Jacobi Forms of Degree One
and Weil Representations

Nils-Peter Skoruppa

Abstract
We discuss the notion of Jacobi forms of degree one with matrix index, we state dimension formulas, give explicit examples, and indicate how closely their theory is connected to the theory of invariants of Weil representations associated to finite quadratic modules.

1 Jacobi forms of degree one

Jacobi forms of degree one with matrix index $F$ gained recent interest, mainly due to applications in the theory of Siegel and orthogonal modular forms, and in the geometry of moduli spaces. Of particular interest among these Jacobi forms are those of critical weight, i.e. those whose weight equals $(\text{rank}(F) + 1)/2$. There are no Jacobi forms of of index $F$ and weight strictly less than $\text{rank}(F)/2$, and for weights strictly greater than $(\text{rank}(F) + 3)/2$ we have at least an explicit and easily computable dimension formula (see Theorem 1 below).

From the point of view of algebraic geometry Jacobi forms of degree one are holomorphic functions $\phi(\tau, z)$ of a variable $\tau$ in the Poincaré upper half plane $\mathbb{H}$ and of complex variables $z \in \mathbb{C}^n$ such that, for fixed $\tau$, the function $\phi(\tau, \cdot)$ is a theta function on the algebraic variety $\mathbb{C}^n/\Lambda_\tau$, where $\Lambda_\tau$ denotes the lattice $\mathbb{Z}^n \tau + \mathbb{Z}^n$, and such that, for any $\tau$ and all $A$ in a subgroup of finite index in $\text{SL}(2, \mathbb{Z})$, the theta function $\phi(\tau, \cdot)$ on $\mathbb{C}^n/\Lambda_\tau$ and $\phi(A\tau, \cdot)$ on the isomorphic torus $\mathbb{C}^n/\Lambda_{A\tau}$ are related by a simple transformation formula. Thus, for fixed $\tau$, the function $\phi(\tau, \cdot)$ corresponds to a holomorphic section of a positive line bundle on $\mathbb{C}^n/\Lambda_\tau$. The positive line bundles on $X_\tau = \mathbb{C}^n/\Lambda_\tau$ are (up to translation and isomorphism) parameterized by their Chern classes in $H^2(X_\tau, \mathbb{Z})$. It is not difficult to show that the cone of positive Chern classes in $H^2(X_\tau, \mathbb{Z})$ is in one to one correspondence with the set of symmetric,
positive definite matrices $F$ with entries in $\frac{1}{2}\mathbb{Z}$ via the map

$$F = (f_{p,q}) \mapsto \frac{1}{i\Im(\tau)} \sum_{p,q} f_{p,q} \, dz_p \wedge d\bar{z}_q,$$

where $z_p$ denote the coordinate functions associated to the canonical basis of $\mathbb{C}^n$. This explains the appearance and the nature of the matrix index in the formal definition of a Jacobi form, and it shows also that the usual theory of Jacobi forms omits the case of those indices $F$ where the diagonal entries of $F$ are not necessarily integral (as we shall do too in the following discussion; however, see the remarks after the formal definition of the notion *Jacobi form* in the Appendix). Recall that a symmetric, positive definite matrix $F$ is called *half integral* if it has half integral entries, but integers on the diagonal.

For the formal definition of the space $J_{k,F}(\Gamma, \chi)$ of Jacobi forms of integral weight $k$, of positive-definite half integral index $F$, on a subgroup $\Gamma$ of finite index in $\text{SL}(2, \mathbb{Z})$, and with character $\chi$ we refer the reader to the Appendix. The implicit restrictions made at this point can be relaxed: One can admit half integral $k$, semi-positive definite $F$ with half-integers on the diagonal and vector-valued Jacobi-forms. We suppress the discussion of these possible generalizations for not overloading this presentation by too many technical details; the interested reader is referred to [Sko 07] for a more general and elaborated treatment.

There is an explicit dimension formula for $J_{k,F}(\Gamma, \chi)$ if $k > (n + 3)/2$, where $n$ denotes the rank of $F$. For understanding why such a formula exists note that, for fixed $\tau$, the dimension of the space of holomorphic sections of the line bundle on $\mathbb{C}^n/\Lambda_\tau$ corresponding to Jacobi forms of index $F$ has dimension $\det(2F)$. A basis for the space of theta functions corresponding to this line bundle is given by the special theta functions

$$\vartheta_{F,x}(\tau, z) = \sum_{r \in \mathbb{Z}^n \mod 2F\mathbb{Z}^n} e\left(\tau \frac{1}{4} F^{-1} [r] + r^t z\right) \quad (x \in \mathbb{Z}^n).$$

Note that $\vartheta_{F,x}$ depends only on $x$ modulo $2F\mathbb{Z}^n$. Thus any Jacobi $\phi$ form of index $F$ can be written in the form

$$\psi(\tau, z) = \sum_{x \in \mathbb{Z}^n/2F\mathbb{Z}^n} h_x(\tau) \vartheta_{F,x}(\tau, z)$$

with functions $h_x$ which are holomorphic in $\mathbb{H}$. The invariance of $\phi$ under $\Gamma$ is, in this representation, reflected by the fact that the $h_x$ are the coordinates
of a vector valued modular form associated to the (dual of the) Weil representation $W(F)$ which we shall explain in section 2. If the weight of $\phi$ is $k$ then the weight of the corresponding vector valued modular form is $k - \frac{n}{2}$. This shows already that there are no nontrivial Jacobi forms of index $F$ and weight $k < \frac{3}{4}$. In any case we can apply the Eichler-Selberg trace formula or Shimura’s variant (based on the Lefschetz fixed point theorem) to obtain a dimension formula for vector valued modular forms [El-S 95, Sec. 4.3], [Sko 85, p. 100], and we can then deduce from this a general dimension formula for Jacobi forms.

**Theorem 1 ([Sko 07]).** Let $F$ be a half integral positive definite $n \times n$ matrix, let $k \in \mathbb{Z}$, let $\Gamma$ be a subgroup of $\text{SL}(2, \mathbb{Z})$ and let $\chi$ be a linear character of $\Gamma$ of finite order. Then one has

$$\dim J_{k,F}(\Gamma, \chi) = \dim M^{\text{cusp}}_{2, \frac{k}{2}} \otimes \mathbb{C}[\text{Mp}(2, \mathbb{Z})] \cdot X(i^{n-2k})^c$$

$$= \frac{k - \frac{n}{2} - 1}{12} \dim X(i^{n-2k}) + \frac{1}{4} \text{Re} \left( e^{\pi i (k - \frac{3}{2})/2} \text{tr}((S, w_S), X(i^{n-2k})) \right)$$

$$+ \frac{2}{3\sqrt{3}} \text{Re} \left( e^{\pi i (2k-n+1)/6} \text{tr}((ST, w_{ST}), X(i^{n-2k})) \right) - \sum_{j=1}^{r} (\lambda_j - \frac{1}{2}).$$

Here $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and, for any $A$ in $\text{SL}(2, \mathbb{Z})$, we use $(A, w_A)$ for the corresponding element of $\text{Mp}(2, \mathbb{Z})$ (whose precise definition is given in the Appendix). Moreover, $X(i^{n-2k})$ denotes the $\text{Mp}(2, \mathbb{Z})$-submodule of all $v$ in $W(F)^c \otimes \text{Ind}_{\Gamma}^{\text{SL}(2, \mathbb{Z})} \mathbb{C}(\chi)$ such that $(-1, i)v = i^{n-2k}v$, and the $\lambda_j$ are rational numbers $0 \leq \lambda_j < 1$ such that $\prod_{j=1}^{r}(t - e^{2\pi i \lambda_j}) \in \mathbb{C}[t]$ equals the characteristic polynomial of the automorphism of $X(i^{n-2k})$ given by $v \mapsto (T, 1)v$.

(There are some more technical explanations necessary if the reader wants to apply the theorem. In its statement we used the following notations from representation theory: $\mathbb{C}(\chi)$ is the $\Gamma$-module with underlying vector space $\mathbb{C}$ and with the action given by $(A, z) \mapsto \chi(A)z$, and $\text{Ind}_{\Gamma}^{\text{SL}(2, \mathbb{Z})} \mathbb{C}(\chi)$ denotes the $\text{SL}(2, \mathbb{Z})$-module induced by $\mathbb{C}(\chi)$. Moreover, $W(F)^c$ is the $\text{Mp}(2, \mathbb{Z})$-module with underlying vector space equal to the dual vector space of $W(F)$, and with the $\text{Mp}(2, \mathbb{Z})$-action $(A, f) \mapsto Af$, where $(Af)(v) = f(A^{-1}v)$ for all $v$ in $W(F)$. If the action of $\text{Mp}(2, \mathbb{Z})$ on $W(F)^c$ does not factor through $\text{SL}(2, \mathbb{Z})$ then, for forming the tensor product considered in the theorem, the induced representation has to be considered as $\text{Mp}(2, \mathbb{Z})$-module; otherwise we can form the tensor product with respect to $\mathbb{C}[\text{SL}(2, \mathbb{Z})]$. The action of $\text{Mp}(2, \mathbb{Z})$ on $W(F)$ factors through $\text{SL}(2, \mathbb{Z})$ if and only if $n$ is even since then
the sigma-invariant of the determinant module $D_F$ is a fourth root of unity by Milgram’s formula; cf. section 2 for an explanation of these terms.)

If $k \geq \frac{n}{2} + 2$ the theorem gives us a ready to compute formula since then the second term on the left hand side vanishes. There remain two weights where the theorem is of no help: $k = \frac{n}{2}$, $k = \frac{n}{2} + 1$ if $n$ is even, and $k = \frac{n+1}{2}$ (critical weight), $k = \frac{n+3}{2}$ if $n$ is odd. For the case $k = \frac{n}{2} + 1$ with even $n$ we do not know of any approach to set up a general trace formula; in fact, here the underlying vector valued modular forms are of weight one. The case $k = \frac{n+3}{2}$ with odd $n$ can be treated by methods similar to the one for critical weight, which we shall explain in a moment. The cases $\frac{n}{2}$ and $\frac{n+1}{2}$ for even and odd weights, respectively, can be reduced to purely representation theoretic questions concerning the Weil representation $W(F)$ (for the critical weight case we even have to assume additionally that the kernel of $\chi$ is a congruence subgroup).

Assume first of all that $n$ is even and $k = \frac{n}{2}$. Then the $h_x$ introduced above have weight 0 and are consequently constants. This leads to the following theorem.

**Theorem 2 ([Sko 07]).** There exists a natural isomorphism

$$J_{\frac{n}{2},F}(\Gamma, \chi) \cong W(F)^* \otimes_{\mathbb{C}[SL(2, \mathbb{Z})]} \text{Ind}_{\Gamma}^{SL(2, \mathbb{Z})} \mathbb{C}(\chi).$$

(Here $W(F)^*$ denotes the $\text{Mp}(2, \mathbb{Z})$-right module whose underlying vector space is the dual vector space of $W(F)$ and the $\text{Mp}(2, \mathbb{Z})$-action is given by $(f, A) \mapsto A^{-1} f$; cf the remark after Theorem 1.) Note that, for trivial $\chi$ and, say, $\Gamma$ equal to the full modular group, the right hand side of the stated isomorphism is isomorphic to the subspace $\text{Inv}(W(F))$ of $SL(2, \mathbb{Z})$-invariants of the representation $W(F)$. The question of describing these invariants or computing the dimension of the space of invariants is in general unsolved. We shall discuss this question in the second part of this article.

For odd $n$ and $k = \frac{n+1}{2}$, the critical weight, the description of $J_{k,F}(\Gamma, \chi)$ becomes even more subtle. Here the $h_x$ are modular forms of weight $\frac{1}{2}$. If we assume that the kernel of $\chi$ (and hence $\Gamma$) is a congruence subgroup the $h_x$ are modular forms on congruence subgroups. By a theorem of Serre-Stark the only modular forms of weight $\frac{1}{2}$ on congruence subgroups are theta series (more precisely, linear combinations of the null values $\vartheta_{m,x}(\tau, 0)$, where $x$ and $m$ run through the integers and positive integers, respectively). Based on this description, the decomposition of the $SL(2, \mathbb{Z})$-module of all modular forms of weight $\frac{1}{2}$ (on congruence subgroups) was explicitly derived in [Sko 85], [Sko 07]. From this description we can then finally deduce the following theorem.
Theorem 3 ([Sk07]). Let $\Gamma$ be a subgroup of finite index in $\text{SL}(2, \mathbb{Z})$, let $\chi$ be a linear character of $\Gamma$, and let $F$ be half integral of size $n$ and level $f$. Assume that, for some $m$, the group $\chi$ is trivial on $\Gamma(4m)$, and that $f$ divides $4m$. Then there is a natural isomorphism

$$J_{n/2,F}(\Gamma, \chi) \longrightarrow \bigoplus_{l|m, l/\text{squarefree}} \left(W(l \oplus F)^\dagger \right)^* \otimes_{\mathbb{C}[\text{SL}(2, \mathbb{Z})]} \text{Ind}_{\text{SL}(2, \mathbb{Z})}^\Gamma \mathbb{C}(\chi)$$

Here $W(l \oplus F)^\dagger$ denotes the $+1$-eigenspace of the involution $\iota$ on $W(l \oplus F)$ induced by the automorphism $(x,y) \mapsto (-x,y)$ of $D(l \oplus F)$.

(Recall that the level $f$ of $F$ is the smallest positive integer such that $f(2F)^{-1}$ is again half integral. For the notion of the determinant module $D_G$ associated to a half integral $G$ see section 2. By $l \oplus F$ we denote the matrix of size $(n+1) \times (n+1)$ consisting of the two blocks $l$ and $F$ centered on the diagonal and having zero entries otherwise.) We observe again that, for trivial $\chi$, the calculation of $J_{n/2,F}(\Gamma)$ is basically equivalent to the calculation of the subspaces $\text{Inv}(W(l \oplus F))$.

We conclude this section by some remarks concerning the graded algebra

$$J_{*,F}(\Gamma, \chi) = \bigoplus_{k \in \mathbb{Z}} J_{k,F}(\Gamma, \chi).$$

Here $J_{*,F}(\Gamma, \chi)$ may be considered as the subspace of functions on $\mathbb{H} \times \mathbb{C}^n$ spanned by the Jacobi forms in $J_{k,F}(\Gamma, \chi)$, where $k$ runs through all integers, and the proof of the direct sum decomposition is then an easy exercise. It is not hard to see from the dimension formula of Theorem 1 that the Hilbert-Poincaré series of this graded algebra is of the form

$$\sum_{k \in \mathbb{Z}} \dim J_{k,F}(\Gamma, \chi) x^k = \frac{p_F(x)}{(1 - x^4)(1 - x^6)},$$

where $p_F(x)$ is a polynomial. Indeed, for $k \geq \frac{n}{2} + 2$ the Hilbert-Poincaré series splits up into four sums $S_j = \sum_k a_j(k) x^k$ ($j = 1, 2, 3, 4$) according to the four terms on the right hand side of Theorem 1. Here $a_1(k)$ is a linear function in $k$ times a sequence which depends only on $k$ modulo 2, whence $S_1 = \text{polynomial}/(1 - x^2)^2$. Moreover $a_2(k)$, $a_3(k)$ and $a_4(k)$ depend only on $k$ modulo 4, 6 and 2, respectively, which yields $S_j = \text{polynomial}/(1 - x^{j'})$ with $j' = 4, 6, 2$, respectively. From this argument it is also easy to see that the degree of $p_F(x)$ is less than or equal to 12. We note that $p_F(1) = \det(2F) \cdot [\text{SL}(2, \mathbb{Z}) : \Gamma]$. Namely, multiplying the Hilbert-Poincaré series by
The value \( \dim X(i^n) + \dim X(i^n - 2) = \dim W(F)^c \otimes \text{Ind } \mathbb{C}(\chi) \). On the other hand, \( p_F(x)(1 - x)^2/(1 - x^4)(1 - x^6) \) becomes \( p_F(1)/24 \) for \( x = 1 \).

The explanation for the shape of the Hilbert-Poincaré series of \( J_{s,F}(\Gamma, \chi) \) is as follows. Multiplication by usual elliptic modular forms turns \( J_{s,F}(\Gamma, \chi) \) into a graded module over the graded ring \( M_\ast = \mathbb{C}[E_4, E_6] \), where \( E_4 \) and \( E_6 \) are the usual Eisenstein series of weight 4 and 6 on the full modular group. One may copy the proof in \cite{E-Z 85} to show that, in fact, \( J_{s,F}(\Gamma, \chi) \) is free over \( M_\ast \). If \( S \) is a system of homogeneous generators of \( J_{s,F}(\Gamma, \chi) \) as module over \( M_\ast \), and if \( s_j \) denotes the number of generators in \( S \) of weight \( j \) then \( p_F(x) = s_0 + s_1 x + \cdots + s_{12} x^{12} \). In particular, the rank of \( J_{s,F}(\Gamma, \chi) \) over \( M_\ast \) equals \( p_F(1) = \det(2F) | SL(2, \mathbb{Z}) : \Gamma | \).

As an example we consider \( J_{k,F} := J_{k,F}(SL(2, \mathbb{Z})) \) for half integral \( 2 \times 2 \) matrices with \( \det(2F) = p \) for primes \( p \neq 3 \). The dimension formula, for \( k \geq 3 \), becomes then

\[
\dim J_{k,F} = \frac{k - 2}{12} \cdot \frac{(p + (-1)^k)}{2} - \frac{1}{4} \left( \frac{-4}{k - 2} \right) \left( \frac{-2}{p} \right)
- \frac{1}{3} \left( \frac{k - 2}{3} \right) \left( \frac{\chi}{3} \right) + \frac{(-1)^k}{2} - \frac{h(p)}{2} + \frac{1 + (-1)^k}{4},
\]

where \( h(p) \) denotes the class number of the field \( \mathbb{Q}(\sqrt{-p}) \). The occurrence of the class number is due to the sum of the \( \lambda_j \neq 0 \) in the general dimension formula, which, for the binary \( F \) in question, becomes a sum over \( r_j/p \), where \( 0 < r_j < p \) runs through all quadratic residues modulo \( p \). For \( p = 3 \) we have to replace the third term on the right by \( 0, 0, -\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3} \) accordingly as \( k \) modulo 6 equals \( 0, 1, 2, 3, 4, 5 \), respectively, and \( h(p) \) has to be replaced by \( h(3) := \frac{1}{3} \). It is not hard to show that \( \dim J_{1,F} = 0 \) (by Theorem 2 and since \( W(F) \) decomposes into two nontrivial irreducible characters of dimension \( (p \pm 1)/2 \)).

Table \cite{Hilbert-Poincaré} lists the Hilbert-Poincaré polynomials \( p_F \) for the first primes \( p \), more precisely, it lists the polynomials \( \widetilde{p_F} = p_F - x^2 \dim J_{2,F} \). For the dimension of the Jacobi forms of weight 2 we have no clue to make a general statement. It can be computed, however, for each \( F \) as long as \( p \) is not too large.

Note that, for a fixed \( k \), the dimensions \( \dim J_{k,F} \) do only depend on \( \det(2F) \). This reflects the fact that, for binary \( F \) with prime discriminant, the isomorphism class of the determinant module \( D_F \), hence also the isomorphism class of \( W(F) \), depend only on \( \det(2F) \). Indeed, \( D_F \) is isomorphic to the quadratic module \( \mathbb{F}_p \) equipped with the quadratic form \( Q(x) = x^2/p \). See
section 2 for the terminology used in this paragraph.

For deducing the given dimension formula from Theorem 1 it is useful to note that the SL(2, \mathbb{Z})-action on \( W(F) \) factors through SL(2, \mathbb{F}_p) and \( W(F) \) decomposes into a \((p + 1)/2\) and \((p - 1)/2\) dimensional representation, which can be easily identified by consulting a character table of SL(2, \mathbb{F}_p).

For \( p = 3 \) we find

\[
p_F(x) = \tilde{p}_F(x) + cx^2(1 - x^4)(1 - x^6) = cx^{12} + x^9 - cx^8 + (c + 1)x^6 + x^4 + cx^2,
\]

which implies \( c = \dim J_{2,F} = 0 \). Accordingly, we have

\[
J_{s,F} = M_s \Psi_4 \oplus M_s \Psi_6 \oplus M_s \Psi_9.
\]

for the (up to a constant factor) unique Jacobi forms \( \Psi_k \in J_{k,F} \) \((k = 4, 6, 9)\).

To become more specific, choose \( 2F = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \). Then one has:

\[
\Psi_9(\tau, z_1, z_2) = \vartheta(\tau, z_1) \vartheta(\tau, z_1 + z_2) \vartheta(\tau, z_2) \eta^{15}(\tau),
\]

where

\[
\vartheta = q^{\frac{1}{4}}(\zeta^{\frac{1}{2}} - \zeta^{-\frac{1}{2}}) \prod_{n \geq 1} (1 - q^n)(1 - q^n \zeta)(1 - q^n \zeta^{-1}), \quad \eta = q^{\frac{1}{12}} \prod_{n > 1} (1 - q^n).
\]

Here we use \( q(\tau) = e^{2\pi i \tau} \) and \( \zeta(\tau) = e^{2\pi iz} \). The formula for \( \Psi_9 \) follows immediately from the transformation laws of the fundamental Jacobi form \( \vartheta \) (see the Appendix). The nice product formula is due to the fact that, for fixed \( \tau \), the theta function \( \Psi_9(\tau, \cdot) \) has to be a multiple of \( \vartheta_{F,e}(\tau, \cdot) - \vartheta_{F,-e}(\tau, \cdot) \), where \( \epsilon \in \mathbb{Z}^2 \) and \( \epsilon \not\in 2F \mathbb{Z}^2 \), since \( W(F)^c = (+1) \) is one dimensional. From this it is then

| \( p \) | \( \tilde{p}_F(x) \) |
|---|---|
| 3 | \( x^9 + x^6 + x^4 \) |
| 7 | \( x^{11} + x^{10} + x^9 + x^8 + x^7 + x^6 + x^4 \) |
| 11 | \( x^{11} + x^{10} + 2x^9 + 2x^8 + x^7 + 2x^6 + x^5 + x^4 \) |
| 19 | \( x^{11} + 2x^{10} + 3x^9 + 3x^8 + 3x^7 + 3x^6 + 2x^5 + 2x^4 \) |
| 23 | \( x^{12} + 3x^{11} + 3x^{10} + 4x^9 + 4x^8 + 3x^7 + 3x^6 + x^5 + x^4 \) |
| 31 | \( x^{12} + 3x^{11} + 4x^{10} + 5x^9 + 5x^8 + 5x^7 + 4x^6 + 2x^5 + 2x^4 \) |
| 43 | \( 2x^{11} + 4x^{10} + 6x^9 + 7x^8 + 7x^7 + 7x^6 + 5x^5 + 4x^4 + x^3 \) |
| 47 | \( 2x^{12} + 5x^{11} + 6x^{10} + 8x^9 + 8x^8 + 7x^7 + 6x^6 + 3x^5 + 2x^4 \) |

Table 1: Hilbert-Poincaré polynomials for binary \( F \) with \( \det(2F) = p \).
clear that $Ψ_9(τ, ·)$ vanishes along $z_1 = 0$, $z_1 + z_2 = 0$ and $z_2 = 0$, respectively, and hence has to be divisible (in the ring of holomorphic functions) by $η(τ, z_1)η(τ, z_1 + z_2)η(τ, z_2)$.

For $Ψ_4$ and $Ψ_6$ we find the closed formulas

$$Ψ_k(τ, z_1, z_2) = \sum_{n,a,b} q^n e^{2πi(az_1+bz_2)} ν_{a^2-bb^2} \sum_{st=3n-a^2-ab-b^2} s^{k-2} \left[ \left( \frac{s}{3} \right) - \left( \frac{t}{3} \right) \right].$$

In the last formula we have $ν_N = \frac{1}{2}$ or $ν_N = 1$ accordingly as $N$ is 1 or 0 modulo 3. The outer sum is over all integers $n, a, b$ such that

$$n - (a^2 - ab + b^2) = n - F[(a, b)^t]/4 \geq 0,$$

the inner sum is over all positive integers $s, t$ satisfying the given identity, and for $a_k(N) := \sum_{st=N} s^{k-2} \left[ \left( \frac{s}{3} \right) - \left( \frac{t}{3} \right) \right]$ we use the convention $a_k(0) = \frac{1}{2}L(2-k, \left( \frac{N}{3} \right))$, in particular, $a_4(0) = -\frac{1}{8}$ and $a_6(0) = \frac{1}{3}$.

The formulas for $Ψ_4$ and $Ψ_6$ can be deduced as follows. It is well-known \cite{Kl46} Thm. 1, p. 333] that $η_{F,0}$ lies in $J_{1,F}(Γ_0(3), \left( \frac{N}{3} \right))$, hence the sum $η_{F,0} + η_{F,e} + η_{F,-e} (e \in \mathbb{Z}^2, e \not\equiv 2F\mathbb{Z}^2)$ defines an element of $J_{1,F}(Γ^0(3), \left( \frac{N}{3} \right))$ (as follows from $S^{-1}Γ_0(3)S = Γ^0(3)$, and from the action of the $S$-matrix on the Weil representation $W(F)$ (see section 2)). But then the function $h := h_0 + h_e + h_{-e}$ derived from the decomposition $Ψ_k = h_0η_{F,0} + h_eη_{F,e} + h_{-e}η_{F,-e}$ is a modular form of weight $k - 1$ on $Γ^0(3)$ of nebentypus $\left( \frac{N}{3} \right)$ with a Fourier development of the form $h = \sum_{n=0, -1 \mod 3} a(N) q^{N/3}$. For $k = 4, 6$ the spaces of such modular forms are one dimensional, respectively; in fact, these spaces are spanned by the Eisenstein series with Fourier coefficients $a_k(N)$.

(Actually, for even $k$, the map $\sum h_x η_{F,x} \mapsto \sum h_x$ defines an isomorphism of $J_{k,F}$ and the space of modular forms of weight $k$ on $Γ^0(3)$ with real quadratic nebentypus and with the aforementioned special Fourier development. This will be proved elsewhere. In particular, the formulas for $Ψ_k$ define an element of $J_{k,F}$ for all even $k$.) For calculating $Ψ_k$ from $h$ note that $h_e = h_{-e}$ as follows from the evenness of $Ψ_k(τ, z)$ in $z$, as follows in turn from the invariance of $Ψ_k$ under the modular transformation $-1$.

### 2 Weil representations

The metaplectic cover $Mp(2, \mathbb{Z})$ of $SL(2, \mathbb{Z})$ acts on $Θ_F$, the space spanned by the $\det(2F)$ many theta series $η_{F,x}$ (see the Appendix for the definition of the metaplectic cover). This is simply the algebraic restatement of the well-known transformation formulas for theta series relating, for any $A$ in the
modular group, the series $\vartheta_{F,x}(A\tau,z)$ to the series $\vartheta_{F,x}(\tau,z)$. These transformation formulas stem back to Jacobi and were treated by many authors. However, a good reference suited to our discussion is [Kl 46]. To focus on the algebraic nature of the associated representation of $\text{Mp}(2,\mathbb{Z})$ we use the notion of Weil representations associated to finite quadratic modules.

Recall that a finite quadratic module $M$ is a a finite abelian group $M$ endowed with a quadratic form $Q_M : M \rightarrow \mathbb{Q}/\mathbb{Z}$. Here a quadratic form is, by definition, a map such that $Q_M(ax) = a^2Q_M(x)$ for all $x \in M$ and all integers $a$, and such that the application $B_M(x,y) := Q_M(x+y) - Q_M(x) - Q_M(y)$ defines a $\mathbb{Z}$-bilinear map $B_M : M \times M \rightarrow \mathbb{Q}/\mathbb{Z}$. All quadratic modules occurring in the sequel will be assumed to be non degenerate, i.e. we assume that $B_M(x,y) = 0$ for all $y$ is only possible for $x = 0$.

**Definition (Weil representation).** The Weil representation $W(M)$ associated to a finite quadratic module $M$ is the $\text{Mp}(2,\mathbb{Z})$-module whose underlying vector space is $\mathbb{C}[M]$, the complex vector space of all formal linear combinations $\sum x \lambda(x) e_x$, where $e_x$, for $x \in M$, is a symbol, where $\lambda(x)$ is a complex number and where the sum is over all $x$ in $M$, and where the action of $(T,w_T)$ and $(S,w_S)$ on $\mathbb{C}[M]$ is given by the formulas

$$(T, w_T) e_x = e(Q_M(x)) e_x$$

$$(S, w_S) e_x = \sigma |M|^{-\frac{1}{2}} \sum_{y \in M} e_y e(-B_M(y,x)),$$

respectively, where

$$\sigma = \sigma(M) = |M|^{-\frac{1}{2}} \sum_{x \in M} e(-Q_M(x)).$$

That the action of the generators

$$(T, w_T) = \left( \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, 1 \right), \quad (S, w_S) = \left( \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \sqrt{\tau} \right)$$

of $\text{Mp}(2,\mathbb{Z})$ can indeed be extended to an action of $\text{Mp}(2,\mathbb{Z})$ on $\mathbb{C}[M]$ is discussed in more detail in [Sko 07]. For the moment we do not know any reference for this fact and shall explain its (simple) proof elsewhere [Sko 08].

The action of $\text{Mp}(2,\mathbb{Z})$ factors through $\text{SL}(2,\mathbb{Z})$ if and only if $\sigma^4 = 1$ [Sko 08]; in general, $\sigma$ is an eighth root of unity. That the actions of $(S, w_S)$ and $(T, w_T)$ (or rather $S$ and $T$) define an action of $\text{SL}(2,\mathbb{Z})$ if $\sigma^4 = 1$ is well-known (cf. e.g. [N 76]).

The standard example for a quadratic module is the determinant module $D_F$ of a symmetric non degenerate half integral matrix $F$. The quadratic
module has \( D_F = \mathbb{Z}^n / 2F\mathbb{Z}^n \) as underlying abelian group, and the quadratic form on \( D_F \) is the one induced by the quadratic form \( \mathbb{Z}^n \ni x \mapsto \frac{1}{4} F^{-1}[x] \).

The \( \text{Mp}(2, \mathbb{Z}) \)-module \( W(F) := W(D_F) \) is the one occurring in the formulas of the first section. The associated right module \( W(F)^* \) is isomorphic (as \( \text{Mp}(2, \mathbb{Z}) \)-module) to \( \Theta_F \). An isomorphism is given by the map

\[
W(F)^* \rightarrow \Theta_F, \quad \lambda \mapsto \sum_{x \in \mathbb{Z}^n / 2F\mathbb{Z}^n} \lambda(e_x) \vartheta_{F,x}.
\]

That this map is a \( \text{Mp}(2, \mathbb{Z}) \)-right module isomorphism is clear from from the formulas for the action of \( (S, w_S) \) and \( (T, w_T) \) on \( W(F) \) as given above and on \( \Theta_F \) as given in [Ki 46, (1.12), p. 320].

In our context, the most important notion related to a quadratic module is its associated space of invariants.

**Definition** (Invariants associated to a quadratic module). For a quadratic module \( M \), we use \( \text{Inv}(M) \) for the subspace of elements in \( W(M) \) which are invariant under the action of \( \text{Mp}(2, \mathbb{Z}) \).

We already saw in section [1] that \( J_{\frac{n}{2}, F}(\text{SL}(2, \mathbb{Z})) \), for index \( F \) of even size \( n \), is isomorphic to \( \text{Inv}(D_F) \) (cf. the remark after Theorem [2]) and that a similar statement holds true for the critical weight case \( J_{\frac{n+1}{2}, F}(\text{SL}(2, \mathbb{Z})) \) if \( n \) is odd. In fact, it can be shown that, for any \( \Gamma \) and character \( \chi \) which is trivial on a congruence subgroup, the spaces \( J_{\frac{n}{2}, F}(\Gamma, \chi) \) or \( J_{\frac{n+1}{2}, F}(\Gamma, \chi) \), accordingly as \( n \) is even or odd, are isomorphic to certain natural subspaces of invariants of quadratic modules [Sko 07]. Thus, the clue to Jacobi forms of critical weight (and weight \( \frac{n}{2} \) for even \( n \)) is the study of invariants of Weil representations. A rather involved illustration for this statement can be found in [Sko 07], where the spaces \( J_{1, m}(\text{SL}(2, \mathbb{Z}), \varepsilon^8) \) are explicitly determined for all \( m \) (see the Appendix for the character \( \varepsilon \)).

The question for the invariants of a quadratic module seems to be subtle. We do not even know a reasonable (say, easily computable) criterion to answer in general the question which quadratic modules possess invariants and which not. However, it seems that the nature of invariants depend on the Witt class of the underlying quadratic module.

For stating this more precisely note that, for an isotropic submodule \( N \) of a quadratic module \( M \), the dual module \( N^\perp = \{ y \in M : B_Q(N, y) = 0 \} \) contains \( N \) and \( Q_M \) induces a quadratic form \( Q_M \) on \( N^\perp / N \). In fact, \( N^\perp / N \), equipped with the quadratic form induced by \( Q_M \) becomes a (non degenerate)

\(^1\)N is called *isotropic submodule of M* if \( Q_M(x) = 0 \) for all \( x \in N \).
quadratic module. We call $M$ Witt-zero if $M$ contains an isotropic self-dual module $N$ i.e. an isotropic module $N$ such that $N = N^\perp$.

Two quadratic modules $M_1$ and $M_2$ are called Witt-equivalent if they contain isotropic submodules $N_1$ and $N_2$, respectively, such that $N_1^\perp/N_1$ and $N_2^\perp/N_2$ are isomorphic as quadratic modules. It is not hard to check that this defines indeed an equivalence relation. The set of equivalence classes $[M]$ of all (non degenerate finite) quadratic modules $M$ becomes a group via the operation $[M] + [N] := [M \perp N]$, where $M \perp N$ is the quadratic module with underlying group $M \oplus N$ and quadratic form $Q(x \oplus y) = Q_M(x) + Q_N(y)$. Adopting the notation in [Sch 84] we denote this group by $WQ$. Similarly, for any prime $p$, we can define the Witt group $WQ(p)$ of all (finite non degenerate) quadratic modules whose order is a power of $p$. If $p$ is odd, the group $WQ(p)$ is of order 4, whereas $WQ(2)$ is of order 16 [Sko 08], [Sch 84] (This is not hard to prove: If $N$ is a maximal isotropic submodule of $M$ then $P := N^\perp/N$ is anisotropic, i.e. satisfies $Q_P(x) \neq 0$ for all $x \neq 0$, and $M$ and $P$ span the same Witt class; thus it suffices to count the isomorphism classes of anisotropic quadratic modules which are $p$-groups).

Every quadratic module $M$ possesses the primary decomposition

$$M = \perp_p M(p).$$

Here $p$ runs through all primes and $M(p)$ denotes the $p$-part of $M$, i.e. the submodule of all $x$ in $M$ whose order is a power of $p$, equipped with the restriction of $Q_M$. The primary decomposition gives rise to a canonical isomorphism

$$WQ \cong \prod_p WQ(p).$$

Moreover, via the functorial identity $W(M \perp N) = W(M) \otimes W(N)$ it implies the following lemma.

**Lemma.**

$$\text{Inv}(M) \cong \bigotimes_{p|l} \text{Inv} \left( M(p) \right).$$

Thus, to study Weil invariants of quadratic modules we can restrict to quadratic modules which are $p$-groups. For these we have the following theorem [N-R-S 06].

**Theorem 4** (N-R-S 06). Let $M$ be a quadratic of prime power order whose Witt class vanishes. Then $\text{Inv}(M)$ is different from zero. Moreover, $\text{Inv}(M)$ is generated by all $I_U = \sum_{x \in U} e_x$, where $U$ runs through the isotropic self-dual subgroups of $M$.

2Some authors use the terminology weakly metabolic instead.
However, for most of the cases which we are interested in the assumption of the last theorem fails for some prime $p$. Indeed, for a positive half integral $F$, the module $D_F$ is Witt-zero if and only if $\det(2F)$ is a square and $\text{rank}(F) \equiv 0 \mod 8$ (this is e.g. explained in [Sko 07] and follows from Milgram’s formula and the theory of $\sigma$-invariants explained in loc.cit.). But if $D_F$ is not Witt-zero, then one of its primary constituent must not be Witt-zero either and we cannot apply the last theorem.

In [Sko 07] the interested reader finds some more results on the Weil invariants of quadratic modules. However, we do not yet have a complete theory. On the other hand it seems to us that the characterization of the Weil invariants of quadratic modules and the search for a dimension formula for the space spanned by the Weil invariants is a a major problem in the context of Jacobi forms of small weight and their application to the theory of Siegel and orthogonal modular forms.

Appendix

For the reader’s convenience we recall in this section those technical details concerning Jacobi forms which we tacitly used in this article.

The metaplectic double cover $\text{Mp}(2, \mathbb{Z})$ of $\text{SL}(2, \mathbb{Z})$, consists of all pairs $(A, \pm w_A)$, where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{Z})$, and where $w_A$ is the holomorphic function on $\mathbb{H}$ such that $w_A(\tau)^2 = c\tau + d$ and such that $-\pi/2 < \text{Arg}(w_A(\tau)) \leq \pi/2$. The composition law is given by $(A, w) \cdot (B, v) = (AB, w(B\tau)v(\tau))$.

If $F$ is a real symmetric, half integral $n \times n$ matrix and $k$ a half integral integer we have the action of $\text{Mp}(2, \mathbb{Z})$ on functions $\phi$ defined on $\mathbb{H} \times \mathbb{C}^n$ given by the formula:

$$\phi|_{k, F}(A, w)(\tau, z) = \phi \left( A\tau, \frac{z}{c\tau + d} \right) w(\tau)^{-2k} e \left( \frac{-cF[z]}{c\tau + d} \right).$$

If $k$ is integral this action factors through an action of $\text{SL}(2, \mathbb{Z})$ and we write $\phi|_{k,F}A$ for $\phi|_{k,F}(A, w)$. Furthermore, if $F$ is half integral, we have the action of $\mathbb{Z}^n \times \mathbb{Z}^n$ on functions on $\mathbb{H} \times \mathbb{C}^n$ defined by

$$\phi|_{k,F}[\lambda, \mu](\tau, z) = \phi(\tau, z + \lambda \tau + \mu) e \left( \tau F[\lambda] + 2z^t F\lambda \right).$$

Definition. (Jacobi forms) Let $F$ be a symmetric, half integral positive definite $n \times n$ matrix, let $k$ be an integer and let $\Gamma$ be a subgroup of finite index in $\text{SL}(2, \mathbb{Z})$ and $\chi$ be a linear character of $\Gamma$. A Jacobi form of weight $k$ and index $F$ on $\Gamma$ with character $\chi$ is a holomorphic function $\phi : \mathbb{H} \times \mathbb{C}^n \to \mathbb{C}$ such that the following holds true:
(i) For all $A \in \Gamma$ one has $\phi|_{k,F} A = \chi(A) \phi$, and for all $g \in \mathbb{Z}^n \times \mathbb{Z}^n$ one has $\phi|_{k,F} g = \phi$.

(ii) For all $A \in \text{SL}(2, \mathbb{Z})$ the function $\phi|_{k,F} A$ possesses a Fourier expansion of the form

$$
\phi|_{k,F} A = \sum_{l \in \mathbb{Q}, r \in \mathbb{Z}^n, 4l - F - 1 [r] \geq 0} c(l, r) q^l e(z^t r).
$$

The space of these Jacobi forms is denoted by $J_{k,F}(\Gamma, \chi)$.

Similarly, we can define Jacobi forms of half integral weight $k$ on subgroups $\Gamma$ of $\text{Mp}(2, \mathbb{Z})$. As pointed out earlier there should also be a theory of Jacobi forms of index $F$ where the diagonal elements of $F$ are half integral. However, then the transformation law with respect to $\mathbb{Z}^n \times \mathbb{Z}^n$ would involve a character of $\mathbb{Z}^n \times \mathbb{Z}^n$ of order 2. Moreover, the theory of these forms is, in a sense, included in the theory of Jacobi forms of half integral index by the map $\phi(\tau, z) \mapsto \phi(\tau, 2z)$, which maps Jacobi forms of index $F$ (not necessarily with integral diagonal) to Jacobi forms of index $2F$.

The simplest nontrivial Jacobi form is the function

$$
\vartheta(\tau, z) = \sum_{n \in \mathbb{Z}} \left( -\frac{4}{n} \right) q^{n^2} \zeta^2 = \frac{Q}{2} \left( \zeta^2 - \zeta^{-2} \right) \prod_{n \geq 1} (1 - q^n)(1 - q^n \zeta)(1 - q^n \zeta^{-1})
$$

(the second identity follows from the Jacobi triple product identity). In fact, for $(A, w)$ in $\text{Mp}(2, \mathbb{Z})$ and integers $\lambda, \mu$, this function satisfies the following transformation formulas [Sko 92, p. 145]:

$$
\vartheta|_{\frac{1}{2}, \frac{1}{2}}(A, w) = e^3(A, w) \vartheta
$$
$$
\vartheta|_{\frac{1}{2}, \frac{1}{2}}(\lambda, \mu) = (-1)^{\lambda+\mu} \vartheta,
$$

where $\epsilon(A, w) = \eta(A \tau)/w(\tau) \eta(\tau)$. This is not a Jacobi form in the strict sense as defined above, but one of index $\frac{1}{2}$. In any case, $\vartheta(\tau, 2z)$ is an element of $J_{\frac{1}{2}, \frac{1}{2}}(\text{Mp}(2, \mathbb{Z}), e^3)$.

Finally, we remark, that a Jacobi form of index $F$ defines indeed a holomorphic section of a positive line bundle on $\mathbb{C}/\mathbb{Z}^n \tau + \mathbb{Z}^n$ with the Chern class associated to $F$ as explained in section [1]. A hermitian form on this line bundle is induced by the function $H(\tau, z) = e^{-4\pi F[z]/3(\tau)}$. In fact, if $\phi$ and $\psi$ are Jacobi forms of index $F$, then $\phi H \psi$ is invariant under $z \mapsto z + \lambda \tau + \mu$. If we write $F = (f_{p,q})$, the Chern class of the line bundle in question is thus given by

$$
\frac{1}{2\pi i} \partial \bar{\partial} \log H = \frac{1}{3(\tau)} \sum_{p,q} f_{p,q} dz_p \wedge d\bar{z}_q.
$$
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Nils-Peter Skoruppa
Fachbereich Mathematik, Universität Siegen
Walter-Flex-Straße 3, 57068 Siegen, Germany
http://www.countnumber.de