An approach to universality using Weyl $m$-functions

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joint work with  
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Baylor Analysis Fest, ”From Operator Theory to Orthogonal Polynomials, Combinatorics, and Number Theory”
Christoffel–Darboux kernel

- Let $\mu$ be a probability measure on $\mathbb{R}$ with all finite moments,

$$\int |\xi|^n \, d\mu(\xi) < \infty, \quad \forall n \in \mathbb{N}.$$ 

Assume that $\mu$ has infinite support (in sense of cardinality).

- We obtain **orthonormal polynomials** $\{p_j(z)\}_{j=0}^{\infty}$ by the Gram–Schmidt process from the sequence of monomials $\{z^j\}_{j=0}^{\infty}$ in $L^2(\mathbb{R}, d\mu)$.

- The **Christoffel–Darboux (CD) kernel** is

$$K_n(z, w) = \sum_{j=0}^{n-1} p_j(z)p_j(w).$$

Reproducing kernel for subspace $\text{span}\{1, z, \ldots, z^{n-1}\} \subset L^2(\mathbb{R}, d\mu)$. 
Universality limits of CD kernels are double scaling limits

\[
\lim_{n \to \infty} \frac{1}{\tau_n} K_n \left( \xi + \frac{z}{\tau_n}, \xi + \frac{w}{\tau_n} \right)
\]

for an appropriate sequence \( \tau_n \to \infty \) and \( z, w \in \mathbb{C}, \xi \in \mathbb{R} \).

They are called universality limits because the limit is often found to be a standard kernel and does not depend on the exact measure we started with: the most common phenomenon is bulk universality, with limiting kernel

\[
\sin \pi(w - z) \\
\pi(w - z)
\]

Interpretation: Kernel for Gaussian unitary ensemble, Paley Wiener Spaces, and ...
Local zero spacing

- Denote by $\xi_j^{(n)}$ for $j \in \mathbb{Z}$ the zeros of $p_n$ counted from $\xi$, i.e.,
  \[ \ldots < \xi_{-2}^{(n)} < \xi_{-1}^{(n)} < \xi < \xi_0^{(n)} < \xi_1^{(n)} < \ldots \]

- **Freud–Levin theorem**: The bulk universality limit implies
  \[ \lim_{n \to \infty} \tau_n(\xi_{j+1}^{(n)} - \xi_j^{(n)}) = 1 \quad \forall j \in \mathbb{Z}. \]

  Statements of this type are commonly described as “clock behavior”.

- Universality limits were first studied in the setting of random matrices, where this universal eigenvalue spacing was observed.
Previous results on bulk universality

\[
\lim_{n \to \infty} K_n \left( \xi + \frac{z}{\tau_n}, \xi + \frac{w}{\tau_n} \right) = \frac{\sin(\pi(\overline{w} - z))}{\pi(\overline{w} - z)},
\]

with appropriate \( \tau_n \).

- 1971 Freud: on \([-1, 1]\) with \( d\mu(\xi) = w(\xi)d\xi \) and strong conditions on \( w \)
- For Gaussian measure \( d\mu(\xi) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\xi^2} d\xi \) follows from properties of Hermite polynomials
- 1990s Deift–Kriecherbauer–McLaughlin–Venakides–Zhou: Riemann–Hilbert techniques for measures
  \[ d\mu = e^{-Q(\xi)} d\xi \]
- 2009 Lubinsky (Annals): Stahl–Totik regular measures \( d\mu \) with local Lebesgue point/local Szegő conditions at \( \xi \); with extensions by Findley, Simon and Totik.
**A local criterion for bulk universality**

**Theorem (E.–Lukić–Simanek)**

Let $\mu$ be a probability measure on $\mathbb{R}$ with infinite support and finite moments, corresponding to a determinate moment problem. Let

$$m(z) = \int \frac{1}{x - z} \, d\mu(x), \quad z \in \mathbb{C}_+.$$ 

Let $\xi \in \mathbb{R}$ and assume that for some $0 < \alpha < \pi/2$,

$$f_\mu(\xi) := \frac{1}{\pi} \lim_{z \to \xi} \frac{\text{Im} \, m(z)}{\alpha \leq \arg(z - \xi) \leq \pi - \alpha} \in (0, \infty).$$

Then uniformly on compact regions of $(z, w) \in \mathbb{C} \times \mathbb{C}$,

$$\lim_{n \to \infty} \frac{K_n \left( \xi + \frac{z}{f_\mu(\xi) K_n(\xi, \xi)}, \xi + \frac{w}{f_\mu(\xi) K_n(\xi, \xi)} \right)}{K_n(\xi, \xi)} = \frac{\sin(\pi(w - z))}{\pi(w - z)}.$$
Nontangential limits of \( m(z) \)

- The nontangential limit

\[
  f_\mu(\xi) := \frac{1}{\pi} \lim_{z \to \xi} \text{Im} \, m(z)
\]

exists for Lebesgue-a.e. \( \xi \in \mathbb{R} \)
- Pointwise, it exists at every Lebesgue point of the measure \( \mu \)
- This limit recovers the a.c. part of the measure:

\[
  d\mu(\xi) = f_\mu(\xi)d\xi + d\mu_s(\xi)
\]

- The essential support for a.c. spectrum is the set

\[
  \Sigma_{ac}(\mu) = \{ \xi \in \mathbb{R} \mid f_\mu(\xi) \in (0, \infty) \}
\]

In particular, this solves a conjecture of Avila–Last–Simon:

**Corollary**

*Bulk universality holds almost everywhere on \( \Sigma_{ac}(\mu) \).*
Rescaling and decoupling

- If \( \lim_{n \to \infty} \frac{1}{\tau_n} K_n \left( \xi + \frac{z}{\tau_n}, \xi + \frac{w}{\tau_n} \right) = K(z, w), K(0, 0) \neq 0 \), then
  \[
  \lim_{n \to \infty} \frac{K_n(\xi, \xi)}{\tau_n} \in (0, \infty)
  \]

- Conversely, if
  \[
  \lim_{n \to \infty} \frac{K_n(\xi, \xi)}{\tau_n} \in (0, \infty)
  \]
  one scale can be replaced by the due to local uniform convergence.

- **Christoffel functions** for compactly supported measures:
  \[
  \lim_{n \to \infty} \frac{K_n(\xi, \xi)}{n} = c > 0,
  \]

  - \( \mu \) is Stahl–Totik regular (global)
  - local Szegő class condition (local)
  - Lebesgue point conditions (local)
  (Máté–Nevai–Totik 1991 Annals for \( E = [-2, 2] \), generalized by Totik)
**Transfer matrices**

- Define **transfer matrices** by

\[ B(n, z) = A(a_n, b_n; z) \cdots A(a_1, b_1; z), \quad A(a_j, b_j; z) = \begin{pmatrix} \frac{z - b_j}{a_j} & -\frac{1}{a_j} \\ a_j & 0 \end{pmatrix} \]

- The **Jacobi recursion**

\[ zp_n(z) = a_n p_{n-1}(z) + b_{n+1} p_n(z) + a_{n+1} p_{n+1}(z) \]

is equivalent to

\[ \begin{pmatrix} p_n(z) \\ a_n p_{n-1}(z) \end{pmatrix} = B(n, z) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \]

- **Second kind polynomials** are obtained by

\[ \begin{pmatrix} q_n(z) \\ a_n q_{n-1}(z) \end{pmatrix} = B(n, z) \begin{pmatrix} 0 \\ -1 \end{pmatrix} \]
Matrix CD kernel

- Matrix version of Christoffel–Darboux kernel defined by

\[ K_n(z, w) = \begin{pmatrix} \sum_{j=0}^{n-1} p_j(z) \overline{p_j(w)} & \sum_{j=0}^{n-1} q_j(z) \overline{p_j(w)} \\ \sum_{j=0}^{n-1} p_j(z) q_j(w) & \sum_{j=0}^{n-1} q_j(z) \overline{q_j(w)} \end{pmatrix} \]

Note that \( K_n(z, w)_{11} = K_n(z, w) \)
Limits of $m$-function

We say $m : \mathbb{C}_+ \to \mathbb{C}_+$ has a normal limit at $\xi$, if

$$\eta = \lim_{y \downarrow 0} m(\xi + iy)$$

Clearly $\eta \in \overline{\mathbb{C}_+} := \mathbb{C}_+ \cup \mathbb{R} \cup \{\infty\}$

For $\eta \in \overline{\mathbb{C}_+}$, define

$$\hat{H}_\eta := \frac{1}{1 + |\eta|^2} \begin{pmatrix} 1 & -\text{Re}\eta \\ -\text{Re}\eta & |\eta|^2 \end{pmatrix} \quad \eta \in \mathbb{C}_+ \cup \mathbb{R}$$

$$\hat{H}_\infty := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Denote also $j = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $h_\eta = \frac{\text{Im}\eta}{1 + |\eta|^2}$ and define

$$\hat{K}_\eta(z, w) = \frac{j(\cos(h_\eta(\overline{w} - z) - 1)) + \hat{H}_\eta}{h_\eta} \sin(h_\eta(\overline{w} - z)) \overline{w - z}$$
Theorem (E.–Lukić–Simanek)

Denote $\tau(n) = \text{tr} \, K_n(\xi, \xi)$. The following are equivalent:

1. $m$ has a normal limit at $\xi$,
   \[
   \lim_{y \downarrow 0} m(\xi + iy) = \eta \in \mathbb{C}_+
   \]

2. The (matrix) universality limit exists on the diagonal:
   \[
   \lim_{n \to \infty} \frac{1}{\tau(n)} K_n(\xi, \xi) = H_\infty
   \]

3. The (matrix) universality limit exists:
   \[
   \lim_{L \to \infty} \frac{1}{\tau(n)} K_n \left( \xi + \frac{z}{\tau(n)}, \xi + \frac{w}{\tau(n)} \right) = K_\infty(z, w).
   \]

Moreover, in this case, $H_\infty = ˚H_\eta$ and $K_\infty(z, w) = ˚K_\eta(z, w)$. 
A connection to subordinacy theory

A special case of the previous theorem is a result from subordinacy theory: Using that

$$\tau(n) = \sum_{j=0}^{n-1} p_j(\xi)^2 + \sum_{j=0}^{n-1} q_j(\xi)^2$$

we get

$$\lim_{y \downarrow 0} m(\xi + iy) = \infty \iff \lim_{n \to \infty} \frac{\sum_{j=0}^{n-1} p_j(\xi)^2}{\sum_{j=0}^{n-1} q_j(\xi)^2} = 0$$
Let $H : [0, \infty) \to \text{Mat}(2, \mathbb{R})$ be locally integrable and
\[ H(x) \geq 0, \]
for Lebesgue-a.e. $x$.

Let $T : [0, \infty) \times \mathbb{C} \to \text{Mat}(2, \mathbb{C})$ be the solution of the initial value problem
\[ j \partial_x T(x, z) = -zH(x)T(x, z), \quad T(0, z) = I_2. \]

$H$ is called the Hamiltonian of the canonical system.

Assume the limit point case
\[ \text{tr} \int_0^\infty H(x) \, dx = \infty \quad (1) \]

Due to (1) the Weyl $m$-function can be introduced. Let $\tau \in \overline{\mathbb{C}_+}$ and define
\[ m(z) = \lim_{x \to \infty} T(x, z)^{-1} * \tau \]
in the projective sense. This definition is independent of $\tau$. 
Define

\[ \mathcal{K}_L(z, w) = \int_0^L T(x, w)^* H(x) T(x, z) \, dx, \]

and note that \( \tau_L = \text{tr} \mathcal{K}_L(0, 0) = \text{tr} \int_0^L H(x) \, dx \)

The map

\[ H \mapsto m \]

is onto the set \( \{\text{Herglotz functions}\} \cup \{f(z) \equiv c : c \in \mathbb{R} \cup \{\infty\}\} \) but not one-to-one. It is one-to-one up to ”reparametrization” of \( \mathbb{R} \). Thus:

- Canonical systems

\[ j\partial_x T(x, z) = -zH(x) T(x, z), \quad T(0, z) = I_2 \]

- Reparametrize \( x \) to impose \( \text{tr} \, H = 1 \) a.e.

- de Branges (uniqueness): map \( H \mapsto m \) is a bijection

- The correspondences between \( H, m, M, K \) are homeomorphisms
Scaling operation

- Consider a trace-parametrized canonical system

\[ j \partial_t T(t, z) = -z H(t) T(t, z), \quad T(0, z) = I_2 \]

with Weyl function \( m(z) \) and kernel \( K_t(z, w) \)

- For \( r > 0 \), a scaling operation

\[
\begin{align*}
  m_r(z) &= m(z/r) \\
  H_r(t) &= H(rt) \\
  M_r(t, z) &= M(rt, z/r) \\
  (K_r)_t(z, w) &= \frac{1}{r} K_{rt}(z/r, w/r)
\end{align*}
\]

found by Kasahara for Krein strings and used by Eckhardt–Kostenko–Teschl and Langer–Pruckner–Woracek for canonical systems to investigate large energy asymptotics of the \( m \)-function

- We use the scaling operation to “zoom in” towards \( \xi \in \mathbb{R} \)
Proofs of Theorems

Proof of Theorem 2: WLOG $\xi = 0$

- Start from transfer matrices $T(L, z)$ with Weyl function $m(z)$
- Consider family of canonical systems corresponding to Weyl functions
  
  \[ m_r(z) = \begin{cases} 
  m(z/r) & r \in [1, \infty) \\
  \eta & r = \infty 
  \end{cases} \]

- Characterize continuity of this family in terms of $H, m, M, K$

Proof of Theorem 1:

- In addition, use a translation trick and consider the family
  
  \[ \tilde{m}_r(z) = \begin{cases} 
  m(z/r) - \text{Re} \, m(i/r) & r \in [1, \infty) \\
  i f_{\mu}(0) & r = \infty 
  \end{cases} \]
Thank you for your attention!