EMBEDDING DELIGNE–MUMFORD STACKS INTO GIT QUOTIENT STACKS OF LINEAR REPRESENTATIONS

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ABSTRACT. We study how to use a suitably ample locally free sheaf of rank $r$ over a proper Deligne–Mumford stack to furnish an embedding of the stack into a geometric invariant theory (GIT) quotient stack $[\mathbb{A}^n/\text{GL}_r]$ where $\mathbb{A}$ is a finite dimensional representation of the general linear group $\text{GL}_r$.

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1. Introduction

Given a polarized compact complex manifold $(X, L)$, a classical result, which is often called the Kodaira embedding theorem, asserts that for all sufficiently large $m$, the natural evaluation map for sections of $L^m$ defines a closed embedding from $X$ into the projectivization of the dual of the space of global sections. More generally, if a locally free sheaf $V$ of rank $r$ over $X$ is also specified, then for all sufficiently large $m$, the evaluation map for sections of $V \otimes L^m$ defines a closed embedding from $X$ into the Grassmann variety of $r$-planes in the dual of the space of global sections.

An extension of the rank one result has been recently obtained for orbifold singularities which are cyclic. Precisely, let $X$ denote a compact complex orbifold with cyclic stabilizers. Let $N$ be the minimal positive integer such that the order of every element in every stabilizer group divides $N$. (The number $N$ is called the order of $X$ in [17] and the index of $X$ in [2].) Ross and Thomas [17] define an orbifold polarization on $X$ to be an invertible sheaf $L$ on $X$ such that stabilizer groups act faithfully on fibers of $L$ and that $L^N$ descends to an ample line bundle on underlying space $X$. These conditions can also be stated in algebro-geometric language. In fact, the orbifold $X$ is a proper Deligne–Mumford stack defined over the field $\mathbb{C}$ of complex numbers, with cyclic geometric stabilizer groups. Let $\pi : X \to X$ be the morphism to the coarse moduli space. Then an orbifold polarization $L$ consists of a $\pi$-ample invertible sheaf such that $L^N = \pi^* M$ where $M$ is an ample invertible sheaf on $X$. Ross and Thomas use global sections of $L^{m+j}$, where $m \gg 0$ and
0 ≤ j ≤ N, to construct a closed embedding of the stack \( \mathcal{X} \) into a weighted projective stack. In algebro-geometric language, Abramovich and Hassett [2] prove a similar result for a proper cyclotoic stack \( f : \mathcal{X} \to B \) over a general base scheme \( B \) of finite type, equipped with a \( \pi \)-ample invertible sheaf \( \mathcal{L} \) such that \( \mathcal{L}^N = \pi^* M \) where \( M \) is an invertible sheaf on the coarse moduli \( f : X \to B \) which is \( f \)-ample.

The result described by Ross and Thomas corresponds to the case \( B = \text{Spec} \mathbb{C} \) and \( f : \mathcal{X} \to \text{Spec} \mathbb{C} \) is smooth.

The purpose of this note is to describe a natural generalization of the result obtained by Ross and Thomas to the situation where the geometric stabilizers are not necessarily cyclic. We consider a possibly singular Deligne-Mumford stack \( \mathcal{X} \) over a field \( k \) which is algebraically closed and of characteristic zero. Let \( \pi : \mathcal{X} \to X \) be the morphism to the coarse moduli. We assume that

(i) \( p : \mathcal{X} \to \text{Spec} k \) is proper
(ii) \( \bar{p} : X \to \text{Spec} k \) is projective, and
(iii) \( \mathcal{X} \) possesses a \( \pi \)-ample locally free sheaf \( \mathcal{V} \).

By assumption (iii), all of the geometric stabilizer groups act faithfully on the fibers of \( \mathcal{V} \). At the same time, our assumptions guarantee that there exists a \( \pi \)-ample locally free sheaf \( \mathcal{V} \) such that \( (\det \mathcal{V})^N = \pi^* M \) for some ample invertible sheaf \( M \) on \( X \), where \( N \) is the minimal positive integer such that the order of each geometric stabilizer group divides \( N \). We use global sections of \( (\det \mathcal{V})^m \otimes \text{Sym}^j \mathcal{V} \), for \( m \gg 0 \) and \( j \leq N \), to obtain the following main result.

**Theorem 1.1.** Under assumptions (i) through (iii), there is an embedding of \( \mathcal{X} \) into a geometric invariant theory (GIT) quotient stack

\[
[\mathbb{A}^{\text{ss}}/GL_r]
\]

where \( r \) is the rank of the locally free sheaf \( \mathcal{V} \), \( \mathbb{A} \) is a linear representation of the general linear group \( GL_r \), and \( \mathbb{A}^{\text{ss}} \) is the semistable locus determined by the determinant character.

The assumptions we impose are natural and arguably offer another answer to a problem posed by Kresch in [12], namely, to understand when a Deligne–Mumford stack should be called “(quasi-)projective.” There, he offers the following equivalences.

**Theorem 1.2** (Kresch). Let \( \mathcal{X} \) be a proper Deligne–Mumford stack over a field \( k \) of characteristic zero. Then the following are equivalent:

1. \( \mathcal{X} \) has a projective coarse moduli space and possesses a \( \pi \)-ample locally free sheaf, where \( \pi : \mathcal{X} \to X \) is the morphism to the coarse moduli.
2. \( \mathcal{X} \) has a projective coarse moduli space and possesses a generating sheaf.
3. \( \mathcal{X} \) admits a closed embedding into a smooth proper Deligne–Mumford stack with projective coarse moduli space.

In Section 2 we recall useful definitions and results concerning GIT and stacks, including various notions of moduli spaces and stability.

We identify in Section 3 a certain class of stacks of the form (1) which we call twisted Grassmannians because they generalize both the construction of ordinary Grassmann varieties and the construction of weighted projective spaces.

Our main result is stated and proved in Sections 4 and 5. As mentioned earlier, the result extends known results of a similar flavor. Indeed when the geometric
stabilizers are cyclic and rank $V = 1$, we obtain an embedding of $\mathcal{X}$ into a weighted projective stack (with coarse moduli that is a weighted projective space). At the same time, when $\mathcal{X} = X$ is a projective scheme and the rank of $V$ is $r$, we obtain an embedding of $\mathcal{X} = X$ into a Grassmann variety of $r$-planes in a suitable vector space, which reduces to a projective space when $r = 1$.

In future work, we plan to generalize our construction to a Deligne–Mumford stack $f : \mathcal{X} \to B$ over a base scheme $B$ of finite type, assuming (i) $f : \mathcal{X} \to B$ is flat and proper, (ii) the coarse moduli $\bar{f} : \mathcal{X} \to B$ is flat and projective, (iii) $\mathcal{X}$ possesses a $\pi$-ample locally free sheaf $V$, where $\pi : \mathcal{X} \to X$ is the morphism to the coarse moduli, such that $(\det V)^N = \pi^* M$ for some $f$-ample invertible sheaf on $X$. We seek to find additional hypotheses that we may impose in order to obtain an embedding of $\mathcal{X}$ into certain twisted Grassmann bundle defined over $B$.

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## 2. Preliminaries

Throughout, we will work over a fixed field $k$ of characteristic zero that we assume to be algebraically closed. This means in particular that any scheme that appears will be understood to be a scheme over $k$ and that the fiber product of schemes will be over $k$, unless otherwise indicated. In addition, for a positive integer $n$, the notation $\mathbb{A}^n$ means affine $n$-space over $k$ and similarly for the general linear group $GL_n$.

**Remark 2.1.** There are two primary reasons that we assume $k$ has characteristic zero and is algebraically closed.

1. While $GL_n$ is linearly reductive in characteristic zero, it is not linearly reductive in positive or mixed characteristics. On a closely related note, if $GL_n$ acts on an affine scheme $\text{Spec}(R)$, then the natural morphism

   $$[\text{Spec}(R)/GL_n] \to \text{Spec}(R^{GL_n})$$

   will only be an adequate moduli space, but not a good moduli space\footnote{The notion of a good moduli space was introduced by Alper [4]. The precise definition will be given in Definition 2.18.} in general. See [5, Remark 2.5].

2. For a proper scheme $X$ over an algebraically closed field $k$, a classical result says that an invertible sheaf is very ample if and only if it separates points and tangent vectors (see, e.g. [9, Proposition II.7.3]). The proof exploits the Nullstellensatz to assert that all residue fields must be equal to $k$. Our intention in this note is to obtain a generalization of this result, though there may even be suitable extensions relative to fields which are not algebraically closed or even relative to more general Noetherian schemes $B$.

Nevertheless, we conjecture it is possible to obtain results similar to those in this note whilst working relative to any field or, even, relative to an arbitrary base scheme $B$, provided some additional hypotheses are imposed.
2.1. Linearizations.

**Example 2.2.** Any morphism $G \to GL_n$ of group schemes (over $k$) determines an action of $G$ on $k^n$ through the fundamental (left) action of $GL_n$ on $k^n$. More generally, for a scheme $X$, if $\mathbb{A}^n_X$ denotes the base change to $X$, then any morphism $\rho : G \to GL_n$ of group schemes determines an action of $G$ on $\mathbb{A}^n_X$ (over $X$) through base change:

$$\Sigma_\rho : G \times \mathbb{A}^n_X \longrightarrow \mathbb{A}^n_X.$$

**Definition 2.3 ([7, II, Definition 1.7.8]).** Let $X$ be a scheme. Let $E$ be a locally free sheaf over $X$. The vector bundle associated to $E$ is the scheme $V(E) = \text{Spec}_X(\text{Sym}(E))$ together with the affine morphism $V(E) \to X$. It is important to note that the functor $V$ is a contravariant functor from the category of locally free sheaves to the category of vector bundles over $X$. (This is because the functor $\text{Sym}$ is covariant and the functor $\text{Spec}_X$ is contravariant.)

As an alternative characterization, consider the functor which associates to any scheme $f : T \to X$ the set of morphisms of $O_T$-modules $\text{Hom}(f^*E, O_T)$.

This functor is representable by the scheme $V(E)$.

**Example 2.4.** Let $X$ be a scheme. For a positive integer $n$,

$$V(O^n_X) = \text{Spec}_X(\text{Sym}(O^n_X)) \simeq \mathbb{A}^n_X$$

is affine $n$-space over $X$. Indeed for any scheme $f : T \to X$, the $T$-points are

$$V(O^n_X)(T) = \text{Hom}(f^*(O^n_X), O_T) \simeq \text{Hom}(O^n_T, O_T) \simeq \Gamma(T, O_T)^n = \mathbb{A}^n_X(T).$$

**Definition 2.5 ([13, Definition 1.6]).** Let $X$ be a scheme. Let $G$ be a group scheme. Let $\sigma : G \times X \to X$ be an action of $G$ on $X$. Let $E$ be a locally free sheaf over $X$. By a $G$-linearization is meant an isomorphism of sheaves

$$\phi : \sigma^*E \longrightarrow p^*_2E$$

over $G \times X$ satisfying the cocycle relation

$$p^*_2\phi \circ (1_G \times \sigma)^*\phi = (\mu \times 1_X)^*\phi$$

over $G \times G \times X$, where $\mu : G \times G \to G$ denotes the group law.

**Lemma 2.6 ([13, Section 1.3]).** Let $X$ be a scheme. Let $G$ be a group scheme. Let $\sigma : G \times X \to X$ be an action of $G$ on $X$. Let $E$ be a locally free sheaf over $X$. The data of a $G$-linearization of $E$ is equivalent to the data of an action

$$\Sigma : G \times V(E) \longrightarrow V(E)$$

of $G$ on $V(E)$ such that

$$\begin{array}{ccc}
G \times V(E) & \xrightarrow{E} & V(E) \\
\downarrow & & \downarrow \\
G \times X & \xrightarrow{\sigma} & X
\end{array}$$

commutes and such that $\Sigma$ is a bundle isomorphism of the vector bundles $G \times V(E)$ over $G \times X$ and $V(E)$ over $X$. 
Example 2.7. Let $X$ be a scheme. Let $G$ be a group scheme. Let $G$ act on $X$ by the trivial action

$$\sigma = p_2 : G \times X \to X.$$  

Any morphism $\rho : G \to GL_n$ of group schemes determines a $G$-linearization of $O_X^n$ that can be described as follows. The vector bundle corresponding to $O_X^n$ can be identified with affine $n$-space over $X$

$$\mathbb{V}(O_X^n) = \mathbb{A}_X^n.$$  

Example 2.2 describes an action

$$\Sigma_\rho : G \times \mathbb{A}_X^n \to \mathbb{A}_X^n$$  

of $G$ on $\mathbb{A}_X^n$ (over $X$). This action determines a $G$-linearization of $O_X^n$.

Lemma 2.8 ([13, Section 1.3]). The tensor product of two $G$-linearized sheaves enjoys the structure of an induced $G$-linearization.

Lemma 2.9. The direct sum of two $G$-linearized sheaves enjoys the structure of an induced $G$-linearization.

2.2. Algebraic spaces, stacks, and inertia. We follow closely the notation and terminology of [4].

Definition 2.10 ([4, Section 2]). An algebraic space (over $k$) is a sheaf of sets

$$X : (\text{Sch}/k)^{pp}_{\text{etale}} \to \text{Set}$$

such that

1. The diagonal $\Delta_{X/k} : X \to X \times_k X$ is representable by schemes and quasi-compact.
2. There exists an étale, surjective morphism $U \to X$ where $U$ is a scheme (over $k$).

Definition 2.11 ([4, Section 2]). An Artin stack (over $k$) is a category

$$p : \mathcal{X} \to (\text{Sch}/k)_{\text{etale}}$$

over $k$ with the following properties.

1. The category $\mathcal{X}$ is a stack in groupoids over $(\text{Sch}/k)_{\text{etale}}$.
2. The diagonal $\Delta_{\mathcal{X}/k} : \mathcal{X} \to \mathcal{X} \times_k \mathcal{X}$ is representable by algebraic spaces, separated, and quasi-compact.
3. There exists a smooth, surjective map $U \to \mathcal{X}$ where $U$ is a scheme over $k$.

If the smooth, surjective morphism $U \to \mathcal{X}$ can be taken to be étale, then we call $\mathcal{X}$ a Deligne–Mumford (DM) stack.

Definition 2.12. Let $p : \mathcal{X} \to \text{Spec } k$ be an Artin stack. The inertia stack $\mathcal{I}\mathcal{X}$ is the fibre product

$$\xymatrix{ \mathcal{I}\mathcal{X} \ar@{->}[r] & \mathcal{X} \ar@{->}[d]_{\Delta_{\mathcal{X}/k}} \ar@{->}[d] \ar@{->}[r] & \mathcal{X} \ar@{->}[d]_{\Delta_{\mathcal{X}/k}} \ar@{->}[r] & \mathcal{X} \times_k \mathcal{X} \ar@{->}[d]_{\Delta_{\mathcal{X}/k}} }$$
Definition 2.13 \((\text{[4 Section 2.1]})\). Let \(p : \mathcal{X} \to \text{Spec} \, k\) be an Artin stack. For a geometric point \(x : \text{Spec} \, K \to \mathcal{X}\) of \(\mathcal{X}\), let \(G_x\) denote the fibre product
\[
\text{Aut}_x = \text{Spec} \, K \times_{\mathcal{X}} \mathcal{I} \mathcal{X}.
\]
Because \(\Delta_{\mathcal{X}/k}\) is separated, \(G_x\) is a group scheme (see Lemma 99.19.1 of \([1]\)). We call \(G_x\) the stabilizer group associated to \(x\).

2.3. Moduli. In this section, we consider the following situation.

Situation 2.14. Let \(\mathcal{X}\) be an Artin stack. Let \(X\) be an algebraic space. Let \(\pi : \mathcal{X} \to X\) be a morphism of stacks (over \(\text{Spec} \, k\)).

Definition 2.15 \((\text{[6 Section 1]}\)). In Situation 2.14, the morphism \(\pi\) is called a coarse moduli space if the following properties are satisfied.

1. The morphism \(\pi\) is initial among all morphisms over \(\text{Spec} \, k\) to algebraic spaces over \(\text{Spec} \, k\).
2. The induced map
\[
[\mathcal{X}(k)] \to X(k)
\]

is bijective, where \([\mathcal{X}(k)]\) denotes the set of isomorphism classes of objects in the small category \(\mathcal{X}(k)\). (Remember that \(k\) is algebraically closed.)

Here is a well-known situation in which a coarse moduli space exists.

Situation 2.16. Let \(p : \mathcal{X} \to \text{Spec} \, k\) be an Artin stack. Assume the following hypotheses.

1. The morphism \(p\) is separated and locally of finite presentation.
2. The projection \(\mathcal{I} \mathcal{X} \to \mathcal{X}\) is finite.

Theorem 2.17 \((\text{[6 11]}\)). In Situation 2.16 there is a coarse moduli space \(\pi : \mathcal{X} \to X\). Moreover, it satisfies the following additional properties.

1. The structure morphism \(\bar{p} : X \to \text{Spec} \, k\) is separated.
2. The morphism \(\pi\) is proper and quasi-finite.

There are other types of moduli, such as good and tame moduli, studied by Alper \([4]\).

Definition 2.18 \((\text{[4 Definition 4.1]}\)). In Situation 2.14 the morphism \(\pi\) is called a good moduli space if the following properties are satisfied.

1. The morphism \(\pi\) is cohomologically affine.
2. The natural morphism \(\mathcal{O}_X \to \pi_* \mathcal{O}_\mathcal{X}\) is an isomorphism.

Definition 2.19 \((\text{[4 Definition 7.1]}\)). In Situation 2.14 the morphism \(\pi\) is called a tame moduli space if the following properties are satisfied.

1. The morphism \(\pi\) is a good moduli space.
2. The induced map
\[
[\mathcal{X}(k)] \to X(k)
\]

is a bijection of sets.

By imposing a suitable tameness condition, one can ensure that a coarse moduli space is always tame.
Definition 2.20 ([3 Definition 3.1]). In Situation 2.10 we say that $\mathcal{X}$ is tame if the pushforward functor
$$\pi_* : \text{QCoh}\mathcal{X} \to \text{QCoh}X$$
is exact, where $\pi : \mathcal{X} \to X$ denotes the coarse moduli space.

Lemma 2.21 ([3 Theorem 3.2]). In Situation 2.14 the following are equivalent.

1. The stack $\mathcal{X}$ is tame.
2. For each geometric point $x : \text{Spec } K \to \mathcal{X}$, the group scheme $G_x$ is linearly reductive.

Lemma 2.22 ([4 Remark 7.3]). In Situation 2.16, if $\mathcal{X}$ is tame, then the coarse moduli space $\pi : \mathcal{X} \to X$ is a tame moduli space.

Lemma 2.23 ([14, Lemma 2.8] or [16]). In Situation 2.16, write $\pi : \mathcal{X} \to X$ for the coarse moduli space. Let $F$ be a quasi-coherent sheaf over $\mathcal{X}$ and $G$ a coherent sheaf over $X$. Assume $\mathcal{X}$ is DM and tame. Then there is a natural isomorphism
$$\pi_*(\pi^*G \otimes F) \simeq G \otimes \pi_* F.$$
There is an open subscheme $V \subset X$ such that $\pi^{-1}(V) = X_L^\times$ and $\pi_{X_L^\times} : X_L^\times \to V$ is a tame moduli space.

If $X^\times_L$ is quasi-compact, then there exists an ample line bundle $M$ on $X$ such that $\pi^* M \simeq \mathcal{L}^N$ for some $N$.

2.5. Frame bundles.

**Definition 2.28** ([1, Situation 08JT]). Let $p : X \to \text{Spec } k$ be an Artin stack. Let $E, F$ be quasi-coherent sheaves of $\mathcal{O}_X$-modules. Let $\text{Hom}_X(\mathcal{E}, \mathcal{F})$ denote the functor

$$\text{Hom}_X(\mathcal{E}, \mathcal{F}) : (\text{Sch}/k)^{\text{op}} \to \text{Set}$$

defined as follows. If $T$ is a scheme, then an element of $\text{Hom}_X(\mathcal{E}, \mathcal{F})(T)$ consists of a pair $(h, u)$, where $h$ is a morphism $h : T \to X$ of stacks and $u : h^* \mathcal{E} \to h^* \mathcal{F}$ is an $\mathcal{O}_T$-module morphism. The functor $\text{Hom}_X(\mathcal{E}, \mathcal{F})$ is equipped with a natural morphism to $X$ described by sending a pair $(h, u)$ to the morphism $h$.

**Example 2.29.** Let $p : X \to \text{Spec } k$ be a scheme. Let $E, F$ be quasi-coherent sheaves of $\mathcal{O}_X$-modules. The functor $\text{Hom}_X(E, F)$ is representable by the scheme $V(\text{Hom}(E, F))$.

**Example 2.30.** Given a quasi-coherent sheaf $\mathcal{E}$ over $X$, the functor $\text{Hom}_X(\mathcal{E}, \mathcal{O}_X^r)$ is naturally a $GL_r$-torsor, where the action is left multiplication.

**Lemma 2.31.** Let $p : X \to \text{Spec } k$ be an Artin stack. Let $V$ be a locally free sheaf of rank $r$ over $X$. Set $\mathcal{L} = \text{det } V$. For each pair of positive integers $i, m$, there is a natural morphism

$$\text{Hom}_X(V, O_X^r) \to \text{Hom}_k(p_*(\text{Sym}^i V \otimes \mathcal{L}^m), \text{Sym}^i(O_X^r))$$

over $\text{Spec } k$. Moreover, the morphism is equivariant if the right-hand side is viewed as a $GL_r$-torsor where the action is described by $\text{Sym}^i \rho \otimes (\text{det } \rho)^{\otimes m}$ if $\rho$ denotes the fundamental representation of $GL_r$.

**Proof.** For notational convenience, write

$$\psi_{i,k} : p^* p_*(\text{Sym}^i V \otimes \mathcal{L}^m) \to \text{Sym}^i V \otimes \mathcal{L}^m$$

for the natural adjunction morphism. Let $T$ be a scheme over $S$. Let $h : T \to X$ be a morphism of stacks. Let $u : h^* V \to O_T^r$ be morphism of $\mathcal{O}_T$-modules. Note that for each $i$, the morphism $u$ determines a morphism

$$h^*(\text{Sym}^i V) \simeq \text{Sym}^i(h^* V) \xrightarrow{u} \text{Sym}^i(O_T^r).$$

In addition, $u$ determines a morphism

$$h^* \mathcal{L} = h^* \text{det } V \simeq \text{det } h^* V \xrightarrow{u} \text{det}(O_T^r) \simeq O_T$$

and so for each $m$ a morphism

$$h^* \mathcal{L}^m \xrightarrow{u} O_T$$

as well. Using this, we see that we have

- a morphism $p \circ h : T \to \text{Spec } k$ depicted by

$$T \xrightarrow{h} X \xrightarrow{p} \text{Spec } k$$
• and an $\mathcal{O}_T$-module morphism
  $$(p \circ h)^* p_* (\text{Sym}^i V \otimes L^m) \longrightarrow (p \circ h)^* (\text{Sym}^i(\mathcal{O}_{\text{Spec} k}))$$
depicted by
  $$h^* p_* (\text{Sym}^i V \otimes L^m) \xrightarrow{\text{whitney}} \text{Sym}^i(\mathcal{O}_T^r) \simeq (p \circ h)^* (\text{Sym}^i(\mathcal{O}_{\text{Spec} k})).$$
Some details omitted. □

**Definition 2.32.** Let $p : \mathcal{X} \to \text{Spec} k$ be an Artin stack. Let $\mathcal{E}, \mathcal{F}$ be quasi-coherent sheaves of $\mathcal{O}_\mathcal{X}$-modules. We can consider the subfunctor

$$\text{Isom}_{\mathcal{X}}(\mathcal{E}, \mathcal{F}) \subset \text{Hom}_{\mathcal{X}}(\mathcal{E}, \mathcal{F})$$
whose value on a scheme $T$ consists of the set of pairs $(h, u)$ where $h$ is a morphism $h : T \to \mathcal{X}$ of stacks and $u : h^* \mathcal{E} \to h^* \mathcal{F}$ is an $\mathcal{O}_T$-module isomorphism.

**Definition 2.33.** Let $p : \mathcal{X} \to \text{Spec} k$ be an Artin stack. Let $\mathcal{V}$ be a locally free sheaf over $\mathcal{X}$ of rank $r$. The **frame bundle of $\mathcal{V}$** is defined to be the functor

$$\text{Fr}(\mathcal{V}) = \text{Isom}_{\mathcal{X}}(\mathcal{V}, \mathcal{O}_\mathcal{X}^r)$$
over $\mathcal{X}$.

**Lemma 2.34.** Let $p : \mathcal{X} \to \text{Spec} k$ be an Artin stack. Let $\mathcal{V}$ be a locally free sheaf of rank $r$ over $\mathcal{X}$. Set $L = \det \mathcal{V}$. For each pair of positive integers $i, m$, there is a natural morphism

$$\text{Fr}(\mathcal{V}) \longrightarrow \text{Hom}_k(p^* (\text{Sym}^i \mathcal{V} \otimes L^m), \text{Sym}^i(\mathcal{O}_{\text{Spec} k})).$$
over $\text{Spec} k$. Moreover, the morphism is equivariant if the right-hand side is viewed as a $\text{GL}_r$-torsor where the action is described by $\text{Sym}^i \rho \otimes (\det \rho)^{\otimes m}$ if $\rho$ denotes the fundamental representation of $\text{GL}_r$.

**Proof.** Follows immediately from Lemma 2.31. □

2.6. **Ample and generating sheaves.** In this section, we consider the following situation.

**Situation 2.35.** In Situation 2.16, write $\pi : \mathcal{X} \to X$ for the coarse moduli space. Let $\mathcal{E}$ denote a locally free sheaf over $\mathcal{X}$. Make the following assumptions.

(1) The stack $\mathcal{X}$ is DM.

(2) There is a positive integer $N$ such that $N$ is a multiple of the order of each geometric stabilizer group $G_x$ for $x : \text{Spec} k \to \mathcal{X}$. Moreover, it is sufficient to select $N$ to be minimal with this property.

**Definition 2.36.** In Situation 2.35 we say that $\mathcal{E}$ is **generating** if for each quasi-coherent sheaf $\mathcal{F}$ over $\mathcal{X}$, the natural morphism

$$\pi^* \pi_* \mathcal{H}om(\mathcal{E}, \mathcal{F}) \otimes \mathcal{E} \longrightarrow \mathcal{F}$$
is surjective.

**Lemma 2.37** ([12 Section 5.2]). In Situation 2.35 the sheaf $\mathcal{E}$ is generating if and only if for each geometric point of $\mathcal{X}$, with $G$ the geometric stabilizer group, the representation of $G$ on the fibre of $\mathcal{E}$ at the geometric point contains every irreducible representation of $G$. 

Definition 2.38 ([14, Definition 2.1]). In Situation 2.35 we say that $\mathcal{E}$ is $\pi$-ample if for each geometric point of $\mathcal{X}$, with $G$ the geometric stabilizer group, the representation of $G$ on the fibre of $\mathcal{E}$ at the geometric point is faithful.

The next result follows from a theorem of Burnside, which enjoys several extensions, such as one due to Steinberg [18, Theorem 7], which states that every irreducible representation of a finite group is contained in some symmetric power of a faithful representation.

Lemma 2.39. In Situation 2.35, if $\mathcal{E}$ is $\pi$-ample, then $\bigoplus_{i=0}^{N} \text{Sym}^{i}\mathcal{E}$ is a generating sheaf.

The next result can be found in the proof of [8, Lemma 2.12].

Lemma 2.40. In Situation 2.35, if $\mathcal{E}$ is $\pi$-ample, then the frame bundle $\text{Fr}(\mathcal{E})$ is representable by an algebraic space.

Lemma 2.41. In Situation 2.35, assume $\mathcal{E}$ has rank $r$ and is $\pi$-ample. Then the natural morphism $\text{Fr}(\mathcal{E}) \to \mathcal{X}$ descends to an isomorphism

$$[\text{Fr}(\mathcal{E})/\text{GL}_r] \sim \to \mathcal{X}$$

of stacks over $S$.

2.7. Amplitude and properness. In this section, we consider the following situation.

Situation 2.42. In Situation 2.16, write $\pi: \mathcal{X} \to X$ for the coarse moduli space. Assume the following.

1. The stack $\mathcal{X}$ is DM.
2. The morphism $p: \mathcal{X} \to \text{Spec } k$ is proper. (In particular, it is of finite type and of finite presentation.)

Lemma 2.43. In Situation 2.42, the following are true.

1. The stack $\mathcal{X}$ is tame.
2. There is a positive integer $N$ such that $N$ is a multiple of the order of each geometric stabilizer group $G_x$ for $x: \text{Spec } k \to \mathcal{X}$. Moreover, it is sufficient to select $N$ to be minimal with this property.

Proof. Statement (1) is because the characteristic of $k$ is zero. Statement (2) follows because the morphism $\mathcal{L} \to \mathcal{X}$ is finite and $\mathcal{X} \to \text{Spec } k$ is proper.

Lemma 2.44. In Situation 2.42, the following are true.

1. The morphism $\bar{p}: X \to \text{Spec } k$ is proper.
2. The stack $\mathcal{X}$ is Noetherian.
3. The space $X$ is Noetherian.
4. If $\mathcal{L}$ is an invertible sheaf on $\mathcal{X}$, then there is an invertible sheaf $L$ on $X$ and an isomorphism

$$\mathcal{L}^N \simeq \pi^* L$$

of sheaves over $\mathcal{X}$.
5. The pushforward functor $\pi_*$ maps coherent sheaves to coherent sheaves.
6. The pullback functor $\pi^*$ maps coherent sheaves to coherent sheaves.
Proof. Statement (1) follows because $\pi : X \to X$ is a universal homeomorphism. Statement (2) follows because $p$ is of finite type over $\text{Spec } k$. Statement (3) follows because $\overline{p}$ is of finite type over $\text{Spec } k$. A proof of Statement (4) is found in [2, Lemma 2.3.7] in the case where all of the stabilizer groups are cyclic, but the same argument extends mutatis mutandis to our general setting. Statement (5) follows because $\pi$ is proper and $X$ is Noetherian (see [7, III, Theorem 3.2.1]). Statement (6) follows because $X$ is Noetherian (see e.g. [19, Theorem 16.3.7 (3)]).

Recall the following result concerning amplitude with respect to proper morphisms (over a Noetherian base).

Lemma 2.45 (Serre’s vanishing criterion [7, III, Proposition 2.6.1]). Let $f : X \to \text{Spec } k$ be a proper morphism of schemes. Let $L$ be an invertible sheaf over $X$. The following conditions are equivalent.

1. The sheaf $L$ is $f$-ample.
2. For each coherent $\mathcal{O}_X$-module $F$, there is an integer $m_0$ such that $H^q(X, F \otimes L^m) = 0$ for $q > 0$ and $m \geq m_0$.

Here is an extension to Situation 2.42.

Lemma 2.46. In Situation 2.42, let $L$ be an invertible sheaf over $X$. The following are equivalent.

1. The sheaf $L$ is $\overline{p}$-ample.
2. The algebraic space $X$ is a scheme and for each coherent sheaf $F$ over $X$, there is an integer $m_0$ such that $H^q(X, F \otimes L^m) = 0$ for $q > 0$ and $m \geq m_0$.

Proof. The implication (2) implies (1) is immediate from Lemma 2.45. Assume (1). Then $X$ is a scheme by [1, Lemma 0D32]. The rest of (2) then follows from Lemma 2.45 again.

Lemma 2.47. In Situation 2.42, let $L$ be an invertible sheaf over $X$. Assume $L$ is $\overline{p}$-ample. For each locally free sheaf $\mathcal{E}$ over $X$, there is an integer $m_0$ satisfying $H^q(X, \pi_* \mathcal{E} \otimes L^m) = 0$ for each $q > 0$ and $m \geq m_0$.

Proof. The sheaf $\mathcal{E}$ is coherent over $X$, and so the sheaf $\pi_* \mathcal{E}$ is coherent over $X$ by Lemma 2.44. It follows that the sheaves $\pi_* \mathcal{E} \otimes L^m$ are coherent over $X$. The statement follows from Lemma 2.46.

3. Twisted Grassmannians

Let $r, n$ be positive integers. Let $\mu = (\mu_0, \ldots, \mu_n)$ be a sequence of nonnegative integer multiplicities with at least one multiplicity strictly positive. Consider the functor which associates to any scheme $T$ the additive group

$$\bigoplus_{i=0}^{n} \text{Hom}_k(\Gamma(T, \mathcal{O}_T)^{\mu_i}, \text{Sym}^i(\Gamma(T, \mathcal{O}_T)^*))).$$

This is representable by an affine scheme which we denote by $\mathbb{A}(r, \mu)$. Moreover, the affine scheme $\mathbb{A}(r, \mu)$ is an additive group scheme over $k$.

The affine group scheme $GL_r$ naturally acts on $\mathbb{A}(r, \mu)$ over $k$ via a morphism of schemes

$$\sigma(r, \mu) : GL_r \times \mathbb{A}(r, \mu) \to \mathbb{A}(r, \mu).$$
satisfying the property that for each scheme $T$, the corresponding map of sets

$$GL_r(T) \times \mathbb{A}(r, \mu)(T) \longrightarrow \mathbb{A}(r, \mu)(T)$$

is induced by viewing each

$$\text{Hom}_k(\Gamma(T, \mathcal{O}_T)^{\mu_i}, \text{Sym}^i(\Gamma(T, \mathcal{O}_T)^r)) \cong \left(\Gamma(T, \mathcal{O}_T)^{\mu_i}\right)^\vee \otimes_k \text{Sym}^i(\Gamma(T, \mathcal{O}_T)^r)$$

as a $GL_r(T)$-set via the trivial action of $GL_r(T)$ on $\Gamma(T, \mathcal{O}_T)^{\mu_i}$ and the action of $GL_r(T)$ on $\text{Sym}^i(\Gamma(T, \mathcal{O}_T)^r)$ induced by the fundamental action of $GL_r(T)$ on $\Gamma(T, \mathcal{O}_T)^r$.

In addition, for each nonnegative integer $m$, there is an action

$$\sigma(r, \mu, m) : GL_r \times \mathbb{A}(r, \mu) \longrightarrow \mathbb{A}(r, \mu)$$

which is obtained from $\sigma(r, \mu)$ by twisting by $(\det)^{\otimes m}$.

Let $A(r, \mu, m)$ denote the resulting quotient stack

$$A(r, \mu, m) = [\mathbb{A}(r, \mu)/GL_r]$$

over $k$.

The determinant morphism $\det : GL_r \rightarrow G_m$ determines a $GL_r$-linearization of the structure sheaf $\mathcal{O}_{\mathbb{A}(r, \mu)}$ via Example 2.7. Let $\mathcal{L}_0$ denote the corresponding invertible sheaf over $A(r, \mu, m)$. Denote by $A(r, \mu, m)_{\mathcal{L}_0}^{ss}$ the open substack corresponding to the locus of semistable geometric points as in Definition 2.2.

**Definition 3.1.** Define the twisted Grassmann stack associated to the data $(r, \mu)$ to be the open substack

$$G(r, \mu, m) = A(r, \mu, m)_{\mathcal{L}_0}^{ss}$$

of $A(r, \mu, m)$ over $k$.

**Lemma 3.2.** The twisted Grassmann stack $p : G(r, \mu, m) \rightarrow \text{Spec } k$ admits a good moduli space

$$\pi : G(r, \mu, m) \longrightarrow G(r, \mu, m)$$

where $G(r, \mu, m)$ is an open subscheme of $\text{Proj } \bigoplus_{d \geq 0} H^0(G(r, \mu, m), \mathcal{L}_0^d)^{GL_r}$.

**Proof.** See Lemma 2.27 (a). $\square$

**Example 3.3.** For example, if $r = 1$, then $G(1, \mu, m)$ is the weighted projective stack

$$\mathbb{P}(m, \ldots, m, m+1, \ldots, m+n)$$

as in [2]. The good moduli space $G(1, \mu, m)$ is the weighted projective space

$$G(1, \mu, m) = \mathbb{P}(m, \ldots, m, m+1, \ldots, m+n)$$

**Example 3.4.** As another example, suppose $n = 1$, $\mu_0 = 0$, and $r < \mu_1$. Then the affine scheme $A(r, \mu)$ is the one whose $k$-valued points can be identified with the vector space $\text{Hom}_k(k^{\mu_1}, k^r)$. For $m = 0$, at the level of $k$-points, the action of $GL_r$ is just post-composition, and the semistable locus consists of those linear maps with full rank. It follows that the quotient stack $G(r, \mu, 0)$ can be identified with the Grassmann variety of $r$-planes in $A^{\mu_1}$ (see e.g. [10, Example 2.14]).
4. Embedding statement

In this section, we consider the following situation.

**Situation 4.1.** In Situation 2.42 let $\mathcal{V}$ be a locally free sheaf of rank $r$ over $X$. Make the following assumptions.

1. The sheaf $\mathcal{V}$ is $\pi$-ample.
2. There is a $\bar{\pi}$-ample invertible sheaf $\mathcal{L}$ over $X$ such that $(\det \mathcal{V})^N \simeq \pi^* \mathcal{L}$, where $N$ is the smallest positive integer which is a multiple of the order of each stabilizer group $G_x$ for $x : \text{Spec } k \to X$.

Then, because $\mathcal{V}$ is $\pi$-ample, the frame bundle $\text{Fr}(\mathcal{V})$ is representable by an algebraic space (Lemma 2.40). In fact, because $\mathcal{L}$ pulls back to an ample sheaf over $\text{Fr}(\mathcal{V})$, the frame bundle is representable by a scheme (according to [1, Lemma 69.14.12]). Call this scheme $Y$. It enjoys a natural morphism $q : Y \to X$.

For each suitably large positive integer $m$, there is a morphism of schemes

$$f_m : Y \to \mathcal{A}(r, \mu(m)),$$

where $\mathcal{A}(r, \mu(m))$ is a certain affine scheme as described as in (2). Indeed, the sheaf

$$\bigoplus_{i=0}^N \text{Sym}^i \mathcal{V}$$

is generating (see Lemma 2.39). For each $0 \leq i \leq N$, define the sheaf

$$V_i(m) := p_* (X, \text{Sym}^i \mathcal{V} \otimes \pi^* \mathcal{L}^m) = \bar{p}_* \pi_* (\text{Sym}^i \mathcal{V} \otimes \mathcal{L}^m)$$

over Spec $k$, where the second equality follows from Lemma 2.23. The sheaf $V_i(m)$ corresponds to a free $k$-module of rank $\mu(m)$, or equivalently a $\mu(m)$-dimensional vector space over $k$, for some non-negative integer $\mu(m)$. There is an isomorphism of affine spaces over $k$

$$\mathcal{V} \left( \bigoplus_{i=0}^N \mathcal{H}om(V_i(m), \text{Sym}^i \mathcal{O}_{\text{Spec } k}) \right) \simeq \mathcal{A}(r, \mu(m))$$

where $\mu(m) = (\mu_0(m), \ldots, \mu_N(m))$ and where $\mathcal{A}(r, \mu(m))$ is defined as in (2). Then using Lemma 2.34 we obtain the morphism (4).

According to Lemma 2.31 the morphism is $GL_r$-equivariant, and thus it descends to a morphism of stacks

$$F_m : X \to \mathcal{A}(r, \mu(m))$$

over Spec $k$, where $\mathcal{A}(r, \mu(m)) = [\mathcal{A}(r, \mu(m))/GL_r]$ as in [3]. In summary, we have, for each $m$, a Cartesian square of stacks

$$\begin{array}{ccc}
Y & \xrightarrow{f_m} & \mathcal{A}(r, \mu(m)) \\
\downarrow q & & \downarrow \\
X & \xrightarrow{F_m} & \mathcal{A}(r, \mu(m))
\end{array}$$

over Spec $k$. 

The next section will be devoted to proving that the morphism (4) is an embedding of schemes for \( m \) sufficiently large, as stated in the following lemma.

**Lemma 4.2.** In Situation 4.1, the morphism (4) of schemes is an embedding for \( m \) sufficiently large.

Assuming Lemma 4.2 for now, we obtain the following main result.

**Theorem 4.3.** In Situation 4.1, for \( m \) sufficiently large, the morphism (5) of stacks factors through the semistable locus and thus determines an embedding of stacks \( X \rightarrow G(r, \mu(m)) \).

**Proof.** According to Example 2.26, it suffices to show that the morphism (4) factors through the semistable locus. Because there is a \( \bar{p} \)-ample invertible sheaf \( L \) over \( X \) such that \((\det V)^m \cong \pi^* L\) we can guarantee that, for \( m \) suitably large, the pushforward \( V_0(m) = p_*((\det V)^m) \) is nontrivial, say of rank \( \ell \). Decompose the morphism \( f_m \) into

\[
 f_m = (f_{m,0}, \ldots, f_{m,N})
\]

where

\[
 f_{m,i} : Y \rightarrow \mathcal{H}om(V_i(m), \text{Sym}^i \mathcal{O}_{\text{Spec} k}).
\]

In particular, we may identify \( f_{m,0} \) with

\[
 f_{m,0} : Y \rightarrow \mathbb{A}^\ell.
\]

The group scheme \( \text{GL}_r \) acts on \( \mathbb{A}^\ell \) via multiplication by \((\det \rho)^m \), where \( \rho \) denotes the fundamental representation of \( \text{GL}_r \). If \( z_1, \ldots, z_\ell \) denote coordinates on \( \mathbb{A}^\ell \), then each describes a \( \text{GL}_r \)-equivariant section of the equivariant line bundle described by the structure sheaf together with the \( \text{GL}_r \)-linearization determined by the determinant character.

Let \( y \) be a point of \( Y \). Let \( x \) denote the image of \( y \) under the quotient morphism \( q : Y \rightarrow X \). There is a section \( s \) of \((\det V)^m \) such that \( s(x) \neq 0 \). As a result, the image \( f_{m,0}(y) \) is nonzero. There is a coordinate \( z_j \) such that \( z(f_{m,0}(y)) \neq 0 \). The subset \( \mathbb{A}(r, \mu(m))_{z_j} \) is affine. Hence \( f_m(y) \) is semistable in \( \mathbb{A}(r, \mu(m)) \). □

5. Embedding proof

This section is devoted to proving Lemma 4.2. Throughout, we assume we are in Situation 4.1.

Let us set up some convenient notation. Let \( x : \text{Spec} k \rightarrow X \) be a closed point of \( X \) with stabilizer group \( G_x \). Then \( V_x \) is a faithful representation of \( G_x \), and, as a result, \( G_x \) acts on the right on the \( k \)-module

\[
 \text{Hom}_k(V_x, k^r).
\]

On the other hand, this \( k \)-module also enjoys a left action by the group scheme \( \text{GL}_r \) described by post-composition.

At the same time, the point \( x \) determines a Cartesian square of stacks

\[
 \text{Isom}_k(V_x, k^r)/G_x \rightarrow Y.
\]

\[
 BG_x \rightarrow X.
\]
Let $J_x$ denote the ideal sheaf of $O_X$ determined by $BG_x$. For any locally free sheaf $E$ over $X$, there is a long exact sequence in cohomology, a portion of which may be identified with

$$H^0(X, E) \to H^0(BG_x, E_x) \to H^1(X, J_x \otimes E).$$

**Lemma 5.1.** There is a positive integer $m_0$ such that for each geometric point $x : \text{Spec } k \to X$, we have

$$H^1(X, J_x \otimes (\det V)^m \otimes \text{Sym}^i V) = 0 \quad 0 \leq i \leq N \text{ and } m \geq m_0.$$  

**Proof.** For each geometric point $x$, Lemma 2.47 guarantees the existence of a positive integer $m_0(x)$ satisfying

$$H^1(X, J_x \otimes (\det V)^m \otimes \text{Sym}^i V) = 0 \quad 0 \leq i \leq N \text{ and } m \geq m_0(x).$$

The fact that $X$ is proper implies that we may choose $m_0$ uniformly. □

For a closed point $x : \text{Spec } k \to X$, define the affine scheme

$$A_x(m) = \bigoplus_{i=0}^N A_i^x(m)$$

where $A_i^x(m)$ is the affine scheme whose $k$-points can be identified with

$$\text{Hom}_k([(\det V_x)^m \otimes \text{Sym}^i V_x]^{G_x}, \text{Sym}^i(k^r)).$$

The group scheme $GL_r$ acts on $A_i^x(m)$ on the left by the action described by $(\det \rho)^m \otimes \rho^i$ where $\rho$ is the fundamental representation. In particular, for $i = 0$, we have an identification of $k$-modules

$$[(\det V_x)^m \otimes \text{Sym}^0 V_x]^{G_x} \simeq k,$$

and so the affine scheme $A_0^x(m)$ is identified with 1-dimensional affine space $A^1$.

**Lemma 5.2.** There is a positive integer $m_0$ such that for each geometric point $x : \text{Spec } k \to X$, the natural morphisms

$$A_x(m) \to \mathbb{A}(r, \mu(m)) \quad m \geq m_0$$

are injective.

**Proof.** There is a positive integer $m_0$ such that the natural maps

$$H^0(X, (\det V)^m \otimes \text{Sym}^i V) \to H^0(BG_x, (\det V)^m \otimes \text{Sym}^i V)$$

are surjective for $0 \leq i \leq N$ and $m \geq m_0$. At the level of $k$-points, we have an identification

$$A_i^x(m) = \text{Hom}_k(H^0(BG_x, (\det V)^m \otimes \text{Sym}^i V)), \text{Sym}^i(k^r))$$

and so the lemma follows. □

As a result, for each geometric point $x : \text{Spec } k \to X$, the restriction of the morphism $[4]$ to the fiber over $x$ fits into a commutative diagram

$$\text{Isom}_k(V_x, k^r)/G_x \xrightarrow{f_{m,x}} A_x(m) \quad \xrightarrow{f_m} \mathbb{A}(r, \mu(m))$$
Lemma 5.3. There is a positive integer $m_0$ such that for each geometric point $x : \text{Spec } k \to X$, the morphisms

$$f_{m,x} : \text{Isom}_k(V_x, k^r)/G_x \to A_x(m) \quad m \geq m_0$$

are locally closed immersions.

Proof. The affine scheme $V_x^\vee$ can be identified with $\text{Spec } k[p_1, \ldots, p_r]$. In this way, we have an identification of $k$-algebras

$$\text{Sym}^\bullet V_x \simeq k[p_1, \ldots, p_r].$$

The action of $G_x$ on $V_x$ corresponds to one on $\text{Spec } k[p_1, \ldots, p_r]$. The invariant subring is finitely generated, so there are a finite number of polynomials $t_\ell = t_\ell(p_1, \ldots, p_r)$ such that $1 \leq \ell \leq R$

$$k[p_1, \ldots, p_r]^{G_x} \simeq k[t_1, \ldots, t_R].$$

Moreover, we may arrange that each $t_\ell$ is homogeneous of degree $d_\ell$ and the degrees are arranged in ascending order $d_1 \leq \cdots \leq d_R$ with the highest degree being at most the order of $G_x$ (see [15]), which is a number that divides $N$.

As a result, each $t_\ell$ can be considered as an element of $(\text{Sym}^d V_x)^{G_x}$. Moreover, for each $d$, a basis for $(\text{Sym}^d V_x)^{G_x}$ may be identified with a subset of monomials of the form

$$t^\alpha := t_1^{\alpha_1} \cdots t_R^{\alpha_R}$$

with $\alpha_\ell \geq 0$ satisfying

$$d_1 \alpha_1 + \cdots + d_R \alpha_R = d.$$  

(The basis may not consist of the full set of monomials satisfying (6) because there could be relations among the $t_\ell$s.) In addition, we may ensure that the basis for $(\text{Sym}^d V_x)^{G_x}$ includes the monomial $t_\ell$.

The action of $G_x$ on $\text{Hom}_k(V_x, k^r)$ is compatible with the decomposition

$$\text{Hom}_k(V_x, k^r) \simeq \bigoplus_{j=1}^r V_x^{\vee}$$

of $k$-modules. For each $j = 1, \ldots, r$, identify the $j$th summand with

$$\text{Spec } k[p_{i1}, \ldots, p_{ij}]$$

so that the $p_{ij}$ describe global coordinates on the affine scheme $\text{Hom}_k(V_x, k^r)$. For each $j$, the group $G_x$ acts on the affine scheme $\text{Spec } k[p_{i1}, \ldots, p_{ij}]$. If we set

$$t_{ij} = t_\ell(p_{i1}, \ldots, p_{ij}),$$

then we have an identification

$$k[p_{ij}]^{G_x} = k[t_{ij}].$$

The subscheme $\text{Isom}_k(V_x, k^r)$ can be identified with the affine scheme

$$\text{Spec } k[p_{ij}, 1/\det(p_{ij})].$$
There is a smallest positive integer $a$ such that $(\det p_{ij})^a$ is invariant under $G_x$. Moreover, the number $a$ divides $N$. There is a polynomial $D = D(t_{ij})$ such that $D = (\det p_{ij})^a$. We have an identification of affine schemes

$$\text{Isom}_k(V_x, k^r)/G_x \simeq \text{Spec } k[t_{ij}, 1/D].$$

There is a positive integer $b$ such that $ab = N$ so that $D^{bm} = (\det p_{ij})^{mN}$. Notice that $D^{bm}$ can be identified with an element of the $k$-module $(\det V_x)^{mN}$. Let $e_1, \ldots, e_r$ denote a dual basis for $k^r$. A dual basis for $\text{Sym}^d(k^r)$ can then be described by

$$e^\beta := e_1^{\beta_1} \cdots e_r^{\beta_r}$$

where $\beta_j \geq 0$ satisfy

$$\beta_1 + \cdots + \beta_r = d. \quad (7)$$

Consider the dual space of $A_x(m)$. This space can be identified with the $k$-module

$$\bigoplus_{d=0}^N (\det V_x)^{mN} \otimes \text{Hom}_k(\text{Sym}^d(k^r), (\text{Sym}^d V_x)^{G_x}).$$

With our notation, the $d$th piece admits a basis described by

$$D^{bm} \otimes e^\beta \otimes t^\alpha$$

where (6) and (7) hold. In particular, the zeroth piece is one-dimensional spanned by the element

$$z^0 := D^{bm} \otimes e^0 \otimes t^0.$$

Let $U_0$ denote the open subset of $A_x(m)$ where $z^0$ does not vanish. According to the proof of Theorem 4.3, the morphism

$$f_{m,x} : \text{Isom}_k(V_x, k^r)/G_x \longrightarrow A_x(m)$$

factors through $U_0$.

If we restrict our attention to this subscheme, where $z^0$ is invertible, then we may obtain each $t_{ij}$ as a linear combination of

$$f_{m,x}'(D^{bm} \otimes e^{\beta(j, \ell)} \otimes t^{\alpha(\ell)})$$

where

$$\beta(j, \ell) = (0, \ldots, 0, d_{\ell}, 0, \ldots, 0)$$

and

$$\alpha(\ell) = (0, \ldots, 0, 1, 0, \ldots, 0).$$

The result follows.

**Lemma 5.4.** Given distinct closed points $y, y' : \text{Spec } k \rightarrow Y$ of $Y$, there is a positive integer $m_0(y, y')$ such that for each $m \geq m_0(y, y')$ the morphism

$$f_m : Y \longrightarrow \mathbb{A}(r, \mu(m))$$

satisfies $f_m(y) \neq f_m(y')$. In addition, there are open neighborhoods $U$ and $U'$ of $q(y)$ and $q(y')$ respectively in $\mathcal{X}$, such that for each pair of distinct points $z \in q^{-1}(U)$ and $z' \in q^{-1}(U')$ and for each $m \geq m_0(y, y')$, we have $f_m(z) \neq f_m(z')$. 

□
Proof. If \( y, y' \) lie in the same \( GL_r \)-orbit, then they map to the same point of \( \mathcal{X} \), and we may choose \( m_0(y, y') \) from Lemma 5.5. Suppose \( y, y' \) do not lie in the same orbit. Then they determine distinct points \( x, x' \) of \( \mathcal{X} \). Let \( J_{x,x'} \) denote the ideal sheaf of \( \mathcal{O}_X \) corresponding to their union. For each \( i = 0, \ldots, N \), there is a positive integer \( m_0(y, y', i) \) such that

\[
H^1(\mathcal{X}, J_{y,y'} \otimes \mathcal{L}^{mN} \otimes \text{Sym}^i \mathcal{V}) = 0 \quad m \geq m_0(y, y', i).
\]

Let \( m_0(y, y') \) denote the maximum of these integers. Then the map

\[
H^0(\mathcal{X}, (\text{det} \mathcal{V})^{mN} \otimes \text{Sym}^i \mathcal{V}) \to (\text{Sym}^i \mathcal{V}_x)^{G_x} \oplus (\text{Sym}^i \mathcal{V}_{x'})^{G_{x'}}
\]

is surjective for each \( i = 0, \ldots, N \). Because \( \mathcal{V} \) is \( \pi \)-ample, there is an \( i \) such that

\[
(\text{Sym}^i \mathcal{V}_x)^{G_x} \neq 0.
\]

It follows that there is a section \( s \) of \( \mathcal{L}^{mN} \otimes \text{Sym}^i \mathcal{V} \) such that \( s(x) \neq 0 \) but \( s(x') = 0 \). This means that \( f_m(y) \neq f_m(y') \) for each \( m \geq m_0(y, y') \).

In fact, notice that the choice of \( m(y, y') \) depends only on the images \( x, x' \) of \( y, y' \) in \( \mathcal{X} \). The other statement of the lemma then follows because \( f_m \) is continuous. \( \square \)

**Lemma 5.5.** For sufficiently large \( m \), the morphism

\[
f_m : Y \to \mathbb{A}(r, \mu(m))
\]

is injective on closed points.

**Proof.** Using Lemma 5.4, the properness of \( \mathcal{X} \) allows us to cover \( \mathcal{X} \times \mathcal{X} \) by a finite number of open neighborhoods \( U_1, \ldots, U_{\ell} \) together with a finite number of positive integers \( m_{0,1}, \ldots, m_{0,\ell} \) such that the corresponding morphisms \( f_1, \ldots, f_{\ell} \) determined by these integers are injective on \( U_1, \ldots, U_{\ell} \) respectively. Note that if the morphism \( f \) determined by \( m \) separates \( x \) and \( y \), then any morphism determined by \( m' \geq m \) will also separate \( x \) and \( y \). As a result, we may take \( m_0 \) to be the maximum of \( m_{0,1}, \ldots, m_{0,\ell} \). \( \square \)

**Lemma 5.6.** Given a closed point \( y : \text{Spec} \, k \to Y \) of \( Y \), there is a positive integer \( m_0(y) \) such that for each \( m \geq m_0(y) \), the morphism of stalks

\[
f^i_{m,y} : \mathcal{O}_{\mathcal{X}(r, \mu(m)), f_m(y)} \to f_{m,*} \mathcal{O}_{\mathcal{Y}, y}
\]

is surjective. In addition, there is a neighborhood \( U \) of \( q(y) \) in \( \mathcal{X} \) such that for each \( z \in q^{-1}(U) \) and each \( m \geq m_0(y) \), the morphism \( f^i_{m,z} \) is surjective.

**Proof.** Let \( \mathfrak{m}_y \) denote the maximal ideal in \( \mathcal{O}_{\mathcal{Y}, y} \). Let \( x \) denote the image of \( y \) in \( \mathcal{X} \), and let \( J_x \) denote the corresponding ideal sheaf of \( \mathcal{O}_X \). For each locally free sheaf \( \mathcal{E} \) over \( \mathcal{X} \), there is an exact sequence in cohomology, a portion of which is

\[
H^0(\mathcal{X}, J_x \otimes \mathcal{E}) \to (\mathfrak{m}_y/\mathfrak{m}_y^2 \otimes \mathcal{E}_x)^{G_x} \to H^1(\mathcal{X}, J_x^2 \otimes \mathcal{E}).
\]

Using \( \mathcal{E} = (\text{det} \mathcal{V})^{mN} \otimes \text{Sym}^i \mathcal{V} \), Lemma 2.47 gives, for each \( i \), a positive integer \( m_0(y, i) \) such that

\[
H^1(\mathcal{X}, J_x^2 \otimes (\text{det} \mathcal{V})^{mN} \otimes \text{Sym}^i \mathcal{V}) = 0 \quad m \geq m_0(y, i).
\]

Let \( m_0(y) \) denote the maximum of the \( m_0(y, i) \). Because \( G_x \) is linearly reductive, we may decompose \( \mathfrak{m}_y/\mathfrak{m}_y^2 \) into irreducible representations

\[
\mathfrak{m}_y/\mathfrak{m}_y^2 = \bigoplus_{\ell} W_\ell.
\]
Using that $\mathcal{V}$ is $\pi$-ample, for each $\ell$, there is a positive integer $i_{\ell}$ such that

$$(W_\ell \otimes (\det \mathcal{V})^{mN} \otimes \text{Sym}^i \mathcal{V}_x)^G_x \neq 0.$$ 

As a result, upon writing $a = f_m(y)$, the morphism of $k$-modules

$$m_{a}/m_{a}^2 \longrightarrow m_{y}/m_{y}^2$$

is surjective. It follows that the morphism of local rings

$$\mathcal{O}_{k(r,\mu(m))}.a \longrightarrow f_m,^*\mathcal{O}_{Y,y}$$

is surjective too (see e.g. [1, Lemma 0E8M]).

The other statement of the lemma follows from the fact that $f_{m,y}^*$ being surjective represents an open condition on $X$. □

**Lemma 5.7.** For sufficiently large $m$, the morphism

$$f_m^*: \mathcal{O}_{k(r,\mu(m))} \longrightarrow f_m,^*\mathcal{O}_{Y}$$

is surjective.

**Proof.** The properness of $X$ allows us to cover $X$ by a finite number of open neighborhoods $U_1, \ldots, U_\ell$ together with a finite number of positive integers $m_{0,1}, \ldots, m_{0,\ell}$ such that the corresponding morphisms $f_1, \ldots, f_\ell$ satisfy the property that $f_{m_j,y}^*$ is surjective for each $y \in U_j$. Note that if the morphism $f_{m,y}^*$ is surjective, then $f_{m',y}^*$ is surjective for each $m' \geq m$. As a result, we may take $m_0$ to be the maximum of $m_{0,1}, \ldots, m_{0,\ell}$. □

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