$H_T$- Vertex Algebras
and
the Infinite Toda Lattice
M.J. Bergvelt
## Contents

Chapter 1. Introduction 5  
1. Classical Vertex Algebras and the Hopf Algebra $H_D$ 5  
2. Gelfand-Dickey Structures and the Hopf Algebra $H_D$ 7  

Chapter 2. The Hopf Algebra $H_T$ and Sequences 9  
1. Introduction 9  
2. The Hopf Algebra $H_T$ 10  
3. The Dual of $H_T$ and Sequences 10  
4. The Hopf Dual of $H_T$ and Difference Equations 11  
5. The antipodal sequences $\tau(\ell)$ 14  
6. The Dual Difference Basis 15  
7. Localization for $C_Z^{pol}$ 16  
8. $\hat{H}_T$ and $\hat{H}_T^+$ 17  
9. Action of $C_Z[[\sigma]]$ on $\hat{H}_T$ 18  
10. Action of $C_Z[[\sigma]]$ on $K_{\text{Sing}}$ 19  
11. Trace for $K$ 21  
12. Orthogonality Relations 22  
13. Expansions in $K$ 23  
14. (Twisted) Coproduct and (Twisted) Exponential Operators 23  
15. $H$-Covariance of (Twisted) Exponentials 25  
16. Multiplicativity of (Twisted) Exponentials 26  
17. $H_T$-Leibniz Algebras and Exponentials 26  
18. Inverses of exponentials. 27  
19. Adjoint action of exponentials 28  
20. Multivariable expansions 29  
21. Extension of Exponential Distributions 30  
22. Localization of (Twisted) Coproducts and Dirac Distributions 33  
23. Difference Operators 34  
24. Expansions of Distributions 35  
25. Some Properties of Distributions 37  
26. Distributions of the form $P^{\frac{1}{2}}_{K_T}$ 39  
27. Rationality of Distributions on $K$ 41  
28. Rationality of Multivariable Distributions 42
Chapter 3. Singular Hamiltonian Structures 45
1. Overview 45
2. Spectrum and Singular Points 45
3. Classical Fields 46
4. Expansions of Classical Fields 47
5. Classical Fields and Affinization 47
6. Multivariable Classical Fields 48
7. Derivations and Poisson Structures 49
8. Evolutionary Derivations and Singular Poisson Brackets 50
9. Projection from $V(\tau_1, \tau_2)$ to $V$ 53
10. Free Difference Algebras 55

Chapter 4. [ 59
1. The Finite Toda Lattice 59
2. Lax Operator and $r$-Matrix 60

Chapter 5. $H_T$-Conformal Algebras 67
1. Introduction 67
2. $H_T$-Conformal Algebras and $H_T$-Vertex Poisson Algebras. 67
3. The Lie Algebra of a Conformal Algebra 71
4. Singular Vertex operators 72
5. Holomorphic $H_T$-vertex algebras 74
6. Extension of $H_T$-conformal Structure to the Affinization 75

Chapter 6. $H_T$-Vertex Algebras. 81
1. Introduction 81
2. Fields 81
3. Normal Ordered Products and Dong’s Lemma 83
4. State-Field Correspondence and Vacuum Axioms 86
5. Definition of $H_T$-Vertex Algebras 87
6. First Properties of $H_T$-vertex Algebras 88
7. Uniqueness and Normal Ordered Products 91
8. Alternative Axiomatization 92
9. Existence 94
10. Affine $H_T$-vertex algebras 94
11. Toda Vertex Algebra 95

Bibliography 97
CHAPTER 1

Introduction

1. Classical Vertex Algebras and the Hopf Algebra $H_D$

In this section we recall what a vertex algebra in the usual sense is and indicate how it is related to a Hopf algebra $H_D$. See e.g., Frenkel-Lepowsky-Meurman ([FLM88]) or Kac ([Kac98] for more details on classical vertex algebras and Borcherds ([Bor98]) on the idea of emphasizing the role of the Hopf algebra, cf., also Snydal ([Sny]).

Briefly, a vertex algebra structure on a vector space $V$ is a map $Y: V \otimes V \to V((z))$ associating to a pair of vectors $a, b$ in $V$ a formal Laurent series $Y(a \otimes b, z)$ with values in $V$. As usual we write $Y(a \otimes b, z) = Y(a, z)b$ to obtain the vertex operator $Y(a, z) = \sum a(n)z^{-n-1}$ with the $a(n)$ linear maps $V \to V$. Also an essential ingredient is the infinitesimal translation operator $D: V \to V$, such that $[D, Y(a, z)] = \partial_z Y(a, z)$. These ingredients satisfy a number of axioms that we will not recall at this point. One of the properties of a vertex algebra is that two vertex operators satisfy a commutator formula of the form

$$[Y(a, z), Y(b, w)] = \sum_{n=0}^{N} Y(a(n)b, w) \partial_{w}^{(n)} \delta(z, w),$$

where $\delta(z, w) = \sum_{k \in \mathbb{Z}} z^{k}w^{-k-1}$ is the formal Dirac delta function, and for any linear operator $D$ we write $D^{(n)} = \frac{D^n}{n!}$.

The simplest nontrivial example is given by the vertex algebra of the Heisenberg algebra, the Lie algebra spanned by a central element $c$ and elements $b(n), n \in \mathbb{Z}$, with relations

$$[b(m), b(n)] = m \delta_{m+n} c.$$  

The vector space $V = \mathbb{C}[b(-n-1)]_{n \geq 0}$ is then a vertex algebra, with basic vertex operator the free boson $b(z) = Y(b(-1), z) = \sum_{n \in \mathbb{Z}} b(n)z^{-n-1}$. All other vertex operators in $V$ are obtained by taking linear combinations of normal ordered products of the free boson. The commutator of the free boson with itself is

$$[b(z), b(w)] = \partial_{w} \delta(z, w).$$
Let $H_D = \mathbb{C}[D]$ be the algebra of polynomials in the linear operator $D$. So by definition every vertex algebra is a module over $H_D$. We can think of $H_D$ as the universal enveloping algebra of the 1-dimensional Abelian Lie algebra with basis element $D$. As such $H_D$ is a Hopf-algebra, with coproduct $\Delta_H : D \mapsto D \otimes 1 + 1 \otimes D$, antipode $S : D \mapsto -D$ and counit $\varepsilon : D \mapsto 0$. Now the dual Hopf algebra of $H_D$ is the Hopf algebra $H_D^* = \mathbb{C}[[t]]$ of formal power series in a variable $t$, the functions on the formal disk. Note that the map $Y$ in a vertex algebra structure takes values in formal Laurent series, which are the singular functions on the formal disk. In the same spirit note that the commutator is singular, in the sense that we cannot put $z = w$ in this formula. The delta function can be obtained from the difference of two expansions of the basic singularity $\frac{1}{z-w}$.

The occurrence of $H_D$ and the singularities in its dual is no coincidence but the tip of an iceberg. Borcherds, [Bor98] has a general construction of what he calls $G$-vertex algebras. The construction is based on the notion of a vertex group $G$, which is, roughly speaking,

- the choice of a (cocommutative) Hopf algebra $H$,
- the choice of singularities; this is in examples some localization $K$ of the dual (commutative) algebra $H^*$.

The classical notion of a vertex algebra corresponds to the choice of $H = H_D$ and $K = K_D = \mathbb{C}[[t]][t^{-1}]$. Borcherds’ work and examples are very interesting, but lacks the level of detail we know about the usual vertex algebras, as developed for instance in Frenkel, Lepowsky and Meurman ([FLM88]) or Kac ([Kac98]).

In this paper we study in detail a new class of vertex algebras. These are non-trivial examples of $G$-vertex algebras of the sort envisioned by Borcherds. We choose as a Hopf algebra not, as in the classical case, the universal enveloping algebra $H_D = \mathbb{C}[L]$ of an 1-dimensional Abelian Lie algebra $L = \mathbb{C}D$, but the group algebra

$$H_T = \mathbb{C}A = \mathbb{C}[T, T^{-1}]$$

(1.3)

of a free rank one Abelian group $A = \langle T \rangle$ generated by an element $T$. The dual algebra is the algebra $\mathbb{C}_\mathbb{Z}$ of functions on the integers, and we have found a natural way to introduce singularities for $\mathbb{C}_\mathbb{Z}$ by localizing to get an algebra $K$. This leads to a class of what we call $H_T$-vertex algebras. These occurs naturally in the theory of the infinite Toda Lattice. To explain this we first recall some facts about Gelfand-Dickey Hamiltonian structures, and how they are related to classical vertex algebras.
2. Gelfand-Dickey Structures and the Hopf Algebra $H_D$

In the simplest case one wants to describe an infinite hierarchy of differential equations for a single unknown $v$ depending on spatial variable $x$ and infinitely many time variables $t_i$, $i > 0$. For instance, the Modified Korteweg-de Vries (mKdV) hierarchy is a system of differential equations of the form

$$\partial_t v = f_i(v,v',\ldots), \quad v' = \partial_x v, \quad i > 0,$$

where $f_i$ is a differential polynomial, i.e., an element of the polynomial algebra $V = \mathbb{C}[v^{(n)}]_{n \geq 0}$. This algebra is a module over $H_D$ via $Dv^{(n)} = v^{(n+1)}$. $V$ is a commutative algebra, and its (maximal) spectrum $\text{Specm}(V) = \text{Hom}_{\mathbb{C}-\text{alg}}(V, \mathbb{C})$ can be identified with $\mathbb{C}[[t]]$. Indeed, a formal power series $s \in \mathbb{C}[[t]]$ acts on an element $f = f(v,v^{(1)},\ldots) \in V$ by

$$s(f) = \text{Res}\left(\frac{f(s,s',\ldots)}{t}\right), \quad s' = \partial_t s,$$

and we have $s(fg) = s(f)s(g)$. Of course, usually we will think of $f$ as a function on the phase space $\text{Specm}(V)$, and we will write $f[s] = s(f)$. More generally, one can define functionals

$$(2.2) \quad f_{(n)}[s] = \text{Res}(f(s,s',\ldots)t^n)$$

so that $f[s] = f_{(-1)}[s]$. For $n \geq 0$ the functionals $f_{(n)}$ are identically zero on $s \in \mathbb{C}[[t]]$, so we will from now on allow $s \in \mathbb{C}((t))$. The reader will be excited to note that here the same space of singular functions on the formal disk appears as in the previous section in the context of vertex algebras. But there is more . . .

Define the Classical Field $C(f,z)$ of $f \in V$ as the generating series of functionals

$$C(f,z) = \sum_{n \in \mathbb{Z}} f_{(n)}z^{-n-1}.$$ 

A Gelfand-Dickey Hamiltonian structure on $V$ is then a prescription to define the Poisson brackets of functionals $f_{(n)}$, or, equivalently, for the corresponding classical fields. For a precise definition (of the Poisson bracket $\{f_{(0)},g_{(0)}\}$ of the residues of the classical fields) see Dickey’s book ([Dic03]).

The simplest nontrivial example is, perhaps, the Hamiltonian structure for the modified Korteweg-de Vries hierarchy. In this case the Poisson bracket for two functionals is given by

$$\{f_{(m)},g_{(n)}\} = \text{Res}\left(D\left(\frac{\delta f_{(m)}}{\delta v}\right)\frac{\delta g_{(n)}}{\delta v}\right),$$
where the variational derivative of a functional is given by
\[
\frac{\delta f(m)}{\delta v} = \sum_{i=0}^{\infty} (-D)^i \left( \frac{\partial f}{\partial v(i)} \otimes t^m \right), \quad D = D_V + \partial_t.
\]
Then one checks that the Poisson bracket of the basic classical field \( C(v,z) \) is given by
\[
\{ C(v,z), C(v,w) \} = \partial_w \delta(z,w).
\]
More generally, the Poisson bracket of two classical fields will be a linear combination of derivatives of delta functions, with classical fields as coefficients. Of course, this is very similar to the commutator formula (1.2) for free bosons.

The basic reason \( H_D \) appears in the theory of equations of the form (2.1) is that if \( v(x) \) solves this system, also the infinitesimal translate \( v(x + \delta) \) will solve it. In the theory of Toda lattices one studies equations of the following form (or the obvious multicomponent generalization)
\[
\partial_t x_n = f_j(\ldots, x_{n-1}, x_n, x_{n+1}, \ldots),
\]
where \( x = (x_n(t))_{n \in \mathbb{Z}} \) is a sequence (of functions of the times \( t_j, j > 0 \)) and \( f_j \) is a polynomial in the entries of the sequence \( x \). In this case we get as basic symmetry invariance under shifts: if \( x(t) \) solves (2.3) then also the shifted sequence \( T x := (x_{n+1}(t))_{n \in \mathbb{Z}} \) is a solution. In this case the symmetry algebra is the Hopf algebra \( H_T \) of (1.3).

First we will develop some of the machinery of the Hopf algebra \( H_T \) in the next chapter. Then we use this to define in chapter 3 the notion of a Singular Hamiltonian structure and the associated Poisson brackets. As an example we discuss the Hamiltonian structure for the infinite Toda lattice in chapter 4. Then we define a generalization of singular Hamiltonian structures called \( H_T \)-conformal algebras in 5. Finally we define a class of vertex algebras associated to \( H_T \) in chapter 6 and discuss some simple examples.
CHAPTER 2

The Hopf Algebra $H_T$ and Sequences

1. Introduction

The goal of this chapter is to indicate how one can construct for the Hopf algebra $H_T$ analogs of well known structures related to formal series used in classical vertex algebras. Recall that the dual of the symmetry algebra $H_D = \mathbb{C}[D]$ is just the power series ring $\mathbb{C}[[t]]$. Among the structures we have in mind here are

- The localization of the dual. This gives the formal Laurent series algebra $K_D = \mathbb{C}((t))$ from $H^*_D = \mathbb{C}[[t]]$ by inverting $t$.
- The residue map $\text{Res}_0 : K_D \to \mathbb{C}$, characterized by $\text{Res}_0(Df) = 0$ and the normalization $\text{Res}_0(\frac{1}{t}) = 1$. Note that the counit $\varepsilon$ on $H_D$ is given by $\varepsilon(D) = 0$, so that we can write $\text{Res}_0(Df) = \varepsilon(D) \text{Res}_0(f)$.
- The expansion maps (in regions $|x| > |y|$, resp. $|y| > |x|$)

\[
\frac{1}{x-y} \mapsto e^{-y\partial_y} \frac{1}{x}, \quad \frac{1}{x-y} \mapsto e^{-x\partial_x} \frac{1}{-y}.
\]

(1.1)

- The formal delta function, the difference of the two expansions:

\[
\delta(x, y) = e^{-y\partial_y} \frac{1}{x} + e^{-x\partial_x} \frac{1}{y}.
\]

For $H_T = \mathbb{C}[T, T^{-1}]$ the dual Hopf algebra is the algebra $\mathbb{C}_\mathbb{Z}$ of (arbitrary) functions from the integers to $\mathbb{C}$ as will be explained below. In other words the elements of $\mathbb{C}_\mathbb{Z}$ are sequences $s = (s_n)_{n \in \mathbb{Z}}$, with $s_n \in \mathbb{C}$. Let $\delta_n$ be the Kronecker sequence with value 1 at $n$ and zero elsewhere, so that $s = \sum_{n \in \mathbb{Z}} s_n \delta_n$. Note that Kronecker sequences are zero divisors in $\mathbb{C}_\mathbb{Z}$, as are all non invertible elements of $\mathbb{C}_\mathbb{Z}$.

Because of the zero divisors in $\mathbb{C}_\mathbb{Z}$ localizing is a somewhat delicate matter. In general, we want to choose a multiplicative set $M \subset \mathbb{C}_\mathbb{Z}$ of elements that can be inverted, but we must be careful not to have $f, g \in M$ with $f.g = 0$, because in that case $1/f = g/f.g$. In particular inverting more than one of the Kronecker sequences $\delta_n$ would not be a useful strategy. It turns out that we need to find appropriate analogs in $\mathbb{C}_\mathbb{Z}$ of the powers of $t$ occurring in the dual $\mathbb{C}[[t]]$ of $H_D$. These are the sequences $\tau(\ell)$ introduced in section 4.
2. The Hopf Algebra $H_T$

Let $H_T = \mathbb{C}[T, T^{-1}]$, as before, so that $H_T$ is a commutative and cocommutative Hopf algebra and we have the following structures on $H_T$:

- The unit $i: \mathbb{C} \to H_T$, $1 \mapsto T^0$ and multiplication map $m: H_T \otimes H_T \to H_T$, given by $T^m \otimes T^n \mapsto T^{m+n}$, $m, n \in \mathbb{Z}$.
- The coproduct $\pi: H_T \to H_T \otimes H_T$, given by $T^m \mapsto T^m \otimes T^m$ and the counit $\varepsilon: H_T \to \mathbb{C}$, $T^m \mapsto 1$.
- The antipode $S: H_T \to H_T$, $T^m \mapsto T^{-m}$.

These maps satisfy a large number of compatibility conditions that we will not recall here, see e.g., [CP95].

The powers of the translation $T$ form a basis $\{T^m\}_{m \in \mathbb{Z}}$ for $H_T$. Another basis will also be useful for us. Instead of translations we can use differences as basis: let $\Delta = T - 1$ and $\bar{\Delta} = 1 - T^{-1}$. The (divided) powers of $\Delta$ and $\bar{\Delta}$ form the difference operator basis $\{\Delta[m]\}_{m \in \mathbb{Z}}$, where

$$\Delta[m] = \begin{cases} \Delta^m / m! & m \geq 0, \\ \bar{\Delta}^k / k! & m = -k < 0. \end{cases}$$

Note that $\bar{\Delta} = -S\Delta$. Then we get for the antipode acting on this basis:

$$S\Delta[m] = \begin{cases} (-\bar{\Delta})^m / m! & m \geq 0 \\ (-\Delta)^k / k! & m = -k < 0 \end{cases} = (-1)^m \Delta[-m].$$

The coproduct of these basis elements becomes

$$\pi(\Delta[n]) = \sum_{s=0}^{n} \Delta[s] \otimes T^s \Delta[n-s] = \sum_{s=0}^{n} \Delta[s] T^{n-s} \otimes \Delta[n-s].$$

3. The Dual of $H_T$ and Sequences

$H_T$ is the group algebra (over the complex numbers) of the free Abelian group of rank 1, i.e., of the additive group $\mathbb{Z}$. Therefore the linear dual $H_T^*$ is the space of complex-valued maps on $\mathbb{Z}$. So let $\mathbb{C}_\mathbb{Z}$ be the vector space of two sided infinite sequences $s = (s_n)_{n \in \mathbb{Z}}$, $s_n \in \mathbb{C}$. Then the sequence $s$ defines a function on $\mathbb{Z}$ with value $s_n$ at $n \in \mathbb{Z}$. Let $\delta_n$ be the Kronecker sequence with a 1 on the $n$th position and for the rest zeroes, so that any sequence $s$ has an expansion

$$s = \sum_{n \in \mathbb{Z}} s_n \delta_n.$$

Define an action of $H_T$ on $\mathbb{C}_\mathbb{Z}$ by $T \delta_k = \delta_{k-1}$ so that if $s, \hat{s}$ are sequences with $\hat{s} = Ts$ then $\hat{s}_n = s_{n+1}$. Define a pairing

$$\langle s, \rangle: H_T \otimes \mathbb{C}_\mathbb{Z} \to \mathbb{C}, \ P(T) \otimes s \mapsto \langle P(T)s \rangle_{n=0},$$
4. The Hopf Dual of $H_T$ and Difference Equations

Recall that the Hopf dual of a Hopf algebra is the subspace $H^\circ$ of the full linear dual $H^*$ consisting of $\phi$ such that

$$m^*(\phi) \in H^* \otimes H^* \subset (H \otimes H)^*.$$ 

One proves that in fact in this case

$$m^*(\phi) \in H^\circ \otimes H^\circ,$$

and that $H^\circ$ is alternatively characterized as the subspace of elements $\phi$ such that the $H$-submodule generated by $\phi$ is finite dimensional. $H^\circ$ is then a Hopf algebra in its own right, see [Abe80] for details.

In case of $H = H_T$ the Hopf dual $H_T^\circ$ is the space of sequences $s \in \mathbb{C}_Z$ such that

$$H_T . s \simeq H_T / \text{Ann}_s$$
is finite dimensional, where $\text{Ann}_s$ is the annihilator of $s$, the ideal of elements of $H_T$ killing $s$. Since $H_T$ has Krull dimension 1 the submodule $H_T.s$ is finite dimensional whenever $\text{Ann}_s$ is not zero. So $H^o_T$ is the space of sequences that solve at least one non trivial homogeneous constant coefficient difference equation

$$P(T)s = 0, \quad P(T) \neq 0 \in H_T.$$  

Solutions of these equations are easy to describe explicitly. First note that without loss of generality we can assume that

$$P(T) = \sum_{n=0}^N p_n T^n, \quad p_0 \neq 0, p_N \neq 0.$$  

The space of solutions of (4.1) is then $N$ dimensional. Now we can factor $P(T) = \prod_n (T - \lambda_i)^{d_i}$ and solving (4.1) reduces to solving $(T - \lambda)^d s = 0$.

Define for $\lambda \neq 0$ an exponential sequence

$$E_\lambda = \sum_{n} \lambda^n \delta_n,$$  

so that we have

$$(T - \lambda)E_\lambda = 0.$$  

For any sequence $f$ we have

$$(T - \lambda)fE_\lambda = \lambda(\Delta F)E_\lambda, \quad \Delta = T - 1,$$  

and so $fE_\lambda$ satisfies $(T - \lambda)^d fE_\lambda = 0$ if and only if $\Delta^d f = 0$. To find such $f$ we introduce polynomial sequences $\tau(\ell) \in H^o_T, \ell \geq 0$.

First put $\tau(0) = \sum_{n \in \mathbb{Z}} \delta_n$, the sequence with 1 in all positions; this is the identity $1_{C\mathbb{Z}}$ in the $C\mathbb{Z}$-algebra $C\mathbb{Z}$. Then define recursively sequences $\tau(\ell)$, as the unique solutions of the difference equations

$$\Delta \tau(\ell) = \ell \tau(\ell - 1),$$  

satisfying the initial conditions $\tau(\ell)0 = 0$. We have an explicit formula for the $\tau(\ell)$. First of all

$$\tau(1) = \sum_{n \in \mathbb{Z}} n \delta_n.$$  

We abbreviate $\tau = \tau(1)$. Then

$$\tau(\ell) = \prod_{j=0}^{\ell-1} T^{-j} \tau = \sum_{n \in \mathbb{Z}} (n)_{\ell} \delta_n,$$  

where $(n)_{\ell}$ is the Jordan factorial: $(n)_{\ell} = n(n - 1) \ldots (n - \ell + 1)$, for $n, \ell \in \mathbb{Z}, \ell \geq 0$. In particular $\tau(\ell)$ is a sequence with 0, 1, \ldots, $\ell - 1$ as its only zeroes. The formula (4.4) will follow from (4.7) below.
The space
\[ S_{\lambda,d} = \bigoplus_{\ell=0}^{d-1} \mathbb{C} \tau(\ell) E_{\lambda} \]
is the \(d\)-dimensional solution space of the equation \((T - \lambda)^d s = 0\). Define
\[ (4.5) \quad \mathbb{C}^{\text{pol}}_{\mathbb{Z}} = \bigoplus_{\ell \geq 0} \mathbb{C} \tau(\ell). \]
Then
\[ (4.6) \quad H^\circ_T = \bigoplus_{\lambda \in \mathbb{C}^\times} \mathbb{C}^{\text{pol}}_{\mathbb{Z}} E_{\lambda}. \]

An element \(f E_{\lambda}\) of \(H^\circ_T\) is a function on \(\mathbb{Z}\) obtained by restricting the function \(f(x) \lambda^x\), with \(f(x) \in \mathbb{C}[x]\), from \(\mathbb{C}\) to \(\mathbb{Z}\), the sequence \(\tau(1) = \tau\) corresponding to the polynomial \(x\).

Note that the sequences \(\tau(\ell)\) are analogs for \(H_T\) (or \(\mathbb{C}[\Delta]\)) of the powers \(t^\ell \in H^*_D = \mathbb{C}[t]\), satisfying the differential recursion
\[ D t^\ell = \ell t^{\ell-1}, \quad t^\ell |_0 = 0. \]

Of course, the multiplicative properties of the \(\tau(\ell)\)'s are more complicated. We have
\[ (4.7) \quad \tau(\ell) T^{-\ell} \tau(m) = \tau(\ell + m), \]
and hence we can calculate products in \(\mathbb{C}^{\text{pol}}_{\mathbb{Z}}\):
\[ (4.8) \quad \tau(\ell) \tau(m) = \tau(\ell) T^{-\ell} \tau(m) = \tau(\ell) T^{-\ell} (1 + \Delta)^\ell \tau(m) \]
\[ = \sum_{k=0}^{\ell} \binom{\ell}{k} (m)_k \tau(\ell + m - k). \]

We see that the product of two basis elements of \(\mathbb{C}^{\text{pol}}_{\mathbb{Z}}\) is a finite (integral) linear combination of basis elements and \(\mathbb{C}^{\text{pol}}_{\mathbb{Z}}\) is a subalgebra of \(H^\circ_T\).

To derive (4.7) we use the Leibniz rule for the difference of a product: by the coproduct formula (2.3)
\[ (4.9) \quad \Delta[n](fg) = \sum_{s=0}^{n} \Delta[s] (f) T^s \Delta[n-s] g. \]

We have
\[ \Delta(\tau T^{-1} \tau) = \Delta(\tau) \tau + \tau(T^{-1} \Delta \tau) = 2 \tau, \quad (\tau T^{-1} \tau)_0 = 0 \]
so that by uniqueness $\tau(2) = \tau T^{-1} \tau$. Assuming that (4.7) is true for all $\ell, m$ such that $\ell + m = n$ then
\[
\Delta \left( \tau(\ell + 1) T^{-\ell - 1} \tau(m) \right) = (\ell + 1) \tau(\ell) T^{-\ell} \tau(m) + m \tau(\ell + 1) T^{-\ell - 1} \tau(m - 1)
\]
\[
= (\ell + m + 1) \tau(\ell + m),
\]
by induction and so by uniqueness of solutions of (4.3) (4.7) is true for $\ell + 1, m$. A similar calculation shows the same for $\ell, m + 1$.

5. The antipodal sequences $\overline{\tau(\ell)}$

We defined the sequences $\tau(\ell)$ as solutions of the difference equation (4.3), using $\Delta$. We can instead use $\Delta$ to define $\overline{\tau(\ell)}$ by $\overline{\tau(0)} = 1$ and
\[
\Delta \overline{\tau(\ell)} = \ell \overline{\tau(\ell - 1)}, \quad \overline{\tau(0)} = 0.
\]
We see that
\[
\overline{\tau(1)} = \tau(1) = \tau,
\]
and
\[
(5.1) \quad \overline{\tau(\ell)} = \prod_{j=0}^{\ell-1} T^j \tau = \sum_{n \in \mathbb{Z}} (n)_{-\ell} \delta_n,
\]
where $(n)_{-\ell} = n(n+1) \ldots n + \ell - 1$. Introduce notation
\[
(5.2) \quad \tau[m] = \begin{cases} 
\tau(m), & m \geq 0, \\
\overline{\tau(k)}, & m = -k < 0,
\end{cases}
\]
so that
\[
(5.3) \quad \tau[m] = \sum_{n \in \mathbb{Z}} (n)_m \delta_n.
\]
Using
\[
S \tau = S \sum_n n \delta_n = \sum_n n \delta_{-n} = -\tau,
\]
we find
\[
(5.4) \quad S \tau[m] = (-1)^m \tau[-m], \quad m \in \mathbb{Z}.
\]
In case either $0 \geq s, \ell$ or $0 \leq s, \ell$ we have the factorization
\[
(5.5) \quad \tau[s + \ell] = \tau[s] T^{-s} \tau[\ell].
\]
6. The Dual Difference Basis

The difference operator basis \( \{\Delta[n]\}_{n \in \mathbb{Z}} \) of (2.1) has as dual the set of sequences \( \Delta^*[n] \) defined by \( \langle \Delta[n], \Delta^*[m] \rangle = \delta_{mn} \). We refer to \( \{\Delta[n]\}_{n \in \mathbb{Z}} \) as the dual difference basis. Introduce notation

\[
\theta(k) = \begin{cases} 
1, & k \geq 0, \\
-1, & k < 0.
\end{cases}
\]

**Lemma 6.1.** For all \( \ell \in \mathbb{Z} \)

\[
\Delta^*[\ell] = \sum_{\theta(\ell)[n-\ell] \geq 0} (n)_{\ell} \delta_n.
\]

**Proof.** For any \( \phi \in \mathbb{C}_\mathbb{Z} \) we have \( \phi = \sum_n \langle T^n, \phi \rangle \delta_n \). Now \( T = 1 + \Delta, \ T^{-1} = 1 - \Delta \), so that if \( n,k > 0 \)

\[
\langle T^n, \Delta^*[-k] \rangle = 0 = \langle T^{-n}, \Delta^*[k] \rangle.
\]

By the binomial formula we then obtain

\[
\Delta^*[\ell] = \sum_{n \geq \ell} \langle T^n, \Delta^*[\ell] \rangle \delta_n = \sum_{n \geq 0} (n)_{\ell} \delta_n,
\]

and

\[
\Delta^*[-\ell] = \sum_{n \leq -\ell} \langle T^n, \Delta^*[-\ell] \rangle \delta_n = \sum_{n \leq 0} (n)_{-\ell} \delta_n.
\]

\( \square \)

Comparing (6.2) with (5.3) we see that for \( m \neq 0 \) \( \Delta^*[m] \) is a projection of \( \tau[m] \):

\[
\Delta^*[m] = \Pi_{\theta(m)} \tau[m] = \begin{cases} 
\Pi_+ \tau(m) & m > 0, \\
\Pi_- \tau(-m) & m < 0,
\end{cases}
\]

where \( \Pi_+ \) is the restriction of functions on \( \mathbb{Z} \) to \( \mathbb{Z}_{\geq 0} \) and \( \Pi_- \) is the restriction to \( \mathbb{Z}_{<0} \).

We can conversely express the polynomials \( \tau[k] \) in terms of the dual difference basis. We have by calculating \( \tau[k] = \sum_n \langle \Delta[n], \tau[k] \rangle \Delta^*[k] \) in general

\[
\tau[\pm \ell] = \Delta^*[\pm \ell] + \sum_{s < 0} c_{s\ell} \Delta^*[\pm s],
\]

for certain integers \( c_{s\ell} \), see section 22 for an explicit formula. In particular

\[
\tau[\pm 1] = \Delta^*[1] + \Delta^*[-1].
\]
7. Localization for $\mathbb{C}_{\mathbb{Z}}^{\text{pol}}$

Let $M \subset \mathbb{C}_{\mathbb{Z}}^{\text{pol}}$ be the multiplicative set generated by 1 and the translates $T^k \tau$ of $\tau$, $k \in \mathbb{Z}$. Elements of $M$ correspond to polynomial functions on $\mathbb{C}$ with only zeroes at the integers. Define

$$K = M^{-1}\mathbb{C}_{\mathbb{Z}}^{\text{pol}}.$$ 

So an element of $K$ is of the form $q = \frac{f}{g}$, with $f \in \mathbb{C}_{\mathbb{Z}}^{\text{pol}}$, $g \in M$.

**Remark 7.1.** We could also consider the localization $K^\circ = M^{-1}H_T^\circ$.

This does not seem to give an essential different theory, so we will restrict ourselves to considering $K$.

Since $M$ is stable under the action of the group generated by $T$ we can extend the action of $H_T$ on $\mathbb{C}_{\mathbb{Z}}^{\text{pol}}$ to $K$ via

$$T^k \frac{f}{g} = \frac{T^k f}{T^k g}, \quad f \in \mathbb{C}_{\mathbb{Z}}^{\text{pol}}, g \in M.$$

Using the Leibniz rule (4.9) we find the quotient rule

$$\Delta^\pm \left[ \frac{f}{g} \right] = \frac{\Delta^\pm \left[ f \right] g - f \Delta^\pm \left[ g \right]}{g T^\pm 1 g}.$$

It follows from this that

$$\frac{(S\Delta)^\ell}{\ell!} \frac{1}{\tau} = \frac{1}{\tau(\ell + 1)}, \quad \frac{(S\Delta)^\ell}{\ell!} \frac{1}{\tau} = \frac{1}{\tau(\ell + 1)}.$$

Then we can summarize (7.2), using the notations (6.1), (5.2),

$$S\Delta[k] \frac{1}{\tau[\theta(k)]} = \frac{1}{\tau[\theta(k) + k]}, \quad k \in \mathbb{Z}.$$

Recall that elements of $\mathbb{C}_{\mathbb{Z}}^{\text{pol}}$ correspond to polynomial functions on $\mathbb{C}$, via the algebra homomorphism

$$\Phi: \mathbb{C}[x] \rightarrow \mathbb{C}_{\mathbb{Z}}^{\text{pol}}, \quad f(x) \mapsto \sum_{n \in \mathbb{Z}} f(n) \delta_n.$$

Also recall that by the *division algorithm* in $\mathbb{C}[x]$ we can write, uniquely, for $f, g \in \mathbb{C}[x], g \neq 0$,

$$\frac{f}{g} = q + \frac{r}{g}, \quad q, r \in \mathbb{C}[x],$$

where in case $r \neq 0$ we have $\deg r < \deg g$. We can and will assume that $r$ and $g$ have no common zeroes. Furthermore we have a *partial fraction*
expansion for $\frac{g}{f}$: if $g$ has zeroes $\{n_i\}_{i \in I}$ with $n_i \in \mathbb{Z}$ and with multiplicities $d_i$ then
\[
\frac{r}{g} = \sum_{i \in I} \sum_{k=0}^{d_i-1} \frac{a_{i,k}}{(x-n_i)^{-k+1}}, \quad a_{i,k} \in \mathbb{C}.
\]
Translating this result from $\mathbb{C}[x]$ to $\mathbb{C}_Z^\text{pol}$, using the map $\Phi$ of (7.4), we see that any $y \in K$ can be uniquely written as
\[
y = y^\text{pol} + y_-, \quad (7.5)
\]
with $y^\text{pol} \in \mathbb{C}_Z^\text{pol}$ and
\[
y_- = \sum_{i \in I} \sum_{k=0}^{d_i-1} \frac{a_{i,k}}{T^{n_i} \tau^{k+1}}. \quad (7.6)
\]
Here the $n_i$ are integers (as $y_-$ is supposed to come, via $\Phi$, from a fraction $\frac{g}{f}$ where $r, g \in \mathbb{C}[x]$ and $g$ has only integral zeroes). Hence
\[
K = \mathbb{C}_Z^\text{pol} \bigoplus K_{\text{Sing}}, \quad K_{\text{Sing}} = \bigoplus_{n \in \mathbb{Z}, k \geq 0} \mathbb{C} T^n \frac{1}{\tau^{k+1}}. \quad (7.7)
\]
We have then a projection
\[
\text{Sing}: K \to K_{\text{Sing}}. \quad (7.8)
\]

8. $\hat{H}_T$ and $\hat{H}^*_T$

From (7.7) it is clear that $K_{\text{Sing}}$ is freely generated by $\frac{1}{\tau}$ as a module over the Hopf algebra
\[
\hat{H}_T = H_T[\partial_\tau] = \mathbb{C}[\Delta, \bar{\Delta}, \partial_\tau],
\]
where $\partial_\tau$ commutes with $\Delta, \bar{\Delta}$, has antipode $S(\partial_\tau) = -\partial_\tau$, coproduct $\pi(\partial_\tau) = \partial_\tau \otimes 1 + 1 \otimes \partial_\tau$ and counit $\varepsilon(\partial_\tau) = 0$. We have an action of $\hat{H}_T$ on $\mathbb{C}_Z^\text{pol}$ by putting
\[
\partial_\tau = -\log(T^{-1}) = \sum_{k=1}^{\infty} \frac{\Delta^k}{k}, \quad (8.1)
\]
and this gives $K$ an $\hat{H}_T$-module structure.

A basis for $\hat{H}_T$ is given by
\[
e_{k,\ell} = \Delta[k] \partial_\tau^{(\ell)}, \quad k, \ell \in \mathbb{Z}, \quad \ell \geq 0. \quad (8.2)
\]
Consider the algebra $\mathbb{C}_Z[[\sigma]]$ of formal power series in a variable $\sigma$ with coefficients in $\mathbb{C}_Z$, the dual of $H_T$. We define an action of $\hat{H}_T$ on $\mathbb{C}_Z[[\sigma]]$ by extending the action of $H_T$ on $\mathbb{C}_Z$ by putting:
\[
\partial_\tau \Delta^k[k] = 0, \quad \Delta[k] \sigma^\ell = \delta_{0,k} \sigma^\ell, \quad \partial_\tau \sigma^\ell = \ell \sigma^{\ell-1}.
\]
An element \( f \in \mathbb{C}[[\sigma]] \) has an expansion
\[
f = \sum f_{n,k} \Delta^*[n] \sigma^k,
\]
and we extend the counit of \( \mathbb{C}[[\sigma]] \) to a map \( \varepsilon : \mathbb{C}[[\sigma]] \to \mathbb{C} \) by putting
\[
\varepsilon(f) = f_{0,0}.
\]
We use this to define a pairing
\[
\langle \cdot, \cdot \rangle : \hat{H}_T \otimes \mathbb{C}[[\sigma]] \to \mathbb{C}
\]
\[
P(\Delta, \partial_{\tau}) \otimes f \mapsto \varepsilon(P(\Delta, \partial_{\sigma}) f).
\]
This identifies \( \mathbb{C}[[\sigma]] \) with the dual \( \hat{H}_T^* \), and we get functions dual to \( e_{k,\ell} \) by defining
\[
e_{k,\ell}^* = \Delta^*[k] \sigma^\ell.
\]

9. Action of \( \mathbb{C}[[\sigma]] \) on \( \hat{H}_T \)

The duality (8.3) allows us to define an action
\[
\mathbb{C}[[\sigma]] \otimes \hat{H}_T \to \hat{H}_T,
\]
by putting, for \( e^* \in \mathbb{C}[[\sigma]] \), \( e \in \hat{H}_T \)
\[
e^* \cdot e = \sum \langle e^*, e_{n,\ell}^* \rangle e_{n,\ell}.
\]
One easily checks the following Lemma

**Lemma 9.1.**

(a)
\[
\sigma \cdot e_{n,\ell} = \begin{cases} e_{n,\ell-1} & \ell > 0 \\ 0 & \ell = 0 \end{cases}
\]

(b)
\[
\Delta^*[1] \cdot e_{n,\ell} = \begin{cases} 0 & n \leq 0 \\ T e_{n-1,\ell} & n > 0 \end{cases}
\]

(c)
\[
\Delta^*[-1] \cdot e_{n,\ell} = \begin{cases} T^{-1} e_{n+1,\ell-1} & n < 0 \\ 0 & n \geq 0 \end{cases}
\]

(d)
\[
\Delta^*[k] \cdot \Delta[-\ell] = \Delta^*[-k] \cdot \Delta[\ell] = 0, \quad k, \ell > 0.
\]

(9.2)
10. Action of $\mathbb{C}_Z[[\sigma]]$ on $K_{\text{Sing}}$

Let $\overline{K}_{\text{Sing}} = K / \mathbb{C}_Z^{\text{pol}}$. As a vector space and $\hat{H}_T$-module this is the same as $K_{\text{Sing}}$ but $\overline{K}_{\text{Sing}}$ is a $\mathbb{C}_Z^{\text{pol}}$-module, which $K_{\text{Sing}}$ is not.

Now $\mathbb{C}_Z^{\text{pol}}$ is a subalgebra of $\mathbb{C}_Z[[\sigma]]$, via the action of $\hat{H}_T$ on $\mathbb{C}_Z^{\text{pol}}$ (see (8.1)): we identify $f \in \mathbb{C}_Z^{\text{pol}}$ with $\sum e_n,\ell f|\theta_n,\ell$. In particular

$$\tau \mapsto \Delta^*[1] + \Delta^*[-1] + \sigma = e^*_1,0 + e^*_{-1,0} + e_{0,1}.$$

We have an $\hat{H}_T$-module isomorphism

$$\alpha: \hat{H}_T \rightarrow \overline{K}_{\text{Sing}}$$

$$P(\Delta, \partial_\tau) \mapsto S(P(\Delta, \partial_\tau)) \frac{1}{\tau}.$$ 

Domain and range of $\alpha$ are $\mathbb{C}_Z^{\text{pol}}$-modules.

**Lemma 10.1.**

(a) In $\hat{H}_T$ we have

$$\tau.e_{k,\ell} = \begin{cases} Te_{k-1,\ell} + e_{k,\ell-1} & k > 0, \\ e_{0,\ell-1} & k = 0, \\ T^{-1}e_{k+1,\ell} + e_{k,\ell-1} & k < 0. \end{cases}$$

(b) In $\overline{K}_{\text{Sing}}$ we have

$$\tau.S(e_{k,\ell}) \frac{1}{\tau} = \begin{cases} S(Te_{k-1,\ell} + e_{k,\ell-1}) \frac{1}{\tau} & k > 0 \\ S(e_{0,\ell-1}) \frac{1}{\tau} & k = 0 \\ S(T^{-1}e_{k+1,\ell} + e_{k,\ell-1}) \frac{1}{\tau} & k < 0. \end{cases}$$

(c) Hence the map $\alpha$ of (10.2) is also an isomorphism of $\mathbb{C}_Z^{\text{pol}}$-modules.

**Proof.** Part (a) follows from Lemma (9.1) and (10.1). Part (b) follows from the observation that in any $\hat{H}_T$- and $\mathbb{C}_Z^{\text{pol}}$-module we have

$$[\Delta[k], \tau] = \Delta[k - \theta(k)] T^{\theta(k)}, \quad [\partial_{\tau}(\ell), \tau] = \partial_{\tau}^{(\ell-1)},$$

recalling the notation (6.1). Then part (c) follows from the other parts. \qed

We use $\alpha$ to extend the $\mathbb{C}_Z^{\text{pol}}$ action on $\overline{K}_{\text{Sing}}$ to an action of $\mathbb{C}_Z[[\sigma]]$ on $\overline{K}_{\text{Sing}}$. In particular the product $e_{n,k}^* F$ is well defined for all $F \in \overline{K}_{\text{Sing}}$, and
for instance we have, by (9.2),
\[
\Delta^* [k], \frac{1}{\tau[\ell+1]} = \Delta^* [-k], \frac{1}{\tau[\ell+1]} = 0, \quad k, \ell > 0.
\]
\[(10.3)\]
\[
\Delta^* [\ell], \frac{1}{\tau(m+1)} = \tau(\ell) \frac{1}{\tau(m+1)} , \quad \ell, m \geq 0
\]
\[
\Delta^* [-\ell], \frac{1}{\tau(m+1)} = \tau(\ell) \frac{1}{\tau(m+1)} , \quad \ell, m \geq 0.
\]

**Remark 10.2.** The action of \( \mathbb{C}^{\text{pol}}_Z \) on \( K_{\text{Sing}} \) can be written as
\[(10.4)\]
\[
f.(\frac{g}{h}) = \frac{fg}{h} - (\frac{fg}{h})_{\text{Hol}}, \quad f \in \mathbb{C}^{\text{pol}}_Z, \quad \frac{g}{h} \in K_{\text{Sing}}.
\]
Here we use that \( K \) is the localization \( M^{-1} \mathbb{C}^{\text{pol}}_Z \) (with \( M \) the multiplicative set of translates of \( \tau \)), and that \( K \) has a decomposition in singular and holomorphic elements. Now we can localize also the full dual \( \mathbb{C}_Z = H_T^* \): define
\[
K_Z = M^{-1} \mathbb{C}_Z.
\]
Now, maybe, one expects that similarly the \( \mathbb{C}_Z \)-action on \( K_{\text{Sing}} \) can be written in the form (10.4), or equivalently that the \( \mathbb{C}_Z \)-action on \( K_{\text{Sing}} \) comes from a projection of the natural action of \( \mathbb{C}_Z \) on \( K_Z \). However, this is not the case. Indeed, let \( f = \delta_k \in \mathbb{C}_Z \), the Kronecker sequence with support at \( k > 0 \), for example. Then we have
\[
\delta_k \tau(k+1) = 0, \quad \text{see (4.4)},
\]
so that for all \( \frac{g}{h} \in K \) (or \( K_Z \)) we have
\[
\frac{\delta_k g}{h} = \frac{\delta_k \tau(k+1)g}{\tau(k+1)h} = 0 \in K_Z.
\]
On the other hand we have by (9.1)
\[
\frac{\delta_k}{\tau - \ell} = \frac{\delta_k, \ell}{\tau - \ell} \neq 0.
\]
In fact, inside \( K_Z \) there is no distinction between holomorphic and singular elements. To see this note that we have an exact sequence
\[
0 \to \mathbb{C}_Z^{\text{tor}} \to \mathbb{C}_Z \xrightarrow{\beta} K_Z \to \mathbb{C}_Z
\]
where \( \beta : \mathbb{C}_Z \to K_Z \) is the canonical map \( f \mapsto \frac{f}{\tau} \), and
\[
\mathbb{C}_Z^{\text{tor}} = \bigoplus_{k \in \mathbb{Z}} \mathbb{C} \delta_k.
\]
Define \( \mathbb{C}_Z^{\text{ff}} = \mathbb{C}_Z / \mathbb{C}_Z^{\text{tor}} \). Then we see that also
\[
K_Z = M^{-1} \mathbb{C}_Z^{\text{ff}}.
\]
where we identify $M$ with its image in $\mathbb{C}_Z^\mathrm{tf}$, as $M \cap \mathbb{C}_Z^\mathrm{pol} = \emptyset$. Now if $m \in M$ then $m$ has at most a finite number of zeroes (as function on the integers), so modulo $\mathbb{C}_Z^\mathrm{tor}$, $m$ is invertible. Hence as subset of $\mathbb{C}_Z^\mathrm{tf}$, $M$ consists of units and so we find $K_Z \simeq \mathbb{C}_Z^\mathrm{tf}$. There are no singular elements in $\mathbb{C}_Z^\mathrm{tf}$.

11. Trace for $K$

Now $\hat{H}_T$, being a Hopf algebra, has a counit, a multiplicative map $\varepsilon: \hat{H}_T \to \mathbb{C}$ given on the basis (8.2) by

$$\varepsilon(e_k, \ell) = \varepsilon(\Delta(k) \partial(\ell)) = \delta_{k,0} \delta_{\ell,0}.$$ 

The counit induces, via the isomorphism $\alpha$, a map $\overline{K}_{\text{Sing}} \to \mathbb{C}$, which we extend (by zero) to all of $K$ to get a map, called the Trace

$$(11.1) \quad \text{Tr}: \ K \to \mathbb{C}, \quad \frac{f}{g} \mapsto \varepsilon(\alpha^{-1}(\text{Sing}(\frac{f}{g}))),$$

where Sing is the projection (7.8)

**Lemma 11.1.** (a) If $F \in K$, $h \in \hat{H}_T$, then

$$\text{Tr}(hF) = \varepsilon(h) \text{Tr}(F).$$

(b) In particular we have

$$\text{Tr}(T^kF) = \text{Tr}(F), \quad \text{Tr}(\Delta F) = \text{Tr}(\Delta F) = \text{Tr}(\partial F) = 0.$$ 

(c) (Partial Summation/Integration) For all $F, G \in K$ and $h \in \hat{H}_T$ we have

$$(11.2) \quad \text{Tr}(h\langle F \rangle G) = \text{Tr}(FS(h)\langle G \rangle).$$

Recall that an element $F \in K$ corresponds to a rational function $F(x)$ on $\mathbb{C}$ with at most a finite number of finite order poles at the integers. Let $F(x)dx$ be the corresponding rational 1-form on $\mathbb{C}$. Then the trace of $F$ is the sum of residues of this 1-form.

**Lemma 11.2.** If $F \in K$ then

$$(11.3) \quad \text{Tr}(F) = \sum_{n \in \mathbb{Z}} \text{Res}_{x=n}(F(x)dx).$$

Note that if $f \in \mathbb{C}_Z^\mathrm{pol}$ the value of $f$ at 0 can be expressed as a trace

$$(11.4) \quad f|_0 = \text{Tr}(\frac{f}{\tau}).$$

The terminology Trace for the map (11.1) is used to indicate the analogy with the trace of finite matrices. The Hamiltonians of the finite Toda lattice are traces of Lax matrices. Similarly the Hamiltonians for the infinite Toda lattice are traces in the sense of (11.1). This is discussed in more detail at the end of section 2.
Remark 11.3. The trace on $K$ cannot reasonably extended to a trace on the localization $K_Z$ introduced in Remark 10.2. As explained there, in $K_Z$ there is no distinction between singular and holomorphic elements.

However, what we can and will do is define a pairing, by abuse of notation also called trace, between $C_Z$ and $K_{\text{Sing}}$:

$$C_Z \otimes K_{\text{Sing}} \to C, \quad f \otimes g \mapsto \text{Tr}(f.(g/h))$$

We warn the reader that in general, for $g/h \in K_{\text{Sing}}$

$$\text{Tr}(f.(g/h)) \neq \text{Tr}((mf).(g/mh)), \quad m \in M,$$

unless $f \in C_Z^{\text{pol}}$, see Lemma 10.1, part c.

12. Orthogonality Relations

We introduce notation: for $\ell \geq 0$ we put

$$\tau(-\ell-1) = \frac{1}{\tau(\ell+1)}, \quad \tau(-\ell-1) = \frac{1}{\tau(\ell+1)}.$$

Lemma 12.1. For all $m,n \in \mathbb{Z}$

$$\text{Tr}(\tau(m)\tau(n)) = \delta_{m,-n-1}, \quad \text{Tr}(\tau(m)\tau(n)) = \delta_{m,-n-1}.$$

Proof. We consider the left hand identity first. In case $m,n \geq 0$ the trace is zero because $\tau(m)\tau(n)$ is nonsingular. In case $m,n \leq -1$ the one form $1/dx$ corresponding to $1/\tau(-m)\tau(-n)$ is regular at $\infty$ and has only poles at the integers, so we can, using (11.3), replace the trace by the sum over the residues of $1/dx$ at all points of $\mathbb{P}^1$, which is zero by the residue theorem.

So assume $m \geq 0$ and $n = -k - 1$ for $k \geq 0$. Then we get using the quotient rule (7.2), the partial summation rule (11.2), (4.3), and (4.7) and (11.4)

$$\text{Tr}(\tau(m)\tau(n)) = \text{Tr} \left( \tau(m) \left( \frac{-\Delta^k}{k!} \right) \frac{1}{\tau} \right) = \text{Tr} \left( \frac{\Delta^k}{k!} \left( \tau(m) \frac{1}{\tau} \right) \right) = \delta_{m,k}.$$

The rest of the proof is similar. \qed

Recall the action of $C_Z[[\sigma]]$ on $K_{\text{Sing}}$ described by combining Lemma 9.1 with the isomorphism $\alpha$.

Lemma 12.2. For all $n,m,\ell,k \in \mathbb{Z}$, $\ell,k \geq 0$ we have

$$\text{Tr}(e_{n,\ell}^* \cdot S(e_{m,k}) \frac{1}{\tau}) = \delta_{n,m} \delta_{\ell,k}.$$
13. Expansions in $K$

A basis for $K$ is given by

(13.1) \[ \{ \tau(k), k \geq 0 \} \cup \{ S(e_{n,\ell}) \frac{1}{\tau}, n, \ell \in \mathbb{Z}, \ell \geq 0 \}. \]

So any $F \in K$ has a (finite) expansion

(13.2) \[ F = \sum F_{\{ \tau(k+1) \}} \tau(k) + F_{\{ e^*_{n,\ell} \}} S(e_{n,\ell}) \frac{1}{\tau}. \]

The coefficients of $F$ are traces:

\[ F_{\{ \tau(k+1) \}} = \text{Tr}(\frac{1}{\tau(k+1)} F), \quad F_{\{ e^*_{n,\ell} \}} = \text{Tr}(e^*_{n,\ell} F), \]

using the action of $\mathbb{C}_{\mathbb{Z}}[[\sigma]]$ on $K_{\text{Sing}}$, see section 10.

14. (Twisted) Coproduct and (Twisted) Exponential Operators

Let $H$ be a commutative and cocommutative Hopf algebra, for simplicity, and let $H^*$ be its dual. Fix a basis $\{ e_i \}_{i \in I}$ for $H$ and let $e^*_j$ be linear functions on $H$ such that

\[ e^*_j(e_i) = \delta_{ij}. \]

Any $\phi \in H^*$ can then be expanded as

(14.1) \[ \phi = \sum \phi(e_i) e^*_i, \]

and similarly we have for $\omega \in (H \otimes H)^*$

\[ \omega = \sum \omega(e_i \otimes e_j) e^*_i \otimes e^*_j. \]

Let now $A, B$ be two linear maps $H \to H$. (Later on $A, B$ will be either the identity or the antipode $S$.) Define, if $m$ is the usual multiplication of $H$, the $A \otimes B$-twisted multiplication on $H$ by

\[ m_{A \otimes B}: H \otimes H \to H, \quad x \otimes y \mapsto m(Ax \otimes By). \]

Dual to the twisted multiplication we get a map

\[ m^*_{A \otimes B}: H^* \to (H \otimes H)^*. \]

**Lemma 14.1.** We have for all $\phi \in H^*$

\[ m^*_{A \otimes B}(\phi) = L_{A \otimes B}(\phi) = R_{A \otimes B}(\phi), \]

where

\[ L_{A \otimes B}(\phi) = \sum_{i \in I} e^*_i \otimes B^* ((Ae_i).\phi), \]

\[ R_{A \otimes B}(\phi) = \sum_{i \in I} A^* ((Be_i).\phi) \otimes e^*_i. \]
2. THE HOPF ALGEBRA $H_T$ AND SEQUENCES

PROOF. If, for instance, $B : H \to H$ is a linear map, then the dual is given by $B^* \phi(e) = \phi(Be)$ for $\phi \in H^*$, $e \in H$. Then we have

$$m_{A \otimes B}^*(\phi) = \sum m_{A,B}^*(\phi)(e_i \otimes e_j)e_i^* \otimes e_j^* = \sum \phi(A(e_i)B(e_j))e_i^* \otimes e_j^* = \Delta_{A \otimes B}(\phi)$$

By a similar calculation we find the expression of the $A \otimes B$ twisted coproduct in terms of $\mathcal{R}_{A \otimes B}$. \qed

Because of Example 14.2 below we call $\mathcal{L}_{A \otimes B}, \mathcal{R}_{A \otimes B}$ twisted exponential operators. In case $A = B = 1$ we write $\mathcal{L}_{1 \otimes 1} = \mathcal{L}, \mathcal{R}_{1 \otimes 1} = \mathcal{R}$ and refer to $\mathcal{L}, \mathcal{R}$ as untwisted exponential operators. We will mainly interested in the case $A = 1, B = S$, and we will write for brevity

$$\mathcal{L}^S = \mathcal{L}_{1 \otimes S}, \quad \mathcal{R}^S = \mathcal{R}_{1 \otimes S},$$

EXAMPLE 14.2. Take $H = H_D$, with as basis the divided powers $e_i = D^{(i)} / i!$. The dual is $H_D^* = \mathbb{C}[t]$, with $e_i^* = t^i$. The action of $H_D$ on $H_D^*$ is given by $D = \partial_t$. If $f(t) \in \mathbb{C}[t]$ then the untwisted comultiplication is $m^*(f) = f(t_1 + t_2)$. Here and below we will write $t_1, t_2$ for $t \otimes 1, 1 \otimes t$. We have two expansions:

$$f(t_1 + t_2) = \exp(t_2 \partial_{t_1}) f(t_1) = \exp(t_1 \partial_{t_2}) f(t_2).$$

In this case the untwisted exponential operators are

$$\mathcal{L} = \sum_i e_i^* \otimes e_i = \exp(t_1 \partial_{t_2}), \quad \mathcal{R} = \sum_i e_i \otimes e_i^* = \exp(t_2 \partial_{t_1}).$$

The $1 \otimes S$-twisted comultiplication is

$$m_{1 \otimes S}^*(f(t)) = f(t_1 - t_2),$$

and

$$\mathcal{L}^S = S_2 \exp(t_1 \partial_2), \quad \mathcal{R}^S = \exp(-t_2 \partial_1).$$ \qed

EXAMPLE 14.3. Now take $H = H_T$, with as basis the difference operators $\{e_n = \Delta[n]\}$ of (2.1) and as dual basis the sequences $\{e^*_n = \Delta^*[n]\}$ introduced in section 6. If $\phi \in \mathbb{C}_Z$ then $\phi$ is a function on the integers. The coproduct $m^*(\phi)$ and the twisted coproduct $m_{1 \otimes S}^*(\phi)$ are functions on the product $\mathbb{Z} \times \mathbb{Z}$ given by

$$(14.2) \quad m^*(\phi)(m, n) = \phi(m + n), \quad m_{1 \otimes S}^*(\phi)(m, n) = \phi(m - n).$$
The untwisted and twisted exponential operators for $H_T$ using the difference operator basis are
\[
\mathcal{L} = \sum_{n \in \mathbb{Z}} \Delta^*[n] \otimes \Delta[n], \quad \mathcal{L}^S(-) = \sum_{n \in \mathbb{Z}} \Delta^*[n] \otimes S(\Delta[n](-)),
\]
(14.3)
\[
\mathcal{R} = \sum_{n \in \mathbb{Z}} \Delta[n] \otimes \Delta^*[n], \quad \mathcal{R}^S(-) = \sum_{n \in \mathbb{Z}} S(\Delta[n])(-) \otimes \Delta^*[n],
\]
so that $m^*(\phi) = \mathcal{L}(\phi) = \mathcal{R}(\phi)$ and $m^*_{1 \otimes S}(\phi) = \mathcal{L}^S(\phi) = \mathcal{R}^S(\phi)$.
\[\Box\]

15. $H$-Covariance of (Twisted) Exponentials

The domain $H^*$ and the range $(H \otimes H)^*$ of the twisted comultiplications admit actions of $H$ and $H \otimes H$, and one can expect that for suitable $A, B$ the twisted exponentials relate these actions.

We will write $h_1 = h \otimes 1$ and $h_2 = 1 \otimes h$ and similar for other operators acting on tensor products.

**Lemma 15.1.** For all $\phi \in H^*$, $h \in H$

(a) $\mathcal{L}(h\phi) = (h_1 \mathcal{L}(\phi) + h_2 \mathcal{L}(\phi))$.

(b) $\mathcal{R}(h\phi) = (h_1 \mathcal{R}(\phi) + h_2 \mathcal{R}(\phi))$.

(c) $\mathcal{L}^S(h\phi) = h_1 \mathcal{L}^S(\phi) = (Sh)_2 \mathcal{L}^S(\phi)$.

(d) $\mathcal{R}^S(h\phi) = h_1 \mathcal{R}^S(\phi) = (Sh)_2 \mathcal{R}^S(\phi)$.

**Proof.** We have by the commutativity of $H$
\[
\mathcal{L}(h\phi) = \sum_i e_i^* \otimes e_i h\phi = 1 \otimes h \sum_i e_i^* \otimes e_i \phi = h_2 \mathcal{L}(\phi).
\]

On the other hand, introducing constants $h_i^j \in \mathbb{C}$ describing the action of $h \in H$ on $H^*$ and $H$:
\[
he_i = \sum_j h_i^j e_j, \quad he_i = \sum_i h_i^j e_i^*,
\]
we see that
\[
\mathcal{L}(h\phi) = \sum_i e_i^* \otimes e_i h\phi = \sum_i e_i^* \otimes \sum_j h_i^j e_j \phi = \sum_j he_j \otimes e_j \phi = h_1 \mathcal{L}(\phi).
\]
The proof of part (b) is the same.

Similarly we find
\[
\mathcal{L}^S(h\phi) = \sum_i e_i^* \otimes S(e_i h\phi) = 1 \otimes S(h) \sum_i e_i^* \otimes S(e_i \phi) = (Sh)_2 \mathcal{L}^S(\phi).
\]

On the other hand
\[
\mathcal{L}^S(h\phi) = \sum_{i,j} e_i^* \otimes S(h_i^j e_j \phi) = \sum_{j} he_j^* \otimes S(e_j \phi) = (h_1) \mathcal{L}^S(\phi).
\]

Part (d) is proved similarly. \[\Box\]
Remark 15.2. Let $M$ be an $H$-module. Then we can define for $m \in M$ the action of, say, twisted exponentials by

\[ L_M^S(m) = \sum e_i^* \otimes S_M(e_im), \]
\[ R_M^S(m) = \sum S_H(e_i)(m) \otimes e_i^*, \]

where $S_M : M \to M$ is supposed to be a linear map such that

\[ S_M(hm) = S_H(h)S_M(m), \quad h \in H_T, \quad m \in M. \]

In this situation Lemma 15.1 will still be true, with the same proof. We will frequently omit the subscript $M$ on the exponentials in case the module they are acting on should be clear.

16. Multiplicativity of (Twisted) Exponentials

The domain $H^*$ and range $(H \otimes H)^*$ of twisted exponentials are algebras so one can expect, for suitable twistings $A, B$, that these algebra structures are related. We will need the following cases.

Lemma 16.1. The exponentials $\mathcal{L}, \mathcal{L}^S, \mathcal{R}, \mathcal{R}^S$ are multiplicative: we have for all $\phi, \beta \in H^*$

(a) $\mathcal{L}(\phi \beta) = \mathcal{L}(\phi)\mathcal{L}(\beta), \mathcal{R}(\phi \beta) = \mathcal{R}(\phi)\mathcal{R}(\beta)$.

(b) $\mathcal{L}^S(\phi \beta) = \mathcal{L}^S(\phi)\mathcal{L}^S(\beta), \mathcal{R}^S(\phi \beta) = \mathcal{R}^S(\phi)\mathcal{R}^S(\beta)$.

Proof. Introduce structure constants $C^k_{ij} \in \mathbb{C}$ for the multiplication in $H^*$ and the comultiplication in $H$:

\[ e_i^* e_j^* = \sum_k C^k_{ij} e_k^*, \quad \pi(e_k) = \sum_{i,j} C^k_{ij} e_i \otimes e_j. \]

Then, for instance,

\[ \mathcal{L}^S(\phi \beta) = \sum_k e_k^* \otimes S(e_k(\phi \beta)) = \sum_k e_k^* \otimes S(C^k_{ij} e_i(\phi) e_j(\beta)) \]
\[ = \sum_{k,i,j} C^k_{ij} e_k^* \otimes S(e_i(\phi) e_j(\beta)) = \sum_{i,j} e_i^* e_j^* \otimes S(e_i(\phi)) S(e_j(\beta)) \]
\[ = \mathcal{L}^S(\phi) \mathcal{L}^S(\beta). \]

Similar computations gives the multiplicativity of the other exponentials. \qed

17. $H_T$-Leibniz Algebras and Exponentials

If $H$ is a Hopf algebra, then an $H$-Leibniz algebra is a associative algebra in the category of $H$-modules. So $V$ is both an $H$-module and an
algebra, such that the unital algebra structure maps
\[ i: \mathbb{C} \to V \quad \quad m: V \otimes V \to V \]
\[ \lambda \mapsto \lambda 1_V \quad \quad a \otimes b \mapsto ab \]
are \( H \)-module morphisms (giving \( \mathbb{C} \) the trivial \( H \)-module structure): if \( \varepsilon \) is the counit and \( \pi(h) = \sum h' \otimes h'' \) is the coproduct, then
\begin{equation}
hi(\lambda) = i(\varepsilon(h)\lambda), \quad hm(a \otimes b) = \sum m(h'a \otimes h''b).
\end{equation}
In case \( H = H_D = \mathbb{C}[D] \) a (commutative) \( H_D \)-Leibniz algebra is just a (com- mutative) differential algebra, and the last equation of (17.1) is the Leibniz rule of differentiation of a product. For any cocommutative Hopf algebra the dual \( H^* \) is a commutative \( H \)-Leibniz algebra. We will assume cocommutativity from now on, being interested in the case \( H = H_T \).

Examples of \( H_T \)-Leibniz algebras \( V \) are \( \mathbb{C}_\mathbb{Z} \), \( \mathbb{C}_\mathbb{Z}_{\text{pol}} \), \( H^o \) and the localiza-
tions \( K, K^o \).

**Remark 17.1.** We noted in Remark 15.2 that (twisted) exponentials act on any \( H \)-module and that they will have then the \( H \)-covariance properties of Lemma 15.1. In case \( V \) is a \( H_T \)-Leibnitz algebra (twisted) exponentials will act on \( V \), and in this situation furthermore the multiplicativity properties of Lemma 16.1 will still hold, with the same proof.

**18. Inverses of exponentials.**

The twisted exponential operators \( \mathcal{L}^S, \mathcal{R}^S \) are, in a sense, inverses of the untwisted ones. Indeed, recall that for \( \phi \in \mathbb{C}_\mathbb{Z} \), i.e., if \( \phi \) is a function on the integers, then the coproducts \( m^*(\phi) \) and \( m^*_{1 \otimes S} \) are functions on \( \mathbb{Z} \times \mathbb{Z} \) given by (14.2). We can define functions on \( \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \) by
\[ m^*(m^*_{1 \otimes S}(\phi))(p, q, r) = \phi(p - q + r), \]
and restricting to the diagonal \( q = r \) we get \( \phi \) back. This holds not only in \( \mathbb{C}_\mathbb{Z} \) but is an identity for exponential operators acting on any \( H_T \)-module. More precisely, we have the following Lemma.

**Lemma 18.1.** Let \( V \) be an \( H_T \)-module. Then
\[ \mathcal{R} \circ \mathcal{R}^S = \mathcal{R}_{1 \otimes S} \circ \mathcal{R} = 1_V \otimes 1_{H^o_T}, \]
\[ \mathcal{L} \circ \mathcal{L}^S = \mathcal{L}_{1 \otimes S} \circ \mathcal{L} = 1_{H^o_T} \otimes 1_V. \]

**Proof.** For instance
\[ \mathcal{R}^S \circ \mathcal{R}(v) = \sum_{i,j} S(e_i)e_j(v) \otimes e_i^*e_j^* = \sum_{i,j,k} \gamma^S_{i,j,k} e_k(v) \otimes e_i^*e_j^*. \]
Now \( \gamma_{ij}^{S,k} = \langle S(e_i) e_j, e_k^* \rangle = \langle S(e_i) \otimes e_j, m^*(e_k^*) \rangle \), where \( m^*(e_k^*) = \sum e_k^* \otimes e_k^* \) is the coproduct on \( H^* \) dual to the product on \( H \). Hence

\[
\mathcal{R}^S \circ \mathcal{R}(v) = \sum_{i,j,k} e_k(v) \otimes \gamma_{ij}^{S,k} e_j e_i^* = \sum_k e_k(v) \otimes \mu_{S \otimes 1}(m^*(e_k)),
\]

where \( \mu_{S \otimes 1} \) is the twisted multiplication on \( H^* \), so

\[
\mathcal{R}^S \circ \mathcal{R}(v) = \sum_k e_k(v) \otimes \sum S(e_k^*) e_k^* = \sum_k e_k(v) \otimes i^*(e_k^*) I_{H^*},
\]

using the connection \( i^*(\phi) 1 = \sum S(\phi') \phi'' \) between counit, antipode and coproduct on a Hopf algebra, hence

\[
\mathcal{R}^S \circ \mathcal{R}(v) = \sum_k i^*(e_k^*) e_k(v) \otimes I_{H^*} = v \otimes 1_{H^*},
\]

since \( i^*(e_k^*) e_k = 1_H \). Similarly one proves the other parts.

**Theorem 19.** Adjoint action of exponentials

Define for \( h \in H \) and \( X \in \text{End}(V) \), for some \( H \)-module \( V \), the adjoint action of \( h \) by

\[
\text{ad}_h(X) = \sum h' \circ X \circ S(h'').
\]

The twisted coproduct \( \pi_{1 \otimes S} : h \mapsto h' \otimes S(h'') \) is dual to the twisted multiplication \( \mu_{1 \otimes S} \) on \( H^* \). Extend the adjoint action termwise to exponential operators:

\[
\text{ad}_{\mathcal{R}^S}(X) = \sum \text{ad}_{e_i}(X) \otimes e_i^* , \quad \text{ad}_{\mathcal{R}^S}(X) = \sum \text{ad}_{s_i}(X) \otimes e_i^* .
\]

**Lemma 19.1.** For all \( X \in \text{End}(V) \) we have

\[
\text{ad}_{\mathcal{R}^S}(X) = \mathcal{R} \circ X \circ \mathcal{R}^S, \quad \text{ad}_{\mathcal{R}^S}(X) = \mathcal{R}^S \circ X \circ \mathcal{R}.
\]

**Proof.** We have

\[
\pi_{1 \otimes S} (\mathcal{R}) = \sum_i \pi_{1 \otimes S}(e_i) \otimes e_i^* = \sum_i \gamma_{jk} e_j \otimes e_k \otimes e_i^*,
\]

where

\[
\gamma_{jk} = \langle \pi_{1 \otimes S}(e_i), e_j^* \otimes e_k^* \rangle = \langle e_i, e_j^* S(e_k^*) \rangle .
\]

So

\[
\pi_{1 \otimes S} (\mathcal{R}) = \sum e_j \otimes e_k \otimes e_j^* S(e_k^*) = \mathcal{R} \otimes \mathcal{R}^S,
\]

where we use

\[
\mathcal{R}^S(f) = \sum S(e_i)(f) \otimes e_i^* = \sum e_j(f) \otimes S(e_j^*).
\]

Hence

\[
\text{ad}_{\mathcal{R}^S}(X) = \mathcal{R} \circ X \circ \mathcal{R}.
\]

The other part is proved similarly. \( \square \)
Lemma 19.2. If $V$ is an $H$-module, the map $H \to \text{End}(\text{End}(V))$, $h \mapsto \text{ad}_h : \text{End}(V) \to \text{End}(V)$, $X \mapsto \sum h' \circ X \circ S(h'')$ gives $\text{End}(V)$ the structure of $H$-Leibniz algebra. In particular
\[
\text{ad}_h(X \circ Y) = \sum \text{ad}_{h'}(X) \circ \text{ad}_{h''}(Y).
\]

Corollary 19.3.
\[
\text{ad}_{\mathcal{D}}(X \circ Y) = \text{ad}_{\mathcal{D}}(X) \circ \text{ad}_{\mathcal{D}}(Y).
\]

20. Multivariable expansions

Let $H$ be a Hopf algebra, commutative and cocommutative for simplicity. If $M$ is an $H$-module, and $m \in M$, then we get a distribution $\mathcal{R}_M(m)$ on $H$ and the twisted version $\mathcal{R}_M^S(m)$. We think of these distributions as expandable in a basis $e_i^*$ of $H^*$, and to emphasize this we write $\mathcal{R}_M(e^*)(m)$, $\mathcal{R}_M^S(e^*)(m)$. We will also consider distributions on $H \otimes H$ and $H \otimes H \otimes H$. For instance if $\mathcal{D}$ is a distribution on $H^{\otimes 3}$ it will be expandable in $e_{i,1}^* = e_i^* \otimes 1 \otimes 1$, $e_{i,2}^* = 1 \otimes e_i^* \otimes 1$, $e_{i,3}^* = 1 \otimes 1 \otimes e_i^*$, and we write $\mathcal{D}(e_1^*, e_2^*, e_3^*)$. In this section we discuss how the invertibility properties of exponential operators, see section 18, leads to identities between distributions in several variables.

First we need some notation. If $e^* \in H^*$ we write $e_{i,23}^* = m_{1 \otimes S}(e^*) = \sum e'^* \otimes e^*''$, $e_{13}^* = \sum e'^* \otimes 1 \otimes e^*'''$ and $e_{23}^* = 1 \otimes e^* \otimes e^*''$. We can consider exponential operators in these variables, for instance
\[
\mathcal{R}_M(e_{23}^*) m = \sum S(e_i) m \otimes e_{i,23}^*.
\]

Lemma 20.1. For all $m \in M$ we have
\[
\mathcal{R}_M(e_1^*) \mathcal{R}_M^S(e_2^*)(m) = \mathcal{R}_M(e_{12}^*) m.
\]

Proof. We have
\[
\mathcal{R}_M(e_1^*) \mathcal{R}_M^S(e_2^*)(m) = \sum e_i S(e_j)(m) \otimes e_i^* \otimes e_j^*
= \sum c_{ij}^k e_k(m) \otimes e_i^* \otimes e_j^*
\]
where $\sum e_i S(e_j) = \sum c_{ij}^k e_k$. We also have $m_{1 \otimes S}(e_k^*) = \sum c_{ij}^k e_i^* \otimes e_j^*$, so that
\[
\sum e_k(m) \otimes m_{1 \otimes S}(e_k^*) = \mathcal{R}_M(e_{12}^*) (m).
\]

Then, by section 18 and Lemma 20.1, we get identities like
\[
\mathcal{R}_M(e_1^*) (m) = \mathcal{R}_M(e_2^*) \mathcal{R}_M^S(e_3^*) \mathcal{R}_M(e_1^*) (m) = \mathcal{R}_M(e_2^*) \mathcal{R}_M(e_{12}^*) (m).
\]

We now specialize to the case $H = \mathbb{C}[\Delta]$ and $M = \mathbb{C}_Z^\text{pol}$. 

LEMMA 20.2. Then for all \( \ell \geq 0 \) we have
\[
\tau_{12}[\ell] = \sum_{s=0}^{\ell} \binom{\ell}{s} \tau_{13}[\ell-s] \tau_{32}[s].
\]
Here \( \tau_{32}[s] = R_{S}(\tau_{2}) \tau_{3}[s] \).

PROOF. We have
\[
\tau_{12}[\ell] = R_{S}(\tau_{2})(\tau_{1}[\ell])
= R_{S}(\tau_{3})R_{S}(\tau_{3})(\tau_{2})(\tau_{1}[\ell]) \quad \text{Lemma 18.1}
= R_{S}(\tau_{23})R_{S}(\tau_{3})(\tau_{1}[\ell]) \quad \text{Lemma 20.1}
= R_{S}(\tau_{23})(\tau_{13}[\ell]) = \sum_{s=0}^{\ell} \binom{\ell}{s} \tau_{13}[\ell-s] \tau_{32}[s].
\]
\(\square\)

21. Extension of Exponential Distributions

Let \( W, H \) be vector spaces. A \( W \)-valued distribution on \( H \) is just a linear map \( H \to W \). In this section we describe the following phenomenon. Assume that \( H \) is an algebra and let \( H \subset H' \) be an inclusion of \( H \) in a bigger algebra \( H' \). Let \( \{e_i\} \) be a basis for \( H \), with \( e_i^* \in H^* \) dual functions. Let \( M \) be an \( H' \)-module and consider for \( m \in M \) the action by the exponential for \( H \) on \( m \):
\[
R_{H}(m) = \sum e_i(m)e_i^*.
\]
So \( R_{H}(M) : H \to M \) is the \( M \)-valued distribution given by
\[
R_{H}(m)(h) = h.m, \quad h \in H.
\]
Now it can happen that the infinite series (21.1) makes sense, for all \( m \in M \), as a distribution on the bigger algebra \( H' \), in such a way that
\[
R_{H}(m)(h') = h'.m, \quad h' \in H'.
\]
In this case we say that the distributions \( R_{H}(m) \) (on \( H \)) can be extended to \( H' \). This means that we can replace on \( M \) the exponential operator \( R_{H'} \) by the simpler \( R_{H} \).

The basic example we are interested in is where \( H = \mathbb{C}[\Delta] \) and \( H' \) is either \( H_T \) or \( \hat{H}_T \). Here \( \mathbb{C}[\Delta] \) is the semigroup algebra of the semigroup generated by \( T \), with basis \( \Delta[\ell] \) and dual functions \( \tau[\ell], \ell \geq 0 \). So the exponential
operators of $\mathbb{C}[\Delta]$ on $M$ are defined by

\begin{align*}
\mathcal{L}_\Delta(f) &= \sum_{\ell \geq 0} \tau[\ell] \otimes \Delta[\ell](f), \\
\mathcal{R}_\Delta(f) &= \sum_{\ell \geq 0} \Delta[\ell](f) \otimes \tau[\ell], \\
\mathcal{L}_{1 \otimes S}(f) &= \sum_{\ell \geq 0} \tau[\ell] \otimes S(\Delta[\ell](f)), \\
\mathcal{R}_{1 \otimes S}(f) &= \sum_{\ell \geq 0} S(\Delta[\ell])(f) \otimes \tau[\ell],
\end{align*}

although $S(\Delta[\ell])$ does not belong to $\mathbb{C}[\Delta]$, but exists as an operator on $M$.

**Lemma 21.1.** Let $H'$ be either $H_T$ or $\hat{H}_T$, with basis $\{e_i\}$ and dual functions $\{e_i^*\}$ and let $M$ be an $H'$-module such that the $H'$ generators acting on $M$ can be expanded as

\begin{equation}
(21.3) \quad e_{i,M} = \sum_{\ell \geq 0} c_{i,\ell} \Delta_M[\ell], \quad c_{i,\ell} = \text{Tr} \left( \frac{e_i \tau(\ell)}{\tau} \right).
\end{equation}

Then, for all $m \in M$, we have

\begin{align*}
\mathcal{L}_\Delta(m) &= \mathcal{L}^{H'}(m), \\
\mathcal{R}_\Delta(m) &= \mathcal{R}^{H'}(m), \\
\mathcal{L}_{1 \otimes S}(m) &= \mathcal{L}^{H'}_{1 \otimes S}(m), \\
\mathcal{R}_{1 \otimes S}(m) &= \mathcal{R}^{H'}_{1 \otimes S}(m).
\end{align*}

**Proof.** We have an inclusion of $\mathbb{C}^\text{pol}_Z$ in $(H')^*$ given by

\[ \tau(\ell) = \sum_i c_{i,\ell} e_i^*, \quad c_{i,\ell} = \text{Tr} \left( \frac{e_i \tau(\ell)}{\tau} \right), \]

so that $c_{i,\ell}$ are the same coefficients that appear in (21.3). Then the proof of the lemma is a simple computation. For instance,

\[ \mathcal{R}_{1 \otimes S}^{\Delta}(m) = \sum_{\ell \geq 0} S(\Delta[\ell])(m) \tau(\ell) = \sum_{\ell \geq 0} S(\Delta[\ell])(m) \sum_i c_{i,\ell} e_i^* \]

\[ = \sum_i S \left( \sum_{\ell \geq 0} c_{i,\ell} \Delta[\ell] \right)(m) e_i^* = \sum_i S(e_i)(m) e_i^* = \mathcal{R}^{H'}_{1 \otimes S}(m). \]

The rest is proved similarly. \hfill \Box

**Example 21.2.** Consider the $H_T$-module $M = \mathbb{C}^\text{pol}_Z$. This satisfies the conditions of Lemma 21.1. Indeed, on $\mathbb{C}^\text{pol}_Z$ the action of $\Delta$ is locally nilpotent and we have on $\mathbb{C}^\text{pol}_Z$

\[ \Delta = (1 - T^{-1}) = (1 - \frac{1}{1 + \Delta}) = -\sum_{k=1}^{\infty} (-\Delta)^k, \]

\[ \partial = \text{log}(T) = \text{log}(1 + \Delta) = \sum_{k=1}^{\infty} (-\Delta)^k / k. \]
From this one easily checks the condition (21.3). So on $\mathbb{C}^\text{pol}_Z$ we can replace the exponential operators (14.3) for $H_T$ by the ones for $\hat{\mathbb{C}}[\Delta]$ in (21.2). In particular we can calculate the coproduct on $\mathbb{C}^\text{pol}_Z$ use these simpler exponentials. For instance

$$m^*_1 \otimes S(\tau) = \hat{\mathbb{R}}^S(\tau) = \tau \otimes 1 - 1 \otimes \Delta^*[1] - 1 \otimes \Delta^*[-1]$$

can be written as

$$(21.4) \quad m^*_1 \otimes S(\tau) = \hat{\mathbb{R}}^{\mathbb{C}[\Delta]}_1 \otimes S(\tau) = \tau \otimes 1 - 1 \otimes \tau.$$ 

More generally, on $\mathbb{C}^\text{pol}_Z$ we have an action of $\hat{H}_T$, so we have a twisted exponential for $\hat{H}_T$ on $\mathbb{C}^\text{pol}_Z$:

$$\hat{\mathbb{R}}^\tau(f) = \sum S(e_{n,k})(f)e_{n,k},$$

and we calculate the twisted coproduct on $\mathbb{C}^\text{pol}_Z$ by thinking of it as a subspace of the dual of $\hat{H}_T$. For instance, in this way we get

$$m^*_1 \otimes S(\tau) = \tau \otimes 1 - 1 \otimes e_{1,0}^* - 1 \otimes e_{-1,0}^* - 1 \otimes e_{0,1}^*,$$

which reduces to the previous result (21.4) by (10.1).

\begin{example}
Now take $M = K_{\text{Sing}}$. We think of it as an $H = \mathbb{C}[\Delta]$ or $H'$-module by the $S$-twisted action: if $F \in K$

$$P(\Delta, \partial_\tau) F = S(P(\Delta, \partial_\tau)) F.$$

It is no longer true, as on $\mathbb{C}^\text{pol}_Z$, that $\Delta$ acts locally nilpotent on $K_{\text{Sing}}$. We would like to write

$$(21.5) \quad \partial_\tau = -\log(T^{-1}) = -\sum_{k>0} (\Delta)^k / k,$$

but the infinite sum on the right does not make sense in $K_{\text{Sing}}$. However, we can think of $K_{\text{Sing}}$ as consisting of distributions on $\mathbb{C}^\text{pol}_Z$, via the embedding $F \in K \mapsto \mathcal{D}_F$, where

$$\mathcal{D}_F(G) = \text{Tr}(FG), \quad G \in \mathbb{C}^\text{pol}_Z.$$ 

Now we have for any distribution on $\mathbb{C}^\text{pol}_Z$ the expansion

$$\mathcal{D} = \sum_{\ell \geq 0} \mathcal{D}(\tau[\ell]) \frac{1}{\tau[\ell+1]},$$

and (21.5) makes sense in the dual of $\mathbb{C}^\text{pol}_Z$ and similarly we can expand

$$\Delta = 1 + \Delta = 1 + \frac{1}{1 - \Delta}.$$
as a power series in $\Delta$. This is sufficient to check (21.3) and the Lemma 21.1 applies to $K_{\text{Sing}}$. Now $K = \mathbb{C}_{\mathbb{Z}}^{\text{pol}} \oplus K_{\text{Sing}}$, so by combining with Example 21.2 we see that we can replace on $K$ the exponential operators for $H_T$ or $\hat{H}_T$ by the simpler ones for $\mathbb{C}[\Delta]$. □

**Example 21.4.** We can generalize Example 21.2 as follows. Let $M$ be any $H_T$-module that is locally finite, i.e., $M$ decomposes into generalized eigenspaces for $T$ and $T^{-1}$:

$$ M = \bigoplus_{\lambda \in \mathbb{C}^\times} M_{\lambda}, $$

where

$$ M_{\lambda} = \{ m \in M \mid (T - \lambda)^k m = 0, \ k \gg 0 \}. $$

Then by a slight variation of the proof of Lemma 21.1 we see that the exponential operators of $\mathbb{C}[\Delta]$ and $H_T$ or $\hat{H}_T$ on locally finite $M$ give the same distributions (on $H_T$ or $\hat{H}_T$). □

**Example 21.5.** Let $M = H_T B$ be the free $H_T$-module generated by a single element $B$. In this case we cannot have (21.3), in particular $\Delta$ cannot be expressed in terms of $\Delta$. Now let $N = \mathbb{C}[\Delta] B$, the free $\mathbb{C}[\Delta]$-module generated by $B$. Then $N$ has a completion $\tilde{N}$ consisting of infinite sums $\sum_{k \geq 0} c_k \Delta^k B$. Then we can define in $\tilde{N}$ an action of $H_T$ or of $\hat{H}_T$, such that the conditions of Lemma 21.1 are satisfied, and we can extend the exponential distributions from $\mathbb{C}[\Delta]$ to $H_T$ or $\hat{H}_T$. □

### 22. Localization of (Twisted) Coproducts and Dirac Distributions

We defined the twisted coproduct of $\phi \in H^*$ as the distribution on $H \otimes H$ given by

$$ m_{1 \otimes \mathcal{S}}(\phi)(h_1 \otimes h_2) = \phi(h_1 Sh_2). \quad (22.1) $$

We are interested in localizations $K$ of $H^*$, and we would like to extend the definition of $m_{1 \otimes \mathcal{S}}^*$ to $K$. We can not use (22.1), as $\phi \in K$ is in general a singular function on $H$, and not defined on all $h \in H$.

Now we expressed the twisted coproduct $m_{1 \otimes \mathcal{S}}^*$ on $H^*$ in terms of twisted exponentials $\mathcal{L}^S, \mathcal{R}^S$. These expressions make sense for $F \in K$, $K$ being an $H$-module, but one finds that the two calculations give in this case in general a different result. So define for $F \in K$ the obstruction of extending $m_{1 \otimes \mathcal{S}}^*$ to $F$ as the Dirac Distribution associated to $F$ by

$$ \delta(F) = \mathcal{R}^S(F) - \mathcal{L}^S(F). $$

In case $F \in H^*$ we have $\delta(F) = 0$, of course.
EXAMPLE 22.1. We continue the example 14.2. We consider the localization \( K = \mathbb{C}[t][t^{-1}] \). Then
\[
\delta \left( \frac{1}{t} \right) = \exp(-t_2 \partial_1) \frac{1}{t_1} - S \exp(t_1 \partial_2) \frac{1}{t_2} = \sum_{k \geq 0} t_1^k t_2^{k+1} + \sum_{k \geq 0} t_1^k t_2^{k+1} = \sum_{n \in \mathbb{Z}} t_1^n t_2^{-n-1}.
\]
This is the usual formal delta distribution, usually denoted by \( \delta(t_1 - t_2) \) or \( \delta(t_1, t_2) \), see [Kac98].

EXAMPLE 22.2. We continue with Example 14.3 and consider the localization \( K = M^{-1}\mathbb{C}_{\mathbb{Z}}^{\text{pol}} \) of \( \mathbb{C}_{\mathbb{Z}}^{\text{pol}} \subset \mathbb{C}_{\mathbb{Z}} \). By Example 21.3 the exponential operators of \( \mathbb{C}[\Delta] \) and of \( H' = H_T, \hat{H_T} \) give the same distribution on \( K \). So define
\[
\rho\left( \frac{1}{\tau} \right) = \mathcal{A}_{1 \otimes \mathcal{S}} \left( \frac{1}{\tau} \right) = \sum_{\ell \geq 0} \frac{\tau_2(\ell)}{\tau_1(\ell + 1)} \quad \lambda\left( \frac{1}{\tau} \right) = \mathcal{L}_{1 \otimes \mathcal{S}} \left( \frac{1}{\tau} \right) = -\sum_{\ell \geq 0} \frac{\tau_1(\ell)}{\tau_2(\ell + 1)}.
\]
then we also have
\[
\rho\left( \frac{1}{\tau} \right) = \sum_{n \in \mathbb{Z}} S(\Delta[n]) \frac{1}{\tau} \otimes \Delta^*[n] = \sum_{n,k \in \mathbb{Z}, k \geq 0} S(e_{n,k}) \frac{1}{\tau} \otimes e_{n,k}^*,
\]
\[
\lambda\left( \frac{1}{\tau} \right) = -\sum_{n \in \mathbb{Z}} \Delta^*[n] \otimes S(\Delta[n]) \frac{1}{\tau} = -\sum_{n,k \in \mathbb{Z}, k \geq 0} e_{n,k}^* \otimes S(e_{n,k}) \frac{1}{\tau},
\]
and so we get 3 forms of the Dirac distribution \( \delta\left( \frac{1}{\tau} \right) = \rho\left( \frac{1}{\tau} \right) - \lambda\left( \frac{1}{\tau} \right) \):
\[
\delta\left( \frac{1}{\tau} \right) = \sum_{n \in \mathbb{Z}} \tau_1(n) \tau_2(-n-1)
\]
\[
= \sum_{n \in \mathbb{Z}} S(\Delta[n]) \frac{1}{\tau} \otimes \Delta^*[n] + \sum_{n \in \mathbb{Z}} \Delta^*[n] \otimes S(\Delta[n]) \frac{1}{\tau}
\]
\[
= \sum_{n,k \in \mathbb{Z}, k \geq 0} S(e_{n,k}) \frac{1}{\tau} \otimes e_{n,k}^* + \sum_{n,k \in \mathbb{Z}, k \geq 0} e_{n,k}^* \otimes S(e_{n,k}) \frac{1}{\tau}.
\]
\]

23. Difference Operators

Let \( H \) be a Hopf algebra and let \( V \) be vector space. Denote by \( VH \) the vector space of \( V \)-valued \( H \)-operators: elements of \( VH \) are finite sums of the form
\[
P = \sum_i p_i e_i, \quad p_i \in V,
\]
(23.1)
using a basis \( \{ e_i \} \) for \( H \). Then \( VH \) is an \( H \)-module: if \( h \in H, P \in V \) then we have

\[
h . P = \sum h' (P) h'' ,
\]
where the coproduct of \( h \) is given by \( \sum h' \otimes h'' \).

In case \( H = HT \) we will call elements of \( VH_T \) difference operators with values in \( V \). In case \( V \) is an \( HT \)-Leibniz algebra, see section 17, \( VH_T \) will be an algebra in the obvious way, but we will not need this at this point.

The following operations make sense for any space \( VH \). First we have the adjoint map of \( VH \) given, for \( P = \sum p_i e_i \), by

\[
P^* = \sum S(e_i) . p_i .
\]
In case \( V \) happens to be an \( H \)-Leibniz algebra this is an anti-involution. Secondly there is the antipodal operator

\[
P^S = \sum p_i S(e_i).
\]
Thirdly we will need later the map

\[
VH \rightarrow V, \quad P \mapsto P_V = \sum_i S(e_i) (p_i).
\]

### 24. Expansions of Distributions

Let \( W \) be a vector space. We denote by \( W_n \) the space of all \( W \)-valued distributions on \( K^{\otimes n} \). For simplicity consider first \( \mathcal{D} \in W_1 \). It has a decomposition

\[
\mathcal{D} = \mathcal{D}_{\text{Hol}} + \mathcal{D}_{\text{Sing}},
\]
where

\[
\mathcal{D}_{\text{Hol}} : K_{\text{Sing}} \rightarrow W, \quad \mathcal{D}_{\text{Sing}} : C_{\mathbb{Z}}^{\text{pol}} \rightarrow W,
\]
i.e., the holomorphic part of the distribution vanishes on \( C_{\mathbb{Z}}^{\text{pol}} \) and the singular part on \( K_{\text{Sing}} \).

Now we have a basis \( \{ \tau[\ell] \}_{\ell \geq 0} \) for \( C_{\mathbb{Z}}^{\text{pol}} \) and a basis \( \{ S(e_{n,k}) \frac{1}{\tau} \} \) for \( K_{\text{Sing}} \). This implies that we can represent the distribution \( \mathcal{D} \) and its components by kernels \( \mathcal{D}(\tau) = \mathcal{D}_{\text{Hol}}(\tau) + \mathcal{D}_{\text{Sing}}(\tau) \), where

\[
\mathcal{D}_{\text{Hol}}(\tau) = \sum \mathcal{D} \left( S(e_{n,k}) \frac{1}{\tau} \right) e_{n,k}^* ,
\]

\[
\mathcal{D}_{\text{Sing}}(\tau) = \sum \mathcal{D}(\tau[\ell]) \frac{1}{\tau[\ell + 1]}.
\]
Then we have for all \( F \in K \)

\[
\mathcal{D}(F) = \text{Tr}(\mathcal{D}(\tau) F).
\]
To make sense of this formula, let
\[(24.2) \hat{\mathcal{K}} = \hat{\mathbb{C}}_{\text{pol}}^\mathbb{Z} \oplus \mathbb{K}_{\text{Sing}}, \quad \hat{\mathbb{C}}_{\text{pol}}^\mathbb{Z} = \bigoplus C_{\tau(n)} e_{n,k} \subset (\hat{\mathcal{H}}_{\text{T}})^*.
\]

Then the trace extends to \(\hat{\mathcal{K}}\). Now \(\mathcal{D}(\tau)F\) is an infinite sum of terms in \(W \otimes \hat{\mathcal{K}}\), of which at most a finite number have a second component with nonvanishing trace. The trace then also extends then to such infinite sums. One also sees that we can let \(F \in \hat{\mathcal{K}}\), and \(\mathcal{D}(F)\) is well defined in that case.

Now note that in the expression (24.1) for \(\mathcal{D}_{\text{Hol}}\) the \(\hat{\mathcal{H}}_{\text{T}}\) form of the distribution \(\rho(\frac{1}{\tau})\) appears, see Example 22.2. So we can write \(\mathcal{D}_{\text{Hol}}\) in the form adapted to \(C[\Delta]\):
\[
\mathcal{D}_{\text{Hol}}(\tau) = \sum_{\ell \geq 0} \mathcal{D}(\frac{1}{\tau(\ell+1)}) \tau(\ell),
\]
where we think of \(\mathcal{D}_{\text{Hol}}\) as a distribution on \(C[\Delta] \simeq S(C[\Delta])^1\) that can be extended to a distribution on all of \(\hat{\mathcal{H}}_{\text{T}} \simeq K_{\text{Sing}}\), as discussed in section 21. This means that we have a convenient uniform expression for the kernel of any distribution in one variable:
\[(24.3) \mathcal{D}(\tau) = \sum_{n \in \mathbb{Z}} \mathcal{D}(\tau(n)) \tau(-n-1),\]
using the notation (12.1). If \(F \in \mathcal{K}\) it defines a rational distribution \(\mathcal{D}_F\) with \(F\) as kernel:
\[(24.4) \mathcal{D}_F(G) = \text{Tr}(FG).\]
This means that any \(F \in \mathcal{K}\) (or rather its image in \(K^*\)) has a formal expansion
\[(24.5) F = \sum_{n=-N}^{\infty} F_n \tau(-n-1), \quad F_n = \text{Tr}(F \tau(n)).\]
For example we have the following infinite expansion of the rational function \(\frac{1}{\tau^2} \in K_{\text{Sing}}\):
\[
\frac{1}{\tau^2} = \sum_{k \geq 0} \frac{k!}{\tau(k+2)},
\]
if we think of \(\frac{1}{\tau^2}\) as a distribution (on \(\hat{\mathbb{C}}_{\text{pol}}^\mathbb{Z}\)).

In the same way we have for \(\mathcal{D} \in W_2\) a kernel
\[(24.6) \mathcal{D}(\tau_1, \tau_2) = \sum_{m,n \in \mathbb{Z}} \mathcal{D}(\tau(m) \otimes \tau(n)) \tau_1(-m-1) \tau_2(-n-1),\]
with for all \(F, G \in \mathcal{K}\)
\[
\mathcal{D}(F \otimes G) = \text{Tr}_{\tau_1} \text{Tr}_{\tau_2}(\mathcal{D}(\tau_1, \tau_2)F(\tau_1)G(\tau_2)).
\]
We will frequently identify a distribution with its kernel. We will also write
\[ \rho\left(\frac{1}{\tau}\right) = \rho(\tau_1, \tau_2), \quad \lambda\left(\frac{1}{\tau}\right) = \lambda(\tau_1, \tau_2), \quad \delta\left(\frac{1}{\tau}\right) = \delta(\tau_1, \tau_2), \]
and more generally we will use the notation \( a(\tau_i, \tau_j) \) for a distribution depending on variables \( \tau_i(n), \tau_j(n) \), where
\[ \tau_i(n) = 1 \otimes \cdots \otimes \tau(n) \otimes \cdots 1, \]
with \( \tau(n) \) on the \( i \)-th position.

Similarly any distribution \( D \in W_n \) has an expansion in \( \tau_i, i = 1, \ldots, n \). We will write for this reason \( W_n = W[[\tau_1^{\pm 1}, \tau_2^{\pm 1}, \ldots, \tau_n^{\pm 1}]] \), and we will denote by \( W[[\tau]] \subset W[[\tau^{\pm 1}]] \) the subspace of holomorphic distributions (i.e., distributions on \( K_{\text{Sing}} \)), and by \( W[[\tau^{-1}]] \tau^{-1} \) the subspace of singular distributions, those on \( \mathbb{C}^{\text{pol}}_{\mathbb{Z}} \).

25. Some Properties of Distributions

Distributions on \( K^{\otimes n} \) can be multiplied by difference operators, via the action of difference operators on \( K \). For instance if \( P \in VH_T \) and \( D \in \mathbb{C}[[\tau_1^{\pm 1}, \tau_2^{\pm 1}]] \) then we define
\[ P_1 D(F \otimes G) = D(P F \otimes G), \]
\[ P_2 D(F \otimes G) = D(F \otimes P G), \]
using the adjoint operation on \( VH_T \), see (23.2). Here we extend the \( \mathbb{C} \)-valued distribution \( \mathcal{D} \) on \( K \otimes K \) to one on \( V \otimes K \otimes K \) or \( K \otimes V \otimes K \) by linearity in the obvious way, to obtain a \( V \)-valued distribution.

We can not multiply distributions except in special cases. Any distribution \( \mathcal{D} \in W_n \) can be multiplied by a rational distribution in \( n \) ways. For example, for \( n = 2, F, L, M \in K \) and \( \mathcal{D}_F \) the rational distribution (24.4) we have
\[ (\mathcal{D}_F)_1 \mathcal{D}(L \otimes M) = \mathcal{D}(FL \otimes M), \quad (\mathcal{D}_F)_2 \mathcal{D}(L \otimes M) = \mathcal{D}(L \otimes FM). \]

In general we would like to define the product of a distribution \( \mathcal{D} \) in one variable with a two variable distribution \( \mathcal{E} \) by
\[ (\mathcal{D}_\mathcal{F})_1 \mathcal{E}(F \otimes G) = \text{Tr}_{\tau_1} \text{Tr}_{\tau_2} (\mathcal{D}(\tau_1) \mathcal{E}(\tau_1, \tau_2) F(\tau_1) G(\tau_2)). \]
(25.1)

However, for general \( \mathcal{D} \) and \( \mathcal{E} \) the products of kernels doesn’t make sense. We have the special two variable distributions \( \rho\left(\frac{1}{\tau}\right), \lambda\left(\frac{1}{\tau}\right) \) and \( \delta\left(\frac{1}{\tau}\right) \). The claim now is that in case \( \mathcal{E} = \rho, \lambda \) or \( \delta \) the definition (25.1) does make sense for any \( \mathcal{D} \). This can be easily checked.

To investigate the properties of such products we need the notion of the trace of a distribution in several variable with respect to one of the variables.
For instance, let \( \mathcal{D}(\tau_1, \tau_2) \) be a \( W \)-valued distribution on \( K \otimes K \). We can produce distributions in a single variable from \( \mathcal{D} \) by
\[
\text{Tr}_{\tau_1}(\mathcal{D}(\tau_1, \tau_2))(G) = \mathcal{D}(\tau_1, \tau_2)(1_K \otimes G),
\text{Tr}_{\tau_2}(\mathcal{D}(\tau_1, \tau_2))(F) = \mathcal{D}(\tau_1, \tau_2)(F \otimes 1_K).
\]
For instance
\[
\text{Tr}_{\tau_1}(\delta(\tau_1, \tau_2)) = \text{Tr}_{\tau_2}(\delta(\tau_1, \tau_2)) = 1,
\]
where \( 1 = 1_K \) is the rational distribution on \( K \) with kernel 1, so
\[
\text{Tr}_{\tau_1}(\delta(\tau_1, \tau_2))(F) = \text{Tr}(F) = 1_K(F).
\]

**Lemma 25.1.** Let \( \mathcal{D}(\tau) \) be a \( W \)-valued distribution on \( K \). Then
\[
\text{Tr}_{\tau_1}(\mathcal{D}(\tau_1)\rho(\tau_1, \tau_2)) = \mathcal{D}(\text{Hol}(\tau_2)),
\text{Tr}_{\tau_2}(\mathcal{D}(\tau_2)\rho(\tau_1, \tau_2)) = \mathcal{D}(\text{Sing}(\tau_1)),
\text{Tr}_{\tau_1}(\mathcal{D}(\tau_1)\lambda(\tau_1, \tau_2)) = -\mathcal{D}(\text{Sing}(\tau_2)),
\text{Tr}_{\tau_2}(\mathcal{D}(\tau_2)\lambda(\tau_1, \tau_2)) = -\mathcal{D}(\text{Hol}(\tau_1))
\]
so that
\[
\text{Tr}_{\tau_1}(\mathcal{D}(\tau_1)\delta(\tau_1, \tau_2)) = \mathcal{D}(\tau_2),
\text{Tr}_{\tau_2}(\mathcal{D}(\tau_2)\delta(\tau_1, \tau_2)) = \mathcal{D}(\tau_1).
\]

**Proof.** This follows from expanding \( \mathcal{D} \) as in (24.6) and using the explicit forms for \( \rho, \lambda, \delta \), see section 22, combined with orthogonality relations from section 12.

We use this to give explicit formulas for the product of an arbitrary distribution \( \mathcal{D} \) with \( \rho, \lambda, \delta \).

**Lemma 25.2.** For all \( F, G \in K \) we have
\[
\begin{align*}
\text{(a)} & \quad \rho(\frac{1}{\tau})(F \otimes G) = \text{Tr}(F_{\text{Hol}}G_{\text{Sing}}), \lambda(\frac{1}{\tau})(F \otimes G) = -\text{Tr}(F_{\text{Sing}}G_{\text{Hol}}), \\
\text{(b)} & \quad \delta(\frac{1}{\tau})(F \otimes G) = \text{Tr}(FG), \\
\text{(c)} & \quad \text{For all } \mathcal{D} \in W[[\tau^{\pm 1}]]
\end{align*}
\]
\[
\mathcal{D}(\tau_1)\rho(\frac{1}{\tau})(F \otimes G) = \mathcal{D}(FG_{\text{Sing}}), \quad \mathcal{D}(\tau_2)\rho(\frac{1}{\tau})(F \otimes G) = \mathcal{D}(F_{\text{Hol}}G)
\]
\[
\mathcal{D}(\tau_1)\lambda(\frac{1}{\tau})(F \otimes G) = -\mathcal{D}(FG_{\text{Hol}}), \quad \mathcal{D}(\tau_2)\lambda(\frac{1}{\tau})(F \otimes G) = -\mathcal{D}(F_{\text{Sing}}G)
\]
\[
\mathcal{D}(\tau_1)\delta(\frac{1}{\tau})(F \otimes G) = \mathcal{D}(FG), \quad \mathcal{D}(\tau_2)\delta(\frac{1}{\tau}) = \mathcal{D}(FG)
\]

**Proof.** Part (a) follows from the explicit forms for \( \rho, \lambda \) in Example 22.2, combined with the expansions of \( F \) and \( G \) (see (24.5)) and with the orthogonality relations of section 12. The part (b) follows from this. For
part (c) we calculate for instance

\[ D.\rho(\frac{1}{\tau})(F \otimes G) = \text{Tr}_{\tau_1} \text{Tr}_{\tau_2} (D(\tau_1)\rho(\tau_1, \tau_2)F(\tau_1)G(\tau_2)) \]

\[ = \text{Tr}_{\tau_1} (D(\tau_1)F(\tau_1)G_{\text{Sing}}(\tau_1)) \]

\[ = D(FG_{\text{Sing}}), \]

since by Lemma 25.1 we have

\[ \text{Tr}_{\tau_2} (\rho(\tau_1, \tau_2)G(\tau_2)) = G_{\text{Sing}}(\tau_1), \]

identifying \( G \) with the rational distribution of which it is the kernel. The other identities follow similarly. \( \square \)

We can not only multiply \( \rho(\frac{1}{\tau}), \lambda(\frac{1}{\tau}) \) and \( \delta(\frac{1}{\tau}) \) by distributions, but also by distribution valued difference operators. For instance, let \( V = W[[\tau^{\pm 1}]] \) and consider distributions defined by, if \( P = \sum P_n(\tau)\Delta[n] \in VH_T, \)

\[ P(\tau_1)\delta(\tau_1, \tau_2)(F \otimes G) = \sum_{n \in \mathbb{Z}} P_n(F\Delta[n]G), \]

(25.2)

\[ P(\tau_2)\delta(\tau_1, \tau_2)(F \otimes G) = \sum_{n \in \mathbb{Z}} P_n(\Delta[n] \langle F \rangle G). \]

**Lemma 25.3.** For \( P(\tau) \in VH_T, \) where \( V = W[[\tau^{\pm 1}]] \) we have

(25.3) \[ P(\tau_1)\delta(\tau_1, \tau_2) = P^*(\tau_2)\delta(\tau_1, \tau_2). \]

**26. Distributions of the form \( P_1^{\frac{1}{\tau}} \)**

We will need, if \( P \in VH_T, \) expressions of the form

(26.1) \[ P^* \frac{1}{\tau} = \sum_n P_n S(\Delta[n]) \frac{1}{\tau}. \]

This is a \( V \)-valued distribution on \( K \) (in fact on \( \mathbb{C}_H^{\text{pol}} \)). Since \( V \) is an \( H_T \)-module, we can let the exponential operator \( R \) of \( V \) act on expressions (26.1). Recall from (10.3)

(26.2) \[ \Delta^*[m].S(\Delta[n]) \frac{1}{\tau} = 0, \]

in case of opposite sign \( m > 0 > n \) or \( m < 0 < n \). In case of the same sign \((m, n \geq 0) \) or \( m, n \leq 0 \) we have

(26.3) \[ \Delta^*[m].S(\Delta[n]) \frac{1}{\tau} = \tau|m|S(\Delta[n]) \frac{1}{\tau}. \]
We use this to note that for \( n \neq 0 \) we have

\[
R V P_n S(\Delta[n]) \frac{1}{\tau} = \begin{cases} 
\sum_{k \geq 0} \Delta V[k] P_n \tau[k] S(\Delta[n]) \frac{1}{\tau} & n > 0 \\
\sum_{k \geq 0} \Delta V[k] P_n \tau[k] S(\Delta[n]) \frac{1}{\tau} & n < 0
\end{cases}
\]

Recall, see section 24, the notion of the singular part of a distribution \( \mathcal{D} \), We write \( \text{Sing}(\mathcal{D}) = \mathcal{D}_{\text{Sing}} \).

**Lemma 26.1.**

\[
(P^*)^S \frac{1}{\tau} = \text{Sing} \left( R_V (\tau) P^1 \frac{1}{\tau} \right),
\]

\[
(P^*)^L \frac{1}{\tau} = \text{Sing} \left( R_V^S (\tau) P^S \frac{1}{\tau} \right).
\]

**Proof.** For \( P = P_0 \Delta[0] \) (26.5) is clear. Let \( P_+ = P_n \Delta[n] \), for \( n > 0 \). Then, using the coproduct (2.3), the factorization (5.5) and (26.4) we find

\[
(P^*)^S \frac{1}{\tau} = (S(\Delta[n]) P_n)^S \frac{1}{\tau} = (-1)^n \sum_{s=0}^{n} \Delta V[s] P_n (T^{-s} \Delta[n-s])^S \frac{1}{\tau}
\]

\[
= (-1)^n \sum_{s=0}^{n} \Delta V[s] P_n T^{s} S(\Delta[n-s]) \frac{1}{\tau}
\]

\[
= (-1)^n \sum_{s=0}^{n} \Delta V[s] P_n T^{s} \frac{1}{\tau(1+n-s)}
\]

\[
= \sum_{s=0}^{n} \Delta V[s] P_n \tau[s] \Delta[n] \frac{1}{\tau}
\]

\[
= \text{Sing}(R_V P_+ \frac{1}{\tau}).
\]

In the same way one calculates (26.5) for \( P_- = P_n \Delta[n] \) for \( n < 0 \). The proof of (26.6) is similar. \( \square \)

**Corollary 26.2.** For all \( a, b \in \mathbb{Z} \) we have

\[
T^{-b} \langle T^{-a} \frac{1}{\tau_1}, \frac{1}{\tau_2} \rangle = \text{Sing}_{\tau_2} \left( R_W (\tau_2) (T^{-a} \frac{1}{\tau_1} T^b) \frac{1}{\tau_2} \right),
\]

where \( W \) is the \( H_T \)-module generated by \( \frac{1}{\tau_1} \).

**Proof.** This is (26.5) in case \( P = T^{a} \frac{1}{\tau_1} T^b \). \( \square \)
27. Rationality of Distributions on \( K \)

Recall the notion of a rational distribution, see (24.4). We say that a distribution \( D \) on \( K \) has rational singularities in case the singular part \( D_{\text{Sing}} \) is rational, i.e., there is \( \mathcal{K}_{\text{Sing}}^{D} \in \mathcal{K}_{\text{Sing}} \) such that for all \( G \in C_{\mathbb{Z}}^{\text{pol}} \)

\[
D_{\text{Sing}}(G) = \text{Tr}(\mathcal{K}_{\text{Sing}}^{D} G).
\]

**Lemma 27.1.** A \( W \)-valued formal distribution \( D \) has rational singularities if and only if \( F. D_{\text{Sing}} = 0 \) for some \( F = \prod_{(n,d)} (\tau - n)^d \in C_{\mathbb{Z}}^{\text{pol}} \).

**Proof.** If \( D \) has rational singularities with singular kernel \( \mathcal{K}_{\text{Sing}}^{D} \in \mathcal{K}_{\text{Sing}} \), then there is some \( F = \prod_{(n,d)} (\tau - n)^d \in C_{\mathbb{Z}}^{\text{pol}} \) such that \( F. \mathcal{K}_{\text{Sing}}^{D} G = 0 \) for all \( G \in C_{\mathbb{Z}}^{\text{pol}} \), since the argument of \( \text{Tr} \) is here nonsingular.

Conversely, let

\[
D_{\text{Sing}} = \sum_{k \geq 0} \frac{d_k}{\tau(k+1)}, \quad d_k \in W,
\]

such that \( F. D_{\text{Sing}} = 0 \) (as a distribution on \( C_{\mathbb{Z}}^{\text{pol}} \)), for

\[
F = \prod_{(n,d)} (\tau - n)^d = \tau(N) + \sum_{n=1}^{N} f_k \tau(N-k).
\]

Now the action of \( C_{\mathbb{Z}}^{\text{pol}} \) on singular distributions is given by

\[
\tau(m+\ell). \frac{1}{\tau(\ell)} = T^{-\ell} \frac{1}{\tau(m)} = 0 \pmod{C_{\mathbb{Z}}^{\text{pol}}}
\]

(27.1)

\[
\tau(m). \frac{1}{\tau(\ell+\ell)} = T^{-\ell} \frac{1}{\tau(m)} = (1 - \Delta)^{\ell} \frac{1}{\tau(m)}.
\]

By using the last equation of (27.1) we can write \( F. D_{\text{Sing}} = 0 \) as a collection of recursion equations for the coefficients of \( D_{\text{Sing}} \):

\[
d_{N+p} + \sum_{k=1}^{N} \gamma_{p,k} d_{N+p-k} = 0, \quad p \geq 0, \quad \gamma_{p,k} \in \mathbb{C}.
\]

This leaves \( d_0, d_1, \ldots, d_{N-1} \) arbitrary, and determines uniquely \( d_{p+N} \), for \( p \geq 0 \). So the subspace of distributions that solve \( F. D = 0 \) is \( n \) dimensional.

Now consider the rational element

\[
A = \sum_{(n,d)} \sum_{i=1}^{d} \frac{a_{(n,i)}}{(\tau - n)^i},
\]

(27.2)
containing \( N = \deg(F) \) arbitrary constants \( a_{(n,i)} \). In the summation \((n,d)\) runs over the same set of pairs that occur in \( F \). Then \( F.A = 0 \mod \mathbb{C}_\mathbb{Z}^{\text{pol}} \). We can expand \( A \) (or rather its image in \( K^* \)) as

\[
A = \sum_{n \geq 0} a_{(n)} \frac{1}{\tau(n1)},
\]

and the first \( N \) coefficients are determined by a system of equations of the form

\[
a_{(j)} = \sum_{(n,d)} d \sum_{i=1}^d c_{(n,i)}^{(j)} a_{(n,i)}, \quad j = 0, 1, \ldots, N - 1,
\]

where \( c_{(n,i)}^{(j)} \in \mathbb{C} \). One easily checks that the \( N \times N \) matrix \( (c_{(n,i)}^{(j)}) \) is invertible, so that any solution of the equation \( F.D_{\text{Sing}} = 0 \) determines uniquely a rational function of the form (27.2). \( \square \)

28. Rationality of Multivariable Distributions

We define

(28.1) \( \tau_1 \otimes S[\ell] = m_1^* \otimes S(\tau[\ell]). \)

**Lemma 28.1.** Let \( P(\tau) = \sum_n P_n(\tau)\Delta[n] \) be a difference operator with values in the space of \( W \)-valued distributions on \( K \). For all \( m \geq 0 \) we have

(28.2) \( \text{Tr}(\tau_1 \otimes S[m] P(\tau_2) \delta(\tau_1, \tau_2)) = P_m(\tau_2). \)

**Proof.** Let \( X \) be the left hand side in (28.2). We have

\[
X = \sum_{\ell \geq 0} \text{Tr}(\tau_1 \otimes S[m] P_\ell(\tau_2) S(\Delta_1[\ell]) \delta(\tau_1, \tau_2))
= \sum_{\ell \geq 0} \text{Tr}(\Delta_1[\ell] \langle \tau_1 \otimes S[m] \rangle P_\ell(\tau_2) \delta(\tau_1, \tau_2)).
\]

Now by using Lemma 15.1 we have

\[
\Delta_1[\ell] \tau_1 \otimes S[m] = m_1^* \otimes S(\Delta[\ell] \tau[m]),
\]

which is zero for \( \ell > m \). In this case there is no contribution to \( X \). So assume \( m = \ell + s, s \geq 0 \). Then

\[
\Delta[\ell] \tau[\ell + s] = \gamma \tau[s],
\]

for some non zero constant \( \gamma \). Now, by the multiplicativity of exponentials, see section 16, we have

\[
\tau_1 \otimes S[s] \delta(\tau_1, \tau_2) = \delta\left(\frac{\tau[s]}{\tau}\right),
\]

which is zero unless \( s = 0 \). \( \square \)
28. RATIONALITY OF MULTIVARIABLE DISTRIBUTIONS

Recall the formal expansion of \( f \in K \), see (24.5). Similarly we will often consider distributions that have a formal expansion

(28.3) \[ a(\tau_1, \tau_2) = \sum_{j=0}^{\infty} a_j(\tau_2) \Delta_2[j] \delta(\tau_1, \tau_2). \]

The coefficients of such an expansion are traces by Lemma 28.1:

(28.4) \[ a_j(\tau_2) = \text{Tr}_{\tau_1} (a(\tau_1, \tau_2) \tau_1 \otimes S(j)). \]

Let \( V_{\text{conv}}[[\tau_1^+, \tau_2^\pm]] \) be the space of \( V \)-valued distributions \( a(\tau, \tau_2) \) such that

\[ \pi(a(\tau_1, \tau_2)) = \sum_{j=0}^{\infty} a_j(\tau_2) \Delta_2(j) \delta(\tau_1, \tau_2) \]

converges, where \( a_j(\tau) \) is given by (28.4).

Recall from section 8 that the singular part \( K_{\text{Sing}} \) is generated by \( \frac{1}{\tau} \) over \( \hat{H}_T = H_T[\partial_\tau] \). If \( \kappa = (-\partial_\tau)^d S(\Delta[n]) \frac{1}{\tau} \in K_{\text{Sing}} \), then, by Lemma 15.1,

(28.5) \[ \delta(\kappa) = \Delta_2(n)(\partial_\tau)^d \delta(\tau_1, \tau_2). \]

Recall also the notion of an element \( \sigma \in V[[\tau^\pm]] \) having rational singularities, see section 13. Similarly we say that a distribution \( a(\tau_1, \tau_2) \in V_{\text{conv}}[[\tau_1^+, \tau_2^\pm]] \) has rational singularities (or is rational) if there is a \( f \in \mathbb{C}_Z^{\text{pol}} \) such that

\[ f_1 \otimes S a(\tau_1, \tau_2) = 0, \]

where \( f_1 \otimes S = m^1_1 \otimes S(f) \in \mathbb{C}_Z^{\text{pol}} \otimes \mathbb{C}_Z^{\text{pol}} \).

**Lemma 28.2.** A distribution \( a(\tau_1, \tau_2) \in V_{\text{conv}}[[\tau_1^+, \tau_2^\pm]] \) is rational if and only if it is a finite sum

(28.6) \[ a(\tau_1, \tau_2) = \sum_{n \in \mathbb{Z}, d \geq 0} c_{n,d}(\tau_2) \Delta_2(n)(\partial_\tau)^d \delta(\tau_1, \tau_2). \]

The proof of this Lemma is similar to that of Lemma 27.1.

Note that the expansion (28.3) of a rational \( a(\tau_1, \tau_2) \) will in general be infinite.
Singular Hamiltonian Structures

1. Overview

Let $V$ be a commutative $H_T$-Leibniz algebra, see section 17. We will show in section 2 that the affine variety of $V$ can be identified with the set $V(H^*) = \text{Hom}_{H-\text{alg}}(V, H^*)$ of homomorphisms of $H$-Leibniz algebras from $V$ to $H^*$. We refer to $V(H^*)$ as the set of singular points of $V$ or rather of the variety associated to $V$). If we replace $H^*$ by some other $H$-Leibniz algebra $L$ we obtain a set $V(L) = \text{Hom}_{H-\text{alg}}(V, L)$, which we refer to as a set of singular points of $V$. The notion of singularity is distinct from the usual concept of a singular point of an algebraic variety and motivated by the example of $H = H_D$. In this case, if $V$ is a free differential algebra with one generator, so that $V = \mathbb{C}[v(i)], Dv^i = v^{i+1}$, the set $V(H^*)$ is isomorphic to $H^* = \mathbb{C}[[t]]$, the set of nonsingular functions on the formal disk, whereas if we take $K = \mathbb{C}[[t]][t^{-1}]$ the set $V(K)$ is isomorphic to $K$, a set of singular functions on the formal disk.

In section 3 we will define classical fields as generating series of functions on $V(L)$, for the case $H = H_T$. The rest of this chapter develops then the theory of (singular) Poisson brackets between classical fields.

2. Spectrum and Singular Points

Let $V$ be a commutative $H$-Leibniz algebra. The $H$-action on $V$ induces an $H$-action on $V^*$ and by duality we obtain a coaction

$$m^*_V : V \rightarrow (H \otimes V^*)^*, \quad v \mapsto \{h \otimes v^* \mapsto v^*(hv)\}.$$  

We can expand $m^*_V(v)$ in an infinite sum of elements of $V \otimes H^*_T$. As we discussed in section 14 we can calculate $m^*_V$ using the exponential operator $\mathcal{R}$ introduced there: introducing a basis $\{e_i\}$ for $H$ and functions $e^*_i$ such that $e^*_i(e_j) = \delta_{ij}$ we have

$$m^*_V(v) = \mathcal{R}(v) = \sum e_i(v) \otimes e^*_i.$$  

Then $m^*_V$ is a multiplicative map (see Lemma 16.1) and satisfies $H$-covariance (see Lemma 15.1). Since we assume $V$ to be a commutative algebra we can define its spectrum $\text{Spec}_m(V)$: a point $\sigma \in \text{Spec}_m(V)$ is a $\mathbb{C}$-algebra homomorphism $\sigma : V \rightarrow \mathbb{C}$. Given such $\sigma$ we construct, using the exponential
map \( m^*_V \), a \( H \)-Leibniz algebra homomorphism
\[
\hat{\sigma} : V \to H^*, \quad v \mapsto \sigma m^*_V (v) = \sum \sigma(e_i^* v) e_i^*.
\]
Indeed, this is an algebra homomorphism:
\[
\hat{\sigma}(v_1 v_2) = (\sigma \otimes 1) m^*_V (v_1 v_2) = (\sigma \otimes 1) m^*_V (v_1) m^*_V (v_2),
\]
\[
\hat{\sigma}(v_1) \hat{\sigma}(v_2).
\]
Similarly, using part (a) of Lemma 15.1, we see that \( \hat{\sigma} \) is an \( H \)-module morphism:
\[
\hat{\sigma}(hv) = (\sigma \otimes 1)(1 \otimes h_{H^*}) m^*_V (v) = (1 \otimes h_{H^*})(\sigma \otimes 1) m^*_V (v) = h_{H^*} \sigma(v).
\]
Denote by \( V(H^*) \) the set of all \( H \)-Leibniz algebra morphisms \( \tau : V \to H^* \). So we have constructed a map \( \text{Specm}(V) \to V(H^*), \sigma \mapsto \hat{\sigma} \). Conversely we get a map \( V(H^*) \to \text{Specm}(V) \) by composing \( \tau \in V(H^*) \) with the canonical algebra homomorphism
\[
i^* : H^* \to \mathbb{C},
\]
dual to the unit \( i : \mathbb{C} \to H \). Then one easily checks that these maps are inverses of each other:
\[
(i^* \circ \tau) = \tau, \quad i^* \circ \hat{\sigma} = \sigma.
\]
Hence as sets we find an isomorphism
\[
V(H^*) \simeq \text{Specm}(V),
\]
and we refer to \( V(H^*) \) as the set of nonsingular points of the (variety associated to the) \( H \)-Leibniz algebra \( V \).

### 3. Classical Fields

From now on we restrict ourselves for convenience to the case \( H = H_T \) and consider the localization \( K = M^{-1} \mathbb{C}_Z^{\text{poly}} \) described in section 7.

Let \( V \) be a commutative \( H_T \)-Leibniz algebra and let \( V(K) \) be the set of \( H_T \)-Leibniz algebra homomorphisms \( \hat{\sigma} : V \to K \). (By taking the nonsingular part of \( \hat{\sigma} \) we get a map \( \hat{\sigma}_{\text{hol}} : V \to H_T^*, \) i.e., \( \hat{\sigma}_{\text{hol}} \) corresponds to a point of \( \text{Specm}(V) \).)

For \( f \in V \) we define the classical field \( C(f) \) as the distribution on \( K \) with values in the space of functions \( V(K) \to \mathbb{C} \), defined tautologically by:
\[
C(f)(F)(\hat{\sigma}) = \text{Tr}(\hat{\sigma}(f) F), \quad F \in K, \hat{\sigma} \in V(K).
\]
For each \( \hat{\sigma} \) we get a rational distribution \( C(f)(\hat{\sigma}) \) on \( K \) (with values in \( \mathbb{C} \)). Rational distributions can be multiplied, see Section 25. This allows us to define the (commutative, associative) product \( C(f).C(g) \) of classical fields associated to \( f, g \in V \) by
\[
C(f).C(g)(\hat{\sigma}) = C(f)(\hat{\sigma})C(g)(\hat{\sigma}).
\]
Then we have
\[ C(f)C(g) = C(fg). \]
Also we have an \( H_T \) action on distributions and this turns the space \( V(\tau) \) of classical fields (spanned by \( C(f), f \in V \)) into a commutative \( H_T \)-Leibniz algebra.

4. Expansions of Classical Fields

Any distribution has an expansion (24.3). In the case of classical fields this becomes:
\[ C(f) = \sum_{n \in \mathbb{Z}} f(n) \tau(-n - 1), \]
where \( f(n) = C(f)(\tau(n)) \) is the function
\[(4.1) \quad f(n) : V(K) \to \mathbb{C}, \quad \hat{\sigma} \mapsto \text{Tr}(\hat{\sigma}(f) \tau(n)). \]
We will write \( f(\tau_i) \) for the expansion of \( C(f) \) in the variable \( \tau_i \).

Consider the holomorphic part \( C(f)_{\text{Hol}} = \sum_{k \geq 0} f_{(-k-1)}(k) \) of the classical field \( C(f) \). We can identify the components \( f_{(-k-1)} \) of \( C(f)_{\text{Hol}} \) with the elements \( \Delta[k]f \in V \). Indeed, for any \( \hat{\sigma} \in V(K) \) we have
\[ f_{(-k-1)}(\hat{\sigma}) = \text{Tr}\left(\hat{\sigma}(f)\frac{1}{\tau(k+1)}\right) = \text{Tr}\left(\hat{\sigma}_{\text{Hol}}(f) S(\Delta[k] \frac{1}{\tau})\right) \]
\[ = \text{Tr}\left(\hat{\sigma}_{\text{Hol}}(\Delta[k]f) \frac{1}{\tau}\right) \]
\[ = \sigma(\Delta[k]f), \]
if \( \sigma \in \text{Specm}(V) \) corresponds to \( \hat{\sigma}_{\text{Hol}} \in V(H_T^+) \). This mean that we can identify the holomorphic part of the classical field \( C(f) \) with the distribution \( \mathcal{R}_V(f) = \sum_{k \geq 0} \Delta[k]f \otimes \tau(k) \). In particular the constant term of a classical fields \( C(f) = f(\tau) \) can be identified with the element \( f \in V \):
\[(4.2) \quad f_{(-1)} = f_{(1)} = \text{Tr}(C(f) \frac{1}{\tau}) \simeq f. \]

5. Classical Fields and Affinization

If \( M \) is an \( H_T \)-module, its \textit{affinization} is the \( H_T \)-module \( LM = M \otimes K \), where the tensor product is given the \( H_T \)-module structure by using the coproduct: for \( h \in H_T, m \in M \) and \( k \in K \),
\[ h(m \otimes k) = \sum (h'm) \otimes (h''k). \]
Recall the counit \( \varepsilon : H_T \to \mathbb{C} \). This is an algebra homomorphism, so that
\[ m = \varepsilon^{-1}(0) \subset H_T \]
is an ideal (called the augmentation ideal); it is the ideal generated by \( \Delta \),
or equivalently the ideal generated by \( \overline{\Delta} \). Define for any \( H_T \)-module \( M \) the module of coinvariants

\[
M_m = M / mM.
\]

So \( M_m \) is the quotient by total differences. Note that \( M_m \) is a trivial \( H_T \)-module. Therefore we will denote the canonical projection by

\[
\text{Tr}_M : M \to M_m.
\]

Often we will suppress the subscript \( M \) and write just \( \text{Tr} \). Denote by \( L \) the quotient \( LM / mL \). Elements of \( L \) will be written as

\[
f \{ p \} = \text{Tr}(f \otimes p), \quad f \in M, \quad p \in K.
\]

We have an action of \( \hat{H} \) on \( L \): if \( h \in \hat{H} \) we put

\[
h \cdot f \{ p \} = f \{ S(h) p \}.
\]

In this section we consider \( M = V \), where \( V \) is a commutative \( H_T \)-Leibniz algebra. Consider \( f \otimes p \in LV \). It defines a function on \( V(K) \) via

\[
f \otimes p[\hat{\sigma}] = \text{Tr}_K(\hat{\sigma}(f)p(\tau)).
\]

Suppose that \( f \otimes p \in mL \), so that

\[
f \otimes p = \Delta \sum f_i \otimes k_i = \sum \Delta' f_i \otimes \Delta'' k_i.
\]

Then, for any \( \hat{\sigma} \in V(K) \),

\[
f \otimes p[\hat{\sigma}] = \text{Tr}(\hat{\sigma}(f)p(\tau)) = \text{Tr}(\sum \sigma(\Delta' f_i) \Delta'' k_i)
\]

\[
= \text{Tr}(\Delta(\sum \sigma(f_i) k_i)) = 0.
\]

Hence we see that in fact any representative in \( LV \) of the element \( f \{ p \} \in L \) will define the same function on \( V(K) \). Also we note that \( f \{ \tau(n) \} \) defines the same function on \( V(K) \) as the coefficient \( f(\tau(n)) \) of the classical field \( C(f) \), see (4.1). More generally \( f(\tau) \) defines the same function as the value \( C(f)(p) \) of the classical field, where we think of the classical field as a distribution on \( K \) with values in functions on \( V(K) \). We will from now on identify the coefficients of \( C(f) \) with elements of \( L \) so that classical fields will be generating sequences for elements of \( L \).

## 6. Multivariable Classical Fields

A classical field \( C(f) \) is a distribution on \( K \), so we can multiply it by the Dirac distribution, in two ways, see Section 25. More generally, we can consider classical field valued difference operators \( P = \sum_{n \in \mathbb{Z}} P_n(\tau) \Delta^n \),
where $P_n(\tau) \in V(\tau)$, with $V(\tau)$ the space of classical fields. Then we get products $P_1 \delta(\tau_1, \tau_2), P_2 \delta(\tau_1, \tau_2)$ and we have

\begin{equation}
P_1 \delta(\tau_1, \tau_2) = P_2^* \delta(\tau_1, \tau_2),
\end{equation}

using the involution (23.2). Let $V(\tau_1, \tau_2)$ be the $V(\tau)H_T \otimes V(\tau)H_T$-module generated by $\delta(\tau_1, \tau_2)$ (cf., section 27). We think of elements of $V(\tau_1, \tau_2)$ as classical fields depending on two variables. Similarly we define the module of three variable classical fields $V(\tau_1, \tau_2, \tau_3)$ as the $VH_T^\otimes 3$ module generated by

\[
\delta(\tau_1, \tau_2, \tau_3) = \delta(\tau_1, \tau_2)\delta(\tau_2, \tau_3) = \delta(\tau_1, \tau_3)\delta(\tau_2, \tau_3).
\]

We have then relations between the various $VH_T$ actions similar to (6.1), for instance

\[
P_1 \delta(\tau_1, \tau_2, \tau_3) = (P^*_1)2 \delta(\tau_1, \tau_2, \tau_3) = (P^*_3)3 \delta(\tau_1, \tau_2, \tau_3).
\]

7. Derivations and Poisson Structures

Let $A$ be a commutative associative unital algebra, $M$ an $A$-module. Denote by $\text{Der}(A,M)$ the space of all derivations from $A$ to $M$; this are linear maps $X : A \to M$ such that $X(ab) = aX(b) + bX(a)$.

There is an $A$-module $\Omega^1(A)$ together with a universal derivation $d : A \to \Omega^1(A)$, such that for any $X \in \text{Der}(A,M)$ there exists a unique $A$-module morphism $\gamma_X : \Omega^1(A) \to M$ making the following diagram commute:

\[
\begin{array}{ccc}
A & \xrightarrow{\gamma_X} & \Omega^1(A) \\
\downarrow{\mathrlap{\kappa}} & & \downarrow{\mathrlap{\kappa}} \\
M & \xrightarrow{\gamma_X} & \Omega^1(A)
\end{array}
\]

The module of differentials $\Omega^1(A)$ is generated by elements $da, a \in A$, subject to the relations $d(ab) = adb + bda$, for $a, b \in A$. There is a pairing

\[
\langle \cdot, \cdot \rangle : \Omega^1(A) \otimes \text{Der}(A,M) \to M,
\]

given by $\omega \otimes X \mapsto \langle \omega, X \rangle = \gamma_X(\omega)$. In particular $\langle da, X \rangle = X(a)$.

The space $\text{Der}(A,M)$ is an $A$-module via

\[
a.X(b) = a(X(b)), \quad a, b \in A.
\]
Then consider derivations $\Gamma : A \to \text{Der}(A, M)$. Define the bracket associated to $\Gamma$ by

$$\{a, b\} = \Gamma(a)(b), \quad a, b \in A.$$  

Then we call $\Gamma$ Hamiltonian if

- $\Gamma$ is skew-symmetric: $\{a, b\} = -\{b, a\}$,
- $\Gamma$ is a Lie map: $\Gamma(\{a, b\}) = [\Gamma(a), \Gamma(b)]$.

In this case we call the bracket of $\Gamma$ the Poisson bracket; $(A, \{\})$ is then a Lie algebra. The bracket factors through the differentials:

$$\{a, b\} = \langle \gamma^\Gamma(da), db \rangle.$$  

In case $A$ is a polynomial algebra we get simple explicit formulae for the Poisson bracket in terms of generators: let $A = \mathbb{C}[v_\alpha]_{\alpha \in I}$. Then

$$\{a, b\} = \sum_{i, j \in I} \frac{\partial a}{\partial v_i} \frac{\partial b}{\partial v_j} \{v_i, v_j\},$$

so that the Poisson bracket is uniquely determined by the Poisson bracket between the generators.

### 8. Evolutionary Derivations and Singular Poisson Brackets

Let $V$ be an $H_T$-Leibniz algebra. The module of differentials $\Omega^1(V)$ for any algebra $V$ a $V$-module, but it is in this case in fact a $V_{HT}$-module, via

$$h(fdg) = \sum (h'f)d(h''g), \quad h \in H_T, f, g \in V.$$  

We call, for $M$ an $V_{HT}$-module, a derivation $X : V \to M$ evolutionary in case

$$X(hf) = hX(f), \quad h \in H_T, f \in V.$$  

In this case the $V$-module morphism $\gamma_X : \Omega^1(V) \to M$ of (7.1) is in fact a morphism of $V_{HT}$-modules.

Now we would like to generalize the notion of a Hamiltonian map to the case of evolutionary derivations. An obstacle is that the space $\text{Der}^e(V, V)$ of evolutionary derivations (from $V$ to itself) is not (in an obvious way) a module over $V$ or $V_{HT}$: if $X$ is evolutionary then

$$f.X(hg) = f.(hX(g)) \neq h.f.X(g),$$

as $h \in H, f \in V$ do not commute in $V_{HT}$, in general. In case $V$ is free as $H_T$-module there is a $V$-module structure on $\text{Der}(V, V)$, using a basis for $V$, see section (10.1) below.

Therefore we consider instead the space $D_2 = \text{Der}^e(V, V(\tau_1, \tau_2))$ of evolutionary derivations from $V$ to the module $V(\tau_1, \tau_2)$ of two variable
classical fields introduced in section 6. So an element $X \in \mathcal{D}_2$ acts on $f \in V$ by
\[ X(f) = \sum_{n \in \mathbb{Z}} X(f)_n(\tau_2)\Delta[n]2\delta(\tau_1, \tau_2), \quad X(f)_n \in V, \]
and satisfies
\[ X(fg) = f(\tau_2)X(g) + g(\tau_2)X(f), \quad f, g \in V, \]
\[ X(hf) = hX_2(f), \quad h \in H_T. \] (8.1)

In other words, we use the $1 \otimes V H_T$-module structure on the target to define the words evolutionary and derivation.

Now note that we can define a $V H_T$-module structure on $\mathcal{D}_2$ by using the other $V H_T$ module structure on the target: if $X$ is an evolutionary derivation then define for $P \in V H_T$
\[ P.X(f) = P_1(X(f)), \quad f \in V. \]
The two actions of $V H_T$ on $V(\tau_1, \tau_2)$ commute:
\[ P_1P_2X(f) = P_2P_1X(f), \]
from which we see that $P.X$ is again evolutionary:
\[ P.X(hf) = P_1hX_2(F) = hP.X(f). \]

Any $X \in \mathcal{D}_2$ factors through $\Omega^1(V)$:
\[ \begin{array}{ccc}
V & \xrightarrow{d} & \Omega^1(V) \\
\downarrow & & \downarrow \gamma^T \\
X & & V(\tau_1, \tau_2) \\
\end{array} \] (8.2)
where $\gamma^T : \Omega^1(V) \to V(\tau_1, \tau_2)$ is a morphism of $V H_T$-modules (for the $1 \otimes V H_T$ structure on the target).

Instead of considering derivations with target $V(\tau_1, \tau_2)$ we could use distributions in any pair of variables as target. In this case we will write
\[ X(f)_{ij} = \sum_n X(f)_n(\tau_j)\Delta_j(n)\delta(\tau_i, \tau_j). \]
We can extend any $X \in \text{Der}^\text{ev}(V, V(\tau_i, \tau_k))$ to a map
\[ X_{ijk} : V(\tau_j, \tau_k) \to V(\tau_i, \tau_j, \tau_k), \]
by
\[
\sum_{n \in \mathbb{Z}} p_n(\tau_k)\Delta_k(n)\delta(\tau_j, \tau_k) \mapsto \sum_{n \in \mathbb{Z}} X(p_n)_{jk}\Delta_k(n)\delta(\tau_j, \tau_k) = \sum_{n \in \mathbb{Z}} X(p_n)_{jkm}\Delta_k(m)\delta(\tau_i, \tau_k)\Delta_k(n)\delta(\tau_j, \tau_k).
\]

Then \(X_{ijk}\) is an evolutionary derivation, for the \((\mathcal{V}H_T)_k\)-module structures on \(V(\tau_j, \tau_k)\) and \(V(\tau_i, \tau_j, \tau_k)\).

Now we will define the evolutionary analog of the Hamiltonian maps introduced in section 7. Consider a map \(\Gamma: V \to \text{Der}^\text{ev}(V, V(\tau_1, \tau_2))\), and introduce the Poisson bracket of \(f, g \in V\) by
\[
\{f(\tau_1), g(\tau_2)\} = \Gamma(f)(g) \in V(\tau_1, \tau_2).
\]

**Definition 8.1.** We call \(\Gamma\) a singular Hamiltonian map in case
(a) \(\Gamma\) is an evolutionary derivation,
(b) \(\Gamma\) is skew-symmetric: \(\{f(\tau_1), g(\tau_2)\} = -\{g(\tau_2), f(\tau_1)\}\),
(c) \(\Gamma\) is a Lie map:
\[
\{f(\tau_1), \{g(\tau_2), k(\tau_3)\}\} = \{\{f(\tau_1), g(\tau_2)\}, k(\tau_3)\}
+ \{g(\tau_2), \{f(\tau_1), k(\tau_3)\}\}.
\]

Here we extend in (c) the derivation \(\{f(\tau_1), \cdot\} = \Gamma(f)\) from a map on \(V\) to a map \(\Gamma(f)_{123}: V(\tau_2, \tau_3) \to V(\tau_1, \tau_2, \tau_3)\), as discussed above, and similar for the other terms.

We call, in case \(\Gamma\) is a singular Hamiltonian map for a \(H_T\)-Leibniz algebra \(V\), the pair \((V, \Gamma)\) an \(H_T\)-Poisson algebra.

The singular Poisson bracket \(\{f(\tau_1), g(\tau_2)\}\) is an evolutionary derivation in both \(f\) and \(g\) and so factors through \(\Omega^1(V)^{\otimes 2}\). We can write
\[
\{f(\tau_1), g(\tau_2)\} = \langle \varpi(\text{df}), \text{dg} \rangle,
\]
where \(\langle \cdot, \cdot \rangle: \mathcal{D}_2 \otimes \Omega^1(V) \to V(\tau_1, \tau_2)\) is the canonical pairing, and \(\varpi\) is the unique \(VH_T\)-module morphism \(\Omega^1(V) \to \mathcal{D}_2\) making the diagram
\[
\begin{array}{ccc}
V & \xrightarrow{\varpi} & \Omega^1(V) \\
\downarrow & & \downarrow \varpi \\
\mathcal{D}_2 & \xrightarrow{\Gamma} & \Omega^1(V)
\end{array}
\]

commute.
9. Projection from $V(\tau_1, \tau_2)$ to $V$

Let $F(\tau_1, \tau_2) \in V(\tau_1, \tau_2)$ be a two variable classical field, see section 6. It can be uniquely expanded as

$$F(\tau_1, \tau_2) = \sum_{k \geq 0} F_k(\tau_2) \Delta[k] \delta(\tau_1, \tau_2),$$

see (28.3). Define a map

$$\Pi: V(\tau_1, \tau_2) \to V, \quad F(\tau_1, \tau_2) \mapsto (F_0)_{(-1)}$$

so that

$$\Pi(F(\tau_1, \tau_2)) = \text{Tr}_{\tau_1, \tau_2}(F(\tau_1, \tau_2) \frac{1}{\tau_2}),$$

see Lemma 28.1 and Section 4.

**Lemma 9.1.** Let $P, Q \in VH_T$, so that

$$P = \sum_{n \in \mathbb{Z}} p_n \Delta[n], \quad Q = \sum_{n \in \mathbb{Z}} q_n \Delta[n].$$

Then

(a) $\Pi(P_2Q_2 \delta(\tau_1, \tau_2)) = P(q_0)$.  
(b) $\Pi(P_1Q_2 \delta(\tau_1, \tau_2)) = Q(P_V)$.

Here we write for instance $P_i$ for the classical field valued difference operator $P_i = \sum_{n \in \mathbb{Z}} p_n(\tau_i) \Delta_i[n]$, for $i = 1, 2$.

In particular, it follows from part (b) that

$$\Pi(\Delta_1 F(\tau_1, \tau_2)) = 0, \quad \forall F(\tau_1, \tau_2) \in V(\tau_1, \tau_2).$$

**Proof.** We use two facts about $H_T$. First of all the product of elements of the difference operator basis has an expansion

$$\Delta[n] \Delta[m] = \sum_{k \in \mathbb{Z}} a^k_{nm} \Delta[k], \quad a^k_{nm} \in \mathbb{Z},$$

with the coefficient $a^0_{nm}$ of $\Delta[0] = 1$ zero, unless $n = m = 0$. Secondly the coproduct of $\Delta[n]$ has the form

$$\pi(\Delta[m]) = \Delta[m] \otimes 1 + \sum_i h'_i \otimes h''_i,$$
with \( h''_i \in m \). It suffices to check part (a) in case \( P = p\Delta[m], Q = q\Delta[n] \), and in this case we have

\[
\Pi (P_2Q_2\delta(\tau_1, \tau_2)) = \Pi (p(\tau_2)\Delta_2(n)[q(\tau_2)\Delta_2(m)\delta(\tau_1, \tau_2)])
\]

\[
= \Pi \left( p(\tau_2) \left[ \Delta_2(n)(q(\tau_2))\Delta_2(m) + \sum_i h'_{i,2}(q(\tau_2))h''_{i,2} \right] \delta(\tau_1, \tau_2) \right)
\]

\[
= \delta_{m0}p\Delta[n](q)
\]

\[
= P(Q_0).
\]

Then part (b) follows from (a): we have

\[
\Pi (P_1Q_2\delta(\tau_1, \tau_2)) = \Pi (Q_2(P^S)_2\delta(\tau_1, \tau_2)) = Q(P^S_0),
\]

but

\[
P^S_0 = \sum S\Delta[n](p_n) = P_V.
\]

\[\square\]

**Lemma 9.2.** Let \( X \) be an evolutionary derivation \( X : V \rightarrow V(\tau_1, \tau_2) \). Then the composition

\[X_0 = \Pi \circ X : V \rightarrow V\]

is an evolutionary derivation of \( V \).

**Proof.** We have for \( f, g \in V \) by definition of evolutionary derivation with values in \( V(\tau_1, \tau_2) \), see (8.1),

\[X_0(fg) = \Pi (f(\tau_2)X(g) + g(\tau_2)X(f)) = fX_0(g) + gX_0(f),\]

using Part a of Lemma 9.1 with \( P = f, Q = 1 \). In case \( h \in H_T \) similarly with \( P = h \) and \( Q = 1 \)

\[X_0(h.f) = \Pi (h_2X(f)) = hX_0(f).\]

\[\square\]

So we obtain a map

(9.4) \( \Pi^v : \text{Der}^{\text{ev}}(V, V(\tau_1, \tau_2)) \rightarrow \text{Der}^{\text{ev}}(V, V), \quad X \mapsto X_0. \)

Now let \( \Gamma : V \rightarrow \text{Der}^{\text{ev}}(V, V(\tau_1, \tau_2)) \) be a singular Hamiltonian structure, see section 8. Recall that if \( P \in VH_T \) then \( \Gamma(P.f)(g) = P_1\Gamma(f)(g) \). We define a map by composition with \( \Pi^v \):

\[\Gamma_0 = \Pi^v \circ \Gamma : V \rightarrow \text{Der}^{\text{ev}}(V, V), \quad \Gamma_0(f) = (\Gamma(f))_0.\]

Recall that \( \Gamma \) factors through the differentials: if \( f \in V \) then

\[\Gamma(f) = \gamma_T(df),\]
where $\gamma_1 : \Omega^1(V) \to \text{Der}^{ev}(V, V(\tau_1, \tau_2))$ is a morphism of $VH_T$-modules. In particular, if $f \in mV$ (i.e., $f$ is a total difference, $f = \Delta g$), then

$$\Gamma_0(f) = \Pi \circ \gamma_1(df) = \Pi \circ \gamma_1(\Delta dg) = \Pi (\Delta_1 \gamma_1(df)) = 0,$$

by (9.1). This means that $\Gamma_0$ factors through, $\Omega^1(V)_m$, the module of coinvariants. Then we obtain the following commutative diagram

$$\begin{array}{ccc}
V & \xrightarrow{d} & \Omega^1(V) \\
\downarrow{\Gamma_0} & & \downarrow{\Pi \circ \gamma_1} \\
\text{Der}^{ev}(V, V) & \xleftarrow{B} & \Omega^1(V)_m
\end{array}$$

We write $\delta f = \text{Tr}_{\Omega^1(V)}(df)$ for the map $V \to \Omega^1(V)_m$ (variational differential). For $V$ a free $H_T$-Leibniz algebra $B : \Omega^1(V)_m \to \text{Der}^{ev}(V, V)$ is a matrix of difference operators, i.e., an matrix with coefficients in $VH_T$, as we will explain in the next section.

### 10. Free Difference Algebras

Let $V$ be the free difference algebra generated by $v_\alpha, \alpha \in \mathcal{A} = \{1, 2, \ldots, k\}$. So $V$ is the polynomial algebra

$$V = \mathbb{C}[v_\alpha^{(n)} ; \alpha \in \mathcal{A}, n \in \mathbb{Z}],$$

where $v_\alpha^{(0)} = v_\alpha$ and the action of $H_T$ is determined by

$$\Delta[n]v_\alpha = v_\alpha^{(n)},$$

together with linearity and the Leibniz rule for $H_T$. In this case $\text{Der}^{ev}(V, V)$ and $\Omega^1(V)_m$ are free $V$-modules of rank $k = |\mathcal{A}|$, which simplifies the description of singular Hamiltonian structures on $V$.

Any $X \in \text{Der}^{ev}(V, V)$ is uniquely determined by its values $X_\alpha = X(v_\alpha)$ on the generators of $V$. Introduce notation

$$X_\alpha \frac{\delta}{\delta v_\alpha} = \sum_{n \in \mathbb{Z}} \Delta[n](X_\alpha) \frac{\partial}{\partial v_\alpha^{(n)}} \in \text{Der}^{ev}(V, V),$$
so that

\[ X = \sum_{\alpha \in \mathcal{A}} X_\alpha \frac{\delta}{\delta v_\alpha}. \]

Define an action of \( V \) on evolutionary derivations by

\[ f.X = \sum_{\alpha \in \mathcal{A}} (fX_\alpha) \frac{\delta}{\delta v_\alpha}, \quad f \in V. \]

Then \( \text{Der}^{ev}(V, V) \) is a free \( V \)-module:

\[ \text{Der}^{ev}(V, V) = \bigoplus_{\alpha \in \mathcal{A}} V \frac{\delta}{\delta v_\alpha}, \]

generated by the (evolutionary) derivations \( \frac{\delta}{\delta v_\alpha} \).

The module of differentials of \( V \) is a free \( V \)-module:

\[ \Omega^1(V) = \bigoplus_{\alpha \in \mathcal{A}, n \in \mathbb{Z}} V d\!v^{(n)}_\alpha. \]

Now the \( H_T \)-module structure on \( \Omega^1(V) \) is such that

\[ d\!v^{(n)}_\alpha = \Delta[n]d\!v_\alpha, \]

so that \( \Omega^1(V) \) is also a free \( VH_T \)-module:

\[ \Omega^1(V) = \bigoplus_{\alpha \in \mathcal{A}} VH_T d\!v_\alpha. \]

In particular for \( f \in V \) we can express the differential in terms of difference operators:

\[ df = \sum_{\alpha \in \mathcal{A}} d\alpha f d\!v_\alpha, \]

where the \( d\alpha f \) are difference operators

\[ d\alpha f = \sum_{n \in \mathbb{Z}} \frac{\partial f}{\partial d\!v^{(n)}_\alpha} \Delta[n] \in VH_T. \]

Inside \( \Omega^1(V) \) we have a \( V \)-submodule

\[ \Omega^1(V)_0 = \bigoplus_{\alpha \in \mathcal{A}} V d\!v_\alpha. \]

We have a pairing

\[ \langle \cdot, \cdot \rangle : \text{Der}^{ev}(V, V) \otimes \Omega^1(V)_0 \to V, \quad X \otimes \omega \mapsto \sum X_\alpha \omega_\alpha, \]

when \( X = \sum X_\alpha \frac{\delta}{\delta v_\alpha}, \omega = \sum \omega_\alpha d\!v_\alpha \). It is known (see [Kup85], chapter 2) that

\[ \Omega^1(V)_0 \cap m\Omega^1(V) = \{0\}. \]
Recall the module of coinvariants $\Omega^1(V)_m = \Omega^1(V)/m\Omega^1(V)$. We have for all $P \in VH_T$

$$Pdv_\alpha \equiv P_Vdv_\alpha \mod m\Omega^1(V),$$

(where $P_V$ is defined in (23.4)). This implies, combined with (10.2), that

$$\Omega^1(V) = m\Omega^1(V) \bigoplus \Omega^1(V)_0.$$  

So we have a splitting of the exact sequence

$$0 \longrightarrow m\Omega^1(V) \overset{s}{\longrightarrow} \Omega^1(V) \overset{p}{\longrightarrow} \Omega^1(V)_m \longrightarrow 0,$$

where the splitting map $s: \Omega^1(V)_m \rightarrow \Omega^1(V)$ is

$$[\sum P_\alpha dv_\alpha] \mapsto \sum (P_\alpha)_Vdv_\alpha.$$  

In particular we obtain a variational differential $\delta = s \circ p \circ d: V \rightarrow \Omega^1(V)_0$ given by

$$\delta f = \sum_\alpha \frac{\partial f}{\partial v_\alpha}dv_\alpha,$$

where the variational derivative is

$$\tag{10.3} \frac{\delta f}{\delta v_\alpha} = (d_\alpha f)_V = \sum_n S\Delta[n] \left( \frac{\partial f}{\partial v_\alpha^{(n)}} \right).$$

The variational derivative $\delta f$ is the unique element of $\Omega^1(V)_0$ such that for all $X \in \text{Der}^{ev}$

$$X(f) \equiv \langle X, \delta f \rangle \mod mV,$$

using the pairing (10.1). Indeed, if $\delta = \sum \delta_\alpha dv_\alpha \in \Omega^1(V)_0$ also satisfies this then we have for all $X \in \text{Der}^{ev}$ $\langle X, \delta f - \delta \rangle \equiv 0 \mod mV$, or for each $\alpha$

$$X_\alpha(\frac{\delta f}{\delta v_\alpha} - \delta_\alpha) \in mV, \quad \forall X_\alpha \in V,$$

but it is known that this forces $\frac{\delta f}{\delta v_\alpha} = \delta_\alpha$, see [Kup85], Lemma 17.

The module of coinvariants is a free $V$-module:

$$\Omega^1(V)_m = \bigoplus_\alpha Vdv_\alpha.$$  

We can therefore represent an element $\omega = \sum_\alpha \omega_\alpha dv_\alpha$ of $\Omega^1(V)_m$ as a column vector $\omega = \left( \begin{array}{c} \omega_1 \\ \vdots \\ \omega_k \end{array} \right)$ and similarly if $X = \sum_\alpha X_\alpha \frac{\delta f}{\delta v_\alpha} \in \text{Der}^{ev}(V, V)$ then
we write this as a column vector $X = \begin{pmatrix} X_1 \\ \vdots \\ X_k \end{pmatrix}$. Now suppose we are given a singular Hamiltonian structure $\Gamma: V \to \text{Der}^{ev}_V(V, V)_{\tau_1, \tau_2}$ on $V$. We have seen (see section 9) that this gives rise to a map $B: \Omega^1(V)_m \to \text{Der}^{ev}_V(V, V)$. In our case $B$ is a map between free rank $k$ $V$-modules, and we will now see that $B$ is given by an $k \times k$ matrix of difference operators.

Introduce difference operators $B_{\beta \alpha} \in VH_T$ by

$$\Gamma(v_\alpha)(v_\beta) = B_{\beta \alpha, 2} \delta(\tau_1, \tau_2) \in V(\tau_1, \tau_2).$$

Let $X_f = \Pi \circ \Gamma(f)$ be the evolutionary derivation on $V$ associated to $f$. The we have $X_f = \sum_\alpha X_{f, \alpha (v_\alpha)}$ and

$$X_{f, \alpha} = X_f(v_\alpha) = \Pi(\Gamma(f)(v_\alpha)) = \Pi((df_\beta)_1\Gamma(v_\beta)(v_\alpha)) = \Pi((df_\beta)_1 B_{\alpha \beta, 2} \delta(\tau_1, \tau_2)) = \sum_\beta B_{\alpha \beta} \left( \frac{\delta f}{\delta v_\beta} \right),$$

by Lemma 9.1 and (10.3). So

$$X_f \begin{pmatrix} v_1 \\ \vdots \\ v_k \end{pmatrix} = B \begin{pmatrix} \frac{\delta f}{\delta v_1} \\ \vdots \\ \frac{\delta f}{\delta v_k} \end{pmatrix}, \quad B_{\alpha \beta} = B_{\alpha \beta},$$

so that $B$ is the matrix of the map $B$. 

Infinite Toda Hamiltonian Structure for Infinite Toda

1. The Finite Toda Lattice

Recall, see [Tod89], that the finite non periodic Toda lattice with \( N \) particles is a Hamiltonian system on a phase space with global coordinates \( p_i, q_i \) for \( i = 1, \ldots, N \) and Hamiltonian
\[
H = \frac{1}{2} \sum_{i=1}^{N} p_i^2 + \sum_{k=2}^{N} \exp(q_{k-1} - q_k).
\]
So the equations of motion are
\[
\dot{p}_i = \exp(q_{i-1} - q_i) - \exp(q_i - q_{i+1}),
\]
\[
\dot{q}_i = p_i.
\]
Introduce new variables \( B_i = p_i, i = 1, \ldots, N \) and \( C_j = \exp(q_{j-1} - q_j) \), \( j = 2, \ldots, N \). Then we obtain the following system in the new variables:
\[
\begin{align*}
\dot{B}_i &= C_i - C_{i+1} & i = 1, \ldots, N, C_{N+1} &= 0 \\
\dot{C}_j &= C_j(B_{j-1} - B_j) & j = 2, \ldots, N.
\end{align*}
\]
Introduce a Lax matrix
\[
L = \begin{pmatrix}
B_1 & 1 & 0 & 0 & 0 & 0 \\
C_2 & B_2 & 1 & \cdots & \cdots & 0 \\
0 & C_3 & B_3 & \cdots & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \cdots & C_{N-1} & B_{N-1} & 1 \\
0 & 0 & 0 & 0 & C_N & B_N
\end{pmatrix}
\]
Then the system (1.1) is equivalent to the Lax equation
\[
\dot{L} = [L_-, L],
\]
where \( L_- \) is the strictly lower triangular part of \( L \). We also have for the Hamiltonian
\[
H = \frac{1}{2} \text{Tr}(L^2) = \frac{1}{2} \sum_{i=1}^{N} B_i^2 + \sum_{j=2}^{N} C_j.
\]
Now the system (1.1) make still sense if we let \( i, j \) run over all integers, and we can introduce a doubly infinite Lax matrix such that this infinite system
is still given by (1.3). Now, however, the meaning of (1.4) becomes less clear (but see [Ike94]). Kuperschmidt, for instance, advocates in [Kup85] an other way of dealing with infinite systems, where one replaces the infinite Lax matrix by a difference operator. We follow in this chapter Kuperschmidt’s approach, expressed in the formalism of singular Hamiltonian maps, and also indicating the connection with the formalism of $r$-matrices.

2. Lax Operator and $r$-Matrix

Consider the difference operator (“Lax Operator”)

$$L = AT + B + CT^{-1} \in VH_T,$$

where $V$ is the $H_T$-Leibniz algebra freely generated by $A, B, C$. So $V$ is the polynomial algebra

$$V = \mathbb{C}[A^{(n)}, B^{(n)}, C^{(n)}], \quad Y^{(n)} = \Delta[n]Y, \quad Y = A, B, C.$$

We will also write the Lax operator as

$$L = \sum_{\alpha} v^\alpha T^\alpha,$$

so that

$$v_1 = A, v_0 = B, v_{-1} = C.$$

Any evolutionary derivation $X$ of $V$ defines a time-flow on the Lax operator by

$$\dot{L} = X L = X(A)T + X(B) + X(C)T^{-1}.$$

Conversely we will use $L$ to define singular Hamiltonian structures on $V$.

We have a map, if $P = \sum_{\alpha} p^\alpha v^\alpha \in VH_T$, such that

$$\text{Sp}: VH_T \rightarrow V, \quad P \rightarrow p_0.$$

For all $P, Q \in VH_T$ we have

$$\text{Sp}([P, Q]) \equiv 0 \mod mV.$$

Indeed, it suffices to check this for $P = pT^\alpha, Q = qT^{-\alpha}$, for $\alpha \in \mathbb{Z}, p, q \in V$.

But in that case

$$[pT^\alpha, qT^{-\alpha}] = (T^\alpha - 1)(qT^{-\alpha}(p)),$$

and $T^\alpha - 1 \in m$.

Recall the pairing (10.1). In our situation we have encoded the evolutionary derivation $X$ in the difference operator $X_L$. Similarly we can encode an element $\omega = \sum_{\alpha} \omega_\alpha dv^\alpha \in \Omega^1(V)_0$ in a difference operator $\omega_L = \sum T^{-\alpha} \omega_\alpha$. Then we have

$$\langle X, \omega \rangle = \text{Sp}(X_L \omega_L).$$

For $f \in V$ we get a difference operator $\delta_L f = \sum T^{-\alpha} \frac{\delta f}{\delta v^\alpha}$ such that for all $X \in \text{Der}^\text{ev}$$

$$X(f) \equiv \text{Sp}(X_L \delta_L f) \mod mV.$$
Note that the difference operator $\delta_L f$ is not uniquely determined by this condition: we are free to add to $\delta_L f$ any difference operator of the form $\delta_{\geq 2} + \delta_{\leq -2}$, where $\delta_{\geq 2} = \sum_{\alpha \geq 2} T^\alpha \delta_\alpha$ and $\delta_{\leq -2} = \sum_{\alpha \leq -2} T^\alpha \delta_\alpha$, since $\delta_{\geq 2}$ and $\delta_{\leq -2}$ are orthogonal to $X_L$ with respect to $Sp$.

We have projections

$$\Pi_{\leq k}, \Pi_{\geq k}, \Pi_{> k}, \Pi_{< k}, \Pi_k : VH_T \to VH_T,$$

where, if $P = \sum_{\alpha \in \mathbb{Z}} p_\alpha T^\alpha \in VH_T$, for instance

$$\Pi_{\leq k}(P) = \sum_{s \leq k} p_\alpha T^\alpha, \quad \Pi_k P = p_k,$$

and similar for the other projections.

We use $Sp$ also to define the dual of a map $\sigma : VH_T \to VH_T$ by

$$Sp(\sigma(P)) = Sp(P\sigma^*(Q)).$$

Then, for instance, we have

$$\Pi_{\geq 0}^* = \Pi_{\leq 0}, \quad \Pi_{< 0}^* = \Pi_{> 0}.$$

Define the r-matrix $\rho : VH_T \to VH_T$ for $VH_T$ by $P \mapsto (\Pi_{\geq 0} - \Pi_{< 0})(P)$.

Then $\rho$ is not skew-adjoint, but we have $\rho = S + A$, where $A = \Pi_{> 0} - \Pi_{< 0}$ is skew and $S = \Pi_0$ is symmetric, so that $\rho^* = S - A$.

Besides the Lax operator $L$ (with coefficients in $V$) we also have the Lax operator $L(\tau_i)$ with coefficients classical fields depending on $\tau_i$:

$$L(\tau_i) = A(\tau_i)T_i + B(\tau_i) + C(\tau_i)T_i^{-1}.$$

Define distributions associated to the r-matrix $\rho$ and its adjoint $\rho^*$:

$$r(\tau_1, \tau_2) = \sum_{\alpha, m \in \mathbb{Z}} \tau_1(-m - 1)T^\alpha_1 \rho(T^\alpha_2) \tau_2(m) = \rho(\tau_1 T^\alpha_2) \delta(\tau_1, \tau_2)T^\alpha_1$$

$$r^*(\tau_1, \tau_2) = \sum_{\alpha, m \in \mathbb{Z}} \rho(T^{-\alpha}_1) \tau_1(m) \tau_2(-m - 1)T^\alpha_2 = \rho(T^{-\alpha}_1) \delta(\tau_1, \tau_2)T^\alpha_2.$$

We define commutators of the (field-valued) Lax operator with these distributions:

$$[L_2, r] = [L(\tau_2), \rho(T^{-\alpha}_2) \delta(\tau_1, \tau_2)]T^\alpha_1,$$

and similarly

$$[L_1, r^*] = \sum_{\alpha, m \in \mathbb{Z}} [L(\tau_1), \rho(T^{-\alpha}_1) \delta(\tau_1, \tau_2)]T^\alpha_2.$$

If $h \in H_T, x \in V$, we write $h(x)$ for the action of $h$ on $x$, and $hx$ for the multiplication in $VH_T$. For instance, $T^k x = T^k(x)T^k$. 

LEMMA 2.1.
\[[L_2, r] - [L_1, r^*] = C(\tau_2)(T_2^{-1} - 1)\langle \delta(\tau_1, \tau_2) \rangle T_2^{-1} + \langle T_2 - 1 \rangle C(\tau_2)\delta(\tau_1, \tau_2)T_1^{-1}\).

PROOF. First of all
\[\begin{align*}
[L(\tau_1), T_i^k \delta(\tau_1, \tau_2)] &= \left(A(\tau_1)T_i^{k+1}\langle \delta(\tau_1, \tau_2) \rangle - T_i^k A(\tau_1)\delta(\tau_1, \tau_2)\right)T_i^{k+1} \\
&\quad + \left(B(\tau_1)T_i^k\langle \delta(\tau_1, \tau_2) \rangle - T_i^k B(\tau_1)\delta(\tau_1, \tau_2)\right)T_i^k \\
&\quad + \left(C(\tau_i)T_i^{-k-1}\langle \delta(\tau_1, \tau_2) \rangle - T_i^{-k} C(\tau_i)\delta(\tau_1, \tau_2)\right)T_i^{-k-1}.
\end{align*}\]

We will write all expressions in the form \(\sum_{\alpha, \beta} \ell_{\alpha, \beta} T_1^\alpha T_2^\beta\), i.e., with the powers of \(T\) to the right. Consider the coefficient of \(T_1^{-k}T_2^{k+1}\) for \(k > 0\) in \([L_2, r] - [L_1, r^*]\). It is
\[A(\tau_2)T_2^{k+1}\langle \delta(\tau_1, \tau_2) \rangle - T_2^k A(\tau_2)\delta(\tau_1, \tau_2) + A(\tau_1)T_1^{-k}\langle \delta(\tau_1, \tau_2) \rangle - T_1^{-k-1} A(\tau_1)\delta(\tau_1, \tau_2) = 0\]
by the property (6.1) of the action of \(VH_T \otimes VH_T\) on two variable classical fields. In the same manner one checks that the coefficients of \(T_1^{-k}T_2^k, T_1^kT_2^{-k}\), for \(k \geq 0\) and of \(T_1^{-k}T_2^{k-1}, T_1^kT_2^{1-k}, T_1^kT_2^{-k-1}\) for \(k > 0\) all vanish. There remain only 2 nonzero coefficients. For \(T_1^0 T_2^1\) the coefficient is
\[C(\tau_2)T_2^{-1}\delta(\tau_1, \tau_2) - C(\tau_2)\delta(\tau_1, \tau_2) = C(\tau_2)(T_2^{-1} - 1)\delta(\tau_1, \delta_2)\]
Similarly the coefficient of \(T_1^{-1}T_2^0\) is found to be
\[\tau(-m - 1)[C(\tau_1)T_1^{-1}\langle \tau_1(m) \rangle - C(\tau_1)\tau_1(m)] = C(\tau_1)T_1^{-1}\delta(\tau_1, \delta_2) - C(\tau_1)\delta(\tau_1, \tau_2) = (T_1^{-1} - 1)C(\tau_1)\delta(\tau_1, \delta_2) = C(\tau_2)(T_2 - 1)\delta(\tau_1, \delta_2)\]

Given a singular Hamiltonian structure on \(V\) we define the Poisson bracket of the field Lax operator \(L(\tau) = \sum_\alpha v_\alpha(\tau)T^\alpha\) by
\[\{L(\tau_1), L(\tau_2)\} = \sum_{\alpha, \beta} \{v_\alpha(\tau_1), v_\beta(\tau_2)\} T_1^\alpha T_2^\beta\]
Conversely we define on \(V\) a singular Hamiltonian structure by putting
\[\{L(\tau_1), L(\tau_2)\} = [L_2, r] - [L_1, r^*]\]
By Lemma 2.1 this means that we have specified the Poisson brackets of the generators of $V$ to be

\begin{align}
\{B(\tau_1), C(\tau_2)\} &= C(\tau_2)(T_2^{-1} - 1)\delta(\tau_1, \tau_2), \\
\{C(\tau_1), B(\tau_2)\} &= (1 - T_2)C(\tau_2)\delta(\tau_1, \tau_2),
\end{align}

and all other Poisson brackets are zero, in particular $\{A(\tau_1), F(\tau_2)\} = 0$, for all $F \in V$. Then one checks that this can be extended consistently to a Poisson brackets on all of $V$, essentially since

$$\{B(\tau_1), \{B(\tau_2), C(\tau_3)\}\} + \{B(\tau_2), \{C(\tau_3), B(\tau_1)\}\} + \{C(\tau_3), \{B(\tau_1), C(\tau_2)\}\} = 0.$$ 

The projection (9.4) of $\Gamma(f) \in \text{Der}^{ev}(V, V(\tau_1, \tau_2))$ gives an evolutionary derivation $X(f) \in \text{Der}^{ev}(V, V)$ which reads in column form

$$X(f) = \begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & (1 - T)C \\ 0 & C(T^{-1} - 1) & 0 \end{pmatrix} \begin{pmatrix} \delta f \\ \delta \alpha \\ \delta \beta \end{pmatrix}.$$ 

In terms of the Lax operator we have

$$X_L(f) = 0T + (1 - T)(C(\delta f/\delta c))T^0 + C(T - 1)(\delta f/\delta b)T^{-1}.$$ 

We can express this in terms of the difference operator $\delta_L f$ and the classical $r$-matrix.

**Lemma 2.2.**

$$X_L(f) = [L, \rho(\delta_L f)] + \rho^*[L, \delta_L f].$$

**Proof.** Writing $L = \sum v_\alpha T^\alpha$ we have

$$\Gamma(f)(L(\tau_2)) = \{f(\tau_1), v_\alpha(\tau_2)\} T_2^\alpha = (d_\beta f)_1 \{v_\beta(\tau_1), v_\alpha(\tau_2)\} T_2^\alpha.$$ 

Now $\{v_\beta(\tau_1), v_\alpha(\tau_2)\} T_2^\alpha$ is the $T_1^\beta$-component of $\{L(\tau_1), L(\tau_2)\}$, so is given by (2.3). Therefore we can write (2.6) as a sum of two terms (involving $r$ and $r^*$ respectively). The first term is (suppressing the summation over the repeated indices here and below)

$$\begin{align}
(d_\beta f)_1 [L(\tau_2), \rho(T_2^{-\beta}) \delta(\tau_1, \tau_2)] \\
= (d_\beta f)_1 \left( v_\alpha(\tau_2) T_2^{\alpha - \beta} \langle \delta(\tau_1, \tau_2) \rangle - T_2^{-\beta} \langle v_\alpha(\tau_2) \delta(\tau_1, \tau_2) \rangle \right) T_2^\alpha \rho(T_2^{-\beta}).
\end{align}$$

Now recall that the projection $\Pi$ from section 9 satisfies for all difference operators $P, Q$ the relation $\Pi(P_1 Q_2 \delta(\tau_1, \tau_2)) = Q(P_1)$ and that we have
\((d_\beta f)_V = \frac{\delta f}{\delta v^\beta}\). Therefore, by applying \(\Pi\) to the coefficients of (2.7) we get as first contribution to \(X_f(L)\) (writing \(T\) for \(T_2\))

\[
(v_\alpha T^{\alpha-\beta} \left( \frac{\delta f}{\delta v^\beta} \right) - T^{-\beta} \langle v_\alpha \frac{\delta f}{\delta v^\beta} \rangle) T^\alpha \rho(T^{-\beta}) = [L, \rho(T^{-\beta}) \frac{\delta f}{\delta v^\beta}]
= [L, \rho(\delta_L f)].
\]

Next consider

\[
[r^*, L(\tau)] = [\rho(T_1^{1-\alpha}) \delta(\tau_1, \tau_2), v_\alpha T_1^{\alpha-\beta}]
= \left( \rho(T_1^{1-\alpha}) \langle v_\alpha(\tau) \delta(\tau_1, \tau_2) \rangle - v_\alpha(\tau) T_1^{\alpha} \rho(T_1^{1-\alpha}) \langle \delta(\tau_1, \tau_2) \rangle \right) T_1^{\beta} T_2^{\alpha-\beta}.
\]

Taking the coefficient of \(T^\beta_1\) of the last expression we see that the second term of (2.6) is

(2.8)

\[
(d_\beta f)_1 \left( \rho(T_1^{1-\alpha}) \langle v_\alpha(\tau) \delta(\tau_1, \tau_2) \rangle - v_\alpha(\tau) T_1^{\alpha} \rho(T_1^{1-\alpha}) \langle \delta(\tau_1, \tau_2) \rangle \right) T_2^{\alpha-\beta}.
\]

Now the \(r\)-matrix satisfies

\[
\rho(T_1^k) \delta(\tau_1, \tau_2) = \rho^*(T_2^{-k}) \delta(\tau_1, \tau_2),
\]

so that we can rewrite (2.8) as

\[
(d_\beta f)_1 \left( v_\alpha(\tau) \rho^*(T_2^{\alpha-\beta}) \langle \delta(\tau_1, \tau_2) \rangle - T_2^{-\alpha} \rho^*(T_2^{\alpha-\beta}) \langle v_\alpha(\tau) \delta(\tau_1, \tau_2) \rangle \right) T_2^{\alpha-\beta}.
\]

Applying \(\Pi\) to the coefficients we get as second contribution to \(X_f(L)\) (again writing \(T\) for \(T_2\))

\[
(v_\alpha T^{\alpha-\beta} \left( \frac{\delta f}{\delta v^\beta} \right) - T^{-\beta} \langle v_\alpha \frac{\delta f}{\delta v^\beta} \rangle) \rho^*(T^{\alpha-\beta})
= \rho^*[v_\alpha T^{\alpha}, T^{-\beta} \frac{\delta f}{\delta v^\beta}] = \rho^*[L, \delta_L f],
\]

so that indeed

\[
X_f(L) = [L, \rho(\delta_L f)] + \rho^*[L, \delta_L f].
\]

\[\square\]

Recall that in (2.2) the difference operator \(\delta_L f\) was determined up to addition of terms \(\delta_{\geq 2}\) and \(\delta_{\leq -2}\). We have the same freedom in the vector
field $X_f(L)$: we have
\[
[L, \rho(\delta_{\geq 2})] + \rho^*[L, \delta_{\geq 2}] = 0
\]
\[
[L, \rho(\delta_{\leq -2})] + \rho^*[L, \delta_{\leq -2}] = 0,
\]
so that $X_f(L)$ is independent of addition of these terms.

Next consider the Hamiltonians
\[
H_n = \text{Sp}(L^n/2n) \in V.
\]
We have for any $X \in \text{Der}^\text{ev}(V, V)$
\[
X(H_n) \equiv \frac{1}{2} \text{Sp}(X(L)L^{n-1}) \mod mV,
\]
so that we can represent the differential $\delta_L H_n$ of $H_n$ by the difference operator $L^n - 1/2$. Therefore the Hamiltonian vector field of $H_n$ is given by the Lax-equation
\[
\dot{X}(H_n) = [L, \rho(L^n - 1/2)] + \rho^*[L, L^n - 1/2]
\]
\[
= [L, \rho(L^n - 1/2)] = [L/2, L_{0, +}^{n-1} - L_{-}^{n-1}]
\]
\[
= [L, L_{0, +}^{n-1}].
\]
In particular for $n = 2$ we get the following equations:
\[
\dot{A} = 0, \quad \dot{B} = CT^{-1}A - ATC, \quad \dot{C} = C(T^{-1} - 1)B.
\]

Putting $A = 0$ we obtain an infinite analog of the finite Toda equation (1.1):
\[
(2.9) \quad \dot{B} = (1 - T)C, \quad \dot{C} = C(T^{-1} - 1)B.
\]

We want now clarify the motivation for the terminology $\text{Tr}$ for the maps $K \to C$, (11.1) and $LV \to \mathcal{Z}V$, see section 5. In the finite Toda lattice the variables $q_i, p_i$ or $B_i, C_i$ give local functions on the phase space (attached to a site $i$ of the finite lattice, roughly speaking), and the Hamiltonians, global functions on phase space, are obtained as traces over the powers of the Lax matrix (1.2). Similarly, for the infinite Toda lattice the elements of $V$, difference polynomials in $B, C$, define densities of functions on the phase space $V(K)$, and to obtain honest functions one has to take the trace.
CHAPTER 5

$H_T$-Conformal Algebras

1. Introduction

In chapter 3 we introduced the notion of a singular Hamiltonian structure on an $H_T$-Leibniz algebra $V$; this was given by an evolutionary derivation $\Gamma: V \to \text{Der}^e(V, V(\tau_1, \tau_2))$ satisfying the conditions of Definition 8.1. By projecting using the map $\Pi: V(\tau_1, \tau_2) \to V$, see section 9, we obtain for all $f \in V$ an evolutionary derivation $\Pi \circ \Gamma(f) = X_f: V \to V$.

In this chapter we will see that $X_f$ is just one of an infinite collection of products labeled by $h^* \in H^*_T$:

$$\{h^*\}: V \otimes V \to V, \quad f \otimes g \mapsto f_{\{h^*\}}g.$$

In this notation $X_f = f_{\{1\}}$, where $1 = 1_{H^*_T}$, the identity of the algebra $H^*_T = \mathbb{C}Z$, see section 3. The collection of all conformal products $f_{\{h^*\}}g$ for all $h^* \in H^*_T$ will give $V$ the structure of what we will call an $H_T$-Conformal Algebra. These are analogs of the conformal algebras from the usual theory of vertex algebras, see [Kac98], cf., also [Li] where they are called vertex Lie algebras.

We will define for any $H_T$-conformal algebra $C$ a singular vertex operator $Y_{\text{Sing}}(f, \tau)$ for all $f \in C$, see Section 4 below. This is a generating series for the conformal products $f_{\{h^*\}}g$ and we will formulate the axioms of an $H_T$-conformal algebra both in terms of the components $f_{\{h^*\}}$ and also in terms of the singular vertex operator.

The singular vertex operators will be generalized in the next Chapter 6 to the vertex operators of an $H_T$-vertex algebra.

2. $H_T$-Conformal Algebras and $H_T$-Vertex Poisson Algebras.

If $\Gamma$ is a singular Hamiltonian map for an $H_T$-Leibniz algebra $V$ we obtain for $f, g \in V$ a distribution $\Gamma(f)(g) \in V(\tau_1, \tau_2)$ that we can uniquely write as

$$\Gamma(f)(g) = D(f \otimes g)(\tau_2)\delta(\tau_1, \tau_2),$$

where $D(f \otimes g)$ is a difference operator in $VH_T$. 

67
The properties of the Poisson brackets \( \{ f(\tau_1), g(\tau_2) \} = \Gamma(f)g \) of a singular Hamiltonian structure can be expressed in terms of the difference operators \( D(f \otimes g) \). The fact that \( \Gamma \) is evolutionary implies that

\[
D(h.f \otimes g) = D(f \otimes g)S(h), \quad h \in H_T,
\]

and the skew symmetry translates to

\[
D(f \otimes g) = -D(g \otimes f)^*,
\]

using the adjoint in the algebra \( VH_T \), see (23.2). The Lie property of \( \Gamma \) gives a more complicated condition on the \( D(f \otimes g) \) that we will not need explicitly in the sequel.

To get explicit formulas it is convenient to choose a basis for \( H_T \). We can either use the translation operator basis \( \{ T^n \} \) of \( H_T \), with dual basis \( \{ \delta_n \} \) \( n \in \mathbb{Z} \) or the difference operator basis \( \{ \Delta[n] \} \), with dual basis \( \{ \Delta^*[n] \} \) \( n \in \mathbb{Z} \).

We expand

\[
D(f \otimes g) = \sum_{n \in \mathbb{Z}} f_n g T^n = \sum_{n \in \mathbb{Z}} f_{\{\delta_n\}} g \Delta[n],
\]

with \( f_n, f_{\{\delta_n\}} g \in V \). Here we use abbreviations:

\[
f_n = f_{\{\delta_n\}}, \quad f_{\{\delta_n\}} = f_{\{\Delta^*[n]\}},
\]

as the coefficients of the expansion (2.4) should be labeled by elements of the dual of \( H_T \).

Since \( \Gamma(f)(-) \) is a derivation, we obtain in this way derivations

\[
f_n, f_{\{\delta_n\}} : V \rightarrow V,
\]

for all \( n \in \mathbb{Z} \). We think of the operations \( n, \langle n \rangle : V \otimes V \rightarrow V \) as giving (non-commutative, non-associative) multiplications on \( V \). The properties of the multiplications are simplest in terms of the translation operator basis, since the multiplication and comultiplication of \( H_T \) is simplest in that basis.

**Lemma 2.1.** For a singular Hamiltonian structure \( \Gamma \) on a commutative \( H_T \)-Leibniz algebra \( V \) the operations \( n : V \otimes V \rightarrow V \) satisfy

(a) (Finiteness) For all \( f, g \in V \) \( f_n g = 0 \) for all but a finite number of \( n \in \mathbb{Z} \).

(b) (\( H_T \)-covariance) For all \( f \in V, n, p \in \mathbb{Z} \) we have \( (T^n f)_n = f_{n+p} \).

(c) (Skew-symmetry) For all \( f, g \in V, n \in \mathbb{Z} \) we have \( f_n g = -T^n (g_{-n} f) \).

(d) (Commutator) we have \( [f_m, g_n] = (f_{m-n} g_n) \) for all \( f, g \in V, m, n \in \mathbb{Z} \).

**Proof.** The finiteness property of \( f_n g \)'s is clear since \( D(f \otimes g) \in VH_T \) and any element of \( VH_T \) has a finite expansion in \( T^n \).
Taking \( h = T^p \) in (2.2) we calculate
\[
D(T^p f \otimes g) = \sum_{n \in \mathbb{Z}} (T^p f)_n g^{T^n} = \sum_{n \in \mathbb{Z}} f_n g^{T^n-p} = \sum_{n \in \mathbb{Z}} f_{n+p} g^{T^n},
\]
so that
\[
(T^p f)_n g = f_{n+p} g.
\]
For skew-symmetry we use (2.3). Now
\[
D(g \otimes f)^* = \sum_{n \in \mathbb{Z}} T^{-n} g_n f = \sum_{n \in \mathbb{Z}} T^{-n} (g_n f) T^{-n} = \sum_{n \in \mathbb{Z}} T^n (g_{-n} f) T^n,
\]
so that
\[
f_{n} g = -T^n (g_{-n} f).
\]
For the commutator formula we calculate some Poisson brackets. For \( f, g, k \in V \) we have
\[
\{ f(\tau_1), \{ g(\tau_2), k(\tau_3) \} \} = \sum_{n,p \in \mathbb{Z}} f_n (g_p k)(\tau_3) T^n_{\tau_1} T^{p}_{\tau_2} \delta(\tau_1, \tau_3) T^p_{\tau_2} \delta(\tau_2, \tau_3)
\]
\[
= \sum_{n,p \in \mathbb{Z}} f_n (g_p k)(\tau_3) T^{p}_{\tau_1} T^{n}_{\tau_2} \delta(\tau_1, \tau_3) \delta(\tau_2, \tau_3).
\]
\[
\{ g(\tau_2), \{ f(\tau_1), k(\tau_3) \} \} = \sum_{n,p \in \mathbb{Z}} g_p (f_n k)(\tau_3) T^{p}_{\tau_1} T^{n}_{\tau_2} \delta(\tau_1, \tau_3) \delta(\tau_2, \tau_3).
\]
\[
\{ \{ f(\tau_1), g(\tau_2) \}, k(\tau_3) \} = \sum_{q,r \in \mathbb{Z}} (f_q g)_r T^{q}_{\tau_1} T^{r}_{\tau_2} \delta(\tau_1, \tau_2) \delta(\tau_2, \tau_3)
\]
\[
= \sum_{q,r \in \mathbb{Z}} (f_q g)_r T^{q}_{\tau_1} T^{r}_{\tau_2} \delta(\tau_1, \tau_2) \delta(\tau_2, \tau_3).
\]
Making the substitution \( q + r = n, r = p \) we obtain
\[
= \sum_{n,p \in \mathbb{Z}} (f_{n-p} g)_p T^{n}_{\tau_1} T^{p}_{\tau_2} \delta(\tau_1, \tau_2) \delta(\tau_2, \tau_3),
\]
so that by the Lie property of \( \Gamma \) we find
\[
[f_n, g_p] = (f_{n-p} g)_p.
\]

We now observe that the operations \( f_n \) (which are derivations of the commutative algebra structure of \( V \)) with the properties (a)-(d) of Lemma 2.1 make sense also if \( V \) is just an \( H_T \)-module, not an \( H_T \)-Leibniz algebra. So we define:

**Definition 2.2.** An \( H_T \)-Conformal algebra is an \( H_T \)-module \( C \) with operations
\[
n : C \otimes C \to C, \quad n \in \mathbb{Z},
\]
satisfying
(a) **(Finiteness)** For all \( f, g \in C \), \( f_n g = 0 \) for all but a finite number of \( n \in \mathbb{Z} \).

(b) **\( H_T \)-covariance** For all \( f \in C \), \( n, p \in \mathbb{Z} \) we have \( (T^n f)_n = f_{n+p} \).

(c) **(Skew-symmetry)** For all \( f, g \in C \), \( n \in \mathbb{Z} \) we have \( f_n g = -T^n (g_{-n} f) \).

(d) **(Commutator)** We have \([f_m, g_n] = (f_{m-n} g)_n\) for all \( f, g \in C \), \( m, n \in \mathbb{Z} \).

As a first result we have the Leibniz rule for conformal products.

**Lemma 2.3.** Let \( C \) be an \( H_T \)-conformal algebra, and \( f, g \in C \). If \( h \in H_T \) has coproduct \( \sum h' \otimes h'' \) then for all \( n \in \mathbb{Z} \) we have

\[
h(f_n g) = \sum (h' f)_n (h'' g).
\]

**Proof.** By \( H_T \)-covariance we have \( f_n g = (T^n f)_0 g \) so it suffices to check (2.6) in case \( n = 0 \). Also, it suffices to check for \( h = T^p \), \( p \in \mathbb{Z} \), in which case the coproduct is just \( T^p \otimes T^p \). Then

\[
T^p(f_0 g) = T^p ((T^p f)_p g) = - (g_p (T^p f)) \quad = - (T^p g)_0 (T^p f) = (T^p f)_0 (T^p g),
\]

using skew-symmetry and \( H_T \)-covariance repeatedly. \(\square\)

**Example 2.4.** Let \( CToda \) be the free \( H_T \)-module generated by two elements \( B \) and \( C \). We define products \( x_n y \) for \( x, y \in CToda, n \in \mathbb{Z} \) by putting on generators

\[
B_n B = C_n C = 0, \quad n \in \mathbb{Z},
\]

and

\[
B_0 C = -C, \quad B_{-1} C = C \quad C_0 C = C, \quad C_1 B = -TC,
\]

and all other products \( B_n C \) and \( C_n B \) are zero. Since the fundamental Poisson brackets (2.4) define a singular Hamiltonian structure on \( V = \mathbb{C}[B^{(n)}, C^{(n)}] \) the products (2.7), (2.8) and (2.9) define an \( H_T \)-conformal structure on \( CToda \), called the *first Toda conformal structure* on \( CToda \), cf., [Kup85].

We plan to discuss higher Toda conformal structures in a future paper.

Note that \( CToda \) is a finite rank \( H_T \)-conformal subalgebra of \( V \) and in a sense \( V \) is the symmetric \( H_T \)-conformal algebra of \( CToda \). Also note that \( CToda \) is a non trivial extension: if we let \( CC, CB \) be the rank 1 free Abelian \( H_T \)-conformal algebras generated \( C, \) resp \( B \) then we have an exact sequence

\[
0 \to CC \to CToda \to CB \to 0.
\]

\(\square\)
EXAMPLE 2.5. Let \( g \) be a Lie algebra, and let \( Cg \) be the free \( HT \)-module generated by \( g \). Then define, for \( x, y \in g \) and \( p, q, n \in \mathbb{Z} \) products

\[
(T^p x)_n (T^q y) = T^q [x, y] \delta_{n+p-q,0}.
\]

Then one checks that this gives \( Cg \) the structure of \( HT \)-conformal algebra, the commutator axiom follows from the Jacobi identity for \( g \). We call \( Cg \) the affine \( HT \)-conformal algebra of the Lie algebra \( g \). □

The notion of a singular Hamiltonian structure on an \( HT \)-Leibniz algebra gives rise to a special class of \( HT \)-conformal algebras.

DEFINITION 2.6. An \( HT \)-Vertex Poisson Algebra is a (commutative) \( HT \)-Leibniz algebra \( V \) with a compatible \( HT \)-conformal algebra structure: the operations \( f_n \) for \( f \in V \), \( n \in \mathbb{Z} \) are derivations of the algebra structure of \( V \).

So a singular Hamiltonian map \( \Gamma : V \to \text{Der}^{ev}(V, V(\tau_1, \tau_2)) \) gives an \( HT \)-Leibniz algebra \( V \) the structure of \( HT \)-vertex Poisson algebra. Conversely, let \( V \) be an \( HT \)-vertex Poisson algebra. Then we can define an evolutionary derivation \( \Gamma : V \to \text{Der}^{ev}(V, V(\tau_1, \tau_2)) \) by

\[
\Gamma(f)(g) = \sum_{n \in \mathbb{Z}} f_n g(\tau_2) T^n \delta(\tau_1, \tau_2).
\]

One then checks that \( \Gamma \) is in fact a singular Hamiltonian map.

3. The Lie Algebra of a Conformal Algebra

To obtain from a singular Hamiltonian structure \( \Gamma \) on \( V \) a conformal algebra structure we used the expansion of the difference operators \( D(f \otimes g) \) in the basis \( \{ T^n \} \) of \( HT \). The operations \( \langle n \rangle : V \otimes V \to V \) related to the difference basis, see (2.4), satisfy properties similar to those of \( n \), but they are more complicated. We will not need explicitly the operations \( \langle n \rangle \) in the sequel, except for the zeroeth one, which is given by

\[
f_{(0)} g = \sum_{n \in \mathbb{Z}} f_n g.
\]

In the notation of (2.5) we have \( f_{(0)} = f_{1_{H^*}} \).

Recall the augmentation ideal \( m \subset HT \) and the submodule \( mM \) of total differences of an \( HT \)-module \( M \), see Section 5.

LEMMA 3.1. In any \( HT \)-conformal algebra \( V \) the operation \( \langle 0 \rangle \) given by (3.1) satisfies

(a) \( (T f)_{(0)} = f_{(0)}. \)
(b) \( f_{(0)} (T^q g) = T^q (f_{(0)} g). \)
(c) \( f_{(0)} g - g_{(0)} f \in mM. \)
(d) \( [f_{(0)}, g_{(0)}] = (f_{(0)} g)_{(0)}. \)
5. \( H_T \)-CONFORMAL ALGEBRAS

PROOF. We have by \( H_T \) covariance
\[
(Tf)_{(0)}g = \sum_{n \in \mathbb{Z}} (Tf)_n g = \sum_{n \in \mathbb{Z}} (f)_{n+1} g = f_{(0)}g.
\]

By skew-symmetry and \( H_T \)-covariance we get
\[
f_{(0)}(Tg) = \sum_{n \in \mathbb{Z}} f_n (Tg) = - \sum_{n \in \mathbb{Z}} T^n \langle (Tg)_{-n} f \rangle = - \sum_{n \in \mathbb{Z}} TT^{n-1} \langle g_{1-n} f \rangle
\]
\[
= T \sum_{n \in \mathbb{Z}} \langle f_{n-1} g \rangle = T \langle f(0) g \rangle.
\]

Using the commutator we find
\[
[f_{(0)}, g_{(0)}] q = \sum_{m,n \in \mathbb{Z}} [f_m, g_n] = \sum_{m,n \in \mathbb{Z}} (f_{m-n} g)_n = \sum_{p,q \in \mathbb{Z}} (f_p g) q = (f(0) g)_{(0)}.
\]

□

Recall the trace map \( \text{Tr} : M \to M/mM \), see section 5, and write \( \tilde{f} \) for the image \( \text{Tr}(f) \) in \( C_m \) of \( f \in C \), see (5.2). Then the lemma shows that in case \( C \) is an \( H_T \)-conformal algebra the bilinear map
\[
(3.2) \quad [\, , \,] : C_m \otimes C_m \to C_m, \quad \tilde{f}, \tilde{g} = \tilde{f(0)} g
\]
is well defined, and defines a Lie algebra structure on \( C_m \).

EXAMPLE 3.2. In case of the Toda conformal algebra \( C_{\text{Toda}} \) of Example 2.4 the module of coinvariants is two dimensional:
\[
C_{\text{Toda}} m = C_{\text{Toda}}/mC_{\text{Toda}} = C\tilde{B} \oplus C\tilde{C},
\]
with vanishing Lie bracket: \([\tilde{B}, \tilde{C}] = 0\).

EXAMPLE 3.3. For the affine \( H_T \)-conformal algebra \( Cg \) of a Lie algebra \( g \), see 2.5, the module of coinvariants \( Cg_m \simeq g \) and the Lie bracket (3.2) is just the standard bracket of \( g \).

□

4. Singular Vertex operators

In this section we reformulate the axioms of an \( H_T \)-conformal algebra \( C \), see Definition 2.2, in terms of the generating series of the conformal products of \( f \in C \). Define the singular vertex operator of \( f \) by
\[
Y_{\text{Sing}}(f, \tau) = \sum_{n \in \mathbb{Z}} f_{(n)} S(\Delta[n]) \frac{1}{\tau} = \sum_{n \in \mathbb{Z}} f_n S(T^n) \frac{1}{\tau}.
\]

LEMMA 4.1. Let \( C \) be an \( H_T \)-conformal algebra. The singular vertex operator for \( f \in C \) satisfies
(a) (Finiteness) \( Y_{\text{Sing}}(f, \tau) g \) belongs to \( CH_T \frac{1}{\tau} \) for all \( g \in C \).
(b) (\( H_T \)-covariance) For all \( h \in H_T \) we have \( \tilde{Y}_{\text{Sing}}(hf, \tau) = hY_{\text{Sing}}(f, \tau) \).
(c) **(Skew-symmetry)** For all $g \in C$

$$Y_{\text{Sing}}(f, \tau)g = -\text{Sing}_{\tau}[\mathcal{R}_{C}(\tau)Y_{\text{Sing}}^{S}(g, \tau)f].$$

(d) **(Commutator)** For all $g \in C$

$$[Y_{\text{Sing}}(f, \tau_1), Y_{\text{Sing}}(g, \tau_2)] = \text{Sing}_{\tau_2} \left( Y_{\text{Sing}}(\mathcal{R}_{W}^{S}(\tau_2), Y_{\text{Sing}}(f, \tau_1)g, \tau_2) \right),$$

where $W = (H_{T})^{1}_{\tau_1}$.

**PROOF.** Finiteness is clear. For $H_{T}$-covariance note that

$$Y_{\text{Sing}}(f, \tau)g = D(f \otimes g)^{1}_{\tau}.$$ 

So by (2.2) we have

$$Y_{\text{Sing}}(hf, \tau)g = D(hf \otimes g)^{1}_{\tau} = D(f \otimes g)^{1}_{\tau}h = h_{K}Y_{\text{Sing}}(f, \tau)g.$$ 

For skew-symmetry we use (2.3) to calculate:

$$Y_{\text{Sing}}(f, \tau)g = D(f \otimes g)^{1}_{\tau} = -(D(g \otimes f)^{*})^{1}_{\tau} $$

$$= -\text{Sing}(\mathcal{R}_{V}(\tau)D(g \otimes f)^{1}_{\tau}) $$

$$= -\text{Sing}(\mathcal{R}_{V}(\tau)Y_{\text{Sing}}^{S}(g, \tau)f),$$

using Lemma 26.1, where

$$Y_{\text{Sing}}^{S}(g, \tau)f = D(g \otimes f)^{1}_{\tau}.$$ 

Finally

$$[Y_{\text{Sing}}(f, \tau_1), Y_{\text{Sing}}(g, \tau_2)] = \sum_{m,n} [f_{m} \cdot g_{n}] T^{-m}(\frac{1}{\tau_1}) T^{-n}(\frac{1}{\tau_2})$$

$$= \sum_{n,m} (f_{m} \cdot g_{n}) T^{-m}(\frac{1}{\tau_1}) T^{-n}(\frac{1}{\tau_2}) $$

$$= \sum_{a,b} (f_{a}g_{b}) T^{-a}(\frac{1}{\tau_1}) T^{-b}(\frac{1}{\tau_2}) $$

$$= \sum_{a,b} (f_{a}g_{b}) P_{a,b}^{*}(\frac{1}{\tau_2}).$$
where \( P_{a,b} = T^{-a} \left( \frac{1}{\tau_1} \right) T^b \). Then by Corollary 26.2 we have

\[
= \text{Sing}_{\tau_2} \left( R^S_W (\tau_2) (f_{a\bar{g}} b) P^S_{a,b} \frac{1}{\tau_2} \right)
\]

\[
= \text{Sing}_{\tau_2} \left( R^S_W (\tau_2) (f_{a\bar{g}} b) (T_1^{-a} \frac{1}{\tau_1} T_2^{-b} \frac{1}{\tau_2}) \right)
\]

\[
= \text{Sing}_{\tau_2} (Y_{\text{Sing}} (R^S_W (\tau_2) Y_{\text{Sing}} (f, \tau_1 ; g, \tau_2)));
\]

\( \square \)

Of course, we can reverse our procedure, and define an \( H_T \)-conformal algebra as an \( H_T \)-module \( C \) together with for all \( f \in C \) a singular vertex operator \( Y_{\text{Sing}}(f, \tau) \), satisfying the properties of Lemma 4.1. Expanding the singular vertex operator we find that the components \( f_n \) satisfy the properties of Definition 2.2.

5. Holomorphic \( H_T \)-vertex algebras

Let \( V \) be a commutative \( H_T \)-Leibniz algebra. For \( f \in V \) define the holomorphic vertex operator by

\[
Y_{\text{Hol}}(f) = R(f) = \sum_{n \in \mathbb{Z}} \Delta[n](f) \otimes \Delta^*[n],
\]

where \( R \) is the untwisted exponential operator for \( H_T \) introduced in section 14, see (14.3), so that \( Y_{\text{Hol}} \) is just a suggestive notation for the coaction map \( m^*_V : V \to V \otimes H^*_T \) introduced in section 2. In particular \( Y_{\text{Hol}} \) is multiplicative, see section 16. We define

\[
Y_{\text{Hol}}^S(f) = \sum \Delta[n](f) \otimes S(\Delta^*[n]) = R^S(f_1).
\]

By Lemma 18.1 we have

\[
Y_{\text{Hol}}^S(Y_{\text{Hol}}(f)) = Y_{\text{Hol}}(Y_{\text{Hol}}^S(f)) = f \otimes 1_{H^*_T}.
\]

It then follows that we have Skew-symmetry:

\[
Y_{\text{Hol}}(f) g = R \left( Y_{\text{Hol}}^S(g) f \right).
\]

The \( H_T \)-covariance of exponential operators, see section 15, implies that

\[
Y_{\text{Hol}}(h f) = h_{H^*_T} (Y_{\text{Hol}}(f)), \quad Y_{\text{Hol}}^S(h f) = S(h)_{H^*_T} (Y_{\text{Hol}}^S(f)).
\]

**Definition 5.1.** A holomorphic \( H_T \)-vertex algebra is an \( H_T \)-module, together with a distinguished element \( 1_V \in V \) and a map

\[
Y_{\text{Hol}} : V \otimes V \to V \otimes H^*_T, \quad f \otimes g \mapsto Y_{\text{Hol}}(f) g,
\]

such that
6. Extension of $H_T$-conformal Structure to the Affinization

Recall the notion of the affinization $LM = M \otimes K$ of an $H_T$-module $M$ and the canonical trace map $\text{Tr}: LM \to \mathcal{L}M = LM/mLM$, see Section 5. In this section we take $M = C$, where $C$ is an $H_T$-conformal algebra, and show that the affinization $LC$ inherits an $H_T$-conformal structure from the $H_T$-conformal structure of $C$ and the holomorphic vertex algebra structure of $K$. In particular this gives $\mathcal{L}C$ a Lie algebra structure, by Section 3. In case $C = V$ is an $H_T$-vertex Poisson algebra, see Section 2, the Lie bracket on $\mathcal{L}V$ is the Poisson bracket of the components of the classical fields of Chapter 3, see Example 6.4.

For $f \in C$ we have the singular vertex operator
\[ Y_{\text{Sing}}(f, \tau)g = \sum_{n \in \mathbb{Z}} f(n) g S(\Delta[n]) \frac{1}{\tau}. \]

For $p \in K$ we have the holomorphic vertex operator $Y_{\text{Hol}}(p)$, see (5.1), and by combining $Y_{\text{Sing}}$ on $C$ with $Y_{\text{Hol}}$ on $K$ we define a singular vertex operator on $LC$:
\[ Y_{\text{Sing}}^{LC}(f \otimes p, \tau) := \text{Sing} \left( Y_{\text{Sing}}(f, \tau) \otimes Y_{\text{Hol}}(p, \tau) \right). \]

Here $\otimes$ is a combination of tensor product and multiplication: we have explicitly
\begin{equation} \label{eq:6.1}
Y_{\text{Sing}}(f, \tau) \otimes Y_{\text{Hol}}(p, \tau) = \sum_{m,n \in \mathbb{Z}} \left( f[m] \otimes \Delta[n] \langle p \rangle \Delta^*[n] S(\Delta[m]) \frac{1}{\tau} \right).
\end{equation}
Now by (26.2) we have a vanishing contribution to (6.1), and hence to $Y^L_{\text{Sing}}$, from the terms in which $m, n$ have opposite sign. In case $0 \leq m < n$

$$\Delta^*[n]S(\Delta[m]) \frac{1}{\tau}, \quad \Delta^*[-n]S(\Delta[m]) \frac{1}{\tau}$$

is nonsingular, so these terms will also not contribute to $Y^L_{\text{Sing}}$. Finally, by (26.3) and the factorization (4.7) we have, if $m = n + s, \ n, s \geq 0$

$$\Delta^*[n]S(\Delta[n+s]) \frac{1}{\tau} = \frac{1}{\tau[n+n+s+1]} = T^{-n}s(\Delta[m]) \frac{1}{\tau}$$

$$\Delta^*[-n]S(\Delta[-n-s]) \frac{1}{\tau} = \frac{1}{\tau[n+n+s+1]} = T^nS(\Delta[-s]) \frac{1}{\tau}$$

Hence

(6.2) $Y^L_{\text{Sing}}(f \otimes p, \tau) = \sum_{m > 0} \sum_{\ell = 0}^{m} (f_{(m)} \otimes \Delta[\ell]p) \otimes T^{-\ell}s(\Delta[m - \ell]) \frac{1}{\tau} +$

$$\sum_{m > 0} \sum_{\ell = 0}^{m} (f_{(-m)} \otimes \Delta[\ell]p) \otimes T^\ell s(\Delta[m - \ell]) \frac{1}{\tau}.$$

**Lemma 6.1.** $Y^L_{\text{Sing}}$ gives LC the structure of $H_T$-conformal algebra.

**Proof.** We need to check the properties of Lemma 4.1. Finiteness is manifest from (6.2), as in any $H_T$-conformal algebra $f_{(m)}g = 0$ for all but a finite number of $m \in \mathbb{Z}$.

For $h \in H_T$ we have, if $\pi(h) = \sum h' \otimes h''$ is the coproduct, using the $H_T$-covariance of $Y_{\text{Sing}}$ and $Y_{\text{Hol}},$

$$Y^L_{\text{Sing}}(h.(f \otimes p), \tau) = \sum \text{Sing}_\tau \left( Y_{\text{Sing}}(hf, \tau) \otimes Y_{\text{Hol}}(h''p, \tau) \right)$$

$$= \sum \text{Sing}_\tau \left( h_k Y_{\text{Sing}}(f, \tau) \otimes h_k Y_{\text{Hol}}(p, \tau) \right)$$

$$= \sum \text{Sing}_\tau \left( h_k Y_{\text{Sing}}(f, \tau) \otimes Y_{\text{Hol}}(p, \tau) \right)$$

$$= h_k Y^L_{\text{Sing}}(f \otimes p, \tau),$$

since for any distribution we have

$$\text{Sing}(h \mathcal{D}) = h \text{Sing}(\mathcal{D}).$$

This proves $H_T$-covariance for $Y^L_{\text{Sing}}$.

For skew-symmetry we need the following simple fact about the projection $\text{Sing}$, cf., [Li]: let $X, Y$ be vector spaces and let $A \in X \otimes K$ and $B \in Y \otimes H_T$, then

(6.3) $\text{Sing}(A \otimes B) = \text{Sing}(\text{Sing}(A) \otimes B)$. 

Then we have

\[ Y_{\text{Sing}}^{LC}(f \otimes p, \tau)(g \otimes q) = \text{Sing} \left( Y_{\text{Sing}}(f, \tau)g \otimes Y_{\text{Hol}}(p, \tau)q \right), \]

and by skew-symmetry in the conformal algebra \( C \) and the holomorphic vertex algebra \( K \):

\[
\begin{align*}
= \text{Sing} \left( \mathcal{R}_C(\tau)Y_{\text{Sing}}^S(g, \tau)f \otimes \mathcal{R}_K(\tau)Y_{\text{Hol}}(q, \tau)p \right), \\
= \text{Sing} \left( \mathcal{R}_C(\tau)Y_{\text{Sing}}^S(g, \tau)f \otimes \mathcal{R}_K Y_{\text{Hol}}^S(q, \tau_2)p \right), \quad \text{by (6.3)}
\end{align*}
\]

and by multiplicativity of exponentials, Section 16,

\[
\begin{align*}
= \text{Sing} \left( \mathcal{R}_{LC}(\tau)(Y_{\text{Sing}}^S(g, \tau)f \otimes Y_{\text{Hol}}^S(q, \tau)p) \right), \\
= \text{Sing} \left( \mathcal{R}_{LC}(\tau)\text{Sing} \left( Y_{\text{Sing}}^S(g, \tau)f \otimes Y_{\text{Hol}}^S(q, \tau)p \right) \right), \quad \text{again by (6.3)}
\end{align*}
\]

\[ = \text{Sing} \left( \mathcal{R}_{LC}(Y_{\text{Sing}}^{LC, S}(g \otimes q, \tau)f \otimes p) \right), \]

proving skew-symmetry for \( Y_{\text{Sing}}^{LC} \).

To check commutator for \( LC \) we first note that it follows from Lemma 20.1 that

\[ \mathcal{R}_K(\tau_1)\langle p \rangle \mathcal{R}_K(\tau_2)\langle q \rangle = \mathcal{R}_K(\tau_2) \left( \mathcal{R}_K^{S, K_{\tau_1}}(\tau_2) \mathcal{R}_K(\tau_1)\langle p \rangle \right) \langle q \rangle. \]

Here \( p, q \in K \) are rational functions of \( \tau, \) say. Then \( \mathcal{R}_K(\tau_1)p \) is rational in \( \tau \) and holomorphic in \( \tau_1, \) and \( \mathcal{R}_K^{S, K_{\tau_1}} \) is the exponential operator acting on rational functions of \( \tau_1. \) We will use similar notation below without further elaboration.

Then we calculate

\[
\begin{align*}
[ Y_{\text{Sing}}^{LC}(f \otimes p, \tau_1), Y_{\text{Sing}}^{LC}(g \otimes q, \tau_2)] &= \text{Sing}_{\tau_1, \tau_2} \left( [Y_{\text{Sing}}^C(f, \tau_1), Y_{\text{Sing}}^C(g, \tau_2)] \otimes \mathcal{R}_K(\tau_1)\langle p \rangle \mathcal{R}_K(\tau_2)\langle q \rangle \right) \\
&= \text{Sing}_{\tau_1, \tau_2} \left( Y_{\text{Sing}}(\mathcal{R}_K^{S, K_{\tau_1}}(\tau_2)Y_{\text{Sing}}^C(f, \tau_1)g, \tau_2) \otimes \mathcal{R}_K(\tau_1)\langle p \rangle \mathcal{R}_K(\tau_2)\langle q \rangle \right) \\
&= \text{Sing}_{\tau_1, \tau_2} \left( Y_{\text{Sing}}^C(\mathcal{R}_K^{S, K_{\tau_1}}(\tau_2))Y_{\text{Sing}}(f, \tau_1)g, \tau_2) \otimes \mathcal{R}_K(\tau_2) \left( \mathcal{R}_K^{S, K_{\tau_1}}(\tau_2) \mathcal{R}_K(\tau_1)\langle p \rangle \right) \langle q \rangle \right) \\
&= \text{Sing}_{\tau_2} \left( Y_{\text{Sing}}^{LC}(\mathcal{R}_K^{S, K_{\tau_1}}(\tau_2))Y_{\text{Sing}}^C(f \otimes p, \tau_1)g \otimes q, \tau_2) \right).
\end{align*}
\]

The Lie bracket of \( LC \) is induced by the \( \langle 0 \rangle \) bracket on \( LC. \) This is the coefficient of \( \sum_{n \in \mathbb{Z}} T^n \frac{1}{\tau} \) in (6.2), hence

\[ (f \otimes p)\langle 0 \rangle g \otimes q = \sum_{n \in \mathbb{Z}} f(n)g\Delta[n]\langle p \rangle \langle q \rangle. \]
Write \( f_{(p)} = \text{Tr}(f \otimes p) \in \mathcal{L}C \). The Lie bracket on \( \mathcal{L}C \) is then given by
\[
[f_{(p)}, g_{(q)}] = \sum_{n \in \mathbb{Z}} f_{(n)}g_{(\Delta[n](p)q)}.
\]

Define the generating series of elements of \( \mathcal{L}C \) ("currents") for \( f \in C \) by
\[
f(\tau) = \text{Tr}_{\tau_{0}}(f \otimes \delta(\tau_{0}, \tau)) = \sum_{n \in \mathbb{Z}} f_{(\tau(n))}(\tau(-n - 1)).
\]

The currents are \( \hat{H}_{T} \)-covariant: for all \( h \in \hat{H}_{T} \)
\[
h_{\mathcal{L}C}.f(\tau) = \text{Tr}_{\tau_{0}}(f \otimes S(h)_{0}\delta(\tau_{0}, \tau)) = \text{Tr}_{\tau_{0}}(f \otimes h_{1}\delta(\tau_{0}, \tau)) = h_{K}f(\tau).
\]

The current commutator is given by
\[
[f(\tau_{1}), g(\tau_{2})] = \left[ \text{Tr}_{\tau_{0}}(f \otimes \delta(\tau_{0}, \tau_{1})), \text{Tr}_{\tau_{0}}(g \otimes \delta(\tau_{0}, \tau_{2})) \right]
= \sum_{n \in \mathbb{Z}} \text{Tr}_{\tau_{0}}(f_{(n)}g \otimes \Delta[n]_{0}\delta(\tau_{0}, \tau_{1})\delta(\tau_{0}, \tau_{2}))
= \sum_{n \in \mathbb{Z}} f_{(n)}g(\tau_{2})\Delta[n]_{2}\delta(\tau_{1}, \tau_{2}).
\]

**Example 6.2.** Let \( Cg \) be the \( H_{T} \)-conformal algebra of the Lie algebra \( g \), see Example 2.5. Then let \( \mathcal{L}g = \mathcal{L}Cg \) be the corresponding Lie algebra. If \( \{X^{\alpha}\} \) is a basis for \( g \), then \( \mathcal{L}g \) is spanned by elements
\[
X_{(p)}^{\alpha} = \text{Tr}(X^{\alpha} \otimes p), \quad p \in K,
\]
If \( X, Y \in g \) the commutator in \( \mathcal{L}g \) is
\[
[X_{(p)}, Y_{(q)}] = [X, Y]_{(pq)}.
\]

The corresponding currents have commutator
\[
[X(\tau_{1}), Y(\tau_{2})] = [X, Y](\tau_{2})\delta(\tau_{1}, \tau_{2}).
\]

\( \square \)

**Example 6.3.** Let \( CToda \) be the Toda conformal algebra, see Example 2.4. Let \( \mathcal{L}Toda = \mathcal{L}CToda \). Then elements of \( \mathcal{L}Toda \) are linear combinations of expressions of the form \( X_{(p)} \), where \( X = B, C \) and \( p \in K \), with commutator
\[
[B_{(p)} , C_{(q)}] = -C_{(\Delta (p) q)},
\]
and current commutator
\[
[B(\tau_{1}), C(\tau_{2})] = -C(\tau_{2})\Delta_{2}\delta(\tau_{1}, \tau_{2}).
\]
Of course this looks like (2.4) and in the next example we explain the connection. \( \square \)
Example 6.4. Let $V$ be an $H_T$-vertex Poisson algebra. This is in particular an $H_T$-conformal algebra, and so we obtain a Lie algebra $\mathcal{L}V$. Elements of $\mathcal{L}V$ are of the form $f\{p\}$, for $f \in V$ and $p \in K$. The commutator of two such elements is given by

$$[f\{p\}, g\{q\}] = \sum_{n \in \mathbb{Z}} f\{n\} g(\Delta[n]\{p\}q).$$

The corresponding current is given by (6.6).

Recall from Section 5 that the coefficients of the classical field $C(f)$, for $f \in V$, can be identified with elements of $\mathcal{L}V$:

$$C(f)(p) \simeq f\{p\} = \text{Tr}(f \otimes p), \quad p \in K.$$ 

Under this identification the Poisson bracket (8.3) is nothing but the commutator (6.6) in the conformal algebra $V$. 

CHAPTER 6

$H_T$-Vertex Algebras.

1. Introduction

Recall that in chapter 3 we defined the classical field $C(f)$, for $f \in V$, where $V$ is a commutative $H_T$-Leibniz algebra. This is a function $V(L) \rightarrow L$, for some other $H_T$-Leibniz algebra $L$, where $V(L) = \text{Hom}_{H_T\text{-alg}}(V,L)$. Recall the localization $K$ of section 7. In case $L = K$, which we will assume from now on, the classical field has an expansion

$$
C(f) = f(\tau) = \sum_{n \in \mathbb{Z}} f(n) \tau(-n-1).
$$

The coefficients $f(n)$ are $\mathbb{C}$-valued functions (on $V(K)$). In particular these coefficients generate a commutative $\mathbb{C}$-algebra. We can also think of a classical field as a distribution with values in functions on $V(K)$: for each $F \in K$ we get the function $f_F = C(f)(F)$ given by $f_F(\sigma) = \text{Tr}(\sigma(f)F)$. In particular $f(n) = f(\tau(n))$, $n \in \mathbb{Z}$.

In this chapter we will quantize this situation, and define fields (which will be called vertex operators) that have expansions like (1.1), but now the coefficients will be linear maps on a vector space $V$, and they will generate a non-commutative algebra.

This leads to the definition of an $H_T$-vertex algebra, which will be a quantization of the singular Hamiltonian structures of chapter 3, which are the $H_T$-vertex Poisson algebras of chapter 5 and a generalization of the $H_T$-conformal algebras of chapter 5.

2. Fields

Let $W$ be a vector space. Recall the vector space $W[[[\tau^{\pm 1}]]]$ of $W$-valued distributions $\mathcal{D}$ on $K$ in the variable $\tau$ with decomposition $\mathcal{D} = \mathcal{D}_{\text{Hol}} + \mathcal{D}_{\text{Sing}}$ in holomorphic and singular part, see section 24.

Let $V$ be a vector space. We will call $f(\tau) \in \text{End}(V)[[[\tau^{\pm 1}]]]$ a field on $V$ if the action of $f_{\text{Sing}}(\tau)$ on $V$ is rational: for all $v \in V$ the $V$-valued distribution obtained by applying $f_{\text{Sing}}$ to $v$ is has rational kernel: we have

$$
f_{\text{Sing}}(\tau)v(G) = \text{Tr}(F_vG), \quad G \in K
$$
for some $F_v \in V \otimes K_{\text{Sing}}$. So we can expand in a finite sum:

$$f_{\text{Sing}}(\tau)v = \sum_{n,k} v_{n,k} S(e_{n,k}) \frac{1}{\tau}, \quad v_{n,k} \in V.$$  

Note that the rationality condition does \textit{not} imply that for all $v \in V$ we have in the expansion (24.3) of the $V$-valued distribution $f_{\text{Sing}}(\tau)v$ a finite sum for the singular part (as is the case, mutatis mutandis, for fields in the usual vertex algebras based on $H_D$).

In case $V$ is an $\hat{H}_T$-module and $f(\tau)$ is a field on $V$ the End($V$)-valued distributions $\text{ad}_{\hat{h}} f(\tau)$ and $h.f(\tau)$ are in fact again fields on $V$, for all $h \in \hat{H}_T$.

We expand the field $f(\tau)$ as in (24.3):

$$f(\tau) = \sum_{n \in \mathbb{Z}} f((-n-1)\tau(n)), \quad f((-n-1) \in \text{End}(V).$$

Consider $f_{\text{Hol}}(\tau)$. This is an End($V$)-valued distribution on $K_{\text{Sing}}$. By (10.2) we can consider it a distribution on $\hat{H}_T$. So for each $h \in \hat{H}_T$ we get a linear map

$$f_{[h]} : V \to V, \quad f_{[h]} = \langle h, f_{\text{Hol}}(\tau) \rangle = h_K f_{\text{Hol}}(\tau)|_{\tau=0}.$$  

We can also interpret $f_{\text{Hol}}(\tau)$ as a distribution on $\mathbb{C}[\Delta]$ or $H_T$ that extends to $\hat{H}_T$, so we have expansions

$$f_{\text{Hol}}(\tau) = \sum_{k \geq 0} f_{[\Delta[k]]} \tau(k) = \sum_{n \in \mathbb{Z}} f_{[\Delta[n]]} \Delta^* \tau(n) = \sum_{n, \ell} f_{[e_{n,\ell}]} e_{n,\ell}^*,$$

see section 21.

If $\mathcal{D}$ is a holomorphic field, i.e., a distribution that vanishes on $\mathbb{C}_Z^\text{pol}$, then we define

$$\mathcal{D}|_{\tau=0} = \mathcal{D}(\frac{1}{\tau}),$$

i.e., we define the \textit{constant term} of the holomorphic $\mathcal{D}$ as the value of $\mathcal{D}$ at $\frac{1}{\tau}$.

Since the field $f(\tau)$ is an End($V$)-valued distribution on $K$ we get for each $F \in K$ a linear map

$$f_{\{F\}} : V \to V, \quad f_{\{F\}} = \text{Tr}_{\tau}(f(\tau)F(\tau)).$$

So we have three different notations for the holomorphic coefficients of $f(\tau)$, i.e., the coefficients of $f_{\text{Hol}}(\tau)$:

$$f_{(-k-1)} = f_{\{\frac{1}{(\tau+\tau^{-1})}\}} = f_{[\Delta[k]]} = \text{Tr}\left(f(\tau)\frac{1}{(\tau^2+1)}\right).$$

Similarly for the singular components we have two notations:

$$f_{(k)} = f_{\{\tau(k)\}} = \text{Tr}(f(\tau)\tau(k)).$$
In fact we can let $F \in \hat{K}$, see (24.2), for instance
$$f_{\text{Sing}}(\tau) = \sum_{n,k} f(\nu_{n,k}) S(e_{n,k} \frac{1}{\tau}),$$
so that in (2.1) we have $v_{n,k} = f(\nu_{n,k}) v$.

If $f(\tau), g(\tau)$ are fields on $V$, they are in particular $\text{End}(V)$-valued distributions, and we can calculate their commutator distribution: this is the distribution on $K \otimes K$ with value at $F \otimes G$
$$[f(\tau_1), g(\tau_2)](F \otimes G) = (f(F)g(G) - g(G)f(F)).$$
If we have an expansion (infinite, in general) as in (28.3),
$$[f(\tau_1), g(\tau_2)] = \sum_{\ell \geq 0} c_\ell(\tau_2) \Delta[\ell] 2 \delta(\tau_1, \tau_2),$$
then the coefficients $c_\ell(\tau)$ are also fields, by (28.4).

We say that fields are mutually rational if the distribution $[f(\tau_1), g(\tau_2)]$ has rational singularities, i.e., is killed by multiplication by some $m^{\text{pol}}_1 \otimes S(F)$, for $F \in \mathbb{C}^\text{pol}_Z$. By Lemma 28.2 this implies that we have a finite sum
$$(2.7) [f(\tau_1), g(\tau_2)] = \hat{D} (f \otimes g)(\tau_2) \delta(\tau_1, \tau_1) = \sum c_{n,k}(\tau_2)(e_{n,k})_2 \delta(\tau_1, \tau_2),$$
where $\hat{D}(f \otimes g)$ is a $W$-valued differential difference operator (i.e., an element of $\hat{WH_T}$) where $W = \text{End}(V)[[\tau^{\pm 1}]]$ and the right hand side sum is finite with the $c_{n,k}(\tau)$ $\text{End}(V)$-valued distributions (in fact fields) on $K$.

3. Normal Ordered Products and Dong’s Lemma

In this section we collect some properties of fields on a vector space $V$ that we will later need to investigate $H_T$-vertex algebras (see Definition 5.1 below) and construct examples.

**Lemma 3.1.** Let $f(\tau), g(\tau)$ be fields on $V$. Then also
$$(3.1) f_{\text{Hol}}(\tau)g_{\text{Hol}}(\tau), f_{\text{Sing}}(\tau)g_{\text{Sing}}(\tau), f_{\text{Hol}}(\tau)g_{\text{Sing}}(\tau)$$
are fields on $V$.

**Proof.** By the product formula (4.8) we see that $f_{\text{Hol}}(\tau)g_{\text{Hol}}(\tau)$ is a well defined element of $\text{End}(V)[[\tau^{\pm 1}]]$. The singular part of this distribution is zero so this product is a field for trivial reasons.

Now for any $v \in V$ the $V$-valued distribution $g_{\text{Sing}}(\tau)v$ is rational, and we can multiply it with any $\text{End}(V)$-valued distribution, see section 24. So both $f_{\text{Hol}}(\tau)g_{\text{Sing}}(\tau)v$ and $f_{\text{Sing}}(\tau)g_{\text{Sing}}(\tau)v$ are well defined $V$-valued distributions on $K$. Their singular part is clearly rational. \hfill $\square$

**Remark 3.2.** In general the product $f_{\text{Sing}}g_{\text{Hol}}$ is not well defined.
We use these results on products of holomorphic and singular parts to define normal ordered products.

**Definition 3.3.** Let \( f(\tau) \) and \( g(\tau) \) be fields on \( V \), and \( F \in K \). The \( F \)-th normal ordered product of \( f(\tau) \) and \( g(\tau) \) is

\[
(3.2) \quad f(\tau_2)_{\{F\}} g(\tau_2) = \text{Tr}_{\tau_1} \left( f(\tau_1) g(\tau_2) \mathcal{R}^S(\tau_2)(F(\tau_1)) - g(\tau_2) f(\tau_1) \mathcal{L}^S(\tau_1)(F(\tau_2)) \right).
\]

We define in particular the normal ordered product, using Lemma 25.1 and the definitions of \( \rho\left(\frac{1}{\ell}\right) \) and \( \lambda\left(\frac{1}{\ell}\right) \), (22.2), as

\[
(3.3) \quad : f(\tau_2) g(\tau_2) : = f(\tau_2)_{\{\frac{1}{\ell}\}} g(\tau_2) = \text{Tr}_{\tau_1} \left( f(\tau_1) g(\tau_2) \rho_{12} - g(\tau_2) f(\tau_1) \lambda_{12} \right) = \text{Tr}_{\tau_1} \left( f(\tau_2) g(\tau_2) + f(\tau_2) g(\tau_2) \right).
\]

More generally, for \( h \in H_T \) define

\[
(3.4) \quad f(\tau_2)_{[h]} g(\tau_2) = f(\tau_2)_{\{S(\frac{1}{\ell})\}} g(\tau_2) = \text{Tr}_{\tau_1} \left( f(\tau_1) g(\tau_2) h_{2\rho_{12}} - g(\tau_2) f(\tau_1) h_{2\lambda_{12}} \right) = h_{\mathcal{K}} \left( f(\tau_2) g(\tau_2) + f(\tau_2) g(\tau_2) \right).
\]

So

\[
(3.5) \quad f(\tau_2)_{[h]} g(\tau_2) = h_{\mathcal{K}} \left( f(\tau_2) g(\tau_2) \right).
\]

We also refer to \( f(\tau_2)_{[h]} g(\tau_2) \) as the \( h \)-th normal ordered product of \( f \) and \( g \).

In case \( F \in \mathbb{C}^\text{pol}_{Z} \) the \( F \)-th normal ordered product is a component of the commutator. Indeed for \( F \in \mathbb{C}^\text{pol}_{Z} \) we have

\[
\mathcal{R}^S(\tau_1)(F(\tau_2)) = \mathcal{L}^S(\tau_2)(F(\tau_1)) = m_1^\ast \otimes \Delta(F),
\]

so that in this case

\[
f(\tau_2)_{\{F\}} g(\tau_2) = \text{Tr}_{\tau_1} \left( [f(\tau_1), g(\tau_2)] m_1^\ast \otimes \Delta(F) \right).
\]

We have in particular, see Lemma 28.1,

\[
(3.6) \quad [f(\tau_1), g(\tau_2)] = \sum_{\ell \geq 0} f(\tau_2)_{\{\ell\}} g(\tau_2) \Delta_2[\ell] \delta(\tau_1, \tau_2).
\]

To get the components of the finite expansion (2.7) in case \( f(\tau_1) \) and \( g(\tau_2) \) are mutually rational we need to take \( F \) in \( \mathbb{C}^\text{pol}_{Z} \), see (24.2): we have

\[
(3.7) \quad c_{n,k}(\tau) = f(\tau)_{\{e_{n,k}\}} g(\tau).
\]

By Lemma 3.1 the \( h \)-th normal ordered product of fields is again a field, for all \( h \in \hat{H}_T \). For \( F \in \mathbb{C}^\text{pol}_{Z} \) the \( F \)-th normal ordered products are components of commutators of fields so also fields.
LEMMA 3.4. Let $V$ be an $H_T$-module and $f(\tau), g(\tau)$ fields on $V$, $F \in K$ and $h \in H_T$ with coproduct $\sum h' \otimes h''$. Then

1. $\text{ad}^h_h' \left( f(\tau)_{(F)} g(\tau) \right) = \sum \left( \text{ad}^h_h' f(\tau) \right)_{(F)} \left( \text{ad}^h_h' g(\tau) \right)$. 
2. $h_k \left( f(\tau)_{(F)} g(\tau) \right) = \sum \left( h_k^0 f(\tau) \right)_{(F)} \left( h_k^0 g(\tau) \right)$.

LEMMA 3.5. (Dong’s Lemma) Let $a(\tau), b(\tau), c(\tau)$ be fields on $V$ that are mutually rational. Then also the $F$th normal ordered product $a(\tau)_{(F)} b(\tau)$ and $c(\tau)$ are mutually rational, for all $F \in \hat{K}$.

We first need a simple Lemma on polynomials.

LEMMA 3.6. Let $F \in \mathbb{C}[u]$, $H \in \mathbb{C}[v]$. Put $x = u + v$, $y = u - v$. Then there are non zero $q, r \in \mathbb{C}[u, v]$ such that $K = F(u)q(u, v) + H(v)r(u, v) \in \mathbb{C}[x]$, i.e., such that $K$ is independent of $y$.

PROOF. Write for $A \in \mathbb{C}[u, v] = \mathbb{C}[x, y]$

$$A = \sum_{i \geq 0} A_i(x)y^i, \quad A_i \in \mathbb{C}[x],$$

so that

$$F(u)q(u, v) + H(v)r(u, v) = \sum_{i, j \geq 0} \left( F_i(x)q_j(x) + H_i(x)r_j(x) \right)y^{i+j}. \quad (3.8)$$

Let $I \subset \mathbb{C}[x]$ be the ideal generated by $F_0(x), H_0(x)$. The $y^0$-term of (3.8) reads $F_0(x)q_0(x) + H_0(x)r_0(x) \in I$.

We will assume that also $q_0, r_0$ belong to $I$, but for the rest they are arbitrary. Then assume we have found $q_j(x), r_j(x) \in I$ such that the coefficient of $v^j$ in (3.8) vanishes, for $j = 1, 2, \ldots, s - 1$. Then the $v^s$ term reads

$$F_0 q_s + H_0 r_s + \sum_{i=1}^{s} \sum_{j=0}^{s-1} F_i q_j + H_i r_j. \quad (3.9)$$

By assumption the double sum belongs to $I$, so that we can find $q_s, r_s$ making (3.9) vanish. By multiplying the $q_j, r_j$, $j < s$, by an element of $I$, if necessary, we can assume $q_s, r_s \in I$, too, and the lemma follows by induction. □

PROOF (OF DONG’S LEMMA, cf., [Kac98], Lemma 3.2). Recall the following notation: if $f \in \mathbb{C}^{\text{pol}}_{Z_+}$ we write $f_{12} = m^*_{1 \otimes S}(f) \in \mathbb{C}[\tau_1, \tau_2]$. Now by multiplicativity of $m^*_{1 \otimes S}$ and the fact that $m^*_{1 \otimes S}(\tau) = \tau_{12} = \tau_1 - \tau_2$ we
have $f_{12} = f(\tau_{12}) \in \mathbb{C}[\tau_{12}]$. In the same way we define for $h, g \in \mathbb{C}^\text{pol}_Z$ the elements $h_{13} \in \mathbb{C}[\tau_{13}], g_{23} \in \mathbb{C}[\tau_{23}],$ with $\tau_{ij} = \tau_i - \tau_j$.

By definition of rationality there are $f, g, h \in \mathbb{C}^\text{pol}_Z$ such that
\begin{equation}
(3.10) \quad f_{12}[a(\tau_1), b(\tau_2)] = 0, \quad g_{23}[b(\tau_2), c(\tau_3)] = 0, \quad h_{13}[a(\tau_1), c(\tau_3)] = 0.
\end{equation}

We need to find $k \in \mathbb{C}^\text{pol}_Z$ such that
\begin{equation}
(3.11) \quad k_{23}[a(\tau_2)_F b(\tau_2), c(\tau_3)] = 0.
\end{equation}

It suffices to show that
\begin{equation}
(3.12) \quad k_{23}A = k_{23}B,
\end{equation}
where
\[
A = \left( a(\tau_1)b(\tau_2)\mathcal{R}^S(\tau_2)F(\tau_1) - b(\tau_2)a(\tau_1)\mathcal{L}^S(\tau_1)F(\tau_2) \right) c(\tau_3)
\]
\[
B = c(\tau_3) \left( a(\tau_1)b(\tau_2)\mathcal{R}^S(\tau_2)F(\tau_1) - b(\tau_2)a(\tau_1)\mathcal{L}^S(\tau_1)F(\tau_2) \right).
\]

Indeed, taking $\text{Tr}_{\tau_1}$ of (3.12) gives (3.11).

Now note that there is a $G \in \mathbb{C}^\text{pol}_Z$ such that $FG \in \mathbb{C}^\text{pol}_Z$ and in that case we have
\[
\mathcal{R}^S(\tau_2)(F(\tau_1)G(\tau_1)) = \mathcal{L}^S(\tau_1)(F(\tau_2)G(\tau_2)) = m^1_{1 \otimes S}(FG) = (FG)_{12}.
\]

This implies that
\begin{equation}
(3.13) \quad (fG)_{12}A = f_{12}[a(\tau_1), b(\tau_2)]c(\tau_3)(FG)_{12} = 0,
\end{equation}
\begin{equation}
(3.13) \quad (fG)_{12}B = c(\tau_3)f_{12}[a(\tau_1), b(\tau_2)](FG)_{12} = 0,
\end{equation}

Also note that
\begin{equation}
(3.14) \quad h_{13}g_{23}A = h_{13}g_{23}B.
\end{equation}

Next we use Lemma 3.6, with $u = -\tau_{12} = \tau_2 - \tau_1$ and $v = \tau_{13} = \tau_1 - \tau_3$, $F(u) = (fG)_{12}$ and $H(v) = h_{13}$. Then there are $q$ and $r$ such that
\[
K(\tau_{23}) = (fG)_{12}q + h_{13}r \in \mathbb{C}[\tau_{23}].
\]

and (3.12) follows from (3.13) and (3.14) if we take $k = Kg$, so that $k_{23} = K_{23}g_{23}$. \qed

4. State-Field Correspondence and Vacuum Axioms

Let $V$ be a vector space. A State-Field correspondence for $V$ is a map that associates to all $f \in V$ a field $Y(f, \tau)$ on $V$, called the vertex operator of $f$. We also write $f(\tau)$ for $Y(f, \tau)$. 
5. Definition of $H_T$-Vertex Algebras

Now let $V$ contain a distinguished vector $1_V$, called the vacuum of $V$. We say that a state-field correspondence satisfies the Vacuum Axioms (with respect to $1_V$) in case

\[(4.1)\quad Y(1_V, \tau) = 1_{\text{End}(V)}, \quad Y(f, \tau)1_V = f_{\text{Hol}}(\tau)1_V,\]

with the constant term (see (2.5))

\[(4.2)\quad f_{\text{Hol}}(\tau)1_V|_{\tau=0} = f.\]

So we have $f(-1)1_V = f$, for all $f \in V$.

We fix from now on a state-field correspondence satisfying the vacuum axioms for some $1_V \in V$. Let $f \in V$ and define for all $h \in \hat{H}_T$ an operator $h_V : V \to V$ by

\[(5.1)\quad hVf = f[h]1_V, \quad f \in V.\]

At this point we don’t know that, given a state-field correspondence, $h \mapsto h_V$ gives an $\hat{H}_T$-module structure to $V$. This is an extra assumption about the state-field correspondence.

**5. Definition of $H_T$-Vertex Algebras**

We will need the notion of an antipodal vertex operator $Y^S(f, \tau)$ associated to $f \in V$. More generally if $\mathcal{D}$ is a distribution on $K$ we put

\[(5.2)\quad \mathcal{D}^S(F) = \mathcal{D}(S(F)),\]

using the antipodal homomorphism on $K$ given by $S(\tau) = -\tau$.

**Definition 5.1.** An $H_T$-vertex algebra is a vector space $V$ with a vacuum vector $1_V \in V$ and a state-field correspondence $f \mapsto f(\tau) = Y(f, \tau)$, satisfying the vacuum axioms (4.1), (4.2) and such that furthermore

- (H$_T$-Covariance) For all $f \in V$ and $h \in H_T$

  \[(5.3)\quad hKY(f, \tau) = Y(h_V f, \tau).\]

- (Skew-Symmetry)

- (Mutual Rationality) The vertex operators $Y(f, \tau_1)$ and $Y(g, \tau_2)$ are for all $f, g \in V$ mutually rational.

Note that from (5.2) it follows that the map $h \in H_T \mapsto h_V \in \text{End}(V)$ gives $V$ the structure of $H_T$-module.

The simplest examples of $H_T$-vertex algebras are of course the holomorphic vertex algebras introduced in section 5, corresponding to commutative $H_T$-Leibniz algebras, with as vertex operator $Y(f, \tau) = Y_{\text{Hol}}(f, \tau) = R_V(\tau).f$. 

REMARK 5.2. We define similarly the notion of $\mathcal{H}_T$-vertex algebra, by imposing instead of (5.2) a $\mathcal{H}_T$-covariance axiom. It seems that the non holomorphic $\mathcal{H}_T$-vertex algebras are all in fact $\hat{\mathcal{H}}_T$-vertex algebras.

6. First Properties of $\mathcal{H}_T$-vertex Algebras

From now on $V$ is an $\mathcal{H}_T$-vertex algebra, with vacuum $1_V$.

**Lemma 6.1.**

1. For all $h \in \mathcal{H}_T$
   \[ hV1_V = \varepsilon(h)1_V. \]

2. For all $f \in V$
   \[ f_{\text{Hol}}(\tau)1_V = \mathcal{R}_V(\tau)f. \]

**Proof.** From the vacuum axioms and the definition (2.4) it follows that
\[
Y(1_V, \tau)1_V = \sum (1_V)[n]1_V\Delta^*[n] = 1_V.
\]
Hence by definition (4.3) we find
\[
\Delta[n]V1_V = \begin{cases} 1_V & \text{if } n = 0 \\ 0 & \text{if } n \neq 0 \end{cases} = \varepsilon(\Delta[n])1_V,
\]
and part 1. follows from the linearity of the counit $\varepsilon$ and linearity of distributions.

Similarly, from (2.4) and (4.3) it follows that
\[
f_{\text{Hol}}(\tau) = \sum_{n \in \mathbb{Z}} \Delta[n]Vf\Delta^*[n] = \mathcal{R}_V(\tau)f,
\]
by definition of the exponential operator, see (14.3).

**Lemma 6.2. (Leibniz Rule) For all $f, g \in V$ and $h \in \mathcal{H}_T$

\[ hV(Y(f, \tau)g) = \sum Y(h'Vf, \tau)h''Vg. \]

**Proof.** First some general remarks. The $\mathcal{H}_T$-covariance of the antipodal vertex operators is
\[
Y^S(hf, \tau) = S(h)K Y^S(f, \tau).
\]
For the exponential operator $\mathcal{R}_V(\tau)$ (acting on a space of $V$-valued distributions, say), we have the following identities:
\[
h_K\mathcal{R}_V(\tau) = \mathcal{R}_V(\tau)\sum h'_V h''_K,
\]
\[
h_V\mathcal{R}_V(\tau) = \mathcal{R}_V(\tau)h_V,
\]
the first by the Leibniz rule in $K$ and Lemma 15.1, and the second by the commutativity of $\mathcal{H}_T$. 
Then, by the $H_T$-covariance of $Y(f, \tau)$ and skew symmetry we have

$$\sum Y(h'_V f, \tau) h''_V g = \sum h'_K Y(f, \tau) h''_V g$$

$$= \sum h'_K \mathcal{R}_V(\tau) Y^S(h''_V g, \tau) f$$

$$= \sum h'_K \mathcal{R}_V(\tau) S(h''_V) K Y^S(g, \tau) f \quad \text{by (6.1)}$$

$$= \mathcal{R}_V(\tau) \sum h'_V h''_K S(h''_V) K Y^S(g, \tau) f, \quad \text{by (6.2)}$$

$$= \mathcal{R}_V(\tau) h_V Y^S(g, \tau) f,$$

$$= h_V Y(f, \tau) g,$$

by elementary identities in the Hopf algebra $H_T$ and (6.3). \hfill \Box

As $Y(f, \tau)$ is an $\text{End}(V)$-valued distribution on $K$, we get for all $F \in K$ (in fact $\hat{K}$) a linear map

$$f_{\{F\}} : V \to V, \quad f_{\{F\}} = \text{Tr}(Y(f, \tau) F(\tau)).$$

We call $f_{\{F\}} g$ the $F$-product of $f, g$. Then Lemma 6.2 implies the Leibniz rule for $F$-products:

$$h_V (f_{\{F\}} g) = \sum (h'_V)_{\{F\}} (h''_V g).$$

Another reformulation of the Leibniz rule is the following ad-covariance of vertex operators.

**Lemma 6.3.** For all $f \in V, h \in H_T$

$$h_K Y(f, \tau) = \text{ad}_h^V (Y(f, \tau)).$$

**Proof.** For any $g \in V$ we have by $H_T$-covariance and elementary identities

$$h_K Y(f, \tau) g = Y(h_V f, \tau) g$$

$$= \sum Y(h'_V f, \tau) h''_V S(h'''_V) V g$$

$$= \sum h'_V (Y(f, \tau) S(h''_V) V g) \quad \text{by Lemma 6.2},$$

$$= \text{ad}_h^V (Y(f, \tau)) g.$$

\hfill \Box

**Lemma 6.4.** For all $f \in V, h \in H_T$

$$h_K f_{\text{Hol}}(\tau) = \mathcal{R}_V(\tau) \circ f_{\{h\}} \circ \mathcal{R}^S(\tau).$$

**Proof.** By taking $\tau = 0$ in the holomorphic part of Lemma 6.3 we find

$$f_{\{h\}} = \text{ad}_h^V (f_{\{1\}}).$$
Hence, by $H_T$-covariance and the definitions
\[
h_K f_{\text{Hol}}(\tau) = \text{ad}^V_h \left( f_{\text{Hol}}(\tau) \right)
= \text{ad}^V_h \left( \sum f_{[\Delta[n]]} \Delta^*[n] \right)
= \sum \text{ad}^V_{\Delta[n]} f_{[h]} \Delta^*[n] \quad \text{by commutativity of } H_T
= \text{ad}^V_{\Delta[n]} f_{[h]} \Delta^*[n]
= \mathcal{R}_V(\tau) \circ f_{[h]} \circ \mathcal{R}_V^S(\tau),
\]
by Lemma 19.1.

**Lemma 6.5.** For all $f \in V$
\[
\mathcal{R}^K(\tau_1) Y(f, \tau_2) = \mathcal{R}_V(\tau_1) \circ Y(f, \tau_2) \circ \mathcal{R}_V^S(\tau_1).
\]

**Proof.** This follows from Lemma 6.3 and Lemma 19.1.

**Lemma 6.6.** For all $f \in V$, $h \in H_T$ and $F \in \hat{K}$ we have
\[
(\text{hv}f)_{\{F\}} = f_{\{S(h)F\}} = \text{ad}^V_h f_{\{F\}}.
\]

**Proof.** By (2.6) and $H_T$-covariance (5.2) we have
\[
Y(\text{hv}f, \tau)(F) = (\text{hv}f)_{\{F\}} = h_K Y(f, \tau)(F) = Y(f, \tau)(S(h)F)
\]
by the definition of the $H_T$-action on distributions, and so
\[
(\text{hv}f)_{\{F\}} = f_{\{S(h)F\}},
\]
proving the first equality. For the second use Lemma 6.3:
\[
(\text{hv}f)_{\{F\}} = Y(\text{hv}f, \tau)(F) = h_K Y(f, \tau)(F) = \text{ad}^V_h Y(f, \tau)(F) = \text{ad}^V_h f_{\{F\}}.
\]

**Lemma 6.7.** Let $F \in \mathcal{C}^{\text{pol}}_Z$ and let $f(\tau) = Y(f, \tau)$ be a vertex operator. Then
\[
\text{Tr}_{\tau_1} \left( f(\tau_1) m^*_1 \otimes S(F) \right) = \mathcal{R}_V(\tau_2) \circ f_{\{F\}} \circ \mathcal{R}_V^S(\tau_2) = (\mathcal{R}_V(\tau_2)f)_{\{F\}}.
\]

**Proof.** We have $m^*_1 \otimes S(F) = \mathcal{R}_K^S(\tau_2)(F(\tau_1))$. So
\[
\text{Tr}_{\tau_1} \left( f(\tau_1) m^*_1 \otimes S(F) \right) = \text{Tr}_{\tau_1} \left( f(\tau_1) \mathcal{R}_K^S(\tau_2)(F(\tau_1)) \right)
= f_{\{\mathcal{R}_K^S(\tau_2)F\}} \quad \text{by (2.6)}
= \text{ad}_{\mathcal{R}_V(\tau_2)} f_{\{F\}} \quad \text{by Lemma 6.6}
= \mathcal{R}_V(\tau_2) f_{\{F\}} \circ \mathcal{R}_V^S(\tau_2) \quad \text{by Lemma 19.1}
= (\mathcal{R}_V(\tau_2)f)_{\{F\}} \quad \text{again by Lemma 6.6}.
\]
LEMMMA 6.8. For all \( f, g \in V, F \in \hat{H}_T \),
\[
f(\tau)_F g(\tau)_{1V} = \mathcal{R}_V(\tau) f_F g.
\]

PROOF. We distinguish two cases: \( F \in \hat{C}_Z^{\text{pol}} \) and \( F \in K_{\text{Sing}} \). For \( F \in K_{\text{Sing}} \) we write \( F = S(h) \frac{1}{\tau} \), so that
\[
f(\tau)_{\{F\}} g(\tau) = f(\tau)_{[h]} g(\tau) =: h_K \langle f(\tau) \rangle \langle g(\tau) \rangle,
\]
by (3.5). Using the vacuum axioms and (3.3) we find
\[
f(\tau)_{\{F\}} g(\tau)_{1V} = h_K \langle f_{\text{Hol}}(\tau) \rangle g_{\text{Hol}}(\tau)_{1V} = h_K \langle f_{\text{Hol}}(\tau) \rangle g_{\text{Hol}}(\tau)_{1V} = f(\tau)_{\{F\}} g_{\text{Hol}}(\tau)_{1V}.
\]
For \( F \in \hat{C}_Z^{\text{pol}} \) we write \( F_{12} = m^*_{\otimes S}(F) \) and we have
\[
f(\tau)_{\{F\}} g(\tau)_{1V} = \text{Tr}_{\tau} ([f(\tau_1), g(\tau_2)] F_{12})_{1V}
\]
\[
= [f(\tau_1), g(\tau_2)]_{1V} \quad \text{by Lemma 6.7}
\]
\[
= f(\tau_2)_{12} g(\tau_2)_{1V} \quad \text{as } f(\tau)_{1V} = 0, G \in \hat{C}_Z^{\text{pol}}
\]
\[
= \mathcal{R}_V(\tau_2) f(\tau)_{\{F\}} \mathcal{R}_V^S(\tau_2) g_{\text{Hol}}(\tau_2)_{1V} \quad \text{by vacuum axioms}
\]
\[
= \mathcal{R}_V(\tau_2) f(\tau)_{\{F\}} g_{\text{Hol}}(\tau_2)_{1V} \quad \text{by Lemma 18.1}.
\]

COROLLARY 6.9. For all \( f, g \in V, F \in \hat{K} \)
\[
f(\tau)_{\{F\}} g(\tau)_{1V|_{\tau=0}} = f(\{F\}) g.
\]

PROOF. This follows from Lemma 6.8 and (2.5). \( \square \)

7. Uniqueness and Normal Ordered Products

THEOREM 7.1. (Goddard’s Uniqueness) Let \( G(\tau) \) be a field on an \( H_T \)-vertex algebra that is mutually rational to all vertex operators \( Y(f, \tau) \) of \( V \) and such that
\[
G(\tau)_{1V} = \mathcal{R}_V(\tau) g,
\]
for some \( g \in V \). Then
\[
G(\tau) = Y(g, \tau).
\]

The proof is the same as in the usual case, using the vacuum axioms, skew symmetry and rationality, cf., [Kac98], Thm. 4.4.

Now we will express the vertex operators of an \( H_T \)-vertex algebra in terms of normal ordered products, see Definition 3.3.
Lemma 7.2. For \( f, g \in V \) and \( F \in \hat{K} \) we have
\[
Y(f_{\{F\}}g, \tau) = f(\tau)_{\{F\}}g(\tau) .
\]
More generally, if \( f^1, f^2, \ldots, f^k \in V \) and \( F^1, F^2, \ldots, F^k \in \hat{K} \), then
\[
Y(f^1_{\{F^1\}}f^2_{\{F^2\}} \cdots f^k_{\{F^k\}}1_V, \tau) = f^1(\tau)_{\{F^1\}}f^2(\tau)_{\{F^2\}} \cdots f^k(\tau)_{\{F^k\}}1_V .
\]
We define the normal ordered product of more factors as usually from right to left:
\[
f^1(\tau)_{\{F^1\}}f^2(\tau)_{\{F^2\}} \cdots f^k(\tau)_{\{F^k\}}g(\tau) = \left( f^1(\tau)_{\{F^1\}}\left( f^2(\tau)_{\{F^2\}} \cdots \left( f^k(\tau)_{\{F^k\}}g(\tau) \right) \right) \right) .
\]

Proof. We have, by the vacuum axioms, Lemma 6.1 and Lemma 6.4
\[
Y(f_{\{F\}}g, \tau)1_V = \mathcal{R}_V(\tau)f_{\{F\}}g = f(\tau)_{\{F\}}g(\tau)1_V ,
\]
so that the first part of the Lemma follows from Goddard’s uniqueness. The second part follows by induction. \( \square \)

Lemma 7.3. (Borcherds OPE) Let \( V \) be an \( H_T \)-vertex algebra, \( f, g \in V \). Then
\[
[Y(f, \tau_1), Y(g, \tau_2)] = \sum_{n,k} f_{\{e_{n,k}\}}g(\tau_2)(e_{n,k})_2 \delta(\tau_1, \tau_2)
\]
\[
= \sum_{\ell \geq 0} f_{\{\tau(\ell)\}}g(\tau_2)(\Delta[\ell])_2 \delta(\tau_1, \tau_2) .
\]
Note that the first sum on the RHS in the Lemma will be finite, the second infinite, in general.

Proof. The first equality follows from the mutual rationality of vertex operators by combining (2.7), (3.7) and the previous Lemma 7.2. The second equality follows from (3.6) combined with Lemma 7.2. \( \square \)

8. Alternative Axiomatization

We took as one of the axioms of an \( H_T \)-vertex algebra the \( H_T \)-covariance axiom
\[
Y(h_V f, \tau) = h_K Y(f, \tau) ,
\]
and derived in Lemma 6.3 the ad-covariance (8.1)
\[
Y(h_V f, \tau) = \text{ad}_h^V Y(f, \tau) .
\]
This gives a set of axioms similar to those for an \( H_T \)-vertex Poisson algebra, see Definition 2.6, and of an \( H_T \)-conformal algebra, see Definition 2.2. However, these axioms are not very efficient, in particular we assumed the skew-symmetry (5.3) property. In this section we give a different set of axioms, including the ad-covariance (8.1), which are easier to check and
are analogous to those in [Kac98]. In particular we don’t assume skew-symmetry as an axiom, but derive it.

**Proposition 8.1.** Let $V$ be an $H_T$-module with vacuum vector $1_V$ and a state-field correspondence satisfying the vacuum axioms (4.1), (4.2), and such that

(a) *(Compatibility of State-Field Correspondence with $H_T$-action)* For all $f \in V$ and $h \in H_T$ we have

$$h \cdot f = f_{[h]} 1_V,$$

where the right hand side is defined by (2.3).

(b) *(ad$_h$-covariance)* For all $f \in V$ and $h \in H_T$

$$\text{ad}^V_h (f, \tau) = h_K \cdot Y(f, \tau).$$

(c) *(Mutual Rationality)* The fields $Y(f, \tau_1)$ and $Y(g, \tau_2)$ are for all $f, g \in V$ mutually rational.

Then $V$ is an $H_T$-vertex algebra as defined in Definition 5.1.

**Proof.** We need to prove skew-symmetry (5.3) and $H_T$-covariance (5.2). First we note that

$$Y(f, \tau) 1_V = \mathcal{R}_V(\tau) f,$$

because the proof of Lemma 6.1 still works. We also have by $\text{ad}_h$-covariance (8.1) and Lemma 19.1

$$\mathcal{R}^k_\mathcal{K}(\tau_1) Y(g, \tau_2) = \mathcal{R}^S_V(\tau_1) Y(g, \tau_2) \mathcal{R}_V(\tau_1).$$

Finally, for any holomorphic distribution $\mathcal{D}_{\text{Hol}}$ we have

$$\mathcal{D}^S(\tau_1) \mathcal{D}_{\text{Hol}}(\tau_2)|_{\tau_1=0} = \mathcal{D}^S_{\text{Hol}}(\tau_1),$$

where $\mathcal{D}^S$ is the antipodal distribution defined by (5.1).

Now by mutual rationality we have, for some $F \in \mathbb{C}^\text{pol}_Z$

$$F_{12} Y(f, \tau_1) Y(g, \tau_2) 1_V = F_{12} Y(g, \tau_2) Y(f, \tau_1) 1_V$$

$$= F_{12} Y(g, \tau_2) \mathcal{R}_V(\tau_1) f$$

by (8.2)

$$= F_{12} \mathcal{R}^S_K(\tau_1) \mathcal{R}_V(\tau_1) Y(g, \tau_2) f$$

by (8.3).

Now the left hand side is manifestly holomorphic in $\tau_2$, whereas in the right hand side $F_{12} \mathcal{R}^S_K(\tau_1) Y(g, \tau_2) f$ can be assumed to holomorphic by suitable choice of $F$. So we can put $\tau_2 = 0$ and cancel $F_{12}|_{\tau_2=0}$ so get skew-symmetry (5.3).

Next note that we have

$$(f(\tau) \{F\} g(\tau)) 1_V = \mathcal{R}_V(\tau) f_{\{F\}} g,$$
as the proof of 6.7 just uses ad$_h$-covariance of fields. Now we use Goddard’s uniqueness in our situation: if $V$ has a state-field correspondence $Y$ as in the statement of the theorem and we have a field $G(\tau)$ mutually rational to all $Y(f, \tau)$ such that for some $g \in V$

$$Y(g, \tau) 1_V = G(\tau) 1_V,$$

then in fact $G(\tau) = Y(g, \tau)$. We use this to conclude that for all $f, g \in V$

(8.4) $$Y(f \{F\} g, \tau) = f(\tau) \{F\} g(\tau).$$

In particular, for $F = S(h)\frac{1}{\tau}, h \in H_T$ and $g = 1_V$ we find on the one hand, by compatibility of state-field correspondence with the $H_T$-action

$$Y(f \{S(h)\frac{1}{\tau}\} 1_V, \tau) = Y(f h \{F\} 1_V, \tau),$$

and on the other hand, by (8.4)

$$Y(f \{S(h)\frac{1}{\tau}\} 1_V, \tau) = h\langle f(\tau) \rangle Y(1_V, \tau),$$

so that $H_T$-covariance (5.2) follows from the vacuum axioms. □

9. Existence

**Proposition 9.1.** [FKRW95] Let $V$ be an $H_T$-module with distinguished vector $1_V$ and a collection of fields $f^\alpha(\tau)$ such that

(a) $\text{ad}_h^V f^\alpha(\tau) = h f^\alpha(\tau), h \in H_T.$

(b) $h 1_V = \varepsilon(h) 1_V, h \in H_T.$

(c) $f^\alpha(\tau) 1_V = f^\alpha_{\text{Hol}}(\tau) 1_V$, and, defining $f^\alpha \in V$ by $f^\alpha(\tau) 1_V|_{\tau=0} = f^\alpha$, the map $f^\alpha(\tau) \mapsto f^\alpha$ is injective.

(d) All fields $f^\alpha(\tau)$ are mutually rational.

(e) $V$ is spanned by $f^\alpha_{\{F_1\}} f^\alpha_{\{F_2\}} \ldots f^\alpha_{\{F_k\}} 1_V$, for $F_1, F_2, \ldots, F_K \in \hat{K}$.

Then $V$ is a vertex algebra, with state field correspondence given by $F$-normal ordered products:

$$Y(f^\alpha_{\{F_1\}} f^\alpha_{\{F_2\}} \ldots f^\alpha_{\{F_k\}} 1_V, \tau) = f^\alpha(\tau)_{\{F_1\}} f^\alpha_{\{F_2\}} \ldots f^\alpha_{\{F_k\}} (\tau)_{\{F_k\}} 1_V.$$

The proof in [Kac98], Thm. 4.5 works in our situation, mutatis mutandis, using the alternative axioms of section 8.

10. Affine $H_T$-vertex algebras

Let $\mathfrak{g}$ be a Lie algebra. Recall from Example 6.2 the notion of the conformal affinization $\mathcal{L} \mathfrak{g}$ associated to the conformal algebra $C \mathfrak{g}$, see Example 2.5. Define a decomposition $\mathcal{L} \mathfrak{g} = \mathcal{L} \mathfrak{g}_{\text{cr}} \oplus \mathcal{L} \mathfrak{g}_{\text{ann}}$ in creation and
annihilation Lie subalgebras (and \( \hat{H}_T \)-submodules), where
\[
\mathcal{L}_{\text{g cr}} = \text{Span}(X_{(p)}; X \in \mathfrak{g}, p \in K_{\text{Sing}}), \\
\mathcal{L}_{\text{g ann}} = \text{Span}(X_{(p)}; X \in \mathfrak{g}, p \in \mathbb{C}^\text{pol}_Z).
\]
Define a 1-dimensional trivial \( \mathcal{L}_{\text{g ann}} \) module \( \mathbb{C}_0 = \mathbb{C}m_0 \) and let \( V(\mathfrak{g}) \) be the induced \( \mathcal{L}_{\mathfrak{g}} \)-module:
\[
V(\mathfrak{g}) = \mathcal{U}(\mathcal{L}_{\mathfrak{g}}) \otimes \mathcal{U}(\mathcal{L}_{\text{g ann}}) \mathbb{C}_0 \simeq \mathcal{U}(\mathcal{L}_{\text{g cr}})v_0, \quad v_0 = 1 \otimes m_0.
\]
Introduce an action of \( H_T \) on \( V(\mathfrak{g}) \) by putting \( hv_0 = \epsilon(h)v_0 \) and extending the \( H_T \) action (5.3) on \( \mathcal{L}_{\mathfrak{g}} \) to \( \mathcal{U}(\mathcal{L}_{\mathfrak{g}}) \) by the Leibniz rule. Define for \( \{X^\alpha\} \) a basis for \( \mathfrak{g} \) currents \( X^\alpha(\tau) \) as in (6.5). These are fields on \( V(\mathfrak{g}) \) that are mutually rational, and the action of the coefficients of these fields on the vacuum \( v_0 \) obviously produces a spanning set for \( V(\mathfrak{g}) \). Also note that, if \( X^\alpha(\tau)v_0|_{\tau=0} = X^\alpha_0 \), the the map \( X^\alpha(\tau) \mapsto X^\alpha_0 \) is injective. Furthermore, we have \( \text{ad}_h \)-covariance of these fields.

**Lemma 10.1.** For all \( h \in H_T \) and all \( X^\alpha \) we have \( \text{ad}_h^{V(\mathfrak{g})}X^\alpha(\tau) = h_KX^\alpha(\tau) \).

**Proof.** For all \( v \in V = V(\mathfrak{g}) \) we have
\[
\text{ad}_h^{V(\mathfrak{g})}X^\alpha(\tau_1) = \sum h'_v \circ X^\alpha(\tau_1) \circ S(h''_v)v \\
= \sum \text{Tr}_0(X^\alpha \otimes S(h'_v)\delta(\tau_0, \tau_1)) h'_v S(h''_v)v \\
= \sum \text{Tr}_0(X^\alpha \otimes S(h'_v)\delta(\tau_0, \tau_1)) \epsilon(h''_v)1_v \\
= \sum \text{Tr}_0(X^\alpha \otimes S(h)\delta(\tau_0, \tau_1))v \\
= \text{ad}_hX^\alpha(\tau_1)v \\
= h_KX^\alpha(\tau_1)v \\
\]

Then it follows from Proposition 9.1 that \( V(\mathfrak{g}) \) is an \( H_T \)-vertex algebra, called the affine \( H_T \)-vertex algebra of \( \mathfrak{g} \).

### 11. Toda Vertex Algebra

Recall from Example 6.3 the Lie algebra \( \mathcal{L}_{\text{Toda}} \) associated to the Toda conformal algebra of Example 2.4. Define as before a decomposition \( \mathcal{L}_{\text{Toda}} = \mathcal{L}_{\text{Toda cr}} \oplus \mathcal{L}_{\text{Toda ann}} \) in creation and annihilation Lie subalgebras (and \( \hat{H}_T \)-submodules), where
\[
\mathcal{L}_{\text{Toda cr}} = \text{Span}(B_{(p)}, C_{(p)}; p \in K_{\text{Sing}}), \\
\mathcal{L}_{\text{Toda ann}} = \text{Span}(B_{(p)}, C_{(p)}; p \in \mathbb{C}^\text{pol}_Z).
\]
Define a 1-dimensional trivial $\mathcal{L}^{\text{Toda}_{\text{ann}}}$ module $\mathbb{C}_0 = \mathbb{C}m_0$ and let $\mathcal{V}\text{Toda}$ be the induced $\mathcal{L}^{\text{Toda}}$-module:

$$\mathcal{V}\text{Toda} = \mathcal{U}(\mathcal{L}^{\text{Toda}}) \otimes \mathcal{U}(\mathcal{L}^{\text{Toda}_{\text{ann}}}) \mathbb{C}_0 \simeq \mathcal{U}(\mathcal{L}^{\text{Toda}_{\text{cr}}}) v_0, \quad v_0 = 1 \otimes m_0.$$ 

In exactly the same as for the affine vertex algebra we see that $\mathcal{V}\text{Toda}$ is an $H_T$-vertex algebra, called the Toda vertex algebra. It is a quantization of the Toda vertex Poisson algebra discussed in Chapter 4.

Note that $\mathcal{V}\text{Toda}$ is non commutative, for the multiplication $f \otimes g \mapsto f_{(1)} g$, because the subalgebra $\mathcal{L}^{\text{Toda}_{\text{cr}}}$ is non-Abelian, see (6.7).
Bibliography

[Abe80] Eiichi Abe, *Hopf algebras*, Cambridge Tracts in Mathematics, vol. 74, Cambridge University Press, Cambridge, 1980, Translated from the Japanese by Hisae Kinoshita and Hiroko Tanaka. MR 83a:16010

[Bor98] Richard E. Borcherds, *Vertex algebras*, Topological field theory, primitive forms and related topics (Kyoto, 1996), Birkhäuser Boston, Boston, MA, 1998, pp. 35–77. MR 1 653 021

[CP95] Vyjayanthi Chari and Andrew Pressley, *A guide to quantum groups*, Cambridge University Press, Cambridge, 1995, Corrected reprint of the 1994 original. MR 96h:17014

[Dic03] L. A. Dickey, *Soliton equations and Hamiltonian systems*, second ed., Advanced Series in Mathematical Physics, vol. 26, World Scientific Publishing Co. Inc., River Edge, NJ, 2003. MR MR1964513 (2004c:37160)

[FKRW95] Edward Frenkel, Victor Kac, Andrey Radul, and Weiqiang Wang, $\mathcal{W}_{1+\infty}$ and $\mathcal{W}(\mathfrak{gl}_N)$ with central charge $N$, Comm. Math. Phys. 170 (1995), no. 2, 337–357. MR MR1334399 (96i:17024)

[FLM88] Igor Frenkel, James Lepowsky, and Arne Meurman, *Vertex operator algebras and the Monster*, Academic Press Inc., Boston, MA, 1988. MR 90h:17026

[Ike94] Kaoru Ikeda, *The Poisson structure on the coordinate ring of discrete Lax operator and Toda lattice equation*, Adv. in Appl. Math. 15 (1994), no. 4, 379–389. MR MR1304086 (96g:58076)

[Kac98] Victor Kac, *Vertex algebras for beginners*, second ed., American Mathematical Society, Providence, RI, 1998. MR 99f:17033

[Kup85] B. A. Kuperschmidt, *Discrete Lax equations and differential-difference calculus*, Astérisque (1985), no. 123, 212. MR 86m:58070

[Li] Haisheng Li, *Vertex algebras and vertex poisson algebras*, arXiv:math.QA/0209310.

[Sny] Craig T. Snydal, *Equivalence of Borcherds G-Vertex Algebras and Axiomatic Vertex Algebras*, arXiv:math/QA/9904104.

[Tod89] Morikazu Toda, *Theory of nonlinear lattices*, second ed., Springer Series in Solid-State Sciences, vol. 20, Springer-Verlag, Berlin, 1989. MR MR971987 (89h:58082)