Article

Forcing total outer connected monophonic number of a graph

K. Ganesamoorthy, S. Lakshmi Priya

Abstract. For a connected graph $G = (V, E)$ of order at least two, a subset $T$ of a minimum total outer connected monophonic set $S$ of $G$ is a forcing total outer connected monophonic subset for $S$ if $S$ is the unique minimum total outer connected monophonic set containing $T$. A forcing total outer connected monophonic subset for $S$ of minimum cardinality is a minimum forcing total outer connected monophonic subset of $S$. The forcing total outer connected monophonic number $f_{tom}(S)$ in $G$ is the cardinality of a minimum forcing total outer connected monophonic subset of $S$. The forcing total outer connected monophonic number of $G$ is $f_{tom}(G) = \min\{f_{tom}(S)\}$, where the minimum is taken over all minimum total outer connected monophonic sets $S$ in $G$. We determine bounds for it and find the forcing total outer connected monophonic number of a certain class of graphs. It is shown that for every pair $a, b$ of positive integers with $0 \leq a < b$ and $b \geq a + 4$, there exists a connected graph $G$ such that $f_{tom}(G) = a$ and $cm_{to}(G) = b$, where $cm_{to}(G)$ is the total outer connected monophonic number of a graph.

Keywords: total outer connected monophonic set, total outer connected monophonic number, forcing total outer connected monophonic subset, forcing total outer connected monophonic number

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Forcing total outer connected monophonic number of a graph

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Аннотация. Для связного графа $G = (V, E)$ с числом вершин не менее 2 подмножество $T$ минимального общего внешне связанного монофонического множества $S$ графа $G$ является сильным общим внешне связаным монофоническим подмножеством для $S$, если $S$ есть единственное минимальное общее внешне связное монофоническое множество, содержащее $T$. Сильное общее внешне связанное монофоническое подмножество для $S$ с минимальным числом элементов есть минимальное сильное общее внешне связанное монофоническое подмножество $S$. Сильное общее внешне связанное монофоническое число $f_{tom}(S)$ в $G$ есть число элементов минимального сильного общего внешне связанного монофонического подмножества $S$. Сильное общее внешне связанное монофоническое число графа $G$ есть $f_{tom}(G) = \min\{f_{tom}(S)\}$, где минимум принимается над всеми минимальными общими внешними связными монофоническими множествами $S$ в $G$. Мы определяем его границы и находим сильное общее внешне связанное монофоническое число некоторых классов графов. Показывается, что для каждой пары $a$, $b$ положительных целых с $0 \leq a < b$ и $b \geq a + 4$ существует связный график $G$ такой, что $f_{tom}(G) = a$ и $cm_{to}(G) = b$, где $cm_{to}(G)$ является общим внешне связным монофоническим числом графа.

Ключевые слова: общее внешне связанное монофоническое множество, общее внешне связанное монофоническое число, сильное общее внешне связанное монофоническое множество, сильное общее внешне связанное монофоническое число

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Introduction

By a graph $G = (V, E)$ we mean a finite simple undirected connected graph. The order and size of $G$ are denoted by $p$ and $q$, respectively. For basic graph theoretic terminology we refer to Harary [1, 2]. The distance $d(x, y)$ between two vertices $x$ and $y$ in a connected graph $G$ is the length of a shortest $x - y$ path in $G$. An $x - y$ path of length $d(x, y)$ is called an $x - y$ geodesic. A vertex $v$ of a connected graph $G$ is called an endvertex of $G$ if its degree is 1. A vertex $v$ of a connected graph $G$ is called a support vertex if it is adjacent to an endvertex of $G$. The neighborhood of a vertex $v$ is the set $N(v)$ consisting of all vertices $u$ which are adjacent with $v$. A vertex $v$ is an extreme vertex if the subgraph induced by its neighbors is complete. A chord of a path $P$ is an edge joining two non-adjacent vertices of $P$. A path $P$ is called a monophonic path if it is a chordless path. A set $S$ of vertices of $G$ is a monophonic set of $G$ if each vertex $v$ of $G$ lies on a $x - y$ monophonic path for some $x$ and $y$ in $S$. The minimum cardinality of a monophonic set of $G$ is the monophonic number of $G$ and is denoted by $m(G)$. The monophonic number of a graph, an algorithmic aspect of monophonic concepts was introduced and studied in [3–7]. A total monophonic set of a graph $G$ is a monophonic set $S$ such that the subgraph $G[S]$ induced by $S$ has no isolated vertices. The minimum cardinality of a total monophonic set of $G$ is the total monophonic number of $G$ and is denoted by $m_t(G)$. The total monophonic number of a graph and its related concepts were studied in [8–10]. A set $S$ of vertices in a graph $G$ is said to be an outer connected monophonic set if $S$ is a monophonic set of $G$ and either $S = V$ or the subgraph induced by $V - S$ is connected. The minimum cardinality of an outer connected monophonic set of $G$ is the outer connected monophonic number of $G$ and is denoted by $m_{oc}(G)$. The outer connected monophonic number of a graph was introduced in [11]. Very recently, outer connected monophonic concepts have been widely investigated in graph theory, such as a connected outer connected monophonic number [12], extreme outer connected monophonic graphs [13], and so on. A total outer connected monophonic set $S$ of $G$ is an outer connected monophonic set such that the subgraph induced by $S$ has no isolated vertices. The minimum cardinality of a total outer connected monophonic set of $G$ is the total outer connected monophonic number of $G$ and is denoted by $cm_{oc}(G)$.

The authors of this article introduced and studied the general externally total outer connected monophonic number of a graph and proved the following theorems\(^1\), which will be used further.

**Theorem 1.** Each extreme vertex and each support vertex of a connected graph $G$ belong to every total outer connected monophonic set of $G$.

**Theorem 2.** For the complete graph $K_p (p \geq 2)$, $cm_{oc}(K_p) = p$.

**Theorem 3.** For any non-trivial tree $T$, the set of all endvertices and support vertices of $T$ is the unique minimum total outer connected monophonic set of $G$.

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Theorem 4. For any connected graph $G$, $cm_{to}(G) = 2$ if and only if $G = K_2$.

Throughout this paper, $G$ denotes a connected graph with at least two vertices.

1. Main Results

Definition 1. Let $S$ be a minimum total outer connected monophonic set of $G$. A subset $T$ of $S$ is a forcing total outer connected monophonic subset for $S$ if $S$ is the unique minimum total outer connected monophonic set containing $T$. A forcing total outer connected monophonic subset for $S$ of minimum cardinality is a minimum forcing total outer connected monophonic subset of $S$. The forcing total outer connected monophonic number $f_{tom}(S)$ in $G$ is the cardinality of a minimum forcing total outer connected monophonic subset of $S$. The forcing total outer connected monophonic number of $G$ is $f_{tom}(G) = \min\{f_{tom}(S)\}$, where the minimum is taken over all minimum total outer connected monophonic sets $S$ in $G$.

Example 1. For the graph $G$ in Fig. 1, it is clear that $S_1 = \{v_1, v_2, v_4, v_5\}$, $S_2 = \{v_1, v_4, v_5, v_8\}$, $S_3 = \{v_1, v_2, v_5, v_6\}$ and $S_4 = \{v_1, v_5, v_6, v_8\}$ are the minimum total outer connected monophonic sets of $G$. It is clear that no minimum total outer connected monophonic set $S_i (i = 1, 2, 3, 4)$ is the unique minimum total outer connected monophonic set containing any of its 1-element subsets. It is easy to see that $\{v_2, v_4\}$ is a forcing total outer connected monophonic subset contained in $S_1$ and $f_{tom}(S_1) = 2$. Hence, we have $f_{tom}(G) = 2$. By Theorem 3, for any non-trivial tree $T$, the set of all endvertices and support vertices of $T$ is the unique minimum total outer connected monophonic set of $T$ and so $f_{tom}(T) = 0$.

Theorem 5. For any connected graph $G$ of order $p$, $0 \leq f_{tom}(G) \leq cm_{to}(G) \leq p$.

Proof. By the definition of the forcing total outer connected monophonic number of a graph, it is clear that $f_{tom}(G) \geq 0$. Let $S$ be a minimum total outer connected monophonic set of $G$. Clearly, $f_{tom}(S) \leq |S| = cm_{to}(G)$ and $f_{tom}(G) = \min \{f_{tom}(S)\}$, where the minimum is taken over all minimum total outer connected monophonic sets $S$ in $G$. Hence $0 \leq f_{tom}(G) \leq cm_{to}(G) \leq p$. \qed

Remark 1. The bounds in Theorem 5 are sharp. By Theorem 3, for any non-trivial tree $T$, the set of all endvertices and support vertices of $T$ is the unique minimum total outer connected monophonic set of $T$ and so $f_{tom}(T) = 0$. By Theorem 2, for the complete graph $K_p (p \geq 2)$, $cm_{to}(K_p) = p$. Also all the inequalities in Theorem 5 can be strict. For the graph $G$ given in Fig. 1 of order 8, it is clear that no 2-element subset or 3-element subset of $V(G)$ is a total outer connected monophonic set of $G$. The minimum total outer connected monophonic sets of $G$ are $S_1 = \{v_1, v_2, v_4, v_5\}$, $S_2 = \{v_1, v_4, v_5, v_8\}$, $S_3 = \{v_1, v_2, v_5, v_6\}$ and $S_4 = \{v_1, v_5, v_6, v_8\}$ so that $cm_{to}(G) = 4$. It is clear that $f_{tom}(S_i) = 2(i = 1, 2, 3, 4)$ and so $f_{tom}(G) = 2$. Thus $0 < f_{tom}(G) < cm_{to}(G) < p$.

The following theorem characterizes graphs $G$ for which the lower bound in Theorem 5 is attained and also characterizes graphs $G$ for which $f_{tom}(G) = 1$ and $f_{tom}(G) = cm_{to}(G)$.
**Theorem 6.** Let $G$ be a connected graph. Then

(i) $f_{tom}(G) = 0$ if and only if $G$ has the unique minimum total outer connected monophonic set;

(ii) $f_{tom}(G) = 1$ if and only if $G$ has at least two minimum total outer connected monophonic sets, one of which is the unique minimum total outer connected monophonic set containing one of its elements;

(iii) $f_{tom}(G) = cm_{to}(G)$ if and only if no minimum total outer connected monophonic set of $G$ is the unique minimum total outer connected monophonic set containing any of its proper subsets.

**Proof.** (i) Let $f_{tom}(G) = 0$. Then, by the definition, $f_{tom}(S) = 0$ for some minimum total outer connected monophonic set $S$ of $G$ so that the empty set $\varnothing$ is the minimum forcing subset for $S$. Since the empty set $\varnothing$ is a subset of every set, it follows that $S$ is the unique minimum total outer connected monophonic set of $G$. The converse is clear.

(ii) Let $f_{tom}(G) = 1$. Then by (i), $G$ has at least two minimum total outer connected monophonic sets. Since $f_{tom}(G) = 1$, there is a 1-element subset $T$ of a minimum total outer connected monophonic set $S$ of $G$ such that $T$ is not a subset of any other minimum total outer connected monophonic set of $G$. Thus $S$ is the unique minimum total outer connected monophonic set containing one of its elements. The converse is clear.

(iii) Let $f_{tom}(G) = cm_{to}(G)$. Then $f_{tom}(S) = cm_{to}(G)$ for every minimum total outer connected monophonic set $S$ in $G$. Since any total outer connected monophonic set of $G$ needs at least two vertices, $cm_{to}(G) \geq 2$ and hence $f_{tom}(G) \geq 2$. Then by (i), $G$ has at least two minimum total outer connected monophonic sets, and so the empty set $\varnothing$ is not a forcing subset for any minimum total outer connected monophonic set of $G$. Since $f_{tom}(G) = cm_{to}(G)$, no proper subset of $S$ is a forcing subset of $S$. Thus no minimum total outer connected monophonic set of $G$ is the unique minimum total outer connected monophonic set containing any of its proper subsets.

Conversely, the data implies that $G$ contains more than one minimum total outer connected monophonic set and no subset of any minimum total outer connected monophonic set $S$ other than $S$, is a forcing subset for $S$. Hence it follows that $f_{tom}(G) = cm_{to}(G)$. \hfill \Box

**Definition 2.** A vertex $v$ of $G$ is said to be a total outer connected monophonic vertex if $v$ belongs to every minimum total outer connected monophonic set of $G$.

**Remark 2.** If $G$ has the unique minimum total outer connected monophonic set $S$, then every vertex in $S$ is a total outer connected monophonic vertex of $G$. Also, if $x$ is an extreme vertex or a support vertex of $G$, then $x$ is a total outer connected monophonic vertex of $G$. For the graph $G$ given in Fig. 1, $v_1$ and $v_5$ are the total outer connected monophonic vertices of $G$.

The next theorem and corollary are an immediate consequence of the definitions of total outer connected monophonic vertex and a forcing total outer connected monophonic subset of $G$.

**Theorem 7.** Let $G$ be a connected graph and let $\Psi_{tom}$ be the set of relative complements of the minimum forcing total outer connected monophonic subsets in their respective minimum total outer connected monophonic sets in $G$. Then $\bigcap_{F \in \Psi_{tom}} F$ is the set of all total outer connected monophonic vertices of $G$. 

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Corollary 1. Let $S$ be a minimum total outer connected monophonic set of $G$. Then no total outer connected monophonic vertex of $G$ belongs to any minimum forcing total outer connected monophonic subset of $S$.

Theorem 8. Let $M$ be the set of all total outer connected monophonic vertices of $G$. Then $f_{\text{tom}}(G) \leq cm_{\text{to}}(G) - |M|$.

Proof. Let $S$ be any minimum total outer connected monophonic set of $G$. Then $cm_{\text{to}}(G) = |S|$, $M \subseteq S$, and $S$ is the unique minimum total outer connected monophonic set containing $S - M$. Hence $f_{\text{tom}}(G) \leq |S - M| = |S| - |M| = cm_{\text{to}}(G) - |M|$.

Corollary 2. If $G$ is a connected graph with $l$ extreme vertices and $k$ support vertices, then $f_{\text{tom}}(G) \leq cm_{\text{to}}(G) - (l + k)$.

Remark 3. The bound in Theorem 8 is sharp. For the graph $G$ given in Fig. 1, the minimum total outer connected monophonic sets of $G$ are $S_1 = \{v_1, v_2, v_4, v_5\}$, $S_2 = \{v_1, v_4, v_5, v_8\}$, $S_3 = \{v_1, v_2, v_5, v_6\}$ and $S_4 = \{v_1, v_5, v_6, v_8\}$ so that $cm_{\text{to}}(G) = 4$. It is clear that $f_{\text{tom}}(S_i) = 2(i = 1, 2, 3, 4)$ and so $f_{\text{tom}}(G) = 2$. Also, $M = \{v_1, v_3\}$ is the set of all total outer connected monophonic vertices of $G$ and so $f_{\text{tom}}(G) = cm_{\text{to}}(G) - |M|$. The inequality in Theorem 8 can be strict. For the graph $G$ given in Fig. 2, the minimum total outer connected monophonic sets of $G$ are $M_1 = \{v_1, v_2, v_3, v_6\}$, $M_2 = \{v_3, v_4, v_5, v_6\}$, $M_3 = \{v_2, v_3, v_4, v_6\}$ and so $cm_{\text{to}}(G) = 4$. It is clear that $f_{\text{tom}}(M_i) = 1$ $(i = 1, 2)$, and so $f_{\text{tom}}(G) = 1$. Also, the vertices $v_3$ and $v_6$ are the total outer connected monophonic vertices of $G$, we have $f_{\text{tom}}(G) < cm_{\text{to}}(G) - |M|$.

Theorem 9. If $G$ is a connected graph with $cm_{\text{to}}(G) = 2$, then $f_{\text{tom}}(G) = 0$.

Proof. If $cm_{\text{to}}(G) = 2$ then by Theorem 4, we have $G = K_2$. Hence $V(G)$ is the unique minimum total outer connected monophonic set of $G$. Also, by Theorem 6(i), $f_{\text{tom}}(G) = 0$.

Remark 4. The converse of Theorem 9 need not be true. For the path $P_4$ of order 4, the vertex set $V(P_4)$ is the unique minimum total outer connected monophonic set of $G$ and so $cm_{\text{to}}(P_4) = 4$. By Theorem 6 (i), $f_{\text{tom}}(P_4) = 0$.

Theorem 10. For the complete bipartite graph $G = K_{m,n}(2 \leq m \leq n)$,

$$f_{\text{tom}}(G) = \begin{cases} m + n - 1 & \text{if } 2 = m \leq n, \\ 4 & \text{if } 3 \leq m \leq n. \end{cases}$$

Proof. Let $U = \{u_1, u_2, \ldots, u_m\}$ and $W = \{w_1, w_2, \ldots, w_n\}$ be the partite sets of $G$, where $m \leq n$. We prove this theorem by considering two cases.

Case 1. If $m = 2$, then it is clear that any minimum total outer connected monophonic sets of $G$ is of the form $V(G) - \{w_i\}(1 \leq i \leq n)$ or $V(G) - \{u_j\}(1 \leq j \leq m)$. It is easy to verify that, no minimum total outer connected monophonic set of $G$ is the unique
minimum total outer connected monophonic set containing any of its proper subsets. Then by Theorem 6 (iii), we have \( f_{tom}(G) = m + n - 1 \).

**Case 2.** If \( 3 \leq m \leq n \), then any minimum total outer connected monophonic set of \( G \) is obtained by choosing any two elements from \( U \) as well as \( W \), and \( G \) has at least two minimum total outer connected monophonic sets. Hence \( cm_{to}(G) = 4 \). Clearly, no minimum total outer connected monophonic set of \( G \) is the unique minimum total outer connected monophonic set containing any of its proper subsets. Then by Theorem 6 (iii), we have \( f_{tom}(G) = cm_{to}(G) = 4 \). \(

Theorem 11. For any cycle \( C_n(n \geq 3) \), \( f_{tom}(C_n) = \begin{cases} 0 & \text{if } n = 3, \\ 3 & \text{if } n = 4, \\ 2 & \text{if } n \geq 5. \end{cases} \)

**Proof.** Let \( C_n : v_1, v_2, \ldots, v_n, v_1 \) be a cycle of order \( n \). We prove this theorem by considering two cases.

**Case 1:** \( n = 3 \). Since \( C_3 \) is the complete graph of order 3, \( V(C_3) \) is the unique minimum total outer connected monophonic set of \( C_3 \). By Theorem 6 (i), \( f_{tom}(C_3) = 0 \).

**Case 2:** \( n \geq 4 \). It is clear that no 2-element subset of \( V(C_n) \) is a total outer connected monophonic set of \( C_n \). It is easy to verify that any minimum total outer connected monophonic set of \( C_n \) consists of three consecutive vertices of \( C_n \) so that \( cm_{to}(C_n) = 3 \). For \( n = 4 \), it is clear that no minimum total outer connected monophonic set of \( C_4 \) is the unique minimum total outer connected monophonic set containing any of its proper subsets. Thus by Theorem 6 (iii), we have \( f_{tom}(C_4) = 3 \). For \( n \geq 5 \), it is clear that the set of two non-adjacent vertices of any minimum total outer connected monophonic set \( S \) of \( C_n \) is a minimum forcing total outer connected monophonic subset of \( S \) and so \( f_{tom}(S) = 2 \). Hence \( f_{tom}(C_n) = 2 \). \(

Theorem 12. For the wheel \( W_n = K_1 + C_{n-1}(n \geq 5) \), \( f_{tom}(W_n) = \begin{cases} 3 & \text{if } n = 5, \\ 2 & \text{if } n \geq 6. \end{cases} \)

**Proof.** It is clear that no 2-element subset of \( V(W_n) \) is a total outer connected monophonic set of \( W_n \). It is easy to observe that any minimum total outer connected monophonic set of \( W_n \) consists of three consecutive vertices of \( C_{n-1} \) so that \( cm_{to}(W_n) = 3 \). For \( n = 5 \), it is clear that no minimum total outer connected monophonic set of \( W_5 \) is the unique minimum total outer connected monophonic set containing any of its proper subsets. Thus by Theorem 6 (iii), we have \( f_{tom}(W_5) = 3 \). For \( n \geq 6 \), it is clear that the set of two non-adjacent vertices of any minimum total outer connected monophonic set \( S \) of \( W_n \) is a minimum forcing total outer connected monophonic subset of \( S \) and so \( f_{tom}(S) = 2 \). Hence \( f_{tom}(W_n) = 2 \). \(

Theorem 13. For any complete graph \( G = K_p(p \geq 2) \) or any non-trivial tree \( G = T \), \( f_{tom}(G) = 0 \).

**Proof.** Let \( G = K_p \). By Theorem 2, the set of all vertices of \( G \) is the unique minimum total outer connected monophonic set of \( G \) and so by Theorem 6 (i), \( f_{tom}(G) = 0 \). If \( G \) is a non-trivial tree, then by Theorem 3, the set of all endvertices and support vertices of \( G \) is the unique minimum total outer connected monophonic set of \( G \) and by Theorem 6 (i), \( f_{tom}(G) = 0 \). \(

Theorem 14. For every pair \( a, b \) of integers such that \( 0 \leq a < b \) and \( b \geq a + 4 \), there is a connected graph \( G \) with \( f_{tom}(G) = a \) and \( cm_{to}(G) = b \).
Proof. If $a = 0$, let $G = K_a$. Then by Theorem 13, $f_{tom}(G) = 0$, and by Theorem 2, $cm_{to}(G) = b$. Now, assume that $0 < a < b$. The required graph $G$ is obtained from the star $K_{1,4}$ having the vertex set $\{z_1, z_2, z_3, z_4, z_5\}$ with $z_3$ as the cut-vertex by adding $a + b - 2$ new vertices $w_1, w_2, \ldots, w_a, v_1, v_2, \ldots, v_a, u_1, u_2, \ldots, u_{b-a-3}, x$ and joining each $w_i (1 \leq i \leq a)$ to the vertices $z_2, z_1$ and $z_4$; and joining each $v_i (1 \leq i \leq a)$ to the vertices $z_2, z_4$ and $z_5$, and joining each $u_i (1 \leq i \leq b-a-3)$ to the vertex $z_5$; and also joining the vertex $x$ to the vertex $z_1$, the vertex $z_1$ to the vertex $z_5$, and the vertex $z_2$ to the vertex $z_4$. The graph $G$ is shown in Fig. 3. Let $S = \{u_1, u_2, \ldots, u_{b-a-3}, x, z_1, z_5\}$ be the set of all endvertices and support vertices of $G$. By Theorem 1, every total outer connected monophonic set of $G$ contains $S$. It is clear that $S$ is not a total outer connected monophonic set of $G$. We observe that every minimum total outer connected monophonic set of $G$ contains exactly one vertex from the set $\{v_i, w_i\}$ for every $(1 \leq i \leq a)$. Thus $cm_{to}(G) \geq b$. Since $S_1 = S \cup \{w_1, w_2, \ldots, w_a\}$ is a total outer connected monophonic set of $G$, it follows that $cm_{to}(G) = b$.

Next, we show that $f_{tom}(G) = a$. Since every minimum total outer connected monophonic set of $G$ contains $S$, it follows from Theorem 8 that $f_{tom}(G) \leq cm_{to}(G) - |S| = b - (b - a) = a$. It is clear that every minimum total outer connected monophonic set $S'$ of $G$ is of the form $S \cup \{x_1, x_2, \ldots, x_a\}$, where $x_i \in \{v_i, w_i\}$ for every $(1 \leq i \leq a)$. Let $T$ be any proper subset of $S'$ with $|T| < a$. Then there is a vertex $x \in S' - S$ such that $x \notin T$. If $x = v_i (1 \leq i \leq a)$, then $S'' = (S' - \{v_i\}) \cup \{w_i\}$ is a minimum total outer connected monophonic set of $G$ containing $T$. Similarly, if $x = w_j (1 \leq j \leq a)$, then $S''' = (S' - \{w_j\}) \cup \{v_j\}$ is a minimum total outer connected monophonic set of $G$ containing $T$. Thus $S'$ is not the unique minimum total outer connected monophonic set containing $T$ and so $T$ is not a forcing total outer connected monophonic subset of $S'$. This is true for all minimum total outer connected monophonic sets of $G$ and so $f_{tom}(G) = a$. 

Fig. 3. A graph $G$ with $f_{tom}(G) = a > 0$ and $cm_{to}(G) = b > a$

References

1. Buckley F., Harary F. Distance in Graphs. Redwood City, CA, Addison-Wesley, 1990. 335 p.
2. Harary F. Graph Theory. Addison-Wesley, 1969. 274 p.
3. Costa E. R., Dourado M. C., Sampaio R. M. Inapproximability results related to monophonic convexity. Discrete Applied Mathematics, 2015, vol. 197, pp. 70–74. https://doi.org/10.1016/j.dam.2014.09.012
4. Dourado M. C., Protti F., Szwarcfiter J. L. Algorithmic aspects of monophonic convexity. Electronic Notes in Discrete Mathematics, 2008, vol. 30, pp. 177–182. https://doi.org/10.1016/j.endm.2008.01.031
5. Dourado M. C., Protti F., Szwarcfiter J. L. Complexity results related to monophonic convexity. Discrete Applied Mathematics, 2010, vol. 158, pp. 1268–1274. https://doi.org/10.1016/j.dam.2009.11.016

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6. Paluga E. M., Canoy S. R. Monophonic numbers of the join and composition of connected graphs. *Discrete Mathematics*, 2007, vol. 307, iss. 9–10, pp. 1146–1154. https://doi.org/10.1016/j.disc.2006.08.002

7. Santhakumaran A. P., Titus P., Ganesamoorthy K. On the monophonic number of a graph. *Journal of Applied Mathematics & Informatics*, 2014, vol. 32, iss. 1–2, pp. 255–266. https://doi.org/10.14317/JAMI.2014.255

8. Ganesamoorthy K., Murugan M., Santhakumaran A. P. Extreme-support total monophonic graphs. *Bulletin of the Iranian Mathematical Society*, 2021, vol. 47, pp. 159–170. https://doi.org/10.1007/s41980-020-00485-4

9. Ganesamoorthy K., Murugan M., Santhakumaran A. P. On the connected monophonic number of a graph. *International Journal of Computer Mathematics: Computer Systems Theory*, 2022, vol. 7, iss. 2, pp. 139–148. https://doi.org/10.1080/23799927.2022.2071765

10. Santhakumaran A. P., Titus P., Ganesamoorthy K., Murugan M. The forcing total monophonic number of a graph. *Proyecciones*, 2021, vol. 40, iss. 2, pp. 561–571. https://doi.org/10.22199/issn.0717-6279-2021-02-0031

11. Ganesamoorthy K., Lakshmi Priya S. The outer connected monophonic number of a graph. *Ars Combinatoria*, 2020, vol. 153, pp. 149–160.

12. Ganesamoorthy K., Lakshmi Priya S. Further results on the outer connected monophonic number of a graph. *Transactions of National Academy of Sciences of Azerbaijan. Series of Physical-Technical and Mathematical Sciences, Issue Mathematics*, 2021, vol. 41, iss. 4, pp. 51–59.

13. Ganesamoorthy K., Lakshmi Priya S. Extreme outer connected monophonic graphs. *Communications in Combinatorics and Optimization*, 2022, vol. 7, iss. 2, pp. 211–226. https://dx.doi.org/10.22049/cc.o.2021.27042.1184