TOPICS IN HIDDEN SYMMETRIES. II

DENIS V. JURIEV

"Thalassa Aitheria" Research Center for Mathematical Physics and Informatics,
ul. Miklukho-Maklaya 20-180, Moscow 117437, Russia.
E-mail: denis@juriev.msk.ru

E–version: q-alg/yymmxxx

Abstract. This short paper being devoted to some aspects of the inverse problem of the representation theory briefly treats the interrelations between the author’s approach to the setting free of hidden symmetries and the researches of D.P.Zhelobenko in the generalized Mickelsson algebras and related topics.

This short paper being a continuation of the first part [1] is a collection of examples illustrating the general ideology presented in the review [2]. The examples should emphasize the interrelations between the material of the previous papers (see e.g. [3] and [1:§2]) on the setting free of hidden symmetries and the researches of D.P.Zhelobenko on the generalized Mickelsson algebras [4,5] and related objects of the representation theory, so the material may be considered also as an illustrative commentary to the book [5].

1. Topic Four: The generalized Mickelsson $S$–algebra $S(\widehat{RW}(\mathfrak{sl}(2,\mathbb{C})),\mathfrak{sl}(2,\mathbb{C}))$ and the Zhelobenko $Z$–algebra $Z(\widehat{RW}(\mathfrak{sl}(2,\mathbb{C})),\mathfrak{sl}(2,\mathbb{C}))$ of the central extension $\widehat{RW}(\mathfrak{sl}(2,\mathbb{C}))$ of the Racah–Wigner algebra $RW(\mathfrak{sl}(2,\mathbb{C}))$ for the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$

This topic is devoted to the generalized Mickelsson $S$–algebra and the Zhelobenko $Z$–algebra of the central extension $\widehat{RW}(\mathfrak{sl}(2,\mathbb{C}))$ of the Racah–Wigner algebra $RW(\mathfrak{sl}(2,\mathbb{C}))$, which was introduced in [6:§2.2;3] (see also [2:§§1.2,1.3;7:§1.2]).

The algebra $\widehat{RW}(\mathfrak{sl}(2,\mathbb{C}))$ is generated by the nine elements $l_i$ ($i = -1, 0, 1$), $w_i$ ($i = -2, -1, 0, 1, 2$) and $\varrho$, the least is a central element. The commutation relations have the form

\[
[l_i, l_j] = (i - j)l_{i+j},
\]

\[
[l_i, w_j] = (2i - j)w_{i+j},
\]

\[
[w_{-1}, w_{-2}] = 2l_{-1} \circ w_{-2},
\]

\[
[w_0, w_{-2}] = \frac{4}{3}(2l_{-1} \circ w_{-1} + l_0 \circ w_{-2}),
\]

\[
[w_0, w_{-1}] = \frac{1}{6}(-l_0 \circ w_2 + 10l_0 \circ w_{-1} + 3l_{-1} \circ w_0 - 3\varrho l_{-1}),
\]
\[ [w_1, w_{-2}] = l_1 \circ w_{-2} + 2l_0 \circ w_{-1} + 3l_{-1} \circ w_0 + q l_{-1}, \]
\[ [w_1, w_{-1}] = \frac{1}{2}(l_1 \circ w_{-1} + 6l_0 \circ w_0 + l_{-1} \circ w_1 - q l_0), \]
\[ [w_2, w_{-2}] = 4(l_1 \circ w_{-1} + l_1 \circ w_1), \]
\[ [w_2, w_{-1}] = 2l_{-1} \circ w_2 + 2l_0 \circ w_1 + 3l_1 \circ w_0 + q l_1, \]
\[ [w_1, w_0] = \frac{1}{6}(-l_{-1} w_2 + 10l_0 \circ w_1 + 3l_1 \circ w_0 - 3q l_1), \]
\[ [w_2, w_0] = \frac{4}{3}(2l_1 \circ w_1 + l_0 \circ w_2), \]
\[ [w_2, w_1] = 2l_1 \circ w_2, \]

where \( X \circ Y = \frac{1}{2}(XY + YX) \). Note that one may use the right or the left ordering for the products of \( l_i \) and \( w_j \) in the r.h.s. instead of their Weyl ordering (it should be marked also that the formulas in [6:§2.2] contain misprints in two commutators, which were corrected in [3]). The hypothesis of [6:§2.2] suggests that the spectrum of the degenerate Verma modules of the finitely generated associative algebra \( \hat{\mathcal{WW}}(\mathfrak{sl}(2, \mathbb{C})) \) of polynomial growth is deeply related to the Kac spectrum for the Virasoro algebra \([8,9]\) and possibly may be used for the description of some models ("quasi-minimal") of two-dimensional field theory with broken infinite-dimensional Virasoro symmetries (cf.[10]).

**Remark 1.** The associative algebra \( \hat{\mathcal{WW}}(\mathfrak{sl}(2, \mathbb{C})) \) is neither contragredient nor even finitely generated locally triangular algebra in sense of D.P.Zhelobenko [5:App.A] (see also [11]).

The generators \( l_i \) form the Lie algebra \( \mathfrak{sl}(2, \mathbb{C}) \) so \( \hat{\mathcal{WW}}(\mathfrak{sl}(2, \mathbb{C})) \) is an enveloping algebra for \( \mathfrak{sl}(2, \mathbb{C}) \), moreover, it admits an embedding of \( \mathcal{U}(\mathfrak{sl}(2, \mathbb{C})) \). The pair \( (\hat{\mathcal{WW}}(\mathfrak{sl}(2, \mathbb{C})), \mathfrak{sl}(2, \mathbb{C})) \) obeys the conditions of D.P.Zhelobenko [4,5:§7.3], so one may construct the generalized Mickelson S–algebra \( S(\hat{\mathcal{WW}}(\mathfrak{sl}(2, \mathbb{C})), \mathfrak{sl}(2, \mathbb{C})) \) and the Zhelebofenko Z–algebra \( Z(\hat{\mathcal{WW}}(\mathfrak{sl}(2, \mathbb{C})), \mathfrak{sl}(2, \mathbb{C})) \) in lines of [4,5].

Let us denote the span of \( l_0 \) by \( \mathfrak{h} \), \( \mathcal{U}(\mathfrak{h}) \) and \( \mathcal{D}(\mathfrak{h}) \) are the universal enveloping algebra of \( \mathfrak{g} \) and its algebra of quotients [12], \( \hat{\mathcal{WW}}(\mathfrak{sl}(2, \mathbb{C})) = \hat{\mathcal{WW}}(\mathfrak{sl}(2, \mathbb{C})) \otimes \mathcal{U}(\mathfrak{h}) \) \( \mathcal{D}(\mathfrak{h}) \). Then, the generalized Mickelson S–algebra \( Z(\hat{\mathcal{WW}}(\mathfrak{sl}(2, \mathbb{C})), \mathfrak{sl}(2, \mathbb{C})) \) is the algebra over \( \mathcal{D}(\mathfrak{h}) \) (in sense of D.P.Zhelobenko) generated by the six generators \( v_i \) \((i = -2, -1, 0, 1, 2)\) and \( r \), the least is central one. The generators are the operators \( p w_i \) and \( pq \) in \( \hat{\mathcal{WW}}(\mathfrak{sl}(2, \mathbb{C})) \) reduced to the quotient \( \hat{\mathcal{WW}}(\mathfrak{sl}(2, \mathbb{C})) / I_+ \), where \( I_+ \) is the left ideal generated by \( l_1 \), and \( p \) is the extremal projector for \( \mathfrak{sl}(2, \mathbb{C}) \) [5:§3.2,3.3]. The extremal projector has the form:

\[
p = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(2l_0 - 2) \ldots (2l_0 - n - 1)} l_{-1}^n l_1^n, \quad \text{so that} \quad l_1 p = pl_{-1} = 0,
\]

(this formula for \( p \) slightly differs from one of D.P.Zhelobenko in view of the different choices of basises in \( \mathfrak{sl}(2, \mathbb{C}) \)).

**Theorem 1.** The Zhelebofenko Z–algebra \( Z(\hat{\mathcal{WW}}(\mathfrak{sl}(2, \mathbb{C})), \mathfrak{sl}(2, \mathbb{C})) \) is defined by the quadratic (over \( \mathcal{D}(\mathfrak{h}) \)) relations between the generators \( v_i \) \((i = -2, -1, 0, 1, 2)\) and \( r \) (the least is central one). The relations between \( v_i \) have the form:

\[
v_{-1} v_{-2} = (1 - \frac{2}{n-2}) v_{-2} v_{-1},
\]
\[v_0v_{-2} = (1 - \frac{6}{(\eta-2)(2\eta-3)})v_{-2}v_0 - \frac{4}{\eta-1}v_0^2 - \frac{4\eta}{3}v_{-2},\]
\[v_0v_{-1} = (1 - \frac{3}{(\eta-1)}v_{-1}v_0 + \frac{2(\eta-3)}{(\eta-2)(2\eta-3)}v_{-2}v_1 + \frac{5\eta-2}{3}v_{-1},\]
\[v_1v_{-2} = (1 - \frac{6}{(\eta-1)(\eta-1)}v_{-2}v_1 + \frac{12(\eta-2)}{(\eta-1)(\eta-1)}v_{-1}v_0 + \frac{2(\eta-1)(\eta-2)}{\eta}v_{-1},\]
\[v_1v_{-1} = (1 - \frac{9}{(\eta-1)(\eta-1)}v_{-1}v_1 + \frac{(\eta-2)(\eta-2)}{(\eta-1)(\eta-2)(2\eta-3)}v_{-2}v_2 - \frac{9}{2\eta} + \frac{3(2\eta+1)}{2}v_0 + \frac{\eta}{2}r,\]
\[v_2v_{-2} = (1 - \alpha)v_{-2}v_2 - \frac{2(\eta-3)(\eta+2)}{(\eta-1)(\eta+1)}v_{-1}v_1 + \frac{36}{(\eta-1)(\eta+1)}v_0^2 - \frac{12\eta}{\eta+1}v_0 + \frac{\eta(\eta-3)}{\eta+1}r,\]
\[v_2v_{-1} = (1 - \frac{6}{(\eta+1)}v_{-1}v_2 + \frac{12(\eta-1)}{(\eta+1)(\eta+1)}v_0v_1 + \frac{2\eta-1}{\eta+1}v_1,\]
\[v_1v_0 = (1 - \frac{3}{\eta})v_0v_0 + \frac{2(\eta-2)}{(\eta-1)(\eta-1)}v_{-1}v_2 + \frac{5\eta-3}{3}v_1,\]
\[v_2v_0 = (1 - \frac{6}{\eta(\eta+1)})v_0v_2 - \frac{4}{\eta+1}v_0^2 + \frac{4(\eta+2)}{3}v_2,\]
\[v_2v_1 = (1 - \frac{2}{\eta+1})v_1v_2,\]

where \(\eta = l_0 \in \mathcal{D}(h)\) (note that \([\eta, v_i] = -iv_i\)) and \(\alpha = \frac{4}{(\eta+1)(\eta-2)}(1 - \frac{9}{2(\eta-1)(2\eta-3)}).\)

The proof is just in lines of the proof of Theorem 4.2.4 in [5:§4.2] with slight but evident modifications. The concrete computations are made in the standard way (see [5:§4.2]) here.

**Remark 2.** The described construction of the generalized Mickelsson S–algebra \(S(\mathcal{R}\mathcal{W}(\mathfrak{sl}(2,\mathbb{C})), \mathfrak{sl}(2,\mathbb{C}))\) and the Zhelobenko Z–algebra \(Z(\mathcal{R}\mathcal{W}(\mathfrak{sl}(2,\mathbb{C})), \mathfrak{sl}(2,\mathbb{C}))\) admits a superanalogue. The role of the Racah–Wigner algebras \(\mathcal{R}\mathcal{W}(\mathfrak{sl}(2,\mathbb{C}))\) for the Lie algebra \(\mathfrak{sl}(2,\mathbb{C})\) is played by the Racah–Wigner algebra \(\mathcal{R}\mathcal{W}(\mathfrak{sl}(2\vert 1,\mathbb{C}))\) for the Lie superalgebra \(\mathfrak{sl}(2\vert 1,\mathbb{C})\) [6:§2.3].

### 2. Topic Five: The Generalized Mickelsson S–algebras \(S(\mathcal{U}(\mathfrak{g}, \pi), \mathfrak{g})\) and the Zhelobenko Z–algebras \(Z(\mathcal{U}(\mathfrak{g}, \pi), \mathfrak{g})\) of the Mho–algebras \(\mathcal{U}(\mathfrak{g}, \pi)\) for the Complex Semisimple Lie Algebras \(\mathfrak{g}\)

This topic is devoted to the generalized Mickelsson S–algebras and the Zhelobenko Z–algebras of the mho–algebras \(\mathcal{U}(\mathfrak{g}, \pi)\) for the complex semisimple Lie algebras \(\mathfrak{g}\) and their finite–dimensional representations, which were introduced and discussed in [1:§2] (see also [2:§1.4]).

**Definition.**

**A.** Let \(\mathfrak{g}\) be a Lie algebra and \(\pi\) be its (irreducible) representation. \(\text{Mho–algebra } \mathcal{U}(\mathfrak{g}, \pi)\) is an associative algebra such that (1) \(\mathcal{U}(\mathfrak{g})\) is a subalgebra of \(\mathcal{U}(\mathfrak{g}, \pi)\) and, hence, \(\mathfrak{g}\) naturally acts in \(\mathcal{U}(\mathfrak{g}, \pi)\), (2) there is defined a \(\mathfrak{g}\)-equivariant embedding of \(\pi\) into \(\mathcal{U}(\mathfrak{g}, \pi)\), so \(\pi\) may be considered as a subspace of \(\mathcal{U}(\mathfrak{g}, \pi)\), (3) the \(\mathfrak{g}\)-equivariant embedding of \(\pi\) into \(\mathcal{U}(\mathfrak{g}, \pi)\) is extended to a \(\mathfrak{g}\)-equivariant embedding of \(S'(\pi)\) into \(\mathcal{U}(\mathfrak{g}, \pi)\), defined by the Weyl symmetrization, and, therefore, \(S'(\pi)\) may be considered as a subspace of \(\mathcal{U}(\mathfrak{g}, \pi)\); (4) \(\mathfrak{g}\)-modules \(\mathcal{U}(\mathfrak{g}, \pi)\) and \(S'(\mathfrak{g})\otimes S'(\pi)\) are isomorphic, here the isomorphism of subalgebra \(\mathcal{U}(\mathfrak{g})\) of the algebra \(\mathcal{U}(\mathfrak{g}, \pi)\) and \(S'(\mathfrak{g})\) as \(\mathfrak{g}\)-modules is used; (5) in an arbitrary basis \(w_i\) in \(\pi\) the commutator of two elements of the basis in the algebra \(\mathcal{U}(\mathfrak{g}, \pi)\) may be represented in the form \([w_i, w_j] = f^k_{ij}w_k\), where the ”noncommutative structural functions” \(f^k_{ij}\) are the elements of the algebra \(\mathcal{U}(\mathfrak{g})\).

**B.** Let \(\mathfrak{g}\) be a Lie algebra and \(\pi\) be its (irreducible) representation. \(\text{Affine mho–algebra } \mathcal{U}(\mathfrak{g}, \pi)\) is an associative algebra such that the conditions (1)–(4) above hold.
and (5') in an arbitrary basis $w_i$ in $\pi$ the commutator of two elements of the basis in the algebra $\hat{\mathcal{O}}(g, \pi)$ may be represented in the form $[w_i, w_j] = f_{ij}^k w_k + g_{ij}$, where the "noncommutative structural functions" $f_{ij}^k$ and $g_{ij}$ are the elements of the algebra $\mathcal{U}(g)$.

Notations $\mathcal{O}(g, \pi)$ and $\hat{\mathcal{O}}(g, \pi)$ emphasize an analogy between mho–algebras and affine mho–algebras and the universal enveloping algebras. The examples of the mho–algebra for $g = sl(2, \mathbb{C})$ were considered in [1; §2] (see also [2; §1.4]). For instance, the natural semi–direct product $\mathcal{U}(sl(2, \mathbb{C})) \times \mathcal{U}(sl(2, \mathbb{C}))$ and the Racah–Wigner algebra $\mathcal{RW}(sl(2, \mathbb{C}))$ are mho–algebras, whereas $\mathcal{U}(sl(2, \mathbb{C}) \oplus sl(2, \mathbb{C}))$ with the diagonal embedding of $\mathcal{U}(sl(2, \mathbb{C}))$ and $\mathcal{RW}(sl(2, \mathbb{C}))$ are affine mho–algebras. The mho-algebra $\mathcal{O}(sl(2, \mathbb{C}), \pi_3)$ ($\pi_3$ is the seven–dimensional representation of $sl(2, \mathbb{C})$) was considered in [1; §2] (see also [2; §1.4]).

Theorem 2. Let $g$ be a complex semisimple Lie algebra, and $\mathcal{O}(g, \pi)$ be an arbitrary mho-algebra over $g$, then the pair $(\mathcal{O}(g, \pi), g)$ obeys the conditions of D.P.Zhelobenko [4;5;§7.3], the generalized Mickelsson $S$–algebra $S(\mathcal{O}(g, \pi), g)$, the Zhelobenko $Z$–algebra $Z(\mathcal{O}(g, \pi), g)$ may be constructed as in [4,5], the least is generated by the elements $px, x \in \pi, p$ is the extremal projector for $g$, and it is a quadratic algebra over $D(h)$ in sense of D.P.Zhelobenko ($h$ is the Cartan subalgebra of $g$).

This statement holds true for affine mho–algebras $\hat{\mathcal{O}}(g, \pi)$ also.

The simplest way to restore the proof of this theorem for the reader is to calculate explicitely the quadratic relations in the Zhelobenko $Z$–algebra $Z(\mathcal{O}(sl(2, \mathbb{C}), \pi_3), sl(2, \mathbb{C}))$ (the commutation relations in $\mathcal{O}(sl(2, \mathbb{C}), \pi_3)$ were written by the author in [1; §2], see also [2; §1.4]).

Remark 3. One may consider the reductive Lie algebra $g$ instead of the semisimple one. In this case the mentioned above conditions of D.P.Zhelobenko should be checked to mantain the statement of the theorem 2.

Remark 4. One may also consider the generalized Mickelsson $S$–algebras $S(\mathcal{O}(g, \pi), f)$ and the Zhelobenko $Z$–algebras $Z(\mathcal{O}(g, \pi), f)$, where $f$ is a reductive algebra reductively embed into $g$ such that $Res_f^g(\pi)$ is completely reducible (Res denotes the restriction functor here).
References

[1] Juriev D., Topics in hidden symmetries. I.: E-print [hep-th/9405050] (1994).
[2] Juriev D., An excursion into the inverse problem of representation theory [in Russian]: Report RCMPI-95/04 (August 1995) [e-version: mp_arc/96-477 (1996)].
[3] Juriev D., Setting hidden symmetries free by the noncommutative Veronese mapping. J.Math.Phys. 35(9) (1994) 5021-5024.
[4] Zhelobenko D.P., Extremal projectors and generalized Mickelsson algebras over reductive Lie algebras [in Russian]. Izvestiya AN SSSR. Ser.matem. 52(4) (1988) 758-773.
[5] Zhelobenko D.P., Representations of the reductive algebras. Moscow, Nauka, 1994.
[6] Juriev D., Complex projective geometry and quantum projective field theory [in Russian]. Teor.Matem.Fiz. 101(3) (1994) 331-348 [English transl.: Theor.Math.Phys. 101 (1994) 1387-1403].
[7] Juriev D., Infinite dimensional geometry and quantum field theory of strings. III. Infinite dimensional W-geometry of a second quantized free string. J.Geom.Phys. 16 (1995) 275-300 [e-version: hep-th/9401026].
[8] Kac V.G., Infinite dimensional Lie algebras. Cambridge, Cambridge Univ. Press, 1990.
[9] Feigin B.L., Fuchs D.B., Representations of the Virasoro algebra. In "Representations of infinite dimensional Lie algebras". Gordon and Breach, 1991.
[10] Fradkin E.S., Palchik M.Ya., Exactly solvable models of conformal–invariant quantum field theory in D-dimensional space. J.Geom.Phys. 5(4) (1988) 601-629 [reprinted in "Geometry and physics. Essays in honor of I.M.Gelfand", Eds.S.Gindikin and I.M.Singer, Bologna, Pitagora Editrice and Amsterdam, Elsevier Sci. Publ., 1991].
[11] Zhelobenko D.P., Contragredient algebras. J.Group Theory Phys. 1(1) (1993) 201-233.
[12] Dixmier J., Algèbres enveloppantes. Paris, Villars, 1973.