Applicability of Mean Field Approximation to Numerical Experiments of Thermal Convection during the Cooling Period of Lake Biwa

Takashi Hosoda¹ and Frederick Malembeka²

¹Department of Urban Management, Graduate School of Engineering, Kyoto University, Kyoto 615-8540, Japan
²Department of Civil Engineering, Arusha Technical College, Arusha 296, Tanzania

Received: April 12, 2013 / Accepted: July 16, 2013 / Published: December 31, 2013.

Abstract: This paper describes the applicability of a stochastic model to the numerical experiments of thermal convection carried out under the condition of the northern part of Lake Biwa, Shiga Prefecture, Japan. It was shown in the previous study that the temporal changes of vertical water temperature distributions during the cooling period between September and February can be reproduced by a simple 3D-CFD model. It was also pointed out that the spatial distributions of cooled water body sinking to the bottom due to water surface cooling represent similar features of forest gap distribution, which can be clarified by a stochastic model. The basic features of numerical experiments on thermal convection such as the spatial distribution of cooled water body are firstly shown with several cooling rates at water surface. Then, a stochastic model, which was originally introduced to explain forest gap dynamics, is shown with its MFA (mean field approximation) as first approximation of stochastic model. It is pointed out through the comparison of theoretical results by MFA with tuned model constants to numerical experiments that MFA with some refinement can be applicable to reproduce the basic features of simulated results to some extent, although further investigations are required to clarify the applicability of the model to more detailed mechanism of thermal convection such as size distribution of cooled water body, phase change of flow pattern, etc...

Key words: Thermal convection, lake hydrodynamics, Lake Biwa, stochastic model.

1. Introduction

Lake Biwa is the largest monomictic lake located in the central part of Japan, having a total surface area of about 670 (km²) and 104 (m) at maximum depth. Its role as the source of water is paramount in supporting the people’s life in nearby prefectures such as Osaka, Kyoto and Shiga. Its value as an ancient lake offering habitat for rare indigenous species warranted it to be vested as UNESCO Ramsar’s site. Alarming deterioration of water quality is a main concern with climate change such as increased air temperature being noted to be linked with cycles of warm winters which are related to low levels of DO (dissolved oxygen) around lake’s bottom layer [1].

Fig. 1 shows the monthly changes of water temperature distribution in vertical direction during both the heating period from March to August and the cooling period from September to February taken from the open data source of Shiga Prefecture. It can be pointed out that the uniform distributions of temperature near the water surface are clearly observed in the temporal changes from September and February due to thermal convection, and the thermo-cline reaches the bottom in January.

Fig. 2 shows the temporal variations of DO observed by the former Lake Biwa Research Institute at the bottom off the shore of Ohmi-Imazu with about...
Applicability of Mean Field Approximation to Numerical Experiments of Thermal Convection during the Cooling Period of Lake Biwa

Fig. 1 Monthly change of vertical water temperature distributions (from open data source in Shiga Prefecture: (a) heating period and (b) cooling period).

Fig. 2 Temporal variations of DO and temperature observed 1 m above the bottom of the northern part of Lake Biwa between December 2001 and March 2002 by the former Research Institute of Lake Biwa, Shiga Prefectural Government.

100 (m) in depth. The dark line indicates the water temperature near the bottom, and the red line is DO. It should be noted that the discontinuous increase of DO from 5 (mg/L) to 12 (mg/L) is caused by the vertical mixing due to thermal convection in the end of January when the thermo-cline reaches the bottom. The water temperature at the bottom also increases from 7.0 (deg) to 7.8 (deg), and then temperature decreases due to further cooling in February.

Several researches have been conducted to address water quality issues associated with vertical mixing due to thermal convection. Ref. [2] shows that the distributions of water quality indices are influenced, with meteorological conditions likely to intensify DO depletion at the bottom layer. When warm winter was reported, the delay of DO supply to the bottom was detected by more than one month compared to normal years.

A one-dimensional simplified hydrodynamic model with water quality model was applied to elucidate these dynamic processes of water quality indices [3]. The results show that the observed characteristics of DO and temperature distributions can be reproduced quantitatively under the meteorological inputs of Lake Biwa. The large scale circulation with gyres in the lake during cooling period was successfully simulated.
using a 3-D hydrostatic model [4]. The vertical mixing mechanism due to thermal convection was also studied using non-hydrostatic 3-D CFD model considering the effects of surface processes lumped as water surface cooling rate [5]. The basic features of flow patterns induced by thermal convection can be stated based on simulated results that water bodies cooled locally at the surface goes down to the bottom with compensating upward flows around the sinking water body, inducing the interfacial mixing between upper and lower layers.

In this paper, a stochastic model which is similar to forest gap model [6] is applied to explain the basic features of numerical experiments conducted under the cooling condition of the lake. The simple theoretical equation of the ratio of cooled water body is derived applying the MFA (mean field approximation) to a stochastic model, and is verified qualitatively by numerical experiments.

2. Numerical Experiments

2.1 Model Description

Flow mechanism in the lake undergoing thermal cooling is simulated considering the conservation of mass, momentum and heat. The basic equations are described in Eqs. (1)-(5):

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} = 0$$

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} = - \frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right)$$

$$\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} = - \frac{1}{\rho} \frac{\partial p}{\partial y} + \nu \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right)$$

$$\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} = -g - \frac{1}{\rho} \frac{\partial p}{\partial z} + \nu \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right)$$

$$\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} + w \frac{\partial T}{\partial z} = \frac{1}{\rho C_p} \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right)$$

where, \((x, y, z) = \text{spatial coordinates}, t = \text{time}, (u, v, w) = \text{components of velocity vectors}, p = \text{pressure}, T = \text{temperature}, g = \text{gravity acceleration}, \nu = \text{dynamic molecular viscosity}, \lambda = \text{molecular diffusivity}.$

Computation was conducted under 3-D Cartesian domain of 40 cells on each direction with computational domain of dimension 80 m × 2 km × 2 km as shown in Fig. 3. Flow variables are defined at a staggered grid system in the framework of finite volume method with H-SMAC algorism as the pressure iteration [7]. QUICK scheme is used as the finite difference scheme of convective terms of Eqs. (2)-(5) [8].

![Fig. 3 Side view (top) and plan view (bottom) of the computational domain with initial temperature conditions.](image-url)
Applicability of Mean Field Approximation to Numerical Experiments of Thermal Convection during the Cooling Period of Lake Biwa

Considering the thermal condition in January, the initial water temperature distributions in the vertical direction are 9 (deg) for 20 (m) < z < 80 (m), and 7 (deg) for z ≤ 20 (m). The cooling process at the surface is represented as the surface boundary condition, Eq. (6):

$$\lambda \frac{\partial T}{\partial z} = \frac{Q_0}{\rho c_p}$$

(6)

where, $Q_0 = \text{cooling rate at the surface}$ and $c_p = \text{specific heat}$.

2.2 Results of Numerical Experiments

To understand the basic features of vertical mixing under different surface cooling processes, several cooling rates at the surface were tested. For easy identification, cooling rates are denoted as $(Q_0)_{\text{rate}}$, where the subscript rate indicates a value of surface cooling rate in cal/cm²/s × 10⁴.

Fig. 4 shows the side views of water temperature and flow field induced by thermal convection with $(Q_0)_{\text{rate}} = 50$ and 100 after 5 days. It can be described that water bodies cooled locally are sinking with the compensating upward flow around the sinking region, promoting the interfacial mixing between the upper and the lower layers and the descent of interface.

Fig. 5 shows the plan views of water temperature and flow field with $(Q_0)_{\text{rate}} = 50$ and 100 after 5 days which are taken at 12 (m) from the water surface. The blue areas indicate the cooled water body with large descending velocity. These flow features shown in Figs. 4 and 5 seem to represent basic flow patterns during cooling period in the lake. Fig. 6 is the satellite image showing the lake surface temperature on January 9, 2004 in Lake Biwa. The spatial variations of surface water temperature shown in this figure seem to be considered as an illustration of the existence of thermal convection.

Statistical analysis was done by using plan views of numerical experiments. The horizontal area is divided into two regions of cooled water area and others using the descending velocity as a criterion.

![Fig. 4 Side view of simulated flow after 5 days with (a) $(Q_0)_{\text{rate}} = 50$ and (b) $(Q_0)_{\text{rate}} = 100$.](image)

Fig. 7 shows the spatial distributions of cooled water body cells (cold sites) classified using the descending velocity $\Omega = 5$ (mm/s) and 0 (mm/s), which are depicted using the data with $(Q_0)_{\text{rate}} = 50$ after 5 days. Cold sites are shown with black squares, while hot sites with white squares. Smaller values of descending velocity allow more cells to be categorized as cold sites.

The densities for cold and hot sites, which are denoted as $\rho_0$ and $\rho_+$, respectively, are defined as the fraction of each site area in the whole horizontal domain. Fig. 8 shows the relation between $\rho_0$ and surface cooling rates $(Q_0)_{\text{rate}}$ with three descending velocities.
3. Application of Stochastic Model

3.1 Description of Stochastic Model

The spatial distributions of cold sites shown in Fig. 7 seem to be analogous to the forest gap distribution. Since a stochastic model was applied to explain the dynamics of gap recovery in forests [6], and the applicability of a similar model was tested to explain the characteristics of numerical experiments on thermal convection. In the case of a stochastic model for forest dynamics, a canopy site surrounded by gap sites changes to a gap site with the rate proportional to the number of gap sites in its neighborhood. A gap site changes to a canopy site with constant rate due to growing.

In the case of thermal convection, a cooled water body site (cold site) is likely to sink further with increasing the number of hot sites in its neighborhood, and is transformed into a hot site due to compensating upward flows. Fig. 9 shows spatial patterns of cold/hot sites using Neumann neighborhood. The probability of transformation from cold site to hot site in the type (a) is the largest, and the type (e) has the smallest probability. It is considered that a hot site changes to a cold site with constant rate due to cooling.

The model mentioned above can be approximated by the ordinary differential Eq. (7), applying mean field approximation, where the number of hot sites around a cold site can be replaced by the fraction of cold cells in the whole domain, \( \rho_0 \).

\[
\frac{d\rho_0}{dt} = b(1 - \rho_0) - \rho_0 \{d + \delta(1 - \rho_0)\} \tag{7}
\]
Applicability of Mean Field Approximation to Numerical Experiments of Thermal Convection during the Cooling Period of Lake Biwa

Fig. 7  Spatial distributions of cooled water body cells (cold site) classified by means of descending velocity (top: $\Omega = 5$ (mm/s), bottom: $\Omega = 0$ (mm/s)).

Fig. 8  Relation between fraction of cold sites and cooling rate.

where $b, d, \delta = \text{model constants.}$ Eq. (7) was used as the first approximation in the forest gap dynamics [6]. The first term on the right of Eq. (7) represents the transformation from hot site to cold site due to cooling, and the second term represents the transformation from cold site to hot site due to sinking.

$$\frac{dp_0}{dt} = b(1 - p_0)^n - \frac{p_0 \left( \Omega + \delta (1 - p_0)^n \right)}{(1 + \varepsilon p_0)^\rho}$$

Fig. 9  Classification of spatial patterns of cold/hot cells used for a stochastic model.

Assuming the steady state by setting the left side to be zero, $\rho_0$ in the equilibrium state can be evaluated. $\rho_0$ approaches to 1 with increasing cooling rate, $b$ as shown as Line B in Fig. 10. But, $\rho_0$ in numerical experiments can not become to 1 as shown in Fig. 8, because the whole area can not sink due to the constraints from the continuity Eq. (1).

Considering the constraints of continuity, Eq. (8) is proposed with the revised form in the second term on the right of Eq. (8), where the rate of transformation from cold site to hot site is reduced with increasing the fraction of cold site.

$\rho_0$ becomes to a constant value less than 1. These results indicate that some basic features of numerical experiments on thermal convection can be reproduced by using the simple stochastic model, although more detailed comparisons between numerical experiments and theoretical results with various flow conditions are required to assess the stochastic model.
Applicability of Mean Field Approximation to Numerical Experiments of Thermal Convection during the Cooling Period of Lake Biwa

4. Conclusions

Fundamental characteristics of the flow patterns induced thermal convection during the cooling period in Lake Biwa were studied using 3D simulation model. Basic characteristics of thermal convection were investigated using the results of numerical experiments under various surface cooling rates. Statistical analysis of the numerical results showed that the fraction of cold sites in the whole horizontal domain increases to an equilibrium value with the increase of cooling rates. Application of a stochastic model, which is similar to the model applied to forest gap dynamics [6], is tested to explain the basic features of thermal convection such as the relation between the fraction of cold sites and cooling rates. The revised equation with mean field theory applied to a stochastic model is proposed considering the constraints from the continuity equation. The results of analysis showed that the stochastic model and its associated model equation can capture the basic features of thermal convection, although further investigations are required to assess the applicability of the stochastic model. Since the relation between forest dynamics model and Ising model, a well-known model of magnetism in statistical mechanics, was made clear exactly [9], the flow pattern changes with a phase shift and the other characteristics under various thermal conditions may be explained to some extent through the examination of the stochastic model.

References

[1] M. Kumagai, W.F. Vincent, K. Ishikawa, Y. Aota, Lessons from Lake Biwa and other Asian Lakes, in: M. Kumagai, W.F. Vincent (Eds.), Freshwater Management, Springer, 2003, pp. 1-22.
[2] C. Yoshimizu, K. Yoshiyama, I. Tayasu, T. Koitabashi, T. Nagata, Vulnerability of a large monomictic lake (Lake Biwa) to warm winter event, Limnology 11 (3) (2010) 233-239.
[3] T. Hosoda, T. Hosomi, A simplified model for long term prediction on vertical distributions of water qualities in Lake Biwa, in: N. Afgan, Z. Bogdan, N. Duic (Eds.), Sustainable Development of Energy, Water and Environment Systems, Balkema, 2002, pp. 357-365.
[4] K. Akitomo, M. Kurogi, M. Kumagai, Numerical study of a thermally induced gyre system in Lake Biwa, Limnology 5 (2) (2004) 103-114.
[5] T. Hosoda, Numerical experiments on thermal convection during cooling period in the northern part of Lake Biwa, in: Proceedings of 15th International Symposium on Environmental Hydraulics, Tempe, Arizona, Dec. 4-7, 2007.
[6] T. Kubo, Y. Iwasa, N. Furumoto, Forest spatial dynamics with gap expansion: Total gap area and gap size distribution, Journal of Theoretical Biology 180 (3) (1996) 229-246.
[7] C.W. Hirt, J.L. Cook, Calculating three-dimensional flows around structures and over rough terrain, J. Computational Physics 10 (1972) 324-340.
[8] B.P. Leonard, A stable and accurate convective modeling procedure based on quadratic upstream interpolation, Computer Methods in Applied Mechanics and Engineering 19 (1979) 59-98.
[9] S. Kizaki, M. Katori, Analysis of canopy-gap structures of forest by Ising-Gibbs states-equilibrium and scaling property of real forests, Journal of the Physical Society of Japan 68 (8) (1999) 2553-2560.