General Linear and Symplectic Nilpotent Orbit Varieties

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Abstract

The condition of nilpotency is studied in the general linear Lie algebra \(\mathfrak{gl}_n(\mathbb{K})\) and the symplectic Lie algebra \(\mathfrak{sp}_{2n}(\mathbb{K})\) over an algebraically closed field of characteristic 0. In particular, the conjugacy class of nilpotent matrices is described through nilpotent orbit varieties \(O_\lambda\) and an algorithm is provided for computing the closure \(\overline{O_\lambda} \cong \text{Spec}(\mathbb{K}[X]/J_\lambda)\). We provide new generators for the ideal \(J_\lambda\) defining the affine variety \(\overline{O_\lambda}\) which show that the generators provided in [Wey89] are not minimal. Furthermore, we conjecture the existence of local weak Néron models for nilpotent orbit varieties based on bounding \(p\) in the polynomial ring with \(p\)-adic integer coefficients for which the equations defining \(O_\lambda\) can embed.

1 Introduction

Let \(\mathbb{K}\) be an algebraically closed field of characteristic zero. We are interested in geometrically describing the condition of nilpotency in the general linear Lie algebra \(\mathfrak{gl}_n(\mathbb{K})\) through associating varieties with conjugacy classes of nilpotent elements in \(\mathfrak{gl}_n(\mathbb{K})\). Let \(X\) be an \(n \times n\) matrix in the nilpotent cone or nullcone \(\mathcal{N}(n) := \mathfrak{gl}_n^{\text{nilp}}(\mathbb{K}) = \{X \in \mathfrak{gl}_n(\mathbb{K}) \mid X^k = 0, \exists k \in \mathbb{N}\}\), and denote the conjugacy class (similarity class) of \(X\), i.e., the orbit of \(X\) under the action of conjugation, by \(C_X = \{P^{-1}XP \mid P \in \mathfrak{gl}_n(\mathbb{K})\}\). We denote the origin of the nilpotent cone by \(\mathcal{N}_0(n) := \{x_{ij} = 0 \mid 1 \leq i,j \leq n\}\). By the Jordan normal form theorem, \(\exists P \in \mathfrak{gl}_n(\mathbb{K})\) so that \(Y = P^{-1}XP\) has Jordan blocks of sizes determined by an integer partition \(\lambda_Y = [\lambda_1,...,\lambda_l]\) of \(n\) with \(\lambda_1 \geq \cdots \geq \lambda_l\). Thus, the map \(C_X \mapsto \lambda_Y\) is a bijection between the set of nilpotent conjugacy classes and the set of partitions of \(n\). Letting \(\lambda = [\lambda_1,...,\lambda_l]\) and \(\lambda' = [\lambda'_1,...,\lambda'_s]\) be partitions of the integer \(n\) listed in a non-increasing sequence, the dominance order \(\preceq\) on the set of partitions of a positive integer \(n\) is defined by \(\lambda \preceq \lambda'\) if \(\sum_{i=1}^k \lambda_i \leq \sum_{i=1}^k \lambda'_i\) for all \(k \leq \max\{l,s\}\). If \(l > s\) then we add \(l-s\) zeros to end of the partition \(\lambda'\) and if \(s > l\) then we add \(s-l\) zeros to end of the partition \(\lambda\) for this definition to be well-defined. Through this bijection, the dominance ordering of integer partitions partially orders the set of nilpotent conjugacy classes. The nilpotent orbit variety \(O_\lambda\) associated with the nilpotent conjugacy class in bijection with the partition \(\lambda\) is shown to be given by exact conditions on ranks of powers of matrices, where \(\mathbb{K}[X] := \mathbb{K}[x_{ij} \mid 1 \leq i,j \leq n]\). Thus,

\[
O_\lambda = \{f \in \mathbb{K}[X] \mid \text{rank}(X^k) = r, \forall (k,r) \in U_\lambda\}
\]

with

\[
U_\lambda = \{(k,r) \mid 1 \leq k \leq \max\{\lambda_1,...,\lambda_l\}, r = \sum_{i=1}^l f^k(\lambda_i)\}
\]

and the rank counting function \(f\) defined by

\[
f(x) = \begin{cases} 
  x - 1 & \text{if } x > 0 \\
  0 & \text{if } x \leq 0
\end{cases}
\]

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We remark that \(\text{rank}(X^k) = \sum_{i=1}^t f^k(\lambda_i)\) because the entries in the first upper diagonal of \(X\) in Jordan normal form pass to the second upper diagonal of \(X^2\) and so on until the nilpotency of \(X\) ends this marching of the entries away from the main diagonal. From this observation, the non-zero entries in the Jordan blocks of \(X\) are then naturally kept track of by powers of the rank counting function.

We have that the Zariski closure of a nilpotent orbit variety \(\overline{O}_\lambda\) associated with the nilpotent conjugacy class in bijection with the partition \(\lambda\) is defined by upper bounds on ranks of powers of matrices. Thus,
\[
\overline{O}_\lambda = \{ f \in \mathbb{K}[X] \mid \text{rank}(X^k) \leq r, \forall (k, r) \in U_\lambda \}
\]

Using the dominance ordering of integer partitions and thus nilpotent orbit varieties, we express the closure of a nilpotent orbit variety in terms of nilpotent orbit varieties by
\[
\overline{\sigma}_\lambda = O_\lambda \cup \bigcup_{\mu \triangleleft \lambda} O_\mu
\]

We can visualize the nilpotent cone \(\mathcal{N}(n)\) as the union of all nilpotent orbit varieties as seen in Figure 1.

![Figure 1: A representation of the nilpotent cone with each region denoting a unique nilpotent orbit variety.](image)

\section{Nilpotent Orbit Varieties and Ideal Generators}

Since \(\overline{O}_\lambda\) is an affine variety, it is defined by an ideal \(J_\lambda\) associated with the partition \(\lambda\) by \(\overline{O}_\lambda \cong \text{Spec} \left( \mathbb{K}[X]/J_\lambda \right)\). We use a more recent rephrasing of Theorem 4.6 of \cite{Wey89} given by Theorem 5.4.3 of \cite{KLMW07} regarding the generators of \(J_\lambda\)

\textbf{Theorem 1.} The ideal \(J_\lambda\) is generated by \(V_{0,p}(1 \leq p \leq n)\) and \(V_{i,\lambda(i)}(1 \leq i \leq n)\) where \(\lambda(i) = \lambda_1 + \cdots + \lambda_i - i + 1\) and \(V_{i,p}\) is defined as a span of linear combinations
\[
V_{i,p} := \text{span} \left\{ \sum_{|J|=p-i} X(P, J|Q, J) \mid P, Q \subset \{1, \ldots, n\}, |P| = |Q| = i, (P \cup Q) \cap J = \emptyset \right\}
\]

where \(X(P|Q)\) denotes the minor of \(X \in \mathfrak{gl}_n(\mathbb{K})\) with rows indexed by \(P\) and columns indexed by \(Q\).

\textbf{Proof.} This is a restatement of Theorem 4.6 of \cite{Wey89} using the alternative definition of \(V_{i,p}\) given on page 30 of \cite{KLMW07}. In \cite{Wey89},
\[
V_{i,p} \cong \bigwedge^i V^* \otimes \bigwedge^i V
\]
with elements given from a basis \(e_1, ..., e_n\) of the vector space \(V\) by \(e_{p_1}^* \wedge e_{p_2}^* \wedge \cdots \wedge e_{p_r}^* \otimes e_{q_1} \wedge e_{q_2} \wedge \cdots \wedge e_{q_i}\). Whereas, in [KLMW07]
\[
V_{i,p} = \text{span} \left\{ \sum_{|J|=p-i} X(P,J|Q,J) \mid P,Q,J \subset \{1, ..., n\}, |P| = |Q| = i, (P \cup Q) \cap J = \emptyset \right\}
\]
with the proof that \(J_{\lambda}\) is generated by \(V_{0,p}(1 \leq p \leq n)\) and \(V_{i,\lambda(j)}(1 \leq i \leq n)\) given in [Wey89] using Lascoux resolution of complexes, Schur functors used to define irreducible representations of \(\mathfrak{sl}_n\); spectral sequences of filtrations, and induction on the length of the partition.

In order to recover the nilpotent orbit variety \(O_{\lambda}\) from the closure \(\overline{O}_{\lambda}\), we construct the set
\[
H_{\lambda} = \{ h \in \mathbb{K}[X] \mid \text{rank}(X^k) \geq r, \forall (k, r) \in U_{\lambda} \}
\]
and use localization. Since \(\text{rank}(X^k) \geq r\) is guaranteed by the existence of an \(r \times r\) minor of \(X\) with non-zero determinant, we construct another set
\[
H^k_{\lambda} = \left\{ X(P|Q) \neq 0 \mid P, Q \subseteq \{1, ..., n\}, |P| = |Q| = r_k := \text{rank}(X^k) = \sum_{i=1}^{r_k} j^k(\lambda_i) \right\}
\]
which indexes the \(r_k \times r_k\) minors of \(X^k\). Then since there are \(\binom{n}{r}^2\) minors of \(X\) with size \(r \times r\),
\[
H_{\lambda} = \bigcup_{k=1}^{\max\{\lambda\}} H^k_{\lambda} = \left\{ h_{j,k} \in H^k_{\lambda} \mid 1 \leq k \leq \max\{\lambda\}, 1 \leq j \leq \binom{n}{r_k}^2 \right\}
\]
where \(\max\{\lambda\} = \max\{\lambda_1, ..., \lambda_l\}\). We now take unions of localizations of nilpotent orbit variety closures by \(h_{j,k} \in H_{\lambda}\) and obtain
\[
O_{\lambda} = \bigcup_{h \in H_{\lambda}} \left( \overline{O}_{\lambda} \right)_h = \bigcup_{k=1}^{\max\{\lambda\}} \left( \sum_{j=1}^{\binom{n}{r_k}^2} \left( \overline{O}_{\lambda} \right)_{h_{j,k}} \right) \cong \bigcup_{k=1}^{\max\{\lambda\}} \left( \sum_{j=1}^{\binom{n}{r_k}^2} \text{Spec} \left( \left( \mathbb{K}[X]/J_{\lambda} \right)_{h_{j,k}} \right) \right) \cong \bigcup_{k=1}^{\max\{\lambda\}} \left( \sum_{j=1}^{\binom{n}{r_k}^2} \text{Spec} \left( \mathbb{K}[X]/J_{\lambda}(h_{j,k}, t) \right) \right)
\]
where \((\cdot)_h\) denotes localization at \(h\). We remark that the transition maps for this atlas are induced by the isomorphism
\[
\left( \text{Spec} \left( \mathbb{K}[X]/J_{\lambda} \right) \right)_{h_h} \cong \left( \text{Spec} \left( \mathbb{K}[X]/J_{\lambda} \right) \right)_{h'h}
\]
where \(h, h' \in H_{\lambda}\).

### 3 Computing Nilpotent Orbits in \(\mathfrak{gl}_n\)

To gain some intuition for what \(V_{i,p}\) represents in the formulation in [Wey89] and in [KLMW07] we present an example which illustrates both. We first remark that the condition that \((P \cup Q) \cap J = \emptyset\) ensures that the minor \(X(P, J|Q, J)\) is square and thus has a well-defined determinant. With this in mind, we compute the nilpotent orbit variety \(O_{[2,1]}\) in \(\mathcal{N}(3) := \mathfrak{gl}^\text{nilp}(\mathbb{K})\) using a simple construction which yields generators for \(J_{[2,1]}\) which are more minimal than in Theorem \([2,1]\) before presenting this case in the harder to understand language of \(V_{i,p}'s\). We conjecture that for small values of \(n\) the generators presented in our algorithm are less minimal than those constructed by Weyman.

\[3\]
We begin with the bijection between integer partitions and nilpotent orbit varieties,

\[ [2,1] \mapsto \mathcal{O}_{[2,1]} = \{ f \in \mathbb{K}[X] \mid \text{rank}(X) = 1, X^2 = 0 \} \supset [0 \ 0 \ 0]

\begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}

where \( U_{[2,1]} = \{(1,1),(2,0)\} \). We now compute the nilpotent orbit variety closure \( \overline{\mathcal{O}_{[2,1]}} \) by using a lemma which upper bounds the rank of a matrix by conditions on the determinants of minors of the matrix.

Lemma 1. If \( X \in \mathfrak{gl}_n \) and \( \det(M) = 0 \) for every \( (r+1) \times (r+1) \) minor \( M \) of \( X \), then \( \text{rank}(X) \leq r \). That is, if \( X(P,Q) = 0 \) for every \( P,Q \subset \{1,\ldots,n\} \) with \( |P| = |Q| = r + 1 \), then \( \text{rank}(X) \leq r \).

Proof. The rank of a matrix can be equivalently defined as the dimension of the largest minor whose determinant is not zero. Hence, if the determinant of every \( (r+1) \times (r+1) \) minor of \( X \) is zero then \( \text{rank}(X) \leq r \). \( \square \)

From computing nilpotent orbit variety closures we can recover the nilpotent orbit variety in this case by using

\[ \overline{\mathcal{O}_{[2,1]}} = \mathcal{O}_{[2,1]} \cup \mathcal{O}_{[1,1,1]} \]

since \( \mathcal{O}_{[1,1,1]} = \{ f \in \mathbb{K}[X] \mid X = 0 \} = \{ x_{11} = 0, \ldots, x_{33} = 0 \} = \mathcal{N}_0(3) \). Now,

\[ \overline{\mathcal{O}_{[2,1]}} = \{ f \in \mathbb{K}[X] \mid \text{rank}(X) \leq 1, X^2 = 0 \}

we have that \( \text{rank}(X) \leq 1 \) is satisfied when every \( 2 \times 2 \) minor of \( X \) has determinant zero and that \( X^2 = 0 \) is satisfied when every \( 1 \times 1 \) minor of \( X^2 \) has determinant zero, that is, when each entry of \( X^2 \) is zero. Thus,

\[ \overline{\mathcal{O}_{[2,1]}} = \{ x_{12}x_{33} - x_{13}x_{32}, x_{11}x_{32} - x_{12}x_{31}, x_{11}x_{22} - x_{12}x_{21}, x_{12}x_{23} - x_{13}x_{22}, x_{21}x_{32} - x_{22}x_{31},
\]

\[ x_{22}x_{33} - x_{23}x_{32}, x_{11}x_{23} - x_{13}x_{21}, x_{12}x_{21} - x_{13}x_{22}, x_{11}x_{33} - x_{13}x_{31}, x_{11}^2 + x_{12}x_{21} + x_{13}x_{31},
\]

\[ x_{11}x_{12} + x_{12}x_{22} + x_{13}x_{32}, x_{21}x_{32} - x_{22}x_{31} + x_{23}x_{31}, x_{21}x_{21} + x_{22}x_{21} + x_{23}x_{21} + x_{23}x_{31},
\]

\[ x_{21}x_{12} + x_{22}^2 + x_{23}x_{32}, x_{21}x_{13} + x_{22}x_{23} + x_{23}x_{33}, x_{21}x_{11} + x_{22}x_{21} + x_{23}x_{31},
\]

\[ x_{31}x_{12} + x_{32}x_{22} + x_{33}x_{32}, x_{31}x_{13} + x_{32}x_{23} + x_{33}^2 \}

which is a system of 18 polynomial equations in \( \mathbb{K}[x_{11},x_{12},x_{13},x_{21},x_{22},x_{23},x_{31},x_{32},x_{33}] \). We then have that \( \overline{\mathcal{O}_{[2,1]}} = \mathcal{O}_{[2,1]} \cup \mathcal{N}_0(3) \), where \( \mathcal{N}_0(3) \) denotes the origin of the nilpotent cone in \( \mathfrak{gl}_3 \). In general, we refer to Algorithm 1 for computing nilpotent orbit variety closures in terms of \( \lambda \).

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**Algorithm 1** \( \mathfrak{gl}_n \) Nilpotent Orbit Variety Closure

**Require:** \( \lambda = [\lambda_1, \ldots, \lambda_l] \), where \( \sum_{i=1}^l \lambda_i = n \) and \( \lambda_i \in \mathbb{N}, \forall i \in \{1,\ldots,l\} \).

Set \( \overline{\mathcal{O}_\lambda} = \emptyset \).

for all \( k \in \{1,\ldots,n\} \) do

Set \( r = \text{rank}(X^k) = \sum_{i=1}^l f^k(\lambda_i) \)

if \( r \geq 0 \) then

for all \( P,Q \subset \{1,\ldots,n\} \) do

if \( |P| = |Q| = r + 1 \) then

Set \( \overline{\mathcal{O}_\lambda} = \overline{\mathcal{O}_\lambda} \cup \{ X^k(P,Q) = 0 \} \).

end if

end for

end if

end for

return \( \overline{\mathcal{O}_\lambda} \)
In the formalism presented by Weyman we have that
\[ \overline{O}_{[2,1]} \cong \text{Spec } (\mathbb{K}[X]/J_{[2,1]}) \]
where \( J_{[2,1]} = (V_{0,1}, V_{0,2}, V_{0,3}, V_{1,2}, V_{2,2}, V_{3,1}) \), which as we will see reduces to \( (V_{0,1}, V_{0,2}, V_{0,3}, V_{1,2}, V_{2,2}) \) since \( V_{i,p} \) is trivial for \( i > p \). The function \( \lambda(i) = \lambda_1 + \cdots + \lambda_i - i + 1 \) is used to apply Theorem 4 to this example as follows. For the partition \( \lambda = [2, 1] \), we append \( i - |\lambda| \) additional zeroes if required to define \( V_{i,p} \) for a specific \( p = \lambda(i) \). In this case we have \( \lambda(1) = 2, \lambda(2) = 2, \) and \( \lambda(3) = 1 \) are the values of \( p \) for each non-zero \( i \). Then,

\[
V_{0,1} = \text{span} \left\{ \sum_{|J|=1} X(J|J) \mid J \subset \{1, 2, 3\} \right\} = \{x_{11} + x_{22} + x_{33}\}
\]

\[
V_{0,2} = \text{span} \left\{ \sum_{|J|=2} X(J|J) \mid J \subset \{1, 2, 3\} \right\} = \text{span}\{X(1, 2|1, 2) + X(2, 3|2, 3) + X(1, 3|1, 3)\}
\]

\[
= \{x_{11}x_{22} - x_{21}x_{12} + x_{22}x_{33} - x_{23}x_{32} + x_{11}x_{33} - x_{13}x_{31}\}
\]

\[
V_{0,3} = \text{span} \left\{ \sum_{|J|=3} X(J|J) \mid J \subset \{1, 2, 3\} \right\} = \{\det(X)\}
\]

\[
V_{1,2} = \text{span} \left\{ \sum_{|J|=1} X(P, J|Q, J) \mid P, Q \subset \{1, 2, 3\}, |P| = |Q| = 1, (P \cup Q) \cap J = \emptyset \right\}
\]

\[
= \text{span}\{X(2, 1|2, 1) + X(3, 1|3, 1) + X(2, 1|3, 1) + X(3, 1|2, 1),
X(1, 2|1, 2) + X(3, 2|3, 2) + X(1, 2|3, 2) + X(3, 2|1, 2),
X(1, 3|1, 3) + X(2, 3|2, 3) + X(1, 3|2, 3) + X(2, 3|1, 3)\}
\]

\[
V_{2,2} = \text{span}\{X(P|Q) \mid P, Q \subset \{1, 2, 3\}, |P| = |Q| = 2\}
\]

\[
= \text{span}\{X(1, 2|1, 2), X(1, 3|1, 3), X(2, 3|2, 3), X(1, 2|1, 3), X(1, 2|2, 3), X(1, 3|2, 3), X(1, 3|1, 2), X(2, 3|1, 3),
X(2, 3|1, 2), X(1, 3|1, 2), X(2, 3|1, 2), X(2, 3|1, 3), X(1, 2|1, 3), X(1, 3|2, 3), X(1, 2|2, 3)\}
\]

It is difficult to find reductions in the span of a system of equations as opposed to the direct computation provided by Algorithm 1. Thus, linear hulls of subsets of 21 polynomial equations generate \( J_{[2,1]} \).

4 Computing Nilpotent Orbits in \( \mathfrak{sp}_{2m} \)

A symplectic matrix is a \( 2m \times 2m \) matrix \( M \) with entries from \( \mathbb{K} \) which satisfies \( M^T \Omega M = \Omega \), where \( \Omega \) is a fixed \( 2m \times 2m \) invertible (nonsingular) and skew-symmetric \( (M^T = -M) \) matrix, where typically

\[
\Omega = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & -1 & 0 \\
& & & & & \\
0 & 1 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

The symplectic group of degree \( 2m \) over a field \( \mathbb{K} \) is denoted by \( Sp(2m, \mathbb{K}) \) and is the group of all symplectic matrices with matrix multiplication as the group operation. The symplectic Lie algebra \( \mathfrak{sp}_{2m} \) is the Lie algebra of the Lie group \( Sp(2m, \mathbb{K}) \) and is the set of all matrices \( M \) such that \( e^{tM} \in Sp(2m, \mathbb{K}) \). Equivalently, \( \mathfrak{sp}_{2m} \) can be thought of as the tangent space to \( Sp(2m, \mathbb{K}) \) at the identity. We now want to compute nilpotent orbit varieties in \( \mathfrak{sp}_{2m} \), which can be indexed by partitions of \( 2m \) for which each odd integer appears with even multiplicity due to a theorem of Gerstenhaber presented in Section 5.1 of [CM93].
Lemma 2. Let \( X \in \mathbb{K}[x_{ij} \mid 1 \leq i, j \leq 2m] \). Then \( X \) is symplectic when \( X^T \Omega X = \Omega \), which is when the equations in the following sets are satisfied.

\[
\Lambda_{2m}^{sp}(2q + 1, n - 2q) = \left\{ 1 + \sum_{k=1}^{2m} (-1)^k x_{2m+1-k,i,x_{k,j}} = 0 \mid i = 2q + 1, j = n - 2q, q \in \mathbb{N}, q < m \right\}
\]

\[
\Lambda_{2m}^{sp}(2q, n - 2q + 1) = \left\{ 1 + \sum_{k=1}^{2m} (-1)^{k+1} x_{2m+1-k,i,x_{k,j}} = 0 \mid i = 2q, j = n - 2q + 1, q \in \mathbb{N}, q < m \right\}
\]

\[
\Lambda_{2m}^{sp}(r, s) = \left\{ \sum_{k=1}^{2m} (-1)^k x_{2m+1-k,i,x_{k,j}} = 0 \mid 3q \in \mathbb{N}, (i = r = 2q + 1 \land j = s = n - 2q) \lor (i = r = 2q \land j = s = n - 2q + 1), 1 \leq r, s \leq 2m \right\}
\]

Furthermore, \( |\Lambda_{2m}^{sp}(2q + 1, n - 2q)| = |\Lambda_{2m}^{sp}(2q, n - 2q + 1)| = m \) and \( |\Lambda_{2m}^{sp}(r, s)| = 4m^2 - 2m \).

We now call

\[
\Lambda_{2m}^{sp} = \bigcup_{q=0}^{m-1} \Lambda_{2m}^{sp}(2q + 1, n - 2q) \cup \bigcup_{q=1}^{m} \Lambda_{2m}^{sp}(2q, n - 2q + 1) \cup \bigcup_{(r,s)} \Lambda_{2m}^{sp}(r, s)
\]

and note that \( |\Lambda_{2m}^{sp}| = 4m^2 \). We can compute nilpotent orbit varieties closures in \( \mathfrak{sp}_{2m} \) with Algorithm 2.

**Algorithm 2** \( \mathfrak{sp}_{2m} \) Nilpotent Orbit Variety Closure

**Require:** \( \lambda = [\lambda_1, \ldots, \lambda_l] \), where \( \sum_{i=1}^{l} \lambda_i = n \) and \( \lambda_i \in \mathbb{N}, \forall i \in \{1, \ldots, l\} \).

Set \( \overline{\Omega}_\lambda = \emptyset \).

for all \( k \in \{1, \ldots, n\} \) do

Set \( r = \text{rank}(X^k) = \sum_{i=1}^{l} f^k(\lambda_i) \)

if \( r \geq 0 \) then

for all \( P, Q \subset \{1, \ldots, n\} \) do

if \( |P| = |Q| = r + 1 \) then

Set \( \overline{\Omega}_\lambda = \overline{\Omega}_\lambda \cup \{X^k(P|Q) = 0\} \).

end if

end if

end for

end if

end for

Set \( \overline{\Omega}_\lambda^{sp} = \overline{\Omega}_\lambda \cap \Lambda_{2m}^{sp} \)

return \( \overline{\Omega}_\lambda^{sp} \)

For computing symplectic nilpotent orbit varieties we intersect the general linear nilpotent orbit variety with \( \mathfrak{sp}_{2m} \) and obtain

\[
\overline{\Omega}_\lambda^{sp} = \bigcup_{h \in H_\lambda} \left( \overline{\Omega}_\lambda^{sp} \right)_h.
\]
5 Néron Models and Future Research Directions

Let $R$ be a Dedekind domain, that is, an integral domain in which every nonzero proper ideal factors into a product of prime ideals, with field of fractions $K$ and let $R_K$ be an abelian variety over $K$ (which is that $R_K$ is a projective algebraic variety that is also an algebraic group). A Néron model is a universal separated smooth scheme $A_R$ over $R$ with a rational map to $A_K$; equivalently, Néron models are commutative quasi-projective group schemes over $R$. Motivation for studying Néron models can come from understanding good reduction of elliptic curves over $\mathbb{Q}$ or for understanding the Birch and Swinnerton-Dyer Conjecture which involves the Tate-Shafarevich group that is defined in terms of a Néron model over $\mathbb{Z}$ for an abelian variety over $\mathbb{Q}$. For further references regarding Néron models, consult the seminal work [BLR90].

We conjecture the existence of a local weak Néron model for a nilpotent orbit variety determined by a partition $\lambda$ of $n$, we find the maximum coefficient of the polynomials in $H_\lambda$ and $F_\lambda$ defined by

$$H_\lambda = \bigcup_{k=1}^{\max(\lambda)} H^k_\lambda = \left\{ h_{j,k} \in H^k_\lambda : \exists x \in \mathbb{K}[X] \left| x(P)|Q) \neq 0 \right| \mid P,Q \subseteq \{1, ..., n\}, |P| = |Q| = r_k \right\}$$

$$F_\lambda = \bigcup_{k=1}^{\max(\lambda)} F^k_\lambda = \left\{ f_{j,k} \in F^k_\lambda : \exists x \in \mathbb{K}[X] \left| x^k(P)|Q) = 0 \right| \mid P,Q \subseteq \{1, ..., n\}, |P| = |Q| = r_k + 1 \right\}$$

We define the coefficient projection function $\pi_t : \mathbb{K}[X] \to \mathbb{K}$ by $\pi_t(x) = c_{t,j,k}$, where

$$g(X) = g(x_{11}, ..., x_{nn}) = \sum_{t=1}^{\Omega_g} c_{t,j,k} \prod_{u=1}^{n} p_{t,u,v}^{x_{uv}}$$

is an arbitrary polynomial function with $c_{t,j,k} \in \mathbb{K}, p_{t,u,v} \in \mathbb{N} \cup \{0\}$ and

$$\Omega_g = \sum_{d=1}^{\deg(g)} \binom{d + n - 1}{n - 1}$$

For indexing the variables $x_{uv}$ in the polynomial ring $\mathbb{K}[X]$, we remark that $uv$ denotes the concatenation of $u$ and $v$ as natural numbers including zero, not the product of $u$ and $v$. We now define the set of coefficients of a polynomial $g \in \mathbb{K}[X]$ by

$$C_g = \{ \pi_t(g(X)) \mid 1 \leq t \leq \Omega_g \}$$

and remark that the problem of determining the maximum coefficient of the polynomials in $H_\lambda$ and $F_\lambda$ is then defined by

$$p > \max \left\{ \left( \max(\lambda) \binom{n}{r_k} \right)^2 \bigg| \prod_{k=1}^{\max(\lambda)} C_{h_{j,k}} \right\}$$

As such, the problem of bounding the value of $p$ in $\mathbb{Z}_p$ is reduced to evaluating this maximum. In order to solve this problem we present a lemma.
Lemma 3. Let $X$ be an $n \times n$ matrix. Then for each $i, j \in \{1, ..., n\}$ there are $(n-1)!$ occurrences of $x_{ij}$ in $\det(X)$.

Proof. We use the Leibniz formula for the determinant of an $n \times n$ matrix

$$\det(X) = \sum_{\sigma \in S_n} \text{sgn}(\sigma) \prod_{i=1}^{n} X_{i, \sigma(i)}$$

Let $x_{ij}$ be an arbitrary entry in $X$ and observe that for a fixed $\sigma \in S_n$ the entry $x_{i, \sigma(i)}$ appears exactly once in $\det(X)$. Then, since there are $(n-1)!$ permutations $\sigma \in S_n$ with the property that $\sigma(i) = j$ we have that $x_{ij}$ appears $(n-1)!$ times in $\det(X)$. Alternatively, since there are $n$ multiplicative terms in each additive term and $n!$ additive terms, there are $(n-1)!$ appearances of variables $x_{ij}$ for varying $i, j \in \{1, ..., n\}$. Since each $x_{ij}$ appears an equal number of times in $\det(X)$ we have that each particular $x_{ij}$ occurs $(n+1)!/n^2 = (n-1)!$ times in $\det(X)$.

With this fact we have the following corollary regarding embedding determinant equations in a polynomial ring with $p$-adic integer coefficients.

Corollary 1. For an $n \times n$ matrix with entries $x_{ij}$ in a field $\mathbb{K}$, we have $\det(X) \in \mathbb{Z}_p[X]$ with $p > (n-1)!$.

Proof. By Lemma 3 each $x_{ij}$ appears $(n-1)!$ times in $\det(X)$ and so there can be at most a coefficient of $(n-1)!$ for any $x_{ij}$ which implies that the image of $\det(X)$ is invariant under the map $\mathbb{K} \rightarrow \mathbb{Z}_p$ with $p > (n-1)!$. Hence, $\det(X) \in \mathbb{Z}_p[X]$ for $p > (n-1)!$.

Since the equations $h_{j,k} \in H_{\lambda}$ are expressed in terms of $r_k \times r_k$ minors and the equations in $f_{j,k} \in F_{\lambda}$ are expressed in terms of $(r_k + 1) \times (r_k + 1)$ minors, we immediately have that

$$\max \left\{ \max_{\lambda} \{ \binom{n}{r_k+1}^2 \} \right\} > \max \left\{ \max_{\lambda} \{ \binom{n}{r_k}^2 \} \right\}$$

and that

$$\max \{ r_k \mid 1 \leq k \leq \max_{\lambda} \} > \max \left\{ \max_{\lambda} \{ \binom{n}{r_k+1}^2 \} \right\}$$

since each $f_{j,k}$ is an $(r_k + 1) \times (r_k + 1)$ determinant function with the property by Corollary 1 that it embeds in $\mathbb{Z}_p[X]$ with $p > (r_k + 1)! = r_k!$. Therefore, we can bound the value of $p$ by

$$p > \max \{ r_k \mid 1 \leq k \leq \max_{\lambda} \}!$$

where $r_k = \text{rank}(X^k) = \sum_{i=1}^{l} f^k(\lambda_i)$ and

$$f(x) = \begin{cases} x - 1 & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$$

Future work will focus on the explicit construction of local weak Néron models for nilpotent orbit varieties, applying the Greenberg transform to these models, thus producing pro-schemes over finite fields with a remarkable property: the set of rational points on these pro-schemes is canonically identified with the set of rational points on nilpotent orbit varieties appearing in Lie algebras over local fields and global fields.
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