Three-dimensional AdS black holes in massive-power-Maxwell theory

B. Eslam Panah\textsuperscript{1,2,3*}, K. Jafarzade\textsuperscript{1,2†}, A. Rincón\textsuperscript{4,5‡}

\textsuperscript{1} Department of Theoretical Physics, Faculty of Basic Sciences, University of Mazandaran, P. O. Box 47416-95447, Babolsar, Iran
\textsuperscript{2} ICRANet-Mazandaran, University of Mazandaran, P. O. Box 47416-95447, Babolsar, Iran
\textsuperscript{3} ICRANet, Piazza della Repubblica 10, I-65122 Pescara, Italy
\textsuperscript{4} Departamento de Física Aplicada, Universidad de Alicante, Campus de San Vicente del Raspeig, E-03690 Alicante, Spain
\textsuperscript{5} Sede Esmeralda, Universidad de Tarapacá, Avda. Luis Emilio Recabarren 2477, Iquique, Chile

Recently, it was shown that the power-Maxwell (PM) theory could remove the singularity of the electric field \cite{1}. Motivated by a great interest in three-dimensional black holes and a surge of success in studying massive gravity from both the cosmological and astrophysical points of view, we investigate three-dimensional black hole solutions in de Rham, Gabadadze, and Tolley (dRGT) massive theory of gravity in the presence of PM electrodynamics. First, we extract exact three-dimensional solutions in this theory of gravity. Then we study the geometrical properties of these solutions. Calculating conserved and thermodynamic quantities, we check the validity of the first law of thermodynamics for these black holes. We also examine the stability of these black holes in the context of the canonical ensemble. We continue calculating this kind of black hole’s optical features, such as the photon orbit radius, the energy emission rate, and the deflection angle. Considering these optical quantities, finally, we analyze the effective role of the parameters of models on them.

I. INTRODUCTION

Considering General Relativity (GR) in three-dimensional spacetime, Banados, Teitelboim, and Zanelli (BTZ) have found a black hole solution \cite{2}, which is known as a BTZ black hole. The study of these black holes opened different aspects of physics in three-dimensional spacetime, such as the existence of specific relations between the BTZ black holes and effective action in string theory \cite{3, 4}, providing simplified machinery for studying different features of black holes such as thermodynamic ones \cite{5, 6}, contributing to our understanding of gravitational systems and their interactions in lower dimensions \cite{7, 8}, the possible existence of gravitational Aharonov-Bohm effect due to the non-commutative BTZ black holes \cite{9}, AdS/CFT correspondence \cite{10, 11}, quantum aspect of three-dimensional gravity, entanglement, and quantum entropy \cite{12, 13}, holographic aspects of BTZ black hole solutions \cite{14, 15}, and anti-Hawking phenomena of BTZ black holes \cite{16, 17}. It is essential to point out that gravity in three-dimensional spacetime is a vibrant field of research in part because the absence of propagating degrees of freedom makes things more straightforward than in four dimensions, in particular, when dealing with the challenge of formulating a quantization of this theory. Also, the BTZ black hole is currently a seminal toy model to study different effects beyond GR. On the other hand, by considering various theories of gravity coupled with different matter fields, three-dimensional black hole solutions and their thermodynamic properties have been studied in literature \cite{18, 19, 20, 21, 22}.

One of the most challenging problems of modern cosmology is related to the fact that our Universe is expanding with acceleration. Some candidates have been proposed to explain this acceleration, such as the existence of a positive cosmological constant, dark energy, and modified theories of gravity. Among different candidates of modified theories of gravity, massive theories of gravity have attracted much attention lately due to a wide variety of motivations in various aspects of physics \cite{23, 24}. From a cosmological point of view, one can point out interesting features such as describing the accelerating expansion of our Universe without requiring any dark energy \cite{25, 26}, explaining the current observations related to dark matter \cite{27, 28}, suitable description of rotation curves of the Milky Way, spiral galaxies, and low surface brightness galaxies \cite{29}. The most important achievements in the astrophysics context are: the existence of white dwarfs more than the Chandrasekhar limit \cite{30} and massive neutron stars with a maximum mass more than three times the solar mass \cite{31}. To name a few in the point of black hole physics, one can mention: the existence of van der Waals-like behavior in extended phase space for non-spherical black holes \cite{32, 33}, triple points and N-fold reentrant phase transitions \cite{34}, the existence of a remnant for a black hole which may help to ameliorate the information paradox \cite{35, 36}, etc.

In recent years, a new version of the theory of massive gravity has been proposed by de Rham, Gabadadze, and

\* email address: eslampanah@umz.ac.ir
† email address: khadije.jafarzade@gmail.com
‡ email address: angel.rincon@ua.es
Tolley (dRGT) [51, 52], which is known as dRGT massive gravity. The dRGT massive gravity’s action contains a nonlinear interaction term that admits the Vainshtein mechanism, and it is free from van Dam-Veltman-Zakharov discontinuity [53, 54], and Boulware-Deser ghost [53, 54] in arbitrary dimensions (which appears in Fierz-Pauli theory of massive gravity [57]). Notably, the dRGT theory of massive gravity requires a fiducial reference metric \( f_{\mu\nu} \) in addition to the dynamical metric \( (g_{\mu\nu}) \) to define a mass term for graviton by introducing some non-derivative potential terms \( \mathcal{U}_i \). Also, modification in the introduced reference metric leads to a special family of dRGT massive gravity \([55, 56]\). So, different reference metrics could lead to a variety of new solutions. In this regard, it was shown that the dRGT massive gravity is ghost-free by considering different reference metrics such as Minkowski and degenerate (singular) reference metrics (see Refs. [58, 60], for more details). Asymptotically flat and (A)dS black holes in the context of massive gravity have been obtained by considering the flat (Minkowski) reference metric or on a degenerate (spatial) and singular reference metric [61, 66].

One of the interesting cases of theories of massive gravity is related to the AdS black hole solutions with the degenerate (spatial) reference metric, which is singular and has important applications in gauge/gravity duality (see Ref. [60], for more details). In this theory of massive gravity, graviton may behave like a lattice and exhibit a Drude peak [66]. It was indicated that this theory of massive gravity is stable and ghost-free [59]. Black hole solutions in the context of this massive gravity have been studied in Refs. [61, 66]. Study on black holes in this theory has attracted extensive attention recently, ranging from heat engine and Joule-Thomson expansion [69, 70], quasinormal modes [71, 72], van der Waals-like phase transition [46, 48, 69, 73], reentrant phase transitions and triple points [74], phase transition and entropic force [75], thermodynamics and geometrical thermodynamics [76–80], and also the existence correspondence between black hole solutions of conformal and massive theories of gravity [81]. In the present work, motivated by the interesting properties of three-dimensional black hole solutions, we study BTZ black holes in this specific massive gravity theory which has a lot of applications in AdS/CMT [82, 83], QCD [45], quantum information [84] and studies of black hole information paradox.

Another fascinating subject that has gained significant attention is the coupling of theories of gravity with nonlinear electrodynamics (NED). The power-Maxwell (PM) theory is one of the interesting and special classes of NED that Hassaine and Martinez presented in 2007 [85]. Recently, it was shown that PM theory, similar to Born-Infeld theory, could remove the singularity of the electric field at the origin [1]. The Lagrangian of PM theory is an arbitrary power of Maxwell Lagrangian, where it is invariant under the conformal transformation \( g_{\mu\nu} \rightarrow \Omega^2 g_{\mu\nu} \) (where \( g_{\mu\nu} \) is metric tensor) and \( A_\mu \rightarrow A_\mu \), see Ref. [86], for more details. It is worth mentioning that the PM theory reduces to linear Maxwell theory when the power of Maxwell is unit [86]. Another marvelous feature of PM is related to its conformal invariance. As we know, the Maxwell action enjoys conformal invariance in four dimensions, but it does not possess this symmetry in higher dimensions. However, the Lagrangian of PM theory extends the conformal invariance in higher dimensions if the power is chosen as \( s = (\text{dimensions of spacetime})/4 \) (where \( s \) is the power of PM theory). This leads to black hole solutions, which are inverse square electric fields in arbitrary dimensions (the so-called Coulomb law). In this regard, some interesting properties of black hole solutions coupled with the PM theory have been studied in Refs. [87, 93]. Generalization of GR with a massive spin=−2 field Lagrangian minimally coupled to a PM \( U(1) \) gauge field in four and higher dimensional spacetime have been investigated in Refs. [48, 94], which led to some novel and interesting properties of black hole physics.

Taking into account the mentioned motivations, in this paper, we will extract three-dimensional black hole solutions by considering three generalizations, a massive spin=−2 field, a PM \( U(1) \) gauge field, and the cosmological constant to the Einstein-Hilbert Lagrangian. Then we study their properties from various perspectives.

This paper is organized as follows: In Sec. II, we consider the three-dimensional action of Einstein-dRGT massive gravity in the presence of PM electrodynamics and obtain field equations. Solving the field equations analytically, we obtain exact three-dimensional solutions in the PM-dRGT massive gravity and discuss the main properties of the solution in Sec. III. Thermodynamic behaviors and the stability of the black hole solutions are investigated in Sec. IV. Moreover, it is shown that the first law is valid for the obtained solutions. In Sec. V, we determine the null geodesics equations and radius of the photon orbit. We find the allowed regions of the model parameters to have acceptable optical behavior. The energy emission rate and deflection angle for this type of solution are investigated in this section. Finally, We end the paper with remarkable results in Sec. VI.

II. BASIC EQUATIONS

The three-dimensional action of Einstein-dRGT massive gravity coupled with the PM nonlinear electrodynamics is given by [94]

$$ I = -\frac{1}{16\pi} \int d^3 x \sqrt{-g} \left[ \mathcal{R} - 2\Lambda + (-F)^s + m^2 g \varepsilon \mathcal{U}_i (g,f) \right], \quad (1) $$
where $R$, and $\Lambda$ are the scalar curvature and the cosmological constant, respectively. Also, $F = F_{\mu\nu}F^{\mu\nu}$ is the Maxwell invariant (where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $A_\mu$ are the Faraday tensor and the gauge potential, respectively). Here, $s$ is related to the power of PM theory. It is straightforward to show that the term $(-F)^s$ reduces to the standard Maxwell Lagrangian for $s = 1$. In the above action, $m_g$ and $f$ are related to the graviton mass and a fixed symmetric tensor, respectively. Note that $\varepsilon_i$’s are some constants and also $U_i$’s are self-interaction potentials of graviton constructed from the building blocks $K_\mu^\nu = \sqrt{g^{\mu\alpha}f_{\alpha\nu}}$ which can be written as follows

\[
U_1 = [\mathcal{K}], \quad U_2 = [\mathcal{K}]^2 - [\mathcal{K}^2], \quad U_3 = [\mathcal{K}]^3 - 3[\mathcal{K}] [\mathcal{K}^2] + 2[\mathcal{K}^3],
\]

\[
U_4 = [\mathcal{K}]^4 - 6 [\mathcal{K}^2] [\mathcal{K}^2] + 8 [\mathcal{K}^3] [\mathcal{K}] + 3 [\mathcal{K}^2]^2 - 6 [\mathcal{K}^4].
\]

The square root in $\mathcal{K}$ means $(\mathcal{K})^{\frac{1}{2}}(\mathcal{K})^{\frac{1}{2}} = \mathcal{K}_\mu^\nu$ and the rectangular brackets denote traces, $[\mathcal{K}] = \mathcal{K}_\mu^\mu$. Taking into account the action $[1]$ and using the variational principle, we can extract the field equations related to the gravitation and gauge fields as $[94]$

\[
G_{\mu\nu} + \Lambda g_{\mu\nu} + m_g^2 \chi_{\mu\nu} = \frac{1}{2}g_{\mu\nu} (-F)^s + 2s (-F)^{s-1} F_{\mu\nu} F^\nu, \quad (2)
\]

\[
\partial_\mu \left( \sqrt{-g} (-F)^{s-1} F^{\mu\nu} \right) = 0, \quad (3)
\]

where $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$, is Einstein’s tensor. Also $\chi_{\mu\nu}$ is the massive term with the following form

\[
\chi_{\mu\nu} = -\frac{\varepsilon_1}{2} (U_1 g_{\mu\nu} - K_{\mu\nu}) - \frac{\varepsilon_2}{2} (U_2 g_{\mu\nu} - 2U_1 K_{\mu\nu} + 2K_{\mu\nu}) - \frac{\varepsilon_3}{2} (U_3 g_{\mu\nu} - 3U_2 K_{\mu\nu} + 6U_1 K_{\mu\nu} - 6K_{\mu\nu}) - \frac{\varepsilon_4}{2} (U_4 g_{\mu\nu} - 4U_3 K_{\mu\nu} + 12U_2 K_{\mu\nu} - 24U_1 K_{\mu\nu} + 24K_{\mu\nu}). \quad (4)
\]

### III. BLACK HOLE SOLUTIONS

In this section, we are interested in studying the three-dimensional static black holes with (A)dS asymptotes in the presence of PM theory and Einstein-dRGT-massive gravity. In this regard, we consider the metric of three-dimensional static spacetime with the following explicit form

\[
ds^2 = -g(r)dt^2 + g^{-1}(r)dr^2 + r^2 d\varphi^2, \quad (5)
\]

where $g(r)$ is an arbitrary function of the radial coordinate.

To obtain exact solutions, we should choose the reference metric. We consider the following ansatz metric $[83]$

\[
f_{\mu\nu} = \text{diag}(0, 0, c^2), \quad (6)
\]

where $c$ is a positive constant. Such a choice of reference metric depends only on the spatial components; meaning that the diffeomorphism invariance is preserved in the $t$ and $r$ coordinates (the breakdown of diffeomorphism invariance leads to BD ghost. In the ADM language, the BD ghost is a consequence of the absence of the Hamiltonian constraint $[95]$). To solve this problem, one has to show that the Hamiltonian constraint is preserved and thus eliminates the sixth degree of freedom (BD ghost) $[96]$. In Ref. $[90]$, Vegh has presented a version of this proof for the case of a degenerate reference metric $f_{\mu\nu} = \text{diag}(0, 0, 1, 1)$ and found that BD ghost eliminates because the corresponding diffeomorphism remains unbroken in the $t$ and $r$ coordinates) but is broken in the spatial dimensions. One can imagine a more general reference metric such that the diffeomorphism invariance in the $r^{- direction}$ is broken. For instance, in four dimensions, to preserve rotational invariance on the sphere and general time parametritization invariance, a reference metric as $f_{\mu\nu} = \text{diag}(0, 1, c^2, c^2 \sin^2 \theta)$ was considered as a natural ansatz $[97]$. Another choice was a different generalization of $f_{\mu\nu}$, with $\sin^2 \theta f_{\theta\theta} = f_{\varphi\varphi} = F(r)$ such that other components were zero $[97]$. This can lead to an ability to add arbitrary polynomial terms in $r$ to the emblackening factor.

Using the metric ansatz $[9]$, $U(g, f)$ is given by $[83]$

\[
U(g, f) = \frac{c}{r}, \quad (7)
\]

It is notable that in three-dimensional spacetime, the term $m_g^2 \varepsilon_i U_i(g, f)$ in the action $[1]$ turns to $m_g^2 \varepsilon U(g, f)$ or $m_g^2 \varepsilon f/c$. Also, the massive term in Eq. $[3]$ reduces to

\[
\chi_{\mu\nu} = -\frac{\varepsilon}{2} (g_{\mu\nu} U(g, f) - K_{\mu\nu}). \quad (8)
\]
Since we are going to study electrically charged black holes, we consider a radial electric field its related gauge potential is

\[ A_\mu = h(r) \delta^\mu_r. \]  

(9)

Using the metric (5) and the PM field equation (3), one finds the following differential equation

\[ r h''(r) + \Psi_1 = 0, \]  

(10)

where

\[ \Psi_1 = \begin{cases} h'(r) & s = 1 \\ 2h'(r) & s = \frac{3}{4} \\ -h'(r) - 2srh''(r) & \text{otherwise} \end{cases}, \]  

(11)

where the prime and double primes are the first and the second derivatives versus \( r \), respectively. It is easy to find the solution of Eq. (11) as

\[ h(r) = \begin{cases} \frac{q}{l} \ln \left( \frac{r}{l} \right) & s = 1 \\ -\frac{q^{2/3}}{r} & s = \frac{3}{4} \\ \frac{(2s-1)(qr^{-2s})^{1-s/2}}{2(s-1)} & \text{otherwise} \end{cases}, \]  

(12)

where \( q \) is an integration constant related to the electric charge and \( l \) is an arbitrary constant with length dimension, which comes from the fact that the logarithmic arguments should be dimensionless.

It is notable that the electromagnetic field tensor is given by

\[ F_{tr} = E(r) = \begin{cases} \frac{q}{lr} & s = 1 \\ -\frac{q^{2/3}}{r^2} & s = \frac{3}{4} \\ \left( qr^{-2s} \right)^{1-s/2} & \text{otherwise} \end{cases}. \]  

(13)

Worth mentioning that the electromagnetic gauge potential (12) and the electromagnetic field (13), should be finite at infinity. These constraints impose the following restriction on the nonlinearity parameter (s) as

\[ \frac{1}{2} < s \leq 1. \]  

(14)

By considering the above constraint, hereafter, the expression "otherwise" belongs to the range \( \{ \frac{1}{2} < s < \frac{3}{4} \} \cup \{ \frac{3}{4} < s < 1 \} \), or \( s \in \left( \frac{1}{2}, \frac{3}{4} \right) \cup \left( \frac{3}{4}, 1 \right) \).

Now, we would like to obtain exact solutions. For this purpose, by employing Eq. (8), the metric ansatz (6), and the metric (5), we obtain the massive terms in the following forms

\[ \chi_{tt} = \frac{c \varepsilon g(r)}{2r}, \]

\[ \chi_{rr} = -\frac{c \varepsilon g(r)}{2rg(r)}, \]

\[ \chi_{\phi\phi} = 0, \]

(15)

where \( \chi_{tt}, \chi_{rr}, \) and \( \chi_{\phi\phi} \) are corresponding to \( tt, rr, \) and \( \phi\phi \) components of Eq. (7), respectively. Now, we calculate
the non-zero components of $G_{\mu\nu} + \Lambda g_{\mu\nu} = 0$, by using the metric \((19)\), which are

$$G_{tt} + \Lambda g_{tt} = -\frac{g(r) [g'(r) + 2\Lambda r]}{2r},$$

$$G_{rr} + \Lambda g_{rr} = \frac{g'(r) + 2\Lambda}{2rg(r)},$$

$$G_{\varphi\varphi} + \Lambda g_{\varphi\varphi} = \frac{r^2}{2} g''(r) + \Lambda r^2,$$

by considering Eqs. \((15)\) and \((16)\), we can obtain the non-zero components of Eq. \((2)\), as

$$E_{tt} = E_{rr} = rg'(r) + 2\Lambda r^2 - m_g^2 \varepsilon r + 2^s (2s - 1) \Psi_2 = 0,$$

$$E_{\varphi\varphi} = \frac{r^2 g''(r)}{2} + \Lambda r^2 - 2^{s - 1} \Psi_2 = 0,$$

where $\Psi_2$ is

$$\Psi_2 = \begin{cases} \frac{2^s}{2} & s = 1 \\ \frac{2^s}{4} & s = \frac{3}{4} \\ \frac{2^s}{4} \frac{2^{(1-s)(s-1)}}{s} & s \in \left(\frac{1}{2}, \frac{3}{4}\right) \cup \left(\frac{3}{4}, 1\right) \end{cases},$$

which $E_{tt}$, $E_{rr}$ and $E_{\varphi\varphi}$ are corresponding to $tt$, $rr$ and $\varphi\varphi$ components of Eq. \((2)\), respectively. After some manipulations, one can obtain the following metric function

$$g(r) = -m_0 - \Lambda r^2 + m_g^2 \varepsilon r + \begin{cases} \frac{-2^s}{2} \ln \left(\frac{r}{r_c}\right) & s = 1 \\ \frac{2^s}{2^{s+1}} \ln \left(\frac{r}{r_c}\right) & s = \frac{3}{4} \\ \frac{2^s}{4} \frac{2^{(1-s)(s-1)}}{s} & s \in \left(\frac{1}{2}, \frac{3}{4}\right) \cup \left(\frac{3}{4}, 1\right) \end{cases},$$

where $m_0$ is an integration constant related to the black hole’s total mass. We should note that the obtained metric function simultaneously satisfies all field equation components \((2)\).

To examine the geometrical structure of these solutions, first, we look for essential singularity(ies) by calculating the Ricci and Kretschmann scalars. We obtain these scalars in the following forms

$$R = 6\Lambda - \frac{2m_g^2 \varepsilon}{r} + \begin{cases} \frac{2^s}{2^{s+1}} & s = 1 \\ 0 & s = \frac{3}{4} \\ \frac{2^s}{4} \frac{2^{(1-s)(s-1)}}{s} & s \in \left(\frac{1}{2}, \frac{3}{4}\right) \cup \left(\frac{3}{4}, 1\right) \end{cases},$$

$$R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} = 12\Lambda^2 - \frac{8\Lambda m_g^2 \varepsilon}{r} + \frac{2m_g^4 \varepsilon^2}{r^2} + \begin{cases} \frac{8\Lambda^2 r}{r^2} - \frac{8\Lambda m_g^2 \varepsilon}{r^2} + \frac{12\Lambda^2}{r^2} & s = 1 \\ -\frac{2^{7/2} \varepsilon^2 m_g^2 \varepsilon r}{r^2} + \frac{3\sqrt{2} m_g^2}{r^2} & s = \frac{3}{4} \\ 2^{s+2} \left(\frac{r}{r_c}\right)^{\frac{4(s-1)(s-2)}{4s-1}} \left(\frac{\Lambda(4s-3)r^2 - (2s-1)m_g^2 \varepsilon r}{r^2}\right)^{\frac{2^{(1-s)(s-1)}}{s}} + 2^{s+1} \left(s^2 - s + \frac{3}{4}\right) & s \in \left(\frac{1}{2}, \frac{3}{4}\right) \cup \left(\frac{3}{4}, 1\right) \end{cases}.$$
It is evident that the Ricci and Kretschmann scalars diverge at the origin as

\[
\lim_{r \to 0} R = \infty, \\
\lim_{r \to 0} R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} = \infty,
\]

so, there is a curvature singularity at \( r = 0 \).

Also, for large values of radial coordinate, \( r \to \infty \), the Ricci and Kretschmann scalars are

\[
\lim_{r \to \infty} R = 6\Lambda, \\
\lim_{r \to \infty} R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} = 12\Lambda^2,
\]

in which confirm that the solution’s asymptotical behavior is dS for \( \Lambda > 0 \) and AdS for \( \Lambda < 0 \). However, there is a study on three-dimensional black holes that shows that classical black holes do not exist in three-dimensional dS spacetime \[98\]. Therefore, we consider the AdS case in our work.

In order to study the effects of massive gravitons \( (m_g) \), the parameter of PM theory \( (s) \), electrical charge \( (q) \), and the cosmological constant \( (\Lambda) \), one can investigate the metric function (20). Regarding various terms of \( g(r) \), it is worthwhile to mention that \( q \)-term (the fourth term in Eq. (20)) is dominant near the origin \( (r \to 0) \). Therefore, one can conclude that the singularity is timelike. In addition, for large distance \( (r \to \infty) \), \( \Lambda \)-term (the second term in Eq. (20) ) is dominant, which confirms that the solutions can be asymptotically AdS. As one can see, the behavior of \( g(r) \) is highly sensitive to the massive graviton, the parameter of PM theory, the electrical charge, and the cosmological constant (see Fig. 1 for more details). It is evident that for specific values of different parameters, the metric function could have two roots, one extreme root or no root (see up panels in Fig. 1). In addition, the three-dimensional solutions are covered by an event horizon, which confirms the existence of black hole solutions in three-dimensional spacetime.

IV. THERMODYNAMICS

Now, we intend to calculate the conserved and thermodynamic quantities of these black hole solutions and show thermodynamic quantities of this theory satisfy the first law of thermodynamics. For studying the thermodynamic properties of three-dimensional black holes in massive-power-Maxwell theory, it is necessary to express the mass \( (m_0) \) in terms of the radius of the event horizon \( r_e \), electrical charge \( (q) \), and the cosmological constant \( (\Lambda) \), mass of graviton

\[\text{FIG. 1: } g(r) \text{ versus } r \text{ for } \varepsilon = 1, c = 1 \text{ and } m_0 = 3.\]
(m_g), and the parameters of massive gravity (c, and ε). Equating \( g(r) = 0 \) (Eq. 20), we obtain

\[
m_0 = -\Lambda r_c^2 + m_g^2 c \varepsilon r_c - \left\{ \begin{array}{ll}
2q^2 \ln(\frac{r}{r_s}), & s = 1 \\
-\frac{2^{3/4}g}{2r_s}, & s = \frac{3}{4} \\
2^{s-1}(2s-1)^2(\frac{q}{r_s})^{2(1-s)} & s \in (\frac{1}{2}, \frac{3}{4}) \cup (\frac{3}{4}, 1) 
\end{array} \right.
\] (25)

The Hawking temperature is defined as

\[
T = \frac{\kappa}{2\pi}
\] (26)

where \( \kappa \) denotes the surface gravity by

\[
\kappa = \sqrt{-\frac{1}{2} \left( \nabla_{\mu} \chi_{\nu} \left( \nabla^{\mu} \chi^{\nu} \right) \right)},
\] (27)

in which \( \chi \) is Killing vector. For the mentioned spacetime, the Killing vector is \( \chi = \partial_t \). Therefore, by considering the metric (5), we can obtain the surface gravity in the following form

\[
\kappa = \frac{1}{2} \frac{dg(r)}{dr} \bigg|_{r = r_c},
\] (28)

by considering the metric function \( g(r) \) (20), the surface gravity (28), and the Hawking temperature (26), we find

\[
T = -\frac{\Lambda r_c}{2\pi} + \frac{m_g^2 c \varepsilon}{4\pi} - \left\{ \begin{array}{ll}
\frac{q^2}{2^{2s} r_c} & s = 1 \\
\frac{q}{2^{3/4} r_c} & s = \frac{3}{4} \\
2^{s-2}(2s-1)\left(\frac{q}{r_c}\right)^{2(1-s)} & s \in (\frac{1}{2}, \frac{3}{4}) \cup (\frac{3}{4}, 1) 
\end{array} \right.
\] (29)

According to Gauss’s law, the electric charge, \( Q \), can be found by calculating the flux of the electric field at infinity. For example, we apply Gauss’s law for Maxwell theory (\( s = 1 \)) which yields

\[
Q_{Maxwell} = \frac{1}{4\pi} \int_0^{2\pi} F_{tr_{Maxwell}} \sqrt{g} \varepsilon \varphi d\varphi \bigg|_{r = r_c} = \frac{q}{4\pi l} \int_0^{2\pi} d\varphi = \frac{q}{2l}
\] (30)

where \( F_{tr_{Maxwell}} = \frac{q}{l} \) is the Maxwell electromagnetic field \([13]\), and \( g_{\varphi\varphi} = r^2 \) is \( \varphi \) component of the metric tensor \( (g_{\mu\nu}) \). In addition, the electric charge of the PM theory is obtained in Ref. [99], which is

\[
Q = \left\{ \begin{array}{ll}
\frac{3q^{1/3}}{2^{2s} r_c} & s = \frac{3}{4} \\
2^{s-2} q \varepsilon & s \in (\frac{1}{2}, \frac{3}{4}) \cup (\frac{3}{4}, 1)
\end{array} \right.
\] (31)

To obtain the total mass of the solutions, we follow the obtained result in Ref. [67]. Indeed, the ADM mass can be obtained through the Hamiltonian approach which is given by \( M = \frac{(d-2)\omega_d m_0}{16\pi} \) \([67]\), which \( d \) and \( \omega_d \) are related to the dimension of spacetime, and unit volume in \( d \)-dimension, respectively. For three-dimensional spacetime, we have \( d = 3 \), and \( \omega_{d=3} = \int_0^{2\pi} d\varphi = 2\pi \). So, the total mass of the solutions is given by

\[
M = \frac{m_0}{8}
\] (32)

in which by evaluating metric function on the horizon \( (g(\varphi = r_c) = 0) \), we obtain

\[
M = -\frac{\Lambda r_c^2 + m_g^2 c \varepsilon r_c}{8} + \left\{ \begin{array}{ll}
\frac{q^2 \ln(\frac{r}{r_s})}{4\pi} & s = 1 \\
\frac{q}{2^{3/4} r_c} & s = \frac{3}{4} \\
2^{s-4}(2s-1)^2(\frac{q}{r_c})^{2(1-s)} & s \in (\frac{1}{2}, \frac{3}{4}) \cup (\frac{3}{4}, 1)
\end{array} \right.
\] (33)
To obtain the entropy of black holes in the presence of Einstein-massive gravity, one can use the area law proposed by Hawking and Bekenstein in the following form

\[ S = \frac{A}{4}, \]  
\[ \text{(34)} \]

where \( A \) is the horizon area and for three-dimensional spacetime is defined

\[ A = \int_0^{2\pi} \sqrt{g_{\varphi \varphi}} \, d\varphi \bigg|_{r=r_e} = 2\pi r_e, \]  
\[ \text{(35)} \]

by replacing the horizon area \( 35 \) within Eq. \( 34 \), we obtain the entropy as

\[ S = \pi r_e. \]  
\[ \text{(36)} \]

The electric potential, \( U \), is defined through the gauge potential in the following form

\[ U = A_{\mu} \chi^\mu \bigg|_{r \rightarrow \text{reference}} - A_{\mu} \chi^\mu \bigg|_{r \rightarrow r_e} = \begin{cases} 
-\frac{q}{r_e} \ln \left( \frac{r_e}{T} \right) & s = 1 \\
\frac{2^{2/3}}{r_e} & s = \frac{3}{4} \\
\left( \frac{(2s-1)(qr_e^{-2s})}{2(1-s)} \right)^{1/3} & s \in \left( \frac{1}{2}, \frac{3}{4} \right) \cup \left( \frac{3}{4}, 1 \right) 
\end{cases}, \]  
\[ \text{(37)} \]

where \( A_{\mu} = h(r) \delta_{\mu}^t \) is obtained in Eq. \( 12 \).

Having conserved and thermodynamic quantities at hand, we are in a position to check whether thermodynamic quantities satisfy the first law of thermodynamics. It is easy to show that by using thermodynamic quantities such as electric charge \( 31 \), mass \( 32 \), and entropy \( 36 \), with the first law of black hole thermodynamics

\[ dM = TdS + UdQ, \]  
\[ \text{(38)} \]

one can define the intensive parameters conjugate to \( S \) and \( Q \). These quantities are the temperature and the electric potential

\[ T = \left( \frac{\partial M}{\partial S} \right)_Q \quad \& \quad U = \left( \frac{\partial M}{\partial Q} \right)_S, \]  
\[ \text{(39)} \]

which are the same as those calculated for the temperature \( 29 \) and the electric potential \( 37 \). Evidently, the obtained thermodynamic quantities could satisfy the first law of thermodynamics.

### A. Thermal stability in the canonical ensemble

Here, we study thermal stability criteria and the effects of different parameters on them. The stability conditions in canonical ensemble are based on the sign of the heat capacity. This change of sign could happen when heat capacity meets root(s) or divergency(ies). The root of heat capacity (or temperature) indicates a bound point, which separates physical solutions (positive temperature) from non-physical ones (negative temperature). The heat capacity divergencies (the roots of the denominator of heat capacity) represent phase transition points. The negativity of heat capacity represents unstable solutions that may undergo a phase transition and acquire a stable state. To get a better picture and enrich our study’s results, we investigate temperature and heat capacity simultaneously.

The heat capacity is given by the following traditional relation

\[ C_Q = \frac{T}{\left( \frac{\partial^2 M}{\partial S^2} \right)_Q} = \frac{T}{\left( \frac{\partial T}{\partial S} \right)_Q}. \]  
\[ \text{(40)} \]

Considering Eqs. \( 29 \) and \( 36 \), it is a matter of the calculation to show that

\[ C_Q = \frac{(2\Lambda r_e^2 - m_g^2 e r_e + 2^s (2s-1) \Psi_3) \pi r_e}{4\Lambda r_e^2 - 2s+1 \Psi_3}, \]  
\[ \text{(41)} \]
where

\[
\Psi_3 = \begin{cases} 
\frac{y^2}{r^2} & s = 1 \\
\frac{y}{r^2} & s = \frac{3}{4} \\
\left(\frac{y}{r^2}\right)^{2(1-s)} & s \in \left(\frac{1}{2}, \frac{3}{4}\right) \cup \left(\frac{3}{4}, 1\right)
\end{cases}
\] (42)

Notably, the obtained temperature \(29\) and the heat capacity \(41\) include three terms: cosmological constant, electric charge, and massive terms.

According to the obtained heat capacity for cases \(s = 1\) and \(s = \frac{3}{4}\), we are in a position to find the exact bound and phase transition points of AdS black holes. Solving the numerator and denominator of the heat capacity concerning the horizon’s radius leads to the following solutions for bound points \((r_b)\) and phase transition points \((r_p)\), respectively,

\[
r_b = \begin{cases} 
m_e x c - \sqrt{m_e x c^2 - 16\Lambda^2} & s = 1 \\
\Psi_4 + \frac{m_e x c^2 + m_x c^2 \Psi_4}{4\Lambda} & s = \frac{3}{4}
\end{cases}
\] (43)

\[
r_p = \begin{cases} 
\sqrt{-\frac{q}{r^3\Lambda}} & s = 1 \\
\left(\frac{2^{11/4}q^2\Lambda^2}{2\Lambda}\right)^{1/3} & s = \frac{3}{4}
\end{cases}
\] (44)

where \(\Psi_4 = \left(-3^{23/4}q\Lambda^2 + m^6 c^3 \varepsilon^3 + 3^{3/2}\Lambda \sqrt{23/4q \left(3^{23/4}q\Lambda^2 - 2m^6 c^3 \varepsilon^3\right)}\right)^{1/3}\). Interestingly, phase transition cannot exist for AdS black holes, and it is independent of the massive term (Eq. 44). Another interesting result is related to the effects of massive term and the electric charge on the bound points of AdS black holes for cases \(s = 1\) and \(s = \frac{3}{4}\). Considering Eq. 43, it is clear that the bound points shift to the larger (smaller) horizon radius by increasing the electric charge (massive term). Indeed, the physical area decreases for higher charged AdS black holes (see the middle and right panels in Fig. 2). For different values of \(s\), the physical area increases by increasing \(s\) (see the left panel in Fig. 2).

![Graphs showing T (Bold lines) and CQ (thin lines) versus r+ for different s (left panel), different q (middle panel) and different m_g (right panel).](image)

**FIG. 2:** \(T\) (Bold lines) and \(C_Q\) (thin lines) versus \(r_+\) for different \(s\) (left panel), different \(q\) (middle panel) and different \(m_g\) (right panel).

### V. OPTICAL FEATURES

In this section, we carefully study the optical features of AdS black holes in three-dimensional PM-massive theory, such as the photon orbit, the energy emission rate, and the deflection angle. Considering these optical quantities, we investigate the influence of parameters of the theory on the black hole solutions.
A. Null geodesics and photon orbit

Here, we would like to obtain the radius of the photon orbit and critical impact parameter for the corresponding black hole and show how they are affected by solution parameters. To this purpose, we employ the Hamilton-Jacobi method for a photon in the black hole spacetime as \[100, 101\]

\[
\frac{\partial S}{\partial \sigma} + H = 0, \quad (45)
\]

where \(S\) and \(\sigma\) are the Jacobi action and affine parameter along the geodesics, respectively. The geodesic motion of a massless photon in the static spherically symmetric spacetime can be controlled by the following Hamiltonian

\[
H = \frac{1}{2} g^{ij} p_i p_j = 0. \quad (46)
\]

Taking into account Eq. (5), the above equation can be written as

\[
\frac{1}{2} \left[ -\frac{p_t^2}{g(r)} + g(r)p_r^2 + \frac{p_\varphi^2}{r^2} \right] = 0, \quad (47)
\]

from which we deduce

\[
\dot{p}_t = -\frac{\partial H}{\partial t} = 0, \quad \dot{p}_\varphi = -\frac{\partial H}{\partial \varphi} = 0, \quad (48)
\]

This shows that the Hamiltonian is independent of the coordinates \(t\) and \(\varphi\). So, one can consider \(p_t\) and \(p_\varphi\) as constants of motion. We define \(-p_t \equiv E\) and \(p_\varphi \equiv L\) where \(E\) and \(L\) are, respectively, the energy and angular momentum of the photon.

Using the Hamiltonian formalism, the equations of motion are given by

\[
\dot{t} = \frac{\partial H}{\partial p_t} = -\frac{p_t}{g(r)}, \quad \dot{r} = \frac{\partial H}{\partial p_r} = p_r g(r), \quad \dot{\varphi} = \frac{\partial H}{\partial p_\varphi} = \frac{p_\varphi}{r^2}, \quad (49)
\]

where \(p_r\) is the radial momentum and the overdot denotes a derivative with respect to the affine parameter \(\sigma\). These equations and two conserved quantities provide a complete description of the dynamics by taking into account the orbital equation of motion as follows

\[
r^2 + V_{\text{eff}}(r) = 0, \quad (50)
\]

where \(V_{\text{eff}}\) is the effective potential of the photon, given by

\[
V_{\text{eff}}(r) = g(r) \left[ \frac{L^2}{r^2} - \frac{E^2}{g(r)} \right]. \quad (51)
\]

It should be noted that the photon orbits are circular and unstable, associated with the maximum value of the effective potential. The following conditions can obtain such a maximum

\[
V_{\text{eff}}(r_{ph}) = 0, \quad \& \quad V'_{\text{eff}}(r_{ph}) = 0, \quad \& \quad V''_{\text{eff}}(r_{ph}) < 0, \quad (52)
\]

where the first two conditions determine the critical angular momentum of the photon orbit \((L_p)\) and the photon orbit radius \((r_{ph})\), respectively, also, the third condition ensures that the photon orbits are unstable. The impact parameter which is defined by the ratio of angular momentum and energy of the photon is obtained as

\[
b = \frac{L}{E}, \quad (53)
\]

and the critical impact parameter is defined as \(b_c = \frac{L}{E}\). An ingoing photon from infinity with \(b < b_c\) falls into the black hole without reaching the observer, whereas it bounces back if \(b > b_c\) and can be observed by the observer located at infinity. An interesting phenomenon is related to the critical impact parameter for which \(\max(V_{\text{eff}}) = 0\). In this case, the ingoing photon loses both its radial velocity and acceleration at \(r = r_{\text{max}}\) completely. But due to its non-vanishing transverse velocity orbits the black hole. For spherically symmetric black holes, the critical value of the
impact parameter corresponds to photons in the unstable circular orbits, filling the photon sphere. The geometrical cross-section of the photon sphere (so-called capture cross-section of the black hole) is directly related to the critical impact parameter. For spherically symmetric black holes, the critical impact parameter corresponds to the area of shadow. The optical shadow around the event horizon describes the visual boundary that light cannot escape from the event horizon by viewers. Since the spacetime under consideration is three-dimensional, one cannot speak about the area of shadow, instead, $b_c$ represents the radius of the capture cross-section [102]. Three-dimensional black holes have been widely studied in the context of photon orbit and energy emission rate [103, 104], bending of light [105–107], and geodesic structure [108–111].

Taking into account the metric functions (20) and the effective potential (51), $V'_{\text{eff}}(r_{ph}) = 0$ leads to the following relations

$$m_g^2 c^2 r_{ph} - \frac{4q^2}{l^2} \ln\left(\frac{r_{ph}}{l}\right) + \frac{2q^2}{l^2} - 2m_0 = 0, \quad s = 1,$$

$$m_g^2 c^2 r_{ph}^2 + \frac{3q}{2} - 2m_0 r_{ph} = 0, \quad s = \frac{3}{4},$$

$$\left(m_g^2 c^2 r_{ph} - 2m_0\right) (s - 1) + s^2 \left(1 - 2s\right) \left(\frac{q}{r_{ph}}\right)^{2\left(\frac{s}{2}^s\right)} = 0, \quad s \in \left(\frac{1}{2}, \frac{3}{4}\right) \cup \left(\frac{3}{4}, 1\right).$$

Now, we examine each of the above relations separately.

### I. Three-dimensional black holes in Maxwell-massive gravity

We consider black holes in the Maxwell-massive theory, i.e., $s = 1$. Solving Eq. (54) results into the following solution

$$r_{ph} = l \exp\left(\frac{1}{2} - \frac{m_0 l^2}{2q^2} - \text{LambertW}\left[-\frac{m_g^2 c^2 l^3}{4q^2} \exp\left(\frac{q^2 - m_0 l^2}{2q^2}\right)\right]\right).$$

As was already mentioned, by evaluating the metric function on the horizon ($g(r = r_\text{e}) = 0$), one can obtain the parameter $m_0$ as a function of $r_\text{e}$. Inserting the obtained $m_0$ into Eq. (57), $r_{ph}$ can be rewritten in terms of $r_\text{e}$. According to our analysis, $r_{ph}$ has a maximum in

$$r_{\text{e, max}} = \frac{m_g^2 c^2 l}{4\Lambda l},$$

for $r_\text{e} < r_{\text{e, max}}$, the radius of the photon orbit is larger than the horizon radius, whereas for $r_\text{e} > r_{\text{e, max}}$, the photon orbit radius is smaller than the horizon radius, which is not physically acceptable. To have a better understanding of the acceptable regions of parameters, we have plotted Fig. 3 which shows the behavior of $\frac{r_{ph}}{r_\text{e}}$ versus $r_\text{e}$. Since an acceptable optical result can be observed for $\frac{r_{ph}}{r_\text{e}} > 1$ [112], only for limited regions of $r_\text{e}$, this condition is satisfied.

As a next step in the analysis, we investigate the critical impact parameter for the corresponding black hole which is obtained as [113]

$$b_c = \frac{L_p}{E} = \frac{r_{ph}}{\sqrt{g(r_{ph})}}.$$

where $L_p$ is the critical angular momentum of the photon orbit.

According to Eq. (59), a real positive value of $b_c$ is obtained for $g(r_{ph}) > 0$. By rewriting $g(r_{ph})$ in terms of $r_\text{e}$, one finds

$$G = m_g^2 c^2 l e^\Gamma_1 - \Lambda l^2 e^{2\Gamma_1} + \frac{q^2 (2\Gamma_2 - 1)}{l^2},$$

where

$$\Gamma_1 = \frac{q^2 + 2q^2 \ln(\frac{r_\text{e}}{l}) + \Lambda l^2 r_\text{e}^2 - m_g^2 c^2 l^2 r_\text{e} - 2q^2 \Gamma_2}{2q^2},$$

$$\Gamma_2 = \text{LambertW}\left(-\frac{m_g^2 c^2 l^3 \exp\left(\frac{q^2 + 2q^2 \ln(\frac{r_\text{e}}{l}) + \Lambda l^2 r_\text{e}^2 - m_g^2 c^2 l^2 r_\text{e}}{2q^2}\right)}{4q^2}\right).$$
FIG. 3: The ratio between the photon orbit radius and the horizon radius \( \frac{r_{ph}}{r_e} \) versus \( r_e \) for \( s = 1 \), \( c = l = 1 \) and different values of black hole parameters.

Our analysis shows that \( g(r_{ph}) \) is negative for all values of black hole parameters. So, according to Eq. (59), the critical impact parameter is imaginary, indicating that an acceptable optical behavior cannot be observed for three-dimensional black holes in the Maxwell-massive theory of gravity.

2. Three-dimensional black holes in conformal invariant Maxwell-massive gravity

For the second case, we examine Eq. (55) to study the radius of the photon orbit for three-dimensional charged AdS black holes in massive gravity for conformally invariant Maxwell (\( s = \frac{3}{4} \)). Solving the equation (55), one can obtain the photon orbit radius in the following form

\[
r_{ph} = m_0 + \sqrt{m_0^2 - \frac{3m_g^2c\varepsilon q}{2l^4}}.
\]  

(63)

To investigate the ratio \( \frac{r_{ph}}{r_e} \), we need to determine the horizon radius, which is the root of the metric function. Our analysis shows that the metric function can admit up to three roots as
\[ r^{(1)} = \frac{2\sqrt{3\Psi_5}}{\sqrt{3}} \sin \left[ \frac{1}{3} \sin^{-1} \left( \frac{3\sqrt{3}\Psi_6}{2(\sqrt{3}\Psi_5)^3} \right) \right] + \frac{m_2^2 c \varepsilon}{3\Lambda}, \]  
\[ r^{(2)} = -\frac{2\sqrt{3\Psi_5}}{\sqrt{3}} \sin \left[ \frac{1}{3} \sin^{-1} \left( \frac{3\sqrt{3}\Psi_6}{2(\sqrt{3}\Psi_5)^3} \right) + \frac{\pi}{3} \right] + \frac{m_2^2 c \varepsilon}{3\Lambda}, \]  
\[ r^{(3)} = \frac{2\sqrt{3\Psi_5}}{\sqrt{3}} \cos \left[ \frac{1}{3} \sin^{-1} \left( \frac{3\sqrt{3}\Psi_6}{2(\sqrt{3}\Psi_5)^3} \right) + \frac{\pi}{6} \right] + \frac{m_2^2 c \varepsilon}{3\Lambda}, \]

in which

\[ \Psi_5 = -\frac{m_0^2 c^2 \varepsilon^2}{3\Lambda^2} + \frac{m_0}{\Lambda}, \]  
\[ \Psi_6 = -\frac{2m_0^2 c^3 \varepsilon^3}{27\Lambda^3} + \frac{m_0^2 c \varepsilon m_0}{3\Lambda^2} - \frac{q}{2\Lambda}. \]

According to our analysis, \( r^{(2)} \) is always negative, and \( r^{(3)} \) given by Eq. (60) is the largest positive root. Inserting

**TABLE I**: The event horizon \( r_e \), photon orbit radius \( r_{ph} \) and critical impact parameter \( b_c \) for the variation of \( m_g, q, \varepsilon \) and \( \Lambda \) for \( c = m_0 = 1 \) and \( s = \frac{3}{4} \).

| \( m_g \) | 0.53 | 0.6 | 0.7 | 0.9 |
| --- | --- | --- | --- | --- |
| \( r_e (\Lambda = -0.01, q = 0.2, \varepsilon = 2) \) | 1.5436 | 1.1702 | 0.7991 | 0.3 + 0.09i |
| \( r_{ph} (\Lambda = -0.01, q = 0.2, \varepsilon = 2) \) | 3.2867 | 2.4971 | 1.7459 | 0.8810 |
| \( b_c (\Lambda = -0.01, q = 0.2, \varepsilon = 2) \) | 3.2774 | 2.5926 | 1.9074 | 1.1136 |
| \( r_{ph} > r_e \) | ✓ | ✓ | ✓ | × |
| \( b_c > r_{ph} \) | × | ✓ | ✓ | ✓ |

| \( q \) | 0.04 | 0.2 | 0.3 | 0.4 |
| --- | --- | --- | --- | --- |
| \( r_e (\Lambda = -0.01, m_g = 0.7, \varepsilon = 2) \) | 0.9755 | 0.7991 | 0.5453 | 0.5 + 0.28i |
| \( r_{ph} (\Lambda = -0.01, m_g = 0.7, \varepsilon = 2) \) | 1.9890 | 1.7459 | 1.5390 | 1.1279 |
| \( b_c (\Lambda = -0.01, m_g = 0.7, \varepsilon = 2) \) | 1.9833 | 1.9074 | 1.8449 | 1.7481 |
| \( r_{ph} > r_e \) | ✓ | ✓ | ✓ | × |
| \( b_c > r_{ph} \) | × | ✓ | ✓ | ✓ |

| \( \varepsilon \) | −0.5 | 1.1 | 1.5 | 3.4 |
| --- | --- | --- | --- | --- |
| \( r_e (\Lambda = -0.01, m_g = 0.7, q = 0.2) \) | 28.0443 | 1.6136 | 1.1425 | 0.3 + 0.07i |
| \( r_{ph} (\Lambda = -0.01, m_g = 0.7, q = 0.2) \) | −8.4081 | 3.4383 | 2.4397 | 1.2228 |
| \( b_c (\Lambda = -0.01, m_g = 0.7, q = 0.2) \) | −6.3614 | 3.4037 | 2.5413 | 1.2481 |
| \( r_{ph} > r_e \) | × | ✓ | ✓ | × |
| \( b_c > r_{ph} \) | × | × | ✓ | ✓ |

| \( \Lambda \) | −0.01 | −0.04 | −0.06 | −0.07 |
| --- | --- | --- | --- | --- |
| \( r_e (\varepsilon = 2, m_g = 0.7, q = 0.2) \) | 0.7991 | 0.7743 | 0.7590 | 0.7517 |
| \( r_{ph} (\varepsilon = 2, m_g = 0.7, q = 0.2) \) | 1.7459 | 1.7459 | 1.7459 | 1.7459 |
| \( b_c (\varepsilon = 2, m_g = 0.7, q = 0.2) \) | 1.9074 | 1.8111 | 1.7545 | 1.7218 |
| \( r_{ph} > r_e \) | ✓ | ✓ | ✓ | ✓ |
| \( b_c > r_{ph} \) | ✓ | ✓ | ✓ | × |

Eq. (63) into Eq. (60), we can calculate the critical impact parameter. To have acceptable optical behavior, we need
to examine the condition \( r_e < r_{ph} < b_c \) where \( r_e \) is the horizon radius. Several values of \( r_e, r_{ph}, \) and \( b_c \) listed in Table. It can be seen that the increase of the electric charge, graviton mass, and parameter \( \varepsilon \) lead to an imaginary event horizon. Also, for small values of these parameters, \( b_c \) is smaller than the photon orbit radius, which is physically not acceptable. This shows that an acceptable optical result can be obtained only for limited regions of these three parameters. A remarkable point regarding the parameter \( \varepsilon \) is that for negative values of this parameter, both photon orbit radius and critical impact parameter are negative, which is a non-physical result. Regarding the effect of the cosmological constant on \( r_{ph} \) and \( b_c \), we notice that an acceptable optical behavior can be observed only for small values of \( |\Lambda| \). Notably, there are some acceptable optical behaviors for suitable values of \( m_g, q, \) and \( \varepsilon \).

3. Three-dimensional black holes in PM-massive gravity

We want to investigate the optical properties of the corresponding black hole for the total value of \( s \). Nevertheless, according to the equation, we cannot find an exact solution for the total \( s \). For this purpose, we must consider a value for \( s \). Here we are interested in considering \( s = \frac{5}{4} \). Considering \( s = \frac{5}{4} \) in Eq. (56), we have

\[
5m_g^2c\varepsilon r_{ph}^{5/3} - 10m_0r_{ph}^{2/3} + 12 \times 2^{4/5}q^{2/3} = 0, \tag{69}
\]

We can find the photon orbit radius by solving the above equation. Since this equation is complicated to solve analytically, we employ numerical methods to obtain the horizon radius, the photon orbit radius, and the critical impact parameter. In this regard, we list several values of these three quantities in Table II. We see that an acceptable optical result is obtained only for limited regions of these parameters due to the imaginary event horizon. Regarding the cosmological constant, just in a very low curvature background, the photon orbit radius would be smaller than the critical impact parameter, which is physically acceptable. From this table, it can also be seen that all parameters have a decreasing effect on the event horizon, photon orbit radius, and critical impact parameter.

B. Energy emission rate

In this subsection, we are interested in studying the associated energy emission rate. It has been known that the absorption cross-section oscillates around a limiting constant value \( \sigma_{lim} \) at very high energies which is defined in the following form for an arbitrary dimensional spacetime \[114\]

\[
\sigma_{lim} = \frac{\pi^{d/2}b_{c}^{d/2}}{\Gamma\left(\frac{d}{2}\right)} \tag{70}
\]

The energy emission rate for three-dimensional spacetime is expressed as \[115\]

\[
\frac{d^2E(\omega)}{dt d\omega} = \frac{4\pi^2\omega^2b_{c}}{e^{\frac{2T}{\omega}} - 1}, \tag{71}
\]

in which \( \omega \) is the emission frequency, and \( T \) denotes the Hawking temperature.

Conformal invariant Maxwell case: for conformal invariant Maxwell field, we have to consider \( s = \frac{3}{4} \). In this case, Hawking’s temperature is calculated as

\[
T = \frac{1}{4\pi} \left( m_g^2c\varepsilon - 2\Lambda r_e - \frac{q^2}{21/4r_e^2} \right). \tag{72}
\]

The qualitative behavior of the energy emission rate is illustrated in Fig. 4 as a function of \( \omega \) for different values of parameters. Looking at this figure, one can see that there exists a peak of the energy emission rate, which decreases and shifts to the low frequency with the increase (decrease) of the electric charge and parameter \( \varepsilon \) (values of \( |\Lambda| \)). As one can see from Fig. 4(a), the electric charge decreases the energy emission, meaning that the evaporation process would be slower for a black hole located in a more powerful electric field. The effect of graviton mass on this optical quantity is a little different such that for \( 0.6 < m_g < 0.7 \), the energy emission rate increases with the increase of the graviton mass, whereas for \( 0.7 < m_g < 0.8 \) the energy emission reduces with the growth of \( m_g \) (see Figs. 4(b) and 4(c)). Regarding the effect of \( \varepsilon \), Fig. 4(d) displays that increasing this parameter results in a decrease in energy emission. In other words, decreasing this parameter implies a fast emission of particles. To examine the influence of
TABLE II: The event horizon \( r_e \), photon orbit radius \( r_{ph} \) and critical impact parameter \( b_c \) for the variation of \( m_g, q, \varepsilon \) and \( \Lambda \) for \( c = m_0 = 1 \) and \( s = \frac{4}{5} \).

| \( m_g \) | \( 0.49 \) | \( 0.5 \) | \( 0.7 \) | \( 0.8 \) |
|---|---|---|---|---|
| \( r_e (\Lambda = -0.01, q = 0.1, \varepsilon = 1.5) \) | 3.1041 | 1.9922 | 0.8318 | 0.41 + 0.05f |
| \( r_{ph} (\Lambda = -0.01, q = 0.1, \varepsilon = 1.5) \) | 4.6568 | 4.4453 | 1.9312 | 1.2933 |
| \( b_c (\Lambda = -0.01, q = 0.1, \varepsilon = 1.5) \) | 4.6221 | 4.4688 | 2.3515 | 1.7555 |
| \( r_{ph} > r_e \) | ✓ | ✓ | ✓ | × |
| \( b_c > r_{ph} \) | × | ✓ | ✓ | ✓ |

| \( q \) | 0.09 | 0.15 | 0.2 | 0.25 |
|---|---|---|---|---|
| \( r_e (\Lambda = -0.01, m_g = 0.5, \varepsilon = 1.5) \) | 2.0335 | 1.7787 | 1.5266 | 1.0 + 0.12f |
| \( r_{ph} (\Lambda = -0.01, m_g = 0.5, \varepsilon = 1.5) \) | 4.5140 | 4.1066 | 3.7562 | 3.3634 |
| \( b_c (\Lambda = -0.01, m_g = 0.5, \varepsilon = 1.5) \) | 4.4877 | 4.3748 | 4.2765 | 4.1672 |
| \( r_{ph} > r_e \) | ✓ | ✓ | ✓ | × |
| \( b_c > r_{ph} \) | × | ✓ | ✓ | ✓ |

| \( \varepsilon \) | -0.5 | 1.4 | 2.0 | 3.8 |
|---|---|---|---|---|
| \( r_e (\Lambda = -0.01, m_g = 0.5, q = 0.1) \) | 17.8308 | 2.1457 | 1.4264 | 0.41 + 0.03f |
| \( r_{ph} (\Lambda = -0.01, m_g = 0.5, q = 0.1) \) | 0.2938 | 4.8118 | 3.1646 | 3.1362 |
| \( b_c (\Lambda = -0.01, m_g = 0.5, q = 0.1) \) | -0.5633 | 4.7317 | 3.4547 | 1.7766 |
| \( r_{ph} > r_e \) | × | ✓ | ✓ | × |
| \( b_c > r_{ph} \) | × | × | ✓ | ✓ |

| \( \Lambda \) | -0.005 | -0.01 | -0.012 | -0.015 |
|---|---|---|---|---|
| \( r_e (\varepsilon = 1.5, m_g = 0.5, q = 0.1) \) | 2.0531 | 1.9922 | 1.9695 | 0.7311 |
| \( r_{ph} (\varepsilon = 1.5, m_g = 0.5, q = 0.1) \) | 4.4453 | 4.4453 | 4.4453 | 4.4453 |
| \( b_c (\varepsilon = 1.5, m_g = 0.5, q = 0.1) \) | 4.7102 | 4.4688 | 4.3822 | 4.2612 |
| \( r_{ph} > r_e \) | ✓ | ✓ | ✓ | ✓ |
| \( b_c > r_{ph} \) | ✓ | ✓ | × | × |

the cosmological constant, we depict Fig. 4(e), which illustrates that this parameter has an increasing contribution to the emission rate, unlike the electric charge. In fact, by increasing \( |\Lambda| \), the energy emission rate grows. This reveals that the evaporation process would be faster when the black hole is located in a high curvature background. From what was expressed, one can find that the black hole has a longer lifetime when it is located in a low curvature background or a strong electric field.

**PM case:** for PM NED we must consider a value for \( s \). Here we consider \( s = 4/5 \). Our investigation of the energy emission rate for \( s = 4/5 \) shows that the effect of parameters on the energy emission is similar to the conformal invariant Maxwell case (i.e., \( s = 3/4 \)), with this difference that the emission rate increases by increasing the graviton mass for \( 0.5 < m_g < 0.6 \) and decreases for \( 0.6 < m_g < 0.7 \). To avoid repetition, we omit plotting the figures.

C. Deflection angle

Here, we study the deflection angle of light using the null geodesics method [116,119]. The total deflection \( \Theta \) can be determined by the following relation

\[
\Theta = 2 \int_{b}^{\infty} \frac{|d\varphi|}{dr} dr - \pi, \tag{73}
\]

in which \( b \) is the impact parameter, defined as \( b \equiv L/E \). Using equations of motion [19], we have

\[
|\frac{d\varphi}{dr}| = \frac{\dot{\varphi}}{r} = \frac{b}{r^2} \left( 1 - \frac{b^2 g(r)}{r^2} \right)^{-\frac{1}{2}}. \tag{74}
\]
Conformal invariant Maxwell case: Considering $s = \frac{3}{4}$, $c = m_0 = 1$ and different values of black hole parameters.

To show the effects of different parameters on the deflection angle, we have depicted Fig. 5, which displays the variation of the deflection angle $\Theta$ as a function of the parameter $b$ for different values of black hole parameters. As one can see, all curves reduce to a minimum value with an increase of $b$ and then gradually grow as the impact parameter increases more. In other words, they have a global minimum value, meaning that there is a finite value of the impact parameter $b$ for which the light deflection is very small. According to the relation $b \equiv L/E$, this finite value is dependent on the values of angular momentum and energy of the photon. Fig. 5(a) illustrates the increasing effect of the electric charge on the deflection angle. This shows that the light deflection will be very high in a strong electric field. To examine the impact of graviton mass, we have plotted Fig. 5(b), indicating that photons deflected more than their straight path in the presence of massive gravitons. Regarding the effect of parameter $\varepsilon$, the increase of this parameter leads to the increase of the deflection angle (see Fig. 5(c) for more details). Studying the $\Lambda$ effect, we observe that as $|\Lambda|$ increases, the deflection angle increases as well. This shows that the light deflection is low in the background with lower curvature. Comparing all the panels in Fig. 4, we notice that the effect of electric charge is notable for small values of the impact parameter, whereas other parameters have a significant effect for large values of $b$.
**PM case:** Since according to our analysis, for the case of \( s = 4/5 \), the behavior of the deflection angle under varying parameters is the same as \( s = 3/4 \), we omit drawing figures.

![Graphs showing the behavior of \( \Theta \) with respect to the impact parameter \( b \) for different values of BH parameters.](image1)

(a) \( m_g = 0.7, \varepsilon = 2 \) and \( \Lambda = -0.01 \)  
(b) \( q = 0.2, \varepsilon = 2 \) and \( \Lambda = -0.01 \)  
(c) \( q = 0.2, m_g = 0.7 \) and \( \Lambda = -0.01 \)  
(d) \( q = 0.2, m_g = 0.7 \) and \( \varepsilon = 2 \)

**FIG. 5:** The behavior of \( \Theta \) with respect to the impact parameter \( b \) for \( s = 4/5 \), \( c = m_0 = 1 \) and different values of BH parameters.

**VI. CONCLUSIONS**

In this paper, we have considered three-dimensional charged black holes in the PM-massive gravity context. After a short introduction, we computed the exact black hole solution and investigated several physical properties of this black hole. Studying the thermodynamic behavior of the solution, we examined its thermal stability and phase transition by calculating the heat capacity in a canonical ensemble. Then, we performed an in-depth analysis of the optical features of the corresponding black hole, including the photon orbit radius, energy emission rate, and deflection angle, and inspected the influence of the model’s parameters on the considered optical quantities.

In studying the black hole’s photon orbit and critical impact parameter, we noticed that an acceptable optical behavior could not be observed for three-dimensional black holes in the Maxwell-massive theory. Regarding the three-dimensional charged black holes in PM-massive gravity, our analysis showed that an admissible optical result could be obtained for special regions of black hole parameters. Worth mentioning that such an admissible optical result is observable only for intermediate values of the nonlinearity parameter \( s \).

Then, we studied the energy emission rate and explored the effect of black hole parameters on the radiation process. The results indicated that as the parameter \( \varepsilon \) increases, the emission of particles around the black hole decreases. This revealed that the radiation rate grows when the effect of this parameter gets weaker. Studying \( m_g \) effect showed that depending on the value of the graviton mass, this parameter has an increasing/decreasing contribution to the energy...
emission rate. Such that for $0.6 < m_g < 0.7$, the energy emission rate increases with the increase of the graviton mass, whereas for $0.7 < m_g < 0.8$ the energy emission reduces with the growth of $m_g$. Regarding the effects of electric charge and the cosmological constant, we noticed that the evaporation process would be slow for a black hole located in a powerful electric field or background with lower curvature. In other words, the lifetime of a black hole would be longer under such conditions.

Finally, we presented a study in the context of the gravitational lensing of light around these black holes. Depending on the values of black hole parameters and impact parameters, photons get deflected from their straight path and have different behaviors. For small values of the impact parameter $b$, the deflection angle was a decreasing function of $b$, whereas the opposite behavior was observed for large values. It shows that there exists a global minimum value of the impact parameter $b$, which the deflection of light is very low for it. Relative to the impact of electric charge, we found that it has an increasing contribution to the deflection angle $\Theta$. In other words, the deflection of light in a large electric charge is very high compared to black holes located in a weak electric field. We also noticed that the effects of graviton mass, the parameter $\varepsilon$, and the cosmological constant on the deflection angle are similar to that of the electric charge.

Acknowledgments

BEP thanks University of Mazandaran. Also AR acknowledges Universidad de Tarapacá for financial support.

[1] B. Eslam Panah, Europhys. Lett. 134 (2021) 20005.
[2] M. Banados, C. Teitelboim, and J. Zanelli, Phys. Rev. Lett. 69 (1992) 1849.
[3] E. Witten, Adv. Theor. Math. Phys. 2 (1998) 505.
[4] H. -W. Lee, Y. -S. Myung, and J. -Y. Kim, Phys. Lett. B 466 (1999) 211.
[5] A. Larrranaga, Commun. Theor. Phys. 50 (2008) 1341.
[6] S. Carlip, Class. Quantum Gravit. 12 (1995) 2853.
[7] R. -G. Cai, and J. -H. Cho, Phys. Rev. D 60 (1999) 067502.
[8] A. Ashtekar, J. Wisniewski, and O. Dreyer, Adv. Theor. Math. Phys. 6 (2003) 507.
[9] T. Sarkar, G. Sengupta, and B. Nath Tiwari, J. High Energ. Phys. 11 (2006) 015.
[10] M. Cadoni, and C. Monni, Phys. Rev. D 80 (2009) 024034.
[11] W. Cong, and R. B. Mann, J. High Energ. Phys. 11 (2019) 004.
[12] E. Witten, Three-Dimensional Gravity Revisited, arXiv:0706.3359.
[13] M. A. Anacleto, F. A. Brito, and E. Passos, Phys. Lett. B 743 (2015) 184.
[14] R. Emparan, G.T. Horowitz, and R.C. Myers, J. High Energ. Phys. 01 (2000) 021.
[15] S. Carlip, Class. Quantum Gravit. 22 (2005) R85.
[16] M. R. Setare, Eur. Phys. J. C 49 (2007) 865.
[17] E. Frodden, M. Geiller, K. Noui, and A. Perez, J. High Energ. Phys. 05 (2013) 139.
[18] D. V. Singh, and S. Siwach, Class. Quantum Grav. 30 (2013) 235034.
[19] P. Capu0245ta, V. Jejjila, and H. Soltanpanahi, Phys. Rev. D 89 (2014) 046006.
[20] T. Jurić, and A. Samsarov, Phys. Rev. D 93 (2016) 104033.
[21] R. Emparan, A. M. Frassino, and B. Way, JHEP 11 (2020) 137.
[22] C. Germani, and G. P. Procopio, Phys. Rev. D 74 (2006) 044012.
[23] A. de la Fuente, and R. Sundrum, J. High Energ. Phys. 09 (2014) 073.
[24] V. Ziogas, J. High Energ. Phys. 09 (2015) 114.
[25] L. J. Henderson, R. A. Hennigar, R. B. Mann, A. R. H. Smith, and J. Zhang, Phys. Lett. B 809 (2020) 135732.
[26] L. de Souza Campos, and C. Dappiaggi, Phys. Lett. B 816 (2021) 136198.
[27] M. Cardenas, O. Fuentealba, and C. Martinez, Phys. Rev. D 90 (2014) 124072.
[28] B. Gwak, and B. -H. Lee, Phys. Lett. B 755 (2016) 324.
[29] S. Alsae, Int. J. Mod. Phys. A 32 (2017) 1750076.
[30] K. S. Gupta, T. Jurić, and A. Samsarov, J. High Energ. Phys. 06 (2017) 107.
[31] G. G. L. Nashed, and S. Capozziello, Int. J. Mod. Phys. A 33 (2018) 1850076.
[32] B. Eslam Panah, S. H. Hendi, S. Panahiyani, and M. Hassaine, Phys. Rev. D 98 (2018) 084006.
[33] A. Rincón, and B. Koch, Eur. Phys. J. C 78 (2018) 1022.
[34] S. -T. Hong, Y. -W. Kim, and Y. -J. Park, Phys. Rev. D 99 (2019) 024047.
[35] B. Mu, J. Tao, and P. Wang, Phys. Lett. B 800 (2020) 135098.
[36] P. Caiate, D. Magos, and N. Breton, Phys. Rev. D 101 (2020) 064010.
[37] K. Hinterbichler, Rev. Mod. Phys. 84 (2012) 671.
[38] C. de Rham, Living Rev. Rel. 17 (2014) 7.
[39] Y. Akrami, T. S. Koivisto, and M. Sandstad, J. High Energ. Phys. 03 (2013) 99.
[40] Y. Akrami, S. F. Hassan, F. Knüg, A. Schmidt-May, and A. R. Solomon, Phys. Lett. B 748 (2015) 37.
[41] E. Babichev, L. Marzola, M. Raidal, A. Schmidt-May, F. Urban, H. Veerme, and M. von Strauss, Phys. Rev. D 94 (2016) 084055.
[42] E. Babichev, L. Marzola, M. Raidal, A. Schmidt-May, F. Urban, H. Veerme, and M. von Strauss, J. Cosmol. Astropart. Phys. 09 (2016) 016.
[43] S. Panpanich, and P. Burikham, Phys. Rev. D 98 (2018) 064008.
[44] B. Eslam Panah, and H. L. Liu, Phys. Rev. D 99 (2019) 104074.
[45] S. H. Hendi, G. H. Bordbar, B. Eslam Panah, and S. Panahiyan, J. Cosmol. Astropart. Phys. 07 (2017) 004.
[46] J. Xu, L. M. Cao, and Y. P. Hu, Phys. Rev. D 91 (2015) 124033.
[47] E. Babichev, L. Marzola, M. Raidal, A. Schmidt-May, F. Urban, H. Veerme, and M. von Strauss, Phys. Rev. D 94 (2016) 084055.
[48] E. Babichev, L. Marzola, M. Raidal, A. Schmidt-May, F. Urban, H. Veerme, and M. von Strauss, J. Cosmol. Astropart. Phys. 09 (2016) 016.
[49] S. Panpanich, and P. Burikham, Phys. Rev. D 98 (2018) 064008.
[50] B. Eslam Panah, and H. L. Liu, Phys. Rev. D 99 (2019) 104074.
[51] S. H. Hendi, G. H. Bordbar, B. Eslam Panah, and S. Panahiyan, J. Cosmol. Astropart. Phys. 07 (2017) 004.
[52] J. Xu, L. M. Cao, and Y. P. Hu, Phys. Rev. D 91 (2015) 124033.
[53] S. H. Hendi, R. B. Mann, S. Panahiyan, and B. Eslam Panah, Phys. Rev. D 95 (2017) 021501(R).
[54] A. Dehghani, S. H. Hendi, and R. B. Mann, Phys. Rev. D 101 (2020) 084026.
[55] B. Eslam Panah, S. H. Hendi, and Y. C. Ong, Phys. Dark Universe, 27 (2020) 100452.
[56] S. H. Hendi, G. H. Bordbar, B. Eslam Panah, and S. Panahiyan, J. Cosmol. Astropart. Phys. 07 (2017) 004.
[57] J. Xu, L. M. Cao, and Y. P. Hu, Phys. Rev. D 91 (2015) 124033.
[58] S. H. Hendi, R. B. Mann, S. Panahiyan, and B. Eslam Panah, Phys. Rev. D 95 (2017) 021501(R).
[59] A. Dehghani, S. H. Hendi, and R. B. Mann, Phys. Rev. D 101 (2020) 084026.
[60] B. Eslam Panah, S. H. Hendi, and Y. C. Ong, Phys. Dark Universe, 27 (2020) 100452.
[61] M. -S. Hou, H. Xu, and Y. C. Ong, Eur. Phys. J. C 80 (2020) 1090.
[62] C. de Rham, and G. Gabadadze, Phys. Rev. D 82 (2010) 044020.
[63] C. de Rham, G. Gabadadze, and A. J. Tolley, Phys. Rev. Lett. 106 (2011) 231101.
[64] H. van Dam, and M. Veltman, Nucl. Phys. B 22 (1970) 397.
[65] V. Zakharov, JETP Lett. 12 (1970) 312.
[66] D. G. Boulware, and S. Deser, Phys. Rev. D 6 (1972) 3368.
[67] E. Babichev, L. Marzola, M. Raidal, A. Schmidt-May, F. Urban, H. Veerme, and M. von Strauss, Phys. Rev. D 94 (2016) 084055.
[101] Y. Decanini, A. Folacci, and B. Raffaelli, Class. Quantum Grav. 28 (2011) 175021.
[102] B. Toshmatov, B. Ahmedov, and Z. Stuchlik, Eur. Phys. J. C 83 (2023) 981.
[103] K. Jafarzade, E. Rezaei, and S. H. Hendi, PTEP 2023 (2023) 053E01.
[104] S. Upadhyay, S. Mandal, Y. Myrzakulov, and K. Myrzakulov, Ann. Phys. 450 (2023) 169242.
[105] S. Kala, H. Nandan, P. Sharma, and M. Elmardi, Mod. Phys. Lett. A 36 (2021) 2150224.
[106] P. A. Gonzalez, M. Olivares, E. Papantonopoulos and Y. Vasquez, Phys. Rev. D 101 (2020) 044018.
[107] G. Panotopoulos, and A. Rincon, Ann. Phys. 443 (2022) 168947.
[108] N. Cruz, M. Olivares, and J. R. Villanueva, Eur. Phys. J. C 73 (2013) 2485.
[109] G. Panotopoulos, Gen. Rel. Grav. 52 (2020) 54.
[110] S. Fernando, D. Krug, and C. Curry, Gen. Rel. Grav. 35 (2003) 1243.
[111] N. Cruz, C. Martinez, and L. Pena, Class. Quant. Grav. 11 (1994) 2731.
[112] H. Lu and H. D. Lyu, Phys. Rev. D 101 (2020) 044059.
[113] V. Perlick, O. Y. Tsupko, and G. S. Bisnovatyi-Kogan, Phys. Rev. D 92 (2015) 104031.
[114] S. W. Wei, and Y. X. Liu, J. Cosmol. Astropart. Astropart. 11 (2013) 063.
[115] Y. Decanini, G. Esposito-Farse, and A. Folacci, Phys. Rev. D 83 (2011) 044032.
[116] S. Chandrasekhar, "The Mathematical Theory of Black Holes", Oxford University Press, New York (1983).
[117] S. Weinberg, "Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity", (Wiley, New York, 1972).
[118] P. Kocherlakota, and L. Rezzolla, Phys. Rev. D 102 (2020) 064058.
[119] W. Javed, J. Abbas, and A. ovgun, Ann. Phys. 418 (2020) 168183.