THE UNIFORM BOUNDEDNESS AND DYNAMICAL LANG CONJECTURES FOR POLYNOMIALS

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ABSTRACT. We give a conditional proof of the Uniform Boundedness Conjecture of Morton and Silverman in the case of polynomials over number fields, assuming a standard conjecture in arithmetic geometry. Our technique simultaneously yields a dynamical analogue of Lang’s conjecture on minimal canonical heights for these maps. We obtain similar results for non-isotrivial polynomials over a function field of characteristic zero. When the latter are unicritical of degree at least 5, the results hold unconditionally.

1. INTRODUCTION

In the dynamics of rational functions \( f : \mathbb{P}^1 \to \mathbb{P}^1 \) of degree \( d \geq 2 \) defined over a number field \( K \), two conjectures stipulate that few points of \( \mathbb{P}^1(K) \) have small canonical height \( \hat{h}_f \) relative to \( f \), in a way that depends only on \( d \) and \( K \). The first of these conjectures is the Uniform Boundedness Conjecture of Morton and Silverman [23]. A dynamical generalization of Merel’s theorem on the torsion points of an elliptic curve over a number field, this conjecture can be shown to imply the well-known Torsion Conjecture on abelian varieties [12].

Conjecture 1.1 (Uniform Boundedness Conjecture [23]). Let \( d \geq 2, N \geq 1, \) and let \( K \) be a number field. Let \( f : \mathbb{P}^N \to \mathbb{P}^N \) be a morphism of degree \( d \) defined over \( K \). There is a constant \( B = B(d, N, [K : \mathbb{Q}]) \) such that \( f \) has at most \( B \) preperiodic points in \( \mathbb{P}^N(K) \).

Aside from the special case of Lattès maps and the unicritical maps studied in [20, 24], progress on Conjecture 1.1 has only been obtained by imposing strong local conditions on the dynamics of \( f \), as in [3, 8, 16]. We will forfeit all such local hypotheses, at the expense of assuming a standard conjecture in arithmetic geometry, in order to prove Conjecture 1.1 for the \( K \)-rational preperiodic points of polynomials.

Our first result is the following.
Theorem 1.2. Let \( K \) be a number field or a one-dimensional function field of characteristic zero, and let \( f(z) \in K[z] \) be a polynomial of degree \( d \geq 2 \). Assume the \( abcd \)-conjecture (Conjecture 2.1), and if \( K \) is a function field, assume \( f \) is non-isotrivial. Then there is a constant \( B = B(d, K) \) such that \( f \) has at most \( B \) preperiodic points contained in \( K \).

The second conjecture concerning uniform bounds on points of small canonical height is the Dynamical Lang Conjecture, proposed by Silverman [26, Conjecture 4.98]. The technique used to prove Theorem 1.2 allows us to prove a weaker version of this conjecture in the case of polynomial maps.

Theorem 1.3. Let \( K \) be a number field or a one-dimensional function field of characteristic zero, and let \( f(z) \in K[z] \) be a polynomial of degree \( d \geq 2 \). Assume the \( abcd \)-conjecture (Conjecture 2.1). Let \( h_{\text{crit}}(f) \) be the critical height of \( f \). Then there is a \( \kappa = \kappa(d, K) > 0 \) such that for all \( P \in K \), either \( \hat{h}_f(P) = 0 \), or

\[
\hat{h}_f(P) \geq \kappa \max\{1, h_{\text{crit}}(f)\}.
\]

When \( K \) is a number field, the critical height \( h_{\text{crit}}(f) \approx h_{M_d}(f) \) is a moduli height [17, Theorem 1], and by Northcott’s Theorem (1) becomes \( \hat{h}_f(P) \geq \kappa \max\{1, h_{M_d}(f)\} \). This lines up with the formulation of the conjecture given in [26, Conjecture 4.98].

We remark that the \( abcd \)-conjecture is a consequence of Vojta’s conjecture with truncated counting function, as is shown in [20]. The \( abcd \)-conjecture can be thought of as generalizing the \( abc \)-conjecture to higher dimensions, or alternatively, to linear Diophantine equations with more than two summands. Vojta’s conjecture with truncated counting function implies what is usually referred to as Vojta’s conjecture tout court. This weaker version already has Schmidt’s subspace theorem as a special case.

The strategy for proving Theorems 1.2 and 1.3 is as follows. First we prove a global quantitative equidistribution result (Theorem 3.1) that is uniform across all degree \( d \) polynomials. Specifically, we introduce a geometric notion quantifying a certain kind of equidistribution at each place of bad reduction of \( f \in K[z] \), and then prove an upper bound on the average pairwise logarithmic distance between points of small local canonical height realizing a given failure of equidistribution. This upper bound is uniform across places of bad reduction, and is also uniform across degree \( d \) polynomials over \( K \). We derive from this a theorem stating that large sets of points having small height must be roughly equidistributed at “most” places of bad reduction. The implied constants are again independent of the degree \( d \) map \( f \in K[z] \). This theorem upgrades [20, Corollary 3.8] in passing from unicritical to arbitrary polynomials, as well as from preperiodic points to points of small height.

Theorem 3.1 implies that differences of points of small height typically have most of their prime support contained within the set of places of bad reduction of \( f \). This phenomenon
is articulated precisely in Proposition 5.1. The proof of Theorems 1.2 and 1.3, appearing in §6, uses this fact, combined with the geometric descriptions of these point configurations in the local filled Julia sets, to derive a contradiction of the \(abcd\)-conjecture (Conjecture 2.1) when too many of these points are assumed to lie in \(K\). The contradiction is obtained by considering the prime factorizations of cross-ratios of quadruples of \(K\)-points of small height, and then using the Grassmann–Plücker relations satisfied by these cross-ratios to furnish an \(abcd\)-tuple.

Quantitative equidistribution theorems have served as a tool in studying points of small canonical height across families of dynamical systems; see for example [2, 5, 9, 10]. Many of these have, though not always explicitly, been formulated in terms of the energy pairing introduced in [25], and had [13, Théorème 3] as their underlying substrate. On the other hand, the geometric information leveraged in the proof of Theorems 1.2 and 1.3 does not appear to be readily accessible via equidistribution results given solely in terms of energy pairings. Theorem 3.1 presents a formulation that is well suited both to the main theorems of this article, and to those of [19].

Finally, we note that Theorem 3.1, along with its consequence Proposition 5.1, can be combined with the methods appearing in [20] to prove the following.

**Theorem 1.4.** Let \(K\) be a number field or a one-dimensional function field of characteristic zero, and let \(f(z) = z^d + c \in K[z]\), where \(d \geq 5\). If \(K\) is a number field, assume the \(abc\)-conjecture for \(K\). Then there is a \(\kappa = \kappa(d, K) > 0\) such that for all \(P \in K\), either \(\hat{h}_f(P) = 0\), or

\[
\hat{h}_f(P) \geq \kappa \max\{1, h_{\text{crit}}(f)\}.
\]

In particular, this result holds unconditionally when \(K\) is a function field. The proof is immediate by adapting [20, §7–8] to incorporate Proposition 5.1.

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2. **Background**

2.1. **Notation.** We set the following notation:
If $K$ is a number field, let $\mathcal{O}_K$ denote the ring of integers. If $K$ is a function field, let $\mathcal{O}_K$ be the integral closure of $k[t]$ in $K$. If $K$ is a number field, $n \geq 2$ and $P = (z_1, \ldots, z_n) \in \mathbb{P}^{n-1}(K)$ with $z_1, \ldots, z_n \in K$, let

$$h(P) = \sum_{\text{primes } p \text{ of } \mathcal{O}_K} -\min\{v_p(z_1), \ldots, v_p(z_n)\}N_p + \frac{1}{[K : \mathbb{Q}]} \sum_{\sigma : K \rightarrow \mathbb{C}} \log \max\{|\sigma(z_1)|, \ldots, |\sigma(z_n)|\},$$

where we do not identify conjugate embeddings. (We choose to express the height in this form, which separates the nonarchimedean and archimedean contributions, for convenience in applying the $abcd$-conjecture.) If $K$ is a function field, let

$$h(P) = \sum_{\text{primes } p \text{ of } \mathcal{O}_K} -\min\{v_p(z_1), \ldots, v_p(z_n)\}N_p.$$
The Uniform Boundedness and Dynamical Lang Conjectures

**Conjecture 2.1** (The abcd-conjecture). Let $K$ be a number field or a one-dimensional function field of characteristic zero, and let $n \geq 3$. Let $[Z_1 : \cdots : Z_n]$ be the standard homogeneous coordinates on $\mathbb{P}^{n-1}(K)$, and let $\mathcal{H}$ be the hyperplane given by $Z_1 + \cdots + Z_n = 0$. For any $\epsilon > 0$, there is a proper Zariski closed subset $Z = Z(K, \epsilon, n) \subseteq \mathcal{H}$ and a constant $C_{K,Z,\epsilon,n}$ such that for all $P = (z_1, \ldots, z_n) \in H \setminus Z$ with $z_1, \ldots, z_n \in K^*$, we have

$$h(P) < (1 + \epsilon)\text{rad}(P) + C_{K,Z,\epsilon,n}.$$ 

For a divisor $D \in \text{Div}(X)$ and $v \in M_K$, let $\lambda_{D,v}$ be a $v$-adic local height on $(X \setminus D)(K_v)$ relative to $D$. (For background on local height functions, see [15, Chapter B.8].) For $P \in X(K)$, let $h_D(P) = \sum_{v \in M_K} r_v \lambda_{D,v}(P)$. We say that an effective divisor $D \in \text{Div}(X)$ is a normal crossings divisor if $D = \sum_{i=1}^r D_i$ for distinct irreducible subvarieties $D_i$, and the variety $\cup_{i=1}^r D_i$ has normal crossings.

**Definition.** Let $S \subseteq M_K$ be a finite set of places of $K$ containing $M_\infty^\infty$. For $P \in X(K) \setminus D$, and $\lambda_{D,p}$ a set of local height functions relative to $D$, the arithmetic truncated counting function is

$$N^{(1)}_S(D, P) = \sum_{P \in M_K \setminus S} \chi(\lambda_{D,p}(P))N_p$$

where for $a \in \mathbb{R}$,

$$\chi(a) = \begin{cases} 
0 & \text{if } a \leq 0 \\
1 & \text{if } a > 0.
\end{cases}$$

In [20], it is shown that Conjecture 2.1 is a consequence of the following conjecture [27, Conjecture 2.3].

**Conjecture 2.2.** [27, Conjecture 2.3] Let $K$ be a number field or a one-dimensional function field of characteristic zero, and let $S$ be a finite set of places of $K$ containing the archimedean places. Let $X$ be a smooth projective variety over $K$, let $D$ be a normal crossings divisor on $X$, let $K_X$ be a canonical divisor on $X$, let $A$ be an ample divisor on $X$, and let $\epsilon > 0$. Then there exists a proper Zariski closed subset $Z = Z(K,S,X,D,A,\epsilon) \subseteq X$ such that

$$N^{(1)}_S(D, P) \geq h_{K_X+D}(P) - \epsilon h_A(P) + O(1)$$

for all $P \in X(K) \setminus Z$.

Remark. The version of this conjecture appearing in [27, Conjecture 2.3] is stated for points $P \in X(L)$ where $L$ has bounded degree over $K$, at the expense of a logarithmic discriminant term $d(L/K)$. However, Masser has shown [22] that this form of the conjecture is false. Here we will only require the weaker statement appearing in Conjecture 2.2.
2.3. Nonarchimedean potential theory. For $v \in M_K$, let $\mathbb{C}_v$ be the $v$-adic completion of $\overline{K_v}$. We denote open and closed disks in $\mathbb{C}_v$ as follows:

$$D(a, r) = \{z \in \mathbb{C}_v : |z - a|_v \leq r\},$$
$$D(a, r)^- = \{z \in \mathbb{C}_v : |z - a|_v < r\}.$$

We impose the convention that disks have radius belonging to the value group $|C_v^\times|$. For $f(z) \in \mathbb{C}_v[z]$ and $z \in \mathbb{C}_v$, let

$$\hat{\lambda}_v(z) = \lim_{n \to \infty} \frac{1}{d^n} \log \max\{1, |f^n(z)|_v\}$$

be the standard $v$-adic escape-rate function. (See [26, §3.4, 3.5] for a proof that the limit defining $\hat{\lambda}_v(z)$ exists.) Note that $\hat{\lambda}_v(z)$ obeys the transformation rule

$$\hat{\lambda}_v(f(z)) = d\hat{\lambda}_v(z)$$

for all $z \in \mathbb{C}_v$.

Now suppose $|\cdot|_v$ is nonarchimedean. By [8, Proposition 7.33], if $E$ is the minimal disk containing $K_v$, then for all $m \geq 1$, $f^{-m}(E)$ is a finite union of disjoint closed disks. We refer to the preimage disks as the disk components of $f^{-m}(E)$. If $f^{-1}(E)$ is not a disk, we say that $v$ is a place of bad reduction, or alternatively a bad place. For a bad place $v$, the log of the radius of $E$ will be called the splitting radius of $f$ at $v$. It will be denoted $g_v$.

If $R_f$ is the set of finite critical points of $f$, the $v$-adic critical height is

$$\lambda_{\text{crit}, v}(f) := \max_{a \in R_f} \{\hat{\lambda}_v(a)\}.$$

For $f(z) \in K[z]$, let

$$h_{\text{crit}}(f) = \sum_{v \in M_K} r_v \lambda_{\text{crit}, v}(f).$$

We will be using the fact that when $v \in M_0^\times \setminus \mathcal{S}_d$ is a place of bad reduction for a monic polynomial $f(z) \in K[z]$ of degree $d$, the splitting radius equals $\lambda_{\text{crit}, v}(f)$ [18, Lemmas 2.1 and 2.2]. We retain the two different notations for concepts that coincide over all relevant places in order to make it clear which concept (splitting radius or critical height) is the applicable one.

We will be using a measure of the size of a set of bad places for a given polynomial $f(z) \in K[z]$.

**Definition.** For $0 < \delta < 1$, $f \in K[z]$ and $S \subseteq M_0^\times$, a $\delta$-slice of places $v \in S$ is a set $S'$ of bad places $v \in S$ of $f$ such that

$$\sum_{v \in S'} r_v \lambda_{\text{crit}, v}(f) \geq \delta \sum_{v \in S} r_v \lambda_{\text{crit}, v}(f).$$
Let $A^1_v$ denote the Berkovich affine line over $\mathbb{C}_v$. For $a \in \mathbb{C}_v$, we define open and closed Berkovich disks of radius $r$

$$B(a, r)^- = \{ x \in A^1_v : [T - a]_x < r \},$$
$$B(a, r) = \{ x \in A^1_v : [T - a]_x \leq r \},$$
corresponding to the classical disks $D(a, r)^-$ and $D(a, r)$ respectively. A basis for the open sets of $A^1_v$ is given by sets of the form $B(a, r)^-$ and $B(a, r)$ respectively. A basis for the open sets of $A^1_v$ is given by sets of the form $B(a, r)^-$ and $B(a, r)$ respectively. We consider $A^1_v$ as a measure space whose Borel $\sigma$-algebra is generated by this topology. Let $\delta_v(z, w)$ denote the Hsia kernel relative to infinity (see [6, Section 4.1]).

The Berkovich $v$-adic filled Julia set of $f(z) \in K[z]$ is defined as

$$K_v = \bigcup_{M > 0} \{ x \in A^1_v : [f^n(z)]_x \leq M \text{ for all } n \geq 0 \}.$$

Let $v \in M^0_K$, let $E \subseteq A^1_v$, and let $\nu$ be a probability measure with support contained in $E$. The potential function of $\nu$ is by definition

$$p_\nu(z) = \int_E -\log \delta_v(z, w) d\nu(w),$$
and the energy integral of $\nu$ is

$$I(\nu) = \int_E p_\nu(z) d\nu(z).$$

The integrals here are Lebesgue integrals; the function $\delta_v(z, w)$ is upper semicontinuous ([6, Proposition 4.1(A)]), so $-\log \delta_v(z, w)$ is lower semicontinuous, and hence Borel measurable relative to the $\sigma$-algebra generated by the Berkovich topology. The capacity of $E$ is

$$\gamma(E) := e^{-\inf I(\nu)}.$$

If $E$ is compact and $\gamma(E) > 0$, there is a unique probability measure $\mu_E$ on $E$ for which $I(\mu_E) = \inf \nu I(\nu)$ [6, Proposition 7.21]. This measure $\mu_E$ is called the equilibrium measure for $E$. When $E$ is compact, the capacity coincides with the quantity

$$\lim_{n \to \infty} \sup \left\{ \prod_{i \neq j} \delta_v(z_i, z_j)^{1/(n(n-1))} : z_1, \ldots, z_n \in E \right\},$$
which is known as the transfinite diameter of $E$ [6, Theorem 6.24]. For a set $T \subseteq A^1_v$ of $n$ points $z_1, \ldots, z_n$, let

$$d_\nu(T) := \prod_{i \neq j} \delta_v(z_i, z_j)^{1/(n(n-1))}.$$

For closed Berkovich disks $B_1, B_2$, we write

$$\delta_v(B_1, B_2) = \max_{x \in B_1, y \in B_2} \{ \delta_v(x, y) \}.$$
and for any disk $B \subseteq A_1^1$, let

$$\text{diam}(B) = \delta_v(B,B).$$

### 3. Global Quantitative Equidistribution

The main result of this section is Theorem 3.1, which gives a uniform global quantitative equidistribution theorem over degree $d$ polynomials. The argument shares a similar basic idea as the main equidistribution theorem of [20, Corollary 3.8]: the main difference is that one must account for the many possible large scale structures of the filled Julia set at a place $v \in M_K^* \setminus \mathcal{J}_d$ of bad reduction, in contrast with the unicritical case, where only one such structure occurs.

Let $v \in M_K^* \setminus \mathcal{J}_d$, and let $f(z) \in \mathbb{C}_v[z]$ of degree $d \geq 2$ have bad reduction. Let $g_v$ be the splitting radius of $f$, let $E$ be the unique disk of radius $\exp(g_v)$ containing $K_v$, and let $E_m := f^{-m}(E)$.

Let $B_{1,1}, \ldots, B_{1,d}$ be the disk components of $E_1$, listed with multiplicity. Similarly, for $m \geq 2$, list the disk components $\{B_{m,i}\}_{i=1}^d$ of $E_m$ inductively, so that $B_{m,i} \subseteq B_{1,j}$ for $\left\lfloor \frac{m}{d-1} \right\rfloor = j$, and according to multiplicity.

A wing decomposition of $E_1$ is a partition of $E_1$ into two nonempty disjoint sets (wings) $A$ and $B$ with the following properties:

- $A$ and $B$ are unions of disk components of $E_1$.
- For any disk components $B_{1,i}$ of $A$ and $B_{1,j}$ of $B$, we have $\log \delta_v(B_{1,i}, B_{1,j}) = g_v$.

A wing decomposition is not unique in general. We note that wing decompositions always exist, since the smallest disk containing $K_v$ is $E$, and $\log \text{diam}(E) = g_v$.

**Definition.** Let $\epsilon > 0$. We say that a finite set $T \subseteq \mathbb{C}_v$ is $\epsilon$-equidistributed (at $v$) if for any wing decomposition of $E_1$ as above, we have

$$|T \cap A| > \left( \frac{1 - \epsilon}{d} \right) |T|.$$

Our goal is to prove the following theorem.

**Theorem 3.1** (Global quantitative equidistribution). Let $f(z) \in K[z]$ be a polynomial of degree $d \geq 2$. Let $\epsilon > 0$, and $0 < \delta < 1$. Let $T \subseteq K$ be a finite set. There are constants $N$ and $\kappa > 0$, depending only on $d$, $[K : F]$, $\delta$, and $\epsilon$, such that if $|T| \geq N$ and

$$\frac{1}{|T|} \sum_{P_i \in T} h_f(P_i) \leq \kappa \text{crit}(f),$$

then $T$ is $\epsilon$-equidistributed for a $\delta$-slice of bad places $v \in M_K^* \setminus \mathcal{J}_d$.

By Proposition 4.1, this reduces to proving:
**Theorem 3.2.** Let \( f(z) \in K[z] \) be a polynomial of degree \( d \geq 2 \). Let \( \epsilon > 0 \) and \( 0 < \delta < 1 \). Let \( T \subseteq K \) be a finite set. There are constants \( N, M, \xi, \) and \( \kappa > 0 \) depending only on \( d, [K : F], \delta, \) and \( \epsilon \), such that if \( |T| \geq N \), \( h_{\text{crit}}(f) \geq M \),

\[
\sum_{v \in M_K^0 \setminus \mathcal{S}_d} r_v \lambda_{\text{crit},v}(f) \geq (1 - \xi) h_{\text{crit}}(f),
\]

and

\[
\frac{1}{|T|} \sum_{P_i \in T} \hat{h}_f(P_i) \leq \kappa h_{\text{crit}}(f),
\]

then \( T \) is \( \epsilon \)-equidistributed for a \( \delta \)-slice of bad places \( v \in M_K^0 \setminus \mathcal{S}_d \).

Let

\[
\bar{\vec{k}} = (k_1, \ldots, k_d) \in (\mathbb{Q} \cap [0, 1])^d
\]

with \( \sum_{i=1}^d k_i = 1 \). For \( v \in M_K^0 \) a place of bad reduction for \( f \in C_v[z] \) with filled Julia set \( \mathcal{K}_v \), we say a nonempty finite set \( T \subseteq C_v \) is \( \bar{\vec{k}} \)-distributed if for any \( 1 \leq i \leq d \),

\[
d_i k_i |T| = |T \cap B_{1,i}|,
\]

where \( B_{1,i} \) is a disk component of \( \mathcal{E}_1 \) mapping onto \( \mathcal{E}_0 \) with degree \( d_i \), and the \( B_{1,i} \) are listed with multiplicity.

In proving the potential-theoretic propositions needed in the proof of Theorem 3.1, we will assume \( f(z) \in C_v[z] \) is a monic polynomial having bad reduction, and splitting radius \( g_v \). This hypothesis is mathematically inessential, and merely serves to simplify the presentation.

Let us outline the idea behind the proofs of Propositions 3.5 and 3.6, which are key in proving the local result underlying Theorem 3.1. The goal is to consider various weights attached to the disk components of a large scale structure of the filled Julia set. A key observation is that once a large scale structure is fixed, and weights are given on each disk component in this structure, the energy corresponding to this set of weights is determined. We explain this idea further after Equation (10). From there, one considers only sets of weights corresponding to a failure of \( \epsilon \)-equidistribution. Pairs of what we will call admissible \( 1 \)-structures and weight vectors can be given a topology such that the subset in question is compact. It is the compactness of this space that allows us to reduce a question about an infinite collection of possible structures and weights to standard facts that hold for any single filled Julia set.

**Definition.** Let \( v \in M_K^0 \), and let \( f \in C_v[z] \) be of degree \( d \geq 2 \), with splitting radius \( g_v > 0 \). We say that a union of \( d \) (possibly not disjoint) disks \( F = \bigcup b_i \subseteq A_1^1 \) is admissible if \( \log \max_{i,j} \delta_v(b_i, b_j) = g_v \) and \( 0 \geq \log \text{diam}(b_i) \geq -(d - 2)g_v \) for all \( i \). We impose the convention that no \( b_i \) is properly contained in \( b_j \).
In particular, \( E_1 \) is admissible, as can be seen in the proof of [21, Proposition 4.3].

**Definition.** Let \( v \in M_0^0 \), and let \( f \in \mathbb{C}_v[z] \) be monic of degree \( d \geq 2 \), with splitting radius \( g_v > 0 \). Let \( m_0 \geq 1 \). An \( m_0 \)-structure (with respect to \( f \) and \( v \)) is an element

\[
(r_{1,1}, \ldots, r_{1,d^{m_0}}, \ldots, r_{d^{m_0},1}, \ldots, r_{d^{m_0},d^{m_0}}) \in [-d^{m_0}, 1]^{d^{2m_0}}
\]

such that there is a union of disks

\[
\mathcal{F}_{m_0} = \bigcup_{i=1}^{d^{m_0}} b_{m_0,i} \subseteq A_v^1
\]

with no \( b_{m_0,i} \) properly contained in any \( b_{m_0,j} \), satisfying

\[
\log \frac{\delta_v(b_{m_0,i}, b_{m_0,j})}{g_v} = r_{i,j}
\]

for all \( 1 \leq i, j \leq d^{m_0} \),

\[
\log \frac{\gamma(\mathcal{F}_{m_0})}{g_v} = \frac{1}{d^{m_0}},
\]

and

\[
\mu(b_{m_0,i}) = \frac{d_i}{d^{m_0}},
\]

where \( \mu \) is the equilibrium measure on \( \mathcal{F}_{m_0} \), and \( d_i \) is the number of indices \( 1 \leq j \leq d^{m_0} \) for which \( b_{m_0,i} = b_{m_0,j} \).

We refer to such an \( \mathcal{F}_{m_0} \) as an underlying set of the \( m_0 \)-structure \( \Sigma \). Given \( \Sigma \) an \( m_0 \)-structure, let a \( \Sigma \)-mesh be a sequence \( \{\Sigma_m\}_{m=m_0}^\infty \) of \( m \)-structures having underlying sets

\[
\mathcal{F}_m = \bigcup_{i=1}^{d^m} b_{m,i}
\]

that satisfy

\[
b_{m,i} \subseteq b_{m-1,j}
\]

for \( j = \lceil \frac{i}{d} \rceil \) for all \( m \geq m_0 \), and

\[
\mu_m(b_{m_0,i} \cap \mathcal{F}_m) = \frac{d_i}{d^{m_0}},
\]

where \( \mu_m \) is the equilibrium measure on \( \mathcal{F}_m \). We note that given an \( m_0 \)-structure, such a sequence must exist, which we can prove by induction. Suppose \( m \geq m_0 \), and \( \Sigma \) is an \( m \)-structure with underlying set \( \mathcal{F}_m \). Given a disk component \( b_i \) of \( \mathcal{F}_m \), we can take \( d \) disks \( d_1, \ldots, d_d \) with centers \( c_1, \ldots, c_d \in b_i \cap \mathbb{C}_v \) having pairwise distance \( \text{diam}(b_i) \) from each other, and diameter equal to \( \text{diam}(b_i) \) (so that the disks in fact equal \( b_i \) itself), and then shrink those disks about their centers \( c_j \) until the following condition holds. For each \( i \), let
Let \( d_{m,i} \) be the number of \( b_k \) equalling \( b_i \) in the disk decomposition (8). Let \( \nu \) be the probability measure

\[
\nu = \frac{d_{m,i}}{d^m} \mu_b + \frac{d^m - d_{m,i}}{d^m} \mu_b,
\]

where \( \mu_b \) is the equilibrium measure on \( \bigcup \partial_j \), and \( \mu_b \) is the equilibrium measure on \( \bigcup_{b_k \neq b_i} b_k \).

We shrink the radii of the disks \( \partial_j \) (uniformly, say) until, for each \( x \in \bigcup \partial_j \),

\[
\int \log \delta_v(x, y) d\nu(y) = \frac{g_v}{d^{m+1}}.
\]

The disks so obtained are disjoint, by our condition on the centers. Repeating this procedure for each disk component \( b_i \) of \( F_m \), and taking the union of the resulting shrunk disks, we obtain an \( F_{m+1} \) that is the underlying set of an \((m+1)\)-structure. The condition (9) on the equilibrium measure of \( F_{m+1} \) is obtained by noting that the property of everywhere equal potentials characterizes the equilibrium measure on \( F_{m+1} \). (This can be thought of as a sort of converse to Frostman’s Theorem for this particular set. To prove it, observe that if \( \mu_{m+1} \) is the equilibrium measure on \( F_{m+1} \), then by Frostman’s Theorem and the fact that the equilibrium measure is unique, the \( \mu_{m+1} \)-measure of each wing component in any wing decomposition of \( F_{m+1} \) must be equal to the \( \nu \)-measure. We then apply the same reasoning to an analogous decomposition of each wing, and iterate this process until we have shown that \( \nu = \mu_{m+1} \).) Our construction of \( \nu \) thus ensures that \( \nu \) is in fact the equilibrium measure on \( F_{m+1} \), and that (9) holds, with \( m \) replacing \( m_0 \).

Fix \( \vec{k} \) as in (2). For a 1-structure \( \Sigma \) with \( \Sigma \)-mesh \( \{\Sigma_m\} \) and underlying sets \( \{F_m\} \), define for each \( m \geq 1 \) the unique probability measure \( \mu_{\vec{k},m} = \mu_{\vec{k},m}(\Sigma) \) on \( F_m \) such that for all \( 1 \leq i \leq d \),

\[
\mu_{\vec{k},m}(b_i \cap F_m) = k_i,
\]

and \( \mu_{\vec{k},m} \) is a scalar multiple of the equilibrium measure on \( b_i \cap F_m \). Let

(10)

\[
I_{\vec{k}}(\Sigma) = \lim_{m \to \infty} I(\mu_{\vec{k},m}(\Sigma)).
\]

Note that this limit exists by the monotone convergence theorem, and that \( I_{\vec{k}}(\Sigma)/g_v \) depends only on \( \vec{k} \) and on \( \Sigma \). In particular, it is independent of the choice of mesh \( \{\Sigma_m\} \) and underlying sets \( F_m \). This is because for any \( m \geq 1 \),

\[
\frac{\log \gamma(F_m)}{g_v} = \frac{1}{d^m},
\]

and so by (9) and Frostman’s Theorem [6, Theorem 6.18], specifying the shape given by the 1-structure determines each of the

\[
\frac{\log \gamma(b_i \cap F_m)}{g_v}
\]
for $1 \leq i \leq d$, the full set of which in turn determines $I(\mu_{\vec{k},m}(\Sigma))/g_v$ once $\vec{k}$ is specified. Hence $I_{\vec{k}}(\Sigma)/g_v$ is independent of the mesh $\{\Sigma_m\}$.

Let $\mu_{\vec{k}}(\Sigma)$ be any weak* subsequential limit of the $\mu_{\vec{k},m}(\Sigma)$. We note that

\begin{equation}
I_{\vec{k}}(\Sigma) = I(\mu_{\vec{k}}(\Sigma)),
\end{equation}

which is proved exactly as the Claim in the proof of [20, Proposition 3.7]. Write $\gamma(\mu_{\vec{k}}) = \exp(-I(\mu_{\vec{k}}(\Sigma))) = \exp(-I_{\vec{k}}(\Sigma))$. We introduce a proposition giving a uniform rate of convergence for the limit (10). It is key in the proof of Proposition 3.5.

**Proposition 3.3.** Let $f(z) \in \mathbb{C}_v[z]$ be monic of degree $d \geq 2$ having bad reduction, and let $\Sigma$ be a 1-structure with respect to $f$ and $v$. Let $k$ be as in (2). Then for all $m \geq 1$,

$$-I(\mu_{\vec{k},m}(\Sigma)) + I(\mu_{\vec{k},m+1}(\Sigma)) \leq \left( \frac{1}{d^m} - \frac{1}{d^{m+1}} \right) g_v.$$ 

To prove this proposition, we utilize the following fact.

**Lemma 3.4.** Let $v \in M^0_K \setminus \mathcal{S}_d$, let $f(z) \in \mathbb{C}_v[z]$ be a monic polynomial of degree $d \geq 2$ having bad reduction, and let $g_v$ be the splitting radius of $f$. Then

$$\log \text{diam}(B_{m,i}) \geq -d^m g_v.$$ 

**Proof.** The proof is analogous to that of [20, Lemma 3.3].

**Proof of Proposition 3.3.** Let $\{F_m\}$ be the sequence of underlying sets associated to a $\Sigma$-mesh. For $m \geq 1$, we have

$$-I(\mu_{\vec{k},m}(\Sigma)) + I(\mu_{\vec{k},m+1}(\Sigma)) \leq \max_{1 \leq i \leq d} \{ \log \gamma(F_m \cap B_i) - \log \gamma(F_{m+1} \cap B_i) \}$$

$$\leq d(\log \gamma(F_m) - \log \gamma(F_{m+1}))$$

$$= d \left( \frac{1}{d^m} - \frac{1}{d^{m+1}} \right) g_v,$$

where the final equality follows from Lemma 3.4.

**Proposition 3.5.** Let $v \in M^0_K \setminus \mathcal{S}_d$, let $f(z) \in \mathbb{C}_v[z]$ be a monic polynomial of degree $d \geq 2$ having bad reduction and splitting radius $g_v$, and let $k$ be as in (2). Let $\epsilon > 0$. There exist integers $N = N(\epsilon)$ and $m = m(\epsilon)$ such that for any $k$-distributed set $T \subseteq E_m \cap \mathbb{C}_v$ of order $n \geq N$, and any $w \in T$,

$$d_v(T) \leq \gamma(\mu_{\vec{k}}) e^{\epsilon g_v}.$$
Proof. Let \( \epsilon' > 0, m \geq 1 \), and let \( \vec{k} \) be as in (2).

Claim: There is an \( N = N(\epsilon', m) \) such that if \( T \subseteq E_m \cap C_v \) is a set of \( n \geq N \) elements \( z_1, \ldots, z_n \in E_m \), then

\[
(12) \quad d_v(T) \leq e^{\epsilon' g_v} \exp(-I(\mu_{m,T})).
\]

Proof of claim: Write \( j_{m,i} = \mu_{m,T}(B_{m,i}) \) and

\[
E_m = \bigcup_{i=1}^{s_m} B_{m,i}
\]

for disjoint disks \( B_{m,i} \), and let

\[
r_m = \min_{1 \leq i \leq s_m} \{ \delta_v(B_{m,i}) \}.
\]

For any \( w \in T \cap B_{m,i} \), we have

\[
\exp(-p_{\mu_{m,T}}(w)) \geq \left( \prod_{z_i \in T, z_i \neq w} \delta_v(z_i, w) \left( \frac{1}{r_m} \right)^{(j_{m,i})n-1} (r_m)^{(j_{m,i})n} \right)^{1/n}
\]

\[
= \left( r_m \prod_{z_i \in T, z_i \neq w} \delta_v(z_i, w) \right)^{1/n}.
\]

As \( \log r_m \geq -d^m g_v \) by Lemma 3.4, taking \( n \gg m, \epsilon' \) proves the claim. Since by definition of the capacity

\[
(14) \quad \log(-I(\mu_{m,T})) \leq \log \gamma(E_m),
\]

Proposition 3.3 completes the proof. \( \square \)

Proposition 3.6. Let \( v \in M_0^0 \setminus \mathcal{S}_d \) let \( f(z) \in \mathbb{C}_v[z] \) be a monic polynomial of degree \( d \geq 2 \) having bad reduction, and let \( g_v \) be the splitting radius of \( f \). Let \( \epsilon > 0 \). There is an \( N_0 = N_0(\epsilon) \), \( m = m(\epsilon) \geq 1 \), and an \( \epsilon' = \epsilon'(\epsilon) > 0 \) such that if \( T \subseteq f^{-m}(E_0) \) is a finite set with \( |T| \geq N_0 \), and \( T \) is not \( \epsilon \)-equidistributed, then

\[
\log d_v(T) \leq -\epsilon' \log(e^{g_v}) = -\epsilon' g_v.
\]

Proof. Call a 1-structure admissible if its underlying set \( \mathcal{F}_1 \) is admissible. Let \( \mathcal{W} \) be the set of admissible 1-structures with respect to polynomials \( f \in \mathbb{C}_v[z] \) of degree \( d \). Let \( \mathcal{K} \) be the set of \( d \)-tuples as in (2). Let \( \mathcal{G} \) be the set of elements \( (\Sigma, \vec{k}) \in \mathcal{W} \times \mathcal{K} \) such that any \( \vec{k} \)-distributed set of points contained in any underlying set of \( \Sigma \) fails to be \( \epsilon \)-equidistributed. Let

\[
\psi : \mathcal{G} \to \mathbb{R}
\]
be given by

\[ \psi(\Sigma, \vec{k}) = \frac{I_{\vec{k}}(\Sigma)}{g_v}. \]

**Claim:** \( \mathcal{S} \) is a compact subset of \( \mathcal{W} \times \mathcal{K} \).

Proof of claim: The set \( \mathcal{W} \) of \( d^2 \)-tuples as in (3) with \( m_0 = 1 \) that correspond to sets as in (4) satisfying (5) is closed in \([-d, 1]^d\). Moreover, the subset \( \mathcal{J} \subseteq \mathcal{W} \) corresponding to underlying sets satisfying (6) and (7) is itself closed in \( \mathcal{W} \). Finally, \( \mathcal{W} \) is closed in \( \mathcal{J} \). From this we conclude that \( \mathcal{W} \) is closed in the topology inherited from the Euclidean topology on \([-d, 1]^d\).

To finish proving the claim, we note that every sufficiently small perturbation of the disks (more precisely, perturbation of their radii and relative distances, not simply their centers) in the underlying set \( \mathcal{F}_1 \) of an element of \( \mathcal{W} \) preserves the minimal number of elements that \( \vec{k} \)-distributed sets have in each of the \( A \) and \( B \) components under any wing decomposition of \( \mathcal{F}_1 \). Here we are considering perturbations within \( \mathcal{W} \), not \([-d, 1]^d\), so that the resulting sets \( \mathcal{F}_1 \) actually correspond to a set of disks. We can clearly also alter \( \vec{k} \) around a point \( x \in (\mathcal{W} \times \mathcal{K}) \setminus \mathcal{S} \) while the disks are perturbed, and stay within \( (\mathcal{W} \times \mathcal{K}) \setminus \mathcal{S} \) if both perturbations are sufficiently small. Thus \( \mathcal{S} \) is a closed and bounded subspace of \([-d, 1]^d\), as desired.

Continuing with the proof of Proposition 3.6, we assume \( \epsilon < \frac{1}{d} \) so that \( \mathcal{S} \) is also nonempty. As

\[ I_{\vec{k}}(\Sigma) = \lim_{m \to \infty} I(\mu_{\vec{k}, m}(\Sigma)), \]

it follows that \( \psi \) is a continuous function on a compact set, and so must attain a minimum \( -\eta = -\eta(d, \epsilon) \). Let \( (\Sigma, \vec{k}) \) be such that \( \psi(\Sigma, \vec{k}) = -\eta \), and let \( \{\mathcal{F}_m\} \) be a sequence of underlying sets of \( \Sigma \). From the uniqueness of the equilibrium measure on each \( \mathcal{F}_m \) and condition (7), we see that for all \( m \), \( I(\mu_{\vec{k}, m}(\Sigma)) \) is not equal to the equilibrium energy on the underlying sets \( \mathcal{F}_m \). (It is worth remarking that it is in this very final deduction that the coarseness in the definition of \( \epsilon \)-equidistribution is crucial. A definition that is given in terms of individual disks does not account for the fact that we need to allow disk components of \( \mathcal{F}_1 \) to collide, and a failure of equidistribution can suddenly vanish when two disks with a “skewed” distribution join together.) Thus we conclude that \( \eta > 0 \). On the other hand, by our definition of admissible 1-structures and \( \Sigma \)-meshes, and the fact ([11, Theorem 1.2]) that

\[ \gamma(\mathcal{E}_m) = \frac{1}{dm} g_v \]

for all \( m \geq 1 \), it follows that for any \( f \in \mathbb{C}_v[z] \) of degree \( d \) having splitting radius \( g_v > 0 \), there is an admissible 1-structure and mesh \( \{\Sigma_m\}_{m=1}^\infty \) along with a choice of underlying sets \( \mathcal{F}_m \) such that the filled Julia set \( \mathcal{K}_v \) is contained in \( \mathcal{F}_m \) for all \( m \geq 1 \). Combined with
Proposition 3.3 and the Claim (12), this completes the proof of the proposition (cf. the proof of [20, Proposition 3.7]). □

**Proposition 3.7.** Let \( v \in M_0^0 \setminus \mathcal{S}_d \) let \( f(z) \in \mathbb{C}_v[z] \) be a monic polynomial of degree \( d \geq 2 \) having bad reduction, and let \( g_v \) be the splitting radius of \( f \). Let \( \epsilon > 0 \), and let \( T \subseteq \mathbb{C}_v \) be a finite set. There are constants \( N = N(d, \epsilon), \tau = \tau(d, \epsilon) > 0 \) and \( \epsilon' = \epsilon'(d, \epsilon) > 0 \) such that if \( n = |T| \geq N \) and

\[
\frac{1}{|T|} \sum_{P_i \in T} \hat{\lambda}_v(P_i) \leq \tau g_v,
\]

then

\[
\frac{1}{n(n-1)} \sum_{x \neq y \in T} \log \delta_v(x, y) \leq \epsilon g_v,
\]

and if \( T \) fails to be \( \epsilon \)-equidistributed,

\[
\frac{1}{n(n-1)} \sum_{x \neq y \in T} \log \delta_v(x, y) \leq -\epsilon' g_v.
\]

**Proof.** We first prove (16). Let \( v \in M_0^0 \setminus \mathcal{S}_d \) be a bad place, let \( T \subseteq \mathbb{C}_v \) be a nonempty finite set, and let \( m \geq 1 \) be such that

\[
\frac{1}{|T|} \sum_{P_i \in T} \hat{\lambda}_v(P_i) \leq \frac{1}{d^m g_v}.
\]

We partition \( T \) into three subsets based on local canonical height: \( T_1 \) will correspond to ‘small height points,’ and \( T_2 \) and \( T_3 \) will correspond to ‘large height points.’ Most points will lie in \( T_1 \). The large points are subdivided into \( T_2 \) and \( T_3 \) because the relationship between the local canonical height and the \( v \)-adic absolute value differs based on whether a point lies in \( \mathcal{E} \). Let \( 0 < \zeta < 1 \) be such that \( 1/\zeta < d^{m/2} \). By (17), there is a subset \( T_1 \subseteq T \) of size at least \( (1 - \zeta)|T| \) such that for each \( P_i \in T_1 \),

\[
\hat{\lambda}_v(P_i) \leq \frac{1}{\zeta d^m g_v}.
\]

Let \( T' = T \setminus T_1 \), let \( T_2 \) be the set of elements \( P_i \in T' \) such that

\[
\hat{\lambda}_v(P_i) \leq g_v,
\]

and let \( T_3 \) be the set of elements \( P_i \in T' \) such that

\[
\hat{\lambda}_v(P_i) > g_v.
\]

For disjoint nonempty finite sets \( A, B \subseteq \mathbb{A}_v^1 \), write

\[
(A, B) = \sum_{x \in A} \sum_{y \in B} \log \delta_v(x, y),
\]
and let
\[(A, A) = \sum_{x \neq y \in A} \log \delta_v(x, y).\]

Let \(|T| = n\). Then since \(T_1, T_2, T_3\) are pairwise disjoint,
\[\frac{1}{n(n-1)} \sum_{x \neq y \in T} \log \delta_v(x, y) = \frac{1}{n(n-1)} \sum_{i,j=1}^{3} (T_i, T_j).\]

As \(v \notin \mathcal{L}_d\) and hence \(\gamma(\mathcal{E}_m) = \exp(g_v/d^m)\), the proof of Proposition 3.5 (in particular, the Claim (12) along with (14)) implies that for any \(\delta > 0\), if \(n \gg_{m,d} 1\), then
\[\frac{1}{(\max\{1, |T_2|\})(|T_1 \cup T_2| - 1)} \sum_{i=1}^{2} (T_2, T_i) \leq (\delta + 1)g_v,\]
and thus, since \(|T_2|/n \leq \zeta,\)
\[
\frac{1}{n(n-1)} \sum_{i=1}^{2} (T_2, T_i) \leq \zeta(\delta + 1)g_v. \tag{18}
\]

Moreover, as \(|T_3|/n \leq \zeta,\) (17) implies that
\[
\frac{1}{\max\{1, |T_3|\}} \sum_{P_i \in T_3} \hat{\lambda}_v(P_i) \leq \frac{1}{\zeta} \cdot \frac{1}{d^m} g_v. \tag{19}
\]

We note that for any \(x \in T_3\) and any \(y \in T,\)
\[
\log \delta_v(x, y) \leq \max\{\hat{\lambda}_v(x), \hat{\lambda}_v(y)\} + C, \tag{20}
\]
where \(C = C(d)\) is an absolute constant depending only on \(d\). Therefore, fixing \(\delta > 0\), it follows from (19) and (20) that if \(n \gg_{d,m,\delta} 1\), then
\[
\frac{1}{(n-1) \max\{1, |T_3|\}} \sum_{i=1}^{3} (T_3, T_i) \leq \frac{2(\delta + 1)}{\zeta d^m} g_v
\]
and so
\[
\frac{1}{n(n-1)} \sum_{i=1}^{3} (T_3, T_i) \leq (2\delta + 2)\zeta \left(\frac{1}{\zeta} \cdot \frac{1}{d^m} g_v\right) = (2\delta + 2) \frac{1}{d^m} g_v(f). \tag{21}
\]

If \(\zeta\) is sufficiently small and \(m\) is sufficiently large, then if \(T\) is not \(\epsilon\)-equidistributed, \(T_1\) fails to be \(2\epsilon\)-equidistributed. In that case it follows from Proposition 3.6 that if \(m \gg_{d,\epsilon} 1\), there is an \(\epsilon_2 = \epsilon_2(d, \epsilon, m) > 0\) such that if \(n \gg_{m,d,\epsilon} 1\), then
\[
\frac{1}{|T_1|(|T_1| - 1)} (T_1, T_1) \leq -\epsilon_2 g_v,
\]
and so

\[(22) \quad \frac{1}{n(n-1)} (T_1, T_1) \leq -\epsilon_2 (1 - \zeta)^2 g_v. \]

Combining (18), (21), and (22), we obtain

\[
\frac{1}{n(n-1)} \sum_{x \neq y \in T} \log \delta_v(x, y) \leq 2 \left( (2\delta + 2) \left( \zeta + \frac{1}{d^m} \right) - \epsilon_2 (1 - \zeta)^2 \right) g_v.
\]

Letting \(m \gg_{\delta, \epsilon, \epsilon_1} 1\) and \(\zeta \ll_{\delta, \epsilon, m} 1\), \(\delta \ll_{\delta, \epsilon, m} 1\) completes the proof. In general, the proof of Proposition 3.5 implies that if \(m \gg_{d, \epsilon} 1\) and \(n \gg_{d, \epsilon} 1\), then

\[
\frac{1}{|T_1|(|T_1| - 1)} (T_1, T_1) \leq \epsilon g_v,
\]

and so

\[
\frac{1}{n(n-1)} (T_1, T_1) \leq \epsilon(1 - \zeta)^2 g_v.
\]

This yields

\[
d_v(T) = \frac{1}{n(n-1)} \sum_{x \neq y \in T} \log \delta_v(x, y) \leq 2 \left( (\delta + 1) \left( \zeta + \frac{1}{d^m} \right) + \epsilon(1 - \zeta)^2 \right) g_v,
\]

and so the proof is concluded by taking \(\zeta \ll_{d, \epsilon} 1\) and \(m \gg_{d, \epsilon} 1\).

\[\square\]

**Proof of Theorem 3.1.** By Proposition 4.1, it suffices to prove Theorem 3.2. Without loss of generality suppose \(f\) is monic, let \(T \subseteq K\) be a nonempty finite set, let \(\xi\) be such that

\[
\sum_{v \in M_0^0 \setminus S_d} r_v \lambda_{\text{crit},v}(f) \geq (1 - \xi) h_{\text{crit}}(f)
\]

and let \(\kappa > 0\) be such that

\[
\frac{1}{|T|} \sum_{P_i \in T} \hat{h}_f(P_i) \leq \kappa h_{\text{crit}}(f).
\]

Let \(S_1\) be the set of places \(v \in M_K^0 \setminus S_d\) of bad reduction for \(f\) such that

\[
\frac{1}{|T|} \sum_{P_i \in T} \hat{\lambda}_v(P_i) \leq \frac{\kappa}{\xi} \lambda_{\text{crit},v}(f),
\]

and let \(S_2 = M_K \setminus S_1\). Let \(S_{1,1}\) be the set of bad places \(v \in M_K^0 \setminus S_d\) where \(T\) fails to be \(\epsilon\)-equidistributed, and let \(S_{1,2} = S_1 \setminus S_{1,1}\). Let \(\epsilon_1 > 0\). By Proposition 3.7, if \(\kappa \ll_{d, \epsilon, \epsilon_1} 1\), \(|T| \gg_{d, \epsilon, \epsilon_1} 1\), then for all \(v \in S_{1,1},\)

\[
\log d_v(T) \leq \epsilon_1 g_v,
\]

and there is an \(\epsilon_2 = \epsilon_2(d, \epsilon, m_0, \kappa)\) such that for all \(v \in S_{1,2},\)

\[
\log d_v(T) \leq -\epsilon_2 g_v.
\]
From this we obtain

\begin{equation}
0 = \sum_{v \in M_K} r_v \log d_v(T) \leq \sum_{v \in \mathcal{S}_{1,1}} \epsilon_1 r_v g_v + \sum_{v \in \mathcal{S}_{1,2}} -\epsilon_2 r_v g_v + \sum_{v \in \mathcal{S}_2} r_v \log d_v(T).
\end{equation}

On the other hand, for all $v \in M_K$,

$$
\log |P_i - P_j|_v \leq \hat{\lambda}_v(P_i) + \hat{\lambda}_v(P_j) + \lambda_{\text{crit},v}(f) + 2C_v,
$$

where $C_v$ is an $M_K$-constant supported only on $v \in M_K^\infty \cup \mathcal{S}_d$. Thus

\begin{equation}
\sum_{v \in \mathcal{S}_2} r_v \log d_v(T) \leq \sum_{v \in \mathcal{S}_2} r_v \cdot \frac{1}{|T|} \sum_{P_i \in T} \left(2\hat{\lambda}_v(P_i) + \lambda_{\text{crit},v}(f)\right) + \sum_{v \in \mathcal{S}_2} 2r_v C_v
\leq 2\kappa h_{\text{crit}}(f) + \xi h_{\text{crit}}(f) + \eta.
\end{equation}

If $\epsilon_1, \kappa, \xi \ll d, \epsilon, 1$, and $h_{\text{crit}}(f) \gg \epsilon_1, \kappa, \xi, 1$, then from (24) we see that the right-hand side of (23) is negative, a contradiction.

\[ \square \]

4. AN EASY CASE

In this section we show that when one restricts consideration to maps $f \in K[z]$ satisfying certain local hypotheses, proving Theorems 1.2 and 1.3 is relatively straightforward. In particular, given $s \in \mathbb{Z}_{>0}$ and $0 < \xi < 1$, if a map $f \in K[z]$ of degree $d \geq 2$ has a set of at most $s$ places that are either bad or archimedean, and contribute a total of at least $\xi h_{\text{crit}}(f)$ to $h_{\text{crit}}(f)$, then $f$ has at most $B = B(d, s, \xi)$ preperiodic points contained in $K$. We apply this idea to the special case where the set of places in question is $\mathcal{S}_d \cup M_K^\infty$.

**Proposition 4.1.** Let $f(z) \in K[z]$ be of degree $d \geq 2$. If $K$ is a function field, assume $f$ is non-isotrivial. Let $T \subseteq K$ be a finite set. Then for any $0 < \xi < 1$ and $M \in \mathbb{R}_{>0}$, there is an $N = N(d, [K : F], \xi, M)$ and a $\kappa = \kappa(d, [K : F], \xi, M) > 0$ such that if $|T| \geq N$ and

$$
\frac{1}{|T|} \sum_{P_i \in T} \hat{\lambda}_f(P_i) \leq \kappa h_{\text{crit}}(f),
$$

then $h_{\text{crit}}(f) \geq M$ and

$$
\sum_{v \in M_K^0 \setminus \mathcal{S}_d} r_v \lambda_{\text{crit},v}(f) \geq (1 - \xi)h_{\text{crit}}(f).
$$

To prove Proposition 4.1, we require a result governing the minimal distance of a point of small local canonical height to a point of $f^{-3}(0)$, assuming $f$ takes an appropriate form.

**Proposition 4.2.** [21, Corollary 3.4, Proposition 4.3] Let $f(z) \in K[z]$ be a monic polynomial of degree $d \geq 2$, and let $v \in M_K$ be a place of bad reduction for $f$. Suppose $0 \in \mathcal{K}_v$, and let $\alpha \in \mathcal{C}_v$. There is a constant $\eta$ depending only on $d$ such that $\alpha \in D(0, 2e^{\lambda_{\text{crit},v}(f)})$ implies that

\[ \square \]
every \( y \in f^{-3}(\alpha) \) satisfies
\[
\min_{\beta \in f^{-3}(0)} \log |y - \beta|_v \leq -\frac{1}{d-1} \lambda_{\text{crit},v}(f) + (\eta)\kappa.
\]

Remark. The normal form for \( f \) used in [21] has 0 as a fixed point of \( f \), and lead coefficient \( 1/d \). However, the proof is readily adapted to the hypotheses of Proposition 4.2.

Proof of Proposition 4.1. Without loss of generality, assume \( f \) is monic. Suppose \( f \) satisfies
\[
\sum_{v \in S} r_v \lambda_{\text{crit},v}(f) \geq \xi h_{\text{crit}}(f)
\]
for some \( \xi > 0 \) and a nonempty set \( S \) of places of \( K \) with \( |S| \leq s \). Then for some \( v_0 \in S \),
\[
r_{v_0} \lambda_{\text{crit},v_0}(f) \geq \frac{\xi}{s} h_{\text{crit}}(f).
\]
Let \( S_0 = M_K^\infty \cup J_d \cup \{v_0\} \), with \( |S_0| = s_0 \). A fortiori,
\[
(25) \sum_{v \in S_0} r_v \lambda_{\text{crit},v}(f) \geq \frac{\xi}{s} h_{\text{crit}}(f).
\]
Let \( \epsilon > 0 \), let \( \kappa > 0 \), and let \( T \subseteq K \) be a finite set such that
\[
\frac{1}{|T|} \sum_{P_i \in T} \hat{h}_f(P_i) \leq \kappa h_{\text{crit}}(f).
\]
Proposition 4.2 and the pigeonhole principle imply that if \( |T| \geq s_0 \), there is a nonempty \( T' \subseteq T \) such that for each \( v \in S_0 \), \( T' \) is contained in a disk of radius at most \( \exp((\eta)\kappa - \frac{1}{d-1} \lambda_{\text{crit},v}(f)) \) in \( C_v \), and \( |T'| \geq n/d^{s_0} \). By Proposition 3.5, there is an \( N = N(\epsilon) \) such that if \( |T'| \geq N \), then
\[
\log d_v(T') \leq \epsilon \lambda_{\text{crit},v}(f)
\]
for all \( v \in M_K \setminus S_0 \). This yields
\[
(26) \sum_{v \in M_K} r_v \log d_v(T') \leq \sum_{v \in S_0} \left( -\frac{1}{d-1} r_v \lambda_{\text{crit},v}(f) + (\eta)\kappa \right) + \sum_{v \in M_K \setminus S_0} r_v \epsilon \lambda_{\text{crit},v}(f).
\]
Suppose \( K \) is a number field. If \( \epsilon \) is chosen to be sufficiently small, and \( h_{\text{crit}}(f) \gg_{\epsilon,s,s_0,1} \), then (25) and (26) give
\[
\sum_{v \in M_K} r_v \log d_v(T') \leq \sum_{v \in S_0} \left( -\frac{1}{d-1} r_v \lambda_{\text{crit},v}(f) + \eta \right) + \sum_{v \in M_K \setminus S_0} r_v \epsilon \lambda_{\text{crit},v}(f)
\]
\[
\leq \left( -\frac{1}{d-1} + \epsilon \right) \frac{\xi}{s} h_{\text{crit}}(f) + \sum_{v \in M_K \setminus S_0} r_v \epsilon \lambda_{\text{crit},v}(f)
\]
\[
\leq \left( -\frac{1}{d-1} + \epsilon \right) \frac{\xi}{s} h_{\text{crit}}(f) + \left( 1 - \frac{\xi}{s} \right) \epsilon h_{\text{crit}}(f)
\]
\[
< 0,
\]
contradicting the product formula. If $K$ is a function field, then $S_0 = \{v_0\}$ and so (26) becomes

$$\sum_{v \in M_K} r_v \log d_v(T') \leq -\frac{1}{d-1} r_{v_0} \lambda_{\text{crit},v_0}(f) + \sum_{v \neq v_0 \in M_K} r_v \epsilon \lambda_{\text{crit}}(f)$$

(27)

$$\leq -\frac{1}{d-1} \xi h_{\text{crit}}(f) + \left(1 - \frac{\xi}{s}\right) \epsilon h_{\text{crit}}(f).$$

Since $f$ is non-isotrivial by assumption, and hence, by [1, Theorem 1.9] $h_{\text{crit}}(f) > 0$, the right-hand side of (27) is strictly less than 0 when $\epsilon$ is sufficiently small, contradicting the product formula. Noting that $|T'| \geq N = N(\epsilon)$ when $|T| \gg_{s_0, \epsilon} 1$ completes the proof. □

5. DIFFERENCES OF SMALL POINTS

This section is devoted to showing that elements of the form $P_i - P_j$, where $P_i, P_j$ have small canonical height relative to $f$, tend to have their prime support mostly contained within the set of places of bad reduction for $f$. (Of course, one must take the lead coefficient of $f$ into account.) In the special case where $P_i, P_j$ are preperiodic, this is a dynamical analogue of a parallel phenomenon in the setting of groups: for instance, $n$-th roots of unity remain distinct modulo primes not dividing $n$, and $n$-torsion points on elliptic curves remain distinct modulo primes of good reduction not dividing $n$.

Definition. Let $\epsilon > 0$, and let $f(z) \in K[z]$ be of degree $d \geq 2$ with lead coefficient $a_d$. Let $S_{2,1}$ be the set of places of good reduction for $f$, and let $S_{2,2} = \mathcal{S}_d \cup M_K^\infty$. For $v \in M_K^0$, let $v : K^* \to \mathbb{Z}$ denote the standard $v$-adic valuation. We say $\alpha \in K^*$ is $\epsilon$-adically good if

$$\sum_{v \in S_{2,1}} v(a_d^{1/(d-1)} \alpha) N_v \leq \epsilon h_{\text{crit}}(f),$$

and

$$\sum_{v \in S_{2,2}} r_v \log |a_d^{1/(d-1)} \alpha|_v \geq -\epsilon h_{\text{crit}}(f).$$

Proposition 5.1. Let $f(z) \in K[z]$ be a polynomial of degree $d \geq 2$, and let $\epsilon > 0$. There is an $N = N(d, \epsilon, [K : F])$ and a $\kappa = \kappa(d, \epsilon, [K : F]) > 0$ such that if $T \subseteq K$ is a set of $n \geq N$ points such that

$$\frac{1}{|T|} \sum_{P_i \in T} \hat{h}_f(P_i) \leq \kappa h_{\text{crit}}(f),$$

then at least $(1 - \epsilon)n^2$ elements $(P_i, P_j) \in T^2$ have the property that $P_i - P_j$ is $\epsilon$-adically good.
Proof. Let $\epsilon > 0$, suppose without loss that $f(z) \in K[z]$ of degree $d \geq 2$ is monic, and let $1 > \xi > 0$ be such that

$$\sum_{v \in M_\infty K \cup S_d} r_v \lambda_{\text{crit}, v}(f) \leq \xi h_{\text{crit}}(f).$$

By Proposition 3.5, there is an $N = N(\epsilon)$ such that if $n \geq N$ and $v \in S_1 := M_K \setminus (S_{2,1} \cup S_{2,2})$, then

$$\frac{1}{n^2} \sum_{P_i \neq P_j \in T} \log |P_i - P_j|_v \leq \frac{\epsilon^2}{2} \sum_{v \in S_1} r_v \lambda_{\text{crit}, v}(f),$$

and hence

$$\sum_{v \in S_1} r_v \lambda_{\text{crit}, v}(f) \leq \xi h_{\text{crit}}(f).$$

Assume $n \geq N$. On the other hand, for any $P_i \neq P_j$,

$$\sum_{v \in S_1} r_v \log |P_i - P_j|_v \geq -\log 4 - \xi h_{\text{crit}}(f).$$

Indeed,

$$\sum_{v \in M_\infty K \cup S_d} r_v \lambda_{\text{crit}, v}(f) \leq \xi h_{\text{crit}}(f),$$

so

$$\sum_{v \in M_\infty K \cup S_d} r_v \log |P_i - P_j|_v \leq \sum_{v \in M_\infty K \cup S_d} r_v (\lambda_{\text{crit}, v}(f) + \log 4) \leq \log 4 + \xi h_{\text{crit}}(f).$$

From (29), one sees that if $h_{\text{crit}}(f) \gg \xi 1$, then

$$\sum_{v \in S_1} r_v \log |P_i - P_j|_v \geq -(\xi + \epsilon) h_{\text{crit}}(f).$$

It follows that when $n \gg \epsilon 1$, at least $(1 - \epsilon)n^2$ of the elements $(P_i, P_j) \in T^2$ must satisfy

$$\sum_{v \in S_1} r_v \log |P_j - P_i|_v \leq \epsilon \sum_{v \in S_1} r_v \lambda_{\text{crit}, v}(f);$$

otherwise, by (30), we have for all $h_{\text{crit}}(f) \gg \xi 1$ and $\xi \ll \epsilon 1$ that

$$\frac{1}{n^2} \sum_{P_i \neq P_j \in S_1} r_v \log |P_j - P_i|_v > \sum_{v \in S_1} r_v \lambda_{\text{crit}, v}(f) \left(\epsilon^2 - (\xi + \epsilon)(1 - \epsilon)\right) > \frac{\epsilon^2}{2} \sum_{v \in S_1} r_v \lambda_{\text{crit}, v}(f),$$

contradicting (28). The product formula and Proposition 4.1 complete the proof. 

6. Proof of Theorems 1.2 and 1.3

In this section, we prove Theorems 1.2 and 1.3. These theorems follow from the following two statements.
Theorem 6.1. Let $f(z) \in K[z]$ be of degree $d \geq 2$. Assume the $abcd$-conjecture (Conjecture 2.1). There is a $\kappa = \kappa(d, K) > 0$ and a $B = B(d, K)$ such that if a finite set $T \subseteq K$ satisfies
\[
\frac{1}{|T|} \sum_{P_i \in T} \hat{h}_f(P_i) \leq \kappa h_{\text{cm}}(f),
\]
then either $K$ is a function field and $f$ is isotrivial, or $|T| \leq B$.

Lemma 6.2. Let $K/k(t)$ be a function field, and let $f \in K[z]$ of degree $d \geq 2$ be conjugate by $\mu \in \text{PGL}_2(K)$ to $f \in \mathbb{F}[z]$. Then there is a $\kappa = \kappa(d, [K : k(t)]) > 0$ such that for all $P \in K$, either $\hat{h}_f(P) = 0$, or
\[
\hat{h}_f(P) \geq \kappa.
\]

Proof. Let $f \in K[z]$ of degree $d$ be isotrivial, i.e., assume that $\mu f \mu^{-1} \in \mathbb{F}[z]$ for some $\mu \in \text{PGL}_2(K)$. For each $v \in M_K$, the Newton polygon $N(f, v)$ of $f$ at $v$ must be a line segment, as $f$ has potential good reduction at all $v$. If $s_v$ is the slope of this segment, we clearly have $v(s_d) \in \mathbb{Z}$. Let $\zeta(z) = \alpha z \in \mathbb{K}$ and $g = \zeta f \zeta^{-1} \in L[z]$ be such that for each $v \in M_L$, $N(g, v)$ is a line segment with slope 0. Assume $L$ is minimal among the extensions of $K$ such that $g \in L[z]$ and $\zeta(K) \subseteq L$. Let $\kappa > 0$, and assume $P \in L$ is such that $0 < \hat{h}_g(P) \leq \kappa$. Then there is a $v \in M_L$ such that $0 < u_v \hat{\lambda}_v(P) \leq \kappa$, where $u_v = \frac{|L_v : k(t)_v|}{|K_v : k(t)|}$ and $\hat{\lambda}_v(P)$ is the $v$-adic canonical height with respect to $g$. Since $g$ has good reduction at $v$, $\hat{\lambda}_v(P) = \log \max\{1, |P|_v\}$, so in fact $0 < u_v \log |P|_v \leq \kappa$. Finally, noting that by our hypothesis on $L$,
\[
u_v \in \mathbb{Z} \left\lfloor \left\lfloor \frac{K_v : k(t)_v}{K : k(t)} \cdot d! \right\rfloor \right\rfloor,
\]
we conclude that if $\kappa < d, [K : k(t)]$, then $P \notin L$, a contradiction. 

Proposition 6.3. Let $\epsilon > 0$, let $d \geq 2$, let $f(z) \in K[z]$ be a degree $d \geq 2$ polynomial, and let $v \in M_K \setminus \mathcal{S}_d$ be a place of bad reduction for $f$. There are integers $k = k(d, \epsilon)$ and $N = N(d, \epsilon)$ such that if $T \subseteq K$ is $\epsilon$-equidistributed, then at least $(1 - \epsilon)|T|^d$ choices of
\[
(a_1, b_1, c_1, d_1, \ldots, a_k, b_k, c_k, d_k) \in T^d
\]
have the property that for some $1 \leq i \leq k$, there are disk components $B_{1,j}$ and $B_{1,l}$ of $E_1$ with $\log \delta_v(B_{1,j}, B_{1,l}) = g_v$ and $a_i, b_i \in B_{1,j}$ and $c_i, d_i \in B_{1,l}$.

Proof. Let $\epsilon > 0$, and let $T$ be $\epsilon$-equidistributed at $v$. Let $A$ and $B$ be the wings of a wing decomposition of $E_1$. By the definition of $\epsilon$-equidistribution and the pigeonhole principle, there is a disk component $B_{1,j}$ of $A$ such that at least $(1 - \epsilon)/(d(d - 1))$ elements of $T$ lie in $B_{1,j}$, and at least $((1 - \epsilon)/(d(d - 1)))^2$ elements of $T^2$ lie in $A$. Similarly, there is a disk component $B_{1,l}$ of $B$ such that at least $(1 - \epsilon)/(d(d - 1))$ elements of $T$ lie in $B_{1,l}$. Thus, at least $((1 - \epsilon)/(d(d - 1)))^4$ elements $(a_i, b_i, c_i, d_i) \in T^4$ satisfy $a_i, b_i \in B_{1,j}$ and $c_i, d_i \in B_{1,l}$.
From this it follows that at least \(1 - (1 - ((1 - \epsilon)/(d(d-1)))^k)\) choices of
\[
(a_1, b_1, c_1, d_1, \ldots, a_k, b_k, c_k, d_k) \in \mathcal{T}^4k
\]
have the property that for some \(1 \leq i \leq k\),
\[
a_i, b_i \in \mathcal{E}_{1,j} \quad \text{and} \quad c_i, d_i \in \mathcal{E}_{1,l}.
\]
Taking \(k \gg \epsilon, \delta\) completes the proof. \(\Box\\)

**Corollary 6.4.** Let \(\epsilon > 0\), let \(d \geq 2\), and let \(f(z) \in K[z]\) be a degree \(d\) polynomial. There exist constants \(\kappa = \kappa(d, \epsilon) > 0\), \(N = N(d, \epsilon)\), and \(k = k(d, \epsilon)\) such that if \(T \subseteq K\) is a finite set with \(|T| \geq N\), and
\[
\frac{1}{|T|} \sum_{P_i \in T} \hat{h}_f(P_i) \leq \kappa \max\{1, h_{\text{crit}}(f)\},
\]
then at least \((1 - \epsilon)N^{4k}\) elements \((a_1, b_1, c_1, d_1, \ldots, a_k, b_k, c_k, d_k) \in \mathcal{T}^4k\) have the property that there is a \((1 - \epsilon)\)-slice \(S\) of bad places \(v \in M_K^0 \setminus \mathcal{S}_d\) such that for any \(v \in S\), there is some \(1 \leq i \leq k\) such that
\[
a_i, b_i \in B_{1,j} \quad \text{and} \quad c_i, d_i \in B_{1,l},
\]
for some disk components \(B_{1,j}, B_{1,l}\) of \(\mathcal{E}_1\) satisfying \(\log \delta_v(B_{1,j}, B_{1,l}) = \lambda_{\text{crit},v}(f)\).

**Proof.** For \(v \in M_K^0 \setminus \mathcal{S}_d\) a place of bad reduction, \(k\) as in Proposition 6.3, and
\[
\bar{x} = (a_1, b_1, c_1, d_1, \ldots, a_k, b_k, c_k, d_k) \in \mathcal{T}^4k
\]
write \(\chi_v(\bar{x}) = 1\) if for some \(1 \leq i \leq k\) we have \(a_i, b_i \in B_{1,j}\) and \(c_i, d_i \in B_{1,l}\) for disk components \(B_{1,j}, B_{1,l}\) such that \(\log \delta_v(B_{1,j}, B_{1,l}) = \lambda_{\text{crit},v}(f)\). Proposition 6.3 says that if \(T\) is \(\epsilon\)-equidistributed at \(v\), then
\[
\frac{1}{|T|^{4k}} \sum_{\bar{x} \in \mathcal{T}^4k} \chi_v(\bar{x}) \geq (1 - \epsilon).
\]
From Theorem 3.1, we deduce that if \(|T| \gg \epsilon, d\) and \(\kappa \ll \epsilon, d\), then
\[
\frac{1}{|T|^{4k}} \sum_{v \in M_K^0 \setminus \mathcal{S}_d} \sum_{\bar{x} \in \mathcal{T}^4k} r_v \chi_v(\bar{x}) \lambda_{\text{crit},v}(f) \geq (1 - \epsilon) \sum_{v \in M_K^0 \setminus \mathcal{S}_d} r_v \lambda_{\text{crit},v}(f).
\]
In other words, a \((1 - \epsilon)\)-slice \(S\) of bad places \(v \in M_K^0 \setminus \mathcal{S}_d\) has the desired property. \(\Box\\)

**Proof of Theorem 6.1.** Let \(\epsilon > 0\), and let \(T \subseteq K\) be a finite set with \(n = |T|\) and
\[
\frac{1}{n} \sum_{P_i \in T} \hat{h}_f(P_i) \leq \kappa \max\{1, h_{\text{crit}}(f)\}.
\]
Let \(k = k(d, \epsilon)\) be as in Corollary 6.4, and let \(Z = Z(K, \epsilon, 2k + 1)\). Let \(Y_1, \ldots, Y_{2k+1}\) be the standard homogeneous coordinates on \(\mathbb{P}^{2k}\), let
\[
\mathcal{H} : Y_1 + \cdots + Y_{2k+1} = 0,
\]
and suppose $Z \subseteq H$ is contained in the hypersurface defined by $g(Y_1, \ldots, Y_{2k+1}) = 0$ for some $g \in \overline{K}[Y_1, \ldots, Y_{2k+1}]$. We note that $Z$ is nonempty, and so by symmetry, it is clear that $g$ is non-constant in each of the variables $Y_1, \ldots, Y_{2k+1}$. Since on $Z$ we have $Y_{2k+1} = -\sum_{i=1}^{2k} Y_{2i}$, there is a $g_1$ such that $g(Y_1, \ldots, Y_{2k+1}) = g_1(Y_1, \ldots, Y_{2k})$ on $Z$. For each $1 \leq j \leq k$, write $Y_{2j-1} = X_j$ and $Y_{2j} = M_j - X_j$. Substituting these expressions for the $Y_j$, we have

$$g_1(Y_1, \ldots, Y_{2k}) = g_2(X_1, M_1, \ldots, X_k, M_k),$$

where $g_2$ is a polynomial over $\overline{K}$ non-constant in each of the variables $M_1, \ldots, M_k$, and non-constant in the variables in some (a priori possibly empty) subset $\{X_{i_1}, \ldots, X_{i_s}\} \subseteq \{X_1, \ldots, X_k\}$. All but finitely many choices of $M_1 = m_1 \in \mathbb{Z}_+$ yield $g_2(X_1, m_1, X_2, M_2, \ldots, X_k, M_k)$ non-constant in each of the variables $X_{i_1}, X_{i_2}, \ldots, X_{i_s}, M_2, \ldots, M_k$. Choose such an $m_1$, and make successive choices of $m_2, \ldots, m_k \in \mathbb{Z}_+$ such that

$$g_2(X_1, m_1, \ldots, X_k, m_k)$$

is nonzero, and non-constant in each of the variables $X_{i_1}, \ldots, X_{i_s}$. By Corollary 6.4, if $n \gg_{d, \epsilon} 1$ and $\kappa \ll_{d, \epsilon} 1$, then at least $(1 - \epsilon)n^{4k}$ choices of

$$\left(a_1, b_1, c_1, d_1, \ldots, a_k, b_k, c_k, d_k\right) \in T^{4k}$$

have the property that there is a $(1 - \epsilon)$-slice $S$ of bad places $v \in M_0^0 \setminus \mathcal{S}_d$ such that for each $v \in S$, there are disk components $B_{1,j}, B_{1,l} \subseteq \mathcal{E}_1$ such that

$$a_i, b_i \in B_{1,j} \text{ and } c_i, d_i \in B_{1,l},$$

and $\log \delta_v(B_{1,j}, B_{1,l}) = \lambda_{\text{crit},v}(f)$. We note, for use in the third to last inequality in (36), that

$$\log \text{diam}(B_{1,j}), \log \text{diam}(B_{1,l}) \leq 0$$

which can be seen in the proof of [21, Proposition 4.3]. It follows that under these hypotheses, at least $(1 - 2\epsilon)n^{4k}$ choices of tuples as in (31) have this property, as well as the property that if we take

$$x_1 = m_1 \frac{(a_1 - c_1)(d_1 - b_1)}{(c_1 - d_1)(b_1 - a_1)}, \ldots, x_k = m_k \frac{(a_k - c_k)(d_k - b_k)}{(c_k - d_k)(b_k - a_k)}$$

for such $\mathbf{x} \in T^{4k}$, then $g_2(x_1, m_1, \ldots, x_k, m_k) \neq 0$. Moreover, by the Plücker identity

$$m_i \frac{(a_i - c_i)(d_i - b_i)}{(c_i - d_i)(b_i - a_i)} - m_i = m_i \frac{(d_i - a_i)(c_i - b_i)}{(d_i - c_i)(b_i - a_i)},$$

(34)
and Propositions 4.1 and 5.1, if $n \gg_{\epsilon,k} 1$ and $\kappa \ll_{\epsilon,d} 1$, we can assume that $x_1, m_1 - x_1, \ldots, x_k, m_k - x_k$ are $\epsilon$-adically good. Choose such $x_1, \ldots, x_k$, let

$$P = \left( x_1, m_1 - x_1, \ldots, x_k, m_k - x_k, -\sum_{j=1}^{k} m_j \right) \in \mathbb{P}^{2k}(K),$$

and let $S$ be a $(1 - \epsilon)$-slice of bad places as in (32). Note that

$$x_1 + (m_1 - x_1) + x_2 + (m_2 - x_2) + \cdots + x_k + (m_k - x_k) = \sum_{j=1}^{k} m_j,$$

so $P \in \mathcal{H}$ where

$$\mathcal{H} : Y_1 + \cdots + Y_{2k+1} = 0$$

in $\mathbb{P}^{2k}$. Let $S'$ be the set of places dividing $\sum_{i=1}^{k} m_i$ along with those places $v$ such that the valuation of some $x_1, m_1 - x_1, \ldots, x_k, m_k - x_k$ is nonzero. If $n \gg_{\epsilon,d} 1$ and $\kappa \ll_{\epsilon,d} 1$, then by Corollary 6.4

(35) \[ \sum_{v \in S' \setminus S} r_v \lambda_{\text{crit},v}(f) - N_v \geq -\epsilon h_{\text{crit}}(f). \]

Write

$$\eta_v(P) = \log \max \left\{ |x_1|_v, |m_1 - x_1|_v, \ldots, |m_k - x_k|_v, -\sum_{j=1}^{k} m_j \right\}.$$ 

Combining (35) with (32) and (33) yields

$$h(P) - \text{rad}(P) \geq \sum_{v \in S} r_v \eta_v(P) - N_v + \sum_{v \in (S' \setminus S)} r_v \eta_v(P) - N_v$$

$$\geq \left( \sum_{v \in S} r_v \eta_v(P) - N_v \right) - \epsilon h_{\text{crit}}(f)$$

$$\geq \left( \sum_{v \in S} \frac{1}{2} r_v \lambda_{\text{crit},v}(f) \right) - \epsilon h_{\text{crit}}(f)$$

$$\geq \left( \frac{1 - \epsilon}{2} \right) \sum_{v \in M_K \setminus M_d} r_v \lambda_{\text{crit},v}(f) - \epsilon h_{\text{crit}}(f)$$

$$\geq \frac{(1 - 4\epsilon)}{2} h_{\text{crit}}(f)$$

(36)

when $n \gg_{\epsilon,d} 1$ and $\kappa \ll_{\epsilon,d} 1$. For $h_{\text{crit}}(f) \gg_{\epsilon} 1$, this contradicts Conjecture 2.1.

Finally, suppose $h_{\text{crit}}(f)$ lies below this bound. Then $f$ has a bounded number of places of bad reduction. If $K$ is a number field, then by [17], $h_{\text{crit}}(f) \asymp h_{M_d}(f)$, where $h_{M_d}$ is the height associated to an embedding of the moduli space $M_d$ of degree $d$ rational functions into projective space. Thus Northcott’s Theorem completes the proof. If $K$ is a function
field, and $f$ has at least one place of bad reduction, [8, Main Theorem] completes the proof. Otherwise, $f$ is isotrivial [1, Theorem 1.9]. □

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