Measuring risk based on skewed t distribution approach

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Abstract. This paper analyses a Skewed t Distribution approach to estimate Value at Risk (VaR) as a tool that can measure a risk investment. The method can estimate an investment risk that can overcome the shortcoming of classical VaR, which cannot capture the existence of fat tail and skewness. The application of the method was utilized to evaluate the individual risk of four stocks taken from the NYSE Index, namely Advance Micro Devices Inc (AMD), The Coca-Cola Company (KO), Pfizer Inc. (PFE), and Walmart Inc (WMT). It can be summarized from the result of the analysis that VaR (in several confidence levels) based on the distribution approach is powerful in risk measurement and can give an alternative to the investor for estimating the risk.

1. Introduction

Value at Risk (VaR) is defined as an estimation method of maximum loss over a specific period in normal market condition at specific confidence levels. Even though VaR can estimate the loss that can be suffered by the investor over time, \( t \), with a specified confidence level, \( \alpha \), return data is frequently asymmetric distributed. Return data distributions are commonly fat tailed and skewed [1]. So, it is difficult to explain the behavior of fat tail and skewness on the return distribution. Because of that facts, a risk measure based on a distribution approach that can capture excess kurtosis and excess skewness of return data is needed.

Several distributions can be utilized to explain the behavior of fat tail and skewness. Aas and Haff [2] explained the Generalized Hyperbolic Skewed Student’s t Distribution and presented empirical evidence of tail behavior of skewed financial data. Aas and Haff [2] also showed the importance of exponential/polynomial tail behavior through VaR based on the distribution. Sukono et al. [3] used Skewed Student t Distribution to propose Modified Value at Risk, a new model to measure the risk. Theodossiou [4] derived a formula to compute Value at Risk for asset returns based on Skewed Generalized t Distribution.

Leinwander and Aziz [5] utilized Skewed t Distribution (STD) to describe the number of claims in insurance. Leinwander and Aziz [5] stated that Skewed t Distribution (STD) is regarded as an appropriate model that enables the users to control skewness and kurtosis of a claim distribution. Then, Hu and Kercheval [1] applied STD as a distribution approach to examine an optimal portfolio located on the efficient frontier. Their analysis showed that STD could better fit real return data than the Normal Distribution due to portfolio optimization.

Because of the mentioned eminences of the STD, this paper will analyze STD in order to measure the risk of return assets that are often founded asymmetrically. This study refers to VaR based on STD.
constructed by Dokov, Stoyanov, and Rachev [6]. Because [6] only focused on the theoretical analysis, this paper attempts to present the VaR application in real financial data and assess its performance in measuring the financial asset’s risk.

The research paper is organized as follows: VaR based on STD is presented in Section 2. The application of VaR based on STD and assessing VaR’s performance based on STD is given in Section 3. Finally, section 4 contains the conclusion of the research paper.

2. VaR Based on Skewed t Distribution

In this section, this paper will provide characteristics of STD, also known as Asymmetric Student-t Distribution. This section also presents the characteristics of VaR based on STD.

STD belongs to the family of Skewed Elliptical Distributions [7]. STD also is a special case (subclass) of Generalized Hyperbolic Distribution (GHD) [8]. GHD is a mixture of Normal Multivariate Distribution with Generalized Inverse Gaussian Distribution. A random variable, \( Y \), follows a GHD when the variable has a density function as follows:

\[
\mu + \Sigma^{1/2} \text{BesselI}(\lambda, \psi' \Sigma^{-1} \psi) \frac{d}{d \lambda},
\]

where \( \psi \) represents a location parameter, and \( \Sigma \) is a \( d \times d \) semidefinite covariance matrix [9], \( \alpha \) is a degree of freedom, and \( \gamma \) is a parameter that controls the distribution asymmetry. The positive value of \( \gamma \) signifies that the distribution of asset return will be skewed to the right. Conversely, the negative value of \( \gamma \) indicates that the distribution of asset return will be skewed to the left.

The density function of STD is a density function of GHD when \( d = 1, \alpha = \nu, \lambda = \frac{1}{2} \nu \), and \( \psi = 0 \). Therefore, STD’s density function can be written as follows [6]

\[
f(y) = \frac{\nu^{\nu/2} \lambda^{\nu/2} \Gamma(\frac{\nu}{2})}{\sqrt{\pi} \nu} \int_{0}^{\infty} e^{-\lambda y^2 - \frac{\nu y^2}{4t}} e^{-\frac{\nu y^2}{4t}} \, dt, \quad -\infty < y < \infty.
\]

The value of \( d = 1 \) denotes that the asset return is considered as univariate required to assess the VaR [6].

VaR is an \( \alpha \) -quantile of distribution in some specific confidence levels. VaR of asset return that is assumed Skewed t distributed written in Equation (2) at confidence level \( \alpha \) can be defined as the smallest number \( y_0 \) such that the probability of the loss \( Y \) exceeds \( y_0 \) is not greater than \( 1 - \alpha \) [6]

\[
\text{VaR}_\alpha(Y) = \inf\{y_0: P(Y > y_0) \leq 1 - \alpha\} = \inf\{y_0: F(y_0) \geq \alpha\} = F^{-1}_Y(\alpha),
\]

where \( F(.) \) symbolizes the cumulative distribution function (CDF) of \( Y \), while \( F^{-1}_Y \) represents the inverse function of \( F \), and the last equality will hold for a continuous distribution.

VaR for a random variable that is Skewed t distributed can be determined by Equation (3).

\[
\text{VaR}_\alpha = y_0,
\]

where the value of Equation (3) is coming as the unique zero of Equation (4) and Equation (5) [6].

A function \( g(y_0) \) for the Skewed t, which has negative skewness \( \gamma < 0 \), is given in the following equation [6]
\[ g(y_0) = 1 - \alpha + \frac{2C\sqrt{\pi}}{\gamma} \int_0^\infty t^{-(p+2)/2} e^{-\frac{y^2}{4t}} \Phi \left( \frac{y\sqrt{\gamma}}{\sqrt{2t}} - \sqrt{2t} \right) dt = 0, \tag{4} \]

where \( C = \frac{\gamma^{3/2}}{\sqrt{\pi} \Gamma \left( \frac{\gamma}{2} \right)} \).

The estimation of \( y_0 \) as a unique zero of \( g(y_0) \) will be derived by substituting the integral of \( g(y_0) \) into Equation (5).

So, it can be resulted

\[ g(y_0) = -\alpha + \int_0^{t_0(y_0)} t^{-(p+2)/2} e^{-\frac{y^2}{4t}} \Phi \left( \frac{y\sqrt{\gamma}}{\sqrt{2t}} - \sqrt{2t} \right) + R(t_0(y_0)). \tag{5} \]

Therefore, \( y_0 \) can be obtained on these intervals [6]

\[ \begin{cases} \left[ -\frac{\alpha}{\gamma}, 0 \right], & \text{if } \alpha < \alpha_0, \gamma > 0 \\ [0, \infty), & \text{if } \alpha > \alpha_0, \gamma > 0 \\ (-\infty, 0], & \text{if } \alpha < \alpha_0, \gamma < 0 \\ [0, -\frac{\alpha}{\gamma}], & \text{if } \alpha > \alpha_0, \gamma < 0 \end{cases} \tag{6} \]

Meanwhile, \( \alpha_0 \) in Equation (6) can be derived by Equation (7).

\[ \alpha_0 = \begin{cases} \frac{2C\sqrt{\pi}}{\gamma} \int_0^{18} t^{-(p+2)/2} e^{-\frac{y^2}{4t}} \Phi(-\sqrt{2t}) dt, & \text{if } \gamma > 0 \\ 1 + \frac{2C\sqrt{\pi}}{\gamma} \int_0^{18} t^{-(p+2)/2} e^{-\frac{y^2}{4t}} \Phi(-\sqrt{2t}) dt, & \text{if } \gamma < 0 \end{cases} \tag{7} \]

Then, a function \( g(y_0) \) for the Skewed t, which has positive skewness (\( \gamma > 0 \)) is given in the following equation [6]

\[ g(y_0) = -\alpha + \frac{2C\sqrt{\pi}}{\gamma} \int_0^\infty t^{-(p+2)/2} e^{-\frac{y^2}{4t}} \Phi \left( \frac{y\sqrt{\gamma}}{\sqrt{2t}} - \sqrt{2t} \right) dt = 0. \tag{8} \]

### 3. Application of VaR Based on Skewed t Distribution

In this section, VaR based on STD was applied to several stocks, namely Advance Micro Devices Inc (AMD), The Coca-Cola Company (KO), Pfizer Inc. (PFE), and Walmart Inc (WMT). The analyzed data ranges from 1st July 2019 until 30th June 2020. The computation of VaR based on STD (VaR STD) in this section was conducted by developing an R program.

Before conducting the analysis, the ten-period data of close-price stocks accessed from https://finance.yahoo.com/ was transformed previously into profit/loss data. The summary statistics of the profit loss are given in Table 1.

**Table 1. Summary statistics of profit/loss data**

| Stock | Min | 1st Qu. | Mean | 3rd Qu. | Max. | Skewness | Kurtosis |
|-------|-----|---------|------|---------|------|----------|----------|
| AMD   | -6.690 | -0.100 | 0.017 | 0.120 | 4.930 | -0.383 | 24.320 |
| KO    | -5.049 | -0.189 | 0.008 | 0.229 | 3.110 | -1.176 | 15.935 |
| WMT   | -10.670 | -0.390 | 0.028 | 0.450 | 12.500 | 0.704 | 30.526 |
| PFE   | -2.730 | -0.370 | -0.044 | 0.310 | 2.690 | -0.531 | 3.319 |
Next, using Maximum Likelihood Estimation, parameter estimations of STD for profit/loss of AMD, KO, WMT, and PFE over the period 1st July 2019 until 30th June 2020 were summarized in Table 2.

| Stock | $\hat{\mu}$ | $\hat{\sigma}$ | $\hat{\nu}$ | $\hat{\gamma}$ |
|-------|-------------|----------------|-------------|--------------|
| AMD   | 0.091       | 3.459          | 2.122       | 0.977        |
| KO    | -0.032      | 8.270          | 2.007       | 0.942        |
| WMT   | -0.038      | 23.232         | 2.003       | 0.964        |
| PFE   | -0.031      | 0.849          | 2.733       | 0.955        |

Table 2 shows that the estimated mean of profit/loss for asset AMD, KO, WMT, and PFE successively is 0.091, -0.032, -0.038, and -0.031. The table also reports that there is a big discrepancy of the estimated standard deviation ($\hat{\sigma}$) of WMD’s profit/loss with the other profit/loss’ standard deviation. Then, Table 2 also provides the estimated skewness parameter ($\hat{\gamma}$) for the corresponding profit/loss of each stock.

Based on Table 2, VaR STD’s parameter estimation for the four assets at 99 percent, 97.5 percent, 95 percent, 90 percent, and 80 percent confidence levels can be obtained as tabulated in Table 3.

| Stock | $1 - \alpha$ | VaR STD | $1 - \alpha$ | VaR STD | $1 - \alpha$ | VaR STD |
|-------|--------------|---------|--------------|---------|--------------|---------|
| AMD   | 0.010        | 5.304   | 0.050        | 2.388   | 0.200        | 0.969   |
| KO    | 3.182        | 1.361   | 0.900        | 0.514   |              |         |
| WMT   | 5.733        | 2.429   | 0.900        | 0.514   |              |         |
| PFE   | 2.035        | 1.017   | 0.900        | 0.514   |              |         |

Table 3 also indicates that AMD and WMT have a greater risk than other stocks in Table 3. Meanwhile, PFE has the smallest risk among AMD, KO, and WMT. Moreover, similar to characteristic classical VaR, Table 3 reports that the higher confidence level utilized in VaR STD computation produces the higher VaR STD.

A risk measurement method is considered to be well-specified when it can appropriately fulfill the required theoretical statistical properties [11]. Whether or not a method meets this requirement can be determined by investigating the proportion of exceedances, namely the profit/loss values of assets that are greater than the VaR values of the proposed model. In this section, we analyzed the VaR STD method’s performance using Kupiec backtesting, as proposed by Kupiec [12]. The backtesting method involves the examination of how many times a risk measure is exceeded over a given time interval. Table 4 presents the backtesting results of VaR STD for AMD, KO, WMT, and PFE for the determined confidence levels.
Table 4. Results of Kupiec backtesting for VaR STD for analyzed assets

| Asset | \( \alpha \) (%) | Number of Outliers | Percentage of Loss | P-Value |
|-------|------------------|--------------------|--------------------|---------|
| AMD   | 80               | 39                 | 15.538             | 0.957   |
| KO    | 80               | 53                 | 21.116             | 0.297   |
| WMT   | 80               | 52                 | 20.717             | 0.353   |
| PFE   | 80               | 53                 | 21.116             | 0.297   |
| AMD   | 90               | 26                 | 10.359             | 0.374   |
| KO    | 90               | 25                 | 9.960              | 0.455   |
| WMT   | 90               | 28                 | 11.155             | 0.233   |
| PFE   | 90               | 30                 | 11.952             | 0.129   |
| AMD   | 95               | 12                 | 4.781              | 0.488   |
| KO    | 95               | 18                 | 7.171              | 0.051   |
| WMT   | 95               | 14                 | 5.578              | 0.276   |
| PFE   | 95               | 15                 | 5.976              | 0.192   |
| AMD   | 97.5             | 9                  | 3.586              | 0.101   |
| KO    | 97.5             | 11                 | 4.382              | 0.026   |
| WMT   | 97.5             | 8                  | 3.187              | 0.180   |
| PFE   | 97.5             | 6                  | 2.390              | 0.438   |
| AMD   | 99               | 2                  | 0.799              | 0.459   |
| KO    | 99               | 4                  | 1.594              | 0.109   |
| WMT   | 99               | 2                  | 0.797              | 0.459   |
| PFE   | 99               | 6                  | 2.390              | 0.014   |

The results of Kupiec Backtesting in Table 4 show that VaR STD’s performance with the several confidence levels effectively measures the risk of AMD, KO, WMT, and PFE. The VaR STD’s effectiveness is indicated by the values of the fifth column of Table 4, which are greater than the corresponding \( 1 - \alpha \). Moreover, the empirical performance of the VaR STD in this section complemented the research of Dokov, Stoyanov, and Rachev [6], who only examined the performance of the VaR STD theoretically.

4. Conclusions

STD distribution has a skewness parameter that can capture the fat tail of asset return. This characteristic is substantial in constructing a risk measure such as VaR because we are often to find return asset that is asymmetrically distributed. Based on the empirical analysis conducted in this study and the VaR’s performance assessed by Kupiec Backtesting, we summarized that the VaR STD approach could be utilized to quantify the risk of return assets in various confidence levels.

Acknowledgments

The authors would like to thank the anonymous referee for the valuable suggestions that improve the paper. The authors also acknowledge the Faculty of Mathematics and Natural Sciences, Universitas Tanjungpura, that finance the authors’ participation at the 10th ISNPINSA 2020 conducted virtually.

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