Image objects detection with local topological characteristics, obtained by two-dimensional variations

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Abstract. I proposed method for producing local topological image characteristics on the base of two-dimensional variations. This method uses variations-based estimates and forms the metrical parameters of the objects of observed scene. I offered new concepts of the object size index and the convexity amplitude indices up and down. I stated algorithm for calculation of these characteristics. Application of these indices renders the new approach for object detection in an image. I demonstrated results of object detection in real gray scale and color images.

1. Introduction

Variation is simple but informative metric characteristic of variability and complexity of 1D-function. Considering that function \( f(x) \) reflects some real physical value (e.g., energy or density), it is natural to assume that \( f(x) \) is bounded in observed interval \([a, b]\) and has a finite number of first-order discontinuity points. Then, \( f(x) \) is a function of limited total variation, which, by definition, is the following value:

\[
V^x_0(f(x)) = \sup_{D \in [a,b]} \sum_{k=0}^{K-1} \| f(x_{k+1}) - f(x_k) \|,
\]

i.e., the exact upper boundary of partitioning \( D \) in the interval \([a, b]\).

In the two-dimensional (and all the more, in the \( n \)-dimensional) case, the difficulties appear with the formulation of variation [1][2]. There were proposed many definitions for multivariable variation (total variation): Arzelá, Frechet, Tonelli, Vitali, et al. [3][4][6]. These variations are constructed as some generalizations of 1D-variation and for the analyzed function give only one value, basing, somehow or other, on the modulo of gradient of function at a point. Therefore, despite of the differences in the definitions, the behaviors and the values of above variances are similar.

Kronrod [5] proposed an alternative variation definition, based on the concept of the level set of a function. For \( n \)-dimensional functions this approach was extended by Vitushkin [6]. They demonstrate that a function of \( n \) variables should be characterized not one, but \( n \) independent functionals. An application of Kronrod variations in image processing tasks was studied in [7][8], where advantage of this approach was demonstrated.

One of the substantial properties of two-dimensional variations is that they are global function indices in analyzed domain. This drawback of two-dimensional variations complicates their use for the
decision of the problems requiring local estimates. Nevertheless the analysis of two-dimensional variations reveals that some useful intermediate data arise during the calculation. These data can be separated, modified, and used for the analysis of the local features and objects of an image. This paper is aimed at solution of this problem.

Below we demonstrate the possibility of application of the two-dimensional variation apparatus for formation and analysis of not only global but also local topological characteristics of two-dimensional signals (images). We introduce the concept of indices of the size and the amplitude of convexities upwards and downwards and also propose an approach to formation of these indices. We demonstrate the techniques for application of these indices for detection of noise and objects of different dimensions in images. Theoretical conclusions are confirmed experimentally.

### 2. 2D-variations as an image analysis instrument

Somehow or other, the variations are based on the absolute value of the function gradient at a point. Therefore, their values and behavior are similar. All of them come to the following principle. Some unique functional is defined, which finiteness guarantees that 2D-function satisfies some properties, similar to ones of 1D-function [6]. However, the list of these properties is much poor then the collection of 1D-function properties. Moreover, depending on the selected coordinate system, some ambiguity of multivariate variations appears.

The generalization of conclusions and theorems, successfully formulated for continuous functions on the base of mentioned above total variations, have led to the following deduction. The function of \( n \) variables should be characterized not by one, but by \( n \) functionals, being independent in a certain sense. Kronrod [5] substantiated this thesis in studying two-variable functions. He proposed using two functionals, that base on the concept of the level sets of a function and not use its gradient characteristics. These functionals are defined as follows:

\[
\begin{align*}
  w_1(f) &= \int_{-\infty}^{\infty} v_0(e_t) \, dt \\
  w_2(f) &= \int_{-\infty}^{\infty} v_1(e_t) \, dt .
\end{align*}
\]

(2)

Here, set \( e_t \) is the \( t \)-level of function \( f(x,y) \), i.e., the set of the points \((x,y)\), where \( f(x,y) = t \); \( v_0(e_t) \) is the number of components of the set \( e_t \); and \( v_1(e_t) \) is the length of \( e_t \) set (according to Hausdorff). The \( w_1(f) \) variation, in certain sense, is not metric but topological characteristic of the function \( f \); it bases on the concept of connectivity and remains unchanged under the homeomorphism. In a case of finite \( v_0(e_t) \), the \( v_1(e_t) \) value may be interpreted as the total length of \( e_t \) components. It may be demonstrated that the estimate \( w_2(f) \) is analogous to other discussed total variations. Further Vitushkin [6] formulates felicitous definition for variation of a set, and this approach, also based on the consideration of the level set, was successfully expanded to multivariable functions.

Therefore, Kronrod variation (2) stand apart in a row of variations of multidimensional functions because it does not use any gradient characteristics of a function and, as a result, gives the values of several (rather than one) functionals.

Concerning the variation value \( w_2(f) \), it was proved in [9] that for continuously differentiable function \( f(x,y) \) the following equality is true:

\[
  w_2(f) = \int_{D} \left| \text{grad}(f(x,y)) \right| \, dx \, dy ,
\]

i.e. \( w_2(f) \) is the integral of modulo of the gradient of the function \( f(x,y) \) on the definitional domain.

For continuous functions multidimensional variations are formulated by means of the least upper bounds on the set of possible partitioning of the function domain by secant hyper planes into elementary parallelepipeds. If the function is discrete, its minimum partitioning is bounded below by the accuracy of representation. Therefore, formulas of variations for discrete functions are only the analogues and approximations of the variations, formulated for continuous functions. In some cases, this may lead to certain inaccuracies. Furthermore, not all of the operations can be applicable to a discrete function without losing the information, e.g., rotation of the function definition area through an arbitrary angle, or compression either in the domain area or in the space of values.

In a case of discrete function \( f(i,j) \), 2D-variations (2) are expressed by the formulas:
\[ w_1(f) = \sum_{i=1}^{T} v_0(e_i)/T \quad \text{and} \quad w_2(f) = \sum_{i=1}^{T} v_1(e_i)/T , \]

where \( T \) is the total number of possible values of function \( f(i,j) \); for an image, it is the number of brightness levels. The calculation of \( w_1 \) and \( w_2 \) values on bounded domain \( D \) assumes that domain \( D \) itself is also taken into account as a separate component. This leads to the following contradiction. As obvious, for a constant value function \( f(i,j) = \text{const} \), the \( w_1 = w_2 = 0 \) condition must be true, but, according to (3), \( w_1(f) = 1 \) and \( w_2(f) = P(D) \), i.e. equals to the perimeter of domain \( D \). To eliminate this contradiction, the formulas (3) should be modified as follows:

\[ w_1(f) = \left( \sum_{i=1}^{T} v_0(e_i)T \right) - 1 \quad \text{and} \quad w_2(f) = \left( \sum_{i=1}^{T} v_1(e_i)T \right) - P(D) . \]  

### 3. Generation of image topological characteristics

Let continuous function \( f(x,y) \) is specified on two-dimensional variety \( D \) and has finite number of components of the set \( e_i \) in each of its \( t \)-level. Assume the \( t \)-level set of this function \( e_i \), as containing \( v_i(e_i) \) nonintersecting components. Let consider inner regions \( c_i(i) \) of every \( i \)-component in \( t \)-level: \( c_i(i), 1 \leq i \leq v_i(e_i) \).

We introduce the convexity index parameter \( z(c_i(i)) \) for the region \( c_i(i) \). It is attached to every image point \((x,y)\) as the value \( z(c_i(i)) = z(x,y) \left( x \right) \left( y \right) \chi_{c_i(i)} \), on the assumption that

\[ z(x,y) \left( x \right) \left( y \right) \chi_{c_i(i)} = 1, \quad \text{if} \quad (f(x,y) - t) \geq 0, \quad \text{and} \]
\[ z(x,y) \left( x \right) \left( y \right) \chi_{c_i(i)} = 0, \quad \text{if} \quad (f(x,y) - t) < 0. \]

Let some function \( a(c_i(i)) \geq 0 \) reflects the size of the region \( c_i(i) \), and \( a(x,y) = a(c_i(i)) \) for \( (x,y) \chi_{c_i(i)} \). Note, that “size” may be any of quantifiable region estimates, for example: an area, border length, maximum diameter, etc. Also introduce some nonnegative function of size \( s(a) \geq 0 \) and construct in \( D \) the following functions \( p(x,y) \) and \( n(x,y) \) of positive and negative convexity:

\[ p_{x,y} = z(x,y)s(a(x,y)) \quad \text{and} \quad n_{x,y} = (1-z(x,y))s(a(x,y)). \]  

As one can see, \( p_{x,y} > 0 \) for the components where \( f(x,y) \) is convex upwards, and \( n_{x,y} = 0 \) for the components where \( f(x,y) \) is convex downwards. On the contrary, \( n_{x,y} > 0 \) for the components, where \( f(x,y) \) is convex downwards, and \( n_{x,y} = 0 \) for the components, where \( f(x,y) \) is convex upwards. Integrating the functions \( p_{x,y} \) and \( n_{x,y} \) on \( t \), define the convexity indices upward and downward at each point \((x,y) \chi D \) as:

\[ p(x,y) = \int_{-\infty}^{\infty} p_{x,y} \, dt \quad \text{and} \quad n(x,y) = \int_{-\infty}^{\infty} n_{x,y} \, dt . \]  

Or, for the discrete case:

\[ p(x,y) = \sum_{i=1}^{T} p_{x,y} / T = \sum_{i=1}^{T} z_{x,y} s(a(x,y)) / T , \quad \text{and} \]
\[ n(x,y) = \sum_{i=1}^{T} n_{x,y} / T = \sum_{i=1}^{T} (1-z_{x,y}) s(a(x,y)) / T . \]  

The functions \( p(x,y) \) and \( n(x,y) \) are, per se, metric characteristics of local topological features of analyzed function \( f(x,y) \), basing on the function of sizes \( s(a) \). The obtained functions can be named: \( p(x,y) \) as the size and amplitude index of upward convexity, and \( n(x,y) \) as the size and amplitude index of downward convexity.

It is important to note, that in contrast to two-dimensional functionals (2)–(4), which are global characteristics of \( f(x,y) \), the functions \( p(x,y) \) and \( n(x,y) \) are local characteristics of the function, and these values relates to every coordinate point \((x,y)\). Therefore these estimates can be used for the analysis of local image features, in particular, for detecting the objects and the noise [10].
4. Detection of the noise and the objects

4.1. Pulse noise definition

Real images can be distorted by various types of noise. Many explorations were assigned to the problem of pulse noise detection and removing; the overall question goes far beyond the border of this work. Now we limit our consideration only on the task of detecting this kind of defects [10].

The model of image distortion by pulse noise is enough simple. The value of each image element \( f(x,y) \) accidentally (with the probability \( p \)) is replaced by random value \( \xi(x,y) \). Therefore, the distortion function is specified as follows:

\[
\begin{align*}
    f(x,y) &= f_0(x,y) \quad \text{with probability } (1-p) \text{ for undistorted pixel;} \\
    &= \xi(x,y) \quad \text{with probability } p \text{ for distorted pixel.}
\end{align*}
\]

(8)

Without essential limitation one may consider that the pulse noise values \( \xi(x,y) \) are distributed uniformly in the range \([1,T]\). As it was noted in [10], the task of pulse noise filtering consists in two stages: detection of distorted pixels and correction of these elements. The most difficult and crucial is the noise detection stage.

Note, that pulse noise appears in \( t \)-level set \( e_t \), as independent components of one-pixel size, so they may be considered as the components with minimal possible dimension and perimeter.

4.2. Object and noise detection algorithm

An extremely important role in the problem of noise and object detection is played by size function \( s(a) \) in (5–7). The form of this function allows one to control selection of objects by their size (both on linear and on area estimates).

The simplest way to construct the function of sizes \( s(a) \) is threshold splitting of the range of possible values to nonzero and zero elements. For example:

\[
s(a) = 1, \quad \text{if } a \leq r, \quad \text{and} \quad s(a) = 0 \quad \text{otherwise.}
\]

(9)

The choice of \( r = 1 \) means the pulse noise detection algorithm.

More general is three-range variant of \( s(a) \) function:

\[
s(a) = 1, \quad \text{if } r_1 \leq a \leq r_2, \quad \text{and} \quad s(a) = 0, \quad \text{if } a < r_1 \text{ or } a > r_2.
\]

(10)

The boundaries \([r_1,r_2]\) define the range of object sizes to be detected.

The selection of \( r_1 \) and \( r_2 \) values and the clipping threshold for \( p(x,y) \) and \( n(x,y) \) functions allows to construct the indicator function for detecting the noise and the objects. Of course, it is possible to build more complicated function of sizes \( s(a) \), for example multi-range ones.

4.3. Application to color and multispectral images

Solution of the object detection problem in grayscale images can be expanded to color and multispectral images. There are available two variants of application of obtained above results.

(i) Compose the most informative image component, for example, the first term in principal components decomposition (Hotelling transformation) [11][13], or the brightness (or some another) component of appropriate color image presentation. According to given function of sizes \( s(a) \) in (10), mark the necessary objects (regions) in this component and remove from the image all others regions.

(ii) Detect the objects in every component of multispectral image separately, using only the lower limitation in size \( r_1 \) of function \( s(a) \) in (10). Combine the boundaries, obtained for all components, to common map of regions. Remove from this map all the areas which dimensions are less then \( r_1 \) or greater then \( r_2 \) in (10).

The further operations are the same. Find the average values of color (multispectral) vectors for every of detected regions. Having the reference color characteristic (color vector) for the regions of interest or obtaining it from some chosen typical region, separate the detected areas. This means the removal of the regions, which color characteristics are distanced far from the etalon color vector.
5. Detection examples

5.1. Pulse noise detection
Figure 1 demonstrates the example of noise detection. Image (a) is an image distorted by pulse noise (8) with probability \( p = 0.03 \); MSD of distorted image from the original one equals to 13.99 gray scales. In image (b) the pixels, detected by the algorithm (9) with \( r = 1 \), were restored; MSD of restored image from the original one equals to 0.87 gray scales. Image (c) represents 4-times contrasted difference between restored (b) and non distorted source images.

In the image (c), one can see only a small portion of undetected noisy pixels. One of the reasons, that the algorithm (9) did not detect these pixels, is the following. Due to the randomness of noisy pixels location, some of them can be neighbor and combine groups. If the values \( \xi \) in a group are close, these pixels constitute some false object of two or more pixels, which area exceed the area limit, and therefore they cannot be detected. This defect can be avoided with iterative process, when after the first detection step with \( r = 1 \), the second iteration accomplishes with \( r = 2 \) or more. The other reason is that some bugs tightly adjoin to sharp edge of lengthy objects. If the difference between the bug and object values is small, then such bug, per se, spatially joins to this object, distorting its shape. In such a case, this bag cannot be detected without morphological analysis.

![Figure 1. Pulse noise detection: (a) distorted image (noise 3%); (b) detected noisy pixels restored; (c) the difference between (b) and not distorted source image (4 times contrasted).](image)

5.2. Object detection
Basing on three-range variant of \( s(a) \) function in (10), it is possible to construct global area-based object detection algorithm as an alternative to local-based analysis one [12]. In figure 2 the examples of object detection are demonstrated with different ranges \( (r_1 \text{ and } r_2) \) of the function \( s(a) \).

Image (a) is source image (air photo) of 512x512 pixels, and (b)–(f) are indicator functions of detected objects. Small-sized objects (range 20–80 pixels) with downward convexity are displayed in the picture (b); objects of the same range with upward convexity are in the picture (c). Middle-sized objects (range 1000–3000 pixels) are, analogously, in the pictures (d) and (e). Large-sized objects of downward convexity (range 12000–20000 pixels) are in the picture (f).

5.3. Object detection in color image
Figure 3 demonstrates the example of object detection in a color image from Adobe album. The map of spots in giraffe’s skin is unique. The problem to solve is formation of a catalogue of animal spot patterns for identification of individuals. The object detection method, discussed above, can be used for its decision. Figure 3, a is the brightness component of source color image; image (b) is the detected set of downward convexity areas in the range 25–2000 pixels.

The further separation of the detected areas carries out using the probability distribution of image elements depending on their chromaticity-saturation parameters. These distributions in Maxwell triangle space [13][14] for the images (a)–(c) in figure 3 are presented in pictures (d)–(f). The power of darkness in (d)–(f) is proportional to the probability of image pixels with corresponding chromaticity-saturation value. The separation was realized by localizing in distribution (e) the region, corresponding
to average parameters of desired spots, and subsequent removing from the image (b) those areas, which color mapping of average value exceed the boundaries of established color region.

Picture (f) is the resulting distribution for residuary areas. Image in figure 3,c demonstrates the final set of detected areas after the separation on corresponding color characteristics.

![Figure 2. Detection of objects basing on their area: (a) source image; small range (20–80), downward (b) and upward (c) convexities; middle range (1000–3000), downward (d) and upward (e) convexities; (f) large range (12000–20000), downward convexity.](image1)

![Figure 3. Object detection in color image: (a) brightness component; (b) detected areas of downward convexity in the range 25–2000 pixels; (c) the result of separation; (d), (e), and (f) are corresponding chromaticity–saturation distributions (R^G_B) for the images (a)–(c).](image2)

6. Conclusions
We investigated the possibilities of application of two-dimensional variations for the analysis of both global and local image characteristics. A technique for formation of the metric parameters of local topological features of images has been developed. The concept of indices of the object size and the amplitude of the convexity upwards and downwards has been introduced. The algorithm for calculation of these characteristics has been developed.
Methods for using the convexity size and amplitude indices for detecting the objects by their area have been proposed. The theoretical conclusions have been illustrated by the experiments on pulse noise and object detection in grayscale and color images.

The obtained results demonstrate high efficiency of the approach based on two-dimensional variations for constructing image analysis algorithms.

7. References
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