QKZ-Ruijsenaars correspondence revisited

A. Zabrodin*  A. Zotov†

April 2017

Abstract

We discuss the Matsuo-Cherednik type correspondence between the quantum Knizhnik-Zamolodchikov equations associated with $GL(N)$ and the $n$-particle quantum Ruijsenaars model, with $n$ being not necessarily equal to $N$. The quasiclassical limit of this construction yields the quantum-classical correspondence between the quantum spin chains and the classical Ruijsenaars models.

1 Introduction

The quantum Knizhnik-Zamolodchikov (qKZ) equations [8] is a system of holonomic difference equations

$$e^{\eta \hbar \partial_x} |\Phi\rangle = K_i^{(h)} |\Phi\rangle, \quad i = 1, \ldots, n$$

(1)

for the vector $|\Phi\rangle = |\Phi\rangle (x_1, \ldots, x_n)$ belonging to the tensor product $\mathcal{V} = V \otimes V \otimes \ldots \otimes V = V^\otimes n$ of the vector spaces $V = \mathbb{C}^N$. The operator $K_i^{(h)}$ in the right hand side is constructed as a chain product of quantum $R$-matrices:

$$K_i^{(h)} = R_{i,i-1}(x_i-x_{i-1}+\eta \hbar) \ldots R_{i,1}(x_i-x_1+\eta \hbar) g^{(i)} R_{n,i}(x_i-x_n) \ldots R_{i,i+1}(x_i-x_{i+1}).$$

(2)

Here $R_{ij}(x)$ is the $R$-matrix acting in the $i$-th and $j$-th tensor factors (it has to satisfy the unitarity condition $R_{ij}(x)R_{ji}(-x) = \text{id}$), $g = \text{diag}(g_1, \ldots, g_N)$ is a diagonal $N \times N$ matrix and $g^{(i)}$ is the operator in $\mathcal{V}$ acting as $g$ on the $i$-th factor (and identically on all other factors). For example, the rational $R$-matrix is of the form

$$R_{ij}(x) = \frac{x I + \eta P_{ij}}{x + \eta},$$

(3)
where $x$ is the spectral parameter, $\mathbf{I}$ is the identity operator and $\mathbf{P}_{ij}$ is the permutation of the $i$-th and $j$-th tensor factors. Compatibility of the qKZ equations follows from the Yang-Baxter equation for the $R$-matrix and from the commutativity $[\mathfrak{g} \otimes \mathfrak{g}, R(x)] = 0$.

The remarkable correspondence of the qKZ equations with the Macdonald type difference operator

$$\hat{\mathcal{H}} = \sum_{i=1}^{n} \left( \prod_{j \neq i} \frac{x_i - x_j + \eta}{x_i - x_j} \right) e^{\eta \hbar \partial x_i}$$  \hspace{1cm} (4)$$

was discussed in [5, 11, 14, 17] in the case $N = n$. In this case solutions to (1) can be found in the form

$$\begin{align*}
|\Phi\rangle &= \sum_{\sigma \in S_n} \Phi_{\sigma} |e_{\sigma}\rangle, & |e_{\sigma}\rangle &= e_{\sigma(1)} \otimes e_{\sigma(2)} \otimes \ldots \otimes e_{\sigma(n)},
\end{align*}$$

where $e_a$ are standard basis vectors in $V = \mathbb{C}^N = \bigoplus_{a=1}^{N} \mathbb{C} e_a$ and $S_n$ is the symmetric group. If such $|\Phi\rangle$ is a solution to the qKZ equations, then the function $\Psi = \sum_{\sigma \in S_n} \Phi_{\sigma}$ solves the spectral problem for the operator $\hat{\mathcal{H}}$:

$$\sum_{i=1}^{n} \prod_{j \neq i} \frac{x_i - x_j + \eta}{x_i - x_j} \Psi(x_1, \ldots, x_i + \eta \hbar, \ldots, x_n) = E \Psi(x_1, \ldots, x_n), \quad E = \sum_{a=1}^{N} g_a.$$  \hspace{1cm} (5)

This is referred to as the Matsuo-Cherednik type correspondence [13, 4]. A similar correspondence holds true for qKZ equations with trigonometric $R$-matrices.

The operator $\hat{\mathcal{H}}$ is essentially the Hamiltonian of the quantum Ruijsenaars model [15] which is a relativistic version of the Calogero model (more precisely, the operator $\hat{\mathcal{H}}$ and the Ruijsenaars Hamiltonian are connected by a similarity transformation, see [10]). The parameter $\eta$ plays the role of the inverse velocity of light.

We will give a simple proof of the correspondence valid also in the case when $n$ is not necessarily equal to $N = \text{dim } V$. In this form it looks like a quantum deformation of the quantum-classical correspondence [11, 2, 9, 21, 3] between the quantum XXX or XXZ spin chains and the classical Ruijsenaars models.

The spectral problem for the spin chain appears as a “quasiclassical” limit of qKZ as $\hbar \to 0$. Indeed, as $\hbar \to 0$ the qKZ solutions have the asymptotic form [18, 19]

$$|\Phi\rangle = \left( |\phi_0\rangle + \hbar |\phi_1\rangle + \ldots \right) e^{S/\hbar},$$

where $S$ is some scalar function. Upon substitution to the qKZ equations (1), this leads, in the leading order, to the joint eigenvalue problems

$$K_{i}^{(0)} |\phi_0\rangle = e^{\eta p_i} |\phi_0\rangle, \quad p_i = \frac{\partial S}{\partial x_i}, \quad i = 1, \ldots, n,$$  \hspace{1cm} (6)

for the commuting operators $K_{i}^{(0)}$. They are Hamiltonians of the inhomogeneous quantum spin chain. They can be diagonalized using the algebraic Bethe ansatz. In the quantum-classical correspondence, the $p_i$’s are identified with momenta of the Ruijsenaars
particles. The parameter $\hbar$ becomes the true Planck constant after quantization of the corresponding Ruijsenaars model.

The paper is organized as follows. In section 2, we review the inhomogeneous XXX spin chain and the associated qKZ equations. In section 3 the qKZ-Ruijsenaars correspondence is established by a simple direct calculation. In section 4 we extend the result to higher Ruijsenaars Hamiltonians. Section 5 is devoted to the qKZ equations with trigonometric $R$-matrices. Finally, in section 6 we discuss the interpretation of the results as a “quantum” deformation of the quantum-classical correspondence. The link to spin chains and their solutions by means of the algebraic Bethe ansatz appears naturally because of the usage of $R$-matrices in fundamental representation. It is alternative to the approach based on the affine Hecke algebras $[4, 5, 6]$, where similar results were originally presented. Explicit relationship between these two approaches deserves further consideration.

2 The XXX spin chain and qKZ equations

Let $e_{ab}$ be the standard basis in the space of $N \times N$ matrices: the matrix $e_{ab}$ has only one non-zero element (equal to 1) at the place $ab$: $(e_{ab})_{a'b'} = \delta_{aa'}\delta_{bb'}$. Note that $I = \sum_a e_{aa}$ is the unity operator and $P = \sum_{ab} e_{ab} \otimes e_{ba}$ is the permutation operator in the space $\mathbb{C}^N \otimes \mathbb{C}^N$. We embed $e_{ab}$ into $\text{End}(V^\otimes n)$ in the usual way: $e_{ab} \mapsto I^\otimes(n-i) \otimes e_{ab} \otimes I^\otimes(n-i)$. It is clear that $e_{ab}$, $e_{a'b'}$ commute for any $i \neq j$ because they act non-trivially in different spaces. Similarly, for any matrix $g \in \text{End}(\mathbb{C}^N)$ we define $g(i)$ acting in the tensor product $V^\otimes n$: $g(i) = I^\otimes(i-1) \otimes g \otimes I^\otimes(n-i) \in \text{End}(V)$. In this notation, the permutation operator of the $i$-th and $j$-th tensor factors in $V = \mathbb{C}^N \otimes \ldots \otimes \mathbb{C}^N$ is $P_{ij} = \sum_{a,b} e_{ab}^{(i)} e_{ba}^{(j)}$.

Let $g \in GL(N)$ be a diagonal matrix $g = \text{diag}(g_1, g_2, \ldots, g_N) = \sum_{a=1}^N g_a e_{aa}$. We call it the twist matrix with twist parameters $g_a$. It is used for the construction of an integrable spin chain with twisted boundary conditions. Together with the $R$-matrix $[3]$, we introduce another rational $R$-matrix which differs from the $R(x)$ by a scalar factor:

$$
\tilde{R}(x) = \frac{x + \eta}{x} R(x) = I + \frac{\eta}{x} P.
$$

(7)

It obeys the same Yang-Baxter equation and commutes with $g \otimes g$. The transfer matrix of the inhomogeneous spin chain (or, equivalently, of the associated statistical vertex model on the 2D lattice) is defined in the standard way as a trace of the chain product of $R$-matrices in the auxiliary space $V = \mathbb{C}^N$ with index 0:

$$
T(x) = \text{tr}_0(\tilde{R}_{0n}(x-x_n) \ldots \tilde{R}_{02}(x-x_2) \tilde{R}_{01}(x-x_1)(g \otimes I)),
$$

(8)

where $x_1, x_2, \ldots, x_n$ are inhomogeneity parameters. (We assume that they are in general position meaning that $x_i \neq x_j$ and $x_i \neq x_j \pm \eta$ for all $i \neq j$.) As is known, the Yang-Baxter equation for the $R$-matrix implies that the transfer matrices with fixed inhomogeneity and twist parameters commute: $[T(x), T(x')] = 0$.

The dynamical variables of the model (we call them “spins” in analogy with the rank 1 case) are vectors in the vector representation of $GL(N)$ realized in the spaces $V = \mathbb{C}^N$
attached to each site. Non-local commuting Hamiltonians $H_j$ are defined as residues of $T(x)$ at $x = x_j$:

$$T(x) = \text{tr} \, g \cdot I + \sum_{j=1}^{n} \frac{\eta H_j}{x - x_j}. \quad (9)$$

From (8) it follows that their explicit form is

$$H_i = \tilde{R}_{i,i-1}(x_i - x_{i-1}) \cdots \tilde{R}_{i1}(x_i - x_1)g^{(i)}(\tilde{R}_{in}(x_i - x_n) \cdots \tilde{R}_{i,i+1}(x_i - x_{i+1}). \quad (10)$$

We obviously have

$$H_i = K_i^{(0)} \prod_{j \neq i} \frac{x_i - x_j + \eta}{x_i - x_j}, \quad (11)$$

where $K_i^{(0)}$ is the operator [2] at $\hbar = 0$.

Let us introduce the operators

$$M_a = \sum_{l=1}^{n} \epsilon_{aa}^{(l)}. \quad (12)$$

They commute among themselves and with the Hamiltonians: $[M_a, M_b] = [H_i, M_a] = 0$. We call them weight operators. Clearly, $\sum_a M_a = nI$. Comparing the expansion of (8) as $x \to \infty$,

$$T(x) = \text{tr}_0 \left[ (I + \frac{\eta P_{0n}}{x - x_n}) \cdots (I + \frac{\eta P_{01}}{x - x_1})g^{(0)} \right]$$

$$= \text{tr} \, g \cdot I + \frac{\eta}{x} \sum_{i=1}^{n} \text{tr}_0 \left( P_{0i}g^{(0)} \right) + \ldots$$

$$= \text{tr} \, g \cdot I + \frac{\eta}{x} \sum_{i=1}^{n} g^{(i)} + \ldots,$$

with that of (9), we conclude that

$$\sum_{i=1}^{n} H_i = \sum_{i=1}^{n} g^{(i)} = \sum_{a=1}^{N} \eta a M_a, \quad (13)$$

so the system has $n + N - 2$ independent integrals of motion. The joint spectral problem is

$$\left\{ \begin{array}{l}
H_i |\phi\rangle = H_i |\phi\rangle \\
M_a |\phi\rangle = M_a |\phi\rangle
\end{array} \right.$$

The common eigenstates of the Hamiltonians can be classified according to eigenvalues of the operators $M_a$.

Let

$$\mathcal{V} = V^\otimes n = \bigoplus_{M_1, \ldots, M_N} \mathcal{V} \{ \{ M_a \} \} \quad (14)$$

be the weight decomposition of the Hilbert space of the spin chain, $\mathcal{V}$, into the direct sum of eigenspaces for the operators $M_a$ with the eigenvalues $M_a \in \mathbb{Z}_{\geq 0}, a = 1, \ldots, N$. 
(recall that $M_1 + \ldots + M_N = n$). The common eigenstates of the operators $M_a$ and $H_i$ belong to the spaces $\mathcal{V}(\{M_a\})$. The dimension of $\mathcal{V}(\{M_a\})$ is

$$\dim \mathcal{V}(\{M_a\}) = \frac{n!}{M_1! \ldots M_N!}.$$ 

The basis vectors in $\mathcal{V}(\{M_a\})$ are $| J \rangle = e_{j_1} \otimes e_{j_2} \otimes \ldots \otimes e_{j_n}$, where the number of indices $j_k$ such that $j_k = a$ is equal to $M_a$ for all $a = 1, \ldots, N$. We also introduce dual vectors $\langle J |$ such that $\langle J | J' \rangle = \delta_{J,J'}$. Matrix elements of an operator $O$ are $\langle J | O | J' \rangle$.

Associated with the inhomogeneous XXX spin chain is the system of qKZ equations $e^{\eta h \partial_{x_i}} \Phi = K_i^{(h)} \Phi$ [7] with the operators $K_i^{(h)}$ given by [2]. The compatibility condition

$$\left( e^{\eta h \partial_{x_i}} K_j^{(h)} \right) K_i^{(h)} = \left( e^{\eta h \partial_{x_i}} K_i^{(h)} \right) K_j^{(h)}$$

follows from the Yang-Baxter equation for the $R$-matrix. The operators $K_i^{(h)}$ respect the weight decomposition [14], hence the solutions to the qKZ system belong to the weight subspaces $\mathcal{V}(\{M_a\})$.

### 3 The qKZ-Ruijsenaars correspondence in the rational case

Let $| \Phi \rangle = \sum_J \Phi_J | J \rangle$ be any solution of the qKZ equations belonging to the weight subspace $\mathcal{V}(\{M_a\})$. We claim that the function

$$\Psi = \sum_J \Phi_J$$

(15)

is an eigenfunction of the Macdonald operator $\hat{H}$ with the eigenvalue $E = \sum_{a=1}^N M_a g_a$:

$$\sum_{i=1}^n \prod_{j \neq i} \frac{x_i - x_j + \eta}{x_i - x_j} \Psi(x_1, \ldots, x_i + \eta h, \ldots, x_n) = E \Psi(x_1, \ldots, x_n).$$

(16)

For the proof we consider the covector $\langle \Omega |$ equal to the sum of all basis (dual) vectors:

$$\langle \Omega | = \sum_J \langle J |,$$

then $\Psi = \langle \Omega | \Phi \rangle$. It is important to note that $\langle \Omega | P_{ij} = \langle \Omega |$, and, therefore, $\langle \Omega | R_{ij}(x) = \langle \Omega |$. It then follows that $\langle \Omega | K_i^{(h)} = \langle \Omega | K_i^{(0)}$, so the projection of the $i$-th qKZ equation onto the covector $\langle \Omega |$ reads

$$e^{\eta h \partial_{x_i}} \langle \Omega | \Phi \rangle = e^{\eta h \partial_{x_i}} \Psi = \langle \Omega | K_i^{(h)} | \Phi \rangle = \langle \Omega | K_i^{(0)} | \Phi \rangle.$$
Therefore, multiplying by $\prod_{j \neq i}^{n} \frac{x_i - x_j + \eta}{x_i - x_j}$ and summing over $i$, we get:

$$\sum_{i=1}^{n} \left( \prod_{j \neq i}^{n} \frac{x_i - x_j + \eta}{x_i - x_j} \right) e^{\eta \partial_{x_i}} \Psi = \sum_{i=1}^{n} \prod_{j \neq i}^{n} \frac{x_i - x_j + \eta}{x_i - x_j} \left\langle \Omega \middle| K_i^{(0)} \right| \Phi \right\rangle$$

$$= \sum_{i=1}^{n} \left\langle \Omega \middle| H_i \right| \Phi \right\rangle = \sum_{i=1}^{n} \left\langle \Omega \middle| g^{(i)} \right| \Phi \right\rangle = \sum_{a=1}^{N} g_a \left\langle \Omega \middle| M_a \right| \Phi \right\rangle = \left( \sum_{a=1}^{N} g_a M_a \right) \Psi,$$

where we have used (11) and (13).

In the next section we show that $\Psi$ is the common eigenfunction for all higher Ruijsenaars Hamiltonians $\hat{H}_d$ with the eigenvalues $e_d(g_1, \ldots, g_1, \ldots g_N, \ldots, g_N)$, where $e_d$ is the elementary symmetric polynomial of $n$ variables.

## 4 Higher Hamiltonians

Here we show that $\Psi$ is an eigenfunction of the higher rational Macdonald-Ruijsenaars Hamiltonians $\hat{H}_d$ defined by

$$\hat{H}_d = \sum_{I \subset \{1, \ldots, n\}, |I| = d} \left( \prod_{s \in I, r \notin I} \frac{x_s - x_r + \eta}{x_s - x_r} \right) \prod_{i \in I} e^{\eta \partial_{x_i}} \tag{17}$$

or

$$\hat{H}_d = \sum_{1 \leq i_1 < \cdots < i_d \leq n} \prod_{k=1}^{d} \prod_{r \neq i_k}^{n} \frac{x_{i_k} - x_r + \eta}{x_{i_k} - x_r} \prod_{1 \leq \alpha < \beta \leq d} \left( 1 - \frac{\eta^2}{(x_{i_\alpha} - x_{i_\beta})^2} \right)^{-1} \prod_{k=1}^{d} e^{\eta \partial_{x_{i_k}}} \tag{18}$$

For example, for $d = 2$

$$\hat{H}_2 = \sum_{i < j}^{n} \left( \prod_{k \neq i, j}^{n} \frac{x_i - x_k + \eta}{x_i - x_k} \prod_{l \neq i, j}^{n} \frac{x_j - x_l + \eta}{x_j - x_l} \right) e^{\eta \partial_{x_i}} e^{\eta \partial_{x_j}} =$$

$$= \sum_{i < j}^{n} \left( \prod_{k \neq i}^{n} \frac{x_i - x_k + \eta}{x_i - x_k} \prod_{l \neq j}^{n} \frac{x_j - x_l + \eta}{x_j - x_l} \right) \left( 1 - \frac{\eta^2}{(x_i - x_j)^2} \right)^{-1} e^{\eta \partial_{x_i}} e^{\eta \partial_{x_j}}. \tag{19}$$

**Proposition 4.1** The operators $K_j^{(h)}$ from (4) and the wave function (13) $\Psi = \left\langle \Omega \right| \Phi \right\rangle$ satisfy

$$\prod_{s=1}^{d} e^{\eta \partial_{x_{i_s}}} \Psi = \left\langle \Omega \middle| K_{i_1}^{(0)} \ldots K_{i_d}^{(0)} \right| \Phi \right\rangle \text{ for } i_k \neq i_m. \tag{20}$$

For the proof we introduce the notation $R_{ab} = R_{ab}(x_a - x_b)$ and $R_{ab}^+ = R_{ab}(x_a - x_b + \eta \hbar)$, so that

$$K_j^{(h)} = R_{jj-1}^+ \ldots R_{jj-1}^+ g^{(j)} R_{jn} \ldots R_{jj+1}. $$
Consider first the case \( d = 2 \). For \( i < j \) we have
\[
e^{\eta \hbar \partial_{x_i}} e^{\eta \hbar \partial_{x_j}} \Psi = \left\langle \Omega \right| K^{(h)}_j (x_i + \eta \hbar) K^{(h)}_i | \Phi \rangle = \left\langle \Omega \right| R^+_{j-1} \cdots R^+_{j+i+1} R^+_j R^+_i \cdots R^+_j R^+_j \cdots R^+_j g^{(j)} R_{j+i} \cdots R_{j+i+1} | \Phi \rangle \tag{21}
\]

\[
\times \left( R^+_{i-1} \cdots R^+_i g^{(i)} R_{i-1} \cdots R_{i+i+1} | \Phi \rangle \right.
\]

The “underbraced” expressions in (21) consist of commuting products of \( R \)-matrices thanks to \( i < j \). Therefore, these expressions can be permuted. Using then the property \( \left\langle \Omega \right| R_{kl}(x) = \left\langle \Omega \right| \) we conclude that all shifts by \( \eta \hbar \) in the \( R \)-matrix arguments can be removed. Finally, we can permute the products of \( R \)-matrices coming back to the initial order (but with non-shifted arguments).

For \( d = 3 \) and \( i < j < k \) we have
\[
e^{\eta \hbar \partial_{x_i}} e^{\eta \hbar \partial_{x_j}} e^{\eta \hbar \partial_{x_k}} \Psi = \left\langle \Omega \right| K^{(h)}_k (x_i + \eta \hbar, x_j + \eta \hbar) K^{(h)}_j (x_i + \eta \hbar) K^{(h)}_i | \Phi \rangle = \left\langle \Omega \right| R^+_{k-1} \cdots R^+_{k+j+1} R^+_j R^+_i \cdots R^+_j R^+_j \cdots R^+_j g^{(k)} R_{k+i} \cdots R_{k+i+1}
\]
\[
\times \left( R^+_{j-1} \cdots R^+_{j+i+1} R^+_j R^+_i \cdots R^+_j R^+_j \cdots R^+_j g^{(j)} R_{j+i} \cdots R_{j+i+1} \right.
\]
\[
\times \left( R^+_{i-1} \cdots R^+_i g^{(i)} R_{i-1} \cdots R_{i+i+1} | \Phi \rangle \right)
\tag{22}
\]

Again, since \( i < j < k \) we see that “underbraced” expressions commute as well as the “overbraced” expressions. Permute first the “underbraced” expressions. Then we get
\[
e^{\eta \hbar \partial_{x_i}} e^{\eta \hbar \partial_{x_j}} e^{\eta \hbar \partial_{x_k}} \Psi = \left\langle \Omega \right| K^{(h)}_k (x_i + \eta \hbar, x_j + \eta \hbar) K^{(h)}_j (x_i + \eta \hbar) K^{(h)}_i | \Phi \rangle = \left\langle \Omega \right| R^+_{k-1} \cdots R^+_{k+j+1} R^+_j R^+_i \cdots R^+_j R^+_j \cdots R^+_j g^{(k)} R_{k+i} \cdots R_{k+i+1}
\]
\[
\times \left( R^+_{j-1} \cdots R^+_{j+i+1} R^+_j R^+_i \cdots R^+_j R^+_j \cdots R^+_j g^{(j)} R_{j+i} \cdots R_{j+i+1} \right.
\]
\[
\times \left( R^+_{i-1} \cdots R^+_i g^{(i)} R_{i-1} \cdots R_{i+i+1} | \Phi \rangle \right)
\tag{23}
\]

Now the “overbraced” expression from the second line of (23) commutes with the whole third line. By permuting them we get the product where all \( R \)-matrices with shifted arguments \( (R^+) \) are to the left from twist matrices and act on \( \left\langle \Omega \right| \) (to the left). Then we can apply the previously used reasoning and remove all the shifts of arguments.

It is easy to see that the same proof holds true for arbitrary \( d \). The choice of ordering \( i_1 < \ldots < i_d \) is convenient but the final answer is independent of it since \( [K^{(0)}_i, K^{(0)}_j] = 0 \) (because \( K^{(0)}_i \propto H_1 \)).
Multiplying both parts of (20) by the products transforming \( R(x) \) to \( \overline{R}(x) \) \((\text{7}), (\text{11})\), we get

\[
\prod_{k=1}^{d} \prod_{r \neq k}^{n} \frac{x_{ik} - x_{ir} + \eta}{x_{ik} - x_{ir}} \prod_{k=1}^{d} e^{\eta \frac{g_{ik}}{x_{ik}}} \Psi = \langle \Omega | H_{i_{1}} \ldots H_{i_{d}} | \Phi \rangle
\]  

(24)

A comparison between (18) and (24) shows that

\[
\hat{H}_{d} \Psi = \sum_{1 \leq i_{1} < \ldots < i_{d} \leq n} \langle \Omega | H_{i_{1}} \ldots H_{i_{d}} | \Phi \rangle \prod_{1 \leq \alpha < \beta \leq d} \left(1 - \frac{\eta^{2}}{(x_{i_{\alpha}} - x_{i_{\beta}})^{2}}\right)^{-1}
\]  

(25)

We are now in a position to use the following operator relation (see [20, eq. (4.2)]):

\[
\det_{1 \leq i, j \leq n} \left(z \delta_{ij} - \frac{\eta H_{i}}{x_{j} - x_{i} + \eta}\right) = \prod_{a=1}^{N} (z - g_{a})^{M_{a}}.
\]  

(26)

The both sides are polynomials in \( z \). Equating the coefficients in front of the powers of \( z \), we have

\[
\sum_{1 \leq i_{1} < \ldots < i_{d} \leq n} H_{i_{1}} \ldots H_{i_{d}} \prod_{1 \leq \alpha < \beta \leq d} \left(1 - \frac{\eta^{2}}{(x_{i_{\alpha}} - x_{i_{\beta}})^{2}}\right)^{-1} = e_{d}(\{P_{j}\}).
\]  

(27)

where \( e_{d}(\{P_{j}\}) \) are elementary symmetric functions defined by the generating function as

\[
\exp \left(- \sum_{k \geq 1} \frac{z^{k}}{k} P_{k} \right) = \sum_{d=0}^{n} (-1)^{d} z^{d} e_{d}(\{P_{j}\}),
\]

and \( P_{k} = \sum_{a} M_{a} g_{a}^{k} \). In particular, for \( k = 1, 2, 3 \)

\[
e_{1} = \sum_{a=1}^{N} M_{a} g_{a},
\]

\[
e_{2} = \frac{1}{2} \left(\sum_{a=1}^{N} M_{a} g_{a}\right)^{2} - \frac{1}{2} \sum_{a=1}^{N} M_{a} g_{a}^{2},
\]  

(28)

\[
e_{3} = \frac{1}{6} \left(\sum_{a=1}^{N} M_{a} g_{a}\right)^{3} - \frac{1}{2} \left(\sum_{a=1}^{N} M_{a} g_{a}^{2}\right) \left(\sum_{b=1}^{N} M_{b} g_{b}\right) + \frac{1}{3} \sum_{a=1}^{N} M_{a} g_{a}^{3}.
\]

The vector \( | \Phi \rangle \) is an eigenvector for these operators with the eigenvalues given by the same formulas with \( M_{a} \rightarrow M_{a} \). Therefore, plugging (27) into (25) we get \( \hat{H}_{d} \Psi = E_{d} \Psi \) with the eigenvalue

\[
E_{d} = e_{d}(\sum_{a} M_{a} g_{a}^{k})
\]  

(29)

It is easy to see that the right hand side is the elementary symmetric polynomial of \( n \) variables \( e_{d}(g_{1}, \ldots, g_{1}, \ldots, g_{N}, \ldots, g_{N}) \), with \( M_{1} \) variables \( g_{1}, \ldots, g_{1} \) and \( M_{N} \) variables \( g_{N}, \ldots, g_{N} \)
5 The qKZ-Ruijsenaars correspondence in the trigonometric case

The trigonometric (hyperbolic) analog of the $R$-matrix (3) which participates in the qKZ equations (1) reads

$$R(x) = \sum_{a=1}^{N} e_{aa} \otimes e_{aa} + \frac{\sinh x}{\sinh(x+\eta)} \sum_{a \neq b}^{n} e_{ab} \otimes e_{ba} + \frac{\sinh \eta}{\sinh(x+\eta)} \sum_{a<b}^{n} (e^x e_{ab} \otimes e_{ba} + e^{-x} e_{ba} \otimes e_{ab}).$$

(30)

(In this section we use the same notation for analogous objects in the trigonometric and rational cases.) After some algebra it can be represented in the form

$$R_{12}(x) = P_{12} + \frac{\sinh x}{\sinh(x+\eta)} \left(I - P_{q}^{12}\right),$$

(31)

where

$$P_{12} = \sum_{a=1}^{N} e_{aa} \otimes e_{aa} + q \sum_{a>b}^{n} e_{ab} \otimes e_{ba} + q^{-1} \sum_{a<b}^{n} e_{ab} \otimes e_{ba}, \quad q = e^{\eta},$$

is the $q$-permutation operator acting as follows:

$$P_{12} e_{a} \otimes e_{b} = \begin{cases} q e_{b} \otimes e_{a}, & a < b \\ q^{-1} e_{b} \otimes e_{a}, & a > b \\ e_{b} \otimes e_{a}, & a = b \end{cases}$$

(32)

Note that $P_{ij} = P_{j/i}^{1/q}$. The unitarity condition can be easily checked.

We also introduce the $R$-matrix

$$\tilde{R}_{12}(x) = \frac{\sinh(x+\eta)}{\sinh x} R_{12}(x) = I - P_{q}^{12} + \frac{\sinh(x+\eta)}{\sinh x} P_{12},$$

(33)

and the transfer matrix $T(x) = \text{tr}_{0} \left( \tilde{R}_{0n}(x-x_{n}) \ldots \tilde{R}_{01}(x-x_{1}) g^{(0)} \right)$ of the inhomogeneous XXZ spin chain with twisted boundary conditions. Similarly to the rational case, the commuting Hamiltonians are defined by the pole expansion

$$T(x) = C + \sinh \eta \sum_{k=1}^{n} H_{k} \coth(x-x_{k}), \quad H_{k} = (\sinh \eta)^{-1} \text{res}_{x=x_{k}} T(x).$$

They are expressed through the $R$-matrices by the same formula (10). We have (see [2]):

$$T(\pm \infty) = C \pm \sinh \eta \sum_{k}^{n} H_{k} = \sum_{a=1}^{N} g_{a} e^{\pm \eta M_{a}},$$

where $C$ is some operator and the weight operators $M_{a}$ are defined in the same way as before. Hence

$$\sum_{k=1}^{n} H_{k} = \sum_{a=1}^{N} g_{a} \frac{\sinh(\eta M_{a})}{\sinh \eta}.$$

(34)

Similarly to (11), we have:

$$H_{i} = K_{i}^{(0)} \prod_{j \neq i}^{n} \frac{\sinh(x_{i} - x_{j} + \eta)}{\sinh(x_{i} - x_{j})}.$$

(35)
As in the rational case, the trigonometric operators $K_i^{(h)}$ respect the weight decomposition \([14]\). Let $|\Phi\rangle = \sum_{J} \Phi_J |J\rangle$ be any solution of the trigonometric $q$KZ equations belonging to the weight subspace $\mathcal{V} \{M_a\}$. Let $\ell(J)$ be the minimal number of elementary permutations $\sigma_{i \rightarrow i+1} \in S_n$ which are required to get the multi-index $J = (j_1, j_2, \ldots, j_n)$ from the “minimal” one\(^1\), where the $j_k$’s are ordered as $1 \leq j_1 \leq j_2 \leq \ldots \leq j_n \leq N$. In the case $n = N$, $M_1 = M_2 = \ldots = M_N = 1$ the $\ell(J)$ is what is called length of the permutation $(12 \ldots N) \rightarrow (j_1 j_2 \ldots j_N)$. We claim that the function

$$
\Psi = \sum_{J} q^{\ell(J)} \Phi_J \quad (36)
$$

is an eigenfunction of the trigonometric Macdonald operator $\hat{\mathcal{H}}$ with the eigenvalue $E = \sum_{a=1}^{N} g_a \sinh(\eta M_a) \over \sinh \eta$:

$$
\sum_{i=1}^{n} \prod_{j \neq i} \sinh(x_i - x_j + \eta) \Psi(x_1, \ldots, x_i + \eta h, \ldots, x_n) = E\Psi(x_1, \ldots, x_n). \quad (37)
$$

The idea of the proof is the same as in the rational case. We consider the covector

$$
\langle \Omega_q | = \sum_{J} q^{\ell(J)} \langle J |,
$$

then $\Psi = \langle \Omega_q | \Phi \rangle$. It is not difficult to see that $\langle \Omega_q | P_{i,i-1}^q = \langle \Omega_q | E \rangle$ This implies the important relation

$$
\langle \Omega_q | R_{i,i-1}(x) = \langle \Omega_q | P_{i,i-1}, \quad i = 2, \ldots, n \quad (38)
$$

(the second term in \([31]\) disappears). Using the relation $P_{i,i-1} P_{i,i-2}^q = P_{i,i-2}^q P_{i,i-1}$, one can show by induction that

$$
\langle \Omega_q | R_{i,i-1}(x_i - x_{i-1} + \eta h) \ldots R_{i1}(x_i - x_1 + \eta h) = \langle \Omega_q | P_{i,i-1} \ldots P_{i1}
$$

for all $i = 2, \ldots, n$. Since the right hand side does not depend on the spectral parameters, we can substitute each $R$-matrix in the left hand side by the same one with $h = 0$ and conclude that $\langle \Omega_q | K_i^{(h)} = \langle \Omega_q | K_i^{(0)}$. The projection of the $i$-th $q$KZ equation onto the covector $\langle \Omega_q |$ reads

$$
e^{\eta h \partial_{x_i}} \langle \Omega_q | \Phi = e^{\eta h \partial_{x_i}} \Psi = \langle \Omega_q | K_i^{(h)} | \Phi \rangle = \langle \Omega_q | K_i^{(0)} | \Phi \rangle.$$

\[^1\]Put it differently, any $|J\rangle$ can be constructed from the minimal one $|J_{\min}\rangle = e_{1}^{\otimes M_1} \otimes \ldots \otimes e_{N}^{\otimes M_N}$ in $\ell(J)$ steps by applying elementary permutations of neighboring tensor components $\sigma_{k_i,k_i+1}$:

$$
\[J_{\min}\rangle = |J^{(0)}\rangle \sigma_{1,2} \sigma_{2,3} \ldots \sigma_{k_i,k_i+1} \ldots \sigma_{k_{\ell(J)}-1,k_{\ell(J)}} |J^{(\ell(J))}\rangle = |J\rangle,
$$

where $|J^{(i)}\rangle = e_{j_i}^{(i)} \otimes \ldots \otimes e_{j_n}^{(i)}$ and each time $j_{k_i}^{(i-1)} < j_{k_i+1}^{(i-1)}$.

\[^2\]Indeed, from \([32]\) and construction of $|J\rangle$ from $|J_{\min}\rangle$ one gets $P_{i,i-1}^q |\Omega_q\rangle = |\Omega_q\rangle$. The matrix transposition of this equality gives $\langle \Omega_q | = \langle \Omega_q | (P_{i,i-1}^q)^T = \langle \Omega_q | P_{i,i-1}^{1/q} = \langle \Omega_q | P_{i,i-1}^q$. 

10
In the same way as in the rational case, we multiply this by \( \prod_{j \neq i} \frac{\sinh(x_i - x_j + \eta)}{\sinh(x_i - x_j)} \) and sum over \( i \) to get:

\[
\sum_{i=1}^{n} \left( \prod_{j \neq i} \frac{\sinh(x_i - x_j + \eta)}{\sinh(x_i - x_j)} \right) e^{\eta \partial x_i} \Psi = \sum_{i=1}^{n} \prod_{j \neq i} \frac{\sinh(x_i - x_j + \eta)}{\sinh(x_i - x_j)} \langle \Omega_q | K_i^{(0)} | \Phi \rangle
\]

Here we have used (34) and (35).

Similarly to the rational case, the correspondence can be extended to the higher trigonometric Macdonald-Ruijsenaars operators.

6 Conclusion and discussion

We have established the correspondence between solutions to the qKZ equations (11) in different weight subspaces of \( V^\otimes n \) and solutions to the spectral problem for the \( n \)-body Ruijsenaars model with Planck’s constant \( \hbar \), with \( V \) being the space of \( N \)-dimensional vector representation of \( GL(N) \). The proof appears to be even simpler than in the case of the correspondence between the differential KZ equations and the quantum Calogero model (see [13, 4, 7] and [22]). In the limit \( \hbar \to 0 \) we obtain the quantum-classical correspondence [1, 9, 20] between the quantum spin chain (XXX or XXZ) and the classical Ruijsenaars system of particles [16] (rational or trigonometric).

The Hamiltonian of the classical Ruijsenaars system has the form

\[
H = \sum_{i=1}^{n} e^{\eta p_i} \prod_{j \neq i} \frac{x_i - x_j + \eta}{x_i - x_j}
\]

with the usual Poisson brackets \( \{ p_i, x_j \} = \delta_{ij} \) (for simplicity we consider the rational case). The model is integrable, with the Lax matrix

\[
L_{ij} = \frac{\dot{x}_j}{x_i - x_j + \eta},
\]

where

\[
\dot{x}_i = \frac{\partial H}{\partial p_i} = \eta e^{\eta p_i} \prod_{j \neq i} \frac{x_i - x_j + \eta}{x_i - x_j}
\]

is the velocity of the \( i \)-th particle. The higher Hamiltonians in involution are coefficients of the characteristic polynomial of the Lax matrix:

\[
\det(z \delta_{ij} - L_{ij}) = \sum_{d=0}^{n} (-1)^d z^{n-d} H_d, \quad H_1 = H
\]

The correspondence with the quantum spin chain goes as follows. Let the eigenvalues of the \( n \times n \) Lax matrix be the twist parameters \( g_a \) with multiplicities \( M_a \) (recall that...
$M_1 + \ldots + M_a = n$). This means that we consider the level set of all the classical Hamiltonians

$$\mathcal{H}_d = e_d(g_1, \ldots, g_1, \ldots g_N, \ldots, g_N) \quad d \geq 1, \quad M_a \in \mathbb{Z}_{\geq 0},$$

with fixed coordinates $x_i$. Then the admissible values of velocities, $\dot{x}_i$, are equal to $\eta H_i$, where the $H_i$'s are eigenvalues of the spin chain Hamiltonians $H_i$ in the weight subspace $\mathcal{V}(\{M_a\})$ for the model with the inhomogeneity parameters $x_i$ and the twist matrix $g = \text{diag}(g_1, \ldots, g_N)$. In fact the admissible values of $\dot{x}_i$'s obey a system of algebraic equations (see [20]). Different solutions of this system correspond to different eigenstates of the spin chain Hamiltonians.

In the trigonometric case eigenvalues of the Lax matrix $L_{ij}^{\text{trig}} = \frac{\dot{x}_j}{\sinh(x_i - x_j + \eta)}$ should form “multiplicative strings” of lengths $M_a$ centered at $g_a$:

$$g_a^{(\alpha)} = g_a e^{-(M_a-1)\eta + 2\eta\alpha}, \quad \alpha = 0, 1, \ldots, M_a - 1.$$

Then $\dot{x}_i/\eta$ are eigenvalues of the XXZ spin chain Hamiltonians. The eigenvalue $E = \sum_{a=1}^N g_a \sinh(\eta M_a) \sinh \eta$ of the Ruijsenaars operator agrees with this since it is clear that

$$E = \sum_{a=1}^n \sum_{\alpha=0}^{M_a-1} g_a^{(\alpha)}.$$

Comparing this with (6) and taking into account (11), (40), we see that in the limit $\hbar \to 0$ of the qKZ system (which is the spectral problem for $H_i$'s or $K^{(0)}_i$'s) the other side of the correspondence becomes the classical Ruijsenaars model. In other words, we can say that the quantization of the classical Ruijsenaars system of particles with the Planck constant $\hbar$ ($p_i \to \hbar \partial_{x_i}$) corresponds to the passage from the spectral problem for the spin chain to the system of qKZ equations with the step parameter $\eta \hbar$.

**Acknowledgments**

We thank A.Liashyk for discussions. The work of A. Zabrodin has been funded by the Russian Academic Excellence Project ‘5-100’. It was also supported in part by RFBR grant 15-01-05990 and by joint RFBR grant 15-52-50041 YaF. The work of A. Zotov was supported by RFBR grant 15-01-04217 and by joint RFBR project 15-51-52031 HHC.

**References**

[1] A. Alexandrov, V. Kazakov, S. Leurent, Z. Tsuboi and A. Zabrodin, Classical tau-function for quantum spin chains, JHEP 09 (2013) 064.

[2] M. Beketov, A. Liashyk, A. Zabrodin and A. Zotov, Trigonometric version of quantum-classical duality in integrable systems, Nucl. Phys. B, B903 (2016) 150-163.
[3] K. Bulycheva, A. Gorsky, *BPS states in the Omega-background and torus knots*, JHEP 04 (2014) 164.

[4] I. Cherednik, *Integration of quantum many-body problems by affine Knizhnik-Zamolodchikov equations*, Advances in Mathematics, 106 (1994) 65-95.

[5] I. Cherednik, *Double affine Hecke algebras and Macdonald’s operators*, Internat. Math. Res. Notices No. 9 (1992) 171-180.

[6] I. Cherednik, *Lectures on Knizhnik-Zamolodchikov equations and Hecke algebras*, Mathematical Society of Japan Memoirs, 1, (1998) 1–96.

[7] G. Felder and A. Veselov, *Shift operators for the quantum Calogero-Sutherland problems via Knizhnik-Zamolodchikov equation*, Commun. Math. Phys. 160 (1994) 259-273.

[8] I. Frenkel and N. Reshetikhin, *Quantum affine algebras and holonomic difference equations*, Commun. Math. Phys. 146 (1992) 1-60.

[9] A. Gorsky, A. Zabrodin and A. Zotov, *Spectrum of Quantum Transfer Matrices via Classical Many-Body Systems*, JHEP 01 (2014) 070.

[10] K. Hasegawa, *Ruijsenaars Commuting Difference Operators as Commuting Transfer Matrices*, Commun. Math. Phys. 187 (1997) 289-325.

[11] S. Kato, *R-Matrix Arising from Affine Hecke Algebras and its Application to Macdonald’s Difference Operators*, Commun. Math. Phys. 165 (1994) 533-553.

[12] I. Macdonald, *Orthogonal polynomials associated with root systems*, in “Orthogonal polynomials”, 311-318, P.Nevai (ed.), Kluwer Academic Publishers (1990).

[13] A. Matsuo, *Integrable connections related to zonal spherical function*, Inventiones Mathematicae, 110 (1992) 95-121.

[14] K. Mimachi, *A solution to quantum Knizhnik-Zamolodchikov equations and its application to eigenvalue problems of the Macdonald type*, Duke Math. J. 85 (1996) 635-658.

[15] S.N.M. Ruijsenaars, *Complete integrability of relativistic Calogero-Moser systems and elliptic function identities*, Commun. Math. Phys. 110 (1987) 191-213.

[16] S.N.M. Ruijsenaars and H. Schneider, *A new class of integrable systems and its relation to solitons*, Ann. Phys. 146 (1986) 1-34.

[17] J.V. Stokman, *Quantum affine Knizhnik-Zamolodchikov equations and quantum spherical functions, I*, Int. Math. Res. Not. 2011 (2011) no. 5 1023-1090.

[18] V. Tarasov and A. Varchenko, *Jackson integral representations for solutions to the quantized Knizhnik-Zamolodchikov equation*, St.Petersburg Math. J. 6 (1994) 275-313, arXiv:hep-th/9311040.
[19] V. Tarasov and A. Varchenko, *Asymptotic solutions to the quantized KZ equation and Bethe vectors*, Amer. Math. Soc. Transl.(2) **174** (1996) 235-273, [arXiv:hep-th/9406060](https://arxiv.org/abs/hep-th/9406060).

[20] Z. Tsuboi, A. Zabrodin and A. Zotov, *Supersymmetric quantum spin chains and classical integrable systems*, JHEP **05** (2015) 086.

[21] A. Zabrodin, *Quantum spin chains and integrable many-body systems of classical mechanics*, Springer Proceedings in Physics, Volume **163** (2015) 29-48, [arXiv:1409.4099](https://arxiv.org/abs/1409.4099).

[22] A. Zabrodin and A. Zotov, *KZ-Calogero correspondence revisited*, J. Phys. A: Math. Theor. **50** (2017) 205202; [arXiv:1701.06074](https://arxiv.org/abs/1701.06074).