UNIFORM BOUNDEDNESS ON EXTREMAL SUBSETS IN ALEXANDROV SPACES

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Abstract. In this paper, we study extremal subsets in $n$-dimensional Alexandrov spaces with curvature $\geq \kappa$ and diameter $\leq D$. We show that the number of extremal subsets in an Alexandrov space, the Betti number of an extremal subset, and the volume of an extremal subset are uniformly bounded above by some constant depending only on $n$, $\kappa$, and $D$. The proof is an application of essential coverings introduced by T. Yamaguchi.

1. Introduction

Alexandrov spaces are metric spaces which have the notion of a lower curvature bound in the sense of comparison theorems in Riemannian geometry. Namely, geodesic triangles in these spaces are thicker than corresponding triangles in the plane of constant curvature with the same sidelengths. Alexandrov spaces naturally arise as limits of infinite sequences of Riemannian manifolds, or quotient spaces of Riemannian manifolds by isometric group actions. Hence, they have singular points in general. Extremal subsets are singular point sets in Alexandrov spaces defined by Perelman and Petrunin [PP1]. The following are typical examples: a point at which all angles are not greater than $\pi/2$; the projection of the fixed point set in the above quotient space of a Riemannian manifold; and the boundary of an Alexandrov space. To illustrate them, consider the quotient space of the three-dimensional closed unit ball by the $\pi$-rotation around the $z$-axis. In this case, the projections of the north pole, the south pole, the $z$-axis, and the boundary are extremal. Extremal subsets are closely related to stratifications of Alexandrov spaces. Thus, the study of extremal subsets is important to understand the singular structure of Alexandrov spaces. Although an extremal subset equipped with the induced intrinsic metric do not have a lower curvature bound generally, several important theorems on Alexandrov spaces also hold for extremal subsets (for instance, see [PP1], [Pet1], [K]).

The most remarkable advantage of considering Alexandrov spaces is that the family of Alexandrov spaces with dimension $\leq n$, curvature $\geq \kappa$ and diameter $\leq D$ is compact with respect to the Gromov-Hausdorff distance. Therefore, we can expect that there are various uniform bounds independent of each space. The main results of this paper are several uniform boundedness theorems for extremal subsets. Let $A(n, \kappa, D)$ denote the family of all isometry classes of $n$-dimensional Alexandrov spaces with curvature $\geq \kappa$ and diameter $\leq D$.
Theorem 1.1. For given \(n, \kappa\) and \(D\), there exists a constant \(C = C(n, \kappa, D)\) such that the following hold for any \(M \in \mathcal{A}(n, \kappa, D)\):

1. The number of extremal subsets in \(M\) is not greater than \(C\).
2. The total Betti number of any extremal subset \(E\) of \(M\) is not greater than \(C\).
3. The \(m\)-dimensional Hausdorff measure of any extremal subset \(E\) of \(M\) is not greater than \(C\), where \(m = \dim E\).

Our main tool is essential coverings and isotopy covering systems introduced by Yamaguchi [Y]. Let us illustrate them (note that we use a slightly stronger version for our applications). Consider a thin rectangle with length 1 and width \(\varepsilon \ll 1\). If one wants to cover this rectangle by metric balls on which each distance function from the center has no critical points, then one needs about \([\varepsilon^{-1}]\) balls. However, we can cover it by a few metric balls having a similar property as follows: First, cover this rectangle by the two metric balls of radius \(2/3\) centered at the midpoints of the short sides. Then, each distance function from the center has no critical points on the ball with the concentric ball of radius \(2\varepsilon/3\) removed. Second, cover these small balls of radius \(2\varepsilon/3\) by the four metric balls of radius \(5\varepsilon/6\) centered at the vertices. Then, these four balls are free of critical points of the respective distance functions from the centers. Such a multi-step covering is called an isotopy covering system, and also the four balls centered at the vertices are called an essential covering of this rectangle. The number of steps is called depth. Yamaguchi [Y] proved that for any \(M \in \mathcal{A}(n, \kappa, D)\), the minimal number of metric balls forming an essential covering of \(M\) with depth \(\leq n\) is uniformly bounded above by \(C(n, \kappa, D)\).

As regards Theorem 1.1(1), it is already known that every compact Alexandrov space has only finitely many extremal subsets ([PP1, 3.6]). Combining the original proof of this fact with the above properties of isotopy covering systems, we obtain uniform boundedness of the number of extremal subsets. We remark that this result is cited in [A] as a private communication of A. Petrunin.

Theorem 1.1(2) is an analog of uniform boundedness of the Betti numbers of Alexandrov spaces. Historically, Gromov [G1] first proved uniform boundedness of the Betti numbers of Riemannian manifolds. Liu and Shen [LS] generalized it to Alexandrov spaces, and Yamaguchi [Y] gave another proof using essential coverings. This proof is based only on Perelman’s stability theorem and fibration theorem. Thus it also works well for extremal subsets, since those two theorems hold for extremal subsets ([K §9], [PP1 §2]).

The proof of Theorem 1.1(3) is an application of isotopy covering systems and gradient exponential maps constructed in [PP2], which are extensions of exponential maps having comparison properties.

The optimal constants of Theorem 1.1 are partially known:

1. ([Per3 4.3]) The number of extremal points in an \(n\)-dimensional compact Alexandrov space with nonnegative curvature is at most \(2^n\). The classification of the maximal case was given by Lebedeva [L].
2. ([Pet2 3.3.5]) The volume of the boundary of an \(n\)-dimensional Alexandrov space with curvature \(\geq 1\) is not greater than that of the standard unit sphere of dimension \(n - 1\). It is not known whether the boundary of an Alexandrov space equipped with the induced intrinsic metric is an Alexandrov space (with the same lower curvature bound).
We remark that it is necessary to fix the dimension of Alexandrov spaces in Theorem 1.1 and it is not sufficient to fix only the dimension of extremal subsets. For example, consider the following $n$-dimensional Alexandrov space of curvature $\geq 1$:

$$M = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1^2 + \cdots + x_{n+1}^2 = 1, \ x_1, \ldots, x_n \geq 0\}.$$ 

Then, the minimal geodesic $\gamma_i$ between $(0, \ldots, 0, 1)$ and $(0, \ldots, 0, -1)$ passing through $(0, \ldots, \hat{i}, \ldots, 0, 0)$ is a one-dimensional extremal subset for each $1 \leq i \leq n$. Thus, (1) does not hold when $n \to \infty$, and also (2) and (3) do not hold for $E = \bigcup_{i=1}^{n} \gamma_i$.

For nonnegatively curved spaces, Theorem 1.1 (1) and (2) hold without the upper diameter bound $D$.

**Corollary 1.2.** For given $n$, there exists a constant $C(n)$ such that the following hold for any Alexandrov space $M$ with nonnegative curvature:

1. The number of extremal subsets in $M$ is not greater than $C(n)$.
2. The total Betti number of any extremal subset $E$ of $M$ is not greater than $C(n)$.

We can uniformly bound the number of $\varepsilon$-discrete points in an extremal subset with respect to the induced intrinsic metric, as well as Theorem 1.1 (3). As a corollary, we obtain a precompactness theorem for the family of extremal subsets with the induced intrinsic metrics. Let $\mathcal{E}(n, \kappa, D)$ denote the family of all isometry classes of connected extremal subsets of Alexandrov spaces in $A(n, \kappa, D)$ equipped with the induced intrinsic metrics.

**Corollary 1.3.** $\mathcal{E}(n, \kappa, D)$ is precompact with respect to the Gromov-Hausdorff distance.

From the above corollary, an infinite sequence of extremal subsets with the induced intrinsic metrics has a convergent subsequence. It is known that the limit is also an extremal subset with the induced intrinsic metric when the sequence of ambient spaces does not collapse: if Alexandrov spaces $M_i$ converge to $M$ without collapse and extremal subsets $E_i \subset M_i$ converge to $E \subset M$ as subsets, then the induced intrinsic metrics of $E_i$ converge to that of $E$ (Pet1, 1.2). However, it is unknown what the limit space is in the collapsing case.

**Organization.** The organization of this paper is as follows: In §2 we recall the basics of Alexandrov spaces and extremal subsets. In §3 we recall the definitions of essential coverings and isotopy covering systems, and review the main result of [Y]. In §4 we prove uniform boundedness of the numbers of extremal subsets in Alexandrov spaces (Theorem 1.1(1) and Corollary 1.2(1)). In §5 we prove uniform boundedness of the Betti numbers of extremal subsets (Theorem 1.1(2) and Corollary 1.2(2)). In §6 we prove uniform boundedness of the volumes of extremal subsets (Theorem 1.1(3) and Corollary 1.3).

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2. **Preliminaries**

2.1. **Alexandrov spaces.** We refer to [BGP] and [BBI] for the basics of Alexandrov spaces. Let us first recall the definition of Alexandrov spaces.
A geodesic space is a metric space such that every two points can be joined by a minimal geodesic. We assume that every minimal geodesic is parametrized by arclength. For \( \kappa \in \mathbb{R} \), \( \kappa \)-plane is the simply-connected complete surface of constant curvature \( \kappa \). For three points \( p, q \) and \( r \) in a geodesic space \( M \), consider a triangle on \( \kappa \)-plane with sidelengths \( |pq|, |pr| \) and \( |qr| \). We denote by \( \angle qpr \) the angle opposite to \( |qr| \) and call it a comparison angle at \( p \). A complete geodesic space \( M \) is called an Alexandrov space with curvature \( \geq \kappa \) if every point has a neighborhood \( U \) such that for any two minimal geodesics \( \gamma \) and \( \sigma \) in \( U \) starting at the same point \( p \), \( \angle \gamma(t)p\sigma(s) \) is nonincreasing in both \( t \) and \( s \). In this paper, we only deal with finite-dimensional Alexandrov spaces in the sense of Hausdorff dimension. It is known that the Hausdorff dimension of a finite-dimensional Alexandrov space is an integer.

From now, \( M \) denotes an \( n \)-dimensional Alexandrov space.

For two minimal geodesics \( \gamma \) and \( \sigma \) starting at \( p \), \( \lim_{t,s \to 0} \angle \gamma(t)p\sigma(s) \) always exists from the above monotonicity. It is called the angle between \( \gamma \) and \( \sigma \) and denoted by \( \angle(\gamma, \sigma) \). The angle \( \angle \) is a pseudo-distance on the space \( \Gamma_p \) consisting of all minimal geodesics starting at \( p \). The completion of the metric space induced by \( (\Gamma_p, \angle) \) is called the space of directions at \( p \) and denoted by \( \Sigma_p \). \( \Sigma_p \) is a compact \((n-1)\)-dimensional Alexandrov space with curvature \( \geq 1 \). For \( p, q \in M \), we denote by \( \gamma_p' \in \Sigma_p \) one of the directions of minimal geodesics from \( p \) to \( q \). Moreover, for a closed subset \( A \subset M \), we denote by \( A' \subset \Sigma_p \) the set of all directions of minimal geodesics from \( p \) to \( A \). The Euclidean cone \( K(\Sigma_p) \) over \( \Sigma_p \) is called the tangent cone at \( p \) and denoted by \( T_p \). (\( T_p, o \)) is also equal to the pointed Gromov-Hausdorff limit \( \lim_{\lambda \to \infty} (\lambda M, p) \), where \( o \) denotes the vertex of the cone. \( T_p \) is an \( n \)-dimensional Alexandrov space with nonnegative curvature.

The Gromov-Hausdorff limit of an infinite sequence of \( n \)-dimensional Alexandrov spaces with curvature \( \geq \kappa \) is an Alexandrov space with dimension \( \leq n \) and curvature \( \geq \kappa \). Let \( \mathcal{A}(n) \) denote the family of all isometry classes of \( n \)-dimensional Alexandrov spaces with curvature \( \geq -1 \), and \( \mathcal{A}(n, D) \) its restriction to all elements with diameter \( \leq D \). Moreover, let \( \mathcal{A}_p(n) \) denote the family of all isometry classes of \( n \)-dimensional pointed Alexandrov space with curvature \( \geq -1 \). The following property plays an important role in this paper.

**Theorem 2.1** ([BGP] §8, [BBI] 10.7.3, cf. [G2] 5.3). \( \mathcal{A}(n, D) \) (resp. \( \mathcal{A}_p(n) \)) is pre-compact with respect to the Gromov-Hausdorff distance (resp. the pointed Gromov-Hausdorff topology).

### 2.2. Extremal subsets

We refer to [PP1] and [Pet2] §4 for the basics of extremal subsets. Let us recall the definition of extremal subsets. For a point \( x \in M \), \( \text{dist}_x \) denotes the distance function \( d(x, \cdot) \).

**Definition 2.2.** A closed subset \( E \) of an Alexandrov space \( M \) is said to be extremal if the following condition is satisfied:

\[(*) \text{ If } \text{dist}_q |E \text{ has a local minimum at } p \in E \text{ for } q \notin E, \text{ then } p \text{ is a critical point of } \text{dist}_q, \text{ i.e. } \angle qpx \leq \pi/2 \text{ for all } x \in M. \]

Note that this definition includes the cases \( E = \emptyset, M \).

Moreover, only when we remark that \( \Sigma \) has curvature \( \geq 1 \) (e.g. a space of directions), a closed subset \( F \) of \( \Sigma \) is said to be extremal if it satisfies the following in addition to \((*)\):

1. If \( F = \emptyset \) then \( \text{diam } \Sigma \leq \pi/2 \);
2.3. Semiconcave functions, gradient curves and radial curves. The contents of this section are only needed to prove uniform boundedness of the volumes of extremal subsets in §6. We refer to [Pet2 §1–3], [PP2 §3] and [AKP] for more details.
Let $M$ be an Alexandrov space and $\Omega$ an open subset of $M$. First, suppose $M$ has no boundary. A (locally Lipschitz) function $f : \Omega \to \mathbb{R}$ is said to be $\lambda$-concave if for any minimal geodesic $\gamma(t)$ parametrized by arclength, $f(\gamma(t) - (\lambda/2)t^2)$ is concave. When $M$ has nonempty boundary, $f$ is said to be $\lambda$-concave if its tautological extension to the double of $M$ is $\lambda$-concave in the above sense. A function $f$ is said to be semiconcave if for any $p \in \Omega$, there exists $\lambda_p \in \mathbb{R}$ such that $f$ is $\lambda_p$-concave in some neighborhood of $p$. For example, $\text{dist}_x$ is a semiconcave function on $M \setminus \{x\}$.

For a semiconcave function $f : \Omega \to \mathbb{R}$ and $p \in \Omega$, the differential $d_p f : T_p \to \mathbb{R}$ of $f$ at $p$ is defined by $d_p f := \lim_{\lambda \to \infty} \lambda (f - f(p))$, where $\lambda (f - f(p))$ is defined on $\lambda M$ and the limit is taken under the convergence $(\lambda M, p) \to (T_p, 0)$. For example, $d_p \text{dist}_x(v) = -|v| \cos \min_{px} \angle (x', v)$ for $v \in T_p$, where $px$ runs over all minimal geodesics from $p$ to $x$. Since $d_p f$ is concave and positively homogeneous, $|d_p f|_{\Sigma_p}$ has a unique maximum at some $\xi_{\max} \in \Sigma_p$ if $\max d_p f|_{\Sigma_p} > 0$. Thus, we can define the gradient $\nabla_p f \in T_p$ of $f$ at $p$ by

$$\nabla_p f := \begin{cases} d_p f(\xi_{\max}) \xi_{\max} & \text{if } \max d_p f|_{\Sigma_p} > 0, \\ o & \text{if } \max d_p f|_{\Sigma_p} \leq 0. \end{cases}$$

A point $p$ is called a critical point of $f$ if $\nabla_p f = o$. Note that $|\nabla f|$ is lower semicontinuous. Hence the limit of critical points is a critical point.

For a semiconcave function $f : M \to \mathbb{R}$, a curve $\alpha(t)$ in $M$ satisfying

$$\alpha^+(t) = \nabla_{\alpha(t)} f$$

is called a gradient curve of $f$. It is known that for any $\lambda$-concave function $f$ and $p \in M$, there exists a unique gradient curve $\alpha_p : [0, \infty) \to M$ of $f$ with $\alpha_p(0) = p$. We define the gradient flow $\Phi^t f : M \to M$ of a semiconcave function $f$ by $\Phi^t f(p) := \alpha_p(t)$, where $t \geq 0$ (note that $\Phi^t f$ is not defined on all of $M$ in general). $\Phi^t f$ is locally Lipschitz continuous and clearly satisfies $\Phi^t f \circ \Phi^s f = \Phi^{t+s} f$. If $p$ belongs to an extremal subset $E \subset M$, then $\nabla_p f \in T_p E$. As a result, any gradient curve starting at a point of $E$ lies in $E$, that is, $\Phi^t f(E) \subset E$ for any $f$. Conversely, every subset having such a property is extremal.

Below we assume $\kappa = -1$ for simplicity. For $p \in M$ and $\xi \in \Sigma_p$, consider a curve $\beta_\xi : [0, \infty) \to M$ satisfying the following differential equation:

$$\beta_\xi^+(s) = \frac{\tanh |p\beta_\xi(s)|}{\tanh s} \nabla_{\beta_\xi(s)} \text{dist}_p,$$

$$\beta_\xi(0) = p, \quad \beta^+_\xi(0) = \xi.$$  

We call it the radial curve starting at $p$ in the direction $\xi$. It is known that there exists a unique radial curve for any initial data $(p, \xi)$. If there is a minimal geodesic, then the radial curve starting in the same direction coincides with it. Radial curves starting at $p$ are reparametrizations of gradient curves of the semiconcave function $f = \cosh \circ \text{dist}_p - 1$. Let $t(t)$ be the gradient curve of $f$ starting at $x$ and $\beta(s)$ the radial curve starting at $p$ in a direction $x'_p$. Then the relation between both parameters is described by

$$\frac{dt}{ds} = \frac{1}{\tanh s \cosh |p\beta(s)|},$$

where $t \geq 0$ and $s \geq |px|$. If $p$ belongs to an extremal subset $E \subset M$, then the radial curve starting at $p$ in any direction $\xi \in \Sigma_p E$ lies in $E$. 

To explain the properties of radial curves, let us define comparison angles for 1-Lipschitz curves. For a 1-Lipschitz curve $c$, consider a triangle on $\kappa$-plane with sidelengths $|pc(t_1)|, |t_2 - t_1|$ and $|pc(t_2)|$. We denote by $\tilde{Z}pc(t_1) - c(t_2)$ the angle opposite to $|pc(t_2)|$. Furthermore, for 1-Lipschitz curves $c_1$ and $c_2$ with $c_1(0) = c_2(0) = p$, consider a triangle on $\kappa$-plane with sidelengths $|t_1|, |t_2|$ and $|c_1(t_1)c_2(t_2)|$. We denote by $\tilde{Z}c_1(t_1) - p - c_2(t_2)$ the angle opposite to $|c_1(t_1)c_2(t_2)|$. In case such a triangle does not exist, we assume the comparison angle is equal to 0. Then, the following monotonicity of comparison angles holds for radial curves.

**Proposition 2.4 ([PP2 3.3, 3.3.3]).**  
(1) For the radial curve $\beta_\xi$ starting at $p$ in a direction $\xi \in \Sigma_p$ and $q \in M$, $\tilde{Z}qp - \beta_\xi(s)$ is nonincreasing in $s$. In particular, $\tilde{Z}qp - \beta_\xi(s) \leq \min_{pq} \tilde{Z}(q', \xi)$.

(2) For two radial curves $\beta_1$ and $\beta_2$ starting at $p$ such that $\beta_1|[0,a_1]$ and $\beta_2|[0,a_2]$ are minimal geodesics, we have $\tilde{Z}\beta_1(s_1) - p - \beta_2(s_2) \leq \tilde{Z}\beta_1(a_1)p\beta_2(a_2)$ whenever $s_1 \geq a_1$ and $s_2 \geq a_2$.

Now, we define the gradient exponential map $\exp_p, T_p \to M$ at $p$ by $\exp_p(s\xi) := \beta_\xi(s)$. Then, $\exp_p$ is the extension of $\exp_p$ (defined by minimal geodesics), and is surjective. Moreover, by the second statement of the above proposition, $\exp_p$ is a 1-Lipschitz map from $(T_p, h)$ to $M$. Here, $h$ denotes the metric on $T_p$ defined by the hyperbolic law of cosines instead of the Euclidean one (i.e. the elliptic cone over $\Sigma_p$). If $p$ belongs to an extremal subset $E \subset M$, then $\exp_p(T_pE) \subset E$. However, $\exp_p|T_pE : T_pE \to E$ is not surjective in general. Later we will give a sufficient condition for surjectivity (see Proposition 6.2).

**Remark 2.5.** There is another definition of radial curves when $\kappa = -1$. Namely, we can replace the differential equation (2.1) by a simpler (and slower) one

$$\beta_\xi^+(s) = \frac{\sinh |p\beta_\xi(s)|}{\sinh s} \nabla_{\beta_\xi(s)} \text{dist}_p.$$

Then, Proposition 2.4 also holds for these curves (see [AKP]). However, we need the faster one (2.1) for our application.

3. Essential coverings and isotopy covering systems

From now on, we assume that the lower curvature bound is equal to $-1$. In this section, we recall the notion of essential coverings and isotopy covering systems introduced by Yamaguchi [Y]. The following theorem plays a key role throughout this paper. For $0 < r_1 < r_2$, $A(p;r_1,r_2)$ denotes the closed annulus $B(p,r_2) \setminus B(p,r_1)$.

**Theorem 3.1 ([Y 3.2]).** Let $(M_i, p_i) \in \mathcal{A}_p(n)$ converge to an Alexandrov space $(X,p)$ with dimension $\geq 1$. Then, for sufficiently small $r > 0$, there exists $\hat{p}_i \in M_i$ converging to $p$ such that either (1) or (2) holds:

(1) There is a subsequence $\{j\} \subset \{i\}$ such that $\text{dist}_{\hat{p}_j}$ has no critical points on $B(\hat{p}_j,r) \setminus \{\hat{p}_j\}$.

(2) There exists a sequence $\delta_i \to 0$ such that

(i) for any $\lambda > 1$ and sufficiently large $i$, $\text{dist}_{\hat{p}_i}$ has no critical points on $A(\hat{p}_i; \lambda \delta_i, r)$;

(ii) for any limit $(Y, y_0)$ of $(\frac{1}{\lambda_i}M_i, \hat{p}_i)$, we have $\dim Y \geq \dim X + 1$.

In particular, if $\dim X = n$, then (1) holds for all sufficiently large $i$. 

**Proof.**
We remark that $\delta_i$ is the maximum distance between $\hat{p}_i$ and critical points of $\text{dist}_{\hat{p}_i}$ in $\hat{B}(\hat{p}_i, r) \setminus \{\hat{p}_i\}$ for sufficiently large $i$.

**Example 3.2.** Let $S^1_\varepsilon$ denote the circle of length $\varepsilon$. Consider a collapsing sequence $(K(S^1_\varepsilon), \partial) \xrightarrow{\text{GH}} (\mathbb{R}_+, 0)$ as $\varepsilon \to 0$. If we take $p = 0 \in \mathbb{R}_+$, then we can take $\hat{p}_i = \partial \in K(S^1_\varepsilon)$ so that (1) holds. On the other hand, for a collapsing sequence $\mathbb{R} \times S^1_\varepsilon \xrightarrow{\text{GH}} \mathbb{R}$, we can take $\delta_\varepsilon = \varepsilon/2$ so that (2) holds.

We omit the proof of Theorem 3.1 but later we will prove a somewhat strong version of it (see Theorem 6.4).

**Remark 3.3.** The choice of $r$ depends only on the limit space $X$ (and the dimension $n$), but not the sequence $M_i$.

Now, we give the definitions of essential coverings and isotopy covering systems. Although the definitions below are slightly stronger than the original ones, they are essentially obtained in [Y]. Note that we use the same terminology as in [Y] with different meanings.

Let $M$ be an Alexandrov space. For an open metric ball $B \subset M$ centered at $p$, we call a concentric open metric ball $\hat{B} \subset B$ an isotopic subball of $B$ if $\text{dist}_p$ has no critical points on the annulus $\hat{B} \setminus \hat{B}$. Consider a family of open metric balls $B = \{B_{\alpha_1 \cdots \alpha_k}\}$, where

$$1 \leq \alpha_1 \leq N_1, \quad 1 \leq \alpha_2 \leq N_2(\alpha_1), \quad \ldots, \quad 1 \leq \alpha_k \leq N_k(\alpha_1 \cdots \alpha_{k-1})$$

and $1 \leq k \leq l$ for some $l$ depending on $\alpha_1, \alpha_2, \ldots$. We call $N_1$ the first degree of $B$ and $N_k(\alpha_1 \cdots \alpha_{k-1})$ the $k$-th degree of $B$ with respect to $\alpha_1 \cdots \alpha_{k-1}$. Let $A$ be the set of all multi-indices $\alpha_1 \cdots \alpha_k$ such that $B_{\alpha_1 \cdots \alpha_k} \in B$, and $\hat{A}$ the set of all maximal multi-indices in $A$. Here, $\alpha_1 \cdots \alpha_i$ is maximal if there are no $\alpha_{i+1}$ with $\alpha_1 \cdots \alpha_i \alpha_{i+1} \in A$. For each $\alpha = \alpha_1 \cdots \alpha_k \in \hat{A}$, we put $|\alpha| := k$.

**Definition 3.4.** Let $X$ be a subset of $M$. We call $B$ an isotopy covering system of $X$ if it satisfies the following conditions:

1. $\{B_{\alpha_1}\}_{\alpha_1=1}^{N_1}$ covers $X$;
2. $\{B_{\alpha_1 \cdots \alpha_k}\}_{\alpha_k=1}^{N_k(\alpha_1 \cdots \alpha_{k-1})}$ covers an isotopic subball $\hat{B}_{\alpha_1 \cdots \alpha_{k-1}}$ of $B_{\alpha_1 \cdots \alpha_{k-1}}$;
3. for each $\alpha \in \hat{A}$, $\text{dist}_{p_\alpha}$ has no critical points on $\hat{B}_\alpha \setminus \{p_\alpha\}$, where $p_\alpha$ is the center of $B_\alpha$;
4. there is a uniform bound $d$ such that $|\alpha| \leq d$ for all $\alpha \in \hat{A}$.

In this case, we also call $U = \{B_\alpha\}_{\alpha \in \hat{A}}$ an essential covering of $X$. In addition, we call $d_0 = \max_{\alpha \in \hat{A}} |\alpha|$ the depth of both $B$ and $U$.

**Remark 3.5.** Perelman’s stability theorem and fibration theorem show that the definition above is stronger than that of an “isotopy covering system modeled on $\mathcal{C}(n)$” in [Y]. In particular, $B_\alpha$ is contractible if $\alpha \in \hat{A}$ and is homeomorphic to $\hat{B}_\alpha$ if $\alpha \in A \setminus \hat{A}$ (cf. Theorem 5.1).

For a positive integer $d$, we denote by $\tau_d(X)$ the minimal number of metric balls forming an essential covering of $X$ with depth $\leq d$. Furthermore, for an open metric ball $B \subset M$ having a proper isotopic subball, we set

$$\tau^*_d(B) := \min_{\hat{B}} \tau_d(\hat{B}),$$
where \( \hat{B} \) runs over all isotopic subballs of \( B \). In addition, if \( \text{dist}_p \) has no critical points on \( \hat{B} \setminus \{p\} \), where \( p \) is the center of \( B \), we put \( \tau^*_0(B) := 1 \); otherwise \( \tau^*_0(B) := \infty \). Then, if \( X \) is covered by open metric balls \( \{B_{\alpha_i}\}_{\alpha_i=1}^{N_i} \) having proper isotopic subballs, we have

\[
\tau_d(X) \leq \sum_{\alpha_i=1}^{N_i} \tau^*_d(B_{\alpha_i})
\]

for any \( d \geq 1 \).

**Example 3.6.** For \( 0 < \varepsilon \ll 1 \), consider a thin \( n \)-dimensional rectangular parallellepiped

\[
I^n_\varepsilon = [0, 1] \times [0, \varepsilon] \times [0, \varepsilon^2] \times \cdots \times [0, \varepsilon^{n-1}].
\]

Then, as we have seen in [14] metric balls of radii slightly less than \( \varepsilon^{n-1} \) centered at the vertices form an essential covering of \( I^n_\varepsilon \) with depth \( n \). Thus \( \tau_n(I^n_\varepsilon) \leq 2^n \) for any \( \varepsilon \), whereas \( \lim_{\varepsilon \to 0} \tau_{n-1}(I^n_\varepsilon) = \infty \). Note that the faces of each dimension are extremal subsets.

The following theorem is the main result of [Y].

**Theorem 3.7 ([Y, 4.4]).** For given \( n \) and \( D \), there exists a constant \( C(n, D) \) such that for any \( M \in A(n) \) and \( p \in M \), we have \( \tau_n(B(p, D)) \leq C(n, D) \).

**Remark 3.8.** [Y, 4.4] states further that there exists an isotopy covering system of \( B(p, D) \) whose first degree is bounded above by \( C(n, D) \) and whose other higher degrees are bounded above by some constant \( C(n) \) independent of \( D \). However, we only need the above weaker version for our applications.

Let us review the proof.

**Proof.** Fix \( n \) and take \( 1 \leq k \leq n \). We prove the following two statements by the reverse induction on \( k \).

\( (P_k) \) Let \( (M_i, p_i) \in A_p(n) \) converge to a \( k \)-dimensional Alexandrov space \( (X, p) \). Then, we have

\[
\lim_{i \to \infty} \inf \tau_{n-k+1}(B(p_i, D)) < \infty.
\]

\( (Q_k) \) Let \( (M_i, p_i) \in A_p(n) \) converge to a \( k \)-dimensional Alexandrov space \( (X, p) \). Then, for sufficiently small \( r > 0 \), there exists a sequence \( \hat{p}_i \in M_i \) converging to \( p \) such that

\[
\lim_{i \to \infty} \inf \tau_{n-k}^*(B(\hat{p}_i, r)) < \infty.
\]

We remark that the radius \( r \) in \( (Q_k) \) is the one in Theorem 3.1 and thus depends only on the limit space \( X \). Note that \( (P_k) \) is global whereas \( (Q_k) \) is local. The proof is carried out in the order of \( (Q_n), (P_n), (Q_{n-1}), (P_{n-1}), \ldots, (Q_1), (P_1) \).

(\( Q_n \)) trivially follows from Theorem 3.1. Let us prove \( (Q_k) \Rightarrow (P_k) \). Suppose that \( (P_k) \) does not hold. Then, there exists \( (M_i, p_i) \in A_p(n) \) converging to \( (X, p) \) with \( \text{dim } X = k \) such that \( \lim_{i \to \infty} \tau_{n-k+1}(B(p_i, D)) = \infty \). By compactness, we can cover \( B(p, D) \) by finitely many balls \( \{B(x_\alpha, r_\alpha/2)\}_{\alpha=1}^N \), where \( r_\alpha \) is the one in \( (Q_k) \). Then, there exist a subsequence \( \{j\} \) and a constant \( C \) such that \( \tau_{n-k}^*(B(\hat{x}_j, r_\alpha)) \leq C \) for every \( \alpha \) and some \( \hat{x}_j \to x_\alpha \). Since \( \{B(\hat{x}_j, r_\alpha)\}_{\alpha=1}^N \) is a covering of \( B(p_j, D) \) for sufficiently large \( j \), we have \( \tau_{n-k+1}(B(p_j, D)) \leq NC \). This contradicts the assumption.
Next we prove \((P_n), \ldots, (P_{k+1}) \Rightarrow (Q_k)\). Let \((M_i, p_i) \in \mathcal{A}_p(n)\) converge to \((X, p)\) with \(\dim X = k\). By Theorem 3.1 for sufficiently small \(r > 0\), there exists \(\hat{p}_i \to p\) such that either (1) or (2) holds. When (1) holds, the claim is trivial. When (2) holds, there exists \(\delta_i \to 0\) satisfying both (i) and (ii). Passing to a subsequence \(\{j\}\), we may assume that \((\frac{1}{\delta_j} M_j, \hat{p}_j)\) converges to \((Y, y_0)\). Then we have \(l := \dim Y \geq \dim X + 1\). Applying \((P_l)\) to \(\frac{1}{\delta_j}B(\hat{p}_j, 2\delta_j)\) and passing to a subsequence again, we have

\[
\tau_{n-l+1}(\frac{1}{\delta_j}B(\hat{p}_j, 2\delta_j)) \leq C
\]

for some constant \(C\). Since \(B(\hat{p}_j, 2\delta_j)\) is an isotopic subball of \(B(\hat{p}_j, r)\) for sufficiently large \(j\), we obtain

\[
\tau_{n-k}^i(B(\hat{p}_j, r)) \leq \tau_{n-k}(B(\hat{p}_j, 2\delta_j)) \leq C.
\]

Now, Theorem 3.7 clearly follows from \((P_1), \ldots, (P_n)\) by contradiction. Note that the case \(\dim X = 0\) follows from the case \(\dim X \geq 1\) by rescaling \(M_i\) with the reciprocal of its diameter. \(\square\)

4. NUMBERS OF EXTREMAL SUBSETS IN ALEXANDROV SPACES

In this section, we prove Theorem 1.1(1) and Corollary 1.2(1).

For a subset \(X\) of an Alexandrov space \(M\), we define \(\nu(X)\) as follows:

\[
\nu(X) := \# \left( \{ E : \text{an extremal subset of } M \} / \sim \right),
\]

where \(E \sim E' \iff X \cap E = X \cap E'\), i.e. the number of extremal subsets in \(M\) counted by ignoring the differences outside \(X\). If \(X\) is covered by \(\{X_\alpha\}_{\alpha=1}^N\), then clearly

\[
\nu(X) \leq \prod_{\alpha=1}^N \nu(X_\alpha).
\]

The following lemma was essentially proved in [PP1] to show boundedness of the number of extremal subsets in a compact Alexandrov space. It controls the behavior of \(\nu\) on balls of isotopy covering systems.

**Lemma 4.1** (cf. [PP1] 3.6). Let \(M\) be an Alexandrov space and \(p \in M\).

1. If \(\text{dist}_p\) has no critical points on \(B(p, r) \setminus \{p\}\), then \(\nu(B(p, r)) \leq \nu(\Sigma_p) + 1\). Here, \(\nu(\Sigma_p)\) denotes the number of extremal subsets in \(\Sigma_p\) regarded as a space of curvature \(\geq 1\) (see Definition 2.9).

2. If \(\text{dist}_p\) has no critical points on \(A(p; r_1, r_2)\), then \(\nu(B(p, r_1)) = \nu(B(p, r_2))\).

**Proof.** First we show (2). Suppose \(\nu(B(p, r_1)) < \nu(B(p, r_2))\) and take two extremal subsets \(E, F \subset M\) such that

\[
B(p, r_1) \cap E = B(p, r_1) \cap F \text{ and } B(p, r_2) \cap E \neq B(p, r_2) \cap F.
\]

We may assume that \(B(p, r_2) \cap (E \setminus F) \neq \emptyset\). Then, \(G := E \setminus F\) is an extremal subset satisfying \(B(p, r_1) \cap G = \emptyset\) and \(B(p, r_2) \cap G \neq \emptyset\). Therefore, a closest point \(q \in G\) from \(p\) lies in \(A(p; r_1, r_2)\). However, by extremality of \(G\), \(q\) must be a critical point of \(\text{dist}_p\). This contradicts the assumption.

Next we show (1). Observe that by the assumption and (2), every extremal subset intersecting \(B(p, r)\) contains \(p\). Let \(E, F \subset M\) be two extremal subsets
intersecting $B(p, r)$ such that $\Sigma_p E = \Sigma_p F$. Then $\Sigma_p (E \setminus F) = \Sigma_p E \setminus \Sigma_p F = \emptyset$ and $\Sigma_p (F \setminus E) = \emptyset$ (see \ref{22}). Therefore, $E$ and $F$ coincide on a sufficiently small neighborhood of $p$. Now again by \ref{22}, we see that $B(p, r) \cap E = B(p, r) \cap F$. Thus we can conclude $\nu(B(p, r)) \leq \nu(\Sigma_p) + 1$. Note that there are extremal subsets not intersecting $B(p, r)$. □

**Remark 4.2.** The equality in \ref{1} does not hold generally.

Theorem \ref{37} and Lemma \ref{41} imply uniform boundedness of the numbers of extremal subsets.

**Theorem 4.3.** For given $n$ and $D$, there exists a constant $C(n, D)$ such that for any $M \in \mathcal{A}(n)$ and $p \in M$, we have $\nu(B(p, D)) \leq C(n, D)$.

**Proof.** We use the induction on $n$. By Theorem \ref{37}, there exists an isotopy covering system $\mathcal{B} = \{B_{\alpha_1\cdots\alpha_k}\}$ of $B(p, D)$ with depth $\leq n$ whose degrees are bounded above by $C(n, D)$. Let $\mathcal{U} = \{B_{\alpha}\}_{\alpha \in \tilde{A}}$ be the essential covering associated with $\mathcal{B}$.

First, we prove by the reverse induction on $k$ that $\nu(B_{\alpha_1\cdots\alpha_k}) \leq C(n, D)$ for $\alpha = \alpha_1 \cdots \alpha_l \in \tilde{A}$ and $1 \leq k \leq l$. In the case $k = l$, this follows from the hypothesis of the induction on $n$ and Lemma \ref{41}. Consider the case $k < l$. Recall that $\{B_{\alpha_1\cdots\alpha_{k+1}}\}_{\alpha_{k+1} = 1}^{\alpha_{k+1}}$ is a covering of an isotopic subball $\tilde{B}_{\alpha_1\cdots\alpha_k}$ of $B_{\alpha_1\cdots\alpha_k}$. On the other hand, $\nu(B_{\alpha_1\cdots\alpha_{k+1}}) \leq C(n, D)$ for every $1 \leq \alpha_{k+1} \leq N_{k+1}(\alpha_1 \cdots \alpha_k)$ by the hypothesis of the reverse induction. Therefore, by Lemma \ref{41}, we have

$$\nu(B_{\alpha_1\cdots\alpha_k}) = \nu(\tilde{B}_{\alpha_1\cdots\alpha_k}) \leq \prod_{\alpha_{k+1} = 1}^{N_{k+1}(\alpha_1 \cdots \alpha_k)} \nu(B_{\alpha_1\cdots\alpha_{k+1}}) \leq C(n, D)^{C(n, D)}.$$

Finally, since $\{B_{\alpha_1}\}_{\alpha_1 = 1}$ is a covering of $B(p, D)$, we obtain $\nu(B(p, D)) \leq C(n, D)$. □

Corollary \ref{41} immediately follows from the following lemma.

**Lemma 4.4.** Let $M$ be a noncompact Alexandrov space with nonnegative curvature and $p \in M$. Then, dist$_p$ has no critical point on $M \setminus B(p, R)$ for sufficiently large $R > 0$.

**Proof.** For nonnegatively curved space, we have $\lim_{\lambda \to 0}(\lambda M, p) = (K(M(\infty)), o)$, where $M(\infty)$ denotes the ideal boundary of $M$ (see \cite{Sh} 1.1). Since dist$_p$ has no critical points on $K(M(\infty)) \setminus \{o\}$, so does dist$_p$ on $M \setminus B(p, \lambda^{-1})$ for sufficiently small $\lambda$. □

**Proof of Corollary \ref{41}.** Let $M$ be an $n$-dimensional (noncompact) Alexandrov space with nonnegative curvature and $p \in M$. By rescaling, we can take the constant in Theorem \ref{43} independent of $D$. Namely, there exists $C(n)$ such that $\nu(B(p, D)) \leq C(n)$ for any $D > 0$. On the other hand, Lemma \ref{43} and Lemma \ref{41} imply that the number of extremal subsets does not increase outside sufficiently large $B(p, R)$. Thus we have $\nu(M) \leq C(n)$. □

**Remark 4.5.** As stated in \ref{41}, the number of extremal points in an $n$-dimensional compact Alexandrov space with nonnegative curvature is at most $2^n$ \cite{Per3 4.3}. On the other hand, it is conjectured that $\tau_n \leq 2^n$ for such spaces \cite{Y 4.8}. Note that by our Definition \ref{43} if an extremal point exist, then it must be the center of a metric ball of an essential covering.
5. Betti numbers of extremal subsets

In this section, we prove Theorem 1.1(2) and Corollary 1.2(2). We need the
stability theorem and the fibration theorem for extremal subsets (see §2.2). The
following is a special case of these two theorems.

**Theorem 5.1** ([K] §9, PPI §2, cf. Per1, Per2). Let $M$ be an Alexandrov space,
$E \subset M$ an extremal subset, and $p \in M$.

1. If $\text{dist}_p$ has no critical points on $\overline{B}(p,r) \setminus \{p\}$, then $B(p,r) \cap E$ is home-
omorphic to $T_pE$. Note that if $B(p,r) \cap E \neq \emptyset$, then $p \in E$ (see Lemma
§4.1).

2. If $\text{dist}_p$ has no critical points on $A(p;r_1,r_2)$, then $A(p;r_1,r_2) \cap E$ is home-
omorphic to $\partial B(p,r_1) \cap E \times [0,1]$.

The following lemma was used in the original work [G1] of Gromov on the Betti
numbers of extremal subsets.

**Lemma 5.2** ([G1, Appendix], §5.4). Let $B^i_\alpha$, $1 \leq \alpha \leq N$, $0 \leq i \leq n+1$, be open
subsets of a topological space $X$ with $B^i_\alpha \subset B^{i+1}_\alpha$. Set $A^i = \bigcup_{\alpha=1}^N B^i_\alpha$. Below we
only consider homology groups of dimension $\leq n$. For each $\mu = (\alpha_1, \ldots, \alpha_m)$, let $f^i_\mu : H_*(B^{i+1}_m \cap \cdots \cap B^i_m) \to H_*(B^{i+1}_m \cap \cdots \cap B^i_m)$ be the inclusion homeomorphism.
Then, the rank of the inclusion homeomorphism $H_*(A^0) \to H_*(A^{n+1})$ is bounded
above by the sum

$$\sum_{0 \leq i \leq n, \mu} \text{rank} f^i_\mu.$$

Note that if $B^i_{\alpha_1} \cap \cdots \cap B^i_{\alpha_m} = \emptyset$, then we put $\text{rank} f^i_\mu = 0$.

Using the above theorem and lemma, we show uniform boundedness of Betti
numbers of extremal subsets. The proof is exactly the same as for Alexandrov
spaces in [K] §5]. Let $\beta(\ ; \mathcal{F})$ denotes the total Betti number $\sum_{i=0}^\infty b_i(\ ; \mathcal{F})$ with a
coefficient field $\mathcal{F}$.

**Theorem 5.3.** For given $n$ and $D$, there exists a constant $C(n,D)$ such that for
any $M \in A(n)$ and extremal subset $E \subset M$ with diameter $\leq D$, we have $\beta(E; \mathcal{F}) \leq C(n,D)$, where $\mathcal{F}$ is an arbitrary field.

Note that $b_i(E) = 0$ for all $i > m = \dim E$. Hence, we only consider homology
groups of dimension $\leq m$ in the following proof. For an open metric ball $B$ of radius
$r$, $\lambda B$ denotes a concentric open metric ball of radius $\lambda r$. In addition, for a subset
$X$ of $M$, $X|_E$ denotes $X \cap E$.

**Proof.** By Theorem 3.7 we can take an isotopy covering system $\mathcal{B} = \{B_{\alpha_1, \ldots, \alpha_k}\}$
of $E$ with depth $\leq n$ such that $N_k \leq C(n,D)$ for all $k$. Put $\lambda_k := 10^k$ and
$B^i_{\alpha_1, \ldots, \alpha_k} := \lambda_k B_{\alpha_1, \ldots, \alpha_k}$ for $0 \leq i \leq m + 1$. In view of Theorem 3.1(2)(i) and
the proof of Theorem 5.7 we may assume in addition that

- $B^{m+1}_{\alpha_1, \ldots, \alpha_k} \subset B_{\alpha_1, \ldots, \alpha_{k-1}}$ for $1 \leq \alpha_k \leq N_k(\alpha_1 \cdots \alpha_{k-1})$;
- $B^i_{\alpha_1, \ldots, \alpha_k}$ is an isotopic subball of $B^{i+1}_{\alpha_1, \ldots, \alpha_k}$ for $0 \leq i \leq m$.

Let $\mathcal{U} = \{B_\alpha\}_{\alpha \in \bar{A}}$ be the essential covering associated with $\mathcal{B}$.

We first prove by the reverse induction on $k$ that

$$\beta(B_{\alpha_1, \ldots, \alpha_k}|_E) \leq C(n,D).$$
for \( \alpha = \alpha_1 \cdots \alpha_l \in \hat{A} \) and \( 1 \leq k \leq l \). The case \( k = l \) is clear from Theorem 5.1(1). Consider the case \( k \leq l - 1 \). Recall that \( \{B_{\alpha_1 \cdots \alpha_{k+1}}\}_{k+1} \) is a covering of an isotopic subball \( \hat{B}_{\alpha_1 \cdots \alpha_k} \) of \( \hat{B}_{\alpha_1 \cdots \alpha_k} \). Fix \( (\alpha_1, \ldots, \alpha_k) \) and put

\[
B := B_{\alpha_1 \cdots \alpha_k}, \quad \hat{B} := \hat{B}_{\alpha_1 \cdots \alpha_k}, \quad B_a := B_{\alpha_1 \cdots \alpha_k a}, \quad B_\alpha^i := \lambda_i B_a
\]

for each \( 1 \leq \alpha \leq N_{k+1} \) and \( 0 \leq i \leq m + 1 \). Set \( A^i := \bigcup_{\alpha=1}^{N_{k+1}} B_\alpha^i \). From the inclusions \( \hat{B}|_E \subset A^0|_E \subset A^{m+1}|_E \subset B|_E \) and Theorem 5.1(2), we have

\[
\beta(\hat{B}|_E) = \beta(B|_E) \leq \text{rank } [H_*(A^0|_E) \to H_*(A^{m+1}|_E)].
\]

Now, we estimate the right hand side of the above inequality. Take \( \mu = (\gamma_1, \ldots, \gamma_\ell) \) such that \( B_{\gamma_1}^i \cap \cdots \cap B_{\gamma_\ell}^i \neq \emptyset \). Let \( B_{\gamma_\ell} \) have minimal radius among \( \{B_{\gamma_\ell}^i\}_{i=1}^l \). Then, the following inclusions hold:

\[
B_{\gamma_1}^i \cap \cdots \cap B_{\gamma_\ell}^i \subset B_{\gamma_\ell}^{i+1} \subset B_{\gamma_1}^{i+1} \cap \cdots \cap B_{\gamma_\ell}^{i+1}.
\]

Let \( f_{\mu} : H_*(\{B_{\gamma_1}^i \cap \cdots \cap B_{\gamma_\ell}^i\}|_E) \to H_*(\{B_{\gamma_1}^{i+1} \cap \cdots \cap B_{\gamma_\ell}^{i+1}\}|_E) \) be the inclusion homeomorphism. Then, we have

\[
\text{rank } f_{\mu} \leq \text{rank } [H_*(B_{\gamma_1}^{i+1}|_E) \to H_*(B_{\gamma_\ell}^{i+1}|_E)] = \beta(B_{\gamma_\ell}|_E) \leq C(n, D)
\]

by Theorem 5.1(2) and the induction hypothesis. Therefore, by Lemma 5.2 we obtain

\[
\text{rank } [H_*(A^0|_E) \to H_*(A^{m+1}|_E)] \leq (m + 1)2^{C(n, D)}.\]

Finally, since \( E = \bigcup_{\alpha=1}^{N_1} B_{\alpha_1}|_E = \bigcup_{\alpha=1}^{N_1} B_{\alpha_1}^{m+1}|_E \), applying Lemma 5.2, we conclude \( \beta(E) \leq C(n, D) \).

Corollary 1.2(2) immediately follows from Lemma 4.4.

Proof of Corollary 1.2(2). Let \( M \) be an \( n \)-dimensional Alexandrov space with non-negative curvature, \( E \subset M \) a (noncompact) extremal subset, and \( p \in M \). Then, we can show that

\[
\text{rank } [H_*(B(p, D) \cap E) \to H_*(E)] \leq C(n)
\]

for any \( D > 0 \). Indeed, take an isotopy covering system of \( B(p, D) \) instead of \( E \), and repeat the above argument. Only the last part of the proof is slightly different: we estimate the rank of the inclusion homeomorphism \( H_*(B(p, D) \cap E) \to H_*(E) \) from the inclusions \( B(p, D) \cap E \subset \bigcup_{\alpha_1=1}^{N_1} B_{\alpha_1}|_E \subset \bigcup_{\alpha_1=1}^{N_1} B_{\alpha_1}^{m+1}|_E \subset E \). Moreover, the constant \( C(n) \) can be chosen independently from \( D \) by rescaling. On the other hand, Lemma 4.4 and the fibration theorem imply that the inclusion \( B(p, R) \cap E \hookrightarrow E \) is a homotopy equivalence for sufficiently large \( R \). Thus we have \( \beta(E) \leq C(n) \).

6. Volumes of extremal subsets

In this section, we prove Theorem 1.1(3) and Corollary 1.3. We remark that the Hausdorff measure of an extremal subset with respect to the induced intrinsic metric is equal to the one with respect to the original metric of the ambient space \( (\mathbb{F}) \). We denote by \( \text{vol}_m \) the \( m \)-dimensional Hausdorff measure.

First, we study surjectivity of the restriction of a gradient exponential map to an extremal subset. Note that \( \text{gexp}_p : T_p E \to E \) is not surjective generally. Property 3 in Pet2 §2.2 states local surjectivity of gradient flows. We need its generalization for extremal subsets.
Lemma 6.1 (cf. [Pet2 §2.2 property 3]). Let \( f : M \to \mathbb{R} \) be a semiconcave function on an Alexandrov space \( M \) and \( \Phi_f^t : M \to M \) its gradient flow (we suppose that \( \Phi_f^t \) is defined on all \( x \in M \) and \( t \geq 0 \)). Then for any \( y \in M \), there exists \( x \in M \) and \( t > 0 \) such that \( \Phi_f^t(x) = y \). Moreover, if \( y \) belongs to an extremal subset \( E \subset M \), then \( x \) can be taken from \( E \).

Proof. We prove it by the induction on \( \dim E \). If \( \dim E = 0 \), the claim is clear since every gradient flow fixes extremal points. Suppose that the claim holds for dimension \( \leq m - 1 \) and that \( \dim E = m \). Let \( E^{(k)} \) be the \( k \)-dimensional stratum of a canonical stratification of \( E \). Then \( E^{(k)} \) is also extremal (see [2.2]). Hence, by the induction hypothesis, the claim holds for all \( y \in E \setminus E^{(m)} \). Suppose that the claim does not hold for some \( y \in E^{(m)} \). Then, \( \Phi_f^t \) maps \( E \) into \( E \setminus \{y\} \) for all \( t > 0 \).

Since \( \Phi_f^t \) is homotopic to \( \Phi_f^0 = \text{id}_M \), the following commutative diagram holds:

\[
\begin{array}{ccc}
H_*(E, E \setminus \{y\}) & \xrightarrow{\text{id}} & H_*(E, E \setminus \{y\}) \\
\downarrow \Phi_f^t & & \uparrow \downarrow \\
H_*(E \setminus \{y\}, E \setminus \{y\}) & & H_*(E \setminus \{y\}, E \setminus \{y\})
\end{array}
\]

However, since \( y \in E^{(m)} \), the local homology group \( H_m(E, E \setminus \{y\}) \) equals to \( \mathbb{Z} \). This contradicts the diagram. \( \square \)

Using the above lemma, we give a sufficient condition for surjectivity of a gradient exponential map restricted to an extremal subset. Note that we use the gradient exponential map of the case \( \kappa = -1 \) (see [2.2]).

Proposition 6.2. Let \( M \) be an Alexandrov space with curvature \( \geq -1 \), \( E \) an extremal subset of \( M \), and \( p \in M \).

1. If \( \text{dist}_p \) has no critical points on \( \overline{B(p, r)} \setminus \{p\} \), then for sufficiently large \( R > 0 \), we have

\[
\text{gexp}_p(B(o, R) \cap T_p E) \supset B(p, r) \cap E.
\]

Note that if \( B(p, r) \cap E \neq \emptyset \), then \( p \in E \) (see Lemma 4.7).

2. If \( \text{dist}_p \) has no critical points on \( A(p; r_1, r_2) \), then for sufficiently large \( R > 0 \), we have

\[
\text{gexp}_p(A; r_1, R) \cap K((\partial B(p, r_1) \cap E)') \supset A(p; r_1, r_2) \cap E,
\]

where \( K((\partial B(p, r_1) \cap E)') \) is a subcone of \( T_p \).

Furthermore, if we define a map \( G_p^{(r_1, R)} : B(p, r_1) \to B(p, R) \) by

\[
G_p^{(r_1, R)}(x) := \text{gexp}_p\left(\frac{R}{r_1}p|x_x'\right)
\]

for the above \( R \), then we have

\[
G_p^{(r_1, R)}(B(p, r_1) \cap E) \supset B(p, r_2) \cap E.
\]

Proof. First we show (1). By the lower semicontinuity of \( |\nabla \text{dist}_p| \), there exists a constant \( c > 0 \) such that \( |\nabla \text{dist}_p| > c \) on \( \overline{B(p, r)} \setminus \{p\} \). Consider the semiconcave function \( f = \cosh \circ \text{dist}_p - 1 \) (note that \( f \) satisfies the assumption of Lemma 6.1) and take \( z \in B(p, r) \cap E \). It follows from Lemma 6.1 by contradiction that there is a sequence \( y_i \in E \) converging to \( p \) such that \( \Phi_f^{t_i}(y_i) = z \) for some \( t_i \). Therefore,
Thus, $G$ of radial curves. It controls the speeds of radial curves. The following technique was used in \cite[3.3]{PP2} to prove the convergence of parameters $\beta_i(s) = \exp_p(s(y_i)_p)$. From the differential equation \eqref{eq:2.1}, we have

$$|p\beta_i(s)|' = \frac{\tanh |p\beta_i(s)|}{\tanh s} \cdot |\nabla_{\beta_i(s)} \text{dist}_p|^2.$$ 

Together with the assumption $|\nabla \text{dist}_p| > c$, this implies

$$\frac{|p\beta_i(s)|'}{\tanh |p\beta_i(s)|} \geq \frac{c^2}{\tanh s}.$$ 

Integrating this inequality over the interval $[\sigma, s_i]$ for some fixed $\sigma > 0$, we obtain

$$\log \sinh s_i \leq c^{-2} (\log \sinh r - \log \sinh |p\beta_i(\sigma)|) + \log \sinh \sigma.$$ 

Since $|p \exp_p(\sigma \cdot t)|$ is a positive continuous function on $\Sigma_p$, $s_i$ is uniformly bounded above.

Next we show (2). The first statement follows from Lemma \ref{lem:6.1} in the same way as (1). Let us show the second. Take $z \in B(p, r_2) \cap E$. If $z \in A(p; r_1, r_2) \cap E$, then by the first statement,

$$\Phi_f^{\tau_0}(y_0) = \exp_p(s_0(y_0)_p) = z$$

for some $y_0 \in \partial B(p, r_1) \cap E$. If $z \in B(p, r_1) \cap E$, we obtain the same equation holds for $y_0 = z$, $t_0 = 0$, and $s_0 = |pz|$. It follows from Lemma \ref{lem:6.1} by contradiction that for any $T > 0$, there exists $x_0 \in E$ such that $\Phi_f^T(x_0) = y_0$. Let $T$ be sufficiently large and consider the gradient curve $\alpha(t) = \Phi_f^T(x_0)$. Note that $G_p(\alpha(t))$ lies on the curve $\alpha$. Then, $G_p(\alpha(t))$ is before $z$ on $\alpha$ when $t = 0$, but beyond $z$ when $t = T$. Indeed, we can express $G_p(x_0) = \Phi_f^{\tau_0}(x_0)$ by the reparametrization \eqref{eq:2.1}, where

$$\tau_0 = \int_{|px_0|}^{\frac{|px_0|}{\tanh s \cosh |p \exp_p(s(x_0)_p)|}} \frac{ds}{\tanh s}$$

$$\leq \int_{|px_0|}^{\frac{|px_0|}{\tanh s \cosh |p \exp_p(s(x_0)_p)|}} \frac{ds}{\tanh s} \leq \log \frac{\sinh r}{\sinh r_1}.$$ 

Thus, $G_p(x_0)$ is before $z = \Phi_f^{T+\tau_0}(x_0)$ if $T$ is sufficiently large. On the other hand, $G_p(y_0) = \exp_p(\frac{|py_0|}{\tanh s \cosh |p \exp_p(s(y_0)_p)|})$ is beyond $z = \exp_p(s_0(y_0)_p)$ since $\frac{|py_0|}{\tanh s \cosh |p \exp_p(s(y_0)_p)|} \geq s_0$. Hence, the claim follows from the intermediate value theorem.

Next, we estimate such $R$ in Proposition \ref{prop:6.2} independently from each space. The following technique was used in \cite[3.3]{PP2} to prove the convergence of parameters of radial curves. It controls the speeds of radial curves.

**Lemma 6.3.** Let $M$ be an Alexandrov space with curvature $\geq -1$ and $p \in M$.

1. Assume that there exists $c > 0$ such that $|\nabla \text{dist}_p| > c$ on $B(p, r) \setminus \{p\}$. Assume further that there exist $\rho > 0$ and $\theta < \pi/2$ such that $(\partial B(p, \rho))'_p$ is $\theta$-dense in $\Sigma_p$. Then, there exists $R = R(r, c, \rho, \theta) > 0$ (depending only on $r$, $c$, $\rho$ and $\theta$) such that

$$\exp_p^{-1}(B(p, r)) \subset B(o, R).$$
(2) Assume that there exists \( c > 0 \) such that \( |\nabla \text{dist}_p| > c \) on \( A(p; r_1, r_2) \). Assume further that there exist \( \rho > 0 \) and \( \theta < \pi/2 \) such that for any \( y \in \partial B(p, r_1) \) there is \( x \in \partial B(p, \rho) \) with \( \angle xpy < \theta \). Then, there exists \( R = R(r_2, c, \rho, \theta) > 0 \) (independent of \( r_1 \)) such that
\[
\text{gexp}_p^{-1}(B(p, r_2)) \cap K((\partial B(p, r_1))^\rho_p) \subset B(o, R).
\]

**Proof.** (1) follows from (2) by taking \( r_2 = r \) and \( r_1 \to 0 \). Let us show (2). Consider the radial curve \( \beta(s) = \text{gexp}_p(sy_p') \) for \( y \in \partial B(p, r_1) \). We must show that if \( \beta(s) \in B(p, r_2) \) then \( s < R(r_2, c, \rho, \theta) \). Fix \( \sigma > 0 \). We may assume \( r_1 < \sigma \) since \( |\nabla \text{dist}_p| > c \) on \( A(p; r_1, r_2) \). Then the same inequality as (6.1) holds:
\[
\log \sinh s \leq c^{-2} (\log \sinh r_2 - \log \sinh |p\beta(\sigma)|) + \log \sinh \sigma.
\]
Therefore, it is enough to show that \( |p\beta(\sigma)| \) has a positive uniform lower bound for some \( \sigma \) depending only on \( \rho \) and \( \theta \). Take \( x \in \partial B(p, \rho) \) with \( \angle xpy < \theta \). Then, Proposition 2.4(1) implies \( \angle x \beta(\sigma) < \theta \). Therefore, for sufficiently small \( \sigma \), we have
\[
|p\beta(\sigma)| \geq \rho - |x\beta(\sigma)|
= \cos \angle x \beta(\sigma) \cdot \sigma + o_{\rho}(\sigma)
\geq \cos \theta \cdot \sigma + o_{\rho}(\sigma)
\geq \text{const}(\rho, \theta, \sigma) > 0
\]
since \( \theta < \pi/2 \). This completes the proof. \( \square \)

In view of Lemma 6.3, we modify Theorem 3.1.

**Theorem 6.4.** Let \( (M_i, p_i) \in \mathcal{A}_p(n) \) converge to an Alexandrov space \((X, p)\) with dimension \( \geq 1 \). Then, for sufficiently small \( r > 0 \) and \( c > 0 \), there exists \( \tilde{p}_i \in M_i \) converging to \( p \) such that either (1) or (2) holds:

1. There is a subsequence \( \{j_i\} \subset \{i\} \) such that \( |\nabla \text{dist}_{\tilde{p}_i}| > c \) on \( \bar{B}(\tilde{p}_j, r) \setminus \{\tilde{p}_j\} \) and \( (\partial B(\tilde{p}_j, r))^\rho_{\tilde{p}_j} \) is \( (\pi/2 - c) \)-dense in \( \Sigma_{\tilde{p}_j} \).
2. There exists a sequence \( \delta_i \to 0 \) such that
   - (i) for any \( \lambda > 1 \) and sufficiently large \( i \), \( |\nabla \text{dist}_{\tilde{p}_i}| > c \) on \( A(\tilde{p}_i; \lambda \delta_i, r) \) and for any \( y \in \partial B(\tilde{p}_i, \lambda \delta_i) \), there is \( x \in \partial B(\tilde{p}_i, r) \) with \( \angle x \tilde{p}_i y < \pi/2 - c \);
   - (ii) for any limit \( (Y, y_0) \) of \( \left\{ \frac{1}{\delta_i} M_i, \tilde{p}_i \right\} \), we have \( \dim Y \geq \dim X + 1 \).

In particular, if \( \dim X = n \), then (1) holds for all sufficiently large \( i \).

Note that (1) (resp. (2)) satisfies the assumption of Lemma 6.3(1) (resp. (2)) independently from \( j \) (resp. \( i \)).

**Proof.** We argue along with the original proof of [Y1 3.2]. For positive numbers \( 0 < \varepsilon \ll \theta \leq \pi/100 \), take sufficiently small \( 0 < r < 1/100 \) so that
- \( \angle xpy - \angle xpy < \varepsilon \) for every \( x, y \in \partial B(p, 2r) \);
- \( (\partial B(p, 2r))^\rho_p \) is \( \varepsilon \)-dense in \( \Sigma_p \).

Note that the latter implies that \( |\nabla \text{dist}_p| > 1/10 \) on \( \bar{B}(p, r) \setminus \{p\} \). Let \( \{x_\alpha\}_\alpha \) be a maximal \( \theta \)-discrete set in \( \partial B(p, 2r) \). Furthermore, for each \( \alpha \), let \( \{x_{\alpha, \beta}\}_{\beta=1}^{N_\alpha} \) be a maximal \( \varepsilon r \)-discrete set in \( B(x_\alpha, \theta r) \cap \partial B(p, 2r) \). Then, the Bishop-Gromov
inequality implies that

\[
N_\alpha \geq \text{const} \cdot \left( \frac{\theta}{\varepsilon} \right)^{\dim X - 1}.
\]

Define functions \( f_\alpha \) and \( f \) on \( X \) by

\[
f_\alpha(x) := \frac{1}{N_\alpha} \sum_{\beta=1}^{N_\alpha} d(x, \alpha_\beta, x), \quad f(x) := \min_{\alpha} f_\alpha(x).
\]

Then, it is easy to see that \( f \) has a strict maximum at \( p \) on \( B(p, r) \) (see [11 3.3]).

Fix a \( \mu \)-Hausdorff approximation \( \varphi_i : B(p, 1/\mu) \to B(p_i, 1/\mu) \) with \( \varphi_i(p) = p_i \), where \( \mu_i \to \infty \) as \( i \to \infty \). Put \( x'_{\alpha\beta} := \varphi_i(x_{\alpha\beta}) \) and define functions \( f'_\alpha \) and \( f' \) on \( M_i \) by

\[
f'_\alpha(x) := \frac{1}{N_\alpha} \sum_{\beta=1}^{N_\alpha} d(x', \alpha_\beta, x), \quad f'(x) := \min_{\alpha} f'_\alpha(x).
\]

Note that \( f'_\alpha \) and \( f' \) converge to \( f_\alpha \) and \( f \) respectively. Let \( \hat{p}_i \) be a maximum point of \( f' \) on \( B(\tilde{p}_i, r) \). Then \( \hat{p}_i \) converges to \( p \), the unique maximum point of \( f \). Now, put \( c := \sin(\varepsilon/2N) \), where \( N = \max_\alpha N_\alpha \). Suppose that (1) does not hold for these \( r \) and \( c \). Then for any sufficiently large \( i \), there exists \( y \in B(\tilde{p}_i, r) \setminus \{\hat{p}_i\} \) such that

(a) \( |\nabla_y \text{dist}_{\hat{p}_i}| \leq c \) or;
(b) \( \hat{p}_i r_y \geq \pi/2 - c \) for all \( x \in \partial B(\tilde{p}_i, r) \).

Let \( \hat{q}_i \in B(\tilde{p}_i, r) \setminus \{\hat{p}_i\} \) be a farthest point from \( \hat{p}_i \) satisfying either (a) or (b), and let \( \delta_i \) be the distance between \( \hat{p}_i \) and \( \hat{q}_i \). Then, (2)(i) is obvious. Moreover, \( \delta_i \to 0 \) since \( |\nabla \text{dist}_p| > 1/10 \) on \( B(p, r) \setminus \{p\} \) and also \( (\partial B(p, 2r))^\prime_p \) is \( \varepsilon \)-dense in \( \Sigma_p \).

Let us show (2)(ii). Suppose that \( (\frac{1}{\pi} M_i, \tilde{p}_i) \) converges to an Alexandrov space \((Y, y_0)\) of nonnegative curvature. Passing to a subsequence, we may assume that \( \hat{q}_i \) converges to \( z_0 \in Y \). We may further assume that minimal geodesics \( \tilde{p}_i \hat{q}_i \) and \( \tilde{p}_i x'_{\alpha\beta} \) converge to a minimal geodesic \( y_0 z_0 \) and a ray \( \gamma_{\alpha\beta} \) from \( y_0 \), respectively. Let \( v_i, v'_{\alpha\beta} \in \Sigma_{\tilde{p}_i} \) denote the directions of \( \tilde{p}_i \hat{q}_i \) and \( \tilde{p}_i x'_{\alpha\beta} \), and let \( v, v_{\alpha\beta} \in \Sigma_{y_0} \) be the directions of \( y_0 z_0 \) and \( \gamma_{\alpha\beta} \), respectively. Note that

\[
\angle(v, v_{\alpha\beta}) \geq \angle(x_{\alpha\beta} p x_{\alpha\beta'}) \geq \frac{\pi}{4}
\]

for every \( 1 \leq \beta \neq \beta' \leq N_\alpha \).

First we show that

\[
\angle(v, v_{\alpha\beta}) \geq \frac{\pi}{2} - \frac{\varepsilon}{2N}
\]

for every \( \alpha \) and \( \beta \). If (a) holds for infinitely many \( \hat{q}_i \), then by lower semicontinuity of \( |\nabla| \), we have \( |\nabla_{x_0} \text{dist}_{y_0}| \leq \sin(\varepsilon/2N) \). This implies that \( \angle y_0 z_0 x_{\alpha\beta}(\infty) \leq \pi/2 + \varepsilon/2N \), where \( x_{\alpha\beta}(\infty) \) denotes the element of the ideal boundary of \( Y \) defined by the ray \( \gamma_{\alpha\beta} \). Thus we obtain \( \angle(v, v_{\alpha\beta}) \geq \angle y_0 z_0 x_{\alpha\beta}(\infty) \geq \pi/2 - \varepsilon/2N \). On the other hand, if (b) holds for infinitely many \( \hat{q}_i \), then by monotonicity of angles, we have

\[
\angle(v, v_{\alpha\beta}) \geq \lim_{i \to \infty} \angle \hat{q}_i \hat{p}_i x'_{\alpha\beta}(r) \geq \frac{\pi}{2} - c \geq \frac{\pi}{2} - \frac{\varepsilon}{2N},
\]

where \( x'_{\alpha\beta}(r) \) denotes the point on the minimal geodesic \( \hat{p}_i x'_{\alpha\beta} \) at distance \( r \) from \( \hat{p}_i \).
Fix $\alpha$ such that $(f^i)_p^i(v_i) = (f^i)_p^i(v_i)$ for infinitely many $i$. Since $f^i$ has a local maximum at $\hat{p}_i$, the first variation formula implies that

$$0 \geq (f^i)_p^i(v_i) = \frac{1}{N_\alpha} \sum_{\beta=1}^{N_\alpha} - \cos \angle(v_i, v^i_{\alpha\beta})$$

(choose $v^i_{\alpha\beta}$ so that the first variation formula holds for $v_i$). Passing to the limit, we have

$$0 \geq \frac{1}{N_\alpha} \sum_{\beta=1}^{N_\alpha} - \cos \angle(v, v_{\alpha\beta})$$

by lower semicontinuity of angles. Therefore, combining (6.3) and (6.4), we obtain

$$\left| \angle(v, v_{\alpha\beta}) - \frac{\pi}{2} \right| \leq \varepsilon.$$ 

Hence, $\{v_{\alpha\beta}\}_{\beta=1}^{N_\alpha}$ is an $\varepsilon/4$-discrete set of $A(v; \pi/2 - \varepsilon, \pi/2 + \varepsilon)$. Since there exists a noncontracting map from $\Sigma_{h_0}$ to the unit sphere $S^{\dim Y - 1}$ preserving the distance from $v$, we have

$$N_\alpha \leq \text{const}(n) \cdot \varepsilon^{-(\dim Y - 2)}.$$ 

Thus, combining (6.2) and (6.5) and taking sufficiently small $\varepsilon$, we can conclude $\dim Y \geq \dim X + 1$. \hfill \Box

Now, we can prove uniform boundedness of the volumes of extremal subsets.

**Theorem 6.5.** For given $n$ and $D$, there exists a constant $C(n, D)$ satisfying the following: Let $M \in A(n)$, $p \in M$, and $E \subset M$ be an $m$-dimensional extremal subset. Then, we have $\text{vol}_m(B(p, D) \cap E) \leq C(n, D)$.

**Proof.** We use the induction on $n$. Suppose that the conclusion does not hold. Take an infinite sequence of Alexandrov spaces $(M_i, p_i) \in A_p(n)$ and $m$-dimensional extremal subsets $E_i \subset M_i$ such that $\text{vol}_m(B(p_i, D) \cap E_i) \to \infty$ as $i \to \infty$. We may assume that $(M_i, p_i)$ converges to an Alexandrov space $(X, p)$. Set $k = \dim X$.

We prove by the reverse induction on $k$ that there exists a constant $C$ such that $\text{vol}_m(B(p_i, D) \cap E_i) \leq C$ for some subsequence. This is a contradiction.

First suppose $k = n$. Take a finite covering $\{B(x_\alpha, r_\alpha/2)\}_{\alpha=1}^{N}$ of $B(p, D)$, where $r_\alpha$ is the one in Theorem 6.4. Then, there exists $\hat{x}^i_\alpha \to x_\alpha$ for each $\alpha$ such that Theorem 6.4(1) holds for sufficiently large $i$. Therefore, by Proposition 6.2(1) and Lemma 6.3(1), there exists $R_\alpha$ independent of $i$ such that

$$\text{gexp}_{\hat{x}^i_\alpha}(B(a, R_\alpha) \cap T_{\hat{x}^i_\alpha}E_i) \supset B(\hat{x}^i_\alpha, r_\alpha) \cap E_i.$$ 

Since $\text{gexp}_{\hat{x}^i_\alpha}$ is a 1-Lipschitz map from the elliptic cone $(T_{\hat{x}^i_\alpha}, h)$ over $\Sigma_{\hat{x}^i_\alpha}$, we have

$$\text{vol}_m(B(\hat{x}^i_\alpha, r_\alpha) \cap E_i) \leq \int_0^{R_\alpha} \sinh^{m-1} r \cdot \text{vol}_{m-1}(\Sigma_{\hat{x}^i_\alpha}E_i) \, dr \leq C.$$ 

Here, the second inequality follows from the hypothesis of the induction on $n$. Since $\{B(\hat{x}^i_\alpha, r_\alpha)\}_{\alpha=1}^{N}$ is a covering of $B(p_i, D)$ for sufficiently large $i$, we obtain

$$\text{vol}_m(B(p_i, D) \cap E_i) \leq \sum_{\alpha=1}^{N} \text{vol}_m(B(\hat{x}^i_\alpha, r_\alpha) \cap E_i) \leq C.$$
Next suppose $1 \leq k \leq n - 1$. Cover $\tilde{B}(p, D)$ by $\{B(x_\alpha, r_\alpha/2)\}_{\alpha=1}^N$ as above. Then, there exists $\beta_i \to x_\alpha$ for each $\alpha$ such that either (1) or (2) in Theorem 6.4 holds. If (1) holds, then we have $\text{vol}_m (\tilde{B}(\beta_i, r_\alpha) \cap E_i) \leq C$ for some subsequence as above. Suppose that (2) holds for some $\alpha$. We fix this $\alpha$ and omit it below. Then, there exists $\delta_i \to 0$ such that both (i) and (ii) holds. Passing to a subsequence, we may assume that $(\frac{1}{\delta_i}M_i, \tilde{x}^i) \xrightarrow{GH} (Y, y_0)$. Then we have $\dim Y \geq \dim X + 1$.

Applying the hypothesis of the reverse induction to $\frac{1}{\delta_i}B(\tilde{x}^i, 2\delta_i)$ and $\frac{1}{\delta_i}E_i$, we have

$$\text{vol}_m (B(\tilde{x}^i, 2\delta_i) \cap E_i) \leq C \delta^m_i$$

for some subsequence. On the other hand, by Proposition 6.2(2) and Lemma 6.3(2), there exists $R$ independent of $i$ such that

$$G^{(2\delta_i, R)}(B(\tilde{x}^i, 2\delta_i) \cap E_i) \supset B(\tilde{x}^i, r) \cap E_i.$$  

In addition, Proposition 2.3(2) states that $G^{(2\delta_i, R)}$ is $\frac{\sinh R}{\sinh 2\delta_i}$-Lipschitz. Therefore, together with the above inequality, it implies

$$\text{vol}_m (B(\tilde{x}^i, r) \cap E_i) \leq \left(\frac{\sinh R}{\sinh 2\delta_i}\right)^m \cdot C \delta^m_i \leq C.$$

Since $\{B(\tilde{x}_\alpha, r_\alpha)\}_{\alpha=1}^N$ is a covering of $B(p_i, D)$ for sufficiently large $i$, we obtain $\text{vol}_m (B(p_i, D) \cap E_i) \leq C$.

Finally, if $k = 0$, then the claim follows from the case $k \geq 1$ by rescaling $M_i$ with the reciprocal of its diameter.

For a metric space $(X, d)$ and a positive number $\varepsilon$, we denote by $N_\varepsilon(X, d)$ the maximal number of $\varepsilon$-discrete points in $X$. Note that we allow $d = \infty$.

**Theorem 6.6.** For given $n$ and $D$, there exists a constant $C(n, D)$ satisfying the following: Let $M \in \mathcal{A}(n)$, $p \in M$, and $E \subset M$ be an $m$-dimensional extremal subset. Then, for any $\varepsilon > 0$, we have

$$N_\varepsilon (B(p, D) \cap E, d_E) \leq \frac{C(n, D)}{\varepsilon^m},$$

where $d_E$ denotes the induced intrinsic metric of $E$.

The proof is similar as that of Theorem 6.5. Indeed, we can repeat the same argument by considering $\varepsilon^m N_\varepsilon(\cdot, d_E)$ instead of $\text{vol}_m (\cdot)$. Corollary 1.3 follows from Theorem 6.6 together with Gromov’s precompactness theorem ([G2 5.2], [BBI 7.4.15]).

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