Eigenvectors distribution and quantum unique ergodicity for deformed Wigner matrices

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Abstract
We analyze the distribution of eigenvectors for mesoscopic, mean-field perturbations of diagonal matrices, in the bulk of the spectrum. Our results apply to a generalized $N \times N$ Rosenzweig-Porter model. We prove that the eigenvectors entries are asymptotically Gaussian with a specific variance, localizing them onto a small, explicit, part of the spectrum. For a well spread initial spectrum, this variance profile universally follows a heavy-tailed Cauchy distribution. In the case of smooth entries, we also obtain a strong form of quantum unique ergodicity as an overwhelming probability bound on the eigenvectors probability mass.

The proof relies on a priori local laws for this model as given in [31, 29, 11], and the eigenvector moment flow from [12, 13].

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1. Introduction
In the study of large interacting quantum systems, Wigner conjectured that empirical results are well approximated by statistics of eigenvalues of large random matrices. This vision has not been shown for correlated quantum systems but is regarded to hold for numerous models. For instance, the Bohigas-Giannoni-Schmit conjecture in quantum chaos [7] connects eigenvalues distributions in the semiclassical limit to the Gaudin distribution for GOE statistics. These statistics also conjecturally appear for random Schrödinger operators [4] in the delocalized phase. Most of these hypotheses are unfortunately far from being proved with mathematical rigor. It is, however, possible to study systems given by large random matrices. One of the most important models of this type is the Wigner ensemble, random Hermitian or symmetric matrices whose elements are, up to the symmetry, independent and identically distributed zero-mean random variables. For this ensemble, local statistics of the spectrum only depend on the symmetry class and not on the laws of the elements (see [19, 36, 22, 20, 10]). The Wigner-Dyson-Mehta conjecture was solved for numerous, more general mean-field models such as the generalized Wigner matrices, random matrices for which the laws of the matrix elements can have distinct variances (see [21] and references therein).
The statistics of eigenvectors were not used in Wigner’s original study but localization, or delocalization, has been broadly studied in random matrix theory. For Wigner matrices, it has been shown in [18] that eigenvectors are completely delocalized in the following sense: denote \( u_1, \ldots, u_N \) the \( L^2 \)-normalized eigenvectors of a \( N \times N \) Wigner matrix, we have with very high probability,

\[
\sup_\alpha |u_i(\alpha)| \leq C (\log N)^{9/2} \sqrt{\frac{1}{N}}.
\]

Thus, eigenvectors cannot concentrate onto a set of size larger than \( N(\log N)^{-9/2} \) (see [23] for similar estimates for generalized Wigner matrices). In the GOE and GUE cases, the distribution of the matrix is orthogonally invariant and eigenvectors are distributed according to the Haar measure on the orthogonal group. In particular, the entries of bulk eigenvectors are asymptotically normal:

\[
\sqrt{N} u_i(\alpha) \xrightarrow{N \to \infty} \mathcal{N},
\]

where \( \mathcal{N} \) is a standard Gaussian random variable. Asymptotic normality was first proved for Wigner matrices in [28, 37] under a matching condition on the first four moments of the entries using Green’s function comparison theorems introduced in [36]. These conditions were later removed in [12] where asymptotic normality holds for generalized Wigner matrices. Beyond mean-field models, conjectures of interest, for example for band matrices, are still yet to be proved. A sharp transition is conjectured to occur when the band width \( W \) cross the critical value \( \sqrt{N} \). For \( W \ll \sqrt{N} \), eigenvectors are expected to be exponentially localized on \( O(W^2) \) sites and eigenvalue statistics are Poisson, while for \( W \gg \sqrt{N} \) eigenvectors would be completely delocalized and one would get Wigner-Dyson-Mehta statistics for the eigenvalues.

In this paper, we consider a generalized Rosensweig-Porter model, of mean-field type, which also interpolates between delocalized and localized (or partially delocalized) phases, but always with GOE/GUE statistics. It is defined as a perturbation of a potential, consisting of a deterministic diagonal matrix, by a mean field noise, given by a Wigner random matrix, scaled by a parameter \( t \). This model follows two distinct phase transitions. When \( t \ll 1/N \), eigenvalue statistics coincide with \( t = 0 \) and eigenvectors are localized on \( O(1) \) sites [39], while when \( t \gg 1 \), local statistics fall in the Wigner-Dyson-Mehta universality class [31] with fully delocalized eigenvectors [30]. For \( 1/N \ll t \ll 1 \), it has been shown in [29] that eigenvalue statistics are in the Wigner-Dyson-Mehta universality class and in [40] that eigenvectors are not completely delocalized when the noise is Gaussian. In this intermediate phase, also called the “bad metal regime” (see [24] or [38] for instance), eigenstates are partially delocalized over \( Nt \) sites, a diverging number as \( N \) grows but a vanishing fraction of the whole spectrum. The existence of this regime for more intricate models is only conjectured or even debated in the physics literature though progress has been made recently, for instance for the Anderson model on the Bethe lattice and regular graph in [16].

Our results give the asymptotic distribution of the eigenvectors for this model, giving a rather complete understanding of this regime for the Rosensweig-Porter model. We show that bulk eigenvectors are asymptotically Gaussian with a specific, explicit variance depending on the initial potential, the parameter \( t \) and the position in the spectrum. For a well-spread initial condition, this variance is heavy-tailed and follows a Cauchy distribution. This shape appeared in a non-rigorous way in [1], where the Gaussian distribution of eigenvectors (Corollary 1.5) was conjectured, in the case of Gaussian entries, the eigenvector distribution has been exhibited in the physics literature in [24] using the resolvent flow and in [38] using supersymmetry techniques.

Another strong form of delocalization of eigenfunctions is quantum ergodicity. It has been proved for the Laplace-Beltrami operator on negative curved compact Riemannian manifold by Shnirel’mann [35], Colin de Verdière [15] and Zelditch [41] but also for regular graphs by Anantharaman-Le Masson [2]. In [34], Rudnick-Sarnack conjectured a stronger form of delocalization for eigenfunctions of the Laplacian called the quantum unique ergodicity. More precisely, denote \( \langle \phi_k \rangle_{k \geq 1} \) the eigenfunctions of the Laplace operator on any negatively curved compact Riemannian manifold \( \mathcal{M} \), then they supposedly become equidistributed with respect to the volume measure \( \mu \) in the following sense: for
any open set $A \in \mathcal{M}$
\[
\int_A |\phi_k|^2 d\mu \xrightarrow{k \to \infty} \int_A d\mu.
\]
This conjecture has not been proved for all negatively curved compact Riemannian manifold but has been rigorously shown for arithmetic surfaces (see [25, 26, 32]).

A probabilistic form of quantum unique ergodicity exists for eigenvectors of large random matrices. It first appeared in [12] for generalized Wigner matrices, large symmetric or Hermitian matrices whose entries are independent up to the symmetry and zero mean random variables but with varying variances. It is stated as a high-probability bound showing that eigenvectors are asymptotically flat in the following way: let $(u_k)_{1 \leq k \leq N}$ be the eigenvectors of a $N \times N$ generalized Wigner matrix, then for any deterministic $N$-dependent set $I \subseteq [1, N]$ such that $|I| \to +\infty$ and any $\delta > 0$,
\[
P\left(\frac{N}{|I|} \sum_{\alpha \in I} \left( u_k(\alpha)^2 - \frac{1}{N} \right) > \delta \right) \leq \frac{N^{-\varepsilon}}{\delta^2}
\]
for some $\varepsilon > 0$. Similar high-probability bounds were proved for different models of random matrices such as $d$-regular random graphs in [5], or band matrices in [9, 13]. In these last papers on band matrices, it was seen that quantum unique ergodicity is a useful property to study non mean-field models. In [13], a stronger form of probabilistic quantum unique ergodicity has been found, showing that the eigenvectors mass is asymptotically flat with overwhelming probability (the probability decreases faster than any polynomial). Our result adapts the method introduced in [13] to show a strong deformed quantum unique ergodicity for eigenvectors of a class of deformed Wigner matrices. Indeed, the probability mass is not flat but concentrate onto an explicit and deterministic profile with a quantitative error.

The key ingredient for this analysis is the Bourgade-Yau eigenvector moment flow [12], a multi-particle random walk in a random environment given by the trajectories of the eigenvalues. This method was used for generalized Wigner matrices [12] and sparse random graphs [11], and both settings correspond to equilibrium or close to equilibrium situations. Our main contribution consists in treating the non-equilibrium case, which implies additional difficulties made explicit in the next section.

1.1. Main Results

Consider a deterministic diagonal matrix $D = \text{diag}(D_1, \ldots, D_N)$. The eigenvalues (or diagonal entries) need to be regular enough on a window of size $r$ in the following way first defined in [29].

**Definition 1.1.** Let $\eta_*$ and $r$ two $N$-dependent parameters satisfying
\[
N^{-1} \leq \eta_* \leq N^{-\varepsilon'}, \quad N^{\varepsilon'} \eta_* \leq r \leq N^{-\varepsilon'}
\]
for some $\varepsilon' > 0$. A deterministic diagonal matrix $D$ is said to be $(\eta_*, r)$-regular at $E_0$ if there exists $c_D > 0$ and $C_D > 0$ such that for any $E \in [E_0 - r, E_0 + r]$ and $\ell \leq \eta \leq 10$, we have
\[
c_D \leq \text{Im} m_D(E + i\eta) \leq C_D,
\]
where $m_D$ is the Stieltjes transform of $D$:
\[
m_D(z) = \frac{1}{N} \sum_{k=1}^N \frac{1}{D_k - z}.
\]

We want to study the perturbation of such a diagonal matrix, notably the eigenvectors, by a mesoscopic Wigner random matrix. We will now suppose that $D$ is $(\eta_*, r)$-regular at $E_0$, a fixed energy point. Let $\kappa > 0$, we will denote in the rest of the paper the bulk of the spectrum as
\[
\mathcal{I}_\kappa = [E_0 - (1 - \kappa)r, E_0 + (1 - \kappa)r],
\]
\[
\mathcal{A}_\kappa = \{i \in [1, N], D_i \in \mathcal{I}_\kappa\}.
\]

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We will also use the following time and spectral domains. For the time domain, we will need the perturbation to be mesoscopic but smaller than the energy window size $r$, define then for any small positive $\omega$,

$$\mathcal{T}_\omega = [N^{-1+\omega}, N^{-\omega}r].$$

For the spectral domain, take first $t \in \mathcal{T}_\omega$ and note that we will consider only $\text{Im}(z) := \eta$ smaller than $t$ but most results such as local laws holds up to macroscopic $\eta$, let $\vartheta > 0$ an arbitrarily small constant and

$$D^\vartheta,\eta = \{z = E + i\eta, E \in \mathcal{T}_\omega^c, N^\vartheta/N \leq \eta \leq N^{-\vartheta}t\}.$$

Hereafter is our assumptions on our Wigner matrix

**Definition 1.2.** A Wigner matrix $W$ is a $N \times N$ Hermitian/symmetric matrix satisfying the following conditions

(i) The entries $(W_{i,j})_{1 \leq i,j \leq N}$ are independent.

(ii) For all $i,j$, $E[W_{i,j}] = 0$ and $E[|W_{i,j}|^2] = N^{-1}$.

(iii) For every $p \in \mathbb{N}$, there exists a constant $C_p$ such that $\left\|\sqrt{N}W_{ij}\right\|_p \leq C_p$.

Let $W$ be a Wigner matrix and define the following $t$-dependent matrix for $t \in \mathcal{T}_\omega$

$$W_t = D + \sqrt{t}W. \quad (1.1)$$

The eigenvectors of $D$ are exactly the vectors of the canonical basis since the matrix is diagonal. However, if $t$ were of order one instead of being in $\mathcal{T}_\omega$, the local statistics of $W_t$ would become universal and would be given by local statistics from the Gaussian ensemble. In particular, the eigenvectors would be completely delocalized [31]. Our model consists in looking at the diffusion of universal and would be given by local statistics from the Gaussian ensemble. In particular, the coordinates of bulk eigenvectors are time and position dependent Gaussian random variables. Before stating our result, we first define the asymptotic distribution of the eigenvalues of the matrix $W_t$ which is the free convolution of the semicircle law (coming from $W$) and the empirical distribution of $D$. We will define this distribution through its Stieltjes transform $m_t(z)$ as the solution to the following self-consistent equation

$$m_t(z) = \frac{1}{N} \sum_{k=1}^{N} \frac{1}{D_k - z - tm_t(z)}. \quad (1.2)$$

It is known that this equation has a unique solution with positive imaginary part and is the Stieltjes transform of a measure with density denoted by $\rho_t$ (see [6] for more details). Define the quantiles $(\gamma_{i,t})_{0 \leq i \leq N}$ of this measure by

$$\int_{-\infty}^{\gamma_{i,t}} \rho_t(x)dx = \frac{i}{N}. \quad i = 1, \ldots, N$$

We can now state our main results, denoting $u_1(t), \ldots, u_N(t)$ the $L^2$-normalized eigenvectors of $W_t$ (we will often omit the $t$-dependence for $u$).

**Theorem 1.3.** (Gaussianity of bulk eigenvectors) Let $\kappa \in (0,1)$ and $\omega$ a small positive constant, for instance $\omega < \varepsilon'/10$. Let $t \in \mathcal{T}_\omega$ and $I \subset \mathcal{A}_N^\kappa$ be a deterministic ($N$-dependent) set of $m$ elements. Let $W$ as in Definition 1.2 and $W_t$ as in (1.1). Write $I = \{k_1, \ldots, k_m\}$, take a deterministic $q \in \mathbb{R}^N$ such that $\|q\|_2 = 1$, and define for $\eta \in [N^{-1+\omega}, N^{-\omega} \sqrt{\frac{1}{N}}]$, 

$$\sigma^2_t(q, k_i, \eta) := \sum_{k=1}^{N} \frac{q^2_k}{(D_k - \gamma_{k,t})^2 + (\text{Im} m_t(\bar{\gamma}_{k,t} + i\eta))^2}. \quad (1.3)$$
Then we have
\[
\left( \frac{N}{\sigma_l^2(q, k, \eta)} |(q, u_{k_i})| \right)_{i=1}^{m} \xrightarrow{N \to \infty} (|\mathcal{N}_i|)_{i=1}^{m}
\]
\[
\left( \frac{2N}{\sigma_l^2(q, k, \eta)} |(q, u_{k_i})| \right)_{i=1}^{m} \xrightarrow{N \to \infty} (|\mathcal{N}_i^{(1)} + i\mathcal{N}_i^{(2)}|)_{i=1}^{m}
\]
in the sense of convergence of moments, where all $\mathcal{N}_i$, $\mathcal{N}_i^{(1)}$, and $\mathcal{N}_i^{(2)}$ are independent Gaussian random variables with variance 1. The convergence is uniform in $I$ and $q$.

**Remark 1.4.** Note that the convergence holds for any $\eta$ in the spectral domain since we have $|\partial_r u_I(z)| \leq 1/t$. Indeed, we can take any $\eta$ as long as it is of smaller order than $t$. However in the proof, $\eta$ will be of the same order as the time $\tau$ which corresponds to the time we make our matrix undergo the Dyson Brownian motion. In order to use the continuity argument from Section 4.1, we need $\tau$ to be of order smaller than $\sqrt{t/N}$.

One can deduce joint weak convergence of eigenvector entries from the previous convergence of moments because $q$ is arbitrary in $S^{N-1}$ (see [12, Section 5.3]). However, since the eigenvectors are defined up to a phase, we first need to define the following equivalence relation: $u \sim v$ if and only if $u = \pm v$ in the symmetric case and $u = e^{i\omega}v$ for some $\omega \in \mathbb{R}$ in the Hermitian case.

**Corollary 1.5.** Let $\kappa \in (0, 1)$ and $I \subseteq \mathbb{N}$, let $W$ as in Definition 1.2 and $W_t$ as in 1.1. Then for any deterministic $k \in \mathcal{A}_t^\kappa$ and $J \subseteq [1, N]$ such that $|J| = I$ we have
\[
\left( \frac{N}{\sigma_l(e, k, \eta)^2} u_k(\alpha) \right)_{\alpha \in J} \xrightarrow{N \to \infty} (\mathcal{N}_i^{(1)})_{i=1}^{m}
\]
in the sense of convergence of moments modulo $\sim$, where all $\mathcal{N}_i$, $\mathcal{N}_i^{(1)}$, and $\mathcal{N}_i^{(2)}$ are independent Gaussian random variables with variance 1.

This result states that the entries of bulk eigenvectors are asymptotically independent Gaussian random variables with variance $\sigma_l^2$ which answers a conjecture from [1, Section 3.2], stated in the more restrictive case where $W$ is GOE. When $q$ is a vector from the canonical basis, $u_k(\alpha)$ has the following asymptotic variance
\[
\frac{1}{N} \frac{t}{(D_k - \gamma_{\alpha,k})^2 + (Ct)^2}.
\]
For regularly spaced $D_k$’s, this is heavy-tailed with Cauchy shape. It localizes the entries onto a subset of the spectrum of size $Nt \leq N^{1-s-\varepsilon'}$: a fraction of the spectrum vanishing as $N$ grows. The eigenvector is then delocalized over a small fraction of the spectrum. Such a partial localization appears in [40] for $W$ GOE-distributed.

The asymptotic normality of the eigenvectors gives the following weak form of quantum unique ergodicity.

**Corollary 1.6.** (Weak Quantum Unique Ergodicity) Let $W$ as in 1.2 and $W_t$ as in Definition (1.1). There exists $\vartheta > 0$ such that for any $c > 0$, there exists $C > 0$ such that the following holds: for any $I \subseteq [1, N]$ and $k \in \mathcal{A}_t^\kappa$, we have
\[
P \left( \frac{Nt}{|I|} \sum_{\alpha \in I} |u_k(\alpha)|^2 - \frac{1}{N} \sum_{\alpha \in I} \sigma_l^2(\alpha, k) > c \right) \leq C(N^{-\vartheta} + |I|^{-1}).
\]

This high probability bound is not the strongest form of quantum unique ergodicity one can obtain for random matrices. Indeed, if we consider the Gaussian ensembles for which the eigenbasis
is Haar-distributed on the orthogonal group and each eigenvector is uniformly distributed on the sphere, one can get that for any $\varepsilon$ and $D$ positive constants,
\[
P \left( \left| \sum_{\alpha \in I} |u_k(\alpha)|^2 - \frac{|I|}{N} \right| \geq N^{\varepsilon} \frac{\sqrt{T}}{N} \right) \leq N^{-D}.
\]

In this paper, we will obtain a similar overwhelming probability bound on the probability mass of a single eigenvector with an explicit error for a more restrictive model of matrices: deformed random matrix with smooth entries given by the following definition or from a Gaussian divisible ensemble.

**Definition 1.7.** A smooth Wigner matrix $W$ is a $N \times N$ Hermitian/symmetric matrix with the following conditions

(i) The matrix entries $(W_{ij})_{1 \leq i, j \leq N}$ are independent and identically distributed random variables following the distribution $N^{-1/2} \nu$ where $\nu$ has mean zero and variance $1$.

(ii) The distribution $\nu$ has a positive density $\nu(x) = e^{-\Theta(x)}$ such that for any $j$, there are constants $C_0$ and $C_1$ such that
\[
|\Theta^{(j)}(x)| \leq C_0(1 + x^2)^{C_1}
\]

(iii) The tail of the distribution $\nu$ has a subexponential decay. In other words, there exists $C$ and $q$ two positive constants such that
\[
\int_{\mathbb{R}} 1_{|x| \geq y} d\nu(x) \leq C \exp(-y^q)
\]

We need the smoothness assumptions on $W$ in order to use the reverse heat flow techniques from [17, 19]. Indeed, our result is an overwhelming probability bound on the eigenvectors of $W_t$. Using the three-step strategy, we are able with the first two steps to obtain the result for any matrices of the type $D + \sqrt{t}W + \sqrt{t} GSOE$ for a more general class of $W$, for instance matrices whose entries have all finite moments. The smoothness assumptions appear in the third step of the strategy, the removal of the small Gaussian components. We think, however, that this property holds for a larger matrix ensembles and that the smoothness property is simply technical.

In the following, since the eigenvectors are concentrated on $Nt$ sites, it is relevant to define the following notation for any set (which can be $N$-dependent) $A$, denote
\[
\hat{A} = \frac{|A|}{Nt} \wedge 1.
\]

Indeed, having errors involving $\hat{A}$ allows us to get bounds improving for $|A| \leq Nt$ but still holding for $|A| \gg Nt$.

**Theorem 1.8.** Let $\kappa \in (0, 1)$, $\omega$ a small positive constant, for instance $\omega < \varepsilon'/10$. Let $t \in T_\omega$, $I \subset [1, N]$ be a deterministic ($N$-dependent) set, $W$ as in Definition 1.7 and $W_t$ as in (1.1). Define now
\[
\Xi = \frac{\tilde{f}}{(Nt)^{1/3}} \text{ and } \sigma^2(\alpha, k) := \sigma^2(\alpha, k, \eta_0) \text{ with } N\eta_0 = \frac{\tilde{f}^2}{\Xi^2}.
\]

Then we have, for any $\varepsilon > 0$ (small) and $D > 0$ (large) and for $k, \ell \in A^*_{\omega}$ with $k \neq \ell$, in the symmetric case
\[
P \left( \left| \sum_{\alpha \in I} \left( u_k(\alpha)^2 - \frac{1}{N} \sigma^2(\alpha, k) \right) \right| + \left| \sum_{\alpha \in I} u_k(\alpha) u_\ell(\alpha) \right| \right) \geq N^{\varepsilon} \Xi \leq N^{-D}
\]
and in the Hermitian case,
\[
P \left( \left| \sum_{\alpha \in I} \left( u_k(\alpha)^2 - \frac{1}{2N} \sigma^2(\alpha, k) \right) \right| + \left| \sum_{\alpha \in I} u_k(\alpha) \bar{u}_\ell(\alpha) \right| \right) \geq N^{\varepsilon} \Xi \leq N^{-D}
\]

**Remark 1.9.** The choice of $\eta_0$ depends on our proof and is the one we should take to optimize our error $\Xi$. However, this error and the choice of $\eta_0$ do not seem optimal since we actually expect to have some form of Gaussian fluctuations around this deterministic profile.


1.2. Method of Proof

Our proof is based on the three-step strategy from [17, 19] (see [21] for a recent book presenting this method). The first step is to have an optimal, local control of the spectral elements of the matrix ensemble given by a local law on the resolvent. The second step is to obtain the wanted result for a relaxation of the model by a small Gaussian perturbation. Finally, the third and last step consists of removing this Gaussian part. We will give the proof of Theorem 1.3, Corollary 1.6, Theorem 1.8 only in the symmetric case and refer the reader to [12, 13] for the tools needed in the Hermitian case.

First step: local laws for our model. In [29], Landon-Yau showed a local law for the Dyson Brownian motion with a diagonal initial condition at all times. This result gives us an averaged local law on the Stieltjes transform but also an entrywise anisotropic local law for the resolvent. Since we want to look at any projection of the eigenvectors, we will also need a local law on the bilinear form \( \langle q, Gq \rangle \). This control of the resolvent for mesoscopic perturbation has been showed in [11]. Note that these results were done in the Gaussian case but can easily be generalized to the Wigner case with the right assumptions on moments.

Second step: short time relaxation. The second step consists of perturbing \( W_t \) by a small Gaussian component. We will obtain this perturbated model by making \( W_t \) undergo the Dyson Brownian motion given by the following definition.

**Definition 1.10.** Here is our choice of Dyson Brownian motion. Let \( B \) be a \( N \times N \) symmetric matrix such that \( B_{ij} \) for \( i < j \) and \( B_{ii}/\sqrt{2} \) are independent standard brownian motions. The \( N \times N \) symmetric Dyson Brownian motion with initial condition \( H_0 \) is defined as

\[
H_s = H_0 + \frac{1}{\sqrt{N}} B_s. \tag{1.14}
\]

We also give the dynamics followed by the eigenvalues and the eigenvectors of such matrices.

**Definition 1.11.** Let \( \lambda_0 \) be in the simplex \( \Sigma_N = \{ \lambda_1 < \cdots < \lambda_N \} \), \( u_0 \) be an orthogonal \( N \times N \) matrix, and \( B \) as in (1.14). Consider the dynamics

\[
d\lambda_k = \frac{1}{\sqrt{N}} \sum_{\ell \neq k} \frac{d s}{\lambda_k - \lambda_\ell},
\]

\[
du_k = \frac{1}{\sqrt{N}} \sum_{\ell \neq k} \frac{d B_{k\ell}}{\lambda_k - \lambda_\ell} u_\ell - \frac{1}{2N} \sum_{\ell \neq k} \frac{d s}{(\lambda_k - \lambda_\ell)^2} u_k. \tag{1.16}
\]

This eigenvector flow was first computed in different contexts such as [3] for GOE/GUE matrices, [14] for real Wishart processes and [33] for Brownian motion on ellipsoids.

**Remark 1.12.** If \( \lambda_0 \) and \( u_0 \) are the eigenvalues and eigenvectors of a fixed matrix \( H_0 \), then the dynamics (1.16) is followed by the eigenvalues and eigenvectors of

\[
H_s = H_0 + \sqrt{s} G
\]

with \( G \) a \( N \times N \) renormalized GOE matrix. In this paper, taking \( W \) to be such a Gaussian matrix, we study the eigenvectors of the Dyson Brownian motion with a diagonal initial condition after a mesoscopic time.

We will then need to study the eigenvectors of \( H_\tau \) for a small \( \tau \ll t \). The convergence of joint moments of eigenvectors projections will be obtained by the maximum principle technique introduced in [12]. It is based on analyzing the dynamics followed by these moments. We will now recall notations and results on this eigenvector moment flow.

Take \( q \in \mathbb{R}^N \) such that \( \| q \|_2 = 1 \) a fixed direction (in the bulk of the spectrum) onto which we will project our eigenvectors. For \( u_1^H, \ldots, u_N^H \) the eigenvectors of the matrix (1.14), define

\[
z_k(s) = \sqrt{N} \langle q, u_k^H(s) \rangle. \tag{1.17}
\]
Now for $m \in [1, N]$, denote by $j_1, \ldots, j_m$ positive integers and let $i_1, \ldots, i_m$ in $[1, N]$ be distinct indices. We will consider the following normalized polynomials

$$Q_{i_1, \ldots, i_m}^{j_1, \ldots, j_m} = \prod_{l=1}^{m} \frac{z_{i_l}^{2j_l}}{a(2j_l)} \quad \text{where} \quad a(n) = \prod_{k \leq n, k \text{ odd}} k. \quad (1.18)$$

Note that $a(2n) = \mathbb{E}[N^{2n}]$ with $N$ a standard Gaussian random variables.

Consider a configuration of particles $\eta : [1, N] \to \mathbb{N}$ where $\eta_j := \eta(j)$ is seen as the number of particles at the site $j$. We denote $N(\eta) = \sum_j \eta_j$ the total number of particles in the configuration $\eta$.

Define $\eta^{i,j}$ to be the configuration by moving one particle from $i$ to $j$. If there is no particle in $i$ then $\eta^{i,j} = \eta$. It is clear that we can map $\{(i_1, j_1), \ldots, (i_m, j_m)\}$ with distinct $i_k$’s and positive $j_k$’s summing to a $n > 0$ to a configuration $\eta$ with $\eta_{i_k} = j_k$ and $\eta_l = 0$ if $l \notin \{i_1, \ldots, i_m\}$.

Define now, given this map,

$$f_{\lambda, s}(\eta) := \mathbb{E} \left[ Q_{s,i_1,\ldots,i_m}^{j_1,\ldots,j_m} | \lambda \right], \quad (1.19)$$

a $n$-th joint moment of the coordinates of $u_k$. The next theorem gives the eigenvector moment flow that $f_{\lambda, s}$ undergoes.

**Theorem 1.13** ([12]). Suppose that $u$ is the solution of the symmetric Dyson vector flow (1.16) and $f_{\lambda, s}(\eta)$ is given by (1.19) with the polynomials $Q_s$. Then it satisfies the equation

$$\partial_s f_{\lambda, s}(\eta) = \frac{1}{N} \sum_{i \neq j} 2 \eta_i (1 + 2 \eta_j) \left( f_{\lambda, s}(\eta^{i,j}) - f_{\lambda, s}(\eta) \right) \left( \lambda_i - \lambda_j \right)^2. \quad (1.20)$$

![Figure 1: Example of the symmetric eigenvector moment flow with a configuration of 7 particles.](image)

Now that we have the expression of the eigenvector moment flow, we can give an heuristic for the apparition of a Cauchy profile in the variance (1.3). Indeed, the single particle case $m = 1$ gives us the variance of an entry of an eigenvector. To understand this result, consider the diagonal entries of the matrix $D$ to be the quantiles of the semicircle law for instance, it is then interesting to look at the following continuous dynamics, define the operator $K$ acting on smooth functions on $[-2, 2]$ as

$$(Kf)(x) = \int_{-2}^{2} \frac{f(x) - f(y)}{(x-y)^2} \, d\rho(y). \quad (1.21)$$

The differential equation $\partial_t f = Kf$ can be seen as a deterministic and continuous equivalent of (1.20) because of the rigidity property of the Dyson Browian motion eigenvalues. We then get the following lemma from [10]

**Lemma 1.14** ([10]). Let $f$ be smooth with all derivatives uniformly bounded. For any $x, y \in (-2, 2)$, denote $x = 2 \cos \theta$, $y = 2 \cos \phi$ with $\theta, \phi \in (0, \pi)$. Then

$$(e^{-tK} f)(x) = \int p_t(x, y) f(y) \, d\rho(y) \quad (1.22)$$

where the kernel is given by

$$p_t(x, y) := \frac{1 - e^{-t}}{\left| e^{i(\theta + \phi)} - e^{-t/2} \right|^2 \left| e^{i(\theta - \phi)} - e^{-t/2} \right|^2}. \quad (1.23)$$
Now at our small time-scale, we have
\[ p_t(x, y) \sim \frac{t}{(x - y)^2 + t^2}. \]
Hence (1.3) where \( m = 1 \) can be considered as a result of stochastic homogenization in a non-equilibrium setting when we look at the dynamics (1.20) in the bulk.

For Theorem 1.8, we will study another observable which follows the same dynamics as in Theorem 1.13. This new observable has been analyzed in [13] to obtain universality for a class of band matrices. Define now the centered eigenvectors overlaps for symmetric matrices,
\[ p_{ij} = \sum_{\alpha \in I} u_i(\alpha)u_j(\alpha), \quad i \neq j \in I, \]
\[ p_{ii} = \sum_{\alpha \in I} u_i(\alpha)^2 - C_0, \quad i \in I \]
where \( u \) are the eigenvectors of \( H_s \) and \( C_0 \) is any constant in the sense that it does not depend on \( i \) but can depend on \( N \).

Now for \( \eta \) a configuration of \( n \) particles on \( N \) sites, define the following set
\[ V_\eta = \{(i, a), 1 \leq i \leq N, 1 \leq a \leq 2\eta_i \}. \]
The set \( V \) will be a set of vertices. Consider now \( G_\eta \) the set of perfect matchings on \( V_\eta \). For any edge on \( G, e = \{(i, a), (j, b)\}, \) define \( p(e) = p_{ij}, \) \( P(G) = \prod_{e \in E(G)} p(e) \) and finally
\[ F_{\lambda, s}(\eta) = \frac{1}{\mathcal{M}(\eta)} \mathbb{E} \left[ \sum_{G \in \mathcal{G}_\eta} P(G) \right] \lambda \]
where \( \mathcal{M}(\eta) = \prod_{i=1}^{N} (2\eta_i)!! \), with \( (2m)!! \) being the number of perfect matchings of the complete graph on \( 2m \) vertices. Note that this quantity depend on the eigenvalues trajectories \( \lambda \).

(a) A configuration \( \eta \) with \( V(\eta) = 6 \) particles
(b) An example of a perfect matching
\[ G \in \mathcal{G}_\eta^{(i)} \text{ with } P^{(i)}(G) = p_{11,2}p_{22,3}p_{23,2} \]

The previous quantity follows the eigenvector moment flow.

**Theorem 1.15 ([13]).** Suppose that \( u \) is the solution of the symmetric Dyson vector flow (1.16) and \( F_{\lambda, s}(\eta) \) is given by (1.26). Then it satisfies the equation
\[ \partial_s F_{\lambda, s}(\eta) = \frac{1}{N} \sum_{i \neq j} 2\eta_i(1 + 2\eta_j) \left( F_{\lambda, s}(\eta^{i,j}) - F_{\lambda, s}(\eta) \right) \frac{(\lambda_i - \lambda_j)^2}{(\lambda_i - \lambda_j)^2}. \]

**Third step: invariance of local statistics.** The third and last step will be to obtain the result for the matrix \( W_t \) without any Gaussian part. We will do so using a variant of the dynamical method introduced in [12, Appendix A] which will show the continuity of resolvent statistics along the trajectory. This method was also used in [27] to study sparse matrices. In order to remove the Gaussian component for Theorem 1.8, we will use the reverse heat flow and will need smoothness of the entries of our original matrix.
The next section will state the local laws proved in different papers ([29] and [11]). The third section is dedicated to prove Theorem 1.3, Corollary 1.6 and Theorem 1.8 for a short time relaxation of the matrix \( W_t \). We use a maximum principle on \( f_s \) or \( F_s \), a basic tool for the analysis of parabolic equations. However, since we want a local result, remember that the variance depends on the position of the spectrum, we need to localize the maximum principle. We finish with an induction on the number of particles in the multi-particle random walk or a multi-scale argument. For the third step, we will need a continuity result for the Dyson Brownian motion which will be shown in Subsection 4.1 and we will give the reverse heat flow technique in Subsection 4.2. We will then conclude by combining the three steps in Section 5.

In Appendix A, we will study more the case where the initial condition is given by a semi-circular profile. Indeed, using the dynamics followed by the resolvent, it is possible to get an entrywise local law on the scale \( (N\eta)^{-1} \) instead of the usual \( (N\eta)^{-1/2} \). Indeed, combining both the equations (1.14) and (1.16), the singularities \((\lambda_i - \lambda_j)^{-1}\) and \((\lambda_i - \lambda_j)^{-2}\) exactly cancel out for the dynamics followed by the resolvent

\[
(q, G_t(z)q) := \frac{1}{N} \sum_{k=1}^{N} \frac{z_k(t)^2}{\lambda_k(t) - z}.
\]

This resolvent flow has been studied in the literature to show the stability of the Dyson Brownian motion at below microscopic scale in [39] or the localization of the eigenvectors of the Gaussian Rosensweig-Porter model in [40]. A similar equation was found for other observables in [8].

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\section{Local laws}

In this section, we focus on the different local laws result for \( W_t \). These local laws are high probability bounds, for simplicity we will now introduce the following notation for stochastic domination. For

\[
X = (X_N(\omega), N \in \mathbb{N}, \omega \in \Omega_N)
\]

\[
Y = (Y_N(\omega), N \in \mathbb{N}, \omega \in \Omega_N)
\]

two families of nonnegative random variables depending on \( N \) (note that \( \Omega_N \) can also depend on \( N \)), we will say that \( X \) is stochastically dominated by \( Y \) uniformly in \( \omega \), and write \( X \prec Y \), if for all \( \tau > 0 \) and \( D > 0 \) we have

\[
\sup_{\omega \in \Omega_N} P(X_N(\omega) > N^\tau Y_N(\omega)) \leq N^{-D}
\]

for \( N \) large enough. If we have \(|X| \prec Y\) for some family \( X \), we will write \( X = \mathcal{O}_\prec (Y) \).

Define now the resolvent of \( W_t \),

\[
G(z) = (W_t - z)^{-1} = \sum_{k=1}^{N} \frac{|u_k|^2}{\lambda_k - z},
\]

and denote \( G_{ij}(z) \) the \((i, j)\) entry of the resolvent matrix. In the rest of the section we will omit the dependence in \( t \) of the resolvent since we are not looking at its dynamics.

\subsection{Anisotropic local law for deformed Wigner matrices}

An averaged local law was proved in [29]. The proof relies on Schur’s complement formula, large deviations bounds and interlacing formula in order to first state a weak local law on the resolvent entries and the Stieltjes transform. The result then follows from a fluctuation averaging lemma in
order to go from the scale \((N\eta)^{-1/2}\) to \((N\eta)^{-1}\). We first give the definition of the limiting Stieltjes transform as the solution \(m_t(z)\) of the following equation

\[
m_t(z) = \frac{1}{N} \sum_{k=1}^{N} \frac{1}{D_i - z - tm_t(z)} = \frac{1}{N} \sum_{k=1}^{N} g_i(t, z)
\]  

(2.1)

where we defined

\[
g_i(t, z) := \frac{1}{D_i - z - tm_t(z)}.
\]

We will also need the following lemma on the Stieltjes transform claiming that its imaginary part is of order one.

**Lemma 2.1** ([29]). Let \(\vartheta > 0\) a small constant and \(\kappa \in (0, 1)\). Take \(z \in \mathcal{D}_r^{\vartheta, \kappa}\) for \(N\) large enough, the following bounds holds

\[
c \leq \text{Im} \ m_t(z) \leq C.
\]

Moreover

\[
c t \leq |D_i - z - tm_t(z)| \leq C.
\]

Note that the constants above do not depend on any parameter.

Here is the averaged local law taken from [29].

**Theorem 2.2** (Theorem 3.3, [29]). Let \(W_t\) be as in Definition 1.1, \(\vartheta > 0\) and \(\kappa \in (0, 1)\), for any \(z = E + i\eta \in \mathcal{D}_r^{\vartheta, \kappa}\), we have

\[
|s_t(z) - m_t(z)| \ll \frac{1}{N\eta} \tag{2.2}
\]

for \(N\) large enough.

The proof of Theorem 2.2 also gives the following entrywise local law from [29] also properly stated in [11].

**Theorem 2.3** ([29, 11]). Let \(W_t\) be as in Definition 1.1 and \(\vartheta > 0\), \(\kappa \in (0, 1)\). Uniformly in \(z = E + i\eta \in \mathcal{D}_r^{\vartheta, \kappa}\), we have for the diagonal entries

\[
|G_{ii}(z) - g_i(t, z)| \ll \frac{t}{\sqrt{N\eta}} |g_i(t, z)|^2, \tag{2.3}
\]

and for the off-diagonal entries

\[
|G_{ij}(z)| \ll \frac{1}{\sqrt{N\eta}} \min\{|g_i(t, z)|, |g_j(t, z)|\}. \tag{2.4}
\]

In order to study \(\langle q, u_k \rangle\), we will need the following local law for \(\langle q, G(z)q \rangle\) proved in [11].

**Theorem 2.4** ([11]). Let \(\vartheta > 0\), \(\kappa \in (0, 1)\) and \(q\) a \(L^2\)-normalized vector of \(\mathbb{R}^N\), for \(z \in \mathcal{D}_r^{\vartheta, \kappa}\) we have

\[
|\langle q, G(z)q \rangle - \sum_{k=1}^{N} q_k^2 g_k(t, z)| \ll \frac{1}{\sqrt{N\eta}} \text{Im} \left( \sum_{k=1}^{N} q_k^2 g_k(t, z) \right).
\]

This theorem also gives us control of the resolvent as a bilinear form by polarization. We will give the proof of this corollary for completeness.

**Corollary 2.5**. Let \(\vartheta > 0\), \(\kappa \in (0, 1)\), let \(v\) and \(w\) two \(L^2\)-normalized vectors of \(\mathbb{R}^N\), for \(z \in \mathcal{D}_r^{\vartheta, \kappa}\), we have

\[
|\langle v, G(z)w \rangle - \sum_{i=1}^{N} v_i w_i g_i(t, z)| \ll \frac{1}{\sqrt{N\eta}} \sqrt{\text{Im} \left( \sum_{i=1}^{N} v_i^2 g_i(t, z) \right) \text{Im} \left( \sum_{i=1}^{N} w_i^2 g_i(t, z) \right)}.
\]
Proof. Let \( \mu \in \mathbb{R} \), a parameter fixed later. Consider

\[
((v + \mu w), G(v + \mu w)) = \langle v, Gv \rangle + \mu^2 \langle w, Gw \rangle + 2\mu \langle v, Gw \rangle,
\]

by linearity and symmetry of the resolvent \( G \). On one hand, using Theorem 2.4 on the first two terms of the right hand side of (2.5), we get the equation

\[
((v + \mu w), G(v + \mu w)) = 2\mu \langle v, Gw \rangle + \sum_{i=1}^{N} v_i^2 g_i(t, z) + \mu^2 \sum_{i=1}^{N} w_i^2 g_i(t, z)
\]

\( + \mathcal{O}_\prec \left( \frac{1}{\sqrt{N} \eta} \left( \text{Im} \left( \sum_{i=1}^{N} v_i^2 g_i(t, z) + \mu^2 \sum_{i=1}^{N} w_i^2 g_i(t, z) \right) \right) \right) . \) (2.6)

On the other hand, using Theorem 2.4 on the left hand side of (2.5), we obtain

\[
((v + \mu w), G(v + \mu w)) = \sum_{i=1}^{N} (v_i + \mu w_i) g_i(t, z) + \mathcal{O}_\prec \left( \frac{1}{\sqrt{N} \eta} \text{Im} \left( \sum_{i=1}^{N} (v_i + \mu w_i) g_i(t, z) \right) \right). \) (2.7)

Finally, combining (2.6) and (2.7) and choosing

\[
\mu = \frac{\text{Im} \left( \sum_{i=1}^{N} v_i^2 g_i(t, z) \right)}{\text{Im} \left( \sum_{i=1}^{N} w_i^2 g_i(t, z) \right)}
\]

we get the final result. \( \square \)

This control of the resolvent allows us to give an upper bound for the moments of the eigenvectors of \( W_t \). Defining, for a fixed \( q \in S^{N-1} \),

\[
\varphi_{\lambda, t}(\eta) = E \left[ \prod_{k=1}^{N} \frac{(\sqrt{N}(q, u_k))^2 a_k}{a^2(2\eta_k)} \right] \lambda^{q_k}
\]

with \( u_1, \ldots, u_N \) the eigenvectors of \( W_t \) and \( \lambda \) the trajectories up to time \( t \) of its eigenvalues, we have the following corollary. Note that in the part of the spectrum where we are interested, the typical size of \( \sigma^2 \) and subsequently \( \sqrt{N}(q, u_k) \) is of order \( t^{-1} \), the following corollary gives us a high-probability bound of right order on the \( n \)-moment \( \varphi_t \).

**Corollary 2.6.** Let \( n \in \mathbb{N} \), \( \kappa \in (0, 1) \) then

\[
\|\varphi_{\lambda, t}\|_{\infty, n} := \sup_{\eta \in \mathcal{A}_\prec, N(\eta) = n} \varphi_{\lambda, t}(\eta) \propto \frac{1}{t^n} . \] (2.9)

**Proof.** Consider \( \eta = N^{1-\theta} \ll t \) and \( k \in \mathcal{A}_\prec^c \), we have the following first high probability bound with \( z_k = \lambda_k + \eta \)

\[
\frac{1}{\eta} \left( \sqrt{N}(q, u_k) \right)^2 = \frac{\left( \sqrt{N}(q, u_k) \right)^2}{(\lambda_k - \text{Re}(z_k))^2 + \eta^2} \leq N \text{Im} \left( (q, G(z_k)q) \right)
\]

\[
= N \text{Im} \left( \sum_{i=1}^{N} \frac{q_i^2}{D_i - z_k - tm_t(z_k)} \right) + \mathcal{O}_\prec \left( \frac{1}{\sqrt{N} \eta} \right) \left( \text{Im} \left( \sum_{i=1}^{N} \frac{q_i^2}{D_i - z_k - tm_t(z_k)} \right) \right) .
\]

We can then write,

\[
\left( \sqrt{N}(q, u_k) \right)^2 \propto N \eta \text{Im} \left( \sum_{i=1}^{N} \frac{q_i^2}{D_i - z_k - tm_t(z_k)} \right) \propto \frac{1}{t^{\kappa}}
\]

where we used the definition of \( \prec \) and that \( \vartheta \) is as small as we want. We finish the proof by definition of \( \varphi_t \). \( \square \)
The bound from Theorem 2.4 is at the core of the proof of our main result and is the reason why \( \sigma_t \) has a Cauchy profile. It can be seen as an averaged version of Theorem 1.3. Indeed, let \( z := E + i\eta \in D_{r,\kappa} \) then

\[
\text{Im} \left( \frac{1}{N} \sum_{k=1}^{N} \frac{\varphi_{\lambda_k}(k)}{\lambda_k - z} \right) = \text{Im} \left( \sum_{k=1}^{N} \frac{q_k^2}{D_k - z - tm_t(z)} \right) + O \left( \frac{1}{\sqrt{N\eta t}} \right)
\]

\[
= \sum_{k=1}^{N} \left( \frac{q_k^2 \text{Im} m_t(z)}{(D_k - E)^2 + (\text{Im} m_t(z))^2} \right) + O(z) \left( \frac{1}{\sqrt{N\eta t}} \right). \tag{2.10}
\]

The local law gives us a strong control on the spectral elements of our matrix ensemble. As \( W_t \) will undergo the Dyson Brownian motion, these quantities will still be controlled through a local law up to a small error coming from the time of the relaxation. This is the statement of the following lemma.

**Lemma 2.7.** Denote

\[
G(\tau, z) = (H_\tau - z)^{-1}, \quad \text{and} \quad s_\tau(z) = \frac{1}{N} \text{Tr} G(\tau, z),
\]

we have the following overwhelming-probability bounds,

(i) Uniformly for \( z \in S_\varepsilon \),

\[
\sup_{0 \leq \tau \leq tN^{-a}} |s_\tau(z) - m_t(z)| \prec \frac{1}{N\eta} + \frac{\tau}{t} \quad \text{and} \quad \sup_{0 \leq \tau \leq tN^{-a}} |\lambda_i(t + \tau) - \gamma_{i,t}| \prec N^{-1}
\]

uniformly for indices in \( A_\kappa \).

(ii) Conditionally on the trajectory \( \lambda \)

\[
\sup_{0 \leq \tau \leq tN^{-a}} \left| G(\tau, z)_{ii} - \frac{1}{D_i - z - tm_t(z)} \right| \prec \frac{1}{\sqrt{N\eta}} + \frac{\tau}{t},
\]

\[
\sup_{0 \leq \tau \leq tN^{-a}} \left| \sum_{\alpha \in I} G(\tau, z)_{\alpha\alpha} - \frac{1}{N} \sum_{\alpha \in I} \frac{1}{D_{\alpha} - z - tm_t(z)} \right| \prec \frac{\sqrt{\tau}}{\sqrt{N\eta}} + \frac{\tau^2}{t}. \tag{2.11}
\]

**Remark 2.8.** Note that the error of \( \frac{\sqrt{\tau}}{\sqrt{N\eta}} \) is not optimal when, for instance, \( |I| \) is of order \( N \) since the averaged local law holds and we obtain an error \( 1/N\eta \). Actually, this bound holds for \( |I| \) down to order \( Nt \ll N \) via a careful bookkeeping from the proof in [29]. We will not follow this path in this paper as this improved bound conflicts with an other error of larger order in the next section.

### 3. Short time relaxation

In this section, we are going to prove Theorems 1.3 and 1.8 for the Dyson Brownian motion starting from \( W_t \) using maximum principles.

Recall the dynamics of the eigenvector moment flow with \( n \) particles for \( H_\tau \).

\[
\begin{align*}
\partial_\tau f_\tau(\eta) &= \frac{1}{N} \sum_{i \neq j} 2\eta_i(1 + 2\eta_j) \left( f_\tau(\eta^{i,j}) - f_\tau(\eta) \right) \quad \text{=:} \ (\mathcal{B}_\tau f_\tau)(\eta), \\
fo(\eta) &= \varphi(\eta).
\end{align*}
\tag{3.1}
\]
where we noted $\lambda_i(t + \tau)$ the eigenvalues of $H_\tau$. Note that in the case of a single particle in $k$, we can write the dynamics

$$
\begin{cases}
\partial_\tau f_\tau(k) = \frac{2}{N} \sum_{j=1}^{N} \frac{f_\tau(j) - f_\tau(k)}{(\lambda_j(t + \tau) - \lambda_k(t + \tau))^2}, \\
f_0(k) = \varphi_\tau(k).
\end{cases}
$$

(3.2)

We cut the dynamics into two parts: the short range where most of the information will be and the long range. This decomposition in this context was first introduced in [20]. Let $1 \ll \ell \ll N$ be a parameter that we will choose later, we then define

$$
(\mathcal{F}_\tau f_\tau)(\eta) = \frac{1}{N} \sum_{|j-k| \leq \ell} \frac{2\eta_j(1 + 2\eta_j) (f_\tau(\eta^{\ell j}) - f(\eta))}{(\lambda_j - \lambda_j)^2},
$$

(3.3)

$$
(\mathcal{L}_\tau f_\tau)(k) = \frac{1}{N} \sum_{|j-k| > \ell} \frac{2\eta_j(1 + 2\eta_j) (f_\tau(\eta^{\ell j}) - f_\tau(\eta))}{(\lambda_j - \lambda_j)^2}.
$$

(3.4)

Denote by $U_{\mathcal{F}}(s, \tau)$ the semigroup associated with $\mathcal{F}$ from time $s$ to $\tau$:

$$
\partial_\tau U_{\mathcal{F}}(s, \tau) = \mathcal{F} U_{\mathcal{F}}(s, \tau)
$$

(3.5)

for any $s \leq \tau$. We will denote in the same way $U_{\mathcal{L}}$. The short range dynamics is then a good approximation of the global dynamics. It is stated as a lemma in [12],

**Lemma 3.1** ([12]). Assume that for some (small) fixed parameter $\varepsilon > 0$ there is a constant $C$ such that for any $|i - j| \geq N^\varepsilon$ and $0 \leq s \leq 1$, we have

$$
\frac{1}{N} \sum_{i,j} \frac{1}{(\lambda_i(s) - \lambda_j(s))^2} \leq C N^\varepsilon.
$$

(3.6)

Let $\ell > N^\varepsilon$, then for any $s \geq 0$, we have

$$
\|U_{\mathcal{L}}(0, s) - U_{\mathcal{F}}(0, s)\|_1 \leq C N s \frac{1}{\ell}.
$$

(3.7)

Now, it has been proved in [12] that the parabolic short range dynamics has a finite speed of propagation in the following sense, write

$$
p_s(\eta, \xi) = (U_{\mathcal{F}}(0, s)\delta_\eta)(\xi),
$$

and define the following distance on the set of configurations with $n$ particles

$$
d(\eta, \xi) = \sum_{\alpha=1}^{n} |x_\alpha - y_\alpha|.
$$

(3.8)

where $(x_1, \ldots, x_n)$ are the positions of the particles in nondecreasing order of $\eta$ and $y_\alpha$ of $\xi$. The following lemma then states that if two configurations are far from each other, the short-range dynamics started at one and evaluated at the other is exponentially small with high-probability.

**Lemma 3.2** ([12]). Choose $\ell \geq N \tau$, let $\varepsilon > 0$ be a small constant and $\kappa \in (0, 1)$. Uniformly in $\eta$ supported on $A_\kappa^\varepsilon$ and $\tau > 0$, if $d(\eta, \xi) \geq L N^\varepsilon$, we have

$$
P \left( p_s(\eta, \xi) > e^{-N^\varepsilon/2} \right) = \mathcal{O} \left( N^{-D} \right)
$$

(3.9)

for any $D > 0$.

The two previous lemmas will be very useful tools to prove Theorem 1.4. Indeed, the finite speed of propagation in Lemma 3.2 allows us to localize our problem but is a property of the short range dynamics, Lemma 3.1 then tells us that most of the information of the global dynamics is in this short range part.
3.1. Analysis of the moment observable

To prove Theorem 1.3 we will prove the following intermediary proposition,

**Proposition 3.3.** Let $\kappa \in (0,1)$, $\varepsilon$ a small positive constant and $n$ an integer, if $q \in \mathbb{S}^{N-1}$, for any $\eta \subset \mathcal{A}_\kappa^\varepsilon$ such that $N(\eta) = n$ we have for any

$$f_\tau(\eta) = \sum_{k=1}^N \sigma_k(q,k,\tau)^{2\eta_k} + O_{\prec} \left( \left( \frac{1}{\sqrt{\tau}} + \left( \frac{\tau}{7} \right)^{1/3} \right) t^{-n} \right). \quad (3.10)$$

where $\sigma_k(q,k,\tau)$ is given by (1.3).

The $1/3$ exponent that we give here in the error is not optimal. We are not able to reach an optimal error because of the strong dichotomy we do between the short range and the long range dynamics. Using a multi-scale partition of the dynamics could improve the error term. Note also that as said in Remark 1.4, the choice of the parameter $\eta$ corresponds to $N^{-\varepsilon} \tau$ (where we consider the $N^{-\varepsilon}$ in the definition of $\prec$ which optimize our error term.

Let $\varepsilon > 0$ be a small constant such that (3.6) holds. Recall that $t \in T_\omega$ and $\tau \ll t$. First, the following lemma gives us a local law for $f_\tau$, in the case of a single particle, deduced from both the entwary local law for $W_1$.

**Lemma 3.4.** For $z \in D^{b,\kappa}_\tau$, we have

$$\text{Im} \left( \frac{1}{N} \sum_{k=1}^N \frac{f_\tau(k)}{\lambda_k(t+\tau) - z} \right) = \text{Im} \left( \sum_{k=1}^N q_k^2 g_k(t,z) \right) + O_{\prec} \left( \frac{1}{\sqrt{\tau}} \right) \text{Im} \left( \sum_{k=1}^N q_k^2 g_k(t,z) \right).$$

**Proof.** See first that, by definition of $f_\tau$, we have

$$\frac{1}{N} \sum_{k=1}^N \frac{f_\tau(k)}{\lambda_k(t+\tau) - z} = E \left[ \langle q, G^{H^\tau}(z) q \rangle | \lambda \right]$$

where $G^{H^\tau} := (H^\tau - z)^{-1}$ is the resolvent of $H^\tau$. Now, the law of $H^\tau$ is $D + \sqrt{\tau} W + \sqrt{\tau} \text{GOE} \overset{d}{=} D + \sqrt{\tau + t} W'$ for some $W'$ a Wigner matrix. We can use the local law from [11] for this matrix and write

$$\text{Im} \left( \langle q, G^{H^\tau} q \rangle \right) = \text{Im} \left( \sum_{k=1}^N q_k^2 g_k(t+\tau,z) \right) + O_{\prec} \left( \frac{1}{\sqrt{\tau}} \right) \text{Im} \left( \sum_{k=1}^N q_k^2 g_k(t+\tau,z) \right)$$

with $m_{t+\tau}(z)$ the solution with positive imaginary part of the following self-consistent equation,

$$m_{t+\tau}(z) = \frac{1}{N} \sum_{k=1}^N \frac{1}{D_k - z - (t+\tau)m_{t+\tau}(z)}.$$

Note that we have, for $z \in D^{b,\kappa}_\tau$, $0 < \text{Im}(m_{t+\tau}(z)) \leq C$ for some constant $C$ so that, by a Taylor expansion in $\tau \ll t$ (remember that $\eta \ll t$ for $z \in D^{b,\kappa}_\tau$), we obtain, recalling the definition of $g_k(t,z),$

$$\text{Im} \left( \sum_{k=1}^N q_k^2 g_k(t+\tau,z) \right) = \text{Im} \left( \sum_{k=1}^N \frac{q_k^2}{D_k - z - tm_{t+\tau}(z)} \right) + O_{\prec} \left( \frac{1}{t} \right) \text{Im} \left( \sum_{k=1}^N \frac{q_k^2}{D_k - z - tm_{t+\tau}(z)} \right).$$

We get the final result with the bound $|\partial t m_t(z)| \leq C \log N/t$ deduced from the time evolution of $m_t,

$$\partial_t m_t(z) = \partial_z (m_t(z)(m_t(z) + z))$$

combined with the estimates $|\partial z m_t(z)| \leq C/t$ and $|m_t(z)| \leq \log N.$
Let $\xi_w \subset A_\sigma^+$ be a fixed configuration, we want to use a maximum principle on a window centered around $\xi_w$, of size $u$. Since we make a small perturbation $\tau \ll t$, in order to be able to see anything in this window, we need to have $u \gg N\tau$. Furthermore, we want to look in the part of the spectrum where the eigenvector will be of typical size $1/t$, to localize the dynamics in this small part of the spectrum. We then need to take $u \ll Nt$.

We define the following flattening and averaging operator, for $a > 0$

$$ (\text{Flat}^2_{\xi_w} f)(\eta) = \begin{cases} f(\eta) \\ \prod_{k=1}^N \sigma_t(q,k)^{2\xi_w k} \end{cases} \text{ if } d(\eta, \xi_w) \leq a, \quad \text{if } d(\eta, \xi_w) > a $$

(3.11)

and

$$ (\text{Avg}_{\xi_w} f)(\eta) = \frac{1}{u} \int_{u/2}^u (\text{Flat}^2_{\xi_w} f)(\eta) da. $$

(3.12)

Notice that for every $\eta$, there exists $a_\eta \in [0,1]$ such that

$$ \text{Avg}_{\xi_w} f(\eta) = a_\eta f(k) + (1 - a_\eta) \prod_{k=1}^N \sigma_t(q,k)^{2\xi_w k}. $$

(3.13)

To show the result (3.10) by induction, we will first prove in the case of one particle with the dynamics (3.2).

**Proof in the case of a single particle.** We first prove

$$ f_t(k) = \sigma_t(q,k)^2 + O_\prec \left( \left( \frac{1}{\sqrt{N\eta}} + \left( \frac{\tau}{t} \right)^{1/3} \right) t^{-1} \right). $$

(3.14)

To do so, we want to use a localized maximum principle centered around $k_w$ which is the position of the particle for the configuration $\xi_w$ in this case. However, we need to know that the maximum stays in that window, that is why we first flatten and average $f_t(k)$ and use it as an initial condition for the dynamics (1.20). We will then make the short range dynamics work on $f_t(k)$ during a time $\tau \ll t$. Since we use the short range dynamics for a time $\tau$, we will be able to use the finite speed of propagation (3.2) and we should choose $t \gg N\tau$ for the range cut-off.

Consider $g_t$,

$$ \partial_t g_t(k) = \frac{1}{N} \sum_{|j-k| \leq \ell} \frac{g_t(j) - g_t(k)}{(\lambda_j(t + \tau) - \lambda_k(t + \tau))^2} \quad \text{with} \quad g_0(k) = (\text{Avg}_{k_w} \varphi_t)(k). $$

First note that, in order to prove (3.14), we will first show that

$$ g_t(k) = \sigma_t(q,k)^2 + O_\prec \left( \frac{\ell}{u} + \frac{N\tau}{\ell} + \frac{1}{\sqrt{N\eta}} + \frac{\tau}{u} + \frac{u}{N\eta} \right) t^{-1}, $$

where the parameter $\eta$, the spectral resolution, follows $N^{-1} \ll \eta \ll t$. Indeed we have,

$$ |f_t(k) - g_t(k)| = \left| (U_{\varphi_t}(0,t)\varphi_t)(k) - (U_{\varphi_t}(0,t)\text{Avg}_{k_w} \varphi_t)(k) \right| $$

$$ = \left| (U_{\varphi_t}(0,t) - U_{\varphi_t}(0,t))\varphi_t \right| \left( U_{\varphi_t}(0,t)(\text{Id} - \text{Avg}_{k_w})\varphi_t \right)(k). $$

(3.15)

By Lemma 3.1 and $\|\varphi_t\|_{\infty,1} \ll 1/t$, we get

$$ \left| (U_{\varphi_t}(0,t) - U_{\varphi_t}(0,t))\varphi_t \right| \left( U_{\varphi_t}(0,t)(\text{Id} - \text{Avg}_{k_w})\varphi_t \right)(k) = O_\prec \left( \frac{N\tau}{\ell t} \right). $$

(3.16)

By Lemma 3.1 and $\|\varphi_t\|_{\infty,1} \ll 1/t$, we get

$$ \left| (U_{\varphi_t}(0,t) - U_{\varphi_t}(0,t))\varphi_t \right| \left( U_{\varphi_t}(0,t)(\text{Id} - \text{Avg}_{k_w})\varphi_t \right)(k) = O_\prec \left( \frac{N\tau}{\ell t} \right). $$

(3.17)
Now, for the second term in (3.16). Since \((\varphi_t - \text{Av}_{k_w} \varphi_t)(k) = 0\) for \(k \in [k_w - u/2, k_w + u/2]\), and \(u \gg \ell N^\varepsilon\) by definition, looking at \(k \in [k_w - u/3, k_w + u/3]\) for instance, Lemma 3.2 tells us that the term is exponentially small. Thus, we obtain

\[
f_r(k) = g_r(k) + O_{\prec} \left( \frac{N\tau}{\ell t} \right).
\]

We will first prove the following equation which can be seen as an averaged version of the result. For \(k_0 \in [k_w - u, k_w + u]\), set \(z(k_0) = \gamma_{k_0,t} + i\eta \in \mathcal{D}_{\rho^k}\) we will show

\[
\left| \text{Im} \left( \frac{1}{N} \sum_{k=1}^{N} \frac{g_r(k)}{\lambda_k(t) - z(k_0)} \right) - \text{Im}(m_t(z(k_0)))\right| \sigma_t(q, k_w)^2 \lesssim \left( \frac{\ell}{u} + \frac{N\tau}{\ell} + \frac{1}{\sqrt{N\eta}} + \frac{\tau}{\eta} + \frac{u}{N\ell} \right) \frac{1}{\ell}.
\]

(3.18)

First, decompose the left hand side term into three different terms:

\[
\left| \text{Im} \left( \frac{1}{N} \sum_{k=1}^{N} \frac{(U_{\varphi}(0, \tau)\text{Av}_{k_w} \varphi_t)(k) - (\text{Av}_{k_w} U_{\varphi}(0, \tau) \varphi_t)(k)}{\lambda_k - z(k_0)} \right) \right| \quad (3.19)
\]

\[
+ \left| \text{Im} \left( \frac{1}{N} \sum_{k=1}^{N} \frac{(\text{Av}_{k_w} U_{\varphi}(0, \tau) \varphi_t)(k) - \text{Av}_{k_w} f_r(k)}{\lambda_k - z(k_0)} \right) \right| \quad (3.20)
\]

\[
+ \left| \text{Im} \left( \frac{1}{N} \sum_{k=1}^{N} \frac{\text{Av}_{k_w} f_r(k)}{\lambda_k - z(k_0)} \right) - \text{Im}(m_t(z(k_0)))\right| \sigma_t(q, k_w)^2 . \quad (3.21)
\]

To bound (3.19), we write

\[
(U_{\varphi}(0, \tau)\text{Av}_{k_w} \varphi_t)(k) - (\text{Av}_{k_w} U_{\varphi}(0, \tau) \varphi_t)(k) = \frac{2}{u} \int_{u/2}^{u} \Omega^g_{k_w}(\varphi_t)(k)da
\]

with

\[
\Omega^g_{k_w}(\varphi_t)(k) = (U_{\varphi}(0, \tau)\text{Flat}^g_{k_w} \varphi_t)(k) - (\text{Flat}^g_{k_w} U_{\varphi}(0, \tau) \varphi_t)(k).
\]

Look now at what happens around \(k_w - a\), the other boundary of the window \(k_w + a\) can be bounded exactly the same way.

By finite speed of propagation, for \(k > k_w - a - \ell N^\varepsilon\), we easily get

\[
(U_{\varphi}(0, \tau)\text{Flat}^g_{k_w} \varphi_t)(k) = (U_{\varphi}(0, \tau)\text{Flat}^g_{k_w} \varphi_t)(k) + O_{\prec} \left( e^{-N^\varepsilon/2} \right).
\]

The same equality is true for \(k < k_w - a + \ell N^\varepsilon\) using the same argument.

For \(k_w - a - \ell N^\varepsilon \leq k \leq k_w - a + \ell N^\varepsilon\), since the operator \(U_{\varphi}\) is bounded in \(L_\infty\), we have

\[
|\Omega^g_{k_w}(\varphi_t)(k)| \lesssim 2 \|\varphi_t\|_{L_1} \lesssim \frac{1}{\ell}.
\]

(3.22)

We can then bound (3.19) by

\[
(3.19) \lesssim \frac{\ell}{ut} \text{Im} \left( m_t(z(k_0)) \right) \lesssim \frac{\ell}{ut},
\]

(3.23)

where we used that \(\text{Im}(m_t(z(k_0))) \ll 1\) in the bulk of the spectrum.

To bound (3.20), noting that \(f_r = U_{\varphi}(0, \tau) \varphi_t\),

\[
|(\text{Av}_{k_w} U_{\varphi}(0, \tau) \varphi_t)(k) - (\text{Av}_{k_w} U_{\varphi}(0, \tau) \varphi_t)(k)| \leq ||(U_{\varphi}(0, \tau) - U_{\varphi}(0, \tau)) \varphi_t\| \lesssim \frac{N\tau}{\ell t}
\]

where we used the fact that \(\|\varphi_t\|_{L_1} \lesssim \frac{1}{\ell}\) and applied Lemma 3.1.

Thus we have

\[
(3.20) \lesssim \frac{CN\tau}{\ell t} \text{Im} \left( \frac{1}{N} \sum_{k=1}^{N} \frac{1}{\lambda_k - z(k_0)} \right) \lesssim \frac{N\tau}{\ell t}
\]

(3.24)

where we used the averaged local law and the fact that in the bulk we have \(\text{Im}(m_t(z(k_0))) \ll 1\).
To bound (3.21), we want to use (2.10) which comes from the local law. Recall that \( z^{(k_0)} = \gamma_{k_0} + i\eta \), then

\[
\text{Im} \left( \frac{1}{N} \sum_{k=1}^{N} \frac{(Av_{k,\sigma}(0, \tau) \varphi_t)(k)}{\lambda_k - z^{(k_0)}} \right) = \text{Im} \left( \frac{1}{N} \sum_{|k-k_0| \leq N \sqrt{\eta}} \frac{(Av_{k,\sigma}(0, \tau) \varphi_t)(k)}{\lambda_k - z^{(k_0)}} \right) + O_\prec \left( \frac{\eta}{\ell^2} \right).
\]

If we use the notation (3.13), we obtain

\[
\text{Im} \left( \frac{1}{N} \sum_{k=1}^{N} \frac{(Av_{k,\sigma}(0, \tau) \varphi_t)(k)}{\lambda_k - z^{(k_0)}} \right) = \text{Im} \left( \frac{1}{N} \sum_{|k-k_0| \leq N \sqrt{\eta}} \frac{(a_k f_r(k) + (1 - a_k) \sigma_t(q, k_w)^2)}{\lambda_k - z^{(k_0)}} \right) + O_\prec \left( \frac{\sqrt{\eta}}{\ell} \right).
\]

Combining (3.23), (3.24) and (3.30), we get the final result

\[
\text{Im} \left( \frac{1}{N} \sum_{k=1}^{N} \frac{g_r(k)}{\lambda_k - z^{(k_0)}} \right) - \text{Im}(m_t(z^{(k_0)})) \sigma_t(q, k_w)^2 = O_\prec \left( \left( \frac{\ell}{u} + \frac{N \tau}{\ell} + \frac{1}{\sqrt{\eta}} + \frac{\tau + u}{N \ell} \right) \frac{1}{\ell} \right).
\]

Note now that

\[
\left| a_{k_0} \text{Im} \left( \frac{1}{N} \sum_{k=1}^{N} \frac{f_r(k)}{\lambda_k - z^{(k_0)}} \right) \right| + (1 - a_{k_0}) \sigma_t(q, k_w)^2 \text{Im}(m_t(z^{(k_0)})) + O_\prec \left( \frac{1}{N^2 \eta} + \frac{\tau}{\ell^2} \right)
\]

so that, by Lemma 3.4 and the averaged local law, we obtain

\[
(3.27) = a_{k_0} \text{Im}(m_t(z^{(k_0)})) \sigma_t(q, k_0)^2 + (1 - a_{k_0}) \sigma_t(q, k_w)^2 \text{Im}(m_t(z^{(k_0)})) + O_\prec \left( \frac{1}{\sqrt{\eta}} + \frac{\tau}{\ell^2} + \frac{u}{N \ell^2} \right)
\]

where we used that by definition of \( \sigma_t(q, \cdot) \) and since \( k \) is in \([k_w - u, k_w + u]\),

\[
|\sigma_t(q, k_w)^2 - \sigma_t(q, k)^2| \leq \frac{1}{\ell^2} |\gamma_{k_w, t} - \gamma_{k, t}| \leq \frac{u}{\ell^2 N}.
\]

Finally, with the elementary property \( |a_i - a_k| \leq \frac{C|t-k|}{N} \), we get that (3.28) \( \leq C \frac{\sqrt{\eta}}{\ell^2} \). Putting these estimates together, we obtain

\[
(3.21) = O_\prec \left( \frac{1}{\sqrt{\eta}} + \frac{\tau}{\ell^2} + \frac{u}{N \ell^2} \right).
\]

Combining (3.23), (3.24) and (3.30), we get the final result

\[
\left| \text{Im} \left( \frac{1}{N} \sum_{k=1}^{N} \frac{g_r(k)}{\lambda_k - z^{(k_0)}} \right) - \text{Im}(m_t(z^{(k_0)})) \sigma_t(q, k_w)^2 \right| = O_\prec \left( \left( \frac{\ell}{u} + \frac{N \tau}{\ell} + \frac{1}{\sqrt{\eta}} + \frac{\tau + u}{N \ell} \right) \frac{1}{\ell} \right).
\]

Now, we just need to prove that

\[
|g_r(k) - \sigma_t(q, k)| = O_\prec \left( \left( \frac{N \eta}{\ell} + \frac{\ell}{u} + \frac{N \tau}{\ell} + \frac{1}{\sqrt{\eta}} + \frac{\tau + u}{N \ell} \right) \frac{1}{\ell} \right).
\]
Let \( k_m \) be the index such that \( g_r(k_m) = \max_k g_r(k) \) and \( z = \lambda_{k_m} + i\eta \). If we have that
\[
|g_r(k_m) - \sigma_t(q, k_m^2)| \leq N^{-10},
\] there is nothing to prove. Now if the left hand side is greater than \( N^{-10} \), by finite speed of propagation, \( k_m \) is in the interval \([k_w - u, k_w + u]\). Indeed, if it is not then the difference in (3.33) would be exponentially small. We then have
\[
\partial_t g_r(k_m) = \frac{1}{N} \sum_{|j-k_m| \leq \ell, j \neq k_m} g_r(j) - g_r(k_m)
\]
\[
\leq \frac{1}{N\eta} \sum_{|j-k_m| \leq \ell, j \neq k_m} \frac{\eta g_r(j)}{(\lambda_j - \lambda_{k_m})^2 + \eta^2} = \frac{g_r(k_m)}{N\eta} \sum_{|j-k_m| \leq \ell, j \neq k_m} \frac{\eta}{(\lambda_j - \lambda_{k_m})^2 + \eta^2}
\]
\[
\leq \frac{1}{\eta} \Im \left( \frac{1}{N} \sum_{k=1}^N \frac{g_r(k)}{\lambda_k - z} \right) = \frac{g_r(k_m)}{\eta} \Im(m(z)) + O_{\prec} \left( \frac{N}{\ell t} + \frac{1}{N\eta^2 t} \right)
\]
\[
\leq \frac{C}{\eta} (\sigma_t(q, k_w^2 - g_r(k_m))) + O_{\prec} \left( \frac{N}{\ell t} + \frac{1}{\sqrt{N\eta}} + \frac{1}{\sqrt{N\eta}} + \frac{1}{\tau t} + \frac{1}{N t} \right) (1) \quad (3.34)
\]
where we used in the first inequality that \( g_r(k_m) \) is the maximum, in the second inequality that extending the sum to all \( j \) adds an error \( N^{1+\epsilon}\eta/\ell t \). Finally in the last inequality we used (3.18), \( c \leq \Im(m(z)) \leq C \) in the bulk and that the rigidity errors that appears from changing \( \lambda_{k_m} \) into \( \gamma_{k_m,t} \) are smaller than the other terms. Injecting (3.29) in (3.34), and denoting \( S_r = g_r(k_m) - \sigma_t(q, k_m^2) \), we get
\[
\partial_S S_r \leq - \frac{C}{\eta} S_r + O_{\prec} \left( \left( \frac{N\eta}{\ell t} + \frac{\ell}{u} + \frac{N\tau}{\ell t} + \frac{1}{\sqrt{N\eta}} + \frac{1}{\tau t} + \frac{1}{N t} \right) \frac{1}{\eta t} \right).
\]
Using Gronwall’s lemma, we have
\[
S_r = O_{\prec} \left( \left( \frac{N\eta}{\ell t} + \frac{\ell}{u} + \frac{N\tau}{\ell t} + \frac{1}{\sqrt{N\eta}} + \frac{1}{\tau t} + \frac{1}{N t} \right) \frac{1}{t} \right).
\]
We can do the same reasoning with the infimum. Finally taking the following parameters
\[
\eta = N^{-\tau}, \quad u = (N\tau(N\tau)^2)^{1/3}, \quad \ell = \sqrt{N\tau u} \quad (3.35)
\]
we get the result for a single particle.

Proof in the case of \( n \) particles. In the previous part of the proof, we looked only at the second moment \( E[N(q, u_k(t))^2|X] \) which corresponds to a single particle in the site \( k \). Now, we will do the proof of (3.10) by induction on the number of particles.

We can first define the same objects as the single particle case: we will look at the short range dynamics for a small time \( \tau \ll t \) with initial condition an average of the eigenvectors moment of \( W_t \) localized onto a specific window. More precisely define, with \( \xi_w \) being the configuration with \( n \) particles that lies at the center of our window of size \( u \),
\[
\partial_t g_r(q) = \frac{1}{N} \sum_{|i-j| \leq \ell, i \neq j} g_r(q^{ij}) - g_r(q)
\]
\[
g_0(q) = \langle \lambda \xi_w f_t \rangle(q).
\]
By the same reasoning as for the first particle case, using Lemmas 3.1 and 3.2 with \( n \) particles, we get
\[
|f_r(q) - g_r(q)| \prec \frac{CN\tau}{\ell t n}. \quad (3.38)
\]
To reason by induction on the number of particles, we need to show the following equation, similar to (3.18) in the case of one particle. For \( k_r \in A^*_r \), define \( z^{(k_r)} = \gamma_{k_r} + i\eta \) and let \( \eta \) a configuration of \( n \) particles with at least one particle in \( k_r \), we need to show

\[
\text{Im} \left( \frac{1}{N} \sum_{k=1}^{N} \frac{g_{\tau_r} (\eta^{k_r,k})}{\lambda_k(t + \tau) - z^{(k_r)}} \right) - \left( a_{\eta} f_{\tau_r} (\eta \setminus k_r) \sigma_t(q, k_r)^2 + (1 - a_{\eta}) \prod_{k=1}^{N} \sigma_t(q, k)^{2\xi_{u_k}} \right)
= \mathcal{O}_{\lesssim} \left( \left( \frac{\ell}{u} + \frac{N\tau}{\ell} + \frac{1}{\sqrt{N\eta}} + \frac{\tau}{\ell^{\eta}} \right) \frac{1}{\ell^n} \right),
\]

(3.39)

where \( \eta \setminus k_r \) denote the configuration where we removed one particle in \( k_r \) from \( \eta \).

We apply the same decomposition in three terms as in the single particle case, the first two terms can be bounded the same way and we can bound the left hand side of (3.39) by

\[
\left| \text{Im} \left( \frac{1}{N} \sum_{k=1}^{N} \left( \text{Av}_{\xi_{\omega}} f_{\tau_r} (\eta^{k_r,k}) \right) \frac{(\eta^{k_r,k})}{\lambda_k - z^{(k_r)}} \right) - \left( a_{\eta} f_{\tau_r} (\eta \setminus k_r) \text{Im}(n_t(z^{(k_r)}))\sigma_t(q, k_r)^2 + (1 - a_{\eta})\text{Im}(n_t(z^{(k_r)})) \prod_{k=1}^{N} \sigma_t(q, k)^{2\xi_{u_k}} \right) \right|
= \mathcal{O}_{\lesssim} \left( \frac{\ell}{u} + \frac{N\tau}{\ell} \right) \frac{1}{\ell^n} \quad (3.40)
\]

Now we need to see that,

\[
\text{Av}_{\xi_{\omega}} f_{\tau_r} (\eta^{k_r,k}) = a_{\eta^{v_r,k}} f_{\tau_r}(\eta^{k_r,k}) + (1 - a_{\eta^{v_r,k}}) \prod_{k=1}^{N} \sigma_t(q, k)^{2\xi_{u_k}}
= \left( a_{\eta} f_{\tau_r}(\eta^{k_r,k}) + (1 - a_{\eta}) \prod_{k=1}^{N} \sigma_t(q, k)^{2\xi_{u_k}} \right)
+ \left( (a_{\eta^{v_r,k}} - a_{\eta}) f_{\tau_r}(\eta^{k_r,k}) + (a_{\eta} - a_{\eta^{v_r,k}}) \prod_{k=1}^{N} \sigma_t(q, k)^{2\xi_{u_k}} \right)
= \left( a_{\eta} f_{\tau_r}(\eta^{k_r,k}) + (1 - a_{\eta}) \prod_{k=1}^{N} \sigma_t(q, k)^{2\xi_{u_k}} \right) + \mathcal{O}_{\lesssim} \left( \frac{d(\eta^{k_r,k}, \eta^{v_r,k})}{N} \right).
\]

We can use the same decomposition into \( |k - k_r| \leq N\sqrt{\eta} \) and the averaged local law to get

\[
\text{Im} \left( \frac{1}{N} \sum_{k=1}^{N} \text{Av}_{\xi_{\omega}} f_{\tau_r}(\eta^{k_r,k}) \frac{f_{\tau_r}(\eta^{k_r,k})}{\lambda_k - z^{(k_r)}} \right) = a_{\eta} \text{Im} \left( \frac{1}{N} \sum_{k=1}^{N} f_{\tau_r}(\eta^{k_r,k}) \frac{f_{\tau_r}(\eta^{k_r,k})}{\lambda_k - z^{(k_r)}} \right)
+ (1 - a_{\eta})\text{Im}(n_t(z^{(k_r)})) \prod_{k=1}^{N} \sigma_t(q, k)^{2\xi_{u_k}} + \mathcal{O}_{\lesssim} \left( \frac{1}{N\eta t^n} \right).
\]

(3.41)

Look now at the sum in the right hand side, recall that there’s at least one particle in \( k_r \) and denote \( k_1, \ldots, k_m \) with \( m \leq n \), the sites where there is at least one particle in the configuration \( \eta \).
Recall that $z^{(k_r)} = \gamma_{k_r, t} + i\eta$,

\[
\frac{1}{N} \sum_{k=1}^{N} \frac{\eta f_r(\eta^{k_r,k})}{(\gamma_{k_r, t} - \lambda_k)^2 + \eta^2} = \frac{1}{N} \sum_{k \not\in \{k_1, \ldots, k_m\}} \frac{\eta f_r(\eta^{k_r,k})}{(\gamma_{k_r, t} - \lambda_k)^2 + \eta^2} + O\left(\frac{1}{N\eta^n}\right) \tag{3.42}
\]

\[
= \frac{1}{N} \sum_{k \not\in \{k_1, \ldots, k_m\}} \eta E \left[ \prod_{1 \leq r' \leq m} \frac{z_{r',r}}{\eta^{2(r'-1)}} \times \frac{2^{z_r^{(r-1)}}}{a(2(j_r - 1))} \right] + O\left(\frac{1}{N\eta^n}\right) \tag{3.43}
\]

\[
= E \left[ \prod_{1 \leq r' \leq m} \frac{z_{r',r}}{a(2(j_r - 1))} \frac{\eta z_{r}^2}{(\gamma_{k_r, t} - \lambda_k)^2 + \eta^2} \right] + O\left(\frac{1}{N\eta^n}\right) \tag{3.44}
\]

By Lemma 3.4, we have,

\[
\frac{1}{N} \sum_{j \not\in \{k_1, \ldots, k_m\}} \frac{\eta z_{j}^2}{(\gamma_{k_r, t} - \lambda_j)^2 + \eta^2} = \frac{1}{N} \sum_{k=1}^{N} \frac{\eta z_{k}^2}{(\gamma_{k_r, t} - \lambda_k)^2 + \eta^2} + O\left(\frac{1}{N\eta^l}\right), \tag{3.45}
\]

\[
= \text{Im} \left( \frac{1}{N} \sum_{k=1}^{N} \frac{z_{k}^2}{\lambda_k - z} \right) + O\left(\frac{1}{N\eta^l}\right), \tag{3.46}
\]

\[
= \text{Im}(m_t(z^{(k_r)}))\sigma_t(q, k_r)^2 + O\left(\frac{1}{\sqrt{N\eta^l}} + \frac{\tau}{\nu + 1}\right). \tag{3.47}
\]

Combining (3.47) and (3.44), we get

\[
\frac{1}{N} \sum_{k=1}^{N} \frac{\eta f_r(\eta^{k_r,k})}{(\gamma_{k_r, t} - \lambda_k)^2 + \eta^2} = \text{Im}(m_t(z^{(k_r)}))\sigma_t(q, k_r)^2 f_r(\eta \setminus k_r) + O\left(\frac{1}{\sqrt{N\eta^l}} + \frac{\tau}{\nu + 1}\right) \tag{3.48}
\]

Finally, combining (3.48) and (3.41), we have

\[
\text{Im} \left( \frac{1}{N} \sum_{k=1}^{N} \frac{\eta f_r(\eta^{k_r,k})}{\lambda_k - z^{(k_r)}} \right) = a \eta f_t(\eta \setminus k_r)\text{Im}(m_t(z^{(k_r)}))\sigma_t(q, k_r)^2
\]

\[
+ (1 - a) \prod_{k=1}^{N} \text{Im}(m_t(z^{(k_r)}))\sigma_t(q, k_r)^2 \xi_{k_r} + O\left(\frac{1}{\sqrt{N\eta^l}} + \frac{\tau}{\nu + 1}\right)
\]

which, combined with (3.40), gives us (3.39).

We now follow the same proof as in the case of one particle: we state a maximum principle on the flattened and averaged moment. First define

\[
\xi_m = \max_{\eta \in \mathcal{M}} g_t(\eta), \tag{3.49}
\]

and let $k_1, \ldots, k_m$ be the positions of the particles of the configuration $\xi_m$ with $m \leq n$. We are
going to use our induction hypothesis in the maximum principle inequalities by (3.39).

\[ \partial_t g_r(\mathbf{\xi}_m) \leq \frac{C}{N} \sum_{|i-j| \leq 1 \atop i \neq j} \frac{g_r(\mathbf{\xi}_m^{i,j}) - g_r(\mathbf{\xi}_m)}{(\lambda_i - \lambda_j)^2} \]

\[ \leq \frac{C}{N} \sum_{r=1}^{m} \left( \frac{1}{\eta} \sum_{|k_r| \leq 1} \frac{\eta g_r(\mathbf{\xi}_m^{k_r})}{(\lambda_j - \lambda_{k_r})^2 + \eta^2} - \frac{g_r(\mathbf{\xi}_m)}{\eta} \sum_{|k_r| \leq 1 \atop j \neq k_r} \frac{\eta}{(\lambda_j - \lambda_{k_r})^2 + \eta^2} \right) \]

\[ \leq \frac{C}{\eta} \sum_{r=1}^{m} \left[ \text{Im} \left( \frac{1}{N} \sum_{j=1}^{N} \frac{g_r(\mathbf{\xi}_m^{k_r,j})}{\lambda_j - z^{(k_r,j)}} \right) - g_r(\mathbf{\xi}_m) \text{Im} \left( s_r(z^{(k_r,j)}) \right) \right] + O \left( \frac{1}{l^n} \right) \]

\[ \leq C \sum_{r=1}^{m} \left( a_{\xi_m} f_r(\mathbf{\xi}_m \setminus k_r) \text{Im}(m_t(z^{(k_r)})) \sigma_t(q, k_r)^2 + (1 - a_{\xi_m}) \text{Im}(m_t(z^{(k_r)})) \prod_{k=1}^{N} \sigma_t(q, k)^2 \right) \]

\[ - C \sum_{r=1}^{m} g_r(\mathbf{\xi}_m) \text{Im}(m_t(z^{(k_r)})) + O \left( \frac{1}{\sqrt{N\eta l^n}} + \frac{\tau}{\eta l^{n+1}} + \frac{\ell}{u l^n} + \frac{N\tau}{\ell l^n} + \frac{N}{\ell l^n} + \frac{1}{N\eta l^n} \right). \]

Now, we use the induction assumption on \( f_r(\mathbf{\xi}_m \setminus k_r) \) which is a \((n-1)\)th moment and obtain

\[ f_r(\mathbf{\xi}_m \setminus k_r) \sigma_t(q, k_r)^2 = \prod_{k=1}^{N} \sigma_t(q, k)^2 \xi_\mathfrak{w} + O \left( \left( \frac{1}{\sqrt{N\tau}} + \left( \frac{\tau}{\ell} \right)^{1/3} \right)^{1/2} \right) \]

Besides, using the same reasoning as in (3.29), if \( x_1 \leq \cdots \leq x_n \) are the positions of the particles in the configuration \( \mathbf{\xi}_w \) and \( y_1 \leq \cdots \leq y_N \) the positions in \( \mathbf{\xi}_m \) and writing

\[ \prod_{k=1}^{N} \sigma_t(q, k)^2 \xi_\mathfrak{w} = \prod_{\alpha=1}^{n} \sigma_t(q, x_\alpha)^2, \]

we can easily see that, since \( d(\mathbf{\xi}_w, \mathbf{\xi}_m) \leq 2u \)

\[ \left| \prod_{\alpha=1}^{n} \sigma_t(q, x_\alpha)^2 - \prod_{\alpha=1}^{n} \sigma_t(q, y_\alpha)^2 \right| \leq C \frac{u}{N l^{n+1}}. \]

Now, injecting (3.51) and (3.53) in (3.50), we get

\[ \partial_t \left( g_r(\mathbf{\xi}_m) - \prod_{k=1}^{N} \sigma_t(q, k)^2 \xi_\mathfrak{w} \right) \leq -C \sum_{r=1}^{m} \left( g_r(\mathbf{\xi}_m) - \prod_{k=1}^{N} \sigma_t(q, k)^2 \xi_\mathfrak{w} \right) \]

\[ + O \left( \left( \frac{N\tau}{\ell} + \frac{\ell}{u} + \frac{1}{\sqrt{N\eta}} + \frac{\tau}{\ell} + \frac{\eta}{\ell} + \frac{u}{N\tau} + \left( \frac{\tau}{\ell} \right)^{1/3} \right)^{1/2} \right) \frac{1}{l^n}. \]

Doing the same reasoning as in the proof for one particle, we get, by applying Gronwall’s lemma,

\[ g_r(\mathbf{\xi}_m) = \prod_{k=1}^{N} \sigma_t(q, k)^2 \xi_\mathfrak{w} + O \left( \left( \frac{N\tau}{\ell} + \frac{\ell}{u} + \frac{1}{\sqrt{N\eta}} + \frac{\tau}{\ell} + \frac{\eta}{\ell} + \frac{u}{N\tau} + \left( \frac{1}{\sqrt{N\tau}} + \left( \frac{\tau}{\ell} \right)^{1/3} \right)^{1/2} \right) \frac{1}{l^n} \right) \]

We can again do the same reasoning with the infimum and choosing the parameters as in (3.35) the claim from Proposition 3.3 follows. \( \Box \)
Consider now a deterministic set of indices $I \subset [1,N]$. Note that in the definition of the centered overlaps $p_{ii}$, we can only center by a constant not depending on $i$. However in Theorem 1.8, one can see that the expectation of the probability mass of the $i$th-eigenvector on $I$ clearly depends on $i$. Thus, we will need to localize our perfect matching observables onto a window of size $w$ chosen later and show that these $p_{ii}$ are, up to an error depending on $w$, centered around the same constant. More precisely, we will fix an integer $i_0 \in \mathcal{A}^c_{\eta}$ and look at the set of indices

$$\mathcal{A}_\eta^{(i)}(i_0) = \{ i \in [1,N], \gamma_{t,i} \in [\gamma_{t,i_0} - (1 - \kappa)w, \gamma_{t,i_0} + (1 - \kappa)w] \}$$

so that we will take for our centered diagonal overlaps

$$p_{ii} = \sum_{\alpha \in I} u_\alpha(\alpha)^2 - C_w$$

with $C_w = \frac{1}{N} \sum_{\alpha \in I} \sigma^2_\alpha(e_\alpha, i_0)$, \hspace{1cm} (3.54)

the overlaps for $i \neq j$ will not change. We will now prove the result from Theorem 1.8 for a Gaussian divisible ensemble for $p_{ii,i_0}$. We will need the following technical lemma allowing us to bound the $p_{ij}$ by the perfect matching observables.

**Lemma 3.5 ([13]).** Take an even integer $n$, there exists $C > 0$ depending on $n$ such that for any $i < j$ and any time $s$ we have

$$\mathbb{E} [p_{ij}(s)^n | \lambda] \leq C \left( F_s(\eta^{(1)}) + F_s(\eta^{(2)}) + F_s(\eta^{(3)}) \right)$$

where $\eta^{(1)}$ is the configuration of $n$ particles in the site $i$, $\eta^{(2)}$ particles in the site $j$, and $\eta^{(3)}$ an equal number of particles between the site $i$ and $j$.

The purpose of this section is to prove Theorem 1.8 for another matrix ensemble: a deformed Wigner matrix perturbed by a small Gaussian component. More precisely, we state it as the following theorem.

**Theorem 3.6.** Consider $a$ and $\omega$ two small positive constants and $\kappa \in (0,1)$, take $t \in T_{\omega}$, $D$ a deterministic diagonal matrix given by Definition 1.1 and $W$ a Wigner matrix given by Definition 1.2. Let $\tau \in \mathcal{T}_a := [N^{-1-\sigma}, N^{-\sigma}]$, then if $u_1, \ldots, u_N$ are the eigenvectors of the matrix

$$H_\tau = D + \sqrt{t}W + \sqrt{\tau} \text{GOE},$$

define the error

$$\Xi(\tau) = \frac{\hat{t}}{\sqrt{N\tau}} + \hat{\tau} \frac{t}{t},$$

we have, for any $k, \ell \in \mathcal{A}^c$ with $k \neq \ell$ and any $\varepsilon > 0$ and $D > 0$,

$$\mathbb{P} \left( \left| \sum_{\alpha \in I} \left( u_\alpha(\alpha)^2 - \frac{1}{N} \sigma^2_\alpha(\alpha,k,\tau) \right) \right| + \left| \sum_{\alpha \in I} u_\alpha(\alpha)w_\alpha(\alpha) \right| \geq N^\varepsilon \Xi(\tau) \right) \leq N^{-D}.$$
Lemma 3.7. For any intervals $J_{in} \subset A_w(i_0)$ and $J_{out} = \{i, d(i, J_{in}) \leq N^\varepsilon \ell \} \subset A_w^c(i_0)$ since we will take $\ell \ll Nw$, any configuration $\xi$ such that $N(\xi) = n$ supported on $J_{in}$ and any $N^{-1} \ll \tau \ll w$ we have

$$
|((U_{\mathcal{S}}(0, \tau) - U_{\mathcal{S}}(0, \tau))F_{\tau}(\xi)| < \frac{N \tau}{\ell} \left( S^{0, \tau}_{J_{out}} + \frac{\ell}{\tau} S^{0, \tau}_{J_{out}} \cdot \frac{\alpha^2}{\tau} + \frac{\ell}{\tau} S^{0, \tau}_{J_{out}} \cdot \frac{\alpha^2}{\tau} \right)
$$

(3.56)

where $F_{\tau}$ is the perfect matching observable defined in 1.26.

This bound is used in this form so we can obtain information on a box in space by extracting information from a larger box. Iterating this bound will give us Theorem 3.6.

Proof. We will follow the proof from [9]. Define the following flattening operator. For $f$ a function on configurations of $n$ particles and $\eta$ such a configuration,

$$(\text{Flat}_a f)(\eta) = \begin{cases} f(\eta) & \text{if } \eta \subset \{i, d(i, J_{in}) \leq a\}, \\ 0 & \text{otherwise.} \end{cases}$$

(3.57)

We make the functions vanish outside of a certain interval. We use now Duhamel’s formula and write

$$
((U_{\mathcal{S}}(0, \tau) - U_{\mathcal{S}}(0, \tau))F_{\tau}(\xi) = \int_0^\tau U_{\mathcal{S}}(s, \tau) \mathcal{L}(s) F_s(\xi).
$$

Now, see that, by definition of the flattening operator and the fact that $\xi$ is supported on $J_{in}$,

$$
d(\text{Supp} (\mathcal{L}(s) F_s - \text{Flat}_{N^\varepsilon \ell}(\mathcal{L}(s) F_s)), \xi) \geq N^\varepsilon \ell.
$$

With this bound, we can use the finite speed of propagation from Lemma 3.2 and obtain, using that $U_{\mathcal{S}}$ is a contraction in $L^\infty$,

$$
|U_{\mathcal{S}}(s, \tau) \mathcal{L}(s) F_s(\xi)| \leq \max_{\eta \subset J_{out}} |(\text{Flat}_{N^\varepsilon \ell}(\mathcal{L}(s) F_s))(\eta)| + O(e^{-cN^\varepsilon/2}).
$$

Thus we need to control $|((\mathcal{L}(s) F_s)(\tilde{\eta})|$ for $\tilde{\eta} = \{(i_1, j_1), \ldots, (i_m, j_m)\}$ a configuration of $n \geq m$ particles supported in $J_{out}$.

We have, by definition of $\mathcal{L}$,

$$
|\mathcal{L}(s) F_s(\tilde{\eta})| \leq \sum_{1 \leq p \leq m} \sum_{|i_p - k| \geq \ell} \frac{F_s(\tilde{\eta}^{i_p, k})}{N(\lambda_{i_p} - \lambda_k)^2} + |F_s(\tilde{\eta})| \sum_{1 \leq p \leq m} \sum_{|i_p - k| \geq \ell} \frac{1}{N(\lambda_{i_p} - \lambda_k)^2}.
$$

For the second term in the previous inequality, we can use the averaged local law and a dyadic decomposition and see that

$$
\sum_{\substack{k, |i_p - k| \geq \ell \ni \lambda_k \neq 0}} \frac{1}{N(\lambda_{i_p} - \lambda_k)^2} = \frac{N}{\ell^2},
$$

(3.58)

so that we have the bound

$$
|\mathcal{L}(s) F_s(\tilde{\eta})| \lesssim \sum_{1 \leq p \leq m} \sum_{|i_p - k| \geq \ell \ni \lambda_k = 0} \frac{F_s(\tilde{\eta}^{i_p, k})}{N(\lambda_{i_p} - \lambda_k)^2} + \frac{N}{\ell^2} S^{0, \tau}_{J_{out}}.
$$

(3.59)

For the first sum in (3.59), we will first restrict it to the sites $k$ such that there are no particles in the configuration $\tilde{\eta}$, so that we will have $\tilde{\eta}_k = 1$. Note that, if we have $\tilde{\eta}_k \neq 0$ then by definition of $\tilde{\eta}$ supported on $J_{out}$, $\tilde{\eta}^{i_p, k}$ is also supported on $J_{out}$. This gives us the bound

$$
\sum_{\substack{k, |i_p - k| \geq \ell \ni \lambda_k \neq 0}} \frac{F_s(\tilde{\eta}^{i_p, k})}{N(\lambda_{i_p} - \lambda_k)^2} \lesssim \sum_{\substack{k, |i_p - k| \geq \ell \ni \lambda_k = 0}} \frac{F_s(\tilde{\eta}^{i_p, k})}{N(\lambda_{i_p} - \lambda_k)^2} + S^{0, \tau}_{J_{out}} \sum_{\substack{k, |i_p - k| \geq \ell \ni \lambda_k \neq 0}} \frac{1}{N(\lambda_{i_p} - \lambda_k)^2}.
$$

$$
\lesssim \sum_{\substack{k, |i_p - k| \geq \ell \ni \lambda_k = 0}} \frac{F_s(\tilde{\eta}^{i_p, k})}{N(\lambda_{i_p} - \lambda_k)^2} + \frac{N}{\ell^2} S^{0, \tau}_{J_{out}}.
$$

(3.59)
where in the second inequality we used the rigidity of the eigenvalues, which gives us \((\lambda_{i_p} - \lambda_k)^2 \geq C(\ell/N)^2\) for \(|i_p - k| > \ell\), and the fact that there is at most \(m\) sites \(k\) such that \(\eta_k \neq 0\). By definition of the perfect matching observables, we can write

\[
\sum_{k, |i_p - k| > \ell} \frac{F_s(\hat{q}^{i_p,k})}{N(\lambda_{i_p} - \lambda_k)^2} = C(n)E \left[ \sum_{k, |i_p - k| > \ell} \sum_{\eta_k=0} \frac{\prod_{c \in E(G)} p(c)}{N(\lambda_{i_p} - \lambda_k)^2} \right].
\]

In order to control this term, we will consider two different types of perfect matchings. Define the following partition of \(G_\eta\) into two subsets

\[
G^{(1)}_\eta = \{ G \in G_\eta, \{(k,1), (k,2)\} \in E(G)\}, \quad \text{(3.60)}
\]

\[
G^{(2)}_\eta = \{ G \in G_\eta, \{(k,1), (k,2)\} \notin E(G)\}. \quad \text{(3.61)}
\]

We will begin by bounding the contribution from (3.60). Note first that, for \(G \in G^{(1)}_\eta\), we have

\[
\prod_{e \in E(G)} p(e) = p_{kk} \times Q_1((p(e))_{e \in E(G)})
\]

with

\[
Q_1((p(e))_{e \in E(G)}) = \prod_{e \in E(G) \setminus \{(k,1), (k,2)\}} p(e).
\]

See that \(Q_1\) is a monic monomial of degree \(n - 1\) so that we can use Lemma 3.5 and obtain

\[
E \left[ \sum_{G \in G^{(1)}_\eta} Q_1((p(e))_{e \in E(G)}) \right] \leq C \sup_{0 \leq \eta, N(\eta) = n-1} |F_s(\eta)| \leq C \left( S_{j=1}^{(0,\tau)} \right)^{\frac{n}{n-1}}. \quad \text{(3.63)}
\]

Combining (3.62) and (3.63), we now only need to bound

\[
\sum_{k, |k-i_p| > \ell} \frac{p_{kk}}{N(\lambda_{i_p} - \lambda_k)^2} = \sum_{k, |k-i_p| > \ell} \frac{p_{kk}}{N(\lambda_{i_p} - \lambda_k)^2} + O\left( \frac{N}{\ell^2} \right).
\]

In order to bound the sum from the right hand side of the previous equation, first define the following functions, for \(|z - \lambda_{i_p}| \leq N^{-\epsilon} \ell/N\),

\[
f(z) = \sum_{k, \gamma, k \notin \{E_1^-, E_2^+\}} \frac{p_{kk}}{N(z - \lambda_k)},
\]

\[
g(z) = \sum_{k, \gamma, k \notin \{E_1^-, E_2^+\}} \frac{p_{kk}}{N(z - \lambda_k)}
\]

where \(E_1 = \gamma_{t,i_p-\ell}, E_1^- = \gamma_{t,i_p-\ell-N'}, E_2^- = \gamma_{t,i_p-\ell-N'}, E_2^+ = \gamma_{t,i_p+\ell}, E_2^+ = \gamma_{t,i_p+\ell+N'}\). Let also \(\Gamma\) be the rectangle with vertices \(E_1 \pm i \ell / N\) and \(E_2 \pm i \ell / N\). We therefore want to bound

\[
\sum_{k, |k-i_p| > \ell} \frac{p_{kk}}{N(\lambda_{i_p} - \lambda_k)^2} = \partial_{z} f(z) \bigg|_{z = \lambda_{i_p}}
\]
Finally, putting all the contributions together and coming back to (3.64), we obtain
\[ \frac{\partial f(\lambda_{ip})}{\partial z} = \frac{1}{2\pi i} \int_{C_{ip}} \frac{f(z)}{(z - \lambda_{ip})^2} \, dz. \]
By using another Cauchy integral formula on the contour \( \Gamma \) for \( f \) and seeing that for \( \lambda_{int}, z \) inside the contour and \( \lambda_{ext} \) outside of the contour we have, by a residue calculus,
\[ \int_{\Gamma} \frac{d\xi}{(\xi - \lambda_{int})(\xi - \lambda_{ip})} = \int_{\Gamma} \frac{d\xi}{(\xi - \lambda_{ext})(\xi - \lambda_{ip})} = 0 \]
we can write
\[ |\partial f(\lambda_{ip})| = \left| \frac{1}{2\pi i} \int_{\Gamma} \frac{g(\xi)}{(\xi - \lambda_{ip})^2} \, d\xi \right| = O\left( \frac{N}{\ell} \int_{\Gamma} \frac{\Im g(\xi)}{\xi - \lambda_{ip}} \, d\xi \right). \tag{3.64} \]
We will first control the part of the contour closest to the real axis. Consider \( \Gamma_1 = \{ z = E + i\eta \in \Gamma, |\eta| < N^{\varepsilon}/N \} \), as in [13] and bounding \( p_{kk} \) by \( \hat{T} \), we obtain
\[ \left| \int_{\Gamma_1} \frac{\Im g(\xi)}{\xi - \lambda_{ip}} \, d\xi \right| = O\left( \frac{\hat{T}}{\ell} \right). \]
Now for the rest of the contour, note that we can add the missing eigenvalues to the total sum in \( g \) up to adding an error of order \( N^\varepsilon \hat{T}/\ell \). Finally we just have to bound
\[ \frac{N}{\ell} \int_{\Gamma \setminus \Gamma_1} \left| \Im \sum_{k=1}^{N} \frac{p_{kk}}{N(\xi - \lambda_k)} \right| \, d\xi = \frac{N}{\ell} \int_{\Gamma \setminus \Gamma_1} \left| \Im \sum_{\alpha \in \mathcal{L}} G_{\alpha\alpha}(\xi) - \frac{\Im m_{\alpha}(\xi)}{\Im m_{\alpha}(z_0)} \frac{1}{N} \sum_{\alpha \in \mathcal{L}} \Im g_{\alpha}(t,z_0) \right| \, d\xi \tag{3.65} \]
where we used the definition of \( p_{kk} \) and of \( \sigma_t \) and defined \( z_0 = \gamma_{t,0} + i\eta_0 \), with \( \eta_0 \ll t \) is the center of our window of size \( w \ll t \) with positive imaginary part. Now, using the partial local law and expanding between \( z_0 \) and \( \xi \) since \( |\xi - z_0| \ll w \), we have
\[ \left| \Im \frac{1}{N} \sum_{\alpha \in \mathcal{L}} G_{\alpha\alpha}(\xi) - \frac{\Im m_{\alpha}(\xi)}{\Im m_{\alpha}(z_0)} \frac{1}{N} \sum_{\alpha \in \mathcal{L}} \Im g_{\alpha}(t,z_0) \right| \leq \left| \Im \frac{1}{N} \sum_{\alpha \in \mathcal{L}} (G_{\alpha\alpha}(\xi) - g_{\alpha}(t,\xi)) \right| + \left| \Im g_{\alpha}(t,\xi) - \frac{\Im m_{\alpha}(\xi)}{\Im m_{\alpha}(z_0)} \Im g_{\alpha}(t,z_0) \right| \]
\[ \leq \frac{T}{\sqrt{N}|\Im \xi|} + \frac{\hat{T}w}{\ell}. \]
Injecting this bound in the contour integral, we get the following bound.
\[ (3.65) \times \frac{\hat{T}w}{\ell} + \frac{T}{N} \log N. \]
Finally, putting all the contributions together and coming back to (3.64), we obtain
\[ |\partial f(\lambda_{ip})| \times \frac{N}{\ell} \left( \frac{\hat{T}}{\ell} + \frac{\hat{T}w}{\ell} \right). \tag{3.66} \]
Consider now the contribution from (3.61), first see that for \( G \in \mathcal{G}^{(2)}_\eta \), there exists \( q_1 \) and \( q_2 \) in \( \{1, \ldots, m\} \), such that
\[ \prod_{e \in \mathcal{E}(G)} p(e) = p_{kq_1}p_{kq_2} \times Q_2((p(e))_{e \in \mathcal{E}(G)}) \]
26
with \(Q_2\) a monic monomials of degree \(n - 2\). Then using Lemma 3.5, we can bound

\[
E \left[ \sum_{G \in G^\prime} Q_2((p(c))_{c \in E(G)}) \right] = O \left( \left( S_{\text{out}}^{(0,\tau)} \right)^{\frac{n-2}{n}} \right).
\]

Besides, we can bound the term with the cross-edges in the following way

\[
\sum_{k, |t_k - t_{k+1}| > \epsilon} \frac{p_{t_k} p_{t_{k+1}}}{N (\lambda_{t_k} - \lambda_{t_{k+1}})^2} \leq \frac{N}{T^2} \sum_{k=1}^{N} (p_{t_k}^2 + p_{t_{k+1}}^2) = \frac{N}{T^2} \sum_{\alpha \in I} (u_{t_k}(\alpha)^2 + u_{t_{k+1}}(\alpha)^2) \leq \tilde{T}^{\frac{N}{T^2}}.
\]

Putting everything together, we get

\[
|(U_{\text{av}}(0, \tau) - U_{\text{av}}(0, \tau)) F_t(\eta)| \leq \frac{N\tau}{\ell} \left( S_{\text{out}}^{(0,\tau)} + \tilde{T}^\frac{w}{\ell} \left( S_{\text{out}}^{(0,\tau)} \right)^{\frac{n-1}{n}} + \frac{T}{\ell} \left( S_{\text{out}}^{(0,\tau)} \right)^{\frac{n-2}{n}} \right)
\]

which is exactly the result wanted. \(\square\)

We will first prove the following proposition in order to deduce Theorem 1.8.

**Proposition 3.8.** For any \(\varepsilon\) and \(N\) large enough the following holds. For any intervals \(J_{in} \subset A_{\varepsilon}^c\) and \(J_{out} \subset \{ i, d(i, J_{in}) \leq N^{-\varepsilon} Nw \}\) we have

\[
S_{\text{in}}^{(0,\tau)} < \left( \frac{\ell}{Nw} + \frac{N\tau}{\ell} + \frac{1}{N\tau} \right) S_{\text{out}}^{(0,\tau)} + \left( \frac{\tilde{T}}{\sqrt{N\tau}} + \tilde{T}^\frac{w}{\ell} \right) \left( S_{\text{out}}^{(0,\tau)} \right)^{\frac{n-1}{n}}
\]

\[
+ \left( \frac{T}{\ell} + \frac{\tilde{T}^\frac{N\tau}{\ell}}{Nw} \right) \left( S_{\text{out}}^{(0,\tau)} \right)^{\frac{n-2}{n}}. \tag{3.67}
\]

**Proof.** We will use the short-range dynamics and its finite speed of propagation property in order to localize the maximum principle in \(J_{in}\). We will then use the local laws and Lemma 3.7 in order to get a Gronwall type bound. Define the following averaging operator. For \(f\) a function on configurations and \(\eta\) a configuration,

\[
\text{Av}(f) = \frac{3N^\varepsilon}{Nw} \sum_{\frac{Nw}{2} \leq a < \frac{Nw}{2}} \text{Flat}_a(f).
\]

Note that, if \(\eta\) is not included in \(J_{out}\), by definition of the flattening operator, \((\text{Av}(f))(\eta) = 0\). The purpose of these operators is to change the initial condition in order to remove the particles far from the initial interval \(J_{in}\). Note also that we can write, for any \(\eta\),

\[
\text{Av}(f)(\eta) = a_{\eta} F_t(\eta),
\]

with \(a_{\eta} \in [0, 1]\). Note that we have the elementary bound \(|a_{\eta} - a_{\xi}| \leq C N^\varepsilon / Nw\). Define now the following dynamics

\[
\begin{cases}
\partial_s \Gamma_s = \mathcal{A}(s) \Gamma_s, & 0 \leq s \leq \tau \\
\Gamma(0)(\eta) = (\text{Av} F_t)(\eta).
\end{cases}
\]

\(\tag{3.69}
\]

Now, if you take a configuration \(\eta\) supported on \(J_{in}\), it suffices to show the bound in Proposition 3.8 for \(\Gamma\). Indeed

\[
|\Gamma_t(\eta) - \Gamma(\eta)| \leq |(U_{\text{av}}(0, \tau) F_t - U_{\text{av}}(0, \tau) F_t(\eta))| + |U_{\text{av}}(0, \tau)(F_t - \text{Av} F_t)(\eta)|
\]

\[
< \frac{C N^\varepsilon}{\ell} \left( S_{\text{out}}^{(0,\tau)} + \tilde{T}^\frac{w}{\ell} \left( S_{\text{out}}^{(0,\tau)} \right)^{\frac{n-1}{n}} + \frac{T}{\ell} \left( S_{\text{out}}^{(0,\tau)} \right)^{\frac{n-2}{n}} \right) + \exp(-c N^\varepsilon)
\]

where we bounded the first term by using Lemma 3.7 and the second term using the finite speed of propagation. Indeed, since \(\eta\) is supported on \(J_{in}\), \((\text{Id} - \text{Av}) F_t\) vanishes for any configuration
supported on $J_{out}$. Note that we can use Lemma 3.7 since we will take $\ell/N \ll w$. In the rest of the proof, we will prove the bound from Proposition 3.8 for $\Gamma_\tau$. If we already have for some $C > 0$,

$$\Gamma_\tau(\eta_m) := \sup_{\eta, N(\eta) = n} \Gamma_\tau(\eta) \leq N^{-C}$$

then we have nothing to prove by the argument above and the definition of $F_\tau$. However, if this supremum is greater than $N^{-C}$, then by the finite speed of propagation of $\mathcal{S}$, we know that $\eta_m$ will be supported in, for instance, $\{i, d(i, J_m) \leq \frac{3Nw}{2N}\}$.

Consider now, a parameter $\eta$ that we will choose later and denote also $m \leq n$ the number of sites with at least a particle and $j_1, \ldots, j_m$ those sites. Then, we can write

$$\partial_t \Gamma_\tau(\eta_m) = \sum_{0 < |j - k| \leq \ell} \frac{2\eta_m,k(1 + 2\eta_m,j) \left( \Gamma_\tau(\eta_m^{k,j}) - \Gamma_\tau(\eta_m) \right)}{N(\lambda_k(t + \tau) - \lambda_j(t + \tau))^2}$$

(3.70)

$$\leq \frac{C}{N\eta} \sum_{0 \leq |j - k| \leq \ell} \frac{\eta \left( \Gamma_\tau(\eta_m^{k,j}) - \Gamma_\tau(\eta_m) \right)}{(\lambda_k - \lambda_j)^2 + \eta^2}$$

(3.71)

$$\leq \frac{C}{N\eta} \sum_{0 \leq |j - k| \leq \ell} \Im \left( \frac{\Gamma_\tau(\eta_m^{k,j})}{\lambda_k - z_{j_p}} \right) - \frac{1}{N\eta} \Gamma_\tau(\eta_m) \sum_{0 \leq |j - k| \leq \ell} \Im \left( \frac{1}{z_{j_p} - \lambda_k} \right)$$

(3.72)

with $z_{j_p} = \lambda_{j_p} + i\eta$. For the second term, see that for $p \in \{1, m\}$, if we choose $\eta$ to be smaller than $\ell/N$,

$$\# \{k, 0 < |j - k| \leq \ell \} \geq C\ell \geq C N \eta \geq C \# \{k, |\lambda_{j_p} - \lambda_k| \leq \eta \} \geq C' N \eta$$

where we used, in the two last inequalities, the rigidity of the eigenvalues. Now we can write

$$\sum_{0 \leq |j - k| \leq \ell} \Im \left( \frac{1}{z_{j_p} - \lambda_k} \right) \geq \sum_{0 \leq |j - k| \leq \ell} \frac{\eta}{(\lambda_{j_p} - \lambda_k)^2 + \eta^2} \geq CN.$$

We now need to control the first term in (3.72). To do so, we will split it in the three following terms:

$$\Im \left( \sum_{0 \leq |j - k| \leq \ell} \frac{(U_{\mathcal{S}}(0, \tau)AvF_t)(\eta_m^{j_p,k}) - (AvU_{\mathcal{S}}(0, \tau)F_t)(\eta_m^{j_p,k})}{N(z_{j_p} - \lambda_k)} \right)$$

(3.73)

$$+ \Im \left( \sum_{0 \leq |j - k| \leq \ell} \frac{(AvU_{\mathcal{S}}(0, \tau)F_t)(\eta_m^{j_p,k}) - (AvU_{\mathcal{S}}(0, \tau)F_t)(\eta_m^{j_p,k})}{N(z_{j_p} - \lambda_k)} \right)$$

(3.74)

$$+ \Im \left( \sum_{0 \leq |j - k| \leq \ell} \frac{(AvU_{\mathcal{S}}(0, \tau)F_t)(\eta_m^{j_p,k})}{N(z_{j_p} - \lambda_k)} \right)$$

(3.75)

To bound the first term, we will use the finite speed of propagation property of $\mathcal{S}$. Indeed, we can write

$$(U_{\mathcal{S}}(0, \tau)AvF_t)(\eta_m^{j_p,k}) - (AvU_{\mathcal{S}}(0, \tau)F_t)(\eta_m^{j_p,k}) = \frac{3N\varepsilon}{NW} \sum_{\frac{3}{2Nw} \leq q \leq \frac{3}{2Nw}} \mathcal{U}_q(\eta_m^{j_p,k})$$

with

$$\mathcal{U}_q = U_{\mathcal{S}}(0, \tau)Flat_qF_t - Flat_qU_{\mathcal{S}}(0, \tau)F_t.$$
Fix \( a \) and consider three cases, if \( \eta_m^{j,p,k} \) is supported on \( \{ i, d(i, J_m) > a + N^\varepsilon \ell \} \) then by definition of \( \text{Flat}_a \) we have
\[
\text{Flat}_a U_{\varepsilon} (0, \tau) F_t = 0,
\]
and by finite speed of propagation we have
\[
|U_{\varepsilon} (0, \tau) \text{Flat}_a F_t| \leq \exp \left(- \frac{c N^\varepsilon}{2} \right).
\]
Now if \( \eta_m^{j,p,k} \) is supported on \( \{ i, d(i, J_m) \leq a - N^\varepsilon \ell \} \) then again by definition of \( \text{Flat}_a \),
\[
\text{Flat}_a \left( U_{\varepsilon} (0, \tau) F_t \right) \left( \eta_m^{j,p,k} \right) = \left( U_{\varepsilon} (0, \tau) F_t \right) \left( \eta_m^{j,p,k} \right).
\]
Thus
\[
\left| \mathcal{Z}_a \left( \eta_m^{j,p,k} \right) \right| \leq \left| U_{\varepsilon} (0, \tau) (F_t - \text{Flat}_a F_t) \left( \eta_m^{j,p,k} \right) \right| \leq \exp \left(- \frac{c N^\varepsilon}{2} \right).
\]
Finally, if \( \eta_m^{j,p,k} \) is supported on \( \{ i, d(i, J_m) \leq a + \ell N^\varepsilon \} \) such a \( a \), then one can see that we can use the finite speed of propagation if we remove particle away from \( a \) at distance \( 2N^\varepsilon \) for instance, then
\[
\left| \mathcal{Z}_a \left( \eta_m^{j,p,k} \right) \right| \leq \left| \text{Flat}_a U_{\varepsilon} (0, \tau) F_t \right| + \left| \text{Flat}_a U_{\varepsilon} (0, \tau) \text{Flat}_a + 2\ell N^\varepsilon F_t \right| + \left| \text{Flat}_a U_{\varepsilon} (0, \tau) F_t - \text{Flat}_a + 2\ell N^\varepsilon \right|,
\]
\[
\leq \| \text{Flat}_a F_t \|_\infty + \| \text{Flat}_a + 2\ell N^\varepsilon \|_\infty + \exp \left(- \frac{c N^\varepsilon}{2} \right),
\]
where we used that \( U_{\varepsilon} \) is a contraction in \( \| \cdot \|_\infty \). Finally we can bound (3.73),
\[
(3.73) \lesssim \frac{\ell}{N^\varepsilon} S^{(0,\tau)}_{J_{\text{out}}} + \exp \left(- \frac{c N^\varepsilon}{2} \right)
\]
where we used the fact that
\[
\left| \frac{1}{N} \text{Im} \left( \sum_{k, 0 < |j_p - k| < \ell} \frac{1}{z_{j_p} - \lambda_k} \right) \right| \lesssim \left| s_{\tau} \left( z_{j_p} \right) \right| \lesssim 1
\]
(3.77)
For (3.74), we will use Lemma 3.7. Indeed, first note that in the short-range regime, the set of \( k \) such that \( |j_p - k| \leq \ell \) is included in \( \mathcal{A}_0^{\varepsilon,\ell} \). Then we can bound
\[
(3.74) \lesssim \frac{N^\tau}{\ell} \left( S^{(0,\tau)}_{J_{\text{out}}} + \frac{\ell^W}{\ell} \left( S^{(0,\tau)}_{J_{\text{out}}} \right)^{\frac{n-1}{2}} + \frac{1}{\ell} \left( S^{(0,\tau)}_{J_{\text{out}}} \right)^{\frac{n-2}{n}} \right)
\]
where we used the fact that \( Av \) is a contraction and (3.77).
Finally, in order to bound the third term (3.75), we will use the local law for \( F_t \). First write
\[
\text{Im} \left( \sum_{k, 0 < |j_p - k| \leq \ell} \frac{\text{Av} U_{\varepsilon} (0, \tau) F_t \left( \eta_m^{j,p,k} \right)}{N(z_{j_p} - \lambda_k)} \right) = \text{Im} \left( \sum_{k, 0 < |j_p - k| \leq \ell} \frac{a_{\eta_m} F_t \left( \eta_m^{j,p,k} \right)}{N(z_{j_p} - \lambda_k)} \right)
\]
\[
= \text{Im} \left( \sum_{k, 0 < |j_p - k| \leq \ell} \frac{a_{\eta_m} F_t \left( \eta_m^{j,p,k} \right)}{N(z_{j_p} - \lambda_k)} \right) + \text{Im} \left( \sum_{k, 0 < |j_p - k| \leq \ell} \frac{(a_{\eta_m} F_t - a_{\eta_m}) F_t \left( \eta_m^{j,p,k} \right)}{N(z_{j_p} - \lambda_k)} \right)
\]
\[
= \text{Im} \left( \sum_{k, 0 < |j_p - k| \leq \ell} \frac{a_{\eta_m} F_t \left( \eta_m^{j,p,k} \right)}{N(z_{j_p} - \lambda_k)} \right) + O_{\varepsilon} \left( \frac{1}{|j_p - k| \leq \ell} \left| a_{\eta_m} - a_{\eta_m} \right| S^{(0,\tau)}_{J_{\text{out}}} \right).
\]
But we have the bound, for $|j_p - k| \leq \ell$,

$$\left| a_{\eta_m} - a_{\eta_m^{j_p-k}} \right| \leq \frac{N^\eps d (\eta_m^{j_p-k}, \eta_m)}{N w} \leq \frac{N^\eps \ell}{N w}. $$

In order to bound the last term in (3.81), we can use the local law. First write,

$$\text{Im} \left( \sum_{k, 0 \leq |j_p - k| \leq \ell, k \notin \{j_1, \ldots, j_p\}} a_{\eta_m} F_\ell \left( \frac{\eta_m^{j_p-k}}{N(z_{j_p} - \lambda_k)} \right) \right) = \text{Im} \left( \sum_{k, 0 \leq |j_p - k| \leq \ell, k \notin \{j_1, \ldots, j_p\}} a_{\eta_m} F_\ell \left( \frac{\eta_m^{j_p-k}}{N(z_{j_p} - \lambda_k)} \right) \right) + o \left( \frac{1}{N \eta} S_{J_{out}}^{(0, \tau)} \right).$$

Recall now the definition of $F_\ell$ from (1.26) and write,

$$\sum_{k, 0 \leq |j_p - k| \leq \ell} F_\ell \left( \frac{\eta_m^{j_p-k}}{N(z_{j_p} - \lambda_k)} \right) \frac{1}{M \left( \eta_m^{j_p-k} \right)} \sum_{k, 0 \leq |j_p - k| \leq \ell, k \notin \{j_1, \ldots, j_p\}} \sum_{\eta_m^{j_p-k}} \Pi_{\ell \notin \{j_1, \ldots, j_p\}} p(e) \frac{\prod_{\ell \notin \{j_1, \ldots, j_p\}} \left( p(e) \right) \frac{\eta_m^{j_p-k}}{N(z_{j_p} - \lambda_k)}}{N(z_{j_p} - \lambda_k)}$$

First consider the contribution of (3.60) in the sum in (3.80), denote $e_k = \{(k, 1), (k, 2)\}$ and write

$$\sum_{k, 0 \leq |j_p - k| \leq \ell} \sum_{k \notin \{j_1, \ldots, j_p\}} \sum_{\eta_m^{j_p-k}} \Pi_{\ell \notin \{j_1, \ldots, j_p\}} p(e) \frac{\prod_{\ell \notin \{j_1, \ldots, j_p\}} \left( p(e) \right) \frac{\eta_m^{j_p-k}}{N(z_{j_p} - \lambda_k)}}{N(z_{j_p} - \lambda_k)}$$

To control the last term in (3.81), we can use the local law. First write,

$$\sum_{k, 0 \leq |j_p - k| \leq \ell} \sum_{k \notin \{j_1, \ldots, j_p\}} \frac{p_{kk}}{N(z_{j_p} - \lambda_k)} = \sum_{k=1}^{N} \frac{p_{kk}}{N(z_{j_p} - \lambda_k)} + \sum_{k, |j_p - k| > \ell} \frac{p_{kk}}{N(z_{j_p} - \lambda_k)} + o \left( \frac{\tilde{\gamma}}{N \eta} \right).$$

where we used the bound $|p_{kk}| < \tilde{\gamma}$. Now, recall the definition of $p_{kk}$ from (3.54) so that we have

$$\left| \text{Im} \sum_{k=1}^{N} \frac{p_{kk}}{N(z_{j_p} - \lambda_k)} \right| = \left| \text{Im} \sum_{a \in I} \frac{G_{aa}(z_{j_p}) - \text{Im} \eta_s(x_{j_p})}{\text{Im} \eta_s(x_{j_p})} \frac{1}{N} \sum_{a \in I} \text{Im} \eta_s(x_{j_p}) \right|$$

$$\leq \left| \text{Im} \sum_{a \in I} \left( G_{aa}(z_{j_p}) - \eta_s(x_{j_p}) \right) \right| + \left| \sum_{a \in I} \left( \text{Im} \eta_s(x_{j_p}) \right) \right|$$

$$\leq \frac{\tilde{\gamma}}{\sqrt{N \eta}} + \frac{\tilde{\gamma}}{t}$$

where $z_0 := \gamma_{\eta, \tau_0} + i \eta$. Note that we used the fact that $\eta_m$ is supported in $J_{out}$, so that $|z_0 - z_{j_p}| = |\gamma_{\eta, \tau_0} - \lambda_{j_p}| \leq w$. We can then use Lemma 3.5 and bound

$$\sum_{G \notin \eta_m \cap \eta_r} \prod_{\ell \notin \{j_1, \ldots, j_p\}} p(e) = O \left( \sup_{\eta \in J_{out}, N(\eta) = n-1} |F_\ell(\eta)| \right) = O \left( \left( \frac{S_{J_{out}}^{(0, \tau)}}{N \eta} \right)^{\frac{\tilde{\gamma}}{t}} \right).$$
Now, consider the contribution of (3.61) in the sum from (3.80). Note that for any graph $G$ in $G_{\eta}^{(2)}$, there exists $q$ and $q'$ in $\{1, \ldots, m\}$, such that $e_q := \{(k, 1), (q, a)\}$ and $e_{q'} := \{(k, 2), (q', b)\}$ are edges in $G$. We can then write

$$\sum_{k, 0 < \lvert \mathbf{p} - k \rvert \leq \ell} \sum_{G \in G_{\eta}^{(2)}} \frac{\prod_{e \in E(G)} p(e)}{N(z_{jp} - \lambda_k)} = \sum_{k, 0 < \lvert \mathbf{p} - k \rvert \leq \ell} \sum_{G \in G_{\eta}^{(2)}} \left( \prod_{e \in E(G) \setminus \{e_q, e_{q'}\}} p(e) \right) \frac{p_{i_1, k} p_{i_2, k}}{N(z_{jp} - \lambda_k)},$$

where we defined the set of graphs $H_{\eta}^{q, q'}$ to be the set of perfect matching of the complete graph on the set of vertices $V_q$ where we removed a single particle at the site $i_q$ and $i_{q'}$. Note that for any graph $G \in H_{\eta}^{q, q'} \prod_{e \in E(G)} p(e)$ is a monomial of degree $n - 2$.

Now, we can bound the imaginary part of the second sum in (3.83),

$$\left| \text{Im} \left( \sum_{k, 0 < \lvert \mathbf{p} - k \rvert \leq \ell} \sum_{G \in G_{\eta}^{(2)}} \frac{p_{i_1, k} p_{i_2, k}}{N(z_{jp} - \lambda_k)} \right) \right| \leq \frac{C}{N\eta} \sum_{k=1}^{N} \left( p_{i_1, k}^2 + p_{i_2, k}^2 \right) = O_{\prec} \left( \frac{\tilde{t}}{N\eta} \right).$$

(3.84)

For the last inequality, we used the following identity on eigenvectors

$$\sum_{k=1}^{N} p_{i, k}^2 = \sum_{\alpha \in \ell} u_{\alpha}^2,$$

and that for any $\varepsilon > 0$, using the entrywise local law on a diagonal entry of the resolvent,

$$u_{\alpha}^2 \leq N^{1+\varepsilon} \text{Im} \left( G(\tau, \lambda_k + i N^{-1+\varepsilon})_{\alpha, \alpha} \right) \leq \frac{1}{N \tilde{t}}.$$

Again, we can bound the other term from (3.83) using Lemma 3.5,

$$\sum_{G \in H_{\eta}^{q, q'} \prod_{e \in E(G)}} p(e) = O_{\prec} \left( \sup_{\eta \in J_{\text{out}}} \left| F_t(\eta) \right| \right) = O_{\prec} \left( \frac{S_{J_{\text{out}}}^{(0, \tau)}}{N^{\frac{a-2}{2}}} \right).$$

Finally, putting all these estimates together, we get the Gronwall-type inequality,

$$\partial_t \Gamma_{\tau}(\eta_m) \leq -\frac{1}{\eta} \Gamma_{\tau}(\eta_m) + O_{\prec} \left( \frac{1}{\eta} \left( \frac{\ell}{Nw} + \frac{N\tau}{\tilde{t}} + \frac{1}{N\eta} \right) S_{J_{\text{out}}}^{(0, \tau)} \right)$$

$$+ \left( \frac{\tilde{t}}{\sqrt{N\eta}} + \frac{\tilde{w}}{\tilde{t}} \right) \left( S_{J_{\text{out}}}^{(0, \tau)} \right)^{\frac{a-1}{2}} + \left( \frac{\tilde{t}}{N\tau} + \frac{N\tau}{\tilde{t}} \right) \left( S_{J_{\text{out}}}^{(0, \tau)} \right)^{\frac{a-2}{2}}$$

(3.85)

In order to get a proper bound using Gronwall’s lemma, we need to take $\eta \ll \tau$ but to get the best estimates possible, we also have to take $\eta$ as large as possible. Hence, consider $\eta = N^{-\varepsilon} \tau$ with $\varepsilon$ as small as we want, we then have the bound, considering the definition of $\prec$,

$$S_{J_{\text{in}}}^{(0, \tau)} \leq \left( \frac{\ell}{Nw} + \frac{N\tau}{\tilde{t}} + \frac{1}{N\eta} \right) S_{J_{\text{out}}}^{(0, \tau)} + \left( \frac{\tilde{t}}{\sqrt{N\eta}} + \frac{\tilde{w}}{\tilde{t}} \right) \left( S_{J_{\text{out}}}^{(0, \tau)} \right)^{\frac{a-1}{2}}$$

$$+ \left( \frac{\tilde{t}}{N\tau} + \frac{N\tau}{\tilde{t}} \right) \left( S_{J_{\text{out}}}^{(0, \tau)} \right)^{\frac{a-2}{2}}$$

(3.86)

which gives the Proposition 3.8. □
Now that we have the bound from Proposition 3.8, we are able to get a bound on the $p_{ij}$ using Lemma 3.5. To do so, we can use a sequence of set of indices with decreasing size and apply recursively Proposition 3.8. We will also need to choose the right parameters $\ell$, $w$ and $\tau$.

**Proof of Theorem 3.6.** Consider first any $\varepsilon$ small enough, such that if we write $t = N^{-1+\delta}$ (recall that $t \in T_\omega$ so that $t \gg N^{-1}$) we have $\varepsilon < 4e/3$, and a large $D > 0$. Then we can take the following parameters:

$$w = N^{\varepsilon} \tau \quad \text{and} \quad \ell = N\sqrt{\tau w}, \quad (3.87)$$

note that we have the right bounds between these parameters: $N^{-1} \ll \tau \ll \ell/N \ll w \ll t$, and define the following sequence of sets of indices, defined implicitely,

$$J_0 = A^\infty w (i_0), \quad J_i = \{i, d(i, J_{i+1}) \leq N^{-\varepsilon} N w\}.$$

From Proposition 3.8 we have the following bound holding with overwhelming probability,

$$S_{J_{i+1}}^{(0, \tau)} \leq N^{-\varepsilon/2} S_{J_i}^{(0, \tau)} + \left( \frac{\hat{I}}{\sqrt{\tau w}} + \frac{\sqrt{\tau \tau}}{t} \right) (S_{J_i}^{(0, \tau)})^{\alpha - 1} + \frac{1}{N^2} \left( S_{J_i}^{(0, \tau)} \right)^{\alpha - \frac{1}{2}}.$$

Now see that as long as we have

$$(S_{J_i}^{(0, \tau)})^{1/n} \geq CN^{3\varepsilon/2} \Xi(\tau)$$

with $\Xi$ given in (1.11), we obtain the recursive bound

$$S_{J_{i+1}}^{(0, \tau)} \leq N^{-\varepsilon/2} S_{J_i}^{(0, \tau)}.$$  

But if we take a very large $i$ so that the previous bound cannot hold, for instance $i = [3\varepsilon^{-1}]$, then it means that for such a $i$ we have the bound

$$(S_{J_i}^{(0, \tau)})^{1/n} \leq CN^{3\varepsilon/2} \Xi.$$  

Now using the definition of $p_{ii}$, we have for $i \in A^\infty w (i_0)$,

$$\sum_{\omega \in I} \left( u_i(\omega)^2 - \frac{1}{N} \sigma_i^2(\omega, i) \right) \leq |p_{ii}| + \hat{I} \frac{w}{t} \leq |p_{ii}| + \Xi(\tau). \quad (3.88)$$

Finally, using Markov’s inequality, taking for instance $n = [3D/\varepsilon]$, and using Lemma 3.5 to bound the $p_{ij}$ by $S^{(0, \tau)}$ we have

$$\mathbb{P}\left(|p_{ii}| + |p_{ij}| \geq N^\varepsilon \Xi(\tau)\right) \leq N^{-D}. \quad (3.89)$$

The result then follows from combining (3.88) and (3.89). 

\[\square\]

### 4. Approximation by a Gaussian divisible ensemble

#### 4.1. Continuity of the Dyson Brownian motion

In Subsection 3.1 we showed that the moments of the eigenvectors of the matrix $H_\tau$ are asymptotically those of a Gaussian random variable with variance $\sigma_i^2$. If we would have taken the time $t - \tau = O(t)$ from the start, the previous section gives us (3.10) for $W$ a matrix from the Gaussian Orthogonal Ensemble. Now, since $\tau$ is a small time, recall that $\tau \ll t$, we can use the continuity of the Dyson Brownian motion to show that $H_\tau$ and $H_0 = W_t$ have the same local statistics. In order to state a proper continuity lemma we need to have a dynamics with constant second moments and vanishing expectation.

First see that the variance of the centered model is

$$\mathbb{E}\left[ (W_{t,ij} - D_{ij})^2 \right] = \frac{t}{N}. \quad (32)$$
Consider, for $0 \leq s \leq \tau$, the following variance-preserving dynamics on symmetric matrices.

\[
\begin{align*}
    d(\tilde{H}(s) - D) &= \frac{dB}{\sqrt{N}} - \frac{1}{2t} (\tilde{H}(s) - D) \, ds, \\
    \tilde{H}(0) &= W = D + \sqrt{t} W.
\end{align*}
\]

The following lemma gives us a continuity argument between $\tilde{H}(\tau)$ and $W_t$. It is similar to Lemma A.2 in [12] or Lemma 4.3 in [27]. We will later use this lemma on the resolvent entries.

**Lemma 4.1.** Denote $\partial_{ij} = \partial_{\tilde{H}_{ij}}$. Take $F$ a smooth function of the matrix entries satisfying

\[
\sup_{\theta \in [0, s]} \mathbb{E}\left[ \left| \frac{1}{10} \sum_{i<j} \left( N \left| (\tilde{H}(s) - D)_{ij} \right|^3 + \left| (\tilde{H}(s) - D)_{ij} \right| \right) \right| \right] \leq M \tag{4.1}
\]

where $(\theta H)_{ij} = \theta_{ij} H_{ij}$ with $\theta_{kl} = 1$ for $k, l \neq i, j$ and $\theta_{ij} \in [0, 1]$. Then

\[
\mathbb{E}[F(\tilde{H}(\tau))] - \mathbb{E}[F(\tilde{H}(0))] = O(\tau) \, M.
\]

**Proof.** By Itô's formula we have

\[
\partial_s \mathbb{E}[F(\tilde{H}(s))] = -\frac{1}{2N} \sum_{i<j} N \mathbb{E}[ (\tilde{H}(s) - D)_{ij} \partial F(\tilde{H}_s) ] - \mathbb{E}[\partial^2_{ij} F(\tilde{H}_s)].
\]

Using Taylor expansions, we can write, forgetting the dependence in time for clarity,

\[
\begin{align*}
    \mathbb{E}[ (\tilde{H} - D)_{ij} \partial_{ij} F(\tilde{H}) ] &= \mathbb{E}[ (\tilde{H} - D)_{ij} \partial_{ij} F(\tilde{H}_s) ] + \mathbb{E}[ (\tilde{H} - D)_{ij}^2 \partial^2_{ij} F(\tilde{H}_s) ] \\
    &\quad + O\left( \sup_{\theta} \mathbb{E}\left[ \left| (\tilde{H} - D)_{ij} \partial_{ij} F(\theta \tilde{H}) \right| \right] \right), \\
    &= \frac{t}{N} \partial^2_{ij} F(\tilde{H}_s) + O\left( \sup_{\theta} \mathbb{E}\left[ \left| (\tilde{H} - D)_{ij}^2 \partial^2_{ij} F(\theta \tilde{H}) \right| \right] \right).
\end{align*}
\]

and

\[
\begin{align*}
    \mathbb{E}[\partial^2_{ij} F(\tilde{H}_s)] &= \mathbb{E}[\partial^2_{ij} F(H_{ij} = D_{ij})] + O\left( \sup_{\theta} \mathbb{E}\left[ \left| (\tilde{H} - D)_{ij} \partial^2_{ij} F(\theta \tilde{H}) \right| \right] \right).
\end{align*}
\]

Putting everything together the claim follows. \(\square\)

This continuity property of the Dyson Brownian motion gives us a control over the eigenvalues and eigenvectors of $\tilde{H}(0)$ and $\tilde{H}_s$.

**Corollary 4.2.** Let $\Theta : \mathbb{R}^{2m} \to \mathbb{R}$ be a smooth function satisfying

\[
\sup_{k \in [0, 1], x \in \mathbb{R}^{2m}} \|\Theta^{(k)}(x)(1 + |x|)^{-C} < \infty,
\]

for some $C > 0$. Denote $\tilde{u}_1(s), \ldots, \tilde{u}_N(s)$ the eigenvectors of $\tilde{H}(s)$ associated with the eigenvalues $\tilde{\lambda}_1(s), \ldots, \tilde{\lambda}_N(s)$. Define, for a small $a$ the time domain

\[
\mathcal{T}_a' = \left[ N^a, N^{-a} \sqrt{\frac{t}{N}} \right],
\]

then for any $\tau \in \mathcal{T}_a'$, there exists $a > 0$ depending on $\Theta$, $a$, such that

\[
\sup_{t \in \mathcal{T}_a', |l| = m, |q| = 1} \left| (\mathbb{E}^{\tilde{H}_s} - \mathbb{E}^{\tilde{H}_0}) \Theta \left( \left( N(\tilde{\lambda}_k - \gamma_{k,l}) \frac{N}{\sigma_t^2} \frac{N}{(q, k)} (q, \tilde{u}_k)^2 \right)_{k \in I} \right) \right| \leq N^{-p} \tag{4.2}
\]
Proof. In order to prove this result on the eigenvalues and eigenvectors, we will first use Lemma 4.1 on a product of traces of resolvent to get continuity of the eigenvalues along the trajectory. This first continuity estimate will allow us to deduce gap universality for the eigenvalue of \( W_t \). We will then use this gap universality result to obtain a level repulsion estimate for the eigenvalue of the matrix \( W_t \). We will finally finish the proof of (4.2), which takes the eigenvectors in account, by combining this level repulsion estimate with a control of the resolvent entries in order to use the contour integral techniques from [28]. We can then split the proof into two results we need to show:

(i) A level repulsion estimate on the eigenvalues for both matrix ensembles in the following way

\[
P( |\lambda_i | \in [E - N^{-1-\xi}, E + N^{-1-\xi}] | \geq 2) \leq N^{-\xi - \theta}.
\]

(ii) Comparison of the resolvent below microscopic scales, for any smooth function \( F \) of polynomial growth, there exists a \( \delta > 0 \) such that, denoting

\[
z_k := E_k + i\eta \in \{ z = E + i\eta, E \in T^\alpha, N^{-1-\xi} < \eta < t \},
\]

for some small \( \xi > 0 \), we have

\[
\sup_{||q||_{L^2} = 1} \left| \left( E_{W_t} - E_{W_s} \right) F \left( \left( \frac{1}{\text{Im} \left( \sum_{i=1}^N q_i^2 g_i(t, z_k) \right)} \langle q, G(z_k) \rangle \right)^m \right) \right| \leq N^{-\xi}
\]

We will first prove (i) for the eigenvalues of \( W_t \). This property can be deduced from gap universality, note that gap universality for Gaussian perturbation of size \( t \in T^\alpha \) has been shown in [29] (a stronger level repulsion estimate can also be found in [29, Section 5]). In [29, Subsection 2.4], Landon-Yau explains that they can deduce universality for deformed Wigner ensembles. However, they state the result for an initial condition such that \( r^2 \gg t \). It has been confirmed by the authors that it is a simple typographical error and should be read as \( r \gg t \). We will nonetheless give an idea of the proof of (i) for the sake of completeness.

As said earlier, we will first apply Lemma 4.1 to

\[
F(\tilde{H}_s) = \frac{1}{N} \text{Tr}(\tilde{H}_s - z)^{-1} \text{ for } z \in \{ z = E + i\eta, E \in T^\alpha, N^{-1-\xi} < \eta < t \}
\]

for \( \xi > 0 \) arbitrarily small. Note that by definition of \( \tilde{H} \), we have \( |(\tilde{H} - D)_{ij}| \sim \frac{\sqrt{t}}{N} \) so that we can bound the left hand side of 4.1 by

\[
\frac{1}{N} \sqrt{\frac{T}{N} \sum_{i \neq j} |\partial^3_{ij} F(\tilde{H}_s)|}.
\]

Taking the third derivative of \( F \) with respect to an entry, we obtain, writing \( G = (\tilde{H}_s - z)^{-1} \) for simplicity

\[
\partial^3_{ij} F(\tilde{H}_s) = -\frac{1}{N} \sum_{k=1}^N \sum_{\alpha, \beta} G_{k\alpha i} G_{\beta j \alpha} G_{\beta 2a} G_{\beta 3k}
\]

where \( \{\alpha, \beta\} = \{i, j\} \) for \( \ell = 1, 2, 3 \). To bound the sum in the previous equation, we will need the following high probability bounds, already stated in Section 2,

\[
|G_{ii}(z) - g_i(t, z)| \prec \frac{t}{\sqrt{N\eta}} |g_i(t, z)|^2,
\]

\[
|G_{ij}(z)| \prec \frac{1}{\sqrt{N\eta}} \min(|g_i(t, z)|, |g_j(t, z)|) \leq \frac{1}{\sqrt{N\eta}} \sqrt{|g_i(t, z)g_j(t, z)|}.
\]

Note that these bounds holds for \( \eta \gg N^{-1} \), we will first consider such \( \eta \). However, since we want to work below microscopic scales, the terms in power of \( (N\eta)^{-1/2} \) can simply be bounded by \( N^{\xi/2} \).
Finally we can bound (4.4),

\[
\frac{1}{N} \sqrt{\frac{t}{N}} \sum_{i \leq j} |\partial_{ij}^3 F(\tilde{H}_s)| < \frac{N^{2\epsilon} \sqrt{t}}{N^2} \sqrt{\frac{t}{N}} \sum_{i \leq j} \sum_{k=1}^{N} |g_k| (|g_{\alpha_1} g_{\alpha_2} g_{\alpha_3} g_{\beta_1} g_{\beta_2} g_{\beta_3}|)^{1/2}
\]  

(4.7)

Now, from Lemma 7.5 of [29], we have

\[
\frac{1}{N} \sum_{k=1}^{N} |g_k(t, z)| \leq C \log N.
\]

(4.8)

Besides, in the last product of (4.7), there are, by definition of \(\alpha\) and \(\beta\), three occurrences of \(g_i\) and three occurrences of \(g_j\). Thus,

\[
\frac{CN^{2\epsilon} \log N}{N} \sqrt{\frac{t}{N}} \sum_{i \leq j} |g_i(t, z)g_j(t, z)|^{3/2} \leq \frac{CN^{2\epsilon} \log N}{N} \sqrt{\frac{t}{N}} \left( \sum_{k=1}^{N} |g_k(t, z)| \right)^2 \leq N^{2\epsilon} \log^3 N \sqrt{\frac{N}{t}}
\]

(4.9)

(4.10)

where we used (4.8) in the first inequality, the fact that \(|g_j| \leq C t^{-1}\) in the second and (4.8) again in the final inequality. In order to go below microscopic scales, recall that we used local laws that holds down to mesoscopic scales, we can use the following identity, for \(y \leq \eta\),

\[
\text{Im} \left( \frac{1}{N} \text{Tr} G(E + iy) \right) \leq \frac{\eta}{y} \text{Im} \left( \frac{1}{N} \text{Tr} G(E + i\eta) \right).
\]

Finally, using Lemma 4.1, we get,

\[
\sup_{E \in \mathbb{R}_{\tau}} \left| \left( \mathbb{E}^{H_{\tau}} - \mathbb{E}^{W_{\tau}} \right) \left[ \frac{1}{N} \text{Tr} G(z) \right] \right| \leq N^5 \tau \sqrt{\frac{N}{T}} \leq N^{-\epsilon}
\]

(4.11)

for some \(\epsilon > 0\) by taking \(\tau \in T_a^\beta\) for \(a > 5\xi\). We can easily generalize this result to a product of trace, indeed taking

\[F(\tilde{H}_s) = \prod_{k=1}^{m} F_k \quad \text{with} \quad F_k = \frac{1}{N} \text{Tr} G(z_k), \]

we can take the third derivative and write

\[
\partial_{ij}^3 F = \sum_{k_1=1}^{m} \partial_{ij}^3 F_k \prod_{k \neq k_1}^{m} F_k + 3 \sum_{k_1=1}^{m} \sum_{k_2 \neq k_1}^{m} \partial_{ij}^2 F_k \partial_{ij} F_{k_2} \prod_{k \neq k_1, k_2}^{m} F_k
\]

\[+ \sum_{k_1=1}^{m} \sum_{k_2 \neq k_1}^{m} \sum_{k_3 \neq k_1, k_2} \partial_{ij} F_{k_1} \partial_{ij} F_{k_2} \partial_{ij} F_{k_3} \prod_{k \neq k_1, k_2, k_3}^{m} F_k. \]

(4.12)

Then, using the first and second derivative of \(F_k\),

\[
\partial_{ij} F_k = \frac{1}{N} \sum_{k=1}^{N} \sum_{\{\alpha, \beta\} = \{i, j\}} G_{k, \alpha} G_{\beta, k},
\]

\[
\partial_{ij}^2 F_k = \frac{1}{N} \sum_{k=1}^{N} \sum_{\{\alpha, \beta\} = \{i, j\}} G_{k\alpha} G_{\beta\alpha} G_{\beta k},
\]

we can bound (4.12) in a similar way and finishing the bound by Lemma 4.1. Again, now that we have any polynomial of fixed degree, we can also extend to any smooth function \(F\) with polynomial growth.
Now, a consequence of these uniform bounds in \( \text{Re}(z) \) between \( \tilde{H}_0 = W_t \) and \( \tilde{H}_r \) for \( \tau \in T_a \) for some small \( \alpha \) gives us a comparison of the gap distribution between these two matrix ensembles (see [22] for instance). Namely, there exists \( c_1 > 0 \) such that for any \( O \) a smooth test function of \( n \) variables and any index \( i \) such that \( \gamma_{i,t} \in T_r \), we have for \( N \) large enough and \( i_1, \ldots, i_n \) indices such that \( i_k \leq N^{c_1} \),

\[
\left| \left( E^{W_t} - E^{\tilde{H}_r} \right) \left[ N \rho_{N}(\gamma_{i,t}) (\gamma_{k,t} - \gamma_{i_1}), \ldots, N \rho_{N}(\gamma_{i,t}) (\gamma_{k,t} - \gamma_{i_n}) \right] \right| \leq N^{-c_1} \tag{4.13}
\]

But \( \tilde{H}_r \) is a matrix with a small Gaussian component following the conditions of [29], so that we have, for this matrix ensemble, gap universality. Hence, combining this gap universality with the continuity of the Green’s function, we obtain gap universality of the matrix ensemble \( D + \sqrt{W} \). In other words, there exists \( c_2 > 0 \) such that, taking the same assumptions as for (4.13), we can write,

\[
\left| E^{W_t} \left[ O \left( \rho_{N}(\gamma_{i,t})(\gamma_{k,t} - \gamma_{i_1}), \ldots, \rho_{N}(\gamma_{i,t})(\gamma_{k,t} - \gamma_{i_n}) \right) \right] - E^{\text{GOE}} \left[ O \left( \rho_{N}(\mu_t)(\gamma_{k,t} - \gamma_{i_1}), \ldots, \rho_{N}(\mu_t)(\gamma_{k,t} - \gamma_{i_n}) \right) \right] \right| \leq N^{-c_2} \tag{4.14}
\]

where \( \rho_{sc} \) is the density of Wigner’s semicircular law and \( \mu_t \) its quantiles defined by

\[
\rho_{sc}(x) = \sqrt{4 - x^2} \mathbb{1}_{[-2,2]}, \quad \int_{-\infty}^{\mu_t} \text{d}\rho_{sc}(E) = \frac{i}{N}. \tag{4.15}
\]

Finally, (4.14) combined with rigidity (Theorem 3.5 of [29]) gives us the level repulsion estimate \((i)\) for the matrix \( W_t \), indeed consider \( E \in T_r^\varepsilon \), and \( \ell \) the index such that

\[
|\gamma_{\ell,t} - E| \leq \min_{k \in I^\varepsilon} |\gamma_{k,t} - E|,
\]

then, for any \( \varepsilon > 0 \), we have

\[
\mathbb{P} \left( \{ |\ell, \lambda_t | E - N^{-1-\varepsilon}, E + N^{-1-\varepsilon} \} \geq 2 \right) \leq \sum_{|k-\ell| \leq N^\varepsilon} \mathbb{P}^{W_t} ( |\lambda_k - \lambda_{k+1} | < N^{-1-\varepsilon} ) \leq \sum_{|k-\ell| \leq N^\varepsilon} \mathbb{P}^{\text{GOE}} ( |\lambda_k - \lambda_{k+1} | < N^{-1-\varepsilon} ) + N^{-c_3 \varepsilon} \leq N^{-2\varepsilon + \varepsilon} + N^{-c_3 \varepsilon} \leq N^{-\varepsilon - \delta}
\]

for some \( \delta > 0 \) by taking \( \varepsilon \) and \( \xi > 0 \) small enough. Note that we used rigidity in the first inequality, gap universality in the second and the Gaudin law for the eigenvalue gaps of GOE in the third inequality.

In order to get the resolvent estimate \((ii)\), we will use Lemma 4.1. To do so, we will first explain how to get the bound \( M \) for

\[
F(\tilde{H}_r) = \frac{1}{\text{Im} \left( \sum_{i=1}^{N} q_i^2 g_i(t,z) \right)} (q, \langle \tilde{H} - z \rangle^{-1} q)
\]

for \( z \in \mathbb{C} \) down to below microscopic scales. To get the right bound, we will first need to use local laws which holds up to mesoscopic scales \( \eta = N^{-1+\varepsilon} \).

Now for the third derivative of \( F \), first write

\[
|\partial_j^3 F(\tilde{H}_r)| = \frac{1}{\text{Im} \left( \sum_{i=1}^{N} q_i^2 g_i(t,z) \right)} \sum_{1 \leq a,b \leq N} \sum_{\alpha,\beta} q_a G_{\alpha_1 \alpha_2} G_{\beta_2 \alpha_3} G_{\beta_3 b} q_b \tag{4.16}
\]

where \( \{\alpha_k, \beta_k\} = \{i, j\} \) for \( k = 1, 2, 3 \). In order to bound the four terms coming up in the previous equation, following the high probability bound

\[
|v_i g_i(t,z) - \sum_{i=1}^{N} v_i w_i g_i(t,z) | \leq \frac{1}{\sqrt{N\eta}} \text{Im} \left( \sum_{i=1}^{N} v_i^2 g_i(t,z) \right) \text{Im} \left( \sum_{i=1}^{N} w_i^2 g_i(t,z) \right). \tag{4.17}
\]
Writing this identity for $v = q$ and $w = e_i$, we obtain
\[
\langle q, Ge_i \rangle = q_i g_i(t, z) + O_{\infty} \left( \frac{1}{\sqrt{Nt}} \right) \left[ \Im \left( \sum_{k=1}^{N} q_k^2 g_k(t, z) \right) \Im (g_i(t, z)) \right].
\]

Note that since we want a bound up to microscopic scales, the error terms has to be taken into account and we can also bound every $(Nt)^{1/2}$ by $N^{2/3}$. In the following computations, we will not bound the errors coming cross terms for simplicity, they can be bounded in a similar way.

We can divide the sum in (4.16) in three parts. The first case consists in $\{\beta_1, \alpha_2\} = \{\beta_2, \alpha_3\} = \{i, j\}$. In this case, note that, necessarily, $\{\alpha_1, \beta_3\} = \{i, j\}$ and write
\[
\sum_{1 \leq a, b \leq N} q_a G_{\alpha_1} G_{\beta_1, \alpha_2} G_{\beta_2, \alpha_3} q_b = \langle q, Ge_{\alpha_1} \rangle G_{\beta_1, \alpha_2} G_{\beta_2, \alpha_3} \langle e_{\beta_3}, Gq \rangle
\]
\[\prec \frac{1}{N^2} \min |g_i(t, z)|, |g_j(t, z)|^2 \left\langle q, Ge_{\alpha_1} \right\rangle \langle e_{\beta_3}, Gq \rangle
\]
\[\prec N^{2\xi} \left( \min |g_i(t, z)|, |g_j(t, z)|^2 \right) q_i g_i(t, z) q_j g_j(t, z) \] \hspace{1cm} (4.18) \hspace{1cm} \[+ \min |g_i(t, z)|, |g_j(t, z)|^2 \Im \left( \sum_{k=1}^{N} q_k^2 g_k(t, z) \right) \left| g_i(t, z) g_j(t, z) \right| \] \hspace{1cm} (4.19)

Putting the leading order (4.18) in the sum of (4.4), we have the bound
\[
\sqrt{\frac{t}{N}} \frac{1}{N \Im \left( \sum_{k=1}^{N} q_k^2 g_k(t, z) \right)} \sum_{1 \leq i \leq j \leq N} (4.18) \] \hspace{1cm} (4.20) \hspace{1cm} \[\leq \sqrt{\frac{t}{N}} \frac{1}{N \Im \left( \sum_{k=1}^{N} q_k^2 g_k(t, z) \right)} \sum_{1 \leq i \leq j \leq N} |q_i q_j| |g_i(t, z) g_j(t, z)|^2 \] \hspace{1cm} (4.21) \hspace{1cm} \[\leq \sqrt{\frac{t}{N}} \frac{1}{N \Im \left( \sum_{k=1}^{N} q_k^2 g_k(t, z) \right)} \sum_{1 \leq i \leq j \leq N} (q_i^2 + q_j^2) |g_i(t, z) g_j(t, z)|^2 \] \hspace{1cm} (4.22) \hspace{1cm} \[\leq \sqrt{\frac{t}{N}} \frac{1}{N \Im \left( \sum_{k=1}^{N} q_k^2 g_k(t, z) \right)} \left( \sum_{i=1}^{N} q_i^2 |g_i(t, z)|^2 \right) \left( \sum_{j=1}^{N} |g_j(t, z)|^2 \right) \] \hspace{1cm} (4.23)

Note that by definition of $g_i(t, z) = (D_t - z - t m_{1}(z))^{-1}$, the fact that $\Im m_{1}(z) \approx 1$ and $\eta \leq t$, we can write
\[
|g_i(t, z)|^2 \geq \frac{1}{t} \Im (g_i(t, z)).
\] \hspace{1cm} (4.24)

Besides we also have (from Lemma 7.5 of [29]),
\[
\sum_{i=1}^{N} |g_i(t, z)|^2 \leq \frac{C}{t} \sum_{i=1}^{N} |g_i(t, z)| \leq \frac{CN}{t} \log N.
\] \hspace{1cm} (4.25)

Injecting now (4.24) and (4.24) in (4.20), we get the bound
\[
\sqrt{\frac{t}{N}} \frac{1}{N \Im \left( \sum_{k=1}^{N} q_k^2 g_k(t, z) \right)} \sum_{1 \leq i < j \leq N} (4.18) \leq \sqrt{\frac{t}{N}} \frac{N^{2\xi} N}{N^2} \log N \] \hspace{1cm} (4.26) \hspace{1cm} \[\leq N^{3\xi} \frac{\sqrt{N}}{N^2} \sqrt{\frac{N}{t}}.
\] \hspace{1cm} (4.27)
Looking now at the error term (4.19) and injecting it in the sum (4.4), we obtain
\[
\sqrt{\frac{T}{N}} \text{Im} \left( \frac{1}{N^2} \sum_{k=1}^{N^2} g_k(t, z) \right) \sum_{1 \leq i < j \leq N} (4.19) \leq \sqrt{\frac{T}{N}} \sum_{1 \leq i < j \leq N} \left| g_i(t, z)g_j(t, z) \right|^{3/2} \leq \sqrt{\frac{T}{N}} \sum_{1 \leq i < j \leq N} \left| g_i(t, z) \right| \leq N^{3\xi} \sqrt{\frac{N}{T}}. (4.28)
\]

The second case are the terms where one term is diagonal and the other is an off-diagonal term. More precisely the set \( \alpha \) and \( \beta \) such that \( \beta_1 = \alpha_2 \) and \( \beta_2 \neq \alpha_3 \) or \( \beta_1 \neq \alpha_2 \) and \( \beta_2 = \alpha_3 \). Note that necessarily, in that case, \( \alpha_1 = \beta_3 \). For instance consider the term
\[
\langle q, Ge_{\alpha_1} \rangle G_{\beta_1, \alpha_2} G_{\beta_2, \alpha_3} \langle e_{\beta_3}, Gq \rangle = \langle q, Ge_{\alpha_1} \rangle G_{\beta_1, \alpha_2} G_{\beta_2, \alpha_3} \langle e_{\beta_3}, Gq \rangle. (4.29)
\]
Putting all the leading terms from (4.5), (4.6) and (4.17), we obtain the bound
\[
\langle q, Ge_{\alpha_1} \rangle G_{\beta_1, \alpha_2} G_{\beta_2, \alpha_3} \langle e_{\beta_3}, Gq \rangle \leq \frac{1}{(N\eta)^{3/2}} \text{Im} \left( \sum_{k=1}^{N} q_k^2 g_k(t, z) \right) \sqrt{\left| g_i(t, z)g_j(t, z) \right| \min(\left| g_i(t, z) \right|, \left| g_j(t, z) \right|)} \leq N^{2\xi} \left( q_i^2 |g_i(t, z)| \right)^{3/2} + \text{Im} \left( \sum_{k=1}^{N} q_k^2 g_k(t, z) \right) \left| g_i(t, z)g_j(t, z) \right|^{3/2}. (4.30)
\]

Then injecting the bounds (4.24) and (4.25) in the sum (4.4), one gets
\[
\sqrt{\frac{T}{N}} \text{Im} \left( \sum_{k=1}^{N^2} g_k(t, z) \right) \sum_{1 \leq i < j \leq N} q_i^2 \left| g_i g_j \right|^2 \leq \frac{C N^{3\xi}}{NT} \sqrt{\frac{N}{T}}. (4.33)
\]
and for the second term,
\[
\sqrt{\frac{T}{N}} \sum_{1 \leq i < j \leq N} \left| g_i(t, z)g_j(t, z) \right|^{3/2} \leq N^{3\xi} \sqrt{\frac{N}{T}}. (4.34)
\]

The final case consists of \( \alpha \) and \( \beta \) such that \( \{\beta_1, \alpha_2\}, \{\beta_2, \alpha_3\} = \{i, j\}, \{i, j\} \}. Note that, in this case, we necessarily have \( \alpha_1 \neq \beta_3 \). For instance, consider the term
\[
\langle q, Ge_{\alpha_1} \rangle G_{\beta_1, \alpha_2} G_{\beta_2, \alpha_3} \langle e_{\beta_3}, Gq \rangle = \langle q, Ge_{\alpha_1} \rangle G_{\beta_1, \alpha_2} G_{\beta_2, \alpha_3} \langle e_{\beta_3}, Gq \rangle. (4.35)
\]
Again, taking the leading terms from the local laws,
\[
\langle q, Ge_{\alpha_1} \rangle G_{\beta_1, \alpha_2} G_{\beta_2, \alpha_3} \langle e_{\beta_3}, Gq \rangle \propto \left| q_i q_j \right|^2 \left| g_i(t, z)g_j(t, z) \right|^2 + N^\xi \left| g_i(t, z)g_j(t, z) \right|^{3/2} \text{Im} \left( \sum_{k=1}^{N} q_k^2 g_k(t, z) \right). (4.36)
\]

Then using similar bounds as the first case one gets
\[
\sqrt{\frac{T}{N}} \text{Im} \left( \sum_{k=1}^{N^2} g_k(t, z) \right) \sum_{1 \leq i < j \leq N} (4.36) \propto N^{2\xi} \left( \frac{1}{NT} \sqrt{\frac{N}{T}} + \sqrt{\frac{N}{T}} \right). (4.37)
\]
Finally, putting together (4.27), (4.28), (4.33), (4.34) and (4.37), we get the bound, for \( \eta = N^{-1+\xi} \).

\[
\sqrt{\frac{T}{N}} \frac{1}{N} \sum_{1 \leq i < j \leq N} \partial^3_{ij} F(\tilde{H}_s) \times N^{3\xi} \sqrt{\frac{N}{I}}.
\] (4.38)

In order to get a bound for \( \eta \) below microscopic scales, we can use the following inequality, for any \( y \leq \eta \),

\[
|\langle q, G(E + iy)q \rangle| \leq C \log N \eta y \text{Im} \langle q, G(E + i\eta)q \rangle.
\]

Thus, uniformly in \( E \in T_\kappa \) and \( N^{-1-\xi} \leq \eta \leq t \), we have

\[
M = O \left( N^{5\xi} \sqrt{\frac{N}{I}} \right).
\] (4.39)

Using now Lemma 4.1, we can make \( W_t \) undergo the dynamics \( \tilde{H}_s \) up to a time \( \tau \ll N^{-5\xi} \sqrt{\frac{N}{I}} \) with \( \xi \) arbitrarily small in order to get the right bound.

For a product of resolvent entries, one can do similar computations and bounds. Indeed consider \( m \geq 0 \), and

\[
F(\tilde{H}_s) = \prod_{k=1}^{m} F_k(\tilde{H}_s) \quad \text{with} \quad F_k(\tilde{H}_s) = \langle q, G(z_k)q \rangle,
\]

then one can write the third derivative of \( F \) as (4.12) and using the fact that

\[
\partial^3_{ij} F_k = -\sum_{\{\alpha, \beta\} = \{i, j\}} \langle q, Ge_\alpha \rangle \langle e_\beta, Gq \rangle, \quad (4.40)
\]

\[
\partial^2_{ij} F_k = \sum_{\alpha, \beta} \langle q, Ge_\alpha \rangle G_{\beta_1, \alpha_2} \langle e_{\beta_2}, Gq \rangle \quad (4.41)
\]

where \( \{\alpha_i, \beta_i\} = \{i, j\} \) and using the same type of bounds as for (4.38), we obtain the result (4.3) since the extension to any smooth function with polynomial growth is also clear.

4.2. Reverse heat flow

In Subsection 3.2, we showed Theorem 3.6, which corresponds to our main result for the matrix \( H_t = W_t + \sqrt{\tau GOE} \) with a general Wigner matrix \( W \) in the definition of \( W_t \). Thus, the overwhelming probability bound holds for the eigenvectors of this matrix \( H_t \) giving us a strong form of quantum unique ergodicity for the deformed Gaussian divisible ensemble. In order to remove the small Gaussian component in the matrix, we will use the reverse heat flow technique from [17, 19] which allows us to obtain an error as small as we want in total variation between two matrix ensembles. In order to use this technique, we need the smoothness assumption on the matrix \( W \) given by Definition 1.7. We first introduce some notation for this section.

As in earlier, we will denote the distribution of the matrix \( W \) entries as \( \nu \) with a density \( \varphi \) with respect to the Gaussian distribution with mean zero and variance one denoted \( \varrho \): \( d\nu = \varphi d\varrho \). The reverse heat flow technique gives the existence of a probability distribution \( \tilde{\nu}_s \) for any \( s \) small enough such that making \( \tilde{\nu}_s \) undergo the Ornstein-Uhlenbeck process

\[
A := \frac{1}{2} \frac{\partial^2}{\partial x^2} - \frac{x \partial}{2 \partial x}
\]

approaches the distribution \( \nu \) in total variation.

This process on all the matrix entries induces the Dyson Brownian motion process on the eigenvalues. Thus the following proposition tells us that there exists a distribution of a matrix from the Gaussian divisible process of the form

\[
\tilde{W}_s = \sqrt{1-s} \tilde{W} + \sqrt{s GOE}
\]
that approximates as close as polynomially possible a smooth Wigner matrix $W$. The precise statement is written in the following proposition.

**Proposition 4.3** ([19]). Let $K$ be a positive integer and $\nu = \varphi \rho$ a distribution smooth in the sense that it follows the conditions (ii) and (iii) of Definition 1.7. Then there exists $s_K$ a small positive constant depending on $K$ such that for any $0 < s \leq s_K$, there exists a probability density $\psi_s$ with mean zero and variance one such that we have the inequality

$$\int |e^{sA}\psi_s - \varphi|\,d\rho \leq Cs^K$$

for some positive constant $C$ depending only on $K$. Besides we also have the inequality for the joint probability of all matrix entries in the following sense,

$$\int |e^{sA^{\otimes N^2}}\psi^{\otimes N^2} - \varphi^{\otimes N^2}|\,d\rho \leq CN^2s^K$$

Now, see that this proposition holds for any fixed $K$ so that, taking $s = N^{-\epsilon}$ for some small $\epsilon$ we can choose a large $K$ only depending on $\epsilon$ (and not on $N$) so that we can obtain any polynomial bound between the two matrix ensembles. This property allows us to get overwhelming probability bounds on the eigenvectors since the total variation distance of the distribution of the eigenvector entries is smaller than the total variation distance between the joint probability of the matrix entries.

5. Proofs of main results

Now that we have the result for the Gaussian divisible ensemble $H_\tau$ with $\tau \ll t$ by Section 3, combining it with the continuity argument from the last subsection, we are able to prove Theorem 1.3 and Corollary 1.6. These two results are a consequence of the following proposition showing the convergence of moments for the eigenvectors of $W_t$.

**Proposition 5.1.**
Let $\kappa \in (0,1)$ and $m$ an integer, for a set of indices $I \subset A^\kappa_n$, such that $|I| = m$, we have

$$E \left[ P \left( \frac{N}{\sigma_t(q,k)} |(q, u_k)^2| \right) \right] \xrightarrow{N \to \infty} E \left[ P \left( (N_k^2)^m_{k=1} \right) \right]$$

with $(N_k)_k$ a family of independent normal random variables.

See now the proof of Theorem 1.3 and Corollary 1.6 given by Proposition 3.3.

**Proof of Theorem 1.3.** Proposition 5.1 exactly gives us that the joint moments of the renormalized eigenvectors converge to those of independent normal random variables which is the result of Theorem 1.3.

**Proof of Corollary 1.6.** By Theorem 1.3, we have the following inequality, for some $\varepsilon > 0$,

$$E \left[ |u_k(\alpha)|^2 \right] = \frac{1}{N} \sigma_t^2(\alpha,k) + O \left( \frac{N^{-\varepsilon}}{Nt} \right).$$

By Markov’s inequality, we can write

$$P \left( \frac{Nt}{|A|} \sum_{\alpha \in A} |u_k(\alpha)|^2 - \frac{1}{N} \sum_{\alpha \in A} \sigma_t^2(\alpha,k) > \epsilon \right) \leq \frac{N^2t^2}{c^2|A|^2} E \left[ \left( \sum_{\alpha \in A} |u_k(\alpha)|^2 - \frac{1}{N} \sum_{\alpha \in A} \sigma_t^2(\alpha,k) \right)^2 \right],$$

$$\leq \frac{N^2t^2}{c^2|A|^2} (A - 2B + C)$$

(5.3)
Finally, we can apply Proposition 3.3 to \( H_t \) so that (5.6) applies and combining it with (5.5) we get the convergence of moments for the eigenvectors of \( W_t \).
Proof of Theorem 1.8. Let $\varepsilon$ and $D$ two positive constants and consider $s = \tau/t$. There exists then a large $K$, which does not depend on $N$, such that by Proposition 4.3 there exists a matrix $\tilde{W}$ such that the total variation distance between the distribution of $W$ and $\sqrt{t-s}W + \sqrt{NsGOE}$ is smaller than $N^{-D}$.

Denote $u_1, \ldots, u_N$ the $L^2$-normalized eigenvectors of $W_t = D + \sqrt{t}W$ and $\tilde{u}_1, \ldots, \tilde{u}_N$ the normalized eigenvectors of $\tilde{W}_t(s) = D + \sqrt{t(1-s)}W + \sqrt{tsGOE}$. Now, since we have in the overwhelming probability bound (1.12) the $N^\varepsilon$ degree of liberty, we can do the scaling $t' = \sqrt{t(1-s)}$ as $s \ll 1$ and still get (1.12) for the deformed Gaussian divisible ensemble $\tilde{W}_t(s)$, thus one can write

$$
\mathbb{P}\left(\sum_{\alpha \in I} \left( u_k(\alpha)^2 - \frac{1}{N} \sigma_k^2(\alpha, k) \right) \geq N^\varepsilon \Xi(\tau) \right) \\
\leq \mathbb{P}\left( \sum_{\alpha \in I} \left( \tilde{u}_k(\alpha)^2 - \frac{1}{N} \sigma_k^2(\alpha, k) \right) \geq N^{\varepsilon/2} \Xi(\tau) \right) + \mathbb{P}\left( \sum_{\alpha \in I} (u_k(\alpha)^2 - \tilde{u}_k(\alpha)) \geq N^{\varepsilon/2} \Xi(\tau) \right) \\
\leq N^{-D}
$$

where for the last inequality we used the quantum unique ergodicity proved in Theorem 3.6 for the deformed Gaussian divisible ensemble of which $\tilde{u}$ are the eigenvectors and Proposition 4.3 in order.

Now, in order to get the error $\Xi$ we now need to optimize the error

$$
\Xi(\tau_0) = \frac{\tilde{T}_0 \tau_0}{N^\varepsilon} + \frac{\tilde{T}_0 \tau_0}{(N^\varepsilon)^{1/3}} = \Xi \quad \text{with} \quad \tau_0 = \left(\frac{t^2}{N}\right)^{1/3}.
$$

We can do the same thing for the quantity $\sum_{\alpha \in I} u_k(\alpha) w(\alpha)$ and get the final result. \hfill \Box

A. The stochastic advection equation method

In this section, we will consider a small perturbation of a Wigner semicircle profile by a Gaussian noise. First, consider a diagonal matrix $D_{sc}$ such that rigidity according to the semicircle density holds for its diagonal entries in the following way, for any $\varepsilon$ a positive constant, and any $k \in [1, N]$, 

$$|D_{k,sc} - \mu_k| \leq N^{-2/3} + \varepsilon N^{-1/3} \quad (A.1)$$

where $(\mu_k)_{1 \leq k \leq N}$ are the quantiles of the semicircular law defined in (4.15).

We then consider the Ornstein-Uhlenbeck Dyson Brownian motion at time $\tau = N^{-1+\delta}$ for any $\delta \in (0, 1)$ with initial condition $D_{sc}$,

$$H_{sc}(\tau) = e^{-\tau/2}D_{sc} + \sqrt{1-e^{-\tau}}G_\beta$$

where $G_\beta$ is an element of the Gaussian Orthogonal Ensemble for $\beta = 1$ and of the Gaussian Unitary Ensemble for $\beta = 2$. In the next subsection, we will see that using the dynamical nature of this model allows us to get a stronger local law than Theorem 2.3 in our specific case when initial rigidity holds. This relies on a stochastic advection equation, as in [8, 24, 40], for which we have optimally small error estimates.

A.1. Improved local law

Denote the Stieltjes transform of the empirical spectral measure and of the semicircle law $\rho_{sc}$

$$s_\tau(z) = \frac{1}{N} \sum_{k=1}^{N} \frac{1}{\lambda_k(\tau) - z} \quad \text{and} \quad m(z) = \int \frac{d\rho_{sc}(x)}{x-z} = \frac{-z + \sqrt{z^2 - 4}}{2}$$

and the renormalized resolvent for $z \in \mathbb{C}$ such that $\text{Im}(z) > 0$,

$$G_\tau(z) = \frac{e^{-\tau/2}}{N} \sum_{k=1}^{N} \frac{z_k(\tau)^2}{\lambda_k(\tau) - z} \quad (A.3)$$

where $z_k(\tau)$ is given by (1.17) and $(\lambda_k(\tau), u_k(\tau))$ denotes the eigenvalues and eigenvectors of $H_\tau$. 

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It was observed in [8] in the context of characteristic polynomials of random matrices (see also [24, 40] for other use of this resolvent flow) that for any \( \text{Im}(z) \neq 0 \), we have

\[
dG_\tau(z) = \left( s_\tau(z) + \frac{\tau}{2} \right) \partial_\tau G_\tau(z) d\tau + \frac{2}{\beta} \left( \frac{1}{2N} \right) \partial_{zz} G_\tau(z)
- \frac{1}{\sqrt{\beta N}} \sum_{k,l=1}^{N} \frac{z_k z_l}{(\lambda_k(\tau) - z)(\lambda_l(\tau) - z)} dB_{k,l}.
\]

(A.4)

This a simple application of the Itô formula. Now, because our initial condition is semicircular, by (A.2), for \( \eta \gg N^{-1} \), the first term of the right hand side of (A.4) is approximately \( \sqrt{z^2 - 4}/2 \) and the other terms are expected to be of smaller order. One can then see that the equation (A.4) is an approximation of the two dimensional advection equation

\[
\partial_\tau h(\tau, z) = \frac{\sqrt{z^2 - 4}}{2} \partial_z h(\tau, z).
\]

(A.5)

This equation is exactly solvable using the method of characteristics as in [8]: define for any fixed \( z \) such that \( \text{Im}(z) > 0 \),

\[
z_\tau = \frac{1}{2} \left( e^{\tau/2}(z + \sqrt{z^2 - 4}) + e^{-\tau/2}(z - \sqrt{z^2 - 4}) \right).
\]

(A.6)

Then the trajectory \( z_\tau \) corresponds to the characteristics for the two dimensional advection equation (A.5). In other words, for any analytic initial condition \( h(0, \cdot) \) on \( \text{Im}(z) > 0 \), we have \( h(\tau, z) = h(0, z_\tau) \).

The following proposition from [8] states that, indeed, with overwhelming probability, \( G_\tau(z) \) is very close to \( G_0(z_\tau) \). It gives us an anisotropic local law with a parameter control of \( (N\eta)^{-1} \) in the bulk which is better than the usual entrywise local laws control parameter of \( (N\eta)^{-1/2} \).

More precisely, first take \( \kappa \) and \( \varepsilon \) to be any small positive constants and define

\[
D_\varepsilon^\kappa = \left\{ z = E + i\eta, -2 + \kappa < E < 2 - \kappa, \frac{N\varepsilon}{N} < \eta < 1 \right\}.
\]

By combining the result in [8] with (A.4), we have the following.

**Proposition A.1** ([8]). Consider the dynamics (A.4) and assume (A.1). Let \( \varepsilon, \kappa > 0 \) be a (small) fixed constant. For any \( 0 < \tau < N^{-\varepsilon} \) and \( z = E + i\eta \in D_\varepsilon^\kappa \), we have

\[
G_\tau(z) = G_0(z_\tau) + \mathcal{O}_\varepsilon \left( \frac{1}{N\eta} \right)
\]

(A.7)

**Remark A.2.** If one looks at the proof of Proposition 3.3, we used the local law on \( (q, Gq) \) that gives an error term of \( (N\eta)^{-1/2} \). Here, the entrywise local law on the perturbation of a Wigner profile from Lemma A.1 gives an error \( (N\eta)^{-1} \) and we can get convergence of moments with a better error term. Take \( N^{-1} \ll \tau \ll 1 \), define \( f_{sc}(\tau) \) as in (1.19) for \( H_\tau \) and see that the variance in this specific case is

\[
\sigma_\tau(k_0, k)^2 := \frac{(\mu_k - \mu_{k_0})^2 + (\tau \text{Im} m(\mu_k + i\eta_\varepsilon))^2}{(\mu_{k_0})^2},
\]

(A.8)

then for any configuration \( \eta \subset [\alpha N, (1 - \alpha)N] \) such that \( \mathcal{N}(\eta) = n \) we have

\[
f_{sc}(\tau, \eta) = \prod_{i=1}^{N} \sigma_\tau(k_0, k)^2 q_\eta + \mathcal{O}_\varepsilon \left( \frac{1}{(N\tau)^{1/2}} \right).
\]

(A.9)

Note that we get a better error, still not optimal, by taking the following parameters in the proof: we make the matrix \( H_\tau \) undergo the Dyson Brownian motion for a time \( \tau' = (N\tau)^{1/3}/N \ll \tau \), choose the spectral resolution \( \eta = N^{-\tau'/2} \), take the short-range parameter \( \ell = \sqrt{uN\tau'} \) where we localize the dynamics onto a window of size \( u = ((N\tau)^2 N\tau')^{1/3} \).
A.2. Multi-particle advection equation

We looked at the following quantity

\[ G_t(z) = N(q, e^{-t/2}Gq) = e^{-t/2} \sum_{k=1}^{N} \frac{|z_k|^2(t)}{\lambda_k(t) - z}. \]  \hspace{1cm} (A.10)

so that \( E[G_t(z)|\lambda] \) encodes the eigenvector moment flow observable for \( \eta \) consisting of one particle. It is interesting to try to encode the information of all sizes of the configurations \( \eta \) into such quantities. We will try to do that by looking instead at the exponential of the renormalized quantity \( G_t(z) \).

On the one hand, we can do the following elementary combinatorics, first look at the Hermitian case,

\[
\exp \left( e^{-t/2} \sum_{k=1}^{N} \frac{|z_k|^2}{2(\lambda_k - z)} \right) = \prod_{k=1}^{N} \sum_{n \geq 0} e^{-nt/2} \frac{|z_k|^{2n}}{2^n(\lambda_k - z)^n n!} = \sum_{n \geq 0} e^{-nt/2} \sum_{\mathcal{N}(\eta) = n} \prod_{k=1}^{N} \frac{|z_k|^{2\eta_k}}{(\lambda_k - z)^{\eta_k} 2^{\eta_k} \eta_k!}
\]

Taking the conditional expectation over \( \lambda \), we get

\[
E \left[ \exp \left( e^{-t/2} \sum_{k=1}^{N} \frac{|z_k|^2}{2(\lambda_k - z)} \right) \right] \lambda = \sum_{n \geq 0} e^{-nt/2} \sum_{\mathcal{N}(\eta) = n} \frac{f^{(h)}(\eta)}{\prod_{k=1}^{N} (\lambda_k - z)^{\eta_k}}.
\]

Now, for the polynomials in the Hermitian case, we can write

\[
E \left[ \exp \left( e^{-t/2} \sum_{k=1}^{N} \frac{|z_k|^2}{2(\lambda_k - z)} \right) \right] \lambda = \sum_{n \geq 0} e^{-nt/2} \sum_{\mathcal{N}(\eta) = n} \frac{f^{(s)}(\eta) \pi^{(s)}(\eta)}{\prod_{k=1}^{N} (\lambda_k - z)^{\eta_k}}
\]

If we now consider the symmetric case, we get a correction term, noted \( \pi^{(s)} \), and we have

\[
E \left[ \exp \left( e^{-t/2} \sum_{k=1}^{N} \frac{|z_k|^2}{2(\lambda_k - z)} \right) \right] \lambda = \sum_{n \geq 0} e^{-nt/2} \sum_{\mathcal{N}(\eta) = n} \frac{f^{(s)}(\eta) \pi^{(s)}(\eta)}{\prod_{k=1}^{N} (\lambda_k - z)^{\eta_k}}
\]

where

\[
\pi^{(s)}(\eta) = \prod_{k=1}^{N} \frac{a(2\eta_k)}{2^{\eta_k} \eta_k!} = \prod_{k=1}^{\eta_k} \prod_{l=1}^{1} \left(1 - \frac{1}{2l}\right).
\]

With the convention \( \pi^{(h)}(\eta) = 1 \) for all configurations \( \eta \), we can write the following identity for both the Hermitian and the symmetric case:

\[
E \left[ \exp \left( \frac{1}{2} N \langle q, e^{-t/2}Gq \rangle \right) \right] \lambda = \sum_{n \geq 0} e^{-nt/2} \sum_{\mathcal{N}(\eta) = n} \frac{f^{(s)}(\eta) \pi^{(s)}(\eta)}{\prod_{k=1}^{N} (\lambda_k - z)^{\eta_k}}. \hspace{1cm} (A.11)
\]

Interestingly, the measures on the configuration space defined above \( \pi^{(s)} \) and \( \pi^{(h)} \) are the reversible measures for the dynamics (1.20) and its Hermitian equivalent respectively, see [12, Prop 3.2].

On the other hand, by Lemma A.1, we have

\[
\exp \left( \frac{1}{2} G_t(z) \right) = \exp \left( \frac{1}{2} G_0(z_t) + O_\infty \left( \frac{1}{N\eta} \right) \right).
\]
Together with (A.11), we obtain the following identity

\[
\sum_{n \geq 0} e^{-nt/2} \sum_{\eta} \frac{f_t(\eta)\pi(\eta)}{\prod_{k=1}^n (\lambda_k - z)^{\eta_k}} = \exp \left( O_\prec \left( \frac{1}{N\eta} \right) \right) \sum_{n \geq 0} \sum_{\eta} \frac{f_0(\eta)\pi(\eta)}{\prod_{k=1}^n (\lambda_k - z)^{\eta_k}}
\]

We believe this observation on the Laplace transform of the resolvent gives a path to the analysis of high moments of the eigenvectors.

References

[1] R. Allez, J. Bun, and J.-P. Bouchaud. The eigenvectors of Gaussian matrices with an external source. arXiv preprint arXiv:1412.7108, 2014.
[2] N. Anantharaman and E. Le Masson. Quantum ergodicity on large regular graphs. Duke Math. J., 164(4):723–765, 2015.
[3] G. W. Anderson, A. Guionnet, and O. Zeitouni. An introduction to random matrices, volume 118 of Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 2010.
[4] P. Anderson. Absences of diffusion in certain random lattices. Phys. Rev., pages 1492–1505, 1958.
[5] R. Bauerschmidt, J. Huang, and H.-T. Yau. Local Kesten–McKay law for random regular graphs. arXiv preprint arXiv:1609.09052, 2016.
[6] P. Biane. On the free convolution with a semi-circular distribution. Indiana Univ. Math. J., 46(3):705–718, 1997.
[7] O. Bohigas, M.-J. Giannoni, and C. Schmit. Characterization of chaotic quantum spectra and universality of level fluctuation laws. Phys. Rev. Lett., 52(1):1–4, 1984.
[8] P. Bourgade. Extreme gaps between eigenvalues of Wigner matrices. In Preparation, 2018+.
[9] P. Bourgade, L. Erdös, H.-T. Yau, and J. Yin. Universality for a class of random band matrices. Adv. Theor. Math. Phys., 21(3):739–800, 2017.
[10] P. Bourgade, L. Erdös, H.-T. Yau, and J. Yin. Fixed energy universality for generalized Wigner matrices. Comm. Pure Appl. Math., 69(10):1815–1881, 2016.
[11] P. Bourgade, J. Huang, and H.-T. Yau. Eigenvector statistics of sparse random matrices. Electron. J. Probab., 22:Paper No. 64, 38, 2017.
[12] P. Bourgade and H.-T. Yau. The eigenvector moment flow and local quantum unique ergodicity. Comm. Math. Phys., 350(1):231–278, 2017.
[13] P. Bourgade, H.-T. Yau, and J. Yin. Random band matrices in the delocalized phase, I: Quantum unique ergodicity and universality. arXiv preprint arXiv:1807.01559, 2018.
[14] M.-F. Bru. Diffusions of perturbed principal component analysis. J. Multivariate Anal., 29(1):127–136, 1989.
[15] Y. Colin de Verdière. Ergodicité et fonctions propres du laplacien. Comm. Math. Phys., 102(3):497–502, 1985.
[16] V. E. Kravtsov, B. L. Altshuler, and L. Ioffe. Non-ergodic delocalized phase in Anderson model on Bethe lattice and regular graph. Ann. Phys., 389, 12 2017.
[17] L. Erdös, S. Péché, J. A. Ramírez, B. Schlein, and H.-T. Yau. Bulk universality for Wigner matrices. Comm. Pure Appl. Math., 63(7):895–925, 2010.
[18] L. Erdös, B. Schlein, and H.-T. Yau. Local semicircle law and complete delocalization for Wigner random matrices. Comm. Math. Phys., 287(2):641–655, 2009.
[19] L. Erdös, B. Schlein, and H.-T. Yau. Universality of random matrices and local relaxation flow. Invent. Math., 185(1):75–119, 2011.
[20] L. Erdős and H.-T. Yau. Gap universality of generalized Wigner and $\beta$-ensembles. *J. Eur. Math. Soc. (JEMS)*, 17(8):1927–2036, 2015.

[21] L. Erdős and H.-T. Yau. *A dynamical approach to random matrix theory*, volume 28 of *Courant Lecture Notes in Mathematics*. Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 2017.

[22] L. Erdős, H.-T. Yau, and J. Yin. Bulk universality for generalized Wigner matrices. *Probab. Theory Related Fields*, 154(1-2):341–407, 2012.

[23] L. Erdős, H.-T. Yau, and J. Yin. Rigidity of eigenvalues of generalized Wigner matrices. *Adv. Math.*, 229(3):1435–1515, 2012.

[24] D. Facoetti, P. Vivo, and G. Biroli. From non-ergodic eigenvectors to local resolvent statistics and back: A random matrix perspective. *EPL (Europhysics Letters)*, 115(4):47003, 2016.

[25] R. Holowinsky. Sieving for mass equidistribution. *Ann. of Math. (2)*, 172(2):1499–1516, 2010.

[26] R. Holowinsky and K. Soundararajan. Mass equidistribution for Hecke eigenforms. *Ann. of Math. (2)*, 172(2):1517–1528, 2010.

[27] J. Huang, B. Landon, and H.-T. Yau. Bulk universality of sparse random matrices. *J. Math. Phys.*, 56(12):123301, 2015.

[28] A. Knowles and J. Yin. Eigenvector distribution of Wigner matrices. *Probab. Theory Related Fields*, 155(3-4):543–582, 2013.

[29] B. Landon and H.-T. Yau. Convergence of local statistics of Dyson Brownian motion. *Comm. Math. Phys.*, 355(3):949–1000, 2017.

[30] J. O. Lee and K. Schnelli. Local deformed semicircle law and complete delocalization for Wigner matrices with random potential. *J. Math. Phys.*, 54(10):103504, 2013.

[31] J. O. Lee, K. Schnelli, B. Stetler, and H.-T. Yau. Bulk universality for deformed Wigner matrices. *Ann. Probab.*, 44(3):2349–2425, 2016.

[32] E. Lindenstrauss. Invariant measures and arithmetic quantum unique ergodicity. *Ann. of Math. (2)*, 163(1):165–219, 2006.

[33] J. R. Norris, L. C. G. Rogers, and D. Williams. Brownian motions of ellipsoids. *Trans. Amer. Math. Soc.*, 294(2):757–765, 1986.

[34] Z. Rudnick and P. Sarnak. The behaviour of eigenstates of arithmetic hyperbolic manifolds. *Comm. Math. Phys.*, 161(1):195–213, 1994.

[35] A. I. Šnirel’mann. Ergodic properties of eigenfunctions. *Uspehi Mat. Nauk*, 29(6(180)):181–182, 1974.

[36] T. Tao and V. Vu. Random matrices: universality of local eigenvalue statistics. *Acta Math.*, 206(1):127–204, 2011.

[37] T. Tao and V. Vu. Random matrices: universal properties of eigenvectors. *Random Matrices Theory Appl.*, 1(1):1150001, 27, 2012.

[38] K. Truong and A. Ossipov. Eigenvectors under a generic perturbation: Non-perturbative results from the random matrix approach. *EPL (Europhysics Letters)*, 116(3):37002, 2016.

[39] P. von Soosten and S. Warzel. Local stability of the resolvent flow under Dyson Brownian motion. *arXiv preprint arXiv:1705.00923*, 2017.

[40] P. von Soosten and S. Warzel. Non-ergodic delocalization in the Rosenzweig-Porter model. *arXiv preprint arXiv:1709.10313*, 2017.

[41] S. Zelditch. Uniform distribution of eigenfunctions on compact hyperbolic surfaces. *Duke Math. J.*, 55(4):919–941, 1987.