Paley Graphs and Their Generalizations

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Introduction

Paley graphs are named after Raymond Paley (7 January 1907 – 7 April 1933). He was born in Bournemouth, England. He won a Smith’s Prize in 1930 and was elected a fellow of Trinity College, Cambridge, where he showed himself as one of the most brilliant students among a remarkable collection of fellow undergraduates.

In Paley graphs, finite fields form their sets of vertices. So to understand Paley graphs we will start our work with classification of finite fields and study their properties. We will show that any finite field \( \mathbb{F} \) has \( p^n \) elements, where \( p \) is prime and \( n \in \mathbb{N} \). Moreover, for every prime power \( p^n \) there exists a field with \( p^n \) elements, and this field is unique up to isomorphism.

In the last section of Chapter 1 we will see how one can construct the finite field for any prime power \( p^n \), and we will give the explicit construction of the fields of 9, 16, and 25 elements.

To construct a Paley graph, we fix a finite field and consider its elements as vertices of the Paley graph. Two vertices are connected by an edge if their difference is a square in the field. In the first section of Chapter 2 we will give some basic definitions and properties from graph theory which we will use in the study of the Paley graphs. In the second section we will give the definition of the Paley graph and we will give examples of the Paley graphs of order 5, 9, and 13. Finally, we will study some important properties of the Paley graphs.

In particular, we will show that the Paley graphs are connected, symmetric, and self-complementary. In [13], Peisert proved that the Paley graphs of prime order are the only self-complementary symmetric graphs of prime order; furthermore, in [12], he proved that any self-complementary and symmetric graph is isomorphic to a Paley graph, a \( \mathcal{P}^* \)-graph, or the exceptional graph \( G(23^2) \) with 23² vertices.

Also we will show that the Paley graph of order \( q \) is \( \frac{q-1}{2} \)-regular, and every two adjacent vertices have \( \frac{q-1}{4} \) common neighbors, and every two non-adjacent vertices have \( \frac{q-1}{4} \) common neighbors, which means that the Paley
graphs are strongly regular with parameters \((q, \frac{q-1}{2}, \frac{q-5}{4}, \frac{q-1}{4})\), see also [5].

**Fig. 0.1.** The Paley graph of order 17

In [5], Ananchuen and Caccetta proved that for every 3-element subset \(S\) of the vertices of the Paley graph with at least 29 vertices, and for every subset \(T\) of \(S\), there is a vertex \(x \notin S\) which is joined to every vertex in \(T\) and to no vertex in \(S \setminus T\); that is, the Paley graphs are 3-existentially closed.

Paley graphs are generalized by many mathematicians. In the first section of Chapter 3 we will see three examples of these generalizations and some of their basic properties.

In [1], Ananchuen introduced two of these generalizations. The cubic Paley graphs, in which pairs of elements of a finite field are connected by an edge if and only if they differ in a cubic residue, and the quadruple Paley graphs, in which pairs of elements of a finite field are connected by an edge if and only if they differ in a quadruple residue. The third generalization is called the generalized Paley graphs, in this family of graphs, pairs of elements of a finite field are connected by an edge if and only if their difference belongs to a subgroup \(S\) of the multiplicative group of the field. This generalization is given by Lim and Praeger in [10].

In the second section of Chapter 3 we will define a new generalization of the Paley graphs, in which pairs of elements of a finite field are connected by an edge if and only if there difference belongs to the \(m\)-th power of the multiplicative group of the field, for any odd integer \(m > 1\), and we call them the \(m\)-Paley graphs.

Since the cubic Paley graphs are 3-Paley graphs, we can say that the cubic Paley graphs are a special case of the family of \(m\)-Paley graphs. Also, we will give some examples of this family.
In the third section we will show that the $m$-Paley graph of order $q$ is complete if and only if $\gcd(m, q - 1) = 1$ and when $d = \gcd(m, q - 1) > 1$, the $m$-Paley graph is $\frac{q-1}{d}$-regular.

Also we will prove that the $m$-Paley graphs are symmetric but not self-complementary. In particular, $m$-Paley graphs are not in the Peisert’s list. Since strongly regular graphs must be self-complementary, we see that the $m$-Paley graphs are not strongly regular.

We will show also that the $m$-Paley graphs of prime order are connected but the $m$-Paley graphs of order $p^n$, $n > 1$ are not necessary connected, for example they are disconnected if $\gcd(m, p^n - 1) = \frac{p^n-1}{2}$. 

Chapter 1

Finite Fields

This chapter provides an introduction to some basic properties of finite fields and their structure. This introduction will be useful for understanding the properties of Paley graphs in which the elements of a finite field represent the set of vertices.

1.1 Basic definitions and properties

**Definition:** A field $\mathbb{F}$ is a set of at least two elements, with two operations $\oplus$ and $\ast$, for which the following axioms are satisfied:

1. The set $\mathbb{F}$ under the operation $\oplus$ forms an abelian group (whose identity is denoted by 0).
2. The set $\mathbb{F}^* = \mathbb{F} \setminus \{0\}$ under the operation $\ast$ forms an abelian group (whose identity is denoted by 1).
3. Distributive law: For all $a, b, c \in \mathbb{F}$, we have $(a \oplus b) \ast c = (a \ast c) \oplus (b \ast c)$.

Note that it is important to allow 0 to be an exceptional element with no inverse, because if 0 had an inverse then $1 = 0^{-1} \ast 0 = 0$, it would follow that $x = x \ast 1 = 0$ for all $x \in \mathbb{F}$; therefore, $\mathbb{F}$ would consist of only one element 0.

**Examples**

1. $\mathbb{Q}, \mathbb{R},$ and $\mathbb{C}$ are fields with respect to the usual addition and multiplication.

2. The subring $\mathbb{Q}[i] := \{a + bi \in \mathbb{C} : a, b \in \mathbb{Q}\}$ of $\mathbb{C}$ is a field, called the field of Gaussian rationals.
3. The ring of integers modulo $p$, $\mathbb{Z}_p$, is a field if $p$ is a prime number.

4. A commutative division ring (a ring in which every nonzero element has a multiplicative inverse) is a field.

5. A finite integral domain (a commutative ring with no zero divisors) is a field.

   Indeed, let $R$ be an integral domain and $0 \neq a \in R$. The map $x \to ax$, $x \in R$, is injective because $R$ is an integral domain ($a(x_1 - x_2) = 0 \iff (x_1 - x_2) = 0$). If $R$ is finite, the map is surjective as well, so that $ax = 1$ for some $x$, i.e., every nonzero element $a$ has an inverse.

6. The quotient ring $R/M$ such that $R$ is a commutative ring and $M$ is a maximal ideal, is a field.

   Indeed, since $R$ is commutative, the ring $R/M$ is commutative, where $(x+M)(y+M) = xy + M = yx + M = (y+M)(x+M)$. It also has an identity $1_{R/M} = 1_R + M$. Moreover, $1_R \notin M$ because if $1_R \in M$, then $M = R$, which contradicts the definition of maximal ideal. In order to show that $R/M$ is a field, it remains to prove that every nonzero element $x + M \in R/M$ has an inverse. So we fix $x \notin M$ and consider the set $I = M + xR = \{m + xr : m \in M, r \in R\}$. First we need to check that $I$ is an ideal. Let $m_1, m_2 \in M$ and $r_1, r_2, s \in R$, then $0 = 0 + x0 \in I$, $s((m_1 + xr_1) - (m_2 + xr_2)) = s(m_1 - m_2) + s(x(r_1 - r_2)) = s(m_1 - m_2) + xs(r_1 - r_2) \in I$. Hence $I$ is an ideal. Now $M \subseteq I$ because $x \in I$ but $x \notin M$. Since $M$ is maximal, it follows that $I = R$, and in particular $1_R \in I$. So there exist $m \in M$ and $y \in R$ such that $1_R = m + xy$. Then $(x + M)(y + M) = xy + M = (1_R - m) + M = 1_R + M$, so $x + M$ has an inverse in $R/M$.

**Definition:** The characteristic of a field $F$ (denoted by $\text{Char } F$) is the smallest positive integer $n$ such that $n1 = 0$, where $n1$ is an abbreviation for $1 + 1 + \cdots + 1(n \text{ ones})$. If $n1$ is never 0, we say that $F$ has characteristic 0.

Note that the characteristic of a field can never be equal to 1, since $1 = 1 \ast 1 \neq 0$. If $\text{Char } F \neq 0$, then $\text{Char } F$ must be a prime number. For if $\text{Char } F = n = rs$ where $r$ and $s$ are positive integers greater than 1, then $(r1)(s1) = n1 = 0$, so either $r1$ or $s1$ is 0, which contradicts the minimality of $n$.

**Definition:** If $F$ and $E$ are fields and $F \subseteq E$, we say that $E$ is an extension of $F$ and $F$ is a subfield of $E$, and we write $F \leq E$.

Note that if $F \leq E$ then we can consider $E$ as a vector space over $F$, because
if we consider the elements of $E$ as vectors and the elements of $F$ as scalars, then the axioms of a vector space are satisfied. The dimension of this vector space is called the degree of the extension and is denoted by $[E : F]$. If $[E : F] = n$ with $n < \infty$, we say that $E$ is a finite extension of $F$.

**Definition:** A minimal subfield $F_p$ of a field $F$ with $\text{Char} F = p$ is called a prime field.

Note that the only subfield of the prime field $F_p$ is $F_p$ itself. Let $\text{Char} F = p$ then $\{1, 1 + 1, 1 + 1 + 1, \ldots, 0 = 1 + 1 + \cdots + 1(p \text{ ones})\}$ form a subfield of $F$ which is a prime field isomorphic to $\mathbb{Z}_p$, i.e., every finite field with characteristic $p$ has a prime subfield which is isomorphic to $\mathbb{Z}_p$.

**Definition:** A nonzero polynomial $f(x)$ of degree $m$ over a field $F$ is an expression of the form

$$f(x) = f_0 + f_1 x + f_2 x^2 + \cdots + f_m x^m$$

where $f_i \in F$, $0 \leq i \leq m$, and $f_m \neq 0$. The degree of $f(x)$ is denoted by $\deg f(x)$. The polynomial $f(x) = 0$ is called the zero polynomial and the set of all polynomials over $F$ is denoted by $F[x]$.

**Definition:** A nonzero polynomial $f(x)$ over $F$ with $f_m = 1$ is called monic.

Note that the set $F[x]$ is a ring, its additive identity is the zero polynomial $f(x) = 0$ and its multiplicative identity is $f(x) = 1$. However $F[x]$ is not a field, because the polynomials of degree greater than 0 have no inverse.

**Definition:** A nonzero polynomial $f(x) \in F[x]$ is called irreducible over $F$ if $\deg f(x) \geq 1$ and $f(x) = g(x)h(x)$ with $g(x), h(x) \in F[x]$ gives either $\deg g(x) = 0$ or $\deg h(x) = 0$.

In other words, up to a nonzero constant factor the only divisors of $f(x)$ are $f(x)$ itself and 1.

**Definition:** Let $f(x)$ be a nonzero polynomial of $F[x]$. A finite extension $E$ of $F$ is called the splitting field of $f(x)$ if $E$ is the smallest extension field of $F$ in which $f(x)$ can be written as

$$\lambda(x - \alpha_1) \cdots (x - \alpha_k)$$

for some $\alpha_1, \ldots, \alpha_k \in E$ and $\lambda$ in $F$.

Note that the splitting field $E$ can be written as $E = F[\alpha_1, \ldots, \alpha_k]$ which
denotes the field generated by $\alpha_1, \ldots, \alpha_k$ over $F$, because $F[\alpha_1, \ldots, \alpha_k]$ is the smallest field containing $F$ and $(\alpha_1, \ldots, \alpha_k)$.

**Example:** The field of complex numbers is the splitting field of $x^2 + 1$ over the field of real numbers.

### 1.2 Classification of the finite fields

**Theorem 1.2.1.** The number of elements of a finite field $F$ is equal to $p^n$, where $p$ is prime and $n \in \mathbb{N}$.

**Proof.** Let $\text{Char} F = p$ where $p$ is a prime number, then $F_p$ is a subfield of $F$, so we can consider $F$ as a vector space over $F_p$. Since $F$ is finite, the dimension of $F$ over $F_p$ is finite.

Let $[F : F_p] = n$ then there exists a basis $v_1, v_2, \ldots, v_n$ in $F$. With respect to this basis every element $x \in F$ can be written uniquely as

$$x = a_1v_1 + a_2v_2 + \cdots + a_nv_n, \text{ where } a_1, a_2, \ldots, a_n \in F_p.$$

Since $|F_p| = p$, every $a_i$, $i = 1, 2, \ldots, n$, can be equal to one of $p$ values. Therefore, $F$ has $p^n$ elements. \qed

Now we will consider the following, more difficult, question: Does a field with $p^n$ elements exist, for each prime $p$ and each $n \in \mathbb{N}$? This question will be answered affirmatively in Theorem 1.2.4. In order to prove this theorem, we need Theorems 1.2.2 and 1.2.3.

**Theorem 1.2.2.** Let $f(x)$ be an irreducible polynomial with degree $\geq 1$ in $F[x]$. Then a splitting field for $f(x)$ over $F$ exists and any two such splitting fields are isomorphic.

**Proof.** First we will prove that there exists an extension field of $F$ in which $f(x)$ has a root.

Since $f(x)$ is irreducible, the principal ideal $(f(x))$ is a maximal ideal in the ring $F[x]$. To prove this, let $I$ be an ideal in $F[x]$ with $(f(x)) \subsetneq I \subseteq F[x]$ and let $g(x) \in I \setminus (f(x))$. Since $f(x)$ is irreducible and $f(x) \nmid g(x)$, we have $(f(x), g(x)) = 1$. Therefore, there exist two polynomials $h(x), k(x) \in F[x]$ with $f(x)h(x) + g(x)k(x) = 1$. We see that $1 \in I$ and thus $I = F[x]$, so $(f(x))$ is a maximal ideal in the commutative ring $F[x]$. We conclude that $F[x]/(f(x))$ is a field.
Consider the homomorphism

$$\sigma : \mathbb{F} \rightarrow \mathbb{F}[x]/(f(x))$$

defined by

$$\sigma(a) = \overline{a} = a + (f(x)), \ a \in \mathbb{F},$$

which is injective. Indeed, \( \mathbb{F} \) is a field, which means that \( \ker \sigma \) equals \( \mathbb{F} \) or \( (0) \) but \( \ker \sigma \neq \mathbb{F} \) because \( \mathbb{T} \neq \mathbb{U} \). So \( \ker \sigma = (0) \) and \( \mathbb{F} \) is isomorphic to \( \sigma(\mathbb{F}) \).

Now we can consider \( \mathbb{F}[x]/(f(x)) \) as a finite extension of \( \mathbb{F} \), which has a root \( \alpha = x + (f(x)) \) of \( f(x) \). Thus for a field \( \mathbb{F} \) and an irreducible polynomial \( f(x) \) over \( \mathbb{F} \) there exists an extension in which \( f(x) \) has a root.

Now we will prove by induction on \( \deg f(x) \) the existence of the splitting field of \( f(x) \).

If \( \deg f(x) = 1 \) then \( f(x) \) has one root. Thus, as we have shown, \( \mathbb{F} \) has an extension field \( \mathbb{E} \) which contains a root \( \alpha \) of \( f(x) \), so the field \( \mathbb{F}[\alpha] \) is the splitting field for \( f(x) \).

Now assume that for each irreducible polynomial \( g(x) \) of degree \( m < n \) there exists a splitting field of \( g(x) \).

Let \( \deg f(x) = n \), we have shown that there exists an extension field \( \mathbb{E}_1 \) of \( \mathbb{F} \) with root \( \alpha_1 \) of \( f(x) \). So in \( \mathbb{E}_1 \), \( f(x) \) can be written as

$$f(x) = (x - \alpha_1)g(x), \text{ with } \deg g(x) = n - 1.$$ 

By induction hypothesis, there exists a splitting field in which \( g(x) \) can be written as

$$g(x) = c(x - \alpha_2) \cdots (x - \alpha_n), \text{ with } c \in \mathbb{F}.$$ 

Thus the field \( \mathbb{E} = \mathbb{F}[\alpha_1, \alpha_2, \ldots, \alpha_n] \) is the splitting field of \( f(x) \) over \( \mathbb{F} \) in which all the roots of \( f(x) \) are contained.

We have proved the existence of the splitting field of \( f(x) \), let us now prove the uniqueness up to isomorphism.

Let \( \mathbb{E} \) and \( \mathbb{K} \) be two splitting fields of \( f(x) \), then there exists a nontrivial isomorphism \( \theta : \mathbb{F} \rightarrow \mathbb{F} \) such that \( \mathbb{E} = \mathbb{F}[\alpha_1, \ldots, \alpha_n] \) and \( \mathbb{K} = \theta \mathbb{F}[\beta_1, \ldots, \beta_n] \) with \( f(x) = c_1(x - \alpha_1) \cdots (x - \alpha_n) \) over \( \mathbb{E} \) and \( \theta f(x) = c_2(x - \beta_1) \cdots (x - \beta_n) \) over \( \mathbb{K} \) and \( c_1, c_2 \in \mathbb{F} \).

We will prove by induction on \( \deg f(x) \), that \( \theta \) can be extended to an isomorphism \( \overline{\theta} : \mathbb{E} \rightarrow \mathbb{K} \).

If \( \deg f(x) = 1 \) then \( \mathbb{E} = \mathbb{F}[\alpha] \) and \( \mathbb{K} = \theta \mathbb{F}[\beta] \). Thus the mapping \( \theta_1 : \mathbb{E} \rightarrow \mathbb{K} \) defined by \( \theta_1(\alpha) = \beta \) and \( \theta_1(c) = \theta(c) \) for all \( c \in \mathbb{F} \), is an isomorphism.

If \( \deg f(x) = n \), then for a root \( \alpha_1 \) of \( f(x) \) and a root \( \beta_1 \) of \( \theta f(x) \), there exists an isomorphism \( \theta_1 : \mathbb{F}[\alpha_1] \rightarrow \theta \mathbb{F}[\beta_1] \). So we can write \( f(x) \) as
1.2. CLASSIFICATION OF THE FINITE FIELDS

Let \( F \) be a field and \( f(x) \) be a polynomial over \( F \). Then \( f(x) \) has no roots with multiplicity \( \geq 2 \) if and only if the greatest common divisor of a polynomial \( f(x) \) and \( f'(x) \) has degree 0. i.e., \( f(x) \) and \( f'(x) \) has no common root.

Proof. Assume that the greatest common divisor of \( f(x) \) and \( f'(x) \) has degree 0, we will prove that \( f(x) \) has no roots with multiplicity \( \geq 2 \) by contradiction. Let \( f(x) \) have at least one root with multiplicity \( r \geq 2 \), then we can write \( f(x) \) as

\[
f(x) = c(x - a_1)^r(x - a_2)\cdots(x - a_{n-r})
\]

where \( n = \deg f(x), c \in F \) and \( a_1, a_2, \ldots, a_{n-r} \) are all roots of \( f(x) \) such that \( a_1 \) is repeated \( r \) times. Then

\[
f'(x) = cr(x - a_1)^{r-1}(x - a_2)\cdots(x - a_{n-r}) + c(x - a_1)^r(x - a_3)\cdots(x - a_{n-r}) + \cdots + c(x - a_1)^r(x - a_2)\cdots(x - a_{n-r-1}) = c(x - a_1)^{r-1}[r(x - a_2)\cdots(x - a_{n-r}) + (x - a_1)(x - a_3)\cdots(x - a_{n-r}) + \cdots + (x - a_1)(x - a_2)\cdots(x - a_{n-r-1})].
\]

It follows that \((x - a_1)^{r-1} \mid f(x)\) and \((x - a_1)^{r-1} \mid f'(x)\), so the greatest common divisor of \( f(x) \) and \( f'(x) \) has degree \( \geq r - 1 \neq 0 \) because \( r \geq 2 \).

Now assume that \( f(x) \) has no roots with multiplicity \( \geq 2 \), we prove that the greatest common divisor of a polynomial \( f(x) \) and \( f'(x) \) has degree 0. We can write \( f(x) \) as

\[
f(x) = c(x - a_1)^r(x - a_2)\cdots(x - a_n)
\]

where \( n = \deg f(x), c \in F \) and \( a_1, a_2, \ldots, a_n \) are all roots of \( f(x) \). Then

\[
f'(x) = c(x - a_2)\cdots(x - a_n) + c(x - a_1)(x - a_3)\cdots(x - a_n) + \cdots + c(x - a_1)(x - a_2)\cdots(x - a_{n-1}) = c[(x - a_2)\cdots(x - a_n) + (x - a_1)(x - a_3)\cdots(x - a_n) + \cdots + (x - a_1)(x - a_2)\cdots(x - a_{n-1})].
\]

It follows that \( c \) is the only common divisor of \( f(x) \) and \( f'(x) \). \( \square \)
Theorem 1.2.4. For every prime \( p \) and \( n \in \mathbb{N} \) there is a field with \( p^n \) elements.

Proof. Consider the polynomial \( f(x) = x^{p^n} - x \) over a field \( \mathbb{F} \). If \( f(x) \) is irreducible, then by Theorem 1.2.2 there exists a splitting field \( \mathbb{E} \) of \( f(x) \) which is unique up to isomorphism. If \( f(x) \) is not irreducible, set \( \mathbb{E} = \mathbb{F} \).

Let \( K \) be the set of all zeros of \( f(x) \) in \( \mathbb{E} \), then \( K = \{ a \in \mathbb{E} : a^{p^n} = a \} \).

Since \( f'(x) = p^n x^{(p^n - 1)} - 1, f(x) \) and \( f'(x) \) has no common zero, where \( f(a) = 0 \Rightarrow a^{p^n} = a \Rightarrow a^{p^n - 1} = 1 \Rightarrow f'(a) = p^n a^{p^n - 1} - 1 = p^n - 1 \neq 0 \).

It follows from Theorem 1.2.3 that all zeros of \( f(x) \) are distinct. Thus the set \( K \) has \( p^n \) elements.

We will prove that \( K = \mathbb{E} \). In order to show that, we need to prove that \( K \) is a field, in which each element is a root of \( f(x) \), then \( K \) is a splitting field of \( f(x) \) with \( p^n \) elements.

It is clear that \( K \subseteq \mathbb{E} \) and by using Theorem 1.2.2 \( K \) isomorphic to \( \mathbb{E} \). Thus \( K = \mathbb{E} \).

Clearly \( 0, 1 \in K \). Let \( a, b \in K \), to prove that \( K \) is closed under addition, we need to prove that \( (a + b)^{p^n} = a^{p^n} + b^{p^n} = a + b \). In the polynomial expansion

\[
(a + b)^{p^n} = \sum_{i=0}^{p^n} \binom{p^n}{i} a^i b^{p^n-i} = \frac{p^n!}{i!(p^n-i)!}
\]

we can see that all binomial coefficients are divisible by \( p^n \) except the first and the last, and since the finite field \( \mathbb{E} \) has \( \text{Char} \mathbb{E} = p \), all binomial coefficients are 0 except the first and the last. That is,

\[
(a + b)^{p^n} = a^{p^n} + b^{p^n} = a + b.
\]

Clearly, \( K \) is closed under multiplication or \( (ab)^{p^n} = a^{p^n} b^{p^n} \), because \( \mathbb{E} \) is commutative. Since \( \forall a \neq 0, a^{p^n} = a \), we have \( (a^{-1})^{p^n} = (a^{p^n})^{-1} = a^{-1} \). So the inverse of any element in \( K \) belongs to \( K \).

Thus \( K \) is subfield of \( \mathbb{E} \) or \( K = \mathbb{E} \) is a field with \( p^n \) element. \( \square \)

Notation: We will denote by \( \mathbb{F}_{p^n} \) the field with \( p^n \) elements.

Clearly \( (\mathbb{F}_{p^n})^* = \mathbb{F}_{p^n} \setminus \{0\} \) and \( (\mathbb{F}_{p^n})^2 = \{ a^2 : a \in \mathbb{F}_{p^n} \} \) form a group under multiplication. In general, \( (\mathbb{F}_{p^n})^m = \{ a^m : a \in \mathbb{F}_{p^n}, m \in \mathbb{N} \} \), form a group under multiplication.
Theorem 1.2.5. The multiplicative group \( \mathbb{F}_{p^n}^* \) is cyclic.

Proof. Let \( p^n \geq 3 \) and \( h = p^n - 1 = p_1^{r_1}p_2^{r_2} \cdots p_m^{r_m} \) be the prime factorization of \( |\mathbb{F}_{p^n}^*| \). Let \( a_i \), for every \( 1 \leq i \leq m \), be an element in \( \mathbb{F}_{p^n} \) with \( a_i^{\frac{h}{p_i}} \neq 1 \).

To prove the existence of \( a_i \), consider the polynomial \( x^{\frac{h}{p_i}} - 1 \) which has at most \( \frac{h}{p_i} \) roots in \( \mathbb{F}_{p^n} \). Since \( \frac{h}{p_i} < h \), it follows that there exists \( a_i \in \mathbb{F}_{p^n} \) with \( a_i^{\frac{h}{p_i}} 
eq 1 \).

Now set \( b_i = a_i^{\frac{h}{p_i r_i}} \), it follows that \( b_i 
eq 1 \) and \( b_i^{p_i} = 1 \), then the order of \( b_i \) must be in the form \( p_i^s \) with \( 1 \leq s \leq r_i \). Since \( b_i^{p_i} = (a_i^{\frac{h}{p_i r_i}})^{p_i} = a_i^{\frac{h}{p_i}} 
eq 1 \), the order of \( b_i \) is \( p_i^{r_i} \).

Now we will prove that \( b = b_1 b_2 \cdots b_m \) is a generator of the group \( \mathbb{F}_{p^n}^* \), that is the order of \( b \) is \( h \). We know that \( b^h = b_1 b_2 \cdots b_m = 1 \).

Assume that the order of \( b \) is not \( h \), then the order of \( b \) is a proper divisor of \( h \). So the order of \( b \) is a divisor of at least one of the \( m \) integers \( \frac{h}{p_i} \) and \( 1 \leq i \leq m \), say \( \frac{h}{p_1} \). Thus

\[
 b^{\frac{h}{p_1}} = b_1^{\frac{h}{p_1}} b_2^{\frac{h}{p_1}} \cdots b_m^{\frac{h}{p_1}} = 1. 
\]

Since \( p_i^{r_i} \mid \frac{h}{p_1} \) for every \( 2 \leq i \leq m \), it follows that \( b_i^{\frac{h}{p_1}} = 1 \) for every \( 2 \leq i \leq m \). Thus \( b_1^{\frac{h}{p_1}} = 1 \), which is impossible, because the order of \( b_1 \) is \( p_1^{r_1} \mid \frac{h}{p_1} \).

So the order of \( b \) is \( h \), in other words, the multiplication group \( \mathbb{F}_{p^n}^* \) is cyclic. \( \square \)

We remark here that \( \mathbb{F}_{p^n} \) with \( n > 1 \) is never \( \mathbb{Z}_{p^n} \), in the following section we construct some of such fields.

1.3 Construction of the finite fields

To construct a field with \( p^n \) elements, we use an irreducible monic polynomial \( f(x) \in \mathbb{Z}_p[x] \) with \( \deg f(x) = n \). The elements of the field \( \mathbb{Z}_p[x]/(f(x)) \) can be written in the form

\[
 a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1}, \text{ where } a_i \in \mathbb{Z}_p \text{ for all } i = 0, 1, \ldots, n - 1. 
\]

Since there are \( p \) possible values for each \( a_i \), the field \( \mathbb{Z}_p[x]/(f(x)) \) has \( p^n \) elements.
To find an irreducible polynomial, we list all possible monic polynomials of degree \( n \) (\( p^n \) possible monic polynomials), which is not always an easy process especially for large \( n, p \).

Clearly, any polynomial without a constant term is not irreducible (\( x \) is a factor), so these \( p^n-1 \) polynomial will not be considered.

For each of the remaining \( p^n-p^{n-1} \) polynomials, we could substitute one by one all the field elements for \( x \). If none of these substitutions is equal to zero, the polynomial is irreducible (i.e., it has no root in the field).

Let \( a \) be a zero of the chosen polynomial, then the elements of \( \mathbb{Z}_p[x]/(f(x)) \) can be written in its vector form representation using the basis \( \{1, a, a^2, \ldots, a^{n-1}\} \).

We can also generate a multiplicative representation of the field by using the fact that the multiplicative group of the field is cyclic. So if we can find a primitive element (i.e., a generator of the cyclic group), we will have a representation of the elements.

**Example 1**: We will construct a field of \( 16 = 2^4 \) elements, here \( p = 2, n = 4 \). We start with a field of order 2 which is \( \mathbb{Z}_2 = \{0, 1\} \) and an irreducible polynomial over \( \mathbb{Z}_2 \) of degree 4. We can easily list all possible polynomials of degree 4 over \( \mathbb{Z}_2 \). There are 16 of them:

\[
x^4, \quad x^4 + 1, \quad x^4 + x, \quad x^4 + x^2, \quad x^4 + x^3, \quad x^4 + x + 1, \quad x^4 + x^2 + 1, \\
x^4 + x^3 + 1, \quad x^4 + x^2 + x, \quad x^4 + x^3 + x, \quad x^4 + x^3 + x^2, \quad x^4 + x^2 + x + 1, \\
x^4 + x^3 + x + 1, \quad x^4 + x^3 + x^2 + 1, \quad x^4 + x^3 + x^2 + x, \quad x^4 + x^3 + x^2 + x + 1.
\]

Every polynomial without a constant term has root 0. So we will consider just the following polynomials:

\[
x^4 + 1, \quad x^4 + x + 1, \quad x^4 + x^2 + 1, \quad x^4 + x^3 + 1, \quad x^4 + x^2 + x + 1, \\
x^4 + x^3 + x + 1, \quad x^4 + x^3 + x^2 + 1, \quad x^4 + x^3 + x^2 + x, \quad x^4 + x^3 + x^2 + x + 1.
\]

In this set, every polynomial with even number of terms has root 1. So we will consider just the following polynomials:

\[
x^4 + x + 1, \quad x^4 + x^2 + 1, \quad x^4 + x^3 + 1, \quad x^4 + x^3 + x^2 + x + 1.
\]

All of them have no roots in \( \mathbb{Z}_2 \), then all of them are irreducible polynomials over \( \mathbb{Z}_2 \).

Consider one of them, say \( x^4 + x + 1 \). Let \( a \) be a root of \( x^4 + x + 1 \), then the elements of the field \( \mathbb{Z}_2[x]/(x^4 + x + 1) \) can be obtained by two methods:

The first method is additive, in which we construct all linear combinations of \( 1, a, a^2 \) and \( a^3 \). They are:
0, 1, a, \ a^2, \ a^3, \ a+1, \ a^2+1, \ a^3+1, \ a^2+a, \ a^3+a, \\
\ a^3+a^2, \ a^2+a+1, \ a^3+a+1, \ a^3+a^2+1, \ a^3+a^2+a, \\
\ a^3+a^2+a+1.

The second method is multiplicative. Since \( a^4 = -a - 1 = a + 1 \), we can write down the powers of \( a \) as the following:

\[
\begin{align*}
  a^1 &= a \\
  a^2 &= a^2 \\
  a^3 &= a^3 \\
  a^4 &= a + 1 \\
  a^5 &= a^2 + a \\
  a^6 &= a^3 + a^2 \\
  a^7 &= a^3 + a + 1 \\
  a^8 &= a^2 + 1 \\
  a^9 &= a^3 + a \\
  a^{10} &= a^2 + a + 1 \\
  a^{11} &= a^3 + a^2 + a \\
  a^{12} &= a^3 + a^2 + a + 1 \\
  a^{13} &= a^3 + a^2 + 1 \\
  a^{14} &= a^3 + 1 \\
  a^{15} &= 1,
\end{align*}
\]

which means that \( a \) is a generator of the cyclic group 

\[(\mathbb{Z}_2[x]/(x^4 + x + 1))^* = \mathbb{Z}_2[x]/(x^4 + x + 1) \setminus \{0\}.\]

Notice also that the terms on the right are all the possible terms that can be written as linear combinations of the basis \( \{1, a, a^2, a^3\} \) over \( \mathbb{Z}_2 \). When working with finite fields it is convenient to have both of the above representations, since the terms on the left are easy to multiply and the terms on the right are easy to add.

Now suppose we had chosen a root of the second irreducible polynomial \( x^4 + x^2 + 1 \), say, \( b \). We would then have \( b^4 = b^2 + 1 \) and the powers of \( b \) will be

\[
\begin{align*}
  b^1 &= b \\
  b^2 &= b^2 \\
  b^3 &= b^3 \\
  b^4 &= b^2 + 1 \\
  b^5 &= b^3 + b \\
  b^6 &= b^4 + b^2 = b^2 + 1 + b^2 = 1,
\end{align*}
\]

which means that \( b \) cannot be a generator of the group \((\mathbb{Z}_2[x]/(x^4 + x^2 + 1))^*\).

**Example 2:** Now we will construct a field of \( 9 = 3^2 \) elements, that is, \( p = 3, n = 2 \). We start with a field of order 3 which is \( \mathbb{Z}_3 = \{0, 1, 2\} \) and an irreducible polynomial over \( \mathbb{Z}_3 \) of degree 2. We can easily list all possible monic polynomials over \( \mathbb{Z}_3 \). They are:

\[
x^2, \ x^2+1, \ x^2+2, \ x^2+x, \ x^2+2x, \ x^2+x+1, \ x^2+x+2, \ x^2+2x+1, \ x^2+2x+2.
\]

Every polynomial without a constant term has root 0. So we will consider just the following polynomials

\[
x^2 + 1, \ x^2 + 2, \ x^2 + x + 1, \ x^2 + x + 2, \ x^2 + 2x + 1, \ x^2 + 2x + 2.
\]
In this set, $x^2 + 2$ and $x^2 + x + 1$ have root 1 and $x^2 + 2x + 1$ has root 2. So we will consider just the following polynomials

$$x^2 + 1, \quad x^2 + x + 2, \quad x^2 + 2x + 2.$$ 

All of them have no roots in $\mathbb{Z}_3$, then all of them are irreducible polynomials over $\mathbb{Z}_3$.

Consider one of them, say $x^2 + x + 2$. Let $a$ be a root of $x^2 + x + 2$, then the elements of the field $\mathbb{Z}_2[x]/(x^2 + x + 2)$ can be obtained by two methods:

The first method is additive, in which we construct all linear combinations of 1 and $a$. They are:

$0, \quad 1, \quad 2, \quad a, \quad 2a, \quad a + 1, \quad 2a + 1, \quad 2a + 2.$

The second method is multiplicative. Since $a^2 = -a - 2 = 2a + 1$, we can write out the powers of $a$ as follows:

$$a^1 = a \quad a^2 = 2a + 1 \quad a^3 = 2a + 2 \quad a^4 = 2$$

In other words, $a$ is a generator of the cyclic group

$$(\mathbb{Z}_2[x]/(x^2 + x + 2))^* = (\mathbb{Z}_2[x]/(x^2 + x + 2)) \setminus \{0\}.$$ 

Example 3: Finally, we will construct a field of $25 = 5^2$ elements. Here, $p = 5, n = 2$, a field of order 5 is $\mathbb{Z}_5 = \{0, 1, 2, 3, 4\}$. To find an irreducible polynomial over $\mathbb{Z}_5$ of degree 2, we list all possible monic polynomials of degree 2 over $\mathbb{Z}_5$:

$x^2, \quad x^2 + 1, \quad x^2 + 2, \quad x^2 + 3, \quad x^2 + 4, \quad x^2 + x, \quad x^2 + 2x, \quad x^2 + 3x, \quad x^2 + 4x, \quad x^2 + x + 1, \quad x^2 + x + 2, \quad x^2 + x + 3, \quad x^2 + x + 4, \quad x^2 + 2x + 1, \quad x^2 + 2x + 2, \quad x^2 + 2x + 3, \quad x^2 + 2x + 4, \quad x^2 + 3x + 1, \quad x^2 + 3x + 2, \quad x^2 + 3x + 3, \quad x^2 + 3x + 4, \quad x^2 + 4x + 1, \quad x^2 + 4x + 2, \quad x^2 + 4x + 3, \quad x^2 + 4x + 4.$

Every polynomial without a constant term has root 0,

$x^2 + 4, \quad x^2 + x + 3, \quad x^2 + 2x + 2$ and $x^2 + 3x + 1$ have root 1,

$x^2 + 1, x^2 + 3x, x^2 + x + 4, x^2 + 2x + 2$ and $x^2 + 4x + 3$ have root 2,

$x^2 + 1, x^2 + 2x, x^2 + x + 3, x^2 + 3x + 2$ and $x^2 + 4x + 4$ have root 3,

and $x^2 + 4, x^2 + x, x^2 + 2x + 1, x^2 + 3x + 2$ and $x^2 + 4x + 3$ have root 4.

So we will consider the remaining polynomials:

$x^2 + 2, \quad x^2 + 3, \quad x^2 + x + 1, \quad x^2 + x + 2, \quad x^2 + 2x + 3, \quad x^2 + 2x + 4, \quad x^2 + 3x + 3, \quad x^2 + 3x + 4, \quad x^2 + 4x + 1, \quad x^2 + 4x + 2.$
All of them have no roots in $\mathbb{Z}_5$, then all of them are irreducible polynomials over $\mathbb{Z}_5$.

Consider one of them, say $x^2 + 2$. Let $a$ be a root of $x^2 + 2$, then the elements of the field $\mathbb{Z}_5[x]/(x^2 + 2)$ are all linear combinations of 1 and $a$. They are:

$$0, 1, 2, 3, 4, a, 2a, 3a, 4a, a + 1, a + 2, a + 3, a + 4, 2a + 1, 2a + 2, 2a + 3, 2a + 4, 3a + 1, 3a + 2, 3a + 3, 3a + 4, 4a + 1, 4a + 2, 4a + 3, 4a + 4.$$
Chapter 2

Paley Graph

In this chapter we will give some basic definitions and properties of graph theory and we will study in details Paley graphs and some of its properties.

2.1 Basic definitions and properties

Definition: A graph $G$ is a pair $(V, E)$ of sets satisfying $E \subseteq P_2(V)$, where $P_2(V)$ is the set of all subsets of $V$ with two elements. The elements of $V$ are called vertices and the elements of $E$ are called edges.

Note that the set of vertices of a graph $H = (W, F)$ is denoted by $V(H)$ and the set of edges is denoted by $E(H)$. An edge $e = \{x, y\}$ is sometimes written as $xy$.

Definition: The order of a graph $G$ is the number of its vertices and is denoted by $|G|$. A graph $G$ is called a finite graph or an infinite graph depending on the order of $G$. If $|G| = 0$ then $G$ is called the empty graph.

In order to draw a graph $G$, we can represent its vertex set $V(G)$ by dots, we join two of these dots by a line if and only if the two corresponding vertices form an edge in $E(G)$.

$$
\begin{array}{c}
\text{Fig. 2.1.1. A graph } G \text{ with } V(G) = \{1, 2, 3, 4, 5, 6\} \text{ and } \\
E(G) = \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{1, 5\}, \{2, 3\}, \{2, 4\}, \{2, 5\}\}.
\end{array}
$$
**Definition:** Two vertices \( x, y \) of \( G \) are *adjacent*, or *neighbors*, if \( xy \) is an edge of \( G \), and the set of all neighbors of \( x \) is denoted by \( N(x) \).

The order of the set \( N(x) \) is called the *degree* of \( x \) and is denoted by \( d(x) \).

We say that a vertex \( x \) is *isolated* if \( d(x) = 0 \).

Note that for each graph \( G = (V, E) \) we have

\[
2|E(G)| = \sum_{x \in V(G)} d(x).
\]

**Definition:** A graph \( G \) is called *\( k \)-regular* if all the vertices of \( G \) have the same degree \( k \).

If \( |V(G)| = n \) and \( G \) is an (\( n-1 \))-regular graph then \( G \) is called a *complete* graph. We will denote by \( K_n \) the complete graph on \( n \) vertices.

Note that in a \( k \)-regular graph \( G \) we have \( 2|E(G)| = k|V(G)| \), it follows that \( k \) or \( |V(G)| \) is even.

**Definition:** A *path* of length \( n \) in a graph \( G \) is the sequence

\[
x_1e_1x_2 \cdots e_{n-1}x_n \text{ with } x_i \in V(G), \ e_i = \{x_i, x_{i+1}\} \in E(G)
\]

for all \( i \in \{1, 2, \ldots, n-1\} \), and \( x_i \neq x_j \) for all \( i \neq j \).

Let \( x_1e_1x_2e_2 \cdots x_{n-1}e_{n-1}x_n \) be a path then the sequence

\[
x_1e_1x_2e_2 \cdots x_{n-1}e_{n-1}x_nex_1 \text{ with } e = \{x_n, x_1\} \in E(G)
\]

is called a *cycle* of length \( n \) which will be denoted by \( C_n \).

As an example, in figure 2.1.2 we see the cycle and the complete graph of 5 vertices.

![C_5 and K_5](image-url)
**Definition:** Let $G$ be a $k$-regular graph with $|G| = n$. If there are two integers $\lambda, \mu$ such that

- every two adjacent vertices have $\lambda$ common neighbors and
- every two non-adjacent vertices have $\mu$ common neighbors,

then $G$ is called a **strongly regular graph** with parameters $(n, k, \lambda, \mu)$ and is denoted by $\text{srg}(n, k, \lambda, \mu)$.

Clearly, every strongly regular graph is regular, but not vice versa. For example $C_6$ is 2-regular but not a strongly regular graph.

Note that in any strongly regular graph $\text{srg}(n, k, \lambda, \mu)$, its parameters are related by

$$\mu(n - k - 1) = k(k - \lambda - 1).$$

\(^\ast\)

In order to show that, consider a vertex $x \in V(G)$. We remind that $N(x)$ is the set of all neighbors of $x$. Let $N'(x)$ be the set of all non-adjacent vertices of $x$, then $n = |N(x)| + 1 + |N'(x)|$. We will prove that both sides of $\ast$ are equal to the number of edges between $N(x)$ and $N'(x)$.

Since $G$ is $k$-regular, $|N'(x)| = n - |N(x)| - 1 = n - k - 1$. Let $y \in N'(x)$, then $y$ and $x$ have $\mu$ common neighbors, that is $\mu$ equals the number of edges between $y$ and $N(x)$. It follows that $\mu(n - k - 1)$ is the number of edges between $N(x)$ and $N'(x)$.

Let $z \in N(x)$ then $z$ and $x$ have $\lambda$ common neighbors. Since $|N(x)| = k$, it follows that the number of neighbors of $z$ which are not adjacent to $x$ is equal to $k - \lambda - 1$. Thus $k - \lambda - 1$ is the number of edges between $z$ and $N'(x)$, so $k(k - \lambda - 1)$ equals the number of edges between $N(x)$ and $N'(x)$. Therefore, $\ast$ is proved.

**Definition:** A graph $G$ is called **connected** if every two vertices are connected by a path.

Note that every complete graph is connected, regular, and strongly regular. Both strongly regular and regular graphs are not necessary connected and also connected graphs are not necessary complete, strongly regular, or regular.

For example, a cycle is a connected graph which is not complete, a path is a connected graph which is not regular, and the graph $G$ with $V(G) = \{1, 2, 3, 4\}$ and $E(G) = \\{\{1, 2\}, \{3, 4\}\}$ is strongly regular which is not connected.
2.1. BASIC DEFINITIONS AND PROPERTIES

Definition: Let $G = (V, E)$ and $G' = (V', E')$ be two graphs. $G$ is isomorphic to $G'$ if there is a bijection $f: V \rightarrow V'$ such that $xy \in E$ if and only if $f(x)f(y) \in E'$; we denote this by $G \cong G'$.

An isomorphism from a graph $G$ to itself is called an automorphism. The set of all automorphismus of a graph $G$ form a group under composition, and it is denoted by $\text{Aut}(G)$.

Definition: Let $G = (V, E)$ be a finite graph. The complementary graph of $G$ is a graph $\overline{G}$ with $V(\overline{G}) = V(G)$ and $E(\overline{G}) = P_2(V) \setminus E(G)$. That is, $xy \in E(\overline{G})$ if and only if $xy \notin E(G)$.

A graph $G$ is called self-complementary if it is isomorphic to its complement.

For example $C_5$ is self-complementary, see figure 2.1.4.

Definition: Let $G$ be a group and let $X$ be a non-empty set. We say that $G$ acts on $X$ if there is a map $\phi: G \times X \rightarrow X$ such that the following conditions hold for all $x \in X$:

1. $\phi(e, x) = x$ where $e$ is the identity element of $G$.
2. $\phi(g, \phi(h, x)) = \phi(gh, x)$ $\forall g, h \in G$. 

Fig. 2.1.3.

Fig. 2.1.4.
In this case, \( G \) is called a transformation group, \( X \) is a called a \( G \)-set, and \( \phi \) is called the group action.

The group action is called transitive (we also say that \( G \) acts transitively on \( X \)) if for every \( x, y \in X \), there exists \( g \in G \) such that \( \phi(g, x) = y \).

**Definition:** A graph \( G \) is called symmetric if its automorphism group acts transitively on the vertices and edges.

For example every cycle graph or complete graph is symmetric graph and every symmetric graph is a regular graph, but not vice versa.

\[
\begin{array}{c}
\begin{array}{c}
\text{3} \\
\text{2} \\
\text{1} \\
\text{4} \\
\text{5} \\
\text{6}
\end{array} \\
\begin{array}{c}
\text{2} \\
\text{1} \\
\text{4} \\
\text{5} \\
\text{6} \\
\text{7}
\end{array}
\end{array}
\]

**Fig. 2.1.5.** Symmetric graph which is regular and regular graph which is not symmetric.

### 2.2 Paley graphs

#### 2.2.1 Definition and examples

Before we define the Paley graph, we need the following definition.

**Definition:** Let \( q \) and \( r \) be two positive integers with \( \gcd(q, r) = 1 \), then \( r \) is a quadratic residue of \( q \) if and only if \( x^2 \equiv r \pmod{q} \) has a solution, and \( r \) is a quadratic nonresidue of \( q \) if and only if \( x^2 \equiv r \pmod{q} \) has no solution.

**Definition:** Let \( p \) be a prime number and \( n \) be a positive integer such that \( p^n \equiv 1 \pmod{4} \). The graph \( P = (V, E) \) with \( V(P) = \mathbb{F}_{p^n} \) and \( E(P) = \{ \{x, y\} : x, y \in \mathbb{F}_{p^n}, x - y \in (\mathbb{F}_{p^n}^*)^2 \} \) is called the Paley graph of order \( p^n \).

Note that the set \( E(P) \) in the definition of Paley graph is well defined because \( x - y \in (\mathbb{F}_{p^n}^*)^2 \) if and only if \( y - x \in (\mathbb{F}_{p^n}^*)^2 \). Since \( x - y = -1(y - x) \), we need only to show that \(-1 \in (\mathbb{F}_{p^n}^*)^2\).
2.2. PALEY GRAPHS

We have \( p^n \equiv 1 \pmod{4} \), so \( 4 \mid (p^n - 1) \). Let \( g \) be a generator of the group \( \mathbb{F}_p^\ast \), then \( p^n - 1 \) is the least positive integer such that \( g^{p^n-1} = 1 \). We can rewrite this as \( g^{p^n-1} - 1 = (g^{p^n-2} - 1)(g^{p^n-1} + 1) = 0 \). Since \( g^{p^n-2} \) cannot be equal to 1, it follows that \( g^{p^n-1} = (g^{p^n-2})^2 = -1 \) which means that \( g^{p^n-1} \) is a square root of \(-1\).

Note that if the Paley graph has prime order \( p \), then we can consider the field of integers modulo \( p \), \( \mathbb{Z}_p \), as its vertex set.

However, we cannot consider \( \mathbb{Z}_{p^n} \) with \( n > 1 \) as a vertex set of the Paley graph of order \( p^n \), because as we have seen in the previous chapter, there exists a unique field \( \mathbb{F}_{p^n} \) of order \( p^n \) which is not \( \mathbb{Z}_{p^n} \), such a field will represent the set of vertices of the Paley graph. To get this field \( \mathbb{F}_{p^n} \) we need the construction that was developed in the previous chapter.

The list of integers which can be considered as an order of the Paley graph starts with 5, 9, 13, 17, 25, 29, 37, 41. In the following examples, we show the Paley graphs explicitly for the first three cases.

**Example 1:** The Paley graph of order 5 is the cycle \( C_5 \).

In order to see that, let \( P = (V, E) \) be the Paley graph of order 5 then \( V(P) = \mathbb{Z}_5 = \{0, 1, 2, 3, 4\} \) and \( (\mathbb{Z}_5)^2 = \{1, 4\} \), it follows that

\[
E(P) = \{\{0, 1\}, \{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 0\}\}.
\]

**Example 2:** Let \( P = (V, E) \) be the Paley graph of order 9 = \( 3^2 \). Here we have \( p = 3, n = 2 \), then \( V(P) = \mathbb{F}_{3^2} \), the field of order 9, can be written as

\[
\mathbb{F}_{3^2} = \{0, 1, 2, a, 2a, 1 + a, 1 + 2a, 2 + a, 2 + 2a\} \cong \mathbb{Z}_3[x]/(x^2 + 1)
\]

where \( a \) is a root of \( x^2 + 1 \). Since

\[
\begin{align*}
1^2 &= 1, & 2^2 &= 1, & a^2 &= -1 = 2, \\
(2a)^2 &= 2, & (1+a)^2 &= 2a, & (1+2a)^2 &= a, \\
(2+a)^2 &= a, & (2+2a)^2 &= 2a,
\end{align*}
\]

we have \( (\mathbb{F}_{3^2})^2 = \{1, 2, a, 2a\} \). Thus \( E(G) = \{\{0, 1\}, \{0, 2\}, \{0, a\}, \{0, 2a\}, \{1, 2\}, \{1, 1+a\}, \{1, 1+2a\}, \{2, 2+a\}, \{2, 2+2a\}, \{a, 1+a\}, \{a, 2+a\}, \{a, 2a\}, \{2a, 1+2a\}, \{2a, 2+2a\}, \{1+a, 2+a\}, \{1+a, 1+2a\}, \{1+2a, 2+2a\}, \{2+a, 2+2a\}\}. \)
2.2.2 Properties

In the previous examples we can see that the Paley graphs of order 5, 9, and 13 are connected, symmetric, self-complementary, and strongly regular.
2.2. PALEY GRAPHS

The following Propositions prove that these properties are true for every order $q$.

**Proposition 2.2.1.** The Paley graphs are symmetric.

**Proof.** Let $P$ be the Paley graph of order $q = p^n$. To prove that $P$ is symmetric we need to prove that the automorphism group $\text{Aut}(P)$ acts transitively on $V(P)$ and $E(P)$. In other words, we need to prove that

- for every two vertices $x, y \in V(P)$ there exists $\phi \in \text{Aut}(P)$ such that $\phi(x) = y$, and
- for every two edges $\{x_1, y_1\}, \{x_2, y_2\} \in E(P)$ there exists $\theta \in \text{Aut}(P)$ such that $\theta(x_1) = x_2, \theta(y_1) = y_2$.

Fix $a, b \in V(P)$ with $a \in (\mathbb{F}_{p^n}^*)^2$ and define the nontrivial function

$$\phi : V(P) \to V(P) \text{ with } \phi(x) = ax + b \forall x \in V(P).$$

We show that $\phi$ is an automorphism. Easily, we can see that $\phi$ is one to one, because

$$\phi(x_1) - \phi(x_2) = 0 \iff (ax_1 + b) - (ax_2 + b) = 0 \iff a(x_1 - x_2) + b - b = 0$$

$$\iff x_1 - x_2 = 0.$$ 

Since for every $y \in V(P)$, we have

$$a^{-1}y - a^{-1}b = x \in V(P) \text{ with } \phi(x) = a(a^{-1}y - a^{-1}b) + b = y.$$ 

Thus $\phi$ is onto.

Since $\{x, y\} \in E(P) \iff x - y \in (\mathbb{F}_{p^n}^*)^2 \iff a(x - y) + b - b \in (\mathbb{F}_{p^n}^*)^2 \iff (ax + b) - (ay + b) \in (\mathbb{F}_{p^n}^*)^2 \iff \phi(x) - \phi(y) \in (\mathbb{F}_{p^n}^*)^2 \iff \{\phi(x), \phi(y)\} \in E(P),$ 

this proves that $\phi \in \text{Aut}(P)$.

Moreover, for every two vertices $x, y \in V(P)$, take $a = 1 \in (\mathbb{F}_{p^n}^*)^2$ and $b = y - x \in V(P)$, the mapping $\phi : V(P) \to V(P)$ defined by $\phi(x) = ax + b$ is an automorphism with $\phi(x) = y$. Thus $\text{Aut}(P)$ acts transitively on $V(P)$.

Finally, for every two edges $\{x_1, y_1\}, \{x_2, y_2\} \in E(P)$ we can find

$$a = (x_2 - y_2)(x_1 - y_1)^{-1} \in (\mathbb{F}_{p^n}^*)^2 \text{ and } b = x_2 - ax_1 \in V(P)$$

so that $\theta : V(P) \to V(P)$ with $\theta(x) = ax + b$ is an automorphism with $\theta(x_1) = x_2, \theta(y_1) = y_2$. Thus $\text{Aut}(P)$ acts transitively on $E(P)$. $\square$
Proposition 2.2.2. Let $P$ be the Paley graph of order $q = p^n$, then $P$ is a self-complementary graph.

Proof. Let $r$ be a quadratic nonresidue modulo $q$, consider the function

$$ f : V(P) \longrightarrow V(P) \text{ defined by } f(x) = rx. $$

The function $f$ is well defined, because

$$ \{x, y\} \in E(P) \iff (x - y) \in (\mathbb{F}_p^*)^2 \iff f(x) - f(y) = rx - ry = r(x - y) \notin (\mathbb{F}_p^*)^2 $$

$$ \iff \{f(x), f(y)\} \in E(P). $$

Now we prove that $f$ is a bijection. Clearly, $f$ is injective, since

$$ (x - y) = 0 \iff 0 = r(x - y) = rx - ry = f(x) - f(y). $$

Since $\gcd(r, q) = 1$, there exist $a, b \in \mathbb{Z}$ with $1 = qa + rb \iff rb \equiv 1 \pmod{q}$.

Thus $f(bx) = rbx = x$, so $f$ is surjective. \end{proof}

Proposition 2.2.3. Let $P$ be the Paley graph of order $q = p^n$, then $P$ is a strongly regular graph with parameters

$$ (q, \frac{q - 1}{2}, \frac{q - 5}{4}, \frac{q - 1}{4}). $$

Proof. First, we prove that each vertex has degree $\frac{q-1}{2}$.

Fix $x \in V(P)$, we have $N(x) = \{z \in V(P) : x - z = s \in (\mathbb{F}_p^*)^2\}$. If $x - z_1 = s, x - z_2 = s$ then $z_1 = x - s = z_2$, so for all $s \in (\mathbb{F}_p^*)^2$ there exists a unique $z \in V(P)$ such that $x - z = s$.

Thus there exists a one to one correspondence between the number of elements of $N(x)$ and the number of elements of $(\mathbb{F}_p^*)^2$, so all vertices have the same degree $d(x) = |N(x)| = |(\mathbb{F}_p^*)^2|$. Now we calculate $|(\mathbb{F}_p^*)^2|$, we have $|\mathbb{F}_p^*| = q - 1$ and if $x \neq y \in \mathbb{F}_p^*$ then $x^2 = y^2 \iff 0 = x^2 - y^2 = (x - y)(x + y) \iff x = -y$. Thus $|(\mathbb{F}_p^*)^2| = \frac{q-1}{2}$.

Second, we prove that every two adjacent vertices have $\frac{q-5}{4}$ common neighbors and every two non-adjacent vertices have $\frac{q-1}{4}$ common neighbors.

Let $x \in V(P) = (\mathbb{F}_p^*), A = N(x), B = V(P) \setminus (A \cup \{x\})$. If $y \in A$ and
2.2. PALEY GRAPHS

\( z \in B \) we want to prove that \( |A \cap N(y)| = \frac{q-5}{4} \) and \( |A \cap N(z)| = \frac{q-1}{4} \).

Because \( P \) is symmetric, we can assume that there is an integer \( l \) with every vertex \( y \in A \) is joined to \( l \) vertices in \( B \) (\( |N(y) \cap B| = l \)). Moreover, because \( P \) is self-complementary, every vertex \( z \in B \) is not joined to \( l \) vertices in \( A \ (|V(P) \setminus N(z)) \cap A| = l \).

To find \( l \) we calculate \( |A||B| \) from two sides. First \( |A| = |N(x)| = \frac{q-1}{2} \), \( |B| = |V(P)| - |A \cup \{x\}| = q - \left( \frac{q-1}{2} + 1 \right) = q - \frac{q-1}{2} = 2q - q - 1 = \frac{q-1}{2} \), which means that \( |A||B| = \left( \frac{q-1}{2} \right)^2 \).

Second \( |A||B| = |A \times B| = |\{(a, b) : a \in A, b \in B \text{ and } \{a, b\} \in E(P)\}| + |\{(a, b) : a \in A, b \in B \text{ and } \{a, b\} \notin E(P)\}| = \frac{q-1}{2} l + l \frac{q-1}{2} = 2l \frac{q-1}{2} \).

So \( |A||B| = \left( \frac{q-1}{2} \right)^2 = 2l \frac{q-1}{2} \), which gives \( 2l = \frac{q-1}{2} \) or \( l = \frac{q-1}{4} \).

Now we can calculate \( |A \cap N(y)| \) and \( |A \cap N(z)| \). Since

\[ \frac{q-1}{2} - \frac{q-1}{4} - \frac{q-3}{4} = \frac{2q-2q-3}{4} = \frac{q-5}{4}, \]

we have \( |A \cap N(y)| = \frac{q-1}{2} - \frac{q-3}{4} = \frac{2q-2q-3}{4} = \frac{q-5}{4} \).

Since \( \frac{q-1}{2} = |A| = |(V(P) \setminus N(z)) \cap A| + |N(z) \cap A| = l + |N(z) \cap A| = \frac{q-1}{4} + |N(z) \cap A|, \)

hence \( |N(z) \cap A| = \frac{q-1}{2} - \frac{q-1}{4} = \frac{q-1}{4} \).

Then \( P \) is a strongly regular graph with parameters

\[ (q, \frac{q-1}{2}, \frac{q-5}{4}, \frac{q-1}{4}). \]

\[ \square \]

Corollary 2.2.1. The Paley graphs are connected.

Proof. Let \( P \) be the Paley graph of order \( q = p^n \). Let \( x, y \) be two vertices in \( V(P) \), then \( x, y \) are adjacent or non-adjacent. If \( x, y \) are adjacent, then there exists a path of length 1 connected \( x \) and \( y \).

If \( x, y \) are non-adjacent, then \( x \) and \( y \) have at least one common neighbor \( z \) because \( q \geq 5 \) means that \( \frac{q-1}{4} \geq 1 \). So there exists a path \( x e_1 z e_2 y \) with \( e_1 = \{x, z\}, e_2 = \{z, y\} \) of length 2 connected \( x \) and \( y \).

Thus in all cases every two vertices in \( V(P) \) are connected by a path. \( \square \)
Now we know that the Paley graphs are self-complementary symmetric graphs. So the question now is: Are there any self-complementary symmetric graphs other than Paley graphs?

Peisert proved in [13] that the Paley graphs of prime order are the only self-complementary symmetric graphs of prime order and he proved in [12] that a graph $G$ is self-complementary and symmetric if and only if $|G| = p^n$ for some prime $p$, $p^n \equiv 1 \pmod{4}$, and $G$ is a Paley graph or a $P^*$-graph or is isomorphic to the exceptional graph $G(23^2)$.

The $P^*$-graph is a graph with $V(P^*) = \mathbb{F}_{p^n}$ and two vertices are adjacent if their difference belongs to the set $M = \{g^j : j \equiv 0, 1 \pmod{4}\}$, where $g$ is a primitive root of the field. The graph $G(23^2)$ has $23^2$ vertices and is described in Section 3 in [12].

One of the properties of the Paley graphs is the 3-existentially closed property. As in [7], for a fixed integer $n \geq 1$, a graph $G$ is $n$-existentially closed, if for every $n$-element subset $S$ of the vertices, and for every subset $T$ of $S$, there is a vertex $x \not\in S$ which is joined to every vertex in $T$ and to no vertex in $S \setminus T$.

The $n$-existentially closed graphs were first studied in [8], where they were called graphs with property $P(n)$. Ananchuen and Caccetta, in [3], proved that all Paley graphs with at least 29 vertices are 3-existentially closed, and before [7] they were the only known examples of strongly regular 3-existentially closed graphs. Now in [7] we can find a new infinite family of 3-existentially closed graphs, that are strongly regular but not Paley graphs. For further background on $n$-existentially closed graphs the reader is directed to [6].

Another property of the Paley graphs is also interesting. To understand it we need the following definition.

**Definition:** If $m, n \in \mathbb{N} \cup \{0\}$ and $k \in \mathbb{N}$, a graph $G$ is said to have the property $P(m, n, k)$, if for any disjoint subsets $A$ and $B$ of $V(G)$ with $|A| = m$ and $|B| = n$ there exist at least $k$ other vertices, each of which is adjacent to every vertex in $A$ but not adjacent to any vertex in $B$.

The set of graphs which have the property $P(m, n, k)$ is denoted by $\mathcal{G}(m, n, k)$.

In [1] and [4], it has been proved that the Paley graph

$$P_q \in \mathcal{G}(1, n, k) \text{ for every } q > \left((n - 2)2^n + 2\right)\sqrt{q} + (n + 2k - 1)2^n - 2n - 1;$$
\( P_q \in \mathcal{G}(n, n, k) \) for every \( q > ((2n-3)2^{2n-1}+2)\sqrt{q}+(n+2k-1)2^{2n-1}-2n^2-1; \)
and \( P_q \in \mathcal{G}(m, n, k) \) for every \( q > ((t-3)2^{t-1}+2)\sqrt{q}+(t+2k-1)2^{t-1}-1, \)
where \( t \geq m + n. \)
Chapter 3

Generalizations of The Paley Graphs

There are many generalizations of the Paley graphs. We will see some examples of these generalizations and some of its properties in the following section. In the second section we will define a new generalization, and we will study some of its properties in the third section.

3.1 Examples of some generalizations

Since two vertices in the Paley graphs are adjacent if and only if their difference is a quadratic residue, we can generate other classes of graphs by using higher order residues. For example, in [1], by using the cubic and quadruple residues Ananchuen has defined the cubic Paley graphs and the quadruple Paley graphs.

3.1.1 The cubic and the quadruple Paley graphs

**Definition:** Let $q = p^n$ with odd prime $p$, $n \in \mathbb{N}$, and $q \equiv 1 \pmod{3}$. The graph $G^{(3)}_q$ with

$$V(G^{(3)}_q) = \mathbb{F}_q \text{ and } E(G^{(3)}_q) = \{\{x, y\} : x, y \in \mathbb{F}_q, x - y \in (\mathbb{F}_q^*)^3\}$$

is called the cubic Paley graph.

Note that the set $E(G^{(3)}_q)$ in the definition is well defined because: $-1 = -1^3 \in (\mathbb{F}_q^*)^3$ implies that, $\{x, y\}$ is defined to be an edge if and only if $\{y, x\}$ is defined to be an edge.
**Definition:** Let \( q = p^n \) with odd prime \( p \), \( n \in \mathbb{N} \), and \( q \equiv 1 \pmod{8} \). The graph \( G_q^{(4)} \) with

\[
V(G_q^{(4)}) = \mathbb{F}_q \quad \text{and} \quad E(G_q^{(4)}) = \{ \{x, y\} : x, y \in \mathbb{F}_q, x - y \in (\mathbb{F}_q^*)^4 \}
\]

is called the quadruple Paley graph.

Note that the set \( E(G_q^{(4)}) \) in the definition is well defined because: We have \( q \equiv 1 \pmod{8} \), so \( 8 \mid (q - 1) \). If \( g \) is a generator of the group \( \mathbb{F}_q^* \) then

\[
g^{\frac{q-1}{2}} = (g^{\frac{q-1}{8}})^4 = -1
\]

which means that \(-1 \in (\mathbb{F}_q^*)^4 \). Thus \( \{x, y\} \) is defined to be an edge if and only if \( \{y, x\} \) is defined to be an edge.

The following figure gives an example:

![Graph Example](image)

**Fig. 3.1.1.** The cubic Paley graph \( G_{13}^{(3)} \) and the quadruple Paley graph \( G_{17}^{(4)} \)

In [1], Ananchuen has proved that the cubic Paley graphs

\[
G_q^{(3)} \in \mathcal{G}(2, 2, k) \quad \text{for every} \quad q > \frac{1}{4}(79 + 3\sqrt{36k + 701})^2;
\]

\[
G_q^{(3)} \in \mathcal{G}(m, n, k) \quad \text{for every} \quad q > (t2^{t-1} - 2^t + 1)2^m \sqrt{q} + (m + 2n + 3k - 3)2^{-n}3^{t-1},
\]

where \( t \geq m + n \); and the quadruple Paley graphs

\[
G_q^{(4)} \in \mathcal{G}(m, n, k) \quad \text{for every} \quad q > (t2^{t-1} - 2^t + 1)3^m \sqrt{q} + (m + 3n + 4k - 4)3^{-n}4^{t-1},
\]

where \( t \geq m + n \).

Ananchuen and Caccetta have proved in [2] that the cubic Paley graphs are \( n \)-existentially closed whenever \( q \geq n^22^{4n-2} \) and the quadruple Paley graphs are \( n \)-existentially closed whenever \( q \geq 9n^66^{2n-2} \).
3.1.2 The generalized Paley graphs

In [10], Lim and Praeger have defined the following generalization of the Paley graphs.

Definition: Let $\mathbb{F}_q$ be a finite field of order $q$, and let $k$ be a divisor of $q - 1$ such that $k \geq 2$, and if $q$ is odd then $\frac{q - 1}{k}$ is even. Let $S$ be the subgroup of order $\frac{q - 1}{k}$ of the multiplicative group $\mathbb{F}_q^*$. Then the generalized Paley graph $GPaley(q, \frac{q - 1}{k})$ is the graph with vertex set $\mathbb{F}_q$ and edges all pairs $\{x, y\}$ such that $x - y \in S$.

Note that in the definition they require $\frac{q - 1}{k}$ to be even when $q$ is odd, and hence in all cases $S = -S$, so that the adjacency relation is symmetric ($x - y \in S$ if and only if $y - x \in S$).

Also we can see that if $q \equiv 1 \mod 4$ and $k = 2$, then $GPaley(q, \frac{q - 1}{k})$ is the Paley graph $P_q$.

As an example consider $q = 11$ then $q - 1 = 10$, and since $\frac{q - 1}{k}$ should be even and $k \geq 2$, we have only one choice $k = 5$. Then $|S| = \frac{10}{5} = 2$ and $S = \{1, 10\}$, it follows that $GPaley(11, 2)$ is the cycle $C_{11}$.

Moreover, they have studied in [10] the automorphism groups of this generalized Paley graphs, and in some cases, compute their full automorphism groups. Moreover they have determined precisely when these graphs are connected.

3.2 Definition and examples

Now we will give a new generalization of the Paley graphs.

Definition: Let $q = p^n$ with odd prime $p$, $n \in \mathbb{N}$, and $m \geq 3$ be an odd integer. We will denote by $m-P_q$, the graph with $V(m-P_q) = \mathbb{F}_q$ and $E(m-P_q) = \{\{x, y\} : x, y \in \mathbb{F}_q, x - y \in (\mathbb{F}_q^*)^m\}$. Such a graph will be called $m$-Paley graph.

Note that the set $E(m-P_q)$ in the definition is well defined because : $-1 = -1^m \in (\mathbb{F}_q^*)^m$ implies that $x - y \in (\mathbb{F}_q^*)^m$ if and only if $y - x \in (\mathbb{F}_q^*)^m$.

The list of integers which can be considered as the order of $m$-Paley graph starts with 3,5,7,9,11,13,17,19,23,25,27,29,31.
3.2. **DEFINITION AND EXAMPLES**

In the following examples, we will show the $m$-Paley graphs explicitly for the first three cases.

**Example 1:** The $m$-Paley graph of order 3, for every odd integer $m \geq 3$, is the cycle $C_3$.

In order to see that, let $m-P_3 = (V, E)$ be the $m$-Paley graph of order 3 then $V(m-P_3) = \mathbb{Z}_3 = \{0, 1, 2\}$ and $(\mathbb{Z}_3)^m = \{1, 2\}$.

It follows that $E(m-P_3) = \{\{0, 1\}, \{1, 2\}, \{2, 0\}\}$.

**Example 2:** The $m$-Paley graph of order 5, for every odd integer $m \geq 3$, is the complete graph $K_5$.

In order to see that, let $m-P_3 = (V, E)$ be the $m$-Paley graph of order 5 and $m = 2k + 1$ with positive integer $k$, then $V(m-P_5) = \mathbb{Z}_5 = \{0, 1, 2, 3, 4\}$ and $1^m = 1, 4^m = -1^m = -1$. To calculate $2^m$, we have two cases:

- If $k$ is odd then $2^m = 2^{2k+1} = (2^2)^k 2 = (-1)^k 2 = 3$ and if $k$ is even then $2^m = 2^{2k+1} = (-1)^k 2 = 2$. To calculate $3^m$, we have also two cases:
  - If $k$ is odd then $3^m = 3^{2k+1} = (3^2)^k 3 = (-1)^k 3 = 2$ and if $k$ is even then $3^m = 3^{2k+1} = (-1)^k 3 = 3$.

Which gives $(\mathbb{Z}_5)^m = \{1, 2, 3, 4\}$ for every odd integer $m$, it follows that $E(m-P_5) = \{\{0, 1\}, \{0, 2\}, \{0, 3\}, \{0, 4\}, \{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}\}$.

**Example 3:** Let $m-P_7 = (V, E)$ be the $m$-Paley graph of order 7, then $V(m-P_7) = \mathbb{Z}_7 = \{0, 1, 2, 3, 4, 5, 6\}$.

Consider $m = 3$, then $1^3 = 1, 2^3 = 1, 3^3 = 6, 4^3 = 1, 5^3 = 6, 6^3 = 6$. It follows that $(\mathbb{Z}_7)^3 = \{1, 6\}$ and $E(3-P_7) = \{\{0, 1\}, \{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 5\}, \{5, 6\}, \{6, 0\}\}$.

Consider $m = 5$, then $1^5 = 1, 2^5 = 4, 3^5 = 5, 4^5 = 2, 5^5 = 3, 6^5 = 6$. It follows that $(\mathbb{Z}_7)^5 = \{1, 2, 3, 4, 5, 6\}$ and $5-P_7$ is the complete graph $K_7$.

Consider $m = 7$, then $1^7 = 1, 2^7 = 2, 3^7 = 3, 4^7 = 4, 5^7 = 5, 6^7 = 6$. It follows that

$(\mathbb{Z}_7)^7 = \{1, 2, 3, 4, 5, 6\}$ and $7-P_7$ is the complete graph $K_7$.

Consider $m = 9$, then $1^9 = 1, 2^9 = 1, 3^9 = 6, 4^9 = 1, 5^9 = 6, 6^9 = 6$. It follows that $(\mathbb{Z}_7)^9 = \{1, 6\}$ and
\[ E(9-P_7) = \{\{0, 1\}, \{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 5\}, \{5, 6\}, \{6, 0\}\}. \]

We can see that if \( \gcd(m, 6 = 7 - 1) = 1 \) then \( m-P_7 \) is the complete graph \( K_7 \), and if \( \gcd(m, 6 = 7 - 1) = 3 \) then \( m-P_7 \) is the cycle \( C_7 \).

### 3.3 Properties

We have seen in the previous examples that the \( m \)-Paley graphs \( m-P_q \) are the complete graph \( K_q \) in some cases and in other cases are not. The question, which we consider now, is: When are the \( m \)-Paley graphs complete? The answer to this question can be found in the following theorem.

**Theorem 3.3.1.** Let \( m-P_q = (V, E) \) be the \( m \)-Paley graph of order \( q \) and \( d = \gcd(m, q - 1) \), then

\[ m-P_q \text{ is complete if and only if } d = 1. \]

**Proof.** (\( \Rightarrow \)) If \( m-P_q \) is complete then \( d = 1 \): Assume that \( m-P_q \) is complete, then for all \( x \neq y \in \mathbb{F}_q \) we have \( \{x, y\} \in E(m-P_q) \), it follows \( x - y \in (\mathbb{F}_q^*)^m \).

Clearly \( (\mathbb{F}_q^*)^m \subseteq \mathbb{F}_q^* \). If \( a \in \mathbb{F}_q^* \) then we can find \( x, y \in \mathbb{F}_q \) with \( a = x - y \in (\mathbb{F}_q^*)^m \). It follows that \( (\mathbb{F}_q^*)^m = \mathbb{F}_q^* \), which means that for all \( a \in \mathbb{F}_q^* \) there exists \( b \in \mathbb{F}_q^* \) with \( a = b^m \).

By Theorem \ref{thm:2.2.5} we know that \( \mathbb{F}_q^* \) is cyclic. Let \( g \) be a generator of \( \mathbb{F}_q^* \), then for all \( i \in \{1, 2, \ldots, q - 1\} \) there exists \( j \in \{1, 2, \ldots, q - 1\} \) with \( g^i = (g^j)^m \) or \( g^{i-jm} = 1 \). Since order \( g \) is \( q - 1 \), we have \( q - 1 \mid i - jm \), so \( i - mj = (q - 1)k \) for some \( k \). Thus \( i = mj + (q - 1)k \).

Now to prove that \( d = 1 \), assume the contrary. Let \( d > 1 \), then

\[ m = dm_1, \quad (q - 1) = dk_1 \text{ for some } m_1, k_1. \]
So we can write \( i = dm_1j + dk_1k = d(m_1j + k_1k) \), which means that \( i \) must be a multiple of \( d \), but \( i \) is an arbitrary element of the set \( \{1, 2, \ldots, q-1\} \). Thus we have a contradiction, then \( d = 1 \).

\((\iff)\) If \( d = 1 \) then \( mP_q \) is complete: Assume that \( d = 1 \) and \( mP_q \) is not complete, then in the set \( \{g^m, (g^2)^m, \ldots, (g^{q-1})^m = 1\} \) there exists two equal elements. Let \((g^i)^m = (g^j)^m \) with \( i, j \in \{1, 2, \ldots, q-1\}, i > j \), then \( g^{(i-j)m} = 1 \), so \( q - 1 \mid (i - j)m \).

From the assumption \( \gcd(m, q - 1) = d = 1 \Rightarrow q - 1 \mid i - j \), which is impossible because \( i - j < q - 1 \). Then \( mP_q \) is complete. \( \square \)

Note that the completeness of the \( m \)-Paley graph means that \((\mathbb{F}_q^*)^m = \mathbb{F}_q^*\). So as an application of Theorem 3.3.1, we have the following corollary.

**Corollary 3.3.1.** In the field \( \mathbb{F}_q \), the equation \( x^m = a, a \in \mathbb{F}_q \) has exactly one solution if and only if \( d = \gcd(m, q - 1) = 1 \).

Note that \( d \) is an odd integer because \( m \) is an odd integer. Now the question is: How does the \( m \)-Paley graphs, if \( \gcd(m, q - 1) = d > 1 \), look like? The following proposition has the answer of this question.

**Proposition 3.3.1.** Let \( mP_q = (V, E) \) be the \( m \)-Paley graph of order \( q \), if \( d = \gcd(m, q - 1) > 1 \), then \( mP_q \) is \( \frac{q-1}{d} \)-regular.

**Proof.** Let \( x \) be any vertex in \( V(mP_q) \), then \( y \in V(mP_q) \) is adjacent to \( x \) if and only if there exists \( z \in (\mathbb{F}_q^*)^m \) with \( x - y = z \). Which means that \( |N(x)| = |(\mathbb{F}_q^*)^m| \) for all \( x \in V(mP_q) \), then \( mP_q \) is \( |(\mathbb{F}_q^*)^m| \)-regular.

Let us now prove that \( |(\mathbb{F}_q^*)^m| = \frac{q-1}{d} \). Let \( g \) be a generator of the group \( \mathbb{F}_q^* \) and \((g^i)^m = (g^j)^m \) for some \( i, j \in \{1, 2, \ldots, q-1\} \), then \( g^{(i-j)m} = 1 \), which implies that \( q - 1 \mid (i - j)m \).

Since \( d \mid q - 1 \), \( d \mid m \), it follows that \( \frac{q-1}{d} \mid \frac{q-1}{d} \mid (i - j) \). Since \( \gcd\left(\frac{q-1}{d}, \frac{m}{d}\right) = 1 \), we have \( \frac{q-1}{d} \mid (i - j) \iff i \equiv j \pmod{\frac{q-1}{d}} \). So the following \( d \) elements

\[ g^i, g^{i + \frac{q-1}{d}}, g^{i + 2\frac{q-1}{d}}, \ldots, g^{i + (d-1)\frac{q-1}{d}} \]

are the same. Thus \( |(\mathbb{F}_q^*)^m| = \frac{q-1}{d} \) and \( mP_q \) is \( \frac{q-1}{d} \)-regular. \( \square \)

**Corollary 3.3.2.** In the field \( \mathbb{F}_q \), if \( d = \gcd(m, q - 1) > 1 \), then the equation \( x^m = a, a \in \mathbb{F}_q \) has exactly \( d \) solutions.
CHAPTER 3. GENERALIZATIONS OF THE PALEY GRAPHS

As a special case, if \( d = m \) and \( a = 1 \) we get Lagrange’s lemma.

We can see that the \( m \)-Paley graphs are not strongly regular in general. In example 3 we have seen that \( 3-P_7, 9-P_7 \) are \( C_7 \) which is not strongly regular. So the question now is: Are the \( m \)-Paley graphs symmetric or self-complementary?

**Proposition 3.3.2.** The \( m \)-Paley graphs are symmetric.

**Proof.** By the same proof of Proposition 2.2.1 with replacing \((\mathbb{F}_q^*)^2\) by \((\mathbb{F}_q^*)^m\) we get the result. \( \square \)

**Proposition 3.3.3.** The \( m \)-Paley graphs are not self-complementary.

**Proof.** Clearly, a self-complementary graph of order \( q \) should have \( \frac{q(q-1)}{4} \) edges. Let \( m-P_q = (V, E) \) be the \( m \)-Paley graph of order \( q \).

If \( d = \gcd(m, q-1) > 1 \), then by Proposition 3.3.1

\[
|E(m-P_q)| = \frac{1}{2} \sum_{x \in V(m-P_q)} d(x) = \frac{1}{2} q \frac{q-1}{d}.
\]

Since \( d \geq 3 \), it follows that \( |E(m-P_q)| < \frac{q(q-1)}{4} \).

If \( d = \gcd(m, q-1) = 1 \), then by Proposition 3.3.1

\[
|E(m-P_q)| = \frac{q(q-1)}{2} > \frac{q(q-1)}{4}.
\]

Thus the \( m \)-Paley graphs are not self-complementary. \( \square \)

Now let us ask the following question: Are the \( m \)-Paley graphs connected? Clearly, the \( m \)-Paley graph of order \( q \) with \( d = \gcd(m, q-1) = 1 \), which is the complete graph \( K_q \), is connected. So the case which we will study is the \( m \)-Paley graph of order \( q \) with \( d = \gcd(m, q-1) > 1 \).

Note that \( d \neq q - 1 \) because \( q - 1 \) is even and \( d \) must be odd. So

\[
\frac{q-1}{2} \geq d = \gcd(m, q-1) \geq 1.
\]

**Proposition 3.3.4.** Let \( m-P_q = (V, E) \) be the \( m \)-Paley graph of order \( q \), if \( d = \gcd(m, q-1) > 1 \) and \( q \) is prime, then \( m-P_q \) is connected.

**Proof.** Since \( q \) is prime, \( \mathbb{F}_q = \mathbb{Z}_q \). For every \( x < y \in \mathbb{F}_q \), we have the sequence \( x, x+1, x+2, \ldots, x+(y-x-1) = y-1, x+(y-x) = y \in \mathbb{F}_q \). So \( \{x, x+1\}, \{x+1, x+2\}, \ldots, \{y-1, y\} \in E(m-P_q) \), because \( 1 \in (\mathbb{F}_q^*)^m \) for each
odd integer \( m \). Thus \( x \{ x, x + 1 \} x + 1 \{ x + 1, x + 2 \} x + 2 \cdots y - 1 \{ y - 1, y \} y \) is a path in \( m-P_q \) between \( x \) and \( y \). So \( m-P_q \) is connected.

Note that as a special case of Proposition 3.3.4, if \( d = \gcd(m, q - 1) = \frac{q - 1}{2} \) and \( q \) is prime, then \( m-P_q \) is the cycle \( C_q \), because \( m-P_q \) is connected and 2-regular.

**Proposition 3.3.5.** Let \( m-P_q = (V, E) \) be the \( m \)-Paley graph of order \( q \), if \( d = \gcd(m, q - 1) = \frac{q - 1}{2} \) and \( q \) is not prime, then \( m-P_q \) is disconnected.

**Proof.** Let \( q = p^n, n > 1 \), then \( \mathbb{F}_q = \mathbb{Z}_p[x]/(f(x)) \) where \( f(x) \) is an irreducible polynomial of degree \( n \) over \( \mathbb{Z}_p \).

Since \( 1 \in (\mathbb{F}_q^*)^m \), the path \( \{0, 1\}1\{1, 2\}2\cdots p-1\{p-1, 0\}0 \) in \( m-P_q \) form a cycle, say \( C_p \). By using Proposition 3.3.1 with \( d = \gcd(m, q - 1) = \frac{q - 1}{2} \), it follows that \( m-P_q \) is 2-regular.

Since \( n > 1 \), it follows that the set \( \mathbb{F}_q \setminus \mathbb{Z}_p \) is not empty. Let \( a \in \mathbb{F}_q \setminus \mathbb{Z}_p \), then \( a \) is not adjacent to any vertex in \( C_p \). So we cannot find a path in \( m-P_q \) between any vertex in \( \mathbb{F}_q \setminus \mathbb{Z}_p \) and any vertex in \( C_p \).

Thus \( m-P_q \) is disconnected.

**Example:** Take \( q = 27, m = 13 \), then \( d = \gcd(13, 27 - 1) = 13 = \frac{27 - 1}{2} \). The following figure shows that the 13-\( P_{27} \) is disconnected.

![fig-3.3.1](image)

**Fig. 3.3.1.** The 13-\( P_{27} \) graph, where \( a \) is a root of an irreducible polynomial \( f(x) \) of degree 3 over \( \mathbb{Z}_3 \)

So now we know that the \( m \)-Paley graphs are not always connected and the case : \( q = p^n, n > 1 \) and \( 1 < d = \gcd(m, q - 1) < \frac{q - 1}{2} \) is still open.
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Hiermit versichere ich, die vorliegende Arbeit selbstständig verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt zu haben.

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