FIXED POINTS OF NORMAL COMPLETELY POSITIVE MAPS
ON $B(\mathcal{H})$

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Abstract. Given a sequence of bounded operators $a_j$ on a Hilbert space $\mathcal{H}$ with $\sum_{j=1}^{\infty} a_j^* a_j = 1 = \sum_{j=1}^{\infty} a_j a_j^*$, we study the map $\Psi$ defined on $B(\mathcal{H})$ by $\Psi(x) = \sum_{j=1}^{\infty} a_j^* x a_j$ and its restriction $\Phi$ to the Hilbert-Schmidt class $C^2(\mathcal{H})$. In the case when the sum $\sum_{j=1}^{\infty} a_j^* a_j$ is norm-convergent we show in particular that the operator $\Phi - 1$ is not invertible if and only if the $C^*$-algebra $\mathcal{A}$ generated by $\{a_j\}_{j=1}^{\infty}$ has an amenable trace. This is used to show that $\Psi$ may have fixed points in $B(\mathcal{H})$ which are not in the commutant $\mathcal{A}'$ of $\mathcal{A}$ even in the case when the weak* closure of $\mathcal{A}$ is injective. However, if $\mathcal{A}$ is abelian, then all fixed points of $\Psi$ are in $\mathcal{A}'$ even if the operators $a_j$ are not positive.

1. Introduction and notation

It is well known that all normal (= weak* continuous) completely positive maps on $B(\mathcal{H})$ (the algebra of all bounded operators on a separable Hilbert space $\mathcal{H}$) are of the form

\[ \Psi_a(x) = \sum_{j=1}^{\infty} a_j^* x a_j = \sum_{j=1}^{\infty} a_j^* x^{(\infty)} a_j, \]

where $a_j \in B(\mathcal{H})$ are such that the column $a := (a_j)$ represents a bounded operator from $\mathcal{H}$ to $\mathcal{H}^{\infty}$, and $x^{(\infty)}$ denotes the block-diagonal operator matrix with $x$ along the diagonal. The sum $a^* a = \sum_{j=1}^{\infty} a_j^* a_j$ is convergent in the strong (weak, weak*,...) operator topology. If $a^* a = 1$ (the identity operator on $\mathcal{H}$), then the map $\Psi_a$ is unital. $\Psi_a$ is dual to the map $\Psi_a^*$ defined on the trace class $T(\mathcal{H})$ by

\[ \Psi_{a^*}(t) = \sum_{j=1}^{\infty} a_j t a_j^*. \]

So, if we assume in addition that the sum $\sum_{j=1}^{\infty} a_j a_j^*$ is convergent in the strong operator topology, then the map $\Psi_a$ itself preserves $T(\mathcal{H})$. If moreover $\sum_{j=1}^{\infty} a_j a_j^* = 1$, then the map $\Psi_a |_{T(\mathcal{H})}$ preserves the trace (that is, $Tr(\Psi_a(t)) = Tr(t)$ for all $t \in T(\mathcal{H})$). Such maps are called unital quantum channels in quantum computation theory [20]. A selfadjoint operator $x \in B(\mathcal{H})$ which is fixed by $\Psi_a$ (that is, $\Psi_a(x) = x$) represents a physical quantity that passes unchanged through the quantum channel, so it is important to know the set $\mathcal{F}_a$ of all fixed points of $\Psi_a$. The structure of the set $\mathcal{F}_a$ is studied in several papers (see e.g. [3], [6], [21], [20]).

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But in these cases every operator that commutes with a $C'$ is a perturbation of an element of $A$. Other (non-closed) ideals, the simplest of which is the Hilbert–Schmidt class $C'$ adjoint on $B(H)$ (containing $A_K$ in the case $\dot{\mathcal{K}}$). Since $\dot{\mathcal{L}}$ can find $\mathcal{L}$ (1.3), the condition for $\mathcal{L}$ is norm convergent) on the quotient $B(H)$ is so large that $\dot{\mathcal{L}}$ is irreducible. Now let $\dot{\Psi}$ be the map induced by $\Psi := \Psi$ on $B(H)/K$ and let $\dot{x} \in A'$ be such that $\dot{x}$ can not be lifted to an element in $A'$. Then $\dot{\Psi}(\dot{x}) = \dot{x}$, hence, denoting by $x$ any lift in $B(H)$ of $\dot{x}$,

\begin{equation}
\dot{y} := \dot{\Psi}(\dot{x}) - x \in K.
\end{equation}

Since $\dot{x}$ can not be lifted to $A'$, it follows that $x + z \notin A'$ for all $z \in K$. So, if we can find $z \in K$ such that $\dot{\Psi}(x + z) = x + z$, then we will have $x + z \in \mathcal{F}_a \setminus A'$. Using (1.3), the condition for $z$ is that

\[ (1 - \dot{\Psi})(z) = y. \]

We could then find such a $z$ if we knew that the map $(1 - \dot{\Psi})|K$ is invertible. But in the case $K = K(H)$ the operator $(1 - \dot{\Psi})|K$ can not be invertible since its second adjoint on $B(H)$ (the bidual of $K(H)$) is just $1 - \dot{\Psi}$, which has nontrivial kernel (containing $A'$). Similarly $(1 - \dot{\Psi})|T(H)$ is not invertible. So we have to consider other (non-closed) ideals, the simplest of which is the Hilbert–Schmidt class $C^2(H)$. But in these cases every operator that commutes with a $C^*$-algebra $A$ modulo $C^2(H)$ is a perturbation of an element of $A'$ by an element of $C^2(H)$ (see [19]). So we will
have to consider operators that commute with all $a_j$ modulo $C^2(\mathcal{H})$, but do not commute modulo $C^2(\mathcal{H})$ with the whole $C^*$-algebra $A$ generated by the operators $a_j$. (This is possible since the space $C^2(\mathcal{H})$ is not closed in the usual operator norm.)

In Section 2 we shall see that an operator $\Psi_a$ of the form (1.1) (with the sums $\sum_{j=1}^{\infty} a_j^*a_j = 1 = \sum_{j=1}^{\infty} a_ja_j^*$ weak* converging) always preserves $C^2(\mathcal{H})$, so we may consider the restriction $\Phi_a := \Psi_a|C^2(\mathcal{H})$. We shall prove that if the operator $\Phi_a - 1$ is not invertible then there exists a state $\rho$ on $B(\mathcal{H})$ such that
\[ \rho\left(\sum_{j=1}^{\infty} a_j^* a_j\right) = \rho\left(\sum_{j=1}^{\infty} a_j b_j\right) \]
for all operators $b_j \in B(\mathcal{H})$ such that the two series $\sum_{j=1}^{\infty} b_j b_j^*$ and $\sum_{j=1}^{\infty} b_j^* b_j$ are weak* convergent. Conversely, if there exists a state $\rho$ on $B(\mathcal{H})$ such that $\rho(d) = \rho(dc)$ for all $d \in B(\mathcal{H})$ and all $c$ in the $C^*$-algebra $A$ generated by $\{a_j\}_{j=1}^{\infty} \cup \{1\}$, then the map $\Phi_a - 1$ is not invertible. Thus, in the case when the series $\sum_{j=1}^{\infty} a_j^* a_j$ is norm convergent, $\Phi_a - 1$ is not invertible if and only if $A$ has an amenable trace in the sense of [7], [8]. This result is then used in Section 3 to study fixed points of $\Psi_a$ on $B(\mathcal{H})$.

In the beginning of Section 4 we will present some general observations on the spectra of maps on $B(\mathcal{H})$ of the form $\Theta : x \mapsto \sum_{j=1}^{\infty} a_j x b_j$, where $(a_j)$ and $(b_j)$ are two commutative sequences of normal operators such that the sums $\sum_{j=1}^{\infty} a_j a_j^*$ and $\sum_{j=1}^{\infty} b_j^* b_j$ are weak* convergent. We observe that the spectrum of $\Theta$ in the Banach algebra $CB(B(\mathcal{H}))$ of all completely bounded maps on $B(\mathcal{H})$ is the same as the spectrum of $\Theta$ in certain natural subalgebras of $CB(B(\mathcal{H}))$. (Here some facts from the theory of operator spaces will be needed, but these results are not used in the rest of the paper.) The spectrum of such a map can be much larger than the closure of the set $\sigma$ of all sums $\sum_{j=1}^{\infty} \phi(a_j) \psi(b_j)$, where $\phi$ and $\psi$ are characters on the $C^*$-algebras generated by $(a_j)$ and $(b_j)$, respectively, but all eigenvalues of $\Theta$ are contained in $\sigma$.

At the end of Section 4 we will provide a short proof of the fact that if the $C^*$-algebra $A$ generated by the operators $(a_j)$ is abelian, then the fixed points of $\Phi_a$ are contained in $A'$. For positive operators $a_j$ this was proved in [32] and also in [21], but our proof is different even in this case.

### 2. Amenable Traces and the Spectrum of $\Phi_a$

Throughout the section $a = (a_j)$ is a bounded operator from a separable Hilbert space $\mathcal{H}$ to the direct sum $\mathcal{H}^{\infty}$ of countably many copies of $\mathcal{H}$, such that the components $a_j \in B(\mathcal{H})$ satisfy
\begin{align}
  a^* a = \sum_{j=1}^{\infty} a_j^* a_j &= 1 = \sum_{j=1}^{\infty} a_j a_j^*,
\end{align}
Similarly, \( H \) is not invertible (in \( B(B(\mathcal{H})) \)).

In the following remark.

\( \Box \)

(iii). well-known properties of the trace we have

\[
\sum_{j=1}^{\infty} a_j^* x_{a_j} = a^* x^{(\infty)} a.
\]

By \( C^2(\mathcal{H}) \) we denote the ideal of all Hilbert–Schmidt operators on \( \mathcal{H} \), and \( \| x \|_2 \) denotes the Hilbert–Schmidt norm of an element \( x \in C^2(\mathcal{H}) \), which is defined by \( \| x \|_2 = \sqrt{\text{Tr} (x^* x)} \).

**Proposition 2.1.** (i) \( \Psi(C^2(\mathcal{H})) \subseteq C^2(\mathcal{H}) \) and the restriction \( \Phi := \Psi|C^2(\mathcal{H}) \) is a contraction, that is \( \| \Phi(x) \|_2 \leq \| x \|_2 \) for all \( x \in C^2(\mathcal{H}) \).

(ii) For all \( x \in C^2(\mathcal{H}) \) the inequalities

\[
\| a x - x^{(\infty)} a \|_2^2 \leq 2 \| x - \Phi(x) \|_2 \| x \|_2 \quad \text{and} \quad \| \Phi(x) - x \|_2 \leq \| a x - x^{(\infty)} a \|_2
\]

hold.

(iii) The operator \( \Phi - 1 \) is not invertible if and only if there exists a sequence of selfadjoint elements \( x_k \in C^2(\mathcal{H}) \) with \( \| x_k \|_2 = 1 \) such that

\[
\lim_{k \to \infty} \| \Phi(x_k) - x_k \|_2 = 0.
\]

**Proof.** (i) Since \( \| a \| = 1 \), we have that \( aa^* \leq 1 \) (the identity operator on \( \mathcal{H}^{\infty} \)). Using this and the equality \( \sum_{j=1}^{\infty} a_j a_j^* = 1 \), we compute that for each \( x \in C^2(\mathcal{H}) \)

\[
\| \Phi(x) \|_2^2 = \text{Tr} \left( a^* x^{(\infty)} a a^* x^{(\infty)} a \right) \leq \text{Tr} \left( a^* x^{(\infty)} \right) = \sum_{j=1}^{\infty} \text{Tr} \left( x a_j a_j^* x \right) = \text{Tr} \left( x x^* \right) = \| x \|_2^2.
\]

(ii) Using the relations \( a^* a = 1 \), \( aa^* \leq 1 \), \( \text{Tr} (\Phi(x^* x)) = \text{Tr} (x^* x) \) and the well-known properties of the trace we have

\[
\| a x - x^{(\infty)} a \|_2^2 = \text{Tr} \left( (ax - x^{(\infty)} a)^* (ax - x^{(\infty)} a) \right)
\]

\[
= \text{Tr} \left( x^* x + \Phi(x^* x) - \Phi(x)^* x - x^* \Phi(x) \right)
\]

\[
= \text{Tr} \left( (x - \Phi(x))^* x + x^* (x - \Phi(x)) \right)
\]

\[
\leq 2 \| x \|_2 \| x - \Phi(x) \|_2
\]

Similarly

\[
\| \Phi(x) - x \|_2 = \| a^* x^{(\infty)} a - ax \|_2 \leq \| a^* \| \| x^{(\infty)} a - ax \|_2 = \| ax - x^{(\infty)} a \|_2
\]

(iii) The existence of a sequence \( (x_k) \) as in (iii) clearly implies that the map \( \Phi - 1 \) is not invertible (in \( B(B(\mathcal{H})) \)). Conversely, if \( \Phi - 1 \) is not invertible, then 1 is a boundary point of the spectrum of \( \Phi \) since \( \| \Phi \| \leq 1 \). But all boundary points of the spectrum are approximate eigenvalues ([9], p. 215), so there exists a sequence of elements \( x_k \in C^2(\mathcal{H}) \) such that \( \| x_k \|_2 = 1 \) and \( \lim \| \Phi(x_k) - x_k \|_2 = 0 \). By passing to an appropriate subsequence of real or imaginary parts of \( x_k \) and normalizing we can obtain a sequence of selfadjoint elements in \( C^2(\mathcal{H}) \) satisfying the condition in (iii).

\( \Box \)

In the proof of the main result of this section we will need two simple facts stated in the following remark.
**Remark 2.2.** If \( x = (x_j) \) and \( y = (y_j) \) are two operators from \( \mathcal{H} \) to \( \mathcal{H}^\infty \) of the Hilbert–Schmidt class (so that in particular \( x_j, y_j \in C^2(\mathcal{H}) \)) then:

(i) \( \|x\|_2 = \|x^T\|_2 \), where \( x^T \) is the row \([x_j]\) regarded as the Hilbert–Schmidt operator from \( \mathcal{H}^\infty \) to \( \mathcal{H} \).

(ii) \( \text{Tr} (x^* y) = \sum_{j=1}^\infty \text{Tr} (x_j^* y_j) \), where the series converges absolutely.

Part (i) is immediate. To prove (ii), we choose an orthonormal basis \((\xi_k)\) of \( \mathcal{H} \) and compute that

\[
\text{Tr} (x^* y) = \sum_{k=1}^\infty \sum_{j=1}^\infty \langle x_j^* y_j \xi_k, \xi_k \rangle = \sum_{j=1}^\infty \sum_{k=1}^\infty \langle x_j^* y_j \xi_k, \xi_k \rangle = \sum_{j=1}^\infty \text{Tr} (x_j^* y_j),
\]

where the change of the order of summation is permissible since

\[
\sum_{j,k=1}^\infty |\langle x_j^* y_j \xi_k, \xi_k \rangle| \leq \left( \sum_{j,k=1}^\infty \|y_j \xi_k\|^2 \right)^{1/2} \left( \sum_{j,k=1}^\infty \|x_j \xi_k\|^2 \right)^{1/2} = \|x\|_2 \|y\|_2 < \infty.
\]

Recall that a trace on a \( C^* \)-subalgebra \( A \subseteq B(\mathcal{H}) \) is called amenable if it can be extended to a state \( \rho \) on \( B(\mathcal{H}) \) such that \( \rho(cd) = \rho(dc) \) for all \( c \in A \) and \( d \in B(\mathcal{H}) \) [7], [8]. We also recall the Powers-Störmer inequality: \( \|x - y\|_2^2 \leq \|x^2 - y^2\|_1 \) for all positive \( x, y \in C^2(\mathcal{H}) \) (A proof can be found for example in [7]. Usually the inequality is used in the form \( \|xu - ux\|_2^2 \leq \|x^2 u - ux^2\|_1 \) for positive \( x \in C^2(\mathcal{H}) \) and a unitary \( u \).

**Theorem 2.3.** Let \( A \) be the \( C^* \)-algebra generated by the identity and the operators \( a_j \in B(\mathcal{H}) \) satisfying (2.1) and let \( \Phi = \Phi_\alpha \) be the restriction to \( C^2(\mathcal{H}) \) of the map \( \Phi \) defined by (2.2). If \( \Phi - 1 \) is not invertible then there exists a state \( \rho \) on \( B(\mathcal{H}) \) such that

\[
\rho(b^T a) = \rho(a^T b) \quad \text{for all} \quad b = (b_j) \in B(\mathcal{H}, \mathcal{H}^\infty) \quad \text{such that} \quad b^T \in B(\mathcal{H}^\infty, \mathcal{H}).
\]

Conversely, if \( \rho \) is a state on \( B(\mathcal{H}) \) such that \( \rho(cd) = \rho(dc) \) for all \( c \in A \) and \( d \in B(\mathcal{H}) \) and

\[
\sum_{j=1}^\infty \rho(a_j^* a_j) = 1,
\]

then the map \( \Phi - 1 \) is not invertible.

Thus, if at least one of the series in (2.1) is norm convergent, then the map \( \Phi - 1 \) is not invertible if and only if \( A \) has an amenable trace.

**Proof.** If \( \Phi - 1 \) is not invertible then by Proposition 2.1 there exists a sequence of selfadjoint elements \( x_k \) in \( C^2(\mathcal{H}) \) with \( \|x_k\|_2 = 1 \) and

\[
\lim_{k \to \infty} \|ax_k - x_k^{(\infty)} a\|_2 = 0.
\]

Let \( \rho_k \) be the state on \( B(\mathcal{H}) \) defined by \( \rho_k(d) = \text{Tr}(dx_k^2) \) and let \( \rho \) be a weak* limit point of the sequence \( \{\rho_k\} \). Note that for each \( x \in C^2(\mathcal{H}) \) and \( b = (b_j) \in B(\mathcal{H}, \mathcal{H}^\infty) \) we have \( \text{Tr}(a^T b x^2) = \text{Tr}(x a^T b x) \) and (by Remark 2.2(ii)) since \( ax \) and \( (x^* b_j) \) are in \( C^2(\mathcal{H}, \mathcal{H}^\infty) \)

\[
\text{Tr}(b^T x^{(\infty)} ax) = \sum_{j=1}^\infty \text{Tr}(b_j x a_j x) = \sum_{j=1}^\infty \text{Tr}(a_j x b_j x) = \text{Tr}(a^T x^{(\infty)} b x).
\]
Therefore suitable convex combinations of operators

\[
\rho \text{ is a weak* limit point of } (\rho_k), \text{ this implies that } \rho(b^T a) = \rho(a^T b). \text{ In particular } \rho(a_j d) = \rho(d a_j) \text{ for all } a_j \text{ and } d \in B(\mathcal{H}), \text{ which implies that } \rho A \text{ is an amenable trace.}
\]

Suppose now conversely, that \( \rho \) is a state on \( B(\mathcal{H}) \) satisfying (2.4) and \( \rho(cd) = \rho(dc) \) for all \( c \in A \) and \( d \in B(\mathcal{H}) \). Since the series in (2.4) is convergent, given \( \varepsilon > 0 \), there exists \( m \in \mathbb{N} \) such that

\[
(2.5) \quad \sum_{j=m+1}^{\infty} \rho(a_j a_j^*) = \sum_{j=m+1}^{\infty} \rho(a_j^* a_j) < \frac{\varepsilon}{8}.
\]

Since normal states are weak* dense in the state space of \( B(\mathcal{H}) \), there exists a net of positive operators \( y_k \in T(\mathcal{H}) \) with the trace norm \( \|y_k\|_1 = 1 \) such that the states \( \rho_k(d) := Tr(dy_k) \ (d \in B(\mathcal{H})) \) weak* converge to \( \rho \). By passing to a subnet we may assume that

\[
(2.6) \quad |(\rho_k - \rho) \left( \sum_{j=m+1}^{\infty} (a_j a_j^* + a_j^* a_j) \right) | < \frac{\varepsilon}{4}.
\]

Let \( a_{(m)} = (a_1, \ldots, a_m) \in B(\mathcal{H}, \mathcal{H}^m) \). Observe that the trace class operators \( a_{(m)} y_k - y_k^{(m)} a_{(m)} \in T(\mathcal{H}, \mathcal{H}^m) \) converge weakly to 0 since for all \( d = [d_1, \ldots, d_m] \in B(\mathcal{H}^m, \mathcal{H}) \) we have (denoting by \( y^{(m)} \) the direct sum of \( m \) copies of an operator \( y \))

\[
\begin{align*}
Tr(d(a_{(m)} y_k - y_k^{(m)} a_{(m)})) = & \sum_{j=1}^{m} Tr(d_j(a_j y_k - y_k a_j)) \\
= & \sum_{j=1}^{m} Tr((d_j a_j - a_j d_j) y_k) \quad k \to \sum_{j=1}^{m} \rho(d_j a_j - a_j d_j) = 0.
\end{align*}
\]

Therefore suitable convex combinations of operators \( a_{(m)} y_k - y_k^{(m)} a_{(m)} \) must converge to 0 in norm; thus, replacing the \( y_k \)'s by suitable convex combinations, we may assume that

\[
\|a_{(m)} y_k - y_k^{(m)} a_{(m)}\|_1 \quad k \to 0.
\]

Let \( x_k = y_k^{1/2} \). It follows from the Powers–Störmer inequality (by expressing the components \( a_j \) of \( a_{(m)} \) as linear combinations of unitaries) that

\[
(2.7) \quad \sum_{j=1}^{m} \|a_j x_k - x_k a_j\|_2 \quad k \to 0.
\]
Now we can estimate
\[\|ax_k - x_k(\infty) a\|_2^2 = \sum_{j=1}^{m} \|a_j x_k - x_k a_j\|_2^2 + \sum_{j=m+1}^{\infty} \|a_j x_k - x_k a_j\|_2^2\]
\[\leq \sum_{j=1}^{m} \|a_j x_k - x_k a_j\|_2^2 + 2 \sum_{j=m+1}^{\infty} (\|a_j x_k\|_2^2 + \|x_k a_j\|_2^2)\]
\[= \sum_{j=1}^{m} \|a_j x_k - x_k a_j\|_2^2 + 2 \sum_{j=m+1}^{\infty} (\text{Tr}(a_j x_k a_j^* + a_j^* x_k a_j))\]
\[= \sum_{j=1}^{m} \|a_j x_k - x_k a_j\|_2^2 + 2 \rho_k(\sum_{j=m+1}^{\infty} a_j a_j^*).\]

Using (2.5) and (2.6) it follows now that
\[\|ax_k - x_k(\infty) a\|_2^2 < \sum_{j=1}^{m} \|a_j x_k - x_k a_j\|_2^2 + \varepsilon,\]

hence (2.7) implies that \(\|ax_k - x_k(\infty) a\|_2^2 < \varepsilon\) for some \(k\). Since \(\varepsilon > 0\) was arbitrary, Proposition 2.1 tells us that the map \(\Phi - 1\) is not invertible.

If the series \(\sum_{j=1}^{\infty} a_j^* a_j\) is norm convergent to \(1\) then the condition (2.4) is automatically satisfied for any state \(\rho\). If \(\rho\) is tracial then the same conclusion holds if we assume the norm convergence of the series \(\sum_{j=1}^{\infty} a_j a_j^* = 1\). Finally, observe that for any state \(\rho\) on \(B(H)\) satisfying \(\rho(cd) = \rho(dc)\) for all \(c \in A\) and \(d \in B(H)\) the condition (2.4) implies (2.3) since
\[|\rho(\sum_{j=m}^{\infty} b_j a_j)| \leq \sum_{j=m}^{\infty} |\rho(b_j a_j)| \leq \sum_{j=m}^{\infty} \rho(b_j b_j^*)^{1/2} \rho(a_j^* a_j)^{1/2}\]
\[\leq \|b\| (\sum_{j=m}^{\infty} \rho(a_j^* a_j))^{1/2} \xrightarrow{m \to \infty} 0\]

and similarly \(\rho(\sum_{j=m}^{\infty} a_j b_j)\) \(\xrightarrow{m \to \infty} 0\). \(\square\)

**Corollary 2.4.** If the von Neumann algebra \(\overline{A}\) generated by the operators \(a_j\) (satisfying (2.1)) is finite and injective then the operator \(\Phi_a - 1\) is not invertible.

**Proof.** Let \(E : B(H) \to \overline{A}\) be a conditional expectation, \(\tau\) any normal tracial state on \(\overline{A}\) and \(\rho = \tau E\). The state \(\rho = \tau E\) satisfies the condition (2.4) and \(\rho(cd) = \rho(dc)\) for all \(c \in A\) and \(d \in B(H)\) (since \(E\) is an \(\overline{A}\)-bimodule map), hence the map \(\Phi_a - 1\) is not invertible. \(\square\)

Given an arbitrary von Neumann algebra \(R \subseteq B(H)\), Haagerup [17] proved that if the norm of every elementary operator on \(C^2(H)\) of the form \(x \mapsto \sum_{j=1}^{n} u_j x u_j^*\), where the coefficients \(u_j \in R\) are unitary, is equal to \(1\), then \(R\) is finite and injective. The author does not know if the same conclusion holds under the assumption that the norm of elementary operators of the form \(x \mapsto \sum_{j=1}^{n} a_j x a_j\) is equal to \(1\) for all positive \(a_j \in R\).

**Remark 2.5.** We have seen in the proof of Theorem 2.3 that the condition (2.4) implies (2.3). But the two conditions are not equivalent. Indeed, let \(R \subseteq B(H)\) be
an abelian infinite dimensional von Neumann algebra, \( \omega \) any non-normal state on \( R \) and \( E : B(\mathcal{H}) \to R \) a conditional expectation. Since \( \omega \) is not normal there exists in \( R \) a sequence \( (a_j) \) of mutually orthogonal projections with the sum 1 such that \( \sum_{j=1}^{\infty} \omega(a_j) < 1 \). Let \( \rho = \omega E \), a state on \( B(\mathcal{H}) \). Even though \( E \) is not necessarily weak* continuous the equalities
\[
E(b^T a) = \sum_{j=1}^{\infty} E(b_j)a_j = \sum_{j=1}^{\infty} a_j E(b_j) = E(a^T b)
\]
hold for all \( b = (b_j) \) (\( b_j \in B(\mathcal{H}) \)) such that the two sums \( \sum_{j=1}^{\infty} b_j^* b_j \) and \( \sum_{j=1}^{\infty} b_j b_j^* \) are weak* convergent. (This is so because \( E \) is a completely positive \( \mathcal{A} \)-bimodule map; see [14] or [18].) Hence \( \rho(b^T a) = \omega(E(b^T a)) = \omega(E(a^T b)) = \rho(a^T b) \). But \( \sum_{j=1}^{\infty} \rho(a_j^* a_j) = \sum_{j=1}^{\infty} \omega(a_j) < 1 \).

We show now by an example that the condition (2.3) is not automatically fulfilled by states satisfying \( \rho(cd) = \rho(dc) \) for all \( c \in A \) and \( d \in B(\mathcal{H}) \).

**Example 2.6.** Choose an orthonormal basis \( (\xi_j) \) \( (j = 1, 2 \ldots) \) of \( \mathcal{H} \) and let \( a_j \) be the rank 1 orthogonal projection onto \( \mathbb{C} \xi_j \). Then the \( C^* \)-algebra \( A \) generated by \( (a_j) \) and 1, is the \( C^* \)-algebra of all convergent sequences acting as diagonal operators. For each \( j \) let \( b_j \) be a rank 1 partial isometry such that \( b_j b_j^* = a_{2j} \) and \( b_j^* b_j = a_j \). Then
\[
a^T b = \sum_{j=1}^{\infty} a_j b_j = 0,
\]
while
\[
b^T a = \sum_{j=1}^{\infty} b_j a_j = \sum_{j=1}^{\infty} b_j =: v
\]
is an isometry with the range projection \( p = vv^* = \sum_{j=1}^{\infty} a_{2j} \) of infinite rank. Let \( q : B(\mathcal{H}) \to B(\mathcal{H})/K(\mathcal{H}) = C(\mathcal{H}) \) be the quotient map, \( \theta \) a state on \( C(\mathcal{H}) \) such that \( \theta(q(v)) \neq 0 \), and \( \rho := \theta q \). Then \( q(c) \) is a scalar for each \( c \in A \), hence for each \( d \in B(\mathcal{H}) \)
\[
\rho(cd) = \theta(q(c)q(d)) = \theta(q(c))\theta(q(d)) = \rho(dc).
\]
But nevertheless \( \rho(b^T a) = \rho(v) \neq 0 = \rho(a^T b) \).

**Problem.** Is the necessary condition (2.3) also sufficient for the conclusion of Theorem 2.3? In other words, may the stronger condition (2.4) be replaced by (2.3)?

The answer is affirmative at least when \( a = (a_j) \) is such that the operator \( x^{(\infty)}a \in B(\mathcal{H}, \mathcal{H}^{\infty}) \) is of trace class for a dense set of trace class operators \( x \in T(\mathcal{H}) \). Namely, in this case we can modify the proof of Theorem 2.3 as follows. First we approximate the state \( \rho \) in Theorem 2.3 by normal states coming from operators \( y_k \in T(\mathcal{H}) \) such that the operators \( y_k^{(\infty)}a \in B(\mathcal{H}, \mathcal{H}^{\infty}) \) are of trace class. Then we verify that the sequence \( (y_k^{(\infty)}a - ay_k) \) converges weakly to 0. Finally we show that \( \|\sqrt{y_k} \rightarrow 0 \) for the last step we need the following consequence of the Powers-Störmer inequality.
Proposition 2.7. For all operators \( b \in B(K, H) \) and positive operators \( x \in T(H) \), \( y \in T(K) \) the inequality
\[
||by - xb||^2 \leq \gamma ||by^2 - x^2b|| ||b||
\]
holds, where \( \gamma = \frac{8}{9}\sqrt{3} \).

Proof. By considering the operator
\[
\begin{bmatrix}
0 & b \\
b^* & 0
\end{bmatrix}
\]
instead of \( b \) and
\[
\begin{bmatrix}
x & 0 \\
0 & y
\end{bmatrix}
\]
instead of both \( x \) and \( y \), the proof can be reduced immediately to the case when \( b = b^* \) and \( y = x \). (Further, in this case we may replace \( b \) by \( b + s1 \) for a suitable scalar \( s \) so that we may assume that both \( ||b|| \) and \( -||b|| \) are in the spectrum of \( b \).)

Denote \( \beta = ||b|| \) and for \( t \in \mathbb{R} \setminus \{0\} \) let
\[
u_t = (b - ti)(b + ti)^{-1}, \quad \text{so that} \quad b = ti(1 + u_t)(1 - u_t)^{-1}.
\]
Since \( u_t \) is unitary, we have by the Powers-Störmer inequality
\[
||u_t x - xu_t||^2 \leq ||u_t x^2 - x^2 u_t||_1,
\]
which can be rewritten as
\[
||(b + ti)^{-1}z_t(b + ti)^{-1}||^2 \leq ||2ti(b + ti)^{-1}(bx^2 - x^2b)(b + ti)^{-1}||_1,
\]
where \( z_t := 2ti(bx - xb) \). Since \( ||z_t||_2 \leq ||b + ti||^2 ||(b + ti)^{-1}z_t(b + ti)^{-1}||_2 \), (2.9) implies that
\[
||z_t||^2 \leq 2t||b + ti||^4 ||(b + ti)^{-1}||^2 ||bx^2 - x^2b||_1.
\]
Thus (since \( ||(b + ti)^{-1}||^2 \leq t^{-2} \) and \( ||b + ti||^2 \leq \beta^2 + t^2 \))
\[
||bx - xb||^2 \leq \frac{t \beta^2 + t^2}{2t^3} ||bx^2 - x^2b||_1.
\]
Taking the minimum over \( t \) of the right-hand side of this inequality, we obtain the desired estimate (2.8). \( \square \)

3. On the Fixed Points of the Map \( \Psi_a \)

As we indicated already in the Introduction, Theorem 2.3 implies the following corollary.

Corollary 3.1. With the notation as in Theorem 2.3, suppose that the \( C^* \)-algebra \( A \) has no amenable traces and that the two series \( \sum_{j=1}^{\infty} a_j^*a_j = 1 = \sum_{j=1}^{\infty} a_j a_j^* \) are norm convergent. If there exists an operator \( y \in B(H) \) such that the operator \( y^{(\infty)}a - ay \) is in the Hilbert-Schmidt class and \( y \) is not in \( A' + C^2(H) \), then the operator \( \Psi = \Psi_a \) defined on \( B(H) \) by \( \Psi_a(x) = \sum_{j=1}^{\infty} a_j^*xa_j \) has fixed points which are not in \( A' \).

Proof. Observe that \( y - \Psi(y) \in C^2(H) \) since
\[
y - \Psi(y) = a^*(ay - y^{(\infty)}a)
\]
and \( y^{(\infty)}a - ya \) is in the Hilbert-Schmidt class by the hypothesis. By Theorem 2.3 the map \((\Psi - 1)|C^2(H)\) is invertible, hence there exists a \( z \in C^2(H) \) such that
\((\Psi - 1)(z) = y - \Psi(y)\). This means that \(\Psi(y + z) = y + z\). Hence \(x := y + z\) is a fixed point of \(\Psi\), and \(x\) is not in \(A'\) since \(y \notin A' + C^2(\mathcal{H})\) and \(z \in C^2(\mathcal{H})\).

Now we give an example which satisfies the conditions of Corollary 3.1 and solves a problem left open in [3].

**Example 3.2.** Let \(v_i \ (i = 1, 2)\) be the isometries defined on \(\mathcal{H} = l^2(\mathbb{N})\) by

\[
v_1 e_j = e_{2j} \quad \text{and} \quad v_2 e_j = e_{2j+1} \quad (j = 0, 1, 2, \ldots),
\]

where \((e_j)\) is an orthonormal basis of \(\mathcal{H}\). Then \(v_1 v_1^* + v_2 v_2^* = 1\) and the \(C^*\)-algebra \(A\) generated by \(\{v_1, v_2\}\) is the Cuntz algebra \(O(2)\) (defined in [11] or [16]), which has no tracial states (and is nuclear).

To show that \(A\) is irreducible, choose any \(d \in A'\) and let

\[
d e_0 = \sum_{j=0}^{\infty} \alpha_j e_j \quad (\alpha_j \in \mathbb{C}).
\]

Then

\[
\sum_{j=0}^{\infty} \alpha_j e_j = d e_0 = d v_1^* e_0 = v_1^* d e_0 = \sum_{j=0}^{\infty} \alpha_{2j} e_j,
\]

which implies that \(\alpha_j = \alpha_{2j}\) for all \(j\). Similarly, from \(0 = d v_2^* e_0 = v_2^* d e_0 = \sum_{j=0}^{\infty} \alpha_{2j+1} e_j\) we see that \(\alpha_{2j+1} = 0\) for all \(j\). It follows that \(\alpha_j = 0\) for all \(j > 0\). Thus \(d e_0 = \alpha_0 e_0\) and consequently \(d (e_{k_1}^* e_{k_2}^* \ldots) e_0 = (e_{k_1}^* e_{k_2}^* \ldots) e_0 = e_0\) for any sequence \(k_1, k_2, \ldots\) in \(\mathbb{N}\). Since the linear span of vectors of the form \((e_{k_1}^* e_{k_2}^* \ldots) e_0\) is dense in \(\mathcal{H}\), it follows that \(d = \alpha_0 1\).

We will show that there exists a positive diagonal operator \(y \in B(\mathcal{H})\) such that

\[
y v_2 = v_2 y, \quad y v_1 - v_1 y \in C^2(\mathcal{H}), \quad \text{but} \quad y \notin \mathcal{C}1 + C^2(\mathcal{H}) = A' + C^2(\mathcal{H}).
\]

Let \(y e_j = t_j e_j\), where \(t_j\) are nonnegative scalars to be specified. The condition \(y v_2 = v_2 y\) means that

\[
(3.1) \quad t_{2j+1} = t_j \quad (j = 0, 1, 2, \ldots).
\]

On the other hand, the condition \(y v_1 - v_1 y \in C^2(\mathcal{H})\) means that

\[
(3.2) \quad \sum_{j=0}^{\infty} (t_{2j} - t_j)^2 < \infty.
\]

To satisfy these two conditions, choose \(t_j\), for example, as follows. If \(j\) is of the form \(j = 2^k\) \((k \in \mathbb{N})\) let \(t_j = (k + 1)^{-1/2}\). If \(j\) is not a power of 2 define \(t_j\) recursively by

\[
t_j = \begin{cases} t_{j/2} & \text{if } j \text{ is even;} \\ t_{(j-1)/2} & \text{if } j \text{ is odd.} \end{cases}
\]

Then \(t_{2j+1} = t_j\) for all \(j \in \mathbb{N}\), so (3.1) holds. Further, \(t_{2j} = t_j\) for all \(j\) which are not powers of 2, hence the sum in (3.2) reduces to

\[
\sum_{k=0}^{\infty} \left( \frac{1}{\sqrt{k+1}} - \frac{1}{\sqrt{k+2}} \right)^2 < \infty.
\]

The so defined operator \(y\) is not in \(\mathcal{C}1 + C^2(\mathcal{H})\) since the series \(\sum_{j=0}^{\infty} (t_j + \alpha)^2\) diverges for all \(\alpha \in \mathbb{C}\).
Finally, we write $v_1$ and $v_2$ as linear combinations of positive elements $a_j \in A (j = 1, \ldots, 8)$ such that $\sum_{j=1}^{8} a_j^2 \leq 1$. Define $a_0 = (1 - \sum_{j=1}^{8} a_j^2)^{1/2}$ and $a = (a_0, \ldots, a_8)$. Then $\Psi_a$ is a Lüders operator for which not all fixed points are in $A' (= \mathbb{C}1)$, since $y$ commutes modulo $C^2(\mathcal{H})$ with all $a_j$ and does not commute with all $a_j$.

4. The case of commuting operators

In this section we study the spectrum and fixed points of normal completely bounded maps on $\mathrm{B}(\mathcal{K}, \mathcal{H})$, where $\mathcal{H}$ and $\mathcal{K}$ are separable Hilbert spaces. We denote by $\mathrm{CB}(\mathrm{B}(\mathcal{K}, \mathcal{H}))$ the space of all completely bounded maps on $\mathrm{B}(\mathcal{K}, \mathcal{H})$.

Given $C^*$-subalgebras $A \subseteq \mathrm{B}(\mathcal{H})$ and $B \subseteq \mathrm{B}(\mathcal{K})$, we let $A \odot B$ be the Banach subalgebra of $\mathrm{CB}(\mathrm{B}(\mathcal{K}, \mathcal{H}))$ consisting of all maps $\Theta$ that can be represented in the form

\begin{equation}
\Theta(x) := \sum_{j=1}^{\infty} c_j x d_j,
\end{equation}

where $c_j \in A$ and $d_j \in B$ are such that the row $c = [c_j]$ and the column $d = (d_j)$ represent bounded operators in $\mathrm{B}(\mathcal{H}^\infty, \mathcal{H})$ and $\mathrm{B}(\mathcal{K}, \mathcal{K}^\infty)$, respectively. Thus the sums

\begin{equation}
\sum_{j=1}^{\infty} c_j c_j^* \quad \text{and} \quad \sum_{j=1}^{\infty} d_j^* d_j
\end{equation}

converge in the strong operator topology. We will write such a map $\Theta$ simply as

$$
\Theta = c \odot d = \sum_{j=1}^{\infty} c_j \otimes d_j.
$$

The space $A \odot B$ coincides with the extended Haagerup tensor product (defined in [5], [15], [23]), but we shall not need this fact. The subspace $A \odot^h B$ of $A \odot B$, consisting of elements $c \odot d \in A \odot B$ for which the two sums in (4.2) are norm convergent, is a Banach subalgebra of $A \odot B$, and can be identified with the Haagerup tensor product, but again we shall not need this last fact. If $M$ and $N$ are von Neumann algebras then $M \odot^h N$ coincides with the space $\mathrm{NCB}_{M', N'}(\mathrm{B}(\mathcal{K}, \mathcal{H}))$ of all normal completely bounded $M', N'$-bimodule endomorphisms of $\mathrm{B}(\mathcal{K}, \mathcal{H})$ (see [30] or [22, 1.2]; here $M'$ denotes the commutant of $M$). It is well known that a weak* continuous map $\Theta$ between Banach spaces is invertible if and only if its preadjoint map $\Theta^*$ is invertible [9]. Thus, if $\Theta \in M \odot^h N$ is invertible, then so is $\Theta^*$ (as a bounded map on $T(\mathcal{H}, \mathcal{K})$), hence $\Theta^{-1} = ((\Theta^*)^{-1})^*$ is weak* continuous.

Since $\Theta^{-1}$ is also an $M', N'$-bimodule map, it follows that $M \odot^h N$ is an inverse-closed subalgebra of $\mathrm{CB}(\mathrm{B}(\mathcal{K}, \mathcal{H}))$. The spectrum of an element $c$ in a Banach algebra $A$ is denoted by $\sigma_A(c)$. We summarize the above discussion in the following proposition.

**Proposition 4.1.** If $M$ and $N$ are von Neumann subalgebras of $\mathrm{B}(\mathcal{H})$ and $\mathrm{B}(\mathcal{K})$ (respectively) then

\begin{equation}
\sigma_{M \odot^h N}^e(\Theta) = \sigma_{\mathrm{CB}(\mathrm{B}(\mathcal{K}, \mathcal{H}))}^e(\Theta)
\end{equation}
for each $\Theta \in M \otimes N$.

In many cases the above Proposition can be sharpened to the identity $(M \otimes N)^{cc} = M \otimes N$. Namely, it is known (see [14] or [18]) that the commutant $(M \otimes N)^{c}$ of $M \otimes N$ inside $\text{CB}(B(K, H))$ is the algebra $\text{CB}_{M,N}(B(K, H))$ of all completely bounded $M, N$-bimodule endomorphisms of $B(K, H)$, which we will denote simply by $M' \otimes N'$, thus

\[(4.3) \quad (M \otimes N)^{c} = M' \otimes N'.\]

(We remark that the notation $M' \otimes N'$ usually means the normal Haagerup tensor product as defined in [13], [15], [4, p. 41], but the two algebras $M' \otimes N'$ and $\text{CB}_{M,N}(B(K, H))$ are naturally completely isometrically and weak* homeomorphically isomorphic by [13] (a simpler proof of a more general fact is in [24, 4.4]).)

By a surprising result of Hoffmeier and Wittstock [18] the commutant of $M' \otimes N'$ in $\text{CB}(B(K, H))$ consists only of weak* continuous maps, if $M$ and $N$ do not have central parts of type $I_{\infty,n}$ for $n \in \mathbb{N}$, that is

\[(4.4) \quad (M' \otimes N')^{c} = M \otimes N.\]

(In [18] only the case $N = M$ is considered, but the usual argument with the direct sum $M \oplus N$ reduces the general situation to this case.) This holds in particular when $M$ and $N$ are abelian, thus, in this case we deduce from (4.3) and (4.4) that

\[(M \otimes n)^{cc} = M \otimes N.\]

Proposition 4.2. Suppose that $c_j \in B(H)$, $d_j \in B(K)$ are positive and such that $\sum_{j=1}^{\infty} \|c_j\|\|d_j\| < \infty$. If for each $j$ at least one of the operators $c_j$, $d_j$ is compact then all eigenvalues of the operator $\Theta = c \otimes d$ defined by (4.1) on $B(K, H)$ are in $\mathbb{R}^+$.\]

Proof. By [29, 6.10] each eigenvector corresponding to a nonzero eigenvalue $\lambda$ of $\Theta$ is nuclear, hence in particular in the Hilbert-Schmidt class $C^2(K, H)$. Since the restriction $\Theta|C^2(K, H)$ is a positive operator on a Hilbert space, its spectrum is contained in $\mathbb{R}^+$, hence $\lambda \in \mathbb{R}^+$.\]

We denote by $\Delta(A)$ the spectrum (that is, the space of all multiplicative linear functionals) of a commutative Banach algebra $A$. If $A$ and $B$ are commutative operator algebras then it is easy to see that

\[(4.5) \quad \Delta(A \otimes B) = \Delta(A) \times \Delta(B).\]

For the spectrum of $A \otimes B$, however, there is no such simple formula. In the case when $M$ and $N$ are (abelian) von Neumann algebras there is an injective contraction from $M \otimes N$ into $M \overline{\otimes} N$ (which will be regarded as inclusion and is
dual to the natural contraction $M_* \overset{h}{\otimes} N_* \to M_* \otimes N_* \ [4, 1.5.13], \ [15, 6.1])$, and one might conjecture that the spectrum of an element of $M \otimes N$ is the same as the spectrum of its image in $M \otimes N$, but this is not always true even in the special case $M = \ell^\infty(N) = N$. In this case $C := M \otimes N$ is the von Neumann algebra $\ell^\infty(N \times N)$ of all bounded sequences on $N \times N$. Further, $D := M \otimes h$ is the algebra of all Schur multipliers on $B(\ell^2(N))$ (see [26], Theorem 5.1), which consists of all sequences $d \in \ell^\infty(N \times N)$ such that the double sequence $[d_{i,j}]$ is a matrix of a bounded operator on $\ell^2(N)$ for every $[x_{i,j}]$ representing a bounded operator on $\ell^2(N)$. Such an element $d \in D$ is invertible in $C$ if and only if the closure of the set $\{d_{i,j}\}$ in $C$ does not contain $0$, but this does not guarantee invertibility of $d$ in $D$. To see this, we consider the following example suggested to us by Milan Hladnik and Victor Shulman.

**Example 4.3.** Let $D_0$ be the subalgebra of $D$ consisting of Toeplitz-Schur multipliers, that is, Schur multipliers $d = [d_{i-j}]$ that are constant along the diagonals. If $d$ is invertible in $D$, then $d^{-1}$ is also the inverse of $d$ in $C$, hence $d^{-1}$ consists of the double sequence $[d_{i-j}^{-1}]$, which is in $D_0$, so $D_0$ is inverse-closed in $D$. On the other hand, it is known that the entries of each Toeplitz-Schur multiplier $[d_{i-j}]$ are the Fourier coefficients of a complex regular Borel measure $\mu$ on the unit circle $\mathbb{T}$ (that is, $d_k = \int_{\mathbb{T}} e^{ikx} \, d\mu$) and conversely; that is, $D_0$ is isomorphic to the measure algebra $M(\mathbb{T})$ for the convolution. (A proof of this can be found in [1].) But by [28, 5.3.4] there exists a noninvertible measure $\mu \in M(\mathbb{T})$ such that the Fourier coefficients of $\mu$ are all real and $\geq 1$, so the corresponding Schur multiplier is invertible in $C$ but not in $D$. Moreover, by [28, Theorem 6.4.1] the spectrum of such a multiplier $[d_{i-j}]$ can contain any point in $\mathbb{C}$ even if $d_k \in [-1,1]$ for all $k$.

We remark that the spectra of elementary operators $x \mapsto \sum_{j=1}^m c_j x d_j$, where $m$ is finite and $c = (c_j), d = (d_j) \subseteq B(\mathcal{H})$ are two commutative families, have been intensively studied in the past (see [12] and the references in [12] and in [2]), but the results do not apply to the case of infinite $m$, where the two series $\sum_{j=1}^\infty c_j c_j^*$ and $d_j^* d_j$ converge in the weak* topology. Even if we assume that all the components $c_j$ and $d_j$ are normal operators, the above example suggests that the spectrum of $c \otimes d$ cannot be described in terms of spectra of $c_j$ and $d_j$ in the same way as for finite $m$-tuples.

If $A$ is an abelian Banach algebra and $c = (c_j)$ is a sequence of elements in $A$ we set

$$\sigma_A(c) = \{ (\rho(c_1), \rho(c_2), \ldots) : \rho \in \Delta(A) \}.$$ 

**Lemma 4.4.** If $c = (c_j)$ is a sequence in a commutative unital $C^*$ algebra $A \subseteq B(\mathcal{H})$ such that the series $\sum_{j=1}^\infty c_j c_j^*$ is norm convergent, then $\sigma_A(c)$ is a norm compact subset of $\ell^2$. If this sum is merely weak* convergent, then $\sigma_A(c)$ is a weakly compact subset of $\ell^2$.

**Proof.** For any character $\rho \in \Delta(A)$ and any finite $n$ we have

$$\sum_{j=1}^n |\rho(c_j)|^2 = \rho(\sum_{j=1}^n c_j^* c_j) \leq \|c\|^2,$$

which implies that $(\rho(c_j)) \in \ell^2$ with $\| (\rho(c_j)) \| \leq \|c\|$. It is easy to prove that the map $\rho \mapsto (\rho(c_j))$ from $\Delta(A)$ to $\ell^2$ is weak* to weak continuous, so its range $\sigma_A(c)$
is a weakly compact set since $\Delta(A)$ is weak* compact. If the series $\sum_{j=1}^{\infty} c_j^*c_j$ is norm convergent, then the same map is weak* to norm continuous, hence $\sigma_A(c)$ is a norm compact set in this case.

Given two elements $\lambda = (\lambda_j)$ and $\mu = (\mu_j)$ in $\ell_2$ we denote

$$\lambda \cdot \mu := \sum_{j=1}^{\infty} \lambda_j \mu_j.$$

Further, for two subsets $\sigma_j \subseteq \ell^2$, we denote

$$\sigma_1 \cdot \sigma_2 := \{ \lambda \cdot \mu : \lambda \in \sigma_1, \mu \in \sigma_2 \}.$$

Since the map $(\lambda, \mu) \mapsto \lambda \cdot \mu$ is continuous, $\sigma_1 \cdot \sigma_2$ is a compact subset of $\mathbb{C}$ if $\sigma_1$ and $\sigma_2$ are norm compact subsets of $\ell^2$.

**Proposition 4.5.** Let $(c_j)$ and $(d_j)$ be two commutative families of normal operators in $B(\mathcal{H})$ and $B(\mathcal{K})$ (respectively) such that the two series (4.2) are weak* convergent. Let $A$ and $B$ be the $C^*$ algebras generated by $\{1\} \cup (c_j)$ and $\{1\} \cup (d_j)$, respectively, and $\overline{A}$, $\overline{B}$ their weak* closures, so that the map $\Theta = c \otimes d$ is an element of $\overline{A} \otimes \overline{B}$.

(i) If the two series (4.2) are norm convergent (that is, if $\Theta \in A \overset{h}{\otimes} B$) then

$$\sigma_{CB}(B(\mathcal{H})) = \sigma_A(c) \cdot \sigma_B(d).$$

(ii) In general the point spectrum of $\Theta$ is contained in $\sigma_A(c) \cdot \sigma_B(d)$.

**Proof.** The spectrum of an element $\Theta$ in a unital commutative Banach algebra $D$ is always equal to $\{ \rho(\Theta) : \rho \in \Delta(D) \}$. This applies to our element $\Theta = c \otimes d$ in $D = \overline{A} \otimes \overline{B}$. Given $\rho \in \Delta(D)$, denote $\phi = \rho|_{\overline{A} \otimes 1}$ and $\psi = \rho|_{1 \otimes \overline{B}}$. Then $\phi \in \Delta(A), \psi \in \Delta(B)$ and (by the norm continuity) $\rho|_{\overline{A} \otimes \overline{B}} = \phi \otimes \psi$.

Conversely, any two characters $\phi \in \Delta(\overline{A})$ and $\psi \in \Delta(\overline{B})$ define the character $\phi \otimes \psi$ by $$(\phi \otimes \psi)(\sum_{j=1}^{\infty} x_j \otimes y_j) = \sum_{j=1}^{\infty} \phi(x_j) \psi(y_j).$$ So, if the two series (4.2) are norm convergent then $\sigma_B(\Theta) = \sigma_A(c) \cdot \sigma_B(d)$. Since all characters on $A$ and $B$ extend to characters on $\overline{A}$ and $\overline{B}$, respectively, it also follows that $\sigma_A(c) \cdot \sigma_B(d)$ is contained in $\sigma_{CB}(B(\mathcal{H}))(\Theta)$. By Proposition 4.1 we have that $\sigma_{CB}(B(\mathcal{K}, \mathcal{H}))(\Theta) = \sigma_D(\Theta)$ for each $\Theta \in D$. This concludes the proof of (i).

To prove (ii), let $\lambda$ be an eigenvalue of $\Theta$ and $x \in B(\mathcal{K}, \mathcal{H})$ a corresponding nonzero eigenvector, so that $\Theta(x) = \lambda x$. By a variant of Egoroffs theorem [31, p. 85] in any neighborhoods of the identity 1 (in the strong operator topology) there exists projections $e \in \overline{A}$ and $f \in \overline{B}$ such that the two series $\sum_{j=1}^{\infty} c_j e^* e$ and $\sum_{j=1}^{\infty} d_j^* f$ converge uniformly. We may choose $e$ and $f$ so that $exf \neq 0$. Since

$$(c)(ef)(df) = \lambda exf,$$

$\lambda$ is an eigenvalue of $(c) \otimes (df)$, hence $\lambda \in \sigma_A(c) \cdot \sigma_B(d)$ by (i). Since $\phi(e) \in \{0, 1\}$ for each $\phi \in \Delta(\overline{A})$ and similarly for $f$, it follows that $\lambda \in \sigma_A(c) \cdot \sigma_B(d) = \sigma_A(c) \cdot \sigma_B(d)$ if $\lambda \neq 0$. If $\lambda = 0$, we apply the result just obtained to the map $\Theta + 1 = \tilde{c} \otimes \tilde{d}$, where $\tilde{c} = [1, c_1, c_2, \ldots]$ and $\tilde{d} = (1, d_1, d_2, \ldots)$ and the eigenvalue 1 of this map.

**Theorem 4.6.** Let $a = (a_j)$ and $b = (b_j)$ be two commutative sequences of normal operators on (separable) Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ (respectively) such that $\sum_{j=1}^{\infty} a_j^* a_j = 1$ and $\sum_{j=1}^{\infty} b_j^* b_j = 1$, where the sums are weak* convergent. Then
the fixed points of the map $\Theta = a \otimes b = \sum_{j=1}^{\infty} a_j \otimes b_j$ on $\mathcal{B}(\mathcal{H})$ are precisely the operators $x \in \mathcal{B}(\mathcal{H})$ that intertwine $a$ and $b^*$ (that is, $a_jx = xb_j^*$ for all $j$).

**Proof.** Clearly the intertwiners of $a$ and $b^*$ are fixed points of $\Theta$ since $\sum_{j=1}^{\infty} b_j^*a_j = 1$, so only the converse needs a proof. By considering

$$
\begin{pmatrix}
  a_j & 0 \\
  0 & b_j^*
\end{pmatrix},
\begin{pmatrix}
  a_j^* & 0 \\
  0 & b_j
\end{pmatrix}
$$

instead of $a_j$, $b_j$ and $x$ (respectively), the proof can easily be reduced to the case where $b_j = a_j^*$. So we assume that $b_j = a_j^*$ for all $j$ and we have to prove that each fixed point $x$ of $\Theta$ commutes with all $a_j$. Let $A$ be the $C^*$-algebra generated by 1 and $(a_j)$ and let $e$ be the spectral measure on $\Delta := \Delta(A)$ such that

$$
c = \int_{\Delta} \hat{c}(\phi) \, dc(\phi)
$$

for all $c \in A$, where $\hat{c}$ is the Gelfand transform of $c$ [9, p. 266]. It suffices to show that $xe(K) = e(K)x$ for each compact subset $K$ of $\Delta$ or, equivalently, that $e(K)^{-1}xe(K) = 0$, where $e(K)^{-1} = 1 - e(K)$. Since $e(K)^{-1} = e(\Delta \setminus K)$ is the join of all the projections $e(H)$ for compact subsets $H$ of $K^c := \Delta \setminus K$, it suffices to show that $e(H)xe(K) = 0$ for all such $H$. Assume the contrary, that $e(H)xe(K) \neq 0$ for some compact $H \subseteq K^c$. Consider the orthogonal decomposition

$$
\mathcal{H} = e(H)\mathcal{H} \oplus e(K)\mathcal{H} \oplus e(H^c \cap K^c)^{\perp}\mathcal{H}
$$

and let $x = [x_{k,l}]$ be the corresponding representation of $x$ by a $3 \times 3$ operator matrix. With respect to the decomposition (4.6) each operator $a_j$ is represented by a diagonal matrix $a_j = c_j \oplus d_j \oplus f_j$ (where, for example, $c_j = a_j e(H)\mathcal{H}$). Then the $(1,2)$ entry of the matrix $\Theta(x) = \sum_{j=1}^{\infty} a_jx_j$ is $\sum_{j=1}^{\infty} c_j x_{1,2}d_j^*$, where $x_{1,2} = e(H)xe(K) \neq 0$. From $\Theta(x) = x$ we have

$$
\sum_{j=1}^{\infty} c_j x_{1,2}d_j^* = x_{1,2},
$$

which means that 1 is an eigenvalue of the map $\Theta_{c,d^*} := \sum_{j=1}^{\infty} c_j \otimes d_j^*$. By Proposition 4.5

$$
1 = \langle \lambda, \mu \rangle \quad \text{for some } \lambda \in \sigma_{\mathcal{A}(H)}(c), \ \mu \in \sigma_{\mathcal{A}(K)}(d).
$$

Since $\sum_{j=1}^{\infty} c_j e_j = e(H)$ and $\sum_{j=1}^{\infty} d_j d_j^* = e(K)$, it follows that $\|\lambda\| \leq 1$ and $\|\mu\| \leq 1$, hence (4.7) implies that $\mu = \lambda$. Therefore

$$
\sigma_{\mathcal{A}(H)}(c) \cap \sigma_{\mathcal{A}(K)}(d) \neq \emptyset.
$$

On the other hand, the map $\hat{a} : \phi \mapsto (\phi(a_1), \phi(a_2), \ldots)$ from $\Delta$ into $l^2$ is injective. Since the $C^*$-algebra $\mathcal{A}(H)$ is isomorphic to $C(H)$ (complex valued continuous functions on $H$) by Tietze’s theorem, $\Delta(\mathcal{A}(H)) \cong H$. (That is, all characters of $\mathcal{A}(H)$ are evaluations at points of $H$.) Hence $\sigma_{\mathcal{A}(H)}(c) = \sigma_{\mathcal{A}(H)}(ae(H)) = \hat{a}(H)$. Similarly $\sigma_{\mathcal{A}(K)}(d) = \hat{a}(K)$. Since $H$ and $K$ are disjoint and $\hat{a}$ is injective, $\sigma_{\mathcal{A}(H)}(c)$ and $\sigma_{\mathcal{A}(K)}(d)$ must also be disjoint, but this is in contradiction with (4.8). $\square$
Problem. Does the conclusion of Theorem 4.6 still hold if, instead of commutativity, we assume that each of the two sequences \((a_j)\) and \((b_j)\) is contained in a finite von Neumann algebra?

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