Precoding for $2 \times 2$ Doubly–Dispersive WSSUS Channels

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Abstract

Optimal link adaption to the scattering function of wide sense stationary uncorrelated scattering (WSSUS) mobile communication channels is still an unsolved problem despite its importance for next-generation system design. In multicarrier transmission such link adaption is performed by pulse shaping which in turn is equivalent to precoding with respect to the second order channel statistics. In the present framework a translation of the precoder optimization problem into an optimization problem over trace class operators is used [1], [2]. This problem which is also well-known in the context of quantum information theory is unsolved in general due to its non-convex nature. However in very low dimensions the problem formulation reveals an additional analytic structure which admits the solution to the optimal precoder and multiplexing scheme. Hence, in this contribution the analytic solution of the problem for the $2 \times 2$ doubly–dispersive WSSUS channel is presented.

1 Introduction

It is well known, that channel information at the transmitter increases link capacity. However, since future mobile communication is expected to operate in fast varying channels, it is not realistic to assume perfect channel knowledge at the transmitter. On the other hand statistical information can be used which does not change in the rapid manner as the channel itself. In multicarrier communications this can be employed for the design of transmitter and receiver pulse shapes. However, the problem of optimal signaling in this context is still an unsolved problem.

In this paper the most basic case of precoder and equalizer optimization with respect to the statistics of a doubly–dispersive channel is considered. Hence, the focus is on a model in which two random complex symbols (iid and zero mean distributed) are transmitted parallel over a $2 \times 2$ random channel. Before transmission a special kind of linear precoding is performed which is motivated from Weyl–Heisenberg signaling scheme. The channel itself is a randomly weighted superposition of four possible channel operations, which are

1) do not change anything
2) permutation in the time domain
3) permutation in the frequency domain
4) permutation in the time and frequency domain

Each action is distributed independently from the others and scaled with a certain amount of power. What is the optimal precoding and multiplexing scheme? It will be shown that this is the formulation of the WSSUS pulse shaping problem for Weyl–Heisenberg signaling in the lowest possible dimension.

The paper is organized as follows. In the first part Weyl–Heisenberg (or Gabor) signaling in $L_2(\mathbb{R})$ and $\mathbb{C}^L$ is introduced. Then the main optimization functional according to [1], [2] is established. It will open up the relation to the pure state channel fidelity optimization which is an ongoing research topic in quantum information theory. In the main part the problem is solved for $\mathbb{C}^2$.

2 Signal Model

2.1 Weyl–Heisenberg Signaling on $L_2(\mathbb{R})$

In this article a transmit baseband signal $s(t)$ is considered which is a superposition of time–frequency translates of a single prototype function $\gamma(t)$ with $\|\gamma\|_2 = 1$. The translations are according to a subset of a lattice $\Lambda \mathbb{F}^2$ in the time–frequency plane, i.e.

$$s(t) = \sum_{n \in \mathbb{Z}} x_n (S_{\lambda n} \gamma)(t)$$

(1)

where in this constellation $\mathbb{F} = \mathbb{Z}$ and $\Lambda$ denotes the $2 \times 2$ generator matrix. The indices $n = (n_1, n_2)$ range over the doubly-countable set $I \subset \mathbb{F}^2$, referring to the data burst to be transmitted. Let

$$(S_{\mu f})(t) \stackrel{\text{def}}{=} e^{i 2\pi \mu t} \gamma(t - \mu_1)$$

(2)

denote the time-frequency shift operator (or phase space displacement operator). It is well-known that the operators $S_{\mu}$ establish an unitary representation of the polarized Heisenberg group and up to phase factors they are equal to the Weyl operators (see [3]). The complex data symbols to transmit are $x_n$ where $n_1$ is the time instant and $n_2$ is the subcarrier index. The transmit signal is passed through a linear time-variant channel denoted by the operator $\mathcal{H}$ and further distorted by an additive white Gaussian noise process $n(t)$. Hence, the
received signal is then
\[ r(t) = (\mathcal{H}s)(t) + n(t) = \int \Sigma(\mu)(S_\mu s)(t)d\mu + n(t) \]  
(3)
with \( \Sigma(\mu) \) being a realization of the channel spreading function. In practice \( \Sigma(\mu) \) is causal and has finite second order statistics of \( \Sigma(\mu) \) is
\[ \mathbb{E}(\Sigma(\mu)\Sigma(\nu^*)^*) = C(\mu)\delta(\mu - \nu) \]  
(4)
where \( C(\mu) \) is the scattering function with \( ||C||_1 = 1 \) (no overall path-loss). To obtain the data symbol \( \bar{x}_m \) the receiver does the projection
\[ \bar{x}_m = \langle S_{\Lambda m}g, r \rangle = \int \langle S_{\Lambda m}g(t)r(t)dt \]  
(5)
onto a time-frequency shifted version of the function \( g(t) \) (i.e. an equalization filter) with \( ||g||_2 = 1 \). Let
\[ H_{m,n} \overset{\text{def}}{=} \langle S_{\Lambda m}g, \mathcal{H}S_{\Lambda n}\gamma \rangle \]  
(6)
be the elements of the channel matrix \( H \in \mathbb{C}^{I \times I} \) and the noise \( n_m \overset{\text{def}}{=} \langle S_{\Lambda m}g, n \rangle \) the transmission scheme can be formulated as the linear equation \( \bar{x} = Hx + n \).

2.2 Weyl–Heisenberg Signaling on \( \mathbb{C}^L \)

With the connection of shift operators to unitary representations of the Weyl-Heisenberg group it is straightforward to pass over to finite dimensional models, i.e. (cyclic) shift operators on \( \mathbb{C}^L \) which are given as unitary representation of finite Heisenberg groups. Hence, let \( \mathcal{M}(L, \mathbb{C}) \) be the algebra of \( L \times L \) matrices over \( \mathbb{C} \). Then the operators \( S_\mu \in \mathcal{M}(L, \mathbb{C}) \) are given as the matrices
\[ (S_\mu)_{m,n} = \delta_{m,n+\mu}e^{i\frac{\pi}{\mu^2}(m-1)} \]  
(7)
where all index–arithmetic is modulo \( L \). The finite Heisenberg group is then
\[ \mathbb{H}_L = \{S_{(m,n)} | m, n \in \mathbb{F}_L \} \]  
(8)
where \( \mathbb{F}_L = \{0 \ldots L-1 \} \).

3 Formulation of the Problem

In the view of multicarrier transmission only single carrier equalization is considered. Interference cancellation is not used in this field due to complexity reasons. Hence it is then naturally to require \( a \) (the channel gain of the lattice point \( m \in I \)) to be maximal and the interference power \( b \) from all other lattice points to be minimal as possible, where
\[ a \overset{\text{def}}{=} |H_{m,m}|^2 \quad \text{and} \quad b \overset{\text{def}}{=} \sum_{n \neq m} |H_{m,n}|^2 \]  
(9)
This addresses the concept of pulse shaping, hence to find jointly good pulses \( \{g, \gamma\} \) (or precoders and equalizers) achieving maximum channel gain and minimum interference power. A comprehensive framework for the optimization of redundant precoders and equalizers with respect to instantaneous time-invariant channel realizations (assumed to be known at the transmitter) is given in [4]. However in certain scenarios it is much more realistic to adapt the pulses only to the second order statistics, given by \( C(\mu) \) and not to a particular realization \( \Sigma(\mu) \). This is in the sense of defining for the considered time-frequency slot \( m \)
\[ \text{SINR}(g, \gamma, \Lambda) \overset{\text{def}}{=} \frac{\mathbb{E}_\mathcal{H}\{a\}}{\sigma^2 + \mathbb{E}_\mathcal{H}\{b\}} \geq \frac{\mathbb{E}_\mathcal{H}\{a\}}{\sigma^2 + B_\gamma - \mathbb{E}_\mathcal{H}\{a\}} \]  
(10)
as a long term performance measure. Optimal signaling via (1) and (3) maximizing (10) independent of \( \Sigma(\cdot) \) is of central relevance for band efficient and low-complexity multicarrier implementation. For example, results on joint multipath and Doppler diversity critically rely on so called approximate \( \Sigma \)-independent basis expansions proposed in [5] of \( \mathcal{H} \). But a general approach how to obtain the "best basis" for \( \mathcal{H} \) being WSSUS operators, i.e. maximizing (10), is still unknown. Nevertheless, some iterative methods are contained in [6].

The derivation of the lower bound in (10) can be found in [7] and in the context of pulse shaping in [2]. Equality is achieved if the set \( G(\gamma, \Lambda, \mathbb{F}^2) \overset{\text{def}}{=} \{S_{\Lambda,m}\gamma|n \in \mathbb{F}^2\} \) (called a Gabor set or family) establishes a tight (Gabor or Weyl–Heisenberg) frame [8], [2]. The constant \( B_\gamma \) is called the Bessel bound of \( G(\gamma, \Lambda, \mathbb{F}^2) \) and related to its redundancy. For \( G(\gamma, \Lambda, \mathbb{F}^2) \) being an ONB it follows \( B_\gamma = 1 \).

Straightforward computation yields the channel fidelity (or averaged gain term) given as
\[ \mathbb{E}_\mathcal{H}\{a\} = \int C(\mu)|\langle g, S_\mu \gamma \rangle|^2d\mu \]  
(11)
and the averaged interference power
\[ \mathbb{E}_\mathcal{H}\{b\} = \sum_{m \neq 0} \int C(\mu)|\langle g, S_{\Lambda,m+\mu} \gamma \rangle|^2d\mu \]  
(12)
Hence, the SINR\((g, \gamma, \Lambda)\) is independent of \( m \). Let \( S_\mu^* \) denote the hermitian adjoint operator of \( S_\mu \) with respect to \( \langle \cdot, \cdot \rangle \). Then the channel fidelity can be rewritten as
\[ \mathbb{E}_\mathcal{H}\{a\} = \langle g, \left[ \int C(\mu)S_\mu^*G_{\Lambda}S_\mu d\mu \right] \rangle \overset{\text{def}}{=} \text{Tr}A(\Gamma)G \]  
(13)
where \( G \) (and \( \Gamma \)) is the (rank-one) orthogonal projector onto \( g \) (and \( \gamma \)), i.e. \( Gf \overset{\text{def}}{=} \langle g, f \rangle g \). Similarly
\[ \mathbb{E}_\mathcal{H}\{b\} = \text{Tr} \left[ \sum_{m \neq 0} S_{\Lambda,m}A(\Gamma)S_{\Lambda,m}^\dagger \right] G \overset{\text{def}}{=} \text{Tr}C(\Gamma)G \]  
(14)
where \( A(\cdot) \) and \( C(\cdot) \) are affine maps acting on linear operators. The definition of \( A(\cdot) \) in particular is also known as Kraus representation of a completely positive map (see for example [9]) which establishes a relation to quantum channels. Due to \( ||C|_1 = 1 \) and \( S_\mu \).
being unitary operators, the following properties can be verified

\[ A \text{ is unital } \iff A(1) = 1 \]
\[ A \text{ is trace preserving } \iff \text{Tr}A(X) = \text{Tr}X \]
\[ A \text{ is hermiticity preserving } \iff A(X^*) = A(X)^* \]
\[ A \text{ is entropy increasing } \iff A(X) < X \] (15)

where \(<\) is in the finite case the partial order due to
eigenvalue majorization. Thus, \(A(\cdot)\) flatten the
eigenvalue distribution of its input (increasing its entropy).
After application of \(A(\cdot)\) onto a rank-one projector \(\Gamma\),
the "output in the averaged sense" (over an ensemble of
WSSUS channels) \(A(\Gamma) < \Gamma\) is not (in general)
rank-one. In this picture so called additional eigen modes
occur which can not be collected together using a rank-
one equalizer (a single equalization filter).

With \(D(\Gamma) \overset{\text{def}}{=} C(\Gamma + \sigma^2 I)\) the SINR optimization
problem reads

\[ \max_{G, \Gamma \in Z, \Lambda} \text{SINR}(G, \Gamma) = \max_{G, \Gamma \in Z, \Lambda} \frac{\text{Tr}A(\Gamma)G}{\text{Tr}D(\Gamma)G} \] (16)

The maximization is performed over possible lattices \(\Lambda\)
and \(G, \Gamma \in Z\), where \(Z\) denotes the set of orthogonal
rank-one projectors, i.e.

\[ M_1 \overset{\text{def}}{=} \{ z \mid \text{Tr} z = 1, z^* = z, z \geq 0 \} \]
\[ Z \overset{\text{def}}{=} \{ z \mid z \in M_1, z^2 = z \} \]

Note that the convex hull of \(Z\) is the subset \(M_1\) of
positive–semidefinite trace class operators.

The maximizing \(G\) for fixed \(\Gamma\) in (16) is achieved
by an orthogonal projection onto the generalized
eigenspace corresponding to the maximal generalized
eigenvalue \(\lambda_{\text{max}}(A(\Gamma), D(\Gamma))\). Thus it remains the
"transmitter–side only" optimization:

\[ \max_{\Gamma \in Z, \Lambda} \text{SINR}(\Gamma) = \max_{\Gamma \in Z, \Lambda} \lambda_{\text{max}}(A(\Gamma), D(\Gamma)) \] (17)

With the definition of an adjoint channel it also possible to obtain a "receiver–side only" optimization [2]. Due to joint quasi-convexity of the function \(\lambda_{\text{max}}(\cdot, \cdot)\) (see for example [10]) the constraint \(\Gamma \in Z\) can be relaxed
to the convex set \(M_1\). Thus, the optimization problem is
identified as convex constrained quasi-convex maximization.
Furthermore, if the inverse of \(D(\Gamma)\) exists, the
problem can be rewritten as a classical eigenvalue problem.
Note that this is not non-convex optimization.

Now, the lower bound in (10) suggests the maximization
of the channel fidelity \(\text{Tr}A(\Gamma)G\) only, i.e.

\[ \max_{G, \Gamma \in Z} \text{Tr}A(\Gamma)G = \max_{\Gamma \in M_1} \lambda_{\text{max}}(A(\Gamma)) \leq 1 \] (18)

which does not depend on the lattice \(\Lambda\). The derivation
from the left to the right side in (18) is again due to
convexity of \(\lambda_{\text{max}}(\cdot)\), linearity of \(A(\cdot)\) and unitarity of
\(S_{\mu}\). It can be shown that this formulation is now equivalent
to the problem of maximizing the quantum channel
fidelity [11] for \(G\) being a pure state (rank-one). Where
the solution of (18) for single–dispersive channels is
straightforward, the general case of this optimization
problem – convex constrained convex maximization – is
unsolved in general. For \(C(\mu)\) being a two–dimensional Gaussian
the solution was found in [12]. Another proof
which additional gives the uniqueness of the solution
is in [13].

Already in [2] it is conjectured that with a proper
selection of a basis for the \(L^2\) dimensional real vector
space of hermitian operators on \(\mathbb{C}^L\) the left side of (18)
could be rewritten as a bilinear program over so called
\(L^2–1\) dimensional Bloch manifolds \(B^{1}(L)\) [14], i.e.

\[ \max_{X, Y \in Z} \text{Tr}A(\Gamma)Y = \max_{x, y \in B^{1}(L)} \langle x, ay \rangle \] (19)

where \((a_{ij}) \in \mathcal{M}(L^2, \mathbb{R})\) is then the matrix representation
of \(A(\cdot)\) in this basis. Indeed – this parameterization
will be used in the next section.

Finally, if the Gabor set \(G(\Gamma, \Lambda, \mathbb{F}^2)\) with the "channel
fidelity"–optimal \(\Gamma\) would establish a tight frame
for some lattice \(\Lambda\), the solution maximizes \(\text{SINR}(\Gamma)\)
too. In this case "channel fidelity"–maximization equals
minimization of the averaged interference. Normally
this is not the case and it remains lattice optimization
with \(\text{det} \Lambda = \text{const}\). In pulse shaping procedures then
a so called orthogonalization with respect to \(\Lambda\) has to be
applied on \(\Gamma\) to minimize the Bessel bound \(B_2 \geq 1\)
[15], [2]. But it will turn out that this step is not needed
here for the \(\mathbb{C}^2\) case (at \(\text{det} \Lambda = 1\)).

4 The \(2 \times 2\) WSSUS Channel

For this simple toy model \(\mathbb{C}^2\) as the underlying
Hilbert space is assumed, i.e. \(L = 2\). The corresponding finite Heisenberg group is
\(\mathbb{H}_2 = \{ S_{(0,0)}, S_{(1,0)}, S_{(0,1)}, S_{(1,1)} \},\)
where

\[ S_{(0,0)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, S_{(0,1)} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \]
\[ S_{(1,0)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, S_{(1,1)} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \] (20)

As already intended in the introduction these matrices represent four basic channel operations which occur in
a randomly weighted superposition. The matrix \(S_{(1,0)}\) represents the only cyclic shift that exists, hence it
switches the input samples. The element \(S_{(0,1)}\) does the
same in frequency domain. With \(S_{(1,1)}\) both operations
occur simultaneously and \(S_{(0,0)}\) does not change the
input at all. The following relations are important

\[ S_{(0,0)} = \sigma_0, S_{(0,1)} = \sigma_3 = FS_{(1,0)}F^* \]
\[ S_{(1,0)} = \sigma_1, S_{(1,1)} = i\sigma_2 = S_{(0,1)}S_{(1,0)} \] (21)

where the \(\sigma_i\) are the well known Pauli-matrices and \(F\)
is the 2x2 Fourier matrix given as

\[ F \overset{\text{def}}{=} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \] (22)
Furthermore we define the following constants

\[
p_0 \overset{\text{def}}{=} C(0, 0), \quad p_1 \overset{\text{def}}{=} C(1, 0), \quad p_2 \overset{\text{def}}{=} C(1, 1), \quad p_3 \overset{\text{def}}{=} C(0, 1)
\]

(23)
given by the four possible values of scattering function \( C(\mu) \). This allows us to write

\[
A(X) = \int S_\mu X S_\mu^* C(\mu) d\mu = \sum_{i=0}^{3} p_i \sigma_i X \sigma_i^*
\]

(24)
The Pauli-matrices establish an orthogonal basis for the real vector space of hermitian 2x2 matrices with inner product \( \langle X, Y \rangle = \text{Tr} X^* Y \). Thus, every 2x2 hermitian matrix \( X \) has a decomposition \( X = \frac{1}{2} \sum_{i=0}^{3} x_i \sigma_i \) with \( x_i \in \mathbb{R} \). Furthermore they establish up to factors the finite Weyl–Heisenberg group itself as shown in (21). This additional property will admit the direct solution of the problem. The following properties are useful to verify the calculations later on:

\[
\sigma_i^2 = \sigma_0 \\
\text{Tr} \sigma_j = 2 \delta_{ij} \\
\det \sigma_i = -1 \quad \text{for} \quad i = 1, 2, 3 \\
\sigma_i \sigma_j = \begin{cases} 
\sigma_i & j = 0 \\
\sigma_j & i = 0 \\
\epsilon_{ijk} \sigma_k + \delta_{ij} \sigma_0 & i, j \neq 0
\end{cases}
\]

(25)

\[
\text{Tr} \sigma_i \sigma_j = 2 \delta_{ij}
\]

where \( \epsilon_{ijk} \) is the Levi-Civita symbol. Note that the Pauli-matrices are unitary and hermitian.

### 4.1 The Channel Fidelity

Recall the equivalent problem formulation

\[
\max_{X,Y \in \mathcal{Z}} \text{Tr} A(X) Y = \max_{x,y \in \mathcal{B}^1(2)} \langle x, ay \rangle
\]

(26)
where \( \mathcal{B}^1(2) = \{ x \mid \sum_i x_i \sigma_i \in \mathcal{Z} \} \) is a 3-dimensional sub-manifold in \( \mathbb{R}^4 \) — the Bloch manifold for 2x2. The matrix \( (a_{ij}) \in \mathcal{M}(4, \mathbb{R}) \) is the corresponding matrix representation of \( A(\cdot) \) with elements

\[
a_{ij} = \frac{1}{4} \text{Tr} A(\sigma_i) \sigma_j
\]

(27)
The Bloch parameterization is a well known tool in quantum physics which admits for \( L = 2 \) the simple interpretation of 3-dimensional Bloch vectors. Because it is not very common in this context a short overview will be given. First let us evaluate the conditions for a vector \( x \in \mathbb{R}^4 \) to be in \( \mathcal{B}^1(2) \). We will adopt the notation \( \vec{x} \) which means \( \vec{x} = (x_1, x_2, x_3) \) for \( x = (x_0, x_1, x_2, x_3) \in \mathbb{R}^4 \).

**Lemma 1:** A real vector \( x \in \mathbb{R}^4 \) is in \( \mathcal{B}^1(2) \) iff \( x_0 = 1 \) and \( \| \vec{x} \|_2 = 1 \).

Thus, it is the fact that the Bloch manifold

\[
\mathcal{B}^1(2) = \{ x = (x_0, \vec{x}) \mid x_0 = 1 \quad \text{and} \quad \| \vec{x} \|_2 = 1 \}
\]

is a 2-sphere in \( \mathbb{R}^4 \) with origin \((1, 0, 0, 0)\). To visualize why it is not easy to extent this concept to higher dimensions we give a short proof.

**Proof:** We have to proof \( \text{Tr} X = \text{Tr} X^2 = 1 \) and \( X \geq 0 \).

The trace normalization represents the \( l_2 \) normalization of the precoders and equalizers. The condition

\[
\text{Tr} X = \frac{1}{2} \text{Tr} \sum_i x_i \sigma_i = \frac{1}{2} \sum_i x_i \text{Tr} \sigma_i = \frac{1}{2} x_0 \cdot 2
\]

(28)
is fulfilled if \( x_0 = 1 \). The second trace requirement is the rank-one constraint if \( X \geq 0 \), i.e. the condition

\[
\text{Tr} X^2 = \frac{1}{4} \sum_{i,j} \text{Tr} (x_i \sigma_i)(x_j \sigma_j) = \frac{1}{2} \| x \|_2^2
\]

(29)
is fulfilled if \( \| x \|_2^2 = 2 \). Using \( \| x \|_2^2 = x_0^2 + \| \vec{x} \|_2^2 = 1 + \| \vec{x} \|_2^2 = 2 \) gives the requirement:

\[
\| \vec{x} \|_2 = 1
\]

(30)
To ensure \( X \geq 0 \) we need conditions on the determinants. Firstly

\[
\det X = \frac{1}{4} \det \sum_i x_i \sigma_i
\]

(31)
which is automatically fulfilled due to \( x_0 = 1 \) and \( \| \vec{x} \| = 1 \). Secondly the upper left sub-determinant is

\[
\frac{1}{4} \det (x_0 + x_3) = \frac{1}{4} (x_0 + x_3) \geq 0 \Rightarrow x_3 \geq -1
\]

(32)
Its non-negativity is also automatically fulfilled due to \( \| \vec{x} \| = 1 \).

**Remark:** The condition \( X \geq 0 \) is only in the \( L = 2 \) case automatically fulfilled.

Next we will explicitly compute the matrix representation \( (a_{ij}) \) for the completely positive map \( A(\cdot) \) in terms of the Pauli basis.

**Lemma 2:** The matrix representation of the completely positive map \( A(\cdot) \) is the diagonal matrix

\[
(a_{ij}) = \text{diag} \left( \frac{1}{2}, (p_0 + p_1) - \frac{1}{2}, (p_0 + p_2) - \frac{1}{2}, (p_0 + p_3) - \frac{1}{2} \right)
\]

(33)

**Proof:** The matrix elements \( A(\cdot) \) with respect to the Pauli basis are given as

\[
a_{kl} = \frac{1}{4} \text{Tr} A(\sigma_k) \sigma_l
\]

(33)
optimal vectors. This can be diagonalized simultaneously which is achieved if $k = 0$. But if $k = 0$ follows $F = p_0 + p_n$ where $n = \arg \max_{m=1,2,3} \{ p_m \}$, achieved with $x^{(\text{opt})}(n)$. A closer inspection shows, that if $k \neq 0$ the solution will depend on some ordering property of the scattering powers. The optimal precoder depends now explicitly on multiple values $p_m$.

(3) **doubly-dispersive “underspread” channels**: If in general only $p_k > \frac{1}{3}$ for some $k = 0 \ldots 3$ we have only $F > \frac{1}{2}$. But if $k = 0$ follows $F = p_0 + p_n$ where $n = \arg \max_{m=1,2,3} \{ p_m \}$, achieved with $x^{(\text{opt})}(n)$. This is the worst case scenario.

It is quite interesting what happens if we fix the scattering power $p_0$, i.e. to consider $F$ as the function $F(p_0, \vec{p})$ where $\vec{p} = (p_1, p_2, p_3)$

$$F(p_0, \vec{p}) = \frac{1}{2} (1 + \max_{k=1,2,3} \{ |2(p_0 - p_k) - 1| \})$$

Clearly, $F(p_0, \vec{p})$ is jointly and separately convex in $p_0$ and $\vec{p}$. Furthermore is $F(p_0, \vec{p}) = F(p_0, \Pi \vec{p})$ for every permutation $\Pi$, so $F$ is Schur-convex in the second argument (see for example [16]), i.e. for a fixed $p_0$ follows

$$\vec{p}_1 > \vec{p}_2 \Rightarrow F(p_0, \vec{p}_1) \geq F(p_0, \vec{p}_2)$$

Using Schur-convexity for every fixed $p_0$ follows:

(5) **the worst case channel**: is given for $p_k = \frac{1-p_0}{3}$ for all $k = 1,2,3$ yielding

$$\min_{\vec{p}_1 = 1-p_0} F(p_0, \vec{p}) = F(p_0, \frac{1-p_0}{3}, 1, 1, 1)$$

$$= \frac{1}{2} (1 + \frac{1}{3} |4p_0 - 1|) = \frac{1}{2} + \frac{2}{3} |p_0 - \frac{1}{4}|$$

Note that in the quantum context this corresponds to the general depolarizing channel or Lie algebra channel [14].

(6) **the best case channel**: is given for $\vec{p}_k = (1-p_0) \vec{e}_k$, where $\vec{e}_k$ is a standard basis vector and $k \in \{1,2,3\}$ yielding

$$\max_{\vec{p}_1 = 1-p_0} F(p_0, \vec{p}) = F(p_0, (1-p_0) \vec{e}_k)$$

$$= \frac{1}{2} (1 + \max \{ |2p_0 - 1|, 1\}) = 1$$

achieved with $x^{(\text{opt})}(k)$. This is again a single-dispersive channel because $\sigma_0$ commutes with $\sigma_k$.

### 4.1.2 Construction of the precoders

In this part we will explicitly calculate the corresponding matrix representation $X^{(\text{opt})}(n)$ from the Bloch parameterizations $x^{(\text{opt})}(n)$ using $X = \frac{1}{2} \sum_{i} x_i \sigma_i$ which gives

$$X^{(\text{opt})}(n) = \frac{1}{2} (\sigma_0 + \sigma_n)$$

$$Y^{(\text{opt})}(n) = \frac{1}{2} (\sigma_0 \pm \sigma_n)$$

The simple solution (37) is well-suited to discuss several cases which occur also in the general WSSUS pulse shaping problem.

(1) **non-dispersive channels**: If $p_k = 1$ for some $k = 0 \ldots 3$, this will yield $F = 1$ which is achieved with any $x^{(\text{opt})}(n)$. This is the so called flat fading (non-selective) channel. The maximal channel fidelity ("$F = 1$") can be achieved. No precoding is needed.

(2) **single-dispersive channels**: If $p_k = p_l = 0$ for some $k, l = 0 \ldots 3$ and $k \neq l$ yields again $F = 1$ which achieved with $x^{(\text{opt})}(n)$ where $n \neq k$ and $n \neq l$. This is for example for $k = 2$ and $l = 3$ the frequency selective, time–invariant channel. All contributions $S_{\mu}$ can be diagonalized simultaneously which is achieved for example in OFDM. The maximal channel fidelity ("$F = 1$") is achieved again (in practice there is still a

4.1.1 **Discussion**

Using the properties of the Pauli matrices one can compute

$$\text{Tr} a_{nkl} = 2 \begin{cases} -\delta_{kl} & 0 \neq n \neq l \neq 0 \\ +\delta_{kl} & \text{else} \end{cases}$$

which gives then

$$(a_{ij} = \frac{1}{2} \text{diag}(1, p_0 + p_1 - p_2 - p_3, p_0 - p_1 + p_2 - p_3, p_0 - p_1 - p_2 + p_3)$$

Using now the normalization, i.e. $p_0 = 1 - p_1 - p_2 - p_3$ gives the desired result.

**Theorem 1**: The solution of the problem in (26) is

$$\max_{x, y \in B^1(2)} \langle x, ay \rangle = \frac{1}{2} (1 + \max_{k=1,2,3} \{ |2(p_0 + p_k) - 1| \})$$

**Proof**: Using Lemma 1 and 2 gives explicitly:

$$1 \geq F = \max_{x, y \in B^1(2)} \langle x, ay \rangle$$

$$= \frac{1}{2} (\varphi_{x} y_{0} + \max_{\|x\|=\|y\|=1} \langle x, b y \rangle)$$

$$= \frac{1}{2} (1 + \max \{2(p_0 + p_1) - 1, |2(p_0 + p_2) - 1|, \frac{2(p_0 + p_3) - 1)}{x^{(\text{opt})}(1)=(1,1,0,0), x^{(\text{opt})}(2)=(1,0,1,0), x^{(\text{opt})}(3)=(1,0,1)\}) \geq \frac{1}{2}$$

where $(b_{ij})$ is the lower right $3 \times 3$ sub–matrix of $a$. Because $(b_{ij})$ is a diagonal matrix, only the three optimal vectors $x^{(\text{opt})}(n)$ given in (37) are possible.

Furthermore the $k$th component of the optimal equalizer is

$$g^{(\text{opt})}(n)_k = \text{sign} \{2(p_0 + p_n) - 1\} x^{(\text{opt})}(n)_k$$

(38)
These matrices are the rank-one projectors onto the optimal precoders in the original problem. Hence, turning back to precoders and equalizers \( x(n), y(n) \in \mathbb{C}^2 \) it can be verified that \( (X^{(opt)})(n) \) similarly

\[
Y^{(opt)}(1) = \frac{1}{\sqrt{2}} \begin{pmatrix} \pm 1 \\ 1 \end{pmatrix} \begin{pmatrix} \pm 1 \\ 1 \end{pmatrix}
\]

\[
Y^{(opt)}(2) = \frac{1}{\sqrt{i}} \begin{pmatrix} \pm 1 \\ i \end{pmatrix} \begin{pmatrix} \pm 1 \\ i \end{pmatrix}
\]

\[
Y^{(opt)}(3) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1/(2(1 \mp 1)) \\ 1/(2(1 \pm 1)) \end{pmatrix} \begin{pmatrix} 1/(2(1 \pm 1)) \\ 1/(2(1 \mp 1)) \end{pmatrix}
\]

In the following I will use w.l.o.g. the \( \pm \)-version, hence \( X^{(opt)}(n) = Y^{(opt)}(n) \). The following precoders are then solutions of the optimization problem

\[
x(1) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} F x(1) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}
\]

\[
x(2) = \frac{1}{\sqrt{i}} \begin{pmatrix} 1 \\ i \end{pmatrix} F x(2) = \frac{1}{2} \begin{pmatrix} 1 + i \\ 1 - i \end{pmatrix}
\]

\[
x(3) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} F x(3) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}
\]

The solution \( x(1) \) is maximally localized in the frequency domain, i.e. completely spread out in the time domain. The reverse case holds for \( x(3) \). To understand \( x(2) \) one has to perform a rotation in the time-frequency plane.

### 4.2 The Multiplexing Scheme

Solving the channel fidelity problem as done in the previous section is the first step toward optimal signatures for a given WSSUS channel statistics. In this part we will discuss now how multiplexing has been performed. In \( \mathbb{C}^2 \)-case at spectral efficiency \( \text{det} \Lambda^{-1} = 1 \) this means multiplexing of a second data stream only. Recall that with Gabor (Weyl–Heisenberg) signaling the same channel fidelity is achieved for the second data stream. It remains first to select the lattice with minimum averaged interference and then perform some kind of orthogonalization ("tighten") procedure. But the \( \mathbb{C}^2 \) case admits already the following transmitter side orthogonality relations

\[
\langle x(n), S_{(0,1)} x(n) \rangle = \frac{1}{2} \text{Tr} \sigma_3 (\sigma_0 + \sigma_n) = \delta_{n3}
\]

\[
\langle x(n), S_{(1,0)} x(n) \rangle = \frac{1}{2} \text{Tr} \sigma_1 (\sigma_0 + \sigma_n) = \delta_{n1}
\]

\[
\langle x(n), S_{(1,1)} x(n) \rangle = \frac{i}{2} \text{Tr} \sigma_2 (\sigma_0 + \sigma_n) = i \delta_{n2}
\]

Hence, for each channel optimal pulse \( "n" \) there are several schemes \( "n" \) for multiplexing a second data stream which admits transmitter-side orthogonality, i.e. no further orthogonalization is required. The "channel fidelity"–optimal precoder is also SINR–optimal. If for example the channel optimal precoder is \( x(3) \), "time-division multiplexing" via \( S_{(0,1)} \) is one of the optimal schemes. For \( x(1) \) in turn "frequency-division multiplexing" is the right scheme.

### 5 Conclusions

In this article new insight into the WSSUS pulse shaping problem are given from a precoding viewpoint. The precoding problem is solved for a very simple class of random \( 2 \times 2 \) channels under the assumption that the transmitter has only knowledge of the second order statistics and the receiver has full knowledge of the channel. It is observed that optimality in the channel–fidelity sense is achieved with concentration in a certain domain similarly as one would expect from the continuous case. Unfortunately the direct extension of this approach to higher dimension is problematic due to the difficulties arising with the Bloch parameterization. But a staggered extension could be conceivably and will probably studied in future.

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