Quantum indistinguishability from general representations of $SU(2n)$

JM Harrison† ‡,‡ and JM Robbins2 †

† School of Mathematics, University of Bristol, University Walk, Bristol BS8 1TW, UK
‡ Abteilung Theoretische Physik, Universität Ulm, Albert-Einstein-Allee 11,
D-89069 Ulm, Germany

Abstract

A treatment of the spin-statistics relation in nonrelativistic quantum mechanics due to Berry and Robbins [Proc. R. Soc. Lond. A (1997) 453, 1771-1790] is generalised within a group-theoretical framework. The construction of Berry and Robbins is re-formulated in terms of certain locally flat vector bundles over $n$-particle configuration space. It is shown how families of such bundles can be constructed from irreducible representations of the group $SU(2n)$. The construction of Berry and Robbins, which leads to a definite connection between spin and statistics (the physically correct connection), is shown to correspond to the completely symmetric representations. The spin-statistics connection is typically broken for general $SU(2n)$ representations, which may admit, for a given value of spin, both bose and fermi statistics, as well as parastatistics. The determination of the allowed values of the spin and statistics reduces to the decomposition of certain zero-weight representations of a (generalised) Weyl group of $SU(2n)$. A formula for this decomposition is obtained using the Littlewood-Richardson theorem for the decomposition of representations of $U(m + n)$ into representations of $U(m) \times U(n)$.

---

1E-mail address: jon.harrison@physik.uni-ulm.de
2E-mail address: j.robbins@bristol.ac.uk
1 Introduction

In nonrelativistic quantum mechanics, the spin-statistics relation specifies the behaviour of many-body wavefunctions for indistinguishable particles under the exchange of a pair of particle labels, and asserts that the wavefunctions either remain the same or change sign according to whether the spin of the particles, \( s \), is integral or half-odd-integral. Nonrelativistic quantum mechanics can be formulated in a logically consistent way without the spin-statistics relation, or else, with the wrong (ie, physically incorrect) spin-statistics relation. Therefore, if one is to derive the spin-statistics relation from within a nonrelativistic theory, the nonrelativistic theory must be reformulated, with postulates different from the standard ones. Whether such a reformulation serves to explain the spin-statistics relation is, to some extent, a matter of judgement, and depends on the naturalness and simplicity of the assumptions introduced.

Such a reformulation was presented by Berry and Robbins [3] (referred to in what follows as BR). In BR, the representation of spin was made to depend on position so that, in contrast to the standard formulation, the \( n \)-particle wavefunction was single-valued on configuration space. The statistics of the wavefunction was determined by a topological property of this position-dependent spin representation. A calculation showed that the statistics were in accord with the physically correct spin-statistics relation. The construction was based on Schwinger’s representation of spin as number states of harmonic oscillators. Its implementation assumed without proof the solution of a certain topological problem; a solution was subsequently found by Atiyah [2]. The extension to relativistic wave equations was discussed by Anandan [1].

To be compelling, a derivation of the spin-statistics relation should be based on general physical and mathematical principles, rather than a particular construction. In BR, it was suggested that certain properties of the construction introduced therein might be sufficient to ensure the correct spin-statistics relation. Later, it was shown that this is not the case [4]), as alternative constructions exist which possess these properties but yield the wrong statistics. Thus, a nonrelativistic derivation of the spin-statistics relation from general principles remains to be established along these lines. For a discussion of other nonstandard approaches to the spin-statistics relation, see [5, 6]. Our purpose here is to investigate a certain group-theoretical generalisation of the construction in BR. We begin in Section 2 by framing the underlying requirement, namely that wavefunctions be single-valued, in a geometrical context. The setting for the quantum description of \( n \) indistinguishable particles are certain vector bundles over configuration space, which we call \( n \)-spin bundles. \( n \)-spin bundles carry a representation of the spin-statistics group \( \Sigma(n) \), which is (nearly) the group generated by permutations and independent rotations of \( n \) spinors (the precise definition is given in Section 2.1). The particular representation of \( \Sigma(n) \) characterises the spin and statistics of the particles. The statistics are then embodied in a topological property of the \( n \)-spin bundle, namely the monodromy of its flat connection. This formulation is in the spirit of earlier treatments by Leinaas & Myrheim [13] and Sorkin [17].

In Section 3 it is shown that \( n \)-spin bundles can be constructed from irreducible repre-
sentations $\Gamma^f$ of the group $SU(2n)$. The construction in BR is seen to be a particular case, corresponding to the completely symmetric representations of $SU(2n)$. For the completely symmetric representations, one obtains a definite connection between spin and statistics, indeed the physically correct connection. In contrast, an arbitrary representation of $SU(2n)$ does not necessarily engender a definite relation between spin and statistics; whether or not it does depends on the decomposition of certain representations of the spin-statistics group constructed from $\Gamma^f$.

This decomposition is carried out in Section 4. The calculation involves the evaluation of integrals over characters of the spin-statistics group, and makes use of the Littlewood-Richardson formula for the decomposition of representations of $U(k+l)$ into representations of $U(k) \times U(l)$. It turns out that for an arbitrary representation of $SU(2n)$ and a given value of spin, various choices of statistics may be realised, including parastatistics (which correspond to representations of the symmetric group of dimension greater than one).

Section 5 contains a summary and discussion of the results. A connection to a more general problem in representation theory is described in the Appendix.

Throughout this paper we will use the following notation: Given $n$ elements $a_1, \ldots, a_n$ of a set $A$, we let $A$ denote the ordered $n$-tuple $(a_1, \ldots, a_n)$. The action of a permutation $\sigma \in S_n$ on $A$ is denoted by $\sigma \cdot A$, and defined by

$$\sigma \cdot A = (a_{\sigma^{-1}(1)}, \ldots, a_{\sigma^{-1}(n)}).$$

(1.1)

Many of the results presented here are discussed in greater detail in Harrison [10].

2 Bundle description of $n$-particle quantum mechanics

The configuration space $C_n$ for $n$ particles in three-dimensional space is the set of $n$-tuples $R = (r_1, \ldots, r_n)$. We will suppose the particles cannot coincide, so that $r_j \neq r_k$. If the particles are indistinguishable, then permuted configurations $R$ and $\sigma \cdot R$ are to be regarded as being the same. We describe here a framework for quantum mechanics in which wavefunctions of identical particles are single-valued on configuration space; that is, the wavefunction at permuted configurations is the same.

We first introduce in Section 2.1 the particular irreducible representations of the spin-statistics group $\Sigma(n)$, denoted by $Q^{s\lambda}$, which correspond to $n$ identical spins. $n$-spin-$s$ bundles with statistics $\lambda$ are defined in Section 2.2. These are flat, hermitian vector bundles over configuration space whose fibres carry an irreducible representation of the spin-statistics group equivalent to $Q^{s\lambda}$. This representation is required to be compatible with indistinguishability and the flat connection. Wavefunctions are taken to be sections of the bundle, and operators representing quantum observables are defined on them. The relation to the standard formulation of quantum mechanics, as well as that of BR, is discussed.
2.1 Representations of the spin-statistics group for identical spinors

Let \( S_n \) denote the symmetric group. The irreducible representations, \( \Lambda^\lambda \), of \( S_n \) are characterised by Young tableaux, \( \lambda \), of \( n \) boxes (equivalently, partitions of \( n \)). Let \( d_\lambda \) denote the dimension of the representation \( \Lambda^\lambda \). Let \( |a\rangle \), \( a = 1, \ldots, d_\lambda \) denote an orthonormal basis for \( \mathbb{C}^{d_\lambda} \) (with respect to the standard inner product). For \( \sigma \in S_n \), we write

\[
\Lambda^\lambda(\sigma)|a\rangle = \Lambda^\lambda_{a',a}(\sigma)|a'\rangle
\]

where here and elsewhere a sum over repeated indices is implied. We may take \( \Lambda^\lambda \) to be unitary, so that \( \Lambda^\lambda_{a',a}(\sigma) \) is a unitary matrix.

Let \( SU(2)^n = \underbrace{SU(2) \times \cdots \times SU(2)}_{n \text{ times}} \)

(2.2)
denote the direct product of \( n \) copies of \( SU(2) \). \( SU(2)^n \) describes the independent rotations of \( n \) spinors. Denote elements of \( SU(2)^n \) by \( U = (u_1, \ldots, u_n) \), with \( u_j \in SU(2) \). States of \( n \) spinors, all of spin \( s \), are unchanged if pairs of spinors are rotated through \( 2\pi \), regardless of whether \( s \) is integral or half-odd-integral. Let \( \text{Nul}(n) \subset SU(2)^n \) denote the subgroup generated by pairs of \( 2\pi \)-rotations. It consists of elements of the form

\[
U_0 = ((-1)^{e_1}I_2, \ldots, (-1)^{e_n}I_2), \quad \text{where } (-1)^{e_1} \cdots (-1)^{e_n} = 1
\]

(2.3)

\( I_2 \) is the \( 2 \times 2 \) identity matrix). The \( n \)-spin group, denoted by \( \text{Spn}(n) \), is defined by

\[
\text{Spn}(n) = SU(2)^n / \text{Nul}(n),
\]

(2.4)

and represents in a one-to-one fashion the independent rotations of \( n \) spinors of the same spin. Given \( U \in SU(2)^n \), let

\[
\overline{U} = U \text{Nul}(n)
\]

(2.5)
denote the corresponding element of \( \text{Spn}(n) \) (that is, \( \overline{U} \) is the coset of \( SU(2)^n \) containing elements which differ from \( U \) by an even number of \( 2\pi \) rotations).

The spin-statistics group,

\[
\Sigma(n) = \text{Spn}(n) \rtimes S_n,
\]

(2.6)
is the semidirect product of the \( n \)-spin group and the symmetric group. Elements are denoted by \( (\overline{U}, \sigma) \), where \( U \in SU(2)^n \) and \( \sigma \in S_n \), and multiplication is given by

\[
(\overline{U}, \sigma)(\overline{U'}, \sigma') = (\overline{U}(\sigma \cdot U'), \sigma \sigma').
\]

(2.7)

(It is easy to check that the right-hand side of (2.7) is unchanged if \( U \) and \( U' \) are multiplied by an even number of \( 2\pi \) rotations.) For brevity, when \( \text{Spn}(n) \) and \( S_n \) are to be regarded as subgroups of \( \Sigma(n) \), we will denote their elements simply by \( \overline{U} \) and \( \sigma \) respectively, rather than by \( (\overline{U}, I_{S_n}) \) and \( (I_{\text{Sp}(n)}, \sigma) \).

The complete set of irreducible representations of the spin-statistics group can be obtained from the general representation theory of semidirect products (see, eg, Mackey [16]).
Here we shall only be interested in representations whose restriction to \( \text{Spin}(n) \) describes \( n \) spinors all of spin \( s \), where \( s \) is integral or half-odd-integral. It is easily established that each such irreducible representation of \( \Sigma(n) \) is characterised by \( s \) and, additionally, by an irreducible representation \( \lambda \) of \( S_n \). We denote this representation by \( Q^{s\lambda} \), and describe it in the following.

\( Q^{s\lambda} \) acts on the \((2s+1)^n d_\lambda\)-dimensional vector space \( V^{s\lambda} \) given by

\[
V^{s\lambda} = \bigotimes_{n \text{ times}} \mathbb{C}^{2s+1} \rightarrow \oplus \mathbb{C}^{d_\lambda}.
\]

Let

\[
|M, a\rangle = |m_1\rangle \otimes \cdots \otimes |m_n\rangle \otimes |a\rangle,
\]

where \( M = (m_1, \ldots, m_n) \) and \( m_j \) ranges between \(-s\) and \( s\) in integer steps, denote a basis for \( V^{s\lambda} \) orthonormal with respect to the standard inner products on \( \mathbb{C}^{2s+1} \) and \( \mathbb{C}^{d_\lambda} \). For \( \overline{U} \in \text{Spin}(n) \), \( Q^{s\lambda}(\overline{U}) \) is given by

\[
Q^{s\lambda}(\overline{U})|M, a\rangle = D^{s}_{m', m}(u_1) \cdots D^{s}_{m_n, m_n}(u_n)|M, a\rangle,
\]

where \( D^{s}_{m, m'}(u) \) denotes the standard spin-s representation of \( SU(2) \) on \( \mathbb{C}^{2s+1} \), and \( M' = (m'_1, \ldots, m'_n) \). (It is easy to check that the right-hand side of (2.10) is unchanged if \( U = (u_1, \ldots, u_n) \) is multiplied by an element of \( \text{Nul}(n) \).) For \( \sigma \in S_n \), \( Q^{s\lambda}(\sigma) \) is given by

\[
Q^{s\lambda}(\sigma)|M, a\rangle = \Lambda^{\lambda}_{a', a}|\sigma \cdot M, a'\rangle.
\]

That is, the spin labels \( M \) are permuted while \( |a\rangle \) transforms according to the representation \( \Lambda^{\lambda} \) of \( S_n \). For a general element \((\overline{U}, \sigma) \in \Sigma(n)\), the expression for \( Q^{s\lambda}(\overline{U}, \sigma) \) follows from (2.10) and (2.11) and the multiplication law (2.7).

### 2.2 \( n \)-spin-\( s \) bundles with statistics \( \lambda \)

For our purposes, a \( k \)-dimensional hermitian vector bundle, \( \mathcal{E} \), over the configuration space \( C_n \) will be regarded as a field of \( k \)-dimensional subspaces, \( \mathcal{E}_R \), of a finite-dimensional Hilbert space, \( \mathcal{V} \), depending smoothly on \( R \in C_n \). \( \mathcal{E}_R \) is called the fibre of \( \mathcal{E} \) at \( R \). A hermitian inner product on \( \mathcal{E}_R \) is induced by the hermitian inner product on \( \mathcal{V} \). A section of \( \mathcal{E} \) is a function \( |\Psi(R)\rangle \) on configuration space taking values in \( \mathcal{E}_R \). The inner product of two sections is given by

\[
\int_{C_n} \langle \Psi(R) | \Phi(R) \rangle \, dR.
\]

The space of square-integrable sections forms a Hilbert space.

To represent spin \( s \) and statistics \( \lambda \), each fibre \( \mathcal{E}_R \) must carry a representation of the spin-statistics group unitarily equivalent to \( Q^{s\lambda} \). Denote this representation by \( L_R \). We require that \( L_R \) depend smoothly on \( R \).

Operators representing spin, position and momentum may be defined on wavefunctions as follows. We consider the spin operators, denoted \( S^{op} = (s^{op}_1, \ldots, s^{op}_n) \), first. Consider the
rotation of the $r^{th}$ spinor about an axis $\hat{e}_a$ by an angle $t$ holding the other spinors fixed. This is described by $U_{(r,a)}(t) = (u_1(t), \ldots, u_n(t)) \in SU(2)^n$, where

$$u_j(t) = \begin{cases} \exp(-it\sigma_a/2), & j = r \\ 1_2, & \text{otherwise} \end{cases} \quad (2.13)$$

(here $\sigma_1$, $\sigma_2$, $\sigma_3$ are the Pauli matrices). Then $s_{r,a}^{\text{op}}$, the $a$th component of $s_r^{\text{op}}$, is given by

$$\left| (s_{r,a}^{\text{op}} \Psi)(R) \right| = \frac{1}{i} \left. \frac{d}{dt} \right|_{t=0} L_R(U_{(r,a)}(t)) |\Psi(R)\rangle. \quad (2.14)$$

As the representation $L_R$ is unitary, $s_r^{\text{op}}$, as defined by (2.14), is self-adjoint. The representation property of $L_R$ implies that the standard commutation relations for spin are satisfied.

Position operators, $R^{\text{op}} = (r_1^{\text{op}}, \ldots, r_n^{\text{op}})$, are defined component-wise by

$$\left| (r_{j,a}^{\text{op}} \Psi)(R) \right| = r_{j,a} |\Psi(R)\rangle. \quad (2.15)$$

$r_j^{\text{op}}$ is hermitian with respect to the inner product (2.12), self-adjoint on a suitable domain, and the position operators commute amongst each other and with the spin operators.

The definition of momentum operators requires a hermitian connection on $E$. A hermitian connection associates to piecewise smooth paths $R(t) \in C_n$ a family of unitary maps between the fibres $E_{R(t)}$. These unitary maps describe the parallel transport of spinors along $R(t)$. Momentum operators may be defined in terms of the covariant derivative with respect to this connection.

A characteristic property of a connection is its curvature, which describes parallel transport around infinitesimal closed paths. Nonvanishing curvature corresponds physically to the presence of gauge (eg, magnetic) fields. In order that our theory be capable of describing physics in the absence of fields, we shall require that $E$ admit a flat connection. This condition is not automatically satisfied; the existence of a flat connection depends on the topology of the bundle (just as the fact that a two-torus admits a flat Riemannian metric, while a two-sphere does not, is a consequence of their different Euler characteristics).

For a flat connection, parallel transport around a closed path is trivial, provided the path is contractible. In $C_n$, every closed path is contractible ($C_n$ is simply connected). Therefore, parallel-transport with respect to a flat connection on $E$ is path-independent, and depends only on the endpoints of the path. Therefore, a flat hermitian connection on $E$ is characterised by unitary maps $T_{R' \leftarrow R} : E_R \to E_{R'}$ describing parallel transport from $R$ to $R'$. Path independence then implies that

$$T_{R'' \leftarrow R'} T_{R' \leftarrow R} = T_{R'' \leftarrow R}. \quad (2.16)$$

Momentum operators $P^{\text{op}} = (p_1^{\text{op}}, \ldots, p_n^{\text{op}})$ are defined as follows. Let

$$E_{(j,a)} = (0, \ldots, 0, \hat{e}_a, 0, \ldots, 0) \quad (2.17)$$
denote the tangent vector in configuration space on which the $j^{th}$ particle moves with unit velocity in the direction $\hat{e}_a$ while the other particles stay fixed. Then the $a^{th}$ component of $p_{j,\alpha}^{op}$ is given by

$$\langle (p_{j,\alpha}^{op}\Psi)(R) \rangle = \frac{d}{dt} \left| T_{R\leftarrow R+tE_{(j,a)}} |\Psi(R + tE_{(j,a)})\rangle \right|_0.$$ (2.18)

$p_{j,\alpha}^{op}$ is hermitian with respect to the inner product (2.12) and is self-adjoint on a suitable domain. From (2.15) it is easily verified that the position and momentum operators satisfy the standard commutation relations. That the momentum operators commute amongst themselves follows from the path independence (2.16) of the connection, provided the displacements $tE$ and $uF$ are small enough so as not to make the particles coincide.

The requirement that spin and momentum commute is equivalent to the requirement that parallel transport be compatible with the representation $L_R$. That is, we should have that $L_{R'}(\mathcal{U},\sigma)T_{R'\leftarrow R} = T_{R'\leftarrow R}L_{R'}(\mathcal{U},\sigma)$.

As a basis for subsequent discussion, let us formulate the standard description of $n$-particle quantum mechanics within the framework described above. (In this case, the vector bundle description is unnecessary, of course, and appears artificial.) For this, take the fibres $E_R$ to be everywhere equal to the fixed vector space $V^{s\lambda}$. Take $L_R$, the representation of the spin-statistics group, to be everywhere equal to the standard representation $Q^{s\lambda}$. Parallel transport is everywhere taken to be trivial; ie $T_{R'\leftarrow R}$ is just the identity map on $V^{s\lambda}$. Then $\mathcal{E}$ is just the cartesian product $C_n \times V^{s\lambda}$, and wavefunctions $|\Psi(R)\rangle$ are just $V^{s\lambda}$-valued functions on $C_n$. Wavefunctions may be expanded in the standard basis $|M,a\rangle$ (cf (2.9)),

$$|\Psi(R)\rangle = \sum_M \sum_{a=1}^{d\lambda} \psi_{M,a}(R) |M,a\rangle,$$ (2.20)

The definitions (2.14), (2.15) and (2.18) of the position, spin and momentum operators yield the standard operations on the coefficients $\psi_{M,a}(R)$,

$$r_{j,\alpha}^{op}\psi_{M,a}^S(R) = r_{j,\alpha}^S\psi_{M,a}^S(R),$$ (2.21)

$$p_{j,\alpha}^{op}\psi_{M,a}^S(R) = -i\nabla_{r_{j,\alpha}}\psi_{M,a}^S(R),$$ (2.22)

$$e^{-i\theta_{j,\alpha}^{op}}\psi_{M,a}^S(R) = \sum_{m'=-s}^{s} D_{m,j,m'}^s(e^{-i\theta_{\alpha}}) \psi_{M,a}^{S}(r_{j},\alpha)(R),$$ (2.23)

where, in (2.23), $M'$ differs from $M$ only in the $j^{th}$ component, in which $m_j$ is replaced by $m'$.

We now introduce the requirement, basic to the formulation in BR, that for indistinguishable particles, the values of the wavefunction at permuted configurations should be the same. That is, we require that

$$|\Psi(\sigma \cdot R)\rangle = |\Psi(R)\rangle, \quad \sigma \in S_n.$$ (2.24)
(Note that for this condition to be sensible, the fibres at \( R \) and \( \sigma \cdot R \) must be the same.) In this case, the wavefunction is single-valued as a function of configurations in which the particles are no longer labeled. Wavefunctions in the standard description are not single-valued in this sense. Indeed, in the standard description, the coefficients of the wavefunction at permuted configurations are related by

\[
\psi_{M,a}(\sigma \cdot R) = \Lambda_{a,a'}^{\lambda} \psi_{\sigma^{-1} \cdot M,a'}(R),
\]

so that the wavefunctions themselves satisfy

\[
|\Psi(\sigma \cdot R)\rangle = L_{\sigma \cdot R}(\sigma)|\Psi(R)\rangle.
\]

Descriptions based on single-valued wavefunctions, but physically equivalent to the standard description, are obtained by re-writing \( \text{(2.26)} \) as

\[
|\Psi(\sigma \cdot R)\rangle = L_{\sigma \cdot R}(\sigma)T_{\sigma \cdot R \leftarrow R}\Psi(R)\rangle.
\]

In the standard description, \( \text{(2.27)} \) is the same as \( \text{(2.26)} \), since \( T_{\sigma \cdot R \leftarrow R} \) is just the identity in this case. In contrast, For single-valued wavefunctions, \( \text{(2.27)} \) becomes

\[
L_{\sigma \cdot R}(\sigma)^{-1} = T_{\sigma \cdot R \leftarrow R}.
\]

Thus, for a description in terms of single-valued wavefunctions to be equivalent to the standard one, parallel transport is necessarily a nontrivial operation; between permuted configurations, parallel transport induces the corresponding permutation of spins.

Let us now formalise the preceding considerations. An \( n \)-spin-\( s \) bundle with statistics \( \lambda \), denoted by \( \mathcal{E}^{s\lambda} \), is defined to be a \((2s + 1)d_\lambda\)-dimensional hermitian vector bundle over the configuration space \( C_n \) endowed with the following properties:

A) There exists a smooth family, \( L_R \), of unitary irreducible representations of the spin-statistics group \( \Sigma(n) \) acting on the fibres \( \mathcal{E}_R \), unitarily equivalent to \( Q^{s\lambda} \).

B) The fibres at permuted configurations are the same, ie

\[
\mathcal{E}_{\sigma \cdot R} = \mathcal{E}_R.
\]

C) There exists a flat hermitian connection on \( \mathcal{E}^{s\lambda} \), characterised by unitary maps \( T_{R' \leftarrow R} \) describing parallel transport from \( R \) to \( R' \), satisfying the composition rule

\[
T_{R' \leftarrow R'}T_{R' \leftarrow R} = T_{R'' \leftarrow R}.
\]

Parallel transport is compatible with the representation \( L_R \) in the sense that

\[
L_{R'}(\overline{U}, \sigma)T_{R' \leftarrow R} = T_{R' \leftarrow R}L_R(\overline{U}, \sigma).
\]
D) Parallel transport between permuted fibres induces permutations, ie

\[ T_{\sigma \cdot R \leftarrow R} = L_{\sigma \cdot R}(\sigma^{-1}). \]  

(2.32)

The Hilbert space \( \mathcal{H} \) of wavefunctions describing \( n \) indistinguishable particles of spin \( s \) and statistics \( \lambda \) is the space of sections of \( \mathcal{E}^{s \lambda} \) with inner product (2.12) satisfying the single-valuedness condition

\[ |\Psi(\sigma \cdot R\rangle) = |\Psi(R)\rangle. \]  

(2.33)

Observables are generated by combinations of the position, momentum and spin operators, \( r_j^{op} \), \( p_j^{op} \) and \( s_j^{op} \), given by (2.15), (2.22) and (2.14) respectively, which are invariant under permutations. These permutation-invariant operators preserve the single-valuedness condition (2.33).

To establish explicitly the equivalence between this formulation and the standard one, as well as the treatment in BR, it is useful to introduce a parallel-transported basis for the fibres \( \mathcal{E}_R \). To this end, we fix a reference configuration \( R_0 \in C_n \). Since \( L_{R_0} \) is unitarily equivalent to \( Q^{s \lambda} \), there exists an orthonormal basis \( |M, a(R_0)\rangle \) of \( \mathcal{E}_{R_0} \) for which

\[ L_{R_0}(U, \sigma)|M, a(R_0)\rangle = Q^{s \lambda}_{M', a', Ma}(U, \sigma)|M', a'(R_0)\rangle. \]  

(2.34)

A basis for \( \mathcal{E}_R \) is defined via parallel transport as follows:

\[ |M, a(R)\rangle = T_{R \leftarrow R_0}|M, a(R_0)\rangle. \]  

(2.35)

Because the representation \( L_R \) is compatible with the flat connection, it follows that (2.34) holds for all \( R \).

Wavefunctions \( |\Psi(R)\rangle \) may be expanded in terms of this basis as

\[ |\Psi(R)\rangle = \psi_{M, a}(R)|M, a(R)\rangle. \]  

(2.36)

From the definitions (2.15), (2.18) and (2.14), it is readily verified that the position, momentum and spin operators act on the components \( \psi_{M, a}(R) \) as the standard operators (2.21) – (2.23). The condition (2.32) implies that the components at permuted configurations are related as in (2.25), in accord with the standard formulation.

Apart from allowing parastatistics, the framework described here is equivalent to the one given in Section 2 of BR. There are, however, some differences in the formulation. In BR, properties A) – D) are expressed directly in terms of the parallel-transported basis. For example, instead of property C), BR require that the parallel-transported basis satisfy

\[ \langle M'(R) | \nabla_{r_j} M(R) \rangle = 0. \]  

(2.37)

In this way, the formalism and terminology of vector bundles is avoided.

An advantage of the present formulation is that properties required by physical considerations are distinguished from those which depend on convention. For example, (2.37) implies the existence of a flat connection, but it implies, additionally, that it is a particular connection which is flat- namely, the connection induced by the inner product on \( \mathcal{V} \),
according to which vectors are parallel-transported by translating them to an infinitesimally displaced fibre and there projecting them perpendicularly. This choice of connection, while convenient, is nevertheless a matter of convention, and is not required by physical considerations.

Finally, we note that we could, if we wished, impose the single-valuedness condition more directly by taking \( n \)-particle configuration space to be the identified configuration space \( \mathcal{C}_n = C_n/S_n \) consisting of (unordered) sets \( X = \{ r, s, \ldots, t \} \) of \( n \) distinct points in \( \mathbb{R}^3 \). (This is the point of view taken by Leinaas and Myrheim (1977).) Then wavefunctions would become functions of \( X \), or, more precisely, sections of an \( n \)-spin bundle over the identified configuration space \( \mathcal{C}_n \). However, such a reformulation involves some additional mathematical complication, and, for this reason, we will confine our consideration of it to the following informal remarks.

The complication is due to the fact that there are no global Euclidean coordinates on \( \mathcal{C}_n \); it is no longer sensible to refer to position, spin and momentum operators for a particular particle. In place of individual momenta, for example, one must introduce generalised momentum operators, which are related to covariant derivatives along smooth vector fields on \( \mathcal{C}_n \). A formulation in terms of \( \mathcal{C}_n \) does have some attractive aspects, though. Parallel transport between permuted fibres in \( C_n \) becomes transport from a single fibre to itself around a non-contractible closed path in \( \mathcal{C}_n \). In this way, the statistics of the particles is reflected in the monodromy of the flat connection, a topological property of the bundle.

### 3 Spin bundles \( SU(2n) \) representations

In this section we describe the construction of \( n \)-spin bundles from representations of \( SU(2n) \). The construction is based on a connection between \( SU(2n) \) and the spin-statistics group \( \Sigma(n) \) (Section 3.1), which associates a representation \( \Delta^f \) of \( \Sigma(n) \) to an irreducible representation \( \Gamma^f \) of \( SU(2n) \) (Section 3.2). In general, the representation \( \Delta^f \) is reducible. \( n \)-spin bundles are constructed from the representations \( \Gamma^f \) and \( \Delta^f \) and an \( S_n \)-equivariant map from \( C_n \) to \( SU(n)/T(n) \) (Section 3.3).

Whether \( \Gamma^f \) determines a spin-statistics relation is discussed in Section 3.4. The question is related to the decomposition of \( \Delta^f \) into its irreducible components. A definite statistics for a given value of spin requires that \( \Delta^f \) should contain only one irreducible representation of \( \Sigma(n) \) with that spin. This is the case for the completely symmetric representations, which correspond to the construction in BR. The general case is discussed in Section 4.

#### 3.1 The spin-statistics group and \( SU(2n) \)

Consider \( SU(2n) \), the group of \( 2n \)-dimensional unitary matrices of unit determinant. \( SU(2)^n \) may be identified as a subgroup of \( SU(2) \), with \( U = (u_1, \ldots, u_n) \in SU(2)^n \).
identified with the matrix

\[ U = \begin{pmatrix} u_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & u_n \end{pmatrix}. \] (3.1)

(For simplicity, we’ll use the same symbol, in this instance \( U \), for both an element of \( SU(2)^n \) and for the corresponding matrix in \( SU(2n) \), and will do the same for some other subgroups of \( SU(2n) \) to be introduced below. Taken in context this usage should not introduce any ambiguity). Similarly, \( SU(n) \), the group of \( n \)-dimensional unitary matrices with unit determinant, may be identified with a subgroup of \( SU(2n) \), with \( g \in SU(n) \), with components denoted by \( g_{rt}, 1 \leq r, t \leq n \), identified with the \( SU(2n) \)-matrix

\[ g = \begin{pmatrix} g_{11}I_2 & \cdots & g_{1n}I_2 \\ \vdots & \ddots & \vdots \\ g_{n1}I_2 & \cdots & g_{nn}I_2 \end{pmatrix}. \] (3.2)

Finally, we let \( T(n) \) denote the subgroup of diagonal matrices in \( SU(n) \). From (3.2), \( T(n) \) may be identified as the subgroup of \( SU(2n) \) consisting of diagonal matrices of the form

\[ t(\Theta) = \begin{pmatrix} e^{i\theta_1}I_2 & 0 \\ 0 & e^{i\theta_n}I_2 \end{pmatrix} \] (3.3)

where \( \Theta = (\theta_1, \ldots, \theta_n) \) is an \( n \)-tuple of phases satisfying

\[ e^{i\theta_1} \cdots e^{i\theta_n} = 1. \] (3.4)

Note that \( SU(2)^n \cap T(n) \) is just the subgroup \( \text{Nul}(n) \) of null rotations, which consists of \( SU(2n) \)-matrices of the form

\[ U_0 = \begin{pmatrix} (-1)^{e_1}I_2 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & (-1)^{e_n}I_2 \end{pmatrix}, \] (3.5)

where \( (-1)^{e_1} \cdots (-1)^{e_n} = 1. \)

Let \( N(n) \subset SU(n) \) denote the normaliser of \( T(n) \) in \( SU(n) \), ie the subgroup of \( SU(n) \) which leaves \( T(n) \) invariant under conjugation. It is straightforward to show that elements of \( N(n) \) may be parameterised by a permutation \( \sigma \in S_n \) and an \( n \)-tuple of phases \( \Phi = (\phi_1, \ldots, \phi_n) \) satisfying

\[ e^{i\phi_1} \cdots e^{i\phi_n} = \text{sgn}(\sigma), \] (3.6)
(here \(\text{sgn}(\sigma)\) denotes the parity of \(\sigma\)), and are of the form
\[
y_{rt}(\sigma, \Phi) = \delta_{r,\sigma(t)} e^{i\phi_t}. \quad (3.7)
\]
Multiplication in \(N(n)\) is given by \(y(\sigma, \Phi) y(\sigma', \Phi') = y(\sigma\sigma', \sigma'^{-1} \cdot \Phi + \Phi')\), so that, formally, \(N(n)\) may be regarded as the semidirect product, \(S_n \rtimes T(n)\). The quotient \(N(n)/T(n)\), the Weyl group of \(SU(n)\), is isomorphic to \(S_n\).

Let \(M(n)\) denote the normaliser of \(T(n)\) in \(SU(2n)\), ie the subgroup of \(SU(2n)\) which leaves \(T(n)\) invariant under conjugation. Clearly \(M(n)\) contains \(N(n)\) as a subgroup. \(M(n)\) also contains the \(SU(2n)\)-centraliser of \(T(n)\), denoted by \(Z(n)\), ie the subgroup of \(SU(2n)\) whose elements commute with all elements of \(T(n)\). It is straightforward to show that elements of \(Z(n)\) are of the form \(U t(\Theta)\), where \(U \in SU(2)\) and \(t(\Theta) \in T(n)\). It is then straightforward to show that elements of \(M(n)\) can be expressed as products of elements of \(Z(n)\) and \(N(n)\), and thus are of the form
\[
x(U, \sigma, \Phi) = U y_{rt}(\sigma, \Phi), \quad (3.8)
\]
where the phases \(\Phi\) satisfy (3.6). Multiplication in \(M(n)\) is given by
\[
x(U, \sigma, \Phi) x(U', \sigma', \Phi') = x(U\sigma \cdot U', \sigma\sigma', \sigma'^{-1} \cdot \Phi + \Phi'). \quad (3.9)
\]

The parameterisation \(x(U, \sigma, \Phi)\) of (3.8) is not unique. If \(U\) is replaced by \(UU_0\), with \(U_0 \in \text{Nul}(n)\) given by (3.5), and \(\Phi\) is replaced by \(\Phi'\), where \(\phi'_j = \phi_j + e_j \pi\), then \(x(U, \sigma, \Phi)\) is unchanged. In this way, we see that, formally, \(M(n)\) is isomorphic to \(SU(2)^n \rtimes N(n)/\text{Nul}(n)\).

From these considerations, it follows that the quotient \(M(n)/T(n)\) is isomorphic to the spin-statistics group, ie
\[
\Sigma(n) = \text{Spn}(n) \rtimes S_n \cong M(n)/T(n). \quad (3.10)
\]
The isomorphism is given explicitly by
\[
(\overline{U}, \sigma) \mapsto x(U, \sigma, \Phi) T(n), \quad (3.11)
\]
where \(x(U, \sigma, \Phi) T(n)\) denotes a coset in \(M(n)/T(n)\). This association between \(SU(2n)\) and the spin-statistics group is the basis of the constructions to follow.

### 3.2 Representations of the spin-statistics group from representations of \(SU(2n)\)

Let \(\Gamma^\mathbf{f}\) denote a unitary irreducible representation of \(SU(2n)\), labeled by a Young tableau \(\mathbf{f} = (f_1, \ldots, f_{2n})\) of up to \(2n\) rows (in fact, the last row of \(\mathbf{f}\) may be taken to be empty). Let \(\mathcal{V}\) denote the hermitian inner product space on which \(\Gamma^\mathbf{f}\) acts. (Of course, \(\mathcal{V}\) depends on the choice of representation, but to simplify the notation we will not indicate this explicitly.)
Under the restriction of $\Gamma^f$ to $T(n)$, $\mathcal{V}$ may be decomposed into a direct sum of orthogonal subspaces, $\mathcal{V}^K$, on which $\Gamma^f(t(\Theta))$ is represented by the phase factor $\exp(iK \cdot \Theta)$. Here $K = (k_1, \ldots, k_n)$ is an $n$-tuple of integers, and $K \cdot \Theta = \sum_j k_j \theta_j$. The subspace $\mathcal{V}^0$, corresponding to $K = (0, \ldots, 0)$, consists of vectors which are invariant under $\Gamma^f(T(n))$.

Let us determine the action of $M(n)$ on the subspaces $\mathcal{V}^K$. Given $x(U, \Phi, \sigma) \in M(n)$, it follows from (3.9) that

$$\Gamma^f(x(U, \Phi, \Phi)) \cdot \mathcal{V}^K = \Gamma^f(x(U, \Phi, \Phi)) \Gamma^f(t(\sigma^{-1} \cdot \Theta)) \cdot \mathcal{V}^K = e^{i(K \cdot (\sigma^{-1} \cdot \Theta))} \Gamma^f(x(U, \Phi, \Phi)) \cdot \mathcal{V}^K = e^{i(\sigma \cdot K) \cdot \Theta} \Gamma^f(x(U, \sigma, \Phi)) \cdot \mathcal{V}^K.$$ (3.12)

Thus, under the action of $M(n)$, the subspaces $\mathcal{V}^K$ are mapped into one another according to

$$\Gamma^f(x(U, \Phi, \sigma)) \cdot \mathcal{V}^K = \mathcal{V}^{\sigma \cdot K}. \quad (3.13)$$

It follows from (3.13) that $\mathcal{V}^0$ is invariant under $M(n)$. Therefore $\Gamma^f$ restricts to a representation of $M(n)$ on $\mathcal{V}^0$. Since $T(n) \subset M(n)$ belongs to the kernel of this representation (as $T(n)$ leaves vectors in $\mathcal{V}^0$ invariant), $\Gamma^f(M(n))$ restricts to a representation of the quotient $M(n)/T(n)$, which we denote by $\Delta^f$. Since $M(n)/T(n) \cong \Sigma(n)$ (cf (3.10)), $\Delta^f$ is in fact a representation of the spin-statistics group. From (3.11), $\Delta^f$ is given by

$$\Delta^f(\mathcal{U}, \sigma) = \Gamma^f(x(U, \Phi, \sigma)). \quad (3.14)$$

In general, the representation $\Delta^f$ of $\Sigma(n)$ is reducible. Let $\nu(f, s\lambda)$ denote the multiplicity with which the irreducible representation $Q^{s\lambda}$, given by eq:Q(U), appears in the decomposition of $\Delta^f$. This multiplicity will play a central role in what follows.

### 3.3 Construction of $n$-spin bundles

Let $\Xi : C_n \rightarrow SU(n)/T(n)$ denote a smooth map from $n$-particle configuration space $C_n$ to the coset space $SU(n)/T(n)$. Such a map may be represented by $g(R)$, an $SU(n)$-valued function on $C_n$ which is smooth up to right multiplication by an element of $T(n)$. (That is, discontinuities in $g(R)$ can be removed locally by multiplying on the right by a discontinuous $T(n)$-valued function.) The symmetric group $S_n$ acts on $C_n$ as permutations (ie, $R \mapsto \sigma \cdot R$) and on $SU(n)/T(n)$ as the Weyl group (ie, for $y(\sigma, \Phi) \in N(n)$, $g T(n) \mapsto gy^{-1}(\sigma, \Phi) T(n)$). $\Xi$ is said to be equivariant with respect to $S_n$ if, for all $\sigma \in S_n$, $\Xi \circ \sigma = \sigma \circ \Xi$. In terms of $g(R)$, $S_n$-equivariance is equivalent to

$$g(\sigma \cdot R) T(n) = g(R)y^{-1}(\sigma, \Phi) T(n). \quad (3.15)$$

Atiyah (2000) has shown that there exist continuous (and therefore smooth) $S_n$-equivariant maps from $C_n$ to $SU(n)/T(n)$. Let $g(R)$ represent any such equivariant map (the results which follow do not depend on the particular choice of $g(R)$).

As in Section 3.2 let $\Gamma^f$ be an irreducible representation of $SU(2n)$, and $\Delta^f$ the associated representation of the spin-statistics group $\Sigma(n)$. Suppose $\nu(f, s\lambda) > 0$, ie $\mathcal{V}^0$ contains a
subspace, which we denote by \( V^{s\lambda} \), which transforms under \( \Delta^f \) according to the irreducible representation \( Q^{s\lambda} \) of \( \Sigma(n) \). We may then construct an \( n \)-spin-\( s \) bundle \( \mathcal{E}^{s\lambda} \) as follows. The fibres \( \mathcal{E}^{s\lambda}_R \subset V \) are given by

\[
\mathcal{E}^{s\lambda}_R = \Gamma^f(g(R)) \cdot \mathcal{V}^{s\lambda}.
\]  

(3.16)

Since \( g(R) \) is smooth up to right multiplication by a \( T(n) \)-valued function and \( \mathcal{V}^{s\lambda} \) is invariant under \( \Gamma^f(T(n)) \), it follows that \( \mathcal{E}^{s\lambda}_R \) depends smoothly on \( R \).

Let us verify that \( \mathcal{E}^{s\lambda} \) has the properties A) – D) listed in Section 2.2. For A), we define the representation \( L^{\sigma}_R \) on \( \mathcal{E}^{s\lambda}_R \) by

\[
L^{\sigma}_R(U,\sigma) = \Gamma^f(g(R)) \Delta^f(U,\sigma) \Gamma^f(g(R)),
\]  

(3.17)

where \( \Delta^f(U,\sigma) \) is the representation of \( \Sigma(n) \) given by (3.14). By assumption, \( \Delta^f \) is unitarily equivalent to \( Q^{s\lambda} \) on \( \mathcal{V}^{s\lambda} \), so it is evident from (3.17) that \( L^{\sigma}_R \) is unitarily equivalent to \( Q^{s\lambda} \) for all \( R \).

Since the right-hand side of (3.17) is unchanged if \( g(R) \) is multiplied on the right by a (possibly discontinuous) \( T(n) \)-valued function, it is clear that \( L^{\sigma}_R \) depends smoothly on \( R \).

For B), from the definition (3.16) and the equivariance property (3.15), we have that

\[
\mathcal{E}^{s\lambda}_{\sigma \cdot R} = \Gamma^f(g(\sigma \cdot R)) \Delta^f(\mathcal{U},\sigma) \Gamma^f(g(R)) \mathcal{V}^{s\lambda}.
\]  

(3.18)

Thus the fibres at permuted configurations are the same.

For C), we define the unitary maps \( T^{R' \leftarrow R}_{\sigma} \) describing flat parallel transport between the fibres at \( R \) and \( R' \) by

\[
T^{R' \leftarrow R}_{\sigma} = \Gamma^f(g(R')) \Gamma^f(g(R)).
\]  

(3.19)

The right-hand side is unchanged if \( g(R) \) is multiplied on the right by a \( T(n) \)-valued function, so \( T^{R' \leftarrow R}_{\sigma} \) is well defined and depends smoothly on \( R \) and \( R' \). The composition law (2.30) is easily verified. Compatibility with the representations \( L^{\sigma}_R \) (cf (2.31)) follows from the definition of \( L^{\sigma}_R \) in (3.17) and the representation property of \( \Gamma^f \).

For D), from (3.19), parallel transport \( T_{\sigma \cdot R \leftarrow R} \) between permuted fibres \( R \) and \( \sigma \cdot R \) is given by \( \Gamma^f(g(\sigma \cdot R)) \Gamma^f(g(R)) \). The equivariance condition (3.15) implies this is equal to \( \Gamma^f(g(R)) \Gamma^f(y^{-1}(\sigma, \Phi)) \Gamma^f(g(R)) \). From (3.17), the condition (2.32) follows.

### 3.4 Spin-statistics relations from \( SU(2n) \) representations?

Given an irreducible representation \( \Gamma^f \) of \( SU(2n) \), the preceding construction determines the statistics for spin \( s \) unambiguously, provided that there is just one representation \( Q^{s\lambda} \) with spin \( s \) in the decomposition of \( \Delta^f \); equivalently, given \( f \) and \( s \), the multiplicity \( \nu(f, s, \lambda) \) should vanish for all but one \( \lambda \).

This is the case for the completely symmetric representations of \( SU(2n) \). The completely symmetric representations correspond to Young tableaux with a single row. Let \( \Gamma^d \)
denote the representation for a single row of $d$ boxes. $\Gamma^d$ may be realised on the space $V$ of homogeneous polynomials of degree $d$ in $2n$ variables, $z = (z_1, \ldots, z_{2n}) \in \mathbb{C}^{2n}$, and is given by $\Gamma^d(f) \cdot P(z) = P(f^{-1} \cdot z)$ for $f \in SU(2n)$. $\Gamma^d$ is unitary with respect to the inner product

$$
\langle P, Q \rangle = \int_{\mathbb{C}^{2n}} e^{-z^* \cdot z/2} P^*(z) Q(z) \, d^{4n}z
$$

on $V$.

An orthogonal basis for $V$ is given by the monomials

$$
\prod_{r=1}^{n} z_{a_r}^{s_r} z_{b_r}^{s_r},
$$

where the sum of the exponents $a_j$ and $b_j$ is given by $d$. The subspace $V^0$, whose vectors are invariant under $T(n)$, consists of polynomials which are invariant under $z \mapsto t^{-1}(\Theta) \cdot z$, where

$$
t^{-1}(\Theta) \cdot z = (e^{-i\theta_1} z_1, e^{-i\theta_2} z_2, e^{-i\theta_{2n-1}} z_{2n-1-1}, e^{-i\theta_{2n-2}} z_{2n})
$$

Such polynomials are linear combinations of the monomials for which $a_r + b_r$ is independent of $r$, so that $a_r + b_r = d/n$. Thus, for $V^0$ to be nontrivial, $d$ must be divisible by $n$.

Let us assume this is the case, so that $d/n$ is integral. Then $s = d/2n$ is either integral or half-odd-integer. Let $m_r = a_r - s$. Then $a_r = s + m_r$ and $b_r = s - m_r$. It follows that $V^0$ is spanned by the $(2s + 1)^n$ monomials

$$
\prod_{r=1}^{n} z_{a_r}^{s + m_r} z_{b_r}^{s - m_r},
$$

where $-s \leq m_r \leq s$ and $s \pm m_r$ is integral. Under $Sp(n)$, $V^0$ transforms as $n$ spin-$s$ spinors. As the dimension of $V^0$ is $(2s + 1)^n$, it follows that there is a single irreducible representation $Q^{s\lambda}$ with multiplicity one in the decomposition of $\Delta^d$, and that $d_\lambda = 1$, ie $\lambda$ is either the completely symmetric or the completely antisymmetric representation of $S_n$.

$\lambda$ may be determined by considering the action of permutations on an element of $V^0$, for example

$$
P_s(z) \overset{\text{def}}{=} z_{2}^{2s} z_{4}^{2s} \cdots z_{2n}^{2s}.
$$

From (3.7), under $y^{-1}(\sigma, \Phi)$, the even components transform as $z_{2j} \mapsto e^{-i\phi_{2j}} z_{2\sigma(j)}$. Thus, under $y(\sigma, \Phi)$, $P_s(z)$ is multiplied by the phase factor $e^{2is\phi_1} \cdots e^{2is\phi_n}$. From (3.6), this phase factor is just $\text{sgn}^{2s}(\sigma)$. Thus, the completely symmetric representations of $SU(2n)$ of dimension $d = 2ns$ lead to $n$-spin-bundles with spin $s$ and bose or fermi statistics according to the parity of $2s$, in accord with the physically correct spin-statistics relation. This is precisely the result obtained in BR.
4 Calculation of the multiplicities

Given an arbitrary representation $\Gamma^f$ of $SU(2n)$, we wish to determine for which spins an $n$-spin-$s$ bundle can be constructed, and, for those spins, whether the constructions have a definite type of statistics. To this end, we calculate the multiplicities $\nu(f, s\lambda)$ with which the irreducible representation $Q^{s\lambda}$ of $\Sigma(n)$ appears in the representation $\Delta^f$. This is given by the following integral:

$$\nu(f, s\lambda) = \int_{\Sigma(n)} d\mu_{\Sigma(n)} X^{s\lambda^*}(U, \sigma) X^f(U, \sigma). \quad (4.1)$$

Here $X^{s\lambda}$ and $X^f$ denote the characters of the $\Sigma(n)$-representations $Q^{s\lambda}$ and $\Delta^f$ respectively, and $d\mu_{\Sigma(n)}$ denotes the normalised Haar measure on $\Sigma(n)$. Before evaluating the integral (4.1) in Section 4.2, we first introduce some background material and notation.

4.1 Preliminaries

4.1.1 Character formula for $U(k)$

Irreducible representations of the $k$-dimensional unitary group $U(k)$ are labeled by Young tableaux $\alpha = (\alpha^1, \ldots, \alpha^k)$ of $k$ rows (some of which may be empty), where $\alpha^1 \geq \cdots \geq \alpha^k \geq 0$ specify the number of boxes in each row. Let

$$|\alpha| = \alpha^1 + \cdots + \alpha^k \quad (4.2)$$

denote the number of boxes in the tableau. Denote the eigenvalues of matrices in $U(k)$ by $(\exp i\xi_1, \ldots, \exp i\xi_k)$, and their eigenphases by $\xi = (\xi_1, \ldots, \xi_k)$. The characters $K^\alpha_k$ of the irreducible representations are functions of $\xi$, and are given by the Weyl character formula,

$$K^\alpha_k(\xi) = \begin{vmatrix} e^{i(\alpha^1+k-1)\xi_1} & e^{i(\alpha^1+k-1)\xi_2} & \cdots & e^{i(\alpha^1+k-1)\xi_k} \\ e^{i(\alpha^2+k-2)\xi_1} & e^{i(\alpha^2+k-2)\xi_2} & \cdots & e^{i(\alpha^2+k-2)\xi_k} \\ \vdots & \vdots & \ddots & \vdots \\ e^{i(\alpha^k)\xi_1} & e^{i(\alpha^k)\xi_2} & \cdots & e^{i(\alpha^k)\xi_k} \\ \end{vmatrix} \quad (4.3)$$

Irreducible representations of $SU(k)$ are obtained by restriction. On $SU(k)$, the representation $\alpha + r$, which is obtained by adding $r$ columns of $k$ boxes to $\alpha$, is equivalent to the representation $\alpha$. $SU(k)$ representations can be uniquely labeled by Young tableaux of $k - 1$ rows (some of which may be empty).
4.1.2 The Littlewood-Richardson theorem

Given an irreducible representation $\gamma$ of $U(k+l)$, its restriction to the subgroup $U(k) \times U(l)$ is, in general, reducible, and may be decomposed into a sum of tensor products of irreducible representations $\alpha$ and $\beta$ of $U(k)$ and $U(l)$, respectively. In terms of characters, this decomposition takes the form

$$K^\gamma_{k+l}(\xi, \eta) = \sum_{\alpha, \beta} Y_{\alpha \beta}^\gamma K^\alpha_k(\xi) K^\beta_l(\eta), \quad (4.4)$$

where $\xi$ and $\eta$ denote the eigenphases of elements of $U(k)$ and $U(l)$, respectively. The coefficients $Y_{\alpha \beta}^\gamma$ in the decomposition are given by the Littlewood-Richardson theorem, according to which $Y_{\alpha \beta}^\gamma$ is the number of times that the tableau $\gamma$ can be constructed from $\alpha$ and $\beta$ by the following procedure: Boxes from the first row of $\beta$ are added to $\alpha$ so as to produce a new tableau, with the condition that no two boxes are placed in the same column of the new tableau. This is repeated with the second row of $\beta$, with the additional condition that, on counting added boxes in the new tableau column-wise from right to left, and row-wise from top to bottom, the number of added boxes from the first row of $\beta$ must always be greater than or equal to the number of added boxes from the second row. The procedure is continued for the other rows until all boxes from $\beta$ have been added to $\alpha$. It is evident that $\gamma$ can be constructed in this way only if the number of boxes in $\gamma$ equals the number of boxes in $\alpha$ and $\beta$ together; that is, $Y_{\alpha \beta}^\gamma$ vanishes unless $|\gamma| = |\alpha| + |\beta|$.

Eq. (4.4) generalises to the decomposition of irreducible representations $\gamma$ of $U(k_1 + \cdots + k_c)$ restricted to the subgroup $U(k_1) \times \cdots \times U(k_n)$, as follows:

$$K^\gamma_{k_1 + \cdots + k_c}(\xi_1, \cdots, \xi_c) = \sum_{\alpha_1, \cdots, \alpha_c} Y_{\alpha_1,\cdots,\alpha_c}^\gamma \prod_{b=1}^c K_{k_b}^{\alpha_b}(\xi_b). \quad (4.5)$$

Here the $\alpha_b$’s are tableaux labeling irreducible representations of $U(k_b)$, and the $\xi_b$ denote the eigenphases of elements of $U(k_b)$. The $c$-fold coefficients $Y_{\alpha_1,\cdots,\alpha_c}^\gamma$ may be obtained from the two-fold coefficients $Y_{\alpha \beta}^\gamma$ by performing the $c$-fold decomposition inductively.

With (4.5), the $(k_1 + \cdots + k_c)$-fold determinants in the Weyl character formula (4.3) are reduced to sums of products of ratios of smaller, $k_b$-fold determinants. However, this simplification comes at a price; the Littlewood-Richardson coefficients are not easily calculated, and closed-form expressions for them are not known.

The original statement of the Littlewood-Richardson theorem appears in [14]. A modern version with proof may be found in Macdonald [15]. The application of the Littlewood-Richardson theorem to the unitary groups is discussed by Hagen and MacFarlane [8] and Itzykson and Nauenberg [11]. A more detailed discussion of the rules for multiplying Young tableau can be found in Hamermesh [9].

4.1.3 Characters for $U(2)$ and $SU(2)$

We will need some results and notation particular to the groups $SU(2)$ and $U(2)$. Irreducible characters of $SU(2)$ are denoted by $\chi_{SU(2)}(s \psi)$, where $s$ is the spin and $e^{\pm i\psi}$ denotes...
the eigenvalues of elements of \( SU(2) \), and are given by
\[
\chi_{SU(2)}^s(\psi) = \frac{\sin((2s+1)\psi)}{\sin(\psi)}.
\] (4.6)

Irreducible representations of \( U(2) \) are labeled by tableau \( \alpha = (\alpha^1, \alpha^2) \) of two rows. The \( U(2) \)-characters \( K_2^\alpha(\xi_1, \xi_2) \) are related to the \( SU(2) \)-characters \( \chi_{SU(2)}^s(\psi) \) by
\[
K_2^\alpha(\psi + \theta, -\psi + \theta) = e^{i\langle \alpha | \theta \rangle} \chi_{SU(2)}^S(\alpha)(\psi),
\] (4.7)
where
\[
S(\alpha) = (\alpha^1 - \alpha^2)/2
\] (4.8)
denotes the value of spin associated with the \( U(2) \)-representation \( \alpha \).

The Clebsch-Gordan coefficients \( C(s_1, s_2, s_3) \) for \( SU(2) \) are defined by
\[
C(s_1, s_2, s_3) = \frac{1}{\pi} \int_0^{2\pi} d\psi \sin^2(\psi) \chi_{SU(2)}^{s_1}(\psi) \chi_{SU(2)}^{s_2}(\psi) \chi_{SU(2)}^{s_3}(\psi)
\] (4.9)
(note that \( \sin^2(\psi/2) \) is the Haar measure with respect to classes of \( SU(2) \)), and give the multiplicity of the trivial representation in the decomposition of the tensor product of three \( SU(2) \)-representations with spins \( s_1, s_2 \) and \( s_3 \). It is an elementary result that \( C(s_1, s_2, s_3) \) equals one if \(|s_1 - s_2| \leq s_3 \leq s_1 + s_2\), and is zero otherwise. We define the \( r \)-fold Clebsch-Gordan coefficients by
\[
C(s_1, \ldots, s_r) = \frac{1}{\pi} \int_0^{2\pi} d\psi \sin^2(\psi) \chi_{SU(2)}^{s_1}(\psi) \cdots \chi_{SU(2)}^{s_r}(\psi).
\] (4.10)
These are given inductively by
\[
C(s_1, \ldots, s_r, s_{r+1}) = \sum_s C(s_1, \ldots, s_{r-1}, s) C(s, s_r, s_{r+1}).
\] (4.11)

4.1.4 Cycle decomposition of permutations

The following notations will be used for permutations. Let \( \sigma \in S_n \). Denote the factorisation of \( \sigma \) into disjoint cycles by
\[
\sigma = \hat{\sigma}_1 \cdots \hat{\sigma}_{c(\sigma)},
\] (4.12)
where \( c(\sigma) \) denotes the number of cycles in the factorisation. Denote the length of a cycle in the decomposition, say \( \hat{\sigma}_b \), by \( |\hat{\sigma}_b| \).

Let \( \sigma \in S_n \) and \( U = (u_1, \ldots, u_n) \in SU(2)^n \). For each cycle \( \hat{\sigma}_b \) in the factorisation of \( \sigma \), let \( \hat{u}_b \) denote the product of the corresponding components of \( U \), taken in the reverse order. That is, if \( \hat{\sigma}_b = (jk \cdots l) \), then
\[
\hat{u}_b = u_l \cdots u_k u_j.
\] (4.13)
Similarly, given an \( n \)-tuple of phases, \( \Theta = (\theta_1, \ldots, \theta_n) \), let
\[
\hat{\theta}_b = \theta_l + \cdots + \theta_k + \theta_j.
\] (4.14)
Clearly
\[
\theta_1 + \cdots + \theta_n = \hat{\theta}_1 + \cdots + \hat{\theta}_{c(\sigma)}.
\] (4.15)
4.2 Evaluation of the integral

To evaluate the character integral (4.1), it will be convenient to regard $X^{s\lambda}$ and $X^f$ as characters on $SU(2)^n \rtimes \Sigma(n)$, i.e. as functions of $U$ and $\sigma$ rather than $\overline{U}$ and $\sigma$. Then

$$\nu(f, s\lambda) = \frac{1}{n!} \sum_{\sigma \in S_n} \int_{SU(2)^n} du_1 \cdots du_n X^{s\lambda}(U, \sigma)^* X^f(U, \sigma),$$

(4.16)

where $du_j$ denotes the normalised Haar measure on $SU(2)$.

The character $X^{s\lambda}(U, \sigma)$ may be evaluated as follows. From (2.10) and (2.11),

$$X^{s\lambda}(U, \sigma) = \sum_{a=1}^{d_\lambda} \sum_M \langle M, a | Q^{s\lambda}(U, \sigma) | M, a \rangle$$

$$= \left( \sum_{a=1}^{d_\lambda} \Lambda_{a, a}(\sigma) \right) \left( \sum_M D^s_{m_1(1), m_1}(u_1) \cdots D^s_{m_{\sigma(n)}, m_n}(u_n) \right).$$

(4.17)

The sum over $a$ yields $\chi^\lambda_{S_n}(\sigma)$, the character of the $S_n$-representation $\Lambda^\lambda$. The sum over $M$ factorises into a product over the disjoint cycles $\hat{\sigma}_b$ of $\sigma$, and yields

$$\text{Tr} D^s(\hat{u}_1) \cdots \text{Tr} D^s(\hat{u}_{c(\sigma)}),$$

(4.18)

where $\hat{u}_b \in SU(2)$ is given by (4.13). Let $e^{\pm i\xi_b}$ denote the eigenvalues of $\hat{u}_b$. Then

$$X^{s\lambda}(U, \sigma) = \chi^\lambda_{S_n}(\sigma) \prod_{b=1}^{c(\sigma)} \chi^s_{SU(2)}(\xi_b).$$

(4.19)

We note that $X^{s\lambda}(U, \sigma)$ is real, since the characters of $S_n$ and $SU(2)$ are real.

The character $X^f(U, \sigma)$ in (4.10) may be expressed as

$$X^f(U, \sigma) = \text{Tr} \left( \Gamma^f(x(U, \sigma, \Phi)) P^0 \right).$$

(4.20)

Here the trace is taken over the carrier space $\mathcal{V}$ of the representation $\Gamma^f$, $\Phi$ denotes phases satisfying $e^{i\phi_1} \cdots e^{i\phi_n} = \text{sgn}(\sigma)$, and $P^0$ denotes the hermitian projection onto the subspace $\mathcal{V}^0$ given by

$$P^0 = \frac{1}{(2\pi)^{n-1}} \int d^{n-1} \Theta' \Gamma^f(t(\Theta')),$$

(4.21)

where the $\Theta'$-integral is taken over $0 \leq \theta'_j \leq 2\pi$ subject to the condition that $e^{i\theta'_1} \cdots e^{i\theta'_n} = 1$. Substituting (4.21) into (4.20) we get that

$$X^f(U, \sigma) = \frac{1}{(2\pi)^{n-1}} \int d^{n-1} \Theta' \text{Tr} \Gamma^f(x(U, \sigma, \Theta' + \Phi))$$

$$= \frac{1}{(2\pi)^{n-1}} \int d^{n-1} \Theta \text{Tr} \Gamma^f(x(U, \sigma, \Theta)),$$

(4.22)
where the integral over $\Theta = (\theta_1, \ldots, \theta_n)$ in the last expression is restricted to $e^{i\theta_1} \cdots e^{i\theta_n} = \text{sgn}(\sigma)$. It is convenient to incorporate this restriction using the identity
\[
\frac{1}{(2\pi)^{n-1}} \int d^{n-1}\Theta = \sum_{q = -\infty}^{\infty} (\text{sgn}\sigma)^q \frac{1}{(2\pi)^n} \int e^{i(q\theta_1 + \cdots + \theta_n)} d\Theta. \tag{4.23}
\]

We note that as $\Theta$ on the right-hand side of (4.23) is unconstrained, $x(U, \sigma, \Theta)$ is, in general, an element of $U(2n)$ rather than $SU(2n)$. Let $\mu$ denote the eigenphases of $x(U, \sigma, \Theta)$. Then $\text{Tr} \Gamma^f(x(U, \sigma, \Theta)) = K^f_{2n}(\mu)$, and (4.22) becomes
\[
X^f(U, \sigma) = \sum_{q = -\infty}^{\infty} (\text{sgn}\sigma)^q \frac{1}{(2\pi)^n} \int_0^{2\pi} d\Theta e^{-i(q\theta_1 + \cdots + \theta_n)} K^f_{2n}(\mu). \tag{4.24}
\]

To determine the eigenphases $\mu$ of $x(U, \sigma, \Theta)$, it is convenient to represent vectors in $C^{2n}$ as linear combinations of terms $|v_j\rangle \otimes |j\rangle$, where $|v_j\rangle \in C^2$ and $|j\rangle$ is an orthonormal basis for $C^n$. From (3.7) and (3.8), the action of $x(U, \sigma, \Theta)$ is then given by
\[
x(U, \sigma, \Theta) \sum_{j=1}^{n} |\xi_j\rangle \otimes |j\rangle = \sum_{j=1}^{n} e^{i\theta_j} (u_{\sigma(j)}|\xi_j\rangle) \otimes |\sigma(j)\rangle. \tag{4.25}
\]

Let $\hat{\sigma}_b = (jk \cdots l)$ be a cycle in $\sigma$, and, as above, let $e^{\pm i\hat{\theta}_b}$ denote the eigenvalues of $\hat{u}_b$. Let $|\pm w_b\rangle \in C^2$ denote the associated eigenvectors of $\hat{u}_b$. It is readily verified that $|\pm w_b\rangle \otimes |l\rangle$ are eigenvectors of $x^m(U, \sigma, \Theta)$, with eigenvalues $e^{im(\hat{\xi}_b + \hat{\theta}_b)/|\sigma_b|}$, if and only if $m$ is a multiple of $|\sigma_b|$. In general, if $|v\rangle$ is an eigenvector of some positive integer power $|\hat{\sigma}_b\rangle$ of a matrix $M$, with $\rho$ the associated eigenvalue of $M|\hat{\sigma}_b\rangle$, and if $|v\rangle$ is not an eigenvector of any smaller positive power of $M$, then $M$ has eigenvalues $e^{2\pi i p/|\sigma_b|} \rho^{1/|\sigma_b|}$, where $p = 1, \ldots, |\sigma_b|$. From these considerations we may deduce that to each cycle $\hat{\sigma}_b$ in $\sigma$ are associated $2|\hat{\sigma}_b|$ eigenphases of $x(U, \sigma, \Theta)$, denoted by $\eta_b = (\eta_{b,1}, \ldots, \eta_{b,|\hat{\sigma}_b|})$ and given explicitly by
\[
\eta_{b,p} = \left(\frac{+\hat{\xi}_b + \hat{\theta}_b + 2\pi p}{|\hat{\sigma}_b|}, \frac{-\hat{\xi}_b + \hat{\theta}_b + 2\pi p}{|\hat{\sigma}_b|}\right), \quad p = 1, \ldots, |\hat{\sigma}_b|. \tag{4.26}
\]

The full set of eigenvalues of $x(U, \sigma, \Theta)$ is
\[
\mu = (\eta_1, \ldots, \eta_{c(\sigma)}) \tag{4.27}
\]

From (4.27) it is apparent that $x(U, \sigma, \Theta) \in U(2n)$ is unitarily equivalent to the element of $U(2|\sigma_1|) \times \cdots \times U(2|\sigma_{c(\sigma)}|)$ with eigenphases $\eta_1, \ldots, \eta_{c(\sigma)}$ for the factors. From the Littlewood-Richardson formula (4.5), the character $K^f_{2n}(\mu)$ in (4.24) is given by
\[
K^f_{2n}(\mu) = K^f_{2n}(\eta_1, \ldots, \eta_{c(\sigma)}) = \sum_{\beta_1, \ldots, \beta_{c(\sigma)} \geq 0} Y^f_{\beta_1, \ldots, \beta_{c(\sigma)}} \prod_{b=1}^{c(\sigma)} K^f_{|\sigma_b|}(\eta_b). \tag{4.28}
\]
where the $\beta_b$'s are tableaux labeling representations of $U(2|\hat{\sigma}_b|)$. As noted in Section 4.1.2, the sum in (4.28) may restricted to those $\beta_b$ satisfying

$$\sum_{b=1}^{c(\sigma)} |\beta_b| = |f|.$$  

(4.29)

From (4.26), the characters $K^{\beta_b}_{|\hat{\sigma}_b|}(\eta_b)$ can themselves be expressed as a product of $U(2)$-characters by applying the Littlewood-Richardson theorem once more, as follows:

$$K^{\beta_b}_{|\hat{\sigma}_b|}(\eta_b) = K^{\beta_b}_{|\hat{\sigma}_b|}(\eta_{b,1}, \ldots, \eta_{b,|\hat{\sigma}_b|}) = \sum_{\alpha_1, \ldots, \alpha_{|\hat{\sigma}_b|}} Y^{\beta_b}_{\alpha_1, \ldots, \alpha_{|\hat{\sigma}_b|}} \prod_{p=1}^{|\hat{\sigma}_b|} K^{\alpha_p}_2(\eta_{b,p}),$$  

(4.30)

Here the $\alpha_p$'s are tableaux labeling representations of $U(2)$, and the sum in (4.30) may be restricted to those $\alpha_p$ satisfying

$$\sum_{p=1}^{|\hat{\sigma}_b|} |\alpha_p| = |\beta_b|.$$  

(4.31)

Finally, the $U(2)$-characters are given explicitly (cf (4.7) and (4.26)) by

$$K^{\alpha_p}_2(\eta_{b,p}) = e^{i|\alpha_p|(2\pi p + \hat{\theta}_b)/|\hat{\sigma}_b|} \chi^{S(\alpha_p)}_{SU(2)}(\hat{\xi}_b/|\hat{\sigma}_b|).$$  

(4.32)

To proceed, we substitute the expression (4.19) for $X^s(\lambda, U, \sigma)$ and the expressions (4.24) and (4.28) - (4.32) for $X^f(U, \sigma)$ into the integral (4.16). The integration over $SU(2)^n$ can be arranged so that $\hat{u}_1, \ldots, \hat{u}_{c(\sigma)} \in SU(2)$ are amongst the integration variables. Since the integrand depends only on the $\hat{u}_b$'s, any remaining $SU(2)$-integrals are trivially evaluated. Moreover, since the integrand depends only on the eigenphases $\hat{\xi}_b$, we can make the replacement

$$\int_{SU(2)} d\hat{u}_b \rightarrow \frac{1}{\pi} \int_0^{2\pi} d\hat{\xi}_b \sin^2(\hat{\xi}_b).$$  

(4.33)

Similarly, the $\Theta$-integral in (4.16) can be arranged so that $\hat{\theta}_1, \ldots, \hat{\theta}_{c(\sigma)}$ are amongst the variables of integration; in view of (4.15), the integrand depends only on the $\hat{\theta}_b$'s, so the integrals over any remaining components of $\Theta$ are trivially evaluated. Finally, as the terms in the $S_n$-sum in (4.16) depend only the conjugacy class of $\sigma$ and not on $\sigma$ itself, we may make the replacement

$$\sum_{\sigma \in S_n} \rightarrow \sum_{[\sigma] \in S_n} \Omega_{[\sigma]},$$  

(4.34)

where $[\sigma]$ denotes the conjugacy class of $\sigma$ and $\Omega_{[\sigma]}$ denotes the number of elements in $[\sigma]$ ($\Omega_{[\sigma]}$ may be explicitly expressed in terms of the cycle lengths $|\hat{\sigma}_1|, \ldots, |\hat{\sigma}_{c(\sigma)}|$). In this way,
Eq. (4.16) for the multiplicities may be expressed
\[
\nu(f, s\lambda) = \frac{1}{n!} \sum_{|\sigma| \in S_n} \Omega_{|\sigma|} \chi_{S_n}^\lambda (|\sigma|) \sum_{q=-\infty}^{\infty} (\text{sgn} \sigma)^q \sum_{\beta_1 \cdots ; \beta_{c(\sigma)}} Y_{\beta_1 \cdots ; \beta_{c(\sigma)}}^\dagger \prod_{b=1}^{c(\sigma)} I_b J_b \tag{4.35}
\]

The factor \(I_b\), which contains the integral over \(\theta_b\), is given by
\[
I_b = \frac{1}{2\pi} \int_0^{2\pi} d\hat{\theta}_b e^{i(|\beta_b|/|\hat{\sigma}_b| - q)\hat{\theta}_b}, \tag{4.36}
\]
where we have used (4.31). The factor \(J_b\), which contains the integral over \(\hat{\xi}_b\) and the sum over the \(U(2)\)-tableaux \(\alpha_{p_b}\), is given by
\[
J_b = \sum_{\alpha_1, \cdots, \alpha_{|\sigma_b|}} Y_{\alpha_1, \cdots, \alpha_{|\sigma_b|}} \left(\frac{2\pi i (\sum_{p=1}^{p=1} |p| \alpha_p)}{|\hat{\sigma}_b|} \right) \times \frac{1}{\pi} \times \int_0^{2\pi} d\hat{\xi}_b \sin^2 (\hat{\xi}_b) \chi_{SU(2)}^\alpha (\hat{\xi}_b) \chi_{SU(2)}^{S(\alpha_1)} (\hat{\xi}_b/|\hat{\sigma}_b|) \cdots \chi_{SU(2)}^{S(\alpha_{|\sigma_b|})} (\hat{\xi}_b/|\hat{\sigma}_b|) \tag{4.37}
\]

The \(\hat{\theta}_b\)-integral in (4.36) is trivial, and vanishes unless
\[
q |\hat{\sigma}_b| = |\beta_b|. \tag{4.38}
\]
Summing over \(b\) in (4.38) and using (4.29), we get that
\[
q = |f|/n. \tag{4.39}
\]
Since \(q\) is an integer, it follows that at least one of the \(I_b\)'s must vanish (and, therefore, \(\nu(f, s\lambda)\) must vanish) unless \(n\) divides \(|f|\). Assuming this to be so, the sum over \(q\) in (4.35) collapses to \(q = |f|/n\).

We consider next the expression for \(J_b\). Since the integrand is \(2\pi\)-periodic in \(\hat{\xi}_b\), we can make the replacement
\[
\int_0^{2\pi} d\hat{\xi}_b \rightarrow \int_0^{2\pi} d\hat{\psi}_b, \tag{4.40}
\]
where \(\hat{\psi}_b = \hat{\xi}_b/|\hat{\sigma}_b|\). Using the identity
\[
\sin^2 (r\hat{\psi}_b) \chi_{SU(2)}^\alpha (r\hat{\psi}_b) = \sin^2 (\hat{\psi}_b) \chi_{SU(2)}^{r^s \cdot (r-1)/2} (\hat{\psi}_b) \chi_{SU(2)}^{(r-1)/2} (\hat{\psi}_b), \tag{4.41}
\]
which follows from the definition (4.6), the resulting integral over \(\hat{\psi}_b\) is of the form (4.10), and yields a \((|\hat{\sigma}_b| + 2)\)-fold Clebsch-Gordan coefficient. We obtain
\[
J_b = \sum_{\alpha_1, \cdots, \alpha_{|\sigma_b|}} Y_{\alpha_1, \cdots, \alpha_{|\sigma_b|}} \left(\frac{2\pi i (\sum_{p=1}^{p=1} |p| \alpha_p)}{|\hat{\sigma}_b|} \right) \times C \left(|\hat{\sigma}_b| + \frac{1}{2} (|\hat{\sigma}_b| - 1), \frac{1}{2} (|\hat{\sigma}_b| - 1), S(\alpha_1), \ldots, S(\alpha_{|\sigma_b|}) \right). \tag{4.42}
\]
From (4.35), (4.39) and (4.42), we get our main result for the multiplicities,

\[ \nu(f, s\lambda) = \frac{1}{n!} \sum_{[\sigma] \in S_n} \Omega_{[\sigma]} \chi^\lambda_{S_n}(\sigma)(\text{sgn}\sigma)^{|f|/n} A_{[\sigma]}, \quad (4.43) \]

where

\[ A_{[\sigma]} = \sum_{\beta_1, \ldots, \beta_n(\sigma)} Y_{\beta_1 \cdots \beta_n(\sigma)}^f \times \prod_{b=1}^{c(\sigma)} \sum_{\alpha_1, \ldots, \alpha_{|\beta_b|}} Y_{\alpha_1 \cdots \alpha_{|\beta_b|}} e^{2\pi i \left( \sum_{p=1}^{|\beta_b|} p(\alpha_p) \right) |\beta_b|/|f|} \times \]

\[ \times C \left( |\beta_b| s + \frac{1}{2} (|\beta_b| - 1), \frac{1}{2} (|\beta_b| - 1), S(\alpha_1), \ldots, S(\alpha_{|\beta_b|}) \right). \quad (4.44) \]

For \( \sigma = I = (1) \cdots (n) \) (ie, the identity element in \( S_n \)), the expression (4.44) simplifies considerably. In this case, \( c(\sigma) = n \) and \( |\sigma_b| = 1 \), so that each \( \beta_b \) is a \( U(2) \)-tableaux with \( |\beta_b| = |f|/n \). The sum over \( \alpha_p \) collapses to \( \alpha = \beta_b \), and the corresponding Clebsch-Gordan coefficient, \( C(s, 0, S(\beta_b)) \), vanishes unless \( S(\beta_b) = s \). It follows that the \( \beta_b \) must all coincide with the \( U(2) \)-tableaux \( \beta(f, n, s) \) given by

\[ \beta(f, s) = (|f|/2n + s), (|f|/2n - s)) \quad (4.45) \]

We then obtain

\[ A_{[f]} = \frac{Y_{\beta(f, s), \ldots, \beta(f, s)}^f}{n \text{times}}. \quad (4.46) \]

Some simplification in (4.44) also occurs for \( n \)-cycles in \( S_n \), eg \( \sigma = (12 \cdots n) \). In this case \( c(\sigma) = 1 \) and \( |\sigma| = n \), so that the sum over \( \beta \) collapses to \( \beta = f \). We obtain

\[ A_{[(12 \cdots n)]} = \sum_{\alpha_1, \ldots, \alpha_n} Y_{\alpha_1 \cdots \alpha_n}^f e^{2\pi i \left( \sum_{p=1}^n p(\alpha_p) \right)/n} \times \]

\[ \times C \left( ns + \frac{1}{2} (n - 1), \frac{1}{2} (n - 1), S(\alpha_1), \ldots, S(\alpha_n) \right). \quad (4.47) \]

The sum \( \sum_\lambda \nu(f, s\lambda) \) gives the number of \( n \)-spin-\( s \) representations, regardless of statistics. Using the character relation (see, eg, [2]),

\[ \sum_\lambda \chi^\lambda(\sigma) = \begin{cases} \ n! & \text{if } \sigma = 1, \\ 0 & \text{otherwise}, \end{cases} \quad (4.48) \]

we obtain from (4.43) and (4.46) a simple expression for these summed multiplicities,

\[ \sum_\lambda \nu(f, s\lambda) = \frac{Y_{\beta(f, s), \ldots, \beta(f, s)}^f}{n \text{ times}}. \quad (4.49) \]
### 4.3 Examples

In Section 3.4 we considered the case of completely symmetric representations, for which $f = (d)$. There we showed that $\nu((d), s\lambda)$ vanishes unless $d = 2ns$ and unless $\lambda$ is the trivial (resp. alternating) representation according to whether $s$ is integral (resp. half-odd-integral). This particular case is readily obtained from the general formulas (4.43) and (4.44). Instead of doing so, we sketch below the analogous calculation for the completely antisymmetric representations of $SU(2n)$. The result turns out to be rather different from the symmetric case, in that only $s = \frac{1}{2}$ constructions are supported; the completely antisymmetric representations do not provide a systematic description for all spins. The $s = \frac{1}{2}$ statistics turn out to be bosonic in this case.

Completely antisymmetric representations of $SU(2n)$ correspond to single-columned tableaux of between 1 and $2n - 1$ rows (the $2n$-rowed tableau is equivalent to the trivial representation). Denote the $d$-rowed representation by $f = (1)^d$. From (4.39), $\nu((1)^d, s\lambda)$ vanishes unless $d = n$ and, from (4.45) and (4.49), unless $s = \frac{1}{2}$. In this case, the expressions (4.43) and (4.44) simplify considerably. The sums over the $\alpha$’s collapse to the single term where all the $|\alpha_p|$’s are equal to one, and the sums over the $\beta$’s collapse to the single term where $\beta_b = (1)^{|\beta_b|}$. For these terms, the Littlewood-Richardson coefficients and Clebsch-Gordan coefficients appearing in (4.44) are all equal to one. We get that

$$A_{[\sigma]} = \prod_{b=1}^{c(\sigma)} e^{2\pi i (\sum_{p=1}^{c(\sigma)} p)/|\beta_b|} = \prod_{b=1}^{c(\sigma)} (-1)^{|\beta_b|+1} = \prod_{b=1}^{c(\sigma)} \text{sgn}(\beta_b) = \text{sgn}(\sigma).$$

Substituting the preceding into (4.43), we obtain

$$\nu((1)^n, \frac{1}{2}, \lambda) = \frac{1}{n!} \sum_{|\sigma|\in S_n} \Omega_{|\sigma|} \chi^\lambda_{S_n}(\sigma),$$

which vanishes unless $\lambda$ is the trivial representation of $S_n$.

This result can also be obtained by following the calculation of Section 3.4 and regarding $z = (z_1, \ldots, z_{2n})$ as Grassmann variables. Equivalently, this may be regarded as the $n$-particle version of the ‘anti-Schwinger’ construction of [4], wherein the raising/lower operators of [3] are made to satisfy anticommutation relations. The anticommutation relations are responsible for the restriction to $s = \frac{1}{2}$.

Next, we consider general representations for the case of two particles, $n = 2$. For simplicity, we label the even and odd representations of $S_2$ by $\lambda = +$ and $\lambda = -$, respectively, with characters $\chi^\pm_{S_2}(\sigma) = \pm \text{sgn}(\sigma)$. As in (4.45), let

$$\beta(f, s) = ((|f|/4 + s), (|f|/4 - s)).$$

From (4.52) we can deduce that the multiplicities $\nu(f, s\pm)$ vanish unless $|f|$ is even and $|f|/4 - s$ is a nonnegative integer. From (4.43), (4.46) and (4.47), we get

$$\nu(f, s\pm) = \frac{1}{2} Y_{\beta(f, s)\beta(f, s)\pm} + \frac{(-1)^{2s}}{2} \sum_{\alpha_1, \alpha_2} Y^{\pm}_{\alpha_1, \alpha_2} (-1)^{|\alpha_1|} C(2s + \frac{1}{2}, \frac{1}{2}, S(\alpha_1), S(\alpha_2)).$$

24
The simplest cases are the tableaux $\mathbf{f} = (2, 0)$ and $\mathbf{f} = (1, 1)$ with $|\mathbf{f}| = 2$. Then $s = \frac{1}{2}$ is the only permitted value of the spin, and $\beta(\mathbf{f}, s)$ contains a single box. The tableau $(2, 0)$ corresponds to a completely symmetric representation. As discussed in Section 3.4, this yields fermi statistics for spin-$\frac{1}{2}$, so that $\nu(\mathbf{f}, \frac{1}{2}+) = 0$ and $\nu(\mathbf{f}, \frac{1}{2}-) = 1$. (4.53) is readily evaluated for the tableau $(1, 1)$, and one finds the opposite result, namely bose statistics for spin-$\frac{1}{2}$ (this case is equivalent to the “anti-Schwinger construction” discussed in [4]).

For larger tableaux one typically finds that for each permitted value of spin, both multiplicities $\nu(\mathbf{f}, s+)$ and $\nu(\mathbf{f}, s-)$ are nonzero. Constructions based on such tableaux do not determine a spin-statistics relation. The smallest tableau where this occurs is $\mathbf{f} = (3, 2, 1)$ with $s = \frac{1}{2}$. Evaluation of (4.53) shows there is one fermi and one bose representation. In fact, for larger tableaux one typically finds that $\nu(\mathbf{f}, s+)$ and $\nu(\mathbf{f}, s-)$ are nearly equal, and there are arguments to suggest they are either exactly equal, or else differ by 1, according to whether their sum, given by $Y_{\beta(\mathbf{f}, s), \beta(\mathbf{f}, s')}$, is even or odd. It turns out that most of the terms in the sum over tableaux in (4.53) cancel. First, since the summand is symmetric in $\alpha_1$ and $\alpha_2$, apart from the sign factor $(-1)^{[\alpha_1]}$, only terms for which $|\alpha_1|$ and $|\alpha_2|$ have the same parity contribute. Amongst these remaining terms, the sign factor is responsible for additional cancellations, due to the following fact: If $\alpha_1 + \varepsilon$ is the tableau obtained by adding one box to the first row of $\alpha_1$, and $\alpha_2 - \varepsilon$ is the tableau obtained by removing one box from the first row of $\alpha_2$, then the rules for multiplying tableaux imply that

$$Y_{\alpha_1, \alpha_2}^\mathbf{f} = Y_{\alpha_1 + \varepsilon, \alpha_2 - \varepsilon}^\mathbf{f}. \quad (4.54)$$

Details may be found in [10].

To demonstrate the possibility of parastatistics, we consider the simplest case of three particles and the smallest $SU(6)$ tableau, $\mathbf{f} = (2, 1)$. From (4.35), it follows that $s = \frac{1}{2}$ is the only permitted value of spin, and that $\beta(\mathbf{f}, \frac{1}{2}) = (1)$ consists of a single box. Let $\lambda = E$ denote the two-dimensional representation of $S_3$ (corresponding to the tableau $(2, 1)$), with characters

$$\chi_{S_3}^E(I) = 2, \ \chi_{S_3}^E((12)) = 0, \ \chi_{S_3}^E((123)) = -1, \quad (4.55)$$

and classes

$$\Omega[[1]] = 1, \ \Omega[[12]] = 3, \ \Omega[[123]] = 2 \quad (4.56)$$

(see, eg [9]). From (4.43), (4.46) and (4.47), we get

$$\nu((2, 1), \frac{1}{2}E) = \frac{1}{6} Y_{(1),(1),(1)}^{(2,1)} - \frac{1}{6} \times 2 \sum_{\alpha_1, \alpha_2, \alpha_3} Y_{\alpha_1, \alpha_2, \alpha_3}^{(2,1)} e^{i\frac{2\pi}{3}(|\alpha_1|+|\alpha_2|+|\alpha_3|)} C\left(\frac{5}{2}, 1, S(\alpha_1), S(\alpha_2), S(\alpha_3)\right). \quad (4.57)$$

There are four sets of $U(2)$-tableaux $\alpha_1$, $\alpha_2$ and $\alpha_3$ which may be multiplied to obtain the $SU(6)$ tableau $(2, 1)$ (including cases where one or more of the $\alpha_j$ are empty, which we denote by $\alpha_j = 0$). For each combination we determine the Littlewood-Richardson coefficient $Y_{\alpha_1, \alpha_2, \alpha_3}^{(2,1)}$, the Clebsch-Gordan coefficient $C\left(\frac{5}{2}, 1, S(\alpha_1), S(\alpha_2), S(\alpha_3)\right)$, and the
phase factor $e^{i\frac{2\pi}{3}(\alpha_1+2\alpha_2)}$. The results are summarised in the table below.

| $\alpha_1$ | $\alpha_2$ | $\alpha_3$ | $Y$ | $C$ | $e^{i\frac{2\pi}{3}}$ |
|------------|------------|------------|-----|-----|----------------------|
| (2, 1)     | 0          | 0          | 1   | 0   | 1                    |
| (1, 1)     | (1)        | 0          | 1   | 0   | $e^{i\frac{2\pi}{3}}$ |
| (2)        | (1)        | 0          | 1   | 1   | $e^{i\frac{2\pi}{3}}$ |
| (1)        | (1)        | (1)        | 2   | 1   | 1                    |

Substituting (4.58) into (4.57), we obtain

$$\nu((2, 1), \frac{1}{2}E) = 1.$$  \hspace{5cm} (4.59)

That is, a 3-spin-1/2 bundle with parastatistics may be constructed from the representation $\Gamma^{(2,1)}$ of $SU(6)$.

### 5 Discussion

We have reformulated the quantum kinematics of BR for indistinguishable spinning particles in terms of vector bundles over $n$-particle configuration space. Within this geometrical framework, our main results concern a representation-theoretic generalisation of the construction in BR. We have shown that $n$-spin bundles can be constructed from irreducible representations $\Gamma^f$ of the group $SU(2n)$. The construction makes use of representations $\Delta^f$ of the spin-statistics group $\Sigma(n)$ associated to $\Gamma^f$, as well the existence of a continuous, $S_n$-equivariant map from $SU(n)/T(n)$ to configuration space $C_n$.

The construction in BR is based on particular representations of $SU(2n)$, namely the completely symmetric representations. For a given number, $n$, of indistinguishable particles with spin $s$, there is a unique completely symmetric representation of $SU(2n)$, namely the $2ns$-fold symmetric tensor product of $SU(2n)$ with itself, which leads to a description of the quantum kinematics (ie, which supports an $n$-spin bundle with spin $s$). The statistics is necessarily in accord with the physically correct spin-statistics relation.

Representations of $SU(2n)$ other than the completely symmetric representations, corresponding to Young tableaux $f$ of more than one row, typically support multiple values of spin $s$, and for a given spin may support distinct values of the statistics $\lambda$, including parastatistics. The values of spin and statistics supported by a given representation $\Gamma^f$ are determined by the multiplicities $\nu(f, s, \lambda)$ of the irreducible representations of the spin-statistics group, $\Sigma(n)$, in the decomposition of $\Delta^f$.

Our main calculation is an evaluation of the multiplicities $\nu(f, s, \lambda)$ using character methods. Eqs. (4.43)–(4.44) give the multiplicities as a finite sum over characters of the symmetric group $S_n$, the $n$-fold Clebsch-Gordan coefficients of $SU(2)$, and the Littlewood-Richardson coefficients for the decomposition of representations of $U(n + m)$ into representations of $U(n) \times U(m)$.

Our calculation is related to a more general problem in representation theory, namely the decomposition of zero-weight representations of the Weyl group $W$ of a compact,
connected Lie group $G$ associated with an irreducible representation of $G$. It would be interesting to see if alternative methods could be brought to bear on our calculation, as well as whether the methods used here might prove useful in other contexts. Our construction of $n$-spin-bundles is similarly related to the construction of flat zero-weight bundles over the coset space $G/T$ (where $T$ is a maximal torus of $G$), whose decomposition into a direct sum of sub-bundles irreducible under monodromy leads to the decomposition problem described above.

Concerning the spin-statistics relation, in the first instance our results are similar to those of [3]. Within the group-theoretical framework considered here, the requisite properties introduced in Section 2 do not determine a connection between spin and statistics. When general representations of $SU(2n)$ are admitted alongside the completely symmetric representations, the spin-statistics relation is lost.

As argued in the Introduction, a derivation of the spin-statistics relation from a reformulation of quantum mechanics should be based on principles whose physical motivation is clear. The role played by the group $SU(2n)$ in our considerations is not well motivated in this respect. One could offer as motivation the fact that $SU(2n)$ incorporates both rotations of $n$ spins (ie, $SU(2)^n$) and the permutations $S_n$, but of course it is not the only group which does so.

However, the role played by the completely symmetric representations deserves further consideration. They provide, at least as far as we have discerned, the only systematic means, within the given framework, of associating a representation to a particular value of spin. It is suggestive, too, that the scheme which works treats the spins in a completely symmetrical way; this would seem appropriate for indistinguishable particles. Indeed, characteristic aspects of the completely symmetric representations may indicate a different approach to this nonrelativistic treatment of the spin-statistics relation, which we hope to report on in future.

Acknowledgements. We thank Michael Berry, Roe Goodman, Aidan Schofield and David Thouless for helpful discussions. JMH was supported by the EPSRC and the European Commission under the Research Training Network (Mathematical Aspects of Quantum Chaos) no. HPRN-CT-2000-00103 of the IHP Programme. JMR acknowledges the hospitality of the Mathematical Sciences Research Institute while the manuscript was being completed.

A General Setting

The construction of $n$-spin bundles described in Section 3 is closely related to the following general problem (see, eg, [12]). Let $\Gamma$ be an irreducible unitary representation on a vector space $V$ of a compact Lie group $G$ with maximal torus $T$. $V$ may be decomposed into weight spaces, $V^\mu$, labeled by weights $\mu$ of $T$. The zero-weight space $V^0$ carries a representation $\Delta$ of the Weyl group, $W$, of $G$, which is, in general, reducible. One can then ask for the decomposition of this representation of the Weyl group into its irreducible components.
To each weight space $V^\mu$ of the representation $\Gamma$ is associated a hermitian vector bundle $E^\mu$ over $G/T$ with Abelian $G$-invariant hermitian connection. The curvature of this connection depends linearly on $\mu$, and therefore vanishes on the zero-weight bundle $E^0$. If $G$ is simply connected, then $E^0$ is trivial. However, the quotient bundle $E^0 = E^0/W$ over the quotient space $(G/T)/W$, while locally flat, may be nontrivial. For simply connected $G$, the fundamental group of the quotient space is just the Weyl group $W$, and the monodromy of the flat connection yields a representation of $W$, which is precisely the representation $\Delta$ described above.

This setting can be further generalised by regarding $G$ as a subgroup of a Lie group $F$, and regarding $\Gamma$ as the restriction to $G$ of a representation of $F$. In this case, the natural structure group for the zero-weight bundle is the generalised Weyl group $V = M/T$, where $M$ is the $F$-normaliser of $T$.

The construction of Section 3 is an example belonging to this more general setting, with $F = SU(2n)$, $G = SU(n)$, and $T = T(n)$. The Weyl group of $SU(n)$ is just $S_n$, and the generalised Weyl group $V$ is the spin-statistics group, $\Sigma(n) = SU(2)^n \rtimes S_n/Nul(n)$. The $n$-spin bundles are pullbacks, via the $S_n$-equivariant map $\Xi : C_n \rightarrow SU(n)/T(n)$, of zero-weight bundles over $SU(n)/T(n)$.

This point of view is elaborated below.

A.1 Generalised Weyl group

Let $G$ be a compact, connected semisimple Lie group with maximal torus $T$. Let $N$ denote the normaliser of $T$, and $W = N/T$ the Weyl group of $G$. Suppose $G$ is a subgroup of a compact, connected Lie group $F$. Let $M$ denote the $F$-normaliser of $T$, ie the subgroup of $F$ which leaves $T$ invariant under conjugation. We call

$$V = M/T$$

(A.1)

the generalised Weyl group of $G$. Clearly the Weyl group $W$ is a subgroup of $V$.

Let $Z$ denote the $F$-centraliser of $T$, ie the subgroup of $F$ whose elements commute with all elements of $T$. $Z$ is a normal subgroup of $M$. Therefore, the group $ZN$, consisting of products $zy$ of $z \in Z$ and $y \in N$, is a subgroup of $M$, and

$$ZN/T \cong Z/T \rtimes W,$$

(A.2)

where in the semidirect product $Z/T \rtimes W$, an element $yT \in W$ acts on $zT \in Z/T$ according to $zT \rightarrow (yzy^{-1})T$. The isomorphism (A.2) follows from consideration of the map

$$zy \in ZN \mapsto (zT, yT) \in Z/T \rtimes W.$$  

(A.3)

To show that this map is well defined, we need to check that $zy = z'y'$ implies that $zT = z'T$ and $yT = y'T$. But $zy = z'y'$ implies that $z' = z\tau$ and $y' = \tau^{-1}y$, where $\tau \in Z \cap N$. Since $G$ is compact and connected, $Z \cap N = T$, so $\tau \in T$ as required. It is evident that the map preserves multiplication and that it is surjective. The kernel of the
map consists of elements $zy$ where $z, y \in T$, and therefore is just $T$ itself, so that (A.2) follows.

The spin-statistics group $\Sigma(n)$ is an example of a generalised Weyl group, with $F = SU(2n)$, $G = SU(n)$, and $T = T(n)$. The Weyl group $W$ is isomorphic to $S_n$, $Z/T$ is isomorphic to $Sp_n(n)$, and $V$, the generalised Weyl group, is isomorphic to the semidirect product $Sp_n(n) \rtimes S_n$, which is just $\Sigma(n)$.

### A.2 Zero-weight representations of the generalised Weyl group

Given $T \subset G \subset F$ and $N \subset M$ as above. Let $\mathfrak{f}$ denote the (real) Lie algebra of $F$, $\text{Exp} : \mathfrak{f} \to F$ the exponential map, and $\text{ad}$ the adjoint representation of $F$ on $\mathfrak{f}$. Let $\mathfrak{g} \subset \mathfrak{f}$ denote the Lie algebra of $G$, and $\mathfrak{t} \subset \mathfrak{g}$ the Lie algebra of $T$, (ie, the Cartan subalgebra of $G$), with dual $\mathfrak{t}^\ast$. Denote the pairing between $\mu \in \mathfrak{t}^\ast$ and $\tau \in \mathfrak{t}$ by $\mu \cdot \tau$. The adjoint representation restricts to a representation of $M$ on $\mathfrak{t}$, denoted $\text{ad} (M)$. The co-adjoint representation of $M$ on $\mathfrak{t}^\ast$, denoted $\text{ad}^\ast(M)$, is defined by

$$ (\text{ad}^\ast(x) \cdot \mu) \cdot \tau = \mu \cdot (\text{ad}(x) \cdot \tau). \quad (A.4) $$

Let $\ker_{1}(\text{Exp})$ denote the lattice in $\mathfrak{t}$ mapped to the identity in $T$. A weight $\mu$ of $T$ is an element of $\tau^\ast$ which is integer-valued on $\ker_{1}(\text{Exp})$. Irreducible representations of $T$ are labeled by weights, and are given explicitly by $\text{Exp} \tau \mapsto \exp(2\pi i \mu \cdot \tau)$. (In case $F = G$, this is the usual weight-space decomposition of $\mathcal{V}$.) Let $\mathcal{V}^0$ denote the zero-weight space, ie the subspace of vectors invariant under $T$, for which $\mu = 0$.

For $x \in M$ and $\text{Exp} \tau \in T$, we have that

$$ \Gamma(\exp \tau) \cdot (\Gamma(x) \cdot \mathcal{V}^\mu) = \Gamma(x) \cdot (\Gamma(\exp (x^{-1} \cdot \tau)) \cdot \mathcal{V}^\mu) $$

$$ = \exp(2\pi i \mu \cdot (\text{ad}(x^{-1} \cdot \tau))) \Gamma(x) \cdot \mathcal{V}^\mu $$

$$ = \exp(2\pi i (\text{ad}^\ast(x^{-1}) \cdot \mu) \cdot t) (\Gamma(x) \cdot \mathcal{V}^\mu), \quad (A.5) $$

so that

$$ \Gamma(x) \cdot \mathcal{V}^\mu = \mathcal{V}^{\text{ad}^\ast(x^{-1}) \cdot \mu}. \quad (A.6) $$

It follows that the zero-weight space, $\mathcal{V}^0$, is invariant under $M$, so that $\Gamma$ restricts to a representation of $M$ on $\mathcal{V}^0$. Since $T$ is contained in the kernel, $\Gamma(M)$ reduces to a representation of the generalised Weyl group $V = M/T$ on $\mathcal{V}^0$. Denote this representation by $\Delta^\Gamma$. In general, $\Delta^\Gamma$ is reducible. Let $\Delta$ denote an irreducible representation of $\mathcal{V}$, and let $\nu(\Gamma, \Delta)$ denote the multiplicity of $\Delta$ in the decomposition of $\Delta^\Gamma$ into its irreducible components. The multiplicities $\nu(\Gamma, \Delta)$ are naturally associated with a pair of irreducible representations $\Gamma$ and $\Delta$ of a compact connected Lie group $F$ and the generalised Weyl group $V$. A natural question is how to compute them. In case $F = G = SU(n)$, this question has been discussed by Kostant [12].
A.3 Weight bundles over $G/T$

Associated to the weight space $V^\mu$ is a vector bundle $E^\mu$ over $G/T$. $E^\mu$ is a sub-bundle of the trivial bundle $G/T \times V$, with fibres $E_{gT}$ given by $\Gamma(g) \cdot V^\mu$. By virtue of its embedding in the trivial bundle, there is an induced $G$-invariant connection on $E^\mu$, according to which a vector $|\psi(t)\rangle \in E_{g(t)T}$ is parallel transported along a curve $g(t)T \in G/T$ if and only if $|\dot{\psi}(t)\rangle$ is orthogonal (with respect to the inner product on $V$) to the fibre $E_{gT}$.

It is straightforward to derive an explicit formula for parallel transport along a one-parameter subgroup,

$$g(t) = \text{Exp}(t\xi),$$

(A.7)

where $\xi \in \mathfrak{g}$. We note that $\Gamma$ gives a representation on $V$ of $\mathfrak{f}$, and, by restriction, of $\mathfrak{g}$, by anti-hermitian linear transformations. We denote these Lie-algebra representations by $\Gamma$ as well. Let $t^\perp \subset \mathfrak{g}$ denote the orthogonal complement of $\mathfrak{t}$ in $\mathfrak{g}$ with respect to the Killing form (as $G$ is compact and semisimple, the Killing form is negative definite). It is a standard result (see eg [7]) that $\Gamma(t^\perp)$ maps $V^\mu$ into a direct sum of orthogonal subspaces $V^{\mu'}$ (the difference $\mu' - \mu$ is, in fact, a root of $\mathfrak{g}$). Given $\xi \in \mathfrak{g}$, let $\xi^t + \xi^t\perp$ denote its (unique) decomposition into components in $\mathfrak{t}$ and $\mathfrak{t}^\perp$ respectively. Then, for $|\psi\rangle \in V^\mu$, we have that

$$\Gamma(\xi)|\psi\rangle = 2\pi i(\mu \cdot \xi^t)|\psi\rangle + \{\text{vectors orthogonal to } V^\mu\}$$

(A.8)

It follows from (A.8) that the parallel transport of $|\psi\rangle$ along $g(t)$ is given by

$$|\psi(t)\rangle = \exp(-2\pi i\mu \cdot \xi^t\Gamma(g(t)))|\psi\rangle.$$  

(A.9)

It follows that the induced connection (A.9) is Abelian; under parallel transport around a closed curve in $G/T$, a vector in $E^\mu$ returns to itself up to a phase factor.

Let

$$X_\xi(gT) = \frac{d}{dt} \text{Exp}(t\xi)gT|_0,$$

$$X_\eta(gT) = \frac{d}{dt} \text{Exp}(t\eta)gT|_0$$

denote tangent vector fields at $gT$ generated by the left action of $G$. From (A.9) one can deduce that the scalar-valued curvature two-form $\Omega^\mu$ on $X_\xi, X_\eta$ is given by

$$\Omega^\mu(X_\xi, X_\eta)(gT) = i\mu \cdot ([\xi, \eta]^t - [\xi^t, \eta]).$$

(A.10)

Since the left-invariant vector fields span the tangent bundle of $G/T$, (A.10) determines $\Omega^\mu$. The curvature form, like the connection, is invariant under the action of $G$.

A.4 Zero-weight bundle and representations of the generalised Weyl group

Suppose the representation $\Gamma$ of $F$ has a nontrivial zero-weight space $V^0$. From (A.10), the curvature of the associated zero-weight bundle $E^0$ vanishes, so that induced connection on
is flat. In this case, parallel transport with respect to a flat connection depends only on the homotopy class of the path in $G/T$. If $G/T$ is simply connected, parallel transport is path independent, and $E^0$ is globally flat, and therefore trivial. This is the case if $G$ itself is simply connected, as we will assume from now on. As $G$ is compact and connected, $g(t) ∈ G$ can be expressed $\text{Exp} \left( t\xi(t) \right)$ for some $\xi(t) ∈ g$. It follows from (A.9) that parallel transport along $E$ along $g(t)T$ is given by

$$|φ(t)⟩ = Γ(g(t))|φ⟩.$$ (A.11)

The zero-weight bundle $E^0$, in contrast to weight bundles with non-zero weights, descends from a bundle over $G/T$ to a bundle over $G/N$. We denote this reduced bundle by $\bar{E}^0$. $\bar{E}^0$ is a sub-bundle of the trivial bundle $G/N × V$, with fibres given by $\bar{E}_{gN} = Γ(g)V_0$. (Note that since $N$ leaves $V^0$ invariant, this expression does not depend on the choice of representative $g$ for $gN$.) The flat connection on $E$ passes to $\bar{E}$.

In general, the $G/N$ is not simply connected; its fundamental group is isomorphic to the Weyl group $W = N/T$, as follows from the fact that $G/N = (G/T)/(N/T) = (G/T)/W$, and $G/T$ is simply connected by assumption. An isomorphism between $W$ and $π_1(G/N, N)$, the fundamental group based at the identity coset $IN$, is given explicitly as follows. Let $g(t)N$ denote a closed path in $G/N$ beginning and ending at $N$. For definiteness, take $0 ≤ t ≤ 1$ and $g(0) = I$. Then $g(1)N = IN$ implies that $g(1) ∈ N$. The map

$$g(t)N ↦ g(1)T$$ (A.12)

depends only on the homotopy class of $g(t)N$. It is easily verified that (A.12) preserves group multiplication, and is $1−1$ (since $T$ is connected and $G$ is simply connected) and onto (since $G$ is connected).

Because $G/N$ is not simply connected, parallel transport with respect to the flat connection need not be trivial, and can depend on the homotopy class of the path. For closed paths, parallel transport generates a unitary representation of the fundamental group, the monodromy of the connection. In view of the preceding, the monodromy at the identity coset $IN$ is naturally regarded as a representation of the Weyl group $W$. We denote this representation by $ΔΓ$, and compute it as follows. Given $y ∈ N$, let $g(t) ∈ G$ be a smooth path in $G$ with $g(0) = I$ and $g(1) = y$. From (A.11), parallel transport in $\bar{E}$ along $g(t)N$ is given by

$$|ψ(t)⟩ = Γ(g(t))|ψ⟩.$$ (A.13)

$ΔΓ$ is obtained from parallel transport at $t = 1$, so that

$$ΔΓ(y) = Γ(y).$$ (A.14)

This is just the restriction to $N$ of the representation $ΔΓ$ of the generalised Weyl group $V$ on $\hat{E}_N$.

At an arbitrary fibre $\hat{E}_{gN}$ of the quotient bundle we can define a unitary representation of the generalised Weyl group $V$, which we denote by $L_{gN}(y)$. For example, we can take $L_{gN}(y) = Γ(g)Γ(y)Γ²(g)$. A different choice of representative $g$ for $gN$ would yield a
different but equivalent representation. Therefore, there is a well-defined decomposition of $\mathcal{E}$ into a direct sum of sub-bundles $\mathcal{E}^\alpha$ whose fibres transform according to irreducible representations $\Delta$ of $\mathcal{V}$. This decomposition is determined by the multiplicities $\nu(\Gamma, \Delta)$ discussed in Section A.1.

References

[1] J. Anandan. Spin-statistics connection and relativistic Kaluza-Klein space-time. *Physics Lett.*, A248:124–130, 1998.

[2] M. F. Atiyah. The geometry of classical particles. *Surveys in Differential Geometry (International Press)*, 7:1, 2001.

[3] M. V. Berry and J. M. Robbins. Indistinguishability for quantum particles: spin, statistics and the geometric phase. *Proc. Roy. Soc.*, A453:1771–1790, 1997.

[4] M. V. Berry and J. M. Robbins. Indistinguishability for quantum particles: further considerations. *J. Phys. A: Math. Gen.*, 33:L207–L214, 2000.

[5] I. Duck and E. C. G. Sudarshan. *Pauli and the Spin-Statistics Theorem*. World Scientific, 1997.

[6] I. Duck and E. C. G. Sudarshan. Towards an understanding of the spin-statistics theorem. *Am. J. Phys.*, 66:284–303, 1998.

[7] J. J. Duistermaat and J. Kolk. *Lie Groups*. Springer-Verlag, 2000.

[8] C. R. Hagen and A. J. MacFarlane. Reduction of representations of $su(m + n)$ with respect to the subgroup $su(m) \otimes su(n)$. *J. Math. Phys.*, 6:1366–1371, 1965.

[9] M. Hamermesh. *Group Theory and its Application to Physical Problems*. Addison-Wesley, 1962.

[10] J. M. Harrison. *Group Representations and the Quantum Statistics of Spins*. PhD thesis, University of Bristol, 2001.

[11] C. Itzykson and M. Nauenberg. Unitary groups: Representations and decompositions. *Reviews of Modern Physics*, 38:95–120, 1966.

[12] B. Kostant. On macdonald’s $\eta$-function formula, the laplacian and generalized exponents. *Adv. in Math.*, 20:179–212, 1976.

[13] J. M. Leinaas and J. Myrheim. On the theory of identical particles. *Nuovo Cim.*, 37B:1–23, 1977.

[14] D. E. Littlewood and A. R. Richardson. Group characters and algebra. *Phil. Trans. A*, 233:99–141, 1934.
[15] I. G. Macdonald. *Symmetric Functions and Hall Polynomials*. Clarendon Press, 1979.

[16] G. W. Mackey. *Unitary group representations in physics, probability and number theory*. Reading: Benjamin-Cummings, 1978.

[17] R. Sorkin. Particle statistics in three dimensions. *Phys. Rev. D*, 27:1787–1792, 1983.