Realizing metrics of curvature $\leq -1$ on closed surfaces in Fuchsian anti-de Sitter manifolds

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Abstract

We prove that any metric with curvature $\leq -1$ (in the sense of A. D. Alexandrov) on a closed surface of genus $> 1$ is isometric to the induced intrinsic metric on a space-like convex surface in a Lorentzian manifold of dimension $(2 + 1)$ with sectional curvature $-1$. The proof is done by approximation, using a result about isometric immersion of smooth metrics by Labourie–Schlenker.

1 Introduction

In the following, $S$ is a closed connected oriented surface. When we speak about a metric with curvature $\leq k$ or $\geq k$, this means that $S$ is endowed with a distance $d$ satisfying a curvature bound in the sense of A. D. Alexandrov, see e.g. [BB10] or Section 5. This metric notion of curvature bound was initially introduced in the 40’s to characterize the induced metric on the boundary of convex bodies of the Euclidean space [Ale06]. (In the present article, the word metric is used for distance, and induced metric means the induced intrinsic distance.) While introducing this seminal notion, Alexandrov proved the following statement.

Theorem 1.1. Let $d$ be a metric of curvature $\geq 0$ on the sphere $S$. Then there exists a convex surface in the Euclidean space whose induced metric is isometric to $(S,d)$.

Theorem 1.1 was generalized in many ways. Some of them are contained in the following statement, see the introduction of [FIV16] for details.

Theorem 1.2. Let $k \in \mathbb{R}$ and let $d$ be a metric of curvature $\geq k$ on a closed surface $S$. Then there exists a Riemannian manifold $R$ homeomorphic to $S \times \mathbb{R}$ of constant sectional curvature $k$ which contains a convex surface whose induced metric is isometric to $(S,d)$.

In 2017, F. Fillastre and D. Slutsky proved an analogous results for metrics with curvature bounded from above [FS19].

Theorem 1.3. Let $d$ be a metric of curvature $\leq 0$ on a closed surface $S$ of genus $> 1$. Then there exists a flat Lorentzian manifold $L$ homeomorphic to $S \times \mathbb{R}$ which contains a space-like convex surface whose induced metric is isometric to $(S,d)$.

A natural question is to know if an analog of Theorem 1.2 holds for metrics with curvature bounded from above. The case $k = 0$ is given by Theorem 1.3. In the present article, we solve the case when $k$ is negative. Up to a homothety, this reduces to the case $= -1$. So the main result of the present paper is the following theorem.

Theorem 1.4. Let $d$ be a metric with curvature $\leq -1$ on a closed surface $S$ of genus $> 1$. Then there exists a Lorentzian manifold $L$ of sectional curvature $-1$ homeomorphic to $S \times \mathbb{R}$ which contains a space-like convex surface whose induced metric is isometric to $(S,d)$.
The proof of Theorem 1.4 will be given by a classical approximation procedure, following the main lines of [FS19]. The proof relies on the smooth analogue of Theorem 1.4 proved by F. Labourie and J.-M. Schlenker, see Theorem 4.1. We will prove Theorem 1.4 showing that the universal cover of $\mathcal{S}_d$ is isometric to a convex surface in anti-de Sitter space (see Section 2), invariant under the action of a discrete group of isometries leaving invariant a totally geodesic hyperbolic surface. Such groups are usually called Fuchsian, and the quotient of a suitable part of anti-de Sitter space by such a group may be called a Fuchsian anti-de Sitter manifold. The main issues in our case, comparing to [FS19], is that we lost the vector space structure given by the Minkowski space—it is the Lorentzian analogue of the problem to go from Euclidean space to hyperbolic space. Also, the analogue of an approximation result that is straightforward in the flat case occupies the whole Section 5 here.

Let us describe more precisely the content of the present article. In Section 2, we recall some definitions related to anti-de Sitter space, and define the induced metric on convex surfaces in this space. In Section 3 we look at surfaces invariant under the action of Fuchsian groups, and prove several compactness results. In section 5 we check that any metric with curvature $\leq -1$ on $S$ can be approximated by a sequence of distances given by Riemannian metrics with sectional curvature $< -1$. In Section 4, all the elements are put together to provide a proof of Theorem 1.4.

The case with a positive $k$ is still missing to obtain a Lorentzian analogue of Theorem 1.2. An issue is that it is not clear if the approximation results used in Section 5 can be applied in the $\leq 1$ curvature case.

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2 Convex surfaces in anti-de Sitter space

2.1 Anti-de Sitter space

In the following we will describe a geometric model of anti-de Sitter space (of dimension 3) we are most interested in, and illustrate some of its features. Good references for this material are [Mes07], [BST12], [BS10] and [O’N83].

Let us consider the symmetric bilinear form
\[ b(x, y) = -x_0y_0 - x_1y_1 + x_2y_2 + x_3y_3. \]
of signature $(2, 2)$ on $\mathbb{R}^4$.

**Definition 2.1.** We define $\widetilde{AdS^3}$ as
\[ \widetilde{AdS^3} = \{(x_0, x_1, x_2, x_3) \in \mathbb{R}^4 | b(x, x) = -1\}, \]
endowed with the pseudo-Riemannian metric induced by the restriction of the bilinear form $b$ to its tangent spaces.

Hence $\widetilde{AdS^3}$ is a Lorentzian manifold, and it can be checked that its sectional curvature is $-1$.

A tangent vector $v$ to $\widetilde{AdS^3}$ at a point $x$ is called:
\[
\begin{align*}
&\text{space-like } \text{ if } b(v, v) > 0, \\
&\text{time-like } \text{ if } b(v, v) < 0, \\
&\text{light-like } \text{ if } b(v, v) = 0.
\end{align*}
\]
Now let $x, y \in \mathbb{R}^4$. We say that $x \sim y$ if and only if there exists $\lambda \in \mathbb{R}^*$ such that $x = \lambda y$.

**Definition 2.2.** We define the anti-de Sitter space of dimension 3 as follows:

$$\text{AdS}^3 = \hat{\text{AdS}}^3 / \sim$$

endowed with the quotient metric.

It is easy to see that $\hat{\text{AdS}}^3$ is a double cover of $\text{AdS}^3$. The pseudo-Riemannian metric induced on $\hat{\text{AdS}}^3$ goes down to the quotient.

By definition $\text{AdS}^3$ is a subset of the projective space. In order to better visualize it, we look at its intersection with an affine chart and see its image in $\mathbb{R}^3$. Let $\varphi_0 : \mathbb{RP}^3 \setminus \{x_0 = 0\} \to \mathbb{R}^3$ be an affine chart of $\mathbb{RP}^3$ defined by:

$$\varphi_0([x_0, x_1, x_2, x_3]) = \left(\frac{x_1}{x_0}, \frac{x_2}{x_0}, \frac{x_3}{x_0}\right) = (\bar{x}_1, \bar{x}_2, \bar{x}_3).$$

Then $\varphi_0(\text{AdS}^3 \setminus \{x_0 = 0\})$ gives,

$$-x_0^2 - x_1^2 + x_2^2 + x_3^2 < 0 \Rightarrow -1 - \left(\frac{x_1}{x_0}\right)^2 + \left(\frac{x_2}{x_0}\right)^2 + \left(\frac{x_3}{x_0}\right)^2 < 0$$

so in this affine chart $\text{AdS}^3$ fills the domain

$$-\bar{x}_1^2 + \bar{x}_2^2 + \bar{x}_3^2 < 1,$$

which is the interior of a one-sheeted hyperboloid. Notice that $\text{AdS}^3$ is not contained in a single affine chart. In the affine chart $\varphi_0$ we are missing a totally geodesic plane at infinity, corresponding to $\{x_0 = 0\}$.

In all the article, we will denote by $\mathbb{D}$, the disc $\begin{cases} \bar{x}_2^2 + \bar{x}_3^2 < 1 \\ \bar{x}_1 = 0 \end{cases}$ in the affine chart $\varphi_0$ (see Figure 1).

![Figure 1: Image of AdS^3 in the affine chart \(\varphi_0\).](image)

It is clear from the construction that in the affine chart $\varphi_0$, geodesics (resp. totally geodesic planes) are given by the intersection between affine lines (resp. affine planes) in
$\mathbb{R}^3$ with the interior of the one sheeted hyperboloid described above. A plane $P$ is space-like if the restriction of the induced metric on $P$ is positive-definite. A convex space-like surface in anti-de Sitter space is a surface which is convex in an affine chart and which has only space-like planes as support planes. The boundary at infinity of $AdS^3$ is given by

$$\{ [x] \in \mathbb{R}P^3 : b(x, x) = 0 \}/ \sim$$

and we will denote it by $\partial_\infty AdS^3$, (and by $\varphi_0(\partial_\infty AdS^3)$ the boundary in the affine chart $\varphi_0$). We can distinguish the type of geodesics in the image of anti-de Sitter space in the affine chart as follows (see Figure 2):

- A geodesic in $AdS^3$ is space-like if it meets $\partial_\infty AdS^3$ in two different points.
- A geodesic in $AdS^3$ is light-like if it meets $\partial_\infty AdS^3$ in only one point.
- A geodesic in $AdS^3$ is time-like if it is strictly contained in the hyperboloid.

![Figure 2: Geodesics in an affine model of $AdS^3$.](image)

Note that $AdS^3 \cap \{ x \in \mathbb{R}^4 | x_1 = 0 \} =: H_0$ is isometric to the hyperbolic plane. We use this fact to define the following map. Let $\Psi : \mathbb{H}^2 \times \mathbb{R} \rightarrow AdS^3$ be the map defined by $\Psi(x, t) = \exp_x(tV)$ where

- $\Psi(\mathbb{H}^2, 0) = H_0$, and $x \mapsto \Psi(x, 0)$ is an isometry,
- $V$ is a choice of a unit vector field orthogonal to $H_0$, for the anti-de Sitter metric.

Indeed, we have $\Psi(x, t) = \cos(t)x + \sin(t)V$ with $V = (0, -1, 0, 0)$. For a given $x$, $t \mapsto \Psi(x, t)$ is a time-like geodesic loop with time-length $2\pi$. We will call $AdS$ cylinder the cylinder $\mathbb{H}^2 \times [0, \pi/2]$ endowed with the Lorentzian metric $g_{AdS}$, which is the pull back of the anti-de Sitter metric by $\Psi$. Let us denote $AdS^3 \cap \{ x \in \mathbb{R}^4 | x_1 = r \} =: H_r$. The induced metric onto $H_r$ is homothetic to the hyperbolic metric with factor $(1 - r^2)$, and clearly $\Psi(\mathbb{H}^2, t) = H_{\sin(t)}$. In turn,

$$g_{AdS}(x, t) = \cos^2(t)g_{\mathbb{H}^2}(x) - dt^2$$
where $g_{\mathbb{H}^2}$ is the metric on the hyperbolic plane.

It will be suitable to work with the image of $\tilde{\Psi}$ in the affine chart considered above. Let us denote $\psi = \varphi_0 \circ \tilde{\Psi}$. The set $\Psi(\mathbb{H}^2 \times [0, \pi/2])$ is indeed a Euclidean half-cylinder in $\mathbb{R}^3$ (see Figure 3). We have $\Psi(\mathbb{H}^2, 0) = \mathbb{D}$ and for $x \in \mathbb{H}^2$, $t \mapsto \Psi(x, t)$ is a vertical half line from $\mathbb{D}$. We will call affine AdS cylinder the image of $\mathbb{H}^2 \times [0, \pi/2]$ by $\Psi$. For convexity reasons, we will need only to consider a half cylinder.

![Figure 3: The AdS cylinder and a convex surface inside.](image)

**2.2 Convex functions**

For a function $u : \mathbb{H}^2 \to [0, \pi/2]$, we denote

$$S_u = \{(x, u(x)) | x \in \mathbb{H}^2\}.$$  

For every $x \in \mathbb{H}^2$ we denote by $\bar{x} = \Psi(x, 0)$ the corresponding point on the disc $\mathbb{D}$, where $\Psi$ is the map introduced in the previous section. The image of $S_u$ in the affine AdS cylinder is the graph of a function over $\mathbb{D}$, that we will denote by $\bar{u}$. We will denote by $S_{\bar{u}}$ the image of $S_u$. Hence, $\bar{u} : \mathbb{D} \to \mathbb{R}$ and

$$(\bar{x}, \bar{u}(\bar{x})) = \Psi(x, u(x)).$$

For a point $\bar{x} \in \mathbb{D}$ we use the notation $\bar{x} = (\bar{x}_2, \bar{x}_3)$ for its Euclidean coordinates, and its Euclidean norm is $\|\bar{x}\| = \sqrt{\bar{x}_2^2 + \bar{x}_3^2}$. By the considerations of the preceding section, we immediately obtain the following relation.

**Lemma 2.3.** With the notations above $\bar{u}(\bar{x}) = -\tan(u(x))\sqrt{1 - \|\bar{x}\|^2}$.

**Definition 2.4.** Let $u : \mathbb{H}^2 \to \mathbb{R}$ be a function. We say that $u$ is C-convex if

- $u \geq 0$ and there is $R < \pi/2$ such that $u \leq R < \pi/2$;
- the corresponding function $\bar{u}$ is convex.

It is worth noting that for $R \geq 0$, if $u = R$, then the graph of the map defined by $\bar{u}(\bar{x}) = -\tan(R)\sqrt{1 - \|\bar{x}\|^2}$ is a half ellipsoid. Also, $|\bar{u}(\bar{x})| \leq \tan(R)\sqrt{1 - \|\bar{x}\|^2}$. It follows that if $u$ is C-convex, then $\bar{u}$ is bounded and satisfies $\bar{u}|_{\partial \mathbb{D}} = 0$. 

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It is also clear that a bounded convex function \( \bar{u} : D \to \mathbb{R} \) vanishes everywhere if it vanishes in a point of the open disc \( D \). So we have \( \bar{u} \leq 0 \) by definition, and \( \bar{u} < 0 \) or \( \bar{u} = 0 \).

Let us note the following.

**Lemma 2.5.** In the image of \( AdS^3 \) by \( \varphi_0 \),

1. Every time-like line passes through the disc \( D \).
2. Every light-like line which doesn’t pass through the boundary \( \partial_\infty D \) must pass through the disc \( D \).
3. A cone with basis the disc \( D \) and with apex in the affine cylinder is a convex space-like surface.

**Proof.** The proofs of the two first points are almost immediate. For the third point, either a support plane of the cone does not meet the closure of \( D \), hence it is space-like, or a support plane of the cone contains a half-line of the cone, then it meets the boundary of the disc, but by assumption this half-line is not vertical, hence not light-like, so the plane is space-like. \( \square \)

**Lemma 2.6.** Let \( u : \mathbb{H}^2 \to [0, \pi/2] \) be a C-convex function. Then the surface \( S_u \) is space-like.

**Proof.** Let \( p \) be a point on the image of \( S_u \) in the affine half cylinder, and let \( C_p \) be the cone with basis the disc \( D \) and apex \( p \). By definition, this cone is contained in the affine half cylinder. By convexity, a support plane to the surface at \( p \) is a support plane of the cone, so by Lemma 2.5 it must be space-like. \( \square \)

We say that a sequence \( (u_n)_n \) of C-convex functions is uniformly bounded if there is \( R < \pi/2 \) such that for any \( n \), \( u_n < R \).

**Lemma 2.7.** Let \( (u_n)_n \) be a sequence of uniformly bounded C-convex functions. Up to extracting a subsequence, \( (u_n)_n \) converges to a C-convex function \( u \), uniformly on compact sets.

**Proof.** This is a classical property of the corresponding convex functions \( \bar{u}_n \), [Roc97, Theorem 10.9], in the special case when the surfaces vanish on the boundary of the disc \( D \). \( \square \)

Let \( u_n, \ n > 1 \), be uniformly bounded C-convex functions converging to a C-convex function \( u = u_0 \). Let \( c : I \to \mathbb{H}^2 \) be a Lipschitz curve and \( \bar{c} : I \to \mathbb{D} \) be its image by \( \Psi \). Then \( \bar{u} \circ \bar{c} \), \( \bar{u}_n \circ \bar{c} \) are Lipschitz —the Lipschitz nature of \( \bar{c} \) is independent of a choice of a Riemannian metric on the disc. By Rademacher Theorem, there exists a set \( I_0 \) of Lebesgue measure 0 in \( I \) such that for all \( n \in \mathbb{N} \), \( \bar{u}_n \) is differentiable on \( I \setminus I_0 \).

**Lemma 2.8.** Let \( u_n : \mathbb{H}^2 \to \mathbb{R} \) be uniformly bounded C-convex functions converging to a C-convex function \( u \), and let \( c : I \to \mathbb{H}^2 \) be a Lipschitz curve. Up to extracting a subsequence, for almost all \( t \),

\[
(u_n \circ c)'(t) \to (u \circ c)'(t).
\]

**Proof.** The following proof is a straightforward adaptation of [FS19, Lemma 3.6]. We first prove the Lemma for the corresponding functions \( \bar{u}_n \) and \( \bar{u} \), then we deduce the proof for \( u_n \) and \( u \) using continuity and Lemma 2.3. We consider that \( \bar{c} \) is parameterized by arc-length.

Let \( \langle \cdot , \cdot \rangle \) be the the Euclidean metric on the affine cylinder, and we use the notation \( (a, b) \), with \( a \in \mathbb{D} \) and \( b \in \mathbb{R} \). Let \( t \) be such that the derivatives exist. Let \( X \) be the unit
vector \((0\ 1)\) and \(Y\) the unit vector \(\left(\hat{c}(t)\ \hat{0}\right)\), we have \(\langle X, Y \rangle = 0\). The tangent vector to the
curve \(\left(\hat{c} \ u_n \circ \hat{c}\right)\) at every point \(\left(\hat{u}_n \circ \hat{c}\right)(t)\) is given by
\[
V_n = (\hat{u}_n \circ \hat{c})(t)^{\top}X + Y
\]
and in the plane \(P\) spanned by \(X\) and \(Y\), the vector
\[
N_n = (\hat{u}_n \circ \hat{c})(t)^{\top}Y - X
\]
is orthogonal to \(V_n\) for \(\langle \cdot,\cdot \rangle\). Now because \(\hat{u}_n\) and \(\hat{u}\) are equi-Lipschitz on any compact
set of \(\mathbb{D}\) [see [Roc97] Theorem 10.6)] then there exists \(k\) such that \(|(\hat{u}_n \circ \hat{c})(t)| \leq k\) for all
\(n \in \mathbb{N}\), then
\[
\|N_n\| \leq \|\langle \hat{u}_n \circ \hat{c}\rangle^{\top}(t)\|Y\| + \|X\| \leq \|\langle \hat{u}_n \circ \hat{c}\rangle^{\top}(t)\| + 1 \leq k + 1
\]
so \(\|N_n\|\) are uniformly bounded. Hence, up to extracting a subsequence \((N_n)_n\) converges
to a vector \(N\). Note that \(N\) is not the zero vector, otherwise \(\langle N_n, X \rangle\) would converge to
0, that is impossible because \(\langle N_n, X \rangle = -1\).

Let \(T_n\) be the intersection of the convex surface \(S_{\bar{u}_n}\) defined by \(\bar{u}_n\) and the plane \(P\).
The set \(T_n\) is a convex set in \(P\), and \(V_n\) is a tangent vector, hence by convexity for any
\(\bar{y} \in \mathbb{D} \cap P\),
\[
\langle N_n, \left(\hat{u}_n \circ \hat{c}\right)(0) - \left(\bar{y} / \bar{u}_n(\bar{y})\right) \rangle \geq 0,
\]
and passing to the limit we get
\[
\langle N, \left(\hat{u} \circ \hat{c}\right)(0) - \left(\bar{y} / \bar{u}(\bar{y})\right) \rangle \geq 0,
\]
is says that \(N\) is a normal vector to \(T\) (the intersection of \(S_{\bar{u}}\) with \(P\)), hence
\[
\langle N, (\hat{u} \circ \hat{c})^{\prime}(t) \left(\begin{array}{c} 0 \\ 1 \end{array}\right) \rangle + \left(\hat{c}(t) \ 0\right) = 0.
\]
So there exists \(\lambda\) such that
\[
(\hat{u} \circ \hat{c})^{\prime}(t) \left(\begin{array}{c} 0 \\ 1 \end{array}\right) + \left(\hat{c}(t) \ 0\right) = \lambda \lim_{n \to \infty} (\hat{u}_n \circ \hat{c})^{\prime}(t) \left(\begin{array}{c} 0 \\ 1 \end{array}\right) + \left(\hat{c}(t) \ 0\right).
\]

By identification it follows that \(\lambda = 1\), hence \((\hat{u}_n \circ \hat{c})^{\prime}(t)\) must converge to \((\hat{u} \circ \hat{c})^{\prime}(t)\).
The functions \(\hat{u}_n \circ \hat{c}\) and \(u_n \circ \hat{c}\) are defined from \(I \subset \mathbb{R}\) to \(\mathbb{R}\), by Lemma 2.3
\[
u_n \circ \hat{c}(t) = \arctan \left(\frac{\hat{u}_n \circ \hat{c}(t)}{h(t)}\right)
\]
where \(h(t) = -\sqrt{1 - \|\hat{c}(t)\|^2}\), hence \(u_n \circ \hat{c}\) is clearly differentiable almost everywhere
for all \(n \in \mathbb{N}\) and
\[
(u_n \circ \hat{c})^{\prime}(t) = \frac{(\hat{u}_n \circ \hat{c})^{\prime}(t)h(t) - (\hat{u}_n \circ \hat{c})(t)h^{\prime}(t)}{h^2(t) + (\hat{u}_n \circ \hat{c})^2(t)} \tag{2}
\]
also we have (by hypothesis) for almost all \(t\), that
\[
(u_n \circ \hat{c})(t) \xrightarrow{n \to \infty} (u \circ \hat{c})(t)
\]
hence by continuity (in the relation given by Lemma 2.3) it is clear that,
\[
(\hat{u}_n \circ \hat{c})(t) \xrightarrow{n \to \infty} (\hat{u} \circ \hat{c})(t)
\]
then by the preceding arguments and by continuity again in \(2\) and passing to the
limit, it follows that \((u_n \circ \hat{c})^{\prime}(t)\) converge to \((u \circ \hat{c})^{\prime}(t)\).

Let \(u : \mathbb{H}^2 \to \mathbb{R}\) be a C-convex function. For \(c : [0, 1] \to \mathbb{H}^2\) a Lipschitz curve, \((c, u \circ \hat{c})\)
is a curve on $S_u$, and its length for the anti-de Sitter metric is
\[ L_u(c) = \int_0^1 \sqrt{\cos^2(u \circ c(t))||c'(t)||^2_{H^2} - (u \circ c'(t))^2} \, dt. \] (3)

By Lemma 2.8 above and using the dominated convergence Theorem, we get the following proposition.

**Proposition 2.9.** Let $u_n : \mathbb{H}^2 \to \mathbb{R}$ be uniformly bounded C-convex functions converging to a C-convex function $u$, and let $c : I \to \mathbb{H}^2$ be a Lipschitz curve. Up to extracting a subsequence, $L_{u_n}(c) \to L_u(c)$.

The induced (intrinsic) metric $d_{S_u}$ on $S_u$ is the pseudo-distance induced by $L_u$: for $x, y \in S_u$, $d_{S_u}(x, y)$ is the infimum of the lengths of Lipschitz curves between $x$ and $y$ contained in $S_u$. Note that as the AdS cylinder has a Lorentzian metric, the induced distance between two distinct points on $S_u$ may be equal to 0, that is a major difference with the case of induced metrics on surfaces in a Riemannian space.

**Definition 2.10.** We denote by $d_u$ the pull-back of $d_{S_u}$ on $\mathbb{H}^2$, so that for every point $x, y \in \mathbb{H}^2$
\[ d_u(x, y) = d_{S_u}((x, u(x)), (y, u(y))). \]

From (3), as $\cos \leq 1$, we clearly have the following.

**Lemma 2.11.** With the notations above, for $x, y \in \mathbb{H}^2$, $d_u(x, y) \leq d_{\mathbb{H}^2}(x, y)$.

### 3 Fuchsian invariance

**3.1 Convergence of surfaces implies convergence of metrics**

The aim of this section is to state Proposition 3.9. The arguments are quite general and close to the ones of [FS19]. The main point is Lemma 3.4 below, that is the AdS analogue of Corollary 3.11 in [FS19].

Recall that a Fuchsian group is a discrete group of orientation preserving isometries acting on the hyperbolic plane. In the present article, we will restrict this definition to the groups acting moreover freely and cocompactly.

**Definition 3.1.** A Fuchsian C-convex function is a couple $(u, \Gamma)$, where $u$ is a C-convex function and $\Gamma$ is a Fuchsian group such that for all $\sigma \in \Gamma$ we have $u \circ \sigma = u$.

We will often abuse terminology, speaking about Fuchsian for a single function $u$, so that the Fuchsian group will remain implicit.

**Definition 3.2.** Let $(\Gamma_n)_n$ be a sequence of discrete groups. $(\Gamma_n)_n$ converges to a group $\Gamma$ if there exist isomorphisms $\tau_n : \Gamma \to \Gamma_n$ such that for all $\sigma \in \Gamma$, $\tau_n(\sigma)$ converge to $\sigma$.

**Definition 3.3.** We say that a sequence of Fuchsian C-convex functions $(u_n, \Gamma_n)_n$ converges to a pair $(u, \Gamma)$, if $u$ is a C-convex function, $\Gamma$ is a Fuchsian group such that $(u_n)_n$ converges to $u$ and $(\Gamma_n)_n$ converges to $\Gamma$.

It is easy to see that if $(u_n, \Gamma_n)$ is a sequence of Fuchsian C-convex functions that converges to a pair $(u, \Gamma)$, then $(u, \Gamma)$ is a Fuchsian C-convex function, see e.g. [FS19 Lemma 3.17]. Recall the definition of the distance $d_u$ from Definition 2.10. Recall also that a C-convex function is differentiable almost everywhere. At a point where $u$ is differentiable, we denote by $\| \cdot \|_u$ the norm induced by the ambient anti-de Sitter metric on the tangent of $S_u$ at this point.
Lemma 3.4. Let $u$ be a $C$-convex function. Let $K := \inf(\|v\|/\|v\|_{\mathbb{H}^2})$, and let $d_{\mathbb{H}^2}$ be the distance given by the hyperbolic metric (for instance, $d_{\mathbb{H}^2} = d_u$ for $u = 0$). Then $d_u(x, y) \geq K d_{\mathbb{H}^2}(x, y)$.

Moreover, if $u$ is Fuchsian, then $K > 0$.

Proof. Let $c$ be a Lipschitz curve between two points $x, y \in \mathbb{H}^2$. Let $v$ be the tangent vector field of $(c, u \circ c)$ whenever it exists. We have

$$L_u(c) = \int_a^b \|v\| u \geq K \int_a^b \|v\|_{\mathbb{H}^2} \geq K d_{\mathbb{H}^2}(x, y)$$

and the first result follows as by definition $d_u(x, y)$ is an infimum of lengths.

Now let us suppose that $u$ is Fuchsian. Let us suppose that $K = 0$, i.e. there is a sequence $(x_n)_n$ such that $u$ is differentiable at each $x_n$, and $v_n \neq 0$ in $T_{x_n} \mathbb{H}^2$ such that $\|v_n\|_{\mathbb{H}^2} \rightarrow 0$. Without loss of generality, let us consider that $\|v_n\|_{\mathbb{H}^2} = 1$. Let $\sigma_n$ be isometries of $\mathbb{H}^2$ that send $(x_n, v_n)$ to a given pair $(x, v)$, and let $u_n := u \circ \sigma_n$. As $u$ is Fuchsian, there exists $\beta < \pi/2$ such that $u \leq \beta$, and in turn $u_n \leq \beta$. By Lemma 2.7 up to consider a subsequence, $(u_n)_n$ converges to a $C$-convex function $u_0$. As we supposed that $\|v_n\|_{\mathbb{H}^2} \rightarrow 0$, then $S_{u_0}$ must have a light-like support plane, that contradicts Lemma 2.6.

Note that Lemma 3.4 indicates that in the Fuchsian case, $d_u$ is a distance and not only a pseudo-distance.

Let us recall the following classical result, see e.g. Lemma 3.14 in [FS19]. The homeomorphisms in the statement below could also be constructed by hand, for example using canonical polygons as fundamental domains for the Fuchsian groups, see Section 6.7 in [Bus10].

Lemma 3.5. Let $(\Gamma_n)_n$ be a sequence of Fuchsian groups converging to a group $\Gamma$ and $\tau_n$ the isomorphisms given in Definition 3.2. There exist homeomorphisms $\phi_n : \mathbb{H}^2/\Gamma \rightarrow \mathbb{H}^2/\Gamma$ whose lifts $\tilde{\phi}_n$ satisfy for any $\sigma \in \Gamma$,

$$\tilde{\phi}_n \circ \sigma = \tau_n(\sigma) \circ \tilde{\phi}_n$$

and such that $(\tilde{\phi}_n)_n$ converges to the identity map uniformly on compact sets i.e.

$$\forall x \in \mathbb{H}^2, \tilde{\phi}_n(x) \xrightarrow{n \rightarrow \infty} x$$

Now, let $u$ be a $C$-convex function and $S_u$ the surface described by $u$. The length structure $L_u$ given by (3) induces a (pseudo-)distance $d_{S_u}$. In turn, $d_{S_u}$ induces a length structure denoted by $L_{d_{S_u}}$ defined in the following way: the length of a curve $(c, u \circ c) : [0, 1] \rightarrow S_u$ is defined as

$$L_{d_{S_u}}(c, u \circ c) = \sup_\delta \sum_{i=1}^n d_{S_u}(c(t_i), c(t_{i+1}))$$

$$= \sup_\delta \sum_{i=1}^n d_u(c(t_i), c(t_{i+1})) = L_d(c) \quad \text{(see definition 2.10)}$$

where the sup is taken over all the decompositions

$$\delta = \{(t_1 \ldots t_n) | t_1 = 0 \leq t_2 \leq ... \leq t_n = 1\}$$

We have the following proposition

Proposition 3.6. Let $(u_n)_n$ be a sequence of convex functions such that:

- $d_{u_n}$ is a complete distance with Lipschitz shortest paths,
- $L_{u_n} = L_{d_{u_n}}$ on the set of Lipschitz curves,
- There exists $0 < R < \pi/2$ with $0 \leq u_n < R$,
Then, up to extracting a subsequence, \((u_n)\) converges to a convex function \(u\) and \((d_{u_n})\) converges to \(d_u\) uniformly on compact sets.

**Proof.** The proof of this proposition is similar as the one done in [PS19, Proposition 3.12]. The proof was done using proposition 2.9, the only difference is to use Lemma 2.11 and 3.4 instead of [PS19, corollary 3.11].

We recall that in this paper we are using approximation by smooth surfaces. We note also that by Lemma 3.4 and Lemma 2.11, \(d_{u_n}\) are complete distances on \(\mathbb{H}^2\), also we have \(\mathcal{L}_{u_n} = L_{d_{u_n}}\) (because of smoothness, see [Bur15] for more details), we deduce the following

**Lemma 3.7.** Let \((u_n, \Gamma_n)\) be Fuchsian C-convex functions such that:

- \((u_n, \Gamma_n)\) converges to a pair \((u, \Gamma)\),
- There exist \(0 < R < \pi/2\) with \(0 \leq u_n < R\),
- \(d_{u_n}\) are distances with Lipschitz shortest paths,
- \(d_{u_n}\) converge to \(d_u\), uniformly on compact sets.

Then on any compact set of \(\mathbb{H}^2\), \(d_{u_n}(\tilde{\phi}_n(\cdot), \tilde{\phi}_n(\cdot))\) uniformly converge to \(d_u\), where \(\tilde{\phi}_n\) is given by Lemma 3.7.

**Proof.** By Lemma 3.4 and Lemma 2.11, the topology induced by \(d_u\) onto \(\mathbb{H}^2\) is the topology for the hyperbolic metric. It follows that for the maps \(\tilde{\phi}_n\) of Lemma 3.5, we have that on compact sets, the maps \(x \mapsto d_{u_n}(\tilde{\phi}_n(x), x)\) uniformly converge to 0. By the triangle inequality we have,

\[
d_{u_n}(\tilde{\phi}_n(x), \tilde{\phi}_n(y)) - d_u(x, y) \leq d_{u_n}(\tilde{\phi}_n(x), x) + d_{u_n}(\tilde{\phi}_n(y), y) + d_{u_n}(x, y) - d_u(x, y)
\]

by the preceding arguments and proposition 3.6, for \(n\) sufficiently large the right-hand side is uniformly less than any \(\epsilon > 0\). On the other hand, by triangle inequality again we have,

\[
d_u(x, y) - d_{u_n}(\tilde{\phi}_n(x), \tilde{\phi}_n(y)) = d_u(x, y) - d_{u_n}(x, y) + d_{u_n}(x, y) - d_{u_n}(\tilde{\phi}_n(x), \tilde{\phi}_n(y)) \\
\leq d_u(x, y) - d_{u_n}(x, y) + d_{u_n}(x, \tilde{\phi}_n(x)) + d_{u_n}(y, \tilde{\phi}_n(y)) \\
+ d_{u_n}(\tilde{\phi}_n(x), \tilde{\phi}_n(y)) - d_{u_n}(\tilde{\phi}_n(x), \tilde{\phi}_n(y))
\]

which is uniformly less than any \(\epsilon > 0\) for \(n\) sufficiently large (by the same arguments). \(\square\)

By definition, if \((u, \Gamma)\) is a Fuchsian C-convex function, then \(\Gamma\) acts by isometries on \(d_u\). In turn, \(d_u\) defines a distance on the compact surface \(\mathbb{H}^2/\Gamma\).

**Definition 3.8.** For a Fuchsian C-convex function \((u, \Gamma)\), we denote by \(\tilde{d}_u\) the distance defined by \(d_u\) on \(\mathbb{H}^2/\Gamma\).

The reason to introduce the maps \(\tilde{\phi}_n\) from Lemma 3.5 is the following Corollary of Lemma 3.7. Its proof is formally the same as the one of Proposition 3.19 in [PS19]. (The definition of uniform convergence of metric spaces is recalled in Definition 5.1.)

**Proposition 3.9.** Let \((u_n, \Gamma_n)\) be Fuchsian C-convex functions converging to a pair \((u, \Gamma)\). Up to extracting a subsequence, \((\mathbb{H}^2/\Gamma_n, \tilde{d}_{u_n})\) uniformly converges to \((\mathbb{H}^2/\Gamma, \tilde{d}_u)\).
3.2 Convergence of metrics implies convergence of groups

The aim of this section is to prove Proposition 3.10 that may be seen as a kind of converse of Proposition 3.9. The distance $\bar{d}_u$ was defined in Definition 3.8.

**Proposition 3.10.** Let $(S, d)$ be a metric of curvature $\leq -1$ and let $(u_n, \Gamma_n)$ be smooth Fuchsian $C$-convex functions, such that the sequence $(\mathbb{H}^2/\Gamma_n, \bar{d}_{u_n})_n$ uniformly converges to $(S, d)$. Up to extracting a subsequence,

- $(\Gamma_n)_n$ converges to a Fuchsian group $\Gamma$;
- there exists $0 < \beta < \pi/2$ such that $0 \leq u_n < \beta$.

Under the hypothesis of Proposition 3.10, let’s first prove the convergence of groups. We first have a consequence of simple hyperbolic geometry, see [FST9, Corollary 4.2].

**Lemma 3.11.** There exists $G > 0$ and $N > 0$ such that for any $n > N$, for any $x \in \mathbb{H}^2$, for every element $\sigma_n \in \Gamma_n \setminus \{0\}$

$$d_{u_n}(x, \sigma_n(x)) \geq G.$$ 

**Proposition 3.12.** Under the hypothesis of Proposition 3.10, up to extracting a subsequence, the sequence $(\Gamma_n)_n$ converges to a Fuchsian group $\Gamma$.

**Proof.** First by Lemma 2.11 we have that for all $x, y \in \mathbb{H}^2$,

$$d_{u_n}(x, y) \leq d_{\mathbb{H}^2}(x, y),$$

and by Lemma 3.11, we have that there exists $G > 0$ and $N > 0$ such that for any $n > N$ and for any $x \in \mathbb{H}^2$:

$$G \leq d_{u_n}(x, \sigma_n(x)) \leq d_{\mathbb{H}^2}(x, \sigma_n(x)),$$

in particular if

$$L_{\sigma_n} = \min_{x \in \mathbb{H}^2} d_{\mathbb{H}^2}(x, \sigma_n(x)),$$

we have

$$G \leq L_{\sigma_n}.$$

The length is uniformly bounded from below, hence by a classical result of Mumford [Mum71] we can deduce that up to extracting a subsequence, the sequence of groups converges.

**Lemma 3.13.** Under the assumptions of Proposition 3.10 there exists $M < \pi/2$ such that for all $n$, there is $x_n \in \mathbb{H}^2$ such that $u_n(x_n) < M$.

**Proof.** Suppose that the result is false: for a sequence $M_k \to \pi/2$, there is $n_k$ such that $u_{n_k} \geq M_k$. By the definition of the length structure (3), it follows that $d_{u_{n_k}} \leq \cos M_k d_{\mathbb{H}^2}$. In turn, $(\mathbb{H}^2/\Gamma_n, \bar{d}_{u_n})_n$ has a subsequence converging to 0, that is a contradiction.

**Proposition 3.14.** Under the hypothesis of Proposition 3.10, there exists $0 < \beta < \pi/2$ such that, for any $n \in \mathbb{N}$, for any $x \in \mathbb{H}^2$,

$$u_n(x) < \beta.$$

**Proof.** Let us consider the affine model of anti-de Sitter space. As the sequence of groups converges, there exists a compact set $C \subset \mathbb{D}$, which contains a fundamental domain for $\Gamma_n$ for all $n$. Hence the points $x_n$ given by Lemma 3.13 can be chosen to all belong to $C$. The result follows because the convex maps $\bar{u}_n$ on the disc are zero on the boundary, so for any compact set $C$ in the interior of the disc, the difference between the minimum and the maximum of $\bar{u}_n$ on $C$ cannot be arbitrary large.

Proposition 3.10 is now proved.
4 Proof of Theorem 1.4

The proof relies on the two following results.

Theorem 4.1 ([LS00]). Let \((S, d)\) be a metric induced by a Riemannian metric of sectional curvature \(< -1\). Then there exists a \(C^\infty\) Fuchsian C-convex \(u : \mathbb{H}^2 \to [0, \pi/2]\) such that \(\bar{d}_u\) is isometric to \(d\).

Theorem 4.2. Let \((S, d)\) be a metric of curvature \(\leq -1\). Then there exists a sequence \((S_n, d_n)\) converging uniformly to \((S, d)\), where \(S_n\) are homeomorphic to \(S\) and \(d_n\) are induced by Riemannian metrics with sectional curvature \(< -1\).

Although Theorem 4.2 may seem well-known, we didn’t find any reference for it, so we will prove it in Section 5. Note that we are not aware if the analogue of Theorem 4.2 is unique, up to isometries [BBI01], so \(\lim\) of \(\bar{d}_n\) is isometric to \(d\).

Let \(d\) be a metric of curvature \(\leq -1\) on \(S\). From Theorem 4.2, there exists a sequence \((d_n)_n\) of metrics induced by Riemannian metrics with sectional curvature \(< -1\) on \(S\) that converges uniformly to \(d\). By Theorem 4.1, for each \(n \in \mathbb{N}\) there exists a Fuchsian C-convex pair \((u_n, \Gamma_n)\) such that \(\bar{d}_u\) is isometric to \(d\) and \(u_n\) is smooth. By Proposition 3.10 there is a subsequence of \((\Gamma_n)_n\) converging to a Fuchsian group \(\Gamma\), and \(\beta < \pi/2\) such that \(0 \leq u_n < \beta\).

So Lemma 2.7 and Proposition 3.9 applies: up to extracting a subsequence, there is a function \(u\) such that the induced distance on \(\bar{d}_u\) (the quotient of \(d_u\) by \(\Gamma\)) is the uniform limit of \((\mathbb{H}^2/\Gamma_n, \bar{d}_u)_n\), i.e. the uniform limit of \((S, d_n)\). The limit for uniform convergence is unique, up to isometries [BB01], so \(\bar{d}_u\) is isometric to \(d\). Theorem 1.4 is proved, with \(L\) the quotient of the AdS cylinder of Section 2 by \(\Gamma\).

5 Approximation by smooth metrics

In the following we will use the uniform convergence, so let’s recall its definition.

Definition 5.1. We say that a sequence of metric spaces \((S_n, d_n)_n\) converges uniformly to the metric space \((S, d)\) if there exist homeomorphisms \(f_n : S \to S_n\) such that

\[
\sup_{x,y \in S} |d_n(f_n(x), f_n(y)) - d(x, y)| \xrightarrow{n \to \infty} 0 .
\]

If \(S_n = S\) and \(f_n = id\), then this is the usual definition of uniform convergence of distance functions.

We want to check that a metric of curvature \(\leq -1\) on the closed surface \(S\) can be approximated (in the sense of the uniform convergence) by distances induced by Riemannian metrics with sectional curvature \(< -1\). We will first approximate by hyperbolic metrics with conical singularities of negative curvature. Then we will “smooth” those cone metrics.

5.1 Approximation of metrics by polyhedral metrics

Let \((X, d_0)\) be a metric space such that every pair of points can be joined by a shortest path. A (geodesic) triangle \(\Delta\) of \(X\) consists of three points \(x, y, z \in X\) and shortest paths \([x, y], [y, z]\) and \([z, x]\). A hyperbolic comparison triangle for \(\Delta\) is a geodesic triangle \(\bar{\Delta}\) in the hyperbolic space with vertices \(\tilde{x}, \tilde{y}\) and \(\tilde{z}\), such that \(d_0(x, y) = d_{\mathbb{H}^2}(\tilde{x}, \tilde{y})\), \(d_0(y, z) = d_{\mathbb{H}^2}(\tilde{y}, \tilde{z})\), \(d_0(x, z) = d_{\mathbb{H}^2}(\tilde{x}, \tilde{z})\). The interior angle of \(\bar{\Delta}\) at \(\tilde{x}\) is called the comparison angle at \(x\) of the triangle \(\Delta\).

Definition 5.2. Let \(\gamma, \gamma'\) be two non trivial shortest paths issued from the same point \(x\). Let \(\bar{\gamma}(\gamma(t)x\gamma'(t'))\) be the angle at \(\tilde{x}\) of the comparison triangle \(\bar{\Delta}\) with vertices \(\gamma(t), \tilde{x}\) and
\( \gamma'(t') \) in the hyperbolic plane corresponding to the triangle \( \Delta(\gamma(t)x\gamma'(t')) \) in \( X \). Then the upper angle at \( x \) of \( \gamma \) and \( \gamma' \) is defined by
\[
\limsup_{t,t' \to 0} \angle(\gamma(t)x\gamma'(t')).
\]
(4)

**Definition 5.3.** We say that an intrinsic metric space \((X,d_0)\) is \( \text{CAT}(-1) \) if the upper angle between any couple of sides of every geodesic triangle with distinct vertices is not greater than the angle between the corresponding sides of its comparison triangle in the hyperbolic plane.

Let \( B_{d_0}(x,r) \) be the ball of center \( x \) and radius \( r \) in \((X,d_0)\).

**Definition 5.4.** An intrinsic metric space \((X,d_0)\) has curvature \( \leq -1 \) (in the Alexandrov sense), if for any \( x \) there exists \( r \) such that \( B_{d_0}(x,r) \) endowed with the induced (intrinsic) distance is \( \text{CAT}(-1) \).

Let us remind the notion of bounded integral curvature [AZ67, Chapter I, p. 6]. A simple triangle is a triangle bounding an open set homeomorphic to a disc, consisting of three distinct points (the vertices of the triangle) and three shortest paths joining these points, and which is convex relative to the boundary, i.e. no two points of the boundary of the triangle, can be joined by a curve outside the triangle, which is shorter than a suitable part of the boundary joining the points, (see [AZ67] for more details).

**Definition 5.5.** An intrinsic distance \( d_0 \) on a surface \( S \) is said to be of bounded integral curvature (in short, \( \text{BIC} \)), if \((S,d_0)\) verifies the following property:

For every \( x \in S \) and every neighborhood \( N_x \) of \( x \) homeomorphic to an open disc, for any finite system \( T \) of pairwise non-overlapping simple triangles \( T \) belonging to \( N_x \), the sum of the excesses
\[
\delta_0(T) = \tilde{\alpha}_T + \tilde{\beta}_T + \tilde{\gamma}_T - \pi,
\]
of the triangles \( T \in T \) with upper angles \((\tilde{\alpha}_T,\tilde{\beta}_T,\tilde{\gamma}_T)\) is bounded from above by a number \( C \) depending only on the neighborhood \( N_x \), i.e.
\[
\sum_{T \in T} \delta_0(T) \leq C.
\]

The main tool for our approximation result is the following.

**Theorem 5.6** ([AZ67 Theorem 2 p. 59]). Let \( \epsilon > 0 \). A compact \( \text{BIC} \) surface admits a triangulation by a finite number of arbitrary non overlapping simple triangles of diameter \( < \epsilon \).

We will also need the following result to prove that the sum of the angles in a cone point is not less than \( 2\pi \), it corresponds to Theorem 11 in [AZ67, Chapter II, p. 47].

**Lemma 5.7.** Let \( p \) be a point on a \( \text{BIC} \) surface such that there is at least one shortest arc containing \( p \) in its interior. Then for any decomposition of a neighborhood of \( p \) into sector convex relative to the boundary formed by geodesic rays issued from \( p \) such that the upper angles between the sides of these sectors exist and do not exceed \( \pi \), the total sum of those angles is not less than \( 2\pi \).

To get a triangulation of our surface, we will use some properties of \( \text{BIC} \) surfaces, so let’s consider the following Lemma.

**Lemma 5.8.** A metric of curvature \( \leq -1 \) is a \( \text{BIC} \) surface.
Theorem 5.9. Let \((S, d)\) be a metric of curvature \(\leq -1\) on the closed surface \(S\). Then there exists a sequence \((S_n, d_n)\) converging uniformly to \((S, d)\), where \(S_n\) is homeomorphic to \(S\) and \(d_n\) is the metric induced by a hyperbolic metric with conical singularities of negative curvature on \(S_n\).

The remainder of this section is devoted to the proof of Theorem 5.9. Applying Theorem 5.6 and Lemma 5.8, we obtain a triangulation \(\mathcal{T}_\epsilon\) of our surface in which every simple triangle has diameter \(< \epsilon\). Replace the interiors of the triangles of \(\mathcal{T}_\epsilon\) by the interiors of the hyperbolic comparison triangles. We obtain \((\bar{S}_\epsilon, d_\epsilon)\) which is a hyperbolic metric with conical singularities, corresponding to the vertices of the triangles. By construction, \(\bar{S}_\epsilon\) is endowed with a triangulation \(\mathcal{T}_\epsilon\).

Lemma 5.10. The total angles around the conical singularities of \(d_\epsilon\) are not less than \(2\pi\).

Proof. By a property of the CAT\((-1)\) spaces, we have that every vertex of \(\mathcal{T}_\epsilon\) lies in the interior of some geodesic in \((S, d)\) \([BH99] II.5.12\). Applying Lemma 5.7, we immediately get that the sum of the sector angles \(\alpha_i\) at any vertex \(V\) of the triangulation \(\mathcal{T}_\epsilon\) in \((S, d)\) is not less than \(2\pi\). By definition of the CAT\((-1)\) spaces, we have that the angles \(\alpha_{-1,i}\) of the comparison triangles in the hyperbolic space are not less than the corresponding angles at every vertex \(V\) in the triangulation \(\mathcal{T}_\epsilon\) in \((S, d)\). It follows that

\[2\pi \leq \sum_i \alpha_i \leq \sum_i \alpha_{-1,i}.
\]

We want to prove that the finer the triangulation is, the closer \(d_\epsilon\) is from \(d\) (for the uniform convergence between metric spaces). This relies on a series of lemmas.

Lemma 5.11. Let \(\alpha\) be the angle at a vertex of a triangle \(T\) in a surface of curvature \(\leq -1\), and let \(\alpha_{-1}\) be the corresponding angle in a comparison triangle \(T_{-1}\) in the hyperbolic space then,

\[\alpha_{-1} - \alpha \leq -\text{area}(T_{-1}) - \delta_0(T).
\]

Proof. If \(\beta\) and \(\lambda\) are the other angles of \(T\) and \(\beta_{-1}, \lambda_{-1}\) the corresponding angles in \(T_{-1}\), then we have

\[\alpha_{-1} - \alpha \leq \alpha_{-1} - \alpha + \beta_{-1} - \beta + \lambda_{-1} - \lambda = \delta_0(T_{-1}) - \delta_0(T) = -\text{area}(T_{-1}) - \delta_0(T).
\]

Lemma 5.12. If \(\mathcal{T}\) is a triangulation of a compact surface \((S, d)\) with curvature \(\leq -1\) by non overlapping simple triangles, then

\[\sum_{T \in \mathcal{T}} \delta_0(T) \geq 2\pi \chi(S),
\]

with \(\chi(S)\) the Euler characteristic of \(S\).

Proof. Let \(|T|\) be the number of triangles, \(|E|\) the number of edges and \(|N|\) the number of vertices in our geodesic triangulation. We have \(|E| = \frac{3}{2}|T|\), so that the Euler formula

\[|T| - |E| + |N| = \chi(S)
\]
implies
\[ 2|N| - |T| = 2\chi(S). \] (5)

If we denote by \( \theta_i \) the sum of the angles of the triangles around a vertex, then using (5) it follows that
\[ \sum_{T \in \mathcal{T}} \delta_0(T) = \sum_{i=1}^{N} \theta_i - |T| \pi = \sum_{i=1}^{N} (\theta_i - 2\pi) + 2\pi \chi(S) \]

The proof follows because \( \theta_i - 2\pi \geq 0 \) for all \( i \).

**Lemma 5.13.** Let \( T \) be an isosceles triangle in the hyperbolic space with diameter less than a given \( \epsilon \) and with edges length \( x, x, l \) and \( \theta \) the angle opposite to the edge of length \( l \), then
\[ l \leq \sinh(\epsilon)\theta. \]

**Proof.** By the hyperbolic cosine law,
\[ \cosh(l) = \cosh^2(x) - \sinh^2(x) \cos(\theta), \]
that is equivalent to
\[ 1 + 2\sinh^2\left(\frac{l}{2}\right) = \cosh^2(x) - \sinh^2(x)(1 - 2\sin^2\left(\frac{\theta}{2}\right)), \]
so
\[ \frac{l}{2} \leq \sinh\left(\frac{l}{2}\right) = \sinh(x)\sin\left(\frac{\theta}{2}\right) \leq \sinh(\epsilon)\frac{\theta}{2}. \]

**Lemma 5.14.** Let \( \epsilon > 0 \). Let \( T \) be a simple triangle in \((S, d)\) of diameter \( < \epsilon \) with vertices \( OXY \). Let \( A \) (resp. \( B \)) be on the edge \( OX \) (resp. \( OY \)) and at distance \( a \) (resp. \( b \)) from \( O \). Let \( T_{a,1}^1 \) be a comparison triangle for \( T \) in the hyperbolic space, with vertices \( O'X'Y' \). Let \( A' \) (resp. \( B' \)) be the corresponding point of \( A \) (resp. \( B \)) (i.e. on the edge \( O'X' \) (resp. \( O'Y' \)) and at distance \( a \) (resp. \( b \)) from \( O' \), then
\[ 0 \leq d_{\mathbb{H}^2}(A', B') - d(A, B) \leq -\delta_0(T)\sinh(\epsilon). \]

**Proof.** The first inequality comes from the fact that we are in a CAT\((-1)\) neighborhood ([BH99], page 158). Let \( T_{a,1}^2 \) be the comparison triangle for \( OAB \) in the hyperbolic space drawn such that the edge of length \( a \) is identified with \( O'A' \) (see Figure 4). Let \( B'' \) be the corresponding comparison point for \( B \) in \( T_{a,1}^2 \) (i.e. \( B'' \) satisfies \( d(A, B) = d_{\mathbb{H}^2}(A', B'') \)) and \( d(O, B) = d_{\mathbb{H}^2}(O', B'') \). By the triangle inequality we have
\[ d_{\mathbb{H}^2}(A', B') - d(A, B) = d_{\mathbb{H}^2}(A', B'') - d_{\mathbb{H}^2}(A', B'') \leq d_{\mathbb{H}^2}(B', B''). \] (6)

Let \( \theta_1 \) be the angle at \( O' \) of \( T_{a,1}^1 \) (i.e. the angle at \( O' \) of \( O'A'B' \)), and let \( \theta_2 \) be the angle at \( O' \) of \( T_{a,1}^2 \) (i.e. \( O'AB'' \)).

We have \( \theta_1 - \theta_2 \) is the angle at \( O' \) of \( O'B'B'' \) which is isosceles so by inequality (6) and Lemma 5.13 it follows that
\[ d_{\mathbb{H}^2}(A', B') - d(A, B) \leq \sinh(\epsilon)(\theta_1 - \theta_2). \]

If \( \beta \) is the angle of \( T \) at \( O \), then both \( \theta_1 \) and \( \theta_2 \) are angles corresponding to \( \beta \) in the different comparison triangles, so by Lemma 5.11
\[ \theta_1 - \theta_2 = \theta_1 - \beta + \beta - \theta_2 \leq \theta_1 - \beta \leq -\text{area}(T_{a,1}^1) - \delta_0(T) \]
that leads to the result. \qed
Now, let’s describe a homeomorphism between \((S, d)\) and \((\bar{S}_\epsilon, \bar{d}_\epsilon)\) in the following way. The triangle \(T_i\) do not degenerate into segments, since the sum of every two sides is greater than the third. Therefore, the triangles \(T_i\) can be mapped homeomorphically onto the corresponding triangles \(\bar{T}_i\), such that the vertices are sent to vertices and the homeomorphism restricts to an isometry along the edges. We consider any homeomorphism from the interior of the triangles that extend the homeomorphism on the boundary. As the surfaces are triangulated by such triangles, this gives a homeomorphism from \(S\) to \(\bar{S}_\epsilon\).

For two points \(H\) and \(J\) on \(S\), we denote by \(H', J'\) the corresponding points on \(\bar{S}_\epsilon\).

**Fact 5.15.** With the notations above, 
\[-2\epsilon \leq \bar{d}_\epsilon(H', J') - d(H, J) \leq 2\epsilon - 2\pi \chi(S) \sinh(\epsilon).

**Proof.** The idea of this proof is the same as \cite{Ale06} Lemma 2, page 263. Let’s prove the first inequality. Let \(H', J' \in \bar{S}_\epsilon\) and \(\gamma'\) a shortest path joining \(H'\) and \(J'\) and \(\gamma\) be a path joining \(H\) and \(J\) such that the intersection with every triangle \(T\) is a shortest path (i.e. each connected piece of \(\gamma'\) meeting a triangle \(T'\) from a point \(A'\) to a point \(B'\) on the boundary of \(T'\) is associated in \(T\) the shortest path joining the corresponding (in the sense of Lemma 5.14) points \(A\) and \(B\)).

Let us denote by \(\gamma'_i, i = 0, \ldots, m + 1\) the decomposition of \(\gamma'\) given by the triangles it crosses, and by \(l(\gamma'_i)\) their lengths.

As \((S, d)\) is \(\text{CAT}(-1)\), the length of a connected component of the intersection of \(\gamma'\) with \(T'\) joining two points of the boundary is greater than the length of the corresponding component of \(\gamma\) in \(T\) \cite{BH99}, page 158. Now, because the diameters are not greater than \(\epsilon\) then \(l(\gamma_0) + l(\gamma_{m+1}) \leq 2\epsilon\) and \(l(\gamma'_0) + l(\gamma'_{m+1}) \leq 2\epsilon\). It follows that
\[d(H, J) \leq \sum_{i=1}^{m} l(\gamma_i) + 2\epsilon \leq \sum_{i=1}^{m} l(\gamma'_i) + 2\epsilon \leq \bar{d}_\epsilon(H', J') + 2\epsilon\]
then
\[-2\epsilon \leq \bar{d}_\epsilon(H', J') - d(H, J)\]

The first inequality is now proved.

Let’s now prove the second inequality. For that, consider a shortest path \(\gamma\) joining \(H\) and \(J\) in \(S\) and \(\gamma'\) be a path in \(\bar{S}_\epsilon\) joining \(H'\) and \(J'\) such that the intersection with every triangle \(T'\) is a shortest path (i.e. each connected piece of \(\gamma\) meeting a triangle \(T\) from a point \(A\) to a point \(B\) on the boundary of \(T\) is associated in \(T'\) the shortest path joining the corresponding (in the sense of Lemma 5.14) points \(A'\) and \(B'\)).

Let us denote by \(\gamma_i, i = 0, \ldots, m + 1\) the decomposition of \(\gamma\) given by the triangles it crosses, and by \(l(\gamma_i)\) their lengths, we find
\[ \bar{d}_\epsilon(H', J') - d(H, J) \leq l(\gamma'_0) + l(\gamma'_{m+1}) + \sum_{i=1}^{m} l(\gamma'_i) - l(\gamma_i). \]

Since \( l(\gamma'_0) \) and \( l(\gamma'_{m+1}) \) are not greater than \( \epsilon \) then \( l(\gamma'_0) + l(\gamma'_{m+1}) \leq 2\epsilon \). By Lemma 5.14 it follows that
\[ \bar{d}_\epsilon(H', J') - d(H, J) \leq 2\epsilon - \sum_{i=1}^{m} \delta_0(T_i) \sinh(\epsilon), \]

But \( \delta_0(T_i) \) are non positive and moreover the triangles \( T \) are relative convex, so \( \gamma \) meets each triangle at most once (because, if the shortest path \( \gamma \) meets the (geodesic) triangle more than once, then there will be two points on the boundary of the triangle joined by a shortest path lying outside of the triangle, that contradicts the fact that the triangles are convex relative to the boundary), so \( -\sum_{i=1}^{m} \delta_0(T_i) \) is less than \( -\sum_{T} \delta_0(T) \) for all the triangles of the triangulation of \( S \), which is less than \( -2\chi(S) \) by Lemma 5.12. The second inequality is now proved. This fact is now proved.

The lemmas above imply the uniform convergence. Theorem 5.9 is now proved.

5.2 Approximation of polyhedral metrics by smooth metrics

Proposition 5.16. Let \( d \) be the metric induced by a hyperbolic metric with conical singularities of negative curvature on the closed surface \( S \). Then there exists a sequence \((S_n, d_n)\) converging uniformly to \((S, d)\), where \( S_n \) is homeomorphic to \( S \), \( d_n \) is metric induced by a Riemannian metric of sectional curvature \( < -1 \).

We use the same method as that in [Slu13, Lemma 3.9], but we choose the cone in anti-de Sitter space (Figure 5), rather than the hyperbolic space \( \mathbb{H}^3 \).

Proof. Let \( p \in S \) be a singular point of the polyhedral hyperbolic metric \( d \). Consider a neighborhood \( U_p \) of \( p \) which doesn’t contain any other singular point of \( d \). As the curvature is supposed to be negative, the neighborhood \( U_p \) equipped with the restriction of the metric \( d \) will be isometric to the neighborhood of a space-like circular cone \( C_p \) in the affine model of the anti-de Sitter space, such that the singularity \( p \) corresponds to the apex of \( C_p \). Consider a sequence of smooth convex functions, whose graphs coincide with the cone \( C_p \) outside a neighborhood of the apex, and converging to \( C_p \) (this is very classical, see e.g. [Slu13, Lemma 3.9]).

Using Gauss formula, one can easily check that the sectional curvature for the induced metric on the smooth approximating surfaces is \( \leq -1 \). We can multiply those metrics by any constant \( \lambda > 1 \) to get the sectional curvature \( < -1 \). As the surfaces differ only on a compact set, and as the approximating sequence is smooth, it follows from [3] that the induced distances are uniformly bi-Lipschitz to the hyperbolic metric. From this and Proposition 2.9 it is classical to deduce that the induced distances converges locally uniformly (hence uniformly in this case), see e.g. the proof of Proposition 3.12 in [FS19].

The proposition follows by applying this procedure simultaneously to all singular points of the metric \( d \).
Let $d$ be any metric of curvature $\leq -1$ on a compact surface $S$. We obtain a sequence $(d_n)_n$ from Theorem 5.9 and for each $d_n$, a sequence $(d_n)_k$ from Proposition 5.16. Theorem 4.2 follows from a diagonal argument.

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