An Ore-type condition for hamiltonicity in tough graphs and the extremal examples

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November 1, 2022

Abstract. Let $G$ be a $t$-tough graph on $n \geq 3$ vertices for some $t > 0$. It was shown by Bauer et al. in 1995 that if the minimum degree of $G$ is greater than $\frac{2n}{t+1} - 1$, then $G$ is hamiltonian. In terms of Ore-type hamiltonicity conditions, the problem was only studied when $t$ is between 1 and 2, and recently the author proved a general result. The result states that if the degree sum of any two nonadjacent vertices of $G$ is greater than $2n\frac{t}{t+1} + t - 2$, then $G$ is hamiltonian. It was conjectured in the same paper that the “$+t$” in the bound $2n\frac{t}{t+1} + t - 2$ can be removed. Here we confirm the conjecture. The result generalizes the result by Bauer et al.. Furthermore, we characterize all $t$-tough graphs $G$ on $n \geq 3$ vertices for which $\sigma_2(G) = 2\frac{n}{t+1} - 2$ but $G$ is non-hamiltonian.

Keywords. Ore-type condition; toughness; hamiltonian cycle.

1 Introduction

We consider only simple graphs. Let $G$ be a graph. Denote by $V(G)$ and $E(G)$ the vertex set and edge set of $G$, respectively. Let $v \in V(G)$, $S \subseteq V(G)$, and $H \subseteq G$. Then $N_G(v)$ denotes the set of neighbors of $v$ in $G$, $d_G(v) := |N_G(v)|$ is the degree of $v$ in $G$, and $\delta(G) := \min \{d_G(v) : v \in V(G)\}$ is the minimum degree of $G$. Define $\text{deg}_G(v, H) = |N_G(v) \cap V(H)|$, $N_G(S) = (\bigcup_{x \in S} N_G(x)) \setminus S$, and we write $N_G(H)$ for $N_G(V(H))$. Let $N_H(v) = N_G(v) \cap V(H)$ and $N_H(S) = N_G(S) \cap V(H)$. Again, we write $N_H(R)$ for $N_H(V(R))$ for any subgraph $R$ of $G$. We use $G[S]$ and $G - S$ to denote the subgraphs of $G$ induced by $S$ and $V(G) \setminus S$, respectively. For notational simplicity we write $G - x$ for $G - \{x\}$. Let $V_1, V_2 \subseteq V(G)$ be two disjoint vertex sets. Then $E_G(V_1, V_2)$ is the set of edges in $G$ with one endvertex in $V_1$ and the other endvertex in $V_2$. For two integers $a$ and $b$, let $[a, b] = \{i \in \mathbb{Z} : a \leq i \leq b\}$.

∗Partially supported by NSF grant DMS-2153938.
Throughout this paper, if not specified, we will assume $t$ to be a nonnegative real number. The number of components of a graph $G$ is denoted by $c(G)$. The graph $G$ is said to be $t$-tough if $|S| \geq t \cdot c(G - S)$ for each $S \subseteq V(G)$ with $c(G - S) \geq 2$. The toughness $\tau(G)$ is the largest real number $t$ for which $G$ is $t$-tough, or is $\infty$ if $G$ is complete. This concept was introduced by Chvátal [7] in 1973. It is easy to see that if $G$ has a hamiltonian cycle then $G$ is 1-tough. Conversely, Chvátal [7] conjectured that there exists a constant $t_0$ such that every $t_0$-tough graph is hamiltonian. Bauer, Broersma and Veldman [1] have constructed $t$-tough graphs that are not hamiltonian for all $t < \frac{9}{4}$, so $t_0$ must be at least $\frac{9}{4}$ if Chvátal’s toughness conjecture is true.

Chvátal’s toughness conjecture has been verified for certain classes of graphs including planar graphs, claw-free graphs, co-comparability graphs, and chordal graphs [2]. The classes also include 2$K_2$-free graphs [6, 14, 12], ($P_2 \cup P_3$)-free graphs [15], and $R$-free graphs for $R \in \{P_2 \cup P_3, P_3 \cup 2P_1, P_2 \cup kP_1\}$ [9, 15, 16], where $k \geq 4$ is an integer. In general, the conjecture is still wide open. In finding hamiltonian cycles in graphs, sufficient conditions such as Dirac-type and Ore-type conditions are the most classic ones.

**Theorem 1.1** (Dirac’s Theorem [8]). If $G$ is a graph on $n \geq 3$ vertices with $\delta(G) \geq \frac{n}{2}$, then $G$ is hamiltonian.

Define $\sigma_2(G) = \min\{d_G(u) + d_G(v) : u, v \in V(G) \text{ and they are nonadjacent}\}$ if $G$ is noncomplete, and define $\sigma_2(G) = \infty$ otherwise. Ore’s Theorem, as a generalization of Dirac’s Theorem, is stated below.

**Theorem 1.2** (Ore’s Theorem [11]). If $G$ is a graph on $n \geq 3$ vertices with $\sigma_2(G) \geq n$, then $G$ is hamiltonian.

Analogous to Dirac’s Theorem, Bauer, Broersma, van den Heuvel, and Veldman [4] proved the following result by incorporating the toughness of the graph.

**Theorem 1.3** (Bauer et al. [4]). Let $G$ be a $t$-tough graph on $n \geq 3$ vertices. If $\delta(G) > \frac{n}{t+1} - 1$, then $G$ is hamiltonian.

A natural question here is whether we can find an Ore-type condition involving the toughness of $G$ that generalizes Theorem 1.3. Various theorems were proved prior to Theorem 1.3 by only taking $\tau(G)$ between 1 and 2 [10, 3, 5]. Let $G$ be a $t$-tough graph on $n \geq 3$ vertices. The author in [13] showed that if $\sigma_2(G) > \frac{2n}{t+1} + t - 2$, then $G$ is hamiltonian. It was also conjectured in [13] that $\sigma_2(G) > \frac{2n}{t+1} - 2$ is the right bound. In this paper, we confirm the conjecture. For any odd integer $n \geq 3$, the complete bipartite graph $G := K_{\frac{n-1}{2}, \frac{n-1}{2}}$ is $\frac{n-1}{n+1}$-tough and satisfies $\sigma_2(G) = n - 1 = \frac{2n}{1+\frac{2}{n+1}} - 2$. However, $G$ is not hamiltonian. Thus, the degree sum condition that $\sigma_2(G) > \frac{2n}{t+1} - 2$ is best possible for a $t$-tough graph on at least three vertices to be hamiltonian. In fact, for any odd integers $n \geq 3$, any graph from the family $\mathcal{H} = \{H_{\frac{n-1}{2}} + \overline{K}_{\frac{n+1}{2}} : H_{\frac{n-1}{2}} \text{ is any graph on } \frac{n-1}{2} \text{ vertices}\}$ is an extremal graph, where “+” represents the join of two graphs. We also show that $\mathcal{H}$ is the only family of extremal graphs.
**Theorem 1.** Let $G$ be a $t$-tough graph on $n \geq 3$ vertices. Then the following statements hold.

(a) If $\sigma_2(G) > \frac{2n}{t+1} - 2$, then $G$ is hamiltonian.

(b) If $\sigma_2(G) = \frac{2n}{t+1} - 2$ and $G$ is not hamiltonian, then $G \in \mathcal{H}$.

The remainder of this paper is organized as follows: in Section 2, we introduce some notation and preliminary results, and in Section 3, we prove Theorem 1.

## 2 Preliminary results

Let $G$ be a graph and $\lambda$ be a positive integer. Following [17], a cycle $C$ of $G$ is a $D_\lambda$-cycle if every component of $G - V(C)$ has order less than $\lambda$. Clearly, a $D_1$-cycle is just a hamiltonian cycle. We denote by $c_\lambda(G)$ the number of components of $G$ with order at least $\lambda$, and write $c_1(G)$ just as $c(G)$. Two subgraphs $H_1$ and $H_2$ of $G$ are remote if they are disjoint and there is no edge of $G$ joining a vertex of $H_1$ with a vertex of $H_2$. For a subgraph $H$ of $G$, let $d_G(H) = |N_G(H)|$ be the degree of $H$ in $G$. We denote by $\delta_\lambda(G)$ the minimum degree of a connected subgraph of order $\lambda$ in $G$. Again $\delta_1(G)$ is just $\delta(G)$.

**Lemma 1 ([15]).** Let $t > 0$ and $G$ be a non-complete $n$-vertex $t$-tough graph. Then $|W| \leq \frac{n}{t+1}$ for every independent set $W$ in $G$.

The following lemma provides a way of extending a cycle $C$ provided that the vertices outside $C$ have many neighbors on $C$. The proof follows from Lemma 1 and is very similar to the proof of Lemma 10 in [15].

**Lemma 2.** Let $t > 0$ and $G$ be an $n$-vertex $t$-tough graph, and let $C$ be a non-hamiltonian cycle of $G$. If $x \in V(G) \setminus V(C)$ satisfies $d_G(x, C) > \frac{n}{t+1} - 1$, then $G$ has a cycle $C'$ such that $V(C') = V(C) \cup \{x\}$.

Let $C$ be an oriented cycle. We assume that the orientation is clockwise throughout the rest of this paper. For $x \in V(C)$, denote the immediate successor of $x$ on $C$ by $x^+$ and the immediate predecessor of $x$ on $C$ by $x^-$. For $u, v \in V(C)$, $u \overset{C}{\rightarrow} v$ denotes the segment of $C$ starting at $u$, following $C$ in the orientation, and ending at $v$. Likewise, $u \overset{C}{\leftarrow} v$ is the opposite segment of $C$ with endpoints as $u$ and $v$. Let dist$_C(u, v)$ denote the length of the path $u \overset{C}{\rightarrow} v$. For any vertex $u \in V(C)$ and any positive integer $k$, define

$$L^+_u(k) = \{v \in V(C) : \text{dist}_C(u, v) \in [1, k]\}$$

to be the set of $k$ consecutive successors of $u$. Hereafter, all cycles under consideration are oriented.
A path $P$ connecting two vertices $u$ and $v$ is called a $(u, v)$-path, and we write $uPv$ or $vPu$ in order to specify the two endvertices of $P$. Let $uPv$ and $xQy$ be two paths. If $uv$ is an edge, we write $uPuvQy$ as the concatenation of $P$ and $Q$ through the edge $uv$.

For an integer $\lambda \geq 1$, if a graph $G$ contains a $D_{\lambda+1}$-cycle $C$ but no $D_{\lambda}$-cycle, then $V(G) \setminus V(C) \neq \emptyset$. Furthermore, $G - V(C)$ has a component of order $\lambda$. The result below with $d_G(H)$ replaced by $\delta(G)$ and $H$ replaced by any component of $G - V(C)$ with order $\lambda$ was proved in [4, Corollary 7(a)].

**Lemma 3** ([13]). Let $G$ be a t-tough 2-connected graph of order $n$. Suppose $G$ has a $D_{s+1}$-cycle but no $D_s$-cycle for some integer $s \geq 1$. Let $C$ be a $D_{s+1}$-cycle of $G$ such that $C$ minimizes $c_p(G - V(C))$ prior to minimizing $c_q(G - V(C))$ for any $p, q \in \{1, s\}$ with $p > q$. Then $n \geq (t + |V(H)|)(d_G(H) + 1)$ for any component $H$ of $G - V(C)$.

The lemma below is the key to get rid of the “+t” in the lower bound $\frac{2n}{t+1} + t - 2$ on $\sigma_2(G)$ as proved in Theorem 4 from [13].

**Lemma 4.** Let $G$ be a t-tough 2-connected graph of order $n$. Suppose that $G$ has a $D_{\lambda+1}$-cycle but no $D_{\lambda}$-cycle for some integer $\lambda \geq 1$. Let $C$ be a cycle of $G$. Suppose each component of $G - V(C)$ either has order at most $\lambda - 1$ or is a path of order at least $\lambda$. Then $G - V(C)$ has a path-component $H$ with order at least $\lambda$ such that $\deg_G(x,C) \leq \frac{n}{t+1} - \lambda$ for some $x \in V(H)$.

**Proof.** Since $G$ has no $D_{\lambda}$-cycle, it is clear that $G - V(C)$ has a path-component of order at least $\lambda$. We suppose to the contrary that for each path-component $P$ with order at least $\lambda$ of $G - V(C)$ and each $x \in V(P)$, we have $\deg_G(x,C) > \frac{n}{t+1} - \lambda$. Among all cycles $C'$ of $G$ that satisfies the two conditions below, we may assume that $C$ is one that minimizes $c_p(G - V(C))$ prior to minimizing $c_q(G - V(C))$ for any $p \geq \lambda$ and any $q$ with $q < p$.

1. each component of $G - V(C)$ either has order at most $\lambda - 1$, or
2. the component is a path $P$ of order at least $\lambda$ such that for each $x \in V(P)$, we have $\deg_G(x,C) > \frac{n}{t+1} - \lambda$.

We take a path-component $P$ with order at least $\lambda$ and assume that $N_C(P)$ has size $k$ for some integer $k \geq 2$, and that the $k$ neighbors are $v_1, \ldots, v_k$ and appear in the same order along $C$. Note that $k > \frac{n}{t+1} - \lambda$ by our assumption. For each $i \in [1, k]$, and each $v \in V(v_i^+\overrightarrow{C}v_{i+1}^-)$, where $v_{k+1} := v_1$, we let $\mathcal{C}(v)$ be the set of components of $G - V(C)$ that have a vertex joining to $v$ by an edge in $G$. As $N_C(P) \cap V(v_i^+\overrightarrow{C}v_{i+1}^-) = \emptyset$, we have $P \notin \mathcal{C}(v)$. Let $w_i^* \in V(v_i^+\overrightarrow{C}w_{i+1}^-)$ be the vertex with $\dist_{\overrightarrow{C}}(v_i, w_i^*)$ minimum such that

$$
\sum_{D \in \bigcup_{v \in V(v_i^+\overrightarrow{C}w_{i+1}^-)} \mathcal{C}(v)} |V(D)| + |V(v_i^+\overrightarrow{C}w_i^*)| \geq \lambda.
$$
If such a vertex \( w^*_i \) exists, let \( L^*_v(\lambda) \) be the union of the vertex set \( V(v^+_i \mathrel{\overset{\sim}{\rightarrow}} C w^*_i) \) and all those vertex sets of graphs in \( \bigcup_{v \in V(v^+_i \mathrel{\overset{\sim}{\rightarrow}} C w^*_i)} \mathcal{C}(v) \); if such a vertex \( w^*_i \) does not exist, let \( L^*_v(\lambda) = L^+_v(\lambda) \). Note that when \( w^*_i \) exists, by its definition, \( w^*_i \in V(v^+_i \mathrel{\overset{\sim}{\rightarrow}} C v^*_{i+1}) \). Thus \( V(v^+_i \mathrel{\overset{\sim}{\rightarrow}} C w^*_i) \cap V(v^+_i \mathrel{\overset{\sim}{\rightarrow}} C w^*_j) = \emptyset \) if both \( w^*_i \) and \( w^*_j \) exist for distinct \( i, j \in [1, k] \).

We will show that we can make the following assumptions:

(a) If for some \( i \in [1, k] \), it holds that \( L^*_v(\lambda) = L^+_v(\lambda) \), then dist\(_C^-(v_i, v_j) \geq \lambda + 1 \) for any \( j \in [1, k] \) with \( j \neq i \); and

(b) For distinct \( i, j \in [1, k] \), \( v \in L^*_v(\lambda) \cap V(v^+_i \mathrel{\overset{\sim}{\rightarrow}} C v^*_{i+1}) \) and \( u \in L^*_v(\lambda) \cap V(v^+_j \mathrel{\overset{\sim}{\rightarrow}} C v^*_{j+1}) \), we have \( \mathcal{C}(v) \cap \mathcal{C}(u) = \emptyset \).

(c) \( L^*_v(\lambda) \) and \( L^*_v(\lambda) \) are pairwise remote for any distinct \( i, j \in [1, k] \).

With Assumptions (a), (b) and (c), we can reach a contradiction as follows: note that \( L^*_v(\lambda) \) and \( L^*_v(\lambda) \) are remote for any distinct \( i, j \in [1, k] \) and \( P \) and \( L^*_v(\lambda) \) are remote for any \( i \in [1, k] \). Let \( S = V(G) \setminus \left( \left( \bigcup_{i=1}^k L^*_v(\lambda) \right) \cup V(P) \right) \). Then \( |S| \leq n - (k + \lambda) \) and \( c(G - S) = k + 1 \). As \( G \) is \( t \)-tough, we get

\[
n - (k + 1) \lambda \geq |S| \geq t \cdot c(G - S) = t(k + 1),
\]
giving \( k \leq \frac{n}{t+\lambda} - 1 \). Since \( n \geq \lambda + (t + 1) \) by Lemma 3 \((G \) has a \( D_{\lambda+1} \)-cycle \( C' \) such that \( G - V(C') \) has a component \( H \) of order \( \lambda \) and \( d_C(H) \geq 2t \) by \( G \) being \( t \)-tough), we get \( k \leq \frac{n}{t+\lambda} - 1 \leq \frac{n}{t+1} - \lambda \). This gives a contradiction to \( k > \frac{n}{t+1} - \lambda \). Thus we are only left to show Assumptions (a), (b) and (c). We show that if any one of the assumptions is violated, then we can decrease \( c_p(G - V(C)) \) for some \( p \geq \lambda \).

For Assumption (a), if \( L^*_v(\lambda) = L^+_v(\lambda) \) for some \( i \in [1, k] \) but dist\(_C^-(v_i, v_j) \leq \lambda \) for some \( v_j \in NC(P) \) with \( j \neq i \), then there must exist two consecutive indices \( i, j \in [1, k] \) such that dist\(_C^-(v_i, v_j) \leq \lambda \). Thus we may just assume \( j = i+1 \), where the index is taken modulo \( k \). Let \( v^*_i, v^*_i \in V(P) \) such that \( v_i v^*_i, v_{i+1} v^*_i \in E(G) \). Let \( C_1 = v_i \mathrel{\overset{\sim}{\rightarrow}} C v^*_i \mathrel{\overset{\sim}{\rightarrow}} C v_{i+1} \mathrel{\overset{\sim}{\rightarrow}} C v^*_i \mathrel{\overset{\sim}{\rightarrow}} C v_i \). Note that the component of \( G - V(C_1) \) containing \( v^*_i \mathrel{\overset{\sim}{\rightarrow}} C v^*_{i+1} \) has order at most \( \lambda - 1 \) by the assumption that \( L^*_v(\lambda) = L^+_v(\lambda) \) and dist\(_C(v_i, v_{i+1}) \leq \lambda \). Thus any vertex from a component of \( G - V(C) \) with order at least \( \lambda \) is not adjacent in \( G \) to any vertex from \( v^*_i \mathrel{\overset{\sim}{\rightarrow}} C v^*_{i+1} \). Furthermore, any vertex from each component of \( P - V(v^*_i \mathrel{\overset{\sim}{\rightarrow}} C v^*_{i+1}) \) is not adjacent in \( G \) to any vertex from \( v^*_i \mathrel{\overset{\sim}{\rightarrow}} C v^*_{i+1} \). Hence each component of \( G - V(C_1) \) either has order at most \( \lambda - 1 \) or is a path-component of order at least \( \lambda \) such that each vertex from the component has in \( G \) more than \( \frac{n}{t+1} - \lambda \) neighbors on \( C_1 \). Furthermore, for any \( w \in V(P - V(v^*_i \mathrel{\overset{\sim}{\rightarrow}} C v^*_{i+1})) \), \( \deg_G(w, C_1) > \frac{n}{t+1} - \lambda \). However, \( c_{P}(G - V(C_1)) < c_{P}(G - V(C)) \) and \( c_{q}(G - V(C_1)) = c_{q}(G - V(C)) \) for any \( q > |V(P)| \), contradicting the choice of \( C \). Therefore we have Assumption (a).
For Assumption (b), if for some distinct \(i, j \in [1, k]\), \(v \in L_{v_i}(\lambda) \cap V(v_j^+Cv_{i+1})\) and \(u \in L_{v_j}(\lambda) \cap V(v_j^+Cv_{j+1})\), we have \(C(v) \cap C(u) \neq \emptyset\), we then further choose \(v\) closest to \(v_i\) and \(u\) closest to \(v_j\) along \(\tilde{C}\) with the property. Thus for any \(w_i \in V(v_i^+Cv^-)\) and any \(w_j \in V(v_j^+Cu^-)\), it holds that \(C(w_i) \cap C(w_j) = \emptyset\). Let \(D \in C(v) \cap C(u)\) and \(v', u' \in V(D)\) such that \(v', u' \in E(G)\), and \(P'\) be a \((v', u')\)-path of \(D\). Let \(v_i^*, v_j^* \in V(P)\) such that \(v_iv_i^*, v_jv_j^* \in E(G)\). Then \(C_1 = v_iv_i^*Pu_j^*v_jPu'v'\) is a cycle. Since each of \(V(v_i^+Cv^-)\) and \(V(v_j^+Cu^-)\) contain at most \(\lambda - 1\) vertices and they are proper subsets of \(L_{v_i}(\lambda) \cap V(v_i^+Cv_{i+1})\) and \(L_{v_j}(\lambda) \cap V(v_j^+Cu_{j+1})\) respectively, by Assumption (a) above, we have \(N_C(P) \cap (V(v_i^+Cv^-) \cup V(v_j^+Cu^-)) = \emptyset\). By the choices of \(v\) and \(u\), the components of \(G - V(C_1)\) that respectively contain \(v_i^+Cv^-\) and \(v_j^+Cu^-\) are disjoint. Since \(V(v_i^+Cv^-)\) is a proper subset of \(L_{v_i}(\lambda) \cap V(v_i^+Cv_{i+1})\) and \(V(v_j^+Cu^-)\) is a proper subset of \(L_{v_j}(\lambda) \cap V(v_j^+Cu_{j+1})\), it follows by the definitions of \(L_{v_i}\) and \(L_{v_j}\) that the components of \(G - V(C_1)\) that respectively contain \(v_i^+Cv^-\) and \(v_j^+Cu^-\) have order at most \(\lambda - 1\). By the same reasoning as in proving Assumption (a), we know that each component of \(G - V(C_1)\) has order at most \(\lambda - 1\) or is a path-component such that each vertex from the component has in \(G\) more than \(\frac{\lambda}{\lambda + 1} - \lambda\) neighbors on \(C_1\). However, \(c_{V(P)}(G - V(C_1)) < c_{V(P)}(G - V(C))\) and \(c_{q}(G - V(C_1)) = c_{q}(G - V(C))\) for any \(q \geq |V(P)|\), contradicting the choice of \(C\). Thus we have Assumption (b).

For Assumption (c), assume to the contrary that \(E_G(L_{v_i}(\lambda), L_{v_j}(\lambda)) \neq \emptyset\) for some distinct \(i, j \in [1, k]\). Applying Assumption (b), we know that \(L_{v_i}(\lambda) \cap L_{v_j}(\lambda) = \emptyset\). Since there is no edge between any two components of \(G - V(C)\), \(E_G(L_{v_i}(\lambda), L_{v_j}(\lambda)) \neq \emptyset\) implies that there exist \(y \in L_{v_i}(\lambda) \cap V(v_i^+Cv_{i+1})\) and \(z \in L_{v_j}(\lambda) \cap V(v_j^+Cu_{j+1})\) such that \(yz \in E(G)\). We choose \(y \in L_{v_i}(\lambda) \cap V(v_i^+Cv_{i+1})\) with dist\(_G\)(\(v_i, y\)) minimum and \(z \in L_{v_j}(\lambda) \cap V(v_j^+Cu_{j+1})\) with dist\(_G\)(\(v_j, z\)) minimum such that \(yz \in E(G)\). By this choice of \(y\) and \(z\), it follows that \(E_G(V(y_i^+Cy^-), V(y_j^+Cz^-)) = \emptyset\). Let \(v_i^*, v_j^* \in V(P)\) such that \(v_i^*v_i^*, v_j^*v_j^* \in E(G)\), and let \(C_1 = v_i^*Cyzv_j^*Pu_j^*v_j^*\). Note that no vertex of \(P\) is adjacent in \(G\) to any vertex of \(v_i^+Cy^-\) or \(v_j^+Cz^-\) by the fact that \(V(v_i^+Cy^-) \subseteq L_{v_i}(\lambda) \cap V(v_i^+Cv_{i+1})\) and \(V(v_j^+Cz^-) \subseteq L_{v_j}(\lambda) \cap V(v_j^+Cu_{j+1})\) and Assumption (a). By Assumption (b) and the definitions of \(L_{v_i}(\lambda)\) and \(L_{v_j}(\lambda)\), we know that \(v_i^+Cy^-\) and \(v_j^+Cz^-\) are respectively contained in distinct components of \(G - V(C_1)\) that each of order at most \(\lambda - 1\). By the same reasoning as in proving Assumption (a), we know that each component of \(G - V(C_1)\) has order at most \(\lambda - 1\) or is a path-component such that each vertex from the component has in \(G\) more than \(\frac{\lambda}{\lambda + 1} - \lambda\) neighbors on \(C_1\). However, \(c_{V(P)}(G - V(C_1)) < c_{V(P)}(G - V(C))\) and \(c_{q}(G - V(C_1)) = c_{q}(G - V(C))\) for any \(q \geq |V(P)|\), contradicting the choice of \(C\). Thus we have Assumption (c).

The proof is now completed. \(\square\)
3 Proof of Theorem 1

We may assume that $G$ is not a complete graph. Thus $G$ is $2\lceil t \rceil$-connected as it is $t$-tough. Suppose to the contrary that $G$ is not hamiltonian. By Theorem 1.3, we have $\delta(G) \leq \frac{n}{t+1} - 1$. Since $\delta(G) \geq 2\lceil t \rceil$, we get

$$n \geq (t + 1)(2\lceil t \rceil + 1).$$

Claim 1. We may assume that $G$ is 2-connected.

Proof. Since $t > 0$, $G$ is connected. Assume to the contrary that $G$ has a cutvertex $x$. By considering the degree sum of two vertices respectively from two components of $G - x$, we know that $\sigma_2(G) \leq n - 1$. On the other hand, $G$ has a cutvertex implies $t \leq \frac{1}{2}$ and so $\sigma_2(G) \geq \frac{4n}{3} - 2$. If $\sigma_2(G) > \frac{4n}{3} - 2$, then we get a contradiction to $\sigma_2(G) \leq n - 1$ as $n \geq 3$. Thus we assume $\sigma_2(G) = \frac{4n}{3} - 2$, which contradicts $\sigma_2(G) \leq n - 1$ if $n \geq 4$. Thus $n = 3$ and so $G = P_3$, but this implies $G \in \mathcal{H}$. 

Since $G$ is 2-connected, $G$ contains cycles. We choose $\lambda \geq 0$ to be a smallest integer such that $G$ admits no $D_\lambda$-cycle but a $D_{\lambda+1}$-cycle. Then we choose $C$ to be a longest $D_{\lambda+1}$-cycle that minimizes $c_p(G - V(C))$ prior to minimizing $c_q(G - V(C))$ for any $p, q \in [1, \lambda]$ with $p > q$. As $G$ is not hamiltonian, we have $\lambda \geq 1$. Thus $V(G) \setminus V(C) \neq \emptyset$. Since $\lambda$ is taken to be minimum, $G - V(C)$ has a component $H$ of order $\lambda$. Let

$$W = N_C(H) \quad \text{and} \quad \omega = |W|.$$ 

Since $G$ is a connected $t$-tough graph, it follows that $\omega \geq 2\lceil t \rceil$. On the other hand, Lemma 3 implies that $\omega \leq \frac{n}{t+\lambda} - 1$.

Claim 2.

$$\begin{cases} 
\lambda + \omega < \frac{n}{t+1} & \text{if } \lambda \geq 2, \\
\lambda + \omega \leq \frac{n}{t+1} & \text{if } \lambda = 1.
\end{cases}$$

Proof. Assume to the contrary that the statement does not hold. If $\lambda = 1$, then $\lambda + \omega > \frac{n}{t+1}$ gives $\omega > \frac{n}{t+1} - 1$. By Lemma 2, we can find a cycle $C'$ with $V(C') = V(C) \cup V(H)$, contradicting the choice of $C$.

Thus $\lambda \geq 2$ and $\lambda + \omega \geq \frac{n}{t+1}$. Since $2t \leq \omega \leq \frac{n}{t+\lambda} - 1 \leq \frac{n}{t+2} - 1$, we have $n \geq (t+2)(2t+1)$. 

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By Lemma 3, we have
\[ n \geq (\lambda + t)(\omega + 1) \geq \left( \frac{n}{t+1} - \omega + t \right)(\omega + 1) = \left( \frac{n}{t+1} \right)(\omega + 1) - \omega - t \geq \frac{n}{t+1} \lambda + \omega \geq \frac{n}{t+1} \quad \text{by our assumption.} \]

\[ \geq \begin{cases} 
\left( \frac{n}{t+1} - 2t + t \right)(2t + 1), & \text{if } f(\omega) = (\frac{n}{t+1} - \omega + t)(\omega + 1) \text{ is increasing;} \\
\left( \frac{n}{t+1} - \frac{n}{t+1} - 1 + t \right) \frac{n}{t+1}, & \text{if } f(\omega) = (\frac{n}{t+1} - \omega + t)(\omega + 1) \text{ is decreasing;} \\
n + \frac{tn}{t+1} - 2t^2 - t \geq n + t(2t + 1) - 2t^2 - t > n + t(2t + 1) - 2t^2 - t = n, \\
\frac{n}{(t+1)(t+2)} \frac{n}{t+2} + \frac{(t+1)n}{t+2} \geq \frac{n}{(t+1)(t+2)} \frac{(t+2)(2t+1)}{t+2} + \frac{(t+1)n}{t+2} > \frac{n}{t+2} + \frac{(t+1)n}{t+2} = n,
\end{cases} \]

reaching a contradiction. □

**Claim 3.** If \( \sigma_2(G) \geq \frac{2n}{t+1} - 2 \), then \( H \) is the only component of \( G - V(C) \).

**Proof.** Suppose \( H^* \neq H \) is another component of \( G - V(C) \). Since \( \sigma_2(G) \geq \frac{2n}{t+1} - 2 \), Claim 2 implies that \( |V(H^*)| + |N_C(V(H^*))| > \sigma_2(G) - (\frac{n}{t+1} - 1) + 1 \geq \frac{n}{t+1} \) if \( \lambda \geq 2 \).

Repeating exactly the same argument for \( |V(H^*)| + |N_C(V(H^*))| \) as in the proof of Claim 2 leads to a contradiction.

Thus we assume \( \lambda = 1 \). We get the same contradiction as above if \( \sigma_2(G) > \frac{2n}{t+1} - 2 \) or \( \lambda + \omega < \frac{n}{t+1} \). Thus we have \( \sigma_2(G) = \frac{2n}{t+1} - 2 \) and \( \omega = \frac{n}{t+1} - 1 \) by Claim 2. Then \( H^* \) contains also only one vertex, say \( y \). We first claim that the vertex \( y \) is adjacent in \( G \) to at most one vertex from \( W^+ \). For otherwise, suppose there are distinct \( u, v \in W^+ \) such that \( uv, vy \in E(G) \). Let \( V(H) = \{ x \} \). Then \( C^* = u \vec{C}v \vec{C}u \vec{C}v \vec{x} \vec{u} \) is a \( D_{\lambda+1} \)-cycle of \( G \) with \( c_\lambda(G - V(C^*)) < c_\lambda(G - V(C)) \). This contradicts the choice of \( C \).

We then claim that the set \( W^+ \) is an independent set in \( G \). For otherwise, suppose there are distinct \( u, v \in W^+ \) such that \( uv \in E(G) \). Let \( V(H) = \{ x \} \). Then \( C^* = u \vec{C}v \vec{C}u \vec{C}v \vec{x} \vec{u} \) is a \( D_{\lambda+1} \)-cycle of \( G \) with \( c_\lambda(G - V(C^*)) < c_\lambda(G - V(C)) \). This contradicts the choice of \( C \).

Now let \( S = V(G) \setminus (W^+ \cup V(H) \cup V(H^*)) \). Then \( c(G - S) \geq \omega + 1 \). However
\[
\frac{|S|}{c(G - S)} \leq \frac{n - \omega - 2}{\omega + 1} = \frac{n}{t+1} - 1 < t,
\]
a contradiction.

Therefore, \( H \) is the only component of \( G - V(C) \). □

Since \( H \) is the only component of \( G - V(C) \), every vertex \( v \in V(C) \setminus W \) is only adjacent in \( G \) to vertices on \( C \). As vertices from \( V(C) \setminus W \) are nonadjacent in \( G \) with vertices from \( H \), we have
\[ \deg_C(v, C) \geq \sigma_2(G) - (\omega + \lambda - 1) \quad \text{for any } v \in V(C) \setminus W. \]  
(1)

We construct the vertex sets \( L_u^+ \left( \frac{n}{t+1} - \omega \right) \) for each \( u \in W \). For notation simplicity, we use \( L_u^+ \) for \( L_u^+ \left( \frac{n}{t+1} - \omega \right) \).
Claim 4. (a) If \( \sigma_2(G) \geq \frac{2n}{t+1} - 2 \), then for any two distinct vertices \( u, v \in W \), we have \( \text{dist}_C^-(u, v) \geq \frac{n}{t+1} - \omega + 1 \) and \( E_G(L_u^+, L_v^+) = \emptyset \).

(b) If \( \sigma_2(G) > \frac{2n}{t+1} - 2 \), then for any two distinct vertices \( u, v \in W \), we have \( \text{dist}_C^-(u, v) > \frac{n}{t+1} - \omega + 1 \) and \( E_G(L_u^+, L_v^+) = \emptyset \).

Proof. We only show Claim 4(a), as the proof for Claim 4(b) follows the same argument by just using the strict inequality. Let \( u^* \in N_H(u) \), \( v^* \in N_H(v) \) and \( P \) be a \((u^*, v^*)\)-path of \( H \). For the first part of the statement, it suffices to show that when we arrange the vertices of \( W \) along \( C \), for any two consecutive vertices \( u \) and \( v \) from the arrangement, we have \( \text{dist}_C^-(u, v) \geq \frac{n}{t+1} - \omega + 1 \). Note that \( V(u^+ \text{C} v^-) \cap W = \emptyset \) for such pairs of \( u \) and \( v \). Assume to the contrary that there are distinct \( u, v \in W \) with \( V(u^+ \text{C} v^-) \cap W = \emptyset \) and \( \text{dist}_C^-(u, v) < \frac{n}{t+1} - \omega + 1 \). Let \( C^* = u \text{C} vv^* Pu^*u \). Since \( H \) has order \( \lambda \) and \( V(u^+ \text{C} v^-) \cap W = \emptyset \), \( H - V(P) \) is a union of components of \( G - V(C^*) \) that each is of order at most \( \lambda - 1 \) and \( u^+ \text{C} v^- \) is a path-component of \( G - V(C^*) \) of order less than \( \frac{n}{t+1} - \omega \) but at least \( \lambda \) (\( G \) has no \( D_\lambda \)-cycle). By (1), for each vertex \( x \in V(u^+ \text{C} v^-) \), \( \text{deg}_G(x, C^*) > \sigma_2(G) - \omega - 1 - \frac{n}{t+1} - \lambda \). This shows a contradiction to Lemma 4.

For the second part of the statement, we assume to the contrary that \( E_G(L_u^+, L_v^+) \neq \emptyset \). Applying the first part, we know that \( \text{dist}_C^-(u, v) \geq \frac{n}{t+1} - \omega + 1 \) and \( \text{dist}_C^-(v, u) \geq \frac{n}{t+1} - \omega + 1 \) (exchanging the role of \( u \) and \( v \)). Thus \( L_u^+ \cap L_v^+ = \emptyset \). We choose \( x \in L_u^+ \) with \( \text{dist}_C^-(u, x) \) minimum and \( y \in L_v^+ \) with \( \text{dist}_C^-(v, y) \) minimum such that \( xy \in E(G) \). By this choice of \( x \) and \( y \), it follows that \( E_G(V(u^+ \text{C} x^-), V(v^+ \text{C} y^-)) = \emptyset \). Let \( C^* = u \text{C} yx \text{C} vv^* Pu^*u \). Since \( H \) is of order \( \lambda \) and no vertex of \( H \) is adjacent in \( G \) to any vertex of \( u^+ \text{C} x^- \) or \( v^+ \text{C} y^- \) by the first part of the statement, \( H - V(P) \) is a union of components of \( G - V(C^*) \) that each is of order at most \( \lambda - 1 \). Also \( u^+ \text{C} x^- \) and \( v^+ \text{C} y^- \) are path-components of \( G - V(C^*) \) that each is of order less than \( \frac{n}{t+1} - \omega \) but at least one of them has order at least \( \lambda \).

Since \( E_G(V(u^+ \text{C} x^-), V(v^+ \text{C} y^-)) = \emptyset \), by (1), for each vertex \( w \in V(u^+ \text{C} x^-) \cup V(v^+ \text{C} y^-) \), \( \text{deg}_G(w, C^*) \geq \frac{n}{t+1} - \lambda \). This shows a contradiction to Lemma 4. \( \square \)

By Claim 4, \( L_u^+ \) and \( L_v^+ \) are remote for any two distinct \( u, v \in W \). Furthermore, \( H \) is remote with \( L_u^+ \) for any \( u \in W \). Let \( S = V(G) \setminus (\bigcup_{u \in W} L_u^+ \cup V(H)) \). Then \( e(G - S) = \omega + 1 \) and

\[
|S| < n - \omega \left( \frac{n}{t+1} - \omega \right) - \lambda \quad \text{if} \quad \sigma_2(G) > \frac{2n}{t+1} - 2,
\]

\[
|S| \leq n - \omega \left( \frac{n}{t+1} - \omega \right) - \lambda \quad \text{if} \quad \sigma_2(G) \geq \frac{2n}{t+1} - 2.
\]
As $G$ is $t$-tough and so $|S| \geq tc(G - S) = t(\omega + 1)$, we get

$$
\begin{align*}
\begin{cases}
n > \omega \left( \frac{n}{t + 1} - \omega + t \right) + \lambda + t & \text{if } \sigma_2(G) > \frac{2n}{t+1} - 2, \\
n \geq \omega \left( \frac{n}{t + 1} - \omega + t \right) + \lambda + t & \text{if } \sigma_2(G) \geq \frac{2n}{t+1} - 2.
\end{cases}
\end{align*}
$$

(2) \quad (3)

Claim 5. It holds that $\sigma_2(G) = \frac{2n}{t+1} - 2$, $\lambda = 1$, and $\omega = \frac{n}{t+1} - 1$.

Proof. Suppose to the contrary that $\sigma_2(G) > \frac{2n}{t+1} - 2$, $\lambda \geq 2$, or $\omega < \frac{n}{t+1} - 1$. Assume first that the function $f(\omega) = \omega \left( \frac{n}{t+1} - \omega + t \right) + \lambda + t$ is increasing. Then as $\omega \geq 2t$ and $n \geq (t + \lambda)(\omega + 1) \geq (t + \lambda)(2t + 1)$ by Lemma 3, we have

$$
\begin{align*}
\begin{cases}
n > f(\omega) \geq f(2t) = \frac{2tn}{t+1} - 2t^2 + \lambda + t \geq n & \text{if } \sigma_2(G) > \frac{2n}{t+1} - 2, \\
n \geq f(\omega) \geq f(2t) = \frac{2tn}{t+1} - 2t^2 + \lambda + t > n & \text{if } \sigma_2(G) = \frac{2n}{t+1} - 2 \text{ and } \lambda \geq 2, \\
n \geq f(\omega) \geq f(2t) = \frac{2tn}{t+1} - 2t^2 + \lambda + t > n & \text{if } \sigma_2(G) = \frac{2n}{t+1} - 2, \lambda = 1, \text{ and } \omega < \frac{n}{t+1} - 1,
\end{cases}
\end{align*}
$$

where note that $\omega < \frac{n}{t+1} - 1$ implies $n > (t + 1)(\omega + 1) \geq (t + 1)(2t + 1)$. The above inequalities give a contradiction.

Thus $f(\omega)$ is decreasing. Note by Claim 2 that $\omega < \frac{n}{t+1} - \lambda$ when $\lambda \geq 2$ and $\omega \leq \frac{n}{t+1} - \lambda$ when $\lambda = 1$. Then as $n \geq (t + \lambda)(\omega + 1) \geq (t + \lambda)(2t + 1)$ by Lemma 3, we have

$$
f\left( \frac{n}{t+1} - \lambda \right) = \left( \frac{n}{t+1} - \lambda \right)(\lambda + t) + \lambda + t \geq n.
$$

Thus

$$
\begin{align*}
\begin{cases}
n > f(\omega) \geq f\left( \frac{n}{t+1} - \lambda \right) \geq n & \text{if } \sigma_2(G) > \frac{2n}{t+1} - 2, \\
n \geq f(\omega) > f\left( \frac{n}{t+1} - \lambda \right) \geq n & \text{if } \sigma_2(G) = \frac{2n}{t+1} - 2 \text{ and } \lambda \geq 2, \\
n \geq f(\omega) > f\left( \frac{n}{t+1} - \lambda \right) \geq n & \text{if } \sigma_2(G) = \frac{2n}{t+1} - 2, \lambda = 1, \text{ and } \omega < \frac{n}{t+1} - 1.
\end{cases}
\end{align*}
$$

The inequalities above again give a contradiction.

By Claim 5, Theorem 1(a) holds. In the rest of the proof, we show Theorem 1(b). Let $W^* = W^+ \cup V(H)$.

Since $u^+ \in L_u^+$ for each $u \in W$, Claim 4 implies that $W^*$ is an independent set in $G$.

Claim 6. Every vertex in $V(G) - W^*$ is adjacent in $G$ to at least two vertices from $W^*$.

Proof. Suppose to the contrary that there exists $x \in V(G) - W^*$ such that $x$ is adjacent in $G$ to at most one vertex from $W^*$. Let $S = V(G) \setminus (W^* \cup \{x\})$. Then $c(G - S) \geq \omega + 1$. However

$$
\frac{|S|}{c(G - S)} \leq \frac{n - \omega - 2}{\omega + 1} = \frac{\frac{tn}{t+1} - 1}{n+1} < t,
$$

a contradiction. 

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Claim 7. For every $v \in W^+$, we have $\deg_G(v, C) = \frac{n}{t+1} - 1$ and $v$ is not adjacent in $G$ to any two consecutive vertices on $C$.

Proof. Since $\sigma_2(G) = \frac{2n}{t+1} - 2$, we have $\deg_G(v, C) \geq \frac{n}{t+1} - 1$ for every $v \in W^+$. As $W^*$ is an independent set in $G$, $v^* \not\in W^*$. By Claim 6, $v^*$ is adjacent in $G$ to another vertex $u$ from $W^*$. If $\{u\} = V(H)$, then $C^* = v^*Cv^+uv^-$ is a $D_{\lambda+1}$-cycle of $G$ with $v$ being the only component of $G - V(C^*)$. Assume then that $u \in W^+$. Let $V(H) = \{x\}$. Then $C^* = v^+uCu^-Cv^+$ is a $D_{\lambda+1}$-cycle of $G$ with $v$ being the only component of $G - V(C^*)$.

Again, since $G$ has no $D_{\lambda}$-cycle, it follows that $\deg_G(v, C^*) = \frac{n}{t+1} - 1$ and $v$ is not adjacent in $G$ to any two consecutive vertices on $C^*$. The claim follows as $\deg_G(v, C) = \deg_G(v, C^*)$ and two neighbors $v$ that are consecutive on $C$ will also be consecutive on $C^*$.

Our goal is to show that $N_C(W^+) = N_C(H)$. To do so, we investigate how vertices in $N_C(W^+)$ are located along $C$. We start with some definitions. A chord of $C$ is an edge $uv$ with $u, v \in V(C)$ and $uv \not\in E(C)$. Two chords $ux$ and $vy$ of $C$ that do not share any endvertices are crossing if the four vertices $u, v, x, y$ appear along $C$ in the order $u, v, x, y$ or $u, y, x, v$. For two distinct vertices $x, y \in N_C(W^+)$, we say $x$ and $y$ form a crossing if there exist distinct vertices $u, v \in W^+$ such that $ux$ and $vy$ are crossing chords of $C$.

Claim 8. For any two distinct $x, y \in N_C(W^+)$ with $xy \in E(C)$, it follows that $x$ and $y$ do not form any crossing.

Proof. Suppose to the contrary that for some distinct $x, y \in N_C(W^+)$ with $xy \in E(C)$, the two vertices $x$ and $y$ form a crossing. Let $u, v \in W^+$ such that $yu, yv \in E(G)$. Assume, without loss of generality, that the four vertices $u, v, x, y$ appear in the order $u, v, x, y$ along $C$. Let $V(H) = \{w\}$. Then $uxCyCu^-wv^-Cu$ is a hamiltonian cycle of $G$, a contradiction to our assumption that $G$ is not hamiltonian.

Claim 9. For any vertex $v \in W^+$ and any two distinct $x, y \in N_C(v)$, $xCy$ contains a vertex from $W^+$.

Proof. By Claim 7, $xCy$ has at least three vertices. Suppose to the contrary that $xCy$ contains no vertex from $W^+$. We furthermore choose $x$ and $y$ so that $xCy$ contains no other vertex from $N_C(v) \setminus \{x, y\}$. Assume that the three vertices $v, x, y$ appear in the order $v, x, y$ along $C$. By Claim 6, each internal vertex of $xCy$ is adjacent in $G$ to a vertex from $W^+$. Then by our selection of $x$ and $y$, we know that each internal vertex of $xCy$ is adjacent in $G$ to a vertex from $W^+ \setminus \{v\}$. Applying Claim 8, $x^+$ does not form a crossing with $x$, and so forms a crossing with $y$. Similarly, $x^{++}$ does not form a crossing with $x^+$, and so forms a crossing with $y$. Continuing this argument for all the internal vertices of $x^{++}Cy$, we know that $y^-$ forms a crossing with $y$, a contradiction to Claim 8.
We assume that the $\omega$ neighbors of the vertex from $V(H)$ on $C$ are $v_1, \ldots, v_\omega$ and they appear in the same order along $C$. For each $i \in [1, \omega]$, let $I_i = V(v_iCv_{i+1}) \setminus \{v_{i+1}\}$, where $v_{\omega+1} := v_1$.

**Claim 10.** For every $v \in W^+$, it holds that $N_C(v) = W$.

**Proof.** Since $\deg_G(v, C) = \omega = |W^+|$ by Claim 7 and $xC y$ contains a vertex from $W^+$ for any two distinct $x, y \in N_C(v)$, it follows that $v$ is adjacent in $G$ to exactly one vertex from $I_i$ for each $i \in [1, \omega]$.

Assume to the contrary that $N_C(v) \neq W$. Let $i \in [1, \omega]$ be the index such that dist$_C(v, v_i)$ is largest and $vv_i \notin E(G)$. By the choice of $i$, we have $vv_{i+1} \in E(G)$. Note that the index $i$ exists since $vv^- \in E(G)$, where $v^- \in \{v_1, \ldots, v_\omega\}$ and dist$_C(v, v^-) > $ dist$_C(v, v_i)$ for any $v_i \neq v^-$. As $v$ is adjacent to one vertex from $I_i$ and $vv_i \notin E(G)$, it follows that $v$ is adjacent to a vertex from $v_i^{++}Cv_{i+1}$. As $vv_{i+1} \in E(G)$, we then know that $v$ is adjacent in $G$ to at least two vertices from $v_i^{++}Cv_{i+1}$. However, since $v_i^{++}Cv_{i+1}$ contains no vertex from $W^+$, we get a contradiction to Claim 9.

Claim 10 implies that $N_C(W^*) = W$. Thus every vertex from $W^*$ is adjacent in $G$ to every vertex from $W$. Therefore $t \leq \tau(G) \leq \frac{|W|}{|W^*|}$. Consequently, $|W| \geq t|W^*| = \frac{n}{t+1}$ and so $W = V(G) \setminus W^*$ by noticing $|W^*| = \frac{n}{t+1}$. Thus $G$ contains a spanning complete bipartite graph between $W^*$ and $W$. On the other hand, since $|W^*| = |W| = \frac{n}{t+1} - 1$ and $V(G) = W^* \cup W = (W^* \cup V(H)) \cup W$, we know that $2(\frac{n}{t+1} - 1) + 1 = n$ and so $t = \frac{n-1}{2}$. Thus $|W| = \frac{n-1}{2}$ and $|W^*| = \frac{n-1}{2} + 1 = \frac{n+1}{2}$. Therefore, $G \in \mathcal{H}$. The proof of Theorem 1 is now complete. \hfill $\Box$

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