ON JORDAN DOUBLES OF SLOW GROWTH OF LIE SUPeralgebras

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Abstract. To an arbitrary Lie superalgebra $L$ we associate its Jordan double $\mathcal{J}or(L)$, which is a Jordan superalgebra. This notion was introduced by the second author before. Now we study further applications of this construction.

First, we show that the Gelfand-Kirillov dimension of a Jordan superalgebra can be an arbitrary number $\{0\} \cup [1, +\infty]$. Thus, unlike associative and Jordan algebras [23, 24], one hasn’t an analogue of Bergman’s gap (1, 2) for the Gelfand-Kirillov dimension of Jordan superalgebras.

Second, using the Lie superalgebra $\mathbb{R}$ of [5], we construct a Jordan superalgebra $J = \mathcal{J}or(\mathbb{R})$ that is nil finely $\mathbb{Z}$-graded, in contrast with non-existence of such examples (roughly speaking, analogues of the Grigorchuk and Gupta-Sidki groups) of Lie algebras in characteristic zero [24] and Jordan algebras in characteristic not 2 [25]. Also, $J$ is just infinite but not hereditary just infinite. A similar Jordan superalgebra of slow polynomial growth was constructed before [26]. The virtue of the present example is that it is of linear growth, of finite width 4, namely, its $\mathbb{N}$-gradation by degree in the generators has components of dimensions $\{0, 2, 3, 4\}$, and the sequence of these dimensions is non-periodic.

Third, we review constructions of Poisson and Jordan superalgebras of [38] starting with another example of a Jordan superalgebra appeared in [29]. We discuss the notion of self-similarity for Lie, associative, Poisson, and Jordan superalgebras. We also discuss the notion of a wreath product in case of Jordan superalgebras.

1. Introduction: Superalgebras, constructions

Denote $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$. By $K$ denote the ground field of characteristic $\text{char} K \neq 2$, $(S)_K$ a linear span of a subset $S$ in a $K$-vector space.

1.1. Associative and Lie superalgebras. Superalgebras appear naturally in physics and mathematics [17, 35, 3]. Put $\mathbb{Z}_2 = \{0, 1\}$, the group of order 2. A superalgebra $A$ is a $\mathbb{Z}_2$-graded algebra $A = A_0 \oplus A_1$. The elements $a \in A_\alpha$ are called homogeneous of degree $|a| = \alpha \in \mathbb{Z}_2$. The elements of $A_0$ are even, those of $A_1$ odd. In what follows, if $|a|$ enters an expression, then it is assumed that $a$ is homogeneous of degree $|a| \in \mathbb{Z}_2$, and the expression extends to the other elements by linearity. Let $A, B$ be superalgebras, a tensor product $A \otimes B$ is a superalgebra whose space is the tensor product of the spaces $A$ and $B$ with the induced $\mathbb{Z}_2$-grading and the product satisfying Kaplansky’s rule:

$$(a_1 \otimes b_1)(a_2 \otimes b_2) = (-1)^{|b_1||a_2|}a_1a_2 \otimes b_1b_2, \quad a_1, a_2 \in A, \quad b_1, b_2 \in B.$$ 

An associative superalgebra $A$ is just a $\mathbb{Z}_2$-graded associative algebra $A = A_0 \oplus A_1$. Let $V = V_0 \oplus V_1$ be a $\mathbb{Z}_2$-graded vector space. Then $\text{End}(V)$ is an associative superalgebra, where $\text{End}(V)_a = \{ \phi \in \text{End}(V) | \phi(V_b) \subset V_{a+b}, b \in \mathbb{Z}_2\}, a \in \mathbb{Z}_2$. In case $\dim V_0 = m, \dim V_1 = k$ this superalgebra is denoted by $M(m|k)$. One has an isomorphism of superalgebras $M(a|b) \otimes M(c|d) \cong M(ac + bd|ad + bc)$ for all $a, b, c, d \geq 0$.

A Lie superalgebra is a $\mathbb{Z}_2$-graded algebra $L = L_0 \oplus L_1$ with an operation $[, ,]$ satisfying the axioms (char $K \neq 2, 3$):

- $[x, y] = -(-1)^{|x||y|}[y, x], \quad$ (super-anticommutativity);
- $[x, [y, z]] = [[x, y], z] + (-1)^{|x||y|}[y, [x, z]], \quad$ (super Jacobi identity).

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1.2. Poisson superalgebras. A \( \mathbb{Z}_2 \)-graded vector space \( A = A_0 \oplus A_1 \) is called a Poisson superalgebra provided that, beside the addition, \( A \) has two \( K \)-bilinear operations as follows:

- \( A = A_0 \oplus A_1 \) is an associative superalgebra with unit whose multiplication is denoted by \( a \cdot b \) (or \( ab \)), where \( a, b \in A \). We assume that \( A \) is supercommutative, i.e. \( a \cdot b = (-1)^{|a||b|} b \cdot a \), for all \( a, b \in A \).
- \( A = A_0 \oplus A_1 \) is a Lie superalgebra whose product is traditionally denoted by the Poisson bracket \( \{a, b\} \), where \( a, b \in A \).
- These two operations are related by the super Leibnitz rule:

\[
\{a \cdot b, c\} = a \cdot \{b, c\} + (-1)^{|b||c|} \{a, c\} \cdot b, \quad a, b, c \in A.
\]

Let \( L \) be a Lie superalgebra, \( \{U_n|n \geq 0\} \) the natural filtration of its universal enveloping algebra \( U(L) \) by degree in \( L \). Consider the symmetric algebra \( S(L) = \bigvee_{n=0}^\infty U_n/U_{n+1} \) (see [19]). Recall that \( S(L) \) is identified with the supercommutative superalgebra \( K[v_i | i \in I] \otimes \Lambda(w_j, | j \in J) \), where \( \{v_i | i \in I\}, \{w_j | j \in J\} \) are bases of \( L_0, L_1 \), respectively. Define a Poisson bracket by setting \( \{v, w\} = [v, w] \), where \( v, w \in L \), and extending to the whole of \( S(L) \) by linearity and the Leibnitz rule. Then, \( S(L) \) is turned into a Poisson superalgebra, called the symmetric algebra of \( L \). Let \( L(X) \) be the free Lie superalgebra generated by a graded set \( X \), then \( S(L(X)) \) is a free Poisson superalgebra [21].

Let us consider one more example. Let \( H_n = \Lambda(x_1, \ldots, x_n, y_1, \ldots, y_n) \) be the Grassmann superalgebra supplied with a bracket determined by: \( \{x_i, y_j\} = \delta_{i,j}, \{x_i, x_j\} = \{y_i, y_j\} = 0 \) for \( 1 \leq i, j \leq n \). We obtain the simple Hamiltonian Poisson superalgebra with the bracket:

\[
\{f, g\} = (-1)^{|f|} \sum_{i=1}^n \left( \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial y_i} + \frac{\partial f}{\partial y_i} \frac{\partial g}{\partial x_i} \right), \quad f, g \in H_n.
\]

Let \( A, P \) be Poisson superalgebras, their tensor product \( A \otimes P \) is a Poisson superalgebra with operations:

- \( (a \otimes v) \cdot (b \otimes w) = (-1)^{|a||b|} ab \otimes vw, \)
- \( (a \otimes v, b \otimes w) = (-1)^{|a||b|} \{a, b\} \otimes vw + ab \otimes \{v, w\} \), for all \( a, b \in A, v, w \in P \).

1.3. Jordan superalgebras. While studying Jordan (super)algebras we always assume that \( \text{char} \ K \neq 2 \).

A Jordan algebra is an algebra \( J \) satisfying the identities

- \( ab = ba; \)
- \( a^2 = ba. \)

A Jordan superalgebra is a \( \mathbb{Z}_2 \)-graded algebra \( J = J_0 \oplus J_1 \) satisfying the graded identities:

- \( ab = (-1)^{|a||b|} ba; \)
- \( (ab)(cd) + (-1)^{|b||c|}(ac)(bd) + (-1)^{|b||c|+|d|}(ad)(bc) = ((ab)c)d + (-1)^{|b||c|+|d|+|c||d|}(ad)c) + b + (-1)^{|a||b||c|+|d|+|c||d|}(bd)c)a. \)

Let \( A = A_0 \oplus A_1 \) be an associative superalgebra. The same space supplied with the product \( a \circ b = \frac{1}{4}(ab + (-1)^{|a||b|} ba) \) is a Jordan superalgebra \( A^{(+)} \). A Jordan superalgebra \( J \) is called special if it can be embedded into a Jordan superalgebra of the type \( A^{(+)} \). Also, \( J \) is called \( i \)-special (or weakly special) if it is a homomorphic image of a special one.

I.L. Kantor suggested the following doubling process, which is applied to a Poisson (super)algebra \( A \) and the result is a Jordan superalgebra \( \mathcal{K}an(A) \) [20]. The \( K \)-space \( \mathcal{K}an(A) \) is the direct sum \( A \oplus A \), where \( \bar{A} \) is a copy of \( A \), let \( a \in A \) then \( \bar{a} \) denotes the respective element in \( \bar{A} \). Also, \( \bar{A} \) is supplied with the opposite \( \mathbb{Z}_2 \)-grading, i.e., \( |\bar{a}| = 1 - |a| \) for a \( \mathbb{Z}_2 \)-homogeneous \( a \in A \). The multiplication \( \circ \) on \( \mathcal{K}an(A) \) is defined by:

\[
a \circ \bar{b} = ab,
\bar{a} \circ b = (-1)^{|b|} ab,
\bar{a} \circ \bar{b} = \bar{a} \circ b,
\bar{a} \circ b = (-1)^{|b|} \{a, b\}, \quad a, b \in A.
\]

This construction is important because it yielded a new series of finite dimensional simple Jordan superalgebras \( \mathcal{K}an(A(n)), n \geq 2 \), where \( \Lambda(n) \) is the Grassmann algebra in \( n \) variables [20 21].

Simple finite dimensional nontrivial Jordan superalgebras over an algebraically closed field of characteristic zero were classified [18 20]. Infinite-dimensional simple \( \mathbb{Z} \)-graded Jordan superalgebras with a unit element
over an algebraically closed field of characteristic zero which components are uniformly bounded are classified in \[19\]. Recently, just infinite Jordan superalgebras were studied in \[40\].

1.4. Growth. We recall the notion of growth. Let \(A\) be an associative (or Lie) algebra generated by a finite set \(X\). Denote by \(A^{(X,n)}\) the subspace of \(A\) spanned by all monomials in \(X\) of length not exceeding \(n\), \(n \geq 0\). In case of a Lie superalgebra of char \(K\) \(= 2\) we also consider formal squares of odd monomials of length at most \(n/2\) \[8 \, 30\]. One defines an \((\text{ordinary})\) growth function:

\[
\gamma_A(n) = \gamma_A(X,n) = \dim_K A^{(X,n)}, \quad n \geq 0.
\]

Let \(f, g : \mathbb{N} \to \mathbb{R}^+\) be eventually increasing and positive valued functions. Write \(f(n) \preceq g(n)\) if and only if there exist positive constants \(N, C\) such that \(f(n) \leq C g(n)\) for all \(n \geq N\). Introduce equivalence \(f(n) \sim g(n)\) if and only if \(f(n) \preceq g(n)\) and \(g(n) \preceq f(n)\). Different generating sets of an algebra yield equivalent growth functions \[23\].

It is well known that the exponential growth is the highest possible growth for finitely generated Lie and associative algebras. A growth function \(\gamma_A(n)\) is compared with polynomial functions \(n^\alpha, \alpha \in \mathbb{R}^+\), by computing the \textit{upper} and \textit{lower Gelfand-Kirillov dimensions} \[23\]:

\[
\text{GKdim} \, A = \lim_{n \to \infty} \frac{\ln \gamma_A(n)}{\ln n} = \inf \{\alpha > 0 \mid \gamma_A(n) \preceq n^\alpha\};
\]

\[
\text{GKdim}^\ast A = \lim_{n \to \infty} \frac{\ln \gamma_A(n)}{\ln n} = \sup \{\alpha > 0 \mid \gamma_A(n) \succeq n^\alpha\}.
\]

By Bergman’s theorem, the Gelfand-Kirillov dimension of an associative algebra cannot belong to the interval \((1, 2)\) \[23\]. Similarly, Martinez and Zelmanov proved that there are no finitely generated Jordan algebras with Gelfand-Kirillov dimension strictly between 1 and 2 \[24\]. But such a gap does not exist for Lie algebras, the Gelfand-Kirillov dimension of a finitely generated Lie algebra can be arbitrary number \([0] \cup [1, \infty)\) \[27\].

Suppose that \(L\) is a Lie (super)algebra and \(X \subset L\). By \(\text{Lie}(X)\) denote the subalgebra of \(L\) generated by \(X\). In case of associative, Poisson, and Jordan (super)algebras we use notations \(\text{Alg}(X)\), \(\text{Poisson}(X)\), and \(\text{Jord}(X)\), respectively. A grading of an algebra is called \textsl{fine} if it cannot be splitted by taking a bigger grading group (see e.g. \[4\]).

Pro-p-groups and \(N\)-graded Lie algebras cannot be simple. Instead, one has another important notion. A group (algebra) is \textsl{just infinite} if and only if it has no non-trivial normal subgroups (ideals) of infinite index (codimension). A group (algebra) is said \textsl{hereditary just infinite} if and only if any normal subgroup (ideal) of finite index (codimension) is just infinite. The Gupta-Sidki groups were the first in the class of periodic groups to be shown to be just infinite \[10\]. The Grigorchuk group is also just infinite but not hereditary just infinite \[14\].

2. JORDAN DOUBLE OF LIE SUPERALGEBRA

First, we recall the construction of a double of a Lie superalgebra suggested by the second author \[42\]. The goal of the present paper is to study its different applications.

Let \(L\) be an arbitrary Lie superalgebra. Its symmetric algebra \(S(L)\) has the structure of a Poisson superalgebra. Observe, that the subspace \(H \subset S(L)\) spanned by all tensors of length at least two is its ideal. Thus, one obtains a (rather trivial) Poisson superalgebra \(P(L) = S(L)/H\), which equivalently can be obtained as a vector space endowed with two Poisson products which are nontrivial in the following cases:

\[
P(L) = \langle 1 \rangle \oplus L, \quad 1 \cdot x = x, \quad \{x, y\} = [x, y], \quad x, y \in L.
\]  

Using Kantor’s double, define a Jordan superalgebra \(\mathcal{J}or(L) = \text{Kan}(P(L))\). Equivalently, one can just take a vector space supplied with a product \(\bullet\) which is nontrivial in the following cases (see the example at the end \[42\]):

\[
\mathcal{J}or(L) = \langle 1 \rangle \oplus L \oplus \langle \bar{1} \rangle \oplus \bar{L}, \quad x \bullet \bar{y} = [x, y], \quad x \bullet \bar{1} = (-1)^{|x|} \bar{1} \bullet x = \bar{x}, \quad x, y \in L; \quad 1 \text{ the unit.}
\]

If an associative superalgebra \(A\) is just infinite then the related Jordan superalgebra \(A^{(+)}\) is just infinite as well \[10\]. We establish a similar fact, for convenience of the reader we repeat our arguments, see \[30\].

**Lemma 2.1.** Let \(L\) be a Lie superalgebra, consider the Jordan superalgebra \(\mathcal{J}or(L)\).

i) \(\mathcal{J}or(L)\) is just infinite if and only if \(L\) is just infinite.

ii) The ideal without unit \(\mathcal{J}or^0(L) = \langle 1 \rangle \oplus \bar{1} \oplus \bar{L}\) is solvable of length 3.
iii) \((a^2)^2 = 0\) for any \(a \in \mathcal{J}or^a(L)\).

**Proof.** Let \(L\) be not just infinite. Then there exists an ideal of infinite codimension \(0 \neq I \triangleleft L\) and \(I \oplus \bar{I}\) is a nontrivial ideal of infinite codimension in \(\mathcal{J}or(L)\). Therefore, \(\mathcal{J}or(L)\) is not just infinite.

Conversely, suppose that \(L\) is just infinite. By way of contradiction, assume that \(0 \neq H \subset \mathcal{J}or(L)\) is an ideal of infinite codimension. Then \(H = H \cap (L \oplus \bar{L}) \subset \mathcal{J}or(L)\) is also an ideal of infinite codimension. Denote by \(H_0\) and \(H_1\) the projections of \(H\) onto \(L, \bar{L}\), respectively (\(H_1\) being the copy of a subspace \(H_1 \subset L\)).

Since \(H\) is an ideal, \(\bar{1} \cdot H = \bar{H}_0 \subset H_1\) and \(L \cdot H = [L, H_1] \subset H_0\) and we get \([L, H_1] \subset H_0 \subset H_1 \subset L\). Hence \(H_0 \subset L\) is an ideal, which must be either zero or of finite codimension by our assumption. Let \(H_0 \subset L\) be of finite codimension then \(H \subset \mathcal{J}or(L)\) is of finite codimension, a contradiction. Now assume that \(H_0 = 0\). Then \([L, H_1] = 0\) and \(H_1\) is central. By taking \(0 \neq z \in H_1\), we get an ideal \((z) \subset L\) of infinite codimension, a contradiction. Thus, \(\mathcal{J}or(L)\) is just infinite.

To prove the second claim we repeat the arguments of \(\text{(12)}\). Denote \(J = \mathcal{J}or^a(L)\). Then \(J^2 \subset L \oplus \bar{L}\), \((J^2)^2 \subset L\), and \((J^2)^2)^2 = 0\). Thus, \(J\) is solvable of length 3.

To prove the last claim let \(a \in J\), then \(a = \alpha \bar{1} + u_0 + u_1 + \bar{v}_0 + \bar{w}_1\), where \(\alpha \in K\), \(u_0, w_0 \in L_0, u_1, w_1 \in L_1\). Then \(a^2 = \{w_1, w_1\} + 2 \alpha \bar{u}_0 \in L_0 + \bar{L}_0\). By the same computations, \((a^2)^2 = 0\). \(\square\)

### 3. On Gel’fand-Kirillov Dimension of Jordan superalgebras

In \(\text{(36)}\) we constructed a Jordan superalgebra \(K\) which Gel’fand-Kirillov dimension belongs to \((1, 2)\), that is not possible for associative and Jordan algebras \(\text{(23, 24)}\). Let us prove a more general fact that the gap \((1, 2)\) does not exist for Jordan superalgebras.

**Lemma 3.1.** Let \(K\) be a field, \(\text{char} K \neq 2\). Suppose that a Lie superalgebra \(L\) is \(\mathbb{Z}^k\)-graded by multidegree in a generating set \(X = \{x_1, \ldots, x_k\}\) and \(L = \bigoplus_{n=1}^{\infty} L_n\) the \(\mathbb{N}\)-gradation by total degree in the generators.

Consider the Jordan double \(J = \mathcal{J}or(L)\). Then

i) one has \(J = \bigoplus_{n=0}^{\infty} J_n\), the \(\mathbb{N}\)-gradation by degree in the generating set \(\bar{X} = X \cup \{\bar{1}\}\).

ii) \(J_0 = \{1\}, J_1 = L_1 \oplus \{\bar{1}\}\), the remaining components are as follows:

\[
J_{3n-2} = L_n, \quad J_{3n-1} = \bar{L}_n, \quad J_{3n} = 0, \quad n \geq 1.
\]

iii) \(J\) is \(\mathbb{Z}^{k+1}\)-graded by multidegree in \(\bar{X}\).

**Proof.** The Jordan superalgebra \(J = \mathcal{J}or(L)\) is generated by \(\bar{X} = X \cup \{\bar{1}\}\). Clearly, the \(\mathbb{N}\)-gradation of \(L\) by degree in \(X\) extends to \(J\) as well. Let \(J_{n,k}\) denote the space of Jordan monomials that include \(n\) letters from \(X\) and \(k\) letters \(1\), where \(n, k \geq 0\). Let us prove that

\[
J_{n,k} = \begin{cases} 
\{1\}, & n = k = 0; \\
\{\bar{1}\}, & n = 0, k = 1; \\
\delta_{k,2n-2}L_n, & k \text{ even}; \\
\delta_{k,2n-1}L_n, & k \text{ odd}; 
\end{cases} \quad n, k \geq 0. \tag{4}
\]

We proceed by induction on \(l = n + k\). The base of induction is \(l = 0\) and \(l = 1\), in which cases we have \(J_{0,0} = \{1\}, J_{1,0} = L_1 = \langle X \rangle\) and \(J_{0,1} = \{1\}\). Assume that \(l \geq 2\). Observe that to have a nonzero component we need at least one letter \(1\), thus \(k \geq 1\). Let \(k \geq 1\) be odd. Since \(\bar{1}\) yields a \(\mathbb{Z}_2\)-grading we have \(J_{n,k} \subset L\), such elements can appear only as the products \(1 \cdot L\). Using the inductive assumption,

\[
J_{n,k} = \bar{1} \cdot J_{n,k-1} = \bar{1} \cdot \delta_{k-1,2n-2}L_n = \delta_{k-1,2n-2}\bar{L}_n = \delta_{k,2n-1}\bar{L}_n.
\]

Let \(k \geq 2\) be even. As above, \(J_{n,k} \subset L\) and such elements can appear only as the products \(L \cdot L\). Using the inductive assumption, we have

\[
J_{n,k} = \sum_{n_1 + n_2 = n \atop k_1 + k_2 = k} \bar{L}_{n_1} \cdot \bar{L}_{n_2} = \sum_{n_1 + n_2 = n \atop k_1 + k_2 = k} [L_{n_1}, L_{n_2}] = \delta_{k,2n-2}L_n.
\]

The inductive step is proved.
By (1), we obtain a direct sum \( J = \bigoplus_{n,k \geq 0} J_{n,k} \), which is a \( \mathbb{Z}^2 \)-gradation of \( J \) by the respective multidegree in \( X \cup \{1\} \). By setting \( J_m = \bigoplus_{n+k=m} J_{n,k} \), \( m \geq 0 \), we get the claimed \( \mathbb{N}_0 \)-gradation. Counting the total degree in (1), we get (ii). The multidegree \( \mathbb{Z}^{k+1} \)-gradation is proved similarly.

Define generating functions:
\[
\mathcal{H}(L,t) = \sum_{n=1}^{\infty} \dim L_n t^n, \\
\mathcal{H}(J,t_1,t_2) = \sum_{n,m=0}^{\infty} \dim J_{n,m} t_1^n t_2^m, \\
\mathcal{H}(J,t) = \sum_{n=0}^{\infty} \dim J_n t^n = \mathcal{H}(J,t,t).
\]

**Corollary 3.2.** Using notations above

i) \( \mathcal{H}(J,t_1,t_2) = 1 + t_2 + \left( \frac{1}{t_2} + \frac{1}{t_2^2} \right) \mathcal{H}(L,t_1 t_2^2); \)
\[ \mathcal{H}(J,t) = 1 + t + \left( \frac{1}{t} + \frac{1}{t^2} \right) \mathcal{H}(L,t^3). \]

ii) We obtain an equivalent growth function: \( \gamma_J(\bar{X},n) \sim \gamma_L(X,n). \)

iii) \( J \) has the same Gelfand-Kirillov dimensions as those for \( L \) (provided that they exist).

iv) Let \( \gamma_L(X,n) \approx Cn^{r}, n \to \infty \) for some constants \( C > 0, r \geq 1 \), moreover assume that \( C_1 n^{-r} \leq \dim L_n \leq C_2 n^{-r+1}, n \geq n_0, \) for some constants \( n_0, C_1, C_2 \). Then \( \gamma_J(\bar{X},n) \approx 2C(n/3)^r, n \to \infty. \)

**Proof.** Using (1) we get the formulas for the generating functions. We have the growth functions \( \gamma_L(n,X) = \sum_{k=1}^{n} \dim L_n, n \geq 1 \), and \( \gamma_J(n,\bar{X}) = \sum_{k=1}^{n} \dim J_n, n \geq 0 \). By Lemma 3.1 (ii), we get
\[
\gamma_J(\bar{X},3m) = \gamma_J(\bar{X},3m-1) = 2 + 2\gamma_L(X,m); \\
\gamma_J(\bar{X},3m-2) = 2 + 2\gamma_L(X,m) - \dim L_m, \quad m \geq 1.
\]

Now it remains to use the polynomial estimates on the growth of \( L \).

**Theorem 3.3.** Let \( K \) be a field, \( \text{char} \: K \neq 2 \). Fix any real number \( r \geq 1 \). There exists a three generated Jordan superalgebra \( J \) with the following properties.

i) \( \text{GKdim} \: J = \text{GKdim} \: L = r; \)

ii) \( J \) is graded by degree in the generators, we have \( J = \bigoplus_{n=0}^{\infty} J_n \) where \( J_{3n} = 0 \) for all \( n \geq 1; \)

iii) its ideal without unit \( J^0 \) is solvable of length \( 3 \).

**Proof.** The first author constructed a 2-generated Lie algebra \( L \) such that \( \text{GKdim} \: L = \text{GKdim} \: L = r \) and \( L \) is \( \mathbb{N}_0^2 \)-graded by multidegree in the generators, moreover, we have estimates \( C_1 n^{-r} \leq \dim L_n \leq C_2 n^{-r+1}, n \geq 1, \) for some constants \( C_1, C_2 \) [7]. Also \( (L^2)^3 = 0 \), this condition can be written as \( L \in \mathbb{N}_2A \) using notations of varieties of Lie algebras [2].

Now we consider \( J = Jor(L) \) and apply Lemma 3.1, Corollary 3.2, and Lemma 2.1.

4. Just infinite nil Jordan superalgebra of finite width

The Grigorchuk and Gupta-Sidki groups play fundamental role in modern group theory [13, 14]. They are natural examples of self-similar finitely generated periodic groups. We discuss their analogues in different classes of algebras.

Let \( L \) be a group and \( G = G_1 \supseteq G_2 \supseteq \cdots \) its lower central series. One constructs a related \( \mathbb{N} \)-graded Lie algebra \( L_K(G) = \bigoplus_{i \geq 1} L_i, \) where \( L_i = G_i/G_{i+1} \otimes K, i \geq 1 \). A product is given by \( [aG_{i+1}, bG_{j+1}] = (a,b)G_{i+j+1}, \) where \( a \in G_i, b \in G_j, \) and \( (a,b) = a^{-1}b^{-1}ab \) the group commutator.

A residually \( p \)-group \( G \) is said of *finite width* if all factors \( G_i/G_{i+1} \) are finite groups with uniformly bounded orders. The Grigorchuk group \( G \) is of finite width, namely, \( \dim_{F_2} G_i/G_{i+1} \in \{1, 2\} \) for \( i \geq 2 \) [7].
In particular, the respective Lie algebra $L = L_K(G) = \oplus_{i \geq 1} L_i$ has a linear growth. Bartholdi presented $L_K(G)$ as a self-similar restricted Lie algebra and proved that the restricted Lie algebra $L_K(G)$ is nil while $L_K(G)$ is nil graded, namely, for any homogeneous element $x \in L_i, i \geq 1$, the mapping $\text{ad} x$ is nilpotent, because the group $G$ is periodic. Lie algebras of finite width over a field of positive characteristic and possibility of their classification under some additional conditions are discussed in [39, 40].

Infinite dimensional $\mathbb{N}$-graded Lie algebras $L = \bigoplus_{n=1}^{\infty} L_n$ with one-dimensional components in characteristic zero were classified by Fialowski [11].

The first author constructed an analogue of the Grigorchuk group in case of restricted Lie algebras of characteristic 2 [28]. Shestakov and Zelmanov extended this construction to an arbitrary positive characteristic [43]. Thus, we have examples of finitely generated restricted Lie algebras with a nil $p$-mapping. See further constructions in [32, 34, 35]. A family of restricted Lie algebras of slow growth with a nil $p$-mapping was constructed in [30], in particular, we construct a continuum subfamily of such algebras with Gelfand-Kirillov dimension one but their growth is not linear.

Since the Grigorchuk group is of finite width, probably, a "right analogue" of it should be a Lie algebra of finite width having ad-nil elements, in the next result the components are of bounded dimension and consist of ad-nil elements. Informally speaking, there are no "natural analogues" of the Grigorchuk and Gupta-Sidki groups in the world of Lie algebras of characteristic zero, we say this strictly in terms of the following result.

**Theorem 4.1** (Martinez and Zelmanov [25]). Let $L = \oplus_{\alpha \in \Gamma} L_{\alpha}$ be a Lie algebra over a field $K$ of characteristic zero graded by an abelian group $\Gamma$. Suppose that

i) there exists $d > 0$ such that $\dim_K L_\alpha \leq d$ for all $\alpha \in \Gamma$;

ii) every homogeneous element $a \in L_\alpha, \alpha \in \Gamma$, is ad-nilpotent.

Then the Lie algebra $L$ is locally nilpotent.

But there are natural analogues of the Grigorchuk and Gupta-Sidki groups in the world of Lie superalgebras of an arbitrary characteristic [29], where two Lie superalgebras were constructed, see further examples [8, 36]. In all these examples (actually, four examples), $\text{ad} a$ is nilpotent, $a$ being an even or odd element with respect to the $\mathbb{Z}_2$-gradings as Lie superalgebras. This property is an analogue of the periodicity of the Grigorchuk and Gupta-Sidki groups. The second Lie superalgebra $Q$ in [29] has a natural fine $\mathbb{Z}_2$-gradation with at most one-dimensional components. In particular, $Q$ is a nil finely $\mathbb{Z}_2$-graded Lie superalgebra, which shows that an extension of Theorem [11] for the Lie superalgebras of characteristic zero is not valid.

Strictly in terms of the next result, we say again that there are no "natural analogues" of the Grigorchuk and Gupta-Sidki groups in the class of Jordan algebras too.

**Theorem 4.2** (Zelmanov, private communication [45]). Jordan algebras satisfy a verbatim analogue of Theorem 4.1 over a field $K$, char $K \neq 2$.

On the other hand, the Jordan superalgebra $K$ constructed in [56] shows that an extension of Theorem 4.2 for the Jordan superalgebras in characteristic zero is not valid. Now, we provide a similar but "smaller" example, namely, a nil-graded Jordan superalgebra of finite width. These facts resemble those for Lie algebras and superalgebras mentioned above.

Both Lie superalgebras of [29] and the Lie superalgebra of [30] are of infinite width. We shall use a Lie superalgebra of finite width constructed in [8].

Let $\Lambda = \Lambda(x_i | i \geq 0)$ be the Grassmann algebra. The Grassmann letters and respective superderivatives $\{x_i, \partial_i | i \geq 0\}$ are odd elements of the superalgebra $\text{End} \Lambda$. Define so called pivot elements:

$$v_i = \partial_i + x_i x_{i+1} (\partial_{i+2} + x_{i+2} x_{i+3} (\partial_{i+4} + x_{i+4} x_{i+5} (\partial_{i+6} + \ldots ))) \in \text{Der} \Lambda,$$

$$i \geq 0.$$ (5)

Define the shift mappings:

$$\tau(x_i) = x_{i+1}, \quad \tau(\partial_i) = \partial_{i+1}, \quad \tau(v_i) = v_{i+1}, \quad i \geq 0.$$

We define the Lie superalgebra $R = \text{Lie}(v_0, v_1) \subset \text{Der} \Lambda$ and its associative hull $A = \text{Alg}(v_0, v_1) \subset \text{End} \Lambda$. We formulate their main properties, for more details see the original paper [8].

**Theorem 4.3** (O. de Morais Costa, V. Petrogradsky [8]). Consider the Lie superalgebra $R = \text{Lie}(v_0, v_1)$ and its associative hull $A = \text{Alg}(v_0, v_1)$, where char $K \neq 2$. Then

i) $R$ has a monomial basis consisting of standard monomials of two types.
ii) $\mathbb{R}$ and $\mathbb{A}$ are $\mathbb{Z}^2$-graded by multidegree in the generators. Also, $\mathbb{R}$ has the degree $\mathbb{N}$-gradation, which components are also the factors of the lower central series.

iii) We put the basis monomials of $\mathbb{R}$ and $\mathbb{A}$ on lattice points of plane $\mathbb{Z}^2 \subset \mathbb{R}^2$ using the multidegree. These monomials are in regions of plane bounded by pairs of logarithmic curves.

iv) The components of the $\mathbb{Z}^2$-grading of $\mathbb{R}$ are at most one-dimensional, thus, the $\mathbb{Z}^2$-grading of $\mathbb{R}$ is fine.

v) $\operatorname{GKdim} \mathbb{R} = \overline{\operatorname{GKdim}} \mathbb{R} = 1$, moreover, $\mathbb{R}$ has a linear growth and the growth function satisfies $\gamma_\mathbb{R}(m) \approx 3m$ as $m \to \infty$.

vi) Moreover, $\mathbb{R}$ is of finite width 4. Namely, let $\mathbb{R} = \bigoplus_{n=1}^\infty \mathbb{R}_n$ be the $\mathbb{N}$-grading by degree in the generators, where also $\mathbb{R}_n \cong \mathbb{R}^n/\mathbb{R}^{n+1}$, $n \geq 1$, are the lower central series factors. Then the coefficients $(\dim \mathbb{R}_n | n \geq 1)$, are $\{2, 3, 4\}$. This sequence is eventually non-periodic.

vii) $\overline{\operatorname{GKdim}} \mathbb{A} = \operatorname{GKdim} \mathbb{A} = 2$.

viii) Homogeneous elements of the grading $\mathbb{R} = \mathbb{R}_0 \oplus \mathbb{R}_1$ are ad-nilpotent.

ix) $\mathbb{R}$ is just infinite but not hereditary just infinite.

x) $\mathbb{R}$ again shows that an extension of Theorem 4.2 of Zelmanov [25] to the Lie superalgebras of characteristic zero is not valid.

Remark 1. The first counterexample of a nil finely $\mathbb{Z}^3$-graded Lie superalgebra of slow polynomial growth in any characteristic was suggested before (the second Lie superalgebra $\mathbb{Q}$ in [29]). The virtue of the nil finely $\mathbb{Z}^2$-graded Lie superalgebra $\mathbb{R}$ above is that it is of linear growth, moreover, of finite width 4, and just infinite. Claim (vi) is analogous to the fact that the Grigorchuk group is of finite width [37, 7]. Thus, $\mathbb{R}$ is a "more appropriate" analogue of the Grigorchuk group than the second Lie superalgebra $\mathbb{Q}$ of [29] or the Lie superalgebra considered recently in [36], both being of infinite width.

Actually, the non-periodicity of the sequence above (claim (vi)) was proved in case $p = 2$ [8], but the proof of non-periodicity for other characteristics is the same.

Now, we construct the following Jordan superalgebra and describe its properties.

**Theorem 4.4.** Let $\operatorname{char} K \neq 2$ and $\mathbb{R} = \operatorname{Lie}(v_0, v_1)$ be the Lie superalgebra of Theorem 4.3 Consider the Jordan superalgebra $\mathbb{J} = J_{\mathbb{R}}(\mathbb{R})$ and its subalgebra without unit $\mathbb{J}^o$. They have the following properties.

i) $\mathbb{J}$ is $\mathbb{Z}^3$-graded by multidegree in $\mathbb{X} = \{v_0, v_1, 1\}$.

ii) We put monomials of $\mathbb{J}$ at lattice points of plane $\mathbb{Z}^2 \subset \mathbb{R}^2$ using the partial multidegree in $\{v_0, v_1\}$. These monomials are bounded by a pair of logarithmic curves.

iii) The components of the $\mathbb{Z}^3$-grading of $\mathbb{J}$ are at most one-dimensional.

iv) $\operatorname{GKdim} \mathbb{J} = \overline{\operatorname{GKdim}} \mathbb{J} = 1$, moreover, $\mathbb{J}$ is of linear growth and $\gamma_{\mathbb{J}}(\mathbb{X}, m) \approx 2m$, as $m \to \infty$.

v) Let $\mathbb{J} = \bigoplus_{n=0}^\infty \mathbb{J}_n$ be the $\mathbb{N}_0$-grading by degree in $\mathbb{X}$. Then $\mathbb{J}$ is of finite width 4, and the coefficients $(\dim \mathbb{J}_n | n \geq 1)$ are $\{0, 2, 3, 4\}$, where the trivial components are $\mathbb{J}_{3m} = 0$, $m \geq 1$. This sequence is eventually non-periodic.

vi) $\mathbb{J}$ is just infinite but not hereditary just infinite.

vii) $(a^2)^2 = 0$ for any $a \in \mathbb{J}^o$.

viii) Let $a \in \mathbb{J}^o$ and $a \in J_{m,k}$, $m, n, k \geq 0$. Then $a^2 a = aa^2 = 0$ (i.e. $\mathbb{J}^o$ is nil $\mathbb{Z}^3$-graded).

ix) $\mathbb{J}$ again shows that an extension of Theorem 4.2 of Zelmanov [15] to the Jordan superalgebras of characteristic zero is not valid.

Proof. Almost all statements follow from the properties of $\mathbb{R}$ described in Theorem 4.3 by applying Lemma 2.1, Lemma 2.2, and Corollary 2.2. The fact that $\mathbb{J}$ is not hereditary just infinite is proved as the same property of the second Jordan superalgebra $\mathbb{K}$ in [36, Theorem 13.4]. Let $a \in J_{m,k}$, recall that the latter component is one-dimensional. If $a \in L$ then $a^2 = 0$. Let $a \in L$, then the square is again zero except the pivot elements, namely, let $a = \tilde{v}_n$ then $a^2 = \tilde{v}_n \cdot \tilde{v}_n = \{v_n, v_n\} = x_{n+1}v_{n+2} \in L$ (see [S]) and $a^2 a = aa^2 = 0$. □

Remark 2. A similar example of a just-infinite nil finely $\mathbb{Z}^4$-graded Jordan superalgebra of slow polynomial growth was suggested before, see the second Jordan superalgebra $\mathbb{K}$ in [36]. But, the present example is a more "appropriate analogue" of the Grigorchuk group, because the Jordan superalgebra $\mathbb{J}$ is of linear growth, moreover, of finite width 4. This property resembles the finite width of the Grigorchuk group.
Remark 3. The example J above shows that just infinite Z-graded Jordan superalgebras of finite width can have a complicated structure unlike the classification of such simple algebras over an algebraically closed field of characteristic zero [19].

5. ON SELF-SIMILARITY OF SUPERALGEBRAS

5.1. Self-similarity of Lie superalgebras. We say that an algebra is fractal provided that it contains infinitely many copies of itself. In this section we discuss the notion of self-similarity for our superalgebras. The notion of self-similarity plays an important role in group theory [14, 26]. The Fibonacci Lie algebra introduced by the first author is "self-similar" [32] but not in terms of the definition of self-similarity given by Bartholdi [6]. Namely, a Lie algebra L is called self-similar if it affords a homomorphism [6]:

\[ \psi : L \to \text{Der} A \rtimes (A \otimes L), \]

where A is a commutative algebra, Der A the Lie algebra of derivations, the product of the right hand side is defined via the natural action of Der A on A. The first author constructed a family of two-generated restricted Lie algebras with a nil p-mapping determined by two infinite sequences, if these sequences are periodic we get self-similar restricted Lie algebras [30]. Recently, self-similar Lie algebras are studied in [12].

This definition easily extends to Lie superalgebras by setting A to be a supercommutative associative superalgebra and Der A the Lie superalgebra of superderivations. We have two original examples of ad-nil self-similar Lie superalgebras of slow polynomial growth over an arbitrary field [29].

Recall the construction of the second Lie superalgebra of [29]. Let char K ≠ 2 and Λ = A[x_i, y_i, z_i | i ≥ 0] the Grassmann superalgebra, denote the respective partial superderivatives as \{\partial_{x_i}, \partial_{y_i}, \partial_{z_i} | i ≥ 0\}. Define series of elements, called the pivot elements, of the Lie superalgebra of superderivations Der Λ:

\[
\begin{align*}
 a_i &= \partial_{x_i} + y_i x_i (\partial_{x_{i+1}} + y_{i+1} x_{i+1} (\partial_{x_{i+2}} + y_{i+2} x_{i+2} (\partial_{x_{i+3}} + \cdots))) , \\
 b_i &= \partial_{y_i} + z_i y_i (\partial_{y_{i+1}} + z_{i+1} y_{i+1} (\partial_{y_{i+2}} + z_{i+2} y_{i+2} (\partial_{y_{i+3}} + \cdots))) , \\
 c_i &= \partial_{z_i} + x_i z_i (\partial_{z_{i+1}} + x_{i+1} z_{i+1} (\partial_{z_{i+2}} + x_{i+2} z_{i+2} (\partial_{z_{i+3}} + \cdots))) ,
\end{align*}
\]

Define the shift mapping \( \tau : \Lambda \to \Lambda \) and its natural extensions to the elements defined above:

\[
\tau(x_1) = x_{i+1}, \quad \tau(y_1) = y_{i+1}, \quad \tau(z_1) = z_{i+1},
\]

\[
\tau(\partial_{x_i}) = \partial_{x_{i+1}}, \quad \tau(\partial_{y_i}) = \partial_{y_{i+1}}, \quad \tau(\partial_{z_i}) = \partial_{z_{i+1}}, \quad i ≥ 0.
\]

Define the Lie superalgebra \( Q = \text{Lie}(a_0, b_0, c_0) \subset \text{Der} \Lambda \) and its associative hull \( A = \text{Alg}(a_0, b_0, c_0) \subset \text{End} \Lambda \). For more details see the original paper [29]. We observe only the following.

**Lemma 5.1.** The Lie superalgebra \( Q = \text{Lie}(a_0, b_0, c_0) \) defined above is self-similar with a natural self-similarity embedding:

\[ \psi : Q \hookrightarrow \langle \partial_{x_0}, \partial_{y_0}, \partial_{z_0} \rangle_K \rtimes A[x_0, y_0, z_0] \otimes \tau(Q). \]

**Proof.** By (7) we obtain a recursive presentation:

\[
\begin{align*}
 a_0 &= \partial_{x_0} + y_0 x_0 \tau(a_0), \\
 b_0 &= \partial_{y_0} + z_0 y_0 \tau(b_0), \\
 c_0 &= \partial_{z_0} + x_0 z_0 \tau(c_0).
\end{align*}
\]

Observe that it is sufficient to find a desired presentation in the form (5) for the generators. Indeed, we extend presentation (5) to an arbitrary \( a \in Q = \text{Lie}(a_0, b_0, c_0) \), using that \( \tau : Q \to Q \) is a shift monomorphism. \( \square \)

**Conjecture 1.** Let \( R = \text{Lie}(v_0, v_1) \) be the Lie superalgebra, where \( v_0, v_1 \) are defined by [35] (i.e. the example of [8]). Then \( R \) is not self-similar. We conjecture that the Lie superalgebra of [35] is not self-similar as well.

Indeed, recall that \( R \) is fractal. But, a self-similarity embedding for \( R \) might look like:

\[ \psi : R \hookrightarrow \langle \partial_{v_1} \rangle_K \rtimes \Lambda(x_0) \otimes \tau(R). \]

Recall that \( R \) is generated by \{\( v_0, v_1 \)\}, where the second generator is of the required form \( v_1 = \tau(v_0) \in \tau(R) \). By (7), one has \( v_0 = \partial_0 + x_0 \cdot x_1 v_2 \) where \( x_1 v_2 \notin \tau(R) \). Indeed, \( x_1 v_2 = \tau(x_0 v_1) \) where \( x_0 v_1 \notin R \) (see [8]). It seems that we cannot split our variables in any similar way.
5.2. **Self-similarity of associative superalgebras.** Similar to the case of associative algebras \([5, 14, 33]\), we say that an associative superalgebra \(A\) is **self-similar** provided that there exists a **self-similarity embedding**:

\[
\psi : A \hookrightarrow M(n|m) \otimes A,
\]

for some matrix superalgebra \(M(n|m)\), the tensor product being that of associative superalgebras.

**Lemma 5.2.** The associative superalgebra \(A = \text{Alg}(a_0, b_0, c_0)\) defined above \(7\) is self-similar with a self-similarity embedding:

\[
\psi : A \hookrightarrow M(4|4) \otimes A.
\]

**Proof.** As above, we use presentation \(5\) and get desired presentations for all elements of \(A\). Observe that \(\text{Alg}(\partial_{x_0}, x_0) \cong M(1|1)\), where \(M(1|1)\) is the superalgebra of \(2 \times 2\)-matrices, even part consists of diagonal matrices, and odd part consists of off-diagonal elements. Next, we use that \(\text{Alg}(\partial_{x_0}, x_0, \partial_{y_0}, y_0, \partial_{z_0}, z_0) \cong M(1|1) \otimes M(1|1) \otimes M(1|1) \cong M(4|4)\). \(\square\)

**Conjecture 2.** Consider the associative superalgebra \(A = \text{Alg}(v_0, v_1)\) corresponding to the Lie superalgebra considered above \(R = \text{Lie}(v_0, v_1)\), where \(v_0, v_1\) are defined by \(5\) (i.e. the example of \(3\)). Then \(A\) is not self-similar. We conjecture that the respective associative superalgebra of \(36\) is not self-similar as well.

5.3. **Self-similarity of Poisson superalgebras.** Let us call a Poisson superalgebra \(P\) **self-similar** if there exist a Poisson superalgebra \(H\) and an embedding:

\[
\psi : P \hookrightarrow H \otimes P,
\]

the tensor product being that of Poisson superalgebras.

Following constructions of \(36\), let us suggest a Poisson superalgebra related to \(Q\) above (this was not done in the original paper \(23\), we introduced the Poisson and Jordan superalgebras in case of another example in \(36\)). Consider the Grassmann superalgebra \(H = \Lambda[x_i, y_i, z_i, X_i, Y_i, Z_i | i \geq 0]\) and supply it with the Poisson bracket, which nontrivial products are \(\{X_i, x_i\} = 1, \{Y_i, y_i\} = 1, \{Z_i, z_i\} = 1\) for all \(i \geq 0\). Thus, \(H\) is turned into a Poisson superalgebra. We formally substitute big letters instead of respective derivatives in \(7\):

\[
\begin{align*}
A_i &= x_i + y_i x_i (x_{i+1} + y_{i+1} x_{i+1} (x_{i+2} + y_{i+2} x_{i+2} (x_{i+3} + \cdots))), \\
B_i &= y_i + z_i y_i (y_{i+1} + z_{i+1} y_{i+1} (y_{i+2} + z_{i+2} y_{i+2} (y_{i+3} + \cdots))), \\
C_i &= Z_i + x_i z_i (Z_{i+1} + x_{i+1} z_{i+1} (Z_{i+2} + x_{i+2} z_{i+2} (Z_{i+3} + \cdots))).
\end{align*}
\]

We define the shift endomorphism \(\tau : H \to H\) as above. Actually, we need to consider our elements in a completion of the Poisson superalgebra \(H\), see \(36\). Now we define a Poisson superalgebra \(P = \text{Poisson}(A_0, B_0, C_0)\).

**Lemma 5.3.** The Poisson superalgebra \(P = \text{Poisson}(A_0, B_0, C_0)\) is self-similar with the self-similarity embedding:

\[
P \hookrightarrow H_3 \otimes \tau(P).
\]

**Proof.** By \(11\) we obtain a recursive presentation:

\[
\begin{align*}
A_0 &= x_0 + y_0 x_0 \tau(A_0), \\
B_0 &= y_0 + z_0 y_0 \tau(B_0), \\
C_0 &= Z_0 + x_0 z_0 \tau(C_0).
\end{align*}
\]

Also, we observe that \(\text{Poisson}(x_0, y_0, z_0, X_0, Y_0, Z_0) \cong H_3\), see \(11\). \(\square\)

**Remark 4.** Consider a "small" Poisson superalgebra related to \(Q\) defined by \(2\), namely, \(P(Q) = \langle 1 \rangle \otimes Q\). This algebra is fractal. But, it seems that it is not self-similar according to our definition. Namely, there is a problem with a homomorphism for the associative product.
5.4. Self-similarity and wreath products of Jordan superalgebras. We have the notion of self-similarity for Lie superalgebras \([8]\) (a modification of that of Bartholdi \([9]\)), associative superalgebras \([9]\), and Poisson superalgebras \([10]\). But we lack a similar notion of self-similarity for Jordan superalgebras.

We start with an observation. Let \(P\) be a Poisson superalgebra and \(J = \text{Kan}(P) = P \oplus \bar{P}\) its Kantor double, which is a Jordan superalgebra. Define a mapping

\[
D : J \to J, \quad D(a) = 0, \quad a \in P, \quad D(\bar{a}) = (-1)^{|a|}a, \quad \bar{a} \in \bar{P}.
\]

One checks that \(D\) is an odd superderivation of \(J\) and \(D^2 = 0\).

Assume that we have a self-similar Poisson superalgebra \(P\) with an embedding \([10]\) \(\psi : P \hookrightarrow H \otimes P_1\), where \(H\) is a Poisson superalgebra, and \(P_1 \cong P\). Denote the Kantor double \(J_1 = \text{Kan}(P_1)\). We apply the Kantor double to both algebras in the embedding \(\psi\) above

\[
J = \text{Kan}(P) = P \oplus \bar{P} \hookrightarrow \text{Kan}(H \otimes P_1) = H \otimes P_1 \oplus \overline{H \otimes P_1}
\]

\[
\cong H \otimes (P_1 \oplus P_1) \cong H \otimes \text{Kan}(P_1) = H \otimes J_1,
\]

where the isomorphisms in the last line are of vector spaces. Let us express the product \(\bullet\) of the last space, which is identified with the Jordan superalgebra \(\text{Kan}(H \otimes P_1)\), in terms of the Jordan product \(\circ\) of \(J_1 = \text{Kan}(P_1)\) and two products \((\cdot, \{\cdot, \cdot\})\) of the Poisson superalgebra \(H\). Recall that the product \(\bullet\) on \(\text{Kan}(H \otimes P_1)\) was defined by Kantor’s construction. Take homogeneous \(x, y \in H\) and \(a, b \in P_1\). Consider four cases, where the unspecified products are the associative supercommutative products of the Poisson superalgebra \(H \otimes P_1\)

\[
xa \bullet yb = xayb = (-1)^{|a||y|}(xy)(ab) = (-1)^{|a||y|}(x \cdot y)(a \circ b);
\]

\[
xa \bullet yb = xay\bar{b} = (-1)^{|a||y|}(xy)(\bar{ab}) = (-1)^{|a||y|}(x \cdot y)(a \circ \bar{b});
\]

\[
x\bar{a} \bullet yb = (-1)^{|b||a|}x\bar{a}y\bar{b} = (-1)^{|a||y|+|b|+|a||y|}(xy)(\bar{ab}) = (-1)^{|a||y|+|a||y|}(x \cdot y)(\bar{a} \circ b);
\]

\[
x\bar{a} \bullet yb = (-1)^{|b||a|}\{xa, yb\} = (-1)^{|y||b|+|a||y|}(xy\{a, b\} + \{x, y\}ab)
\]

\[
= (-1)^{|y||b|+|a||y|} \left( (-1)^{|b|}(x \cdot y)(\bar{a} \circ \bar{b}) + (-1)^{|a||b|}\{x, y\}(D(\bar{a}) \circ D(\bar{b})) \right)
\]

\[
= (-1)^{|b||y|} \left( (x \cdot y)(\bar{a} \circ \bar{b}) + (-1)^{|a||y|+1}\{x, y\}(D(\bar{a}) \circ D(\bar{b})) \right).
\]

Now let \(f, g \in J_1 = \text{Kan}(P_1)\), belonging to either \(P_1\) or \(\bar{P}_1\) and \(x, y \in H\). We combine four cases above:

\[
(x \otimes f) \bullet (y \otimes g) = (-1)^{|f||y|} \left( (x \cdot y \otimes f \circ g + (-1)^{|f|+1}\{x, y\} \otimes D(f) \circ D(g) \right).
\]

By these arguments we have the following statement.

**Lemma 5.4.** Let \(P\) be a self-similar Poisson superalgebra, i.e. there exist a Poisson superalgebra \(H\), whose products being \((\cdot, \{\cdot, \cdot\})\), and a self-similarity embedding \([10]\). Then the Kantor double \(J = \text{Kan}(P)\), which is a Jordan superalgebra with a product \(\circ\), enjoys the self-similarity embedding

\[
J \hookrightarrow H \otimes J,
\]

where the right hand side is a Jordan superalgebra that product \(\bullet\) satisfies \([13]\), and the superderivative \(D : J \to J\) was defined above.

**Corollary 5.5.** Consider the Poisson superalgebra \(P = \text{Poisson}(A_0, B_0, C_0)\) related to \(Q\) as above. Let \(J = \text{Kan}(P)\) be its Kantor double. Then the Jordan superalgebra \(J\) has a self-similarity embedding

\[
J \hookrightarrow H_3 \otimes J.
\]

The notion of the wreath product plays an important role in group theory \([22, 26]\). Analogous notion of a wreath product was defined for arbitrary two Lie algebras \([31]\), see also \([8, 12]\). Similarly, the notion of the wreath products of associative algebras has many applications, see a recent paper \([1]\). The observations above allow us to suggest that there exists a notion of a wreath product of a Jordan superalgebra with a Poisson superalgebra as follows.
Conjecture 3. Let $J_1$ be a Jordan superalgebra with the product $\circ$, $D : J_1 \to J_1$ an odd superderivative such that $D^2 = 0$. Let $H$ be a Poisson superalgebra with products $(\cdot, \{, \})$. Supply $J = H \otimes J_1$ with the product $[\cdot, \{, \}]$. Is it true that $J$ is a Jordan superalgebra? In this case, $J$ should be called the wreath product of $J_1$ with $H$.

On the other hand consider a "small" Jordan superalgebra related to the Lie superalgebra $Q$ above. Namely, by (3) set $K = \text{For}(Q) = (1) \oplus Q \oplus (\bar{1}) \oplus Q$. Then it seems that $K$ does not have a self-similarity embedding. On the other hand, $K$ is fractal.

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