SECTIONS OF SURFACE BUNDLES

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Abstract. Let \( p : E \to B \) be a bundle projection with base \( B \) and fibre \( F \) aspherical closed connected surfaces. We review what algebraic topology can tell us about such bundles and their total spaces and then consider criteria for \( p \) to have a section. In particular, we simplify the cohomological obstruction, and show that the transgression \( d_{2,0}^2 \) in the homology LHS spectral sequence of a central extension is evaluation of the extension class. We also give several examples of bundles without sections.

Let \( p : E \to B \) be a bundle projection with base \( B \) and fibre \( F \) aspherical closed connected surfaces, and let \( \pi = \pi_1(E) \), \( \beta = \pi_1(B) \) and \( \phi = \pi_1(F) \). The exact sequence of homotopy for \( p \) reduces to an extension of fundamental groups

\[
\xi(p) : 1 \to \phi \to \pi \to \beta \to 1,
\]

which determines \( p \) up to bundle isomorphism over \( B \). Sections \( \S2-\S4 \) are a summary of our talk at the Bonn conference, reviewing what else algebraic topology can tell us about such bundles and their total spaces. (We refer to [10] for the arguments and other sources.)

The main part of this work (\( \S5-\S8 \)) considers criteria for the existence of sections. Such a bundle \( p \) has a section if and only if \( \xi(p) \) splits, and this is so if and only if the action of \( \beta \) through outer automorphisms of \( \phi \) lifts, and a cohomology class is trivial. We shall simplify the latter condition, and show that the transgression \( d_{2,0}^2 \) in the homology LHS spectral sequence of a central extension is evaluation of the extension class. It is relatively easy to find examples of torus bundles without sections, but seems more difficult to construct such examples with hyperbolic fibre. We thank H.Endo for the example with base and fibre of genus 3 given in \( \S9 \).

We conclude with a short list of questions arising from issues considered here.

1991 Mathematics Subject Classification. 20K35,57N13.

Key words and phrases. factor set, surface bundle, section.
1. NOTATION

Let $\zeta_G$, $G'$ and $I(G)$ denote the centre, the commutator subgroup and the isolator subgroup of a group $G$, respectively. (Thus $G' \leq I(G)$ and $G/I(G)$ is the maximal torsion-free quotient of the abelianization $G_{ab} = G/G'$.). If $H$ is a subgroup of $G$ let $C_G(H)$ be the centralizer of $H$ in $G$. Let $c_g$ denote conjugation by $g$, for all $g \in G$.

We shall assume throughout that “surface” means aspherical closed connected 2-manifold, unless otherwise qualified. A group $G$ is a $PD_2$-group if $G \cong \pi_1(X)$ for some such surface $X$, and it is a $PD_2^+$-group if $X$ is orientable. If $G$ is a $PD_2$-group, with orientation character $w = w_1(G) = w_1(X)$, let $G^+ = \text{Ker}(w)$ and let $X^+$ be the associated orientable covering space of $X$.

2. BUNDLES AND GROUP EXTENSIONS

The classification of surface bundles follows from the deep result of [6], that if $X$ is a hyperbolic surface then the identity component of $\text{Diff}(X)$ is contractible. (See [9] for a proof using only differential topology, which applies also to the based case.) The flat surfaces $X = T$ or $Kb$ have circle actions, and $\text{Diff}(X)_o \sim (S^1)^r$, where $r = 2$ or $1$ is the rank of the centre of $\pi_1(X)$. In all cases, the inclusions of $\text{Diff}(X)$ into $\text{Homeo}(X)$ and into the group of self homotopy equivalences $E(X)$ are homotopy equivalences.

The numbers in parentheses before the statements of theorems in this section refer to the corresponding theorems in [10].

Theorem (5.2). Let $p : E \to B$ be a bundle projection with base $B$ and fibre $F$ aspherical closed connected surfaces. Then $p$ is determined up to bundle isomorphism over $B$ by the group extension $\xi(p)$. Conversely, every such extension is realized by some bundle.

Conjugation in $\pi$ determines a homomorphism $\theta$ from $\beta$ to the outer automorphism group $\text{Out}(\phi)$. If $\chi(F) < 0$ (i.e., if $\zeta \phi = 1$), the extension is determined by the action alone; in general, extensions corresponding to a given action $\theta$ are classified by characteristic cohomology classes in $H^2(B; \zeta \phi^\theta) = H^2(\beta; \zeta \phi^\theta)$. If we allow change of coordinates in base and fibre, we may divide out by the actions of automorphisms of base and fibre.

When is a closed 4-manifold $M$ “equivalent” to the total space of such a bundle, and, if so, in how many ways? If equivalent means “homotopy equivalent” or “$TOP$ $s$-cobordant”, there is a satisfactory answer, but little is known about diffeomorphism, except when $M$ has additional structure.
Theorem (3.5.1). Let $M$ be a closed 4-manifold such that $\pi = \pi_1(M)$ is an extension of $\pi_1(B)$ by $\pi_1(F)$. Then $M$ is aspherical if and only if $\chi(M) = \chi(B)\chi(F)$.

Although 4-dimensional TOP surgery techniques are not yet available if $\pi$ has non-abelian free subgroups, 5-dimensional surgery often suffices to construct s-cobordisms.

Theorem (6.15). A closed 4-manifold $M$ is TOP s-cobordant to the total space $E$ of an $F$-bundle over $B$, if and only if $\pi_1(M) \cong \pi_1(E)$ and $\chi(M) = \chi(E)$. If so, then $\tilde{M}$ is homeomorphic to $R^4$.

When $\pi$ is virtually solvable s-cobordism implies homeomorphism, and $M$ is then homeomorphic to an $E^4$, $Nil^4$, $Nil^3 \times E^1$- or $S\tilde{L} \times E^1$-manifold. Conversely, if $M$ has one of these geometries and $\beta_1(M) \geq 2$ then $M$ fibres over $T$. (See Chapter 8 of [10].) The other geometries realized by total spaces of surface bundles are $H^2 \times E^2$, $H^3 \times E^1$, $S\tilde{L} \times E^1$ and $H^2 \times H^2$. (See Chapters 7 and 13 of [10].) Hamenstädt has announced that no such bundle space has geometry $H^4$. Finally, $H^2(C)$ may be excluded as a consequence of the next theorem (due independently to Kapovich, Kotschick and Hillman) and the fact that quotients of the unit ball in $C^2$ do not submerse holomorphically onto complex curves [15].

Theorem (13.7). Let $S$ be a complex surface. Then $S$ admits a holomorphic submersion onto a complex curve, with base and fibre of genus $\geq 2$, if and only if $S$ is homotopy equivalent to the total space of a bundle with base and fibre hyperbolic surfaces.

In this case homotopy equivalence implies diffeomorphism!

3. THE GROUP DETERMINES THE BUNDLE UP TO FINITE AMBIGUITY

If $\chi(B) < 0$, then there are only finitely many ways of representing $\pi$ as an extension of $PD_3$-groups, up to “change of coordinates” [13]. Let $\pi$ be a $PD_4$-group with a normal subgroup $K$ such that $K$ and $\pi/K$ are surface groups with trivial centre. Johnson showed that whether

(I) $\text{Im}(\theta)$ is infinite and $\text{Ker}(\theta) \neq 1$;
(II) $\text{Im}(\theta)$ is finite; or
(III) $\theta$ is injective

depends only on $\pi$ and not on the subgroup $K$. In case I there is an unique such normal subgroup, and in case II there are two, and $\pi$ is virtually a product. In case III we can only say that there are finitely many such normal subgroups. These assertions follows ultimately from the facts that nontrivial finitely generated normal subgroups of hyperbolic
surface groups have finite index, and the Euler characteristic increases on passage to such subgroups. (See [13], or Theorems 5.5 and 5.6 of [10].)

The examples of Kodaira, Atiyah and Hirzebruch of surface bundles with nonzero signature are of type III, and each have at least two such normal subgroups [4]. It is noteworthy that in each case one of the subgroups satisfies the condition $\chi(K)^2 \leq \chi(\pi)$. It is a straightforward consequence of Johnson’s arguments that this extra condition holds for at most one such subgroup $K$, if $\pi$ is of type III. (In particular, if $\chi(\pi) = 4$ there is at most one such $K$ with $K$ and $\pi/K$ both orientable.)

The Johnson trichotomy extends to the case when $\pi/K$ has a centre, but is inappropriate if $\zeta K \neq 1$, as there are then nontrivial extensions with trivial action ($\theta = 1$). Moreover $Out(K)$ is then virtually free, and so $\theta$ is never injective.

However the situation is very different if $\pi/K \cong \mathbb{Z}^2$, $\chi(K) < 0$ and $\beta_1(\pi) > 2$. For then there are epimorphisms from $\pi$ to $\mathbb{Z}^2$ with kernel a surface group of arbitrarily high genus [5].

4. ORBIFOLDS, SEIFERT FIBRATIONS AND VIRTUAL BUNDLES

In this section we allow the base $B$ to be an aspherical 2-orbifold. An orbifold bundle with general fibre $F$ over $B$ is a map $f : M \to B$ which is locally equivalent to a projection $G\backslash(F \times D^2) \to G\backslash D^2$, where $G$ acts freely on $F$ and effectively and orthogonally on $D^n$. We shall also say that $f : M \to B$ is an $F$-orbifold bundle (over $B$) and $M$ is an $F$-orbifold bundle space.

A virtual bundle space is a manifold with a finite cover which is the total space of a bundle. The total spaces of orbifold bundles are virtual bundle spaces, but the converse is not true. A Seifert fibration (in dimension 4) is an orbifold bundle with general fibre $T$ or $Kb$.

Aspherical orbifold bundles (with 2-dimensional base and fibre) are determined up to fibre-preserving diffeomorphism by their fundamental group sequences. In many cases they are determined up to diffeomorphism (among such spaces) by the group alone [17]. See [11] for a discussion of geometries and geometric decompositions of the total spaces of such orbifold bundles.

Johnson’s trichotomy extends to groups commensurate with extensions of a surface group by a surface group with trivial centre, but it is not known whether all torsion-free such groups are realized by aspherical 4-manifolds.
5. EXTENSIONS OF GROUPS

An extension $\xi$ of a group $\beta$ with kernel $\phi$ splits if and only if the action $\theta : \beta \to \text{Out}(\phi)$ induced by conjugation in the “ambient group” $\pi$ factors through a homomorphism $\tilde{\theta} : \beta \to \text{Out}(\phi)$ and the cohomology class $[\xi] \in H^2(\beta; \zeta \phi)$ of the extension is 0. (Here $\zeta \phi$ is the centre of $\phi$, considered as a $\mathbb{Z}[\beta]$-module via the action $\theta$.) If so, then $\pi$ is a semidirect product $\phi \rtimes \tilde{\theta} \beta$.

There is a natural restriction homomorphism from $\text{Aut}(G)$ to $\text{Aut}(\zeta G)$, which factors through $\text{Out}(G)$. In particular, if $\theta : \beta \to \text{Out}(\phi)$ is a homomorphism then composition with restriction defines a natural $\mathbb{Z} [\beta]$-module structure on $\zeta \phi$. The extensions with given action $\theta$ may be parametrized by $H^2(\beta; \zeta \phi)$. (In general, there is an obstruction in $H^2(\beta; \zeta \phi)$ for there to be such an extension, but this obstruction group is trivial when $\beta$ is a surface group. See Chapter IV of [3].) If $\theta$ factors through $\text{Aut}(\phi)$ then the semidirect product corresponds to $0 \in H^2(\beta; \zeta \phi)$.

**Lemma 1.** If $\zeta \phi = 1$ then $\xi$ splits if and only if the action $\theta$ factors through $\text{Aut}(\phi)$.

**Proof.** If $\phi$ has trivial centre then the extension is determined by the action, since $H^2(\beta; \zeta \phi) = 0$. Thus if the action factors $\pi$ must be a semidirect product, i.e., $p \ast$ splits. The converse is clear. □

The exact sequence of low degree for the extension has the form

$$H_2(\pi; \mathbb{Z}) \to H_2(\beta; \mathbb{Z}) \to H_0(\beta; H_1(\phi; \mathbb{Z})) \to H_1(\pi; \mathbb{Z}) \to H_1(\beta; \mathbb{Z}) \to 0.$$  

The second homomorphism in this sequence is the transgression $d^2_{2,0}$ from the homology LHS spectral sequence for the extension. If the extension splits this gives an isomorphism

$$\pi^{ab} \cong (\phi^{ab}/(I - \theta^{ab})\phi^{ab}) \oplus \beta^{ab} = (\phi/\pi, \phi) \oplus \beta^{ab},$$

where $\theta^{ab}$ is the automorphism of $\phi^{ab}$ induced by $\theta$. This apparently innocuous observation gives the most practical test for whether an extension $\xi$ splits, both when $\phi' = 1$, as in the next lemma, and when $\zeta \phi = 1$, as considered in §9 below.

**Lemma 2.** Let $G$ be a group with a finitely generated abelian normal subgroup $A$ such that $\beta = G/A$ is a PD$_2^+$-group. Then the canonical projection from $G$ to $\beta$ has a section if and only if

$$G^{ab} \cong A/[G, A] \oplus \beta^{ab}.$$  

**Proof.** Let $\overline{A} = A/[G, A]$ and $\overline{G} = G/[G, A]$. Then $\overline{G}$ is a central extension of $\beta$ by $\overline{A}$, and $G^{ab} = \overline{G}^{ab}$. Since c.d.$\beta = 2$, the epimorphism
from $A$ to $\overline{A}$ induces an epimorphism from $H^2(\beta; A)$ to $H^2(\beta; \overline{A})$. Since $\beta$ is a $PD_2$-group, $H^2(\beta; A) \cong H_0(\beta; A)$ and $H^2(\beta; \overline{A}) \cong H_0(\beta; \overline{A})$.

These are each isomorphic to $A$, and so the natural homomorphism from $H^2(\beta; A)$ to $H^2(\beta; \overline{A})$ is an isomorphism. Therefore $G$ splits as a semidirect product if and only if the same is true for $\overline{G}$. Since $\overline{A}$ is central in $\overline{G}$, this is so if and only if $\overline{G} \cong A \times \beta$, and this is equivalent to $\overline{G}^{ab} \cong A \oplus \beta^{ab}$.

6. THE EXTENSION CLASS FOR ACTIONS WHICH LIFT

Let $\beta$ have a finite presentation $\langle X | R \rangle$, with associated epimorphism $q : F(X) \to \pi$. After introducing new generators $x'$ and new relators $x'x$, if necessary, we may assume that the exponents of the generators in each relator are all positive. Let $\varepsilon : Z[\beta] \to Z$ be the augmentation homomorphism, and let $\partial : Z[F(X)] \to Z[\beta]$ be the composite of the Fox free derivative with the linear extension of $q$, for each $x \in X$. Then $\varepsilon_x(v) = \varepsilon \partial_x(v)$ is the exponent sum of $x$ in the word $v$. The presentation determines a Fox-Lyndon partial resolution

$$C^{FL}_*(\beta) : Z[\beta]^R \to Z[\beta]^X \to Z[\beta] \to Z \to 0,$$

where the differentials map the basis elements by

$$\partial c_0 = 1, \quad \partial c^r_1 = x - 1 \quad \text{and} \quad \partial c^r_2 = \sum_{y \in X} \partial y \partial c^{y}_1,$$

for all $r \in R$ and $x \in X$.

Let $h_\ast : C^{FL}_*(\beta) \to C^{\bar{\text{bar}}}_*(\beta)$ be the chain morphism to the normalized bar resolution $C^{\bar{\text{bar}}}_\ast$ given by the identity on $C^{FL}_0(\beta) = Z[\beta] = C^{\bar{\text{bar}}}_0$, the natural inclusion of $C^{FL}_1(\beta) = Z[\beta]^X$ into $C^{\bar{\text{bar}}}_1$, and which sends the generator $c_2^0$ of $C^{FL}_2(\beta) = Z[\beta]^R$ to $\sum_{x \in X} \partial_x r | x \in C^{\bar{\text{bar}}}_2$, for all $r \in R$. (See Exercises II.5.3 and II.5.4 of [3].) If $c.d. \beta \leq 2$ then $C^{FL}_*(\beta)$ is a resolution and $h$ is a chain homotopy equivalence.

Suppose that $\theta$ factors through a homomorphism $\overline{\theta} : \beta \to Aut(\phi)$. Let $\sigma : \beta \to \pi$ be a set-theoretic section such that $\sigma(1) = 1$ and $c_{\sigma(g)} = \overline{\theta}(g)$, for all $g \in \beta$, and define a linear function $f$ on $C^{\bar{\text{bar}}}_2$ by

$$\sigma(g)\sigma(h) = f([g|h])\sigma(gh), \quad \text{for all} \ g, h \in \beta.$$

Then $f$ takes values in $\zeta \phi$, since $\overline{\theta}$ is a homomorphism, and is a 2-cocycle, which represents the extension class $[\xi] \in H^2(\beta; \zeta \phi)$. (See §3 and §6 of Chapter IV of [3].) Let $\xi_\ast$ be the image of $[\xi]$ under the change of coefficients and evaluation homomorphisms

$$H^2(\beta; \zeta \phi) \to H^2(\beta; H_0(\beta; \zeta \phi)) \to Hom(H_2(\beta; Z), H_0(\beta; \zeta \phi)).$$
For each \( r = \Pi_{i=1}^c x_i \) in \( R \), let \( I_k(r) = \Pi_{i=1}^{k-1} x_i \), for \( 1 \leq k \leq c \). Then 
\[
\partial_x r = \Sigma_{x_i=x} I_i(r), \text{ for all } x \in X, \text{ and so }
\]
\[
f(h_2(c_2)) = f(\Sigma_{x \in X} [\partial_x r | x]) = f(\Sigma_{i=1}^c [I_i(r) | x_i]) = \Sigma_{i=1}^c f([I_i(r) | x_i]).
\]
On the other hand, if \( s : F(X) \to \pi \) is the homomorphism defined by \( s(x) = \sigma(q(x)) \), for all \( x \in X \), then 
\[
s(r) = \Pi_{i=1}^c s(x_i) = \Pi_{i=1}^c \sigma(q(x_i)).
\]
A simple induction shows that this is 
\[
\Pi_{i=1}^c f([I_i(r) | x_i]) \sigma(q(r)) = \Pi_{i=1}^c f([I_i(r) | x_i])\sigma(1) = \Pi_{i=1}^c f([I_i(r) | x_i]).
\]
In additive notation, this is just \( f(h_2(c_2)) \). It follows that 
\[
\xi_*(\{z\}) = s(\Pi r^{nr}),
\]
for any 2-cycle \( z = \Sigma_{r \in R} r c'_2 \) of \( \mathbb{Z} \otimes \beta c_{*}^{FL} (\beta) \). (With a little more effort, we could avoid the assumption that the exponents in the relators are all positive.)

Suppose now that \( \beta \) is a \( PD_2 \)-group with orientation character \( w = w_1(\beta) \). Let \( \varepsilon_w : \mathbb{Z}[\beta] \to \mathbb{Z} \) be the \( w \)-twisted augmentation, defined by the linear extension of \( w : \beta \to \mathbb{Z}^\times \), and let \( J_w = \text{Ker}(\varepsilon_w) \). If \( A \) is any left \( \mathbb{Z}[\beta] \)-module then \( H^2(\beta; A) \cong A/(\partial_x r | x \in X)A \), since \( c.d. \beta = 2 \). This is isomorphic to \( H_0(\beta; \overline{A}) = A/J_w A \), by Poincaré duality, and so \( J_w \) is also the ideal generated by \( \{ \partial_x r | x \in X \} \). Then we may recapitulate the above discussion as follows.

**Lemma 3.** If \( \theta \) factors through \( \text{Aut}(\phi) \) then \( s(r) \) is in \( \zeta \phi \), and its image \( [s(r)] \) is well defined. The epimorphism \( p_* \) splits if and only if \( [s(r)] = 0 \).

**Proof.** The first assertion holds since \( q(r) = 1 \). If \( \sigma' \) is another such set-theoretic section and \( s' = \sigma' q \) then \( s'(x) = u(q(x)) s(x) \) for some function \( u : q(X) \to \zeta q \). (Conversely, every such function \( u \) arises in this way.) A simple induction shows that \( s'(r) = s(r) + \Sigma_{x \in X} \partial_x r. u(q(x)) \), and so \( [s(r)] \) is independent of the choice of section.

If \( \sigma : \beta \to \pi \) splits \( p_* \) then we may take \( s = \sigma q \), and so \( s(r) = 1 \) in \( \phi \). Hence \( [s(r)] = 0 \). Conversely, if \( [s(r)] = 0 \) then we may choose \( s \) so that \( s(r) = 1 \), and so \( p_* \) splits. \( \Box \)

### 7. Abelian Extensions and Transgression

We shall show that if \( \xi \) is an extension with abelian kernel \( \phi \) then \( \xi_* \) is the transgression homomorphism \( d_{2,0}^2 \) in the exact sequence of low degree. This appears to be “folklore”, but we have not found a published proof. (Theorem 4 of [12] gives the cohomological analogue.)
Our argument uses naturality of the constructions arising (cf. [1, 2]) to reduce to a special case.

**Theorem 4.** If $\phi$ is abelian then $\xi_* = d_{2,0}^2$.

**Proof.** We shall reduce to the situation when $\beta \cong \mathbb{Z}^2$, $\phi$ is infinite cyclic and central, and $[\xi]$ generates $H^2(\beta; \mathbb{Z}) \cong \mathbb{Z}$. (Thus $\pi \cong F(2)/F(2)[3]$ is the free 2-generator nilpotent group of class 2.)

Let $\xi$ be the extension $0 \to \phi/\pi, \phi \to \pi/[\pi, \phi] \to \beta \to 1$ obtained by factoring out $[\pi, \phi]$. Then the projection of $\pi$ onto $\pi/[\pi, \phi]$ induces isomorphisms of the 5-term exact sequences corresponding to the extensions $\xi$ and $\xi$, and so we may assume that $\phi$ is central.

Secondly, every class in $H_2(\beta; \mathbb{Z})$ is the image of the fundamental class of an aspherical orientable surface. (This is most easily seen topologically, by assembling pairwise the 2-simplices of a representative 2-cycle for $H_2(K(\beta, 1); \mathbb{Z})$, or by using orientable bordism, since $\Omega_2(X) = H_2(X; \Omega_0)$ for any cell complex $X$. However, there is an algebraic argument in [18].) Thus if $\xi \in H_2(\beta; \mathbb{Z})$ there is a $PD_2^+$-group $\beta$ with fundamental class $[\beta] \in H_2(\beta; \mathbb{Z})$ and a homomorphism $f : \beta \to \beta$ such that $f_*([\beta]) = [\xi]$. On passing to the extension $f^* \xi$, we may assume that $\beta$ is a $PD_2^+$-group.

Thirdly, suppose that $\beta$ is the $PD_2^+$-group of genus $g$ and $\eta_\beta$ is the central extension of $\beta$ by $\phi = \mathbb{Z}^2$ corresponding to a generator of $H^2(\beta; \mathbb{Z})$. It is easy to see that $d_{2,0}^2$ and $\eta_\beta$ are isomorphisms of infinite cyclic groups, and so $d_{2,0}^2 = \pm \eta_\beta^*$. There is a natural morphism of extensions from $\eta_\beta$ to $\xi$, and it follows that $d_{2,0}^2 = \pm \xi_*$ whenever $\beta$ is a $PD_2^+$-group.

This is enough to show that $d_{2,0}^2([\xi]) = \pm \xi_*([z])$ in general. However to prove the theorem we must actually calculate $d_{2,0}^2$ for the special case $\eta_\beta$. This is a little tedious, but is not difficult. We shall make one more reduction. Let $h : \beta \to \mathbb{Z}^2$ be a degree-1 homomorphism. Then $\eta_\beta = \pm h^* \eta_1$, and so it is enough to prove the claim for $\eta_1$.

Let $\Gamma = \mathbb{Z}[\pi]$ and $\Lambda = \mathbb{Z}[\beta]$, and let $I_\beta = \text{Ker}(\epsilon : \Lambda \to \mathbb{Z})$. Although $\pi = F(2)/F(2)[3]$ has a presentation $\langle x, y \mid x, y \Rightarrow [x, y] \rangle$, we shall use

$$\langle u, x, y \mid u[x, y], ux = xu, uy = yu \rangle$$

instead. We shall use the same notation $x$ and $y$ for generators of $\beta = \pi/\langle [x, y] \rangle = \mathbb{Z}^2$, for simplicity of reading. The presentation of $\pi$ determines a (partial) resolution

$$P_* : \Gamma^3 \to \Gamma^3 \to \Gamma \to \mathbb{Z} \to 0$$
of the $\mathbb{Z}[\pi]$-augmentation module, with bases $p_1^x$, $p_1^y$ and $p_1^z$ for $P_1$, and $p_2^{ur}$, $p_2^x$ and $p_2^y$ for $P_2$. Let $\mathcal{P}_* = \Lambda \otimes_T P_*$, and let $\partial'$ and $\partial''$ be the differentials of $C_*^{FL}(\beta)$ and $\mathcal{P}_*$, respectively.

The homology LHS for the extension is based on the bicomplex $K_{p,q} = C_p^{FL}(\beta) \otimes_\Lambda \mathcal{P}_q$, with differential $d = \partial' \otimes 1 + (-1)^p 1 \otimes \partial''$. (Since $\Lambda$ is commutative we may view the left module structure on $C_*^{FL}(\beta)$ as also being a right module structure.) The associated total complex $K_{n}^{tot} = \oplus_{p+q=n}K_{p,q}$ has an increasing filtration $F_iK_{n}^{tot}$, given by

$$F_0K_{n}^{tot} = K_{0,n}, \quad F_1K_{n}^{tot} = K_{0,n} \oplus K_{1,n-1} \quad \text{and} \quad F_2K_{n}^{tot} = K_{n}^{tot}.$$  

(See Chapter VII of [3].)

The generator of $H_2(\beta; \mathbb{Z})$ is represented by the 2-cycle $c_2^* \in C_2^{FL}(\beta)$. Let $z = c_2^* \otimes 1 - c_2^x \otimes p_1^x + c_2^y \otimes p_1^z$. Then $z \in Z_2^{ur}$, and $z$ represents a generator of $E_2^{2} = H_2(C_*^{FL}(\beta) \otimes_\beta \mathbb{Z}) = H_2(\beta; \mathbb{Z})$. Now

$$d_2^2([z]) = [(1-x)p_1^y + (y-1)p_1^z]$$

in $E_{0,1}^{2} = H_0(\beta; H_1(\mathcal{P}_*)) \cong \phi$. On the other hand,

$$\partial'' p_2^{ur} = \partial_u u[x,y]p_1^u + \partial_x u[x,y]p_1^x + \partial_y u[x,y]p_1^y$$

$$= p_1^u + (1-xyx^{-1})p_1^x + (x-1)p_1^y = p_1^u + (1-y)p_1^x + (x-1)p_1^y,$$

since $u = 1$ and $uxyx^{-1} = y$ in $\Lambda$. Thus

$$d_2^2([z]) = [p_1^u - \partial'' p_2^{ur}] = [p_1^u],$$

which corresponds to the generator $u$ of $\phi$. Hence $d_2^2 = \eta_1$, proving the theorem.

It is easy to see that $\text{Ker}(d_2^2) \leq \text{Ker}(\xi_*)$ and $\text{Im}(\xi_*) \leq \text{Im}(d_2^2)$, without such reductions or calculation. The diagonal $\Delta : \pi \to \pi \times_{\beta} \pi$ splits the pullback $p^* \xi$ of $\xi$ over $p$. Hence $p^*[\xi] = 0$, and so

$$\text{Ker}(d_2^2) = \text{Im}(H_2(p)) \leq \text{Ker}(\xi_*).$$

On the other hand, if $z = \Sigma_{\tau \in \tau} c_2^\tau$ is a 2-cycle of $\mathbb{Z} \otimes_\beta C_*^{FL}(\beta)$, then $\Sigma_{\tau} \epsilon_x(\tau) = 0$, for all $x \in X$. Let $j$ be the inclusion of $\phi$ as a subgroup of $\pi$. Since $j(s(\Pi r))$ is a product of terms $\sigma(q(x))$ with exponent sum $0$, for each $x \in X$, it lies in $\pi'$. Since $\xi_*(z) = s(\Pi r)$, we see that

$$\text{Im}(\xi_*) \leq \text{Ker}(H_1(j)) = \text{Im}(d_2^2).$$
8. SURFACE BUNDLES WITH FLAT FIBRE

When the base is an orientable surface and the fibre $F$ is the torus $T$ or the Klein bottle $Kb$ the class $[s(r)]$ is the only obstruction to a section.

**Theorem 5.** Let $p : E \to B$ be a bundle with aspherical base and fibre the torus. Then $p$ has a section if and only if $[s(r)] = 0$. If $B$ is orientable then $p$ has a section if and only if $H_1(E; \mathbb{Z}) \cong H_0(B; H_1(F; \mathbb{Z})) \oplus H_1(B; \mathbb{Z})$. The $\phi$-conjugacy classes of sections are parametrized by $H^1(\beta; \zeta \phi)$.

**Proof.** Since the action of $Aut(\phi) = GL(2, \mathbb{Z})$ on $\phi = \mathbb{Z}^2$ is induced by the natural (based) action of $GL(2, \mathbb{Z})$ on $T = \mathbb{R}^2/\mathbb{Z}^2$, every semidirect product $\mathbb{Z}^2 \rtimes_\theta \beta$ is realized by a $T$-bundle over $B$ with a section. Therefore $p$ has a section if and only if $p_*$ splits, since bundles are determined by the associated extensions. This in turn holds if and only if $[s(r)] = 0$, since $Aut(\phi) = Out(\phi)$.

The second assertion follows from Lemma 2.

If $p_*$ splits and $\sigma$ and $\sigma'$ are two sections determining the same lift $\tilde{\theta}$ then $\sigma'(g) \sigma(g)^{-1}$ is in $\zeta \phi$, for all $g \in \beta$. Therefore the sections are parametrized (up to conjugation by an element of $\phi$) by $H^1(\beta; \zeta \phi)$. (See Proposition IV.2.3 of [3].) $\square$

If $p$ has a section then so does the pullback over $B^+$. The converse also holds if $H^2(\beta; \phi) \cong H_0(\beta; \mathbb{Z}^w \otimes \phi)$ has no 2-torsion. For then restriction to $H^2(\beta^+; \phi)$ is injective, since composition with the transfer is multiplication by 2. (See §9 of Chapter III of [3].)

The situation is a little more complicated when $F = Kb$. We may view $Kb$ as the quotient of $\mathbb{R}^2$ by a glide-reflection $x$ along the $X$-axis and a unit translation $y$ parallel to the $Y$-axis. Then $\kappa = \pi_1(Kb)$ has presentation $\langle x, y \mid xyx^{-1} = y^{-1} \rangle$, and $\zeta \kappa$ is generated by the image of $x^2$. Let $\alpha$ and $\gamma$ be the automorphisms determined by $\alpha(x) = x^{-1}$, $\gamma(x) = xy$ and $\alpha(y) = \gamma(y) = y$. Then $Aut(\kappa)$ is generated by $\alpha$, $\gamma$ and $c_x$, and $\gamma^2 = c_y$. It is easily verified that $\alpha \gamma = \gamma \alpha$, and so $Out(\kappa) \cong (\mathbb{Z}/2\mathbb{Z})^2$ is the image of an abelian subgroup $\langle \alpha, \gamma \rangle \cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} < Aut(\kappa)$.

**Theorem 6.** Let $p : E \to B$ be a bundle with aspherical base and fibre the Klein bottle. Then $p$ has a section if and only if $\theta$ factors through $Aut(\kappa)$ and $[s(r)] = 0$. If $B$ is an orientable surface then $p$ has a section if and only if $[s(r)] = 0$.

**Proof.** The automorphism $\alpha$ is induced by a reflection across a circle of fixed points (the image of the $Y$-axis), while the outer automorphism
class of $\gamma$ is induced by a half-unit translation parallel to the $Y$-axis. This may be isotoped through homeomorphisms that commute with $\alpha$ to fix the $Y$-axis also. Thus $\langle \alpha, \gamma \rangle$ lifts to a group of based self-homeomorphisms of $Kb$. Hence $p$ has a section if and only if $p_*$ splits. This in turn holds if and only if $\theta$ factors through $\text{Aut}(\kappa)$ and $[s(r)] = 0$. In particular, if $B$ is orientable then $p$ has a section if and only if $[s(r)] = 0$. □

If $\beta$ acts on $\zeta \phi$ through $w_1(\beta)$ we can make the condition $[s(r)] = 0$ more explicit. For then $H^2(\beta; \zeta \phi)$ maps injectively to $H^2(\beta^+; \zeta \phi) \cong \mathbb{Z}$ under passage to $\beta^+$. Thus $p_*$ splits if and only if $\theta$ factors through $\text{Aut}(\kappa)$ and the restriction to $p^{-1}_*(\beta^+)$ splits. Since $\zeta \phi$ maps injectively to $\phi/I(\phi) \cong \mathbb{Z}$, $H^2(\beta^+; \zeta \phi)$ in turn maps injectively to $H^2(\beta^+; \phi/I(\phi))$. The image of $[\xi(p)]$ is the class of the extension

$$1 \to \phi/I(\phi) \to \hat{\pi}/I(\phi) \to \beta^+ \to 1,$$

where $\hat{\pi}$ is the preimage of $\beta^+$. Hence the extension is trivial if $\beta_1(\hat{\pi})$ is odd, by Lemma 2.

**Examples.** Let $\pi$ be a discrete cocompact subgroup of $Nil^3 \times \mathbb{R}$. Then $\zeta \pi \cong \mathbb{Z}^2$ and $\pi/\zeta \pi \cong \mathbb{Z}^2$, and so the coset space $E = \pi \setminus Nil^3 \times \mathbb{R}$ is the total space of a $T$-bundle over $T$. The action is trivial, and so the split extension is the product $\mathbb{Z}^4$. Thus the bundle projection for this coset space has no section. (In fact, $\pi/\pi'$ has rank 2, rather than 4, and so the criterion of (1) fails.) Similarly, coset spaces of discrete cocompact subgroups of $Nil^4$ are $T$-bundles over $T$ without sections.

The group with presentation

$$\langle u, v, x, y \mid u, v \Rightarrow x, y, [u, v] = x^2, xyx^{-1} = y^{-1} \rangle$$

is the group of a $Nil^3 \times \mathbb{E}^1$-manifold which fibres over $T$ with fibre $Kb$. The base group acts trivially on the fibre, but $\beta_1(\pi) = 2$, rather than 3, and so the bundle does not have a section.

The group with presentation

$$\langle u, v, x, y \mid u \Rightarrow x, y, vxx^{-1} = x^{-1}, vy = yv, [u, v] = x^2, xyx^{-1} = y^{-1} \rangle$$

is the group of a flat 4-manifold which fibres over $T$ with fibre $Kb$. In this case $H^2(\beta; \zeta \phi) = \mathbb{Z}/2\mathbb{Z}$, but $[s(r)] \neq 0$, and so the bundle does not have a section.

Similar examples over a hyperbolic base $B$ may be constructed from these by pullback over a degree-1 map to $T$.

**9. Bundles with hyperbolic fibre**

Suppose now that $\chi(F) < 0$ or, equivalently, that $\zeta \phi = 1$. If an $F$-bundle $p : E \to B$ has a section then the action $\theta$ factors through
\(\text{Aut}(\phi)\). Conversely, every semidirect product \(\phi \rtimes \beta\) is realized by a bundle with a section. This follows from the work in \([9]\) extending \([6]\) to the based cases. We shall not use this, as our concern here is merely to give examples of such bundles without sections.

We may construct the extension corresponding to an action \(\theta : \beta \to \text{Out}(\phi)\) as follows. Let \(\langle X | r \rangle\) be a 1-relator presentation for \(\beta\), with associated epimorphism \(q : F(X) \to \pi\). Let \(\psi : F(X) \to \text{Aut}(\phi)\) be a lift of \(\theta q\). Then \(\psi(r) = c_g\), for some \(g \in \phi\), which is uniquely determined by \(\psi\), since \(\zeta \phi = 1\). Let \(G = \phi \rtimes \psi F(X)\). Then \(\pi = G/\langle\langle rg^{-1}\rangle\rangle\) is an extension of \(\beta\) by \(\phi\) which realizes the action \(\theta\). In particular, \(\pi\) is a semidirect product if \(g = 1\). However, \(g\) depends on the choice of \(\psi\). We need a condition which does not depend on this choice.

If such a bundle has a section then so does the associated Jacobian bundle, with base \(B\), fibre the Jacobian of \(F\) and group \(\pi/\phi'\). Lemma 2 renders more explicit a result of Morita \([16]\). He showed that if \(F\) is oriented and of genus \(g \geq 2\) then the Jacobian bundle has a section if and only if \(\theta^* \mu = 0\), where \(\mu\) is a class in \(H^2(\mathcal{M}_g; H^1(\phi; \mathbb{Z}))\). Examining his construction, we see that if \(f\) is the 2-cocycle with values in \(\phi^{ab}\) associated to a set-theoretic section \(\sigma : \beta \to \pi/\phi'\), as in §6 above, then \(\theta^* \mu\) is the image of \([f]\) under the change of coefficient isomorphism induced by the Poincaré duality isomorphism \(\phi^{ab} \cong H^1(\phi; \mathbb{Z})\). Thus if base and fibre are orientable the Jacobian bundle has a section if and only if \(\pi^{ab} \cong (\phi/[\pi, \phi]) \oplus \beta^{ab}\). This is so if and only if \(g \in [\pi, \phi]\), where \(c_g = \psi(r)\), for some (and hence all) \(\psi\) as in the preceding paragraph.

Endo has suggested the following example of a surface bundle, with base and fibre of genus 3, which has no section \([7]\). Let \(D_1, D_2, D_3\) be disjoint small discs in the interior of the standard unit disc \(D^2\), and let \(\Sigma = D^2 \setminus \bigcup_{j \leq 3} D_j\) be the 4-punctured sphere, with the standard planar orientation. Let \(F = T_3 = \partial(\Sigma \times [0, 1]) \cong \Sigma_0 \cup \Sigma_1\), where \(\Sigma_0\) and \(\Sigma_1\) are collar neighbourhoods of \(\Sigma \times \{0\}\) and \(\sigma \times \{1\}\), respectively, meeting along \(N = \partial \Sigma \times \{1/2\}\). Let \(j_0\) and \(j_1\) be the natural identifications of \(\Sigma\) with \(\Sigma_0\) and \(\Sigma_1\), respectively. Orient \(F\) so that \(j_0\) is orientation preserving. Then \(j_1\) is orientation reversing.

Let \(b_1, b_2, b_3, b_4\) be the boundary components of \(\Sigma\), and \(x, y, z\) be simple closed curves in the interior of \(\Sigma\), as in Figure 5.1 of \([8]\). Let \(d_1, \ldots, d_4\) be simple closed curves parallel to \(b_1, \ldots, b_4\) in the interior of \(\Sigma\). The left hand Dehn twists \(t_{d_1}, \ldots, t_z\) about these curves fix \(\partial \Sigma\). The lantern relation asserts that

\[t_xt_yt_z = t_{d_1}t_{d_2}t_{d_3}t_{d_4},\]

up to isotopy rel \(\partial \Sigma\). (See Chapter 5 of \([8]\).)
Let $x_i = j_i(x)$, and so on. Then $t_{x_0} = j_0 t_x j_0^{-1}$, etc, while $t_{x_1} = j_1 t_x j_1^{-1}$, since the notions of left and right Dehn twist are interchanged under an orientation reversing homeomorphism. Hence the lantern relation gives two equations

$$t_{x_0} t_{y_0} t_{z_0} = t_{d_{10}} t_{d_{20}} t_{d_{30}} t_{d_{40}},$$

and

$$t_{x_1}^{-1} t_{y_1}^{-1} t_{z_1}^{-1} = t_{d_{11}}^{-1} t_{d_{21}}^{-1} t_{d_{31}}^{-1} t_{d_{41}},$$

up to isotopy in $F \rel N$. Combining the last two equations and using the commutativity of Dehn twists about disjoint curves gives

$$t_{x_0} t_{y_0} t_{z_0} t_{x_1}^{-1} t_{y_1}^{-1} t_{z_1}^{-1} = t_{d_{10}} (t_{d_{11}})^{-1} t_{d_{20}} (t_{d_{21}})^{-1} t_{d_{30}} (t_{d_{31}})^{-1} t_{d_{40}} (t_{d_{41}})^{-1}.$$

Let $f$ be the hyperelliptic involution of $F$ which maps $\Sigma_0$ onto $\Sigma_1$ and induces an orientation-reversing involution of $N$, with two fixed points in each component of $N$. Then $f$ is orientation-preserving, and $f(x_0) = x_1$, so $t_{x_1} = f t_{x_0} f^{-1}$, etc. Let $* \in \Sigma_0$ be one of the two fixed points of $f$ on $b_1$, and let $g = [b_1] \in \phi = \pi_1(F,*)$. Then $t_{d_{i0}} (t_{d_{i1}})^{-1}$ induces $c_g$ on $\phi$, while $t_{d_{i1}}$ is isotopic to $t_{d_{i0}} \rel *$, for $i \geq 2$. The equation becomes

$$[t_{x_0}, t_{y_0} t_{z_0} f][t_{y_0}, t_{z_0} f][t_{z_0}, f] = c_g$$

in $\Aut(\phi)$, the mapping class group of $(F,*)$. The left hand side is a product of three commutators, and so we may define an action $\theta : \beta \to \Out(\phi)$ which sends the standard generators to $t_{x_0}$, $t_{y_0} t_{z_0} f$, $t_{y_0}$, $t_{z_0} f$, $t_{z_0}$ and $f$, respectively. (Thus $\Im(\theta)$ is generated by $t_{x_0}, t_{y_0}, t_{z_0}$ and $f$.) It is not hard to see that the image of $g$ in $\phi/\langle \pi, \phi \rangle \cong (\mathbb{Z}/2\mathbb{Z})^4$ is nontrivial. Hence $\pi_{ab}$ is a proper quotient of $\phi/\langle \pi, \phi \rangle \oplus \beta_{ab}$, and so the associated Jacobian bundle does not have a section. Hence the $F$-bundle determined by $\theta$ does not have a section either.

If we use the reflection $\rho$ of $F$ across $N$ instead of the hyperelliptic involution then $t_{x_1} = \rho t_{x_0}^{-1} \rho^{-1}$, etc, and so we get the equation

$$t_{x_0} t_{y_0} t_{z_0} \rho t_{x_0} t_{y_0} t_{z_0} \rho^{-1} = c_g.$$

This gives rise to an $F$-bundle over $Kb$, with monodromy generated by $t_{x_0} t_{y_0} t_{z_0}$ and $\rho$, and non-orientable total space.

Are there any such examples with fibre of genus 2, or with hyperbolic fibre and base $T$?

10. SOME QUESTIONS

The following questions mostly arise from the considerations of §2-§4 above.

(1) Is there a $PD_4$-group which is an extension of surface groups in more than two inequivalent ways?
(2) If a torsion-free group is virtually the group of a surface bundle, is it realized by an aspherical 4-manifold?
(3) Is every iterated extension of $k \geq 3$ surface groups realized by an aspherical $2k$-manifold?
(4) If a symplectic 4-manifold is homotopy equivalent to the total space $E$ of a surface bundle is it diffeomorphic to $E$?
(5) Which bundle groups are realized by complex surfaces?
(6) Do Kodaira fibrations have holomorphic sections?

We note finally that bundles $p : E \to B$ with spherical base and aspherical fibre $F$ are easily handled by the results of Chapter 5 of [10]. If $B = S^2$ then $p$ has a section if and only if it is trivial. If $B = \mathbb{R}P^2$ then $p$ has a section if and only if $\pi = \pi_1(E)$ has an element of order 2, $\pi_2(E) \cong \mathbb{Z}$ and $\text{Ker}(u) \cong \phi = \pi_1(F)$, where $u$ is the natural action of $\pi$ on $\pi_2(E)$. 
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