Min-Max decoding for non binary LDPC codes

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Abstract—Iterative decoding of non-binary LDPC codes is currently performed using either the Sum-Product or the Min-Sum algorithms or slightly different versions of them. In this paper, several low-complexity quasi-optimal iterative algorithms are proposed for decoding non-binary codes. The Min-Max algorithm is one of them and it has the benefit of two possible LLR domain implementations: a standard implementation, whose complexity scales as the square of the Galois field’s cardinality and a reduced complexity implementation called selective implementation, which makes the Min-Max decoding very attractive for practical purposes.

Index Terms—LDPC codes, graph codes, iterative decoding.

I. INTRODUCTION

Although already proposed by Gallager in his PhD thesis [1], [2] the use of non binary LDPC codes is still very limited today, mainly because of the decoding complexity of these codes. The optimal iterative decoding is performed by the Sum-Product algorithm [3] at the price of an increased complexity, computation instability, and dependence on thermal noise estimation errors. The Min-Sum algorithm [3] performs a sub-optimal iterative decoding, less complex than the Sum-Product decoding, and independent of thermal noise estimation errors. The sub-optimality of the Min-Sum decoding comes from the overestimation of extrinsic messages computed within the check-node processing.

The Sum-Product algorithm can be efficiently implemented in the probability domain using binary Fourier transforms [4] and its complexity is dominated by $O(q \log_2 q)$ sum and product operations for each check node processing, where $q$ is the cardinality of the Galois field of the non-binary LDPC code. The Min-Sum decoding can be implemented either in the log-probability domain or in the log-likelihood ratio (LLR) domain and its complexity is dominated by $O(q^2)$ sum operations for each check node processing. In the LLR domain, a reduced selective implementation of the Min-Sum decoding, called Extended Min-Sum, was proposed in [5], [6]. Here “selective” means that the check-node processing uses the incoming messages concerning only a part of the Galois field elements. Non binary LDPC codes were also investigated in [7], [8], [9].

In this paper we propose several new algorithms for decoding non-binary LDPC codes, one of which is called the Min-Max algorithm. They are all independent of thermal noise estimation errors and perform quasi-optimal decoding — meaning that they present a very small performance loss with respect to the optimal iterative decoding (Sum-Product). We also propose two implementations of the Min-Max algorithm, both in the LLR domain, so that the decoding is computationally stable: a "standard implementation" whose complexity scales as the square of the Galois field’s cardinality and a reduced complexity implementation called “selective implementation". That makes the Min-Max decoding very attractive for practical purposes.

The paper is organized as follows. In the next section we briefly review several realizations of the Min-Sum algorithm for non binary LDPC codes. It is intended to keep the paper self-contained but also to justify some of our choices regarding the new decoding algorithms introduced in section III. The implementation of the Min-Max decoder is discussed in section IV. Section V presents simulation results and section VI concludes this paper.

The following notations will be used throughout the paper.

Notations related to the Galois field:

- $GF(q) = \{0, 1, \ldots, q-1\}$, the Galois field with $q$ elements, where $q$ is a power of a prime number. Its elements will be called symbols, in order to be distinguished from ordinary integers.
- $a, s, x$ will be used to denote $GF(q)$-symbols.
- $a, s, x$ will be used to denote vectors of $GF(q)$-symbols. For instance, $a = (a_1, \ldots, a_i) \in GF(q)^i$, etc.

Notations related to LDPC codes:

- $H \in M_{M \times N}(GF(q))$, the $q$-ary check matrix of the code.
- $C$, set of codewords of the LDPC code.
- $C_n(a)$, set of codewords with the $n^{th}$ coordinate equal to $a$; for given $1 \leq n \leq N$ and $a \in GF(q)$.
- $x = (x_1, x_2, \ldots, x_N)$ a $q$-ary codeword transmitted over the channel.

The Tanner graph associated with an LDPC code consists of $N$ variable nodes and $M$ check nodes representing the $N$ columns and the $M$ lines of the matrix $H$. A variable node and a check node are connected by an edge if the corresponding element of matrix $H$ is not zero. Each edge of the graph is labeled by the corresponding non zero element of $H$.

Notations related to the Tanner graph:

- $\mathcal{H}$, the Tanner graph of the code.
- $n \in \{1, 2, \ldots, N\}$ a variable node of $\mathcal{H}$.
- $m \in \{1, 2, \ldots, M\}$ a check node of $\mathcal{H}$.
- $\mathcal{H}(n)$, set of neighbor check nodes of the variable node $n$.
- $\mathcal{H}(m)$, set of neighbor variable nodes of the check node $m$.
- $\mathcal{L}(m)$, set of local configurations verifying the check node $m$; i.e. the set of sequences of $GF(q)$-symbols $a = (a_n)_{n \in \mathcal{H}(m)}$, verifying the linear constraint:

$$\sum_{n \in \mathcal{H}(m)} h_{m,n} a_n = 0$$

- $\mathcal{L}(m \mid a_n = a)$, set of local configurations $a$ verifying $m$, such that $a_n = a$; for given $n \in \mathcal{H}(m)$ and $a \in GF(q)$.

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An iterative decoding algorithm consists of an initialization step followed by an iterative exchange of messages between variable and check nodes connected in the Tanner graph.

Notations related to an iterative decoding algorithm:
- $\gamma_n(a)$, the a priori information of the variable node $n$ concerning the symbol $a$.
- $\bar{\gamma}_n(a)$, the a posteriori information of the variable node $n$ concerning the symbol $a$.
- $\alpha_{m,n}(a)$, the message from the check node $m$ to the variable node $n$ concerning the symbol $a$.
- $\beta_{m,n}(a)$, the message from the variable node $n$ to the check node $m$ concerning the symbol $a$.

II. Realizations of the Min-Sum Decoding for Non Binary LDPC Codes

A. Min-Sum decoding

The Min-Sum decoding is generally implemented in the log-probability domain and it performs the following operations:

Initialization
- A priori information: $\gamma_n(a) = -\ln(\Pr(x_n = a | \text{channel}))$

Variable node messages: $\alpha_{m,n}(a) = \gamma_n(a)$

Iterations
- Check node processing
  \[ \beta_{m,n}(a) = \min_{(a_{n'}, a' \in H(m) \setminus \{n\})} \left( \sum_{a' \in H(m) \setminus \{n\}} \alpha_{m,n'}(a_{n'}) \right) \]

- Variable node processing
  \[ \alpha_{m,n}(a) = \bar{\gamma}_n(a) + \sum_{m' \in H(n) \setminus \{m\}} \beta_{m',n}(a) \]

- A posteriori information
  \[ \bar{\gamma}_n(a) = \gamma_n(a) + \sum_{m \in H(n)} \beta_{m,n}(a) \]

For practical purposes, messages $\alpha_{m,n}(a)$ and $\beta_{m,n}(a)$ should be normalized in order to avoid computational instability (otherwise they could “escape” to infinity). The check node processing, which dominates the decoding complexity, can be implemented using a forward-backward computation method.

Assuming that the Tanner graph is cycle free, the a posteriori information converges after finitely many iterations to [3]:

\[ \bar{\gamma}_n(a) = \min_{{a \in c}} \left( \sum_{k=1}^{N} \gamma_k(a_k) \right) \]

Moreover, if $s_n = \arg \max_{a \in GF(q)}(\bar{\gamma}_n(a))$ is the most likely symbol according to the a posteriori information computed above, then the sequence $s = (s_1, s_2, \ldots, s_N)$ is a codeword (this is no longer true if the Tanner graph contains cycles) and it can be easily verified that $s = \arg \max_{s \in GF(q)}(\Pr(x = s | \text{channel}))$.

Thus, under the cycle free assumption the Min-Sum decoding always outputs a codeword and the above equality looks rather like a maximum likelihood decoding, which is in contrast with the sub-optimality property. In [3], the author concludes that the Min-Sum decoding is optimal in terms of block error probability, but our explanation is quite different. What really happens is that probabilities from the above equality do not take into account the dependence relations between codeword’s symbols (or equivalently between graph’s variable nodes). Taking into account only the channel observation and no prior information about dependence relations between codeword’s symbols, we have $\Pr(x = a | \text{channel}) = \prod_{n=1}^{N} \Pr(x_n = a_n | \text{channel})$, therefore even a sequence $x$ which is not a codeword has a non zero probability. In some sense, the Min-Sum decoding converges to a maximum likelihood decoding, distorted by dependence relations between codeword’s symbols.

B. Equivalent iterative decoders

The term of equivalent (iterative) decoders will be employed several times through this paper. We begin this section by providing its rigorous definition. Consider the a posteriori information available at a variable node $n$ after the $l$th decoding iteration: it defines an order between the symbols of the Galois field, starting with the most likely symbol and ending with the least likely one. Note that the most likely symbol may correspond to the minimum or to the maximum of the a posteriori information, depending on the decoding algorithm. We call this order the a posteriori order of variable node $n$ at iteration $l$.

We say that two decoders are equivalent if, for each variable node $n$, they both induce the same a posteriori order at any iteration $l$. In particular, assuming that the hard decoding corresponds to the most likely symbol, both decoders output the same sequence of $GF(q)$-symbols.

Two iterative decoders, which are equivalent to the Min-Sum decoder are presented below.

1) Min-Sum decoding: The Min-Sum decoding performs the following operations.

Initialization
- A priori information
  \[ \gamma_n(a) = \ln(\Pr(x_n = 0 | \text{channel})/\Pr(x_n = a | \text{channel})) \]

Variable node messages: $\alpha_{m,n}(a) = \gamma_n(a)$

Iterations
- Check node processing: same as for Min-Sum (II-A)
- Variable node processing: same as for Min-Sum (II-A)

A posteriori information: same as for Min-Sum (II-A)

We note that the exchanged messages represent log-likelihood ratios with respect to a fixed symbol (here, the symbol $0 \in GF(q)$ is used, but obviously other symbols may be considered). Its main advantage with respect to the classical

\[ \text{We assume that all codewords are sent equally likely.} \]
Min-Sum decoding is that it is computationally stable, so that there is no need to normalize the exchanged messages. This algorithm is also known as the Extended Min-Sum algorithm [5, 6].

2) Min-Sum decoding: The Min-Sum decoding performs the following operations.

Initialization

- A priori information
  \[ \gamma_n(a) = \ln \left( \frac{\Pr(x_n = s_n \mid \text{channel})}{\Pr(x_n = a \mid \text{channel})} \right) \]
  where \( s_n \) is the most likely symbol for \( x_n \).
- Variable node messages: \( \alpha_{m,n}(a) = \gamma_n(a) \)

Iterations

- Check node processing: same as for Min-Sum [II-A]
- Variable node processing
  \[ \alpha'_{m,n}(a) = \gamma_n(a) + \sum_{m' \in \mathcal{H}(m) \setminus \{m\}} \beta_{m',n}(a) \]
  \[ \alpha_{m,n} = \min_{a \in \text{GF}(q)} \alpha'_{m,n}(a) \]
  \[ \alpha_{m,n}(a) = \alpha'_{m,n}(a) - \alpha_{m,n} \]
- A posteriori information: same as for Min-Sum [II-A]

As the Min-Sum decoding, the Min-Sum decoding also performs in the LLR domain and is computationally stable. The main difference is that the exchanged messages represent log-likelihood ratios with respect to the most likely symbol, which may vary from a variable node to another, or within a fixed variable node, from an iteration to the other.

Theorem 1: The Min-Sum, Min-Sum_0 and Min-Sum_\infty decoders are equivalent.

It follows that the Min-Sum decoding does not present any practical interest: the equivalent Min-Sum_0 decoding is already computationally stable and less complex (it does not require the minimum computation in the variable node processing step). However, the Min-Sum motivates the decoding algorithms that will be introduced in the following section. The fundamental observation is that messages concerning most likely symbols are always equal to zero, and messages concerning the other symbols are positive. Therefore, exchanged messages can be seen as metrics indicating how far is a given symbol from the most likely one. This will be developed in the next section.

III. MIN-NORM DECODING FOR NON BINARY LDPC CODES

According to the discussion in the above section, we interpret variable node messages as metrics indicating the distance between a given symbol and the most likely one. Consider a check node \( m \) and a variable node \( n \in \mathcal{H}(m) \). Using the received extrinsic information, i.e. metrics received from variable nodes \( n' \in \mathcal{H}(m) \setminus \{n\} \), the check node \( m \) has to evaluate:

- the most likely symbol for variable node \( n \),
- how far the other symbols are from the most likely one.

To simplify notation, we set \( \mathcal{H}(m) = \{n, n_1, \ldots, n_d\} \) and let \( s_i \) be the most likely symbols corresponding to variable node \( n_i \), \( i = 1, \ldots, d \). Since the linear constraint corresponding to the check node \( m \) has to be satisfied, the most likely symbol for the variable node \( n \) is the unique symbol \( s \in \text{GF}(q) \), such that:

\[ (s, s_1, \ldots, s_d) \in \mathcal{L}(m) \]

On the other hand, each symbol \( a \in \text{GF}(q) \) corresponds to a set of \( d \)-tuples \( (a_1, \ldots, a_d) \), such that \( (a, a_1, \ldots, a_d) \in \mathcal{L}(m) \):

\[ \mathcal{L}_a(m) = \{(a_1, \ldots, a_d) \mid (a, a_1, \ldots, a_d) \in \mathcal{L}(m)\} \]

Thus, identifying \( s \equiv (s_1, \ldots, s_d) \) and \( a \equiv \mathcal{L}_a(m) \), the distance between the most likely symbol \( s \) and the symbol \( a \) can be evaluated as the distance from the sequence \( (s_1, \ldots, s_d) \) to the set \( \mathcal{L}_a(m) \). As usual, we consider that the distance from a point to a set is equal to the minimum distance between the given point and the points of the given set. In addition, we have to specify a rule that computes the distance between two sequences \( (s_1, \ldots, s_d) \) and \((a_1, \ldots, a_d)\), taking into account the received “marginal distances” between \( s_i \) and \( a_i \)’s. This can be done by using the distance associated with one of the \( p \)-norms \( (p \geq 1) \), or the infinity (also called maximum) norm on an appropriate multidimensional vector space. Precisely:

\[ \beta_{m,n}(a) = \min \{ \{ s_1, \ldots, s_d \} \} \text{dist}(\{(s_1, \ldots, s_d), (a_1, \ldots, a_d)\}) \]

\[ = \min_{(a_1, \ldots, a_d) \in \mathcal{L}_a(m)} \text{dist}(s_1, a_1, \ldots, \text{dist}(s_d, a_d) \|_p \]

\[ = \min_{(a_1, \ldots, a_d) \in \mathcal{L}_a(m)} \alpha_1(a_1), \ldots, \alpha_d(a_d) \|_p \]

Note that for \( p = 1 \) we obtain the Min-Sum_\infty decoding from the above section. Taking into account that:

\[ \| \|_1 \geq \| \|_2 \geq \cdots \geq \| \|_\infty \]

it follows that using any of the \( \| \|_p \) or \( \| \|_\infty \) norms would reduce the value of check node messages, which is desirable, since these messages are actually overestimated by the Min-Sum algorithm. One could think that the infinity norm would excessively reduce check node messages values, but this is not true. In fact, the distance between \( \| \|_1 \) and the infinity norm is less than twice the distance between \( \| \|_1 \) and the Euclidian \( (p = 2) \) norm. In practice, it turns out that the infinity norm induces a more accurate computation of check node messages than \( \| \|_1 \).

We focus now on the Euclidean and the infinity norms. The derived decoding algorithms are called Euclidean decoding, respectively Min-Max decoding. They perform the same initialization step, variable nodes processing, and a posteriori information update as the Min-Sum, decoding; therefore only the check node processing step will be described below.

A. Euclidean decoding

- Check nodes processing

\[ \beta_{m,n}(a) = \min_{(a_{n'}) \in \mathcal{H}(m) \setminus \{n\}} \left( \sum_{n' \in \mathcal{H}(m) \setminus \{n\}} \alpha_{m,n'}^2(a_{n'}) \right)^{1/2} \]
B. Min-Max decoding

• Check nodes processing
  \[ \beta_{m,n}(a) = \min_{a' \in H(m) \setminus \{a\}, n'} \max_{n' \in H(m) \setminus \{n\}, a} \alpha_{m,n'}(a) \]

**Theorem 2:** Over GF(2), any decoding algorithm using any one of the \(p\)-norms, with \(p \geq 1\), or the infinity-norm is equivalent to the Min-Sum decoding. In particular, the Min-Sum, Min-Max, and Euclidian decodings are equivalent.

IV. IMPLEMENTATION OF THE MIN-MAX DECODER

In this section we give details about the implementation of the check node processing within the Min-Max decoder. First, we give a standard implementation using a well-known forward-backward computation technique. Next, we show that the computations performed within the standard implementation do not need to use the information concerning all symbols of the Galois field and, consequently, we derive the so-called selective implementation of the Min-Max decoder.

A. Standard implementation of the Min-Max decoder

Let \(m\) be a check node and \(H(m) = \{n_1, n_2, \ldots, n_d\}\) be the set of variable nodes connected to \(m\) in the Tanner graph.

We recursively define forward and backward metrics, \((F_i)_{i=1,d-1}\) and respectively \((B_i)_{i=2,d}\) as follows:

**Forward metrics**

\[ F_i(a) = \alpha_{m,n_i}(h_{m,n_i}^{-1} a) \]
\[ F_i(a) = \min_{a' \in GF(q) \setminus \{a\}, a'' \in GF(q) \setminus \{a\}} \max(F_{i-1}(a'), \alpha_{m,n_i}(a'')) \]

**Backward metrics**

\[ B_i(a) = \alpha_{m,n_i}(h_{m,n_i}^{-1} a) \]
\[ B_i(a) = \min_{a' \in GF(q) \setminus \{a\}, a'' \in GF(q) \setminus \{a\}} \max(B_{i+1}(a'), \alpha_{m,n_i}(a'')) \]

Then check node messages can be computed as follows:

\[ \beta_{m,n_i}(a) = B_2(a) \]
\[ \beta_{m,n_i}(a) = B_{d-1}(a) \]
\[ \beta_{m,n_i}(a) = \min_{a' \in GF(q) \setminus \{a\}} \max(F_{i-1}(a'), B_{i+1}(a'')) \]

B. Selective implementation of the Min-Max decoder

In this section we focus on the building blocks of the standard implementation, which are min-max computations of the following type:

\[ f(a) = \min_{h_a' + h_a'' \in GF(q) \setminus \{0\}} \max(f'(a'), f''(a'')) \]

The computation of \(f\) requires \(O(q^2)\) comparisons. Our goal is to reduce this complexity by reducing the number of symbols \(a'\) and \(a''\) involved in the min-max computation. We start with the following proposition.

**Proposition 3:** Let \(\Delta', \Delta'' \subseteq GF(q)\) be two subsets of the Galois field, such that \(\text{card}(\Delta') + \text{card}(\Delta'') \geq q + 1\). Then for any \(a \in GF(q)\), there exist \(a' \in \Delta'\) and \(a'' \in \Delta''\) such that:

\[ ha = h_a' + h_a'' a'' \]

**Corollary 4:** Let \(\Delta', \Delta'' \subseteq GF(q)\) be such that the set \(\{f'(a') \mid a' \in \Delta'\} \cup \{f''(a'') \mid a'' \in \Delta''\}\) contains the \(q + 1\) lowest values of the set \(\{f'(a') \mid a' \in GF(q)\} \cup \{f''(a'') \mid a'' \in GF(q)\}\) Then for any \(a \in GF(q)\) the following equality holds:

\[ f(a) = \min_{h_a' + h_a'' \in GF(q) \setminus \{0\}} \max(f'(a'), f''(a'')) \]

**Example 5:** Assume that we have to compute the values of \(f\) and the base Galois field is \(GF(8)\). We proceed as follows:

• Determine the 9 smallest values between the 16 values of the set \(\{f'(0), f'(1), \ldots, f'(7), f''(0), f''(1), \ldots, f''(7)\}\).

Let us assume for instance that the 9 smallest values are \(\{f'(1), f'(2), f'(3), f'(4), f'(6), f''(7), f''(0), f''(1), f''(5)\}\).

• Set \(\Delta' = \{1, 2, 3, 4, 6, 7\}\) and \(\Delta'' = \{0, 1, 5\}\), then compute the values of \(f\) using only the symbols in \(\Delta'\) and \(\Delta''\).

In this way the computation of the 8 values of \(f\) takes only \(6 \times 3 = 18\) comparisons instead of 64.

In general, let \(q' = \text{card}(\Delta')\) and \(q'' = \text{card}(\Delta'')\), where the subsets \(\Delta'\) and \(\Delta''\) are assumed to satisfy the conditions of the above corollary (then \(q' + q'' \geq q + 1\)). If \(q' + q'' = q + 1\), then the complexity of the “min-max” computation can be reduced by at least a factor of four, from \(q^3\) to \(q'q'' \leq q^2 (q + 1)\).

The main problem we encounter is that we should sort the \(2q\) values of \(f'\) and \(f''\) in order to figure out which symbols participate in the sets \(\Delta'\) and \(\Delta''\). In order to avoid the sorting process, we generate the subsets:

\[ \Delta'_k = \{a' \in GF(q) \mid [f'(a')] = k\} \]
\[ \Delta''_k = \{a'' \in GF(q) \mid [f''(a'')] = k\} \]

where \([x]\) denotes the integer part of \(x\). We then define:

\[ \Delta' = \Delta'_0 \cup \cdots \cup \Delta'_t \]
\[ \Delta'' = \Delta''_0 \cup \cdots \cup \Delta''_t \]

starting with \(t = 0\) and increasing its value until we get \(\text{card}(\Delta') + \text{card}(\Delta'') \geq q + 1\).

The use of the sets \(\Delta'\) and \(\Delta''\) has a double interest:

• it reduces the number of symbols required by (and then the complexity of) the min-max computation,

• most of the maxima computations become obsolete. Indeed, \(\max(f'(a'), f''(a''))\) is calculated only if symbols \(a'\) and \(a''\) belong to sets of the same index (i.e. \(a' \in \Delta'_k\) and \(a'' \in \Delta''_k\) with \(k' = k'')\). Otherwise, the maximum corresponds to the symbol belonging to the set of maximal index.

Finally, the proposed approach is carried out in practice as follows:

• The received a priori information is normalized such that its average value is equal to a predefined constant \(A1\) for “Average a priori Information”). This constant is chosen such that the integer part provides a sharp criterion to distinguish between the symbols of the Galois field, meaning that sets \(\Delta'_k\) and \(\Delta''_k\) only contain few elements.

• We use a predefined threshold \(\text{cot}\) (for “Cut Off Threshold”) representing the maximum value of variable node messages. Thus, incoming variable node messages greater than or
Min-Sum decoding of non binary LDPC codes may be seen equal to the predefined threshold will not participate in the min-max computations for check node processing. It follows that the rank \( k \) of subsets \( \Delta_k^L \) and \( \Delta_k^R \) ranges from 0 to \( \text{COT} - 1 \). In this case, the subsets \( \Delta_k^L \) and \( \Delta_k^R \) may contain all together less than \( q+1 \) symbols, in which case the min-max complexity may be significantly reduced. In practice, this generally occurs during the last iterations, when decoding is quite advanced and doubts persist only about a reduced number of symbols.

Constants \( A1 \) and \( \text{COT} \) have to be determined by Monte Carlo simulation and they only depend on the cardinality of the Galois field.

V. SIMULATION RESULTS

We present Monte-Carlo simulation results for coding rate 1/2 over the AWGN channel. Fig. 1 presents the performance of \( \text{GF}(16) \)-LDPC codes with the 16-QAM modulation (Galois field symbols are mapped to constellation symbols). We note that the Euclidian and the Min-Max (standard and selective implementations) algorithms achieve nearly the same decoding performance, and the gap to the Sum-Product decoding performance is only of 0.2 dB. For the selective implementation we used constants \( A1 = 12 \) and \( \text{COT} = 31 \).

Fig. 2 presents the decoding complexity in number of operations per decoded bit (all decoding iterations included) of the Min-Sum, the standard and the selective Min-Max decoders over \( \text{GF}(16) \). The Min-Sum and the standard Min-Max decoders perform nearly the same number of operations per iteration, but the second decoder has better performance and needs a smaller number of iterations to converge. On the contrary, the standard and the selective Min-Max decoders have the same performance and they perform the same number of decoding iterations. Simply, the selective decoder takes a smaller number of operations to perform each decoding iteration, which explains its lower complexity (by a factor of 4 with respect to the standard implementation).

VI. CONCLUSIONS

We shown that the extrinsic messages exchanged within the Min-Sum decoding of non binary LDPC codes may be seen as metrics indicating the distance between a given symbol and the most likely one. By using appropriate metrics, we have derived several low-complexity quasi-optimal iterative algorithms for decoding non-binary codes. The Euclidian and the Min-Max algorithms are ones of them. Furthermore, we gave a canonical selective implementation of the Min-Max decoding, which reduces the number of operations taken to perform each decoding iteration, without any performance degradation. The quasi-optimal performance together with the low complexity make the Min-Max decoding very attractive for practical purposes.

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APPENDIX I

ALTERNATIVE REALIZATIONS OF THE MIN-SUM ALGORITHM

In this section we introduce two alternative realizations, called Min-Sum0 and Min-Sum1, of the Min-Sum algorithm. We prove that they are equivalent to the MSA algorithm of section and, over \( \text{GF}(2) \), the Min-Sum1 is equivalent to the Min-Max algorithm. We assume that the Tanner graph \( \mathcal{H} \) is cycle free thought this section.

\textsuperscript{2}Nevertheless, this assumption can be removed by replacing \( \mathcal{H} \) with computation trees and “codewords” with “pseudo-codewords”
Notation.
- \( n \in \mathcal{N} = \{1, 2, \ldots, N\} \) a variable node of \( \mathcal{H} \).
- \( m \in \mathcal{M} = \{1, 2, \ldots, M\} \) a check node of \( \mathcal{H} \).
- For each check node \( m \in \mathcal{M} \) and each sequence \( (a_n)_{n \in \mathcal{H}(m)} \) of \( \mathbb{G}(q) \)-symbols indexed by the set of variable nodes connected to \( m \), we note:
  \[
  m(\langle a_n \rangle_{n \in \mathcal{H}(m)}) = \sum_{n \in \mathcal{H}(m)} h_{m,n} a_n
  \]

The sequence \( (a_n)_{n \in \mathcal{H}(m)} \) is said to satisfy the check node \( m \) if \( m(\langle a_n \rangle_{n \in \mathcal{H}(m)}) = 0 \). Thus
  \[
  m(\langle a_n \rangle_{n \in \mathcal{H}(m)}) = 0 \iff (a_n)_{n \in \mathcal{H}(m)} \in \mathcal{L}(m)
  \]

The Min-Sum\(_0\) decoding performs the following computations:

**Initialization**
- A priori information
  \[
  \gamma_n(a) = \ln \left( \frac{\Pr(x_n = 0 \mid \text{channel observation})}{\Pr(x_n = a \mid \text{channel observation})} \right)
  \]

- Variable to check messages initialization
  \[
  \alpha_{m,n}(a) = \gamma_n(a)
  \]

**Iterations**
- Check to variable messages
  \[
  \beta_{m,n}(a) = \min_{a' \in \mathcal{C}, (a_n)_{n \in \mathcal{H}(m)} \in \mathcal{L}(m) \setminus \{n\}} \left( \sum_{n' \in \mathcal{H}(m) \setminus \{n\}} \alpha_{m,n'}(a_{n'}) \right)
  \]

- Variable to check messages
  \[
  \alpha'_{m,n}(a) = \gamma_n(a) + \sum_{n' \in \mathcal{H}(m) \setminus \{m\}} \beta_{m',n}(a)
  \]
  \[
  \alpha_{m,n}(a) = \alpha'_{m,n}(a) - \alpha_{m,n}(0)
  \]

- A posteriori information
  \[
  \tilde{\gamma}_n(a) = \gamma_n(a) + \sum_{m \in \mathcal{H}(n)} \beta_{m,n}(a)
  \]

**Theorem 6:** The Min-Sum\(_0\) decoding converges after finitely many iterations to

\[
\tilde{\gamma}_n(a) = \min_{a \in \mathcal{C}} a \cdot \mathcal{V}^{(1)} \min_{a \in \mathcal{C}} a \cdot \mathcal{V}^{(2)} \min_{a \in \mathcal{C}} a \cdot \mathcal{V}^{(3)} \ldots \min_{a \in \mathcal{C}} a \cdot \mathcal{V}^{(N)} \gamma_k(a) - \gamma_k(a) - \sum_{k \in \mathcal{N}} \gamma_k(a)
\]

where \( \mathcal{V}^{(1)} := \bigcup_{m \in \mathcal{H}(n)} \mathcal{H}(m) \) is the set of variable nodes separated by at most 1 check node from \( n \) (including \( n \)).

**Proof:** The proof is derived in same manner as that of theorem 3.1 in [3]. The critical point is that we have to control the effect of withdrawing the \( \alpha_{m,n}(0) \) message from all the others \( \alpha_{m,n}(a) \) messages. Fix a variable node \( n \) and let:

\[
\begin{align*}
  f(a) &= \min_{a \in \mathcal{C}, a_n = a} \sum_{k \in \mathcal{N}} \gamma_k(a_k) \\
  f &= \min_{a \in \mathcal{C}, a_n = a} \sum_{k \in \mathcal{N}} \gamma_k(a_k) \\
  \tilde{f}(a) &= f(a) - f
\end{align*}
\]

We have to prove that, after finitely many iterations, the equality \( \tilde{\gamma}_n(a) = \tilde{f}(a) \) holds. Since the Tanner graph \( \mathcal{H} \) is cycle free, it can be seen as a tree graph rooted at \( n \). Let \( H(n) = \{m_1, m_2, \ldots, m_d\} \) and, for \( j = 1, 2, \ldots, d \), let \( H_j \) be the sub-graph of \( \mathcal{H} \) emanating from the check node \( m_j \), as represented below:

![Graphical representation of sub-graphs \( \mathcal{H}_j \)](image)

Let also \( C_j \) be the linear code corresponding to the sub-graph \( \mathcal{H}_j \cup \{n\} \). Since \( \gamma_k(0) = 0 \) for any \( k = 1, \ldots, N \), we have

\[
\begin{align*}
  f &= \sum_{j=1}^{d} f_j, \quad \text{where} \quad f_j = \min_{a \in C_j} \sum_{k \in H_j} \gamma_k(a_k). \text{ Therefore,}
  \end{align*}
\]

\[
\begin{align*}
  \tilde{f}(a) &= \min_{a \in C, a_n = a} \sum_{k \in H_j} \gamma_k(a_k) - f
  &= \gamma_n(a) + \sum_{j=1}^{d} \left( \min_{a \in C_j} \sum_{k \in H_j} \gamma_k(a_k) \right) - \sum_{j=1}^{d} f_j
  \end{align*}
\]

Denoting \( g_{m_j}(a) = \min_{a \in C_j} \left( \sum_{k \in H_j} \gamma_k(a_k) \right) - f_j \) we get

\[
\begin{align*}
  \tilde{f}(a) &= \gamma_n(a) + \sum_{j=1}^{d} g_{m_j}(a)
\end{align*}
\]

This formula has the same structure as the update rule of the a posteriori information \( \tilde{\gamma}_n(a) \). It follows that the equality \( \tilde{\gamma}_n(a) = \tilde{f}(a) \) holds provided that \( \beta_{m,n}(a) = g_{m_j}(a) \). Due to the symmetry of the situation it suffices to carry out the case \( j = 1 \). For this, let \( H(m_1) = \{n, n_1, n_2, \ldots, n_d\} \) and, for \( j = 1, 2, \ldots, d \), let \( H_{1,j} \) be the sub-graph of \( \mathcal{H}_1 \) emanating from the variable node \( n_j \), as represented below:

![Graphical representation of sub-graphs \( \mathcal{H}_{1,j} \)](image)
Let also $C_{1,j}$ be the linear code corresponding to the sub-graph $H_{1,j}$ and define $t_j(a_j) = \min_{a \in C_{1,j}} \left( \sum_{k \in H_{1,j}} \gamma_k(a_k) \right)$. Then:

$$\min_{a \in C_{1,j}} \left( \sum_{k \in H_{1,j}} \gamma_k(a_k) \right) = \min_{(a_1, \ldots, a_d) \in \mathcal{C}(q) \cap \mathcal{C}(m)} \left( \sum_{j=1}^{d} t_j(a_j) \right)$$

and

$$f_1 = \min_{a \in C_{1}} \sum_{k \in H_{1,j}} \gamma_k(a_k) = \min_{(a_1, \ldots, a_d) \in \mathcal{C}(q) \cap \mathcal{C}(m)} \left( \sum_{j=1}^{d} \sum_{a \in C_{1,j}} \gamma_k(a_k) \right)$$

$$= \sum_{j=1}^{d} t_j(0)$$

Therefore:

$$g_{m_1}(a) = \min_{(a_1, \ldots, a_d) \in \mathcal{C}(q) \cap \mathcal{C}(m)} \left( \sum_{j=1}^{d} t_j(a_j) \right) - \sum_{j=1}^{d} t_j(0)$$

$$= \min_{(a_1, \ldots, a_d) \in \mathcal{C}(q) \cap \mathcal{C}(m)} \left( \sum_{j=1}^{d} \left( t_j(a_j) - t_j(0) \right) \right)$$

Defining $t = \min_{a \in C_{1,j}} \left( \sum_{k \in H_{1,j}} \gamma_k(a_k) \right)$,

$$f'_{n_j}(a_j) = \min_{a \in C_{1,j}} \sum_{k \in H_{1,j}} \gamma_k(a_k) - \min_{a \in C_{1,j}} \sum_{k \in H_{1,j}} \gamma_k(a_k)$$

$$= t_j(a_j) - t_j(0)$$

and $f_{n_j}(a_j) = f'_{n_j}(a_j) - f'_{n_j}(0) = t_j(a_j) - t_j(0)$ we get

$$g_{m_1}(a) = \min_{(a_1, \ldots, a_d) \in \mathcal{C}(q) \cap \mathcal{C}(m)} \left( \sum_{j=1}^{d} f_{n_j}(a_j) \right)$$

This formula has the same structure as the update rule of the check to variable messages $\beta_{m_1,n}(a)$. It follows that the equality $\beta_{m_1,n}(a) = g_{m_1}(a)$ holds, provided that $\alpha_{m_1,n_j}(a) = f_{n_j}(a)$. This derives from the fact that $f_{n_j}(a)$ verifies the same update rule as $\alpha_{m_1,n_j}(a)$ or, equivalently, $f'_{n_j}(a)$ verifies the same update rule as $\alpha'_{m_1,n_j}(a)$. The proof can be proceeded in same manner as for $f(a)$, and then will be omitted.

This process is repeated recursively until the leaf variable nodes are reached. Finally, we remark that if $n_j$ where a leaf variable node, then $f_{n_j}(a_j) = \gamma_{n_j}(a_j) = \alpha_{m_1,n_j}(a_j)$ and so we are done.

The Min-Sum, decoding performs the following computations:

**Initialization**

- **A priori information**

$$\gamma_n(a) = \ln \left( \frac{\operatorname{Pr}(x_n = s_n \mid \text{channel observation})}{\operatorname{Pr}(x_n = a \mid \text{channel observation})} \right)$$

where $s_n$ is the most likely symbol for $x_n$.

- **Variable to check messages initialization**

$$\alpha_{m,n}(a) = \gamma_n(a)$$

- **Iterations**

  - **Check to variable messages**

$$\beta_{m,n}(a) = \min_{(a_1, \ldots, a_d) \in \mathcal{C}(m)} \left( \sum_{n' \in \mathcal{H}(m) \setminus \{m\}} \alpha_{m',n}(a_{n'}) \right)$$

  - **Variable to check messages**

$$\alpha'_{m,n}(a) = \gamma_n(a) + \sum_{n' \in \mathcal{H}(n) \setminus \{n\}} \beta_{m',n}(a_{n'})$$

$$\alpha_{m,n}(a) = \alpha'_{m,n}(a) - \alpha'_n$$

- **A posteriori information**

$$\tilde{\gamma}_n(a) = \gamma_n(a) + \sum_{m \in \mathcal{H}(n)} \beta_{m,n}(a)$$

Fixe a variable node $n$, note $\mathcal{H}_* = \mathcal{H} \setminus \{(n) \cup \mathcal{H}(n)\}$ and let $C_*$ be the linear code associated with $\mathcal{H}_*(a)$ (when a node of $\mathcal{H}$ is removed we also remove all the edges that are incident to the node). With this notation we have the following theorem.

**Theorem 7:** The Min-Sum, decoding converges after finitely many iterations to

$$\tilde{\gamma}_n(a) = \min_{a \in C_*(\mathcal{H}_*)} \sum_{k \in \mathcal{H}_*} \gamma_k(a_k) - \min_{a \in C_*(\mathcal{H}_*)} \sum_{k \in \mathcal{H}_*} \gamma_k(a_k)$$

The proof can be derived in same manner os that of the above theorem, and then will be omitted.

We also note that the convergence formula of the MSA decoder is (see theorem 3.1 in [3]):

$$\tilde{\gamma}_n(a) = \min_{a \in C_*(\mathcal{H}_*)} \sum_{k \in \mathcal{H}_*} \gamma_k(a_k)$$

For each of the Min-Sum and Min-Sum, decoders the convergence formula is obtained by withdrawing from the right hand side term of the above formula a term that is independent of $a$. Note also that similar formulas hold after any fixed number of iterations - to see this, it suffices to cut off the tree graph $\mathcal{H}_*$ at the appropriate depth. Therefore, we get the following:

**Corollary 8:** The MSA, Min-Sum and Min-Sum, decoders are equivalent.

We prove now that the MSA and the MMA decoders are equivalent over GF(2) (in a similar manner can be proved that over GF(2) the MSA decoder is equivalent to any other Min-$$\parallel$$ decoder.)

**Proof:** Since MSA and Min-Sum, decoder are equivalent, it suffices to prove that Min-Sum, and MMA decoders are equivalent over GF(2). In this case, the variable to check

3Note also that the a priori informations $\gamma_n(a)$ are not computed in the same way for the three decoders but they only differ by a term independent of $a$. Thus, for the MSA decoder $\gamma_n(a) = -\ln \left( \operatorname{Pr}(x_n = a) \right)$, for the Min-Sum0 decoder $\gamma_n(a) = -\ln \left( \operatorname{Pr}(x_n = a) \right) + \ln \left( \operatorname{Pr}(x_n = s_n) \right)$ and for the Min-Sum, decoder $\gamma_n(a) = -\ln \left( \operatorname{Pr}(x_n = a) \right) + \ln \left( \operatorname{Pr}(x_n = s_n) \right)$, all probabilities being conditioned by the channel observation.
messages \( \alpha_{m,n}(a) \) are non negative and they concern only two symbols \( a \in \{0, 1\} \). Moreover, for one of these symbols the corresponding variable to check message in zero (precisely, for the symbol realizing the minimum of \( \{ \alpha'_{m,n}(a), a = 0, 1 \} \)). Fix a check node \( m \), a variable node \( n \) and a symbol \( a \in \text{GF}(2) \), and let \( (\bar{a}_{n'})_{n' \in \mathcal{H}(m)\setminus\{n\}} \), such that \( m((\bar{a}_{n'})_{n'}, a) = 0 \), be the sequence realizing the minimum:

\[
\min_{(\alpha'_{n'})_{n' \in \mathcal{H}(m)}} \left( \sum_{n' \in \mathcal{H}(m)\setminus\{n\}} \alpha_{m,n'}(a_{n'}) \right)
\]

Then, for the Min-Sum decoder

\[
\beta_{m,n}(a) = \sum_{n' \in \mathcal{H}(m)\setminus\{n\}} \alpha_{m,n'}(\bar{a}_{n'})
\]

Suppose that there are two symbols, say \( \bar{a}_{n'} \) and \( \bar{a}_{n'} \), of the sequence \( (\bar{a}_{n'})_{n' \in \mathcal{H}(m)\setminus\{n\}} \), whose corresponding variable to check messages are not zero, i.e. \( \alpha_{m,n'}(\bar{a}_{n'}) > 0 \), \( i = 1, 2 \). Then, consider a new sequence \( (\bar{a}_{n'})_{n' \in \mathcal{H}(m)\setminus\{n\}} \), such that \( \bar{a}_{n'} = \bar{a}_{n'} \) if \( n' \neq n' \) and \( \bar{a}_{n'} = \bar{a}_{n'} + 1 \mod 2 \). This sequence still satisfies the check \( m \), i.e. \( m((\bar{a}_{n'})_{n'}, a) = 0 \) and

\[
\sum_{n' \in \mathcal{H}(m)\setminus\{n\}} \alpha_{m,n'}(\bar{a}_{n'}) < \sum_{n' \in \mathcal{H}(m)\setminus\{n\}} \alpha_{m,n'}(\bar{a}_{n'})
\]

as \( \alpha_{m,n'}(\bar{a}_{n'}) = 0 < \alpha_{m,n'}(\bar{a}_{n'}) \), what contradicts the minimality of the sequence \( (\bar{a}_{n'})_{n' \in \mathcal{H}(m)\setminus\{n\}} \). It follows that there is at most one symbol \( \bar{a}_{n'} \) whose corresponding variable to check message \( \alpha_{m,n'}(\bar{a}_{n'}) \neq 0 \). Consequently, the sequence \( (\bar{a}_{n'})_{n' \in \mathcal{H}(m)\setminus\{n\}} \) also realizes the minimum

\[
\min_{(\alpha'_{n'})_{n' \in \mathcal{H}(m)}} \left( \max_{n' \in \mathcal{H}(m)\setminus\{n\}} \alpha_{m,n'}(a_{n'}) \right)
\]

so the update rules for check to variable messages computes the same messages for both decoders.

\[\square\]

**APPENDIX II**

**PROOF OF PROPOSITION 5**

*Proof:* Let \( a \in \text{GF}(q) \). Consider the functions \( \varphi, \psi : \text{GF}(q) \rightarrow \text{GF}(q) \), defined by \( \varphi(x) = ha + h'x \), and \( \psi(x) = h''x \). Since \( \varphi \) and \( \psi \) are injective, we have:

\[
\text{card}(\varphi(\Delta')) + \text{card}(\psi(\Delta'')) = \text{card}(\Delta') + \text{card}(\Delta'') \geq q + 1
\]

It follows that \( \varphi(\Delta') \cap \psi(\Delta'') = \emptyset \), so there are \( a' \in \Delta' \), and \( a'' \in \Delta'' \) such that

\[
\varphi(a') = \psi(a'') \Leftrightarrow ha + h'a' = h''a'' \Leftrightarrow a = h^{-1}(h'a' + h''a'')
\]

\[\square\]