Higgs mass in Noncommutative Geometry

A. Devastato\textsuperscript{1,2}, F. Lizzi\textsuperscript{1,2,3}, and P. Martinetti\textsuperscript{1,2},

\textsuperscript{1} Dipartimento di Fisica, Università di Napoli \textit{Federico II} \\
\textsuperscript{2} INFN, Sezione di Napoli \\
\textsuperscript{3} Departament de Estructura i Constituents de la Matèria, Universitat de Barcelona

In the noncommutative geometry approach to the standard model, an extra scalar field $\sigma$ - initially suggested by particle physicists to stabilize the electroweak vacuum - makes the computation of the Higgs mass compatible with the 126 GeV experimental value. We give a brief account on how to generate this field from the Majorana mass of the neutrino, following the principles of noncommutative geometry.

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The idea that the Higgs field is somehow related to a noncommutativity of spacetime emerged in the late 80'- early 90'\textsuperscript{[10,14]} and reached its full achievement with Connes and al\textsuperscript{[3,9]} description of the Standard Model of elementary particles [SM] in terms of \textit{noncommutative geometry} [NCG]. The latter is a generalization of Riemannian geometry, that allows to incorporate in a single geometrical object the gravitational degrees of freedom (the commutative part of the geometry) and the quantum ones (the noncommutative part). The central object in this description is a generalized Dirac operator $D$, whose components are the usual Dirac matrices, the Yukawa couplings of the fermions and the mixing parameters for quarks and neutrinos. The SM Lagrangian minimally coupled to the Einstein-Hilbert action (in Euclidean signature) is retrieved from one single action formula. Furthermore, the mass of the Higgs boson comes as a function of the other parameters of the theory, thus can be calculated. The model has been through various improvements\textsuperscript{[6]}, but the prediction was always around $m_H = 170$ GeV, a value ruled out by Tevratron in 2008.

The discovery of a 126 GeV Higgs boson in summer 2012 at LHC raised the question of the stability of the electroweak vacuum: the quartic selfcoupling in the Higgs potential becomes negative at high energy, indicating the vacuum is not stable but metastable (fig. 1). There does not seem to be a consensus whether this is a real problem or not: on the one hand the lifetime of this metastable state is longer than the age of the universe, on the other hand metastability may have some cosmological consequence, it seems unlikely that at early age the Higgs field has been trapped everywhere in the false vacuum, and in some firewall scenario the metastability might even have catastrophic consequences (there is a vast literature on the subject, see e.g.\textsuperscript{[12]} and references within for a recent account). Furthermore, 126 GeV is very close to the stability zone (fig. 2), suggesting that new physics may be around the corner. This instability can be cured by a new scalar field $\sigma$ coupled to the Higgs (e.g.\textsuperscript{[15]}). As a bonus, by taking into account this field in the description of the SM in NCG, it is possible to bring the value of $m_H$ to 126 GeV, without modifying the fermion contain of the SM\textsuperscript{[5]} (using an extra scalar field in NCG to lower $m_H$ was already in\textsuperscript{[19]}, but required new fermions). The point is thus to understand how to obtain $\sigma$ intrinsically within the NCG framework.

In\textsuperscript{[5]} this is done by turning into a field the (constant) Majorana mass term $k_R$ of the neutrino, which is one of the component of the generalized Dirac operator $D$. However the substitution $k_R \rightarrow \sigma k_R$ is somehow ad-hoc: why should this - and only this - component of $D$ become a field? In this proceeding we give a non-technical summary of our recent proposal\textsuperscript{[13]} on how to obtain $\sigma$ within the NCG framework. We also discuss our result, in particular in the light of another proposal made almost simultaneously in\textsuperscript{[7,8]}. 

agostino.devastato@na.infn.it, fedele.lizzi@na.infn.it, martinetti@na.infn.it

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The coupling only runs logarithmically slowly with energy, shows clearly that for the lighter Higgs masses that the term \( \mu \) in turn causing sufficiently large Higgs mass, the positive self interaction \( m = 116 \) GeV (lower curve), \( E_s = \sqrt{m} \). The top quark Yukawa coupling risks the Planck scale, as can be deduced from the behavior of the wavefunction renormalization factor. If we think of the field value \( v \) that to any spectral triple \( \mathcal{A} \) that acts faithfully on a Hilbert space \( \mathcal{H} \), together with an operator \( D \) on \( \mathcal{H} \) such that \([D, a]\) is bounded and \(a[D – \kappa]^{-1}\) is compact for all \(a \in \mathcal{A} \) and \( \kappa \notin \text{Sp} D \). With some extra-conditions that are the algebraic version of the geometrical properties of a Riemannian manifold, and that include the definition of two more operators (a chirality \( \Gamma \) and a real structure \( J \)), Connes showed that to any spectral triple \( (\mathcal{A}, \mathcal{H}, D) \) with \( \mathcal{A} \) unital and commutative is associated a compact Riemannian spin manifold \( \mathcal{M} \) such that \( \mathcal{A} = C^\infty(\mathcal{M}) \). These conditions easily adapt to the noncommutative case. The ones that are important for the present work are the grading, the order 0 and the order 1 conditions:

\[
[\Gamma, a] = 0, \quad [a, JbJ^{-1}] = 0, \quad [[D, a], JbJ^{-1}] = 0 \quad \forall a, b \in \mathcal{A}.
\]

The spectral triple of the SM is the product of the spectral triple \( (C^\infty(\mathcal{M}), L^2(\mathcal{M}, S), \bar{\partial}) \) of a compact Riemannian manifold \( \mathcal{M} \) (here \( L^2(\mathcal{M}, S) \) is the Hilbert space of spinors and \( \bar{\partial} \) is the usual Dirac operator) by a finite dimensional spectral triple

\[
\mathcal{A}_{\text{sm}} = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C}), \quad \mathcal{H}_F = \mathbb{C}^{N \times 32} = \mathcal{H}_R \oplus \mathcal{H}_L \oplus \mathcal{H}_R^c \oplus \mathcal{H}_L^c.
\]

\[
D_F = \begin{pmatrix}
0_{8N} & M & M_R & 0_{8N} \\
M^T & 0_{8N} & 0_{8N} & 0_{8N} \\
M_R^T & 0_{8N} & 0_{8N} & M \\
0_{8N} & 0_{8N} & M^T & 0_{8N}
\end{pmatrix} \quad N = \# \text{ generations.}
\]
The Hilbert space $\mathcal{H}_R = \mathbb{C}^{N \times 8}$ is the space of right fermions (6 colored quarks + 1 lepton + 1 neutrino), $\mathcal{H}_L$ is the space of left fermions and $\mathcal{H}_R^c, \mathcal{H}_L^c$ are the ones of antifermions. The matrix $\mathcal{M}$ contains the quarks, leptons, neutrinos (Dirac) mass with CKM mixing; $\mathcal{M}_R$ contains the Majorana neutrinos mass $k_R$. The chirality is $\chi_F = \text{diag}(\mathbb{I}_{8N}, -\mathbb{I}_{8N}, -\mathbb{I}_{8N}, \mathbb{I}_{8N})$ and the real structure is an antidiagonal matrix with entries $\mathbb{I}_{16N}, \mathbb{I}_{16N}$. The total spectral triple is

$$A = C^\infty(\mathcal{M}) \otimes A_{sm}, \quad \mathcal{H} = L^2(\mathcal{M}, S) \otimes \mathcal{H}_F, \quad D = \phi \otimes \mathbb{I}_{N \times 32} + \gamma^5 \otimes D_F$$

with $\Gamma = \gamma^5 \otimes \gamma_F$ and $\mathcal{J} = \mathcal{J} \otimes J_F$, where $\mathcal{J}$ is the charge conjugation operator.

The gauge fields of the SM (including the Higgs) are obtained by fluctuation of the Dirac operator substituting $D$ with

$$D_A = D + A + JAJ^{-1}$$

where $A$ is a generalized gauge potential, i.e. a selfadjoint element of the form $\sum_i a_i[D, b_i]$ with $a_i, b_i \in \mathcal{A}$. This amounts to turning the constant components of $\mathcal{M}$ in $D_F$ into fields on the manifold $\mathcal{M}$. The spectral action $\text{Tr}f(D^\Lambda) / \text{vol}$ with $f$ a smooth approximation of the characteristic function of the interval $[0, 1]$ and $\Lambda$ an energy scale, then yields the bosonic part of the Lagrangian of the SM minimally coupled with Einstein-Hilbert action. It requires a unique unification scale $\Lambda$. With $\Lambda = 10^{17}\text{GeV}$, the running of the Higgs quartic selfcoupling $\lambda$ under the big desert hypothesis yields $m_H = \sqrt{2\lambda v_{EW}} \simeq 170\text{ GeV}$. After having turned into a field $k_{\nu R}$ the neutrino Majorana mass $k_R$, the same procedure allows to push back $k_M$ to 126 GeV.

One may think to obtain $\sigma$ as the other bosonic fields, i.e. thanks to a fluctuation of the metric. Unfortunately the order 1 condition prevents this. Denoting $D_R$ the part of $D_F$ that contains only the neutrino mass, one has

$$[[D_R, a], JbJ^{-1}] = 0 \quad \forall a, b \in A_{sm} \implies [D_R, a] = 0. \quad (6)$$

In other terms, because of the first order condition there is no way to obtain $\sigma$ by a fluctuation of the Majorana part of the finite dimensional Dirac operator $D_F$.

In [13] we proposed to obtain $\sigma$ starting from a bigger algebra than the one of the SM. Under natural assumptions (irreducibility of the representation, existence of a cyclic vector), technical requirements of the NCG model (there is a representation of the opposite algebra that commutes with the action of the algebra and is implemented by an operator that commutes with the chirality) and a hypothesis on the role of quartenion, one has that the most general finite dimensional algebra satisfying the axiom of noncommutative manifolds is of the form $[4]: M_a(\mathbb{H}) \oplus M_{2a}(\mathbb{C}), \quad a \in \mathbb{N}$, and acts on an Hilbert space of dimension $d = 2 \times (2 \times a)^2$. The case $a = 1$ is too small to get the gauge group of the SM as the group of unitaries of $M(\mathbb{H}) \oplus M_2(\mathbb{C})$. The next choice $a = 2$ yields $d = 32$, that is the number of fermions per generation. As explained in [4], the grading condition imposes the reduction

$$M_2(\mathbb{H}) \oplus M_4(\mathbb{C}) \longrightarrow \mathcal{A}_{LR} = \mathbb{H}_L \oplus \mathbb{H}_R \oplus M_4(\mathbb{C}). \quad (7)$$

The order 1 condition and neutrino mass further imposes

$$\mathcal{A}_{LR} \longrightarrow \mathbb{H}_L \oplus \mathbb{H}_R \oplus M_3(\mathbb{C}) \otimes \mathbb{C} \longrightarrow \mathbb{H}_L \oplus \mathbb{C}' \oplus M_3(\mathbb{C}) \otimes \mathbb{C} \quad (8)$$

with $\mathbb{C} = \mathbb{C}'$. Hence the the reduction of $\mathcal{A}_F$ to the algebra $A_{sm}$ of the standard model.

The case $a = 3$ yields $d = 72$. There is no obvious relation with the 32 particles/generation of the SM. Interestingly, $a = 4$ yields $d = 128$, which is 4 times the number of particles/generation. Viewing 4 as the number of components of a Dirac spinor on a 4-dimensional manifold, one can thus decompose the total Hilbert space (for 1 generation), using the fermion doubling of the model [13], as

$$\mathcal{H} = L^2(\mathcal{M}, S) \otimes \mathcal{H}_F = L^2(\mathcal{M}) \otimes \mathcal{H}_F \quad \text{where} \quad \mathcal{H}_F = \mathbb{C}^4 \otimes \mathcal{H}_F = \mathbb{C}^4 \otimes \mathbb{C}^{128} = \mathbb{C}^{128}. \quad (9)$$
In writing (9) we ignore global obstruction and assume that the r.h.s. equality of the first equation above holds on a local trivialization of the spin bundle. The idea we want to promote is that by mixing the spin degrees of freedom \((s = l, r)\) for the left, right components of a Dirac spinor, \(s = 0, 1\) for the (anti)-particles ones) with the internal degrees of freedom

\[
C = p, a \quad \text{(particle, antiparticles),} \quad I = 0, 1, 2, 3 \quad \text{(lepto-color),}
\]

\[
\alpha = u_R, d_R, u_L, d_L \quad (I = 1, 2, 3), \quad e_R, \nu_R, e_L, \nu_L \quad (I = 0) \quad \text{(flavor)},
\]

then the Hilbert space \(\mathcal{H}\) of the standard model is big enough to represent the grand algebra \((a = 4)\)

\[
\mathcal{A}_G = M_4(\mathbb{H}) \oplus M_8(\mathbb{C})
\]

tensorized by \(C^\infty(\mathcal{M})\)) without touching the SM particle contents, and satisfying the order 0 condition.

Explicitly the representation is as follows. We denote a spinor in \(\mathcal{H}\) as \(\Psi^{G1}_{\alpha \beta}\) and view both \(Q \in \mathcal{M}_4(\mathbb{H})\) and \(M \in \mathcal{M}_8(\mathbb{C})\) as \(2 \times 2\) block matrices, with block \(4 \times 4\) complex matrices:

\[
Q = \begin{pmatrix} Q_{00}^{\beta} & Q_{0\beta}^{\alpha} \\ Q_{\alpha0}^{-\beta} & Q_{\alpha\beta}^{-\alpha} \end{pmatrix}, \quad M = \begin{pmatrix} M_{ll}^{ij} & M_{rl}^{ij} \\ M_{rl}^{-ij} & M_{rr}^{-ij} \end{pmatrix}.
\]

Viewing the components of the matrices in (13) as functions on \(\mathcal{M}\), an element \(A = (Q, M)\) in \(C^\infty(\mathcal{M}) \otimes \mathcal{A}_G\) acts on \(\mathcal{H}\) as

\[
A_{sA}^{\alpha ijG_{\beta}l_{\delta}} = \left( \delta_{\alpha \beta} \delta_{\delta \alpha} \delta_{\beta \delta} + \delta_{\alpha \beta} M_{ll}^{ij} \right). 
\]

The indices \(\beta, I\) run on the same set as \(\alpha, J\). The Dirac operator, chirality and real structure are unchanged.

The grading condition imposes the reduction

\[
\mathcal{A}_G = M_4(\mathbb{H}) \oplus M_8(\mathbb{C}) \longrightarrow \mathcal{A}_G' = (M_2(\mathbb{H})_L \oplus M_2(\mathbb{H})_R) \oplus (M_4(\mathbb{C})_L \oplus M_4(\mathbb{C})_R).
\]

A solution of the first-order condition of the Majorana Dirac operator only, \(\gamma^5 \otimes D_R\), is

\[
\mathcal{A}_G' \longrightarrow \mathcal{A}_G'' = (\mathbb{H}_L \oplus \mathbb{H}_L' \oplus \mathbb{C}_R \oplus \mathbb{C}_R') \oplus (\mathbb{C}_I \oplus M_3(\mathbb{C})_L \oplus \mathbb{C}_r \oplus M_3(\mathbb{C})_r)
\]

with \(\mathbb{C}_R = \mathbb{C}_r = \mathbb{C}_I\). The main result of [13] is that for \(A \in \mathcal{A}_G''\)

\[
[\gamma^5 \otimes D_R, A] \text{ is not necessarily zero.}
\]

In other terms, starting from the grand algebra one can generate the field \(\sigma\) by a fluctuation of the Majorana mass term which respects the first order condition imposed by the Majorana mass term.

The further reduction to the standard model, that is \(\mathbb{C}_R' = \mathbb{C}_r, \mathbb{H}_L' = \mathbb{H}_L, M_3(\mathbb{C})_L = M_3(\mathbb{C})_r\), is obtained by the first order condition on the non-Majorana part of the Dirac operator.

Let us now discuss our result. The representation (13) of \(C^\infty(\mathcal{M}) \otimes \mathcal{A}_G\) together with the Dirac operator in [14] do not yield a spectral triple, because whatever \(A \in C^\infty(\mathcal{M}) \otimes \mathcal{A}_G\), the operator

\[
[\not\!\mathcal{D} \otimes \mathbb{I}_{32}, A] = P + T^\mu \partial_\mu
\]

\((P \text{ and } T^\mu\text{ are matrices whose explicit form is given in [13]})\) is not bounded. It becomes bounded if \(T^\mu = 0\) for all \(\mu = 1, \ldots, 4\), but this precisely means that the action of the grand algebra should be diagonal on the spinorial indices, meaning the reduction of \(\mathcal{A}_G\) to \(\mathcal{A}_\text{sm}\). Thus the mixing of the spin and internal indices has two consequences:

- the bounded bosonic operators \(B\) generated by a fluctuation of the Dirac operator include the field \(\sigma\);
- there is a “deeper alteration of spacetime”, encoded within the unbounded operator \(T^\mu \partial_\mu\).
This last point is the most open to drastic changes which may be imagined, for example like dropping out the associativity of the algebra [16]. Alternatively, one may imagine a cosmological scenario beginning with a “pre-geometric phase”, described by the grand algebra and the finite dimensional Dirac operator $\gamma^5 \otimes D_R$. The right neutrino would then play the role of a “primary elementary particle”, that generates the field $\sigma$. Then usual geometry (encoded within the free Dirac operator $\bar{\theta} \otimes 1$) emerges at a later stage, and provokes the reduction to the SM. This makes and interesting echo to a recent inflationary interpretation of the field $\sigma$ [2]. Moreover very recent data [1] seem to indicate an inflationary scale at a scale of $10^{16}$ GeV, a scale in broad agreement with the unification of the coupling constant required by this approach. From a more mathematical point of view, we stress that the triple $(C^\infty(M) \otimes A_G, \mathcal{H}, \gamma^5 \otimes D_R)$ satisfies the bounded commutator condition, but $\gamma^5 \otimes D_R$ has no compact resolvent.

To summarize, the grand algebra transfers the problem of generating $\sigma$ from the noncommutative to the commutative part of the geometry: with the algebra of the standard model, $C^\infty(M) \otimes A_{sm}$, the first order condition is always satisfied by the free Dirac operator, the problem is all in $D_R$. Using the grand algebra, we have that $\gamma^5 \otimes D_R$ both generates the field $\sigma$ and satisfies the first-order condition. But the free Dirac does not satisfy this condition (neither the bounded commutator one). Of course this is not satisfactory but this suggests interesting path to explore.

Another question is whether the reduction to the SM imposed by the first order condition can be understood dynamically (i.e. by a minimization of the spectral action), as in the model of Chamseddine, Connes and van Suijlekom where $\sigma$ is generated from $A_{sm}$ by a fluctuation without first order condition.

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