ON SETS IN $\mathbb{R}^d$ WITH DC DISTANCE FUNCTION

DUŠAN POKORNÝ AND LUDĚK ZAJÍČEK

ABSTRACT. We study closed sets $F \subset \mathbb{R}^d$ whose distance function $d_F := \text{dist} (\cdot, F)$ is DC (i.e., is the difference of two convex functions on $\mathbb{R}^d$). Our main result asserts that if $F \subset \mathbb{R}^2$ is a graph of a DC function $g : \mathbb{R} \to \mathbb{R}$, then $F$ has the above property. If $d > 1$, the same holds if $g : \mathbb{R}^{d-1} \to \mathbb{R}$ is semiconcave, however the case of a general DC function $g$ remains open.

1. INTRODUCTION

Let $F \neq \emptyset$ be a closed subset of $\mathbb{R}^d$ and let $d_F := \text{dist} (\cdot, F)$ be its distance function. Recall that a function on $\mathbb{R}^d$ is called DC, if it is the difference of two convex functions. It is well-known (see, e.g., [1, p. 976]) that

(1.1) the function $(d_F)^2$ is DC but $d_F$ need not be DC.

However, the distance function of some interesting special $F \subset \mathbb{R}^d$ is DC; it is true for example for $F$ from Federer’s class of sets with positive reach, see (4.5).

Our article was motivated by [1] and by the following question which naturally arises in the theory of WDC sets (see [8, Question 2, p. 829] and [7, 10.4.3]).

Question. Is $d_F$ a DC function if $F$ is a graph of a DC function $g : \mathbb{R}^{d-1} \to \mathbb{R}$?

Note that WDC sets form a substantial generalization of sets with positive reach and still admit the definition of curvature measures (see [11] or [7]) and $F$ as in Question is a natural example of a WDC set in $\mathbb{R}^d$.

Our main result (Theorem 3.3) gives the affirmative answer to Question in the case $d = 2$; the case $d > 2$ remains open. However, known results relatively easily imply that the answer is positive if $g$ in Question is semiconcave (Corollary 4.5).

In [13] we show that our main result has some interesting consequences for WDC subsets of $\mathbb{R}^2$, in particular that these sets have DC distance functions.

In Section 2 we recall some notation and needed facts about DC functions. In Section 3 we prove our main result (Theorem 3.3). In last Section 4, we prove a number of further results on the system of sets in $\mathbb{R}^d$ which have DC distance function, including Corollary 4.5 mentioned above.

We were not able to prove a satisfactory complete characterisation of sets $F \subset \mathbb{R}^2$ with DC distance function, but we believe that our methods and results should lead to such a characterisation. However, in our opinion, the case of $F \subset \mathbb{R}^d$, $d \geq 3$, needs some new ideas.

2. PRELIMINARIES

In any vector space $V$, we use the symbol $0$ for the zero element. We denote by $B(x, r)$ ($U(x, r)$) the closed (open) ball with centre $x$ and radius $r$. The boundary and the interior of a set $M$ are denoted by $\partial M$ and $\text{int} M$, respectively. A mapping
is called $K$-Lipschitz if it is Lipschitz with a (not necessarily minimal) constant $K \geq 0$.

In the Euclidean space $\mathbb{R}^d$, the norm is denoted by $| \cdot |$ and the scalar product by $\langle \cdot , \cdot \rangle$. By $S^{d-1}$ we denote the unit sphere in $\mathbb{R}^d$.

If $x, y \in \mathbb{R}^d$, the symbol $[x, y]$ denotes the closed segment (possibly degenerate). If also $x \neq y$, then $l(x, y)$ denotes the line joining $x$ and $y$.

The distance function from a set $A \subset \mathbb{R}^d$ is $d_A := \text{dist}(\cdot, A)$ and the metric projection of $z \in \mathbb{R}^d$ to $A$ is $\Pi_A(z) := \{ a \in A : \text{dist}(z, A) = |z - a| \}$.

If $f$ is defined in $\mathbb{R}^d$, we use the notation $f'_+(x, v)$ for the one sided directional derivative of $f$ at $x$ in direction $v$.

Let $f$ be a real function defined on an open convex set $C \subset \mathbb{R}^d$. Then we say that $f$ is a DC function, if it is the difference of two convex functions. Special DC functions are semiconvex and semiconcave functions. Namely, $f$ is a semiconvex (resp. semiconcave) function, if there exist $a > 0$ and a convex function $g$ on $C$ such that

$$f(x) = g(x) - a \| x \|^2 \quad (\text{resp. } f(x) = a \| x \|^2 - g(x)), \quad x \in C.$$ 

We will use the following well-known properties of DC functions.

**Lemma 2.1.** Let $C$ be an open convex subset of $\mathbb{R}^d$. Then the following assertions hold.

(i) If $f : C \to \mathbb{R}$ and $g : C \to \mathbb{R}$ are DC, then (for each $a \in \mathbb{R}, b \in \mathbb{R}$) the functions $af + bg, \max(f, g)$ and $\min(f, g)$ are DC.

(ii) Each locally DC function $f : C \to \mathbb{R}$ is DC.

(iii) Each DC function $f : C \to \mathbb{R}$ is Lipschitz on each compact convex set $Z \subset C$.

(iv) Let $f_i : C \to \mathbb{R}, i = 1, \ldots, m$, be DC functions. Let $f : C \to \mathbb{R}$ be a continuous function such that $f(x) \in \{ f_1(x), \ldots, f_m(x) \}$ for each $x \in C$.

Then $f$ is DC on $C$.

(v) Each $C^2$ function $f : C \to \mathbb{R}$ is DC.

**Proof.** Property (i) follows easily from definitions, see e.g. [17, p. 84]. Property (ii) was proved in [9]. Property (iii) easily follows from the local Lipschitzness of convex functions. Assertion (iv) is a special case of [13, Lemma 4.8] (“Mixing lemma”). To prove (v) observe that (e.g. by [2, Proposition 1.1.3 (d)]) each $C^2$ function is locally semiconcave and therefore locally DC, hence, DC by (ii).

By well-known properties of convex and concave functions, we easily obtain that each locally DC function $f$ on an open set $U \subset \mathbb{R}^d$ has all one-sided directional derivatives finite and

$$g'_+(x, v) + g'_-(x, -v) \leq 0, \quad x \in U, v \in \mathbb{R}^d, \quad \text{if } g \text{ is locally semiconcave on } U.$$ 

Recall that if $\varnothing \neq A \subset \mathbb{R}^d$ is closed, then $d_A$ need not be DC; however (see, e.g., [2, Proposition 2.2.2]),

$$d_A \text{ is locally semiconcave (and so locally DC) on } \mathbb{R}^d \setminus A.$$ 

3. **Main result**

In the proof of Theorem 3.3 below we will use the following simple “concave mixing lemma”.

**Lemma 3.1.** Let $U \subset \mathbb{R}^d$ be an open convex set and let $\gamma : U \to \mathbb{R}$ have finite one-sided directional derivatives $\gamma'_+(x, v)$, $(x \in U, v \in \mathbb{R}^d)$. Suppose that

$$\gamma'_+(x, v) + \gamma'_+(x, -v) \leq 0, \quad x \in U, v \in \mathbb{R}^d.$$
and that
\begin{equation}
\text{(3.2)} \quad \text{graph } \gamma \text{ is covered by graphs of a finite number of concave functions defined on } U.
\end{equation}

Then \( \gamma \) is a concave function.

\textbf{Proof.} Since \( \gamma \) is clearly concave if each function \( t \mapsto \gamma(a + tv), \ (a \in C, v \in S^{d-1}) \) is concave on its domain, it is sufficient to prove the case \( d = 1, C = (a, b) \). Set \( h(x) := -\gamma(x), \ x \in (a, b) \); we need to prove that \( h \) is convex. Observe that (3.1) easily implies the condition
\begin{equation}
\text{(3.3)} \quad h'_-(x) \leq h'_+(x), \quad x \in (a, b),
\end{equation}
and (3.2) implies that there exists a finite set \( \{ h_\alpha : \alpha \in A \} \) of convex functions on \((a, b)\) such that graph \( h \subset \bigcup \{ \text{graph } h_\alpha : \alpha \in A \} \). To prove the convexity of \( h \), it is sufficient to show that the function \( h'_- \) is nondecreasing on \((a, b)\) (see e.g. [10, Chap. 5, Prop. 18, p. 114]); equivalently (it follows e.g. from [10, Chap. IX, §7, Lemma 1, p. 266]) to prove that
\begin{equation}
\text{(3.4)} \quad \forall x_0 \in (a, b) \ \exists \delta > 0 \ \forall x : \ (x \in (x_0, x_0 + \delta) \Rightarrow h'_+(x) \geq h'_+(x_0))
\end{equation}
\begin{equation}
\quad \wedge \ (x \in (x_0 - \delta, x_0) \Rightarrow h'_-(x) \leq h'_-(x_0)).
\end{equation}

So suppose, to the contrary, that (3.4) does not hold; then there exists a sequence \( x_n \to x_0 \) such that either
\begin{equation}
\text{(3.5)} \quad x_n < x_0 \text{ and } h'_+(x_n) > h'_+(x_0) \text{ for each } n \in \mathbb{N}
\end{equation}
or
\begin{equation}
\text{(3.6)} \quad x_n > x_0 \text{ and } h'_-(x_n) < h'_-(x_0) \text{ for each } n \in \mathbb{N}
\end{equation}

Since \( h \) is clearly continuous, each set \( F_\alpha := \{ x \in (a, b) : h_\alpha(x) = h(x) \} \), \( \alpha \in A \), is closed in \((a, b)\). Since \( A \) is finite, it is easy to see that for each \( n \in \mathbb{N} \) there exists \( \alpha(n) \in A \) such that \( x_n \in F_{\alpha(n)} \) and \( x_n \) is a right accumulation point of \( F_{\alpha(n)} \). Using finiteness of \( A \) again, we can suppose that there exists \( \alpha \in A \) such that \( \alpha(n) = \alpha, \ n \in \mathbb{N} \) (otherwise we could consider a subsequence of \((x_n)\)).

Now suppose that (3.5) holds. Since \( x_n \in F_\alpha, \ n = 0, 1, \ldots, \), we obtain that
\[ h'_+(x_n) = (h_\alpha)'_+(x_n), \ n \in \mathbb{N}, \] and \( h'_-(x_n) = (h_\alpha)'_-(x_n) \). Using also the convexity of \( h_\alpha \) and (3.3), we obtain
\[ h'_+(x_n) = (h_\alpha)'_+(x_n) \leq (h_\alpha)'_-(x_0) = h'_-(x_0) \leq h'_-(x_n) \]
which contradicts (3.5). Since the case when (3.6) holds is quite analogous, neither (3.5) nor (3.6) is possible and so we are done. \( \square \)

We will need also the following easy lemma.

\textbf{Lemma 3.2.} Let \( V \) be a closed angle in \( \mathbb{R}^2 \) with vertex \( v \) and measure \( 0 < \alpha < \pi \). Then there exist an affine function \( A \) on \( \mathbb{R}^2 \) and a concave function \( \psi \) on \( \mathbb{R}^2 \) which is Lipschitz with constant \( \sqrt{2}\tan(\alpha/2) \) such that \( |z - v| + \psi(z) = A(z), \ z \in V \).

\textbf{Proof.} We can suppose without any loss of generality that \( v = (0, 0) \) and
\[ V = \{ (x, y) : x \geq 0, |y| \leq x \tan(\alpha/2) \}. \]
Then \(|z - v| = \sqrt{x^2 + y^2}\) for \( z = (x, y) \). Define the convex function
\[ \varphi(x, y) := \sqrt{x^2 + y^2} - x, \ (x, y) \in V. \]
We will show that
\begin{equation}
\text{(3.7)} \quad \varphi \text{ is Lipschitz with constant } \sqrt{2}\tan(\alpha/2).
\end{equation}
Theorem 3.3. Let $d_n(3.9)$

Proof. By (2.2), convex extension $\tilde{\varphi}$ to $\mathbb{R}^2$ which is also Lipschitz with constant $\sqrt{2} \tan(\alpha/2)$ (see, e.g., [H Theorem 1]). Now we can put $\psi := -\tilde{\varphi}$, since $\sqrt{x^2 + y^2} + \psi(x, y) = x := A(x, y), (x, y) \in V$. □

Theorem 3.3. Let $f : \mathbb{R} \to \mathbb{R}$ be a DC function. Then the distance function $d := \text{dist}(\cdot, \text{graph } f)$ is DC on $\mathbb{R}^2$.

Proof. By (2.2), $d$ is locally DC on $\mathbb{R}^2 \setminus \text{graph } f$. So, by Lemma 2.1 (ii), it is sufficient to prove that, for each $z \in \text{graph } f$, the distance function $d$ is DC on a convex neighbourhood of $z$. Since we can clearly suppose that $z = (0, f(0))$, it is sufficient to prove that

$$d \text{ is DC on } U := U((0, f(0)), 1/10).$$

Write $f = g - h$, where $g, h$ are convex functions on $\mathbb{R}$. For each $n \in \mathbb{N}$, consider the equidistant partition $D_n = \{x^n_i = -1 < x^n_i < \cdots < x^n_0 = 1\}$ of $[-1, 1]$. Let $g_n, h_n$ be the piece-wise linear function on $[-1, 1]$ such that $g_n(x^n_i) = g(x^n_i)$, $h_n(x^n_i) = h(x^n_i)$ ($0 \leq i \leq n$) and $g_n, h_n$ are affine on each interval $[x^n_{i-1}, x^n_i]$ ($1 \leq i \leq n$). Put $f_n := g_n - h_n$ and $d_n := \text{dist}(\cdot, \text{graph } f_n)$. Choose $L > 0$ such that both $g |[-1,1]$ and $h|[-1,1]$ are $(L/2)$-Lipschitz and observe that all $g_n, h_n, f_n$ are $L$-Lipschitz. Since $f_n$ uniformly converge to $f$ on $[-1, 1]$, we easily see that $d_n \to d$ on $U$.

Choose an integer $n_0$ such that

$$n_0 \geq 6 \text{ and } |f_n(0) - f(0)| < \frac{1}{10} \text{ for each } n \geq n_0. \quad (3.9)$$

We will prove that there exist $L^* > 0$ and concave functions $c_n (n \geq n_0)$ on $\overline{U}$ such that

$$\text{each } c_n \text{ is Lipschitz with constant } L^* \text{ and } \quad (3.10)$$

$$c^n_* := d_n + c_n \text{ is concave on } \overline{U}. \quad (3.11)$$

Then we will done, since (3.10) and (3.11) easily imply (3.8). Indeed, we can suppose that $c_{n_0}((0, f(0))) = 0$ and, using Arzelà-Ascoli theorem, we obtain that there exists an increasing sequence of indices $(n_k)$ such that $c_{n_k} \to c$, where $c$ is a continuous concave function on $\overline{U}$. So $d_{n_k} + c_{n_k} \to d + c =: c^*$ on $\overline{U}$. Using (3.11), we obtain that $c^*$ is concave and thus $d = c^* - c$ is DC on $U$.

To prove the existence of $L^*$ and $(c_n)$, fix an arbitrary $n \geq n_0$. For brevity denote $\Pi := \Pi_{\text{graph } f_n}$ and put $x_i := x^n_i, z_i := (x_i, f_n(x_i)), i = 0, \ldots, n$. For $i = 1, \ldots, n-1$, let $0 \leq \alpha_i < \pi$ be the angle between the vectors $z_i - z_{i-1}$ and $z_{i+1} - z_i$. Denote

$$s_i := \frac{f_n(x_{i+1}) - f_n(x_i)}{x_{i+1} - x_i} \quad \text{and } \beta_i := \arctan s_i, \quad i = 0, \ldots, n - 1.$$ 

Then clearly $\alpha_i = |\beta_i - \beta_{i-1}|$. One of the main ingredients of the present proof is the easy fact that

$$\quad \sum_{i=1}^{n-1} |s_i - s_{i-1}| \leq 4L. \quad (3.12)$$
It immediately follows from the well-known estimate of (the “convexity”) $K_n^h(f_n)$ (see [13] p. 24, line 5). To give, for completeness, a direct proof, denote

$$\tilde{s}_i := g_n(x_{i+1}) - h_n(x_i), \quad \tilde{s}_i^* := h_n(x_{i+1}) - h_n(x_i), \quad i = 0, \ldots, n - 1,$$

and observe that the finite sequences $(\tilde{s}_i), (\tilde{s}_i^*)$ are nondecreasing. Consequently

$$\sum_{i=1}^{n-1} |\tilde{s}_i - \tilde{s}_{i-1}| = \tilde{s}_n - \tilde{s}_1 \leq 2L \text{ and } \sum_{i=1}^{n-1} |\tilde{s}_i^* - \tilde{s}_{i-1}^*| = \tilde{s}_n^* - \tilde{s}_1^* \leq 2L.$$

Since $s_i = \tilde{s}_i - \tilde{s}_i^*$, (3.12) easily follows.

Since

$$\alpha_i = |\beta_i - \beta_{i-1}| \leq |\tan(\beta_i) - \tan(\beta_{i-1})| = |s_i - s_{i-1}|,$$

we obtain

$$\sum_{i=1}^{n-1} \alpha_i \leq 4L.$$  \hspace{1cm} (3.13)

Since $|\beta_i| \leq \arctan L$, we have $\alpha_i/2 \leq \arctan L$. Further, since the function $\tan$ is convex on $[0, \pi/2)$, the function $s(x) = \tan x/\pi$ is increasing on $(0, \pi/2)$. These facts easily imply

$$\tan \left( \frac{\alpha_i}{2} \right) \leq \frac{\alpha_i}{2} \cdot \frac{L}{\arctan L}. $$

Thus we obtain by (3.13)

$$\sum_{i=1}^{n-1} \sqrt{2} \tan \left( \frac{\alpha_i}{2} \right) \leq \frac{2\sqrt{2}L^2}{\arctan L} =: M. $$  \hspace{1cm} (3.14)

Further observe that each $d_n$ is DC on $\mathbb{R}^2$ and consequently

$$(d_n)'_+(x, v) \in \mathbb{R} \quad \text{exists for every } x, v \in \mathbb{R}^2.$$  \hspace{1cm} (3.15)

Indeed, since each segment $[z_{i-1}, z_i]$ is a convex set, by the well known fact the distance functions $\text{dist}(\cdot, [z_{i-1}, z_i]), i = 1, \ldots, n,$ are convex and consequently $d_n$ is DC by (4.3) below.

If $\alpha_i = 0$, set

$$V_i := \{ z \in \mathbb{R}^2 : \langle z - z_i, z_{i+1} - z_i \rangle \leq 0, \langle z - z_i, z_{i-1} - z_i \rangle \leq 0 \},$$

which is clearly a closed angle with vertex $z_i$ and measure $\alpha_i$. Let $\psi_i$ and $A_i$ be the (concave and affine) functions on $\mathbb{R}^2$ which correspond to $V_i$ by Lemma 3.2. If $\alpha_i = 0$, put $\psi_i(z) := 0, \quad z \in \mathbb{R}^2$.

Now set

$$\eta_n := \sum_{i=1}^{n-1} \psi_i.$$  \hspace{1cm} (3.16)

Then $\eta_n$ is a concave function on $\mathbb{R}^2$ and Lemma 3.2 with (3.14) imply that

$$|z - z_i| + \psi_i(z) = A_i(z), \quad z \in V_i.$$  \hspace{1cm} (3.17)

The concave function $c_n$ with properties (3.10), (3.11) will be defined as $c_n(x) := \eta_n(x) + \xi_n(x), \quad x \in \overline{U}$, where the concave function $\xi_n$ on $A := (-1,1) \times \mathbb{R}$ will be defined to “compensate the non-concave behaviour of $d_n$ at points of graph $f_n$” in the sense that, for each point $z \in A \cap \text{graph } f_n$,

$$(d_n + \xi_n)'_+(z, v) + (d_n + \xi_n)'_+(z, -v) \leq 0 \text{ whenever } v \in \mathbb{R}^2.$$  \hspace{1cm} (3.18)
We set, for \((x,y) \in A\),
\[
\xi_n(x,y) := -\max(2g_n(x) - y, 2h_n(x) + y) \quad \text{and} \quad p_n(x,y) := |f_n(x) - y|.
\]
Obviously,
\[
(3.19) \quad \xi_n \text{ is concave and Lipschitz with constant } 2L + 1.
\]
Further, for \((x,y) \in A\),
\[
p_n(x,y) = \max(g_n(x) - h_n(x) - y, h_n(x) - g_n(x) + y)
= \max(2g_n(x) - y, 2h_n(x) + y) - h_n(x) - g_n(x),
\]
which shows that \(p_n\) is a DC function and \(p_n + \xi_n\) is concave. Consequently, for each \(z \in A\) and \(v \in \mathbb{R}^2\),
\[
(3.20) \quad (p_n)'_+(z,v) + (\xi_n)'_+(z,v) + (p_n)'_+(z,-v) + (\xi_n)'_+(z,-v) \leq 0.
\]
Since, for each point \(z \in \text{graph } f_n \cap A\), we have \(d_n(z) = p_n(z) = 0\) and for each \((x,y) \in A\) clearly \(d_n(x,y) \leq |(x,y) - (x,f_n(x))| = p_n(x,y)\), we easily obtain (for each \(v \in \mathbb{R}^2\))
\[
(3.21) \quad (p_n)'_+(z,v) + (\xi_n)'_+(z,-v) \geq (d_n)'_+(z,v) + (d_n)'_+(z,-v),
\]
which, together with \((3.20)\), implies \((3.18)\).

Now set
\[
c_n(x) := \eta_n(x) + \xi_n(x), \quad x \in \overline{U}.
\]
By \((3.16)\) and \((3.19)\) we obtain that \((3.10)\) holds with \(L^* := M + 2L + 1\).

To prove \((3.11)\), it is clearly sufficient to show that \(\gamma = c_n^* := d_n + c_n\) is concave on \(U\); we will prove it by Lemma \(3.1\).

First we verify the validity of \((3.1)\) for each \(z \in U\). If \(z \notin \text{graph } f_n\), then \((3.1)\) holds by \((2.1)\), since \(\gamma = d_n + \eta_n + \xi_n\) on \(U\), \(d_n\) is locally semiconcave on \(\mathbb{R}^2 \setminus \text{graph } f_n\) and \(\eta_n + \xi_n\) is concave on \(U\). If \(z \in \text{graph } f_n\), then \((3.1)\) follows by \((3.18)\) and the concavity of \(\eta_n\) on \(U\).

So it is sufficient to verify \((3.2)\). To this end, first define on \(U\) the functions
\[
\omega_i := \text{dist}(\cdot, \ell(z_i, z_{i+1})) \quad \text{and} \quad \mu_i := \omega_i + \eta_n + \xi_n, \quad i = 1, \ldots, n - 1.
\]
Since each graph \(\omega_i\) is covered by graphs of two affine functions, we see that
\[
(3.22) \quad \mu_i \text{ is covered by graphs of two concave functions.}
\]
Now consider an arbitrary \(z \in U\) and choose a point \(z^* \in \Pi(z)\). Since \(d_n(z) \leq 1/5\) by \((3.9)\) and \(n \geq n_0 \geq 6\), we obtain \(z^* \in \bigcup_{i=2}^{n-2}[z_i, z_{i+1}]\).

If \(z^* = z_i\) for some \(1 \leq i \leq n - 1\) with \(\alpha_i \neq 0\), then we easily see that \(z \in V_i\) and
\[
d_n(z) = |z - z_i|, \quad \text{and consequently}
\gamma(z) = (|z - z_i| + \psi_i(z)) + \sum_{1 \leq j \leq n-1, j \neq i} \psi_j(z) + \xi_n(z) = \nu_i(z),
\]
where
\[
\nu_i(z) := A_i(z) + \sum_{1 \leq j \leq n-1, j \neq i} \psi_j(z) + \xi_n(z), \quad z \in U,
\]
is concave on \(U\).

If \(z^* = z_i\) and \(\alpha_i = 0\), or \(z^* \in [z_i, z_{i+1}] \setminus \{z_i, z_{i+1}\}\) for some \(1 \leq i \leq n - 1\), then clearly \(d_n(z) = \omega_i(z)\) and so \(\gamma(z) = \mu_i(z)\).

So we have proved that the graph of \(\gamma = c_n^*\) is covered by graphs of functions \(\nu_i, 1 \leq i \leq n - 1, \alpha_i \neq 0\), and functions \(\mu_i, 1 \leq i \leq n - 1\). Using \(3.22\), we obtain \((3.2)\) and Lemma \(3.1\) implies that \(\gamma = c_n^*\) is concave. \(\square\)
4. Other results

We finish the article with a number of additional results on the systems

\[ \mathcal{D}_d := \{\emptyset\} \cup \{\emptyset \neq A \subset \mathbb{R}^d : A \text{ is closed and } d_A \text{ is DC}\}, \quad d = 1, 2, \ldots. \]

First we observe that the description of \( \mathcal{D}_1 \) is very simple since

\( (1.1) \) \( A \subset \mathbb{R} \) belongs to \( \mathcal{D}_1 \) if and only if the system of all components of \( A \) is locally finite.

Indeed, if the system of all components of \( \emptyset \neq A \subset \mathbb{R} \) is locally finite, then Lemma 2.1(ii) easily implies that \( d_A \) is DC.

If the system of all components of \( A \) is not locally finite, then there exists a sequence \((c_n)\) of centres of components of \( \mathbb{R} \setminus A \) converging to a point \( a \in A \). Therefore \( d_A \) is not one-sidedly strictly differentiable at \( a \), since \( (d_A)^{1'}(c_n) = \mp 1 \).

Consequently \( d_A \) is not DC, since each DC function on \( \mathbb{R} \) is one-sidedly strictly differentiable at each point (see [18, Note 3.2] or [19, Proposition 3.4(i)] together with Remark 3.2).

From this characterisation easily follows that \( \mathcal{D}_1 \) is closed with respect to finite unions and intersections and that, for a closed set \( M \subset \mathbb{R} \),

\( (4.2) \quad M \in \mathcal{D}_1 \iff \partial M \in \mathcal{D}_1. \)

Concerning \( d \geq 2 \) further observe that

\( (4.3) \quad \mathcal{D}_d \) is closed with respect to finite unions.

Indeed, if \( \emptyset \neq A \subset \mathcal{D}_d \) and \( \emptyset \neq B \subset \mathcal{D}_d \), then \( d_{A \cup B} = \min(d_A, d_B) \) and so \( d_{A \cup B} \) is DC by Lemma 2.1(i).

Example 4.1 below shows that already \( \mathcal{D}_2 \) is not closed with respect to finite intersections. Equivalence (4.2) does not generalize already to dimension 2 either (see again Example 4.1), however, one can see that, for a closed set \( M \subset \mathbb{R}^d \),

\( (4.4) \quad \partial M \in \mathcal{D}_d \iff (M \in \mathcal{D}_d \text{ and } \mathbb{R}^d \setminus M \in \mathcal{D}_d), \quad d \in \mathbb{N}. \)

To prove one implication suppose \( \partial M \in \mathcal{D}_d \). If \( x \notin M \) then clearly \( \Pi_M(x) \in \partial M \) and so \( d_M(x) = d_{\partial M}(x) \).

Consequently, for each \( x \in \mathbb{R}^d \), \( d_M(x) \in \{0, d_{\partial M}(x)\} \) and so \( M \in \mathcal{D}_d \) by Lemma 2.1(iv).

Similarly, if \( x \notin \mathbb{R}^d \setminus M \) then \( \Pi_{\mathbb{R}^d \setminus M}(x) \in \partial(\mathbb{R}^d \setminus M) \subset \partial M \) so again \( d_{\partial M}(x) \in \{0, d_{\partial M}(x)\} \) and \( \mathbb{R}^d \setminus M \in \mathcal{D}_d \) follows.

To prove the opposite implication it is enough to show that \( d_{\partial M} = \max(d_M, d_{\mathbb{R}^d \setminus M}) \) if \( \partial M \neq \emptyset \). Clearly \( d_{\partial M} \geq \max(d_M, d_{\mathbb{R}^d \setminus M}) \), since \( \partial M = M \cap \mathbb{R}^d \setminus M \). To prove the opposite inequality suppose to the contrary that

\[ r := d_{\partial M}(x) > \max(d_M(x), d_{\mathbb{R}^d \setminus M}(x)) \]

for some \( x \in \mathbb{R}^d \). Consequently \( U(x, r) \cap M \neq \emptyset \) and \( U(x, r) \cap \mathbb{R}^d \setminus M \neq \emptyset \). Then also \( U(x, r) \cap \mathbb{R}^d \setminus M \neq \emptyset \) and thus \( U(x, r) \cap \partial M \neq \emptyset \) which is a contradiction.

Before presenting the following example we first observe that the function \( g(x) = x^5 \cos \frac{x}{2}, \quad x \neq 0, \quad g(0) = 0 \), is \( C^2 \) on \( \mathbb{R} \) and therefore DC by Lemma 2.1(v). Indeed, a direct computation shows that \( g''(x) = x (8 \pi x \sin \frac{x}{2} - (\pi^2 - 20 x^2) \cos \frac{x}{2}) \), \( x \neq 0 \), and \( g''(0) = 0 \).

**Example 4.1.** There are sets \( A, B \in \mathcal{D}_2 \) such that \( A \cap B \notin \mathcal{D}_2 \). Further, there is a set \( K \in \mathcal{D}_2 \) such that \( \partial K \notin \mathcal{D}_2 \) and \( \mathbb{R}^2 \setminus K \notin \mathcal{D}_2 \).

**Proof.** Define \( g(x) = x^5 \cos \frac{x}{2}, \quad x \neq 0, \quad g(0) = 0 \), and \( f(x) = 0, \quad x \in \mathbb{R} \). Put

\[ A = \{(x, y) : y \geq f(x)\}, \quad B = \{(x, y) : y \leq g(x)\}, \quad H = \{x : f(x) \leq g(x)\}. \]

Since both \( f \) and \( g \) are DC, we obtain that \( A, B \in \mathcal{D}_2 \) by Theorem 3.3 and (4.3).

Put \( M = A \cap B \) and \( K = \mathbb{R}^2 \setminus M \). Clearly also \( M = \mathbb{R}^2 \setminus K \). First note that
$M \notin D_2$ since the function $x \mapsto d_M(x, 0)$ is equal to $d_H$, but clearly $H \notin D_1$ by (1.1).

We obtain that $K \in D_2$ by (4.3), since $K = C \cup D$, where $C = \{(x,y) : y \leq f(x)\}$ and $D = \{(x,y) : y \geq g(x)\}$, and $C,D \in D_2$ by Theorem 3.3 and (4.4). Finally, $\partial K \notin D_2$ by (4.4) applied to $K$. \hfill \Box

Now we will show that equivalence (4.2) holds for sets $M$ of positive reach (cf. (4.5)). We first recall their definition.

If $A \subset \mathbb{R}^d$ and $a \in A$, we define

$$\text{reach}(A,a) := \sup\{r \geq 0 : \Pi_A(z) \text{ is a singleton for each } z \in U(a,r)\}$$

and the reach of $A$ as

$$\text{reach}\ A := \inf_{a \in A} \text{reach}(A,a).$$

Note that each set with positive reach is clearly closed.

As mentioned in Introduction, it is essentially well-known that

$$(4.5) \quad \text{if } A \subset \mathbb{R}^d \text{ has positive reach, then } A \in D_d.$$ 

Indeed, for each $a \in A$ [3] Proposition 5.2 implies that $d_A$ is semiconvex on $U(a,\text{reach}(A)/2)$, which with (2.2) and Lemma 2.1 (ii) implies that $d_A$ is DC.

**Proposition 4.2.** Let $\emptyset \neq A \subset \mathbb{R}^d$ be a set with positive reach and $B := \mathbb{R}^d \setminus A$. Then both $B$ and $\partial A$ belong to $D_d$.

*Proof.* By (4.3) and (4.4) it is sufficient to prove that $B \in D_d$. Since $d_B$ is locally DC on $\mathbb{R}^d \setminus B$ (see (2.2)) and on $\text{int} B$ (trivially), by Lemma 2.1 (ii) it is sufficient to prove that

$$(4.6) \quad \text{for each } a \in \partial B \text{ there exists } \rho > 0 \text{ such that } d_B \text{ is DC on } U(a,\rho).$$

To prove (4.6), choose $0 < r < \text{reach} A$ and denote $A_r := \{x : \text{dist}(x,A) = r\}$. We will first prove that

$$(4.7) \quad \text{dist}(x,B) + r = \text{dist}(x,A_r), \quad \text{whenever } x \in \mathbb{R}^d \setminus B = \text{int} A.$$ 

To this end, choose an arbitrary $x \in \text{int} A$. Obviously, there exists $y \in \partial B \subset \partial A$ such that $\text{dist}(x,B) = |x-y|$. Since $A$ has positive reach and $y \in \partial A$, there exists $z \in A_r$ such that $|y-z| = r$ (It follows, e.g., from [14] Proposition 3.1 (v),(vi)). Therefore

$$\text{dist}(x,A_r) \leq |x-z| \leq |x-y| + |y-z| = \text{dist}(x,B) + r.$$ 

To prove the opposite inequality, choose a point $z^* \in A_r$ such that $\text{dist}(x,A_r) = |x-z^*|$. Obviously, on the segment $[x,z^*]$ there exists a point $y^* \in \partial A \subset C$. Then

$$\text{dist}(x,A_r) = |x-z^*| = |x-y^*| + |y^* - z^*| \geq \text{dist}(x,B) + r,$$

and (4.7) is proved.

Now let $a \in \partial B \subset \partial A$ be given. Then $a \notin A_r$ and so by (2.2) there exists $\rho > 0$ such that $\text{dist}(\cdot, A_r)$ is DC on $U(a,\rho)$. For $x \in U(a,\rho)$, $d_B(x) = \text{dist}(x,A_r) - r$ if $x \in \text{int} A$ (by (4.7)) and $d_B(x) = 0$ if $x \notin \text{int} A$. Thus Lemma 2.1 (iv) implies that $d_B$ is DC on $U(a,\rho)$, which proves (4.6). \hfill \Box

Further recall that our main result (Theorem 3.3) asserts that

$$(4.8) \quad \text{graph} g \in D_2 \quad \text{whenever } \quad g : \mathbb{R} \rightarrow \mathbb{R} \quad \text{is DC}.$$ 

Motivated by a natural question, for which non DC functions $g$ (4.8) holds, we present the following result, whose proof is implicitly contained in the proof of [12] Proposition 6.6; see Remark 4.4 below.

**Proposition 4.3.** If $g : \mathbb{R}^{d-1} \rightarrow \mathbb{R}$ ($d \geq 2$) is locally Lipschitz and $A := \text{graph} g \in D_d$, then $g$ is DC.
Theorem 2.3.7] implies that there exist \( p \) (see [3, Proposition 2.1.5 (d)]) there exists \( \delta > 0 \) such that \( \forall z \in S \), \( d(z, A) < \delta \).

Remark 4.4. One implication of [12, Proposition 6.6] gives that if \( A \) is as in Proposition [13] (or, more generally, \( A \) is a Lipschitz manifold of dimension \( 0 < k < d \); see [13, Definition 2.4] for this notion) and \( A \) is WDC, then \( g \) is DC (or is a DC manifold of dimension \( 0 < k < d \), respectively). The proof of this implication works with an aura \( f = f_M \) of a set \( M \), but under the assumption that \( A \in \mathcal{D}_d \), the proof clearly also works, if we use the distance function \( d_A \) instead of \( f \). So we obtain not only Proposition 4.3, but also the following more general result.

If \( A \subset \mathbb{R}^d \) is a Lipschitz manifold of dimension \( 0 < k < d \) and \( A \in \mathcal{D}_d \), then \( A \) is a DC manifold of dimension \( k \).

Recall that it is an open question, whether \( \text{graph} \, g \in \mathcal{D}_d \), whenever \( g : \mathbb{R}^{d-1} \to \mathbb{R} \) is a DC function. However, using Proposition 4.3 we easily obtain:

Corollary 4.5. If \( g : \mathbb{R}^{d-1} \to \mathbb{R} \) is a semiconcave function then \( \text{graph} \, g \in \mathcal{D}_d \).

Proof. The set \( S := \{(a,b) \in \mathbb{R}^{d-1} \times \mathbb{R} : b \leq g(a)\} \) has positive reach by [6, Theorem 2.3] and consequently \( d_{\text{graph} \, g} = d_{\partial S} \) is DC by Proposition 4.3.

Remark 4.6. Let \( M \subset \mathbb{R}^d \) be a closed set whose boundary can be locally expressed as a graph of a semiconvex function (i.e., for each \( a \in \partial M \) there exist a semiconvex function \( g : \mathbb{R}^{d-1} \to \mathbb{R} \), \( \delta > 0 \) and an isometry \( \varphi : \mathbb{R}^d \to \mathbb{R}^d \) such that \( \partial M \cap U(a, \delta) = \varphi(\text{graph} \, g) \cap U(a, \delta) \). Then \( d_{\partial M} \) is locally DC (and therefore DC). By Corollary 1.2 and (2.2) and so \( M \in \mathcal{D}_d \) by Lemma 2.1 (iv) and 4.3.

Before the next results, we present the following definitions: we say that a set \( A \subset \mathbb{R}^d \) is a DC hypersurface, if there exist a vector \( v \in S^{d-1} \) and a DC function (i.e. the difference of two convex functions) \( g \) on \( W := (\text{span} \, v)^\perp \) such that \( A = \{w + g(w)v : w \in W\} \). A set \( P \subset \mathbb{R}^2 \) will be called a DC graph if it is a rotated copy of \( \text{graph}(f) \) for a DC function \( f : \mathbb{R} \to \mathbb{R} \) and some compact (possibly degenerated) interval \( \emptyset \neq I \subset \mathbb{R} \). Note that \( P \) is a DC graph if and only if it is a nonempty connected compact subset of a DC hypersurface in \( \mathbb{R}^2 \).

Proposition 4.7. Let \( d \geq 2 \) and \( F \in \mathcal{D}_d \). Then each bounded set \( C \subset \partial F \) can be covered by finitely many DC hypersurfaces.

Proof. By our assumptions, \( f := \text{dist} (\cdot, F) \) is a DC function on \( \mathbb{R}^d \) and \( f(x) = 0 \) for every \( x \in C \). So by [12, Crollary 5.4] it is sufficient to prove that for each \( x \in C \) there exists \( y^* \in \partial f(x) \) with \( |y^*| > \varepsilon := 1/4 \), where \( \partial f(x) \) is the Clarke generalized gradient of \( f \) at \( x \) (see [5, p. 27]). To this end, suppose to the contrary that \( x \in C \) and \( \partial f(x) \subset B(0,1/4) \). Since the mapping \( x \mapsto \partial f(x) \) is upper semicontinuous (see [5, Proposition 2.1.5 (d)]), there exists \( \delta > 0 \) such that \( \partial f(u) \subset U(0,1/2) \) for each \( u \in U(x, \delta) \). Since \( x \in \partial F \), we can choose \( z \in U(x, \delta/2) \setminus F \) and \( p \in \Pi_F(z) \). Then \( p \in U(x, \delta) \), \( f(z) - f(p) = |z - p| \) and Lebourg’s mean-value theorem (see [5, Theorem 2.3.7]) implies that there exist \( u \in U(x, \delta) \) and \( \alpha \in \partial f(u) \) such that

\[
\langle \alpha, z - p \rangle = f(z) - f(p) = |z - p|.
\]

Therefore \( |\alpha| \geq 1 \), which is a contradiction.

The above proposition easily implies the following fact.

Corollary 4.8. If \( F \in \mathcal{D}_2 \) then \( \partial F \) is a subset of the union of a locally finite system of DC graphs.

Using Theorem 3.3 we obtain the following easy result.

Proposition 4.9. If \( A \subset \mathbb{R}^2 \) is the union of a locally finite system of DC graphs then \( A \in \mathcal{D}_2 \).
Proof. First note that it is enough to prove that any DC graph $P$ belongs to $D_2$. Indeed, if $M$ is a locally finite system of DC graphs and each DC graph belongs to $D_2$, then $d_M$ is locally DC by (1.5) (and so DC) and $M ∈ D_2$.

So assume that $A$ is a DC graph. Without any loss of generality we may assume that $A = graph f_{[0,p]}$ for some DC function $f : ℝ → ℝ$. If $p = 0$ then $d_A = |·|$ is even convex, so assume that $p > 0$. We may also assume that $f(0) = 0$.

First note that (by Theorem 3.3 and (2.2)) $d_A$ is locally DC on $ℝ^2 \setminus \{(0, 0), (p, f(p))\}$. It remains to prove that $d_A$ is DC on some neighbourhood of $(0, 0)$ and $(p, f(p))$.

We will prove only the case of the point $(0, 0)$, the other case can be proved quite analogously. By Lemma 2.1 (iii) we can choose $L > 0$ such that $f$ is $L$-Lipschitz on $[0, p]$. Define

$$f_±(x) := \begin{cases} f(x) & \text{if } 0 ≤ x ≤ p, \\ f(p) & \text{if } p < x, \\ ±2Ly & \text{if } x < 0. \end{cases}$$

It is easy to see that both $f_+$ and $f_-$ are continuous and so they are DC by Lemma 2.1 (iv).

Put

$$M_0 := \{(u, v) ∈ ℝ^2 : u ≥ 0, v = f_+(u)\},$$

$$M_1 := \{(u, v) ∈ ℝ^2 : u ≥ 0, f_+(u) < v\} ∪ \{(u, v) ∈ ℝ^2 : u < 0, -\frac{u}{2L} < v\},$$

$$M_2 := \{(u, v) ∈ ℝ^2 : u ≥ 0, f_-(u) > v\} ∪ \{(u, v) ∈ ℝ^2 : u < 0, \frac{u}{2L} > v\}$$

and

$$M_3 := \{(u, v) ∈ ℝ^2 : \frac{u}{L} < v < -\frac{u}{L}\}.$$ 

Clearly $ℝ^2 = M_0 ∪ M_1 ∪ M_2 ∪ M_3$ and $M_1, M_2, M_3$ are open.

Set $d := \text{dist (·, } M_0)$ and, for each $y ∈ ℝ^2$, define $d_0(y) = 0$, $d_1(y) := \text{dist (y, graph } f_+)$, $d_2(y) := \text{dist (y, graph } f_-)$, $d_3(y) := |y|$. Functions $d_1$ and $d_2$ are DC on $ℝ^2$ by Theorem 3.3. $d_0$ and $d_3$ are even convex on $ℝ^2$.

Using (for $K = 1/L, -1/L, 1/(2L), -1/(2L)$) the facts that the lines with the slopes $K$ and $-1/K$ are orthogonal and $M_0 ⊂ \{(u, v) : u ≥ 0, -Lu ≤ v ≤ Lu\}$, easy geometrical observations show that

$$(4.9)\quad d_i(y) = d_0(y)\quad \text{if } y ∈ M_i, \ 0 ≤ i ≤ 3,$$

and so Lemma 2.1 (iv) implies that $d$ is DC. To finish the proof it is enough to observe that $d_A = d$ on $U(0, \frac{p}{2})$. \hfill $\square$

However, the following example shows that the opposite implication does not hold even for nowhere dense sets $A$.

**Example 4.10.** There is a nowhere dense set $A ∈ D_2$ which is not the union of a locally finite system of DC graphs.

**Proof.** Define

$$f(x) = \max(x, 0), \ x ∈ ℝ, g(x) = x^5 \cos \frac{π}{x}, \ x ∈ ℝ, \ \text{and } g_k := g \mid_{[\frac{1}{k+1}, \frac{1}{k}]}, \ k ∈ ℕ.$$

Put $A^± := \text{graph } (±f)$, $A_k := \text{graph } g_k, \ k ∈ ℕ$, and

$$A := A^+ ∪ A^- ∪ \bigcup_{k ∈ ℕ} A_k.$$

$A$ is clearly closed and nowhere dense, and it is not the union of a locally finite system of DC graphs since every DC graph $B ⊂ A$ can intersect at most one of the sets $A_k$. It remains to prove that $A ∈ D_2$. 


First we will describe all components of \( \mathbb{R}^2 \setminus A \). To this end, for each \( k \in \mathbb{N} \), define

\[
U_0(x) = \begin{cases} 
  g(x), & x \in [1/3, 1/2], \\
  f(x), & x \in [1/2, \infty), 
\end{cases} \quad U_k(x) = \begin{cases} 
  g(x), & x \in \left[ \frac{1}{2k+3}, \frac{1}{2k+2} \right], \\
  f(x), & x \in \left( \frac{1}{2k+2}, \frac{1}{2k} \right), 
\end{cases}
\]

\[
L_0(x) = -f(x), x \in [1/3, \infty), \quad L_k(x) = \begin{cases} 
  -f(x), & x \in \left[ \frac{1}{2k+3}, \frac{1}{2k+1} \right], \\
  g(x), & x \in \left( \frac{1}{2k+1}, \frac{1}{2k} \right). 
\end{cases}
\]

Set \( G_k := \{(x, y) : L_k(x) < y < U_k(x)\}, k = 0, 1, 2, \ldots \). Then

\[
G^+ := \{(x, y) : f(x) < y\}, G^- := \{(x, y) : y < -f(x)\} \quad \text{and} \quad G_0, G_1, \ldots
\]

are clearly all components of \( \mathbb{R}^2 \setminus A \).

Recall that both \( U_k \) and \( L_k \) is defined on \( D_k \), where \( D_k = \left[ \frac{1}{2k+3}, \frac{1}{2k} \right] \) for \( k \in \mathbb{N} \) and \( D_0 = [1/3, \infty) \). Using the facts that \( D_k \) and \( D_{k+2} \) are disjoint \((k = 0, 1, \ldots)\),

\[
U_k \left( \frac{1}{2k+3} \right) = L_k \left( \frac{1}{2k+3} \right) = g \left( \frac{1}{2k+3} \right), \quad U_k \left( \frac{1}{2k} \right) = L_k \left( \frac{1}{2k} \right) = g \left( \frac{1}{2k} \right)
\]

and \( U_0(1/3) = L(1/3) = g(1/3) \), it is easy to see that there exist unique functions \( U, \bar{U} \) which are continuous on \( \mathbb{R} \), \( U \) (resp. \( \bar{U} \)) extends all \( U_k, k = 0, 2, 4, \ldots \) (resp. \( k = 1, 3, 5, \ldots \)) and \( U \) (resp. \( \bar{U} \)) equals to \( g \) at all points at which no \( U_k, k = 0, 2, 4, \ldots \) (resp. \( k = 1, 3, 5, \ldots \)) is defined. Quite analogously a continuous function \( L \) (resp. \( \bar{L} \)) extending all \( L_k, k = 0, 2, 4, \ldots \) (resp. \( k = 1, 3, 5, \ldots \)) is defined. Since the functions \( g, f, -f \) are DC, Lemma \( \text{(2.1)} \) (iv) implies that the functions \( U, \bar{U}, L, \bar{L} \) are DC. So Theorem \( \text{(3.3)} \) implies that the distance functions

\[
d_A^+, d_A^-, d_{\text{graph } U}, d_{\text{graph } \bar{U}}, d_{\text{graph } L}, d_{\text{graph } \bar{L}}
\]

are DC.

Obviously \( d_A(x) = 0 \) for \( x \in A \), \( d_A(x) = d_A^+(x) \) for \( x \in G^+ \) and \( d_A(x) = d_A^-(x) \) for \( x \in G^- \). Further, if \( x \in G_k \) with \( k = 2, 4, 6, \ldots \), then

\[
d_A(x) \in \{d_{\text{graph } U}(x), d_{\text{graph } \bar{L}}(x)\},
\]

which easily follows from the facts that

\[
\partial G_k \subset (\text{graph } U \cup \text{graph } L) \quad \text{and} \quad (\text{graph } U \cup \text{graph } L) \cap G_k = \emptyset.
\]

Similarly we obtain that, if \( x \in G_k \) with \( k = 1, 3, 5, \ldots \), then

\[
d_A(x) \in \{d_{\text{graph } \bar{U}}(x), d_{\text{graph } L}(x)\}.
\]

Thus, using \( \text{(4.10)} \) and Lemma \( \text{(2.1)} \) (iv), we obtain that \( d_A \) is DC.

\[
\square
\]

It seems that there does not exist an essentially simpler example. Iterating the construction of the example we can obtain nowhere dense sets in \( D_2 \) of quite complicated topological structure.

In our opinion, using Proposition \( \text{(4.7)} \) and Theorem \( \text{(3.3)} \) it is possible to give an optimal complete characterisation of sets in \( D_2 \), but it appears to be a rather hard task. We believe that we succeeded to find some characterisation, however, it is not quite satisfactory and our current proof is very technical. We aim to find a better characterisation, hopefully with a simpler proof.
References

[1] M. Bačák, J. Borwein, On difference convexity of locally Lipschitz functions, Optimization 60 (2011), 961–978.
[2] P. Cannarsa, C. Sinestrari, Semiconcave Functions, Hamilton-Jacobi Equations, and Optimal Control, Progress in Nonlinear Differential Equations and their Applications 58, Birkhäuser, Boston, 2004.
[3] F. Clarke, Optimization and Nonsmooth Analysis. SIAM, Philadelphia, 1990.
[4] S. Cobzas, C. Mustata, Norm-preserving extension of convex Lipschitz functions, J. Approx. Theory, 24 (1978) 236–244.
[5] A. Colesanti, D. Hug, Hessian measures of semi-convex functions and applications to support measures of convex bodies, Manuscripta Math. 101 (2000) 209-238.
[6] J.H.G. Fu, Tubular neighborhoods in Euclidean spaces, Duke Math. J. 52 (1985) no. 4 1025–1046.
[7] J.H.G. Fu, Integral geometric regularity, In: Kiderlen, M., Vedel Jensen, E.B. (eds.) Tensor Valuations and Their Applications in Stochastic Geometry and Imaging, 261–299, Lecture Notes in Math. 2177, Springer, 2017.
[8] J.H.G. Fu, D. Pokorný, J. Rataj, Kinematic formulas for sets defined by differences of convex functions, Adv. Math. 311 (2017) 796–832.
[9] P. Hartman, On functions representable as a difference of convex functions. Pacific J. Math. 9 (1959) 707-713.
[10] I.P. Natanson, Theory of functions of a real variable, vol. I, Frederick Ungar Publishing Co., New York, 1955.
[11] D. Pokorný, J. Rataj, Normal cycles and curvature measures of sets with d.c. boundary. Adv. Math. 248, (2013), 963–985.
[12] D. Pokorný, J. Rataj, L. Zajíček, On the structure of WDC sets. Math. Nachr. (published online 29 March 2019).
[13] D. Pokorný, L. Zajíček, Remarks on WDC sets, preprint available at http://arxiv.org/abs/1905.12709
[14] J. Rataj, L. Zajíček, On the structure of sets with positive reach. Math. Nachr. 290 (2017) 1806–1829.
[15] A.W. Roberts, D. Varberg, Convex Functions, Pure and Applied Mathematics, vol. 57, Academic Press, New York-London, 1973.
[16] H.L. Royden, Real analysis, third edition, Macmillan Publishing Company, New York, 1988.
[17] H. Tuy, Convex Analysis and Global Optimization, 2nd ed., Springer Optimization and Its Applications 110, Springer, 2016.
[18] L. Veselý, L. Zajíček, Delta-convex mappings between Banach spaces and applications, Dissertationes Math. (Rozprawy Mat.) 289 (1989), 52 pp.
[19] L. Veselý, L. Zajíček, On vector functions of bounded convexity, Math. Bohem. 133 (2008) 321–335.

E-mail address: dpokorny@karlin.mff.cuni.cz
E-mail address: zajicek@karlin.mff.cuni.cz