INTRODUCTION

Let $S = K[x_1, \ldots, x_n, y_1, \ldots, y_n]$ denote the polynomial ring in $2n$ variables over the field $K$. In general, given any subset $F$ of $\{x_1, \ldots, x_n\}$, we define the squarefree monomial $u_F$ of $S$ of degree $n$ by setting

$$u_F = \prod_{x_i \in F} x_i \prod_{x_j \in \{x_1, \ldots, x_n\} \setminus F} y_j.$$ 

Given a collection $S$ of subsets of $\{x_1, \ldots, x_n\}$ one defines the ideal $I_S$ generated by the monomials $u_F$ with $F \in S$. Of particular interest are collections of sets which naturally arise in combinatorics.

The first example of this kind which appeared in the literature is the following: let $P = \{x_1, \ldots, x_n\}$ be a finite partially ordered set. A poset ideal of $P$ is a subset $\alpha$ of $P$ with the property that if $x_i \in \alpha$ and $x_j \leq x_i$, then $x_j \in \alpha$. In particular the empty set as well as $P$ itself is a poset ideal of $P$. The squarefree monomial ideal $I(P) \subset S$ which is generated by those monomials $u_\alpha$ for which $\alpha$ is a poset ideal of $P$ has played an important role in combinatorial and computational commutative algebra ([8], [5]). The ideal $I(P)$ has the remarkable property that it has a linear resolution [9, Theorem 9.1.8].

Now let $I_A(P) \subset S$ be the squarefree monomial ideal which is generated by those monomials $u_\beta$ for which $\beta$ is an antichain of $P$. (Recall that an antichain of $P$ is a subset $\beta \subset P$ for which any two elements $x_i$ and $x_j$ with $i \neq j$ belonging to $\beta$ are incomparable in $P$.) The toric ring generated by $u_\alpha$ with $u_\alpha \in I(P)$ and the toric ring generated by $u_\beta$ with $u_\beta \in I_A(P)$ have similar properties. In particular, both are algebras with straightening laws on suitable distributive lattices, see [10] and [12]. Thus one may expect that $I(P)$ and $I_A(P)$ have similar properties as well.

Similarly we may also consider the squarefree monomial ideal $I_C(P) \subset S$ which is generated by those monomials $u_\gamma$ for which $\gamma$ is a chain of $P$. (Recall that a chain of $P$ is a totally ordered subset of $P$.) As expected, each of the antichain ideal $I_A(P)$ and the chain ideal $I_C(P)$ has linear quotients will be shown in Theorem 3.1.
However, far beyond the study of antichain ideals and chain ideals of partially ordered sets, the study of the present paper will be done in a much more general situation. Since the set of antichains of $P$ as well as the set of chains of $P$ is clearly a simplicial complex on the vertex set $\{x_1, \ldots, x_n\}$, it is natural to study more generally ideals $I_S$ where $S$ is the set of faces of any simplicial complex. Thus we introduce the face ideal $J_\Delta$ of a simplicial $\Delta$ on $\{x_1, \ldots, x_n\}$. In other words, the face ideal $J_\Delta$ is the ideal of $\mathcal{S}$ which is generated by the monomials $u_F$ with $F \in \Delta$.

Theorem 1.1 gives the structure of the Alexander dual $(J_\Delta)^\vee$ of $J_\Delta$. Somewhat surprisingly, it turns out that $(J_\Delta)^\vee$ is a whisker complex, which is a generalization of whisker graphs, first introduced by Villarreal [15] and further studied and generalized in [1], [7], [11] and [14]. A simple polarization argument shows that $(J_\Delta)^\vee$ is Cohen–Macaulay. Hence, again, $J_\Delta$ has a linear resolution. We also describe the explicit minimal free resolution of $J_\Delta$ (Theorem 1.3) and compute the Betti numbers of $J_\Delta$ (Corollary 1.4). We would like to mention that whisker complexes are special classes of grafted complexes as introduced by Faridi [6].

In Section 2, we show that the face ideal $J_\Delta$ of a simplicial complex $\Delta$ has linear quotients. This fact implies that the independence complex of the whisker complex of an arbitrary simplicial complex is shellable (Corollary 2.2). Recall that if $\Gamma$ is a simplicial complex and $I(\Gamma)$ its facet ideal, then the simplicial complex $\Delta$ with $I_\Delta = I(\Gamma)$ is called the independence complex of $\Gamma$. Here $I_\Delta$ denotes the Stanley–Reisner ideal of $\Delta$. The faces of $\Delta$ are those subsets of the vertex set of $\Gamma$ which do not contain any facet of $\Gamma$.

Dochtermann and Engström [3] even showed that the independence complex of whisker graphs are pure and vertex decomposable, which in particular implies that the independence complex of any whisker graph is shellable, see also [11].

Finally, in Section 4, we introduce the concept of higher dimensional whisker complexes, which is a generalization of whisker complexes introduced in Section 1. Theorem 4.1 guarantees that the independence complex of the higher dimensional whisker complex of an arbitrary simplicial complex is shellable. Thus our Theorem 4.1 generalizes the result of Dochtermann and Engström regarding shellability.

A. Engström kindly informed us that his student Lauri Loiskekoski in his Master Thesis [13] has also introduced what we call the face ideal of a simplicial complex, and, among other results, has shown that face ideals have linear resolutions, cf. our Corollary 1.2.

1. Face ideals and whisker complexes

Let $S = K[x_1, \ldots, x_n, y_1, \ldots, y_n]$ denote the polynomial ring in $2n$ variables over the field $K$ and $\Delta$ a simplicial complex on the vertex set $\{x_1, \ldots, x_n\}$.

The whisker complex of $\Delta$ is the simplicial complex $W(\Delta)$ on the vertex set $\{x_1, \ldots, x_n, y_1, \ldots, y_n\}$ which is obtained from $\Delta$ by adding the facets $\{x_i, y_i\}$, called whiskers, for $i = 1, \ldots, n$.

For each face $F \in \Delta$ we associate the monomial $u_F \in S$ defined by

$$u_F = x_F y_F^c,$$
where
\[ x_F = \prod_{x_i \in F} x_i \quad \text{and} \quad y_F = \prod_{x_j \in \{x_1, \ldots, x_n\}\setminus F} y_j. \]

The ideal of \( S \) generated by those squarefree monomials \( u_F \) with \( F \in \Delta \) is called the face ideal of \( \Delta \) and is denoted by \( J_\Delta \). As usual we write \( I_\Delta \subset K[x_1, \ldots, x_n] \) for the Stanley–Reisner ideal (\[8\] p. 16) of \( \Delta \) and \( I(\Delta) \subset K[x_1, \ldots, x_n] \) for the facet ideal of \( \Delta \).

Let, in general, \( I \subset S \) be a squarefree monomial ideal of \( S \) with \( I = \bigcap_{k=1}^m P_k \) where the ideals \( P_k \) are the minimal prime ideals of \( I \). Each \( P_k \) is generated by variables. The Alexander dual \( I^\vee \) of \( I \) is defined to be the ideal of \( S \) generated by the squarefree monomials \( u_1, \ldots, u_m \), where \( u_k = (\prod_{x_i \in P_k} x_i)(\prod_{y_j \in P_k} y_j) \). In particular, \( (J_\Delta)^\vee = I_{\Delta^\vee} \) where \( \Delta^\vee \) is the Alexander dual of \( \Delta \).

**Theorem 1.1.** Let \( \Delta \) be a simplicial complex on \( \{x_1, \ldots, x_n\} \) and \( J_\Delta \subset S \) the face ideal of \( \Delta \). Then one has
\[ (J_\Delta)^\vee = I(W(\Gamma)), \]
where \( \Gamma \) is the simplicial complex on \( \{y_1, \ldots, y_n\} \) with
\[ I_{\Delta'} = I(\Gamma), \]
where \( \Delta' \) is the copy of \( \Delta \) on \( \{y_1, \ldots, y_n\} \), i.e., \( \Delta' = \{\{y_i : x_i \in F\} : F \in \Delta\} \).

**Proof.** Let \( \Delta^\sharp \) denote the simplicial complex on \( \{x_1, \ldots, x_n, y_1, \ldots, y_n\} \) with \( J_\Delta = I(\Delta^\sharp) \). It then follows that a squarefree monomial \( x_{i_1} \cdots x_{i_t} y_{j_1} \cdots y_{j_t} \) belongs to the minimal system of monomial generators of \( J_\Delta^\sharp \) if and only if \( \{x_{i_1}, \ldots, x_{i_t}, y_{j_1}, \ldots, y_{j_t}\} \) is a minimal vertex cover (\[9\] p. 156) of \( \Delta^\sharp \).

Since each facet of \( \Delta^\sharp \) contains either \( x_i \) or \( y_i \) for \( 1 \leq i \leq n \), it follows that \( \{x_i, y_i\} \) is a minimal vertex cover of \( \Delta^\sharp \) for \( 1 \leq i \leq n \).

We claim \( F = \{y_j : j \in B\} \), where \( B \subset [n] = \{1, \ldots, n\} \), is a vertex cover of \( \Delta^\sharp \) if and only if \( F \not\subset \Delta' \). In fact, if \( F \in \Delta' \), then \( \{x_j : j \in B\} \cup \{y_i : i \in [n] \setminus B\} \) is a facet of \( \Delta^\sharp \). Hence \( F \) cannot be a vertex cover of \( \Delta^\sharp \). Conversely suppose that \( F \not\subset \Delta' \). Then \( \{x_j : j \in C\} \cup \{y_i : i \in [n] \setminus C\} \) be a facet of \( \Delta^\sharp \). Then \( \{x_j : j \in C\} \) is a facet of \( \Delta \). Since \( F \notin \Delta' \), it follows that \( B \notin C \). Thus \( B \cap ([n] \setminus C) \neq \emptyset \). Hence \( F \) is a vertex cover of \( \Delta^\sharp \).

Finally, suppose that \( G = \{x_i : i \in A\} \cup \{y_j : j \in B\} \) is a minimal vertex cover of \( \Delta' \), where \( A \subset [n], B \subset [n] \) with \( A \cap B = \emptyset \). We claim \( A = \emptyset \). In fact, if \( A \neq \emptyset \), then \( F = \{y_j : j \in B\} \) cannot be a vertex cover of \( \Delta^\sharp \). Hence \( F \) must be a face of \( \Delta' \). Then \( \{x_i : i \in B\} \cup \{y_j : j \in [n] \setminus B\} \) is a facet of \( \Delta^\sharp \). Since \( A \cap B = \emptyset \), it follows that \( G \) cannot be a vertex cover of \( \Delta' \).

In consequence, the minimal vertex covers of \( \Delta^\sharp \) are either \( \{x_i, y_i\} \) for \( 1 \leq i \leq n \) or the minimal nonfaces of \( \Delta' \). Since \( I_{\Delta'} = I(\Gamma) \), the the minimal nonfaces of \( \Delta' \) coincides with the facets of \( \Gamma \). It then follows that \( J_\Delta^\sharp = I(W(\Gamma)) \), as desired.

**Corollary 1.2.** Let \( \Delta \) be a simplicial complex. Then \( J_\Delta \) has a linear resolution.

**Proof.** By applying the Eagon-Reiner Theorem \[8\] (see also \[9\] Theorem 8.1.9) it suffices to show that \( I(W(\Gamma)) \) is a Cohen–Macaulay ideal. This is a well-known fact: Notice that \( I(W(\Gamma)) \) is the polarization of \( L = (I(\Gamma), y_1^2, \ldots, y_n^2) \). Since
dim \( K[y_1, \ldots, y_n]/L = 0 \). It follows that \( L \) is a Cohen-Macaulay ideal. Hence by [9, Corollary 1.6.3], \( I(W(\Gamma)) \) is a Cohen-Macaulay ideal as well.

The next result describes the precise structure of the resolution of \( S/J \). For a simplicial complex \( \Delta \) on \([n] = \{1, \ldots, n\}\) we let \( \overline{\Delta} = \{[n] \setminus F \colon F \in \Delta\} \).

**Theorem 1.3.** Let \( \Delta \) be a simplicial complex of dimension \( d - 1 \) on the vertex set \([n]\). For each integer \( j \geq 1 \), let \( F_j \) be the free \( S \)-module with basis elements \( e_{G,H} \) indexed by \( G \in \Delta \) and \( H \in \overline{\Delta} \) satisfying the condition that \( |G \cap H| = j - 1 \) and \( G \cup H = [n]\). Furthermore, we set \( F_0 = S \). For each \( j = 2, \ldots, d \) we define the \( S \)-linear map

\[
\partial_j : F_j \to F_{j-1}, \quad \text{with} \quad \partial_j(e_{G,H}) = \sum_{i \in G \cap H} (-1)^{\sigma(G \cap H,i)}(x_ie_{G \setminus \{i\},H} - y_ie_{G,H \setminus \{i\}}),
\]

where \( \sigma(G \cap H, i) = |\{j \in G \cap H \colon j < i\}| \). Then

\[
\mathbb{F}_\Delta : 0 \longrightarrow F_d \overset{\partial_d}{\longrightarrow} F_{d-1} \overset{\partial_{d-1}}{\longrightarrow} \cdots \overset{\partial_2}{\longrightarrow} F_1 \overset{\partial_1}{\longrightarrow} F_0 \longrightarrow 0,
\]

is the graded free resolution of \( S/J_\Delta \), where \( \partial_1(e_{G,H}) = x_G y_H \) for all \( e_{G,H} \in F_1 \).

**Proof.** We first show that \( \mathbb{F} \) is a complex. One immediately verifies that \( \partial_1 \circ \partial_2 = 0 \). Now let \( j > 2 \) and \( e_{G,H} \in F_j \). Set \( L = G \cap H \). Then

\[
\partial_j(\partial_{j-1}(e_{G,H})) = \partial_{j-1} \left( \sum_{i \in L} (-1)^{\sigma(L,i)}(x_i e_{G \setminus \{i\},H} - y_i e_{G,H \setminus \{i\}}) \right)
= \sum_{i \in L} (-1)^{\sigma(L,i)}[x_i \left( \sum_{k \in L \setminus \{i\}} (-1)^{\sigma(L \setminus \{i\},k)}(x_k e_{G \setminus \{i,k\},H} - y_k e_{G,H \setminus \{i,k\}}) \right)]
- y_i \left( \sum_{k \in L \setminus \{i\}} (-1)^{\sigma(L \setminus \{i\},k)}(x_k e_{G \setminus \{i,k\},H \setminus \{i\}} - y_k e_{G,H \setminus \{i,k\}}) \right) + \sum_{i,k \in L, i < k} \sigma_{ik} \left( x_i x_k e_{G \setminus \{i,k\},H} - x_i y_k e_{G \setminus \{i,k\},H \setminus \{i\}} + y_i y_k e_{G,H \setminus \{i,k\}} \right).
\]

where \( \sigma_{ik} = (-1)^{\sigma(L,i) + \sigma(L \setminus \{i\},k) + \sigma(L \setminus \{k\},i)} \). Since \( \sigma_{ik} = 0 \) for all \( i < k \) it follows that \( \partial_j(\partial_{j-1}(e_{G,H})) = 0 \), as desired.

We will prove the acyclicity of \( \mathbb{F} \) by induction on \(|\Delta|\). The assertion is trivial if \( \Delta = \emptyset, \{1\}\). Now let \( n > 1 \), and let \( F \) be a facet of \( \Delta \). We set \( \Gamma = \Delta \setminus \{F\} \).

We may assume that \( \Gamma \) is a simplicial complex on the vertex set \([n']\) where \( n' \leq n \). By induction hypothesis, \( \mathbb{F}_\Gamma \) is a graded minimal free \( S' \)-resolution of \( S'/J_\Gamma \), where \( S' = K[x_1, \ldots, x_{n'}, y_1, \ldots, y_{n'}] \). Thus \( \mathbb{G} = \mathbb{F}_\Gamma \otimes_{S'} S \) is a graded minimal free \( S \)-resolution of \( S/J_\Gamma S \), and we obtain an exact sequence of complexes

\[
(1) \quad 0 \longrightarrow \mathbb{G} \overset{\varphi}{\longrightarrow} \mathbb{F}_\Delta \overset{\epsilon}{\longrightarrow} \mathbb{K} \longrightarrow 0.
\]

For all \( H \subseteq [n'] \) we set \( H' = H \cup \{n' + 1, \ldots, n\} \). Then \( \varphi : \mathbb{G} \to \mathbb{F}_\Delta \) is defined by \( \varphi(e_{G,H}) = e_{G,H'} \) for all \( e_{G,H} \in \mathbb{G} \). Furthermore, \( \mathbb{K} = F_\Delta / \varphi(\mathbb{G}) \).

One verifies that \( K_j \) is a free module admitting the basis \( \bar{e}_{F,H} = \epsilon(e_{F,H}) \) with \( H \subseteq [n] \) such that \( |F \cap H| = j - 1 \) and \( F \cup H = [n] \). Denote by \( \delta \) the differential of \( \mathbb{K} \). Then

\[
\delta_j(\bar{e}_{F,H}) = - \sum_{i \in F \cap H} (-1)^{\sigma(F \cap H,i)} y_i \bar{e}_{F,H \setminus \{i\}}.
\]
Thus we see that \( K \) is isomorphic to the Koszul complex attached to the sequence \((y_i)_{i \in F}\), homologically shifted by 1. In particular, it follows that \( H_j(\mathbb{K}) = 0 \) for \( j \neq 1 \), while \( H_1(\mathbb{K}) = S/(y_i : i \in F) \).

Thus from the long exact sequence attached to (1) we obtain that \( H_j(F \Delta) = 0 \) for \( j > 1 \) by using that \( G \) is acyclic by our induction hypothesis. Furthermore, we obtain the exact sequence
\[
0 \to H_1(F C) \to H_1(K) \to H_0(G) \to H_0(F C) \to 0.
\]
Since \( H_0(G) = S/J \Gamma S \) and \( H_0(F \Delta) = S/J \Delta S \) we see that \( \text{Ker}(H_0(G) \to H_0(F C)) = J \Delta/J \Gamma S \). Now \( J \Delta/J \Gamma S = S/J \Gamma S y_F \) and \( J \Gamma S \) is isomorphic to the Koszul complex attached to the sequence \((y_i)_{i \in F}\), homologically shifted by 1. In particular, it follows that \( H_j(K) = 0 \) for \( j \neq 1 \), while \( H_1(K) = S/(y_i : i \in F) \).

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Since \( H_0(G) = S/J \Gamma S \) and \( H_0(F \Delta) = S/J \Delta S \) we see that \( \text{Ker}(H_0(G) \to H_0(F C)) = J \Delta/J \Gamma S \). Now \( J \Delta/J \Gamma S = S/J \Gamma S y_F \) and \( J \Gamma S \) is isomorphic to the Koszul complex attached to the sequence \((y_i)_{i \in F}\), homologically shifted by 1. In particular, it follows that \( H_j(K) = 0 \) for \( j \neq 1 \), while \( H_1(K) = S/(y_i : i \in F) \).

\( \Box \)

Corollary 1.4. Let \( \Delta \) be a simplicial complex of dimension \( d - 1 \) with \( f \)-vector \((f-1, f_0, \ldots, f_{d-1})\). Then
\[
\beta_j(J \Delta) = \sum_{i=-1}^{d-1} \binom{i+1}{j} f_i.
\]
In particular, \( \text{proj dim } J \Delta = \dim \Delta + 1 \).

2. Whisker complexes and shellability

In this section we show that the face ideal of a simplicial complex does not only have a linear resolution but even linear quotients.

Theorem 2.1. Let \( \Delta \) be a simplicial complex. Then \( J \Delta \) has linear quotients.

Proof. We choose any total order of the generators of \( J \Delta \) with the property that \( u_G > u_F \) if \( G \subset F \), and claim that \( J \Delta \) has linear quotients with respect to this order of the generators.

Indeed, let \( F \in \mathcal{F}(\Delta) \), and let \( J(F) \) be the ideal generated by all \( G \) with \( u_G > u_F \). For any \( G \in \mathcal{F}(\Delta) \) one has
\[
(u_G : u_F) = \frac{u_G}{\gcd(u_G, u_F)} = x_G y_F y_{F \setminus G}.
\]
Now let \( u_G \in J(F) \). Then \( F \setminus G \neq \emptyset \). Let \( x_j \in F \setminus G \), and let \( H = F \setminus \{x_j\} \).

Then \( u_H > u_F \) and \((u_H) : u_F = (y_j)\). Since \( y_j \) divides \( y_{F \setminus G} \) it follows that \( y_j \) divides \((u_H) : u_F \), as desired. \( \Box \)

Corollary 2.2. Let \( \Gamma \) be a simplicial complex. Then the independence complex of \( W(\Gamma) \) is shellable.

Theorem 2.1 can be generalized as follows.

Theorem 2.3. Let \( S \) be a non-empty collection of subsets of \( \{x_1, \ldots, x_n\} \) satisfying the following conditions:
\begin{itemize}
  \item [(i)] if \( F, G \in S \), then \( F \cap G \in S \);
  \item [(ii)] for \( F, G \in S \) with \( G \subset F \) there exists \( x_i \in F \setminus G \) such that \( F \setminus \{x_i\} \in S \).
\end{itemize}
Then \( I_S \) has linear quotients.
Proof. We choose any total order of the generators of \( J_\Delta \) with the property that 
\( u_G > u_F \) if \( G \subset F \), and claim that \( J_S \) has linear quotients with respect to this order
of the generators. Let \( u_G > u_F \), and let \( \mathcal{N} = \{ i : F \setminus \{ x_i \} \in S \} \). Then, by (ii), 
\( \mathcal{N} \neq \emptyset \). We claim that 
\[(u_G : u_G > u_F) = (y_i : i \in \mathcal{N}).\]
To see why this is true, let \( u_G > u_F \). Then, by (i), \( H = G \cap F \) belongs to \( S \), and 
by (ii) there exists \( i \in \mathcal{N} \) such that \( x_i \in F \setminus H \). Then \( x_i \not\in G \), and hence \( y_i \) divides 
\( (u_G) : u_F \). Since \( (u_F, \{ x_i \}) : u_F = (y_i) \), we are done. \( \square \)

3. Chain ideals and antichain ideals

In this section we consider special classes of face ideals arising from finite partially
ordered sets (posets, for short). Let \( P = \{ p_1, \ldots, p_n \} \) be a finite poset, and \( S = K[x_1, \ldots, x_n, y_1, \ldots, y_n] \) the polynomial ring in \( 2n \) variables over the field \( K \). For
each chain \( \alpha \) of \( P \) we define the monomial \( u_\alpha \in S \) by setting
\[ u_\alpha = (\prod_{p_i \in \alpha} x_i)(\prod_{p_j \notin \alpha} y_j), \]
and let \( I_C(P) \) be the ideal generated by the monomials \( u_\alpha \) where \( \alpha \) is a chain of \( P \).
We call \( I_C(P) \) the chain ideal of \( P \).

Similarly, for each antichain \( \beta \) of \( P \) we define the monomial \( u_\beta \in S \) by setting
\[ u_\beta = (\prod_{p_i \in \beta} x_i)(\prod_{p_j \notin \beta} y_j), \]
and let \( I_A(P) \) be the ideal generated by the monomials \( u_\beta \) where \( \beta \) is an antichain of \( P \). We call \( I_A(P) \) the antichain ideal of \( P \).

Recall that the comparability graph of \( P \) is a finite simple graph \( \mathcal{C}(P) \) on \([n]\) whose
edges are those subsets \( \{ i, j \} \) such that \( p_i \) and \( p_j \) are comparable in \( P \), i.e., \( \{ p_i, p_j \} \) is
a chain of \( P \). Similarly, the incomparability graph of \( P \) is a finite simple graph \( \mathcal{A}(P) \) on \([n]\) whose
edges are those subsets \( \{ i, j \} \) such that \( p_i \) and \( p_j \) are incomparable in \( P \), i.e., \( \{ p_i, p_j \} \) is an antichain of \( P \).

Theorem 3.1. Let \( P \) be a finite poset.

(a) The Alexander dual of the chain ideal \( I_C(P) \) is the edge ideal of the whisker
graph of the incomparability graph of \( P \).

(b) The Alexander dual of the antichain ideal \( I_A(P) \) is the edge ideal of the whisker
graph of the comparability graph of \( P \).

Proof. (a) Let \( \Delta \) be a simplicial complex on \( \{ y_1, \ldots, y_n \} \) whose faces are those
\( F \subset \{ y_1, \ldots, y_n \} \) with \( \{ p_i : y_i \in F \} \) is a chain of \( P \). Theorem [11] says that the
Alexander dual \( I_C(P)^{\vee} \) of \( I_C(P) \) coincides with \( I(W(\Gamma)) \), where \( \Gamma \) is a simplicial
complex on \( \{ y_1, \ldots, y_n \} \) with \( \Delta = I(\Gamma) \). A subset \( F \subset \{ y_1, \ldots, y_n \} \) is nonface of
\( \Delta \) if and only if an antichain of \( P \) is contained in \( F \). Thus the minimal nonfaces of
\( \Delta \), which coincides with the facets of \( \Gamma \) are those subset \( \{ y_i, y_j \} \) such that \( \{ p_i, p_j \} \) is
an antichain of \( P \). Thus \( I(\Gamma) \) is the edge ideal of the incomparability graph of \( P \). Hence
\( I(W(\Gamma)) \) is the edge ideal of the whisker graph of the incomparability graph of \( P \),
as desired.
(b) Let \( \Delta \) be a simplicial complex on \( \{y_1, \ldots, y_n\} \) whose faces are those \( F \subset \{y_1, \ldots, y_n\} \) with \( \{p_i : y_i \in F\} \) is an antichain of \( P \). Theorem 1.1 says that the Alexander dual \( I_A(P)^\vee \) of \( I_A(P) \) coincides with \( I(W(\Gamma)) \), where \( \Gamma \) is a simplicial complex on \( \{y_1, \ldots, y_n\} \) with \( \Delta = I(\Gamma) \). A subset \( F \subset \{y_1, \ldots, y_n\} \) is nonface of \( \Delta \) if and only if a chain of \( P \) is contained in \( F \). Thus the minimal nonfaces of \( \Delta \), which coincides with the facets of \( \Gamma \) are those subset \( \{y_i, y_j\} \) such that \( \{p_i, p_j\} \) is a chain of \( P \). Thus \( I(\Gamma) \) is the edge ideal of the comparability graph of \( P \). Hence \( I(W(\Gamma)) \) is the edge ideal of the whisker graph of the comparability graph of \( P \), as desired. \( \Box \)

The Dilworth number of a finite poset \( P \) is the least number of chains into which \( P \) can be partitioned. Dilworth's theorem [2] guarantees that the Dilworth number of \( P \) is equal to the maximal cardinality of the antichains of \( P \).

**Corollary 3.2.** The chain ideal and the antichain ideal of a finite poset have a linear resolution. Moreover, \( \text{proj dim } I_C(P) = \text{rank } P + 1 \) while \( \text{proj dim } I_A(P) \) is the Dilworth number of \( P \).

### 4. Higher dimensional whiskers

The purpose of this section is to generalize Corollary 2.2. Let \( \Delta \) be a simplicial complex on \( \{x_1, \ldots, x_n\} \). Given positive integers \( k_1, \ldots, k_n \) and \( d_1, \ldots, d_n \) with \( d_i \leq k_i \) for all \( i \), we define the **higher dimensional whisker complex** \( W_{d_1, \ldots, d_n}^{k_1, \ldots, k_n}(\Delta) \) of \( \Delta \) to be the simplicial complex on the vertex set

\[
x_1, x_1^{(1)}, \ldots, x_1^{(k_1)}, x_2, x_2^{(1)}, \ldots, x_2^{(k_2)}, \ldots, x_n, x_n^{(1)}, \ldots, x_n^{(k_n)},
\]

whose facets are the facets of \( \Delta \) together with all subsets of cardinality \( d_i + 1 \) of \( \{x_i, x_i^{(1)}, \ldots, x_i^{(k_i)}\} \) for \( i = 1, \ldots, n \). These subsets are called the whiskers of \( \Delta \).

Note that the whisker complex of \( \Delta \) as defined in Section 1 is just the complex \( W_{1,\ldots,1}^{1,\ldots,1}(\Delta) \). See Figure 1 for an example of a higher whisker complex.

**Theorem 4.1.** The independence complex of a higher whisker complex is shellable.

**Proof.** Let \( W_{d_1, \ldots, d_n}^{k_1, \ldots, k_n}(\Gamma) \) be the whisker complex, \( \Delta \) its independence complex and \( I = I_{\Delta^c} \). We will show that \( I \) has linear quotients. This is equivalent to say that \( \Delta \) is shellable. Note that the generators of \( I \) correspond bijectively to the vertex covers of \( W_{d_1, \ldots, d_n}^{k_1, \ldots, k_n}(\Gamma) \). Indeed, if

\[
C \subset \{x_1, x_1^{(1)}, \ldots, x_1^{(k_1)}, x_2, x_2^{(1)}, \ldots, x_2^{(k_2)}, \ldots, x_n, x_n^{(1)}, \ldots, x_n^{(k_n)}\}
\]

is a minimal vertex cover of \( W_{d_1, \ldots, d_n}^{k_1, \ldots, k_n}(\Gamma) \), then the product of the elements in \( C \) is the corresponding generator of \( I \).

We claim, that \( C \) is a minimal vertex cover of \( W_{d_1, \ldots, d_n}^{k_1, \ldots, k_n}(\Gamma) \) if and only if the following conditions are satisfied:

1. \( C \cap \{x_1, \ldots, x_n\} \) is a vertex cover of \( \Gamma \);
2. \( C \cap \{x_i, x_i^{(1)}, \ldots, x_i^{(k_i)}\} \) is a minimal vertex cover of the \( d_i \)-skeleton of the simplex on \( \{x_i, x_{i1}, \ldots, x_{ik_i}\} \).
Clearly any set \( C \) of vertices satisfying (1) and (2) is a minimal vertex cover.

We denote by \( u_C \in I \) the monomial corresponding to the vertex cover \( C \) of \( W_{d_1, \ldots, d_n}(\Gamma) \). Then \[ u_C = x_{C_0}x_{C_1} \cdots x_{C_n}, \]
where \[ x_{C_0} = \prod_{x_j \in C} x_i \quad \text{and} \quad x_{C_j} = \prod_{\{i: x_j^{(i)} \in C\}} x_j^{(i)} \]
for \( j = 1, \ldots, n \).

We now define a total order of the monomial generators of \( I \) as follows: let \( u_C = x_{C_0}x_{C_1} \cdots x_{C_n} \) and \( u_D = x_{D_0}x_{D_1} \cdots x_{D_n} \). Then \( u_C > u_D \) if either \( x_{C_0} > x_{D_0} \) with respect to the degree lexicographic order, or \( x_{C_0} = x_{D_0} \) and \( x_{C_1} \cdots x_{C_n} > x_{D_1} \cdots x_{D_n} \) with respect to the lexicographic order induced by

\[ x_1^{(1)} > \ldots > x_1^{(k_1)} > x_2^{(1)} > \ldots > x_2^{(k_2)} > \ldots > x_n^{(1)} > \ldots > x_n^{(k_n)}. \]

We claim that with this ordering of the generators, \( I \) has linear quotients. We first note that all generators of \( I \) have the same degree, namely, \( \sum_{i=1}^n (k_i - d_i) + n \). We must show that for all \( u_C \) the colon ideal \((u_D: u_D > u_C) : u_C \) is generated by variables. Let \( u_D > u_C \) with \( x_{D_0} \neq x_{C_0} \). Then \( x_{D_0} > x_{C_0} \), and hence there exists \( x_j \) such that \( x_j \) divides \( x_{D_0} \) but does not divide \( x_{C_0} \). Let \( u_E \) be the generator with \( x_{E_0} = x_{C_0}x_j, \ x_{E_i} = x_{C_i} \) for \( i \neq j \) and \( x_{E_j} = x_{C_j}/x_j^{(i)} \) where \( x_j^{(i)} \) divides \( x_{C_i} \). Then \( u_E > u_C \) and \((u_E) : u_C = (x_j) \). Since \( x_j \) divides \((u_D) : u_C \), and we are done in this case.
We now consider the case $x_{D_0} = x_{C_0}$. Let $\mathcal{A} = \{u_E : x_{E_0} = x_{C_0}\}$. For each $i \geq 1$, let $\mathcal{A}_i = \{u_E : u_E \in \mathcal{A}\}$. If $x_i$ divides $x_{C_0}$, then $\mathcal{A}_i$ is the set of all monomials of degree $k_i - d_i$ in the variables $x^{(i)}_1, \ldots, x^{(i)}_{k_i}$, and if $x_i$ does not divide $x_{C_0}$, then $\mathcal{A}_i$ is the set of all monomials of degree $k_i - d_i + 1$ in the same set of variables. Note that $\mathcal{A}_i$ generates a matroidal ideal. Moreover the product of matroidal ideals in pairwise disjoint sets of variables is again a matroidal ideal. It is known, see [9, Theorem 12.6.2], that matroidal ideals have linear quotients with respect to the lexicographic order of the generators. This completes the proof of the theorem. □

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