Comment on “Phase Reduction of Stochastic Limit Cycle Oscillators”

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In a recent Letter, Yoshimura and Arai [1] claimed that the conventional phase stochastic differential equation (SDE) used in [2, 3, 4] does not give a proper approximation to limit-cycle oscillators driven by noise, and proposed a modified phase SDE. Here we argue that their claim is not always correct; both SDEs are valid depending on the situation.

Since physical noise has an associated time scale and all oscillators have a characteristic rate of attraction, which of the two SDEs is appropriate depends on the relative sizes of these two scales. As a simple example, let us consider the Stuart-Landau (SL) model used in [1, 2] driven by a colored noise generated by the Ornstein-Uhlenbeck process (OUP) [5], which is rescaled such that the amplitude relaxation time explicitly appears while keeping the limit cycle and its isochrons invariant,

\[ W(t) = \{T^{-1}(1 + ic) + i\omega\}W - T^{-1}(1 + ic)|W|^2W + \sqrt{2\varepsilon}\xi(t), \]

where \( W \) is a complex variable representing the oscillator state, \( T \) is the relaxation time of the amplitude, \( c \) and \( \omega \) are parameters, \( \varepsilon \) is the noise intensity, and \( \xi(t) \) is OUP noise that is applied only to the real component of \( W \) for simplicity. \( \xi(t) \) is Gaussian-distributed, and its correlation function is given by \( \langle \xi(t)\xi(s) \rangle = \exp(-|t-s|/\tau)/(2\tau) \), which converges to \( \delta(t) \) as \( \tau \to 0 \). Thus, \( \xi(t) \) gives a colored-noise approximation to the Wiener process [5].

Introducing the amplitude \( R = |W| \) and the isochron phase \( \phi = \arg(W - c\ln|W|) \) [1, 2], Eq. (1) can be written as

\[ \dot{R}(t) = T^{-1}(R - R^3) + \sqrt{2\varepsilon} \cos(\phi + c\ln R) \xi(t), \]

\[ \dot{\phi}(t) = \omega - \sqrt{2\varepsilon}R^{-1}\{\sin(\phi + c\ln R) + c\cos(\phi + c\ln R)\} \xi(t). \]

It is now clear that \( T \) actually determines the relaxation time of the amplitude \( R \). The limit cycle in the absence of the noise (\( \varepsilon = 0 \)) is simply \( \dot{R}(t) \equiv 1 \) and \( \dot{\phi}(t) = \omega t + \text{const} \).

Two different SDEs have been previously derived describing this and other noisy oscillators. The non-agreement is due to the order in which the white-noise limit and the phase limit are taken [6]. The “conventional” model obtained by taking the phase limit in the first has the form:

\[ d\phi(t) = [\omega + \varepsilon Z(\phi)Z'(\phi)] dt + \sqrt{2\varepsilon}Z(\phi)dw(t), \]

where the phase sensitivity (or response) function \( Z(\phi) = -\sin\phi - c\cos\phi \) in the present example. Yoshimura and Arai’s modified phase model obtained by taking the white-noise limit first is given by

\[ d\phi(t) = [\omega + \varepsilon \{Z(\phi)Z'(\phi) + Y(\phi)\}] dt + \sqrt{2\varepsilon}Z(\phi)dw(t), \]

where the extra term \( Y(\phi) = (1 + c^2)\sin(2\phi)/2 \) [1].

To see which of the two reduced phase SDEs [1, 5] approximates the original noisy SL model Eqs. (2) [3] better, we compare the stationary phase probability density functions (PDFs) obtained by direct Langevin simulations of Eqs. (2) [3] for different pairs of \( (\tau, T) \) with the PDFs obtained from the two phase SDEs [1, 5] by numerically solving the corresponding Fokker-Planck equations. We fix \( \omega = 1 \), \( c = 2 \), \( \varepsilon = 0.01 \), and vary \( \tau \) and \( T \) keeping \( \tau T = 0.001 \) constant.

Figure 1(a) shows the stationary phase PDFs obtained for two typical cases, \( T = 0.01 \ll \tau = 0.1 \) and \( \tau = 0.01 \ll T = 0.1 \). The conventional phase SDE [1] nicely fits the original model when \( T \ll \tau \), whereas the modified phase SDE [5] is better when \( \tau \ll T \). Figure 1(b) shows mean-square errors of the approximate PDFs yielded by...
FIG. 1: (a) Comparison of stationary phase PDFs obtained directly from Eqs. (2, 3) with those obtained from the phase SDEs (4, 5) for \((T, \tau) = (0.1, 0.01)\) and \((0.01, 0.1)\). (b) Mean-square errors of the approximate phase PDFs of SDEs (4, 5) from the original PDFs of Eqs. (2, 3) plotted as functions of the amplitude relaxation time \(T (= 0.001/\tau)\).