A Heavy Traffic Theory of Matching Queues

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Motivated by emerging applications in online matching platforms and marketplaces, we study a matching queue. Customers and servers that arrive into a matching queue depart as soon as they are matched. It is known that a state-dependent control is needed to ensure the stability of a matching queue. However, analytically studying the steady-state behaviour of a matching queue, in general, is challenging. Therefore, inspired by the heavy-traffic regime in classical queueing theory, we study a matching queue in an asymptotic regime where the state-dependent control decreases to zero. It turns out that there are two different ways the control can be sent to zero, and we model these using two parameters viz., $\epsilon$ is the magnitude scaling parameter that goes to zero and $\tau$ is the time scaling parameter that goes to infinity. Intuitively, $\epsilon$ scales the magnitude of the state-dependent control, and $\tau$ is the threshold after which we modulate the control.

We show that depending on the relative rates of $\epsilon$ and $\tau$, there is a phase transition in the limiting regime. We christen the regime when $\epsilon \tau \to 0$, the time dominates magnitude (TDM) regime, and the limiting behaviour is a Laplace distribution. The phase transition starts in the regime when, $\epsilon \tau \to \infty$, the limiting distribution converges to a uniform distribution and we call this the magnitude dominates time (MDT) regime. These results are established using two related proof techniques. Both the proof techniques generalize the characteristic function method. The first one exploits the underlying structure by engineering complex exponential Lyapunov functions, and the second is a novel inverse Fourier transform method.

**Key words:** Dynamic Pricing, Stationary Distribution, Transform Method, Matching Platforms

1. **Introduction** Since the work of Erlang [23] in the context of telecommunication systems, more than a century ago, queueing theory has emerged as a well-established discipline that has had an impact on a large number of applications including wired and wireless networks, cloud computing, manufacturing systems, transportation systems, etc. The central building block of queueing theory is a single server queue, which has a fixed server, customers that wait until their service, and then depart immediately thereafter. In addition, there is a queue or a waiting space for the customers to wait, and a stochastic model of the arrivals and services. While the single server queue is well-understood when the arrivals and service are memory-less, there is no closed-form expression for the stationary distribution of the queue length for general distributions. Therefore, queueing systems are studied in various asymptotic regimes, including the heavy-traffic regime, where the arrival rate approaches the service rate.

More precisely, suppose $\epsilon$ denotes the difference of the service rate and arrival rate, in heavy-traffic one studies the queue in the limit when $\epsilon \downarrow 0$. When $\epsilon = 0$, the queue becomes unstable (null-recurrent). However, it is known [46] that the limiting distribution of the queue length multiplied by $\epsilon$ is an exponential distribution. Moreover, the mean of the exponential depends only on the variance of the inter-arrival and service distributions, but not on the whole distribution. Recent
work [24, 43, 28] has also characterized the rate of convergence to the exponential, thus enabling us to approximate the stationary queue length when $\epsilon$ is not small.

Recent developments in online platforms and matching markets such as ride-hailing, food delivery services, etc. have led to an interest in the study of matching queues, also referred as two-sided queues. In this paper, we will use the two terms interchangeably. In a matching queue, both servers and customers arrive, wait until they are matched, and then immediately depart the system. The behaviour of matching queues is different from that of classical queues. In particular, a matching queue is never stable without external control. To see this, note that if the arrival rates on the two sides don't match, the system is clearly unstable. But when the rates match, it is analogous to a symmetric random walk on a line, which is null-recurrent [74]. Therefore, matching queues have to be always studied under an external control that modulates the arrival rates in a state-dependent manner with levers such as prices in online platforms. In contrast to a large amount of literature on classical queues, there is comparatively very little work on matching queues.

The goal of this paper is to develop a heavy-traffic theory of matching queues, that will enable us to completely characterize the queueing behaviour in an appropriately defined asymptotic regime. Analogous to classical queues, except in special cases, it is hard to obtain the exact stationary distribution of queue length in a matching queue. In fact, further difficulties arise exclusively in the case of matching queues. We point out two of them here. First, as discussed in the previous paragraph, a matching queue is naturally unstable. Due to this difficulty, most of the previous work involving matching queues either characterizes the transient behaviour [31, 38], or considers a simplified model [18, 2]. We overcome this challenge by considering state dependent arrivals which results in a stable system. Secondly, there is no natural notion of ‘heavy-traffic’ here. To quote from Gurvich and Ward [31], “As there are no processing resources, there is no obvious notion of heavy traffic”. We overcome this challenge as follows. Suppose that the uncontrolled arrivals have an equal rate on both sides. Now, we study the system in the regime when the state-dependent control goes to zero. As mentioned before, in the regime when the control is zero, the system is null-recurrent, and the regime is reminiscent of the heavy-traffic regime of a single-server queue.

To further motivate the aforementioned asymptotic regime, we consider the revenue management viewpoint. A popular class of pricing policies considered in the literature (see e.g. [45, 70, 73, 8, 7]) are either static or its perturbed version. These pricing policies are of interest as they are shown to be near-optimal. A perturbed pricing policy corresponds to a small perturbation of the state-independent control and is equivalent to the pricing control studied in this paper. Moreover, as the perturbation converges to zero, it naturally leads to heavy traffic and they are studied in [70, 73]. Thus, our analysis is of relevance to this line of work and can be possibly extended to other models considered in the literature.

In matching queues, there are two ways in which the state-dependent control can be sent to zero. The first is clearly to scale the magnitude of the control by $\epsilon$ which is sent to zero. The second is to apply the control only at larger and larger values of queues, which we control using a parameter $\tau$ that is sent to $\infty$. These parameters are more precisely defined in (2). It turns out that the relative speed at which these two parameters go to their asymptote plays an important role in the limiting behaviour of the matching queue. In particular, the primary contribution of this paper is to demonstrate that there is a phase transition in the limiting behaviour of the matching queue depending upon the limiting value of $\epsilon \tau$. We christen the regime when $\epsilon \tau \downarrow 0$ the time dominates magnitude (TDM) regime and the regime when $\epsilon \tau \uparrow \infty$ the magnitude dominates time (MDT) regime for reasons outlined in Section 2. The phase transition happens in the intermediary regime when $\epsilon \tau$ goes to a nonzero real number, which we call the hybrid regime.

1.1. Main Contributions The main contributions of this paper are the following.
To illustrate the phase-transition behaviour, we first, consider the simplest matching queue with the simplest possible control, viz., the one with Bernoulli arrivals that is controlled with a two-price policy. Analogous to an $M/M/1$ (more precisely, a $Geo/Geo/1$) queue, one can obtain an exact stationary distribution here. By explicitly taking the asymptotic limit of this distribution, we show that the appropriately scaled queue length converges to a Laplace distribution in TDM regime, a Uniform distribution in the MDT regime and a hybrid of the above two in the hybrid regime. This is presented in Section 3.

The motivation to study the asymptotic behaviour is, of course, its utility when we can't find the exact stationary distribution. Thus, in Section 4, we study a general matching queue with general arrival distributions and pricing policy. We show that the behaviour in TDM and MDT regime continues to be Laplace and Uniform distribution respectively, but the behaviour in the hybrid regime generalizes to a Gibbs distribution. The results are summarized in Table 1.

The third contribution of the paper is methodological. We present two different generalizations of the characteristic function method \[43\] to study the three regimes. In particular, directly applying the characteristic function method results in an implicit equation as opposed to a closed form expression for the characteristic function.

— To study the hybrid regime, we tackle the implicit equation by using the theory of inverse Fourier transforms. In particular, inverse Fourier transforms allows us to solve the implicit equation and prove uniqueness of the solution.

— To establish the result in the other two regime, we circumvent the implicit equation by exploiting symmetry in the underlying process. In particular, we engineer multiple complex exponential Lyapunov functions that together results in a closed form expression of the characteristic function.

The above results show that the heavy-traffic behaviour in matching queues is much richer than that of a classical single-server queue. This is due to the state-dependent control in a matching queue, while classical heavy-traffic theory focuses on the case when the arrival rates in a single server queue are fixed. In Section 5 we show that even the classical single server queue exhibits a phase-transition behaviour if the arrival rates of customers are modulated in a state-dependent manner. We obtain these results by using the generalizations of the characteristic function method that we develop which shows the generality of the technique.

1.2. Literature Review

In this section, we present several lines of work that either relates to the applications of our model, or to our proof techniques.

| Bernoulli Arrivals, Two Price Policy (Proposition 1) | Time Dominates | Hybrid Regime | Magnitude Dominates |
|--------------------------------------------------|----------------|---------------|---------------------|
| Magnitude Regime $\epsilon \tau \downarrow 0$ | $\epsilon \tau \rightarrow (0, \infty)$ | $\epsilon \tau \uparrow \infty$ |
| General Arrival and Pricing Theorem 2 | Laplace | Gibbs Distribution Theorem 1 | Uniform Theorem 3 |
| Proof Technique. Complex exponential Lyapunov functions | (i) Exploiting symmetry around zero (ii) Auxiliary Lyapunov functions | Inverse Fourier transforms | (i) Exploiting bounded support (ii) Auxiliary Lyapunov functions |
1.2.1. Applications of Matching Queues  In this section, we first present literature specifically on matching queues and position our paper. Then, we present a brief overview of the vast literature on pricing and revenue management.

There has been a surge of interest in matching queues [31] in recent years with applications that includes ride-hailing and online markets [7, 9, 44, 73, 70], kidney exchange [4, 3], matching markets [38], etc. In addition, there are several models in the literature that are closely related to two sided queues like dynamic matching models [2, 18, 1, 17, 76], assemble-to-order systems [31, 57, 58, 22, 62, 64, 63, 65], and several similar formulations of matching queues [38, 53, 5, 54, 55, 13], etc. Most of the outlined work focuses on transient analysis of a matching queue as it is inherently unstable without an external control. Steady state analysis under external control has been done in [53, 13, 70, 73] but the focus is on finding upper and lower bounds on the mean delay and profit. In addition, [20] proves a product form distribution of the delay for a special case of matching queues with reneging under FCFS. On the other hand, our focus is on pricing, or equivalently state dependent arrivals as the external control. To the best of our knowledge, we are the first ones to provide the limiting distribution of a matching queue under state dependent arrivals.

As our analysis can be adopted to analyze pricing policies in the context of online marketplaces, we provide a brief literature survey on revenue management. The book [67] provides a thorough summary of the classical results in revenue management. Pricing in the context of queueing was studied in [50, 21, 56, 47] and several structural results of the optimal pricing policy were proved. Recently, [45] considers pricing in a single server queue and presented near-optimal pricing policies. Pricing in ride-hailing systems was considered by [10, 12, 19, 29, 16] in a static/non-queueing setting and insights into surge pricing were outlined. Furthermore, [7, 9, 44] studies the ride-hailing system as a closed queueing network and proposed near-optimal static and dynamic pricing policies. Other controls, like matching [9], relocation [15, 37], and joint pricing and matching [55, 54, 72, 73] are also analyzed in the literature. Most of the past work on pricing focused on either structural properties of the optimal policy or characterizing a near-optimal pricing policy with objectives like throughput, revenue, or social benefits. Such an analysis generally involves characterizing mean delay and bounding the tail delay. On the other hand, our focus is to characterize the complete distribution of the delay for a broad class of pricing policies.

1.2.2. Technical Novelty  In this section, we first discuss phase transition and the queueing models that exhibits phase transition. Next, we discuss the proof techniques to analyze queueing systems and position our methodological contributions in the literature.

Phase transition is of course a widespread phenomenon in many systems in general and a large class of queueing systems. One example is the behaviour of many server queues in heavy traffic, where the famous Halfin-Whitt phase transition was presented in [33]. Load balancing systems also are known to exhibit phase transitions in the many servers heavy traffic regime [49, 42], even though the behaviour is not yet completely characterized. In the case of a single server queue with state independent control, we observe a trivial phase transition. In particular, the queue is stable when under-loaded and unstable when over-loaded. Now, consider a single server queue with state dependent control which can be achieved by either considering abandonment, or pricing. A single server queue with abandonment is stable even when it is over-loaded. Such a system is known to exhibit a phase transition in the limiting distribution of queue length, as it moves from under-loaded to over-loaded [75, 39, 36, 48, 6]. We show that phase transition is also observed when pricing is used as a state dependent control. We also show such a behaviour in a matching queue as well. To the best of our knowledge, such a phase transition with pricing or equivalently, state dependent arrivals as the external control hasn’t been observed in the literature.

Single server queue in heavy traffic has been extensively studied in the literature. A popular approach is to use diffusion limits and study the resultant Brownian control problem. This was
first done in literature by Kingman [46]. Even though they analyze the waiting time in a $G/G/1$ queue in continuous time, it is equivalent to the queue lengths in a single server queue in discrete-time. Later, this method was generalized to further analyze settings like heterogeneous customers [34], parallel servers [35, 52], generalized switch [66], generalized Jackson networks [25], etc. More recently, direct methods which work with the original system, as opposed to the diffusion limit have been developed. One of these methods is the drift method introduced in [24] which analyzes the single server queue. This method was further generalized to analyze switch [51], flexible load balancing [78], generalized switch [41], etc. Other methods that directly works with the original system include, transform method [43], basic adjoint relationship (BAR) method [14], and Stein’s method [30]. In addition, a single-server queue with state-dependent control was studied in [45].

In this paper, we adopt the transform method introduced in [43] and used in the context of stochastic processing network comprising of classical queues. This method provides an explicit formula for the limiting moment generating function (MGF) for classical queues. However, in the case of matching queues, we get an implicit equation involving the MGF due to the state dependent control. Solving this implicit equation is a major challenge and we develop the inverse Fourier transform method to address this difficulty in the hybrid regime. In the TDM and MDT regimes, we generalize the characteristic function method by using multiple Lyapunov functions exploiting the underlying symmetry to obtain a closed form expression of the characteristic function. Thus, the aforementioned methods generalizes the transform method and so, it is widely applicable.

1.3. Notation We denote the set of non-negative integers (including 0) by $\mathbb{Z}_+$. The imaginary number $\sqrt{-1}$ is denoted by $j$. For a real number $x \in \mathbb{R}$, we denote its positive part by $[x]^+ \triangleq \max\{x, 0\}$ and negative part by $[x]^− \triangleq \max\{-x, 0\}$. In addition, we denote the smallest integer greater than or equal to $x$ by $[x]$ and the largest integer smaller than or equal to $x$ by $\lfloor x \rfloor$. For a set $A \subseteq \mathbb{R}$, we denote its indicator function $\mathbb{1}_{x \in A}$ by $\mathbb{1}\{A\}$. In addition, we define the sign function by $\text{sgn}(x)$ which is equal to 1 for all $x \geq 0$ and -1 otherwise. Fourier transform of a function $\varphi$ is either denoted by $\mathcal{F}(\varphi)$ or $\hat{\varphi}$ and inverse Fourier transform is denoted by either $\mathcal{F}^{-1}(\varphi)$ or $\check{\varphi}$. We denote the variable in Fourier transform by $\omega \in \mathbb{R}$. Note that $\omega$ does not represent an element in a probability space. Any function $f, g$ of $\epsilon$ such that $\lim_{\epsilon \to 0} \frac{|f(\epsilon)|}{|g(\epsilon)|} = 0$ is denoted by $f(\epsilon) = o(g(\epsilon))$. We denote the space of infinitely differentiable functions by $C_{pol}^{\infty}$, where pol stands for “polynomial like”. A sequence of random variable $\{X_n\}$ converging in distribution to $X$ is denoted by $X_n \overset{D}{\to} X$.

2. Model We consider a matching queue operating in discrete-time with customers and servers both arriving in the system. At a given time epoch $k$, let $q^c(k)$ and $q^s(k)$ be the number of customers and servers waiting in the queue respectively. A waiting customer is matched to a server (and vice versa) as soon as possible, and the pair instantaneously departs from the system. Therefore, both servers and customers cannot be waiting at the same time, and so the main quantity of interest is the imbalance in the queue defined by $z(k) \triangleq q^c(k) − q^s(k)$. Thus, for any $k \in \mathbb{Z}_+$, we have $q^c(k)q^s(k) = 0$ with probability 1. Thus, it suffices to consider the imbalance $z(k)$ as the state descriptor for the queue as $q^c(k) = [z(k)]^−$ and $q^s(k) = [z(k)]^+$. Consider a matching queue with customers and servers both arriving in the system with exogenous arrival rates $\lambda^*$ and $\mu^*$ respectively. We take $\lambda^* = \mu^*$ to balance the arrival rates, otherwise one of the queue lengths will go to infinity. Unfortunately, $\lambda^* = \mu^*$ is not a sufficient condition for stability as the system will be null recurrent in this case (refer to [74, Section III B] for more detailed explanation). Thus, we need additional external control to stabilize the system. In general, we consider state-dependent arrival rates. It is often the case with matching platforms that the system operator can use pricing to influence the arrival rate of customers and servers. For example, the system operator can increase the customer price to reduce its arrival rate as fewer customers
would be willing to accept the higher price and similarly, increase the server price to increase its arrival rate as more servers would be willing to serve for a higher price offered.

Given the imbalance \(z\), let \(\tilde{\phi}^c(z)\) and \(\tilde{\phi}^s(z)\) be the state-dependent control applied to the arrival rate of the customers and servers respectively. In particular, the effective arrival rate of the customers and servers are \(\lambda^* + \tilde{\phi}^c(z)\) and \(\mu^* + \tilde{\phi}^s(z)\) respectively. We will refer to this additional control as a pricing policy and we are interested in analyzing the imbalance given a pricing policy. In particular, \(\tilde{\phi}^c(\cdot)\) and \(\tilde{\phi}^s(\cdot)\) captures the effect of price on the arrival rate. After defining the arrival rates, we will now define the arrival process. Given a time epoch \(k\) and the imbalance of the queue \(z(k) = z\), the customer and server arrivals are denoted by random variables \(a^c(z,k)\) and \(a^s(z,k)\) respectively with \(\mathbb{E}[a^c(z,k)] = \lambda^* + \tilde{\phi}^c(z)\) and \(\mathbb{E}[a^s(z,k)] = \mu^* + \tilde{\phi}^s(z)\) for all \(z \in \mathbb{Z}\) and \(k \in \mathbb{Z}_+\). Moreover, the variances are a function of the mean and is denoted by \(\mathbb{E}[a^c(z,k)] \leq \lambda^* + \tilde{\phi}^c(z)\) and \(\mathbb{E}[a^s(z,k)] \leq \mu^* + \tilde{\phi}^s(z)\) for some continuous, bounded functions \(|\sigma^c(\cdot)| \leq \sigma_{\text{max}}\) and \(|\sigma^s(\cdot)| \leq \sigma_{\text{max}}\).

We assume that there exists an \(A_{\text{max}}\) such that \(|a^c(z,k)| \leq A_{\text{max}}\) and \(|a^s(z,k)| \leq A_{\text{max}}\) with probability 1 for all \(z,k\). In addition, we also assume that the arrivals are independent across time and given the imbalance, the customer and server arrivals are independent of each other.

Now, we are ready to define imbalance as a discrete-time Markov chain (DTMC) denoted as \(\{z(k) : k \in \mathbb{Z}_+\}\). The imbalance evolves as follows: at the start of the time epoch, the system operator observes the imbalance \(z(k)\) and set the customer and server price which leads to customer and server arrivals given by \(a^c(z(k),k)\) and \(a^s(z(k),k)\) respectively. Mathematically, the evolution equation is given by

\[
z(k+1) = z(k) + a^c(z(k),k) - a^s(z(k),k). \tag{1}
\]

We make the following mild assumption on the arrival distributions which ensures that the DTMC \(\{z(k) : k \in \mathbb{Z}_+\}\) is irreducible.

**Assumption 1.** There exists \(p_{\text{min}} > 0\) such that for all \(z \in \mathbb{Z}\) and \(k \in \mathbb{Z}_+\), we have

\[
\mathbb{P}(a^c(z,k) > a^s(z,k)) \geq p_{\text{min}}, \quad \mathbb{P}(a^c(z,k) < a^s(z,k)) \geq p_{\text{min}}.
\]

Intuitively, given a time epoch \(k\) and imbalance \(z\), there is a non-zero probability that the DTMC will transition to a higher or a lower value of imbalance. In addition, without loss of generality (WLOG), we assume that the greatest common divisor (GCD) of the support (except 0) of the random variable \(|a^c(z,k) - a^s(z,k)|\) is 1. It is WLOG as we can scale all the random variables appropriately to ensure the same. This implies that the DTMC governing the imbalance is aperiodic. To summarize, the DTMC governing the imbalance is irreducible (over \(\mathbb{Z}\)) and aperiodic. Therefore, given the pricing policy, if the DTMC is positive recurrent, there exists a unique stationary distribution and we say that the DTMC is stable. We denote the imbalance in steady state with a bar on top, i.e. \(\bar{z}\).

Ideally, one would like to analytically obtain the exact distribution of the imbalance in the steady state. However, this is not possible in general, and so we study the matching queue in an asymptotic regime. We are interested in the performance of pricing policies such that the external control vanishes, analogous to the heavy traffic regime in a single server queue as explained in the introduction. In particular, we consider a sequence of pricing policies parametrized by \(\eta\) and restrict ourselves to the following family of policies characterized by two parameters, \(\epsilon_\eta > 0\) and \(\tau_\eta > 0\).

\[
\lambda_\eta(z) = \lambda^* + \epsilon_\eta \tilde{\phi}^c \left(\frac{z}{\tau_\eta}\right), \quad \mu_\eta(z) = \mu^* + \epsilon_\eta \tilde{\phi}^s \left(\frac{z}{\tau_\eta}\right) \quad \forall z \in \mathbb{Z}, \ \forall \eta > 0. \tag{2}
\]

Here, \(\tilde{\phi}^c(\cdot)\) and \(\tilde{\phi}^s(\cdot)\) are fixed bounded functions. In particular, there exists a \(\phi_{\text{max}} > 0\) such that \(\tilde{\phi}^c(x) \leq \phi_{\text{max}}\) and \(\tilde{\phi}^s(x) \leq \phi_{\text{max}}\) for all \(x \in \mathbb{R}\). We christen \(\tilde{\phi}^c(\cdot)\) and \(\tilde{\phi}^s(\cdot)\) as the control curves.
This class of state-dependent controls are not only general, but are shown in the literature to have good performance in terms of delay and profit. Such a class of controls was first introduced in [45] in the context of a classical single server queue. They presented a near-optimal static, two-price, and a dynamic pricing policies that are of the form (2). This form of state-dependent control was further shown to be near-optimal in the context of two-sided queues in [72, 70].

The parameter $\epsilon_\eta$ modulates the magnitude of the control, and so we call it the magnitude scaling parameter. By picking it such that $\lim_{\eta \to \infty} \epsilon_\eta = 0$, we let the control vanish. The influence of the parameter $\tau_\eta$ is more subtle. It lets us tune the scale of the imbalance $z$ at which we apply the control. In other words, by doubling $\tau_\eta$, we apply the same control only when the imbalance is doubled. With a larger $\tau$, the rate of change of $\bar{z}_\eta/\tau$ decreases in time and so, we call it the time scaling parameter. If we let $\lim_{\eta \to \infty} \tau_\eta = \infty$, we end up applying no state-dependent control, and so this is equivalent to removing the control. Thus, we will study the matching queue when $\lim_{\eta \to \infty} \epsilon_\eta = 0$ and/or $\lim_{\eta \to \infty} \tau_\eta = \infty$. The parameter $\epsilon_\eta$ is similar to the heavy-traffic parameter in a classical single server queue. The parameter $\tau_\eta$ is new in this context and it appears because we use state-dependent control.

Whenever the DTMC $\{z_\eta(k): k \in \mathbb{Z}_+\}$ is positive recurrent, let $\bar{z}$ denote a random variable with distribution same as its stationary distribution. In the asymptotic regime when the control goes to zero, the imbalance $\bar{z}_\eta$ also blows up because we know that the system is null recurrent when there is no external control. Therefore, we need to scale it by the rate at which it blows up in order to study its limiting behaviour. A striking feature of the matching queue is that this rate as well as the limiting behaviour crucially depend on the rate at which $\epsilon_\eta$ and $\tau_\eta$ converges to 0 and $\infty$ respectively. In particular, we define $l \triangleq \lim_{\eta \to \infty} \epsilon_\eta \tau_\eta$ and consider three cases. When $l = 0$, we will see that $\bar{z}_\eta = \Theta(1/\epsilon_\eta)$ and so we will study the limiting behaviour of $\epsilon_\eta \bar{z}_\eta$ as $\eta \uparrow \infty$. We call this the time dominates magnitude (TDM) regime as the time scaling parameter $(1/\tau_\eta)$ dominates the magnitude scaling parameter $(\epsilon_\eta)$ when $\epsilon_\eta \tau_\eta \downarrow 0$. When $l = \infty$, we will see that $\bar{z}_\eta = \Theta(\tau_\eta)$ and so we study the limiting behaviour of $\bar{z}_\eta/\tau_\eta$ as $\eta \uparrow \infty$. Similarly, we call this the magnitude dominates time (MDT) regime as the magnitude scaling parameter $(\epsilon_\eta)$ dominates the time scaling parameter $(1/\tau_\eta)$. Lastly, in the other case when $l \in (0, \infty)$, we will see that $\bar{z}_\eta = \Theta(1/\epsilon_\eta) = \Theta(\tau_\eta)$, and we study the limiting behaviour of $\epsilon_\eta \bar{z}_\eta$ as $\eta \uparrow \infty$, which is same as that of $\bar{z}_\eta/\tau_\eta$ up to the multiplicative factor $l$. We call this the hybrid regime. The objective of this paper is to characterize the limiting distribution of appropriately scaled imbalance $(\epsilon_\eta \bar{z}_\eta$ or $\bar{z}_\eta/\tau_\eta$) in the steady state as $\eta \uparrow \infty$ for any given $\epsilon_\eta$, $\tau_\eta$, $\phi^\epsilon$, and $\phi^\tau$. We will demonstrate a phase transition in the limiting distribution across the three regimes described above. In the next section, we present a simple example to illustrate this behaviour.

In the further sections, all the quantities concerned with the $\eta^{th}$ system is sub-scripted by $\eta$. In addition, for the simplicity of notations, we omit the $\eta$ dependence on $\epsilon$ and $\tau$ everywhere and also omit $\eta$ dependence whenever it is clear from the context.

3. An Illustrative Example: Bernoulli Matching Queue The goal of this section is to illustrate the phase transition phenomenon exhibited by the limiting distribution of the scaled imbalance by considering a simple system. We consider a matching queue operating under the two-price policy and when the arrivals are Bernoulli. First, we define the two-price policy formally.

\textbf{Definition 1.} The two price policy is a special case of (2) with $\phi^\epsilon(x) = -1\{x > 1\}$ and $\phi^\tau(x) = -1\{x < -1\}$. Specifically,

$$
\lambda_\eta(z) = \lambda^* - \epsilon 1\{z > \tau\}, \quad \mu_\eta(z) = \mu^* - \epsilon 1\{z < -\tau\} \quad \forall z \in \mathbb{Z}, \forall \eta > 0.
$$

(3)

In words, when there are too many customers in the system ($z > \tau$), we increase the price for the customers which leads to the reduction in the arrival rate by $\epsilon$. Similarly, when there are too many servers ($z < -\tau$), we decrease the price offered to the servers which leads to an $\epsilon$ reduction in the
arrival rate. Thus, two-price policy is a simple intuitive pricing policy, where $\epsilon$ is the perturbation of the arrival rates when the imbalance is outside a threshold $\tau$. This example also illustrates the need for two different parameters $\epsilon$ and $\tau$, and how they can be separately tuned to make the control vanish. We also present an illustration of the policy in Fig. 1. It has been shown in [70] and [73] that two-price policy is near-optimal in terms of the profit earned by the system operator and delay experienced by the customers and servers.

Consider a matching queue operating under the two-price policy given by Definition 1 such that $\lambda^* < 1$ and $\lambda^* - \epsilon > 0$. The arrivals are Bernoulli i.e. $a_\eta(z,k) = 1$ with probability $\lambda_\eta(z)$ and 0 otherwise. Similarly, $a_\eta(z,k) = 1$ with probability $\mu_\eta(z)$ and 0 otherwise. We will use this example to illustrate that the imbalance exhibits phase transition for $l = 0$ and $l = \infty$. In particular, we show that appropriately scaled imbalance converges to Laplace distribution for $l = 0$, Uniform distribution for $l = \infty$ and a hybrid of Laplace and Uniform distribution for $l \in (0, \infty)$. Mathematically, the CDF of Hybrid$(b,c)$ is as follows:

$$F_{\text{Hybrid}}(x) = \begin{cases} \frac{b}{2(b+c)} e^{\frac{b}{2(b+c)} x} & \text{if } x < -c \\ \frac{1}{2(b+c)} (x + c) + \frac{b}{2(b+c)} & \text{if } x \in [-c, c) \\ 1 - \frac{b}{2(b+c)} e^{-\frac{b}{2(b+c)} x} & \text{if } x \geq c. \end{cases}$$

In essence this distribution is obtained by stitching together the pdfs of a continuous Uniform distribution between $-c$ and $c$ and a Laplace$(0,b)$ distribution. A Laplace distribution with parameters 0 and $b$ is a two-sided exponential distribution with mean $b$ centered at 0. The hybrid distribution essentially flattens the parts between $-c$ and $c$ and as shown in Figure 4. Note that, when $c = 0$, the hybrid distribution is same as Laplace$(0,b)$ and when $b \to 0$, the hybrid distribution is approximately a uniform distribution with support $[-c,c]$, denoted by $\mathcal{U}([-c,c])$. Now, we present the phase transition formally.

**Proposition 1.** Let $\{\epsilon_\eta\}_{\eta > 0}$ and $\{\tau_\eta\}_{\eta > 0}$ be such that $\lim_{\eta \uparrow \infty} \epsilon_\eta \tau_\eta = l \in [0, \infty]$. Consider a matching queue operating under the two-price policy given by Definition 1. In addition, also assume that $a_\eta(z,k) \sim \text{Bernoulli}(\lambda_\eta(z(k)))$ and $a_\eta(z,k) \sim \text{Bernoulli}(\mu_\eta(z(k)))$ for all $z \in \mathbb{Z}, k \in \mathbb{Z}_+, \eta > 0$. Then, as $\eta \uparrow \infty$, we have

1. When $l = 0$, we have $\epsilon_\eta \tilde{z}_\eta \xrightarrow{D} \text{Laplace} \left(0, \frac{\lambda^*(1-\lambda^*) + \mu^*(1-\mu^*)}{2} \right)$

2. When $l \in (0, \infty)$, we have $\epsilon_\eta \tilde{z}_\eta \xrightarrow{D} \text{Hybrid} \left(\frac{\lambda^*(1-\lambda^*) + \mu^*(1-\mu^*)}{2}, l \right)$

3. When $l = \infty$, we have $\tilde{z}_\eta \xrightarrow{D} \mathcal{U}([-1,1])$
The proof of the proposition is presented in Appendix A. The key idea is that the resulting DTMC governing the imbalance is a simple discrete-time birth and death process. This enables us to explicitly evaluate the stationary distribution for every \( \eta \). Taking appropriate limits we get the three cases presented in the proposition.

The above proposition presents a phase transition from a Laplace distribution to a uniform distribution. An illustration of the limiting distribution for various values of \( l \) is presented in Fig. 4. For this, we consider \( \lambda^* = \mu^* = 0.5 \) and plot the limiting distribution of \( \varepsilon \bar{z}_\eta \) and \( \bar{z}_\eta \tau \) for different values of \( l \in [0, \infty) \). When \( l = 0 \), the PDF flattens out for \( \varepsilon \bar{z}_\eta \), which means that the probability mass escapes to infinity. Thus, to get a meaningful limit when \( l = 0 \), we need to scale \( \varepsilon \bar{z}_\eta \) by \( \epsilon \) as \( \epsilon \) decays to zero faster than \( 1/\tau \). It can be seen in the Fig. 2 that the limiting PDF of \( \varepsilon \bar{z}_\eta \) is a Laplace distribution for \( l = 0 \). Similarly, as \( l \) becomes very large, the PDF of \( \varepsilon \bar{z}_\eta \) flattens out to zero which means that the probability mass escapes to infinity. Thus, to get a meaningful limit when \( l = \infty \), we need to scale \( \bar{z}_\eta \tau \) by \( \tau \) as \( 1/\tau \) decays to zero faster than \( \epsilon \). It can be seen in the Fig. 3 that the limiting PDF of \( \bar{z}_\eta \tau \) is nearly a uniform distribution for \( l = 10 \).

Another way to interpret this phase transition is as follows: The limiting distribution of \( \varepsilon \bar{z}_\eta \) in the hybrid regime is \( \text{Hybrid}(\lambda^*(1 - \lambda^*), l) \). Now, if we let \( l \to 0 \), then the Hybrid distribution converges to Laplace distribution which is the limiting distribution of \( \varepsilon \bar{z}_\eta \) in the TDM regime. In addition, in the hybrid regime, the limiting distribution of \( \bar{z}_\eta / \tau \) is \( \text{Hybrid}(\lambda^*(1 - \lambda^*)/l, 1) \). Now, if we let \( l \to \infty \), then the Hybrid distribution converges to Uniform distribution which is the limiting distribution of \( \bar{z}_\eta / \tau \) in the MDT regime. We will later see that such a phase transition holds in more generality.

To intuitively understand the TDM regime, consider the case when \( \tau \) is a constant. Then, \( \epsilon \tau \downarrow 0 \) as \( \epsilon \downarrow 0 \). In this case, \( \epsilon \) acts similar to the heavy traffic parameter in a single server queue. The result we obtain is also analogous. In particular, the limiting distribution of scaled queue length in a single server queue is exponential and we obtain a Laplace distribution for two-sided queues which is a two-sided exponential distribution. This is because imbalance is a signed random variable. The proposition says that the limiting distribution is invariant to the growth rate of \( \tau \) as long as \( l = 0 \).

Next, to intuitively understand the MDT regime, consider \( \epsilon \) to be a constant. Then, \( \epsilon \tau \uparrow \infty \) as \( \tau \uparrow \infty \). Outside the threshold, we have a drift towards zero which is always bounded away from zero. Due to this, the mass of the limiting distribution of imbalance is concentrated between the two thresholds. Inside the threshold, all the states are identical to each other which leads to a uniform distribution between the thresholds. The proposition proves that even when \( \epsilon \downarrow 0 \), we will observe such a distribution as long as \( l = \infty \).
Lastly, we observe a mixed behaviour when \( l \in (0, \infty) \). In particular, (corresponding to \( \epsilon \tilde{z}_\eta \))
when \( l \downarrow 0 \), the parameter \( c \) in the hybrid distribution converges to 0 which makes it the Laplace distribution and when \( l \uparrow \infty \), it converges to \( \infty \) which results in an ill-defined distribution, because it appears to be an infinitely spread uniform distribution. This is because \( \tilde{z}_\eta = \Theta(\tau) \) and so, if the imbalance is scaled by \( \tau \) instead of \( \epsilon \), we obtain a uniform distribution between \((-1, 1)\).

The primary reason for considering the limiting regimes is to understand the stationary behaviour even when we are unable to explicitly find it. Therefore, in the next section, we consider the matching queue under general arrivals and a general pricing policy.

4. Phase Transition: General Arrivals and Pricing Policy

In this section, we will extend the result from the previous section to the general case i.e. given any \( \phi^c \) and \( \phi^s \) and the arrival process governed by an arbitrary distribution. First, we will show that under suitable conditions on the pricing policy, the DTMC is positive recurrent. Then, we first introduce the inverse Fourier transform method by generalizing the characteristic function method \([43]\) to analyze the imbalance given \( \epsilon \tau \rightarrow l \in (0, \infty) \) in the hybrid regime. Next, we observe that as \( l \downarrow 0 \) and \( l \uparrow \infty \), we obtain the result for the TDM and MDT regimes respectively and then introduce a second generalization of the characteristic function method \([43]\) to rigorously prove it.

4.1. Positive Recurrence

To ensure positive recurrence, we need a drift that pushes the imbalance towards zero. In particular, if imbalance is a very large positive value, then there are a lot of customers in the queue and a sensible pricing policy will either reduce the customer arrival rate or increase the server arrival rate. Similarly, if the imbalance is a very large negative value, increasing the customer arrival rate or decreasing the server arrival rate would be sensible. We present the following condition on the control curves which ensures the same.

**Condition 1 (Negative Drift).** There exists \( \delta > 0 \) and \( K > 0 \) such that for all \( x > K \), \( \phi^c(x) - \phi^s(x) < -\delta \) and for all \( x < -K \), \( \phi^c(x) - \phi^s(x) > \delta \).

Now, under this condition, we will show that the underlying DTMC is positive recurrent.

**Proposition 2.** Under Condition 1, for all \( \eta > 0 \), the DTMC \( \{z_\eta(k) : k \in \mathbb{Z}_+\} \) is positive recurrent. Moreover,

\[
-\mathbb{E} \left[ \tilde{z}_\eta \left( \phi^c \left( \frac{\tilde{z}_\eta}{\tau} \right) - \phi^s \left( \frac{\tilde{z}_\eta}{\tau} \right) \right) \right] \leq \frac{2A_{\max}^2}{\epsilon}.
\]

In turn, the above implies that

\[
\mathbb{E} \left[ \|\tilde{z}_\eta\| \right] \leq \frac{2A_{\max}^2 + \epsilon \eta \tau \eta K(2\phi_{\max} + \delta)}{\epsilon \eta \delta}.
\]

The aforementioned bounds will be useful for the analysis later. The details of the proof is deferred to Appendix B and here we mention the idea of the proof. The proof is based on using the Foster-Lyapunov theorem. We analyze the drift of the quadratic test function, \( z^2 \), and show that it is negative outside a finite set, which immediately implies positive recurrence. The bounds on the imbalance follows from a well-known corollary of Foster-Lyapunov theorem \([32, Proposition 6.13]\).

4.2. Main Result: Phase Transition

We first consider the case when \( l \in (0, \infty) \). In this regime, it turns out that the limiting distribution explicitly depends on the control curves \( \phi^c(\cdot) \) and \( \phi^s(\cdot) \). To characterize this distribution, first define

\[
g_{b,c}(x) = \Delta \frac{2b}{\sigma^c(\lambda^*) + \sigma^s(\mu^*)} \left( \phi^s \left( \frac{x}{c} \right) - \phi^c \left( \frac{x}{c} \right) \right).
\]
Intuitively, $g_{b,c}(z)$ characterizes the drift towards zero for imbalance equal to $z$. Then, define the Gibbs distribution $Gibbs(g)$ that corresponds to a function $g(\cdot)$ by the PDF
\[
e^{-f_0^z g(y)dy} \int_{-\infty}^{\infty} e^{-f_0^z g(y)dy} dx.
\] (7)

We need the following technical condition to state the limiting behaviour in this regime.

**CONDITION 2 (Smoothness).** $\phi^c(\cdot)$ and $\phi^s(\cdot)$ belongs to $C_{pol}^\infty$, i.e. they are infinitely differentiable.

Now, we present the main theorem in this regime.

**THEOREM 1 (Hybrid Regime).** Let $\{\epsilon_{\eta}\}_{\eta>0}$ and $\{\tau_{\eta}\}_{\eta>0}$ be such that $\lim_{\eta \uparrow \infty} \epsilon_{\eta} \tau_{\eta} = l$. Consider the positive recurrent DTMC $\{z_{\eta}(k) : k \in \mathbb{Z}_+\}$ for any $\eta > 0$ and for any $\phi^c(\cdot)$ and $\phi^s(\cdot)$ satisfying Condition 1 and let $\tilde{z}_{\eta}$ denote its steady state random variable. If in addition, Condition 2 is satisfied, then as $\eta \uparrow \infty$, we have,
\[
\epsilon_{\eta} \tilde{z}_{\eta} \overset{D}{\rightarrow} Gibbs(g_{1,1}), \quad \frac{\tilde{z}_{\eta}}{\tau_{\eta}} \overset{D}{\rightarrow} Gibbs(g_{1,1}).
\] (8)

In order to understand the Gibbs distribution, consider the special case of two-price policy from Definition 1. Here, we get $g_{1,1}(z) = -1\{z > l\} + 1\{z < -l\}$ and so,
\[
\int_0^\infty g_{1,1}(y)dy = 1\{x > l\}(l - x) + 1\{x < -l\}(l + x).
\]

Therefore, $Gibbs(g)$ as defined in (7) is the hybrid distribution given by (4). While this is consistent with the result in Proposition 1, note that we cannot apply Theorem 1 in this case because the control curves under two price policy have jumps, and so do not satisfy the Condition 2. This suggests that the Condition 2 may not be necessary in Theorem 1, and is possibly an artefact of our proof. While relaxing Condition 2 is future work, note that it is not too restrictive because all polynomials satisfy it. Any continuous control curve can be then approximated by a polynomial arbitrarily well due to the Stone-Weierstrass theorem.

Now, we consider the limiting distribution of $Gibbs(g_{1,1})$ defined above in the Theorem 1 as $l \downarrow 0$. Note that, we have
\[
e^{-\frac{2^2}{\sigma^2(c)}f_0^z (\phi^c(\cdot) - \phi^c(\cdot))} \rightarrow \begin{cases} e^{-x \frac{2^2}{\sigma^2(c)} f_0^z (\phi^c(\cdot) - \phi^c(\cdot))} & \text{if } x > 0 \\ e^{-x \frac{2^2}{\sigma^2(c)} f_0^z (\phi^c(\cdot) - \phi^c(\cdot))} & \text{if } x < 0. \end{cases}
\]

Under the condition $\phi^c(-\infty) - \phi^c(-\infty) = \phi^c(\infty) - \phi^c(\infty)$, the right hand side is the PDF of a Laplace distribution. This illustrates that we should expect the limiting distribution of $\epsilon \tilde{z}_{\eta}$ to be Laplace distribution in the TDM regime. However, note that this is not a formal proof due to unjustified limit interchanges. We will now first state the result formally and then rigorously prove it using a different approach. First, assume that the control curves satisfy the following condition:

**CONDITION 3 (Symmetry).** The limits $\lim_{x \uparrow \infty} \phi^i(x)$ and $\lim_{x \downarrow -\infty} \phi^i(x)$ exists and are denoted by $\phi^i(\infty)$ and $\phi^i(-\infty)$ respectively for $i \in \{c, s\}$. In addition, $\phi^s(\infty) - \phi^c(\infty) = \phi^c(-\infty) - \phi^s(-\infty) = 1$.

Another way to interpret the above condition is as follows: Since we are considering the regime where $\epsilon \tau \downarrow 0$, value of $\phi^c(\cdot)$ and $\phi^s(\cdot)$ at $\pm \infty$ will be crucial as (loosely speaking) $z/\tau$ will escape to infinity. Condition 3 imposes a symmetry condition on the values of $\phi^s(\cdot)$ and $\phi^c(\cdot)$ at $\pm \infty$ so that we can expect $\epsilon \tilde{z}_{\eta}$ to converge to a Laplace distribution which is symmetric around 0. Without loss of generality, we can assume $\phi^s(\infty) - \phi^c(\infty) = 1$ and re-define $\epsilon$ by scaling it with $\phi^s(\infty) - \phi^c(\infty)$. Now, we present the result below which extends the Case 1 of Proposition 1.
The proof is based on the novel inverse Fourier transform method, which we outline here. The complete proof is presented in Section 6.

Sketch of Proof of Theorem 1 The proof is presented in three steps.

Step 1: The first step is to establish the tightness of the family of random variables \( \Pi \overset{\Delta}{=} \{ \epsilon \bar{z}_\eta : \eta > 0 \} \), so that every sequence of random variables in this family has a sub-sequence that converges in distribution. With a slight abuse of notation, let \( \bar{z}_\infty \) denote a random variable with distribution of this limit. Tightness can be easily established using the bound on the imbalance in (5) of Proposition 2.

4.3. Sketch of Proofs

4.3.1. Hybrid Regime: Theorem 1 The proof is based on the novel inverse Fourier transform method, which we outline here. The complete proof is presented in Section 6.

Next, we consider the MDT regime. Note that the limiting distribution of \( \bar{z}_\eta/\tau \) is Gibbs(\( g_{\lambda,1} \)) in the hybrid regime with the temperature of the distribution proportional to \( l \). Now, if we let \( l \uparrow \infty \), then the PDF of Gibbs(\( g_{\lambda,1} \)) will vanish everywhere except where \( \int_0^x (\phi^*(t) - \phi^*(t))dt \) attains its minimum. This is same as soft-min converging to exact minimum in the limit. Thus, we expect the limiting distribution of \( \bar{z}_\eta/\tau \) to be uniform over the set of arguments that minimizes \( \int_0^x (\phi^*(t) - \phi^*(t))dt \).

Due to technical reasons, we consider a class of control curves satisfying the following conditions, which ensures that the minimum is attained over a unique interval.

Condition 4 (Monotonicity). The function \( \phi^*(\cdot) \) is monotonically decreasing and \( \phi^*(\cdot) \) is monotonically increasing. Moreover, \( \Phi^* \overset{\Delta}{=} \{ x : \phi^*(x) - \phi^*(x) = 0 \} = (t_*, t^*) \) for some \( t_* < 0 < t^* \).

Intuitively, as the number of customers increases relative to the servers, the arrival rate of customer defined by \( \phi^*(\cdot) \) should be non-increasing and the arrival rate of server defined by \( \phi^*(\cdot) \) should be non-decreasing. This motivates the monotonicity condition which implies existence of a unique interval for which \( (\phi^*(\cdot) - \phi^*(\cdot)) = 0 \). We also assume that \( \phi^*(0) - \phi^*(0) = 0 \) for the ease of exposition, but it can be relaxed. In addition, we exclude the case when \( t_* = t^* \), i.e. there exists a point as opposed to an interval where \( (\phi^*(\cdot) - \phi^*(\cdot)) = 0 \). We expect the limiting distribution in this case to be Dirac-delta. Such a result requires special treatment and is a part of future work.

Note that, under Condition 4, \( \Phi^* \) is exactly the set that minimizes \( \int_0^x (\phi^*(t) - \phi^*(t))dt \). Now, we present the result below which extends the Case III of Proposition 1.

Theorem 3 (MDT Regime). Let \( \{ \epsilon \eta \}_{\eta > 0} \) and \( \{ \tau \eta \}_{\eta > 0} \) be such that \( \lim_{\eta \uparrow \infty} \epsilon \eta \tau \eta = \infty \). Consider the positive recurrent DTMC \( \{ z^*_k : k \in Z_+ \} \) for any \( \eta > 0 \) and for any \( \phi^*(\cdot) \) and \( \phi^*(\cdot) \) satisfying Condition 1 and let \( \bar{z}_\eta \) denote its steady state random variable. If in addition, Condition 4 is satisfied, then as \( \eta \uparrow \infty \), we have

\[
\bar{z}_\eta \overset{\mathcal{D}}{\rightarrow} \mathcal{U}(\Phi^*).
\]

With the above result, the limiting distribution of imbalance is completely characterized for all the three regimes - TDM, hybrid, and MDT. We highlight the occurrence of phase transition for general class of control curves and arrival distribution. Now, we present the sketch of the proofs in the next section.
Step 2: The key idea in the proof is to use \( e^{j\omega z} \) as the test function for \( \omega \in \mathbb{R} \) and set its drift to zero in steady state. While this step is similar to the transform method in [43], the key challenge is that when we let \( \eta \uparrow \infty \), we do not get an explicit expression for the characteristic function of \( \tilde{z}_\infty \). We instead get that the limit of every convergent sub-sequence satisfies the following implicit equation:

\[
\mathbb{E} \left[ e^{j\omega \bar{z}_\infty} g_{1,l}(\bar{z}_\infty) \right] = j\omega \mathbb{E} \left[ e^{j\omega \bar{z}_\infty} \right]. \tag{9}
\]

Existence of a limit (of the sub-sequence) from Step 1 plays a crucial role in obtaining this equation. Suppose we show that there is a unique distribution that solves this equation, then using standard arguments on convergence, it follows that the family \( \bar{z}_\eta \) also converges to the same distribution. This completes the proof.

Step 3: To complete the proof, the focus of this step is to show that (9) has a solution, which is the Gibbs \((g_{1,l})\) distribution, and more importantly, to show uniqueness. While existence of the solution can be easily verified by plugging the Gibbs \((g_{1,l})\) distribution into (9), we use the following argument, which helps us prove uniqueness. Suppose that \( \bar{z}_\infty \) has a continuously differentiable PDF \( \rho_{\bar{z}_\infty} \), whose Fourier transform exists. Then, (9) can be interpreted as

\[
\mathcal{F} \left( \rho_{\bar{z}_\infty} g_{1,l} + \rho_{\bar{z}_\infty}' \right) = 0,
\]

since the differentiation theorem of Fourier transform gives \( j\omega \mathcal{F}(\rho_{\bar{z}_\infty}) = -\mathcal{F}(\rho_{\bar{z}_\infty}') \). Applying inverse Fourier transform, we get the differential equation,

\[
\rho_{\bar{z}_\infty} g_{1,l} + \rho_{\bar{z}_\infty}' = 0,
\]

solving which we get the Gibbs distribution. However, one cannot assume that \( \bar{z}_\infty \) even exhibits a PDF. We use the theory of inverse Fourier transforms based on tempered distributions to make the above argument formal without assuming even the existence of a PDF. The details of the proof and theory of tempered distribution is provided in Section 4.2.

Transform method was first introduced in [43], and was used to study queues under static arrival rates. Consequently, one directly obtains a closed form expression for the characteristic function of the limiting distribution in [43] which immediately establishes convergence in distribution to an exponential distribution. In contrast, due to the dynamic arrivals, we obtain an implicit equation (9). A major methodological contribution in this section is the introduction of the use of inverse Fourier transform to solve the implicit equation to obtain the limiting distribution. Moreover, due to the implicit equation, we have to separately establish the guarantee that our family converges in distribution. We believe that our proposed method will enable one to use transform techniques in a large class of stochastic network beyond the ones studied in [43].

4.3.2. Time Dominates Magnitude Regime: Theorem 2 We prove Theorem 2 by again building upon the characteristic function method from [43] and set the drift of complex exponential test function to zero. Similar to Theorem 1 in Section 4.2, we obtain an implicit equation due to the state dependent arrivals with \( g_{1,l}(z) = \text{sgn}(z) \). As \( g(x) \) does not satisfy Condition 2, the inverse Fourier transform method is not directly applicable here.

So, we adopt a different technique to resolve this regime. We construct more test functions by exploiting the symmetry in the limiting distribution. In particular, we expect the limiting distribution to be Laplace, which is symmetric around zero. So, we use \( \text{sgn}(z)e^{j\omega |z|} \) as a test function and establish symmetry. Then, we work with \( e^{j\omega |z|} \) as a test function and show that \( |\tilde{z}_\eta| \) converges to an exponential random variable in distribution. Using the already established symmetry of \( \tilde{z}_\eta \), we conclude that \( \tilde{z}_\eta \) has a Laplace distribution in the limit.
Sketch of proof of Theorem 2 Positive recurrence under Condition 1 is already shown in Proposition 2. The key idea in proving the convergence result is the following. It is known from the Levy’s continuity theorem (e.g. see: [77, Chapter 18]) that convergence in distribution is equivalent to convergence of characteristic functions. So, we will focus on finding the characteristic function of the limiting imbalance. The proof consists of two key steps.

Step 1: First, we show that \( |\epsilon \tilde{\eta}| \) converges to an exponential distribution. We do this by setting the drift of the test function, \( e^{j\omega |\tilde{\eta}|} \), to zero in steady state and obtaining a bound on \( E[e^{j\omega |\tilde{\eta}|}] \). Then, taking the limit as \( \eta \to \infty \), we show that \( \lim_{\eta \to \infty} E[e^{j\omega |\tilde{\eta}|}] \) converges to the characteristic function of an exponential distribution. While this step broadly follows the characteristic function method introduced in [43] (but applied to \( |z| \) instead of \( z \)), we overcome several technical challenges that arise due to the state-dependent control in a matching queue. In essence, one needs to work with a sequence of functions as opposed to numbers.

Step 2: In Step 2, we show that \( \tilde{\eta} \) is indeed symmetric around origin, thus completing the proof of the theorem. We first set the drift of the test function, \( \text{sgn}(\tilde{\eta})e^{j\omega |\tilde{\eta}|} \), to zero and show that the expectation, \( E[\text{sgn}(\tilde{\eta})e^{j\omega |\tilde{\eta}|}] \), converges to zero as \( \eta \to \infty \). This establishes symmetry of the limiting \( \tilde{\eta} \).

Step 3: Putting together the results from Step 1 and Step 2, we get that the limiting distribution of \( \epsilon |\tilde{\eta}| \) is two-sided exponential, viz., the Laplace distribution. We do this by considering the following functional identity.

\[
e^{j\omega |x|} + e^{-j\omega |x|} = e^{j\omega x} + e^{-j\omega x} = 2e^{j\omega x} + \text{sgn}(x)e^{-j\omega x\text{sgn}(x)} - \text{sgn}(x)e^{j\omega x\text{sgn}(x)}.
\]

Now taking expectation of these under the distribution \( \tilde{\eta} \) and taking the limit as \( \eta \to \infty \), we get

\[
\lim_{\eta \to \infty} E\left[ e^{j\omega |\tilde{\eta}|} + e^{-j\omega |\tilde{\eta}|} \right] = \lim_{\eta \to \infty} E\left[ 2e^{j\omega \tilde{\eta}} + \text{sgn}(\tilde{\eta})e^{-j\omega \tilde{\eta}\text{sgn}(\tilde{\eta})} - \text{sgn}(\tilde{\eta})e^{j\omega \tilde{\eta}\text{sgn}(\tilde{\eta})} \right].
\]

Both the terms on LHS are characterized from Step 1. We know that the last two terms on the RHS are zero from Step 2. Thus, we have an explicit form for \( \lim_{\eta \to \infty} 2E[e^{j\omega \tilde{\eta}}] \), which we show is the characteristic function of \( \text{Laplace}\left( 0, \frac{\epsilon(\lambda')}{\epsilon'\gamma(\lambda')} \right) \). This completes the proof.

4.3.3. Magnitude Dominates Time Regime: Theorem 3 Similar to the previous section, we prove Theorem 3 by building upon the Characteristic function method presented in [43]. Applying the Transform method as is, does not work for this problem as we get an implicit equation, due to the state-dependent arrival rates. However, unlike in Theorem 1, the implicit equation does NOT have a unique solution. This suggests that working with \( e^{j\omega z} \) alone as a test function is insufficient. So, we construct more test functions.

The key idea is to separately analyze the region where the probability mass vanishes and where it does not. We engineer separate test functions for the two regions and then combine the results to obtain the limiting distribution. In particular, we use \( e^{j\omega z} \{ z \in \Phi^* \} \) as the test function to establish uniform distribution between the thresholds. Then, we use \( z^2 \) as the test function to establish vanishing mass outside the threshold.

Sketch of proof of Theorem 3 Similar to the proof of Theorem 2, we focus on finding the characteristic function of the limiting distribution. The proof consists of two key steps.

Step 1: Loosely speaking, the random variable \( \tilde{\eta}/\tau \) restricted to the set \( \Phi^* \) converges to a uniform distribution. In particular, we show that \( \tilde{\eta}/\tau \{ \tilde{\eta} / \tau \in \Phi^* \} \) converges to a uniform distribution. We do this, by setting the drift of the test function, \( e^{j\omega \tilde{\eta}/\tau} \{ \tilde{\eta} / \tau \in \Phi^* \} \), to zero in steady-state. Then, taking the limit as \( \eta \to \infty \), we show that \( E\left[ e^{j\omega \tilde{\eta}/\tau} \{ \tilde{\eta} / \tau \in \Phi^* \} \right] \) converges to the characteristic function of Uniform distribution weighted by the probability \( P(\tilde{\eta} / \tau \in \Phi^*) \). While this step broadly follows the characteristic function method introduced in [43] (but applied to \( z \{ z \in \Phi^* \} \) instead
of \( z \), we overcome several technical challenges that arise due to the state-dependent control in a matching queue. In essence, one needs to work with a sequence of functions as opposed to numbers.

**Step 2:** In this step, we show that \( \bar{z}_n/\tau \) vanishes outside the set \( \Phi^* \), thus completing the two main pieces of the proof. We do this by first bounding the required probability \( P(\bar{z}_n/\tau \notin \Phi^*) \) by the expectation of the absolute imbalance using the Markov’s inequality. Then, we set the drift of the test function, \( \bar{z}_n^2 \) to zero in steady state to obtain a useful upper bound on \( E[|\bar{z}_n|] \).

**Step 3:** Putting together the results from Step 1 and Step 2, we get that the limiting distribution of \( \bar{z}_n/\tau \) is a uniform distribution over the set \( \Phi^* \). We do this by considering the following functional identity.

\[
e^{jwz} = e^{jwz}1 \{ z \in \Phi^* \} + e^{jwz}1 \{ z \notin \Phi^* \}
\]

Now taking expectation of these under the distribution \( \bar{z}_n \) and taking the limit as \( \eta \to \infty \), we get,

\[
\lim_{|\eta|\to\infty} E \left[ e^{jw\bar{z}_n/\tau} \right] = \lim_{|\eta|\to\infty} E \left[ e^{jw\bar{z}_n/\tau} 1 \{ \bar{z}_n/\tau \in \Phi^* \} \right] + \lim_{|\eta|\to\infty} E \left[ e^{jw\bar{z}_n/\tau} 1 \{ \bar{z}_n/\tau \notin \Phi^* \} \right]
\]

By Step 1, we know that the first term in the RHS converges to the characteristic function of the Uniform distribution and by Step 2, the second term in the RHS vanishes. Thus, we have an explicit form for \( \lim_{|\eta|\to\infty} E \left[ e^{jw\bar{z}_n/\tau} \right] \) which we show is the characteristic function of \( \mathcal{U}(\Phi^*) \). This completes the proof. \( \square \)

**4.3.4. Discussion** In order to analyze the drift of complex exponential test functions, we analyze the drift of several auxiliary test functions in the appendix. The resultant bounds allow us to complete Step 1 and 2 in the proof of Theorem 2 and Theorem 3.

In the proof of Theorem 3, it is noteworthy that the Step 2 which shows bounded support is a form of state space collapse (SSC). In particular, we say that the imbalance stays within the thresholds with a high probability for a finite, large enough \( \eta \). Proving such an SSC is a crucial step in obtaining the complete distribution of imbalance. In addition, showing that the imbalance has a symmetrical distribution in the proof of Theorem 2 is technically not an SSC but reminiscent to it.

**5. Classical Single Server Queue** The heavy-traffic limiting behaviour of a matching queue studied in the previous section exhibits a much richer phase transition behaviour than that of a classical single server queue studied in the literature. This is primarily because most of the literature focuses on a constant arrival rate for a single server queue, whereas we studied a matching queue under state-dependent control. Single server queue with state-dependent control also exhibits phase transition as studied in the previous sections.

Consider a sequence of single server queue \( \{q_\eta(k) : k \in \mathbb{Z}_+\} \) for \( \eta > 0 \) with state-dependent arrival \( a_\eta(q_\eta) \) with expectation given by \( E[a_\eta(q_\eta)] = \lambda_\eta(q_\eta) = \lambda^* + \phi^*(\frac{q_\eta}{\tau}) \epsilon \), and variance given by \( \text{Var}[a_\eta(q_\eta)] = \sigma^2(\lambda_\eta(q_\eta)) \). Similarly, the state-dependent potential service \( s_\eta(q_\eta) \) has expectation \( E[s_\eta(q_\eta)] = \mu_\eta(q_\eta) = \mu^* + \phi^*(\frac{q_\eta}{\tau}) \epsilon \), and variance \( \text{Var}[s_\eta(q_\eta)] = \sigma^2(\mu_\eta(q_\eta)) \). Without loss of generality, we assume that \( \lambda^* = \mu^* \) and \( \phi^*(\infty) = \phi^*(\infty) = 1 \). Next, we assume that there exists \( A_{\max} > 0 \) and \( \sigma_{\max} > 0 \) such that \( |a_\eta(q_\eta)| \leq A_{\max}, |s_\eta(q_\eta)| \leq A_{\max}, \) and \( |\sigma^2(\lambda_\eta(q_\eta))| \leq \sigma_{\max}, |\sigma^2(\mu_\eta(q_\eta))| \leq \sigma_{\max} \) with probability 1. Now, we can write the queue evolution equation as follows:

\[
q_\eta(k+1) = q_\eta(k) + a_\eta(q_\eta(k)) - s_\eta(q_\eta(k)) + u_\eta(q_\eta(k)),
\]

where \( u_\eta(q_\eta(k)) \) is the unused service if there are not enough customers waiting in the queue to be served and a fraction of the potential service \( s_\eta(q_\eta(k)) \) is not utilized. This implies that

\[
q_\eta(k+1)u_\eta(q_\eta(k)) = 0.
\]
Before presenting the general result, to illustrate the phase transition, we consider a single server queue operating in discrete-time under Bernoulli arrivals, Bernoulli service, and a two-price policy. In particular, assume that $\lambda^* - \epsilon > 0$, $\lambda^* < 1$, and the control curves are such that $\lambda_n(q) = \lambda^* - \epsilon\{q > \tau\}$, and $\mu_n(q) = \mu^*$ for all $q \in \mathbb{Z}_+$. Lastly, the arrival and service distribution are given by $a_n(q_n) \sim \text{Bernoulli}(\lambda_n(q_n))$, and $s_n(q_n) \sim \text{Bernoulli}(\mu_n(q_n))$. As the arrival and service distribution is Bernoulli, the single server queue is a birth-death process as shown in Fig. 5 with $m \overset{\Delta}{=} \lambda^*(1 - \mu^*)$. Now, we present the phase transition result below:

**Proposition 3.** Let $\{\varepsilon_\eta\}_{\eta>0}$ and $\{\tau_\eta\}_{\eta>0}$ be such that $\lim_{\eta \uparrow \infty} \varepsilon_\eta \tau_\eta = l$. Consider a single server queue operating under the two-price policy for customer arrival. Then, we have the following results.

1. When $l = 0$, we have
   \[
   \varepsilon_\eta \tilde{q}_\eta \overset{D}{\rightarrow} \text{Exp}\left(\frac{\lambda^*(1 - \lambda^*) + \mu^*(1 - \mu^*)}{2}\right)
   \]

2. When $l \in (0, \infty)$, we have $\varepsilon_\eta \tilde{q}_\eta \overset{D}{\rightarrow} \tilde{q}_\infty$ such that
   \[
   \mathbb{P}(\tilde{q}_\infty \leq q) = \left\{ \begin{array}{ll}
   1 - \frac{m}{l + m} e^{-\frac{q-l}{m}} & \text{if } q \geq l \\
   \frac{q}{l + m} & \text{if } q \in [0, l).
   \end{array} \right.
   \]

   The distribution mentioned above is a one sided hybrid distribution.

3. When $l = \infty$, we have
   \[
   \frac{\tilde{z}_\eta}{\tau_\eta} \overset{D}{\rightarrow} \mathcal{U}([0, 1]).
   \]

Thus, a single server queue also exhibits the phase transition behaviour under state-dependent control. For general arrivals, service and pricing policies, one can obtain results similar to the ones in Section 4, by essentially following the same methods. We present the result below. First, we introduce a negative drift condition analogous to Condition 1 to ensure positive recurrence for a general pricing policy.

**Condition 5 (Negative Drift).** There exists $\delta > 0$ and $K > 0$ such that for all $x > K$, we have $\phi^c(x) - \phi^s(x) < -\delta$.

Next, we show that $\varepsilon \tilde{q}_\eta$ converges in distribution to an exponential distribution in the TDM regime, uniform distribution in the MDT regime, and to Gibbs distribution on the non negative axis, denoted by $\text{Gibbs}_+(\cdot)$ in the hybrid regime. In particular, the PDF of $\text{Gibbs}_+(g)$ is given by

\[
\frac{1}{\int_0^\infty e^{-\int_0^t g(s)ds}dt} e^{-\int_0^t g(s)ds}1\{x \geq 0\}.
\]

Now, we present the result below.

**Theorem 4.** Consider the positive recurrent DTMC $\{q_\eta(k) : k \in \mathbb{Z}_+\}$ for any $\eta > 0$ and for any $\phi^c(\cdot)$ and $\phi^s(\cdot)$ satisfying Condition 5 and let $\tilde{q}_\eta$ denote its steady state random variable.

![Figure 5. Single server queue with Bernoulli arrivals and two-price policy with $m \overset{\Delta}{=} \lambda^*(1 - \mu^*)$](image-url)
1. If $\epsilon \eta \downarrow 0$, then we have,

$$\epsilon \bar{q} \sim \text{Exp}\left(\frac{\sigma^c(\lambda^*) + \sigma^s(\mu^*)}{2}\right).$$

2. If $\epsilon \eta \tau \rightarrow l$ for $l \in (0, \infty)$ and Condition 2 is satisfied, then we have

$$\epsilon \bar{q} \sim \text{Gibbs}_+(g_{1,l})$$

where $g_{1,l}$ is given by (6).

3. If $\epsilon \eta \tau \uparrow \infty$ and Condition 4 is satisfied, then we have

$$\frac{\bar{q}}{\tau} \sim \mathcal{U}([0, t^*]).$$

Transform method introduced in [43] provided the stationary distribution of a single server queue with static arrival and service rate. In particular, they obtain a closed form expression for the characteristic function of the limiting distribution which immediately establishes convergence in distribution. In contrast, to prove part (2) of the theorem, we obtain an implicit equation ((46) in Appendix), to solve which, we use the inverse Fourier transform method. Therefore, the method presented in [43] is inadequate to obtain the limiting distribution for a single server queue with dynamic arrival rates.

In the further sections, we outline the proof of Theorem 1, Theorem 2, and Theorem 3. The proof of Theorem 4 is analogous to these theorems and thus, we defer all the proof details to the Appendix H.

6. Proof of Theorem 1: Inverse Fourier Transform Method In this section, we will first use Proposition 2 and show tightness to complete Step 1. Next, we will carry out the drift analysis to obtain an implicit equation involving characteristic function of imbalance which will complete Step 2 and then present a key lemma to prove uniqueness of the implicit equation which will complete Step 3 and finally, we will present the proof of Theorem 1.

6.1. Step 1: Tightness

Lemma 1. Under Condition 1, for the choice of $\epsilon$, $\tau$ such that $\epsilon \eta \tau \rightarrow l \in (0, \infty)$ as $\eta \uparrow \infty$, for every sequence in the family of random variables $\Pi = \{\epsilon \eta \bar{z} : \eta > 0\}$ such that $\eta \uparrow \infty$, there exists a sub-sequence that converges in distribution.

The proof of the Lemma is presented in Appendix C.1. Now, we consider an arbitrary sequence in the family of random variables $\Pi$. By the above lemma, there exists a sub sequence that converges in distribution. Denote by $\bar{z}_\infty$ the limit of this convergent sub sequence. We will work with this sub sequence in the further subsections.

6.2. Step 2: Drift Analysis In this section, we consider $e^{j\omega z}$ as the test function and set its drift to zero in steady state. First, we present the update equation for the imbalance in steady state. Denote by $a^c(\bar{z})$ and $a^s(\bar{z})$ as the effective arrivals in steady state when the imbalance is $\bar{z}$. The imbalance after one transition $\bar{z}_{\eta}^+$ is governed by the following evolution equation:

$$\bar{z}_{\eta}^+ = \bar{z}_{\eta} + a^c(\bar{z}_{\eta}) - a^s(\bar{z}_{\eta}).$$

We formally define the drift of a test function $V(z)$. This is similar to the definition given in [43, Definition 1] and we present it below for completeness.
\textbf{Definition 2 (Drift of a Function).} Let $V: \mathbb{R} \to \mathbb{C}$ be a function. We define the drift of $V$ at $z$ as
\[ \Delta V(z) = (V(z(k+1)) - V(z(k))) \mathbbm{1}\{z(k) = z\}. \]

If $\mathbb{E}[|V(\bar{z})|] < \infty$, then we say that we set the drift of $V$ to zero when we use the property
\[ \mathbb{E}[\Delta V(\bar{z})] = \mathbb{E}[V(\bar{z}^+) - V(\bar{z})] = 0. \]

Setting the drift of $e^{i\omega z}$ to zero, we get the following lemma, where $g_{1, l}(\cdot)$ is defined in 6.

\textbf{Lemma 2.} Under the same setup as in Theorem 1, we have
\[ \mathbb{E}[e^{i\omega \bar{z}_{\infty}}g_{1, l}(\bar{z}_{\infty})] = j\omega \mathbb{E}[e^{i\omega \bar{z}_{\infty}}]. \]

\textit{Proof of Lemma 2} For $\omega \in \mathbb{R}$, we define the test function
\[ W(z) \triangleq e^{i\omega z}. \]

We now analyze its drift in steady state.

\[ \begin{align*}
\mathbb{E}[\Delta W(z)] &= \mathbb{E}[e^{i\omega \bar{z}_{\infty}^+} - e^{i\omega \bar{z}_{\infty}^-}] \\
&= \mathbb{E}[e^{i\omega (\bar{z}_{\eta} + a_{\eta}^*(\bar{z}_{\eta}) - a_{\eta}^*(\bar{z}_{\eta}))} - e^{i\omega \bar{z}_{\eta}^-}] \\
&= \mathbb{E}[e^{i\omega \bar{z}_{\eta}^-} \left( e^{i\omega (a_{\eta}^*(\bar{z}_{\eta}) - a_{\eta}^*(\bar{z}_{\eta}))} - 1 \right)] \\
&= j\epsilon^2 \mathbb{E}[e^{i\omega \bar{z}_{\eta}^-} \left( \phi^c \left( \frac{\bar{z}_{\eta}}{\epsilon} \right) - \phi^c \left( \frac{\bar{z}_{\eta}}{\epsilon} \right) \right)] - \frac{j}{2} \epsilon^2 \mathbb{E} \left[ e^{i\omega \bar{z}_{\eta}^-} \left( \sigma_c^c(\lambda(\bar{z}_{\eta})) + \sigma^c(\mu(\bar{z}_{\eta})) \right) \right] + o(\epsilon^2) \\
&= j\epsilon^2 \mathbb{E}[e^{i\omega \bar{z}_{\eta}^-} \left( \phi^c \left( \frac{\bar{z}_{\eta}}{\epsilon} \right) - \phi^c \left( \frac{\bar{z}_{\eta}}{\epsilon} \right) \right)] - \frac{j}{2} \epsilon^2 \mathbb{E} \left[ e^{i\omega \bar{z}_{\eta}^-} \left( \sigma_c^c(\lambda(\bar{z}_{\eta})) + \sigma^c(\mu(\bar{z}_{\eta})) \right) \right] + o(\epsilon^2)
\end{align*} \]

where (a) follows by Taylor’s Theorem. The reader can refer to Lemma 14 for precise analysis. Next, (b) follows by the tower property of expectation and (2), and (c) follows by the tower property of expectation and using the definition of arrivals. For precise calculations, refer to (32). Now, as $|e^{i\omega z}| = 1$ and so $\mathbb{E}[|W(z)|] < \infty$ is trivially true, by setting $\mathbb{E}[\Delta W(z)] = 0$, dividing by $je^{i\omega}$, and rearranging the terms, we get
\[ \begin{align*}
\mathbb{E}\left[ e^{i\omega \bar{z}_{\eta}} \left( \phi^c \left( \frac{\bar{z}_{\eta}}{\epsilon} \right) - \phi^c \left( \frac{\bar{z}_{\eta}}{\epsilon} \right) \right) \right] &= \frac{j\omega}{2} \left( \sigma^c(\lambda^*) + \sigma^c(\mu^*) \right) \mathbb{E}\left[ e^{i\omega \bar{z}_{\eta}} \right] + o(1) \\
&= \frac{j\omega}{2} \mathbb{E}\left[ e^{i\omega \bar{z}_{\eta}} \left( \sigma_c^c(\lambda(\bar{z}_{\eta})) + \sigma^c(\mu(\bar{z}_{\eta})) - \sigma^c(\lambda^*) - \sigma^c(\mu^*) \right) \right].
\end{align*} \]

Now, taking the limit as $\eta \uparrow \infty$, the last term in RHS disappears as $\epsilon \tau = l$. Thus, $\bar{z}_{\eta}/\tau$ converges to $\bar{z}_{\infty}/l$, and so, we expect a functional equation of the following form.

\textbf{Claim 1.} By taking the limit as $\eta \uparrow \infty$ in (11), we get
\[ \mathbb{E}\left[ e^{i\omega \bar{z}_{\infty}}g_{1, l}(\bar{z}_{\infty}) \right] = j\omega \mathbb{E}\left[ e^{i\omega \bar{z}_{\infty}} \right]. \]

However, this is not straightforward as one needs to show limits of functions as opposed to numbers due to the state dependent control. The proof of the claim is presented in Appendix C.2. This completes the proof of the Lemma. \hfill \Box
6.3. Step 3: Uniqueness In this section, we show that Gibbs$(g_{1,l})$ distribution is a unique solution of the functional equation obtained in Lemma 2.

**Lemma 3.** Under the same setup as in Theorem 1, let $\varepsilon z_\eta \overset{D}{\to} z_\infty$ and assume that $z_\infty$ satisfy the following:

$$
E[e^{j\omega z_\infty}g_{1,l}(z_\infty)] = j\omega E[e^{j\omega z_\infty}].
$$

(13)

Then $z_\infty$ has Gibbs$(g_{1,l})$ distribution.

As discussed in Section 4.3.1, to prove the above lemma, we take the inverse Fourier transform on both sides. Assuming a continuously differentiable PDF, one obtains a differential equation characterizing the PDF. However, one cannot assume the existence of PDF and we overcome this difficulty by interpreting expectation as a tempered distribution. Intuitively, the space of tempered distribution generalizes the notion of a function. We first verify that Gibbs$(\cdot)$ is a solution, then introduce the basics of tempered distribution, and then use it to show uniqueness of the solution in Appendix C.3.

6.4. Proof of Theorem 1

**Proof of Theorem 1 Step 1:** By Proposition 2, the DTMC $\{z_\eta(k) : k \in \mathbb{Z}_+\}$ is positive recurrent for all $\eta > 0$. Consider an arbitrary sequence in the family of random variables II and a sub-sequence that converges to a random variable $\bar{z}_\infty$ in distribution. Existence of such sub-sequence is guaranteed by Lemma 1.

**Step 2:** By Lemma 2, we have

$$
E[e^{j\omega z_\infty}g_{1,l}(z_\infty)] = j\omega E[e^{j\omega z_\infty}] .
$$

**Step 3:** By using Condition 2, we have $g_{1,l}(\cdot) \in C_{pol}(\mathbb{R})$. Thus, by Lemma 3, the above equation has Gibbs$(g_{1,l})$ as the unique solution. Thus, $\bar{z}_\infty$ has Gibbs$(g_{1,l})$ distribution.

Now, consider any sequence in II. Any sub-sequence of this sequence has a further sub-sequence that converges to $\bar{z}_\infty$ in distribution. Thus, by [11, Theorem 2.6], any sequence in II, converges to $\bar{z}_\infty$ in distribution. Finally, note that, by Slutsky’s Theorem, we have

$$
\frac{\bar{z}_\eta}{\tau} = \frac{\varepsilon \bar{z}_\eta}{\varepsilon \tau} \overset{D}{\to} \frac{z_\infty}{l} .
$$

Thus, as $\bar{z}_\infty \sim$ Gibbs$(g_{1,l})$, by simple variable substitution, we have $\bar{z}_\infty/l \sim$ Gibbs$(g_{1,l})$ which implies that $\bar{z}_\eta/\tau \overset{D}{\to}$ Gibbs$(g_{1,l})$. This completes the proof. □

7. Preliminary Lemmas for the Proof of Theorem 2 and 3 The proof of Theorem 2 and 3 are based on using complex exponential test functions. In order to bound certain terms in the process, we need the following lemmas which are obtained by setting the drift of several auxiliary test functions to zero. The proof of all the lemmas are deferred to Appendix D.

7.1. Auxiliary Bounds on the Imbalance Under Condition 1, we have the following results.

**Lemma 4.** Let $g_\eta(\cdot)$ be a function such that there exists $G_\eta > 0$ with $|g_\eta(x)| \leq G_\eta$ for all $x \in \mathbb{R}$. Then, for any $\eta > 0$, we have

$$
E[\varepsilon \bar{z}_\eta^+(g_\eta(\bar{z}_\eta^+) - g_\eta(\bar{z}_\eta))] = -\varepsilon E\left[\left(\phi^\circ \left(\frac{\bar{z}_\eta}{\tau}\right) - \phi^\circ \left(\frac{\bar{z}_\eta}{\tau}\right)\right)g_\eta(\bar{z}_\eta)\right] .
$$

(14)

The proof of the above lemma follows by setting the drift of the test function $zg(z)$ to zero in steady state. Now, we prove a high probability bound on imbalance being in any state $x \in \mathbb{R}$ below.
Lemma 5. For any \( x \in \mathbb{Z} \), we have
\[
\mathbb{P}(\bar{z}_n = x) \leq \frac{e}{P_{min}} \mathbb{E} \left[ \left| \phi^c \left( \frac{\bar{z}_n}{\tau} \right) - \phi^s \left( \frac{\bar{z}_n}{\tau} \right) \right| \right]
\]

The proof of the above lemma make use of two test functions. In particular, we set the drift of the test functions \( z \mathbb{1} \{ \text{sgn}(x)z > \text{sgn}(x)x \} \) and \( \mathbb{1} \{ \text{sgn}(x)z > \text{sgn}(x)x \} \) to zero in steady state.

7.2. Convergence of State Dependent Random Variables  Due to state dependent control, limits of certain quantities in the proof are non-trivial. So, we need the following two lemmas.

Lemma 6. Assume the same setup as in Theorem 2. For any function \( g \) such that \( \lim_{x \to \infty} g(x) = \lim_{x \to -\infty} g(x) = c_\infty \), and there exists a \( g_{\max} > 0 \) such that \( |g(x)| \leq g_{\max} \) for all \( x \in \mathbb{R} \), we have
\[
\lim_{n \to \infty} \mathbb{E} \left[ g \left( \frac{\bar{z}_n}{\tau} \right) \right] = c_\infty.
\]

Lemma 7. For some \( c > 0 \), let \( g_n \) be a sequence of functions such that
\[
\lim_{n \to \infty} g_n(x) = c \quad \text{uniformly } \forall x \in \mathbb{R}.
\]

Then for any sequence of random variables \( \{Y_n\} \), we have
\[
\lim_{n \to \infty} \mathbb{E} \left[ g_n(Y_n) \right] = c.
\]

8. Proof of Theorem 2: Characteristic Function Method  In this section, we prove the Theorem 2. We will present two key lemmas each completing Step 1 and Step 2 respectively. Finally, we will present the proof of the Theorem.

8.1. Step 1: Limiting Distribution of Absolute Imbalance  In this sub-section, we consider the test function \( e^{j\omega |\bar{z}_\eta|} \), which allows us to characterize the limiting distribution of \( \epsilon |\bar{z}_\eta| \).

Lemma 8. Under the same setup as in Theorem 2, we have
\[
\lim_{\eta \to \infty} \mathbb{E} \left[ e^{j\omega |\bar{z}_\eta|} \right] = \frac{1}{1 - j\omega \sigma^2(\lambda^*) + \sigma^2(\mu^*)}.
\]

Proof For \( \omega \in \mathbb{R} \), we define the test function
\[
V(z) \Delta e^{j\omega |z|} = e^{j\omega \epsilon \text{sgn}(z)}.
\]

Now, consider the one step drift of \( V(z) \).
\[
\Delta V(\bar{z}_\eta) = V(\bar{z}_\eta^+) - V(\bar{z}_\eta^-) = e^{j\omega \epsilon \text{sgn}(\bar{z}_\eta^+)} - e^{j\omega \epsilon \text{sgn}(\bar{z}_\eta^-)}
\]
\[
= e^{j\omega \epsilon \text{sgn}(\bar{z}_\eta^+)} - e^{j\omega \epsilon \text{sgn}(\bar{z}_\eta^-)} + e^{j\omega \epsilon \text{sgn}(\bar{z}_\eta^+)} - e^{j\omega \epsilon \text{sgn}(\bar{z}_\eta^-)}.
\]

We will analyze each term separately. To simplify \( T_1 \), directly use Taylor’s Theorem and consider the expansion up to the second order term.
\[
T_1 = e^{j\omega \epsilon \text{sgn}(\bar{z}_\eta^+)} - e^{j\omega \epsilon \text{sgn}(\bar{z}_\eta^-)}
\]
\[
= e^{j\omega \epsilon \text{sgn}(\bar{z}_\eta^+)} - e^{j\omega \epsilon \text{sgn}(\bar{z}_\eta^-)} 1\{|\bar{z}_\eta^+| \leq A_{\max}\}
\]
\[
= e^{j\omega \epsilon \text{sgn}(\bar{z}_\eta^+)} 1\{|\bar{z}_\eta^+| \leq A_{\max}\} - e^{j\omega \epsilon \text{sgn}(\bar{z}_\eta^-)} 1\{|\bar{z}_\eta^-| \leq A_{\max}\}
\]
\[
= e^{j\omega \epsilon \text{sgn}(\bar{z}_\eta^+)} 1\{|\bar{z}_\eta^+| \leq A_{\max}\} - e^{j\omega \epsilon \text{sgn}(\bar{z}_\eta^-)} 1\{|\bar{z}_\eta^-| \leq A_{\max}\} + o(\epsilon^2)
\]
\[
= j\omega \epsilon \text{sgn}(\bar{z}_\eta^+) 1\{|\bar{z}_\eta^+| \leq A_{\max}\} + o(\epsilon^2),
\]
where (a) follows as \(\text{sgn}(\tilde{z}^+) = \text{sgn}(\tilde{z})\) if \(|\tilde{z}^+| > A_{\text{max}}\) as the arrivals are bounded by \(A_{\text{max}}\) with probability 1. Next, (b) follows by Taylor’s theorem. For a more precise argument, the reader should refer to Lemma 14. Note that the second order term in the expansion is exactly equal to zero as \(\text{sgn}(\tilde{z}_r^+)^2 = \text{sgn}(\tilde{z}_r)^2 = 1\). Now, the above can be simplified by Lemma 4 to get

\[
E[T_1] = -j\epsilon^2 \omega E \left[ (\phi c \left( \frac{\tilde{z}_n}{\tau} \right) - \phi^s \left( \frac{\tilde{z}_n}{\tau} \right) ) \text{sgn}(\tilde{z}_n) \right] + o(\epsilon^2)
\]

Now, to simplify \(T_2\), we will first use the update equation of imbalance given by (10). Then, we will use Taylor’s Theorem to expand and consider up to the second order term. After using some basic properties about expectations, we get the following:

**Claim 2.**

\[
E[T_2|\tilde{z}_n] = j\epsilon^2 \omega e^{j\epsilon \omega \tilde{z}_n} \text{sgn}(\tilde{z}_n) \left( \phi c \left( \frac{\tilde{z}_n}{\tau} \right) - \phi^s \left( \frac{\tilde{z}_n}{\tau} \right) \right) \text{sgn}(\tilde{z}_n) \]

\[
\frac{1}{2} \epsilon^2 \omega^2 e^{j\epsilon \omega \tilde{z}_n} \text{sgn}(\tilde{z}_n) \left( \sigma^c(\lambda(\tilde{z}_n)) + \sigma^s(\mu(\tilde{z}_n)) \right) + o(\epsilon^2).
\]

The proof of the claim has been deferred to Appendix E.2 and here we continue with the proof of Lemma 8. As \(|V(\tilde{z})| = 1\), its expectation in steady state is finite. Thus, we set the drift of the above defined test function to zero in steady state to get \(E[T_1 + T_2] = 0\). Now, by substituting \(T_1\) and \(T_2\) and dividing by \(-j\epsilon^2\omega\), we get

\[
- \frac{j\omega}{2} \left( 1 + \phi c \left( \frac{\tilde{z}_n}{\tau} \right) - \phi^s \left( \frac{\tilde{z}_n}{\tau} \right) \right) \text{sgn}(\tilde{z}_n) = \frac{j\omega}{2} \left[ e^{j\epsilon \omega |\tilde{z}_n|} \left( \sigma^c(\lambda(\tilde{z}_n)) + \sigma^s(\mu(\tilde{z}_n)) \right) \right]
\]

\[
+ \frac{j\omega}{2} \left[ e^{j\epsilon \omega |\tilde{z}_n|} \left( \sigma^c(\lambda(\tilde{z}_n)) + \sigma^s(\mu(\tilde{z}_n)) - \sigma^c(\lambda(\tilde{z}_n)) - \sigma^s(\mu(\tilde{z}_n)) \right) \right] + o(1).
\]

Rearranging the above terms, we get

\[
\frac{j\omega}{2} \left[ \left( 1 + \phi c \left( \frac{\tilde{z}_n}{\tau} \right) - \phi^s \left( \frac{\tilde{z}_n}{\tau} \right) \right) \text{sgn}(\tilde{z}_n) \right] = \frac{j\omega}{2} \left[ e^{j\epsilon \omega |\tilde{z}_n|} \left( \sigma^c(\lambda(\tilde{z}_n)) + \sigma^s(\mu(\tilde{z}_n)) \right) \right]
\]

\[
+ \frac{j\omega}{2} \left[ e^{j\epsilon \omega |\tilde{z}_n|} \left( \sigma^c(\lambda(\tilde{z}_n)) + \sigma^s(\mu(\tilde{z}_n)) - \sigma^c(\lambda(\tilde{z}_n)) - \sigma^s(\mu(\tilde{z}_n)) \right) \right] + o(1).
\]

Now, we will show that \(E[T_3] \to -1\), \(E[T_4] \to 0\), and \(E[T_5] \to 0\) which will give us the result. This is easy to show for state independent control. Due to state dependent control, we have dependence on \(\tilde{z}_n\) in the above terms which makes it non-trivial. We use Lemma 6 and Lemma 7 to bound these terms. We start by analyzing \(T_3\). By Condition 3, we have \((\phi c(x) - \phi^s(x))\text{sgn}(x) \to -1\) as \(x \to \pm \infty\). Also, \(|(\phi c(x) - \phi^s(x))\text{sgn}(x)| \leq 2\phi_{\text{max}}\). Thus, by Lemma 6, we have

\[
\lim_{\eta \to \infty} E[T_3] = \lim_{\eta \to \infty} E \left[ (\phi c \left( \frac{\tilde{z}_n}{\tau} \right) - \phi^s \left( \frac{\tilde{z}_n}{\tau} \right) ) \text{sgn}(\tilde{z}_n) \right] = -1.
\]

Now, we present the following claim to bound \(T_4\) and \(T_5\).

**Claim 3.**

\[
\lim_{\eta \to \infty} E \left[ \left( 1 + \phi c \left( \frac{\tilde{z}_n}{\tau} \right) - \phi^s \left( \frac{\tilde{z}_n}{\tau} \right) \right) \text{sgn}(\tilde{z}_n) \right] = 0
\]

\[
\lim_{\eta \to \infty} \frac{\omega}{2} E \left[ \sigma^c(\lambda(\tilde{z}_n)) + \sigma^s(\mu(\tilde{z}_n)) - \sigma^c(\lambda(\tilde{z}_n)) - \sigma^s(\mu(\tilde{z}_n)) \right] = 0.
\]
The proof of the claim has been deferred to Appendix E.2 and here we continue with the proof of Lemma 8. By the above claim, we have

$$\lim_{\eta \uparrow \infty} \mathbb{E}[T_4] \leq \lim_{\eta \uparrow \infty} \mathbb{E}[|T_4|] = \lim_{\eta \uparrow \infty} \mathbb{E}\left[1 + \left(\phi^c\left(\frac{\bar{z}}{\tau}\right) - \phi^s\left(\frac{\bar{z}}{\tau}\right)\right) \text{sgn}(\bar{z}_\eta)\right] = 0$$

$$\lim_{\eta \uparrow \infty} \mathbb{E}[T_5] \leq \lim_{\eta \uparrow \infty} \mathbb{E}[|T_5|] = \lim_{\eta \uparrow \infty} \frac{|\omega|}{2} \mathbb{E}\left[\sigma^c(\lambda^*) + \sigma^s(\mu^*) - \sigma^c(\lambda(\bar{z}_\eta)) - \sigma^s(\mu(\bar{z}_\eta))\right] = 0$$

This completes the proof. \(\square\)

### 8.2. Step 2: Imbalance has Symmetrical Distribution

The previous lemma shows that the limiting distribution of \(\epsilon|\bar{z}_\eta|\) is exponential. Now, we will show that the limiting distribution of \(\epsilon\bar{z}_\eta\) is symmetrical, which will complete the Step 2 of proof of the Theorem 2.

**Lemma 9.** Under the same conditions as in Theorem 2, we have

$$\lim_{\eta \uparrow \infty} \mathbb{E}\left[\text{sgn}(\bar{z}_\eta)e^{j\omega|\varphi|}\right] = 0.$$  

To prove the lemma, we analyze the drift of \(\text{sgn}(\bar{z}_\eta)e^{j\omega|\varphi|}\). The technical details of the proof is similar to the proof of Lemma 8 and thus, we defer it to Appendix E.1.

### 8.3. Proof of Theorem 2

**Step 3: Proof of Theorem 2** Note that Step 1 is completed by Lemma 8 and Step 2 is completed by Lemma 9. Now, we will combine these parts together to complete the proof of the Theorem 2. For all \(x \in \mathbb{R}\), we have

$$2e^{j\omega x} + \text{sgn}(x)e^{-j\omega x \text{sgn}(x)} - \text{sgn}(x)e^{j\omega x \text{sgn}(x)} = e^{j\omega x} + e^{-j\omega x} = e^{j\omega x} + e^{-j\omega x}$$

By substituting \(x = \epsilon\bar{z}_\eta\) and taking expectation on both sides, we get

$$2\mathbb{E}\left[e^{j\omega \bar{z}_\eta}\right] + \mathbb{E}\left[\text{sgn}(\bar{z}_\eta)e^{-j\omega \bar{z}_\eta \text{sgn}(\bar{z}_\eta)}\right] = \mathbb{E}\left[\text{sgn}(\bar{z}_\eta)e^{j\omega \bar{z}_\eta \text{sgn}(\bar{z}_\eta)}\right] - \mathbb{E}\left[\text{sgn}(\bar{z}_\eta)e^{j\omega \bar{z}_\eta \text{sgn}(\bar{z}_\eta)}\right] = \mathbb{E}\left[e^{j\omega |\varphi|}\right] + \mathbb{E}\left[e^{-j\omega |\varphi|}\right]$$

Now, by taking the limit as \(\eta \uparrow \infty\) on both sides and using Lemma 8 for the RHS and Lemma 9 for the LHS, we get

$$2\lim_{\eta \uparrow \infty} \left(\mathbb{E}\left[e^{j\omega \bar{z}_\eta}\right]\right) = \frac{1}{1 - \frac{\omega^2}{2} \sigma^c(\lambda^*) + \sigma^s(\mu^*)} + \frac{1}{1 + \frac{\omega^2}{2} \sigma^c(\lambda^*) + \sigma^s(\mu^*)}$$

Note that \(\frac{1}{1 + \frac{\omega^2}{2} (\sigma^c(\lambda^*) + \sigma^s(\mu^*))^2}\) is the characteristic function of a Laplace distribution with mean equal to zero and variance equal to \((\sigma^c(\lambda^*) + \sigma^s(\mu^*))^2/2\). Thus, by Levy’s continuity theorem (e.g. see [77, Chapter 18]), \(\epsilon\bar{z}_\eta\) converges to a Laplace random variable. This completes the proof. \(\square\)

### 9. Proof of Theorem 3: Characteristic Function Method

In this section, we will present the proof the Theorem 3. We first present two key lemmas which completes Step 1 and 2 and then we present the proof of the Theorem.
9.1. Step 1: Uniform Distribution Within the Thresholds

**Lemma 10.** Under the same conditions as in Theorem 3, we have

\[
\lim_{\eta \to 0} \frac{E}{C} \left[ e^{\frac{\phi(e)}{\tau} \frac{z}{\tau}} - e^{\phi(\frac{z}{\tau})} \right] = 0.
\]

**Proof of Lemma 10** To prove the lemma, we will analyze the drift of the Lyapunov function defined as follows:

\[
V(z) = e^{\frac{\phi(e)}{\tau} \frac{z}{\tau}} - e^{\phi(\frac{z}{\tau})}.
\]

As \(|V(z)| \leq 1\), its expectation in steady state is finite. Thus, we set the drift of the above defined test function to zero in steady state.

\[
0 = \frac{E}{C} \left[ e^{\frac{\phi(e)}{\tau} \frac{z}{\tau}} 1 - e^{\phi(\frac{z}{\tau})} 1 \right]
\]

Now, we will analyze the two terms - \(T_1\) and \(T_2\) separately. First, we consider \(T_1\). Note that, \(\{\phi(\frac{z}{\tau}) - \phi(x) = 0\} = (t_*, t^*)\). Thus, we have

\[
T_1 = \frac{E}{C} \left[ e^{\frac{\phi(e)}{\tau} \frac{z}{\tau}} 1 \right]
\]

To analyze the above two terms, we first use Taylor’s series expansion and then show that the first order term is dominating. We present the following claim:

**Claim 4.** For any \(t \in \mathbb{R}\), we have

\[
\frac{E}{C} \left[ e^{\frac{\phi(e)}{\tau} \frac{z}{\tau}} 1 \right] = -\frac{j\omega t}{\tau} \frac{E}{C} \left[ e^{\phi(\frac{z}{\tau})} \right] + \omega^2 \frac{1}{\tau^2} o(\omega^2).
\]

We defer the proof of the above claim to the Appendix F and continue with the proof of Lemma 10 here. Now, we present the following claim that characterizes \(T_2\).
Claim 5. Let \( \hat{\lambda}^* = \lambda^* + \phi_c(0) \lim_{\eta \uparrow \infty} \epsilon \) and \( \hat{\mu}^* = \mu^* + \phi^*(0) \lim_{\eta \uparrow \infty} \epsilon \). Then,

\[
\mathcal{F}_2 = -\frac{\omega^2}{2\tau^2} (\sigma^c(\hat{\lambda}^*) + \sigma^*(\hat{\mu}^*)) \mathbb{E} \left[ \left\{ \phi^c\left(\frac{\bar{z}_n}{\tau}\right) - \phi^s\left(\frac{\bar{z}_n}{\tau}\right) = 0 \right\} e^{j\omega \bar{z}_n/\tau} \right] + \omega^2 o \left(\frac{1}{\tau^2}\right) + o \left(\frac{1}{\tau^2}\right) o (\omega^2).
\]

We defer the details of the proof of the claim to the Appendix F. Combining everything, we get

\[
-\frac{\omega^2}{2\tau^2} (\sigma^c(\hat{\lambda}^*) + \sigma^*(\hat{\mu}^*)) \mathbb{E} \left[ \left\{ \phi^c\left(\frac{\bar{z}_n}{\tau}\right) - \phi^s\left(\frac{\bar{z}_n}{\tau}\right) = 0 \right\} e^{j\omega \bar{z}_n/\tau} \right] + \omega^2 o \left(\frac{1}{\tau^2}\right) + o \left(\frac{1}{\tau^2}\right) o (\omega^2) = e^{j\omega t^*} \frac{j \omega}{\tau} \mathbb{E} \left[ \left\{ \frac{\bar{z}_n}{\tau} \leq t^* \right\} \left( \phi^c\left(\frac{\bar{z}_n}{\tau}\right) - \phi^s\left(\frac{\bar{z}_n}{\tau}\right) \right) \right] - e^{j\omega t^*} \frac{j \omega}{\tau} \mathbb{E} \left[ \left\{ \frac{\bar{z}_n}{\tau} \leq t^* \right\} \left( \phi^c\left(\frac{\bar{z}_n}{\tau}\right) - \phi^s\left(\frac{\bar{z}_n}{\tau}\right) \right) \right].
\]

Noting that \( \mathbb{E} \left[ \left\{ \frac{\bar{z}_n}{\tau} \leq t^* \right\} \left( \phi^c\left(\frac{\bar{z}_n}{\tau}\right) - \phi^s\left(\frac{\bar{z}_n}{\tau}\right) \right) \right] = \mathbb{E} \left[ \left\{ \frac{\bar{z}_n}{\tau} \leq t^* \right\} \left( \phi^c\left(\frac{\bar{z}_n}{\tau}\right) - \phi^s\left(\frac{\bar{z}_n}{\tau}\right) \right) \right] \) as \( \phi^c(x) - \phi^s(x) = 0 \) for \( x \in (t^*, \tau] \) and dividing both sides by \( -\frac{\omega^2}{2\tau^2} (\sigma^c(\hat{\lambda}^*) + \sigma^*(\hat{\mu}^*)) \), we get

\[
\mathbb{E} \left[ \left\{ \phi^c\left(\frac{\bar{z}_n}{\tau}\right) - \phi^s\left(\frac{\bar{z}_n}{\tau}\right) = 0 \right\} e^{j\omega \bar{z}_n/\tau} \right] = -j \frac{2\epsilon \tau}{\omega (\sigma^c(\hat{\lambda}^*) + \sigma^*(\hat{\mu}^*))} \mathbb{E} \left[ \left\{ \frac{\bar{z}_n}{\tau} \leq t^* \right\} \left( \phi^c\left(\frac{\bar{z}_n}{\tau}\right) - \phi^s\left(\frac{\bar{z}_n}{\tau}\right) \right) \right] e^{j\omega t^*} - e^{j\omega t_*} + o_\tau(1) o_\omega(1) + o_\tau(1).
\]

Note that, as we are keeping track of the order in terms of \( \tau \) and \( \omega \), we make it explicit by adding a sub-script whenever necessary. Now, we take the limit as \( \omega \downarrow 0 \) on both sides. As the characteristic function is upper bound by 1, we use dominated convergence theorem to interchange the limit and expectation to get

\[
\mathbb{P} \left( \phi^c\left(\frac{\bar{z}_n}{\tau}\right) - \phi^s\left(\frac{\bar{z}_n}{\tau}\right) = 0 \right)
= -j \frac{2\epsilon \tau}{\omega (\sigma^c(\hat{\lambda}^*) + \sigma^*(\hat{\mu}^*))} \mathbb{E} \left[ \left\{ \frac{\bar{z}_n}{\tau} \leq t^* \right\} \left( \phi^c\left(\frac{\bar{z}_n}{\tau}\right) - \phi^s\left(\frac{\bar{z}_n}{\tau}\right) \right) \right] \lim_{\omega \downarrow 0} \frac{e^{j\omega t^*} - e^{j\omega t_*}}{\omega} + o_\tau(1)
= \frac{1}{(\sigma^c(\hat{\lambda}^*) + \sigma^*(\hat{\mu}^*))} \mathbb{E} \left[ \left\{ \frac{\bar{z}_n}{\tau} \leq t^* \right\} \left( \phi^c\left(\frac{\bar{z}_n}{\tau}\right) - \phi^s\left(\frac{\bar{z}_n}{\tau}\right) \right) \right] (t^* - t_*) + o_\tau(1).
\]

Substituting (17) in (16), we get

\[
\mathbb{E} \left[ \left\{ \phi^c\left(\frac{\bar{z}_n}{\tau}\right) - \phi^s\left(\frac{\bar{z}_n}{\tau}\right) = 0 \right\} e^{j\omega \bar{z}_n/\tau} \right] = -j \frac{1}{\omega (t^* - t_*)} \mathbb{P} \left( \phi^c\left(\frac{\bar{z}_n}{\tau}\right) - \phi^s\left(\frac{\bar{z}_n}{\tau}\right) = 0 \right) e^{j\omega t^*} - e^{j\omega t_*} + o_\tau(1) o_\omega(1) + o_\tau(1)
= \frac{1}{j\omega (t^* - t_*)} \mathbb{P} \left( \phi^c\left(\frac{\bar{z}_n}{\tau}\right) - \phi^s\left(\frac{\bar{z}_n}{\tau}\right) = 0 \right) e^{j\omega t^*} - e^{j\omega t_*} + o_\tau(1) o_\omega(1) + o_\tau(1).
\]

Taking the limit as \( \eta \uparrow \infty \) on both sides, and noting that \( o_\tau(1) o_\omega(1) + o_\tau(1) \to 0 \) as \( \tau \uparrow \infty \), we get

\[
\lim_{\eta \uparrow \infty} \frac{1}{j\omega (t^* - t_*)} \mathbb{P} \left( \phi^c\left(\frac{\bar{z}_n}{\tau}\right) - \phi^s\left(\frac{\bar{z}_n}{\tau}\right) = 0 \right) e^{j\omega \bar{z}_n/\tau} = \frac{1}{j\omega (t^* - t_*)} \mathbb{P} \left( \phi^c\left(\frac{\bar{z}_n}{\tau}\right) - \phi^s\left(\frac{\bar{z}_n}{\tau}\right) = 0 \right) e^{j\omega t^*} - e^{j\omega t_*} + o_\tau(1) o_\omega(1) + o_\tau(1).
\]

This completes the proof. \( \square \)
9.2. Step 2: No Mass Outside the Thresholds

**Lemma 11.**

\[
\limsup_{n \to \infty} \mathbb{P}\left( \phi^c\left( \frac{z_n}{\tau} \right) - \phi^s\left( \frac{z_n}{\tau} \right) \neq 0 \right) = 0.
\]

**Proof** For the ease of notation, define \( \phi^c(\cdot) - \phi^s(\cdot) = \phi(\cdot) \). As \( \{ \phi(x) \neq 0 \} = \{ x \in (-\infty, t_*) \cup (t^*, \infty) \} \), we have

\[
\mathbb{P}\left( \phi\left( \frac{z_n}{\tau} \right) \neq 0 \right) = \mathbb{P}\left( \frac{z_n}{\tau} \in (-\infty, t_*) \cup (t^*, \infty) \right).
\]

Now, define \( h_{\eta, t_*} = t_* - \sup\{ x : \phi(x) \geq \frac{1}{\sqrt{\epsilon t}} \} \) and \( h^*_\eta = \inf\{ x : \phi(x) \leq -\frac{1}{\sqrt{\epsilon t}} \} - t^* \). Using Condition 4, note that \( h_{\eta, t_*} \geq 0 \) and \( h^*_\eta \geq 0 \). Moreover, noting that \( \epsilon t \uparrow \infty \), we have \( \lim_{\tau \to \infty} (h_{\eta, t_*} + h^*_\eta) = 0 \). Now, we proceed by dividing the above probability into the following two terms.

\[
\mathcal{T}_3 = \mathbb{P}\left( \frac{z_n}{\tau} \in (t_* - h_*, t_*) \cup (t^*, t_* + h^*) \right)
\]

\[
\mathcal{T}_4 = \mathbb{P}\left( \frac{z_n}{\tau} \in (-\infty, t_* - h_* \cup |t^* + h^*, \infty) \right)
\]

Note that, \( \mathcal{T}_3 \) comprises of the region close to the thresholds. We will upper bound it using Lemma 5. We get

\[
\mathcal{T}_3 = \sum_{k=[(t_* - h_*)\tau]}^{(t_* + h^*)\tau} \mathbb{P}(\frac{z_n}{\tau} = k) + \sum_{k=[(t_* - h_*)\tau]}^{(t_* + h^*)\tau} \mathbb{P}(\frac{z_n}{\tau} = k) \leq (h_* + h^*)\tau + 2 \frac{\epsilon}{p_{\min}} \mathbb{E}\left[ \phi^c\left( \frac{z_n}{\tau} \right) - \phi^s\left( \frac{z_n}{\tau} \right) \right]
\]

\[
\leq -((h_* + h^*)\tau + 2) \frac{\epsilon}{p_{\min}} \min\{t_* + t^*\} \mathbb{E}\left[ \phi^c\left( \frac{z_n}{\tau} \right) - \phi^s\left( \frac{z_n}{\tau} \right) \right]
\]

\[
\leq \frac{2A^2_{\max}}{p_{\min}} \min\{-t_*, t^*\},
\]

where (a) follows by Lemma 5. Next, (b) follows as \( \phi^c(x) - \phi^s(x) \text{sgn}(x) \leq 0 \) and \( \phi^c(x) - \phi^s(x) = 0 \) for \( t_* < 0 \leq t^* \). Lastly, (c) follows by Lemma 2. Next, to upper bound \( \mathcal{T}_4 \), note that, as \( \phi(\cdot) \) is monotonic, by the definition of \( h^* \) and \( h_* \), we have

\[
\phi(x) \leq -\frac{1}{\sqrt{\epsilon t}} \quad \forall x \geq t^* + h^*
\]

\[
\phi(x) \geq \frac{1}{\sqrt{\epsilon t}} \quad \forall x \leq t_* - h_*.
\]

Thus, we have

\[
\mathbb{P}\left( \frac{z_n}{\tau} \in (-\infty, t_* - h_* \cup [t^* + h^*, \infty) \right) \leq \mathbb{P}\left( \left| \phi^c\left( \frac{z_n}{\tau} \right) - \phi^s\left( \frac{z_n}{\tau} \right) \right| \geq \frac{1}{\sqrt{\epsilon t}} \right)
\]

\[
\leq \sqrt{\epsilon \tau} \mathbb{E}\left[ \phi^c\left( \frac{z_n}{\tau} \right) - \phi^s\left( \frac{z_n}{\tau} \right) \right]
\]

\[
\leq -\sqrt{\epsilon \tau} \min\{-t_*, t^*\} \mathbb{E}\left[ \phi^c\left( \frac{z_n}{\tau} \right) - \phi^s\left( \frac{z_n}{\tau} \right) \right]
\]

\[
\leq \frac{2A^2_{\max}}{\sqrt{\epsilon \tau} \min\{-t_*, t^*\}},
\]
where (a) follows by the Markov’s inequality. Next, (b) follows as \((\phi^c(x) - \phi^s(x))\text{sgn}(x) \leq 0\) and \(\phi^c(x) - \phi^s(x) = 0\) for \(t_* < 0 \leq t^*\). Lastly, (c) follows by Lemma 2. Combining everything, we get

\[
P \left( \phi^c \left( \frac{\tilde{z}}{\tau} \right) - \phi^s \left( \frac{\tilde{z}}{\tau} \right) \neq 0 \right) \leq \frac{2A_{\max}^2}{\min \{-t_*, t_*\} \min \{h_*, h^* + \frac{2}{\tau}\}} + \frac{2A_{\max}^2}{\sqrt{\epsilon \tau \min \{-t_*, t_*\}}}.
\]

Taking limit supremum on both sides as \(\eta \uparrow \infty\), and noting that \(\epsilon \tau \uparrow \infty\) and \(\tau \uparrow \infty\), we get

\[
\limsup_{\eta \uparrow \infty} \mathbb{P} \left( \phi^c \left( \frac{\tilde{z}}{\tau} \right) - \phi^s \left( \frac{\tilde{z}}{\tau} \right) \neq 0 \right) \leq \frac{2A_{\max}^2}{\min \{-t_*, t_*\} \min \{h_*, h^*\}} \limsup_{\eta \uparrow \infty} (h_* + h^*) = 0.
\]

This completes the proof. \(\square\)

9.3. Step 3: Proof of Theorem 3

Proof To prove the theorem, we show that the characteristic function converges to the characteristic function of the uniform distribution. We divide the characteristic function into two regions: i.e. when \(\phi^c(\cdot) - \phi^s(\cdot)\) is equal to zero and non-zero respectively.

\[
\mathbb{E} \left[ e^{j\omega \tilde{z}/\tau} \right] = \mathbb{E} \left[ e^{j\omega \tilde{z}/\tau} 1_{\mathcal{T}_5} \right] + \mathbb{E} \left[ e^{j\omega \tilde{z}/\tau} 1_{\mathcal{T}_6} \right] + \mathbb{E} \left[ e^{j\omega \tilde{z}/\tau} 1_{\mathcal{T}_6^c} \right].
\]

We characterize the limit of \(\mathcal{T}_6\) using Lemma 11 as follows:

\[
\limsup_{\eta \uparrow \infty} \left| \limsup_{\eta \uparrow \infty} \left| \mathbb{E} \left[ e^{j\omega \tilde{z}/\tau} 1_{\mathcal{T}_6} \right] \right| \right| = \limsup_{\eta \uparrow \infty} \mathbb{P} \left( \phi^c \left( \frac{\tilde{z}}{\tau} \right) - \phi^s \left( \frac{\tilde{z}}{\tau} \right) \neq 0 \right) = 0.
\]

As the \(\limsup\) of absolute value of \(\mathcal{T}_6\) is upper bounded by 0, we have \(\lim_{\eta \uparrow \infty} \mathcal{T}_6 = 0\). Next, we analyze the limit of \(\mathcal{T}_5\) by using the Lemma 10 as follows:

\[
\lim_{\eta \uparrow \infty} \mathcal{T}_5 = \lim_{\eta \uparrow \infty} \frac{e^{j\omega t^*} - e^{j\omega t_*}}{j\omega (t^* - t_*)} \mathbb{P} \left( \phi^c \left( \frac{\tilde{z}}{\tau} \right) - \phi^s \left( \frac{\tilde{z}}{\tau} \right) = 0 \right) + o(\tau, \omega) (1)
\]

\[
= \frac{e^{j\omega t^*} - e^{j\omega t_*}}{j\omega (t^* - t_*)} \lim_{\eta \uparrow \infty} \frac{e^{j\omega t^*} - e^{j\omega t_*}}{e^{j\omega t^*} - e^{j\omega t_*}} \mathbb{P} \left( \phi^c \left( \frac{\tilde{z}}{\tau} \right) - \phi^s \left( \frac{\tilde{z}}{\tau} \right) \neq 0 \right)
\]

\[
= \frac{e^{j\omega t^*} - e^{j\omega t_*}}{j\omega (t^* - t_*)}.
\]

Using (18) and (19), we get

\[
\lim_{\eta \uparrow \infty} \mathbb{E} \left[ e^{j\omega \tilde{z}/\tau} \right] = \frac{e^{j\omega t^*} - e^{j\omega t_*}}{j\omega (t^* - t_*)}.
\]

Note that the RHS is the characteristic function of \(\mathcal{U}(t_*, t^*)\). Thus, by Levy’s continuity theorem (e.g. see [77, Chapter 18]), \(\tilde{z}/\tau\) converges to a uniform random variable. This completes the proof of the theorem. \(\square\)
10. Conclusion and Future Work  In this paper, we develop a heavy traffic theory of a matching or a two-sided queue. We consider a general, state dependent arrivals and define heavy traffic as the limit such that the external control goes to zero. There are two different ways in which the control can be sent to zero which is modeled by the magnitude parameter, $\epsilon$ that goes to zero, and position parameter, $\tau$ that goes to infinity. Based on the relative speeds at which these parameters converges to their asymptotes, we observe a phase transition on the distribution of the imbalance from Laplace to Uniform. A similar phase transition is also observed in a single server queue under state dependent controls. To obtain these results, we develop a novel inverse Fourier transform method which generalizes the known transform method [43] in the literature.

This paper analyzes the simplest matching queue, which is a building block of matching networks that arise in blockchain [74], ride hailing [72, 71], quantum switch [69, 68], assemble-to-order systems [57, 58], etc. Future work includes studying heavy traffic behaviour of these systems. The key ingredient in such an endeavor is to establish an appropriate state space collapse. Another line of future work building upon our result on state dependent control in a classical single server queue is to study stochastic processing networks (e.g. load balancing [24]) under state dependent control.

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Appendix A: Proof of Proposition 1

Proof In this case, the underlying DTMC governing the imbalance is just a birth and death process. Thus, we can evaluate it’s stationary distribution exactly. In particular, the imbalance will increase by 1 if a customer arrives and no server arrives. That occurs with probability \( \lambda_\eta(\bar{z}_\eta)(1 - \mu_\eta(\bar{z}_\eta)) \). Similarly, the imbalance will decrease by 1 if a server arrives and no customer arrives. That occurs with probability \( (1 - \lambda_\eta(\bar{z}_\eta))\mu_\eta(\bar{z}_\eta) \). If both customer and server arrives or if no one arrives, the imbalance will stay the same. Define \( m \triangleq \mu^*(1 - \lambda^*) = \lambda^*(1 - \mu^*) \) and the transition probability matrix is given by

\[
P_{i,j} = \begin{cases} 
  m & \text{if } j = i + 1, |i| \leq |\tau| \\
  m + \epsilon \mu^* & \text{if } j = i - \text{sgn}(i), |i| > |\tau| \\
  m - \epsilon(1 - \mu^*) & \text{if } j = i + \text{sgn}(i), |i| > |\tau| \\
  1 - 2m & \text{if } j = i, |i| \leq |\tau| \\
  1 - 2m + \epsilon(1 - 2\mu^*) & \text{if } j = i, |i| > |\tau| \\
  0 & \text{otherwise.}
\end{cases}
\]

As this is a birth and death process, it is a reversible DTMC. Thus, by solving the detailed balance equations, we get

\[
\pi_i = \begin{cases} 
  \frac{1}{2|\tau|+1+2m/\epsilon} & \text{if } |i| \leq |\tau| \\
  \frac{1}{2|\tau|+1+2m/\epsilon} \left( 1 - \frac{\epsilon}{m+\epsilon \mu^*} \right)^{|i|-|\tau|} & \text{otherwise.}
\end{cases}
\]

Now, we will calculate the moment generating function of \( \epsilon \bar{z}_\eta \) for Case 1 and 2 and \( \bar{z}_\eta/\tau \) for Case 3. We fix a \( t \) such that \( t < 1/m \). Then, for all \( \epsilon > 0 \), we have \( (1 - \epsilon/(m + \epsilon \mu^*))e^{\epsilon t} < 1 \). We have

\[
\mathbb{E}[e^{\epsilon \bar{z}_\eta}] = \sum_{i=-\infty}^{-[\tau]-1} \pi_i e^{\epsilon t i} \\
= \sum_{i=-\infty}^{-[\tau]-1} \pi_i e^{\epsilon t i} + \sum_{i=-[\tau]}^{[\tau]} \pi_i e^{\epsilon t i} + \sum_{i=-[\tau]+1}^{[\tau]+1} \pi_i e^{\epsilon t i} \\
= \frac{1}{2|\tau|+1+2m/\epsilon} \sum_{i=-\infty}^{-[\tau]-1} \left( 1 - \frac{\epsilon}{m+\epsilon \mu^*} \right)^{|i|-|\tau|} e^{\epsilon t i} + \sum_{i=-[\tau]}^{[\tau]} e^{\epsilon t i} \\
+ \frac{m}{m+\epsilon \mu^*} \sum_{i=[\tau]+1}^{[\tau]+1} \left( 1 - \frac{\epsilon}{m+\epsilon \mu^*} \right)^{i-[\tau]-1} e^{\epsilon t i} \\
= \frac{\epsilon}{2|\tau| \epsilon + \epsilon + 2m} e^{-\epsilon t ([\tau]+1)} \left( \frac{m}{m+\epsilon \mu^*} \frac{e^{-\epsilon t [\tau]+1}}{1 - e^{-\epsilon t} + \frac{\epsilon e^{\epsilon t}}{m+\epsilon \mu^*}} + \frac{e^{-\epsilon t [\tau]} - e^{\epsilon t ([\tau]+1)}}{1 - e^{-\epsilon t}} \right) \\
+ \frac{m}{m+\epsilon \mu^*} \frac{e^{\epsilon t ([\tau]+1)}}{1 - e^{-\epsilon t} + \frac{\epsilon e^{\epsilon t}}{m+\epsilon \mu^*}}
\]

(20)

![Figure 6. Single link matching queue with Bernoulli arrivals](image)
Now, we will take the limit as \( \eta \uparrow \infty \) for the above equation for Case 1 and 2 separately.

**Case 1:** Note that, in this case, we have \( \tau \downarrow 0 \) and \( \epsilon \downarrow 0 \). In addition, note that \( |\epsilon \tau - \epsilon \tau| \leq \epsilon \), which gives us \( \epsilon \tau \downarrow 0 \). Now, we will take the limit for each of three terms in (20) separately. In particular, we resort to Taylor series expansion of the exponential terms and then take the limit as \( \eta \uparrow \infty \).

\[
\frac{e^{-\epsilon \tau} - e^{-\epsilon \tau + 1}}{1 - e^{-\epsilon \tau}} = \epsilon - \epsilon \tau - \epsilon (\tau + 1) + o(\epsilon \tau) = -\epsilon \tau - \epsilon (\tau + 1) + o(\epsilon \tau) \to 0 \quad (21a)
\]

\[
\frac{e^{-\epsilon \tau} + \frac{e^{-\epsilon \tau}}{m+\epsilon \mu^*} + o(\epsilon)}{1 - e^{-\epsilon \tau}} = \epsilon - t + \frac{\epsilon}{m+\epsilon \mu^*} + o(\epsilon) = -t + \frac{\epsilon}{m+\epsilon \mu^*} + o(1) \to 1/m - t \quad (21b)
\]

\[
\frac{e^{-\epsilon \tau} + \frac{e^{-\epsilon \tau}}{m+\epsilon \mu^*} + o(\epsilon)}{1 - e^{-\epsilon \tau}} = \epsilon - t + \frac{\epsilon}{m+\epsilon \mu^*} + o(\epsilon) = t + \frac{\epsilon}{m+\epsilon \mu^*} + o(1) \to 1/m + t \quad (21c)
\]

Now, by using (21) to simplify (20) we get

\[
\mathbb{E}[e^{\epsilon \tau \eta}] \to \frac{1}{2m} \left( \frac{1}{1/m + t} + \frac{1}{1/m - t} \right) = \frac{1}{1 - m^2 t^2} \quad \text{as} \quad \eta \uparrow \infty.
\]

Note that the above is the MGF of a Laplace distribution with parameters \((0, m)\). Thus, by Levy’s continuity theorem [77, Chapter 18], the proof of Case 1 is complete.

**Case 2:** Note that, in this case, we have \( \epsilon \tau \to l \in (0, \infty) \), \( \epsilon \downarrow 0 \) and \( \tau \uparrow \infty \). In addition, note that \( |\epsilon \tau - \epsilon \tau| \leq \epsilon \) which gives us \( \epsilon \tau \to l \). Now, we will evaluate the limit of (20) by using Taylor series expansion.

\[
\frac{e^{-\epsilon \tau} - e^{-\epsilon \tau + 1}}{1 - e^{-\epsilon \tau}} = \epsilon - \epsilon \tau - \epsilon (\tau + 1) + o(\epsilon \tau) = -\epsilon \tau - \epsilon (\tau + 1) + o(\epsilon \tau) \to e^{-lt} - e^{-lt} \quad (22a)
\]

\[
\frac{e^{-\epsilon \tau} + \frac{e^{-\epsilon \tau}}{m+\epsilon \mu^*} + o(\epsilon)}{1 - e^{-\epsilon \tau}} = \epsilon - t + \frac{\epsilon}{m+\epsilon \mu^*} + o(\epsilon) = -t + \frac{\epsilon}{m+\epsilon \mu^*} + o(1) \to 1/m - t \quad (22b)
\]

\[
\frac{e^{-\epsilon \tau} + \frac{e^{-\epsilon \tau}}{m+\epsilon \mu^*} + o(\epsilon)}{1 - e^{-\epsilon \tau}} = \epsilon - t + \frac{\epsilon}{m+\epsilon \mu^*} + o(\epsilon) = t + \frac{\epsilon}{m+\epsilon \mu^*} + o(1) \to e^{-lt} \quad (22c)
\]

Now, by using (22) to simplify (20) we get

\[
\mathbb{E}[e^{\epsilon \tau \eta}] \to \frac{1}{2l + 2m} \left( \frac{e^{-lt}}{1/m + t} + \frac{e^{lt} - e^{-lt}}{t} + \frac{e^{lt}}{1/m - t} \right) \quad (23)
\]

The above is the MGF of the hybrid distribution with parameters \((m, l)\). We can verify it as follows for \( X \sim \text{Hybrid}(b, c) \) as follows:

\[
\mathbb{E}[e^{\epsilon X}] = \int_{-\infty}^{\infty} \rho_{\text{Hybrid}}(x) e^{\epsilon x} dx = \frac{1}{2(b+c)} \left( \int_{-\infty}^{-c} e^{\frac{x}{b+c}} e^{\epsilon x} dx + \int_{-c}^{c} e^{\epsilon x} dx + \int_{c}^{\infty} e^{-\frac{tx}{b+c}} e^{\epsilon x} dx \right)
\]

\[
= \frac{1}{2(b+c)} \left( \frac{e^{c/b} e^{(1+b)t/c} - e^{c} + e^{c}}{t} \right) + \frac{e^{c} e^{c/(b+1/t-1/b)}}{t - 1/b}.
\]

Note that, the limiting value of \( \mathbb{E}[e^{\epsilon X}] \) can be similarly calculated. We omit the details here as they are repetitive. Intuitively, by substituting \( t \to t/l \) in (23), we get

\[
\mathbb{E}[e^{\epsilon \tau \eta}] \to \frac{1}{2 + 2m/l} \left( \frac{e^{-lt}}{l/m + t} + \frac{e^{lt} - e^{-lt}}{t} + \frac{e^{lt}}{l/m - t} \right)
\]
The above is the MGF of the Hybrid distribution with parameters \((m/l, 1)\). Thus, by Levy’s continuity theorem [77, Chapter 18], the proof of Case 2 is complete.

**Case 3:** In this case, we will consider the MGF of \(\frac{\tilde{z}_n}{\eta}\) and carry out similar calculations as in (20). First, let \(\eta > \eta_0\) such that \(\epsilon > 1\) which implies that \((1 - \epsilon/(m + \epsilon \mu^*))e^{t/\tau} \leq (1 - \epsilon/(m + \epsilon \mu^*))e^{t} < 1\). We skip some steps as they are repetitive and directly write the MGF below.

\[
\mathbb{E} \left[ e^{\frac{\tilde{z}_n}{\eta}} \right] = \frac{\epsilon}{2[\tau]e + \epsilon + 2m} e^{\frac{\epsilon}{m + \epsilon \mu^*}} \left( e^{-\frac{\epsilon}{m + \epsilon \mu^*}} + \frac{\epsilon}{m + \epsilon \mu^*} + o(1) \right) + \frac{\epsilon}{2[\tau]e + \epsilon + 2m} e^{-\frac{t}{\tau} - e^{\frac{\epsilon}{m + \epsilon \mu^*}} + o(1)} \right)
\]

(24)

Note that, in this case, we have \(\epsilon \tau \to \infty\) and \(\tau \to \infty\). In addition, note that \(|\epsilon|/\tau - \epsilon \leq \epsilon\) which gives us \(\epsilon \tau \to \infty\). In addition, as \(\tau \to \infty\), we have \(\tau \to 1\). Now, we will evaluate the limit of (20) by using Taylor series expansion.

\[
\frac{\epsilon}{2[\tau]e + \epsilon + 2m} e^{\frac{\epsilon}{m + \epsilon \mu^*}} = \frac{\epsilon}{2[\tau]e + \epsilon + 2m} e^{\frac{\epsilon}{m + \epsilon \mu^*}} - \frac{\epsilon}{2[\tau]e + \epsilon + 2m} + \frac{\epsilon}{2[\tau]e + \epsilon + 2m} + o(1) \to 0 \quad (25a)
\]

\[
\frac{\epsilon}{2[\tau]e + \epsilon + 2m} e^{-\frac{\epsilon}{m + \epsilon \mu^*}} - e^{\frac{\epsilon}{m + \epsilon \mu^*}} + o(1) \to 0 \quad (25b)
\]

\[
\frac{\epsilon}{2[\tau]e + \epsilon + 2m} e^{-\frac{t}{\tau} - e^{\frac{\epsilon}{m + \epsilon \mu^*}} + o(1)} \to e^{-e^{-t}} \quad (25c)
\]

Now, by using (25) in (24), we get

\[
\mathbb{E} \left[ e^{\frac{\tilde{z}_n}{\eta}} \right] \to \frac{e^{t} - e^{-t}}{2t} \Rightarrow \frac{\tilde{z}_n}{\tau} \to \mathcal{U}([-1, 1]),
\]

where the last assertion follows by the Levy’s continuity theorem [77, Chapter 18]. This completes the proof of Case 3.

**Appendix B: Positive Recurrence**

**Proof of Proposition 2** We will analyze the drift of the test function \(z^2\) and show that it is negative outside a finite set.

\[
\begin{align*}
\mathbb{E} \left[ z^2_n(k) + 1 - z^2_n(k) | z_n(k) = z \right] &
\overset{(a)}{=} \mathbb{E} \left[ (z_n(k) + a_n^*(z_n(k), k) - a_n^*(z_n(k), k))^2 - z^2_n(k) | z_n(k) = z \right]
\overset{(b)}{=} \mathbb{E} \left[ (a_n^*(z_n(k), k) - a_n^*(z_n(k), k))^2 + 2z_n(k)(a_n^*(z_n(k), k) - a_n^*(z_n(k), k)) | z_n(k) = z \right]
\overset{(c)}{=} \mathbb{E} \left[ (a_n^*(z, k) - a_n^*(z, k))^2 + 2z \mathbb{E} \left[ a_n^*(z, k) - a_n^*(z, k) \right] \right]
\leq 4A_{max}^2 + 2z \mathbb{E} \left[ a_n^*(z, k) - a_n^*(z, k) \right]
\end{align*}
\]

(26)

\[
\begin{align*}
\mathbb{E} \left[ z^2_n(k) + 1 - z^2_n(k) | z_n(k) = z \right] &
\overset{(a)}{=} \mathbb{E} \left[ (z_n(k) + a_n^*(z_n(k), k) - a_n^*(z_n(k), k))^2 - z^2_n(k) | z_n(k) = z \right]
\overset{(b)}{=} \mathbb{E} \left[ (a_n^*(z_n(k), k) - a_n^*(z_n(k), k))^2 + 2z_n(k)(a_n^*(z_n(k), k) - a_n^*(z_n(k), k)) | z_n(k) = z \right]
\overset{(c)}{=} \mathbb{E} \left[ (a_n^*(z, k) - a_n^*(z, k))^2 + 2z \mathbb{E} \left[ a_n^*(z, k) - a_n^*(z, k) \right] \right]
\leq 4A_{max}^2 + 2z \mathbb{E} \left[ a_n^*(z, k) - a_n^*(z, k) \right]
\end{align*}
\]
\[
\begin{align*}
&\leq 4A_{\text{max}}^2 + 4\epsilon\tau K\phi_{\text{max}} - 2\epsilon|z|\delta \mathbb{1}\{|z| > K\tau\} \\
&\leq -4A_{\text{max}}^2 - 4\epsilon\tau K\phi_{\text{max}} - 2\epsilon|z|\delta \mathbb{1}\{|z| > K\tau\},
\end{align*}
\]

where (a) follows by the update equation of imbalance given by (1). Next, (b) follows as \(|a^e(z, k)| \leq A_{\text{max}}\) and \(|a^s(z, k)| \leq A_{\text{max}}\) with probability 1 for all \(z \in \mathbb{Z}\) and \(k \in \mathbb{Z}_+\). Now, (c) follows by the definition of \(\lambda(\cdot)\) and \(\mu(\cdot)\) given by (2) and noting that \(\lambda^* = \mu^*\) by definition. Finally, (d) follows as \(\phi^e(\cdot)\) and \(\phi^s(\cdot)\) are bounded and satisfy the negative drift condition given by Condition 1.

So, for any given \(\epsilon > 0\) and \(\tau > 0\), we have negative drift outside a finite set. Thus, for any \(\eta > 0\), by the Foster-Lyapunov theorem, \(\{z_\eta(k) : k \in \mathbb{Z}_+\}\) is positive recurrent. By Moment bound theorem \([32, \text{Proposition 6.14}]\) and (27), we have

\[
-2\epsilon \delta \mathbb{E}[|\bar{z}_\eta|] \leq 4A_{\text{max}}^2 + 4\epsilon\tau K(2\phi_{\text{max}} + \delta)
\]

Lastly, by Moment bound theorem \([32, \text{Proposition 6.14}]\) and (26), we get

\[
-\mathbb{E}\left[\bar{z}_\eta \left(\phi^e\left(\frac{\bar{z}_\eta}{\tau}\right) - \phi^s\left(\frac{\bar{z}_\eta}{\tau}\right)\right)\right] \leq \frac{2A_{\text{max}}^2}{\epsilon}
\]

This completes the proof.

C.1. Tightness

Proof of Lemma 1 First, we will show tightness of the sequence of random variables \(\epsilon \bar{z}_\eta\). As \(\epsilon \tau \downarrow l\), there exists \(\tau_{\text{max}} > 0\) such that \(\epsilon \tau \leq \tau_{\text{max}}\) for all \(\eta > 0\). Now, by Proposition 2, we have

\[
\mathbb{E}[\epsilon |\bar{z}_\eta|] \leq \frac{2A_{\text{max}}^2 + \epsilon\tau K(2\phi_{\text{max}} + \delta)}{\delta} \leq \frac{2A_{\text{max}}^2 + \tau_{\text{max}} K(2\phi_{\text{max}} + \delta)}{\delta}
\]

Now, for any \(\omega > 0\), we can pick \(N = \frac{2A_{\text{max}}^2 + \tau_{\text{max}} K(2\phi_{\text{max}} + \delta)}{\delta \omega}\) independent of \(\eta\) to get

\[
\mathbb{P}(\epsilon |\bar{z}_\eta| > N) \leq \frac{1}{N} \mathbb{E}[\epsilon \bar{z}] = \omega \quad \forall \eta > 0.
\]

Thus, the family of random variables \(\Pi = \{\epsilon \bar{z}_\eta\}\) is tight. By \([11, \text{Theorem 5.1}]\), \(\Pi\) is relatively compact. Thus, for any sequence in the family \(\Pi\), there exists a sub-sequence that converges in distribution.

C.2. Drift Analysis

Before proving Claim 1, we need the following technical lemma which we present with proof below.

**Lemma 12.** For some \(c \in \mathbb{R}\), let \(g_n\) be a sequence of functions such that

\[
\lim_{n \uparrow \infty} g_n(x) = c \quad \text{uniformly on compact sets } \forall x \in \mathbb{R}.
\]

Then for any tight sequence of random variables \(\{Y_n\}\), we have

\[
\lim_{n \uparrow \infty} \mathbb{E}[g_n(Y_n)] = c.
\]
Note that the above lemma is similar to Lemma 7 with key differences. Lemma 7 assumed uniform convergence of $g_n(\cdot)$ which is much stronger than uniform convergence over compact sets. This allows one to relax the tightness condition on the sequence of random variables $\{Y_n\}$ in Lemma 7. Now, we present the proof of Lemma 12.

**Proof of Lemma 12** For any $\delta, K > 0$, there exists $n_0$ such that for all $n > n_0$

$$|g_n(x) - c| \leq \delta \quad \forall x \in B[0, K],$$

where $B[0, K]$ is a closed ball with center 0 and radius $K$. As the sequence of random variables $\{Y_n\}$ is tight, for any $\delta' > 0$, there exists $K > 0$ such that $\mathbb{P}(Y_n \notin B[0, K]) \leq \delta'$ for all $n$. Pick such a $K$ for the rest of the proof. Thus for all $n \geq n_0$

$$\mathbb{P}(|g_n(Y_n) - c| > \delta) \leq \mathbb{P}(Y_n \notin B[0, K]) \leq \delta'$$

Thus for all $n \geq n_0$, by using Jensen’s inequality, we have

$$|\mathbb{E}[g_n(Y_n) - c]| \leq \mathbb{E}[|g_n(Y_n) - c|] \leq \delta(1 - \delta') + g_{\text{max}}\delta'$$

Now, by taking $n \uparrow \infty$, we get

$$\lim_{n \uparrow \infty} |\mathbb{E}[g_n(Y_n)] - c| \leq \delta(1 - \delta') + g_{\text{max}}\delta'$$

As the above is true for all $\delta, \delta' > 0$, we get

$$\lim_{n \uparrow \infty} \mathbb{E}[g_n(Y_n)] = c.$$

This completes the proof of the Lemma. \hfill \Box

Now, we will use the above lemma to prove Claim 1.

**Proof of Claim 1** We rewrite the pre-limit equation after dividing both sides by $\frac{2}{\sigma^*(\lambda^*) + \sigma^*(\mu^*)}$.

$$\frac{2}{\sigma^*(\lambda^*) + \sigma^*(\mu^*)} \mathbb{E}[e^{j \omega \bar{z}_n} (\phi^s(\bar{z}_n) - \phi^c(\bar{z}_n))]$$

$$= \frac{j \omega}{\sigma^*(\lambda^*) + \sigma^*(\mu^*)} \mathbb{E}[e^{j \omega \bar{z}_n} (\sigma^c(\lambda^*) \phi^s(\mu^*) - \sigma^c(\lambda^*) - \sigma^s(\mu^*)) + o(1)].$$

Now, the first term in the RHS converges by Levy’s continuity theorem (e.g. see: [77, Chapter 18]). In particular, if a sequence of random variables converges in distribution, the corresponding characteristic functions also converge.

$$\lim_{\eta \uparrow \infty} \mathbb{E}[e^{j \omega \bar{z}_n}] = \mathbb{E}[e^{j \omega \bar{z}_\infty}] \quad (\text{as } \epsilon \bar{z}_\eta \xrightarrow{D} \bar{z}_\infty).$$

Next, the second term in RHS can be bounded as follows:

$$\frac{j \omega}{\sigma^*(\lambda^*) + \sigma^*(\mu^*)} \mathbb{E}[e^{j \omega \bar{z}_n} (\sigma^c(\lambda^*) \phi^s(\mu^*) - \sigma^c(\lambda^*) - \sigma^s(\mu^*))]$$

$$\leq \frac{j \omega}{\sigma^*(\lambda^*) + \sigma^*(\mu^*)} \mathbb{E}[\sigma^c(\lambda^*) \phi^s(\mu^*) - \sigma^c(\lambda^*) - \sigma^s(\mu^*)].$$

Now, by Lemma 12 the upper bound converges to zero, since by the continuity of $\sigma^c(\cdot)$ and $\sigma^s(\cdot)$, we have

$$\lim_{\eta \uparrow \infty} (\sigma^c(\lambda^* + \epsilon \phi^c(x)) + \sigma^s(\mu^* + \epsilon \phi^s(x)) - \sigma^c(\lambda^*) - \sigma^s(\mu^*)) = 0 \quad \text{uniformly } \forall x \in \mathbb{R}. $$
Finally, the LHS can be simplified as follows:

\[
\frac{2}{\sigma^2(\lambda^*) + \sigma^2(\mu^*)} \mathbb{E} \left[ e^{j\omega \tilde{z}_\eta} \left( \phi^* \left( \frac{\tilde{z}_\eta}{\tau} \right) - \phi^* \left( \frac{\tilde{z}_\eta}{\tau} \right) \right) \right] = \mathbb{E} \left[ e^{j\omega \tilde{z}_\eta} \frac{l}{\epsilon \tau} \tilde{z}_\eta \right].
\]

By Heine-Cantor Theorem [61, Theorem 4.19], as \( g_{1,l} \) is continuous, it is uniformly continuous over any compact set. Now, since \( \epsilon \tau \to l \) as \( \eta \to \infty \) and \( g_{1,l} \) is a uniformly continuous on compact sets and bounded, for any \( \delta' > 0 \) and any compact set \( B \), there exists \( \eta_0 \) such that for all \( \eta > \eta_0 \)

\[
\left| g_{1,l} \left( \frac{l}{\epsilon \tau} x \right) - g_{1,l}(x) \right| \leq \delta' \quad \forall x \in B.
\]

Now, multiplying both sides by \( |e^{j\omega x}| = 1 \) on both sides, we get

\[
\left| e^{j\omega x} \left( g_{1,l} \left( \frac{l}{\epsilon \tau} x \right) - g_{1,l}(x) \right) \right| \leq \delta' \quad \forall x \in B.
\]

Thus, \( e^{j\omega x} (g_{1,l} \left( \frac{l}{\epsilon \tau} x \right) - g_{1,l}(x)) \) converges to zero uniformly over compact sets. Note that, \( \epsilon \tilde{z}_\eta \) is tight by Lemma 1. Thus, by Lemma 12 and Jensen’s inequality, we get

\[
\lim_{\eta \to \infty} \mathbb{E} \left[ e^{j\omega \tilde{z}_\eta} g_{1,l} \left( \frac{l}{\epsilon \tau} \tilde{z}_\eta \right) \right] = \mathbb{E} \left[ e^{j\omega \tilde{z}_\infty} g_{1,l}(\tilde{z}_\infty) \right].
\]

Now, since we are working with a convergent sub-sequence such that \( \epsilon \tilde{z}_\eta \xrightarrow{D} \tilde{z}_\infty \) and since \( e^{j\omega x} g_{1,l}(x) \) is a continuous bounded functions, by the definition of weak convergence, we have

\[
\lim_{\eta \to \infty} \mathbb{E} \left[ e^{j\omega \tilde{z}_\eta} g_{1,l} \left( \frac{l}{\epsilon \tau} \tilde{z}_\eta \right) \right] = \mathbb{E} \left[ e^{j\omega \tilde{z}_\infty} g_{1,l}(\tilde{z}_\infty) \right].
\]

This completes the proof. \( \square \)

### C.3. Existence and Uniqueness

#### C.3.1. Existence of a Solution

In this subsection, we will show that Gibbs(\( g \)) is a solution of the implicit equation.

**Lemma 13.** For \( g_{1,l} \) defined in (6), Gibbs(\( g \)) is well defined and it satisfies (13).

**Proof of Lemma 13** By Condition 1, we have

\[
|e^{-f_0^x} g_{1,l}(y) dy| \leq e^{g_{\max} K} \left\{ |x| \leq K \right\} + e^{-\delta'(|x| - K)}
\]

where \( g_{\max} = \frac{4g_{\max}}{\sigma^2(\lambda^*) + \sigma^2(\mu^*)} \) and \( \delta' = 2\delta / (\sigma^2(\lambda^*) + \sigma^2(\mu^*)) > 0 \). By the above equation, we have

\[
\int_{-\infty}^{\infty} |e^{-f_0^x} g_{1,l}(y) dy| \leq 2K e^{g_{\max} K} + \int_{-\infty}^{\infty} e^{-\delta'(|x| - K)}
\]

\[
\leq 2K \left( e^{g_{\max} K} + e^{K \delta'} \right) + \frac{2}{\delta} < \infty.
\]

This implies that Gibbs(\( g_{1,l} \)) is well defined. Now, we show that Gibbs(\( g \)) satisfies (13).

\[
\mathbb{E} \left[ e^{j\omega \tilde{z}_\infty} g_{1,l}(\tilde{z}_\infty) \right] = \frac{1}{C} \int_{-\infty}^{\infty} e^{j\omega x} g_{1,l}(x) e^{-f_0^x} g_{1,l}(y) dy dx.
\]

\[
= \frac{1}{C} \int_{-\infty}^{\infty} e^{j\omega x} \frac{d}{dx} \left( e^{-f_0^x} g_{1,l}(y) \right) dx
\]

\[
= \frac{1}{C} \int_{-\infty}^{\infty} e^{j\omega x} e^{-f_0^x} g_{1,l}(y) dy \left|_{-\infty}^{\infty} + \frac{j\omega}{C} \int_{-\infty}^{\infty} e^{j\omega x} e^{-f_0^x} g_{1,l}(y) dy dx
\]

\[
= \frac{j\omega}{C} \int_{-\infty}^{\infty} e^{j\omega x} e^{-f_0^x} g_{1,l}(y) dy dx
\]

\[
= j\omega \mathbb{E} \left[ e^{j\omega \tilde{z}_\infty} \right],
\]
where (a) follows by the Fundamental theorem of calculus [59, Theorem 7.11], (b) follows by integration by parts, and (c) follows as $|e^{jωx}| = 1$, $\lim_{x \to \infty} e^{-\int_0^x g_{1,l}(y)dy} = 0$ and $\lim_{x \to -\infty} e^{-\int_0^x g_{1,l}(y)dy} = 0$. In particular, we have $\lim_{x \to \pm \infty} \int_0^x g_{1,l}(y)dy = 0$ by Condition 1. This completes the proof. □

C.3.2. Schwartz Space and Tempered Distribution

Before presenting the proof of Lemma 3, we brief on the theory of Schwartz functions and tempered distributions [60, Chapter 7] [40, Chapter 11].

Definition 3. A Schwartz space is a vector space given by
\[ S(\mathbb{R}) = \{ f \in C^\infty(\mathbb{R}) : ||f||_{α,β} < \infty \ \forall \ α, β \in \mathbb{Z}^+ \} \]
where $||f||_{α,β} = \sup_{x \in \mathbb{R}} |x^α f^{(β)}(x)|$ and $f^{(β)}(·)$ is the $β$-th derivative of $f(·)$.

Intuitively, Schwartz functions are infinitely differentiable and each derivative decays faster than any polynomial. Now, we define the dual space of Schwartz functions which are called tempered distributions.

Definition 4 (Tempered Distribution). A linear continuous functional $T : S(\mathbb{R}) \to \mathbb{R}$ from the Schwartz space to the reals is called a Tempered distribution.

Thus, the set of tempered distributions is the dual space of the Schwartz space. A famous tempered distribution is the impulse function $δ[·]$ which operates on Schwartz functions and gives it’s value at 0. In particular, $δ[φ] = φ(0)$. The notion of tempered distribution generalizes the notion of a function. For any continuous and bounded function $f$, we can define it’s ‘corresponding’ tempered distribution as follows:
\[ T_f[φ] \Delta = \int_{\mathbb{R}} f(x)φ(x)dx. \]

Now, we will discuss some operations on tempered distributions which we will use in the proof of Theorem 1. We will present some definitions and theorems without proofs.

Definition 5 (Distributional Derivative). [60, Section 6.12]
\[ T'[φ] \Delta = -T[φ'] \]

The above definition is motivated by considering $T_f$ for some $f$.
\[ T_f'[φ] = \int_{\mathbb{R}} f'(x)φ(x)dx = f(x)φ(x)|_\infty^\infty - \int_{\mathbb{R}} f(x)φ'(x)dx = -T_f'[φ] \]

It can be shown that $T'$ is also a tempered distribution by checking linearity and continuity.

Fourier transform of a Schwartz function is given by
\[ F(φ)(x) = φ(x) = \frac{1}{\sqrt{2π}} \int_{\mathbb{R}} φ(y)e^{-jxy}dy. \] (28)

In addition, the inverse Fourier transform also exists for a Schwartz function and by Fourier inversion theorem, we have
\[ F^{-1}(φ)(x) = \frac{1}{\sqrt{2π}} \int_{\mathbb{R}} φ(y)e^{jxy}dy. \]
**Definition 6.** Fourier transform of tempered distribution is defined as
\[
\hat{T}[\varphi] \triangleq \mathcal{F}(T)[\varphi] = T[\hat{\varphi}]
\]
and the inverse is given by
\[
\check{T}[\varphi] \triangleq \mathcal{F}^{-1}(T)[\varphi] = T[\check{\varphi}].
\]
We will need the following results about tempered distributions which we present without proof.

**Proposition 4.** The following statements are true:
(a) The Fourier transform is a bijection on \(S(\mathbb{R})\) [60, Theorem 7.7].
(b) Multiplication by a test function: For \(\gamma \in C^\infty_{\text{pol}}(\mathbb{R})\), we have \(T[\gamma \varphi] = T[\gamma \varphi] [60, \text{Section 6.15}].\)

**C.3.3. Uniqueness: Proof of Lemma 3**

**Proof** Define a tempered distribution
\[
T_{\tilde{z}_\infty}[\varphi] = \int_{\mathbb{R}} \varphi(x) dF_{\tilde{z}_\infty}(x) = \mathcal{E}[\varphi],
\]
where \(F_{\tilde{z}_\infty}\) is the CDF of \(\tilde{z}_\infty\). It is straightforward to verify that it is a tempered distribution as it is linear and continuous [60, Example 7.12(b)]. Now, we will write (13) in the form of a differential equation involving tempered distribution \(T_{\tilde{z}_\infty}\).

**Claim 6.** The functional equation (13) implies the following:
\[
T_{\tilde{z}_\infty}[\varphi' + g_{1,1} \varphi] = 0 \quad \forall \varphi \in S(\mathbb{R}).
\]
The above equation is similar to the differential equation mentioned in the sketch of the proof in Section 4.3.1. The advantage of using tempered distribution is apparent now as the above equation is valid even if \(\tilde{z}_\infty\) does not exhibit a PDF. Now, to complete the proof, we will show that the above equation has Gibbs\((g_{1,1})\) as the unique solution. This implies that (13) has at most one solution. Combining this with Lemma 13, the proof will be complete.

As Fourier transform is a bijection on \(S(\mathbb{R})\) (Proposition 4 (a)), the above is equivalent to
\[
T_{\tilde{z}_\infty}[\varphi' + g_{1,1} \varphi] = 0 \quad \forall \varphi \in S(\mathbb{R}).
\]
This can be written as the following differential equation in tempered distribution using the definition of derivative of a distribution given in Definition 5 along with Proposition 4 (b).
\[
T'_{\tilde{z}_\infty}[\varphi] = g_{1,1} T_{\tilde{z}_\infty}[\varphi] \quad \forall \varphi \in S(\mathbb{R}).
\]
By [26, Theorem 4] [27, Theorem 3.9], we can directly write the entire solution of the above differential equation to be
\[
T_{\tilde{z}_\infty}[\varphi] = C \int_{-\infty}^{\infty} e^{-G(x)} \varphi(x) dx \quad \forall \varphi \in S(\mathbb{R})
\]
for any constant \(C\) and \(G(x) = \int_0^x g(t) dt\). The steps to solve this differential equation is given in [27, Section IV B]. By the definition of \(T_{\tilde{z}_\infty}\), we conclude that \(\tilde{z}_\infty\) has density
\[
\rho_{\tilde{z}_\infty}(x) = C e^{-G(x)}, \text{ where } C = \frac{1}{\int_{-\infty}^{\infty} e^{-G(x)} dx}.
\]
Thus, the above solution can be the only possible solution of (13). Combining this result with Lemma 13, the proof is complete. Note that the differential equation defined in Claim 6 has multiple solutions but only a unique solution that defines a probability measure. In particular, for any \(C \in \mathbb{R}\), (29) is a solution but \(C\) has to be the normalizing constant to ensure that it defines a probability measure.
\[\square\]
Proof of Claim 6 Replace $\omega$ with $-\omega$ in (13) and note that for any $\varphi \in \mathcal{S}$, we have

$$\int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} \mathbb{E}\left[ e^{-j\omega z_{\infty}} \right] \varphi(\omega) d\omega = \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} g_{1,1}(z) e^{-j\omega z} \varphi(\omega) dF_{z_{\infty}}(z) d\omega$$

where $(a)$ follows by Fubini’s Theorem as $|\varphi(\omega)|$ is integrable with respect to $dF_{z_{\infty}} d\omega$. Next, $(b)$ follows by the definition of Fourier transform given by (28). Finally, $(d)$ follows by the derivative theorem of Fourier transform. In addition, we also have

$$\int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} g_{1,1}(z) e^{-j\omega z} \varphi(\omega) dF_{z_{\infty}}(z) d\omega$$

where $(a)$ follows by Fubini’s Theorem as $|\varphi(\omega)g_{1,1}(z)|$ is integrable with respect to $dF_{z_{\infty}} d\omega$. Next, $(b)$ follows by the definition of Fourier transform given by (28) and $(c)$ follows by the definition of the Tempered distribution $T_{z_{\infty}}$. Multiplying (12) on both sides by $\varphi(\cdot)$ and integrating and using (30) and (31) will imply

$$T_{z_{\infty}} [\varphi' + g_{1,1}\varphi] = 0 \quad \forall \varphi \in \mathcal{S}(\mathbb{R}).$$

This completes the proof of the claim.

Appendix D: Proof of Preliminary Lemmas Required for Theorem 2 and 3

Proof of Lemma 4 We consider $z g_\eta(z)$ as the test function. Note that $\mathbb{E}[|\tilde{z}_\eta g_\eta(\tilde{z}_\eta)|] \leq G_\eta \mathbb{E}[|\tilde{z}_\eta|] < \infty$ by Lemma 2. Thus, we can set the drift of this function to zero in steady state. We have

$$0 = \mathbb{E}\left[ \tilde{z}_\eta^+ g_\eta(\tilde{z}_\eta^+) - \tilde{z}_\eta g_\eta(\tilde{z}_\eta) \right] = \mathbb{E}\left[ \tilde{z}_\eta^+ (g_\eta(\tilde{z}_\eta^+) - g_\eta(\tilde{z}_\eta)) \right] + \mathbb{E}\left[ (\tilde{z}_\eta^+ - \tilde{z}_\eta) g_\eta(\tilde{z}_\eta) \right] = \mathbb{E}\left[ \tilde{z}_\eta^+ (g_\eta(\tilde{z}_\eta^+) - g_\eta(\tilde{z}_\eta)) \right] + \mathbb{E}\left[ (\tilde{a}^\eta(z,k) - \tilde{a}^\eta(\tilde{z}_\eta)) g_\eta(\tilde{z}_\eta) \right]$$

where the last inequality follows by the tower property of expectation and Eq. (2). This completes the proof.

Proof of Lemma 5 For any given $x \in \mathbb{Z}$, consider the test function

$$V_z(z) = z \mathbb{1}\{ \text{sgn}(x)z > \text{sgn}(x)x \}.$$

We will set the drift of above defined test function to zero in steady state to get a bound on $\mathbb{P}(\tilde{z}_\eta = x)$. Alternatively, one can analyze the drift of $z \mathbb{1}\{ z > x \}$ for $x \geq 0$ and $z \mathbb{1}\{ z < x \}$ for $x < 0$. To make the presentation compact, we consider $V_z(z)$ which combines both the cases mentioned
above. For a better understanding of the proof, the reader can consider the case when \( x \geq 0 \) and \( x < 0 \) separately.

Note that, by Lemma 2, we have \( \mathbb{E}[|V_z(\bar{z}_\eta)|] \leq \mathbb{E}[|\bar{z}_\eta|] < \infty \). Thus, setting the drift to zero of \( V_z(\cdot) \) in steady state, we get

\[
0 = \mathbb{E}[\bar{z}_\eta^+ \{ \text{sgn}(x)\bar{z}_\eta^+ > \text{sgn}(x)x \} - \bar{z}_\eta \{ \text{sgn}(x)\bar{z}_\eta > \text{sgn}(x)x \}]
\]

\[
= \mathbb{E}[\bar{z}_\eta^+ \{ \text{sgn}(x)\bar{z}_\eta^+ > \text{sgn}(x)x \} - \bar{z}_\eta^+ \{ \text{sgn}(x)\bar{z}_\eta > \text{sgn}(x)x \}]
+ \mathbb{E}[\bar{z}_\eta \{ \text{sgn}(x)\bar{z}_\eta > \text{sgn}(x)x \} - \bar{z}_\eta \{ \text{sgn}(x)\bar{z}_\eta > \text{sgn}(x)x \}]
\]

\[
= \mathbb{E}[\bar{z}_\eta^+ \{ \text{sgn}(x)\bar{z}_\eta^+ > \text{sgn}(x)x \} - \bar{z}_\eta^+ \{ \text{sgn}(x)\bar{z}_\eta > \text{sgn}(x)x \}]
+ \mathbb{E}[(\phi^c(\frac{\bar{z}_\eta}{\tau}) - \phi^s(\frac{\bar{z}_\eta}{\tau})) \{ \text{sgn}(x)\bar{z}_\eta > \text{sgn}(x)x \}]
\]

In step (a), we add and subtract \( \bar{z}_\eta^+ \{ \text{sgn}(x)\bar{z}_\eta > \text{sgn}(x)x \} \) and \( \bar{z}_\eta \{ \text{sgn}(x)\bar{z}_\eta > \text{sgn}(x)x \} \) and note that the term whose expectation we are calculating is non-negative with probability 1 when \( x \geq 0 \) and non-positive with probability 1 when \( x < 0 \). Lastly, (f) follows by expanding the expectation and only preserving the term corresponding to \( \bar{z}_\eta = x \). Now, using the above equation, for all \( x \in \mathbb{Z} \), by Assumption 1, we have

\[
\mathbb{P}(\bar{z}_\eta = x) \leq \frac{\epsilon}{\mathbb{B}_\min} \mathbb{E}\left[\left|\phi^c\left(\frac{\bar{z}_\eta}{\tau}\right) - \phi^s\left(\frac{\bar{z}_\eta}{\tau}\right)\right|\right].
\]

This completes the proof. \( \square \)

**Proof of Lemma 6** For any given \( \delta > 0 \), there exists \( K > 0 \) such that \( |g(x) - c_\infty| \leq \delta \) and \( |g(-x) - c_\infty| \leq \delta \) for all \( x \geq K \). Thus, we have

\[
\mathbb{P}\left(\left|g\left(\frac{\bar{z}_\eta}{\tau}\right) - c_\infty\right| > \delta\right) = \mathbb{P}\left(\left|g\left(\frac{\bar{z}_\eta}{\tau}\right) - c_\infty\right| > \delta\right| |\bar{z}_\eta| \leq K\tau)\mathbb{P}(|\bar{z}_\eta| \leq K\tau)
+ \mathbb{P}\left(\left|g\left(\frac{\bar{z}_\eta}{\tau}\right) - c_\infty\right| > \delta\right| |\bar{z}_\eta| > K\tau)\mathbb{P}(|\bar{z}_\eta| > K\tau)
= \mathbb{P}\left(\left|g\left(\frac{\bar{z}_\eta}{\tau}\right) - c_\infty\right| > \delta\right| |\bar{z}_\eta| \leq K\tau)\mathbb{P}(|\bar{z}_\eta| \leq K\tau)
\]
\[ \leq \mathbb{P}(|\tilde{z}_n| \leq K\tau) \]
\[ \leq \frac{(2K\tau + 1)\epsilon}{p_{\min}} \mathbb{E}\left[ \left| \phi^c\left(\frac{\tilde{z}_n}{\tau}\right) - \phi^s\left(\frac{\tilde{z}_n}{\tau}\right) \right| \right] \]
\[ \leq \frac{2(2K\tau + 1)\epsilon\phi_{\max}}{p_{\min}}, \]

where the second last inequality follows by Lemma 5. Thus, we have

\[ \mathbb{E}\left[ \left| g\left(\frac{\tilde{z}_n}{\tau}\right) - c_\infty \right| \right] \leq \delta \mathbb{P}\left( \left| g\left(\frac{\tilde{z}_n}{\tau}\right) - c_\infty \right| > \delta \right) + (g_{\max} + c_\infty)\frac{2(2K\tau + 1)\epsilon\phi_{\max}}{p_{\min}} \]

Now, by taking the limit as \( \eta \uparrow \infty \), and noting that \( \epsilon\tau \uparrow \infty \) and \( \epsilon \uparrow \infty \), we get

\[ \lim_{\eta \uparrow \infty} \mathbb{E}\left[ g\left(\frac{\tilde{z}_n}{\tau}\right) \right] - c_\infty \leq \delta. \]

As \( \delta > 0 \) is arbitrary and LHS is independent of \( \delta \), the proof is complete. \( \square \)

**Proof of Lemma 7** For any \( \delta, K > 0 \), there exists \( n_0 \) such that for all \( n > n_0 \)

\[ |g_n(x) - c| \leq \delta \quad \forall x \in \mathbb{R}, \]

By substituting \( Y_n \) for \( x \), taking expectation on both sides, and using Jensen’s inequality, we get

\[ |\mathbb{E}[g_n(x) - c]| \leq \mathbb{E}[|g_n(x) - c|] \leq \delta \]

Now, by taking \( n \uparrow \infty \), we get

\[ \lim_{n \uparrow \infty} \mathbb{E}[g_n(Y_n)] - c \leq \delta \]

As the above is true for all \( \delta > 0 \), the proof is complete.

**Appendix E: Technical Details for the Proof of Theorem 2 and 3**

**E.1. Proof of Lemma 9**

**Proof of Lemma 9** For \( \omega \in \mathbb{R} \), we define the test function

\[ U(z) \overset{\Delta}{=} \text{sgn}(z)e^{j\omega z\text{sgn}(z)} \]

Now, consider the one step drift of the above test function.

\[ \Delta U(\tilde{z}_n) = \text{sgn}(\tilde{z}_n^+)e^{j\omega \tilde{z}_n^+\text{sgn}(\tilde{z}_n^+)} - \text{sgn}(\tilde{z}_n)e^{j\omega \text{sgn}(\tilde{z}_n)} \]
\[ = \underbrace{\text{sgn}(\tilde{z}_n^+)e^{j\omega \tilde{z}_n^+\text{sgn}(\tilde{z}_n^+)} - \text{sgn}(\tilde{z}_n)e^{j\omega \text{sgn}(\tilde{z}_n)}}_{T_6} \]
\[ + \underbrace{\text{sgn}(\tilde{z}_n)e^{j\omega \tilde{z}_n\text{sgn}(\tilde{z}_n)}}_{T_7} \]

We will analyze each of the above terms separately. We simplify \( T_6 \) by using Taylor’s Theorem to get

**Claim 7.**

\[ \mathbb{E}[T_6] = o(\epsilon^2). \]
The proof of the Claim 7 has been deferred to Appendix E.2 and here we continue with the proof of Lemma 9. Now, to simplify $T_7$, note that $T_7 = T_8 \text{sgn}(\bar{z}_n)$, where $T_8$ is defined in (15). Thus, by Claim 2, we have

$$E[|T_7| \bar{z}_n] = e^{j\omega \bar{z}_n \text{sgn}(\bar{z}_n)} \left( \phi^c \left( \frac{\bar{z}_n}{\tau} \right) - \phi^\ast \left( \frac{\bar{z}_n}{\tau} \right) \right) - \frac{1}{2} e^{j\omega \bar{z}_n \text{sgn}(\bar{z}_n)} \left( \phi \left( \frac{\bar{z}_n}{\tau} \right) - \phi^\ast \left( \frac{\bar{z}_n}{\tau} \right) \right) + o(\epsilon^2)$$

Note that, $|E[|U(\bar{z}_n)|]| \leq E[|U(\bar{z}_n)|] = 1$. Thus, by setting the drift of $U(\cdot)$ to zero in steady state, we get $E[T_8] + E[T_7] = 0$. Now, by substituting $T_8$ and $T_7$ and dividing by $j\epsilon^2\omega$, we get

$$E \left[ e^{j\omega \bar{z}_n \text{sgn}(\bar{z}_n)} \left( \phi^c \left( \frac{\bar{z}_n}{\tau} \right) - \phi^\ast \left( \frac{\bar{z}_n}{\tau} \right) \right) + \frac{j\omega}{2} E \left[ \text{sgn}(\bar{z}_n) e^{j\omega \bar{z}_n \text{sgn}(\bar{z}_n)} \left( \phi \left( \frac{\bar{z}_n}{\tau} \right) - \phi^\ast \left( \frac{\bar{z}_n}{\tau} \right) \right) \right] + o(1) = 0.$$

Now, by adding and subtracting certain terms, we get

$$\left( -1 + \frac{j\omega}{2} \left( \sigma^c(\lambda^*) + \sigma^\ast(\mu^*) \right) \right) E \left[ \text{sgn}(\bar{z}_n) e^{j\omega \bar{z}_n \text{sgn}(\bar{z}_n)} \right]$$

$$= - E \left[ e^{j\omega \bar{z}_n \text{sgn}(\bar{z}_n)} \left( 1 + \left( \phi^c \left( \frac{\bar{z}_n}{\tau} \right) - \phi^\ast \left( \frac{\bar{z}_n}{\tau} \right) \right) \text{sgn}(\bar{z}_n) \right]$$

$$+ \frac{j\omega}{2} E \left[ \text{sgn}(\bar{z}_n) e^{j\omega \bar{z}_n \text{sgn}(\bar{z}_n)} \left( \sigma^c(\lambda^*) + \sigma^\ast(\mu^*) - \sigma^c(\lambda(\bar{z}_n)) - \sigma^\ast(\mu(\bar{z}_n)) \right) \right] + o(1).$$

Now, by taking the absolute value on both sides and upper bounding the LHS by using triangle inequality and Jensen’s inequality, we get

$$\left( 1 + \omega \left( \sigma^c(\lambda^*) + \sigma^\ast(\mu^*) \right) \right)^2$$

$$\leq \left\{ \frac{\omega}{2} E \left[ \left( \sigma^c(\lambda^*) + \sigma^\ast(\mu^*) - \sigma^c(\lambda(\bar{z}_n)) - \sigma^\ast(\mu(\bar{z}_n)) \right) \right] \right\} + \frac{\omega}{2} E \left[ \left( \phi^c \left( \frac{\bar{z}_n}{\tau} \right) - \phi^\ast \left( \frac{\bar{z}_n}{\tau} \right) \right) \text{sgn}(\bar{z}_n) \right] + o(1).$$

Lastly, by taking the limit $\eta \uparrow \infty$ and using Claim 3, we get the result.

### E.2. Proof of Claims for Lemma 8 and 9

#### Proof of Claim 2 $T_2$

$T_2$ can be simplified as follows:

$$E[T_2 \bar{z}_n] = E \left[ e^{j\omega \bar{z}_n \text{sgn}(\bar{z}_n)} - e^{j\omega \bar{z}_n \text{sgn}(\bar{z}_n)} \right]$$

$$= E \left[ e^{j\omega \bar{z}_n} - e^{j\omega \bar{z}_n} \right] 1 \{ \bar{z}_n \geq 0 \} + E \left[ e^{-j\omega \bar{z}_n} - e^{-j\omega \bar{z}_n} \right] 1 \{ \bar{z}_n < 0 \}$$

$$= E \left[ e^{j\omega \bar{z}_n} \left( e^{j\omega \bar{a}_n(\bar{z}_n)} - e^{j\omega \bar{a}_n(\bar{z}_n)} \right) - 1 \{ \bar{z}_n \geq 0 \} + e^{-j\omega \bar{z}_n} \left( e^{-j\omega \bar{a}_n(\bar{z}_n)} - e^{-j\omega \bar{a}_n(\bar{z}_n)} \right) - 1 \{ \bar{z}_n < 0 \} \right]$$

$$\leq E \left[ e^{j\omega \bar{z}_n} \left( j \epsilon^2 \omega a^c(\bar{z}_n) - a^c(\bar{z}_n) \right) - \frac{1}{2} \epsilon^2 \omega^2 \left( a^c(\bar{z}_n) - a^c(\bar{z}_n) \right) \right] 1 \{ \bar{z}_n \geq 0 \} + o(\epsilon^2)$$

$$+ E \left[ e^{-j\omega \bar{z}_n} \left( -j \epsilon^2 \omega a^c(\bar{z}_n) - a^c(\bar{z}_n) \right) - \frac{1}{2} \epsilon^2 \omega^2 \left( a^c(\bar{z}_n) - a^c(\bar{z}_n) \right) \right] 1 \{ \bar{z}_n < 0 \} + o(\epsilon^2)$$

$$\leq e^{j\omega \bar{z}_n} \left( j \epsilon^2 \omega \left( \phi^c \left( \frac{\bar{z}_n}{\tau} \right) - \phi^\ast \left( \frac{\bar{z}_n}{\tau} \right) \right) \right) - \frac{1}{2} \epsilon^2 \omega^2 E \left[ \left( a^c(\bar{z}_n) - a^c(\bar{z}_n) \right)^2 | \bar{z}_n \right] \right] 1 \{ \bar{z}_n \geq 0 \} + o(\epsilon^2)$$

$$+ e^{-j\omega \bar{z}_n} \left( -j \epsilon^2 \omega \left( \phi^c \left( \frac{\bar{z}_n}{\tau} \right) - \phi^\ast \left( \frac{\bar{z}_n}{\tau} \right) \right) \right) - \frac{1}{2} \epsilon^2 \omega^2 E \left[ \left( a^c(\bar{z}_n) - a^c(\bar{z}_n) \right)^2 | \bar{z}_n \right] \right] 1 \{ \bar{z}_n < 0 \}$$

$$\leq e^{j\omega \bar{z}_n} \left( j \epsilon^2 \omega \left( \phi^c \left( \frac{\bar{z}_n}{\tau} \right) - \phi^\ast \left( \frac{\bar{z}_n}{\tau} \right) \right) \right) - \frac{1}{2} \epsilon^2 \omega^2 E \left[ \left( a^c(\bar{z}_n) - a^c(\bar{z}_n) \right)^2 | \bar{z}_n \right] \right] + o(\epsilon^2)$$

$$\leq \frac{1}{2} \epsilon^2 \omega^2 e^{j\omega \bar{z}_n \text{sgn}(\bar{z}_n)} E \left[ \left( a^c(\bar{z}_n) - a^c(\bar{z}_n) \right)^2 | \bar{z}_n \right] + o(\epsilon^2)$$
where \((a)\) follows by Lemma 14 where we use Taylor’s Theorem to expand the real and imaginary part of \(E[je^{j\omega z}]\) separately and bound the higher order terms appropriately. Next, \((b)\) follows by using the tower property of expectation and \((2)\). Finally, \((c)\) follows by combining terms and using the definition of \(\text{sgn}(z)\). Now, we will simplify the second term above by calculating the second moment of the arrivals.

\[
\begin{align*}
E \left[ (a_n^\sigma(z_n) - a_n^\rho(z_n))^2 | z_n \right] &= \text{Var} \left[ a_n^\sigma(z_n) \right] + \text{Var} \left[ a_n^\rho(z_n) \right] + E \left[ a_n^\sigma(z_n) - a_n^\rho(z_n) \right]^2 \\
&= \sigma^2(\lambda_n(z_n)) + \sigma^2(\mu_n(z_n)) + \epsilon^2 \left( \phi^\sigma \left( \frac{z_n}{\tau} \right) - \phi^\rho \left( \frac{z_n}{\tau} \right) \right)^2 \\
&= \sigma^2(\lambda_n(z_n)) + \sigma^2(\mu_n(z_n)) + o(\epsilon).
\end{align*}
\]

(32)

This completes the proof.

\textbf{Proof of Claim 3} By Condition 3, we have \(|1 + (\phi^\sigma(x) - \phi^\rho(x))\text{sgn}(x)| \leq 1 + 2\phi_{\max}.\) Thus, by Lemma 6, we have

\[
\lim_{\eta \to \infty} E \left[ \left| 1 + \left( \phi^\sigma \left( \frac{z_n}{\tau} \right) - \phi^\rho \left( \frac{z_n}{\tau} \right) \right) \text{sgn} \left( \frac{z_n}{\tau} \right) \right| \right] = 0.
\]

This completes the first part of the claim. Now, to prove the second part of the claim, note that

\[
|\sigma^2(\lambda^*) + \sigma^2(\mu^*) - \sigma^2(\lambda^* + \epsilon\phi^\sigma(x)) - \sigma^2(\mu^* + \epsilon\phi^\rho(x))| \to 0 \quad \text{uniformly} \quad \forall x \in \mathbb{R}.
\]

Also note that the above function is absolutely bounded by \(4\sigma_{\max}\). Thus, by setting \(Y_n = z_n/\tau\) in Lemma 7, we get

\[
\lim_{\eta \to \infty} \frac{1}{2} E \left[ \left| \sigma^2(\lambda^*) + \sigma^2(\mu^*) - \sigma^2(\lambda(z_n)) - \sigma^2(\mu(z_n)) \right| \right] = 0.
\]

This completes the proof of the claim.

\textbf{Proof of Claim 7}

\[
\begin{align*}
|T_6| &= E \left[ \text{sgn}(z_n^+) e^{j\omega z_n^+} \text{sgn}(z_n^+) - \text{sgn}(z_n) e^{j\omega z_n} \text{sgn}(z_n) \right] \\
&= E \left[ \left( \text{sgn}(z_n^+) e^{j\omega z_n^+} \text{sgn}(z_n^+) - \text{sgn}(z_n) e^{j\omega z_n} \text{sgn}(z_n) \right) \mathbb{I} \{|z_n^+| \leq A_{\max}\} \right] \\
&= E \left[ \left( \text{sgn}(z_n^+) e^{j\omega z_n^+} \text{sgn}(z_n^+) - \text{sgn}(z_n) e^{j\omega z_n} \text{sgn}(z_n) \right) \mathbb{I} \{|z_n^+| \leq A_{\max}\} \mathbb{I} \{|z_n^-| \leq A_{\max}\} \right] \\
&= E \left[ \text{sgn}(z_n^+) - \text{sgn}(z_n) \right] - E \left[ \frac{1}{2} \epsilon^2 \omega^2 (z_n^+)^2 \left( \text{sgn}(z_n^+) - \text{sgn}(z_n) \right) \mathbb{I} \{|z_n^+| \leq A_{\max}\} \right] + o(\epsilon^2) \\
&= E \left[ \text{sgn}(z_n^+) - \text{sgn}(z_n) \right] - E \left[ \frac{1}{2} \epsilon^2 \omega^2 (z_n^-)^2 \left( \text{sgn}(z_n^+) - \text{sgn}(z_n) \right) \right] + o(\epsilon^2) \\
&= -\frac{1}{2} \epsilon^2 \omega^2 E \left[ (z_n^+)^2 \left( \text{sgn}(z_n^+) - \text{sgn}(z_n) \right) \right] + o(\epsilon^2)
\end{align*}
\]

where \((a)\) follows by Lemma 14. Further, \((b)\) follows as \(|\text{sgn}(z)| \leq 1\) and thus, \(E[\Delta(\text{sgn}(z_n))] = 0\). Now, by taking the absolute value on both sides, and using Jensen’s inequality, we get

\[
\Rightarrow \quad |T_6| \leq \frac{1}{2} \epsilon^2 \omega^2 E \left[ (z_n^+)^2 \left( \text{sgn}(z_n^+) - \text{sgn}(z_n) \right) \right] + o(\epsilon^2)
\]

\[
\leq \frac{1}{2} \epsilon^2 \omega^2 A_{\max} E \left[ (z_n^+ \text{sgn}(z_n^+) - \text{sgn}(z_n)) \right] + o(\epsilon^2)
\]

\[
\leq \frac{1}{2} \epsilon^2 \omega^2 A_{\max} E \left[ (z_n^+ \text{sgn}(z_n^+)) \right] + o(\epsilon^2)
\]

\[
\leq \frac{1}{2} \epsilon^2 \omega^2 A_{\max} \phi_{\max} + o(\epsilon^2) = o(\epsilon^2).
\]

By taking mod on both sides and noting that \(\text{sgn}(z_n^+) = \text{sgn}(z_n)\) when \(|z_n^+| \geq A_{\max}\), we get \((c)\). Next, \((d)\) follows as \(z_n^+ \text{sgn}(z_n^+) - \text{sgn}(z_n) \geq 0\). To see this, note that when \(z_n^+ > 0\), \(\text{sgn}(z_n^+) - \text{sgn}(z_n) \geq 0\) and when \(z_n^+ < 0\), \(\text{sgn}(z_n^+) - \text{sgn}(z_n) \leq 0\). Finally, \((e)\) follows by Lemma 4. \qed
Appendix F: Proof of Claims for Lemma 10 and 11

**Proof of Claim 4**

\[
E \left[ e^{j\omega(\bar{z}_n^+ - \tau t)/\tau} \left( 1 \left\{ \frac{\bar{z}_n^+}{\tau} \leq t \right\} - 1 \left\{ \frac{\bar{z}_n}{\tau} \leq t \right\} \right) \right]
\]

\[
= \left( \frac{j\omega}{\tau} E \left[ \frac{\bar{z}_n^+}{\tau} \right] - 1 \left\{ \frac{\bar{z}_n}{\tau} \leq t \right\} \right) \right) \right] - \frac{\omega^2}{2\tau^2} E \left[ (\bar{z}_n^+ - \tau t)^2 \left( 1 \left\{ \frac{\bar{z}_n^+}{\tau} \leq t \right\} - 1 \left\{ \frac{\bar{z}_n}{\tau} \leq t \right\} \right) \right] + o \left( \frac{1}{\tau^2} \right) o(\omega^2)
\]

\[
= \left( \frac{j\omega}{\tau} E \left[ \frac{\bar{z}_n^+}{\tau} \right] - 1 \left\{ \frac{\bar{z}_n}{\tau} \leq t \right\} \right) \right) \right] - \frac{\omega^2}{2\tau^2} E \left[ (\bar{z}_n^+ - \tau t)^2 \left( 1 \left\{ \frac{\bar{z}_n^+}{\tau} \leq t \right\} - 1 \left\{ \frac{\bar{z}_n}{\tau} \leq t \right\} \right) \right] + o \left( \frac{1}{\tau^2} \right) o(\omega^2)
\]

\[
= \left( \frac{j\omega}{\tau} E \left[ \frac{\bar{z}_n^+}{\tau} \right] - 1 \left\{ \frac{\bar{z}_n}{\tau} \leq t \right\} \right) \right) \right] - \frac{\omega^2}{2\tau^2} E \left[ (\bar{z}_n^+ - \tau t)^2 \left( 1 \left\{ \frac{\bar{z}_n^+}{\tau} \leq t \right\} - 1 \left\{ \frac{\bar{z}_n}{\tau} \leq t \right\} \right) \right] + o \left( \frac{1}{\tau^2} \right) o(\omega^2)
\]

where (a) follows as \( 1 \left\{ \frac{\bar{z}_n^+}{\tau} \leq t \right\} - 1 \left\{ \frac{\bar{z}_n}{\tau} \leq t \right\} = 0 \) for all \( |\bar{z}_n^+ - \tau t| > A_{\text{max}} \). Next, (b) follows by Taylor series expansion with the precise argument given in Lemma 14. Note that we add \( 1 \left\{ \frac{\bar{z}_n^+}{\tau} - t \leq \frac{A_{\text{max}}}{\tau} \right\} \) in step (a) to allow us to use Lemma 14 (Taylor’s theorem) as it is only valid for bounded random variables. Furthermore, (c) follows by noting that

\[
E \left[ \left( \frac{\bar{z}_n}{\tau} \leq t \right) - 1 \left\{ \frac{\bar{z}_n}{\tau} \leq t \right\} \right] = 0
\]

as it is the drift of \( 1 \left\{ \frac{\bar{z}_n}{\tau} \leq t \right\} \) in steady state. Now, (d) follows by the following claim:

**Claim 8.** For any \( t \in \mathbb{R} \), we have

\[
E \left[ (\bar{z}_n^+ - \tau t)^2 \left( 1 \left\{ \frac{\bar{z}_n}{\tau} \leq t \right\} - 1 \left\{ \frac{\bar{z}_n}{\tau} \leq t \right\} \right) \right] = o(1).
\]

We defer the proof of the claim to the end of this section and continue with the proof of Lemma 10. Lastly, (e) follows by Lemma 4 with \( g_{\eta}(\bar{z}) = 1 \left\{ \frac{\bar{z}}{\tau} \leq t \right\} \). This completes the proof.

**Proof of Claim 5** We simplify \( \mathcal{T}_2 \) as follows:

\[
\mathcal{T}_2 = E \left[ 1 \left\{ \phi^c \left( \frac{\bar{z}_n}{\tau} \right) - \phi^s \left( \frac{\bar{z}_n}{\tau} \right) = 0 \right\} \right]
\]

\[
= \left( \frac{j\omega}{\tau} E \left[ \frac{\bar{z}_n}{\tau} \right] - 1 \left\{ \frac{\bar{z}_n}{\tau} \leq t \right\} \right) \right) \right] - \frac{\omega^2}{2\tau^2} E \left[ (\bar{z}_n^+ - \tau t)^2 \left( 1 \left\{ \frac{\bar{z}_n^+}{\tau} \leq t \right\} - 1 \left\{ \frac{\bar{z}_n}{\tau} \leq t \right\} \right) \right] + o(1)
\]

\[
= \left( \frac{j\omega}{\tau} E \left[ \frac{\bar{z}_n}{\tau} \right] - 1 \left\{ \frac{\bar{z}_n}{\tau} \leq t \right\} \right) \right) \right] - \frac{\omega^2}{2\tau^2} E \left[ (\bar{z}_n^+ - \tau t)^2 \left( 1 \left\{ \frac{\bar{z}_n^+}{\tau} \leq t \right\} - 1 \left\{ \frac{\bar{z}_n}{\tau} \leq t \right\} \right) \right] + o(1)
\]

\[
= \left( \frac{j\omega}{\tau} E \left[ \frac{\bar{z}_n}{\tau} \right] - 1 \left\{ \frac{\bar{z}_n}{\tau} \leq t \right\} \right) \right) \right] - \frac{\omega^2}{2\tau^2} E \left[ (\bar{z}_n^+ - \tau t)^2 \left( 1 \left\{ \frac{\bar{z}_n^+}{\tau} \leq t \right\} - 1 \left\{ \frac{\bar{z}_n}{\tau} \leq t \right\} \right) \right] + o(1)
\]

\[
= \left( \frac{j\omega}{\tau} E \left[ \frac{\bar{z}_n}{\tau} \right] - 1 \left\{ \frac{\bar{z}_n}{\tau} \leq t \right\} \right) \right) \right] - \frac{\omega^2}{2\tau^2} E \left[ (\bar{z}_n^+ - \tau t)^2 \left( 1 \left\{ \frac{\bar{z}_n^+}{\tau} \leq t \right\} - 1 \left\{ \frac{\bar{z}_n}{\tau} \leq t \right\} \right) \right] + o(1)
\]

\[
= \left( \frac{j\omega}{\tau} E \left[ \frac{\bar{z}_n}{\tau} \right] - 1 \left\{ \frac{\bar{z}_n}{\tau} \leq t \right\} \right) \right) \right] - \frac{\omega^2}{2\tau^2} E \left[ (\bar{z}_n^+ - \tau t)^2 \left( 1 \left\{ \frac{\bar{z}_n^+}{\tau} \leq t \right\} - 1 \left\{ \frac{\bar{z}_n}{\tau} \leq t \right\} \right) \right] + o(1)
\]
\[ \begin{align*}
(\sigma) & \quad \frac{\omega^2}{2\tau^2} \mathbb{E} \left[ \left\{ \phi^\epsilon \left( \frac{\tilde{z}_n}{\tau} \right) - \phi^s \left( \frac{\tilde{z}_n}{\tau} \right) = 0 \right\} e^{i\omega \tilde{z}_n / \tau} (\sigma^e(\lambda(\tilde{z}_n)) + \sigma^s(\mu(\tilde{z}_n))) + o(\omega^2) = o \left( \frac{1}{\tau^2} \right) \\
(\mu) & \quad \frac{\omega^2}{2\tau^2} (\sigma^e(\bar{\lambda}^*) + \sigma^s(\bar{\mu}^*)) \mathbb{E} \left[ \left\{ \phi^e \left( \frac{\tilde{z}_n}{\tau} \right) - \phi^s \left( \frac{\tilde{z}_n}{\tau} \right) = 0 \right\} e^{i\omega \tilde{z}_n / \tau} + \omega^2 o \left( \frac{1}{\tau^2} \right) + o(\omega^2) = o \left( \frac{1}{\tau^2} \right) ,
\end{align*} \]

where (a) follows by the queue evolution equation given by (1). Next, (b) follows by Taylor series expansion and the argument is made precise by Lemma 14. Further, (c) follows by using the law of total expectation (tower property) and using (2). Note that (d) follows by dropping the first term as it is equal to zero. Next, (e) is obtained by computing the second moment of the difference of the arrivals as follows:

\[ \begin{align*}
\mathbb{E} \left[ (\alpha^e(\tilde{z}_n) - \alpha^s(\tilde{z}_n))^2 | \tilde{z}_n \right] 1 \left\{ \phi^e \left( \frac{\tilde{z}_n}{\tau} \right) - \phi^s \left( \frac{\tilde{z}_n}{\tau} \right) = 0 \right\} = \\
\left( \sigma^e(\lambda(\tilde{z}_n)) + \sigma^s(\mu(\tilde{z}_n)) + \mathbb{E} \left[ (\alpha^e(\tilde{z}_n) - \alpha^s(\tilde{z}_n)| \tilde{z}_n \right]^2 \right) 1 \left\{ \phi^e \left( \frac{\tilde{z}_n}{\tau} \right) - \phi^s \left( \frac{\tilde{z}_n}{\tau} \right) = 0 \right\}. \end{align*} \]

Lastly, by Condition 4, as \( \phi^e(\cdot) \) and \( \phi^s(\cdot) \) are monotonically decreasing and increasing respectively, there exists \( c^* \) such that \( \phi^e(z) = \phi^s(z) = c^* \) for all \( z \in (t_\tau, t_\tau^\tau) \). Now, define \( \bar{\lambda} = \lambda^* + \lim_{\eta \to \infty} \epsilon c^* \) and \( \bar{\mu} = \mu^* + \lim_{\eta \to \infty} \epsilon c^* \) and note that

\[ \left| \frac{\omega^2}{2\tau^2} \mathbb{E} \left[ \left\{ \phi^e \left( \frac{\tilde{z}_n}{\tau} \right) - \phi^s \left( \frac{\tilde{z}_n}{\tau} \right) = 0 \right\} e^{i\omega \tilde{z}_n / \tau} (\sigma^e(\bar{\lambda}^*) + \sigma^s(\bar{\mu}^*)) - \sigma^e(\lambda(\tilde{z}_n)) + \sigma^s(\mu(\tilde{z}_n)) \right\} \right| \]

\[ \leq \frac{\omega^2}{2\tau^2} \mathbb{E} \left[ \left\{ \phi^e \left( \frac{\tilde{z}_n}{\tau} \right) - \phi^s \left( \frac{\tilde{z}_n}{\tau} \right) = 0 \right\} \right] \]

\[ \leq \frac{\omega^2}{2\tau^2} \mathbb{E} \left[ \left\{ \phi^e \left( \frac{\tilde{z}_n}{\tau} \right) - \phi^s \left( \frac{\tilde{z}_n}{\tau} \right) = 0 \right\} \right] = \omega^2 o \left( \frac{1}{\tau^2} \right) , \]

where the last equality holds as \( \sigma^e(\bar{\lambda}^*) + \sigma^s(\bar{\mu}^*) - \sigma^e(\lambda^* + \epsilon c^*) - \sigma^s(\mu^* + \epsilon c^*) \to 0 \) as \( \sigma^e(\cdot) \) and \( \sigma^s(\cdot) \) are continuous functions. This completes the proof.

**Proof of Claim 8** Let \( t \in \mathbb{R} \). Now, we prove the claim as follows:

\[ \begin{align*}
(\alpha) & \quad \left| \mathbb{E} \left[ \left( \frac{\tilde{z}_n}{\tau} - t \right)^2 \right] 1 \left\{ \frac{\tilde{z}_n}{\tau} \leq t \right\} \right| \\
(\beta) & \quad \left| \mathbb{E} \left[ \left( \frac{\tilde{z}_n}{\tau} - t \right)^2 \right] 1 \left\{ \frac{\tilde{z}_n}{\tau} \leq t \right\} \right| \\
(\gamma) & \quad \left| \mathbb{E} \left[ \left( \frac{\tilde{z}_n}{\tau} - t \right)^2 \right] \right| \\
(\delta) & \quad \left| \mathbb{E} \left[ \left( \frac{\tilde{z}_n}{\tau} - t \right)^2 \right] \right|
\end{align*} \]
Lemma 14. Consider the set of mutually independent random variables denoted by \( \{ Y_\eta : \eta > 0 \} \) and \( \{ X_\eta(y) : \eta > 0, y \in \mathbb{Z} \} \). Also, there exists \( x_{\text{max}} \) such that \( X_\eta(y) \leq x_{\text{max}} \) with probability 1 for all \( \eta, y \). Now, define functions \( f, h : \mathbb{R} \to \mathbb{C} \) given by \( f = f_1 + jf_2 \) and \( h = h_1 + jh_2 \). Also, \( f_1, f_2, h_1, h_2 \) are thrice continuously differentiable and \( |h_1'''(x)| \leq 1, |h_2''(x)| \leq 1, |f_1(x)| \leq 1, |f_2(x)| \leq 1 \) for all \( x \in \mathbb{R} \). Then, for any \( \omega \in \mathbb{R} \), we have

\[
\left| \frac{1}{\epsilon^2} f(Y_\eta)h(\epsilon\omega X_\eta(Y_\eta)) - f(Y_\eta)h(0) - j\epsilon \omega f(Y_\eta)h'(0)X_\eta(Y_\eta) - \frac{1}{2} \epsilon^2 \omega^2 f(Y_\eta)h'''(0)X_\eta^2(Y_\eta) \right|
\leq \frac{1}{3} |\omega|^3 x_{\text{max}}^3 \quad \text{w.p. 1.} \tag{33}
\]

With a slight abuse of notation, we write

\[
f(Y_\eta)h(\epsilon\omega X_\eta(Y_\eta)) = f(Y_\eta)h(0) + j\epsilon \omega f(Y_\eta)h'(0)X_\eta(Y_\eta) + \frac{1}{2} \epsilon^2 \omega^2 f(Y_\eta)h'''(0)X_\eta^2(Y_\eta) + o(\epsilon^2).
\]

Proof We will analyze the real and imaginary part separately. By Taylor’s Theorem, we have

\[
h_i(x) = h_i(0) + h_i'(0)x + \frac{1}{2} h_i''(0)x^2 + \frac{1}{6} h_i'''(\tilde{x})x^3 \quad \text{for some } \tilde{x} \in (0, x) \ \forall i \in \{1, 2\}.
\]

Thus,

\[
\left| h_i(x) - h_i(0) - h_i'(0)x - \frac{1}{2} h_i''(0)x^2 \right| \leq \frac{1}{6} |x|^3 \quad \forall x \in \mathbb{R}, \forall i \in \{1, 2\}.
\]

Now, substitute \( x \) by \( \epsilon\omega X_\eta(Y_\eta) \) and note that \( |X_\eta(Y_\eta)| \leq x_{\text{max}} \) with probability 1 to get

\[
\left| h_i(\epsilon\omega X_\eta(Y_\eta)) - h_i(0) - \epsilon\omega X_\eta(Y_\eta)h_i'(0) - \frac{1}{2} \epsilon^2 \omega^2 X_\eta(Y_\eta)^2 h_i''(0) \right| \leq \frac{1}{6} |\epsilon\omega x_{\text{max}}|^3 \quad \text{w.p.1 } \forall i \in \{1, 2\}
\]

Now, as \( |f_1(Y_\eta)| \leq 1 \) and \( |f_2(Y_\eta)| \leq 1 \) with probability 1, multiplying on both sides, we get

\[
\left| f_k(Y_\eta)h_i(\epsilon\omega X_\eta(Y_\eta)) - f_k(Y_\eta)h_i(0) - f_k(Y_\eta)\epsilon\omega X_\eta(Y_\eta)h_i'(0) - \frac{1}{2} f_k(Y_\eta)\epsilon^2 \omega^2 X_\eta(Y_\eta)^2 h_i''(0) \right| \leq \frac{1}{6} |\epsilon\omega x_{\text{max}}|^3 \quad \text{w.p.1 } \forall i \in \{1, 2\} \ \forall k \in \{1, 2\}.
\]

Now, by adding \((i, k) = (1, 1)\) and \((i, k) = (2, 2)\) and using triangle inequality will give bound on the real part of \((33)\) and by adding \((i, k) = (1, 2)\) and \((i, k) = (2, 1)\) and using triangle inequality will give bound on the imaginary part of \((33)\). Now, by using the fact that for any vector \( x, \| x \|_1 \geq \| x \|_2 \), we get the lemma. \( \square \)
Appendix H: Classical Single Server Queue

H.1. Proof of Proposition 3

Proof The transition probabilities for the underlying DTMC governing the single server queue operating under the two-price policy is given by

\[ P_{ij} = \begin{cases} 
  m & \text{if } j = i + 1, 0 < i \leq \tau \text{ or } (i, j) = (0, 1) \\
  1 - 2m & \text{if } j = i, 0 < i \leq \tau \\
  1 - m & \text{if } j = i = 0 \\
  m + \epsilon \mu^* & \text{if } j = i - 1, i > \tau \\
  m - \epsilon(1 - \mu^*) & \text{if } j = i + 1, i > \tau \\
  1 - 2m + \epsilon(1 - 2\mu^*) & \text{if } j = i, i > \tau \\
  0 & \text{otherwise.} 
\end{cases} \]

As the DTMC is a birth and death process, it is reversible. Thus, we can solve the local balance equations to get:

\[ \pi_i = \begin{cases} 
  \frac{1}{\tau + 1 + m/\epsilon} & \text{if } i \leq \lfloor \tau \rfloor \\
  \frac{1}{\tau + 1 + m/\epsilon} \frac{m}{m + \epsilon \mu^*} \left( 1 - \frac{\epsilon}{m + \epsilon \mu^*} \right)^{i - \lfloor \tau \rfloor - 1} & \text{if } i > \lfloor \tau \rfloor. 
\end{cases} \]

Note that for large \( \eta \), the stationary distribution of \( \bar{q}_\eta \) is almost the same as the stationary distribution of \( |\tilde{z}_\eta| \). Indeed, the proof follows exactly as the proof of Proposition 1. We present it below for completeness.

Now, we will calculate the moment generating function of \( \epsilon \bar{q}_\eta \) for Case 1 and 2 and \( \bar{q}/\tau \) for Case 3. We fix a \( t \) such that \( t < 1/m \). Then for all \( \epsilon > 0 \), we have \( (1 - \epsilon/(m + \epsilon \mu^*))e^{\epsilon t} < 1 \). Now, we calculate the MGF as follows:

\[
\mathbb{E}[e^{\epsilon t \bar{q}_\eta}] = \sum_{i=0}^{\infty} \pi_i e^{\epsilon t i} \\
= \sum_{i=0}^{\lfloor \tau \rfloor} \pi_i e^{\epsilon t i} + \sum_{i=\lfloor \tau \rfloor + 1}^{\infty} \pi_i e^{\epsilon t i} \\
= \frac{1}{\tau + 1 + m/\epsilon} \sum_{i=0}^{\lfloor \tau \rfloor} e^{\epsilon t i} + \frac{m}{m + \epsilon \mu^*} \sum_{i=\lfloor \tau \rfloor + 1}^{\infty} \left( 1 - \frac{\epsilon}{m + \epsilon \mu^*} \right)^{i - \lfloor \tau \rfloor - 1} e^{\epsilon t i} \\
= \frac{\epsilon}{\tau \epsilon + \epsilon + m} \left( 1 - e^{\epsilon t (\lfloor \tau \rfloor + 1)} + \frac{m}{m + \epsilon \mu^*} \frac{e^{\epsilon t (\lfloor \tau \rfloor + 1)}}{1 - e^{\epsilon t + \frac{\epsilon}{m + \epsilon \mu^*}}} \right) \quad (34)
\]

Now, we will take the limit as \( \eta \uparrow \infty \) for the above equation for Case 1 and 2 separately.

**Case 1:** Similar to the proof of Proposition 1, we will have

\[
\epsilon \frac{1 - e^{\epsilon t (\lfloor \tau \rfloor + 1)}}{1 - e^{\epsilon t}} = \epsilon \frac{-et(\lfloor \tau \rfloor + 1) + o(\epsilon \tau)}{-et + o(\epsilon)} = \frac{-et(\lfloor \tau \rfloor + 1) + o(\epsilon \tau)}{-et + o(\epsilon)} \to 0 \quad (35a)
\]

\[
\frac{1 - e^{\epsilon t + \frac{\epsilon}{m + \epsilon \mu^*}}}{1 - e^{\epsilon t}} = \epsilon \frac{-et + \frac{\epsilon}{m + \epsilon \mu^*} + o(\epsilon)}{-et + o(\epsilon)} = \frac{-et + o(1)}{-et + o(\epsilon)} \to \frac{1}{1/m - t} \quad (35b)
\]

Now, by using (35) to simplify (20) we get

\[
\mathbb{E}[e^{\epsilon t \tilde{z}_\eta}] \to \frac{1}{1 - t/m} \quad \text{as } \eta \uparrow \infty.
\]
This completes Case 1 as the above is the MGF of an exponential distribution with parameter \(m\).

**Case 2:** Similar to the proof of Proposition 1, we have
\[
\begin{align*}
\epsilon \left( \frac{1 - e^{\epsilon t([\tau]+1)}}{1 - e^{\epsilon t}} \right) &= \epsilon \left( \frac{1 - e^{\epsilon t([\tau]+1)}}{1 - e^{\epsilon t}} \right) = 1 - e^{\epsilon t([\tau]+1)} = 1 - e^{\epsilon t} + o(\epsilon) = -t + o(1), \\
\epsilon \left( \frac{1 - e^{\epsilon t} + \frac{e^{\epsilon t}}{m+\epsilon \mu}}{1 - e^{\epsilon t}} \right) &= \epsilon \left( \frac{1 - e^{\epsilon t} + \frac{e^{\epsilon t}}{m+\epsilon \mu}}{1 - e^{\epsilon t}} \right) = -t + \frac{1}{m+\epsilon \mu} + o(\epsilon) \to \frac{e^t}{1/m - t}. 
\end{align*}
\]
(36a, 36b)

Now, by using (36) to simplify (34) we get
\[
\mathbb{E}[e^{t\tilde{q}_n}] \to \frac{1}{1/m - t} \left( \frac{e^t - 1}{1/m} + \frac{e^t}{1/m - t} \right)
\]

The above is the MGF of the non-negative hybrid distribution with parameters \((m,l)\). This can be verified similar to the proof of Proposition 1. We skip the steps here as they are repetitive. This completes the proof for Case 2.

**Case 3:** For large enough \(\eta\) and \(t < 1/m\), the MGF of \(\tilde{q}/\tau\) can be calculated similar to (34) to get
\[
\mathbb{E}\left[ e^{t\tilde{q}/\tau} \right] = \frac{\epsilon}{\tau \epsilon + \epsilon + m} \left( \frac{1 - e^{t([\tau]+1)/\tau}}{1 - e^{t/\tau}} \right) + m \frac{e^{t([\tau]+1)/\tau}}{m + \epsilon \mu} \left( 1 - e^{t/\tau} + \frac{e^{t/\tau}}{m + \epsilon \mu} \right)
\]
(37)

Similar to the proof of Proposition 1, we have
\[
\begin{align*}
\frac{\epsilon}{\tau \epsilon + \epsilon + m} \frac{e^{t([\tau]+1)/\tau}}{1 - e^{t/\tau} + \frac{e^{t/\tau}}{m + \epsilon \mu}} &\to 0, \\
\frac{\epsilon}{\tau \epsilon + \epsilon + m} \frac{1 - e^{t([\tau]+1)/\tau}}{1 - e^{t/\tau}} &= \frac{\epsilon}{\tau \epsilon + \epsilon + m} \frac{1 - e^{t([\tau]+1)/\tau}}{1 - e^{t/\tau} + o(1/\tau)} = -t[\tau]/\tau + o(1) \to \frac{e^t - 1}{t}.
\end{align*}
\]
(38a, 38b)

Now, by using (38) in (37), we get
\[
\mathbb{E}\left[ e^{t\tilde{q}/\tau} \right] \to \frac{e^t - 1}{t} \Rightarrow \tilde{q}/\tau \stackrel{D}{\to} \mathcal{U}([0,1]).
\]
This completes the proof of the theorem.

\[\square\]

**H.2. Preliminaries: Single Server Queue** Similar to Proposition 2, we can show that if the functions \(\phi^*(\cdot), \phi^*\cdot\) satisfy Condition 5, then the single server queue is positive recurrent. We omit the details of this proof as it is intuitive and the steps are analogous to the proof of Proposition 2. To prove the theorem, we will first introduce the following notation: \(\tilde{q}_n\) is the random variable that has the same distribution as the stationary distribution of \(\{ q_n(k) : k \in \mathbb{Z}_+ \}\) and \(\tilde{q}_n^+\) is the queue length one time epoch after \(\tilde{q}_n\). In addition, given the steady state queue length \(\tilde{q}_n\), let the arrival, potential service and unused service be denoted by \(\tilde{a}_n, \tilde{s}_n,\) and \(\tilde{u}_n\) respectively. We omit the dependence of \(\tilde{q}_n\) on \(\tilde{a}_n, \tilde{s}_n,\) and \(\tilde{u}_n\) for simplicity of notation. Thus, we have
\[
\tilde{q}_n^+ = \tilde{q}_n + \tilde{a}_n - \tilde{s}_n + \tilde{u}_n.
\]
(39)
H.3. Proof of Theorem 4: Case I
Proof of Theorem 4: ετ → 0 We will use the characteristic function of \( \bar{q}_n \) as the test function to prove the theorem. Firstly, note that

\[
\left( e^{j\omega \bar{q}_n^+} - 1 \right) \left( e^{-j\omega \bar{a}_n} - 1 \right) = 0
\]
as \( \bar{q}_n^+ > 0 \) implies that \( \bar{u}_n = 0 \) and vice versa. Expanding the above expression and using the queue evolution equation results in

\[
e^{j\omega \bar{q}_n^+} - e^{j\omega (\bar{q}_n^+ + \bar{a}_n - \bar{s}_n)} = 1 - e^{-j\omega \bar{a}_n}.
\]

Now, by taking the expectation with respect to the stationary distribution of \( \{q_n(k) : k \in \mathbb{Z}_+\} \) and noting that \( E[e^{j\omega \bar{q}_n}] = E[e^{j\omega \bar{q}_n^+}] \) by stationarity, we get

\[
E \left[ e^{j\omega \bar{q}_n^+} - e^{j\omega (\bar{q}_n^+ + \bar{a}_n - \bar{s}_n)} \right] = 1 - E \left[ e^{-j\omega \bar{a}_n} \right].
\]

The LHS of the above equation can be simplified to get

\[
\begin{align*}
&= E \left[ e^{j\omega \bar{q}_n^+} (1 - e^{j\omega (\bar{a}_n - \bar{s}_n)}) \right] \\
&= E \left[ e^{j\omega \bar{q}_n} \left( -j\omega (\bar{a}_n - \bar{s}_n) + \frac{\omega^2 \epsilon^2}{2} (\bar{a}_n - \bar{s}_n)^2 \right) \right] + o(\epsilon^2) \\
&= E \left[ e^{j\omega \bar{q}_n} \left( j\omega \epsilon^2 \left( \phi^s \left( \frac{\bar{q}_n}{\tau} \right) - \phi^c \left( \frac{\bar{q}_n}{\tau} \right) \right) + \frac{\omega^2 \epsilon^2}{2} \left( \sigma^s_n(\lambda_n(\bar{q}_n)) + \sigma^\alpha_n(\mu_n(\bar{q}_n)) \right) \right) \right] + o(\epsilon^2),
\end{align*}
\]

where (a) follows from Taylor’s theorem and the reader can refer to Lemma 14 for precise argument. Next, (b) follows by the tower property of expectation and lastly, (c) follows by the definition of queue length dependent expectation and variance of \( \bar{a}_n \) and \( \bar{s}_n \).

Now, to simplify the RHS of (41), first consider \( \bar{q}_n \) as a test function and using \( E[\bar{q}_n^+] = E[\bar{q}_n] \) will give us

\[
E \left[ \bar{u}_n \right] = E \left[ \bar{s}_n - \bar{a}_n \right] = E \left[ \bar{s}_n - \bar{a}_n \bar{q}_n \right] = E \left[ \phi^s \left( \frac{\bar{q}_n}{\tau} \right) - \phi^c \left( \frac{\bar{q}_n}{\tau} \right) \right]
\]

Now, we can simplify the RHS using Taylor’s theorem (Lemma 14) to get

\[
1 - E \left[ e^{-j\omega \bar{a}_n} \right] = j\omega E \left[ \bar{u}_n \right] + o(\epsilon^2) = j\omega \epsilon^2 E \left[ \phi^s \left( \frac{\bar{q}_n}{\tau} \right) - \phi^c \left( \frac{\bar{q}_n}{\tau} \right) \right] + o(\epsilon^2).
\]

Now, using (42) and (44), we get

\[
E \left[ e^{j\omega \bar{q}_n^+} \left( j\omega \epsilon^2 \left( \phi^s \left( \frac{\bar{q}_n}{\tau} \right) - \phi^c \left( \frac{\bar{q}_n}{\tau} \right) \right) + \frac{\omega^2 \epsilon^2}{2} \left( \sigma^s_n(\lambda_n(\bar{q}_n)) + \sigma^\alpha_n(\mu_n(\bar{q}_n)) \right) \right) \right] = j\omega \epsilon^2 E \left[ \phi^s \left( \frac{\bar{q}_n}{\tau} \right) - \phi^c \left( \frac{\bar{q}_n}{\tau} \right) \right] + o(\epsilon^2).
\]

Dividing on both sides by \( j\omega \epsilon^2 \) and rearranging the terms, we get

\[
\begin{align*}
&= \left( 1 - \frac{j\omega}{2} (\sigma^c(\lambda^*) + \sigma^s(\mu^*)) \right) E \left[ e^{j\omega \bar{q}_n} \right] \\
&= \left( E \left[ \phi^s \left( \frac{\bar{q}_n}{\tau} \right) - \phi^c \left( \frac{\bar{q}_n}{\tau} \right) \right] \right) - E \left[ e^{\phi \bar{q}_n} \left( \phi^s \left( \frac{\bar{q}_n}{\tau} \right) - \phi^c \left( \frac{\bar{q}_n}{\tau} \right) - 1 \right) \right] \\
&+ \frac{j\omega}{2} E \left[ e^{j\omega \bar{q}_n} \left( \sigma^s_n(\lambda_n(\bar{q}_n)) + \sigma^\alpha_n(\mu_n(\bar{q}_n)) - \sigma^c(\lambda^*) - \sigma^\alpha(\mu^*) \right) \right] + o(1).
\end{align*}
\]
The terms $T_8, T_9, T_{10}$ are analogous to the terms $T_3, T_4, T_5$ in the proof of Theorem 2. It can be similarly shown using Lemma 6 and the fact that $\phi^c(\infty) - \phi^c(\infty) = 1$ and $\bar{q}_\eta \geq 0$, we get $T_8 \rightarrow 1$, $T_9 \rightarrow 0$. In addition, by using Lemma 7, we get $T_{10} \rightarrow 0$. Thus, by taking the limit as $\eta \uparrow \infty$, we get

$$E \left[ e^{j\omega \bar{q}_\eta} \right] = \frac{1}{1 - \frac{j\omega}{\tau} \left( \sigma^c(\lambda^*) + \sigma^s(\mu^*) \right)}.$$ 

This implies that $\epsilon \bar{q}_\eta$ converges to a random variable in distribution having exponential distribution with mean $0.5(\sigma^c(\lambda^*) + \sigma^s(\mu^*))$.

\[ \square \]

**H.4. Proof of Theorem 4: Case II**

*Proof of Theorem 4: $\epsilon \tau \rightarrow l$* The first step is to show tightness of the family of random variables $\{\epsilon \bar{q}_\eta\}$ which can be proved similar to the proof of Lemma 1. We omit the details here for brevity. Now, the next step is to analyze the drift of the characteristic function of $\bar{q}_\eta$ and then obtain a functional equation by taking the limit as $\eta \uparrow \infty$. Recall that by (45), we have

$$E \left[e^{j\omega \bar{q}_\eta} \left( \phi^c \left( \frac{\bar{q}_\eta}{\tau} \right) \right) - \phi^c \left( \frac{\bar{q}_\eta}{\tau} \right) \right] = E \left[ \phi^c \left( \frac{\bar{q}_\eta}{\tau} \right) \right] + o(1).$$

By taking the limit as $\eta \uparrow \infty$ and following the steps same as in the proof of Claim 1, we get

$$E \left[ e^{j\omega \bar{q}_\infty} g(\bar{q}_\infty) \right] - j\omega E \left[ e^{j\omega \bar{q}_\infty} \right] = E \left[ g(\bar{q}_\infty) \right] \Delta C_1. \quad (46)$$

Note that, this is same as the functional equation (9) if the RHS is zero. In particular, we get a non-zero RHS because of the unused service $\bar{u}_\eta$ which is responsible to ensure that the queue length is non-negative. Now, we will write (46) as a differential equation in the space of tempered distribution which is analogous to taking the inverse Fourier transform. Define the following tempered distribution

$$T_{\bar{q}_\infty}[\varphi] = \int_{\mathbb{R}} \varphi(x) dF_{\bar{q}_\infty}(x) = E \left[ \varphi \right] \quad \forall \varphi \in \mathcal{S}(\mathbb{R}).$$

Now, by using (30) and (31), the LHS of (46) can be written as follows:

$$E \left[ e^{j\omega \bar{q}_\infty} g(\bar{q}_\infty) \right] - j\omega E \left[ e^{j\omega \bar{q}_\infty} \right] = -\sqrt{2\pi} T_{\bar{q}_\infty}[\hat{\varphi} + g\hat{\varphi}].$$

The RHS can be written in terms of the impulse distribution $\delta(\cdot)$. In particular, denote the constant tempered distribution by $T_c$, i.e. $T_c[\varphi] = \int_{-\infty}^\infty \varphi(x) dx$. Then, we have

$$\int_{-\infty}^\infty \varphi(x) dx = T_c[\varphi] \overset{(a)}{=} \hat{T}_c[\hat{\varphi}] \overset{(b)}{=} \delta[\hat{\varphi}],$$

where (a) follows by [60, Definition 7.14] and (b) follows by [60, Example 7.16]. In particular, the Fourier transform of the constant tempered distribution is the impulse distribution. Thus, we can write (46) in terms of distribution functions as follows:

$$T_{\bar{q}_\infty}[\hat{\varphi} + g\hat{\varphi}] = -\frac{C_1}{\sqrt{2\pi}} \delta[\hat{\varphi}] \quad \forall \varphi \in \mathcal{S}(\mathbb{R}).$$

As Fourier transform is a bijection on $\mathcal{S}(\mathbb{R})$ (Proposition 4 (a)), the above is equivalent to

$$T_{\bar{q}_\infty}[\varphi + g\varphi] = -\frac{C_1}{\sqrt{2\pi}} \delta[\varphi] \quad \forall \varphi \in \mathcal{S}(\mathbb{R}).$$
This can be written as the following differential equation in tempered distribution using the definition of derivative of a distribution given by Definition 5.

\[ T_{\bar{\varphi}}[\varphi] = g T_{\bar{\varphi}}[\varphi] = \frac{C_1}{\sqrt{2\pi}} \delta[\varphi] \quad \forall \varphi \in S(\mathbb{R}). \]

By [26, Theorem 4] [27, Theorem 3.9], we can directly write the entire solution of the above differential equation to be

\[ T_{\bar{\varphi}}[\varphi] = C \int_{-\infty}^{\infty} e^{-G(x)} \bar{\varphi}(x) dx - \frac{C_1}{\sqrt{2\pi}} \delta \left[ e^{G(x)} \int_{-\infty}^{x} e^{-G(t)} (\varphi(t) - a_0 \varphi_0(t)) dt \right] \quad \forall \varphi \in S(\mathbb{R}) \]

for any constant C and we have \( G(x) = \int_0^x g(t) dt, \int_{-\infty}^{\infty} \varphi_0(x) e^{-G(x)} dx = 1, \) and we have \( a_0 = \int_{-\infty}^{\infty} \varphi_0(x) e^{-G(x)} dx. \) The steps to solve this differential equation is given in [27, Section IV B].

Now, the second term in the above equation can be simplified as follows:

\[ \delta \left[ e^{G(x)} \int_{-\infty}^{x} e^{-G(t)} (\varphi(t) - a_0 \varphi_0(t)) dt \right] \]

\[ \overset{(a)}{=} e^{G(0)} \int_{-\infty}^{0} e^{-G(t)} (\varphi(t) - a_0 \varphi_0(t)) dt \]

\[ \overset{(b)}{=} \int_{-\infty}^{0} e^{-G(t)} (\varphi(t) - a_0 \varphi_0(t)) dt \]

\[ \overset{(c)}{=} \int_{-\infty}^{0} e^{-G(t)} (\varphi(t) - a_0 \varphi_0(t)) dt - C_2 \int_{-\infty}^{\infty} \varphi(t) e^{-G(t)} dt, \]

where (a) follows by the definition of impulse tempered distribution, (b) follows as \( G(0) = 0 \) by definition, and (c) follows by defining \( C_2 \overset{\Delta}{=} \int_{-\infty}^{0} e^{-G(t)} \varphi_0(t) dt. \) Thus, we get

\[ T_{\bar{\varphi}}[\varphi] = \int_{-\infty}^{\infty} e^{-G(x)} \left( C - \frac{C_1}{\sqrt{2\pi}} \frac{1}{\sqrt{2\pi}} \mathbb{1}\{x < 0\} \right) \varphi(x) dx \]

By the definition of \( T_{\bar{\varphi}} \), we conclude that \( \bar{\varphi} \) has density

\[ \rho_{\bar{\varphi}}(x) = e^{-G(x)} \left( C + \frac{C_1}{\sqrt{2\pi}} \frac{1}{\sqrt{2\pi}} \mathbb{1}\{x < 0\} \right). \]

By the definition of \( \bar{\varphi} \), we will first impose the condition that \( \rho_{\bar{\varphi}}(x) = 0 \) for all \( x < 0 \) which implies that \( C = \sqrt{2\pi} C/(1 - C_2). \) This gives us

\[ \rho_{\bar{\varphi}}(x) = \frac{C}{1 - C_2} e^{-G(x)} \mathbb{1}\{x \geq 0\}. \]

Now, we impose the condition that \( \int_{-\infty}^{\infty} \rho_{\bar{\varphi}}(x) dx = 1 \) which implies that \( C/(1 - C_2) = \frac{1}{\int_0^{\infty} e^{-G(t)} dt}. \) This gives us

\[ \rho_{\bar{\varphi}}(x) = \frac{1}{\int_0^{\infty} e^{-G(t)} dt} e^{-G(x)} \mathbb{1}\{x \geq 0\}. \]

Now, we need to verify that the above is indeed a solution of (46) which is omitted as this part of the proof is similar to the proof of Theorem 1. This completes the proof. □
H.5. Proof of Theorem 4: Case III

Proof of Theorem 4: \( \varepsilon t \uparrow \infty \) By using (40) and multiplying both sides by \( 1 \{ \frac{q_n^+}{T} \leq t^* \} \), we get

\[
1 \left\{ \frac{q_n^+}{T} \leq t^* \right\} \left( e^{j\omega \bar{q}_n^+/T} - e^{j\omega (\bar{q}_n + \bar{s}_n - \bar{s}_n)/T} \right) = (1 - e^{-j\omega \bar{u}_n/T}) 1 \left\{ \frac{\bar{q}_n}{T} \leq t^* \right\}.
\]

Now, by taking the expectation with respect to the stationary distribution of \( \{ q_n(k) : k \in \mathbb{Z}_+ \} \) and noting that \( E \left[ e^{j\omega \bar{q}_n/\tau} 1 \left\{ \frac{\bar{q}_n}{T} \leq t^* \right\} \right] = E \left[ e^{j\omega \bar{q}_n^+/\tau} 1 \left\{ \frac{\bar{q}_n^+}{T} \leq t^* \right\} \right] \) by stationarity, we get

\[
E \left[ e^{j\omega \bar{q}_n/\tau} 1 \left\{ \frac{\bar{q}_n}{T} \leq t^* \right\} \right] - E \left[ e^{j\omega (\bar{q}_n + \bar{s}_n - \bar{s}_n)/\tau} 1 \left\{ \frac{\bar{q}_n^+}{T} \leq t^* \right\} \right] = E \left[ 1 - e^{-j\omega \bar{u}_n/\tau} 1 \left\{ \frac{\bar{q}_n}{T} \leq t^* \right\} \right] \quad (47)
\]

Now, we will simplify the LHS and RHS separately. Define LHS as the sum of \( T_3 \) and \( T_4 \) defined as follows:

\[
T_3 = E \left[ e^{j\omega (\bar{q}_n + \bar{s}_n - \bar{s}_n)/\tau} 1 \left\{ \frac{\bar{q}_n}{T} \leq t^* \right\} - e^{j\omega (\bar{q}_n + \bar{s}_n - \bar{s}_n)/\tau} 1 \left\{ \frac{\bar{q}_n^+}{T} \leq t^* \right\} \right]
\]

\[
T_4 = E \left[ e^{j\omega \bar{q}_n/\tau} 1 \left\{ \frac{\bar{q}_n}{T} \leq t^* \right\} - e^{j\omega (\bar{q}_n + \bar{s}_n - \bar{s}_n)/\tau} 1 \left\{ \frac{\bar{q}_n^+}{T} \leq t^* \right\} \right]
\]

We can now simplify \( T_3 \) analogous to \( T_1 \) in the proof of Lemma 10. We omit some of the steps that are similar to the proof of Lemma 10.

\[
T_3 = e^{j\omega t^*} E \left[ e^{j\omega (\bar{q}_n + \bar{s}_n - \bar{s}_n)/\tau} \left( 1 \left\{ \frac{\bar{q}_n}{T} \leq t^* \right\} - 1 \left\{ \frac{\bar{q}_n^+}{T} \leq t^* \right\} \right) \right]
\]

\[
(\rightarrow) j\omega e^{j\omega t^*} E \left[ \bar{q}_n + \bar{u}_n - \bar{s}_n \left( 1 \left\{ \frac{\bar{q}_n}{T} \leq t^* \right\} - 1 \left\{ \frac{\bar{q}_n^+}{T} \leq t^* \right\} \right) \right] + \omega^2 o \left( \frac{1}{T^2} \right) + o \left( \frac{1}{T^2} \right) o(\omega^2)
\]

\[= \frac{j\omega}{T} e^{j\omega t^*} E \left[ \bar{q}_n \left( 1 \left\{ \frac{\bar{q}_n}{T} \leq t^* \right\} - 1 \left\{ \frac{\bar{q}_n^+}{T} \leq t^* \right\} \right) \right] + \omega^2 o \left( \frac{1}{T^2} \right) + o \left( \frac{1}{T^2} \right) o(\omega^2)
\]

\[= \frac{j\omega}{T} e^{j\omega t^*} E \left[ \bar{q}_n \left( \bar{u}_n - \bar{u}_n + \bar{u}_n \right) \right] + \omega^2 o \left( \frac{1}{T^2} \right) + o \left( \frac{1}{T^2} \right) o(\omega^2)
\]

\[= \frac{j\omega}{T} e^{j\omega t^*} E \left[ \bar{q}_n \left( \bar{u}_n - \bar{u}_n + \bar{u}_n \right) \right] + \omega^2 o \left( \frac{1}{T^2} \right) + o \left( \frac{1}{T^2} \right) o(\omega^2)
\]

\[= \frac{j\omega}{T} e^{j\omega t^*} E \left[ \phi^c \left( \bar{q}_n \right) - \phi^c \left( \bar{q}_n^+/T \right) \right] + \omega^2 o \left( \frac{1}{T^2} \right) + o \left( \frac{1}{T^2} \right) o(\omega^2),
\]

where \( (a) \) follows by Taylor’s series expansion and showing that the second order term is \( \omega^2 o(1/T^2) \) similar to the proof of Claim 8. Next, \( (b) \) follows by the queue evolution equation given by (39) and noting that \( 1 \left\{ \frac{\bar{q}_n}{T} \leq t^* \right\} - 1 \left\{ \frac{\bar{q}_n^+}{T} \leq t^* \right\} \neq 0 \) only when \( |\bar{q}_n^+ - \tau t^*| \leq A_{\max} \) which implies \( \bar{q}_n^+ > 0 \Rightarrow \bar{u} = 0 \) for \( \eta \) large enough. Further, \( (c) \) follows by a result analogous to Lemma 4 for a single server queue. In particular,

\[
0 = E \left[ \bar{q}_n^+ \left\{ \frac{\bar{q}_n^+}{T} \leq t^* \right\} - \bar{q}_n \left\{ \frac{\bar{q}_n}{T} \leq t^* \right\} \right]
\]

\[= E \left[ 1 \left\{ \frac{\bar{q}_n}{T} \leq t^* \right\} - \left( \bar{q}_n^+/T \right) \right] + E \left[ \bar{q}_n \left( 1 \left\{ \frac{\bar{q}_n^+}{T} \leq t^* \right\} - 1 \left\{ \frac{\bar{q}_n}{T} \leq t^* \right\} \right) \right].
\]
Next, (d) follows by the queue evolution equation given by (39), (e) follows by noting that $\mathbb{E}[\bar{q}_n|\bar{q}_n] = \mathbb{E}[\bar{s}_n|\bar{q}_n]$ for $\bar{q}_n \leq \tau t^*$, and $f$ follows as $\mathbb{E}[\bar{q}_n/\tau > t^*] = 0$ as $\bar{q}_n/\tau > t^*$ implies that $\bar{q}_n/\tau > t^* - A_{\max} > 0$ for $n$ large enough. Lastly, (g) follows by (43). Now, we simplify $\mathcal{T}_4$ below which is analogous to $\mathcal{T}_2$ in Lemma 10.

$$
\mathcal{T}_4 = \mathbb{E} \left[ \mathbb{1} \left\{ \frac{\bar{q}_n}{\tau} \leq t^* \right\} e^{j\omega \bar{q}_n/\tau} \left( 1 - e^{j\omega (\bar{a}_n - \bar{s}_n)/\tau} \right) \right]
$$

\[
\begin{align*}
\text{(a)} & \quad = -j\omega \mathbb{E} \left[ \mathbb{1} \left\{ \frac{\bar{q}_n}{\tau} \leq t^* \right\} e^{j\omega \bar{q}_n/\tau} (\bar{a}_n - \bar{s}_n) \right] + \frac{\omega^2}{2\tau^2} \mathbb{E} \left[ \mathbb{1} \left\{ \frac{\bar{q}_n}{\tau} \leq t^* \right\} e^{j\omega \bar{q}_n/\tau} (\bar{a}_n - \bar{s}_n)^2 \right] + o \left( \frac{1}{\tau^2} \right) o(\omega^2) \\
\text{(b)} & \quad = \frac{\omega^2}{2\tau^2} \mathbb{E} \left[ \mathbb{1} \left\{ \frac{\bar{q}_n}{\tau} \leq t^* \right\} e^{j\omega \bar{q}_n/\tau} (\bar{a}_n - \bar{s}_n)^2 \right] + o \left( \frac{1}{\tau^2} \right) o(\omega^2) \\
\text{(c)} & \quad = \frac{\omega^2}{2\tau^2} \left( \sigma^c(\lambda^*) + \sigma^s(\mu^*) \right) \mathbb{E} \left[ \mathbb{1} \left\{ \frac{\bar{q}_n}{\tau} \leq t^* \right\} e^{j\omega \bar{q}_n/\tau} \right] + \omega^2 o \left( \frac{1}{\tau^2} \right) + o \left( \frac{1}{\tau^2} \right) o(\omega^2),
\end{align*}
\]

where (a) follows by Taylor’s series expansion with the precise argument given by Lemma 14. Next, (b) follows by noting that the first term is 0 as $\mathbb{E}[\bar{a}_n|\bar{q}_n] = \mathbb{E}[\bar{s}_n|\bar{q}_n]$ for $\bar{q}_n \leq \tau t^*$. Lastly, (c) follows similar to (f) in the proof of Claim 5. As $\phi^c(\cdot)$ and $\phi^s(\cdot)$ are monotonically decreasing and increasing respectively, there exists $c^*$ such that $\phi^c(q) = \phi^s(q) = c^*$ for all $q \leq t^*$. Now, we define $\lambda^*=\lambda^* + c^* \lim_{\tau \to \infty} \epsilon$ and $\mu^*=\mu^* + c^* \lim_{\tau \to \infty} \epsilon$. As these steps are analogous to the proof of Lemma 10, we omit the details. Now, we simplify the RHS of (47) below.

$$
\mathbb{E} \left[ (1 - e^{-j\omega \bar{q}_n/\tau}) \mathbb{1} \left\{ \frac{\bar{q}_n}{\tau} \leq t^* \right\} \right] \equiv \mathbb{E} \left[ \frac{j\omega \bar{u}_n}{\tau} + \frac{\omega^2 \bar{u}_n^2}{2\tau^2} + o \left( \frac{1}{\tau^2} \right) o(\omega^2) \right] \mathbb{1} \left\{ \frac{\bar{q}_n}{\tau} \leq t^* \right\}
$$

\[
\begin{align*}
\text{(a)} & \quad = \mathbb{E} \left[ \frac{j\omega \bar{u}_n}{\tau} + \frac{\omega^2 \bar{u}_n^2}{2\tau^2} + o \left( \frac{1}{\tau^2} \right) o(\omega^2) \right] \mathbb{1} \left\{ \frac{\bar{q}_n}{\tau} \leq t^* \right\} \\
\text{(b)} & \quad = \mathbb{E} \left[ \frac{j\omega \bar{q}_n}{\tau} + \frac{\omega^2 \bar{q}_n^2}{2\tau^2} + o \left( \frac{1}{\tau^2} \right) o(\omega^2) \right] + \omega^2 o \left( \frac{1}{\tau^2} \right) + o \left( \frac{1}{\tau^2} \right) o(\omega^2),
\end{align*}
\]

where (a) follows by Taylor’s series expansion with the precise argument given by Lemma 14. Next, (b) follows as $\mathbb{E} \left[ \bar{u}_n \mathbb{1} \left\{ \frac{\bar{q}_n}{\tau} > t^* \right\} \right] = 0$ for large enough $\eta$. Now, (c) holds due to the following.

$$
0 \leq \mathbb{E} [\bar{u}_n^2] \leq S_{\max} \mathbb{E} [\bar{u}_n] \mathbb{E} \left[ \phi^c \left( \frac{\bar{q}_n}{\tau} \right) - \phi^c \left( \frac{\bar{q}_n}{\tau} \right) \right] \leq \epsilon S_{\max} \mathbb{E} \left[ \phi^c \left( \frac{\bar{q}_n}{\tau} \right) - \phi^c \left( \frac{\bar{q}_n}{\tau} \right) \right] = o(1),
$$

where the last equality follows by showing a result similar to Lemma 2. Lastly, (d) follows by (43). Substituting the above along with $\mathcal{T}_3$ and $\mathcal{T}_4$ in (47), we get

$$
\mathbb{E} \left[ \mathbb{1} \left\{ \frac{\bar{q}_n}{\tau} \leq t^* \right\} e^{j\omega \bar{q}_n/\tau} \right]
$$

\[
= \frac{2\epsilon \tau t^*}{\omega (\sigma^c(\lambda^*) + \sigma^s(\mu^*))} \mathbb{E} \left[ \phi^c \left( \frac{\bar{q}_n}{\tau} \right) - \phi^c \left( \frac{\bar{q}_n}{\tau} \right) \right] \left( 1 - e^{j\omega t^*} \right) + o_\tau(1) + o_\tau(1) o_\omega(1).
\]

Now, by letting $\omega \downarrow 0$, we get

$$
\mathbb{P} \left( \frac{\bar{q}_n}{\tau} \leq t^* \right) = \mathbb{P} \left( \frac{\bar{q}_n}{\tau} \leq t^* \right) \left( \frac{e^{j\omega t^*} - 1}{j\omega t^*} \right) + o_\tau(1) + o_\tau(1) o_\omega(1).
$$

Substituting that back again, we get

$$
\mathbb{E} \left[ \mathbb{1} \left\{ \frac{\bar{q}_n}{\tau} \leq t^* \right\} e^{j\omega \bar{q}_n/\tau} \right] = \mathbb{E} \left[ \mathbb{1} \left\{ \frac{\bar{q}_n}{\tau} \leq t^* \right\} \left( \frac{e^{j\omega t^*} - 1}{j\omega t^*} \right) + o_\tau(1) + o_\tau(1) o_\omega(1). \right)
$$

(48)
Similar to Lemma 11, we can further show that

$$\limsup_{\eta \uparrow \infty} \mathbb{P}\left(\frac{\bar{q}_\eta}{\tau} > t^*\right) = 0.$$  \hspace{1cm} (49)$$

As the steps are analogous to the proof of Lemma 11, we omit the details here. Now, by using (48) and (49), we get

$$\lim_{\eta \uparrow \infty} \mathbb{E}\left[e^{j\omega \bar{q}_\eta/\tau}\right] = \lim_{\eta \uparrow \infty} \mathbb{E}\left[e^{j\omega \bar{q}_\eta/\tau} 1\{\frac{\bar{q}_\eta}{\tau} \leq t^*\}\right] + \lim_{\eta \uparrow \infty} \mathbb{E}\left[e^{j\omega \bar{q}_\eta/\tau} 1\{\frac{\bar{q}_\eta}{\tau} > t^*\}\right]$$

$$= \lim_{\eta \uparrow \infty} \mathbb{E}\left[e^{j\omega \bar{q}_\eta/\tau} 1\{\frac{\bar{q}_\eta}{\tau} \leq t^*\}\right] + \lim_{\eta \uparrow \infty} \mathbb{P}\left(\frac{\bar{q}_\eta}{\tau} \leq t^*\right) \left(\frac{e^{j\omega t^*} - 1}{j\omega t^*}\right)$$

By noting that the above is a characteristic function of a uniform distribution, the proof is complete by Levy’s continuity theorem [77, Chapter 18].