Wave motion in a fluid under an inhomogeneous ice cover

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Abstract. This paper studies steady waves in fluid and in semi-infinite ice cover generated by a constant pressure distribution with a rectangular planform moving uniformly along the edge of ice cover at fixed distance. This load simulates the air-cushion vehicle (ACV). We consider two cases: (i) the surface of fluid is free outside of ice sheet, (ii) fluid is bounded by a solid vertical wall and the edge of ice cover can be either clamped or free. The fluid is assumed to be ideal incompressible and of finite depth. The ice sheet is modelled by elastic thin plate. The solution of linear hydroelastic problem is obtained by two methods: the Wiener-Hopf technique and matched eigenfunction expansions. The deflection of ice sheet and free surface elevation, as well as wave forces acting on ACV are investigated for different speeds of motion.

1. Introduction

Intensive development of the Arctic causes interest in solving new problems related to the interaction of dynamic perturbations in a fluid and in an ice cover. Previously considerable attention has been paid to three-dimensional flexural oscillations of an ice cover due to a moving pressure area [1], [2], and the ice cover was treated as infinitely extended isotropic, uniform elastic thin plate. In reality, the sea ice is strongly inhomogeneous. Cracks, polynyas and hummocks are the characteristic irregularities of dense ice [3]. The effect of such complex boundary conditions on the wave motion is in the initial stage of the study. The takeoff and landing of an airplane on the very large floating platform was investigated in [4]. The influence of the vertical wall at the clamped edge of the ice cover on the wave motion under moving load has been studied in [5]. The response of ice cover to an external load moving along a frozen channel is presented in [6].

In this paper, the solution of the steady problem for the fluid and semi-infinite ice cover under the action of a rectangle external load moving with uniform speed along the edge is given. Two cases are considered: (i) the surface of the fluid is free outside of the ice sheet, (ii) the fluid is bounded by a solid vertical wall and the edge of the ice sheet can be either clamped or free. The problem is formulated within linear hydroelastic theory. The fluid motion is potential. For the first case, the solutions obtained by two different methods (the Wiener-Hopf technique and matched eigenfunction expansions) are compared.

2. Mathematical formulation

Let us consider the statement of the problem in the case (i). The pressure distribution $P(x, y)$ moves with constant velocity $U$ along the edge of the plate. The plate draft is ignored. We
consider the moving Cartesian coordinate system \((x, y, z)\) with \(x\)-axis directed perpendicular to the edge, the \(y\)-axis along the edge, and the \(z\)-axis vertically upwards. It is assumed that the plate is in contact with the water at all points.

The boundary-value problem for the velocity potential \(\varphi(x, y, z)\) and the plate deflection or the free surface elevation \(w(x, y)\) can be written as

\[
\Delta_3 \varphi = 0 \quad (\infty < x, y < \infty, -H \leq z \leq 0), \quad \Delta_3 \equiv \Delta_2 + \partial^2 / \partial z^2, \quad \Delta_2 \equiv \partial^2 / \partial x^2 + \partial^2 / \partial y^2, \quad (1)
\]

\[
gw - U \frac{\partial \varphi}{\partial y} \bigg|_{z=0} = 0 \quad (x < 0), \quad D \Delta_2 w + \rho U^2 \frac{\partial^2 w}{\partial y^2} + g \rho_0 w - \rho_0 U \frac{\partial \varphi}{\partial y} \bigg|_{z=0} = -P(x, y) \quad (x > 0), \quad (2)
\]

\[
\frac{\partial \varphi}{\partial z} \bigg|_{z=0} = -U \frac{\partial w}{\partial y}, \quad \frac{\partial \varphi}{\partial z} \bigg|_{z=-H} = 0, \quad (3)
\]

\[
\left( \frac{\partial^2}{\partial x^2} + \nu \frac{\partial^2}{\partial y^2} \right) w_+ = 0, \quad \frac{\partial}{\partial x} \left[ \frac{\partial^2}{\partial x^2} + (2 - \nu) \frac{\partial^2}{\partial y^2} \right] w_+ = 0, \quad w_+ = w|_{x=0+}. \quad (4)
\]

Here \(D = Eh^3/[12(1 - \nu^2)]\); \(E, \nu, h, \rho\) are the Young’s modulus, the Poisson’s ratio, the thickness and the density of the ice sheet, respectively; \(\rho_0\) is the fluid density; \(H\) is water depth; \(g\) is the acceleration due to gravity. In the far field a radiation condition should be imposed that requires the radiated waves to be outgoing.

For the case (ii), the fluid is restricted at the left by the rigid wall. Consequently, we have the boundary condition: \(\partial \varphi / \partial x = 0\) at \(x = 0\). The edge of the ice sheet can be free or frozen to the fixed vertical structure, then \(w_+ = \partial w / \partial x|_{x=0+} = 0\).

We restrict our consideration to the constant pressure distribution in the rectangular planform: \(P(x, y) = P_0\) in the domain \((x - x_0) < a, |y| < b, x_0 > a\) and zero otherwise. The forces \(F_x\) (side force) and \(F_y\) (wave resistance) acting on ACV and its non-dimensional values \(A_x, A_y\) are determined by formulas

\[
(F_x, F_y) = -P_0 \int_{-b}^{b} \int_{x_0-a}^{x_0+a} (w_x, w_y) dx dy, \quad (A_x, A_y) = -\frac{g \rho_0}{2a P_0} (F_x, F_y).
\]

3. Method of solution
The dimensionless variables and parameters are introduced

\[
(x', y', z', a', b', x'_0) = \frac{1}{H}(x, y, z, a, b, x_0), \quad \beta = \frac{D}{\rho_0 g H^4}, \quad F = \frac{U}{\sqrt{g H}}, \quad \sigma = \frac{\rho h}{\rho_0 H^2}, \quad P' = \frac{P_0}{\rho_0 g H}.
\]

Below, the primes are omitted. We will seek the velocity potential and the displacement in the form \(\varphi = U H \phi(x, y, z), \quad w = H W(x, y)\).

3.1. The Wiener-Hopf technique
We use the Fourier transform to the variables \(x\) and \(y\) in the form

\[
\Phi_-(\alpha, s, z) = \int_{-\infty}^{\infty} e^{-isy} dy \int_{-\infty}^{0} \phi(x, y, z) e^{i\alpha x} dx, \quad \Phi_+(\alpha, s, z) = \int_{-\infty}^{\infty} e^{-isy} dy \int_{0}^{\infty} \phi(x, y, z) e^{i\alpha x} dx.
\]

From the Laplace equation (1) and the no-flux bottom condition (3), we have

\[
\Phi(\alpha, s, z) = \Phi_- + \Phi_+ = C(\alpha, s) Z(\alpha, s, z), \quad Z = \cosh[(z + 1)\sqrt{\alpha^2 + s^2}] / \cosh \sqrt{\alpha^2 + s^2}, \quad (5)
\]

where \(C(\alpha, s)\) is unknown function. We introduce the functions \(D_\pm(\alpha, s), G_\pm(\alpha, s)\) in following manner:

\[
D_- = \int_{-\infty}^{\infty} e^{-isy} dy \int_{-\infty}^{0} \left( \phi_z + F^2 \phi_{yy} \right) z = 0 e^{i\alpha x} dx, \quad D_+ = \int_{-\infty}^{\infty} e^{-isy} dy \int_{0}^{\infty} \left( \phi_z + F^2 \phi_{yy} \right) z = 0 e^{i\alpha x} dx,
\]

\[
G_- = \int_{-\infty}^{\infty} e^{-isy} dy \int_{-\infty}^{0} \left( \phi_z + F^2 \phi_{yy} \right) z = 0 e^{i\alpha x} dx, \quad G_+ = \int_{-\infty}^{\infty} e^{-isy} dy \int_{0}^{\infty} \left( \phi_z + F^2 \phi_{yy} \right) z = 0 e^{i\alpha x} dx.
\]
We take the values of these roots from the upper half-plane. and as a result we obtain the equation

\[ K_+ = \int_{-\infty}^{\infty} e^{-isg} dy \int_0^\infty \left( (\beta \Delta_2^2 + 1 + \sigma F^2 \partial^2 / \partial y^2) \phi_z + F^2 \phi_{yy} \right)_{z=0} e^{i\alpha x} dx, \]

\[ G_+ = \int_{-\infty}^{\infty} e^{-isg} dy \int_0^\infty \left( (\beta \Delta_2^2 + 1 + \sigma F^2 \partial^2 / \partial y^2) \phi_z + F^2 \phi_{yy} \right)_{z=0} e^{i\alpha x} dx. \]

The functions with the indexes +/- are analytical on \( \alpha \) in the upper/lower half-plane, respectively. From the boundary conditions (2), we have

\[ D_-(\alpha, s) = 0, \quad G_+(\alpha, s) = isQ(\alpha, s), \quad Q(\alpha, s) = 4P_0 e^{iax_0} \sin(\alpha a) \sin(sb)/(\alpha s), \]

where \( Q(\alpha, s) \) is the Fourier-transform of the function \( P(x, y) \). Using (5), we can write

\[ D(\alpha, s) = D_- + D_+ = C(\alpha, s)K_1(\alpha, s), \quad G(\alpha, s) = G_- + G_+ = C(\alpha, s)K_2(\alpha, s), \]

where \( K_1(\alpha, s) \) and \( K_2(\alpha, s) \) are the dispersion functions for the free surface waves and the flexural-gravity ones in a moving coordinate system, respectively,

\[ K_1(\alpha, s) = \sqrt{\alpha^2 + s^2} \tanh \sqrt{\alpha^2 + s^2} - F^2 s^2, \]

\[ K_2(\alpha, s) = \left[ \beta(\alpha^2 + s^2)^2 + 1 - \sigma F^2 s^2 \right] \sqrt{\alpha^2 + s^2} \tanh \sqrt{\alpha^2 + s^2} - F^2 s^2. \]

The dispersion relation for the free surface waves \( K_1(\gamma) \equiv \gamma \tanh \gamma - F^2 s^2 = 0 \) has two real roots \( \pm \gamma_0 \) and the countable set of imaginary roots \( \pm \gamma_m (m = 1, 2, \ldots) \). The dispersion relation for the flexural-gravity waves \( K_2(\mu) \equiv \beta \mu^4 + 1 - \sigma F^2 s^2 \mu \mu_0 - F^2 s^2 = 0 \) has two real roots \( \pm \mu_0 \), four complex roots \( \pm \mu_{-1}, \pm \mu_{-2}, \mu_{-2} = -\mu_{-1} \) (the bar denotes complex conjugation), and the countable set of imaginary roots \( \pm \mu_m (m = 1, 2, \ldots) \). Then the roots of the dispersion relations \( K_n(\alpha, s) = 0 (n = 1, 2) \) are \( \chi_m(s) = \sqrt{\gamma_m^2(s) - s^2} \) \( n = 1 \), \( \alpha_m(s) = \sqrt{\mu_m^2(s) - s^2} \) \( n = 2 \). We take the values of these roots from the upper half-plane.

From the relations (6) and (7), we obtain

\[ G_-(\alpha, s) + isQ(\alpha, s) = D_+(\alpha, s)K(\alpha, s), \quad K(\alpha, s) = K_2(\alpha, s)/K_1(\alpha, s). \]

In accordance with the Wiener-Hopf technique, we factorize the function \( K(\alpha, s) \):

\[ K(\alpha, s) = K_-(\alpha, s)K_+(\alpha, s), \quad K_\pm(\alpha, s) = \frac{(\alpha \pm \alpha_{-1})(\alpha \pm \alpha_{-2})}{\mu_{-1}\mu_{-2}} \prod_{j=0}^{\infty} \frac{(\alpha \pm \alpha_j)\gamma_j}{\mu_j(\alpha \pm \chi_j)}, \]

where the functions \( K_\pm \) are analytical in the upper/lower parts of the complex plane \( \alpha \), respectively. By dividing (8) into \( K_-(\alpha, s) \), we have

\[ \frac{G_-(\alpha, s)}{K_-(\alpha, s)} + 2P_0 \sin(sb) \frac{\psi(\alpha)}{K_-(\alpha, s)} = D_+(\alpha, s)K_+(\alpha, s), \quad \psi(\alpha) = \frac{e^{ia(x_0 + a)} - e^{ia(x_0 - a)}}{\alpha}. \]

Then we use the representation

\[ \frac{\psi(\alpha)}{K_-(\alpha, s)} = L_-(\alpha, s) + L_+(\alpha, s), \quad L_\pm(\alpha, s) = \pm \frac{1}{2\pi i} \int_{-\infty+i\lambda}^{\infty+i\lambda} \frac{\psi(\zeta) d\zeta}{K_-(\zeta, s)(\zeta - \alpha)}, \]

and as a result we obtain the equation

\[ G_-(\alpha, s)/K_-(\alpha, s) + 2P_0 \sin(sb)L_-(\alpha, s) = D_+(\alpha, s)K_+(\alpha, s) - 2P_0 \sin(sb)L_+(\alpha, s). \]
The functions on the left and right sides of this equation are analytical in the lower and upper parts of the complex plane $\alpha$, respectively. Then we have analytical function over the entire complex plane $\alpha$. By the Liouville’s theorem, this function is a polynomial. The degree of the polynomial is determined by the behavior of this function as $|\alpha| \to \infty$ and is equal to one. Consequently, we can write

$$D_+(\alpha, s)K_+(\alpha, s) - 2P_0 \sin(sb)L_+(\alpha, s) = 2P_0 \sin(sb)[a_1(1) + a_2(\alpha)],$$

where $a_1(s)$ and $a_2(s)$ are unknown functions which are defined from the edge conditions. Then we have

$$\Phi(x, s, z) = \frac{P_0 \sin(sb)}{\pi} \int_{-\infty}^{\infty} e^{-i\alpha x} [a_1 + a_2 \alpha + L_+(\alpha, s)]Z(\alpha, s) d\alpha.$$

Using the free-edge conditions (4), we obtain the system of two linear algebraic equations to define the coefficients $a_1(s)$ and $a_2(s)$. All integrals on $\alpha$ are evaluated by the residue method.

After solving this system, we find the free surface elevations and the ice deflections by performing inverse Fourier transform:

for $x < 0$

$$W(x, y) = -\frac{P_0 F^2}{\pi} \int_{-\infty}^{\infty} e^{isy} \sin(sb) \sum_{j=0}^{\infty} \frac{e^{-iyx j\gamma_j} [a_1 + a_2 \chi_j + L_+(\chi_j, s)]}{\chi_j K_+(\chi_j, s) K_1(\gamma_j)} ds,$$

for $x > 0$

$$W(x, y) = -\frac{P_0}{2i\pi^2} \int_{-\infty}^{\infty} e^{isy} \sin(sb) ds \int_{-\infty}^{\infty} e^{-i\alpha x} \psi(\alpha) \sqrt{\alpha^2 + s^2} \tanh \sqrt{\alpha^2 + s^2} d\alpha \frac{K_2(\alpha, s)}{K_2'(\alpha, s)}$$

$$= \frac{P_0}{\pi} \int_{-\infty}^{\infty} e^{isy} \sin(sb) \sum_{j=-2}^{\infty} \frac{e^{iyx j\gamma_j} [a_1 - a_2 \alpha_j - L_-(\alpha_j, s)] K_1(\alpha_j, s) \mu_j \tanh \mu_j}{K_2'(\alpha_j, s)} ds,$$

where the prime denotes the derivative with respect to the first variable. Integrals on $s$ are evaluated numerically.

3.2. Method of matched eigenfunction expansions

The solution of the boundary-value problem (1)-(4) at $x > 0$ is sought in the form

$$\phi(x, y, z) = \phi_0^+(x, y, z) + \phi_0^-(x, y, z) + \psi(x, y, z), \quad w(x, y) = w_0^+(x, y) + w_0^-(x, y) + v(x, y).$$

Here the functions $\phi_0^+$ and $w_0^+$ are the solution for the problem of the action of a rectangular pressure region centered at the point $(x = \pm x_0, \ y = 0)$ on an infinitely extended elastic plate. For the rectangular external load, the solutions for $\phi_0^+$ and $w_0^+$ have the form

$$\phi_0^+ = -\frac{4P_0 F}{\pi^2} \int_0^{\pi/2} \frac{d\theta}{\sin \theta} \int_0^{\infty} \cosh[k(z + 1)] \sin(ak \sin \theta) \sin(bk \cos \theta) \cos[k(x \mp x_0) \sin \theta] \sin(ky \cos \theta) dk,$$

$$w_0^+ = -\frac{4P_0}{\pi^2} \int_0^{\pi/2} \frac{d\theta}{\cos \theta \sin \theta} \int_0^{\infty} \tanh \frac{k}{Z(k, \theta)} \sin(ak \sin \theta) \sin(bk \cos \theta) \cos(k(x \mp x_0) \sin \theta) \cos(ky \cos \theta) dk,$$

$$Z(k, \theta) = (1 + \sigma k \tanh k)[\omega^2(k) - F^2 k^2 \cos^2 \theta], \ \omega(k) = \sqrt{k \tanh k (3k^4 + 1)/(\sigma k \tanh k + 1)}.$$
The function $\omega(k)$ is a dispersion relation for flexural-gravity waves [1]. It is known that the phase velocity of flexural-gravity waves $c(k) = \omega(k)/k$ has the local minimum $c_m$. For a fluid of finite depth, we have always $c_m < 1$. The integrands in (9)-(12) have no singularities at the velocities of the load motion $F < c_m$ (subcritical case). This means that for an infinitely extended ice cover the wave motion is absent far from the load in this case. For $F > c_m$ (supercritical case), there are the rooted singularities in the integrands. By change of integration variable integrands become regular. In this case flexural-gravity waves are spreading in ice cover.

Using the Fourier transform to the variable $y$, the solutions for unknown functions $\phi(x, s, z)$, $w(x, s)$ at $x < 0$ and $\psi(x, s, z)$, $v(x, s)$ at $x > 0$ are sought in the form:

$$\phi(x, s, z) = -isF \sum_{m=0}^{\infty} \frac{E_m(s)}{\gamma_m \sinh \gamma_m} e^{-ixmz} \cos[\gamma_m(z + 1)],$$

$$w(x, s) = \sum_{m=0}^{\infty} E_m(s)e^{-ixmz},$$

$$\psi(x, s, z) = -isF \sum_{m=-2}^{\infty} \frac{V_m(s)e^{ixmz}}{\mu_m \sinh \mu_m} \cos[\mu_m(z + 1)],$$

$$v(x, s) = \sum_{m=-2}^{\infty} V_m(s)e^{ixmz}.$$

Unknown functions $E_m(s)$ and $V_m(s)$ are determined by the method of integral fitting from matching conditions of continuity of horizontal velocity and pressure on the plane $x = 0$:

$$\partial \phi/\partial x|_{x=0-} = \partial \phi/\partial x|_{x=0+}, \quad \partial \phi/\partial y|_{x=0-} = \partial \phi/\partial y|_{x=0+} \quad (|y| < \infty, -1 \leq z \leq 0).$$

### 4. Numerical results

The following input data are used for the ice sheet and the external load: $E = 5GPa$, $\nu = 1/3$, $\rho_0 = 10^3$ kg/m$^3$, $\rho = 900$ kg/m$^3$, $h = 2.5$ m, $P_0 = 10^5$ N/m$^2$, $a = 10$ m, $b = 20$ m, $x_0 = 50$ m, $H = 350$ m. The minimal phase velocity for these parameters is equal to $c_m \approx 21.86$ m/s. The critical velocity for the wave surfaces is equal to $U_c \approx \sqrt{gH} \approx 58.60$ m/s. Calculations were performed for values of speed 10, 15, 20, 30, 50 and 60 m/s. Speeds of load 10, 15, 20 m/s are subcritical. At a speed of load 10 m/s waves aren’t present neither in fluid, nor in a plate. Deformation of a free surface of fluid is small. At a speed of load 15 m/s, there are waves in the fluid which extend behind moving loading, but in a plate there are no waves. At a speed of load 20 m/s, wave amplitudes in fluid become greater and they excite the waves in a plate decaying with the distance from the edge.

Figure 1 (a)-(c) shows the deflection of the ice sheet $w(x, 0)$ for cases (i) and (ii) at the different speeds of load motion. The smallest and largest deflections take place for the vertical wall with the clamped and free edge, respectively. In the presence of a free surface, deflections of the ice cover take intermediate values between the values for the infinitely extended ice cover and the case of a vertical wall with a free edge. The isolines of $w(x, y)$ for the plate deflection and the free surface elevation at $U = 15$m/s are shown in Figure 1 (d). The deflections of the ice cover are practically symmetrical along the axis $y$. However in the region of pure water, a wave pattern is produced at $y < 0$. The dashed line shows the wedge within which, according to the linear theory of surface waves, there is a wave system at large distances downstream. At the given values of the load speed and the depth of the fluid, the half-angle of the aperture practically approaches 19.5°, that corresponds to the case of an infinitely deep fluid [8].

Figure 2 (a)-(d) shows the three-dimensional plots of wave surface at speeds $U = 20$, 30, 50, 60 m/s. At $U = 20$ m/s (subcritical case) waves in fluid generate waves in the ice cover near the edge which are damped away from it. The wave length is not changed. For supercritical case $U = 30$, 50, 60 m/s short elastic wave spread ahead and long gravity ones appear behind the loading region in the plate. As the speed of motion increases, the waves ahead of load become shorter and ones behind it become longer. At $U = 60$ m/s $> U_c$, the shadow zone is appeared behind the load region [7], that is the waves in this zone are absent. The wave wake in fluid is generated by flexural-gravity waves in ice cover.
Figure 1. (a)-(c) Ice-sheet deflections along the line $y = 0$ at $U = 10, 15, 20\text{ m/s}$ for infinitely extended ice cover (1), in the presence of a vertical wall with the clamped (2) and free (3) edge and in the presence of the free surface (4, 5) calculated by Wiener-Hopf technique and matched eigenfunction expansions, respectively. (d) Isolines of $w(x, y)$ in millimeters for the plate deflection ($x > 0$) and the free surface elevation ($x < 0$) at $U = 15\text{ m/s}$.

Figure 2. (a)-(d) Wave patterns of the free surface ($x < 0$) and the ice cover ($x > 0$) at speeds $U = 20, 30, 50, 60\text{ m/s}$.
Figure 3 (a) shows vertical displacements of ice cover and the free surface at $x = 0$ (curve 1 and 2), and the free surface at the distance 50 m from the edge (curve 3) as functions of the coordinate $y$. It is seen that wave length in fluid is changed, short waves ahead of loading region become shorter and of less amplitude, but waves behind the loading zone become of great amplitude.

Non-dimensional values of wave forces acting on moving vehicle are presented on Figure 3 (b) as functions of the load speed. The greatest difference of the value $A_y$ from its value for infinite ice cover is observed at near-critical values of speed $U \approx c_m$. The coefficient $A_x$ changes the sign. This is due to the fact that minimal values of plate deflection are displaced back and the character of dependence of the plate deflection on the coordinate $x$ is changed as the speed increased (Figure 3 (c)). Maximal values of the vertical displacements of ice cover are observed at near-critical values of speed $U \approx c_m$.

![Figure 3](chart.png)

**Figure 3.** (a) The deflection of ice cover at the edge $w(0+, y)$ (1), free surface elevation at the edge $w(0-, y)$ (2), and at the distance $x_0$ from the edge $w(-x_0, y)$ (3) as functions of the coordinate $y$. (b) The non-dimensional values of wave forces $A_y$ (1) and $A_x$ (2) for semi-infinite ice cover in the presence of free surface and corresponding values of wave resistance for infinite ice sheet (3) as functions of the load speed. (c) The ice cover deflection $w(x, 0)$ for different values of speed versus $x$: curves (1-6) correspond to $U = 10$, 15, 20, 30, 50, 60 m/s.

5. Conclusion
The 3-D linear hydroelastic problem on the waves induced by the uniformly moving load is solved for a semi-infinite ice sheet in two configurations: (i) the surface of the fluid is free outside of the sheet, (ii) an ice sheet is in contact with the fixed rigid vertical wall and the edge of the ice cover can be either clamped or free. The solution is obtained by two different methods: the Wiener-Hopf technique and matched eigenfunction expansions. Good agreement between these results is demonstrated. The wave pattern and magnitude of the ice-sheet deflection and free surface elevation are studied for different speeds of the load as well as the forces (wave resistance and side force) acting in horizontal directions on a moving vehicle.

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