Property testing of quantum measurements

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In this paper, we study the following question: given a black box performing some unknown quantum measurement on a multi-qudit system, how do we test whether this measurement has certain property or is far away from having this property. We call this task property testing of quantum measurement. We first introduce a metric for quantum measurements, and show that it possesses many nice features. Then we show that, with respect to this metric, the following classes of measurements can be efficiently tested: 1. the stabilizer measurements, which play a crucial role for quantum error correction; 2. the 1-local measurements, i.e. measurements whose outcomes depend on a subsystem of at most k qudits; 3. the permutation-invariant measurements, which include those measurements used in quantum data compression, state estimation and entanglement concentration. In fact, all of them can be tested with query complexity independent of the system’s dimension. Furthermore, we also present an algorithm that can test any finite set of measurements.

Finally, we consider the following natural question: given two black-box measurement devices, how do we estimate their distance? We give an efficient algorithm for this task, and its query complexity is also independent of the system’s dimension. As a consequence, we can easily test whether two unknown measurements are identical or very different.

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I. INTRODUCTION

Quantum measurements are ubiquitous and play a pivotal role in the quantum information science. Many tasks require not only the most general quantum measurements, but also the entangled ones on many particles. For example, several quantum data compression [1,2] and communication protocols [3] work by performing some entangled measurement related to the representation theory of symmetric group on a multi-particle state. For another instance, certain non-abelian hidden subgroup problems are reduced to the problem of efficiently performing certain joint measurement on multiple coset states [4,5]. In order to guarantee the success of these tasks in practice, it is crucial to make sure that the required measurements are implemented with sufficiently high fidelity.

Now imagine that someone builds a quantum measurement device and claims that it performs some measurement that we need. How do we check if this is true? Of course, we can use the quantum tomography [13–19] to completely characterize the measurement implemented by the device. However, this method is very inefficient, especially when the dimension of the system is very large. Specifically, a general quantum measurement with k possible outcomes on a D-dimensional system is described by a collection \{M_1, M_2, \ldots, M_k\} of measurement operators satisfying the completeness equation

$$\sum_{i=1}^{k} M_i^\dagger M_i = I.$$  \hspace{1cm} (1)

Here $M_i$ is a bounded linear operator on the D-dimensional Hilbert space and it corresponds to the i-th outcome. In order to reconstruct the $M_i$’s, we need to determine $\Theta(kD^2)$ parameters. If the system consists of n d-dimensional particles (or qudits), then $D = d^n$ and hence we must access the device $\Omega(kd^{2n})$ times. So this approach is practical only for systems of moderate sizes.

Given the great difficulty of fully characterizing a black-box measurement, we need to come up with a better method. Suppose our desired measurement has certain property, but the unknown measurement is shown to be very different from any measurement possessing this property, then we know that it is surely not what we want. Therefore, we can consider devising a test that separates the measurements possessing certain property from those “far away” from them. Of course, in order to claim that two measurements are far away, we need to introduce a metric (or distance function) that quantifies their difference in the first place. We call this task property testing of quantum measurements.

Generally speaking, property testing [20, 22] is the task of deciding whether an object has certain property or is far away from any object possessing this property, given the promise that it is one of the two cases. For example, given a boolean function $f : \{0,1\}^n \rightarrow \{0,1\}$ as an oracle (which receives the input $x \in \{0,1\}^n$ and returns the value of $f(x)$), we may want to determine whether this function is linear or far away from any linear functions, by querying the oracle as few times as possible. The property testing of classical objects, such as boolean functions and graphs, has been extensively studied in computer science, and it plays an important role in probabilistically checkable proofs (PCP) [23]. Remarkably, many properties of boolean functions and graphs are found to be testable with very few queries. In fact, sometimes
the query complexity is even independent of the object’s size. The property testing of quantum objects, including quantum states [24] and operations [24 29], has been addressed only recently. It was found that many interesting classes of quantum states and operations, such as the product pure states [24] and Clifford operations [24 29], can also be tested with very few copies or queries.

In this paper, we initiate the study of property testing of general multi-qiudit measurements. First, we introduce a distance function for quantum measurements and show that it has many nice features. In particular, if two measurements are close with respect to this metric, then they behave similarly on most input states. Then, we present efficient algorithms for testing three interesting classes of measurements: 1. the stabilizer measurements, which play a crucial role for quantum error correction [30 32]; 2. the k-local measurements, i.e. measurements whose outcomes depend only on a subsystem of at most k qudits; 3. the permutation-invariant measurements, which includes those measurements used in quantum data compression [1–5], state estimation [33–35] and entanglement concentrations [36 37]. In fact, all of them can be tested with query complexity independent of the system’s dimension. Furthermore, we also present an algorithm that can test any finite set of measurements. Finally, we consider the following natural question: given two black-box measurement devices, how do we estimate their distance? We give an efficient algorithm for this task, and its query complexity is also independent of the system’s dimension. As a corollary, we can easily test whether two unknown measurements are identical or very different.

The remainder of this paper is organized as follows. In Sec.II, we introduce a metric for quantum measurements, formally describe our problem and also present several useful tools for our work. Then, in Sec. III, IV and V, we study the testing of stabilizer measurements, k-local measurements and permutation-invariant measurements. Next, in Sec. VI we give an algorithm that tests any finite set of measurements. After that, in Sec. VII we present an algorithm for estimating the distance between any two measurements. Finally, Sec. VIII concludes this paper.

II. PRELIMINARIES

A. A Metric for Quantum Measurements

Consider a D-dimensional quantum system. Let \( \mathcal{H}_D \) be its Hilbert space, and let \( B(\mathcal{H}_D) \) be the set of bounded linear operators on \( \mathcal{H}_D \). Then any measurement with \( k \) possible outcomes on this system can be described by \( M = \{M_1, M_2, \ldots, M_k\} \) where \( M_i \in B(\mathcal{H}_D) \) and \( \sum_{i=1}^{k} M_i^† M_i = I \). If we perform \( M \) on the state \( \rho \), then the probability of obtaining outcome \( i \) is

\[
p(i) = \text{tr}(M_i \rho M_i^†),
\]

and accordingly the state of the system after the measurement becomes

\[
\rho_i = \frac{M_i \rho M_i^†}{\text{tr}(M_i \rho M_i^†)},
\]

It is worth noting that each \( M_i \) is determined only up to a phase. Namely, if we replace \( M_i \) by \( \beta_i M_i \) for some \( \beta_i \in \mathbb{C} \), \( |\beta_i| = 1 \), we are still describing the same measurement. One can easily see this from Eqs. (2) and (3).

In order to conveniently deal with this equivalence relation, we make the following definitions. For any \( A \in B(\mathcal{H}_D) \), let \[ A = \{\beta A : \beta \in \mathbb{C}, |\beta| = 1\} \]. Furthermore, for any \( M = \{M_i\}_{1 \leq i \leq k} \), let \[ [M] = \{\{\beta_i M_i\}_{1 \leq i \leq k} : \beta_i \in \mathbb{C}, |\beta_i| = 1\} \]. Then \( M \) and \( N \) correspond to the same measurement if and only if \([M] = [N]\).

For some applications, the post-measurement state is of little interest. In such cases, the measurement can be more conveniently described by a positive-operator valued measure (POVM) \( \{E_1, E_2, \ldots, E_k\} \) where \( E_i := M_i^† M_i \). But in this paper we care a lot about the post-measurement state. We would say that two measurements are identical if both the distributions of outcomes and the post-measurement states are exactly the same for the two measurements performed on arbitrary state.

We want to be able to compare any two measurements that may have different number of outcomes. In order to achieve this, we append infinitely many zero operators to \( M = \{M_1, M_2, \ldots, M_k\} \) and rewrite it as \( \{M_i\}_{i \in \mathbb{N}} \) where \( M_i = 0 \) for any \( i > k \). In addition, we use \([M] = k\) to denote the number of nonzero \( M_i \)’s.

Now let \( \Omega \) be the set of all measurements on the \( D \)-dimensional system. We want to define a metric (or distance function) \( \Delta : \Omega \times \Omega \to \mathbb{R} \) that should satisfy the following natural conditions: for any \( M = \{M_i\}_{i \in \mathbb{N}}, N = \{N_i\}_{i \in \mathbb{N}}, L = \{L_i\}_{i \in \mathbb{N}} \in \Omega \),

1. \( \Delta(M, N) \geq 0 \);
2. \( \Delta(M, N) = 0 \) if and only if \([M] = [N]\);
3. \( \Delta(M, N) = \Delta(N, M) \);
4. \( \Delta(M, L) \leq \Delta(M, N) + \Delta(N, L) \).

Furthermore, note that a measurement \( M = \{M_i\}_{i \in \mathbb{N}} \) on the system A can be equivalently viewed as a joint measurement \( M \otimes I := \{M_i \otimes I\}_{i \in \mathbb{N}} \) on the system \( A \otimes B \), where \( B \) is any ancilla system, and similarly for \( N = \{N_i\}_{i \in \mathbb{N}} \). Since the distance between \( M \) and \( N \) should be independent of whether an ancilla system is appended, or what kind of ancilla system is appended, we need

5. \( \Delta(M, N) = \Delta(M \otimes I, N \otimes I) \), where \( I \) is the identity operation on any finite-dimensional Hilbert space.

Finally, since we want to use \( \Delta \) for property testing, it needs to be normalized, i.e.

6. \( \Delta(M, N) \leq 1 \).
Our idea is that $\Delta(M, N)$ should reflect the total differences between each pair of $M_i$ and $N_i$. First, we quantify the distance between any $A, B \in B(\mathcal{H}_D)$ as:

$$\Delta(A, B) := \inf_{A' \in [A], B' \in [B]} \frac{1}{\sqrt{2D}} \|A' - B'\|_F$$

$$= \inf_{\theta \in [0, 2\pi]} \frac{1}{\sqrt{2D}} \|A - e^{i\theta}B\|_F,$$

where

$$\|A\|_F := \sqrt{\text{tr}(A^\dagger A)} = \sqrt{\sum_{i,j=1}^{D} |a_{i,j}|^2}$$

is the Frobenius norm (or Hilbert-Schmidt norm) for $A = [a_{i,j}]$ (fixing an orthonormal basis for $\mathcal{H}_D$, any operator $A \in B(\mathcal{H}_D)$ is represented by a $D \times D$ matrix). This metric has been used in Ref. for the property testing of unitary operations. It possesses the following nice features:

(i) $\Delta(A, B) \geq 0$;

(ii) $\Delta(A, B) = 0$ if and only if $[A] = [B]$;

(iii) $\Delta(A, B) = \Delta(B, A)$;

(iv) $\Delta(A, C) \leq \Delta(A, B) + \Delta(B, C)$;

(v) $\Delta(A, B) = \Delta(A \otimes I, B \otimes I)$, where $I$ is the identity operation on any finite-dimensional Hilbert space.

Furthermore, note that

$$\Delta^2(A, B) = \frac{1}{2D} (\langle A, A \rangle + \langle B, B \rangle - 2|\langle A, B \rangle|),$$

where

$$\langle A, B \rangle := \text{tr}(A^\dagger B)$$

is the Hilbert-Schmidt inner product.

Now we define the distance between $M = \{M_i\}_{i \in \mathbb{N}}$ and $N = \{N_i\}_{i \in \mathbb{N}}$ as

$$\Delta(M, N) := \frac{1}{\max\{|M_i|, |N_i|\}} \sum_{i \in \mathbb{N}} \Delta^2(M_i, N_i)$$

$$= \sqrt{\sum_{i=1}^{\max\{|M_i|, |N_i|\}} \Delta^2(M_i, N_i)}.$$  

Properties (i), (ii), (iii) and (v) of $\Delta(A, B)$ immediately imply that $\Delta(M, N)$ satisfies conditions 1, 2, 3 and 5. In addition, $\Delta(M, N)$ also fulfills condition 6 because, by Eq. (8) and the completeness equation,

$$\Delta^2(M, N) = \frac{1}{2D} \sum_{i \in \mathbb{N}} \{(M_i, M_i) + (N_i, N_i) - 2|\langle M_i, N_i \rangle|\}$$

$$\leq 1 - \frac{D}{2} \sum_{i \in \mathbb{N}} |(M_i, N_i)|.$$

Moreover, $\Delta(M, N)$ also satisfies condition 3. To prove this, we use property (iv) of $\Delta(A, B)$ and the following lemma:

**Lemma 1.** If $a_i$, $b_i$ and $c_i$ are non-negative numbers such that $c_i \leq a_i + b_i$ for $i = 1, 2, \ldots, k$, then

$$\sum_{i=1}^{k} c_i^2 \leq \sum_{i=1}^{k} a_i^2 + \sum_{i=1}^{k} b_i^2 + 2 \sqrt{\sum_{i=1}^{k} a_i^2 \sum_{i=1}^{k} b_i^2}. \quad (10)$$

**Proof:** We need to prove

$$\sum_{i=1}^{k} c_i^2 \leq \sum_{i=1}^{k} a_i^2 + \sum_{i=1}^{k} b_i^2 + 2 \sqrt{\sum_{i=1}^{k} a_i^2 \sum_{i=1}^{k} b_i^2}. \quad (11)$$

Since $0 \leq c_i \leq a_i + b_i$, we have

$$\sum_{i=1}^{k} c_i^2 \leq \sum_{i=1}^{k} (a_i + b_i)^2 = \sum_{i=1}^{k} (a_i^2 + b_i^2 + 2a_ib_i). \quad (12)$$

So it is sufficient to show

$$\sum_{i=1}^{k} a_ib_i \leq \sqrt{\sum_{i=1}^{k} a_i^2 \sum_{i=1}^{k} b_i^2}, \quad (13)$$

which is exactly the Cauchy-Schwarz inequality.

Plugging $a_i = \Delta(M_i, N_i)$, $b_i = \Delta(N_i, L_i)$, $c_i = \Delta(M_i, L_i)$ for $1 \leq i \leq \max\{|M|, |N|, |L|\}$ into lemma (1) we obtain

$$\Delta(M, L) \leq \Delta(M, N) + \Delta(N, L). \quad (14)$$

Now we show that $\Delta(M, N)$ reflects the average difference between the “behaviors” of $M$ and $N$ on a random input state. Here by “behavior” we mean both the distribution of measurement outcomes and the post-measurement states. Specifically, let $|\psi\rangle$ be a random pure state chosen according to the normalized Haar measure. If we perform $M = \{M_i\}_{i \in \mathbb{N}}$ or $N = \{N_i\}_{i \in \mathbb{N}}$ on $|\psi\rangle$, then the unnormalized post-measurement states would be $\{M_i|\psi\rangle\}_{i \in \mathbb{N}}$ or $\{N_i|\psi\rangle\}_{i \in \mathbb{N}}$. We quantify the total difference between the two sets of states as

$$\sum_{i \in \mathbb{N}} \|M_i|\psi\rangle - N_i|\psi\rangle\|^2 = \sum_{i \in \mathbb{N}} \langle M_i^\dagger M_i + N_i^\dagger N_i - M_i^\dagger N_i - N_i^\dagger M_i |\psi\rangle^2.$$  

Integrating this quantity over $|\psi\rangle$ and using the fact

$$\int |\psi\rangle \langle \psi|\ d\psi = \frac{I}{D} \quad (16)$$

and the completeness equation, we obtain

$$\int \sum_{i \in \mathbb{N}} \|M_i|\psi\rangle - N_i|\psi\rangle\|^2\ d\psi = 2 - \frac{2}{D} \sum_{i \in \mathbb{N}} \text{Re}(M_i, N_i). \quad (17)$$
By multiplying each $N_i$ by an appropriate phase (without changing the measurement being described), we can make $\langle M_i, N_i \rangle$ real and non-negative. Then by Eq. (19),

$$\int \sum_{i \in \mathbb{N}} \| M_i(\psi) - N_i(\psi) \|^2 \, d\psi = 2 - \frac{2}{D} \sum_{i \in \mathbb{N}} | \langle M_i, N_i \rangle | \geq 2 \Delta^2(M, N).$$

Thus, if $\Delta(M, N) = \epsilon$ is small, then the expectation of $\sum_{i \in \mathbb{N}} \| M_i(\psi) - N_i(\psi) \|^2 = 2\epsilon^2$, which implies $M_i(\psi)$ and $N_i(\psi)$ are close on average. Furthermore, let $p_i(\langle \psi \rangle) = \| M_i(\psi) \|^2$ and $q_i(\langle \psi \rangle) = \| N_i(\psi) \|^2$ be the probability of obtaining outcome $i$ when performing $M$ and $N$ on $| \psi \rangle$ respectively. Then by the fact

$$\| u - v \| \geq \| u \| - \| v \| \quad \forall u, v \in \mathbb{C}^D,$$  

we have

$$\sum_{i \in \mathbb{N}} \| M_i(\psi) - N_i(\psi) \|^2 \geq \sum_{i \in \mathbb{N}} (\| M_i(\psi) \| - \| N_i(\psi) \|)^2 \geq \sum_{i \in \mathbb{N}} (\sqrt{p_i(\langle \psi \rangle)} - \sqrt{q_i(\langle \psi \rangle)})^2 = 2 - 2F(p(\langle \psi \rangle), q(\langle \psi \rangle)),$$

where

$$F(p(\langle \psi \rangle), q(\langle \psi \rangle)) := \sum_{i \in \mathbb{N}} \sqrt{p_i(\langle \psi \rangle)} q_i(\langle \psi \rangle)$$

is the fidelity of $p(\langle \psi \rangle) := (p_i(\langle \psi \rangle))_{i \in \mathbb{N}}$ and $q(\langle \psi \rangle) := (q_i(\langle \psi \rangle))_{i \in \mathbb{N}}$. Taking the expectation of Eq. (20), then we know from Eq. (18) that $F(\tilde{p}, \tilde{q})$ is at least $1 - 2\Delta^2(M, N)$ on average. So if $\Delta(M, N)$ is small, then $\tilde{p}$ and $\tilde{q}$ are also close on average.

Note that by using the Markov inequality

$$\Pr(\{X > a\}) \leq \frac{E(\{X\})}{\alpha}, \quad \forall \alpha > 0,$$

we can estimate the fraction of “good” input states on which $M$ and $N$ behave similarly. Specifically, let $\Delta(M, N) = \epsilon$. Then setting $X = \sum_{i \in \mathbb{N}} \| M_i(\psi) - N_i(\psi) \|^2$ and $\alpha = 10\epsilon^2$ in Eq. (22) yields

$$\Pr(\sum_{i \in \mathbb{N}} \| M_i(\psi) - N_i(\psi) \|^2 \geq 10\epsilon^2) \leq \frac{1}{\alpha}.$$

So for at least $4/5$ fraction of $\langle \psi \rangle$’s, $M(\psi)$ and $N(\psi)$ are close, provided that $\epsilon$ is small enough. Similarly, setting $X = 1 - F(p(\langle \psi \rangle), q(\langle \psi \rangle))$ and $\alpha = 10\epsilon^2$ in Eq. (22) yields

$$\Pr(1 - F(p(\langle \psi \rangle), q(\langle \psi \rangle)) \leq 10\epsilon^2) \leq \frac{1}{\alpha}.$$

So for at least $4/5$ fraction of $\langle \psi \rangle$’s, $p(\langle \psi \rangle)$ and $q(\langle \psi \rangle)$ are close, provided that $\epsilon$ is small enough.

### B. Property Testing of Quantum Measurements

The task of property testing can be formally described as follows. Suppose $\Omega$ is a class of mathematical objects equipped with a metric $\Delta$. A property is just a subset $A \subset \Omega$. For any $A \in \Omega$, if $A \in S$, then we say that $A$ has property $S$; otherwise, if $\Delta(A, S) \geq \epsilon$, i.e. $\Delta(A, B) \geq \epsilon$ for any $B \in S$, then we say that $A$ is $\epsilon$-far from property $S$. An algorithm $\epsilon$-tests property $S$ with query complexity $q(|\Omega|, \epsilon)$ if on any input $A \in \Omega$, it makes at most $q(|\Omega|, \epsilon)$ queries to $A$ and behaves as follows:

- if $A$ has property $S$, then the algorithm accepts $A$ with probability at least $2/3$;
- if $A$ is $\epsilon$-far from property $S$, then the algorithm accepts $A$ with probability at most $1/3$.

Note that by repeating this algorithm many times and choosing the majority answer, we can exponentially decrease the probability of making an erroneous decision. So the completeness error $1 - 2/3 = 1/3$ here and soundness error $1/3$ here can be replaced by arbitrarily small constants.

In this paper, we study the property testing of quantum measurements on finite-dimensional systems with respect to the metric $\Delta$ defined by Eq. (8). Specifically, fix a $D$-dimensional system, and let $\Omega$ be the set of all measurements on this system. Suppose $S \subset \Omega$ is the property to be tested. Our input is a black box performing some unknown measurement $M \in \Omega$. We can access it by preparing the $D$-dimensional system (denoted by $A$) in some known state $\rho_A$, applying $M$ on the state, and obtaining an outcome $i$ as well as the post-measurement state $M_i\rho AM_i^\dagger$ (up to a normalization). Moreover, we can also introduce an ancilla system $B$ and prepare the joint system $A \otimes B$ in some entangled state $\rho_{AB}$, then apply $M$ on subsystem $A$ of this state, and get an outcome $i$ as well as the post-measurement state $(M_i \otimes I)\rho_{AB}(M_i^\dagger \otimes I)$ (up to a normalization). An algorithm $\epsilon$-tests $S$ with query complexity $q(D, \epsilon)$ if for any input $M \in \Omega$, it accesses the black box in the above way at most $q(D, \epsilon)$ times and makes a correct decision about whether $M \in S$ or $\Delta(M, S) \geq \epsilon$ with probability at least $2/3$. Note that we allow the algorithm to extract useful information about $M$ from both the outcome statistics and post-measurement states. Our goal is to devise such a testing algorithm with the minimal query complexity. In addition, we also prefer this algorithm to be efficiently implementable. Namely, we want its time complexity to be polynomial in $\log(D)$ and $1/\epsilon$, assuming one query to the black box takes a unit time.

#### C. Useful Tools

The following tools will be very useful for our work.

1. **Probability Theory**

Due to the probabilistic nature of quantum measurements, our testing algorithms heavily depend on analyz-
ing the outcome statistics. So the following results from probability theory will be very helpful.

Suppose \( p = (p_i)_{i \in \mathbb{N}} \) and \( q = (q_i)_{i \in \mathbb{N}} \) are two probability distributions over \( \mathbb{N} \). Let

\[
D(p, q) := \frac{1}{2} \sum_{i \in \mathbb{N}} |p_i - q_i|
\]

be the variational distance of \( p \) and \( q \), and let

\[
F(p, q) := \sum_{i \in \mathbb{N}} \sqrt{p_i q_i}.
\]

be the fidelity of \( p \) and \( q \). It is easy to see that \( 0 \leq D(p, q), F(p, q) \leq 1 \). In addition, they satisfy the following relation:

\[
1 - F(p, q) \leq D(p, q) \leq \sqrt{1 - F^2(p, q)}.
\]

Meanwhile, the Chernoff-Hoeffding bound \([38, 39]\) is another important and useful result from probability theory. It gives an exponentially decreasing bounds on tail distributions of sums of independent random variables. Specifically, let \( X_1, X_2, \ldots, X_n \in [0, 1] \) be independent random variables and let \( S = \sum_{i=1}^n X_i \). Then for any \( 0 < \epsilon < E[S]/n \),

\[
\Pr \left[ \left| \frac{S - E[S]}{n} \right| > \epsilon \right] \leq 2e^{-2n\epsilon^2}.
\]

In particular, if the \( X_i \)'s are independent and identically distributed (i.i.d.) as \( X \), then

\[
\Pr \left[ \left| \frac{S - E[X]}{n} \right| > \epsilon \right] \leq 2e^{-2n\epsilon^2}.
\]

By choosing \( n = O(\log(1/\delta)/\epsilon^2) \) we can make the right-hand side smaller than \( \delta \). Then \( E[X] \) can be approximated by \( S/n \) with precision \( \epsilon \) and confidence \( 1 - \delta \).

Now imagine that \( X \) is a \( \{0, 1\} \)-valued random variable indicating whether a particular outcome occurs in an experiment. Namely, \( X = 1 \) if a this outcome occurs, and \( X = 0 \) otherwise. Then \( E[X] \) is the probability for this outcome in this experiment. By Eq. (29), we can approximate this probability with precision \( \epsilon \) and confidence \( 1 - \delta \) by repeating the experiment \( O(\log(1/\delta)/\epsilon^2) \) times.

### 2. The Choi-Jamiolkowski Isomorphism

The Choi-Jamiolkowski isomorphism \([40, 41]\) states that there is a duality between quantum operations and quantum states. Specifically, let

\[
|\Phi_D^+\rangle := \frac{1}{\sqrt{D}} \sum_{i=1}^D |i\rangle|i\rangle
\]

be the \( D \)-dimensional maximally entangled state. For any \( A \in B(H_D) \), define

\[
|v(A)\rangle := (A \otimes I)|\Phi_D^+\rangle,
\]

where \( A \) acts on the first subsystem. Then for any \( A, B \),

\[
\langle v(A)|v(B)\rangle = \frac{1}{D}\langle A, B\rangle.
\]

Namely, the inner product between \( |v(A)\rangle \) and \( |v(B)\rangle \) is proportional to the Hilbert-Schmidt product of \( A \) and \( B \). In particular, if \( A = B \), then we get

\[
p(A) := |||v(A)|||^2 = \frac{1}{D}\|A\|^2.
\]

So \( |v(A)\rangle \) is not normalized unless \( \|A\|_F = \sqrt{D} \). We use \( |\tilde{v}(A)\rangle \) to denote the normalized version of \( |v(A)\rangle \), i.e.

\[
|\tilde{v}(A)\rangle := \frac{|v(A)\rangle}{\|v(A)\|}.
\]

Now suppose that we perform a measurement \( M = \{M_i\}_{i \in \mathbb{N}} \) on the first subsystem of \( |\Phi_D^+\rangle \), then the unnormalized post-measurement states would be \( \{|v(M_i)\rangle\}_{i \in \mathbb{N}} \). Namely, the outcome \( i \) occurs with probability

\[
p(M_i) = \||v(M_i)|||^2 = \frac{1}{\sqrt{D}\|M_i\|_F^2},
\]

and the corresponding post-measurement state is

\[
|\tilde{v}(M_i)\rangle = \frac{1}{\sqrt{p(M_i)}} |v(M_i)\rangle.
\]

This property will be crucial for every testing algorithm given in this paper.

#### 3. Pauli Decomposition

Let

\[
\sigma_{00} = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_{10} = X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},
\]

\[
\sigma_{01} = Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_{11} = Y = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}
\]

be the Pauli operators. For any \( x = (x_1, x_2, \ldots, x_n), z = (z_1, z_2, \ldots, z_n) \in \mathbb{Z}_2^n \), let

\[
\sigma_{x,z} = \sigma_{x_1, z_1} \otimes \sigma_{x_2, z_2} \otimes \cdots \otimes \sigma_{x_n, z_n}.
\]

Then for any \( a, b, c, d \in \mathbb{Z}_2^n \), we have

\[
[\sigma_{a,b}\sigma_{c,d}] = [\sigma_{a\oplus c, b\oplus d}].
\]

Here ‘\( \oplus \)’ denotes the bitwise addition modulo 2. So there exists \( \beta_{a,b,c,d} \in \mathbb{C} \) with \( |\beta_{a,b,c,d}| = 1 \) such that

\[
\sigma_{a,b}\sigma_{c,d} = \beta_{a,b,c,d}\sigma_{a\oplus c, b\oplus d}.
\]

Moreover, \( \{\sigma_{x,z}\}_{x,z \in \mathbb{Z}_2^n} \) form an orthogonal basis for \( B(\mathcal{H}_D^{2n}) \) with respect to the Hilbert-Schmidt product. So for any \( A \in B(\mathcal{H}_D^{2n}) \) we can write it as

\[
A = \sum_{x,z \in \mathbb{Z}_2^n} \mu_{x,z}(A) \sigma_{x,z}.
\]
where

$$\mu_{x,z}(A) := \frac{1}{2^n} \langle \sigma_{x,z}, A \rangle. \quad (42)$$

Note that

$$\|A\|_F^2 = 2^n \sum_{x,z \in \mathbb{Z}_2^d} |\mu_{x,z}(A)|^2. \quad (43)$$

Furthermore, by the Choi-Jamiolkowski isomorphism,

$$|v(A)\rangle = \sum_{x,z \in \mathbb{Z}_2^d} \mu_{x,z}(A) |v(\sigma_{x,z})\rangle, \quad (44)$$

where \{\{|v(\sigma_{x,z})\rangle\}_{x,z \in \mathbb{Z}_2^d}\} form an orthonormal basis for \(\mathcal{H}_n^d\). So

$$p(A) = \| |v(A)\rangle\|^2 = \sum_{x,z \in \mathbb{Z}_2^d} |\mu_{x,z}(A)|^2. \quad (45)$$

It follows that

$$|\tilde{v}(A)\rangle = \frac{1}{\sqrt{p(A)}} |v(A)\rangle = \sum_{x,z \in \mathbb{Z}_2^d} \mu_{x,z}(A) |v(\sigma_{x,z})\rangle. \quad (46)$$

So if we measure \(|\tilde{v}(A)\rangle\) in the basis \{\{|v(\sigma_{x,z})\rangle\}_{x,z \in \mathbb{Z}_2^d}\}, then the outcome \((x,z)\) occurs with probability

$$q_{x,z}(A) := \frac{|\mu_{x,z}(A)|^2}{p(A)}. \quad (47)$$

Now consider any measurement \(M = \{M_i\}_{i \in \mathbb{N}}\) on an \(n\)-qubit system. By plugging \(M_i = \sum_{x,z \in \mathbb{Z}_2^d} \mu_{x,z}(M_i) \sigma_{x,z}\) into \(\sum_{i \in \mathbb{N}} M_i^\dagger M_i = I\), we obtain

$$\sum_{i \in \mathbb{N}} \sum_{x,z \in \mathbb{Z}_2^d} |\mu_{x,z}(M_i)|^2 = 1. \quad (48)$$

Furthermore, if we measure \(|\tilde{v}(M_i)\rangle\) in the basis \{\{|v(\sigma_{x,z})\rangle\}_{x,z \in \mathbb{Z}_2^d}\}, then the outcome \((x,z)\) occurs with probability

$$q_{x,z}(M_i) = \frac{|\mu_{x,z}(M_i)|^2}{p(M_i)}. \quad (49)$$

These properties will be very useful for the testing of the stabilizer and \(k\)-local measurements.

Finally, the above facts can be straightforwardly generalized to the qudit case. We mainly need to replace the Pauli operators by their higher-dimensional analogues

$$\sigma_{x,z} = \sum_{j=0}^{d-1} \omega^{jz} |j \oplus x\rangle \langle j|, \quad (50)$$

for \(x,z \in \mathbb{Z}_d\). Here \(\omega = e^{2\pi i/d}\) and \(\oplus\) denotes the addition modulo \(d\). The reader can easily work out the rest of the details.

### III. Testing the Stabilizer Measurements

Equipped with the notations introduced above, we begin with the testing of the stabilizer measurements. This kind of measurements have played a crucial role in quantum error correction \([31, 32]\). They are closely related to the stabilizer codes \([42]\), which are defined by choosing a set of commuting operators from the Pauli group on \(n\) qubits. Specifically, let \(P_n := \{\pm \sigma_{x,z} : x, z \in \mathbb{Z}_2^n\}\). Then any operator \(g \in P_n\) has eigenvalues \(+1\) and \(-1\), and its \((+1)\)-eigenspace and \((-1)\)-eigenspace are both \(2^{n-1}\)-dimensional. Now if we choose a set of commuting operators \(g_1, g_2, \ldots, g_k \in P_n\) such that \(-I \notin \langle g_1, g_2, \ldots, g_k\rangle\), then the simultaneous \((+1)\)-eigenspace of \(g_1, g_2, \ldots, g_k\) defines the coding space of a stabilizer code. The error-correction for this code is quite simple: we simply measure each of \(g_1, g_2, \ldots, g_k\), and from the outcomes we can determine the error, and then we perform the appropriate recovery operation.

A stabilizer measurement is the projective measurement onto the \((+1)\)-eigenspaces of \(\sigma_{a,b}\) for any \(a, b \in \mathbb{Z}_2^n\). Formally, it is denoted by \(P(a, b) = \{P_1(a, b), P_2(a, b)\}\) where

$$P_1(a, b) := \frac{I + \sigma_{a,b}}{2}, \quad (51)$$
$$P_2(a, b) := \frac{I - \sigma_{a,b}}{2}. \quad (52)$$

So \(M = \{M_i\}_{i \in \mathbb{N}}\) is a stabilizer measurement if and only if there exist \(a, b \in \mathbb{Z}_2^n\) such that

$$\mu_{0,0}(M_1) = \mu_{a,b}(M_1) = \mu_0,0(M_2) = -\mu_{a,b}(M_2) = 1/2, \quad \mu_{x,y}(M_i) = 0, \quad \forall i \geq 3 \text{ or } (x, y) \neq (0, 0), (a, b). \quad (53)$$

The following lemma says that if the above condition is approximately fulfilled, then \(M = \{M_i\}_{i \in \mathbb{N}}\) is close to a stabilizer measurement.

**Lemma 2.** For any \(0 \leq \gamma, \delta \leq 1/4\) and \(a, b \in \mathbb{Z}_2^n\), if a measurement \(M = \{M_i\}_{i \in \mathbb{N}}\) on \(n\) qubits satisfies

$$|\mu_{0,0}(M_1)|, |\mu_{a,b}(M_1)| \in \left[\frac{1}{4} - \gamma, \frac{1}{4} + \gamma\right], \quad (54)$$
$$|\mu_{0,0}(M_2)|, |\mu_{a,b}(M_2)| \in \left[\frac{1}{4} - \gamma, \frac{1}{4} + \gamma\right], \quad (55)$$

and

$$\text{Re}(\mu_{0,0}(M_1)a_{a,b}(M_1)) \geq \frac{1}{4} - \delta, \quad (56)$$
$$\text{Re}(\mu_{0,0}(M_2)a_{a,b}(M_2)) \leq \frac{1}{4} + \delta, \quad (57)$$

then

$$\Delta(M, P(a, b)) \leq \sqrt{8\gamma + 2\delta}. \quad (58)$$
Proof: See Appendix A.

With the help of lemma 2, we obtain the following result on testing the stabilizer measurements.

**Theorem 1.** The stabilizer measurements on $n$ qubits can be $\epsilon$-tested with query complexity $O(1/\epsilon^4)$. Furthermore, the testing algorithm can be efficiently implemented.

**Proof:** Given a black box performing some unknown measurement $M = \{M_i\}_{i \in \mathbb{N}}$ on $n$ qubits, we run the following test on it:

**Algorithm 1** Testing the stabilizer measurements

1. Let $D = 2^n$, $L = \left\lfloor \frac{20000}{\epsilon^4} \right\rfloor$, $N = \left\lfloor \left( \frac{1}{2} - \frac{\epsilon^2}{64} \right) L \right\rfloor$, $T = \left\lfloor 0.99N \right\rfloor$, $W = \left\lfloor \frac{12}{\epsilon^2} \right\rfloor$.
2. Prepare $L$ copies of $\sqrt{D} \sum_{(i_1,i_2,\ldots,i_n) \in \mathbb{Z}_2^n} |i_1,i_2,\ldots,i_n\rangle |i_1,i_2,\ldots,i_n\rangle$.
3. Perform $M = \{M_i\}_{i \in \mathbb{N}}$ on the first subsystem of each copy of $|\phi_D^+\rangle$. If we obtain only outcomes 1 and 2 in these measurements, and the fraction of outcome 1 is between $\frac{1}{2} - \frac{\epsilon^2}{64}$ and $\frac{1}{2} + \frac{\epsilon^2}{64}$, then continue; otherwise, reject $M$ and quit.
4. Now we should have at least $N$ copies of $|\tilde{v}(M_1)\rangle$ and $|\tilde{v}(M_2)\rangle$ which are the post-measurement states corresponding to the outcome 1 and 2 respectively.
5. Choose $T$ copies of $|\tilde{v}(M_1)\rangle$ and $|\tilde{v}(M_2)\rangle$, and measure each of them in the basis $\{|\sigma_{x,a}\rangle\}_{x,a \in \mathbb{Z}_2}$. If we obtain only two different outcomes (0,0) and (a,b) for some $a,b \in \mathbb{Z}_2$ in these measurements, and the fraction of outcome (0,0) is between $\frac{1}{2} - \frac{\epsilon^2}{64}$ and $\frac{1}{2} + \frac{\epsilon^2}{64}$ for both those measurements on $|\tilde{v}(M_1)\rangle$ and those measurements on $|\tilde{v}(M_2)\rangle$, then continue; otherwise, reject $M$ and quit.
6. Select another $W$ copies of $|\tilde{v}(M_1)\rangle$ and $|\tilde{v}(M_2)\rangle$. For each copy, do the following:
   (a) For $j = 1, 2, \ldots, n$, if $a_j = b_j = 0$, then set $s_j = 1$; otherwise, measure the $j$-th qubit of the first subsystem in the $\sigma_{a_j,b_j}$ basis and set $s_j$ to be the outcome +1 or -1.
   (b) If the measured state is $|\tilde{v}(M_1)\rangle$ and $\prod_{j=1}^{n} s_j = -1$, or the measured state is $|\tilde{v}(M_2)\rangle$ and $\prod_{j=1}^{n} s_j = 1$, then reject $M$ and quit.
7. Now $M$ has passed all the above tests. Accept $M$.

For correctness, we need to prove:
(1) if $M$ is a stabilizer measurement, then this algorithm accepts $M$ with probability at least 2/3;
(2) on the other hand, if this algorithm accepts $M$ with probability at least 1/3, then $M$ is $\epsilon$-close to some stabilizer measurement. (Taking the contrapositive, we get that if $M$ is $\epsilon$-far away from any stabilizer measurement, then this algorithm accepts it with probability at most 1/3.)

Before proving the two statements, observe that:

1. In step 3, for $i = 1, 2$, the outcome $i$ should occur with probability $p(M_i)$. So, by Eq. (20) and our choice of $L$, the fraction of outcome $i$ is $\epsilon^2/64$-close to $p(M_i)$ with probability at least 0.95.
2. In step 5, for $i = 1, 2$, if the measured state is $|\tilde{v}(M_i)\rangle$, then the outcome (0,0) (or (a,b)) should occur with probability $q_{0,0}(M_i)$ (or $q_{a,b}(M_i)$). So, by Eq. (20) and our choice of $T$, the fraction of outcome (0,0) (or (a,b)) is $\epsilon^2/64$-close to $q_{0,0}(M_i)$ (or $q_{a,b}(M_i)$) with probability at least 0.95.
3. In step 6, for $i = 1, 2$, if the measured state is $|\tilde{v}(M_i)\rangle$, then the $s_j$’s depend only on the reduced state of the first subsystem of $|\tilde{v}(M_i)\rangle$, which is

$$\rho_i = \frac{M_i M_i^\dagger}{\text{tr}(M_i M_i^\dagger)}.$$ (59)

Then it is easy to see

$$\text{Pr}\left[ \prod_{j=1}^{n} s_j = 1|\rho_1\right] = \frac{\text{tr}(P_1(a,b)\rho_1)}{\text{tr}(M_1 M_1^\dagger)}.$$ (60)

and

$$\text{Pr}\left[ \prod_{j=1}^{n} s_j = -1|\rho_2\right] = \frac{\text{tr}(P_2(a,b)\rho_2)}{\text{tr}(M_2 M_2^\dagger)}.$$ (61)

If $M$ passes the test with probability at least 1/3, then we must have

$$\text{Pr}\left[ \prod_{j=1}^{n} s_j = 1|\rho_1\right] \geq 1 - \frac{\epsilon^2}{4},$$ (62)

$$\text{Pr}\left[ \prod_{j=1}^{n} s_j = -1|\rho_2\right] \geq 1 - \frac{\epsilon^2}{4}$$ (63)

because, if otherwise, step 6 would reject $M$ with probability at least $1 - (1 - \epsilon^2/4)^W \geq 1 - e^{-\epsilon^2 W/4} \geq 0.9$ by our choice of $W$.

Now let us prove statement (1). Suppose $M = P(a,b)$ for some $a,b \in \mathbb{Z}_2^n$. Since

$$p(P_1(a,b)) = p(P_2(a,b)) = 1/2,$$ (64)
by observation 1 step 3 rejects $P(a, b)$ with probability at most 0.1. Next, since

$$q_{0,0}(P_1(a, b)) = q_{a,b}(P_1(a, b)) = \frac{1}{2},$$

$$q_{0,0}(P_2(a, b)) = q_{a,b}(P_2(a, b)) = \frac{1}{2},$$

by observation 2 step 5 rejects $P(a, b)$ with probability at most 0.1. Then, by observation 3 step 6 never rejects $P(a, b)$. So, overall, algorithm \( \Pi \) accepts $P(a, b)$ with probability at least 0.8.

Next, we prove statement (2). Suppose $M$ is accepted by algorithm \( \Pi \) with probability at least 1/3. Then we claim that:

(i) $M$ satisfies Eqs. (60) and (61) with $\gamma = \epsilon^2/16$;
(ii) $M$ satisfies Eqs. (62) and (63) with $\delta = \epsilon^2/4$.

Then, by lemma 4, $M$ is $\epsilon$-close to $P(a, b)$.

To prove (i), it is sufficient to show

$$p(M_1), p(M_2) \in \left[ \frac{1}{2} - \frac{\epsilon^2}{32}, \frac{1}{2} + \frac{\epsilon^2}{32} \right]$$

and

$$q_{0,0}(M_1), q_{a,b}(M_1) \in \left[ \frac{1}{2} - \frac{\epsilon^2}{32}, \frac{1}{2} + \frac{\epsilon^2}{32} \right],$$

$$q_{0,0}(M_2), q_{a,b}(M_2) \in \left[ \frac{1}{2} - \frac{\epsilon^2}{32}, \frac{1}{2} + \frac{\epsilon^2}{32} \right],$$

because if they are true, then

$$|\mu_{x,z}(M_i)|^2 = p(M_i) q_{x,z}(M_i) \in \left[ \frac{1}{2} - \frac{\epsilon^2}{32}, \frac{1}{2} + \frac{\epsilon^2}{32} \right]^2 \subseteq \left[ \frac{1}{4} - \frac{\epsilon^2}{16}, \frac{1}{4} + \frac{\epsilon^2}{16} \right],$$

$$\forall i = 1, 2, \forall (x, z) = (0, 0), (a, b).$$

Eq. (67) holds because, if otherwise, then by observation 1 step 3 would reject $M$ with probability at least 0.95, contradicting our assumption. Similarly, Eqs. (68) and (69) also hold because, if otherwise, then by observation 2 step 5 would reject $M$ with probability at least 0.95, also contradicting our assumption.

Now it only remains to prove (ii). Plugging

$$M_i = \sum_{x,z \in \mathbb{Z}_2} \mu_{x,z}(M_i) \sigma_{x,z}$$

into Eqs. (60) and (61) gives

$$\Pr[\prod_{j=1}^n s_j = 1|\rho_1] = \frac{1}{2} + \frac{2\text{Re}(\mu_{0,0}(M_i)\mu_{a,b}(M_i)) + \lambda_1}{2p(M_1)},$$

$$\Pr[\prod_{j=1}^n s_j = -1|\rho_2] = \frac{1}{2} - \frac{2\text{Re}(\mu_{0,0}(M_2)\mu_{a,b}(M_2)) + \lambda_2}{2p(M_2)},$$

where

$$\lambda_i := \sum_{(x,z) \neq (0,0), (a,b)} \beta_{x,z,x\oplus a,z\oplus b}(M_i) \mu_{x\oplus a,z\oplus b}^*(M_i)$$

for $i = 1, 2$. Note that

$$|\lambda_i| \leq \sum_{(x,z) \neq (0,0), (a,b)} |\mu_{x,z}(M_i)| \cdot |\mu_{x\oplus a,z\oplus b}(M_i)| \leq \sum_{(x,z) \neq (0,0), (a,b)} |\mu_{x,z}(M_i)|^2$$

$$= p(M_i) - |\mu_{0,0}(M_i)|^2 - |\mu_{a,b}(M_i)|^2$$

$$\leq \left( \frac{1}{2} + \frac{\epsilon^2}{32} \right)^2 - 2 \cdot \left( \frac{1}{4} - \frac{\epsilon^2}{16} \right)$$

$$\leq 3\epsilon^2/16,$$

where in the second last step we use Eqs. (67) and (70). Now by Eqs. (62), (67), (2) and (75), we obtain

$$\text{Re}(\mu_{0,0}^*(M_1)\mu_{a,b}(M_1)) \geq \frac{1}{4} - \frac{\epsilon^2}{4},$$

Similarly, by Eqs. (62), (67), (2) and (75), we get

$$\text{Re}(\mu_{0,0}^*(M_2)\mu_{a,b}(M_2)) \leq \frac{1}{4} + \frac{\epsilon^2}{4},$$

as desired.

Finally, algorithm \( \Pi \) has query complexity $O(1/\epsilon^4)$. Moreover, besides querying the black box, it requires $O(n/\epsilon^4)$ quantum operations including: 1. preparing the state $|\Phi_D^+\rangle$, which is equivalent to preparing the $j$-th and $(n+j)$-th qubits in the Bell state for $j = 1, 2, \ldots, n$; 2. measuring a 2n-qubit state in the basis $\{|v(\sigma_{x,z})\rangle\}_{x,z \in \mathbb{Z}_2}$, which is equivalent to measuring the $j$-th and $(n+j)$-th qubit in the Bell basis for $j = 1, 2, \ldots, n$; 3. measuring any Pauli operator on a qubit. In addition, the classical processing is easy. So algorithm \( \Pi \) can be efficiently implemented.

The stabilizer code has been generalized to the qudit case where $d$ is any prime number. So we can also consider testing the stabilizer measurements on $n$ qudits, which are the projective measurements onto the $d$ eigenspaces of $\sigma_{x,z}$ for any $x,z \in \mathbb{Z}_d^2$. Algorithm \( \Pi \) can be straightforwardly generalized to test these measurements. We basically follow the same pattern, except that now we have $d$ equally likely outcomes instead of two. The reader can easily work out the rest of the details. This generalized algorithm still has query complexity $O(1/\epsilon^4)$, and still can be efficiently implemented.

Finally, note that algorithm 1 does not only test whether $M$ is close to a stabilizer measurement, but also learns which stabilizer measurement $M$ is closest to. This is fortunate, but not generic. Usually learning an object is much harder than testing it, as we need to acquire more structural information about the object.
IV. TESTING THE k-LOCAL MEASUREMENTS

Given a measurement on an n-qudit system, it is natural ask whether this measurement truly involves all of the n qudits. Namely, its outcome might depends only on a small subsystem. Specifically, Let \( |n| := \{1, 2, \ldots, n\} \). For any \( T = \{j_1, j_2, \ldots, j_m\} \subseteq |n| \), we use \( T \) to denote the subsystem consisting of the \( j_1 \)-th, \( j_2 \)-th, \ldots, \( j_m \)-th qudits. We say that a measurement \( M = \{M_i\}_{i \in \mathbb{N}} \) is \( k \)-local if there exists some \( T \subseteq |n| \) with \( |T| = k \) such that

\[
M_i = \tilde{M}_{i|T} \otimes I_{|T^c}.
\] (88)

for any \( i \in \mathbb{N} \). Here \( \tilde{M}_i \) acts on the subsystem \( T \) and \( I \) acts on the complementary subsystem \( T^c \). In this section, we will study the testing of these \( k \)-local measurements.

For convenience, we introduce the following notations. For any \( x = (x_1, x_2, \ldots, x_n), z = (z_1, z_2, \ldots, z_n) \in \mathbb{Z}_q^a \), let

\[
\text{supp}(x, z) = \{i \in |n| : x_i \neq 0 \text{ or } z_i \neq 0\}
\] (79)

be the support of \((x, z)\). Then let

\[
\Gamma_T = \{(x, z) : x, z \in \mathbb{Z}_q^a, \text{ supp}(x, z) \subseteq T\}.
\] (80)

Then for any \( A \in \mathcal{B}(\mathcal{H}_d^\otimes n) \), we can write it as

\[
A = f_T(A) + g_T(A),
\] (81)

where

\[
f_T(A) := \sum_{(x, z) \in \Gamma_T} \mu_{x, z}(A) \sigma_{x, z},
\] (82)

\[
g_T(A) := \sum_{(x, z) \in \Gamma_T} \mu_{x, z}(A) \sigma_{x, z}.
\] (83)

While \( f_T(A) \) acts non-trivially only on the subsystem \( T \), \( g_T(A) \) does not. Also, \( f_T(A) \) and \( g_T(A) \) are orthogonal with respect to the Hilbert-Schmidt product. Therefore,

\[
\|A\|_F^2 = \|f_T(A)\|_F^2 + \|g_T(A)\|_F^2.
\] (84)

With the above notations, a measurement \( M = \{M_i\}_{i \in \mathbb{N}} \) is \( k \)-local if and only if there exists some \( T \subseteq |n| \) with \( |T| = k \) such that \( M_i = f_T(M_i) \) for any \( i \in \mathbb{N} \). The following lemma says that if this condition is approximately fulfilled, then \( M \) is close to a \( |T| \)-local measurement:

**Lemma 3.** For any \( 0 < \delta < 1 \), if a measurement \( M = \{M_i\}_{i \in \mathbb{N}} \) on \( n \) qudits satisfies

\[
\sum_{i \in \mathbb{N}} \|f_T(M_i)\|_F^2 \geq D(1 - \delta^2),
\] (85)

where \( D = d^n \), then it is \( \delta \)-close to a \(|T|\)-local measurement.

**Proof:** See Appendix [3]  

With the help of lemma [3] we have the following result on testing the \( k \)-local measurements.

**Theorem 2.** The \( k \)-local measurements on \( n \) qudits can be \( \epsilon \)-tested with query complexity \( O(k \log(k/\epsilon)/\epsilon^2) \). Furthermore, the testing algorithm can be efficiently implemented.

**Proof:** Given a black box performing some unknown measurement \( M = \{M_i\}_{i \in \mathbb{N}} \) on \( n \) qudits, we run the following test on it:

**Algorithm 2.** Testing the \( k \)-local measurements

1. Let \( D = d^n, L = \left\lfloor \frac{1200k \ln(k/\epsilon)}{\epsilon^2} \right\rfloor \).
2. Prepare \( L \) copies of \( \Phi_D^+ = \frac{1}{\sqrt{D}} \sum_{(i_1, i_2, \ldots, i_n) \in \mathbb{Z}_d^n} |i_1, i_2, \ldots, i_n\rangle |i_1, i_2, \ldots, i_n\rangle \).
3. For \( j = 1, 2, \ldots, L \), perform \( M = \{M_i\}_{i \in \mathbb{N}} \) on the first subsystem of the \( j \)-th copy of \( \Phi_D^+ \), then, we use \( |\tilde{v}(\sigma_{x, z}\rangle) \rangle \) to denote the corresponding post-measurement state is in the basis \( \{|v(\sigma_{x, z}\rangle)\rangle_{x, z} \in \mathbb{Z}_d^n\} \). Let \( (x_j, z_j) \) be the outcome.
4. Accept \( M \) if and only if \( |\bigcup_{j=1}^L \text{ supp}(x_j, z_j)| \leq k \).

For correctness, it is sufficient to show:

1. if \( M \) is \( k \)-local, then this algorithm accepts \( M \) with certainty;
2. if \( \delta \)-close to some \( k \)-local measurement.

Before prove the two statements, observe that when we perform \( M = \{M_i\}_{i \in \mathbb{N}} \) on the first subsystem of \( |\Phi_D^+\rangle \), with probability \( p(M_i) \) we obtain outcome \( i \) and the corresponding post-measurement state is \( \tilde{v}(M_i) \). If this post-measurement state is measured in the basis \( \{|v(\sigma_{x, z})\rangle\rangle_{x, z} \in \mathbb{Z}_d^n\} \), then the outcome \( (x, z) \) occurs with probability \( Q_{x, z}(M_i) \). So, in step 3, the probability of \( (x_j, z_j) \) being from \( \Gamma_T \) is

\[
Q_{x, z}(M) := \sum_{i \in \mathbb{N}} p(M_i) Q_{x, z}(M_i) = \sum_{i \in \mathbb{N}} |\mu_{x, z}(M_i)|^2.
\] (86)

Hence, the probability of \( (x_j, z_j) \) being from \( \Gamma_T \) is

\[
\sum_{(x, z) \in \Gamma_T} |\mu_{x, z}(M)|^2 = \frac{1}{D} \sum_{i \in \mathbb{N}} \|f_T(M_i)\|_F^2.
\] (87)

This holds for any \( j = 1, 2, \ldots, L \).

Now let us prove statement (1). Suppose \( M \) is \( k \)-local, i.e. \( M_i \)'s satisfy Eq. \( (88) \) for some \( T \subseteq |n| \) with \(|T| = k \).
Then for any \( i \in \mathbb{N} \), \( \mu_{x,z}(M_i) \) is nonzero only for \((x, z) \in \Gamma_T \). This implies that in step 3 we have \((x_j, z_j) \in \Gamma_T \) for any \( j \). So \( \bigcup_{j=1}^L \text{supp}(x_j, z_j) \subseteq T \) and \( M \) is always accepted by algorithm 2.

Next, we prove statement (2). Suppose \( M \) is accepted by the algorithm with probability at least 1/3. Then we claim that there exists \( T \subseteq [n] \) with \(|T| = k \) such that

\[
\eta_T(M) \geq 1 - \varepsilon^2. \tag{88}
\]

Suppose on the contrary that for any such \( T \), \( \eta_T(M) < 1 - \varepsilon^2 \). Then \( M \) must be rejected by the algorithm with probability at least 0.9. The reason is as follows. The algorithm accepts \( M \) only if \((x_1, z_1, \ldots, x_L, z_L) \) satisfy \(|\bigcup_{j=1}^L \text{supp}(x_j, z_j)| \leq k \). For any such \((x_j, z_j) \)'s, we can find \( k \) numbers \( 1 \leq j_1 < j_2 < \cdots < j_k \leq L \) such that

\[
\bigcup_{j=1}^L \text{supp}(x_j, z_j) = \bigcup_{i=1}^k \text{supp}(x_{j_i}, z_{j_i}). \tag{89}
\]

(For each \( a \in \bigcup_{j=1}^L \text{supp}(x_j, z_j) \), we choose a \((x_{j_i}, z_{j_i}) \) such that \( \text{supp}(x_{j_i}, z_{j_i}) \) contains \( a \). Since there are at most \( k \) elements in \( \bigcup_{j=1}^L \text{supp}(x_j, z_j) \), \( k \) such \((x_{j_i}, z_{j_i}) \)'s are sufficient.) Now there can be \( \binom{k}{k} \) possibilities for \( j_1, j_2, \ldots, j_k \), and when the \((x_{j_i}, z_{j_i}) \)'s are determined, the total support \( W := \bigcup_{j=1}^L \text{supp}(x_j, z_j) \) is fixed, and then any \((x_j, z_j) \) for \( j \neq j_1, \ldots, j_k \) must be chosen from \( \Gamma_W \), with probability at most \( 1 - \varepsilon^2 \) by assumption. Thus, the total probability of getting a valid sequence of \((x_1, z_1, \ldots, x_L, z_L) \)'s is at most \( \binom{k}{k}(1 - \varepsilon^2)^{L-k} \leq L^k e^{-\varepsilon^2(L-k)} \leq 0.1 \) by our choice of \( L \).

Now, by Eqs. (57) and (68),

\[
\sum_{i \in [k]} \|f_T(M_i)\|_F^2 \geq D(1 - \varepsilon^2) \tag{90}
\]

for \(|T| = k \). Then, by lemma 8, \( M \) is \( \varepsilon \)-close to a \( k \)-local measurement.

Finally, algorithm 2 obviously has query complexity \( O(k \log(k/\varepsilon)/\varepsilon^2) \) as claimed. Moreover, besides querying the black box, it requires \( O(k \log(k/\varepsilon)/\varepsilon^2) \) quantum operations including: 1. preparing the state \( |\Phi_D^+\rangle \), which is equivalent to preparing the \( j \)-th and \((n + j)\)-th qudits in the \( d \)-dimensional maximally entangled state for \( j = 1, 2, \ldots, n \); 2. measuring a \( 2n \)-qudit state in the basis \( \{|v(\sigma_x, z)\rangle\}_{x,z \in \mathbb{Z}_d} \), which is equivalent to measuring the \( j \)-th and \((n + j)\)-th qubits in the basis \( \{|v(\sigma_x, z)\rangle\}_{x,z \in \mathbb{Z}_d} \) for \( j = 1, 2, \ldots, n \). In addition, the classical processing is also efficient. Thus, this algorithm can be efficiently implemented.

**V. TESTING THE PERMUTATION-INVARIANT MEASUREMENTS**

In this section, we consider testing a class of multi-qudit measurements possessing certain symmetry with respect to the permutations of the qudits. This kind of measurements have been widely used in many tasks such as quantum data compression [1,3], state estimation [33–35] and entanglement concentration [36,37]. Usually we perform such measurements on \( \rho^{\otimes n} \) to extract certain information from it, or to transform it into a better form, where \( \rho \) is some mixed state that we are interested in. For example, in quantum data compression, suppose an i.i.d. quantum source produces a \( d \)-dimensional mixed state \( \rho \) each time. We first collect \( n \) copies of this state, and then compress \( \rho^{\otimes n} \) into a smaller Hilbert space whose dimension is approximately \( 2^n S(\rho) \). Here \( S(\rho) \) is the von Neumann entropy of \( \rho \), i.e. if \( \rho \) has the spectral decomposition

\[
\rho = \sum_{x=0}^{d-1} p(x)|x\rangle \langle x|, \tag{91}
\]

then

\[
S(\rho) = -\sum_{x=0}^{d-1} p(x) \log_2 p(x). \tag{92}
\]

The typical method [1] is that we perform a projective measurement \{\( \Pi_x, I - \Pi_x \)\} on \( \rho^{\otimes n} \), and if the outcome corresponds to \( \Pi_x \), then we succeed. Here \( \Pi_x \) is the projection operator onto the so-called “\( e \)-typical subspace”. This subspace is spanned by all \(|x_1, x_2, \ldots, x_n\rangle \)'s satisfying

\[
\left| \frac{1}{n} \log_2 \left( \frac{1}{p(x_1)p(x_2)\cdots p(x_n)} \right) - S(\rho) \right| \leq \epsilon. \tag{93}
\]

Note that this condition depends only on how many \( x_i \)'s are \( 0, 1, \ldots, d - 1 \) respectively. Therefore, this \( e \)-typical subspace is invariant under any permutation of the \( n \) qudits. In other words, the measurement \{\( \Pi_x, I - \Pi_x \)\} treats every qudit equally. This property is shared by many other measurements, including those used in other quantum data compression methods [33,34], state estimation [33,35] and entanglement concentration [36,37]. Thus, as a preliminary step of testing these measurements, we can test whether an unknown measurement is permutation-invariant in the first place.

Formally, let \( S_n \) be the symmetric group on \( n \) elements. For any \( \tau \in S_n \), it can be viewed as a unitary operation on \( n \) qudits as follows:

\[
\tau|\phi_1, \phi_2, \ldots, \phi_n\rangle = |\phi_{\tau^{-1}(1)}, \phi_{\tau^{-1}(2)}, \ldots, \phi_{\tau^{-1}(n)}\rangle, \tag{94}
\]

where the \(|\phi_i\rangle \)'s are arbitrary qudit states. We say an operator \( A \in \mathcal{B}(\mathcal{H}^{\otimes n}_d) \) is permutation-invariant if

\[
A = \tau A \tau^\dagger, \quad \forall \tau \in S_n. \tag{95}
\]

Note that this condition is equivalent to

\[
A = \sum_{k} \mu_k E^k_{\tau^\dagger}, \tag{96}
\]
for some $\mu_k \in \mathbb{C}$ and $E_k \in \mathcal{B}(\mathcal{H}_d)$. A measurement $M = \{M_i\}_{i \in \mathbb{N}}$ on $n$ qudits is said to be permutation-invariant if $M_i$ is permutation-invariant for each $i \in \mathbb{N}$. In this case, if we first apply a permutation to the qudits, then perform $M$, and at last apply the inverse permutation, then the effect is equivalent to directly performing $M$.

A. Representation Theory Background

Our algorithm for testing the permutation-invariant measurements depends crucially on the following results from the representation theory of symmetric group. Let $d, n$ be arbitrary integers. Let

$$I_{d,n} = \{ \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_d) : \lambda_i \in \mathbb{Z}, \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_d \geq 0, \sum_{i=1}^{d} \lambda_i = n. \}$$

be the set of partitions of $n$ into at most $d$ parts. A Young diagram of shape $\lambda \in I_{d,n}$ (denoted by $Y(\lambda)$) is a finite collection of boxes arranged in left-justified rows, with the $i$-th row having $\lambda_i$ boxes for $i = 1, 2, \ldots, d$. For example, Fig. 1 illustrates the Young diagram of shape $(5, 3, 1)$.

![Fig. 1: The Young diagram of shape $(5, 3, 1)$.](image1)

Each $\lambda \in I_{d,n}$ (or the Young diagram $Y(\lambda)$) corresponds to an irreducible representation $V_\lambda$ of the symmetric group $S_n$, and also corresponds to an irreducible representation $W_{\lambda,d}$ of the general linear group $GL(d)$ of $d \times d$ invertible matrices. Furthermore, we can calculate the dimension of $V_\lambda$ and $W_{\lambda,d}$ as follows. For a box in the $i$-th row and $j$-th column of a Young diagram $Y(\lambda)$, we define its hook length $\text{hook}(i, j)$ to be the number of boxes that are in the same row to the right of it or in the same column below it plus one. For example, Fig. 2 illustrates the hook lengths of each box in the Young diagram of shape $(5, 3, 1)$.

![Fig. 2: The hook lengths of each box in the Young diagram of shape $(5, 3, 1)$](image2)

Then the dimension of $V_\lambda$ is given by

$$v_\lambda := \prod_{(i,j) \in Y(\lambda)} \frac{n!}{\text{hook}(i,j)},$$

while the dimension of $W_{\lambda,d}$ is given by

$$w_{\lambda,d} := \prod_{(i,j) \in Y(\lambda)} \frac{d + j - i}{\text{hook}(i,j)}.$$  \hspace{1cm} (99)

For more details about $V_\lambda$ and $W_{\lambda,d}$, please see Refs. \[43\] \[44\].

The Schur-Weyl duality [45–47] states that there exists an orthonormal basis for $\mathcal{H}_{d}^{\otimes n}$, which is called the Schur basis and labelled by $|\lambda\rangle|a_\lambda\rangle|b_\lambda\rangle_{\text{Sch}}$ for $\lambda \in I_{d,n}$, $0 \leq a_\lambda \leq w_{\lambda,d} - 1$, $0 \leq b_\lambda \leq v_\lambda - 1$, such that for any $\tau \in S_n$ and $E \in GL(d)$ we have

$$\tau|\lambda\rangle|a_\lambda\rangle|b_\lambda\rangle_{\text{Sch}} = |\lambda\rangle|a_\lambda\rangle(V_\lambda(\tau)|b_\lambda\rangle)_{\text{Sch}}, \quad (100)$$

$$E^{\otimes n}|\lambda\rangle|a_\lambda\rangle|b_\lambda\rangle_{\text{Sch}} = |\lambda\rangle|a_\lambda\rangle(W_{\lambda,d}(E)|a_\lambda\rangle)_{\text{Sch}}.$$  \hspace{1cm} (101)

In other words, the space $\mathcal{H}_d^{\otimes n}$ can be decomposed as

$$\mathcal{H}_d^{\otimes n} = \bigoplus_{\lambda \in I_{d,n}} (W_{\lambda,d} \otimes V_\lambda)_{\text{Sch}},$$

where $W_{\lambda,d}$ is $w_{\lambda,d}$-dimensional and $V_\lambda$ is $v_\lambda$-dimensional, such that any $\tau \in S_n$ and $E \in GL(d)$ satisfy

$$\tau = \bigoplus_{\lambda \in I_{d,n}} (I \otimes V_\lambda(\tau))_{\text{Sch}},$$  \hspace{1cm} (103)

$$E^{\otimes n} = \bigoplus_{\lambda \in I_{d,n}} (W_{\lambda,d}(E) \otimes I)_{\text{Sch}}.$$  \hspace{1cm} (104)

Since $W_{\lambda,d}$ is a continuous mapping from $GL(d)$ to $GL(w_{\lambda,d})$, and $GL(d)$ is a dense subset of the $d \times d$ matrices, we can naturally extend $W_{\lambda,d}$ to all $d \times d$ matrices. Then for any permutation-invariant $A \in \mathcal{B}(\mathcal{H}_d^{\otimes n})$, if

$$A = \sum_k \mu_k E_k^{\otimes k},$$

let

$$\hat{A}_\lambda := \sum_k \mu_k W_{\lambda,d}(E_k).$$

Then

$$A|\lambda\rangle|a_\lambda\rangle|b_\lambda\rangle_{\text{Sch}} = |\lambda\rangle|\hat{A}_\lambda|a_\lambda\rangle|b_\lambda\rangle_{\text{Sch}},$$  \hspace{1cm} (107)

and

$$A = \bigoplus_{\lambda \in I_{d,n}} (\hat{A}_\lambda \otimes I)_{\text{Sch}}.$$  \hspace{1cm} (108)

On the other hand, it is easy to see that any $A \in \mathcal{B}(\mathcal{H}_d^{\otimes n})$ satisfying Eq. (108) commutes with any $\tau \in S_n$ (which satisfies Eq. (109)), since they act on disjoint subsystems.
Hence, \( A \in \mathcal{B}(\mathcal{H}_d^{\otimes n}) \) is permutation-invariant if and only if it has the block-diagonal form in Eq. (108) with respect to the Schur basis.

Finally, let \( \Lambda_{d,n} := \left\{ (\lambda, a_\lambda, b_\lambda) : \lambda \in \mathcal{I}_{d,n}, 0 \leq a_\lambda \leq w_{\lambda,d} - 1, 0 \leq b_\lambda \leq v_{\lambda} - 1 \right\} \), and fix a bijection \( h : \mathbb{Z}_d^n \rightarrow \mathcal{I}_{d,n} \). Then the standard basis state \(|i_1, i_2, \ldots, i_n\rangle\) can also be labelled as \(|\lambda, a_\lambda, b_\lambda\rangle\), provided that \( h(i_1, i_2, \ldots, i_n) = (\lambda, a_\lambda, b_\lambda) \). We use \( \mathcal{U}_{\text{Sch}} \) to denote the unitary operation that converts the standard basis to the Schur basis, and call it the Schur transform \([48]\).

Namely, if \( h(i_1, i_2, \ldots, i_n) = (\lambda, a_\lambda, b_\lambda) \), then
\[
\mathcal{U}_{\text{Sch}}|i_1, i_2, \ldots, i_n\rangle = |\lambda, a_\lambda, b_\lambda\rangle_{\text{Sch}}.
\] (109)

Its inverse \( \mathcal{U}^{-1}_{\text{Sch}} \) is called the inverse Schur transform. Both \( \mathcal{U}_{\text{Sch}} \) and \( \mathcal{U}^{-1}_{\text{Sch}} \) can be (approximately) implemented with time complexity polynomial in \( n, d \) and \( \log(1/\epsilon) \), where \( \epsilon \) is the accuracy parameter \([48]\).

### B. Our Result on Testing the Permutation-Invariant Measurements

It will be convenient to establish the following notations. For any \( A \in \mathcal{B}(\mathcal{H}_d^{\otimes n}) \), it can be written as
\[
A = \sum_{\lambda, \lambda' \in \mathcal{I}_{d,n}} \langle \lambda | \lambda' \rangle \otimes A_{\lambda, \lambda'}_{\text{Sch}}
\] (110)
for some \( A_{\lambda, \lambda'} \)'s. We are particularly interested in the diagonal term \( A_{\lambda, \lambda} \in \mathcal{B}(\mathcal{W}_\lambda \otimes \mathcal{V}_\lambda) \). Let \( \{ g_{\lambda, j} \}_{0 \leq j \leq v_\lambda - 1} \) be the generalized Pauli operators on \( \mathcal{V}_\lambda \) such that \( g_{\lambda, 0} = I \). Then we can expand \( A_{\lambda, \lambda} \) as
\[
A_{\lambda, \lambda} = \mathcal{A}_\lambda \otimes I + \sum_{j \neq 0} \mathcal{A}_{\lambda, j} \otimes g_{\lambda, j}
\] (111)
for some \( \mathcal{A}_\lambda \) and \( \mathcal{A}_{\lambda, j} \)'s. So \( A \) can be written as the sum of three terms:
\[
A = (\mathcal{A} + \mathcal{A} + \mathcal{A})_{\text{Sch}},
\] (112)
where
\[
\mathcal{A} = \bigoplus_{\lambda \in \mathcal{I}_{d,n}} \mathcal{A}_\lambda \otimes I,
\] (113)
\[
\mathcal{A} = \bigoplus_{\lambda \in \mathcal{I}_{d,n}} \left( \sum_{j \neq 0} \mathcal{A}_{\lambda, j} \otimes g_{\lambda, j} \right),
\] (114)
\[
\mathcal{A} = \sum_{\lambda \neq \lambda'} |\lambda \rangle \langle \lambda' | \otimes A_{\lambda, \lambda'}.
\] (115)

While \( \mathcal{A} \) is permutation-invariant, \( \mathcal{A} \) and \( \mathcal{A} \) are not. Furthermore, \( \mathcal{A} \), \( \mathcal{A} \), and \( \mathcal{A} \) are mutually orthogonal with respect to the Hilbert-Schmidt product. Thus,
\[
\|A\|_F^2 = \|\mathcal{A}\|_F^2 + \|\mathcal{A}\|_F^2 + \|\mathcal{A}\|_F^2.
\] (116)

With these notations, a measurement \( M = \{ M_i \}_{i \in \mathbb{N}} \) is permutation-invariant if and only if \( M_i = (\mathcal{M}_i)_{\text{Sch}} \) for any \( i \in \mathbb{N} \). The next lemma shows that if this is approximately true, then \( M \) is close to a permutation-invariant measurement.

**Lemma 4.** For any \( 0 < \delta < 1 \), if a measurement \( M = \{ M_i \}_{i \in \mathbb{N}} \) on \( n \) qubits satisfies
\[
\sum_{i \in \mathbb{N}} \| \mathcal{M}_i \|_2^2 \geq D(1 - \delta^2),
\] (117)
where \( D = d^n \), then \( M \) is \( \delta \)-close to a permutation-invariant measurement.

**Proof:** See Appendix [C].

With the help of lemma [4] we obtain the following result on testing the permutation-invariant measurements.

**Theorem 3.** The permutation-invariant measurements on \( n \) qubits can be \( \epsilon \)-tested with query complexity \( O(1/\epsilon^2) \). Furthermore, the testing algorithm can be efficiently implemented.

**Proof:** Given a black box performing some unknown measurement \( M = \{ M_i \}_{i \in \mathbb{N}} \) on \( n \) qubits, we run the following test on it:

**Algorithm 3** Testing the permutation-invariant measurements

1. Let \( D = d^n \), \( L = \left\lceil \frac{5}{\epsilon^2} \right\rceil \).
2. Repeat the following test \( L \) times:
   (a) Prepare
   \[
   \Phi_B^+ = \frac{1}{\sqrt{D}} \sum_{(i_1, \ldots, i_n) \in \mathbb{Z}_d^n} |i_1, i_2, \ldots, i_n\rangle |i_1, i_2, \ldots, i_n\rangle.
   \]
   (b) Perform the Schur transform on the first subsystem of \( \Phi_B^+ \).
   (c) Perform \( M \) on the first subsystem.
   (d) No matter which measurement outcome occurs in step (c), perform the inverse Schur transform on the first subsystem.
   (e) Re-label the standard basis of both subsystems as \( |\lambda, a_\lambda, b_\lambda\rangle \)'s via the bijection \( h : \mathbb{Z}_d^n \rightarrow \mathcal{I}_{d,n} \).
   (f) Measure the two \(|\lambda\rangle\) registers. If the two outcomes are not equal, then reject \( M \) and quit.
   (g) Measure the two \(|b_\lambda\rangle\) registers in the basis \( \{ |g_{\lambda, j} \rangle \}_{0 \leq j \leq v_\lambda - 1} \). If the outcome corresponds to \( j \neq 0 \), then reject \( M \) and quit.
3. Now \( M \) has passed all the above tests. Accept \( M \).

For correctness, we claim that this algorithm accepts \( M \) with probability \( (\frac{1}{D} \sum_{i \in \mathbb{N}} \| \mathcal{M}_i \|_2^2)^L \). Assuming this is true, then:

(1) if \( M \) is permutation-invariant, then \( M_i = (\mathcal{M}_i)_{\text{Sch}} \) for any \( i \in \mathbb{N} \). Then by \( \sum_{i \in \mathbb{N}} M_i M_i = I \), we obtain
\[
\sum_{i \in \mathbb{N}} \| \mathcal{M}_i \|_2^2 = \sum_{i \in \mathbb{N}} \| M_i \|_2^2 = D,
\] (118)
which implies that algorithm $\mathcal{A}$ accepts $M$ with certainty; (2) on the other hand, if $M$ is accepted by algorithm $\mathcal{B}$ with probability at least $1/3$, then by our choice of $L$ we must have

$$\sum_{i \in \mathbb{N}} \| \tilde{M}_i \|^2_F \geq D(1 - e^2)$$

(119)

(because if otherwise, then the probability of $M$ being accepted is at most $(1 - e^2)L \leq e^{-e^2L} < 0.1$, contradicting our assumption). Then, by lemma 4, $M$ must have probability $M$ which implies that algorithm 3 accepts

where

In the first step we have switched the order of the $j$th and $\lambda$th qudits in the $d$-dimensional maximally entangled state; (2) quantum Schur transform and its inverse, which can be efficiently implemented; (3) measuring the two $|b\rangle$ registers in the basis $\{|v_i\rangle\}_{0 \leq j \leq N^2 - 1}$, which can be accomplished efficiently with the circuit in Fig. 3.

Note that $\tilde{M}_i$ changes $|\lambda\rangle$ and $\tilde{M}_i$ changes $|b\rangle$. Only $\tilde{M}_i$ leaves both $|\lambda\rangle$ and $|b\rangle$ intact. In step 2.(f), we do not reject $M$ only if the measurements on the two $|\lambda\rangle$ registers yield the same outcome. In this case, assuming the outcome is $\lambda$, then the unnormalized state of the rest four registers becomes

$$\tilde{M}_i |\lambda\rangle |a\rangle |b\rangle \quad |\lambda\rangle (\tilde{M}_i |\lambda\rangle |a\rangle |b\rangle),$$

(121)

$$\tilde{M}_i |\lambda\rangle |a\rangle |b\rangle = \sum_{j \neq 0} |\lambda\rangle (\tilde{M}_i |\lambda,j\rangle |a\rangle |b\rangle),$$

(122)

$$\tilde{M}_i |\lambda\rangle |a\rangle |b\rangle = \sum_{\lambda' \neq \lambda} |\lambda'\rangle (\tilde{M}_i |\lambda'\rangle |a\rangle |b\rangle).$$

(123)

Summing this probability over all $\lambda$’s and $\lambda$’s, we know that $M$ passes one iteration of step 2 with probability

$$\sum_{i \in \mathbb{N}} \sum_{\lambda} \frac{v}{D} \| (\tilde{M}_i) |\lambda\rangle \|^2_F = \frac{1}{D} \sum_{i \in \mathbb{N}} \| \tilde{M}_i \|^2_F,$$

(126)

as claimed.

Finally, this algorithm obviously has query complexity $O(1/e^6)$. Furthermore, besides querying the black box, it needs $\text{poly}(n, 1/e)$ number of quantum operations including: (1) preparing the state $|\Phi_D\rangle$, which is equivalent to preparing the $j$th and $(n + j)$th qudits in the $d$-dimensional maximally entangled state; (2) quantum Schur transform and its inverse, which can be efficiently implemented; (3) measuring the two $|b\rangle$ registers in the basis $\{|v_i\rangle\}_{0 \leq j \leq N^2 - 1}$, which can be accomplished efficiently with the circuit in Fig. 3.

In addition, the classical processing is also easy. So algorithm $\mathcal{B}$ can be efficiently implemented. 

VI. TESTING ANY FINITE SET OF MEASUREMENTS

So far we have studied testing three particular classes of quantum measurements. In this section, we present a general result about property testing of quantum measurements. Specifically, we give an algorithm that can test any finite set of measurements on any finite-dimensional system.

Before stating our result, it is helpful to consider the following question. Suppose $M = \{M_i\}_{i \in \mathbb{N}}$ and $N = \{N_i\}_{i \in \mathbb{N}}$ are two measurements on a $D$-dimensional system. If a black box performs one of them, how do we know which one it really implements? We solve this problem by repeating the following experiment sufficiently many times: each time we prepare a copy of $|\Phi_D\rangle$ and apply the unknown measurement to its first subsystem, obtaining an outcome and post-measurement state. If the unknown measurement is $M$ (or $N$), then the outcome $i$ occurs with probability $p(M_i)$ (or $p(N_i)$) and the corresponding post-measurement state should be $|\tilde{b}(M_i)\rangle$ (or $|\tilde{b}(N_i)\rangle$). We claim that as long as $\Delta(M, N)$ is large, we do not need to repeat this experiment many times.
to successfully identify the unknown measurement. The reason is as follows.

Let $p(M) := (p(M_i))_{i \in \mathbb{N}}$ and $p(N) := (p(N_i))_{i \in \mathbb{N}}$ be the distributions of outcomes for $M$ and $N$ respectively. Consider their variational distance

$$\Delta(p(M), p(N)) = \frac{1}{2} \sum_{i \in \mathbb{N}} |p(M_i) - p(N_i)|. \quad (127)$$

By Eqs. (9), (27), (32) and the Cauchy-Schwarz inequality,

$$\Delta^2(M, N) = 1 - \frac{1}{2} \sum_{i \in \mathbb{N}} (|p(M_i)|^2 + |p(N_i)|^2 - 2p(M_i)p(N_i)) \geq 1 - \frac{1}{2} \sum_{i \in \mathbb{N}} \sqrt{p(M_i)p(N_i)} \geq 1 - \sqrt{F(p(M), p(N))} \geq \frac{1}{2} D^2(p(M), p(N)).$$

Thus

$$\Delta(M, N) \geq \frac{1}{\sqrt{2}} D(p(M), p(N)). \quad (129)$$

This means that if $D(p(M), p(N))$ is large, then $\Delta(M, N)$ is also large. In this case, we can simply distinguish $M$ and $N$ from the outcome statistics.

But what if $\Delta(M, N)$ is large but $D(p(M), p(N))$ is small? In this case, we need to use the post-measurement states. Specifically, suppose $|M|, |N| \leq k$. We repeat the above experiment $L$ times, and suppose the outcome $i$ occurs $L_i$ times for $i = 1, 2, \ldots, k$. Note that with high probability we have $L_i \approx p(M_i)L \approx p(N_i)L$ for each $i$. Then let $L = (L_1, L_2, \ldots, L_k)$ and

$$\chi_L(M) := \bigotimes_{i = 1}^k (\tilde{\psi}(M_i) \otimes L_i), \quad (130)$$

$$\chi_L(N) := \bigotimes_{i = 1}^k (\tilde{\psi}(N_i) \otimes L_i). \quad (131)$$

The following lemma says that if $\Delta(M, N)$ is large, then with high probability $\chi_L(M)$ and $\chi_L(N)$ have small overlap for some small $L$, and thus can be easily distinguished.

**Lemma 5.** Let $L = (L_1, L_2, \ldots, L_k)$ be a set of non-negative integers with $\sum_{i = 1}^k L_i = L$. Suppose $M = \{M_1, M_2, \ldots, M_k\}$ and $N = \{N_1, N_2, \ldots, N_k\}$ are two measurements such that $\Delta(M, N) \geq \delta$ for some $0 < \delta < 1$. If for any $i$ such that $L_i \geq 0.1 \delta^2 L/k$, we have

$$p(M_i), p(N_i) \geq (1 - 0.1 \delta^2) \frac{L_i}{L}, \quad (132)$$

then

$$|\langle \chi_L(M) | \chi_L(N) \rangle| \leq (1 - 0.6 \delta^2)^L. \quad (133)$$

**Proof:** See Appendix $[\text{D}]$.

Now let us return to the problem of testing any finite set of measurements on a finite-dimensional system. Suppose $S = \{M^{(1)}, M^{(2)}, \ldots, M^{(m)}\}$ is the set of measurements to be tested, where $M^{(i)} = \{M_j^{(i)}\}_{j \in \mathbb{N}}$. Given an unknown measurement $M$, we want to know whether $M$ is in $S$ or far away from $S$. Our basic idea is to repeat the above experiment sufficiently many times. If $M$ is from $S$, then with high probability the outcome statistics is well-behaved and the post-measurement states lie in certain subspace. On the other hand, if $M$ is far from any $M^{(i)}$, then we divide $S$ into two subsets: for one subset, the outcome statistics are quite different for $M$ and any $M^{(i)}$ in it, so we can easily distinguish $M$ from this subset; for the other subset, the outcome statistics are similar for $M$ and any $M^{(i)}$ in it, so we must utilize the post-measurement states. Lemma 5 says that $\chi_L(M)$ and $\chi_L(M^{(i)})$ are almost orthogonal for some small $L$. Then by the following lemma, $\chi_L(M)$ has a small projection onto the subspace spanned by these $\chi_L(M^{(i)})$’s. So by a projective measurement, we can get to know whether $M$ is in $S$ or far away from $S$.

**Lemma 6.** Suppose $|\phi_1, |\phi_2, \ldots, |\phi_m\rangle \in \mathcal{H}_D$ (where $D$ is arbitrary) satisfy $|\langle \psi | \phi_i \rangle| \leq 1/(5m)$ for any $i$ and $|\langle \phi_i | \phi_j \rangle| \leq 1/(5m)$ for any $i \neq j$. Let $\Pi$ be the projection operator onto the subspace spanned by $|\phi_1, |\phi_2, \ldots, |\phi_m\rangle$. Then $\langle \psi | \Pi | \psi \rangle \leq 0.1$.

**Proof:** See Appendix $[\text{E}]$.

Now we formally state our result on testing any finite set of measurements on any finite-dimensional system.

**Theorem 4.** Suppose $S = \{M^{(1)}, M^{(2)}, \ldots, M^{(m)}\}$ is a set of measurements on a $D$-dimensional system, where $M^{(i)} = \{M_1^{(i)}, M_2^{(i)}, \ldots, M_k^{(i)}\}$ has at most $k$ possible outcomes for any $i$, and $\Delta(M^{(i)}, M^{(j)}) \geq \gamma$ for any $i \neq j$, for some $0 < \gamma < 1$. Then $S$ can be $\epsilon$-tested with query complexity $O(\max(k^2 \log(k)/\epsilon^2, \log(m)/\epsilon^2))$ where $a = \min\{\epsilon, \gamma\}$.

**Proof:** Given a black box performing some unknown measurement $M = \{M_j\}_{j \in \mathbb{N}}$, we run the following test on it:
Algorithm 4 Testing $\mathcal{S} = \{M^{(1)}, M^{(2)}, \ldots, M^{(m)}\}$

1. Let $L = \max \left\{ \frac{5000k^2 \log(20k)}{a^s}, \frac{2\ln(5m)}{a^2} \right\}$.

2. Prepare $L$ copies of $|\Phi^+\rangle = \frac{1}{\sqrt{D}} \sum_{i=0}^{D-1} |i\rangle |i\rangle$.

3. Perform $M = \{M_i\}_{i \in \mathbb{N}}$ on the first subsystem of each copy of $|\Phi^+\rangle$. Let $L_j$ be the number of occurrences of outcome $j$ for any $j \in \mathbb{N}$, and let $L = (L_1, L_2, \ldots, L_k)$.

4. If $L_j > 0$ for some $j > k$, then reject $M$ and quit; otherwise, define $\mathcal{T} \subseteq \mathcal{S}$ as follows: $M^{(i)} \in \mathcal{T}$ if and only if for any $j$ such that $L_j \geq 0.1a^2L/k$, we have $p(M^{(i)}) \geq (1 - 0.1a^2)L_j/L$. If $\mathcal{T} = \emptyset$, then reject $M$ and quit.

5. Now suppose $\mathcal{T} = \{M^{(i_1)}, M^{(i_2)}, \ldots, M^{(i_t)}\}$ for some $1 \leq t \leq m$. Let $\Pi$ be the projection operator onto the subspace spanned by $\chi_L(M^{(i_1)}), \chi_L(M^{(i_2)}), \ldots, \chi_L(M^{(i_t)})$. Perform the projective measurement $\{\Pi, I - \Pi\}$ on the state $\chi_L(M)$. If the outcome corresponds to $\Pi$, then accept $M$; otherwise, reject $M$.

Before prove the correctness of this algorithm, observe that, in step 3, we should obtain outcome $j$ with probability $p(M_j)$, for any $j$. Then by Eq. (29) and our choice of $L$,

$$\Pr \left[ \left| \frac{L_j}{L} - p(M_j) \right| \leq \frac{a^4}{100k} \right] \geq 1 - \frac{1}{10k}. \quad (134)$$

for any $j$. Thus, with probability at least 0.9, for any $j$ satisfying $L_j/L \geq 0.1a^2/k$, we have

$$p(M_j) \geq \frac{L_j}{L} - \frac{a^4}{100k} \geq (1 - 0.1a^2)\frac{L_j}{L}. \quad (135)$$

Now suppose that $M = M^{(i)}$ for some $i$. By the above observation, $M^{(i)}$ is in the set $\mathcal{T}$ with probability at least 0.9. Then, $\chi_L(M) = \chi_L(M^{(i)})$ is obviously in the subspace spanned by the $\chi_L(M^{(i)})$'s and hence the measurement in step 5 always produces the outcome corresponding to $\Pi$. Therefore, $M = M^{(i)}$ is accepted by algorithm 4 with probability at least 0.9.

On the other hand, suppose $M$ is accepted by algorithm 4 with probability at least 1/3. We need to prove that $M$ is $c$-close to some $M^{(i)}$. Suppose on the contrary that this is not true. Then, still by the above observation, with probability at least 0.9, for any $j$ satisfying $L_j/L \geq 0.1a^2/k$, we have $p(M_j) \geq (1 - 0.1a^2)L_j/L$. Furthermore, by the definition of $\mathcal{T}$, any $M^{(i)} \in \mathcal{T}$ also satisfies $p(M^{(i)}) \geq (1 - 0.1a^2)L_j/L$ for such $j$'s. Then by lemma 5 and our choice of $L$, we obtain

$$|\langle \chi_L(M) | \chi_L(M^{(i)}) \rangle| \leq (1 - 0.6a^2)L \leq e^{-0.6a^2L} \leq \frac{1}{5m}. \quad (136)$$

for any $l$, and similarly

$$|\langle \chi_L(M^{(i)}) | \chi_L(M^{(i_2)}) \rangle| \leq \frac{1}{5m}. \quad (137)$$

for any $l \neq l'$. Thus, by lemma 6 in step 5 we obtain the outcome corresponding to $\Pi$ with probability at most 0.1. Overall, $M$ can be accepted by algorithm 4 with probability at most 0.2, contradicting our assumption.

Finally, the query complexity of algorithm 4 is $O(\max\{k^2 \log(k)/a^s, \log(m)/a^2\})$ as claimed. In particular, if the system consists of $n$ qubits, then the query complexity is polynomial (in $n$) as long as $k$ and $1/a$ are at most polynomial and $m$ is at most exponential (in $n$).

In general, algorithm 4 may be not efficiently implementable. But from an information-theoretic point of view, theorem 3 shows that we do not need to query the black box too many times to know whether it belongs to some finite set or is far away from this set, as long as the set is not too large (i.e. $m$ is not too large), the measurements in the set are well-separated (i.e. $\gamma$ is not too small), and any measurement in the set does not have too many possible outcomes (i.e. $k$ is not too large).

Finally, let us apply theorem 3 to the set of stabilizer measurements. This set contains $4^n$ elements. The distance between any two of them is $1/2$, i.e. $\Delta(P(a, b), P(c, d)) = 1/2$ for any $(a, b) \neq (c, d)$. Furthermore, each stabilizer measurement has only 2 possible outcomes. Thus, by theorem 3 the stabilizer measurements can be $\epsilon$-tested with query complexity $O(\max\{1/e^8, n/\epsilon^2\})$. This query complexity is only slightly worse than the $O(1/\epsilon^4)$ given by theorem 3. This example shows that although algorithm 4 uses only certain distance information about the measurements to be tested, it still can achieve a quite good efficiency.

VII. ESTIMATING THE DISTANCE OF TWO UNKNOWN MEASUREMENTS

In the previous sections, we have studied testing several properties of a single measurement. In this section, we consider a different scenario. Suppose we are given two black-box measurement devices, how do we know whether they perform the same measurement? And if not, how do we estimate their distance? We will give an efficient algorithm for this task. Surprisingly, its query complexity does not depend on the dimension of the system, but depends only on the proximity parameter and the number of possible outcomes for the two measurements.

Our basic idea is as follows. Suppose $M = \{M_i\}_{1 \leq i \leq k}$ and $N = \{N_i\}_{1 \leq i \leq k}$ are two measurements with at most $k$ possible outcomes. Since

$$\Delta^2(M, N) = 1 - \sum_{i=1}^{k} |\langle v(M_i) | v(N_i) \rangle|. \quad (138)$$

So we can estimate $\Delta^2(M, N)$ by estimating each $|\langle v(M_i) | v(N_i) \rangle|$. Note that

$$|\langle v(M_i) | v(N_i) \rangle| = \sqrt{p(M_i)p(N_i)} |\langle \tilde{v}(M_i) | \tilde{v}(N_i) \rangle|. \quad (139)$$
Thus, we can estimate \(|\langle v(M_i) \rangle| v(N_i))\) by estimating \(p(M_i), p(N_i)\) and \(|\langle \tilde{v}(M_i) \rangle| v(N_i))\). Recall that \(p(M_i)\) (or \(p(N_i)\)) is the probability of obtaining outcome \(i\) when we perform \(M\) (or \(N\)) on the first subsystem of \(|\Phi^{+}_{D}\rangle\), and \(|\tilde{v}(M_i)\rangle\) (or \(|\tilde{v}(N_i)\rangle\)) is the corresponding post-measurement state. So we repeat this experiment many times. From the outcome statistics, we can estimate \(p(M_i)\) and \(p(N_i)\) with good precision. As to \(|\langle \tilde{v}(M_i) \rangle| v(N_i))\), we can estimate it by performing the swap test on the states \(|\tilde{v}(M_i)\rangle\) and \(|\tilde{v}(N_i)\rangle\). In fact, the swap test can be used to estimate the overlap between any two pure states, as stated by the following lemma.

**Lemma 7.** Given \(O(\log(1/\delta)/\epsilon^4)\) copies of \(|\phi\rangle\) and \(|\psi\rangle\), we can estimate \(|\langle \phi \mid \psi \rangle|\) with precision \(\epsilon\) and confidence \(1 - \delta\). Furthermore, the estimation algorithm can be efficiently implemented.

**Proof:** Given any \(|\phi\rangle\) and \(|\psi\rangle\), the swap test is the standard procedure to estimate \(|\langle \phi \mid \psi \rangle|\). It works as follows: first, we prepare a qubit in the state \(|+\rangle = 1/\sqrt{2}(|0\rangle + |1\rangle)\); then, we perform a controlled-swap gate on \(|\phi\rangle\) and \(|\psi\rangle\), using \(|+\rangle\) as the control qubit (a swap gate is the operation \(|\phi\rangle\langle \psi | \rightarrow |\psi\rangle\langle \phi |\)); finally, we apply a Hadamard gate on the control qubit and measure it in the standard basis. The circuit for this procedure is illustrated in Fig. 4.

![FIG. 4: The swap test](image)

It is easy to see that the final state before the measurement is

\[
\frac{1}{2}(|0\rangle(|\phi\rangle\langle \psi |) + |1\rangle(|\psi\rangle\langle \phi |)) + \frac{1}{2}(|1\rangle(|\phi\rangle\langle \psi |) - |\psi\rangle\langle \phi |)).
\]

Thus, the measurement on the first qubit yields outcome 0 with the probability \((1 + |\langle \phi \mid \psi \rangle|^2)/2\).

Now consider the following estimation algorithm:

**Algorithm 5** Estimating \(|\langle \phi \mid \psi \rangle|\)

1. Assume we have \(L = 2\log(2/\delta)/\epsilon^4\) copies of \(|\phi\rangle\) and \(|\psi\rangle\). We run the swap test on each pair of \(|\phi\rangle\) and \(|\psi\rangle\). Let \(\tilde{p}_0\) be the fraction of outcome 0 among the outcomes obtained. Then return \(\sqrt{2\tilde{p}_0 - 1}\) as the estimate of \(|\langle \phi \mid \psi \rangle|\).

By Eq. (23) and our choice of \(L\), \(\tilde{p}_0\) will be \(\epsilon^2/2\)-close to \((1 + |\langle \phi \mid \psi \rangle|^2)/2\) with probability at least \(1 - \delta\). This implies that \(2\tilde{p}_0 - 1\) is \(\epsilon^2\)-close to \(|\langle \phi \mid \psi \rangle|^2\) with probability at least \(1 - \delta\), and hence \(\sqrt{2\tilde{p}_0 - 1}\) is \(\epsilon\)-close to \(|\langle \phi \mid \psi \rangle|\) with probability at least \(1 - \delta\) (note that \(|a - b|^2 \leq |a|^2 - |b|^2\) for any \(a, b \in [0, 1]\)).

The swap test requires three kinds of quantum operations including the Hadamard gate, controlled-swap gate and the measurement in the standard basis. If \(|\phi\rangle\) and \(|\psi\rangle\) are \(n\)-qubit states, then the controlled-swap gate can be implemented with \(O(n)\) basic quantum gates. Algorithm 5 repeats the swap test \(O(\log(1/\delta)/\epsilon^4)\) times. So it can be efficiently implemented.

So, if we have sufficiently many copies of \(|\tilde{v}(M_i)\rangle\) and \(|\tilde{v}(N_i)\rangle\), then we can estimate their overlap with good precision. But what if \(p(M_i)\) or \(p(N_i)\) is so small that the outcome \(i\) almost never occurs, unless we repeat the experiment extremely many times? Note that for such \(i\)’s, \(|M_i\rangle\langle \psi|_F\) and \(|N_i\rangle\langle \psi|_F\) must be small. The next lemma shows that the total contribution of such \(|\langle v(M_i) \rangle| v(N_i))|\)’s to \(\Delta^2(M, N)\) is small, and hence we can safely ignore them.

**Lemma 8.** Suppose \(M = \{M_1, M_2, \ldots, M_k\}\) and \(N = \{N_1, N_2, \ldots, N_k\}\) are two measurements. Let \(I_{\delta} = \{i : p(M_i) \leq \delta\) or \(p(N_i) \leq \delta\}\) for any \(0 < \delta < 1\). Then

\[
\sum_{i \in I_{\delta}} |\langle v(M_i) \rangle| v(N_i))| \leq 2\sqrt{d k}.
\]  

\[\sum_{i \in A} |\langle v(M_i) \rangle| v(N_i))| \leq \sqrt{\delta k},\]  

\[\sum_{i \in B} |\langle v(M_i) \rangle| v(N_i))| \leq \sqrt{\delta k}.
\]

To prove Eq. (142), we use the Cauchy-Schwarz inequality and the definition of \(A\),

\[
\sum_{i \in A} |\langle v(M_i) \rangle| v(N_i))| \leq \sum_{i \in A} \|\langle v(M_i) \rangle\| \cdot \|v(N_i))\|
\]

\[
= \sum_{i \in A} \sqrt{p(M_i)p(N_i)} \leq \sqrt{\delta} \sum_{i \in A} \sqrt{p(N_i)} \leq \sqrt{\delta k},
\]

where the last step follows from \(\sum_{i \in A} p(N_i) \leq 1\) and \(|A| \leq k\). Eq. (143) can be proved similarly.

Now we are ready to formally describe our result.

**Theorem 5.** Suppose \(M = \{M_1, M_2, \ldots, M_k\}\) and \(N = \{N_1, N_2, \ldots, N_k\}\) are two measurements with at most \(k\) possible outcomes on a \(D\)-dimensional system. Then we can estimate their distance \((i.e. \Delta(M, N))\) with precision \(\epsilon\) and confidence \(0.8\) by using them \(O(k^3 \log(k)/\epsilon^{12})\) times. Furthermore, the estimation algorithm can be efficiently implemented provided that \(k\) is polylogarithmic in \(D\).
Proof: We run the following algorithm on $M$ and $N$:

**Algorithm 6 Estimating $\Delta(M, N)$**

1. Let $L = \left[ \frac{50000k^2 \ln(40k)}{\epsilon^2} \right], T = \left[ \left( \frac{e^4}{16k} - \frac{e^4}{36k^2} \right) L \right]$. 
2. Prepare $L$ copies of $|\Phi^+_{B}⟩ = \frac{1}{\sqrt{D}} \sum_{i=0}^{D-1} |i⟩|i⟩$. Perform $M$ on the first subsystem of each copy. Let $a_i$ be the fraction of outcome $i$ among the $L$ outcomes obtained, for $i = 1, 2, \ldots, k$. Let $A = \{i : a_i \geq \frac{e^4}{16k} - \frac{e^4}{36k^2} \}$. 
3. Prepare another $L$ copies of $|\Phi^+_{B}⟩$. Perform $N$ on the first subsystem of each copy. Let $b_i$ be the fraction of outcome $i$ among the $L$ outcomes obtained, for $i = 1, 2, \ldots, k$. Let $B = \{i : b_i \geq \frac{e^4}{16k} - \frac{e^4}{36k^2} \}$. 
4. For any $i \in A \setminus B$, we have at least $T$ copies of $|\tilde{v}(M_i)⟩$ and $|\tilde{v}(N_i)⟩$. Then we run algorithm [8] on them, and get an estimate $\lambda_i$ of $|⟨\tilde{v}(M_i)|\tilde{v}(N_i)⟩|$. 
5. Return $\hat{\Delta} := \sqrt{\sum_{i \in A \setminus B} \sqrt{a_i b_i \lambda_i}}$ as an estimate of $\Delta(M, N)$.

Now we prove the correctness of this algorithm. Note that in step 2, by Eq. (29) and our choice of $L$, $a_i$ is $e^4/(36k^2)$-close to $p(M_i)$ with probability at least $1 - 1/(20k)$ for each $i$. Therefore, with probability at least 0.95, every $i$ satisfying $p(M_i) \geq e^4/(16k)$ is in $A$. Similarly, in step 3, with probability at least 0.95, every $i$ satisfying $p(N_i) \geq e^4/(16k)$ is in $B$. Furthermore, by the proof of lemma [7] and our choice of $T$, for each $i \in A \setminus B$, the $\lambda_i$ produced by algorithm [8] is $e^2/(6k)$-close to $|⟨\tilde{v}(M_i)|\tilde{v}(N_i)⟩|$ with probability at least $1 - 1/(10k)$. Thus, by a union bound, with probability at least 0.8, we simultaneously have:

1. $|a_i - p(M_i)| \leq \frac{e^4}{36k^2}, \forall i$;
2. $|b_i - p(N_i)| \leq \frac{e^4}{36k^2}, \forall i$;
3. $p(M_i) \leq \frac{e^4}{16k}, \forall i \notin A$;
4. $p(N_i) \leq \frac{e^4}{16k}, \forall i \notin B$;
5. $|\lambda_i - |⟨\tilde{v}(M_i)|\tilde{v}(N_i)⟩|| \leq \frac{e^2}{6k}, \forall i \in A \cap B$.

Using these facts and lemma [8] we obtain

$$
\hat{\Delta}^2 - \Delta^2(M, N) = \left( \sum_{i \in A \cap B} |⟨\tilde{v}(M_i)|\tilde{v}(N_i)⟩| \right) - \left( \sum_{i \in A \cap B} \sqrt{a_i b_i \lambda_i} \right) \\
\leq \sum_{i \in A \cap B} |⟨\tilde{v}(M_i)|\tilde{v}(N_i)⟩| - \sqrt{a_i b_i \lambda_i} + \frac{e^2}{2} \tag{145}
$$

where in the third step we have used

$$
\leq \sqrt{p(M_i)}|v(N_i)| - \sqrt{a_i b_i \lambda_i} + \sqrt{p(N_i)} - \sqrt{b_i} + |⟨\tilde{v}(M_i)|\tilde{v}(N_i)⟩| - \lambda_i \\
\leq \frac{e^2}{2} + \frac{e^2}{2} + \frac{e^2}{2} \tag{146}
$$

(Note that for any $0 \leq \alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2 \leq 1$, we have $|\alpha_1 \beta_1 \gamma_1 - \alpha_2 \beta_2 \gamma_2| \leq |\alpha_1 - \alpha_2| + |\beta_1 - \beta_2| + |\gamma_1 - \gamma_2|$). Thus,

$$
|\hat{\Delta} - \Delta(M, N)| \leq \epsilon \tag{147}
$$

as desired.

Finally, this algorithm obviously has query complexity $O(k^5 \log(k)/\epsilon^2)$. Moreover, beside querying the black boxes, it requires two kinds of quantum operations: 1. preparing $O(k^5 \log(k)/\epsilon^2)$ copies of $|\Phi^+_{B}⟩$; 2. running algorithm [8] $(O(k)$ times. $|\Phi^+_{B}⟩$ can be easily prepared, and algorithm [8] is also efficient. In addition, the classical processing is also efficient. So as long as $k$ is polylogarithmic in $D$, algorithm [8] can be efficiently implemented.

As a corollary of theorem [8] we can easily determine whether two unknown measurements are the same or quite different.

**Corollary 1.** Suppose $M = \{M_1, M_2, \ldots, M_k\}$ and $N = \{N_1, N_2, \ldots, N_k\}$ are two measurements with $k$ possible outcomes on a $D$-dimensional system. Assuming that they are either the same or $\epsilon$-far away, we can know which case it is with probability at least 0.8 by using them $O(k^5 \log(k)/\epsilon^2)$ times. Furthermore, the testing algorithm can be efficiently implemented provided that $k$ is polylogarithmic in $D$.

**Proof:** We simply use algorithm [8] to estimate $\Delta(M, N)$ with precision $\epsilon/2$. If it is smaller than $\epsilon/2$, then we conclude that $M$ and $N$ are the same; otherwise, we conclude that $M$ and $N$ are $\epsilon$-far away. By theorem [8] we succeed with probability at least 0.8. In addition, the query complexity and time complexity are as claimed.
VIII. CONCLUSION

To summarize, we have introduced a metric $\Delta$ for quantum measurements on finite-dimensional systems. This metric indicates the average difference between the behaviors of two measurements on a random input state. Then we show that, with respect to this metric, the stabilizer measurements, $k$-local measurements and permutation-invariant measurements can be all efficiently tested with query complexity independent of the system’s dimension. Moreover, we also present an algorithm for testing any finite set of measurements. Finally, we give an efficient algorithm for estimating the distance of two unknown measurements, and its query complexity is also independent of the system’s dimension. As a consequence, we can easily test whether two unknown measurements are identical or quite different.

It is worth noting that entanglement plays a crucial role in all of our testing algorithms. Namely, we need to prepare the maximally entangled state $|\Phi^+_n\rangle$, then “imprint” the measurement operator $M_i$ on it, obtaining the state $\langle \nu(M_i) \rangle = (M_i \otimes I)|\Phi^+_n\rangle$, and finally extract information about $M_i$ from this state. It seems that by utilizing entanglement, we can somehow gain a better understanding of the global property of a measurement. Meanwhile, combining Eqs. (48), (54) and (55) yields

$$\frac{1}{4} - \gamma \leq \alpha_1, \alpha_2, \beta_1, \beta_2 \in \left[\frac{1}{4} - \gamma, \frac{1}{4} + \gamma\right]$$

and $\theta_1, \theta_2 \in [0, 2\pi)$. Plugging this into Eqs. (50) and (57) we obtain

$$\cos(\theta_1) \geq \frac{1}{4} - \frac{\delta}{1 + \gamma} \geq 1 - 4\delta - 4\gamma,$$

$$\cos(\theta_2) \leq -\frac{1}{4} - \frac{\delta}{1 + \gamma} \leq -1 + 4\delta + 4\gamma.$$  \hfill (A2)

$$\sum_{i=1,2} \sum_{(x,z)\neq (0,0),(a,b)} \left|\mu_{x,z}(M_i)\right|^2 + \sum_{i\geq 3} \sum_{x,z\in Z_2} \left|\mu_{x,z}(M_i)\right|^2 \leq 4\gamma.$$  \hfill (A3)

Now consider the distance between $M_i$ and $P_i(a, b)$ for different $i$’s:

$$\Delta^2(M_1, P_1(a, b)) \leq \frac{1}{2D} \left\|M_1 - P_1(a, b)\right\|^2_F,$$

$$= \frac{1}{2} \left[ \frac{\alpha_1 - 1}{2} + \frac{\beta_1 e^{i\theta_1} - 1}{2} \right] + \frac{1}{2} \sum_{(x,z)\neq (0,0),(a,b)} \left|\mu_{x,z}(M_1)\right|^2,$$  \hfill (A5)

$$\Delta^2(M_2, P_2(a, b)) \leq \frac{1}{2D} \left\|M_2 - P_2(a, b)\right\|^2_F,$$

$$= \frac{1}{2} \left[ \frac{\alpha_2 - 1}{2} + \frac{\beta_2 e^{i\theta_2} + 1}{2} \right] + \frac{1}{2} \sum_{(x,z)\neq (0,0),(a,b)} \left|\mu_{x,z}(M_2)\right|^2.$$  \hfill (A6)

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**Appendix A: Proof of Lemma 2**

Without loss of generality, we can assume that $\mu_{0,0}(M_1)$ and $\mu_{0,0}(M_2)$ are both real and non-negative (if not, we can multiply $M_1$ or $M_2$ by appropriate phases to make this true). Then by Eqs. (54) and (55) we can assume $\mu_{0,0}(M_1) = \alpha_1, \mu_{0,0}(M_2) = \alpha_2, \mu_{a,b}(M_1) = \beta_1 e^{i\theta_1}$ and $\mu_{a,b}(M_2) = \beta_2 e^{i\theta_2}$ where

$$\alpha_1, \alpha_2, \beta_1, \beta_2 \in \left[\frac{1}{4} - \gamma, \frac{1}{4} + \gamma\right]$$

and $\theta_1, \theta_2 \in [0, 2\pi)$. Plugging this into Eqs. (50) and (57) we obtain

$$\cos(\theta_1) \geq \frac{1}{4} - \frac{\delta}{1 + \gamma} \geq 1 - 4\delta - 4\gamma,$$

$$\cos(\theta_2) \leq -\frac{1}{4} - \frac{\delta}{1 + \gamma} \leq -1 + 4\delta + 4\gamma.$$  \hfill (A2)

Meanwhile, combining Eqs. (48), (54) and (55) yields

$$\sum_{i=1,2} \sum_{(x,z)\neq (0,0),(a,b)} \left|\mu_{x,z}(M_i)\right|^2 + \sum_{i\geq 3} \sum_{x,z\in Z_2} \left|\mu_{x,z}(M_i)\right|^2 \leq 4\gamma.$$  \hfill (A3)

Now consider the distance between $M_i$ and $P_i(a, b)$ for different $i$’s:
\[ \Delta^2(M_i, P_i(a, b)) = \frac{1}{2D} \| M_i \|_F^2 \]
\[ = \frac{1}{2} \sum_{x, z \in \mathbb{Z}_2} |\mu_{x,z}(M_i)|^2, \quad \forall i \geq 3. \]

Here \( D = 2^n \). Taking the sum of Eqs. (A6) and (A7) for all \( i \geq 3 \) and using Eq. (A4), we obtain

\[ \Delta^2(M, P(a, b)) \leq \frac{1}{2} \left| \alpha - \frac{1}{2} \right|^2 + \frac{1}{2} \left| \beta_1 e^{i\theta_1} - \frac{1}{2} \right|^2 \]
\[ + \frac{1}{2} \left| \alpha_2 - \frac{1}{2} \right|^2 + \frac{1}{2} \left| \beta_2 e^{i\theta_2} + \frac{1}{2} \right|^2 + 2\gamma. \quad \text{(A8)} \]

Now by Eq. (A1) we have

\[ \left| \alpha - \frac{1}{2} \right|^2 \leq \left| \alpha^2 - \frac{1}{4} \right| \leq \gamma, \quad \text{(A9)} \]
\[ \left| \alpha_2 - \frac{1}{2} \right|^2 \leq \left| \alpha_2^2 - \frac{1}{4} \right| \leq \gamma. \quad \text{(A10)} \]

Moreover, by Eqs. (A1), (A2), and (A3), we have

\[ \left| \beta_1 e^{i\theta_1} - \frac{1}{2} \right|^2 = \left( \frac{1}{4} + \gamma \right) + \frac{1}{4} - \sqrt{\frac{1}{4} - \gamma}, \quad \text{(A11)} \]
\[ \leq 5\gamma + 2\delta, \]
\[ \left| \beta_2 e^{i\theta_2} + \frac{1}{2} \right|^2 = \left( \frac{1}{4} + \gamma \right) + \frac{1}{4} + \sqrt{\frac{1}{4} - \gamma}, \quad \text{(A12)} \]
\[ \leq 5\gamma + 2\delta. \]

Plugging Eqs. (A9)–(A12) into Eq. (A8) yields

\[ \Delta^2(M, P(a, b)) \leq 8\gamma + 2\delta \quad \text{(A13)} \]
as desired.

**Appendix B: Proof of Lemma 3**

Suppose \(|M| = k\), i.e. \( M = \{M_1, M_2, \ldots, M_k\} \). We first prove that

\[ \sum_{i=1}^{k} f^i_T(M_i) \tilde{f}_T(M_i) \leq I. \quad \text{(B1)} \]

(When we write \( A \preceq B \) for two matrices \( A, B \), we mean that \( B - A \) is positive semidefinite.) Note that by the definition of \( f_T(M_i) \), we can write it as

\[ f_T(M_i) = \tilde{f}_T(M_i) \otimes I \quad \text{(B2)} \]

where \( \tilde{f}_T(M_i) \) is some operator on \( T \), and \( I \) is the identity operator on \( T^c \). Then Eq. (B1) is equivalent to

\[ \sum_{i=1}^{k} f^i_T(M_i) \tilde{f}_T(M_i) \leq I. \quad \text{(B3)} \]

Let \(|\psi\rangle \) be an arbitrary pure state on \( T \), and \( \rho \) be the uniformly mixed state on \( T^c \). Then by plugging

\[ M_i = \tilde{f}_T(M_i) \otimes I + g(M_i) \quad \text{(B4)} \]

into

\[ 1 = \text{tr}(\langle \psi | \psi \rangle \otimes \rho) \]
\[ = \text{tr}(\{ \sum_{i=1}^{k} M_i^\dagger M_i \} |\psi\rangle \langle \psi | \otimes \rho) \]
\[ = \langle \psi | \{ \sum_{i=1}^{k} \tilde{f}_T^\dagger(M_i) \tilde{f}_T(M_i) \} |\psi\rangle \langle \psi | \otimes \rho), \quad \text{(B5)} \]

and noting that \( \text{tr}(g(M_i)) = 0 \) for any \( i \), we obtain

\[ 1 = \langle \psi | \{ \sum_{i=1}^{k} \tilde{f}_T^\dagger(M_i) \tilde{f}_T(M_i) \} |\psi\rangle \langle \psi \rangle \]
\[ + \text{tr}(\{ \sum_{i=1}^{k} g_i^\dagger(M_i) g_i(M_i) \} |\psi\rangle \langle \psi | \otimes \rho) \]
\[ \geq \langle \psi | \{ \sum_{i=1}^{k} \tilde{f}_T^\dagger(M_i) \tilde{f}_T(M_i) \} |\psi\rangle. \quad \text{(B6)} \]

Since \(|\psi\rangle \) is arbitrary, we get

\[ \sum_{i=1}^{k} \tilde{f}_T^\dagger(M_i) \tilde{f}_T(M_i) \leq I \quad \text{(B7)} \]
as desired.

Now let \( N = \{N_1, N_2, \ldots, N_{k+1}\} \), where

\[ N_i = f_T(M_i), \quad \forall i = 1, 2, \ldots, k; \quad \text{(B8)} \]
\[ N_{k+1} = \sqrt{I - \sum_{i=1}^{k} f^i_T(M_i) f_T(M_i).stem} \quad \text{(B9)} \]

(For a positive semidefinite matrix \( A \) with spectral decomposition \( A = \sum_{x} \lambda_{x} |x\rangle \langle x| \), we define \( \sqrt{A} := \sum_{x} \sqrt{\lambda_{x}} |x\rangle \langle x| \).) Then \( N \) is a valid measurement. Moreover, since each \( f_T(M_i) \) acts non-trivially only on subsystem \( T \), so does each \( N_i \). Hence \( N \) is a \(|T|\)-local measurement.

Now, the distance between \( M_i \) and \( N_i \) satisfies

\[ \Delta^2(M_i, N_i) \leq \frac{1}{2D} \| M_i - N_i \|_F^2 \]
\[ = \frac{1}{2D} \| g_T(M_i) \|_F^2 \]
\[ = \frac{2}{2D} \| (\| M_i \|_F^2 - \| f_T(M_i) \|_F^2) \|_F^2, \quad \text{(B10)} \]

for \( i = 1, 2, \ldots, k \), and

\[ \Delta(M_{k+1}, N_{k+1}) = \frac{1}{2D} \| N_{k+1} \|_F^2 \]
\[ = \frac{1}{2} - \frac{1}{2D} \sum_{i=1}^{k} \| f_T(M_i) \|_F^2. \quad \text{(B11)} \]
So the distance between $M$ and $N$ satisfies
\[
\Delta^2(M, N) = \sum_{i=1}^{k+1} \Delta^2(M_i, N_i) \\
\leq 1 - \frac{1}{D} \sum_{i=1}^{k} \|f_T(M_i)\|^2_F \\
\leq \delta^2.
\] (B12)

Here we have used the fact that
\[
\sum_{i=1}^{k} \|M_i\|^2_F = \text{tr} \left( \sum_{i=1}^{k} M_i^\dagger M_i \right) = \text{tr}(I) = D. 
\] (B13)

**Appendix C: Proof of Lemma 4**

The proof of this lemma is similar to that of lemma 3. Suppose $|M| = k$, i.e. $M = \{M_1, M_2, \ldots, M_k\}$. We first show that
\[
\sum_{i=1}^{k} \tilde{M}_i^\dagger \tilde{M}_i \leq I, 
\] (C1)
or equivalently,
\[
\sum_{i=1}^{k} (\tilde{M}_i)^\dagger (\tilde{M}_i)_\lambda \leq I, \quad \forall \lambda \in \mathcal{I}_{d,n}. 
\] (C2)

Fix any $\lambda \in \mathcal{I}_{d,n}$. Let $|\psi\rangle$ be an arbitrary pure state in $\mathcal{W}_{\lambda,d}$, and $\rho$ be the uniformly mixed state in $\mathcal{V}_\lambda$. Then by plugging
\[
M_i = (\tilde{M}_i + \tilde{M}_i + \bar{M}_i)_{\text{Sch}} 
\] (C3)
where
\[
\tilde{M}_i = \bigoplus_{\lambda \in \mathcal{I}_{d,n}} (\tilde{M}_i)_\lambda \otimes I, 
\] (C4)
\[
\bar{M}_i = \bigoplus_{\lambda \in \mathcal{I}_{d,n}, j \neq 0} (\sum_{i} (\tilde{M}_i)_{\lambda,j} \otimes g_{\lambda,j}), 
\] (C5)
\[
\bar{M}_i = \sum_{\lambda \neq \lambda'} |\lambda\rangle \langle \lambda'| \otimes (\tilde{M}_i)_{\lambda,\lambda'} 
\] (C6)
into
\[
1 = \text{tr}[|\lambda\rangle \langle \lambda| \otimes |\psi\rangle \langle \psi| \otimes \rho]_{\text{Sch}} \leq \text{tr} \left[ \left( \sum_{i=1}^{k} M_i^\dagger M_i \right) |\lambda\rangle \langle \lambda| \otimes |\psi\rangle \langle \psi| \otimes \rho \right]_{\text{Sch}}, 
\] (C7)
and noting that $\text{tr}(g_{\lambda,j}) = 0$ for $j \neq 0$, we obtain
\[
1 = \langle \psi| (\sum_{i=1}^{k} (\tilde{M}_i)^\dagger (\tilde{M}_i)_\lambda) |\psi\rangle \\
+ \langle \psi| (\sum_{i=1}^{k} \sum_{j \neq 0} (\tilde{M}_i)^\dagger (M_i)_{\lambda,j}) |\psi\rangle \\
+ \text{tr}(\sum_{i=1}^{k} M_i^\dagger \bar{M}_i) (|\lambda\rangle \langle \lambda| \otimes |\psi\rangle \langle \psi| \otimes \rho) \\
\geq \langle \psi| \sum_{i=1}^{k} (\tilde{M}_i)^\dagger (\tilde{M}_i)_\lambda |\psi\rangle. 
\] (C8)

Since $|\psi\rangle$ is arbitrary, we get
\[
\sum_{i=1}^{k} (\tilde{M}_i)^\dagger (\tilde{M}_i)_\lambda \leq I, 
\] (C9)
as desired.

Now let $N = \{N_1, N_2, \ldots, N_{k+1}\}$ where
\[
N_i = (\tilde{M}_i)_{\text{Sch}}, \quad \forall i = 1, 2, \ldots, k; 
\] (C10)
\[
N_{k+1} = \sqrt{I - \sum_{i=1}^{k} (\tilde{M}_i)_{\text{Sch}}}. 
\] (C11)

Then $N$ is a valid measurement. In addition, since each $M_i$ is permutation-invariant, so is each $N_i$. Thus, $N$ is a permutation-invariant measurement.

Now the distance between $M_i$ and $N_i$ satisfies
\[
\Delta^2(M_i, N_i) \leq \frac{1}{2D} \|M_i - N_i\|^2_F \\
\leq \frac{1}{2D} \|\tilde{M}_i + \bar{M}_i\|^2_F \\
\leq \frac{1}{2D} (\|\tilde{M}_i\|^2_F + \|\bar{M}_i\|^2_F) = \frac{1}{2}\frac{1}{2D} (\|M_i\|^2_F - \|\bar{M}_i\|^2_F),
\] (C12)
for $i = 1, 2, \ldots, k$, and
\[
\Delta^2(M_{k+1}, N_{k+1}) \leq \frac{1}{2D} \|N_{k+1}\|^2_F = \frac{1}{2} - \frac{1}{2D} \sum_{i=1}^{k} \|M_i\|^2_F 
\] (C13)
So the distance between $M$ and $N$ satisfies
\[
\Delta^2(M, N) \leq \sum_{i=1}^{k+1} \Delta^2(M_i, N_i) \leq 1 - \frac{1}{D} \sum_{i=1}^{k} \|M_i\|^2_F \\
\leq \delta^2. 
\] (C14)

Here we have used the fact that
\[
\sum_{i=1}^{k} \|M_i\|^2_F = \text{tr} \left( \sum_{i=1}^{k} M_i^\dagger M_i \right) = \text{tr}(I) = D. 
\] (C15)

**Appendix D: Proof of Lemma 5**

For convenience, let $\mathcal{I} = \{i : L_i \geq \omega \text{log} L/k\}$ and $\xi_i = \langle |\tilde{v}(M_i)\rangle |\tilde{v}(N_i)\rangle \rangle$ for any $i$. Then by $\Delta(M, N) \geq \delta$ we get
\[
\delta^2 \leq \Delta^2(M, N) \leq 1 - \sum_{i=1}^{k} \|v(M_i)v(N_i)\| \\
= 1 - \sum_{i=1}^{k} \sqrt{p(M_i)p(N_i)\xi_i}. 
\] (D1)
It follows that
\[
1 - \delta^2 \geq \sum_{i \in I} \sqrt{p(M_i)p(N_i)} |\xi_i| \ 
\geq \sum_{i \in I} \sqrt{p(M_i)p(N_i)} |\xi_i| \ 
\geq (1 - 0.1\delta^2) \sum_{i \in I} \frac{L_i |\xi_i|}{L},
\]
(D2)
since \(p(M_i), p(N_i) \geq (1-0.1\delta^2)L_i/L\) for any \(i \in I\). Hence
\[
\sum_{i \in I} L_i |\xi_i| \leq (1 - 0.1\delta^2) \frac{L}{1 - 0.1\delta^2} \leq (1 - 0.9\delta^2)L.
\]
(D3)
Meanwhile, for any \(i \notin I\), we have \(L_i < 0.1\delta^2L/k\), and there are at most \(k\) such \(i\)’s, so
\[
\sum_{i \in \mathbb{Z}} L_i \geq (1 - 0.1\delta^2)L.
\]
(D4)
Furthermore, by the fact that the arithmetic mean of a set of non-negative real numbers is no less than their geometric mean, we get
\[
\prod_{i \in \mathbb{Z}} \xi_i^L \leq \left(\sum_{i \in \mathbb{Z}} L_i |\xi_i|\right)^{\frac{1}{L}} \sum_{i \in \mathbb{Z}} L_i \ 
\leq \left(1 - 0.9\delta^2\right)^{1 - 0.1\delta^2} \frac{L}{1 - 0.1\delta^2} \ 
\leq \left(1 - 0.8\delta^2\right)^{0.9L} \ 
\leq \left(1 - 0.6\delta^2\right)^{L}.
\]
(D6)

\textbf{Appendix E: Proof of Lemma 6}

Without loss of generality we assume that \(|\phi_1\rangle, |\phi_2\rangle, \ldots, |\phi_m\rangle\) are linearly independent. Then the subspace spanned by them is \(m\)-dimensional. Let \(|\varphi_1\rangle, |\varphi_2\rangle, \ldots, |\varphi_m\rangle\) be an orthonormal basis for this subspace, where
\[
|\varphi_i\rangle = \sum_{j=1}^{m} \lambda_{i,j} |\phi_j\rangle.
\]
(E1)
for some \(\lambda_{i,j}\)’s. Then
\[
m = \sum_{i=1}^{m} \langle \varphi_i |\varphi_i\rangle
\]
\[
= \sum_{i=1}^{m} \sum_{j=1}^{m} \lambda_{i,j}^* \lambda_{i,j} \langle \phi_j |\phi_j\rangle
\]
\[
= \sum_{i=1}^{m} \sum_{j=1}^{m} |\lambda_{i,j}|^2 + \sum_{i=1}^{m} \sum_{j=1}^{m} \lambda_{i,j}^* \lambda_{i,j} \langle \phi_j |\phi_j\rangle
\]
\[
\geq \sum_{i=1}^{m} \sum_{j=1}^{m} |\lambda_{i,j}|^2 - \sum_{i=1}^{m} \sum_{j \neq j'} |\lambda_{i,j}|^2 + |\lambda_{i,j'}|^2 \cdot \frac{1}{5m}
\]
\[
\geq \frac{1}{2} \sum_{i=1}^{m} \sum_{j=1}^{m} |\lambda_{i,j}|^2.
\]
(E2)
Hence
\[
\langle \psi |\Pi |\psi\rangle = \sum_{i=1}^{m} |\langle \varphi_i |\psi\rangle|^2
\]
\[
= \sum_{i=1}^{m} \sum_{j=1}^{m} \lambda_{i,j}^* \lambda_{i,j} |\langle \phi_j |\psi\rangle|^2
\]
\[
= \sum_{i=1}^{m} \sum_{j=1}^{m} |\lambda_{i,j}|^2 |\langle \phi_j |\psi\rangle|^2
\]
\[
\leq \sum_{i=1}^{m} \sum_{j=1}^{m} |\lambda_{i,j}|^2 \cdot \left(\frac{1}{5m}\right)^2 \ 
\leq \frac{1}{25m^2} \sum_{i=1}^{m} \sum_{j=1}^{m} |\lambda_{i,j}|^2 + |\lambda_{i,j'}|^2 \ 
\leq \frac{m}{25m} \sum_{i=1}^{m} \sum_{j=1}^{m} |\lambda_{i,j}|^2 \ 
\leq 0.1.
\]
(E3)
