Graded Geometric Structures Underlying F-Theory Related
Defect Theories

V. K. Oikonomou∗
Max Planck Institute for Mathematics in the Sciences
Inselstrasse 22, 04103 Leipzig, Germany
August 2, 2013

Abstract
In the context of F-theory, we study the related eight dimensional super-Yang-
Mills theory and reveal the underlying supersymmetric quantum mechanics algebra
that the fermionic fields localized on the corresponding defect theory are related to.
Particularly, the localized fermionic fields constitute a graded vector space, and in
turn this graded space enriches the geometric structures that can be built on the
initial eight-dimensional space. We construct the implied composite fibre bundles,
which include the graded affine vector space and demonstrate that the composite
sections of this fibre bundle are in one-to-one correspondence to the sections of the
square root of the canonical bundle corresponding to the submanifold on which the
zero modes are localized.

Introduction
String theory has proven to be the most promising theory towards the unified description
of all forces and matter in nature. Particularly, it encompasses in its theoretical frame-
work gravity and a large number of field theoretic features such as, supersymmetry, chiral
matter and spontaneous symmetry breaking. The appealing attribute of string theory
is that it can provide consistent UV completion of many field theoretic models because
it can accommodate quite successfully supersymmetric grand unified theories. M-theory
embraces all the different string theories that describe independently various features of
the UV completions of the Standard Model (SM hereafter), with the various branches of
M-theory being connected with dualities, a strong tool towards a non-perturbative de-
scription. However, certain branches of M-theory prove to be more efficient in realizing
SM phenomenological features, than others. Particularly, type IIB string theory and the
strongly coupled version of it, F-theory (for an important stream of papers on F-theory
see [1–25] and for reviews on F-theory see [4–6]) embody many phenomenological fea-
tures of the SM. The most appealing feature of these theories is that these allow gauge
interactions, existence and propagation of matter in a way that is independent from the full vacuum configuration, with the last being one of the most difficult problems in string theory, because there is a huge number of possible vacua. In the type IIB–F-theory framework, gauge interactions, such as Yukawa couplings, and chiral matter are localized on sub-manifolds of the total space, and thus depend only on the local geometry to some extent. $D^7$ branes are necessary to describe chiral matter and in type IIB theories this type of matter is realized on the intersection of such objects. The use of these low dimensional D-branes, enables one to describe low energy physics in a bottom-up way, because only local configurations of these $D^7$ branes are considered and these are localized, as we already mentioned, to some region of the compact dimensions. Thus, the IIB–F-theory description is a successful to some extend, bottom-up approach to the problem of finding an appropriate low energy string compactification. In the F-theory context, gravity can be decoupled from SM physics, since $D^7$ branes that contain a GUT group, wrap certain classes of complex surfaces $S$ with four real dimensions, and if the infinite volume limit of these surfaces is taken, the SM physics is obtained but decoupled from gravity. Particularly, an $N = 1$ supersymmetric GUT theory with a singularity enhanced gauge group can either result from M-theory on $G_2$-holonomic seven dimensional manifolds, or from F-theory compactified on elliptic Calabi-Yau fourfolds. In the case of F-theory compactifications, the Kähler geometry provides freedom to use numerous geometric techniques in order to accommodate many phenomenological features of the Standard Model into the theoretical outcomes of such theories. Moreover, non-zero Yukawa couplings exist for fields that reside on intersecting $D^7$ branes. Actually, most of the most interesting SM phenomenology can be deduced from an eight dimensional super-Yang-Mills-Higgs theory living on $D^7$ branes. The Yukawas are obtained from the overlap of the chiral zero modes on the seven-branes (see [10–14] and [7–9]).

F-theory on a local Calabi-Yau fourfold is described at low energies by the twisted super-Yang-Mills theory living on the worldvolume of the seven brane that wraps $S \times R^{3,1}$. Let a seven brane wrapping the complex dimension two surface $S$, intersect with another seven brane that wraps $S'$. We denote their intersection $\Sigma = S \cap S'$, a complex dimension one curve. The physics of the charged fields that reside on the intersection $\Sigma \times R^{3,1}$ can be consistently described by a twisted six dimensional defect theory coupled to the bulk theories $S \times R^{3,1}$ and $S' \times R^{3,1}$. We assume that $\Sigma$ is irreducible, that is, it does not consist of several components. As we already mentioned, light degrees of freedom are localized on the manifold $\Sigma \times R^{3,1}$. It is exactly these degrees of freedom that are described by an effective defect theory coupled to the bulk $S \times R^{3,1}$ and $S' \times R^{3,1}$ super-Yang Mills theories. The structure of the defect theory $\Sigma \times R^{3,1}$ can be determined directly from the $S \times R^{3,1}$ super-Yang Mills theory itself. Starting from a seven brane with world volume gauge group $\Gamma_S$, wrapping the complex surface $S$, in order to obtain an intersection of seven branes, we can allow the scalar field $\phi$ belonging to the $N = 1$ supersymmetric $S \times R^{3,1}$ theory to have a non-trivial holomorphic vacuum expectation value. The scalar field vanishes on some points of the curve $\Sigma$. On the points in which $\phi$ vanishes, the bulk gauge group $G_S$ remains unbroken, but away from the singularity, the gauge group becomes some $\Gamma_S \subset G_S$, times a $U(1)$. The structure of the defect theory can be captured by examining what happens in the $N = 1$ supersymmetric $S \times R^{3,1}$ theory, when $\phi$ acquires
a non-trivial holomorphic expectation value. The defect theory on \( \Sigma \) can be viewed from the point of view of the complex-dimension two surface \( S \), as a global string associated to the vanishing of the holomorphic mass term that is induced by the scalar \( \phi \). Hence, we expect (and this is exactly the case) to find bosonic and fermionic zero modes trapped along \( \Sigma \), thus resulting to massless, chiral matter on \( \mathbb{R}^{3,1} \). Thereby, identifying which fermionic zero modes are trapped in the submanifold \( \Sigma \), simultaneously determines which bosonic zero modes are localized.

In this paper the focus is on exactly these fermionic zero modes that are trapped on the complex dimension one curve \( \Sigma \), with \( \Sigma \) being considered as a defect of the theory on \( S \). We establish the result that these fermionic zero modes are associated to an one dimensional \( N = 2 \) supersymmetric quantum mechanics (SUSY QM hereafter) algebra \([26-39]\). Particularly, the zero modes localized on \( \Sigma \) are in bijective correspondence with the vectors of the graded Hilbert space that describes the SUSY QM vector space of quantum states (for an similar case related to superconducting strings and to Chern-Simons gauge theories in \( (2 + 1) \)-dimensions, see \([40,41]\)). Moreover, the underlying unbroken SUSY QM algebra makes the total eight dimensional space \( S \times \mathbb{R}^{3,1} \) a graded manifold \((X,A)\), with body \( X \equiv S \times \mathbb{R}^{3,1} \) and structure sheaf \( A \). Therefore the total space \( S \times \mathbb{R}^{3,1} \) has a potentially rich variety of geometric structures that can be constructed on it. We are particularly interested on the composite fibre bundles that can be constructed on this space, in which the graded manifold belongs. As we shall see, from all the sections of the \( G_S \)-twisted spin bundle upon \( S \times \mathbb{R}^{3,1} \), only those trapped on \( \Sigma \) are connected to the SUSY QM algebra. This implies that the covariant differential of the \( G_S \)-twisted spin bundle over \( S \times \mathbb{R}^{3,1} \) is reducible to some covariant differential of a composite fibre bundle over some subbundle of \( S \times \mathbb{R}^{3,1} \), with this bundle containing the graded bundle in its substructure. The bundle reducibility is realized in terms of the corresponding connections of the composite fibre bundles. In the following sections we shall study these issues in detail, and reveal the SUSY QM generated underlying geometric structures.

This paper is organized as follows. In section 1 we present the details of the \( S \times \mathbb{R}^{3,1} \) super-Yang Mills theory, and find which modes of the initial eight-dimensional theory are trapped to move along the submanifold \( \Sigma \). In addition, we substantiate the result that these modes are associated to an unbroken SUSY QM algebra, which we describe briefly. We also study how holomorphic perturbations to the Euclidean metric affect the SUSY QM algebra. In section 2 we present the additional geometric structures over the space \( S \times \mathbb{R}^{3,1} \), which is due to the unbroken \( N = 2 \) SUSY QM algebra and discuss the features of the fermionic geometric structures over \( S \times \mathbb{R}^{3,1} \). The concluding remarks follow in the end of the paper.

1 Defect Theory on the Submanifold \( \Sigma \), Localized Zero Modes and \( N = 2 \) SUSY QM

In this section we shall present the details of the \( S \times \mathbb{R}^{3,1} \) super-Yang Mills theory, working locally in \( \Sigma \subset S \). We adopt the notation and conventions of \([1,9]\). Parameterizing the space \( S \) with two holomorphic coordinates \((s^1,s^2)\), and since the field \( \phi \) vanishes on \( \Sigma \),
the field \( \phi \) and its conjugate, \( \bar{\phi} \), can take the form:

\[
\phi = ts^2 ds^1 \wedge ds^2, \quad \bar{\phi} = \bar{t}s^2 ds^1 \wedge ds^2
\]  

Consequently, the complex dimension one curve \( \Sigma \), is defined to be the submanifold of \( S \), for which \( s^2 = 0 \). Therefore, \( s^1 \) parameterizes tangent directions on the curve, while \( s^2 \) normal directions to \( \Sigma \). It is important for the geometric constructions we shall present in the next section, to en-visualize this geometric structure, in the way that \( s^2 \) shows directions that belong to \( S \) but not in \( \Sigma \). The fact that the canonical bundle of the curve \( \Sigma \) consists of the differential \( ds^1 \), has important consequences, as we shall see. The part of the whole eight-dimensional super-Yang Mills action, that is relevant for the fermionic equations of motion is:

\[
I_F = \int_{S \times R^3} dx^4 Tr \left( \chi^a \wedge \partial_A \psi_a + \bar{\chi}^a \wedge \partial_A \bar{\psi}^a + 2 i \sqrt{2} \omega \wedge \partial_A \eta^a \wedge \psi_a \right. \\
\left. + 2 i \sqrt{2} \omega \wedge \partial_A \bar{\eta}^a \wedge \bar{\psi}^a - \frac{1}{2} \bar{\eta}^a \wedge \bar{[\phi, \bar{\psi}^a]} + \frac{1}{2} \psi^a \wedge [\phi, \psi_a] + \sqrt{2} \eta_a \bar{[\phi, \bar{\chi}^a]} + \sqrt{2} \eta^a [\phi, \bar{\chi}^a] \right)
\]  

Thereby, the fermionic equations of motion read,

\[
\partial_A \psi^a - \sqrt{2} [\bar{\phi}, \eta^a] = 0, \quad \partial_A \bar{\psi}^a + \sqrt{2} [\phi, \bar{\eta}^a] = 0
\]  

\[
\omega \wedge \partial_A \eta^a + \frac{i \sqrt{2}}{4} [\bar{\phi}, \bar{\psi}^a] = 0, \quad \omega \wedge \partial_A \bar{\eta}^a + \frac{i \sqrt{2}}{4} [\phi, \psi^a] = 0 = 0
\]  

for the \((\eta, \psi)\) fermion pairs, while for the \((\chi, \psi)\) pair, we have:

\[
\omega \wedge \partial_A \psi^a + \frac{i}{2} [\bar{\phi}, \chi^a] = 0, \quad \omega \wedge \partial_A \bar{\psi}^a - \frac{i}{2} [\bar{\phi}, \bar{\chi}^a] = 0
\]  

\[
\partial_A \chi^a - [\phi, \psi^a] = 0, \quad \partial_A \bar{\chi}^a - [\bar{\phi}, \bar{\psi}^a] = 0 = 0
\]  

The partial derivatives are \( \partial_A = \frac{\partial}{\partial s^2} \), and \( \omega \) is the canonical Euclidean form near \( \Sigma \):

\[
\omega = \frac{i}{2} (ds^1 \wedge d\bar{s}^1 + ds^2 \wedge d\bar{s}^2)
\]  

With \( \omega \) being of this form, no gravitational back-reaction is taken to account. Later on we shall perturb this form in order to study metric back-reaction effects on the SUSY QM algebra. From all the equations appearing in (3) and (4), only those of relation (4) give normalized localized solutions near \( \Sigma \), with the solutions exponentially decaying as:

\[
\Psi \sim \exp(-|s^2|^2)
\]  

Substituting \( \phi \) from equation (1), the equations of motion (4) take the general form (ignoring irrelevant constants):

\[
\frac{\partial \Psi}{\partial s^2} + s^2 \dot{\Psi} = 0
\]  

\[
\frac{\partial \dot{\Psi}}{\partial s^2} + s^2 \dot{\Psi} = 0
\]
with $\Psi$ and $\tilde{\Psi}$ denote any pair $\psi$ and $\chi$ respectively, or their conjugates. Particularly, as discussed in [1–3], each of the fermions $\chi$ and $\psi$ and their conjugates, have exactly one zero mode that behaves as (6). Since all our results hold for both pairs of fermions, we focus for simplicity on the fermions ($\psi, \chi$).

Thereby, we can form the operator $D_F$, corresponding to the equations of (7) (with $\partial_2 = \partial/\partial s^2$),

$$D_F = \begin{pmatrix} \partial_2 & s^2 \\ s^2 & \partial_2 \end{pmatrix}$$

acting on the vector:

$$|\Psi_F\rangle = \begin{pmatrix} \Psi \\ \tilde{\Psi} \end{pmatrix}.$$  (9)

Therefore, the equations (7) can be written as follows:

$$D_F|\Psi_F\rangle = 0$$  (10)

The solutions of equation (10) correspond to the zero modes of the operator $D_F$. Since we know that there is exactly one zero mode for each $\chi$ and $\psi$, we have that

$$\dim \ker D_F = 1$$  (11)

Moreover, the adjoint of the operator $D_F$, namely $D_F^\dagger$, is:

$$D_F^\dagger = \begin{pmatrix} \partial_2 & s^2 \\ s^2 & \partial_2 \end{pmatrix}$$

The corresponding kernel of the adjoint operator is null, that is:

$$\dim \ker D_F^\dagger = 0$$  (13)

Since the kernel of the operator $D_F$ is finite, this property renders it a Fredholm operator.

Using the operator $D_F$, we can construct an unbroken $N = 2, d = 1$ supersymmetry for the system ($\chi, \psi$) of fermions. Indeed, we can form the supercharges and the quantum Hamiltonian of the $N = 2, d = 1$ SUSY algebra:

$$Q_F = \begin{pmatrix} 0 & D_F \\ 0 & 0 \end{pmatrix}, \quad Q_F^\dagger = \begin{pmatrix} 0 & 0 \\ D_F^\dagger & 0 \end{pmatrix}, \quad \mathcal{H}_F = \begin{pmatrix} D_F D_F^\dagger & 0 \\ 0 & D_F^\dagger D_F \end{pmatrix}$$  (14)

These operators satisfy the $N = 2, d = 1$ SUSY QM algebra:

$$\{Q_F, Q_F^\dagger\} = \mathcal{H}_F, \quad Q_F^2 = 0, \quad Q_F^{\dagger 2} = 0$$  (15)

There is an operator $W$, the Witten parity, that commutes with the total Hamiltonian and anti-commutes with the supercharges,

$$[W, \mathcal{H}_F] = 0, \quad \{W, Q_F\} = \{W, Q_F^\dagger\} = 0$$  (16)
Moreover, $W$, satisfies the following identity,

$$ W^2 = 1 $$  \hspace{1cm} (17)

The total Hilbert space of the supersymmetric quantum mechanical system, $\mathcal{H}$, is rendered an $\mathbb{Z}_2$ graded vector space, with the grading provided by the operator $W$, which is actually an involution operator and decomposes as:

$$ \mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^- $$  \hspace{1cm} (18)

with the vectors belonging to the subspaces $\mathcal{H}^\pm$, classified to even and odd parity states according to their Witten parity, that is:

$$ \mathcal{H}^\pm = \mathcal{P}^\pm \mathcal{H} = \{ |\psi\rangle : W|\psi\rangle = \pm|\psi\rangle \} $$  \hspace{1cm} (19)

The corresponding Hamiltonians of the $\mathbb{Z}_2$ graded spaces are:

$$ \mathcal{H}_+ = D_F D_F^\dagger, \quad \mathcal{H}_- = D_F^\dagger D_F $$  \hspace{1cm} (20)

In our case, the operator $W$, has the corresponding representation:

$$ W = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} $$  \hspace{1cm} (21)

The eigenstates of the operator $\mathcal{P}^\pm$, that is, $|\psi^\pm\rangle$ (positive and negative parity eigenstates), satisfy the following relation:

$$ \mathcal{P}^\pm|\psi^\pm\rangle = \pm|\psi^\pm\rangle $$  \hspace{1cm} (22)

Using (21) for the Witten parity operator, the parity eigenstates are the vectors,

$$ |\psi^+\rangle = \begin{pmatrix} |\phi^+\rangle \\ 0 \end{pmatrix}, \quad |\psi^-\rangle = \begin{pmatrix} 0 \\ |\phi^-\rangle \end{pmatrix} $$  \hspace{1cm} (23)

with $|\phi^\pm\rangle \in \mathcal{H}^\pm$. It is easy to verify that:

$$ |\Psi_F\rangle = |\phi^-\rangle = \begin{pmatrix} \Psi \\ \bar{\Psi} \end{pmatrix} $$  \hspace{1cm} (24)

Therefore, the corresponding even and odd parity SUSY QM states are:

$$ |\psi^+\rangle = \begin{pmatrix} |\Psi_F\rangle \\ 0 \end{pmatrix}, \quad |\psi^-\rangle = \begin{pmatrix} 0 \\ |\Psi_F\rangle \end{pmatrix} $$  \hspace{1cm} (25)

Supersymmetry is considered unbroken if the Witten index is a non-zero integer. The Witten index when the operators are Fredholm operators is:

$$ \Delta = n_- - n_+ $$  \hspace{1cm} (26)

with $n_\pm$ the number of zero modes of $\mathcal{H}_\pm$ in the subspace $\mathcal{H}^\pm$. 
When $\Delta$ is zero and also if $n_+ = n_- = 0$, then supersymmetry is broken. Nevertheless, if $n_+ = n_- \neq 0$ the system retains unbroken supersymmetry.

The Fredholm index of the operator $D_F$ is connected to the Witten index, as:

$$\Delta = \dim \ker H_- - \dim \ker H_+ = \dim \ker D^\dagger_F D_F - \dim \ker D_F D_F^\dagger = \dim \ker \mathcal{D}_F$$

$$\text{ind} D_F = \dim \ker D_F - \dim \ker D_F^\dagger$$

In our case, the Witten index is:

$$\Delta = 1$$

Hence, some of the fermions of the eight dimensional super-Yang Mills theory at hand, are related to the graded vectors of an underlying $N = 2$, $d = 1$ unbroken supersymmetry.

We could say that this is in virtue of the fact that the initial system has an $N = 1$ spacetime supersymmetry, so the Hilbert space of the zero modes states inherits an $N = 2$, $d = 1$ supersymmetric quantum algebra. But this is not true since global spacetime supersymmetry in $d > 1$ dimensions and $d = 1$ supersymmetry (supersymmetric quantum mechanics), are not the same both conceptually and quantitatively. Indeed, the SUSY QM supercharges do not generate any transformations between fermionic and bosonic fields. Moreover, these supercharges provide a $Z_2$-grading to the corresponding Hilbert space. It is obvious that the $Z_2$-grading that the SUSY QM algebra provides to some of the fermion states, has nothing to do with the initial supersymmetry. But this grading has a direct impact on the geometry of the associated bundles corresponding to the space $S \times R^{3,1}$.

Before we present the geometric implications of the underlying SUSY QM algebra, let us see whether this algebra remains intact under holomorphic gravitational back-reaction.

So far we supposed that the canonical form for the metric that describes $S$ had the Euclidean form. It is interesting to examine what is the impact of a deformation of the metric on the SUSY QM algebra we presented earlier. The complex dimension two surface $S$ is more like a base space of the Calabi-Yau threefold and not a divisor [7–9]. Since the approach we adopted describes $S$ locally and not globally, there is no consistent way to know the full form of the metric that fully describes the base space $S$. The freedom that stems from this local approach, enables us to use a deformed form of the Euclidean metric we used in order to describe $S$. As we already mentioned, the canonical Euclidean form perfectly describes a gravity decoupled Super Yang-Mills theory. We shall exploit the freedom in selecting the local metric, and use a metric that incorporates some gravitational back-reaction of the complex surface $S$, which is of the form:

$$ds^2 = (1 + \epsilon f_1(s^1))ds^1 \otimes d\bar{s}^1 + (1 + \epsilon f_2(s^2))ds^2 \otimes d\bar{s}^2$$

Accordingly, the Kähler form $\omega$ can be cast as,

$$\omega = \frac{i}{2} (1 + \epsilon f_1(s^1))ds^1 \wedge d\bar{s}^1 + \frac{i}{2} (1 + \epsilon f_2(s^2))ds^2 \wedge d\bar{s}^2$$

In addition, we further assume that $f_1(s^1)$ and $f_2(s^2)$ vanish at $s^1 = 0$ and $s^2 = 0$ respectively, so that the complex curve $\Sigma$ can be formally defined as a divisor of the complex surface $S$, and consequently we do not alter the definition of $\Sigma$. Moreover, the
functions $f_1, f_2$ are assumed to be holomorphic functions of their coordinates, in order the solutions of the equations of motion are sections of holomorphic line bundles along the loci $s^2 = 0$ \[7\] \[9\]. We shall not be interested in the particular form of the solutions the equations of motion, since we are interested only in perturbations of the Witten index. Furthermore, we assume that the functions $f_1$ and $f_2$ are decreasing functions of their arguments. The form of the localized solutions around the complex matter curve $\Sigma$ will have a more evolved form in reference to the non-perturbed metric ones. Let us examine the perturbation around the complex curve $\Sigma$, that is around $s^2 = 0$. The equations of motion corresponding to the $s^2 = 0$ case, are (ignoring again some irrelevant constants):

\[
(1 + \epsilon f_1(s^1)) \partial_2 \psi^a + \tilde{s}^2 \chi^a = 0 \\
\partial_2 \chi^a + s^2 \psi^a = 0
\]

which easily comes to be,

\[
\partial_2 \psi^a + \frac{\tilde{s}^2}{(1 + \epsilon f_1(s^1))} \chi^a = 0 \\
\partial_2 \chi^a + s^2 \psi^a = 0
\]

Expanding in powers of $\epsilon$ and keeping linear terms, we get:

\[
\partial_2 \psi^a + \tilde{s}^2 (1 - \epsilon f_1(s^1)) \chi^a = 0 \\
\partial_2 \chi^a + s^2 \psi^a = 0
\]

The zero modes of equation (33), are the zero modes of the operator:

\[
D_\epsilon = \left( \begin{array}{cc} \partial_2 & \tilde{s}^2 (1 - \epsilon f_1(s^1)) \\ \tilde{s}^2 & \partial_2 \end{array} \right)
\]

We can write $D_\epsilon = D_F + K$, with $D_F$ as in equation (8) and $K$ is the operator:

\[
K = \left( \begin{array}{cc} 0 & \tilde{s}^2 \epsilon f_1(s^1) \\ 0 & 0 \end{array} \right)
\]

We know from the mathematical literature, that compact perturbations of the index of Fredholm operators, leave the index invariant, that is:

* For $Q$ be a Fredholm operator and $K$ be a compact odd operator, then operator $Q + K$ is a Fredholm operator and also:

\[
\text{ind}(Q + K) = \text{ind}Q
\]

The operator $K$ is a compact odd operator. Odd, since it anti-commutes with $W$ and compact since we assumed that the functions $f_i$ are rapidly decreasing functions of their arguments. Thereby we have the result that:

\[
\text{ind}(D_F + K) = \text{ind}D_F
\]
Hence the Witten index of the operator $\mathcal{D}_\epsilon$ is equal to the Witten index of the operator $\mathcal{D}_F$. We can conclude that the SUSY QM algebra remains invariant under rapidly decaying perturbations of the metric. This result is valid, even if we include higher values of $\epsilon$ in the operator $\mathcal{K}$. This can be easily verified, since the operator $\mathcal{K}$ would be equal to:

$$\mathcal{K} = \left( \begin{array}{ccc} 0 & s^2 \epsilon f_1(s^1) + s^2 \epsilon f_2(s^1) + \cdots \\ 0 & 0 \end{array} \right)$$

(38)

This operator is still a compact odd operator and satisfies the aforementioned theorem.

2 Geometrical Implications of the $N = 2$ SUSY QM Algebras

An underlying $N = 2, d = 1$ supersymmetric quantum algebra related to some fermions of the eight dimensional super-Yang Mills theory, implies some additional geometric structures over $S \times R^{3,1}$, and particularly to the associated fibre bundles, to which fermions are sections.

It would be helpful to discuss in short what is our motivation to study such extra geometric structures over $S \times R^{3,1}$, what is our aim and moreover present briefly the main results of this study. The main motivation to our study comes from the fact that some of the sections of the spin bundle over $S \times R^{3,1}$ (the ones localized on the complex dimension one curve $\Sigma$), are related to an $N = 2, d = 1$ supersymmetric quantum algebra, and particularly, the fermions are directly related to the vectors of the graded Hilbert space corresponding to the SUSY QM algebra. This fact suggests that the fermionic sections are directly related to some underlying affine vector bundle that has an intrinsic graded structure. In turn, this in conjunction with the fact that only the $\Sigma$-localized fermions are directly correlated to the SUSY QM algebra, strongly suggests that the total spin bundle over $S \times R^{3,1}$ is reducible (but not projectable) to some subbundle that consists of the affine $Z_2$ graded vector bundle. This reducibility is materialized in terms of the corresponding connections and covariant differentials. We shall formally address these issues in the following subsections. For some references on connections and covariant differentials see for example [42–50] while for issues regarding graded manifolds see [42–47, 51].

2.1 Geometric Structures for the Fermionic Sector

In order to simplify our notation, we denote $X$ the space $S \times R^{3,1}$ upon which the SUSY Yang-Mills model we described in this paper, is built on. The fermionic fields are actually sections of the $G_S$--twisted fibre bundle $P \times S \otimes G_S$, with $S$ the Spin group $Spin(8)$ representation and $P$, the double cover of the principal $SO(8)$ bundle on the tangent manifold $TX$.

As we already mentioned, the $\Sigma$-localized fermions, are elements of graded the vector space of the supersymmetric quantum algebra. The graded vector space $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$, that some of the sections of the total fibre bundle $P \times S \otimes G_S$ belong, strongly suggests
a new geometric structure on the manifold $X$. Particularly $X$ becomes a graded manifold $(X, \mathcal{A})$.

The $N = 2$ SUSY QM algebra, $W, Q_F, Q_F^\dagger$ and particularly the involution $W$, generates the $Z_2$ graded vector space $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$. The subspace $\mathcal{H}^+$ contains $W$-even vectors while $\mathcal{H}^-$, $W$-odd vectors. This grading is an additional algebraic structure on $X$, with $\mathcal{A} = \mathcal{A}^+ \oplus \mathcal{A}^-$ an $Z_2$ graded algebra. Formally, $\mathcal{A}$ is a total rank $m$ ($m = 2$ for our case) sheaf of $Z_2$-graded commutative $R$-algebras. In addition, the sheaf $\mathcal{A}$ underlies the vector space $H = H^+ \oplus H^-$. The subspace $H^+$ contains $W$-even vectors while $H^-$, $W$-odd vectors. This grading is an additional algebraic structure on $X$, with $\mathcal{A} = \mathcal{A}^+ \oplus \mathcal{A}^-$ an $Z_2$ graded algebra. Formally, $\mathcal{A}$ is a total rank $m$ ($m = 2$ for our case) sheaf of $Z_2$-graded commutative $R$-algebras. In addition, the sheaf $\mathcal{A}$ underlies the vector space $H$ and makes the space $H$ an $Z_2$-graded $\mathcal{A}$-module. This can be verified from the fact that:

$$A_+ \cdot H_+ \subset H_+, \quad A_- \cdot H_- \subset H_-, \quad A_- \cdot H_+ \subset H_-, \quad A_+ \cdot H_- \subset H_+$$ (39)

The sheaf $\mathcal{A}$ contains the endomorphism $W$ (which is the involution of the SUSY quantum algebra), $W : \mathcal{H} \to \mathcal{H}$, with $W^2 = I$, providing the $Z_2$-grading on the vector space $\mathcal{H}$, i.e.:

$$W^\mathcal{H}^\pm = \pm 1$$ (40)

Thereby, $\text{End}(\mathcal{H}) \subseteq \mathcal{A}$. $\mathcal{A}$ is called a structure sheaf of the graded manifold $(X, \mathcal{A})$, while $X$ is the body of $(X, \mathcal{A})$.

Since we shall make use of connections on graded manifolds, it is useful to see how these are constructed. Given an open neighborhood $U$ of $x \in X$, we have that locally (bearing in mind that in our case $m = 2$):

$$\mathcal{A}(U) = C^\infty(U) \otimes \wedge^R m$$ (41)

Therefore, the structure sheaf $\mathcal{A}$ is isomorphic to the sheaf $C^\infty(U) \otimes \wedge^R m$ of some exterior affine vector bundle $\wedge^\mathcal{H}^* = U \times \wedge^R m$, with $\mathcal{H} \mathcal{E}$ the affine vector bundle with fiber the vector space $\mathcal{H}$. The structure sheaf $\mathcal{A} = C^\infty(U) \otimes \wedge\mathcal{H}$, is isomorphic to the sheaf of sections of the exterior vector bundle $\wedge^\mathcal{H}^* = R \oplus (\oplus_{k=1}^m \wedge^k) H^*$. As we shall see, the bundle $\mathcal{H} \mathcal{E}$ is a crucial component of the composite bundle we shall construct.

The connections of the graded manifold $(X, \mathcal{A})$ compose an affine space modeled on the space of sections of the vector bundle $TX^* \otimes \wedge \mathcal{H} \mathcal{E}^* \otimes \mathcal{H} \mathcal{E}$. The sections of the bundle $TX^* \otimes \wedge \mathcal{H} \mathcal{E}^* \otimes \mathcal{H} \mathcal{E}$ are operators that belong to the sheaf $\mathcal{A}$ with $\mathcal{H}$-valued forms as elements in some specific representation, the dimension of which depends on the rank of the sheaf. In the present case, these are $2 \times 2$ matrices with $\mathcal{H}$-valued forms as matrix elements. The graded connection will serve as an auxiliary object, since we are interested in the composition of this auxiliary connection with another connection we shall introduce soon.

The Dirac covariant differential, denoted as $\nabla^\gamma$ corresponding to the connection $\gamma_s$ of the total bundle, $P \times S \otimes G_S$, results to the equations (3) and (4). But since only the set of equations (3) are connected to an unbroken $N = 2$ SUSY QM algebra, suggests that the connection $\gamma_s$ is reducible to a connection $\gamma_C$, which in some way is related to the graded manifold $(X, \mathcal{A})$. This reducibility implies that the total covariant differential of the fibre bundle $P \times S \otimes G_S$, is reducible to one covariant differential that when it acts on integral sections of the fibre bundle $P \times S \otimes G_S$ (in this paper we mainly focused on the integral sections of the fibre bundles), we get the equations of (4).
The $N = 2$ SUSY QM algebra suggests an underlying geometric structure that is pictured by the following diagram:

\[
\begin{array}{c}
X \\
\downarrow \gamma_s \\
S \times P \otimes G \Sigma \\
\downarrow \gamma_E \\
TX^* \otimes \Lambda \mathcal{H}^* \Sigma \mathcal{H}
\end{array}
\]

with the arrows showing the directions of the connections of the corresponding fibre bundles and not the direction of the projective maps of the bundles. The connections appearing above are morphisms of the following fibre maps:

\[
\begin{align*}
\gamma_s &: P \times S \otimes G \Sigma \rightarrow J^1(P \times S \otimes U(1)), \\
& \text{(Bundle map, } \pi_s : P \times S \otimes G \Sigma \rightarrow X) \\
\gamma_E &: TX^* \otimes \Lambda \mathcal{H}^* \Sigma \mathcal{H} \rightarrow J^1(TX^* \otimes \Lambda \mathcal{H}^* \Sigma \mathcal{H}) \\
& \text{(Bundle map, } \pi_E : TX^* \otimes \Lambda \mathcal{H}^* \Sigma \mathcal{H} \rightarrow X) \\
\gamma_{SE} &: (P \times S \otimes G \Sigma)_G \rightarrow J^1((P \times S \otimes G \Sigma)_G) \\
& \text{(Bundle map, } \pi_{SE} : P \times S \otimes G \Sigma \rightarrow TX^* \otimes \Lambda \mathcal{H}^* \Sigma \mathcal{H})
\end{align*}
\]

In the above, $J^1Y_i$ stands for the jet bundle of the bundle $Y_i$. The underlying geometrical structure is actually a composite fibre bundle over the manifold $X$, of the form:

\[
P \times S \otimes G \Sigma \xrightarrow{\pi_{SE}} TX^* \otimes \Lambda \mathcal{H}^* \Sigma \mathcal{H} \xrightarrow{\pi_E} X
\]

The composite connection corresponding to the composite fibre bundle (43) is defined as follows:

\[
\gamma_C = \gamma_{SE} \circ \gamma_E
\]

which obviously is the composition of the connections $\gamma_{SE}$ and $\gamma_E$. The composite connections appear in the first order differential operators and the corresponding covariant differentials, thus altering their final form. It worths remembering the definition of the first order differential and of the covariant differential corresponding to a connection. Let $Y \rightarrow X$, be an arbitrary fibre bundle, with $s_Y : X \rightarrow Y$ a section, and a connection $\gamma_Y : Y \rightarrow J^1Y$. The first order differential is defined to be:

\[
\mathcal{D}_{\gamma_Y} : J^1Y \rightarrow TX^* \otimes VY
\]

In the above relation, $VY$ denotes the vertical subbundle of $Y$, which when $Y$ is a vector bundle, we have $VY = Y \times Y$. Accordingly, the covariant differential corresponding to $\gamma_Y$, denoted $\nabla^{\gamma_Y}$, is:

\[
\nabla^{\gamma_Y} = \mathcal{D}_{\gamma_Y} \circ J^1s_Y : X \rightarrow TX^* \otimes VY
\]
Therefore, the total covariant differential of the bundle $P \times S \otimes G_S \to X$, is equal to:

$$\nabla^\gamma = D_{\gamma_S} \circ J^1 s : X \to TX^* \otimes V(P \times S \otimes G_S) \equiv TX^* \otimes P \times S \otimes G_S$$  \hspace{1cm} (47)

When this covariant differential acts on integral sections of $P \times S \otimes G_S \to X$, results to equations (3) and (4). Before proceeding, it worths discussing something very crucial for the rest of the analysis. We denote by $s_E$ and $s_{SE}$ the sections of the following bundles:

$$s_E : X \to TX^* \otimes \bigwedge^h E^* \otimes h_E$$  \hspace{1cm} (48)
$$s_{SE} : TX^* \otimes \bigwedge^h E^* \otimes h_E \to P \times S \otimes G_S$$

The subbundle $Y_h$, is the restriction $Y_h = s_E^*(P \times S \otimes G_S)$ of the fibre bundle $\pi_{SE} : P \times S \otimes G_S \to TX^* \otimes \bigwedge^h E^* \otimes h_E$, to the submanifold $s_E(X) \subset P \times S \otimes G_S$, through the inclusion map

$$i_h : Y_h \hookrightarrow P \times S \otimes G_S$$  \hspace{1cm} (49)

The composition of the sections $s_E$ and $s_{SE}$ is $s_C = s_{SE} \circ s_E$. The section $s_C$, is a section of the fibre bundle $\pi_s : P \times S \otimes G_S \to X$, with $s_C(X) \subset P \times S \otimes G_S$. Accordingly, the covariant differential for the composite bundle corresponding to the connection (44), which we denote $\nabla^\gamma_C$, follows easily:

$$\nabla^\gamma_C = D_{\gamma_C} \circ J^1 s_C : X \to TX^* \otimes VY_h \equiv TX^* \otimes Y_h$$  \hspace{1cm} (50)

When the covariant differential is applied to integral sections of $Y_h$, which form a subset of the set of integral sections of $P \times S \otimes G_S$, we obtain the set of equations of relation (4). Therefore, we may conclude that the covariant differential, $\nabla^\gamma$ of the total bundle $P \times S \otimes G_S$ is reducible to the covariant differential $\nabla^\gamma_C$. This implies that the connection $\gamma_s$ is reducible to $\gamma_C$ (but in any case not projectable), in such a way so that the following diagram is commuting:

$$
\begin{array}{ccc}
P \times S \otimes G_S & \xrightarrow{\gamma_s} & J^1 Y_h \\
\downarrow i_h & & \downarrow J^1 i_h \\
Y_h & \xrightarrow{\gamma_C} & J^1 P \times S \otimes G_S
\end{array}
$$

Recall that the inclusion map $i_h$ is the one appearing in relation (49) and $J^1 i_h$ the corresponding jet prolongation of this inclusion map.

### 2.2 Sections of Canonical Bundles and Sections of the Composite Bundle

Using the geometric constructions we just presented, we can directly connect the sections of the associated fibre bundles corresponding to the underlying graded geometry, to the local geometry of the complex curve. Recall that $\Sigma$ is defined to be the complex curve inside $S$ with $s^2 = 0$. We mentioned earlier that only the localized fields around the curve $\Sigma$ can be associated to the underlying graded structure, but let us give a more formal description of the localized fields around $\Sigma$ in terms of line bundles of the curve. The
normal bundle to the curve $\Sigma$, which we denote $N_{\Sigma/S}$ is parametrized by the coordinate $s^1$ and satisfies:

$$N_{\Sigma/S} \otimes N_{\Sigma/S} \cong K_{\Sigma/S}^{-1}$$

with $K_{\Sigma/S}^{-1}$ the anti-canonical bundle of the manifold $\Sigma$. This obviously implies that the normal bundle $N_{\Sigma/S}$ is equivalent to $K_{\Sigma/S}^{-1/2}$, which means that a section of $N_{\Sigma/S}$, transforms as an anti-holomorphic $1/2$ form on $\Sigma$, that is $(-1/2,0)$. This fact defines a spin structure on $\Sigma$, since the sections of the inverse normal bundle $K_{\Sigma/S}^{-1/2}$ are holomorphic $(1/2,0)$ forms on $\Sigma$. This could be deduced directly from equation (51), by exploiting the fact that the canonical bundle $K_{\Sigma/S}$ in our case is a line bundle, and hence relation (51), reveals a local spin structure around $\Sigma$. So geometrically the localized fermions around $\Sigma$ are actually the sections of the inverse normal bundle $N_{\Sigma/S}^{-1}$ or equivalently the sections of the square root of the canonical line bundle (in our case) $K_{\Sigma/S}^{1/2}$. Using the terminology and notation of the previous section, we can state that the sections $s_C = s_{SE} \circ s_E$ of the composite fibre bundle $\pi_s : P \times S \otimes G_S \xrightarrow{\xi_{SE}} TX^* \otimes \mathcal{H}_E^* \otimes \mathcal{H}_E \xrightarrow{\xi_E} X$, are in bijective (one-to-one) correspondence to the sections of the square root of the canonical bundle $K_{\Sigma/S}^{1/2}$, corresponding to the complex curve $\Sigma$.

**Concluding Remarks**

In this paper we studied an eight dimensional super-Yang Mills theory, compactified on $S \times R^{3,1}$. Working locally in $\Sigma \subset S$ we established the result that some sections of the spin bundle over $S \times R^{3,1}$, and particularly those localized on $\Sigma$, are related to an unbroken $N = 2$ SUSY QM algebra. This SUSY QM algebra provides the fermionic sector with an additional geometric structure over the $S \times R^{3,1}$ space. Particularly, this geometric structure consists of a composite bundle over the space $S \times R^{3,1}$, with a graded manifold over it being a basic ingredient of the composite bundle. Specifically, we demonstrated that the covariant differential $\nabla^{\gamma_s}$ is reducible to $\nabla^{\gamma_c}$ corresponding to some subbundle $Y_0$ of the initial total $G_S$-twisted spin bundle. Moreover, we have shown that the sections of the composite fibre bundle $\pi_s : P \times S \otimes G_S \xrightarrow{s_{SE}} TX^* \otimes \mathcal{H}_E^* \otimes \mathcal{H}_E \xrightarrow{s_E} X$, are in bijective correspondence to the sections of the square root of the canonical bundle $K_{\Sigma/S}^{1/2}$, which are the localized zero modes on the curve $\Sigma$.

The $N = 1$ spacetime supersymmetry of the initial theory on $S \times R^{3,1}$, implies that we can construct a supermanifold on $S \times R^{3,1}$. Interestingly enough, apart from the $N = 1$ supermanifold which we can construct over $S \times R^{3,1}$, that manifold is also a $Z_2$-graded manifold, with the grading structure provided by the involution $W$, the Witten parity of the SUSY QM system. It worths examining if there is any direct correlation between the $N = 1$ supermanifold and the graded manifold (which is not a supermanifold) motivated by the fact that every graded manifold defines a DeWitt supermanifold. Moreover, it would be of some importance to look for higher order one dimensional extended supersymmetries, possibly with non-zero supercharge $[52,53]$. There is an indication of such a structure, since the $N = 1$ supersymmetry implies that the zero modes of the fermions are accompanied by bosonic zero modes. In turn, a SUSY QM algebra can be
built for the bosonic sector (or the corresponding bosonic fluctuations). Hence it probable
that the two $N = 2$ SUSY QM algebras can combine to give a higher order extended $d = 1$
supersymmetry.

Before closing, we have to comment that the results presented in this article are mostly
relevant to mathematical problems. Particularly, in view of the fact that the complexity
of a generally difficult problem is reduced by finding symmetries or regularities of a system
in general, we established the result that there is a one-to-one correspondence between the
sections of the composite fibre bundle related the SUSY QM structure and the sections of
the square root of the F-theory canonical bundle. This is of particular importance since the
graded manifold defines locally a DeWitt supermanifold, and hence we may extend the line
of research we adopted to the global supersymmetric structure of the theory. Apart from
these mathematical applications, there is a probable physical application. Particularly,
the results we presented might shed some light on the local supersymmetry structure of
the theory, especially if the gravitational back-reaction to the local metric is taken into
account. But these tasks are outside the scope of this article.

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