The dilaton Wess–Zumino action in six dimensions from Weyl gauging: local anomalies and trace relations

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Received 7 December 2013, revised 7 March 2014
Accepted for publication 28 March 2014
Published 29 April 2014

Abstract
We extend a previous analysis on the derivation of the dilaton Wess–Zumino (WZ) action in $d = 4$, based on the method of Weyl gauging, to six dimensions. As in the previous case, we discuss the structure of the same action in dimensional regularization using six-dimensional Weyl invariants, extracting the dilaton interactions in the most general scheme, with the inclusion of the local anomaly terms. As an application, we present the WZ action for the $(2,0)$ tensor multiplet, which has been investigated in the past in the context of the $AdS_7/CFT_6$ holographic anomaly matching. We then extend to $d = 6$ the investigation of fully traced correlation functions of energy–momentum tensor’s, formerly presented in $d = 4$, showing that their hierarchy is functionally related only to the first six correlators. We give the explicit expressions of these in the most general scheme, up to rank-4.

Keywords: anomalies, gravitation, effective actions, Weyl symmetry, conformal symmetry
PACS numbers: 11.10.−z, 11.10.Ef

1. Introduction

Anomaly-induced actions play a considerable role among effective field theories. Simple instances of these types of actions are theories with chiral fermions in the presence of anomalous Abelian symmetries [1–4], other examples involve conformal [5–8] and superconformal anomalies [9].

Direct computations of these actions can be performed in ordinary perturbation theory by the usual Feynman expansion at one loop, but alternative approaches are also possible.
fact, an action which reproduces the same anomaly at low energy can be constructed quite directly, just as a variational solution of the anomaly condition, without any reference to the diagrammatic expansion. In gravity, typical examples are anomaly actions such as the Riegert action [10], or the Wess–Zumino (WZ) dilaton action [11], which reproduce the anomaly either with a nonlocal (Riegert) or with a local (WZ) effective operator, using a dilaton field in the latter case [12]. These types of actions are not unique, since possible contributions which are conformally invariant are not identified by the variational procedure. It should also be mentioned that a prolonged interest in these actions has been and is linked to the study of the irreversibility of the renormalization group (RG) flow in various dimensions (see for instance [13–17] and of the trace anomaly matching [18], since Zamolodchikov’s proof of his $c$-theorem in $d = 2$ [19].

A salient feature of some of these anomaly actions, particularly if formulated in a local form, as in the WZ case, is the inclusion of extra degrees of freedom compared to the original tree-level action. In the case of the chiral anomaly this additional degree of freedom is the axion ($\theta(x)$), which is linearly coupled to the anomaly functional in the form of a $(\theta/M)\tilde{F}F$ term—the anomaly coupling—with $F$ and $\tilde{F}$ denoting the field strength of the gauge field and its dual respectively. The anomaly interaction is accompanied by a new scale ($M$). This is the scale at which the anomalous symmetry starts to play a role in the effective theory. A large value of $M$, for instance, is then associated with a decoupling of the anomaly in the low energy theory. In the one-particle irreducible effective action this is obtained—in the chiral case—by allowing the mass of the fermions ($\sim M$) that run in the anomaly loops to grow large. The underlying idea of keeping the anomaly interaction in the form of a local operator at low energy—such as the $(\theta/M)\tilde{F}F$ term—while removing part of the physical spectrum, is important in the study of the RG flows of large classes of theories, both for chiral and for conformal anomalies.

In the case of conformal anomalies [5], which is the case of interest in this work, the pattern is similar to the chiral case, with the introduction of a dilaton field in place of the axion in order to identify the structure of the corresponding WZ action, and the inclusion of a conformal scale ($\Lambda$). As in the chiral case, one of the significant features of the WZ conformal anomaly action is the presence of a linear coupling of the Goldstone mode of the broken symmetry (the dilaton) to the anomaly functional, but with a significant variant. In this case, in fact, this linear term has to be corrected by additional contributions, due to the non invariance of the anomaly functional under a conformal transformation.

This procedure, which allows to identify the structure of WZ action, goes under the name of the Noether method (see for instance [20, 21]) and has to be iterated several times, due to the structure of the anomaly functional, before reaching an end. Given the fact that anomaly functional takes a different form in each space-time dimension, the anomaly action will involve interactions of the dilaton field of different orders in each dimension.

1.1. Weyl gauging and the anomaly action

In a previous paper [20] we have investigated an alternative approach, useful for the computation of this action, which exploits the structure of the counterterms in dimensional regularization and their Weyl gauging, bypassing altogether the Noether procedure. This approach has been discussed in $d = 4$ by several authors [11, 22], and in a cohomological context in [23].

In particular, we have shown how the complete hierarchy of correlators involving traces ($T$) of the energy–momentum tensor’s (EMT’s) of any conformal field theory (CFT) in four dimensions is functionally related only to the first four correlators, with ranks from 1 to 4
(T, T^2, T^3, T^4), of the same theory. These are completely determined by the anomaly. It is the order of the dilaton interaction which determines the maximum rank of independent traced correlators necessary to fix the entire hierarchy. In \( d = 6 \), as we are going to show, the traces of the first six correlators \((T, T^2, \ldots, T^6)\) are sufficient for this goal.

The extension of this construction to higher dimensions is interesting for several reasons. The WZ action in \( d = 6 \) plays an important role in the study of the irreversibility of the RG flow of CFT’s from the ultraviolet to the infrared [16, 21] also for this specific dimensions. At the same time it plays an equally important role in the study of the AdS/CFT correspondence. An example is the investigation of the anomaly matching between conformal tensor multiplets on the six dimensional boundary and a stack of M5 branes of \( AdS_7 \) supergravity in the bulk [24, 25]. We will present, as an application of our formalism, the expression of the WZ action for this specific CFT realization in \( d = 6 \).

Our work, in the study that we present, follows rather closely the layout of our previous derivation of the WZ action by Weyl gauging in \( d = 4 \), that we extend to six dimensions. As in the four-dimensional case, we derive the dilaton effective action by taking into account all the possible counterterms in the construction, which are identified in dimensional regularization within a general subtraction scheme. We also mention that general results on the structure of the action in any even dimensions have been presented in [26], using the general form of the Euler density and its conformal variation, which is sufficient to identify the nonlocal structure of the anomaly in a specific scheme, as we will specify below. However, the identification of the local contributions to the anomaly requires a separate effort, that we undertake in this work for \( d = 6 \).

This more general approach allows us to set a distinction between the nonlocal and local contributions to the anomaly and hence to the effective action, as in our former analysis of the \( d = 4 \) case. We will start our investigation by reviewing the method of Weyl gauging, together with a brief discussion of the structure of correlation functions of traces of the EMT for a generic CFT. In the past, the gauging has been discussed in various ways both in the context of extensions of the Standard Model [27, 28] and in cosmology, where it has been shown that the introduction of an extra scalar brings to a dynamical adjustment of the cosmological constant [22]. Recent discussions of the role of the dilaton in quantum gravity can be found in [29, 30].

The extension of the gauging procedure from four to six dimensions, in a general scheme, is quite demanding from the technical side, and is addressed starting from section 2, where we fix our conventions for the structure of the anomaly functional. This is expressed in terms of the generic coefficients \((c_1, c_2, c_3)\) and \(a\), which describe the anomaly of any CFT in \( d = 6 \). We thus identify the operators in the effective action that are responsible for the local anomaly, which are the analogue of the \( R^2 \) curvature term in \( d = 4 \), explicitly establishing their connection to the part proportional to total derivatives. We then move to the analysis of the structure of the traced correlators and of their hierarchy, showing how to solve it in terms of the first six correlation functions. We have left to appendix A a discussion of some of the more technical steps. Appendix B includes the consistency checks of the recursion relations satisfied by the traced correlators in \( d = 2 \) and \( d = 4 \), presenting the expressions of the first traced Green functions up to rank-6 in the two cases.

## 2. Definitions and conventions

In this section we establish our conventions, which will be used throughout our computations, before coming to a description of the anomaly-induced action in \( d = 6 \). We define the generating functional of the theory \( \mathcal{W} \) as
\[ \mathcal{W}[g] = \int \mathcal{D}\Phi \, e^{-S}, \]  
where \( S \) is the generic euclidean action depending on the set of all the quantum fields \( \Phi \) and on the background metric \( (g) \). The vacuum expectation value (vev) of the EMT is given by

\[ \langle T^\mu_\nu(x) \rangle = \frac{2}{\sqrt{g}} \frac{\delta \mathcal{W}[g]}{\delta g_{\mu\nu}(x)} = - \frac{2}{\sqrt{g}} \frac{\delta}{\delta g_{\mu\nu}(x)} \int \mathcal{D}\Phi \, e^{-S} \]

which is symmetric and traceless for a conformal invariant theory in absence of a trace anomaly. In (2) \( g_\mu(\mathcal{A}) \) is the determinant of the metric tensor. For ordinary field theories, investigated in ordinary Minkowski space, this approach allows to identify a symmetric expression of the EMT, which is traceless in the presence of a scale invariant Lagrangian, modulo standard improvement terms for scalar fields that we will discuss next. In the case of the Standard Model, this approach has been used to fix the entire structure of the EMT in the \( R^\xi \) gauge [31].

The equation of the conformal anomaly is expressed in terms of a functional \( A[g] \) which depends on the metric background \( g \)

\[ g_\mu(T^\mu_\nu) = A[g], \]

which holds in any even dimensions.

The general structure of the trace anomaly equation for general even dimension \( d \), restricted to operators of dimension \( d \), is given by [7]

\[ A[g] = \sum_i c_i (\mathcal{I}_i + \nabla_\mu J^\mu_i) - (-1)^{d/2} a E_d, \]

where \( \sqrt{8} \mathcal{I}_i \) are conformal invariants, the analogous of the Weyl tensor squared in 4 dimensions, whose number increases with the dimension, whereas \( E_d \) is the Euler density in \( d \) dimensions. They are both defined in appendix A.1.

The contribution coming from the Euler density is usually denoted as the \( A \) part of the anomaly, while the rest is called the \( B \) part. The total derivative terms \( \nabla_\mu J^\mu_i \) are known under the name of local anomaly contributions and are sometimes omitted, as they are scheme-dependent and absent if the \( \mathcal{I}_i \)'s are expressed in generic \( d \) dimensions. They can be removed also by adding some local counterterms to the action, which are an intrinsic ambiguity of the dimensional regularization scheme.

The specific expression of (4) for \( d = 6 \) takes the form

\[ A[g] = \sum_{i=1}^{3} c_i (\mathcal{I}_i + \nabla_\mu J^\mu_i) + a E_6, \]

where \( \sqrt{8} \mathcal{I}_i \), \( (i = 1, 2, 3) \), are the three conformal invariants available in six dimensions. Our goal will be to determine the structure of the dilaton WZ action in the most general case, with the inclusion of the contributions related to these three conformal invariant terms.

### 2.1. The conformal invariants and the Euler density in \( d = 6 \)

To characterize the expansion of the scalars appearing in the trace anomaly equation, we introduce the basis of scalars obtained from the Riemann tensor, its contractions and derivatives, which is given by

\[
\begin{align*}
K_1 &= R^\mu_\nu \\
K_2 &= R^\mu_\nu R_\mu\nu \\
K_3 &= R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} \\
K_4 &= R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\alpha} R^{\alpha\beta} \\
K_5 &= R_{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} \\
K_6 &= R_{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} \\
K_7 &= R_{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} \\
K_8 &= R_{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} \\
K_9 &= R_{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} \\
K_{10} &= R_{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} \\
K_{11} &= R_{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} \\
K_{12} &= \partial_{\mu} R_{\mu\nu\rho\sigma} \\
K_{13} &= \nabla_\mu R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} \\
K_{14} &= \nabla_\mu R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} \\
K_{15} &= \nabla_\mu R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}
\end{align*}
\]
in terms of which the Euler density takes the form
\[ E_6 = K_1 - 12K_2 + 3K_3 + 16K_4 - 24K_5 - 24K_6 + 4K_7 + 8K_8. \] (6)

Defining a Weyl transformation of the metric in the form
\[ g_{\mu\nu}(x) \rightarrow e^{2\sigma(x)} g_{\mu\nu}(x), \] (7)
the three Weyl invariants (modulo a \( \sqrt{g} \) factor) in \( d = 6 \), restricted to operators of dimension \( 6, I_i, i = 1, 2, 3 \), are given by the expressions (see appendix A for their definitions in terms of the Weyl and Riemann tensors)
\[ I_1 = \frac{19}{135} K_1 - \frac{9}{135} K_2 + \frac{2}{15} K_3 + \frac{7}{15} K_4 - \frac{3}{5} K_5 - \frac{3}{7} K_6 + K_8, \]
\[ I_2 = \frac{16}{675} K_1 - \frac{2}{675} K_2 + \frac{3}{90} K_3 + \frac{3}{25} K_4 - \frac{1}{7} K_5 - 3K_6 + K_7, \]
\[ I_3 = \frac{1}{675} K_1 - \frac{2}{675} K_2 + \frac{5}{36} K_3 + \frac{1}{6} K_9 - 3K_{10} + 2K_{11} + K_{13} + K_{14} - 2K_{15}. \] (8)

It is easy to prove that for the three scalars defined above the products \( \sqrt{g} I_i \) are Weyl invariant in six dimensions, i.e., denoting with \( \delta_W \) the operator implementing an infinitesimal Weyl transformation,
\[ \delta_W I_i = -6\sigma I_i. \] (9)

We introduce the Green function of \( n \) EMT’s in flat space in the completely symmetric way
\[ \langle T^{\mu_1 \nu_1} (x_1) \ldots T^{\mu_n \nu_n} (x_n) \rangle \equiv \frac{2^n}{\sqrt{S_L} \ldots \sqrt{S_L}} \left. \frac{\delta^n V[g]}{\delta g_{\mu_1 \nu_1} (x_1) \ldots \delta g_{\mu_n \nu_n} (x_n)} \right|_{g_{\mu\nu} = \delta_{\mu\nu}}. \] (10)

We denote the functional derivatives with respect to the metric of generic functionals, in the limit of a flat background, as
\[ [f(x)]^{\mu_1 \nu_1 \ldots \mu_n \nu_n} (x_1, \ldots, x_n) \equiv \left. \frac{\delta^n f(x)}{\delta g_{\mu_1 \nu_1} (x_n) \ldots \delta g_{\mu_n \nu_n} (x_1)} \right|_{g_{\mu\nu} = \delta_{\mu\nu}} \] (11)
and the corresponding expression with traced indices
\[ [f(x)]^{\mu_1 \ldots \mu_n}_{\mu_1 \ldots \mu_n} (x_1, x_2, \ldots, x_n) \equiv \delta_{\mu_1 \nu_1} \ldots \delta_{\mu_n \nu_n} \left[ f(x) \right]^{\mu_1 \nu_1 \ldots \mu_n \nu_n} (x_1, \ldots, x_n), \] (12)
where the curved Euclidean metric \( g_{\mu\nu} \) is replaced by \( \delta_{\mu\nu} \).

By functional differentiations of the anomaly equation (3), one generates an infinite hierarchy of equations satisfied by the correlation functions of multiple traces of the EMT in the form
\[ \langle T(k_1) \ldots T(k_{n+1}) \rangle = 2^n [\sqrt{g} A]^{\mu_1 \ldots \mu_n}_{\mu_1 \ldots \mu_n} (k_1, \ldots, k_{n+1}) \]
\[ -2 \sum_{i=1}^{n} \langle T(k_1) \ldots T(k_{i-1}) T(k_{i+1}) \ldots T(k_{n+1} + k_i) \rangle, \] (13)
which indicate the existence of an open hierarchy. As we have shown in our analysis of the \( d = 4 \) case, this hierarchy can be completely identified just by a certain number of correlators, which in this case corresponds only to the first 6. However, as we have pointed out above, the number of traced correlators required to identify the hierarchy is related to the order of the dilaton interaction in the effective action.

In the expression above we have introduced the notation \( T \equiv T^\mu_{\mu} \) to denote the trace of the EMT. All the momenta characterizing the vertex are taken as incoming, as specified in appendix A.

The identity (13) relates a \( (n + 1) \)-point correlator to correlators of order \( n \), together with the completely traced derivatives of the anomaly functionals \( \sqrt{g} I_i, \sqrt{g} E_6 \) and \( \sqrt{g} \nabla_{\mu} I^\mu \).

For \( \sqrt{g} I_i \), which is a conformal invariant, they are identically zero. For \( \sqrt{g} E_6 \) these are non
vanishing at any arbitrary order $n + 1 \geq 4$, while $\sqrt{g} \nabla_{\mu} I_{\mu}^{n}$ contribute also to the trace of lower order functions. In particular, as shown above, $\nabla_{\mu} J_{\mu}^{\nu}$ are at least quadratic in the Riemann tensor, so that they give non-vanishing contributions from order 3 onwards, whereas $\nabla_{\mu} J_{\mu}^{3}$ contains a term which is linear in $R$ and thus contributes a non-vanishing trace to the two-point function.

### 3. Weyl gauging : overview

#### 3.1. Weyl gauging for scale invariant theories

The procedure of Weyl gauging defines a consistent framework useful to identify the coupling of a dilaton to the fields of a given Lagrangian. It can be implemented starting from a Lagrangian defined on a flat metric background, but written in a diffeomorphic invariant way (i.e. by using curvilinear coordinates), and introducing an appropriate new field which takes a role similar to an Abelian gauge field. This allows to define a new Lagrangian which is diffeomorphic and Weyl invariant in curved space. At a second stage this new degree of freedom can be made dynamical with the inclusion of a kinetic term. As we are going to see, the transformation property of this new field, which can be traded for the gradient of a dilaton, together with the requirement of Weyl invariance, forces its kinetic term to a unique form. This is obtained performing a nonlinear field redefinition in the Lagrangian of a conformally coupled scalar. The approach brings to the construction of a Weyl invariant Lagrangian which is conformally invariant in the flat limit. We are going to summarize these points below, illustrating explicitly the method in the simpler case of a scalar theory.

For a Lagrangian in flat space written in a diffeomorphic invariant form, scale invariance is equivalent to global Weyl invariance. The equivalence can be shown quite straightforwardly [32] by rewriting a scale transformation acting on the coordinates of flat space and the fields

\[
\begin{align*}
    x^\mu &\rightarrow x'^\mu = e^\sigma x^\mu, \\
    \Phi(x) &\rightarrow \Phi'(x') = e^{-d_\Phi} \Phi(x)
\end{align*}
\]  

(14)

in terms of a rescaling of the vielbein and of the matter fields

\[
\begin{align*}
    V_\mu(x) &\rightarrow e^\sigma V_\mu(x), \\
    \Phi(x) &\rightarrow e^{-d_\Phi} \Phi(x)
\end{align*}
\]  

(15)

but leaving the coordinates of the field $\Phi$ invariant. We have denoted with $d_\Phi$ its scaling dimension. Obviously, once we move to a curved metric background, it is natural to promote the global scaling parameter $w = e^\sigma$ to a local function, and modify the theory so that the transformation laws of the vielbein and matter fields ($\Phi$)

\[
\begin{align*}
    V'_\mu(x) = e^{\sigma(x)} V_\mu(x), \\
    \Phi'(x) = e^{-d_\Phi(x)} \Phi(x)
\end{align*}
\]  

(16)

leave the fundamental Lagrangian invariant. For a free scalar theory

\[
\frac{1}{2} \int \sqrt{g} \, d^d x \, \partial_\mu \phi \partial_\nu \phi,
\]  

(17)

the derivative terms are modified as for an Abelian gauge field

\[
\partial_\mu \rightarrow \partial_\mu^W = \partial_\mu - d_\phi W_\mu,
\]  

(18)

where $W_\mu$ is a vector gauge field that transforms under Weyl scaling as

\[
W_\mu \rightarrow W_\mu - \partial_\mu \sigma.
\]  

(19)
In the case of a covariant derivative acting on a spin-1 field $v_\mu$, the Weyl and diffeomorphic covariant derivative is found by adding to (18) the modified Christoffel connection
\[ \hat{\Gamma}^{\lambda}_{\mu\nu} = \Gamma^{\lambda}_{\mu\nu} + \delta^\lambda_\mu W_\nu + \delta^\lambda_\nu W_\mu - g_{\mu\nu} W^\lambda, \]
(20)
which is Weyl invariant. The method follows closely the gauging of a typical Abelian theory, by defining
\[ \nabla_W^\mu v_\nu = \partial_\mu v_\nu - \delta_{\mu} v_\nu - \hat{\Gamma}^{\lambda}_{\mu\nu} v_\lambda \]
(21)
The extension to the fermion case is obtained by the relation
\[ \nabla_\mu \rightarrow \nabla_W^\mu = \nabla_\mu - \partial_\mu \psi W_\mu + 2 \Sigma^{\mu\nu} W_\nu, \quad \Sigma^{\mu\nu} \equiv V_a^{\mu\nu} V_b^{\nu\mu} \Sigma_{ab}, \]
(22)
where we have denoted with $d_\psi$ the scaling dimension of the spinor field ($\psi$) and with $\Sigma_{ab}$ the spinor generators of the Lorentz group.

If we Weyl-gauge the scalar action (17) according to the prescriptions in (18) and (19) we obtain
\[ S_{\phi,W} = \frac{1}{2} \int d^d x \sqrt{|g|} \partial_\mu \phi \partial_\nu \phi - \frac{d-2}{2} W_\mu \partial_\nu \phi, \]
(23)
which, using $\partial_\mu \phi \partial_\nu \phi = \frac{1}{2} \partial_\mu \phi^2$ and integrating by parts, can be written as
\[ S_{\phi,W} = \frac{1}{2} \int d^d x \sqrt{|g|} \left( \partial_\mu W_\nu \partial_\nu \phi + \phi W_\mu \partial_\nu \phi - \frac{d-2}{2} \Omega_{\mu\nu}(W) \right), \]
(24)
where we have introduced
\[ \Omega_{\mu\nu}(W) = \nabla_\mu W_\nu - W_\mu W_\nu + \frac{1}{2} g_{\mu\nu} W^2. \]
(25)
The result of this procedure is a Weyl invariant Lagrangian in which the Weyl variation of the ordinary kinetic term of $\phi$ is balanced by the variation of the $\Omega$ term. One can also render $W_\mu$ dynamical by the inclusion of a kinetic term built out of an appropriate field strength
\[ F_{\mu\nu}^{W} \equiv \partial_\mu W_\nu - \partial_\nu W_\mu \]
(26)
which is manifestly Weyl invariant.

A question that arises is whether it is possible to build a Weyl invariant theory without having to introduce an additional gauge field $W_\mu$ at all. As discussed in [32], this is possible if and only if, having performed the Weyl gauging, $W_\mu$ appears in the gauged action only in the combination given by $\Omega_{\mu\nu}(W)$. In fact, having observed that under a finite Weyl transformation ($\delta_W$) the variation of $\Omega_{\mu\nu}(W)$ coincides, modulo a factor, with the variation of a particular combination of the Ricci tensor and the scalar curvature, i.e.
\[ \delta_W \Omega_{\mu\nu}(W) = \frac{1}{2 - d} \delta_W S_{\mu\nu}, \quad S_{\mu\nu} = \frac{1}{2 - d} \left( R_{\mu\nu} - \frac{1}{d-1} g_{\mu\nu} R \right), \]
(27)
one obtains a new Weyl invariant action via the replacement
\[ \Omega_{\mu\nu}(W) \rightarrow \frac{1}{2 - d} S_{\mu\nu}. \]
(28)
Doing so in the Weyl gauged action of the scalar field (24), the latter takes the form
\[ S_{\phi,\text{imp}} = \frac{1}{2} \int d^d x \sqrt{|g|} \left( g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{1}{4} \frac{d-2}{d-1} R \phi^2 \right), \]
(29)
which is the action of a conformally coupled scalar. The procedure of rendering the theory Weyl
invariant through such supplementary couplings to the Ricci tensor is called Ricci gauging
[32].

A second possibility is to maintain the expression of \( W_{\mu} \), with new interactions induced
by the Weyl gauging, but now identified with the gradient of a dilaton field,
\[
W_{\mu}(x) = \frac{\partial_{\mu} \tau(x)}{\Lambda}. \tag{30}
\]
This second choice offers an interesting physical interpretation—in the flat limit—in
connection with the breaking of the conformal symmetry, related to the conformal scale \( \Lambda \),
as we will shortly point out below. Notice that in this second case the \( \Omega(\partial_{\mu} \tau/\Lambda) \) term
generates non trivial cubic and quartic interactions between the original scalar and the dilaton
\[
\Omega \left( \frac{\partial \tau}{\Lambda} \right) = \nabla_{\mu} \partial_{\nu} \tau + \frac{1}{2} g_{\mu \nu} \left( \frac{\partial \tau}{\Lambda} \right)^2, \tag{31}
\]
which bring (24) to the form
\[
S_{\phi, \partial \tau} = \frac{1}{2} \int d^d x \sqrt{g} g^{\mu \nu} \left( \nabla_{\mu} \phi \nabla_{\nu} \phi + \frac{d - 2}{2} \phi^2 \Box \tau + \frac{1}{2} \left( \frac{d - 2}{2} \right)^2 \phi^2 \left( \frac{\partial \tau}{\Lambda} \right)^2 \right). \tag{32}
\]
As the field strength \( F^W \) in (26), on account of (30) is obviously zero, the standard way to
give a kinetic term to the dilaton is by introducing a conformally coupled scalar field \( \chi \) and
imposing the field redefinition
\[
\chi(\tau) \equiv \Lambda^{d/2} e^{-\frac{d-2}{2}\tau}. \tag{33}
\]
At this point, the dynamics of the combined scalar/dilaton/graviton system is described by the
Weyl invariant action
\[
S = S_{\chi(\tau), \text{imp}} + S_{\phi, \partial \tau}, \tag{34}
\]
having combined (29), where \( \phi \) is replaced by \( \chi \), and (32). The kinetic action for \( \chi \), \( S_{\chi(\tau), \text{imp}} \),
takes the form
\[
S_{\chi(\tau), \text{imp}} = \frac{\Lambda^{d-2}}{2} \int d^d x \sqrt{g} e^{-\frac{d-2}{2}\tau} \left( \frac{(d - 2)^2}{4\Lambda^2} g^{\mu \nu} \partial_{\mu} \tau \partial_{\nu} \tau - \frac{1}{4} \frac{d - 2}{d - 1} R \right). \tag{35}
\]
which, for the particular case \( d = 6 \) of interest in this work, reduces to
\[
S_{\chi(\tau), \text{imp}} = \int d^d x \sqrt{g} e^{-\frac{d}{2}\tau} \left( 2\Lambda^2 g^{\mu \nu} \partial_{\mu} \tau \partial_{\nu} \tau - \frac{\Lambda^4}{10} R \right). \tag{36}
\]
The Weyl gauging, as we have described it so far, is possible only when we take as a starting
point a scale invariant Lagrangian, with dimensionless constants. Things are different when
an action is not scale invariant in flat space, and in that case the same gauging requires
some extra steps. We illustrate this point below and discuss the modification of the procedure
outlined above, by considering again a scalar theory as an example. This approach exemplifies
a situation which is typical in theories with spontaneous breaking of the ordinary gauge
symmetry, such as the Standard Model.

3.2. Weyl gauging for non scale invariant theories

We consider a free scalar theory with a mass term
\[
S_2 = \frac{1}{2} \int d^d x \sqrt{g} \left( g^{\mu \nu} \partial_{\mu} \phi \partial_{\nu} \phi + m^2 \phi^2 \right). \tag{37}
\]
Scale invariance is lost, but it can be recovered. There are two ways to promote this action to a scale invariant one. The first is simply to render the mass term dynamical

\[ m \rightarrow m \frac{\Sigma}{\Lambda}, \]

using a second scalar field, \( \Sigma \). The action (37), with the replacement (38), can be extended with the inclusion of the kinetic term for \( \Sigma \). The action (37), with the replacement (38), can be extended with the inclusion of the kinetic term for \( \Sigma \). The inclusion of \( \Sigma \) and the addition of two conformal couplings (i.e. of two Ricci gaugings) both for \( \phi \) and \( \Sigma \) brings to the new action

\[ S^\Sigma_2 = \int d^4x \sqrt{|g|} \left[ \frac{1}{2} g^{\mu\nu} \left( \partial_\mu \phi \partial_\nu \phi + \partial_\mu \Sigma \partial_\nu \Sigma \right) + \frac{1}{2} m^2 \frac{\Sigma^2}{\Lambda^2} \phi^2 + \frac{1}{4} \frac{d - 2}{4} R \left( \phi^2 + \Sigma^2 \right) \right]. \]

(39)

which is Weyl invariant in curved space. These types of actions play a role in the context of Higgs-dilaton mixing in conformal invariant extension of the Standard Model, where \( \phi \) is replaced by the Higgs doublet and \( \Sigma \) is assumed to acquire a vev which coincides with the conformal breaking scale \( \Lambda \) (see for instance [33]). The mixing is induced by a simple extension of (39), where the mass term is generated via the scale invariant potential

\[ S_{\text{pot}} = \lambda \int d^4x \sqrt{|g|} \left( \phi^2 - \frac{\mu^2 \Sigma^2}{\Lambda^2} \right), \]

(40)

(with \( m = \mu \)). This choice provides a clear example of a Weyl invariant Lagrangian that allows a spontaneous breaking of the \( Z_2 \) symmetry of the scalar sector \( \phi \), following the breaking of the conformal symmetry (\( \langle \Sigma \rangle = \Lambda \), with \( \langle \tau \rangle = 0 \)). The theory is obviously Weyl invariant (see the discussion in [33]), but the contributions proportional to the Ricci scalar \( R \) do not survive, obviously, in the flat limit.

The approach to Weyl gauging of a non scale invariant Lagrangian briefly described above is not unique. In fact, a second alternative in the construction of a Weyl invariant Lagrangian in curved space, starting from (37), is to use the compensation procedure, which amounts to the replacements

\[ m \rightarrow m e^{-\tau/\Lambda}, \]
\[ g_{\mu\nu} \rightarrow \hat{g}_{\mu\nu} \equiv g_{\mu\nu} e^{-2\tau/\Lambda}, \]
\[ \phi \rightarrow \hat{\phi} \equiv \phi e^{\tau/\Lambda}, \]
\[ \partial_\mu \phi \rightarrow \partial_\mu \hat{\phi} = e^{\tau/\Lambda} \partial_\mu \phi, \quad \text{with} \quad W_\mu = \frac{\partial_\mu \tau}{\Lambda}. \]

(41)

giving an action of the form

\[ \hat{S}_2 = \int d^4x \sqrt{|\hat{g}|} \left[ g^{\mu\nu} \partial_\mu \hat{\phi} \partial_\nu \phi + g^{\mu\nu} \Omega_{\mu\nu} \left( \frac{\partial \tau}{\Lambda} \right) \phi^2 + m^2 e^{-2\tau/\Lambda} \phi^2 \right]. \]

(42)

where \( \Omega (\partial \tau / \Lambda) \) was defined in (31). Also in this case, the compensator \( \tau \) becomes a dynamical dilaton field by adding to \( \hat{S}_2 \) the kinetic contribution of a conformally coupled scalar (35), obtaining the total action

\[ S_T \equiv \hat{S}_2 + S_{\text{pot}(\tau),\text{imp}}. \]

(43)

Notice that in this case we choose not to require the Ricci gauging of the \( \Omega (\partial \tau / \Lambda) \) term in \( \hat{S}_2 \), but we leave it as it is, thereby generating additional interactions between the dilaton and the scalar \( \phi \) in flat space. Obviously, also following this second route, we can incorporate spontaneous breaking of the \( Z_2 \) symmetry of the \( \phi \) field after the breaking of conformal invariance (with \( \langle \Sigma \rangle = \Lambda \)). This is obtained, as before, by the inclusion of the potential (40).

In this second approach the \( \Omega (\partial \tau / \Lambda) \) terms are essential in order to differentiate between the two residual dilaton interactions in flat space. In the context of Weyl invariant extensions of the Standard Model, such terms are naturally present in the analysis of [27].
4. Weyl gauging of the renormalized action

The WZ anomaly action, as we have already mentioned above, is derived from the Weyl gauging of the renormalized action, defined as

$$\hat{\Gamma}_{\text{ren}}[g, \tau] \equiv \Gamma_{0}[g, \tau] + \Gamma_{\text{CL}}[\hat{g}],$$

(44)
in terms of a Weyl invariant contribution $\Gamma_{0}[g, \tau]$ and of a local counterterm $\Gamma_{\text{CL}}[\hat{g}]$, where $\hat{g}$ was defined in (41). The WZ action is then identified from the relation

$$\hat{\Gamma}_{\text{ren}}[g, \tau] = \Gamma_{\text{ren}}[g, \tau] - \Gamma_{\text{WZ}}[g, \tau].$$

(45)

Here $\Gamma_{\text{WZ}}[g, \tau]$ is the WZ action, whose Weyl variation equals the trace anomaly. Notice that $\hat{\Gamma}_{\text{ren}}[g, \tau]$, as one can immediately realize, is Weyl invariant by construction, being a functional only of $\hat{g}$. The Weyl invariant terms may take the form of any scalar contraction of $\hat{R}^{\mu\nu\rho\sigma}, \hat{R}^{\mu\nu}$ and $\hat{R}$ and can be classified by their mass dimension, such as

$$J_{n}[\hat{g}] \sim \frac{1}{\Lambda^{2n-d}} \int d^{d}x \sqrt{\hat{g}} R^{n},$$

(46)

and so forth. In principle, all these terms can be included into $\Gamma_{0}[\hat{g}] \equiv \Gamma_{0}[g, \tau]$ which describes the non anomalous part of the renormalized action

$$\Gamma_{0}[\hat{g}] \sim \sum_{n} J_{n}[\hat{g}].$$

(47)

Here we recall the structure of the operators that are at most marginal from the renormalization group viewpoint. The first term that can be included is trivial, corresponding to a cosmological constant contribution

$$S_{(n)}^{(1)} = \Lambda^{6} \int d^{6}x \sqrt{\hat{g}} = \Lambda^{6} \int d^{6}x \sqrt{g} e^{\alpha \tau}/\Lambda^{1}. $$

(48)

Here the superscript number in round brackets in $S^{(2n)}$ denotes the order of the contribution in the derivative expansion, so to distinguish the scaling behaviour of the various terms under the variation of the length scale.

For $n = 1$ we obtain the operator which reproduces the kinetic term of the dilaton extensively discussed above. Here we just mention that (36) can be derived from a general, d-dimensional formula for Weyl gauging the Einstein–Hilbert action,

$$S_{(n)}^{(2)} = \frac{\Lambda^{d-2}(d-2)}{8(d-1)} \int d^{4}x \sqrt{\hat{g}} R = \frac{\Lambda^{d-2}(d-2)}{8(d-1)} \int d^{4}x \sqrt{g} e^{\alpha \tau}/\Lambda^{2} \left[ R - (d-1)(d-2) \right.$$

$$\times \frac{(\dot{\tau})^{2}}{\Lambda^{2}} \left. \right]ight],$$

(49)

which is exactly (36) for $d = 6$.

The possible four-derivative terms ($n = 2$) are

$$\int d^{6}x \sqrt{g}(\alpha \hat{R}^{\mu\nu\rho\sigma} \hat{R}_{\mu\nu\rho\sigma} + \beta \hat{R}^{\mu\nu} \hat{R}_{\mu\nu} + \gamma \hat{R}^{2} + \delta \hat{\Box} \hat{R}).$$

(50)

The $\Box \hat{R}$ contribution in this expression can be obviously omitted, being a total derivative. We can also replace the Riemann tensor with the Weyl tensor squared (see (A.3)) and remain with only two (as $\sqrt{\hat{g}} \hat{C}^{\mu\nu\rho\sigma} = \sqrt{\hat{g}} C^{\mu\nu\rho\sigma} C_{\mu\nu\rho\sigma} \hat{\tau}^{2}$) non trivial contributions, $\hat{R}^{\mu\nu} \hat{R}_{\mu\nu}$ and $\hat{R}^{2}$. We present here the expression of (50) for a conformally flat metric, while the result for a general gravitational background can be computed exploiting the Weyl gauged tensors given
in appendix A.1,  
\[ S^{(4)}_\tau = \int d^6 x \sqrt{g} \left( \alpha \hat{R}^{\mu\nu} \hat{R}_{\mu\nu} + \beta \hat{R}^2 \right) \]
\[ = \int d^6 x e^{-\frac{2}{\Lambda}} \left[ 100\alpha \left( \frac{\Box \tau}{\Lambda} - \frac{2 (\partial \tau)^2}{\Lambda^2} \right)^2 + 2\beta \left( 15 \frac{(\Box \tau)^2}{\Lambda^2} - \frac{68}{\Lambda^3} \Box (\partial \tau)^2 + 72 \frac{(\partial \tau)^4}{\Lambda^4} \right) \right]. \]  
(51)

The last contributions that are significant down to the infrared regime are the marginal ones, i.e. the six-derivative operators. To derive them we follow the analysis in [21]. We use the basis of diffeomorphic invariants of order 6 in the derivatives, on which the \( I_i \)’s are expanded (see equation (8)). It is made of 11 elements, six of which contain the Riemann tensor, that can be traded for a combination of the Weyl tensor and the Ricci tensor and scalar, so that we are left with only the five terms in \( (K_1 - K_{11}) \) (see section (2.1)) that do not contain the Riemann tensor. As we are going to write down the result only in the flat limit, we can exploit two additional constraints. Indeed in [35] it was shown that, in this case, the integral of
\[ R^3 = 11RR^{\mu\nu} R_{\mu\nu} + 30R^{\mu\nu} R_\sigma R_{\mu\nu} - 6R \Box R + 20R^{\mu\nu} \Box R_{\mu\nu} \]  
(52)

vanishes, so that we can use this result to eliminate \( R^{\mu\nu} \Box R_{\mu\nu} \).

Then, as the Euler density can be written in the form
\[ E_6 = \frac{21}{100} R^3 - \frac{21}{20} R^{\mu\nu} R_{\mu\nu} + \frac{1}{2} R_{\alpha\beta} R_{\alpha\beta} + 4C_{\mu
u\rho\sigma} C^{\mu\nu\rho\sigma} - 8C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} - 6R_{\mu\nu} C^{\mu\nu\rho\sigma} C_{\mu\nu\rho\sigma} + \frac{6}{5} R C^{\mu\nu\rho\sigma} C_{\mu\nu\rho\sigma} - 3R_{\mu\nu} R^{\mu\nu} C_{\mu\nu\rho\sigma}, \]
(53)

it is apparent that only the three terms are non vanishing on a conformally flat metric. Now, as in the effective action these contributions are integrated and the Euler density is a total derivative, one can thereby replace \( R_{\alpha\beta} R_{\alpha\beta} \) for \( R^3 \) and \( R R^{\mu\nu} R_{\mu\nu} \). In the end, Weyl gauging \( R^3 \), \( R^{\mu\nu} R_{\mu\nu} \) and \( R \Box R \) is sufficient to account for all the possible six-derivative terms of the dilaton effective action which do not vanish in the flat space limit. After some integrations by parts, one can write the overall contribution as  
\[ S^{(6)}_\tau = \int d^6 x \sqrt{g} \left[ \gamma \hat{R}^3 + \delta \hat{R}^{\mu\nu} \hat{R}_{\mu\nu} + \xi \hat{R} \Box \hat{R} \right] \]
\[ = \int d^6 x e^{-\frac{2}{\Lambda}} \left[ \frac{1}{\Lambda^3} \left( 5\xi \Box \tau (\partial \tau)^3 - (5\gamma + 7\delta - 30\xi) (\Box \tau)^2 \right) \right. \]
\[ + \frac{1}{\Lambda^4} \left( 50(6\gamma + \delta - 2\xi) (\Box \tau)^2 (\partial \tau)^2 \right. \]
\[ + 16(\delta + 5\xi) \Box (\partial \tau)^2 (\partial \tau)^2 - 120 \frac{\Lambda}{\Lambda^4} \left( 5\gamma + \delta - \xi \right) \Box (\partial \tau)^4 \]
\[ \left. \left. + 80 \frac{\Lambda}{\Lambda^6} (5\gamma + \delta - \xi) (\partial \tau)^6 \right] \right]. \]  
(54)

We have introduced the compact notation \( (\partial \tau)^n \equiv (\partial_x (\partial \tau)^2)^n / \Lambda, (\partial \tau)^2 \equiv \partial_x \partial_{\tau} \partial \tau \partial_{\tau} \) to denote multiple derivatives of the dilaton field. The Weyl invariant part of the dilaton effective action is then given by
\[ \Gamma_0[g, \tau] = S^{(0)}_\tau + S^{(2)}_\tau + S^{(4)}_\tau + S^{(6)}_\tau + \cdots, \]
(55)
where the ellipsis denote all the possible higher-order, irrelevant terms.
4.1. The counterterms and the anomaly

As we have discussed above, we construct the effective action by applying the Weyl gauging procedure to the renormalized effective action, which breaks scale invariance via the anomaly. First we must introduce the one-loop counterterm action, which is given, following [5] and [23], by the integrals of all the possible Weyl invariants and of the Euler density continued to d dimensions

\[ \Gamma_{\text{C}}[g] = -\frac{\mu^{-\epsilon}}{\epsilon} \int d^d x \sqrt{g} \left( \sum_{i=1}^{3} c_i I_i + aE_6 \right), \quad \epsilon = 6 - d, \]  

(56)

where \( \mu \) is a regularization scale. It is this form of \( \Gamma_{\text{C}} \), which is part of \( \Gamma_{\text{ren}} \), to induce the anomaly relation

\[ \frac{2}{\sqrt{g}} \frac{\delta \Gamma_{\text{ren}}[g]}{\delta g_{\mu \nu}} \bigg|_{d\rightarrow 6} = \frac{2}{\sqrt{g}} \frac{\delta \Gamma_{\text{C}}[g]}{\delta g_{\mu \nu}} \bigg|_{d\rightarrow 6} = A[g]. \]  

(57)

In the derivation of the equation above, we have exploited the Weyl invariance of the non anomalous action \( \Gamma_{\text{C}}[g] \) in six dimensions

\[ \frac{\delta \Gamma_{\text{C}}[g]}{\delta g_{\mu \nu}} \bigg|_{d\rightarrow 6} = 0, \]  

(58)

while the anomaly is generated entirely by the counterterm action \( \Gamma_{\text{C}}[g] \), due to the relations

\[ \frac{2}{\sqrt{g}} \frac{\delta \Gamma_{\text{C}}[g]}{\delta g_{\mu \nu}} \bigg|_{d\rightarrow 6} \]  

(59)

\[ \frac{2}{\sqrt{g}} \frac{\delta \Gamma_{\text{C}}[g]}{\delta g_{\mu \nu}} \bigg|_{d\rightarrow 6} = -\epsilon (I_i + \nabla_\mu J_i^\mu), \]  

(60)

so that from (57) we find

\[ \langle T \rangle = \frac{2}{\sqrt{g}} \frac{\delta \Gamma_{\text{C}}[g]}{\delta g_{\mu \nu}} \bigg|_{d\rightarrow 6} = \sum_{i=1}^{3} c_i (I_i + \nabla_\mu J_i^\mu) + aE_6. \]  

(61)

The explicit expressions of the derivative terms \( \nabla_\mu J_i^\mu \) in equation (59) can be obtained using the functional variations listed in appendix A.2. They are given by

\[ \nabla_\mu J_1^\mu = -\frac{1}{300} \nabla_\mu \left[ -5(4R^{\rho \sigma \nu} \nabla_\rho R_{\nu \sigma} - 50R_{\rho \sigma} \nabla_\nu R^{\rho \sigma} - 3R^{\rho \sigma \nu} \partial_\rho R - 4R_{\rho \sigma \nu} \nabla_\mu R^{\rho \sigma \nu} + 40R^{\rho \sigma \nu \mu} \nabla_\nu R_{\rho \sigma} ) + 19R \partial_\mu R \right] \]  

\[ \nabla_\mu J_2^\mu = -\frac{1}{300} \nabla_\mu \left[ -5(4R^{\rho \sigma \nu} \nabla_\rho R_{\nu \sigma} + 10R_{\rho \sigma} \nabla_\nu R^{\rho \sigma} + 7R^{\rho \sigma \nu} \partial_\rho R - 4R_{\rho \sigma \nu} \nabla_\mu R^{\rho \sigma \nu} - 40R^{\rho \sigma \nu \mu} \nabla_\nu R_{\rho \sigma} ) + 9R \partial_\mu R \right] \]  

\[ \nabla_\mu J_3^\mu = \frac{1}{2} \nabla_\mu \left[ 10(2\partial_\rho R - 5\nabla_\rho \nabla_\nu R^{\nu \rho} + R_{\rho \sigma} \nabla_\mu R^{\rho \sigma} - 2R^{\rho \sigma \nu} \partial_\rho R - R_{\rho \sigma \nu} \nabla_\mu R^{\rho \sigma \nu} - 10R^{\rho \sigma \nu \mu} \nabla_\nu R_{\rho \sigma} ) - 3R \partial_\mu R \right]. \]  

(62)

The terms above are renormalization prescription dependent and are not present if, instead of the counterterms \( \sqrt{g} I_i \), one chooses scalars that are conformal invariant in d dimensions, i.e. the \( \tilde{f}_i^\mu \)’s defined in appendix A. Notice that the inclusion of \( d \)-dimensional counterterms simplifies considerably the computation of the dilaton WZ action, as shown in [26]. In fact, in this scheme, the contribution of the \( \tilde{f}_i^\mu \)’s to the same action is just linear in the dilaton field and can be derived from the counterterm

\[ \Gamma_{\text{C}}^d[g] = -\frac{\mu^{-\epsilon}}{\epsilon} \int d^d x \sqrt{g} \left( \sum_{i=1}^{3} c_i \tilde{f}_i^\mu + aE_6 \right). \]  

(63)
It can be explicitly checked that by expanding (63) around \( d = 6 \) and computing the order \( O(\epsilon) \) contribution to the vev of the traced EMT one obtains the relation
\[
\frac{2}{\sqrt{g}} g_{\mu\nu} \frac{\delta}{\delta g_{\mu\nu}} \int d^d x \sqrt{g} F_{d} = -\epsilon I_{d} + O(\epsilon^2).
\] (64)

In this simplified scheme it is possible to give the structure of the WZ action in any even dimension \([26]\), just by adding to the contribution of such invariants the one coming from the Euler density \( E_d \), being the total derivative terms \( \nabla_{\mu} I_{d}^{\mu} \) absent.

### 4.2. General scheme-dependence of the trace anomaly

In this section we establish a connection between the two schemes used to derive the dilaton WZ action, with the inclusion of invariant counterterms of \( B \) type which are either \( d \) or six-dimensional, in close analogy with the four-dimensional case \([20]\), that we now briefly review.

In this case one introduces the counterterm action \([5]\)
\[
\Gamma_{Ct}[g] = -\frac{\mu^{-\epsilon}}{\epsilon} \int d^d x \sqrt{g} (\beta_a F + \beta_b G), \quad \epsilon = 4 - d,
\] (65)
where \( \mu \) is a regularization scale. It is this form of \( \Gamma_{Ct} \) to induce the anomaly condition
\[
\frac{2}{\sqrt{g}} g_{\mu\nu} \frac{\delta \Gamma_{Ct}[g]}{\delta g_{\mu\nu}} \bigg|_{d \to 4} = \frac{2}{\sqrt{g}} g_{\mu\nu} \frac{\delta \Gamma_{Ct}[g]}{\delta g_{\mu\nu}} \bigg|_{d \to 4} = \mathcal{A}[g].
\] (66)
where \( G \) is the Euler density in four dimensions,
\[
G = R^\alpha\beta\gamma\delta R_{\alpha\beta\gamma\delta} - 4 R^\alpha\beta R_{\alpha\beta} + R^2,
\] (67)
whereas \( F \) is the squared Weyl tensor, which reads, for generic dimension,
\[
F_d \equiv C^\alpha\beta\gamma\delta C_{\alpha\beta\gamma\delta} = R^\alpha\beta\gamma\delta R_{\alpha\beta\gamma\delta} - \frac{4}{d-2} R^\alpha\beta R_{\alpha\beta} + \frac{2}{(d-1)(d-2)} R^2.
\] (68)
Its \( d = 4 \) realization, called simply \( F \), appears in the trace anomaly equation (3). From the well known relations
\[
\frac{2}{\sqrt{g}} g_{\mu\nu} \frac{\delta}{\delta g_{\mu\nu}} \int d^d x \sqrt{g} F = -\epsilon \left( F - \frac{2}{3} \Box R \right),
\] (69)
\[
\frac{2}{\sqrt{g}} g_{\mu\nu} \frac{\delta}{\delta g_{\mu\nu}} \int d^d x \sqrt{g} G = -\epsilon G,
\] (70)
it follows that the explicit form of the trace anomaly equation (66) is
\[
\langle T \rangle = \frac{2}{\sqrt{g}} g_{\mu\nu} \frac{\delta \Gamma_{Ct}[g]}{\delta g_{\mu\nu}} \bigg|_{d \to 4} = \beta_a \left( F - \frac{2}{3} \Box R \right) + \beta_b G.
\] (71)
The \( \Box R \) term in equation (69) is prescription dependent and can be avoided if the \( F \)-counterterm is chosen to be conformal invariant in \( d \) dimensions, i.e. using the square \( F_d \) of the Weyl tensor in \( d \) dimensions in (68),
\[
\Gamma_{Ct}^d[g] = -\frac{\mu^{-\epsilon}}{\epsilon} \int d^d x \sqrt{g} (\beta_a F_d + \beta_b G). \tag{72}
\]
In fact, expanding (72) around \( d = 4 \) and computing the \( O(\epsilon) \) contribution to the vev of the traced EMT it is found that
\[
\int d^d x \sqrt{g} F_d = \int d^d x \sqrt{g} \left[ F - \epsilon \left( R^\alpha\beta R_{\alpha\beta} - \frac{5}{18} R^2 \right) + O(\epsilon^2) \right], \tag{73}
\]
\[ \frac{2}{3} \Box R = \frac{2}{\sqrt{8}} g_{\mu\nu} \frac{\delta}{\delta g_{\mu\nu}} \int d^4 x \sqrt{g} \left( R^\alpha_{\mu\nu} R_{\alpha\beta} - \frac{5}{18} R^2 \right). \]  

(74)

These formulae, combined with (69), give
\[ \frac{2}{\sqrt{8}} g_{\mu\nu} \frac{\delta}{\delta g_{\mu\nu}} \int d^4 x \sqrt{g} F_{\alpha\beta} = -\epsilon F + O(\epsilon^2) \]  

(75)
in which the \( \Box R \) term is now absent.

For \( d = 6 \) we proceed in a similar way. We expand the \( d \)-dimensional counterterms around \( d = 6 \) to identify the finite contributions as
\[ I_1^d = I_1 + (d - 6) \frac{\partial I_1^d}{\partial d} \bigg|_{d=6} = I_1 - \epsilon \frac{\partial I_1^d}{\partial d} \bigg|_{d=6}. \]  

(76)

Using (76) in the \( d \)-dimensional counterterms, we have
\[ -\frac{1}{\epsilon} \int d^4 x \sqrt{g} I_1^d = -\frac{1}{\epsilon} \int d^4 x \sqrt{g} I_1 + \int d^4 x \sqrt{g} \frac{\partial I_1^d}{\partial d} \bigg|_{d=6} \]  

(77)

This implies, due to (59) and (64), that
\[ \frac{1}{\epsilon} \int d^4 x \sqrt{g} I_1^d = I_1 - \frac{2}{\sqrt{8}} g_{\mu\nu} \frac{\delta}{\delta g_{\mu\nu}} \int d^4 x \sqrt{g} \frac{\partial I_1^d}{\partial d} \bigg|_{d=6} \]  

(78)

and hence
\[ \frac{2}{\sqrt{8}} g_{\mu\nu} \frac{\delta}{\delta g_{\mu\nu}} \int d^4 x \sqrt{g} \frac{\partial I_1^d}{\partial d} \bigg|_{d=6} = -\nabla_{\mu} J_{\mu}^d. \]  

(79)

This clearly identifies the local counterterms that we can add to (56) in order to arbitrarily vary the coefficients \( c_i \) in (61). They are given by the derivatives of the \( d \)-dimensional terms \( I_1^d \) evaluated at \( d = 6 \), linearly combined with arbitrary coefficients \( c'_i \)
\[ \Gamma'_{\text{Cl}}[g] = -\frac{\mu^x}{\epsilon} \int d^4 x \sqrt{g} \left( \sum_{i=1}^{3} c_i I_1 + aE_6 \right) + \int d^4 x \sqrt{g} \sum_{i=1}^{3} c'_i \frac{\partial I_1^d}{\partial d} \bigg|_{d=6} \]  

(80)

which gives
\[ \langle T' \rangle = \frac{2}{\sqrt{8}} g_{\mu\nu} \frac{\delta \Gamma'_{\text{Cl}}[g]}{\delta g_{\mu\nu}} \bigg|_{d=4} = \sum_{i=1}^{3} c_i I_1 + aE_6 + \sum_{i=1}^{3} (c'_i - c_i) \nabla_{\mu} J_{\mu}^d. \]  

(81)

The choice \( c'_i = c_i \) in (80) then allows to move back to the alternative scheme in which the local anomaly contribution is not present. We list the three local counterterms of (80). They are given by
\[ \frac{\partial I_1^d}{\partial d} \bigg|_{d=6} = \frac{1}{16000} (-307 K_1 + 3465 K_2 - 540 K_3 - 3750 K_4 + 6000 K_5 + 3000 K_6), \]
\[ \frac{\partial I_2^d}{\partial d} \bigg|_{d=6} = \frac{1}{4000} (-167 K_1 + 1965 K_2 - 540 K_3 - 2750 K_4 + 3000 K_5 + 3000 K_6), \]
\[ \frac{\partial I_3^d}{\partial d} \bigg|_{d=6} = \frac{1}{500} (-18 K_1 + 140 K_2 - 90 K_3 - 70 K_9 + 500 K_{10} - 250 K_{11} + 25 K_{12} - 625 K_{13} + 750 K_{15}). \]  

(82)

Finally, in general one might also be interested to generate an anomaly functional in which the derivative terms appear in combinations that are different from those in the trace anomaly equation (5). For this goal, one should use proper linear combinations of the \( K_i \) according to the relations listed in (appendix A.2).
4.3. Gauging six-dimensional counterterms

At this point we follow the same approach as in [20] to Weyl-gauge the renormalized effective action.

We expand the gauged counterterms in a double power series with respect to $\epsilon = 6 - d$ and $\kappa_A \equiv 1/\Lambda$ around $(\epsilon, \kappa_A) = (0, 0)$. Denoting generically with $A$ either the Euler density $E_6$ or the three invariants $I_i$'s, the expansion then takes the form

$$
- \frac{1}{\epsilon} \int d^6 x \sqrt{g_A} = - \frac{1}{\epsilon} \int d^6 x \sum_{i,j=0}^{\infty} \frac{1}{n!} \epsilon^i (\kappa_A) j \frac{\partial^{i+j} (\sqrt{g_A})}{\partial \epsilon^i \partial \kappa_A^j},
$$

only the $O(\epsilon)$ contributions are significant, due to the $1/\epsilon$ factor in front of the counterterms. On the other hand, similarly to the case in four dimensions, the condition

$$
\frac{\partial^n (\sqrt{g_A})}{\partial \kappa_A^n} = O(\epsilon^2), \quad n \geq 7
$$

holds, as the Euler density and the three conformal invariants are at most cubic in the Riemann tensor and in its double covariant derivatives. Besides, there are no terms with more than two dilatons in the gauged Riemann tensor (see appendix A). All the terms which are of $O(1/\epsilon)$ in (83) and are different from $I_i$'s, or, respectively $E_6$, are found to vanish after some integrations by parts. Therefore, after gauging the counterterms we end up with the general result

$$
- \frac{\mu^{I_i}}{\epsilon} \int d^6 x \sqrt{g_A} = - \frac{\mu^{I_i}}{\epsilon} \int d^6 x \sqrt{g_A} + \Sigma_A + O(\epsilon).
$$

where each $\Sigma_A$ term is related to the corresponding specific invariant $A$. For instance, if $A = I_1$, then the corresponding $\Sigma$ term is $\Sigma_1$, and so on for each of the $I_i$’s, whereas for the contribution of the Euler density we have $A = E_6$ and $\Sigma_A = \Sigma_d$.

Their explicit expressions are

$$
\Sigma_1 = \int d^6 x \sqrt{g} \left\{ - \frac{\tau}{\Lambda} (I_1 + \nabla_\mu J^\mu) 
+ \frac{1}{\Lambda^2} \left[ \frac{3}{4} R^{\mu \nu \rho \sigma} R_{\mu \nu \rho \sigma} \partial_\mu \tau \partial_\nu \tau - \frac{3}{40} R^{\mu \nu \rho \sigma} R_{\mu \nu \rho \sigma} (\partial_\tau)^2 - \frac{3}{10} R (\nabla \partial_\tau)^2 
+ \frac{9}{4} R^{\mu \rho \tau \nu} R_{\mu \rho \tau \nu} \partial_\mu \tau \partial_\nu \tau - 3 R^{\mu \nu \rho \sigma} \nabla_\mu \partial_\nu \partial_\rho \partial_\sigma \tau - \frac{57}{800} R^2 (\partial_\tau)^2 
- \frac{21}{16} R^{\mu \nu \rho} \partial_\rho \partial_\mu \tau \partial_\nu \tau - \frac{9}{4} R^{\mu \nu \rho \tau} \nabla_\mu \partial_\nu \partial_\rho \partial_\tau \tau + \frac{57}{160} R^{\mu \nu \rho \sigma} R_{\mu \nu \rho \sigma} (\partial_\tau)^2 
+ \frac{3}{2} R^{\mu \nu \rho \tau} \nabla_\mu \partial_\nu \partial_\rho \partial_\tau \tau + \frac{57}{80} R^{\mu \nu \rho \sigma} \nabla_\mu \partial_\nu \partial_\rho \partial_\sigma \tau 
+ \frac{1}{\Lambda^3} \left[ - \frac{7}{16} (\nabla_\tau)^3 + \frac{3}{2} (\nabla \partial_\tau)^2 \nabla_\tau - 6 R^{\mu \nu \rho \sigma} \partial_\mu \tau \partial_\nu \partial_\rho \partial_\sigma \tau 
+ 3 R^{\mu \nu \rho} \nabla_\mu \partial_\nu \tau \partial_\rho \partial_\tau \tau - \frac{9}{4} R^{\mu \nu \rho \sigma} \nabla_\mu \partial_\nu \partial_\rho \partial_\sigma \tau - \frac{3}{5} R \partial_\rho \partial_\mu \partial_\nu \partial_\sigma \tau 
+ \frac{3}{2} R^{\mu \nu \rho \sigma} \nabla_\mu \partial_\nu \partial_\rho \partial_\sigma \tau - \frac{3}{8} R^{\mu \nu \rho \sigma} (\nabla_\tau)^2 + \frac{3}{4} R (\nabla \partial_\tau)^2 \nabla_\tau - \frac{3}{20} R (\partial_\tau)^2 \right] \right\},
$$

for $I_1$. 

15
\[
\Sigma_2 = \int d^6x \sqrt{g} \left\{ -\frac{\tau}{\Lambda} \left( I_2 + \nabla_{\mu} J^\mu \right) + \frac{1}{\Lambda^2} \left[ 3R^{\mu\rho\sigma\alpha} R_{\rho\sigma\alpha\tau} \partial_{\tau} \partial_\tau - \frac{27}{40} R(\Box \tau)^2 - \frac{6}{5} R(\nabla \partial \tau)^2 - \frac{27}{200} R^2(\partial \tau)^2 \right] + \frac{1}{\Lambda^4} \left[ 3R^{\mu\rho\sigma\alpha} R_{\rho\sigma\alpha\tau} \partial_{\tau} \partial_\tau + 3R^{\mu\rho\sigma\alpha} R_{\rho\sigma\alpha\tau} \partial_{\tau} \partial_\tau - \frac{15}{4} R^{\mu\rho} R_{\rho\tau} \partial_{\tau} \partial_\tau \right] - 3R^{\mu\rho} \nabla_\rho \partial_\mu \partial_\tau \partial_\tau + \frac{27}{40} R^{\mu\nu} R_{\mu\nu} \partial_{\tau} \partial_\tau + 6R^{\mu\nu} \nabla_\rho \partial_\mu \partial_\tau \partial_\tau + \frac{27}{200} R^{\mu\nu} \partial_\mu \partial_\tau \partial_\tau \right] \right. \\
+ \frac{1}{\Lambda^3} \left[ \frac{11}{4} (\Box \tau)^2 - 6(\nabla \partial \tau)^2 - 8R^{\mu\nu} \partial_\mu \partial_\tau \partial_\nu \partial_\tau + 6R^{\mu\nu} \nabla_\nu \partial_\mu \tau \partial_\tau \right] \\
+ \frac{1}{\Lambda^5} \left[ 6(\partial \tau)^2 \nabla^2 (\partial \tau)^2 - \frac{9}{2} (\partial \tau)^2 (\Box \tau)^2 - 3\partial^2(\partial \tau)^2 \right] - \frac{3}{5} R(\partial \tau)^2 \right. \\
+ \left. \frac{1}{\Lambda^6} \left[ 6(\partial \tau)^2 \nabla^2 (\partial \tau)^2 - \frac{4}{5} R(\partial \tau)^4 \right] \right\}, \tag{87} \]

for the second invariant \( I_2 \) and

\[
\Sigma_3 = \int d^6x \sqrt{g} \left\{ -\frac{\tau}{\Lambda} \left( I_3 + \nabla_{\mu} J^\mu \right) + \frac{1}{\Lambda^2} \left[ -\frac{3}{25} R^2(\partial \tau)^2 + \frac{13}{10} R^{\mu\nu} R_{\mu\nu} \partial_{\tau} \partial_\tau - \frac{2}{5} R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} \partial_{\tau} \partial_\tau \right] + \frac{9}{10} R(\Box \tau)^2 - \frac{3}{10} R^2(\partial \tau)^2 - \frac{12}{5} R(\nabla \partial \tau)^2 - 5R^{\mu\rho} R^{\nu\sigma} \partial_\mu \partial_\nu \partial_\sigma \partial_\tau + 7R^{\mu\nu} \nabla_\nu \partial_\mu \partial_\tau \partial_\tau - 9R^{\mu\nu} \partial_\rho \partial_\nu \partial_\mu \partial_\rho \partial_\tau + \frac{2}{5} R^{\mu\nu} R_{\mu\nu} \partial_{\tau} \partial_\tau \right. \\
+ \frac{1}{\Lambda^3} \left[ 2(\Box \tau)^2 - 8(\nabla \partial \tau)^2 \Box \tau - \frac{16}{5} R^2(\partial \tau)^2 \nabla_\mu \partial_\mu \partial_\tau + 8R^{\mu\nu} \partial_\mu \partial_\nu \partial_\tau \partial_\tau \right. \\
+ \frac{1}{\Lambda^4} \left[ 4(\partial \tau)^2 (\Box \tau)^2 - 4\partial^2(\partial \tau)^2 \nabla_\mu \partial_\mu \partial_\tau + 16(\partial \tau)^2 (\nabla \partial \tau)^2 - \frac{4}{5} R(\partial \tau)^4 \right] \right. \right\}, \tag{88} \]

for the third invariant \( I_3 \), while the contribution from the integrated Euler density is

\[
\Sigma_\mu = \int d^6x \sqrt{g} \left\{ -\frac{\tau}{\Lambda} E_\mu + \frac{1}{\Lambda^2} \left[ 12R^{\mu\rho\sigma\alpha} R_{\rho\sigma\alpha\tau} \partial_{\tau} \partial_\tau \partial_\mu \tau \partial_\tau - 3R^{\mu\nu\rho\sigma} R_{\nu\rho\sigma\tau} \partial_{\tau} \partial_\tau \partial_\mu \partial_\tau \partial_\mu \partial_\tau \right] \\
+ \frac{1}{\Lambda^3} \left[ 12R^{\mu\rho\sigma\alpha} R_{\rho\sigma\alpha\tau} \partial_{\tau} \partial_\tau \partial_\mu \partial_\tau \partial_\mu \partial_\tau - 3R^{\mu\nu\rho\sigma} R_{\nu\rho\sigma\tau} \partial_{\tau} \partial_\tau \partial_\mu \partial_\tau \partial_\mu \partial_\tau \right] - \frac{12}{\Lambda^4} \left[ 16R^{\mu\nu\rho\sigma} \partial_{\mu} \partial_{\tau} \partial_{\tau} \partial_{\nu} \partial_{\sigma} \partial_{\rho} \partial_{\tau} - 16R^{\mu\nu} \nabla_\mu \partial_{\tau} \partial_{\tau} \partial_{\nu} \partial_{\sigma} \partial_{\rho} \partial_{\tau} \right] + \frac{1}{\Lambda^5} \left[ 2(\partial \tau)^2 (\nabla \partial \tau)^2 - 24(\partial \tau)^2 (\Box \tau)^2 - 6R(\partial \tau)^4 \right] + \frac{36}{\Lambda^6} (\partial \tau)^4 \right\}, \tag{89} \]
The derivation of (86)–(89) is very involved and we have used several integration by parts to simplify the previous expressions. The WZ effective action is then obtained from (45) and, in a general gravitational background, it is just given by the combination of (86)–(89) with the proper coefficients, up to a minus sign, i.e.

\[
\Gamma_{\text{WZ}}[g, \tau] = -\left( \sum_{i=1}^{3} c_i \Sigma_i + a \Sigma_a \right). \tag{90}
\]

Before presenting the expression of the dilaton WZ action, we pause for a comment. It is clear from our analysis that the form of this action is not unique, due to the renormalization scheme dependence of the counterterms which are chosen before performing the Weyl gauging. As we have seen in section 4.2, this ambiguity manifests in the coefficients \(c_i\) which parameterize the local terms of the anomaly, proportional to the derivatives of the currents \(J^\mu_i\). Obviously, it is preferable to be able to characterize this ambiguity in a more complete way, and the analysis of the relation between counterterms in six and in \(d\) dimensions serves this purpose. In fact this allows to identify the functionals whose variation generates the local anomaly terms. By proceeding in this way one is able to identify a three-parameter class of renormalization schemes, related to the coefficients \(c_i\), which become free parameters in the anomaly action. Notice also that the anomaly which is reproduced in ordinary perturbation theory in the Feynman expansion—such as in the dimensional regularization scheme with modified minimal subtraction \(\overline{MS}\)—coincides with the functional form given in (5). We refer to [38] for an illustration of this point in \(d = 4\).

For this purpose we exploit the relations

\[
-\frac{\mu^{d-4}}{\epsilon} \int d^d x \sqrt{g} \frac{\partial I_i}{\partial d} = -\frac{\mu^{d-4}}{\epsilon} \int d^d x \sqrt{g} \frac{\partial I_i}{\partial d} \bigg|_{d=6} = -\frac{\mu^{d-4}}{\epsilon} \int d^d x \sqrt{g} \frac{\partial I_i}{\partial d} \bigg|_{d=6} = -\frac{\mu^{d-4}}{\epsilon} \int d^6 x \sqrt{g} \frac{\partial I_i}{\partial d} - \int d^6 x \sqrt{g} \frac{\partial I_i}{\partial d} = -\int d^6 x \sqrt{g} \frac{\partial I_i}{\partial d} - \int d^6 x \sqrt{g} \frac{\partial I_i}{\partial d} = -\int d^6 x \sqrt{g} \frac{\partial I_i}{\Lambda} \bigg|_{d=6}.
\tag{91}
\]

where the last line follows from the transformations properties of \(I_i\) in \(d\) dimensions under Weyl scaling. From (85) and (91) we infer that

\[
\int d^6 x \sqrt{g} \frac{\partial I_i}{\partial d} - \int d^6 x \sqrt{g} \frac{\partial I_i}{\partial d} = -\int d^6 x \sqrt{g} \frac{\partial I_i}{\partial d} - \int d^6 x \sqrt{g} \frac{\partial I_i}{\partial d} = -\int d^6 x \sqrt{g} \frac{\partial I_i}{\Lambda} \bigg|_{d=6}.
\tag{92}
\]

Then can immediately use (92) to infer that the WZ action corresponding to the modified effective action (80) is given by

\[
\Gamma_{\text{WZ}}[g, \tau] = -\left( \sum_{i=1}^{3} (c_i - c_i') \Sigma_i + a \Sigma_a + \sum_{i=1}^{3} c_i' \int d^6 x \sqrt{g} \frac{\tau}{\Lambda} I_i \right).
\tag{93}
\]

In particular, it is clear that, choosing the counterterms with \(I_i \rightarrow I_i'\), as in equation (63), we get just the so-called universal terms, as done in [26] for general even dimensions

\[
\Gamma_{\text{WZ}}[g, \tau] = -\left( \sum_{i=1}^{3} c_i \int d^6 x \sqrt{g} \frac{\tau}{\Lambda} I_i + a \Sigma_a \right).
\tag{94}
\]

In the flat space limit \((g_{\mu\nu} \rightarrow \delta_{\mu\nu})\) there are obvious simplifications and (90) takes the form

\[
\Gamma_{\text{WZ}}[\delta, \tau] = -\int d^6 x \sqrt{g} \left\{ -\frac{c_3}{\Lambda^2} \Box \tau \frac{\Box \tau}{\Lambda} + \frac{1}{\Lambda^3} \left[ \left( -\frac{7}{16} c_1 + \frac{11}{4} c_2 + 2 c_3 \right) \left( \Box \tau \right)^3 + \left( \frac{3}{2} c_1 - 6 c_2 - 8 c_3 \right) (\partial \tau)^2 \Box \tau \right] + \frac{1}{\Lambda^3} \left[ \left( -\frac{3}{2} c_1 + 6 c_2 + 16 c_3 + 24 a \right) \right] \right\}.
\]
\[
\begin{align*}
\times (\partial \tau)^2 (\partial \partial \tau)^2 - \left( \frac{3}{8} c_1 + \frac{9}{2} c_2 + 4 c_3 + 24 a \right) (\partial \tau)^2 (\partial \partial \tau)^2 \\
+ \left( \frac{3}{4} c_1 - 3 c_4 + 4 c_3 \right) \partial \mu (\partial \tau)^2 \partial \nu (\partial \partial \tau)^2 \left\{ \frac{1}{\Lambda^3} \left( \frac{3}{2} c_1 + 6 c_2 + 36 a \right) (\partial \tau)^4 \square \tau \\
- \frac{1}{\Lambda^6} (c_1 + 4 c_2 + 24 a) (\partial \partial \tau)^4 \right\}.
\end{align*}
\]

(95)

The structures of the flat space limit of (93) and (94) follow trivially. Having obtained the most general form for the WZ action for conformal anomalies in six dimensions, we now turn to discuss one specific example in \( d = 6 \), previously studied within the AdS/CFT correspondence. This provides an application of the results of the previous sections.

### 4.4. The WZ action action for a free CFT: the (2,0) tensor multiplet

In this section we are going to determine the coefficients of the WZ action for the (2,0) tensor multiplet in \( d = 6 \), which has been investigated in the past in the context of the AdS\(_7\)/CFT\(_6\) holographic anomaly matching.

Free field realizations of CFT’s are particularly useful in the analysis of the anomalies and their matching between theories in regimes of strong and weak coupling, allowing to relate free and interacting theories of these types. In this respect, the analysis of correlation functions which can be uniquely fixed by the symmetry is crucial in order to compute the anomaly for theories characterized by different field contents in general spacetime dimensions. This is the preliminary step in order to investigate the matching with other realizations which share the same anomaly content. These are correlation functions which contain up to three EMT’s and that can be determined uniquely, in any dimensions, modulo a set of coefficients, such as the number of fermions, scalars and/or spin 1, which can be fixed within a specific field theory realization [36, 37].

While in \( d = 4 \) these correlation functions can be completely identified by considering a generic theory which combines free scalar, fermions and gauge fields [36–38], in \( d \) dimensions scalars and fermions need to be accompanied not by a spin 1 (a one-form) but by a \( \kappa \)-form (\( d = 2 \kappa + 2 \)). In \( d = 6 \) this is a 2-form, \( B_{\mu \nu} \) [25].

Coming to specific realizations and use of CFT’s in \( d = 6 \), we mention that, for instance, the dynamics of a single M5 brane is described by a free \( N = (2,0) \) tensor multiplet which contains five scalars, two Weyl fermions and a two-form whose strength is anti-selfdual. For \( N \) coincident M5 branes, at large \( N \) values, the anomaly matching between the free field theories realizations and the interacting \( (2,0) \) CFT’s, investigated in the AdS\(_7\)\( \times \)S\(_4\) supergravity description, has served as an interesting test of the correspondence between the \( A \) and \( B \) parts of the anomalies in both theories [24, 25, 41].

We have summarized in table 1 the coefficients of the WZ anomaly action in the case of a scalar, a fermion and a non-chiral \( B_{\mu \nu} \) form, which are the fields appearing in the (2,2)CFT. Anomalies in the (2,0) and the (2,2) theories are related just by a factor \( 1/2 \), after neglecting the gravitational anomalies related to the imaginary parts of the (2,0) multiplet [25].

We have extracted the anomaly coefficients in table 1 from [25], having performed a redefinition of the third invariant \( I_3 \) in the structure of the anomaly functional (5). We choose to denote with \( \tilde{I}_1, \tilde{I}_2 \) and \( \tilde{I}_3 \) the anomaly operators and coefficients in [25]

\[
\tilde{I}_1 = I_1, \quad \tilde{I}_2 = I_2, \quad \tilde{I}_3 = 3 I_3 + 8 I_1 - 2 I_2.
\]

(96)

Actually in [25] the third conformal invariant, whose implicit expression can be found in [39], is given by

\[
\tilde{I}_3 \equiv C^\gamma_{\gamma \rho \sigma} \left( \delta_\alpha^\gamma \square - 4 R_\alpha^\gamma + \frac{6}{5} \delta_\alpha^\gamma R \right) C_{\rho \gamma \rho \sigma} + \left( 8 \delta_\alpha^\gamma \delta_\beta^\lambda - \frac{1}{2} g_{\alpha \beta} g^{\lambda \rho} \right) \nabla_\gamma \nabla_\lambda C^\gamma_{\gamma \rho \sigma} C^\rho_{\gamma \rho \sigma}.
\]

(97)
Table 1. Anomaly coefficients for a conformally coupled scalar (S), a Dirac fermion (F), a two-form field (B) and the chiral \((2, 0)\) tensor multiplet (T), to be normalized by an overall \(1/(7!(4\pi)^3)\).

| \(I\) | \(c_1 \times 7!(4\pi)^3\) | \(c_2 \times 7!(4\pi)^3\) | \(c_3 \times 7!(4\pi)^3\) | \(a \times 7!(4\pi)^3\) |
|------|-----------------|-----------------|-----------------|-----------------|
| \(S\) | \(20/7\) | \(-2/3\) | 6 | \(-5\) |
| \(F\) | \(64/7\) | \(-112\) | 120 | \(-191\) |
| \(B\) | \(-3688/7\) | \(-3458\) | 540 | \(-221\) |
| \(T\) | \(-560\) | \(-700\) | 420 | \(-245\) |

which differs from our choice, reported in appendix A. The relation in (96) between the third invariant \(\tilde{I}_3\) and \(I_3\) can be derived expanding (97) on the basis of the \(K\)-scalars given in section 2.1 and comparing it to the third of (8).

In light of (96), as the conformal anomalies depend only on the field content of the theory, i.e.

\[
\mathcal{A}[g] = \sum_{i=1}^{3} c_i (\tilde{I}_i + \nabla_\mu J^\mu_i) = \sum_{i=1}^{3} \tilde{c}_i (\tilde{I}_i + \nabla_\mu \tilde{J}^\mu_i),
\] (98)

by replacing (96) on the r.h.s. of (98), we conclude that the relations between the anomaly coefficients \(c_i\) and \(\tilde{c}_i\) are

\[
c_1 = \tilde{c}_1 + 8 \tilde{c}_3, \quad c_2 = \tilde{c}_2 - 2 \tilde{c}_3, \quad c_3 = 3 \tilde{c}_3.
\] (99)

The WZ action can be derived from equation (90) by inserting the expressions of the \(c_i\)'s and \(a\) extracted from table 1. These can be specialized to the scalar (S), fermion (F) and to the two-form (B) cases, thereby generating via (98) the corresponding anomaly functionals. For the \((2, 0)\) tensor multiplet this is obtained from the relation

\[
\mathcal{A}^T[g] = \frac{1}{2} (10\mathcal{A}^S[g] + 2\mathcal{A}^F[g] + \mathcal{A}^B[g]).
\] (100)

5. Dilaton interactions and constraints from \(\Gamma_{\text{WZ}}\)

Having extracted the structure of the WZ action and thus of the anomaly-related dilaton interactions via the Weyl gauging of the effective action, we now follow a perturbative approach in the inverse conformal scale \(\kappa/\Lambda\). We proceed with a Taylor expansion of the gauged metric, which is given by

\[
\hat{g}_{\mu\nu} = g_{\mu\nu} e^{-2\kappa/\Lambda} = (\delta_{\mu\nu} + \kappa h_{\mu\nu}) e^{-2\kappa/\Lambda} = (\delta_{\mu\nu} + \kappa h_{\mu\nu}) \sum_{n=0}^{\infty} \frac{(-2)^n}{n!} (\kappa/\Lambda)^n,
\] (101)

where \(\kappa\) is the gravitational coupling constant in six dimensions and we are using Euclidean conventions. As we are considering only the dilaton contributions, we focus on the functional expansion of the renormalized and Weyl-gauged effective action \(\hat{\Gamma}_{\text{ren}}[g, \tau]\) with respect to \(\kappa/\Lambda\).

This is easily done using the relation

\[
\frac{\partial \hat{\Gamma}_{\text{ren}}[g, \tau]}{\partial \kappa/\Lambda} = \int d^6x \frac{\delta \hat{\Gamma}_{\text{ren}}[g, \tau]}{\delta \hat{g}_{\mu\nu}(x)} \frac{\partial \hat{g}_{\mu\nu}(x)}{\partial \kappa/\Lambda}.
\] (102)

Applying (102) repeatedly and taking (101) into account, the perturbative series takes the form
\[ \hat{\Gamma}_{\text{ren}}[g, \tau] = \Gamma_{\text{ren}}[g, \tau] + \frac{1}{2!\Lambda^2} \int d^d x_1 d^d x_2 \frac{\delta^2 \hat{\Gamma}_{\text{ren}}[g, \tau]}{\delta \hat{g}_{\mu_1 \nu_1}(x_1) \delta \hat{g}_{\mu_2 \nu_2}(x_2)} \frac{\partial \hat{g}_{\mu_1 \nu_1}(x_1)}{\partial \kappa_\Lambda} \frac{\partial \hat{g}_{\mu_2 \nu_2}(x_2)}{\partial \kappa_\Lambda} \\
+ \frac{1}{3!\Lambda^3} \left( \int d^d x_1 d^d x_2 d^d x_3 \frac{\delta^3 \hat{\Gamma}_{\text{ren}}[g, \tau]}{\delta \hat{g}_{\mu_1 \nu_1}(x_1) \delta \hat{g}_{\mu_2 \nu_2}(x_2) \delta \hat{g}_{\mu_3 \nu_3}(x_3)} \frac{\partial \hat{g}_{\mu_1 \nu_1}(x_1)}{\partial \kappa_\Lambda} \right) \times \frac{\partial \hat{g}_{\mu_2 \nu_2}(x_2)}{\partial \kappa_\Lambda} + 3 \int d^d x_1 d^d x_2 \frac{\delta^2 \hat{\Gamma}_{\text{ren}}[g, \tau]}{\delta \hat{g}_{\mu_1 \nu_1}(x_1) \delta \hat{g}_{\mu_2 \nu_2}(x_2)} \frac{\partial^2 \hat{g}_{\mu_1 \nu_1}(x_1)}{\partial \kappa_\Lambda^2} \\
\times \frac{\partial \hat{g}_{\mu_2 \nu_2}(x_2)}{\partial \kappa_\Lambda} + \cdots \right) . \tag{103} \]

As we are interested in the flat space limit of the dilaton action, we write (103) by taking the limit of a conformally flat background metric \((\hat{g}_{\mu \nu} \rightarrow \hat{\delta}_{\mu \nu} \equiv \delta_{\mu \nu} e^{-2\kappa_\Lambda x^\mu x^\nu})\) obtaining

\[ \hat{\Gamma}_{\text{ren}}[\delta, \tau] = \Gamma_{\text{ren}}[\delta, \tau] + \frac{1}{2!\Lambda^2} \int d^d x_1 d^d x_2 (T(x_1) T(x_2)) \tau(x_1) \tau(x_2) \\
- \frac{1}{3!\Lambda^3} \left[ \int d^d x_1 d^d x_2 d^d x_3 (T(x_1) T(x_2) T(x_3)) \tau(x_1) \tau(x_2) \tau(x_3) \right] \\
+ 6 \int d^d x_1 d^d x_2 (T(x_1) T(x_2)) (\tau(x_1))^2 \tau(x_2) \right] + \cdots , \tag{104} \]

where we have used equation (10) in the definition of the correlators of the EMT’s and the obvious relation

\[ \left. \frac{\partial^n \hat{g}_{\mu \nu}(x)}{\partial \kappa_\Lambda^n} \right|_{\kappa_\Lambda=\delta_{\mu \nu}, \kappa_\Lambda=0} = (-)^n (\tau(x))^n \delta_{\mu \nu}. \tag{105} \]

From (104) one may identify the expression of the flat limit of the WZ action \(\Gamma_{\text{WZ}} = \Gamma_{\text{ren}}[\delta, \tau] - \hat{\Gamma}_{\text{ren}}[\delta, \tau]\) written in terms of the traced \(n\)-point correlators of EMT’s. This expression has to coincide with equation (95), and by comparing the dilaton vertices extracted from (104) and (95) one can easily obtain some consistency conditions between the two forms of the vertices. In particular, in any even \(d\) dimensions the first \(d\) correlators follow rather directly from the expressions of the first \(d\) dilaton interactions. These are the only non vanishing ones. At the same time, any correlator of rank-\(n\) with \(n > d\) can be evaluated by requiring that all the vertices with more than \(d\) dilatons vanish identically, thus allowing to extract recursively all the EMT’s Green functions of the corresponding rank.

We denote with \(\mathcal{Z}_n(x_1, \ldots, x_n)\) the dilaton vertices obtained by the functional differentiations of the WZ action, expressed in the coordinate space

\[ \mathcal{Z}_n(x_1, \ldots, x_n) = \left. \frac{\delta^n (\hat{\Gamma}_{\text{ren}}[\delta, \tau] - \Gamma_{\text{ren}}[\delta, \tau])}{\delta \tau(x_1) \ldots \delta \tau(x_n)} \right|_{\delta \tau(x_1) = \ldots = \delta \tau(x_n)} = - \frac{\delta^n \mathcal{Z}_{\text{WZ}}[\delta, \tau]}{\delta \tau(x_1) \ldots \delta \tau(x_n)} \tag{106} \]

which can be promptly transformed to momentum space. The expressions of such vertices up to the sixth order in \(\kappa_\Lambda\) in momentum space are given by

\[ \mathcal{I}_2(k_1, -k_1) = \kappa_\Lambda^2 (T(k_1) T(-k_1)) , \]

\[ \mathcal{I}_3(k_1, k_2, k_3) = -\kappa_\Lambda^3 \left[ \langle T(k_1) T(k_2) T(k_3) \rangle + 2 \sum_{i=1}^{3} \langle T(k_i) T(-k_1) \rangle \right] , \]

\[ \mathcal{I}_4(k_1, k_2, k_3, k_4) = \kappa_\Lambda^4 \left[ \langle T(k_1) T(k_2) T(k_3) T(k_4) \rangle + 2 \sum_{\tau \{4, (k_1, k_2)\}} \langle T(k_i + k_j) T(k_{i+j}) \rangle \right] \\
+ 4 \left( \frac{1}{2} \sum_{\tau \{4, (k_1, k_2)\}} \langle T(k_i + k_j + k_k) \rangle + \sum_{i=1}^{4} \langle T(k_i) T(-k_i) \rangle \right) . \]

20
\[ \mathcal{I}_5(k_1, k_2, k_3, k_4, k_5) = -\kappa_\Lambda^5 \left[ \langle T(k_1) T(k_2) T(k_3) T(k_4) T(k_5) \rangle 
+ 2 \sum_{T[5,(k_i,k_j)]} \langle T(k_i + k_j) T(k_{i+1}) T(k_{i+1}) T(k_{i+1}) \rangle 
+ 4 \left( \sum_{T[5,(k_i,k_j,k_n)]} \langle T(k_{i+1} + k_{i+2} + k_{i+1}) T(k_{i+2}) T(k_{i+2}) \rangle
+ \frac{1}{2} \sum_{T[6,(k_i,k_j,k_n,k_m)]} \langle T(k_{i+1} + k_{i+2} + k_{i+2}) T((-k_{i+1} - k_{i+2} - k_{i+2})) \rangle \right) 
+ 8 \left( \sum_{T[5,(k_i,k_j,k_n)]} \langle T(k_i + k_j) T(-k_{i+2} + k_{i+2}) \rangle 
+ \sum_{T[6,(k_i,k_j,k_n,k_m)]} \langle T(k_i + k_j) T(k_{i+2} + k_{i+2}) \rangle \right) \right]. \] (107)

These results can be easily extended to any higher order. The recipe, in this respect, is quite simple. To construct the vertex at order \( n \) one has to sum to the \( n \)-point function all the lower order functions in the hierarchy, down to \( n = 2 \), partitioning the momenta in all the possible ways and symmetrising each single contribution. The normalization factor in front of the correlator of order-\( k \) is always \( 2^{n-k} \), while the factor in front of the vertex of order \( n \) is \( (-\kappa_\Lambda)^n \).

Notice that, for \( n \) even, we have an additional \( 1/2 \) factor in front of the contributions from the two-point functions in which each EMT carries \( n/2 \) momenta, to avoid double counting. All the expressions in (107) have been thoroughly checked in two dimensions, as illustrated in appendix B, being their expression valid for any dimension.
We briefly recall the meaning of the notation used in (107) to organize the momenta. The symbol $\mathcal{T} \{n, \ldots \}$ is used to denote groups of momenta in the $n$-point function. For example $\mathcal{T} \{4, (k_i, k_i)\}$ denotes, in the four point functions, the six possible pairs of distinct momenta $\mathcal{T} \{4, (k_i, k_i)\} = \{(k_1, k_2), (k_1, k_3), (k_2, k_3), (k_1, k_4), (k_2, k_4), (k_3, k_4)\}$, (108)

where we are combining the four momenta $k_1, \ldots, k_4$ into all the possible pairs, for a total of $\frac{4^4}{4!}$ terms. Moving to higher orders, the description of the momentum dependence is more complicated and we need to distribute the external momenta into several groups. For instance, the notation $\mathcal{T} \{5, [(k_i, k_i), (k_i, k_i)]\}$ denotes the set of independent paired couples which can be generated out of five momenta. Their number is 15 and they are given by

$$\mathcal{T} \{5, [(k_i, k_i), (k_i, k_i)]\} =$ [\{(k_1, k_2), (k_3, k_4), (k_1, k_5), (k_2, k_3), (k_1, k_4), (k_2, k_5), (k_1, k_3), (k_2, k_4), (k_2, k_5), (k_3, k_4), (k_3, k_5), (k_4, k_5)\}

(109)

At this point we move on to the evaluation of dilaton interactions and, consequently, of the first six traced correlators, being clear from (107) that a direct computation of $\mathcal{I}_2 - \mathcal{I}_3$ from the anomaly action (95) allows to extract the structure of these Green functions. Thus the dilaton interactions are

$$\mathcal{I}_2(k_1, -k_1) = \frac{2}{\Lambda^2} c_i k_i^6,$$

$$\mathcal{I}_3(k_1, k_2, k_3) = \frac{1}{\Lambda^4} \left[ \left( \frac{21}{8} c_1 - \frac{33}{2} c_2 - 12 c_3 \right) k_1^2 k_2^2 k_3^2 + \left( -3c_1 + 12c_2 + 16c_3 \right) (k_1^2 (k_2 \cdot k_3)^2 + k_2^2 (k_1 \cdot k_3)^2 + k_3^2 (k_1 \cdot k_2)^2) \right],$$

$$\mathcal{I}_4(k_1, k_2, k_3, k_4) = \frac{1}{\Lambda^4} \left[ \left( 6 c_1 - 24c_2 - 64c_3 - 96a \right) \sum_{(4,[k_i,k_i])} k_i \cdot k_j (k_i \cdot k_j)^2 \right.$$

$$\left. + \left( \frac{3}{2} c_1 + 18c_2 + 16c_3 + 96a \right) \sum_{(4,[k_i,k_i])} k_i \cdot k_j k_i^2 k_j^2 \right.$$

$$\left. + \left( -6c_1 + 24c_2 + 32c_3 \right) \sum_{(4,[k_i,k_i],[k_i,k_i])} (k_i + k_j) \cdot (k_i + k_j) k_i \cdot k_j k_i \cdot k_j \right],$$

$$\mathcal{I}_5(k_1, k_2, k_3, k_4, k_5) = -\frac{12}{\Lambda^6} (c_1 + 4c_2 + 24a)$$

$$\times \sum_{(5,[k_i,k_i],[k_i,k_i],[k_i,k_i],[k_i,k_i])} k_1^2 (k_i \cdot k_j k_i \cdot k_i + k_i \cdot k_j k_i \cdot k_i + k_i \cdot k_j k_i \cdot k_i),$$

$$\mathcal{I}_6(k_1, k_2, k_3, k_4, k_5, k_6) = \frac{48}{\Lambda^6} (c_1 + 4c_2 + 24a) \sum_{(6,[k_i,k_i],[k_i,k_i],[k_i,k_i],[k_i,k_i],[k_i,k_i],[k_i,k_i])} k_i \cdot k_j k_i \cdot k_j k_i \cdot k_i,$$

(110)

(with $k_i^m \equiv (k_i^2)^{m/2}$).

These vertices can be used together with the relations (107) in order to extract the structure of the traced correlators. We find that the first two of them are given by
\begin{align*}
\langle T(k_1)T(-k_1) \rangle &= 2c_3 k_1^6, \\
\langle T(k_1)T(k_2)T(k_3) \rangle &= (3c_1 - 12c_2 - 16c_3)(k_1^2(k_2 \cdot k_3)^2 + k_2^2(k_3 \cdot k_1)^2 + k_3^2(k_1 \cdot k_2)^2) \\
&\quad - \left( \frac{3}{2} c_1 - \frac{21}{2} c_2 - 12 c_3 \right) k_2^2 k_3^2 - 4c_3 (k_1^6 + k_2^6 + k_3^6). \quad (111)
\end{align*}

The structure of the four point Green function is much more complicated and is summarized in the expression
\begin{align*}
\langle T(k_1)T(k_2)T(k_3)T(k_4) \rangle &= \left[ 4c_3 (7 f^{2i,2i} + 6 f^{2i,2j} + 3 f^{2i,2j} + 12 f^{2j,2j} + 12 f^{2j,2j} \\
&\quad + 8 f^{j,j,j,j}) + (-18c_1 + 72c_2 + 96c_3)f^{2i,ik,jk} + 4(24a + 3c_1 - 12c_2 - 8c_3) \\
&\quad f^{2i,kl} - 6(16a + c_1 - 4c_2)f^{i,kl,kl} + \left( \frac{23}{2} c_1 - 99 c_2 - 72 c_3 \right) f^{2i,2j,2k} \\
&\quad + (-6c_1 + 24c_2 + 32c_3)(2f^{2i,ik,jl} + f^{i,jk,il}) \right]. \quad (112)
\end{align*}

Here we have introduced a compact notation for the basis of the 12 scalar functions \( f \cdot (k_1, k_2, k_3, k_4) \) on which the correlator is expanded, leaving their dependence on the momenta implicit not to make the formula clumsy. As each term in the Green function is necessarily made of three scalar products of momenta, the role of the tree superscripts on each of the \( f \) scalars is to specify the way in which the momenta are distributed. We present below the expressions of the first four scalar \( f \)'s, from which it should be clear how to derive the explicit forms of all the others. We obtain
\begin{align*}
f^{2i,2i}(k_1, k_2, k_3, k_4) &= \sum_{i=1}^{4} (k_i)^6, \\
f^{2i,2i,j}(k_1, k_2, k_3, k_4) &= \sum_{i=1}^{4} (k_i)^4 \sum_{j \neq i} k_i \cdot k_j, \quad (113) \\
f^{2i,2j,i}(k_1, k_2, k_3, k_4) &= \sum_{i=1}^{4} (k_i)^2 \sum_{j \neq i} k_i^2, \\
f^{2i,j,j,j}(k_1, k_2, k_3, k_4) &= \sum_{i=1}^{4} (k_i)^2 \sum_{j \neq i} (k_i \cdot k_j)^2. \quad (114)
\end{align*}

Notice that each \( f \)-scalar is completely symmetric with respect to any permutation of the momenta, as for the whole correlator. The structure of the five and six-point functions is similar to (111), although they require broader bases of scalar functions to account for all their terms and we do not report them explicitly.

It is clear that the hierarchy in equation (13) can be entirely reexpressed in terms of the first six traced correlators. In fact, one notices that \( \Gamma_{WZ}[\hat{A}] \) is at most of order 6 in \( r \), with
\begin{align*}
I_n(x_1, \ldots, x_n) = 0, \quad n \geq 7. \quad (115)
\end{align*}

Therefore, for instance, the absence of vertices with seven dilaton external lines, which sets \( I_7 = 0 \), combined with the first six fundamental Green functions, are sufficient to completely fix the structure of the seven-point function, and so forth for the vertices of higher orders. In this way one can determine all the others recursively, up to the desired order. The consistency of these relations could be checked, in principle, by a direct comparison with their expression obtained directly from the hierarchy (13). This requires the explicit computation of functional derivatives of the anomaly functional \( A \) up to the relevant order, which is a much more time-consuming task.
6. Conclusions

We have presented a derivation of the WZ conformal anomaly action using the Weyl gauging of the anomaly counterterms in $d = 6$, previously discussed by us in the case of $d = 4$. We have focused our attention on the contributions related to the local part of the anomaly. This result adds full generality to the analysis of dilaton effective actions, which carry an intrinsic regularization scheme dependence, due to the appearance in the anomaly functional of extra (total derivative) terms. In general, the extraction of these extra contributions, as one can figure out from our study, is very involved, with a level of difficulty that grows with the dimensionality of the space in which the underlying CFT is formulated. Our main result for the WZ dilaton action in $d = 6$ takes a simple form in flat space (95), and is characterized by four independent parameters $c_i$ and $a$, which express the field (particle) content of a certain conformal realization. Comparing our results with those of the previous literature [24], we have given the form of the WZ action in the case of the CFT of the $(2,0)$ tensor multiplet, which, in the past, has found application in the $AdS_7/CFT_6$ correspondence. We have also shown, in a second part of our analysis, using the structure of the dilaton action, that multiple correlators of traces of the EMT, for a theory with a certain anomaly content, are functionally related to correlation functions up to the sixth order and we have presented their explicit expressions up to rank-4.

Acknowledgments

We thank Antonio Mariano for discussions.

Appendix A. Technical results

The definition of the Fourier transform for equation (10) as well as for any $n$-point correlator is given by

$$\int d^d x_1 \ldots d^d x_n \langle T_{\alpha_1 \beta_1} (x_1) \ldots T_{\alpha_n \beta_n} (x_n) \rangle e^{-i(k_1 \cdot x_1 + \ldots + k_n \cdot x_n)} = (2\pi)^d \delta^{(d)} \sum_{i=1}^n (T_{\alpha_i \beta_i} (k_i) \ldots T_{\alpha_n \beta_n} (k_n)) \quad (A.1)$$

with all the momenta in the vertex taken as incoming. For the Riemann tensor we choose to adopt the sign convention

$$R^{\mu \nu \lambda \kappa} = \partial_{[\lambda} \Gamma_{\nu \kappa]}^{\mu} - \partial_{[\kappa} \Gamma_{\nu \lambda]}^{\mu} + \Gamma_{[\lambda \kappa]}^{\eta} \Gamma_{\nu \eta}^{\mu} - \Gamma_{[\lambda \nu]}^{\eta} \Gamma_{\kappa \eta}^{\mu}. \quad (A.2)$$

The traceless part of the Riemann tensor in $d$ dimensions is the Weyl tensor

$$C_{\alpha \beta \gamma \delta} = R_{\alpha \beta \gamma \delta} - \frac{1}{d-2} (g_{\alpha \gamma} R_{\beta \delta} - g_{\alpha \delta} R_{\beta \gamma} - g_{\beta \gamma} R_{\alpha \delta} + g_{\beta \delta} R_{\alpha \gamma}) + \frac{R}{(d-1)(d-2)} (g_{\alpha \gamma} g_{\beta \delta} - g_{\alpha \delta} g_{\beta \gamma}). \quad (A.3)$$

It is also customary to introduce the Cotton tensor,

$$\tilde{C}_{\alpha \beta \gamma} = \nabla_{\gamma} K_{\alpha \beta} - \nabla_{\beta} K_{\alpha \gamma}, \quad \text{where} \quad K_{\alpha \beta} = \frac{1}{d-2} (R_{\alpha \beta} - R g_{\alpha \beta}). \quad (A.4)$$

The Weyl variations of the Christoffel symbols are

$$\delta W \Gamma_{\beta \gamma}^{\alpha} = -g_{\beta \gamma} \delta_{\alpha}^\sigma \partial_{\sigma} + \delta_{\alpha}^\sigma \partial_{\beta} \sigma + \delta_{\alpha}^\nu \partial_{\beta} \partial_{\nu} \sigma \Rightarrow \delta W \Gamma_{\alpha \beta \gamma}^{\nu} = d \partial_{\nu} \sigma, \quad (A.5)$$

$$\nabla_{\nu} \delta W \Gamma_{\beta \gamma}^{\alpha} = -g_{\beta \gamma} \nabla_{\nu} \delta_{\alpha}^\sigma \partial_{\sigma} + \delta_{\alpha}^\nu \nabla_{\nu} \partial_{\beta} \sigma + \delta_{\alpha}^\sigma \nabla_{\nu} \partial_{\beta} \sigma \Rightarrow \delta W \nabla_{\nu} \Gamma_{\alpha \beta \gamma}^{\mu} = d \nabla_{\nu} \partial_{\beta} \sigma.$$
which, using the Palatini identity
$$\delta R^a_{\mu\nu\rho} = \nabla_\mu (\delta \Gamma^a_{\rho\nu}) - \nabla_\nu (\delta \Gamma^a_{\mu\rho}) \Rightarrow \delta R_{\mu\nu} = \nabla_\mu (\delta \Gamma^a_{\nu\rho}) - \nabla_\nu (\delta \Gamma^a_{\mu\rho})$$ (A.6)
give the following Weyl variations of the Riemann and Ricci tensors
$$\delta_R R^a_{\mu\nu\rho} = g_{\rho\nu} \nabla_\mu \partial^a \sigma - g_{\rho\nu} \nabla_\mu \partial^a \sigma + \delta_\tau^a \nabla_\mu \partial_\rho \sigma - \delta_\tau^a \nabla_\mu \partial_\rho \sigma.$$ (A.7)
Is is also easy to use (A.5)–(A.7) to show that the variation of the Cotton tensor is simply given by
$$\delta_\tau C_{a\rho\sigma} = -\partial_\rho \sigma \tilde{C}^a_{\alpha\beta} g_\gamma.$$ (A.8)
which is expressed in terms of the Weyl tensor.

A.1. Weyl scalars and the Euler density

There are three dimension-6 scalars that are Weyl invariant when multiplied by $\sqrt{g}$. Their choice is not unique at all, as one can always take linear combinations of them. In particular, the literature is full of different choices for the third one, which involves differential operators. In this paper we adopt the definition of $I_3$ given, for general dimensions, in [40]

$$I_3^a = C_{\mu\nu\rho\beta} \epsilon^{\mu\alpha\nu\rho} C_{\rho\sigma}^a = \frac{d^2 + d - 4}{(d - 1)(d - 2)^3} K_1 + \frac{3(d^2 + d - 4)}{(d - 1)(d - 2)^3} K_2 + \frac{3}{2d^2 - 6d - 4} K_3$$
$$+ \frac{6d - 8}{(d - 2)^2} K_4 - \frac{3d}{(d - 2)^2} K_5 - \frac{3}{d - 2} K_6 + K_8$$

$$I_3^b = C_{\mu\nu\rho\beta} \epsilon^{\mu\alpha\nu\rho} C_{\rho\sigma}^b = \frac{8}{(d - 1)(d - 2)^3} K_1 - \frac{72 - 48d}{(d - 1)(d - 2)^3} K_2 + \frac{6}{d^2 - 3d + 2} K_3$$
$$+ \frac{16(d - 1)}{(d - 2)^2} K_4 - \frac{24}{(d - 2)^2} K_5 - \frac{12}{d - 2} K_6 + K_7$$

$$I_3^c = \frac{d - 10}{d - 2} (\nabla_\mu \epsilon^{\mu\nu\rho\sigma} C_{\rho\sigma}^c - 4 - 2(\nabla_\mu \epsilon^{\mu\nu\rho\sigma} \tilde{C}_{\rho\sigma}^c) + \frac{4}{d - 2} \left( \frac{2}{(d - 1)} R \right) C_{\mu\nu\rho\sigma}^c C_{\rho\sigma}^c$$
$$= \frac{16}{(d^2 - 3d - 2)^2} K_1 - \frac{32}{(d - 1)(d - 2)^2} K_2 + \frac{8}{(d^2 - 3d + 2)} K_3 + \frac{16}{(d - 1)(d - 2)^2} K_9$$
$$- \frac{32}{(d - 2)^2} K_10 + \frac{8}{d - 2} K_11 + \frac{8}{(d - 1)(d - 2)^2} K_12 + \frac{8}{(d - 2)^2} K_13 + K_14$$
$$+ \frac{8(d - 10)}{(d - 2)^2} K_15.$$ (A.9)

In the gauging of the counterterms we use the following relations

$$\tilde{R}^a_{\mu\nu\rho} = \Gamma^a_{\mu\rho\nu} + \frac{1}{\Lambda} (\delta_\mu^a \nabla_\nu \tau + \delta_\nu^a \nabla_\rho \tau - g_{\rho\nu} \nabla^a \tau),$$

$$\tilde{R}^{\mu}_{\nu\rho\sigma} = R^{\mu}_{\nu\rho\sigma} + g_{\nu\rho} \left( \nabla_\sigma \frac{\partial^\mu \tau}{\Lambda^2} + \frac{\partial^\mu \tau \partial_\rho \tau}{\Lambda^2} \right) - g_{\nu\rho} \left( \nabla_\sigma \frac{\partial^\mu \tau}{\Lambda} + \frac{\partial^\mu \tau \partial_\rho \tau}{\Lambda^2} \right)$$
$$+ \delta^{\mu}_{\rho} \left( \nabla_\sigma \frac{\partial_\nu \tau}{\Lambda} + \frac{\partial_\nu \tau \partial_\rho \tau}{\Lambda^2} \right) - \delta^{\mu}_{\rho} \left( \nabla_\sigma \frac{\partial_\nu \tau}{\Lambda} + \frac{\partial_\nu \tau \partial_\rho \tau}{\Lambda^2} \right)$$
$$+ (\delta^{\mu}_{\rho} g_{\nu\sigma} - \delta^{\mu}_{\rho} g_{\nu\sigma}) (\partial^\tau)^2 \Lambda^2,$$

$$\tilde{R}_{\mu\nu} = R_{\mu\nu} - g_{\mu\nu} \left( \frac{\partial^\tau}{\Lambda} - (d - 2) \left( \frac{\partial^\tau}{\Lambda} \right)^2 \right) - (d - 2) \left( \frac{\nabla_\mu \partial_\nu \tau}{\Lambda} + \frac{\partial_\mu \tau \partial_\nu \tau}{\Lambda^2} \right).$$

$$\tilde{R} = g^{\mu\nu} \tilde{R}_{\mu\nu} = e^{\frac{\tau}{2}} \left[ R - (2d - 1) \frac{\partial^\tau}{\Lambda} - (d - 1)(d - 2) \left( \frac{\partial^\tau}{\Lambda} \right)^2 \right].$$ (A.10)
A.2. Functional variations

The results for the trace anomaly given in section 4.1, are obtained by computing the functional variations of the integrals of the $K_i$ in dimensional regularization. A simple counting of the metric tensors needed to contract all the indices for any $K_i$ shows that

$$\delta_W \int d^d x \sqrt{g} K_i = \int d^d x \sqrt{g} (-\epsilon K_i + D(K_i)) \sigma, \quad \epsilon = 6 - d,$$

(A.11)

where the second term on the right hand side, $D(K_i)$, is a total derivative contribution. We give the complete list of these terms below. We obtain

$$D(K_1) = 12(d - 1) \nabla_{\mu} (\partial^\mu R)$$

$$D(K_2) = \nabla_{\mu} \left[ 4(d - 1) R_{\mu\nu} \nabla^\mu R^{\nu\rho} + 2(d - 2) R^\mu R_{\rho\nu} \partial^\nu R + (d + 2) \partial^\mu R \right]$$

$$D(K_3) = 4 \nabla_{\mu} \left[ R \partial^\mu R + 2 R^{\mu\nu} \partial^\nu R + (d - 1) R_{\nu\rho\sigma} \nabla^\mu R^{\nu\rho\sigma} \right]$$

$$D(K_4) = 3 \nabla_{\mu} \left[ d - 2 R^{\nu\rho} \partial^\nu R + (d - 2) R_{\nu\rho} \nabla^\mu R^{\nu\rho} + 2 R_{\nu\rho} \nabla^\mu R^{\nu\rho} \right]$$

$$D(K_5) = \nabla_{\mu} \left[ -R \partial^\mu R - R^{\mu\nu} \partial^\nu R + 2(d - 1) R_{\nu\rho} \nabla^\mu R^{\nu\rho} - 2d R_{\nu\rho} \nabla^\mu R^{\nu\rho} \right]$$

$$+ 2(d - 2) R^{\mu\rho\sigma\nu} \nabla^\mu R^{\rho\sigma\nu}$$

$$D(K_6) = \nabla_{\mu} \left[ 2 R^{\mu\nu} \partial^\nu R + 4 R_{\nu\rho} \nabla^\mu R^{\mu\rho} + \frac{d + 2}{2} R_{\nu\rho\sigma\tau} \nabla^\mu R^{\mu\rho\sigma\tau} - 2d R^{\mu\rho\sigma\nu} \nabla^\mu R_{\nu\rho\sigma\tau} \right]$$

$$D(K_7) = 6 \nabla_{\mu} \left[ R_{\nu\rho\sigma\tau} \nabla^\mu R^{\nu\rho\sigma\tau} + 4 R^{\mu\rho\sigma\nu} \nabla\rho R_{\nu\sigma} \right]$$

$$D(K_8) = 3 \nabla_{\mu} \left[ \frac{1}{2} R_{\nu\rho\sigma\tau} \nabla^\mu R^{\nu\rho\sigma\tau} + 2 R^{\mu\rho\sigma\nu} \left( \nabla_{\nu} R_{\nu\rho} - \nabla_{\nu} R_{\nu\rho} \right) \right]$$

$$D(K_9) = \nabla_{\mu} \left[ 4(d - 1) \partial^\mu R - (d - 2) R \partial^\mu R \right]$$

$$D(K_{10}) = \nabla_{\mu} \left[ 2 \partial^\mu R + 2(d - 2) \nabla_{\nu} \square R^{\mu\nu} + 2 R^{\mu\nu} \partial^\nu R - 4 R_{\nu\rho} \nabla^\mu R^{\nu\rho} - (d - 2) R_{\nu\rho} \nabla^\mu R^{\nu\rho} \right]$$

$$D(K_{11}) = \nabla_{\mu} \left[ 8 \nabla_{\nu} R^{\mu\nu} - (d + 2) R_{\nu\rho\sigma\tau} \nabla^\mu R^{\nu\rho\sigma\tau} - 16 R^{\mu\rho\nu\sigma} \nabla^\mu R_{\nu\rho\sigma} \right]$$

$$D(K_{12}) = 4 \nabla_{\mu} \left[ R \partial^\mu R - (d - 1) \partial^\mu R \right]$$

$$D(K_{13}) = 2 \nabla_{\mu} \left[ -\partial^\mu R - (d - 2) \nabla_\nu \square R^{\mu\nu} - R^{\mu\nu} \partial^\nu R + 2 R_{\nu\rho} \nabla^\mu R^{\nu\rho} + 2 R_{\nu\rho} \nabla^\mu R^{\nu\rho} \right]$$

$$D(K_{14}) = 8 \nabla_{\mu} \left[ -\nabla_\nu \square R^{\mu\nu} + R_{\nu\rho\sigma\tau} \nabla^\mu R^{\nu\rho\sigma\tau} + 2 R^{\nu\rho\sigma\nu} \nabla^\mu R_{\nu\rho\sigma} \right]$$

$$D(K_{15}) = \nabla_{\mu} \left[ -\partial^\mu R - 2(d - 2) \nabla_\nu \square R^{\mu\nu} - 3 R^{\mu\nu} \partial^\nu R + 6 R_{\nu\rho} \nabla^\mu R^{\nu\rho} \right]$$

$$+ 2 R_{\nu\rho} \nabla^\mu R^{\nu\rho} - 2(d - 2) R^{\mu\rho\sigma\nu} \nabla^\mu R_{\nu\rho\sigma} \right].$$

(A.12)

Appendix B. The cases $d = 2$ and $d = 4$ as a check of the recursive formul\ae

We briefly recall here how to cross-check the relations (107).

It is clear that the expressions of the dilaton vertices $I_n$ given in (107) do not depend on the working dimensions. Therefore we take $d = 2$ and check the agreement between the correlators that result from (107) and those found by a direct functional differentiation of the anomaly via the hierarchy (13). This provides a strong check of the correctness of (107). In fact, the equation of the trace anomaly in two dimensions takes the form

$$\langle T \rangle = -\frac{c}{24\pi} R,$$

(B.1)

where $c = n_s + n_f$, with $n_s$ and $n_f$ being the numbers of free scalar and fermion fields respectively. It is derived from the counterterm

$$\Gamma_C[g] = -\mu^k \frac{c}{24\pi} \int d^d x \sqrt{g} R, \quad \epsilon = d - 2.$$
The Weyl gauging procedure for the integral of the scalar curvature gives
\[
\frac{-\mu^s}{\epsilon} \int d^4x \sqrt{g} \bar{R} = \frac{-\mu^s}{\epsilon} \int d^4x \sqrt{g} R + \int d^2x \sqrt{g} \left[ \frac{\tau}{\Lambda} R + \frac{1}{\Lambda^2} (\bar{\tau} \tau)^2 \right].
\] (B.3)

The second term in (B.3) is, modulo a constant, the WZ action in two dimensions,
\[
\Gamma_{WZ}[g, \tau] = -\frac{c}{24\pi} \int d^2x \sqrt{g} \left[ \frac{\tau}{\Lambda} R + \frac{1}{\Lambda^2} (\bar{\tau} \tau)^2 \right],
\] (B.4)

from which we can extract the two-dilaton amplitude according to (106)
\[
\mathcal{I}_2(k_1 - k_1) = \frac{1}{\Lambda^2} (T(k_1)T(-k_1)) = \frac{c}{12\pi} k_1^2.
\] (B.5)

Starting from the two-dilaton vertex, which is the only non-vanishing one, exploiting (115) and inverting the remaining relations, we get the Green functions
\[
\langle T(k_1)T(k_2)T(k_3) \rangle = -\frac{c}{6\pi} (k_1^2 + k_2^2 + k_3^2),
\]
\[
\langle T(k_1)T(k_2)T(k_3)T(k_4) \rangle = \frac{c}{\pi} (k_1^2 + k_2^2 + k_3^2 + k_4^2),
\]
\[
\langle T(k_1)T(k_2)T(k_3)T(k_4)T(k_5)T(k_6) \rangle = -\frac{8c}{\pi} (k_1^2 + k_2^2 + k_3^2 + k_4^2 + k_5^2).
\] (B.6)

These results exactly agree with the combinations of completely traced multiple functional derivatives of the anomaly (B.1) that one derives from (13), providing a consistency check of our recursive formulas (107).

### B.1. The first six traced correlators in \(d = 4\)

Here we just report the expressions of the first six correlators in \(d = 4\). Given the anomaly equation in four dimensions (71), we obtain
\[
\langle T(k_1)T(-k_1) \rangle = -4\beta_0 k_1^4,
\]
\[
\langle T(k_1)T(k_2)T(k_3) \rangle = 8 \left\{ - (\beta_a + \beta_b) \left( \sum_{i=1}^{3} k_i^2 \left( \sum_{j} f_3(k_1, k_2, k_3) + f_3(k_2, k_1, k_3) \right) 
+ f_3(k_1, k_1, k_2) \right) + \beta_a \sum_{i=1}^{3} k_i^2 \right\},
\]
\[
\langle T(k_1)T(k_2)T(k_3)T(k_4) \rangle = 8 \left\{ 6(\beta_a + \beta_b) \left[ \sum_{i,j,k,l} k_i \cdot k_j k_l \cdot k_i \right.ight.
+ f_4(k_1 k_2, k_3, k_4) + f_4(k_2 k_1, k_3, k_4) + f_4(k_3 k_1, k_2, k_4) + f_4(k_4 k_1, k_2, k_3) \right] 
+ \beta_a \left( \sum_{i,j,k,l} (k_i + k_l) \cdot (k_i + k_l) \right) \right\},
\]
\[
\langle T(k_1)T(k_2)T(k_3)T(k_4)T(k_5)T(k_6) \rangle = 16 \left\{ 24(\beta_a + \beta_b) \left[ \sum_{i,j,k,l} k_i \cdot k_j k_l \cdot k_i \right.ight.
+ f_5(k_1, k_2, k_3, k_4) + f_5(k_2, k_1, k_3, k_4) + f_5(k_3, k_1, k_2, k_4) + f_5(k_4, k_1, k_2, k_3) \right] 
+ \beta_a \left( \sum_{i,j,k,l} (k_i + k_l) \cdot (k_i + k_l) \right) \right\},
\]
\[
\langle T(k_1)T(k_2)T(k_3)T(k_4)T(k_5)T(k_6)T(k_7)T(k_8) \rangle = 32 \left\{ 48(\beta_a + \beta_b) \left[ \sum_{i,j,k,l,m} k_i \cdot k_j k_l \cdot k_i \right.ight.
+ f_6(k_1, k_2, k_3, k_4, k_5, k_6) + f_6(k_2, k_1, k_3, k_4, k_5, k_6) + f_6(k_3, k_1, k_2, k_4, k_5, k_6) + f_6(k_4, k_1, k_2, k_3, k_5, k_6) + f_6(k_5, k_1, k_2, k_3, k_4, k_7) + f_6(k_6, k_1, k_2, k_3, k_4, k_7) \right]
+ \beta_a \left( \sum_{i,j,k,l,m} (k_i + k_l + k_m) \cdot (k_i + k_l + k_m) \right) \right\}.
\]
\[ + f_5(k_5, k_1, k_2, k_3, k_6) + f_5(k_5, k_1, k_2, k_3, k_6) \]  
\[ + \beta_\alpha \left[ \sum_{T(5, (k_1, k_2, k_3))} (k_{i_1} + k_{i_2} + k_{i_3})^4 + 3 \sum_{T[5, (k_i, k_j)]} (k_{i_1} + k_{i_2})^4 + 12 \sum_{i=1}^5 k_i^4 \right]. \]

\[ \langle T(k_1) T(k_2) T(k_3) T(k_4) T(k_5) T(k_6) \rangle = 32 \left\{ 120 (\beta_\alpha + \beta_\beta) \left[ \sum_{T[6, (k_1, k_2, k_3, k_4, k_5)]} k_{i_1} \cdot k_{i_2} k_{i_3} \cdot k_{i_4} + f_6(k_1, k_2, k_3, k_4, k_5, k_6) + f_6(k_2, k_1, k_3, k_4, k_5, k_6) + f_6(k_3, k_4, k_5, k_6, k_1, k_2) + f_6(k_4, k_5, k_6, k_1, k_2, k_3) + f_6(k_5, k_6, k_1, k_2, k_3, k_4) \right] \right. \]
\[ - \beta_\alpha \left( \sum_{T[6, (k_1, k_2, k_3, k_4, k_5)]} (k_{i_1} + k_{i_2} + k_{i_3})^4 + 4 \sum_{T[6, (k_1, k_2, k_3, k_4, k_5)]} (k_{i_1} + k_{i_2} + k_{i_3})^4 \right. \]
\[ + 11 \sum_{T[6, (k_1, k_2, k_3, k_4, k_5)]} (k_{i_1} + k_{i_2})^4 + 48 \sum_{i=1}^5 k_i^4 \right\}, \quad (B.7) \]

where we have introduced the compact notation
\[ f_3(k_a, k_b, k_c) = k_a^2 k_b \cdot k_c, \]
\[ f_4(k_a, k_b, k_c, k_d) = k_a^2 (k_b \cdot k_c + k_b \cdot k_d + k_c \cdot k_d). \]
\[ f_5(k_a, k_b, k_c, k_d, k_e) = k_a^2 (k_b \cdot k_e + k_b \cdot k_d + k_b \cdot k_c + k_d \cdot k_e + k_c \cdot k_e + k_d \cdot k_e), \]
\[ f_6(k_a, k_b, k_c, k_d, k_e, k_f) = k_a^2 (k_b \cdot k_e + k_b \cdot k_d + k_b \cdot k_c + k_d \cdot k_f + k_c \cdot k_f + k_d \cdot k_f + k_c \cdot k_f + k_e \cdot k_f + k_d \cdot k_f + k_c \cdot k_f + k_e \cdot k_f). \quad (B.8) \]

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