Abstract

Soliton Solutions of Korteweg-de Vries (KdV) were constructed for given degenerate curves $y^2 = (x - c)P(x)^2$ in terms of hyperelliptic sigma functions and explicit Abelian integrals. Connection between sigma functions and tau function were also presented.

§1. Introduction

The modern soliton theories [DJKM, SN, SS], which were developed in ending of last century, are known as the infinite dimensional analysis and gave us fruitful and beautiful results, e.g., relations of soliton equations to universal Grassmannian manifold, Plücker embedding, infinite dimensional Lie algebra, loop algebra, representation theories, Schur functions, Young diagram, and so on. They stemmed from an investigation of the Bäcklund transformations among the soliton solutions, which are expressed by hyperbolic functions [DJKM, SN, SS]. By primitive considerations, the soliton solutions are related to certain degenerate algebraic curves and thus the Bäcklund transformation can be regarded as transformation among certain degenerate curves of different genera [DJKM, SN, SS].

As the modern soliton theories are based on the abstract theory, theories on the Abelian functions established in nineteenth century are very concrete [B1-3, Kl]. They are of given concrete algebraic curves and of Abelian functions and differential equations there. By fixing a hyperelliptic curve, Klein defined the hyperelliptic sigma function, a hyperelliptic version of Weierstrass sigma function for elliptic curve [Kl]. The sigma function is a well-tuned theta function and brings us fruitful information of the curve. In terms of the hyperelliptic sigma functions and bilinear differential operator, Baker discovered the Korteweg-de Vries hierarchy, Kadomtsev-Petviashvili equation and gave their periodic solutions without ambiguous parameters [B1-3, Ma1].

As the new century began, I believe that we should connect both established theories from a novel point of view. Recently the theories in nineteenth century has been re-evaluated in various fields from similar viewpoint [BEL1-5, CEEK, EE, EEL, EEP, Ma1,2, N]. For example, Buchstaber, Enolskii and Leykin connected the hyperelliptic sigma functions with the Paffian in [BEL3] and the Schur-Weierstrass polynomials in [BEL4]. The purpose of this article is also to give a step to a unification of both theories.

In this article, we will focus on the soliton solutions of Korteweg-de Vries equation in terms of the hyperelliptic sigma functions over degenerate hyperelliptic curves. It is needless to say that the

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soliton solutions played important roles in developments of soliton theory and the investigations of the mathematical structure of the soliton solutions are well-established [H1,2, GGKM, L, DKJM]. In fact the similar studies to this article have been already done by Its and Matveev [BBEIM, I, IM1]. They showed an explicit connection between so-called Hirota’s tau functions of the soliton solutions and Riemannian theta functions in terms of the normalized differential of the first kind. Their works [BBEIM, I, IM1, 2] also influenced the development of the modern soliton theory.

As in [BEL1,2, Kr1, Kr2, IM1,2], solutions space of KdV equation is very fruitful and the soliton solutions are in very special locus of the space, I believe that in order to know the solutions space of the KdV equation, it is very important to recognize again how special the soliton solution is.

Further recent studies [BEL1-5, CEEK, EE, EEL, EEP, Ma1,2, N] shows that the unnormalized differentials (one-forms) are more important than the normalized ones used in [BBEIM, I, IM1]. Thus we believe that we should re-consider these studies of forerunners in terms of more clear words.

In other words, this article should be regarded as a revise of the works of Its and Matveev [I, IM1]. For example they found the nice differentials of the first kind (3-6), which is a key of this study, and obtained the most of results as this article. However from viewpoint of algorithmic investigations, their approach is a little bit heuristic. Further it is neither clear why the soliton solutions in different genera are connected whereas those of more general hyperelliptic curves are not found [Kr1, 2, BBEIM]. They neither commented on the importance of the existence of (3-6) nor the specialty of the soliton solutions.

In this article, by employing older fashion as in nineteenth century, we start with a concrete degenerate curve \( y^2 = P(x)^2 x \) and reconstruct the soliton solutions after performing some explicit Abelian integrations. By dealing with the unnormalized differentials of the first kind, we find nice differentials (one-form) \( dv \)’s in (3-6) over the special curve \( y^2 = P(x)^2 x \), which consists only of the data of zero of \( P(x) \) and does not depend upon other global structure of the curves. The existence of the differentials characterizes the soliton solutions in the solution space of the KdV equation and a key of the fact that the soliton solutions are very simple. As shown in lemma 3-6, the existence of the differentials simplifies the Abelian integrations. For examples, the components of the integration matrices do not depend upon the genus of curve. Accordingly we can easily find connections among such special curves with different genera. However from the study of hyperelliptic functions in nineteenth century, such differentials do not exist for general hyperelliptic curves as shown in Remark 3-10. Our concrete computations and the theories in nineteenth century enable us to recognize that such connections among the curves with different genera might have practical meanings only for such special cases. In other words, as I point out there, the Bäcklund transformation might be effective only for the degenerate curves associated with the soliton solutions.

In order to unify the both theories in nineteenth and twentieth centuries, we should also recognize the difference between them. Thus I considered this problem. After submitting this article, I knew the works of Its and Matveev [BBEIM, I, IM1] and of Edelstein, Gómez-Reino and Mariño EGM and revised this. Edelstein, Gómez-Reino and Mariño also dealt with soliton solutions based upon the formula in [BBEIM] in the context of the gauge field theory after considering the hyperelliptic sigma functions. However as their purpose is not to unify the both theories, their treatments are not enough from our viewpoint. Even after knowing the work [I, IM1, EGM], I believe that concrete computations of the soliton solutions in the framework of sigma function theory has very important meanings in this stage. For example, as the coordinate systems (conventions) in the Jacobian are different in both theories, i.e., \( u \) to that of the nineteenth and \( t \) to that of the twentieth centuries, I connect them in (3-19) in terms of a matrix (3-3) for the case of the soliton solutions. Further as the coefficient of these coordinate systems in the theta functions looks different between the theories in
1970’s [Kr1, 2, IM1, 2] and the definition of Klein, it turns out that they are connected by Legendre relation (2-23) and (3-33). As mention it in Remark 3-10, comparison between one-form (3-6) and (3-3) appearing in Baker’s investigation [Ba2] shows how special the soliton solutions are.

§2. Preliminary of Baker’s Hyperelliptic Sigma Functions

In this section, we will review the hyperelliptic sigma function. In this article, we will mainly use the conventions of Ōnishi [Ô1]. As there is a good self-contained paper on theories of hyperelliptic sigma functions [BEL2] besides [B1,2,3], we will give their notations, definitions and propositions without explanations and proves.

We denote the set of complex number by \( \mathbb{C} \) and the set of integers by \( \mathbb{Z} \).

**Notation 2-1.** We deal with a hyperelliptic curve \( \tilde{C}_g \) of genus \( g \) \((g > 0)\) given by an affine equation,

\[
y^2 = f(x) = \lambda_{2g+1}x^{2g+1} + \lambda_{2g}x^{2g} + \cdots + \lambda_1x + \lambda_0 = P(x)Q(x) \tag{2-1}
\]

where \( \lambda_{2g+1} \equiv 1 \) and \( \lambda_j \)'s and

\[
Q(x) = (x - c_1)(x - c_2) \cdots (x - c_g)(x - c), \\
P(x) = (x - a_1)(x - a_2) \cdots (x - a_g) \\
= x^g + \mu_{g-1}x^{g-1} + \cdots + \mu_1x + \mu_0, \tag{2-2}
\]

c_j's, a_j's, c and \( \mu_j \)'s are complex values.

We will express a point P in the curve by the affine coordinate \((x, y)\) if it is not the infinity point.

**Definition 2-2** [B1 p.195, B2 p.314, B3 p.137, BEL2 Chapter 2, Ô1 p.385-6].

(1) Let us denote the homology of a hyperelliptic curve \( \tilde{C}_g \) by

\[
H_1(\tilde{C}_g, \mathbb{Z}) = \bigoplus_{j=1}^{g} \mathbb{Z} \alpha_j \oplus \bigoplus_{j=1}^{g} \mathbb{Z} \beta_j, \tag{2-3}
\]

where these intersections are given as \([\alpha_i, \alpha_j] = 0\), \([\beta_i, \beta_j] = 0\) and \([\alpha_i, \beta_j] = \delta_{i,j}\). These \( \alpha \)'s and \( \beta \)'s are given as illustrated in Fig.1 for the case of genus five. There we construct the hyperelliptic Riemannian surface using twin Riemannian spheres with cuts.

(2) The unnormalized differentials of the first kind are defined by,

\[
du_1 := \frac{dx}{2y}, \quad du_2 := \frac{x \cdot dx}{2y}, \quad \cdots, \quad du_g := \frac{x^{g-1} \cdot dx}{2y}. \tag{2-4}
\]

(3) The unnormalized differentials of the second kind are defined by,

\[
d\tilde{u}_1 := \frac{x^g \cdot dx}{2y}, \quad d\tilde{u}_2 := \frac{x^{g+1} \cdot dx}{2y}, \quad \cdots, \quad d\tilde{u}_g := \frac{x^{2g-1} \cdot dx}{2y}. \tag{2-5}
\]
and \( dr := (dr_1, dr_2, \cdots, dr_g) \),

\[
(dr) := \Lambda \left( \frac{du}{d\mathbf{u}} \right),
\]

(2-6)

where \( \Lambda \) is \( 2g \times g \) matrix defined by

\[
\Lambda = \begin{pmatrix}
0 & \lambda_3 & 2\lambda_4 & 3\lambda_5 & \cdots & (g-1)\lambda_{g+1} & g\lambda_{g+2} & (g+1)\lambda_{g+3} \\
0 & \lambda_5 & 2\lambda_6 & \cdots & (g-2)\lambda_{g+2} & (g-1)\lambda_{g+3} & g\lambda_{g+4} & \\
0 & \lambda_7 & \cdots & (g-3)\lambda_{g+3} & (g-2)\lambda_{g+4} & (g-1)\lambda_{g+5} & \\
& \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \\
0 & \lambda_{2g-2} & 2\lambda_{2g-1} & 3\lambda_{2g+1} & \\
\cdots & (2g-3)\lambda_{2g-1} & (2g-2)\lambda_{2g} & (2g-1)\lambda_{2g+1} & 0 & \\
\cdots & (2g-4)\lambda_{2g} & (2g-3)\lambda_{2g+1} & 0 & \\
\cdots & (2g-5)\lambda_{2g+1} & 0 & \\
\cdots & 0 & \\
0 & \\
\end{pmatrix}.
\]

(2-7)

(4) The unnormalized period matrices are defined by,

\[
\omega' := \left[ \int_{\alpha_i} du_i \right], \quad \omega'' := \left[ \int_{\beta_j} du_j \right], \quad \omega := \left[ \frac{\omega'}{\omega''} \right].
\]

(2-8)

(5) The normalized period matrices are given by,

\[
^t [d\mathbf{u}_1 \cdots d\mathbf{u}_g] := \omega'^{-1} [du_1 \cdots du_g], \quad \tau := \omega'^{-1} \omega'', \quad \omega := \left[ \frac{1_g}{\tau} \right].
\]

(2-9)

(6) The complete hyperelliptic integrals of the second kind are given as

\[
\eta' := \left[ \int_{\alpha_i} dr_i \right], \quad \eta'' := \left[ \int_{\beta_j} dr_j \right].
\]

(2-10)

(7) By defining the Abelian map for \( g \)-th symmetric product of the curve \( \tilde{C}_g \) and for points \( \{Q_i\}_{i=1}^g \) in the curve,

\[
\hat{w} : \text{Sym}^g(\tilde{C}_g) \rightarrow \mathbb{C}^g, \quad \left( \hat{w}_k(Q_i) := \sum_{i=1}^g \int_{Q_i}^\infty d\mathbf{u}_k \right),
\]

\[
w : \text{Sym}^g(\tilde{C}_g) \rightarrow \mathbb{C}^g, \quad \left( w_k(Q_i) := \sum_{i=1}^g \int_{Q_i}^\infty du_k \right),
\]

(2-11)
the Jacobi varieties $\hat{J}_g$ and $J_g$ are defined as complex torus,

\[
\hat{J}_g := \mathbb{C}^g / \hat{\Lambda}, \quad J_g := \mathbb{C}^g / \Lambda.
\] (2-12)

Here $\hat{\Lambda}$ ($\Lambda$) is a lattice generated by $\hat{\omega}$ ($\omega$).

(8) We defined the theta function over $\mathbb{C}^g$ characterized by $\hat{\Lambda}$ or $\tau$,

\[
\vartheta \left[ \begin{array}{c} a \\ b \end{array} \right] (z; \tau) := \sum_{n \in \mathbb{Z}^g} \exp \left[ 2\pi \sqrt{-1} \left\{ \frac{1}{2} (n + a) \tau (n + a) + \vartheta (n + a) (z + b) \right\} \right],
\] (2-13)

for $g$-dimensional vectors $a$ and $b$.

---

**Definition 2-3** ($\wp$-function, Baker) [B1, B2 p.336, p.358, p.370, BEL2 p.35, Ô1 p.386-7].

The coordinate in $\mathbb{C}^g$ for points $(x_i, y_i)_{i=1,\ldots,g}$ of the curve $y^2 = f(x)$ is given by,

\[
u_j := \sum_{i=1}^g \int_{(x_i, y_i)}^{(x_j, y_j)} du_j.
\] (2-14)

(1) Using the coordinate $u_j$, sigma functions, which is a holomorphic function over $\mathbb{C}^g$, is defined by

\[
\sigma(u) = \sigma(u; \tilde{C}_g) := \gamma \exp \left( -\frac{1}{2} t \left[ \begin{array}{c} \delta'' \\ \delta' \end{array} \right] \left[ \begin{array}{c} \omega' \omega'^{-1} \end{array} \right] \vartheta(u) \right).
\] (2-15)

where

\[
\theta(u) := \vartheta \left[ \begin{array}{c} \delta'' \\ \delta' \end{array} \right] (\omega'^{-1} u; \tau),
\] (2-16)

and

\[
\delta' = \left[ \begin{array}{c} \frac{g}{2} \\ \frac{g-1}{2} \\ \frac{1}{2} \end{array} \right], \quad \delta'' = \left[ \begin{array}{c} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{array} \right],
\] (2-17)

$\gamma$ is a certain constant factor.

(2) In terms of these functions, $\wp$-functions are defined by

\[
\wp_{\mu\nu}(u) = -\frac{\partial^2}{\partial u_\mu \partial u_\nu} \log \sigma(u),
\] (2-18)

\[
\tilde{\wp}_{\mu\nu}(u) = -\frac{\partial^2}{\partial u_\mu \partial u_\nu} \log \theta(u).
\] (2-19)

It is easy to find the difference between $\wp$ and $\tilde{\wp}$ as follows.

**Lemma 2-4.**

\[
\tilde{\wp}_{ij}(u) = \wp_{ij}(u) - C_{ij},
\] (2-20)

where

\[
C_{ij} = (\eta' \omega'^{-1})_{ij}.
\] (2-21)
Proposition 2-5 [B3, BEL2 p.52,53, Ma1].

\[ \varphi_{\mu
u} := \frac{\partial \varphi_{\mu\nu}(u)}{\partial u_{\rho}} \quad \text{and} \quad \varphi_{\mu
u\rho\lambda} := \frac{\partial^2 \varphi_{\mu\nu}(u)}{\partial u_{\mu} \partial u_{\nu}}. \]

Then hyperelliptic \( U := \left( 2\varphi_{gg} + \lambda_2g/6 \right) \) obeys the Korteweg-de Vries equations,

\[ 4 \frac{\partial U}{\partial u_{g-1}} + 6U \frac{\partial U}{\partial u_{g}} + \frac{\partial^2 U}{\partial u_{g}} = 0. \quad (2-22) \]

We should note that this proposition is easily proved in the framework of sigma function theories [BEL2, Ma1, Ma3]. In fact, Baker already discovered this fact [B3] around 1898 as mentioned in [Ma1]; his way is very simple as explained in [Ma1, Ma3]. However, as \( \tilde{\varphi} \) and \( \varphi \) differ just by the constant matrix from (2-20), we should mention the works of pioneers [BBEIM, Kr1, IM1, 2]; they directly showed it in terms of \( \tilde{\varphi} \).

Proposition 2-6 [BEL2 p.11].

The Legendre relation is given by

\[ t \omega' \eta'' - t \omega'' \eta' = 2\pi \sqrt{-1}I_g, \quad (2-23) \]

where \( I_g \) is the \( g \times g \)-unit matrix.

§3. Soliton Solutions of Korteweg-de Vries Equations

In this section, we will consider a degenerate curve which is connected with the so-called soliton solution. For a degenerate curve, the differentials (2-4) and (2-5) become singular as we show later, but the sigma function can be defined as a limit of non-singular curves even though it needs some regularizations. In fact, we will show that the sigma function over a certain curve can be associated with tau function after regularizations.

Degenerate Curves 3-1. We deal with a degenerate hyperelliptic curve \( C_g \) of genus \( g \) \((g > 0)\) given by an affine equation,

\[ y^2 = P(x)^2x, \quad (3-1) \]

i.e., \( Q(x) = P(x)x \) in (2-2), or \( c \equiv 0, c_1 \equiv a_1, c_2 \equiv a_2, \cdots, c_g \equiv a_g \) in (2-1) and (2-2).

For later convenience, we introduce parameters \((k_i)_{i=1,...,g}, k_i = \sqrt{a_i}\). Here we note that since the affine equation has symmetries, i.e., translation, dilatation, inversion, in a projective space, it should be regarded as a representation element in an equivalent relations. In this article, I use the representation for simplicity.

Definition 3-2.

(1) For \( g \)-th polynomial \( P(x) \) in (2-2), we define,

\[ \pi_i(x) := \frac{P(x)}{x - a_i} = \chi_{i,g-1}x^{g-1} + \chi_{i,g-2}x^{g-2} + \cdots + \chi_{i,1} + \chi_{i,0}. \quad (3-2) \]

(2) We will introduce \( g \times g \)-matrices

\[ W := \begin{pmatrix} \chi_{1,0} & \chi_{1,1} & \cdots & \chi_{1,g-1} \\ \chi_{2,0} & \chi_{2,1} & \cdots & \chi_{2,g-1} \\ \vdots & \vdots & \ddots & \vdots \\ \chi_{g,0} & \chi_{g,1} & \cdots & \chi_{g,g-1} \end{pmatrix}, \quad M := \begin{pmatrix} 1 & 0 \\ \mu_{g-1} & 1 \\ \mu_{g-2} & \mu_{g-1} & 1 \\ \vdots & \vdots & \vdots & \ddots \\ \mu_1 & \mu_2 & \cdots & \mu_{g-1} & 1 \end{pmatrix}, \]
Lemma 3-3.

(1) $W_{ig} = \mathbf{1}^{t}(1, 1, \cdots, 1)$ and
\[ W = K(0)M. \] (3-4)

(2) The unnormalized differentials (2-4) and (2-5) become given by
\[ du_i = \frac{s^{2i-2} ds}{P(s^2)}, \quad d\tilde{u}_i = \frac{s^{2g+2i-2} ds}{P(s^2)}, \] (3-5)
where $s := \sqrt{x}$.

(3) The other unnormalized differentials defined by
\[ dv_i := \sum_{j=1}^{g} W_{ij} du_j, \quad d\tilde{v}_i := \sum_{j=1}^{g} W_{ij} d\tilde{u}_j, \] (3-6)
are expressed by
\[
(dv_1, dv_2, \cdots, dv_g) = \left( \frac{ds}{s^2 - a_1}, \frac{ds}{s^2 - a_2}, \cdots, \frac{ds}{s^2 - a_g} \right),
\]
\[
(d\tilde{v}_1, d\tilde{v}_2, \cdots, d\tilde{v}_g) = \left( \frac{s^{2g} ds}{s^2 - a_1}, \frac{s^{2g} ds}{s^2 - a_2}, \cdots, \frac{s^{2g} ds}{s^2 - a_g} \right). \] (3-7)

(4) The inverse matrix of $W$ is given by $W^{-1} = \mathcal{P}^{-1}V$, where $V$ is Vandermonde matrix,
\[
V = \begin{pmatrix}
1 & 1 & \cdots & 1 \\
1 & a_1 & a_2 & \cdots & a_g \\
1 & a_1^2 & a_2^2 & \cdots & a_g^2 \\
& \vdots & \vdots & \ddots & \vdots \\
1 & a_1^{g-1} & a_2^{g-1} & \cdots & a_g^{g-1}
\end{pmatrix}. \] (3-8)

(5) The unnormalized differentials of the second kind $dr$’s are
\[ dr_j = \Lambda \begin{pmatrix} W^{-1} & 0 \\ 0 & W^{-1} \end{pmatrix} \begin{pmatrix} d\mathbf{v} \\ d\tilde{\mathbf{v}} \end{pmatrix}. \] (3-9)

Proof. (1) Noting the relation $\pi_i(x)(x - a_i) = P(x)$, we have
\[ \chi_{i,k} = \mu_{k+1} - a_i \chi_{i,k-1}, \] (3-10)
and the initial condition $\chi_{i,g-1} \equiv 1$. Then we have (1). Noting the properties of the inverse matrix of Vandermonde matrix, others are obtained by direct computations. \[ \square \]
Remark 3-4.
(1) For the degenerate curve in (3-1), $du$’s are no more holomorphic over $C_g$ due to the behavior of $dv_i$.
(2) $\frac{1}{k_i}dv_i$ is a differential of third kind i.e. a differential with two singular points whose residues are $+1$ and $-1$ respectively.
(3) $dv$’s are composed only of the data of a single zero of $P(x)$ and do not depend upon the $y$ explicitly. In other words, $dv$’s do not explicitly carry the data in which curve it is defined.

Lemma 3-5.
For the contour $\alpha$’s and $\beta$’s as illustrated in Fig.2 for genus five case, we have the following results.
(1) $\int_{\alpha_j} dv_i = \pi \sqrt{-1} \frac{1}{k_i} \delta_{j,i}$, $\int_{\alpha_j} d\tilde{v}_i = \pi \sqrt{-1} k_i^{2g-1} \delta_{j,i}$. \hfill (3-11)
(2) $\int_{\beta_j} dv_i = \frac{1}{k_i} \log \left| \frac{(k_i - k_j)}{(k_i + k_j)} \right|$, $\int_{\beta_j} d\tilde{v}_i = \frac{1}{g} \sum_{r=0}^{g-1} k_i^{2r} k_j^{2g-2r-1} + k_i^{2g-1} \log \left| \frac{(k_i - k_j)}{(k_i + k_j)} \right|$. \hfill (3-12)

Proof. The contour $\alpha$ contains only one singularity, $\int_{\alpha_j} dv_i = \oint_{s=k_i} \frac{ds}{(s - k_i)(s + k_i)} = 2\pi \sqrt{-1} \frac{1}{2k_i}$. \hfill (3-13)
Similarly we have integration of $d\tilde{v}$. On (2), the contour $\beta$ can be restricted over real line if $a_i$ are real. Real valued integration $(a_j, \infty)$ gives (3-12) and it is also true even if $a_i$ is not real valued. The factor 2 in (3-12) comes from the returned path. \hfill $lacksquare$

Using lemma 3-5, we have $\omega'$, $\omega''$, $\tau$, and $\eta'$ by direct computations following these definitions, (2,8-10) and (3-6), as next lemma.

Lemma 3-6.
(1) $\omega' = W^{-1} \left( \frac{\pi \sqrt{-1}}{k_i} \delta_{i,j} \right)$, $\omega'' = W^{-1} \left( \frac{1}{k_i} \log \left| \frac{(k_i - k_j)}{(k_i + k_j)} \right| \right)$. \hfill (3-14)
(2) All components of diagonal part of $\tau$, which is denoted by $\tau_{\text{diag}}$, are diverge $\tau_{ii} = \sqrt{-1} \infty$ and off diagonal part $\tau_{\text{off}} := \tau - \tau_{\text{diag}}$ is given by $\tau_{ij} = \sqrt{-1} \frac{1}{\pi} \log \left| \frac{(k_i + k_j)}{(k_i - k_j)} \right|$, $i \neq j$. \hfill (3-15)
(3) The components of $\eta'$, $\eta' = (\eta_i(k_j))$, is given by $\eta_i(k) = \frac{\pi \sqrt{-1} k^{2i-3}}{P''(k^2)} \left[ \frac{d}{dx} \left( \frac{f(x)}{x^{2i}} \right) \right]_{x=k^2}$ \hfill (3-16)
where \((\cdot)_+\) is the polynomial part of rational function of \(x\).

(4) The matrix \(C := (C_{ij})\) in (2-21) is equal to,

\[
C = \left( \frac{k_j}{\pi \sqrt{-1}} \eta_i(k_j) \right) W, \tag{3-17}
\]

and especially \(C_{gg} = 1\).

Here we note that the origin of the truncated polynomial in (3-16) is \(A\) matrix in (2-6) \(i.e.,\)

\[
dr_i = \frac{x^i}{2y} \frac{dx}{dx} d\left[ \left( \frac{f(x)}{x^{2i}} \right)_+ \right]. \tag{3-18}
\]

As it might be a digression, one might wonder why such truncated polynomial appears in definition of sigma function. One of its reasons is to concentrate the singularities of the differential of the second kind at the infinity point; a certain residual integral at the infinity point removes the excess part. As in soliton theory, we encounter a truncated differential operator, its origin is essentially the same as this.

Further we remark that due to the properties of \(dv\)'s, the component of matrices \(W \omega'\), \(W \omega'\) and \(\tau\) in (3-14), (3-15) and (3-16) are also composed only of the data of one or two zeroes of \(P(x)\) and do not depend upon other global structures. In other words, for different curves expressed by the form (3-1), the components have the same form if corresponding zeroes are the same.

**Lemma 3-7.**

*By introducing new coordinate in the Jacobian \(J_g,\)*

\[
t := M^{-1}u, \tag{3-19}
\]

where \(u = (u_1, u_2, \cdots, u_g)^t\) and \(t := (t_g, t_{g-1}, \cdots, t_1)^t\), \(i.e.,\) \(u_g = t_1, u_{g-1} = t_2 + \mu_{g-1}t_1, \cdots,\) we have the relations:

(1)

\[
\pi \sqrt{-1} \omega'^{-1} u = K(1) t. \tag{3-20}
\]

(2)

\[
\begin{pmatrix}
\frac{\partial}{\partial u_1} \\
\frac{\partial}{\partial u_2} \\
\vdots \\
\frac{\partial}{\partial u_g}
\end{pmatrix}
= \left( M \right)_{\tau} \begin{pmatrix}
\frac{\partial}{\partial t_g} \\
\frac{\partial}{\partial t_{g-1}} \\
\vdots \\
\frac{\partial}{\partial t_1}
\end{pmatrix}. \tag{3-21}
\]

(3)

\[
\frac{\partial^2}{\partial u_i \partial u_j} \log \theta(u) = \sum_{kl} M_{ki} M_{lj} \frac{\partial^2}{\partial t_k \partial t_l} \log \theta(t), \tag{3-22}
\]

especially,

\[
\varphi_{gg} = -\frac{\partial^2}{\partial u_g \partial u_g} \log \theta(u) - 1 = -\frac{\partial^2}{\partial t_1 \partial t_1} \log \theta(t) - 1. \tag{3-23}
\]
Proof. (1) is obvious from (3-4) and (3-14). We should pay attentions on the fixed parameters in the partial differential in (2). \( \frac{\partial}{\partial u_i} \) means \( \left( \frac{\partial}{\partial u_i} \right)_{u_1, \ldots, u_{i-1}, u_{i+1}, \ldots, u_g} \) where indices are fixed parameters. Since we have
\[
dt_i = \sum_{j=1}^{g} \left( \frac{\partial t_i}{\partial u_j} \right)_{u_1, u_2, \ldots, u_i-1, u_i+1, \ldots, u_g} du_j.
\] (3-24)
and
\[
\left( \frac{\partial}{\partial u_i} \right)_{u_1, \ldots, u_{i-1}, u_{i+1}, \ldots, u_g} = \sum_{j=1}^{g} \left( \frac{\partial t_i}{\partial u_i} \right)_{u_1, \ldots, u_{i-1}, u_{i+1}, \ldots, u_g} \frac{\partial}{\partial t_i}.
\] (3-25)
Comparing (3-24) with the definition (3-19), (2) is proved. Using (2), (3) is shown.

Here we note that from (3-15) \( \theta(u) \) itself vanishes but \( \tilde{\theta}(u) := e^{(\pi \sqrt{-1}t_{\text{off}})\tau_{\text{diag}})/2} \theta(u) \) is reduced to finite summation,
\[
\tilde{\theta}(u) = \sum_{n \in [-1,0]^g} \exp \left[ 2\pi \sqrt{-1} \left\{ \frac{1}{2} (n + \delta') t_{\text{off}} (n + \delta') + (n + \delta') (\omega^{-1}u + \delta') \right\} \right].
\] (3-26)
Introducing the vector
\[
epsilon = ((-1)^{\epsilon_1}, (-1)^{\epsilon_2}, \ldots, (-1)^{\epsilon_g}),
\] (3-27)
where \( \epsilon \in \mathbb{Z}_2 = [0,1] \). (3-26) can be expressed by
\[
\tilde{\theta}(u) = \sum_{\epsilon \in \mathbb{Z}_2^g} \exp \left[ 2\pi \sqrt{-1} \left\{ \frac{1}{8} (\epsilon) t_{\text{off}} (\epsilon) + (n + \delta') (\omega^{-1}u + \delta') \right\} \right].
\] (3-28)

Let us summarize our results.

**Theorem 3-8.**

The \( \tilde{\theta} \) for the degenerate curve, \( y^2 = P(x)^2 x \), is given by
\[
\tilde{\varphi}_{\mu\nu}(u) = -\frac{\partial^2}{\partial u_\mu \partial u_\nu} \log \tilde{\theta}(u),
\] (3-29)
where
\[
\tilde{\theta}(u) = \sum_{\epsilon \in \mathbb{Z}_2^g} \left[ \prod_{i < j} \left( \frac{k_i - k_j}{k_i + k_j} \right)^{\epsilon_i \epsilon_j / 2} \right] \exp(\sum_i \epsilon_i \xi_i(t)),
\] (3-30)
and \( \xi_i \) is a component of a vector \( \xi := (\xi_g, \xi_{g-1}, \ldots, \xi_1) \) given by,
\[
\xi = K(1)t + \pi \sqrt{-1} \delta'.
\] (3-31)

Proof. We note that \( (\epsilon_i \epsilon_j + \epsilon_j \epsilon_i)/4 = \epsilon_i \epsilon_j / 2 \) Since we have relation,
\[
\frac{\partial^2}{\partial u_i \partial u_j} \log \theta(u) = \frac{\partial^2}{\partial u_j \partial u_i} \log \tilde{\theta}(t),
\] (3-32)
above theorem is obvious.
Here we note that the coefficient of the linear term of $t$ in the exponential function in (3-30) is replaced by a hyperelliptic integral of the second kind in [BBEIM, I, IM1], whose integrand behaves $\sqrt{x}$ around the infinite point. On the other hand, ours is roughly the inverse of the integral of the first kind due to (3-20) and (3-31). Both are connected by the Legendre relation (2-23),

$$t \eta'' \equiv 2\pi \sqrt{-1} \omega_1^{-1} + t \eta' \omega_1 \omega_1^{-1}.$$  

We also note that the unnormalized differentials of the second kind (2-5) behave around the infinite point,

$$d\tilde{u}_i|_{\infty} \sim \sqrt{x} + \text{lower order}.$$  

Thus our arguments and that of [BBEIM, I, IM1] are compatible.

Examples 3-8.

For example, we will consider the case of genus $g = 2$.

$$\tilde{\theta}(u) = -\left[ \left( \frac{k_1 - k_2}{k_1 + k_2} \right)^{1/2} e^{-\xi_1 - \xi_2} + \left( \frac{k_1 + k_2}{k_1 - k_2} \right)^{1/2} e^{\xi_1 - \xi_2} \right] + \left( \frac{k_1 + k_2}{k_1 - k_2} \right)^{1/2} e^{-\xi_1 + \xi_2} \left( \frac{k_1 - k_2}{k_1 + k_2} \right)^{1/2} e^{\xi_1 + \xi_2}.$$  

Let us define

$$\theta_r(t) := -\left( \frac{k_1 - k_2}{k_1 + k_2} \right)^{1/2} e^{\xi_1 + \xi_2} \tilde{\theta}(u),$$  

and

$$\xi'_i := \xi_i + \frac{1}{2} \log \left( \frac{k_1 - k_2}{k_1 + k_2} \right).$$  

Here the transformation (3-37) corresponds to a change of the origin of the Jacobi variety $J_g$. Then the new theta function $\theta_r$ is expressed by

$$\theta_r(t) = 1 + e^{2\xi'_1} + e^{2\xi'_2} + \left( \frac{k_1 - k_2}{k_1 + k_2} \right)^2 e^{2\xi'_1 + 2\xi'_2}.$$  

This agrees with the tau function of two soliton solutions [H].

Remark 3-10.

1. The example 3-9 can be easily extended to arbitrary genus $g$ case.
2. It is easy to construct the $\tilde{\theta}$-function over a curve $C_{g+1}$: $Y^2 = P(X)(X - a_{g+1})^2 X$ if one uses the coordinate $t$ and knows the data of $\tilde{\theta}$ over $C_g$: $y^2 = P(x)^2 x$ because all matrices for $t$ appearing in $\tilde{\theta}(t) := \tilde{\theta}(Mt)$ in (3-26) for $C_g$ are sub-matrices of corresponding ones of $C_{g+1}$. The origin of this fact comes from the existence of $dv$ in (3-7) and the degenerate type of curves $y^2 = P(x)^2 x$. As long as we restrict ourselves to consider the such simple degenerate curves, we have natural inclusion of matrices for curves of any genera $g$ and $g'$. Then it might be easy to construct a filter space and its inverse limit of the filter space for set of the degenerate curves. Then one can regard this operation for $C_g$ and $C_{g+1}$ as a transformation in the filter space.
3. If one wishes to extend the transformation in (2) for other type curves, e.g. non-degenerate curves, $C_g : y^2 = f(x)$ and $C_{g+1} : Y^2 = f(X)(X - a_{g+1})(X - c_{g+1})$, he will encounter
some difficulties because in general we cannot find differential forms \( \{dv_i\} \) which contains only a data of zero of \( f(x) \) or \( f(X)(X-a_{g+1})(X-c_{g+1}) \), even for the case of \( a_{g+1} = c_{g+1} \). In other words, these transformations mentioned in (2) does not have any practical effect on the computations for general curves. In fact as the similar variables to \( \{dv_i\} \) of were designed in [B2 p.338],

\[
dV_i = \gamma_i \frac{P(x)}{(x-a_i)} \frac{dx}{y},
\]

(3-39)

for the non-degenerated curve given in (2-1), where \( \gamma_i \) is a constant. \( dV_i \) becomes \( \{dv_i\} \) if the curve is degenerate like (3-1). It behaves as \( dt_i \) for the point \( (a_i,0) \) but its global behavior are complicate like \( \{du_j\} \)'s and thus Abelian integrals in terms of \( dV \)'s do not have simple forms at all. It is obvious that there is no sub-matrices structure between different curves even though their zeroes are common like \( \tilde{C}_g \) and \( \tilde{C}_{g+1} \). Comparing both differentials \( dV_i \) and \( \{dv_i\} \), we can recognize how special the degenerate curve (3-1) is.

(4) Here we will comment on the derivation of Its and Matveev [I, IM1]. They used normalized differentials of "the first kind", \( dv_i/k_i \)'s; (it is noted that due to Remark 3-4, \( dv_i/k_i \)'s are the third kind because in the degenerate curve (3-1), "the kind first" here.) In [I, IM1], noting the fact that their differentials are a linear combination of the standard differentials of the "first kind" \( du \)'s with coefficients of complex numbers \( \mathbb{C} \), they showed the explicit connections between \( dv_i/k_i \)'s and \( du \)'s.

However they connected between \( dv_i/k_i \)'s and \( du \)'s by residual of computations, using the fact that the normalized differentials of the first kind becomes the third kind [I, IM1]. The normalized differentials of the first kind can be constructed by the periodic matrix \( \omega' \) and the standard differentials of the first kind \( du \)'s for any hyperelliptic curves. Since \( du \)'s are holomorphic in general curves from the definition, the residual computation has no meaning. Thus their way is a little bit heuristic.

Further they did not comment how special the form of \( dv_i/k_i \) is. We emphasize that it is not general that the normalized differential of "the first kind" is expressed by only local data. We also emphasize the importance of the comparison between \( dv_i \)'s and \( dV_i \)'s in (3-39), which let us to know how the degenerate curve is special.

Moreover in their definitions, the coefficients of \( t_1, t_2 \) and \( t_3 \) are given by the complete hyperelliptic integrals of second kind around \( \beta \)'s whereas ours are of \( \omega'^{-1} \). Both are connected by Legendre relation (2-23) and (3-33) as I showed around (3-33).

\[ \text{Discussion} \]

We showed that the soliton solutions are given as algebraic functions over degenerate curves \( \{y^2 = P(x)^2x\} \), which are revised version of work of Its and Matveev [BBEIM, I, IM1]. I believe that in physics, quantitative investigations are the most important. For a given problem in physics, to find an explicit solution which can be plotted as graphs must be required, except recent elementary particle physics. I think that discoveries of the soliton solutions and elliptic solutions of nonlinear differential equations are great successes in mathematical physics. By virtue of them, a number of phenomena in physics become clear [L].

However from viewpoint of study of algebraic geometry, these curves expressed by \( y^2 = P(x)^2x \) are very special even in the set of hyperelliptic curves. There are so many hyperelliptic degenerate curves which are not expressed as \( y^2 = P(x)^2x \), e.g., \( y^2 = g(x)^2h(x), (h(x) \neq x) \) and non-degenerate curves [HM]. Even though one can regard a curve \( y^2 = P(x)^2x \), in which the degree of \( P(x) \) is \( g \), as a curve with genus \( g \) from topological consideration, it is obvious that such curve cannot represent
general properties of hyperelliptic curves. In other words, as a function over an algebraic curve has data of a curve itself in general, the soliton solution exhibits only the data of such a degenerate curve except very weak topological property.

Further even though there is a natural inclusions as sub-matrices between the special degenerate curves $C_0$ and $C_{g+1}$ in notations in remark 3-10 as mentioned there, it is not clear that the method is some advantage for general hyperelliptic curves except formal meaning. In other words, theories based upon special degenerate curves might not be effective on the study of algebraic curves except formal and weekly topological aspects [HM p.42], though of course, it is no doubt that such considerations gave us very fruitful and beautiful results as mentioned in the introductions.

This study shows that the soliton solutions are very special solutions. To know the solutions space of the KdV equation requires concrete solutions of more general curves without undetermined parameters. Though some mathematicians avoid quantitative considerations and their theory might satisfy themselves, I believe that quantitative investigations are very important at least, from physical view point, like an excellent work [HI]. In order to do, theories on hyperelliptic functions established in nineteenth century [B1-3, K] are still important in this century [BEL1-5, CEEK, EE, EEL, EEP, EGM, Ma1-2, N].

Finally, we comment on the soliton solutions of Kadomtsev-Petviashvili equation. As it is expected that the soliton solutions should be also constructed from simple integrations over a degenerate algebraic curve, it should be connected with concrete algebraic curves as we did for KdV soliton solutions. In fact, the sigma function was also extended to that for more general curves; the authors in [BEL5] constructed one for the trigonal curves, which is related to Boussomesq equation.

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1. The homology basis in a Riemannian surface of genus five: $(a_i, c_i)$ and $(c, \infty)$ are the cuts for these branching points. The $\beta$'s go to another twin Riemannian sphere through the corresponding cuts.

1. The homology basis in a degenerate Riemannian surface of genus five: $a_i$'s are singular points. The $\beta$'s go to another twin Riemannian sphere through the points by reparametrizations.