb{Q}) \). At least this is true for the extreme cases \( c = 1 \) (abelian \( N \)) or \( j = 1 \) (first homology group case). Also the above proof shows that \( E_1 = F_1 H_j \) is a quotient of \( \bigotimes_{\mathbb{Z}}^j N' \). This gives an evidence for the fact that the bound \( c(j - 1) + 1 \) in Theorem 2.1 is sharp.

Remark 2.4. If \( N \) is a nilpotent group of class \( c \), then the above theorem also is true for \( H_2(N, \mathbb{Z}) \). By this we mean that there exist a natural filtration of \( H_2(N, \mathbb{Z}) \),

\[
0 = E_0 \subseteq E_1 \subseteq \cdots \subseteq E_{l-1} \subseteq E_l = H_2(N, \mathbb{Z}),
\]

such that for any \( 0 \leq k \leq l \), \( E_k/E_{k-1} \) is a natural subquotient of a \( \mathbb{Z} \)-module from the set \( \{ \bigotimes_{\mathbb{Z}}^k (N/N') \}_{0 \leq s \leq c+1} \). This follows from the above proof, using the facts that for an abelian group \( A \), \( H_2(A, \mathbb{Z}) \simeq A \wedge A \) and also for \( 0 \leq i \leq 2 \),

\[
E_{2-i}^2 \simeq H_1(\gamma_c(N), \mathbb{Z}) \otimes H_{2-i}(\gamma_c(N), \mathbb{Z}).
\]

If \( c = 2 \), the complete structure of \( H_2(N, \mathbb{Z}) \) is established in [11]. This description is simple if \( N \) is torsion-free. In this case \( N/\gamma_2(N) \) is torsion-free and we obtain a filtration

\[
0 \subseteq F_1 H_2 \subseteq F_2 H_2 = H_2(N, \mathbb{Z})
\]

such that

\[
F_1 H_2 \simeq \frac{(N/N') \otimes \mathbb{Z} N'}{(xN' \otimes [y, z] + yN' \otimes [z, x] + zN' \otimes [x, y]) \mid x, y, z \in N}
\]

and

\[
F_2 H_2/F_1 H_2 \simeq \ker \left( (N/N') \wedge (N/N') \rightarrow N', xN' \wedge yN' \mapsto [x, y] \right).
\]

Remark 2.5. Let \( N \) be a free nilpotent group of finite rank and of class \( c = 2 \). Then by [14, p. 532], the differential

\[
d_{p,q}^2 : E_{p,q}^2 = \Lambda_{\mathbb{Z}}^p(N/N') \otimes \Lambda_{\mathbb{Z}}^q N' \rightarrow E_{p-2,q+1}^2 = \Lambda_{\mathbb{Z}}^{p-2}(N/N') \otimes \Lambda_{\mathbb{Z}}^{q+1} N'
\]

of the spectral sequence (2.1) is given by the formula

\[
d_{p,q}^2(a_1 N' \wedge \cdots \wedge a_p N' \otimes x_1 \wedge \cdots \wedge x_q) = \sum_{k<l} (-1)^{k+l-1} a_1 N' \wedge \cdots \wedge a_k N' \wedge \cdots \wedge a_l N' \wedge \cdots \wedge a_p N' \otimes [a_k, a_l] \wedge x_1 \wedge \cdots \wedge x_q.
\]

Also in [14, Theorem 4], it is shown that

\[
H_j(N, \mathbb{Z}) \simeq \bigoplus_{i=1}^j E_{i,j-i}^3
\]

(note that \( E_{0,j}^3 = 0 \)). This means that the filtration of \( H_j(N, \mathbb{Z}) \) induced by the spectral sequence,

\[
0 = F_0 H_j \subseteq F_1 H_j \subseteq \cdots \subseteq F_{j-1} H_j \subseteq F_j H_j = H_j(N, \mathbb{Z}),
\]
has the form
\[ F_i H_j / F_{i-1} H_j \simeq E^3_{i,j-i} \subseteq \left( \bigwedge^i_z(N/N') \otimes \mathbb{Z} \bigwedge^{j-i+1} N' \right) / T_{i,j-i}, \]
where \( T_{i,j-i} \) is generated by the elements
\[
\sum_{k<l} (-1)^{k+l-1} y_1 \wedge \cdots \wedge y_k \wedge \cdots \wedge y_l \wedge \cdots y_{i+2} \otimes [y_k, y_l] \wedge x_1 \wedge \cdots \wedge x_{j-i-1},
\]
where \( y_h \in N/N' \), \( x_g \in N' \). This shows that \( F_i H_j \simeq E^3_{1,j-1} \) from the filtration is a quotient of \( \bigotimes Z^{2j-1}(N/N') \) and is non-trivial. So the bound \( 2j - 1 = c(j - 1) + 1 \) in Theorem 2.1 is sharp.

**Corollary 2.6.** Let \( N \rightarrow G \rightarrow Q \) be an exact sequence of groups, where \( N \) is nilpotent of class \( c \). Then there exist a natural filtration of \( QQ \)-submodules of \( H_j(N, Q) \),
\[
0 = E_0 \subseteq E_1 \subseteq \cdots \subseteq E_{l-1} \subseteq E_l = H_j(N, Q),
\]
such that for any \( 0 \leq k \leq l \), \( E_k / E_{k-1} \) is a natural subquotient of a \( QQ \)-module from the set \( \{ \bigotimes Q^s V \}_{0 \leq s \leq c(j-1)+1} \), where \( V := (N/N') \otimes \mathbb{Z} \), and \( \bigotimes Q V \) is considered as a \( QQ \)-module via the diagonal action of \( Q \).

**Proof.** We have a natural action of \( Q \) on \( H_q(\gamma c(N), Q) \) and \( H_p(N/\gamma c(N), Q) \). From these we obtain a natural action of \( Q \) on the Lyndon-Hochschild-Serre spectral sequence
\[
E^2_{p,q} = H_p(N/\gamma c(N), H_q(\gamma c(N), Q)) \Rightarrow H_{p+q}(N, Q).
\]
This means that the groups \( E^2_{p,q} \) are \( QQ \)-modules and the differentials \( d^2_{p,q} \) are homomorphisms of \( QQ \)-modules. This implies that we have a filtration of \( QQ \)-submodules of \( H_j(N, Q) \)
\[
0 = F_{i-1} H_j \subseteq F_0 H_j \subseteq \cdots \subseteq F_{j-1} H_j \subseteq F_j H_j = H_j(N, Q),
\]
such that each \( E^\infty_{i,j-i} \simeq F_i H_j / F_{i-1} H_j, 0 \leq i \leq j \), is an isomorphism of \( QQ \)-modules.

It is also easy to see that if \( \bigotimes Z^c(N/N') \) is considered as \( ZQ \)-module via the diagonal action of \( Q \), then the natural map \( \bigotimes Z^c(N/N') \rightarrow \gamma c(N) \) is a homomorphism of \( ZQ \)-modules. Now if we follow the proof of Theorem 2.1, we see that in all steps of the proof the \( QQ \)-structure is preserved. This means that all subquotients considered in the proof of Theorem 2.1 are \( QQ \)-subquotients (i.e. the subquotient structure commutes with the \( Q \)-action) and the maps are \( QQ \)-homomorphisms, etc. Therefore, as in the proof of Theorem 2.1, we obtain the desired filtration. \( \square \)

3. **Nilpotent action on the homology of nilpotent groups**

We say that a group \( G \) acts nilpotently on a \( G \)-module \( T \), if \( T \) has a filtration of \( G \)-submodules
\[
0 = T_0 \subseteq T_1 \subseteq \cdots \subseteq T_{k-1} \subseteq T_k = T,
\]
such that the action of $G$ on each quotient $T_i/T_{i-1}$ is trivial.

Corollary 2.6 shows that if $Q = G/N$ acts nilpotently on $N/N'$, then it act nilpotently on $H_j(N,Q)$ for any $j \geq 0$. This fact can be generalized as follow.

**Theorem 3.1.** Let $G$ be a group, $N$ a nilpotent normal subgroup of $G$ and let $T$ be a $G$-module. If $G$ acts nilpotently on $N/N'$ and $T$, then, for any $k \geq 0$, $G$ acts nilpotently on $H_k(N,T)$ and $H^k(N,T)$.

**Proof.** We prove the claim for the homology functor. The proof for the cohomology functor is similar. The proof is in three steps.

**Step 1.** $N$ is abelian and $T$ is a trivial $G$-module: Let

$$0 = N_0 \subseteq N_1 \subseteq \cdots \subseteq N_n = N$$

be a filtration of $N$ such that $G$ acts trivially on each quotient $N_i/N_{i-1}$. We prove this step by induction on the length of the filtration of $N$, i.e. on $n$.

If $n = 1$, then the action of $G$ on $N = N_1$ is trivial. So the action of $G$ on $H_k(N,T)$ also is trivial. From the exact sequence of groups

$$N_1 \rightarrow N \rightarrow N/N_1$$

we obtain the Lyndon-Hochschild-Serre spectral sequence

$$E_{p,q}^2 = H_p(N/N_1, H_q(N_1,T)) \Rightarrow H_{p+q}(N,T).$$

By above, $G$ acts trivially (and so nilpotently) on $H_q(N_1,T)$. Since $G/N_1$ acts nilpotently on $N/N_1$ and $N/N_1$ has a filtration of length $n-1$, by induction hypothesis $G/N_1$, and so $G$, acts nilpotently on each $E_{p,q}^2$. Since $E_{p,q}^{\infty}$ is a subquotient of $E_{p,q}^2$, $G$ acts nilpotently on it too. Moreover, $G$ acts naturally on the above spectral sequence which means that each $E_{p,q}^2$ is a $G$-module and the differentials $d_{p,q}^2$ are homomorphisms of $G$-modules. This implies that we have a filtration of $G$-submodules

$$0 = F_{-1}H_k \subseteq F_0H_k \subseteq \cdots \subseteq F_{k-1}H_k \subseteq F_kH_k = H_k(N,T),$$

such that each isomorphism $E_{i,k-1}^{\infty} \simeq F_iH_k/F_{i-1}H_k$ is an isomorphism of $G$-modules. Thus $G$ acts nilpotently on each quotient $F_iH_k/F_{i-1}H_k$. This implies that $G$ acts nilpotently on $H_k(N,T)$.

**Step 2.** $N$ is abelian and $T$ is any $G$-module: Let

$$0 = T_0 \subseteq T_1 \subseteq \cdots \subseteq T_l = T$$

be a filtration of $T$, such that $G$ acts trivially on each quotient $T_i/T_{i-1}$. In this case we prove the theorem by induction on $l$, the length of the filtration of $T$. If $l = 1$, then the action of $G$ on $T = T_1$ is trivial, so we arrive at Step 1. From the exact sequence

$$0 \rightarrow T_1 \rightarrow T \rightarrow T/T_1 \rightarrow 0$$

we obtain the long exact sequence

$$\cdots \rightarrow H_k(N,T_1) \rightarrow H_k(N,T) \rightarrow H_k(N,T/T_1) \rightarrow \cdots.$$
We know that $G$ acts nilpotently on $H_k(N, T_1)$ and by the induction hypothesis $G$ acts nilpotently on $H_k(N, T/T_1)$. Now the above exact sequence implies that $G$ acts nilpotently on $H_k(N, T)$.

**Step 3.** The general case: The proof of this step is by induction on the nilpotent class $c$ of $N$. If $c = 1$, then $N$ is abelian and this is done in Step 2. Now assume that the claim is true for all nilpotent groups of class $d$, $1 \leq d \leq c - 1$. Consider the lower central series of $N$,

$$1 = \gamma_{c+1}(N) \subset \gamma_c(N) \subset \cdots \subset \gamma_2(N) \subset \gamma_1(N) = N.$$ 

Note that $\gamma_c(N) \subseteq Z(N)$. The exact sequence of groups

$$\gamma_c(N) \hookrightarrow N \twoheadrightarrow N/\gamma_c(N),$$

gives us the Lyndon-Hochschild-Serre spectral sequence

$$E^2_{p,q} = H_p(N/\gamma_c(N), H_q(\gamma_c(N), T)) \Rightarrow H_{p+q}(N, T).$$

We have a natural surjective map

$$\bigotimes \mathbb{Z}(N/N') \twoheadrightarrow \gamma_c(N)$$

which is a map of $G$-modules if we consider $\bigotimes \mathbb{Z}(N/N')$ as a $G$-module via the diagonal action \[15, 1.2.11\]. Since $G$ acts nilpotently on $N/N'$, it also acts nilpotently on $\bigotimes \mathbb{Z}(N/N')$. Thus through the above surjective map, $G$ also acts nilpotently on $\gamma_c(N)$. By Step 2, $G$ acts nilpotently on $H_q(\gamma_c(N), T)$. On the other hand, $N/\gamma_c(N)$ is of nilpotent class $c - 1$ and $G$ acts nilpotently on $(N/\gamma_c(N))/(N/\gamma_c(N))' \simeq N/N'$. So by the induction hypothesis, $G$ acts nilpotently on each $E^2_{p,q}$. Finally by the convergence of the spectral sequence, one can show, as in Step 1, that $G$ acts nilpotently on $H_k(N, T)$. This completes the proof of the theorem. \[127x259, Prop. 4.1, Chap. I\].

If $A$ is an abelian normal subgroup of $G$, then one can show that $G$ is nilpotent if and only if $G/A$ is nilpotent and $G$ acts nilpotently on $A$. One side of this fact can be generalized as follow.

**Corollary 3.2.** Let $G$ be a nilpotent group, $N$ a normal subgroup of $G$ and let $T$ be a $G$-module. If $G$ acts nilpotently on $T$, then for any $k \geq 0$, $G/N$ acts nilpotently on $H_k(N, T)$ and $H^k(N, T)$.

**Proof.** Since $G/N'$ is nilpotent and $N/N'$ is abelian, $G/N'$, and so $G$, acts nilpotently on $N/N'$. Now the claim follows from Theorem 3.1. \[127x259, Proposition 4.1, Chap. I\].

**Lemma 3.3.** Let $G$ be a finite group, $R$ a commutative ring and $T$ an $RG$-module such that $G$ acts nilpotently.

(i) If $1/|G| \in R$, then $T$ is a trivial $G$-module.

(ii) If $G$ is nilpotent, $l$-torsion and $1/l \in R$, then $T$ is a trivial $G$-module.

**Proof.** (i) We know that the functor $- \otimes_G \mathbb{Z} = (-)_G$ is right exact. First we show that this is in fact an exact functor if it is considered as a functor
from the category of $RG$-modules to the category of $R$-modules. Consider the maps
\[ \alpha_G : T^G \to T_G, \quad m \mapsto \overline{m}, \]
and
\[ N : T_G \to T^G, \quad \overline{m} \mapsto Nm, \]
where $N := \sum_{g \in G} g \in RG$. Then clearly $N \circ \alpha$ and $\alpha \circ N$ coincide with multiplication by $|G|$. Since $1/|G| \in R$, $\alpha_G$ is an isomorphism. This implies that $(-)_G$ is exact, because $(-)^G$ is left exact. Next, let
\[ 0 = T_0 \subseteq T_1 \subseteq \cdots \subseteq T_k = T \]
be a filtration of $T$ such that $G$ acts trivially on each $T_i/T_{i-1}$. By applying the exact functor $(-)_G$ to the exact sequence $0 \to T_1 \to T_2 \to T_2/T_1 \to 0$ and using the fact that $G$ acts trivially on $T_1$ and $T_2/T_1$, we see that
\[ 0 \to T_1 \to (T_2)_G \to T_2/T_1 \to 0 \]
is exact. Therefore $T_2 \simeq (T_2)_G$ and so the action of $G$ on $T_2$ is trivial. In a similar way and by induction on $i$, one can show that the action of $G$ on each $T_i$ is trivial. Thus the action of $G$ on $T_k = T$ is trivial.

(ii) First we prove that $(-)_G$ is exact and we do this by induction on the size of $G$. We may assume that $G \neq 1$. Since $G$ is nilpotent, $Z(G) \neq 1$. Let $H$ be a nontrivial cyclic subgroup of $Z(G)$. Then the map $\alpha_G$ coincides with the following composition of maps
\[ T^G \xrightarrow{=} (T^H)^{G/H} \xrightarrow{\alpha_H} (T^H)^{G/H} \xrightarrow{\alpha^{G/H} H} (T^H)_{G/H} \xrightarrow{=} T_G. \]
Now the exactness of the functor $(-)_G$ follows from (i) and the induction step. Finally, as in (i) we can prove that $G$ acts trivially on $T$. \hfill \Box

**Corollary 3.4.** Let $G$ be a nilpotent group and $N$ a normal subgroup of $G$ such that $G/N$ is finite and $l$-torsion. Let $R$ be a commutative ring such that $1/|G| \in R$ and let $T$ be an $RG$-module. If $G$ acts nilpotently on $T$, then, for any $k \geq 0$, the natural action of $G/N$ on $H_k(N,T)$ and $H^k(N,T)$ is trivial and therefore the natural maps
\[ \text{cor}^G_N : H_k(N,T) \to H_k(G,T), \quad \text{res}^G_N : H^k(G,T) \to H^k(N,T) \]
are isomorphisms.

**Proof.** The claim follows from Corollary 3.2 and Lemma 3.3. \hfill \Box

**Corollary 3.5.** Let $G$ be a nilpotent group and $N$ a subgroup of $G$ such that $G/N$ is finite and $l$-torsion. Let $R$ be a commutative ring such that $1/|G| \in R$ and let $T$ be an $RG$-module. If $G$ acts nilpotently on $T$, then, for any $k \geq 0$, the natural maps
\[ \text{cor}^G_N : H_k(N,T) \to H_k(G,T), \quad \text{res}^G_N : H^k(G,T) \to H^k(N,T) \]
are isomorphisms.
Proof. It is well-known that $N$ has a subgroup $L$ such that $L$ is normal in $G$ and $[G:L] \leq [G:N]$!. Now by Corollary 3.4, the maps
\[
\text{cor}_L^G : H_k(L,T) \to H_k(G,T) \quad \text{and} \quad \text{cor}_L^N : H_k(L,T) \to H_k(N,T)
\]
are isomorphisms. Therefore $\text{cor}_N^G : H_k(N,T) \to H_k(G,T)$ is an isomorphism. The cohomology case can be treated in a similar way. □

Example 3.6. In general, in Corollary 3.4 the condition that $[G:N] < \infty$ and $1/l \in R$ can not be removed. In fact, if $N$ is a non-central abelian normal subgroup of a nilpotent group $G$, e.g. $G$ a nilpotent group of class $c = 3$ and $N = G'$, then clearly $G$ does not act trivially on $H_1(N,\mathbb{Z}) = N$.

4. Bieri-Strebel invariant

The main condition of our main Theorem 5.4, proved below, is closely related to an invariant, introduced by Bieri and Strebel [6], which has played a prominent role in the study of soluble groups which are finitely presented.

Let $Q$ be a multiplicative finitely generated abelian group. A homomorphism of groups
\[
v : Q \to \mathbb{R}
\]
is called a valuation on $Q$. If $Q$ has rank $n$, then $\text{Hom}_\mathbb{Z}(Q,\mathbb{R}) \cong \mathbb{R}^n$, so $\text{Hom}_\mathbb{Z}(Q,\mathbb{R})$ can be regarded as a topological vector space. Two valuation $v$ and $v'$ on $Q$ are called equivalent if $v' = av$ for some $a \in \mathbb{R}^{>0}$. We denote the equivalence class of $v$ by $[v]$ and the set $S(Q)$ of all equivalence classes of elements of $\text{Hom}_\mathbb{Z}(Q,\mathbb{R}) \setminus \{0\}$ is called the valuation sphere, which can be identified with the unit sphere $S^{n-1} \subset \mathbb{R}^n$. Notice that $S(Q)$ is empty precisely when $n = 0$, that is, $Q$ is finite. For any valuation $v$ on $Q$ define
\[
Q_v := \{q \in Q | v(q) \geq 0\},
\]
which is a submonoid of $Q$.

For a ring $R$, let $RQ_v$ be the monoid ring, which clearly is a subring of $RQ$. For a finitely generated $RQ$-module $A$, define
\[
\Sigma_A(Q) := \{[v] \in S(Q) | A \text{ is finitely generated over } RQ_v\}.
\]
A finitely generated $RQ$-module $A$ is called $m$-tame if for any $m$ elements $v_1,\ldots,v_m \in \text{Hom}_\mathbb{Z}(Q,\mathbb{R}) \setminus \{0\}$ with $v_1 + \cdots + v_m = 0$, there is $1 \leq i \leq m$ such that $[v_i] \in \Sigma_A(Q)$.

**Theorem 4.1.** Let $Q$ be a finitely generated abelian group, $K$ a field, $A$ a finitely generated $KQ$-module and $m \geq 2$ an integer. Then the following statements are equivalent:

(i) $A$ is $m$-tame as $KQ$-module,

(ii) $\bigotimes_K^m A$ is finitely generated as $KQ$-module via the diagonal $Q$-action,

(iii) $\bigotimes_K^i A$ are finitely generated as $KQ$-modules via the diagonal $Q$-action for $i = 2,\ldots,m$,

(iv) $\bigwedge_K^i A$ are finitely generated as $KQ$-modules via the diagonal $Q$-action for $i = 2,3,\ldots,m$,
(v) $\bigwedge^K_m A$ is finitely generated as $KQ$-module via the diagonal $Q$-action.

Proof. See [5, Theorem C] and [12, Corollary B].

Theorem 4.2. Let $A \rightarrow G \rightarrow Q$ be a short exact sequence of groups with both $A$ and $Q$ abelian and $G$ finitely generated. If $G$ is of type $\text{FP}_m$, then $A \otimes Z K$ is $m$-tame as a $KQ$-module for every field $K$.

Proof. See Theorem D in [5].

5. Virtual rational Betti numbers of nilpotent-by-abelian groups

The following two theorems are taken from [7] and [13], respectively which are very important for the study of virtual rational Betti numbers of abelian-by-polycyclic groups. In this section we will use them for the study of virtual rational Betti numbers of nilpotent-by-abelian groups.

Theorem 5.1 (Bridson-Kochloukova). Let $Q$ be a finitely generated abelian group and $B$ a finitely generated $QQ$-module. If $B \otimes QQ M$ is a finitely generated $QQ$-module via the diagonal action of $Q$, then

$$\sup_{M \in AQ} \dim_Q (B \otimes QQ M) < \infty.$$  

Proof. See Theorem 3.1 in [7].

Theorem 5.2 (Kochloukova-Mokari). Let $Q$ be a finitely generated abelian group and $B$ a finitely generated $QQ$-module. If $\sup_{m \geq 1} \dim_Q (B \otimes QQ^m Q) < \infty$, then for any $i \geq 0$,

$$\sup_{m \geq 1} \dim_Q H_i(Q^m, B) < \infty.$$  

Proof. See Theorem 2.4 in [13].

Lemma 5.3. Let $Q$ be a finitely generated abelian group. Let $V$ be a $QQ$-module such that $\bigotimes^n_Q V$ is a finitely generated $QQ$-module via the diagonal action of $Q$. If $\sup_{m \geq 1} \dim_Q \left( \bigotimes^n_Q^Q V \otimes QQ^m Q \right) < \infty$, then for any $QQ$-subquotient $U$ of $\bigotimes^n_Q V$, we have

$$\sup_{m \geq 1} \dim_Q (U \otimes QQ^m Q) < \infty.$$  

Proof. First let us assume that $U$ is a quotient of $\bigotimes^n_Q V$, i.e. $U = (\bigotimes^n_Q V)/T$, for some $QQ$-submodule $T$ of $\bigotimes^n_Q V$. Then clearly

$$\dim_Q (U \otimes QQ^m Q) \leq \dim_Q \left( \bigotimes^n_Q^Q V \otimes QQ^m Q \right)$$

and thus

$$\sup_{m \geq 1} \dim_Q (U \otimes QQ^m Q) \leq \sup_{m \geq 1} \dim_Q \left( \bigotimes^n_Q^Q V \otimes QQ^m Q \right) < \infty.$$
Next let $U$ be a $\mathbb{Q}Q$-submodule of some $W := (\bigotimes^n_Q V)/T$. Then $W/U$ is of the form $(\bigotimes^n_Q V)/T'$ for some $\mathbb{Q}Q$-submodule $T'$ of $\bigotimes^n_Q V$ and so

$$\sup_{m \geq 1} \dim_{\mathbb{Q}}(W \otimes_{\mathbb{Q}Q}^m \mathbb{Q}) < \infty, \quad \sup_{m \geq 1} \dim_{\mathbb{Q}}((W/U) \otimes_{\mathbb{Q}Q}^m \mathbb{Q}) < \infty.$$  

Now from the exact sequence $0 \to U \to W \to W/U \to 0$, we obtain the long exact sequence

$$\cdots \to \text{Tor}^1_{QQ}(W/U, \mathbb{Q}) \to U \otimes_{\mathbb{Q}Q}^m \mathbb{Q} \to W \otimes_{\mathbb{Q}Q}^m \mathbb{Q} \to (W/U) \otimes_{\mathbb{Q}Q}^m \mathbb{Q} \to 0,$$

which implies that

$$(5.1) \quad \dim_{\mathbb{Q}}(U \otimes_{\mathbb{Q}Q}^m \mathbb{Q}) \leq \dim_{\mathbb{Q}} \text{Tor}^1_{QQ}(W/U, \mathbb{Q}) + \dim_{\mathbb{Q}}(W \otimes_{\mathbb{Q}Q}^m \mathbb{Q}).$$

Since $\sup_{m \geq 1} \dim_{\mathbb{Q}}((W/U) \otimes_{\mathbb{Q}Q}^m \mathbb{Q}) < \infty$, by Theorem 5.2 we obtain

$$(5.2) \quad \sup_{m \geq 1} \dim_{\mathbb{Q}}H_i(Q^m, W/U) < \infty.$$  

But $\text{Tor}^1_{QQ}(W/U, \mathbb{Q}) = H_i(Q^m, W/U)$, thus by (5.1) and (5.2) we have

$$\sup_{m \geq 1} \dim_{\mathbb{Q}}(U \otimes_{\mathbb{Q}Q}^m \mathbb{Q}) < \infty.$$

\[\square\]

The next theorem is the main result of this paper.

**Theorem 5.4.** Let $N \hookrightarrow G \twoheadrightarrow Q$ be an exact sequence of groups, where $G$ is finitely generated, $N$ is nilpotent of class $c$ and $Q$ is abelian. If $N/N'$ is $2(c(n - 1) + 1)$-tame, then for any $0 \leq j \leq n$, $\text{vb}_j(G)$ is finite.

**Proof.** Let $G_1$ be a subgroup of finite index in $G$. Let $Q_1$ be the image of $G_1$ in $Q$ and $N_1 := N \cap G_1$. Then clearly $[Q : Q_1] < \infty$, and $[N : N_1] < \infty$.

From the associated Lyndon-Hochschild-Serre spectral sequence

$$E^2_{p,q} = H_p(Q_1, H_q(N_1, \mathbb{Q})) \Rightarrow H_{p+q}(G_1, \mathbb{Q})$$

of the extension $N_1 \hookrightarrow G_1 \twoheadrightarrow Q_1$, we obtain

$$\dim_{\mathbb{Q}} H_j(G_1, \mathbb{Q}) = \sum_{p=0}^j \dim_{\mathbb{Q}} E^\infty_{p,j-p} \leq \sum_{p=0}^j \dim_{\mathbb{Q}} E^2_{p,j-p}.$$  

Since $[N : N_1] < \infty$, by Corollary 3.4, for any $k \geq 0$, we have

$$H_k(N_1, \mathbb{Q}) = H_k(N, \mathbb{Q}).$$

Thus $E^2_{p,q} \simeq H_p(Q_1, H_q(N, \mathbb{Q}))$. On the other hand, since $[Q : Q_1] < \infty$, there exists $m \in \mathbb{N}$ such that $(Q/Q_1)^m = 1$. Hence $Q^m \subseteq Q_1$. Since $Q_1/Q^m$ is finite, we have

$$H_p(Q_1, H_{j-p}(N, \mathbb{Q})) \simeq H_p(Q^m, H_{j-p}(N, \mathbb{Q}))_{Q_1/Q^m}$$

and this implies that

$$\dim_{\mathbb{Q}} H_p(Q_1, H_{j-p}(N, \mathbb{Q})) \leq \dim_{\mathbb{Q}} H_p(Q^m, H_{j-p}(N, \mathbb{Q})).$$
So to prove the theorem it is sufficient to prove that
\[
\sup_{m \geq 1} \dim_Q H_p(Q^m, H_{j-p}(N, Q)) < \infty.
\]

By Corollary 2.6, \(H_{j-p}(N, Q)\) has a natural filtration of \(Q\)-submodules
\[
0 = E_0 \subseteq E_1 \subseteq \cdots \subseteq E_{l-1} \subseteq E_l = H_{j-p}(N, Q),
\]
such that for any \(0 \leq k \leq l\), \(E_k/E_{k-1}\) is a natural subquotient of a \(Q\)\(Q\)-module from the set \(\{\bigotimes Q V\}_{0 \leq s \leq c(j-p-1)+1}\), where \(V := (N/N') \otimes \mathbb{Q}\) and \(\bigotimes Q V\) is considered as a \(Q\)-module via the diagonal action of \(Q\). By Theorem 4.1, \(\bigotimes Q V\) is a finitely generated \(Q\)-module for \(0 \leq s \leq 2c(j-p-1)+2\). Thus by Theorem 5.1,
\[
\sup_{m \geq 1} \dim_Q (\bigotimes Q V) \otimes_Q Q^m \mathbb{Q}) < \infty \text{ for } 0 \leq s \leq c(j-p-1)+1.
\]
Next Lemma 5.3 implies that
\[
\sup_{m \geq 1} \dim_Q ((E_i/E_{i-1}) \otimes_Q Q^m \mathbb{Q}) < \infty,
\]
and by induction on \(i\), one can show that, for any \(1 \leq i \leq j-p\)
\[
\sup_{m \geq 1} \dim_Q (E_i \otimes_Q Q^m \mathbb{Q}) < \infty.
\]
Therefore
\[
\sup_{m \geq 1} \dim_Q (H_{j-p}(N, Q) \otimes_Q Q^m \mathbb{Q}) = \sup_{m \geq 1} \dim_Q (E_l \otimes_Q Q^m \mathbb{Q}) < \infty.
\]
Now by Theorem 5.2, for any \(0 \leq p \leq j\),
\[
\sup_{m \geq 1} \dim_Q H_p(Q^m, H_{j-p}(N, Q)) < \infty.
\]
This completes the proof of the theorem. \(\square\)

**Lemma 5.5.** Let \(G\) be a group and \(H\) a subgroup of finite index in \(G\). Then \(\text{vb}_n(G)\) is finite if and only if \(\text{vb}_n(H)\) is finite. In fact, for any \(n \geq 0\), \(\text{vb}_n(G) = \text{vb}_n(H)\).

**Proof.** If \(H_0\) is a subgroup of finite index in \(H\), then \([G : H_0] = [G : H][H : H_0] < \infty\). So \(\dim_Q H_0(H_0, \mathbb{Q}) \leq \text{vb}_n(G)\) and hence
\[
\text{vb}_n(H) \leq \text{vb}_n(G).
\]
If \(G_0\) is a subgroup of finite index in \(G\), then \([G_0 : G_0 \cap H] \leq [G : H]\). So there is a normal subgroup \(N\) of \(G_0\) such that \(N \subseteq G_0 \cap H\) and \([G_0 : N] < \infty\). Since \(H_n(G_0, \mathbb{Q}) \simeq H_n(N, \mathbb{Q})_{G_0/N}\), \(\dim_Q H_n(G_0, \mathbb{Q}) \leq \dim_Q H_n(N, \mathbb{Q})\). Now from \([H : N] < \infty\), it follows that \(\dim_Q H_n(G_0, \mathbb{Q}) \leq \dim_Q H_n(N, \mathbb{Q}) \leq \text{vb}_n(H)\). Therefore
\[
\text{vb}_n(G) \leq \text{vb}_n(H).
\]
\(\square\)
Corollary 5.6. Let $G$ be a nilpotent-by-abelian-by-finite group, i.e. we have a chain of subgroups $N \leq H \leq G$, where $N$ is nilpotent, $H/N$ is abelian and $[G : H] < \infty$. If $N$ is of class $c$ and $H/N'$ is of type $\mathrm{FP}_{2c(n-1)+2}$, then $\mathrm{vb}_j(G)$ is finite for any $0 \leq j \leq n$.

Proof. Since $H/N'$ is metabelian of type $\mathrm{FP}_{2c(j-p-1)+2}$, by Theorem 4.2 the $Q$-module $(N/N') \otimes_{\mathbb{Z}} Q$ is $2(c(j-p-1)+1)$-tame. Now the claim follows from Lemma 5.5 and Theorem 5.4. □

Remark 5.7. Theorem 5.4 and Corollary 5.6 generalize [7, Theorem 5.3 and Corollary 5.4] to higher homology groups.

For the first virtual rational Betti number we can improve the above result a bit.

Proposition 5.8. Let $N \hookrightarrow G \twoheadrightarrow Q$ be an exact sequence of groups, where $N$ is nilpotent and $Q$ is polycyclic. Let $G/N'$ be of type $\mathrm{FP}_3$ and let $\bigotimes^2_{\mathbb{Z}} N/N'$ be finitely generated as $\mathbb{Z}Q$-module via the diagonal action. Then $\mathrm{vb}_1(G)$ is finite.

Proof. Let $G_1$ be a normal subgroup of finite index in $G$. Let $Q_1$ be the image of the $G_1$ in $Q$ and $N_1 = N \cap G_1$. The associated Lyndon-Hochschild-Serre spectral sequence of $N_1 \hookrightarrow G_1 \twoheadrightarrow Q_1$, i.e.

$$E^2_{p,q} = H_p(Q_1, H_q(N_1, Q)) \Rightarrow H_{p+q}(G_1, Q),$$

implies that

$$\dim_Q H_1(G_1, Q) \leq \dim_Q E^2_{0,1} + \dim_Q E^2_{1,0}$$

$$\quad = \dim_Q H_0(Q_1, H_1(N_1, Q)) + \dim_Q H_1(Q_1, Q).$$

Since any subgroup of a polycyclic group is polycyclic, by [13, Lemma 3.2] we have $\dim_Q H_1(Q_1, Q) \leq h(Q)$, where $h(Q)$ is the Hirsch length of $Q$. Since $[N : N_1] < \infty$, by Corollary 3.5 we have $H_1(N_1, Q) \simeq H_1(N, Q)$. So to prove the claim it is sufficient to prove that

$$\sup_{[Q:Q_1] < \infty} \dim_Q (N/N' \otimes_{Q_1} Q) < \infty.$$ 

Let $A = N/N'$ and $H = G/N'$ and consider the exact sequence $A \hookrightarrow H \twoheadrightarrow Q$. If we put $A_0 = [A, H]$ and $Q_0 = H/A_0$ and if we follow the proof of Theorem A in [13], we obtain

$$\sup_{[Q_0 : Q_2] < \infty} \dim_Q (A_0 \otimes_{Q_2} Q) < \infty.$$ 

From the exact sequence $A_0 \hookrightarrow A \twoheadrightarrow A/A_0$, we obtain the exact sequence

$$A_0 \otimes_{Q_2} Q \hookrightarrow A \otimes_{Q_2} Q \twoheadrightarrow (A/A_0) \otimes_{Q_2} Q \to 0,$$

which implies that

$$\dim_Q (A \otimes_{Q_2} Q) \leq \dim_Q (A_0 \otimes_{Q_2} Q) + \dim_Q ((A/A_0) \otimes_{Q_2} Q).$$
Now consider the exact sequence $A/A_0 \rightarrow Q_0 \rightarrow Q$ and let $Q_1 = \beta(Q_2)$. Since the action of $A/A_0$ over $A$ is trivial, we have $A \otimes Q_1 \simeq A \otimes Q_2 \otimes Q$. Since $A/A_0$ is a finitely generated abelian group,

$$\sup_{[Q_0:Q_2]\leq\infty} \dim_Q ((A/A_0) \otimes Q_2 \otimes Q) < \infty.$$ 

Therefore from the above relations we have

$$\sup_{[Q:Q_1]\leq\infty} \dim_Q (A \otimes Q_1 \otimes Q) < \infty.$$ 

This completes the proof of the theorem.

**Corollary 5.9.** Let $N \rightarrow G \rightarrow Q$ be an exact sequence of groups, where $N$ is nilpotent and $Q$ is nilpotent of class $c \leq 2$. If $G/N'$ is of type $FP_3$, then $vb_1(G)$ is finite.

**Proof.** By Lemma 3.5 in the proof of Corollary B in [13], $\bigotimes_2^Q (A_0 \otimes Z \otimes Q)$ is finitely generated as $QQ$-module via the diagonal action, where $A_0$ is as in the proof of Theorem 5.8. Now we can proceed as in the proof of Theorem 5.8. □

### 6. Some examples

#### 6.1. S-arithmetic groups.

Unfortunately there is no classification of the nilpotent-by-abelian groups of type $FP_n$ even in the case of $n = 2$, though the metabelian case was solved in [6]. In this case type $FP_2$ turns out to be equivalent to finite presentability. Still in the case of soluble $S$-arithmetic groups there is a complete classification of finite presentability [1, Theorem 7.5.2, Remark 4, Chap. VII]. They are finitely presented if and only if are of type $FP_2$. Note that soluble $S$-arithmetic groups are nilpotent-by-abelian-by-finite.

By a theorem of Borel-Serre [1, Theorem 0.4.4], any $S$-arithmetic subgroup of a reductive group is of type $FP_\infty$ and thus for such soluble subgroups the result of Corollary 5.6 is true for any $j \geq 0$. But such a result can be proved for other type of $S$-arithmetic groups.

The following example was considered in [2]: Let $p$ be a prime and

$$\Gamma_n \leq \text{GL}_{n+1}(Z[1/p]),$$

where $\Gamma_n$ is the group of upper triangular matrices $A$ with $A_{1,1} = 1 = A_{n+1,n+1}$. 

**Theorem 6.1.** The group $\Gamma_n$ is of type $FP_{n-1}$, but not of type $FP_n$.

**Proof.** See Theorem A in [2]. □

Let $N_n$ be the subgroup of $\Gamma_n$ containing all elements of $\Gamma_n$, where the main diagonal contains only entries 1. Then $N_n$ is nilpotent and

$$Q_n = \Gamma_n/N_n \simeq Z^{n-1}.$$
In this case the abelianization $V_n = N_n/[N_n, N_n]$ is isomorphic to $\mathbb{Z}[1/p]$, so $V_n \otimes \mathbb{Z} \mathbb{Q} \simeq \mathbb{Q}$ is finite dimensional over $\mathbb{Q}$. Hence all tensor and exterior powers of $V_n$ are finitely generated over $\mathbb{Q}$. Thus Theorem 4.1 implies that $V_n \otimes \mathbb{Q}$ is $m$-tame for any $m \geq 2$. Now by Theorem 5.4 we obtain the following result.

**Proposition 6.2.** For any $j \geq 0$, $\text{vb}_j(\Gamma_n)$ is finite.

6.2. **Groups of finite torsion-free rank.** It is a well-known theorem of Mal’cev that polycyclic groups are nilpotent-by-abelian-by-finite [15, 3.1.14]. On the other hand, for a polycyclic group $G$, the group ring $\mathbb{Z}G$ is (right) Noetherian [15, 4.2.3] and thus $G$ is of type $\text{FP}_\infty$. Now by Corollary 5.6, all virtual rational Betti numbers of $G$ are finite. A direct and much easier proof of this fact is given in [13, Lemma 3.2].

A polycyclic group is a special case of constructible groups. A soluble group is called constructible if and only if it can be built from the trivial group in finitely many steps by taking descending HNN-extensions and finite extensions. It is well-known that the class of constructible soluble groups is closed with respect to taking homomorphic images and subgroups of finite index [4, Proposition 2, Theorem 4]. Moreover, they have finite Prüfer rank [4, 3.3, Remark 2] and thus are nilpotent-by-abelian-by-finite. The last part follows from the proof of [17, Theorem 10.38]. Furthermore, constructible soluble groups are finitely presented and are of type $\text{FP}_\infty$ [4, Proposition 1]. Thus by Corollary 5.6 all virtual rational Betti numbers of these groups are finite.

Kochloukova and the second author gave a good bound for virtual rational Betti numbers of a polycyclic group [13, Lemma 3.2]. Their proof works even for the larger class of groups of finite torsion-free rank. Polycyclic and constructible groups are of finite Prüfer rank and thus they are of finite torsion-free rank.

A group $G$, not necessarily soluble, is said to be of finite torsion-free rank if it has a series of subgroups

$$1 = G_0 \lhd G_1 \lhd \cdots \lhd G_n = G,$$

such that each non-torsion factor $G_i/G_{i-1}$ is infinite cyclic. One can show that the number of infinite cyclic factors is independent of the chosen series (see the proof of [15, 1.3.3]) which it is called either the torsion-free rank or the Hirsch number of $G$ and we denote it by $h(G)$.

**Proposition 6.3.** Let $G$ be a group of finite torsion-free rank. Then for any integer $j \geq 0$, $\dim \mathbb{Q} H_j(G, \mathbb{Q}) \leq \binom{h(G)}{j}$. In particular,

$$\text{vb}_j(G) \leq \binom{h(G)}{j}.$$ 

**Proof.** The proof is similar to the proof of the case of polycyclic groups given in [13, Lemma 3.2].
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Behrooz Mirzaïi,
Institute of Mathematics and Computer Sciences (ICMC),
University of Sao Paulo (USP), Sao Carlos, Brazil.
e-mail: bmirzai@icmc.usp.br,

Fatemeh Yeganeh Mokari,
Institute of Mathematics, Statistics and Scientific Computing (IMECC),
State University of Campinas (Unicamp), Campinas, Brazil.
email: f.mokari61@gmail.com