J-EQUATIONS ON HOLOMORPHIC SUBMERSIONS

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Abstract. In this paper, we prove that there exists a solution of the J-equation on a total space of a holomorphic submersion if there exist solutions of the J-equation on fibers and a base. The method is an adiabatic limit technique. We also partially prove the converse implication. More precisely, if a total space is J-nef, then each fiber is J-nef. In addition, if each fiber has a solution of the J-equation, then a base is also J-nef.

1. Introduction

Finding a canonical metric such as a constant scalar curvature Kähler (cscK, for short) metric in a given Kähler class is a central problem in Kähler geometry. This paper studies when the total space of a holomorphic submersion admits a canonical metric. Fine [Fin04] constructed a cscK metric on a fibered complex surface admitting a holomorphic submersion onto a high genus curve with fibers of genus at least two. Dervan-Sektnan [DS21] generalized the above result to the higher dimensional case, introducing a new concept of relatively Kähler metrics on a fibration called an \textit{optimal symplectic connection}.

In this paper, instead of cscK metrics, we consider solutions of the J-equations on holomorphic submersions. The J-equation is related to cscK metrics. For example, if there exists a solution of the J-equation, then there exists a (twisted) cscK metric, by the properness-solvability equivalence about the (twisted) cscK equation [CC21] and the J-equation [CS17, Propositions 21 and 22]. For a compact Kähler manifold \((X, \chi)\) of dimension \(n\), the \(J_\chi\)-equation is given by

\begin{equation}
\Lambda_\omega \chi = c,
\end{equation}

where \(\Lambda_\omega\) denotes the trace with respect to a Kähler form \(\omega\) and \(c\) is the constant determined by

\[n \int_X \chi \wedge \omega^{n-1} = c \int_X \omega^n.\]

Let us fix our setup.

\textbf{Setup 1.2.} Let \((X, \chi)\) and \(B\) be compact Kähler manifolds and \(\pi : X \to B\) be a holomorphic submersion of relative dimension \(m\), and denote the dimension of \(B\) by \(n\). Suppose that \(\omega_X\) is a relatively Kähler form on \(X\), i.e. a restriction on each fiber is Kähler, and \(\omega_B\) is a Kähler form on \(B\).

In this setup, the tangent space \(TX\) splits as a smooth bundle

\[TX \cong \mathcal{V} \oplus \mathcal{H},\]

where \(\mathcal{V} = \ker d\pi\) denotes the vertical tangent bundle and \(\mathcal{H}\) denotes the horizontal subbundle of \(TX\) defined by

\[\mathcal{H}_x = \{u \in T_xX \mid \omega_X(u,v) = 0 \text{ for all } v \in \mathcal{V}_x\}.\]
By this splitting, the Kähler form \( \chi \) on \( X \) is divided into the purely vertical component \( \chi_V \), the purely horizontal component \( \chi_H \) and the mixed component \( \chi_m \). Denote by \( \chi_b \) the restriction of \( \chi \) to a fiber \( X_b \). Define a \((1,1)\)-form \( \pi_B(\chi_H) \) on \( B \) by fiberwise integral of \( \chi_H \wedge \omega_X^n \) divided by the volume of a fiber \( X_b \), i.e.,

\[
\pi_B(\chi_H)(b) = \frac{V_b^{-1} \pi_*(\chi_H \wedge \omega_X^n)(b)}{V_b},
\]

where \( V_b = \int_{X_b} \omega_b^m \) and \( \omega_b \) is the restriction of \( \omega_X \) to a fiber \( X_b \). See Section 2.2 for the precise definition. Note that \( V_b \) is independent of \( b \) since \( \omega_X \) is closed. Define a constant \( c_b \) by

\[
c_b = \frac{m \int_{X_b} \chi_b \wedge (\omega_b)^{m-1}}{\int_{X_b} (\omega_b)^m}.
\]

Note that \( c_b \) is independent of \( b \) since \( \chi \) and \( \omega_X \) are closed. The main theorem of this paper is an analogue of the following result by Dervan-Sektnan.

**Theorem 1.3** ([DS21, Theorem 1.2]). Suppose that \((B,L)\) admits a twisted cscK metric \( \omega_B \in c_1(L) \) and \( \pi : (X,H) \to (B,L) \) admits an optimal symplectic connection \( \omega_X \in \mathcal{C}_1(H) \). Assume both of the automorphism groups \( \text{Aut}(X,H) \) and \( \text{Aut}(\pi) \) are discrete. Then there exists a cscK metric in the class \( k \pi^* c_1(L) + c_1(H) \) for \( k \gg 0 \).

A relatively Kähler form \( \omega_X \) whose restriction to each fiber is a cscK metric is called an **optimal symplectic connection** if it satisfies a certain equation. If a cscK metric on each fiber \((X_b,H_b)\) is unique, then the optimal symplectic connection condition becomes trivial. Our situation is similar to this situation, since a solution of the \( J \)-equation is unique [Che00, Proposition 2]. We suppress pullbacks via \( \pi \), so if \( \omega_B \) is a form on \( B \), its pullback to \( X \) will also be denoted by \( \omega_B \).

**Theorem 1.4.** In Setup 1.2, assume that the restriction \( \omega_b \) to each fiber \( X_b \) is a solution of the \( J_{\chi_b} \)-equation and \( \omega_B \) is a solution of the \( J_{\pi_B(\chi_H)} \)-equation. Then there exists a solution of the \( J \)-equation in the class \( [\omega_X + k \omega_B] \) for \( k \gg 0 \).

We prove this theorem using arguments similar to the proofs of [DS20, DS21, Fin04]. In particular, in [DS20, Theorem 1.4], they considered the case where a twisting form on a total space is a pullback of a Kähler form on a base. The difference from those earlier results is that we choose an arbitrary reference metric \( \chi \) on a total space \( X \).

We also consider the converse implication of Theorem 1.4. We use a topological condition called **\( J \)-positivity**. The Lejmi-Székelyhidi conjecture [LS15] says that the \( J \)-equation (1.1) is solvable in a Kähler class \( [\omega] \) on a compact Kähler manifold \( X \) of dimension \( n \) if and only if we have

\[
\int_W c \omega^p - p \omega^{p-1} \wedge \chi > 0
\]

for all \( p \)-dimensional subvarieties \( W \subset X \), where \( p = 1, 2, \ldots, n-1 \). A pair \(([\omega],[\chi])\) is said to be \( J \)-positive if the latter condition holds. If it is just nonnegative, a pair is said to be \( J \)-nef. The uniform version of the conjecture is proved by Gao Chen [Che21] and it is also proved that the uniform conditions are equivalent to the uniform \( J \)-stability. The original version of the conjecture is proved by Datar-Pingali [DP21] in a projective case and by Song [Son20] in general.

Now we state the converse implication.
Theorem 1.5. In Setup 1.2, if the pair \((\omega_X + k\omega_B), [\chi]\) is \(J\)-nef for \(k \gg 0\), then the pair \((\omega_b, [\chi])\) is \(J\)-nef for all \(b \in B\). In addition, if the restriction \(\omega_b\) is a solution of the \(J_X\)-equation or \(\pi^*(\pi_B(\omega_X)_H) = (\omega_X)_H\), then the pair \((\omega_B, [\pi_B(\chi_H)]\)) is also \(J\)-nef.

The calculation also shows that if one has a uniform upper bound of a solution of the \(J\)-equation on a total space, then there exists a solution of the \(J\)-equation on a base (see Remark 3.17). This might be related to the work [GPT21], where they showed the \(L^\infty\) estimate for the solutions of the family of Hessian equations with a certain structural condition.

Outline. The paper is organized as follows. In Section 2, we collect the basic materials needed to prove Theorem 1.4. In Section 3, we prove Theorems 1.4 and 1.5. After constructing a family of approximate solutions of the \(J_X\)-equation in Section 3.1, we perturb the approximate solution to a genuine solution by the inverse function theorem and complete the proof of Theorem 1.4 in Section 3.2. In Section 3.3, by a simple calculation, we prove Theorem 1.5.

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2. Preliminaries

2.1. Some Properties of the Trace Operator. We collect some basic properties of the trace operator needed to prove Theorem 1.4. The arguments are in [Has19].

Let \((X, \chi)\) be a compact Kähler manifold of dimension \(n\) and \(\omega\) be a Kähler form on \(X\). By the \(\bar{\partial}\partial\)-lemma, any Kähler form in a class \([\omega]\) can be described as \(\omega_{\phi} = \omega + \sqrt{-1}\bar{\partial}\partial\phi\), where \(\phi \in C^\infty(X, \mathbb{R})\). The \(J_X\)-equation in \([\omega]\) is given by

\[
\Lambda_{\omega_{\phi}}\chi = c,
\]

where \(\Lambda_{\omega_{\phi}}\) denotes the trace with respect to a Kähler metric \(\omega_{\phi}\) and \(c\) is the constant determined by

\[
n \int_X \chi \wedge \omega^{n-1} = c \int_X \omega^n.
\]

Let \(\omega_t = \omega + t\sqrt{-1}\bar{\partial}\partial\phi\). The linearization of the trace operator at \(\omega\) is given by

\[
\frac{d}{dt} \bigg|_{t=0} \Lambda_{\omega_t}\chi = -\langle \chi, \sqrt{-1}\bar{\partial}\partial\phi \rangle_\omega = -g^{m\bar{q}} \partial_m \partial_{\bar{q}} \phi g^{p\bar{p}} \chi_{p\bar{q}},
\]

where \(g\) is the metric tensor corresponding to \(\omega\) and \(\langle \cdot, \cdot \rangle_\omega\) denotes the inner product on the space of forms defined by \(\omega\).

**Definition 2.2.** We define an operator \(F_{\omega, \chi} : C^\infty(X) \to C^\infty(X)\) by

\[
F_{\omega, \chi}(\phi) = -\langle \chi, \sqrt{-1}\bar{\partial}\partial\phi \rangle_\omega - \langle \partial (\Lambda_{\omega}\chi), \bar{\partial}\phi \rangle_\omega.
\]

**Remark 2.3.** The operator \(F_{\omega, \chi}\) becomes the linearization of the trace operator in \([\omega]\) at a solution of the \(J_X\)-equation by Equation (2.1).

**Lemma 2.4.** The operator \(F_{\omega, \chi}\) is a complex self-adjoint second order elliptic linear operator. Moreover, it satisfies

\[
\int_X \phi F_{\omega, \chi}(\psi) \omega^n = \int_X \nabla^q \phi \nabla^p \psi \chi_{p\bar{q}} \omega^n.
\]
In particular, the subspace \( \ker F_{\omega,\chi} \) of \( C^\infty(X) \) consists of constant functions on \( X \).

**Proof.** Since \( \chi \) is a Kähler form, the operator \( F_{\omega,\chi} \) is a second order elliptic linear operator. Recall that

\[
    n\chi \wedge \omega^{n-1} = (\Lambda\omega\chi)\omega^n,
\]

\[
    n(n-1)\chi \wedge \alpha \wedge \omega^{n-2} = ((\Lambda\omega\chi)(\Lambda\omega\alpha) - (\chi,\alpha)\omega)\omega^n
\]

for real \((1,1)\)-forms \( \chi \) and \( \alpha \) (see [Szé14, Lemma 4.7]). For \( \phi,\psi \in C^\infty(X,\mathbb{R}) \), by using Equations (2.5) and integration by parts, we obtain

\[
    \int_X \phi F_{\omega,\chi}(\psi)\omega^n
    = \int_X \phi (n(n-1)\chi \wedge \sqrt{-1}\partial\bar{\partial}\psi) \wedge \omega^{n-2} - \int_X \phi (\Delta\omega\psi)(\Lambda\omega\chi)\omega^n
    - \int_X n\phi (\sqrt{-1}\partial(\Lambda\omega\chi) \wedge \bar{\partial}\psi) \wedge \omega^{n-1}
    = - \int_X n(n-1)(\sqrt{-1}\partial\phi \wedge \bar{\partial}\psi) \wedge \chi \wedge \omega^{n-2} - \int_X n\phi (\Lambda\omega\chi)(\sqrt{-1}\partial\bar{\partial}\psi) \wedge \omega^{n-1}
    - \int_X n\phi (\sqrt{-1}\partial(\Lambda\omega\chi) \wedge \bar{\partial}\psi) \wedge \omega^{n-1}
    = - \int_X n(n-1)(\sqrt{-1}\partial\phi \wedge \bar{\partial}\psi) \wedge \chi \wedge \omega^{n-2} + \int_X n\phi (\Lambda\omega\chi)(\sqrt{-1}\partial\phi \wedge \bar{\partial}\psi) \wedge \omega^{n-1}
    = \int_X (√{-1}\partial\phi \wedge \bar{\partial}\psi,\chi)\omega^n.
\]

In the second equality, we used the fact that \( \chi \) is closed. Thus, the operator \( F_{\omega,\chi} \) is a complex self-adjoint operator and \( \ker F_{\omega,\chi} \) is precisely the set of constant functions since \( \chi \) is a Kähler form. \( \square \)

This implies that \( F_{\omega,\chi} \) has index zero and is an isomorphism on the set of functions with integral zero.

### 2.2. Materials on Holomorphic Submersions.

Let \( X \) and \( B \) be compact Kähler manifolds of dimensions \( m+n \) and \( n \) respectively and \( \pi: X \to B \) be a holomorphic submersion of relative dimension \( m \). Suppose \( \omega_X \) is a relatively Kähler metric, i.e., a closed \((1,1)\)-form whose restriction \( \omega_b \) to each fiber \( X_b = \pi^{-1}(b) \) is a Kähler metric. Denote the vertical bundle \( \ker d\pi \) by \( \mathcal{V} \). Since \( \omega_X \) is non-degenerate on a fiber, by setting

\[
    \mathcal{H}_x = \{ u \in T_xX \mid \omega_X(u,v) = 0 \text{ for all } v \in \mathcal{V}_x \},
\]

we obtain the vertical-horizontal decomposition

\[
    TX = \mathcal{V} \oplus \mathcal{H}.
\]

By this splitting, any tensor on \( X \) decomposes into some terms via vertical and horizontal components. For a tensor \( \chi \), denote the purely vertical part by \( \chi_{\mathcal{V}} \), the purely horizontal part by \( \chi_{\mathcal{H}} \) and the mixed part by \( \chi_{\mathcal{H}} \), so \( \chi = \chi_{\mathcal{V}} + \chi_{\mathcal{H}} + \chi_{\mathcal{H}} \).

Given a \((p,p)\)-form \( \eta \) on \( X \), we can integrate along fibers and associate a \((p-m,p-m)\)-form \( \pi_*\eta \) on \( B \). Indeed, one reduces to the local case by a partition of unity argument on \( B \). Take a small neighborhood \( U \) of \( b \in B \) such that \( X_b|_U \cong Y \times U \). Then, locally on \( \pi^{-1}(U) \), a \((p,p)\)-form \( \eta \) has two types of terms. The first
implies the positivity of $\pi$, where $\theta$ is a $(m, m)$-form on $Y$ and $\kappa$ is a $(p-m, p-m)$-form on $U$. The second type is in the other cases. It is reduced to the case where we have only one term of the first type by linearity. Then, we define

$$\pi_*(\eta) = \left( \int_Y \theta \right) \kappa.$$  

The following notation will be useful:

**Definition 2.6.** For a $(p, p)$-form $\eta$ on $X$, we define a $(p, p)$-form $\pi_B(\eta)$ on $B$ by

$$\pi_B(\eta)(b) = V_b^{-1} \pi_*(\eta \wedge \omega_X^m)(b),$$

where $V_b = \int_{X_b} \omega_X^m$ is the volume of a fiber $X_b$. The volume $V_b$ is independent of $b \in B$ since $\omega_X$ is closed.

In particular, for a function $f$ on $X$,

$$\pi_B(f)(b) = V_b^{-1} \int_{X_b} f|_{X_b} \omega_X^m.$$  

Denote the subspace $\ker(\pi_B : C^\infty(X) \to C^\infty(B))$ by $C^\infty_0(X)$. For any function $f$ on $X$, we have $f = f_B + f_V$, where $f_B = \pi_B(f)$ and $f_V = f - f_B \in C^\infty_0(X)$.

**Lemma 2.7.** Let $\omega_X$ be a relatively Kähler form on $X$ and $\chi$ be a Kähler form on $X$. If $\omega_b$ is a solution of the $J_X$-equation on each fiber $X_b$ or the horizontal part of $\omega_X$ satisfies $\pi^*(\pi_B(\omega_X)_H) = (\omega_X)_H$, then $\pi_B(\chi_H)$ is a Kähler form on $B$.

**Proof.** Note that

$$\pi_B(\chi) = V_b^{-1} \pi_*(\chi \wedge \omega_X^m) = V_b^{-1} \pi_*(m \chi_Y \wedge (\omega_X)^{m-1}_Y \wedge (\omega_X)_H) + V_b^{-1} \pi_*(\chi_H \wedge (\omega_X)_Y^m).$$

The left-hand side is closed since $\chi$ and $\omega_X$ are closed. The first term of the second line is closed if $\omega_b$ is a solution of the $J_X$-equation on each fiber $X_b$ or $\pi^*(\pi_B(\omega_X)_H) = (\omega_X)_H$. Therefore, the second term, which is $\pi_B(\chi_H)$, is closed. The $J_B$-invariance also follows for the same reason, where $J_B$ is the complex structure of $B$. Lastly we prove $\pi_B(\chi_H)$ is positive, that is, $\pi_B(\chi_H)(b)(u, J_B u) > 0$ for a point $b \in B$ and a vector $u \in T_b B$. Take a local trivialization $\psi : X_U \cong Y \times U$ on a neighborhood $U$ of $b$ such that the splitting $TX \cong V \oplus H$ coincides with $TY \oplus TU$ on a fiber $X_b$ via $\psi$. By the definition of $\pi_B$, we have

$$\pi_B(\chi_H)(b)(u, J_B u) = \frac{1}{V_b} \int_Y (\psi^{-1})^*(\chi_H \wedge \omega_X^m)(b)(u, J_B u)$$

$$= \frac{1}{V_b} \int_Y (\psi^{-1})^*(\chi_H)(b)(u, J_B u)(\psi^{-1})^* \omega_X^m$$

$$= \frac{1}{V_b} \int_Y \chi_H(\psi^{-1} u, \psi^{-1} J_B u)(\psi^{-1})^* \omega_X^m$$

$$= \frac{1}{V_b} \int_Y \chi(\psi^{-1} u, \psi^{-1} J_B u)(\psi^{-1})^* \omega_X^m.$$

Note that, by the definition of $H$, we have $J\psi^{-1} u \in H$. Since we also have $\pi_*(\psi^{-1} J_B u - J\psi^{-1} u) = 0$, we get $\psi^{-1} J_B u = J\psi^{-1} u$. Thus the positivity of $\chi$ implies the positivity of $\pi_B(\chi_H)$. \[\square\]
3. Proofs of Theorems

We first prove Theorem 1.4. We freely use the following notations. Let $X$ and $B$ compact Kähler manifolds of dimensions $m + n$ and $n$ respectively. Let $\pi : X \to B$ be a holomorphic submersion of relative dimension $m$. Fix a Kähler form $\chi$ of $X$. Suppose we have a relative Kähler form $\omega$ on $X$ such that the restriction $\omega_b$ on a fiber $X_b$ is a solution of the $J_{\chi_b}$-equation for all $b \in B$. Split the tangent bundle $TX$ with respect to $\omega$. Suppose moreover we have a solution $\omega_B$ of the $J_{\pi_B(\chi_H)}$-equation on $B$. Here, a $(1, 1)$-form $\pi_B(\chi_H)$ is Kähler by Lemma 2.7. We define $\omega_k = \omega + k\omega_B$.

3.1. Approximate solutions. We first construct a family of approximate solutions of the $J_{\chi}$-equation. More precisely, we prove the following:

**Proposition 3.1.** For any $r \in \mathbb{Z}_{\geq 0}$, there exist $\{\phi_{i,B}\}_{i=0}^r \in C^\infty (B)$ and $\{\phi_{i,V}\}_{i=0}^r \in C^\infty_0 (X)$ such that

$$
\omega_{k,r} = \omega_k + \sqrt{-1} \partial \bar{\partial} \left( \sum_{i=0}^r \phi_{i,B} k^{2-i} + \sum_{i=0}^r \phi_{i,V} k^{-i} \right)
$$

satisfies

$$
\Lambda_{\omega_{k,r}} (\chi) = \sum_{i=0}^r k^{-i} c_i + O (k^{-r-1}) ,
$$

where $c_i$ are constants.

We will prove this by induction. The notation $O (k^{-r})$ is only pointwise in this section. We first see that $\omega_k$ is a solution of the $J_{\chi}$-equation up to $O (k^{-1})$.

**Definition 3.2.** For a $(1, 1)$-form $\chi$ on $X$, we define a function $\Lambda_V \chi$ on $X$ by

$$
\Lambda_V \chi = m \frac{\chi_V \wedge \omega_V^{m-1}}{\omega_V^m} .
$$

Also, we define a function $\Lambda_{\omega_B} \chi$ on $X$ by

$$
\Lambda_{\omega_B} \chi = n \frac{\chi_H \wedge (\pi^* \omega_B)^{n-1}}{(\pi^* \omega_B)^n} .
$$

By a simple calculation, we have the following:

**Lemma 3.3.** For a $(1, 1)$-form $\chi$ on $X$, we have

$$
\Lambda_{\omega_k} \chi = \Lambda_V \chi + k^{-1} \Lambda_{\omega_B} \chi + O (k^{-2}) .
$$
Proof. Since \( \omega_X \) has no mixed term by the definition of the splitting, 
\[
\Lambda_{\omega_k} \chi \\
= (m + n) \frac{\chi \wedge (\omega_X + k\omega_B)^{m+n-1}}{(\omega_X + k\omega_B)^{m+n}} \\
= m \frac{\chi \nu \wedge (\omega_X)_V^{m-1} \wedge (\omega_X)_H^{n-1}}{(\omega_X)_V^m \wedge (\omega_X)_H^n} + n \frac{\chi_H \wedge (\omega_X)_V^m \wedge (\omega_X)_H^n}{(\omega_X)_V^m \wedge (\omega_X)_H^n} \\
= \Lambda_{\nu} \chi + n \frac{\chi_H \wedge \sum_{i=0}^{n-1} \binom{n}{i} k^{n-1-i} (\omega_X)_V^i \wedge (\omega_X)_H^{n-i}}{(\omega_X)_V^m \wedge (\omega_X)_H^n} \\
= \Lambda_{\nu} \chi + k^{-1} \Lambda_{\omega_B} \chi + O(k^{-2}).
\]

Since \( \omega_b \) is a solution of the \( J_{\chi_b} \)-equation on each fiber, the lemma implies \( \omega_k \) satisfies 
\[
\Lambda_{\omega_k} \chi = c_0 + O(k^{-1}),
\]
which is the case of \( r = 0 \) of Proposition 3.1.

We will perturb \( \omega_k \) to a solution of the \( J_{\chi_b} \)-equation up to \( O(k^{-2}) \) to prove the case of \( r = 1 \) of Proposition 3.1. Denote the \( k^{-1} \)-term \( \Lambda_{\omega_B} \chi \) of \( \Lambda_{\omega_k} \chi \) by \( f_1 \), and decompose \( f_1 = (f_1)_B + (f_1)_V \) as in the argument below Definition 2.6. Since \( \Lambda_{\omega_B} \) and \( \pi_B \) are commutative by the definition of fiber integral as in Subsection 2.2 and \( \omega_B \) is a solution of \( J_{\pi_B(\chi_b)} \)-equation, the function \( (f_1)_B \) is constant. We denote this constant by \( c_1 \). Hence, we only need to delete the term \( (f_1)_V \). This can be done by adding a fiberwise mean value zero function to \( \omega_k \).

Definition 3.4. For \( \phi \in C^\infty(X) \), we define the operator \( F_{(\omega_X)_V,\chi_V}(\phi) \) by 
\[
\left(F_{(\omega_X)_V,\chi_V}(\phi)\right)|_{\chi_b} = F_{\omega_b,\chi_b}(\phi|_{\chi_b}).
\]

Lemma 3.5. For \( \phi \in C^\infty(X) \), we have 
\[
\frac{d}{dt}\big|_{t=0} \Lambda_{\omega_k + t\sqrt{-1}\partial\bar{\partial}\phi} (\chi) = F_{(\omega_X)_V,\chi_V}(\phi) + O(k^{-1}).
\]

Proof. As in Subsection 2.1,
\[
\frac{d}{dt}\big|_{t=0} \Lambda_{\omega_k + t\sqrt{-1}\partial\bar{\partial}\phi} (\chi) = -g_k^{\mu\bar{\nu}} \partial_\mu \partial_{\bar{\nu}} \phi \left(g_k^{p\bar{\mu}} \chi_{p\bar{\nu}} \right),
\]
where \( g_k \) is the metric tensor corresponding to \( \omega_k \). We expand \( g_k^{-1} \) in \( k \) as a matrix in terms of vertical and horizontal parts,
\[
g_k^{-1} = \begin{bmatrix} (\omega_X)_V & 0 \\ 0 & (\omega_X)_H + k\omega_B \end{bmatrix}^{-1} = \begin{bmatrix} (\omega_X)_V^{-1} & 0 \\ 0 & k^{-1}(k^{-1}(\omega_X)_H + \omega_B)^{-1} \end{bmatrix} \\
= \begin{bmatrix} (\omega_X)_V^{-1} & 0 \\ 0 & 0 \end{bmatrix} + k^{-1} \begin{bmatrix} 0 & 0 \\ 0 & \omega_B^{-1} \end{bmatrix} + O(k^{-2}).
\]
Therefore we see that
\[-(g_k)^{p\bar{q}} \partial_m \partial_n \phi (g_k)^{p\bar{q}} \chi_{p\bar{q}} = -(g_X)^{p\bar{q}} (\partial_m \partial_n \phi) (g_X)^{p\bar{q}} \chi_{p\bar{q}} + O(k^{-1})\]
where \((g_X)^{p\bar{q}}\) the metric tensor on the vertical subbundle \(V\) of \(TX\) corresponding to \((\omega_X)^{p\bar{q}}\).

By Lemma 2.4, there exists a unique function \((\phi_{1,v})_b \in C^\infty_0(X_b)\) for all \(b \in B\) such that
\[F_{\omega_b,\chi_b} ((\phi_{1,v})_b) = -(f_1)_v |_{X_b},\]
where \(C^\infty_0(X_b)\) denotes the set of functions on \(X_b\) with integral zero with respect to \(\omega_X |_{X_b}\). Since the operator \(F_{\omega_X |_{X_b},\chi(X_b)}\) is smooth in \(b\), the implicit function theorem implies that there exists a unique function \(\phi_{1,v} \in C^\infty_0(X)\) such that
\[\phi_{1,v} |_{X_b} = (\phi_{1,v})_b.\]

By combining this with Lemma 3.5 and Lemma 3.3, we have
\[
\Lambda_{\omega_k, k^{-1} \sqrt{-1} \partial \bar{\partial} \phi_{1,v}} (\chi) = \Lambda_{\chi} + k^{-1} \left( c_1 + (f_1)_v + F_{(\omega_X)^{p\bar{q}},\chi} (\phi_{1,v}) \right) + O(k^{-2})
\]
\[= c_0 + c_1 + O(k^{-2}).\]

This completes the proof for the case of \(r = 1\) of Proposition 3.1.

To prove Proposition 3.1, we repeat perturbing in the same manner. Suppose that we have the solution \(\omega_{k,r}\) as in Proposition 3.1 for \(r \geq 1\). We need to calculate the linearization of the trace operator at \(\omega_{k,r}^{-1}\).

**Lemma 3.6.** For a function \(\phi \in C^\infty(X)\), we have
\[
\frac{d}{dt} \bigg|_{t=0} \Lambda_{\omega_{k,r} + t \sqrt{-1} \partial \bar{\partial} \phi} (\chi) = F_{(\omega_X)^{p\bar{q}},\chi} (\phi) + k^{-1} D_1 (\phi) + k^{-2} D_2 (\phi) + O(k^{-3}),
\]
where the operators \(D_1\) and \(D_2\) satisfy \(D_1 (\phi) = 0\) and \(\pi_B \left( D_2 (\phi) \right) = F_{\omega_B, \pi_B (\chi_B)} (\phi)\) for \(\phi \in C^\infty(B)\).

**Proof.** The proof is similar to the one of Lemma 3.5, but we need to deal with \(g_{k,r}\) here instead of \(g_k\). Denote \(\sum_{i=0}^r k^{-i} \phi_{i,v}\) by \((\phi_v)_{k,r}\) and \(\sum_{i=0}^r k^{-i} \phi_{i,B}\) by \((\phi_B)_{k,r}\) for simplicity. Then, we have
\[
g_{k,r}^{-1} = \left[ \begin{array}{cc} (\omega_k)^{p\bar{q}} + \left( \sqrt{-1} \partial \bar{\partial} (\phi_v)_{k,r} \right)^{p\bar{q}} & \left( \sqrt{-1} \partial \bar{\partial} (\phi_v)_{k,r} \right)^{m\bar{n}} \\ \left( \sqrt{-1} \partial \bar{\partial} (\phi_v)_{k,r} \right)^{m\bar{n}} & (\omega_k)^{m\bar{n}} + \left( \sqrt{-1} \partial \bar{\partial} (\phi_v)_{k,r} + (\phi_B)_{k,r} \right)^{m\bar{n}} \end{array} \right]^{-1}.
\]

The \(O(1)\)-term of \(g_{k,r}^{-1}\) is \(\begin{bmatrix} (\omega_X)^{p\bar{q}} & 0 \\ 0 & 0 \end{bmatrix}\) in the same way as the proof of Lemma 3.5, since \((\phi_v)_{k,r}\) is \(O(k^{-1})\). By using the formula of the inverse of a matrix and recalling that \((\phi_v)_{k,r}\) is \(O(k^{-1})\), we see that the \(k^{-1}\)-term of \(g_{k,r}^{-1}\) is \(\begin{bmatrix} 1 & 0 \\ 0 & \omega_B^{-1} \end{bmatrix}\).
Let $\phi \in C^\infty(B)$. As $\left(\sqrt{-1}\partial\bar{\partial}(\pi^*\phi)\right)_\nu = \left(\sqrt{-1}\partial\bar{\partial}(\pi^*\phi)\right)_m = 0$, the operator $D_1$ satisfies $D_1(\pi^*\phi) = 0$. The operator $D_2$ satisfies
\[
\pi_B\left(D_2(\phi)\right) = \pi_B\left(-\left(\chi_H, (\sqrt{-1}\partial\bar{\partial}\pi^*\phi)\right)_\omega_B\right)
= -\left(\pi_B(\chi_H), \sqrt{-1}\partial\bar{\partial}\phi\right)_\omega_B
= F_{\omega_B, \pi_B(\chi_H)}(\phi).
\]

**Proof of Proposition 3.1.** Denote the $k^{-r-1}$-term of $A_{\omega_{k,r}}\chi$ by $f_{r+1} = (f_{r+1})_B + (f_{r+1})_\nu$, where $(f_{r+1})_B \in C^\infty(B)$ and $(f_{r+1})_\nu \in C^\infty_0(X)$. Since $\omega_B$ is a solution of the $J_{\pi_B(\chi_H)}$-equation, Lemma 2.4 implies that there exists a function $\phi_{r+1,B} \in C^\infty(B)$ such that
\[
F_{\omega_B, \pi_B(\chi_H)}(\phi_{r+1,B}) = - (f_{r+1})_B + c_{r+1},
\]
where $c_{r+1} = \left(f_B\omega_B^b\right)^{-1} \left(f_B(f_{r+1})_B \omega_B^b\right)$. Hence Lemma 3.6 implies that we have
\[
A_{\omega_{k,r} + k^{-r-1}\sqrt{-1}\partial\bar{\partial}\phi_{r+1,B}}(\chi) = \sum_{i=0}^r k^{-i} c_i + k^{-r-1} \left( f_{r+1} + D_2(\phi_{r+1,B}) \right) + O(k^{-r-2})
= \sum_{i=0}^r k^{-i} c_i + k^{-r-1} \left( c_{r+1} + (f_{r+1})'_\nu \right) + O(k^{-r-2})
\]
for some function $(f_{r+1})'_\nu \in C^\infty_0(X)$. For the same reason stated below Lemma 3.5, there exists a function $\phi_{r+1,\nu} \in C^\infty_0(X)$ such that
\[
F_{(\omega_{k,r})_B, \chi_B}(\phi_{r+1,\nu}) = - (f_{r+1})'_\nu.
\]
By using Lemma 3.6 again, we obtain
\[
A_{\omega_{k,r} + k^{-r-1}\sqrt{-1}\partial\bar{\partial}\phi_{r+1,B} + k^{-r-1}\sqrt{-1}\partial\bar{\partial}\phi_{r+1,\nu}}(\chi)
= \sum_{i=0}^r k^{-i} c_i + k^{-r-1} \left( c_{r+1} + (f_{r+1})'_\nu + F_{(\omega_{k,r})_B, \chi_B}(\phi_{r+1,\nu}) \right) + O(k^{-r-2})
= \sum_{i=0}^{r+1} k^{-i} c_i + O(k^{-r-2}).
\]
This completes the proof of Proposition 3.1. \qed

3.2. **Proof of Theorem 1.4.** We perturb the approximate solution $\omega_{k,r}$ obtained in the previous section to a genuine solution of the $J_\pi$-equation for $k \gg 0$. A positive integer $r \gg 0$ is fixed and we consider all $k \gg 0$. An important ingredient is the following theorem called the quantitative inverse function theorem [Fin04, Theorem 4.1].

**Theorem 3.7.** We assume the following conditions.

- A map $F : B_1 \to B_2$ is a differentiable map of Banach spaces, whose derivative at $0$, $DF$, is an isomorphism of Banach spaces, with inverse $P$.
- A constant $\delta'$ is the radius of the closed ball in $B_1$, centered at $0$, on which $F - DF$ is Lipschitz, with constant $1/(2||P||)$.
- A constant $\delta$ is defined by $\delta = \delta'/2||P||$.

Then, whenever $y \in B_2$ satisfies $||y - F(0)|| < \delta$, there exists $x \in B_1$ such that $F(x) = y$. Moreover, such an $x$ is unique subject to the constraint $||x|| < \delta'$. 


In this section, if it is not stated explicitly, all Sobolev spaces are considered with respect to $g_{k,r}$. Let $l$ be a positive integer satisfying $l-(m+n)>0$. Then, for a function $\phi \in L^2_{l+2}$, a $(1,1)$-form $\sqrt{-1} \partial \bar{\partial} \phi$ is continuous by the Sobolev embedding. Since the trace operator is analytic in the metric, we can extend the trace operator to the smooth operator $\Lambda_{\omega_{k,r}}(\chi) : L^2_l \to L^2_l$ defined by $\Lambda_{\omega_{k,r}}(\chi)(\phi) = \Lambda_{\omega_{k,r},+\sqrt{-1} \partial \bar{\partial} \phi}(\chi)$. Denote the set of functions in $L^2_l$ with integral zero with respect to $\omega_{k,r}$ by $L^2_{l,0}$, the projection from $L^2_l$ to $L^2_{l,0}$ by $p$, and define $L_{k,r} := p \circ \Lambda_{\omega_{k,r}}(\chi)_{|L^2_{l+2}}$. We will use Theorem 3.7 for the operator $L_{k,r}$. Therefore, we need to check that the linearization $D_0 L_{k,r}$ at 0 is isomorphism and that the norm of the inverse and the radius of the ball as in Theorem 3.7 can be estimated sufficiently to conclude 0 and $L_{k,r}(0)$ are close enough. First, we show that $D_0 L_{k,r}$ is an isomorphism and estimate the norm of the inverse. The following lemma [Fin04, Lemma 6.10] is key:

**Lemma 3.8.** Let $D : B_1 \to B_2$ be a bounded invertible linear map of Banach spaces with bounded inverse $Q$. If $L : B_1 \to B_2$ is another linear map with

$$||L - D|| \leq (2||Q||)^{-1},$$

then $L$ is also invertible with bounded inverse $P$ satisfying $||P|| \leq 2||Q||$.

The operator close to $D_0 L_{k,r}$ will be $F_{\omega_{k,r},\chi}$, which is an isomorphism between $L^2_{l+2,0}$ and $L^2_{l,0}$ by Lemma 2.4

**Lemma 3.9.** There exist a constant $C$ and an integer $A$ such that, for all $\phi \in L^2_{l,0}$ and $k \gg 0$, we have

$$||F_{\omega_{k,r},\chi}^{-1}(\phi)||_{L^2_{l+2,0}} \leq C k^A ||\phi||_{L^2_l}.$$

**Proof.** By Lemma 2.4, for $\phi \in L^2_{l,0}$, we have

$$\int_X \phi F_{\omega_{k,r},\chi}(\phi) \omega_{k,r}^{m+n} = \int_X \left( \nabla^q_{k,r} \phi \right) \left( \nabla^p_{k,r} \phi \right) \chi_{pq} \omega_{k,r}^{m+n},$$

where $\nabla_{k,r}$ denotes the covariant derivative with respect to $\omega_{k,r}$. Since we assume $\chi$ is a Kähler form, there exists a constant $C_1$ such that for all $k \gg 0$,

$$\chi \geq C_1 k^{-1} \omega_{k,r}.$$

Therefore, if we denote the first non-zero eigenvalue of $F_{\omega_{k,r},\chi}$ by $\lambda_1$ and an eigenfunction corresponding to $\lambda_1$ by $\phi_1$, we have

$$\lambda_1 = \left( \int_X \phi_1 F_{\omega_{k,r},\chi}(\phi_1) \omega_{k,r}^{m+n} \right) / ||\phi_1||^2_{L^2}$$

$$= \left( \int_X \left( \nabla^q_{k,r} \phi_1 \right) \left( \nabla^p_{k,r} \phi_1 \right) \chi_{pq} \omega_{k,r}^{m+n} \right) / ||\phi_1||^2_{L^2}$$

$$\geq C_1 k^{-1} \left( \int_X |\nabla_{k,r} \phi_1|^2 \omega_{k,r}^{m+n} \right) / ||\phi_1||^2_{L^2}$$

$$\geq C_2 k^{-2}.$$

In the last inequality, we used [Fin04, Lemma 6.5]. Also, for $\phi \in L^2_{l+2,0}$, by the following lemma, we have

$$||\phi||_{L^2_{l+2,0}} \leq C k^A \left( ||\phi||_{L^2_l} + ||F_{\omega_{k,r},\chi}(\phi)||_{L^2_l} \right).$$

Combining these two estimates gives the desired estimate. □
Lemma 3.10. There exist a constant $C$ and an integer $A$, independent of $k$ and $r$, such that for $\phi \in L^2_{s+2}$, an integer $r \geq A$ and any $k \gg 0$ we have

$$||\phi||_{L^2_{s+2}} \leq C k^A \left(||\phi||_{L^2} + ||F_{k,r,\chi}(\phi)||_{L^2}\right).$$

Remark 3.11. Note that our operator $F_{k,r,\chi}$ depends on a reference metric $\chi$ which is independent of $k$. Thus, when we try to apply the arguments in [Fin04, Sections 5 and 6], the constant of the elliptic estimate on a model space depends on $k$ in our case. Due to this, we cannot deduce the estimate on a local fibered space $X|_D$ from the estimate on a model space $S \times D$, where $S$ is a fiber and $D$ is a small neighborhood on $B$ as in [Fin04, Lemma 5.9]. Instead of the arguments in [Fin04, Sections 5 and 6], we follow the line of the standard proof of the local elliptic estimate in [Tay11, Chapter 5 Theorem 11.1], and then patch them by using a partition of unity on the total space $X$.

Proof. Choose a finite product coordinate system $\{(U_i, \psi_i; z^1, \ldots, z^{m+n})\}_{i=1}^N$ on the total space $X$ such that $z^j$ are base direction for $j = m+1, \ldots, m+n$ and the coordinate can extend to $U'_i$ satisfying $U_i \subset U'_i$. Take also a partition of unity $\{\rho_i\}_{i=1}^N$ subordinate to $\{(U_i, \psi_i)\}_{i=1}^N$. Define a new coordinate system $\{(U_i, \tau_i; w^1, \ldots, w^{m+n})\}_{i=1}^N$, which is given by $w^j = z^j$ for $j = 1, \ldots, m$ and $w^j = \sqrt{k} z^j$ for $j = m+1, \ldots, m+n$. By this scaling and the construction of the approximate solutions $\omega_{k,r}$, the coefficients of the corresponding metric $g_{k,r}$ are $O(1)$ in $C^4$ on this coordinate $(w^1, \ldots, w^{m+n})$ for $k \gg 0$ and each fixed $r$, where $t$ is chosen large enough and fixed for the arguments below. Denote the linearization of the trace operator by $G_{k,r}$, i.e.

$$G_{k,r}(\phi) = -\left(\chi, \sqrt{-1} \bar{\partial} \partial \phi\right)_{\omega_{k,r}}.$$

We will prove the elliptic estimate in the statement for the operator $G_{k,r}$ first. The following claim is the local version of the estimate in the statement:

Claim. On $U_i$, denote the corresponding operator of $G_{k,r}$ via $\tau_i$ on $\tau_i(U_i)$ by $G_{k,r}$ as well. For a real number $s \in \mathbb{R}$ and a function $\phi \in L^2_{s+2}$ with compact support contained in $U_i$, there exist a constant $C$ and $A$, depending on $s$ but not on $k$ or $r$, such that

$$||\phi||_{L^2_{s+2}(\mathbb{R}^{2(m+n)})} \leq C k^A \left(||\phi||_{L^2_{s+1}(\mathbb{R}^{2(m+n)})} + ||G_{k,r}(\phi)||_{L^2_{s+1}(\mathbb{R}^{2(m+n)})}\right),$$

where we suppress $\tau_i$ and consider functions on $U_i$ as on $\tau_i(U_i)$.

Proof of Claim. We basically follow the line of the standard proof of the elliptic estimate in [Tay11, Chapter 5 Theorem 11.1]. The differences between our case and the proof there are that our operator $G_{k,r}$ depends on $k$ and that we consider functions $\phi$ whose compact support gets large as $k$ gets large. We first observe that the principal symbol of $G_{k,r}$ can be estimated from below by $C/k$ for some constant $C$ on $\tau_i(U_i)$. On a coordinate neighborhood $\tau_i(U_i)$, we have

$$g_{k,r}^{-1}\chi g_{k,r}^{-1} \to \begin{bmatrix} (g_{x})_{11}^{-1}(g_{x})_{11}^{-1} & 0 \\ 0 & 0 \end{bmatrix}$$

as $k \to \infty$,

where we consider metrics $g_{k,r}$ and $\chi$ as real $2(m+n) \times 2(m+n)$ matrices on $\tau_i(U_i)$ and $(g_{x})_{11}$ and $\chi_{11}$ are matrices of purely fiber part. This convergence is uniformly on $\tau_i(U_i)$. Thus, eigenvalues of the principal symbol $g_{k,r}^{-1}\chi g_{k,r}^{-1}$ converge.
to $2m$ positive numbers and $2n$ zeros. For the eigenvalue $\lambda$ which converges to zero, consider $k\lambda$. This is the zero of the polynomial

$$P(t) = \det \left( \begin{bmatrix} I & 0 \\ 0 & \sqrt{k}I \end{bmatrix} g_{k,r}^{-1} \chi g_{k,r}^{-1} \begin{bmatrix} I & 0 \\ 0 & \sqrt{k}I \end{bmatrix} - \begin{bmatrix} \lambda I & 0 \\ 0 & tI \end{bmatrix} \right).$$

We have

$$\begin{bmatrix} I & 0 \\ 0 & \sqrt{k}I \end{bmatrix} g_{k,r}^{-1} \chi g_{k,r}^{-1} \begin{bmatrix} I & 0 \\ 0 & \sqrt{k}I \end{bmatrix} \rightarrow \begin{bmatrix} (g\chi)^{-1} & 0 \\ D & g_B^{-1} \end{bmatrix} \chi' \begin{bmatrix} (g\chi)^{-1} & 0 \\ 0 & g_B^{-1} \end{bmatrix}$$

as $k \to \infty$, where $D$ is some matrix and $g_B$ and $\chi'$ denote the corresponding matrices of the metrics $\omega_B$ and $\chi$, respectively, on the coordinate neighborhood $\psi_i(U_i)$ which independent of $k$. The convergence is uniformly on $\tau_i(U_i)$, and thus $k\lambda$ converges to some positive number, as $g\chi, g_B$ and $\chi'$ are positive in each part and $\lambda$ converges to zero. This confirms that we can estimate the eigenvalues of the principal symbol of $G_{k,r}$ from below by $C/k$. We then follow the proof of the elliptic estimate in [Tay11, Chapter 5 Theorem 11.1] with checking dependence of $k$, which causes the constant to depend on $k$ but be written in the form $Ck^A$ for some integer $A$. □

Let $\phi \in L^2_{t+2}$. Then, we have

$$||\phi||_{L^2_{t+2}} \leq \sum_{i=1}^{N} ||\rho_i \phi||_{L^2_{t+2}}$$

$$= \sum_{i=1}^{N} \left( \sum_{j=0}^{l+2} \int_{\tau_i(U_i)} |\nabla^j \rho_i \phi|_{g_{k,r}}^2 \left( \det[g_{k,r}] \right) dw \right)^{1/2}$$

$$\leq C \sum_{i=1}^{N} \left( \sum_{j=0}^{l+2} \int_{\tau_i(U_i)} |D^j \rho_i \phi|^2 dw \right)^{1/2},$$

where $\nabla^j$ is the $j$-th covariant derivative with respect to $g_{k,r}$ and $D^j$ is the $j$-th derivative with respect to the coordinate $(w^1, \ldots, w^{m+n})$. The last inequality follows from the fact that the corresponding matrices $g_{k,r}$ and $g_{k,r}^{-1}$ with respect to the coordinate $(w^1, \ldots, w^{m+n})$ are $O(1)$ in $C^l(\tau_i(U_i))$ for $k \gg 0$ and each fixed $r$.

Using the claim above, we get

$$||\phi||_{L^2_{t+2}} \leq Ck^A \sum_{i=1}^{N} \left( ||\rho_i \phi||_{L^2_{t+2}(\tau_i(U_i))} + ||G_{k,r}(\rho_i \phi)||_{L^2_{t+2}(\tau_i(U_i))} \right).$$

Define the operator $[G_{k,r}, \rho_i] := G_{k,r} \circ \rho_i - \rho_i \circ G_{k,r}$, where $\rho_i$ is the operator of multiplication by $\rho_i$. This operator is a differential operator of the first order. Moreover, the coefficients can be estimated from above by constants in $C^l(\tau_i(U_i))$ Therefore, by introducing a new cutoff function $\rho_i,1$, which is constant 1 on $\text{Supp}(\rho_i)$ and has compact support $\text{Supp}(\rho_i,1)$ contained in $U_i$, we have

$$||\phi||_{L^2_{t+2}} \leq Ck^A \sum_{i=1}^{N} \left( ||\rho_i,1 \phi||_{L^2_{t+2}(\tau_i(U_i))} + ||\rho_i G_{k,r}(\phi)||_{L^2_{t+2}(\tau_i(U_i))} \right).$$

For the second term, note that the derivatives of $\rho_i$ can be estimated from above, and $L^2_{t+2}(\tau_i(U_i))$-norm is equivalent to $L^2$-norm as $g_{k,r}$ is $O(1)$ on this coordinate.
Thus, we get

\[ \|\phi\|_{L^2_{t+2}} \leq CK^A \left( \sum_{i=1}^N \|\rho_{i,1}\phi\|_{L^2_{t+1}(\tau_i(U_i))} \right) + CK^A \|G_{k,r}(\phi)\|_{L^2_t}. \]

We next estimate \( \|\rho_{i,1}\phi\|_{L^2_{t+1}(\tau_i(U_i))} \) by using the claim above and iterate this process. Eventually, we get

\[ \|\phi\|_{L^2_{t+2}} \leq CK^A \left( \|\phi\|_{L^2_t} + \|G_{k,r}(\phi)\|_{L^2_t} \right), \]

where we use the same notation \( A \) for simplicity. To obtain the desired estimate for \( F_{\omega_{k,r},\chi} \), recall that we have

\[ (G_{k,r} - F_{\omega_{k,r},\chi})(\phi) = \left( \partial \Lambda_{\omega_{k,r}}(\chi), \bar{\partial} \phi \right)_{\omega_{k,r}}. \]

Although we only proved that \( \omega_{k,r} \) is an approximate solution of the \( J_\chi \)-equation up to \( O(k^{-r-1}) \) pointwise in the last section, the argument of [Fin04, Lemma 5.7] follows and we have

\[ p(\Lambda_{\omega_{k,r}}) = O(k^{-r-1}) \quad \text{in } C^\ell(g_{k,r}), \]
\[ p(\Lambda_{\omega_{k,r}}) = O(k^{-r-1-n/2}) \quad \text{in } L^2_t(g_{k,r}). \]

Therefore, we have

\[ \|G_{k,r} - F_{\omega_{k,r},\chi}\| \leq Ck^{-r-1} \]

and if we choose \( r \geq A \), we get

\[ \|\phi\|_{L^2_{t+2}} \leq CK^A \left( \|\phi\|_{L^2_t} + \|F_{\omega_{k,r},\chi}(\phi)\|_{L^2_t} \right) \]

for \( k \gg 0 \).

Recall that we have

\[ (D_0 \Lambda_{\omega_{k,r}}(\chi) - F_{\omega_{k,r},\chi})(\phi) = \left( \partial \Lambda_{\omega_{k,r}}(\chi), \bar{\partial} \phi \right)_{\omega_{k,r}}. \]

As \( \omega_{k,r} \) is a solution of the \( J_\chi \)-equation up to \( O(k^{-r-1}) \) and the derivative \( D_0 \mathcal{L}_{k,r} \) is given by \( D_0 \mathcal{L}_{k,r} = p \circ \left( D_0 \Lambda_{\omega_{k,r}}(\chi) \right) \), the same arguments in the proof of [Fin04, Theorem 6.1] imply that we have

\[ \|(D_0 \mathcal{L}_{k,r} - F_{\omega_{k,r},\chi})(\phi)\|_{L^2_{t+2}} \leq ck^{-r-1-\|\phi\|_{L^2_{t+2}}}. \]

Hence, if we choose \( r \geq A \), Lemma 3.8 implies that \( D_0 \mathcal{L}_{k,r} \) is an isomorphism for \( k \gg 0 \) and the operator norm of the inverse \( P \) satisfies \( \|P\|_{op} \leq Ck^A \) for some constant \( C \).

We also estimate the radius of the ball on which \( \mathcal{L}_{k,r} - D_0 \mathcal{L}_{k,r} \) is Lipschitz with constant \( 1/(2\|P\|) \). Denote the nonlinear part of \( \mathcal{L}_{k,r} \) by \( \mathcal{N}_{k,r} \), i.e., \( \mathcal{N}_{k,r} = \mathcal{L}_{k,r} - D_0 \mathcal{L}_{k,r} \). Then, by the mean value theorem,

\[ \|\mathcal{N}_{k,r}(\phi) - \mathcal{N}_{k,r}(\psi)\|_{L^2_t} \leq \sup_{f \in [\phi,\psi]} \|D_f \mathcal{N}_{k,r}\| \|\phi - \psi\|_{L^2_{t+2}}, \]

where \( D_f \mathcal{N}_{k,r} \) denotes the derivative of \( \mathcal{N}_{k,r} \) at \( f \).

**Lemma 3.12.** There exists a constant \( C \) which is independent of \( k \) such that if \( \|f\|_{L^2_{t+2}} \leq \epsilon k^{-n/2} \) for a small constant \( \epsilon \), we have

\[ \|D_f \mathcal{N}_{k,r}\| \leq Ck^{n/2} \|f\|_{L^2_{t+2}}. \]
Proof. For \( \phi \in L^2_{l+2,0} \), by a simple calculation, we have
\[
\| (D_f N_{k,r}) (\phi) \|_{L^2_{l}} = \| (D_f \mathcal{L}_{k,r} - D_0 \mathcal{L}_{k,r}) (\phi) \|_{L^2_{l}} \\
\leq \| - (\chi, \sqrt{-1} \partial \bar{\partial} \phi) \omega_{k,r} + \sqrt{-1} \partial \bar{\partial} f + (\chi, \sqrt{-1} \partial \bar{\partial} \phi) \omega_{k,r} \|_{L^2_{l}} \\
= \| (\partial_p \partial_{\bar{q}} \phi) \chi_m \bar{n} \left( -(g_{k,r,f})^{m\bar{q}} (g_{k,r,f})^{p\bar{m}} + g_{k,r}^{m\bar{q}} g_{k,r}^{p\bar{m}} \right) \|_{L^2_{l}},
\]
where \( g_{k,r,f} \) is the metric tensor corresponding to \( \omega_{k,r} + \sqrt{-1} \partial \bar{\partial} f \). Note that the Sobolev constants with respect to \( g_{k,r} \) are independent of \( k \) by [Fin04, Lemma 5.8]. If \( 2l > 2(m + n) \), by the Sobolev inequality, for any tensors \( T \) and \( T' \) we have \( ||T \cdot T'||_{L^2_{l}} \leq C ||T||_{L^2_{l}} ||T'||_{L^2_{l}} \) for some constant \( C \) which is independent of \( k \), where \( T \cdot T' \) denotes tensor product or contraction. Thus, we have
\[
|| (D_f N_{k,r}) (\phi) ||_{L^2_{l}} \leq C \| \phi \|_{L^2_{l+2}} \| \chi \|_{C^1} \| - g_{k,r,f}^{-1} \otimes g_{k,r,f}^{-1} + g_{k,r}^{m\bar{q}} g_{k,r}^{p\bar{m}} \|_{L^2_{l}}.
\]
If we assume \( ||f||_{L^2_{l+2}} \leq \epsilon k^{-n/2} \) for a small constant \( \epsilon \), since \( ||g_{k,r}^{-1}||_{L^2_{l}} \leq C k^{n/2} \) for some constant \( C \) and \(- (g_{k,r,f})^{m\bar{q}} \otimes g_{k,r,f}^{-1} = (g_{k,r,f})^{m\bar{q}} (\partial_p \partial_{\bar{q}} f) g_{k,r}^{m\bar{q}} \), we have
\[
|| g_{k,r,f}^{-1} ||_{L^2_{l+2}} \leq || g_{k,r,f}^{-1} - g_{k,r}^{-1} ||_{L^2_{l+2}} + C k^{n/2} \leq || g_{k,r,f}^{-1} ||_{L^2_{l+2}} + C k^{n/2}.
\]
Also, the form \( \chi \) is uniformly bounded above with respect to the \( C^l \)-norm ([Fin04, Lemma 5.6]). By combining these estimates, we have the desired estimate. \( \square \)

This implies that \( \mathcal{L}_{k,r} - D_0 \mathcal{L}_{k,r} \) is Lipschitz with constant \( 1/(2 \| P \|) \) on the ball centered at 0 with radius \( C k^{-A-n/2} \) for some constant \( C \). As \( \omega_{k,r} \) is an approximate solution of the \( J_\chi \)-equation, the same arguments as in [Fin04, Lemmas 5.6 and 5.7] imply that we have \( \mathcal{L}_{k,r}(0) = O(k^{-r-1}) \) in \( C^l(g_{k,r}) \) and \( || \mathcal{L}_{k,r}(0) ||_{L^2_{l}} = O(k^{-r+1+n/2}) \). Therefore, if we choose \( r \geq 2A + n \), for all \( k \gg 0 \), we have a function \( \phi \) such that \( \mathcal{L}_{k,r}(\phi) = 0 \). As \( \mathcal{L}_{k,r} \) is an elliptic operator of second order, if we make \( l \) large enough, the regularity theorem implies \( \phi \in C^\infty \). This completes the proof of Theorem 1.4.

3.3. Proof of Theorem 1.5. We also consider the converse implication of Theorem 1.4. Instead of considering a solution of the \( J_\chi \)-equation, we consider the topological condition called \( J \)-positivity and it relates by the following theorem ([Che21, Theorem 1.1]):

**Theorem 3.13.** Fix a Kähler manifold \( X \) of dimension \( n \) with Kähler metrics \( \omega \) and \( \chi \). Let \( c > 0 \) be the constant determined by
\[
c \int_X \omega^n = n \int_X \chi \wedge \omega^{n-1}.
\]
Then, there exists a solution of \( J_\chi \)-equation in the class \( \omega \) if and only if there exists a constant \( \epsilon > 0 \) such that
\[
\int_W (c - (n - p) \epsilon) \omega^p - p \chi \wedge \omega^{p-1} \geq 0
\]
for all \( p \)-dimensional subvarieties \( W \) with \( p = 1, 2, \ldots, n-1 \).

The following definition is from [Son20, Definition 1.1]:

**Definition 3.14.** Let \( X \) be a Kähler manifold of dimension \( n \) with Kähler metrics \( \omega \) and \( \chi \).
(1) The pair \((|\omega|, [\chi])\) is said to be \(J\)-positive if we have
\[
p \frac{\int_X \chi \wedge \omega^{p-1}}{\int_W \omega^p} < n \frac{\int_X \chi \wedge \omega^{n-1}}{\int_X \omega^n}
\]
for any \(p\)-dimensional subvarieties \(W\) of \(X\) with \(p = 1, 2, \ldots, n - 1\).

(2) The pair \((|\omega|, [\chi])\) is said to be \(J\)-nef if we have
\[
p \frac{\int_X \chi \wedge \omega^{p-1}}{\int_W \omega^p} \leq n \frac{\int_X \chi \wedge \omega^{n-1}}{\int_X \omega^n}
\]
for any \(p\)-dimensional subvarieties \(W\) of \(X\) with \(p = 1, 2, \ldots, n - 1\).

(3) The pair \((|\omega|, [\chi])\) is said to be uniformly \(J\)-positive if there exists a constant \(\epsilon > 0\) such that we have
\[
p \frac{\int_X \chi \wedge \omega^{p-1}}{\int_W \omega^p} \leq (n - \epsilon) \frac{\int_X \chi \wedge \omega^{n-1}}{\int_X \omega^n}
\]
for any \(p\)-dimensional subvarieties \(W\) of \(X\) with \(p = 1, 2, \ldots, n - 1\).

**Proof of Theorem 1.5.** We prove by contradiction. Assume that there exists \(b \in B\) such that \((|\omega_b|, [\chi_b])\) is not \(J\)-nef, where \(\omega_b\) and \(\chi_b\) are restrictions to the fiber \(X_b\) of \(\omega_X\) and \(\chi\) in the assumption of Theorem 1.5. Then, there exists a \(p\)-dimensional subvariety \(W_b\) of \(X_b\) such that
\[
c_b = m \frac{\int_{X_b} \chi_b \wedge \omega_b^{m-1}}{\int_{X_b} \omega_b^m} < p \frac{\int_{W_b} \chi_b|_{W_b} \wedge (\omega_b|_{W_b})^{p-1}}{\int_{W_b} (\omega_b|_{W_b})^p}.
\]

By a simple calculation,
\[
(m + n) \frac{\int_X \chi \wedge \omega_k^{m+n-1}}{\int_X \omega_k^{m+n}} = m \frac{\int_X \chi \wedge (\omega_X)_k^{m-1} \wedge (\omega_k)^n}{\int_X \omega_k^{m+n}} + n \frac{\int_X \chi \wedge (\omega_X)_k^{m} \wedge (\omega_k)^{n-1}}{\int_X \omega_k^{m+n}}
\]
\[
= m \frac{\int_X \chi \wedge (\omega_X)_k^{m-1} \wedge \omega_B^n}{\int_X \omega_k^{m} \wedge \omega_B^n} + k^{-1}C_1 + O \left( k^{-2} \right)
\]
\[
+ k^{-1}n \frac{\int_X (\omega_X)_k^m \wedge \chi \wedge \omega_B^{n-1}}{\int_X (\omega_X)_k^m \wedge \omega_B^n} + O \left( k^{-2} \right),
\]
where \(C_1\) is the constant calculated as
\[
C_1 = \frac{\int_X (\Lambda_Y \chi) (\Lambda_{\omega_X} \omega_X) \omega_X^m \wedge \omega_B^n}{\int_X \omega_X^m \wedge \omega_B^n} - \frac{\int_X (\Lambda_Y \chi) \omega_X^m \wedge \omega_B^n}{\int_X \omega_X^m \wedge \omega_B^n} - \frac{\int_X (\Lambda_{\omega_X} \omega_X) \omega_X^m \wedge \omega_B^n}{\int_X \omega_X^m \wedge \omega_B^n}.
\]

Note that
\[
m \frac{\int_X \chi \wedge (\omega_X)_k^{m-1} \wedge \omega_B^n}{\int_X \omega_X^m \wedge \omega_B^n} = m \frac{\int_B V_b \omega_B^n}{\int_B V_b \omega_B^n} = c_b,
\]
where \(V_b\) is the volume of the fiber \(X_b\) with respect to \(\omega_X\). Therefore, the constant \((m + n) \frac{\int_X \chi \wedge \omega_k^{m+n-1}}{\int_X \omega_k^{m+n}} / \frac{\int_X \omega_k^{m+n}}{\int_X \omega_k^{m+n}}\) converges to \(c_b\) as \(k \to \infty\). By (3.15), this implies that the \(p\)-dimensional subvariety \(W_b\) of \(X\) satisfies
\[
(m + n) \frac{\int_X \chi \wedge \omega_k^{m+n-1}}{\int_X \omega_k^{m+n}} < p \frac{\int_{W_b} \chi|_{W_b} \wedge (\omega_b|_{W_b})^{p-1}}{\int_{W_b} (\omega_b|_{W_b})^p}.
\]
for all $k \gg 0$. However, the pair $([\omega_k], [\chi])$ is $J$-nef by assumption and this is a contradiction. In conclusion, the pair $([\omega_k], [\chi])$ on any fiber $X_k$ is $J$-nef.

Similarly, we prove $J$-nefness of the base by contradiction. Define the $(1,1)$-form $\chi_B$ on $B$ by $\chi_B = \pi_B (\chi_H)$. Assume that the pair $([\omega_B], [\chi_B])$ is not $J$-nef. Then, there exists a $p$-dimensional subvariety $W$ of $B$ such that

$$
\frac{n \int_B \chi_B \wedge \omega_B^{n-1}}{\int_B \omega_B^n} < p \frac{\int_W \chi_B \wedge \omega_B^{p-1}}{\int_W \omega_B^p}.
$$

(3.16)

By the assumption that $\Lambda_{\omega} \chi = c_b$ or $\pi^* (\pi_B (\omega_X)_{\mathcal{H}}) = (\omega_X)_{\mathcal{H}}$, the constant $C_1$ vanishes. By the same calculation as above, we have

$$(m + p) \frac{\int_{\pi^{-1}W} \chi \wedge \omega_k^{m+p-1}}{\int_{\pi^{-1}W} \omega_k^{m+p}} = c_b + k^{-1} \frac{\int_{\pi^{-1}W} (\omega_X)^m \chi_H \wedge \omega_B^{p-1}}{\int_{\pi^{-1}W} (\omega_X)^m \chi_H \wedge \omega_B^{p-1}} + O (k^{-2})
$$

$$
= c_b + k^{-1} \frac{\int_W V_B \pi_B (\chi_H) \wedge \omega_B^{p-1}}{\int_W V_B \omega_B^p} + O (k^{-2})
$$

$$
= c_b + k^{-1} \frac{\int_W \chi_B \wedge \omega_B^{p-1}}{\int_W \omega_B^p} + O (k^{-2}).
$$

By (3.16), we have

$$
(m + n) \frac{\int_X \chi \wedge \omega_k^{m+n-1}}{\int_X \omega_k^{m+n}} < (m + p) \frac{\int_{\pi^{-1}W} \chi \wedge \omega_k^{m+p-1}}{\int_{\pi^{-1}W} \omega_k^{m+p}}
$$

for any $k \gg 0$. However, the pair $([\omega_k], [\chi])$ is $J$-nef by assumption and this is a contradiction. In conclusion, the pair $([\omega_B], [\pi_B (\chi_{\mathcal{H}})])$ is also $J$-nef. \qed

Remark 3.17. If we assume the existence of a solution $\omega'_k = \omega_k + \sqrt{-1} \partial \bar{\partial} \phi_k$ of the $J$-$\chi$-equation in the class $[\omega_k]$ for all $k \gg 0$, for small $t > 0$ such that $\chi - t \omega'_k$ is a Kähler form, we have a solution $\omega'_k$ of the $J_{\chi - t \omega'_k}$-equation. By Theorem 3.13, we have

$$
\int_{\pi^{-1}W} c \omega_k^p - p \gamma \wedge \omega_k^{p-1} \geq (m + n - p) t \int_{\pi^{-1}W} \omega_k^p
$$

for all $p$-dimensional subvarieties $W$ of $B$ with $p = 1, 2, \ldots, n - 1$. If a family of the solutions $\omega'_k$ has an order $O(k)$ as $k \to \infty$, we have $t = O (k^{-1})$. Therefore, in the same assumptions of Theorem 1.5, we obtain the $J$-positivity of the pair $([\omega_B], [\pi_B (\chi_{\mathcal{H}})])$ by the above calculation. In the recent work [GPT21], they proved the $L^\infty$ estimate for a solution of the family of Hessian equations with a certain structural condition. Our situation is not included in the class they studied. It is interesting to see if the method can apply to our case.

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