AVERAGE NUMBER OF SQUARES, DIVIDING $mn$

ANDREW V. LELECHENKO

Abstract. We study the asymptotic behaviour of $\sum_{m,n \leq x} \tau_1(mn)$, where $\tau_{1,2}(n) = \sum_{d \mid n} \tau_1(d)$, using multidimensional Perron formula and complex integration method. An asymptotic formula with an error term $O(x^{1.47537})$ is proven.

1. Introduction

Let $f$ be a multiplicative arithmetic function of one variable. The asymptotic behaviour of $\sum_{n \leq x} f(n)$ is a classic problem of analytic number theory, deeply studied for various specific functions and classes. Let us consider the problem of estimating $\sum_{m,n \leq x} f(mn)$.

The divisor function $\tau$ is a simple, but non-trivial case. Applying Busche—Ramanujan identity

$$\tau(mn) = \sum_{d \mid \gcd(m,n)} \tau(m/d) \tau(n/d) \mu(d),$$

we split variables and obtain

$$\sum_{m,n \leq x} \tau(mn) = \sum_{j,k,l} \tau(j) \tau(k) \mu(l) \left( \sum_{j \leq x/l} \tau(j) \right)^2.$$

Using Huxley’s estimate $\sum_{j \leq y} \tau(j) = y \log y + (2\gamma - 1)y + O(y^{\theta + \epsilon})$, where $\theta = 131/416$, we regroup terms and get

$$\sum_{m,n \leq x} \tau(mn) = x^2 \left( \sum_{l=1}^{\infty} \frac{\mu(l) \log l}{l^2} \right) \left( \log^2 x + 2(2\gamma - 1) \log x + (2\gamma - 1)^2 \right) - \left( \sum_{l=1}^{\infty} \frac{\mu(l) \log l}{l^2} \right) \left( 2 \log x + 2(2\gamma - 1) \right) + O(x^{1+\theta+\epsilon}).$$

It is natural to ask whether this result can be derived analytically, by complex integration method. We will not go into details, but note that

$$\sum_{a,b=0}^{\infty} \tau(p^{a+b}) x^a y^b = \sum_{a,b=0}^{\infty} (a+b+1) x^a y^b = \frac{1 - xy}{(1-x)^2(1-y)^2}, \quad |x|, |y| < 1.$$ 

The series $\sum_{m,n=1}^{\infty} \tau(mn)m^{-z}n^{-w}$ converges absolutely for $\Re z, \Re w > 1$, so by multiplicativity in this region we have

$$\sum_{m,n=1}^{\infty} \tau(mn) m^{-z} n^{-w} = \prod_{p} \sum_{a,b=0}^{\infty} \tau(p^{a+b}) \frac{1 - p^{-z-w}}{(1-p^{-z})^2(1-p^{-w})^2} = \frac{\zeta^2(z) \zeta^2(w)}{\zeta(z+w)}.$$

2010 Mathematics Subject Classification. 11A25, 11N37.

Key words and phrases. Average order, asymmetric divisor function, Perron formula.
The right-hand side of (3) allows to compute the coefficient of multiple Laurent series of $x^z w^{-1} w^{-1} \sum_{m,n=1}^{\infty} \tau(mn) m^{-z} n^{-w}$ at $1/(z-1)(w-1)$, which appears coinciding with the main term of (2).

Our paper is devoted to $\sum_{m,n \leq x} \tau_1(mn)$, where $\tau_1(mn) = \sum_{ab=mn} 2$. This function is not as lucky as $\tau$ and does not possess representation like (1), so there is no easy way to split $m$ and $n$.

We prove following theorem.

**Theorem 1.**

$$\sum_{m,n \leq x} \tau_1(mn) = C_1 x^2 + C_2 x^{3/2} + O(x^{\alpha+\varepsilon}), \quad \alpha = \frac{2306}{1563} \approx 1.47537,$$

where $C_1, C_2$ are computable constants, defined below in (25).

One can compare our result with an estimate by Graham and Kolesnik [2]

$$\sum_{n \leq x} \tau_1(n) = \zeta(2)x + \zeta(1/2)x^{1/2} + O(x^{\beta+\varepsilon}), \quad \beta = \frac{1057}{4785} \approx 0.2209.$$

2. **Notations**

Letter $p$ with or without indexes denotes a prime number. We write $f \ast g$ for Dirichlet convolution

$$(f \ast g)(n) = \sum_{d|n} f(d)g(n/d).$$

In asymptotic relations we use $\sim$, $\asymp$, Landau symbols $O$ and $o$, Vinogradov symbols $\ll$ and $\gg$ in their usual meanings. All asymptotic relations are given as an argument (usually $x$) tends to the infinity.

Letter $\gamma$ denotes Euler—Mascheroni constant. Everywhere $\varepsilon > 0$ is an arbitrarily small number (not always the same even in one equation).

As usual $\zeta(s)$ is the Riemann zeta-function. Real and imaginary components of the complex $s$ are denoted as $\sigma := \Re s$ and $t := \Im s$, so $s = \sigma + it$.

For a fixed $\sigma \in [1/2, 1]$ define

$$\mu(\sigma) := \limsup_{t \to \infty} \frac{\log|\zeta(\sigma + it)|}{\log t}.$$

3. **Preliminary estimates**

We say that function is symmetric if any permutation of arguments does not change its value.

**Lemma 1.** Let $f$ be a symmetric arithmetic function of $r$ variables and $(\sigma_1, \ldots, \sigma_r)$ are abscissas of absolute convergence of the associated Dirichlet series $F(s_1, \ldots, s_r)$. Define

$$F^{\sigma}(\sigma, x, T) := \sum_{n_1, \ldots, n_r=1}^{\infty} \frac{|f(n_1, \ldots, n_r)|^{-(\sigma_1 \cdots \sigma_r)}}{\min_{j=1, \ldots, r} \big( T \log(x/n_j) + 1 \big)},$$

and let

$$\sum_{n_1, \ldots, n_r \leq x} f(n_1, \ldots, n_r) = \sum_{n_1, \ldots, n_r \leq x} f(n_1, \ldots, n_r) h(x/n_1) \cdots h(x/n_r),$$

where $h(y) = 0$ for $0 < y < 1$, $h(1) = 1/2$ and $h(y) = 1$ otherwise.
For \( x \geq 2, T \geq 2, \sigma \leq \sigma_a, \delta > 0, \kappa = \sigma_a - \sigma + \delta / \log x, 1 = N_1 \leq \cdots \leq N_r, 1 = M_1 \leq \cdots \leq M_r \) and \( N_0 := N_1 + \cdots + N_r \) we have

\[
(6) \quad \left| \sum_{n_1, \ldots, n_r \leq x} \frac{f(n_1, \ldots, n_r)}{n_1 \cdots n_r} \right|^{N_1 \kappa + M_1 T} - \frac{1}{(2\pi i)^r} \int_{N_1 \kappa - iM_1 T}^{N_r \kappa + iM_r T} \cdots \int F(s + w_1, \ldots, s + w_r) x^{w_1 + \cdots + w_r} dw_1 \cdots dw_r \right| \ll \\
\ll x^N_0(\sigma_a - \sigma) F_T^C(\sigma_a + \delta / \log x, x, T).
\]

**Proof.** This is a result of Balazard, Naimi and Pétermann [1, Prop. 6]. \( \square \)

**Lemma 2.** Let \( f(t) \geq 0 \). If

\[
\int_1^T f(t) \, dt \ll g(T),
\]

where \( g(T) = T^\alpha \log^\beta T, \alpha \geq 1, \) then

\[
I(T) := \int_1^T \frac{f(t)}{t} \, dt \ll \begin{cases} 
\log^{\beta + 1} T & \text{if } \alpha = 1, \\
T^{\alpha - 1} \log^\beta T & \text{if } \alpha > 1.
\end{cases}
\]

**Proof.** Let us divide the interval of integration into parts:

\[
I(T) \leq \sum_{k=0}^{\log_2 T} \int_{T/2^{k+1}}^{T/2^k} f(t) \, dt \ll \sum_{k=0}^{\log_2 T} \frac{1}{T/2^{k+1}} \int_1^{T/2^k} f(t) \, dt \ll \sum_{k=0}^{\log_2 T} g(T/2^k) T/2^{k+1}.
\]

Now the lemma’s statement follows from elementary estimates. \( \square \)

**Lemma 3.** Let \( \eta > 0 \) be arbitrarily small. Then for growing \(|t| \geq 3\)

\[
(7) \quad \zeta(s) \ll \begin{cases} 
|t|^{1/2 - (1 - 2\mu(1/2))\sigma}, & \sigma \in [0, 1/2], \\
|t|^{2\mu(1/2)(1 - \sigma)}, & \sigma \in [1/2, 1 - \eta], \\
|t|^{2\mu(1/2)(1 - \sigma)} \log^{2/3} |t|, & \sigma \in [1 - \eta, 1], \\
\log^{2/3} |t|, & \sigma \in [1, 1 + \eta], \\
1, & \sigma \geq 1 + \eta.
\end{cases}
\]

**Proof.** Estimates follow from Phragmén–Lindelöf principle, exact and approximate functional equations for \( \zeta(s) \) and convexity properties. See Titchmarsh [7, Ch. 5] or Ivić [5, Ch. 7.5] for details. \( \square \)

4. REDUCTION TO COMPLEX INTEGRATION

Applying Lemma 4 with \( r = 2, f(n_1, n_2) = \tau_{1,2}(n_1 n_2), \sigma = s = 0, \sigma_a = 1 + \varepsilon, N_1 = N_2 = M_1 = M_2 = 1, \delta = 1, \log T = \log x \) and writing \((m, n, w, c)\) instead of \((n_1, n_2, w_1, w_2, \kappa)\) for convenience we can rewrite (5) as

\[
(8) \quad \sum_{m, n \leq x} \tau_{1,2}(mn) = \frac{1}{(2\pi i)^2} \int_{c - iT}^{c + iT} F(z, w) \frac{z^{w+\varepsilon}}{z} \, dz \, dw + O \left( x^2 F_2^C (1 + \varepsilon + \delta / \log x, x, T) \right),
\]

where \( c = 1 + 1/\log x \) and

\[
(9) \quad F(z, w) = \sum_{m, n = 1}^{\infty} \frac{\tau_{1,2}(mn)}{m^n n^w}, \quad \Re z, \Re w > 1.
\]
By (11) for non-integer $x$

$$T F_2^\gamma (c, x, T) \ll \sum_{m,n} \frac{\tau_{1,2}(mn)}{(mn)^c \min(|\log \frac{x}{n}|, |\log \frac{x}{m}|)} \ll \sum_{|\log \frac{x}{n}| \leq 1} \frac{\tau_{1,2}(mn)}{(mn)^c} + \sum_{|\log \frac{x}{m}| \leq 1} \frac{\tau_{1,2}(mn)}{(mn)^c |\log \frac{x}{n}|} + \sum_{|\log \frac{x}{m}| \leq 1} \frac{\tau_{1,2}(mn)}{(mn)^c \min(|\log \frac{x}{n}|, |\log \frac{x}{m}|)} := \Sigma_1 + \Sigma_2 + \Sigma_3.$$  

We have $\Sigma_1 \ll \sum_{m,n=1}^{\infty} \frac{\tau_{1,2}(mn)}{(mn)^c} = F(c, c)$ and we will show below in (10) that

$$F(c, c) \ll \frac{1}{(c-1)^2} = \log^2 x.$$  

Further, for $x$ such that $|\log \frac{x}{n}| \leq 1$ we have $|\log \frac{x}{n}| \geq c|x-n|/x$ for $c = 1/(c-1)$. Then

$$\Sigma_2 \ll \sum_{x/e \leq n \leq xe} \sum_{m} \frac{\tau_{1,2}(mn) x}{(mn)^c|x-n|}.$$  

Note that $\tau_{1,2}(mn) \leq \tau(mn) \leq \tau(m)\tau(n)$, because $\tau$ is completely submultiplicative. Thus

$$\Sigma_2 \ll x \sum_{x/e \leq n \leq xe} \frac{\tau(n)}{n^c|x-n|} \sum_{m} \frac{\tau(m)}{m^c}.$$  

Here

$$\sum_{m=1}^{\infty} \frac{\tau(m)}{m^c} = \zeta^2(c) \ll (c-1)^{-2} = \log^2 x.$$  

Let $M(y) = \max_{n \leq y} \tau(n)$. We have

$$\Sigma_2 \ll x M(xe) \log^2 x \sum_{x/e \leq n \leq xe} \frac{1}{n^c|x-n|},$$  

where the last sum is $\ll x^{-c} \log x \ll x^{-1} \log x$, so finally

$$\Sigma_2 \ll M(xe) \log^3 x.$$  

Now consider $\Sigma_3$. Defining $M_{1,2}(y) = \max_{n \leq y} \tau_{1,2}(n)$ we obtain

$$\Sigma_3 \ll \sum_{x/e \leq n \leq m \leq xe} \frac{\tau_{1,2}(mn)x}{(mn)^c \min(|x-n|, |x-m|)} \ll \frac{x M_{1,2}(x^2 e^2)}{xe} \sum_{x/e \leq n \leq m \leq xe} \max(|x-n|^{-1}, |x-m|^{-1}) \ll M_{1,2}(x^2 e^2) \log x.$$  

Standard estimates [3, Th. 315] give $M_{1,2}(y) \leq M(y) \ll y^\epsilon$, so substituting (11), (12) and (13) into (10) we obtain

$$F_2^\gamma (c, x, T) \ll T^{-1} \left( M(xe) \log^3 x + M_{1,2}(x^2 e^2) \log x \right) \ll T^{-1} x^\epsilon.$$  

Note also that by definition

$$\sum_{m,n \leq x} \tau_{1,2}(mn) - \sum_{m,n \leq x} \tau_{1,2}(mn) \ll \sum_{n \leq x} \tau_{1,2}(xn) \ll M(x^2)x.$$
Combining (8), (14) and (15) we get

\[
\sum_{m,n \leq x} \tau_{1,2}(mn) = \frac{1}{(2\pi i)^2} \int_{c-iT}^{c+iT} \int F(z,w) \frac{x^{z+w}}{zw} \, dz \, dw + O(x^{1+\varepsilon} + T^{-1}x^{2+\varepsilon}).
\]

5. Double Dirichlet series for \( \tau_{1,2} \)

Let us return to (9) and extract a product of zeta-functions from \( F(z,w) \). Define

\[
f(x,y) = \sum_{a,b=0}^{\infty} \tau_{1,2}(p^{a+b}) x^a y^b, \quad |x|, |y| < 1.
\]

Using identity

\[
\tau_{1,2}(p^a) - \tau_{1,2}(p^{a-1}) - \tau_{1,2}(p^{a-2}) + \tau_{1,2}(p^{a-3}) = 0
\]

multiply both sides of (17) by \((1-x)(1-x^2)\):

\[
(1-x)(1-x^2)f(x,y) = \sum_{a=3}^{\infty} \sum_{b=0}^{\infty} \left( \tau_{1,2}(p^{a+b}) - \tau_{1,2}(p^{a+b-1}) - \tau_{1,2}(p^{a+b-2}) + \tau_{1,2}(p^{a+b-3}) \right) x^a y^b + \sum_{b=0}^{\infty} y^b \left( (1-x-x^2)\tau_{1,2}(p^b) + (1-x)\tau_{1,2}(p^{b+1})x + \tau_{1,2}(p^{b+2})x^2 \right) = \sum_{b=0}^{\infty} y^b \left( (1-x-x^2)\tau_{1,2}(p^b) + (x-x^2)\tau_{1,2}(p^{b+1})x + x^2 \tau_{1,2}(p^{b+2}) \right)
\]

and further

\[
(1-x)(1-x^2)(1-y)(1-y^2)f(x,y) = (1-x-x^2)((1-y-y^2) + (1-y)y + 2y^2) + (x-x^2)((1-y-y^2) + 2(1-y)y + 2y^2) + x^2 \left( 2(1-y-y^2) + 2(1-y)y + 3y^2 \right) = 1 + xy - x^2 y - xy^2,
\]

which induces

\[
f(x,y) = \frac{1 + xy - x^2 y - xy^2}{(1-x)(1-x^2)(1-y)(1-y^2)} = \frac{1 - x^2 y - xy^2 - x^2 y^2 + x^2 y^3}{(1-x)(1-x^2)(1-y)(1-y^2)(1-x-y)}
\]

Representation (18) immediately implies that

\[
F(z,w) = \prod_p f(p^{-z}, p^{-w}) = \zeta(z)\zeta(2z)\zeta(w)\zeta(2w)G(z+w) = \frac{\zeta(z)\zeta(2z)\zeta(w)\zeta(2w)\zeta(z+w)}{\zeta(2z+w)\zeta(2w+z)}H(z,w),
\]

where series \( H(z,w) \) converges absolutely in the region \( \Re(2z+2w) > 1 \). Definitely \( G(z,w) \) converges absolutely for \( (z,w) \in Q := \{ \Re z \geq 1/3, \Re w \geq 1/3 \} \).

Product of zeta-functions (19) shows that inside of the region \( Q \) function \( F(z,w) \) has poles along lines \( z = 1, z = 1/2, w = 1, w = 1/2 \) and \( z + w = 1 \). All of them are
of the first order, except poles at (1, 1), (1, 1/2), (1/2, 1), which are of the second order, and a pole at (1/2, 1/2), which is of the third order.

Both (3) and (19) are partial cases of a general rule, which will be stated as a lemma.

Lemma 4. Let \( \tau_{1,k}(n) = \sum_{a,b=0}^{\infty} 1 \). Then for \( \Re z, \Re w > 1 \) we have

\[
\sum_{m,n=1}^{\infty} \frac{\tau_{1,k}(mn)}{m^n n^w} = \zeta(z) \zeta(w) \frac{\prod_{l=0}^{k} \zeta(lz + (k-l)w)}{\prod_{l=1}^{k} \zeta(lz + (k+1-l)w)} H_k(z, w),
\]

where the series \( H_k \) converges absolutely for \( \Re z, \Re w > 1/(k+2) \).

Proof. Cases \( k = 1 \) and \( k = 2 \) were proven above, so we consider \( k > 2 \) only. Let

\[
f(x, y) = \sum_{a,b=0}^{\infty} \tau_{1,k}(p_{a+b}) x^a y^b, \quad |x|, |y| < 1.
\]

Let \( |M|f(x, y) \) be a coefficient of monomial \( M \) in series \( f \). Here \( [x]f(x, y) = [y]f(x, y) = 1 \), so let us define

\[
g(x, y) = (1-x)(1-y)f(x, y) = \sum_{a,b=1}^{\infty} \left( \tau_{1,k}(p_{a+b}) - 2\tau_{1,k}(p_{a+b-1}) + \tau_{1,k}(p_{a+b-2}) \right) x^a y^b + \sum_{a=1}^{\infty} \left( \tau_{1,k}(p_{a}) - \tau_{1,k}(p_{a-1}) \right) (x^a + y^a) + 1.
\]

We have

\[
\tau_{1,k}(p^n) = \begin{cases} 
1, & a < k, \\
2, & k \leq a < 2k,
\end{cases}
\]

so one can verify that

\[
[x^a y^b]g(x, y) = \begin{cases} 
0, & a+b < k, \\
1, & a+b = k, \\
0, & a+b = k+1, \ ab = 0 \\
-1, & a+b = k+1, \ ab > 0.
\end{cases}
\]

Thus

\[
f(x, y) = \frac{1}{(1-x)(1-y)} \frac{\prod_{l=1}^{k} (1-x^l y^{k+1-l})}{\prod_{l=0}^{k} (1-x^l y^{k-l})} h(x, y),
\]

where all monomials of the series \( h(x, y) \) has degree at least \( k+2 \).

\[\square\]

6. Path of integration and the main term

Our aim is to translate the domain of integration in (16) from \([c-iT, c+iT]^2\) till \([b-iT, b+iT]^2\), where \( b = 1/3 \). This is trickier than in one-dimensional case, because a hyperrectangle \( R \) with opposite vertices \((b-iT, b+iT)\) and \((c+iT, c+iT)\) has 24 two-dimensional faces. Figure 11 contains a schematic plain projection of \( R \) with 16 vertices and 32 edges marked.

Denote \( L(z, w) = G(z, w) x^z w^{-1} \). This function has the same poles in \( R \) as \( G(z, w) \) has. Note that (on contrary with integration by one-dimensional contour) poles of the first order do not induce divergence of integrals by plane domains: e. g.,

\[
\int_{x^2+y^2 \leq 1} \frac{dz \, dy}{\sqrt{x^2+y^2}} = 2\pi < \infty, \quad \text{however} \quad \int_{x^2 \leq 1} \frac{dz}{x} = \infty.
\]

Only poles of the second and higher orders are worth to pay attention.
Let $E(x)$ be the integral of $L(z, w)$ over all faces of $R$ except $[c - iT, c + iT]^2$.

By residue theorem [6]

$$
\frac{1}{(2\pi i)^2} \int\int_{[c - iT, c + iT]^2} L(z, w) \, dz \, dw = \left( \text{res}_{z=w=1} + \text{res}_{z=1/2, w=1} + \text{res}_{z=1/2, w=1/2} \right) L(z, w) + O(E(x)).
$$

Expanding $L(z, w)$ into Laurent series in two variables we get

$$
\text{res}_{z=w=1} L(z, w) = \zeta(2)^2 G(1, 1) x^2,
$$

$$
\text{res}_{z=1/2, w=1} L(z, w) = \zeta(2) \zeta(1/2) \zeta(3/2) G(1, 1/2) x^{3/2},
$$

$$
\text{res}_{z=1/2, w=1/2} L(z, w) \ll x \log x.
$$

After substitution into $\int$ the residue at $(1/2, 1/2)$ will be absorbed by error term, so it is enough to have only upper bound. Inserting (22), (23) and (24) into (21) we get

$$
\frac{1}{(2\pi i)^2} \int\int_{[c - iT, c + iT]^2} L(z, w) \, dz \, dw =

\pi^6
\pi^2
\zeta(2) \zeta(1/2) \zeta(3/2) G(1, 1/2) x^{3/2}
O(x \log x + E(x)).
$$

Figure 1. The hyperrectangle $R$ with opposite vertices $(b - iT, b - iT)$ and $(c + iT, c + iT)$
7. The error term

Let us estimate $E(x)$. It consists of integrals over 23 faces of $R$, but due to the symmetry many of them can be estimated in the same way.

In computations below we assume $x^{1/2} \ll T \ll x$, the exact value of $T$ will be specified later.

There are 2 faces of form $[b-iT, b+iT] \times [c-iT, c+iT]$. We have

$$I_1 := \int_{b-iT}^{b+iT} \int_{c-iT}^{c+iT} L(z, w) \, dz \, dw \ll \int_1^T \int_1^T \zeta(b + it_1)\zeta(2b + 2it_1) \times \zeta(c + it_2)\zeta(2c + 2it_2)\zeta(b + c + i(t_1 + t_2)) x^{b+c} t_1^{-1} t_2^{-1} dt_1 dt_2.$$  

By (7) we can estimate

$$\zeta(c + it_2)\zeta(2c + 2it_2)\zeta(b + c + i(t_1 + t_2)) \ll \log^{2/3} T \cdot 1 \cdot 1.$$

As soon as $x^{1+1/\log x} \ll 1$ we have $x^{b+c} \ll x^{4/3}$. Also $\int_1^T t_2^{-1} dt_2 \ll \log T$. Thus $I_1$ can be estimated as

$$I_1 \ll x^{4/3} \log^{5/3} T \int_1^T \zeta(b + it)\zeta(2b + 2it) t^{-1} dt.$$  

It is well-known that $\int_1^T \zeta(\sigma + it)^2 dt \ll T$ for $1/2 \leq \sigma < 1$, e.g., see Ivic [5 (1.76)]. Then by Cauchy–Schwarz inequality

$$(26)\quad J := \int_1^T \zeta(b + it)\zeta(2b + 2it) dt \ll \left( \int_1^T \zeta^2(b + it) dt \right)^{1/2} \ll \left( \int_1^T t^{2\mu(1/3)} dt \right)^{1/2} T^{1/2} \ll T^{1 + \mu(1/3)}.$$  

Applying Lemma 2 on (26) we obtain

$$(27)\quad I_1 \ll x^{4/3} T^{\mu(1/3)} \log^{5/3} T.$$  

We will show below in (10) that integrals over other faces (and so $E(x)$ as a whole) are less than either $I_1$ or $x^{2 + e + \mu^{-1}}$, so $T$ should be chosen to equalize this two magnitudes. Inequality (7) implies $\mu(1/3) \leq 1/6 + 2/3\mu(1/2)$, so we choose

$$(28)\quad T = x^{1/(7+4\mu(1/2))}$$  

and substitute it into (10) and (27) to obtain the final error term $x^{2 + e - 4/(7+4\mu(1/2))}$. In the view of Huxley’s result $\mu(1/2) \leq 32/205$ (see (4)), the error term is nothing more than $x^{2306/1563+e}$, which approves the statement of the Theorem 4.

From here and till the end of the section we will omit factors $\ll x^a$ in asymptotic estimates for the brevity: they do not influence the resulting error term.

There are 4 faces of form $[b - iT, b + iT] \times [c \pm iT, c \pm iT]$. We have

$$I_2 := \int_{b-iT}^{b+iT} \int_{b+ iT}^{c+iT} L(z, w) \, dz \, dw \ll \int_1^T \int_1^T \zeta(b + it)\zeta(2b + 2it) \times \zeta(\sigma + iT)\zeta(2\sigma + 2iT)\zeta(b + \sigma + i(t + T)) x^{b+\sigma} t^{-1} T^{-1} \sigma \, dt \ll x^{1/3} T^{T^{-2}} \max_{\sigma \in [b, c]} \left( \zeta(\sigma + iT)\zeta(2\sigma + 2iT)\zeta(b + \sigma + i(t + T)) x^{\sigma} \right) \ll x^{1/3} T^{\mu(1/3) - 1} \max_{\sigma \in [b, c]} \zeta(\sigma + iT)\zeta(\sigma + 1/3 + iT)\zeta(2\sigma + iT) x^{\sigma}.$$  

...
Splitting $[b, c]$ into intervals $[1/3, 1/2], [1/2, 2/3], [2/3, c]$ and estimating $\zeta(\sigma + iT) \times \zeta(\sigma + 1/3 + iT)x^\sigma$ on each of them separately, we get

$$I_2 \ll x^{1/3}T^{\mu(1/3)-1}(T^{\mu(1/3)+2\mu(2/3)}x^{1/2} + T^{\mu(1/2)+\mu(5/6)}x^{2/3} + T^{\mu(2/3)}x).$$

Utilizing rough estimate $\mu(1/2) \leq 1/6$ from [7, Th. 5.5] we get by (7) that

$$(29) \quad \mu(\sigma) \leq \begin{cases} 1/2 - 2\sigma/3, & \sigma \in [0, 1/2], \\ (1 - \sigma)/3, & \sigma \in [1/2, 1] \end{cases}$$

and

$$(30) \quad \mu(1/3) \leq 5/18, \quad \mu(2/3) \leq 1/9, \quad \mu(5/6) \leq 1/18,$$

so

$$(31) \quad I_2 \ll x^{1/3}T^{-13/18}(T^{1/2}x^{1/2} + T^{2/9}x^{2/3} + T^{1/9}x) \ll x^{4/3}. $$

There is 1 face of form $[b - iT, b + iT]^2$. Applying (30) we have

$$I_3 := \iint_{[b - iT, b + iT]^2} L(z, w) \, dz \, dw \ll \iint_{[1, T]^2} \zeta(b + it_1)\zeta(2b + 2it_1) \times \zeta(b + it_2)\zeta(2b + 2it_2)\zeta(2b + i(t_1 + t_2)) x^{2b_1-1}t_1^{-1}dt_1dt_2 \ll \ll x^{2/3} \iint_{[1, T]^2} t_1^{5/18+1/9-1}t_2^{5/18+1/9-1}(t_1 + t_2)^{1/9}dt_1dt_2,$$

which implies

$$(32) \quad I_3 \ll x^{2/3}T^{8/9},$$

which is less than $x^{4/3}$ by our choice of $T$ in (28).

There are 4 faces of form $[c - iT, c + iT] \times [b \pm iT, c \pm iT]$. We have

$$(33) \quad I_4 := \int_{c-iT}^{c+iT} \int_{b+iT}^{c+iT} L(z, w) \, dz \, dw \ll \ll \int_b^{c} \int_t^{c} \zeta(c + it)\zeta(2c + 2it)\zeta(\sigma + iT)\zeta(2\sigma + 2iT)\zeta(c + \sigma + iT) \times x^{c+\sigma}t^{-1}T^{-1}dt \ll xT^{-1} \int_b^{c} \zeta(\sigma + iT)\zeta(2\sigma + 2iT)x^\sigma \, d\sigma.$$

Here

$$\int_b^{c} \zeta(\sigma + iT)\zeta(2\sigma + 2iT)x^\sigma \, d\sigma \ll \max_{\sigma \in [b, c]} \zeta(\sigma + iT)\zeta(2\sigma + iT)x^\sigma.$$

For $\sigma \in [b, 1/2]$ we have

$$(34) \quad \zeta(\sigma + iT)\zeta(2\sigma + iT)x^\sigma \ll T^{\mu(1/3)+\mu(2/3)}x^{1/2} \ll Tx^{1/3}.$$

Taking into account (29) for $\sigma \in [1/2, 1]$ we get

$$(35) \quad \zeta(\sigma + iT)\zeta(2\sigma + iT)x^\sigma \ll T^{\mu(\sigma)}x^\sigma \ll x^{\mu(\sigma)+\sigma} \ll x^{(1+2\sigma)/3} \ll x.$$

Returning to (33) we get

$$(36) \quad I_4 \ll x^{2T^{-1}} + x^{4/3}.$$
There are 4 faces of form $[b \pm iT, c \pm iT]^2$. We have

\begin{equation}
I_5 := \iint_{[b \pm iT, c \pm iT]^2} L(z, w) \, dz \, dw \ll \max_{(z, w) \in [b \pm iT, c \pm iT]^2} L(z, w) \ll \\
\ll \max_{\sigma_1, \sigma_2 \in [b, c]} \zeta(\sigma_1 + iT)\zeta(2\sigma_1 + 2iT)\zeta(\sigma_2 + iT)\zeta(2\sigma_2 + iT)\zeta(\sigma_1 + \sigma_2 + 2iT) \times x^{\sigma_1 + \sigma_2 T^{-2}} \ll T^{2\mu(1/3) + 3\mu(2/3) - 2} \ll x^{2T^{-1}}.
\end{equation}

Finally, there are 8 faces, which are parallel either to $z$- or $w$-plane, of form $[b - iT, c + iT] \times w$, where $w \in W := \{b \pm iT, c \pm iT\}$. We have

\begin{equation}
I_6 := \iint_{b - iT} L(z, b + iT) \, dz \ll \int_1^T \int_b^c \zeta(\sigma + iT)\zeta(2\sigma + 2iT)\zeta(\sigma + b + iT + T) \times \\
\times \zeta(b + iT)\zeta(2b + 2iT)x^{\sigma + b T^{-1}} T^{-1} \, d\sigma \, dt \ll T^{\mu(1/3) + \mu(2/3) - 1} x^{1/3} \times \\
\times \int_1^T \int_b^c \zeta(\sigma + iT)\zeta(2\sigma + 2iT)(\sigma + 1/3 + iT)x^{\sigma T^{-1}} \, d\sigma \, dt.
\end{equation}

Here

\[\zeta(\sigma + iT)\zeta(2\sigma + 2iT)(\sigma + 1/3 + iT)x^{\sigma T^{-1}} \ll T^{\mu(1/3) + \mu(2/3) - 1} x,\]

so

\begin{equation}
I_6 \ll T^{\mu(1/3) + \mu(2/3) - 1} x^{1/3} \int_1^T T^{\mu(1/3) + \mu(2/3) - 1} x \, dt \ll x^{4/3}.
\end{equation}

Also

\begin{equation}
I_7 := \iint_{b - iT} L(z, c + iT) \, dz \ll \int_1^T \int_b^c \zeta(\sigma + iT)\zeta(2\sigma + 2iT) \times \\
\times \zeta(\sigma + c + iT + T)\zeta(c + iT)\zeta(2c + 2iT)x^{\sigma + c T^{-1}} T^{-1} \, d\sigma \, dt \ll \\
\ll xT^{-1} \int_1^T \int_b^c \zeta(\sigma + iT)\zeta(2\sigma + 2iT)x^{\sigma T^{-1}} \, d\sigma \, dt
\end{equation}

We derive from \ref{eq:54} and \ref{eq:35} that

\[\int_b^c \zeta(\sigma + iT)\zeta(2\sigma + 2iT)x^\sigma \, d\sigma \ll tx^{1/3} + x,\]

so

\begin{equation}
I_7 \ll xT^{-1} \int_1^T (x^{1/3} + xT^{-1}) \, dt \ll x^{2T^{-1}} + x^{4/3}.
\end{equation}

Now summing up \ref{eq:27}, \ref{eq:31}, \ref{eq:32}, \ref{eq:36}, \ref{eq:37}, \ref{eq:38}, \ref{eq:39}, we get

\begin{equation}
E(x) \ll x^{4/3} T^{\mu(1/3)} + x^{2 + \epsilon} T^{-1}.
\end{equation}

8. Final remarks

Our result can be slightly improved under Riemann hypothesis. In such case we have $\zeta^\pm(s) \ll x^\varepsilon$ for $\sigma > 1/2$ and $\mu(1/2) = 0$ due to \ref{eq:7} \& \ref{eq:14.2.5}--\ref{eq:14.2.6}). Then \ref{eq:19} immediately induces $F(z, w) \ll x^\varepsilon \zeta(z)\zeta(w)$ for $\mathbb{R}z, \mathbb{R}w > 1/4$ and all
double integrals, incorporated in $E(x)$, can be split and estimated by a product of two one-dimensional integrals. For $b = 1/4 + 1/\log x$ we obtain
\[
\begin{align*}
\int_{b-iT}^{b+iT} \frac{\zeta(z)}{z} dz &\ll x^{1/4+\varepsilon} T^{1/4}, \\
\int_{c-iT}^{c+iT} \frac{\zeta(z)}{z} dz &\ll x^{1+\varepsilon}, \\
\int_{b \pm iT}^{c \pm iT} \frac{\zeta(z)}{z} dz &\ll (x^{1/2+\varepsilon} T^{1/4} + x^{1+\varepsilon})/T.
\end{align*}
\]
Then $E(x) \ll x^{5/4+\varepsilon} T^{1/4}$ and choice $T = x^{3/5}$ provides us with $\alpha = 7/5 = 1.4$ in the statement of Theorem 1.

One should expect in the view of (20) that
\[
\sum_{m,n \leq x} \tau_1(kmn) = D_1 x^2 + D_2 x^{1+1/k} + O(x^{\alpha_k+\varepsilon}).
\]
Translating the domain of integration till $[b-iT, b+iT]^2$, where $b = 1/(k+1)$, leads to the error term at least $x^{1/2+\varepsilon} T^{(1/(k+1))} + x^{2+\varepsilon} T^{-1}$, which corresponds to $\alpha_k = 2 - 2k/(3k+1+4\mu(1/2))$ for the best possible choice of $T$. Under Riemann hypothesis for $b = 1/2k + 1/\log x$ we obtain $\alpha_k = (4k-1)/(3k-1)$. However, for $k > 2$ both this estimate are bigger than $x^{4/3}$ and absorbs the term $D_2 x^{1+1/k}$ in (41). Such result can hardly be reckoned satisfactory.

REFERENCES

[1] Balazard M., Naimi M., Pétermann Y.-F. S. Étude d’une somme arithmétique multiple liée à la fonction de Môbius // Acta Arith. — 2008. — Vol. 132, no. 2. — P. 245–298.
[2] Graham S. W., Kolesnik G. On the difference between consecutive squarefree integers // Acta Arith. — 1988. — Vol. 49, no. 5. — P. 435–447.
[3] Hardy G. H., Wright E. M. An introduction to the theory of numbers / Ed. by D. R. Heath-Brown, J. H. Silverman. — 6th, rev. edition. — New York : Oxford University Press, 2008. — xxii+635 p. — ISBN: 0199219867, 9780199219865.
[4] Huxley M. N. Exponential sums and the Riemann zeta function V // Proc. Lond. Math. Soc. — 2005. — Vol. 90, no. 1. — P. 1–41.
[5] Ivić A. The Riemann zeta-function: Theory and applications. — Mineola, New York : Dover Publications, 2003. — 562 p. — ISBN: 0486428133, 9780486428130.
[6] Shabat B. V. Introduction to complex analysis II: Functions of several variables / Ed. by S. Ivanov. Vol. 110 of Translations of mathematical monographs. — Providence, Rhode Island : American Mathematical Soc., 1992. — xxii+371 p. — ISBN: 082185986X, 9780821859863.
[7] Titchmarsh E. C. The theory of the Riemann zeta-function / Ed. by D. R. Heath-Brown. — 2nd, rev. edition. — New York : Oxford University Press, 1986. — 418 p. — ISBN: 0198543691, 9780198543696.

I. I. Mechnikov Odessa National University
E-mail address: 1@dxdy.ru