CORD RING INVARIANT OF KNOTS IN $S^1 \times S^2$

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Abstract. We generalize Ng’s algebraic 0-th knot contact homology for links in $S^3$ to links in $S^1 \times S^2$, and prove that the resulted link invariant is the same as the cord ring of links. Our main tool is Lin’s generalization of the Markov theorem for braids in $S^3$ to braids in $S^1 \times S^2$. We conjecture that our cord ring is always finitely generated for non-local links.

1. Introduction

The dream of finding new higher categorical quantum invariants of smooth 4-manifolds that can distinguish smooth structures beyond Donaldson/Seiberg-Witten/Heegaard-Floer theory is largely unrealized, despite the spectacular success in 3-dimensions and recent progress in higher category theory. A potentially new quantum invariant would be to promote the relative knot contact homology of knots in $S^3$ in [7] to a $(3 + 1)$-TQFT-type theory (presumably the 0-th part of the BRST cohomology of a topological string theory). One lesson from $(2 + 1)$-dimensions is the emergence of powerful diagram techniques as exemplified by the Kauffman bracket definition of the Jones polynomial, and the subsequently elementary formulation of Turaev-Viro and Reshetikhin-Turaev $(2 + 1)$-TQFTs. We see a striking parallel between the cord ring invariant of knots and the Jones polynomial of knots.

In [7], the 0-th part of the relative knot contact homology in $S^3$ is interpreted using cords and skein relations—the main ingredients of diagram techniques in $(2 + 1)$-dimensions, analogous to Jones polynomial from von Neumann algebra reformulated using knot diagram and the Kauffman bracket. Taking the elementary cord ring invariant of knots in general 3-manifolds $M$ as the main object of interest, we will follow the diagram approach to constructing $(2 + 1)$-TQFTs such as Turaev-Viro and Reshetikhin-Turaev. As a first step, we generalize Ng’s algebraic 0-th knot contact homology for links in $S^3$ to links in $S^1 \times S^2$. We conjecture that our cord ring is always finitely generated for non-local links.

1. Key words and phrases. knot, braid group, knot contact homology, cord ring.

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$S^1 \times S^2$, and prove that the resulted link invariant is the same as the cord ring of links.

In [7] it is conjectured that the cord ring invariant of knots in a general 3-manifold $M$ is the 0-th knot contact homology, we do not prove this conjecture and will not use any knot contact homology theory. Instead we provide an algebraic version of this conjectured 0-th knot contact homology for knots in $S^1 \times S^2$ following [7] and regard our algebraic definition of the cord ring as an effective method to calculate the cord ring invariant of knots. Our long term goal is to understand the higher categories underlying this algebraic formulation with an eye towards a diagram construction of a (3 + 1)-TQFT-type theory.

A second reason for our interest in the cord ring invariant of knots is the conjectured relation between the augmentation polynomial and the Homfly polynomial. A well-known question since the discovery of the Jones polynomial is how to place the Jones polynomial within classical topology (since knots are determined by their complements, so any knot invariant is determined by the homeomorphism type of the knot complement). The cord ring of a knot is basically within classical topology, so the establishment of the conjecture between the augmentation polynomial and the Homfly is one answer to an old question.

To generalize the algebraic 0-th knot contact homology in [7] from $S^3$ to $S^1 \times S^2$, our main tool is Lin’s generalization of the Markov theorem for braids in $S^3$ to braids in $S^1 \times S^2$ [5] developed for defining a Jones polynomial of knots in $S^1 \times S^2$.

The rest of the paper is organized as follows. In Section 2.1 we introduce the Markov theorem for knots in $S^1 \times S^2$, which are represented by the closure of elements in $\mathcal{C}_n$, the Artin group whose Dynkin diagram is $B_n$. In Section 2.2 we give several actions of $\mathcal{C}_n$ on free algebras. We interpret these actions both algebraically and topologically. These actions will be the key ingredients to define the invariant $HC_0$ in Section 3.1. In Section 3.2 - Section 3.4 we compute some specific examples, show some useful propositions, and prove the invariance of $HC_0$ under Markov moves, respectively. Section 4.1 - Section 4.4 are devoted to prove several properties of the $HC_0$ invariant. We study two special classes of knots in $S^1 \times S^2$, torus knots and local knots. Moreover, we derive a family of invariants, called augmentations, from $HC_0$. Finally, in Section 5 we prove that the $HC_0$ invariant has a nice topological interpretation as the cord ring defined in [7].

\footnote{This generalization, eventually rendered unnecessary for the intended application by Witten’s work, finds a similar application in our work. We dedicate our work to X.-S. Lin—an important vanguard in quantum knot theory.}
The first author also created a mathematica package for computer calculations of the $HC_0$ invariant and augmentation numbers. The program can be found at [8] and is partly motivated by the computer package created by Ng, who used it to compute various invariants derived from knot contact homology for knots in $S^3$. To run the program, one need to install the non-commutative algebra package NCAlgebra/NCGB [4].

2. Markov moves and actions of $C_n$ on free algebras

Before introducing the invariant, we provide some background materials and recall some necessary techniques. Links are always oriented.

2.1. Markov moves in $S^1 \times S^2$. In this subsection, we describe a theorem of Markov moves for links in $S^1 \times S^2$. For detailed discussion on this topic, see [5].

Recall that the classical braid group with $n$ strands, $B_n$, is defined by the presentation \( \langle \sigma_1, \cdots, \sigma_n | \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \sigma_i \sigma_j = \sigma_j \sigma_i, |i - j| \geq 2 \rangle \). It is the Artin group whose Dynkin diagram is of type $A_{n-1}$, and can also be viewed as the braid group on the 2-disk $D_2 \subset \mathbb{R}^2$.

It’s well known that any link in $S^3$ can be represented as the closure of some braid in the classical braid group. The Markov theorem says that two braids $B, B'$ give the same link if and only if $B'$ can be obtained from $B$ by a finite sequence of the following operations or their inverses:

1). change $B \in B_n$ to one of its conjugates in $B_n$;
2). change $B \in B_n$ to $B \sigma_i^{\pm 1} \in B_{n+1}$.

In [5], this theorem is generalized to links in $S^1 \times S^2$.

Let $C_n$ be the Artin group corresponding to the Dynkin diagram $B_n$ generated by $\alpha_0, \cdots, \alpha_{n-1}$, with the following generating relations:

1). $\alpha_i \alpha_j = \alpha_j \alpha_i, |i - j| \geq 2$
2). $\alpha_i \alpha_{i+1} \alpha_i = \alpha_{i+1} \alpha_i \alpha_{i+1}, i \geq 1$
3). $\alpha_0 \alpha_1 \alpha_0 \alpha_1 = \alpha_1 \alpha_0 \alpha_1 \alpha_0$.

Clearly, we have natural inclusions $C_1 \subset C_2 \subset \cdots \subset C_n \subset \cdots$. We denote by $\epsilon^-$ these natural embeddings.

It is shown in [2] that $C_n$ is isomorphic to the braid group on the annulus $I \times S^1$, or the 1-punctured disk. Specifically, the isomorphism is illustrated in Figure 1.

It’s clear that a braid on the 1-punctured disk can also be viewed as a braid on the disk. Simply treat $\{\text{puncture}\} \times I$ as the first strand of the new braid. Thus we have an embedding of $C_n$ into $B_{n+1}$. Denote the generators of $B_{n+1}$ by $\sigma_0, \sigma_1, \cdots, \sigma_{n-1}$. Then the embedding from
Figure 1. $\alpha_0$ and $\alpha_k, k \geq 1$

$C_n$ to $B_{n+1}$ is given by the following map:

$$C_n \rightarrow B_{n+1}, \quad \alpha_0 \mapsto \sigma_0^2, \quad \alpha_i \mapsto \sigma_i, i \geq 1.$$

From now on, we will identify $C_n$ with its image in $B_{n+1}$, which is the subgroup consisting of the braids that fix the first strand.

The correspondence between braids on the annulus and links in $S^1 \times S^2$ is obtained via open book decompositions.

Consider the standard open book decomposition of $S^3$ with an unknot $J$ as the binding. Let $K$ be another unknot which is a closed braid with respect to the braid axis $J$. Then

$$M = S^3 \setminus (J \times D^2 \cup K \times D^2)$$

is a fibration over $S^1$ whose fibre is an annulus $I \times S^1$. $S^1 \times S^2$ is obtained by a 0-Dehn surgery along $K$. Thus $S^1 \times S^2 = M \cup f D^2 \times S^1$, where $f$ is the gluing homeomorphism which maps the meridian of the solid torus to $K \times z_0, z_0 \in \partial D^2$. Let $K^*$ be the dual knot of $K$. Then the fibration on $M$ extends to an open book decomposition on $S^1 \times S^2$ with the binding $J \cup K^*$. Note that $S^1 \times S^2 \setminus (J \cup K^*)$ is homeomorphic to the product of the annulus with $S^1$. It’s not hard to see that any link in $S^1 \times S^2$ can be isotoped into $S^1 \times S^2 \setminus (J \cup K^*)$ transversal to each page, and thus becomes a braid on the annulus.

To state the Markov theorem, we need one more lemma.

Define a map $\epsilon^+: C_n \rightarrow C_{n+1}$,
The map $\epsilon^+$ has a nice geometrical interpretation if we view $C_n$ as the braid group on the annulus. The map simply inserts a straight strand right next to the line $\{\text{puncture}\} \times I$. See Figure 2.

Note that the newly inserted line will be labeled by 1, and the other strands' labels will be shifted by 1.

**Lemma 1.** The map $\epsilon^+$ is an injective group morphism.

**Proof** From the geometrical interpretation of the map, it should be clear that it is an injective group morphism. For a rigorous algebraic proof, see [5].

**Remark 1.** Now there are two embeddings of $C_n$ into $C_{n+1}$, namely the natural inclusion $\epsilon^-$ and the map $\epsilon^+$. From the geometric point of view, $\epsilon^-$ is to place a strand on the far right of the braid, while $\epsilon^+$ is to insert a strand right next to the line $\{\text{puncture}\} \times I$.

Here is the statement of the Markov Theorem for links in $S^1 \times S^2$.

**Theorem 1.** The closures of two braids $\beta, \beta' \in \bigcup_{n=1}^{\infty} C_n$ give the same link in $S^1 \times S^2$ if and only if there is a finite sequence of braids, $\beta = \beta_0, \beta_1, \cdots, \beta_k = \beta'$, such that $\beta_{i+1}$ can be obtained from $\beta_i$ by one of the following operations or their inverses:

1). change $\beta_i \in C_n$ to one of its conjugates in $C_n$;
2.2. Actions of \( C_n \) on free algebras. Let \( R \) be the commutative ring \( \mathbb{Z}[\mu, \nu]/(\mu \nu - 2) = \mathbb{Z}[\mu, \frac{2}{\mu}] \). We define several free non-commutative algebras over the ring \( R \) as follows.

\[
\begin{align*}
A_n^+ &= R(a_{ij}^+, 0 \leq i, j \leq n, x \in \mathbb{Z})/\langle a_{ii}^0 - \mu, 0 \leq i \leq n \rangle, \\
A_n^- &= R(a_{ij}^-, 1 \leq i, j \leq n + 1, x \in \mathbb{Z})/\langle a_{ii}^0 - \mu, 1 \leq i \leq n + 1 \rangle, \\
A_n &= R(a_{ij}, 1 \leq i, j \leq n, x \in \mathbb{Z})/\langle a_{ii}^0 - \mu, 1 \leq i \leq n \rangle.
\end{align*}
\]

The algebra \( A_n \) can be embedded into \( A_n^+ \) and \( A_n^- \) in the most natural way. We will always identify \( A_n \) with its images in \( A_n^+ \) and \( A_n^- \).

Now we introduce an action of \( C_n \) on \( A_n \), and extend the action to the larger algebras \( A_n^+ \), \( A_n^- \). Below the action is first presented algebraically and then will be given a topological interpretation.

Recall that the generators \( C_n \) are denoted by \( \alpha_0, \cdots, \alpha_{n-1} \), which satisfy the relation given in Section 2.1. We define a group morphism \( \Phi : C_n \to \text{Aut}(A_n) \) as follows.

For \( 1 \leq k \leq n - 1 \),

\[
(2.2) \quad \Phi(\alpha_k)(a_{ij}^x) = \begin{cases} 
-a_{k+1,j}^x + 2a_{k+1,k}^0a_{k,j}^x & i = k, j \neq k, k + 1 \\
-a_{k+1,k}^x + 2a_{k+1,k}^0a_{k,k}^x & i = k, j = k + 1 \\
\frac{2}{\mu}a_{k+1,k}a_{k,k+1}^x + \frac{4}{\mu^2}a_{k+1,k}^0a_{k,k+1}^x & i = k, j = k \neq k, k + 1 \\
\frac{2}{\mu}a_{k,k}^0a_{k,k+1} & i = k + 1, j \neq k, k + 1 \\
a_{k+1,k}a_{k,k+1}^x & i = k + 1, j = k \\
-a_{i,k+1}^x + 2a_{i,k}^0a_{k+1,k}^0 & i \neq k, k + 1, j = k \neq k + 1 \\
a_{i,k}^x & i \neq k, k + 1, j \neq k, k + 1 \\
a_{i,j}^x & \text{otherwise}
\end{cases}
\]

\[
(2.3) \quad \Phi(\alpha_0)(a_{ij}^x) = \begin{cases} 
\frac{2}{\mu}a_{i,j}^1a_{i,j}^{-1} & i = 1, j = 1 \\
\frac{2}{\mu}a_{i,j}^{-1} + \frac{2}{\mu^2}a_{i,1}^1a_{1,j}^{-1} & i = 1, j \geq 2 \\
\frac{2}{\mu}a_{i,1}^1a_{i,j}^x - \frac{2}{\mu^2}a_{i,1}^1a_{i,j}^{-1} & i \geq 2, j \geq 2 \\
\frac{2}{\mu}a_{i,j}^1a_{i,j}^x - \frac{2}{\mu^2}a_{i,1}^1a_{1,j}^x & i \geq 2, j \geq 2 \\
a_{i,j}^x & \text{otherwise}
\end{cases}
\]

It is not hard, though tedious, to check that \( \Phi \) is well defined, i.e. \( \Phi(\alpha_i) \) satisfies the braid relations that define \( C_n \).
We extend the action of $C_n$ to the algebra $A^+_n$ by furthermore defining the action on $a^x_{i0}, a^x_{0i}, 0 \leq i,j \leq n$. This extended action will be denoted by $\Phi^+$.

$$\Phi^+(\alpha)(a^x_{ij}) = \begin{cases} a^x_{0,0} & i = 0, j = 0 \\ a^x_{0,1} & i = 0, j = 1 \\ -a^x_{0,j} + \frac{2\mu}{x} a^x_{0,1} a^x_{1,j} & i = 0, j \geq 2 \\ a^x_{1,0} & i = 1, j = 0 \\ -a^x_{i,0} + \frac{2\mu}{x} a^x_{1,1} a^x_{i,0} & i \geq 2, j = 0 \end{cases}$$

For $1 \leq k \leq n - 1$, $\Phi^+(\alpha_k)(a^x_{ij})$ are given by the same equations as 2.2 except that $i,j$ are allowed to be zero when they are not $k$ or $k+1$.

Similarly, the action of $C_n$ on $A^-_n$ is defined by Equations 2.2, 2.3 except that the range of $i,j$ now is from 1 to $n+1$. We denote this action by $\Phi^-$.

Again, it can be checked $\Phi^+, \Phi^-$ are both well defined.

A few remarks are in order.

**Remark 2.**

1. From now on, for a braid $\beta \in C_n$, we will write $\Phi_{\beta}, \Phi^+_{\beta}, \Phi^-_{\beta}$ for $\Phi(\beta), \Phi^+(\beta), \Phi^-(\beta)$, respectively.

2. It’s direct from the very definitions that $\Phi_\beta = (\Phi^+_{\beta})_{|A_n} = (\Phi^-_{\beta})_{|A_n}$.

It’s also clear that $\Phi^-_{\beta} = \Phi^-_{\epsilon^{-1}(\beta)}$ if we identify $A^-_n$ with $A_{n+1}$ in the obvious way.

3. From the definition of $C_n$ in Section 2.1, it’s easy to see that the subgroup generated by $\{\alpha_1, \cdots, \alpha_n\}$ is isomorphic to the classical braid group on $n$ strands. We denote this subgroup by $B_n$. In Equation 2.2, if we set $\mu = -2$ and $x = 0$, then $\Phi_{|B_n}$ acting on $\mathbb{Z}\langle a_{ij}^x \rangle$ is exactly the braid group action given in [6]. So our braid group action is a generalization of Ng’s in [6].

The above actions will be less mysterious after we give a topological interpretation.

Let $D$ be a disk in the plane, $D_n$ be the punctured disk with $n+1$ punctures labeled, from left to right, by $p, p_1, \cdots, p_n$. See Figure 3.

![Figure 3. $D_n$](image)
Let $P_n = \{p, p_i, 1 \leq i \leq n\}$ and let $Q_n = \{\gamma : [0, 1] \to D \setminus \{p\} | \gamma$ is continuous, $\gamma^{-1}(P_n) = \{0, 1\}\} / \sim$. Here $\sim$ is the equivalence relation which means two curves $\gamma_1 \sim \gamma_2$ if and only if $\gamma_1$ and $\gamma_2$ are homotopic inside $D \setminus P_n$ relative to their end points. In another way to say it, each curve connects some $p_k$ to some $p_j$ and during the homotopy it is not allowed to pass through any of the punctures and its end points are fixed. $Q_n$ is the set of equivalence classes of such curves.

Let $\tilde{A}_n$ be the free non-commutative algebra over $R$ generated by elements of $Q_n$ modulo the “skein” relations shown in Figure 4. Note that $\otimes$ in Figure 4 means the multiplication in $\tilde{A}_n$.

\begin{itemize}
  \item 1). \hfill \begin{tikzpicture}
    \draw[->, >=latex] (0.5,0) .. controls (0,0.5) and (0,0.5) .. (0,0);
    \draw[->, >=latex] (0,0) .. controls (0,0) and (0,-1) .. (-0.5,0);
    \draw[->, >=latex] (-0.5,0) .. controls (0,0) and (0,0) .. (0,0);
    \node at (-0.5,0) {$p_i$};
    \end{tikzpicture} = \mu
  \end{itemize}

\begin{itemize}
  \item 2). \hfill \begin{tikzpicture}
    \draw[->, >=latex] (-0.5,0) .. controls (0,0) and (0,0) .. (0,0);
    \draw[->, >=latex] (0,0) .. controls (0,0) and (0,1) .. (0.5,0);
    \draw[->, >=latex] (0.5,0) .. controls (0,0) and (0,0) .. (0,0);
    \node at (-0.5,0) {$p_i$};
    \end{tikzpicture} = \frac{2}{\mu} \begin{tikzpicture}
    \draw[->, >=latex] (-0.5,0) .. controls (0,0) and (0,0) .. (0,0);
    \draw[->, >=latex] (0,0) .. controls (0,0) and (0,1) .. (0.5,0);
    \draw[->, >=latex] (0.5,0) .. controls (0,0) and (0,0) .. (0,0);
    \node at (-0.5,0) {$p_i$};
    \end{tikzpicture}
\end{itemize}

\textbf{FIGURE 4. skein relation}

For $1 \leq i,j \leq n$, $x \in \mathbb{Z}$, let $\gamma_{i,j}^x$ be the curve shown in Figure 5, namely $\gamma_{i,j}^x$ starts from $p_i$, winds around $p$ counter clockwise $x$ times if $x \geq 0$, or clockwise $-x$ times if $x < 0$, and finally goes to $p_j$ through the upper half disk.

\textbf{FIGURE 5. $\gamma_{i,j}$}

It’s not hard to check that any curve can be decomposed into a (non-commutative) polynomial in $\gamma_{i,j}^x$ ‘s by repeated applications of the “skein” relations. Therefore $\tilde{A}_n$ is generated by $\gamma_{i,j}^x$ ‘s. Actually they turn out to be free generators after we construct an isomorphism between $\tilde{A}_n$ and $A_n$ below. Of-course, since $\gamma_{ii}^0 = \mu$, this doesn’t count as part of the free generators.

Now pick a base point on the boundary of the disk $D$. To make it explicit, let us pick some $z_0$ on the upper half of the boundary as the base point. The fundamental group of $D \setminus P_n$ is the free group $F_{n+1}$ on $n + 1$ generators, which we denote by $e, e_1, \cdots, e_n$, where $e_i$ is the loop that winds around $p_i$ counter clockwise once and $e$ is the loop that winds around $p$ counter clockwise once. See Figure 6.
First, we define an intermediate non-commutative algebra $B = R\langle e^\pm 1, y_1, y_2, \cdots, y_n \rangle/I$, where $I$ is the two-sided idea generated by $ee^{-1} - 1, e^{-1}e - 1$ and $\frac{4}{\mu}y_i^2 - \frac{4}{\mu}y_i, 1 \leq i \leq n$. Define a multiplicative map from $F_{n+1}$ to $B$ as follows.

$$\tau : F_{n+1} \longrightarrow B$$

(2.5) $\tau(w) = \begin{cases} 2y_i - 1 & w = e_i^\pm 1, 1 \leq i \leq n \\ e^\pm 1 & w = e^\pm 1 \\ 1 & w = 1 \end{cases}$

Clearly $\tau(e_i)\tau(e_i^{-1}) = 1 = \tau(1)$ in $B$. Therefore, we can extend the action of $\tau$ uniquely to arbitrary words to get a well-defined multiplicative map on $F_{n+1}$. Actually $\tau$ extends to an algebra morphism from the group ring $R[F_{n+1}]$ to $B$.

Next, for $1 \leq i, j \leq n$, we define an $R$-linear map $\alpha_{ij} : R\langle e^\pm 1, y_1, y_2, \cdots, y_n \rangle \longrightarrow \mathcal{A}_n$.

$$\alpha_{ij}(e^{i_1}y_{j_1}e^{i_2}y_{j_2} \cdots e^{i_k}y_{j_k}e^{i_{k+1}}) := a_{i_{j_1}j_1}a_{j_1j_2} \cdots a_{j_{k-1}j_k}a_{j_kj_j}^{i_{k+1}}$$

It's easy to check that $\alpha_{ij}$ factors through $I$ because of the fact that $a_{ii}^0 = \mu$. Therefore we get an induced map from $B$ to $\mathcal{A}_n$, which is still denoted by $\alpha_{ij}$.

Finally we can describe the isomorphism between $\tilde{\mathcal{A}}_n$ and $\mathcal{A}_n$.

Let $\gamma_i$ be the straight line from $z_0$ to $p_i$, and $\tilde{\gamma}_i$ be the same line but with direction reversed.

For any curve $\gamma \in \mathcal{Q}_n$ with $\gamma(0) = p_i, \gamma(1) = p_j$, let $\tilde{\gamma} = \gamma_i * \gamma * \gamma_j$, where $*$ means connecting the two adjacent curves. Perturb $\tilde{\gamma}$ a little so that it is off the points $p_i, p_j$, and thus becomes an element in $\pi_1(D_n, z_0) = F_{n+1}$. The perturbed curve is still denoted by $\tilde{\gamma}$.

Define the isomorphism $\psi : \tilde{\mathcal{A}}_n \longrightarrow \mathcal{A}_n$ by $\psi(\gamma) := \alpha_{ij}\tau(\tilde{\gamma})$. 

![Diagram of knot and curves](image)
Theorem 2. The map \( \psi \) defined above is an algebra isomorphism from \( \mathcal{A}_n \) to \( \mathcal{A}_n \).

Proof There are several points where we need to check that \( \psi \) is well-defined. And after that, it will be straightforward to prove \( \psi \) is an isomorphism.

Step 1: \( \psi(\gamma) \) is independent of the perturbation. Let \( \gamma \) be any curve as described above. Note that different perturbations of \( \tilde{\gamma} \) result in words in \( F_{n+1} \) different by some powers of \( e_i \) on the left end and powers of \( e_j \) on the right end of the word that represents \( \tilde{\gamma} \). Therefore, it suffices to show for any word \( w \in F_{n+1} \), \( \alpha_{ij} \tau(e_i w) = \alpha_{ij} \tau(w) = \alpha_{ij} \tau(we_j) \).

We have \( \alpha_{ij} \tau(e_i w) = \alpha_{ij}(\frac{1}{\mu} \gamma_i - 1) \tau(w) = \frac{1}{\mu} \alpha_{ii} \alpha_{ij} \tau(w) - \alpha_{ij} \tau(w) = \alpha_{ij} \tau(w) \), since \( \alpha_{ii} = \mu \).

The other equation can be proved in the same way.

Step 2: \( \psi(\gamma) \) is independent of the choice of \( \gamma \) in the equivalence class. This is trivial as homotopy equivalent \( \tilde{\gamma} \)'s result in the same group element in \( F_{n+1} \).

Step 3: \( \psi \) factors through the “skein” relations.

Clearly, \( \psi(\tilde{\gamma}_i) = \alpha_{ii} = \mu \), so the first “skein” relation holds.

Let \( C_1, C_2 \) denote the two curves passing above and below \( p_k \), respectively, in the definition of the second “skein” relation in Figure 4. They have the same initial and end points, say \( p_i, p_j \). Let \( C_3, C_4 \) be the curves which ends at \( p_k \) and starts at \( p_k \), respectively. So \( C_3 \) starts from \( p_i \) and \( C_4 \) ends at \( p_j \). Let \( w_3, w_4 \) be the words which represent \( \tilde{C}_1, \tilde{C}_2 \) are \( w_3 w_4, w_3 e_k w_4 \).

Therefore, \( \psi(C_1) + \psi(C_2) = \alpha_{ij}(\tau(w_3) \tau(w_4)) + \alpha_{ij}(\tau(w_3)(\frac{1}{\mu} \gamma_i - 1) \tau(w_4)) = \frac{2}{\mu} \alpha_{ij} \tau(w_3) \tau(w_4) + \frac{2}{\mu} \alpha_{ik} \tau(w_3) \alpha_{kj} \tau(w_4) = \frac{2}{\mu} \alpha_{ij} \tau(w), \end{array} \)

which says \( \psi \) factors through the second “skein” relation.

Step 4: The above three steps showed \( \psi \) is a well-defined algebra morphism. It’s clear that \( \psi(\tilde{\gamma}_{ij}) = a_{ij}^x \), and thus \( \psi \) is onto. Define an inverse map \( \psi' : \mathcal{A}_n \rightarrow \mathcal{A}_n \) by sending each \( a_{ij}^x \) to \( \tilde{\gamma}_{ij}^x \). Noting that \( \tilde{\gamma}_{ij}^x \) are generators of \( \mathcal{A}_n \), it’s obvious that \( \psi \psi' = Id \) and \( \psi' \psi = Id \). Therefore, \( \psi \) is an algebra isomorphism.

Now we describe a natural action of \( C_n \) on \( \mathcal{A}_n \).

Recall that the group of isotopy classes of homeomorphisms of \( D_n \) with boundary fixed point-wise is the classical braid group on \( n + 1 \) strands \( \mathcal{B}_{n+1} \). Here we assume the generators are \( \sigma_0, \sigma_1, \ldots, \sigma_{n-1} \).

Note that here \( D_n \) has \( n + 1 \) punctures.
where $\sigma_0$ is the natural Dehn twist that switches $p$ with $p_1$ counter clock-wise and $\sigma_i$ switches $p_i$ with $p_{i+1}$, $1 \leq i \leq n - 1$. Also recall that we identified $C_n$ with the subgroup of $B_{n+1}$ which consists of the braids that fix the first strand. See Section 2.1 for the explicit embedding. Therefore, the elements of $C_n$ fix the puncture $p$ and permute $\{p_i, 1 \leq i \leq n\}$, and thus act on $Q_n$. It’s also easy to see that this action actually preserves the “skein” relations. Therefore, we get a natural action $\Phi$ of $C_n$ on $A_n$.

**Theorem 3.** The algebra isomorphism $\psi : \tilde{A}_n \rightarrow A_n$ preserves the action of $C_n$, i.e. $\psi \Phi = \Phi \psi$, for any $\beta \in C_n$.

**Proof** It suffices to check for any $\beta = \alpha_k$, $\psi \Phi = \Phi \psi$ holds on the generators $\gamma_{ij}^x$. We left this as an exercise. $\square$

**Remark 3.** It’s worth pointing out that when we want to find the image of some complicated curve in $\tilde{A}_n$ under $\psi$, it’s usually more efficient to use the “skein” relations than using the definition directly. Also, instead of memorizing the action of $C_n$ on the $a_{ij}^x$’s, it’s much easier to manipulate the “skein” relations and the Dehn twists. This provides us another way to calculate the action of a braid $\beta$ on $a_{ij}^x$, namely, first use a sequence of Dehn twists representing $\beta$ to map $\gamma_{ij}^x$ to some curve, and then decompose this curve into a polynomial of generators using “skein” relations, finally replace the generators in the polynomial by the corresponding $a_{ij}^x$’s.

For example, to compute $\Phi_{a_{ij}^0}(a_{12}^0)$, we can first compute $\Phi_{a_{ij}^0}(\gamma_{12}^0)$ using Dehn twists that represent $a_{ij}^0$. See Figure 7. Then we decompose the resulting curve using “skein” relations to get the expression:

$$\Phi_{a_{ij}^0}(\gamma_{12}^0) = \gamma_{12}^{-1} - 2 \gamma_{11}^{-1} \gamma_{12} + 2 \gamma_{12}^0 \gamma_{22}^{-1} - \frac{4}{\mu^2} \gamma_{12}^0 \gamma_{21}^{-1} - \frac{4}{\mu^2} \gamma_{12} \gamma_{21}^{-1} \gamma_{12}^{-1} + \frac{8}{\mu^2} \gamma_{12} \gamma_{21} \gamma_{11}^{-1} \gamma_{12}^{-1}$$

Replacing the $\gamma_{ij}^x$’s above with $a_{ij}^x$’s, we obtain the expression for $\Phi_{a_{ij}^0}(a_{12}^0)$.

There are analogous topological interpretations of the extended actions of $C_n$ on $A_n^+$ and $A_n^-$.

The procedure goes the same as above, and we will point out what modifications should be made at each step.

First of all, let $P_n^+ = \{p, p_0, p_1, 1 \leq i \leq n\}$, $P_n^- = \{p, p_i, 1 \leq i \leq n, p_{n+1}\}$, where the puncture $p_0$ is between $p$ and $p_1$ in the disk, and $p_{n+1}$ is on the right of $p_n$. Let $D_n^\pm = D \setminus P_n^\pm$, and $Q_n^\pm$ be the set of equivalence classes of curves with end points in $P_n^\pm \setminus \{p\}$. Define $A_n^\pm$ to be the $R$-algebra generated by elements of $Q_n^\pm$ modulo the “skein” relations:
where $p_\pm = p_0$ in the "+" case and $p_\pm = p_{n+1}$ otherwise.

So we added one more relation when defining $\tilde{A}_n^\pm$, namely, the curves are allowed to pass through the new point $p_0$ ($p_{n+1}$).

The fundamental group of $D_n^\pm$ is the free group $F_{n+2}$ generated by $e, e', e_i, 1 \leq i \leq n$, where $e'$ is the generator that correspond to the new puncture $p_0$ or $p_{n+1}$. We will use the same intermediate algebra $B$, and the map $\tau$ is extended to $F_{n+2}$ by furthermore defining $\tau(e') = 1$.

In the same way as we defined the isomorphism $\psi$ from $\tilde{A}_n$ to $A_n$, we can define an isomorphism $\psi^\pm$ from $\tilde{A}_n^\pm$ to $A_n^\pm$ which sends $\gamma_{ij}^\pm$ to $\alpha_{ij}^x$.

Next, we extend the action of $C_n$ to $\tilde{A}_n^\pm$.

Recall the embedding $\epsilon^+: C_n \rightarrow C_{n+1}$ introduced in Section 2.1. For notational convenience, we denote the generators of $C_{n+1}$ by $\alpha_{-1}, \alpha_0, \ldots, \alpha_{n-1}$. Thus the embedding $\epsilon^+$ sends $\alpha_0$ to $\alpha_0 \alpha_{-1} \alpha_0$ and $\alpha_i$ to $\alpha_i$, $1 \leq i \leq n-1$.

From the geometrical point of view, $\epsilon^+$ simply inserts a strand labeled by $p_0$ right next to $\{p\} \times I$. See the first picture in Figure 9.

It's easy to see any braid in $\epsilon^+(C_n)$ fixes the first two strands (the strands that are labeled by $p$ and $p_0$). Thus it should be clear that via
the embedding \( \epsilon^+ \), the action of \( C_n \) preserves all the “skein” relations defining \( \tilde{A}^+_n \), and therefore induces an action \( \tilde{\Phi}^+ \) on \( \tilde{A}^+_n \).

For the action \( \tilde{\Phi}^- \) of \( C_n \) on \( \tilde{A}^-_n \), we use the other embedding \( \epsilon^- : \mathcal{C}_n \rightarrow \mathcal{C}_{n+1} \). Note that here the generators of \( \mathcal{C}_{n+1} \) are \( \alpha_0, \ldots, \alpha_n \), and \( \epsilon^-(\alpha_i) = \alpha_i, 0 \leq i \leq n-1 \). And the map \( \epsilon^- \) inserts a strand labeled by \( p_{n+1} \) on the right of the braid. See the second picture in Figure 9.

![Figure 9](image)

**Figure 9.** \( \epsilon^+(\alpha_1\alpha_0) \) and \( \epsilon^-(\alpha_1\alpha_0) \)

Here in Figure 9 we use \( i \) to represent \( p_i \).

Again, since elements of \( \epsilon^-(\mathcal{C}_n) \) fix \( p_{n+1} \), they preserve the “skein” relations that define \( \tilde{A}^-_n \). We get an induced action \( \tilde{\Phi}^- \) of \( C_n \) on \( \tilde{A}^-_n \).

\( \tilde{A}_n \) can obviously be embedded as a subalgebra into \( \tilde{A}^\pm_n \).

**Theorem 4.** The maps \( \psi^\pm : \tilde{A}^\pm_n \rightarrow \mathcal{A}^\pm_n \) are algebra isomorphisms and commute with the extended action of \( \mathcal{C}_n \), namely, for any \( \beta \in \mathcal{C}_n \),

\[ \psi^\pm \tilde{\Phi}^\pm_\beta = \Phi^\pm_\beta \psi^\pm. \]

Moreover, \( (\tilde{\Phi}_\beta)|_{\tilde{A}_n} = \tilde{\Phi}_\beta \), and the following diagram commutes:

\[
\begin{array}{ccc}
\tilde{A}^-_n & \xrightarrow{\psi^-} & \mathcal{A}^-_n \\
\downarrow & & \downarrow \\
\tilde{A}^+_n & \xrightarrow{\psi^+} & \mathcal{A}^+_n \\
\end{array}
\]

And each of the maps in the above diagram preserves the action of \( \mathcal{C}_n \).

**Proof** Proofs are analogous to that of Theorem 2. \( \square \)
The action of $\tilde{\Phi}^\pm$ on $\tilde{A}_n^\pm$ and the action of $\tilde{\Phi}$ on $\tilde{A}_n$ can also be visualized as follows.

For a braid $\beta \in C_n$, draw a braid diagram of $\beta$ inside $D_n \times I$. For any curve $\gamma \subset D_n \times \{0\}$ representing some element in $\tilde{A}_n$, push $\gamma$ to $D_n \times \{1\}$ along the braid diagram to get a curve $\gamma'$. Then $\gamma' = \tilde{\Phi}_\beta(\gamma)$.

To visualize $\tilde{\Phi}_\beta^\pm$, we draw a braid diagram of $\epsilon^\pm(\beta)$ inside $D_n^+ \times I$, and push any curve along the braid diagram up to $D_n^+ \times \{1\}$.

With the above observations, we have the following simple but important proposition.

**Proposition 1.** Let $\beta \in C_n$ be a braid.

1). For any two curves $\gamma_1, \gamma_2 \in Q_n^+$, such that $\gamma_1(1) = \gamma_2(0) = p_0$ and $\gamma_1(0) = p_i, \gamma_2(1) = p_j$ for some $1 \leq i, j \leq n$, then $\gamma_1 \ast \gamma_2$ is a curve in $Q_\gamma$ from $p_i$ to $p_j$, and we have $\tilde{\Phi}_\beta^+(\gamma_1 \ast \gamma_2) = \tilde{\Phi}_\beta^+(\gamma_1) \ast \tilde{\Phi}_\beta^+(\gamma_2)$.

2). For any two curves $\gamma_1, \gamma_2 \in Q_n^-$, such that $\gamma_1(1) = \gamma_2(0) = p_{n+1}$ and $\gamma_1(0) = p_i, \gamma_2(1) = p_j$ for some $1 \leq i, j \leq n$, then $\gamma_1 \ast \gamma_2$ is a curve in $Q_\gamma$ from $p_i$ to $p_j$, and we have $\tilde{\Phi}_\beta^-(\gamma_1 \ast \gamma_2) = \tilde{\Phi}_\beta^-(\gamma_1) \ast \tilde{\Phi}_\beta^-(\gamma_2)$.

Here $\ast$ again means connecting the two curves.

**Remark 4.** From now on, we will identify $\tilde{A}_n$ with $A_n$, $\tilde{A}_n^\pm$ with $A_n^\pm$, $\gamma_\beta^x$ with $\alpha_\beta^x$ via the corresponding isomorphisms and identify $\tilde{\Phi}_\beta$ with $\Phi_\beta^\pm$ with $\Phi_\beta^\pm$, respectively. A useful picture to be kept in mind is as follows. $\alpha_\beta^x$ is the arc diagram described in Figure 5. The action $\Phi_\beta(\Phi_\beta^\pm)$ of $\beta$ on some curve is to push that curve along the braid diagram that represents $\beta(\epsilon^\pm(\beta))$ up to $D_n \times \{1\} (D_n^+ \times \{1\})$.

Due to the above remark, we can also define the "$\ast$" operation on some elements of $A_n$.

**Definition 1.** 1). Let $P, Q \in A_n^+$ such that $P = \sum_{x \in \mathbb{Z}} \sum_{i=1}^n P_i^x a_{0,i}^x, \ Q = \sum_{y \in \mathbb{Z}} \sum_{j=1}^n P_i^y Q_j^y, P_i^x, Q_j^y \in A_n$, then $P \ast Q \in A_n$ is defined to be $\sum_{x,y \in \mathbb{Z}} \sum_{i,j=1}^n P_i^x a_{ij}^{x+y} Q_j^y$.

2). Let $P, Q \in A_n^-$ such that $P = \sum_{x \in \mathbb{Z}} \sum_{i=1}^n P_i^x a_{i,n+1}^x, \ Q = \sum_{y \in \mathbb{Z}} \sum_{j=1}^n a_{n+1,j}^y Q_j^y, P_i^x, Q_j^y \in A_n$, then $P \ast Q \in A_n$ is defined to be $\sum_{x,y \in \mathbb{Z}} \sum_{i,j=1}^n P_i^x a_{ij}^{x+y} Q_j^y$.

3). Two elements $P, Q \in A_n^\pm$ are called connectable, if they satisfy the condition in one of the above two definitions.

**Proposition 2.** If $P, Q \in A_n^\pm$ are connectable, then for any $\beta \in C_n$, $\Phi_\beta^\pm(P), \Phi_\beta^\pm(Q)$ are also connectable, and $\Phi_\beta(P \ast Q) = \Phi_\beta^\pm(P) \ast \Phi_\beta^\pm(Q)$.
Proof We will only prove the “+” case. The proof of the other case is analogous.

Let \( P, Q \) be as described in 1) of Definition 1, then for \( \beta \in \mathbb{C}_n \),
\[
\Phi^+_\beta(P) = \sum_{x \in \mathbb{Z}} \sum_{i=1}^n \Phi^+_\beta(P_i^x) \Phi^+_\beta(a_{i0}^x) = \sum_{x \in \mathbb{Z}} \sum_{i=1}^n \Phi^+_\beta(P_i^x) \Phi^+_\beta(a_{i0}^x),
\]
and similarly, \( \Phi^+_\beta(Q) = \sum_{y \in \mathbb{Z}} \sum_{j=1}^n \Phi^+_\beta(a_{0j}^y) \Phi^+_\beta(Q_j^y) \). Clearly, \( \Phi^+_\beta(a_{i0}^x) \) and \( \Phi^+_\beta(a_{0j}^y) \) are connectable, so \( \Phi^+_\beta(P) \) and \( \Phi^+_\beta(Q) \) are connectable. Moreover,
\[
\Phi^+_\beta(P) \ast \Phi^+_\beta(Q) = \sum_{x,y \in \mathbb{Z}} \sum_{i,j=1}^n \Phi^+_\beta(P_i^x) \{\Phi^+_\beta(a_{i0}^x) \ast \Phi^+_\beta(a_{0j}^y)\} \Phi^+_\beta(Q_j^y) = \sum_{x,y \in \mathbb{Z}} \sum_{i,j=1}^n P_i^x a_{i0}^{x+y} Q_j^y = \Phi^+_\beta(P \ast Q).
\]

Proposition 3.1

3. The knot invariant

In this section, first we give the definition of the knot invariant. Since the knot invariant looks very complicated at first glance, we will compute some examples after the definition. We then proceed to give some ancillary results, and finally prove the invariance under Markov moves.

3.1. Definition of the invariant. Here are some notations we will use to define the invariant.

Let \( M_\infty(\mathbb{A}_n) \) denote the group of \( \mathbb{A}_n \times \mathbb{A}_n \) matrices with elements in \( \mathbb{A}_n \), namely, the rows and columns of a matrix in \( M_\infty(\mathbb{A}_n) \) are both indexed by integers. We call a matrix row-finite if there are only finitely many non-zero entries in each row. A column-finite matrix is defined analogously. If \( M, N \) are two matrices in \( M_\infty(\mathbb{A}_n) \), in general the multiplication of them is not well-defined. However, if \( M \) is row-finite or \( N \) is column-finite, then \( M \cdot N \) is well-defined. And the associativity is satisfied whenever multiplications make sense. All throughout the paper, the matrices always satisfy the above condition when they are multiplied together, and for \( x, y \in \mathbb{Z} \), we will use \( M^{xy} \) to refer to the \((x, y)\)-entry of \( M \). Let \( M_n(M_\infty(\mathbb{A}_n)) \) denote the set of \( n \times n \) matrices with entries in \( M_\infty(\mathbb{A}_n) \).

Recall \( \epsilon^\pm : \mathbb{C}_n \rightarrow \mathbb{C}_{n+1} \) are the two embeddings, and for \( \beta \in \mathbb{C}_n \),
\[
(\Phi^\pm_\beta)|_{\mathbb{A}_n} = \Phi_\beta = (\Phi^\pm_\beta)|_{\mathbb{A}_n}.
\]

It’s not hard see (perhaps easier from the topological interpretation) that for \( 1 \leq i \leq n, x \in \mathbb{Z} \), \( \Phi^-_\beta(a_{i,n+1}^x) \) can be written as a finite
linear combinations of $a^z_{k,n+1}, 1 \leq k \leq n, z \in \mathbb{Z}$ with coefficients in $\mathcal{A}_n$. A similar argument holds for $\Phi^-_\beta(a^x_{n+1,i}), \Phi^+_\beta(a^x_{i,0}), \Phi^+_\beta(a^z_{0,i})$. For example, $\Phi^+_\beta(a^x_{0,i})$ is a finite linear combinations of $a^z_{0,k}$ with coefficients in $\mathcal{A}_n$ multiplied on the right. Explicitly, this is how we define $\Phi^-_\beta, \Phi^+_\beta \in M_n(M_\infty(\mathcal{A}_n)).$

For each $\beta \in \mathcal{C}_n, 1 \leq i, j \leq n, x, y \in \mathbb{Z}$, define

$$
\Phi^-_\beta(a^x_{i,n+1}) = \sum_{k=1}^{n} \sum_{z \in \mathbb{Z}} (\Phi^-_\beta)^{xz}_{ik} a^z_{k,n+1}
$$

$$
\Phi^-_\beta(a^y_{n+1,j}) = \sum_{k=1}^{n} \sum_{z \in \mathbb{Z}} a^z_{n+1,k} (\Phi^-_\beta)^{zy}_{kj}
$$

$$
\Phi^+_\beta(a^x_{i,0}) = \sum_{k=1}^{n} \sum_{z \in \mathbb{Z}} (\Phi^+_\beta)^{xz}_{ik} a^z_{k,0}
$$

$$
\Phi^+_\beta(a^y_{0,j}) = \sum_{k=1}^{n} \sum_{z \in \mathbb{Z}} a^z_{0,k} (\Phi^+_\beta)^{zy}_{kj}
$$

where $(\Phi^-_\beta)^{xz}_{ik}$ is the $(x,z)$-entry of the $\infty \times \infty$ matrix $(\Phi^-_\beta)^{xz}_{ik}$ which is the $(i,k)$-entry of the $n \times n$ matrix $\Phi^-_\beta$. So we have $\Phi^-_\beta \in M_n(M_\infty(\mathcal{A}_n))$. A similar statement holds for the other three symbols.

One more notation. Define $a_{ij} \in M_\infty(\mathcal{A}_n)$ by $(a_{ij})^{xy} = a_{ij}^{x+y}$ and define $A \in M_n(M_\infty(\mathcal{A}_n))$ by $A_{ij} = a_{ij}$.

**Lemma 2.** For $\beta \in \mathcal{C}_n, 1 \leq i, j \leq n$, $(\Phi^-_\beta)^{ij}_{ij}, (\Phi^+_\beta)^{ij}_{ij}$ are row-finite and $(\Phi^-_\beta)^{ij}_{ij}, (\Phi^+_\beta)^{ij}_{ij}$ are column-finite.

**Proof** These are direct consequences of the definitions. □

**Remark 5.** Actually, $(\Phi^+_\beta)^{ij}_{ij}, (\Phi^+_\beta)^{ij}_{ij}$ are both row-finite and column-finite. This is due to a careful inspection of the action $\Phi^+_\beta$. Since we will not use this property, we omit its proof.

**Definition 2.** Let $\beta \in \mathcal{C}_n$, the 0-th knot contact homology of $\beta$, $HC_0(\beta)$, is defined to be the $R$-algebra $\mathcal{A}_n$ modulo the two sided idea $\mathcal{I}_\beta$ generated by the entries of the entries of following matrices:

1. $A - \Phi^-_\beta A$
2. $A - A\Phi^-_\beta$
3. $A - \Phi^+_\beta A$
4. $A - A\Phi^+_\beta$
Remark 6. (1) For some matrix $M \in M_n(M_\infty(A_n))$, the phrase “the entries of the entries of $M$” is really awkward. From now on, we will use “the elements of $M$” to stand for “the entries of the entries of $M$”.

(2) Note that $(\Phi_\beta^L A)_{ij}^{xy} = \sum_{k=1}^{\infty} \sum_{z \in \mathbb{Z}} (\Phi_\beta^L)_{ik}^{xz} a_{kj}^{zy} = \sum_{k=1}^{\infty} \sum_{z \in \mathbb{Z}} ((\Phi_\beta^L)_{ik}^{xz} a_{kj}^{zy})*$

\[ a_{n+1,j} = \Phi_\beta(a_{i,n+1}) + a_{n+1,j}, \]

Similarly, we have $(A\Phi_\beta^R)_{ij}^{xy} = a_{i,n+1} * \Phi_\beta(a_{n+1,j}), (\Phi_\beta^L A)_{ij}^{xy} = \Phi_\beta(a_{i,n+1}) * a_{n+1,j}, (\Phi_\beta^L)_{ij}^{xy} = a_{i,n+1} * \Phi_\beta(a_{n+1,j}).$

Since $A_{ij} = a_{ij}^{x+y} = a_{i,n+1} * a_{n+1,j}$. So the relations in $I_\beta$ are the same as the following:

\[ a_{i+1,j} * a_{i,n+1,j} = \Phi_\beta(a_{i,n+1}) * a_{n+1,j}, \]

\[ a_{i+1,n} * a_{n+1,j} = \Phi_\beta(a_{i,n+1}) * a_{n+1,j}, \]

\[ a_{i,0} * a_{i,0} = \Phi_\beta(a_{i,0}) * a_{i,0}, \]

\[ a_{i,0} * a_{i,0} = \Phi_\beta(a_{i,0}) * a_{i,0}, \forall 1 \leq i, j \leq n, x, y \in \mathbb{Z}. \]

The following theorem is our main result.

**Theorem 5.** If $\beta_1, \beta_2$ are two braids in $\prod_{n=1}^{\infty} C_n$ whose closure represent the same knot in $S^1 \times S^2$, then $HC_0(\beta_1)$ and $HC_0(\beta_2)$ are isomorphic as $R$-algebras.

We will give the proof of the theorem in Section 3.4.

Following the theorem, we thus can define the 0-th contact homology, $HC_0(K)$, of a knot $K$ in $S^1 \times S^2$ to be $HC_0(\beta)$ for any $\beta \in C_n$ such that the closure of $\beta$ is $K$.

**Corollary 1.** $HC_0(K)$ as an $R$-algebra is a knot invariant for knots in $S^1 \times S^2$.

Remark 7. As the name “the 0-th knot contact homology” indicates, this invariant is conjectured to be the 0-th Legendrian contact homology of $\Lambda_K$ in $ST^*(S^1 \times S^2)$, where $ST^*(S^1 \times S^2)$ is the unit cotangent bundle of $S^1 \times S^2$ and $\Lambda_K$ is the unit conormal bundle of $K$. As this paper is not relevant to proving this conjecture, the readers can just treat $HC_0$ purely as a name.

3.2. Examples. Before giving the proof of the invariance, let us first look at some examples.

**Unknot.** The most simple example is the unknot represented by the identity element $e$ in $C_1$. In this case, it’s clear that $\Phi_e^L, \Phi_e^R, \Phi_e^L, \Phi_e^R$ are all identity matrices, thus all the relations in $I_e$ become 0, and so $HC_0(\text{unknot}) \simeq R(a_{11}^x, x \in \mathbb{Z})$, the free non-commutative algebra generated by $a_{11}^x, x \in \mathbb{Z}$.

$\alpha_0^2$. Let $\beta = \alpha_0^2$. We first compute $\Phi_\beta^L, \Phi_\beta^R$. Direct calculations show that $\Phi_\beta^L(a_{10}^x) = a_{10}^{x-2}$, $\Phi_\beta^R(a_{01}^y) = a_{01}^{y+2}$. Thus we have $(\Phi_\beta^L)_{11}^{xy} =
Therefore, we have $HC_{x,y} = \mu_0$, then the above two relations both become $a_{11}^{x+y+2}$. So the third and fourth relation defining $I_\beta$ both are $a_{11}^{x+2} - a_{11}^{x-2}, x \in \mathbb{Z}$.

Now we compute $\Phi^L_\beta, \Phi^R_\beta$. By definition, $\Phi^-_\alpha(a_{11}) = a_{11}^{x_1}$, $\Phi^-_\alpha(a_{11}) = -a_{12}^{-1} + 2a_{11}^{-1}$. Therefore,

$$\Phi^-_\alpha(a_{12}) = -\Phi^-_\alpha(a_{12}^{-1}) + \frac{2}{\mu} \Phi^-_\alpha(a_{11}) \Phi^-_\alpha((a_{12}^{-1}) = a_{12}^{-2} - 2a_{12}^{-1}a_{12}^{-1} - 2a_{12}^{-1}a_{12}^{-2} + \frac{4}{\mu^2} a_{11}^{-1}a_{11}^{-1} a_{12}^{-1}.$$

By Part (2) of Remark 6,

$$(\Phi^L_\beta A)^{xy}_{11} - A^{xy}_{11} = a_{11}^{x+y} - 2a_{12}^{x-1}a_{12}^{-1} - a_{11}^{x+y} = a_{11}^{x+y}.$$  

Similarly,

$$(A \Phi^R_\beta)^{xy}_{11} - A^{xy}_{11} = a_{11}^{x+y+2} - 2a_{12}^{x+1}a_{12}^{-1} - a_{11}^{x+y} = a_{11}^{x+y+2} - 2a_{12}^{x+1}a_{12}^{-1}.$$  

Since we have $a_{11}^{2k} = \mu, a_{11}^{2k+1} = a_{11}^1$, only parities of $x$ and $y$ will make a difference in the above two relations. For example, set $x = y = 0$, then the above two relations both become $2\mu - 2a_{11}^1$. It could be checked that other parities of $x, y$ will not add to more relations. Therefore, we have $HC_0(\hat{\beta}) \cong R[X]/(2\mu - \frac{2}{\mu} X^2)$.

It will be shown in Section 4.2 that $\hat{\alpha}_0^2$ is a particular knot in a large family of knots, namely the torus knots. Explicitly, it is the $(1, 2)$-knot. See Section 4.2 for a definition of torus knots and more examples.

3.3. Properties of $\Phi^\pm L, \Phi^\pm R$. We give several propositions which will be used in proving the invariance of $HC_0(K)$. A similar version of these propositions are proved in [6] where the author defined the $HC_0$ invariant for knots in $S^3$.

If $\phi$ is an algebra morphism from $A_n$ to $A_n$, and $M \in M_n(M_\infty(A_n))$, we denote by $\phi(M)$ or $M(\phi)$ the matrix obtained from $M$ by replacing each $a_{ij}$ by $\phi(a_{ij})$. Recall in last subsection, we defined the four matrices $\Phi^L_\beta, \Phi^R_\beta, \Phi^+_L, \Phi^+_R \in M_n(M_\infty(A_n))$ for $\beta \in C_n$.

**Proposition 3.** Let $\beta_1, \beta_2 \in C_n$ be two braids, then we have

$$\Phi^L_{\beta_1 \beta_2} = \Phi^L_{\beta_2}(\Phi_{\beta_1}) \Phi^L_{\beta_1}$$

$$\Phi^R_{\beta_1 \beta_2} = \Phi^R_{\beta_2}(\Phi_{\beta_1})$$

$$\Phi^+_{\beta_1 \beta_2} = \Phi^+_{\beta_2}(\Phi_{\beta_1})$$

$$\Phi^+_{\beta_1 \beta_2} = \Phi^+_{\beta_2}(\Phi_{\beta_1})$$
Proof The proof of the four equalities are straightforward and completely analogous, so we will just prove the first one.

By definition, \( \Phi_{\beta_1\beta_2}(a_{i,n+1}^x) = \sum_{k=1}^{n} \sum_{z \in \mathbb{Z}} (\Phi_{\beta_2}^{-L})_{zk}^x a_{k,n+1}^z \). So

\[
\begin{align*}
\Phi_{\beta_1\beta_2}(a_{i,n+1}^x) &= \Phi_{\beta_1}(a_{i,n+1}^x) \\
\sum_{z=1}^{n} \sum_{\mathbb{Z}} \Phi_{\beta_1}((\Phi_{\beta_2}^{-L})_{zk}^x) \Phi_{\beta_1}(a_{k,n+1}^z) \\
= \sum_{z=1}^{n} \sum_{j=1}^{n} \sum_{Z \in \mathbb{Z}} \Phi_{\beta_2}^{-L}(\Phi_{\beta_1})_{zk}((\Phi_{\beta_2}^{-L})_{zk}^y) a_{j,n+1}^y \\
= \sum_{j=1}^{n} \sum_{y \in \mathbb{Z}} (\Phi_{\beta_2}^{-L}(\Phi_{\beta_1})_{yi}) a_{j,n+1}^y
\end{align*}
\]

On the other hand, by definition, \( \Phi_{\beta_1\beta_2}(a_{i,n+1}^x) = \sum_{j=1}^{n} \sum_{z \in \mathbb{Z}} (\Phi_{\beta_1\beta_2})_{ij} a_{j,n+1}^y \).

Therefore, we have \( (\Phi_{\beta_2}^{-L}(\Phi_{\beta_1})_{yi}) = (\Phi_{\beta_1\beta_2})_{ij} \). \( \square \)

Let \( I_n \in M_n(M_{\infty}(A_n)) \) be the identity matrix, i.e \( (I_n)_{ij} = \delta_{ij}\delta_{x,y} \). Then apparently, for a trivial braid \( \beta \in C_n \), \( \Phi_{\beta}^{-L}, \Phi_{\beta}^{-R}, \Phi_{\beta}^{+L}, \Phi_{\beta}^{+R} \) are all invertible. Explicitly,

\[
(\Phi_{\beta}^{-L})^{-1} = \Phi_{\beta}^{-L}(\Phi_{\beta}), \quad (\Phi_{\beta}^{-R})^{-1} = \Phi_{\beta}^{-R}(\Phi_{\beta}), \\
(\Phi_{\beta}^{+L})^{-1} = \Phi_{\beta}^{+L}(\Phi_{\beta}), \quad (\Phi_{\beta}^{+R})^{-1} = \Phi_{\beta}^{+R}(\Phi_{\beta}).
\]

Proof In Proposition 3 set \( \beta_1 = \beta, \beta_2 = \beta^{-1} \). \( \square \)

The following proposition is central in our proof of invariance.

Proposition 4. For any \( \beta \in C_n \), we have \( \Phi_{\beta}(A) = \Phi_{\beta}^{-L}A\Phi_{\beta}^{-R} = \Phi_{\beta}^{+L}A\Phi_{\beta}^{+R} \).

Proof By Proposition 3 it suffices to show the above equation holds for any \( \alpha_k \in C_n \). This can be verified directly, though maybe tediously.

Here we provide another way to prove it.

\[
\begin{align*}
\Phi_{\beta}(A_{ij}^{x+y}) &= \Phi_{\beta}(a_{i,n+1}^x \ast a_{n+1,j}^y) \\
\sum_{k=1}^{n} \sum_{z \in \mathbb{Z}} (\Phi_{\beta}^{-L})_{zk}^x a_{k,n+1}^z \ast (\sum_{k'=1}^{n} \sum_{z' \in \mathbb{Z}} a_{n+1,k'}^y (\Phi_{\beta}^{-R})_{k'z'}^y)
\end{align*}
\]
Corollary 3. For $\beta \in C_n$, the image of $Id - \Phi_\beta$ is contained in $I_\beta$.

Proof It suffices to show $a_i^x - \Phi_\beta(a_i^x) \in I_\beta$, which follows from the following equations:

$$A - \Phi_\beta(A) = A - \Phi_\beta^{-L}A\Phi_\beta^{-R} = A - \Phi_\beta^{-L}A + \Phi_\beta^{-L}(A - A\Phi_\beta^{-R}).$$

3.4. Invariance proof. In this subsection, we prove Theorem 5 namely, if the closure of two braids $\beta_1, \beta_2 \in C = \sqcup C_n$ represent the same knots, then $HC_0(\beta_1)$ and $HC_0(\beta_2)$ are isomorphic as $R$-algebras. By Theorem 1 we only need to show the three Markov moves given in that theorem preserve the isomorphism class of $HC_0(\beta)$ for $\beta \in C_n$.

3.4.1. Invariance under Markov Move I. Let $\tilde{\beta} = \alpha^{-1}\beta\alpha$, $\alpha, \beta \in C_n$. We define an isomorphism $\varphi : HC_0(\tilde{\beta}) \longrightarrow HC_0(\beta)$ by specifying the image of the generators.

$$\varphi(A) := \Phi_\alpha(A), \text{i.e. } \varphi(a_{ij}^x) := \Phi_\alpha(a_{ij}^x)$$

We need to show $\varphi(I_\beta) \subset I_\beta$.

First of all, by using Proposition 3, we have

$$\Phi_\alpha(\Phi_\beta^{-L}a_{\beta}\alpha^{-1}) = \Phi_\alpha(\Phi_\beta^{-L}(\Phi_\alpha^{-1})(\Phi_\beta^{-L})) = \Phi_\alpha^{-L}(\Phi_\beta^{-L})\Phi_\alpha^{-L}(\Phi_\alpha^{-1})(\Phi_\beta^{-L})\Phi_\alpha^{-L}(\Phi_\alpha^{-1})(\Phi_\beta^{-L})\Phi_\alpha^{-L}(\Phi_\alpha^{-1})(\Phi_\beta^{-L})$$

Therefore, we have

$$\varphi(A - \Phi_\beta^{-L}A) = \varphi(A) - \varphi(\Phi_\beta^{-L}\varphi(A))$$

$$=$$$\Phi_\alpha(A) - \Phi_\alpha(\Phi_\alpha^{-L}\Phi_\alpha^{-L}\Phi_\alpha^{-L})\Phi_\alpha^{-L}(\Phi_\alpha^{-L})\Phi_\alpha^{-L}(\Phi_\alpha^{-L})A\Phi_\alpha^{-L}$$

$$=$$$\Phi_\alpha^{-L}(\Phi_\alpha^{-L})\Phi_\alpha^{-L}(\Phi_\alpha^{-L})\Phi_\alpha^{-L}(\Phi_\alpha^{-L})\Phi_\alpha^{-L}(\Phi_\alpha^{-L})\Phi_\alpha^{-L}(\Phi_\alpha^{-L})A\Phi_\alpha^{-L}$$

$$=$$$\Phi_\alpha^{-L}(\Phi_\alpha^{-L})\Phi_\alpha^{-L}(\Phi_\alpha^{-L})\Phi_\alpha^{-L}(\Phi_\alpha^{-L})\Phi_\alpha^{-L}(\Phi_\alpha^{-L})\Phi_\alpha^{-L}(\Phi_\alpha^{-L})A\Phi_\alpha^{-L}$$

By Corollary 3 elements of $\Phi_\alpha^{-L} - \Phi_\alpha^{-L}(\Phi_\beta)$ are in $I_\beta$. Since elements of $A - \Phi_\beta^{-L}A$ are also in $I_\beta$, this implies $\varphi(A - \Phi_\beta^{-L}A) \subset I_\beta$.

Similarly,
\[ \varphi(A - A\Phi^{-R}_{\beta}) = \varphi(A) - \varphi(A)\varphi(\Phi^{-R}_{\beta}) \]
\[ = \Phi_{\alpha}(A) - \Phi_{\alpha}(A)\Phi_{\alpha}(\Phi^{-R}_{\alpha - 1\beta}) \]
\[ = \Phi_{\alpha}(A) - \Phi_{\alpha}(A)\Phi_{\alpha}(\Phi^{-R}_{\alpha - 1\beta}\Phi^{-R}_{\beta}(\Phi_{\alpha - 1})) \]
\[ = \Phi_{\alpha}^{-L}A\Phi^{-R}_{\alpha} - \Phi_{\alpha}^{-L}A\Phi^{-R}_{\beta}\Phi^{-R}_{\alpha}(\Phi_{\beta}) \]
\[ = \Phi_{\alpha}^{-L}(A - A\Phi^{-R}_{\beta})\Phi^{-R}_{\alpha} + \Phi_{\alpha}^{-L}A\Phi^{-R}_{\beta}(\Phi^{-R}_{\alpha} - \Phi^{-R}_{\alpha}(\Phi_{\beta})) \]

Again by Corollary 3, elements of \( \Phi^{-R}_{\alpha} - \Phi^{-R}_{\alpha}(\Phi_{\beta}) \) and \( A - A\Phi^{-R}_{\beta} \) are in \( I_{\beta} \). This implies \( \varphi(A - A\Phi^{-R}_{\beta}) \subseteq I_{\beta} \).

The proofs of the other two cases \( \varphi(A - \Phi_{\beta}^{+L}A) \), \( \varphi(A - A\Phi^{+R}_{\beta}) \) are completely analogous.

This shows \( \varphi(I_{\beta}) \subseteq \varphi(I_{\beta}) \) and thus induces a well-defined map \( HC_{0}(\tilde{\beta}) \rightarrow HC_{0}(\beta) \). In a similar way, we can define the inverse map \( HC_{0}(\beta) \rightarrow HC_{0}(\tilde{\beta}) \) by sending \( A \) to \( \Phi_{\alpha - 1}(A) \) and show that it is well defined. Thus \( \varphi \) is an isomorphism.

3.4.2. Invariance under Markov Move II. This subsection contains technical details so the impatient readers may skip to the next subsection.

We need to introduce several conventions first. For an \( n \times n \) matrix \( M \), \( M_{k,m} \) means the sub matrix formed by the first \( k \) rows and first \( m \) columns. \( M_{1\ldots k,m} \) means the \( k \times 1 \) matrix whose entries are the first \( k \) elements in the \( m \)-th column of \( M \). Similarly there is the notion \( M_{m,1\ldots k} \). \( M(k,m) \) means the \( (k,m) \)-entry of \( M \). Finally, \( M(\cdot,k) \), \( M(\cdot,m) \) stand for the \( k \)-th row and the \( m \)-th column, respectively.

For any \( \beta \in C_n \), let \( \tilde{\beta} = \epsilon^{-}(\beta)\alpha_n \). We show \( HC_{0}(\tilde{\beta}) \simeq HC_{0}(\beta) \).

**Remark 8.** The proof of \( HC_{0}(\epsilon^{-}(\beta)\alpha^{-1}_n) \simeq HC_{0}(\beta) \) is completely analogous. To save space, we omit its proof here.

**Step 1:** We first write down explicitly all the relations that generate the ideal \( I_{\beta} \).

We have the following formula for \( \Phi^{-L}_{\epsilon^{-}(\beta)} \) and \( \Phi^{-L}_{\alpha_n} \):

\[
\Phi^{-L}_{\epsilon^{-}(\beta)} = \left( \begin{array}{c|c} \Phi^{-L}_{\beta} & 0 \\ \hline (\Phi^{-L}_{\epsilon^{-}(\beta)})_{n+1,1\ldots n} & \Phi^{-L}_{\epsilon^{-}(\beta)}(n + 1, n + 1) \end{array} \right)
\]
\[ \Phi_{\alpha_n}^{-L} = \begin{pmatrix} I_{n-1} & 0 \\ 0 & \frac{2}{\mu} a_{n+1,n}^0 -1 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \]

Note that the entries of \( \Phi_{\alpha_n}^{-L} \) are elements of \( M_\infty(A_n) \), thus each scalar number present in \( \Phi_{\alpha_n}^{-L} \) actually means a scalar \( \infty \times \infty \) matrix.

By using the formula

\[ \Phi_{\beta_1 \beta_2}^{-L} = \Phi_{\beta_2}^{-L}(\Phi_{\beta_1}) \Phi_{\beta_1}^{-L} \]

we can translate \( A = \Phi_{\beta}^{-L}A \) in \( HC_0(\tilde{\beta}) \) to the following equations.

(3.1) \( (\Phi_{\beta}^{-L})_{n-1,n} A_{n,n} = A_{n-1,n} \)

(3.2) \( \Phi_{\beta}^{-L}(n, \cdot) A_{n,n} = A_{n+1,1-\cdots} \)

(3.3) \( \Phi_{\beta}^{-L} A_{1-\cdots,n,n+1} = \begin{pmatrix} A_{1-\cdots,n,n+1} \\ a_{n+1,n+1} \end{pmatrix} \)

(3.4) \( - (\Phi_{\epsilon^{-}(\beta)}^{-L} A)_{n+1,1-\cdots} + \frac{2}{\mu} \Phi_{\epsilon^{-}(\beta)}(a_{n+1,n}^0) A_{n+1,1-\cdots} = A_{n,1-\cdots} \)

(3.5) \( - (\Phi_{\epsilon^{-}(\beta)}^{-L} A)(n+1, n+1) + \frac{2}{\mu} \Phi_{\epsilon^{-}(\beta)}(a_{n+1,n}^0) a_{n+1,n+1} = a_{n,n+1} \)

Analogously, we can compute \( \Phi_{\epsilon^{-}(\beta)}^{-R} \) and \( \Phi_{\alpha_n}^{-R} \)

\[ \Phi_{\epsilon^{-}(\beta)}^{-R} = \begin{pmatrix} \Phi_{\beta}^{-R} & (\Phi_{\epsilon^{-}(\beta)}^{-R})_{1-\cdots,n,n+1} \\ 0 & \Phi_{\epsilon^{-}(\beta)}^{-R}(n+1, n+1) \end{pmatrix} \]

\[ \Phi_{\alpha_n}^{-R} = \begin{pmatrix} I_{n-1} & 0 \\ 0 & \frac{2}{\mu} a_{n,n+1}^0 -1 \\ 0 & 1 \end{pmatrix} \]

And by using the formula
\[
\Phi_{\beta_1 \beta_2}^{-R} = \Phi_{\beta_1}^{-R} \Phi_{\beta_2}^{-R} (\Phi_{\beta_1})
\]

we translate \( A = A \Phi_{\beta}^{-R} \) in \( HC_0(\tilde{\beta}) \) to the following equations.

(3.6) \[
A_{n,n}(\Phi_{\beta}^{-R})_{n,n-1} = A_{n,n-1}
\]

(3.7) \[
A_{n,n} \Phi_{\beta}^{-R}(\cdot, n) = A_{1\ldots n,n+1}
\]

(3.8) \[
A_{n+1,1\ldots n} \Phi_{\beta}^{-R} = (A_{n+1,1\ldots n-1}, a_{n+1,n+1})
\]

(3.9) \[
-(A \Phi_{\epsilon^{-}(\beta)})_{1\ldots n,n+1} + \frac{2}{\mu} A_{1\ldots n,n+1} \Phi_{\epsilon^{-}(\beta)}(a_{n,n+1}^0) = A_{1\ldots n,n}
\]

(3.10) \[
-(A \Phi_{\epsilon^{-}(\beta)})(n+1, n+1) + \frac{2}{\mu} a_{n+1,n+1} A_{1\ldots n,n+1} \Phi_{\epsilon^{-}(\beta)}(a_{n,n+1}^0) = a_{n+1,n}
\]

The equations that correspond to \( A = \Phi_{\beta}^{+L} A \) and \( A = A \Phi_{\beta}^{+R} \) are obtained from the above equations by changing \( \Phi^{-L}, \Phi^{-R} \) to \( \Phi^{+L}, \Phi^{+R} \).

Also notice that \((\Phi_{\epsilon^-}^{-L})_{0x}^{n+1,i} = \delta_{n+1,i}^0, x \) and \((\Phi_{\epsilon^-}^{-R})_{0x}^n = \delta_{n+1,i}^0, x \).

Then by the formula \( \Phi_{\beta}(A) = \Phi_{\beta}^{-L} A \Phi_{\beta}^{-R} \), we have \( \Phi_{\epsilon^{-}(\beta)}(a_{n,n+1}^{x_0}) = (\Phi_{\epsilon^{-}(\beta)} A)_{i,n+1}^{x_0} \) and \( \Phi_{\epsilon^{-}(\beta)}(a_{n+1,i}^{x_0}) = (A \Phi_{\epsilon^{-}(\beta)} A)_{n+1,i}^{x_0} \), for any \( i = 1, \ldots n+1 \). Explicitly, we have the following equations held in \( \mathcal{A}_{n+1} \).

(3.11) \[
\Phi_{\epsilon^-(\beta)}(A)_{1\ldots n,n+1}^{x_0} = (\Phi_{\beta}^{-L} A_{1\ldots n,n+1})^{x_0}
\]

(3.12) \[
\Phi_{\epsilon^-(\beta)}(a_{n+1,n+1}^{x_0}) = (\Phi_{\epsilon^-(\beta)} A)^{x_0}_{n+1,n+1}
\]

(3.13) \[
\Phi_{\epsilon^-(\beta)}(A)_{n+1,1\ldots n}^{0x} = (A_{n+1,1\ldots n} \Phi_{\beta}^{-R})^{0x}
\]

(3.14) \[
\Phi_{\epsilon^-(\beta)}(a_{n+1,n+1})^{0x} = (A A_{\beta}^{-R})^{0x}_{n+1,n+1}
\]

Step 2: Before giving the isomorphism from \( HC_0(\tilde{\beta}) \) to \( HC_0(\beta) \), we deduce the following relations in \( HC_0(\tilde{\beta}) \):

\[ a_{n+1,i}^x = a_{n,i}^x, \quad a_{i,n+1}^x = a_{i,n}^x, \forall i = 1, \ldots n+1 \]
Firstly,

\[
\Phi_{\epsilon-(\beta)}(a_{n,n+1}^x) \stackrel{\text{Equ. (3.11)}}{=} \left(\Phi_{\beta}^{-L}(n, \cdot) A_{1 \ldots n,n+1}\right)^{x0} \stackrel{\text{Equ. (3.3)}}{=} a_{n+1,n+1}^x.
\]

\[
\Phi_{\epsilon-(\beta)}(a_{n+1,n+1}^x) \stackrel{\text{Equ. (3.13)}}{=} \left(A_{n+1,1 \ldots n} \Phi_{\beta}^{-R}(\cdot, n)\right)^{0x} \stackrel{\text{Equ. (3.8)}}{=} a_{n+1,n+1}^x.
\]

Secondly,

Multiplying \(\Phi_{\beta}^{-L}\) on the left to both sides of Equation 3.9, we get

\[
-\Phi_{\beta}^{-L}(A\Phi_{\epsilon-(\beta)})_{1 \ldots n,n+1} + \frac{2}{\mu} \Phi_{\beta}^{-L} A_{1 \ldots n,n+1} \Phi_{\epsilon-(\beta)}(a_{n,n+1}^0) = \Phi_{\beta}^{-L} A_{1 \ldots n,n+1}
\]

It’s easy to see that

\[
\Phi_{\beta}^{-L}(A\Phi_{\epsilon-(\beta)})_{1 \ldots n,n+1} = (\Phi_{\epsilon-(\beta)}^{-L})_{n,n+1} (A\Phi_{\epsilon-(\beta)})_{1 \ldots n+1}
\]

Using the above equality and Equation 3.15, we get

\[
-\Phi_{\epsilon-(\beta)}(A)_{1 \ldots n,n+1} + 2 \Phi_{\beta}^{-L} A_{1 \ldots n,n+1} = \Phi_{\beta}^{-L} A_{1 \ldots n,n+1}
\]

And thus

\[
-\Phi_{\epsilon-(\beta)}(A)_{1 \ldots n,n+1} + 2(\Phi_{\beta}^{-L} A_{1 \ldots n,n+1})^{x0} = (\Phi_{\beta}^{-L} A)(\cdot, n)^{x0}
\]

By Equation 3.11, 3.3, the left hand side of Equation 3.17 becomes

\[
\begin{pmatrix}
A_{1 \ldots n-1,n+1}^{x0} \\
a_{n+1,n+1}^{x0}
\end{pmatrix}
\]

By Equation 3.1, 3.2, the right hand side of Equation 3.17 equals

\[
\begin{pmatrix}
A_{1 \ldots n-1,n}^{x0} \\
a_{n+1,n}^{x0}
\end{pmatrix}
\]

Therefore, we get the equalities:

\[
a_{i,n+1}^x = a_{i,n}^x, \ i \neq n.
\]

Similarly, By multiplying \(\Phi_{\beta}^{-R}\) on the right to both sides of Equation 3.4, and using Equation 3.16, 3.13, 3.8, 3.6, 3.7, we get the equalities:

\[
a_{n+1,j}^x = a_{n,j}^x, \ j \neq n.
\]

From Equation 3.5, 3.11, 3.16, we obtain

\[
-\Phi_{\epsilon-(\beta)}(a_{n+1,n+1}^x) + 2a_{n+1,n+1}^x = a_{n,n+1}^x
\]

And thus
\begin{equation}
\Phi_{\epsilon^{-}(\beta)}(a_{n+1,n+1}^{x}) = a_{n+1,n+1}^{x} = a_{n,n+1}^{x}.
\end{equation}

Lastly, notice that
\[ a_{n,n}^{x} = \Phi_{\tilde{\beta}}(a_{n,n}^{x}) \]
\[ = \Phi_{\epsilon^{-}(\beta)}(a_{n+1,n+1}^{x}) + 2\mu a_{n+1,n}^{x}a_{n,n+1}^{0} + 2\mu a_{n+1,n}^{0}a_{n,n+1}^{x} - 4\mu\beta a_{n+1,n}^{x}a_{n,n+1}^{0} + 1 \]
\[ = 5a_{n+1,n+1}^{x} - 4\Phi_{\tilde{\beta}}(a_{n+1,n+1}^{x}) = a_{n+1,n+1}^{x}. \]

Therefore, \[ a_{n,n}^{x} = a_{n+1,n+1}^{x}. \] This completes Step 2.

**Step 3:** Now we define \( \varphi : HC_{0}(\tilde{\beta}) \rightarrow HC_{0}(\beta), \)
\begin{equation}
\varphi(a_{ij}^{x}) = \begin{cases}
a_{nn}^{x} & i = n + 1, j = n + 1 \\
a_{nj}^{x} & i = n + 1, j \leq n \\
a_{mn}^{x} & i \leq n, j = n + 1 \\
a_{ij}^{x} & i \leq n, j \leq n
\end{cases}
\end{equation}

To show \( \varphi \) is well-defined, we need to prove \( \varphi \) maps Equation 3.1 - 3.10 and the same equations with \( \Phi_{-L}, \Phi_{-R} \) replaced by \( \Phi_{+L}, \Phi_{+R} \) to identities in \( HC_{0}(\beta) \).

Equation 3.1 - 3.3, 3.6 - 3.8 are trivial to check.

Now we check Equation 3.9, 3.10. Equation 3.4, 3.5 can be checked analogously.

First of all, notice that in Equation 3.15, the first equality holds in \( A_{n+1} \), and the second equality is part of Equation 3.3, which are preserved by \( \varphi \). Therefore, \( \varphi(\Phi_{\epsilon^{-}(\beta)}(a_{n+1,n+1}^{x})) = a_{nn}^{x} \). Especially, \( \varphi(\Phi_{x^{-}(\beta)}(a_{n,n+1}^{0})) = a_{nn}^{0} = \mu \). To prove Equation 3.9 is preserved by \( \varphi \), it suffices to show
\[ \varphi((A\Phi_{\epsilon^{-}(\beta)}a_{1\cdots n+1}) = A_{1\cdots n}. \]
\[ A\Phi_{\epsilon^{-}(\beta)} = (\Phi_{\epsilon^{-}(\beta)})^{-1}A\Phi_{\epsilon^{-}(\beta)}^{-1}A \Phi_{\epsilon^{-}(\beta)} = (\Phi_{\epsilon^{-}(\beta)})^{-1}A \Phi_{\epsilon^{-}(\beta)}(A). \]

Denote \( (\Phi_{\epsilon^{-}(\beta)})^{-1} \) by \( C \). Note that \( C \) has the following form:

\[ C = \begin{pmatrix} 
(\Phi_{\epsilon^{-}(\beta)})^{-1} & 0 \\
\ast & \ast \\
\ast & \ast 
\end{pmatrix} \]

Then for any \( 1 \leq i \leq n \),
\[ (A\Phi_{\epsilon^{-}(\beta)})_{i,n+1}^{xy} = \sum_{j=1}^{n+1} \sum_{z=1}^{n} C_{ijz}^{x} \Phi_{\epsilon^{-}(\beta)}(a_{j,n+1}^{z+y}) \]
\[ \sum_{j=1}^{n} \sum_{z=1}^{n} C_{ijz}^{x} (\Phi_{\epsilon^{-}(\beta)}(a_{j,n+1})^{z+y,0} \]
\[ \varphi \to \sum_z \sum_{j=1}^n \varphi(C_{ij})(\Phi_\beta^{-L}(j, \cdot)A_{1,\ldots,n})_z^{y,0} \]

\[ = \sum_z \sum_{j=1}^n \varphi(C_{ij})(\Phi_\beta^{-L}A)_z^{y,0} \]

\[ = \sum_z \sum_{j=1}^n \varphi(C_{ij})(\Phi_\beta^{-L}A)_{j,n}^{y} = (\Phi_\beta^{-L}-1\Phi_\beta^{-L}A)_{jin}^{y} \]

\[ = a_{in}^{xy} = \varphi(a_{in}^{xy}) \]

So Equation 3.9 is preserved under \( \varphi \).

To check Equation 3.10, we need to show \( \varphi((-A\Phi_\beta^{-R}(n+1, n+1)) = a_{nn} \).

\[ \varphi((-A\Phi_\beta^{-R}(n+1, n+1)) = \sum_{j=1}^{n+1} a_{nj}\varphi((-\Phi_\beta^{-R})_{j,n+1}) \]

\[ = \varphi((-A\Phi_\beta^{-R}(n, n+1)) = a_{nn} \text{ as we just showed above.} \]

The last bit of work is to show \( \varphi \) also preserves Equation 3.1 - 3.10 when all the \( \Phi^{-L}, \Phi^{-R} \) are replaced by \( \Phi^{+L}, \Phi^{+R} \). The proof basically proceeds in the same way. So we don’t bother to write out the details.

So now we have a well defined algebra morphism \( \varphi : HC_0(\tilde{\beta}) \to HC_0(\beta) \), which is clearly onto. We can also define an inverse map from \( HC_0(\beta) \) to \( HC_0(\tilde{\beta}) \) which sends \( a_{ij}^{xy} \) to \( a_{ij}^{xy} \), \( 1 \leq i, j \leq n \). It’s straightforward to show this map sends \( I_\beta \) to \( I_{\tilde{\beta}} \). Thus \( HC_0(\tilde{\beta}) \simeq HC_0(\beta) \).

3.4.3. Invariance under Markov move III. Let \( D \) be the disk in the plane of radius 2 centered at the origin. We define a reflection \( r : Int(D) \setminus \{0\} \to Int(D) \setminus \{0\} \), by \( r(x) = 2 \cdot \frac{x}{|x|} - x \). Namely, \( r \) is the reflection with respect to the unit circle. Then \( r \times Id \) defines a reflection \( X = (Int(D) \setminus \{0\}) \times I \). We will still use \( r \) to denote this map.

Recall that \( C_n \) is the braid group on the punctured disk \( Int(D) \setminus \{0\} \) inside \( X \). Therefore, \( r \) induces a group isomorphism from \( C_n \) to itself. Explicitly, the isomorphism, also denote by \( r \), is given by:

\[ r(\alpha_i) = \begin{cases} \alpha_{n-1} \cdots \alpha_1 \alpha_0 \alpha_1 \cdots \alpha_{n-1} & i = 0 \\ \alpha_{n-i}^{-1} & 1 \leq i \leq n-1 \end{cases} \]

**Lemma 3.** The map \( r \) defined above from \( C_n \) to \( C_n \) is a group isomorphism and \( r^2 = Id \).

**Proof** This can be verified purely algebraically. \( \square \)
Also recall that \( P_n = \{ p_i, 1 \leq i \leq n \} \), \( Q_n = \{ \gamma : [0, 1] \mapsto D \setminus \{ p \} | \gamma \text{ continuous, } \gamma^{-1}(P_n) = \{ 0, 1 \} \} / \sim \). Then \( r \) also induces a bijection on \( P_n \) and a bijection on \( Q_n \).

\[
\begin{align*}
r : P_n &\rightarrow P_n, \quad r(p) = p, \quad r(i) = n + 1 - i.
\end{align*}
\]

Note that here we use \( i \) to represent the point \( p_i \).

It could also be checked that \( r \) preserves the “skein” relations that define \( A_n \), thus \( r \) induces an isomorphism \( r : A_n \rightarrow A_n \), given by Figure 10.

**Figure 10.** \( r(a_{ij}^x) \)

Remark 9. \( r \) also extends to a bijection from \( P_n^+ \) to \( P_n^- \), namely, \( r(p) = p, \ r(i) = n + 1 - i \). And \( r \) maps the curves in \( Q_n^+ \) bijectively to those in \( Q_n^- \). Furthermore, \( r \) maps the “skein” relations that define \( A_n^+ \) to the corresponding “skein” relations that define \( A_n^- \). Consequently, we get an isomorphism \( r : A_n^+ \rightarrow A_n^- \). Note that the inverse map is also induced by the reflection \( r \) that maps \( Q_n^- \) to \( Q_n^+ \). For this reason, we will denote the inverse map also by \( r \). In summary, \( r \) is an isomorphism between \( A_n^+ \) and \( A_n^- \), which restricts to an isomorphism on \( A_n \) and which has square \( Id \).

**Lemma 4.** If \( P,Q \in A_n^\pm \) are connectable, then \( r(P), \ r(Q) \) are connectable, and \( r(P \ast Q) = r(P) \ast r(Q) \).

**Proof** It’s clear from the geometrical interpretation of \( a_{ij}^x \) that we have, for \( 1 \leq i, j \leq n, x,y \in \mathbb{Z} \), \( r(a_{ij}^x) \ast r(a_{0j}^y) = r(a_{ij}^{xy}) \) and \( r(a_{i,n+1}^x) \ast r(a_{n+1,j}^y) = r(a_{ij}^{xy}) \).

In general, let \( P = \sum_{x \in \mathbb{Z}} \sum_{i=1}^{n} P_i^x a_{0i}^x, Q = \sum_{y \in \mathbb{Z}} \sum_{j=1}^{n} a_{0j}^y Q_j^y, P_i^x, Q_j^y \in A_n \), then \( r(P) = \sum_{x,y \in \mathbb{Z}} \sum_{i,j=1}^{n} r(P_i^x)r(a_{0i}^x), r(Q) = \sum_{x,y \in \mathbb{Z}} \sum_{i,j=1}^{n} r(a_{0j}^y)r(Q_j^y) \). So \( r(P), r(Q) \) are connectable, and \( r(P \ast r(Q) = \sum_{x,y \in \mathbb{Z}} \sum_{i,j=1}^{n} r(P_i^x)r(a_{0i}^x) \ast r(a_{0j}^y)r(Q_j^y) = \sum_{x,y \in \mathbb{Z}} \sum_{i,j=1}^{n} r(P_i^x)r(a_{ij}^{xy})r(Q_j^y) = r(P \ast Q) \). \( \square \)

**Lemma 5.** If \( \beta \) is a braid in \( C_n \), then we have \( r \circ \Phi_\beta = \Phi_{r(\beta)} \circ r \). More generally, we have \( r \circ \Phi^-_\beta = \Phi^+_\beta \circ r \).
Proof It’s possible, though tedious, to prove it algebraically. For example, it suffices to prove the case for \( \beta = \alpha_k^{\pm 1} \) acting on \( a_{ij} \). Here we give another geometric proof which makes the statement in the lemma almost trivial. Recall that the reflection \( r : C_n \rightarrow C_n \) is induced by the reflection \( r \times Id : (Int(D) \setminus \{0\}) \times I \rightarrow (Int(D) \setminus \{0\}) \times I \). By the Remark 4, \( \beta(\gamma_{ij}^r) \) can be obtained as the curve by pushing \( \gamma_{ij}^r \) in \((Int(D) \setminus \{0\}) \times \{0\}\) along the braid \( \beta \) up to \((Int(D) \setminus \{0\}) \times \{1\}\). The reflection \( r \times Id \) maps \( \gamma_{ij}^r \) to \( r(\gamma_{ij}^r) \), \( \Phi_\beta(\gamma_{ij}^r) \) to \( r \circ \Phi_\beta(\gamma_{ij}^r) \), and \( \beta \) to \( r(\beta) \). Thus \( r \circ \Phi_\beta(\gamma_{ij}^r) \) is obtained by pushing \( r(\gamma_{ij}^r) \) along the braid \( r(\beta) \), and therefore \( \Phi_{r(\beta)} \circ r(\gamma_{ij}^r) = r \circ \Phi_\beta(\gamma_{ij}^r) \).

The more general equation can be proved analogously by using Remark 4 and Lemma 6.

\[ \text{Theorem 6.} \quad \text{For} \quad \beta \in C_n, \text{the map} \quad r : A_n \rightarrow A_n \text{induces an isomorphism from} \quad HC_0(\beta) \quad \text{to} \quad HC_0(r(\beta)). \]

Proof It suffices to show \( r \) maps \( I_\beta \) to \( I_{r(\beta)} \).
\[
r((\Phi_\beta^{-L}A)_{ij}^{xy}) = r(\Phi_\beta^{-1}(a_{i,n+1}^x) * a_{n+1,j}^y) = r(\Phi_\beta^{-1}(a_{i,n+1}^x) * a_{n+1,j}^y) = r(\Phi_\beta^{-1}(a_{i,n+1}^x) * a_{n+1,j}^y).
\]

The first identity in the above equation is by the argument in the Part (2) of Remark 4, the second identity is by Lemma 4, and the third by Lemma 5.

Assume \( r(a_{i,n+1}^x) = \sum P_k^z a_{kk}^0, r(a_{n+1,j}^y) = \sum a_{nk}^\prime Q_k^\prime \), where \( P_k^z, Q_k^\prime \) are elements in \( A_n \). Then
\[
r(\Phi_\beta^{-L}A)_{ij}^{xy} = \sum \Phi_{r(\beta)}(P_k^z) \Phi_{r(\beta)}(a_{kk}^0) * a_{0k}^\prime Q_k^\prime = \sum \Phi_{r(\beta)}(P_k^z) (\Phi_{r(\beta)}^{-L}A)_{kk}^z Q_k^\prime = \sum \Phi_{r(\beta)}(P_k^z) A_{kk}^z Q_k^\prime.
\]

On the other hand,
\[
r(a_{i,n+1}^x) = r(a_{i,n+1}^x + a_{n+1,j}^y) = r(a_{i,n+1}^x) * r(a_{n+1,j}^y) = \sum P_k^z A_{kk}^z Q_k^\prime.
\]
Therefore,
\[
r((\Phi_\beta^{-L}A)_{ij}^{xy}) = \sum (\Phi_{r(\beta)}(P_k^z) - P_k^z) A_{kk}^z Q_k^\prime = 0 \quad \text{in} \quad HC_0(r(\beta)).
\]
The last equality above is by Corollary 3.

This shows \( r(\Phi_\beta^{-L}A - A) \subset I_{r(\beta)}. \)

The other relations are proved in basically the same way. And thus we showed \( r \) is well-defined. The fact that \( r \) is an isomorphism is trivial to check.
Now we prove $HC_0(\beta)$ is invariant under Markov move III. A key observation is the following commuting diagram.

\[
\begin{array}{c}
C_n \xrightarrow{\epsilon^+} C_{n+1} \\
\downarrow r \\
C_n \xleftarrow{\epsilon^-} C_{n+1}
\end{array}
\]

**Lemma 6.** The above diagram commutes, namely $r \circ \epsilon^+ = \epsilon^- \circ r : C_n \rightarrow C_{n+1}$.

**Proof.** We only need to check on the generators.

$r\epsilon^+(\alpha_0) = r(\alpha_1\alpha_0\alpha_1) = \alpha_n^{-1}\alpha_n \cdots \alpha_1\alpha_0\alpha_1 \cdots \alpha_n^{-1} = \alpha_{n-1} \cdots \alpha_1\alpha_0\alpha_1 \cdots \alpha_{n-1} = \epsilon^- r(\alpha_0)$.

For $i \geq 1$, $r\epsilon^+(\alpha_i) = r(\alpha_{i+1}) = \alpha_{n-i}^{-1} = \epsilon^- r(\alpha_i)$.

Let $\beta \in C_n$, then $r(\epsilon^+(\beta)) = r(\epsilon^+(\beta))r(\alpha_1^{\pm 1}) = \epsilon^- (r(\beta))\alpha_1^{\pm 1}$. Therefore,

$HC_0(\epsilon^+(\beta)) \cong HC_0(r(\epsilon^+(\beta))\alpha_1^{\pm 1}) = HC_0(\epsilon^- (r(\beta))\alpha_1^{\pm 1}) \cong HC_0(r(\beta)) \cong HC_0(\beta)$.

The first and last isomorphisms above are due to Theorem 6 and the second isomorphism is the invariance isomorphism under Markov move II.

Now we finished showing $HC_0(\beta)$ is invariant under Markov move III.

4. Properties of the invariant

4.1. Symmetries of the invariant. In Theorem 6 we have shown that for $\beta \in C_n$, we have $HC_0(\beta) \cong HC_0(r(\beta))$. Here we introduce more operations on the braids which preserve the invariant. As a corollary, we prove the invariant $HC_0(K)$, for a knot $K$, is independent of the knot’s orientation.

Let $D$ be the disk centered at the origin, and let $X = (D \setminus \{0\}) \times I$. Define a reflection $inv : D \setminus \{0\} \rightarrow D \setminus \{0\}$ by $inv(x, y) = (x, -y)$. Namely, $inv$ is reflection of the disk about the x-axis. Then $r \times Id$ defines a reflection on $X$, still denoted by $inv$.

Similar to the discussion in Section 3.4.3, the map $inv$ induces an isomorphism on $C_n, A_n$ and $A_n^\pm$. The isomorphisms are given below explicitly.

$inv : C_n \rightarrow C_n, inv(\alpha_i) = \alpha_i^{-1}$.

$inv : A_n \rightarrow A_n$, is shown in Figure 11.

And $inv : A_n^\pm \rightarrow A_n^\pm$ is defined analogously.
Lemma 7. If $P, Q \in \mathcal{A}_n^\pm$ are connectable, then $inv(P)$, $inv(Q)$ are connectable, and $inv(P \ast Q) = inv(P) \ast inv(Q)$.

**Proof** Similar to the proof of Lemma 4. \qed

Lemma 8. For $\beta \in \mathcal{C}_n$, we have $inv \circ \Phi_\beta = \Phi_{inv(\beta)} \circ inv : \mathcal{A}_n \rightarrow \mathcal{A}_n$. More generally, we have $inv \circ \Phi_\beta^\pm = \Phi_{inv(\beta)}^\pm \circ inv : \mathcal{A}_n^\pm \rightarrow \mathcal{A}_n^\pm$.

**Proof** Similar to the proof of Lemma 5. \qed

The following theorem is a summary of the symmetries which our invariant has.

**Theorem 7.** For $\beta \in \mathcal{C}_n$, $HC_0(\beta) \simeq HC_0(r(\beta)) \simeq HC_0(inv(\beta)) \simeq HC_0(\beta^{-1})$ as $R$-algebras.

**Proof** The first isomorphism is the content of Theorem 6, the proof that $HC_0(\beta) \simeq HC_0(inv(\beta))$ is basically the same as that of Theorem 6. The use of Lemma 4 and Lemma 5 in that proof just needs to be replaced by Lemma 7 and Lemma 8 respectively. Everything else proceeds identically. Now we prove the third isomorphism.

Define the isomorphism $HC_0(\beta^{-1}) \rightarrow HC_0(\beta)$ to be the one induced by $\Phi_\beta$. We need to check $\Phi_\beta$ maps $I_{\beta^{-1}}$ to $I_\beta$.

$\Phi_\beta(\Phi_\beta^+L_1A-A) = \Phi_\beta^+L_1(\Phi_\beta)\Phi_\beta(A) - \Phi_\beta(A) = \Phi_\beta^+L_1(\Phi_\beta)\Phi_\beta^+L_1A\Phi_\beta^+R - \Phi_\beta^+L_1A\Phi_\beta^+R = A\Phi_\beta^+R - \Phi_\beta^+L_1A\Phi_\beta^+R = (A - \Phi_\beta^+L_1A)\Phi_\beta^+R$.

The second equality is by Proposition 4 and the third one is by Corollary 2.

The other three relations can be proved analogously. Therefore, $\Phi_\beta$ induces a well-defined algebra map from $HC_0(\beta^{-1})$ to $HC_0(\beta)$. It’s easy to check it’s also an isomorphism. \qed

**Corollary 4.** For a knot $K$ in $S^1 \times S^2$, we have $HC_0(K) \simeq HC_0(\overline{K})$, where $\overline{K}$ is the knot obtained from $K$ by reversing its orientation. Namely, $HC_0(K)$ is independent of the orientation of $K$.

**Proof** If the knot $K$ is represented by a braid $\beta \in \mathcal{C}_n$, then $\overline{K}$ is represented by $inv(\beta^{-1})$. Then the corollary follows from Theorem 7. \qed
4.2. Torus Knots. In this subsection we study some properties of the torus knots in $S^1 \times S^2$.

Let $C$ be the equator of $S^2$, then $S^1 \times C$ is a torus which bounds two solid tori in $S^1 \times S^2$, with $z_0 \times C$ being the meridian and $S^1 \times z_1$ the longitude. In [1], a knot in $S^1 \times S^2$ is called a torus knot if it can be isotoped to a knot in $S^1 \times C$. Fix a meridian $\nu$ and a longitude $\lambda$ in $S^1 \times C$, and let $m, l$ be two relatively prime integers. An $(m, l)$-knot in $S^1 \times S^2$ is a knot which can be isotoped to $m\nu + l\lambda$ in $S^1 \times C$. In general, for a knot $K$, $HC_0(K)$ may not be finitely generated as an $R$-algebra. However, we show below that for torus knots, the invariant indeed is always finitely generated.

We first present a model for $S^1 \times S^2$ which relates to the braid presentation of $C_n$. Let $A$ be an annulus $S^1 \times I$. Also think of $A$ as a punctured disk with $S^1 \times \{0\}$ being the puncture. Let $X = A \times S^1$. See Figure 12. Then $S^1 \times S^2$ is obtained by gluing a solid torus to each torus boundary component in $X$. The gluing maps are given by sending the meridian of each solid torus to $z_0 \times \{0\} \times S^1$ and $z_0 \times \{1\} \times S^1$, respectively for some $z_0 \in S^1$. Equivalently, $S^1 \times S^2$ is obtained by gluing a disk along each of the circles $z \times \{0\} \times S^1$, $z \times \{1\} \times S^1$ for any $z \in S^1$. The torus $S^1 \times C$ is shown in Figure 12 as the dashed cylinder inside the big cylinder, where the meridian and longitude are also shown there. The vertical line in the middle of the cylinder is $\{\text{puncture}\} \times S^1$.

![Figure 12. $S^1 \times C$ in $S^1 \times S^2$](image)

**Theorem 8.** Let $K$ be an $(m,l)$-knot in $S^1 \times S^2$ where $m, l$ are relatively prime integers, then $HC_0(K)$ is finitely generated as an $R$-algebra. Moreover, the minimum number of algebra generators is no more than $l - 1$. 
Proof By the arguments above, an \((m, l)-knot\) is represented by the braid \(\beta(m, l) = (\alpha_0 \cdots \alpha_{m-1})^l\). See Figure 13 for a picture of \((3, 2)-knot\). For simplicity, we still use \(\beta\) to denote \(\beta(m, l)\). Also for reasons that will become clear below, we use the notation \(b_{ij} = a_{i+1,j+1}^{x} - a_{i,j+1}^{x-\lfloor\frac{i+1+l}{m}\rfloor}\).

Then we have \(HC_{0}(\hat{\beta}) = A_{m}/I_{\beta}\). It’s easy to check that the following equation holds:

\begin{align*}
\Phi_{\beta(m, l)}^{+}(a_{i,0}^{x}) = \begin{cases} 
    a_{i+1,0}^{x} & 1 \leq i \leq m - 1 \\
    a_{1,0}^{x-\lfloor\frac{i+1+l}{m}\rfloor} & i = m
\end{cases}
\end{align*}

Then we have \(\Phi_{\beta(m, l)}^{+}(a_{i,0}^{x}) = a_{(i-1+l)(mod\,m)+1,0}^{x-\lfloor\frac{i+1+l}{m}\rfloor}\). Using \(b_{ij}^{x}\) to replace \(a_{i+1,j+1}^{x}\), we get a simpler expression \(\Phi_{\beta(m, l)}^{+}(b_{ij}^{x}) = b_{(i+l)(mod\,m),-1}^{x-\lfloor\frac{i+1+l}{m}\rfloor}\).

Thus by Part 2 of Remark 6, the third relation that defines \(I_{\beta}\) is

\begin{align*}
(4.2) \quad b_{ij}^{x} - b_{(i+l)(mod\,m),j}^{x-\lfloor\frac{i+1+l}{m}\rfloor}, 0 \leq i, j \leq m - 1, x \in \mathbb{Z}.
\end{align*}

Similarly, the fourth relation that defines \(I_{\beta}\) is

\begin{align*}
(4.3) \quad b_{ij}^{x} - b_{(i+l)(mod\,m),j}^{x+\lfloor\frac{i+1+l}{m}\rfloor}, 0 \leq i, j \leq m - 1, x \in \mathbb{Z}.
\end{align*}

Define \(f(i, k) := \sum_{r=0}^{k-1} \lfloor \frac{(i+rl)(mod\,m)+l}{m} \rfloor, 0 \leq i \leq m - 1, 1 \leq k \in \mathbb{Z}\), and define \(f(i, 0) := 0\).

It’s elementary to check that \(f(i, k) = \lfloor \frac{k}{m} \rfloor l + f(i, k mod m)\), and in \(HC_{0}(\hat{\beta})\), we have the equalities \(b_{ij}^{x-f(i,k)} = b_{(i+kl)(mod\,m),j}^{x-f(i,k)} = b_{i,(j+kl)(mod\,m),j}^{x-f(i,k)} \forall k \geq 0\).
0. Especially, we have \( b_{ij}^l = b_{ij}^{x-f(i,m)} = b_{ij}^{x-l} \), so \( b_{ij}^x \) is periodic in \( x \) with period equal to \( l \).

Let \( k_1, k_2 \) be any numbers that satisfy \( k_1 l \pmod{m} = i \), \( k_2 l \pmod{m} = j \), then \( b_{ij}^x = b_{00}^{x+f(0,k_1)-f(0,k_2)} \), and \( b_{00}^l = b_{00}^{x+l} \). Thus all the \( b_{ij}^x \)'s are completely determined by \( b_{00}^l = \mu, b_{00}^{l_1}, \ldots, b_{00}^{l_{l-1}} \) and the condition that \( b_{00}^x = b_{00}^{x+l} \). So \( HC_0(\hat{\beta}) \) is finitely generated and \( \{b_{00}^x, 1 \leq x \leq l-1\} \) is a set of generators.

\[ \square \]

At the end of this subsection, let’s compute some examples of torus knots.

**Example 1.** We will use the same notations as those in the proof of Theorem 8.

1). \((m, 1)\)-knot. The \((m, 1)\)-knot is represented by the braid \( a_0 \cdots a_{m-1} \).

Clearly, by Markov II in Theorem 4 this braid is equivalent to \( a_0 \) representing the \((1, 1)\)-knot. Set \( \beta = a_0 \in \mathcal{C}_1 \). By Theorem 8 \( b_{00}^{x+1} = b_{00}^x \).

Since \( b_{00}^0 = \mu \), all the \( b_{00}^x \)'s are equal to \( \mu \).

By definition, \( \Phi_{\beta}^{-1}(a_{i_1}^{x_1}a_{i_2}^{x_2}) = -a_{i_2}^{x_1} + \frac{2}{\mu} a_{i_1}^{x_1}a_{i_2}^{-1} \); thus \( (\Phi_{\beta}^{-L}A)^{xy}_{11} = -a_{i_1}^{x+y-1} + \frac{2}{\mu} a_{i_1}^{x}a_{i_2}^{-1} \). Since all the \( a_{i_1}^{x} \)'s are equal to \( \mu \), \( (\Phi_{\beta}^{-L}A)^{xy}_{11} = -\mu + \frac{2}{\mu} \mu \mu = \mu = a_{i_1}^{x+y} \). Thus, the first relation in Definition 2 automatically holds. Similar calculations show the second relation also holds. Therefore, \( HC_0(\hat{\beta}) \simeq R \).

2). \((m, 2)\)-knot. By Proposition 2.2 in [1], all the \((m, 2)\)-knots are equivalent to each other with \( m \) odd. This can also been seen directly by Markov moves. Thus, we only need to compute the \((1, 2)\)-knot, which is represented by \( \beta = a_0^2 \). It was shown in the second example in Section 3.2 that \( HC_0(\hat{\beta}) \simeq R[X]/(2\mu - 2X^2) \).

3). \((m, 3)\)-knot. Again by Proposition 2.2 in [1], there are two classes of knots of this type. A representative of each class could be chosen as \((1, 3)\)-knot and \((2, 3)\)-knot. Since the calculations are not difficult but tedious, we just present the result obtained by computer packages. Both of the two knots have the same \( HC_0 \), so are not distinguished by this invariant. See [8] for the computer package which was created by the first author. The \( HC_0 \) invariant of \((m, 3)\)-knots is isomorphic to \( R(X, Y)/(2X - \frac{2Y^2}{M}, 2Y - \frac{2X^2}{M}) \).

4.3. Local knots. Throughout this subsection, we will set \( \mu = -2 \).

A knot is called local if it is contained in a 3-ball. It’s easy to see that a knot in \( S^1 \times S^2 \) is local if and only if it can be represented as the closure of a braid which doesn’t contain \( a_0 \) or \( a_0^{-1} \), i.e. a braid in \( \mathcal{B}_n = \langle a_1, \cdots, a_{n-1} \rangle \subset \mathcal{C}_n \). Note that the braids in \( \mathcal{B}_n \) are closed
under the Markov moves given in Theorem 1 and moreover, Markov move III in this case is a consequence of Markov moves I, II. Since Markov moves I, II are just the classical Markov moves for braids in \( B_n \) representing knots in \( S^3 \), we thus have a one-to-one correspondence between knots in \( S^3 \) and local knots in \( S^1 \times S^2 \).

Let \( K \) be a local knot in \( S^1 \times S^2 \), as noted above, \( K \) can also be viewed as a knot in \( S^3 \). Let \( hc_0(K) \) denote the 0-th knot contact homology in [6], then we have the following decompositions for the \( HC_0 \) invariant of local knots, which relates our invariant to \( hc_0(K) \).

**Proposition 5.** Let \( K \) be a local knot in \( S^1 \times S^2 \), then \( HC_0(K) \cong hc_0(K) + \sum_{0 \neq n \in \mathbb{Z}} H_x \), where all the \( H_x \)'s are isomorphic to each other as subalgebras and there is a surjective algebra morphism from \( H_x \) to \( hc_0(K) \).

**Proof** Let \( \beta \in B_n \) represent the knot \( K \). Let \( D_0 = \mathbb{Z} \langle a_{ij}^0, 1 \leq i, j \leq n \rangle \) and for \( x \neq 0 \), \( D_x = \mathbb{Z} \langle a_{ij}^x, 1 \leq i, j \leq n \rangle \). By Part 3) in Remark 2, \( \Phi_\beta \) restricted on \( D_0 \) is the same as the braid action given in [6] if we identify \( a_{ij}^0 \) with \( a_{ij} \) in that paper. Moreover, it’s easy to see from Equation 2.2 that \( \Phi^-_\beta(a_{i,n+1}^x) = \Phi^-_\beta(a_{i,n+1}^0) \ast a_{n+1,n+1}^x \) and that \( \Phi^+_\beta(a_{i,0}^x) = \Phi^+_\beta(a_{i,0}^0) \ast a_{0,0}^x = \Phi^+_\beta(a_{i,n+1}) \ast a_{n+1,n+1}^x \). Therefore, \( (\Phi^+_\beta A)_{xy} = \Phi^+_\beta(a_{x,y}^0) \ast a_{0,0}^y = \Phi^-_\beta(a_{x,y}^0) \ast a_{y,x}^y \). Similarly, we have \( \Phi^+_\beta A = \Phi^-_\beta A \). Similarly, we have \( A \Phi^+_\beta = A \Phi^-_\beta \). So we only need to consider the relations \( A - \Phi^-_\beta L A, A - \Phi^+_\beta R A \).

\[
(\Phi^+_\beta A)_{xy} - (\Phi^-_\beta A)_{ij} = A_{ij}^0 - (A_{ij}^0 - (\Phi^-_\beta A)_{ij}^0) \ast a_{x,y}^y.
\]

Similarly, we have \( A_{xy}^y - (A \Phi^-_\beta R A)_{ij} = a_{i,j}^y \ast (A_{ij}^0 - (A \Phi^+_\beta R A)_{ij}^0) \).

Here the relations \( A^0_{ij} - (A^0_{ij} - (\Phi^+_\beta A)^0_{ij}) \) and \( A^0_{ij} - (A^0_{ij} - (\Phi^-_\beta R A)_{ij}^0) \) are exactly equal to \( a_{ij} - (\Phi^0_{ij} A)_{ij} \) and \( a_{ij} - (A \Phi^0_{ij} R A)_{ij} \) in the definition of the 0-th knot contact homology in [6]. Therefore, \( hc_0(K) = D_0/(A^0_{ij} - (\Phi^+_\beta A)^0_{ij}, A^0_{ij} - (A \Phi^+_\beta R A)_{ij}, 1 \leq i, j \leq n) \) is a subalgebra of \( HC_0(K) \).

Similarly, \( H_x = D_x/(A^0_{ij} - (\Phi^+_\beta A)^0_{ij}, A^0_{ij} - (A \Phi^+_\beta R A)_{ij}, 1 \leq i, j \leq n) \) is also a subalgebra of \( HC_0(K) \). It’s clear that \( HC_0(K) \cong hc_0(K) + \sum_{0 \neq x \in \mathbb{Z}} H_x \).

From the definition of \( H_x \), we can define an algebra morphism from \( H_x \) to \( hc_0(K) \) by sending \( a_{ij}^x \ast a_{ij}^0 \) to \( a_{ij}^0 \), which is clearly well-defined and onto. \( \square \)
Corollary 5. If $K$ is a local knot in $S^1 \times S^2$, then $HC_0(K)$ is infinitely generated as an $R$-algebra.

We just showed that $HC_0$ is infinitely generated for local knots. On the other hand, Theorem 8 shows $HC_0$ is always finitely generated for torus knots. Through some amount of computer calculations, we find that $HC_0$ is always finitely generated for non-local knots. This motivates us to come up with following conjecture.

Conjecture 1. Let $K$ be a knot in $S^1 \times S^2$, then $HC_0(K)$ is finitely generated as an $R$-algebra if and only if $K$ is not local.

4.4. Augmentations. The invariant, $HC_0$, could be very difficult to compute for general knots, especially when the number of crossings is large. Thus we will deduce a family of invariants from $HC_0$, which are called augmentation numbers and which are relatively easier to compute by computers. The concept of augmentation numbers are introduced in [6] for basically the same reason. We need to make slight changes to the definition of augmentation numbers in order for it to fit in our framework.

Recall that $HC_0(K)$ is an algebra over the ring $R = \mathbb{Z}[\mu, \frac{2}{\mu}]$. Let $d \geq 2$ be an integer and let $\mathbb{Z}_d = \mathbb{Z}/d\mathbb{Z}$. Pick a number $k \in \mathbb{Z}_d$ such that $\frac{2}{k}$ is defined in $\mathbb{Z}_d$. Then $\mathbb{Z}_d$ can be treated as an $R$-mod, with $\mu$ acting by multiplication by $k$. Then $H(K; k, d) := HC_0(K) \otimes_R \mathbb{Z}_d$ is a $\mathbb{Z}_d$-algebra. Assume Conjecture 1 is true, then for non-local knot $K$, $H(K; k, d)$ is a finitely generated $\mathbb{Z}_d$-algebra, and thus has finitely many algebra morphisms into $\mathbb{Z}_d$.

Definition 3. Let $K$ be a knot in $S^1 \times S^2$ such that $HC_0(K)$ is finitely generated as an $R$-algebra, and let $d \geq 2$ be an integer and $k \in \mathbb{Z}_d$ such that $\frac{2}{k}$ exists in $\mathbb{Z}_d$, then $\text{Aug}(K; k, d)$ is defined to be the number of algebra morphisms from $H(K; k, d)$ to $\mathbb{Z}_d$.

For example, denote $T(m, l)$ the $(m, l)$-torus knot, then $\text{Aug}(T(1, 4); 1, 3) = 4$, $\text{Aug}(T(1, 5); 1, 3) = 2$, $\text{Aug}(T(1, 6); 1, 3) = 4$, while $\text{Aug}(T(1, 4); 1, 4) = 16$, $\text{Aug}(T(1, 5); 1, 4) = 32$, $\text{Aug}(T(1, 6); 1, 4) = 128$.

Proposition 6. For any knot $K$ in $S^1 \times S^2$, there is an $R$-algebra morphism from $HC_0(K)$ to $R$ sending each $a_{ij}^\beta$ to $\mu$.

Proof Let $t : A_n \rightarrow R$, $t(a_{ij}^\beta) = \mu$. We first show for $\beta \in C_n$, $t\Phi_{\beta} = t$. Clearly, it suffices to prove $t\Phi_{\beta} = t$, $0 \leq k \leq n - 1$. This can be checked directly from Equations 2.2, 2.3.

Similarly, one can prove $t\Phi_+ = t\Phi_\beta = t$.

Assume $K$ is represented by $\beta \in C_n$. We need to show $t$ factors through $I_\beta$. 


\[ t((\Phi^L_{\beta} A)_{ij}^{xy}) = t(\Phi^L_{\beta}(a_x^{i,n+1} \ast a_y^{n+1,j})) = t(\Phi^L_{\beta}(a_x^{i,n+1})) = t(a_x^{i,n+1}) = \mu = t(A_{ij}^{xy}). \]

The other three relations can be verified analogously. \[ \square \]

Corollary 6. Let \( K \) be a knot in \( S^1 \times S^2 \), and let \( k \in \mathbb{Z}_d \) such that \( 2k \) is defined in \( \mathbb{Z}_d \), then \( \text{Aug}(K; k, d) \geq 1 \).

**Proof** The map \( t \) defined in Proposition 6 naturally induces a map from \( H(K; k, d) \) to \( \mathbb{Z}_d \). \[ \square \]

5. A Topological Interpretation of the Knot Invariant

In this section, we show that the algebraic knot invariant, \( HC_0(K) \), is the same as the cord ring defined in A.4 in [7]. In that paper, this cord ring is conjectured to be the zero-th relative contact homology of \( LK \) in \( ST^*M \). Our invariant can be viewed as a combinatorial realization of the cord ring. The cord ring of a knot in a 3-manifold is defined as follows.

**Definition 4.** [7]

Let \( K \) be a knot in a 3-manifold \( M \),

1). A cord in \( M \) relative to \( K \) is a continuous map \( \gamma : [0, 1] \rightarrow M \), such that \( \gamma^{-1}(K) = \{0, 1\} \). Two cords \( \gamma_1, \gamma_2 \) are said to be equivalent if they are homotopic relative to \( K \). Informally speaking, one can slide \( \gamma_1 \) along \( K \) to reach \( \gamma_2 \).

2). Let \( \mathcal{A}_K \) be the free algebra over \( R \) generated by equivalence classes of cords. The cord ring of \( K \) in \( M \) is defined to be \( \mathcal{A}_K / J_K \), where \( J_K \) is the ideal generated by the relations in Figure 14.

![Figure 14. cord relations](image-url)

In Figure 14, the thinner lines represent the cords and the thicker represent the knot \( K \). And in the second relation, the diagrams are understood to depict some local neighborhood outside of which the diagrams agree. And again the \( \otimes \) represents the multiplication.
Remark 10. If we set $\mu = -2$, then we recover the definition of the cord ring in [7].

Theorem 9. Let $K$ be a knot in $S^1 \times S^2$, then $HC_0(K) \simeq A_K/J_K$ as $R$-algebras.

Proof  Let $\beta \in C_n$ be a braid such that the closure of $\beta$ is $K$. Let $A$ be the punctured disk $D \setminus B_\epsilon(0)$. Let $X = A \times [0, 1]/\{(x, 0) \sim (x, 1), x \in A\}$. Then we can present $\beta$ as a braid diagram inside $X$. See Figure 15. $S^1 \times S^2$ is obtained by gluing a solid torus to each torus boundary component. The gluing maps are given by sending the meridian of each solid torus to $\{a\} \times S^1$ and $\{b\} \times S^1$, respectively. Thus $X$ is a subspace of $S^1 \times S^2$, and it’s clear that any cord in $S^1 \times S^2$ can be homotoped to inside $X$. Then we project the cord along the braid to $A \times \{0\}$ to get an arc which is an element in $Q_n$.

We define a map $\varphi : A_K \rightarrow A_n/I_\beta = HC_0(\beta)$ as follows. For any cord $\gamma$, homotope it so that it is contained in $X$, and then project it to $A \times \{0\}$ along the braid to get an arc $\tilde{\gamma}$. View $\tilde{\gamma}$ as an element in $\tilde{A}_n \simeq A_n$, so we can project it to $A_n/I_\beta$. There are several points where we need to check the map is well-defined.

Step 1: The projection to $A \times \{0\}$ is not unique, and different projections differ by actions of $\Phi_\beta$. By Corollary 3, the image of $Id - \Phi_\beta$ is contained in $I_\beta$. So different projections will lead to the same image in $A_n/I_\beta$.

Step 2: In $S^1 \times S^2$, the cords have more flexibilities to be homotoped than in $X$. Precisely, there are two more type of flexibilities. Let $\gamma_1, \gamma_2$ be two curves in $A$ such that $\gamma_1(1) = \gamma_2(0) = b, \gamma_1(0) = p_i, \gamma_2(1) = p_j$, and let $\delta$ be the loop $\{b\} \times S^1$, then it’s clear that $\gamma_1 \ast \gamma_2, \gamma_1 \ast \delta \ast \gamma_2$ are equivalent curves in $S^1 \times S^2$ but not in $X$. If we project $\gamma_1 \ast \delta \ast \gamma_2$ to $A \times \{0\}$, then we get $\Phi_\beta^L(\gamma_1) \ast \gamma_2$ or $\gamma_1 \ast \Phi_\beta^R(\gamma_2)$. These are precisely the relations $A - \Phi_\beta^{-L}A, A - A \Phi_\beta^{-R}$. See Part (2) of Remark 6.

Similarly, in the above argument, if we replace “b” by “a”, then we get the relations $A - \Phi_\beta^{-L}A, A - A \Phi_\beta^{-R}$. See Part (2) of Remark 6.

Step 3: The two relations that define $J_K$ apparently maps to the two “skein” relations that define $\tilde{A}_n$. So they pass through to $A_n/I_\beta$.

The above three steps showed that $\varphi$ is well-defined. It’s also easy to prove it’s a bijection.

□

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Figure 15. $\beta$

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