On the Generalized Minimal Massive Gravity

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Abstract

In this paper we study the Generalized Minimal Massive Gravity (GMMG) in asymptotically $AdS_3$ background. The generalized minimal massive gravity theory is realized by adding the CS deformation term, the higher derivative deformation term, and an extra term to pure Einstein gravity with a negative cosmological constant. We study the linearized excitations around the $AdS_3$ background and find that at special point (tricritical) in parameter space the two massive graviton solutions become massless and they are replaced by two solutions with logarithmic and logarithmic-squared boundary behavior. So it is natural to proposed that GMMG model could also provide a holographic description for a 3–rank Logarithmic Conformal Field Theory (LCFT). Then we show that General Zwei-Dreibein Gravity (GZDG) model can reduce to GMMG model. Finally by a Hamiltonian analysis we show that the GMMG model has no Boulware-Deser ghosts and this model propagate only two physical modes.

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1
1 Introduction

It is well known that Einstein gravity suffers from the problem that the theory is nonrenormalizable in four and higher dimensions. Adding higher derivative terms such as Ricci and scalar curvature squared terms makes the theory renormalizable at the cost of the loss of unitarity [1]. In the other hand pure Einstein-Hilbert gravity in three dimensions exhibits no propagating physical degrees of freedom [2, 3]. But adding the gravitational Chern-Simons term produces a propagating massive graviton [4]. The resulting theory is called topologically massive gravity (TMG). Including a negative cosmological constant, yields cosmological topologically massive gravity (CTMG). In this case the theory exhibits both gravitons and black holes. Unfortunately there is a problem in this model, with the usual sign for the gravitational constant, the massive excitations of CTMG carry negative energy. In the absence of a cosmological constant, one can change the sign of the gravitational constant, but if $\Lambda < 0$, this will give a negative mass to the BTZ black hole, so the existence of a stable ground state is in doubt in this model [5].

A few years ago [6] a new theory of massive gravity (NMG) in three dimensions has been proposed. This theory is equivalent to the three-dimensional Fierz-Pauli action for a massive spin-2 field at the linearized level. Moreover NMG in contrast with the TMG [4] is parity invariant. As a result, the gravitons acquire the same mass for both helicity states, indicating two massive propagating degrees of freedom. One of common aspects in these two theories is the existence of AdS vacuum solution. So TMG and NMG provide useful models in which to explore the AdS/CFT correspondence. The conformal boundary of a three-dimensional asymptotically anti-de Sitter spacetime is a flat two-dimensional cylinder, and the asymptotic symmetries are described by a pair of Virasoro algebras [7]. So many study have been done along the route of AdS/CFT correspondence in the TMG and NMG setup [8]. Although, it has been shown the compliance of the NMG with the holographic c-theorem [9, 10], both TMG and NMG have a bulk-boundary unitarity conflict. In another term either the bulk or the boundary theory is non-unitary, so there is a clash between the positivity of the two Brown-Henneaux boundary $c$ charges and the bulk energies [11].

There is this possibility to extend NMG to higher curvature theories. One of these extension of NMG has been done by Sinha [9] where he has added the $R^3$ terms to the action. The other modification is the extension to the Born-Infeld type action [12]. But these extensions of NMG did not solve the unitary conflict [9, 12, 13]. The recently constructed Zwei Dreibein Gravity
(ZDG) shows that there is a viable alternative to NMG \cite{14,15}. It is interesting if one combine TMG and NMG as a generalized massive model in 3-dimension, dubbed Generalized Massive Gravity theory (GMG), this work first introduced in \cite{6}, then studied more in \cite{16}. This theory has two mass parameters and TMG and NMG are just two different limits of this generalized theory.

Recently an interesting three dimensional massive gravity introduced by Bergshoeff, et. al \cite{17} which dubbed Minimal Massive Gravity (MMG), which has the same minimal local structure as TMG. The MMG model has the same gravitational degree of freedom as the TMG has and the linearization of the metric field equations for MMG yield a single propagating massive spin-2 field. It seems that the single massive degree of freedom of MMG is unitary in the bulk and gives rise to a unitary CFT on the boundary. During last months some interesting works have been done on MMG model \cite{18}.

In this paper we would like to unify MMG and NMG into a General Minimal Massive Gravity theory (GMMG). The generalized minimal massive gravity theory is realized by adding the CS deformation term, the higher derivative deformation term, and an extra term to pure Einstein gravity with a negative cosmological constant. In the other term we would like to extend Generalized Massive Gravity theory (GMG), by adding an extra term. This theory is expected to have more interesting physics because we can have one more adjustable mass parameter.

Our paper is organized as follows. In section 2 we introduce the GMMG in \(AdS_3\) space. In section 3 we study the linear perturbation around \(AdS_3\) vacuum. Then we obtain the solutions of linearized equation of motion in terms of representations of isometry group \(SL(2,R)_L \times SL(2,R)_R\) of \(AdS_3\) space. Then in section 4 we show that GMMG exhibits not only massless graviton solutions, but also \(\log\) and \(\log^2\) solutions. The \(\log\) and \(\log^2\) modes appear in tricritical points in the parameter space of GMMG model. So in such special point in parameter space all massive gravitons become massless.

In another terms the massive graviton modes that satisfy Brown-Henneaux boundary conditions, in the tricritical points replaced by \(\log\) and \(\log^2\) solutions, which obey \(\log\) and \(\log^2\) boundary conditions toward \(AdS_3\) boundary exactly as what occur in GMG model \cite{16}. Therefore it is natural to proposed that GMMG model could also provide a holographic description for a 3–rank Logarithmic Conformal Field Theory (LCFT). In section 5 we study the relation between our model and General Zwei-Dreibein Gravity (GZDG) model. In another term we show that GMMG model can be obtained from Zwei-Dreibein Gravity (ZDG) plus a Lorentz Chern-Simons term for one of
the two spin-connections. Then in section 6 we study the Hamiltonian analysis of the GMMG model and show that there is not any Boulware-Deser ghosts in the framework of this model. We conclude in section 7 with a discussion of the our results.

2 The Generalized Minimal Massive Gravity

The Lagrangian 3-form of MMG is given by [17]

\[ L_{MMG} = L_{TMG} + \frac{\alpha}{2} e \times h \]  

where \( L_{TMG} \) is the Lagrangian of TMG,

\[ L_{TMG} = -\sigma e.R + \frac{\Lambda_0}{6} \varepsilon \times e + \bar{h}.T(\omega) + \frac{1}{2\mu} (\omega.d\omega + \frac{1}{3} \varepsilon \omega \times \omega) \]  

where \( \Lambda_0 \) is a cosmological parameter with dimension of mass squared, and \( \sigma \) a sign. \( \mu \) is mass parameter of Lorentz Chern-Simons term. \( \alpha \) is a dimensionless parameter, \( e \) is dreibein, \( h \) is the auxiliary field, \( \omega \) is dualised spin-connection, \( T(\omega) \) and \( R(\omega) \) are Lorentz covariant torsion and curvature 2-form respectively. Now we introduce the Lagrangian of GMMG model as

\[ L_{GMMG} = L_{GMG} + \frac{\alpha}{2} e \times h \]  

where

\[ L_{GMG} = L_{TMG} - \frac{1}{m^2} (f.R + \frac{1}{2} e \times f) \]  

here \( m \) is mass parameter of NMG term and \( f \) is an auxiliary one-form field. One can rewrite the Lagrangian 3-form \( L_{GMMG} \) as following

\[ L_{GMMG} = -\sigma e R^a + \frac{\Lambda_0}{6} e^{abc} e_a e_b e_c + h_a T^a + \frac{1}{2\mu} [\omega_a d\omega^a + \frac{1}{3 e^{abc}} \omega_a \omega_b \omega_c] \]
\[ - \frac{1}{m^2} [f_a R^a + \frac{1}{2} e^{abc} e_a f_b f_c] + \frac{\alpha}{2} e^{abc} e_a h_b h_c \]  

The equations of motion of the above Lagrangian by making variation with respect to the fields \( h, e, \omega \) and \( f \) are as following respectively

\[ T(\omega) + \alpha e \times h = 0 \]  

\[ -\sigma R(\omega) + \frac{\Lambda_0}{2} e \times e + D(\omega) h - \frac{1}{2m^2} f \times f + \frac{\alpha}{2} h \times h = 0 \]
$R(\omega) + \mu e \times h - \sigma \mu T(\omega) - \frac{\mu}{m^2} (df + \omega \times f) = 0$ \hspace{1cm} (7)

$R(\omega) + e \times f = 0$ \hspace{1cm} (8)

where the locally Lorentz covariant torsion and curvature 2-forms are

$T(\omega) = de + \omega \times e$, \hspace{1cm} $R(\omega) = d\omega + \frac{1}{2} \omega \times \omega$ \hspace{1cm} (9)

The covariant exterior derivative $D(\omega)$ in Eq.(6) is given by

$D(\omega)h = dh + \omega \times h$ \hspace{1cm} (10)

So by adding extra term $\frac{2}{3} e \cdot h \times h$ to the Lagrangian of generalized massive gravity we obtain Lagrangian of our model. The equation for metric can be obtained by generalizing field equation of MMG. Due to this we introduce GMMG field equation as follows

$\tilde{\sigma}G_{mn} + \tilde{\Lambda}_0 g_{mn} + \frac{1}{\mu} C_{mn} + \frac{\gamma}{\mu^2} J_{mn} + \frac{s}{2m^2} K_{mn} = 0$, \hspace{1cm} (11)

where

$G_{mn} = R_{mn} - \frac{1}{2} R g_{mn}$, \hspace{1cm} $C_{mn} = \epsilon_m^{\ ab} \nabla_a (R_{bn} - \frac{1}{4} g_{bn} R)$

$J_{mn} = R_m^\ a R_{an} - \frac{3}{4} R R_{mn} - \frac{1}{2} g_{mn} (R^\ ab R_{ab} - \frac{5}{8} R^2)$,

$K_{mn} = -\frac{1}{2} \nabla^2 R g_{mn} - \frac{1}{2} \nabla_m \nabla^n R + 2 \nabla^2 R_{mn} + 4 R_{manb} R^{ab}$

$-\frac{3}{2} R R_{mn} - R_{ab} R^{ab} g_{mn} + \frac{3}{8} R^2 g_{mn}$,

where $s$ is sign, $\gamma$, $\tilde{\sigma}$, $\tilde{\Lambda}_0$ are the parameters which defined in terms of cosmological constant $\Lambda = \frac{-1}{l^2}$, $m$, $\mu$, and the sign of Einstein-Hilbert term. Here $G_{mn}$ and $C_{mn}$ denote Einstein tensor and Cotton tensor respectively. Symmetric tensors $J_{mn}$ and $K_{mn}$ are coming from MMG and NMG parts respectively \cite{6, 17}.

2.1 AdS\textsubscript{3} Solution

The field equation (11) admits AdS\textsubscript{3} solution,

$d\bar{s} = \bar{g}_{mn} dx^m dx^n = l^2 ( - \cosh^2 \rho \ d\tau^2 + \sinh^2 \rho \ d\phi^2 + d\rho^2)$
where $l^2 = -\Lambda^{-1}$ fixes with parameters of theory. To show this consider the Ricci tensor, Ricci scalar and Einstein tensor of $AdS_3$

$$R_{mn} = 2\Lambda \bar{g}_{mn}, \quad \bar{R} = 6\Lambda, \quad \bar{G}_{mn} = -\Lambda \bar{g}_{mn}.$$ 

Using these results it is easy to see that

$$\bar{C}_{mn} = 0, \quad \bar{J}_{mn} = \frac{\Lambda^2}{4} \bar{g}_{mn}, \quad \bar{K}_{mn} = -\frac{\Lambda^2}{2} \bar{g}_{mn}.$$ 

Then field equation for $AdS_3$ reduces to an quadratic equation for $\Lambda$

$$\left(\frac{\gamma}{4\mu^2} - \frac{s}{4m^2}\right) \Lambda^2 - \Lambda \bar{\sigma} + \bar{\Lambda} = 0. \quad (12)$$

So,

$$\Lambda = \frac{\bar{\sigma} \pm \sqrt{\bar{\sigma}^2 - \bar{\Lambda} \left(\frac{\gamma}{\mu^2} - \frac{s}{m^2}\right)}}{\frac{1}{2} \left(\frac{\gamma}{\mu^2} - \frac{s}{m^2}\right)} \quad (13)$$

### 3 Linearized Field Equation

In this section we study the linear perturbation around the $AdS_3$ spacetime correspondence to propagation of graviton. We take vacuum background $AdS_3$ metric as $\bar{g}_{mn}$ and perturb it with a small perturbation $h_{mn}$ as

$$g_{mn} = \bar{g}_{mn} + h_{mn}.$$ 

At the first order the field equation (11) reduces to

$$\bar{\sigma} G^{(1)}_{mn} + \bar{\Lambda}_0 h_{mn} + \frac{1}{\mu} C^{(1)}_{mn} + \frac{\gamma}{\mu^2} J^{(1)}_{mn} + \frac{s}{2m^2} K^{(1)}_{mn} = 0, \quad (14)$$

where

$$R^{(1)}_{mn} = \frac{1}{2} \left(-\bar{\nabla}^2 h_{mn} + \bar{\nabla}^a \bar{\nabla}_m h_{an} + \bar{\nabla}^a \bar{\nabla}_n h_{am} - \bar{\nabla}_m \bar{\nabla}_n h\right),$$

$$R^{(1)} = (g^{mn} R_{mn})^{(1)} = -\bar{\nabla}^2 h + \bar{\nabla}_m \bar{\nabla}_n m^{mn} - 2\Lambda h,$$

$$G^{(1)}_{mn} = R^{(1)}_{mn} - \frac{1}{2} (R g_{mn})^{(1)} = R^{(1)}_{mn} - \frac{1}{2} \bar{g}_{mn} R^{(1)} - 3\Lambda h_{mn},$$

$$K^{(1)}_{mn} = -\frac{1}{2} \bar{\nabla}^2 R^{(1)} g_{mn} - \frac{1}{2} \bar{\nabla}_m \bar{\nabla}_n R^{(1)} + 2\bar{\nabla}^2 R^{(1)}_{mn} - 4\Lambda \bar{\nabla}^2 h_{mn} - 5\Lambda R^{(1)}_{mn} + \frac{3}{2} \bar{\Lambda} R^{(1)} g_{mn} + \frac{19}{2} \bar{\Lambda}^2 h_{mn},$$

$$C^{(1)}_{mn} = \epsilon_{mn}^{ab} \bar{\nabla}_a (R^{(1)}_{bn} - \frac{1}{4} \bar{g}_{bn} R^{(1)}) - 2\bar{\Lambda} h_{bn},$$

$$J^{(1)}_{mn} = -\frac{\Lambda}{2} (R^{(1)}_{mn} - \frac{1}{2} \bar{g}_{mn} R^{(1)}) - \frac{5}{4} \bar{\Lambda}^2 h_{mn}.$$
Using the fact that \( J^{(1)}_{mn} = -\frac{\Lambda}{2} C^{(1)}_{mn} - \frac{\Lambda^2}{4} h_{mn} \), we obtain following linearized field equation

\[
\left( \tilde{\sigma} - \frac{\gamma \Lambda}{2 \mu^2} \right) G^{(1)}_{mn} + (\tilde{\Lambda}_0 - \frac{\gamma \Lambda^2}{4 \mu^2}) h_{mn} + \frac{1}{\mu} C^{(1)}_{mn} + \frac{s m^2}{2 m^2} K^{(1)}_{mn} = 0. \tag{15}
\]

Imposing Eq.\((\ref{12})\) this equation reduces to

\[
\frac{\tilde{\mu}}{\mu} G^{(1)}_{mn} + \left( \frac{s}{2 m^2} \tilde{\Lambda} + \frac{1}{\mu} \right) h_{mn} + \frac{1}{\mu} C^{(1)}_{mn} + \frac{s m^2}{2 m^2} K^{(1)}_{mn} = 0, \quad \tilde{\mu} = \tilde{\sigma} \mu - \frac{\gamma \Lambda}{2 \mu} \tag{16}
\]

or in the following form

\[
G^{(1)}_{mn} + (1 + \frac{s}{2 m^2} \tilde{\Lambda}) \Lambda h_{mn} + \frac{1}{\mu} C^{(1)}_{mn} + \frac{s m^2}{2 m^2} K^{(1)}_{mn} = 0, \quad \tilde{m}^2 = \frac{\tilde{\mu}}{\mu} m^2
\]

which is exactly same as linearized field equation of GMG [16] \((s = -1)\). As mentioned in [17, 18] this shows the MMG locally has the same degrees of freedom. After fixing gauge as \( \bar{\nabla}_a h^a_{mn} = 0 = h \), the linearized field equation of GMMG on \( AdS_3 \) becomes [2]

\[
(\bar{\nabla}^2 - 2 \Lambda) \left( \bar{\nabla}^2 h_{mn} + s \frac{\tilde{m}^2}{\tilde{\mu}} \epsilon^a_m \epsilon^b_n \bar{\nabla}_a h^a_{mn} - (s \tilde{m}^2 + \frac{5}{2} \Lambda) h_{mn} \right) = 0. \tag{17}
\]

We can define the following operators which commute with each other as

\[
(D^L/R)^n_{\ m} = \delta^n_m \pm \epsilon^a_m \bar{\nabla}_a,
\]

\[
(D^m)^n_{\ m} = \delta^n_m + \frac{1}{m_i} \epsilon^a_m \bar{\nabla}_a, \quad i = 1, 2. \tag{19}
\]

The equation of motion \(\ref{17}\) can then be written as

\[
(D^L R^D D^m D^m h^a) = 0, \tag{20}
\]

The mass parameters \(m_1, m_2\) appearing in \(\ref{20}\) given by

\[
m_1 = \frac{-s \tilde{m}^2}{2 \tilde{\mu}} + \sqrt{\frac{1}{2 \ell^2} + \tilde{\sigma} s \tilde{m}^2 + \frac{\tilde{m}^4}{4 \mu^2}} \quad m_2 = \frac{-s \tilde{m}^2}{2 \tilde{\mu}} - \sqrt{\frac{1}{2 \ell^2} + \tilde{\sigma} s \tilde{m}^2 + \frac{\tilde{m}^4}{4 \mu^2}} \tag{21}
\]

\footnote{Having products of d'Alembertian operators (or more precisely, field equations with more than second order time derivatives) is usually a sign of ghosts. This would be what is sometimes called an Ostrogradski instability.}
The GMMG has various critical points in its parameter space where some of differential operator in (20) degenerate. Due to the similarity between linearized equation of GMMG and GMG, we can use the result of [16] to find the solution of (20) in terms of representations of isometry group $SL(2, R)_L \times SL(2, R)_R$ of $AdS_3$ space. So one can write the Laplacian acting on tensor $h_{mn}$ in terms of the sum of Casimir operators of $SL(2, R)_L$ and $SL(2, R)_R$, [19] (see also [20, 21]).

$$\nabla^2 h_{mn} = -\frac{2}{l^2}(L^2 + \bar{L}^2) + \frac{6}{l^2}h_{mn} \quad (22)$$

Consider states with weight $(h, \bar{h})$:

$$L_0|\psi_{mn}\rangle = h|\psi_{mn}\rangle, \quad \bar{L}_0|\psi_{mn}\rangle = \bar{h}|\psi_{mn}\rangle \quad (23)$$

Since $|\psi_{mn}\rangle$ are primary states:

$$L_1|\psi_{mn}\rangle = \bar{L}_1|\psi_{mn}\rangle = 0. \quad (24)$$

For highest weight states, $L^2|\psi_{mn}\rangle = -h(h-1)|\psi_{mn}\rangle$. Then for the massless modes we have \cite{3}

$$h(h-1) + \bar{h}(\bar{h}-1) - 2 = 0, \quad (25)$$

for $h = 2 + \bar{h}$, we obtain

$$\bar{h} = \frac{-1 \pm 1}{2}, \quad h = \frac{3 \pm 1}{2} \quad (26)$$

but for $h = -2 + \bar{h}$, we have

$$\bar{h} = \frac{3 \pm 1}{2}, \quad h = \frac{-1 \pm 1}{2} \quad (27)$$

Also for massive mode simply for $h = 2 + \bar{h}$ we obtain

$$\bar{h} = \frac{-2 + \frac{s\tilde{m}^2}{\mu} \pm \sqrt{2 - 4s\tilde{m}^2l^2 + \frac{4s^2\tilde{m}^4l^2}{\mu^2}}}{4}, \quad h = \frac{6 + \frac{s\tilde{m}^2}{\mu} \pm \sqrt{2 - 4s\tilde{m}^2l^2 + \frac{4s^2\tilde{m}^4l^2}{\mu^2}}}{4} \quad (28)$$

\cite{3}The massless graviton in three dimensions has no degrees of freedom, which is why people call three dimensional gravity a topological theory (one can equivalently write it as a Chern-Simons gauge theory). However, imposing suitable boundary conditions can lead to asymptotically defined global charges which can differ from one solution to another. So relevant 'physical states' in 3D GR are characterized by their boundary charge and that’s why people sometimes call them boundary gravitons. Also in the higher-derivative theories one gets in addition to the massless graviton (which is pure gauge), several massive gravitons which all have 2 degrees of freedom each.
and for \( h = -2 + \bar{h} \), we have

\[
\bar{h} = \frac{6 - \frac{s\bar{m}^2}{\mu} \pm \sqrt{2 - 4s\bar{m}^2\mu^2 + \frac{s^2\bar{m}^4\mu^2}{\mu^2}}}{4}, \quad h = \frac{-2 - \frac{s\bar{m}^2}{\mu} \pm \sqrt{2 - 4s\bar{m}^2\mu^2 + \frac{s^2\bar{m}^4\mu^2}{\mu^2}}}{4}
\]

(29)

The solutions with the lower sign will blow up at infinity, thus we consider only the ones with the upper sign.

4 Logarithmic modes and dual LCFT

Now we can easily use the result of [16] and obtain central charges of GMMG as follow

\[
c_L = \frac{3l}{2G}(\bar{\sigma} + \frac{s\mu}{2\bar{m}^2l^2} - \frac{1}{\mu\bar{l}}), \quad c_R = \frac{3l}{2G}(\bar{\sigma} + \frac{s\mu}{2\bar{m}^2l^2} + \frac{1}{\mu\bar{l}}).
\]

(30)

which explicitly in terms of the parameters of GMMG are as following

\[
c_L = \frac{3l}{2G}\left(\bar{\sigma} + \frac{s\mu}{\bar{m}^2l^2} - \frac{l}{2}\right), \quad c_R = \frac{3l}{2G}\left(\bar{\sigma} + \frac{s\mu}{\bar{m}^2l^2} + \frac{l}{2}\right)
\]

(31)

The asymptotic symmetry algebra of \( AdS_3 \) space in GMMG model consists of two copies of the Virasoro algebra with the above central charges.

At the critical line,

\[
\bar{\sigma} = \frac{1}{\mu\bar{l}} - \frac{s}{2\bar{m}^2l^2}
\]

(32)

we have

\[
c_L = 0, \quad c_R = \frac{3}{G\mu} = \frac{3}{G(\bar{\sigma}\mu - \frac{4\Lambda}{2\mu})}.
\]

(33)

In this case the linearized equation of motion (20) becomes

\[
(D^RD^LD^{m_2}h)^n_m = 0,
\]

(34)

in another term the operators \( D^{m_1} \) and \( D^L \) degenerate here. Due to this, massive graviton with mass \( m_1 \) degenerate with left massless graviton. Therefore at \( c_L = 0 \) a new logarithmic solution appear. Logarithmic solution satisfies

\[
(D^L D^{L^*}h^{log})^n_m = 0, \quad (D^L h^{log})^n_m \neq 0.
\]

(35)

In GMMG there is another critical line where the operators \( D^{m_1} \) and \( D^{m_2} \) degenerate. This line can be obtained when \( m_1 = m_2 \):

\[
\bar{\sigma} = \frac{-s\bar{m}^2}{4\bar{\mu}^2} - \frac{1}{2s\bar{m}^2l^2}
\]

(36)
At the intersection of the critical line (32) and (36) one can see a critical point as
\[- s\tilde{m}^2 l^2 = 2\tilde{\mu} l = \frac{3}{2} \bar{\sigma} \tag{37}\]
where the left central charge \( c_L = 0 \), and three operators \( D^L, D^{m_1} \) and \( D^{m_2} \) degenerate. Due to this, the intersection of the critical line (32) and (36) is a tricritical point. More than this, there is another tricritical point in GMMG. \( c_R = 0 \), give us a new critical line as
\[\bar{\sigma} = -\left(\frac{1}{\tilde{\mu} l} + \frac{s}{2\tilde{m}^2 l^2}\right) \tag{38}\]
By intersecting the above line with line (36), we obtain following new tricritical point
\[- s\tilde{m}^2 l^2 = 2\tilde{\mu} l = \frac{3}{2} \bar{\sigma} \tag{39}\]
In this new tricritical point the operators \( D^{m_1} \) and \( D^{m_2} \) degenerate with \( D^R \). So the equations of motion in the first and second mentioned tricritical points are given respectively by
\[(D^R D^L D^L h)^n_m = 0, \tag{40}\]
\[(D^R D^R D^R D^L h)^n_m = 0, \tag{41}\]
Due to the tricritical points, more than the logarithmic solutions, we have the square-logarithmic mode \( h^{log^2}_{mn} \)
\[(D^L D^L D^L h^{log^2})^n_m = 0, \quad (D^L D^L h^{log^2})^n_m \neq 0 \tag{42}\]
Similar to the Eqs. (35), (32), in second tricritical point we have following equations
\[(D^R D^R h^{log})^n_m = 0, \quad (D^R h^{log})^n_m \neq 0. \tag{43}\]
\[(D^R D^R D^R h^{log^2})^n_m = 0, \quad (D^R D^R h^{log^2})^n_m \neq 0 \tag{44}\]
The modes \( h^{log}_{mn}, h^{log^2}_{mn} \) in contrast with modes \( h^{R}_{mn}, h^{L}_{mn} \), do not obey the usual Brown-Henneaux boundary conditions, they satisfy a \( log \) and \( log^2 \) asymptotic behavior at the boundary [16].
Similar to what one can obtain in GMG [22, 23], it seems reasonable to conjecture that for GMMG at the critical line (32), the dual CFT is a
LCFT, where the operators $D^{m_1}$ and $D^{m_2}$ degenerate, from Eq. (21), we have

$$\frac{s\tilde{m}^2}{4\mu^2} + \frac{s}{2\tilde{m}^2l^2} = -\bar{\sigma}$$

(45)

So, as GMG [22, 23] this kind of degeneration allows for the possibility of an LCFT. So the left-moving sector has a two point function for dual logarithmic operators as

$$\langle O^{\log}(z)O^{\log}(0) \rangle = \frac{b_L}{2z^4}$$

(46)

where $b_L = \frac{4\bar{\sigma}}{G}$.

5 Relation of the GMMG model with GZDG

The authors of [24] have obtained the Chern-Simons-like formulation of NMG from ZDG model by field and parameter redefinitions. Similarly in this section we show that GMMG model can be obtained from Zweidreibein Gravity (ZDG) plus a Lorentz Chern-Simons term for one of the two spin-connections. In another term we show General Zweidreibein Gravity (GZDG) [15] can reduce to GMMG. The Lagrangian 3-form of ZDG is [14] (see also [15])

$$L_{ZDG} = -MP[\sigma e_1 R_1 + e_2 R_2 + \frac{m^2}{6}(\alpha_1 e_1 \times e_1 + \alpha_2 e_2 \times e_2) - \frac{m^2}{2}(\beta_1 e_1 \times e_1 + \beta_2 e_1 \times e_2)]$$

(47)

where $R_1^a$ and $R_2^a$ are the dualised Riemann 2-forms constructed from $\omega_1^a$ and $\omega_2^a$ respectively. Also $\alpha_1$ and $\alpha_2$ are two dimensionless cosmological parameters and $\beta_1$ and $\beta_2$ are two dimensionless coupling constants. The authors of [15] generalized the above Lagrangian to GZDG model by adding a Lorentz Chern-Simons term as, but with $\beta_2 = 0$

$$L_{GZDG} = L_{ZDG}(\beta_2 = 0) + \frac{MP}{2\mu}(\omega_1 d\omega_1 + \frac{1}{3}\omega_1 \omega_1 \times \omega_1)$$

(48)

Here we consider the above Lagrangian but with $\beta_2 \neq 0$. Now we consider following field redefinitions

$$e_1 \rightarrow e + xf, \quad e_2 \rightarrow e$$

(49)

$$\omega_1 \rightarrow \omega, \quad \omega_2 \rightarrow \omega + yh.$$

(50)
where $x$ and $y$ are arbitrary parameters. By the above field redefinitions, the Lagrangian (48) reduce to the following

$$L_{GZDG} = M_P \left[-(1 + \sigma)e.R - m^2(\frac{\alpha_1 + \alpha_2}{6} - \frac{\beta_1 + \beta_2}{2})e.e \times eight.$$  

$$- \left[ x\sigma f.R + \frac{m^2 x^2}{2}(\alpha_1 - \beta_1) e \times f \right] - \frac{y^2}{2} e.h \times h$$  

$$+ \frac{1}{2\mu}(\omega.d\omega + \frac{1}{3}\omega.e \times \omega) - y e.D(\omega)h$$  

$$- \frac{x m^2}{2}(\alpha_1 - 2\beta_1 - \beta_2)e.e \times f - \frac{\alpha_1 m^2 x^3}{6} f.f \times f \right]$$

As one can see all terms of the GMMG model are generated, plus a couple last terms which are extra terms. Now by considering $\alpha_1 = 0, \beta_2 = -2\beta_1$, we remove these extra terms. Then by following parameter redefinitions

$$1 + \sigma \to \sigma, \quad \frac{m^2 x^2}{2} \beta_1 \to -\frac{1}{2m^2}, \quad m^2(\frac{\alpha_2}{6} + \frac{\beta_1}{2}) \to \Lambda_0 6$$

$$-\frac{y^2}{2} \to \alpha 2 \quad - x\sigma \to -\frac{1}{m^2} \quad - y \to 1,$$

also since

$$y e.D(\omega)h = y h.T,$$

the Lagrangian (48) reduce to the Lagrangian of GMMG model.

### 6 Hamiltonian Analysis

In this section by a Hamiltonian analysis we show that the GMMG model has no Boulware-Deser ghosts. In section 2 we have written the Lagrangian of GMMG as a Lagrangian 3-form constructed from one form fields and their exterior derivatives. So the GMMG model takes a Chern-Simons-like form. As has been discussed in [13] the Chern-Simons formulation of gravity models is well-adapted to a Hamiltonian analysis. It is important that by a Hamiltonian analysis one can obtain the the number of local degrees of freedom exactly and independent of a linearised approximation. Following the approach of [15] (see also [25]) we can rewrite Lagrangian 3-form of GMMG as

$$L = \frac{1}{2} g_{rs} a^r.d a^s + \frac{1}{6} f_{rst} a^r.(a^s \times a^t)$$

where $g_{rs}$ is a symmetric constant metric on the flavour space which is invertible, so it can be used to raise and lower flavour indices, and the
coupling constants $f_{rst}$, which is totally symmetric flavour tensor. GMMG model has four flavours of one-forms: the dreibein $a^e_a = e^a$, the dualised spin-connection $a^\omega_a = \omega^a$, and and two extra fields $a^f_a = f^a$, $a^h_a = h^a$. By comparing Lagrangian 3-form of GMMG which is given under equation(4) with above Lagrangian, we obtain following nonzero components of flavour-space metric $g_{rs}$ and the structure constants $f_{rst}$,

$$
\begin{align*}
    g_{we} &= -\sigma, \quad g_{eh} = 1, \quad g_{f\omega} = -\frac{1}{m^2}, \quad g_{\omega\omega} = \frac{1}{\mu}, \\
    f_{e\omega\omega} &= -\sigma \quad f_{eh\omega} = 1 \quad f_{eff} = -\frac{1}{m^2}, \quad f_{\omega\omega\omega} = \frac{1}{\mu}, \\
    f_{\omega\omega f} &= -\frac{1}{m^2}, \quad f_{eee} = \Lambda_0 \quad f_{ehh} = \alpha.
\end{align*}
$$

Now we consider following the integrability conditions

$$
A_{qrs}a^r_a a^p_p a^q_q = 0 \quad (53)
$$

where $A_{qrs} = f^{l}_{q[r} f_{s]p}$. Using the above integrability conditions we obtain following 3-form relations,

$$
\begin{align*}
    f^a \left[ \frac{1}{\mu} e.f + (1 + \alpha\sigma) h.e \right] - (1 + \alpha\sigma) h^a e.f + \frac{\alpha}{m^2} f^a f.h &= 0 \quad (54) \\
    e^a \left[ \frac{1}{\mu} e.f + (1 + \alpha\sigma) h.e + \frac{\alpha}{m^2} f.h \right] - \frac{\alpha}{m^2} h^a e.f &= 0 \quad (55) \\
    (1 + \alpha\sigma) e^a e.f - \frac{\alpha}{m^2} f^a e.f &= 0 \quad (56)
\end{align*}
$$

as one expected the above equation reduced to the corresponding relations for GMG in the limit $\alpha \to 0$, where have been obtained in [15].

The secondary constraints are given by

$$
\psi_s = B_{rs} = f^{l}_{q[r} f_{s]p} \Delta^{pq} \quad (57)
$$

where $\Delta^{pq} = e^{ij} a^p_i a^q_j$. Now we assume $(1 + \alpha\sigma) e^a - \frac{\alpha}{m^2} f^a$ to have an inverse. By this assumption we restrict our model, such that Boulwar-Deser ghost does not appear as the degrees of freedom of the model. Then from Eq.(56) we obtain the following secondary constraint

$$
e.f = \Delta^{ef} = 0, \quad (58)$$

\footnote{This assumption of invertibility is similar to the the assumed invertibility of $\beta_1 e_1 + \beta_2 e_2$ in the (G)ZDG [15].}
then by an appropriate linear combination of Eqs. (54) and (55) we have
\[ e.f[C f^a \mu - C(1 + \alpha \sigma) h^a + D(\frac{e^a}{\mu} - \frac{\alpha}{m^2} h^a)] + (C f^a + D e^a)[(1 + \alpha \sigma) h.e + \frac{\alpha}{m^2} f.h] = 0 \] (59)
where \( C \) and \( D \) are arbitrary constants, here we assume \( C = -\frac{\alpha}{m^2} \) and \( D = 1 + \alpha \sigma \). So by considering secondary constraint (58), and since \((1 + \alpha \sigma)e^a - \frac{\alpha}{m^2} f^a\) is invertible we derive the second secondary constraint
\[ h.[(1 + \alpha \sigma)e - \frac{\alpha}{m^2} f] = 0 \] (60)
where again in the limiting case \( \alpha \to 0 \), we obtain the two secondary constraint of GMG in [15]. Now we should obtain the rank of matrix \( P^{pq}_{rs} \) which is defined as following
\[ P^{pq}_{rs} = B_{r s} q^{ab}_{pq} + C^{pq}_{rs} \] (61)
where \( C^{pq}_{rs} = 2A_{rsqp}(V^{ab})^{pq}, \) and \( V^{pq}_{ab} = \epsilon^{ij} a^i_{pa} a^j_{qb} \). Taking into account the two secondary constraints, the first term in \( P^{pq}_{rs} \) omit. So in the basis \((e, \omega, f, h)\), matrix \( P^{pq}_{rs} \) takes following form:
\[
\begin{pmatrix}
\frac{V^{ef}_{\mu}}{\mu} - 2(1 + \alpha \sigma)V^{fh}_{[ab]} & 0 & -\frac{V^{fe}_{\mu}}{\mu} + \frac{\alpha}{m^2} V^{fh}_{ab} & (1 + \alpha \sigma)V^{fe}_{ab} - \frac{\alpha}{m^2} V^{ff}_{ab} \\
0 & 0 & 0 & 0 \\
-\frac{V^{ef}_{\mu}}{\mu} + \frac{\alpha}{m^2} V^{hf}_{ab} + (1 + \alpha \sigma)V^{eh}_{ab} & 0 & \frac{V^{ee}_{\mu}}{\mu} - \frac{\alpha}{m^2} V^{eh}_{[ab]} & -(1 + \alpha \sigma)V^{ee}_{ab} + \frac{\alpha}{m^2} V^{ef}_{ab} \\
(1 + \alpha \sigma)V^{ef}_{ab} - \frac{\alpha}{m^2} V^{ff}_{ab} & 0 & -(1 + \alpha \sigma)V^{ee}_{ab} + \frac{\alpha}{m^2} V^{fe}_{ab} & 0
\end{pmatrix}
\]
when we consider the limit \( \alpha \to 0 \), the above matrix reduce to the corresponding matrix for GMG model [15]. The rank of the above matrix at an arbitrary point in space-time is 4. From following equation, one can obtain the dimension of the physical phase space per space point
\[ D = 6N - 2(3N - \text{rank}P - M) - (\text{rank}P + 2M) = \text{rank}P, \] (62)
where \( N \) is the number of flavours, and \( M \) is the number of secondary constraints. In our case, \( N = 4 \), \( \text{rank}P = 4 \), \( M = 2 \). So,
\[ D = 6 \times 4 - 2(12 - 4 - 2) - (4 + 4) = 4. \] (63)
Therefore GMMG model in non-linear regime has two bulk local degrees of freedom. This is exactly the number of massive graviton which we have obtained in section 3 by a linear analysis. The importance of Hamiltonian analysis is its independence of background. Moreover we see that this model is free of Boulware-Deser ghost.

5 Please note that the basis for matrix \( P^{pq}_{rs} \) in GMG case in [15] is as \((\omega, h, e, f)\).
7 Conclusions

In this paper we have generalized recently introduced Minimal Massive Gravity (MMG) model [17] to Generalized Minimal Massive Gravity (GMMG) in asymptotically $AdS_3$ background. MMG is an extension of TMG, but in contrast to TMG, there is not bulk vs boundary clash in the framework of this new model. Although MMG is qualitatively different from TMG, it has locally the same structure as that of TMG model. Moreover both models have the same spectrum [18]. Parallel to this extension, GMMG is an extension of GMG, so one can obtain GMMG by adding the CS deformation term, the higher derivative deformation term, and an extra term to pure Einstein gravity with a negative cosmological constant. This last extra term is exactly what the authors of [17] have added to the TMG to obtain their interesting model, i.e. MMG. So importance of the work [17] is not only solve the problem of TMG, but also do this by introduction only one parameter.

Here we have studied the linearized excitations around the $AdS_3$ background, and have shown that in contrast to MMG, when GMMG linearized about a $AdS_3$ vacuum, a couple massive graviton modes appear. At a special, so-called tricritical point in parameter space the two massive graviton solutions become massless and they are replaced by two solutions with logarithmic and logarithmic-squared boundary behavior.

We have found a critical line, where the left central charge $c_L = 0$. In this critical line where massive graviton $m_1$ degenerate with left-moving massless graviton, a logarithmic mode $h_{log}^{m_1}$ appear. Another critical line has obtained when $m_1 = m_2$, so in this case the operator $D^{m_1}$ and $D^{m_2}$ degenerate. At the intersection of mentioned critical lines, one can see a critical point, where three operators $D^L$, $D^{m_1}$ and $D^{m_2}$ degenerate, so this critical point is a tricritical point. Similar to this tricritical point there is another point in the parameter space of GMMG model where right central charge $c_R = 0$, and three operators $D^R$, $D^{m_1}$ and $D^{m_2}$ degenerate. Due to the presence of these tricritical points, more than logarithmic modes, we obtained the squared-logarithmic modes $h_{log^2}^{m_1}$ in GMMG. These log and log-squared modes in contrast with left and right moving massless gravitons do not satisfy the Brown-Henneaux boundary conditions, they obey log and log$^2$ asymptotic behavior at the boundary, exactly similar to the corresponding modes in GMG [16]. These arguments support the conjecture that GMMG with log and log$^2$ modes, is dual of a rank-3 LCFT.

Here we should mention that the presence of log-modes, (or log$^2$ modes) is a sign of non-unitarity, as the theory with modified boundary conditions
is expected to be dual to a logarithmic conformal field theory, which are well-known to be non-unitary. Also as we have mentioned in footnote 2, having products of d’Alembertian operators in equation of motion is usually a sign of ghosts, but one thing to keep in mind is that the ghosts arising from the Ostrogradski instability are genuinely different from what is called the Boulware-Deser ghost. This is an additional degree of freedom corresponding to a scalar ghost, which is only manifest in the non-linear theory. So in the higher order derivative theory, there are massive spin-2 ghosts and Boulware-Deser ghosts. The latter can be removed by tuning the precise coefficients in the action, but the former cannot be removed in a higher-derivative theory of gravity [26, 27].

Then we have investigated the relation between GMMG and GZDG models. We have shown that the GMMG model can be obtained from ZDG plus a Lorentz Chern-Simons term for one of the two spin-connections. Hamiltonian formulation allows to count the number of local degrees of freedom in the non-linear regime. In section 6 by Hamiltonian analysis we have shown that the Lagrangian 3-form (3) defines a model describing two bulk degrees of freedom. So fortunately GMMG model is free of Boulwar-Deser ghost. But we should mention that situation here is similar to ZDG model. Our model is without Boulwar-Deser ghost only if we demand that the linear combination \((1 + \alpha \sigma)\sigma_a - \frac{\alpha}{m^2} f^a \) is invertible. As we have shown in section 5, GZDG with some special parameters can reduce to the GMMG. In the other hand from [15] we know that GZDG in contrast with ZDG is free of Boulwar-Deser ghost at all. The point is this, the \(L_{GZDG}\) of [15] is a combination of \(L_{ZDG}(\beta_2 = 0)\) plus Lorentz-Chern-Simons (LCS) term. But \(L_{GZDG}\) in this paper is a combination of \(L_{ZDG}(\beta_2 \neq 0)\) plus LCS term. ZDG model with \(\beta_2 = 0\) is free from Boulwar-Deser ghost, but in the case \(\beta_2 \neq 0\), this model has ghost [28]. If one demand that a linear combination of the dreibeine to be invertible, then ZDG will be free of ghost. This is exactly similar to our model. Therefore GMMG propagate two massive graviton with different masses and is free of Boulwar-Deser ghost by restriction applied to it.

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17
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