Some Inequalities Involving Perimeter and Torsional Rigidity

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Abstract

We consider shape functionals of the form $F_q(\Omega) = P(\Omega)T^q(\Omega)$ on the class of open sets of prescribed Lebesgue measure. Here $q > 0$ is fixed, $P(\Omega)$ denotes the perimeter of $\Omega$ and $T(\Omega)$ is the torsional rigidity of $\Omega$. The minimization and maximization of $F_q(\Omega)$ is considered on various classes of admissible domains $\Omega$: in the class $A_{all}$ of

all domains,

in the class $A_{convex}$ of convex domains, and in the class $A_{thin}$ of thin domains.

Keywords Torsional rigidity · Shape optimization · Perimeter · Convex domains

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1 Introduction

In this paper, given an open set $\Omega \subset \mathbb{R}^d$ with finite Lebesgue measure, we consider the quantities

$$P(\Omega) = \text{perimeter of } \Omega;$$

$$T(\Omega) = \text{torsional rigidity of } \Omega.$$
The perimeter \( P(\Omega) \) is defined according to the De Giorgi formula
\[
P(\Omega) = \sup \left\{ \int_{\Omega} \operatorname{div} \phi \, dx : \phi \in C_0^1(\mathbb{R}^d; \mathbb{R}^d), \|\phi\|_{L^\infty(\mathbb{R}^d)} \leq 1 \right\}.
\]
The \textit{scaling property} of the perimeter is
\[
P(t\Omega) = t^{d-1} P(\Omega) \quad \text{for every } t > 0
\]
and the relation between \( P(\Omega) \) and the Lebesgue measure \(|\Omega|\) is the well-known \textit{isoperimetric inequality}:
\[
\frac{P(\Omega)}{|\Omega|^{(d-1)/d}} \geq \frac{P(B)}{|B|^{(d-1)/d}} \tag{1.1}
\]
where \( B \) is any ball in \( \mathbb{R}^d \). In addition, the inequality above becomes an equality if and only if \( \Omega \) is a ball (up to sets of Lebesgue measure zero).

The torsional rigidity \( T(\Omega) \) is defined as
\[
T(\Omega) = \int_{\Omega} u \, dx
\]
where \( u \) is the unique solution of the PDE
\[
\begin{cases}
-\Delta u = 1 & \text{in } \Omega, \\
u \in H_0^1(\Omega). 
\end{cases} \tag{1.2}
\]
Equivalently, \( T(\Omega) \) can be characterized through the maximization problem
\[
T(\Omega) = \max \left\{ \left[ \int_{\Omega} u \, dx \right]^2 \left[ \int_{\Omega} |\nabla u|^2 \, dx \right]^{-1} : u \in H_0^1(\Omega) \setminus \{0\} \right\}.
\]
Moreover \( T \) is increasing with respect to the set inclusion, that is
\[
\Omega_1 \subset \Omega_2 \implies T(\Omega_1) \leq T(\Omega_2)
\]
and \( T \) is additive on disjoint families of open sets. The scaling property of the torsional rigidity is
\[
T(t\Omega) = t^{d+2} T(\Omega), \quad \text{for every } t > 0,
\]
and the relation between \( T(\Omega) \) and the Lebesgue measure \(|\Omega|\) is the well-known \textit{Saint-Venant inequality} (see for instance [16,17]):
\[
\frac{T(\Omega)}{|\Omega|^{(d+2)/d}} \leq \frac{T(B)}{|B|^{(d+2)/d}}. \tag{1.3}
\]
Again, the inequality above becomes an equality if and only if $\Omega$ is a ball (up to sets of capacity zero). If we denote by $B_1$ the unitary ball of $\mathbb{R}^d$ and by $\omega_d$ its Lebesgue measure, then the solution of (1.2), with $\Omega = B_1$, is

$$u(x) = \frac{1 - |x|^2}{2d}$$

which provides

$$T(B_1) = \frac{\omega_d}{d(d+2)}.$$  \hspace{1cm} (1.4)

We are interested in the problem of minimizing or maximizing quantities of the form

$$P^\alpha(\Omega)T^\beta(\Omega)$$

on some given class of open sets $\Omega \subset \mathbb{R}^d$ having a prescribed Lebesgue measure $|\Omega|$, where $\alpha$, $\beta$ are two given exponents. Similar problems have been considered for shape functionals involving:

– the torsional rigidity and the first eigenvalue of the Laplacian in [2,3,6,8,11,19,20,22];
– the torsional rigidity and the Newtonian capacity in [1];
– the perimeter and the first eigenvalue of the Laplacian in [14];
– the perimeter and the Newtonian capacity in [9,13].

The case $\beta = 0$ reduces to the isoperimetric inequality, and we have, denoting by $\Omega^*_m$ a ball of measure $m$,

$$\left\{ \begin{array}{l}
\min \{ P(\Omega) : |\Omega| = m \} = P(\Omega^*_m) \\
\sup \{ P(\Omega) : |\Omega| = m \} = +\infty.
\end{array} \right.$$  

Similarly, in the case $\alpha = 0$, the Saint Venant inequality yields

$$\max \{ T(\Omega) : |\Omega| = m \} = T(\Omega^*_m) = \frac{m}{d(d+2)} \left( \frac{m}{\omega_d} \right)^{2/d}$$

while

$$\inf \{ T(\Omega) : |\Omega| = m \} = 0.$$  

Indeed if we choose $\Omega_n = \bigcup_{k=1}^{n} B_{n,k}$ where $B_{n,k}$ are disjoint balls of measure $m/n$ each, we get for every $n \in \mathbb{N}$

$$\inf \{ T(\Omega) : |\Omega| = m \} \leq T(\Omega_n) = \frac{m^{(d+2)/d}}{d(d+2)\omega_d^{2/d} n^{-2/d}} \rightarrow 0.$$
The case when $\alpha$ and $\beta$ have a different sign is also immediate; for instance, if $\alpha > 0$ and $\beta < 0$ we have from (1.1) and (1.3)

$$\begin{align*}
\min \left\{ P^\alpha(\Omega)T^\beta(\Omega) : |\Omega| = m \right\} &= P^\alpha(\Omega^*_m)T^\beta(\Omega^*_m), \\
\sup \left\{ P^\alpha(\Omega)T^\beta(\Omega) : |\Omega| = m \right\} &= +\infty,
\end{align*}$$

and similarly, if $\alpha < 0$ and $\beta > 0$ we have

$$\begin{align*}
\inf \left\{ P^\alpha(\Omega)T^\beta(\Omega) : |\Omega| = m \right\} &= 0, \\
\max \left\{ P^\alpha(\Omega)T^\beta(\Omega) : |\Omega| = m \right\} &= P^\alpha(\Omega^*_m)T^\beta(\Omega^*_m).
\end{align*}$$

The cases we will investigate are the remaining ones; with no loss of generality we may assume $\alpha = 1$, so that the optimization problems we consider are for the quantities

$$P(\Omega)T^q(\Omega), \quad \text{with } q > 0.$$ 

In order to remove the Lebesgue measure constraint $|\Omega| = m$ we consider the scaling free functionals

$$F_q(\Omega) = \frac{P(\Omega)T^q(\Omega)}{|\Omega|^{\alpha_q}} \quad \text{with } \alpha_q = 1 + q + \frac{2q - 1}{d}.$$ 

In the following sections we study the minimization and the maximization problems for the shape functionals $F_q$ on various classes of domains. More precisely we consider the cases below.

The class of all domains $\Omega$ (nonempty)

$$\mathcal{A}_{all} = \left\{ \Omega \subset \mathbb{R}^d : \Omega \neq \emptyset \right\}$$

will be considered in Sect. 2; we show that for every $q > 0$ both the maximization and the minimization problems for $F_q$ on $\mathcal{A}_{all}$ are ill posed.

The class of convex domains $\Omega$

$$\mathcal{A}_{convex} = \left\{ \Omega \subset \mathbb{R}^d : \Omega \neq \emptyset, \ \Omega \text{ convex} \right\}$$

will be considered in Sect. 3; we show that for $0 < q < 1/2$ the maximization problem for $F_q$ on $\mathcal{A}_{convex}$ is ill posed, whereas the minimization problem is well posed. On the contrary, when $q > 1/2$ the minimization problem for $F_q$ on $\mathcal{A}_{convex}$ is ill posed, whereas the maximization problem is well posed. In the threshold case $q = 1/2$ the precise value of the infimum of $F_{1/2}$ is provided; concerning the precise value of the supremum of $F_{1/2}$ an interesting conjecture is stated. At present, the conjecture has been shown to be true in the case $d = 2$, while the question is open in higher dimensions.
The class of thin domains $A_{\text{thin}}$, suitably defined, will be considered in Sect. 4. If $h(s)$ represents the asymptotical local thickness of the thin domain as $s$ varies in a $d-1$ dimensional domain $A$, the maximization of the functional $F_{1/2}$ on $A_{\text{thin}}$ reduces to the maximization of a functional defined on nonnegative functions $h$ defined on $A$; this allows us to prove the conjecture for any dimension $d$ on the class of thin convex domains.

2 Optimization in the Class of All Domains

In this section we show that the minimization and the maximization problems for the shape functionals $F_q$ are both ill posed, for every $q > 0$.

**Theorem 2.1** There exist two sequences $(\Omega_{1,n})$ and $(\Omega_{2,n})$ of smooth domains such that for every $q > 0$ we have

$$F_q(\Omega_{1,n}) \to 0 \quad \text{and} \quad F_q(\Omega_{2,n}) \to +\infty.$$  

In particular, we have

$$\left\{ \begin{array}{l}
\inf \left\{ F_q(\Omega) : \Omega \in A_{\text{all}}, \Omega \text{ smooth} \right\} = 0 \\
\sup \left\{ F_q(\Omega) : \Omega \in A_{\text{all}}, \Omega \text{ smooth} \right\} = +\infty.
\end{array} \right.$$  

**Proof** In order to show the sup equality it is enough to take as $\Omega_{2,n}$ a perturbation of the unit ball $B_1$ such that

$$B_{1/2} \subset \Omega_{2,n} \subset B_2 \quad \text{and} \quad P(\Omega_{2,n}) \to +\infty.$$  

Then we have

$$|\Omega_{2,n}| \leq |B_2|, \quad T(\Omega_{2,n}) \geq T(B_{1/2}),$$

where we used the monotonicity of the torsional rigidity. Then

$$F_q(\Omega_{2,n}) \geq \frac{P(\Omega_{2,n})T^q(B_{1/2})}{|B_2|^{\alpha_q}} \to +\infty.$$  

In order to prove the inf equality we take as $\Omega_{c,\epsilon}$ the unit ball $B_1$ from which we remove a periodic array of holes; the centers of two adjacent holes are at distance $\epsilon$ and the radii of the holes are

$$r_{c,\epsilon} = \begin{cases} 
e^{-1/(c\epsilon^2)} & \text{if } d = 2 \\ c\epsilon^{d/(d-2)} & \text{if } d > 2, \end{cases}$$  

where $c$ is a positive constant. It is easy to see that, as $\epsilon \to 0$, we have

$$|\Omega_{c,\epsilon}| \to |B_1| \quad \text{and} \quad P(\Omega_{c,\epsilon}) \to P(B_1).$$
Concerning the torsion $T(\Omega_{c,\varepsilon})$, we have (see [10])

$$T(\Omega_{c,\varepsilon}) \to \int_{B_1} u_c \, dx$$

where $u_c$ is the nonnegative function which solves

$$\begin{cases} 
-\Delta u_c + K_c u_c = 1 & \text{in } B_1 \\
u_c \in H^1_0(B_1),
\end{cases}$$

being $K_c$ the constant

$$K_c = \begin{cases} 
c\pi/2 & \text{if } d = 2 \\
d(d - 2)2^{-d}\omega_d c^{d-2} & \text{if } d > 2.
\end{cases}$$

Since for every $c > 0$ we have that

$$\int_{B_1} |\nabla u_c(x)|^2 + K_c u_c^2(x) \, dx = \int_{B_1} u_c \, dx$$

we get that

$$\int_{B_1} u_c \, dx \leq \frac{\omega_d}{K_c}.$$ 

Therefore, a diagonal argument allows us to construct a sequence $(\Omega_{1,n})$ such that

$$|\Omega_{1,n}| \to |B_1|, \quad P(\Omega_{1,n}) \to P(B_1), \quad T(\Omega_{1,n}) \to 0,$$

which concludes the proof. \hfill \qed

### 3 Optimization in the Class of Convex Domains

In this section we consider only domains $\Omega$ which are convex. A first remark is in the proposition below and shows that in some cases the optimization problems for the shape functional $F_q$ is still ill posed.

**Proposition 3.1** We have

$$\begin{cases} 
\inf \left\{ F_q(\Omega) : \Omega \in A_{\text{convex}} \right\} = 0 & \text{for every } q > 1/2; \\
\sup \left\{ F_q(\Omega) : \Omega \in A_{\text{convex}} \right\} = +\infty & \text{for every } q < 1/2.
\end{cases}$$
Proof Let $A$ be a smooth convex $d - 1$ dimensional set and for every $\varepsilon > 0$ consider the domain $\Omega_\varepsilon \in \mathcal{A}_{\text{convex}}$ given by

$$
\Omega_\varepsilon = A \times ] - \varepsilon / 2, \varepsilon / 2[.
$$

We have (for the torsion asymptotics see for instance [2])

$$
P(\Omega_\varepsilon) \approx 2\mathcal{H}^{d-1}(A),
$$

$$
T(\Omega_\varepsilon) \approx \frac{\varepsilon^3}{12}\mathcal{H}^{d-1}(A),
$$

$$
|\Omega_\varepsilon| = \varepsilon \mathcal{H}^{d-1}(A),
$$

so that

$$
F_q(\Omega_\varepsilon) \approx \frac{2}{12q (\mathcal{H}^{d-1}(A))^{(2q-1)/d}} \varepsilon^{(2q-1)(d-1)/d}. 
$$

(3.1)

Letting $\varepsilon \to 0$ achieves the proof.

We show now that in some other cases the optimization problems for the shape functional $F_q$ is well posed. Let us begin to consider the case $q = 1/2$.

Proposition 3.2 We have

$$
\inf \{ F_{1/2}(\Omega) : \Omega \in \mathcal{A}_{\text{convex}} \} = 3^{-1/2}
$$

(3.2)

and the infimum is asymptotically reached by domains of the form

$$
\Omega_\varepsilon = A \times ] - \varepsilon / 2, \varepsilon / 2[ 
$$
as $\varepsilon \to 0$, where $A$ is any $d - 1$ dimensional convex set.

Proof Thanks to a classical result by Polya ([21], see also Theorem 5.1 of [11]) it holds

$$
T(\Omega) \geq \frac{1}{3} \frac{|\Omega|^3}{(P(\Omega))^2}.
$$

Then

$$
F_{1/2}(\Omega) = \frac{P(\Omega)(T(\Omega))^{1/2}}{|\Omega|^{3/2}} \geq 3^{-1/2}
$$

for any bounded open convex set. Taking into account (3.1), we get (3.2).

Concerning the supremum of $F_{1/2}(\Omega)$ in the class $\mathcal{A}_{\text{convex}}$ we can only show that it is finite.
Proposition 3.3 For every $\Omega \in A_{\text{convex}}$ we have

$$F_{1/2}(\Omega) \leq \frac{2^d d^{3d/2}}{\omega_d} \sqrt{\frac{d}{d+2}}.$$

(3.3)

Proof By the John’s ellipsoid Theorem [18], there exists an ellipsoid that, without loss of generality, we may assume centered at the origin,

$$E_a = \left\{ x \in \mathbb{R}^d : \sum_{i=1}^d \frac{x_i^2}{a_i^2} < 1 \right\}, \quad a = (a_1, \ldots, a_d), \text{ with } a_i > 0$$

such that $E_a \subset \Omega \subset dE_a$. Then we have

$$F_{1/2}(\Omega) \leq \frac{P(dE_a)(T(dE_a))^{1/2}}{|E_a|^{3/2}}.$$

(3.4)

Since the solution of (1.2) for $E_a$ is given by

$$u(x) = \frac{1}{2} \left( \sum_{i=1}^d a_i^{-2} \right)^{-1} \left( 1 - \sum_{i=1}^d \frac{x_i^2}{a_i^2} \right),$$

we obtain

$$T(E_a) = \frac{\omega_d}{d+2} \left( \sum_{i=1}^d a_i^{-2} \right)^{-1} \prod_{i=1}^d a_i,$$

while

$$|E_a| = \omega_d \prod_{i=1}^d a_i.$$

To estimate $P(E_a)$ we notice that $E_a$ is contained in the cuboid $Q_a = \prod_{i=1}^d [a_i, a_i]$, so that

$$P(E_a) \leq P(Q_a) = 2 \sum_{i=1}^d \prod_{j \neq i} (2a_j) = 2^d \left( \sum_{i=1}^d \frac{1}{a_i} \right) \prod_{i=1}^d a_i.$$

Combining these formulas we have from (3.4)

$$F_{1/2}(\Omega) \leq \frac{2^d d^{3d/2}}{\omega_d (d+2)^{1/2}} \left( \sum_{i=1}^d \frac{1}{a_i} \right) \left( \sum_{i=1}^d \frac{1}{a_i} \right)^{-1/2}.$$
and finally, by Jensen inequality,

\[ F_{1/2}(\Omega) \leq \frac{2^d d^{3d/2}}{\omega_d} \sqrt{\frac{d}{d+2}}. \]

as required.

On the precise value of \( \sup \{ F_{1/2}(\Omega) : \Omega \in \mathcal{A}_{\text{convex}} \} \) we make the following conjecture.

**Conjecture 3.4** We have

\[ \sup \{ F_{1/2}(\Omega) : \Omega \in \mathcal{A}_{\text{convex}} \} = d \left( \frac{2}{(d+1)(d+2)} \right)^{1/2} \]

and it is asymptotically reached by taking for instance

\[ \Omega_\varepsilon = \{ (s, t) : s \in A, \ 0 < t < \varepsilon (1 - |s|) \} \]

as \( \varepsilon \to 0 \), where \( A \) is the unit ball in \( \mathbb{R}^{d-1} \).

**Remark 3.5** We recall that Conjecture 3.4 has been shown to be true in the case \( d = 2 \) (see [21,23], and the more recent paper [12]). In Sect. 4 we prove the conjecture above for every \( d \geq 2 \) in the class of convex thin domains.

We show now that for \( F_q \) in the class \( \mathcal{A}_{\text{convex}} \) the minimization problem is well posed when \( q < 1/2 \) and the maximization problem is well posed when \( q > 1/2 \). From the bounds obtained in Propositions 3.2 and 3.3 we can prove the following results.

**Proposition 3.6** We have

\[
\begin{align*}
\inf \{ F_q(\Omega) : \Omega \in \mathcal{A}_{\text{convex}} \} &\geq 3^{-1/2} (d(d+2))^{1/2-q} \omega_d^{(1-2q)/d} \quad \text{for every } q \leq 1/2 \\
\sup \{ F_q(\Omega) : \Omega \in \mathcal{A}_{\text{convex}} \} &\leq 2^d d^{3d/2-q+1} \omega_d \left( \frac{2}{(d+2)(d+3)} \right)^{1/(2q-1)/d} \quad \text{for every } q \geq 1/2.
\end{align*}
\]

**Proof** We have

\[ F_q(\Omega) = F_{1/2}(\Omega) \left( \frac{T(\Omega)}{|\Omega|^{(d+2)/d}} \right)^{q-1/2}. \]

Hence it is enough to apply the bounds (3.2) and (3.3), together with the Saint-Venant inequality (1.3) to get that for every \( \Omega \in \mathcal{A}_{\text{convex}} \)

\[
\begin{align*}
\inf \{ F_q(\Omega) : \Omega \in \mathcal{A}_{\text{convex}} \} &\geq 3^{-1/2} \left( \frac{T(B)}{B^{(d+2)/d}} \right)^{q-1/2} \quad \text{if } q \leq 1/2 \\
\sup \{ F_q(\Omega) : \Omega \in \mathcal{A}_{\text{convex}} \} &\leq \frac{2^d d^{3d/2}}{\omega_d} \sqrt{\frac{d}{d+2}} \left( \frac{T(B)}{B^{(d+2)/d}} \right)^{q-1/2} \quad \text{if } q \geq 1/2.
\end{align*}
\]

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By the expression (1.4) for $T(B)$ we conclude the proof.

We now prove the existence of a convex minimizer when $q < 1/2$ and of a convex maximizer when $q > 1/2$.

**Theorem 3.7** There exists a solution for the following optimization problems:

\[ \begin{align*}
\min \{ F_q(\Omega) : \Omega \in A_{\text{convex}} \} & \quad \text{for every } q < 1/2; \\
\max \{ F_q(\Omega) : \Omega \in A_{\text{convex}} \} & \quad \text{for every } q > 1/2.
\end{align*} \]

**Proof** Suppose $q < 1/2$ and consider $(\Omega_n)$ a minimizing sequence for $F_q(\Omega)$. By the John’s ellipsoid Theorem we can assume that there exists a sequence of ellipsoids $E_{a_n}$ such that

\[ E_{a_n} \subset \Omega_n \subset dE_{a_n}. \]

By rotations, translations and scaling invariance of $F_q$ we can assume without loss of generality that

\[ E_{a_n} = \left\{ x \in \mathbb{R}^d : \sum_{i=1}^d x_i^2 < 1 \right\}, \quad a_n = (a_1, \ldots, a_d), \quad 0 < a_1 \leq \cdots \leq a_d = 1. \]

Observe that this implies that the diameter of $\Omega_n$ is uniformly bounded in $n$. We claim that

\[ a_1 \geq c \quad \text{for every } n \in \mathbb{N} \]

where $c$ is a positive constant. Then the proof is achieved by extracting a subsequence $(\Omega_{a_n})$ which converges both in the sense of characteristic functions and in the co-Hausdorff metric to some open, non empty, convex, bounded set $\Omega^-$ and by using the continuity properties of torsional rigidity, perimeter and volume (see for instance, [7,17]).

To prove the claim we use a strategy similar to the one already used in the proof of Proposition 3.3. Let $Q_{a_n}$ be the cuboid $\prod_{i=1}^d [a_{in} - 1, a_{in} + 1]$. Since

\[ d^{-1/2} Q_{a_n} \subset E_{a_n} \]

we have, for $n$ large enough,

\[ F_q(B_1) \geq F_q(\Omega_n) \geq \frac{1}{d(d-1)/2d^{d\alpha_q}} \frac{T^q(E_{a_n})P(Q_{a_n})}{|E_{a_n}|^{\alpha_q}}. \] (3.5)

An explicit computation shows

\[ \frac{T^q(E_{a_n})P(Q_{a_n})}{|E_{a_n}|^{\alpha_q}} = \frac{2^d \omega_d^{q-d\alpha_q}}{(d + 2)^q} \left( \frac{\sum_{i=1}^d a_{in}^{-1}}{(\sum_{i=1}^d a_{in}^{-2})^{1/2}} \right)^{1-2q} \left( \prod_{i=1}^d a_{in}^{-1/2} \right)^{1/2}. \]

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Observe that, by Cauchy–Schwarz inequality,

\[ 1 \leq \frac{\sum_{i=1}^{d} a_{in}^{-1}}{\left(\sum_{i=1}^{d} a_{in}^{-2}\right)^{1/2}} \leq \sqrt{d}, \tag{3.6} \]

while for the last term it holds

\[ \left(\frac{\sum_{i=1}^{d} a_{in}^{-2}}{\prod_{i=1}^{d} a_{in}^{-1}}\right)^{1/d} = \left(\frac{\sum_{i=1}^{d} a_{in}^{-2}}{\prod_{i=1}^{d} a_{in}^{-1}}\right)^{1/d} \geq \frac{a_{1n}^{-1}}{(a_{1n}^{-1})^{(d-1)/d}} = \left(\frac{1}{a_{1n}}\right)^{1/d}. \tag{3.7} \]

Therefore, putting together (3.5)–(3.7) and using the fact that \( q < 1/2 \) we obtain that, if \( n \) is large enough, the sequence \( a_{1n} \) must be greater than some positive constant \( c \), which proves the claim.

The case \( q > 1/2 \) can be proved in a similar way. If \( (\Omega_n) \) is a maximizing sequence for \( F_q(\Omega) \) and \( E_{an} \) are ellipsoids such that \( E_{an} \subset \Omega_n \subset d E_{an} \), we have

\[ F_q(B_1) \leq F_q(\Omega_n) \leq \frac{P(dE_{an}) T^q (dE_{an})}{|E_{an}|^{aq}} = d^{d-1+q(d+2)} \frac{P(E_{an}) T^q (E_{an})}{|E_{an}|^{aq}}. \tag{3.8} \]

If \( Q_{an} \) is the cuboid \( \prod_{i=1}^{d} a_{in} \), we have \( E_{an} \subset Q_{an} \), so that

\[ P(E_{an}) \leq P(Q_{an}) = 2^d \left(\sum_{i=1}^{d} a_{in}^{-1}\right)^d \prod_{i=1}^{d} a_{in}. \]

Hence (3.8) implies, for a suitable constant \( C_{q,d} \) depending only on \( q \) and on \( d \),

\[ F_q(B_1) \leq C_{q,d} \left(\sum_{i=1}^{d} a_{in}^{-1}\right)^{(2q-1)/d} \leq d^q C_{q,d} \left(\frac{(\prod_{i=1}^{d} a_{in}^{-1})^{1/d}}{\sum_{i=1}^{d} a_{in}^{-1}}\right)^{(2q-1)/d}, \]

where in the last inequality we used the Cauchy–Schwarz inequality (3.6). Finally, since \( a_{in} \leq a_{dn} = 1 \), we obtain

\[ F_q(B_1) \leq d^q C_{q,d} \left( a_{in}^{-1}\right)^{(2q-1)/d} \]

and, since \( q > 1/2 \), the conclusion follows as in the previous case. \( \square \)
4 Optimization in the Class of Thin Domains

In this section we consider the class of thin domains, that we define below through the families of domains

\[ \Omega_\varepsilon = \{(s, t) : s \in A, \varepsilon h_-(s) < t < \varepsilon h_+(s)\} \quad (4.1) \]

where \( \varepsilon \) is a small positive parameter, \( A \) is a (smooth) domain of \( \mathbb{R}^{d-1} \), and \( h_-, h_+ \) are two given (smooth) functions. We denote by \( h(s) \) the local thickness

\[ h(s) = h_+(s) - h_-(s), \]

and we assume that \( h(s) \geq 0 \). More precisely, we call thin domain a family \( (\Omega_\varepsilon)_{\varepsilon > 0} \) as above; in other words a thin domain is characterized by the \( d-1 \) dimensional domain \( A \) and by the local thickness function \( h \).

The following asymptotics hold for the quantities we are interested to (for the torsional rigidity we refer to [5]):

\[
\begin{align*}
P(\Omega_\varepsilon) & \approx 2 \mathcal{H}^{d-1}(A), \\
T(\Omega_\varepsilon) & \approx \frac{\varepsilon^3}{12} \int_A h^3(s) \, ds, \\
|\Omega_\varepsilon| & = \varepsilon \int_A h(s) \, ds,
\end{align*}
\]

which together give the asymptotic formula when \( q = 1/2 \)

\[
F_{1/2}(\Omega_\varepsilon) \approx 3^{-1/2} \mathcal{H}^{d-1}(A) \left[ \int_A h^3(s) \, ds \right]^{1/2} \left[ \int_A h(s) \, ds \right]^{-3/2}
\]

\[= 3^{-1/2} \left[ \left( \int_A h^3(s) \, ds \right) \left( \int_A h(s) \, ds \right) \right]^{1/2} \quad (4.2)\]

where we use the notation

\[
\int_A f(s) \, ds = \frac{1}{\mathcal{H}^{d-1}(A)} \int_A f(s) \, ds.
\]

We then define the functional \( F_{1/2} \) on the thin domain \( (\Omega_\varepsilon)_{\varepsilon > 0} \) associated with the \( d-1 \) dimensional domain \( A \) and the local thickness function \( h \) by

\[
F_{1/2}(A, h) = 3^{-1/2} \left[ \left( \int_A h^3(s) \, ds \right) \left( \int_A h(s) \, ds \right) \right]^{1/2}.
\]

By Hölder inequality we have

\[
F_{1/2}(A, h) \geq 3^{-1/2}
\]
and the value $3^{-1/2}$ is actually reached by taking the local thickness function $h$ constant, which corresponds to $\Omega_\varepsilon$ a thin slab.

A sharp inequality from above is also possible for $F_{1/2}(A, h)$, if we restrict the analysis to convex domains, that is to local thickness functions $h$ which are concave. The following result will be used, for which we refer to [4,15].

**Theorem 4.1** Let $1 \leq p \leq q$. Then for every convex set $A$ of $\mathbb{R}^N$ ($N \geq 1$) and every nonnegative concave function $f$ on $A$ we have

$$\left[ \int_A f^q \, dx \right]^{1/q} \leq C_{p,q} \left[ \int_A f^p \, dx \right]^{1/p},$$

where the constant $C_{p,q}$ is given by

$$C_{p,q} = \left( \frac{N + p}{N} \right)^{1/p} \left( \frac{N + q}{N} \right)^{-1/q}.$$

In addition, the inequality above becomes an equality when $A$ is a ball of radius 1 and $f(x) = 1 - |x|$.

We are now in a position to prove the Conjecture 3.4 for convex thin domains.

**Theorem 4.2** If $(\Omega_\varepsilon)_{\varepsilon > 0}$ is a thin convex domains given by (4.1), we have

$$F_{1/2}(A, h) \leq d \left( \frac{2}{(d + 1)(d + 2)} \right)^{1/2}.$$  \hspace{1cm} (4.3)

In addition, the inequality above becomes an equality taking for instance as $A$ the unit ball of $\mathbb{R}^{d-1}$ and as the local thickness $h(s)$ the function $1 - |s|$.

**Proof** Since the local thickness function $h$ is concave, by Theorem 4.1 with $N = d - 1$, $q = 3$, $p = 1$, we obtain

$$\int_A h^3 \, dx \leq C_{3,3}^3 \left[ \int_A h \, dx \right]^3,$$

so that

$$F_{1/2}(A, h) \leq 3^{-1/2} C_{3/2}^3 = d \left( \frac{2}{(d + 1)(d + 2)} \right)^{1/2}$$

as required. Finally, an easy computation shows that in (4.3) the inequality becomes an equality if $A$ is the unit ball of $\mathbb{R}^{d-1}$ and $h(s) = 1 - |s|$.

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