On the Jackson constants for algebraic approximation of continuous functions

A.G. Babenko, Yu. V. Kryakin

Dedicated to Professor Igor A. Shevchuk on the occasion of his 70th birthday

Abstract We establish new estimates for the constant \( J_a(k, \alpha) \) in the Brudnyi–Jackson inequality for approximation of \( f \in C[-1, 1] \) by algebraic polynomials:

\[
E_a^n(f) \leq J_a(k, \alpha) \omega_k(f, \alpha \pi/n), \quad \alpha > 0
\]

The main result of the paper implies the following inequalities

\[
1/2 < J_a(2k, \alpha) < 10, \quad n \geq 2k(2k - 1), \quad \alpha \geq 2
\]

1. Introduction

In this note we use the relatively new approach (convolution series by C.Neumann \[1\] with the Boman–Shapiro integral operators \[2, 3\]) for the constant problems in the following Brudnyi–Jackson theorem (see \[4, 5\]) for algebraic approximation of \( f \in C[-1, 1] \):

\[
E_a^n(f) \leq J_a(k, \alpha) \omega_k(f, \alpha \pi/n), \quad \alpha > 0.
\] (1.1)

The case of algebraic approximation \( E_{n-1}^a(f) \) of a continuous function \( f \) by algebraic polynomials of degree \( \leq n - 1 \) is in some sense more difficult than the case of trigonometric approximation. Usually the reduction to the trigonometric approximation is used. There are some technical problems in the case of the modulus of \( \omega_k(f, \alpha \pi/n) \) of the order \( k \geq 2 \). The problem of exact constants in this case is a difficult one and we do not have sharp results for \( k > 1 \).

We recall here a result by Korneichuk \[6\], as the corollary of his remarkable theorem on constants in the case of concave modulus of continuity \( \overline{\omega} \):

\[
E_n^a(f) \leq \frac{1}{2} \overline{\omega}(f, \pi/n),
\]

and the new Mironenko’s result \[7\] for the second modulus of continuity:

\[
E_{n-1}^a(f) \leq 5 \omega_2(f, \alpha \pi/n), \quad \alpha = 8^{-1/2}.
\]

In the present paper we prove that for \( n > 2k(2k - 1) \) the constants in (1.1) are bounded by an absolute constant:

\[
J_a(2k, \alpha) < 10, \quad \alpha \geq 2.
\]

It is clear that

\[
\omega_{2k+1}(f, \delta) \leq 2 \omega_{2k}(f, \delta), \quad \delta > 0,
\]

and therefore the main result of this paper states that for \( \alpha \geq 2 \), \( n > k(k - 1) \) we can write constant in \[11\] that do not depend on \( k \).

---

1AMS classification: Primary 41A17, 41A44, 42A10.
2Key words: Algebraic approximation, Brudnyi–Jackson theorem, \( k \)-th modulus of smoothness, estimate of constants.
2. Notation. Auxiliary facts. Main results

In this paper, \( i, j, k, l, m, n \) denote the natural numbers. Let \( A \) be \([-1, 1]\) or \( \mathbb{R} \). We denote by \( W^k(A) \) the space of smooth functions \( f^{(j)} \in C(A) \), \( j = 0, 1, \ldots, k-1 \), \( f^{(k)} \) bounded a.e. on \( A \). We will also use the notation

\[
\|f\| := \|f\|_A := \text{ess sup}_{x \in A} |f(x)|.
\]

We consider the approximation of real functions on \( I = [-1, 1] \) by algebraic polynomials \( p_{n-1}(x) = \sum_{j=0}^{n-1} a_j x^j \) of degree at most \( n - 1 \). We will denote by \( P_{n-1} \) the space of such polynomials. The best approximation of \( f \in C(I) \) by \( p \in P_{n-1} \) is defined by standard way

\[
E_n(f) := \inf_{p \in P_{n-1}} \sup_{x \in I} |f(x) - p_n(x)| = \inf_{p \in P_{n-1}} \|f - p\|.
\]

Smoothness of function \( f \in C(A) \) is measured by modulus of smoothness. Beside the classical \( k \)-th modulus of smoothness

\[
\omega_k(f, \delta) := \sup_{x \in (1-\delta/2)I, 0 < \delta \leq 1} |\Delta_k f(x)|, \quad \Delta_k f(x) := \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} f(x + jh - kh/2),
\]

we will use the special Boman–Shapiro modulus of continuity, which measures the deviation of the function from the special linear combination of Steklov’s means (see \([8, 9, 10]\)). We will use the following convolution notation

\[
(f * g)(x) := \int_{\mathbb{R}} f(t) g(x - t) \, dt,
\]

and the following notation for characteristic function and the convolution square of a characteristic function:

\[
\chi_h(x) := \begin{cases} \frac{1}{h}, & x \in [-h/2, h/2], \\ 0, & x \notin [-h/2, h/2]. \end{cases}
\]

\[
\chi_h^2(x) := (\chi_h * \chi_h)(x) = \begin{cases} \frac{1}{h} (1 - |x|/h), & x \in [-h, h], \\ 0, & x \notin [-h, h]. \end{cases}
\]

Define the special difference operator for a locally integrable function \( f \) in the following way (see \([9, 10]\))

\[
W_{2k}(f, x, \chi_h^2) := (-1)^k \frac{1}{\binom{2k}{k}} \int_{\mathbb{R}} \Delta_{2k}^2 f(x) \chi_h^2(t) \, dt = (f - \Lambda_{2k} * f)(x),
\]

where

\[
\Lambda_{2k}(x) := \Lambda_{2k,h}(x) = 2 \sum_{j=1}^{k} (-1)^{j+1} a_{j, k} \chi_{jh}^2(x), \quad a_{j, k} := \binom{2k}{k+j}/\binom{2k}{k}.
\]

It was proved in \([10]\) that

\[
\int_{\mathbb{R}} |\Lambda_{2k}(t)| \, dt \leq c_k - 1,
\]

with

\[
c_1 = 2, \quad c_2 < 2.18, \quad c_3 < 2.26, \quad c_4 < 2.31, \quad c_k < 3, \quad k \geq 5,
\]

and therefore

\[
|W_{2k}(f, x, \chi_h^2)| \leq c_k \sup_{t \in [x-kh, x+kh]} |f(t)|. \tag{2.2}
\]
We will use the standard notation for the Favard constants

\[ K_k := \frac{4}{\pi} \sum_{j=-\infty}^{\infty} (4j+1)^{-k-1}, \]

\[ K_0 = 1, \quad K_1 = \frac{\pi}{2}, \quad K_2 = \frac{\pi^2}{8}, \quad K_3 = \frac{\pi^3}{24}, \quad K_4 = \frac{5\pi^4}{384}, \quad K_6 = \frac{61\pi^6}{46080}, \quad K_8 = \frac{277\pi^8}{2064384}. \]

We are now ready to state the main results of this paper.

**Theorem 1** Suppose \( f \in C(\mathbb{I}) \). Then for \( n \geq 2k(2k-1), \ k \geq 5 \)

\[ E_n^{a-1}(f) \leq J_a(2k, \alpha) \omega_{2k}(f, \alpha \pi/n), \]

and

\[ 1/2 \leq J_a(2k, \alpha) < 3(2+e^{-2}) \left( 2 \sec(\pi/2\alpha) - 1 - \frac{K_2}{\alpha^2} \right), \quad 1 < \alpha \leq (2k-1)\pi^{-1}. \]

Note that in the trigonometric case we have (see [9][10])

\[ \left( \frac{2k}{k} \right)^{-1} \leq J(2k, \alpha) < \sec(\pi/2\alpha) \left( \frac{2k}{k} \right)^{-1}, \quad \alpha > 1. \]

**Theorem 2** For small \( k = 1, 2, 3, 4, \ \alpha > 0 \), we have the following estimates of the constants

\[ \frac{1}{2} \leq J_a(2, \alpha) \leq \frac{3}{4} \left( 1 + \frac{1}{4\alpha^2} \right), \quad n \geq 2, \]

\[ \frac{1}{2} \leq J_a(4, \alpha) \leq 2.18 \cdot (1 + K_2 \beta_2) + 2K_4 \beta_2^3, \quad n \geq 12, \]

\[ \frac{1}{2} \leq J_a(6, \alpha) \leq 2.26 \cdot (1 + K_2 \beta_3 + 2K_4 \beta_3^2) + 2K_6 \beta_3^3, \quad n \geq 30, \]

\[ \frac{1}{2} \leq J_a(8, \alpha) \leq 2.31 \cdot (1 + K_2 \beta_4 + 2K_4 \beta_4^2 + 2K_6 \beta_4^3) + 2K_8 \beta_4^3, \quad n \geq 56, \]

where

\[ \beta_k = \frac{4\gamma_k}{\pi^2 \alpha^2}, \quad \gamma_2 = \frac{4}{3}, \quad \gamma_3 = \frac{68}{45}, \quad \gamma_4 = \frac{512}{315}. \]

Thus we may write the estimates for \( \alpha = 1, 2 \). The constants are increasing as \( k \) increases for \( \alpha = 1 \):

\[ J_a(2, 1) \leq 0.94, \quad J_a(4, 1) \leq 4.38, \quad J_a(6, 1) \leq 6.71, \quad J_a(8, 1) \leq 8.9, \]

but for \( \alpha = 2 \)

\[ J_a(2, 2) \leq 0.8, \quad J_a(4, 2) \leq 2.59, \quad J_a(6, 2) \leq 2.84, \quad J_a(8, 2) \leq 2.97, \]

and for \( k \geq 5 \) the estimates of Theorem 1 give

\[ J_a(2k, 2) \leq 9.74, \quad n \geq 2k(2k-1). \]

Note that if the Sendov conjecture \( w_k \leq 1 \) is true (see Theorem B below), then we achieve better inequality

\[ J_a(2k, 2) \leq 9.74 \cdot (2 + e^{-2})^{-1} = 4.57, \]

which is near the results in the case of small \( k \).

The proofs of the main inequalities are based on the following fundamental facts. The first important fact is the algebraic variant [15] of the Bohr–Favard–Akhieser–Krein inequality (see [11][12][13][14]):
Theorem A For \( f \in W_m^w(I) \)

\[
E_{n-1}^w(f) \leq \frac{K_m}{m^n} \|f^{(m)}\|, \quad m = 1, 2.
\]

\[
E_{n-1}^w(f) \leq K_m \frac{(n-m)!}{n!} \|f^{(m)}\|, \quad m \geq 3.
\]

The second important fact is the modern variant of Whitney’s theorem, with good estimates of constants (see \cite{16,17,18,19,20,21,22,23,24,25}):

Theorem B Suppose \( f \in C[a, b], \; 0 < a < b \). Then

\[
E_{k-1}^w(f) \leq w_k \omega_k(f, (b-a)/k), \quad (2.3)
\]

with

\[
w_k \leq \begin{cases} 0.5, & k = 1, 2, \\ 1, & k = 3, \ldots, 8, \\ 2, & k \leq 82000, \\ 2 + \exp(-2), & k \geq 82000. \end{cases}
\]

Theorems A,B are the main technical tools for proving Theorem 1 and Theorem 2. Theorem 1 and Theorem 2 follow from Proposition 1 and Proposition 2.

Proposition 1 concerns the problem of continuation of a function from \( I \) to \( \mathbb{R} \). This continuation allows us to use the periodic–case approach.

Proposition 1 Let \( f \in C(I), \; 0 < h < (2k)^{-1} \). Then there exists a function \( g_f \) which is equal to \( f \) on \( I \), continuous on \( \mathbb{R} \setminus I \) and such that

\[
W_{2k}(g_f, h) := \|W_{2k}(g_f, \cdot, \chi_h^2)\|_\mathbb{R} \leq d_k \omega_{2k}(f, h),
\]

\[
\|\Delta_h^{2k} g_f\|_\mathbb{R} \leq d'_k \omega_{2k}(f, h),
\]

with

\[
d_k \leq \begin{cases} 1, & k = 1, \\ c_k, & k = 2, 3, 4, \\ 6, & 4 < k \leq 41000, \\ 3(2 + \exp(-2)), & k > 41000, \end{cases}
\]

where

\[
c_2 < 2.18, \; c_3 < 2.26, \; c_4 < 2.31,
\]

and

\[
d'_k \leq \begin{cases} 1.5, & k = 1, \\ 2^{2k}, & k = 2, 3, 4, \\ 2^{2k+1}, & 4 < k \leq 41000, \\ (2 + \exp(-2)) \cdot 2^{2k}, & k > 41000. \end{cases}
\]
The key-idea is the same as in the periodic case (see [9, 10]): we will use the truncated Neumann convolution series
\[
gf = \sum_{j=0}^{k-1} (gf - gf \ast \Lambda_{2k}) \ast \Lambda_{2k}^j + gf \ast \Lambda_{2k}^k, \quad \Lambda_{2k}^k := \Lambda_{2k}^{j-1} \ast \Lambda_{2k},
\]
with some modification for the algebraic case.

**Proposition 2** Let \( f \in C(I) \), \( gf \equiv f \) on \( I \) and \( gf \in C(\mathbb{R} \setminus I) \). Then for \( k \geq 2 \), \( n \geq 2k(2k-1) \)
\[
E_{n-1}^a(f) \leq \sum_{j=0}^{k-1} \frac{K_{2j}}{n^{2j}} \left\| ((gf - gf \ast \Lambda_{2k}) \ast \Lambda_{2k}^j)^{(2j)} \right\|_\mathbb{R} + 2 \sum_{j=2}^{k-1} \frac{K_{2j}}{n^{2j}} \left\| ((gf - gf \ast \Lambda_{2k}) \ast \Lambda_{2k}^j)^{(2j)} \right\|_\mathbb{R}
\]
\[
+ 2 \frac{K_{2k}}{n^{2k}} \left\| (gf \ast \Lambda_{2k}^k)^{(2k)} \right\|_\mathbb{R}.
\]

For deducing the main theorems from Proposition 2 we present here a new variant of the known estimates (see [9, 10]).

Put
\[
\gamma_k := 2 \sum_{j=1}^{(k+1)/2} \frac{a_{2j-1,k}}{(2j-1)^2} < 2 \sum_{j=1}^{\infty} \frac{1}{(2j-1)^2} = \frac{\pi^2}{8}, \quad k \geq 2.
\]

**Lemma 1** For \( k \geq 2 \), \( j = 0, \ldots, k - 1 \)
\[
\left\| ((gf - gf \ast \Lambda_{2k}) \ast \Lambda_{2k}^j)^{(2j)} \right\|_\mathbb{R} \leq (4\gamma_k)^j h^{-2j} \left\| gf - gf \ast \Lambda_{2k} \right\|_\mathbb{R},
\]
\[
\left\| (gf \ast \Lambda_{2k}^k)^{(2k)} \right\|_\mathbb{R} \leq \gamma_k^k h^{-2k} \left\| \hat{\Delta}_{2k}^h g \right\|_\mathbb{R}.
\]

Note that in comparison with [9, 10] we add new inequality here, which allows to estimate the last term in inequality (2.4).

To prove Lemma 1 we represent \( \Lambda_{2k}(x) \) as the special linear combination of \( \chi_{h_j,h}^2(x) := \chi_{h}^2(x - h_j) \).

**Lemma 2** For \( k \geq 2 \), \( h > 0 \) we have
\[
\Lambda_{2k} = 2 \sum_{j=1}^{k} (-1)^{j+1} a_{j,k} \chi_{h_j}^2 = \sum_{j=-(k-1)}^{k-1} \alpha_{j,k} \chi_{j,h}^2, \quad \sum_{j=-(k-1)}^{k-1} |\alpha_{j,k}| = \gamma_k.
\]

The last part of the paper is organized in the following way. In the next section we give the proofs of auxiliary results. The proofs of main results will appear in the last section.

### 3. Proofs of auxiliary results

The proofs of auxiliary results will be given in the reverse order. First, we will prove Lemma 2. Then, on the basis of Lemma 2, we will prove Lemma 1.

After that, we will use the C. Neumann decomposition to give the proof of Proposition 2, and then, finally we prove Proposition 1.
Proof of Lemma 2. We decompose the characteristic function \( \chi_{jh} \) in the operator

\[
\Lambda_{2k} = 2 \sum_{j=1}^{k} (-1)^{j+1} a_{j,k} \chi_{jh}^2, \quad a_{j,k} = \left( \frac{2k}{k+j} \right) \left( \frac{2k}{k} \right)^{-1},
\]

in the special way

\[
\chi_{jh} = \frac{1}{j} \sum_{i=0}^{j-1} \chi_{-(j-1)h/2+ih,h}, \quad \chi_{t,h}(x):=\chi_h(x-t).
\]

In particular

\[
\begin{align*}
\chi_{2h} &= \frac{1}{2} \left( \chi_{-h/2,h} + \chi_{h/2,h} \right), \\
\chi_{3h} &= \frac{1}{3} \left( \chi_{-h,h} + \chi_{0,h} + \chi_{h,h} \right), \\
\chi_{4h} &= \frac{1}{4} \left( \chi_{-3h/2,h} + \chi_{-h/2,h} + \chi_{h/2,h} + \chi_{3h/2,h} \right), \\
\chi_{5h} &= \frac{1}{5} \left( \chi_{-2h,h} + \chi_{-h,h} + \chi_{0,h} + \chi_{h,h} + \chi_{2h,h} \right).
\end{align*}
\]

It is easy to see that

\[
\chi_{t,h} \ast \chi_{s,h} = \chi_{t+s,h}^2.
\]

Therefore, we have by direct calculation

\[
\chi_{jh}^2 = \chi_{jh} \ast \chi_{jh} = \frac{1}{j} \sum_{i=-j+1}^{j-1} \varphi_{i,j} \cdot \chi_{ih,h}^2, \quad \varphi_{i,j} := \left( 1 - \frac{|j|}{j} \right).
\]

Equality \((3.6)\) implies that this representation is equivalent to the following equalities for the Fejér kernel. For \( j = 2\nu + 1 \),

\[
\frac{1}{(2\nu + 1)^2} \left( \sum_{t=-\nu}^{\nu} e^{it} \right)^2 = \frac{1}{(2\nu + 1)^2} \left( \frac{\sin(2\nu + 1)t/2}{\sin t/2} \right)^2 = \frac{1}{2\nu+1} \sum_{t=-2\nu}^{2\nu} \varphi_{t,2\nu+1} \cdot e^{it},
\]

and for \( j = 2\nu \)

\[
\frac{1}{(2\nu)^2} \left( \sum_{t=-\nu}^{\nu} e^{it/2} \right)^2 = \frac{1}{(2\nu)^2} \left( \frac{\sin \nu t}{\sin t/2} \right)^2 = \frac{1}{2\nu} \sum_{t=-2\nu}^{2\nu-1} \varphi_{t,2\nu} \cdot e^{it}.
\]

The substitution of

\[
\chi_{jh}^2 = \frac{1}{j} \sum_{l=-j+1}^{j-1} \varphi_{l,j} \cdot \chi_{lh,h}^2 = \frac{1}{j^2} \left( j\chi_{0,h}^2 + \sum_{l=1}^{j-l} \varphi_{l,j} \cdot (\chi_{(l-j)h,h}^2 + \chi_{(j-l)h,h}^2) \right)
\]

into \((3.5)\) gives

\[
\Lambda_{2k} = 2 \sum_{j=1}^{k} (-1)^{j+1} a_{j,k} \frac{1}{j^2} \left( j\chi_{0,h}^2 + \sum_{l=1}^{j-l} \varphi_{l,j} \cdot (\chi_{(l-j)h,h}^2 + \chi_{(j-l)h,h}^2) \right)
\]

\[
= 2 \sum_{j=1}^{k} (-1)^{j+1} a_{j,k} \frac{1}{j^2} \chi_{0,h}^2 + 2 \sum_{l=1}^{k-1} \chi_{lh,h}^2 \sum_{j=1}^{k} (-1)^{j+1} a_{j,k} \frac{j-l}{j} \frac{1}{j^2} + 2 \sum_{l=1}^{k-1} \chi_{lh,h}^2 \sum_{j=l}^{k} (-1)^{j+1} a_{j,k} \frac{j-l}{j^2}
\]

\[
= \sum_{l=-k+1}^{k-1} \alpha_{l,k} \chi_{lh,h}^2,
\]

where \( \alpha_{l,k} \) are the coefficients.
with the coefficients
\[
\alpha_{l,k} = 2 \sum_{j=|l|+1}^{k} (-1)^{j+1} \frac{(j-|l|) a_{j,k}}{j^2}.
\] (3.8)

Note that \( \text{sign} \alpha_{l,k} = (-1)^l \) (see [10], Lemma A).

By using (3.8) and the identity
\[
\alpha = (-1)^j + 2 \sum_{j=1}^{l-1} (-1)^j l = \frac{1 + (-1)^{j+1}}{2},
\]
we obtain
\[
\sum_{j=-k+1}^{k-1} \alpha_{j,k} = \sum_{j=-k+1}^{k-1} (1)^j \alpha_{j,k} = \alpha_{0,k} + 2 \sum_{l=1}^{k-1} (1)^j \alpha_{l,k}
\]
\[
= 2 \sum_{l=1}^{k} (-1)^{l+1} \frac{a_{l,k}}{l} + 4 \sum_{l=1}^{k-1} (1)^{l} \sum_{j=|l|+1}^{k} (-1)^{j+1} \frac{(j-1) a_{j,k}}{j^2}
\]
\[
= 2 \sum_{l=1}^{k} (-1)^{l+1} \frac{a_{l,k}}{l} + 2 \sum_{j=1}^{k} (-1)^{l+1} \frac{a_{j,k}}{j^2} + 2 \sum_{l=1}^{k-1} (1)^{l} \sum_{j=|l|+1}^{k} (j-l)
\]
\[
= 2 \sum_{j=1}^{k} \frac{a_{j,k}}{j^2} \left((-1)^{l+1} + 2 \sum_{l=1}^{k} (-1)^{l+1} (j-l)\right) = 2 \sum_{j=1}^{k} \frac{\sigma_j a_{j,k}}{j^2} = 2 \sum_{j=1}^{k} \frac{a_{j,k}}{j^2} = \gamma_k.
\]

**Proof of Lemma 1**

Lemma 2 implies for \( j = 1, \ldots, k \)
\[
\Lambda_{2k}^j = \sum_{l=-j(k-1)}^{j(k-1)} \alpha_{l,k}(j) \chi_{lh,h}, \quad \sum_{l=-j(k-1)}^{j(k-1)} |\alpha_{l,k}(j)| \leq \gamma_j^k.
\]

Indeed, we have
\[
\Lambda_{2k}^2 = \Lambda_{2k} \ast \Lambda_{2k} = \left( \sum_{l=-2(k-1)}^{2(k-1)} \alpha_{l,k} \chi_{lh,h}^2 \right) \ast \left( \sum_{l=-2(k-1)}^{2(k-1)} \alpha_{l,k} \chi_{lh,h}^2 \right)
\]
\[
= \sum_{l=-2(k-1)}^{2(k-1)} \alpha_{l,k}(2) \chi_{lh,h}^4, \quad \sum_{l=-2(k-1)}^{2(k-1)} |\alpha_{l,k}(2)| \leq \left( \sum_{l=-2(k-1)}^{2(k-1)} |\alpha_{l,k}| \right)^2 = \gamma_k^2,
\]
\[
\vdots
\]
\[
\Lambda_{2k}^k = \sum_{l=-k(k-1)}^{k(k-1)} \alpha_{l,k}(k) \chi_{lh,h}^{2k}, \quad \sum_{l=-k(k-1)}^{k(k-1)} |\alpha_{l,k}(k)| \leq \left( \sum_{l=-k(k-1)}^{k(k-1)} |\alpha_{l,k}| \right)^k = \gamma_k^k.
\]

Now, one can apply the identities
\[
(g \ast \chi_h)'(x) = \left( h^{-1} \int_{x-h/2}^{x+h/2} f(t) \, dt \right)' = h^{-1} (f(x + h/2) - f((x - h/2)) = h^{-1} \Delta_h f(x),
\]
\[
(f \ast \chi_h^2)'(2) = ((f \ast \chi_h \ast \chi_h)' = h^{-1} \Delta_h^2 f = h^{-2} \Delta_h^2 f,
\]
\[
(f \ast \chi_h^4)'(4) = ((f \ast \chi_h^2 \ast \chi_h^2)' = h^{-2} \Delta_h^4 f = h^{-4} \Delta_h^4 f,
\]
\[
\vdots
\]
\[
(f \ast \chi_h^{2k})'(2k) = h^{-2k} \Delta_h^{2k} f,
\]
which are true a.e. for the function $f$ continuous on $(-\infty, -1) \cup (-1, 1) \cup (1, +\infty)$ and the inequalities
\[ \|\Lambda_n^j f\|_R \leq 4^j \|f\|_R, \quad j = 1, \ldots, k - 1, \]
to end the proof. \hfill \square

**Proof of Proposition 2** By using the Neumann decomposition of $g_f$
\[ g_f = \sum_{j=0}^{k-1} (g_f - g_f \ast \Lambda_{2k}) \ast \Lambda_{2k}^j + g_f \ast \Lambda_{2k}^k, \quad \Lambda_{2k}^j := \Lambda_{2k}^{j-1} \ast \Lambda_{2k}, \]
one can estimate the approximation of $f$ on $I$ by algebraic polynomials $p \in \mathcal{P}_{n-1}$ of degree $n - 1 \geq 2k(2k - 1) - 1$ :
\[ E_{n-1}^a(f) = E_{n-1}^a(g_f) \leq \sum_{j=0}^{k-1} E_{n-1}^a((g_f - g_f \ast \Lambda_{2k}) \ast \Lambda_{2k}^j) + E_{n-1}^a(g_f \ast \Lambda_{2k}^k). \]
We apply Theorem A for $(g_f - g_f \ast \Lambda_{2k}) \ast \Lambda_{2k}^j$, and the inequality
\[ \frac{n^m}{n(n - 1) \cdots (n - m + 1)} < 2, \quad n \geq m(m - 1), \quad m \geq 4. \quad (3.9) \]
The estimate follows from the inequality
\[ \ln(1 - x) > -2 \ln 2 \cdot x, \quad x \in (0, 1/2]. \]
We have
\[ E_{n-1}^a(f) \leq \sum_{j=0}^{k-1} \frac{K_{2j}}{n^{2j}} \left\| (g_f - g_f \ast \Lambda_{2k}) \ast (\Lambda_{2k}^j)^{(2j)} \right\|_R + 2 \sum_{j=2}^{k-1} \frac{K_{2j}}{n^{2j}} \left\| (g_f - g_f \ast \Lambda_{2k}) \ast (\Lambda_{2k}^j)^{(2j)} \right\|_R \]
\[ + 2 \frac{K_{2k}}{n^{2k}} \left\| (g_f \ast \Lambda_{2k}^k)^{(2k)} \right\|_R. \]
\hfill \square

**Proof of Proposition 1** Suppose that $p_1^{2k-1}, p_2^{2k-1}$ are the polynomials of the best approximation of $f$ on $I^- = [-1, -1 + 2k]$, and $I^+ = [1 - 2k, 1]$ respectively. Define
\[ g_f(x) := \begin{cases} p_1^{2k-1}(x), & x \in (1, +\infty), \\ f(x), & x \in I, \\ p_2^{2k-1}(x), & x \in (-\infty, -1). \end{cases} \]
Theorem B implies
\[ \sup_{x \in I} |f(x) - p_1^{2k-1}(x)| \leq w_{2k} \omega_{2k}(f, h). \]
We will prove that
\[ W_{2k}(g_f, h) = \|g_f - \Lambda_{2k} \ast g_f\|_R \leq c_k w_{2k} \omega_{2k}(f, h), \]
with
\[ c_1 = 1, \quad c_2 < 2.18, \quad c_3 < 2.26, \quad c_4 < 2.31, \quad c_k < 3, \quad k \geq 5. \]
By symmetry, it is sufficient to consider only the cases
1) \( x \in [0, 1 - kh] \),
2) \( x \in (1 - kh, 1] \),
3) \( x \in (1, +\infty) \).

In the case 1) we have
\[
|W_{2k}(gf, x, \chi_1^2)| \leq \left( \frac{2k}{k} \right)^{-1} \omega_{2k}(f, h).
\]

In the second case we have \( x + kh \geq 1 \). The identity \( W_{2k}(p_{2k-1}^+, x, \chi_1^2) \equiv 0 \) yields
\[
W_{2k}(g, x, \chi_1^2) = W_{2k}(g - p_{2k-1}^+, x, \chi_1^2) + W_{2k}(p_{2k-1}^+, x, \chi_1^2) = W_{2k}(g - p_{2k-1}^+, x, \chi_1^2).
\]

By using the inequality (2.2)
\[
|W_{2k}(gf - p_{2k-1}^+, x, \chi_1^2)| \leq c_k \sup_{t \in [x - kh, x + kh]} |gf(t) - p_{2k-1}^+(t)|,
\]
and Whitney’s theorem (2.3)
\[
\sup_{x \in I^n} |f(x) - p_{2k-1}^+(x)| \leq w_{2k} \omega_{2k}(f, h),
\]
we deduce the estimate in the second case.

The third case is similar to the second case, when \( x \in (1, 1 + kh] \). If \( x > 1 + kh \), the \( 2k \)-th difference is equal to zero. Thus the estimate
\[
W_{2k}(gf, h) \leq d_k \omega_{2k}(f, h), \quad 0 < h < (2k)^{-1},
\]
is proved.

The proof of the estimate
\[
\omega_{2k}(gf, h) \leq d_k' \omega_{2k}(f, h), \quad 0 < h < (2k)^{-1},
\]
is the same as in the case \( W_{2k} \). It is sufficient to consider only \( x \geq 0 \) and instead of the inequality
\[
|W_{2k}(f - p_{2k-1}^+, x, h)| \leq c_k \sup_{x \in I^n} |f(x) - p_{2k-1}^+(x)|,
\]
to use in the cases 2) and 3) the following inequalities
\[
|\Delta_{2k}^h (f(x) - p_{2k-1}^+(x))| \leq (2^k - 1) \sup_{x \in I^n} |f(x) - p_{2k-1}^+(x)| \leq (2^k - 1) w_{2k} \omega_{2k}(f, h).
\]

\[\Box\]

4. Proofs of main results

Proof of Theorem 1 Put
\[
\delta_k := \delta_k(h, n) := \frac{4 \gamma_k}{h^2 n^2}.
\]

Proposition 2 and Lemma 1 imply
\[
E_{n-1}^a(f) \leq \left( 2 \sum_{j=0}^{k-1} \delta_k^j \mathcal{K}_{2j} - 1 - \delta_k \mathcal{K}_2 \right) \|W_{2k}(gf, \cdot, h)\|_R + 2 \mathcal{K}_{2k} 4^{-k} \delta_k \|\Delta_{2k}^h gf\|_R.
\]

Now we can apply Proposition 1 and obtain
\[
E_{n-1}^a(f) \leq d_k \left( 2 \sum_{j=0}^{k} \delta_k^j \mathcal{K}_{2j} - 1 - \mathcal{K}_2 \delta_k \right) \omega_{2k}(f, h).
\]
By choosing 
\[ h = \frac{\alpha \pi}{n}, \]
and by using the identity
\[ \sum_{j=0}^{\infty} K_{2j} \rho^{-2j} = \sec \left( \frac{\pi}{2\rho} \right), \quad \rho := \delta^{-1/2} = \alpha \sqrt{\frac{\pi^2}{4\gamma_k}} > 1, \]
we deduce the estimate
\[ J(2k, \alpha) \leq 3 (2 + e^{-2}) \left(2 \sec \left( \frac{\pi}{2\rho} \right) - 1 - \frac{K_2}{\rho^2}\right) \leq 3 (2 + e^{-2}) \left(2 \sec \left( \frac{\pi}{2\alpha} \right) - 1 - \frac{K_2}{\alpha^2}\right), \quad \alpha > 1. \]
The last estimate follows from the fact that \(2 \sec \left( \frac{\pi}{2x}\right) - 1 - \frac{K_2}{x^2}\) is a decreasing function for \(x > 1\). So, the upper estimate for \(J(2k, \alpha)\) is proved.

At last we prove the lower estimate. Consider the function
\[ f_0(x) = \begin{cases} 1, & x = -1, \\ 0, & x \in \mathbb{I}, \quad x \neq -1. \end{cases} \]
The best approximation of this function is \(\geq 1/2\). But the \(k\)-th difference
\[ \Delta^k_h f(x) = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} f(x+jh), \quad h \in (0, (1-x)/k), \]
for \(x = -1\) is equal to \((-1)^k\).
Consider now a regularization of \(f_0\). Suppose that \(k \geq 2, \varepsilon \in (0, 1/k)\) and define
\[ f_{\varepsilon,k}(x) := \left\{ \begin{array}{ll} \frac{(-1)^{k-1}}{\varepsilon} (1 + x - \varepsilon)^{k-1}, & -1 \leq x \leq -1 + \varepsilon, \\ 0, & -1 + \varepsilon < x \leq 1. \end{array} \right. \]
We will use the representation
\[ \Delta^k_h f_{\varepsilon}(x) = \int_0^h du_1 \cdots \int_0^h \Delta^1_h f_{\varepsilon}^{(k-1)}(x+u_1+\cdots+u_k-1) \, du_{k-1}. \]
Note that
\[ |f_{\varepsilon}^{(k-1)}(x)| = \begin{cases} \frac{(k-1)!}{\varepsilon}, & x \in (-1, -1 + \varepsilon), \\ 0, & x \in (-1 + \varepsilon, 1). \end{cases} \]
For \(h > 0\) the first difference
\[ \Delta^1_h f_{\varepsilon}^{(k-1)}(x+u_1+\cdots+u_{k-1}) = f_{\varepsilon}^{(k-1)}(x+h+u_1+\cdots+u_{k-1}) - f_{\varepsilon}^{(k-1)}(x+u_1+\cdots+u_{k-1}) \]
is not equal to zero only if
\[ 0 \leq u_1 + \cdots + u_{k-1} \leq \varepsilon. \]
From
\[ |\Delta^1_h f^{(k-1)}(x)| \leq \frac{(k-1)!}{\varepsilon^{k-1}}, \]
and
\[ \int \cdots \int_{0 \leq u_1 + \cdots + u_{k-1} \leq \varepsilon} \, du_1 \cdots du_{k-1} = \frac{\varepsilon^{k-1}}{(k-1)!}, \]
we obtain for \(x, x + kh \in \mathbb{I}\)
It is clear, that for small $\varepsilon > 0$

$$E^\alpha_{n-1}(f_\varepsilon) \geq 1/2 - \delta(\varepsilon), \quad \lim_{\varepsilon \to 0} \delta(\varepsilon) = 0.$$  

In the case $k = 1$ sufficient to consider the function $f_{\varepsilon, 2}$. 

\textbf{Proof of Theorem 2} In the case $k = 1$ we have

$$E^\alpha_{n-1}(f) \leq W_2(g_f, h) + E^\alpha_{n-1}(g_f * \chi_h^2) \leq 0.5 \omega_2(g_f, h) + K_2 n^{-2} \| (g_f * \chi_h^2)'' \| \leq (0.75 + 1.5 K_2(hn)^{-2}) \omega_2(f, h).$$

In this case the inequalities (see Proposition 1)

$$W_2(g_f, h) \leq 0.5 \omega(g_f, h) \leq 3/4 \omega_2(f, h)$$

are better than inequality

$$W_2(g_f, h) \leq \omega_2(f, h).$$

In the case $k \geq 2$ the estimates $W_{2k}(g_f, h) \leq d_k \omega_{2k}(f, h)$ give better results. If $k = 2, 3, 4$, then Theorem 2 follows from the inequality (see the proof of Theorem 1)

$$E^\alpha_{n-1}(f) \leq \left( 2d_k \sum_{j=0}^{k-1} \left( \delta^j_k K_{2j} - 1 - \delta_k K_2 \right) + 2 d^*_k K_2 K_{2k} 4^{-k} \delta^k_k \right) \omega_{2k}(f, h),$$

and the estimates of $d_k, d^*_k, \gamma_k$ (see Proposition 1 and definition of $\gamma_k$). \hfill \Box

\begin{thebibliography}{99}

[1] Neumann, C.: Untersuchungen über das Logarithmische und Newton’sche potential. Leipzig: Teubner. (1877). 404pp.

[2] Shapiro, H.S.: A Tauberian theorem related to approximation theory. Acta Math. 120, 279–292 (1968)

[3] Boman, J., Shapiro, H.: Comparison theorems for a generalized modulus of continuity. Arkiv för Matematik.

[4] Brudnyi, Yu.A.: The approximation of functions by algebraic polynomials. Mathematics of the USSR-Izvestiya. 2(4), 735–743 (1968)

[5] DeVore, R.A., Lorentz, G.G.: Constructive Approximation. Grundlehren der mathematischen Wissenschaften, vol. 303. Springer, Berlin (1993). 449 pp.

[6] Korneichuk, N.P.: On the best approximation of continuous functions. Izv. Akad. Nauk SSSR Ser. Mat. 27(1), 29–44 (1963)

[7] Mironenko, A.V.: On the Jackson–Stechkin inequality for algebraic polynomials. Trudy Inst. Mat. i Mekh. UrO RAN. 16(4), 246-253 (2010)

[8] Stekloff, W.: Sur les problemes de représentation des fonctions a l’aide de polygones, du calcul approché des intégrales définies, du développement des fonctions en séries infinies suivant les polynomes et de l’interpolation, considérés au point de vue des idées de Tchebycheff. Toronto: Proceeding of ICM, 631–640 (1924)

[9] Foucart, S., Kryakin, Yu., Shadrin, A.: On the exact constant in Jackson-Stechkin inequality for the uniform metric. Constr. Approx. 29(2), 157–179 (2009)

[10] Babenko, A.G., Kryakin Yu.V., Staszałk P.T.: Special Moduli of Continuity and the Constant in the Jackson–Stechkin Theorem. Constr. Approx. 38(3), 339–364 (2013)

[11] Bohr, H.: Ein allgemeiner Satz über die Integration eines trigonometrischen Polynoms. Pr. Mat.-Fiz. 43 273–288 (1935). (Collected Mathematical Works II, C 36)

[12] Favard, J.: Sur l’approximation des fonctions périodiques par des polynomes trigonométriques. Comptes rendus Acad. Sci. Paris. 203 1122-1124 (1936)

[13] Favard, J.: Sur les meilleurs procédés d’approximation de certaines classes de fonctions par des polynomes trigonométriques. 

Bul. Sci. Math. 61, 209–224, 243–256 (1937)

[14] Akhiesier, N.I., Krein, M.G.: On the best approximation of periodic functions. DAN SSSR, 15, 107–112 (1937). (Russian)
[15] Sinwel, H.F.: Uniform Approximation of Differentiable Functions by Algebraic Polynomials. Journal of Approximation Theory. 32, 1–8, (1981)
[16] Whitney, H.: On the functions with bounded $n$–differences. J.Math.Pures.Appl. 36(9), 67–95 (1957)
[17] Sendov, Bl.: On the constants of H.Whitney. C. R. Acad. Bulg. Sci. 35(4), 431 – 434 (1982)
[18] Sendov, Bl.: The constants of H.Whitney are bounded. C. R. Acad. Bulg. Sci. 38(10), 1299 – 1302 (1985)
[19] Kryakin, Yu.V.: On Whitney's theorem and Constants. Sbornik Mathematics. 81(2), 281–295 (1995)
[20] Kryakin, Yu.V.: On functions with bounded $n$–th differences. Izvestiya Mathematics. 61(2), 331–346 (1997)
[21] Kryakin, Yu.V.: Whitney’s constants and Sendov’s conjectures. Math. Balkanica (N.S.). 16(1–4), 235–247 (2002)
[22] Gilewicz, J., Shevchuk, I.A., Kryakin, Yu.V.: Boundedness by 3 of the Whitney Interpolation Constant. Journal of Approximation Theory. 119(2), 271–290 (2002)
[23] Zhelnov, O.: Whitney constants are bounded by 1 for $k = 5, 6, 7$. East J. Approximation. 8, 1–14 (2002)
[24] Zhelnov, O.: Whitney’s Inequality and its generalizations. Ph.D., Kiev (2004). 128 pp.
[25] Dzyadyk, V.K. Shevchuk, I.A.: Theory of Uniform Approximation of Functions by Polynomials. Berlin: Walter de Gruyter (2008)

Alexander Babenko
Institute of Mathematics and Mechanics
Ural Branch of the Russian Academy of Sciences
S. Kovalyovskoi Str. 16,
Ekaterinburg, 620990, Russia,
Ural Federal University,
Ekaterinburg, Russia
babenko@imm.uran.ru

Yuriy Kryakin
Institute of Mathematics
University of Wroclaw
Plac Grunwaldzki 2/4,
50-384 Wroclaw, Poland
kryakin@gmail.com,
kryakin@math.uni.wroc.pl