Josephson junctions detectors for Majorana modes and Dirac fermions

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We demonstrate that the current-voltage (I-V) characteristics of resistively and capacitively shunted Josephson junctions (RCSJ) hosting localized subgap Majorana states provides a phase sensitive method for their detection. The I-V characteristics of such RCSJs, in contrast to their resistively shunted counterparts, exhibit subharmonic odd Shapiro steps. These steps, owing to their subharmonic nature, exhibit qualitatively different properties compared to harmonic odd steps of conventional junctions. In addition, the RCSJs hosting Majorana bound states also display an additional sequence of steps in the devil staircase structure seen in their I-V characteristics; such sequence of steps make their I-V characteristics qualitatively distinct from that of their conventional counterparts. A similar study for RCSJs with graphene superconducting junctions hosting Dirac-like quasiparticles reveals that the Shapiro step width in their I-V curves bears a signature of the transmission resonance phenomenon of their underlying Dirac quasiparticles; consequently, these step widths exhibit a $\pi$ periodic oscillatory behavior with variation of the junction barrier potential. We discuss experiments which can test our theory.

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I. INTRODUCTION

The possibility of realization of Majorana zero modes, particles with anyonic statistics described by real wavefunctions, has attracted tremendous interest in recent years.$^{[1]}$ Several suggestions regarding condensed matter systems which can host such fermions have recently been put forth.$^{[2,3]}$ Out of these, the most promising ones for experimental realization turns out to be those which host Majorana modes as localized subgap states in their superconducting ground state.$^{[4,5]}$ Typically, the occurrence of such states require unconventional superconducting pairing symmetry such as $p$- or $d$-wave pairing.$^{[6,7]}$ However recent proposals have circumvented this requirement; it was shown that such bound states can occur either at the end of one-dimensional (1D) wire in a magnetic field with spin-orbit coupling and in the presence of a proximate $s$-wave superconductor$^{[8]}$ or in superconducting junctions atop a topological insulator surface hosting Dirac fermions on the surface of a topological insulator.$^{[9]}$ Such Majorana fermions leave their signature as a midgap peak in tunneling conductance measurement$^{[10]}$ as well through fractional Josephson effect.$^{[11]}$

Another interesting phenomenon in recent years has been the discovery of materials whose low-energy quasiparticles obey Dirac-like equations. These materials are commonly dubbed as Dirac materials; graphene and topological insulators are common examples of such materials.$^{[12,13]}$ These materials can exhibit superconductivity via proximity effect with Cooper pairing occurring between Dirac electrons with opposite momentum.$^{[14,15]}$ It is well known that transport properties of such superconductors differ from their conventional counterparts and can serve as experimental signatures of the Dirac nature of their constituent quasiparticles.$^{[16,17]}$

The experimental detection of Majorana modes have mainly relied on measurement of either midgap peak$^{[15]}$ or detection of even Shapiro steps in a Josephson junctions of superconductors hosting Majorana mode.$^{[18]}$ The effectiveness of the former set of experiments in detection of Majorana modes has been questioned since the midgap peak did not lead to the expected $2e^2/h$ value of the tunneling conductance and could have also occur from several other effects such as presence of magnetic impurities leading to Kondo effect$^{[19]}$ and impurity induced subgap states$^{[20]}$. In this sense, the presence of even Shapiro steps at $V = n\hbar\omega_J/e$ (and the absence of odd ones at $(2n+1)\hbar\omega_J/2e$) in Josephson current measurement, where $\omega_J$ is the Josephson frequency, $n$ is an integer, and $V$ is the applied external voltage, provide a more definite detection of such fermions since they constitute a phase sensitive signature which is free of effects of disorder.$^{[21]}$ Consequently, theoretical studies of AC Josephson effect for unconventional superconductors which hosts Majorana modes has received a lot of attention lately.$^{[21,22]}$ Theoretical studies of Josephson effect in graphene Josephson junctions has also been carried out.$^{[23,24]}$ It was shown that the critical current of such junctions shows a novel oscillatory dependence on the barrier potential of the junctions. However, the features of AC Josephson effects in either of these systems for a resistively and capacitively shunted Josephson junction (RCSJ) in the presence of an external radiation has not been studied previously. In this context, we mention that the analysis presented in the paper for junctions hosting Majorana subgap states is somewhat idealized in the sense that it does not provide a full treatment of Landau-Zener tunneling and other related quantum effects; however we do provide a qualitative discussion and identify a regime of junction parameters where our analysis is expected to hold qualitatively.

In this work, we study the Josephson junction de-
scribed by a RCSJ model where the superconductors forming the junction either host Majorana modes at the interface or constitutes Dirac-like quasiparticles. In the former case, we show that in contrast to their counterpart in conventional junctions, the I-V characteristics display subharmonic odd Shapiro steps whose width vanishes for resistive junctions (in the limit where the junction capacitance approaches zero). We provide an analytical formula for the step width of both even and odd steps for such junctions, show that the analytic result matches exact numerics closely, and demonstrate, on basis of this analytical result, that the behavior of this ratio is qualitatively different for Josephson junctions with and without Majorana bound states. In particular we demonstrate that the ratio of the odd and the even step width decrease exponentially with the junction capacitance for junctions with Majorana modes; in contrast, this ratio does not vary appreciably for conventional s-wave junctions. Thus it serves as a robust indicator for bound Majorana states in a JJ. Our analytical results, supported by numerical analysis, reproduces the phenomenon of absence of odd Shapiro steps in Josephson junctions with Majorana bound states as a special limiting property of resistive Josephson junctions; thus our work indicates that the absence of odd Shapiro steps, while sufficient, is not a necessary characteristics of Majorana bound states in a Josephson junction. We also find that the I-V characteristics of junctions with Majorana bound states show a qualitatively different devil staircase structure which is distinct from their s-wave counterparts. In particular, they display additional sequence of steps which follow Farey’s sum rule, such sequences are absent in I-V characteristics of conventional s-wave junctions. In the latter case, for junctions of superconductors hosting Dirac quasiparticles, we show that the width of the Shapiro steps display $\pi$ periodic oscillatory dependence on the barrier potential of the junction. Such a behavior is a direct consequence of the transmission resonance phenomenon of the Dirac-like quasiparticles of the superconductors forming the junctions and is qualitatively distinct from conventional junctions hosting quasiparticles which obey Schrodinger equation. We note that our work shows that a RCSJ can act as phase sensitive detection device for both Majorana bound states in topological superconductors and Dirac like quasiparticles in a superconductor; it is therefore expected to be of interest to theorists and experimentalists working on both Majorana modes and Dirac materials.

The plan of the rest of the work is as follows. In Sec. II we provide our analytical and numerical results for junctions which hosts subgap Majorana bound states. This is followed by Sec. III where we present our results on junctions which hosts Dirac-like quasiparticles. Finally, we summarize our main results, provide a discussion of experiments that can test our theory, and conclude in Sec. IV.

II. JUNCTIONS WITH MAJORANA MODES

The basic design of the circuit which we propose to serve as detector is shown in Fig. 1. To analyze the property of this circuit, we first consider the Josephson junction. In our proposal, this comprises of two superconductors with order parameters $\Delta_R$ (for $x > d/2$) and $\Delta_L$ (for $x < -d/2$) separated by a barrier region of width $d$ ($-d/2 \leq x \leq d/2$) characterized by a barrier potential $V_0$ as shown in Fig. 1. The superconductors can either be topological superconductors with (effective) p-wave pairing or superconductors with Dirac-like quasiparticles which has s-wave pair-potentials. In this section, we analyze the former case in details. We note at the outset that the present analysis will hold for topological superconductors in 1D wire geometry provided that the transverse dimension $L$ is set to zero. Josephson junctions shown in Fig. 1 are known to support localized subgap Andreev bound states which can be obtained as solution of the Bogoliubov-de Gennes (BdG) equation. For topological superconductors which support p-wave pairing, the BdG equations reads

$$[(H_\beta + V(x))\tau_z + (\Delta_\beta(x)\tau_+ + \text{h.c.})] \psi_\beta = E\psi_\beta$$

where $\beta = R, L$ for the right and the left superconductors, $\psi_\beta = [\psi_{\beta L}(x, k_\parallel), \psi_{\beta R}^\dagger(x, k_\parallel)]$ is the two component BdG wavefunction, $V(x) = V_0\delta(x)$ is the barrier potential (we take the limit of a thin barrier for which $d \to 0$), $H_\beta = \hbar^2k^2/(2m) - \mu$ denotes the dispersion of the left and right superconductors with $\mu$ being the chemical potential, $m$ the electron mass, and $k^2 = -\partial^2_\parallel + k_F^2$. In what follows, we are going to assume $p_{x}$-wave pairing and write $\Delta_L(x) = \Delta_0k_F/k_F$ and $\Delta_R = \Delta_0k_F/k_F \exp(\delta\phi)$, where $k_F$ are the Fermi mo-

![FIG. 1: (Color online) Schematic representation of the Josephson junction in the RCSJ circuit (see inset). The junction has width $L$ in the transverse direction and the barrier region separating the two superconductors has a thickness $d$ and is modeled by a potential $V_0$. $I_J$, $I_C$ and $I_R$ are the Josephson current, displacement current and quasiparticle currents respectively.](image)
menta of the two superconductors, $\phi$ is the phase difference across the junction, $\Delta_0$ is the amplitude of the superconducting gap, and $k_F$ is the $x$ component of the Fermi momentum. The wavefunctions $\psi_\beta$ satisfy the boundary condition $\psi_L(x = 0) = \psi_R(x = 0)$ and $\partial_x \psi_L(x = 0) - \partial_x \psi_R(x = 0) = k_F \chi_1(k_\parallel) \psi_L(x = 0)$, where $\chi_1(k_\parallel) = \chi = 2U_0/\hbar v_F(k_\parallel)$ is the dimensionless barrier potential for a given transverse momentum of the quasiparticles. The localized subgap solutions of Eq. 1 are given by [23,24]

$$E_1 = -\Delta_0 \cos(\phi/2)/\sqrt{1 + \chi^2}/4. \tag{2}$$

Note that for $\phi = \pi$, $E = 0$ and it has been shown that this state constitutes a realization of Majorana modes [9]. Since these subgap states are the only ones with $\phi$ dependent dispersion, one find the zero temperature Josephson current as

$$I_1(\phi) = \frac{2e}{h} \partial E_1/\partial \phi \int_{-k_F}^{k_F} \frac{dk_n}{2\pi} \frac{e^{\Delta_0} \sin(\phi/2)}{2\hbar \sqrt{1 + \chi^2}/4}. \tag{3}$$

As noted in Ref. [13], the current is $4\pi$ periodic and a substitution $\phi \rightarrow 2eVt/h$ in the presence of a bias voltage $V$ leads to fractional AC Josephson effect [13,22].

We now use Eqs. 3 to obtain the response of an RCSJ circuit constructed out of the superconducting junctions discussed above. The RCSJ model, shown in the inset of Fig. 1, include a resistive component to take into account dissipative process into account which may occur, for example, due to quasiparticle tunneling and a shunting capacitance $C$ which takes into account the displacement currents due to possible charge accumulation in the leads [25]. The current phase-relationship for this model, in the presence of an external radiation, is given by [23,25]

$$\dot{\phi} + \beta \dot{\phi} + I_1(\phi)/I_c = I/I_c + A \sin(\omega t)/I_c, \tag{4}$$

where $I_c$ is the critical current of the junction, $A$ and $\omega$ are the amplitude and frequency of the external radiation, $I_1(\phi) = I_1$ for superconducting junctions with Majorana fermions, $\beta = \sqrt{h/(2eI_c R^2 C_0)}$, $R$ and $C_0$ denote the resistance and capacitance of the junction, and we have scaled $t \rightarrow \beta t$, where $\tau = \sqrt{\hbar C_0/(2eI_c)}$.

Before proceeding further, we note that Eq. 4 holds in the ideal limit where the two subgap Majorana modes at the two ends of each wire do not interlace [23]. Such an interaction leads to hybridization of amplitude $\delta$ between the two Majorana branches: $E_{1yvb} = \pm \sqrt{\delta^2 + E_F^2}$ [23]. The hybridization amplitude $\delta$ arising from such interaction is exponentially suppressed for long wires $\delta \sim \exp[-L/(2\xi)]$, where the coherence length $\xi$, for 1D nanowires, depends on the spin-orbit coupling strength of the wire. In the presence of an external voltage, this leads to a Landau-Zener tunneling probability $P_t = \exp[-2\pi \delta^2/(E_F \hbar \phi)]$, where $E_F \sim \Delta_0$ is the maximal Josephson energy. The manifestation of $4\pi$ periodicity is evident when $P_t \simeq 1$. This occurs when $\delta^2/(E_F \hbar \phi) \ll 1$; a complete determination of frequency and voltage range where $P_t \simeq 1$ requires the self-consistent solution of Eq. 4 and $I_1 = \partial E_{2yvb}/\partial \phi$. We have not attempted this in this work; however, we note that such a regime can always be obtained for long enough wire since $\delta$ is exponentially suppressed in this regime leading to $P_t \simeq 1$. An analysis of these conditions for resistive junctions can be found in Ref. [23] in the rest of this work, we shall assume $P_t \simeq 1$ and work with Eq. 4.

In what follows, we first obtain an approximate analytical solution of Eq. 4 in Sec. II A which demonstrates the existence of subharmonic odd Shapiro steps for junctions hosting Majorana bound states. This is followed by Sec. II B where we carry out a detailed numerical study of Eq. 4 where the analytical results are verified and the devil staircase structure of the Shapiro steps is studied.

### A. Perturbative Analytical Solution

In this section we consider perturbative analysis of Eq. 4 for an unconventional Josephson junction comprised of superconductors hosting subgap Majorana states with $E = E_J \cos(\phi/2)$. We begin from the equation of such a junction given by Eq. 4 and analyze this equation for $\omega, \beta, A >> 1$. The key point regarding this analysis is the observation in the regime mentioned above it is possible to expand $\phi$ as [25]

$$\phi = \sum_n \epsilon^n \phi_n, \quad I = \sum_{n=0}^\infty \epsilon^n I_n \tag{5}$$

where $I_0$ is the applied current and $I_n, \epsilon \ll 1$, and $I_n$ for $n > 0$ determined self-consistently from the condition of absence of additional dc voltage: $\lim_{n \to \infty} I_n/\phi_n = 0$ [21].

The equations for $\phi_n$ can be obtained by equating terms in the same order of $\epsilon$. The procedure is standard and yields

$$\phi_n + \beta \dot{\phi}_n = f_n(t) + I_n, \quad f_0 = A \sin(\omega t), \quad f_1 = -\sin[\phi_0/2], \quad f_2 = \phi_1 \cos[\phi_0/2]/2 \tag{6}$$

Note that the $n = 0$ equation represents the autonomous I-V curve of the junction and is independent of the non-linear sinusoidal term.

To solve these equations, we note that this represent a linear first order differential equation in $\phi_n$; consequently we define $y_n = \phi_n$, and write

$$\dot{y}_n + \beta y_n = I_n + f_n(t). \tag{7}$$

These equations admit a solution

$$y_n(t) = I_n + e^{-\beta t} \int_0^t e^{\beta t'} f_n(t')dt', \quad \phi_n(t) = \int_0^t y_n(t')dt' + \phi_n(0). \tag{8}$$

For $n = 0$, this yields

$$\phi_0(t) = \phi' + I_0 t + \frac{A}{\omega \gamma} \sin(\omega t + \alpha_0), \tag{9}$$
where $\alpha_0 = \arccos(\omega/\gamma)$, $\gamma = \sqrt{\beta^2 + \omega^2}$, and $\phi'$ is the DC phase of the junction. The supercurrent at this order is given by

$$I_s^{(0)} \sim \sin(\phi_0(t)/2) = \Im(e^{i\phi_0(t)/2})$$

$$= \Im \sum_{n=-\infty}^{\infty} J_n(x)e^{i\left[(\ln/2+n\alpha_0)t+n\omega_0+\phi'/2\right]},$$  \hspace{1cm} (11)$$

where $x = A/(2\gamma \omega)$. Thus the Shapiro steps occur when the AC component of the supercurrent vanishes: $I_0 = 2|n|\omega$ (even steps). The width of the $n^{th}$ even step can be read off from Eq. (11) $W_{\text{even}} = \Delta I_s^{(\text{even})} = 2J_n(x)$. These yield expressions for the width of harmonic steps. Note that in contrast to the conventional junctions where $I_s \sim \sin[\phi_0]$, only even harmonic steps occur for junctions which host subgap Majorana steps.

Next, we obtain the solution for $\phi_1$. Substituting Eq. (11) in Eq. 7 we find after some straightforward algebra

$$\phi_1 = \sum_{n=-\infty}^{\infty} J_n(x)(\gamma_n \omega_n)^{-1} \cos(\omega_n t + n\alpha_0 + \delta_0 + n\phi'/2).$$

$$= \sum_{n_1,n_2=-\infty}^{\infty} J_{n_1}(x) J_{n_2}(x)(2\gamma_n \omega_n)^{-1} \times \sin(\omega_n t + n_1(\alpha_0 + \phi'/2) + \delta_{n_1}) \times \cos(\omega_n t + n_2(\alpha_0 + \phi'/2) + \delta_{n_2})$$

$$= \sum_{n_1,n_2} J_{n_1}(x) J_{n_2}(x)(4\gamma_n \omega_n)^{-1} \times \sin([\omega_n + \omega_{n_2}]t + [n_1 + n_2](\alpha_0 + \phi'/2) + \delta_{n_1}) \times \cos([\omega_n + \omega_{n_2}]t + [n_1 + n_2](\alpha_0 + \phi'/2) + \delta_{n_1})$$

At this order, we find that there are additional steps in the DC component of the supercurrent; these steps occur at $|n_1 + n_2|\omega = I_0$ for which the first of the two terms in the right side of Eq. (14) become independent of time. A set of these steps occur at $(n_1 + n_2) = 2n - 1$ for integers $n = 1, 2, \ldots$ and constitute the odd Shapiro steps. Thus we find that the odd steps for a junction of superconductors hosting Majorana ground states are necessarily of subharmonic nature. The width of these steps can be read off from Eq. (14) to be

$$W_{\text{odd}} = \Delta I_s^{(\text{odd})} = \sum_{n_1} J_{n_1}(x) J_{2n-1-n_1}(x) \times \frac{1}{2} \left| \left(2n - 1 - n_1 \right) \omega \right|^2 + \beta^2 \right|^2.$$

We note that when $C_0 \to 0$, $\beta \to \infty$ and the subharmonic Shapiro steps vanish leading to the result that only even harmonic Shapiro steps exists for resistive Josephson junctions hosting subgap Majorana steps. Thus our analysis reproduces the absence of odd Shapiro steps in Josephson junctions with Majorana bound states as a special case. We also note that these odd steps have a completely different origin than the analogous steps discussed in Ref. [23] since they occur without any $\sim \sin(\phi)$ dependence of $I_s$. Finally, we note that the ratio of the $n^{th}$ even and the adjacent odd Shapiro steps for these junctions are given by

$$\eta_n = \frac{W_{\text{even}}}{W_{\text{odd}}} = \frac{2J_n(x)}{\sum_{n_1} J_{n_1}(x) J_{2n-1-n_1}(x)} \times \frac{1}{2} \left| \left(2n - 1 - n_1 \right) \omega \right|^2 + \beta^2 \right|^2.$$  \hspace{1cm} (15)$$

In the next section, we shall compare the theoretical results (Eq. 15) with numerical results obtained by exact numerical solution of Eq. 4.

B. Exact numerical results

To compute the I-V characteristics, we study the temporal dependence of $V = \hbar \delta/(2e)$ obtained by numerical solution of Eq. 4 as a function of time for a fixed bias current $I$. The dc component of the voltage is obtained by standard procedure [23] from $V$ and plotted as a function of $I$ to generate the I-V characteristics.

The central results that we obtain from this analysis are as follows. First, for topological superconductors hosting Majorana subgap states, we find that for a significant range of external coupling and in the underdamped region $\beta < 1$, show both even and odd Shapiro steps as expected from the analytical results obtained in Sec. II.A. The even steps at $V = \hbar \delta/\omega/e$ are enhanced compared to their odd counterparts at $V = (2n + 1)\hbar \omega/e$ as shown in Figs. 3a and 3b, for underdamped ($\beta < 1$) and overdamped ($\beta > 1$) regions respectively. For $\eta > 1$, the exponential dependence of the even steps over the odd ones is characterized by $\eta = \eta (\text{Eq. 15}).$ A plot of $\eta$ as a function of $\beta$ shown in Fig. 3a, demonstrates that $\eta \sim \exp(0.3\beta^2)$ for junctions with Majorana modes. We also note that the theoretical result for $\eta$ obtained from Eq. 15 provides a near-perfect match with the exact numerics demonstrating the accuracy of the analytical solution over a wide range of $\beta$. We further note that the behavior of $\eta$ as a function of $\beta$ is in complete contrast to its counterpart for s-wave superconductors where $\eta$ does not vary appreciably with $\beta$ as shown in Fig. 2a. Thus the exponential dependence of $\eta$ on the junction capacitance $C$ constitutes a phase sensitive signature of the presence of the Majorana modes. Note that it is generally expected that only even Shapiro steps occur in I-V characteristics of Josephson junctions which supports Majorana modes due to 4$\pi$ periodicity of the Josephson current phase, and this is a consequence of analysis of the problem in the limit of zero junction capacitance $C \to 0$ where $\beta \to \infty$ leading to vanishing of the odd steps. However, our study show that underdamped RCSJ of such unconventional Josephson junctions with finite $C$ can display both even and
Fig. 2: (Color online) Top panels: CVC of $p$-wave Josephson junction. Fig. 2a shows the CVC in the underdamped region ($\beta = 0.2$) and for $A = 20$ and $\omega = 2$. While Fig. 2b represents the overdamped region ($\beta = 1.2$) with $A = 5$ and $\omega = 1$ (in appropriate dimensionless units; see text). Bottom Panels: Fig. 2c shows the CVC of $s$-wave junction (all parameters are the same as in Fig. 2b). Fig. 2d shows the ratio of the widths of the even and odd Shapiro steps, $\eta = W_{\text{even}}(2\omega)/W_{\text{odd}}(\omega)$ for the $p$-wave and the $s$-wave junctions (inset) as a function of $\beta$ for $A = 10$ and $\omega = 3$. For $p$-wave, $\eta \sim e^{0.313\beta^2}$ while for $s$-wave, $\eta$ does not vary with $\beta$. Figs. 2a, b, and c have $V$ and $I$ scaled in units of $h/(e\tau)$ and $I_c$ respectively. The red solid (blue dashed) curves in panels (a) and (c) correspond to data for increasing(decreasing) current sweeps; these data coincides in the overdamped region as shown in panel (b).

Odd steps. Thus the presence of odd Shapiro steps do not necessarily signify the absence of Majorana modes specially if the RCSJ is underdamped.

Another qualitative difference between the I-V characteristics of Josephson junctions hosting Majorana states with their conventional counterparts occur in the devil staircase structures of the Shapiro steps occurring within a fixed bias current intervals in these junctions. Such steps are known to occur for conventional $s$-wave junctions and their voltage-frequency relation can be represented by a continued fraction as

$$V = [N \pm 1/(n \pm (1/m \pm 1/(p \pm 1...)))] \omega, \quad (16)$$

where $N, n, m, p$ are integers. The fractions involving $N$ are termed as first level fractions, those with $N$ and $n$ are second level fractions, and so on. To distinguish between $s$-wave and $p$-wave junctions, we investigate the structure of these steps for both $p$- and $s$-wave junctions as shown in Figs. 2a and 2c. We find that for $s$-wave, only second level subharmonics are present and $V = 6\omega$ is approached from below with steps occurring at $V = N(1-1/n)\omega$ for different $N$ and $n$. In contrast, for $p$-wave junctions, one finds that in addition to the steps which occur for the $s$-wave junctions, there are additional steps corresponding to $V = [(N+2)][1+1/(n \pm 1/m)]\omega$ (Figs. 2a and 2c). This difference in structure lead us to hypothesize that in contrast to $s$-wave Josephson junctions, the continued fractions for $p$-wave shows additional fractions all of which are consistent with the Farey sum rule; i.e., the widest phase-locked region (the widest step) between any two resonances $p/q$ and $m/n$ for these fractions is given by $(p + m)/(q + n)$. We therefore suggest that the presence of these additional specific continued fractions may be considered as a signature of Majorana modes.

Two specific examples of this phenomenon is presented in Fig. 3 and 4. In Fig. 3 we find that the continued fraction $V = (N-1/n)\omega$ with $N = 6$ which appears in $s$-wave Josephson junctions at $\beta = 0.2$, $\omega = 0.5$ and $A = 0.8$, is also appeared in the $p$-wave junction at smaller amplitude of radiation $A = 0.6$. However, as clearly demonstrated in Fig. 3, the steps for the $p$-wave junction constitutes an additional sequence of continued fraction $V = [(N-2)][1+1/(n+1/m)]\omega$ with $N = 6$, $n = 2$ and for several $m$. Further, as shown in Ref. 29 the continued fraction $V = (N+1/n)\omega$ with $N = 6$ which appears in $s$-wave Josephson junctions at $A = 0.9$. In contrast, for $p$-wave Josephson junctions, as shown in Fig. 3 there is an additional continued fraction $V = [(N+2)/2][1+1/(n-1/m)]\omega$ with $N = 6$ and $n = 2$. In both cases, additional steps occur when Majorana subgap states are present; thus, although we have not been able to find a precise analytical expression relating the $4\pi$ periodicity of $I_J(\phi)$ and these additional steps, we conjecture that these additional steps are a consequence of the subgap Majorana bound states of the superconductors forming the junctions.

FIG. 2: (Color online) Top panels: CVC of $p$-wave Josephson junction. Fig. 2a shows the CVC in the underdamped region ($\beta = 0.2$) and for $A = 20$ and $\omega = 2$, while Fig. 2b represents the overdamped region ($\beta = 1.2$) with $A = 5$ and $\omega = 1$ (in appropriate dimensionless units; see text). Bottom Panels: Fig. 2c shows the CVC of $s$-wave junction (all parameters are the same as in Fig. 2b). Fig. 2d shows the ratio of the widths of the even and odd Shapiro steps, $\eta = W_{\text{even}}(2\omega)/W_{\text{odd}}(\omega)$ for the $p$-wave and the $s$-wave junctions (inset) as a function of $\beta$ for $A = 10$ and $\omega = 3$. For $p$-wave, $\eta \sim e^{0.313\beta^2}$ while for $s$-wave, $\eta$ does not vary with $\beta$. Figs. 2a, b, and c have $V$ and $I$ scaled in units of $h/(e\tau)$ and $I_c$ respectively. The red solid (blue dashed) curves in panels (a) and (c) correspond to data for increasing(decreasing) current sweeps; these data coincides in the overdamped region as shown in panel (b).

FIG. 3: (Color online) Plots of the self-similar structure for $p$-wave ($A = 0.6$) $s$-wave ($A = 0.8$) Josephson junctions for $D = 0.7$, and $\omega = 0.5$. These additional fractions marked with arrows pointing to the right belong to the additional sequence characteristics of Josephson junctions with Majorana subgap states and obey Farey sum rule. $V$ and $I$ are scaled in units of $h/(e\tau)$ and $I_c$ respectively; see text for details.
and possesses the following qualitative features: it provides a direct manifestation of the transmission resonance condition of the Dirac quasiparticles. Consequently, one expects \( I_J \) and hence \( W \) to oscillate with \( \chi_2 \). This expectation is corroborated in Fig. 5b, where the \( \pi \) periodic oscillation of the step width is plotted as a function of \( \chi_2 \). We note that such an oscillatory behavior is a direct manifestation of the transmission resonance condition of the Dirac quasiparticles; it thus provides a qualitative distinction between Josephson junction hosting Schrodinger and Dirac quasiparticles.

IV. DISCUSSION

In this work we have studied the I-V characteristics of a RCSJ where the superconductors making up the junction either hosts subgap Majorana bound states or have Dirac-like character of the Bogoliubov quasiparticles. The former set of junctions occur for \( p \)-wave superconductors, or Dirac quasiparticles, with strong spin-orbit coupling and transverse magnetic field while graphene superconduction junctions provide an example of the latter class. We find that the I-V characteristics of RCSJs for each of these classes of junctions are qualitatively different from their conventional counterparts. Thus such junctions may serve as phase sensitive detectors.

\[ E_2 = \pm \Delta_0 \sqrt{1 - T(k_y, \chi_2) \sin^2(\phi/2)} \]

where \( T(k_y, \chi_2) = \cos^2(\gamma)/(1 - \cos^2(\gamma) \sin^2(\chi_2)) \) is a measure of transparency of the junction and \( \sin(\gamma) = \hbar v_F k_y/\mu \). Note that Eq. 18 is invariant to the periodicity of the junction.

The corresponding Josephson current at zero temperature is given by \( I_J(\phi) = 2e/\pi \int_{-\pi/2}^{\pi/2} d\gamma \cos(\gamma) \partial E_2/\partial \phi \) and leads to

\[ I_J = I_0 \Delta_0 \int_{-\pi/2}^{\pi/2} d\gamma \cos(\gamma) \sin(\phi) T(\gamma, \chi_2)/|E_2| \]
these numbers one can estimate, \( \omega \) the devil staircase structure can be obtained as follows. Quasiparticles in such superconductors and demonstrates to the transmission resonance of the Dirac-Bogoliubov of the junction. We trace the origin of this phenomenon even with s\(^{\text{tions with and without Majorana subgap states.}}\)

Second, we find that the devil staircase structure of the devil staircase structure is seen at an external radiation frequency of 0.5GHz. The self-similar structure is seen at energy range of 5 \(-\) 6\( \omega \) which is around 2 \(-\) 3GHz. In this context, we note that the required frequency range is large small enough to avoid possible smearing due to 2\( \pi \) periodicity arising due to multimode effect. In addition, this estimate holds for zero-temperature analysis; however it is expected to be qualitatively accurate for \( k_{B}T \ll \Delta_0 \) when quasiparticle poisoning and thermal decoherence rates do not play a significant role. Also, we note that it is possible to model the dissipation and noise in the junction by assuming it to be coupled through a thermal bath using the standard Caldeira-Leggett formalism, the Langevin or saddle point equation corresponding to that analysis at low temperature, where the effects of white noise can be ignored, reduces to Eq. 4 of our work with the resistive term being renormalized by the coefficient of dissipation. This formalism also allows for study of effect of quantum fluctuations and noise in such junctions beyond saddle point approximation which is left as a possible subject of future study.

The experiments to test our theory involves on measurement on RCSJ under an applied radiation with definite amplitude \( A \) and frequency \( \omega \). Such experiments are rather standard for s-wave junctions; more recently, such experiments have been performed for 1D Majorana nanowire setup with resistive junctions. Our specific suggestion involves measurement of \( \eta \) as a function of the effective junction capacitance for JJS with Majorana bound states in a circuit with finite capacitance which can be modeled by a RCSJ; we predict the presence of subharmonic odd Shapiro steps for such junctions whose width depends on the junction capacitance and lead to an exponential dependence of \( \chi \) with the junction capacitance (See Fig. 24). In addition, we suggest the presence of an additional sequence in the devil staircase structure of the Shapiro steps. For junctions with Dirac quasiparticles, which can be made with graphene, we predict that the width of the Shapiro steps will display \( \pi \) periodic oscillatory dependence with the junction barrier potential.

In conclusion, we have studied RCSJ Josephson junction circuits and have shown that they can serve as phase sensitive detectors for both Majorana and Dirac quasiparticles in such junctions. We have charted out the properties of such junctions which are qualitatively distinct from their s-wave counterparts and have suggested experiments which can test our theory.

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