SIMPLE MODULES OVER THE HIGHER RANK SUPER-VIRASORO ALGEBRAS

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Abstract. It is proved that uniformly bounded simple modules over higher rank super-Virasoro algebras are modules of the intermediate series, and that simple modules with finite dimensional weight spaces are either modules of the intermediate series or generalized highest weight modules.

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§1. Introduction

The Virasoro algebra $Vir$, playing a fundamental role in two dimensional conformal quantum field theory, is the universal central extension of the complex Lie algebra of the polynomial vector fields on the circle [1]. The notion of higher rank Virasoro algebras and the notion of higher rank super-Virasoro algebras were introduced in [2,3] and the notions of generalized Virasoro and super-Virasoro algebras were introduced in [4]. Kac [5] conjectured a theorem that a simple $Vir$-module with finite dimensional weight spaces is either a module of the intermediate series or else a highest or lowest weight module. This theorem was partially proved in [6,7,8] and fully proved in [9] and then generalized to the super-Virasoro algebras in [10], and was in some sense generalized to the higher rank Virasoro algebras in [11]. In this paper, we shall obtain a similar result for the higher rank super-Virasoro algebras, thus completing the proof of the theorem conjectured by the author in [3]. In [12,] by using the above mentioned theorem, we gave a theorem on indecomposable $Vir$-modules. Using our result here, we shall have a better understanding on the simple modules over the higher rank super-Virasoro algebras and it may be possible to give in some extent a classification of the uniformly bounded modules over the higher rank super-Virasoro algebras. This is also our motivation to present the result here.

Let $n$ be a positive integer. Let $M$ be an $n$-dimensional $\mathbb{Z}$-submodule of $\mathfrak{g}'$, and let $s \in \mathfrak{g}'$ such that $2s \in M$. The rank $n$ super-Virasoro algebra (or a higher rank super-Virasoro algebra if $n \geq 2$) is the Lie superalgebra $SVir[M,s] = SVir_0 \oplus SVir_1$, where $SVir_0$ has a basis $\{L_\mu, c \mid \mu \in M\}$ and $SVir_1$ has a basis $\{G_\eta \mid \eta \in s + M\}$, with the commutation relations

$$[L_\mu, L_\nu] = (\nu - \mu)L_{\mu+\nu} - \delta_{\mu+\nu,0} \frac{1}{12} (\mu^3 - \mu)c, \quad [L_\mu, G_\eta] = (\eta - \frac{\mu}{2}) G_{\mu+\eta},$$

$$[G_\eta, G_\lambda] = 2L_{\eta+\lambda} - \delta_{\eta+\lambda,0} \frac{1}{3} (\eta^2 - \frac{1}{4}) c,$$

$$[L_\mu, c] = [G_\eta, c] = 0,$$  \hspace{1cm} (1.1)

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for $\mu, \nu \in M$, $\eta, \lambda \in s + M$. Then $SVir_0$ is the higher rank Virasoro algebra $SVir[M]$. It is proved in [4] that a module of the intermediate series (i.e., a module with all the dimensions of the weight spaces of “even” or “odd” part are $\leq 1$) over $SVir[M, s]$ is one of the three series of the modules $SA_{a,b}, SA_{a'}, SB_{a'}$ or their quotient modules for suitable $a, b, a' \in G'$, where $SA_{a,b}, SA_{a'}$ have basis $\{x_\mu | \mu \in M\} \cup \{y_\eta | \eta \in s + M\}$ and $SB_{a'}$ has basis $\{x_\eta | \eta \in s + M\} \cup \{y_\mu | \mu \in M\}$ such that $c$ acts trivially and such that

\begin{align}
SA_{a,b}: \quad & L_\mu x_\nu = (a + \nu + \mu b)x_{\mu + \nu}, \quad L_\mu y_\eta = (a + \eta + \mu (b - \frac{1}{2}))y_{\mu + \eta} \\
& G_\lambda x_\nu = y_{\lambda + \nu}, \quad G_\lambda y_\eta = (a + \eta + 2\lambda (b - \frac{1}{2}))y_{\lambda + \eta} \\
SA_{a'}: \quad & L_\mu x_\nu = (\nu + \mu)x_{\mu + \nu}, \quad \nu \neq 0, \quad L_\mu y_\nu = \mu (\mu + a')y_\mu, \quad L_\mu y_\eta = (\eta + \lambda (\mu + a'))y_{\mu + \eta} \\
& G_\lambda x_\nu = y_{\lambda + \nu}, \quad \nu \neq 0, \quad G_\lambda y_\nu = (2\lambda + \mu + a')y_\lambda, \quad G_\lambda y_\eta = (\eta + \lambda)\nu x_{\lambda + \nu} \\
SB_{a'}: \quad & L_\mu x_\eta = (\eta + \frac{\lambda}{\mu})x_{\mu + \eta}, \quad L_\mu y_\nu = \nu y_{\mu + \nu}, \quad \nu \neq -\mu, \quad L_\mu y_\eta = -\mu (\mu + a')y_\eta, \quad L_\mu y_{-\mu} = -\mu (\mu + a')y_0, \quad L_\mu y_{-\mu} = -\mu (\mu + a')y_0, \\
& G_\lambda x_\eta = y_{\lambda + \mu}, \quad \eta \neq -\lambda, \quad G_\lambda x_{-\lambda} = (2\lambda + a')y_0, \quad G_\lambda y_\nu = \nu x_{\lambda + \nu},
\end{align}

(1.2a) \hspace{2cm} (1.2b) \hspace{2cm} (1.2c)

for $\mu, \nu \in M$, $\lambda, \eta \in s + M$. In this paper, we shall prove the following theorem which was conjectured by the author in [3].

**Theorem 1.1.** A uniformly bounded simple $SVir[M, s]$-module is a module of the intermediate series.

A $SVir[M, s]$-module $V$ is called a *generalized highest module* if it is generated by a weight vector $v$ such that there exists a $\mathbb{Z}$-basis $B = \{d_1, \ldots, d_n\}$ of $M$ such that

$$L_\mu v = G_\lambda v = 0, \forall \mu = \sum_{i=1}^n m_i d_i \in M \setminus \{0\}, \lambda = \sum_{i=1}^n n_i d_i \in s + M \setminus \{0\},$$

(1.3)

with all coefficients $m_i \in \mathbb{Z}_+$, $n_i \in \frac{1}{2} \mathbb{Z}_+$. The vector $v$ is called a *generalized highest weight vector*. Observe that a lowest weight module over the super-Virasoro algebras (i.e., when $n = 1$) is a generalized highest weight module by this definition. In general, if a module $V$ has a weight space decomposition with all the dimensions of weight spaces being finite, then we shall call $V$ a *weight module*. The following theorem generalizes Kac’s conjecture in some sense.

**Theorem 1.2.** A weight $SVir[M, s]$-module is either a module of the intermediate series or a generalized highest weight module.

Thus the problem of the classification of simple weight modules is reduced to the problem of the classification of the generalized highest weight modules.
§2. Uniformly bounded modules

The goal of this section is to prove Theorem 1.1. We shall use induction on the rank of $M$, the results of [11], the structure of modules of the intermediate series defined in (1.2), and the defining relations (1.1) to obtain our result.

Thus suppose that $V$ is a uniformly bounded $SVir[M, s]$-module. Obviously $c$ must act trivially on such a module [3]. Thus we shall ALWAYS omit $c$ in this section. Decompose $V = V^{(0)} \oplus V^{(1)}$ such that

$$SVir_i V^{(j)} \subset V^{(i+j)}$$

for $i, j \in \mathbb{Z}/2\mathbb{Z}$. For $a \in \mathfrak{g}'$, define a submodule

$$V(a) = \sum_{\mu \in M} V^{(0)}_{\mu + \mu} \oplus \sum_{\lambda \in s + M} V^{(1)}_{\lambda + \mu},$$
for $a \in \mathfrak{g}'$. Then $V$ is a direct sum of different $V(a)$. Since we assume that $V$ is simple, we have $V = V(a)$ for some $a \in \mathfrak{g}'$. Since $V^{(i)}, i \in \mathbb{Z}/2\mathbb{Z}$ are $SVir_0$-modules, by [11, Lemma 2.1], there exist nonnegative integers $N^{(0)}, N^{(1)}$ such that

$$\dim V^{(i)}_{a + is + \mu} = N^{(i)}$$
for all $\mu \in M$ with $a + is + \mu \neq 0, i = 0, 1$.

By interchanging $V^{(0)}$ with $V^{(1)}$ if necessary, we always suppose that $N^{(0)} \leq N^{(1)}$.

**Proof of Theorem 1.1.** The result is obvious if $N^{(1)} = 0$. Suppose that $N^{(1)} \geq 1$. If $n = 1$, the result follows from [10]. So assume that $n \geq 2$. Observe that

$$SVir[M, s] \cong SVir[M', s'] \iff \exists \alpha \in \mathfrak{g}' \text{ such that } M' = \alpha M \text{ and } s' - \alpha s \in M'.$$

Thus we can always suppose that $1 \in M$ and furthermore we can suppose that $1$ is a basis element of $M$ whenever necessary. So, we can take a $\mathbb{Z}$-basis $B = \{d_1 = 1, d_2, \cdots, d_{n-1}, d_n = d\}$ of $M$ such that

$$M = M_1 \bigoplus \mathbb{Z}d, \quad M_1 = \mathbb{Z} \bigoplus \sum_{i=2}^{n-1} \mathbb{Z}d_i, \quad 2s \in M_1. \quad (2.1)$$

Let $SVir[M_1, s]$ be the rank $n - 1$ super-Virasoro algebra generated by

$$\{L_\mu, G_\lambda | \mu \in M_1, \lambda \in s + M_1\}.$$

For $k \in \mathbb{Z}$, let

$$V[k] = \sum_{\mu \in M_1} V^{(0)}_{a + kd + \mu} \oplus \sum_{\lambda \in s + M_1} V^{(1)}_{a + kd + \lambda},$$
which is a uniformly bounded $SVir[M_1, s]$-module. Since $\mathbb{Z}d \cap M_1 = \{0\}$, there exists at most one $k \in \mathbb{Z}$ such that $a + kd \in M_1$. Fix a $k_0$ such that
\[ a + kd \not\in M_1, \quad \forall k \geq k_0, \]  
(2.2)
i.e., $a + kd + \nu \neq 0$ for all $k \geq k_0$, $\nu \in M_1$. First we take $k = k_0$. By inductive assumption, we can suppose that Theorem 1.1 holds for the rank $n - 1$ super-Virasoro algebra $SVir[M_1, s]$. Thus there exists a simple $SVir[M_1, s]$-submodule $V[k, 1]$ of $V[k]$ such that $V[k, 1]$ has the form $SA_{a+kd,b}$ in (1.2) (by (2.2), $SA_{a+kd,b}$ is simple), thus, there exists a basis
\[ \{x_\mu \mid \mu \in M_1\} \cup \{y_\lambda \mid \lambda \in s + M_1\}, \]  
(2.3)
of $V[k, 1]$, such that
\[(1.2a) \text{ holds for all } \mu, \nu \in M_1, \lambda, \eta \in s + M_1 \text{ with } a \text{ replaced by } a + kd. \]  
(2.4)
Furthermore, among all simple submodules of $V[k]$, we choose $V[k, 1]$ to be one such that the real part $\text{Re}(b)$ of $b$ is minimum. Take a composition series of the $SVir[M_1, s]$-module $V[k + 1]$: \[ 0 = V[k + 1, 0] \subset V[k + 1, 1] \subset \cdots \subset V[k + 1, N] = V[k + 1], \]  
(2.5)
for some $N$. From this, we can observe that $N = N^{(0)} = N^{(1)}$. We shall choose the composition series (2.5) such that
\[ V[k + 1, i]/V[k + 1, i - 1] \cong SA_{a+(k+1)d,b'}, \]  
(2.6)
for some $b' \in G'$, $i = 1, \cdots N$, where $b'$ may depend on $i$. Then, we have
\[ L_dV[k, 1]^{(0)} \subset L_dV[k, 1] \subset L_dV[k] \subset V[k + 1], \]  
(2.7)
where $V[k, 1]^{(0)}$ is the “even” part of $V[k, 1]$.

We shall divide the proof into some claims.

**Claim 1.** $L_dV[k, 1]^{(0)} \neq 0$.

Suppose $L_dV[k, 1]^{(0)} = 0$. Using the fact that
\[ L_{\mu+md} = \prod_{i=1}^{m-2} (\mu + id)^{-1}(\text{ad} L_d)^m \mu, \quad G_{s+\mu+md} = \prod_{i=1}^{m-2} (s + \mu + \frac{i}{2}d)^{-1}(\text{ad} L_d)^m G_{s+\mu}, \]
we obtain
\[ L_{\mu+md}V[k, 1]^{(0)} = G_{s+\mu+md}V[k, 1]^{(0)} = 0, \quad \forall \mu \in M_1, \; m > 0. \]

Let $L_1$ and $L_2$ be the Lie super-subalgebras of $SVir[M, s]$ generated by
\[ SVir[M_1, s] \cup \{L_d\} \text{ and } \{L_{\mu+md}, G_{s+\mu+md} \mid \mu \in M_1, \; m \in \mathbb{Z}_+ \setminus \{0\}\}, \]
respectively. Then as space, we have $SVir[M, s] = \mathcal{L}_1 \oplus \mathcal{L}_2$. By decomposing the universal enveloping algebra $U(SVir[M, s]) = U(\mathcal{L}_1)U(\mathcal{L}_2)$, we obtain that

$$V = U(\mathcal{L}_1)U(\mathcal{L}_2)V[k, 1]^{(0)} = U(\mathcal{L}_1)V[k, 1]^{(0)}.$$ 

From this, we obtain that $V_{\pi_{\nu}^{(i)}}^{(i)} = 0$ for all $\mu \in M_1$, $m > 0$, $i = 0, 1$. Thus $N = 0$, contradicting the assumption that $N = N^{(1)} \geq 1$. Hence Claim 1 follows.

Now from (2.5), (2.7), we can take $L_d V[k, 1] \subset V[k + 1, K + 1]$. (8) Furthermore, we can choose (2.5) to be a composition series such that $K$, defined by (8), is minimal possible among all composition series of $V[k + 1]$. Then, we can take

$$\{x'_\mu \in V[k + 1, K + 1]^{(0)} \mid \mu \in M_1\},$$

such that

$$(V[k + 1, K + 1]/V[k + 1, K])^{(0)} = \text{span}\{x'_\mu + V[k + 1, K]^{(0)} \mid \mu \in M_1\},$$

and such that there exist some $b' \in \mathfrak{G}'$ and some $a_\nu \in \mathfrak{G}'$ with

$$L_\mu x'_\nu \equiv (a + (k + 1)d + \nu + \mu b')x'_{\mu+\nu}, \quad L_d x'_\nu \equiv a_\nu x'_\nu \pmod{V[k + 1, K]^{(0)}},$$

for all $\mu, \nu \in M_1$.

Now [11, Lemmas 2.3-6] shows that $b' = b$ and that by rescaling $x'_\mu$ if necessary, we can suppose that $a_\nu = a + kd + \nu + bd$, and furthermore $K = 0$, i.e., $V[k + 1, K] = 0$. Thus (10) becomes

$$L_\mu x'_\nu = (a + (k + 1)d + \nu + \mu b')x'_{\mu+\nu}, \quad L_d x'_\nu = (a + kd + \nu + bd)x'_\nu.$$ 

Since $V[k + 1, 1]$ is a $SVir[M_1, s]$-module of the intermediate series, there exists basis $\{y'_\lambda \mid \lambda \in s + M_1\}$ of $V[k + 1, 1]^{(1)}$ such that

(1.2a) holds for all $\mu, \nu \in M_1$, $\lambda, \eta \in s + M_1$ with the symbols $x, y$ replaced by symbols $x', y'$ respectively, and with $a$ replaced by $a + (k + 1)d$. (2.12)

By (8), $L_d V[k, 1]^{(1)} \subset V[k + 1, 1]^{(1)}$. Thus we can suppose

$$L_d y_\lambda = c_\lambda y'_\lambda$$

for some $c_\lambda \in \mathfrak{G}'$, $\lambda \in s + M_1$. (2.13)

Then by [11, Lemma 2.3], there exists nonzero scalar $c \in \mathfrak{G}'$ such that

$$c_\lambda = c(a + kd + \lambda + d(b - 1/2)), \quad \forall \lambda \in s + M_1.$$ 

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Claim 2. One can choose a suitable simple $SVir[M_1, s]$-submodule $V[k, 1]$ of $V[k]$ such that $L_{-d}L_d V[k, 1] \subset V[k, 1]$.

Let $W[k]$ be the direct sum of ALL simple $SVir[M_1, s]$-submodules $V[k, 1]$ of $V[k]$ of the same type: $V[k, 1] \cong SA_{a+kd,b}$ and let $W[k + 1]$ be the direct sum of ALL simple $SVir[M_1, s]$-submodules $V[k + 1, 1]$ of $V[k + 1]$ of the same type: $V[k + 1, 1] \cong SA_{a+(k+1)d,b}$. Then the above discussion shows that $L_dW[k] \subset W[k + 1]$, and so we also have $L_{-d}W[k + 1] \subset W[k]$ by replacing $d$ by $-d$ in the above arguments. By Claim 1, $W[k], W[k + 1]$ must have the same number, say, $r$, of composition factors. Choose a suitable basis $\{x^{(j)}_\nu, y^{(j)}_{s+\nu} | \nu \in M_1, j = 1, ..., r\}$ of $W[k]$ such that $x_0^{(1)}$ is an eigenvector of $L_{-d}L_d$ and

$$L_\mu X_\nu = (a + kd + \nu + \mu b)X_{\mu + \nu}, \quad L_\mu Y_\eta = (a + kd + \eta + \mu(b - \frac{1}{2}))Y_{\mu + \eta}$$

$$G_\lambda X_\nu = Y_{\lambda + \nu}, \quad G_\lambda Y_\eta = (a + kd + \eta + 2\lambda(b - \frac{1}{2}))X_{\lambda + \eta},$$

where

$$X_\nu = \begin{pmatrix} x^{(1)}_\nu \\ \vdots \\ x^{(r)}_\nu \end{pmatrix}, \quad Y_\eta = \begin{pmatrix} y^{(1)}_\eta \\ \vdots \\ y^{(r)}_\eta \end{pmatrix}, \quad \forall \nu \in M_1, \eta \in s + M_1.$$  \hspace{1cm} (2.14)

Then Claim 1, (2.11) and (2.13) allow us to choose a basis $\{x^{(j)}_\nu, y^{(j)}_{s+\nu} | \nu \in M_1, j = 1, ..., r\}$ of $W[k + 1]$ such that

$$L_dX_\nu = (a + kd + \nu + bd)X'_\nu, \quad L_dY_\eta = c(a + kd + \eta + d(b - \frac{1}{2}))Y'_\eta,$$

where $X'_\nu, Y'_\eta$ are the similar notations as in (2.15), and such that (2.14) holds for $k$ replaced by $k + 1$ and $X, Y$ replaced by $X', Y'$. Similarly, with $d$ replaced by $-d$ in the above arguments, we have

$$L_{-d}X'_\nu = (a + (k + 1)d + \nu - bd)AX_\nu, \quad L_{-d}Y'_\eta = c'(a + (k + 1)d + \eta - d(b - \frac{1}{2}))AY_\eta,$$

where $A$ is some constant $r \times r$ matrix and $c' \in \mathcal{C} \setminus \{0\}$. Thus we have

$$L_{-d}L_d X_\nu = (a + kd + \nu + bd)(a + (k + 1)d + \nu - bd)AX_\nu.$$  \hspace{1cm} (2.16)

Since $x_0^{(1)}$ is an eigenvector of $L_{-d}L_d$, on the first row of $A$, by (2.16), there is only one possible nonzero entry in the first position. If we choose

$$V[k, 1] = \text{span}_\mathcal{C} \{x^{(1)}_\nu, y^{(1)}_{s+\nu} | \nu \in M_1\},$$

then we see that $L_{-d}L_d V[k, 1] \subset V[k, 1]$. This proves our Claim 2.

Now we can complete the proof of Theorem 1.1 as follows. Starting from $k = k_0$ and choosing $V[k_0, 1]$ as in Claim 2, by induction on $k > k_0$, we can choose $V[k, 1]$ to be the
simple $SVir[M_1, s]$-submodule generated by $L_dV[k - 1, 1]$. As in the proof of Claim 1, using $L_{-d}L_dV[k_0, 1] \subset V[k_0, 1]$, by induction on $m$, we have

$$L_{\nu + md}V[k, 1] \subset V[k + m, 1], \quad G_{\lambda + md}V[k, 1] \subset V[k + m, 1], \quad (2.17)$$

for all $\nu \in M_1, \lambda \in s + M_1, m, k \in \mathbb{Z}$ such that $k \geq k_0, k + m \geq k_0$. Since $V$ is a simple $SVir[M,s]$-module, it must be generated by $V[k_0,1]$, thus by (2.17),

$$\dim V_{a+\nu+kd}^{(0)} = \dim V_{a+\lambda+kd}^{(1)} = 1, \quad \forall \nu \in M_1, \lambda \in s + M_1, k \geq k_0.$$

This shows that $N = 1$, thus $\dim V_{a+\nu}^{(0)} = \dim V_{a+\lambda}^{(1)}$ for all $\nu \in M, \lambda \in s + M$ with $a + \nu \neq 0 \neq a + \lambda$, from this one sees that we must also have $\dim V_{a+\nu}^{(0)} \leq 1$, $\dim V_{a+\lambda}^{(1)} \leq 1$ if $a + \nu = 0 = a + \lambda$. Thus $V$ must be a module of the intermediate series.

§3. Generalized highest weight modules

The aim of this section is to prove Theorem 1.2. This can be done by three lemmas below. We suppose $n \geq 2$ as for $n = 1$, the result follows from [10].

**Lemma 3.1.** Let $V$ be a simple $SVir[M, s]$-module and let $B = \{d_1, \ldots, d_n\}$ be any $\mathbb{Z}$-basis of $M$. If $0 \neq v \in V$ is a weight vector and $k \in \mathbb{Z}_+$ such that $L_\mu v = 0 = G_\lambda v$ for all

$$\mu \in M_{B}^{(k)} = \{\mu \in M \mid \mu = \sum_{i=1}^{n} m_i d_i, \text{ all } k \leq m_i \in \mathbb{Z}_+\},$$

$$\lambda \in M_{B}^{(k)} = \{\lambda \in s + M \mid \lambda = \sum_{i=1}^{n} n_i d_i, \text{ all } k \leq m_i \in \frac{1}{2} \mathbb{Z}_+\},$$

then $V$ is a generalized highest weight module.

**Proof.** Take

$$d_i' = \sum_{j=1}^{i} (k + i - j + 1) d_j + k \sum_{j=i+1}^{n} d_j, \quad i = 1, \ldots, n.$$

It is straightforward to check that the determinant of coefficients of $\{d_1, \ldots, d_n\}$ is 1, thus $B' = \{d_1', \ldots, d_n'\}$ is also a $\mathbb{Z}$-basis of $M$. Now for any

$$0 \neq \mu' = \sum_{i=1}^{n} m_i' d_i' \in M_{B'}^{(0)} \quad \text{(so all } m_i' \geq 0, \text{ at least one } m_i' > 0),$$

we have $\mu' = \sum_{i=1}^{n} m_i d_i$ with all coefficients

$$m_i = k \sum_{j=1}^{i-1} m_j' + \sum_{j=i}^{n} (k + j + 1 - i) m_j' \geq k,$$

i.e., $\mu' \in M_{B}^{(k)}$, thus $L_{\mu'} v = 0$. Similarly, for any $0 \neq \lambda' \in M_{B'}^{(0)}$, we have $G_{\lambda'} v = 0$. Hence by definition (cf. (1.3)), $v$ is a generalized highest weight vector and $V$ is a generalized highest weight module.
Lemma 3.2. Fix a \( \mathbb{Z} \)-basis \( B = \{d_1, ..., d_n\} \) of \( M \). Let

\[ A = \{ \mu \in M \mid \mu = \sum_{i=1}^{n} m_i d_i, \ \text{all } m_i \in \{-1, 0, 1\}\}, \]

\[ A' = \{ \lambda \in s + M \mid \lambda = \sum_{i=1}^{n} n_i d_i, \ \text{all } n_i \in \{-1, -\frac{1}{2}, 0, \frac{1}{2}, 1\}\}, \]

be two finite sets. For any \( \mu \in L \) and let \( A \) be the Lie super-subalgebra of \( SVir[M, s] \) generated by \( \{L_\nu \mid \nu \in A_\mu \} \cup \{G_\lambda \mid \lambda \in A'_\mu \} \). Then there exists a \( \mathbb{Z} \)-basis \( B' = \{d'_1, ..., d'_n\} \) of \( M \) such that

\[ \mathcal{L} \supset SVir[M, s]^{(k)}_{B'} = \{L_\nu, G_\lambda \mid \nu \in M^{(k)}_{B'_\mu}, \lambda \in M^{(k)}_{B'_\mu}\} \] for some \( k \in \mathbb{Z}_+ \).

Proof. If necessary, by replacing \( d_i \) by \(-d_i\) (this does not change the sets \( A, A' \)), we can suppose \( m_i \geq 0, \ i = 1, ..., n \). First suppose that \( m_1 \neq 0, m_2 \neq 0 \). Take

\[ d'_1 = m_2 \mu + d_1 = (\mu + d_1) + \sum_{i=1}^{m_2-1} \mu = (m_1 m_2 + 1)d_1 + m_2 d_2 + \sum_{i=3}^{n} m_2 m_i d_i, \]

\[ d'_2 = m_1 \mu - d_2 = (\mu - d_2) + \sum_{i=1}^{m_1-1} \mu = m_1 d_1 + (m_1 m_2 - 1)d_2 + \sum_{i=3}^{n} m_1 m_i d_i, \]

\[ d'_i = d'_{i-1} + d_i = (\mu + d_{i-1} + d_i) + \sum_{i=1}^{m_2-1} \mu, \ i = 3, ..., n. \]

We can easily solve \( d_i, i = 1, ..., n, \) as an integral linear combination of \( B' = \{d'_1, ..., d'_n\} \), so \( B' \) is a \( \mathbb{Z} \)-basis of \( M \). From the second equality of \( d'_i \), we see \( L_{d'_i} \in \mathcal{L}, \ i = 1, ..., n \). For example,

\[ [L_\mu, ..., [L_\mu, L_{\mu+d_1}]... \] (\( m_2 - 1 \) copies of \( L_\mu \)) = a L_{d'_i}, \] where \( a = \prod_{i=0}^{m_2-2} (i \mu + d_1) \neq 0, \)

so that the left-hand side is in \( \mathcal{L} \), thus \( L_{d'_i} \in \mathcal{L} \) and so \( \{L_\nu \mid \nu \in M_{B'}^{(0)}\} \subset \mathcal{L} \). Similarly, \( \{G_\lambda \mid \lambda \in M_{B'}^{(0)}\} \subset \mathcal{L} \). Next suppose that some \( m_i = 0, 1 \leq i \leq n, \) say, \( m_1 = 0 \). Take

\[ d'_1 = \mu + d_1, \ \ d'_i = d'_{i-1} + d_i = \mu + d_1 + d_i, \ i = 2, ..., n. \]

Then as above \( B' = \{d'_1, ..., d'_n\} \) is a \( \mathbb{Z} \)-basis of \( M \) and \( SVir[M^{(0)}_{B'}] \subset \mathcal{L} \).

Lemma 3.3. A simple weight \( SVir[M, s] \)-module \( V \) with dimensions of weight spaces not uniformly bounded is a generalized highest weight module.
Proof. Suppose conversely that $V$ is not a generalized highest weight module. For any $\mu \in M$, let $A, A', A_\mu, A'_\mu, \mathcal{L}$ and $B'$ be as in Lemma 3.2. By Lemmas 3.1 and 3.2, for any $0 \neq v \in V$, we have $SVir[M, s]^{(k)}_{\mu'} v \neq 0$, thus, $L_\nu v \neq 0$ for some $\nu \in A_\mu$ or $G_\lambda v \neq 0$ for some $\lambda \in A'_\mu$, i.e.,

$$\bigcap_{\nu \in A_\mu} \ker(L_\nu|_V) \bigcap_{\lambda \in A'_\mu} \ker(G_\lambda|_V) = 0.$$ 

In particular,

$$( \bigoplus_{\nu \in A_\mu} L_\nu \bigoplus_{\lambda \in A'_\mu} G_\lambda )|_{V_{a-\mu}} : V_{a-\mu} \to \bigoplus_{\nu \in A} V_{a+\nu} \bigoplus_{\lambda \in A'} V_{a+\lambda},$$

is an injection. Therefore, $\dim V_{a-\mu} \leq N$, where

$$N = \sum_{\nu \in A} \dim V_{a+\nu} + \sum_{\lambda \in A'} \dim V_{a+\lambda},$$

is a fixed finite integer since both $A$ and $A'$ are finite sets. As $\mu \in M$ is arbitrary, this proves that $V$ is uniformly bounded, a contradiction.

Now Theorem 1.2 follows from Lemma 3.3.

References
1 Gelfand, I. M., Fuchs, D. B.: Cohomologies of the Lie algebra of vector fields on the circle, \textit{Funct. Anal. Appl.}, 2 (1968), 92-39 (English translation 114-126).
2 Patera, J., Zassenhaus, H.: The higher rank Virasoro algebras, \textit{Comm. Math. Phys.}, 136 (1991), 1-14.
3 Su, Y.: Harish-Chandra modules of the intermediate series over the high rank Virasoro algebras and high rank super-Virasoro algebras, \textit{J. Math. Phys.}, 35 (1994), 2013-2023.
4 Su, Y., Zhao, K.: Generalized Virasoro and super-Virasoro algebras and modules of the intermediate series, to appear.
5 Kac, V. G.: Some problems of infinite-dimensional Lie algebras and their representations, in: \textit{Lect. Notes in Math.}, 933, 117-126. Berlin, Heidelberg, New York: Springer, 1982.
6 Chari, V., Pressley, A.: Unitary representations of the Virasoro algebra and a conjecture of Kac, \textit{Compositio Math.}, 67 (1988), 315-342.
7 Martin, C., Piard, A.: Indecomposable modules over the Virasoro Lie algebra and a conjecture of V. Kac, \textit{Comm. Math. Phys.}, 137 (1991), 109-132.
8 Su, Y.: A classification of indecomposable $sl_2(\mathfrak{g})$-modules and a conjecture of Kac on Irreducible modules over the Virasoro algebra, \textit{J. Alg.}, 161 (1993), 33-46.
9 Mathieu, O.: Classification of Harish-Chandra modules over the Virasoro Lie algebra, \textit{Invent. Math.}, 107 (1992), 225-234.
10 Su, Y.: Classification of Harish-Chandra modules over the super-Virasoro algebras, \textit{Comm. Alg.}, 23 (1995), 3653-3675.
11 Su, Y.: Simple modules over the High Rank Virasoro algebras,” \textit{Comm. Alg.}, in press.
12 Su, Y: On indecomposable modules over the Virasoro algebra, \textit{Sciences in China A}, in press.