On the regularity of the solution map of the porous media equation

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Abstract

In this paper we consider the incompressible porous media equation in the Sobolev spaces \( H^s(\mathbb{R}^2), s > 2 \). We prove that for \( T > 0 \) the time \( T \) solution map \( \rho_0 \mapsto \rho(T) \) is nowhere locally uniformly continuous. On the other hand we show that the particle trajectories are analytic curves in \( \mathbb{R}^2 \).

1 Introduction

The initial value problem for the incompressible porous media equation in the Sobolev space \( H^s(\mathbb{R}^2), s > 2 \) is given by

\[
\begin{align*}
\rho_t + (u \cdot \nabla)\rho &= 0 \\
\text{div} u &= 0 \\
u &= -\nabla p - \begin{pmatrix} 0 \\ \rho \end{pmatrix} \\
\rho(0) &= \rho_0
\end{align*}
\]

(1)

where \( \rho : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R} \) is the density, \( u : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2 \) the velocity of the flow, \( p : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R} \) the pressure. Local well-posedness for \( \rho \) lying in \( C^0([0, T]; H^s(\mathbb{R}^2)) \) is known – see [2]. For \( T > 0 \) we denote by \( U_T \subseteq H^s(\mathbb{R}^2) \) the set of initial values \( \rho_0 \) for which the solution of (1) exists longer than time \( T \). We can state our main theorem as
Theorem 1.1. Let $s > 2$ and $T > 0$. Then the time $T$ solution map

$$U_T \to H^s(\mathbb{R}^2), \quad \rho_0 \mapsto \Phi_T(\rho_0) = \rho(T)$$

is nowhere locally uniformly continuous. Here $\rho(T)$ is the value of $\rho$ at time $T$.

Our method relies on a geometric formulation of (1). This approach was made popular by the work of Arnold [1] for the incompressible Euler equations and subsequently by [3]. In the following we will work this out for (1). Taking the divergence in the third equation (Darcy’s law) in (1) we have

$$-\Delta p = \partial_2 \rho$$

Reexpressing $\nabla p$ gives

$$u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} -(-\Delta)^{-1} \partial_1 \partial_2 \rho \\ -(-\Delta)^{-1} \partial_2 \rho - \rho \end{pmatrix} = \begin{pmatrix} -(-\Delta)^{-1} \partial_1 \partial_2 \rho \\ (-\Delta)^{-1} \partial_2 \rho \end{pmatrix} = \begin{pmatrix} -R_1 R_2 \rho \\ R_2^2 \rho \end{pmatrix}$$

where

$$R_k = \partial_k (-\Delta)^{-\frac{1}{2}}, \quad k = 1, 2$$

are the Riesz operators. Applying $-R_1 R_2$ resp. $R_2^2$ to the first equation in (1) gives

$$u_t + (u \cdot \nabla) u = \begin{pmatrix} [u \cdot \nabla, -R_1 R_2] \rho \\ [u \cdot \nabla, R_2^2] \rho \end{pmatrix}$$

(2)

where we use $[A, B] = AB - BA$ for the commutator of operators. We will express (2) in Lagrangian coordinates, i.e. in terms of the flow map of $u$

$$\varphi_t = u \circ \varphi, \quad \varphi(0) = \text{id}$$

where $\text{id}$ is the identity map in $\mathbb{R}^2$. The functional space for $\varphi$ is for $s > 2$ the diffeomorphism group

$$D^s(\mathbb{R}^2) := \{ \varphi : \mathbb{R}^2 \to \mathbb{R}^2 \mid \varphi - \text{id} \in H^s(\mathbb{R}^2; \mathbb{R}^2) \text{ and } \det(d_x \varphi) > 0 \forall x \in \mathbb{R}^2 \}$$

where $H^s(\mathbb{R}^2; \mathbb{R}^2)$ denotes the vector valued Sobolev space. By the Sobolev imbedding theorem $D^s(\mathbb{R}^2)$ consists of $C^1$ diffeomorphisms. Regarding it as an open subset $D^s(\mathbb{R}^2) - \text{id} \subseteq H^s(\mathbb{R}^2; \mathbb{R}^2)$ it is a connected topological group under composition of maps – see [4]. The first equation in (1) in terms of $\varphi$ reads as

$$\rho(t) = \rho_0 \circ \varphi(t)^{-1}$$
Taking the $t$ derivative of $\varphi_t = u \circ \varphi$ is

$$\varphi_{tt} = (u_t + (u \cdot \nabla) u) \circ \varphi$$

Thus we can write (2) as

$$\varphi_{tt} = \left( \left( [\varphi_t \circ \varphi^{-1} \cdot \nabla, -\mathcal{R}_1 \mathcal{R}_2] (\rho_0 \circ \varphi^{-1}) \right) \circ \varphi \right)$$

or as a first order equation in $\mathcal{D}^s(\mathbb{R}^2) \times H^s(\mathbb{R}^2; \mathbb{R}^2)$

$$\frac{d}{dt} \left( \begin{array}{c} \varphi \\ v \end{array} \right) = \left( \begin{array}{c} [v \circ \varphi^{-1} \cdot \nabla, -\mathcal{R}_1 \mathcal{R}_2] (\rho_0 \circ \varphi^{-1}) \end{array} \right) \circ \varphi = \left( \begin{array}{c} v \\ F(\varphi, v, \rho_0) \end{array} \right)$$

We claim that $F(\varphi, v, \rho_0)$ is analytic in its arguments

**Lemma 1.1.** The map

$$F: \mathcal{D}^s(\mathbb{R}^2) \times H^s(\mathbb{R}^2; \mathbb{R}^2) \times H^s(\mathbb{R}^2; \mathbb{R}^2) \to H^s(\mathbb{R}^2; \mathbb{R}^2)$$

$$(\varphi, v, \rho_0) \mapsto F(\varphi, v, \rho_0)$$

is analytic.

In the following we will use the notation $R_\varphi : g \mapsto g \circ \varphi$ for the composition from the right. Note that $R_\varphi^{-1} g = g \circ \varphi^{-1}$.

**Proof of Lemma 1.1.** In [6] it was shown that for $k = 1, 2$

$$\mathcal{D}^s(\mathbb{R}^2) \times H^s(\mathbb{R}^2) \to H^s(\mathbb{R}^2), \quad (\varphi, f) \mapsto (\mathcal{R}_k (f \circ \varphi^{-1})) \circ \varphi = R_\varphi \mathcal{R}_k R_\varphi^{-1} f$$

is analytic and also that

$$\mathcal{D}^s(\mathbb{R}^2) \times \mathcal{D}^s(\mathbb{R}^2; \mathbb{R}^2) \times H^s(\mathbb{R}^2) \to H^s(\mathbb{R}^2)$$

$$(\varphi, w, f) \mapsto \left( \left( [w \circ \varphi^{-1} \cdot \nabla, \mathcal{R}_k] (f \circ \varphi^{-1}) \right) \circ \varphi = R_\varphi [R_\varphi^{-1} w \cdot \nabla, \mathcal{R}_k] R_\varphi^{-1} f \right)$$

is analytic. Using these results we see by writing for $j, k = 1, 2$

$$R_\varphi [R_\varphi^{-1} v \cdot \nabla, \mathcal{R}_j \mathcal{R}_k] R_\varphi^{-1} \rho_0 =$$

$$R_\varphi [R_\varphi^{-1} v \cdot \nabla, \mathcal{R}_j] R_\varphi R_\varphi^{-1} \rho_0 + R_\varphi \mathcal{R}_j [R_\varphi^{-1} v \cdot \nabla, \mathcal{R}_k] R_\varphi^{-1} \rho_0 =$$

$$R_\varphi [R_\varphi^{-1} v \cdot \nabla, \mathcal{R}_j] R_\varphi^{-1} R_\varphi \mathcal{R}_k R_\varphi^{-1} \rho_0 + R_\varphi \mathcal{R}_j R_\varphi^{-1} R_\varphi [R_\varphi^{-1} v \cdot \nabla, \mathcal{R}_k] R_\varphi^{-1} \rho_0$$

that $F(\varphi, v, \rho_0)$ is analytic in its arguments. \qed
By Picard-Lindelöf we get for any \( \rho_0 \in H^s(\mathbb{R}^2) \) local solutions to

\[
\frac{d}{dt} \begin{pmatrix} \varphi \\ v \end{pmatrix} = \begin{pmatrix} v \\ F(\varphi, v, \rho_0) \end{pmatrix}, \quad \varphi(0) = \text{id}, \ v(0) = u_0
\]

(3)

By taking \( u_0 = (-\mathcal{R}_1 \mathcal{R}_2 \rho_0, \mathcal{R}_2^2 \rho_0) \) we claim that we get solutions to (1).

**Proposition 1.2.** Let \( \rho_0 \in H^s(\mathbb{R}^2) \) and let \((\varphi, v)\) be the solution to (3) with \( u_0 = (-\mathcal{R}_1 \mathcal{R}_2 \rho_0, \mathcal{R}_2^2 \rho_0) \) on some time interval \([0, T]\) with \( T > 0 \). Then

\[
u(t) = v(t) \circ \varphi(t)^{-1} \quad \text{and} \quad \rho(t) = \rho_0 \circ \varphi(t)^{-1}
\]

is a solution \((\rho, u)\) \(\in C^0([0, T]; H^s(\mathbb{R}^2) \times H^s(\mathbb{R}^2; \mathbb{R}^2))\) to (1).

**Proof.** By the properties of the composition we clearly have

\[(\rho, u) \in C^0([0, T]; H^s(\mathbb{R}^2) \times H^s(\mathbb{R}^2; \mathbb{R}^2))\]

Define

\[
w = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} -\mathcal{R}_\varphi \mathcal{R}_1 \mathcal{R}_2 \mathcal{R}_\varphi^{-1} \rho_0 \\ \mathcal{R}_\varphi \mathcal{R}_1 \mathcal{R}_2 \mathcal{R}_\varphi^{-1} \rho_0 \end{pmatrix} \in C^\infty([0, T]; H^s(\mathbb{R}^2; \mathbb{R}^2))
\]

Calculating the \( t \) derivative of \( w_1 \) gives (note that by the Sobolev imbedding we have \( C^1 \) expressions)

\[
\frac{d}{dt}w_1 = -\frac{d}{dt}\mathcal{R}_\varphi \mathcal{R}_1 \mathcal{R}_2 \mathcal{R}_\varphi^{-1} \rho_0 = -\mathcal{R}_\varphi \mathcal{R}_1 \mathcal{R}_2 \frac{d}{dt}\mathcal{R}_\varphi^{-1} \rho_0 - (\varphi_t \cdot \mathcal{R}_\varphi \nabla)\mathcal{R}_1 \mathcal{R}_2 \mathcal{R}_\varphi^{-1} \rho_0
\]

\[
= \mathcal{R}_\varphi \mathcal{R}_1 \mathcal{R}_2 (\mathcal{R}_\varphi^{-1} \rho_0 \cdot \mathcal{R}_\varphi^{-1} \mathcal{R}_\varphi^{-1} \rho_0) - (\varphi_t \cdot \mathcal{R}_\varphi \nabla)\mathcal{R}_1 \mathcal{R}_2 \mathcal{R}_\varphi^{-1} \rho_0
\]

\[
= \mathcal{R}_\varphi (\mathcal{R}_1 \mathcal{R}_2, (\mathcal{R}_\varphi^{-1} v \cdot \nabla))\mathcal{R}_\varphi^{-1} \rho_0
\]

showing that \( w_1(t) = v_1(t) \) for \( t \in [0, T] \) as \( w_1(0) = v_1(0) = -\mathcal{R}_1 \mathcal{R}_2 \rho_0 \).

Similarly we have \( w_2 = v_2 \). Thus

\[
u = w \circ \varphi^{-1} = \begin{pmatrix} -\mathcal{R}_1 \mathcal{R}_2 \rho_0 \\ \mathcal{R}_1^2 \rho_0 \end{pmatrix}
\]

showing the claim. \(\square \)
On the other hand consider a solution of (1) in 
\((\rho, u) \in C^0([0, T]; H^s(\mathbb{R}^2) \times H^s(\mathbb{R}^2; \mathbb{R}^2)) \cap C^1([0, T]; H^{s-1}(\mathbb{R}^2) \times H^{s-1}(\mathbb{R}^2; \mathbb{R}^2))\)

We know (see [5]) that there exists a unique \(\varphi \in C^1([0, T]; \mathcal{D}^s(\mathbb{R}^2))\) with 

\[
\varphi_t = u \circ \varphi, \quad \varphi(0) = \text{id}
\]

We claim that \(\varphi\) and \(v = \varphi_t\) is a solution to (3). From (1) we get 

\[
u = \begin{pmatrix}
-\mathcal{R}_1\mathcal{R}_2(\rho_0 \circ \varphi^{-1}) \\
\mathcal{R}_2^2(\rho_0 \circ \varphi^{-1})
\end{pmatrix}
\]

Consider \(u \circ \varphi\). Taking the \(t\) derivative we get pointwise 

\[(u_t + (u \cdot \nabla) u) \circ \varphi = \begin{pmatrix}
[u \cdot \nabla, -\mathcal{R}_1\mathcal{R}_2](\rho_0 \circ \varphi^{-1}) \\
[u \cdot \nabla, \mathcal{R}_2^2](\rho_0 \circ \varphi^{-1})
\end{pmatrix} \circ \varphi
\]

with the righthandside continuous with values in \(H^s(\mathbb{R}^2; \mathbb{R}^2)\). Hence \(v \in C^1([0, T]; H^s(\mathbb{R}^2; \mathbb{R}^2))\) with derivative 

\[
v_t = \begin{pmatrix}
[(v \circ \varphi^{-1}) \cdot \nabla, -\mathcal{R}_1\mathcal{R}_2]\rho_0 \circ \varphi^{-1} \\
[(v \circ \varphi^{-1}) \cdot \nabla, \mathcal{R}_2^2]\rho_0 \circ \varphi^{-1}
\end{pmatrix} \circ \varphi
\]

Hence \((\varphi, v)\) is a solution to (3) showing uniqueness of solutions for (1) by the uniqueness of solutions for ODEs. Together with Proposition 1.2 we get therefore the local well-posedness of (1).

From the ODE formulation (3) we immediately get

**Theorem 1.2.** The particle trajectories of the flow determined by (1) are analytic curves in \(\mathbb{R}^2\).

**Proof.** As (3) is analytic we get by ODE theory that 

\([0, T] \to D^s(\mathbb{R}^2), \quad t \mapsto \varphi(t)\)

is analytic. Thus evaluation at \(x \in \mathbb{R}^2\), giving the trajectory of the particle which is located at \(x\) at time zero, 

\([0, T] \to \mathbb{R}^2, \quad t \mapsto \varphi(t, x)\)

is also analytic.
2 Nonuniform dependence

The goal of this section is to prove Theorem 1.1. For the proof we need some preparation. Note that we have the following scaling property for (1):

Assume that \((\rho(t, x), u(t, x))\) is a solution to (1). Then a simple calculation shows that for \(\lambda > 0\)

\[
(\rho_\lambda(t, x), u_\lambda(t, x)) = (\lambda \rho(\lambda t, x), \lambda u(\lambda t, x))
\]

is also a solution to (1). Thus we have for the domain \(U_T \subseteq H^s(\mathbb{R}^2)\)

\[
U_{\lambda T} = \frac{1}{\lambda} \cdot U_T
\]

and for the solution map \(\Phi_T\)

\[
\Phi_{\lambda T}(\rho_0) = \frac{1}{\lambda} \cdot \Phi_T(\lambda \cdot \rho_0)
\]

Hence we have \(\Phi_T(\rho_0) = \frac{1}{T} \cdot \Phi_1(T \cdot \rho_0)\). Therefore it will be enough to prove Theorem 1.1 for the case \(T = 1\). For \(T = 1\) we introduce

\[
U := U_1 \quad \text{and} \quad \Phi := \Phi_1
\]

Similarly we denote by

\[
\Psi(\rho_0) := \varphi(1; \rho_0)
\]

where \(\varphi(1; \rho_0)\) is the value of the \(\varphi\) component at time 1 of the solution to (3) for the initial values

\[
\varphi(0) = \text{id}, \quad v(0) = (-R_1 R_2 \rho_0, R_2^2 \rho_0)
\]

Thus by analytic dependence on initial values and parameters we have that

\[
\Psi : U \subseteq H^s(\mathbb{R}^2) \rightarrow D^s(\mathbb{R}^2), \quad \rho_0 \mapsto \Psi(\rho_0)
\]

is analytic. For the proof of the main result we will need the following technical lemma

**Lemma 2.1.** There is a dense subset \(S \subseteq U(\subseteq H^s(\mathbb{R}^2))\) with the property that for each \(\rho_\bullet \in S\) we have: the support of \(\rho_\bullet\) is compact and there is \(\bar{\rho} \in H^s(\mathbb{R}^2)\) and \(x^* \in \mathbb{R}^2\) with \(\text{dist}(x^*, \text{supp} \rho_\bullet) > 2\) (i.e. the distance of \(x^*\) to the support of \(\rho_\bullet\) is bigger than 2) and

\[
(d_{\rho_\bullet} \Psi(\bar{\rho}))(x^*) \neq 0
\]

where \(d_{\rho_\bullet} \Psi\) is the differential of \(\Psi\) at \(\rho_\bullet\).
Proof. First note that
\[ \Psi(t \cdot \bar{\rho}) = \varphi(1; t \cdot \bar{\rho}) = \varphi(t; \bar{\rho}) \]
where the last equality follows from the scaling property discussed above. Taking the \( t \) derivative at \( t = 0 \) gives
\[ d_0 \Psi(\bar{\rho}) = \frac{d}{dt} \bigg|_{t=0} \Psi(t \cdot \bar{\rho}) = \frac{d}{dt} \bigg|_{t=0} \varphi(t; \bar{\rho}) = \varphi_t(0, \bar{\rho}) = (-\mathcal{R}_1 \mathcal{R}_2 \bar{\rho}, \mathcal{R}_2^2 \bar{\rho}) \]
Fix an arbitrary \( \rho_\bullet \in U \subseteq H^s(\mathbb{R}^2) \) with compact support. Take \( x^* \in \mathbb{R}^2 \) with \( \text{dist}(x^*, \text{supp}(\rho_\bullet)) > 2 \). Take a \( \tilde{\rho} \in H^s(\mathbb{R}^2) \) with \( -\mathcal{R}_1 \mathcal{R}_2 \tilde{\rho} \neq 0 \). The operator \( -\mathcal{R}_1 \mathcal{R}_2 \) is translation invariant as it is a Fourier multiplier operator. Therefore we can choose \( \tilde{\rho}(\cdot) = \tilde{\rho}(\cdot + \delta x) \) with
\[ (-\mathcal{R}_1 \mathcal{R}_2 \tilde{\rho})(x^*) \neq 0 \]
Now consider the analytic curve
\[ (d_{t \rho_\bullet} \Psi(\tilde{\rho}))(x^*) \]
which at \( t = 0 \) is different from zero. Therefore there exist \( t_n \uparrow 1 \) for \( n \geq 1 \) with
\[ (d_{t_n \rho_\bullet} \Psi(\tilde{\rho}))(x^*) \neq 0 \]
So we can put all these \( t_n \rho_\bullet \) into \( S \). By this construction we see that \( S \) is dense in \( U \).

Theorem \( \boxed{1.1} \) will follow from

**Proposition 2.2.** The time \( T = 1 \) solution map
\[ \Phi : U \subseteq H^s(\mathbb{R}^2) \to H^s(\mathbb{R}^2), \quad \rho_0 \mapsto \Phi(\rho_0) \]
is nowhere locally uniformly continuous.

**Proof.** Let \( S \subseteq U \) be as in Lemma \( \boxed{2.1} \). Take an arbitrary \( \rho_\bullet \in S \). In successive steps we will choose \( R_\bullet > 0 \) and prove that
\[ \Phi : B_R(\rho_\bullet) \subseteq U \to H^s(\mathbb{R}^2) \]
is not uniformly continuous for any \( 0 < R \leq R_\bullet \). Here we denote by \( B_R(\rho_\bullet) \subseteq H^s(\mathbb{R}^2) \) the ball of radius \( R \) around \( \rho_\bullet \). As \( S \) is dense this is clearly sufficient
to prove the proposition.

Fix $x^* \in \mathbb{R}^2$ and $\bar{\rho} \in H^s(\mathbb{R}^2)$ with

$$m := |(d_{\rho_\bullet} \Psi(\bar{\rho}))(x^*)| > 0$$

as guaranteed by Lemma 2.1. Here $| \cdot |$ is the Euclidean norm in $\mathbb{R}^2$. Define $\varphi_\bullet = \Phi(\rho_\bullet)$ and let

$$d := \text{dist}(\varphi_\bullet(\text{supp } \rho_\bullet), \varphi_\bullet(B_1(x^*))) > 0$$

where $B_1(x^*) \subseteq \mathbb{R}^2$. By the Sobolev imbedding we fix $\tilde{C} > 0$ with

$$||f||_{C^1} \leq \tilde{C}||f||_s$$

for all $f \in H^s(\mathbb{R}^2; \mathbb{R}^2)$. Choose $R_1 > 0$ and $C_1 > 0$ with

$$\frac{1}{C_1}||f||_s \leq ||f \circ \varphi_\bullet^{-1}||_s \leq C_1||f||_s$$

for all $f \in H^s(\mathbb{R}^2)$ and for all $\varphi \in \Psi(B_{R_1}(\rho_\bullet))$ which is possible due to the continuity of composition – see [4]. Using the Sobolev imbedding (4) we take

$$0 < R_2 \leq R_1$$

and $L > 0$ with

$$|\varphi(x) - \varphi(y)| < L|x - y| \text{ and } |\varphi(x) - \varphi_\bullet(x)| < d/4$$

for all $x, y \in \mathbb{R}^2$ and $\varphi \in \Psi(B_{R_2}(\rho_\bullet))$.

Consider the Taylor expansion of $\Psi$ around $\rho_\bullet$

$$\Psi(\rho_\bullet + h) = \Psi(\rho_\bullet) + d_{\rho_\bullet} \Psi(h) + \int_0^1 (1-s)d_{\rho_\bullet + sh}^2 \Psi(h, h) \, ds$$

for $h \in H^s(\mathbb{R}^2)$. In order to estimate $d^2 \Psi$ we choose $0 < R_3 \leq R_2$ such that

$$||d^2_{\rho} \Psi(h_1, h_2)||_s \leq K||h_1||_s||h_2||_s$$

and

$$||d^2_{\rho_1} \Psi(h_1, h_2) - d^2_{\rho_2} \Psi(h_1, h_2)||_s \leq K||\rho_1 - \rho_2||_s||h_1||_s||h_2||_s$$

for all $\rho, \rho_1, \rho_2 \in B_{R_3}(\rho_\bullet)$ and for all $h_1, h_2 \in H^s(\mathbb{R}^2)$ which is possible due to the smoothness of $\Psi$. Finally we choose $0 < R_4 \leq R_3$ such that

$$\tilde{C}K||\bar{\rho}||_s R^2_4/4 + \tilde{C}K||\bar{\rho}||_s R_4 < m/4$$
Now fix $0 < R \leq R_\ast$. We will construct two sequences of initial values

$$(\rho^{(n)}_0)_{n \geq 1}, (\check{\rho}^{(n)}_0)_{n \geq 1} \subseteq B_R(\rho_\bullet)$$

with $\lim_{n \to \infty} \|\rho^{(n)}_0 - \check{\rho}^{(n)}_0\|_s = 0$ whereas

$$\limsup_{n \to \infty} \|\Phi(\rho^{(n)}_0) - \Phi(\check{\rho}^{(n)}_0)\|_s > 0$$

Define the radii $r_n = m/8nL$ and choose $w_n \in H^s(\mathbb{R}^2)$ with

$$\operatorname{supp} w_n \subseteq B_{r_n}(x^\ast) \quad \text{and} \quad \|w_n\|_s = R/2$$

We choose the initial values as

$$\rho^{(n)}_0 = \rho_\bullet + w_n \quad \text{and} \quad \check{\rho}^{(n)}_0 = \rho_\bullet + w_n + \frac{1}{n} \bar{\rho}$$

For some $N$ we clearly have

$$(\rho^{(n)}_0)_{n \geq N}, (\check{\rho}^{(n)}_0)_{n \geq N} \subseteq B_R(\rho_\bullet)$$

and $\operatorname{supp} w_n \subseteq B_1(x^\ast)$ for $n \geq N$. By taking $N$ large enough we can also ensure

$$\tilde{C}K \frac{1}{n} \|\rho\|_s < m/4, \quad \forall n \geq N \quad (7)$$

Furthermore

$$\|\rho^{(n)}_0 - \check{\rho}^{(n)}_0\|_s = \|\frac{1}{n} \bar{\rho}\|_s \to 0$$

as $n \to \infty$. We introduce

$$\varphi^{(n)} = \Psi(\rho^{(n)}_0) \quad \text{and} \quad \check{\varphi}^{(n)} = \Psi(\check{\rho}^{(n)}_0)$$

With this we have

$$\Phi(\rho^{(n)}_0) = \rho^{(n)}_0 \circ (\varphi^{(n)})^{-1} \quad \text{and} \quad \Phi(\check{\rho}^{(n)}_0) = \check{\rho}^{(n)}_0 \circ (\check{\varphi}^{(n)})^{-1}$$

Hence

$$\|\Phi(\rho^{(n)}_0) - \Phi(\check{\rho}^{(n)}_0)\|_s = \|\rho_\bullet + w_n \circ (\varphi^{(n)})^{-1} - (\rho_\bullet + w_n + \frac{1}{n} \bar{\rho}) \circ (\check{\varphi}^{(n)})^{-1}\|_s$$
From (5) we conclude
\[
\limsup_{n \to \infty} ||(\rho_\bullet + w_n) \circ (\varphi(n))^{-1} - (\rho_\bullet + w_n + \frac{1}{n} \rho) \circ (\tilde{\varphi}(n))^{-1}||_s = \\
\limsup_{n \to \infty} ||(\rho_\bullet + w_n) \circ (\varphi(n))^{-1} - (\rho_\bullet + w_n) \circ (\varphi(n))^{-1}||_s
\]

Note that by (6) we have for \( n \geq N \)
\[
supp(\rho_\bullet \circ (\varphi(n))^{-1}), supp(\rho_\bullet \circ (\tilde{\varphi}(n))^{-1}) \subseteq \varphi_\bullet(supp \rho_\bullet) + B_{d/4}(0)
\]
and
\[
supp(w_n \circ (\varphi(n))^{-1}), supp(w_n \circ (\tilde{\varphi}(n))^{-1}) \subseteq \varphi_\bullet(B_1(x^*)) + B_{d/4}(0)
\]
where we use \( A + B = \{ a + b \mid a \in A, b \in B \} \). So the \( \rho_\bullet \) and \( w_n \) terms are supported in disjoint sets. This allows us (see [6]) to estimate with a constant \( \bar{C} > 0 \)
\[
\limsup_{n \to \infty} ||(\rho_\bullet + w_n) \circ (\varphi(n))^{-1} - (\rho_\bullet + w_n) \circ (\tilde{\varphi}(n))^{-1}||_s \\
\limsup_{n \to \infty} \bar{C} ||w_n \circ (\varphi(n))^{-1} - w_n \circ (\tilde{\varphi}(n))^{-1}||_s
\]

The goal is to separate the two \( w_n \) expressions by showing that their supports are also disjoint in a suitable way. We have
\[
\varphi^{(n)} = \Psi(\rho_\bullet + w_n) = \Psi(\rho_\bullet) + d_{\rho_\bullet} \Psi(w_n) + \int_0^1 (1 - s) d^2_{\rho_\bullet + sw_n} \Psi(w_n, w_n) \, ds
\]
resp.
\[
\tilde{\varphi}^{(n)} = \Psi(\rho_\bullet + w_n + \frac{1}{n} \bar{\rho}) = \Psi(\rho_\bullet) + d_{\rho_\bullet} \Psi(w_n + \frac{1}{n} \bar{\rho}) \\
+ \int_0^1 (1 - s) d^2_{\rho_\bullet + s(w_n + \frac{1}{n} \bar{\rho})} \Psi(w_n + \frac{1}{n} \bar{\rho}, w_n + \frac{1}{n} \bar{\rho}) \, ds
\]
Thus
\[
\tilde{\varphi}^{(n)} - \varphi^{(n)} = d_{\rho_\bullet} \Psi(\frac{1}{n} \bar{\rho}) + I_1 + I_2 + I_3
\]
where
\[
I_1 = \int_0^1 (1 - s) \left( d^2_{\rho_\bullet + sw_n + \frac{1}{n} \bar{\rho}} \Psi(w_n, w_n) - d^2_{\rho_\bullet + sw_n} \Psi(w_n, w_n) \right) \, ds
\]
and
\[ I_2 = 2 \int_0^1 (1 - s) d^2_{\rho + s(w_n + \frac{1}{n}\bar{\rho})} \Psi(w_n, \frac{1}{n}\bar{\rho}) \, ds \]
and
\[ I_3 = \int_0^1 (1 - s) d^2_{\rho + s(w_n + \frac{1}{n}\bar{\rho})} \Psi(-\frac{1}{n}\bar{\rho}, -\frac{1}{n}\bar{\rho}) \, ds \]

Using the estimates for \( d^2_\Psi \) we have
\[ ||I_1||_s \leq K \frac{1}{n} ||\bar{\rho}||_s R^2 / 4, \quad ||I_2||_s \leq 2K \frac{1}{n} ||\bar{\rho}||_s R / 2, \quad ||I_3||_s \leq K \frac{1}{n^2} ||\bar{\rho}||_s^2 \]

Using (4), (7) and by the choice of \( R_* \) we have
\[ |I_1(x^*)| + |I_2(x^*)| + |I_3(x^*)| < \frac{m}{2n} \]

Hence
\[ |\tilde{\varphi}^{(n)}(x^*) - \varphi^{(n)}(x^*)| \geq |(d_{\rho^*} \Psi(\bar{\rho})) (x^*)| \frac{n}{n - \frac{m}{2n}} = \frac{m}{2n} \]

We have by (6) and the choice of \( r_n \)
\[ \text{supp}(w_n \circ (\tilde{\varphi}^{(n)})^{-1}) \subseteq B_{L_{r_n}}(\tilde{\varphi}^{(n)}(x^*)) = B_{m/(8n)}(\tilde{\varphi}^{(n)}(x^*)) \]
resp.
\[ \text{supp}(w_n \circ (\varphi^{(n)})^{-1}) \subseteq B_{L_{r_n}}(\varphi^{(n)}(x^*)) = B_{m/(8n)}(\varphi^{(n)}(x^*)) \]

which shows that the supports of the two \( w_n \) terms are in such a way disjoint that we can separate the \( H^s \) norms with a constant (see [6]) to get
\[ ||w_n \circ (\varphi^{(n)})^{-1} - w_n \circ (\tilde{\varphi}^{(n)})^{-1}||_s \geq \tilde{K} (||w_n \circ (\varphi^{(n)})^{-1}||_s + ||w_n \circ (\tilde{\varphi}^{(n)})^{-1}||_s) \]
\[ \geq 2\tilde{K} \frac{1}{C_1} ||w_n||_s = \frac{\tilde{K} R}{C_1} \]

where we used (5). Therefore
\[ \limsup_{n \to \infty} ||\Phi(\rho_0^{(n)}) - \Phi(\tilde{\rho}_0^{(n)})||_s \geq \frac{\tilde{K} C R}{C_1} \]

whereas
\[ ||\rho_0^{(n)} - \tilde{\rho}_0^{(n)}||_s \to 0 \quad \text{as} \quad n \to \infty \]
showing that \( \Phi \) is not uniformly continuous on \( B_R(\rho_\bullet) \). As \( 0 < R \leq R_* \) is arbitrary the result follows.
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