Katugampola Fractional Calculus With Generalized $k$–Wright Function

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ABSTRACT. In this article, we present some properties of the Katugampola fractional integrals and derivatives. Also, we study the fractional calculus properties involving Katugampola Fractional integrals and derivatives of generalized $k$–Wright function $\phi_m^k(z)$.

1. Introduction and Preliminaries

In recent years, researchers have introduced new fractional integral and differential operators which are generalizations of the famous definitions of Riemann–Liouville, Caputo, Hadamard, Hilfer, etc. They have made a qualitative contribution to fractional differential equations. For more details, see [1,5-7,9-14] and references therein.

Definition 1.1. [9] Let $\Omega = [a, b]$, the Katugampola fractional integrals $\rho I_{0^+}^{\gamma} \varphi$ and $\rho I_{-}^{\gamma} \varphi$ of order $\gamma \in \mathbb{C}(\Re(\gamma) > 0)$ are defined for $\rho > 0$, $a = 0$ and $b = \infty$ as

\[
(\rho I_{0^+}^{\gamma} \varphi)(s) = \frac{\rho^{1-\gamma}}{\Gamma(\gamma)} \int_0^s \frac{\tau^{\rho-1} \varphi(\tau)}{(s^\rho - \tau^\rho)^{1-\gamma}} d\tau \quad (s > 0),
\]

(1.1)

and

\[
(\rho I_{-}^{\gamma} \varphi)(s) = \frac{\rho^{1-\gamma}}{\Gamma(\gamma)} \int_s^\infty \frac{\tau^{\rho-1} \varphi(\tau)}{(\tau^\rho - s^\rho)^{1-\gamma}} d\tau \quad (s > 0),
\]

(1.2)

the corresponding Katugampola fractional derivatives $\rho D_{0^+}^{\gamma} \varphi$ and $\rho D_{-}^{\gamma} \varphi$ are defined with $(n = 1 + |\Re(\gamma)|)$ as

\[
(\rho D_{0^+}^{\gamma} \varphi)(s) := (s^{1-\rho} \frac{d}{ds})^{1+|\Re(\gamma)|} (\rho I_{0^+}^{1-\gamma+|\Re(\gamma)|} \varphi)(s)
\]

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\[
\frac{\rho^{\gamma - [\Re(\gamma)]}}{\Gamma(1 - \gamma + [\Re(\gamma)])} (s^{1-\rho} \frac{d}{ds})^{1+[\Re(\gamma)]} \int_0^s \frac{\tau^{\rho-1} \varphi(\tau)}{(s^\rho - \tau)\gamma - [\Re(\gamma)]} d\tau \quad (s > 0),
\]

(1.3)

and

\[
(\rho D^{\gamma}_s \varphi)(s) := \left( -s^{1-\rho} \frac{d}{ds} \right)^{1+[\Re(\gamma)]} \left( \rho I_{s^\rho}^{1-\gamma + [\Re(\gamma)]} \varphi \right)(s)
\]

\[
= \frac{\rho^{\gamma - [\Re(\gamma)]}}{\Gamma(1 - \gamma + [\Re(\gamma)])} \left( -s^{1-\rho} \frac{d}{ds} \right)^{1+[\Re(\gamma)]} \int_s^\infty \frac{\tau^{\rho-1} \varphi(\tau)}{(\tau^\rho - s^\rho)\gamma - [\Re(\gamma)]} d\tau \quad (s > 0).
\]

(1.4)

Definition 1.2. [2] The generalized \( K \)-Gamma function \( \Gamma_k(y) \) is defined by

\[
\Gamma_k(y) = \lim_{n \to \infty} \frac{n! k^n (nk)^{y-1}}{(y)_{n,k}} \quad (k > 0; \ y \in \mathbb{C} \setminus k\mathbb{Z}^-),
\]

(1.5)

where \((y)_{n,k}\) is the \( k \)-Pochhammer symbol given as

\[
(y)_{n,k} := \begin{cases} \frac{\Gamma_k(y+nk)}{\Gamma_k(y)} & (k \in \mathbb{R}; \ y \in \mathbb{C} \setminus \{0\}) \\ y(y+k)(y+2k)...(y+(n-1)k) & (n \in \mathbb{N}^+; \ y \in \mathbb{C}) \end{cases}
\]

(1.6)

and for \( \Re(y) > 0 \), the \( K \)-Gamma function \( \Gamma_k(y) \) is defined by the integral

\[
\Gamma_k(y) = \int_0^\infty x^{y-1} e^{-x^k} dx.
\]

(1.7)

This gives a relation with Euler’s Gamma function as

\[
\Gamma_k(y) = k^{y-1} \Gamma\left(\frac{y}{k}\right).
\]

(1.8)

Also, in [8], we have

\[
\Gamma(1 - y)\Gamma(y) = \frac{\pi}{\sin(y\pi)}.
\]

(1.9)

Definition 1.3. [14] The Beta function \( B(\upsilon, \omega) \) is defined as

\[
B(\upsilon, \omega) = \int_0^1 z^{\upsilon-1} (1 - z)^{\omega-1} dz, \quad \Re(\upsilon) > 0, \ \Re(\omega) > 0,
\]

\[
= \frac{\Gamma(\upsilon)\Gamma(\omega)}{\Gamma(\upsilon + \omega)}
\]

(1.10)

Furthermore, we have

\[
\int_{\hat{x}}^{\infty} (z - \hat{x})^{\upsilon-1} (z - \hat{y})^{\omega-1} dz = (\hat{x} - \hat{y})^{\upsilon+\omega-1} B(\upsilon, 1 - \upsilon - \omega),
\]

\[
\hat{x} > \hat{y}, \quad 0 < \Re(\upsilon) < 1 - \Re(\omega).
\]

(1.11)

Recently, the Generalized \( K \)-Wright function introduced by (Gehlot and Prajapati [3]) is defined as follows:
Lemma 2.1. Let $\phi$ be a function satisfying derivatives (1.1), (1.2) and (1.3), (1.4) for the power function $\varphi(s) = s^{\alpha-1}$, for $\alpha \in \mathbb{R}$. If $\alpha, \beta_j \in \mathbb{R}$, $i = 1, 2, ..., n$, $j = 1, 2, ..., m)$ and $(\rho_i + \alpha_i r)$, $(q_j + \beta_j r) \in \mathbb{C} \setminus k\mathbb{Z}^-$, the generalized $k$–Wright function $n \Phi_m^k$ is defined by

$$n \Phi_m^k(z) = \sum_{r=0}^{\infty} \frac{\Gamma_k(p_i + \alpha_i r) z^r}{\prod_{j=1}^{m} \Gamma_k(q_j + \beta_j r)} r!,$$}

with the convergence conditions described as

$$\Delta = \sum_{j=1}^{m} \left( \frac{\beta_j}{k} \right) - \sum_{i=1}^{n} \left( \frac{\alpha_i}{k} \right); \quad \mu = \prod_{i=1}^{n} \frac{\alpha_i}{k} - \sum_{j=1}^{m} \frac{\beta_j}{k}; \quad \nu = \sum_{j=1}^{m} \frac{q_j}{k} - \sum_{i=1}^{n} \frac{p_i}{k} + \frac{n-m}{2}.$$}

Lemma 1.1. [3] For $k \in \mathbb{R}^+$; $z \in \mathbb{C}$; $\rho_i, q_j \in \mathbb{C}$, $\alpha_i, \beta_j \in \mathbb{R}$ $(\alpha_i, \beta_j \neq 0)$; $i = 1, 2, ..., n$; $j = 1, 2, ..., m)$ and $(\rho_i + \alpha_i r)$, $(q_j + \beta_j r) \in \mathbb{C} \setminus k\mathbb{Z}^-$

(1) If $\Delta > -1$, then series (1.12) is absolutely convergent for all $z \in \mathbb{C}$ and generalized $k$–Wright function $n \Phi_m^k(z)$ is an entire function of $z$.

(2) If $\Delta = -1$, then series (1.12) is absolutely convergent for all $|z| < \mu$ and of

$$|z| = \mu, \Re(\mu) > \frac{1}{2}.$$}

2. Properties of Katugampola Fractional Integral and Derivative

In this section, we investigate some properties of the Katugampola fractional integrals and derivatives (1.1), (1.2) and (1.3), (1.4) for the power function $\varphi(s) = s^{\alpha-1}$ and the exponential function $e^{-\lambda s^\rho}$.

Lemma 2.1. Let $\rho > 0, \Re(\gamma) \geq 0$ and $n = 1 + [\Re(\gamma)]$

(1) If $\Re(\alpha) > 0$, then

$$\left( \rho^{1+\alpha-1}_0 \right)^{(\gamma)}(s) = \frac{\rho^{-\gamma} \Gamma(1 + \frac{\alpha-1}{\rho}) s^{\rho\gamma+\alpha-1}}{\Gamma(1 + \frac{\alpha-1}{\rho} + \gamma)} \quad (\Re(\gamma) \geq 0; \Re(\alpha) > 0) \quad (2.1)$$

$$\left( \rho D_0^\gamma \right)^{(\alpha-1)}(s) = \frac{\rho^{-\gamma} \Gamma(1 + \frac{\alpha-1}{\rho}) s^{(\alpha-1)-\rho\gamma}}{\Gamma(1 + \frac{\alpha-1}{\rho} - \gamma)} \quad (\Re(\gamma) \geq 0; \Re(\alpha) > 0) \quad (2.2)$$

(2) If $\alpha \in \mathbb{C}$, then

$$\left( \rho^{1+\alpha-1}_\gamma \right)^{(\gamma)}(s) = \frac{\rho^{-\gamma} \Gamma(1 + \frac{1-\alpha}{\rho} - 1 - \gamma)}{\Gamma(1 + \frac{1-\alpha}{\rho})} s^{\rho\gamma+\alpha-1} \quad (\Re(\gamma) \geq 0; \Re(\gamma + \alpha) < 1) \quad (2.3)$$

$$\left( \rho D_\gamma^\gamma \right)^{(\alpha-1)}(s) = \frac{\rho^{-\gamma} \Gamma(1 + \frac{1-\alpha}{\rho} + \gamma)}{\Gamma(1 + \frac{1-\alpha}{\rho})} s^{(\alpha-1)-\rho\gamma} \quad (\Re(\gamma) \geq 0; \Re(\gamma + \alpha - [\Re(\gamma)]) < 1) \quad (2.4)$$

(3) If $\Re(\lambda) > 0$, then

$$\left( \rho^{1+\alpha-1}_\gamma e^{-\lambda s^\rho} \right)^{(\gamma)}(s) = (\lambda \rho)^{-\gamma} e^{-\lambda s^\rho} \quad (\Re(\gamma) \geq 0) \quad (2.5)$$
\((\rho D^\gamma_+ e^{-\lambda s^\rho})(s) = (\lambda \rho)^\gamma e^{-\lambda s^\rho} \quad (\Re(\gamma) \geq 0)\). \quad (2.6)

**Proof.** To prove this Lemma, let the substitution \(x = \frac{\tau}{s^\rho}\) in parts (1) and (2).

(1) Firstly, by the equation (1.1) and the given substitution, we have

\[(\rho \gamma_+ s^\rho + \alpha - 1)(s) = \frac{\rho^{-\gamma s^\rho + \alpha - 1}}{\Gamma(\gamma)} \int_0^1 \frac{x^{\frac{\alpha - 1}{\rho}}}{(1 - x)^{1 - \gamma}} \, dx \]

\[= \frac{\rho^{-\gamma s^\rho + \alpha - 1}}{\Gamma(\gamma)} \, B(\gamma, 1 + \frac{\alpha - 1}{\rho}).\]

Now, using equation (1.10), we obtain the result (2.1).

Secondly, by the equation (1.3), the given substitution and by using the result (2.1), we have

\[(\rho D^\gamma_+ s^\rho + \alpha - 1)(s) = \left( s^{1 - \rho} \frac{d}{ds} \right)^n \left( \rho \gamma_+ s^\rho + \alpha - 1 \right)(s) \]

\[= \frac{\rho^{-\gamma s^\rho + \alpha - 1}}{\Gamma(\gamma)} \left( s^{1 - \rho} \frac{d}{ds} \right)^n \rho s^{(n - \gamma) + \alpha - 1} \]

\[= \frac{\rho^{-\gamma s^\rho + \alpha - 1}}{\Gamma(\gamma)} \left( s^{(\alpha - 1) - \rho} \right) \gamma.\]

(2) Firstly, by the equation (1.2) and the given substitution, we have

\[(\rho \gamma_+ s^\rho + \alpha - 1)(s) = \frac{\rho^{-\gamma s^\rho + \alpha - 1}}{\Gamma(\gamma)} \int_1^\infty \frac{x^{\frac{\alpha - 1}{\rho}}}{(x - 1)^{\gamma - 1}} \, dx.\]

Now, using the equation (1.11) with \(\tilde{x} = 1\) and \(\tilde{y} = 0\), we obtain

\[(\rho \gamma_+ s^\rho + \alpha - 1)(s) = \frac{\rho^{-\gamma s^\rho + \alpha - 1}}{\Gamma(\gamma)} B(\gamma, 1 - \gamma - (1 + \frac{\alpha - 1}{\rho})).\]

By using equation (1.10), we obtain the result (2.3).

Secondly, by the equation (1.4), the given substitution and by using the result (2.3), we have

\[(\rho D^\gamma_+ s^\rho + \alpha - 1)(s) = \left( - s^{1 - \rho} \frac{d}{ds} \right)^n \left( \rho \gamma_+ s^\rho + \alpha - 1 \right)(s) \]

\[= \frac{(-1)^n \rho^{-\gamma s^\rho + \alpha - 1}}{\Gamma\left(1 - \frac{\alpha - 1}{\rho}\right)} \left( s^{1 - \rho} \frac{d}{ds} \right)^n \rho s^{(n - \gamma) + \alpha - 1} \]

\[= \frac{(-1)^n \rho^{-\gamma s^\rho + \alpha - 1}}{\Gamma\left(1 - \frac{\alpha - 1}{\rho}\right)} \Gamma\left(1 - \frac{\alpha - 1}{\rho}\right) \left( 1 - \gamma - (1 + \frac{\alpha - 1}{\rho}) \right). \] \quad (2.7)

Also, by using (1.9), we have

\[\Gamma\left(1 - \frac{\alpha - 1}{\rho}\right) \Gamma\left(1 - \frac{1 - \alpha}{\rho} + \gamma - n\right) = \frac{\pi}{\sin\left(\frac{\pi}{\rho} \left(\frac{1 - \alpha}{\rho} + \gamma - n\right)\right)} = \frac{(-1)^n \pi}{\sin\left(\gamma - \frac{\alpha - 1}{\rho}\right) \pi} \]

\[= \frac{\pi}{\sin\left(\gamma - \frac{\alpha - 1}{\rho}\right) \pi} \] \quad (2.8)

and

\[\frac{1}{\Gamma\left(1 - \frac{\alpha - 1}{\rho}\right)} = \frac{\Gamma\left(\gamma - \frac{\alpha - 1}{\rho}\right)}{\Gamma\left(\gamma - \frac{\alpha - 1}{\rho}\right) \Gamma\left(1 - \frac{\alpha - 1}{\rho}\right)} = \frac{\Gamma\left(\gamma - \frac{\alpha - 1}{\rho}\right)}{\pi} \sin\left(\gamma - \frac{\alpha - 1}{\rho}\right) \pi \]

\[= \frac{\Gamma\left(\gamma - \frac{\alpha - 1}{\rho}\right)}{\pi} \sin\left(\gamma - \frac{\alpha - 1}{\rho}\right) \pi \] \quad (2.9)

Substituting relations (2.8) and (2.9) in (2.7), we obtain (2.4).
For this part, let the substitution \( x = \tau - s^\rho \).

Firstly, by the equation (1.2) and the given substitution in this part, we have

\[
(\rho I^\gamma e^{-\lambda \tau^\rho})(s) = \frac{\rho^{-\gamma}}{\Gamma(\gamma)} e^{-\lambda s^\rho} \int_0^\infty e^{-\lambda x} x^{\gamma-1} dx,
\]

then by use the substitution \( \vartheta = \lambda x \), we obtain

\[
(\rho I^\gamma e^{-\lambda \tau^\rho})(s) = \frac{\rho^{-\gamma}}{\Gamma(\gamma)} e^{-\lambda s^\rho} \lambda^{-\gamma} \int_0^\infty e^{-\vartheta} \vartheta^{\gamma-1} d\vartheta,
\]

since \( \int_0^\infty e^{-\vartheta} \vartheta^{\gamma-1} d\vartheta = \Gamma(\gamma) [8] \), then the result is satisfied.

Secondly, by the equation (1.4) and by using the result (2.5), we have

\[
(\rho D^\gamma e^{-\lambda \tau^\rho})(s) = (-s^\rho)^n (\rho I^\gamma e^{-\lambda \tau^\rho})(s)
\]

\[
= (-1)^n (s^\rho)^n (\rho I^\gamma e^{-\lambda \tau^\rho})(s)
\]

\[
= (-1)^n s^{(1-\rho)n} (\rho \lambda^\gamma e^{-\lambda s^\rho})
\]

\[
= (\lambda \rho)^\gamma e^{-\lambda s^\rho}.
\]

\[\square\]

**Remark 2.1.** (a) In Lemma 2.1, if the power function is \( \varphi(s) = \left(\frac{s^\rho}{\rho}\right)^{\alpha-1} \), then

1. If \( \Re(\alpha) > 0 \), then

\[
\left(\rho I^\gamma \left(\frac{\tau^\rho}{\rho}\right)^{\alpha-1}\right)(s) = \frac{\Gamma(\alpha)}{\Gamma(\alpha+\gamma)} \left(\frac{s^\rho}{\rho}\right)^{\alpha+\gamma-1} \quad (\Re(\gamma) \geq 0; \ Re(\alpha) > 0)
\]

\[
\left(\rho D^\gamma \left(\frac{\tau^\rho}{\rho}\right)^{\alpha-1}\right)(s) = \frac{\Gamma(\alpha)}{\Gamma(\alpha-\gamma)} \left(\frac{s^\rho}{\rho}\right)^{\alpha-\gamma-1} \quad (\Re(\gamma) \geq 0; \ Re(\alpha) > 0).
\]

2. If \( \alpha \in \mathbb{C} \), then

\[
\left(\rho I^\gamma \left(\frac{\tau^\rho}{\rho}\right)^{\alpha-1}\right)(s) = \frac{\Gamma(1-\gamma-\alpha)}{\Gamma(1-\alpha)} \left(\frac{s^\rho}{\rho}\right)^{\alpha+\gamma-1} \quad (\Re(\gamma) \geq 0; \ Re(\gamma + \alpha) < 1)
\]

\[
\left(\rho D^\gamma \left(\frac{\tau^\rho}{\rho}\right)^{\alpha-1}\right)(s) = \frac{\Gamma(1+\gamma-\alpha)}{\Gamma(1-\alpha)} \left(\frac{s^\rho}{\rho}\right)^{\alpha-\gamma-1} \quad (\Re(\gamma) \geq 0; \ Re(\gamma + \alpha - [\Re(\gamma)]) < 1).
\]

(b) If \( \Re(\alpha) > \Re(\gamma) > 0 \), then

\[
(\rho I^\gamma \tau^{-\alpha})(s) = \frac{\rho^{-\gamma} \Gamma(\frac{\alpha}{\rho} - \gamma)}{\Gamma(\frac{\alpha}{\rho})} s^{\rho \gamma - \alpha}.
\]

(2.10)
3. Katugampola Fractional Integration for Generalized $k$–Wright Function

In this section, we establish the Katugampola fractional integration for generalized $k$–Wright function (1.12).

**Theorem 3.1.** Let $\gamma, \alpha \in \mathbb{C}$ such that $\Re(\gamma) > 0$, $\Re(\alpha) > 0$; $\lambda \in \mathbb{C}$, $\rho > 0$, $\nu > 0$, then for $\Delta > -1$, the Katugampola fractional integration $\rho I_{0+}^{\gamma}$ for generalized $k$–Wright function $\psi^k_m(z)$ is given as

$$
\left( \rho I_{0+}^{\gamma} \left( \Phi^k_m \left( \frac{\alpha_i}{\beta_j} \right) \lambda \tau^\rho \right) \right) (s) = \left( \frac{k}{\rho} \right)^\gamma s^\frac{\alpha + \rho \gamma - 1}{\rho} \psi^k_{m+1} \left( \Phi^k_m \left( \frac{\alpha_i}{\beta_j} \right) \lambda \tau^\rho \right) (s).
$$

**Proof.** According to Lemma 1.1, a generalized $k$–Wright function in both sides of the equation (3.1) exists for $s > 0$. We consider that

$$
M \equiv \left( \rho I_{0+}^{\gamma} \left( \Phi^k_m \left( \frac{\alpha_i}{\beta_j} \lambda \tau^\rho \right) \right) \right) (s).
$$

Using (1.12), we can write the above equation as

$$
M \equiv \left( \rho I_{0+}^{\gamma} \left( \Phi^k_m \left( \frac{\alpha_i}{\beta_j} \lambda \tau^\rho \right) \right) \right) (s).
$$

Now, using the integration of the series term by term, we obtain

$$
M \equiv \sum_{r=0}^{\infty} \Gamma_k \left( \frac{\alpha_i}{\beta_j} \lambda \tau^\rho \right) r! \left( \rho I_{0+}^{\gamma} \left( \Phi^k_m \left( \frac{\alpha_i}{\beta_j} \lambda \tau^\rho \right) \right) \right) (s).
$$

Applying (2.1), the above equation is reduced to

$$
M \equiv \sum_{r=0}^{\infty} \Gamma_k \left( \frac{\alpha_i}{\beta_j} \lambda \tau^\rho \right) r! \left( \rho I_{0+}^{\gamma} \left( \Phi^k_m \left( \frac{\alpha_i}{\beta_j} \lambda \tau^\rho \right) \right) \right) (s).
$$

Using (1.8), we obtain

$$
M \equiv \left( \frac{k}{\rho} \right)^\gamma s^\frac{\alpha + \rho \gamma - 1}{\rho} \psi^k_{m+1} \left( \Phi^k_m \left( \frac{\alpha_i}{\beta_j} \lambda \tau^\rho \right) \right) (s).
$$

**Theorem 3.2.** Let $\gamma, \alpha \in \mathbb{C}$ such that $\Re(\gamma) > 0$, $\Re(\alpha) > 0$; $\lambda \in \mathbb{C}$, $\rho > 0$, $\nu > 0$, then for $\Delta > -1$, the Katugampola fractional integration $\rho I_{-}^{\gamma}$ for generalized $k$–Wright function $\psi^k_m(z)$ is given as

$$
\left( \rho I_{-}^{\gamma} \left( \Phi^k_m \left( \frac{\alpha_i}{\beta_j} \lambda \tau^{-\rho} \right) \right) \right) (s) = \left( \frac{k}{\rho} \right)^\gamma s^\frac{\alpha + \rho \gamma - 1}{\rho} \psi^k_{m+1} \left( \Phi^k_m \left( \frac{\alpha_i}{\beta_j} \lambda \tau^{-\rho} \right) \right) (s).
$$
\[= \left( \frac{k}{\rho} \right)^{\gamma} s^{\rho \gamma - \frac{\omega}{m}} n_{+1} \Phi_{m+1}(\rho, \alpha_i)_{1,n}, \left( \frac{\alpha}{\rho} - k \gamma, \frac{\omega}{\rho} \right) \bigg| {\lambda} s^{-\frac{\omega}{m}} \bigg]. \tag{3.2} \]

**Proof.** According to Lemma 1.1, a generalized \(k\)-\textit{Wright} function in both sides of the equation (3.2) exists for \(s > 0\). We consider that

\[N \equiv \left( \rho \right)^{\gamma} \left( \tau^{\frac{\omega}{m}} n \Phi_{m} k(\rho, \alpha_i)_{1,n} \bigg| {\lambda} \tau^{-\frac{\omega}{m}} \bigg) \right) \equiv (s). \]

Using (1.12), we can write the above equation as

\[N \equiv \sum_{r=0}^{\infty} \prod_{j=1}^{n} \frac{\Gamma(k + \alpha_i r)}{\Gamma(k + \beta_j r)} \frac{(\lambda)^{r}}{r!} \left( \rho \right)^{\gamma} \frac{\Gamma(\frac{\alpha + \nu}{\rho}) - \gamma}{\Gamma(\frac{\alpha + \nu}{\rho})} s^{\rho \gamma - \frac{\omega}{m}}. \]

Applying (2.10), the above equation is reduced to

\[N \equiv \sum_{r=0}^{\infty} \prod_{j=1}^{n} \frac{\Gamma(k + \alpha_i r)}{\Gamma(k + \beta_j r)} \frac{(\lambda)^{r}}{r!} \left( \rho \right)^{\gamma} \frac{\Gamma(\frac{\alpha + \nu}{\rho}) - \gamma}{\Gamma(\frac{\alpha + \nu}{\rho})} s^{\rho \gamma - \frac{\omega}{m}}. \]

Using (1.8), we obtain

\[N \equiv \left( \frac{k}{\rho} \right)^{\gamma} s^{\rho \gamma - \frac{\omega}{m}} n_{+1} \Phi_{m+1}(\rho, \alpha_i)_{1,n}, \left( \frac{\alpha}{\rho} - k \gamma, \frac{\omega}{\rho} \right) \bigg| {\lambda} s^{-\frac{\omega}{m}} \bigg]. \]

\[\square\]

4. **Katugampola Fractional Differentiation for Generalized \(k\)-\textit{Wright} Function**

This section deals with the Katugampola fractional differentiation for generalized \(k\)-\textit{Wright} function (1.12).

**Theorem 4.1.** Let \(\gamma, \alpha \in \mathbb{C}\) such that \(\Re(\gamma) > 0, \Re(\alpha) > 0; \lambda \in \mathbb{C}, \rho > 0, \nu > 0\), then for \(\Delta > -1\), the Katugampola fractional differentiation \(\rho D_{0+}^{\alpha} \) for generalized \(k\)-\textit{Wright} function \(n \Phi_{m}^{k}(z)\) is given as

\[\left( \rho D_{0+}^{\alpha} \left( \tau^{\frac{\omega}{m}} \Phi_{m}(\rho, \alpha_i)_{1,n} \bigg| {\lambda} \tau^{-\frac{\omega}{m}} \bigg) \right) \right) \equiv (s).
\]

\[= \left( \frac{k}{\rho} \right)^{-\gamma} s^{\rho \gamma - \rho \gamma - \frac{\omega}{m}} n_{+1} \Phi_{m+1}(\rho, \alpha_i)_{1,n}, \left( \frac{\alpha}{\rho} - (\rho - 1)k, \frac{\omega}{\rho} \right) \bigg| {\lambda} s^{-\frac{\omega}{m}} \bigg]. \tag{4.1} \]

**Proof.** According to Lemma 1.1, a generalized \(k\)-\textit{Wright} function in both sides of the equation (4.1) exists for \(s > 0\). Let \(n = 1 + [\Re(\gamma)]\). Then, we consider that

\[P \equiv \left( \rho D_{0+}^{\alpha} \left( \tau^{\frac{\omega}{m}} \Phi_{m}(\rho, \alpha_i)_{1,n} \bigg| {\lambda} \tau^{-\frac{\omega}{m}} \bigg) \right) \right) \equiv (s).\]
Using (1.3), we have
\[ P \equiv (s^{1-\rho} \frac{d}{ds})^n \left( \rho \Gamma_{q_0}^{n-\tau} \left( \tau_{q_1}^{\tau-1} n \Phi_m^k \left( \frac{(p_i, \alpha_i)_{1,n}}{(q_j, \beta_j)_{1,m}} | \lambda, \tau^{-\frac{\nu}{\alpha}} \right) \right) \right)(s). \]

Using Theorem 3.1, we obtain
\[ P \equiv (s^{1-\rho} \frac{d}{ds})^n \left( \left( \frac{k}{\rho} \right)^{n-\gamma} s_{\tau}^{\frac{\nu}{\alpha}+\rho(\gamma-1)} n+1 \Phi_{m+1}^k \left[ \left( \frac{(p_i, \alpha_i)_{1,n}}{(q_j, \beta_j)_{1,m}} | \lambda, s^{\frac{\nu}{\alpha}} \right) \right] \right). \]

Using (1.12), we can write the above equation as
\[ P \equiv \left( \frac{k}{\rho} \right)^{n-\gamma} \sum_{r=0}^{\infty} \frac{\prod_{i=1}^{n} \Gamma_k(p_i + \alpha_i r) \Gamma_k(1/\rho(\alpha + (\rho - 1)k) + \frac{\nu}{\rho})}{\prod_{j=1}^{m} \Gamma_k(q_j \beta_j r) \Gamma_k(1/\rho(\alpha + (\rho(n - \gamma + 1) - 1)k) + \frac{\nu}{\rho})} \frac{(\lambda)^r}{r!} \left( s^{1-\rho} \frac{d}{ds} \right)^n \left( s_{\tau}^{\frac{\nu}{\alpha}+\rho(\gamma-1)} \right). \]

Also, the above equation can be written as
\[ P \equiv \left( \frac{k}{\rho} \right)^{n-\gamma} s_{\tau}^{\frac{\nu}{\alpha}+\rho(\gamma-1)} n+1 \Phi_{m+1}^k \left[ \left( \frac{(p_i, \alpha_i)_{1,n}}{(q_j, \beta_j)_{1,m}} | \lambda, s^{\frac{\nu}{\alpha}} \right) \right]. \]

□

**Theorem 4.2.** Let \( \gamma, \alpha \in \mathbb{C} \) such that \( \Re(\gamma) > 0, \Re(\alpha) > 1 + [\Re(\gamma)] - \Re(\gamma); \lambda \in \mathbb{C}, \rho > 0, \nu > 0 \), then for \( \Delta > -1 \), the Katugampola fractional differentiation \( _{\rho}D_\Delta^\gamma \) for generalized \( k \)–Wright function \( n \Phi_{m}^k(z) \) is given as
\[
\left( _{\rho}D_\Delta^\gamma \left( \tau^{\frac{\alpha}{\nu}} n \Phi_m^k \left( \frac{(p_i, \alpha_i)_{1,n}}{(q_j, \beta_j)_{1,m}} | \lambda, \tau^{-\frac{\nu}{\alpha}} \right) \right) \right)(s) = \left( \frac{k}{\rho} \right)^{-\gamma} s^{-\rho} s^{\frac{\nu}{\alpha}} n+1 \Phi_{m+1}^k \left[ \left( \frac{(p_i, \alpha_i)_{1,n}}{(q_j, \beta_j)_{1,m}} | \lambda, s^{\frac{\nu}{\alpha}} \right) \right] \text{ (4.2)}
\]

**Proof.** According to Lemma 1.1, a generalized \( k \)–Wright function in both sides of the equation (4.2) exists for \( s > 0 \). Let \( n = 1 + [\Re(\gamma)] \). Then, we consider that
\[ Q \equiv \left( _{\rho}D_\Delta^\gamma \left( \tau^{\frac{\alpha}{\nu}} n \Phi_m^k \left( \frac{(p_i, \alpha_i)_{1,n}}{(q_j, \beta_j)_{1,m}} | \lambda, \tau^{-\frac{\nu}{\alpha}} \right) \right) \right)(s) \]

Using (1.4), we have
\[ Q \equiv \left( - s^{1-\rho} \frac{d}{ds} \right)^n \left( \rho \Gamma_{q_0}^{n-\tau} \left( \tau^{\frac{\alpha}{\nu}} n \Phi_m^k \left( \frac{(p_i, \alpha_i)_{1,n}}{(q_j, \beta_j)_{1,m}} | \lambda, \tau^{-\frac{\nu}{\alpha}} \right) \right) \right)(s). \]
Using Theorem 3.2, we obtain

\[
Q \equiv \left( -s^{1-\rho} \frac{d}{ds} \right)^n \left( \frac{k}{\rho} \right)^{n-\gamma} s^{\rho(n-\gamma) - \frac{n}{2}} n_{m+1} \Phi_m^k \left[ \left( \rho_i, \alpha_i \right)_{1,n}, \left( \frac{\alpha}{\rho}, k(n-\gamma), \frac{\nu}{\rho} \right) \right] \left( q_j, \beta_j \right)_{1,m}, \left( \frac{\alpha}{\rho}, \frac{\nu}{\rho} \right) \lambda s^{-\frac{n}{2}} \right].
\]

Using (1.12), we can write the above equation as

\[
Q \equiv (-1)^n k^{n-\gamma} \rho^{2} \sum_{r=0}^{\infty} \prod_{i=1}^{n} \Gamma_k(p_i + \alpha_i r) \prod_{j=1}^{m} \Gamma_k(q_j + \beta_j r) \lambda^r \frac{(\lambda)^r}{r!} \left( s^{1-\rho} \frac{d}{ds} \right)^n \left( s^{\rho(n-\gamma) - \frac{n}{2}} \right).
\]

On simplifying the above equation, we obtain

\[
Q \equiv (-1)^n k^{n-\gamma} \rho^{2} \sum_{r=0}^{\infty} \prod_{i=1}^{n} \Gamma_k(p_i + \alpha_i r) \prod_{j=1}^{m} \Gamma_k(q_j + \beta_j r) \lambda^r \frac{(\lambda)^r}{r!} \Gamma(1 + (n-\gamma) - \frac{\alpha}{\rho k} - \frac{\nu}{\rho k} r) \Gamma(1 - (n - \frac{\alpha}{\rho k} + \frac{\nu}{\rho k} r)) \left( s^{-\rho(n-\gamma) - \frac{n}{2}} \right).
\]

Using (1.8), we obtain

\[
Q \equiv (-1)^n k^{n-\gamma} \rho^{2} \sum_{r=0}^{\infty} \prod_{i=1}^{n} \Gamma_k(p_i + \alpha_i r) \prod_{j=1}^{m} \Gamma_k(q_j + \beta_j r) \lambda^r \frac{(\lambda)^r}{r!} \Gamma(1 + (n-\gamma) - \frac{\alpha}{\rho k} - \frac{\nu}{\rho k} r) \Gamma(1 - (n - \frac{\alpha}{\rho k} + \frac{\nu}{\rho k} r)) \left( s^{-\rho(n-\gamma) - \frac{n}{2}} \right).
\]

Using (1.9), we have

\[
\Gamma(n + \frac{\alpha}{pk} + \frac{\nu}{pk} r) \Gamma(1 - (n + \frac{\alpha}{pk} + \frac{\nu}{pk} r)) \frac{\pi}{\sin[(\gamma + \frac{\alpha}{pk} + \frac{\nu}{pk} r)\pi - n\pi]}
\]

\[
= \frac{\pi}{\sin[(\gamma + \frac{\alpha}{pk} + \frac{\nu}{pk} r)\pi - n\pi]}
\]

\[
= \frac{\pi}{\sin[(\gamma + \frac{\alpha}{pk} + \frac{\nu}{pk} r)\pi - n\pi]}
\]

\[
= \frac{(-1)^n \pi}{\sin[(\gamma + \frac{\alpha}{pk} + \frac{\nu}{pk} r)\pi]}
\]

and

\[
\frac{1}{\Gamma(1 - (\gamma + \frac{\alpha}{pk} + \frac{\nu}{pk} r))} = \frac{\Gamma(\gamma + \frac{\alpha}{pk} + \frac{\nu}{pk} r) \sin[(\gamma + \frac{\alpha}{pk} + \frac{\nu}{pk} r)\pi]}{\pi}.
\]

Substituting (4.4) and (4.5) in (3.3) and finally by using (1.8), we obtain

\[
Q \equiv \left( \frac{k}{\rho} \right)^{n-\gamma} s^{-\rho(n-\gamma) - \frac{n}{2}} n_{m+1} \Phi_m^k \left[ \left( \rho_i, \alpha_i \right)_{1,n}, \left( \frac{\alpha}{\rho}, k(n-\gamma), \frac{\nu}{\rho} \right) \right] \left( q_j, \beta_j \right)_{1,m}, \left( \frac{\alpha}{\rho}, \frac{\nu}{\rho} \right) \lambda s^{-\frac{n}{2}} \right].
\]
5. Concluding Remarks

- If $\rho = 1$, then
  Theorems 3.1, 3.2, 4.1 and 4.2, are reduced to Theorems 2, 3, 4 and 5 respectively (see [4]).
- Some general properties of the Katugampola fractional integrals and derivatives for the power function $\varphi(s) = s^{\alpha-1}$ and the exponential function $e^{-\lambda s^\rho}$ are investigated.
- The Katugampola fractional integration $\rho I^\gamma_{0+}$ and $\rho I^\gamma_-$ for generalized $k$–Wright function $n\Phi^k_m(z)$ are established.
- The Katugampola fractional differentiation $\rho D^\gamma_{0+}$ and $\rho D^\gamma_-$ for generalized $k$–Wright function $n\Phi^k_m(z)$ are established.

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