A Short Note on Compact Embeddings of Reproducing Kernel Hilbert Spaces in $L^2$ and Infinite-variate Function Approximation.

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June 16, 2022

Abstract

This note consists of two largely independent parts. In the first part we give conditions on the kernel $k : \Omega \times \Omega \to \mathbb{R}$ of a reproducing kernel Hilbert space $H$ continuously embedded via the identity mapping into $L^2(\Omega, \mu)$, which are equivalent to the fact that $H$ is even compactly embedded into $L^2(\Omega, \mu)$.

In the second part we consider a scenario from infinite-variate $L^2$-approximation. Suppose that the embedding of a reproducing kernel Hilbert space of univariate functions with reproducing kernel $1 + k$ into $L^2(\Omega, \mu)$ is compact. We provide a simple criterion for checking compactness of the embedding of a reproducing kernel Hilbert space with the kernel given by

$$\sum_{u \in \mathcal{U}} \gamma_u \bigotimes_{j \in u} k,$$

where $\mathcal{U} = \{u \subset \mathbb{N} : |u| < \infty\}$, and $(\gamma_u)_{u \in \mathcal{U}}$ is a sequence of non-negative numbers, into an appropriate $L^2$ space.

1 Introduction and Problem Formulation

Throughout the note we assume that all the Hilbert spaces we are considering are infinite-dimensional, since else the results are trivial. We also assume that they are separable.

Let $\Omega \neq \emptyset$ and $\mu$ be a $\sigma$-finite measure on $\Omega$. We are considering a reproducing kernel $k : \Omega \times \Omega \to \mathbb{R}$ with the corresponding reproducing kernel Hilbert space $H = H(k)$. We assume that the identical embedding

$$S : H \to L^2(\Omega, \mu), \quad f \to f$$

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is continuous. This is exactly the case when every \( f \in H \) is a representer of some equivalence class \( \overline{f} \in L^2(\Omega, \mu) \). When no risk of confusion appears we denote \( L^2 := L^2(\Omega, \mu) \).

There is a certain interest in characterizing the situations in which \( S \) is even compact. It stems from the fact that the problem of approximating \( S \) (known as \( L^2 \) function approximation) by deterministic algorithms in the information-based complexity framework is solvable in the worst-case setting exactly when \( S \) is a compact operator. As recently shown in \[5\], the same holds also for the randomized algorithms.

It is well-known that
\[
\int_{\Omega} k(t,t) d\mu(t) < \infty
\]
(2)
is equivalent to \( S \) being a Hilbert-Schmidt operator. Furthermore, there is an ample supply of reproducing kernel Hilbert spaces and \( L^2 \) spaces for which the identical embedding is compact, but has an infinite trace, i.e. the embedding is compact, even though the condition (2) is violated. For details on those and related issues we refer to \[2\], Chapter 26. In Theorem 1.2 we give an if-and-only-if characterisation of the compactness of \( S \) in terms of the reproducing kernel \( k \).

In the recent years there was a surge of interest in investigating numerics of functions depending on infinitely many variables, see e.g. \[4, 3, 1\] and the references therein. One of the most prominent problems is a special case of \( L^2 \) function approximation. The typical starting point is a reproducing kernel Hilbert space \( H_\gamma \) and a space \( L^2(\mathcal{X}, \mu^N) \), such that
\[
S_\gamma : H_\gamma \to L^2(\mathcal{X}, \mu^N), \quad f \mapsto f,
\]
is continuous. Here \( \gamma \) is a sequence of positive numbers, the so called weights, moderating the importance of different coordinates, and \( \mathcal{X} \subset \Omega^N \) is an appropriate subset of the sequence space. The space \( H_\gamma \) is in a certain way build up of reproducing kernel Hilbert spaces \( H \) of “univariate” functions \( f : \Omega \to \mathbb{R} \). Assuming that the “univariate” identical embedding \( S : H \to L^2(\Omega, \mu) \) is compact we show in Theorem 2.1 that \( S_\gamma \) is compact exactly when some sequence depending only on \( \gamma \) and \( \|S\| \) converges to 0. The setting of infinite-dimensional approximation is described in more detail in Section 2.

1.1 Embeddings of Reproducing Kernel Hilbert Spaces into \( L^2 \)

For the convenience of the reader we recall a few useful characterisations of compact operators.

**Proposition 1.1.** Let \( X, Y \) be Hilbert spaces and let \( C : X \to Y \) be a linear operator. The following are equivalent:

1. \( C \) is compact, i.e. the image of the unit ball of \( X \) is precompact in \( Y \),
2. For every bounded sequence \( (x_n) \) the sequence \( (Cx_n) \) admits a convergent subsequence,
3. For each sequence \((x_n)_n\) which converges weakly to 0 the sequence \((Cx_n)_n\) converges strongly to 0.

4. \(C^*\) is compact (this is a special case of Schauder’s Theorem).

We define now integral operators

\[
T_{H,H} : H \to H, \quad T_{H,L^2} : H \to L^2(\Omega, \mu), \quad T_{L^2,L^2} : L^2(\Omega, \mu) \to L^2(\Omega, \mu),
\]

all given by the same formula

\[
f \mapsto \left(s \mapsto \int_{\Omega} k(s, t)f(t) \, d\mu(t)\right).
\]

Our main result is the following.

**Theorem 1.2.** Let \(\Omega \neq \emptyset\) and \(k : \Omega \times \Omega \to \mathbb{R}\) be a reproducing kernel on \(\Omega\), with the corresponding reproducing kernel Hilbert space \(H\). Moreover, let \(\mu\) be a \(\sigma\)-finite measure on \(\Omega\), for which \(H \subset L^2(\Omega, \mu)\). Denote by

\[
S : H \to L^2(\Omega, \mu), \quad f \mapsto f
\]

the identical embedding. Then the following conditions are equivalent:

1. \(S\) is compact,
2. Each (or equivalently: one) of the operators \(T_{H,H}, T_{H,L^2}, T_{L^2,L^2}\) defined in (3) is compact,
3. For each bounded (in the norm) sequence \((f_n)_{n \in \mathbb{N}} \subset H\) converging pointwise to 0 one has

\[
\lim_{n \to \infty} \int_{\Omega} \left(\int_{\Omega} k(s, t)f_n(t) \, d\mu(t)\right)^2 \, d\mu(s) = 0.
\]

**Proof of Theorem 1.2.** First we prove the equivalence (1) \(\iff\) (2). Note that we have

\[
T_{H,H} = S^*S, \quad T_{H,L^2} = SS^*S, \quad T_{L^2,L^2} = SS^*.
\]

Now the equivalence of (1) and (2) follows from the general theory of operators on Hilbert spaces, however, for completeness we present short arguments. \(1) \Rightarrow (2)\) follows immediately from the ideal property of compact operators. Compactness of \(T_{H,H}\) implies the compactness of \(S\), because the singular values of \(S\) are just the square roots of eigenvalues of \(T_{H,H}\), i.e. either both sequences converge to 0, or they both do not.

By Schauder’s Theorem and what we have shown so far, compactness of \(T_{L^2,L^2} = SS^* = (S^*)^*S^*\) is equivalent to the compactness of \(S^*\), and thus also to the compactness of \(S\).
Suppose now that $SS^*S$ is compact. Let $(f_n)_{n \in \mathbb{N}}$ be any sequence from the unit ball of $H$. There exists a weakly convergent subsequence $(f_{n_k})_{k \in \mathbb{N}}$. Denote its weak limit by $f$. Using the Cauchy-Schwarz inequality we obtain
\[
\|S^*Sf_{n_k} - S^*Sf\|^2_H = (SS^*(f_{n_k} - f), S(f_{n_k} - f))_{L^2} \leq 2\|SS^*(f_{n_k} - f)\|_{L^2}\|S\|.
\]
Now $SS^*S$ is a compact operator between Hilbert spaces, so in particular it maps sequences which converge weakly to 0 to sequences which converge strongly to 0. Thus
\[
\lim_{k \to \infty} \|S^*Sf_{n_k} - S^*Sf\|_H = 0,
\]
and the compactness of $S^*S$ follows. This in turn implies the compactness of $S$.

Now we prove $(2) \iff (3)$. We claim that $(f_n)_n \subset H$ converges weakly to 0 exactly when it is bounded and it converges pointwise to 0. As $\text{span}\{k(s, \cdot) : s \in \Omega\}$ is dense in $H$, it follows that $\text{span}\{\delta_s : s \in \Omega\}$, where $\delta_s$ denotes the evaluation functional at the point $s$, is dense in $H'$. Let $Q \in \text{span}\{\delta_s : s \in \Omega\}$. If $\delta_s(f_n) \to 0$ for each $s \in \Omega$, then also $Q(f_n) \to 0$. Let now $\varphi \in H'$ be arbitrary. Given an $\epsilon > 0$ we may find a $Q \in \text{span}\{\delta_s : s \in \Omega\}$ with $\|\varphi - Q\| < \epsilon$, and so
\[
\limsup_{n \to \infty} |\varphi(f_n)| \leq \limsup_{n \to \infty} (|\varphi(f_n) - Q(f_n)| + |Q(f_n)|) \leq \sup_n \|\varphi - Q\| \|f_n\|,
\]
\[i.e. \ (f_n)_n \text{ converges weakly to 0.} \]

The other implication is obvious (recall that by the general theory each weakly convergent sequence is bounded).

Let now $f \in H$. The equivalence $(2) \iff (3)$ follows from
\[
\|T_{H,L^2}f\|^2_{L^2} = \int_{\Omega} \left( \int_{\Omega} k(s,t)f(t)d\mu(t) \right)^2 d\mu(s).
\]

**Corollary 1.3.** With the notation as in Theorem 1.2: If $k \in L^2(\Omega \times \Omega, \mu \otimes \mu)$ then the embedding $S$ is compact.

**Proof.** This follows from the well-known fact that $k \in L^2(\Omega \times \Omega, \mu \otimes \mu)$ implies the compactness of $T_{L^2,L^2}$, and from the equivalence of 1. and 2. in Theorem 1.2. \(\square\)

**Example 1.4.** We show that if $k \notin L^2(\Omega \times \Omega, \mu \otimes \mu)$ then a priori we cannot say anything about the compactness of the embedding $S$. To this end consider two finite measures $\nu = (\nu_i)_{i \in \mathbb{N}}$ and $\mu = (\mu_i)_{i \in \mathbb{N}}$ on $\mathbb{N}$ assigning a positive value to each natural number. We let $H = \ell^2(\nu)$, i.e.

\[S : \ell^2(\nu) \to \ell^2(\mu), \quad f \mapsto f.\]

Note that the reproducing kernel of $H$ is given by
\[
k(i,j) = \frac{\delta_{i,j}}{\nu_i}.\]

One can easily calculate that
• \( k \in L^2(\mathbb{N} \times \mathbb{N}, \mu \otimes \mu) \) if and only if \( \sum_{i=1}^{\infty} \left( \frac{\mu_i}{\nu_i} \right)^2 < \infty \),

• \( S \) is bounded if and only if \( \sup_{i \in \mathbb{N}} \frac{\mu_i}{\nu_i} < \infty \),

• \( S \) is compact if and only if \( \lim_{n \to \infty} \frac{\mu_i}{\nu_i} = 0 \).

An example when \( S \) is continuous but not compact is given e.g. by putting \( \nu = \mu \). On the other hand, putting

\[
\mu_i = \frac{1}{i^2}, \quad \nu_i = \frac{\log(i+1)}{i^2}, \quad i \in \mathbb{N},
\]

we see that

\[
\lim_{i \to \infty} \frac{\mu_i}{\nu_i} = \lim_{i \to \infty} \frac{1}{i \log(i+1)} = 0,
\]

i.e. \( S \) is compact, even though

\[
\sum_{i=1}^{\infty} \left( \frac{\mu_i}{\nu_i} \right)^2 = \sum_{i=1}^{\infty} \frac{1}{\log(i+1)^2} = \infty,
\]

i.e. \( k \notin L^2(\mathbb{N} \times \mathbb{N}, \mu \otimes \mu) \).

\[
\square
\]

2 Infinite-variate function approximation.

We shortly describe the typical setting for infinite-variate function approximation. Let \( \Omega \neq \emptyset \) and let \( \mu \) be a probability measure on \( \Omega \). Let \( k \) be a reproducing kernel such that the corresponding Hilbert space \( H = H(k) \) is compactly embedded into \( L^2(\Omega, \mu) \).

We additionally assume that the only constant function in \( H(k) \) is the zero function. We treat \( H \) as a space of univariate functions. Based on it we build up a space of functions with infinitely many variables. To this end denote

\[
\mathcal{U} = \{ u \subseteq \mathbb{N} : |u| < \infty \}.
\]

For \( u \in \mathcal{U} \) we put

\[
k_u : \Omega^N \times \Omega^N \to \mathbb{R}, \quad (x, y) \mapsto \prod_{j \in u} k(x_j, y_j).
\]

Denote

\[
H_u = H(k_u).
\]

For a family of non-negative numbers \( \gamma = (\gamma_u)_{u \in \mathcal{U}} \), called weights, we set

\[
\mathcal{U}_\gamma = \{ u \in \mathcal{U} : \gamma_u > 0 \}.
\]

Put

\[
\mathcal{X} = \{ x \in \Omega^N : \sum_{u \in \mathcal{U}_\gamma} \gamma_u k_u(x, x) < \infty \}.
\]
We define a reproducing kernel $K$ on $X$ via

$$K : X \times X \to \mathbb{R}, \quad (x, y) \mapsto \sum_{u \in U} \gamma u k u(x, y). \quad (4)$$

The corresponding reproducing kernel Hilbert space is denoted by $H_\gamma$. We have

$$H_\gamma = \bigoplus_{u \in U} H_u.$$ This means that each function $f \in H_\gamma$ admits a unique decomposition

$$f = \sum_{u \in U} f_u, \quad f_u \in H_u.$$ The norm of $f$ is then given by

$$\|f\|_{H_\gamma}^2 = \sum_{u \in U} \frac{1}{\gamma u} \|f_u\|_{H_u}^2.$$ We denote the univariate embedding

$$S : H \to L^2(\Omega, \mu).$$ We are interested in the compactness of the embedding

$$S_\gamma : H_\gamma \to L^2(X, \mu^N). \quad (5)$$

**Theorem 2.1.** Let the univariate embedding $S$ be compact. Then the following conditions are equivalent:

1. $S_\gamma$ is compact,
2. $\limsup_{j \to \infty} \gamma u_j \|S\|^{2|u_j|} = 0$, where $(u_j)_{j \in \mathbb{N}}$ is some enumeration of the elements of $U_\gamma$.

**Proof.** (1) $\Rightarrow$ (2). Suppose that $\limsup_{j \to \infty} \gamma u_j C^{2|u_j|} = \epsilon > 0$. In this case we will expose a sequence $(e_j)_{j \in \mathbb{N}}$ in $H_\gamma$ converging weakly to 0, for which $(S_\gamma e_j)_{j \in \mathbb{N}}$ does not converge strongly to 0. This will mean that $S_\gamma$ cannot be compact. To this end for $j \in \mathbb{N}$ let $e_j$ be any vector from $H_{u_j}$ satisfying

$$\|e_j\|_{H_\gamma} = 1, \quad \|e_j\|_{L^2}^2 > \frac{1}{2} \gamma u_j \|S\|^{2|u_j|}.$$ Such a vector exists, since $\sqrt{\gamma u_j \|S\|^{2|u_j|}}$ is the norm of the identical embedding of $H_{u_j}$ into $L^2(\Omega^{u_j}, \mu^{u_j})$, if we equip $H_{u_j}$ with the norm induced by $H_\gamma$. As $H_\gamma$ is an orthogonal
sum of $H_{u_j}$, $j \in \mathbb{N}$, the sequence $(e_j)_j$ forms an orthonormal system in $H_{\gamma}$. Thus for any $f \in H_{\gamma}$

$$\sum_{j=1}^{\infty} |\langle f, e_j \rangle |^2 < \infty,$$

and so

$$\lim_{j \to \infty} |\langle f, e_j \rangle | = 0.$$

It follows that $(e_j)_j$ indeed converges weakly to 0. On the other hand there is a subsequence $(e_{j_k})_k$ such that the $L^2$-norms of all the $e_{j_k}$ are bounded away from 0 by $\frac{1}{2} \epsilon$, so $(e_j)_j$ cannot converge to 0 in $L^2$.

$(2) \Rightarrow (1)$. Suppose that $(2)$ holds and let $(f_j)_{j \in \mathbb{N}}$ be any sequence in the unit ball of $H_{\gamma}$. We will show that $(S_{\gamma} f_j)_{j \in \mathbb{N}}$ admits a convergent subsequence. First of all note that $(2)$ actually implies $\lim_{n \to \infty} \gamma_{u_j} \|S_{\gamma}^{[u_j]}\| = 0$. Denote

$$H_n := \bigoplus_{j \leq n} H_{u_j},$$

and equip $H_n$ with the norm induced from $H_{\gamma}$. Write

$$P_n : H_{\gamma} \to H_{\gamma}$$

for the orthogonal projection onto $H_n$. Note that from $\sup_j \|f_j\| \leq 1$ we obtain

$$\sup_j \|P_n f_j\| \leq 1.$$

The univariate embedding $S_{\gamma}$ was compact. Moreover, finite sums and tensor products of compact operators are also compact, and so we obtain that for each $n$ the sequence $(S_{\gamma} P_n f_j)_j$ admits a convergent subsequence. Use now the diagonal method to conclude that there is a subsequence $(f_{j_k})_{k \in \mathbb{N}}$ such that $(S_{\gamma} P_n f_{j_k})_k$ converges for all $n$. We claim that $(S_{\gamma} f_{j_k})_k$ is also convergent. Put

$$g_n := \lim_{k \to \infty} S_{\gamma} P_n f_{j_k}, \quad n \in \mathbb{N}.$$

We claim that

1. The sequence $(g_n)_n$ converges to some point $g$ in $L^2(\mathcal{X}, \mu^{\mathbb{N}})$,

2. $\lim_{k \to \infty} S_{\gamma} f_{j_k} = g$.

We start by proving the first statement. Choose an $\epsilon > 0$. For $n, m, k$ (for notational reasons we assume $n > m$) large enough we have

$$\|g_n - g_m\| \leq \|S_{\gamma} P_n f_{j_k} - S_{\gamma} P_m f_{j_k}\| + \epsilon = \|S_{\gamma} (P_n - P_m) f_{j_k}\| + \frac{\epsilon}{2} \leq \sqrt{\gamma_{u_{m+1}}} \|S\|^{[u_{m+1}]} + \frac{\epsilon}{2}.$$

Thus $(g_n)_n$ is a Cauchy sequence, and the first statement follows.
Now to the second statement. Choose an \( \epsilon > 0 \). We can write

\[
\|S_\gamma f_{jk} - g\|_{L^2} \leq \|S_\gamma f_{jk} - S_\gamma P_n f_{jk}\|_{L^2} + \|S_\gamma P_n f_{jk} - g_n\|_{L^2} + \|g_n - g\|_{L^2}.
\]

Take an \( n \) large enough so that \( \|g_n - g\|_{L^2} < \frac{\epsilon}{3} \) and \( \sqrt{\eta_{n+1}} \|S\|_{\|u_{n+1}\|} < \frac{\epsilon}{3} \). This makes the first and the third summand small. Given this \( n \) we may now choose a \( K \) satisfying \( \|S_\gamma P_n f_{jk} - g_n\|_{L^2} < \frac{\epsilon}{3} \) for all \( k \geq K \). Thus we have shown

\[
\|S_\gamma f_{jk} - g\|_{L^2} < \epsilon
\]

for all \( k \) large enough. The second statement follows.

\[\square\]

Acknowledgements

The author would like to thank Michael Gnewuch for helpful comments.

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