K-groups of the quantum homogeneous space

\[ SU_q(n)/SU_q(n-2) \]

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Abstract

Quantum Steiffel manifolds were introduced by Vainerman and Podkolzin in [13]. They classified the irreducible representations of their underlying C*-algebras. Here we compute the K groups of the quantum homogeneous spaces \( SU_q(n)/SU_q(n-2) \), \( n \geq 3 \). Specializing to the case \( n = 3 \) we show that the fundamental unitary for quantum \( SU(3) \) is nontrivial and forms part of a generating set in the \( K_1 \).

1 Introduction

Quantization of mathematical theories is a major theme of research today. The theory of Quantum Groups and Noncommutative Geometry are two prime examples in this program. Both these programs started in the early eighties. In the setting of operator algebras the theory of quantum groups was initiated independently by Woronowicz ([19]); Vaksman and Soibelman ([18]). They studied the case of quantum \( SU(2) \). Later Woronowicz studied the family of compact quantum groups and obtained Tannaka type duality theorems ([20]). Soon the notion of quantum subgroups and quantum homogeneous spaces followed ([14]).

The Noncommutative Differential Geometry program of Alain Connes also started in the eighties ([6]). In his interpretation geometric data is encoded in elliptic operators or more generally in specific unbounded K-cycles, which he called spectral triples. It is natural to expect that for compact quantum groups and their homogeneous spaces there should be associated canonical spectral triples. Chakraborty and Pal showed that ([3]) indeed that is the case for quantum \( SU(2) \). In fact for odd dimensional quantum spheres one can construct finitely summable spectral triples that witnesses Poincare duality ([4]). A natural question in this connection is, are these examples somewhat singular or in general one can construct finitely summable spectral triples with further properties like Poincare duality on quantum groups associated with Lie groups or their homogeneous spaces. Even though there are suggestions to construct such spectral triples ([12]) one does not know their nontriviality as a K-cycle. In
fact there are suggestions that for quantum groups and their homogeneous spaces one should look for a type III formulation of noncommutative geometry. On this formulation also there are currently two points of view, that of Alain Connes and Henri Moscovici ([8]), and that of Carey-Phillips-Rennie ([2]). Therefore to understand the true nature of interplay between noncommutative geometry and quantum homogeneous spaces it makes sense to take a closer look at these algebras. The underlying $C^*$-algebras of these compact quantum groups were analysed by Soibelman ([17]) (also [9]) who described the irreducible representations of these algebras. Exploiting their findings Sheu went on to obtain composition sequences for these algebras. He initially obtained the results for $SU_q(3)$ ([15]) and later extended them for the general $SU_q(n)$ ([16]). In this hierarchy of exploration the next thing to look for, would be the $K$-groups and that is what we are looking for. But instead of concentrating on the quantum groups we consider quantum analogs of the Steiffel manifolds $SU(n)/SU(n - m)$, introduced by Podkoljin and Vainerman ([13]). They have already described the structure of irreducible representations of quantum Steiffel manifolds $SU_q(n)/SU_q(n - m)$. We take up the case of $SU_q(n)/SU_q(n - 2), n \geq 3$. We obtain the composition sequences for these algebras and then utilising them we compute the $K$-groups. More importantly as we remarked earlier applications towards noncommutative geometry requires explicit understanding of generators for these $K$-groups and during our calculation we also achieve that. Specializing to the case of $n = 3$ we get the $K$-groups of quantum $SU(3)$. Here we should remark that probably these K-groups can also be computed using the variant of KK-theory introduced by Nagy in ([11], but here we produce explicit generators and that is essential to test nontriviality of K-cycles by computing the K-theory K-homology pairing. To our knowledge there are not many instances of K-theory calculations for compact quantum groups and other than the paper by Nagy there is another related work by McClanahan ([10]) where he computes the K-groups of the universal $C^*$-algebra generated by a unitary matrix and shows that the associated $K_1$ is generated by the defining unitary itself. This raises the question whether something similar holds for compact matrix quantum groups, namely whether the defining unitary of a compact matrix quantum group is nontrivial in $K_1$. For quantum $SU(2)$ this was remarked by Connes ([7]). Here we not only prove that the defining unitary of quantum $SU(3)$ is non-trivial we show that this is part of a generating set of elements for the $K_1$.

2 The quantum steifel manifolds and their irreducible representations

The quantum steifel manifold $S_q^{n,m}$ was introduced by Vainerman and Podkolzin in ([13]). Throughout we assume that $q \in (0, 1)$. Recall that the $C^*$ algebra $C(SU_q(n))$ is the universal
unital $C^*$ algebra generated by $n^2$ elements $u_{ij}$ satisfying the following condition

$$\sum_{k=1}^{n} u_{ik}^* u_{jk} = \delta_{ij}, \quad \sum_{k=1}^{n} u_{ki} u_{kj}^* = \delta_{ij}$$  \hspace{1cm} (2.1)

$$\sum_{i_1=1}^{n} \sum_{i_2=1}^{n} \cdots \sum_{i_n=1}^{n} E_{i_1 i_2 \cdots i_n} u_{j_1 i_1} \cdots u_{j_n i_n} = E_{j_1 j_2 \cdots j_n}$$  \hspace{1cm} (2.2)

where

$$E_{i_1 i_2 \cdots i_n} := \begin{cases} 0 & \text{if } i_1, i_2, \cdots, i_n \text{ are not distinct} \\ (-1)^{\ell(i_1, i_2, \cdots, i_n)} & \text{otherwise} \end{cases}$$

where for a permutation $\sigma$ on $\{1, 2, \cdots, n\}$, $\ell(\sigma)$ denotes its length. The $C^*$ algebra $C(SU_q(n))$ has a compact quantum group structure with the comultiplication $\Delta$ given by

$$\Delta(u_{ij}) := \sum_{k} u_{ik} \otimes u_{kj}.$$  

Let $1 \leq m \leq n - 1$. Call the generators of $SU_q(n - m)$ as $v_{ij}$. The map $\phi : C(SU_q(n)) \to C(SU_q(n - m))$ defined by

$$\phi(u_{ij}) := \begin{cases} v_{ij} & \text{if } 1 \leq i, j \leq n - m, \\ \delta_{ij} & \text{otherwise.} \end{cases}$$  \hspace{1cm} (2.3)

is a surjective unital $C^*$ algebra homomorphism such that $\Delta \circ \phi = (\phi \otimes \phi) \Delta$. In this way the quantum group $SU_q(n - m)$ is a subgroup of the quantum group $SU_q(n)$. The $C^*$ algebra of the quotient $SU_q(n)/SU_q(n - m)$ is defined as

$$C(SU_q(n)/SU_q(n - m)) := \{ a \in C(SU_q(n)) : (\phi \otimes 1)\Delta(a) = 1 \otimes a \}.$$  

We refer to [13] for the proof of the following proposition.

**Proposition 2.1** The $C^*$ algebra $C(SU_q(n)/SU_q(n - m))$ is generated by the last $m$ rows of the matrix $(u_{ij})$ i.e. by the set $\{ u_{ij} : n - m + 1 \leq i \leq n \}$.

In [13] the quotient space $SU_q(n)/SU_q(n - m)$ is called a quantum steiffel manifold and is denoted by $S_q^{n,m}$. We will also use the same notation from now on.

The irreducible representations of the $C^*$ algebra $C(S_q^{n,m})$ was described in [13]. First we recall the irreducible representations of $C(SU_q(n))$ as in [17]. The one dimensional representations of $C(SU_q(n))$ are paramatrised by the torus $\mathbb{T}^{n-1}$. We consider $\mathbb{T}^{n-1}$ as a subset of $\mathbb{T}^n$ under the inclusion $(t_1, t_2, \cdots, t_n) \to (t_1, t_2, \cdots, t_{n-1}, t_n)$ where $t_n := \prod_{i=1}^{n-1} t_i$. For $t := (t_1, t_2, \cdots, t_n) \in \mathbb{T}^{n-1}$, let $\tau_t : C(SU_q(n)) \to \mathbb{C}$ be defined as $\tau_t(u_{ij}) := t_{n-i+1} \delta_{ij}$. Then $\tau_t$ is a $*$ algebra homomorphism. Moreover the set $\{ \tau_t : t \in \mathbb{T}^{n-1} \}$ forms a complete set of mutually inequivalent one dimensional representations of $C(SU_q(n))$. 

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Let us denote the transposition \((i, i + 1)\) by \(s_i\). The map \(\pi_{s_i} : C(SU_q(n)) \to B(\ell^2(\mathbb{N}))\) defined on the generators \(u_{rs}\) as follows

\[
\pi_{s_i}(u_{rs}) := \begin{cases} 
\sqrt{1 - q^{2N+2}}S & \text{if } r = i, s = i, \\
-q^{N+1} & \text{if } r = i, s = i + 1, \\
q^N & \text{if } r = i + 1, s = i, \\
S^*\sqrt{1 - q^{2N+2}} & \text{if } r = i + 1, s = i + 1, \\
\delta_{ij} & \text{otherwise}
\end{cases}
\]

is a * algebra homomorphism. For any two representations \(\phi\) and \(\xi\) of \(C(SU_q(n))\), let \(\phi * \xi := (\phi \otimes \xi)\Delta\). For \(\omega \in S_n\), let \(\omega = s_{i_1}s_{i_2}\cdots s_{i_k}\) be a reduced expression. Then the representation \(\pi_{\omega} := \pi_{s_{i_1}} * \pi_{s_{i_2}} * \cdots * \pi_{s_{i_k}}\) is an irreducible representation and up to unitary equivalence the representation \(\pi_{\omega}\) is independent of the reduced expression. For \(t \in \mathbb{T}^{n-1}\) and \(\omega \in S_n\) let \(\pi_{t,\omega} := \pi_t * \pi_{\omega}\). We refer to [17] for the proof of the following theorem.

**Theorem 2.2** The set \(\{\pi_{t,\omega} : t \in \mathbb{T}^{n-1}, \omega \in S_n\}\) forms a complete set of mutually inequivalent irreducible representations of \(C(SU_q(n))\).

In [13] the irreducible representations of \(C(S_q^{n,m})\) are studied and we recall them here. We embed \(\mathbb{T}^m\) into \(\mathbb{T}^{n-1}\) via the map \(t = (t_1, t_2, \ldots, t_m) \to (t_1, t_2, \ldots, t_m, 1, 1, \ldots, 1, t_n)\) where \(t_n := \prod_{i=1}^m \bar{t}_i\). For a permutation \(\omega \in S_n\), let \(\omega^s\) be the permutation in the coset \(S_{n-m}\omega\) with the least possible length. We denote the restriction of the representation \(\pi_{t,\omega}\) to the subalgebra \(C(S_q^{n,m})\) by \(\pi_{t,\omega}\) itself. Then we have the following theorem whose proof can be found in [13]

**Theorem 2.3** The set \(\{\pi_{t,\omega^s} : t \in \mathbb{T}^m, \omega \in S_n\}\) forms a complete set of mutually inequivalent irreducible representations of \(C(S_q^{n,m})\).

### 3 Composition sequences

In this section we derive certain exact sequences analogous to that of Theorem 4 in [16]. We then apply the six term sequence in K theory to compute the K groups of \(C(S_q^{n,2})\).

**Lemma 3.1** Let \(t \in \mathbb{T}^m\) and \(\omega := s_{n-1}s_{n-2}\cdots s_{n-k}\). Then Image of \(C(S_q^{n,m})\) under the homomorphism \(\pi_{t,\omega}\) contains the algebra of compact operators \(K(\ell^2(\mathbb{N}^k))\).

**Proof.** Since \(\pi_{t,\omega}(C(S_q^{n,m})) = \pi_{\omega}(C(S_q^{n,m}))\), it is enough to show that \(K(\ell^2(\mathbb{N}^k)) \subset \pi_{\omega}(C(S_q^{n,m}))\). We prove this result by induction on \(n\). Since \(\pi_{\omega}(u_{nn}) := S^*\sqrt{1 - q^{2N+2}} \otimes 1\), it follows that \(S \otimes 1 \in \pi_{\omega}(C(S_q^{n,m}))\). Hence \(K(\ell^2(\mathbb{N})) \otimes 1 \subset \pi_{\omega}(C(S_q^{n,m}))\). Thus the result is true if \(n = 2\). Next observe that for \(1 \leq i \leq n-1\), \((p \otimes 1)\pi_{\omega}(u_{ni,i}) := p \otimes \pi_{\omega}'(v_{n-1,i})\) where \(\omega' := s_{n-2}s_{n-3}\cdots s_{n-k}\) and \((v_{ij})\) denotes the generators of \(C(SU_q(n-1))\). Hence \(\pi_{\omega}(C(S_q^{n,m}))\) contains the algebra \(p \otimes \pi_{\omega}'(C(S_q^{n-1,m}))\). Now by induction hypothesis, it follows that \(\pi_{\omega}(C(S_q^{n,m}))\) contains
Thus it follows that

\[ \text{If } k \leq n, \text{ let } \omega_n := \omega_{n-1} \ldots \omega_1 \omega_n. \]

Define \( \psi_w := \pi_{s_1} * \pi_{s_2} * \ldots * \pi_{s_r} \) and for \( t \in \mathbb{T}^n \), let \( \psi_{t,w} := \tau_t * \psi_w \). Observe that the image of \( \psi_{t,w} \) is contained in \( \tau \otimes r \). We prove that if \( w' \) is a 'subword' of \( w \) then \( \psi_{t,w'} \) factors through \( \psi_{t,w} \).

**Proposition 3.2** Let \( w = w_1 w_2 \) be a word on \( s_1, s_2, \ldots, s_n \). Denote the word \( w_1 w_2 \) by \( w' \).

Let \( t \in \mathbb{T}^m \) be given. Then there exists a \( * \) homomorphism \( \epsilon : \psi_{t,w}(C(S_q^{n,m})) \rightarrow \psi_{t,w'}(C(S_q^{n,m})) \) such that \( \psi_{t,w'} = \epsilon \circ \psi_{t,w} \).

**Proof.** For a word \( u \) on \( s_1, s_2, \ldots, s_n \), let \( \ell(u) \) denote it's length. Then \( \psi_{t,w}(C(S_q^{n,m})) \) is contained in \( \tau \otimes \ell(u) \otimes \tau \otimes \ell(w_2) \). Let \( \epsilon \) denote the restriction of \( 1 \otimes \sigma \otimes 1 \) to \( \psi_{t,w}(C(S_q^{n,m})) \) where \( \sigma : \tau \rightarrow \mathbb{C} \) is the homomorphism for which \( \sigma(S) = 1 \).

\[
\psi_{t,w}(u_{rs}) = \sum_{j_1, j_2} \psi_{t,w_1}(u_{rj_1}) \otimes \pi_{s_k}(u_{j_1 j_2}) \otimes \psi_{w_2}(u_{j_2 s}).
\]

Since \( \sigma(\pi_{s_k}(u_{j_1 j_2})) = \delta_{j_1 j_2} \), it follows that

\[
\epsilon \circ \psi_{t,w}(u_{rs}) = \sum_j \psi_{t,w_1}(u_{rj}) \otimes \psi_{w_2}(u_{j s}) = \psi_{t,w'}(u_{rs}).
\]

This completes the proof. \( \square \)

Let \( w \) be a word on \( s_1, s_2, \ldots, s_n \). Then for \( n - m + 1 \leq i \leq n \) and \( 1 \leq j \leq n \), the map \( \mathbb{T}^m : t \rightarrow \psi_{t,w}(u_{ij}) \in \tau \otimes \ell(w) \) is continuous. Thus we get a homomorphism \( \chi_w : C(S_q^{n,m}) \rightarrow C(\mathbb{T}^m) \otimes \tau \otimes \ell(w) \) such that \( \chi_w(a)(t) = \psi_{t,w}(a) \) for all \( a \in C(S_q^{n,m}) \).

**Remark 3.3** Clearly for a word \( w \) on \( s_1, s_2, \ldots, s_n \) the representations \( \psi_{t,w} \) factors through \( \chi_w \). One can also prove as in lemma 3.2 that if \( w' \) is a 'subword' of \( w \) then \( \chi_{w'} \) factors through \( \chi_w \).

Let us introduce some notations. Denote the permutation \( s_j s_{j-1} \ldots s_i \) by \( \omega_{j,i} \) for \( j \geq i \). If \( j > i \) we let \( \omega_{j,i} := 1 \). For \( 1 \leq k \leq n \), let \( \omega_k := \omega_{n-m,1} \omega_{n-m+1,1} \ldots \omega_{n-1,n-k+1} \).

**Theorem 3.4** The homomorphism \( \chi_{\omega_n} : C(S_q^{n,m}) \rightarrow C(\mathbb{T}^m) \otimes \tau \otimes \ell(\omega_n) \) is faithful.

**Proof.** If \( \omega_0 \in S_n \) then \( \omega_0^\circ \) (the representative in \( S_n - \omega_0 \) with the shortest length) is a 'subword' of \( \omega_n \). Hence by remark 3.3 every irreducible representation of \( C(S_q^{n,m}) \) factors through \( \chi_{\omega_n} \). Thus it follows that \( \chi_{\omega_n} \) is faithful. This completes the proof. \( \square \)

For \( 1 \leq k \leq n \), let \( C(S_q^{n,m,k}) := \chi_{\omega_k}(C(S_q^{n,m})) \). Then \( C(S_q^{n,m,k}) \subset C(S_q^{n,m,1}) \otimes \tau \otimes (k-1) \). For \( 1 \leq k \leq n \), let \( \sigma_k \) denote the restriction of \( 1 \otimes 1 \otimes (k-2) \otimes \sigma \) to \( C(S_q^{n,m,k}) \). Then the image of \( \sigma_k \) is \( C(S_q^{n,m,k-1}) \). We determine the kernel of \( \sigma_k \) in the next proposition. We need the following two lemmas.
Lemma 3.5 The algebra $\chi_{\omega_{n-1,n-k}}(C(S_q^{n,1}))$ contains $C^*(t_1) \otimes K(\ell^2(\mathbb{N}^k))$ which is isomorphic to $C(\mathbb{T}) \otimes K(\ell^2(\mathbb{N}^k))$.

Proof. Note that $\chi_{\omega_{n-1,n-k}}(u_{nm}) = t_1 \otimes S^* \sqrt{1 - q^{2N + 2}} \otimes 1$. Hence it follows that the operator $1 \otimes \sqrt{1 - q^{2N + 2}} \otimes 1 = \chi_{\omega_{n-1,n-k}}(u_{nm}^* u_{nm})$ lies in the algebra $\chi_{\omega_{n-1,n-k}}(C(S_q^{n,1}))$. As $\sqrt{1 - q^{2N + 2}}$ is invertible, one has $t_1 \otimes S^* \otimes 1 \in \chi_{\omega_{n-1,n-k}}(C(S_q^{n,1}))$. Thus the projection $1 \otimes p \otimes 1$ is in the algebra $C(S_q^{n,1,k+1})$. Now observe that for $1 \leq s \leq n - 1$, one has

$$
(1 \otimes p \otimes 1)\chi_{\omega_{n-1,n-k}}(u_{ns}) = t_1 \otimes p \otimes \pi_{\omega_{n-2,n-k}}(v_{n-1,s}) \quad (3.4)
$$

where $(v_{ij})$ are the generators of $C(SU_q(n - 1))$. If $n = 2$ then $k = 1$ and what we have shown is that $C(S_q^{2,1,2})$ contains $t_1 \otimes S^*$ and $t_1 \otimes p$. Hence one has $C^*(t_1) \otimes K$ is contained in the algebra $C(S_q^{2,1,2})$.

Now we can complete the proof by induction on $n$. Equation 3.4 shows that $C^*(t_1) \otimes K \otimes (k-1)$ is contained in the algebra $C(S_q^{n,1,k+1})$ and we also have $t_1 \otimes S^* \otimes 1 \in C(S_q^{n,1,k+1})$. Hence it follows that $C^*(t_1) \otimes K \otimes k$ is contained in the algebra $C(S_q^{n,1,k+1})$. This completes the proof. $\Box$

Lemma 3.6 Given $1 \leq s \leq n$, there exists compact operators $x_s, y_s$ such that $x_s \pi_{\omega_{n-1,n-k}}(u_{js})y_s = \delta_{js}(p \otimes p \otimes \cdots \otimes p)$ where $p := 1 - S^* S$.

Proof. Let $1 \leq s \leq n$ be given. Note that the operator $\omega_{n-1,n-k}(u_{ss}) = z_1 \otimes z_2 \otimes \cdots z_k$ where $z_i \in \{1, \sqrt{1 - q^{2N + 2}}, S^* \sqrt{1 - q^{2N + 2}}\}$. Define $x_i, y_i$ as follows

$$
x_i := \begin{cases} p & \text{ if } z_i = 1, \\ p & \text{ if } z_i = S^* \sqrt{1 - q^{2N + 2}}, \\ (1 - q^2)^{-\frac{1}{2}} p S & \text{ if } z_i = \sqrt{1 - q^{2N + 2}}. \end{cases}
$$

$$
y_i := \begin{cases} p & \text{ if } z_i = 1, \\ (1 - q^2)^{-\frac{1}{2}} S^* p & \text{ if } z_i = S^* \sqrt{1 - q^{2N + 2}}, \\ p & \text{ if } z_i = \sqrt{1 - q^{2N + 2}}. \end{cases}
$$

Then $x_i y_i = p$ for $1 \leq i \leq k$. Now let $x_s := x_1 \otimes x_2 \otimes \cdots x_k$ and $y_s := y_1 \otimes y_2 \otimes \cdots y_k$. Then $x_s \omega_{n-1,n-k}(u_{ss}) = p \otimes p \otimes \cdots \otimes p$. Let $j \neq s$ be given. Then $\chi_{\omega_{n-1,n-k}}(u_{js}) = a_1 \otimes a_2 \otimes \cdots \otimes a_k$ where $a_i \in \{1, \sqrt{1 - q^{2N + 2}}, S^* \sqrt{1 - q^{2N + 2}}, -q^{N+1}, q^N\}$. Since $j \neq s$, there exists an $i$ such that $a_i \in \{q^N, -q^{N+1}\}$. Let $r$ be the largest integer for which $a_r \in \{q^N, -q^{N+1}\}$. Then $z_r \neq 1$. Hence $x_r a_r y_r = 0$. Thus $x_s \omega_{n-1,n-k}(u_{js})y_s = 0$. This completes the proof. $\Box$

Proposition 3.7 Let $2 \leq k \leq n$. Then $C(S_q^{n,m,1}) \otimes K(\ell^2(\mathbb{N})) \otimes (k-1)$ is contained in the algebra $C(S_q^{n,m,k})$. Moreover the kernel of the homomorphism $\sigma_k$ is exactly $C(S_q^{n,m,1}) \otimes K(\ell^2(\mathbb{N})) \otimes (k-1)$.

Thus we have the exact sequence

$$
0 \rightarrow C(S_q^{n,m,1}) \otimes K^{\otimes (k-1)} \rightarrow C(S_q^{n,m,k}) \xrightarrow{\sigma_k} C(S_q^{n,m,k-1}) \rightarrow 0.
$$
Proof. First we prove that \( C(S^{n,m,1}_q) \otimes K^\otimes(k-1) \) is contained in the algebra \( C(S^{n,m,k}_q) \). For \( a \in C(S^{n,1}_q) \) one has \( \chi_{\omega_k}(a) := 1 \otimes \chi_{\omega_{n-1,n-k+1}}(a) \), it follows from lemma 3.5 that \( C(S^{n,m,k}_q) \) contains \( 1 \otimes K(\ell^2(N^{k-1})) \). Let \( n - m + 1 \leq r \leq m \) and \( 1 \leq s \leq n \) be given. Then note that

\[
\chi_{\omega_k}(u_{rs}) = \sum_{j=1}^{n} \chi_{\omega_1}(u_{rj}) \otimes \pi_{\omega_{n-1,n-k+1}}(u_{js}).
\]

Hence by lemma 3.6 there exists \( x_s, y_s \in C(S^{n,m,k}_q) \) such that \( x_s \chi_{\omega_k}(u_{rs}) y_s := \chi_{\omega_1}(u_{rs}) \otimes p^\otimes(k-1) \) where \( p^\otimes(k-1) := p \otimes p \otimes \cdots \otimes p \). Thus we have shown that \( C(S^{n,m,k}_q) \) contains \( 1 \otimes K^\otimes(k-1) \) and \( C(S^{n,m,1}_q) \otimes p^\otimes(k-1) \). Hence \( C(S^{n,m,k}_q) \) contains \( C(S^{n,m,1}_q) \otimes K^\otimes(k-1) \).

Clearly \( \sigma_k \) vanishes on \( C(S^{n,m,1}_q) \otimes K^\otimes(k-1) \). Let \( \pi \) be an irreducible representation of \( C(S^{n,m,k}_q) \) which vanishes on the ideal \( C(S^{n,m,1}_q) \otimes K^\otimes(k-1) \). Then \( \pi \circ \chi_{\omega_k} \) is an irreducible representation of \( C(S^{n,m}_q) \). Hence \( \pi \circ \chi_{\omega_k} = \pi_{t,\omega} \) for some \( \omega \) of the form \( \omega_{n-m,i_1}\omega_{n-m+1,i_2} \cdots \omega_{n-1,i_{n-m}} \) and \( t \in \mathbb{T}^m \). Since \( \pi \circ \chi_{\omega_k}(u_{n,n-k+1}) = 0 \), it follows that \( \pi_{t,\omega}(u_{n,n-k+1}) = 0 \). But one has \( \pi_{t,\omega}(u_{n,n-k+1}) = t_1(1 \otimes \pi_{\omega_{n-1,i_{n-m}}}(u_{n,n-k+1})) \). Hence \( i_{n-m} > n-k+1 \). In other words \( \omega \) is a subword of \( \omega_{k-1} \). Thus \( \pi \circ \chi_{\omega_k} \) factors through \( \chi_{\omega_{k-1}} \). In other words there exists a representation \( \rho \) of \( C(S^{n,m,k-1}_q) \) such that \( \pi \circ \chi_{\omega_k} = \rho \circ \chi_{\omega_{k-1}} \). Since \( \chi_{\omega_{k-1}} = \sigma_k \circ \chi_{\omega_k} \), it follows that \( \pi = \rho \circ \sigma_k \). Thus we have shown that every irreducible representation of \( C(S^{n,m,k}_q) \) which vanishes on the ideal \( C(S^{n,m,1}_q) \otimes K^\otimes(k-1) \) factors through \( \sigma_k \). Hence the kernel of \( \sigma_k \) is exactly the ideal \( C(S^{n,m,1}_q) \otimes K^\otimes(k-1) \). This completes the proof.

We apply the six term exact sequence in K theory to the exact sequence in proposition 3.7 to compute the \( K \) groups of \( C(S^{n,2,1}_q) \) for \( 1 \leq k \leq n \). In the next section we briefly recall the product operation in \( K \) theory.

4 The operation \( P \)

Let \( A \) and \( B \) be \( C^* \) algebras. Then we have the following product maps.

\[
K_0(A) \otimes K_0(B) \to K_0(A \otimes B),
K_1(A) \otimes K_0(B) \to K_1(A \otimes B),
K_0(A) \otimes K_1(B) \to K_1(A \otimes B),
K_1(A) \otimes K_1(B) \to K_0(A \otimes B).
\]

The first map is defined as \([p] \otimes [q] \to [p \otimes q]\). The second one is defined as \([u] \otimes [p] \to [u \otimes p + 1 - 1 \otimes p]\). The third map is defined in the same manner and the fourth one is defined using Bott periodicity and using the first product. In fact we have the following formula for the last product. We refer to Appendix of [5].
Let \( h : \mathbb{T}^2 \to P_1(\mathbb{C}) := \{ p \in Proj(M_2(\mathbb{C}) : \text{trace}(p) = 1 \} \) be a degree one map. Then given unitaries \( u \in M_p(A) \) and \( v \in M_q(B) \) the product \([u] \otimes [v] \) is given by \([h(u,v)] - [e_0] \) where \( e_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \) \( \in M_2(M_{pq}(A \otimes B)) \) and \( h(u,v) \) is the matrix with entries \( h_{ij}(u \otimes 1, 1 \otimes v) \).

We denote the image of \([x] \otimes [y] \) by \([x] \otimes [y] \) itself. Now let \( A \) be a unital commutative \( C^* \) algebra. Then the multiplication \( m : A \otimes A \to A \) is a \( C^* \) algebra homomorphism. Hence we get a map at the K theory level from \( K_1(A) \otimes K_1(A) \to K_0(A) \).

Suppose \( U \) and \( V \) are two commuting unitaries in a \( C^* \) algebra \( B \). Let \( A := C^*(U,V) \). Then \( A \) is commutative. Define

\[
P(U,V) := K_0(m)([U] \otimes [V])
\]

which is an element in \( K_0(A) \) which we can think of as an element in \( K_0(B) \) by composing with the inclusion map. From the formula that we just recalled from [5] the following properties are clear:

1. If \( U \) and \( V \) are commuting unitaries in \( A \) and \( p \) is a rank one projection in \( \mathcal{K} \) we have
\[
P(U \otimes p + 1 - 1 \otimes p, V \otimes p + 1 - 1 \otimes p) := P(U,V) \otimes p.
\]

2. If \( U \) and \( V \) are commuting unitaries and \( p \) is a projection that commutes with \( U \) and \( V \) then
\[
P(U,Vp + 1 - p) = P(Up + 1 - p, Vp + 1 - p).
\]

3. If \( \phi : A \to B \) is a unital homomorphism and if \( U \) and \( V \) are commuting unitaries in \( A \) then
\[
K_0(\phi)(P(U,V)) = P(\phi(U),\phi(V)).
\]

4. If \( U \) is a unitary in \( A \) then \( P(U,U) = 0 \). For \( P_1(\mathbb{C}) \) is simply connected, it follows that the matrix \( h(U,U) \) is path connected to a rank one projection in \( M_2(\mathbb{C}) \). Hence \( P(U,U) = 0 \).

We need the following lemma in the six term computation. Let \( z_1 \otimes 1 \) and \( 1 \otimes z_2 \) be the generating unitaries of \( C(\mathbb{T}) \otimes C(\mathbb{T}) \). Then \( K_0(C(\mathbb{T}^2)) \) is isomorphic to \( \mathbb{Z}^2 \) and is generated by \( 1, P(z_1 \otimes 1, 1 \otimes z_2) \).

**Lemma 4.1** Consider the exact sequence

\[
0 \longrightarrow C(\mathbb{T}) \otimes \mathcal{K} \longrightarrow C(\mathbb{T}) \otimes \tau \longrightarrow C(\mathbb{T}) \otimes C(\mathbb{T}) \longrightarrow 0
\]

and the six term sequence in \( K \) theory.

\[
\begin{array}{cccc}
K_0(C(\mathbb{T}) \otimes \mathcal{K}) & \longrightarrow & K_0(C(\mathbb{T}) \otimes \tau) & \longrightarrow & K_0(C(\mathbb{T}) \otimes C(\mathbb{T})) \\
\delta \downarrow & & \delta \downarrow & & \delta \downarrow \\
K_1(C(\mathbb{T}) \otimes \mathcal{K}) & \longrightarrow & K_1(C(\mathbb{T}) \otimes \tau) & \longrightarrow & K_1(C(\mathbb{T}) \otimes C(\mathbb{T}))
\end{array}
\]

Then the subgroup generated by \( \delta(P(z_1 \otimes 1, 1 \otimes z_2)) \) coincides with the group generated by \( z_1 \otimes p + 1 - 1 \otimes p \) which is \( K_1(C(\mathbb{T}) \otimes \mathcal{K}) \cong \mathbb{Z} \).
Proof. The toeplitz map $\epsilon : \tau \rightarrow C(\mathbb{T})$ induces isomorphism at the $K_0$ level. Thus by Kunneth theorem, it follows that the image of $K_0(1 \otimes \epsilon)$ is $K_0(C(\mathbb{T})) \otimes K_0(C(\mathbb{T}))$ which is the subgroup generated by $[1]$. Now the inclusion $0 \rightarrow \mathcal{K} \rightarrow \tau$ induces the zero map at the $K_0$ level and hence again by Kunneth theorem the inclusion $0 \rightarrow C(\mathbb{T}) \otimes \mathcal{K} \rightarrow C(\mathbb{T}) \otimes \tau$ induces zero map at the $K_1$ level. Hence the image of $\delta$ is $K_1(C(\mathbb{T}) \otimes \tau)$. This completes the proof.

Corollary 4.2 Let

$$0 \rightarrow I \rightarrow A \xrightarrow{\phi} B \rightarrow 0$$

be a short exact sequence of $C^*$ algebras. Consider the six term sequence in $K$ theory.

$$
\begin{array}{c}
K_0(I) \xrightarrow{\delta} K_0(A) \xrightarrow{K_0(\phi)} K_0(B) \\
\downarrow \delta \downarrow \downarrow \downarrow \\
K_1(B) \xrightarrow{K_1(\phi)} K_1(A) \xrightarrow{K_1(\phi)} K_1(I)
\end{array}
$$

Suppose that $U$ and $V$ are two commuting unitaries in $B$ such that there exists a unitary $X$ and an isometry $Y$ such that $\phi(X) = U$ and $\phi(Y) = V$. Also assume that $X$ and $Y$ commute. Then the subgroup generated by $\delta(P(U,V))$ coincides with the subgroup generated by the unitary $X(1-YY^*) + YY^*$ in $K_1(I)$.

Proof. Since $C(\mathbb{T})$ is the universal $C^*$ algebra generated by a unitary and $\tau$ is the universal $C^*$ algebra generated by an isometry, there exists homomorphisms $\Phi : C(\mathbb{T}) \otimes \tau \rightarrow A$ and $\Psi : C(\mathbb{T}) \otimes C(\mathbb{T}) \rightarrow B$ such that

$$\Phi(z_1 \otimes 1) := X,$$
$$\Phi(1 \otimes S^*) := Y,$$
$$\Psi(z_1 \otimes 1) := U,$$
$$\Psi(1 \otimes z_2) := V.$$

Hence we have the following commutative diagram.

$$
\begin{array}{c}
0 \rightarrow C(\mathbb{T}) \otimes \mathcal{K} \xrightarrow{\Phi} C(\mathbb{T}) \otimes \tau \xrightarrow{\Phi} C(\mathbb{T}) \otimes C(\mathbb{T}) \xrightarrow{\Psi} 0 \\
0 \rightarrow I \xrightarrow{\phi} A \xrightarrow{\phi} B \xrightarrow{\psi} 0
\end{array}
$$

Now by the functoriality of $\delta$ and $P$, it follows that $\delta(P(U,V)) = K_1(\Phi)(\delta(P(z_1 \otimes 1, 1 \otimes z_2)))$. Hence by lemma 4.1, it follows that the subgroup generated by $\delta(P(U,V))$ is the subgroup generated by $\Phi(z_1 \otimes p + 1 - 1 \otimes p)$ in $K_1(I)$. Note that $\Phi(z_1 \otimes p + 1 - 1 \otimes p) = X(1-YY^*) + YY^*$. This completes the proof.

5 $K$ groups of $C(S_q^{n,2,k})$ for $k < n$

In this section we compute the $K$ groups of $C(S_q^{n,2,k})$ for $1 \leq k < n$ by applying the six term sequence in $K$ theory to the exact sequence in 3.7. Let us fix some notations. If $q$ is
a projection in $\ell^2(\mathbb{N})$ then $q_r$ denotes the projection $q \otimes q \otimes \cdots \otimes q$ in $\ell^2(\mathbb{N}^r)$. Let us define the unitaries $U_k, V_k, u_k, v_k$ as follows.

$$
U_k := t_1 \otimes 1_{n-2} \otimes p_{k-1} + 1 - 1 \otimes 1_{n-2} \otimes p_{k-1},
$$

$$
V_k := t_2 \otimes p_{n-2} \otimes 1_{k-1} + 1 - 1 \otimes p_{n-2} \otimes 1_{k-1},
$$

$$
u_k := t_1 \otimes p_{n-2} \otimes p_{k-1} + 1 - 1 \otimes p_{n-2} \otimes p_{k-1},
$$

$$
v_k := t_2 \otimes p_{n-2} \otimes p_{k-1} + 1 - 1 \otimes p_{n-2} \otimes p_{k-1}.
$$

First let us note that the operators $U_k, V_k, u_k, v_k$ lies in the algebra $C(S_q^{n,2,k})$. For,

$$
U_k = 1\{1\}(u_{n,n-k+1}u_{n,n-k+1}^*)u_{n,n-k+1} + 1 - 1\{1\}(u_{n,n-k+1}u_{n,n-k+1}^*),
$$

$$
V_k = 1\{1\}(u_{n-1,1}u_{n-1,1}^*)u_{n-1,1} + 1 - 1\{1\}(u_{n-1,1}u_{n-1,1}^*),
$$

$$
u_k = 1\{1\}(u_{n,n-k+1}u_{n,n-k+1}^*)u_{n,n-k+1} + 1 - 1\{1\}(u_{n,n-k+1}u_{n,n-k+1}^*),
$$

$$
v_k = 1\{1\}(u_{n,n-k+1}u_{n,n-k+1}^*)u_{n,n-k+1} + 1 - 1\{1\}(u_{n,n-k+1}u_{n,n-k+1}^*).
$$

Note that the unitaries $U_n, u_n, v_n$ lies in the algebra $C(S_q^{n,2,n})$. We start with the computation of the $K$ groups of $C(S_q^{n,2,1})$.

**Lemma 5.1** The $K$ groups $K_0(C(S_q^{n,2,1}))$ and $K_1(C(S_q^{n,2,1}))$ are both isomorphic to $\mathbb{Z}^2$. In fact, $[U_1]$ and $[V_1]$ form a $\mathbb{Z}$ basis for $K_1(C(S_q^{n,2,1}))$ and $[1]$ and $P(u_1,v_1)$ form a $\mathbb{Z}$ basis for $K_0(C(S_q^{n,2,1}))$.

**Proof.** First note that $C(S_q^{n,2,1})$ is generated by $t_1 \otimes 1_{n-2}$ and $t_2 \otimes \pi_{\omega_{n-2}}(u_{n-1,j})$ where $1 \leq j \leq n-1$. But the $C^*$ algebra generated by $\{t_2 \otimes \pi_{\omega_{n-2}}(u_{n-1,j}) : 1 \leq j \leq n-1\}$ is isomorphic to $C(S_{q^{2n-3}})$. Hence $C(S_q^{n,2,1})$ is isomorphic to $C(\mathbb{T}) \otimes C(S_{q^{2n-3}})$. Also $K_0(C(S_q^{2n-3})$ and $K_1(C(S_q^{2n-3}))$ are both isomorphic to $\mathbb{Z}$ with $[1]$ generating $K_0(C(S_q^{2n-3}))$ and $[t_2 \otimes p_{n-2} + 1 - 1 \otimes p_{n-2}]$ generating $K_1(C(S_q^{2n-3}))$.

Now by the Kunneth theorem for tensor product of $C^*$ algebras(See [2]), it follows that $C(S_q^{n,2,1})$ has both $K_1$ and $K_0$ isomorphic to $\mathbb{Z}^2$ with $[U_1]$ and $[V_1]$ generating $K_1(C(S_q^{n,2,1}))$ and $[1]$ and $P(t_1 \otimes 1_{n-2}, t_2 \otimes p_{n-2} + 1 - 1 \otimes p_{n-2})$ generating $K_0(C(S_q^{n,2,1}))$. Note that the projection $1 \otimes p_{n-2} = 1\{1\}(\chi_{\omega_{n-2}}(u_{n-1,1}u_{n-1,1}^*))$ is in $C(S_q^{n,2,1})$ and commutes with the unitaries $t_1 \otimes 1_{n-2}$ and $t_2 \otimes p_{n-2} + 1 - 1 \otimes p_{n-2}$. Hence

$$
P(t_1 \otimes 1_{n-2}, t_2 \otimes p_{n-2} + 1 - 1 \otimes p_{n-2}) = P(u_1,v_1).
$$

This completes the proof. \quad \square

**Proposition 5.2** Let $1 \leq k < n$ be given. Then the $K_0(C(S_q^{n,2,k}))$ and $K_1(C(S_q^{n,2,k}))$ are both isomorphic to $\mathbb{Z}^2$. In particular, $[U_k]$ and $[V_k]$ form a $\mathbb{Z}$ basis for $K_1(C(S_q^{n,2,k}))$ and $[1]$ and $P(u_k, v_k)$ form a $\mathbb{Z}$ basis for $K_0(C(S_q^{n,2,k}))$.
Proof. We prove this result by induction on $k$. The case $k = 1$ is just lemma 5.1. Now assume the result to be true for $k$. From proposition 3.7 we have the short exact sequence

$$0 \rightarrow C(S_q^{n,2,1}) \otimes K^{\otimes(k)} \rightarrow C(S_q^{n,2,k+1}) \xrightarrow{\sigma_{k+1}} C(S_q^{n,2,k}) \rightarrow 0$$

which gives rise to the following six term sequence in $K$ theory.

$$K_0(C(S_q^{n,2,1}) \otimes K^{\otimes(k)}) \xrightarrow{\partial} K_0(C(S_q^{n,2,k+1})) \xrightarrow{K_0(\sigma_{k+1})} K_0(C(S_q^{n,2,k}))$$

$$\xrightarrow{\delta} K_1(C(S_q^{n,2,k+1}) \otimes K^{\otimes(k)})$$

We determine $\delta$ and $\partial$ to compute the six term sequence. As $\sigma_{k+1}(V_{k+1}) = V_k$, it follows that $\partial([V_k]) = 0$. Since $C(S_q^{n,2,k+1})$ contains the algebra $C(S_q^{n,2,1}) \otimes K^{\otimes(k)}$, it follows that the operator $\tilde{X} := t_1 \otimes 1_{n-2} \otimes q^N \otimes q^N \otimes \cdots q^N \otimes S^*$ is in the algebra $C(S_q^{n,2,1})$ as the difference

$$X - \chi_{\omega_{k+1}}(1_{n,n-k+1}) \text{ lies in the ideal } C(S_q^{n,2,1}) \otimes K^{\otimes(k)}.$$ 

Then $X$ is an isometry such that $\sigma_{k+1}(X) = U_k$. Hence $\partial([U_k]) = [1 - X^*X] - [1 - XX^*]$. Thus $\partial([U_k]) = -[1 \otimes 1_{n-2} \otimes p_k]$. Thus the image of $\partial$ is the subgroup of $K_0(C(S_q^{n,2,1}) \otimes K^{\otimes(k)})$ generated by $[1 \otimes 1_{n-2} \otimes p_k]$ and the kernel is $[V_k]$.

Next we compute $\delta$. Since $\sigma_{k+1}(1) = 1$, it follows that $\delta([1]) = 0$. Let

$$Y := (1 \otimes p_{n-2} \otimes 1_k)(1 \otimes 1_{n-2} \otimes p_{k-1} \otimes 1) \tilde{X} + 1 - 1 \otimes p_{n-2} \otimes p_{k-1} \otimes 1.$$ 

Since $1 \otimes p_{n-2} \otimes 1 = 1_{\{1\}}(\chi_{\omega_2}(1_{n-1,n-1})$ and $1 \otimes 1_{n-2} \otimes p_{k-1} = 1_{\{1\}}(1 \otimes 1_{n-2} \otimes 1)$ it follows that the operator $Y \in C(S_q^{n,2,k+1})$. Also

$$Y = t_1 \otimes p_{n-2} \otimes p_{k-1} \otimes S^* + 1 - 1 \otimes p_{n-2} \otimes p_{k-1} \otimes 1.$$ 

Note that $Y$ is an isometry such that $\sigma_{k+1}(Y) = u_k$. One has $\sigma_{k+1}(v_{k+1}) = v_k$. Note that $Y$ and $v_{k+1}$ commute. Hence by corollary 4.1 it follows that the image of $\delta$ is the subgroup generated by $[v_{k+1}(1 - YY^*) + YY^*] = [V_1 \otimes p_k + 1 - 1 \otimes p_k]$.

Thus the above computation with the six term sequence implies that $K_0(C(S_q^{n,2,k+1}))$ is isomorphic to $Z^2$ and is generated by $P(u_1, v_1) \otimes p_k = P(u_k, v_k)$ and $[1]$ and $K_1(C(S_q^{n,2,k+1}))$ is isomorphic to $Z^2$ and is generated by $[V_{k+1}]$ and $[U_1 \otimes p_k + 1 - 1 \otimes p_k] = [U_{k+1}]$. This completes the proof. \hfill \Box

6 \hspace{1em} $K$ groups of $C(S_q^{n,2})$

In this section we compute the $K$ groups of $C(S_q^{n,2})$. We start with a few observations.

Lemma 6.1 \hspace{1em} In the permutation group $S_n$ one has $\omega_{n-2,1} \omega_{n-1,1} = \omega_{n-1,1} \omega_{n-1,2}$. 

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Proof. First note that \( s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \) and \( s_is_j = s_js_i \) if \(|i - j| \geq 2\). Hence one has \( \omega_{n-1,k} \omega_{n-1,1} = \omega_{n-1,k+1} \omega_{n-1,1} \omega_{n-1} s_{k+1} \). Now the result follows by induction on \( k \). \( \square \)

We denote the representation \( \chi_{\omega_{n-1,1}} \star \pi_{\omega_{n-1,2}} \) by \( \tilde{\chi}_{\omega_n} \). Since \( \omega_{n-1,1} \omega_{n-1,2} \) is a reduced expression for \( \omega_n \) it follows that the representations \( \tilde{\chi}_{\omega_n} \) and \( \chi_{\omega_n} \) are equivalent. Let \( U \) be a unitary such that \( U \chi_{\omega_n}(.) U^* = \tilde{\chi}_{\omega_n}(.) \). It is clear that \( \tilde{\chi}_{\omega_n}(C(S_q^{n,2})) \subset C(\mathbb{T}^m) \otimes \tau \otimes \tau \otimes (\omega_{n-1}) \).

Let \( \tilde{\sigma}_n \) denote the restriction of \( 1 \otimes \sigma \otimes 1 \otimes (2^{n-2}) \) to \( \tilde{\chi}_{\omega_n}(C(S_q^{n,2})) \). Since \( \tilde{\sigma}_n(\tilde{\chi}_{\omega_n}(u_{ij})) = \chi_{\omega_{n-1}}(u_{ij}) \) one has the following commutative diagram

\[
\begin{array}{ccc}
\chi_{\omega_n}(C(S_q^{n,2})) & \xrightarrow{U(.) U^*} & \tilde{\chi}_{\omega_n}(C(S_q^{n,2})) \\
\sigma_n & \searrow & \tilde{\sigma}_n \\
& C(S_q^{n,2,n-1}) &
\end{array}
\]

**Lemma 6.2** There exists a coisometry \( X \in \chi_{\omega_n}(C(S_q^{n,2})) \) such that \( \sigma_n(X) = V_{n-1} \) and \( X^* X = 1 - 1_{\{1\}}(\chi_{\omega_n}(u_{11}^* u_{11})) \).

Proof. By the above commutative diagram, it is enough to show that there exists a coisometry \( \tilde{X} \in \tilde{\chi}_{\omega_n}(C(S_q^{n,2})) \) such that \( \tilde{\sigma}_n(X) = V_{n-1} \) and \( X^* X = 1 - 1_{\{1\}}(\tilde{\chi}_{\omega_n}(u_{11}^* u_{11})) \). Note now that \( \tilde{\chi}_{\omega_n}(u_{n-1,1}^* u_{n-1,1}) = q^2 u_{n1}^* u_{n1} \) is a reduced projection \( 1 \otimes 1 \otimes p_{n-2} \otimes 1_{n-2} = 1_{\{1\}}(\tilde{\chi}_{\omega_n}(u_{n-1,1}^* u_{n-1,1} - q^2 u_{n1}^* u_{n1})) \) is in the algebra \( \tilde{\chi}_{\omega_n}(C(S_q^{n,2})) \). Now let \( Y = (1 \otimes 1 \otimes p_{n-2} \otimes 1_{n-2}) \tilde{\chi}_{\omega_n}(u_{n-1,1}) \). Then \( Y = t_2 \otimes \sqrt{1 - q^{2n+2}} S \otimes p_{n-2} \otimes 1_{n-2} \). Hence the operator \( Z = t_2 \otimes S \otimes p_{n-2} \otimes 1_{n-2} \) is in the algebra \( \tilde{\chi}_{\omega_n}(C(S_q^{n,2})) \).

Then \( \tilde{X} \) is a coisometry such that \( \tilde{\sigma}_n(\tilde{X}) = V_{n-1} \) and \( X^* \tilde{X} = 1 - 1 \otimes p_{n-1} \otimes 1_{n-2} \) which is \( 1 - 1_{\{1\}}(\tilde{\chi}_{\omega_n}(u_{11}^* u_{11})) \). This completes the proof. \( \square \)

Observe that the operator \( \tilde{Z} := t_1 \otimes 1_{n-2} \otimes q^N \otimes q^N \otimes \cdots \otimes q^N \otimes S^* \) lies in the algebra \( C(S_q^{n,2,n}) \) since the difference \( \tilde{Z} - \chi_{\omega_n}(u_{n,2}) \) lies in the ideal \( C(S_q^{n,2,1}) \otimes \mathcal{K}^{\otimes (n-1)} \). Let \( Z := 1_{\{1\}}(\tilde{Z} \tilde{Z}) \tilde{Z} \) and \( Y_n := Z + 1 - Z^* Z \). Then

\[
Z_n = t_1 \otimes 1_{n-2} \otimes p_{n-2} \otimes S^* , \quad \text{(6.5)}
\]
\[
Y_n = t_1 \otimes 1_{n-2} \otimes p_{n-2} \otimes S^* + 1 - 1 \otimes 1_{n-2} \otimes p_{n-2} \otimes 1 . \quad \text{(6.6)}
\]

Hence \( Y \) is an isometry and \( YY^* = 1 - 1_{\{1\}}(\chi_{\omega_n}(u_{11}^* u_{11})) \). Let \( X \) be a coisometry in \( C(S_q^{n,2,n}) \) such that \( \sigma_n(X) = V_{n-1} \) and \( X^* X = 1 - 1_{\{1\}}(\chi_{\omega_n}(u_{11}^* u_{11})) \). The existence of such an \( X \) was shown in lemma 6.2. Then \( XY \) is a unitary.

**Proposition 6.3** The \( K \) groups \( K_0(C(S_q^{n,2})) \) and \( K_1(C(S_q^{n,2})) \) are both isomorphic to \( \mathbb{Z}^2 \). In particular we have the following.
1. The projections \([1]\) and \(P(u_n, v_n)\) generate \(K_0(C(S_q^{n,2}))\).

2. The unitaries \(U_n\) and \(XY_n\) generate \(K_1(C(S_q^{n,2}))\) where \(X\) is a coisometry in \(C(S_q^{n,2})\) such that \(\sigma_n(X) = V_{n-1}\) and \(X^*X = 1 - 1_{\{1\}}(u_n^*u_n)\) and \(Y_n\) is as in equation (6.6).

Proof. By proposition 3.7, we have the following exact sequence.

\[
0 \rightarrow C(S_q^{n,2,1}) \otimes K \otimes (n-1) \rightarrow C(S_q^{n,2,n}) \otimes C(S_q^{n,2,1}) \rightarrow 0.
\]

which gives rise to the following six term sequence in \(K\) theory.

\[
\begin{array}{cccccc}
K_0(C(S_q^{n,2,1}) \otimes K \otimes (n-1)) & \rightarrow & K_0(C(S_q^{n,2,n})) & \rightarrow & K_0(C(S_q^{n,2,k})) \\
\delta & & \delta & & \\
K_1(C(S_q^{n,2,n-1})) & \rightarrow & K_1(C(S_q^{n,2,n})) & \rightarrow & K_1(C(S_q^{n,2,1}) \otimes K \otimes (n-1))
\end{array}
\]

Now we compute \(\partial\) and \(\delta\) to compute the six term sequence. First note that since \([U_{n-1}]\) and \([V_{n-1}]\) generate \(K_1(C(S_q^{n,2,n-1}))\), it follows that \([U_{n-1}]\) and \([V_{n-1}U_{n-1}]\) generate \(K_1(C(S_q^{n,2,n-1}))\).

As \(XY_n\) is a unitary for which \(\sigma_n(XY_n) = V_{n-1}U_{n-1}\), it follows that \(\partial([V_{n-1}U_{n-1}]) = 0\). Next \(Y_n\) is an isometry for which \(\sigma_n(Y_n) = U_{n-1}\). Hence \(\partial([U_{n-1}]) = [1 - Y^*X] - [1 - YY^*]\). Thus \(\partial([U_{n-1}]) = [1 \otimes 1_{n-2} \otimes p_{n-1}]\).

Now we compute \(\delta\). Since \(\sigma_n(1) = 1\), it follows that \(\delta([1]) = 0\). Now one observes that \(p_{n-2} \otimes S_{\pi_{\omega_n}^{-1}}(u_j) = 0\) if \(j > 1\). Hence \(Z_n\omega_n(u_n-1,1) = t_1 t_2 \otimes p_{n-2} \otimes p_{n-2} \otimes \sqrt{1 - q^{2N+2}}\)

where \(Z_n\) is as defined in (6.3). Thus the operator \(R_n := t_1 t_2 \otimes p_{n-2} \otimes p_{n-2} \otimes 1\) lies in the algebra \(C(S_q^{n,2,n})\) as the difference \(R_n - Z_n\omega_n(u_n-1,1)\) lies in the ideal \(C(\mathbb{T}^2) \otimes K \otimes (2n-3))\).

Hence projection \(1 \otimes p_{n-2} \otimes p_{n-2} \otimes 1\) lies in the algebra \(C(S_q^{n,2,n})\). Now define

\[
\begin{align*}
S_n & := R_n + 1 - R_n R_n^* \\
T_n & := (1 \otimes p_{n-2} \otimes p_{n-2} \otimes 1) Z_n + 1 - 1 \otimes p_{n-2} \otimes p_{n-2} \otimes 1.
\end{align*}
\]

Then \(S_n\) is a unitary and \(T_n\) is an isometry such that \(\sigma_n(S_n) = u_{n-1} v_{n-1}\) and \(\sigma_n(T_n) = u_{n-1}\). Moreover \(S_n\) and \(T_n\) commute. Now note that \(P(u_n-1, v_n-1) = P(u_n, u_n-1 v_n-1)\). Hence by corollary (6.4) it follows that the image of \(\delta\) is the subgroup generated by \(S_n(1 - T_n T_n^*) + T_n T_n^*\) in \(K_1(C(S_q^{n,2,1}) \otimes K \otimes (n-1))\).

Now

\[
S_n(1 - T_n T_n^*) + T_n T_n^* = t_1 t_2 \otimes p_{n-2} \otimes p_{n-1} + 1 - 1 \otimes p_{n-2} \otimes p_{n-1}.
\]

Since \(1 \otimes p_{n-2}\) is a trivial in \(K_0(C(S_q^{n,2-n-3}))\) it follows that the unitary \(t_1 \otimes p_{n-2} + 1 - 1 \otimes p_{n-2}\) is trivial in \(K_1(C(S_q^{n,2,1}) = K_1(C(\mathbb{T}) \otimes C(S_q^{n,2-n-3})\). Hence one has \([S_n(1 - T_n T_n^*) + T_n T_n^*] = [V_1 \otimes p_{n-1} + 1 - 1 \otimes p_{n-1}]\) in \(K_1(C(S_q^{n,2,1}) \otimes K \otimes (n-1))\).

Thus the above computation with the exactness of the six term sequence completes the proof. \(\blacksquare\)
7 K groups of quantum SU(3)

In this section we show that for \( n = 3 \) the unitary \( XY_n \) in proposition 6.3 can be replaced by the fundamental \( 3 \times 3 \) matrix \( (u_{ij}) \) of \( C(SU_q(3)) \). First note that for \( n = 3 \) we have \( C(S^{q,2}_n) = C(SU_q(3)) \) since \( C(SU_q(1)) = \mathbb{C} \). The embedding \( SU_q(1) \subseteq SU_q(3) \) is given by the counit. Hence the quotient \( C(SU_q(3)/SU_q(1)) \) becomes isomorphic with \( C(SU_q(3)) \). The algebra \( C(S^{3,2,1}_3) \) is denoted \( C(U_q(2)) \) in [16]. Then \( C(U_q(2)) = C(T) \otimes C(SU_q(2)) \). Let \( ev_1 : C(T) \to \mathbb{C} \) be the evaluation at the point \( '1' \). Then \( \phi = (ev_1 \otimes 1)\sigma_2\sigma_3 \) where \( \phi : C(SU_q(3)) \to C(SU_q(2)) \) is the subgroup homomorphism defined in equation 2.3.

**Proposition 7.1** The \( K \) group \( K_1(C(SU_q(3)) \) is isomorphic to \( \mathbb{Z}^2 \) generated by the unitary \( U_3 := t_1 \otimes p \otimes p + 1 - 1 \otimes p \otimes p \) and the fundamental unitary \( U = (u_{ij}) \).

**Proof.** By proposition 6.3 we know that \( K_1(C(SU_q(3)) \) is isomorphic to \( \mathbb{Z}^2 \) and is generated by \( [U_3] \) and \( [XY_3] \) where \( X \) is a isometry such that \( \sigma_3(X) = V_2 \) and \( X^*X = 1 - 1_{11}(\chi_{w_3}(u_{31}^*u_{31})) \). Now observe that \( \phi(X) = t_2 \otimes p + 1 - 1 \otimes p \) and \( \phi(Y_3) = 1 \). Hence \( \phi(XY_3) = t_2 \otimes p + 1 - 1 \otimes p \).

Also note that \( \phi(U_3) = 0 \) and \( \phi(U) = \begin{bmatrix} u & 0 \\ 0 & 1 \end{bmatrix} \) where \( u \) denote the fundamental unitary of \( C(SU_q(2)) \). Since \( K_1(C(SU_q(2)) \) is isomorphic to \( \mathbb{Z} \) the proof is complete if we show that \( t_2 \otimes p + 1 - 1 \otimes p \) and \( [u] \) represents the same element in \( K_1(C(SU_q(2)) \) which we do in the next lemma. \( \square \)

We denote the \( 2 \times 2 \) fundamental unitary \( u = (u_{ij}) \) of \( C(SU_q(2)) \) by \( u_q \). Consider the representation \( \chi_{s_1} : C(SU_q(2)) \to B(\ell^2(\mathbb{Z}) \otimes \ell^2(\mathbb{N})) \). We let the unitary \( t \) act on \( \ell^2(\mathbb{Z}) \) as the right shift i.e \( te_n = e_{n+1} \). Let \( \{e_{n,m} : n \in \mathbb{Z}, m \in \mathbb{N} \} \) be the standard orthonormal basis for the Hilbert space \( \ell^2(\mathbb{Z}) \otimes \ell^2(\mathbb{N}) \). For an integer \( k \), denote the orthogonal projection onto the closed subspace spanned by \( \{e_{n,m} : n + m \leq k \} \) by \( P_k \) and set \( F_k := 2P_k - 1 \). Note that \( F_k \) is a selfadjoint unitary.

**Proposition 7.2** For any integer \( k \), the triple \( (\chi_{s_1}, \ell^2(\mathbb{Z}) \otimes \ell^2(\mathbb{N}), F_k) \) is an odd Fredholm module for \( C(SU_q(2)) \) and we have the pairing

1. \( \langle [u_q], F_k \rangle = -1 \)

2. \( \langle t \otimes p + 1 - 1 \otimes p, F_k \rangle = -1 \) where \( p = 1 - S^*S \).

**Proof.** It is not difficult to show that \( C(SU_q(2)) \) is generated by \( t \otimes S \) and \( t \otimes p \). Now it is easy to see that \( [t \otimes S, P_k] = 0 \) and \( [t \otimes p, P_k] \) is a finite rank operator. Hence the triple \( (\chi_{s_1}, \ell^2(\mathbb{Z}) \otimes \ell^2(\mathbb{N}), F_k) \) is an odd Fredholm module for \( C(SU_q(2)) \). Since \( C(SU_q(2)) \) is generated by \( t \otimes S \) and \( t \otimes p \) it follows that \( u_p \in C(SU_q(2)) \) for every \( p > 0 \). Also as \( p \to 0 \), \( u_p \) approaches \( u \) in norm where \( u \) is given by

\[
 u := \begin{pmatrix} t \otimes S & 0 \\ i \otimes p & i \otimes S^* \end{pmatrix}
\]
Hence \([u_q] = [u]\) in \(K_1(C(SU_q(2)))\). It is easy to check the following

\[
\langle [u], F_k \rangle = -1
\]

\[
\langle [t \otimes p + 1 - 1 \otimes p], F_k \rangle = -1.
\]

This completes the proof. \(\square\)

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