Twisted boundary states in Kazama-Suzuki models

Hiroshi ISHIKAWA ¹ and Taro TANI ²

Department of Physics, Tohoku University
Sendai 980-8578, JAPAN

We construct Cardy states in the Kazama-Suzuki model $G/H \times U(1)$, which satisfy the boundary condition twisted by the automorphisms of the coset theory. We classify all the automorphisms of $G/H \times U(1)$ induced from those of the $G$ theory. The automorphism group contains at least a $\mathbb{Z}_2$ as a subgroup corresponding to the charge conjugation. We show that in several models there exist extra elements other than the charge conjugation and that the automorphism group can be larger than $\mathbb{Z}_2$. We give the explicit form of the twisted Cardy states which are associated with the non-trivial automorphisms. It is shown that the resulting states preserve the $\mathbb{N}=2$ superconformal algebra. As an illustration of our construction, we give a detailed study for two hermitian symmetric space models $SU(4)/SU(2) \times SU(2) \times U(1)$ and $SO(8)/SO(6) \times U(1)$ both at level one. We also study the action of the level-rank duality on the Cardy states and find the relation with the exceptional Cardy states originated from a conformal embedding.

¹ishikawa@tuhep.phys.tohoku.ac.jp
²tani@tuhep.phys.tohoku.ac.jp
1 Introduction

The understanding of geometry at small distance is one of the fundamental issues in string theory. Due to the existence of the string scale, we are forced to modify the classical notion of geometry in the stringy regime. D-branes are appropriate objects for studying this regime since they can probe the distance smaller than the string scale. From the worldsheet point of view, D-branes are expressed as boundary states in an N=2 superconformal field theory (SCFT) describing the target space of string. One expects that the information about the stringy geometry can be extracted from the study of boundary states in N=2 SCFTs, see for example [1,2,3,4]. The classification of all the boundary states in a given SCFT is therefore a crucial step towards the understanding of stringy geometry.

Every consistent set of boundary states in a rational CFT should satisfy the sewing relations [5,6,7], which include the so-called Cardy condition [5]. Finding a set of Cardy states, i.e., the states satisfying the Cardy condition, is equivalent to finding a non-negative integer matrix representation (NIM-rep) of the fusion algebra [8,9]. In any fusion algebra, there exists at least one NIM-rep, called the regular NIM-rep, since the fusion coefficients themselves form a representation of the fusion algebra. One can construct corresponding Cardy states in the charge conjugation or the diagonal modular invariant, which are called the regular Cardy states [5]. However, it is in general not clear whether there are other Cardy states compatible with the regular ones in the same modular invariant. One way to obtain such states is to twist the boundary condition by the automorphism of the chiral algebra [10,11]. (The regular Cardy states correspond to the trivial automorphism.) It is therefore important to classify all the automorphisms of a given chiral algebra in order to find the Cardy states compatible with the regular states.

The Kazama-Suzuki models [12,13] are a wide class of N=2 SCFTs based on the coset construction, which contain the N=2 minimal models as a special case. The algebraic properties of boundary states such as intersection numbers have been studied in [14,15]. However, the analysis is limited to the states corresponding to the trivial or the charge conjugation automorphisms. It is then natural to raise a question whether there are other automorphisms in the Kazama-Suzuki models. Once we have a non-trivial automorphism in the coset chiral algebra, we can obtain the corresponding Cardy states by the general procedure to construct boundary states in coset CFTs, which are developed in [16,17,18]. (For other works on boundary states in coset theories, see [19,20,21,22,23,24,25,26].)

In this paper, we give a systematic study of twisted Cardy states in the Kazama-Suzuki models $G/H$. We restrict ourselves to the simplest case [28,29], i.e., the models with

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3It should be noted that the Cardy condition is a necessary condition on boundary states. Actually, there is a NIM-rep which does not give rise to any consistent boundary states, such as the tadpole NIM-rep of $su(2)_{2n-1}$ (see e.g. [5]).

4For an earlier attempt, see [27].
a single $U(1)$ factor in $H$ and rank $\mathfrak{g} = \text{rank } \mathfrak{h}$. We first classify automorphisms of $G/H$ induced from those of the $G$ theory. Clearly, the automorphism group of the Kazama-Suzuki models contains at least a $\mathbb{Z}_2$ as a subgroup, since the charge conjugation acts on the N=2 superconformal algebra non-trivially. To obtain the Cardy states other than those from the charge conjugation, therefore, the automorphism group should be larger than $\mathbb{Z}_2$. We show that in several models the automorphism group actually contains non-trivial elements other than the charge conjugation. Among the hermitian symmetric space (HSS) models, two series of models have non-trivial automorphism group, namely, $SU(2n)/SU(n) \times SU(n) \times U(1)$ and $SO(2n)/SO(2n-2) \times U(1)$. We construct the Cardy states subject to the boundary condition twisted by the automorphisms we have found. Since the Kazama-Suzuki models are based on the N=1 super Kac-Moody algebra, it is obvious that the resulting twisted Cardy states keep the N=1 SCA. However, there is a priori no reason that the N=2 SCA is also preserved. We show that all the boundary conditions we have found keep the N=2 SCA.

Some of the Kazama-Suzuki models are related via the level-rank duality [12, 30, 31, 32, 33, 34]. For example, the Grassmannian model $SU(m+n)/SU(m) \times SU(n) \times U(1)$ at level $k$ is equivalent to the model $SU(k+n)/SU(k) \times SU(n) \times U(1)$ at level $m$. It is interesting to examine how this duality acts on the boundary states. The primary fields, and hence the regular Cardy states, are mapped with each other under this duality. However, the correspondence among the twisted Cardy states is not so clear, since the automorphism group we have found is model-dependent. As an example, we take a Grassmannian model $SU(4)/SU(2) \times SU(2) \times U(1)$ at level one, which is equivalent to $SU(3)/SU(2) \times U(1)$ at level two. The automorphism group of the former model is $\mathbb{Z}_2 \times \mathbb{Z}_2$. Therefore, we obtain non-trivial Cardy states other than those from the charge conjugation in the former model. We find that the level-rank duality maps these states to the states in the latter model not originated from the automorphism of the current algebra of $G$. We show that the resulting states are described by a NIM-rep obtained from the conformal embedding $\mathfrak{su}(3)_2 \oplus \mathfrak{su}(2)_3 \subset \mathfrak{su}(6)_1$ by the procedure developed in [17].

The organization of this paper is as follows. We review in section 2 the construction of boundary states in coset theories. In section 3 we determine the automorphism group of the Kazama-Suzuki models and show that the corresponding twisted boundary conditions keep the N=2 SCA. In section 4 we present the explicit construction of twisted Cardy states in two examples: $SU(4)/SU(2) \times SU(2) \times U(1)$ at level one (section 4.2.1) and $SO(8)/SO(6) \times U(1)$ at level one (section 4.2.2). These models belong to the N=2 minimal series, for which all the N=2 Cardy states are known. We check that the twisted Cardy states we have found together with the regular ones reproduce all the Cardy states for the minimal models. The action of the level-rank duality is discussed in section 4.3. Section 5 is devoted to summary and discussion. Some technical details for constructing coset boundary states are
summarized in appendix A. In appendix B we present the superfield description of the N=2 SCA. In appendix C we argue the complex structure of the Kazama-Suzuki models. Field identifications and selection rules [35, 36, 31, 32] used in section 4 are reviewed in appendix D. The explicit form of the twisted Cardy states considered in section 4.2 is given in appendix E.

2 Boundary states in coset theories

In this section, we review the construction of the Cardy states in coset theories developed in [16, 17, 18].

2.1 WZW models

We begin with the WZW models since the coset theories are based on them. The chiral algebra $\mathcal{A}$ of the $G$-WZW model is the affine Lie algebra $g$. We denote by $\mathcal{I}$ the set $P^k_+(g)$ of all the integrable highest-weight representations of $g$ at level $k$. In this section, we mainly consider the bulk theory with the charge conjugation modular invariant

$$Z = \sum_{\lambda \in \mathcal{I}} |\chi_\lambda|^2,$$

where $\chi_\lambda$ is the character of the representation $\lambda \in \mathcal{I}$. The space $\mathcal{H}$ of the states in this theory is decomposed as

$$\mathcal{H} = \bigoplus_{\lambda \in \mathcal{I}} \mathcal{H}_\lambda \otimes \tilde{\mathcal{H}}_\lambda,$$

where $\mathcal{H}_\lambda$ ($\tilde{\mathcal{H}}_\lambda$) is the representation space in the (anti-) chiral sector. The case of the diagonal modular invariant will be considered later in this subsection.

With the presence of boundaries, the chiral and the anti-chiral sectors are related with each other. This is expressed in terms of boundary conditions. The simplest one is

$$J_n + \tilde{J}_{-n} = 0,$$

where $J$ ($\tilde{J}$) is the current of the (anti-) chiral sector. Associated with this boundary condition, we can construct the regular Cardy states $|^\alpha\rangle$

$$|^\alpha\rangle = \sum_{\lambda \in \mathcal{I}} \frac{S_{n\lambda}}{\sqrt{|\delta_{0\lambda}|}} |\lambda\rangle, \quad \alpha \in \mathcal{I}. $$

Here 0 is the vacuum representation and $\{|\lambda\rangle\mid \lambda \in \mathcal{I}\}$ are the Ishibashi states [37] with the normalization

$$\langle \lambda | \tilde{q}^{H_c/2} | \mu \rangle = \delta_{\lambda\mu} \chi_\lambda(-1/\tau).$$

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Table 1: The twisted affine Lie algebras and the corresponding orbit Lie algebras.

\[
\begin{array}{cccccc}
\mathfrak{g}^{(1)} & A_{1}^{(1)} & A_{2}^{(1)} & D_{1}^{(1)} & E_{6}^{(1)} & D_{4}^{(1)} \\
\mathfrak{g}^{(r)} & A_{2}^{(2)} & A_{2}^{(2)} & D_{4}^{(2)} & E_{6}^{(2)} & D_{4}^{(3)} \\
\tilde{\mathfrak{g}}^{(r)} & A_{2}^{(2)} & D_{4}^{(2)} & A_{2}^{(2)} & E_{6}^{(2)} & D_{4}^{(3)}
\end{array}
\]

There are other boundary states in the charge conjugation modular invariant, called the twisted Cardy states, which are associated with the twisted boundary condition

\[\omega(J_n) + \tilde{J}_{-n} = 0.\]  

Here \(\omega\) is an outer automorphism of the affine Lie algebra \(\mathfrak{g}\). The labels of the Ishibashi states satisfying (2.6) are limited to those fixed by \(\omega\), which we denote by \(\mathcal{I}_{\omega}\),

\[\mathcal{I}_{\omega} = \{\lambda \in \mathcal{I} | \omega(\lambda) = \lambda\}.\]  

Since \(\omega\) acts on \(\mathcal{I}\) non-trivially, \(\mathcal{I}_{\omega}\) is a proper subset of \(\mathcal{I}\). Corresponding Cardy states \(|\tilde{\alpha}\rangle_{\omega}\) are written as \([10, 11]\)

\[|\tilde{\alpha}\rangle_{\omega} = \sum_{\lambda \in \mathcal{I}_{\omega}} \tilde{S}_{\tilde{\alpha}\lambda} |\lambda\rangle_{\omega}, \quad \tilde{\alpha} \in \tilde{\mathcal{I}}.\]  

\(\tilde{\mathcal{I}}\) is the set of all the possible representations of the twisted chiral algebra \(\mathcal{A}_{\omega}\), an algebra generated by the currents with the twisted boundary condition \(J(e^{2\pi i} z) = \omega(J(z))\). \(\tilde{S}_{\lambda\lambda}^{\ast} = 1\) (2.9)

Here \(\chi_{\tilde{\lambda}}\) is the character of the representation \(\tilde{\lambda} \in \tilde{\mathcal{I}}\), \(r\) is the order of the automorphism \(\omega\) and \(\chi_{\lambda}^{\omega}\) is the twining character [38]. For \(\mathcal{A} = \mathfrak{g}^{(1)}\), one can take as \(\omega\) the diagram automorphism of the horizontal subalgebra. Then the twisted chiral algebra is the twisted affine Lie algebra \(\mathcal{A}_{\omega} = \mathfrak{g}^{(r)}\) and the twining character \(\chi_{\lambda}^{\omega}\) is given by the character of the orbit Lie algebra \(\tilde{\mathfrak{g}}^{(r)}\) [38] (see Table 1, \(\tilde{\lambda} \in \tilde{\mathcal{I}} = P_{+}^{k}(\mathfrak{g}^{(r)}), \quad \lambda \in \mathcal{I}_{\omega} \cong P_{+}^{k}(\tilde{\mathfrak{g}}^{(r)}).\) (2.10)

The twisted Cardy states are compatible with the regular ones (2.9). In particular, one can show

\[\omega(\tilde{\alpha})q^{H/2}|0\rangle_{\omega = 1} = \chi_{\tilde{\alpha}}(\tau/r).\]  

\[4\]
This is the reason that the twisted Cardy states are labeled by the representations \( \tilde{I} \) of the twisted chiral algebra.

When the affine Lie algebra is not simple, one can consider automorphisms mixing several factors. Suppose that the chiral algebra is a direct sum of some affine Lie algebra \( \mathcal{A} \),

\[
\mathcal{A}^r = \mathcal{A} \oplus \mathcal{A} \oplus \cdots \oplus \mathcal{A}.
\]  

(2.12)

Clearly, the cyclic permutation \( \pi \) of the factors \( \mathcal{A} \) is an automorphism of \( \mathcal{A}^r \). One can use this automorphism \( \pi \) to twist boundary conditions and construct the corresponding twisted Cardy states \([39]\). In this case, both of the set \( \mathcal{I}^\pi \) labeling the Ishibashi states and the set \( \tilde{\mathcal{I}} \) labeling the twisted Cardy states are identified with \( \mathcal{I} \),

\[
\tilde{\lambda} \in \tilde{\mathcal{I}} = \mathcal{I}, \quad \Lambda \in \mathcal{I}^\pi = \{ (\lambda, \lambda, \ldots, \lambda) | \lambda \in \mathcal{I} \}.
\]  

(2.13)

The twining character is given by \( \chi^\pi_\Lambda (-1/\tau) = \chi_\lambda (-r/\tau) \) and the modular transformation matrix \( \tilde{S} \) is equal to that for \( \mathcal{A} \).

Before concluding this subsection, we comment on the construction of the Cardy states in the diagonal modular invariant. The natural boundary condition for the diagonal modular invariant is that twisted by the charge conjugation \( \omega_c \),

\[
\omega_c (J_n) + \tilde{J}_{-n} = 0.
\]  

(2.14)

One can construct the Cardy states corresponding to this boundary condition by replacing the Ishibashi states \( |\lambda\rangle \rangle \) in the regular Cardy states (2.4) with \( |\lambda\rangle \rangle_{\omega_c} \) satisfying the condition (2.14),

\[
|\alpha\rangle = \sum_{\lambda \in \mathcal{I}} \frac{S_{\alpha \lambda}}{\sqrt{S_0 \lambda}} |\lambda\rangle \rangle_{\omega_c}, \quad \alpha \in \mathcal{I}.
\]  

(2.15)

We also call these states the regular Cardy states in the case of the diagonal modular invariant. Accordingly, in this case, we refer to the following boundary condition as the twisted one,

\[
\omega \omega_c (J_n) + \tilde{J}_{-n} = 0,
\]  

(2.16)

where \( \omega \) is an automorphism of the chiral algebra. The corresponding twisted Cardy states take the same form as those for the charge conjugation modular invariant (see eq. (2.8)) except that we have to use the twisted Ishibashi states \( |\lambda\rangle \rangle_{\omega \omega_c} \) instead of \( |\lambda\rangle \rangle_\omega \),

\[
|\tilde{\alpha}\rangle_\omega = \sum_{\lambda \in \mathcal{I}^\omega} \frac{\tilde{S}_{\tilde{\alpha} \lambda}}{\sqrt{S_0 \lambda}} |\lambda\rangle \rangle_{\omega \omega_c}, \quad \tilde{\alpha} \in \tilde{\mathcal{I}}.
\]  

(2.17)

To summarize, the Cardy states in the diagonal modular invariant have exactly the same form as those in the charge conjugation modular invariant. The difference of these two sets is only in the expression of the Ishibashi states, and we can translate one set into the other by taking appropriate Ishibashi states.
2.2 \( G/H \) theories

A coset model \( G/H \) is based on an embedding of the affine Lie algebra \( \mathfrak{h} \) into \( \mathfrak{g} \) [40]. A representation \( \lambda \) of \( \mathfrak{g} \) is decomposed into representations \( \mu \) of \( \mathfrak{h} \) as follows,

\[
\mathcal{H}_\lambda^G = \bigoplus_\mu \mathcal{H}_{(\lambda; \mu)}^H.
\]

(2.18)

Hence a representation of the \( G/H \) theory is labeled by a pair of representations of the \( G \) and the \( H \) theories. However, not all the pairs appear in this decomposition (selection rule) and more than one pairs may give the same representation (field identification) [35, 36, 31]. Consequently, the spectrum of the \( G/H \) theory reads

\[
\hat{\mathcal{I}} = \{ (\mu; \nu) | \mu \in \mathcal{I}^G, \nu \in \mathcal{I}^H, b_\mu^G(J) = b_\nu^H(J'), (J\mu; J'\nu) = (\mu; \nu), \forall (J/J') \in \mathcal{G}_{\text{id}} \}.
\]

(2.19)

Here \( \mathcal{I}^G (\mathcal{I}^H) \) is the set of all the representations in the \( G \) (\( H \)) theory. The condition \( b_\mu^G(J) = b_\nu^H(J') \) expresses the selection rule, while the relation \( (J\mu; J'\nu) = (\mu; \nu) \) stands for the field identification. \( J\mu \) is the fusion of \( \mu \) with \( J \in \mathcal{G}_{\text{sc}}^G \), the simple current of the \( G \) theory [41, 42]. \( b_\mu^G(J) \) is defined as

\[
b_\mu^G(J) = \frac{S_{\mu J}^G S_{0J}^G}{S_{\mu J}^G S_{0J}^H} (J \equiv J0),
\]

(2.20)

and takes values in roots of unity.\(^5\) All the field identifications \((J/J')\) form a subgroup of \( \mathcal{G}_{\text{sc}}^G \times \mathcal{G}_{\text{sc}}^H \), which is called the identification current group \( \mathcal{G}_{\text{id}} \). The modular transformation matrix \( \hat{S} \) of the \( G/H \) theory is given by those of the \( G \) and \( H \) theories,

\[
\hat{S}_{(\mu; \nu)(\mu'; \nu')} = |\mathcal{G}_{\text{id}}| S_{\mu \mu'}^G S_{\nu \nu'}^H,
\]

(2.21)

if the identification current group \( \mathcal{G}_{\text{id}} \) has no fixed points.

One can construct the regular Cardy states of the \( G/H \) theory in the same way as the WZW models,

\[
|\langle \alpha; \beta \rangle \rangle = \sum_{(\lambda; \mu) \in \hat{\mathcal{I}}} \frac{\hat{S}_{(\alpha; \beta)(\lambda; \mu)}}{\sqrt{\hat{S}_{(00)(\lambda; \mu)}}} |\lambda; \mu \rangle \rangle, \quad (\alpha; \beta) \in \hat{\mathcal{I}}.
\]

(2.22)

As we have seen in the previous subsection, if there are automorphisms of the chiral algebra, one can twist boundary conditions and construct the corresponding twisted Cardy states which coexist with the regular ones. Therefore, it is important to know what kind of automorphisms exist in the \( G/H \) theory. As is discussed in [16, 17], an automorphism \( \omega^0 \in \text{Aut}(\mathfrak{g}) \) of the \( G \) theory induces an automorphism \( \hat{\omega} \) of the \( G/H \) theory if \( \omega^0 \) can be restricted on \( \mathfrak{h} \), i.e., \( \omega^0(\mathfrak{h}) = \mathfrak{h} \). (In general, not all automorphisms of the coset theory are

\(^5\) \( b_\mu^G(J) \) is nothing but the exponential of the monodromy charge, \( b_\mu^G(J) = e^{2\pi i Q_J(\mu)} \) [41].
Here we denote by $\text{Aut}(\mathfrak{g})$ the group of automorphisms of $\mathfrak{g}$. $\text{Aut}(\mathfrak{g})$ takes the following form

$$\text{Aut}(\mathfrak{g}) = \text{Ad}(G) \rtimes D(\mathfrak{g}), \quad (2.23)$$

where $\text{Ad}(G)$ is the group of inner automorphisms $X \mapsto \text{Ad}(g)X = gXg^{-1}$ ($g \in G$) and $D(\mathfrak{g})$ is the group of diagram automorphisms. Clearly, the automorphisms that keep $\mathfrak{h} \subset \mathfrak{g}$ invariant form a subgroup of $\text{Aut}(\mathfrak{g})$, which we denote by $\text{Aut}_\mathfrak{h}(\mathfrak{g})$

$$\text{Aut}_\mathfrak{h}(\mathfrak{g}) = \{\omega^g \in \text{Aut}(\mathfrak{g}) | \omega^g(\mathfrak{h}) = \mathfrak{h}\}. \quad (2.24)$$

Each element $\omega^g \in \text{Aut}_\mathfrak{h}(\mathfrak{g})$ induces an automorphism $\hat{\omega}$ of the $G/H$ theory defined as follows,

$$\hat{\omega} : (\mu; \nu) \mapsto (\omega^g(\mu); \omega^h(\nu)), \quad (\mu; \nu) \in \hat{I}, \quad (2.25)$$

where $\omega^h = \omega^g|_h$.

Not all the automorphisms $\omega^g \in \text{Aut}_\mathfrak{h}(\mathfrak{g})$ induce non-trivial automorphisms in the coset theory. Let us take an element $h \in H$ and consider its adjoint action on the currents, $J \mapsto \text{Ad}(h)J = hJh^{-1}$. Although $\omega^g = \text{Ad}(h)$ is a non-trivial element of $\text{Aut}_\mathfrak{h}(\mathfrak{g})$, the corresponding automorphism $\hat{\omega}$ acts trivially on the coset theory. In order to see this, let us note that the character $\chi^G_{\mu}$ of $\mu \in I^G$ with the insertion of $h \in H$ has the following branching rule

$$\chi^G_{\mu}(\tau, z) = \sum_{\nu \in I^H} \chi_{(\mu; \nu)}(\tau)\chi^H_{\nu}(\tau, z), \quad (2.26)$$

where $z$ is a weight of $\mathfrak{h}$ depending only on the conjugacy class of $h \in H$. Since $z$ is a weight of $\mathfrak{h}$, the branching function $\chi_{(\mu; \nu)}$ does not depend on $z$. The insertion of $h$ corresponds to the twist by $\omega^g = \text{Ad}(h)$ and the independence of $\chi_{(\mu; \nu)}$ on $z$ shows that $\text{Ad}(h)$ acts trivially on the coset theory.

All the conjugations $\text{Ad}(h)$ by the elements $h$ of $H$ form a normal subgroup $\text{Ad}(H) \triangleleft \text{Aut}_\mathfrak{h}(\mathfrak{g})$. Since $\text{Ad}(H)$ acts on the $G/H$ theory trivially, we define the group of induced automorphisms of the $G/H$ theory as the quotient of $\text{Aut}_\mathfrak{h}(\mathfrak{g})$ by $\text{Ad}(H)$

$$\text{Aut}(G/H) = \text{Aut}_\mathfrak{h}(\mathfrak{g})/\text{Ad}(H). \quad (2.27)$$

Let $[\omega^g]$ ($\omega^g \in \text{Aut}_\mathfrak{h}(\mathfrak{g})$) be an element of $\text{Aut}(G/H)$. We can take the representative $\omega^g$ such that $\omega^g$ maps the Cartan subalgebra $t_h$ of $\mathfrak{h}$ to itself. In general, $\omega^g \in \text{Aut}_\mathfrak{h}(\mathfrak{g})$ maps $t_h$ to another maximal Abelian subalgebra of $\mathfrak{h}$. However, all the maximal Abelian subalgebras of $\mathfrak{h}$ are conjugate to each other by the action of $\text{Ad}(H)$. Therefore, we can choose an element $h \in H$ so that the following equation holds,

$$\omega^g(t_h) = \text{Ad}(h)(t_h), \quad (2.28)$$

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which means that \( \text{Ad}(h^{-1}) \omega^g \) maps \( t_h \) to itself. Since we consider the quotient by \( \text{Ad}(H) \), \( \omega^g \)
and \( \text{Ad}(h^{-1}) \omega^g \) determine the same element in Aut\((G/H)\), \( [\omega^g] = [\text{Ad}(h^{-1}) \omega^g] \). To determine the group Aut\((G/H)\), it is hence sufficient to consider the automorphisms that map \( t_h \) to itself.

Furthermore, when \( \text{rank } g = \text{rank } h \), we can express the representatives in more concise form. In this case, the Cartan subalgebras of \( g \) and \( h \) coincide \( t_h = t_g \), and we can take the representative of \( [\omega^g] \in \text{Aut}(G/H) \) from automorphisms that map \( t_g \) to itself, which form the group

\[
\text{Aut}_t(g) = \{ \omega^g \in \text{Aut}(g) \mid \omega^g(t_g) = t_g \}. \tag{2.29}
\]

Therefore, the group of induced automorphism of the \( G/H \) theory \( \text{(2.27)} \) can be written as

\[
\text{Aut}(G/H) = \text{Aut}_t(g) \cap \text{Aut}_h(g) / \text{Aut}_t(g) \cap \text{Ad}(H). \tag{2.30}
\]

It is known \([43]\) that there is a homomorphism \( \varphi \) from \( \text{Aut}_t(g) \) onto the automorphism group \( \text{Aut}(Q^g) \) of the root lattice \( Q^g \) of \( g \)

\[
\varphi : \text{Aut}_t(g) \to \text{Aut}(Q^g). \tag{2.31}
\]

The kernel of \( \varphi \) is \( \text{Ad}(T^G) \), where \( T^G \) is the Cartan subgroup of \( G \) corresponding to \( t_g \). The numerator of \( \text{(2.30)} \) is mapped by this homomorphism \( \varphi \) onto the subgroup \( \text{Aut}_{Q^h}(Q^g) \subset \text{Aut}(Q^g) \) that keep the root lattice \( Q^h \) of \( h \) fixed

\[
\text{Aut}_{Q^h}(Q^g) = \{ \omega^g \in \text{Aut}(Q^g) \mid \omega^g(Q^h) = Q^h \}. \tag{2.32}
\]

On the other hand, the denominator is the inverse image of the Weyl group \( W(h) \) of \( h \), i.e., \( \text{Aut}_t(g) \cap \text{Ad}(H) = \varphi^{-1}(W(h)) \). From the homomorphism theorem, one therefore obtains the following isomorphism

\[
\text{Aut}_t(g) \cap \text{Aut}_h(g) / \text{Aut}_t(g) \cap \text{Ad}(H) \cong \text{Aut}_{Q^h}(Q^g) / W(h). \tag{2.33}
\]

In conclusion, when \( \text{rank } g = \text{rank } h \), one can express the group \( \text{Aut}(G/H) \) of induced automorphisms in terms of the automorphism group \( \text{Aut}(Q^g) \) of the root lattice as follows

\[
\text{Aut}(G/H) \cong \text{Aut}_{Q^h}(Q^g) / W(h). \tag{2.34}
\]

The twisted Cardy states corresponding to \( \hat{\omega} \) are written as \([16, 18]\)

\[
|\langle \tilde{\alpha}; \tilde{\beta} \rangle\rangle_{\hat{\omega}} = \sum_{(\lambda, \mu) \in \bar{\mathcal{I}}} \frac{\hat{S}^G_{(\tilde{\alpha}; \tilde{\beta})(\lambda; \mu)}}{\sqrt{\hat{S}^G(0;0)(\lambda; \mu)}} |\langle \lambda; \mu \rangle\rangle_{\hat{\omega}}, \quad (\tilde{\alpha}; \tilde{\beta}) \in \bar{\mathcal{I}}, \tag{2.35}
\]

where the boundary state coefficients \( \hat{S}^G \) consist of those for the twisted Cardy states \( \text{(2.8)} \) associated with the automorphism \( \omega^g \) and \( \omega^h \),

\[
\hat{S}^G_{(\tilde{\alpha}; \tilde{\beta})(\lambda; \mu)} = NS^G_{\tilde{\alpha} \tilde{\lambda}} S^H_{\tilde{\beta} \mu}. \tag{2.36}
\]
For $N$ and other definitions, see below. Similarly to the case of the WZW models \[ (2.7) \], the set labeling the twisted Ishibashi states is the set of all the representations fixed by $\hat{\omega}$ \[ 6 \],

$$\hat{\mathcal{T}}^0 = \{ (\lambda; \mu) \in \hat{\mathcal{T}} | \hat{\omega}((\lambda; \mu)) = (\lambda; \mu) \} = \{ (\lambda; \mu) \in \hat{\mathcal{T}} | \omega^g(\lambda) = \lambda, \omega^h(\mu) = \mu \}. \tag{2.37}$$

As is seen in the definition \[ (2.38) \], the labels of the twisted Cardy states are expressed by pairs of those for the $G$ and the $H$ theories. In the way analogous to the case of the regular Cardy states, we need some selection rule and identification of labels in order to obtain a consistent set of states, namely, the brane selection rule and the brane identification \[ 16, 17 \]. For the case of regular Cardy states, both of the selection rule and the identification are described by the identification current group $G_{id}$ since the labels of the Ishibashi states and the Cardy states belong to the same set $\hat{\mathcal{I}}$. On the other hand, in the case of the twisted Cardy states, the set labeling the Cardy states is distinct from that for the Ishibashi states, and we need two types of identification groups $G_{id}(\mathcal{I}^g)$ and $G_{id}(\mathcal{I})$ to define the set labeling the twisted Cardy states. One can determine these groups once the twisted Cardy states for the $G$ and the $H$ theories are given \[ 16, 17 \]. See appendix \[ A \] for the definition. The set of the labels for the twisted Cardy states is then defined as

$$\hat{\mathcal{I}} = \{ (\tilde{\alpha}; \tilde{\beta}) | \tilde{\alpha} \in \hat{\mathcal{I}}^G, \tilde{\beta} \in \hat{\mathcal{I}}^H, \tilde{b}^G_{\tilde{\alpha}}(J) = \tilde{b}^H_{\tilde{\beta}}(J'), \forall (J/J') \in G_{id}(\mathcal{I}^g); (J\tilde{\alpha}; J'\tilde{\beta}) = (\tilde{\alpha}; \tilde{\beta}), \forall (J/J') \in G_{id}(\mathcal{I}) \}. \tag{2.38}$$

The phase $\tilde{b}^G_{\tilde{\alpha}}(J)$ and the action $\tilde{\alpha} \mapsto J\tilde{\alpha}$ of the simple currents on the Cardy states are also defined in appendix \[ A \] The coefficient $N$ in \[ (2.36) \] is given by the order of the identification groups, namely,

$$N = |G_{id}(\mathcal{I}^g)| = |G_{id}(\mathcal{I})|. \tag{2.39}$$

3 Boundary conditions in Kazama-Suzuki models

3.1 Kazama-Suzuki models

The Kazama-Suzuki models \[ 12, 13 \] are rational $N=2$ superconformal field theories obtained by applying the coset construction to the $N=1$ super Kac-Moody algebras $\mathfrak{h} \subset \mathfrak{g}$. The $N=2$ superconformal symmetry is obtained when the coset space $G/H$ is a Kähler manifold. In this subsection, we review some basic facts about these models and give an argument about the boundary conditions induced from those for the current algebras.

The $N=1$ super Kac-Moody algebra of $G$ at level $\tilde{k}$ is expressed in terms of the superfields

\[ \text{Note that the last equality does not hold in general since } (\omega^g(\mu); \omega^h(\nu)) \text{ may be equal to } (\mu; \nu) \text{ due to the field identification. If this is the case, the construction of the twisted Cardy states suffers from the brane identification fixed points } 18. \]
as follows,

\[ J^A(z_1, \theta_1) J^B(z_2, \theta_2) \sim \frac{1}{z_{12}} \delta^{AB} + \frac{\theta_{12}}{z_{12}} i f^{ABC} J^C(z_2, \theta_2), \tag{3.1} \]

\[(z_{12} \equiv z_1 - z_2 - \theta_1 \theta_2, \quad \theta_{12} \equiv \theta_1 - \theta_2).\]

We take the orthonormal basis with respect to the Killing metric, in which the length of the long root is \( \sqrt{2} \). The current \( J^A \) is expressed in terms of the components as

\[ J^A(z, \theta) = J^A(z) + \theta J^A(z), \tag{3.2} \]

where \( J^A \) is the bosonic current and \( j^A \) is its superpartner. The super stress-energy tensor reads

\[ T_G(z, \theta) = \frac{1}{2k} \left( \partial J^A \partial J^A + \frac{i}{3k} f_{ABC} J^A J^B J^C \right), \tag{3.3} \]

where \( D = \partial/\partial \theta + \theta \partial/\partial z \).

Let \( H (h) \) be a subgroup (subalgebra) of \( G (g) \). We use the following notations for the indices of the currents: \( A, B, \ldots \) for \( g \); \( a, b, \ldots \) for \( h \); \( \bar{a}, \bar{b}, \ldots \) for \( g \setminus h \). The super stress-energy tensor of the \( G/H \) theory is defined as the difference of those of the \( G \) and the \( H \) theories,

\[ T_{G/H} = T_G - T_H. \tag{3.4} \]

The central charge of this theory is given by

\[ c_{G/H} = \frac{3}{2} \left[ \left( 1 - \frac{2h_G^\vee}{3k} \right) \dim G - \left( 1 - \frac{2h_H^\vee}{3k} \right) \dim H \right]. \tag{3.5} \]

In general, the \( N=1 \) superconformal algebra enhances to \( N=2 \) if and only if there exists a weight one superprimary field \( \mathcal{G} \) satisfying the following OPE with itself (see appendix [B] for the detail),

\[ \mathcal{G}(z_1, \theta_1) \mathcal{G}(z_2, \theta_2) \sim \frac{1}{z_{12}^2} \frac{c}{3} + \frac{\theta_{12}}{z_{12}} 2 T(z_2, \theta_2). \tag{3.6} \]

For the \( G/H \) theory, in addition to this, the superprimary field \( \mathcal{G} \) should commute with the \( H \) currents \( J^a \). The most general form of the weight one superfield available in the \( G/H \) theory is written as

\[ \mathcal{G}_{G/H} = i \frac{2k}{i} \left( \epsilon_A J^A + h_{AB} J^A J^B \right). \tag{3.7} \]

For this \( \mathcal{G}_{G/H} \) together with \( T_{G/H} \) to define an \( N=2 \) SCA, the coefficients \( \epsilon_A \) and \( h_{AB} \) have to satisfy the following conditions \[ \underline{[2] 7}, \]

\[ h^b_a h^a_c = - \delta^b_c, \quad h_{ab} = h_{ab} = 0, \]

\[ f^{ab}_{\phantom{ab}d} h_{bc} - f^{ab}_{\phantom{ab}d} h_{bd} = 0, \]

\[ f_{abc} = h^p_a h^q_b f_{pq} + h^p_b h^q_c f_{pq} + h^p_c h^q_a f_{pq}, \]

\[ \epsilon_A = -i f^{bc}_{\phantom{bc}A} h_{bc}. \tag{3.8} \]

\[ \text{Here we use the Killing metric } g_{AB} = \delta_{AB} \text{ to raise and lower the indices.} \]
The first condition means that $h_{\bar{a}\bar{b}}$ is a complex structure on the coset space $G/H$, which is invariant with respect to $\mathfrak{h}$ by the second condition.

The solutions to these conditions are classified in [13,28,29]. We restrict ourselves to the simplest cases, in which rank $\mathfrak{g} = \text{rank } \mathfrak{h}$ and $\mathfrak{h}$ has a single $\mathfrak{u}(1)$ factor. In these cases, $\mathfrak{h}$ takes the form $\mathfrak{h}' \oplus \mathfrak{u}(1)$, where the Dynkin diagram of $\mathfrak{h}'$ is obtained by deleting a node from that of $\mathfrak{g}$. We denote this distinguished node of $\mathfrak{g}$ by a cross $\times$ and the corresponding simple root and the fundamental weight by $\alpha_\times$ and $\Lambda_\times$, respectively. The coset we consider is therefore the form of $G/H' \times U(1)$ and is specified by $\mathfrak{g}$ and the node $\times$ of $\mathfrak{g}$.

The conditions (3.8) become simple if $f_{\bar{a}\bar{b}\bar{c}}$ vanish, for which the corresponding coset space is a hermitian symmetric space (HSS). The Kazama-Suzuki model based on this coset space is called a HSS model. In terms of Dynkin diagrams, the HSS models are characterized by the condition that a node $\times$ has a unit mark.

We turn to the discussion of boundary conditions in the Kazama-Suzuki models. Since we are using the superfield formalism, we have to specify the boundary condition for the supercoordinate,

$$ \theta - i\eta \bar{\theta} = 0, \quad (3.9) $$

where $\eta = \pm 1$ depending on the spin structure of the worldsheet. As we have discussed in section 2.1 the regular boundary condition for the bosonic current $J^A$ in the charge conjugation modular invariant takes the form

$$ J^A + \bar{J}^A = 0. \quad (3.10) $$

The corresponding boundary condition for the supercurrent $\mathbb{J}^A$ is written as

$$ \mathbb{J}^A + i\eta \bar{\mathbb{J}}^A = 0. \quad (3.11) $$

From eqs. (3.3) and (3.7), one can show that this induces the following boundary conditions for the N=2 SCA,

$$ T_{G/H} - i\eta \bar{T}_{G/H} = 0, $$

$$ \mathbb{G}_{G/H} + \bar{\mathbb{G}}_{G/H} = 0, \quad (3.12) $$

which is written in terms of the components as

$$ T - \bar{T} = 0, $$

$$ G^\pm - i\eta \bar{G}^\pm = 0, $$

$$ J + \bar{J} = 0. \quad (3.13) $$

Thus the regular boundary condition for $\mathbb{J}^A$ in the charge conjugation modular invariant yields the B-type boundary condition [1] for the N=2 SCA. On the other hand, as we will show in section 3.3 (see eq. (3.31)), the regular boundary condition (2.14) in the diagonal modular invariant is of the A-type.

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3.2 Automorphisms of the Kazama-Suzuki models

As is explained in section 2.1 if there exists an automorphism of the chiral algebra, one can use it to twist boundary conditions and obtain the corresponding twisted Cardy states compatible with the regular ones. The first thing we have to do is therefore the classification of automorphisms in the Kazama-Suzuki models, which we consider in this subsection.

One can obtain automorphisms of the coset $G/H$ from those of $G$, as we have discussed in section 2.2. When $\text{rank } g = \text{rank } h$, the classification of induced automorphisms is equivalent to the classification of automorphisms $\omega^g$ of the root lattice $Q^g$ that keeps $Q^h$ fixed (see eq.(2.34)),

$$\omega^g \in \text{Aut}(Q^g), \quad \omega^g(Q^h) = Q^h.$$  

The latter condition has a simple expression for the models we consider. As we have mentioned in the previous subsection, the models we consider have the form $G/H' \times U(1)$, i.e., $h = h' \oplus u(1)$. The model in this class is specified by $g$ and a node $\times$ in the unextended Dynkin diagram of $g$, which defines the subalgebra $h' \subset g$. The $u(1)$ direction is perpendicular to the root lattice $Q^{h'}$ of $h'$ and is parallel to the fundamental weight $\Lambda_\times$ corresponding to the node $\times$. This is because $(\alpha_i, \Lambda_\times) = 0$ for $i \neq \times$ and $\{\alpha_i | i \neq \times\}$ form the simple roots of $h'$. Since $\omega^g$ preserves the angle between any two weights, the invariance of the $u(1)$ direction under $\omega^g$ implies that of the root lattice $Q^{h'}$. Therefore, $\omega^g$ is an automorphism of $Q^h$ if and only if it keeps the $u(1)$ direction,

$$\omega^g(\Lambda_\times) = \pm \Lambda_\times.$$

In the following, we find the solutions to this equation (3.15) for any pair of $g$ and $\Lambda_\times$.

The automorphism group $\text{Aut}(Q^g)$ is generated by the elements of the Weyl group $W(g)$ and the diagram automorphisms of $g$. As is seen from (2.34), $\omega^g \in W(h)$ gives a trivial automorphism of $G/H' \times U(1)$. Therefore, we have to consider the following two cases,

(i) $\omega^g \in W(g)$, \quad $\omega^g|_H$ : an outer automorphism of $h$,

(ii) $\omega^g \notin W(g)$.

We denote by $\text{Aut}(G/H' \times U(1))$ the group of induced automorphisms of $G/H' \times U(1)$. The automorphisms of the case (i) form a subgroup of $\text{Aut}(G/H' \times U(1))$, which we call $\text{Aut}_0(G/H' \times U(1))$.

Case (i)

Let us begin with the case of $\omega^g \in W(g)$. As in (3.15), $\omega^g$ has to keep $\Lambda_\times$ fixed up to sign. We first consider the case of $\omega^g(\Lambda_\times) = \Lambda_\times$. The elements of $W(g)$ that fix $\Lambda_\times$ form a stabilizer group $S(\Lambda_\times) \subset W(g)$. One can show that $S(\Lambda_\times)$ is generated by all
the fundamental reflections fixing $\Lambda_\times$ (see, e.g., Proposition 3.12 in [44]). In the present setting, therefore, $S(\Lambda_\times)$ is given by the Weyl group $W(\mathfrak{h}')$ of $\mathfrak{h}'$, and $\omega^g(\Lambda_\times) = \Lambda_\times$ means $\omega^g \in W(\mathfrak{h}')(= W(\mathfrak{h}))$. For $\omega^g$ to be non-trivial, therefore, $\omega^g$ has to map $\Lambda_\times$ to $-\Lambda_\times$. This is possible if and only if $L_{\Lambda_\times}$ is a real representation of $G$, in which the longest element $w_0 \in W(\mathfrak{g})$ maps $\Lambda_\times$ to $-\Lambda_\times$. $w_0$ is non-trivial in the coset theory, since it flips the sign of the $u(1)$ current. All the remaining elements that map $\Lambda_\times$ to $-\Lambda_\times$ are obtained by the action of $W(\mathfrak{h}')$ and equivalent to $w_0$ as automorphisms of the coset theory. To summarize, within this class, the automorphism group $\text{Aut}_0(G/H' \times U(1))$ takes the following form,

$$\text{Aut}_0(G/H' \times U(1)) = \begin{cases} \{1, w_0\} \cong \mathbb{Z}_2 & L_{\Lambda_\times} : \text{real}, \\ 1 & \text{otherwise}. \end{cases} \quad (3.16)$$

Case (ii)

This type ($\omega^g \notin W(G)$) of automorphisms contain non-trivial diagram automorphisms, which exist only for $\mathfrak{g} = A_l, D_l$ and $E_6$:

(a) charge conjugation $\omega_c$ for $A_l, D_{\text{odd}}$ and $E_6$,

(b) chirality flip $\omega_2$ for $D_{\text{even}}$,

(c) triality $\omega_3$ for $D_4$.

In general, $\omega^g$ does not keep the root lattice $Q^{\mathfrak{h}'}$ of $\mathfrak{h}'$ invariant. Thus we have to find an element $w \in W(\mathfrak{g})$ so that the condition (3.15) is satisfied,

$$w \omega^g(\Lambda_\times) = \pm \Lambda_\times. \quad (3.17)$$

It is sufficient to find a particular solution $w$ of this equation; the others are obtained by the composition with the elements of $\text{Aut}_0(G/H' \times U(1))$. For case (a), it is always possible to find $w$ such that $w \omega_c(\Lambda_\times) = -\Lambda_\times$ since $\omega_c(L_{\Lambda_\times}) = L_{\bar{\Lambda}_\times}$ and $L_{\bar{\Lambda}_\times}$ includes $-\Lambda_\times$. For case (b), $\omega_2$ flips the chirality of spinors, $\omega_2 : \Lambda_l \leftrightarrow \Lambda_{l-1}$, where $\Lambda_{l-1}$ and $\Lambda_l$ are spinor weights of $D_l$. Thus $\omega_2(\Lambda_\times) = \Lambda_\times$ if $\Lambda_\times$ corresponds to a tensor representation. If $\Lambda_\times$ corresponds to the spinors, there is no solution to eq. (3.17). For case (c), the triality $\omega_3$ acts on the nodes of $D_4$ as follows: $\Lambda_1 \rightarrow \Lambda_3 \rightarrow \Lambda_4 \rightarrow \Lambda_1, \Lambda_2 \rightarrow \Lambda_2$. Thus eq. (3.17) holds for $\Lambda_2$, whereas there is no solution to eq. (3.17) for $\Lambda_\times \neq \Lambda_2$.

Putting these things together, we obtain the automorphism group $\text{Aut}(G/H' \times U(1))$ of the coset theory. See Table 2 for our result. One can see that the automorphism group always contains a $\mathbb{Z}_2$ as a subgroup, which is nothing but the charge conjugation of the coset theory. In several models, however, the automorphism group has extra elements other than the charge conjugation, and hence is larger than $\mathbb{Z}_2$. We list all the HSS models with extra automorphisms in Table 3. The non-HSS models in [29] with extra automorphisms are also listed.
| \( g \) | \( \Lambda_\times \) | \( \text{Aut}_0(G/H' \times U(1)) \) | \( \text{Aut}(G/H' \times U(1)) \) |
|---|---|---|---|
| \( A_l \) | real \( \Lambda_\times \) otherwise | \( \mathbb{Z}_2 \) \( \times \mathbb{Z}_2 \) | \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) |
| \( D_{\text{odd}} \) | real(\( =\)tensor) spinor | \( \mathbb{Z}_2 \) \( \times \mathbb{Z}_2 \) | \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) |
| \( D_{\text{even}\neq 4} \) | tensor spinor | \( \mathbb{Z}_2 \) \( \times \mathbb{Z}_2 \) | \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) |
| \( D_4 \) | \( \Lambda_2 \) \( \Lambda_1 \) \( \Lambda_3, \Lambda_4 \) | \( \mathbb{Z}_2 \) \( \times \mathbb{Z}_2 \) \( S_3 \times \mathbb{Z}_2 \) | \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) |
| \( E_6 \) | real(\( \Lambda_3, \Lambda_6 \) otherwise | \( \mathbb{Z}_2 \) \( \times \mathbb{Z}_2 \) | \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) |
| \( B_1 \) | any | \( \mathbb{Z}_2 \) | \( \mathbb{Z}_2 \) |
| \( C_l \) | \( E_7, E_8 \) | \( \mathbb{Z}_2 \) | \( \mathbb{Z}_2 \) |
| \( F_4 \) | \( G_2 \) | \( \mathbb{Z}_2 \) | \( \mathbb{Z}_2 \) |

Table 2: Automorphism group \( \text{Aut}(G/H' \times U(1)) \) of the coset theories induced from the automorphisms of the \( G \) theory. We specify coset theories by \( g \) and \( \Lambda_\times \), the fundamental weight corresponding to the node of \( g \) which is not contained in \( h' \). \( \text{Aut}_0(G/H' \times U(1)) \) is the subgroup induced from the inner automorphism of \( g \).

### 3.3 Boundary condition and N=2 superconformal symmetry

In the previous subsection, we have classified the automorphism \( \hat{\omega} \) of the Kazama-Suzuki models \( G/H' \times U(1) \), which is induced from the automorphism \( \omega^g \) of the \( G \) theory. As we have reviewed in section 3.1, the Kazama-Suzuki models have N=2 superconformal symmetry. It is therefore natural to ask how the automorphisms we have found act on the N=2 superconformal algebra.

Let us first consider the action of \( \hat{\omega} \) on the N=1 superconformal algebra. \( \omega^g \) keeps the Killing form and the structure constants of both of the \( G \) and the \( H \) theories (\( H = H' \times U(1) \)). The super stress-energy tensors \( T_G \) and \( T_H \) (see (3.3)) are therefore invariant under the action of \( \omega^g \). Since the super stress-energy tensor \( T_{G/H} \) (3.4) of the coset theory is defined as the difference of \( T_G \) and \( T_H \), it is also invariant by the action of \( \omega^g \), and hence by the induced automorphism \( \hat{\omega} \),

\[
\hat{\omega}(T_{G/H}) = T_{G/H}.
\]

(3.18)

We next turn to the study of the action on the N=2 SCA. The N=2 SCA consists of the super stress-energy tensor \( T \) and the weight one superprimary field \( \mathcal{G} \) (see appendix B). For
Table 3: The Kazama-Suzuki models with non-trivial automorphisms other than the charge conjugation. The list covers all the HSS models together with the non-HSS models studied in [29]. A cross \( \times \) expresses the node \( \times \) defining the model. The arrows stand for the actions of the extra automorphisms.
the Kazama-Suzuki models, \( \mathcal{G} \) takes the form given in eq. (3.19),

\[
\mathcal{G}_{G/H} = \frac{i}{2k} \left( -i f_A^b c^e h_{bc} D \mathcal{J}^A + h_{ab} \mathcal{J}^a \mathcal{J}^b \right).
\]

Since \( \omega^g \) keeps \( H' \times U(1) \) invariant, the action of \( \omega^g \) on the currents can be expressed as follows,

\[
\omega^g : \mathcal{J}^A \mapsto \omega^g(\mathcal{J}^A) = \Omega^A_B \mathcal{J}^B, \quad \Omega^A_B = \begin{pmatrix} \Omega^a_b & 0 & 0 \\ 0 & \Omega^a_b & 0 \\ 0 & 0 & \Omega^0_0 \end{pmatrix},
\]

(3.20)

where we denote the \( u(1) \) part by \( A = 0 \), and the indices \( a, b, \ldots \) stand for \( \mathfrak{h}' \). Then the action of \( \hat{\omega} \) on \( \mathcal{G}_{G/H} \) takes the form

\[
\hat{\omega}(\mathcal{G}_{G/H}) = \frac{i}{2k} \left( -i f_A^b c^e h_{bc} \Omega^A_{A'} \mathcal{J}^{A'} + h_{ab} \Omega^a_{a'} \Omega^b_{b'} \mathcal{J}^a \mathcal{J}^b \right)
\]

(3.21)

where the second equality holds since \( \omega^g \) keeps the Killing form and the structure constants invariant. Comparing with eq. (3.19), one can see that \( \hat{\omega}(\mathcal{G}_{G/H}) \) is obtained from \( \mathcal{G}_{G/H} \) by replacing the complex structure \( h_{ab} \) with its conjugation by \( \Omega \),

\[
h_{ab} \equiv h_{a'b'} \Omega^a_{a'} \Omega^b_{b'}.
\]

(3.22)

This \( h_{ab} \) can be regarded as a complex structure on \( G/H \). Actually, one can check that \( h_{ab} \) is also a solution to the equations (3.8). It is therefore natural to ask how many solutions the equations (3.8) have, \( i.e. \), how many complex structures exist on \( G/H \). This problem is studied in appendix C, in which we show that the equations (3.8) uniquely determines the complex structure up to sign. Therefore, one can conclude that \( h_{ab} \) has to coincide with \( h \) up to sign

\[
h_{ab} = \pm h,
\]

(3.23)

since both of \( h \) and \( h_{ab} \) satisfy the equations (3.8). Substituting this into (3.21), one finds

\[
\hat{\omega}(\mathcal{G}_{G/H}) = \pm \mathcal{G}_{G/H}.
\]

(3.24)

As is seen from the OPE of the superfields \( T \) and \( \mathcal{G} \), it is clear that the non-trivial automorphism of the \( N=2 \) SCA that keeps \( T \) invariant is only the sign change \( \mathcal{G} \to -\mathcal{G} \), which is nothing but the mirror automorphism. Thus \( \hat{\omega} \) acts on the \( N=2 \) SCA of the Kazama-Suzuki model as the mirror automorphism.

The sign in (3.24) is correlated with the sign appearing in the action (3.15) of \( \omega^g \) on \( \Lambda_x \). In order to see this, let us first recall that \( \omega^g \) acts on \( \mathcal{J}^A \) as \( \mathcal{J}^A \mapsto \Omega^A_B \mathcal{J}^B \). From the form (3.20) of \( \Omega \), it is manifest that \( \mathcal{J}^0 \) does not mix with the other components, \( \omega^g(\mathcal{J}^0) = \Omega^0_0 \mathcal{J}^0 \). Considering the term proportional to \( D \mathcal{J}^0 \) in \( \mathcal{G}_{G/H} \) (3.19), one can see that the sign in (3.24)
is given by $\Omega^0_0$. This is compatible with $\Omega^0_0 = \pm 1$, which follows from the fact that $\Omega$ is an orthogonal matrix. Since $J^0$ is proportional to $\Lambda_x \cdot H$, where $H$ are the Cartan currents of $g$, the sign of $\Omega^0_0$ is the same as that appearing in (3.15). We therefore obtain the following relation between the signs for $\hat{\omega}$ and $\omega^g$,

$$\hat{\omega}(G_{G/H}) = \pm G_{G/H} \iff \omega^g(\Lambda_x) = \pm \Lambda_x. \quad (3.25)$$

As we have argued in section 3.1, the regular boundary condition for $J^A$ in the charge conjugation modular invariant is given by eq. (3.11), which yields the B-type boundary condition of the N=2 SCA

$$G_{G/H} + \tilde{G}_{G/H} = 0. \quad (3.26)$$

Once an automorphism $\omega^g$ is given, one can twist boundary conditions for $J^A$ as follows,

$$\omega^g(J^A) + i\eta \tilde{J}^A = 0. \quad (3.27)$$

The corresponding boundary condition for $G_{G/H}$ takes the form

$$\hat{\omega}(G_{G/H}) + \tilde{G}_{G/H} = 0. \quad (3.28)$$

Since $\hat{\omega}$ acts as an automorphism of the N=2 SCA (3.24), one can conclude that the twisted boundary condition (3.27) keeps the N=2 superconformal symmetry,

$$\pm G_{G/H} + \tilde{G}_{G/H} = 0, \quad (3.29)$$

where the sign is determined according to eq. (3.25).

Taking $\omega^g = \omega_c$, the charge conjugation automorphism of $g$, we can obtain the regular boundary condition for $J^A$ in the diagonal modular invariant,

$$\omega_c(J^A) + i\eta \tilde{J}^A = 0. \quad (3.30)$$

Since $\omega_c$ maps $\Lambda_x$ to $-\Lambda_x$, the corresponding boundary condition (3.28) for $G_{G/H}$ reads

$$-G_{G/H} + \tilde{G}_{G/H} = 0, \quad (3.31)$$

which is of the A-type. Similarly to the case of the charge conjugation modular invariant, twisted boundary conditions in the diagonal modular invariant are obtained by using $\omega^g$,

$$\omega^g \omega_c(J^A) + i\eta \tilde{J}^A = 0, \quad (3.32)$$

which also keeps the N=2 superconformal symmetry. The boundary condition for $G_{G/H}$ in this case is

$$-\hat{\omega}(G_{G/H}) + \tilde{G}_{G/H} = 0. \quad (3.33)$$
From the relation (3.25), we find the following correspondence
\[ \omega^0(\Lambda_x) = \Lambda_x \iff \text{A-type}, \]
\[ \omega^0(\Lambda_x) = -\Lambda_x \iff \text{B-type}. \] (3.34)

Obviously, the trivial element of the automorphism group Aut\((G/H' \times U(1))\) corresponds to the regular boundary condition, which is of the A-type. Moreover, the charge conjugation \(\omega_c\) is always an element of Aut\((G/H' \times U(1))\), which yields the B-type boundary condition. When Aut\((G/H' \times U(1))\) contains the non-trivial elements other than \(\omega_c\), we have additional N=2 boundary conditions, the type of which is determined by the rule (3.34).

4 Construction of twisted Cardy states

In this section, we give the explicit construction of the twisted Cardy states in the Kazama-Suzuki models. We restrict ourselves to the case of the diagonal modular invariant and hence the regular boundary condition is of the A-type.

4.1 Bosonic form of the Kazama-Suzuki models

Since we construct the Cardy states in the bosonic form, we first rewrite the supercoset \(G/H' \times U(1)\) as
\[ \frac{G_k \times SO(2m)_1}{H_{I(k+h_G^\vee)-h_{I'}^\vee} \times U(1)_K}, \] (4.1)
where \(2m = \dim G/H = \dim G/H' - 1\), \(I\) is the Dynkin index of the embedding \(h'\) into \(g\) and the level \(K\) of \(U(1)\) is model dependent. The \(SO(2m)\) part consists of the \(2m\) free fermions \(j^a\). The level \(k\) is related to the level \(\tilde{k}\) of the super Kac-Moody algebra (3.1) as \(k = \tilde{k} - h_G^\vee\). The spectrum \(\hat{I}\) of this model takes the form
\[ \hat{I} = \{ \hat{\Lambda} \equiv (\Lambda, \bar{\Lambda}; \lambda, \sigma) | \text{selection rule, field identification} \}, \] (4.2)
where each entry of \(\hat{\Lambda}\) stands for the integrable representation of the constituent theories,
\[ \Lambda \in P_+^k(g), \quad \bar{\Lambda} \in P_+^1(so(2m)) = \{o, v, s, c\}, \]
\[ \lambda \in P_{I(k+h_G^\vee)-h_{I'}^\vee}(h'), \quad \sigma = 0, 1, \ldots, K - 1. \] (4.3)

In the \(SO(2m)\) part, the labels \(o, v, s\) and \(c\) express the vacuum, vector, spinor and cospinor representations, respectively. The explicit forms of the selection rule and the field identification are given in appendix [D]. The examples considered in this section have no fixed points and the matrix \(\hat{S} (2.21)\) of the coset theory is given by
\[ \hat{S}_{\hat{\Lambda}\hat{\Lambda}'} = N_0 S_{\hat{\Lambda}\hat{\Lambda}'}^{Gk} S_{\hat{\Lambda}\hat{\Lambda}'}^{SO(2m)} \frac{H_{I(k+h_G^\vee)-h_{I'}^\vee}}{S_{\hat{\Lambda}\hat{\Lambda}'}^{U(1)_K}} \], (4.4)
where $N_0$ is the order of the field identification group.

To construct twisted Cardy states, we have to clarify how the automorphism $\omega^g$ acts on the $SO(2m)$ part. Since $j^a\bar{a}$ is the component of the supercurrent $\mathbb{J}^a$, $\omega^g$ acts on the free fermions $j^a\bar{a}$ in the same way as $\mathbb{J}^a$,

$$j^a\bar{a} \mapsto \omega^g(j^a\bar{a}) = \Omega^a_{\bar{b}\bar{b}} j^\bar{b},$$

(4.5)

where we used (3.20). Let $\bar{\Omega}$ be a $2m \times 2m$ orthogonal matrix $(\Omega^a_{\bar{b}\bar{b}})$. If $|\bar{\Omega}| = 1$, $\bar{\Omega}$ is an element of $SO(2m)$ and $\omega^g$ acts on the $SO(2m)$ part as an inner automorphism, whereas, if $|\bar{\Omega}| = -1$, $\omega^g$ acts as the outer automorphism $s \leftrightarrow c$.

Whether the action of $\omega^g$ on the $SO(2m)$ part is inner or outer can be seen from its action on $\Lambda$. Let $\bar{\Delta}$ be the roots in $\mathfrak{g} \setminus (\mathfrak{h} \oplus \mathfrak{u}(1))$. This set $\bar{\Delta}$ are decomposed into two sets $\bar{\Delta}^\pm$ according to the sign of the $U(1)$ charge. Each set contains $m$ elements. If $\omega^g(\Lambda) = \Lambda$, $\omega^g$ maps $\bar{\Delta}^\pm$ to itself and $\bar{\Omega}$ has the block diagonal form $\bar{\Omega} = O \oplus O$ on $\bar{\Delta}^+_\pm \oplus \bar{\Delta}^-_\pm$, where $O$ is an orthogonal matrix. Therefore, $|\bar{\Omega}| = |O|^2 = 1$ and $\omega^g$ acts on the $SO(2m)$ part as an inner automorphism. On the other hand, if $\omega^g(\Lambda) = -\Lambda$, $\omega^g$ exchanges $\bar{\Delta}^+_\pm$ and $\bar{\Delta}^-_\pm$. It is sufficient to consider the case that every positive root is mapped to its negative and vice versa, since other cases are obtained by the composition with the elements of $SO(2m)$. In this case, $|\bar{\Omega}| = (-1)^m$, and hence $\omega^g$ acts as an inner automorphism for even $m$ and as an outer automorphism for odd $m$.

### 4.2 Examples

#### 4.2.1 $SU(4)/SU(2) \times SU(2) \times U(1)$

This model is one of the models based on the complex Grassmannian manifold $SU(m + n)/SU(m) \times SU(n) \times U(1)$,

$$
\frac{SU(m+n)_k \times SO(2mn)_1}{SU(m)_{n+k} \times SU(n)_{m+k} \times U(1)_{mn(m+n)(m+n+k)}},
$$

(4.6)

with the central charge

$$c = \frac{3mnk}{m + n + k}.$$  

(4.7)

We consider the case $(m, n, k) = (2, 2, 1)$, i.e., the model

$$SU(4)_1 \times SO(8)_1 / SU(2)_3 \times SU(2)_3 \times U(1)_{80}.$$  

(4.8)

The primary fields of this model are labeled by the following representations (see eq. (4.3)),

$$\Lambda \in P^1_+ (\mathfrak{su}(4)) = \{(0,0,0), (1,0,0), (0,1,0), (0,0,1)\},$$

$$\lambda \in P^3_+ (\mathfrak{su}(2)) \times P^3_+ (\mathfrak{su}(2)) = \{(\lambda^{(1)}, \lambda^{(2)}), | \lambda^{(j)} = 0, 1, 2, 3\}.$$  

(4.9)
Here each representations are expressed by the Dynkin labels. The identification current group is generated by (see (D.3))

\[
J_{(1)} = (J, 1/(J', 1), 10),
J_{(2)} = (J, 1/(1, J'), -10),
\]

(4.10)

where \(J\) and \(J'\) are the generators of the simple current group of \(SU(4)_1\) and \(SU(2)_3\), respectively, while \(\pm 10\) represents the shift of the \(u(1)\) charge \(\sigma \mapsto \sigma \pm 10\). From the relations \(J^4 = 1\) and \(J'^2 = 1\), one obtains

\[
G_{id} \cong \mathbb{Z}_2 \times \mathbb{Z}_8, \quad N_0 = |G_{id}| = 16.
\]

(4.11)

Since this model has no field identification fixed points, the number of the primary fields in this model is calculated as

\[
|\hat{\mathcal{I}}| = \frac{|P^1_+(su(4))| \times |P^1_+(so(8))| \times |P^2_+(su(2))| \times |P^2_+(so(8))| \times |P^{80}_+(u(1))|}{|G_{id}| \times |G_{id}|} = \frac{4 \times 4 \times (4 \times 4) \times 80}{16 \times 16} = 80,
\]

(4.12)

where the factors in the denominator correspond to the selection rule and the field identification. The explicit form of \(\hat{\mathcal{I}}\) reads

\[
\hat{\mathcal{I}} = \{((0, 0, 0), \tilde{\Lambda}; (0, 0), 8j), ((0, 0, 0), \tilde{\Lambda}; (1, 1), 4 + 8j),
((0, 0, 0), \tilde{\Lambda}; (0, 2), 8j), ((0, 0, 0), \tilde{\Lambda}; (2, 0), 8j)| \tilde{\Lambda} \in \{o, v, s, c\}; j = 0, 1, \ldots, 4\}.
\]

(4.13)

The modular transformation matrix takes the form

\[
\hat{S}_{\tilde{\Lambda}\Lambda'} = 16 S_{\Lambda_0 \Lambda_0}^{SU(4)_1} S_{\tilde{\Lambda}\Lambda'}^{SO(8)_1} S_{\Lambda_1 \Lambda_1'}^{SU(2)_3} S_{\Lambda_2 \Lambda_2'}^{SU(2)_3} S_{\sigma \sigma'}^{U(1)_{so}}.
\]

(4.14)

One can construct the corresponding 80 regular Cardy states in the diagonal modular invariant, which are of the \(A\)-type, in the standard manner.

As is shown in Table 3, the automorphism group \(\text{Aut}(SU(4)/SU(2) \times SU(2) \times U(1))\) is \(\mathbb{Z}_2 \times \mathbb{Z}_2\) and contains a non-trivial element that fixes \(\Lambda_x = \Lambda_2\). We denote this automorphism by \(\hat{\omega}_o\). The corresponding automorphism \(\omega_o\) of \(su(4)\) acts as indicated by the arrow in Table 3. One can see that \(\omega_o\) acts on \(su(2) \oplus su(2)\) as the permutation \(\pi\) of two factors. From the argument in the previous subsection, \(\omega_o\) acts on \(so(8)\) as an inner automorphism, and the action on the \(u(1)\) part is trivial, since \(\omega_o(\Lambda_x) = \Lambda_x\). To summarize, the automorphism \(\hat{\omega}_o\) acts in each sector of the coset theory as follows,

\[
\hat{\omega}_o = (\omega_o, 1; \pi, 1).
\]

(4.15)

As we have mentioned above, the regular boundary condition in the diagonal modular invariant is of the \(A\)-type. Since \(\omega_o\) fixes \(\Lambda_x\), the boundary condition twisted by \(\hat{\omega}_o\) is also of
the A-type (see (3.34)). The corresponding twisted Cardy states are constructed following to
the procedure reviewed in section 2.2. The boundary state coefficie
nts $\tilde{S}_G$ for the $G$ theory
is the tensor product of $\tilde{S}^{SU(4)}$ and $\tilde{S}^{SO(8)}$ (see section 2.1 for our notation). Similarly, $\tilde{S}_H$ is the tensor product of $\tilde{S}^{SU(2)_1 \times SU(2)_3}$ and $\tilde{S}^{SU(2)_1}$ (see section 2.1). From the formula in appendix A, we find that $|G_{id}(I^\omega)| = |G_{id}(\tilde{I})| = 8$. Putting these things together, the number of the twisted Cardy states is calculated as follows,

$$|\hat{\tilde{I}}| = \left| \frac{P^1_{+}(A_3^{(2)})}{|G_{id}(I^\omega)|} \times \frac{|P^3_{+}(so(8))|}{|G_{id}(\tilde{I})|} \times \frac{|P^3_{+}(su(2))|}{|P^8_{+}(u(1))|} \right| \times \frac{2 \times 4 \times 4 \times 80}{8 \times 8} = 40.$$  

(4.16)

We give the explicit form of the twisted Cardy states in appendix E.1.

We obtain 40 A-type twisted Cardy states besides 80 regular ones, yielding 120 A-type Cardy states in total. These 120 states have the following natural interpretation. As is seen from the value of the central charge $c = 12/5$, the Grassmannian model $(2, 2, 1)$ is equivalent
to one of the $N=2$ minimal models,

$$SU(2)_8 \times SO(2)_{1_{20}}.$$  

(4.17)

More precisely, we should take the D-type modular invariant in the $SU(2)_8$ theory for the equivalence. The number of the primary fields in this model is $6 \times 4 \times 20/2/2 = 120$, where the first factor stands for the number of the primary fields in the $D_6$ modular invariant of $SU(2)_8$. Correspondingly, by taking the D-type boundary coefficients for the $SU(2)_8$ part, we can construct 120 Cardy states in this minimal model, which satisfy the $N=2$ A-type boundary condition. This is the same number as we have obtained for the Kazama-Suzuki model $SU(4)/SU(2) \times SU(2) \times U(1)$ at level one. Actually, from the comparison of the boundary state coefficients, one can identify 80 of the 120 Cardy states in the minimal model with the regular ones in the Kazama-Suzuki model, and 40 of them with the twisted Cardy states we have obtained above. This result shows that 40 twisted Cardy states are compatible with 80 regular ones in the diagonal modular invariant of the Kazama-Suzuki model, since these two sets are combined into one NIM-rep of the minimal model.

4.2.2 $SO(8)/SO(6) \times U(1)$

The next example is the first non-trivial model in the series of

$$\frac{SO(2n)_k \times SO(4(n - 1))_1}{SO(2n - 2)_{k+2} \times U(1)_{4(k+2(n-1))}}.$$  

(4.18)

We consider the case of $n = 4, k = 1$. The primary fields of this model are labeled by the following representations

$$\Lambda \in P^1_{+}(so(8)) = \{o, v, s, c\},$$

$$\lambda \in P^3_{+}(so(6)) = \{\lambda_1, \lambda_2, \lambda_3| \lambda_1 + \lambda_2 + \lambda_3 \leq 3\}.$$  

(4.19)
Here $\lambda_2$ ($\lambda_3$) corresponds to the cospinor (spinor) node of $\mathfrak{so}(6)$. The identification group of this model is given by (see appendix D.2)

$$G_{\text{id}} \cong \mathbb{Z}_2 \times \mathbb{Z}_4.$$  \hspace{1cm} (4.20)

Since this model has no identification fixed points, the number of the primary fields is calculated as

$$|\hat{\mathcal{I}}| = \frac{|P_+^{1}(\mathfrak{so}(8))| \times |P_+^{1}(\mathfrak{so}(12))| \times |P_+^{3}(\mathfrak{so}(6))| \times |P_+^{28}(\mathfrak{u}(1))|}{|G_{\text{id}}| \times |G_{\text{id}}|}$$

$$= \frac{4 \times 4 \times 20 \times 28}{8 \times 8} = 140.$$ \hspace{1cm} (4.21)

The explicit form of $\hat{\mathcal{I}}$ reads

$$\hat{\mathcal{I}} = \{(o, \tilde{\Lambda}_{\text{NS}}; (\lambda_1, \lambda_2, \lambda_3), 4j), (o, \tilde{\Lambda}_{\text{R}}; (\lambda_1, \lambda_2, \lambda_3), 4j + 2) \mid \tilde{\Lambda}_{\text{NS}} \in \{o, v\}, \tilde{\Lambda}_{\text{R}} \in \{s, c\}; (\lambda_1, \lambda_2, \lambda_3) \in \{(0, 0, 0), (2, 0, 0), (0, 1, 1), (1, 2, 0), (1, 0, 2)\}; j = 0, 1, \ldots, 6\}. \hspace{1cm} (4.22)$$

We can construct corresponding 140 regular Cardy states in the diagonal modular invariant, which are of the A-type.

This model also has an extra automorphism that fixes $\Lambda_\chi = \Lambda_1$ (see Table 3). We denote this automorphism by $\hat{\omega}_2$, the action of which on each sector reads

$$\hat{\omega}_2 = (\omega_2, 1; \omega'_c, 1). \hspace{1cm} (4.23)$$

Here $\omega_2$ is the order two outer automorphism of $\mathfrak{so}(8)$ and $\omega'_c$ is the charge conjugation of $\mathfrak{so}(6)$.

We can construct the corresponding twisted Cardy states in the way parallel to the first example. We find that $|G_{\text{id}}(\mathcal{I}^\omega)| = |G_{\text{id}}(\hat{\mathcal{I}})| = 4$ and the number of the twisted Cardy states is calculated as

$$|\hat{\mathcal{I}}^\omega| = \frac{|P_+^{1}(D_4^{(2)})| \times |P_+^{1}(\mathfrak{so}(12))| \times |P_+^{3}(D_3^{(2)})| \times |P_+^{28}(\mathfrak{u}(1))|}{|G_{\text{id}}(\mathcal{I}^\omega)| \times |G_{\text{id}}(\hat{\mathcal{I}})|}$$

$$= \frac{2 \times 4 \times 6 \times 28}{4 \times 4} = 84.$$ \hspace{1cm} (4.24)

The explicit form of these twisted Cardy states are given in appendix E.2.

We obtain 84 A-type twisted Cardy states in addition to 140 regular ones, yielding 224 A-type Cardy states as a whole. Similarly to the previous example, these states are interpreted in terms of the N=2 minimal model, which is the $k = 12$ minimal model with the $D_8$ modular invariant this time. In this model, there exist $8 \times 4 \times 28/2/2 = 224$ Cardy states of the A-type, and we again see the complete match of the numbers of the Cardy states for both models.
4.3 Action of the level-rank duality

In this subsection, we consider the action of the level-rank duality on the twisted Cardy states we have obtained. Some of the Kazama-Suzuki models are related with each other by exchanging the level and the rank of the WZW models. For example, the Grassmannian models \((m,n,k)\) are invariant with respect to the permutation of \(m, n\) and \(k\). Namely, for \(m = n\), the following identity holds,

\[
SU(2m)_k \times SO(2m^2)_1 \times SU(m)_{k+m} \times U(1)_{2m^3(k+2m)} = SU(k + m)_m \times SO(2km)_1 \times SU(k)_2m \times SU(m)_k \times U(1)_{km(k+m)(k+2m)}.
\]  

(4.25)

The primary fields, and hence the regular Cardy states, for both models are identified by this duality map. As we have found in section 3.2, the automorphism group \(\text{Aut}(SU(2m)/SU(m) \times SU(m) \times U(1)) = \mathbb{Z}_2 \times \mathbb{Z}_2\) and one can construct A-type twisted Cardy states corresponding to non-trivial automorphism other than the charge conjugation. On the other hand, in the right hand side of (4.25), \(\text{Aut}(SU(k + m)/SU(k) \times SU(m) \times U(1)) = \mathbb{Z}_2\) except for \(k = m\), and hence twisted Cardy states cannot be constructed in the way similar to the examples we have given. Then, it is natural to ask what kind of states in the right hand side correspond to the twisted Cardy states in the left hand side via the level-rank duality.

In order to answer to this question, we study the action of the level-rank duality on the twisted Cardy states in \(SU(4)/SU(2) \times SU(2) \times U(1)\) constructed in section 4.2.1. According to eq. (4.25), this model is dual to the model

\[
SU(3)_2 \times SO(4)_1 \times SU(2)_3 \times U(1)_{30},
\]

(4.26)

for which the automorphism group is \(\mathbb{Z}_2\) (see Table 2) and we cannot construct A-type twisted Cardy states in the diagonal modular invariant using the method of the induced automorphism.

A primary field in this model is labeled by the following representations

\[
\Lambda \in P^2_\Lambda(\mathfrak{su}(3)) = \{(0, 0), (2, 0), (0, 2), (1, 1), (0, 1), (1, 0)\}, \\
\lambda \in P^3_\lambda(\mathfrak{su}(2)) = \{0, 1, 2, 3\}.
\]

(4.27)

The identification current group is generated by

\[
(J, \tilde{J}_v/ J', 5),
\]

(4.28)

and \(G_{\mathfrak{su}} := \mathbb{Z}_6\). Here \(J\) and \(J'\) are the generators of the simple current group for \(\mathfrak{su}(3)_2\) and \(\mathfrak{su}(2)_3\), respectively. \(\tilde{J}_v\) is the vector simple current for \(\mathfrak{so}(4)_1\) and 5 means the shift of the \(u(1)\) charge by five. The number of the primary fields is \(6 \times 4 \times 4 \times 30/6/6 = 80\), which is

\(^8\text{For the level-rank duality in the WZW models, see [45, 46].}\)
equal to that for the model $SU(4)/SU(2) \times SU(2) \times U(1)$ at level one and is consistent with
the level-rank duality. The set $\hat{\mathcal{I}}$ of the primary fields reads

$$\hat{\mathcal{I}} = \{((0,0), o; \lambda, 6j + 3\lambda), ((0,0), s; \lambda, 6j + 3 + 3\lambda), \}
((1,1), o; \lambda, 6j + 3\lambda), ((1,1), s; \lambda, 6j + 3 + 3\lambda) | \lambda = 0, 1, 2, 3; j = 0, 1, \ldots, 4\}. \quad (4.29)$$

The modular transformation matrix takes the form

$$\hat{S} = 6 S^{SU(3)_2} S^{SO(4)_1} S^{SU(2)_3} S^{U(1)_{30}}. \quad (4.30)$$

From this $\hat{S}$, one can construct 80 A-type regular Cardy states, which are identified with 80
regular Cardy states in $SU(4)/SU(2) \times SU(2) \times U(1)$.

We have seen in section 4.2.1 that there are 40 twisted Cardy states in $SU(4)/SU(2) \times SU(2) \times U(1)$ at level one besides 80 regular ones. Upon the level-rank duality, these 40 states
should be mapped to 40 states in $SU(3)/SU(2) \times U(1)$ at level two, which are compatible
with the regular ones. In $SU(3)/SU(2) \times U(1)$, these 40 states have to form a NIM-rep of
the fusion algebra. One can easily see that the resulting NIM-rep cannot be constructed by
the method of the induced automorphisms $^9$, since the automorphism originated from $su(3)$
gives no A-type NIM-rep other than the regular one (see Table 2). Therefore, we have to
consider other possibilities in order to explain these states within $SU(3)/SU(2) \times U(1)$. A
systematic procedure to construct NIM-reps in coset theories has been given in $^{17}$. In the
case of $SU(3)_2 \times SO(4)_1/SU(2)_3 \times U(1)_{30}$, one can apply the method of $^{17}$ to obtain a
NIM-rep based on a conformal embedding,

$$su(6)_1 \supset su(3)_2 \oplus su(2)_3. \quad (4.31)$$

As we will see below, this NIM-rep is forty dimensional, and precisely coincides with the
NIM-rep corresponding to the twisted Cardy states in $SU(4)/SU(2) \times SU(2) \times U(1)$.\footnote{The existence of these states has been reported in $^{14}$ based on the comparison of the spectrum with
that of the minimal model.}

The construction of a NIM-rep based on a conformal embedding starts from finding the
branching of the representations,

$$(0,0,0,0,0) \mapsto ((0,0), 0) \oplus ((1,1), 2),$$
$$(1,0,0,0,0) \mapsto ((1,0), 1) \oplus ((0,2), 3),$$
$$(0,1,0,0,0) \mapsto ((2,0), 0) \oplus ((0,1), 2),$$
$$(0,0,1,0,0) \mapsto ((1,1), 1) \oplus ((0,0), 3),$$
$$(0,0,0,1,0) \mapsto ((0,2), 0) \oplus ((1,0), 2),$$
$$(0,0,0,0,1) \mapsto ((0,1), 1) \oplus ((2,0), 3),$$

\footnote{The relation between level-rank dualities and conformal embeddings has been observed before in the context of the bulk theory. See \textit{e.g.} $^{17}$.}

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Figure 1: The NIM-rep graph of the SU(3)$_2$ × SU(2)$_3$ WZW model from the conformal embedding su(3)$_2$ ⊕ su(2)$_3$ ⊂ su(6)$_1$. Six of these states are originated from the regular ones in the SU(6)$_1$ theory. The subgraph in the dashed (solid) lines corresponds to $n_{((0,0),1)}$ ($n_{((1,0),0)}$). For the definition of the NIM-rep graph, see e.g. [17]. The value $\tilde{b}(J,1)$ is expressed as black (1), gray ($\omega$) or white ($\omega^2$) with $\omega = e^{2\pi i/3}$, whereas $\tilde{b}(J,J')$ is expressed as circled (+1) or un-circled (−1). The action of the simple current of SU(3)$_2$ (SU(2)$_3$) is $2\pi/3$ rotations (the reflection about the center of this diagram).

where the left hand sides are the elements of $P^1_+ (su(6))$. We can construct Cardy states in the SU(3)$_2$ × SU(2)$_3$ theory based on this branching. The general procedure has been given in section 2.3.2 of [17] (see also [48]). Since there appear twelve representations in the branching (4.32), we have twelve Ishibashi states, which we label by the set

$$E^e = \{((0,0),0), ((1,0),2), ((2,0),0), ((1,1),2), ((0,2),0), ((0,1),2),$$

$$((0,0),3), ((1,0),1), ((2,0),3), ((1,1),1), ((0,2),3), ((0,1),1)\}. \quad (4.33)$$

We first obtain six Cardy states by reinterpreting six regular states in the SU(6)$_1$ theory. Next, by the fusion in SU(3)$_2$ and SU(2)$_3$, we obtain six additional states, and hence 12 Cardy states in total which we label by the set

$$V^e = \{((0,0),+), ((1,0),+), ((2,0),+), ((1,1),+), ((0,2),+), ((0,1),+),$$

$$((0,0),-), ((1,0),-), ((2,0),-), ((1,1),-), ((0,2),-), ((0,1),-)\}. \quad (4.34)$$

See Fig.1 for our labeling of the states and the corresponding NIM-rep graph.

By using these Cardy states in the SU(3)$_2$ × SU(2)$_3$ theory, we can construct Cardy states in the coset theory SU(3)$_2$ × SO(4)$_1$/SU(2)$_3$ × U(1)$_{30}$. In the same way as the case of the induced automorphisms explained in section 2.2, we need some selection rule and identification of the Cardy states in order to obtain a NIM-rep in the coset theory. One
can show that the selection rule and the identification have the same order as the field identification group, namely,

$$|G_{id}(\mathcal{E})| = |G_{id}(\mathcal{V})| = 6.$$  \hspace{1cm} (4.35)

Therefore, the number of the resulting Cardy states is calculated as

$$|\hat{\mathcal{V}}^e| = \frac{|\mathcal{V}^e| \times |P_1^{30}(\mathfrak{so}(4))| \times |P_{30}^{\mathfrak{u}(1)}|}{|G_{id}(\mathcal{E})| \times |G_{id}(\mathcal{V})|} = \frac{12 \times 4 \times 30}{6 \times 6} = 40.$$  \hspace{1cm} (4.36)

This number coincides exactly with that of the twisted Cardy states in $SU(4)/SU(2) \times SU(2) \times U(1)$, which suggests that this NIM-rep expresses the twisted Cardy states we have obtained. In fact, we can check that the coefficients of the twisted Cardy states are precisely reproduced from this NIM-rep. The explicit form of the NIM-rep reads

$$\psi^e = 6 \psi^e S^{SO(4)}_{SU(3)_{10}} S^{SU(1)_{30}},$$

$$\hat{\mathcal{E}}^e = \{(0, 0), o; 0, 6j\}, ((0, 0), o; 3, 6j + 3), ((0, 0), s; 0, 6j + 3), ((0, 0), s; 3, 6j),$$

$$((1, 1), o; 1, 6j + 3), ((1, 1), o; 2, 6j), ((1, 1), s; 1, 6j), ((1, 1), s; 2, 6j + 3)$$

$$|j = 0, 1, \ldots, 4\},$$

$$\hat{\mathcal{V}}^e = \{(0, 0), o; +, 6j\}, ((0, 0), o; -, 6j + 3), ((0, 0), s; +, 6j + 3), ((0, 0), s; -, 6j),$$

$$((1, 1), o; -, 6j + 3), ((1, 1), o; +, 6j), ((1, 1), s; -, 6j), ((1, 1), s; +, 6j + 3)$$

$$|j = 0, 1, \ldots, 4\},$$

$$\psi^e = \frac{1}{\sqrt{2}} \begin{pmatrix} S^{SU(3)_{2}}_{SU(3)_{2}} & S^{SU(3)_{2}}_{SU(3)_{2}} \\ S^{SU(3)_{2}}_{SU(3)_{2}} & -S^{SU(3)_{2}}_{SU(3)_{2}} \end{pmatrix}.$$  \hspace{1cm} (4.37)

Here $\psi^e$ is the boundary state coefficients for the NIM-rep based on the conformal embedding \((4.31)\) and the rows and columns of $\psi^e$ are ordered as in \((4.33)\) and \((4.34)\). $S^{SU(3)_{2}}$ is the $6 \times 6$ modular transformation matrix whose rows and columns are the $SU(3)_{2}$ part of \((4.33)\) and \((4.34)\).

To summarize, the twisted Cardy states of $SU(4)/SU(2) \times SU(2) \times U(1)$ at level one are mapped via the level-rank duality to those expressed by the NIM-rep based on the conformal embedding $\mathfrak{su}(3)_{2} \oplus \mathfrak{su}(2)_{3} \subset \mathfrak{su}(6)_{1}$. This result shows that the Cardy states compatible with the regular ones are not limited to those obtained from the induced automorphisms. Since the automorphism group of the chiral algebra is considered to be not affected by the level-rank duality, this result suggests that the automorphism group of the coset theory $G/H$ is in general larger than that induced from those of $G$.
5 Summary and Discussion

In this paper, we have studied Cardy states in the Kazama-Suzuki models $G/H' \times U(1)$, which satisfy the boundary conditions twisted by the automorphisms of the coset theory. We have classified automorphisms of $G/H' \times U(1)$ induced from those of the $G$ theory. The automorphism group of the Kazama-Suzuki models contains at least a $\mathbb{Z}_2$ as a subgroup which corresponds to the charge conjugation. We have found that in several models the automorphism group contains non-trivial elements other than the charge conjugation and can be larger than $\mathbb{Z}_2$. Based on the general procedure to construct the Cardy states in coset theories, we have given the explicit form of the twisted Cardy states corresponding to non-trivial automorphisms of the Kazama-Suzuki models. We have shown that the resulting twisted Cardy states preserve the N=2 superconformal algebra, i.e., either of the A-type or of the B-type. As an illustrative example of our construction, we have given a detailed study for two HSS models: $SU(4)/SU(2) \times SU(2) \times U(1)$ and $SO(8)/SO(6) \times U(1)$ both at level one, which have the description as the N=2 minimal models. We have compared our results with those for the minimal models and have shown that the twisted Cardy states together with the regular ones reproduce all the Cardy states for the minimal models. The action of the level-rank duality on the twisted Cardy states has been studied for the simplest case, $SU(4)/SU(2) \times SU(2) \times U(1)$ at level one, which is equivalent to $SU(3)/SU(2) \times U(1)$ at level two. We have shown that the level-rank duality maps the twisted Cardy states in the former model to the states in the latter which are associated with the conformal embedding $su(3)_2 \oplus su(2)_3 \subset su(6)_1$.

We have restricted ourselves to the Kazama-Suzuki models $G/H$ where rank $g = \text{rank } h$ and $H$ contains a single $U(1)$ factor. It is straightforward to extend our analysis to the other cases, namely, models with rank $g \neq \text{rank } h$ or those with more than one $U(1)$ factors. These models may admit extra automorphisms other than those considered in this paper. It is interesting to examine the corresponding twisted boundary condition, in particular its relation to N=2 SCA.

Another interesting problem is the issue of the geometrical interpretation of the twisted Cardy states. For the N=2 minimal models, combined with the N=2 Liouville theory, it has been shown that the regular Cardy states can be interpreted as the cycles in the ALE spaces. It is important to clarify whether the Cardy states in the Kazama-Suzuki models have the similar interpretation for some noncompact varieties.

We have seen in section 4.3 that, in $SU(3)/SU(2) \times U(1)$ at level two, one can obtain the Cardy states other than the regular ones although this model does not admit non-trivial induced automorphisms. This fact implies that the model $SU(3)/SU(2) \times U(1)$ has an extra automorphism which is not obtained from that of $SU(3)$. In order to have all the automorphisms of the coset theory, it is therefore insufficient to consider only the induced...
automorphisms. It is important for the complete analysis of the boundary states to find a systematic way of classifying all the automorphisms of the coset theory.

Acknowledgement

We would like to thank H. Awata, K. Ito, S. Mizoguchi, T. Nakatsu and H. Suzuki for helpful discussions. The work of H. I. is supported in part by the Grant-in-Aid for Young Scientists from the Ministry of Education, Culture, Science, Sports and Technology of Japan. The work of T. T. is supported by the Grant-in-Aid for Scientific Research on Priority Areas (2) 14046201 from the Ministry of Education, Culture, Science, Sports and Technology of Japan.

A Identification current groups for twisted Cardy states

In this appendix, we give the definition of the identification current groups $G_{id}(\mathcal{I}^w)$ and $G_{id}(\tilde{\mathcal{I}})$, which are necessary for determining the set $\tilde{\mathcal{I}}$ labeling the twisted Cardy states in the coset theory $G/H \cong G_{sc}$. See [17] for the detail.

The simple current $J \in G_{sc}^G$ has a natural action on the labels of the Cardy states and the Ishibashi states. In terms of the boundary state coefficients $\tilde{S}^G$, this action can be written as follows:

$$
\tilde{S}^G_{\tilde{\alpha}J\lambda} = \tilde{b}^G_{\tilde{\alpha}}(J) \tilde{S}^G_{\tilde{\alpha}\lambda}, \quad J \in G(\mathcal{I}^w),
$$

$$
\tilde{S}^G_{J\tilde{\alpha}\lambda} = \tilde{S}^G_{\tilde{\alpha}J\lambda} b^G_{J\lambda}(J), \quad J \in G(\tilde{\mathcal{I}}^G).
$$

(A.1)

Here $G(\mathcal{I}^w)$ and $G(\tilde{\mathcal{I}}^G)$ are the groups of the simple currents for the twisted Ishibashi states and the twisted Cardy states in the $G$ theory, respectively,

$$
G(\mathcal{I}^w) = \{J \in G_{sc}^G | J : \mathcal{I}^w \mapsto \mathcal{I}^w, G(\mathcal{I}^w) = G_{sc}/S(\tilde{\mathcal{I}}^G), \quad S(\tilde{\mathcal{I}}^G) = \{J_0 \in G_{sc}^G | J_0 \tilde{\alpha} = \tilde{\alpha}, \forall \tilde{\alpha} \in \tilde{\mathcal{I}}^G\}. \quad (A.2)
$$

One can also define these groups for the $H$ theory in the same way as above, which we denote by $G(\mathcal{I}^h)$ and $G(\tilde{\mathcal{I}}^H)$. The groups $G_{id}(\mathcal{I}^w)$ and $G_{id}(\tilde{\mathcal{I}})$ of the identification currents for the twisted Ishibashi states and the twisted Cardy states in the coset theory are then defined as follows:

$$
G_{id}(\mathcal{I}^w) = G_{id} \cap (G(\mathcal{I}^w) \times G(\mathcal{I}^h)),
$$

$$
G_{id}(\tilde{\mathcal{I}}) = G_{id}/(G_{id} \cap \tilde{S}(\tilde{\mathcal{I}})),
$$

(A.3)

$$
\tilde{S}(\tilde{\mathcal{I}}) = \{(J, J') \in G_{sc} | b^G_{\lambda}(J) = b^H_{\mu}(J') \forall (\lambda, \mu) \in \mathcal{I}^w \otimes \mathcal{I}^h\}.
$$

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The N=2 superconformal algebra consists of the stress-energy tensor $T$, two supercurrents $G^\pm$ and the $U(1)$ current $J$. The N=1 supercurrent $G$ is given by a particular combination of $G^\pm$,

$$G = \frac{1}{\sqrt{2}}(G^+ + G^-). \quad (B.1)$$

The super stress-energy tensor $T$ is then written as

$$T(z, \theta) = \frac{1}{2}G(z) + \theta T(z) = \frac{1}{2\sqrt{2}}(G^+ + G^-)(z) + \theta T(z). \quad (B.2)$$

The remaining generators of the N=2 SCA are arranged in a superfield $G$,

$$G(z, \theta) = J(z) + \theta \frac{1}{\sqrt{2}}(-G^+ + G^-)(z). \quad (B.3)$$

In terms of these superfields, the N=2 SCA can be written as follows,

$$T(z_1, \theta_1)T(z_2, \theta_2) \sim \frac{1}{z_{12}^{3}} \frac{c}{6} + \frac{\theta_{12} z_{12}^{2}}{3} T(z_2, \theta_2) + \frac{1}{z_{12}^{2}} D T(z_2, \theta_2) + \frac{\theta_{12}}{z_{12}} \partial T(z_2, \theta_2),$$

$$T(z_1, \theta_1)G(z_2, \theta_2) \sim \frac{\theta_{12}}{z_{12}^{2}} G(z_2, \theta_2) + \frac{1}{z_{12}^{2}} D G(z_2, \theta_2) + \frac{\theta_{12}}{z_{12}} \partial G(z_2, \theta_2), \quad (B.4)$$

$$G(z_1, \theta_1)G(z_1, \theta_1) \sim \frac{1}{z_{12}^{3}} \frac{c}{3} + \frac{\theta_{12} z_{12}}{2} D(z_2, \theta_2).$$

The second equation shows that $G$ is a superprimary field with weight one.

## C Uniqueness of the complex structure

We present the proof of the uniqueness of the complex structure $h_{ab}$ for the Kazama-Suzuki models $G/H' \times U(1)$ with rank $g = \text{rank } h' + 1$.

Let $\tilde{\Delta}$ be the set of the roots belonging to $g \setminus (h' \oplus u(1))$. This set is decomposed as

$$\tilde{\Delta} = \tilde{\Delta}_+ \oplus \tilde{\Delta}_-,$$

(C.1)

where $+$ ($-$) means that the elements of $\tilde{\Delta}_\pm$ are positive (negative) roots of $g$. We will show that, in the Cartan-Weyl basis, $h = (h^a_b)$ is a diagonal matrix with entries $+i$ for $\tilde{\Delta}_+$ and $-i$ for $\tilde{\Delta}_-$ (up to overall sign) \footnote{In \cite{13}, this fact is used implicitly to derive the corollary from the theorem 1. Here we will give an explicit proof.}. This means the uniqueness of the complex structure up to sign.

The sets $\tilde{\Delta}_\pm$ are further decomposed according to the values of the $U(1)$ charge. As is shown in \cite{13}, the $U(1)$ charge is positive (negative) for the roots in $\tilde{\Delta}_+$ ($\tilde{\Delta}_-$) and zero for
the roots of \( h' \). The simple roots of \( g \) are consisted of the simple roots of \( h' \) and \( \alpha_x \), which is the simple root of \( g \) with non-vanishing \( U(1) \) charge. Therefore, a root in \( \bar{\Delta}_\pm \) has a \( U(1) \) charge which is an integer multiple of the \( U(1) \) charge for \( \alpha_x \). This yields the following decompositions,

\[
\bar{\Delta}_+ = \bar{\Delta}^{(1)} + \bar{\Delta}^{(2)} + \cdots ,
\]

\[
\bar{\Delta}_- = \bar{\Delta}^{(-1)} + \bar{\Delta}^{(-2)} + \cdots ,
\]

where \( \bar{\Delta}^{(n)} \) contains all the roots of \( g \) whose \( U(1) \) charge is \( n \) times that for \( \alpha_x \). For the HSS models, \( \bar{\Delta}^{(n)} \) is empty for \( |n| \geq 2 \). For non-HSS models, however, this does not hold in general, e.g., \( \bar{\Delta}^{(n)} \) is not empty for \(-3 \leq n \leq 3 \) in \( G_2/A_1^7 \oplus \mathfrak{u}(1) \). An important property of the set \( \bar{\Delta}^{(n)} \) is that it forms an irreducible representation of \( h' \). This is because all the roots in \( \bar{\Delta}^{(n)} \) are in the same conjugacy class of \( h' \) and the multiplicity of the adjoint representation of \( g \) is one. Based on this structure of \( \bar{\Delta} \), the proof proceeds in two steps:

(i) \( h \) is \( \pm i \) on each \( \bar{\Delta}^{(n)} \).

(ii) \( h = +i \) on \( \bar{\Delta}^{(n)} \) and \( -i \) on \( \bar{\Delta}^{(-n)} \) for any positive \( n \) (up to overall sign).

To show (i), we use the second condition in (3.8). In the matrix form \(( f^a )_{\bar{b} \bar{c}} = f_a^{\bar{b} \bar{c}} \), this condition is \( f^a h = h f^a \) for all \( a \in h' \). From Schur’s lemma, we see that \( h \) is proportional to the unit matrix on each \( \bar{\Delta}^{(n)} \). The entries of \( h \) are \( +i \) or \( -i \) by the first condition in (3.8),

\[
h^{\bar{a}}_{\bar{b}} = i h_n \delta^{\bar{a}}_{\bar{b}}, \quad (\bar{a} \in \bar{\Delta}^{(n)}),
\]

where \( h_n = \pm 1 \). Then, one can prove (ii) if we show \( h_n = -h_{-n} \) and \( h_{n+1} = h_n \) for positive \( n \). For a diagonal \( h \), the second condition in (3.8) for \( a = 0 \) reads,

\[
f^{\bar{b} \bar{d}}_a h^d_{\bar{d}} = -h^b_{\bar{b}} f^{\bar{b} \bar{d}}_a .
\]

Suppose \( \bar{b} \in \bar{\Delta}^{(n)} \). Since \( f^{\bar{b} \bar{d}}_0 \) is non-zero if and only if \( \bar{b} = -\bar{d} \), the equation (C.4) means that \( h_{-n} = -h_n \). For \( h_{n+1} = h_n \), let \( (\bar{a}, \bar{b}, \bar{c}) \in (\bar{\Delta}^{(1)}, \bar{\Delta}^{(n)}, \bar{\Delta}^{(n+1)}) \) and rewrite the third condition in (3.8) as

\[
f^{\bar{a} \bar{b}}_{\bar{c}} = (h^{\bar{a}}_{\bar{d}} h^b_{\bar{b}} - h^b_{\bar{d}} h^c_{\bar{c}} - h^c_{\bar{d}} h^a_{\bar{a}}) f^{\bar{a} \bar{b}}_{\bar{c}} .
\]

We can choose a combination of \( \bar{a}, \bar{b} \) and \( \bar{c} \) such that \( f^{\bar{a} \bar{b}}_{\bar{c}} \neq 0 \) if the set \( \bar{\Delta}^{(n+1)} \) is not empty. Then the above equation reads,

\[
1 = -h_1 h_n + h_n h_{n+1} + h_{n+1} h_1 .
\]

Setting \( n = 1 \), we find \( h_1 h_2 = 1 \), which means \( h_2 = h_1 \). By the induction, we find that \( h_n = h_1 \) for all positive \( n \). This completes the proof.
D Selection rules and field identifications

In this appendix, we briefly review the selection rules and the field identifications \cite{35,36,31,32} for the models considered in the present paper.

D.1 $SU(m+n)/SU(m) \times SU(n) \times U(1)$

The bosonic form of this model is given in (4.6). A primary field is labeled as $(\Lambda, \tilde{\Lambda}; \lambda, \sigma)$ where

$$\Lambda = (\Lambda_1, \Lambda_2, \ldots, \Lambda_{m+n-1}) \in P^k_+ (\mathfrak{su}(m+n)),
\tilde{\Lambda} \in P^1_+ (\mathfrak{so}(2mn)) = \{o, v, s, c\},
\lambda = (\lambda^{(1)}, \lambda^{(2)}) = ((\lambda^{(1)}_1, \lambda^{(1)}_2, \ldots, \lambda^{(1)}_{m-1}), (\lambda^{(2)}_1, \lambda^{(2)}_2, \ldots, \lambda^{(2)}_{n-1})) \in P^{n+k}_+ (\mathfrak{su}(m)) \times P^{m+k}_+ (\mathfrak{su}(n)),
\sigma \in P^K_+ (\mathfrak{u}(1)) = \{0, 1, \ldots, K - 1\} \quad (K \equiv mn(m+n)(m+n+k)).$$

The selection rule is given by

$$m r_\Lambda + mn(m+n) \epsilon_\tilde{\Lambda}/2 = (m+n) r_{\lambda^{(1)}} + \sigma \quad \text{mod} \ m(m+n),
$$
$$n r_\Lambda + mn(m+n) \epsilon_\tilde{\Lambda}/2 = (m+n) r_{\lambda^{(2)}} - \sigma \quad \text{mod} \ n(m+n),$$

where

$$r_\Lambda = \sum_j j \Lambda_j, \quad r_{\lambda^{(i)}} = \sum_j j \lambda^{(i)}_j,$$

$$\epsilon_\tilde{\Lambda} = \begin{cases} 0 & \tilde{\Lambda} = o, v \quad \text{(NS)}, \\ 1 & \tilde{\Lambda} = s, c \quad \text{(R)}. \end{cases}$$

Correspondingly, the identification group $G_{id}$ has the following two generators,

$$J_{(1)} = (J, \tilde{J}_v^n / (J', 1), -n(m+n+k)),
J_{(2)} = (J, \tilde{J}_v^m / (1, J'), m(m+n+k)),$$

where each simple currents act as follows,

$$J : \Lambda_j \to \Lambda_{j-1}, \quad \tilde{J}_v : \quad (o, v, s, c) \mapsto (v, o, c, s),
J' : \lambda^{(i)}_j \to \lambda^{(i)}_{j-1}, \quad p : \quad \sigma \mapsto \sigma + p \quad (p \in \{0, 1, \ldots, K - 1\}).$$

The monodromy charges are given by

$$b_\Lambda (J) = e^{2\pi i r_\Lambda/(m+n)}, \quad b_\Lambda (\tilde{J}_v) = e^{2\pi i \epsilon_\tilde{\Lambda}/2},
$$
$$b_{\lambda^{(1)}} (J') = e^{2\pi i r_{\lambda^{(1)}}/m}, \quad b_{\lambda^{(2)}} (J') = e^{2\pi i r_{\lambda^{(2)}}/n}, \quad b_\sigma (p) = e^{-2\pi i p \sigma/(mn(m+n)(m+n+k))}.$$
D.2 \( \text{SO}(2n)/\text{SO}(2n-2) \times U(1) \)

The bosonic form of this model is given in (4.18). The selection rule is given by
\[
\begin{align*}
  r^{(v)}_{\Lambda} & = r^{(v)}_{\Lambda} \quad (\text{mod } 1), \\
  r^{(s)}_{\Lambda} + (n-1)\epsilon_{\Lambda}/2 & = r^{(s)}_{\Lambda} + \sigma/4 \quad (\text{mod } 1).
\end{align*}
\]

(D.7)

Here \( r^{(s)}_{\Lambda} \) and \( r^{(v)}_{\Lambda} \) are the inner products of the weight \( \Lambda \) with the spinor weight \( \Lambda_s \) and the vector weight \( \Lambda_v \), respectively,
\[
\begin{align*}
  r^{(s)}_{\Lambda} & = (\Lambda, \Lambda_s) = \frac{1}{2}(\Lambda_1 + 2\Lambda_2 + \cdots + (n-2)\Lambda_{n-2} + \frac{n-2}{2}\Lambda_{n-1} + \frac{n}{2}\Lambda_n), \\
  r^{(v)}_{\Lambda} & = (\Lambda, \Lambda_v) = \Lambda_1 + \Lambda_2 + \cdots + \Lambda_{n-2} + \frac{1}{2}\Lambda_{n-1} + \frac{1}{2}\Lambda_n.
\end{align*}
\]

(D.8)

\( r^{(s)}_{\Lambda} \) and \( r^{(v)}_{\Lambda} \) are defined similarly. The field identification group is generated by
\[
\begin{align*}
  J_{(1)} & = (J_v, 1 / J'_v, 0), \\
  J_{(2)} & = (J_s, \tilde{J}^{n-1}_v / J'_v, k + 2(n-1)),
\end{align*}
\]

(D.9)

where \( J_{\Lambda} \) is the simple current of the representation \( \Lambda \).

E Explicit form of the twisted Cardy states

In this appendix, we give the explicit form of the twisted Cardy states for the two examples considered in section 4.2.

E.1 \( \text{SU}(4)/\text{SU}(2) \times \text{SU}(2) \times U(1) \)

In this case, the non-trivial automorphism is written as \( (\omega_0, 1; \pi, 1) \) (see (4.15)). We first construct twisted Cardy states in two sectors \( \text{SU}(4)_1, \text{SU}(2)_3 \times \text{SU}(2)_3 \). From the formula given in section 2.1 we obtain the following form of the boundary state coefficients \( \tilde{S} \) and the simple current groups \( G(\mathcal{I}^\omega) \) and \( G(\tilde{\mathcal{I}}) \):

\[ \text{SU}(4)_1 \]

The boundary state coefficients are given by the modular transformation matrix for the twisted chiral algebra \( A_3^{(2)} \) at level one,
\[
\begin{align*}
  \tilde{S}^{\text{SU}(4)_1} & = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \\
  \tilde{\mathcal{I}} & = P^1_+ (A_3^{(2)}) = \{(0,0), (1,0)\}, \quad \mathcal{I}^{\omega_0} = P^1_+ (D_3^{(2)}) = \{(0,0), (0,1)\}.
\end{align*}
\]

(E.1)

The simple current group of this theory is
\[
G_{sc} = \{ J^j | j = 0, 1, 2, 3 \} \cong \mathbb{Z}_4.
\]

(E.2)
The twisted NIM-rep graph $n_{(1,0)}$ of the $SU(4)$ WZW model. The black (white) node represents $\tilde{b}(J^2) = 1(-1)$.

The twisted NIM-rep graph $n_{(1,0)} = n_{(0,1)}$ of the $SU(2)_3 \times SU(2)_3$ WZW model. The black (white) nodes represent $\tilde{b}(J',J') = 1(-1)$ ($\langle J',J' \rangle \in G(\tilde{I}^\pi)$). The action of the simple current $(J',1) = (1,J') \in G(\tilde{I})$ is also shown.

The simple current groups for $I_{\omega_o}$ and $\tilde{I}$ are

$$G(I_{\omega_o}) = \{1, J^2\} \cong \mathbb{Z}_2, \quad G(\tilde{I}) = \{1, J\} \cong \mathbb{Z}_2, \quad S(\tilde{I}) = \{1, J^2\}. \quad (E.3)$$

We show the action of the simple current $J$ of $G(\tilde{I})$ and $\tilde{b}(J^2)$ ($J^2 \in G(I_{\omega_o})$) in Fig. 2.

$SU(2)_3 \times SU(2)_3$

The boundary state coefficients are given by the modular transformation matrix for $SU(2)_3$,

$$S^{SU(2)_3 \times SU(2)_3} = S^{SU(2)_3}, \quad \tilde{I} = I = \{\tilde{\lambda} \mid \tilde{\lambda} = 0, 1, 2, 3\}, \quad \tilde{I}^\pi = \{(\lambda, \lambda) \mid \lambda = 0, 1, 2, 3\}, \quad (E.4)$$

and the simple current groups read

$$G_{sc} = \{(J^i, J^j) \mid i, j = 0, 1\} \cong \mathbb{Z}_2 \times \mathbb{Z}_2, \quad G(I^\pi) = \{(J^k, J^k) \in G_{sc} \mid k = 0, 1\} \cong \mathbb{Z}_2, \quad (E.5)$$

$$G(\tilde{I}) = G_{sc}/S(\tilde{I}) \cong \mathbb{Z}_2, \quad S(\tilde{I}) = \{(J^k, J^k) \mid k = 0, 1\}.$$  

We show the action of these groups in Fig. 3.

Then, we can construct twisted Cardy states in the coset theory by applying the procedure reviewed in section 2.2 and in appendix A. The set $\hat{I}_{\omega_o}$ (2.37) is given by

$$\hat{I}_{\omega_o} = \{(0,0), \tilde{\lambda}; (0,0), 8j); ((0,0), \tilde{\lambda}; (1,1), 4+8j) \mid \tilde{\lambda} \in \{o, v, s, c\}; \; j = 0, 1, \ldots, 4\}. \quad (E.6)$$
The twisted NIM-rep graph $n_{(0,0,0,1)}$ of the $SO(8)_1$ WZW model. The black (white) node represents $\tilde{b}(J_v) = 1$ ($-1$).

From the definitions (A.3), the identification current groups read

$$G_{id}(I^\omega) = \{(J_{(1)}^{-1})^k(J_{(1)}J_{(2)})^{k'}| k = 0, 1, 2, 3; k' = 0, 1\} \cong \mathbb{Z}_4 \times \mathbb{Z}_2,$$

$$G_{id}(\tilde{I}) = \{J_{(1)}^i = J_{(2)}^{-i}| i = 0, 1, \ldots, 7\} \cong \mathbb{Z}_8, \quad G_{id}(\tilde{S}(\tilde{I})) = \{(J_{(1)}J_{(2)})^{k'}| k' = 0, 1\}. \quad (E.7)$$

The labels (2.38) of 40 twisted Cardy states are given as follows:

$$\tilde{I} = \{((0, 0), \tilde{\Lambda}; 0, 4j), ((0, 0), 2, 4j)| \tilde{\Lambda} \in \{o, v, s, c\}; j = 0, 1, \ldots, 4\}. \quad (E.8)$$

The boundary state coefficients $\tilde{S}$ (2.36) are given by

$$\tilde{S} = 8 \tilde{S}^{SU(4)_1} \tilde{S}^{SO(8)_1} \tilde{S}^{SU(2)_3 \times SU(2)_3} \tilde{S}^{SU(1)_{so}},$$

since $|G_{id}(I^\omega)| = |G_{id}(\tilde{I})| = 8$.

**E.2 $SO(8)/SO(6) \times U(1)$**

In this case, the non-trivial automorphism is written as $(\omega_2, 1; \omega'_c, 1)$ (see (4.23)). The twisted Cardy states in the $SO(8)_1$ and the $SO(6)_3$ sectors are as follows:

**SO(8)$_1$**

The boundary state coefficients $\tilde{S}$ is the modular transformation matrix of $D_4^{(2)}$ at level one,

$$\tilde{S}^{SO(8)_1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},$$

$$\tilde{I} = P_+^{1}(D_4^{(2)}) = \{(0, 0, 0), (0, 0, 1)\}, \quad I^{\omega_2} = P_+^{1}(A_3^{(2)}) = \{(0, 0, 0), (1, 0, 0)\}.$$

The simple current groups for $I^{\omega_2}$ and $\tilde{I}$ read

$$G_{sc} = \{J_v^i J_v^j| i, j = 0, 1\} \cong \mathbb{Z}_2 \times \mathbb{Z}_2,$$

$$G(I^{\omega_2}) = \{1, J_v\} \cong \mathbb{Z}_2, \quad G(\tilde{I}) = \{1, J_s\} \cong \mathbb{Z}_2. \quad (E.11)$$

The action of these groups are shown in Fig. 4.

**SO(6)$_3$**

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The action of these groups is shown in Fig. 5. In this way, we obtain 84 A-type twisted Cardy states.

The simple current groups are given by
\[
\omega S \tilde{\omega} = \begin{cases} \tilde{\lambda} = (\tilde{\lambda}_1, \tilde{\lambda}_2) = \{(0,0),(0,3),(0,2),(0,1),(1,0),(1,1)\}, \\
\mathcal{I} = \mathcal{P}_+^3(D_3^{(2)}) = \{\lambda = (\lambda_1, \lambda_2 = \lambda_3) = \{(0,0),(3,0),(2,0),(1,0),(0,1),(1,1)\}\}
\]

The simple current groups for the coset theory are given by
\[
G_{sc} = \{J_s^k | k = 0,1,2,3\} \cong \mathbb{Z}_4,
\]
\[
G(\mathcal{I}^\omega) = \{1, J_s^{2,2} = J_o'\} \cong \mathbb{Z}_2,
\]
\[
G(\tilde{\mathcal{I}}) = \{1, J_s'\} \cong \mathbb{Z}_2, \quad S(\tilde{\mathcal{I}}) = \{1, J_s^{2,2}\}
\]  

The action of these groups is shown in Fig 5.

The simple current groups for the coset theory are given by
\[
G_{id}(\mathcal{I}^\omega) = \{J^{i,j}_1, J^{2j}_2 | i, j = 0,1\} \cong \mathbb{Z}_2 \times \mathbb{Z}_2,
\]
\[
G_{id}(\tilde{\mathcal{I}}) = \{J^k_2 | k = 0,1,2,3\} \cong \mathbb{Z}_4, \quad G_{id} \cap S(\tilde{\mathcal{I}}) = \{1, J^{i,j}_1\}
\]

The boundary state coefficients \( \hat{S} \) are then written as
\[
\hat{S} = 4 \hat{S}_{SO(8)} S_{SO(12)} S_{SO(6)_3} S_{U(1)_2 s},
\]
\[
\tilde{\mathcal{I}}^{\omega} = \{(o, \tilde{\lambda}_N ; (\lambda_1, \lambda_2), 4j) ; (o, \tilde{\lambda}_R ; (\lambda_1, \lambda_2), 4j + 2) \}
\]
\[
| \tilde{\lambda}_N \in \{o,v\}, \tilde{\lambda}_R \in \{s,c\}; (\lambda_1, \lambda_2) \in \{(0,0),(2,0),(0,1)\}; j = 0,1,\ldots,6, \}
\]
\[
\tilde{\mathcal{I}} = \{(o, \tilde{\lambda} ; (\tilde{\lambda}_1, \tilde{\lambda}_2), 2j) \}
\]
\[
| \tilde{\lambda} \in \{o,v,s,c\}; (\tilde{\lambda}_1, \tilde{\lambda}_2) \in \{(0,0),(0,2),(1,0)\}; j = 0,1,\ldots,6, \}
\]

In this way, we obtain 84 A-type twisted Cardy states.

Figure 5: The twisted NIM-rep graph \( n_{(0,0,1)} \) in the \( SO(6)_3 \) theory. The black (white) nodes represent \( \tilde{b}(J_s^{2,2}) = 1 (-1) \). The action of the simple current \( J_s' \) is also shown.

The boundary state coefficients \( \hat{S} \) is the modular transformation matrix of \( D_3^{(2)} \) at level three,
\[
\hat{S}_{SO(6)_3} = \begin{pmatrix} x \kappa & -y \kappa & -z \kappa \\
-y \kappa & z \kappa & x \kappa \\
-z \kappa & x \kappa & y \kappa \end{pmatrix}, \quad (E.12)
\]
\[
k = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\
1 & -1 \end{pmatrix}, \quad x = \frac{2}{7}(2c_2 - c_6 - 1), \quad y = \frac{2}{7}(2c_6 - c_4 - 1), \quad z = \frac{2}{7}(2c_4 - c_2 - 1) \quad (c_n \equiv \cos \frac{n \pi}{7}),
\]
\[
\tilde{\mathcal{I}} = \mathcal{P}_+^3(D_3^{(2)}) = \tilde{\lambda} = (\tilde{\lambda}_1, \tilde{\lambda}_2) = \{(0,0),(0,3),(0,2),(0,1),(1,0),(1,1)\},
\]
\[
\mathcal{I}^\omega = \mathcal{P}_+^3(A_3^{(2)}) = \lambda = (\lambda_1, \lambda_2 = \lambda_3) = \{(0,0),(3,0),(2,0),(1,0),(0,1),(1,1)\}.
\]
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