EXERCISES IN THE BIRATIONAL GEOMETRY OF ALGEBRAIC VARIETIES

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The book [KM98] gave an introduction to the birational geometry of algebraic varieties, as the subject stood in 1998. The developments of the last decade made the more advanced parts of Chapters 6 and 7 less important and the detailed treatment of surface singularities in Chapter 4 less necessary. However, the main parts, Chapters 1–3 and 5, still form the foundations of the subject.

These notes provide additional exercises to [KM98]. The main definitions and theorems are recalled but not proved here. The emphasis is on the many examples that illustrate the methods, their shortcomings and some applications.

1. Birational classification of algebraic surfaces

For more detail, see [BPVdV84].

The theory of algebraic surfaces rests on the following three theorems.

Theorem 1. Any birational morphism between smooth projective surfaces is a composite of blow-downs to points. Any birational map between smooth projective surfaces is a composite of blow-ups and blow-downs.

Theorem 2. There are 3 species of “pure-bred” surfaces:

(Rational): For these surfaces the internal birational geometry is very complicated, but, up to birational equivalence, we have only \( \mathbb{P}^2 \). These frequently appear in the classical literature and in “true” applications.

(Calabi-Yau): These are completely classified (Abelian, K3, Enriques, hyperelliptic) and their geometry is rich. They are of great interest to other mathematicians.

(General type): They have a canonical model with Du Val singularities and ample canonical class. Although singular, this is the “best” model to work with. There are lots of these but they appear less frequently outside algebraic geometry.

There are also two types of “mongrels”:

(Ruled): Birational to \( \mathbb{P}^1 \times (\text{curve of genus } \geq 1) \).

(Elliptic): These fiber over a curve with general fiber an elliptic curve.

The “mongrels” are usually studied as an afterthought, with suitable modifications of the existing methods. In a general survey, it is best to ignore them.

Theorem 3. Assume that \( S \) is neither rational nor ruled. Then there is a unique smooth projective surface \( S^{\text{min}} \) birational to \( S \) such that every birational map \( S' \rightarrow S^{\text{min}} \) from a smooth projective surface \( S' \) is automatically a morphism.

Some of these theorems are relatively easy, and some condense a long and hard story into a short statement.
The first aim of higher dimensional algebraic geometry is to generalize these theorems to dimensions three and up.

In these notes we focus only on certain aspects of this project. Let us start with mentioning the parts that we will not cover.

The correct higher dimensional analogs of rational surfaces are rationally connected varieties and the ruled surfaces are replaced by rationally connected fibrations. We do not deal with them here. See [Kol01] for an introduction and [Kol96] for a detailed treatment.

The study of higher dimensional Calabi-Yau varieties is very active, with most of the effort going into understanding mirror symmetry rather than developing a general classification scheme.

It is known that any birational map between smooth projective varieties is a composite of blow-ups and blow-downs of smooth subvarieties [Wl03, AKMW02]. While it is very useful to stay with these easy-to-understand elementary steps, in practice it is very hard to keep track of geometric properties during blow-ups. It is much more useful to factor every birational morphism between smooth projective varieties as a composite of elementary steps. It turns out that smooth blow-ups do not work [22]. From our current point of view, the natural question is to work with varieties with terminal singularities and consider the factorization of birational morphisms as a special case of the MMP. However, the following intriguing problem is still open.

**Question 4.** Let \( f : X \to Y \) be a birational morphism between smooth projective 3-folds. Is \( f \) a composite of smooth blow-downs and flops?

## 2. Naive minimal models

This is a more technical version of my notes [Kol07].

Much of the power of affine algebraic geometry rests on the basic correspondence

\[
\begin{array}{ccc}
\{ \text{affine schemes} \} & \rightarrow & \{ \text{commutative rings} \} \\
\text{spectrum} & & \\
\end{array}
\]

Thus every affine variety is the natural existence domain for the ring of all regular functions on it.

**Exercise 5.** Let \( X \) be a \( \mathbb{C} \)-variety of finite type. Prove that \( X \) is affine iff the following 2 conditions are satisfied:

1. (Point separation) For any two points \( p \neq q \in X \) there is a regular function \( f \) on \( X \) such that \( f(p) \neq f(q) \).
2. (Maximality of domain) For any sequence of points \( p_i \in X \) that does not converge to a limit in \( X \), there is a regular function \( f \) on \( X \) such that \( \lim f(p_i) \) does not exist.

**Exercise 6.** Reformulate and prove Exercise 5 for varieties over arbitrary fields.

7. As we move to more general varieties, this nice correspondence breaks down in two distinct ways.

**Quasi affine varieties.** Let \( X := \mathbb{A}^n \setminus \text{(point)} \) for some \( n \geq 2 \). Check that every regular function on \( X \) extends to a regular function on \( \mathbb{A}^n \). Thus the function
theory of $X$ is rich but the natural existence domain for the ring of all regular functions on $X$ is the larger space $\mathbb{A}^n$. Similarly, if

$$X = (\text{irreducible affine variety}) \setminus (\text{codimension} \geq 2 \text{ subvariety}),$$

then every regular function on $X$ extends to a regular function on the irreducible affine variety.

**Projective varieties.** On a projective variety every regular (or holomorphic) function is constant, hence the regular (or holomorphic) function theory of a projective variety is not interesting.

On the other hand, a projective variety has many interesting rational functions. That is, functions that can locally be written as the quotient of two regular functions. At a point the value of a rational function $f$ can be finite, infinite or undefined. The set of points where $f$ is undefined has codimension $\geq 2$. This makes it hard to control what happens in codimensions $\geq 2$.

Rational functions on a $k$-variety $X$ form a field $k(X)$, called the function field of $X$.

**Exercise 8.** Let $X = (xy - uv = 0) \subset \mathbb{A}^4$ and $f = x/u$. Show that $X$ is normal and $f$ is undefined only at the origin $(0,0,0,0)$.

**Exercise 9.** Let $X$ be a normal, proper variety over an algebraically closed field $k$. Prove that $X$ is projective iff for any two points $p \neq q \in X$ and finite subset $R \subset X$, there is a rational function $f$ on $X$ such that $f(p) \neq f(q)$ and $f$ is defined at all points of $R$.

Following the example of affine varieties we ask:

**Question 10.** How tight is the connection between $X$ and $k(X)$?

Assume that we have $X_1 \subset \mathbb{P}^r$ with coordinates $(x_0 : \cdots : x_r)$, $X_2 \subset \mathbb{P}^s$ with coordinates $(y_0 : \cdots : y_s)$ and an isomorphism $\Psi : k(X_1) \cong k(X_2)$. Then $\phi_i := \Psi(x_i/x_0)$ are rational functions on $X_2$ and $\phi_j^{(-1)} := \Psi^{-1}(y_j/y_0)$ are rational functions on $X_1$. (Note that $\phi_j^{(-1)}$ is not the inverse of $\phi_j.$) Moreover,

$$\Phi : (y_0 : \cdots : y_s) \mapsto (1 : \phi_1(y_0 : \cdots : y_s) : \cdots : \phi_r(y_0 : \cdots : y_s))$$

defines a rational map $\Phi : X_2 \dashrightarrow X_1$ and

$$\Phi^{-1} : (x_0 : \cdots : x_r) \mapsto (1 : \phi_1^{(-1)}(x_0 : \cdots : x_r) : \cdots : \phi_s^{(-1)}(x_0 : \cdots : x_r))$$

defines a rational map $\Phi^{-1} : X_2 \dashrightarrow X_1$ such that $\Psi$ is induced by pulling back functions by $\Phi$ and $\Psi^{-1}$ is induced by pulling back functions by $\Phi^{-1}$. That is, $X_1$ and $X_2$ are birational to each other.

**Exercise 11.** Let $C_1, C_2$ be 1-dimensional, irreducible, projective with all local rings regular. Prove that every birational map $C_1 \dashrightarrow C_2$ is an isomorphism.

The situation is more complicated in higher dimensions. A map with an inverse is usually an isomorphism, but this fails in the birational case since $\Phi$ and $\Phi^{-1}$ are not everywhere defined. The simplest examples are blow-ups and blow-downs.

**12 (Blow-ups).** Let $X$ be a smooth projective variety and $Z \subset X$ a smooth subvariety. Let $B_Z X$ denote the blow-up of $X$ along $Z$ and $E_Z \subset B_Z X$ the exceptional divisor. We refer to $\pi : B_Z X \to X$ as a blow-up if we imagine that $B_Z X$ is created from $X$, and a blow-down if we start with $B_Z X$ and construct $X$ later. Note that
$E_Z$ has codimension 1 and $Z$ has codimension $\geq 2$. Thus a blow-down decreases the Picard number by 1.

By blowing up repeatedly, starting with any $X$ we can create more and more complicated varieties with the same function field. Thus, for a given function field $K = k(X)$, there is no “maximal domain” where all elements of $K$ are rational functions. (The inverse limit of all varieties birational to $X$ appears in the literature occasionally as such a “maximal domain,” but so far with limited success.) On the other hand, one can look for a “minimal domain” or “minimal model.”

As a first approximation, a variety $X$ is a minimal model if the underlying space $X$ is the “best match” to the rational function theory of $X$.

**Example 13.** Let $S$ be a smooth projective surface which is neither rational nor ruled. Explain why it makes sense to say that $S^{\text{min}}$ (as in (3)) is a “minimal domain” for the field $k(S)$.

**Exercise 14.** Let $X$ be a projective variety that admits a finite morphism to an Abelian variety. Prove that every rational map $f : Y \to X$ from a smooth projective variety $Y$ to $X$ is a morphism.

Thus, if $X$ is smooth, it makes sense to say that $X$ is a “minimal domain” of its function field $k(X)$.

Not all varieties have a “minimal domain” with the above strong properties.

**Example 15.** Let $Q^3 \subset \mathbb{CP}^4$ be the quadric hypersurface given by the equation $x^2 + y^2 + z^2 + t^2 = u^2$. Let

$$\pi : (x : y : z : t : u) \mapsto (x : y : z : u - t)$$

be the projection from the north pole $(0 : 0 : 0 : 1 : 1)$ to the equatorial plane ($t = 0$). Its inverse $\pi^{-1}$ is given by

$$(x : y : z : u) \mapsto (2xu : 2yu : 2zu : x^2 + y^2 + z^2 - u^2 : x^2 + y^2 + z^2 + u^2).$$

These maps show that the meromorphic function theory of $Q^3$ is the same as that of $\mathbb{CP}^3$.

Show that $\pi$ contracts the lines $(a\lambda : b\lambda : c\lambda : 1 : 1)$ to the points $(a : b : c : 0)$ whenever $a^2 + b^2 + c^2 = 0$, and $\pi^{-1}$ contracts the plane at infinity $(u = 0)$ to the point $(0 : 0 : 0 : 1 : 1)$. Write $\pi$ as a composite of blow ups and blow downs with smooth centers.

On the other hand, $Q^3$ and $\mathbb{CP}^3$ are quite different as manifolds. Show that they have the same Betti numbers but they are not homeomorphic. Prove that $Q^3$ and $\mathbb{CP}^3$ both have Picard number 1.

A more subtle example is the following.

**Exercise 16.** Let $Y$ be a smooth projective variety of dimension 3 and $f, g, h$ general sections of a very ample line bundle $L$ on $Y$. Consider the hypersurface

$$X := (s^2f + 2stg + t^2h = 0) \subset Y \times \mathbb{P}^1_{s,t}.$$ 

Show that $X$ is smooth and compute its canonical class.

Show that the projection $\pi : X \to Y$ has degree 2; let $\tau : X \to X$ be the corresponding Galois involution. Write it down explicitly in coordinates and decide where $\tau$ is regular.
Show that $X$ contains $(L^3)$ curves of the form $(\text{point}) \times \mathbb{P}^1$ and they are numerically equivalent to each other. (This may need the Lefschetz theorem on the Picard groups of hyperplane sections.)

Assume that $Y$ admits a finite morphism to an Abelian variety. Prove that the following hold:

1. Any smooth projective variety $X'$ that is birational to $X$ has Picard number at least $\rho(X)$.
2. If $X'$ has Picard number $\rho(X)$ then it is isomorphic to $X$.
3. If $(L^3) > 1$ then there are nonprojective compact complex manifolds $Z$ that are bimeromorphic to $X$, have Picard number $\rho(X)$, but are not isomorphic to $X$.

Exercise 17. Let $X$ be a smooth projective variety such that $K_X$ is nef. Let $f : X \rightarrow X'$ be a birational map to a smooth projective variety. Prove that the exceptional set $\text{Ex}(f)$ has codimension $\geq 2$ in $X$. Generalize to the case when $X$ is canonical and $X'$ is terminal (60).

Hint: You should find (107) helpful.

Definition 18. We say that a birational map $f : X_1 \rightarrow X_2$ contracts a divisor $D \subset X_1$ if $f$ is defined at the generic point of $D$ and $f(D) \subset X_2$ has codimension $\geq 2$. The map $f$ is called a birational contraction if $f^{-1}$ does not contract any divisor.

A birational map $f : X_1 \rightarrow X_2$ is called small if neither $f$ nor $f^{-1}$ contracts any divisor.

The simplest examples of birational contractions are composites of blow-downs, but there are many, more complicated, examples.

Exercise 19. Let $f : S_1 \rightarrow S_2$ be a birational contraction between smooth projective surfaces. Show that $f$ is a morphism.

Exercise 20. Let $L, M \subset \mathbb{P}^3$ two lines intersecting at a point. The identity on $\mathbb{P}^3$ induces a rational map $g : B_L B_M \mathbb{P}^3 \rightarrow B_L \mathbb{P}^3$. (With a slight abuse of notation, we also denote by $L$ the birational transform of $L$ on $B_M \mathbb{P}^3$, etc.) Show that $g$ is a contraction but it is not a morphism. Describe how to factor $g$ into a composite of smooth blow-ups and blow-downs.

There is essentially only one way to write a birational morphism between smooth surfaces as a composite of point blow-ups. The next exercise shows that this no longer holds for 3-folds.

Exercise 21. Let $p \in L \subset \mathbb{P}^3$ be a point on a line. Let $C \subset B_L \mathbb{P}^3$ be the preimage of $p$. Show that the identity on $\mathbb{P}^3$ induces an isomorphism $B_C B_L \mathbb{P}^3 \cong B_L B_p \mathbb{P}^3$.

The next exercise shows that not every birational morphism between smooth 3-folds is a composite of smooth blow-ups.

Exercise 22. Let $C \subset \mathbb{P}^3$ be an irreducible curve with a unique singular point which is either a node or a cusp. Show that $B_C \mathbb{P}^3$ has a unique singular point; call it $p$. Check that $B_p B_C \mathbb{P}^3$ is smooth. Prove that $\pi : B_p B_C \mathbb{P}^3 \rightarrow \mathbb{P}^3$ can not be written as a composite of smooth blow-ups.

Write $\pi$ as a composite of two smooth blow-ups and a flop (17).
**Exercise 23.** Let \( f : X \to Y \) be a birational map between smooth, proper varieties. Show that
\[
\rho(X) - \rho(Y) = \#\{\text{divisors contracted by } f\} - \#\{\text{divisors contracted by } f^{-1}\}
\]

We are not yet ready to define minimal models. As a first approximation, let us focus on the codimension 1 part.

**Temporary Definition 24.** Let \( X \) be a smooth projective variety. We say that \( X \) is **minimal in codimension 1** if every birational map \( f : Y \to X \) from a smooth variety \( Y \) is a birational contraction.

In particular, this implies that \( X \) has the smallest Picard number in its birational equivalence class.

**Exercise 25.** 1. Let \( X \) be a smooth projective variety such that \( K_X \) is nef. Prove that \( X \) is minimal in codimension 1.

2. \( \mathbb{P}^3 \) has the smallest Picard number in its birational equivalence class but it is not minimal in codimension 1.

3. Let \( X \subset \mathbb{P}^4 \) be a smooth degree 4 hypersurface. Then \( K_X \) is not nef but, as proved by Iskovskikh-Manin, \( X \) is minimal in codimension 1. (See \[KSC04, Chap.5\] for a proof and an introduction to these techniques.)

**Exercise 26.** Set \( X_0 := (x_1x_2 + x_3x_4 + x_5x_6 = 0) \subset \mathbb{A}^6 \). Let \( L \subset X_0 \) be any 3-plane through the origin. Prove that, after a suitable coordinate change, \( L \) can be given as \((x_1 = x_3 = x_5 = 0)\). Prove that \( B_L \) is smooth.

Let \( Y \) be a smooth projective variety of dimension 3 and \( f_1, g_i \) are general sections of a very ample line bundle \( L \) on \( Y \). Set
\[
X' := \left( \sum_{i=1}^3 f_i(x)g_i(y) = 0 \right) \subset Y_x \times Y_y,
\]
where \( x \) (resp. \( y \)) are the coordinates on the first (resp. second) factor.

Assume that \( Y \) admits a finite morphism to an Abelian variety. Show that \( X' \) is not birational to any smooth proper variety \( X \) that is minimal in codimension 1.

**Exercise 27 (Contractions of products).** \[KL07\] Let \( X, U, V \) be normal projective varieties and \( \phi : U \times V \to X \) a birational contraction. Assume that \( X \) is smooth (or at least has rational singularities). Prove that there are normal projective varieties \( U' \) birational to \( U \) and \( V' \) birational to \( V \) such that \( X \cong U' \times V' \).

In particular, \( U \times V \) is minimal in codimension 1 if \( U \) and \( V \) are both minimal in codimension 1

Hints to the proof. First reduce to the case when \( U, V \) are smooth.

Let \( |H| \) be a complete, very ample linear system on \( X \) and \( \phi^*|H| \) its pull back to \( U \times V \). Using that \( \phi \) is a contraction, prove that \( \phi^*|H| \) is also a complete linear system.

If \( H^1(U, \mathcal{O}_U) = 0 \), then \( \text{Pic}(U \times V) = \pi_U^*\text{Pic}(U) + \pi_V^*\text{Pic}(V) \), thus there are divisors \( H_U \) on \( U \) and \( H_V \) on \( V \) such that \( \phi_*|H| \sim \pi_U^*H_U + \pi_V^*H_V \). Therefore
\[
H^0(U \times V, \mathcal{O}_{U \times V}(\phi^*|H|)) = H^0(U, \mathcal{O}_U(H_U)) \otimes H^0(V, \mathcal{O}_V(H_V)).
\]
Let now \( U' \) be the image of \( U \) under the complete linear system \( |H_U| \) and \( V' \) the image of \( V \) under the complete linear system \( |H_V| \).

The \( H^1(U, \mathcal{O}_U) \neq 0 \) case is a bit harder. Replace \( H \) by a divisor \( H' := H + B \) where \( B \) is a pull back of a divisor from the product of the Albanese varieties of \( U \).
and $V$. Show that for suitable $B$, there are divisors $H_U$ on $U$ and $H_V$ on $V$ such that $\phi_* |H| \sim \pi_U^* H_U + \pi_V^* H_V$. The rest of the argument now works as before.

3. The cone of curves

For details, see [KM98 Chap.3].

**Definition 28.** Let $X$ be a projective variety over $\mathbb{C}$. Any irreducible curve $C \subset X$ has a homology class $[C] \in H_2(X, \mathbb{R})$. These classes generate a cone $\overline{NE}(X) \subset H_2(X, \mathbb{R})$, called the cone of curves of $X$. Its closure is denoted by $\overline{NE}(X) \subset H_2(X, \mathbb{R})$.

If $X$ is over some other field, we can use the vector space $N_1(X)$ of curves modulo numerical equivalence instead of $H_2(X, \mathbb{R})$ to define the cone of curves $\overline{NE}(X) \subset N_1(X)$.

**Exercise 29.** Show that every effective curve in $\mathbb{P}^{a_1} \times \cdots \times \mathbb{P}^{a_n}$ is rationally equivalent to a nonnegative linear combination of lines in the factors. Thus

$$\overline{NE}(\mathbb{P}^{a_1} \times \cdots \times \mathbb{P}^{a_n}) \subset \mathbb{R}^n$$

is the polyhedral cone spanned by the basis elements corresponding to the lines.

**Exercise 30.** Assume that a connected, solvable group acts on $X$ with finitely many orbits. Show that $\overline{NE}(X)$ is the polyhedral cone spanned by the homology classes of the 1-dimensional orbits. (The same holds even for rational equivalence instead of homological equivalence.)

Hint. Use the Borel fixed point theorem: A connected, solvable group acting on a proper variety has a fixed point. Apply this to the Chow variety or the Hilbert scheme parametrizing curves in $X$.

**Exercise 31.** Let $S \subset \mathbb{P}^3$ be a smooth cubic surface. Show that every effective curve is linearly equivalent to a linear combination of lines. Thus

$$\overline{NE}(\mathbb{P}^{a_1} \times \cdots \times \mathbb{P}^{a_n}) \subset \mathbb{R}^3$$

is a “round” cone.

Despite what these examples suggest, the cone of curves is usually extremely difficult to determine. For instance, we still don’t know the cone of curves for the following examples.

1. $C \times C$ for a general curve $C$. (See [Laz04 Sec.1.5] for the known results and references.)

2. The blow up of $\mathbb{P}^n$ at more than a few points, cf. [CT06].
A basic discovery of [Mor82] is that the part of the cone of curves which has negative intersection with the canonical class is quite well behaved. Subsequently it was generalized to certain perturbations of the canonical class. The precise definitions will be given in Section 4. For now you can imagine that $X$ is smooth and $\Delta = \sum a_i D_i$ is a $\mathbb{Q}$-divisor where $\sum D_i$ is a simple normal crossing divisor and $0 < a_i < 1$ for every $i$.

**Theorem 33** (Cone theorem). (cf. [KM98, Thm.3.7.1–2]) Let $(X, \Delta)$ be a projective klt pair with $\Delta$ effective. Then:

1. There are (at most countably many) rational curves $C_j \subset X$ such that $0 < -(K_X + \Delta) \cdot C_j \leq 2 \dim X$ and
   $$\overline{\text{NE}}(X) = \overline{\text{NE}}(X)_{(K_X + \Delta) \geq 0} + \sum \mathbb{R}_{\geq 0}[C_j],$$
   where $\overline{\text{NE}}(X)_{(K_X + \Delta) \geq 0}$ denotes the set of those elements of $\overline{\text{NE}}(X)$ that have nonnegative intersection number with $K_X + \Delta$.

2. For any $\epsilon > 0$ and ample $\mathbb{Q}$-divisor $H$,
   $$\overline{\text{NE}}(X) = \overline{\text{NE}}(X)_{(K_X + \Delta + \epsilon H) \geq 0} + \sum_{\text{finite}} \mathbb{R}_{\geq 0}[C_j].$$

If $-(K_X + \Delta)$ is ample then taking $H = -(K_X + \Delta)$ and $\epsilon < 1$ in (33.2), the first summand on the right is trivial. Hence we obtain:

**Corollary 34.** Let $(X, \Delta)$ be a projective klt pair with $\Delta$ effective and $-(K_X + \Delta)$ ample. There are finitely many rational curves $C_j \subset X$ such that
   $$\overline{\text{NE}}(X) = \sum \mathbb{R}_{\geq 0}[C_j].$$
In particular, $\overline{\text{NE}}(X)$ is a polyhedral cone.

**Warning 35.** If the cone is 3-dimensional, the cone theorem implies that the $(K_X + \Delta)$-negative part of $\overline{\text{NE}}(X)$ is locally polyhedral. This, however, fails for 4-dimensional cones.

Use [322] to show that such an example is given by $\overline{\text{NE}}(E \times E \times \mathbb{P}^1)$ where $E$ is an elliptic curve which does not have complex multiplication.

**Definition 36.** In convex geometry, a closed subcone $F \subset \overline{\text{NE}}(X)$ is called an extremal face if $u, v \in \overline{\text{NE}}(X)$ and $u + v \in F$ implies that $u, v \in F$. A 1-dimensional extremal face is called an extremal ray.

In algebraic geometry, one frequently assumes in addition that intersection product with $K_X$ (or $K_X + \Delta$) gives a strictly negative linear function on $F \setminus \{0\}$.

Thus, extremal rays of $\overline{\text{NE}}(X)$ are precisely those summands $\mathbb{R}_{\geq 0}[C_j]$ in (33.1) that are actually needed.

The next result shows that there are contraction morphisms associated to any extremal face.

**Theorem 37** (Contraction theorem). (cf. [KM98, Thm.3.7.2–4]) Let $(X, \Delta)$ be a projective klt pair with $\Delta$ effective. Let $F \subset \overline{\text{NE}}(X)$ be a $(K_X + \Delta)$-negative extremal face. Then there is a unique morphism $\text{cont}_F : X \to Z$, called the contraction of $F$, such that $(\text{cont}_F)_* O_X = O_Z$ and an irreducible curve $C \subset X$ is mapped to a point by $\text{cont}_F$ iff $[C] \in F$. Moreover,

1. $R^i (\text{cont}_F)_* O_X = 0$ for $i > 0$, and
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(2) if $L$ is a line bundle on $X$ such that $(L \cdot C) = 0$ whenever $[C] \in F$ then there is a line bundle $L_Z$ on $Z$ such that $L \cong \text{cont}_Z^* L_Z$.

**Exercise 38.** Let $Z$ be a smooth, projective variety and $W \subset X$ a smooth, irreducible subvariety of codimension $\geq 2$. Show that $\pi : B_W Z \to Z$ is the contraction of an extremal ray on $B_W Z$.

**Exercise 39.** Let $Z$ be an $n$-dimensional projective variety with a unique singular point $p$. Show that $B_p Z$ is smooth and $\pi : B_p Z \to Z$ is the contraction of an extremal face on $B_p Z$ iff $m < n$. The exceptional divisor is the smooth hypersurface $(x_1^m + \cdots + x_{n+1}^m = 0) \subset \mathbb{P}^n$.

If $n \geq 4$, then by the Lefschetz theorem, $\pi$ is the contraction of an extremal ray. Find examples with $n = 3$ where we do contract a face.

**Exercise 40.** Let $f_i(x_1, \ldots, x_4)$ for $i = m, m + 1$ be homogeneous of degree $i$. Assume that

$$X := \{ x_0 f_m(x_1, \ldots, x_4) + f_{m+1}(x_1, \ldots, x_4) = 0 \} \subset \mathbb{P}^4$$

is smooth away from the origin. Prove that every Weil divisor on $X$ is obtained by intersecting $X$ with another hypersurface.

**Exercise 41.** Let $Z$ be an $n$-dimensional projective variety with a unique singular point $p$ of the form

$$x_1^m + \cdots + x_n^m + x_{n+1}^m + (\text{higher terms}) = 0.$$ 

Show that $B_p Z$ is smooth and $\pi : B_p Z \to Z$ is the contraction of an extremal ray on $B_p Z$ iff $m < n$ and $n \geq 3$. The exceptional divisor is the singular hypersurface $(x_1^m + \cdots + x_n^m = 0) \subset \mathbb{P}^n$.

**Exercise 42.** Let $f_m(x_1, \ldots, x_{n+1})$ be an irreducible, homogeneous degree $m$ polynomial and $g_{m+1}(x_1, \ldots, x_{n+1})$ a general, homogeneous degree $m + 1$ polynomial. Let $Z$ be an $n$-dimensional projective variety with a unique singular point $p$ of the form

$$f_m(x_1, \ldots, x_{n+1}) + g_{m+1}(x_1, \ldots, x_{n+1}) + (\text{higher terms}) = 0.$$ 

Use (51) and (67) to prove that $B_p Z$ has only canonical singularities (60).

Show that $\pi : B_p Z \to Z$ is the contraction of an extremal face on $B_p Z$ iff $m < n$.

Note that the exceptional divisor is the hypersurface $(f_m(x_1, \ldots, x_{n+1}) = 0) \subset \mathbb{P}^n$, which can be quite singular.

**Exercise 43.** Let $Z \subset \mathbb{P}^n$ be defined by $x_0 = f(x_1, \ldots, x_n) = 0$ where $f$ is irreducible. Show that $B_Z \mathbb{P}^n \to \mathbb{P}^n$ is the contraction of an extremal ray on $B_Z \mathbb{P}^n$. Show that $Z$ has only $cA$-type singularities (67). When is $Z$ canonical or terminal (60)?

Note that the exceptional divisor is a $\mathbb{P}^1$-bundle over $Z$, which can be quite singular.

**Exercise 44.** Let $X$ be a smooth, projective variety, $D \subset X$ a smooth hypersurface and $C \subset D$ any curve. Assume that the Picard number of $D$ is 1 and the conormal bundle $N_D^* |_X$ is ample.
Prove that $|C|$ is an extremal ray of $\overline{\text{NE}}(X)$ in the convex geometry sense (36).

When is it a $K_X$-negative extremal ray?

Assume in addition that $-K_D$ is ample. Generalize the proof of Castelnuovo’s theorem (for instance, as in [Har77 V.5.7]) to prove (37) in this case. (That is, there is a contraction $\pi : X \to X'$ that maps $D$ to a point and is an isomorphism on $X \setminus D$.)

Exercise 45. With notation as in (44), assume that $D \cong \mathbb{P}^{n-1}$ and $N_{D/X} \cong \mathcal{O}_{\mathbb{P}^{n-1}}(-m)$. Set $x' := \pi(D)$. Prove that the completion of $X'$ (at $x'$) is isomorphic to the completion (at the origin) of the quotient of $\mathbb{A}^n$ by the $\mathbb{Z}/m$-action $(x_1, \ldots, x_n) \mapsto (\epsilon x_1, \ldots, \epsilon x_n)$ where $\epsilon$ is a primitive $m$th root of 1. (Hint: Use the methods of [Har77 Exrc.II.6.7].)

Exercise 46. Let $Z$ be a smooth, projective variety and $Z \subset X \times \mathbb{P}^m$ a smooth hypersurface such that $X \cap \{z \times \mathbb{P}^m\}$ is a hypersurface of degree $d$ for general $z \in Z$.

Show that the projection $\pi : X \to Z$ is the contraction of an extremal face on $X$ iff $d < m + 1$ and $m \geq 2$.

If $m = 2$ and dim $Z = 2$ then show that every fiber of $\pi : X \to Z$ is either a line (if $d = 1$) or a (possibly singular) conic (if $d = 2$). (This can fail if $X$ has an ordinary double point.)

If $m = 2$ and dim $Z = 3$ then find smooth examples where the general fiber of $\pi : X \to Z$ is a line or a conic but special fibers are $\mathbb{P}^2$.

Exercise 47. If you know some about the deformation theory and the Hilbert scheme of curves on smooth varieties, prove the following. (You will find (37) very helpful.)

Let $\pi : X \to Z$ be an extremal contraction with $X$ smooth where every fiber has dimension $\leq 1$. Then $Z$ is smooth and we have one of the following cases:

1. $X = B_W Z$ for some smooth $W \subset Z$ of codimension 2.
2. $X$ is a $\mathbb{P}^1$-bundle over $Z$.
3. $X$ is a hypersurface in a $\mathbb{P}^2$-bundle over $Z$ and every fiber of $\pi : X \to Z$ is a (possibly singular) conic.

Exercise 48. Let $X \subset \mathbb{P}^4$ be a degree 3 hypersurface with a unique singular point that is an ordinary node. (That is, analytically isomorphic to $(xy - zt = 0)$.)

Let $\pi : Y \to X$ denote the blow up of the node. Prove that its exceptional divisor $E$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ and its normal bundle is $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(-1, -1)$.

Thus $E$ looks like it could have been obtained by blowing up a curve $C \cong \mathbb{P}^1$ with normal bundle $\mathcal{O}_{\mathbb{P}^1}(-1) + \mathcal{O}_{\mathbb{P}^1}(-1)$ in a smooth 3-fold. Nonetheless, use (40) to show that there is no such projective 3-fold.

Example 49. Let $X$ be the $cE_7$-type singularity $(x^2 + y^4 + yg_3(z, t) + h_5(z, t) = 0) \subset \mathbb{A}^4$, where $g_3$ and $h_5$ do not have a common factor. Show that $X$ has an isolated singular point at the origin and its $(3, 2, 1, 1)$-blow up $Y \to X$ has only terminal singularities. (See [KM98 4.56] or [KSC04 6.38] for weighted blow-ups.)

Conclude from this that $X$ itself has a terminal singularity.

One of the standard charts on the blow up is given by the substitutions $x = x_1y_1^3, y = y_1^2, z = z_1y_1, t = t_1y_1$ and the exceptional divisor has equation

$E = (g_3(z_1, t_1) + h_5(z_1, t_1) = 0)/\mathbb{A}^{3}/1, 1, 1) \subset \mathbb{A}^3/1, 1, 1)$. 

\begin{align*}
E & = (g_3(z_1, t_1) + h_5(z_1, t_1) = 0)/\mathbb{A}^{3}/1, 1, 1) \subset \mathbb{A}^3/1, 1, 1) \\
\end{align*}
This gives examples of extremal contractions whose exceptional divisor $E$ has quite complicated singularities.

1. \( x^2 + y^3 + yz^3 + t^5 \). $E$ is singular along \((z_1 = t_1 = 0)\), with a transversal singularity type \( z^3 + t^5 \), that is \( E_8 \).
2. \( x^2 + y^3 + g(z- at)(z - bt)(z - ct) + t^5 \). $E$ has triple self-intersection along \( z_1 = t_1 = 0 \).

**Exercise 50.** Let $X$ be a smooth Fano variety, \( \dim X \geq 4 \). Let $Y \subset X$ be a smooth divisor in \(|-K_X| \) (thus $K_Y = 0$). Show that the natural map $i_* : \overline{NE}(Y) \to \overline{NE}(X)$ is an isomorphism. Thus $\overline{NE}(Y)$ is a polyhedral cone. (See [Bor90, Bor91] for many such interesting examples.)

Steps of the proof.
1. By a theorem of Lefschetz, $i_*$ is an injection. Thus we need to show that for every extremal ray $R$ of $\overline{NE}(X)$ there is a curve $C_R \subset Y$ such that $C_R$ generates $R$ in $\overline{NE}(X)$.
2. Let $f : X \to Z$ be the contraction morphism of $R$. If there is a fiber $F \subset X$ of $f$ whose dimension is at least two then $Y \cap F$ contains a curve $C_R$ which works.
3. If every fiber of $f$ has dimension one then we use (17). We need to show that in these cases $Y$ contains a fiber of $f$.
4. Prove the following lemma. Let $g : U \to V$ be a $\mathbb{P}^1$-bundle over a normal projective variety. Let $V' \subset U$ be an irreducible divisor such that $g : V' \to V$ is finite of degree one (thus an isomorphism). If $V'$ is ample then $\dim V \leq 1$.
5. In the divisorial contraction case apply this lemma to $U := \text{the exceptional divisor of } f$.
6. In the $\mathbb{P}^1$-bundle case apply this lemma to $U := \text{normalization of the branch divisor of } Y \to Z$. (If there is no branch divisor, then to $X \times_Z Y \to Y$.)
7. In the conic bundle case there are two possibilities. If every fiber is smooth, this is like the $\mathbb{P}^1$-bundle case. Otherwise apply the lemma to $U := \text{normalization of the divisor of singular fibers of } Y \to Z$.

**Exercise 51.** Prove the following result of [Kol97, 4.4].

**Theorem.** Let $X$ be a smooth variety over a field of characteristic zero and $|B|$ a linear system of Cartier divisors. Assume that for every $p \in X$ there is a $B(p) \in |B|$ such that $B(p)$ is smooth at $p$ (or $p \notin B(p)$).

Then a general member $B^g \in |B|$ has only $cA$-type singularities [67].

Hint. By Noetherian induction it is sufficient to prove that for every irreducible subvariety $Z \subset X$ there is an open subset $Z^0 \subset Z$ such that a general member $B^g \in |B|$ has only $cA$-type singularities at points of $Z^0$.

If $Z \not\subset \text{Bs}|B|$ then use the usual Bertini theorem.

If $Z \subset \text{Bs}|B|$ and $\text{codim}(Z, X) = 1$, then use the usual Bertini theorem for $|B| - Z$.

If $Z \subset \text{Bs}|B|$ and $\text{codim}(Z, X) > 1$ then restrict to a suitable hypersurface $Z \subset Y \subset X$ and use induction.

**Exercise 52.** Use the following examples to show that the conclusion of [51] is almost optimal:

Let $X = \mathbb{C}^n$ and $f \in \mathbb{C}[x_3, \ldots, x_n]$ such that $(f = 0)$ has an isolated singularity at the origin. Consider the linear system $|B| = (\lambda x_1 + \mu x_2 + \nu f = 0)$. Show that at each point there is a smooth member and the general member is singular at $(0, -\lambda/\mu, 0, \ldots, 0)$ with local equation $(x_1 x_2 + f = 0)$. 

EXERCISES IN THE BIRATIONAL GEOMETRY OF ALGEBRAIC VARIETIES
Consider the linear system $\lambda(x^2 + z y^2) + \mu y^2$. At any point $x \in \mathbb{C}^3$ its general member has a $cA$-type singularity, but the general member has a moving pinch point.

4. Singularities

For details, see [KM98] Chaps.4–5).

We already saw in several examples that even if we start with a smooth variety, the contraction of an extremal ray can lead to a singular variety. It took about 10 years to understand the correct classes of singularities that one needs to consider. Instead of going through this historical process, let us jump into the final definitions.

**Remark 53.** In the early days of the MMP, a lot of effort was devoted to classifying the occurring singularities in dimensions 2 and 3. While it is comforting to have some concrete examples and lists at hand, the recent advances use very little of these explicit descriptions. In most applications, we fall back to the definitions via log resolutions. The key seems to be an ability to work with log resolutions.

**Definition 54.** Let $X$ be a normal scheme and $\Delta$ a $\mathbb{Q}$-divisor on $X$ such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. Let $f : Y \to X$ be a birational morphism, $Y$ normal. Let $E_i \subseteq \operatorname{Ex}(f)$ be the exceptional divisors. If $m(K_X + \Delta)$ is Cartier, then $f^* \mathcal{O}_X(m(K_X + \Delta))$ is defined and there is a natural isomorphism

$$f^* \mathcal{O}_X(m(K_X + \Delta))|_{Y \setminus \operatorname{Ex}(f)} \cong \mathcal{O}_Y(m(K_Y + f_*^{-1}\Delta))|_{Y \setminus \operatorname{Ex}(f)},$$

where $f_*^{-1}\Delta$ denotes the birational transform of $\Delta$. Hence there are integers $b_i$ such that

$$\mathcal{O}_Y(m(K_Y + f_*^{-1}\Delta)) \cong f^* \mathcal{O}_X(m(K_X + \Delta))(\sum b_i E_i).$$

Formally divide by $m$ and write this as

$$K_Y + \Delta_Y \sim_\mathbb{Q} f^*(K_X + \Delta) \quad \text{where} \quad \Delta_Y := f_*^{-1}\Delta - \sum (b_i/m) E_i.$$

The rational number $a(E_i, X, \Delta) := b_i/m$ is called the discrepancy of $E_i$ with respect to $(X, \Delta)$.

The closure of $f(E_i) \subseteq X$ is called the center of $E_i$ on $X$. It is denoted by $\operatorname{center}_X E_i$.

If $f' : Y' \to X$ is another birational morphism and $E'_i := ((f')^{-1} \circ f)(E_i) \subseteq Y'$ is a divisor then $a(E'_i, X, \Delta) = a(E_i, X, \Delta)$ and $\operatorname{center}_X E_i = \operatorname{center}_X E'_i$. Thus the discrepancy and the center depend only on the divisor up to birational equivalence, but not on the particular variety where the divisor appears.

**Definition 55.** Let $X$ be a normal variety. An $\mathbb{R}$-divisor on $X$ is a formal $\mathbb{R}$-linear combination $\sum r_i D_i$ of Weil divisors. We say that two $\mathbb{R}$-divisors $A_1, A_2$ are $\mathbb{R}$-linearly equivalent, denoted $A_1 \sim_\mathbb{R} A_2$, if there are rational functions $f_i$ and real numbers $r_i$ such that $A_1 - A_2 = \sum r_i (f_i)$.

One can pretty much work with $\mathbb{R}$-divisors as with $\mathbb{Q}$-divisors, but some basic properties need to be thought through.

**Exercise 56.** Prove the following about $\mathbb{R}$-divisors and $\mathbb{R}$-linear equivalence.

1. Let $A_1, A_2$ be two $\mathbb{Q}$-divisors. Show that $A_1 \sim_\mathbb{R} A_2$ iff $A_1 \sim_\mathbb{Q} A_2$.
2. Define the pull back of $\mathbb{R}$-divisors and show that it is well defined.
3. Let $A$ be an $\mathbb{R}$-divisor such that $A \sim_\mathbb{R} 0$. Prove that one can write $A = \sum r_i(f_i)$ such that $\operatorname{Supp}((f_i)) \subseteq \operatorname{Supp} A$ for every $i$. 
Exercise 57. Let \( X \) be a normal scheme and \( \Delta \) an \( \mathbb{R} \)-divisor on \( X \) such that \( K_X + \Delta \) is \( \mathbb{R} \)-Cartier. Let \( f : Y \to X \) be a proper birational morphism, \( Y \) normal. Show that there is a unique \( \mathbb{R} \)-divisor \( \Delta_Y \) such that
\[
\begin{align*}
(1) & \quad f_* (\Delta_Y) = \Delta, \\
(2) & \quad K_Y + \Delta_Y \equiv_f f^*(K_X + \Delta), \text{ where } \equiv_f \text{ denotes relative numerical equivalence,} \\
& \quad \text{that is, } (K_Y + \Delta_Y \cdot C) = (f^*(K_X + \Delta) \cdot C) \text{ for every curve } C \subset Y \text{ such} \\
& \quad \text{that } \dim f(C) = 0. \text{ (Note that the latter is just 0.)}
\end{align*}
\]
Use this to define discrepancies for \( \mathbb{R} \)-divisors.

Exercise 58. Formulate (54) in case \( f : Y \to X \) is a birational map which is defined outside a codimension 2 set. (This holds, for instance if \( X \) is proper over the base scheme \( S \).)

Exercise 59 (Divisors and rational maps). Let \( f : X \to Y \) be a generically finite rational map between proper, normal schemes. Define the push forward \( f_* : \text{Div}(X) \to \text{Div}(Y) \) of Weil divisors. Show that if \( f, g \) are morphisms then \((f \circ g)_* = f_* \circ g_* \), but this fails even for birational maps.

Let \( f : X \to Y \) be a dominant rational map between normal schemes, \( Y \) proper. Define the pull back \( f^* : \text{CDiv}(Y) \to \text{Div}(X) \) from Cartier divisors to Weil divisors. Show that if \( f \) is a morphism then we get \( f^* : \text{CDiv}(Y) \to \text{CDiv}(X) \) but not in general. Find examples of birational maps between smooth projective varieties such that \((f \circ g)^* \neq f^* \circ g^* \).

Definition 60. Let \((X, \Delta)\) be a pair where \( X \) is a normal variety and \( \Delta = \sum a_i D_i \) is a sum of distinct prime divisors. (We allow the \( a_i \) to be arbitrary real numbers.) Assume that \( K_X + \Delta \) is \( \mathbb{R} \)-Cartier. We say that \((X, \Delta)\) is
\[
\begin{align*}
terminal \quad & \text{if } a(E, X, \Delta) > 0 \quad \forall E \text{ exceptional,} \\
canonical \quad & \text{if } a(E, X, \Delta) \geq 0 \quad \forall E \text{ exceptional,} \\
klt \quad & \text{if } a(E, X, \Delta) > -1 \quad \forall E, \\
plt \quad & \text{if } a(E, X, \Delta) > -1 \quad \forall E \text{ such that } (X, \Delta) \text{ is not snc at} \\
dlt \quad & \text{the generic point of center}_X(E), \\
lc \quad & \text{if } a(E, X, \Delta) \geq -1 \quad \forall E.
\end{align*}
\]

Here \( klt \) is short for Kawamata log terminal, \( plt \) for purely log terminal, \( dlt \) for divisorial log terminal, \( lc \) for log canonical and \( \text{snc for simple normal crossing} \). (The phrase \((X, \Delta)\) has terminal etc. singularities” may be confusing since it could refer to the singularities of \((X, 0)\) instead.)

Each of these 5 notions has an important place in the theory of minimal models:

1. **Terminal.** Assuming \( \Delta = 0 \), this is the smallest class that is necessary to run the minimal model program for smooth varieties. If \((X, 0)\) is terminal then \( \text{Sing } X \) has codimension \( \geq 3 \). All 3-dimensional terminal singularities are classified, see [71] for some examples. It is generally believed that already in dimension 4 a complete classification would be impossibly long. The \( \Delta \neq 0 \) case appears only infrequently.

2. **Canonical.** Assuming \( \Delta = 0 \), these are precisely the singularities that appear on the canonical models of varieties of general type. Two dimensional canonical singularities are classified, see [66]. There is some structural information in dimension 3 [KM98, 5.3]. This class is especially important for moduli problems.
(3) **Kawamata log terminal.** This is the smallest class that is necessary to run the minimal model program for pairs \((X, \Delta)\) where \(X\) is smooth and \(\Delta\) a simple normal crossing divisor with coefficients \(< 1\).

The vanishing theorems (cf. [KM98, 2.4–5]) seem to hold naturally in this class. In general, proofs that work with canonical singularities frequently work with klt. Most unfortunately, this class is not large enough for inductive proofs.

(4) **Purely log terminal.** This is useful mostly for inductive purposes. \((X, \Delta)\) is plt iff \((X, \Delta)\) is dlt and the irreducible components of \(\lfloor \Delta \rfloor\) are disjoint.

(5) **Divisorial log terminal.** This is the smallest class that is necessary to run the minimal model program for pairs \((X, \Delta)\) where \(X\) is smooth and \(\Delta\) a simple normal crossing divisor with coefficients \(\leq 1\).

By [Sza94], there is a log resolution \(f : (X', \Delta') \to (X, \Delta)\) such that every \(f\)-exceptional divisor has discrepancy \(> -1\) and \(f\) is an isomorphism over the snc locus of \((X, \Delta)\).

While the definition of this class is somewhat artificial looking, it has good cohomological properties and is much better behaved than general log canonical pairs.

If \(\Delta = 0\) then the notions klt and dlt coincide and in this case we say that \(X\) has log terminal singularities (abbreviated as lt).

(6) **Log canonical.** This is the largest class where discrepancy still makes sense and inductive arguments naturally run in this class. There are three major complications though:

- The refined vanishing theorems frequently fail.
- The singularities are not rational and not even Cohen-Macaulay, hence rather complicated from the cohomological point of view; see, for example, (71).
- Several tricks of perturbing coefficients can not be done since a perturbation would go above 1; see, for example, (95).

**Exercise 61.** Let \(f : X \to Y\) be a birational morphism, \(\Delta_X, \Delta_Y \mathbb{R}\)-divisors such that \(f_* \Delta_X = \Delta_Y\) and \(D\) an effective \(\mathbb{R}\)-divisor. Assume that \(K_Y + \Delta_Y\) and \(D\) are \(\mathbb{R}\)-Cartier and

\[
K_X + \Delta_X \sim_{\mathbb{R}} f^*(K_Y + \Delta_Y) + D.
\]

Prove that for any \(E\), \(a(E, X, \Delta_X) \leq a(E, Y, \Delta_Y)\) and the inequality is strict iff center\(X \cap \text{Supp} \; D\).

**Exercise 62.** Show that the assumptions of (61) are fulfilled (for suitable \(\Delta_Y\) and \(D\)) if \(X\) is \(\mathbb{Q}\)-factorial, \(f\) is the birational contraction of a \((K_X + \Delta_X)\)-negative extremal ray and \(\text{Ex}(f)\) has codimension 1.

The following exercise shows why log canonical is the largest class defined.

**Exercise 63.** Given \((X, \Delta)\) assume that there is a divisor \(E_0\) such that \(a(E_0, X, \Delta) < -1\). Prove that \(\inf \{a(E, X, \Delta)\} = -\infty\).

**Exercise 64.** Show that if \((X, \sum a_i D_i)\) is lc (and the \(D_i\) are distinct) then \(a_i \leq 1\) for every \(i\).

**Exercise 65.** Assume that \(X\) is smooth and \(\Delta\) is effective. Show that if \(\text{mult}_x \Delta < 1\) (resp. \(\leq 1\)) for every \(x \in X\) then \((X, \Delta)\) is terminal (resp. canonical).

Prove that the converse holds for surfaces but not in higher dimensions.
Exercise 66 (Du Val singularities). In each of the following cases, construct the minimal resolution and verify that its dual graph is the graph given. Check that these singularities are canonical. (One can see that these are all the 2-dimensional canonical singularities.) See [KM98, Sec. 4.2] or [Dur79] for more information. (The equations below are correct in characteristic zero. The dual graphs are correct in every characteristic.)

\[ A_n: x^2 + y^2 + z^{n+1} = 0, \text{ with } n \geq 1 \text{ curves in the dual graph:} \]
\[ 2 - 2 - \cdots - 2 - 2 \]

\[ D_n: x^2 + y^2z + z^{n-1} = 0, \text{ with } n \geq 4 \text{ curves in the dual graph:} \]
\[ \begin{array}{c}
 2 \\
 2 - 2 - \cdots - 2 - 2 
\end{array} \]

\[ E_6: x^2 + y^3 + z^4 = 0, \text{ with dual graph:} \]
\[ \begin{array}{c}
 2 \\
 2 - 2 - 2 - 2 - 2 - 2 
\end{array} \]

\[ E_7: x^2 + y^3 + yz^3 = 0, \text{ with dual graph:} \]
\[ \begin{array}{c}
 2 \\
 2 - 2 - 2 - 2 - 2 - 2 
\end{array} \]

\[ E_8: x^2 + y^3 + z^5 = 0, \text{ with dual graph:} \]
\[ \begin{array}{c}
 2 \\
 2 - 2 - 2 - 2 - 2 - 2 - 2 
\end{array} \]

Exercise 67 (cA-type singularities). Let \( 0 \in X \) a normal cA-type singularity. That is, either \( X \) is smooth at 0, or, in suitable local coordinates \( x_1, \ldots, x_n \), the equation of \( X \) is \( x_1x_2 + (\text{other terms}) = 0 \).

Show that \( X \) is

1. canonical near 0 iff \( \dim \text{Sing} X \leq \dim X - 2 \), and
2. terminal near 0 iff \( \dim \text{Sing} X \leq \dim X - 3 \).

Hint. First show that being cA-type is an open condition. Then use a lemma of Zariski and Abhyankar (cf. [KM98, 2.45]) to reduce everything to the statements:

3. The exceptional divisor(s) of \( B_0X \to X \) have discrepancy \( \dim X - 2 \), save when \( X \) is smooth.
4. \( B_0X \) has only cA-type singularities.

Exercise 68 (Some simple elliptic singularities). In each of the following cases, construct the minimal resolution. Verify that the exceptional set is a single elliptic curve with self intersection \(-k\).

\( k = 3 \) \( (x^3 + y^3 + z^3 = 0) \). (This is very easy)
\( k = 2 \) \( (x^2 + y^4 + z^4 = 0) \).
\( k = 1 \) \( (x^2 + y^3 + z^6 = 0) \). (This is a bit tricky.)
In general, prove that for any elliptic curve $E$ and any $k \geq 1$ there is a normal singularity whose minimal resolution contains $E$ as the single exceptional curve with self intersection $-k$.

Check that all of these are log canonical.

Use the methods of [Har77, Exc.II.8.6–7] to prove that the completion of the singularity is uniquely determined by $E$.

**Exercise 69.** Construct the minimal resolutions of the following quotients of the singularities in (68). (See (72) for the notation.)

\[
\begin{align*}
(x^3 + y^3 + z^3 &= 0): \frac{1}{3}(1,0,0), \frac{1}{3}(1,1,1).\\
(x^2 + y^4 + z^k &= 0): \frac{1}{4}(1,0,0), \frac{1}{4}(0,0,1).\\
(x^2 + y^3 + z^0 &= 0): \frac{1}{5}(0,0,1).
\end{align*}
\]

**Exercise 70.** Let $X \subset \mathbb{P}^n$ be a smooth variety and $C(X) \subset \mathbb{A}^{n+1}$ the cone over $X$. Show that $C(X)$ is normal iff $H^i(\mathbb{P}^n, O_{\mathbb{P}^n}(m)) \to H^i(X, O_X(m))$ is onto for every $m \geq 0$.

Assume next that $C(X)$ is normal. Let $\Delta$ be an effective $\mathbb{Q}$-divisor on $X$. Prove that

1. $K_{C(X)} + C(\Delta)$ is $\mathbb{Q}$-Cartier iff $K_X + \Delta \sim_{\mathbb{Q}} r \cdot H$ for some $r \in \mathbb{Q}$ where $H \subset X$ is the hyperplane class.
2. If $K_X + \Delta \sim_{\mathbb{Q}} r \cdot H$ then $(C(X), C(\Delta))$ is
   (a) terminal iff $r < -1$ and $(X, \Delta)$ is terminal,
   (b) canonical iff $r \leq -1$ and $(X, \Delta)$ is canonical,
   (c) klt iff $r < 0$ and $(X, \Delta)$ is klt, and
   (d) lc iff $r \leq 0$ and $(X, \Delta)$ is lc.

**Exercise 71.** Notation as in (70). Prove that $C(X)$ has a rational singularity iff $H^i(X, O_X(m)) = 0$ for every $i > 0, m \geq 0$ and a Cohen-Macaulay singularity iff $H^i(X, O_X(m)) = 0$ for every dim $X > i > 0, m \geq 0$. In particular:

1. If $X$ is an Abelian variety and dim $X \geq 2$ then $C(X)$ is log canonical but not Cohen-Macaulay.
2. If $X$ is a K3 surface then $C(X)$ is log canonical, Cohen-Macaulay but not rational.
3. If $X$ is an Enriques surface then $C(X)$ is log canonical and rational.

**72 (Quotient singularities).** Let $G$ be any finite group. A homomorphism $G \to GL_n$ is equivalent to a linear $G$-action on $\mathbb{A}^n$. The resulting quotient singularities $\mathbb{A}^n/G$ are rather special but they provide a very good test class for many questions involving log-terminal singularities.

One can always reduce to the case when the $G$-action on $\mathbb{A}^n$ is effective and fixed point free outside a codimension 2 set. (Unless you are into stacks.) Thus assume this in the sequel.

Show that any such $\mathbb{A}^n/G$ is log terminal.

Show that if $G \subset SL_n$ then the canonical class of $\mathbb{A}^n/G$ is Cartier. In particular, $\mathbb{A}^n/G$ is canonical.

Assume that $G = \langle g \rangle$ is a cyclic group. Any cyclic action on $\mathbb{A}^n$ can be diagonalized and written as

$$g : (x_1, \ldots, x_n) \mapsto (\epsilon^{a_1} x_1, \ldots, \epsilon^{a_n} x_n),$$

where $\epsilon = e^{2\pi i / m}$, $m = |G|$ and $0 \leq a_j < m$. Define the age of $g$ as $\text{age}(g) := \frac{1}{m}(a_1 + \cdots + a_n)$. As a common shorthand notation, we denote the quotient by this
action by

$$\mathbb{A}^n / \frac{1}{m}(a_1, \ldots, a_n).$$

The following very useful criterion tells us when $\mathbb{A}^n / G$ is terminal or canonical.

**Reid-Tai criterion.** $\mathbb{A}^n / G$ is canonical (resp. terminal) iff the age of every non-identity element $g \in G$ is $\geq 1$ (resp. $> 1$).

(This is not hard to prove if you know some basic toric techniques. Otherwise, look up [Rei87].)

As a consequence, prove that the 3-fold quotients $\mathbb{A}^3 / \frac{1}{m}(1, -1, a)$ are terminal if $(a, n) = 1$. (It is a quite tricky combinatorial argument to show that these are all the 3-dimensional terminal quotients, cf. [Rei87].)

By contrast, every “complicated” higher dimensional quotient singularity is terminal. By the results of [KL07, GT08], if the $G$-action on $\mathbb{A}^n$ is irreducible and primitive, then $\mathbb{A}^n / G$ is terminal whenever $n \geq 5$.

## 5. Flips

For more on flips, see [KM98 Chap.6], [Cor07] or [HM05].

The following is the most general definition of flips.

**Definition 73.** Let $f^- : X^- \to Y$ be a proper birational morphism between pure dimensional $S_2$ schemes such that the exceptional set $\text{Ex}(f^-)$ has codimension at least two in $X^-$. Let $H^-$ be an $\mathbb{R}$-Cartier divisor on $X^-$ such that $-H^-$ is $f^-$-ample. A pure dimensional $S_2$ scheme $X^+$ together with a proper birational morphism $f^+ : X^+ \to Y$ is called an $H^-$-flip of $f^-$ if

1. the exceptional set $\text{Ex}(f^+)$ has codimension at least two in $X^+$.
2. the birational transform $H^+$ of $H^-$ on $X^+$ is $\mathbb{R}$-Cartier and $f^+$-ample.

By a slight abuse of terminology, the rational map $\phi := (f^+)^{-1} \circ f^- : X^- \dasharrow X^+$ is also called an $H^-$-flip. We will see in (75) or (90) that a flip is unique and the main question is its existence. A flip gives the following diagram:

$$X^- \xrightarrow{\phi} X^+ \quad \text{(-}H^- \text{ is } f^-\text{-ample)}$$

$$\phantom{X^-} \searrow \quad \swarrow \quad f^- \quad \swarrow \quad f^+ \quad \text{ (}H^+ \text{ is } f^+\text{-ample)} \quad \downarrow \text{Y}$$

**Warning 74.** In the literature the notion of flip is frequently used in more restrictive ways. Here are the most commonly used variants that appear, sometimes without explicit mention.

1. In older papers, flip refers to the case when $X^-$ is terminal and $H = K_{X^-}$. These are the ones needed when we start the MMP with a smooth variety.
2. In the MMP for pairs $(X, \Delta)$ we are interested in flips when $(X^-, \Delta^-)$ is a klt (or dlt or lc) pair and $H = K_{X^-} + \Delta^-$. In older papers this is called a log-flip, but more recently it is called simply a flip.
3. Given $(X^-, \Delta^-)$, a $(K_{X^-} + \Delta^-)$-flip is frequently called a $\Delta^-$-flip.
4. The statement “$n$-dimensional terminal (or canonical, klt, . . . ) flips exist” means that the $H^-$-flip of $f^- : X^- \to Y$ exists whenever $\dim X^- = n$, $H^- = K_{X^-} + \Delta^-$ and $(X^-, \Delta^-)$ is terminal (or canonical, klt, . . . ).
relatively ample divisors

Prove the following result of Matsusaka and Mumford [MM64].

Exercise 75. Prove the following result of Matsusaka and Mumford [MM64].

Let $X_i$ be pure dimensional $S_2$-schemes and $X_i \to S$ projective morphisms with relatively ample divisors $H_i$. Let $U_i \subset X_i$ be open subsets such that $X_i \setminus U_i$ has codimension $\geq 2$ in $X_i$. Let $\phi_U : U_1 \to U_2$ be an isomorphism such that $\phi_U^*(H_1|_{U_1}) = H_2|_{U_2}$.

Then $\phi_U$ extends to an isomorphism $\phi_X : X_1 \to X_2$.

Exercise 76. Notation as in (75). Prove that $f^-_*(H^-)$ is not $\mathbb{R}$-Cartier on $Y$.

We see in (73) that not all flips exist. Currently, the strongest existence theorem is the following.

Theorem 77. [HM05] [BCHM06] Dlt flips exist.

Exercise 78. Let $\phi : X^- \to X^+$ be a $(K_X^- + \Delta^-)$-flip. Prove that for any $E$, $a(E, X^-, X^-) \leq a(E, X^+, X^+)$ and the inequality is strict iff the center of $E$ on $X^-$ is contained in $\text{Ex}(\phi)$.

Definition 79. Let $(X, \Delta)$ be an lc pair and $f : X \to S$ a proper morphism. A sequence of flips over $S$ starting with $(X, \Delta)$ is a sequence of birational maps $\phi_i$ and morphisms $f_i$

$$X_i \xrightarrow{\phi_i} X_{i+1}
\downarrow
\begin{array}{c}
\phi_i \setminus S
f_i
\end{array}
\xrightarrow{f_{i+1}} X_{i+1}$$

(starting with $X_0 = X$) such that for every $i \geq 0$, $\phi_i$ is a $(K_{X_i} + \Delta_i)$-flip where $\Delta_i$ is the birational transform of $\Delta$ on $X_i$.

The basic open question in the field is the following

Conjecture 80. Starting with an lc pair $(X_0, \Delta_0)$, there is a no infinite sequence of flips $\phi_i : (X_i, \Delta_i) \to (X_{i+1}, \Delta_{i+1})$.

This is known in dimension 3, almost known in dimension 4 and known in certain important cases in general; see [BCHM06] or [99] for more precise statements.

Exercise 81. Let $\phi_i : (X_i, \Delta_i) \to (X_{i+1}, \Delta_{i+1})$ be a sequence of flips. Prove that the composite $\phi_n \circ \cdots \circ \phi_0 : X_0 \to X_{n+1}$ can not be an isomorphism.

Problem 82. Let $\phi_i : (X_i, \Delta_i) \to (X_{i+1}, \Delta_{i+1})$ be a sequence of flips. Prove that $(X_n, \Delta_n)$ can not be isomorphic to $(X_0, \Delta_0)$ for $n > 0$. (I do not know how to do this, but it may not be hard.)
By contrast, show that the involution \( \tau \) in (16) is a flop and even a flip for some \( H = K_X + \Delta \) where \((X, \Delta)\) is klt. (Thus \( X_n \) could be isomorphic to \( X_0 \), but the isomorphism should not preserve \( \Delta \).)

**Exercise 83** (Simplest flop). Let \( L_1, L_2 \subset \mathbb{P}^3 \) be two lines intersecting at a point \( p \). Let \( X_1 := B_{L_1}B_{L_2}\mathbb{P}^3 \) and \( X_2 := B_{L_2}B_{L_1}\mathbb{P}^3 \). Set \( Y := B_{L_1+L_2}\mathbb{P}^3 \).

Show that the identity on \( \mathbb{P}^3 \) induces morphisms \( f_i : X_i \to Y \) and a rational map \( \phi : X_1 \dashrightarrow X_2 \). We get a flop diagram

\[
\begin{array}{ccc}
X_1 & \xrightarrow{\phi} & X_2 \\
\downarrow f_1 & & \downarrow f_2 \\
& Y.
\end{array}
\]

Show that neither \( \phi \) nor \( \phi^{-1} \) contracts divisors but neither is a morphism. Describe how to factor \( \phi \) into a composite of smooth blow ups and blow downs.

**Exercise 84** (Non-algebraic flops). Let \( X \subset \mathbb{P}^4 \) be a general smooth quintic hypersurface. It is known that for every \( d \geq 1 \), \( X \) contains a smooth rational curve \( \mathbb{P}^1 \cong C_d \subset X \) with normal bundle \( \mathcal{O}_{\mathbb{P}^1}(-1) + \mathcal{O}_{\mathbb{P}^1}(-1) \) \[\text{[Cle83]}\].

Prove that the flop by \( \phi_d : X \dashrightarrow X_d \) and let \( H_d \in H^2(X_d, \mathbb{Z}) \) be the image of the hyperplane class. Compute the self-intersection \((H_d^2)\). Conclude that the \( X_d \) are not homeomorphic to each other and not projective.

**Exercise 85** (Harder flops). Let \( C_1, C_2 \subset \mathbb{P}^3 \) be two smooth curves intersecting at a single point \( p \) where they are tangent to order \( m \). Let \( X_1 := B_{C_1}B_{C_2}\mathbb{P}^3 \) and \( X_2 := B_{C_2}B_{C_1}\mathbb{P}^3 \). Set \( Y := B_{C_1+C_2}\mathbb{P}^3 \).

Show that the identity on \( \mathbb{P}^3 \) induces morphisms \( f_i : X_i \to Y \), a rational map \( \phi : X_1 \dashrightarrow X_2 \) and we get a flop diagram as before. Describe how to factor \( f \) into a composite of smooth blow ups and blow downs.

**Exercise 86** (Even harder flops). Consider the variety

\[
X := (sx + ty + uz = sz^2 + tx^2 + uy^2 = 0) \subset \mathbb{P}^2_{xyz} \times \mathbb{A}^3_{stu}.
\]

Show that \( X \) is smooth, the projection \( \pi : X \to \mathbb{A}^3 \) has degree 2 and \( C := \text{red} \pi^{-1}(0,0,0) \) is a smooth rational curve. Compute \((C \cdot K_X)\) and the normal bundle of \( C \).

Let \( Y \to \mathbb{A}^3 \) be the normalization of \( \mathbb{A}^3 \) in \( k(X) \). Determine the singularity of \( Y \) sitting over the origin.

As before, the Galois involution of \( Y \to \mathbb{A}^3 \) provides the flop of \( X \to Y \).

It is quite tricky to factor \( f \) into a composite of smooth blow ups and blow downs.

**Exercise 87** (Simplest flips). Fix \( n \geq 3 \) and consider the affine hypersurface

\[
Z := (u^n - u^{n-1}y + x^{n-1}z = 0) \subset \mathbb{A}^4,
\]

which we view as a degree \( n \) covering of the \((x,y,z)\)-space.

Show that \( Z \) is not normal and its normalization has a unique singular point which lies above \((0,0,0)\).

Show that

\[
X^+ := (s^n x - s^{n-1} ty + t^nx^{n-1}z = 0) \subset \mathbb{A}^3_{xyz} \times \mathbb{P}^1_{st}
\]
is a small resolution of $Z$. Write down the morphism $X^+ \to Z$. It has a unique 1-dimensional fiber $C^+ \subset X^+$. Determine the normal bundle of $C^+$ in $X^+$ and the intersection number of $C^+$ with the canonical class.

Construct another small modification $X^- \to Z$ as follows. First blow up the ideal $(z, u^{n-1})$. We get the variety $X_1$ defined by equations

\begin{equation}
(s(y - u) - tx^{n-1} - sz - tu^{n-1} = u^n - u^{n-1}y + x^{n-1}z = 0) \subset \mathbb{A}^4_{xyzw} \times \mathbb{P}^1_{st}.
\end{equation}

Show that the $s \neq 0$ chart is smooth and on the $t \neq 0$ chart we have a complete intersection

\begin{equation}
(w(y - u) - x^{n-1} = wz - u^{n-1} = 0) \subset \mathbb{A}^4_{xyzwu} \quad \text{with } w = s/t.
\end{equation}

Setting $y' := y - u$ we have the local equations for $X_1$

\begin{equation}
wyz' - x^{n-1} = wz - u^{n-1} = 0.
\end{equation}

Write down a $\mathbb{Z}/(n - 1)$-invariant finite morphism to the above local chart on $X_1$ from $\mathbb{A}^3_{xyz}$ with the $\mathbb{Z}/(n - 1)$-action $(p, q, r) \mapsto (ep, eq, e^{-1}r)$, where $e$ is a primitive $(n - 1)$-st root of unity. Let $X^-$ be the normalization of $X_1$. Show that $X^-$ has a single quotient singularity of the above form.

Write down the morphism $X^- \to Z$. It has a unique 1-dimensional fiber $C^- \subset X^-$. Determine the intersection number of $C^-$ with the canonical class.

**Exercise 88.** Let now $Y$ be any smooth 3-fold and $L$ a very ample line bundle on $Y$ with 3 general sections $f, g, h$. Fix $n \geq 3$ and consider the hypersurface

\begin{equation}
Z := (u^n - u^{n-1}g + f^{n-1}h = 0) \subset L^{-1}.
\end{equation}

One small resolution is given by

\begin{equation}
X^+ := (s^nf - s^{n-1}tg + t^n h = 0) \subset Y \times \mathbb{P}^1_{st}.
\end{equation}

Compute its canonical class in terms of $K_Y$ and $L$.

**Exercise 89 (Log terminal flips).** Work out the analog of [87] when we start with

\begin{equation}
X^+ := (s^n x - s^{n-1}ty + t^nz = 0) \subset \mathbb{A}^3_{xyz} \times \mathbb{P}^1_{st}.
\end{equation}

**Exercise 90.** Let $X$ be a Noetherian, reduced, pure dimensional, $S_2$-scheme and $D$ a Weil divisor on $X$ which is Cartier in codimension 1. Prove that the following are equivalent.

1. $\sum_{m \geq 0} \mathcal{O}_X(mD)$ is a finitely generated sheaf of $\mathcal{O}_X$-algebras.
2. There is a proper, birational morphism $\pi : X^+ \to X$ such that the exceptional set $\text{Ex}(\pi)$ has codimension $\geq 2$ and the birational transform $D^+ := \pi^{-1}_*(D)$ is $\mathbb{Q}$-Cartier and $\pi$-ample.

Hint of proof. (2) $\Rightarrow$ (1) is easy.

To see the converse, set $X^+ := \text{Proj}_X \sum_{m \geq 0} \mathcal{O}_X(mD)$. We need to show that $X^+ \to X$ is small. Assume that $E \subset \text{Ex}(\pi)$ is an exceptional divisor. Study the sequence

\begin{equation}
0 \to \mathcal{O}_{X^+}(mD^+) \to \mathcal{O}_{X^+}(mD^+ + E) \to \mathcal{O}_E((mD^+ + E)|_E) \to 0
\end{equation}

to get, for some $m > 0$, a section of $\mathcal{O}_{X^+}(mD^+ + E)$ which is not a section of $\mathcal{O}_{X^+}(mD^+)$. By pushing forward to $X$, we would get extra sections of $\mathcal{O}_X(mD)$. 

**Exercise 91.** Let \((X, \Delta)\) be klt. Let \(f : X \to Y\) be a small \((K_X + \Delta)\)-negative contraction. Show that there is a \(\mathbb{Q}\)-divisor \(D\) on \(X\) such that \((X, \Delta + D)\) is klt and \((K_X + \Delta + D) \sim_{\mathbb{Q}} f^* 0\).

Conclude from this that \((Y, f_* (\Delta + D))\) is klt.

A consequence of the relative MMP is the following finite generation result, which we prove in [109]. By (11), it formally implies the existence of dlt flips.

**Theorem 92.** Let \((X, \Delta)\) be klt and \(D\) a \(\mathbb{Q}\)-divisor on \(X\). Then \(\sum_{m \geq 0} \mathcal{O}_X([mD])\) is a finitely generated sheaf of \(\mathcal{O}_X\)-algebras.

**Exercise 93.** Show that \([A + B] \geq [A] + [B]\) for any divisors \(A, B\), thus, for any divisor \(D\), \(R(X, D) := \sum_{m \geq 0} H^0(X, \mathcal{O}_X([mD]))\) is a ring.

Give examples where \(R^n(X, D) := \sum_{m \geq 0} H^0(X, \mathcal{O}_X([mD]))\) is not a ring. Note, however, that \([A + B] \geq [A] + [B]\), thus \(R^n(X, D)\) is an \(R(X, D)\)-module.

**Exercise 94.** Let \(X\) be normal and \(D\) an \(\mathbb{R}\)-divisor. Show that if \(\sum_{m \geq 0} \mathcal{O}_X([mD])\) is a finitely generated sheaf of \(\mathcal{O}_X\)-algebras then \(D\) is a \(\mathbb{Q}\)-divisor.

The following example shows that (92) fails for lc pairs.

**Exercise 95.** Let \(E \subset \mathbb{P}^2\) be a smooth cubic. Let \(S\) be obtained by blowing up 9 general points on \(E\) and let \(E_S \subset S\) be the birational transform of \(E\). Let \(H\) be a sufficiently ample divisor on \(S\) giving a projectively normal embedding \(S \subset \mathbb{P}^n\). Let \(X \subset \mathbb{A}^{n+1}\) be the cone over \(S\) and \(D \subset X\) the cone over \(E_S\).

Prove that \((X, D)\) is lc yet \(\sum_{m \geq 0} \mathcal{O}_X(mD)\) is not a finitely generated sheaf of \(\mathcal{O}_X\)-algebras.

Hints. First show that \(H^0(X, \mathcal{O}_X(mD)) = \sum_{r \geq 0} H^0(S, \mathcal{O}_S(mE_S + rH))\). Check that \(\mathcal{O}_S(mE_S + rH)\) is very ample if \(r > 0\) but \(\mathcal{O}_S(mE_S)\) has only the obvious section which vanishes along \(mE_S\). Thus the multiplication maps

\[
\sum_{a=0}^{m-1} H^0(S, \mathcal{O}_S(aE_S + H)) \otimes H^0(S, \mathcal{O}_S((m-a)E_S)) \to H^0(S, \mathcal{O}_S(mE_S + H))
\]

are never surjective.

The next exercise shows that log canonical flops sometimes do not exist.

**Exercise 96.** Let \(E\) be an elliptic curve, \(L\) a degree 0 non-torsion line bundle and \(S = \mathbb{P}_E(\mathcal{O}_E + L)\). Let \(C_1, C_2 \subset S\) be the corresponding sections of \(S \to E\). Note that \(K_S + C_1 + C_2 \sim 0\). Let \(0 \in X\) be a cone over \(S\) and \(D_i \subset X\) the cones over \(C_i\). Show that \((X, D_1 + D_2)\) is lc.

Following the method of (95) show that \(\sum_{m \geq 0} \mathcal{O}_X(mD_i)\) is not a finitely generated sheaf of \(\mathcal{O}_X\)-algebras for \(i = 1, 2\).

Let \(F \subset S\) be a fiber of \(S \to E\) and \(B \subset X\) the cone over \(F\). Show that \(\sum_{m \geq 0} \mathcal{O}_X(mB)\) is a finitely generated sheaf of \(\mathcal{O}_X\)-algebras and describe the corresponding small contraction \(\pi : Z \to X\).

Prove that the flip of \(\pi : Z \to X\) does not exist (no matter what \(H\) we choose).

What happens if \(L\) is a torsion element in \(\text{Pic}(E)\)?

**Exercise 97.** Let \(S\) be a Noetherian, reduced, 2-dimensional, \(S_2\)-scheme and \(D\) a Weil divisor on \(S\). Prove that \(\sum_{m \geq 0} \mathcal{O}_S(mD)\) is a finitely generated sheaf of \(\mathcal{O}_S\)-algebras iff \(\mathcal{O}_S(mD)\) is locally free for some \(m > 0\).

Use this to show that the following algebras are not finitely generated.
Conjecture 98.

The minimal model program.

(1) $S$ is a cone over an elliptic curve and $D \subset S$ a general line. State the precise generality condition.

(2) Let $C \subset \mathbb{P}^n$ be a projectively normal curve of genus $\geq 2$ and $S \subset \mathbb{A}^{n+1}$ the cone over $C$. Assume that $\mathcal{O}_C(1)$ is a general line bundle and let $D = K_S$. Again, state the precise generality condition.

(3) Let $S$ be the quadric cone $(xy - z^2 = 0) \subset \mathbb{A}^3$ and the $(u,v)$-plane glued together along the lines $(x = z = 0)$ and $(v = 0)$. (Show that this surface does not embed in $\mathbb{A}^4$ but realize it in $\mathbb{A}^4$ by explicit equations.) Set $D = K_S$.

The following conjecture is known if $x \in H$ is a quotient singularity [KSB88] or when $x \in H$ is a quaduple point [Ste91]. It is quite remarkable that, aside from the case when $x \in H$ is a quotient singularity, the conjecture seems unrelated to the minimal model program.

**Conjecture 98.** [Kol91, 6.2.1] Let $x \in X$ be a 3-dimensional normal singularity and $x \in H \subset X$ a Cartier divisor. Assume that $x \in H$ is a (normal) rational surface singularity. Then $\sum_{m \geq 0} \mathcal{O}_X(mK_X)$ is a finitely generated sheaf of $\mathcal{O}_X$-algebras.

### 6. Minimal models

For more details, see [KM98, 3.7–8] or [BCHM06].

**Definition 99** (Running the MMP). Let $(X, \Delta)$ be a pair such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier and $f : X \to S$ a proper morphism. Assume for simplicity that $X$ is $\mathbb{Q}$-factorial. A running of the $(K_X + \Delta)$-MMP over $S$ yields a sequence

$$(X, \Delta) =: (X_0, \Delta_0) \xrightarrow{\phi_0} (X_1, \Delta_1) \xrightarrow{\phi_1} \cdots \xrightarrow{\phi_{n-1}} (X_r, \Delta_r),$$

where each $\phi_i$ is either the divisorial contraction of a $(K_{X_i} + \Delta_i)$-negative extremal ray or the flip of a small contraction of a $(K_{X_i} + \Delta_i)$-negative extremal ray, $\Delta_{i+1} := (\phi_i)_* \Delta_i$ and all the $X_i$ are $S$-schemes $f_i : X_i \to S$ such that $f_i = f_{i+1} \circ \phi_i$. We say the $(K_X + \Delta)$-MMP stops or terminates with $(X_r, \Delta_r)$ if

1. either $K_{X_r} + \Delta_r$ is $f_r$-nef (and there are no more extremal rays),
2. or there is a Fano contraction $X_r \to Z_r$.

Sometimes we impose a stronger restriction:

(2’) every extremal contraction of $(X_r, \Delta_r)$ is Fano.

Conjecturally, every running of the $(K_X + \Delta)$-MMP stops. This is known if $\dim X \leq 3$ [Kaw92], in many cases in dimension 4 [AHK07] or when the generic fiber of $f$ is of general type [BCHM06] and at each step the extremal rays are chosen “suitably.” Note that the latter includes the case when $f$ is birational (or generically finite), since a point is a 0-dimensional variety of general type.

(Everything works the same if $X$ is not $\mathbb{Q}$-factorial, except in that case it does not make sense to distinguish divisorial contractions and flips.)

**Definition 100.** Let $(X, \Delta)$ be a pair and $f : X \to S$ a proper morphism. We say that $(X, \Delta)$ is an

- $f$-weak canonical, $f$-canonical, $f$-minimal model if $(X, \Delta)$ is\left\{\begin{array}{l}
lc \\
lc \text{ dlt}
\end{array}\right.$ and $K_X + \Delta$ is\left\{\begin{array}{l}
f$-nef \\
f$-ample \\
f$-nef
\end{array}\right.$.
Warning 101. Note that a canonical model \((X, \Delta)\) has log canonical singularities, not necessarily canonical singularities. This, by now entrenched, unfortunate terminology is a result of an incomplete shift. Originally everything was defined only for \(\Delta = 0\). When \(\Delta\) was introduced, its presence was indicated by putting “log” in front of adjectives. Later, when the use of \(\Delta\) became pervasive, people started dropping the prefix “log”. This is usually not a problem. For instance, the canonical ring \(R(X, K_X)\) is just the \(\Delta = 0\) special case of the log canonical ring \(R(X, K_X + \Delta)\).

However, canonical singularities are not the \(\Delta = 0\) special cases of log canonical singularities.

Definition 102. Let \((X, \Delta)\) be a pair such that \(K_X + \Delta\) is \(\mathbb{Q}\)-Cartier and \(f : X \rightarrow S\) a proper morphism. A pair \((X^w, \Delta^w)\) sitting in a diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\phi} & X^w \\
\downarrow f & & \swarrow f^w \\
S & & 
\end{array}
\]

is called a weak canonical model of \((X, \Delta)\) over \(S\) if

1. \(f^w\) is proper,
2. \(\phi\) is a contraction, that is, \(\phi^{-1}\) has no exceptional divisors,
3. \(\Delta^w = \phi_* \Delta\),
4. \(K_{X^w} + \Delta^w\) is \(\mathbb{Q}\)-Cartier and \(f^w\)-nef, and
5. \(a(E, X, \Delta) \leq a(E, X^w, \Delta^w)\) for every \(\phi\)-exceptional divisor \(E \subset X\). Equivalently, \((K_X + \Delta) - \phi^* (K_{X^w} + \Delta^w)\) is effective and \(\phi\)-exceptional.

A weak canonical model \((X^m, \Delta^m) = (X^w, \Delta^w)\) is called a minimal model of \((X, \Delta)\) over \(S\) if, in addition to (1–4), we have

5. \(a(E, X, \Delta) < a(E, X^m, \Delta^m)\) for every \(\phi\)-exceptional divisor \(E \subset X\).

A weak canonical model \((X^c, \Delta^c) = (X^w, \Delta^w)\) is called a canonical model of \((X, \Delta)\) over \(S\) if, in addition to (1–3) and (5) we have

4. \(K_{X^c} + \Delta^c\) is \(\mathbb{Q}\)-Cartier and \(f^c\)-ample.

Exercise 103. Let \((X, \Delta)\) be a pair such that \(K_X + \Delta\) is \(\mathbb{Q}\)-Cartier and \(f : X \rightarrow S\) a proper morphism. Run the MMP:

\[
(X, \Delta) =: (X_0, \Delta_0) \xrightarrow{\phi_0} (X_1, \Delta_1) \xrightarrow{\phi_1} \cdots \xrightarrow{\phi_{n-1}} (X_r, \Delta_r),
\]

and assume that \(K_{X_r} + \Delta_r\) is \(f\)-nef. Show that \((X_r, \Delta_r)\) is a minimal model of \((X, \Delta)\) over \(S\).

Exercise 104. Let \(f : (X, \Delta) \rightarrow S\) be a canonical model. Let \(g : X' \rightarrow X\) be a proper birational morphism with exceptional divisors \(E_i\). When is \(f : (X, \Delta) \rightarrow S\) a canonical model of \((X', g^{-1} \Delta + \sum e_i E_i)\)?

Exercise 105. Let \(\phi : (X, \Delta) \rightarrow (X^w, \Delta^w)\) be a weak canonical model. Prove that

\[
a(E, X^w, \Delta^w) \geq a(E, X, \Delta)
\]

for every divisor \(E\).
Hint. Fix $E$ and consider any diagram

$$
Y \xrightarrow{g} h \xrightarrow{\phi} X^w \\
\downarrow f \phantom{123} \downarrow \phi \phantom{123} \downarrow f^w \phantom{123} \downarrow S
$$

where center $E$ is a divisor. Write $K_Y$ in two different ways and apply (107).

Exercise 106. Let $\phi : (X, \Delta) \to (X^w, \Delta^w)$ be a weak canonical model. Prove that if a curve $C \subset X$ is not contained in $\text{Ex}(\phi)$ then

$$
C \cdot (K_X + \Delta) \geq \phi^*(C) \cdot (K_{X^w} + \Delta^w).
$$

Exercise 107. Let $h : Z \to Y$ be a proper birational morphism between normal varieties. Let $-B$ be an $h$-nef $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on $Z$. Then

1. $B$ is effective iff $h_* B$ is.
2. Assume that $B$ is effective. Then for every $y \in Y$, either $h^{-1}(y) \subset \text{Supp } B$ or $h^{-1}(y) \cap \text{Supp } B = \emptyset$.

Hint. Use induction on $\dim Z$ by passing to a hyperplane section $H \subset Z$. Be careful: $h^*(B \cap H)$ need not be contained in $h_* B$.

Exercise 108 (Q-factorialization). Let $(X, \Delta)$ be klt. Let $f : Y \to X$ be a log resolution with exceptional divisor $E$. For $0 < \epsilon \ll 1$ run the $(Y, f^{-1}E + (1-\epsilon)\Delta)$-MMP over $X$ and assume that it stops. (This is not a restriction by (99).)

Prove that the MMP stops at a small contraction $f_r : Y_r \to X$ such that $Y_r$ is Q-factorial.

It is called a Q-factorialization of $X$.

More generally, prove that Q-factorializations exist if $(X, \Delta)$ is dlt. Find lc examples without any Q-factorialization.

Exercise 109. Notation as in (108). Let $D$ be any Weil divisor on $X$. Prove that there is a Q-factorialization $f_D : Y_D \to X$ such that the birational transform of $D$ on $Y_D$ is $f_D$-nef.

Use this to prove that Q-factorializations are never unique, save when $X$ itself is Q-factorial.

Use this and the contraction theorem to prove (92).

Warning 110. You may have noticed already that we have not defined when a pair $(X', \Delta')$ is birational to another pair $(X, \Delta)$. The problem is: what should the coefficient of a divisor $D \subset X'$ be in $\Delta'$ when the center of $D$ on $X$ is not a divisor.

One approach is to insist that birational pairs have the same canonical rings. Then the next exercise suggests a definition.

It is, however, best to keep in mind that birational equivalence of pairs is a problematic concept.

Exercise 111. Let $f_1 : X_1 \to S$ and $f_2 : X_2 \to S$ be proper morphisms of normal schemes and $\phi : X_1 \to X_2$ a birational map such that $f_1 = f_2 \circ \phi$. Let $\Delta_1$ and $\Delta_2$ be $\mathbb{Q}$-divisors such that $K_{X_1} + \Delta_1$ and $K_{X_2} + \Delta_2$ are $\mathbb{Q}$-Cartier. Prove that

$$
f_1_* \mathcal{O}_{X_1}(mK_{X_1} + \lfloor m\Delta_1 \rfloor) = f_2_* \mathcal{O}_{X_2}(mK_{X_2} + \lfloor m\Delta_2 \rfloor) \quad \text{for } m \geq 0
$$

if the following conditions hold:
(1) $a(E, X_1, \Delta_1) = a(E, X_2, \Delta_2)$ if $\phi$ is a local isomorphism at the generic point of $E$.
(2) $a(E, X_1, \Delta_1) \leq a(E, X_2, \Delta_2)$ if $E \subset X_1$ is $\phi$-exceptional, and
(3) $a(E, X_1, \Delta_1) \geq a(E, X_2, \Delta_2)$ if $E \subset X_2$ is $\phi^{-1}$-exceptional.

Hints: Let $Y$ be the normalization of the closed graph of $\phi$ in $X_1 \times_S X_2$ and $g_i : Y \to X_i$ the projections. We can write
\[ K_Y \sim_{\mathbb{Q}} g_1^* (K_{X_1} + \Delta_1) + \sum_E a(E, X_1, \Delta_1)E, \quad \text{and} \]
\[ K_Y \sim_{\mathbb{Q}} g_1^* (K_{X_2} + \Delta_2) + \sum_E a(E, X_2, \Delta_2)E. \]

Set $b(E) := \max \{-a(E, X_1, \Delta_1), -a(E, X_2, \Delta_2)\}$. Prove that $\sum E (b(E) + a(E, X_1, \Delta_1))E$ is effective and $g_i$-exceptional for $i = 1, 2$. Conclude that
\[
(f_i \circ g_i)_* \mathcal{O}_Y (mK_Y + \sum_E mb(E)E) = f_* g_* \mathcal{O}_Y \left( g_1^* (mK_{X_1} + m\Delta_1) + \sum_E (mb(E) + ma(E, X_1, \Delta_1))E \right)
\]
\[ = f_* \mathcal{O}_{X_i} (mK_{X_i} + m\Delta_i). \]

**Exercise 112.** Let $(X, \Delta)$ be a lc pair with $\Delta \geq 0$, $f : X \to S$ a proper morphism and $f^w : (X^w, \Delta^w) \to S$ a weak minimal model. Prove the following:
(1) $f_* \mathcal{O}_X (mK_X + |m\Delta|) = f^w_* \mathcal{O}_{X^w} (mK_X^w + |m\Delta^w|)$ for every $m \geq 0$.
(2) If a canonical model $(X^c, \Delta^c)$ exists then
\[ X^c = \text{Proj} S \sum_{m \geq 0} f_* \mathcal{O}_X (mK_X + |m\Delta|), \]
and the right hand side is a sheaf of finitely generated algebras. In particular, a canonical model is unique.
(3) Any two minimal models of $(X, \Delta)$ are isomorphic in codimension one. (Hint: Prove this first when $\Delta = 0$ and $(X, 0)$ is terminal. The general case is more subtle.)

**Exercise 113.** Assume that $X$ is irreducible,
\[ R(X, K_X + \Delta) := \sum_{m \geq 0} f_* \mathcal{O}_X (mK_X + |m\Delta|) \]
is a sheaf of finitely generated algebras and
\[ \dim X = \dim \text{Proj} S R(X, K_X + \Delta). \]
Prove that the natural map $\phi : X \dasharrow \text{Proj} S R(X, K_X + \Delta)$ is birational and
\[ (X^c, \Delta^c) := (\text{Proj} S R(X, K_X + \Delta), \phi_* \Delta) \]
is the canonical model of $(X, \Delta)$.

Hint: You should find (114) useful.

**Exercise 114.** Let $X$ be an irreducible and normal scheme, $L$ a Weil divisor on $X$ and $f : X \to S$ a proper morphism, $S$ affine. Write $|L| = |M| + F$ where $|M|$ is the moving part and $F$ the fixed part. Assume that $R(X, L) := \sum_{m \geq 0} f_* \mathcal{O}_X (mL)$ is generated by $f_* \mathcal{O}_X (L)$. Set $Z := \text{Proj} S R(X, L)$ with projection $p : Z \to S$ and let $\phi : X \dasharrow Z$ be the natural morphism. Prove that
(1) $Z \setminus \phi(X)$ has codimension $\geq 2$ in $Z$.
(2) If $\phi$ is generically finite then it is birational and $F$ is $\phi$-exceptional.

(Hint: This is similar to (90).)
Exercise 115 (Chambers in the cone of big divisors). Let $X$ be a normal variety and $D_i$ big $\mathbb{Q}$-divisors. Assume that the rings
$$R(D_i) := \sum_{m \geq 0} H^0(X, \mathcal{O}_X(\lfloor mD_i \rfloor))$$
are finitely generated and the maps $X \rightarrow \text{Proj} R(D_i)$ are birational and independent of $i$. Let $D = \sum a_i D_i$ be a nonnegative $\mathbb{Q}$-linear combination.

Prove that $R(D)$ is finitely generated and $X \rightarrow \text{Proj} R(D)$ is the same map as before.

Conclude that the set of all big $\mathbb{Q}$-divisors with the same $X \rightarrow \text{Proj} R(D)$ forms a convex subcone, called a chamber in the cone of big divisors.

Exercise 116. Develop a relative version of the notion of chambers of divisors for maps. (Note that for birational maps, every divisor is relatively big.) Let $Y \rightarrow X$ be a $\mathbb{Q}$-factorialization of a klt pair $(X, \Delta)$ \cite{HK08}. Prove that there is a one-to-one correspondence between open chambers of $N^1(Y/X)$ and $\mathbb{Q}$-factorializations of $X$.

What kind of maps correspond to the other chambers?

Exercise 117. Let $a_i$ be different complex numbers. Consider the singularity
$$X = X(a_1, \ldots, a_n) := (xy - \prod_i (u - a_i v) = 0) \subset \mathbb{A}^4.$$ 
Find a small resolution of $X$ by repeatedly blowing up planes of the form $(x = u - a_i v = 0)$.

Prove that the class group $\text{Cl}(X)$ of $X$ is generated by the planes $(x = u - a_i v = 0)$, with a single relation $\sum [x = u - a_i v = 0] = 0$.

Describe all small resolutions of $X$ and the corresponding chamber structure on $\text{Cl}(X)$.

(The same method can be used to describe the class group and the chamber structure for any $cA$-type terminal 3-fold singularity, see \cite{Kol91}, 2.2.7. A similarly explicit description is not known for the $cD$ and $cE$-type cases.)

Exercise 118. Let $S := (xy - z^3 = 0) \subset \mathbb{A}^3$ and $f : X \rightarrow S$ its minimal resolution with exceptional curves $D_1, D_2$. Let $D_3, D_4$ be the birational transforms of the lines $(x = z = 0)$ and $(y = z = 0)$. For $0 \leq a_i \leq 1$ describe minimal and canonical models of $(X, \sum a_i D_i)$ over $S$. Describe the chamber decomposition of $[0, 1]^4$.

Exercise 119. Let $S$ be one of the singularities in \cite{99} and $f : X \rightarrow S$ its minimal resolution with exceptional curves $D_i$. For $0 \leq a_i \leq 1$ describe minimal and canonical models of $(X, \sum a_i D_i)$ over $S$ and the corresponding chamber decomposition.

(This is pretty easy for the $\mathbb{Z}/2$-quotient. Some of the others have many curves to check.)

For the theory behind the next exercises, see \cite{KL07}.

Exercise 120. Let $E$ be the projective elliptic curve with affine equation $(y^2 = x^3 - 1)$ and set $\tau : (x, y) \mapsto (x, -y)$. Check that

1. $E/\tau \cong \mathbb{P}^1$.
2. $(E \times E)/(\tau \times \tau)$ has Kodaira dimension 0. It is an example of a Kummer surface. If $u = y_1 y_2$ then it has affine equation
$$u^2 = (x_1^3 - 1)(x_2^3 - 1).$$

Find the singularities using this equation.
(3) For $n \geq 3$, $(E^n)/(\tau, \ldots, \tau)$ has Kodaira dimension 0.

**Exercise 121.** Let $E$ be the projective elliptic curve with affine equation $(y^3 = x^3 - 1)$ and set $\sigma : (x, y) \mapsto (x, \epsilon y)$ where $\epsilon = \sqrt[3]{1}$. Check that

1. $E/\sigma \cong \mathbb{P}^1$.
2. $(E \times E)/(\sigma, \sigma^2)$ has Kodaira dimension 0. It is an example of a K3 surface. If $u = y_1 y_2$ then it has affine equation

$$u^3 = (x_1^3 - 1)(x_2^3 - 1).$$

Find the singularities using this equation.

3. $(E \times E)/(\sigma, \sigma)$. If $v = y_1 y_2^2$ then it has affine equation

$$v^3 = (x_1^3 - 1)(x_2^3 - 1)^2.$$

Find the singularities using this equation.

Prove that this surface is rational in two ways:

(a) Find many rational curves on it as preimages of rational curves of bi-degree $(2,2)$ on $\mathbb{P}^1 \times \mathbb{P}^1$.

(b) Show that it is birational (even over $\mathbb{Z}$) to the cubic surface $y_1^3 - y_2^3 = x_1^3$. 

(4) For $n \geq 3$, $(E^n)/(\tau, \ldots, \tau)$ has Kodaira dimension 0.

**Exercise 122.** Let $E$ be the projective elliptic curve with affine equation $(y^6 = x(x - 1)^2(x + 1)^3)$ and set $\rho : (x, y) \mapsto (x, \epsilon y)$ where $\epsilon = \sqrt[6]{1}$. Check that

1. $E/\rho \cong \mathbb{P}^1$.
2. For $2 \leq n \leq 5$, $(E^n)/(\rho, \ldots, \rho)$ is uniruled, that is, it has a covering family of rational curves. Try to find explicitly such a family. (Such a family exists by [KL07], but I do not know how to construct one.) I don’t know if these examples are rational or unirational.

3. For $6 \leq n$, $(E^n)/(\rho, \ldots, \rho)$ has Kodaira dimension 0.

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