HOW DENNIS REALIZED HE HAD ‘INVENTED’ $L_\infty$- ALGEBRAS
A.K.A. STRONGLY HOMOTOPY LIE ALGEBRAS

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Abstract. This note is an attempt to rediscover how Dennis realized that he had discovered/invented $L_\infty$- algebras.

1. Introduction

$L_\infty$- algebras appeared independently in the mid-1980’s in a supporting role in deformation theory in my work with Mike Schlessinger [6, 5, 4] on rational homotopy theory and in correspondence between Drinfel’d and Schechtman [1, 2] as well as in mathematical physics. They were in fact implicit in Dennis’ models. Here is my attempt to rediscover how Dennis realized what he had discovered.

The philosophy that every deformation problem is controlled by a differential graded Lie algebra, but not uniquely even up to isomorphism, leads to consideration of morphisms ‘up to homotopy’ and hence to $L_\infty$ algebras (originally called ‘strongly homotopy Lie algebras’ or, by Drinfel’d, ‘Lie-Sugawara algebras’).

2. Schechtman-Drinfel’d-Hinich-Sullivan

2.1. Schechtman writes: The starting point of our studies with Volodya Hinich [3] on algebraic higher homotopy was the following example. To define the multiplication in his cohomology theory which he invented to construct higher regulators, Beilinson used some multiplication of complexes which was commutative up to a homotopy. One noticed that this multiplication was a part of a richer structure; namely Beilinsons complexes turned out to possess a structure of a module over a certain algebraic contractible operad (its terms were chain complexes of cubes).
Apparently Volodya Drinfel’d was also interested in these subjects. Drinfel’d wrote that he had some ideas on the subject and asked if I am interested in details. Here is the beginning of his letter dated 09/28/83 [1]:

“I am reading your papers with Hinich you have sent to me. Both of them are very well written. In connection with your little paper (On homotopy limit of homotopy algebras) [3] I have some questions...”

The known letter from Drinfel’d I received in September 1988 [2], just before my first trip to USA. Among other things, we can find the following there:

“A Lie-Sugawara DG-algebra is, by definition, a Z-graded space $g$ plus a degree 1 differential on the cofree cocommutative coalgebra generated by $g$ with the grading shifted below (the square of the differential is 0).”

This is what is called an $L_\infty$-algebra now; of course Drinfeld gives also a definition of a DG-Sugawara (co)commutative (now $C_\infty$) algebra. Concerning our paper with Hinich in the Gelfand seminar, we have discussed these subjects at IHES in the summer 90. Ginzburg and Kapranov were also there; they conceived their famous paper during this summer. At some point, it was recognized that the description of an $L_\infty$-algebra as a coderivation differential on a cofree connected graded symmetric coalgebra identified the $L_\infty$-algebra implicit in Sullivan’s models (not necessarily minimal).

2.2. Drinfel’d comments further:

At the time of my correspondence with Schechtmann I felt that I was trying to understand something known rather than inventing new things. Maybe my feeling was correct. I wouldn’t be surprised if everything from my correspondence with Schechtmann is already in Quillen’s article ”Rational Homotopy Theory” (maybe except the word ”operad”).

2.3. Vladimir Hinich recalls:

I visited Schechtman at Stony Brook in 1992 during my last PhD year at Weizmann Institute. If I remember correctly, he told me about Drinfel’d’s letter to him (written in 1988), and this was, I think, the only ”source” of our paper in Gel’fand volume.
2.4. **Sullivan recalls:**

I invited Schechtman to CUNY in the early nineties because I had heard he had come upon a notion of a global Lie algebra up to homotopy, motivated by work on deforming algebraic varieties where a sheaf up to homotopy of dgLie algebras controlled the deformation theory.

My motivation was trying to discretize the pde for fluid motion while preserving all known properties, energy conservation, helicity conservation, vorticity frozen in the fluid, etc. I had done this except that my discrete version of volume preserving vector fields had a bracket which satisfied Jacobi only up to homotopy. I wondered if there was an analogue in Lie algebra of what you had done for associative H spaces....

I was struck by lightning when Schechtman revealed that his Lie algebra up to homotopy was nothing but a differential-derivation on a free graded commutative algebra.

Then it was clear that the various forms of rational homotopy theory could be viewed as infinity versions of structures here and there: the dgc infinity coalgebra on chains computing homology was Quillen’s differential on the free Lie algebra, the dgLie infinity structure on a Moore complex computing homotopy was Quillen’s pre-dual coalgebra of the free dgc algebra models coming from forms.

To summarize: what was new for me was the familiar structures of rational homotopy theory were just infinity versions of appropriate structures on chain complexes...

**References**

[1] V.G. Drinfeld, *Letter to Schechtman*, 1983.

[2] ———, *Letter to Schechtman*, 1988.

[3] V. A. Khinich and V. V. Shekhtman, *The homotopy limit of homotopy algebras*, Uspekhi Mat. Nauk 41 (1986), no. 3(249), 205–206. MR 854263

[4] M. Schlessinger and J. Stasheff, *Deformation theory and rational homotopy type*, arXiv:1211.1647.

[5] M. Schlessinger and J. D. Stasheff, *The Lie algebra structure of tangent cohomology and deformation theory*, J. of Pure and Appl. Alg. 38 (1985), 313–322.

[6] J. Stasheff, *Rational homotopy theory – obstructions and deformations*, Proc. Conf. on Algebraic Topology, Vancouver, 1977, LNM 673, pp. 7–31.