Variations in $\mathbb{A}^1$ on a Theme of Mohan Kumar

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Abstract. For every prime $p$, Mohan Kumar constructed examples of stably free modules of rank $p$ on suitable $p+1$-dimensional smooth affine varieties. This note discusses how to detect the corresponding unimodular rows by an explicit motivic cohomology group. Using the recent developments in the $\mathbb{A}^1$-obstruction classification of vector bundles, this provides an alternative proof of non-triviality of Mohan Kumar’s stably free modules. The reinterpretation of Mohan Kumar’s examples also allows to produce interesting examples of stably trivial torsors for other algebraic groups.

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1. Introduction

The starting point of this note is the following theorem of Mohan Kumar [MK85] which provides important examples of stably free modules of high rank:

Theorem 1.1 (Mohan Kumar). Let $k$ be an algebraically closed field. For every prime $p$, there exists a $(p+1)$-dimensional smooth affine variety $X = \text{Spec} A$ over $k(T)$ such that there exists a nontrivial stably free $A$-module of rank $p$, given by a unimodular row of length $p+1$.

The main point of the present note is to reinterpret the geometric constructions underlying the results in [MK85]. The unimodular row defining the stably free module can be viewed as a morphism $X \to Q_{2p+1}$. In the setting of $\mathbb{A}^1$-homotopy theory, we have a composition of maps

$[X, Q_{2p+1}]_{\mathbb{A}^1} \to H^0_{\text{Nis}}(X, K^{\text{MW}}_{p+1}) \to H^p_{\text{Nis}}(X, K^{\text{M}}_{p+1}/p!)$.

The first map takes an $\mathbb{A}^1$-homotopy class of maps to the lifting class for the first non-trivial $\mathbb{A}^1$-homotopy group $\pi^1_{\mathbb{A}^1}(Q_{2p+1}) \cong K^{\text{MW}}_{p+1}$, and the second map is induced from the natural projection $K^{\text{MW}}_{p+1} \to K^{\text{M}}_{p+1}/p!$. Now the variety $X$ constructed by Mohan Kumar arises from a Zariski covering of another variety $X'$, and the image of the unimodular row in $H^p_{\text{Nis}}(X, K^{\text{M}}_{p+1}/p!)$ can be checked to be non-trivial because it has non-trivial image under the connecting map $H^p_{\text{Nis}}(X, K^{\text{M}}_{p+1}/p!) \to CH^{p+1}(X')$
of the Mayer–Vietoris sequence. By the recent computations due to Asok and Fasel in \cite{AF14a, AF14b}, we know that there is a natural map

$$H^p_{\text{Nis}}(X, K^M_{p+1}/p!) \to H^p_{\text{Nis}}(X, \pi^A_{\pm}(B \text{SL}_n)),$$

and some exact sequence chasing plus the above information can be used to show that the lifting class of the composition $X \to Q_{2p+1} \to B \text{SL}_p$ yields a non-trivial class in $H^p(X, \pi^A_{\pm}(B \text{SL}_n))$. This provides an alternative $A^1$-homotopical proof of the non-triviality of the stably free modules constructed by Mohan Kumar, cf. Theorem 4.2 and the discussion in Section 3.4.

The relevance of this reformulation comes from the recent approach to torsor classification via $A^1$-homotopy theory: the representability results \cite{AHWI, AHWII} together with obstruction-theoretic methods \cite{Mor12, AF14a} relate the classification of $G$-torsors over smooth affine varieties to Nisnevich cohomology with coefficients in $A^1$-homotopy sheaves of the classifying space of $G$. The above reinterpretation explains exactly how Mohan Kumar’s stably free modules fit into the $A^1$-topological approach to vector bundle classification. Moreover, there are several other classifying spaces of algebraic groups where quotients of Milnor $K$-theory appear in $A^1$-homotopy sheaves. The above $A^1$-topological construction of stably free modules can then be adapted to provide stably trivial torsors for other algebraic groups. For example, a second class of examples of stably trivial torsors is obtained by applying a similar argument to the symplectic groups, cf. Theorem 6.2:

**Theorem 1.2.** Let $k$ be an algebraically closed field of characteristic $\neq 2$. For every odd prime $p$, there exists a $p+1$-dimensional smooth affine variety over $k(T)$ and a stably trivial non-trivial $\text{Sp}_{p-1}$-torsor over it. Clearing denominators, there exists a $p+2$-dimensional smooth affine variety over $k$ with a stably trivial non-trivial $\text{Sp}_{p-1}$-torsor over it.

Taking the underlying projective module and adding a trivial line recovers the stably free modules of Theorem 1.1, i.e., Mohan Kumar’s stably free modules have (stably trivial) symplectic lifts.

1.1. **Acknowledgements.** This note was written during a pleasant stay at Institut Mittag-Leffler, in the program “Algebro-geometric and homotopical methods”, and I thank the institute for its hospitality. I would like to thank Chuck Weibel for comments on an earlier version, Jean Fasel for pointing out some mistakes in the arguments for triviality of connecting morphisms and Aravind Asok for very helpful suggestions related to obstruction theory issues.

2. **Recollection on $A^1$-homotopy and representability.**

In this section, we give a short recollection on the relevant input we use from $A^1$-homotopy theory.

We assume the reader is familiar with the basic definitions of $A^1$-homotopy theory, cf. \cite{Mor12}. Short introductions to those aspects relevant for the obstruction-theoretic torsor classification can be found in papers of Asok and Fasel, cf. e.g. \cite{AF14a, AF14b}. The notation in the paper generally follows the one from \cite{AF14a}. We generally assume that we are working over base fields of characteristic $\neq 2$.

When considering $A^1$-homotopy sheaves, only the simplicial grading will appear, i.e., for a pointed $A^1$-connected space $(\mathcal{X}, x)$, we will denote by $\pi^A_n(\mathcal{X})$ the Nisnevich sheafification of $[S^1_n \wedge U_+, (\mathcal{X}, x)]$. 
2.1. **Representability theorem and obstruction theory.** Now we recall the representability theorem for torsors and introduce some notation for Postnikov towers which are used for the obstruction-theoretic approach to torsor classification in $\mathbb{A}^1$-homotopy.

The following representability theorem, significantly generalizing an earlier result of Morel in [Mor12], has been proved in [AHWII].

**Theorem 2.1.** Let $k$ be an infinite field, and let $X = \text{Spec } A$ be a smooth affine $k$-scheme. Let $G$ be a reductive group such that each absolutely almost simple component of $G$ is isotropic. Then there is a bijection

$$H_{Nis}^1(X; G) \cong [X, B_{Nis}G]_{\mathbb{A}^1}$$

between the pointed set of isomorphism classes of rationally trivial $G$-torsors over $X$ and the pointed set of $\mathbb{A}^1$-homotopy classes of maps $X \to B_{Nis}G$.

In this paper, we will apply this to the cases $G = \text{SL}_n$ (oriented projective modules) and $G = \text{Sp}_{2n}$ (symplectic modules). Note that these groups are special, i.e., all torsors are automatically rationally trivial. Applications to stably trivial torsors for $G_2$ and Spin groups will be discussed elsewhere.

The representability theorem translates questions about $G$-torsor classification into questions about $\mathbb{A}^1$-homotopy classes of maps into an appropriate classifying space. In particular, we can prove that a torsor is non-trivial by exhibiting some $\tau$-

In the other hand, we can deduce the existence of torsors over smooth affine schemes with suitable properties by producing maps $X \to B_{Nis}G$, e.g., by obstruction-theoretic methods.

While the study of $\mathbb{A}^1$-homotopy classes maps into classifying spaces may not seem an easier subject than the torsor classification, the other relevant tool actually allowing to prove some meaningful statements is obstruction theory. The basic statements concerning obstruction theory as applied to torsor classification can be found in various sources, such as [Mor12] or [AF14a, AF15]. We only give a short list of the relevant statements which are enough for our purposes.

Let $(\mathcal{Y}, y)$ be a pointed, $\mathbb{A}^1$-simply connected space. Then there is a sequence of pointed $\mathbb{A}^1$-simply connected spaces, the Postnikov sections $(\tau_{\leq i}(\mathcal{Y}), y)$, with morphisms $p_i : \mathcal{Y} \to \tau_{\leq i}(\mathcal{Y})$ and morphisms $f_i : \tau_{\leq i+1}(\mathcal{Y}) \to \tau_{\leq i}(\mathcal{Y})$ such that

1. $\pi^A_j(\tau_{\leq i}(\mathcal{Y})) = 0$ for $j > i$,
2. the morphism $p_i$ induces an isomorphism on $\mathbb{A}^1$-homotopy group sheaves in degrees $\leq i$,
3. the morphism $f_i$ is an $\mathbb{A}^1$-fibration, and the $\mathbb{A}^1$-homotopy fiber of $f_i$ is an Eilenberg–Mac Lane space of the form $K(\pi^A_{i+1}(\mathcal{Y}), i + 1)$,
4. the induced morphism $\mathcal{Y} \to \tau_{\leq i}$ holim, $\mathcal{Y}$ is an $\mathbb{A}^1$-weak equivalence.

Moreover, $f_i$ is an principal $\mathbb{A}^1$-fibration, i.e., there is a morphism, unique up to $\mathbb{A}^1$-homotopy,

$$k_{i+1} : \tau_{\leq i}(\mathcal{Y}) \to K(\pi^A_{i+1}(\mathcal{Y}), i + 2)$$

called the $i + 1$-th $k$-invariant and an $\mathbb{A}^1$-fiber sequence

$$\tau_{\leq i+1}(\mathcal{Y}) \to \tau_{\leq i}(\mathcal{Y}) \xrightarrow{k_{i+1}} K(\pi^A_{i+1}(\mathcal{Y}), i + 2).$$

From these statements, one gets the following consequence: for a smooth $k$-scheme $X$ and a pointed, $\mathbb{A}^1$-simply connected space $\mathcal{Y}$, a given pointed map $g^{(i)} : X_+ \to \tau_{\leq i}(\mathcal{Y})$ lifts to a map $g^{(i+1)} : X_+ \to \tau_{\leq i+1}(\mathcal{Y})$ if and only if the following composite is null-homotopic:

$$X_+ \xrightarrow{g^{(i)}} \tau_{\leq i}(\mathcal{Y}) \to K(\pi^A_{i+1}(\mathcal{Y}), i + 2),$$
or equivalently, if the corresponding obstruction class vanishes in the cohomology group $H^{i+2}_{Nis}(X; \pi_{i+1}^A(\mathcal{F}))$.

If this happens, then the possible lifts are parametrized by the quotient of the following set of homotopy classes of maps

$$[X_+, K(\pi_{i+1}^A(\mathcal{F}), i+1)]_A \cong H^{i+1}_{Nis}(X; \pi_{i+1}^A(\mathcal{F}))$$

modulo the standard action of $[X_+, \Omega_{\leq i}^A(\mathcal{F})]_A$. This quotient (or sometimes the cohomology group $H^{i+1}_{Nis}(X; \pi_{i+1}^A(\mathcal{F}))$) will usually be called the **lifting set**.

We want to state clearly what this means for the torsor classification over smooth schemes. If we have a torsor, then the map into the classifying space $B$ for $\text{group } H$ or equivalently, if the corresponding obstruction class vanishes in the cohomology group $H^i(X; \pi^A_\ast(\mathcal{F}))$.

We now provide a short recollection on strictly $\mathbb{A}^1$-invariant sheaves and their Nisnevich cohomology. All of the material here can be found in [Mor12]. Recall that a sheaf $\mathcal{A}$ of abelian groups on $\text{Sm} / k$ is called strictly $\mathbb{A}^1$-invariant if for each smooth $k$-scheme $X$ the map $\text{pr}_1^* : H^i_{Nis}(X; \mathcal{A}) \to H^i_{Nis}(X \times \mathbb{A}^1, \mathcal{A})$ is an isomorphism.

If $k$ is an infinite perfect field and $\mathcal{F}$ is an $\mathbb{A}^1$-simply-connected simplicial presheaf over $\text{Sm} / k$, then its $\mathbb{A}^1$-homotopy sheaves $\pi^A_{\ast} \mathcal{F}$ are strictly $\mathbb{A}^1$-invariant sheaves of groups. This is one of the main results of [Mor12]. It allows to actually manipulate these things, compute their cohomology and get obstruction-theoretic consequences. Note, however, that the conditions on $k$ being perfect is essential throughout [Mor12].

**Remark 2.2.** One of the central assumptions for cohomology computations with $\mathbb{A}^1$-homotopy sheaves is that we are working over a perfect base field. This assumption arises in the theory of strictly $\mathbb{A}^1$-invariant Nisnevich sheaves of abelian groups, cf. [Mor12]. In the present paper, we are mostly working over the field $k(T)$ which is not perfect if $k$ has positive characteristic. However, the Nisnevich sheaves we consider ($\mathbb{A}^1$-homotopy sheaves of $\text{BSL}_n$ and $\text{BSp}_{2n}$) are all pulled back from the prime field, which is perfect. By [Hoy15, Lemma A.2, A.4], this implies that Gersten resolutions exist for these objects, and we can freely work with these over $k(T)$.

If $\mathcal{A}$ is a strictly $\mathbb{A}^1$-invariant sheaf of abelian groups on $\text{Sm} / k$, then for any smooth $k$-scheme $U$ the unit $1 \in G_m$ defines a morphism $U \to G_m \times U$, and the contraction of $\mathcal{A}$ is defined to be the sheaf

$$A_{-1}(U) = \ker (\mathcal{A}(G_m \times U) \to \mathcal{A}(U)).$$

This construction can be iterated to yield $A_{-n}$ for $n \in \mathbb{N}$; it is an exact functor on the category of strictly $\mathbb{A}^1$-invariant sheaves on $\text{Sm} / k$. 

If $A$ is a strictly $\mathbb{A}^1$-invariant sheaf on $\text{Sm}/k$, then the Nisnevich cohomology of a smooth $k$-scheme $X$ can be computed in terms of a Gersten-type complex (called Rost–Snijder complex in [Mor12] and [AF14a])

$$0 \to \bigoplus_{x \in X^{(0)}} A(k(x)) \to \bigoplus_{y \in X^{(1)}} A_{-1}(k(y)) \to \cdots \to \bigoplus_{z \in X^{(\dim X)}} A_{-\dim X}(k(z)) \to 0.$$  

We omit mentioning of orientations and $K^M$-actions but note that these are relevant for the definitions of differentials. For the complexes computing Milnor $K$-cohomology, the GW-action reduces to the abelian group structures on Milnor $K$-groups, and the orientations are not relevant for the definition of the differentials. The cases appearing in this paper where orientations do matter are complexes for higher Grothendieck–Witt groups and the reader can consult the literature for Gersten complexes in Witt groups for those. For most of the computations, the boundary maps in the above complex will not matter; several vanishing results will only be proved by making statements about the structure of the reduction of certain strictly $\mathbb{A}^1$-invariant sheaves.

2.3. Mayer–Vietoris sequences. Finally, we need to discuss Mayer–Vietoris sequences for the Nisnevich cohomology of strictly $\mathbb{A}^1$-invariant sheaves which will be needed for some computations in the paper. Let $X = U \cup V$ be a Zariski covering of smooth schemes over an infinite perfect field $k$, and let $A$ be a strictly $\mathbb{A}^1$-invariant Nisnevich sheaf of abelian groups on $\text{Sm}/k$. Then there is an exact sequence of Gersten complexes associated to a Zariski covering $X = U \cup V$:

$$0 \to C^*(X, A) \to C^*(U, A) \oplus C^*(V, A) \to C^*(U \cap V, A) \to 0$$

which induces a long exact Mayer–Vietoris sequence in Nisnevich cohomology

$$\to H^i_{\text{Nis}}(X, A) \to H^i_{\text{Nis}}(U, A) \oplus H^i_{\text{Nis}}(V, A) \to H^i_{\text{Nis}}(U \cap V, A) \to H^{i+1}_{\text{Nis}}(X, A) \to$$

This sequence is functorial in $A$. We shortly recall the standard description of the boundary morphism. A cycle representative of a class in $H^i_{\text{Nis}}(U \cap V, A)$ is given by a finite sum, indexed over codimension $i$ points $x_x$ of $U \cap V$, of elements of $A_{-i}(k(x_x))$. This formal sum can be viewed as a formal sum in $C^i(U; A)$ (which is the summand of the middle complex where the restriction map is id, not $-\text{id}$). This may no longer be a cycle, but we can apply the boundary map of the Gersten complex (which is given in terms of residue maps) to this lifted chain. The result will have trivial image in $C^i(U \cap V; A)$ and therefore the resulting formal sum can be viewed as an element of $C^{i+1}(X; A)$. This will be a cycle representing the image of the boundary map $H^i_{\text{Nis}}(U \cap V, A) \to H^{i+1}_{\text{Nis}}(X, A)$.

3. Cohomological analysis of Mohan Kumar’s construction

In this section, we recall the geometric constructions of [MK85] and explain how they give rise to varieties with interesting cohomology classes. In fact, we will explain how Mohan Kumar’s construction provides varieties $Y \cap Z$ where the following composition is a surjection

$$[Y \cap Z, Q_{2p+1}]_{\mathbb{A}^1} \to H^p(Y \cap Z, K^M_{p+1}/p) \to H^p(Y \cap Z, K^M_{p+1}/p) \to \mathbb{Z}/p\mathbb{Z}.$$  

3.1. Geometric setup. Fix a prime $p$ and a field $k$. The first geometric construction produces a smooth affine variety with non-trivial torsion in the top Chow group. For this, let $f(T)$ be a polynomial of degree $p$ over $k$ such that $f(0) = a \in k^\times$.  

Then there are recursively defined polynomials
\[ F_i(X_0, X_1) = X_1^p f \left( \frac{X_0}{X_1} \right), \]
and
\[ F_{i+1}(X_0, \ldots, X_{i+1}) = F_i \left( F(X_0, \ldots, X_i), a_{i+1} X_{i+1}^{p^n} \right). \]
If \( f(T^{p^{m-1}}) \) is irreducible then, according to [MK85 Claim 1], \( F_n \) is irreducible for \( n \leq m \). In this case, [MK85 Claim 2] states that \( X = \mathbb{P}^n \setminus V(F_n) \) is a smooth affine variety over \( k \), with \( \text{CH}^0(X) \cong \mathbb{Z}/p\mathbb{Z} \), generated by the class of a \( k \)-rational point of \( X \).

The second part of the geometric construction produces a Zariski covering of \( X \) by two affine subvarieties with trivial top Chow groups. The first subvariety is
\[ Y = (\mathbb{P}^n \setminus V(F_{n-1})) \cap X, \]
where we view \( F_{n-1} \) in the obvious way as a polynomial in the variables \( X_0, \ldots, X_n \). Since \( X \setminus Y \) contains the \( k \)-rational point \( x = [0 : 0 : \cdots : 0 : 1] \), we have \( \text{CH}^n(Y) = 0 \) by the localization sequence for Chow groups. The second subvariety is
\[ Z = (\mathbb{P}^n \setminus V(G)) \cap X \]
with the polynomial
\[ G(X_0, \ldots, X_n) = F_{n-1}(X_0, \ldots, X_{n-1}) - a^{n-1} X_1^{p^n-1}. \]
Here the variety \( V(G) \) contains the \( k \)-rational point \( y = [0 : 0 : \cdots : 0 : 1 : 1] \), and again the localization sequence for Chow groups implies \( \text{CH}^n(Z) = 0 \). We have a Zariski covering \( X = Y \cup Z \) because \( F_n \in \langle F_{n-1}, G \rangle \).

Finally, the relevant variety is now the intersection \( Y \cap Z \).

### 3.2. Non-trivial cohomology classes

We first note that the variety \( Y \cap Z \) constructed by Mohan Kumar supports a non-trivial cohomology class with coefficients in Milnor–Witt K-theory. This non-trivial class exists because the class in \( \text{CH}^n(X) \) locally trivializes in the covering \( X = Y \cup Z \).

**Proposition 3.1.** Let \( k \) be a field of 2-cohomological dimension \( \leq 1 \). Let \( p \) be a prime and assume that there exists a degree \( p \) polynomial \( f(T) \) over \( k \) such that \( f(T^{p^n}) \) is irreducible. Consider the situation \( X = Y \cup Z \) outlined above. Then there is a surjection
\[ H^p_{\text{Nis}}(Y \cap Z, K_{p+1}^\text{MW}) \to \text{CH}^{p+1}(X) \cong \mathbb{Z}/p\mathbb{Z}. \]

**Proof.** Use the Mayer–Vietoris sequence for cohomology of \( K_{p+1}^\text{MW} \) associated to the Zariski covering \( X = Y \cup Z \), whose relevant portion is the following
\[ H^p_{\text{Nis}}(Y \cap Z, K_{p+1}^\text{MW}) \to \text{CH}^{p+1}(X) \to \text{CH}^{p+1}(Y) \oplus \text{CH}^{p+1}(Z). \]
By construction the last group of the sequence is trivial, showing that the boundary map in the Mayer–Vietoris sequence is a surjection. Now we consider the exact sequence of strictly \( \mathcal{A}^1 \)-invariant sheaves \( 0 \to \mathcal{I}^{p+2} \to K_{p+1}^\text{MW} \to K_{p+1}^\text{MW} \to 0 \). The induced morphism \( H^p_{\text{Nis}}(Y \cap Z, K_{p+1}^\text{MW}) \to H^p_{\text{Nis}}(Y \cap Z, K_{p+1}^\text{MW}) \) is surjective if we can show \( H^{p+1}_{\text{Nis}}(Y \cap Z, \mathcal{I}^{p+2}) = 0 \). But that follows from [AF14a Proposition 5.2] because by assumption the 2-cohomological dimension of the base field \( k \) is \( \leq 1 \). \( \square \)

**Corollary 3.2.** Let \( k \) be a field of 2-cohomological dimension \( \leq 1 \). Let \( p \) be a prime and assume that there exists a degree \( p \) polynomial \( f(T) \) over \( k \) such that \( f(T^{p^n}) \) is irreducible. Consider the situation \( X = Y \cup Z \) outlined above. Then there is a surjection
\[ [Y \cap Z, Q_{2p+1}^1] \to H^p_{\text{Nis}}(Y \cap Z, K_{p+1}^\text{MW}) \to \text{CH}^{p+1}(X) \cong \mathbb{Z}/p\mathbb{Z}. \]
Proof. By Proposition 3.1, it suffices to show that the first map is a surjection. This follows from [AFT16, Proposition 1.1.10 (1)] since $\mathbb{Q}_{2p+1}$ is $(p-1)$-$\mathbb{A}^1$-connected and $Y \cap Z$ has Krull dimension $p + 1$. □

Remark 3.3. It would be very interesting to have more generally a construction of smooth affine varieties with non-trivial classes in $H_{Nis}^r(X, \mathbb{K}_n^M/m)$ for $r \geq 2$. Possibly this could be done by setting up a Mayer–Vietoris spectral sequence for coverings $X = \bigcup \cdots \cup U_{r+1}$ in Milnor $K$-cohomology and then use that to produce such varieties as intersections $U_i \cap \cdots \cap U_{r+1}$. The next interesting case would be $r = 2$. For this, one would want a variety $X$ with a covering $X = U_1 \cup U_2 \cup U_3$, a non-trivial class in $\text{CH}^{n+2}(X)/m$ whose restriction to $U_i$ vanishes and such that the induced classes in $H_{Nis}^{n+1}(U_i \cap U_j, \mathbb{K}_n^M/m)$ are also trivial at $\infty$ (either by not being a cycle in $E_1$ or by lifting to $V_1$ and thus be killed by the $d^1$-differential). The Mayer–Vietoris spectral sequence would then produce a non-trivial class in $H_{Nis}^m(U_1 \cap U_2 \cap U_3, \mathbb{K}_n^M/m)$. However, at this point I don’t know how to guarantee the latter condition on the vanishing of $H^{n+1}(\mathbb{K}_n^M/m)$.

3.3. Explicit description of a class. We now want to write out an explicit description of a Milnor $K$-cohomology class whose boundary can be detected on the Chow group. Note that the smooth affine variety $X$ is defined as the complement of a hypersurface such that each point on the hypersurface has degree divisible by $p$ (which is exactly the reason for the $p$-torsion in $\text{CH}^{p+1}(X)$). Now the complements of $Y$ and $Z$ in $X$ are given by hypersurfaces which contain rational points (which is the reason why the top Chow groups of $Y$ and $Z$ are trivial). Note however that the hypersurface complements of $Y$ and $Z$ only intersect in the complement of $X$, so they don’t have rational points in common. On $Y \cap Z$ there are hence two reasons for triviality of $\text{CH}^{p+1}(Y \cap Z)$, namely a rational point in the complement of $Y$ or a rational point in the complement of $Z$. The lift in the Mayer–Vietoris sequence is then given by a “homotopy between these two trivializations”: take a line in $\mathbb{P}^{p+1}$ connecting a rational point in $X \setminus Y$ and a rational point in $X \setminus Z$. On this line there is a rational function having divisor exactly the difference of these rational points. The class of this rational function in the $p$-residues of the function field of the line is a cycle on $Y \cap Z$, hence represents a class in $H_{Nis}^1(Y \cap Z, \mathbb{K}_n^M/p)$. Note that this is exactly the geometric situation in [MKS15, Claim 3], and we will see in 3.3 that this is the lifting class of Mohan Kumar’s stably free module.

Now we want to show that the class we described in $H_{Nis}^1(Y \cap Z, \mathbb{K}_n^M/p)$ is actually non-trivial. This non-triviality is detected using the Mayer–Vietoris sequence associated to the covering $X = Y \cup Z$, cf. 2.3. More specifically, we want to show that the above cycle has non-trivial image under the boundary map

$$H_{Nis}^1(Y \cap Z, \mathbb{K}_n^M/p) \to \text{CH}^{p+1}(X)/p.$$ 

To compute the image of the class, recall the description of the boundary map from 2.3. The cycle description of the class above was that we take the rational function with divisor $[y] - [x]$ as an element in the mod $p$ residues of the function field of the line connecting the points $x$ and $y$. To compute the boundary in the Mayer–Vietoris sequence, we first take the Gersten chain on $Y$ given by the very same rational function on the very same line. Now we apply the boundary map in the Gersten complex, which in our situation is given by mapping the rational function on the line $l \cap Y$ to its divisor. The rational function on the line $l$ has a zero at $y \in Y$ and a pole at $x \in X$, hence its divisor on the line $l$ is $[y] - [x]$. However, since $x \not\in Y$, the divisor of the function in the Gersten complex for $Y$ is $[y]$. The final step in the computation of the boundary for the Mayer–Vietoris sequence is to view $[y]$ as a 0-cycle on the whole variety $X$. Since $y$ is a $k$-rational point, the class $[y]$ is a generator of $\text{CH}^{p+1}(X)$. 

\[ \text{Proof.} \]
We have shown the following:

**Proposition 3.4.** Denote by $\sigma \in H_{\text{Nis}}^p(Y \cap Z, K_{p+1}^M)$ the class of a rational function with divisor $[y] - [x]$ supported on the line connecting $x$ and $y$. Then the image of $\sigma$ under the boundary map

$$H_{\text{Nis}}^p(Y \cap Z, K_{p+1}^M) \to CH^{p+1}(X)/p$$

is a generator of $CH^{p+1}(X)/p \cong \mathbb{Z}/p\mathbb{Z}$.

**Remark 3.5.** Another approach to the construction of a non-trivial class in the group $H_{\text{Nis}}^{p-1}(W, K_{p+1}^M)$ could now be to use the construction above. If we can provide a covering $Y \cap Z = W_1 \cup W_2$ on which the Milnor $K$-cohomology class in $H_{\text{Nis}}^p(Y \cap Z, K_{p+1}^M)$ is trivialized, then the Mayer–Vietoris sequence would produce such a class. The Milnor $K$-cohomology class can be represented by a line in $\mathbb{P}^a$ connecting the points $x$ and $y$. Actually, using Milnor $K_2$-classes associated to the coordinate axes in $\mathbb{P}^2$, any line in $\mathbb{P}^{p+1}$ connecting rational points on $X \setminus Y$ and $X \setminus Z$ will represent the same cohomology class. If we now can find two such lines $L_1, L_2$ contained in hypersurfaces $S_1, S_2$ which only meet outside $Y \cap Z$, then the Mayer–Vietoris sequence associated to the covering $Y \cap Z = (Y \cap Z \setminus S_1) \cup (Y \cap Z \setminus S_2)$ would produce the required class. Unfortunately, I don’t know how to construct the appropriate hypersurfaces.

**3.4. Comparison with Mohan Kumar’s construction.** Now we want to compare this to the original construction of stably free modules in [MK85]. In fact, we will show that the unimodular row defining Mohan Kumar’s stably free module maps exactly to cohomology classes as described above.

Recall from [MK85] that the stably free module $\mathcal{P}$ is given by a unimodular row as follows. The point $y$ is a complete intersection in $Y$, i.e., its maximal ideal is of the form $m_y = \{b_1, \ldots, b_n\}$ for a regular sequence of functions $b_1, \ldots, b_n \in O_Y(Y)$. Because $y \notin Z$, we can now consider $(b_1, \ldots, b_n)$ as a unimodular row on $Y \cap Z$.

The stably free module $\mathcal{P}$ over $Y \cap Z$ is now the one defined as the kernel of this unimodular row, cf. p.1441 of [MK85]. Note that, compared to the situation in 3.3, the regular sequence $(b_2, \ldots, b_n)$ defines the line connecting $x$ and $y$ and $b_1$ can be taken to be a function having a simple zero at $y$ and a simple pole at $x$.

It remains to recall the description of the map $[Y \cap Z, Q_{2p+1}]_{\text{MW}} \to H^0(Y \cap Z, K_{p+1}^M)$ sending a map $Y \cap Z \to Q_{2p+1}$ corresponding to a unimodular row of length $p + 1$ to its associated lifting class. Recall that for each unimodular row of length $n+1$ over the ring $R$ there is a morphism $u : Spec R \to Q_{2n+1} \cong A^{n+1} \setminus \{0\}$, well-defined up to $A^1$-homotopy. The first lifting class associated to the morphism $Y \cap Z \to Q_{2p+1}$ is given by the composition

$$Y \cap Z \to Q_{2p+1} \to \tau_{\leq p} Q_{p+1} \cong K(K_{p+1}^M, p).$$

This composition corresponds to a cohomology class in $H_{\text{Nis}}^p(Y \cap Z, K_{p+1}^M)$ and has been evaluated in [Fas11] Theorem 4.1: without loss of generality, we can assume that the unimodular row $(b_1, \ldots, b_{n+1})$ is such that the sequence $(b_2, \ldots, b_{n+1})$ is regular; the first lifting class of the unimodular row in $H_{\text{Nis}}^n(Y \cap Z, K_{n+1}^M)$ is then given by the cycle $(b_1, \{\{b_1\}, b_2, \ldots, b_{n+1}\})$ on the subscheme defined by $(b_2, \ldots, b_{n+1})$.

Reduction of coefficients to $K_{n+1}^M/n!$ means that the class is given by the unit $b_1$ on the closed integral subscheme defined by $(b_2, \ldots, b_{n+1})$.

We formulate the combination of the above statements which is implicitly contained in [Fas11]:

**Proposition 3.6.** Let $k$ be an infinite field, and let $X = Spec R$ be a smooth affine scheme over $k$. Let $(b_1, \ldots, b_{n+1})$ be a unimodular row over $R$ such that the sequence $(b_2, \ldots, b_{n+1})$ is regular and denote by $P$ the associated stably free
module of rank $n$. The first lifting class associated to $P$ in $H^n_{Nis}(X, K^M_{p+1}/n!)$ is

given by the cycle whose underlying codimension $n$ scheme is $R/(b_2, \ldots, b_{n+1})$ and the
associated rational function on it is $b_1$.

Applying this statement to the specific case of Mohan Kumar’s stably free modules, the lifting class is given by a rational function with divisor $[y] - [x]$ viewed as $p$-residue in the function field of the line connecting $x$ and $y$, cf. [MK85] Claim 3.

Note that this is exactly the class discussed in [3.3]. In particular, Proposition 3.4 now implies that the lifting class of Mohan Kumar’s stably free module is a non-trivial class in $H^n_{Nis}(Y \cap Z, K^M_{p+1}/p!)$, mapping to a generator of $\text{CH}^{p+1}(X)/p! \cong \mathbb{Z}/p\mathbb{Z}$ under the boundary map of the Mayer–Vietoris sequence.

4. Stably free modules: Mohan Kumar’s examples at odd primes

In the previous section, we discussed a cohomological reinterpretation of Mohan Kumar’s constructions from [MK85]. This provided, in particular, a smooth affine variety $Y \cap Z$ with a morphism $Y \cap Z \rightarrow Q_{2p+1}$ which is detected in Milnor $K$-cohomology. This morphism can be composed with the natural map $Q_{2p+1} \rightarrow B\text{SL}_p$ which is the inclusion of the $\mathbb{A}^1$-homotopy fiber of the stabilization map $B\text{SL}_p \rightarrow B\text{SL}_{p+1}$. This produces a rank $p$ vector bundle which becomes trivial after adding a free rank one summand; it is an $\mathbb{A}^1$-topological reformulation of the fact that the kernel of a unimodular row of length $p + 1$ is a projective module of rank $p$ which becomes trivial after adding a free rank one module. The morphism $Q_{2p+1} \rightarrow B\text{SL}_p$ induces a morphism $H^p(Y \cap Z, K^M_{p+1}) \rightarrow H^p(Y \cap Z, \pi^1_1 B\text{SL}_p)$.

The main point of the present section will now be to show that the composition

$$[Y \cap Z, Q_{2p+1}]_{\mathbb{A}^1} \rightarrow H^p(Y \cap Z, K^M_{p+1}) \rightarrow H^p(Y \cap Z, \pi^1_1 B\text{SL}_p)$$

sends the unimodular row given by Mohan Kumar to a non-zero element, thus providing an $\mathbb{A}^1$-topological proof that the stably free module is non-trivial. For this, we will need to recall some information on the $\mathbb{A}^1$-homotopy sheaf $\pi^1_1(B\text{SL}_p)$.

For the present section, $p$ will be an odd prime, the case $p = 2$ will be discussed in the next section. This case distinction is due to a structural difference of the relevant $\mathbb{A}^1$-homotopy sheaves; Mohan Kumar’s constructions in [MK85] work for even and odd primes.

4.1. $\mathbb{A}^1$-homotopy groups and lifting classes. First note that $B\text{SL}_n$ is simply connected for all $n$. The stabilization results imply that

$$\pi^1_i(B\text{SL}_n) \cong K^Q_i$$

for $i < n$. The corresponding lifting classes are related to the Chern classes, but the non-uniqueness of lifting classes implies that making this relationship precise is a rather subtle business.

The first unstable $\mathbb{A}^1$-homotopy group of $B\text{SL}_n$ has been computed in [AF14b, Theorem 3.14]; for odd $n$, it is given by an exact sequence

$$0 \rightarrow S_{n+1} \rightarrow \pi^1_n(B\text{SL}_n) \rightarrow K^Q_n \rightarrow 0.$$  

By [AF14b, Corollary 3.11], there is a canonical epimorphism $K^M_{n+1}/n! \rightarrow S_{n+1}$ which becomes an isomorphism after $n - 1$-fold contraction.

4.2. Construction of stably free modules. We now describe how to construct the stably free modules. The standard stabilization morphism

$$\text{SL}_n \rightarrow \text{SL}_{n+1} : M \mapsto \begin{pmatrix} M & 0 \\ 0 & 1 \end{pmatrix}$$
induces a morphism $\text{BSL}_n \to \text{BSL}_{n+1}$ which maps an oriented projective $R$-module $\mathcal{P}$ of rank $n$ to $\mathcal{P} \oplus R$. There is an $\mathbb{A}^1$-fiber sequence $Q_{2n+1} \to \text{BSL}_n \to \text{BSL}_{n+1}$, and the above exact sequence arises from the associated long exact sequence of $\mathbb{A}^1$-homotopy sheaves. In particular, the induced morphism $\pi_n^\mathcal{A} (Q_{2n+1}) \to \pi_n^\mathcal{A} (\text{BSL}_n)$ factors as

$$K_{n+1}^\text{MW} \cong \pi_n^\mathcal{A} (Q_{2n+1}) \twoheadrightarrow K_{n+1}^\text{M} \twoheadrightarrow S_{n+1} \hookrightarrow \pi_n^\mathcal{A} (\text{BSL}_n).$$

To construct a projective module of rank $n$ which becomes trivial upon addition of a free rank one summand, we can proceed as follows. Suppose we have a morphism $\alpha : X \to Q_{2n+1}$ corresponding to a unimodular row of length $n$, we can consider the composition $X \overset{\alpha}{\to} Q_{2n+1} \to \text{BSL}_n$. If we can show that the image of $\alpha$ under the composition

$$[X, Q_{2n+1}] \otimes \mathbb{A}^1 \to H^n(X, K_{n+1}^\text{MW}) \to H^n(X, \pi_n^\mathcal{A} (\text{BSL}_n))$$

is non-trivial, we will obtain a non-trivial stably free module, as required. This procedure was suggested by Aravind Asok to replace an earlier argument which didn’t properly address the non-uniqueness of lifting classes.

The non-triviality of the image of $\alpha$ in $H^n_{\text{Nis}}(X, \pi_n^\mathcal{A} (\text{BSL}_n))$ can now be discussed by means of the above exact sequence describing $\pi_n^\mathcal{A} (\text{BSL}_n)$. It induces a long exact sequence of Nisnevich cohomology groups whose relevant portion is

$$H^n_{\text{Nis}}(X, K_n^\mathcal{Q}) \to H^n_{\text{Nis}}(X, K_n^\mathcal{M}) \to H^n_{\text{Nis}}(X, \pi_n^{\mathcal{A}} (\text{BSL}_n)) \to H^n_{\text{Nis}}(X, K_n^\mathcal{Q}).$$

By construction, the image of $\alpha$ lands in the image of $H^n_{\text{Nis}}(X, K_n^\mathcal{M})$. It should be noted, however, that the map $H^n_{\text{Nis}}(X, K_n^\mathcal{M}) \to H^n_{\text{Nis}}(X, \pi_n^{\mathcal{A}} (\text{BSL}_n))$ is not necessarily injective. In particular, producing a non-trivial class in $K_n^\mathcal{M}$-cohomology doesn’t necessarily lead to a non-trivial stably free module because the class could be in the image of $H^n_{\text{Nis}}^{-1}(X, K_n^\mathcal{Q})$. To deal with this issue, we need the following statement concerning triviality of a boundary map:

**Lemma 4.1.** Let $k$ be a field, let $X$ be a smooth variety and let $n$ be an odd integer. Then the boundary map $\text{CH}^n(X) \to \text{CH}^{n+1}(X)/n!$ induced from the exact sequence

$$0 \to S_{n+1} \to \pi_n^\mathcal{A} (\text{BSL}_n) \to K_n^\mathcal{Q} \to 0$$

is trivial.

**Proof.** We make use of the fact that the natural map $K_{n+1}^\mathcal{M}/n! \to S_n$ is an isomorphism after $(n - 2)$-fold delooping, tacitly behaving like the first term in the exact sequence of sheaves is $K_{n+1}^\mathcal{M}/n!$.

Recall that the boundary map is described as follows: an element in the group $H^n_{\text{Nis}}(X, K_n^\mathcal{Q}) \cong \text{CH}^n(X)$ is given by a $\mathbb{Z}$-linear combination of codimension $n$ points. The exact sequence of sheaves induces an exact sequence of Gersten complexes, which in degree $n$ is

$$0 \to \bigoplus_{x \in X^{(n)}} k(x)^\times/(k(x)^\times)^{n!} \to \bigoplus_{x \in X^{(n)}} \left(\pi_n^\mathcal{A} (\text{BSL}_n)\right)^{n!}_{-n} (k(x)) \to \bigoplus_{x \in X^{(n)}} \mathbb{Z} \to 0.$$

We lift the cycle to an element in the middle term and compute the residue of this element. The result lies in the image of

$$\bigoplus_{x \in X^{(n)}} \mathbb{Z}/n! \mathbb{Z} \twoheadrightarrow \bigoplus_{x \in X^{(n)}} \left(\pi_n^\mathcal{A} (\text{BSL}_n)\right)^{n!}_{-n-1} (k(x))$$

and can then be viewed as a cycle in $\text{CH}^{n+1}(X)/n!$. 
So we need to find a lift and compute its residue. To compute the lift, we note that the exact sequences

\[ 0 \to k(x)^{x}/(k(x)^{x})^n \to \left( \pi_{n}^{n}(B \text{SL}_n) \right)_{n-1}(k(x)) \to \mathbb{Z} \to 0 \]

are functorial in the field \( k(x) \) (because they are obtained by taking sections of an exact sequence of sheaves). Moreover, since \( \mathbb{Z} \) is a constant sheaf, we find that for every field \( k(x) \) and every \( n \in \mathbb{Z} \), we can choose a lift of \( n \) to \( \left( \pi_{n}^{n}(B \text{SL}_n) \right)_{n-1}(k(x)) \) which is actually induced from a lift defined over the base field \( k \).

In the second step, we need to compute the residue. If \( v \) is a discrete valuation on the field \( F \) with residue field \( E \), the residue is a map

\[ \left( \pi_{n}^{n}(B \text{SL}_n) \right)_{n-1}(F) \to \left( \pi_{n}^{n}(B \text{SL}_n) \right)_{n-1}(E). \]

We need to explain how to “compute” this residue. Recall that by the definition of contraction we have

\[ \left( \pi_{n}^{n}(B \text{SL}_n) \right)_{n-1}(E) = \ker \left( \left( \pi_{n}^{n}(B \text{SL}_n) \right)_{n-1}(\mathbb{G}_{m,E}) \to \left( \pi_{n}^{n}(B \text{SL}_n) \right)_{n-1}(E) \right) \]

where the map is induced by the unit section. The residue of a section defined over \( \mathbb{G}_{m,E} \) is defined by first adding a constant section to make the evaluation at 1 trivial, and then the resulting element in the kernel will be the residue. The same works if we have a field \( F \) with discrete valuation \( v \) and residue field \( E \); Nisnevich-locally, we can replace \( \mathcal{O}_{F,v} \setminus \text{Spec } E \) by \( \mathbb{G}_{m,E} \) and do the same construction. From this description of the residue, we deduce the following. Let \( F \) be a field with discrete valuation \( v \), valuation ring \( \mathcal{O}_{v} \) and residue field \( E \), and \( \sigma \in \left( \pi_{n}^{n}(B \text{SL}_n) \right)_{n-1}(F) \) be a section. If the section extends to \( \mathcal{O}_{v} \) then its residue is trivial. If and only if this is (locally in the Nisnevich topology on \( \text{Spec } \mathcal{O}_{v} \)) extended from a constant section on \( \mathbb{G}_{m,E} \).

Now we are ready to show that the boundary map is trivial. We take a \( \mathbb{Z} \)-linear combination of codimension \( n \) points on \( X \). Let \( x \) be such a point with residue field \( F \), and let \( v \) be the discrete valuation for a codimension \( n+1 \) point in the closure of \( x \) with residue field \( E \). As remarked before, we can choose a lift of the multiplicity \( m_{x} \in \mathbb{Z}_{+} \) to \( \left( \pi_{n}^{n}(B \text{SL}_n) \right)_{n-1}(k(x)) \) which is pulled back from the base field \( k \). Since the lift is pulled back from the base field, it obviously extends over \( \mathcal{O}_{v} \). By the previous discussion, its residue is therefore trivial, but this means that the boundary map is trivial as claimed. \( \square \)

Now we can go on to show non-triviality of the stably free modules.

**Theorem 4.2.** Let \( F \) be an algebraically closed field and set \( k = F(T) \). Let \( X \) be an \( n+1 \)-dimensional smooth variety over \( k \) with a covering \( X = Y \cup Z \) such that \( Y \cap Z \) is affine. Assume there is a non-zero class \( c \in \text{CH}^{n+1}(X)/n! \) which restricts trivially to both \( Y \) and \( Z \). Then there exists a non-trivial stably free module over \( Y \cap Z \) whose lifting class maps to \( c \) under the boundary map in the Mayer–Vietoris sequence.

**Proof.** By Proposition 3.1 resp. its corollary, the geometric assumptions imply that there is a surjection

\[ [Y \cap Z, Q_{2n+1}]_{\alpha} \to H_{\text{Nis}}^{n}(Y \cap Z, \text{K}_{n+1}^{MW}) \to \text{CH}^{n+1}(X)/n! \]

We call \( \alpha \) any map \( Y \cap Z \to Q_{2n+1} \) lifting the class \( c \). This map will necessarily induce a non-trivial element in \( H_{\text{Nis}}^{n}(Y \cap Z, \text{K}_{n+1}/n!) \). The canonical epimorphism
But by Lemma 4.1, we know that the boundary map $CH$ is trivial. We have thus proved that the image of $\alpha$ shows that lifts of different classes will yield different lifting classes in $Y \cap Z$.

We need to prove that the class has non-trivial image under the morphism $$H^n_{\text{Nis}}(Y \cap Z, K_{n+1}^M) \rightarrow H^n_{\text{Nis}}(Y \cap Z, \pi_{n}^{A_1}(B \text{SL}_n))$$

To do this, consider the ladder of Mayer–Vietoris sequences (associated to the covering $X = Y \cup Z$) for cohomology with coefficients in $K^Q_2$ and $K_{p+1}/p!$, connected by the boundary map arising from the exact sequence

$$0 \rightarrow S_{n+1} \rightarrow \pi_{n}^{A_1}(B \text{SL}_n) \rightarrow K^Q_n \rightarrow 0.$$ 

This ladder contains the following commutative square

$$\begin{array}{ccc}
H^{n-1}_{\text{Nis}}(Y \cap Z, K^Q_n) & \longrightarrow & CH^n(X) \\
\downarrow \partial & & \downarrow \partial \\
H^n_{\text{Nis}}(Y \cap Z, K_{n+1}^M) & \longrightarrow & CH^{n+1}(X)/n!
\end{array}$$

where the horizontal maps are the boundary maps in the respective Mayer–Vietoris sequences, and the vertical maps are the boundary maps for the above exact sequence of sheaves. Assuming that the image of $\alpha$ in $H^n_{\text{Nis}}(Y \cap Z, K_{n+1}^M)$ is in the image of $\partial$, then its associated class in $CH^{n+1}(X)/n!$ is also in the image of $\partial$. But by Lemma 4.3, we know that the boundary map $CH^n(X) \rightarrow CH^{n+1}(X)/n!$ is trivial. We have thus proved that the image of $\alpha$ in $H^n_{\text{Nis}}(Y \cap Z, S_{n+1})$ obtained from lifting a non-trivial class $c \in CH^{n+1}(X)/n!$ has non-trivial image under the map

$$H^n_{\text{Nis}}(Y \cap Z, S_{n+1}) \rightarrow H^n_{\text{Nis}}(Y \cap Z, \pi_{n}^{A_1}(B \text{SL}_n)).$$

As mentioned before, the element in $[Y \cap Z, Q_{2n+1}]_{A_1}$ can, by [AHWII], be explicitly written as a unimodular row of length $n+1$ over the coordinate ring of $Y \cap Z$. Since the image of the class $\alpha$ in $H^n_{\text{Nis}}(Y \cap Z, \pi_{n}^{A_1}(B \text{SL}_n))$ is non-trivial, this unimodular row gives rise to a non-trivial stably free module as claimed. □

**Remark 4.3.** Take $X \subset \mathbb{P}^n$ to be the complement of a hypersurface such that it has non-trivial Chow group. Now take any two hypersurfaces $F_1, F_2 \subset \mathbb{P}^n$ which intersect only outside $X$ and which each have a $k$-rational point. Then $Y = X \setminus F_1$ and $Z = X \setminus F_2$ provides a covering as required. The fact that such a situation can be arranged over $k(T)$ with $k$ algebraically closed is proved in [MKS]. Clearing denominators this shows the existence of rank $p$ stably free modules over varieties of $p+2$ over algebraically closed fields.

**Remark 4.4.** The same argument, applied to differences of classes in $CH^{p+1}(X)$ shows that lifts of different classes will yield different lifting classes in

$$H^n_{\text{Nis}}(Y \cap Z, \pi_{p}^{A_1}(B \text{SL}_p)).$$

This implies that we can actually get $p-1$ pairwise non-isomorphic stably free modules on $Y \cap Z$.

5. **Stably free modules: Mohan Kumar’s examples at the prime 2**

Now we discuss Mohan Kumar’s stably free modules of rank $2$. While the constructions in [MKS] work for odd and even primes, the $A_1$-topological reinterpretation here is slightly different for the prime 2, due to a difference in structure of the relevant $A_1$-homotopy sheaf. The goal is to get examples of a smooth variety of
We want to show that the class produced above has non-trivial image under \( \tilde{\map} \). This would produce a non-trivial element in the kernel of the natural projection \( X \rightarrow X \). Proposition 5.2.

For the situation of the hypersurface complements of \( Y \) in \( X \), the last group vanishes since the Nisnevich cohomological dimension of \( V \) arising from Voevodsky’s solution of the Milnor conjecture induces a sequence non-trivial. The exact sequence

\[
\ker \left( \tilde{\map}^2(X) \cong \text{H}^2_{\text{Nis}}(X, K_2^{MW}) \rightarrow \text{H}^2_{\text{Nis}}(X, K_2^M) \cong \text{CH}^2(X) \right).
\]

From the discussion in Section 3, we obtain a smooth affine variety \( Y \cap Z \) together with a morphism \( \alpha : Y \cap Z \rightarrow \mathbb{Q}_5 \) which is detected on \( \text{H}^3(Y \cap Z, K_3^{MW}) \). We want to show that the composition \( Y \cap Z \rightarrow \mathbb{Q}_5 \rightarrow B \text{SL}_2 \) of \( \alpha \) with the homotopy fiber of the stabilization morphism is not null-\( \mathbb{A}^1 \)-homotopic.

**Lemma 5.1.** Let \( k \) be an algebraically closed field of characteristic \( \neq 2 \) and let \( X \) be the open subvariety of \( \mathbb{P}^3 \) over \( k(T) \) given by Mohan Kumar’s construction, cf. Section 3. There is a non-trivial class in \( \text{H}^3_{\text{Nis}}(X, I^1) \), detected by \( \text{CH}^3(X)/2 \).

**Proof.** By the results of Mohan Kumar, \( \text{CH}^3(X)/2 \cong \text{H}^3_{\text{Nis}}(X, K_3^{MW}/2) \cong \mathbb{Z}/2\mathbb{Z} \) is non-trivial. The exact sequence

\[
0 \rightarrow I^1 \rightarrow I^3 \rightarrow K_3^M/2 \rightarrow 0
\]

arising from Voevodsky’s solution of the Milnor conjecture induces a sequence

\[
\text{H}^3_{\text{Nis}}(X, I^1) \rightarrow \text{H}^3_{\text{Nis}}(X, K_3^M/2) \rightarrow \text{H}^3_{\text{Nis}}(X, I^1).
\]

The last group vanishes since the Nisnevich cohomological dimension of \( X \) is 3. Therefore, we get a surjection \( \text{H}^3_{\text{Nis}}(X, I^1) \rightarrow \mathbb{Z}/2\mathbb{Z} \).

Now we can consider the Mayer–Vietoris sequence again to lift this non-trivial element to another \( \mathbb{I}^4 \)-cohomology group. We can choose the class in \( \text{H}^3_{\text{Nis}}(X, I^3) \) to be represented by an odd-dimensional form supported on a rational point. Because the hypersurface complements of \( Y \) and \( Z \) contain rational points, this choice of class in \( \mathbb{I}^4 \)-cohomology of \( X \) will trivialize in the covering. This implies the following statement:

**Proposition 5.2.** Let \( k \) be an algebraically closed field of characteristic \( \neq 2 \). In the situation \( p = 2 \) of Mohan Kumar’s example, the boundary map of the Mayer–Vietoris sequence associated to the covering \( X = Y \cap Z \) provides a surjection

\[
\text{H}^2_{\text{Nis}}(Y \cap Z, I^3) \rightarrow \text{H}^2_{\text{Nis}}(X, I^3) \rightarrow \text{CH}^3(X)/2 \cong \mathbb{Z}/2\mathbb{Z}.
\]

This produces a class in \( \text{H}^2_{\text{Nis}}(Y \cap Z, I^3) \) which is detected by the non-trivial \( \mathbb{I}^4 \)-cohomology of \( X \). Now consider the exact sequence

\[
0 \rightarrow I^4 \rightarrow K_2^{MW} \rightarrow K_2^M \rightarrow 0.
\]

We want to show that the class produced above has non-trivial image under

\[
\text{H}^2_{\text{Nis}}(Y \cap Z, I^3) \rightarrow \text{H}^2_{\text{Nis}}(Y \cap Z, K_2^{MW}) \cong \text{CH}^2(Y \cap Z).
\]

This would produce a non-trivial element in the kernel of the natural projection map \( \text{CH}^2(Y \cap Z) \rightarrow \text{CH}^2(Y \cap Z) \). As before, we use the Mayer–Vietoris sequences for the covering \( X = Y \cup Z \). We use the two sequences for coefficients with \( K_2^M \).
Proposition 5.2 induces a non-trivial class in the ladder of Mayer–Vietoris sequences now implies that the element produced in because the left-hand vertical map is trivial, as claimed. The previous square from Remark 5.4.

\[ H_{\text{Nis}}(Y \cap Z, K_2^{\text{MW}}) \text{ } \xrightarrow{\beta} \text{ } H^2_{\text{Nis}}(X, K_2^{\text{MW}}) \]
\[ H^2_{\text{Nis}}(Y \cap Z, \mathbb{P}^3) \xrightarrow{\beta} H^3_{\text{Nis}}(X, \mathbb{P}^3). \]

The horizontal arrows are the boundary maps for the Mayer–Vietoris sequences, the vertical maps are the boundary maps for the exact sequence of sheaves. To check that the left-hand vertical map doesn’t hit the element from Proposition 5.2, it suffices to show that the right-hand vertical map doesn’t hit the non-trivial 2-torsion from Lemma 5.1. So we need to compute the right-hand boundary map, which can be viewed as an integral Bockstein operation. By [Fas13, Proposition 11.6], we know that the Bockstein map \( \beta : CH^2(\mathbb{P}^3) \to H^2_{\text{Nis}}(\mathbb{P}^3, \mathbb{I}) \) is trivial. Since the Bockstein map is compatible with the maps in localization sequences, we get a commutative square

\[ \begin{array}{ccc}
CH^2(\mathbb{P}^3) & \xrightarrow{\beta} & CH^2(X) \\
\downarrow & & \downarrow \\
H^3(\mathbb{P}^3, \mathbb{I}) & \xrightarrow{\beta} & H^3(X, \mathbb{I})
\end{array} \]

where the horizontal arrows are the restriction to open subschemes in the localization sequence. Note that the restriction \( CH^2(\mathbb{P}^3) \to CH^2(X) \) is in fact surjective. This implies, in particular, that the right-hand vertical map \( \beta \) is the zero map, because the left-hand vertical map is trivial, as claimed. The previous square from the ladder of Mayer–Vietoris sequences now implies that the element produced in Proposition 5.2 induces a non-trivial class in

\[ \ker \left( CH^2(Y \cap Z) \to CH^2(Y \cap Z) \right). \]

We can take any such non-trivial element as a class in \( H^3_{\text{Nis}}(Y \cap Z, K_2^{\text{MW}}) \) and lift it along the surjection \( [Y \cap Z, Q_3]_{\text{H}} : H^2_{\text{Nis}}(Y \cap Z, K_2^{\text{MW}}) \). This corresponds to a unimodular row of length 3 which gives rise to a stably free module of rank 2. By construction, its image in \( H^3_{\text{Nis}}(Y \cap Z, \pi_2^A) = BS_{\mathbb{I}}(2) \) is non-trivial. Applying the representability theorem of 2.1, we have then proved the following:

**Theorem 5.3.** Let \( k \) be an algebraically closed field of characteristic \( \neq 2 \). There exists a 3-dimensional smooth affine scheme \( Y \cap Z \) over \( k(T) \) and a non-trivial stably trivial rank 2 vector bundle on \( Y \cap Z \). Clearing denominators, there exists a 4-dimensional smooth affine scheme over \( k \) with a non-trivial stably free rank 2 bundle over it.

**Remark 5.4.** Over other fields \( k \) it could be possible to get other types of examples. Since the hypersurface complement of \( X \) has no rational points, the map \( H^3_{\text{Nis}}(\mathbb{P}^3 \setminus X, \mathbb{P}^3, \mathbb{I}) \to H^3_{\text{Nis}}(\mathbb{P}^3, \mathbb{I}) \) should map a class (given by elements of \( W(k(x)) \) supported on points \( x \) on \( W \)) to the sum of transfers of the classes from \( W(k(x)) \) supported on a rational point. In particular, the ideal generated by transfers of classes from points on \( W \) should be strictly smaller than the fundamental ideal whenever there are non-trivial quaternion algebras over \( k \). Classes not in the image of transfer should then yield non-trivial classes in \( H^3_{\text{Nis}}(X, \mathbb{I}) \) not detected on \( CH^3(X)/2 \). Applying the previous argument would produce stably free modules of a more arithmetic nature on \( X \). This could be quite similar to the examples discussed in [BFS14, Section 3].
Remark 5.5. Elements in \( \ker \left( CH^2(X) \to CH^2(X) \right) \) do not only give rise to stably free rank 2 modules. They also give rise to stably trivial \( \text{Spin}(n) \)-torsors for \( 3 \leq n \leq 5 \). I’ll discuss this elsewhere.

6. Stably trivial symplectic modules

In the following section, we will now provide a construction of non-trivial stably trivial symplectic torsors for symplectic groups, based on the above \( \mathbb{A}^1 \)-topological reinterpretation of Mohan Kumar’s construction. Recall that Section 5 provided a smooth affine variety \( Y \cap Z \) with a morphism \( Y \cap Z \to Q_{2p+1} \) which is detected in Milnor K-cohomology. We can now compose this morphism with the natural map \( Q_{2p+1} \to B\text{Sp}_{p-1} \) which is the inclusion of the \( \mathbb{A}^1 \)-homotopy fiber of the stabilization map \( B\text{Sp}_{p-1} \to B\text{Sp}_{p+1} \). This produces a symplectic vector bundle which becomes trivial after adding a symplectic line. The morphism \( Q_{2p+1} \to B\text{Sp}_{p-1} \) induces a morphism \( H^p(Y \cap Z, K_{MW}^{p+1}) \to H^p(Y \cap Z, \pi_p^{\mathbb{A}^1} B\text{SL}_{p-1}) \). The main point of the present section will now be to show that the composition

\[
[Y \cap Z, Q_{2p+1}]_{\mathbb{A}^1} \to H^p(Y \cap Z, K_{MW}^{p+1}) \to H^p(Y \cap Z, \pi_p^{\mathbb{A}^1} B\text{Sp}_{p-1})
\]

sends the map considered previously to a non-zero element, thus showing that the stably trivial symplectic module is in fact non-trivial. For this, we will need to recall some information on the \( \mathbb{A}^1 \)-homotopy sheaf \( \pi_p^{\mathbb{A}^1}(B\text{Sp}_{p-1}) \).

6.1. Recollection on \( \mathbb{A}^1 \)-homotopy of the symplectic groups. First, recall that the first 2\( n \) \( \mathbb{A}^1 \)-homotopy groups of \( B\text{Sp}_{2n} \) are stable, isomorphic to the respective symplectic K-groups. The first unstable \( \mathbb{A}^1 \)-homotopy sheaf of \( B\text{Sp}_{2n} \) was computed in [AF14a, Theorem 3.3] and is described by the following exact sequences

\[
0 \to T_{2n+2} \to \pi_{2n+1}(B\text{Sp}_{2n}) \to KSp_{2n+1} \to 0,
0 \to D_{2n+3} \to T_{2n+2} \to S_{2n+2} \to 0.
\]

The sheaf \( D_{2n+3} \) is a quotient of \( \mathcal{P}^{n+3} \). For \( n \) even, the natural morphism of sheaves \( K_{2n+2}^M/(2n+1)! \to S_{2n+2} \) induces an isomorphism after \( 2n \)-fold contraction, cf. [AF14a, Lemma 7.1]. For \( n \) odd, the natural morphism \( K_{2n+2}^M/2(2n+1)! \to S_{2n+2} \) induces an isomorphism after \( 2n+1 \)-fold contraction, cf. [AF14a, Lemma 7.2].

Note that the proof of the exact sequences above, cf. [AF14a], implies that the morphism \( \pi_{2n+1}(B\text{Sp}_{2n}) \to KSp_{2n+1} \) appearing in the exact sequence above is the map induced on \( \pi_{2n+1} \) by the stabilization morphism \( B\text{Sp}_{2n} \to B\text{Sp}_{2n+2} \) which adds a trivial symplectic line.

As before, the stabilization morphism sits in an \( \mathbb{A}^1 \)-fiber sequence

\[
Q_{4n+3} \to B\text{Sp}_{2n} \to B\text{Sp}_{2n+2}
\]

We want to construct stably free symplectic modules by composing a morphism \( X \to Q_{4n+3} \) lifting a class in \( H^{2n+1}(X, K_{MW}^{n+2}) \) with the map \( Q_{4n+3} \to B\text{Sp}_{2n} \). The class in Milnor–Witt K-cohomology will be obtained as before from Proposition 3.1 and we need to check that the image of this class in the cohomology group \( H^{2n+1}(X, \pi_{4n+2}^{\mathbb{A}^1}(B\text{Sp}_{2n})) \) is non-trivial.

6.2. Existence of stably trivial symplectic modules. Before we can show the existence of interesting stably trivial symplectic modules, we need a result concerning cohomology of symplectic K-sheaves.

Proposition 6.1. Let \( k \) be an infinite perfect field and let \( X \) be a smooth variety over \( k \), and let \( n \) be a positive odd integer. Then the group \( H^n_{Nis}(X, K_{Sp}) \) is \( 2 \)-torsion.
Proof. Again, we use the Gersten resolution to prove the result. A cohomology class in $H^0_{\text{Nis}}(X, \text{KSp}_n)$ can be represented by a sum, indexed by codimension $n$ integral subvarieties $Z$ of $X$, of elements in $(\text{KSp}_n)_{\text{Nis}}(Z)$. In terms of higher Grothendieck–Witt groups, $\text{KSp}_n \cong GW_2^n$ and the corresponding $n$-th contraction is $GW_0^{2-n}$, cf. [AF14a Proposition 4.4]. In our case, $n$ is odd. Using the 4-periodicity of higher Grothendieck–Witt groups, the groups $(\text{KSp}_n)_{\text{Nis}}(Z)$ are then of the form $GW_0^n(Z)$ or $GW_0^n(k(Z))$, for $n \equiv 1 \mod 4$ and $n \equiv 3 \mod 4$, respectively. By [FRS12 Lemma 2.2], we have $GW_0^n(F) = 0$ for any field of characteristic $\neq 2$, implying that $H^0_{\text{Nis}}(X, \text{KSp}_n) = 0$ for $n \equiv 1 \mod 4$. On the other hand, [FS08 Lemma 4.1] implies $GW_0^n(F) \cong Z/2Z$ for fields of characteristic $\neq 2$, implying that $H^0_{\text{Nis}}(X, \text{KSp}_n) = 0$ is 2-torsion for $n \equiv 3 \mod 4$. \hfill \Box

Now that we can establish the existence of some interesting stably trivial symplectic modules by chasing the non-trivial Milnor K-cohomology classes to the lifting set.

Theorem 6.2. Let $k$ be an algebraically closed field of characteristic $\neq 2$. For every odd prime $p$, there exists a $p + 1$-dimensional smooth affine variety over $k(T)$ and a non-trivial stably trivial $\text{Sp}_{p-1}$-torsor over it. Clearing denominators, there exists a $p + 2$-dimensional smooth affine variety over $k$ and a stably trivial non-trivial $\text{Sp}_{p-1}$-torsor over it.

Proof. Proposition[4,1] provides us with a smooth affine $p + 1$-dimensional variety $X$ over $k(T)$, with an affine open covering $X = Y \cup Z$, such that we have a surjection

$$H^p_{\text{Nis}}(Y \cap Z, K^{MW}_{p+1}) \twoheadrightarrow \text{CH}^{p+1}(X)/p! \cong Z/pZ.$$  

Denote by $\alpha$ a lift of a non-trivial class in $Z/pZ$ along the above surjection. We need to show that the image of $\alpha$ in $H^p_{\text{Nis}}(Y \cap Z, \pi^1_p(B \text{Sp}_{p-1}))$ is non-trivial. By construction, the above surjection factors through

$$H^p_{\text{Nis}}(Y \cap Z, K^{MW}_{p+1}) \twoheadrightarrow H^p_{\text{Nis}}(Y \cap Z, T'_{p+1}) \twoheadrightarrow \text{CH}^{p+1}(X)/p! \cong Z/pZ$$  

which implies that the image of $\alpha$ in the $T'_{p+1}$-cohomology is non-trivial.

Now we want to show that the elements produced so far actually provide non-trivial lifting classes. Recall that the exact sequence

$$0 \to T'_{p+1} \to \pi^1_p(B \text{Sp}_{p-1}) \to \text{KSp}_p \to 0,$$  

induces a long exact sequence in Nisnevich cohomology whose relevant portion looks as follows

$$H^{p-1}_{\text{Nis}}(Y \cap Z, K\text{Sp}_p) \to H^p_{\text{Nis}}(Y \cap Z, T'_{p+1}) \to H^p_{\text{Nis}}(Y \cap Z, \pi^1_p(B \text{Sp}_{p-1})).$$  

We need to show that the $p$-torsion elements produced so far are not in the image of $H^{p-1}_{\text{Nis}}(Y \cap Z, K\text{Sp}_p)$. As in the analysis of the stably free modules in Theorem[4,2] we consider the ladder of Mayer–Vietoris sequences (associated to the covering $X = Y \cup Z$) for cohomology with coefficients in $K\text{Sp}_p$ and $T'_{p+1}$, connected by the boundary map arising from the exact sequence of strictly $A^1$-invariant sheaves above. This ladder contains the following commutative square

$$\begin{array}{ccc}
H^{p-1}_{\text{Nis}}(Y \cap Z, K\text{Sp}_p) & \longrightarrow & H^p_{\text{Nis}}(X, K\text{Sp}_p) \\
\downarrow & & \downarrow \\
H^p_{\text{Nis}}(Y \cap Z, T'_{p+1}) & \longrightarrow & \text{CH}^{p+1}(X)/p!
\end{array}$$
In particular, the composition
\[ \text{where the first map is the natural projection and the second is the hyperbolic map.} \]
allows to replace the lower-right \( H_{\text{Nis}}^{p+1}(X, T_{p+1}) \) by the appropriate Chow group. Assuming our class \( \sigma \in H_{\text{Nis}}^{p+1}(Y \cap Z, T_{p+1}) \) is in the image of \( \hat{\partial} \), then its associated \( p \)-torsion class in \( \text{CH}^{p+1}(X)/p! \) is also in the image of \( \hat{\partial} \). It therefore, suffices to show that the boundary map \( H_{\text{Nis}}^{p-1}(X, K\text{Sp}_p) \rightarrow \text{CH}^{p+1}(X)/p! \) is trivial. This now follows from Proposition 6.1 which shows that \( H_{\text{Nis}}^{p}(X, K\text{Sp}_p) \) is 2-torsion. This implies that the morphism \( H_{\text{Nis}}^{p+1}(X)/p! \cong \mathbb{Z}/p\mathbb{Z} \) is trivial. As a consequence, the non-trivial \( p \)-torsion elements in \( H_{\text{Nis}}^{p-1}(Y \cap Z, T_{p+1}) \) will still be non-trivial \( p \)-torsion elements in \( H_{\text{Nis}}^{p-1}(Y \cap Z, \pi^{A^1}_{p}(B\text{Sp}_{p-1})) \).

As mentioned before, the class \( \alpha \in H^{p}(X, K\text{Sp}^{\text{MW}}_{p+1}) \) lifts to an element in \( [Y \cap Z, Q_{2p+1}]_{A^1} \). Since the image of the class \( \alpha \) in \( H_{\text{Nis}}^{p}(Y \cap Z, \pi^{A^1}_{p}(B\text{Sp}_{p-1})) \) is non-trivial, the composition \( Y \cap Z \rightarrow Q_{2p+1} \rightarrow B\text{Sp}_{p-1} \) gives rise, via the representability theorem 2.1, to a non-trivial stably trivial symplectic module as claimed.

Clearing denominators, this example over \( Y \cap Z \) (which is of dimension \( p + 1 \)) can be extended to an example over a \( p + 2 \)-dimensional smooth affine variety over \( k \), proving the claim. \( \square \)

6.3. Comparison to stably free modules. The symplectic modules described here would appear to only be old examples wearing new clothes (but the new clothes tell us something about the old examples which we possibly didn’t know before).

Consider the composition \( \text{Sp}_{p-1} \rightarrow \text{SL}_{p-1} \rightarrow \text{SL}_{p} \) which is obtained as composition of the standard injection of the symplectic group into the special linear group as stabilizer of the standard symplectic form, followed by the stabilization homomorphism. This corresponds to the morphism of classifying spaces representing the map
\[ H_{\text{Nis}}^{p}(Y \cap Z, \text{Sp}_{p-1}) \rightarrow H_{\text{Nis}}^{p}(Y \cap Z, \text{SL}_{p-1}) \rightarrow H_{\text{Nis}}^{p}(Y \cap Z, \text{SL}_{p}) \]
which first takes the underlying oriented projective module of rank \( p - 1 \) and then adds a trivial line bundle. We can use information from [AF15] to investigate the effect of this map on the stably free symplectic modules.

Recall that \( \text{SL}_{p}/\text{Sp}_{p-1} \) is a space denoted by \( X_{A^1} \) in [AF15]. There is an \( A^1 \)-fiber sequence
\[ \text{SL}_{p}/\text{Sp}_{p-1} \rightarrow B\text{Sp}_{p-1} \rightarrow B\text{SL}_{p} \]
which induces an exact sequence of \( A^1 \)-homotopy groups
\[ \pi^{A^1}_{p}B\text{Sp}_{p-1} \rightarrow \pi^{A^1}_{p}B\text{SL}_{p} \rightarrow \pi^{A^1}_{p-1}B\text{SL}_{p}/\text{Sp}_{p-1} \]
The first homotopy sheaf in this sequence is the first unstable homotopy sheaf of \( B\text{Sp}_{p-1} \) discussed in 6.1. The second homotopy sheaf in this sequence is the first unstable homotopy sheaf of \( B\text{SL}_{p} \) discussed in [4.1]. The third homotopy sheaf in the sequence is \( \pi^{A^1}_{p-1}(X_{A^1}) \cong GW_{p}^{3} \) by [AF15] Proposition 4.2.2. The proof of [AF15] Theorem 4.3.1] shows that there is a factorization
\[ \pi^{A^1}_{p}B\text{SL}_{p} \rightarrow K^{Q}_{p} \xrightarrow{H} GW_{p}^{3} \]
where the first map is the natural projection and the second is the hyperbolic map. In particular, the composition
\[ \text{Sp}_{p+1} \rightarrow \pi^{A^1}_{p}B\text{SL}_{p} \rightarrow \pi^{A^1}_{p-1}B\text{SL}_{p}/\text{Sp}_{p-1} \]
is the zero map. The beginning of [AF15, Section 4.2] shows that there is a cartesian square

\[
\begin{array}{ccc}
\text{Sp}_{2n} & \rightarrow & \text{Sp}_{2n+2} \\
\downarrow & & \downarrow \\
\text{SL}_{2n+1} & \rightarrow & \text{SL}_{2n+2}.
\end{array}
\]

This implies that the stabilization exact sequences describing the first unstable $A^1$-homotopy groups are compatible in the sense that there is a commutative diagram with exact rows:

\[
\begin{array}{cccccccc}
0 & \rightarrow & T'_{2n+2} & \rightarrow & \pi_{A^1}^1 B \text{Sp}_{2n} & \rightarrow & K \text{Sp}_{2n+1} & \rightarrow & 0 \\
0 & \rightarrow & S_{2n+2} & \rightarrow & \pi_{A^1}^1 B \text{SL}_{2n+1} & \rightarrow & K^2 \text{SL}_{2n+1} & \rightarrow & 0
\end{array}
\]

We saw previously that the morphism $S_{2n+2} \rightarrow \pi_{A^1}^1 B \text{Sp}_{p-1} / \text{Sp}_{p-1}$ is trivial, and therefore the left vertical arrow in the above ladder diagram is surjective. The explicit description of the Milnor K-cohomology classes given previously then implies that the image of the lifting classes associated to the stably trivial symplectic modules under the morphism

\[
H_{\text{Nis}}^p(X, \pi_{A^1}^1 (B \text{Sp}_{p-1})) \rightarrow H_{\text{Nis}}^p(X, \pi_{A^1}^1 (B \text{SL}_p))
\]

is non-trivial. In particular, taking the underlying projective modules of the stably trivial symplectic modules and adding a trivial line produces non-trivial stably free modules.

**Corollary 6.3.** Let $k$ be an algebraically closed field of characteristic $\neq 2$ and let $p$ be an odd prime. The stably free modules of rank $p$ of Mohan Kumar split off a trivial line, and the resulting stably free module of rank $p - 1$ has a symplectic structure.

**Remark 6.4.** Since the kernel of the morphism $\pi_{A^1}^1 B \text{Sp}_{p-1} \rightarrow \pi_{A^1}^1 B \text{SL}_p$ is not explicitly known (and cohomology with coefficients in the kernel is not explicitly computed), it’s not quite clear if the surjectivity above suffices to prove that Mohan Kumar’s examples lift to the symplectic group just by exact sequence formalities. Probably that is true, but for now the above argument uses that we have explicit stably trivial symplectic modules lifting the stably free projective modules.

**Remark 6.5.** The above result means in particular, that for every odd prime $p$ and every algebraically closed field $k$ of characteristic $\neq 2$, there is a $p + 2$-dimensional smooth affine $k$-scheme with a stably free module of rank $p - 1$ on it. I am not aware of examples in the literature of stably free modules of rank $d - 3$ over varieties of dimension $d$. Note that the fact that Mohan Kumar’s stably free modules split off a trivial line cannot be deduced from Euler class group methods because their rank is already too low. Finally, it is interesting to note that the non-triviality of the stably free modules of rank $p - 1$ cannot be established using the obstruction-theoretic approach for $B \text{SL}_{p-1}$ because the relevant $A^1$-homotopy group has not been computed yet.

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