The UV prolate spectrum matches the zeros of zeta

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We describe a remarkable property of the self-adjoint extension of the prolate spheroidal operator introduced in 1998 by A.C. The restriction of this operator to the interval whose characteristic function commutes with it is well known, has a discrete positive spectrum, and is well understood. What we have discovered is that the restriction of the prolate differential operator to the complement of the finite interval admits (besides a replica of the above positive spectrum) negative eigenvalues whose ultraviolet (UV) behavior reproduces that of the squares of zeros of the Riemann zeta function. Moreover, we show that their corresponding eigenfunctions belong to the Sonin space. This feature fits with the proof (by A.C. and C. Consani) of Weil’s positivity at the Archimedean place, which uses the compression of the scaling action to the Sonin space. Furthermore, we construct an isospectral family of Dirac operators whose spectra have the same UV behavior as the zeros of the Riemann zeta function.

The prolate spheroidal wave functions play a key role in refs. 1–3 in relation to the Riemann zeta function. In all these applications, they appear as eigenfunctions of the angle operator between two orthogonal projections in the Hilbert space $L^2 (\mathbb{R})$ of even square integrable function on $\mathbb{R}$. These projections depend on a parameter $\lambda > 0$; the projection $P_\lambda$ is given by multiplication with the characteristic function of the interval $[-\lambda, \lambda] \subset \mathbb{R}$. The projection $\tilde{P}_\lambda$ is its conjugate by the Fourier transform $F_{\text{ex}}$ which is the unitary operator in $L^2 (\mathbb{R})$ defined by

$$F_{\text{ex}}(\xi)(y) = \int \xi(x) \exp(-2\pi i x y) dx.$$ 

In all the above applications of prolate spheroidal wave functions, the miraculous existence, discovered by the Bell Labs group (4–6), of a differential operator $W_\lambda$ commuting with the angle operator plays only an auxiliary role. In the present paper, we uncover another “miracle”: a careful study of the natural self-adjoint extension of $W_\lambda$ introduced in ref. 9, Lemma 6 (see also ref. 10, section 3.3) to $L^2 (\mathbb{R})$ shows that it still has a discrete spectrum and that its negative eigenvalues reproduce the ultraviolet (UV) behavior of the squares of zeros of the Riemann zeta function. In a similar way, the positive spectrum corresponds, in the UV regime, to the trivial zeros. This coincidence holds for two values $\lambda = 1$ and $\lambda = \sqrt{2}$. The conceptual reason for this coincidence is the link between the operator

$$W_\lambda \xi(x) = -\partial_x (\lambda^2 - x^2) \partial_x \xi(x) + (2\pi \lambda)^2 x^2 \xi(x)$$  \[1\]

and the square of the scaling operator $S := x \partial_x$. In ref. 2, the compression of $f(S)$ to Sonin’s space (11) (which consists of functions $f \in L^2 (\mathbb{R})$ such that $P_\lambda (f) = f \in L^2 (\mathbb{R})$ was shown to be (for $\lambda = 1$) the root of Weil’s positivity at the Archimedean place on test functions supported in the interval $[2^{-1/2}, 2^{1/2}]$, but, since Sonin’s space is not preserved by scaling, one could not restrict scaling to this space. It turns out that $W_\lambda$ commutes with the orthogonal projection on Sonin’s space. Thus one can restrict $W_\lambda$ to Sonin’s space, and the UV spectral similarity with the squares of nontrivial zeros of zeta suggests that one has spectrally captured the contribution of the Archimedean place to the mysterious zeta spectrum. In fact, using the Darboux process, we construct an isospectral family of Dirac square-root operators of $W_\lambda$ depending on a deformation parameter, whose spectrum has the same UV behavior as the zeros of the Riemann zeta function.

Our paper is organized as follows: In The Self-Adjoint Prolate Wave Operator, we show that there exists a unique self-adjoint extension $W_{\text{sa}}$ of the symmetric operator $W_{\text{min}}$ defined on Schwartz space $S(\mathbb{R})$ by Eq. 1. Moreover, $W_{\text{sa}}$ commutes with Fourier transform and has a discrete spectrum unbounded in both directions. In Sonin Space and

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*As pointed out in ref. 7, this discovery can be traced back to the work of Bateman (8) in 1907.

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Significance

We show that the eigenvalues of the self-adjoint extension (introduced by A.C. in 1998) of the prolate spheroidal operator reproduce the UV behavior of the squares of zeros of the Riemann zeta function, and we construct an isospectral family of Dirac operators whose spectra have the same UV behavior as those zeros.
Negative Eigenvalues, we show that the eigenvectors for negative eigenvalues of $W_{a}$ belong to Sonin’s space. In Semiclassical Approximation and Counting Function, we compute the semiclassical approximation to the number of negative eigenvalues of $W_{a}$ whose absolute value is less than $E^{2}$. In Dirac Operators, we use the Darboux method combined with solutions of a Riccati equation to construct an isospectral family of Dirac operators $D_{\lambda}$ invariant under the parity exchange. But, having two interior singular points, it is not directly treatable by the usual Sturm–Liouville theory. However, its restrictions to each of the intervals $(-\infty, -\lambda)$, $(-\lambda, \lambda)$, and $(\lambda, \infty)$ are standard, in fact quasi-regular, Sturm–Liouville operators.

Henceforth, $W_{\lambda}$ will be simply denoted as $W$ whenever $\lambda$ is a general parameter. To begin with, we regard $W$ as an unbounded operator on $L^{2}(\mathbb{R})$ with core the Schwartz space $S(\mathbb{R})$. As such, $W$ is real, symmetric, and invariant under the parity exchange $x \mapsto -x$. These features are inherited by its closure in the graph norm $W_{\min}$, as well as by $W_{\max} = W_{\min}^{\ast}$, the latter having domain

$$\text{Dom}(W_{\max}) = \{ \xi \in L^{2}(\mathbb{R}) \mid W\xi \in L^{2}(\mathbb{R}) \},$$

with $W\xi$ viewed as a tempered distribution. In addition, $W$ has the remarkable property of commuting with the Fourier transform

$$F_{\xi}(f)(y) := \int_{-\infty}^{\infty} f(x) \exp(-2\pi ixy) \, dx.$$  

Since both the Schwartz space $S(\mathbb{R})$ and its dual are globally invariant under the Fourier transform, the domains $W_{\min}$ and $W_{\max}$ are invariant too; therefore, both $W_{\min}$ and $W_{\max}$ commute with $F_{\xi}$.

**Lemma 1.1.** The deficiency indices of $W_{\max}$ are $(4, 4)$.

Any $\xi \in \text{Dom}(W_{\max})$ satisfying $W\xi = \pm i\xi$ is a piecewise real analytic function and is uniquely specified by six parameters in the complement of the two regular singular points $\pm \lambda$. The known form of the solutions (cf. ref. 12) together with the fact that $W\xi \in L^{2}(\mathbb{R})$ imply that the logarithmic singularities of $\xi$ on the left and the right of $\pm \lambda$ have to match. This reduces the number of parameters to four. Conversely, since all four singular points are LC (limit circle case), any solution of $W\xi = \pm i\xi$ belongs to $\text{Dom}(W_{\max})$; hence, $\dim \text{Ker}(W_{\max} \pm iI) = 4$.

**Lemma 1.2.** Let $\xi \in \text{Dom}W_{\max}$, and denote $a = \pm \lambda$. The distribution $p(x)\partial_{x}\xi$ coincides with a continuous function $f$ in a neighborhood of $a$, and the evaluation map $L(\xi) := f(a)$ defines a nonzero continuous linear form on $\text{Dom}W_{\max}$ which vanishes on the closed subspace $\text{Dom}W_{\min}$.

Let $\psi$ be the distribution $\psi = p(x)\partial_{x}\xi(x)$. Since $\xi \in \text{Dom}W_{\max}$, the derivative of $\psi$ in the sense of distributions belongs locally to $L^{2}$ and hence locally to $L^{1}$. It follows that $\psi$ coincides with the integral of an $L^{1}$ function and hence with a continuous function $f$. The evaluation $L(\xi) := f(a)$ is, by construction, continuous in the graph norm of $W_{\max}$.

For $\xi \in S(\mathbb{R})$, the distribution $\psi = p(x)\partial_{x}\xi(x)$ is a function vanishing at $x = a$, and thus $L(\xi) = 0$. By the density of $S(\mathbb{R})$ in $\text{Dom}W_{\min}$ for the graph norm, it follows that $L$ vanishes on the closed subspace $\text{Dom}W_{\min}$.

Let $\mu_{\lambda}$ be the cutoff projection associated with the interval $[-\lambda, \lambda]$, that is, the multiplication operator by the characteristic function $1_{[-\lambda, \lambda]}$, and let $\mu_{\lambda} = F_{\xi_{\lambda}}F_{\xi_{\lambda}}^{-1}$ denote its conjugate by the Fourier transform.

**Lemma 1.3.** If $\xi \in \text{Dom}W_{\min}$, then $P_{\lambda}\xi \in \text{Dom}W_{\max}$ and $P_{\lambda}\xi \psi = P_{\lambda}W\xi$. The same holds with respect to $P_{\lambda}$.

Let $f \in C_{c}^{\infty}(V)$, where $V$ is a neighborhood of the interval $[-\lambda, \lambda]$. Then $P_{\lambda}f \in \text{Dom}W_{\max}$, and, viewing $W(P_{\lambda}f)$ as distribution, one gets, for any $\phi \in S(\mathbb{R})$,

$$\langle W(P_{\lambda}f), \phi \rangle = \int_{-\lambda}^{\lambda} f(x)(W\phi)(x) \, dx$$

$$= \int_{-\lambda}^{\lambda} -f(x)(2\pi \lambda)(2\pi \lambda - x^{2})\phi(x) \, dx$$

$$+ \int_{-\lambda}^{\lambda} f(x)(2\pi \lambda)(2\pi \lambda + x^{2})\phi(x) \, dx.$$  

Using twice integration by parts, together with the fact that $(\lambda^{2} - x^{2})\phi(x)$ and $(\lambda^{2} - x^{2})f'(x)$ vanish on the boundary, one obtains

$$\langle W(P_{\lambda}f), \phi \rangle = \int_{-\lambda}^{\lambda} f(x)(2\pi \lambda)(2\pi \lambda - x^{2})\phi(x) \, dx$$

$$+ \int_{-\lambda}^{\lambda} f(x)(2\pi \lambda)(2\pi \lambda + x^{2})\phi(x) \, dx$$

$$= \int_{-\lambda}^{\lambda} f(x)(2\pi \lambda)(2\pi \lambda - x^{2})\phi(x) \, dx$$

$$= \int_{-\lambda}^{\lambda} f(x)(2\pi \lambda)(2\pi \lambda + x^{2})\phi(x) \, dx$$

which shows that $W(P_{\lambda}f) = P_{\lambda}Wf$. In particular, the same is true for any $f \in S(\mathbb{R})$, and, by the density of $S(\mathbb{R})$ in $\text{Dom}W_{\min}$, for the graph norm, it follows that

$$\xi \in \text{Dom}W_{\min} \implies P_{\lambda}\xi \in \text{Dom}W_{\max} \text{ and } W_{\max}P_{\lambda}\xi = P_{\lambda}W\xi.$$  

The claim now follows from the fact that $W$ commutes with $F_{\xi}$.

The self-adjoint extensions of $W_{\min}$ are parametrized by self-orthogonal subspaces of $E := \text{Dom}(W_{\max})/\text{Dom}(W_{\min})$ with respect to the antisymmetric sesquilinear form given by the pairing

$$\Omega(\xi, \eta) := \frac{1}{\pi} \left( (W_{\max}\xi, \eta) - (\xi, W_{\max}\eta) \right), \xi, \eta \in \text{Dom}(W_{\max})$$

which descends to a nondegenerate form on $E$.  

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The \( \Omega \)-pairing can be expressed in terms of boundary values as usual. One starts with the Lagrange identity
\[
\frac{d}{dx}[\xi, \eta] = \xi W \eta - \eta W \xi,
\]
where \( \xi, \eta \in C^1(\mathbb{R}) \cap \text{Dom } W_{\text{max}} \) and
\[
[\xi, \eta] := -p \left( \frac{d}{dx} \eta - \frac{d}{dx} \xi \right), \quad p(x) = x^2 - \lambda^2,
\]
is the (generalized) Wronskian. By integrating it on compact subintervals \([a, b] \subset \mathbb{R} \setminus \{ \pm \lambda \} \), one obtains Green's formula
\[
\int_a^b (W(x) \xi - \eta W(x)) \, dx = [\xi, \eta]|_a^b
\]
\[
:= \lim_{x \to b} [\xi, \eta](x) - \lim_{x \to a} [\xi, \eta](x).
\]
Passage to the lateral limits toward the endpoints of the three subintervals partitioning \( \mathbb{R} \setminus \{ \pm \lambda \} \) extends this identity to the whole real line, allowing expression of \( \Omega \) in terms of Lagrange brackets as follows:
\[
i \Omega(\xi, \eta) = [\xi, \eta]|_{-\infty} - [\xi, \eta]|_{\pm \lambda} + [\xi, \eta]|_{\infty}
\]
for all pairs \( \xi, \eta \in \text{Dom } W_{\text{max}} \).

Since \( W \) is invariant under parity exchange, it preserves the orthogonal decomposition \( L^2(\mathbb{R}) = L^2_{\text{even}}(\mathbb{R}) \oplus L^2_{\text{odd}}(\mathbb{R}) \) into even functions, which, in turn, induces corresponding splittings \( W = W^+ \oplus W^- \), \( \Omega = \Omega^+ \oplus \Omega^- \), and \( \mathcal{E} = \mathcal{E}^+ \oplus \mathcal{E}^- \). Note, also, that \( \mathcal{E}^\pm \) are invariant under Fourier transform.

The following auxiliary lemma is a straightforward computation.

Lemma 1.4. (i) Let \( f(x) = (1/2) \log((\lambda^2 - x^2)^{-2}) \) viewed as a tempered distribution. Then the Fourier transform \( \mathcal{F}_{\mathcal{E}^+}f \) is a distribution which coincides outside zero with the function
\[
\tilde{f}(y) = \frac{\cos(2\pi \lambda y)}{|y|}.
\]

(ii) Let \( \beta \) be the characteristic function of the interval \( I = [-\lambda, \lambda] \);
then
\[
\mathcal{F}_{\mathcal{E}^+}1_I(y) = \frac{\sin(2\pi \lambda y)}{\pi y}.
\]

We now proceed to construct a basis of \( \mathcal{E} \). First, for \( \mathcal{E}^+ \), we pick an even function \( \alpha_+ \in C^\infty_c(\mathbb{R}) \) such that \( \alpha_+(x) = \log[\lambda^2 - x^2] \) for \( x \in [(3/4)\lambda, (5/4)\lambda] \) and with support in \((1/2)\lambda, (3/2)\lambda)\). Then we take \( \beta_+(x) = 1_I \), the characteristic function of the interval \( I = [-\lambda, \lambda] \), which belongs to \( P_{\lambda}\mathcal{S}(\mathbb{R}) \) and hence to \( \text{Dom } W_{\text{max}} \). Next, for \( \mathcal{E}^- \), we let \( \alpha_-(x) := x\alpha_+(x) \) and \( \beta_-(x) := x\beta_+(x) \).

Lemma 1.5. The quadruplet \( (\alpha_\pm, \beta_\pm, \tilde{\alpha}_\pm, \tilde{\beta}_\pm) \) forms a basis of \( \mathcal{E}^\pm \).

One checks, using the expression Eq. 10 of the \( \Omega \)-pairing together with Lemma 1.4, that the matrix representation of \( \Omega_+ \) with respect to the given quadruplet has a single nonzero entry in each row and column.

In the odd case, the calculation is similar.

The \( \Omega \)-pairings with the above basis elements yield boundary conditions of Sturm–Liouville type. Denoting, for \( \xi \in \text{Dom } (W_{\text{max}}^\pm) \),
\[
\mathbb{L}_{\alpha_\pm}(\xi) := i\Omega_\pm(\xi, \alpha_\pm), \quad \mathbb{L}_{\tilde{\alpha}_\pm}(\xi) := i\Omega_\pm(\xi, \tilde{\alpha}_\pm),
\]
\[
\mathbb{L}_{\beta_\pm}(\xi) := i\Omega_\pm(\xi, \beta_\pm), \quad \mathbb{L}_{\tilde{\beta}_\pm}(\xi) := i\Omega_\pm(\xi, \tilde{\beta}_\pm),
\]
the minimal domains are characterized in these terms as being the intersection
\[
\text{Dom}(W_{\text{min}}^\pm) = \text{Ker } \mathbb{L}_{\alpha_\pm} \cap \text{Ker } \mathbb{L}_{\beta_\pm} \cap \text{Ker } \mathbb{L}_{\tilde{\alpha}_\pm} \cap \text{Ker } \mathbb{L}_{\tilde{\beta}_\pm},
\]
and the induced functionals on \( \mathcal{E}^\pm = \text{Dom } (W_{\text{max}}^+) / \text{Dom } (W_{\text{min}}^+) \) form a basis of \( \mathcal{E}^\pm \).

By straightforward calculation, using the fact that one can always restrict the computation to \( \mathbb{R}^+ \), one obtains explicit expressions for the boundary functionals. Up to a nonzero constant factor, they are as follows. In the even case,
\[
\mathbb{L}_{\alpha_\pm}(\xi) := \lim_{x \to \lambda} \left( (x - \lambda) \log(\lambda - x) - \partial_\xi(\xi - x) \right) + \lim_{x \to -\lambda} \left( (x - \lambda) \log(\lambda - x) - \partial_\xi(\xi - x) \right) - \lim_{x \to \lambda} \left( (x + \lambda) \log(\lambda + x) - \partial_\xi(\xi + x) \right);
\]
\[
\mathbb{L}_{\beta_\pm}(\xi) := \lim_{x \to \lambda} \left( (x + \lambda) \log(\lambda + x) - \partial_\xi(\xi + x) \right) - \lim_{x \to -\lambda} \left( (x + \lambda) \log(\lambda + x) - \partial_\xi(\xi + x) \right) + \lim_{x \to \lambda} \left( (x - \lambda) \log(\lambda - x) - \partial_\xi(\xi - x) \right) - \lim_{x \to -\lambda} \left( (x - \lambda) \log(\lambda - x) - \partial_\xi(\xi - x) \right).
\]

Note that the existence of the limit defining \( \mathbb{L}_{\beta_\pm}(\xi) \), that is, the equality of the lateral limits, is ensured by Lemma 1.2. Similar formulas define the functionals \( \mathbb{L}_{\alpha_\pm}, \mathbb{L}_{\beta_\pm}, \mathbb{L}_{\tilde{\alpha}_\pm}, \mathbb{L}_{\tilde{\beta}_\pm} \) in the odd case.

Since both Dom(\( W_{\text{min}}^- \)) and Dom(\( W_{\text{max}}^- \)) as well as the symplectic form \( \Omega \), are globally invariant under the Fourier transform, the quotient inherits induced transformations \( f_{\mathcal{E}^\pm} : \mathcal{E}^\pm \to \mathcal{E}^\pm \), which relates the boundary functionals as follows:
\[
\mathbb{L}_{\beta_\pm} = \mathbb{L}_{\beta_\pm} \circ f_{\mathcal{E}^+} \quad \text{and} \quad \mathbb{L}_{\tilde{\beta}_\pm} = \mathbb{L}_{\tilde{\beta}_\pm} \circ f_{\mathcal{E}^+}.
\]

This association gives rise to two distinguished self-orthogonal subspaces, namely,
\[
\mathcal{L}_\beta = \bigcap_\pm \ker \mathbb{L}_{\beta_\pm} \cap \bigcap_\pm \ker \mathbb{L}_{\tilde{\beta}_\pm} = \bigcap_\pm \ker \mathbb{L}_{\alpha_\pm} \cap \bigcap_\pm \ker \mathbb{L}_{\tilde{\alpha}_\pm}.
\]

Definition: We denote by \( W_{\text{sa}} \) the restriction of the operator \( W_{\text{max}} \) to the subspace \( \mathcal{L}_\beta = \bigcap_\pm \ker \mathbb{L}_{\beta_\pm} \cap \bigcap_\pm \ker \mathbb{L}_{\tilde{\beta}_\pm} \). Explicitly, its domain \( \text{Dom } W_{\text{sa}} \) consists of the elements \( \xi \in \text{Dom } W_{\text{max}} \) satisfying the following boundary conditions:
\[
\lim_{x \to \pm \lambda} (\lambda^2 - x^2) \partial_\xi x (x) = 0,
\]
And, at \( \pm \infty \), writing \( \xi = \xi^+ + \xi^- \) with \( \xi^\pm \in \text{Dom } W_{\text{max}}^\pm \),
\[
\lim_{x \to \pm \infty} \left( x \sin(2\pi \lambda x) \partial_\xi x^+ (x) - (2\pi \lambda \cos(2\pi \lambda x) \sin(2\pi \lambda x) \partial_\xi x^- (x) \right) = 0,
\]
\[
\lim_{x \to \pm \infty} \left( -2\pi \lambda \cos(2\pi \lambda x) \sin(2\pi \lambda x) \partial_\xi x^- (x) + (x \cos(2\pi \lambda x) \partial_\xi x^+ (x) \right) = 0.
\]

We are now in a position to establish the main result of this section.

Theorem 1.6. (i) \( W_{\text{sa}} \) is self-adjoint and commutes with the Fourier transform.
(ii) \( W_a \) commutes with the projections \( P_\lambda \) and \( \bar{P}_\lambda \).

(iii) \( W_a \) is the only self-adjoint extension of \( W_{\min} \) commuting with \( P_\lambda \) and \( \bar{P}_\lambda \). (iv) The spectrum of \( W_a \) is discrete and unbounded on both sides; its negative eigenvalues are simple, while the positive eigenvalues (with possibly finitely many exceptions) have multiplicity 2.

(i) \( W_a \) is self-adjoint by construction, and its domain \( \mathcal{L}_\beta \) is invariant under the Fourier transform also by construction.

(ii) Since \( \text{Dom} \ W_{\min} \) is given by Eq. 12, every element of \( \mathcal{L}_\beta \) is a linear combination of an element \( \xi \in \text{Dom} \ W_{\min} \) and the four vectors \( \beta_+, \beta_- \) of Lemma 1.1. Each \( \beta_\pm \) is of the form \( P_\lambda f_\pm \) with \( f_\pm \) smooth with compact support, and thus one has, using Lemma 1.3,

\[
P_\lambda \beta_\pm = \beta_\pm \in \mathcal{S}, \quad W_a P_\lambda \beta_\pm = W_a P_\lambda f_\pm = P_\lambda W f_\pm,
\]

which shows that \( W_a P_\lambda \beta_\pm = P_\lambda W_a P_\lambda \beta_\pm = P_\lambda W_a \beta_\pm \), giving the required commutation for the \( \beta_\pm \).

(iii) The domain of a self-adjoint extension of \( W_{\min} \) commuting with \( P_\lambda \) and \( \bar{P}_\lambda \) must be contained in \( \text{Dom} \ W_{\max} \) and also contain both \( P_\lambda \mathcal{S}(\mathbb{R}) \) and \( \bar{P}_\lambda \mathcal{S}(\mathbb{R}) \). Thus it must contain \( \mathcal{L}_\beta \), and cannot be larger due to self-adjointness.

(iv) The operator \( P_\lambda W_a \) is self-adjoint and positive on \( (-\lambda, \lambda) \) and has a simple spectrum. It coincides with the self-adjoint operator extending \( W \vert_{(-\lambda, \lambda)} \) which corresponds to the boundary condition Eq. 17 and has a simple spectrum consisting of strictly positive eigenvalues (cf. ref. 13). To handle the operator \( W_a' = (I - P_\lambda) W_a' \in \mathcal{H} = (I - P_\lambda) \mathcal{L}^2(\mathbb{R}) \), we use the orthogonal decomposition \( \mathcal{L}^2(\mathbb{R}) = \mathcal{L}^2_{\uparrow}(\mathbb{R}) \oplus \mathcal{L}^2_{\downarrow}(\mathbb{R}) \) into even (resp. odd) functions, which induces the orthogonal decomposition \( \mathcal{H} = \mathcal{H}_\uparrow \oplus \mathcal{H}_\downarrow \) and the splitting \( W_a' = W_a^{\uparrow} \oplus W_a^{\downarrow} \). Under the canonical isomorphisms \( \mathcal{H}_\uparrow \approx \mathcal{L}^2(\lambda, \infty) \) (obtained by restriction), the operators \( W_a^{\uparrow, \downarrow} \) are self-adjoint extensions of the restriction of \( W \) with boundary condition at \( \lambda \) given by Eq. 17 and, at infinity, by Eqs. 18 and 19. Thus, they are covered by standard results in Sturm–Liouville theory (cf. refs. 13–16). Indeed the endpoints are LC in the Weyl classification (see e.g., refs. 15 and 16 for relevant definitions and properties), which can be easily checked by using explicit bases of formal solutions for \( W \xi - \mu \xi = 0, \mu \in \mathbb{R} \), around each singular point (cf., e.g., ref. 12, section 2). The endpoint \( \lambda \) is LCNO (nonoscillatory), while \( +\infty \) is LCO (oscillatory), since the prolate spheroidal wave functions (which provide principal solutions around \( \lambda \)) have infinitely many zeros in the neighborhood of \( +\infty \) (cf. refs. 6–8). The spectrum of \( W^{\pm} \) is discrete (cf. ref. 14, sections 19.2 and 19.3) but not bounded on either side, since \( \infty \) is an LC endpoint (see ref. 16, Theorem 10.7.1). Moreover, their spectrum has multiplicity 1 (see ref. 15, Theorem 10.7). The operators \( W^{\pm} \) inherit the positive eigenvalues corresponding to the Fourier transforms of the (prolate spheroidal) eigenfunctions of \( P_\lambda W_a \) which are eigenvalues for the same positive eigenvalues. Thus the corresponding positive spectrum of the original operator \( W_a \) is of multiplicity 2. Note, finally, that the spectra of the operators \( W_a^{\pm} \) are disjoint because the boundary conditions at \( \infty \) are exclusive of each other.

**Corollary 1.7.** If \( \phi \) is an eigenfunction of \( W_a^{\pm} \), then

(i) \( \phi \) is regular on \([\lambda, \lambda + \epsilon)\) and on \((\lambda - \epsilon, \lambda]\) for some \( \epsilon > 0 \), with a possible discontinuity at \( \lambda \);

(ii) the leading term of the asymptotic expansion of \( \phi \) at \( \infty \) is proportional to \( \sin(2\pi \lambda x)/x \) if \( \phi \) is even, and is proportional to \( \cos(2\pi \lambda x)/x \) if \( \phi \) is odd.

This follows from the above characterization Eqs. 17–19 of the domain of \( W_a \) combined with the known bases of formal solutions for the equation \( W \xi = \mu \xi, \mu \in \mathbb{R} \), around \( \pm \lambda \) and \( \pm \infty \) (cf. ref. 12).

### 2. Sonin Space and Negative Eigenvalues

We translate the requirement that the Fourier transform \( \mathcal{F}_x f \) of an \( f \in \text{Dom} \ W_{\max} \) has no logarithmic singularity at the singular points into a condition on the asymptotic behavior of \( f \) at \( \infty \). For simplicity, we only deal with even functions, and, for notational convenience, we take \( \lambda = 1 \).

We can then find the asymptotic expansion at \( \infty \) using the boundary condition that the leading term has \( \sin(2\pi \lambda y)/y \). We take, for simplicity, \( \lambda = 1 \) and use ref. 12 to get, for the tentative eigenvector for eigenvalue \( \mu \), the expansion at \( \infty \),

\[
\xi_\mu(x) \approx \frac{\sin(2\pi x)}{x} + \frac{(\mu - 4\pi^2)\cos(2\pi x)}{4\pi^2} + \frac{\mu^2 - 8\pi^2 \mu + 2\mu - 16\pi^4 + 8\pi^2}{32\pi^2 x^2} \sin(2\pi x) + O(x^{-1}).
\]

In fact, as shown in Proposition 14 of ref. 12, the coefficients of this expansion are directly related to the coefficients of the expansion of the finite solution at \( \lambda \), and taking for simplicity \( \lambda = 1 \), if the latter is of the form

\[
f_\mu(x) = \sum U_n(\mu)(x - 1)^n, \quad U_0(\mu) = 1, \quad U_1(\mu) = \frac{\mu - 4\pi^2}{2}, \quad U_2(\mu) = \frac{\mu^2 - 8\pi^2 \mu - 2\mu + 16\pi^4 - 8\pi^2}{16}, \ldots
\]

then the asymptotic series at infinity which governs the solution which has a leading term in \( \exp(-2\pi ix)/x \) is equal to

\[
v(x) \approx \sum n! U_n(\mu)(2\pi ix)^{-n}.
\]

When one applies the Borel summation to this series, the first step is to replace it by its Borel transform which is, up to normalization,

\[
B(y) := \sum U_n(\mu) y^n
\]

and is related to \( v(x) \) by \( \int_0^\infty t^n \exp(-zt) \, dt = z^{-n-1} \Gamma(n + 1) \), that is, the Laplace transform

\[
\frac{v(x)}{2\pi ix} = \int_0^\infty \exp(-2\pi ixt) B(t) \, dt.
\]

**Lemma 2.1.** For any \( \mu \in \mathbb{R} \), the asymptotic expansion of the unique solution \( \xi_\mu \) which, at \( \infty \), is asymptotically \( \sim - \sin(2\pi x)/x \), is Borel summable and is equal to the Fourier transform of the unique even solution \( \phi_\mu \) which is zero on \([-1, 1]\) and agrees with \( f_\mu(x) \) for \( x > 1 \).

One has the equality

\[
\frac{v(x)}{2\pi ix} = \int_0^\infty \exp(-2\pi ixt) B(t) \, dt = \int_0^\infty \exp(-2\pi ixt) f_\mu(t + 1) \, dt = \int_1^\infty \exp(-2\pi ix(y - 1)) f_\mu(y) \, dy.
\]
Thus one gets
\[ v(x) \exp(-2\pi ix)/(2\pi ix) = \int_1^\infty \exp(-2\pi ixy)f_\mu(y)\,dy. \]
The function \( \phi_\mu \) is even and vanishes on \([-1, 1]\), so
\[ \int_{-\infty}^\infty \exp(-2\pi ixy)\phi_\mu(y)\,dy = \int_1^\infty \exp(-2\pi ixy)f_\mu(y)\,dy \]
\[ + \int_1^\infty \exp(-2\pi ixy)f_\mu(y)\,dy = v(x) \exp(-2\pi ix)/(2\pi ix) \]
Now these two terms are asymptotic solutions, since \( v(x) \) is real and \( v(x) \exp(-2\pi ix)/(2\pi ix) \) is an asymptotic solution. Moreover, the leading behavior at \( \infty \) is in
\[ \exp(-2\pi ix)/(2\pi ix) - \exp(2\pi ix)/(2\pi ix) = -\frac{\sin(2\pi x)}{\pi x}. \]
Thus it follows that the Fourier transform \( F_{\text{es}}\phi_\mu = \xi_\mu \).

**Corollary 2.2.** With the above notation, assume \( \mu \) is a negative eigenvalue. Then \( \phi_\mu \) belongs to the Sonin space.

The Sonin space coincides with the orthogonal complement of the eigenspace of \( W_a \) associated with the classical proton functions and their Fourier transforms. We should note that, at this point, we do not claim (although this is supported by numerical evidence) that all eigenvalues of the restriction of \( W_a \) to Sonin's space are negative; however, there could be only finitely many exceptions (see Positive Eigenvalues and Trivial Zeros of Zeta).

### 3. Semiclassical Approximation and Counting Function

In this section, we use the semiclassical estimate for the function counting the number of eigenvalues and investigate the negative eigenvalues of the operator \( W_a \). We consider the classical Hamiltonian
\[ H_\lambda(p, q) = (p^2 - \lambda^2)(q^2 - \lambda^2) \]
and use it as a semiclassical approximation of \( W_a \) via the formal relation
\[ W_\lambda \approx -4\pi^2 H_\lambda + 4\pi^2 \lambda^4 \]
using the correspondence \( q \to x \) and \( p \to (1/2\pi i)\partial_x \) associated with the choice of the Fourier transform \( F_{\text{es}}^{-1} \). Sonin's space corresponds to the conditions \( p^2 - \lambda^2 \geq 0 \) and \( q^2 - \lambda^2 \geq 0 \), and the region of interest for the counting of eigenvalues is thus
\[ \Omega_\lambda(E) := \{(q, p) \mid q \geq \lambda, p \geq \lambda, H_\lambda(p, q) \leq \frac{E^2}{2\pi}\lambda^4\}. \]
The area of \( \Omega_\lambda(E) \) is given, with \( a = (E/2\pi)^2 + \lambda^4 \), by the convergent integral
\[ I_\lambda(a) = \int_\lambda^\infty \left( \frac{\sqrt{a + \lambda^2 x^2 - \lambda^4}}{\sqrt{x^2 - \lambda^2}} - \lambda \right) \,dx. \]
One obtains, by a change of variables, the equality
\[ I_\lambda(a) = \lambda^2 I_1(a \lambda^{-4}). \]
We recall that the elliptic integrals \( E(m) \) and \( K(m) \) are defined by
\[ E(m) := \int_0^{\pi/2} \sqrt{1 - m \sin^2 \theta} \,d\theta, \]
\[ K(m) := \int_0^{\pi/2} \frac{1}{\sqrt{1 - m \sin^2 \theta}} \,d\theta. \]
**Lemma 3.1.** The integral \( I(a) = I_1(a) \) is given by the sum of elliptic integrals
\[ I(a) = aK(1-a) - E(1-a) + 1. \]
This follows from a straightforward computation. We thus get the following.

**Proposition 3.2.** The semiclassical approximation to the number of negative eigenvalues \( \xi \) of \( W_a \) with \(-\xi \leq E^2 \) on even functions is the same as on odd functions and is equal to \( 2\sigma(E, \lambda) \), where
\[ \sigma(E, \lambda) \approx \frac{E}{2\pi} \left( \log \left( \frac{E}{2\pi} \right) - 1 + \log(4) - 2 \log(\lambda) \right) + \lambda^2 + o(1). \]
The semiclassical approximation corresponds, for even functions (or to odd functions), to twice the area of \( \Omega_\lambda(E) \) and hence to \( I_\lambda(a) = \lambda^2 I_1(\lambda^{-4}) \), for \( a = (E/2\pi)^2 + \lambda^4 \). One has the asymptotic expansion for \( a \to \infty \),
\[ I(a) \approx \frac{1}{2} \sqrt{a} (\log(a) - 2 + 4 \log(2)) + 1 \]
\[ + \frac{1}{\sqrt{a}} / \sqrt{a} (-\log(a) - 4 \log(2)) + O \left( \frac{1}{a} \right), \]
so that
\[ I_\lambda(a) \approx \frac{1}{2} \sqrt{a} (\log(a) - 2 + 4 \log(2) - 4 \log(\lambda)) + \lambda^2 + o(1). \]
We then use the expansions
\[ \sqrt{a} = \frac{E}{2\pi} + O(1/E), \log(a) = 2 \log \left( \frac{E}{2\pi} \right) + O(1/E^2) \]
and obtain Eq. 24.

### 4. Dirac Operators

The results of Semiclassical Approximation and Counting Function show that, for suitable values of \( \lambda \), the negative spectrum of \( W_a \), or, equivalently, of \( W_{a}^\prime = (I - P_a) W_{a} \), has the same UV behavior as the squares of zeros of the Riemann zeta function. Since \( W_{a} \) is a differential operator of second order, we liken it to the Klein–Gordon operator and construct the analog of the Dirac operator. This can be done by means of the Darboux process (see refs. 4, 17, and 18), which allows factorizing of \( W_{a} \) as a product of two first-order differential operators. It suffices to treat the case of \( W_{a}^\prime \), which, under the canonical isomorphism \( \mathcal{H}^+ \cong L^2(\lambda \geq 0) \), is identified with the self-adjoint operator given by the restriction of \( \mathcal{W} \) to \( (\lambda \geq 0) \) subject to the boundary conditions Eqs. 17 and 18.

**Lemma 4.1.** Let \( p(x) \) and \( q(x) \) be as in Eq. 3. Then the following is a Riccati equation \((\lambda \in (\lambda \geq 0))\):
\[ p^{1/2}(x)\partial w(x) + w(x)^2 = -q(x) + \left( \frac{p''(x)}{4} - \frac{p'(x)^2}{16p(x)} \right). \]
Any solution $w$ of this equation gives rise to a factorization

$$W((λ, ∞)) = (\nabla + w)(\nabla - w), \quad \nabla := p^{1/4}∂_xp^{1/4}. \quad [28]$$

Let $f$ be a smooth function on $\mathbb{R}$ and consider the differential operators $T_1 := f∂_x^2 + x$ and $T_2 := (∂_x^2 + x)^{-1}$. Let us show that $T_1^2 - T_2$ is an operator of order zero: One has

$$T_1^2 = f∂_x^2 f∂_x - f^2 f∂_x^2 + f∂_x^3 f∂_x,$$

so that $T_1^2 - T_2$ is the multiplication by $2f^2 f∂_x - f^3 f''$. Applying this for $f(x) = p(x)^{1/4}$ and the conclusion follows, using Eq. 27.

The solutions of Eq. 27 can be found by the standard reduction to a Bernoulli equation and are as follows.

**Lemma 4.2.** Let $u_1, u_2$ be two linearly independent real valued solutions of the equation $(Wu)(x) = 0, x ∈ (λ, ∞).$

(i) For $z ∈ \mathbb{C}$, the solution $u = u_1 + zu_2$ has no zero in $(λ, ∞)$ if $z ∈ \mathbb{R}$, and has infinitely many zeros otherwise.

(ii) All solutions of the Riccati equation Eq. 27 are of the form

$$w = \frac{\nabla u}{u} \quad [29]$$

with $u = u_1 + zu_2$ and $z ∈ \mathbb{C} \setminus \mathbb{R}$.

(iii) The map $z → w_z$ from $\mathbb{C} \setminus \mathbb{R}$ to the space of solutions of Eq. 27 is a homeomorphism.

**Proposition 4.3.** Let $w$ be a solution of the Riccati equation Eq. 27, and let $\mathcal{D}_w$ be the operator on $\mathcal{H}^+ \oplus \mathcal{H}^+$ (with $\mathcal{H}^+$ another copy of $\mathcal{H}^+$) defined by

$$\mathcal{D}_w = \begin{pmatrix} 0 & \nabla + w \\ \nabla - w & 0 \end{pmatrix} \quad [30]$$

with domain $\text{Dom}\mathcal{D}_w := \{ \left( \begin{array}{c} \xi \\ \bar{\xi} \end{array} \right) : \xi ∈ \text{Dom} W_{sa}^+, (\nabla + w)(\bar{\xi}) ∈ \text{Dom} W_{sa}^+ \}$. \quad [31]

(i) Then

$$\mathcal{D}_w^2 = \begin{pmatrix} W_{sa}^+ & 0 \\ 0 & W_{sa}^+ + 2\nabla w \end{pmatrix} \quad [32]$$

with the diagonal terms isospectral.

(ii) The spectrum of $\mathcal{D}_w$ is $\{ ±√π | μ ∈ \text{Spec} W_{sa}^+ \}$, independently of $w$.

One uses the factorization Eq. 28, which, in particular, implies that $\text{Dom}\mathcal{D}_w$ so defined is contained in the domain of the closure of $\mathcal{D}_w^2$. Let $(\xi_μ)$ be an orthonormal basis of eigenfunctions for $W_{sa}^+$ (indexed by the eigenvalues $μ$). Using the vectors $\xi_μ = (\nabla - w)\xi_μ ∈ \mathcal{H}^+$, the operator $\mathcal{D}_w$ splits as a direct sum of two by two matrices of the form

$$\begin{pmatrix} 0 & μ \\ 1 & 0 \end{pmatrix}$$

whose eigenvalues are $±√π$. We choose the inner product on $\mathcal{H}^+$ for which $μ^{−1/2}\xi_μ$ form an orthonormal basis.

5. UV Behavior of Spectrum of Dirac

In this section, we take $λ = √2$, and consider the operator $2\mathcal{D}$ where $\mathcal{D} = \mathcal{D}_1$ as defined in Proposition 4.3.

**Theorem 5.1.** The operator $2\mathcal{D}$ has discrete simple spectrum contained in $i\mathbb{R}$. Its imaginary eigenvalues are symmetric under complex conjugation, and the counting function $N(E)$ counting those of positive imaginary part less than $E$ fulfills

$$N(E) ≈ \frac{E}{2π} \left( \log \left( \frac{E}{2π} \right) - 1 \right) - \frac{\log E}{2π} + O(1). \quad [33]$$

By Proposition 4.3, the spectrum of $2\mathcal{D}$ consists of the complex numbers of the form $ξ = ±2√π$ where $μ$ varies in the spectrum of $W_{sa}^+$. The latter is real, and, to estimate the number of negative eigenvalues of $W_{sa}^+$ less, in absolute value, than $(E/2)^2$, we apply Heywood’s formula (1.4) of ref. 19, which was subsequently extended to a larger class of potentials (see equation (1.4) in ref. 20, and references therein, and also ref. 21, Theorem 1, for a slightly stronger version). A Liouville transformation shows that $W_{sa}^+$ is unitarily equivalent to the Sturm–Liouville operator $S$ on $(0, ∞)$, where

$$q(y) = -(4π)^2 \cosh(y)^2 - \frac{1}{4} \left( \coth^2(y) - 2 \right). \quad [34]$$

A delicate computation which is a refinement of the semiclassical estimate of Proposition 3.2 gives the additional logarithmic contribution $−(\log E/2π)$ of Eq. 33. One can use the computer to obtain the first eigenvalues of $2\mathcal{D}$ (of positive imaginary part) from those of $W_{1}^+$ and compare them with the zeros of the Riemann zeta function as shown in Figs. 1 and 2.

6. Final Remarks

We gather, in this final section, a number of more speculative remarks.

A. Geometric Meaning of Theorem 5.1. The operator $2\mathcal{D}$ of Theorem 5.1 together with the action by multiplication of smooth functions on the interval $[√2, ∞)$ would define a standard spectral triple if $2\mathcal{D}$ were self-adjoint (or skew adjoint), but its spectrum contains both real and imaginary pieces. Still, the key property that the resolvent is compact is fulfilled, and, moreover, since the leading term $(2√π^2 - 2)∂_x$ of $2\mathcal{D}$ is equivalent to $2x∂_x$ for $x → ∞$, the algebra of functions having bounded

![Fig. 1. Plot of the imaginary part of the n-th eigenvalue (in blue) of 2D and of the n-th zero of zeta (red). When the red dot hides the blue dot, the two values are too close to each other to be distinguished.](https://doi.org/10.1073/pnas.2123174119)
commutator with $2D$ contains smooth functions which are Lipschitz for the scale-invariant metric $dx/x$. The classical metric associated with $2D$ is

$$ds^2 = \frac{1}{4}dx^2/(x^2 - 2) = \frac{1}{\alpha(x)}dx^2, \quad \alpha(x) = -4(x^2 - 2).$$

This $ds^2$ changes sign when crossing the boundary $x = \sqrt{2}$, and this suggests, in order to handle all even functions on $\mathbb{R}$ and also to take into account the real and imaginary eigenvalues of the square of $2D$, that one should look for a two-dimensional metric with signature $(-1, 1)$ of the form

$$ds^2 = -\alpha(x)dx^2 + \frac{1}{\alpha(x)}dx^2.$$

This geometry corresponds to a black hole in two space–time dimensions with horizon at $x = \pm \sqrt{2}$. It fulfills the two-dimensional analog of Einstein's equation with a cosmological constant $\Lambda = 8$ and no source (22).

**B. Positive Eigenvalues and Trivial Zeros of Zeta.** The operator $W^{\pm}_n$ admits, as positive eigenvalues, the eigenvalues $\chi(2n), n \in \mathbb{N}$, of the restriction of $W_n$ to even functions in the interval $[-\lambda, \lambda]$. Moreover, by Heywood's formula (1.5) of ref. 19, one can show that the counting function for the positive eigenvalues of $W^{\pm}_n$ differs only by $O(1)$ from the counting function for the $\chi(2n)$. This implies that there are, at most, finitely many positive eigenvalues of $W^{\pm}_n$ besides the $\chi(2n)$. In fact, we conjecture that there are none. The well-understood asymptotic form of the eigenvalues $\chi(n)$ (ref. 23, Theorem 3.11) implies that, independently of the value of $\lambda$,

$$\chi(2n) = \left(2n + \frac{1}{2}\right)^2 + O(1), \quad n \to \infty.$$

This behavior is the same as that of the squares of the trivial zeros of the Riemann zeta function with the same shift of $1/2$ as for the critical line.

**C. Spectral Truncation.** In order to eliminate the real eigenvalues of $2D$ coming from the positive eigenvalues of $W_n$, one can effect a spectral truncation (24); the algebra of functions acting by multiplication is then replaced by the operator system obtained by compression on Sonin’s space.

**Data Availability.** All study data are included in the article.

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