Universal locally univalent functions and universal conformal metrics with constant curvature

Daniel Pohl* and Oliver Roth†

Department of Mathematics, University of Würzburg
Emil Fischer Straße 40, 97074 Würzburg, Germany

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Abstract We prove Runge–type theorems and universality results for locally univalent holomorphic and meromorphic functions. Refining a result of M. Heins, we also show that there is a universal bounded locally univalent function on the unit disk. These results are used to prove that on any hyperbolic simply connected plane domain there exist universal conformal metrics with prescribed constant curvature.

Keywords: Universal functions, Runge theory, locally univalent functions, conformal metrics, constant curvature

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1 Introduction

Let \( \Omega \) be a domain in the complex plane \( \mathbb{C} \) and let \( \mathcal{H}(\Omega) \) be the space of all holomorphic functions on \( \Omega \). We think of \( \mathcal{H}(\Omega) \) as a (closed) subspace of the Fréchet space \( C(\Omega) \) of all complex–valued continuous functions on \( \Omega \) equipped with the compact–open topology of locally uniform convergence. We denote by \( \text{Aut}(\Omega) \) the group of all conformal automorphisms of \( \Omega \). A function \( f \in \mathcal{H}(\Omega) \) is called universal if the set \( \{ f \circ \phi : \phi \in \text{Aut}(\Omega) \} \) is dense in \( \mathcal{H}(\Omega) \), i.e., as big as it possibly can be.

The concept of universality goes back at least to Birkhoff in 1929, who showed [6] that there exist universal functions in \( \mathcal{H}(\mathbb{C}) \). Another early, related universality result was obtained by Seidel and Walsh in 1941, who proved in [28] that there are universal functions in \( \mathcal{H}(\mathbb{D}) \), where \( \mathbb{D} \) denotes the open unit disc.

*daniel.pohl@mathematik.uni-wuerzburg.de
†roth@mathematik.uni-wuerzburg.de, Phone: +49 931 318 4974
On the other hand, it follows from the maximum principle that there are no universal functions in \( \mathcal{H}(\mathbb{C}\setminus\{0\}) \). In all other cases, so when \( \Omega \) is not conformally equivalent to \( \mathbb{C}\setminus\{0\} \), then universal functions \( f \in \mathcal{H}(\Omega) \) exist if and only if \( \text{Aut}(\Omega) \) is not compact. These results are due to Bernal–González and Montes–Rodríguez [5]. They fully characterize all domains \( \Omega \) in \( \mathbb{C} \) for which universal functions \( f \in \mathcal{H}(\Omega) \) exist.

The notion of universality has been modified for many other classes of holomorphic and meromorphic functions and even beyond. We refer the reader to the survey paper [15], but wish to explicitly point out three specific universality results:

1. (Universal bounded holomorphic functions, Heins [18])
   There are universal bounded holomorphic functions. In fact, there is a universal Blaschke product \( B \) such that \( \{B \circ \phi : \phi \in \text{Aut}(D)\} \) is dense in the set of all holomorphic self-maps of \( D \).

2. (Universal univalent functions, Pommerenke [27])
   Let \( S \) be the class of all univalent holomorphic functions \( f : D \to \mathbb{C} \) normalized by \( f(0) = 0 \) and \( f'(0) = 1 \). Then there is a universal function \( f \in S \) in the sense that
   \[
   \left\{ \frac{f \circ \phi - (f \circ \phi)(0)}{(f \circ \phi)'(0)} : \phi \in \text{Aut}(D) \right\}
   \]
   is dense in \( S \). Note that the imposed normalization condition for the class \( S \) requires passing from \( f \circ \phi \) to the “Koebe transforms” \( [f \circ \phi - (f \circ \phi)(0)]/(f \circ \phi)'(0) \).

3. (Universal meromorphic functions)
   There are Birkhoff-type universality results for meromorphic functions (see e.g. [8]).

The main goals of the present paper are to investigate universal locally univalent holomorphic and meromorphic functions and, in particular, universal constantly curved conformal metrics. A key auxiliary step consists in establishing Runge–type results for locally univalent functions which might be interesting in their own right.

Here is a quick outline of our work. We denote for a set \( M \subseteq \mathbb{C} \) by \( \mathcal{H}_{l.u.}(M) \) the family of all functions which are holomorphic and locally univalent on some open neighborhood of \( M \) in \( \mathbb{C} \) (which might depend on the function).

**Theorem 1.1** (Runge–type theorem for locally univalent holomorphic functions).
Let \( \Omega \) be a domain in \( \mathbb{C} \) and let \( K \) be a compact set in \( \Omega \) such that \( \Omega \setminus K \) has no relatively compact components in \( \Omega \). Then every function \( f \in \mathcal{H}_{l.u.}(K) \) can be approximated uniformly on \( K \) by functions in \( \mathcal{H}_{l.u.}(\Omega) \).

There is an analogue of Theorem 1.1 for meromorphic functions, cf. Theorem 2.1 below.

**Remark 1.2.** It can be shown that Theorem 1.1 also holds more generally on any open Riemann surface. Note that it is already a deep result due to Gunning and Narasimhan [17] that every open Riemann surface carries at least one locally univalent holomorphic function. Recently, Frostnerič [10] extended the Gunning–Narasimhan theorem to Stein...
manifolds and Majcen [23] established a Runge–type theorem for holomorphic 1–forms on Stein manifolds. We shall use some of the ideas of these papers in our proof of Theorem 1.1.

**Theorem 1.3 (Universal locally univalent functions).**

Let $\Omega$ be a simply connected domain in $\mathbb{C}$. Then there exists a function $f \in \mathcal{H}_{l.u.}(\Omega)$ such that $\{f \circ \phi : \phi \in \text{Aut}(\Omega)\}$ is dense in $\mathcal{H}_{l.u.}(\Omega)$.

Hence there exist universal locally univalent holomorphic functions on any simply connected plane domain $\Omega$. In fact, these functions form a dense $G_{\delta}$–subset of $\mathcal{H}_{l.u.}(\Omega)$. A version of Theorem 1.3 also holds for meromorphic functions, see Theorem 2.12 below.

**Remark 1.4.** As it is the case with Theorem 1.1, also Theorem 1.3 can be generalized to Riemann surfaces. We note that Montes–Rodríguez [25] has studied universal holomorphic functions on Riemann surfaces, and his approach can be combined with Theorem 1.3 to investigate universal locally univalent holomorphic functions on Riemann surfaces.

We denote by $B(\Omega)$ the set of all $f \in \mathcal{H}(\Omega)$ such that $|f(z)| \leq 1$ on $\Omega$, and write $B_{l.u.}(\Omega) := B(\Omega) \cap \mathcal{H}_{l.u.}(\Omega)$ for the set of bounded locally univalent functions.

**Theorem 1.5 (Universal bounded locally univalent functions).**

Let $\Omega$ be a simply connected proper subdomain of $\mathbb{C}$. Then there is a function $f \in B_{l.u.}(\Omega)$ such that $\{f \circ \phi : \phi \in \text{Aut}(\Omega)\}$ is dense in $B_{l.u.}(\Omega)$.

Theorem 1.5 is in the spirit of the universality results of Heins and Pommerenke mentioned above. However, Heins [18] considers only bounded, but not necessarily locally univalent functions, and Pommerenke [27] is concerned with univalent functions, which are not necessarily bounded. We note that while the proof of Theorem 1.3 is based on the Runge–type Theorem 1.1, the proof of Theorem 1.5 in Section 2.3 below is considerably different and will be based on Heins’ universality result and the use of universal covering maps, see Theorem 2.15.

Theorem 1.3 and Theorem 1.5 put us in a position to investigate universal conformal metrics with constant curvature. Recall (see Ahlfors [11] §1.5] or Simon [29, Chapter 12]) that a regular conformal metric $\lambda(z) |dz|$ on a domain $\Omega$ is given by a positive $C^2$–function $\lambda$ on $\Omega$, called the density of the metric. Denoting, as usual, by $\Delta := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ the Laplace operator in the standard Cartesian coordinates of the $xy$–plane, the Gauss curvature of such a metric $\lambda(z) |dz|$,\n
$$(1.1) \quad \kappa_\lambda(z) := -\frac{\Delta \log \lambda(z)}{\lambda(z)^2}, \quad z \in \Omega,$$

has an important invariance property: If we define for a locally univalent self–map $\phi$ of $\Omega$ the pullback $\phi^* \lambda(z) |dz|$ of $\lambda(z) |dz|$ via $\phi$ by

$$\phi^* \lambda(z) := (\lambda \circ \phi)(z) |\phi'(z)|, \quad z \in \Omega,$$

then

$$\kappa_{\phi^* \lambda} = \kappa_\lambda \circ \phi.$$
Hence, for conformal metrics $\lambda(z)\,|dz|$ it is more natural to consider the pullback $\phi^*\lambda$ instead of the composition $\lambda \circ \phi$, even though the additional “conformal factor” $|\phi'(z)|$ causes some difficulties.

We denote by $\Lambda_c(\Omega)$ the set (of densities) of all regular conformal metrics with constant curvature $c \in \mathbb{R}$. Then $\Lambda_c(\Omega)$ is a subset of the Fréchet space $C(\Omega)$ which is invariant under pullback in the sense that $\phi^*\lambda \in \Lambda_c$ for all $\lambda \in \Lambda_c$ and all locally univalent self–maps $\phi$ of $\Omega$. We can now state the following theorem, which is maybe the main result of this paper.

**Theorem 1.6 (Universal constantly curved conformal metrics).**

Let $\Omega$ be a simply connected domain in $\mathbb{C}$ and let $c \in \mathbb{R}$. Suppose that $\Omega \neq \mathbb{C}$ if $c < 0$. Then there is an $\Lambda \in \Lambda_c(\Omega)$ such that $\{\phi^*\Lambda : \phi \in \text{Aut} (\Omega)\}$ is dense in $\Lambda_c(\Omega)$.

Hence there exist universal constantly curved conformal metrics on any simply connected domain $\neq \mathbb{C}$. Theorem [1.6] is perhaps the first universality result for conformal metrics, except possibly for the case of constant curvature $c = 0$ which is closely related to universal harmonic functions, see Theorem [2.19] below for more details.

We wish to point out that proving universality results for conformal metrics or – and this amounts in view of (1.1) to the same thing – solutions of the Gauss curvature equation $\Delta u = -ce^{2u}$, a basic nonlinear elliptic PDE in conformal geometry, has been our main initial motivation for proving universality results for locally univalent functions.

The present paper is organized as follows. In Section 2 we discuss in greater generality various Runge–type theorems and universality results for locally univalent holomorphic and meromorphic functions as well as for constantly curved conformal metrics. There are many related open problems and we explicitly discuss a considerable number of them. The proofs of the results are deferred to the final Section 3.

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## 2 Results and open problems

In what follows $\Omega$ is always a domain in $\mathbb{C}$ and $\mathcal{M}(\Omega)$ the set of all meromorphic functions on $\Omega$. We think of $\mathcal{M}(\Omega)$ as a metric space equipped with the (metrizable) topology of locally uniform convergence w.r.t. the chordal metric $\chi$ on the Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ where $\chi$ is defined as usual by

$$\chi(z_1, z_2) := \frac{|z_1 - z_2|}{\sqrt{1 + |z_1|^2} \sqrt{1 + |z_2|^2}}$$

if $z_1, z_2 \in \mathbb{C}$,

and

$$\chi(z_1, \infty) := \chi(\infty, z_1) := \frac{1}{\sqrt{1 + |z_1|^2}}$$

if $z_1 \in \mathbb{C}$.
For $f_n, f \in \mathcal{M}(\Omega)$ we write $f_n \to f$ locally $\chi$–uniformly on $\Omega$ if $\chi(f_n, f) \to 0$ locally uniformly on $\Omega$.

A function $f \in \mathcal{M}(\Omega)$ is called \textit{locally univalent} if $f$ has at most simple poles and $f'(z) \neq 0$ for all $z \in \Omega$ with $f(z) \neq \infty$. For a family $\mathcal{G} \subseteq \mathcal{M}(\Omega)$ we let

$$
\mathcal{G}_{l.u.} := \{ f \in \mathcal{G} : f \text{ is locally univalent} \}.
$$

We will be mainly interested in the families

(i) $\mathcal{G} = \mathcal{H}(\Omega)$, and

(ii) $\mathcal{G} = \mathcal{B}(\Omega) := \{ f \in \mathcal{H}(\Omega) : \sup_{z \in \Omega} |f(z)| \leq 1 \}$.

2.1 Runge–type theorems for locally univalent functions

For a compact set $K$ of $\mathbb{C}$ we denote by $\mathcal{M}_{l.u.}(K)$ the set of all locally univalent meromorphic functions on (the components of) some open neighborhood of $K$.

**Theorem 2.1** (Runge–type theorem for locally univalent functions). Let $\Omega$ be a domain in $\mathbb{C}$ and let $K$ be a compact set in $\Omega$ such that $\Omega \setminus K$ has no relatively compact components in $\Omega$. Then

(a) every $f \in \mathcal{H}_{l.u.}(K)$ can be approximated uniformly on $K$ by functions in $\mathcal{H}_{l.u.}(\Omega)$;

(b) every $f \in \mathcal{M}_{l.u.}(K)$ can be approximated $\chi$–uniformly on $K$ by functions in $\mathcal{M}_{l.u.}(\mathbb{C})$, provided that $\mathbb{C} \setminus K$ is connected.

Note that (b) is a somewhat weaker statement than (a). In fact, we do not know if the analogue of (a) holds for meromorphic functions:

**Problem 2.2.** Let $\Omega$ be a domain in $\mathbb{C}$ and let $K$ be a compact set in $\Omega$ such that $\Omega \setminus K$ has no relatively compact components in $\Omega$. Can every $f \in \mathcal{M}_{l.u.}(K)$ be approximated $\chi$–uniformly on $K$ by functions in $\mathcal{M}_{l.u.}(\mathbb{C})$?

The following example shows that we have to assume that $\Omega \setminus K$ has no relatively compact components in $\Omega$ in Problem 2.2 and also that $\mathbb{C} \setminus K$ is connected in Theorem 2.1 (b). We employ the well–known fact that a function $f \in \mathcal{M}(\Omega)$ is locally univalent if and only if its Schwarzian derivative

$$
S_f := \left( \frac{f''}{f'} \right)' - \frac{1}{2} \left( \frac{f''}{f'} \right)^2
$$

is holomorphic on $\Omega$.

**Example 2.3.** Let $K := \{ z \in \mathbb{C} : 1/2 \leq |z| \leq 2 \}$ and $f(z) := -1/z^2 \in \mathcal{M}_{l.u.}(K)$. Suppose that there is a sequence $(g_n)$ in $\mathcal{M}_{l.u.}(\mathbb{C})$ which converges to $f$ $\chi$–uniformly on $K$. Then we have $S_{g_n} \to S_f$ uniformly on $\partial \mathbb{D}$. But since $S_{g_n} \in \mathcal{H}(\mathbb{C})$ for all $n \in \mathbb{N}$ the
maximum principle implies $S_{g_n} \to h$ uniformly in $D$ for a function $h \in \mathcal{H}(D)$. We have $S_f \equiv h$ on $D \cap K$ and hence on $D \setminus \{0\}$. This, however, contradicts the fact that 0 is a critical point of $f$.

Since Theorem 2.1 (a) is a form of the classical Runge theorem in which one allows only locally univalent functions, it is tempting to ask if there are analogues of Mergelyan’s approximation Theorem [24] and Arakelyan’s Theorem [3] for locally univalent functions:

**Problem 2.4.**

Let $K$ be a compact set in $\mathbb{C}$ with connected complement. Suppose $f : K \to \mathbb{C}$ is continuous on $K$ and locally univalent in the interior $K^o$ of $K$. Can $f$ be approximated by entire locally univalent functions? What if $K$ is only closed but unbounded and in addition $\hat{\mathbb{C}} \setminus K$ is locally connected at $\infty$?

Note that we allow $f$ to have “critical points” on the boundary $\partial K$. Recently, Andersson [2] has posed a similar problem about zero–free approximation.

### 2.2 Universal locally univalent functions

**Definition 2.5.** Let $\Omega$ be a domain in $\mathbb{C}$, $\mathcal{G} \subseteq \mathcal{M}(\Omega)$ and $\Phi$ a family of holomorphic self–maps of $\Omega$. A function $G \in \mathcal{G}$ is called $\Phi$–universal in $\mathcal{G}$ if $\{G \circ \phi : \phi \in \Phi\}$ is dense in $\mathcal{G}$. If $G \in \mathcal{G}$ is $\text{Aut}(\Omega)$–universal in $\mathcal{G}$, we simply call $G$ universal in $\mathcal{G}$.

Note that a $\Phi$–universal function in $\mathcal{G}$ is always supposed to belong to $\mathcal{G}$.

The aim of this section is to provide necessary and also sufficient conditions for the existence of $\Phi$–universal functions for families of locally univalent holomorphic or meromorphic functions on a domain $\Omega$ in $\mathbb{C}$. For this purpose, the following concepts, which have been introduced in [5] and [16], will play a crucial role.

**Definition 2.6.** Let $\Omega$ be a domain in $\mathbb{C}$ and let $(\phi_n)$ be a sequence of holomorphic self–maps of $\Omega$.

(i) We say that $(\phi_n)$ is run–away, if for every compact set $K \subseteq \Omega$ there exists $n \in \mathbb{N}$ with $\phi_n(K) \cap K = \emptyset$.

(ii) We say that $(\phi_n)$ is eventually injective, if for every compact set $K \subseteq \Omega$ there exists $N \in \mathbb{N}$ such that the restriction $\phi_n|_K$ is injective for all $n \geq N$.

These conditions turn out to be necessary for the existence of $\Phi$–universal functions in $\mathcal{H}_{t,u}(\Omega)$:

**Proposition 2.7.**

Let $\Omega$ be a domain in $\mathbb{C}$ and let $\Phi$ be a family of locally univalent self–maps of $\Omega$. Suppose that there is a $\Phi$–universal function in $\mathcal{H}_{t,u}(\Omega)$. Then $\Phi$ contains a run–away and eventually injective sequence.
We next turn to sufficient conditions, but restricting the discussion to the cases when \( \Omega \) is either simply connected or of infinite connectivity. The reason for this is the fact that for domains of finite connectivity \( N > 1 \) there are \( \Phi \)-universal functions \( f \) for \( \mathcal{H}(\Omega) \) such that the family \( \Phi \) of locally univalent self-maps of \( \Omega \) is mainly responsible for the denseness of \( \{ f \circ \phi : \phi \in \Phi \} \) in \( \mathcal{H}(\Omega) \) and not \( f \in \mathcal{H}(\Omega) \), see [16].

For simply connected domains, we have a complete picture:

**Theorem 2.8.**

Let \( \Omega \) be a simply connected domain in \( \mathbb{C} \). Suppose that \( \Phi \) is a family of locally univalent self-maps of \( \Omega \) which contains a run-away and eventually injective sequence. Then there is a \( \Phi \)-universal function in \( \mathcal{H}_{l.u.}(\Omega) \) and the set of all \( \Phi \)-universal functions in \( \mathcal{H}_{l.u.}(\Omega) \) is a dense \( G_\delta \)-subset of \( \mathcal{H}_{l.u.}(\Omega) \).

In order to discuss the case of domains of infinite connectivity, we recall that a compact subset \( K \) of a domain \( \Omega \) in \( \mathbb{C} \) is called \( \mathcal{O} \)-convex if \( \Omega \backslash K \) has no relatively compact components in \( \Omega \).

**Theorem 2.9.**

Let \( \Omega \) be a domain in \( \mathbb{C} \) of infinite connectivity and let \( \Phi \) be a family of locally univalent self-maps of \( \Omega \). Suppose that there exists a sequence \( (\phi_n) \) in \( \Phi \) such that

(i) \( (\phi_n) \) is eventually injective, and

(ii) for every \( \mathcal{O} \)-convex compact set \( K \) in \( \Omega \) and every \( N \in \mathbb{N} \) there exists \( n \geq N \) such that \( \phi_n(K) \cap K = \emptyset \) and \( \phi_n(K) \cup K \) is \( \mathcal{O} \)-convex.

Then there is a \( \Phi \)-universal function in \( \mathcal{H}_{l.u.}(\Omega) \) and the set of all such functions is a dense \( G_\delta \)-subset of \( \mathcal{H}_{l.u.}(\Omega) \).

We now take a closer look at the case \( \Phi \subseteq \text{Aut} (\Omega) \). It has been shown by Bernal-Gonzáles and Montes-Rodríguez [5] that if \( \Omega \) is not conformally equivalent to \( \mathbb{C} \setminus \{0\} \) then there is a \( \Phi \)-universal function in \( \mathcal{H}(\Omega) \) if and only if \( \Phi \) contains a run-away sequence. This result also holds in the setting of locally univalent functions:

**Theorem 2.10.**

Let \( \Omega \) be a domain in \( \mathbb{C} \) which is not conformally equivalent to \( \mathbb{C} \setminus \{0\} \) and let \( (\phi_n) \subseteq \text{Aut} (\Omega) \). Then there is a \( (\phi_n) \)-universal function in \( \mathcal{H}_{l.u.}(\Omega) \) if and only if \( (\phi_n) \) is run-away.

Note that Theorem 1.3 is a special instance of Theorem 2.10.

**Remark 2.11.** If \( \Omega \) is conformally equivalent to \( \mathbb{C} \setminus \{0\} \) then there are no universal functions in \( \mathcal{H}_{l.u.}(\Omega) \). In fact, it was observed in [5] that there are no universal functions for \( \mathcal{H}(\Omega) \) in this case. The argument is based on the maximum principle and stays valid for locally univalent functions.

The final result of this section is concerned with universal locally univalent meromorphic functions. Chan [8] has shown that there exists a meromorphic function \( f \in \mathcal{M}(\mathbb{C}) \) such
that the set \( T_f := \{ f(\cdot + n) : n \in \mathbb{N} \} \) is dense in \( \mathcal{M}(\Omega) \) for every domain \( \Omega \subseteq \mathbb{C} \). In the locally univalent situation we need to restrict the discussion to simply connected domains since the same reasoning as in Example 2.3 shows that if \( T_f \) is dense in \( \mathcal{M}_{l.u.}(\Omega) \) for some \( f \in \mathcal{M}_{l.u.}(\mathbb{C}) \), then \( \Omega \) has to be simply connected.

**Theorem 2.12.**

Let \( \Omega \subseteq \mathbb{C} \) be a simply connected domain and let \( \Phi \) be a family of locally univalent self-maps of \( \Omega \) which contains a run-away and eventually injective sequence \((\phi_n)\). Then there is a \( \Phi \)-universal function in \( \mathcal{M}_{l.u.}(\Omega) \) and the set of all such functions is a dense \( G_\delta \)-subset of \( \mathcal{M}_{l.u.}(\Omega) \).

**Corollary 2.13.**

There exists a function \( f \in \mathcal{M}_{l.u.}(\mathbb{C}) \) such that \( T_f = \{ f(\cdot + n) : n \in \mathbb{N} \} \) is dense in \( \mathcal{M}_{l.u.}(\Omega) \) for every simply connected domain \( \Omega \).

### 2.3 Universal bounded locally univalent functions

Theorem 2.10 says that if \( \Omega \) is a simply connected domain in \( \mathbb{C} \) and \( \Phi \subseteq \text{Aut}(\Omega) \), then there is a \( \Phi \)-universal function in \( \mathcal{H}_{l.u.}(\Omega) \) if and only if \( \Phi \) contains a run-away sequence. The corresponding problem for bounded locally univalent functions is the following:

**Problem 2.14.**

Let \( \Omega \neq \mathbb{C} \) be a simply connected domain in \( \mathbb{C} \) and let \((\phi_n)\) be a run-away sequence in \( \text{Aut}(\Omega) \). Does there exist a \((\phi_n)\)-universal function in \( \mathcal{B}_{l.u.}(\Omega) \) ?

We can only prove the weaker result (see Theorem 1.5) that for any simply connected domain \( \Omega \subseteq \mathbb{C} \) there is an universal function in \( \mathcal{B}_{l.u.}(\Omega) \). The idea is to “remove” the critical points of a universal function in \( \mathcal{B}(\Omega) \) while keeping the function bounded. Such a construction is based on the following considerations for which we initially assume that \( \Omega \) is the unit disk \( \mathbb{D} \).

For any non-constant \( f \in \mathcal{M}(\mathbb{D}) \) we denote by \( \Omega_f \) the set of all non-critical points of \( f \). Then \( \Omega_f \) is a subdomain of \( \mathbb{D} \), so by the Uniformization Theorem there exists a universal covering map \( \Psi \) from \( \mathbb{D} \) onto \( \Omega_f \) which is uniquely determined up to pre-composition with a unit disc automorphism. Clearly, \( f \circ \Psi \) is a locally univalent meromorphic function on \( \mathbb{D} \).

**Theorem 2.15.**

Let \( \mathcal{G} \subseteq \mathcal{M}(\mathbb{D}) \) and let \( G \in \mathcal{M}(\mathbb{D}) \) be a non-constant universal function in \( \mathcal{G} \). Suppose that \( \Psi \) is a universal covering map from \( \mathbb{D} \) onto \( \Omega_G \). Then \( F := G \circ \Psi \in \mathcal{M}_{l.u.}(\mathbb{D}) \) and for each \( f \in \mathcal{G}_{l.u.} \) there is a sequence \((\phi_n)\) in \( \text{Aut}(\mathbb{D}) \) such that \( F \circ \phi_n \to f \) locally uniformly in \( \mathbb{D} \) w.r.t. the chordal metric.

Note that Theorem 2.15 does not assert that \( F \) is \( \Phi \)-universal in \( \mathcal{G}_{l.u.} \), since in general \( F \not\in \mathcal{G} \).
Theorem 1.5 for Ω = D follows immediately from Theorem 2.15 by taking for G a universal function in B(D). One can even take a universal Blaschke product in B(D), see, for instance, [13] and [14]. Using the Riemann Mapping Theorem, the general case of Theorem 1.5 that Ω is a proper simply connected subdomain of C easily reduces to the case of the unit disk.

2.4 Universal conformal metrics with constant curvature

In this section we investigate universal conformal metrics with constant curvature. We shall use the following terminology.

Definition 2.16. Let Ω be a domain in C and let Φ be a family of locally univalent self–maps of Ω. Let c be a real number. We call Λ ∈ Λc(Ω) Φ–universal in Λc(Ω) if the family of pullbacks \{φ∗Λ : φ ∈ Φ\} is dense in Λc. We call Λ universal in Λc(Ω) if Λ is Aut(Ω)–universal in Λc(Ω).

We note that for c < 0 the set Λc(Ω) is non–empty if and only if Ω is a hyperbolic domain, that is, C \ Ω has at least two distinct points.

Theorem 2.17.

Let Ω be a simply connected domain in C and let c be a non–negative real number. Suppose that Φ is a family of locally univalent self–maps of Ω which contains a runaway and eventually injective sequence. Then there exists a Φ–universal Λ ∈ Λc(Ω).

In particular, under the assumptions of Theorem 2.17 there is always an Aut(Ω)–universal element in Λc(Ω). This proves the cases c ≥ 0 of Theorem 1.6. The cases of constant negative curvature appear to be more difficult:

Problem 2.18.

Let Ω be a simply connected proper subdomain of C and c < 0. Which families Φ of locally univalent self–maps of Ω admit Φ–universal elements in Λc(Ω)?

Only in the case Φ = Aut(Ω) we are able to show that the answer to Problem 2.18 is affirmative. This, at least, proves the cases c < 0 of Theorem 1.6.

The proof of Theorem 2.17 and the case Φ = Aut(Ω) of Problem 2.18 below is based on the classical representation theorem of Liouville for constantly curved conformal metrics in terms of locally univalent holomorphic maps and the universality results for these maps from Section 2.2 and Section 2.3. A major drawback of this approach lies in the fact that Liouville's representation theorem only works for simply connected domains.

The case of constant curvature 0 is particularly interesting since a conformal metric λ(z) |dz| has curvature 0 if and only if u := log λ is harmonic. We can therefore also employ the Runge–type theorems for harmonic functions (see for example Theorem 3 in [12] or Theorem 4 in [11]) to prove existence of universal conformal metrics with constant curvature 0, even for not simply connected domains.
Theorem 2.19.
Let \( \Omega \subseteq \mathbb{C} \) be a domain of infinite connectivity and let \( \Phi \) be a family of locally univalent self–maps of \( \Omega \). Suppose that there exists a sequence \((\phi_n)\) in \( \Phi \) such that

(i) \( (\phi_n) \) is eventually injective, and

(ii) for every \( \mathcal{O} \)–convex compact set \( K \subseteq \Omega \) and every \( N \in \mathbb{N} \) there exists \( n \geq N \) such that \( \phi_n(K) \cap K = \emptyset \) and \( \phi_n(K) \cup K \) is \( \mathcal{O} \)–convex.

Then the following hold:
(a) There exists a harmonic function \( u \) on \( \Omega \) such that \( \{u \circ \phi : \phi \in \Phi\} \) is dense in the set of all harmonic functions on \( \Omega \).
(b) There exists a \( \Phi \)–universal element in \( \Lambda_0(\Omega) \).

For other universality results for harmonic functions, see [9], [4] or [7].

Remark 2.20. Suppose that \( u \) is a harmonic function on a domain \( \Omega \) in \( \mathbb{C} \) such that the conclusion (a) of Theorem 2.19 holds. Then one might be inclined to suspect that \( e^u \) is an \( \Phi \)–universal element in \( \Lambda_0(\Omega) \). However, the presence of the “conformal factor” \(|\phi'(z)|\) in the pullback \( \phi^*e^u(z) = e^{u(\phi(z))}|\phi'(z)| \) calls this into question.

Remark 2.21 (SK–metrics). Finally we would like to indicate that it is tempting to replace the classes \( \Lambda_c(\Omega) \) of conformal metrics with constant curvature \( c \) by other classes of conformal metrics which are invariant under pullback. The most prominent example is perhaps the class \( \text{SK}(\Omega) \) of (densities of) SK–metrics as introduced by M. Heins [19]. Now, in order to study universality aspects for \( \text{SK}(\Omega) \) one first needs to specify the underlying topology. In view of the fact that (densities of) SK–metrics are subharmonic functions, there are two natural topologies for this purpose (see [13]):

(T1) Topology of locally uniform convergence;
(T2) Topology of decreasing convergence.

Both topologies have been employed e.g. in [13] for proving universality results for subharmonic functions. While the (T1) topology is naturally restricted to the subclass of continuous subharmonic functions (see the discussion in [13], in particular Section 3), the (T2) topology turns out to be fairly natural for the class of all subharmonic functions (see e.g. Theorem 3.4 in [13]). However, the (T2) topology is not convenient for studying universality for SK–metrics! The reason is that one can show that the strong form of Ahlfors’ Lemma ([19, Theorem 7.1]) implies that w.r.t. the topology of decreasing convergence the density of the hyperbolic metric \( \lambda_\Omega(z)|dz| \) of the domain \( \Omega \) is an isolated point of \( \text{SK}(\Omega) \) and in fact the only candidate for a universal SK–metric in this setting. This, however, is clearly not possible since \( \phi^*\lambda_\Omega = \lambda_\Omega \) for every \( \phi \in \text{Aut}(\Omega) \).

We are therefore led to consider the set \( \text{SK}_c(\Omega) := \text{SK}(\Omega) \cap C(\Omega) \) of all continuous SK–metrics equipped the topology of locally uniform convergence.

Problem 2.22.
Let \( \Omega \subseteq \mathbb{C} \) be a hyperbolic domain. Does there exist a continuous SK–metric \( \Lambda(z)|dz| \) on \( \Omega \) such that \( \{\phi^*\Lambda : \phi \in \text{Aut}(\Omega)\} \) is dense in \( \text{SK}_c(\Omega) \)?
3 Proofs

We first introduce some notation.

We denote by \( \text{tr}(\gamma) \) the trace of a curve \( \gamma \) in \( \mathbb{C} \) and by \( \text{ind}_\gamma(z) \) the winding number of \( \gamma \) around \( z \). Let \( U \) be an open set in \( \mathbb{C} \) and let \( \mathcal{H}_{\neq 0}(U) \) be the set of all functions in \( \mathcal{H}(U) \) with no zeros in \( U \). For a set \( M \) in \( \mathbb{C} \) we write \( f \in \mathcal{H}_{\neq 0}(M) \) if there is an open neighborhood \( U \) of \( M \) such that \( f \in \mathcal{H}_{\neq 0}(U) \). For a compact set \( K \) in \( \mathbb{C} \) we set

\[
||f - g||_K := \max_{z \in K} |f(z) - g(z)|
\]

if \( f \) and \( g \) are holomorphic functions in a neighborhood of \( K \), and

\[
\chi_K(f, g) := \max_{z \in K} \chi(f(z), g(z))
\]

if \( f \) and \( g \) are meromorphic functions in a neighborhood of \( K \).

**Proposition 3.1.**

Let \( \Omega \) be a domain in \( \mathbb{C} \), \( K \) a compact \( O \)-convex set in \( \Omega \) and \( \epsilon > 0 \).

(a) Suppose \( f \in \mathcal{H}_{\neq 0}(K) \). Then there exists a connected compact \( O \)-convex set \( M \) in \( \Omega \) with piecewise differentiable boundary \( \partial M \) such that \( K \subseteq M \) and a function \( g \in \mathcal{H}_{\neq 0}(M) \) with \( ||f - g||_K < \epsilon \).

(b) Suppose \( f \in \mathcal{M}_{l.u.}(K) \). Then there exists a compact \( O \)-convex set \( M \) in \( \Omega \) with connected interior \( M^o \) such that \( K \subseteq M^o \) and a function \( g \in \mathcal{M}_{l.u.}(M) \) with \( \chi_K(f, g) < \epsilon \). If \( f \in \mathcal{M}_{l.u.}(K) \), then \( g \in \mathcal{M}_{l.u.}(M) \) with \( ||f - g||_K < \epsilon \).

**Proof.** We only prove part (a); the proof of part (b) is similar. By the classical theorem of Runge and by Hurwitz’ theorem, there exists a rational function \( g \in \mathcal{H}(\Omega) \cap \mathcal{H}_{\neq 0}(K) \) such that \( ||f - g||_K < \epsilon \). Let \( z_1, \ldots, z_N \) be the zeros of \( g \) in \( \Omega \). Since \( K \) is \( O \)-convex, there exist paths \( \gamma_j : [0, 1] \to \Omega \setminus K \) with \( \gamma_j(0) = z_j, \gamma_j(t) \to \partial \Omega \) for \( t \to 1 \) and such that \( W := \Omega \setminus (\text{tr}(\gamma_1) \cup \cdots \cup \text{tr}(\gamma_N)) \) is connected. Note that \( W \) is open and \( K \subseteq W \). Let \((M_n)\) be a compact exhaustion of \( W \) with connected compact \( O \)-convex sets in \( W \) such that \( \partial M_n \) is piecewise differentiable for each \( n \in \mathbb{N} \). Since a compact set in \( W \) is \( O \)-convex in \( W \) if and only if it is \( O \)-convex in \( \Omega \), we can take \( M = M_n \) with \( n \in \mathbb{N} \) sufficiently large so that \( M = M_n \supseteq K \).

3.1 Proof of Theorem 2.1 (a)

The proof of Theorem 2.1 is based on the following theorem:

**Theorem 3.2.**

Let \( \Omega \) be a domain in \( \mathbb{C} \), \( K \) a \( O \)-convex compact set in \( \Omega \) and \( g \in \mathcal{H}_{\neq 0}(K) \). Then there exists a sequence \((f_m) \subseteq \mathcal{H}_{\neq 0}(\Omega)\) such that \( \lim_{m \to \infty} f_m = g \) uniformly on \( K \) and

\[
(3.1) \quad \int_\gamma f_m(z)dz = \int_\gamma g(z)dz
\]

for every closed curve \( \gamma \subseteq K \) and every \( m \in \mathbb{N} \).
Remark 3.3. Theorem 3.2 is in fact a special case of Theorem 2 in [23], where closed holomorphic 1-forms on Stein manifolds are considered. We give a direct and simpler proof for the one–dimensional situation of Theorem 3.2.

Proof. By Proposition 3.1(a) we may assume that $K$ is connected and $\partial K$ is piecewise differentiable. Let $D_1, \ldots, D_n$ be the bounded connected components of $\mathbb{C} \setminus K$. For $j = 1, \ldots, n$ choose $z_j \in D_j \setminus \Omega$ and let $\gamma_j$ be a parametrization of the positively oriented boundary $\partial D_j$. Then $\text{ind}_{\gamma_j}(z_j) = \delta_{kj}$. The connectedness of $K$ implies that $\Gamma := \bigcup_{k=1}^n \text{tr}(\gamma_k)$ is a compact $O$–convex set in $\Omega$. Since every closed curve in $K$ is homologous to a linear combination of the curves $\gamma_1, \ldots, \gamma_n$ with integer coefficients, it suffices to find a sequence $(f_m) \in \mathcal{H}_{\neq 0}(\Omega)$ such that $\lim_{m \to \infty} f_m = g$ uniformly on $K$ and equation (3.1) holds for $\gamma = \gamma_k$ for every $k = 1, \ldots, n$.

Now for any $j = 1, \ldots, n$, Runge’s Theorem implies that there is a sequence $(w_{j,m})$ in $H(\Omega)$ such that $\lim_{m \to \infty} w_{j,m}(z) = 1$ uniformly on $\Gamma$. In particular,

$$\lim_{m \to \infty} \left( \int_{\gamma_{jk}} w_{j,m}(z) g(z) dz \right)_{k,j=1,\ldots,n} = E_n,$$

where $E_n \in \mathbb{C}^{n \times n}$ is the identity matrix. Hence we can find a $\mu \in \mathbb{N}$ such that the matrix

$$A := \left( \int_{\gamma_{jk}} w_{j,\mu}(z) g(z) dz \right)_{k,j=1,\ldots,n}$$

is non–singular.

By a well–known extension of Runge’s Theorem (Theorem 6.3.1 in [26]) there exists a sequence $(g_m) \in \mathcal{H}_{\neq 0}(\Omega)$ such that $\lim_{m \to \infty} g_m = g$ uniformly on $K$. Consider the functions

$$\psi_k : \mathbb{C}^n \to \mathbb{C}, \quad (s_1, \ldots, s_n) \mapsto \int_{\gamma_k} \exp \left( \sum_{j=1}^n s_j w_{j,\mu}(z) \right) g(z) dz,$$

$$\psi_{k,m} : \mathbb{C}^n \to \mathbb{C}, \quad (s_1, \ldots, s_n) \mapsto \int_{\gamma_k} \exp \left( \sum_{j=1}^n s_j w_{j,\mu}(z) \right) g_m(z) dz,$$

and the entire functions $\psi, \psi_m : \mathbb{C}^n \to \mathbb{C}^n$ defined by $\psi(s) := (\psi_1(s), \ldots, \psi_n(s))$ and $\psi_m(s) := (\psi_{1,m}(s), \ldots, \psi_{n,m}(s))$. Then $\lim_{m \to \infty} \psi_m = \psi$ locally uniformly on $\mathbb{C}^n$ and $D\psi(0) = A$ is non–singular. Hence there exists a sequence $(s_m) = (s_{1,m}, \ldots, s_{n,m})$ in $\mathbb{C}^n$ with $\lim_{m \to \infty} s_m = 0$ and $\psi_m(s_m) = \psi(0)$. This concludes the proof with

$$f_m(z) = \exp \left( \sum_{j=1}^n s_{j,m} w_{j,\mu}(z) \right) g_m(z).$$

$\blacksquare$
Proof of Theorem 2.1(a). By Proposition 3.1 (b) we may assume that \( f \in \mathcal{H}_{l.u.}(M) \) for some connected \( \mathcal{O} \)-convex compact set \( M \) of \( \Omega \) with smooth boundary and such that \( K \subseteq M^o \). Hence we can apply Theorem 3.2 to \( f' \in \mathcal{H}_{\neq 0}(M) \), so there exists a sequence \((g_n) \subseteq \mathcal{H}_{\neq 0}(\Omega)\) with \( \lim_{n \to \infty} g_n = f' \) uniformly on \( M \) and

\[
\int_\gamma g_n(z) \, dz = \int_\gamma f'(z) \, dz = 0
\]

for every closed curve \( \gamma \) in \( M \).

Now we choose a compact exhaustion \((K_k)_k\) of \( \Omega \) by connected \( \mathcal{O} \)-convex sets in \( \Omega \) with smooth boundaries and such that \( K_1 = M \). Suppose we have fixed arbitrary numbers \( \varepsilon > 0 \), \( k \in \mathbb{N} \) and a function \( h \in \mathcal{H}_{\neq 0}(\Omega) \) with \( \int_\gamma h(z) \, dz = 0 \) for every closed curve \( \gamma \) in \( K_{k+1} \). From this fact and an obvious induction argument, we can deduce that there exists a sequence \((v_{n,k})_k\) in \( \mathcal{H}(\Omega) \) with \( \|v_{n,k}\|_{K_k} < \frac{\varepsilon}{2^k} \) and such that for every closed curve \( \gamma \) in \( K_k \) we have

\[
\int_\gamma e^{\sum_{j=1}^k v_{n,j}(z)} g_n(z) \, dz = 0.
\]

We define a holomorphic function \( w_n \in \mathcal{H}(\Omega) \) by

\[
w_n(z) := \sum_{j=1}^{\infty} v_{n,j}(z).
\]

Clearly we have \( \|w_n\|_K < \frac{1}{n} \) and

\[
\int_\gamma e^{w_n(z)} g_n(z) \, dz = 0
\]

for every closed curve \( \gamma \) in \( \Omega \). This means that for fixed \( z_0 \in K \) and for each \( n \) there is an anti-derivative \( G_n \in \mathcal{H}(\Omega) \) of \( e^{w_n}g_n \) with \( G_n(z_0) = f(z_0) \). By construction, \( G_n \in \mathcal{H}_{l.u.}(\Omega) \). Since \( M \) is connected and \( \lim_{n \to \infty} G''_n = f' \) uniformly on \( M \) we conclude \( \lim_{n \to \infty} G_n = f \) uniformly on \( M \) and hence on \( K \).

\[\blacksquare\]

3.2 Proof of Theorem 2.1 (b)

By Proposition 3.1 (b) we may assume that \( f \in \mathcal{M}_{l.u.}(M) \) for some compact \( \mathcal{O} \)-convex set \( M \) in \( \mathbb{C} \) whose interior \( G := M^o \) is connected and contains \( K \). Since \( f \) is locally univalent in a neighborhood of \( M \), its Schwarzian derivative \( S_f \) is holomorphic there, so \( S_f \in \mathcal{H}(M) \). According to some basic facts about complex differential equations, see e.g. [21, Theorem 6.1], we can recover \( f \) from \( S_f \) by writing \( f \) as the quotient,

\[
f = \frac{u_1}{u_2},
\]
of two linearly independent solutions \( u_1, u_2 \in \mathcal{H}(\Omega) \) of the homogeneous linear differential equation

\[
(3.2) \quad w'' + \frac{1}{2} S f \cdot w = 0 .
\]

Since \( S f \in \mathcal{H}(M) \) and \( \mathbb{C} \setminus M \) has no bounded components, the classical Runge theorem shows that there exist polynomials \( p_n : \mathbb{C} \to \mathbb{C} \) such that

\[
p_n \to S f \quad \text{uniformly on } M .
\]

We now consider the homogeneous linear differential equations corresponding to the polynomials \( p_n \). Fix \( z_0 \in G \) with \( u_2(z_0) \neq 0 \), and let \( v_n \in \mathcal{H}(\mathbb{C}) \) be the unique solution of the initial value problem

\[
v_n'' + \frac{1}{2} p_n \cdot v_n = 0, \quad v_n(z_0) = u_1(z_0), \quad v_n'(z_0) = u_1'(z_0) .
\]

Then we clearly have

\[
v_n(z) = u_1(z_0) + u_1'(z_0)(z - z_0) - \frac{1}{2} \int_{z_0}^{z} (z - \xi)p_n(\xi)v_n(\xi) \, d\xi, \quad z \in \mathbb{C} .
\]

Hence a standard application of Gronwall's lemma (cf. [21, Lemma 5.10]) shows that the sequence \( (v_n) \) is locally bounded in \( G \). We are therefore in a position to apply Montel’s theorem and conclude that \( \{v_n : n \in \mathbb{N}\} \) is a normal family. Clearly, every subsequential limit function \( v \in \mathcal{H}(G) \) of \( (v_n) \) is a solution of \( (3.2) \) with \( v(z_0) = u_1(z_0) \) and \( v'(z_0) = u_1'(z_0) \) in \( G \). By uniqueness of this solution, we conclude \( v = u_1 \). Consequently, we have

\[
v_n \to u_1 \quad \text{locally uniformly in } G .
\]

For the unique solution \( w_n \in \mathcal{H}(\mathbb{C}) \) of the initial value problem

\[
w_n'' + \frac{1}{2} p_n \cdot w_n = 0, \quad w_n(z_0) = u_2(z_0), \quad w_n'(z_0) = u_2'(z_0) ,
\]

we arrive in a similar way at

\[
w_n \to u_2 \quad \text{locally uniformly in } G .
\]

We claim that \( v_n \) and \( w_n \) are linearly independent for large \( n \). For this purpose we consider the Wronskian

\[
W(h, g) = hg' - h'g \in \mathcal{H}(G) \quad \text{for } g, h \in \mathcal{H}(G) .
\]

Since \( u_1 \) and \( u_2 \) are solutions of the linear differential equation \( (3.2) \), there is a constant \( \lambda \in \mathbb{C} \) such that \( W(u_1, u_2)(z) = \lambda \) for all \( z \in G \), see [21, Proposition 1.4.8]. In a similar way, we see that for each \( n \in \mathbb{N} \) there is \( \lambda_n \in \mathbb{C} \) such that \( W(v_n, w_n)(z) = \lambda_n \) for all \( z \in \mathbb{C} \). By what we have already shown, \( \lambda_n \to \lambda \) as \( n \to \infty \). Since \( u_1 \) and \( u_2 \) are linearly
independent, we have $\lambda \neq 0$, see [21] Proposition 1.4.2. Hence $\lambda_n \neq 0$, so $v_n$ and $w_n$ are linearly independent for all but finitely many $n \in \mathbb{N}$.

We can therefore apply Theorem 6.1 in [21] which implies that

$$g_n := \frac{v_n}{w_n} \in \mathcal{M}_{l.u.}(\mathbb{C}).$$

Since $v_n \to u_1$ and $w_n \to u_2$ locally uniformly in $G$, we see that $g_n \to u_1/u_2 = f$ locally uniformly in $G$ w.r.t. the chordal metric, so in particular $\chi$–uniformly on $K$. ■

### 3.3 Proof of Proposition 2.7

Suppose that $u \in \mathcal{H}_{l.u.}(\Omega)$ is $\Phi$–universal in $\mathcal{H}_{l.u.}(\Omega)$, and let $K$ be a compact set in $\Omega$. Choose a compact set $L$ in $\Omega$ which contains $K$ in its interior and which is the closure of its interior. Let

$$\delta := \frac{1}{2} \text{dist}(K, \partial L) > 0, \quad M := \sup_{z \in L} |z|.$$ 

Then $f(z) := z + 2M + 2\delta$ belongs to $\mathcal{H}_{l.u.}(\Omega)$. Since $u$ is $\Phi$–universal in $\mathcal{H}_{l.u.}(\Omega)$ there exists $\phi \in \Phi$ such that

$$(3.3) \quad \|u \circ \phi - f\|_L < \delta.$$ 

This in particular implies $|u(\phi(z))| \geq |f(z)| - \delta \geq M + \delta$ for all $z \in K$, so

$$\min_{z \in K} |u(\phi(z))| \geq M + \delta > M \geq \max_{\phi(z) \in K} |\phi(z)|,$$

and thus $\phi(K) \cap K = \emptyset$. Next, we fix $z_0 \in K$. Then the estimate (3.3) shows that for every $z \in \partial L$ we have

$$|[u(\phi(z_0)) - u(\phi(z))] - [z_0 - z]| < 2\delta \leq |z_0 - z|.$$ 

Hence, by Rouché’s theorem, $u(\phi(z_0)) - u(\phi(z))$ and $z_0 - z$ have the same numbers of zeros in $L^\circ$. This implies that $\phi$ is injective on $K$.

Finally let $(K_n)$ be a compact exhaustion of $\Omega$. We can apply the reasoning above to each $K_n$ to obtain a run–away and eventually injective sequence in $\Phi$.

### 3.4 Proofs of Theorems 2.8 and 2.12, and Corollary 2.13

We are going to apply a fairly standard universality criterion. Let $\mathcal{T}$ be a collection of continuous self–maps of a topological space $X$. We say that $\mathcal{T}$ acts transitively on $X$ if for every pair of open sets $U$ and $V$ in $X$ there is an $\tau \in \mathcal{T}$ such that $\tau(U) \cap V \neq \emptyset$. An element $u \in X$ is called universal for $\mathcal{T}$ if the orbit $\{\tau(u) : \tau \in \mathcal{T}\}$ is dense in $X$.

**Theorem 3.4** (Birkhoff transitivity criterion).

Let $X$ be a second countable Baire–space and $\mathcal{T}$ a family of continuous self–maps of $X$. Suppose that $\mathcal{T}$ acts transitively on $X$. Then there exists an universal element for $\mathcal{T}$ and the set of all universal elements for $\mathcal{T}$ is a dense $G_\delta$–subset of $X$. 

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For a proof see for instance [15, Theorem 1]. For later use, we note that if $X$ is a separable metric space, then a collection $T$ of continuous self–maps of $X$ acts transitively on $X$ if and only if for every pair of points $v$ and $w$ in $X$ there exist a sequence $(v_n)$ in $X$ and a sequence $(\tau_n)$ in $T$ such that $v_n \to v$ and $\tau_n(v_n) \to w$.

We now apply these concepts to investigate universality for holomorphic and meromorphic functions. Let $\Omega$ be a domain in $\mathbb{C}$. We associate to any holomorphic self–map $\phi$ of $\Omega$ the composition operator

$$C_\phi : \mathcal{H}(\Omega) \to \mathcal{H}(\Omega), \quad f \mapsto f \circ \phi.$$  

If $\phi$ is locally univalent, then $C_\phi$ maps $\mathcal{H}_{l.u.}(\Omega)$ into $\mathcal{H}_{l.u.}(\Omega)$. Since the union of $\mathcal{H}_{l.u.}(\Omega)$ with all constant functions is a complete metric space and $\mathcal{H}_{l.u.}(\Omega)$ is an open subset of this space, $\mathcal{H}_{l.u.}(\Omega)$ is a Baire–space.

Proof of Theorem 2.8: In view of Theorem 3.4 it suffices to show that the family \{ $C_\phi : \phi \in \Phi$ \} acts transitively on $\mathcal{H}_{l.u.}(\Omega)$. Let $f, g \in \mathcal{H}_{l.u.}(\Omega)$. Since $\Omega$ is simply connected there is an exhaustion $(K_n)$ of $\Omega$ with compact sets $K_n$ in $\Omega$ such that each $K_n$ has connected complement. By assumption there is a sequence $(\phi_n)$ in $\Phi$ such that $\phi_n$ is injective on $K_n$ and $\phi_n(K_n) \cap K_n = \emptyset$ for each $n \in \mathbb{N}$. Define $L_n := K_n \cup \phi_n(K_n)$ and $h_n \in \mathcal{H}_{l.u.}(L_n)$ by

$$(3.4) \quad h_n(z) := \begin{cases} f(z), & z \in K_n \\ g(\phi_n^{-1}(z)), & z \in \phi_n(K_n). \end{cases}$$

Note that each $L_n$ has connected complement, hence by Theorem 2.1 (a) there exists a function $f_n \in \mathcal{H}_{l.u.}(\Omega)$ with $\|f_n - h_n\|_{K_n} \leq 1/n$. This implies $f_n \to f$ and $f_n \circ \phi_n \to g$ locally uniformly in $\Omega$.

The proof of Theorem 2.12 is identical except for that we need to apply part (b) of Theorem 2.1 instead of part (a).

Proof of Corollary 2.13. By Theorem 2.12 there exists a universal function $f \in \mathcal{M}_{l.u.}(\mathbb{C})$ such that $T_f$ is dense in $\mathcal{M}_{l.u.}(\mathbb{C})$. Let $\Omega \subseteq \mathbb{C}$ be a simply connected domain. Then, as a consequence of Theorem 2.1 (b), $\mathcal{M}_{l.u.}(\mathbb{C})$ is dense in $\mathcal{M}_{l.u.}(\Omega)$. This fact together with the universality of $f$ implies, that $T_f$ is dense in $\mathcal{M}_{l.u.}(\Omega)$.

3.5 Proofs of Theorems 2.9, 2.10 and 2.19

Except for Theorem 2.19 (b), the proofs are similar to the proof of Theorem 2.8 so we only indicate the modifications that are necessary.

Proof of Theorem 2.9. We start with an exhaustion $(K_n)$ of $\Omega$ with compact sets $K_n$ in $\Omega$. We can assume that each $K_n$ is $O$–convex in $\Omega$. By hypothesis, there is a sequence $(\phi_n)$ in $\Phi$ that has all the properties we need in order to proceed as in the proof of Theorem 2.8.
Proof of Theorem 2.10. By Proposition 2.7 we only need to show the “if”-part. Since by hypothesis, Ω is not conformally equivalent to C \ {0}, the existence of a run-away sequence in Aut(Ω) implies that Ω is either simply connected or of infinite connectivity, see the discussion following Lemma 2.9 in [5, p. 51–52]. In the first case, when Ω is simply connected, we can simply apply Theorem 2.8. In the second case, when Ω is of infinite connectivity, we start with an exhaustion \( (K_n) \) of Ω with 0-convex compact sets \( K_n \) in Ω. Since \( \phi_n \) is run-away we may assume that \( \phi_n(K_n) \cap K_n = \emptyset \) for each \( n \in \mathbb{N} \). Then, by a key observation (see [5, Lemma 2.12]), it follows that each \( \phi_n(K) \cup K \) is 0-convex in Ω. We are thus in a position to apply Theorem 2.9. ■

Proof of Theorem 2.19 (a). The proof is identical to the proof of Theorem 2.9 except that one has to apply a Runge-type theorem for harmonic functions (see for example Theorem 3 in [12] or Theorem 4 in [11]) instead of Theorem 2.1 (a). ■

Proof of Theorem 2.19 (b). We show that the collection of continuous maps
\[
\Lambda_0(\Omega) \to \Lambda_0(\Omega), \quad \lambda \mapsto \phi^* \lambda , \quad \phi \in \Phi,
\]
acts transitively on \( \Lambda_0(\Omega) \) and then apply Theorem 3.4. Let \( (K_n) \) be a compact exhaustion of Ω such that each \( K_n \) is 0-convex. Then by the assumptions on \( \Phi \) we can find a sequence \( (\phi_n) \) in \( \Phi \) such that each \( \phi_n \) is injective in an open neighborhood of \( K_n \), \( K_n \cap \phi_n(K_n) = \emptyset \) and \( L_n := K_n \cup \phi_n(K_n) \) is 0-convex. Let \( \lambda, \mu \in \Lambda_0(\Omega) \) and define \( h_n : L_n \to \mathbb{R} \) by
\[
h_n(z) := \begin{cases} 
\log \lambda(z), & z \in K_n \\
\log [ (\phi_n^{-1})^* \mu(z) ], & z \in \phi_n(K_n)
\end{cases}
\]
Since \( h_n \) is harmonic in a neighborhood of \( L_n \), we can find for each \( \delta_n > 0 \) a harmonic function \( u_n : \Omega \to \mathbb{R} \) such that \( \| u_n - h_n \|_{L_n} \leq \delta_n \) (see Theorem 4 in [11]). We choose \( \delta_n > 0 \) so small that
\[
(3.5) \quad \| e^{u_n} - e^{h_n} \|_{L_n} \leq \min \left\{ \frac{1}{n}, \frac{\| \phi_n^* \|_{K_n}}{n} \right\}.
\]
Define \( \lambda_n := e^{u_n} \in \Lambda_0(\Omega) \). Then we have \( \| \lambda_n - \lambda \|_{K_n} \leq 1/n \). On the other hand note that \( \mu = \phi_n^*(e^{h_n}) \) on \( K_n \), so
\[
\| \phi_n^* \lambda_n - \mu \|_{K_n} = \| \phi_n^* \cdot (e^{u_n} \circ \phi_n - e^{h_n} \circ \phi_n) \|_{K_n} \leq \| \phi_n^* \|_{K_n} \cdot \| e^{u_n} - e^{h_n} \|_{L_n} \leq \frac{1}{n}.
\]
We conclude \( \lambda_n \to \lambda \) and \( \phi_n^* \lambda_n \to \mu \) locally uniformly in Ω. ■

3.6 Proof of Theorem 2.15

We start with a simple observation. Let \( f \in \mathcal{M}(D) \) and let \( 0 \in \Omega_f \), i.e., 0 is a non-critical point of \( f \). Then there is unique universal covering map \( \Psi_f \) from \( D \) onto \( \Omega_f \) such that \( \Psi_f(0) = 0 \) and \( \Psi_f'(0) > 0 \). We call \( \Psi_f \) the normalized universal covering of \( \Omega_f \). Note that \( \Psi_f = \text{id} \) if \( f \) is locally univalent. If \( (f_n) \) is a sequence in \( \mathcal{M}(D) \) which converges locally \( \chi \)-uniformly in \( D \) to \( f \), then 0 is also a non-critical point of \( f_n \) for all but finitely many \( n \), so \( 0 \in \Omega_{f_n} \).
Proposition 3.5.
Let \( f \in \mathcal{M}(\mathbb{D}) \) such that \( 0 \in \Omega_f \) and suppose that \( (f_n)_n \subseteq \mathcal{M}(\mathbb{D}) \) converges locally \( \chi \)-uniformly in \( \mathbb{D} \) to \( f \). Then \( f_n \circ \Psi_{f_n} \to f \circ \Psi_f \) locally \( \chi \)-uniformly in \( \mathbb{D} \).

Proof. A straightforward application of Hurwitz’s theorem shows that \( \Omega_f \) is the kernel of the sequence of domains \( \Omega_{f_n} \) (with respect to the point \( 0 \)), that is, \( \Omega_f \) is the largest domain \( D \) in \( \mathbb{C} \) containing \( 0 \) such that each compact subset \( K \) of \( D \) is contained in all but finitely many of the domains \( \Omega_{f_n} \). A well-known result of Hejhal \([20]\) then implies \( \Psi_{f_n} \to \Psi_f \) locally uniformly in \( \mathbb{D} \). Since \( f_n \to f \) locally \( \chi \)-uniformly in \( \mathbb{D} \), we can deduce \( f_n \circ \Psi_{f_n} \to f \circ \Psi_f \) locally \( \chi \)-uniformly in \( \mathbb{D} \). \( \blacksquare \)

Proof of Theorem 2.17. Let \( G \in \mathcal{M}(\mathbb{D}) \) be a non–constant universal function for \( \mathcal{G} \). By precomposing \( G \) with a disk automorphism we may assume that \( 0 \) is not a critical point of \( G \), so \( 0 \in \Omega_G \). Let \( \Psi_G \) denote the normalized universal covering of \( \Omega_G \).

(i) Let \( F := G \circ \Psi_G \in \mathcal{M}_{l.u.}(\mathbb{D}) \) and \( f \in \mathcal{G}_{l.u.} \). We show that there is a sequence \( (\phi_n) \) in \( \text{Aut}(\mathbb{D}) \) such that \( F \circ \phi_n \to f \) locally \( \chi \)-uniformly on \( \mathbb{D} \).

In fact, since \( G \) is universal for \( \mathcal{G} \) and \( f \in \mathcal{G} \) there is a sequence \( (\alpha_n) \) in \( \text{Aut}(\mathbb{D}) \) such that \( G \circ \alpha_n \to f \) locally \( \chi \)-uniformly in \( \mathbb{D} \). By our preliminary discussion we may assume that \( 0 \in \Omega_{\alpha_n} \) for each \( n \in \mathbb{N} \). Let \( \Psi_{\alpha_n} \) denote the normalized universal covering for \( \Omega_{\alpha_n} \). By Proposition 3.5 we see that

\[
(3.6) \quad G \circ \alpha_n \circ \Psi_{\alpha_n} \to f \circ \Psi_f = f
\]

locally \( \chi \)-uniformly on \( \mathbb{D} \). Now we observe that \( \alpha_n \circ \Psi_{\alpha_n} \) is a universal covering map from \( \mathbb{D} \) onto \( \Omega_G \) since \( \alpha_n(\Omega_{\alpha_n}) = \Omega_G \). This implies that there is \( \phi_n \in \text{Aut}(\mathbb{D}) \) such that

\[
\alpha_n \circ \Psi_{\alpha_n} = \Psi_G \circ \phi_n.
\]

Using \((3.6)\) we get that

\[
F \circ \phi_n = G \circ \Psi_G \circ \phi_n = G \circ \alpha_n \circ \Psi_{\alpha_n} \to f
\]

locally \( \chi \)-uniformly on \( \mathbb{D} \).

(ii) Now let \( \Psi \) be any universal covering from \( \mathbb{D} \) onto \( \Omega_G \). Then \( \Psi = \Psi_G \circ T \) for some \( T \in \text{Aut}(\mathbb{D}) \). Hence, if \( f \in \mathcal{G}_{l.u.} \), then by (i) there is a sequence \( (\phi_n) \) in \( \text{Aut}(\mathbb{D}) \) such that \( G \circ \Psi_G \circ \phi_n \to f \) locally \( \chi \)-uniformly in \( \mathbb{D} \), so

\[
G \circ \Psi \circ (T^{-1} \circ \phi_n) = G \circ \Psi_G \circ \phi_n \to f
\]

locally \( \chi \)-uniformly in \( \mathbb{D} \) with \( T^{-1} \circ \phi_n \in \text{Aut}(\mathbb{D}) \) for each \( n \in \mathbb{N} \). \( \blacksquare \)

3.7 Proof of Theorem 2.17 and Theorem 1.6 (Case \( c < 0 \))

We first need to briefly recall the standard way of generating regular conformal metrics \( \lambda(z) |dz| \) with constant curvature \( c \in \mathbb{R} \) on a domain \( \Omega \) in \( \mathbb{C} \). Note that scaling \( \lambda(z) |dz| \) by \( |c| \), we get the metric \( |c| \lambda(z) |dz| \), where

\[
c^2 \kappa_{|c| \lambda} = \kappa_{\lambda}.
\]
In what follows we might therefore restrict ourselves without loss of generality to the normalized cases $c \in \{-1, 0, +1\}$. For these cases the canonical constantly curved metrics are given by

$$
\lambda_{D_c}(z) |dz| := \begin{cases} 
\frac{2}{1 - |z|^2} |dz| & D_c = \mathbb{D} \text{ and } c = -1 \quad \text{(hyperbolic case)}, \\
1 |dz| & D_c = \mathbb{C} \text{ and } c = 0 \quad \text{(Euclidean case)}, \\
\frac{2}{1 + |z|^2} |dz| & D_c = \hat{\mathbb{C}} \text{ and } c = +1 \quad \text{(spherical case$^1$)}.
\end{cases}
$$

Then for any $f \in \mathcal{M}_{l.u.}(\Omega)$ with $f(\Omega) \subseteq D_c$ the pullback of $\lambda_{D_c}(z) |dz|$ by $f$ provides us with a conformal metric $f^* \lambda_{D_c}(z) |dz|$ on $\Omega$ with constant curvature $c$. Liouville [22] discovered that the converse statement holds for simply connected domains:

**Theorem 3.6 (Liouville).**

Let $\Omega$ be a simply connected domain in $\mathbb{C}$ and $c \in \{-1, 0, 1\}$. Then for any $\lambda \in \Lambda_c(\Omega)$ there exists a function $f \in \mathcal{M}_{l.u.}(\Omega)$ with $f(\Omega) \subseteq D_c$ such that

$$
\lambda = f^* \lambda_{D_c}.
$$

In addition, $f$ is uniquely determined by $\lambda$ up to postcomposition with a holomorphic rigid motion $T$ of $D_c$.

Recall that the holomorphic rigid motions of $D_c$ are

(a) the conformal automorphisms of $\mathbb{D}$ for $c = -1$;

(b) the direct Euclidean motions of $\mathbb{C}$ (i.e., the maps $z \mapsto z + b$, $b \in \mathbb{C}$) for $c = 0$;

(c) the rotations of the Riemann sphere $\hat{\mathbb{C}}$ for $c = +1$.

Hence Liouville’s theorem gives us, for simply connected domains $\Omega$, a bijection from the set of all locally univalent holomorphic mappings from $\Omega$ to $D_c$ (modulo the action of the rigid motions of $D_c$) onto the set $\Lambda_c(\Omega)$ of all conformal metrics with constant curvature $c$. The next result is an immediate consequence of Liouville’s theorem and shows that this map is “universality preserving”:

**Proposition 3.7.**

Let $\Omega$ be a simply connected domain in $\mathbb{C}$ and $c \in \{-1, 0, 1\}$. Suppose that $\lambda \in \Lambda_c(\Omega)$ and $f \in \mathcal{M}_{l.u.}(\Omega)$ with $f(\Omega) \subseteq D_c$ such that

$$
\lambda = f^* \lambda_{D_c}.
$$

If $f$ is $\Phi$–universal in $\{g \in \mathcal{M}_{l.u.}(\Omega) : g(\Omega) \subseteq D_c\}$, then $\lambda$ is $\Phi$–universal in $\Lambda_c(\Omega)$.

Note that Theorem 2.17 follows directly from Proposition 3.7 and

---

$^1$Here, we have to consider local coordinates.
(i) Theorem 2.8 if \( c = 0 \);
(ii) Theorem 2.12 if \( c > 0 \).

Proposition 3.7 and Theorem 1.5 also prove the cases \( c < 0 \) of Theorem 1.6.

**Proof of Proposition 3.7.** Let \( \mu \in \Lambda_c(\Omega) \). By Liouville’s theorem there exists a map \( g \in M_{l.u.}(\Omega) \) with \( g(\Omega) \subseteq D_c \) such that \( \mu = g^*\lambda_{D_c} \). Since \( f \) is \( \Phi \)-universal in \( \{ g \in M_{l.u.}(\Omega) : g(\Omega) \subseteq D_c \} \) there is a sequence \( (\phi_n) \) in \( \Phi \) with the property that

\[
\phi_n \lambda(z) = \lambda_{D_c} (f \circ \phi_n(z)) \quad |(f \circ \phi_n)'(z)| \rightarrow \lambda_{D_c}(g(z)) \quad |g'(z)| = \mu(z)
\]

locally \( \chi \)-uniformly in \( \Omega \). This clearly implies

\[
f \circ \phi_n \rightarrow g
\]

locally \( \chi \)-uniformly in \( \Omega \). ■

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