Martingale transformations of Brownian motion with application to functional equations

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\textbf{ABSTRACT}

We describe the classes of functions $f = (f(x), x \in \mathbb{R})$, for which processes $f(W_t) - Ef(W_t)$ and $f(W_t)/Ef(W_t)$ are martingales. We apply these results to give a martingale characterization of general solutions of the quadratic and D’Alembert functional equations. We study also the time-dependent martingale transformations of a Brownian motion.

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\section{Introduction}

It is well known that if for a function $f = (f(x), x \in \mathbb{R})$ the transformed process $(f(W_t), t \geq 0)$ of a Brownian Motion $W$ is a right-continuous martingale, then $f$ is a linear function (see Theorem 5.5 from [4]). It is also known that the time-dependent function $f = (f(t,x), t \geq 0, x \in \mathbb{R})$ is a linear function of $x$ if and only if the transformed process $(f(t,\sigma W_t), t \geq 0)$ is a martingale for any martingale $M$ and to this end to require the martingale property of $f(t,\sigma W_t)$ for two different $\sigma_1 \neq \sigma_2, \sigma_1 \neq 0, \sigma_2 \neq 0$ is sufficient (see Corollary 1 of Theorem 2 from [9]).

In this paper, we give simple generalizations of these results. We describe the classes of functions $f$ for which the processes $f(W_t) - Ef(W_t)$ and $f(W_t)/Ef(W_t)$ (for $f(x) > 0$) are martingales. We prove that the process

$$f(W_t) - Ef(W_t) \quad \text{(resp.} f(W_t)/Ef(W_t)), \quad t \geq 0$$

is a right-continuous martingale if and only if the function $f(x)$ is of the form

$$ax^2 + bx + c \quad \text{(resp.} ae^{\lambda x} + be^{-\lambda x}),$$

for some constants $a$, $b$, $c$ and $\lambda$. Besides, we show that if $f(W_t) - Ef(W_t)$ (resp. $f(W_t)/Ef(W_t)$) is only a martingale (without assuming the regularity of paths), then $f(x)$ is equal to some polynomial of order 2 (resp. to the function $ae^{\lambda x} + be^{-\lambda x}$ for some $a$, $b$, $\lambda$) almost everywhere with respect to the Lebesgue measure.

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Our main motivation to consider such martingale transformations of a Brownian Motion was their relations with functional equations. We show that if the function \( f = (f(x), x \in \mathbb{R}) \) is a measurable solution of the quadratic functional equation
\[
f(x + y) + f(x - y) = 2f(x) + 2f(y) \quad \text{for all } x, y \in \mathbb{R},
\]
then the difference \( f(W_t) - Ef(W_t) \) is a martingale and if \( f \) is a strictly positive solution of the D’Alembert functional equation
\[
f(x + y) + f(x - y) = 2f(x)f(y) \quad \text{for all } x, y \in \mathbb{R},
\]
then a martingale will be the process \( f(W_t)/Ef(W_t) \). The above-mentioned descriptions of martingale functions enable us to give equivalent characterization of the general measurable solution of Equations (1) and (2) in martingale terms.

We consider also time-dependent functions \( (f(t, x), t \geq 0, x \in \mathbb{R}) \) for which the transformed processes
\[
f(t, \sigma W_t) - Ef(t, \sigma W_t) \quad \text{and} \quad f(t, \sigma W_t)/Ef(t, \sigma W_t)
\]
are martingales, where \( \sigma \) is a constant. To obtain simple structural properties for such functions, as for the case of functions \( f = (f(x), x \in \mathbb{R}) \), one needs some types of growth conditions on the function \( f \), or one should require the martingale property for transformed processes (3) at least for two different \( \sigma \neq 0 \). Corresponding assertions (Theorems 4.1–4.3) are given in Section 4.

2. Martingale transformations of a Brownian motion

Let \( W = (W_t, t \geq 0) \) be a standard Brownian Motion defined on a complete probability space \( (\Omega, \mathcal{F}, P) \) with filtration \( \mathcal{F} = (\mathcal{F}_t, t \geq 0) \) generated by the Brownian Motion \( W \).

**Theorem 2.1:** Let \( f = (f(x), x \in \mathbb{R}) \) be a measurable strictly positive function, such that \( f(W_t) \) is integrable for every \( t \geq 0 \). Then

(a) the process
\[
N_t = \frac{f(W_t)}{Ef(W_t)}, \quad t \geq 0,
\]
is a right-continuous (P-a.s.) martingale if and only if the function \( f \) is of the form
\[
f(x) = a e^{\lambda x} + b e^{-\lambda x},
\]
for some constants \( a \geq 0, b \geq 0, ab \neq 0 \) and \( \lambda \in \mathbb{R} \).
(b) the process \( N_t \) is a martingale if and only if the function \( f(x) \) coincides with the function \( a e^{\lambda x} + b e^{-\lambda x} \) (for some constants \( a \geq 0, b \geq 0, ab \neq 0 \) and \( \lambda \in \mathbb{R} \)) almost everywhere with respect to the Lebesgue measure.
Proof: 1. Let the process $N_t$ be a right-continuous ($P$-a.s.) martingale and let

$$ g(t) \equiv Ef(W_t) = \int_{\mathbb{R}} f(y) \frac{1}{\sqrt{2\pi t}} e^{-\frac{y^2}{2t}} \, dy. $$

Since $E|f(W_t)| < \infty$ for all $t \geq 0$, the function $g(t)$ will be continuous for any $t > 0$. Since $N_t$ is right-continuous ($P$-a.s.) and $g(t)$ is continuous, the process $f(W_t)$ will be also right-continuous $P$-a.s. This implies that the function $f(x)$ is continuous (see Theorem 5.5 from [4] or Lemma A1 from [11]).

Let

$$ F(t, x) = \frac{f(x)}{g(t)}, \quad t \geq 0, \quad x \in \mathbb{R}. $$

Since $F(t, W_t)$ is a martingale, we have that

$$ F(t, W_t) = \frac{f(W_t)}{g(t)} = \frac{1}{g(T)} E(f(W_T)|\mathcal{F}_t) \quad (5) $$

$P$-a.s. for all $t \leq T$. Let

$$ u(t, x) = E(f(W_T)|W_t = x). $$

Since $f$ is positive, $u(t, x)$ will be of the class $C^{1,2}$ on $(0, T) \times \mathbb{R}$ and satisfies the "backward" heat equation (see, e.g. [7, p. 257])

$$ \frac{\partial u}{\partial t} + \frac{1}{2} \frac{\partial^2 u}{\partial x^2} = 0, \quad 0 < t < T, \quad x \in \mathbb{R}. \quad (6) $$

By the Markov property of $W$

$$ u(t, W_t) = E(f(W_T)|\mathcal{F}_t) $$

and from (5) we have that

$$ f(W_t) = \frac{g(t)}{g(T)} u(t, W_t) \quad \text{a.s.} $$

Therefore,

$$ \int_{\mathbb{R}} \left| f(x) - \frac{g(t)}{g(T)} u(t, x) \right| \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} \, dx = 0 $$

which implies that for any $0 < t \leq T$

$$ f(x) = \frac{g(t)}{g(T)} u(t, x) \quad (7) $$

almost everywhere with respect to the Lebesgue measure. Since $f$ and $u$ are continuous, we obtain that for any $0 < t < T$ equality (7) is satisfied for all $x \in \mathbb{R}$.

Since $f(x)$ is strictly positive, we have that $g(t) > 0$ and $u(t, x) > 0$ for all $t \geq 0$ and $x \in \mathbb{R}$. This and equality (7) imply that $g(t)$ is differentiable, $f(x)$ is two-times differentiable and

$$ u(t, x) = \frac{g(T)}{g(t)} f(x) \quad \text{for all } x \in \mathbb{R} \quad (8) $$

for any $0 < t < T$. 
Therefore, it follows from (6) and (8) that
\[
\frac{1}{2} g(T) f''(x) - \frac{g(T)g'(t)}{g^2(t)} f(x) = 0,
\]
which implies that
\[
\frac{f''(x)}{f(x)} = \frac{2g'(t)}{g(t)}.
\tag{9}
\]
Since the left-hand side of (9) does not depend on \( t \) and the right-hand side on \( x \), both parts of (9) are equal to a constant, which we denote by \( c \). If \( c < 0 \), then the general solution of equation \( f''(x) = cf(x) \) leads to \( f(x) \) which, with necessity, changes its sign, hence \( c = \lambda^2 \) for some \( \lambda \in \mathbb{R} \). Therefore, we obtain that
\[
f''(x) = \lambda^2 f(x) \quad \text{and} \quad g'(t) = \frac{\lambda^2}{2} g(t).
\]
for some constant \( \lambda \in \mathbb{R} \). Thus
\[
f(x) = a \, e^{\lambda x} + b \, e^{-\lambda x}, \quad g(t) = E f(W_t) = (a + b) \, e^{\frac{\lambda^2}{2} t}.
\]
Since the function \( f \) should be strictly positive, we shall have that \( a \geq 0, b \geq 0, ab \neq 0 \).

Now let us assume that the function \( f \) is of the form (4). Then \( g(t) = E(a e^{\lambda W_t} + b e^{-\lambda W_t}) = (a + b) \, e^{\frac{\lambda^2}{2} t} \) and the process
\[
\frac{f(W_t)}{Ef(W_t)} = \frac{a}{a + b} \, e^{\lambda W_t - \frac{\lambda^2}{2} t} + \frac{b}{a + b} \, e^{-\lambda W_t - \frac{\lambda^2}{2} t}
\]
is a martingale, as a linear combination of two exponential martingales.

2. Let \( N_t \) be a martingale and let
\[
\tilde{f}(x) = \frac{g(t_0)}{g(T)} u(t_0, x),
\]
for some \( t_0 > 0 \). It follows from (7) that
\[
\lambda(x : f(x) \neq \tilde{f}(x)) = 0,
\tag{10}
\]
where \( \lambda \) is the Lebesgue measure and by definition of \( u(t, x) \) the function \( \tilde{f}(x) \) is continuous (moreover, it is two times differentiable). It follows from (10) that \( P(f(W_t) = \tilde{f}(W_t)) = 1 \) for any \( t \geq 0 \) and since \( Ef(W_t) = E\tilde{f}(W_t) \), we obtain that for any \( t \geq 0 \)
\[
P \left( \frac{f(W_t)}{Ef(W_t)} = \frac{\tilde{f}(W_t)}{E\tilde{f}(W_t)} \right) = 1.
\]
This implies that the process \( \tilde{f}(W_t)/E\tilde{f}(W_t) \) is a continuous martingale and it follows from part (a) of this theorem that \( \tilde{f}(x) \) is of the form (4). Therefore, \( f(x) \) coincides with the function \( a \, e^{\lambda x} + b \, e^{-\lambda x} \) almost everywhere with respect to the Lebesgue measure.

The converse is proved similarly to the part 1 of this theorem. \( \blacksquare \)
**Theorem 2.2:** Let \( f = (f(x), x \in \mathbb{R}) \) be a measurable function, such that \( f(W_t) \) is integrable for every \( t \geq 0 \). Then

(a) the process

\[
M_t = f(W_t) - Ef(W_t), \quad t \geq 0,
\]

is a right-continuous \((P\text{-a.s.) martingale if and only if the function } f \text{ is of the form}

\[
f(x) = ax^2 + bx + c \quad \text{for some constants } a, b \quad \text{and} \quad c \in \mathbb{R}, \quad (11)
\]

(b) the process \( M_t \) is a martingale if and only if \( f(x) \) coincides with the function \( ax^2 + bx + c \) (for some constants \( a, b, c \in \mathbb{R} \)) almost everywhere with respect to the Lebesgue measure.

**Proof:** 1. Let the process \( M_t \) be a right-continuous \((P\text{-a.s.) martingale and let } g(t) \equiv Ef(W_t) = \int \frac{1}{\sqrt{2\pi t}} e^{-\frac{y^2}{2t}} dy. \) Using the same arguments as in the proof of Theorem 1, the process \( f(W_t) \) will be also right-continuous \(P\text{-a.s.}, which implies that the function } f(x) \text{ is continuous.}

Let

\[
F(t, x) = f(x) - g(t), \quad t \geq 0, \quad x \in \mathbb{R}.
\]

Since \( F(t, W_t) \) is a martingale, we have that

\[
F(t, W_t) = f(W_t) - g(t) = E(f(W_T)|\mathcal{F}_t) - g(T) \quad (12)
\]

\(P \text{-a.s. for all } t \leq T. \) Let

\[
u(t, x) = E(f(W_T)|W_t = x).
\]

Since

\[
E|f(W_t)| = \int |f(y)| \frac{1}{\sqrt{2\pi t}} e^{-\frac{y^2}{2t}} dy < \infty \quad \text{for all } t \geq 0, \quad (13)
\]

\(u(t, x)\) will be of the class \(C^{1.2} \) on \((0, T) \times \mathbb{R}\) and satisfies the "backward" heat equation (see, e.g. [7])

\[
\frac{\partial u}{\partial t} + \frac{1}{2} \frac{\partial^2 u}{\partial x^2} = 0, \quad 0 < t < T, \quad x \in \mathbb{R}, \quad (14)
\]

on the strip \((0, T) \times \mathbb{R}\).

Note that, using the same arguments as in the proof of Theorem 3.6 from Karatzas and Shreve [7], the positivity assumption on the function \( f \) by integrability condition (13) can be replaced, to guarantee that (14) is satisfied.

Similarly to the proof of Theorem 2.1 one can show that for any \( 0 < t \leq T \)

\[
f(x) = u(t, x) - g(T) + g(t) \quad (15)
\]

almost everywhere with respect to the Lebesgue measure. By continuity of \( f \) and \( u \), we obtain that for any \( 0 < t \leq T \) the equality (15) is satisfied for all \( x \in \mathbb{R} \).
This implies that \( g(t) \) is differentiable, \( f(x) \) is two-times differentiable and it follows from (14) and (15) that
\[
\frac{1}{2} f''(x) = g'(t). \tag{16}
\]
Since the left-hand side of (16) does not depend on \( t \) and the right-hand side on \( x \), both parts of (16) are equal to a constant. Therefore, we obtain that
\[
f''(x) = 2a \quad \text{and} \quad g'(t) = a \quad \text{for some } a \in \mathbb{R}. \tag{17}
\]
The general solutions of these equations are
\[
f(x) = ax^2 + bx + c \quad \text{and} \quad g(t) = at + c \tag{18}
\]
respectively, for some \( a, b, c \in \mathbb{R} \).

Conversely, if the function \( f \) is of the form (11), then
\[
f(W_t) = aW_t^2 + bW_t + c, \quad Ef(W_t) = at + c \tag{20}
\]
and the process \( f(W_t) - Ef(W_t) = a(W_t^2 - t) \) is a martingale.

2. Let
\[
\tilde{f}(x) = u(t_0, x) + g(t_0) - g(T).
\]
for some \( t_0 > 0 \). It follows from (15) that
\[
\lambda(x : f(x) \neq \tilde{f}(x)) = 0, \tag{19}
\]
where \( \lambda \) is the Lebesgue measure and by definition of \( u(t, x) \) the function \( \tilde{f}(x) \) is continuous (moreover, it is two times differentiable). It follows from (19) that \( P(f(W_t) = \tilde{f}(W_t)) = 1 \) for any \( t \geq 0 \) and since \( Ef(W_t) = E\tilde{f}(W_t) \), we obtain that the processes \( M_t = f(W_t) - Ef(W_t) \) and \( \tilde{M}_t = \tilde{f}(W_t) - E\tilde{f}(W_t) \) are equivalent, which implies that the process \( \tilde{M}_t \) is a continuous martingale. Therefore, it follows from part (a) of this theorem that \( \tilde{f}(x) \) is of the form (11) and hence, \( f(x) \) coincides with the function (11) almost everywhere with respect to the Lebesgue measure.

The converse is proved similarly to the part (a) of this theorem. \( \blacksquare \)

**Corollary 2.1:** Let \( f = (f(x), x \in \mathbb{R}) \) be a function of one variable.

(a) If the process \( (f(W_t), \mathcal{F}_t, t \geq 0) \) is a right-continuous martingale, then
\[
f(x) = bx + c \quad \text{for all } x \in \mathbb{R} \tag{20}
\]
for some constants \( b, c \).

(b) If the process \( (f(W_t), \mathcal{F}_t, t \geq 0) \) is a martingale, then \( f(x) = bx + c \) almost everywhere with respect to the Lebesgue measure for some constants \( b, c \in \mathbb{R} \).

**Proof:** If the process \( \tilde{f}(W_t) \) is a martingale, then \( g(t) = Ef(W_t) \) is constant and the coefficient \( a \) in (18) is equal to zero. Therefore, this corollary follows from Theorem 2.2. \( \blacksquare \)

Based on the earlier version [10] of this section, the authors of [8] have generalized Theorem 2.1 and 2.2 to Levy processes; the methods are completely independent and different from ours.
3. Application to functional equations

It was proved in [12] (see also [11] for multidimensional case) that if the function \( f = (f(x), x \in R) \) is a measurable solution of the Cauchy additive functional equation

\[
f(x + y) = f(x) + f(y), \quad \text{for all } x, y \in R,
\]

then the transformed process \( f(W_t), t \geq 0 \) is a right-continuous martingale, which (by Corollary 2.1 of Theorem 2.2) implies that \( f \) is a linear function. Here we propose a similar characterization of solutions of the quadratic functional equation

\[
f(x + y) + f(x - y) = 2f(x) + 2f(y) \quad \text{for all } x, y \in R,
\]

where for a measurable solution of this equation the process \( f(W_t) \) is no longer a martingale, but a martingale will be the difference \( f(W_t) - Ef(W_t) \).

It is well known (see, e.g. [2,14]) that the general measurable solution of Equation (21) is the function \( f(x) = ax^2, a \in R \). Moreover, in [2] Equation (21) has been solved without any assumptions using the Hamel basis. We consider only the Lebesgue measurable solutions of (21) and using Theorem 2.2 we characterize the general measurable solution of Equation (21) in terms of martingales, which gives also a probabilistic proof of this assertion.

Let first prove the following two lemmas.

**Lemma 3.1:** Let \( f = (f(x), x \in R) \) be a measurable function and let \( f(W_t) \) be integrable. Then for \( s \leq t \)

\[
E(f(2W_s - W_t)|\mathcal{F}_s) = E(f(W_t)|\mathcal{F}_s) \quad P \text{ - a.s.} \tag{22}
\]

In particular,

\[
Ef(W_t - 2W_s) = Ef(2W_s - W_t) = Ef(W_t) \quad \text{for } s \leq t. \tag{23}
\]

**Proof:** First note that \( 2W_s - W_t \sim W_t \), where as usual, the sign \( \sim \) means ‘has the same law’. Indeed, since \( W_t - W_s \) is symmetrically distributed and independent of \( W_s \)

\[
2W_s - W_t = W_s - (W_t - W_s) \sim W_s - (W_s - W_t) = W_t.
\]

Thus, \( f(2W_s - W_t) \) is also integrable. Since \( \mathcal{F}_t \) is generated by the Brownian Motion \( W \), \( W_t - W_s \) is independent of \( \mathcal{F}_s \). Therefore, using conditional calculus and the Markov property we obtain the validity of equality (22)

\[
E(f(2W_s - W_t)|\mathcal{F}_s) = E(f(W_s + (W_s - W_t))|W_s) = E(f(W_t)|W_s) = E(f(W_t)|\mathcal{F}_s) \quad P \text{ - a.s.} \]

This proof of Lemma 3.1, which is simpler and better than was ours, was suggested by the Referee.

It is easy to show that if \( f \) is a solution of (21), then (see, e.g. [14])

\[
f(rx) = r^2f(x) \quad \text{and} \quad |f(rx)| = r^2|f(x)| \tag{24}
\]

for each rational \( r \).
In particular, (24) implies that
\[ f(r) = ar^2, \quad \text{with } a = f(1) \]
for any rational \( r \).

It is evident that if \( f \) is continuous, then (24) is satisfied for all real \( r \), but we don’t assume continuity of the solution beforehand.

**Lemma 3.2:** If \( (f(x), x \in \mathbb{R}) \) is a measurable solution of (21), then the random variable \( f(\eta) \) is integrable for any random variable \( \eta \) with normal distribution. In particular, for a Brownian Motion \( W \)
\[ E|f(x + W_t)| < \infty, \quad \text{for each } t \geq 0 \quad \text{and} \quad x \in \mathbb{R}. \]  

**Proof:** Let \( r_n \) be sequence of rational numbers with \( r_n \downarrow t, \ t \neq 0 \) and let \( \xi \) be a random variable with standard normal distribution \( (\xi \in N(0, 1)) \). By the dominated convergence theorem, we have
\[
E\varphi(\xi) e^{-r_n^2|f(\xi)|} \to E\varphi(\xi) e^{-t^2|f(\xi)|} \quad \text{and} \\
E\varphi(\xi) e^{-|f(r_n \xi)|} = \int_{\mathbb{R}} \varphi \left( \frac{1}{r_n} x \right) e^{-|f(x)|} \frac{e^{-x^2/2r_n}}{\sqrt{2\pi r_n}} \, dx
\]
\[
\to \int_{\mathbb{R}} \varphi \left( \frac{1}{t} x \right) e^{-|f(x)|} \frac{e^{-x^2/2t}}{\sqrt{2\pi t}} \, dx = E\varphi(\xi) e^{-|f(t \xi)|},
\]
for each bounded continuous function \( \varphi \). Therefore, by (24) these limits should coincide

\[ E\varphi(\xi) ( e^{-t^2|f(\xi)|} - e^{-|f(t \xi)|} ) = 0 \]
and by arbitrariness of \( \varphi \) we get

\[ |f(t \xi)| = t^2|f(\xi)|, \quad P - \text{a.s. for each } t \in \mathbb{R}. \]  

(26)

Let \( \eta \) be a gaussian random variable with mean \( \mu \) and variance \( \sigma^2 \), independent from \( \xi \). Then, it follows from (26) that
\[
E \left( e^{-\eta^2|f(\xi)|} - e^{-|f(\eta \xi)|} \right)^2 = \int_\mathbb{R} \frac{e^{-\gamma^2|f(\xi)|}}{2\pi \sigma} - E \left( e^{-\gamma^2|f(\xi)|} - e^{-|f(\gamma \xi)|} \right)^2 \, dy = 0.
\]

Thus \( \eta^2|f(\xi)| = |f(\eta \xi)| \) and \( (\mu^2 + \sigma^2)|f(\xi)| = E(|f(\eta \xi)|/\xi) \) P-a.s.. Finally we get
\[ E|f(x \eta)| = (\mu^2 + \sigma^2)|f(x)| < \infty, \quad -\text{a.e.} \]  

(27)

with respect to the Lebesgue measure. This also implies that \( E|f(\eta)| < \infty \).

Indeed, it follows from (27) that there exists \( \gamma > 1 \) such that
\[ E|f(\gamma \eta)| < \infty. \]  

(28)

Therefore, after changing densities and taking the maximum in the exponent we obtain from (28) that
\[ E|f(\eta)| = E|f(\gamma \eta)| \gamma e^{-\frac{(\gamma \eta - \mu)^2}{2\sigma^2} + \frac{\mu^2}{2\sigma^2}} \]
\[
\leq \gamma e^{\frac{\mu^2 \gamma - 1}{2\sigma^2}} E|f(\gamma \eta)| < \infty.
\]

**Remark 3.1:** Lemma 3.2 implies that any measurable solution of (21) is locally integrable. Similar assertion for Cauchy’s additive functional equation was proved in [15] using the Bernstein theorem on the characterization of the normal distributions.

**Theorem 3.1:** The following assertions are equivalent:

(i) the function \( f = (f(x), x \in \mathbb{R}) \) is a measurable solution of (21),
(ii) \( f = (f(x), x \in \mathbb{R}) \) is a measurable even function with \( f(0) = 0 \) and such that \( f(W_t) \) is integrable for every \( t \) and the process

\[
N_t = f(W_t) - Ef(W_t), \quad t \geq 0,
\]

is a right-continuous martingale,
(iii) the function \( f \) is of the form

\[
f(x) = ax^2,
\]

for some constant \( a \in \mathbb{R} \).

**Proof:** 1 → 2. It is evident that if \( f \) is a solution of (21) then \( f(0) = 0 \) and \( f(x) = f(-x) \) for all \( x \in \mathbb{R} \). Therefore,

\[
f(W_t - 2W_s - x) = f(2W_s - W_t + x)
\]

and substituting \( x = W_t - W_s \) and \( y = x + W_s \) in Equation (21) we have that

\[
f(x + W_t) + f(2W_s - W_t + x) = 2f(W_t - W_s) + 2f(x + W_s).
\]

By Lemma 3.1 the random variables \( W_t \) and \( 2W_s - W_t \) have the same normal distributions and by Lemma 3.2 the random variables \( f(x + W_t) \) and \( f(x + 2W_s - W_t) \) are integrable. So, we may take expectations in (30) to obtain

\[
Ef(x + W_t) = Ef(W_t - W_s) + Ef(x + W_s) \quad s \leq t.
\]

If we take \( s = 0 \) in (31), we get

\[
f(x) = Ef(x + W_t) - Ef(W_t) = \int_{\mathbb{R}} f(x + y) \frac{1}{\sqrt{2\pi t}} e^{-\frac{y^2}{2t}} dy - Ef(W_t)
\]

\[
= \int_{\mathbb{R}} f(y) \frac{1}{\sqrt{2\pi t}} e^{-\frac{(y-x)^2}{2t}} dy - Ef(W_t),
\]

which implies that \( f \) is continuous by (25).

Taking now conditional expectations in (30) for \( x = 0 \), using the independent increment property of \( W \) and then equality (31) we have that \( P \)-a.s.

\[
Ef(W_t)|\mathcal{F}_s) + Ef(W_t - 2W_s)|\mathcal{F}_s)
\]

\[
= 2Ef(W_t - W_s)|\mathcal{F}_s) + 2f(W_s)
\]
\[ = 2Ef(W_t - W_s) + 2f(W_s) = 2Ef(W_t) + 2f(W_s) - 2Ef(W_s). \]  

(33)

On the other hand, it follows from Lemma 3.1 that \( P \)-a.s.
\[ E(f(W_t - 2W_s) | \mathcal{F}_s) = E(f(W_t) | \mathcal{F}_s). \]  

(34)

Therefore the martingale equality
\[ E(f(W_t) - Ef(W_t)) | \mathcal{F}_s) = f(W_s) - Ef(W_s), \quad P \text{-a.s.,} \]
follows from Equations (33) and (34). Thus the process \( N = (f(W_t) - Ef(W_t), t \geq 0) \) is a martingale with \( P \)-a.s. continuous paths.

2 \( \rightarrow \) 3. Since \( f(W_t) - Ef(W_t) \) is a martingale, Theorem 2.2 implies that the function \( f \) should be of the form
\[ f(x) = ax^2 + bx + c, \quad a, b, c \in \mathbb{R}. \]

Since \( f \) is even we have that \( b = 0 \) and \( c = 0 \) since \( f(0) = 0 \). Thus, \( f(x) = ax^2 \) for some \( a \in \mathbb{R} \).

The proof of implication 3 \( \rightarrow \) 1 is evident.

Now we give an application of Theorem 2.1.

Let consider the D’Alembert functional equation
\[ f(x + y) + f(x - y) = 2f(x)f(y) \quad \text{for all } x, y \in \mathbb{R}, \]  

(35)

This equation possesses the following measurable solutions and only these: \( f(x) = 0, f(x) = \cosh \lambda x, f(x) = \cos \lambda x \), where \( \lambda \) is some constant. The last two also contain (for \( \lambda = 0 \)) the constant solution \( f(x) = 1 \) (see, e.g. [1,14]).

In the following theorem, we give a martingale characterization of measurable strictly positive solutions of Equation (35).

**Theorem 3.2:** The following assertions are equivalent:

(i) the function \( f = (f(x), x \in \mathbb{R}) \) is a measurable strictly positive solution of (35),

(ii) \( f = (f(x), x \in \mathbb{R}) \) is a strictly positive even function with \( f(0) = 1 \), such that \( f(W_t) \) is integrable for every \( t \) and the process
\[ N_t = \frac{f(W_t)}{Ef(W_t)}, \quad t \geq 0, \]

is a right-continuous martingale,

(iii) the function \( f \) is of the form
\[ f(x) = \cosh(\lambda x) = \frac{1}{2} (e^{\lambda x} + e^{-\lambda x}), \]  

(36)

for some constant \( \lambda \in \mathbb{R} \).

**Proof:** 1 \( \rightarrow \) 2. With \( y = 0 \), it follows from (35) that \( f(x) = f(x)f(0) \), which implies \( f(0) = 1 \), since we consider only solutions with \( f(x) > 0 \). It is also evident that \( f \) is an even function, since taking \( x = 0 \) from (35) we have \( f(y) + f(-y) = 2f(0)f(y) \), hence \( f(y) = f(-y) \).
Substituting $x = W_t - W_s$ and $y = x + W_s$ in Equation (35) we have that
\[ f(x + W_t) + f(W_t - 2W_s - x) = 2f(W_t - W_s)f(x + W_s). \] (37)
Since $f(x)$ is positive, expectations below have a sense and using the independent increment property of $W$ and Lemma 3.1, we obtain from (37) that
\[ Ef(x + W_t) = Ef(W_t - W_s)Ef(x + W_s) \quad s \leq t. \] (38)

Let $g(t) = Ef(W_t)$. Then $Ef(W_t - W_s) = g(t - s)$ and it follows from (38) that $g$ satisfies the Cauchy exponential functional equation
\[ g(t) = g(t - s)g(s), \quad s \leq t \]
on $\mathbb{R}^+$. It is well known (see, e.g. [1]) that any bounded from bellow solution of this equation is of the form
\[ g(t) = e^{ct} \quad \text{for some constant } c \in \mathbb{R}. \] (39)
Therefore, $f(W_t)$ is integrable for any $t \geq 0$ and $Ef(W_t) = e^{ct}$.

If we take $s = 0$ in (38), we obtain
\[ f(x) = \frac{Ef(x + W_t)}{Ef(W_t)} = \frac{1}{Ef(W_t)} \int_{\mathbb{R}} f(x + y) \frac{1}{\sqrt{2\pi t}} e^{-\frac{y^2}{2t}} \ dy \]
\[ = \frac{1}{Ef(W_t)} \int_{\mathbb{R}} f(y) \frac{1}{\sqrt{2\pi t}} e^{-\frac{(y-x)^2}{2t}} \ dy, \] (40)
which implies that $f$ is continuous, since $f(W_t)$ is integrable.

Taking now conditional expectations in (37) for $x = 0$, using the independent increment property of $W$ and then equality (38) we have that $P$-a.s.
\[ Ef(f(W_t)|\mathcal{F}_s) + Ef(f(W_t - 2W_s)|\mathcal{F}_s) = 2Ef(f(W_t - W_s)|\mathcal{F}_s)f(W_s) \]
\[ = 2f(W_s)Ef(W_t - W_s) = 2f(W_s) \frac{Ef(W_t)}{f(W_s)}. \] (41)

On the other hand, it follows from Lemma 3.1 that $P$-a.s.
\[ Ef(f(W_t - 2W_s)|\mathcal{F}_s) = Ef(2W_s - W_t)|\mathcal{F}_s) = Ef(f(W_t)|\mathcal{F}_s). \] (42)
Therefore the martingale equality
\[ Ef \left( \frac{f(W_t)}{Ef(W_t)} | \mathcal{F}_s \right) = f(W_s) \frac{Ef(W_t)}{Ef(W_s)}, \quad P \text{- a.s.,} \]
follows from Equations (41) and (42). Thus the process $N = (\frac{f(W_t)}{Ef(W_t)}, t \geq 0)$ is a martingale with $P$-a.s. continuous paths.

2 $\rightarrow$ 3 It follows from Theorem 2.1 that $f(x)$ is of the form
\[ f(x) = ae^{\lambda x} + be^{-\lambda x} \]
for some constants $a, b, \lambda \in \mathbb{R}$. Since the function $f$ is even we have that $a = b$ and $a + b = 1$ by equality $f(0) = 1$. Therefore, $a = b = 1/2$, which implies representation (4).

The proof of implication $3 \rightarrow 1$ is evident. \qed
4. On time-dependent martingale transformations of a Brownian motion

In this section, we consider time-dependent functions \( f(t, x), t \geq 0, x \in \mathbb{R} \) for which the transformed processes

\[
f(t, \sigma W_t) - Ef(t, \sigma W_t) \quad \text{and} \quad f(t, \sigma W_t)/Ef(t, \sigma W_t)
\]

are martingales, where \( \sigma \) is a constant. To obtain simple structural properties for such functions, as for the case of functions \( f = f(x), x \in \mathbb{R} \) in Section 2, we need some type of growth conditions on the function \( f \), or one should require the martingale property for transformed processes (43) for at least two different \( \sigma \neq 0 \).

Recall that a heat polynomial is any polynomial solution of the heat equation \( u_t + \frac{1}{2} u_{xx} = 0 \).

Denote by \( \langle M \rangle \) the square characteristic of the continuous martingale \( M \) and let \( g(t) \equiv Ef(t, W_t) \).

**Theorem 4.1:** Let \( f : (0, \infty) \times \mathbb{R} \to \mathbb{R} \) be a continuous function, such that \( f(t, W_t) \) is integrable for every \( t \geq 0 \) and \( n \geq 1 \) is an integer. Then \( f(t, W_t) - Ef(t, W_t) \) is a martingale satisfying condition:

(C) for some \( C > 0 \) the process

\[
\langle f(\cdot, W) - g \rangle_t - C \int_0^t (1 + s + W_s^2)^{n-1} ds \quad \text{is non-increasing,}
\]

if and only if the function \( f(t, x) \) is of the form

\[
f(t, x) = P(t, x) + c(t)
\]

for some heat polynomial \( P \) of degree \( n \) and a deterministic continuous function \( c(t), t \geq 0 \).

**Proof:** Let us prove the sufficiency, the necessity part of this theorem is evident. The martingale property of the process \( f(t, W_t) - g(t) \), the Markov property of \( W \) and the continuity of the function \( f \) imply that

\[
f(t, x) - g(t) = \int_R (f(T, y) - g(T)) \frac{1}{\sqrt{2\pi (T-t)}} e^{-\frac{(y-x)^2}{2(T-t)}} dy
\]

for all \( t \in [0, T] \) and \( x \in \mathbb{R} \).

It is evident that \( f - g \) is a weak solution of the heat equation. Indeed, for every infinitely differentiable finite (on \( (0, \infty) \times \mathbb{R} \)) function \( \varphi \), with compact support \( supp \varphi \subset (0, T) \times \mathbb{R} \) for some \( T > 0 \) we have from (45) that

\[
\int_0^\infty \int_R (f(s, x) - g(s)) \left( \frac{\partial \varphi}{\partial s} (s, x) - \frac{1}{2} \varphi_{xx}(s, x) \right) dx ds
\]

\[
= \int_0^T \int_R (f(T, y) - g(T)) \frac{1}{\sqrt{2\pi (T-t)}} \left( \frac{\partial \varphi}{\partial s} (s, x) - \frac{1}{2} \varphi_{xx}(s, x) \right) dy dx ds
\]
\[ \int_0^T \int_R (f(T, y) - g(T)) \varphi(s, x) \left( \frac{\partial}{\partial s} + \frac{1}{2} \frac{\partial^2}{\partial x^2} \right) \frac{e^{-\frac{(x-y)^2}{2(T-s)}}}{\sqrt{2\pi(T-s)}} \, dx \, dy \, ds = 0. \]

By hypoellipticity property of the heat equation, \( f(t, x) - g(t) \) coincides with an infinitely differentiable function a.e. (see [16, § 14.6]). Since \( f(t, x) \) is continuous, the function \( f(t, x) - g(t) \) will be infinitely differentiable itself and will satisfy the heat equation

\[ \left( \frac{\partial}{\partial t} + \frac{1}{2} \frac{\partial^2}{\partial x^2} \right) (f(t, x) - g(t)) = 0. \]  
(46)

Therefore, by the Itô formula we get

\[ f(t, W_t) - g(t) = f(0, 0) - g(0) + \int_0^t f_x(s, W_s) \, dW_s, \]

which implies that \( \langle f(\cdot, W) - g \rangle_t = \int_0^t (f_x(s, W_s))^2 \, ds \). The condition of this theorem gives \( |f_x(s, W_s)|^2 \leq C(1 + s + W_s^2)^{n-1} \), which is equivalent to the inequality \( |f_x(t, x)| \leq C(\sqrt{1 + t + x^2})^{n-1} \). Since by (46)

\[ \left( \frac{\partial}{\partial t} + \frac{1}{2} \frac{\partial^2}{\partial x^2} \right) f_x(t, x) = \frac{\partial}{\partial x} \left( \frac{\partial}{\partial t} + \frac{1}{2} \frac{\partial^2}{\partial x^2} \right) (f(t, x) - g(t)) = 0, \]

\( f_x \) is the classical solution of the heat equation and by the Liouville theorem for heat equations (see, e.g. [5]) it follows that \( f_x(t, x) \) coincides a.e. with a heat polynomial of degree \( n-1 \). Therefore \( f(t, x) \) is of the form (44).

**Corollary 4.1:** Let \( f : (0, \infty) \times R \rightarrow R \) be a continuous function. Then \( f(t, W_t) - Ef(t, W_t) \) is a martingale satisfying condition:

for some \( C > 0 \) the process

\[ \langle f(\cdot, W) - g \rangle_t - C \int_0^t (1 + s + W_s^2) \, ds \]

is non-increasing, if and only if the function \( f(t, x) \) is of the form

\[ f(t, x) = ax^2 + bx + c(t) \]
(47)

for some constants \( a, b \in R \) and deterministic function \( c(t) = f(t, 0), t \geq 0 \).

**Remark 4.1:** If we only assume that the function \( f(t, x) \) is measurable (without assumption continuity of \( f \)), then Equations (44) and (47) will be satisfied almost everywhere with respect to the Lebesgue measure \( dt \times dx \).

**Remark 4.2:** The martingale functions are not even polynomials without some type growth condition on \( \langle f(\cdot, W) \rangle \). Shows this the process \( \exp(W_t - t/2) \), which is martingale and \( g(t) = 1 \). Moreover, it was proved in [13] that any strictly positive martingale function \( f(t, x) \) is expressed as

\[ f(t, x) = \int_{-\infty}^\infty \exp \left( yx - \frac{1}{2} y^2 t \right) \, dV(y), \quad 0 < t < \infty, \quad x \in R, \]

for some non-decreasing function \( V : R \rightarrow R \).
Remark 4.3: Any heat polynomial $P(t, x)$ of degree $n$ admits the decomposition

$$P(t, x) = \sum_{k=0}^{n} C_k H_k(t, x)$$

with respect to Hermite polynomials $H_k(t, x) = \frac{\partial^k}{\partial x^k} e^{x^2-t\alpha^2/2}|_{\alpha=0}$, which can be also expressed as multiple Itô integrals

$$\int_0^t \cdots \int_0^{t_{k-1}} dW_{t_k} \cdots dW_{t_1} = \frac{H_k(t, W_t)}{k!}$$

(see [7] or [6]).

Representation (48) of heat polynomials is well known (see, e.g. [3]). We give here a simple proof:

Since $\frac{\partial^k}{\partial x^k} P(t, x)$ is a martingale function for any $k$ and it coincides with 0 for $k > n$, using the Itô formula several times and equality (49) we obtain the desired representation

$$P(t, W_t) = P(0, 0) + \int_0^t \frac{\partial P(t_1, W_{t_1})}{\partial x} dW_{t_1}$$

$$= P(0, 0) + \frac{\partial P(0, 0)}{\partial x} W_t + \int_0^t \int_0^{t_1} \frac{\partial^2 P(t_2, W_{t_2})}{\partial x^2} dW_{t_2} dW_{t_1} = \cdots$$

$$= \sum_{k=0}^{n} \frac{\partial^k P(0, 0)}{\partial x^k} \int_0^t \cdots \int_0^{t_{k-1}} dW_{t_k} \cdots dW_{t_1} = \sum_{k=0}^{n} \frac{\partial^k P(0, 0)}{\partial x^k} \frac{H_k(t, W_t)}{k!}.$$  

Hence instead of (44) we can write that there exist constants $C_k, k = 0, \ldots, n$ and a continuous deterministic function $c(t)$ such that

$$f(t, x) = \sum_{k=0}^{n} C_k H_k(t, x) + c(t).$$

Now we give another description of the time-dependent martingale functions.

Theorem 4.2: Let $f = (f(t, x), t \geq 0, x \in \mathbb{R})$ be a continuous function. The following assertions are equivalent:

(a) $E|f(t, \sigma W_t)| < \infty$ for every $t \geq 0$ and the process

$$M_t(\sigma) = f(t, \sigma W_t) - Ef(t, \sigma W_t), \quad t \geq 0,$$

is a martingale for all $\sigma \in \mathbb{R},$

(b) $E|f(t, \sigma_1 W_t)| < \infty, E|f(t, \sigma_2 W_t)| < \infty$ for every $t \geq 0$ and the processes

$$M_t(\sigma_1) = f(t, \sigma_1 W_t) - Ef(t, \sigma_1 W_t), \quad M_t(\sigma_2) = f(t, \sigma_2 W_t) - Ef(t, \sigma_2 W_t) \quad t \geq 0,$$

are martingales for two different $\sigma_1 \neq \sigma_2, \sigma_1 \neq 0, \sigma_2 \neq 0.$
(c) the function \( f(t, x) \) is of the form
\[
    f(t, x) = ax^2 + bx + c(t)
\]

for some constants \( a, b \in \mathbb{R} \) and deterministic function \( c(t) = f(t, 0), t \geq 0 \).

**Proof:** \( 2 \rightarrow 3 \). Let
\[
g(t, \sigma) = Ef(t, \sigma W_t) \quad \text{and} \quad u(t, x) = f(t, x) - g(t, \sigma).
\]

Since \( u(t, x) \) is continuous and the process \( u(t, W_t) \) is a martingale, \( u(t, x) \) will be of the class \( C^{1, 2} \) on \( (0, T) \times \mathbb{R} \) and satisfies the "backward" heat equation (see, e.g. [7])
\[
    \frac{\partial u}{\partial t} + \frac{\sigma^2}{2} \frac{\partial^2 u}{\partial x^2} = 0, \quad 0 < t < T, \quad x \in \mathbb{R},
\]
which implies that
\[
    \frac{\partial (f(t, x) - g(t, \sigma))}{\partial t} + \frac{\sigma^2}{2} \frac{\partial^2 f(t, x)}{\partial x^2} = 0, \quad 0 < t < T, \quad x \in \mathbb{R}.
\]
Taking the difference of Equations (52) for \( \sigma_1 \) and \( \sigma_2 \) we have that
\[
    \frac{\sigma_1^2 - \sigma_2^2}{2} \frac{\partial f(t, x)}{\partial x^2} = \frac{\partial (g(t, \sigma_1) - g(t, \sigma_2))}{\partial t}, \quad 0 < t < T, \quad x \in \mathbb{R}.
\]
It follows from the last equation that the second derivative \( f_{xx}(t, x) \) is constant for any fixed \( t \), which implies that \( f(t, x) \) is a polynomial of order 2 with time dependent coefficients \( a(t), b(t), c(t), t \geq 0 \)
\[
    f(t, x) = a(t)x^2 + b(t)x + c(t).
\]
Therefore,
\[
g(t, \sigma) = Ef(t, \sigma W_t) = \sigma^2 ta(t) + c(t)
\]
and substituting expressions (54) and (55) for \( \sigma_1 \) and \( \sigma_2 \) in (53) we obtain that
\[
    (\sigma_1^2 - \sigma_2^2)a(t) = \frac{\partial (\sigma_1^2 - \sigma_2^2)ta(t)}{\partial t},
\]
which implies that \( a'(t)t = 0 \) and hence \( a(t) \) is a constant for any \( t > 0 \).

This, together with (54) and (55), implies that
\[
f(t, \sigma W_t) - Ef(t, \sigma W_t) = a\sigma^2 (W_t^2 - t) + \sigma b(t) W_t
\]
and this process is a martingale if and only if \( b(t) \) is equal to a constant. Thus \( f(t, x) \) will be of the form (50).

\( 3 \rightarrow 1 \) If the function \( f(t, x) \) is of the form (50), then
\[
f(t, \sigma W_t) = a\sigma^2 W_t^2 + b\sigma W_t + c(t),
\]
\[
    Ef(t, \sigma W_t) = a\sigma^2 W_t^2 + b\sigma W_t = a\sigma^2 (W_t^2 - t)
\]
is a martingale. It is evident that the process \( f(t, \sigma W_t) - Ef(t, \sigma W_t) = a\sigma^2 (W_t^2 - t) \) is a martingale.
The implication $1 \rightarrow 2$ is evident. ■

**Corollary 4.2:** Let $f = (f(t, x), t \geq 0, x \in \mathbb{R})$ be a continuous function. The following assertions are equivalent:

(a) the process $f(t, \sigma W_t) \quad t \geq 0$, is a martingale for all $\sigma \neq 0$,

(b) the processes $f(t, \sigma_1 W_t)$ and $f(t, \sigma_2 W_t)$ are martingales for two different $\sigma_1 \neq \sigma_2 \neq 0$,

(c) the function $f(t, x)$ is of the form

$$f(t, x) = bx + c$$

for some constants $b$ and $c$.

**Proof:** If the process $f(t, \sigma W_t)$ is a martingale, then $g(t) = Ef(t, \sigma W_t)$ is constant and from (55) $a(t) = a = 0$ and $c(t)$ is equal to a constant. Therefore, this corollary follows from Theorem 4.2. ■

**Theorem 4.3:** Let $f = (f(t, x), t \geq 0, x \in \mathbb{R})$ be a continuous strictly positive function differentiable at $t$ for any $t \geq 0$. The following assertions are equivalent:

(a) $E|f(t, \sigma W_t)| < \infty$ for every $t \geq 0$ and the process

$$N_t(\sigma) = \frac{f(t, \sigma W_t)}{Ef(t, \sigma W_t)}, \quad t \geq 0,$$

is a martingale for all $\sigma \in \mathbb{R}$.

(b) $E|f(t, \sigma_1 W_t)| < \infty, E|f(t, \sigma_2 W_t)| < \infty$ for every $t \geq 0$ and the processes

$$N_t(\sigma_1) = \frac{f(t, \sigma_1 W_t)}{Ef(t, \sigma_1 W_t)}, \quad N_t(\sigma_2) = \frac{f(t, \sigma_2 W_t)}{Ef(t, \sigma_2 W_t)}, \quad t \geq 0,$$

are martingales for two different $\sigma_1 \neq \sigma_2, \sigma_1 \neq 0, \sigma_2 \neq 0$.

(c) the function $f(t, x)$ is of the form

$$f(t, x) = ac(t) e^{\lambda x} + bc(t) e^{-\lambda x}$$

for some constants $a \geq 0, b \geq 0$ with $a + b = 1, ab \neq 0$ and deterministic function $c(t) = f(t, 0), t \geq 0$.

**Proof:** $2 \rightarrow 3$. Let

$$g(t, \sigma) = Ef(t, \sigma W_t) \quad \text{and} \quad h(t, x) = \frac{f(t, x)}{g(t, \sigma)}.$$

Since $h(t, x)$ is continuous and the process $h(t, W_t)$ is a martingale, $h(t, x)$ will be of the class $C^{1,2}$ on $(0, T) \times \mathbb{R}$ and satisfies the "backward" heat equation (see, e.g. [7])

$$\frac{\partial h}{\partial t} + \frac{\sigma^2}{2} \frac{\partial^2 h}{\partial x^2} = 0, \quad 0 < t < T, \quad x \in \mathbb{R},$$

(58)
which implies that
\[
\frac{\partial (f(t,x)g(t,\sigma))}{\partial t} + \frac{\sigma^2}{2} \frac{\partial^2 f(t,x)}{\partial x^2} \frac{1}{g(t,\sigma)} = 0, \quad 0 < t < T, \quad x \in \mathbb{R}.
\] (59)

Since \(f(t,x)\) is differentiable at \(t\), the function \(g(t,\sigma)\) will be also differentiable and from (59) we have that
\[
\frac{\sigma^2}{2} \frac{\partial^2 f(t,x)}{\partial x^2} + \frac{\partial f(t,x)}{\partial t} - f(t,x) \frac{g'(t,\sigma)}{g(t,\sigma)} = 0, \quad 0 < t < T, \quad x \in \mathbb{R}.
\] (60)

Taking the difference of Equations (60) for \(\sigma_1\) and \(\sigma_2\) we have that
\[
\frac{\sigma_1^2 - \sigma_2^2 f_{xx}(t,x)}{2 f(t,x)} = \frac{g'(t,\sigma_1)}{g(t,\sigma_1)} - \frac{g'(t,\sigma_2)}{g(t,\sigma_2)}, \quad 0 < t < T, \quad x \in \mathbb{R}.
\] (61)

It follows from the last equation that \(f_{xx}(t,x)/f(t,x)\) does not depend on \(x\), i.e. \(f_{xx}(t,x)/f(t,x) = c(t)\) for some function \((c(t), t \geq 0)\), which should be positive for all \(t \geq 0\), since if \(c(t_0) < 0\) for some \(t_0\) then the general solution of equation \(f_{xx}(t_0,x)/f(t_0,x) = c(t_0)\) leads to \(f(t_0,x)\) which can take negative values. Hence
\[
\frac{f_{xx}(t,x)}{f(t,x)} = \lambda^2(t),
\] (62)

for some function \((\lambda(t), t \geq 0)\). For any fixed \(t\) the general solution of Equation (62) is of the form
\[
f(t,x) = a(t) e^{\lambda(t)x} + b(t) e^{-\lambda(t)x},
\] (63)

for some functions of \(t\)---\(a(t), b(t)\) and \(\lambda(t)\).

Let first show that \(\lambda(t) = \lambda\) for all \(t \geq 0\), for some \(\lambda \in \mathbb{R}\).

It follows from (63)
\[
Ef(t,\sigma W_t) = (a(t) + b(t)) e^{\frac{\sigma^2}{2} t}
\] (64)

and it is easy to see that
\[
\frac{g'(t,\sigma)}{g(t,\sigma)} = \frac{a'(t) + b'(t)}{a(t) + b(t)} + \sigma^2 \lambda(t) \lambda'(t) t + \frac{\sigma^2}{2} \lambda^2(t).
\] (65)

Substituting expressions (65) for \(\sigma_1\) and \(\sigma_2\) in (61) we obtain from (62) that \(\lambda(t)\lambda'(t)t = 0\), which implies that
\[
\lambda^2(t) = \lambda^2 \quad \text{for some constant } \lambda \in \mathbb{R}.
\] (66)

Therefore, it follows from (66), (63) and (64) that
\[
N_t(\sigma) = \frac{f(t,\sigma W_t)}{Ef(t,\sigma W_t)} = \frac{a(t)}{a(t) + b(t)} e^{\lambda \sigma W_t - \frac{\sigma^2}{2} t} + \frac{b(t)}{a(t) + b(t)} e^{-\lambda \sigma W_t - \frac{\sigma^2}{2} t}.
\] (67)

Since the processes \(X_t = e^{\lambda \sigma W_t - \frac{\sigma^2}{2} t}\) and \(Y_t = e^{-\lambda \sigma W_t - \frac{\sigma^2}{2} t}\) are martingales and \(P(X_t \neq Y_t) = 1\) for all \(t\), the process \(N_t(\sigma)\) defined by (67) will be a martingale if and only
if
\[ \alpha_t \equiv \frac{a(t)}{a(t) + b(t)} = a, \quad \text{and} \quad \beta_t \equiv \frac{b(t)}{a(t) + b(t)} = b \] (68)
for some constants \( a, b \in \mathbb{R} \).

Indeed, since \( \alpha_t \) is a deterministic function, \( \beta_t = 1 - \alpha_t \) and the process \( \alpha_t X_t + (1 - \alpha_t) Y_t \) is a martingale, then for any \( s \leq t \)
\[ \alpha_s X_s + (1 - \alpha_s) Y_s = E(\alpha_t X_t + (1 - \alpha_t) Y_t / F_s) = \alpha_t X_s + (1 - \alpha_t) Y_s, \]
which implies that \( (\alpha_t - \alpha_s)(X_s - Y_s) = 0 \). Therefore, \( \alpha_t = \alpha_s \) and \( \alpha_t \) is equal to a constant by arbitrariness of \( s \) and \( t \).

Therefore, (66), (63) and (68) imply that
\[ f(t, x) = (a(t) + b(t))[a e^{\lambda x} + b e^{-\lambda x}], \] (69)
where by (68) \( a + b = 1 \) and \( c(t) \equiv a(t) + b(t) = f(t, 0) \) from (69). Besides, \( a \geq 0, b \geq 0 \) and \( ab \neq 0 \), since \( f(t, x) \) is strictly positive. Hence \( f(t, x) \) is of the form (57).

3 \( \rightarrow \) 1 If the function \( f(t, x) \) is of the form (57) then
\[ f(t, \sigma W_t) = f(t, 0)[a e^{\lambda \sigma W_t} + b e^{-\lambda \sigma W_t}] \] (70)
and \( E[f(t, \sigma W_t)] < \infty \) for all \( t \geq 0, \sigma \in \mathbb{R} \). It is evident that \( Ef(t, \sigma W_t) = f(t, 0) e^{\frac{\sigma^2}{2} t} \)
and the process
\[ \frac{f(t, \sigma W_t)}{Ef(t, \sigma W_t)} = a e^{\lambda \sigma W_t - \frac{\sigma^2}{2} t} + b e^{-\lambda \sigma W_t - \frac{\sigma^2}{2} t} \]
is a martingale for any \( \sigma \).

The implication \( 1 \rightarrow 2 \) is evident. \( \blacksquare \)

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