MAHLER MEASURE OF 3D LANDAU-GINZBURG POTENTIALS

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Abstract. We express the Mahler measures of 23 families of Laurent polynomials in terms of Eisenstein-Kronecker series. These Laurent polynomials arise as Landau-Ginzburg potentials on Fano 3-folds, 16 of which define K3 hypersurfaces of generic Picard rank 19, and the rest are of generic Picard rank < 19. We relate the Mahler measure at each rational singular moduli to the value at 3 of the L-function of some weight-3 newform. Moreover, we find 10 exotic relations among the Mahler measures of these families.

Introduction

The (logarithmic) Mahler measure of a Laurent polynomial \( f \in \mathbb{C}[x_1^{±1}, x_2^{±1}, \ldots, x_n^{±1}] \) is the arithmetic average of \( \log |f| \) over the \( n \)-dimensional torus \( \mathbb{T}^n \):

\[
m(f) := \frac{1}{(2\pi i)^n} \int_{\mathbb{T}^n} \log |f(x_1, \ldots, x_n)| \frac{dx_1}{x_1} \cdots \frac{dx_n}{x_n}.
\]

In the mid 1990s, Boyd [9] (after a suggestion of Deninger) found by numerical experiment many identities of the form

\[
m(f(x, y)) = sL'(E, 0),
\]

where the polynomial \( f \) defines the elliptic curve \( E \) and \( s \) is some rational number. He conjectured that such identities should hold under some additional conditions on \( f \). Many conjectural identities were verified but the general cases remain open.

The first powerful idea was introduced by F. Villegas [33]. Motivated by the mirror symmetry, he found appropriate families of Laurent polynomials parametrized by modular functions, then he could express their Mahler measures in terms of certain Eisenstein-Kronecker series. Finally he linked those series to the \( L \)-function of elliptic curves with complex multiplication.

The second more general idea due to Deninger is related to Bloch-Beilinson’s conjecture, which was also explained in [33]. This approach via regulators was further pursued by Mellit, Zudilin and Brunault. Instead of referring their original papers, we recommend the new textbook [11].

After the elliptic curves, Bertin considered the case of K3 surfaces [5, 6]. She treated two families, and later Samart treated four hypergeometric families [29, 28]. They together proved over 16 identities of the similar nature.

We hope to push Villegas’ ideas and Bertin’s work further to other 3-variable Laurent polynomials. One difficulty is that a random choice of families won’t have nice modular parametrization. The first goal of this paper is to find more reasonable families to study. Thanks to the work of Golyshev and others, we find 25 interesting families of Laurent polynomials originated from certain version of mirror symmetry theory (see 2010 Mathematics Subject Classification. Primary 11R06; Secondary 11F67.
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Section 1 serves as an extended introduction to explain the origin of the 25 families of Laurent polynomials. In Section 2, we recall the relation between quantum periods and Mahler measures, and review Villegas’s idea in our setting. In Section 2.3, we prove our first main result – Theorem 2.6 (Mahler measures in Eisenstein-Kronecker series). In Section 3, we start by reviewing some basic material on Hecke characters for imaginary quadratic fields. Then we tailor Schütt’s work on CM newforms with rational coefficients to our setting (Corollary 3.9). In Section 3.3, we link the relevant Hecke $L$-functions to the theta functions in Proposition 3.22. In Section 4, we conjecture a level-$N$ generalization of the Hilbert class polynomials, and explain how we reduce the search for rational singular moduli of the modular function $c(\tau)$ to reasonably small size. In Section 5, we prove our second main result – Theorem 5.6 (the 179 identities). In Section 6, we prove our last main result – Theorem 6.3 (the 10 exotic relations).
Notations and Conventions. In this paper, \( q \) is always set to be \( e^{2\pi i \tau} \) for \( \tau \in \mathbb{C} \). The \( D_x \) \((x=q,t,\text{etc})\) always denotes the differential operator \( x \frac{d}{dx} \). The bold \( x \) denotes a set of variables \((x_1,x_2,\ldots,x_n)\). We write the \( \eta \)-quotient \( \prod_i \eta(a_i \tau)^{d_i} \) in exponential form \( \prod_i a_i^{d_i} (a_i,d_i \in \mathbb{Z}) \).

1. The 25 Families of K3 Hypersurfaces

1.1. Periods of Laurent Polynomials. Given a Laurent polynomial \( f \in \mathbb{C}[x_1^\pm 1,\ldots,x_n^\pm 1] \), one can form the quantum period of \( f \).

**Definition 1.1.** The quantum periods of \( f \) is the following integral over the \( n \)-dimensional torus \( T_n : |x_1| = \cdots = |x_n| = 1 \)

\[
\pi_f(t) = \frac{1}{(2\pi i)^n} \int_{T^n} (1 - tf(x_1,\ldots,x_n))^{-1} \frac{dx_1}{x_1} \cdots \frac{dx_n}{x_n}.
\]

It is a (possibly multivalued) holomorphic function of \( t \), and is annihilated by a Picard-Fuchs operator

\[
L_f = \sum_{i=0}^{k} t^i P_i(D) \in \mathbb{C}[t,D], \quad D = \frac{d}{dt}
\]

The \( G \)-series for a Fano variety \( X \) is a generating function for certain genus-zero Gromov-Witten invariants of \( X \). Since it is irrelevant to the main results of this paper, we will omit its precise definition. One mirror conjecture states that the Laplace transform of \( G \)-series for \( X \) is the solution of Picard-Fuchs differential equation for some pencil of Calabi-Yau varieties that is called the Landau-Ginzburg model mirror dual to \( X \). In its most basic form, the Picard-Fuchs equation is given by the above \( L_f \) for some Laurent polynomial \( f \), and the hypersurfaces defined by \( f = c \) can be compactified to the required pencil of Calabi-Yau varieties. In this case, the Laurent polynomial \( f \) is called a weak Landau-Ginzburg potential.

If the Laurent polynomial \( f \) is mirror to a Fano variety \( X \), then one expects that \( X \) can be constructed from certain smoothing of \( X_f \) where \( X_f \) is the toric variety associated to the Newton polytope of \( f \) (Batyrev’s construction \[3\]). Since a Fano variety can degenerate to many different singular toric varieties, one might expect many Landau-Ginzburg mirrors for a given Fano \( X \).

In \[1\] a class of Laurent polynomials called Minkowski polynomials were constructed as mirror partners to many Fano 3-folds. They defined birational transformations, called mutations, that preserve periods. They showed that any two Minkowski polynomials with the same period are related by a sequence of mutations. We will give a short introduction on that in Section 1.3.

**Remark 1.2.** The quantum periods can be computed by applying the residue theorem \( n \) times:

\[
\pi_f(t) = \sum_{m=0}^{\infty} t^m \frac{1}{(2\pi i)^n} \int_{T^n} f^m \frac{dx_1}{x_1} \cdots \frac{dx_n}{x_n} := \sum_{m=0}^{\infty} b_m t^m.
\]

So \( b_m \) is the constant coefficient of the Laurent polynomial \( f^m \).

The existence of the Picard-Fuchs operator \( L_f \) is equivalent to the recurrence relation

\[
\sum_{i=0}^{k} P_i(m-i)b_{m-i} = 0 \quad \text{for any } m \in \mathbb{N}_0,
\]

which can be guessed from first few terms of \( \pi_f(t) \).
1.2. Modular Picard-Fuchs Equations. In this subsection, we briefly recall the work of V. Golyshev and others.

We recall that a Fano n-fold $X$ is by definition a smooth $n$-dimensional complex variety with ample anticanonical divisor. In dimension 3, according to Mori-Mukai’s classification, there are exactly 105 Fano varieties up to deformation. Prior to Mukai, Iskovskikh had classified those of Picard rank 1: there are exactly 17 families among those 105. The relevant invariants for the classification are

\[ \text{the index } d = [H^2(X, \mathbb{Z}) : \mathbb{Z}c_1] \text{ and the level } N = \frac{1}{2d^2} \langle c_1^3, [X] \rangle, \]

where $c_1$ is the anticanonical divisor. We will label those of index 1 by $V_N$ ($N = 1, \ldots, 9, 11$), and those of index 2 by $B_N$ ($N = 1, \ldots, 5$). The remaining two are a smooth quadric $Q \subset \mathbb{P}^4$ and $\mathbb{P}^3$.

If $X$ is a Fano n-fold, then the adjunction formula implies that its anticanonical divisors are Calabi-Yau. In the case $n = 3$, this will be a family of $K3$ surfaces of Picard rank $20 - \rho$, where $\rho$ is the Picard rank of $X$, which can range from 1 to 6. We recall that for a general $K3$ surface $Y$ the second homology group $H_2(Y, \mathbb{Z})$ is free of rank 22. As a free subgroup, Pic($Y$) can have rank ranging from 1 to 20. If $\rho(Y) = 20$ then we say that the $K3$-surface is singular. If a one-parameter family $Y_t$ of $K3$ surfaces with generic Picard rank $\rho$, then one expect that the associated Picard-Fuchs equation has order $22 - \rho$. This is mostly due to the fact that $\int \omega = 0$ for any $\gamma \in \text{Pic}(Y)$ (see [22, Proposition 5.2] for details). Here, $\omega$ is the unique (up to scalar multiplication) holomorphic differential 2-form on $Y_t$.

Golyshev constructed in [18] a specific collection of 17 pencils of $K3$ surfaces mirror to the 17 smooth Fano 3-folds of Picard rank 1. In particular, he described the corresponding Picard-Fuchs equations and their modular properties (see Section 2.2). He found that all of them are of type $D3$, which is a specific class of determinantal linear differential equations of order 3. They can be written as the following form

\[ D^3 + t(D + \frac{1}{2})(\alpha_3(D^2 + D) + \beta_1) + t^2(D + 1)(\alpha_2(D + 1)^2 + \beta_0) + \alpha_1 t^3(D + 2)(D + \frac{3}{2})(D + 1) \]

\[ + \alpha_0 t^4(D + 3)(D + 2)(D + 1). \]

(1.1)

The corresponding Laurent polynomials for the above 17 families were given by V. Przyjalkowski in [20, Table 1]. We slightly modify them by mutations and shifts to our preferred form and list them together with their Picard-Fuchs equations in the upper part of Table 11 and Table 13.

However, S. Galkin found 8 more Fano 3-folds with Picard rank $> 1$ satisfying $D3$ equations, and verified their modular properties in [17]. Y. Prokhorov and he observed that for some complex structure they admit a finite group action $G \acts X$ such that Pic$^G(X) = \mathbb{Z}c_1(X)$. We list their mirror Laurent polynomials in the lower part of Table 11 and Table 13. We record their Picard ranks here:

$\rho(V_{12a}) = 2, \rho(V_{12b}) = 3, \rho(V_{20}) = 2, \rho(V_{24}) = 4, \rho(V_{28}) = 2, \rho(V_{30}) = 3, \rho(B_{6a}) = 2, \rho(B_{6b}) = 3.$

The corresponding results for the remaining 80 Fano 3-folds of the Mori-Mukai classification were conjectured by T. Coates, A. Corti, S. Galkin and A. Kasprzyk together with Golyshev and were proved in all cases as an application of their Fanosearch Program [12]. Their work gives new explicit descriptions of the Fano varieties as well as a Laurent polynomial defining the mirror family in every case. However, all their Picard-Fuchs equations have order greater than 3.

\footnote{The Laurent polynomials of $V_{24}$ and $B_{6b}$ are what Bertin labelled by $Q$ and $P$ [5][8], and those of $V_2, V_4, V_6, V_8$ are equivalent to what Samart labelled by $D, C, B, A$ in [23][28]. The recurrence relation of the period sequence of $V_{12}$ is the same as the one used in Apéry’s proof of the irrationality of $\zeta(3)$ [8].}
1.3. **Mutation Invariance.** Readers can safely skip this subsection to reach the main results of this article.

Consider a Laurent polynomial \( f = \sum_{i=1}^{l} C_i(x_1, x_2) x_3^i \) (\( k < 0 \) and \( l > 0 \)). A monomial change of variables

\[
(x_1, x_2, x_3) \mapsto (x_1^{a_{11}} x_2^{a_{12}} x_3^{a_{13}}, x_1^{a_{21}} x_2^{a_{22}} x_3^{a_{23}}, x_1^{a_{31}} x_2^{a_{32}} x_3^{a_{33}})
\]

is called a GL\(_3(\mathbb{Z})\)-equivalence if the integral matrix \((a_{ij})\) is invertible.

Suppose that each \( C_i \) is a Laurent polynomial in \( x_1 \) and \( x_2 \) such that \( A(x_1, x_2)^i C_i(x_1, x_2) \) remains Laurent. Then the pullback of \( f \) along the birational transformation \((\mathbb{C}^*)^3 \rightarrow (\mathbb{C}^*)^3\) given by

\[
(x_1, x_2, x_3) \mapsto (x_1, x_2, A(x_1, x_2)x_3)
\]

is another Laurent polynomial

\[
g = \sum_{i=k}^{l} A(x_1, x_2)^i C_i(x_1, x_2)x_3^i.
\]

**Definition 1.3** (I). A mutation is a birational transformation \((\mathbb{C}^*)^3 \rightarrow (\mathbb{C}^*)^3\) given by a composition of:

1. a GL\(_3(\mathbb{Z})\)-equivalence;
2. a birational transformation of the form \((1.2)\);
3. another GL\(_3(\mathbb{Z})\)-equivalence.

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**Table 1.** List of Laurent polynomials and Picard-Fuchs equations for \( d = 1 \)

| label | \( f \) | \(- (\alpha_3, \alpha_2, \alpha_1; \beta_1, \beta_0)\) |
|-------|---------|----------------------------------|
| \( V_2 \) | \((x_1 + x_2 + x_3 + 1)^6/(x_1 x_2 x_3)\) | (48, 36, 0, 0; 5, 0) |
| \( V_4 \) | \((x_1 + x_2 + x_3 + 1)^4/(x_1 x_2 x_3)\) | (16, 16, 0, 0; 3, 0) |
| \( V_6 \) | \((x_1 + 1)^2(x_2 + x_3 + 1)^3/(x_1 x_2 x_3)\) | (12, 9, 0, 0; 2, 0) |
| \( V_8 \) | \((x_1 + 1)^2(x_2 + 1)^2(x_3 + 1)^2/(x_1 x_2 x_3)\) | (16, 4, 0, 0; 1, 0) |
| \( V_{10} \) | \((x_1 + 1)(x_2 + 1)^2(x_3 + 1)(x_1 + x_3 + 1)/(x_1 x_2 x_3)\) | (41, 11, 4, 0; 3, -1) |
| \( V_{12} \) | \((x_1 + 1)(x_3 + 1)(x_2 + x_1 x_2 + 1)(x_3 + x_2 x_3 + 1)/(x_1 x_2 x_3)\) | (34, -1, 0; 10, 0) |
| \( V_{14} \) | \((x_1 + x_1 x_2 + 1)(x_2 + x_2 x_3 + 1)(x_3 + x_1 x_3 + 1)/(x_1 x_2 x_3)\) | (26, 27, 3; 8, 0) |
| \( V_{16} \) | \((x_1 + 1)(x_2 + x_3 x_2 x_3)(x_1 + x_2 + x_1 x_2 + 1)/(x_1 x_2 x_3)\) | (24, -16, 0; 8, 0) |
| \( V_{18} \) | \((x_1 + 1)(x_3 + 1)(x_2 + x_2 x_3 + x_1 x_2 x_3 x_2 + 1)/(x_1 x_2 x_3)\) | (18, 27, 0; 6, 0) |
| \( V_{22} \) | \((x_1 + x_2 + x_1 x_2 + x_2 x_3)(x_1 + x_3 + x_1 x_3 + x_2 x_3)/(x_1 x_2 x_3) + 1/(x_2 x_3)\) | (20, -56, 4; 8, -8) |
| \( V_{24} \) | \((x_1 + 1)(x_2 + 1)(x_2 + 1)(x_3 + 1)^2/(x_1 x_2 x_3)\) | (28, 128, 0; 8, -32) |
| \( V_{26} \) | \((x_1 + 1)(x_2 + 1)(x_3 + 1)^2/(x_1 x_2 x_3)\) | (40, 0, 144; 12, 0) |
| \( V_{28} \) | \((x_1 x_2 x_3 + 1)(x_1^{-1})^i(x_1^1 + 1)(x_3^{-1})^i\) | (12, -144, 0; 4, 36) |
| \( V_{30} \) | \((x_1 + x_2 + x_3 + 1)(x_1^{-1} + x_2^{-1} + x_3^{-1} + 1)\) | (20, -64, 0; 8, 0) |
| \( V_{32} \) | \((x_1 + 2 x_2 + x_3 + (x_1 x_2)^{-1} + x_2 x_3 + x_1 x_2^{-1} + x_2 x_3^{-1} + 2 x_2^{-1} + 1)\) | (6, 47, 28; 2, 4) |
| \( V_{34} \) | \((x_1 + x_2 + x_3 + x_1^{-1} + x_2^{-1} + x_3^{-1} + x_1 x_2^{-1} + x_2 x_3^{-1} + x_3 x_1^{-1} + 3)\) | (14, -29, 60; 6, -4) |

The right column lists the coefficients of the D3 equation \((1.1)\). All of them have \( \alpha_0 = 0 \).
If $f$ and $g$ are Laurent polynomials and $\varphi$ is a mutation such that $\varphi^* f = g$ then we say that $f$ and $g$ are related by the mutation $\varphi$.

**Lemma 1.4** (\cite{1}). *If $f$ and $g$ are related by a mutation, then the quantum periods of $f$ and $g$ coincide.*

**Example 1.5.** Consider the following Laurent polynomials

\[
\begin{aligned}
f_0 &= (1 + x_1)(1 + x_2)(1 + x_1 + x_3)(1 + x_2 x_3 + x_3)/(x_1 x_2 x_3), \\
f_1 &= (1 + x_1)(1 + x_2)(1 + x_3)(1 + x_2 + x_3 + x_1 x_2)/(x_1 x_2 x_3), \\
f_2 &= (1 + x_1)(1 + x_2)(1 + x_1 + x_3)(1 + x_1 + x_2 + x_3)/(x_1 x_2 x_3).
\end{aligned}
\]

Then $f_0$ can be mutated to $f_1$ by the transformation

\[(x_1, x_2, x_3) \mapsto (x_1, x_2, \frac{1 + x_1}{x_3})\]

and $f_1$ can be mutated to $f_2$ by the $\text{GL}_3(\mathbb{Z})$-equivalence $(x_1, x_2, x_3) \mapsto (x_1, x_3^{-1}, x_2)$ followed by the transformation

\[(x_1, x_2, x_3) \mapsto (x_1, x_2, \frac{1 + x_1}{x_3}).\]

According to the appendix of \cite{1}, the mutation-equivalent class of $f_0$ contains at least 71 $\text{GL}_3(\mathbb{Z})$-nonequivalent Laurent polynomials supporting on reflexive polytopes.

### 2. Mahler Measure

#### 2.1. Mahler Measure and Periods

Let $f$ be a Laurent polynomial in $n$ variables.

**Definition 2.1.** The (logarithmic) Mahler measure of $f$ is the arithmetic average of $\log |f|$ over the $n$-dimensional torus $\mathbb{T}^n$:

\[
m(f) := \frac{1}{(2\pi i)^n} \int_{\mathbb{T}^n} \log |f(x_1, \ldots, x_n)| \frac{dx_1}{x_1} \cdots \frac{dx_n}{x_n} = \int_0^1 \cdots \int_0^1 \log \left| f(e^{2\pi i \theta_1}, \ldots, e^{2\pi i \theta_n}) \right| d\theta_1 \cdots d\theta_n.
\]

It is a nontrivial fact \cite{11} Proposition 3.1 that the integral defining $m(f)$ always converges.

From now on we will denote by $\{f_c\}$ the family of Laurent polynomials $f - c$ for $c \in \mathbb{C}$, and by $f_c$ the particular member $f - c$. One of the ideas in \cite{33} is to study $m(f_c)$ as a function of the complex parameter $c$. Sometimes it is more convenient to work with the parameter $t = 1/c$.

Let $K$ be compact region given by the image of the torus $\mathbb{T}^n$ under the map $x \mapsto f(x)$. Then $1 - tf$ does not vanish on $\mathbb{T}^n$ for $t^{-1} \notin K$. For $t^{-1} \in \mathbb{C} \setminus K$, we define the holomorphic function

\[
\tilde{m}(t) := -\log(t) + \frac{1}{(2\pi i)^n} \int_{\mathbb{T}^n} \log(1 - tf) \frac{dx_1}{x_1} \cdots \frac{dx_n}{x_n}
\]

\[
= -\log(t) - \sum_{n=1}^{\infty} \frac{b_n}{n} t^n.
\]  

(2.1)

Here and throughout we take the principal branch of the logarithm.

**Lemma 2.2** (\cite{33}). *We have that $m(f_t) = \text{Re}(\tilde{m}(t))$ for $1/t = c \in \mathbb{C} \setminus K^o$, where $K^o$ is the interior of $K$.**

**Caution 2.3.** In general, $m(f_t)$ may not agree with $\text{Re}(\tilde{m}(t))$ in the interior of $K$. \cite{33} Example 1 contains such an example.
Remark 2.4. In view of the above caution, it worthwhile to give a complete description of the region $\mathcal{K}$ for our 25 Laurent polynomials. This seems not an easy task. However, it is elementary to find the (real) maximum and minimum of $\mathcal{K}$. The maximums are particular easy. Since all Laurent polynomials have positive coefficients, the maximum on $T^n$ are equal to $f(1,1,1)$. We also observe that the polynomials for $V_N \ (N = 8, 12a, 12b, 20, 24), B_4$ and $B_6b$ are reciprocal, i.e.,

$$f(x_1, x_2, x_3) = f(x_1^{-1}, x_2^{-1}, x_3^{-1}).$$

So these polynomials only take real values on $T^n$. In these cases we have that $K^\circ = \emptyset$. Finally, we remark that exactly the same proof as Lemma 1.4 shows that the Mahler measure is also mutation-invariant.

2.2. The Mirror-Moonshine for the 25 Families. We have listed the Picard-Fuchs equations satisfied by the quantum periods of the 25 Laurent polynomials. All of them have the maximal unipotent monodromy at zero. In terms of the Frobenius method, this is equivalent to say that apart from the quantum period $u_0(t) = 1 + \sum_{n=1}^{\infty} b_n t^n$ as a holomorphic solution around $t = 0$, there is a second solution $u_1(t)$ and a third solution $u_2(t)$ of the form

$$u_1(t) = u_0(t) \log(t) + v_1(t),$$

$$u_2(t) = \frac{1}{2} u_0(t) \log^2(t) + v_1(t) \log(t) + v_2(t),$$

where $v_1(t) = \sum_{n \geq 1} b_{1,n} t^n$ and $v_2(t) = \sum_{n \geq 2} b_{2,n} t^n$ are holomorphic around 0. In this paper, $u_2(t)$ is irrelevant to our discussion.

Following the similar argument as in [33], we define

$$\tau = \frac{1}{2\pi i} \frac{u_1}{u_0},$$

then

$$q = e^{2\pi i \tau} = t + \cdots.$$  

So we can locally invert $q$ around 0 and obtain the so-called mirror map

$$t(\tau) = q + \cdots.$$  

Let

$$u_0(\tau) = u_0(t(\tau)) = 1 + \sum_{n \in \mathbb{N}} c_n q^n,$$

$$c(\tau) = u_0(\tau) \frac{D_q t(\tau)}{t(\tau)} = 1 + \sum_{n \in \mathbb{N}} c_n q^n,$$

where $D_q = \frac{d}{dq} = \frac{1}{2\pi i} \frac{d}{d\tau}$ is the usual differential for modular forms.

Finally we notice that the equality (2.1) can be rewritten as

$$\tilde{m}(t) = -\log t - \int_0^t (u_0(s) - 1) \frac{ds}{s}.$$  

By change of variables $s = t(\tau)$ we obtain an expression for $\tilde{m}$ as a function of $\tau$. Note that $t = 0$ corresponds to $\tau = i\infty$.

Theorem 2.5 ([33]). Locally around $\tau = i\infty$, we have

$$\tilde{m}(\tau) = -2\pi i \tau - \sum_{n=1}^{\infty} \frac{c_n q^n}, \quad q = e^{2\pi i \tau}.$$
It is a nontrivial fact that $t(\tau)$ is a modular function (i.e., a meromorphic modular form of weight 0). Before Golyshew, this had been proved by C. Doran in [15] in the context of lattice-polarized $K_3$ surfaces. It then follows from the general theory of modular forms that $u_0(\tau)$ and $e(\tau)$ are meromorphic modular forms of weight 2 and 4 respectively. It turns out that $u_0(\tau)$ and $e(\tau)$ are genuine modular forms except for $V_2$ and $B_1$. In fact, they are all Eisenstein series (see Table 2).

Alternatively one can start with a modular form $u_0(\tau)$ and a modular function $t(\tau)$ in the same row of Table 2 and employ the following general fact. If $f(\tau)$ is an arbitrary modular form of positive weight $k$ and $t(\tau)$ a modular function, then the power series $F(t)$ obtained by expressing $f(\tau)$ locally as a power series in $t(\tau)$ always satisfies a linear differential equation of order $k+1$ with algebraic (if $t(\tau)$ is a Hauptmodul, even polynomial) coefficients. A discussion of this phenomenon in general, and an algorithm to find the explicit linear differential equation, can be found in [35 5.4] and [34].

It worths mentioning that the modular functions $c(\tau) = t(\tau)^{-1}$ are all of moonshine type [14]. Lian and Yau first observed this phenomenon in [22] and formulate their mirror-moonshine conjecture, which roughly says that for any pencil of $K3$ surfaces of generic Picard rank 19, (the reciprocal of) the mirror map $c(\tau) = t(\tau)^{-1}$ is commensurable with some McKay-Thompson series. The conjecture was proved by Doran in [15], and Galkin verified in [17] that the mirror-moonshine also holds for the 8 families in Table 3 (of Picard rank < 19). In fact, for our 25 families the function $c(\tau)$ is itself a McKay-Thompson series.

Let $\Gamma_0(N)$ be the congruence subgroup of $SL_2(\mathbb{Z})$

$$\Gamma_0(N) := \{ (a \ b \ c \ d) \in SL_2(\mathbb{Z}) | c \equiv 0 \mod N \}.$$ 

We recall the definition of the Atkin-Lehner involution $W_n$ for any $n \mid N$ such that $\gcd(n, N/n) = 1$. Note that the number of such divisors of $N$ is $2^{\sigma_0^+(N)}$, where $\sigma_0^+(N)$ is the number of distinct prime factors of $N$. Over $\mathbb{C}$ they may be defined as elements of $SL_2(\mathbb{R})$ as follows. Let $a, b, c, d \in \mathbb{Z}$ be such that $adn - bcN/n = 1$ and define $W_n = \frac{\tau}{\sqrt{n}} (\begin{smallmatrix} n & a \\ b & n \end{smallmatrix})$. This construction is well-defined up to (left and right) multiplication by $\Gamma_0(N)$. Moreover, modulo $\Gamma_0(N)$ we have the following relations

$$W_{n_1}W_{n_2} = W_{n_1n_2/\gcd(n_1,n_2)^2}.$$ 

Let $W(N)$ be the group generated by all $W_n$. Following the notations of [14], we denote by $\Gamma_0^+(N)$ the subgroup of $SL_2(\mathbb{R})$ generated by $\Gamma_0(0)$ and $W_n$, and by $\Gamma_0^+(N)$ the subgroup generated by $\Gamma_0(N)$ and $W(N)$. Then $\Gamma_0(N)$ is a normal subgroup of $\Gamma_0^+(N)$ and the quotient $\Gamma_0^+(N)/\Gamma_0(N)$ is isomorphic to $\mathbb{Z}_2^{\sigma_0^+(N)}$. We see from Table 2 that the genus-0 congruence subgroups of $SL_2(\mathbb{R})$ associated to those Thompson series of level $N$ are all equal to $\Gamma_0^+(N)$ except for $V_{12}$.

2.3. **Expressing Mahler Measure in Eisenstein-Kronecker Series.** Let $E_2$ and $E_4$ be the Eisenstein series of weight 2 and 4 on $\Gamma_0(1)$:

$$E_2(\tau) = 1 + 24 \sum_{n=1}^{\infty} \sigma_1(n)q^n,$$

$$E_4(\tau) = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n)q^n.$$ 

We also set

$$G_{2,d}(\tau) = \sum_{n \geq 1} \sigma_1(n)q^{dn},$$

$$G_{4,d}(\tau) = \sum_{n \geq 1} \sigma_3(n)q^{dn}.$$
Table 2. List of Forms

| label | group | \(c(\tau) = t(\tau)^{-1}\) | \(e(\tau)\) | \(u_0(\tau)\) |
|-------|-------|-----------------|-----|-----|
| \(V_2\) | \(\Gamma_0(1)\) | \(j = (h^8 + (2h^{-2})^8)^3, \ h = \frac{1}{3^3\tau}\) | \(E_4^{1/2}\) | \(E_6E_4^{-1/2}\) |
| \(V_4\) | \(\Gamma_0^+(2)\) | \((h^{12} + 64h^{-12})^2, \ h = \frac{1}{3^3\tau}\) | \(80(-1, 1)\) | \(24(1, -1)\) |
| \(V_6\) | \(\Gamma_0^+(3)\) | \((h^6 + 27h^{-6})^2, \ h = \frac{1}{3^3\tau}\) | \(30(-1, 1)\) | \(12(1, -1)\) |
| \(V_8\) | \(\Gamma_0^+(4)\) | \(h^{24}, \ h = \frac{2^3}{3^3\tau}\) | \(16(-1, 0, 1)\) | \(8(1, 0, -1)\) |
| \(V_{10}\) | \(\Gamma_0^+(5)\) | \(h^6 + 22 + 125h^{-6}, \ h = \frac{4^4}{3^3\tau}\) | \(10(-1, 1)\) | \(6(1, -1)\) |
| \(V_{12}\) | \(\Gamma_0^+(6)\) | \(h^{12}, \ h = \frac{2^3}{3^3\tau}\) | \((-7, 1, -1, 7)\) | \((5, -1, 1, -5)\) |
| \(V_{14}\) | \(\Gamma_0^+(7)\) | \((h^2 + 7h^{-2})^2, \ h = \frac{1}{3^3\tau}\) | \(5(-1, 1)\) | \((4, 1, -1)\) |
| \(V_{16}\) | \(\Gamma_0^+(8)\) | \(h^8, \ h = \frac{2^9}{3^9\tau}\) | \((-4, 1, -1, 4)\) | \((2, -1, 1, -2)\) |
| \(V_{18}\) | \(\Gamma_0^+(9)\) | \(h^6, \ h = \frac{3^2}{3^4\tau}\) | \((3, -1, 0, 1)\) | \((3, 1, 0, -1)\) |
| \(V_{22}\) | \(\Gamma_0^+(11)\) | \((1 + 3b)^2(1 + 3b + h^{-1}), \ h = \frac{3^1}{114\tau}\) | \(2(-1, 1)\) | \((12, 1, -1) + \theta_{11}\) |
| \(V_{22a}\) | \(\Gamma_0^+(6)\) | \((h^8 + 8h^{-3})^2, \ h = \frac{1^7}{3^6\tau}\) | \((6, -1, 1, 1)\) | \((4, 1, -1, -1)\) |
| \(V_{22b}\) | \(\Gamma_0^+(6)\) | \((h^2 + 9h^{-2})^2, \ h = \frac{1^7}{3^6\tau}\) | \((8, -1, 1, -1)\) | \((6, 1, -1, -1)\) |
| \(V_{24}\) | \(\Gamma_0^+(10)\) | \((h^2 + 4h^{-2})^2, \ h = \frac{1^5}{3^9\tau}\) | \((2, -1, 1, 1)\) | \((2, 1, 1, -1)\) |
| \(V_{28}\) | \(\Gamma_0^+(12)\) | \(h^6, \ h = \frac{2^9}{3^9\tau}\) | \((-2, 1, 2, -2, -1, 2)\) | \((4, 1, -1, 1, 1, -1)\) |
| \(V_{30}\) | \(\Gamma_0^+(14)\) | \((h^2 + h^{-2})^2, \ h = \frac{2^9}{13^9\tau}\) | \((-1, -1, 1, 1)\) | \((4, 1, 1, -1, -1) + \theta_{14}\) |
| \(V_{30}\) | \(\Gamma_0^+(15)\) | \((h + 3h^{-1})^2, \ h = \frac{1^5}{3^9\tau}\) | \((-1, -1, 1, 1)\) | \((3, 1, 1, -1, -1) + \theta_{15}\) |

The modular forms \(e(\tau)\) and \(u_0(\tau)\) are given by the coefficients \(a_d\) and \(a_d'\) in (2.3) and (2.4), and \(\theta_N\) is some cusp form of weight 2 and level \(N\).

By explicit calculation, we find for our 25 families except \(V_2\) and \(B_1\) that all \(u_0(\tau)\) are \(\text{(up to a shift)}\)\(^{2}\) Eisenstein series of weight 2 of the form:

\[
1 + \sum_{d \mid N} a_d' d G_{2,d} = \sum_{d \mid N} \frac{a_d'}{24} d E_2(d\tau) \quad (a_d' \in \mathbb{Z}),
\]

and all \(e(\tau)\) are Eisenstein series of weight 4 of the form

\[
1 - \sum_{d \mid N} a_d d^2 G_{4,d} = -\sum_{d \mid N} \frac{a_d}{240} d^2 E_4(d\tau) \quad (a_d \in \mathbb{Z}).
\]

\(^2\)Unlike the form \(e(\tau)\), the modular form \(u_0(\tau)\) is not invariant under a shift of the function value of \(f\). Such a shift of \(f\) will give rise to a shift of \(u_0(\tau)\) by some weight-2 cusp form (of the same level).
Note that the equations imply that
\[(2.5) \quad \sum_{d\mid N} a'_d d = 24, \]
\[(2.6) \quad \sum_{d\mid N} a_d d^2 = 240, \]
which is equivalent to say that both series are of level \(N\). In fact, we can further observe that \(a_d = -a_e\) and \(a'_d = -a'_e\) if \(de = N\). At this moment, we do not need this observation.

Now according to Lemma 2.2 and Theorem 2.5, to compute the Mahler measure of \(f_{c(\tau)}\), we need to compute
\[
\text{Re}\left( -2\pi i\tau - \sum_{n\in\mathbb{N}} \sigma_3(n) dq^{dn} n \right) = \text{Re}\left( -2\pi i\tau + \sum_{d\mid N} a_d \sum_{n\in\mathbb{N}} \sigma_3(n) dq^{dn} n \right)
\]
\[(2.7) \quad = \text{Re}\left( -2\pi i\tau + \sum_{d\mid N} a_d \sum_{n\in\mathbb{N}} d^{-1} D_q^2 (Li_3(q^{dn})) \right). \]
The last equality due to the relation
\[D_q^2 (Li_3(q^{dn})) = (dn)^2 Li_1(q^{dn}) \quad \text{for } n, d \in \mathbb{N}.\]

For this, following [5, 29] we introduce
\[F_d(\xi) = \sum_{n\in\mathbb{N}} Li_3(q^{nd} + \xi).\]

It is known that the Fourier series of \(F_d(\xi)\) converges to \(F_d(\xi)\) at \(\xi = 0\), i.e., \(F_d(0) = \sum_{n\in\mathbb{Z}} \hat{F}_d(n)\), where the Fourier coefficients
\[\hat{F}_d(n) = \begin{cases} \frac{-1}{2\pi i} \sum_{m\geq 1} (m^3(dm\tau - \frac{n}{4}))^{-1} & \text{if } 4 \mid n, \\ 0 & \text{otherwise}. \end{cases} \]

Since \(D_q^2 = -\frac{1}{4\pi^2} \frac{d^2}{d\tau^2}\), we have that
\[(2.8) \quad d^{-1} D_q^2 (F_d(0)) = \frac{1}{4\pi^3 i} \sum_{n\in\mathbb{Z}, m \geq 1} \frac{d}{m (dm\tau - n)^3}. \]

Then we can continue our computation
\[(2.9) \quad = \text{Re}\left( -2\pi i\tau + \sum_{d\mid N} \frac{1}{4\pi^3 i} \sum_{n\in\mathbb{Z}, m \geq 1} \frac{d}{m (dm\tau - n)^3} \right) \]
\[= \text{Im}\left( 2\pi \tau + \frac{1}{8\pi^3} \sum_{d\mid N} \sum_{n\in\mathbb{Z}, m \neq 0} \frac{1}{m (dm\tau - n)^3} \right).
\]

It is straightforward to verify that
\[(2.10) \quad \frac{1}{m} \text{Im} \left( \frac{1}{(m\tau + n)^3} \right) = -\text{Im}\tau \left( \frac{1}{(m\tau + n)^3 (m\tau + n)^3} \right) + \frac{1}{(m\tau + n)^3 (m\tau + n)^3} \right). \]
Moreover,
\begin{equation}
\sum_{d|N} a_d \sum_{n \neq 0, m=0} 2 \text{Re} \left( \frac{d^2}{(dm\tau + n)^3 (m\tau + n)} \right) + \frac{d^2}{(dm\tau + n)^2 (m\tau + n)^2} = \sum_{d|N} a_d d^2 \sum_{n \in \mathbb{Z}} \frac{3}{n^4} = 3 \cdot 240 \cdot \frac{\pi^4}{45}.
\end{equation}

Combining (2.10) and (2.11) with (2.9), we find that (2.9) is equal to
\begin{equation}
\frac{\text{Im} \tau}{(2\pi)^3} \sum_{d|N} a_d d^2 \left( \sum_{m,n} 2 \text{Re} \left( (dm\tau + n)^{-3} (m\tau + n)^{-1} \right) + (dm\tau + n)^{-2} (m\tau + n)^{-2} \right),
\end{equation}
where the summation \( \sum_{m,n} \) means \( \sum_{(m,n) \in \mathbb{Z}^2 \setminus \{0,0\}} \).

We set
\begin{equation}
H_d(\tau) = \sum_{m,n} 2 \text{Re} \left( (dm\tau + n)^{-3} (m\tau + n)^{-1} \right) + (dm\tau + n)^{-2} (m\tau + n)^{-2},
\end{equation}
and thus obtain the following theorem.

**Theorem 2.6.** For each \( f \) of the 25 families except \( V_2 \) and \( B_1 \), let \( \mathcal{F} \) be any fundamental domain of the modular function \( c(\tau) \) containing \( i\infty \). Then for any \( \tau \in \mathcal{F} \) the function \( \text{Re}(\tilde{m}(c(\tau))) \) is equal to
\[ \frac{\text{Im} \tau}{(2\pi)^3} \sum_{d|N} a_d d^2 H_d(\tau), \]
where \( H_d(\tau) \) is defined as above and \( a_d \) is given by the column \( c(\tau) \) of Table 2.

**Caution 2.7.** The equality in Theorem 2.5 only holds locally around \( i\infty \). So it is necessary to choose a fundamental domain containing \( i\infty \). As a simple example, let us consider the family \( V_4 \). In this case, \( c(\tau) = (\frac{1}{2}(\frac{2}{3}))^{12} + 64 \frac{1}{2}(\frac{2}{3})^{12} \) is a Hauptmodul for the monodromy group \( \Gamma^+ \). One can check that \( c(\frac{1}{\sqrt{2}}) = c(\frac{1}{\sqrt{2}} - 1) = 256 \) but the \( \tilde{m}(\frac{1}{\sqrt{2}}) \neq \tilde{m}(\frac{1}{\sqrt{2}} - 1) \). The correct value for the Mahler measure is the former because \( \frac{1}{\sqrt{2}} - 1 \) is in a fundamental domain away from \( i\infty \).

The result for families \( V_{24} \) and \( B_{6} \) were proved in [3], and for families \( V_4, V_6 \) and \( V_8 \) were proved in [29]. It will turn out that for certain choices of \( \tau \), \( H_d(\tau) \) is a sum of the special values of two partial Hecke \( L \)-functions, which are both related to theta functions (see Proposition 3.2 and 5.4).

3. **Generalized Theta Functions and Hecke \( L \)-functions**

3.1. **Imaginary Quadratic Fields and Quadratic Forms.** We review some elementary facts about orders in imaginary quadratic fields and their relation to binary quadratic forms, mostly taken from [13]. Let \( K \) be a quadratic field, then \( K = \mathbb{Q}(\sqrt{m}) \) for a unique squarefree integer. Recall that the discriminant \( d_K \) of \( K \) is defined to be
\[ d_K = \begin{cases} n & \text{if } n \equiv 1 \mod 4 \\ 4n & \text{otherwise}. \end{cases} \]

An order \( \mathcal{O} \) in \( K \) is a subring \( \mathcal{O} \subset K \) which is a finitely generated \( \mathbb{Z} \)-module of rank \( [K : \mathbb{Q}] \). The ring of integer \( \mathcal{O}_K \) of \( K \) is the maximal order of \( K \). Since \( \mathcal{O} \) and \( \mathcal{O}_K \) are free \( \mathbb{Z} \)-modules of rank 2, it follows that \( m = [\mathcal{O}_K : \mathcal{O}] < \infty \). \( m \) is called the conductor of \( \mathcal{O} \), and \( D = m^2 d_K \) is by definition the discriminant of \( \mathcal{O} \). Then we have that
\[ \mathcal{O} = \mathbb{Z} + m \mathcal{O}_K = \text{span}_{\mathbb{Z}} \left( 1, m \frac{d_K + \sqrt{d_K}}{2} \right). \]
Definition 3.1. Given an ideal \( m \) of \( O_K \), an \( O \)-ideal \( a \) is prime to \( m \) if \( a + mO = O \).

In most situations we consider, \( m \) will just be the principal ideal \((m)\) or \((1)\). Being prime to \((m)\) is equivalent to that \( \gcd(N(a), m) = 1 \) ([13, Lemma 7.18]), where \( N(a) = |O/a| \) is the ideal norm of \( a \). Let

\[
I(O, m) := \text{group of invertible fractional } O \text{-ideals prime to } m,
\]

\[
P(O, m) := \text{subgroup of } I(O, m) \text{ generated by principal ideals}.
\]

When \( m = (m) \), we write \( I(O, m) \) and \( P(O, m) \) for \( I(O, (m)) \) and \( P(O, (m)) \). When \( m = (1) \), we write \( I(O) \) and \( P(O) \) for \( I(O, (1)) \) and \( P(O, (1)) \). We call the quotient

\[
Cl(O) := I(O)/P(O)
\]

the ideal class group of \( O \). When \( O \) is the maximal order \( O_K \), \( I(O_K) \) and \( P(O_K) \) will be denoted by \( I_K \) and \( P_K \). The assignment \( a \mapsto aO_K \) gives an isomorphism \( I(O, m) \cong I_K(m) \) with the inverse given by \( a \mapsto a \cap O \).

Lemma 3.2 ([13, Proposition 7.22]). Let \( K \) be an imaginary quadratic field, and let \( O \) be an order of conductor \( m \) in \( K \). Then there are natural isomorphisms

\[
(3.1) \quad Cl(O) \cong I(O, m)/P(O, m) \cong I_K(m)/P_K,m(m),
\]

where \( P_{K.m}(m) := \{ \alpha O_K \mid \alpha \in O_K \text{ and } \alpha \equiv a \mod mO_K \text{ for some integer } a \text{ relatively prime to } m \} \).

We will denote a binary quadratic form \( f(x, y) = ax^2 + bxy + cy^2 \) by the triple \([a, b, c]\). A quadratic form \([a, b, c]\) is called primitive if \( \gcd(a, b, c) = 1 \). We denote by \( Q \) (resp. \( Q^0 \)) the set of all (resp. primitive) positive definite quadratic forms \((p, d, q, f)\), and by \( Q_D \) (resp. \( Q^0_D \)) the subset of \( Q \) (resp. \( Q^0 \)) of all quadratic forms of discriminant \( D = b^2 - 4ac \). Consider the equivalence relation given by the natural \( SL_2(\mathbb{Z}) \)-action on \( Q \):

\[
f_1 \sim f_2 \iff f_1(x, y) = f_2(px + qy, rx + sy) \text{ for some } (p, q, r, s) \in SL_2(\mathbb{Z})\).
\]

Note that the subset \( Q^0 \) and \( Q_D \) are \( SL_2(\mathbb{Z}) \)-invariant.

We denote by \( Cl(D) \) the equivalence classes of \( Q^0_D \). Gauss and Dirichlet defined a composition on \( Cl(D) \) making it an abelian group [13]. We mention that the identity element is the class containing the form

\[
(3.2) \quad [1, 0, -D/4] \quad \text{if } D \equiv 0 \mod 4,
\]

\[
(3.3) \quad [1, 1, (1 - D)/4] \quad \text{if } D \equiv 1 \mod 4,
\]

and the inverse of the class containing \([a, b, c]\) in \( Cl(D) \) is the class containing \([a, -b, c]\).

Theorem 3.3 ([13, Theorem 7.7]). Let \( O \) be the order of discriminant \( D \) in \( K \), and \([a, b, c] \in Q^0_D \). Then the map

\[
(3.4) \quad I : [a, b, c] \mapsto \text{span}_\mathbb{Z} \left( a, \frac{b - \sqrt{D}}{2} \right)
\]

induces an isomorphism \( Cl(D) \cong Cl(O) \).

Hereafter we will freely identify \( Cl(O) \) with \( Cl(D) \). Since \([a, b, c] \sim [c, -b, a] \) in \( Cl(D) \), we also have a twin map \( I : [a, b, c] \mapsto \text{span}_\mathbb{Z} \left( c, \frac{b + \sqrt{D}}{2} \right) \) inducing the same isomorphism.
3.2. CM Newforms with Rational Coefficients. In this subsection, we tailor Schütz’s work [30] to our setting. For unexplained (standard) terminology about modular forms in this section, we refer readers to the textbook [36, 21].

Let \( m \) be an ideal of an imaginary quadratic field \( K \) and \( \ell \in \mathbb{N} \). We denote
\[
P_{K,1×}(m) := \{ \alpha\mathcal{O}_K \mid \alpha \in K^	imes, \, \alpha \equiv 1 \mod{\ell}m \},
\]
where \( \mod{\ell}m \) is the multiplicative congruence. This is a subgroup of \( P_K(m) \).

**Definition 3.4.** A Hecke character \( \psi \) of \( K \) modulo \( m \) with \( \infty \)-type \( \ell \) is a homomorphism
\[
\psi : I_K(m) \to \mathbb{C}^	imes
\]
such that for all \( \alpha \in K_1 \times m \), we have
\[
\psi(\alpha \mathcal{O}_K) = \alpha^\ell.
\]
The ideal \( m \) is called the conductor of \( \psi \) if it is minimal in the sense that if \( \psi \) is defined modulo \( m' \), then \( m|m' \).

The definition requires some elaboration. We denote by \( X^\ell_K(m) \) the set of all Hecke characters of \( K \) modulo \( m \) with \( \infty \)-type \( \ell \). To define an element in \( X^\ell_K(m) \) completely, one need to further specify a character
\[
\phi : I_K(m)/P_{K,\mathbb{Z}}(m) \to \mathbb{C}^	imes
\]
and a character
\[
\eta_\mathbb{Z} : P_{K,\mathbb{Z}}(m)/P_{K,1×}(m) \to \mathbb{C}^	imes.
\]
Recall that for \( m = (m), I_K(m)/P_{K,\mathbb{Z}}(m) \) is isomorphic to the class group \( \text{Cl}(\mathcal{O}) \). The following lemma is well-known (eg., [31, Proposition 1.2]).

**Lemma 3.5.** \( P_K(m)/P_{K,1×}(m) \cong (\mathcal{O}_K/m)\times \) and the subgroup \( P_{K,\mathbb{Z}}(m)/P_{K,1×}(m) \) of \( P_K(m)/P_{K,1×}(m) \) can be identified with
\[
G_\mathbb{Z} := \{ \alpha \in (\mathcal{O}_K/m)\times \mid \alpha \equiv a \mod{m}, \exists a \in \mathbb{Z} \}.
\]
Then we can observe that \( \eta_\mathbb{Z} \) can be recover from \( \psi \) by
\[
a \mapsto \psi(\alpha \mathcal{O}_K)/\alpha^\ell \quad (a \in \mathbb{Z}, \ gcd(a, N(m)) = 1).
\]

**Definition 3.6.** For any Hecke character \( \psi \in X^\ell_K(m) \) we define \( f_\psi(\tau) = \sum_{n \in \mathbb{N}} a_n q^n \) by
\[
f_\psi(\tau) := \sum_{\text{a integral}} \psi(a)q^{\chi_K(a)}.
\]
In this definition we tacitly extended \( \psi \) by 0 for all fractional ideals of \( K \) which are not prime to \( m \).

Let \( \chi_K \) be the quadratic character of conductor \( |d_K| \).

**Theorem 3.7** (Hecke, Shimura). \( f_\psi \) is a Hecke eigenform of weight \( \ell + 1 \), level \( \mathcal{N}(m)|d_K| \) and nebentypus character \( \varepsilon = \chi_K \eta_\mathbb{Z} \):
\[
f_\psi \in \text{S}_{\ell+1}(\Gamma_0(N), \chi_K \eta_\mathbb{Z}).
\]
Moreover, \( f_\psi \) is a newform if and only if \( m \) is the conductor of \( \psi \).

We say \( \psi \) or \( f_\psi \) has rational (or real) coefficients if all Fourier coefficients \( a_n \) are rational (or real). Such condition will impose very strong restriction on the characters \( \phi, \eta_\mathbb{Z} \) and the (imaginary quadratic) field \( K \). But let us first mention ([30, Corollary 1.2]) that if a newform \( f \) has real coefficients and its weight \( k \) is odd, then \( f \) has complex multiplication by its nebentypus \( \varepsilon \), that is,
\[
f = f \otimes \varepsilon = \sum_{n \in \mathbb{N}} \varepsilon(n) a_n q^n.
\]
Suppose that $f_\psi$ has real coefficients, then the character $\eta_\psi$ is automatically determined. More precisely, if $\ell$ is odd, then $\eta_\psi = \chi_K$ so $\varepsilon$ is trivial; if $\ell$ is even, then $\eta_\psi = 1$ and thus $\varepsilon = \chi_K$ ([30 Corollary 1.5]). So the only freedom left is the choice of the character $\phi : Cl(\mathcal{O}) \to \mathbb{C}^\times$ if $m = (m)$. If we further assume that $f_\psi$ has rational coefficients, then [30 Proposition 3.1] says that $e_K \mid \ell$ where $e_K$ is the exponent of the class group $Cl(\mathcal{O}_K)$. We remark that in general $e_\mathcal{O} \nmid \ell$ where $e_\mathcal{O}$ is the exponent of $Cl(\mathcal{O})$.

Conversely, suppose that $e_K \mid \ell$ and $a \in I_K(m)$ has order $n$ in $Cl(\mathcal{O}_K)$ with $a^n = \alpha \mathcal{O}_K$. For each character $\phi : Cl(\mathcal{O}) \to \mathbb{C}^\times$, we define a Hecke character $\Phi \in X^\times_K(m)$ by

$$
\Phi(a) = \phi([a \cap \mathcal{O}]) \alpha^{\ell/n}.
$$

In this case $f_\Phi$ is also denoted by $f_{\phi, \ell}$. We recall an elementary but useful criterion.

**Lemma 3.8** ([30 Lemma 2.2]). Let $\psi$ be a Hecke character of an imaginary quadratic field $K$. Then $f_\psi$ has rational coefficients if and only if $\text{Im} \, \psi \subset \mathcal{O}_K$.

In particular, if the image of $\phi$ is contained in $\mathcal{O}_K^\times$, then we see from [30] that $f_\Phi$ has rational coefficients. For our application, we summarize the discussion so far in the next corollary. The “moreover” part is due to the construction in [30] Lemma 4.1 and Proposition 5.1.

**Corollary 3.9.** Let $K$ be an imaginary quadratic field and $\ell$ is a multiple of the exponent $e_K$ of $Cl(\mathcal{O}_K)$. Let $D = m^2 d_K$ for some $m \in \mathbb{N}$. Then any character $\phi : Cl(D) \to \mathcal{O}_K^\times$ yields a Hecke character $\Phi \in X^\times_K(m)$ such that $f_\Phi = f_{\phi, \ell}$ has rational coefficients. Moreover, if $\ell$ is even, any $f_\psi$ of weight $\ell + 1$ with rational coefficients arises this way.

**Caution 3.10.** (1) In general, the $f_{\phi, \ell}$ in Corollary 3.9 may not be a newform. See Proposition 3.24 for examples. (2) For the case $\ell = 2$ that we mainly concern, the fact that $e_K \mid \ell$ restricts us to imaginary quadratic fields whose class group consists only of 2-torsion. However, higher torsion may appear in the group $Cl(\mathcal{O}) \cong Cl(D)$. But it follows from Remark 1.5 that only $\mathbb{Z}_2^g$ ($g = 0, 1, 2$) can appear as $Cl(D)$ in our application if Conjecture 1.4 holds.

The modular forms are related to L-functions via the Mellin transform. The Hecke L-function of $\psi$ is the Mellin transform of $f_\psi$:

$$L(\psi, s) = \sum_{a \text{ integral}} \frac{\psi(a)}{N(a)^s} Z(\ell, a, s).$$

To any ideal $a$ of $\mathcal{O}$, we associate a partial Hecke series

$$Z(\ell, a, s) = \frac{1}{w} \sum_{\lambda \in a} \lambda^{-s} \ell^{\lambda},$$

where $w$ is the number of units in $\mathcal{O}$. Recall that $w$ is equal to 6 or 4 for $D = -3$ or $-4$, and to 2 otherwise.

The following lemma is a straightforward variation of the well-known statement for maximal orders.

**Lemma 3.11.** Let $\mathcal{O}$ be an order of $K$ of conductor $m$, and $\psi \in X^\times_K(m)$. Then

$$L(\psi, s) = \sum_{[a] \in Cl(\mathcal{O})} \frac{N(a\mathcal{O}_K)^s}{\psi(a\mathcal{O}_K)} Z(\ell, a, s).$$

**Proof.** Recall the isomorphism (3.1). We shall consider the partition of $I_K(m)$ by the classes in $Cl(\mathcal{O})$. For each $[a_0] \in Cl(\mathcal{O})$, we choose some $a \in [a_0]^{-1}$ so that $aa' = \lambda \mathcal{O}$ for any $a' \in [a_0]$. Then $\lambda \in a$ and we have that

$$\sum_{a' \in [a_0]} \frac{\psi(a'\mathcal{O}_K)}{N(a'\mathcal{O}_K)^s} = \sum_{\lambda \in (a \setminus \{0\})/\mathcal{O}_K} \frac{\psi(a^{-1}\mathcal{O}_K)\psi(\lambda)}{N(a\mathcal{O}_K)^{-s}N(\lambda)^s} = \frac{N(a\mathcal{O}_K)^s}{\psi(a\mathcal{O}_K)} \frac{1}{w} \sum_{\lambda \in a} \frac{\psi(\lambda)}{N(\lambda)^s}. $$
for any \( n \in \mathbb{Z} \) the generalized theta function (Definition 3.13).

3.3. Generalized Theta Functions. Let \( A = (a_{ij}) \) be a symmetric positive definite matrix of rank \( r \). The Laplace operator attached to \( A \) is

\[
\Delta_A = \sum_{i,j} a_{ij}^* \frac{\partial^2}{\partial x_i \partial x_j}, \quad \text{where } (a_{ij}^*) = A^{-1}.
\]

A spherical function for \( A \) is a homogeneous polynomial \( P \) in \( r \) variables such that

\[
\Delta_A P = 0.
\]

Now we assume \( A \) is integral and even, which means the diagonal entries of \( A \) are even. So \( n^t A n \) is even for any \( n \in \mathbb{Z}^r \). Let \( N \) be such that \( NA^{-1} \) has the same properties.

Definition 3.13. The generalized theta function attached to \( A \) and \( P \) is defined as

\[
\Theta_{A,P}(\tau) = \sum_{n \in \mathbb{Z}^r} P(n) \exp \left( \frac{1}{2} n^t A n \tau \right).
\]

Clearly, \( \Theta_{A,P} \) is linear w.r.t. \( P \).

Theorem 3.14 (\cite{21} Theorem 10.9)). Let \( P \) be a spherical function for \( A \) of degree \( v \). Then

\[
\Theta_{A,P}(\tau) \in \mathcal{M}_{v+r/2}(\Gamma_0(N), \chi_D),
\]

where \( D = (-1)^{r/2}|A| \) and \( \chi_D \) is the Kronecker character. Moreover, if \( v > 0 \), then \( \Theta_{A,P}(\tau) \) is a cusp form.

Example 3.15. A positive definite quadratic form \( Q = [a, b, c] \) corresponds to an even symmetric positive definite matrix \( (\begin{array}{cc} a & b \\ b & c \end{array}) \). Then

\[
\theta_Q(\tau) := \Theta_{Q,1}(\tau) = \sum_{m,n \in \mathbb{Z}} q^{am^2 + bmn + cn^2}
\]

is the ordinary \( \theta \)-series attached to \( Q \).
Example 3.16. For two positive definite quadratic forms \( Q = [a, b, c] \) and \( P = [a', b', c'] \) satisfying
\[
\langle Q, P \rangle := 2ac' - bb' + 2ac = 0,
\]
we define the theta function
\[
\Theta_{Q,P}(\tau) = \sum_{m,n \in \mathbb{Z}} \frac{1}{2}(a'm^2 + b'mn + c'n^2)q^{am^2 + bmn + cn^2}.
\]
It is a cusp form of weight 3. For any quadratic form \( Q = [a, b, c] \), we denote
\[
\Theta_Q(\tau) := \sum_{m,n \in \mathbb{Z}} \frac{1}{2}(am^2 - cn^2)q^{am^2 + bmn + cn^2}.
\]

Note that \( \Theta_{gQ,gP} = \Theta_{Q,P} \) for \( g \in \text{SL}_2(\mathbb{Z}) \) but in general \( \Theta_{gQ} \neq \Theta_Q \).

**Definition 3.17.** The linear span of \( \Theta_{Q,P} \) for all quadratic forms \( P \) spherical for \( Q \) is denoted by \( \langle \Theta_Q \rangle \). The theta kernel of \( Q \) consists of all \( P \) such that \( \Theta_{Q,P} = 0 \).

**Lemma 3.18.** If \( Q \) is a p.d.q.f. of order at most 2, then the dimension of \( \langle \Theta_Q \rangle \) is at most 1.

**Proof.** It is clear that the dimension of \( \langle \Theta_Q \rangle \) is at most 2. If \( Q = [a, b, c] \) is a form of order at most 2, then \( g \cdot [a, b, c] = [a, -b, c] \) for some \( g \in \text{SL}_2(\mathbb{Z}) \), so there is some nontrivial \( g' = \left( \begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} \right) \) \( g \in \text{GL}_2(\mathbb{Z}) \) such that \( g'Q = Q \). Since \( Q \notin Q^\perp \) and \( g' \) cannot fix the entire \( Q \), \( g' \) cannot fix \( Q^\perp \). So there is some \( P \in Q^2 \) such that \( P' = g'P \neq P \). Then we have that
\[
\Theta_{Q,P} = \Theta_{g'Q,g'P} = \Theta_{Q,P'}.
\]
Hence \( \Theta_{Q,P-P'} = 0 \), and therefore the dimension of \( \langle \Theta_Q \rangle \) is at most 1. \( \square \)

**Lemma 3.19.** For \( k \in \mathbb{Z} \) and \( \gcd(a,c) = 1 \), the form \( Q = [a, ka, c] \) has order at most 2, and its \( \Theta \)-kernel contains \([0, 2, k]\). Moreover, let \( \mathcal{O} \) be an order of discriminant \( D = k^2a^2 - 4ac \) in \( K \), then we have that \( \mathcal{I}([a, ka, c])^2 = a\mathcal{O} \) where \( \mathcal{I} \) is the map in \( \text{(3.4)} \).

**Proof.** We check by straightforward calculation that
\[
g \cdot [a, ka, c] = [a, -ka, c] \quad \text{for} \quad g = \left( \begin{smallmatrix} 1 & -k \\ 0 & 1 \end{smallmatrix} \right).
\]
So the form has order at most 2. We run the simple algorithm in the proof of Lemma 3.18 and find that \( g' = \left( \begin{smallmatrix} 1 & -k \\ 0 & 1 \end{smallmatrix} \right) \) and \( P = [a, 0, -c] \in Q^2 \) is not fixed by \( g' \). Then \( P - g'P = ak[0, 2, k] \) lies in the \( \Theta \)-kernel.

Recall that \( \mathcal{I}([a, ka, c]) = \text{span}_\mathbb{Z}(a, r) \) where \( r = \frac{ka - \sqrt{D}}{2} \). The norm of the ideal \( \mathcal{I}([a, ka, c]) \) can be computed as the GCD of \( (N(a), \text{Tr}(a^2), N(r)) \) which is \( (a^2 - a^2k/2, ac) = a \). Suppose that \( \mathcal{I}([a, ka, c])^2 = (u) \), then \( N(u) = a^2 \). Using \( \gcd(a, c) = 1 \), we check that \( a \) is the only element in \( \mathcal{I}([a, ka, c]) \) with norm \( a^2 \). Hence \( u = a \). \( \square \)

As before, \( K \) is an imaginary quadratic field, and \( \mathcal{O} \) is an order in \( K \) of discriminant \( D = m^2d_K \).

**Lemma 3.20.** Suppose \( [a] \in \text{Cl}(\mathcal{O}) \) is represented by a form \( Q = [a, b, c] \) such that \( a^2 = a\mathcal{O} \) and \( [0, 2a, b] \) is in the \( \Theta \)-kernel of \( Q \). Then for \( \psi \in X_K^\times(m) \) determined by \( \phi : \text{Cl}(\mathcal{O}) \to \mathbb{C}^\times \) as \( \text{(3.3)} \), we have
\[
\frac{N(a\mathcal{O}_K)^a}{\psi(a\mathcal{O}_K)} Z(2, a, s) = \phi([a])L(\Theta_Q, s).
\]

**Proof.** By Remark 3.12 and Theorem 3.3 we may assume that \( a = \text{span}_\mathbb{Z} \left( a, \frac{b - \sqrt{D}}{2} \right) \). Then
\[
Z(2, a, s) = \frac{1}{2} \sum_{m,n \in \mathbb{Z}} \frac{\lambda^2}{(\lambda m)^s} = \frac{1}{2} \sum_{m,n \in \mathbb{Z}} a(am^2 - cn^2) + \frac{1}{2}(b - \sqrt{D})(2amn + bn^2) \quad \frac{a^s(am^2 + bmn + cn^2)^s}{a^s(am^2 + bmn + cn^2)^s}.
\]
By our assumption that $[0, 2a, b]$ is in the $\Theta$-kernel, the last one is equal to
\[
\sum_{m,n \in \mathbb{Z}} \frac{a(am^2 - cn^2)}{2a^s(am^2 + bmn + cn^2)^s} = a^{1-s} L(\Theta_Q, s).
\]
Then by the assumption that $a^2 = a\mathcal{O}$, we have
\[
\frac{N(a\mathcal{O}_K)^s}{\psi(a\mathcal{O}_K)} = \frac{a^s}{\phi([a]a)}.
\]
Hence our desired equality follows. \hfill $\square$

Remark 3.21. (1). The similar proof can show that for any $a$, $Z(\ell, a, s) = L(\Theta_{Q,P}, s)$ where $P$ is some spherical function of degree $\ell$ for $Q$. However, only the above special case appear in our application.

(2). There are similar statement as Lemma 3.19 for the form $Q = [a, kc, c]$. Its $\Theta$-kernel contains $[k, 2, 0]$, and we have that $\mathcal{I}([a, kc, c])^2 = (c)$, where $\mathcal{I}$ is the twin map of $\mathcal{I}$. Moreover, the conclusion of Lemma 3.20 also holds for a satisfying $a^2 = c\mathcal{O}$ and $[b, 2c, 0]$ is in the $\Theta$-kernel of $Q$.

Proposition 3.22. Suppose that the form class group $\text{Cl}(D)$ is isomorphic to $\mathbb{Z}_2^2$ where $D = m^2 d_K$ and $g \in \mathbb{N}_0$. Let $Q_i = [a_i, b_i, c_i] \ (i = 1, 2, \ldots, 2^g)$ be a set of representatives in $\text{Cl}(D)$ such that

the $\Theta$-kernel of $Q_i$ is spanned by $[0, 2a_i, b_i]$ and $\mathcal{I}(Q_i)^2 = (a_i)$. Then the Hecke $L$-series for $\Phi \in X^2_K(m)$ given by (3.5) is equal to

\[
L(\Phi, s) = \sum_{i=1}^{2^g} \phi(Q_i) L(\Theta_{Q_i}, s).
\]

Moreover, any cusp eigenform of weight 3 and level $D$ with rational coefficients is of the form $\sum_i \phi(Q_i) \Theta_{Q_i}$. Proof. Let $Q = [a, b, c]$ be one of those $Q_i$’s, and let $a = \mathcal{I}(Q)$. Then the equality follows from Lemma 3.11 and Lemma 3.20. The last statement is obtained from the inverse Mellin transform and Corollary 3.9. \hfill $\square$

Convention: For cusp forms (of a fixed odd weight) induced from the characters $\text{Cl}(D) \to \mathcal{O}_K^*$, we shall label them by the level $N = m^2 d_K$ and a letter as follows. We first normalize all forms such that the Fourier coefficient of $a$ is 1. For two cusp forms $g_1$ and $g_2$ at the same level, we say $g_1 < g_2$ if $g_1$ is a newform but $g_2$ is not, or both are newforms (or oldforms) and the Fourier coefficient sequences of $g_1$ is less than that of $g_2$ in the lexicographical ordering. Then we assign the letters to those forms at the same level according to this ordering. This labelling is compatible with the one in the LMFDB [24] in the sense that if $g_{N,x}$ is a newform then $g^{N,x}$ is also the LMFDB label if we ignore the first letter (for the Dirichlet character). If $g_{N,x}$ is an oldform, we will add a circle superscript $g^{N,x}$ to indicate it is an oldform.

Example 3.23. There are 4 form classes for $D = -84$ represented by

\[
Q_1 = (1, 14, 70), \quad Q_2 = (2, 14, 35), \quad Q_3 = (7, 14, 10), \quad Q_4 = (14, 14, 5),
\]

which corresponds to $(0, 0), (1, 0), (0, 1), (1, 1)$ in $\mathbb{Z}_2 \times \mathbb{Z}_2$. By Lemma 3.19 the four forms all satisfy the conditions in Proposition 3.22. So all weight-3 newforms of level 84 with rational coefficients are

\[
\begin{align*}
g_{84a} &= \Theta_{Q_1} - \Theta_{Q_2} + \Theta_{Q_3} - \Theta_{Q_4} = q - 2q^2 - 3q^3 + \cdots, \\
g_{84b} &= \Theta_{Q_1} - \Theta_{Q_2} - \Theta_{Q_3} + \Theta_{Q_4} = q - 2q^2 + 3q^3 + \cdots, \\
g_{84c} &= \Theta_{Q_1} + \Theta_{Q_2} + \Theta_{Q_3} + \Theta_{Q_4} = q + 2q^2 - 3q^3 + \cdots, \\
g_{84d} &= \Theta_{Q_1} + \Theta_{Q_2} - \Theta_{Q_3} - \Theta_{Q_4} = q + 2q^2 + 3q^3 + \cdots.
\end{align*}
\]
As we remarked in Proposition 3.24, the construction does not necessarily produce newforms. We make a short list for all such cases appearing in our application. All the identities can be easily verified by Sturm’s theorem.

**Proposition 3.24.** We have the following equalities:

\[ g_{286}^0(\tau) = g_7(\tau) + 3g_7(2\tau) + 8g_7(4\tau), \]
\[ g_{326}^0(\tau) = g_8(\tau) + 2g_8(2\tau) + 8g_8(4\tau), \]
\[ g_{446}^0(\tau) = g_{11}(\tau) + 8g_{11}(4\tau), \]
\[ g_{486}^0(\tau) = g_{12}(\tau) + 8g_{12}(4\tau), \]
\[ g_{606}^0(\tau) = g_{15b}(\tau) - g_{15b}(2\tau) + 8g_{15b}(4\tau), \]
\[ g_{606}^0(\tau) = g_{15a}(\tau) + g_{15a}(2\tau) + 8g_{15a}(4\tau), \]
\[ g_{726}^0(\tau) = g_8(\tau) + 2g_8(3\tau) + 27g_8(9\tau), \]
\[ g_{966}^0(\tau) = g_{24b}(\tau) - 2g_{24b}(2\tau) + 8g_{24b}(4\tau), \]
\[ g_{996}^0(\tau) = g_{11}(\tau) + 5g_{11}(3\tau) + 27g_{11}(9\tau), \]
\[ g_{1126}^0(\tau) = g_7(\tau) + 3g_7(2\tau) + 12g_7(4\tau) + 24g_7(8\tau) + 64g_7(16\tau), \]
\[ g_{1806}^0(\tau) = g_{206}(\tau) + 4g_{206}(3\tau) + 27g_{206}(9\tau), \]
\[ g_{1926}^0(\tau) = g_{12}(\tau) + 8g_{12}(4\tau) + 64g_{12}(16\tau), \]
\[ g_{2406}^0(\tau) = g_{15b}(\tau) - g_{15b}(2\tau) + 12g_{15b}(4\tau) - 8g_{15b}(8\tau) + 64g_{15b}(16\tau), \]
\[ g_{2406}^0(\tau) = g_{15a}(\tau) + g_{15a}(2\tau) + 12g_{15a}(4\tau) + 8g_{15a}(8\tau) + 64g_{15a}(16\tau). \]

Then one can use the obvious relation \( L(g(n\tau), s) = n^{-s}L(g(\tau), s) \) to get relations between the corresponding \( L \)-functions.

**Lattice Sums.** When \( \ell = 0 \), the analogue of Proposition 3.22 reads:

**Observation 3.25.** Let \( Q_i = [a_i, b_i, c_i] \) be a set of representatives in \( \text{Cl}(D) \) with \( D = m^2d_K \). Then the Hecke \( L \)-series for \( \Phi \in X_K^0(m) \) given by (3.30) is equal to

\[ L(\Phi, s) = \sum_i \phi(Q_i)L(\theta_{Q_i}, s). \]

Recall that \( \theta_Q \) is the ordinary theta series attached to \( Q \). The associated \( L \)-function \( L(\theta_Q, s) \) is called the Epstein zeta-function, which is also denoted by \( \zeta_Q(s) \).

As before, let \( \chi_d \) be the Kronecker character. For two discriminants \( d_1 \) and \( d_2 \) with product \( D \), we define

\[ \chi_{d_1,d_2}(Q) = \chi_{d_1}(p) = \chi_{d_2}(p), \]

where \( p \) is any prime represented by \( Q \) and not dividing \( D \). The definition is independent of the choice of \( p \).

If \( \text{Cl}(D) \) has at most 2-torsion, then any \( \phi \) is a quadratic character (the so-called genus character). It is known that \( \phi = \chi_{d_1,d_2} \) for two discriminants \( d_1,d_2 \) with \( d_1d_2 = D \). In general, \( d_1 \) and \( d_2 \) need not be fundamental (i.e., a discriminant of a quadratic field), and \( (d_1, d_2) \) is not unique. Finding a particular pair \( (d_1, d_2) \) is easy and completely elementary.
4.1. We first briefly review the classical class polynomial for the \( J \) function. This more general version is similar. We write both sides as Euler products, then compare each term.

Theorem 3.27 was classically stated for genus characters of \( \text{Cl}_d \) (Dirichlet-Kronecker) \( \chi_d \), and \( \phi = \chi_{14,-7} \).

Similarly, the genus characters given by (1, -1, -1, 1), (1, -1, 1, -1), and (1, 1, 1, 1) are equal to \( \chi_{21,-4}, \chi_{28,-3}, \) and \( \chi_{1,-84} \) respectively.

Theorem 3.27 (Dirichlet-Kronecker). Suppose that \( D = m^2 \delta = d_1 d_2 \) for two fundamental discriminants \( d_1 \) and \( d_2 \). Let \( \phi = \chi_{d_1,d_2} \) be the associated genus character on \( \text{Cl}(D) \) and \( \Phi \in X_0^0(1) \). Then we have the factorization

\[
L(\Phi, s) = \omega(D)L(\chi_{d_1}, s)L(\chi_{d_2}, s),
\]

where \( w(D) = 6 \) if \( D = -3 \), \( w(D) = 4 \) if \( D = -4 \), otherwise \( w(D) = 2 \).

We observe that \( L(\Phi, s) \) is clearly stable under the action of \( \Gamma_0(1) \). For \( \Phi \in X_0^0(1) \), we have that \( \chi_1 = \chi_2 = \chi_3 = \chi_4 = \chi_7 = \chi_8 = \chi_9 = \chi_2 = \chi_7 = \chi_8 = \chi_9 \), and is irreducible.

Theorem 3.27 was classically stated for genus characters of \( \text{Cl}(O_K) \) (eg. [21 Theorem 12.7]).

4. Rational Singular Moduli

4.1. We first briefly review the classical class polynomial for the \( j \)-invariant. For any primitive p.d.q.f. \( Q \), we write \( \tau_Q \) for the (unique) root of the \( Q(x, 1) \) in the upper half plane \( \mathbb{H} \). The value of a modular function \( \vartheta \) at an irrational \( \tau_Q \) is called a singular modulus of \( \vartheta \). They are algebraic integers. Recall the classical (Hilbert) class polynomial for the \( j \)-invariant is

\[
H_D(X) = \prod_{Q \in \mathbb{Q}_0^0} (X - j(\tau_Q)).
\]

Theorem 4.1 ([36]). The polynomial \( H_D(X) \) belongs to \( \mathbb{Z}[X] \) and is irreducible.

In particular, the rational singular moduli of \( j \) are precisely those \( j(\tau_Q) \) where \( Q \) is the unique orbit in \( \mathbb{Q}_0^0(1)/\Gamma_0 \).

Our next goal is to calculate the Mahler measure of \( f_c \) for \( c = c(\tau) \) a rational singular modulus. So we hope to find all \( Q \) such that \( c(\tau_Q) \) is rational and \( \tau_Q \) lie in the fundamental domain of \( c(\tau) \) containing \( i\infty \).

4.2. Throughout \( a, b, c \) are always integers. Following [20], we denote

\[
\mathbb{Q}^0_{D,N} = \{ [aN,b,c] \in \mathbb{Q}_D \mid \text{gcd}(a,b,c) = 1 \},
\]

\[
\mathbb{Q}^0_{D,N,m} = \{ [aN,b,c] \in \mathbb{Q}^0_{D,N} \mid \text{gcd}(N,b,ac) = m \}.
\]

The set \( \mathbb{Q}^0_{D,N} \) is clearly stable under the action of \( \Gamma_0(N) \). Let \( M_{D,N} \) be the set of positive integers \( m \) such that \( \mathbb{Q}^0_{D,N,m} \) is non-empty. Recall the group \( W(N) \) generated by the Atkin-Lehner involutions \( W_n \). Here are three simple observations.

**Observation 4.2.** (1). For \( m \in M_{D,N} \), we have that \( m \mid N, m^2 \mid D \) and \( D/m^2 \) is a discriminant. \( m \) is a product of two coprime numbers \( m_1 = \text{gcd}(N,b,a) \) and \( m_2 = \text{gcd}(N,b,c) \).

(2). If \( N \) is square-free, then \( M_{D,N} \) contains at most one element.

(3). The group \( W(N) \) acts on \( \mathbb{Q}^0_{D,N} / \Gamma_0(N) \) and the action restricts to \( \mathbb{Q}^0_{D,N,m} / \Gamma_0(N) \).
Fix a solution $\beta \mod 2N$ of $\beta^2 \equiv D \mod 4N$. Define the subset of $Q_{D,N}^0$

$$Q_{D,N,\beta}^0 = \{ Q = [a,b,c] \in Q_{D,N}^0 \mid b \equiv \beta \mod 2N \}.$$  

The set $Q_{D,N,\beta}^0$ is also stable under the action of $\Gamma_0(N)$, but not stable under $W(N)$. We have that \([21, (7)]\)

$$W_n : Q_{D,N,\beta}/\Gamma_0(N) \cong Q_{D,N,\beta^*}/\Gamma_0(N), \quad \beta^* \equiv \begin{cases} \beta & \text{mod } 2N/n \\ -\beta & \text{mod } 2n \end{cases}.$$  

Fix a number $m \in M_{D,N}$, we denote $Q_{D,N,m,\beta}^0 := Q_{D,N,m}^0 \cap Q_{D,N,\beta}^0$. Let $t(N, D, m)$ be the number of $\beta \mod 2N$ such that $Q_{D,N,m,\beta}^0$ is nonempty, that is, $\beta^2 \equiv D \mod 4N$ with $\gcd(N, b, ac) = m$.

**Lemma 4.3** ([21] Proposition p.505–507). Suppose that $Q_{D,N,m,\beta}^0$ is non-empty. Fix a decomposition $m = m_1m_2$ with $m_1, m_2 > 0$ and $\gcd(m_1, m_2) = 1$. We define

$$Q_{D,N,m_1,m_2,\beta}^0 := \{ [aN, b, c] \in Q_{D,N,m,\beta}^0 \mid \gcd(N, b, a) = m_1, \gcd(N, b, c) = m_2 \}.$$  

The map $Q_{D,N,m,\beta}/\Gamma_0(N)$ induced by $[aN, b, c] \mapsto [aN, b, cN_2]$. is a bijection. Here, $N_1N_2$ is any decomposition of $N$ into coprime positive factors satisfying $\gcd(m_1, N_2) = \gcd(m_2, N_1) = 1$. Moreover, there is a decomposition

$$Q_{D,N}/\Gamma_0(N) = \bigcup_{\beta^2 \equiv D \mod 4N} Q_{D,N,\beta}/\Gamma_0(N).$$  

The decomposition certainly restricts to $Q_{D,N,m}/\Gamma_0(N)$. In particular, we have that $|Q_{D,N,m}^0| = 2^{\sigma_5^+(m)}h(D)$.

Let $j_N^+$ be a Hauptmodul for $\Gamma_0^+(N)$. We define the level-$N$ classical class polynomial

$$H_{D,N,m}(X) = \prod_{Q \in Q_{D,N,m}^0/\Gamma_0(N)} (X - j_N^+(\tau_Q)).$$  

We conjecture the following analogue of Theorem 3.1.

**Conjecture 4.4.** The polynomial $H_{D,N,m}(X)$ belongs to $\mathbb{Z}[X]$ and is irreducible.

The conjecture would imply that the singular moduli $j_N^+(\tau_Q) \in \mathbb{Q}$ if and only if $j_N^+(\tau_Q) \in \mathbb{Z}$ if and only if $Q$ is the unique orbit in $Q_{D,N,m}^0/\Gamma_0^+(N)$. In particular, if $j_N^+(\tau_Q) \in \mathbb{Q}$, then by Lemma 4.3 we have that

$$t(N, D, m)2^{\sigma_5^+(m)}h(D) \leq 2^{\sigma_5^+(N)}.$$  

**Remark 4.5.** In all our cases (see Table 2), the number $\sigma_5^+(N)$ of prime divisors of $N$ is no greater than 2. So by Lemma 1.3 to search for the discriminant $D$ such that $Q_{D,N}^0$ has only one $\Gamma_0^+(N)$-orbit, it suffices to look at those $D$ with $Cl(D) \cong \mathbb{Z}_2^g$ ($g = 0, 1, 2$). This already reduces the search to a (reasonably small) finite list. Moreover, if $N$ is prime, $g$ has to be 0 or 1. Due to the presence of $t(N, D, m)2^{\sigma_5^+(m)}$, the condition is actually more restrictive. In view of [30] Proposition 7.1, this implies that singular $K3$ surfaces arising from special fibres of these 25 families cannot cover all weight-3 newforms with rational coefficients. The geometric realization of all such newforms was accomplished in [14].

**Example 4.6.** Let $N = 14$. If $D = -84$, then there are 4 classes in $Cl(D)$ represented by elements

$$Q_1 = [70, 14, 1], Q_2 = [42, 42, 11], Q_3 = [154, 42, 3], Q_4 = [14, 14, 5].$$  

Since $W_2(Q_2) = W_7(Q_3) = W_{14}(Q_4) = Q_1$ in $Q_{D,N}/\Gamma_0(N)$, they reduce to a single element in $Q_{D,N}/\Gamma_0^+(N)$.

If $D = -20$, then $Q_{D,N}^0 = Q_{D,N,6}^0 \cup Q_{D,N,-6}^0$, which has representative

$$Q_1 = [14, 6, 1], Q_2 = [42, -22, 3], \text{ and } Q_3 = [42, 22, 3], Q_4 = [14, -6, 1].$$
We have that \( W_2(Q_2) = W_7(Q_3) = W_14(Q_4) = Q_1 \) in \( Q^0_{D,N}/\Gamma_0(N) \). So \( Q^0_{D,N}/\Gamma_0^+(N) \) has only one element.

5. RELATING TO L-FUNCTIONS

5.1. Computing \( d^2 H_d(\tau) \). Let \( Q = [a, b, c] \in Q_D \). Throughout we assume that \( d \mid a \) and \( D = b^2 - 4ac < 0 \).

Let \( \tau = \tau_Q = -\frac{b + \sqrt{D}}{2a} \). Recall the series (2.13) for \( H_d(\tau) \). We have that

\[
d^2 H_d(\tau) = \sum_{m,n} \left( \frac{4d^2(dm + n)^2}{(n^2 + bmn + cd^2m^2)^3} - \frac{d^2}{(n^2 + bmn + \frac{c}{a}d^2m^2)^2} \right) = aD \sum_{m,n} \left( \frac{d^2}{(n^2 + bmn + cd^2m^2)^3} + \frac{3a^2}{(n^2 + bmn + cd^2m^2)^2} \right).
\]

We have a simple but important observation, which can be verified by straightforward calculation.

**Observation 5.1.** For \( Q = [\frac{a}{2}, b, cd] \), \( x = \frac{2a}{D} \) is the unique \( x \) such that \( [0, 0, d] + xQ \) is \( Q \)-spherical.

From this observation, we continue our calculation.

\[
d^2 H_d(\tau) = aD \sum_{m,n} \left( \frac{d^2m^2 + x(\frac{a}{2}n^2 + bmn + cd^2m^2)}{(\frac{a}{2}n^2 + bmn + cd^2m^2)^3} - \frac{1}{(\frac{a}{2}n^2 + bmn + cd^2m^2)^2} \right) + (3a^2 - xaD) \sum_{m,n} \frac{1}{(\frac{a}{2}n^2 + bmn + cd^2m^2)^2},
\]

\[
(5.1)
\]

To proceed further, we focus on the cases \( b = ka \) or \( b = kc \) \((k \in \mathbb{Z})\), that is, the cases covered by Lemma 3.19 (Remark 3.21) and thus Proposition 3.22. For \( b = ka \), we have that

\[
d^2 H_d(\tau) = a \sum_{m,n} \left( \frac{d(ka)^2m^2 + 2a(\frac{a}{2}n^2 + kmn + cd^2m^2)}{(\frac{a}{2}n^2 + kmn + cd^2m^2)^3} \right) + a \sum_{m,n} \frac{a^2}{(\frac{a}{2}n^2 + kmn + cd^2m^2)^2},
\]

\[(\text{Lemma 3.19})\]

\[
= a^2 \left( 4L(\Theta_{Q/a(3)} + \zeta_{Q/a(2)} \right).
\]

Here, for \( Q = [a, b, c] \) with \( d \mid a \), we denote \( Q_{/d} := [\frac{a}{d}, b, cd] \). So we obtain the following.

**Proposition 5.2.** If \( \tau = \tau_Q \) for \( Q = [a, ka, c] \) and \( d \mid a \), then

\[
\text{Im} \frac{\tau}{(2\pi)^3} d^2 H_d(\tau) = \frac{a\sqrt{D}}{(2\pi)^3} \left( 2L(\Theta_{Q/a(3)} + \frac{1}{2}\zeta_{Q/a(2)} \right).
\]

**Remark 5.3.** If \( Q = [a, kc, c] \) with \( d \mid k \), it is easy to check by Remark 3.21 (2) that we have the same equality (5.2).

**Corollary 5.4.** Suppose that the form class group \( \text{Cl}(D) \) is isomorphic to \( \mathbb{Z}_d^2 \) where \( D = m^2d_K \). Suppose that for some \( Q = [a, ka, c] \), \( Q_{/d} \in \delta \) constitutes a set of representatives in \( \text{Cl}(D) \) for a set \( \delta \) of divisors of \( a \). Then for any character \( \phi : \text{Cl}(D) \rightarrow \{ \pm 1 \} \), we have that

\[
\text{Im} \frac{\tau}{(2\pi)^3} \sum_{d \in \delta} \phi(Q_{/d}) d^2 H_d(\tau) = \frac{a\sqrt{D}}{(2\pi)^3} \left( 2L(f_{\phi,3}, 3) + \frac{1}{2}L(f_{\phi,1}, 2) \right).
\]
Proof. By Lemma 8.19 all $Q_{\lambda}$ satisfy the conditions of Proposition 3.22. So $f_{\phi, \lambda} = \sum_{d \in \mathfrak{d}} \phi(Q_{\lambda}) \Theta_{Q_{\lambda}} d$. Then the result follows from Proposition 5.2.

Another important case is when $a = cde$. For this case, we shall compute $-d^2 H_d(\tau) + e^2 H_\epsilon(\tau)$ rather than a single summand. We have from (5.1) that

$$-d^2 H_d(\tau) + e^2 H_\epsilon(\tau) = -a \sum_{m,n} \frac{(a^2 mn^2 + 2an)^2}{(\tau^2 + bmn + cdm^2)^3} + a \sum_{m,n} \frac{e^{-1}(ebn + 2an)^2}{(\tau^2 + bmn + cem^2)^3},$$

$$= -a \sum_{m,n} \frac{(db^2 - 4a^2/c)n^2 - (eb^2 - 4a^2/d)n^2}{(cen^2 + bmn + cdm^2)^3},$$

$$= -aD \sum_{m,n} \frac{dm^2 - en^2}{(cen^2 + bmn + cdm^2)^3},$$

$$= aD \frac{2L(\Theta_{Q_{\lambda}}, 3)}{e}.$$

So we obtain the following.

**Proposition 5.5.** If $\tau = \tau_Q$ for $Q = [cde, b, c]$, then

$$\frac{\text{Im} \tau}{(2\pi)^3} \left( -d^2 H_d(\tau) + e^2 H_\epsilon(\tau) \right) = c^{-1} \left( \frac{\sqrt{D}}{2\pi} \right)^3 L(\Theta_{Q_{\lambda}}, 3).$$

5.2. **Lists.** To better state our next main result, we will rescale some special values of the relevant $L$-functions. Let $g_d$ be a newform of weight 3 and level $d$. We set

$$\tilde{L}(g_d, 3) := \frac{2\sqrt{d}}{(2\pi)^3} L(g_d, 3) = \frac{e}{d} L'(g_d, 0),$$

where the latter equality follows from the functional equation of $L(g_d, s)$ and $\epsilon \in \{\pm 1\}$ is the sign of the functional equation. We also denote

$$I_{d_1, d_2} := \frac{\sqrt{-d_1 d_2}}{(2\pi)^3} L(\chi_{d_1}, 2) L(\chi_{d_2}, 2).$$

It is well-known that if $d > 0$ and the conductor of $\chi_d$ is $d_0$, then $L(\chi_d, 2)$ is a rational number, which can be easily calculated; if $d < 0$ and $\chi_d$ is primitive, then $L(\chi_d, 2) = \frac{\zeta(3)}{(d_0 \pi^2)^2} L'(\chi_d, -1)$. Note that if $d$ is a fundamental discriminant, then $\chi_d$ is primitive. Then it is easy to see that Theorem 5.6 below is equivalent to Theorem 6.2.

**Theorem 5.6.** For all the 25 families except for $V_2$ and $B_1$, and all known rational singular moduli of $c(\tau)$ of discriminant $D$, the value of $\text{Re}(\tilde{m}(c(\tau)))$ is equal to

$$\alpha \tilde{L}(g_d, 3) + \beta I_{d_1, d_2},$$

for some newform $g_d$ of weight 3 with rational coefficients, some fundamental discriminants $d_1$ and $d_2$ with $d_1 d_2 = D$, and $\alpha, \beta \in \mathbb{Q}$. Moreover, $-D/d$ is a square. The complete list is given below.

Proof. The proofs for all entries in the lists are similar. We start with the Eisenstein-Kronecker series in Theorem 2.6, then use Proposition 5.2 (Remark 5.3) or Proposition 5.5 to relate $L$-functions. In this way we cover all but 6 cases (2 rational $c(\tau)$ and 4 quadratic $c(\tau)$ marked by 8 in the remark column). For the 6 exceptional cases, almost the same calculation goes through.
As an illustration, we prove a moderately complicated (but typical) case:

\[ \text{Re}(\tilde{m}(-7, V_{28})) = 14 \tilde{L}(g_{84d}, 3) + 14l_{12,-7}. \]

which is the second row in the table for \( V_{28} \). By the second last row of Table 2 and Theorem 2.6, we need to compute

\[ \frac{\text{Im} \tau}{(2\pi)^3} \left( -1^2 H_1(\tau) - 2^2 H_2(\tau) + 7^2 H_7(\tau) + 14^2 H_{14}(\tau) \right), \]

where \( \tau \) is the root of \( 14x^2 + 14x + 5 = 0 \) in \( \mathbb{H} \). We have explained in Example 4.6 why \( c(\tau) \in \mathbb{Z} \), then numerical calculation finds \( c(\tau) = -7 \).

Since the forms \( Q = [14, 14, 5], Q_{/2} = [7, 14, 10], Q_{/7} = [2, 14, 35], \) and \( Q_{/14} = [1, 14, 70] \) constitute a set of representatives in \( Cl(-84) \), by Corollary 5.4 the above is equal to

\[ \frac{14\sqrt{84}}{(2\pi)^3} \left( 2L(f_{\phi,3}, 3) + \frac{1}{2}L(f_{\phi,1}, 2) \right). \]

where \( \phi \) sends \((Q, Q_{/2}, Q_{/7}, Q_{/14})\) to \((-1, -1, 1, 1)\). We have seen from Example 3.26 that \( \phi = \chi_{12,-7} \) so \( L(f_{\phi,1}, 2) = 2L(\chi_{12}, 2)L(\chi_{-7}, 2) \) by Theorem 3.27. From Example 3.23 we have seen that \( f_{\phi,3} = g_{84d} \). After the rescaling \([5.3] \) and \([5.4]\), we get the desired result.

\[ \square \]

**Remark 5.7.** As we shall see below that in most cases we have that \( \alpha = \beta \) and \( D = d_1d_2 = d \). If this is not the case, then either the quadratic form \( Q \) is not primitive or the relevant \( g_D \) is not a newform so that a relation in Proposition 3.24 is invoked. We will indicate this in the remark columns.

Before displaying the list, we make a few other comments.

1. For the families \( V_2 \) and \( B_1 \), it seems hard to conjecture a relation at any of the 13 rational singular moduli. So far the only conjectured relation is \( m(1728, V_2) = \frac{1}{2}L'(g_{144a}, 0) \) ([28, Conjecture 4.11]). This conjecture remains a big challenge for us.
2. The lists below settle all the conjectures for \( V_4 \), \( V_6 \), and \( V_8 \) in [28] at rational singular moduli.
   Quadratic singular moduli can be treated similarly. As an illustration, we give such results at all regular singular points of Picard-Fuchs equations. We label the regular singular points with a '*' in the column \( c(\tau) \).
3. The results for \( \text{Re}(\tilde{m}(\tau)) \) between every pair of double lines are not guaranteed to be equal to \( m(f_{c(\tau)}) \) by Lemma 2.2 (cf., Remark 2.4).
### $V_4$

| $\tau$ | $c(\tau)$ | $\text{Re}(\tilde{m}(\tau))$ | remark |
|--------|------------|-----------------------------|--------|
| $[2, 2, 19]$ | $-2^{10}3^{17}4$ | $160L(g_{1484}, 3) + 160l_{37.4}$ | |
| $[2, 2, 13]$ | $-2^{14}3^{4}5$ | $160L(g_{1008}, 3) + 160l_{5.29}$ | |
| $[2, 2, 7]$ | $-2^{10}3^{4}$ | $160L(g_{528}, 3) + 160l_{13.4}$ | $[28]$ Conjugate |
| $[2, 2, 5]$ | $-12288$ | $160L(g_{364}, 3) + 160l_{12.3}$ | $[28]$ Conjugate |
| $[1, 1, 2]$ | $-3969$ | $400L(g_{7}, 3) + 80l_{4.7}$ | $[28]$ Conjugate $3.24$ |
| $[2, 2, 3]$ | $-1024$ | $160L(g_{204}, 3) + 160l_{5.4}$ | $[28]$ Conjugate |
| $[1, 1, 1]$ | $-144$ | $160L(g_{12}, 3) + 80l_{4.3}$ | $[28]$ Conjugate |
| $[2, 2, 1]$ | $0$ | $0$ | |
| $[2, 1, 1]$ | $81$ | $280L(g_{7}, 3)$ | $[28]$ Conjugate |
| $[2, 0, 1]$ | $256$ | $320L(g_{8}, 3)$ | $[28]$ Conjugate |
| $[1, 0, 1]$ | $648$ | $160L(g_{16}, 3) + 80l_{4.4}$ | $[28]$ Conjugate |
| $[2, 0, 3]$ | $2304$ | $160L(g_{224}, 3) + 160l_{8.3}$ | $[28]$ |
| $[2, 0, 5]$ | $20736$ | $160L(g_{408}, 3) + 160l_{5.8}$ | $[28]$ |
| $[2, 0, 9]$ | $2^{17}$ | $\frac{160}{L(g_{8}, 3)} + 160l_{24.3}$ | $[28, 3.24]$ |
| $[2, 0, 11]$ | $2^{8}3^{11}2$ | $160L(g_{888}, 3) + 160l_{8.11}$ | |
| $[2, 0, 29]$ | $2^{8}3^{8}11^{4}$ | $160L(g_{2232}, 3) + 160l_{29.8}$ | |

### $V_6$

| $\tau$ | $c(\tau)$ | $\text{Re}(\tilde{m}(\tau))$ | remark |
|--------|------------|-----------------------------|--------|
| $[3, 3, 23]$ | $-2^{10}3^{15}6$ | $90L(g_{2676}, 3) + 90l_{39.3}$ | |
| $[3, 3, 13]$ | $-2^{9}3^{9}7$ | $90L(g_{1479}, 3) + 90l_{21.7}$ | |
| $[3, 3, 11]$ | $-2^{12}3^{13}$ | $90L(g_{1233}, 3) + 90l_{41.3}$ | |
| $[3, 3, 7]$ | $-8640$ | $90L(g_{750}, 3) + 90l_{5.15}$ | |
| $[3, 3, 5]$ | $-1728$ | $90L(g_{518}, 3) + 90l_{17.3}$ | |
| $[1, 1, 4]$ | $-192$ | $90L(g_{27}, 3) + 60l_{6.3}$ | |
| $[3, 3, 2]$ | $-27$ | $90L(g_{156}, 3) + 90l_{5.3}$ | |
| $[3, 3, 1]$ | $0$ | $0$ | |
| $[3, 2, 1]$ | $8$ | $120L(g_{8}, 3)$ | $[28]$ Conjugate |
| $[3, 1, 1]$ | $64$ | $165L(g_{11}, 3)$ | |
| $[3, 0, 1]$ | $108$ | $180L(g_{12}, 3)$ | $[28]$ Conjugate |
| $[3, 0, 2]$ | $216$ | $90L(g_{248}, 3) + 90l_{8.3}$ | $[28]$ |
| $[3, 0, 4]$ | $\frac{224}{3}L(g_{12}, 3) + 180l_{12.4}$ | $[28, 3.24]$ |
| $[3, 0, 5]$ | $3375$ | $180L(g_{156}, 3) + 117l_{20.3}$ | $[28]$ Conjugate $3.24$ |
### $V_8$

| $\tau$ | $c(\tau)$ | $\text{Re}(\tilde{m}(\tau))$ | remark |
|--------|------------|-------------------------------|--------|
| $[4, 4, 5]$ | $-512$ | $64L(g_{64}, 3) + 64l_{8,-8}$ | $[28 \text{ Conj}]$ |
| $[4, 4, 3]$ | $-64$ | $64L(g_{32}, 3) + 64l_{8,-4}$ | $[28 \text{ Conj}]$ |
| $[2, 2, 1]$ | $-8$ | $64L(g_{16}, 3) + 32l_{4,-4}$ | $[28 \text{ Conj}]$ |
| $[4, 3, 1]$ | $1$ | $56L(g_{7}, 3)$ | $[28 \text{ Conj}]$ |
| $[4, 2, 1]$ | $16$ | $96L(g_{12}, 3)$ | $[28 \text{ Conj}]$ |
| $[4, 0, 1]$ | $64\ast$ | $128L(g_{16}, 3)$ | $[28 \text{ Conj}]$ |
| $[4, 0, 3]$ | $256$ | $64L(g_{48}, 3) + 64l_{12,-4}$ | $[28 \text{ Conj}]$ |
| $[4, 0, 7]$ | $4096$ | $64L(g_{112}, 3) + 64l_{28,-4}$ | $[28 \text{ Conj}]$ |

### $V_{10}$

| $\tau$ | $c(\tau)$ | $\text{Re}(\tilde{m}(\tau))$ | remark |
|--------|------------|-------------------------------|--------|
| $[5, 5, 13]$ | $-15228$ | $50L(g_{235a}, 3) + 50l_{5,-47}$ | $[28 \text{ Conj}]$ |
| $[5, 5, 7]$ | $-828$ | $50L(g_{115a}, 3) + 50l_{5,-23}$ | $[28 \text{ Conj}]$ |
| $[5, 5, 3]$ | $-28$ | $50L(g_{35b}, 3) + 50l_{5,-7}$ | $[28 \text{ Conj}]$ |
| $[5, 5, 2]$ | $-3$ | $50L(g_{15a}, 3) + 50l_{5,-3}$ | $[28 \text{ Conj}]$ |
| $[10, 10, 3]$ | $22 - 10\sqrt{5}$ | $100L(g_{20a}, 3) - L(g_{20b}, 3)$ | $[28 \text{ Conj}]$ |
| $[5, 4, 1]$ | $0$ | $0$ | $[28 \text{ Conj}]$ |
| $[5, 3, 1]$ | $4$ | $55L(g_{11}, 3)$ | $[28 \text{ Conj}]$ |
| $[5, 2, 1]$ | $18$ | $80L(g_{16}, 3)$ | $[28 \text{ Conj}]$ |
| $[5, 1, 1]$ | $36$ | $95L(g_{19}, 3)$ | $[28 \text{ Conj}]$ |
| $[5, 0, 1]$ | $22 + 10\sqrt{5}$ | $50L(g_{20a}, 3) + L(g_{20b}, 3)$ | $[28 \text{ Conj}]$ |
| $[5, 0, 2]$ | $72$ | $50L(g_{40a}, 3) + 50l_{5,-8}$ | $[28 \text{ Conj}]$ |
| $[5, 0, 3]$ | $147$ | $125L(g_{15a}, 3) + 65l_{20,-3}$ | $[28 \text{ Conj}]$ |

### $V_{12}$

| $\tau$ | $c(\tau)$ | $\text{Re}(\tilde{m}(\tau))$ | remark |
|--------|------------|-------------------------------|--------|
| $[3, 3, 1]$ | $-1$ | $36L(g_{12}, 3) + 24l_{4,-3}$ | $[28 \text{ Conj}]$ |
| $[30, 24, 5]$ | $17 - 12\sqrt{2}$ | $48L(g_{24a}, 3) - 36L(g_{24b}, 3)$ | $[28 \text{ Conj}]$ |
| $[6, 4, 1]$ | $1$ | $32L(g_{8}, 3)$ | $[28 \text{ Conj}]$ |
| $[6, 0, 1]$ | $17 + 12\sqrt{2}$ | $48L(g_{24a}, 3) + 36L(g_{24b}, 3)$ | $[28 \text{ Conj}]$ |
\[ V_{14} \]

| \( \tau \) | \( c(\tau) \) | Re(\( \tilde{m}(\tau) \)) | remark |
|---|---|---|---|
| [7, 7, 17] | -10648 | \( 35L(g_{127a}, 3) + 35l_{61, -7} \) | |
| [7, 7, 5] | -64 | \( 35L(g_{916}, 3) + 35l_{13, -7} \) | |
| [7, 7, 3] | -8 | \( 35L(g_{35a}, 3) + 35l_{5, -7} \) | |
| [7, 7, 2] | -1* | \( 70L(g_7, 3) \) | |
| [7, 5, 1] | 0 | 0 | |
| [7, 4, 1] | 2 | \( 30L(g_{12}, 3) \) | |
| [7, 3, 1] | 8 | \( \frac{35}{2} L(g_{19}, 3) \) | |
| [7, 1, 1] | 24 | \( \frac{35}{2} L(g_{27}, 3) \) | |
| [7, 0, 1] | 27* | \( 210L(g_7, 3) \) | \[ 3.24 \]
| [7, 0, 4] | 125 | \( \frac{35}{2} L(g_7, 3) + 35l_{28, -4} \) | \[ 3.24 \]

\[ V_{16} \]

| \( \tau \) | \( c(\tau) \) | Re(\( \tilde{m}(\tau) \)) | remark |
|---|---|---|---|
| [8, 8, 3] | -4 | \( 80L(g_8, 3) + 32l_{8, -4} \) | \[ 3.24 \]
| [24, 16, 3] | 12 - 8\( \sqrt{2} \)* | \( 32L(g_{32}, 3) - 80L(g_8, 3) \) | \[ 3.24 \]
| [8, 5, 1] | 1 | \( 21L(g_7, 3) \) | |
| [8, 4, 1] | 4 | \( 32L(g_{16}, 3) \) | |
| [8, 2, 1] | 16 | \( 161L(g_7, 3) \) | \[ 3.24 \]
| [8, 0, 1] | 12 + 8\( \sqrt{2} \)* | \( 32L(g_{32}, 3) + 80L(g_8, 3) \) | \[ 3.24 \]

\[ V_{18} \]

| \( \tau \) | \( c(\tau) \) | Re(\( \tilde{m}(\tau) \)) | remark |
|---|---|---|---|
| [9, 9, 5] | -27 | \( 27L(g_{99}, 3) + 27l_{33, -3} \) | |
| [3, 3, 1] | -3 | \( 27L(g_{27}, 3) + 18l_{9, -3} \) | |
| [18, 18, 5] | 9 - 6\( \sqrt{3} \)* | \( 54(L(g_{36a}, 3) - L(g_{36b}, 3)) \) | |
| [9, 8, 2] | -1 | \( 48L(g_8, 3) \) | |
| [9, 5, 1] | 1 | \( \frac{35}{2} L(g_{11}, 3) \) | |
| [9, 3, 1] | 9 | \( \frac{35}{2} L(g_{27}, 3) \) | |
| [9, 0, 1] | 9 + 6\( \sqrt{3} \)* | \( 27(L(g_{36a}, 3) + L(g_{36b}, 3)) \) | |
| [9, 0, 2] | 27 | \( 27L(g_{72}, 3) + 27l_{24, -3} \) | |
### $V_{22}$

| $\tau$  | $c(\tau)$ | $\text{Re}(\tilde{m}(\tau))$ | remark |
|---------|------------|-------------------------------|--------|
| $[11, 11, 7]$ | $-44$ | $22L(g_{187}, 3) + 22l_{17,-11}$ |        |
| $[11, 11, 5]$ | $-12$ | $\frac{22}{144}L(g_{11}, 3) + 22l_{33,-3}$ | 3.23   |
| $[11, 11, 3]$ | $0$ | $44L(g_{11}, 3)$ |        |
| $[11, 9, 2]$ | $1$ | $28L(g_7, 3)$ | $\S$ |
| $[11, 6, 1]$ | $2$ | $8L(g_8, 3)$ |        |
| $[11, 5, 1]$ | $4$ | $19L(g_{19}, 3)$ |        |
| $[11, 4, 1]$ | $7$ | $84L(g_7, 3)$ | 3.23   |
| $[11, 1, 1]$ | $16$ | $43L(g_{43}, 3)$ |        |
| $[33, 22, 4]$ | $\alpha_2^*$ | $\frac{22}{144}(-\frac{143}{16}L(g_{11}, 3) + (5 - 33)L(h_{440}, 3) + (5 + 33)L(h_{440}', 3))$ | 3.24 $\S$ |
| $[44, 22, 3]$ | $\alpha_2^*$ | $88L(h_{440}, 3) + L(h_{440}', 3)$ | $\S$ |
| $[11, 0, 1]$ | $\alpha_1^*$ | $88L(g_{11}, 3)$ | 3.23   |
| $[11, 0, 2]$ | $22$ | $22L(g_{186}, 3) + 22l_{8,-11}$ |        |

$^{a}$where $\alpha_1, \alpha_2, \alpha_2^*$ are the roots of $x^3 - 20x^2 + 56x - 44$. The coefficient fields of $h_{440}$ and its Galois conjugate $h_{440}'$ are $\Q(\sqrt{33})$, and their full LMFDB labels are 44.3.d.a.

### $V_{12a}$

| $\tau$  | $c(\tau)$ | $\text{Re}(\tilde{m}(\tau))$ | remark |
|---------|------------|-------------------------------|--------|
| $[6, 6, 31]$ | $-1123600$ | $36L(g_{708}, 3) + 36l_{177,-4}$ |        |
| $[6, 6, 17]$ | $-24304$ | $36L(g_{372}, 3) + 36l_{12,-41}$ |        |
| $[6, 6, 11]$ | $-2704$ | $36L(g_{228}, 3) + 36l_{57,-4}$ |        |
| $[6, 6, 7]$ | $-400$ | $36L(g_{132}, 3) + 36l_{33,-4}$ |        |
| $[6, 6, 5]$ | $-112$ | $36L(g_{84}, 3) + 36l_{12,-7}$ |        |
| $[3, 3, 2]$ | $-49$ | $90L(g_{155}, 3) + 54l_{20,-3}$ | 3.24   |
| $[2, 2, 1]$ | $-16$ | $36L(g_{366}, 3) + 24l_{9,-4}$ |        |
| $[3, 3, 1]$ | $-4^*$ | $72L(g_{12}, 3)$ |        |
| $[6, 4, 1]$ | $0$ | $0$ |        |
| $[6, 3, 1]$ | $5$ | $45L(g_{155}, 3)$ |        |
| $[6, 2, 1]$ | $16$ | $60L(g_{208}, 3)$ |        |
| $[6, 0, 1]$ | $32^*$ | $72L(g_{224}, 3)$ |        |
| $[3, 0, 1]$ | $50$ | $117L(g_{12}, 3) + 36l_{12,-4}$ | 3.24   |
| $[2, 0, 1]$ | $96$ | $36L(g_{72}, 3) + 24l_{9,-8}$ |        |
| $[6, 0, 5]$ | $320$ | $36L(g_{120}, 3) + 36l_{8,-15}$ |        |
| $[6, 0, 7]$ | $896$ | $36L(g_{108}, 3) + 36l_{24,-7}$ |        |
| $[6, 0, 13]$ | $10400$ | $36L(g_{302}, 3) + 36l_{8,-39}$ |        |
| $[6, 0, 17]$ | $39200$ | $36L(g_{408}, 3) + 36l_{17,-24}$ |        |
### $V_{12b}$

| $\tau$ | $c(\tau)$ | $\text{Re}(\tilde{m}(\tau))$ | remark |
|--------|------------|--------------------------------|--------|
| [6, 6, 31] | $-1123596$ | $48L(g_{7086}, 3) + 48l_{12,-59}$ |         |
| [6, 6, 17] | $-24300$   | $48L(g_{3726}, 3) + 48l_{12,+3}$ |         |
| [6, 6, 11] | $-2700$    | $48L(g_{2286}, 3) + 48l_{76,-3}$ |         |
| [6, 6, 7]  | $-396$     | $48L(g_{132b}, 3) + 48l_{12,-11}$|         |
| [6, 6, 5]  | $-108$     | $48L(g_{84b}, 3) + 48l_{28,-3}$  |         |
| [3, 3, 2]  | $-45$      | $96L(g_{15a}, 3) + 24l_{4,-15}$  | 3.21   |
| [2, 2, 1]  | $-12$      | $48L(g_{39}, 3) + 48l_{12,-3}$   |         |
| [3, 3, 1]  | 0          | $96l_{4,-3}$                    |         |
| [6, 4, 1]  | 4*         | $64L(g_{8}, 3)$                 |         |
| [6, 3, 1]  | 9          | $60L(g_{15a}, 3)$              |         |
| [6, 2, 1]  | 20         | $80L(g_{20a}, 3)$              |         |
| [6, 0, 1]  | 36*        | $96L(g_{24a}, 3)$              |         |
| [3, 0, 1]  | 54         | $48L(g_{48a}, 3) + 33l_{16,-3}$ |         |
| [2, 0, 1]  | 100        | $192L(g_{8}, 3) + 48l_{24,-3}$  | 3.24   |
| [6, 0, 5]  | 324        | $48L(g_{120b}, 3) + 48l_{40,-3}$|         |
| [6, 0, 7]  | 900        | $48L(g_{168b}, 3) + 48l_{21,-8}$|         |
| [6, 0, 13] | 10404      | $48L(g_{112b}, 3) + 48l_{13,-24}$|         |
| [6, 0, 17] | 39204      | $48L(g_{4086}, 3) + 48l_{136,-3}$|         |

### $V_{20}$

| $\tau$ | $c(\tau)$ | $\text{Re}(\tilde{m}(\tau))$ | remark |
|--------|------------|--------------------------------|--------|
| [10, 10, 11] | $-324$     | $20L(g_{440}, 3) + 20l_{17,-20}$|         |
| [10, 10, 7]  | $-64$      | $20L(g_{206}, 3) + 20l_{12,-15}$| 3.24   |
| [2, 2, 1]   | $-20$      | $20L(g_{100b}, 3) + 16l_{25,-4}$|         |
| [5, 5, 2]   | $-9$       | $60L(g_{15a}, 3) + 30l_{20,-3}$ | 3.24   |
| [10, 10, 3]  | $-4*$      | $40L(g_{206}, 3)$              |         |
| [5, 4, 1]   | $-2$       | $32L(g_{16}, 3)$               |         |
| [10, 6, 1]  | 0          | 0                              |         |
| [10, 5, 1]  | 1          | $15L(g_{15a}, 3)$              |         |
| [10, 4, 1]  | 4          | $24L(g_{24a}, 3)$              |         |
| [10, 2, 1]  | 12         | $36L(g_{36b}, 3)$              |         |
| [10, 0, 1]  | 16*        | $40L(g_{406}, 3)$              |         |
| [10, 0, 3]  | 36         | $20L(g_{120d}, 3) + 20l_{8,-15}$|         |
| [10, 0, 7]  | 196        | $20L(g_{280c}, 3) + 20l_{40,-7}$|         |
| [10, 0, 13] | 1296       | $20L(g_{520c}, 3) + 20l_{65,-8}$|         |
| [10, 0, 19] | 5776       | $20L(g_{760c}, 3) + 20l_{8,-95}$|         |
\[
V_{24}
\]

| \(\tau\) | \(c(\tau)\) | \(\text{Re}(\tilde{m}(\tau))\) | remark |
|-------|-------------|------------------|--------|
| [12, 12, 7] | -32 | \(96L(g_{12}, 3) + 24L_{48,-4}\) | 3.24 |
| [12, 12, 5] | -8 | \(36L(g_{24h}, 3) + 24L_{8,-12}\) | 3.24 |
| [3, 3, 1] | -2 | \(42L(g_{12}, 3) + 24L_{12,-4}\) | 3.24 |
| [12, 9, 2] | 1 | \(150L_{5,-3}\) | \(\mathbb{Q}(-3), \mathbb{S}\) |
| [12, 6, 1] | 4* | \(24L(g_{12}, 3)\) | \(\mathbb{Q}_0\) |
| [12, 4, 1] | 8 | \(80L(g_8, 3)\) | 3.24 |
| [12, 0, 1] | 16* | \(96L(g_{12}, 3)\) | \(\mathbb{Q}_0\) |
| [12, 0, 5] | 64 | \(90L(g_{156}, 3) + 24L_{20,-12}\) | 3.24 |

\[
V_{28}
\]

| \(\tau\) | \(c(\tau)\) | \(\text{Re}(\tilde{m}(\tau))\) | remark |
|-------|-------------|------------------|--------|
| [14, 14, 13] | -175 | \(14L(g_{532}, 3) + 14L_{76,-7}\) | \(\mathbb{Q}_0\) |
| [14, 14, 5] | -7 | \(14L(g_{84a}, 3) + 14L_{12,-7}\) | \(\mathbb{Q}_0\) |
| [7, 7, 2] | -4* | \(98L(g_7, 3)\) | 3.24 |
| [14, 12, 3] | -3 | \(24L(g_{24h}, 3)\) | \(\mathbb{Q}_0\) |
| [7, 5, 1] | -1 | \(12L(g_{12}, 3)\) | \(\mathbb{Q}_0\) |
| [42, 28, 5] | 5 - \(3/2\) | \(14\sqrt{2}(L(h_{56c}, 3) - L(h'_{56c}, 3))\) | \(\mathbb{S}\) |
| [14, 7, 1] | 0 | \(7L(g_{7}, 3)\) | \(\mathbb{Q}_0\) |
| [14, 6, 1] | 1 | \(10L(g_{206}, 3)\) | \(\mathbb{Q}_0\) |
| [14, 4, 1] | 5 | \(20L(g_{408}, 3)\) | \(\mathbb{Q}_0\) |
| [14, 2, 1] | 9 | \(26L(g_{528}, 3)\) | \(\mathbb{Q}_0\) |
| [14, 0, 1] | 5 + \(3/2\) | \(14L(h_{56c}, 3) + L(h'_{56c}, 3)\) | \(\mathbb{Q}_0\) |
| [7, 0, 1] | 14 | \(133L(g_7, 3) + 14L_{28,-4}\) | 3.24 |
| [14, 0, 3] | 21 | \(14L(g_{168b}, 3) + 14L_{24,-7}\) | \(\mathbb{Q}_0\) |
| [14, 0, 5] | 45 | \(14L(g_{280d}, 3) + 14L_{40,-7}\) | \(\mathbb{Q}_0\) |

\(^a\)The coefficient fields of \(h_{56c}\) and its Galois conjugate \(h'_{56c}\) are \(\mathbb{Q}(\sqrt{2})\), and their full LMFDB labels are 56.3.h.c.
We are in a good position to discuss the modularity problem for K3 surfaces. For an algebraic K3 surface $Y$ over $\mathbb{Q}$, we denote by $Y_p$ the reduction of $Y$ modulo $p$. The zeta function of $Y_p$ at “good” primes is of the form

$$Z_{Y_p}(t) = \frac{1}{(1 - t)(1 - p^2t)P_p(T)},$$

where $P_p(T)$ is a polynomial of degree 22 and $P_p(0) = 1$. Furthermore, $P_p(T)$ factors as $P_p(T) = Q_p(T)R_p(T)$, where $Q_p(T)$ and $R_p(T)$ come from the transcendental and algebraic cycles respectively. Let $T(Y) = \text{Pic}(Y)^+ \subset H^2(Y,\mathbb{Z})$ be the transcendental lattice of $Y$. We define

$$L(T(Y), s) := (*) \prod_{p \text{ good}} \frac{1}{Q_p(p^{-s})},$$

(*) is the product of the Euler factors corresponding to the primes of bad reduction.

If $Y$ is singular, then $T(Y)$ is a 2-dimensional lattice, which can be expressed through its intersection form. We define the discriminant of $Y$ as the discriminant of the intersection form.

**Theorem 5.8** (Livné[23]). Let $Y$ be a singular K3 surface defined over $\mathbb{Q}$ with discriminant $d$. Then there exists a newform $g$ of weight 3 with CM by $\mathbb{Q}(\sqrt{d})$ such that $L(T(Y), s) = L(g, s)$.

**Conjecture 5.9.** Each above K3 hypersurface (after desingularization if necessary) is modular with the weight-3 newform given in the column $\text{Re}(\tilde{m}(\tau))$.

**Remark 5.10.** It is not very difficult to verify one by one as did in [4]. But we do not know a conceptional uniform proof. We also suspect that some of them, especially those in $V_{12a}, V_{12b}, V_{20}, V_{24}, V_{28}, V_{30}$ (and $B_{6a}, B_{6b}$), are not singular (i.e., have Picard rank less than 20). In those cases, $Q_p(T)$ would have some trivial factors.

---

| $\tau$ | $c(\tau)$ | $\text{Re}(\tilde{m}(\tau))$ | remark |
|--------|------------|-----------------------------|--------|
| $[15, 15, 17]$ | $-363$ | $15L(g_{795c}, 3) + 15l_{265}, -3$ |        |
| $[15, 15, 13]$ | $-135$ | $15L(g_{555c}, 3) + 15l_{37}, -15$ |        |
| $[15, 15, 11]$ | $-75$ | $15L(g_{435c}, 3) + 15l_{145}, -3$ |        |
| $[15, 15, 7]$ | $-15$ | $15L(g_{195c}, 3) + 15l_{13}, -15$ |        |
| $[3, 3, 1]$ | $-3$ | $15L(g_{75b}, 3) + 12l_{25}, -3$ |        |
| $[15, 15, 4]$ | $0$ | $30L(g_{15a}, 3)$ |        |
| $[15, 7, 1]$ | $1$ | $-\frac{1}{2} L(g_{11}, 3)$ |        |
| $[15, 20, 7]$ | $1 - 2i^*$ | $25(L(g_{20a}, 3) + \frac{1}{2}L(g_{20b}, 3))$ | $\S$ |
| $[15, 10, 2]$ | $1 + 2i^*$ | $25(L(g_{20a}, 3) + \frac{1}{2}L(g_{20b}, 3))$ |        |
| $[15, 6, 1]$ | $3$ | $12L(g_{24a}, 3)$ |        |
| $[15, 5, 1]$ | $5$ | $\frac{3}{2} L(g_{35b}, 3)$ |        |
| $[15, 3, 1]$ | $9$ | $\frac{3}{2} L(g_{51b}, 3)$ |        |
| $[15, 0, 1]$ | $12^*$ | $75L(g_{15a}, 3)$ | $3, 24$ |
| $[15, 0, 2]$ | $15$ | $15L(g_{120d}, 3) + 15l_{40}, -3$ |        |
| $[15, 0, 4]$ | $30$ | $\frac{645}{8} L(g_{15a}, 3) + 15l_{12}, -20$ | $3, 24$ |
6. Exotic Relations

If one family of hypersurface $\tilde{f}_c$ is obtained from another family $f_c$ by pulling back some degree $d$ cover of the base space, then we have the following relation

$$m(\tilde{f}_c) = \frac{1}{d}m(f_c).$$

We call such relations trivial. Other relations among Mahler measures of families are called exotic. As a convention in this section, we shall write $m_i(c)$ for $m(f_c)$ where $f$ is the Laurent polynomial labelled by $V_i$ (or $B_i$ if $i = 6a, 6b$). However, for the function $\tilde{m}$ we will switch to the modular parameter $\tau$, namely, $\tilde{m}_i(\tau) = \tilde{m}_i(c(\tau))$.

M. Rogers proved in [24, Theorem 2.5] the following exotic relations between the Mahler measures of two families:

(6.1) \[ m_{6b}(3(z + z^{-1})) = \frac{1}{20}m_4\left(\frac{9(z^2 + 3)^4}{z^6}\right) + \frac{3}{20}m_4\left(\frac{(3 + z^{-2})^4}{z^{-6}}\right). \]

(6.2) \[ m_{24}(z) = \frac{8}{15}m_6\left(\frac{(z - 4)^3}{z}\right) - \frac{1}{15}m_6\left(\frac{(16 - z)^3}{z^2}\right). \]

We shall see that such relations arise naturally from the modular relations among the modular functions. As the first step, we observe that the following relations are the direct consequence of Theorem 2.5 and the column $c(\tau)$ of Table 2. As a trivial remark, we mention that the relations hold unconditionally if we replace $\tilde{m}$ by the right side of the equation (2.2).

**Corollary 6.1.** We have the following relations for $\tau$ lying in the fundamental domains containing $i\infty$ of the modular functions $c(\tau)$ for both sides.

(6.3) \[ \tilde{m}_8(\tau) = \frac{1}{5}\tilde{m}_4(\tau) + \frac{2}{5}\tilde{m}_4(2\tau), \]

(6.4) \[ \tilde{m}_{12}(\tau) = \frac{1}{2}(\tilde{m}_{12a}(\tau) + \tilde{m}_{12b}(\tau)), \]

(6.5) \[ \tilde{m}_{10}(\tau) = \frac{1}{20}\tilde{m}_4(\tau) + \frac{3}{40}\tilde{m}_4(2\tau) + \frac{1}{5}\tilde{m}_4(4\tau), \]

(6.6) \[ \tilde{m}_{18}(\tau) = \frac{1}{10}\tilde{m}_6(\tau) + \frac{3}{10}\tilde{m}_6(3\tau), \]

(6.7) \[ \tilde{m}_{12a}(\tau) = \frac{1}{5}\tilde{m}_6(\tau) + \frac{2}{5}\tilde{m}_6(2\tau), \]

(6.8) \[ \tilde{m}_{12b}(\tau) = \frac{1}{10}\tilde{m}_4(\tau) + \frac{3}{10}\tilde{m}_4(3\tau), \]

(6.9) \[ \tilde{m}_{20}(\tau) = \frac{1}{5}\tilde{m}_{10}(\tau) + \frac{2}{5}\tilde{m}_{10}(2\tau), \]

(6.10) \[ \tilde{m}_{24}(\tau) = \frac{1}{15}\tilde{m}_6(\tau) - \frac{1}{15}\tilde{m}_6(2\tau) + \frac{4}{15}\tilde{m}_6(4\tau), \]

(6.11) \[ \tilde{m}_{28}(\tau) = \frac{1}{5}\tilde{m}_{14}(\tau) + \frac{2}{5}\tilde{m}_{14}(2\tau), \]

(6.12) \[ \tilde{m}_{30}(\tau) = \frac{1}{10}\tilde{m}_{10}(\tau) + \frac{3}{10}\tilde{m}_{10}(3\tau). \]

To convert the above relations to exotic relations among Mahler measures in the parameter $c$, we need the relations among the corresponding Hauptmoduln. Some of the relations in the following lemma may be known in the literature, and most likely appeared as the defining equations of certain modular curves. We
found these relations by numerical experiment with Matlab [25]. To rigorously verify them, we can multiply some appropriate $\eta$-quotient to make both sides holomorphic, then check the equality using Sturm’s theorem.

It is more convenient to switch to a different $\eta$-quotient representation for the following Hauptmoduln:

\[(h + 4h^{-1})(h + 8h^{-1}), \ h = \frac{1241}{24182} \text{ for } V_{16},\]
\[(h + 4h^{-1})^2, \ h = \frac{1^{131}}{4^{112}} \text{ for } V_{24},\]
\[h^3 + 8h^{-3} + 5, \ h = \frac{1^{171}}{2^{142}} \text{ for } V_{28}.\]

**Lemma 6.2.** We have the following modular equations

\[(6.13) \quad z + 16z^{-1} = y, \quad y = \frac{2^{24}}{3^{12}4^{112}}, \quad z = \frac{4^{3}}{14},\]
\[(6.14) \quad z + 9 + 27z^{-1} = y, \quad z = \frac{3^{12}}{1690}, \quad z = \frac{1^{3}}{93},\]
\[(6.15) \quad z = \frac{y^4}{(y + 4)(y + 8)^2}, \quad y = \frac{2^{44}}{14^{48}}, \quad z = \frac{1^{24}}{2424},\]
\[(6.16) \quad z + 27z^{-1} + 54 = (y + 16)^3y^{-2}, \quad y = \frac{1^{12}}{6^{12}}2^{24}, \quad z = \frac{1^{12}}{6^{12}}2^{12};\]
\[(6.17) \quad z + 27z^{-1} + 54 = (y + 4)^3y^{-1}, \quad y = \frac{1^{12}}{6^{12}}2^{24}, \quad z = \frac{1^{12}}{6^{12}}2^{12};\]
\[(6.18) \quad z + 64z^{-1} = (y^2 + 27)^2y^{-3}, \quad y = \frac{1^{2^{2}}3^{2}}{3^{2}6^{2}}, \quad z = \frac{1^{12}}{24};\]
\[(6.19) \quad z + 125z^{-1} + 22 = (y + 8)^2(y + 4)^{-1}, \quad y = \frac{1^{45}}{2^{14}10^4}, \quad z = \frac{1^{6}}{5^{6}}2^{4};\]
\begin{align*}
\quad & z + 49z^{-1} + 14 = (y + 4)^2y^2 \\
\quad & y = \frac{137^3}{2^{13}4^3}, \quad z = \frac{14}{74}; \\
\quad & z + 49z^{-1} + 14 = (y + 2)^3y^1 \\
\quad & y = \frac{137^3}{2^{14}4^5}, \quad z = \frac{24}{144}; \\
\quad & z + 125z^{-1} + 22 = (y + 9 + 27y^{-1})^2y^{-1} \\
\quad & y = \frac{125^2}{3^213^2}, \quad z = \frac{16}{56}; \\
\quad & z + 125z^{-1} + 22 = (y + 3 + 3y^{-1})^2y \\
\quad & y = \frac{125^2}{3^213^2}, \quad z = \frac{36}{156}.
\end{align*}

Next we translate the condition on \( \tau \) in Corollary \ref{corollary6.1} to the condition on the parameter \( c = c(\tau) \). We need to pick \( \tau \) such that \( \tau \) lie in the fundamental domain of \( c(\tau) \) containing \( i\infty \) for each term \( \tilde{m}_i(c(\tau)) \) in the relation. Note that this is not a problem if each \( c_i \) is sufficiently large.

**Theorem 6.3.** For \( |y| \) (or \( |z| \)) sufficiently large, we have the following relations on Mahler measures

\begin{align*}
(6.22) & \quad m_8 \left( z + 256z^{-1} + 32 \right) = \frac{1}{5} m_4 \left( \frac{(z + 32)^4}{z^2(z + 16)} \right) + \frac{2}{5} m_4 \left( \frac{(z + 8)^4}{z(z + 16)} \right), \\
(6.23) & \quad m_{12}(y) = \frac{1}{2} \left( m_{12a}(y + y^{-1} - 2) + m_{12b}(y + y^{-1} + 2) \right), \\
(6.24) & \quad m_{16}(y + 32y^{-1} + 12) = \frac{1}{20} m_4 \left( \frac{(y^2 + 32y + 128)^4}{y^4(y + 4)(y + 8)^2} \right) + \frac{3}{10} m_4 \left( \frac{(y^2 + 8y + 32)^4}{y^4(y + 4)^2(y + 8)^2} \right) + \frac{1}{5} m_4 \left( \frac{(y^2 + 8y + 8)^4}{y^4(y + 4)^2(y + 8)^2} \right), \\
(6.25) & \quad m_{18}(z + 27z^{-1} + 9) = \frac{1}{10} m_6 \left( \frac{(z + 9)^6}{z^3(z^2 + 9z + 27)} \right) + \frac{3}{10} m_6 \left( \frac{(z + 3)^6}{z(z^2 + 9z + 27)} \right), \\
(6.26) & \quad m_{12a}(y + 64y^{-1} + 16) = \frac{1}{5} m_6 \left( (y + 16)^3y^{-2} \right) + \frac{2}{5} m_6 \left( (y + 4)^3y^{-1} \right), \\
(6.27) & \quad m_{12b}(y + 81y^{-1} + 18) = \frac{1}{10} m_4 \left( (y + 27)^4y^{-3} \right) + \frac{3}{10} m_4 \left( (y + 3)^4y^{-1} \right), \\
(6.28) & \quad m_{20}(y + 16y^{-1} + 8) = \frac{1}{5} m_{10} \left( (y + 4)(1 + 8y^{-1})^2 \right) + \frac{2}{5} m_{10} \left( (y + 2)(1 + 4y^{-1}) \right), \\
(6.29) & \quad m_{24}(y + 16y^{-1} + 8) = \frac{1}{15} m_6 \left( \frac{(y + 8)^6}{y^2(y + 4)} \right) + \frac{1}{15} m_6 \left( \frac{(y + 2)^2(y - 4)^2(y + 8)^2}{y^2(y + 4)^2} \right) + \frac{4}{15} m_6 \left( \frac{(y + 2)^6}{y(y + 4)} \right), \\
(6.30) & \quad m_{28}(y + 8y^{-1} + 5) = \frac{1}{15} m_4 \left( (y + 4)^4y^{-2} \right) + \frac{2}{5} m_4 \left( (y + 2)^3y^{-1} \right), \\
(6.31) & \quad m_{30}(y + 9y^{-1} + 6) = \frac{1}{10} m_{10} \left( (y^2 + 9y + 27)^2y^{-3} \right) + \frac{3}{10} m_{10} \left( (y^2 + 3y + 3)^2y^{-1} \right).
\end{align*}

**Proof.** As an illustration, we only prove the first relation. The others can be verified in a similar fashion. Let \( h_1 = \frac{1^4}{2^2} \) and \( h_2 = \frac{2^4}{4^2} \), then \( h_1^4 = y^{-1}z^3 \) and \( h_2^6 = y^1z^3 \). We have that

\begin{align*}
\quad & c_8(\tau) = y^2, \\
\quad & c_{14}(\tau) = (h_1^3 + 64h_1^{-3})^2 = y^{-1}z^3 + 64^2yz^{-3} + 128, \\
\quad & c_{14}(2\tau) = (h_2^3 + 64h_2^{-3})^2 = y^1z^3 + 64^2y^{-1}z^{-3} + 128.
\end{align*}

Now the relation follows from the first equation of Lemma \ref{lemma6.2} and the first equation of Corollary \ref{corollary6.1}.

In the following table we indicate which exotic relation follows from which relations in Lemma \ref{lemma6.2} and Corollary \ref{corollary6.1}. \[\square\]
Remark 6.4. In particular, Roger’s relation (6.1) is a consequence of (6.22) and the trivial relation (see Proposition 7.1). But our relation for $V_{24}$ and $V_{6}$ is different from Roger’s (6.2) because he used Bertin’s (different) modular parametrization of the family $V_{24}$. The relation (6.2) can be easily obtained from Bertin’s parametrization by the same method.

Example 6.5. Let us examine the first relation when $z = -32$ and $z = 16$. When $z = -32$, the relation reads $m_{8}(-8) = \frac{1}{5}m_{4}(0) + \frac{3}{4}m_{4}(648)$ which agrees with our table. But when $z = 16$, the relation reads $m_{8}(64) = \frac{1}{5}m_{4}(648) + \frac{3}{4}m_{4}(648)$. This does not contradict our table because we are unable to choose $\tau$ lying in both fundamental domains containing $i\infty$. For our choice in the table, $\tau = \frac{i}{2}$ lies in the fundamental domain containing $i\infty$ for $\Gamma^+_0(4)$ but not for $\Gamma^+_0(2)$.

7. Appendix for index $> 1$

For completeness, we will treat those families corresponding to Fano 3-folds of index $> 1$. The statement for their Mahler measures follows easily from the trivial relations below.

| label | $N$ | $f$ | $-(\alpha_2, \alpha_1, \alpha_0; \beta_1, \beta_0)$ | $e(\tau)$ |
|-------|-----|-----|---------------------------------|----------|
| $\mathbb{P}^3$ | 8 | $x_1 + x_2 + x_3 + (x_1x_2x_3)^{-1}$ | $(0, 0, 0, 256; 0, 0)$ | $(0, 0, -1, 1)$ |
| $Q$ | 9 | $x_1 + x_2 + x_3 + (x_1x_2)^{-1} + (x_1x_3)^{-1}$ | $(0, 0, 108; 0, 0)$ | $\frac{40}{3}(0, -1, 1)$ |
| $B_1$ | 2 | $(x_1x_2 + x_2x_3 + x_3x_1 + 1)^3/(x_1x_2x_3)$ | $(1728, 0, 0; 0, 240, 0)$ | $E_4^{1/2}(2\tau)$ |
| $B_2$ | 4 | $(x_1x_2 + x_2x_3 + x_3x_1 + 1)^2/(x_1x_2x_3)$ | $(0, 256, 0; -64, 0, 0)$ | $(0, 0, -1, 1)$ |
| $B_3$ | 6 | $(x_1^{-1} + x_2^{-1})(1 + x_3^{-1})(x_3 + x_1x_2 + 1)$ | $(0, 108, 0; -12, 0, 0)$ | $(0, 1, 0, -1)$ |
| $B_4$ | 8 | $(x_1 + x_1^{-1})(x_2 + x_2^{-1})(x_3 + x_3^{-1})$ | $(0, 64, 0; 0, 0, 0)$ | $(0, -1, 0, 1)$ |
| $B_5$ | 10 | $x_1 + x_2 + x_3 + x_1^{-1} + x_2^{-1} + x_3^{-1} + (x_1x_2x_3)^{-1}$ | $(0, 11, 0; 16, 0, 1)$ | $\frac{5}{2}(0, -1, 0, 1)$ |
| $B_{6a}$ | 12 | $x_1 + x_2 + x_3 + x_1^{-1} + x_2^{-1} + (x_1x_2x_3)^{-1}$ | $(0, 28, 0; 128, 0, 4)$ | $\frac{1}{7}(0, -1, 0, -1, 1, 1)$ |
| $B_{6b}$ | 12 | $x_1 + x_2 + x_3 + x_1^{-1} + x_2^{-1} + x_3^{-1}$ | $(0, 40, 0; 144, 0, 8)$ | $(0, -1, 0, 1, -1, 1)$ |

Proposition 7.1. We have the following trivial relations

\[
\tilde{m}(\tau, B_i) = \frac{1}{2} \tilde{m}(2\tau, V_{2i}),
\]
\[
\tilde{m}(\tau, Q) = \frac{1}{3} \tilde{m}(3\tau, V_{6}),
\]
\[
\tilde{m}(\tau, \mathbb{P}^3) = \frac{1}{4} \tilde{m}(4\tau, V_{4}).
\]

Here, our convention is that $2(6a) = 12a$ and $2(6b) = 12b$. 

Proof. This is rather clear from Theorem 2.6 if we compare the columns $e(\tau)$ of Table 1 and Table 3.

As a sample, we give the corresponding table for $B_{6b}$ because it was intensively studied in the literature [5, 7, 27, 10]. One can compare it with the table for $V_{12b}$. Note also that $m(c) = m(-c)$ for $B_{6b}$.

| $\tau$   | $e(\tau)$ | $\text{Re}(\tilde{m}(\tau))$ | remark |
|----------|-----------|-------------------------------|--------|
| (12, 6, 1) | 0         | $24l_{4,-3}$                |        |
| (24, 8, 1) | 2         | $32L(g_8, 3)$               | [5]    |
| (24, 9, 1) | 3         | $30L(g_{15b}, 3)$           | [5]    |
| (24, 0, 1) | 6         | $48L(g_{24a}, 3)$           |        |
| (24, 0, 3) | 10        | $24L(g_8, 3) + 24l_{24,-3}$ | [7]    |
| (24, 0, 5) | 18        | $24L(g_{12b}, 3) + 12l_{16b,-3}$ | [7] |
| (24, 0, 7) | 30        | $24L(g_{168a}, 3) + 24l_{21,-3}$ | [7] |
| (24, 0, 13) | 102      | $24L(g_{312b}, 3) + 24l_{136,-3}$ | [7] |
| (24, 0, 17) | 198      | $24L(g_{408b}, 3) + 24l_{136,-3}$ | [7] |

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