On de-Sitter geometry in crater statistics

G. W. Gibbons\textsuperscript{1}\footnote{E-mail: G.W.Gibbons@damtp.cam.ac.uk (GWZ); marcus.werner@ipmu.jp (MCW)} and M. C. Werner\textsuperscript{2,3}\footnote{\textsuperscript{2}Kavli Institute for the Physics and Mathematics of the Universe, University of Tokyo, 5-1-5 Kashiwanoha, Kashiwa 277-8583, Japan} \textsuperscript{3}Department of Mathematics, Duke University, Durham, NC 27708, USA

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1 INTRODUCTION

When observations by the spacecraft Dawn enabled the first detailed survey of impact craters on the asteroid 4 Vesta recently, it was noted that the cumulative size–frequency distribution can be approximated by a power law with (double logarithmic) slope $-1.9 \pm 0.1$ for small craters, close to the geometric saturation slope of $-2$ (Marchi et al. 2012). This approximate inverse square power law has also been observed on other planetary bodies in the Solar system, for example on Mars (see e.g. Werner & Tanaka 2011 and references therein). Since size–frequency distributions of impact craters are a powerful tool in planetology to date surfaces and reconstruct planetary evolution (see e.g. Michael & Neukum 2010 and the review by Hartmann & Neukum 2001 for Mars), it is important to understand their origin. The first detailed statistical model of impact cratering was given by Marcus (1964) in the context of lunar exploration. Assuming that craters are circular but may overlap, are Poisson-distributed over the lunar surface subject to a time-dependent production function with probability density $p(r, t)$ for a crater radius to lie in the interval $(r, r + dr)$, he showed in a subsequent work that for any $p(r, t) \propto r^{-\gamma} \exp[-(r/r_0)^2]$ with $\gamma > 2$, the cumulative frequency of craters larger than radius $r$ is indeed approximately proportional to $r^{-2}$ in a steady state at late times (Marcus 1966). Therefore, when this state is reached, the distribution of craters is the same for a class of underlying production functions. Moreover, if a population of craters of radius $r$ reaches geometric saturation, then its frequency is exactly proportional to $r^{-2}$ because of circle packing on the surface (cf. Gault 1970). It may also be recalled that power-law distributions are self-similar (for craters see e.g. Takayasu 1990, pp. 18f and 34ff), but this feature alone does not explain the fairly consistent power of $-2$ itself.

The universality of this property, then, seems to lend itself naturally to a geometrical explanation. In this paper, we present a new geometrical approach to study crater distribution functions, which is inspired by applications of measures to cosmological models of inflation (e.g. Gibbons & Turok 2008). While any size–frequency distribution of craters is, in the absence of other resurfacing, the result of the production function which depends on properties of the impactors creating the craters, it is possible that the size–frequency distribution evolves to a steady state at least partially independent of its creation history, as shown by Marcus’ result. In such a case, one might expect a priori a uniform distribution over the configuration space of craters. In order to investigate this point, we shall start in this paper with the assumption of a uniform distribution of craters over their configuration space and show that this implies in fact an inverse square power law for the size–frequency distribution of small craters. Of course, by contraposition, this means that any deviation from this relationship implies a non-uniformity of the crater distribution over their configuration space, which in turn may have physically interesting implications for the underlying class of production functions.

The structure of this paper is as follows. In Section 2, we consider the three-dimensional configuration spaces of circles on the Euclidean plane and on the unit sphere, and it turns out that these can be given a de-Sitter geometry. By interpreting the configuration space of craters as that of circles and computing its canonical volume measure, we show in Section 3 how an inverse square power law for the cumulative size–frequency distribution of small craters emerges. This model also shows that the global crater distribution is no longer self-similar once crater size becomes comparable to the size of the planetary body. Initially, we shall assume that craters are sparse with negligible overlap in the sense that their area filling factor $F \ll 1$, corresponding to a small fraction of geometric saturation which has filling factor $\pi/(2\sqrt{3}) \approx 0.907$ (cf. Gault 1970). This appears to be a reasonable assumption, as the observed distribution of craters is typically below 10 per cent of geometric saturation (e.g. for the Moon, see again Gault 1970 who compares it to a ‘Mare Exemplum’ created in laboratory experiments). The effect of crater overlap on the distribution function is then considered as well. The conclusions are given in Section 4, where an interpretation of this result and further applications are suggested.

ABSTRACT

The cumulative size–frequency distributions of impact craters on planetary bodies in the Solar system appear to approximate a universal inverse square power law for small crater radii. In this paper, we show how this distribution can be understood easily in terms of geometrical statistics, using a de-Sitter geometry of the configuration space of circles on the Euclidean plane and on the unit sphere. The effect of crater overlap is also considered.

Key words: methods: statistical – planets and satellites: surfaces – cosmology: miscellaneous.
2 DE-SITTER CIRCLE GEOMETRY

We begin by showing that the geometry of the configuration spaces of circles on the Euclidean plane and on the unit sphere in Euclidean space can be regarded as de-Sitter space–times. The first case will be seen in Section 2.1 as a special case of the configuration spaces of \( n \)-spheres in \( n + 1 \)-dimensional Euclidean spaces. The discussion of this more general case will also help to establish the second case in Section 2.2. Then, in both cases, one can easily derive the canonical volume measures for these configuration spaces. The connection with more general sphere geometry is outlined in Appendix A.

### 2.1 On the plane

Consider \( n \)-dimensional unoriented spheres \( S^n_0 \) in the \( n + 1 \)-dimensional Euclidean space \( \mathbb{E}^{n+1} \). Any such sphere is uniquely defined by its centre at \( a \in \mathbb{E}^{n+1} \) and its radius \( r > 0 \). Since these are independent of each other, the configuration space of such spheres \( \mathcal{M}^{n+2}_E = \{ S^n_0 \subset \mathbb{E}^{n+1} \} \) is \( n + 2 \)-dimensional. The special case \( n = 1 \) corresponds, of course, to circles, on the Euclidean plane. We will now see how the configuration space \( \mathcal{M}^{n+2}_E \) can be interpreted as an \( n + 2 \)-dimensional de-Sitter space–time, and this space–time of constant curvature can be described in the usual way as a quadric hypersurface in \( n + 3 \)-dimensional Minkowski space–time \( \mathbb{E}^{n+1,2} \) with metric \( \eta_{ab} = \text{diag}(-1, 1, \ldots, 1) \).

First, one may regard the Euclidean space \( \mathbb{E}^{n+1} \) as the intersection of the null cone through the origin in an \( n + 3 \)-dimensional Euclidean space. This configuration space \( \mathbb{E}^{n+1} \) is the configuration space of circles \( \{ S^n_0 \subset \mathbb{E}^{n+1} \} \) with a null hypersurface, and this subspace of \( \mathbb{E}^{n+1} \). For any arbitrary \( x \in \mathbb{E}^{n+1} \), define

\[
X^a = \left( \frac{x^2 + 1}{2}, -\frac{x^2 - 1}{2} \right) \in \mathbb{E}^{1,n+2},
\]

where the square of vectors in \( \mathbb{E}^{n+1} \) is with respect to the corresponding Euclidean metric, and let

\[
N^a = (1, 0, \ldots, 0, 1) \in \mathbb{E}^{1,n+2}.
\]

Then each \( x \) is identified with a point which is both on the null cone through the origin,

\[
\eta_{ab}X^aX^b = 0,
\]

and on the hypersurface,

\[
\eta_{ab}X^aN^b = -1,
\]

which is null since \( \eta_{ab}N^aN^b = 0 \).

Secondly, consider a space-like vector \( Y^a = (Y^0, Y, Y^{n+2}) \in \mathbb{E}^{1,n+2} \), so that \( \eta_{ab}Y^aY^b > 0 \), which is perpendicular to \( X^a \):

\[
\eta_{ab}X^aY^b = 0.
\]

Writing \( Y \cdot N = \eta_{ab}Y^aY^b = -Y^0 + Y^{n+2} \) for short, one can recast condition (1) as

\[
\left( \frac{x + Y}{Y \cdot N} \right)^2 = \frac{\eta_{ab}Y^aY^b}{(Y \cdot N)^2}.
\]

Hence, for any \( k \neq 0 \), a given \( kY^a \) uniquely defines an \( n \)-sphere in \( \mathbb{E}^{n+1} \) with centre and radius as

\[
a = -\frac{Y}{Y \cdot N}, \quad r = \sqrt{\frac{\eta_{ab}Y^aY^b}{(Y \cdot N)^2}},
\]

respectively. Because of the freedom to rescale \( Y^a \) we may set, without loss of generality,

\[
\eta_{ab}Y^aY^b = 1,
\]

and use equation (3) to express \( Y^a \) in terms of the variables defining the corresponding sphere:

\[
Y^0 = \frac{1}{2} \left( \frac{a^2 + 1}{r} - r \right),
\]

\[
Y = -\frac{a}{r},
\]

\[
y^{n+2} = \frac{1}{2} \left( \frac{a^2 - 1}{r} - r \right).
\]

Hence, this configuration space of spheres in Euclidean space is seen to be an \( n + 2 \)-dimensional de-Sitter space–time \( dS^{n+2} \),

\[
\mathcal{M}_{dS}^{n+2} = \{ Y^a \in \mathbb{E}^{1,n+2}; \eta_{ab}Y^aY^b = 1 \} = dS^{n+2},
\]

as required. The metric on this de-Sitter space–time is induced from the Minkowski metric of \( \mathbb{E}^{1,n+2} \),

\[
ds^2 = -(dY^0)^2 + dY^2 + (dY^{n+2})^2
\]

\[
= \frac{1}{r^2}(-dr^2 + da^2),
\]

using equations (4)–(6). With \( g^{E}_{ab} \) as the metric of the de-Sitter space–time, one can rewrite equation (7) as

\[
dV_{dS}^{n+2} = \sqrt{-\det g^{E}_{ab}} \, da^1 \ldots da^{n+1}
\]

\[
= \frac{1}{r^{n+2}} \, dr \, da^1 \ldots da^{n+1}
\]

(8)

as the volume measure of the de Sitter and hence the configuration space. Finally, as a special case of equation (8) one finds

\[
dV_{dS}^1 = \frac{1}{r^2} \, dr \, da^1 \, da^2
\]

(9)

the volume measure of the three-dimensional configuration space \( \mathcal{M}_dS_0 \) of circles on the Euclidean plane.

### 2.2 On the sphere

In view of the application to the distribution of craters on a planetary surface, we shall proceed by extending the previous case to the configuration space of circles \( S^n_1 \) on the three-dimensional unit sphere \( S^2 \) centred at the origin of \( \mathbb{E}^3 \). Each circle is fully characterized by its angular radius \( \rho \in (0, \pi/2) \) and by its centre \( n \) on the unit 2-sphere in Euclidean 3-space, so that \( n^2 = 1 \) and the two spherical coordinates of the centre can be computed in the usual way. The corresponding configuration space \( \mathcal{M}_S^1 \) is, as before, clearly three-dimensional and can also be considered a de-Sitter space–time.

To see this easily, one can utilize a trick to discuss this problem in terms of the previous one. Each circle on our unit 2-sphere \( S^2 \) may be regarded as the intersection of another 2-sphere \( S^n_2 \) of radius \( r \) centred at \( a \in \mathbb{E}^3 \) with \( S^2 \). As noted in the previous section, the configuration space \( \mathcal{M}^{2}_S \) of 2-spheres in Euclidean 3-space is a four-dimensional de-Sitter space–time defined by the quadric (3) in five-dimensional Minkowski space–time \( \mathbb{E}^{1,4} \) with coordinates (4)–(6). Now, for a given circle of angular radius \( \rho \), there is a family of spheres \( S^n_2 \) whose intersection with the unit 2-sphere produces this circle. Hence, their radius \( r \) depends on the choice of \( a \) which, by symmetry, must be parallel to \( n \). As shown in Fig. 1 (left-hand
we shall now show how this may be applied to the distribution of craters on planetary surfaces. The assumptions of our model shall be that, first, each crater can be represented by a circle and, secondly, the distribution of circles is uniform on their configuration space as defined in Section 2. The latter assumption therefore serves as a null hypothesis about the structure of the configuration space of craters. However, the configuration spaces of circles allow arbitrary overlap whereas, naturally, this is not the case for craters erasing older and smaller ones within. In Section 3.1, we shall consider the case that craters are sparse so that their overlap may be ignored. This condition is relaxed in Section 3.2, where we attempt to indicate the effect of overlap on the distribution function of craters.

3 APPLICATION TO CRATER DISTRIBUTIONS

3.1 Sparse craters

3.1.1 On the plane

Suppose that an infinitesimal area dA in the Euclidean plane contains the centres of dn craters of radii in the interval \([r, r + dr]\). Then the corresponding distribution function \(f_E\) may be written \(dn = f_E\, dr\, dA\) so that, given our assumptions and the result from Section 2.1, \(dn \propto dV_E^3\) and hence

\[
f_E = \frac{C_E}{r^3}
\]

from equation (9), where \(C_E\) is a constant. If this constant is sufficiently small, the assumption of sparse craters will hold. In order to determine this and estimate its value, let \(n_0\) be the total number of craters with radius \(r_{\min} \leq r \leq r_{\max}\) wholly within a disc of radius \(R \geq r_{\max}\). Then the area filling factor of these craters is

\[
F_E = \int_{r_{\min}}^{r_{\max}} \pi r^2 f_E(r) \, dr = \pi C_E \ln \frac{r_{\max}}{r_{\min}}
\]

from equation (14), so that for sparse craters with \(F_E \ll 1\) we must have

\[
C_E \ll \left( \frac{\pi \ln \frac{r_{\max}}{r_{\min}}}{r_{\min}^3} \right)^{-1} \cdot
\]

As \(R \to \infty\) and condition (15) continues to hold, \(n_0/\pi R^2\) tends to the total frequency of craters with this radial range per unit area on the Euclidean plane,

\[
v_E = \int_{r_{\min}}^{r_{\max}} f_E \, dr,
\]

whence, using again equation (14),

\[
C_E = 2 v_E r_{\min}^2 \left( 1 - \frac{r_{\min}^2}{r_{\max}^2} \right)^{-1}.
\]

The cumulative frequency\(v(r)\) of craters with radius greater than \(r\) per unit area as a function of size can also be obtained from the distribution function (14):

\[
v(r) = \int_r^{r_{\max}} f_E(r') \, dr' = \frac{C_E}{2} \left( 1 - \frac{r^2}{r_{\max}^2} \right).
\]

Finally, note that the approximation \(r_{\min} \ll r_{\max} \ll R\) applied to equations (15)–(17), which is reasonable for a crater count, gives \(C_E \approx 2v_E r_{\min}^2 \ll 1\) and an inverse square power law for the cumulative frequency as

\[
v(r) \approx v_E \frac{r_{\min}^2}{r^2}.
\]
3.1.2 On the sphere

When the maximum radius of the craters is not negligible compared to the radius of the planetary body then, given our assumptions, the configuration space of circles on the unit sphere from Section 2.2 is applicable. Let \( R_0 \) be the radius of the planetary body and \( R \) be the radius of a crater measured on the spherical surface so that its angular radius is \( \rho = R/R_0 \in (0, \pi/2) \) as before. Now suppose \( dn \) is the number of the centres of craters with angular radius in the interval \( [\rho, \rho + d\rho] \) within the solid angle element \( d\omega = dA/R_0^2 \), so \( dn = f_\omega \, d\rho \, d\omega \) with distribution function \( f_\omega \propto dV_3 \) given by

\[
f_\omega(\rho) = C_\omega \csc^3 \rho, \tag{19}\]

from equation (13), where \( C_\omega \) is a constant. A crater of angular radius \( \rho \) subtends a solid angle

\[
\omega(\rho) = \int_0^{2\pi} \int_0^\rho \sin \theta \, d\theta \, d\phi = 2\pi(1 - \cos \rho) \tag{20}\]

on the planetary surface, so the area filling factor of craters with angular radius \( \rho_{\text{min}} \leq \rho \leq \pi/2 \) is then

\[
F_\omega = \int_{\rho_{\text{min}}}^{\pi/2} \omega(\rho) \, f_\omega(\rho) \, d\rho = \pi C_\omega \left( \frac{\cos \rho_{\text{min}} \csc \rho_{\text{min}} + \ln \cot \frac{\rho_{\text{min}}}{2}}{1 + \cos \rho_{\text{min}}} \right). \tag{21}\]

For the assumption of sparse craters to hold, we require that \( F_\omega \ll 1 \) as before. Now if \( \rho_{\text{min}} \ll 1 \), that is, the smallest craters counted are much smaller than the radius of the planetary body, one must therefore require that

\[
C_\omega \ll \left( \frac{\pi}{2} + \pi \ln \frac{2}{\rho_{\text{min}}} \right)^{-1}. \tag{22}\]

The total number of craters with radius greater than \( \rho_{\text{min}} \) is

\[
n_\omega = \int_0^{4\pi} \int_{\rho_{\text{min}}}^{\pi/2} f_\omega(\rho) \, d\rho \, d\omega = 2\pi C_\omega \left( \cot \rho_{\text{min}} \csc \rho_{\text{min}} + \ln \cot \frac{\rho_{\text{min}}}{2} \right). \tag{23}\]

Again if \( \rho_{\text{min}} \ll 1 \), then we can approximate equation (22) by the leading monomial to find

\[
C_\omega \approx \frac{n_\omega \rho_{\text{min}}^2}{2\pi}. \tag{24}\]

The total cumulative number of craters with radius greater than \( \rho \) is also obtained from equation (19):

\[
n(\rho) = 4\pi \int_0^{\pi/2} f_\omega(\rho') \, d\rho' = 2\pi C_\omega \left( \cot \rho \csc \rho + \ln \cot \frac{\rho}{2} \right). \tag{25}\]

This function is shown in Fig. 2 for three values of \( C_\omega \), for which condition (21) holds in the range shown. Moreover, if \( \rho_{\text{min}} \ll 1 \), one can use approximation (23) for the constant in equation (24) to find

\[
n(\rho) \approx n_\omega \rho_{\text{min}}^2 \left( \cot \rho \csc \rho + \ln \cot \frac{\rho}{2} \right). \tag{26}\]

This expression clearly recovers equation (18) in the limit of \( \rho \ll 1 \) where \( 4\pi R_0^2 \nu = n \) and \( R_0 = R \approx r \), as expected. Likewise, one can easily convert equation (24) to the cumulative crater frequency per unit area of the planetary body as a function of crater size. Note also that the global cumulative number (24) is given by trigonometric functions and not a power law. Hence, the crater distribution is no longer self-similar, as is the case for small craters or for craters on the Euclidean plane in the limit \( r_{\text{max}} \to \infty \) of equation (17).

3.2 Overlapping craters

The spherical model discussed in Section 2.2 is, of course, the generalization of the planar model of Section 2.1 applicable to sparse craters on a planetary surface. However, a more realistic model for the distribution of craters should take into account the possibility of overlap. In this section, we shall consider how the more general spherical model can thus be extended. For simplicity we shall assume that, first, there is a finite period of crater formation and, secondly, the crater distribution on the planetary surface is always isotropic. Thirdly, we take equation (19) to be the intrinsic distribution of craters produced, which may differ from the distribution observed on the planetary surface due to overlap. Finally, we assume that the centre and radius of a crater can be identified as long as a sector of its perimeter survives, that is, only craters that are wholly overlapped by a later and larger crater are considered to be obliterated.

Hence, one may take the initial distribution to be \( f_\omega(\rho) \) and the final observed distribution \( f_\omega(\rho') \) to be a function of \( \rho \) only. Craters of radius \( \rho \) will have been wholly overlapped by larger craters of radius \( \rho' > \rho \) if their centres are within a solid angle of

\[
\omega(\rho' - \rho) = 2\pi[1 - \cos(\rho' - \rho)] \tag{27}\]

from equation (20), concentric with a larger crater of radius \( \rho' \) in the observed distribution, of which there are \( 4\pi f_\omega(\rho') \, d\rho' \). This is illustrated in Fig. 1 (right-hand panel). In addition, there remain \( 4\pi f_\omega(\rho) \, d\rho \) craters of radius \( \rho \) in the distribution observed on the planetary surface. By assumption, we take the distribution of craters of radius \( \rho \) that are produced to be proportional to \( f_\omega(\rho) \). Then this applies to the overlapped craters as well as to the sum of the observed craters plus the overlapped ones, of course with different constants of proportionality. We therefore expect to have

\[
f_\omega(\rho) + \int_{\rho}^{\pi/2} f_\omega(\rho') f_\omega(\rho) \omega(\rho' - \rho) \, d\rho' \propto f_\omega(\rho). \tag{28}\]
It may be useful to express the distribution function as

\[ f_0(\rho) = C_0 f(\rho), \]

where \( C_0 \) is the scaling factor proportional to the total number of observed craters \( n_0 \) greater than \( \rho_{\text{min}} \), which reduces to \( C_0 \) in the limit of sparse craters,

\[ n_0 = 4\pi C_0 \int_{\rho_{\text{min}}}^{\pi/2} f(\rho) \, d\rho. \tag{28} \]

Hence, using equations (19) and (26), one can rewrite equation (27) as

\[ f(\rho) = g(\rho) + \int_{\rho}^{\pi/2} f(\rho')K(\rho, \rho') \, d\rho', \tag{29} \]

where

\[ g(\rho) = \csc^3 \rho \]

and

\[ K(\rho, \rho') = \csc^3 \rho [1 - \cos(\rho' - \rho)] \]

and \( \lambda < 0 \) is a (by definition negative) constant. The problem of overlapping craters, at least in this simplified form, can therefore be expressed as the integral equation (29), which is the standard form of a linear, inhomogeneous Volterra equation of the second kind. Note that its kernel is of separable form since it may be written as

\[ K(\rho, \rho') = \sum_i k_i(\rho)k_i(\rho'), \]

where \( k_i \) and \( k_i' \) are functions only of \( \rho \) and \( \rho' \), respectively. Alternatively, one can turn the integral equation (29) into a differential equation. Letting

\[ \tilde{f}(\rho) = \frac{f(\rho)}{g(\rho)}, \]

we get

\[ \lambda g \tilde{f} + \frac{d \tilde{f}}{d\rho} + \frac{d^2 \tilde{f}}{d\rho^2} = 0, \tag{30} \]

a linear, homogeneous ordinary differential equation of third order with non-constant coefficients. Furthermore, one can define

\[ \hat{f}(\rho) = \frac{d \ln \tilde{f}}{d\rho} \]

to reduce the order of equation (30) and convert it to

\[ \lambda g + \hat{f} + \hat{f}^3 + 3 \hat{f} \frac{d \hat{f}}{d\rho} + \frac{d^2 \hat{f}}{d\rho^2} = 0, \tag{31} \]

a non-linear, inhomogeneous ordinary differential equation of second order with constant coefficients. While a detailed discussion of the properties of equations (29)-(31) and the numerical comparison with observed crater distributions is beyond the scope of this paper, we shall use the Volterra equation (29) to illustrate the effect of little overlap for craters with small lower limit of the angular radius, \( \rho_{\text{min}} \ll 1 \). By this we mean a small deviation of \( f_0 \) from the sparse crater distribution so that the constant in the Volterra equation is small, \( |\lambda| \ll 1 \). Then an approximate solution of equation (29) for \( f \) can be expressed naturally as the leading terms of a Neumann series expansion of the full solution,

\[ f(\rho) = \sum_{j=0}^{\infty} \lambda^j u_j(\rho), \]

whence the first approximation will be given by

\[ f(\rho) \approx u_0 + \lambda u_1 \]
\[ = \csc^3 \rho + \lambda \int_\rho^{\pi/2} \csc^3 \rho' \csc^3 \rho [1 - \cos(\rho' - \rho)] \, d\rho' \]
\[ = \csc^3 \rho \left( 1 + \frac{\lambda}{2} \left( -\cos \rho + \ln \cot \frac{\rho}{2} \right) \right). \tag{32} \]

Integrating equation (32) in (28), the leading monomial in \( \rho_{\text{min}} \) yields the approximation

\[ n_0 \approx \frac{2\pi C_0}{\rho_{\text{min}}^2} \left( 1 - \frac{\lambda}{4} \left( 3 - 2 \ln \frac{2}{\rho_{\text{min}}} \right) \right). \]

Hence, the constant \( \lambda \) indicates the importance of crater overlap. In the current formulation, it is a free parameter that will depend on the total number of craters produced and on the number \( n_0 \) of craters that remain observable.

**4 CONCLUSION**

Starting from the assumption that impact craters are uniformly distributed over their de-Sitter configuration space, it is shown by equations (18) and (25) that this can indeed explain the approximate inverse square power law for the cumulative size–frequency distribution of small craters, which is widely observed as noted in the Introduction. Moreover, while equation (24) shows that the global crater distribution is not self-similar, Fig. 2 illustrates that the deviation from the inverse square power law due to the finite size of the planetary body is in fact mostly negligible, so that the crater distribution is effectively self-similar. Moreover, since smaller craters are relatively more affected by overlap, we expect that they are systematically undercounted relative to the distribution of sparse craters with negligible overlap. This is the case, as indicated by equation (32) and the dotted curve in Fig. 2 for small \( \rho \).

Hence, the approach promoted in this paper suggests that the de-Sitter configuration space may be a useful prior for the study of cratering histories. Deviations from the inverse square power law correspond to non-uniform distributions of small craters on this configuration space. We suggest that such deviations from uniformity will indicate physically interesting processes like resurfacing due to igneous activity and erosion, as well as properties of the impactors before a steady state of the crater population, as in the formalism of Marcus mentioned in the Introduction, is reached. A first step in this direction would be to classify production functions and resurfacing processes in terms of the deviation of the resulting crater configuration space from this de-Sitter geometry, which will depend on time. Hence, comparing these templates with observed maps of crater configuration spaces may yield new insights into cratering histories.

Finally, this paper notes a perhaps surprising formal connection between the geometry of a cosmological space–time and a problem in planetology. Recall that the three-dimensional configuration space of circles, and hence of craters as discussed, was interpreted as the three-dimensional de-Sitter space–time with the metric given by equation (12) and hence the line element

\[ ds^2 = -d\rho^2 + \cosh^2 \rho \left( d\theta^2 + \sin^2 \theta \, d\phi^2 \right). \]

Now in order to connect this with the de-Sitter space–time familiar from cosmology, one can use the coordinate transformation \( \csc \rho = \cosh t \) so that

\[ ds^2 = -dt^2 + \cosh^2 t \left( d\theta^2 + \sin^2 \theta \, d\phi^2 \right). \]
where \( t \) takes its usual interpretation as coordinate time. Extending our argument from two to three spatial dimensions, one obtains a measure on the space of spheres in three-dimensional Euclidean space from equation (8) for \( n = 2 \). This may be useful to study the statistics of bubbles in the Universe, for instance supernova remnants within the interstellar medium or voids in the large-scale structure. Moreover, eternal inflation treats of bubble universes which are expected to collide (see e.g. Garriga, Guth & Vilenkin 2007), so finding a version of the overlap formula (29) applicable to this problem may further elucidate the collision process of bubble universes.

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APPENDIX A: THE CONNECTION WITH SPHERE GEOMETRY

Here we give a brief discussion of the geometry underlying the construction in Section 2. Circle geometry, beginning with the classical problem of Apollonius to find the circles touching three given ones, is a special case of the sphere geometries developed by Lie and Laguerre. Following partially the treatment by Benz (2012), consider the Euclidean space \( \mathbb{E}^{n+1} \) with its standard Euclidean metric. Then a Laguerre cycle is either a point or an oriented \( n \)-sphere of radius \( r \neq 0 \) centred at \( a \in \mathbb{E}^{n+1} \), with orientation expressed by the sign of \( r \). Laguerre cycles can thus be regarded as elements of a real vector space \( V = \mathbb{R} \oplus \mathbb{E}^{n+1} \). Assigning coordinates \( c = (r, a) \) to a Laguerre cycle is called cyclographic projection, and addition of Laguerre cycle coordinates is defined as for vectors. The zero Laguerre cycle is \( 0 = (0, \mathbf{0}) \). Now given two Laguerre cycles \( c_1 = (r_1, a_1) \), \( c_2 = (r_2, a_2) \), the real number

\[
P(c_1, c_2) = -(r_2 - r_1)^2 + (a_2 - a_1)^2,\tag{A1}
\]

where the product of vectors in \( \mathbb{E}^{n+1} \) is understood to be the inner product with respect to the Euclidean metric, is called power of \( c_1 \) and \( c_2 \). It turns out that \( c_1 \) and \( c_2 \) touch each other respecting orientation if, and only if (Benz 2012, proposition 3.43),

\[
P(c_1, c_2) = 0,
\]

and this is exactly the reflexive and symmetric contact relation studied in sphere geometry. Moreover, \( V \) can be equipped with a product defined as

\[
c_1 c_2 = -r_1 r_2 + a_1 a_2.
\]

Letting \( dc = (dr, da) \), we therefore find that

\[
dc^2 = -dr^2 + da^2 = P(c, c + dc)\tag{A2}
\]

measures the infinitesimal deviation of Laguerre cycles with respect to the contact relation, and this is seen to define a Minkowski metric on \( V \). Note that equation (A2) may be regarded as a line element on \( V \) which is invariant under transformations of Laguerre cycles preserving power (A1). Hence, there is a close connection between the Minkowski space–time of special relativity and the geometry of Laguerre cycles, which appears to have been pointed out first by Timerding (1912). In addition to the set of Laguerre cycles \( \Gamma \), one can introduce the set of oriented Euclidean hyperplanes \( \Sigma \) in \( \mathbb{E}^{n+1} \) and the object infinity \( \Lambda \) which is invariant under transformations of Laguerre cycles, and bijections \( \Lambda : \Gamma \to \Lambda \) that preserve contact relations form the group of Lie transformations of \( \mathbb{E}^{n+1} \). It turns out (Benz 2012, proposition 3.56) that to every Lie cycle a homogeneous Lie cycle coordinate \( L = [L^0, L, L^{n+2}, L^{n+3}] \), where \([L^0, L, L^{n+2}, L^{n+3}] = [kL^0, kL, kL^{n+2}, kL^{n+3}] \) for any \( k \neq 0 \), can be assigned bijectively, which satisfies the Lie quadric

\[
-(L^0)^2 + L^2 + (L^{n+2})^2 - (L^{n+3})^2 = 0.\tag{A3}
\]

For a Laguerre cycle \( c = (r, a) \), the Lie cycle coordinate is (cf. Benz 2012, p. 154, applying a sign change and reordering)

\[
L(c) = \left[ \frac{P(0, c) + 1}{2}, a, \frac{P(0, c) - 1}{2}, -r \right].
\]

By homogeneity of the Lie cycle coordinates, we can set \( L^{n+3} = 1 \) to obtain

\[
L(c) = \left[ -\frac{1}{2} \left( \frac{a^2 + 1}{r} - r \right), -a, \frac{a^2 - 1}{2}, \frac{1}{2} \right],
\]

using equation (A1) and the coordinates (4)–(6) defined in Section 2. Then these are seen to satisfy

\[
-(Y^0)^2 + Y^2 + (Y^{n+2})^2 = 1\tag{A4}
\]

because of the Lie quadric (A3). Suppose now that we consider the line element

\[
ds^2 = -(dY^0)^2 + dY^2 + (dY^{n+2})^2
\]

\[
= \frac{1}{r^2}(-dr^2 + da^2) = \frac{dc^2}{r^2}\tag{A5}
\]

using equation (A2). Clearly, any transformation \((Y^0, Y, Y^{n+2}) \leftrightarrow (Y^0 + dY^0, Y + dY, Y^{n+2} + dY^{n+2})\) that preserves our choice (A4) leaves this line element invariant, and in fact null \( ds^2 = 0 \) so that \( dc^2 = 0 \). Hence, such transformations preserve the contact relations of the corresponding Laguerre cycles, as described above. Under this assumption, then, \((Y^0, Y, Y^{n+2})\) may be regarded as a point on the de-Sitter configuration space of Section 2 defined by the quadric hypersurface (A4) in the Minkowski space–time \( \mathbb{E}^{n+2} \) with line element (A5). A connection between Laguerre sphere geometry and de-Sitter space–time seems to emerge first in Graf (1934), who uses the opposite metric signature.

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