Grids for efficient all sky search of white dwarf binaries in Mock LISA Data Challenge

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Abstract. We present construction of the 3 and 4 dimensional grids in the parameter space for all sky-search of gravitational wave signals from white dwarf binaries with LISA data. The 3 dimensional grid is for search of frequency and the sky position of the source and the 4 dimensional grid includes the spin down parameter. The grid solves the covering problem in the parameter space with the constraint that nodes of the grid coincide with Fourier frequencies (multiples of the inverse of the observation time). This allows the use of the fast Fourier transform (FFT) in the evaluation of the optimal statistic and greatly speeds up the search.

1. Introduction

Matched filtering detection of weak, quasi-monochromatic gravitational wave signals relies on an efficient placement of the templates in the bank. Optimally it should minimize the number of templates for a given allowed loss in the signal-to-noise ratio due to a signal offset. More specifically one considers the space of parameters of the modeled signal together with, generally non-flat, metric provided e.g. by the reduced Fisher matrix. Here we consider flat parameter space for all sky search for monochromatic and nearly monochromatic signals. We construct the grid of templates which is to be restricted in the frequency direction in order to speed the search by making use of FFT.

The paper is organized as follows. In section 2 we sketch a search method that enables fast evaluation of the detection statistic for gravitational wave signals from nearly monochromatic sources by using the FFT algorithm. In section 3 we define the covering problem with constraint. In section 4 we present two constructions of nearly optimal lattices in any number of dimensions that in a good approximation satisfy the constraint. Results are briefly illustrated in section 5 for the case of 3 and 4 dimensions. In Appendix we briefly summarize some basic definitions from the theory of lattices.

2. Grid on the parameter space

The response of the space-based detector LISA to a gravitational wave can be written \cite{1, 2} as a linear combination of 4 functions $h^{(k)}$ depending on time. Each function $h^{(k)}$ depends on parameters $\xi^\mu$ called intrinsic parameters and the constant coefficients $a^{(k)}$ called extrinsic...
parameters:
\[ h(t; a^{(k)}, \xi^\mu) = \sum_{k=1}^{4} a^{(k)} h^{(k)}(t; \xi^\mu). \]  

In the maximum-likelihood search method the estimators \( \hat{a}^{(k)}, \hat{\xi}^\mu \) are found by maximizing the log likelihood function with respect to parameters \( (a^{(k)}, \xi^\mu) \). By solving explicitly the maximum likelihood equation for the extrinsic parameters one can define the so called \( F \)-statistic and reduce the search to the intrinsic parameters space. For a signal from white-dwarf binaries buried in a stationary Gaussian noise and for intrinsic parameters defined by \( (\xi^1, \xi^2, \xi^3, \xi^4) = (\omega, \dot{\omega}, \beta, \lambda) \), where \( \omega \) is the angular frequency of the gravitational wave, \( \dot{\omega} \) is the time derivative of \( \omega \), \( \beta \) and \( \lambda \) are the latitude and the longitude of the source, the detection statistic \( F \) consists of integrals of the form
\[ \int_0^{T_o} y(t) m(t; \omega, \beta, \lambda) \exp \left[ i\varphi_{\text{mod}}(t; \omega, \dot{\omega}, \beta, \lambda) \exp i\omega t \right] dt, \]

where \( y(t) \) are the noisy data and \( m, \varphi_{\text{mod}} \) are amplitude and phase modulation functions. \( T_o \) is the observation time. As the modulation functions \( m \) and \( \varphi_{\text{mod}} \) depend on the frequency \( \omega \) we cannot apply the FFT algorithm directly in calculation of the \( F \)-statistic. In order to do that we first analyze the data in a narrow band over which the slowly varying modulation function \( m \) is evaluated at the mid frequency of the band and, second, we introduce a linear parametrization of the phase
\[ \varphi_{\text{mod}} = \frac{1}{2} \omega t^2 + A \cos \Omega t + B \sin \Omega t, \]

where frequency is absorbed in the new parameters \( A \) and \( B \),
\[ A = \omega R \cos \beta \cos (\lambda - \eta_0) \]
\[ B = \omega R \cos \beta \sin (\lambda - \eta_0), \]

where \( \eta_0 \) is the initial position of the LISA constellation on the orbit around the sun, \( R = 1 \text{AU} \) and \( \Omega = 2\pi/\text{yr} \). This results in the following linear phase model
\[ h(t) = A_0 \cos (\omega t + \frac{1}{2} \dot{\omega} t^2 + A \cos \Omega t + B \sin \Omega t + \phi_0) \]

with extrinsic \( (A_0, \phi_0) \) and intrinsic \( (\omega, \dot{\omega}, A, B) \) parameters. The reduced Fisher matrix \[ \tilde{\Gamma} \] of the model has constant coefficients. We want to distribute the points of the grid in such a way that the distance defined by \( \tilde{\Gamma} \) from any point of the parameter space to the nearest node of the grid is not larger than some fixed value \( r \).

### 3. Covering problem with constraints

The problem of constructing the grid in the parameter space is equivalent to the problem of covering \( d \)-dimensional space with equal overlapping spheres of a given radius. The optimal covering would have minimal possible thickness (see [3] and appendix). The thinnest lattice coverings are known in dimensions up to 5. In what follows we consider only lattice coverings.

In our search scheme the calculation of the detection statistic \( F \) can be interpreted as a Fourier transform and computed efficiently by fast Fourier transform algorithm. For this reason the nodes of the grid must coincide with Fourier frequencies: \( \Delta \omega, 2\Delta \omega, 3\Delta \omega, \ldots \) for some fixed frequency resolution \( \Delta \omega \). This imposes a condition that one of the lattice basis vectors has a fixed length
\[ |\tilde{l}| = \sqrt{\tilde{\Gamma} \left[ (\Delta \omega, 0, \ldots, 0), (\Delta \omega, 0, \ldots, 0) \right]} \]
and forbids an immediate use of the general results of the theory of lattice coverings. Instead one can formulate the covering problem with constraint: to find the thinnest lattice covering of the \( d \)-dimensional space with spheres of radius \( r \) and one of the basis vectors of the lattice having fixed length \( |\vec{l}| \). As far as we know the general solution to the problem is not known. We present two constructions of lattices that in a good approximation satisfy the constraints. They can be viewed as possible solutions to the problem and may serve as starting point for alternative constructions. We illustrate the procedures in 3 and 4 dimensions.

4. Constructions of the grid

Let a vector \( \vec{v}_0 \) define the frequency resolution. We search for a lattice \( \Lambda(w') \) with lattice basis \( w' = \{ \vec{v}_0, \vec{w}_1, \ldots, \vec{w}_{d-1} \} \) and one of the basis vectors of the lattice having fixed length \( |\vec{l}| \). As far as we know the general solution to the problem is not known. We present two constructions of lattices that in a good approximation satisfy the constraints.

4.1. Procedure I

As an initial approximation for the lattice we take \( \Lambda(w) \), \( w = \{ \vec{v}_0, \vec{w}_1, \ldots, \vec{w}_{d-1} \} \), with an arbitrary orthonormal \((d - 1)\)-dimensional basis \( \{ \vec{w}_1, \ldots, \vec{w}_{d-1} \} \). The construction of \( \Lambda(w') \) is based on the following algorithm \( AI \).

Algorithm \( AI \)

Input: \( w = \{ \vec{v}_0, \vec{w}_1, \ldots, \vec{w}_{d-1} \} \).
Output: \( w' = \{ \vec{v}_0, \vec{w}_1', \ldots, \vec{w}_{d-1}' \} \), with covering radius \( R(\Lambda(w')) \approx r \).

AI1. \( w' = w \).
AI2. Repeat
   AI3. \( w' = \{ \vec{v}_0, c\vec{w}_1', \ldots, c\vec{w}_{d-1}' \} \), where \( c = \frac{r}{R(\Lambda(w'))} \).
AI4. Until \( R(\Lambda(w')) \approx r \).
AI5. Return \( w' \).

We notice that the orientation of the constraint vector \( \vec{v}_0 \) has no effect on the optimal constrained lattice, it is only the length of \( \vec{v}_0 \) and assumed value of covering radius that matters (and more precisely: only their ratio because the overall scale can be taken arbitrarily).

To construct an optimal constrained lattice one repeats the algorithm \( AI \) with \( \vec{v}_0 \) having different orientations with respect to \( \{ \vec{w}_1, \ldots, \vec{w}_{d-1} \} \) and chooses \( w' \) having the smallest thickness.

4.2. Procedure II

In the Procedure II one tries to find an optimal constrained lattice starting with a thinnest unconstrained lattice in \( d \)-dimensions. The idea is to shrink the optimal lattice as little as possible such that one of the basis vectors of the resulting lattice coincides with the constraint vector \( \vec{v}_0 \).

We notice first that for a given lattice \( \Lambda \) there always exists the shortest lattice vector \( \vec{l} \) satisfying \( |\vec{v}_0| \leq \vec{l} \) that we denote by \( \vec{l}(\Lambda) \). We define the following algorithm \( AI \).

Algorithm \( AI \)

Input: Lattice \( \Lambda \); vector \( \vec{v}_0 \).
Output: Lattice \( \Lambda' \): \( \vec{l}(\Lambda') = \vec{v}_0 \).
AI1. Find \( \vec{l}(\Lambda) \).
AI2. Contract \( \Lambda \) along \( \vec{l}(\Lambda) \) to obtain \( \Lambda_c \) with \( |\vec{l}(\Lambda_c)| = |\vec{v}_0| \).
AI3. Rotate \( \Lambda_c \) to obtain a lattice \( \Lambda_{rc} \) with \( \vec{l}(\Lambda_{rc}) = \vec{v}_0 \).
AI4. Return \( \Lambda' \).
The contraction of the initial lattice $\Lambda$ is "minimal" in the sense that the thickness of the resulting lattice $\Lambda'$ satisfies $\Theta(\Lambda) \leq \Theta(\Lambda') \leq \|h(\Lambda)\|_1 \Theta(\Lambda)$.

For optimal initial lattices the algorithm $\text{AII}$ defines the following function $f_{\vec{v}_0} : \mathbb{R} \to \mathbb{R}$: for $x = R(\Lambda)$, $f_{\vec{v}_0}(x) = R(\Lambda')$. For a given $r \in \mathbb{R}$ we denote by $r_i$ the value of $x$ for which the function $|f_{\vec{v}_0}(x) - r|$ reaches its minimum. The optimal constrained lattice is obtained by application of the algorithm $\text{AII}$ to an optimal (unconstrained) lattice $\Lambda$ with covering radius $R(\Lambda) = r_i$.

In dimensions $d = 2, 3, 4, 5$ as the initial lattices one takes the so called Voronoi’s principal lattice of the first type, $A^*_d$ [3] but the procedure can be generalized to any number of dimensions by taking as the input the best known lattice covering in a given dimension [3],[4].

5. Results

Figure 1 show results of the application of the two procedures to the model (3) in 3 and 4 dimensions for different covering radii. The resolution vector is $\vec{v}_0 = (2\pi, 0, \ldots)$ and the observation time is 2 years.

**Figure 1.** Thickness of the lattice in 3 and 4 dimensions.

Appendix

In general for any discrete set of points $S = \{\vec{s}_1, \vec{s}_2, \ldots\}$ in $\mathbb{R}^n$ the covering radius $R$ of $S$ is defined as the least upper bound for any point of $\mathbb{R}^n$ to the closest point $\vec{s}_i$:

$$R(S) = \sup_{\vec{x} \in \mathbb{R}^n} \inf_{\vec{s} \in S} |\vec{x} - \vec{s}|.$$  

Then spheres of equal radius $r$ centered at the points $\vec{s}_i$ will cover $\mathbb{R}^n$ only if $r \geq R$.

A lattice $\Lambda$ is a discrete subset of $\mathbb{R}^n$. Any lattice has a basis $\{\vec{l}_1, \ldots, \vec{l}_n\}$ of linearly independent vectors such that the lattice is the set of all linear combinations of $\vec{l}_i$’s with integer coefficients:

$$\Lambda = \left\{ \sum_{i=1}^{n} c_i \vec{l}_i : \ c_i \in \mathbb{Z}, \quad i = 1, 2, \ldots, n \right\}. \quad (A.1)$$  

A lattice basis is not unique, in dimensions $d > 1$ there are infinitely many of them, but all the bases have the same number of elements called the dimension of the lattice.

The parallelootope consisting of points $c_1\vec{l}_1 + \ldots + c_n\vec{l}_1$ with $0 \leq c_i < 1$ is a fundamental parallelootope and is an example of an elementary cell, that is the building block containing one
lattice point which tiles the whole $\mathbb{R}^n$ by translations of lattice vectors. There are infinitely many elementary cells but the volume of each elementary cell is unique for a given lattice $\Lambda$.

The Voronoi cell around any point $\mathbf{v}$ of $\Lambda$ is the set of vectors $\mathbf{x}$ of $\mathbb{R}^n$ which are closer to $\mathbf{v}$ than to any other lattice vector:

$$V(\mathbf{v}) = \{ \mathbf{x} : |\mathbf{x} - \mathbf{w}| \geq |\mathbf{x} - \mathbf{v}| \text{ for all } \mathbf{w} \in \Lambda \}. \quad (A.2)$$

All Voronoi cells of a given lattice are congruent convex polytopes and are another examples of elementary cells sometimes referred to as Wigner-Seitz cells or Brillouin zones.

For the lattice $\Lambda$ having Voronoi cells congruent to polytope $V(\mathbf{v})$, where $\mathbf{v}$ is any of the lattice points, the covering radius is the circumradius $R(\Lambda)$ of $V(\mathbf{v})$ i.e. the largest distance between $\mathbf{v}$ and the vertices of $V(\mathbf{v})$.

The thickness of the lattice covering is given by:

$$\Theta(\Lambda) = \frac{\text{volume of } d\text{-dimensional sphere of radius } R(\Lambda)}{\text{volume of the elementary cell of } \Lambda} \quad (A.3)$$

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