Gravitational Rutherford scattering of electrically charged particles from a charged Weyl black hole

Mohsen Fathi · Marco Olivares · J.R. Villanueva

Received: date / Accepted: date

Abstract Considering electrically charged test particles, we continue our study of the exterior dynamics of a charged Weyl black hole which has been previously investigated regarding the motion of mass-less and (neutral) massive particles. In this paper, the deflecting trajectories of charged particles are designated as being gravitationally Rutherford-scattered and detailed discussions of angular and radial particle motions are presented.

Keywords Particle motion · charged black holes · Rutherford scattering

1 Introduction

The motion of particles in gravitational field of static charged black holes has been studied widely, and has found its way into classic textbooks (see for example Refs. [1, 2] and the reviews in Refs. [3, 4]). Such black holes, known as Reissner-Nordström (RN) black holes in general relativity, have appeared extremely interesting since they are able to offer all kinds of possible orbits to the approaching particles. Applying the available mathematical methods in the investigation of motion in gravitational fields, scientists have been able to study several aspects of static charged black holes, regarding the charged particle orbits (in the contexts of general relativity and alternative gravity, see for example Refs. [5–11] and Refs. [12–15]). Although black holes with net electric charge are still remained as purely theoretical objects, however, studying them can pave the way in understanding physical phenomena like radiation reaction of particles [16, 17] and black hole evaporation [18]. Hence, the interest in investigating particle motion around charged black holes becomes more justified and, as well as in general relativity, it has found its way into alternative theories of gravity.

In the same effort, we investigate the motion and the scattering of charged test particles, as they travel in a particular charged black hole spacetime, which has been obtained as a non-vacuum solution to the Weyl theory of gravity (see Ref. [19] and references therein). Recently, the motion of mass-less and neutral massive particles has been scrutinized in this spacetime [20, 21]. This solution is related to a spherically symmetric massive source of charge $\tilde{q}$, equipped with the vector potential

$$A_\alpha = \left( \frac{\tilde{q}}{r}, 0, 0, 0 \right).$$ (1)
Considering the general metric

\[ ds^2 = -B(r)dt^2 + \frac{dr^2}{B(r)} + r^2(d\theta^2 + \sin^2 \theta d\phi^2) \]  

(2)

in the usual Schwarzschild coordinates \((-\infty < t < \infty, r \geq 0, 0 \leq \theta \leq \pi \text{ and } 0 \leq \phi \leq 2\pi\), the black hole solution is given by \([19]\)

\[ B(r) = 1 - \frac{r^2 - Q^2}{\lambda^2}, \]

(3)
in which

\[ \frac{1}{\lambda^2} = \frac{3 \hat{m}}{r^3} + \frac{2c_1}{3}, \]

(4a)

\[ Q = \sqrt{2q}, \]

(4b)

where \(\hat{m}\) and \(\hat{r}\) are respectively the mass and the radius of the source. When \(\lambda > Q\), the causal structure of the spacetime is confined between an event \((r_+)\) and a cosmological horizon \((r++)\), which are given by \([21]\)

\[ r_+ = \lambda \sin \left( \frac{1}{2} \arcsin \left( \frac{Q}{\lambda} \right) \right), \]

(5a)

\[ r_{++} = \lambda \cos \left( \frac{1}{2} \arcsin \left( \frac{Q}{\lambda} \right) \right). \]

(5b)

Accordingly, the extremal black hole horizon is at \(r_{\text{ex}} = r_+ = r_{++} = \lambda/\sqrt{2}\) and is given for \(\lambda = Q\).

Note that, the transition to RN spacetime is not trivial, because to do so, we need to apply the imaginary transformation \(Q \rightarrow 2iQ_0\), in which \(Q_0\) is the total charge of a spherical massive source. Furthermore, the RN-(Anti) de Sitter spacetime is recovered by letting \(\hat{r}\) to be the free radial distance, \(3\hat{m} \rightarrow 2M\) and \(2c_1 \rightarrow \pm A\) (\(A\) is the cosmological constant).

In this discussion, we focus on studying the possible radial and angular motions of charged test particles that approach the charged Weyl black hole. Our method of study is based on the standard Hamilton-Jacobi formulation of particle motion. In particular, we concern with the deflecting trajectories which are strongly related to the gravitational version of Rutherford scattering of charged particles. This latter has been particularly dealt with in Ref. \([12]\) for an alternative charged black hole. Here, we consider the same concept and apply it to the spacetime under consideration. The equations of motion are those corresponding to hyperbolic orbits and it is found that finding analytical solutions to these equations needs specific mathematical tricks. Aside from this, we also calculate the Rutherford scattering for particles on radial trajectories and the evolution of temporal parameters are derived. Additionally, assuming the congruence deviation of a bundle of infalling world-lines, we discuss the internal interactions between the particles and point out their effects on the kinematical congruence expansion.

The paper is organized as follows: In Sec. 2, the basic equations governing the motion of charged particles in the spacetime generated by the charged Weyl black hole, are given by means of the Hamilton-Jacobi formalism. In particular, the angular motion is analyzed in detail in Sec. 3, which is followed by that for the radial motion in Sec. 4. In these two sections, the gravitational Rutherford scattering is formulated analytically and the trajectories are demonstrated respectively. In Sec. 5, a bundle of particle world-lines is considered and we use its deviation as a tool to discuss the congruence’s internal acceleration and expansion. We close our discussion in Sec. 6. Throughout the paper, we adopt a geometric unit system in which \(G = c = 1\).

## 2 Motion of charged particles around the charged Weyl black hole

The Hamilton-Jacobi method of describing the motion of particles of mass \(m\) and charge \(q\) in an electromagnetic field, is based on the superhamiltonian \([1]\)

\[ H = \frac{1}{2} g^{\mu\nu} p_\mu p_\nu, \]

(6)
in which the 4-momentum \(p\) satisfies \(p_\mu p^\mu = -m^2\) and is defined as

\[ p_\mu = g_{\mu\nu} \frac{dx^\nu}{d\tau} = (\pi_\mu + gA_\mu), \]

(7)
in terms of the affine parameter \(\tau\), the vector potential \(A\) and the generalized momentum \(\pi\), which is given according to the canonical Hamilton equation

\[ \frac{d\pi_\mu}{d\tau} = -\frac{\partial H}{\partial x^\mu}, \]

(8)

Recasting \(H\) in terms of the characteristic Hamilton function (i.e. the Jacobi action)

\[ H = -\frac{\partial S}{\partial \tau}, \]

(9)
we have $\pi_\mu = \partial S/\partial x^\mu$ and the Hamilton-Jacobi equation of the wave crests can be written as
\[
\frac{1}{2} g^{\mu\nu} \left( \frac{\partial S}{\partial x^\mu} + q A_\mu \right) \left( \frac{\partial S}{\partial x^\nu} + q A_\nu \right) + \frac{\partial S}{\partial \tau} = 0.
\] (10)

The generalized momentum $\pi$ is indeed responsible for the possible constants of motion. For stationary spherically symmetric spacetimes, such as that in Eq. (2), these constants are
\[
\pi_t \equiv -E = g_{tt} \frac{dt}{d\tau} - q A_t,
\]
(11a)
\[
\pi_\phi \equiv L = g_{\phi \phi} \frac{d\phi}{d\tau} - q A_\phi.
\]
(11b)

The constant $L$ corresponds to the particles’ angular momentum and for motion in asymptotically flat spacetimes, $E$ is their energy. Now, confining the motion to the equatorial plane ($\theta = \pi/2$) and taking into account the only non-zero term of the vector potential (i.e. $A_t = \frac{qQ}{\sqrt{2}}$), we can specify Eq. (10) as
\[
-\frac{1}{B(r)} \left( \frac{\partial S}{\partial t} + \frac{qQ}{\sqrt{2}r} \right)^2 + B(r) \left( \frac{\partial S}{\partial r} \right)^2 + \frac{1}{r^2} \left( \frac{\partial S}{\partial \phi} \right)^2 + 2 \frac{\partial S}{\partial \tau} = 0.
\]
(12)

Based on the method of separation of variables of the Jacobi action, Eq. (12) can be solved by defining $S_0(r)$ for which, the interpolation in Eq. (12) results in
\[
S_0(r) = \pm \int \frac{dr}{B(r)} \sqrt{(E - V_-)(E - V_+)},
\]
(13)
where the radial potentials are given by
\[
V_\pm(r) = V_q(r) \pm \sqrt{B(r) \left( m^2 + \frac{L^2}{r^2} \right)},
\]
(15a)
\[
V_q(r) = \frac{qQ}{\sqrt{2}r}.
\]
(15b)

Note that, both of the $\pm$ branches of $V_\pm(r)$ converge to the value $E_+ = \frac{qQ}{\sqrt{2}r}$ at $r = r_+$, which can be either positive or negative, depending on the sign of the electric charges, $q$ and $Q$. Here we adopt the condition $qQ > 0$, so that the $V_-$ branch is always negative (in the causal region $r_+ < r < r_{++}$), and we can consider the positive branch as the effective potential, i.e. $V_{eff} = V_+ \equiv V$. Furthermore, applying Eqs. (11) and (14), it is possible to obtain the following three velocities:
\[
u(r) \equiv \frac{dr}{d\tau} = \pm \sqrt{(E - V_+)(E - V)},
\]
(16)
\[
v_t(r) \equiv \frac{dr}{dt} = \pm \frac{B(r)u(r)}{E - V_q(r)},
\]
(17)
\[
v_\phi(r) = \frac{dr}{d\phi} = \pm \frac{r^2 u(r)}{L}.
\]
(18)

The zeros of the above velocities do correspond to the so-called turning points, $r_t$, which are specified by the condition $V(r_t) = E_t$. Additionally, these equations lead to the quadratures that determine the evolution of the trajectories. This is dealt with in the forthcoming sections and the corresponding analytical solutions are obtained. In the next section, we begin with the angular motion of the test particles.

3 Angular Motion

In this section we focus on analyzing the trajectories followed by charged particles with non-zero angular momentum ($L \neq 0$). The effective potential in Eq. (15) has been plotted in Fig. 1, in which the turning points $r_t$ correspond to the values of $E = E_t$ that satisfy $E_t = V(r_t)$. The significance of these points is that they do reveal the possible orbits of the test particles. In fact, according to Fig. 1, three turning points are highlighted; $r_t = r_U$ (the radius of unstable circular orbits), $r_t = r_S$ (the distance from the point of scattering) and $r_t = r_F$ (the point of no return, or the capturing distance).

3.1 Unstable circular orbits

The radius of the unstable circular (critical) orbits, $r_U$, is given by the condition $V'(r) \equiv \frac{dV(r)}{dr} \bigg|_{r_U} = 0$. Hence, from Eq. (15) we get
\[
\left( \sqrt{\frac{G(r;L)}{B(r)}} - \frac{B'(r)}{2B(r)} \right) - \sqrt{\frac{B(r)}{G(r;L)} \frac{L^2}{r^2} + \frac{qQ}{\sqrt{2}r^2}} \bigg|_{r_U} = 0,
\]
(19)
The turning points are determined by the intersection of \( r \) of unstable circular orbits \( \Delta \phi \equiv V(\tau) \). These include the radius of unstable circular orbits \( r_u \), and two other points, \( r_S \) and \( r_F \).

In which the function \( G(r; L) \) is defined as

\[
G(r; L) = m^2 + \frac{L^2}{r^2}.
\]  

In fact, the left hand side of Eq. (19) leads to an incomplete polynomial of twelfth degree in \( r \), and hence, it can be solved only numerically. It is however still possible to calculate the proper \( (T_r) \) and the coordinate \( (T_t) \) periods of these orbits. Combining Eqs. (16), (17) and (18), and the fact that for a complete orbit \( \Delta \phi = 2\pi \), we have

\[
T_r \equiv \Delta \tau = \frac{2\pi r_u^2}{L_U},
\]

\[
T_t \equiv \Delta t = \frac{2\pi r_u^2}{L_U} \frac{E_U - V_u(r_u)}{B(r_u)} = T_r \sqrt{\frac{G_U}{B_U}},
\]

where \( G_U \equiv G(r_U; L_U) \) and \( B_U \equiv B(r_U) \). Moreover, Eq. (19) allows for obtaining an expression for \( L_U \), by solving

\[
aL_U^4 - bL_U^2 + c = 0,
\]

in which

\[
a = \frac{(Q^2 - 2r_u^2)^2}{r_u^2},
\]

\[
b = \frac{2Q^2(1 + q^2)}{r_u^2} - \frac{Q^4(2 + q^2)}{2r_u^2} - \frac{8r_u^2 - 2Q^2(1 - q^2)}{\lambda^4},
\]

\[
c = \frac{Q^4(1 + 2q^2)}{4r_u^2} - 2q^2 Q^2 + \frac{4r_u^2}{\lambda^4} - \frac{2Q^2r_u^2(1 - q^2)}{\lambda^2}.
\]

Solving Eq. (23) for \( L_U \), then yields

\[
L_U = \sqrt{\frac{b - \sqrt{b^2 - 4ac}}{2a}},
\]

as the angular momentum for the circular orbits. Accordingly, one can obtain the proper frequency

\[
\omega_r = \frac{2\pi}{T_r} = \sqrt{\frac{b - \sqrt{b^2 - 4ac}}{2ar_u^2}},
\]

straightly from Eqs. (21) and (25). The coordinate frequency can then be given by the ratio

\[
\omega_r = \sqrt{\frac{G_U}{B_U}}.
\]

These values correspond to the velocity of particles on a surface, where they can maintain a circular orbit before falling into the event horizon or escape from it. In the study of particle trajectories, the critical orbits can locate the innermost possible stable orbits around black holes and therefore are of great importance. The particles, however, can also be scattered at the turning point \( r_S \) and pursue a hyperbolic motion and escape the black hole. For electrically charged particles, this corresponds to the so-called Rutherford scattering. We continue our discussion analyzing by this kind of orbit.

3.2 Orbits of the first kind and the gravitational Rutherford scattering

The particle deflection by the black hole happens when the condition \( E_+ < E < E_U \) is satisfied. This indeed results in two points of approach, \( r_t = r_S \) and \( r_t = r_F \), at which, \( \frac{dr}{d\phi} |_{r_t} = 0 \) or \( E_t = V(r_t) \) (see Fig. 1). The relevant equation of motion can be derived from Eqs. (16) and (18), giving

\[
\frac{d^2r}{d\phi^2} = \frac{\mathcal{P}(r)}{v^2},
\]

where

\[
\mathcal{P}(r) \equiv r^6 + Ar^4 + Br^3 + Cr^2 + D,
\]

\[
v = \frac{L}{m},
\]
The equation of motion \( \mu = v^2 \left( \frac{E^2 - m^2}{L^2} + \frac{1}{\lambda^2} \right) \), needs to solve an equation of sixth degree in \( r \), and values of \( r(\phi) \) can therefore be obtained through numerical methods. To deal with this problem, we pursue the inverse process and find an analytical expression for \( \phi(r) \). The behavior of \( r(\phi) \) can then be demonstrated by means of numerical interpolations.

To proceed with this method, let us consider that \( \mathcal{P}(r) \) has two distinct real roots, corresponding to the turning points \( r_1 = r_S \) and \( r_2 = r_F \), two equal and negative real roots, say \( r_3 = r_4 < 0 \), and finally a complex conjugate pair \( r_5 \) and \( r_6 = r_5^* \). Accordingly, we can recast \( \mathcal{P}(r) \) as

\[
\mathcal{P}(r) = \prod_{j=1}^{6} (r - r_j)
\]

\[
= (r - r_S)(r - r_F)(r - r_3)(r - r_4)(r - r_5^*)(r - r_6^*).
\]

The equation of motion (28) can then be written as

\[
\phi(r) = \frac{1}{6} \int_{r_S}^{r} \frac{dr}{\sqrt{\mathcal{P}(r)}}.
\]

Particles reaching \( r_S \), experience a hyperbolic motion around the black hole and then escape to infinity. This kind of motion, known as orbit of the first kind (OFK) [2, 23], has the significance of gravitational Rutherford scattering, when the test particles are electrically charged. Since the distance of the closest approach happens at \( r_S \), to deal with the integral in Eq. (32), we define the following non-linear change of variable:

\[
u = \frac{1}{r_S - 1},
\]

which reduces Eq. (32) to

\[
\phi(r) = \kappa_0 \left[ \int_{\nu}^{\infty} \frac{d\nu}{\sqrt{\mathcal{P}_3(\nu)}} - u_3 \int_{\nu}^{\infty} \frac{d\nu}{(\nu + u_3)\sqrt{\mathcal{P}_3(\nu)}} \right],
\]

with \( u_3 = \frac{1}{r_S - 1} \), where \( j = \{2, 3, 5, 6\} \), and

\[
\mathcal{P}_3(u) = u^3 + au^2 + bu + c,
\]

where \( a = u_2 + u_5 + u_6 \), \( b = u_2(u_5 + u_6) + u_5u_6 \), \( c = u_2u_5u_6 \).

Defining

\[
k_0 = \frac{\nu}{r_S^2} u_3 \sqrt{u_2u_5u_6},
\]

and applying another change of variable

\[
U = \frac{1}{4} \left( \nu + \frac{a}{3} \right),
\]

we can rewrite Eq. (34) as

\[
\phi(r) = k_0 \left[ \int_{U}^{\infty} \frac{dU}{\sqrt{\mathcal{P}_3(U)}} - \frac{u_3}{4} \int_{U}^{\infty} \frac{dU}{(U + u_3)\sqrt{\mathcal{P}_3(U)}} \right],
\]

given that \( U_3 = \frac{1}{4} \left( u_3 + \frac{a}{3} \right) \), and

\[
\mathcal{P}_3(u) = 4U^3 - g_2U - g_3.
\]

Now, the direct integration of the right hand side of Eq. (39) results in

\[
\phi(r) = k_0 \left[ \beta(U) - \frac{u_3}{4} \bar{\beta}(U) \right],
\]

where \( \beta(x) \equiv \beta(x, g_2, g_3) \) is the inverse Weierstraß \( \varphi \) function, and

\[
\bar{\beta}(U) = \frac{1}{\varphi'(\Omega_S)} \left[ \zeta(\Omega_S)\beta(U) + \ln \left| \frac{\sigma(\beta(U) - \Omega_S)}{\sigma(\beta(U) + \Omega_S)} \right| \right],
\]

where \( \zeta(\Omega_S) \) is the Riemann zeta function.

Gravitational Rutherford scattering of electrically charged particles from a charged Weyl black hole
also contains the Weierstraß Zeta and Sigma functions ($\zeta(y)$ and $\sigma(y)$, respectively)\(^1\). Here, we have defined

\[
U \equiv U(r) = \frac{1}{4 \left( \frac{r}{r_s} - 1 \right)} + \frac{a}{12},
\]

(43a)

\[
\Omega_S = \beta \left( \frac{a}{12} - \frac{1}{4 \left( \frac{r}{r_s} - 1 \right)} \right),
\]

(43b)

\[
g_2 = \frac{a^2}{12} - \frac{b}{4},
\]

(43c)

\[
g_3 = \frac{1}{16} \left( \frac{ab}{3} - \frac{2a^3}{27} - c \right).
\]

(43d)

The scattering angle in Eq. (41) gives the change in the particles’ orientation as they approach and recede the black hole at the scattering point $r_S$. To illustrate their corresponding trajectories, we make a list of points $(r_t, \phi(r_t))$ and then find the numerical interpolating function of $r(\phi)$. The resultant OFK trajectories have been illustrated in Fig. 2 for particles of different values for $E$. As it is observed, the scattering can be formed convexly (approaching) or concavely (receding). This in fact can be justified by measuring the scattering angle. For particles coming from infinity, the scattering angle can be written as\([12, 21]\)

\[
\vartheta = 2\phi_\infty - \pi,
\]

(44)

\(^1\) By definition, we have [24]

\[
\beta(x) \equiv y = \int_x^\infty \frac{dt}{\sqrt{4t^4 - g_2 t - g_3}}
\]

Then the inverse function $x = \varphi(y, g_2, g_3) \equiv \varphi(y)$ defines the elliptic Weierstraß $\varphi$ function with the coefficients $g_2$ and $g_3$, for which

\[
\varphi'(y) = \frac{d}{dy} \varphi(y) = -\sqrt{4\varphi^4(y) - g_2 \varphi(y) - g_3}.
\]

The two other related functions, namely the Weierstraß Zeta and Sigma functions, are defined as

\[
\zeta(y) = -\int \varphi(y)dy,
\]

\[
\sigma(y) = e^{\int \zeta(y)dy}.
\]

\[\text{Fig. 2} \quad \text{The Rutherford scattering plotted for } m = 1, Q = 1, q = 0.5, \lambda = 10 \text{ and } L = 1. \text{ For these values, } r_+ = 0.50, r_U = 0.69 \text{ and } E_U = 1.72. \text{ The trajectories have been plotted for } E_1 = 0.88, E_2 = 1.1, E_3 = 1.3, E_4 = 1.5 \text{ and } E_5 = 1.7, \text{ while their corresponding scattering distance } (r_S) \text{ have been indicated by dashed circles. As it is observed, the condition } E_5 \approx E_U \text{ has made the corresponding shape of the scattering to be of a convex form, showing an appeal to the critical orbits.}
\]

in which $\phi_\infty \equiv \phi(\infty)$. Accordingly, from Eq. (41) we have

\[
\vartheta = -\pi + 2\kappa_0 \left\{ \frac{\beta \left( \frac{a}{12} \right)}{4} + \frac{u_3}{4} \left[ \frac{1}{\sqrt{4a^4 - 18ab + 27u_3 (b - au_3 + u_3^2)}} \right] \right. \times \left[ \zeta(\Omega_S) \frac{\beta \left( \frac{a}{12} \right)}{12} + \ln \left[ \frac{\sigma \left( \frac{\beta \left( \frac{a}{12} \right) - \Omega_S}{\sigma \left( \frac{\beta \left( \frac{a}{12} \right) + \Omega_S} \right)} \right) \right] \right\}. \]

(45)

The value of $\vartheta$ is specified directly by the initial $E$ and the corresponding particular solutions $r_j$, which are determined by the equation $E = V(r)$. These values therefore, cannot be considered to evolve in terms of a single variable. However, one can calculate the scattering angle for each particu-
In this section, we analyzed the angular trajectories of infalling particles and demonstrated the corresponding Rutherford scattering which is the most significant feature of such trajectories. However, scattering can happen, as well, in the absence of the particles’ angular momentum, which is what we deal with in the next section by studying the radial trajectories of the test particles.

4 Radial Trajectories

To discuss the horizon crossing process, as viewed by the comoving and the distant observers, it is common to use a framework, in which, the particles fall radially onto the black hole, without possessing any angular momentum. As well as in the case of angular orbits, purely radial motion can itself be ramified into several kinds. Further in this section, these kinds of motion are discussed regarding the horizon crossing of the comoving observers and the frozen particles as viewed by distant observers (see Refs. [25, 26] for the notion of a frozen infalling object onto a black hole).

The vanishing angular momentum of the radially moving particles, reduces the effective potential in Eq. (15) to

\[ V_r(r) = V_0(r) + m \sqrt{B(r)}, \]

whose behavior has been plotted in Fig. 3. Accordingly, the motion becomes unstable where \( V'(r) = 0 \), which leads to the following equation of eighth degree:

\[ r^8 + \tilde{a} r^4 + \tilde{b} r^2 + \tilde{c} = 0, \]

(54)

with

\[ \tilde{a} = -\frac{Q^2 \lambda^2}{2} \left( 1 - \frac{q^2}{m^2} \right), \]

(55a)

\[ \tilde{b} = -\frac{q^2 Q^2 \lambda^4}{2 m^2}, \]

(55b)

\[ \tilde{c} = \frac{Q^4 \lambda^4}{16} \left( 1 + \frac{2 q^2}{m^2} \right). \]

(55c)

The maximum distance of the unstable motion can then be obtained from Eq. (54) as

\[ r_u = \left[ \tilde{a} - \sqrt{\tilde{a}^2 - \tilde{b}} \right]^{1/2}. \]

(56)
The effective potential for radially moving particles plotted for \( m = 1, Q = 1, q = 0.5 \) and \( \lambda = 10 \). The maximum distance of unstable motion, \( r_e \), and the two turning points \( r_s \) and \( r_f \) have been indicated, in accordance with their corresponding values of \( E \). In particular, the point \( r_s \) is related to the distance at which the particles of the constant of motion \( E_+ \), experience their Rutherford scattering.

\[
\tilde{\alpha} = \sqrt{\tilde{U} - \frac{\tilde{\alpha}}{6}}, \quad (57a)
\]
\[
\tilde{\beta} = 2\tilde{\alpha}^2 + \frac{\tilde{\alpha}}{2} + \frac{\tilde{b}}{4\tilde{\alpha}}, \quad (57b)
\]
given that
\[
\tilde{U} = 2\sqrt{\frac{\tilde{\eta}_2}{3}} \cosh \left( \frac{1}{3} \cosh^{-1} \left( \frac{3}{2} \sqrt{\frac{3}{\tilde{\eta}_2}} \right) \right), \quad (58a)
\]
\[
\tilde{\eta}_2 = \frac{\tilde{\alpha}^2}{48} + \frac{\tilde{c}}{4}, \quad (58b)
\]
\[
\tilde{\eta}_3 = \frac{\tilde{\alpha}^3}{864} + \frac{\tilde{b}^2}{64} - \frac{\tilde{a}\tilde{c}}{24}. \quad (58c)
\]

Accordingly, we have \( E_u \equiv V_i(r_u) \). Same as in the angular case, possible motions are categorized based on the value of \( E \) compared with its critical value, \( E_u \):

- **Critical radial motion**: For \( E = E_u \), the particles can stay on an unstable radial distance of radius \( r = r_u \). Therefore, those coming from the initial distances \( \epsilon_i \) \((r_u < r_i < r_+)\) and \( d_i \) \((r_u < d_i < r_u\) respectively), will ultimately fall on \( r_u \), or cross the horizons.

Now, let us rewrite the radial velocity relations given in Eqs. (16) and (17) as
\[
\frac{dr}{dt} = \frac{m^2 p(r)}{\lambda^2 r^2}, \quad (59)
\]
\[
\frac{d\epsilon}{dt} = \frac{m^2 (r^2 - r_e^2) \epsilon^2 (r^2 - r_e^2)^2 p(r)}{E^2 \lambda^2 r^4 (r - \sqrt{2qQ})^2}, \quad (60)
\]
with
\[
p(r) \equiv r^4 + \tilde{\alpha} r^2 + \tilde{\beta} r + \tilde{c}, \quad (61)
\]
where
\[
\tilde{\alpha} = \frac{(E^2 - m^2) \lambda^2}{m^2}, \quad (62a)
\]
\[
\tilde{\beta} = -\frac{\sqrt{2qQ} E \lambda^2}{m^2}, \quad (62b)
\]
\[
\tilde{c} = \frac{Q^2 (m^2 + 2q^2) \lambda^2}{4 m^2}. \quad (62c)
\]

Applying the above equations, in this section, the radial behavior of the time parameter for the moving particles is calculated, together with that for the distant observers.

### 4.1 Frontal scattering

As it is inferred from the effective potential in Fig. 3, particles can encounter two turning points \( r_s \) and \( r_f \) which are located at either sides of the critical distance \((r_f < r_u < r_s)\). These turning points do lead the trajectories to different fates. Particles with \( E_+ < E < E_u \), however, can only escape the black hole by being scattered at the only possible turning point \( r_s \). Same as discussed in Sec. 3, the turning points are where the particles’ coordinate velocity vanishes, which for the radial trajectories requires \( p(r) = 0 \) in Eq. (61), giving
\[
r_s = \tilde{\alpha} + \sqrt{\tilde{\alpha}^2 - \tilde{\beta}}, \quad (63)
\]
\[
r_f = \tilde{\alpha} - \sqrt{\tilde{\alpha}^2 - \tilde{\beta}}. \quad (64)
\]
where
\[ \bar{\alpha} = \sqrt{U - \frac{\bar{a}}{6}}, \]
\[ \beta = \frac{2}{3} \bar{a}^2 + \frac{\bar{a}}{2} + \frac{\bar{b}}{4}, \]
with
\[ U = 2 \sqrt{\frac{\bar{\eta}_2}{3}} \cosh \left( \frac{1}{3} \cosh^{-1} \left( \frac{3}{2} \sqrt{\bar{\eta}_3} \right) \right). \]
\[ \bar{\eta}_2 = \frac{\bar{a}^2}{3} + \bar{c}, \]
\[ \bar{\eta}_3 = \frac{\bar{a}^3}{84} + \frac{\bar{b}^2}{64} - \frac{\bar{a} \bar{b}}{24}. \]
Having determined the turning points, the polynomial \( p(r) \) can be decomposed accordingly, and Eq. (59) can be recast as
\[ \frac{d}{dr} \left( \frac{1}{\lambda^2 r^2} \right) = \frac{m^2 (r - r_s) p_3(r)}{r^2}, \]
in which
\[ p_3(r) \equiv r^3 + r_s r^2 + (r_s^3 + \bar{a}) r + r_s^3 + r_s \bar{a} + \bar{b}. \]
As described above, the first kind scattering (RSFK) happens when the particles approach at \( r_s \), which is now considered as their initial position. We therefore can rewrite Eq. (67) as
\[ \tau(r) = \frac{\lambda}{m} \int_{0}^{r} \frac{r dr}{\sqrt{(r - r_s) p_3(r)}}. \]
Defining the linear change of variable
\[ z \equiv \frac{r}{r_s} - 1, \]
reduces Eq. (69) to
\[ \tau(z) = \frac{\lambda}{m} \int_{0}^{z} \frac{(z + 1) dz}{\sqrt{2 \bar{\rho}_3(z)}}, \]
with
\[ \bar{\rho}_3(z) \equiv z^3 + 4 z^2 + \gamma_1 z + \gamma_0, \]
and
\[ \gamma_1 = 6 + \frac{\bar{a}}{r_s}, \]
\[ \gamma_0 = 4 + \frac{2 \bar{a}}{r_s^2} + \frac{\bar{b}}{r_s^3}. \]
Now, letting
\[ u \equiv \frac{1}{2}, \]
yields the following reduced integral form of Eq. (71):
\[ \tau(u) = \frac{-\lambda}{m \sqrt{\gamma_0}} \left( \int_{u}^{\infty} \frac{du}{\sqrt{p_3(u)}} + \int_{u}^{\infty} \frac{du}{u \sqrt{p_3(u)}} \right), \]
in which
\[ \bar{p}_3(u) \equiv u^3 + \gamma_1 u^2 + \frac{4}{\gamma_0} u + \frac{1}{\gamma_0}. \]
Applying the last change of variable
\[ u \equiv 4U - \frac{\gamma_1}{3\gamma_0}, \]
we get
\[ \tau(U) = \frac{-\lambda}{m \sqrt{\gamma_0}} \left( \int_{\gamma_0}^{U} \frac{dU}{\sqrt{\bar{p}_3(U)}} + \frac{1}{4} \int_{\gamma_0}^{U} \frac{dU}{(U - \frac{\gamma_1}{12\gamma_0})^{1/2} \sqrt{\bar{p}_3(U)}} \right), \]
where we have defined
\[ \bar{\rho}_3(U) \equiv 4U^3 - \bar{g}_2 U^2 - \bar{g}_3. \]
The direct integration of the elliptic integral in Eq. (78), now results in
\[ \tau(r) = \frac{-\lambda}{m \sqrt{\gamma_0}} \left[ \bar{\rho}(U) + \frac{1}{4} \bar{F}(U) \right], \]
where
\[ \bar{F}(U) = \frac{1}{\bar{g}'(\bar{r}_s)} \left[ \zeta(\bar{r}_s) \bar{\rho}(U) + \ln \left| \frac{\sigma(\bar{\rho}(U) - \bar{\rho}_s)}{\sigma(\bar{\rho}(U) + \bar{\rho}_s)} \right| \right], \]
and the function \( U(r) \) and the Weierstraß coefficients are given as
\[ U(r) = \frac{r_s}{4(r - r_s)} + \frac{\gamma_1}{12\gamma_0}, \]
\[ \bar{\Omega}_s = \bar{\beta} \left( \frac{\gamma_1}{12\gamma_0} \right), \]
\[ \bar{g}_2 = \frac{\gamma_1^2}{12\gamma_0^2} - \frac{1}{\gamma_0}, \]
\[ \bar{g}_3 = \frac{1}{16} \left( \frac{4\gamma_1^3}{3\gamma_0^3} - \frac{2\gamma_1^3}{27\gamma_0^3} - \frac{1}{70} \right). \]
The relation in Eq. (80) measures the radial change of the time parameter for observers comoving with the particles. For distant observers, such measurement is done on the coordinate time, whose evolution can be obtained by exploiting the velocity in Eq. (60). Applying the same method as before, we obtain

$$t(r) = -\delta_0 \left[ \beta(U) + \frac{1}{4} \sum_{k=1}^{4} \delta_k F_k(U) \right], \quad (83)$$

where

$$F_k(U) = \frac{1}{\varphi(\Omega_k)} \left[ \zeta(\Omega_k) \beta(U) + \ln \left| \frac{\sigma(\beta(U) - \Omega_k)}{\sigma(\beta(U) + \Omega_k)} \right| \right], \quad (84)$$

and

$$\Omega_k = \beta \left( \frac{\gamma_1}{12z_0} + \frac{z_k}{4} \right), \quad (85)$$

in which $$z_k \equiv \frac{1}{(r_u/r_s^-)^{1/2}}$$, with $$r_1 \equiv r_+$$, $$r_2 \equiv -r_+$$, $$r_3 \equiv r_+$$, and $$r_4 \equiv -r_+$$, and the coefficients are expressed as

$$\delta_0 = \frac{\lambda^2 E}{m\sqrt{z_0/z_0^2}} z_1 z_2 z_3 z_4, \quad (86a)$$

$$\delta_1 = \frac{\left( z_1 + 1 \right)^2 z_1 (z_1 - z_5)}{(z_1 - z_2)(z_1 - z_3)(z_1 - z_4)}, \quad (86b)$$

$$\delta_2 = \frac{\left( z_2 + 1 \right)^2 z_2 (z_2 - z_5)}{(z_2 - z_1)(z_2 - z_3)(z_2 - z_4)}, \quad (86c)$$

$$\delta_3 = \frac{\left( z_3 + 1 \right)^2 z_3 (z_3 - z_5)}{(z_3 - z_1)(z_3 - z_2)(z_3 - z_4)}, \quad (86d)$$

$$\delta_4 = \frac{\left( z_4 + 1 \right)^2 z_4 (z_4 - z_5)}{(z_4 - z_1)(z_4 - z_2)(z_4 - z_3)}. \quad (86e)$$

Since these trajectories escape the black hole, they will eventually confront the cosmological horizon. In Fig. 4, the temporal relations in Eqs. (80) and (83) have been used to demonstrate the RSFK, as observed by comoving and distant observers, for three different scattering distances. As it is seen, for the comoving observers, a horizon crossing occurs within a finite time after the particles are scattered at $$r_s$$. This is while for the distant observers it takes an infinite amount of time for the particles to pass the horizon; in other words, they appear frozen.

### 4.2 Frontal scattering of the second kind

By switching the scattering distance to $$r_f$$, the particles experience the RSSK and they confront the event horizon. The corresponding equations of motion are the same as those in the case of RSFK and are given by exchanging $$r_s \rightarrow r_f$$ in the relations. The corresponding temporal parameters have been demonstrated in Fig. 5. Same as before, the comoving and distance observers see different fates for the infalling particles, but here, regarding the capture process by the event horizon.

### 4.3 Critical radial motion

In the case that $$E = E_{ul}$$, the unstable (critical) motion of particles depends on whether they approach from $$r_i > r_u$$ or from $$d_i < r_u$$. According to the discontinuity of $$\frac{\delta t}{\delta r}$$ and $$\frac{\delta t}{\delta d}$$ at $$r_i$$ and $$d_i$$, we can expect two different behaviors for the approaching particles, in the sense that they either fall on $$r = r_u$$ (fate I) or be pulled towards the horizons (fate II). These are revealed by integrating the equations of motion for the time parameters. For particles coming from $$r_i$$, we
where the coefficients are given as

\[ \begin{align*}
\omega_0 &= \frac{r_u^3(r_u - r_3)}{(r_1 - r_u)(r_2 - r_u)(r_3 - r_u)(r_4 - r_u)}, \\
\omega_1 &= \frac{r_u r_1^2}{(r_1 - r_u)(r_2 - r_u)(r_3 - r_u)(r_4 - r_u)}, \\
\omega_2 &= \frac{r_u^4}{(r_2 - r_u)(r_3 - r_u)(r_4 - r_u)}, \\
\omega_3 &= \frac{r_u^2}{(r_3 - r_u)(r_4 - r_u)}, \\
\omega_4 &= \frac{r_u}{(r_4 - r_u)}.
\end{align*} \]

in which \( r_0 \equiv r_u, r_1 \equiv r_+, r_2 \equiv -r_+, r_3 \equiv r_{++} \) and \( r_4 \equiv -r_{++} \). The critical radial behavior of temporal parameters, as measured by the comoving and the distant observers, have been plotted in Fig. 6, separately for the initial points \( r_I \) and \( d_i \). In each of the diagrams, the cases \( I \) and \( II \) have been demonstrated and the horizon crossing is shown accordingly. Once again, particles appear frozen to distant observers as they approach the horizons.

So far, both the angular and the radial trajectories of charged particles were studied and possible analytical solutions to the equations of motion were given. It is however worth mentioning that the study of the physical properties of moving particles is not summarized to the evolution of a single particle’s trajectory. In the case that a bundle of trajectories is taken into account, definite kinematical parameters will play important roles in the characterization of a flow of particle trajectories. Accordingly, and in the next section, we consider such a flow of particles and study how it reacts to the internal and external forces acting on the world-lines.

5 A congruence of infalling charged particles

In this section, we consider a bundle of particle trajectories, which together, constitute a congruence of world-lines that fall onto the black hole. Essentially, the congruence kinematics is a tool to inspect the Penrose-Hawking singularity theorems [27–30] and is accurately formulated by the well-known Raychaudhuri equation [31]. This equation formulates the way the congruences would evolve their cross-sectional (transverse) area (for a good review see Ref. [32]).
Fig. 6 The critical radial motion for fates I and II, plotted for comoving (thick line) and distant (thin line) observers, by letting \( m = 1, Q = 1, q = 0.5 \) and \( \lambda = 10 \). The trajectories have been specified for particles approaching from (a) \( r = r_i = 5 \) and (b) \( r = d_i = 0.68 \).

Here, we switch our discussion to the possibility of applying some geometrical methods in order to demonstrate the deviation of a congruence of time-like trajectories while they pass the black hole. For particles passing a RN black hole, this deviation has been studied in detail in Refs. [33, 34].

In the geometric sense, the congruence deviation gives the relative acceleration between the curves that are generated by the tangential vector \( \mathbf{u} \), in terms of the Jacobi (deviation) vector field \( \xi \). This vector field resides on the curves that connect points of equal \( \tau \) on smooth planes of world-lines. These vectors satisfy [35, 36]

\[
\mathcal{L}_u \xi = \mathcal{L}_\xi u, \tag{94}
\]

where \( \mathcal{L}_X \) indicates the Lie differentiation with respect to a vector field \( X \). The above equation therefore can be recast as

\[
\xi^\alpha;_\beta u^\beta = u^\alpha;_\beta \xi^\beta. \tag{95}
\]

In above, the semicolons correspond to covariant differentiation. Note that, the quantity \( \xi \cdot u \) varies along the congruence as [36]

\[
\frac{D}{d\tau} (\xi \cdot u) \equiv (\xi \cdot u);_\beta u^\beta = \frac{1}{2} (u \cdot u);_\beta \xi^\beta + a^\alpha;_\beta \xi^\alpha u^\beta, \tag{96}
\]

where

\[
a^\mu = u^\mu;_\nu u^\nu, \tag{97}
\]

is the 4-acceleration of the non-inertial frames, according to non-gravitational effects. In this regard, a non-zero \( a \) corresponds to a vector field which is not parallel-transported along the world-lines. Accordingly, the congruence deviation equation can then be written as

\[
\mathcal{A}^\alpha = \frac{D^2 \xi^\alpha}{d\tau^2} \equiv (\xi^\alpha;_\beta u^\beta);_\gamma u^\gamma = a^\alpha;_\beta \xi^\beta - R^\alpha;_\beta;_\gamma u^\beta \xi^\gamma u^\delta. \tag{98}
\]

This vector, measures the relative acceleration between two world-lines, as measured by the change in \( \xi \), and connects it to the spacetime curvature [37, 38].

According to the Eqs. (16), (17) and (18), we know that a congruence of charged particles with angular motion, that fall onto the charged black hole, is generated by the following 4-velocity:

\[
w^\mu = \left( \frac{E - V_e(r)}{B(r)}, \sqrt{(E - V_e)(E - V)}, 0, L \right), \tag{99}
\]

which satisfies \( \mathbf{u} \cdot \mathbf{u} = -m^2 \) (we let \( m = 1 \)). The congruence deviation (Jacobi) field, related to the vector field (99), can then take the generic form

\[
\xi^\alpha = (\xi^0(r), \xi^1(r), 0, \xi^3(r)), \tag{100}
\]

\footnote{For two vectors \( \mathbf{x} \) and \( \mathbf{y} \), we notate \( \mathbf{x} \cdot \mathbf{y} = g_{\mu\nu}x^\mu y^\nu \).}
for which, the consideration of the Lie transportation condition (i.e. $L u \xi = 0$), provides

$$\xi^0(r) = 2^{\frac{3}{4}} \lambda^2 \left( \frac{E - rV_q(r)}{4 - 4\lambda^2 + \frac{Q^2}{r^2}} \right) - r \frac{E - V_q(r)}{r^2 \left( 4 - 4\lambda^2 + \frac{Q^2}{r^2} \right)}, \quad (101)$$

$$\xi^1(r) = 2^{\frac{3}{4}} \sqrt{(E - V)(E - V_\pm)}, \quad (102)$$

$$\xi^3(r) = -2^{\frac{3}{4}} L \left( 1 - \frac{r}{r^2} \right). \quad (103)$$

The above vector field results in a non-zero rate of change of $\xi \cdot u$, indicating that the Jacobi field $\xi$ is nowhere orthogonal to the congruence.

The 4-acceleration of the infalling charged particles in electromagnetic fields, obey the following relation [1]:

$$a^\mu = -\frac{q}{m} g^{\mu\nu} F_{\nu\rho} u^\rho, \quad (104)$$

which is given in terms of the field strength tensor

$$F_{\mu\nu} = A_{\nu\mu} - A_{\mu\nu}. \quad (105)$$

Accordingly, the congruence deviation equation (98) can be recast as (see also Refs. [33, 34])

$$\mathfrak{A}^\alpha = -R^\alpha_{\beta\gamma\delta} u^\beta \xi^\gamma u^\delta - \frac{q}{m} g^{\alpha\nu} \left( F_{\nu\sigma;\beta} u^\sigma \xi^\beta + F_{\nu\rho} u^\sigma ;\beta \xi^\beta \right). \quad (106)$$

Since, this acceleration is related to the internal interaction of the world-lines, it naturally affects the expansion of the congruence. This expansion is defined as the fractional rate of change of the transverse subspace of the congruence, and in our case is defined as [36]

$$\Theta = u^\alpha \omega^\alpha. \quad (107)$$

Accordingly, we can compare the behavior of $\mathfrak{A}$ with the congruence expansion as the particle world-lines approach the black hole. For this, we consider the norm of the aforementioned vector field, i.e. $||\mathfrak{A}||^3$, and plot it for a definite range for $E$, inside the causal region. Same is done for the congruence expansion (see Fig. 7). As it is seen in the figures, the norm of a vector $X$ is defined as $||X|| = \sqrt{X \cdot X}$. 

\[Fig. 7\] The behaviors of $||\mathfrak{A}||$ and $\Theta$ for $0.3 < E < 1.9$, considering $Q = 1$, $q = 0.5$, $\lambda = 10$ and $L = 1$. The corresponding event and cosmological horizons are located respectively at 0.5 and 9.98. The contours indicate discrete values for the parameters for specific ranges of $r$ and $E$. In particular, the parameter $\Theta$, beside discrete ones, can have very close values that reside on a line tangent to the contours.
the approaching congruence is of positive expansion, so that its transverse cross-section increases in area and the world-lines recede from each other. This is in agreement with the positive acceleration between the world-lines, as it is shown in the diagram of $|\mathbf{\Omega}|$. As the particles approach the event horizon, the congruence’s internal acceleration merges to a single value at a specific $E$, which indicates that only distinct particles can reach that region and there they will maintain a constant mutual force. In other regions, distant from the event horizon, the particle deflection (and scattering) can happen under positive congruence expansion and positive internal acceleration. According to the figures, for some fixed values of $E$, the internal interactions between the world-lines remain repulsive at all distances, however this repulsion is smaller at regions near the event horizon. This is while the congruence expansion reaches its maximum values for the same initial conditions. This is therefore a signature of scattering, where the expansion of the scattered congruence is a result of interactions with the source. On the other hand, for higher $E$, the relative acceleration and the congruence expansion take their maximum values near the black hole. The expansion in this case is naturally a result of internal interactions between the world-lines. We can therefore infer that the dynamical characteristics of a bundle of infalling world-lines on the black hole, can indicate the effect of such interactions on the way the particles approach and recede the source, through their specific type of orbit.

6 Final remarks

The gravitational effects of the electrical charge constituents of charged black holes are apparent in the motion of neutral particles in their vicinity. Charged test particles on the other hand, are also affected by an additional coulomb potential and this makes it more complicated to analyze their motion. Moving in the exterior geometry of such black holes, test particles feel this electric charge through a scalar potential, which is an external classical parameter. Despite this, the way this parameter is distributed inside the source, can change the total charge in the effective potential and therefore, can affect the particle trajectories. In general relativity, this can be inferred by applying the Einstein-Maxwell equations to obtain interior solutions of a relativistic star. In 1917, one year before proposing his theory of gravity, Weyl discovered a relationship between the metric and electrical potentials and found a class of interior axially symmetric solutions of a static source which had a quadratic dependence on the coulomb potential [39]. This was later proved to be the case even in the absence of axial symmetry and could stabilize the source by balancing the gravitational pull and the coulomb repulsion [40, 41]. These Weyl-type interior solutions, were then classified regarding the total integral of the interior charge density [42, 43], with extensions to higher dimensions [44, 45]. The total charge parameter of the black hole, is therefore proved to be a consequence of the type of the interior solution it obeys and this can be distinguishable regarding the particle trajectories. For example, in Ref. [7], it has been shown that switching between the Weyl-type interior solutions for a RN black hole, can change the intensity and the shape of possible orbits of approaching charged test particles.

The exterior geometry of the static charged black hole proposed by Mannheim and Kazanas in Ref. [46] and that in the current study, are derived from Weyl-Maxwell equations. However, besides a few Schwarzschild-like interior solutions [47, 48], there is not yet a study specified to the interior structure of charged relativistic stars in Weyl conformal gravity. Regarding the general interest of this paper, which was the Rutherford scattering of charged particles, such possible interior solutions could provide the chance of classifying the bending angle and the range of initial conditions to obtain a particular shape of scattering. As shown earlier in this paper, despite the similarity between the electric charge of the black hole and that of the test particles, the scattering at some distances can be convex and attractive, which indicates the unbalance between the gravitational and coulomb potentials felt by the test particles. If the source is endowed with a particular charge density function and definite interior solutions, then the scattering can be categorized in accordance with the total charge integral for each of the solutions. So, an outlook for future studies can be looking for obtaining charged interior solutions to Weyl conformal gravity and matching them with the exterior geometry of respected charged black holes. This way, beside categorizing the types of motion of test particles, we will also be able to investigate them around a star under gravitational collapse.

Acknowledgements M. Fathi has been supported by the Agencia Nacional de Investigación y Desarrollo (ANID) through DOC-TORADO Grant No. 2019-21190382. J.R.V. is partially supported by Centro de Astrofísica de Valparaíso (CAV).
A Solving Rutherford scattering of electrically charged particles from a charged Weyl black hole

We combine the methods of Ferrari and Cardano to solve a depressed quartic equation of the form (originally studied by Cardano in Ref. [49])

\[ x^4 + ax^2 + bx + c = 0, \quad (a, b, c) \in \mathbb{R}. \]  

(108)

This equation can be rewritten as the product of two quadratic equations, as follows:

\[ x^4 + ax^2 + bx + c = (x^2 - 2\alpha x + \beta)(x^2 + 2\alpha x + \gamma) = 0. \]  

(109)

Accordingly, we obtain

\[ a = \beta + \gamma - 4\alpha^2, \]  

(110a)

\[ b = 2\alpha(\beta - \gamma), \]  

(110b)

\[ c = \beta\gamma. \]  

(110c)

Solving the first two equations for \( \beta \) and \( \gamma \), yields

\[ \beta = 2\alpha^2 + \frac{a}{2} - \frac{b}{4\alpha}, \]  

(111a)

\[ \gamma = 2\alpha^2 + \frac{a}{2} + \frac{b}{4\alpha}, \]  

(111b)

which together with Eq. (110c), results in an equation of sixth degree in \( \alpha \):

\[ \alpha^6 + \frac{a}{2} \alpha^4 + \left( \frac{a^2}{16} - \frac{c}{4} \right) \alpha^2 - \frac{B}{64} = 0. \]  

(112)

Applying the change of variable

\[ \alpha^2 = U - \frac{a}{6}, \]  

(113)

we obtain the depressed cubic equation

\[ U^3 - \eta_2 U - \eta_3 = 0, \]  

(114)

where

\[ \eta_2 = \frac{a^2}{16} + \frac{c}{4}, \]  

(115a)

\[ \eta_3 = \frac{a^3}{864} + \frac{b^2}{64} - \frac{ac}{24}. \]  

(115b)

The real solution to this cubic equation is obtained as [50, 51]

\[ U = 2\sqrt[3]{\frac{\eta_2}{3}} \cosh \left( \frac{1}{3} \cosh^{-1} \left( \frac{3}{2} \sqrt{\frac{3}{\eta_2}} \right) \right). \]  

(116)

Therefore, the roots of Eq. (108) are

\[ x_1 = \alpha + \sqrt{\alpha^2 - \beta}, \]  

(117a)

\[ x_2 = \alpha - \sqrt{\alpha^2 - \beta}, \]  

(117b)

\[ x_3 = -\alpha + \sqrt{\alpha^2 - \gamma}, \]  

(117c)

\[ x_4 = -\alpha - \sqrt{\alpha^2 - \gamma}. \]  

(117d)

References

1. C.W. Misner, K.S. Thorne, J.A. Wheeler, Gravitation (Princeton University Press, 2017). URL https://press.princeton.edu/books/hardcover/9780691177793/gravitation.

2. S. Chandrasekhar, The mathematical theory of black holes. Oxford classic texts in the physical sciences (Oxford University Press, 2002). URL https://cds.cern.ch/record/579245.

3. E. Poisson, A. Pound, I. Vega, Living Rev. Rel. 14, 7 (2011). DOI 10.12942/lrr-2011-7. URL https://doi.org/10.12942/lrr-2011-7.

4. L. Blanchet, A. Spallucci, B. Whiting, Mass and Motion in General Relativity. Fundamental Theories of Physics book series (Springer, Dordrecht, 2011). URL https://doi.org/10.1007/978-90-481-3015-3.

5. D. Pugliese, H. Quevedo, R. Ruffini, Phys. Rev. D 83, 104052 (2011). DOI 10.1103/PhysRevD.83.104052. URL https://link.aps.org/doi/10.1103/PhysRevD.83.104052.

6. M. Olivares, J. Saavedra, C. Leiva, J.R. Villanueva, Modern Physics Letters A 26(39), 2923 (2011). DOI 10.1142/S0217732311037261. URL https://doi.org/10.1142/S0217732311037261.

7. M. Fathi, Chinese Physics C 37(2), 025101 (2013). DOI 10.1088/1674-1137/37/2/025101. URL https://doi.org/10.1088/1674-1137/37/2/025101.

8. A. Garca, E. Hackmann, J. Kunz, C. Lmmerzahl, A. Macas, Journal of Mathematical Physics 56(3), 032501 (2015). DOI 10.1063/1.4913882. URL https://doi.org/10.1063/1.4913882.

9. D. Pugliese, H. Quevedo, R. Ruffini, European Physical Journal C 77(4), 206 (2017). DOI 10.1140/epjc/s10052-017-4769-x. URL https://doi.org/10.1140/epjc/s10052-017-4769-x.

10. P. Das, R. Sk, S. Ghosh, European Physical Journal C 77(11), 735 (2017). DOI 10.1140/epjc/s10052-017-5295-6. URL https://doi.org/10.1140/epjc/s10052-017-5295-6.

11. J. Vrba, A. Abdujabbarov, M. Kolos, B. Ahmedov, Z. Stuchlík, J. Rayimbaev, Phys. Rev. D 101, 124039 (2020). DOI 10.1103/PhysRevD.101.124039. URL https://link.aps.org/doi/10.1103/PhysRevD.101.124039.

12. J.R. Villanueva, M. Olivares, European Physical Journal C 75, 562 (2015). DOI 10.1140/epjc/s10052-015-3794-x. URL https://doi.org/10.1140/epjc/s10052-015-3794-x.

13. Z. Stuchlík, J. Rayimbaev, I. Radinschi, T. Grammenos, ScienceDirect, 100648 (2020). DOI https://doi.org/10.1016/j.dark.2020.100648. URL https://link.aps.org/doi/10.1103/PhysRevD.98.124001.

14. Q.Q. Zhao, Y.Z. Li, H. Liu, Phys. Rev. D 98, 124001 (2018). DOI 10.1103/PhysRevD.98.124001. URL https://link.aps.org/doi/10.1103/PhysRevD.98.124001.

15. S. Shyamatov, J. Vrba, D. Malafarina, B. Ahmedov, Z. Stuchlik, Physics of the Dark Universe 30, 100648 (2020). DOI https://doi.org/10.1016/j.dark.2020.100648. URL http://www.sciencedirect.com/science/article/pii/S2212686420301399.
