Classes of exact static spheroidal Einstein-Maxwell solutions.

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Abstract

In this paper we study the spheroidal cases of static charged fluid configurations in general relativity. We consider the effect of the anisotropic stresses of electromagnetic field on the shape of static charged self-gravitating objects. It is shown that electromagnetic fields can have significant effect on the structure and properties of self-gravitating objects.

1 Introduction

Exact solutions of the Einstein-Maxwell field equations are of crucial importance in relativistic astrophysics. These solutions may be utilised to model a charged relativistic star as they are matchable to the Reissner-Nordstrom [1], [2] exterior at the boundary. A wide spread assumption in the study of stellar structure is that the shape of star can be modeled as a spherical symmetry object. This approach has been used extensively in the study of star, star system and galaxies. However, in many systems, deviation from spherical symmetry may play an important role in determining of them properties. Physical situation where unspherical shape may be relevant are very diverse. On the other hand, self-gravitating objects resulting from the coupling electromagnetic field to gravity are a system where anisotropic pressure occurs naturally.

Anisotropy appears as an extra assumption on the behavior of electromagnetic field and on the shape of equilibrium configuration. Since we still

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do not have a formulation of the possible anisotropic stresses is emerging in these or other contexts, we take the approach of finding several exact solutions representing physical situations, modelled by ellipsoid of revolution. Solutions to the equation in spheroidal coordinates have application to a wide range of problems in physics [5]. Our goals here is to find exact spheroidal solution, offering an analysis of the change in the physical properties of the stellar and galaxy models due to presence of electromagnetic field.

2 Einstein-Maxwell Equations and Static spheroidal configurations.

In this paper we study static, spheroidal solutions of the Einstein-Maxwell system featuring a spinless charge configurations. The vacuum Einstein-Maxwell equations, in geometrized units such that $c = 8\pi G = \mu_0 = \varepsilon_0 = 1$, can be written as [3]

\[ R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} = T_{\mu\nu}, \quad (1) \]

\[ F^{\mu\nu} = 0, \quad (2) \]

with the electromagnetic energy-momentum tensor given by

\[ T_{\mu\nu} = F_{\mu\eta}F^{\eta}_{\nu} - \frac{1}{4}g_{\mu\nu}F^{\eta\xi}F^{\mu\xi}, \quad (3) \]

where

\[ F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}, \quad (4) \]

is the electromagnetic field tensor and $A_{\mu}$ is the electromagnetic four potential.

To start with, note that by using coordinate freedom inherent in general relativity any static spheroidal geometry can by put into form where are only two independent metric components typically functions of the coordinates $\xi$. As we have already mentioned we consider the ansatz static spheroidal space-time.

The two-dimensional elliptic coordinate system is defined from the set of all ellipses and all hyperbolas with a common set of two focal points. We denote the separation of the two focal points by $2c$. 

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Oblate spheroidal coordinates are derived from elliptic coordinates by rotating the elliptical coordinate system about the perpendicular bisector of the focal points. The coordinates are often labelled \( \eta, \xi \) and \( \theta \) with the transformation to Cartesian coordinates given by

\[
x = c\eta \xi \sin(\theta),
\]

\[
y = c\sqrt{\left(\xi^2 - 1\right)(1 - \eta^2)},
\]

\[
z = c\eta \xi \sin(\theta) \cos(\theta).
\]

Similarly, one can obtain the prolate spheroidal coordinates by rotating it about the parallel bisector.

\[
x = c\eta \xi,
\]

\[
y = c\sqrt{\left(\xi^2 - 1\right)(1 - \eta^2)} \cos(\theta),
\]

\[
z = c\sqrt{\left(\xi^2 - 1\right)(1 - \eta^2)} \sin(\theta).
\]

Let the spacetime ansatz be described by the spheroidal metric given by

\[
ds^2 = -B(\xi) \, dt^2 + A(\xi) \, d\Omega^2.
\]

For this static spheroidal ansatz of space-time we take the electromagnetic potential as

\[
A_\mu = (\psi, 0, 0, 0),
\]

where it is assumed that the electric potential \( \psi \) depends on \( \xi \) only.

We adopt coordinates that allow us to write spheroidal geometry in prolate form

\[
d\Omega^2 = c^2 \frac{\xi^2 - \eta^2}{\xi^2 - 1} \, d\xi^2 + c^2 \frac{\xi^2 - \eta^2}{1 - \eta^2} \, d\eta^2 + c^2 (\xi^2 - 1)(1 - \eta^2) \, d\theta^2,
\]

and in oblate form

\[
d\Omega^2 = c^2 \frac{\xi^2 - \eta^2}{\xi^2 - 1} \, d\xi^2 + c^2 \frac{\xi^2 - \eta^2}{1 - \eta^2} \, d\eta^2 + (c\xi\eta)^2 \, d\theta^2,
\]

where \( A, B \) are function of \( \xi \) only and \( \xi \geq 1, -1 \leq \eta \leq 1, 0 \leq \theta \leq 2\pi \).
2.1 Prolate spheroidal configurations.

After a bit of algebra, the field equations (11) - (2) are explicitly given in forms of the metric (11) in prolate case.

\[
\frac{A''}{2A} + \frac{A'B'}{2AB} = -\frac{\varepsilon^2}{2(1 - \xi^2)A} = T_{11}, \quad (15)
\]

\[
\frac{\xi^2 - 1}{2(\eta^2 - 1)} \left[ -\frac{A''}{A} - \frac{B''}{B} + \frac{A'^2}{A^2} + \frac{B'^2}{2B^2} \right] = \frac{\varepsilon^2}{2(\xi^2 - 1)(\eta^2 - 1)A} = T_{22}, \quad (16)
\]

\[
\frac{A'}{A} + \frac{B'}{B} = 0 = T_{12}, \quad (17)
\]

\[
\frac{(\xi^2 - 1)(\eta^2 - 1)}{2(\xi^2 - \eta^2)} \left[ -\frac{A''}{A} - \frac{B''}{B} + \frac{A'^2}{A^2} + \frac{B'^2}{2B^2} \right] = -\frac{\varepsilon^2(\eta^2 - 1)}{2(\xi^2 - \eta^2)A} = T_{33}, \quad (18)
\]

\[
\frac{B(\xi^2 - 1)}{c(\xi^2 - \eta^2)} \left[ -\frac{A''}{A^2} + \frac{3A'^2}{4A^3} - \frac{2\xi}{\xi^2 - 1} \frac{A'}{A^2} \right] = \frac{\varepsilon^2 B}{2c(\xi^2 - 1)(\xi^2 - \eta^2)A^2} = T_{00}, \quad (19)
\]

where prime ('') denoting derivative with respect to the \( \xi \) coordinate.

After a simple integration, from (15) - (19) we obtain

\[
A = \frac{1}{8} \left[ a_0 \pm \varepsilon \ln \left( \frac{\xi + 1}{\xi - 1} \right) \right]^2, \quad (20)
\]

\[
B = \frac{b_0}{A}, \quad (21)
\]

where \( a_0, b_0 \) arbitrary constants.

2.2 Oblate spheroidal configurations.

Replacing the line element in the field equations, the oblate set is

\[
\frac{A'^2}{4A^2} + \frac{A'B'}{2AB} = -\frac{\varepsilon^2}{2\xi^2(\xi^2 - 1)A} = T_{11}, \quad (22)
\]
\[
\frac{\xi^2 - 1}{2(\eta^2 - 1)} \left[ -\frac{A''}{A} - \frac{B''}{B} + \frac{A'^2}{A^2} + \frac{B'^2}{2B^2} \right] = -\frac{\varepsilon^2}{2\xi^2(\eta^2 - 1)} = T_{22}, \tag{23}
\]

\[
\frac{A'}{A} + \frac{B'}{B} = 0 = T_{12}, \tag{24}
\]

\[
\frac{\xi^2 \eta^2(\xi^2 - 1)}{2(\xi^2 - \eta^2)} \left[ \frac{A''}{A} + \frac{B''}{B} - \frac{A'^2}{A^2} - \frac{B'^2}{2B^2} \right] = -\frac{\varepsilon^2 \eta^2}{2(\xi^2 - \eta^2)A} = T_{33}, \tag{25}
\]

\[
\frac{B(\xi^2 - 1)}{c(\xi^2 - \eta^2)} \left[ -\frac{A''}{A^2} + \frac{3A'^2}{4A^3} - \frac{(2\xi^2 - 1) A'}{\xi(\xi^2 - 1) A^2} \right] = \frac{B\varepsilon^2}{2c\xi^2(\xi^2 - \eta^2)A^2} = T_{00}. \tag{26}
\]

Then the solutions of the gravitational field equations take in oblate case the form

\[
A = 2 \left( a_0 \pm \varepsilon \arctan \sqrt{\frac{\xi^2 - 1}{\xi^2 + 1}} \right)^2, \tag{27}
\]

\[
B = \frac{b_0}{A}, \tag{28}
\]

where \(a_0, b_0\) arbitrary constants.

### 2.3 Analysis

To see that all these metrics is asymptotically flat Minkowski it is enough to show that the metric components behave in an appropriate way at large \(\xi\)-coordinate values, e.g., \(g_{\mu\nu} = \eta_{\mu\nu} + O(1/\xi)\) as \(\xi \to \infty\). By inspection of the coefficients, we verify that this is so \((a_0 = 1, b_0 = 1)\).

In fact, in the present approach, it is easy to show that, in the case of absent the electromagnetic field \(\varepsilon = 0\), Einstein’s field equations yield only the flat space

\[
A = 1, \tag{29}
\]
Therefore, we see that it is possible to explain the shape of spheroidal configurations by electromagnetic or other fields [4]. This seems to be a remarkable result, although in a way it should be anticipated since the directional components of "equation of state" of electromagnetic field are anisotropic in the oblate and prolate cases. However, in this case there is a contribution from the electromagnetic field that makes $T_{\mu\nu}$ nonzero. On the other hand, it seems natural that we have obtained an "equation of state" that describes vacuum, since we do not have matter.

3 Discussion

In this article we delineated the qualitative features one would expect from spheroidal object. It is demonstrated that our model can successfully predict the spheroidal configuration in terms of a self-gravitating spacetime solution to the Einstein field equations and reproduce the not spherically-symmetric shape in terms of the non-trivial energy density and anisotropic pressure of the electromagnetic field which was absent in the context of empty space.

Hence the approach followed in this paper has proved to be a fruitful avenue for generating new exact solutions for describing the spacetimes of charged configurations.

We believe that following this hypotheses the shape of galaxy and rotation curve may be explained by action of electromagnetic or other fields. The solution presented here could be a first approximation at the galactic spacetime provided the presence of any physical fields. Therefore, it is necessary to study how these results modify the standard method of interpretation rotation data. Further investigation into the nature solutions with view to separating the real rotational effects from the electromagnetic, scalar or other fields anisotropy might be rewarding.

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