Second-order accurate numerical scheme with graded meshes for the nonlinear partial integrodifferential equation arising from viscoelasticity

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Abstract

This paper establishes and analyzes a second-order accurate numerical scheme for the nonlinear partial integrodifferential equation with a weakly singular kernel. In the time direction, we apply the Crank-Nicolson method for the time derivative, and the product-integration (PI) rule is employed to deal with Riemann-Liouville fractional integral. From which, the non-uniform meshes are utilized to compensate for the singular behavior of the exact solution at $t = 0$ so that our method can reach second-order convergence for time. In order to formulate a fully discrete implicit difference scheme, we employ a standard centered difference formula for the second-order spatial derivative, and the Galerkin method based on piecewise linear test functions is used to approximate the nonlinear convection term. Then we derive the existence and uniqueness of numerical solutions for the proposed implicit difference scheme. Meanwhile, stability and convergence are proved by means of the discrete energy method. Furthermore, to demonstrate the effectiveness of the proposed method, we utilize a fixed point iterative algorithm to calculate the discrete scheme. Finally, numerical experiments illustrate the feasibility and efficiency of the proposed scheme, in which numerical results are consistent with our theoretical analysis.

Keywords: Nonlinear partial integrodifferential equation, second-order accurate difference scheme, non-uniform meshes, product-integration rule, existence and uniqueness, stability and convergence.

AMS subject classification (2020). 26A33, 45K05, 65M12, 65M22, 65M60

1. Introduction

In this work, we shall consider a nonlinear partial integrodifferential equation with a weakly singular kernel

$$ u_t(x, t) + u(x, t)u_x(x, t) - \mathcal{I}^{(\alpha)}u_{xx}(x, t) = f(x, t), \quad x \in (0, L), \quad t \in (0, T], $$

which subjects to the following boundary conditions

$$ u(0, t) = u(L, t) = 0, \quad t \in (0, T], $$

and the initial condition

$$ u(x, 0) = u_0(x), \quad x \in [0, L], $$

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where the Riemann-Liouville (R-L) fractional integral is denoted by [8]

\[ I^{(\alpha)}\zeta(t) := \int_0^t \beta(t - \zeta) \zeta d\zeta, \quad 0 < \alpha < 1, \quad (1.4) \]

from which, Abel kernel \( \beta(t) = t^{\alpha-1}/\Gamma(\alpha) \) and \( \Gamma(q) = \int_0^{+\infty} s^{q-1} \exp(-s)ds \) is the Euler’s Gamma function.

Sanz-Serna [10] pointed out that (1.1) affords a simple model which combines the Eulerian derivative, that is, \( \partial u(x, t)/\partial t + u(x, t) \partial u(x, t)/\partial x \), with a viscoelastic effect, just like Burgers equation [1] provides a simple model for the studies of more realistic situations, involving Eulerian derivatives and viscous forces. In addition, problem (1.1)-(1.3) can model the physical phenomena, which involves the viscoelastic forces, population dynamics, viscous plasticity problems, heat transfer materials with memory, nuclear reaction theory and so on [6, 8, 10, 11]. Equation (1.1) has the significant applications in the fields of engineering and science, and it is still worthy of our further study. However, there are no analytic solutions for the problem (1.1)-(1.3), thus we have to yield its numerical solutions.

In fact, for problem (1.1)-(1.3), many studies of numerical fields have been considered in recent years, especially finite difference methods. First, Lopez-Marcos [4] developed a backward Euler (BE) scheme for the time derivative, and the R-L integral term was approximated by the first-order convolution quadrature rule. Then, Tang [12] considered this problem by utilizing a Crank-Nicolson (CN) method for the time derivative, and the R-L integral term was discretized by the trapezoidal product method. Further, Chen and Xu [2] proposed a formally second-order backward differentiation formula (BDF) scheme for the time derivative, and the R-L integral term was dealt with second-order convolution quadrature rule. After that, Zheng et al. [14] considered a CN-type method for the time derivative, and they used the trapezoidal convolution quadrature rule to approximate the R-L integral term. The theoretical analysis regarding the stability and convergence was reported in these articles. Unfortunately, these studies considered the temporal uniform mesh, which is still affected by the singular behavior of the exact solution at \( t = 0 \), so that it is impossible to achieve accurate second-order convergence for time. Furthermore, in some researches above, numerical algorithms and simulations have been not given to illustrate the effectiveness of their numerical schemes, except for certain linear examples in [12] and Crandall’s finite difference scheme in [3]. These facts all urge us to establish a temporal high-order scheme and its algorithm implementation to the numerical solutions of problem (1.1)-(1.3).

The main contributions of this work is presented as follows. (i) Based on the non-uniform meshes, we eliminate the singular behavior of the exact solution at \( t = 0 \), and obtain the exact second-order implicit difference scheme of the non-linear problem (1.1)-(1.3). (ii) For the constructed implicit difference scheme, with some suitable hypotheses, we prove the existence and uniqueness of the numerical solution, and derive the stability and convergence of the numerical scheme. (iii) In order to illustrate the effectiveness of the implicit difference scheme, we adopt an iterative algorithm to calculate and implement that. The numerical results show that the proposed scheme can reach the accurate second order in the space-time directions, which is consistent with our theoretical results.

The rest of this paper is organized as follows. In Section 2, some preliminaries regarding temporal/spatial discretizations are given. Then Section 3 is devoted to the establishment of a fully discrete implicit difference scheme. In Section 4, we prove the existence and uniqueness of numerical solutions and derive the stability and convergence of the proposed scheme. With a fixed point iterative algorithm, numerical experiments are carried out to verify our theoretical results in Section 5. Ultimately, Section 6 gives the concluding remarks.
2. Preliminaries

2.1. Notations for temporal discretizations

Here, we present the temporal levels \( t_0 = t_0 < t_1 < t_2 < \cdots \) and define the symbols

\[ k_n := t_n - t_{n-1}, \quad t_{n-\frac{1}{2}} := \frac{1}{2}(t_n + t_{n-1}), \quad n \geq 1. \]

Furthermore, define the grid function \( W_k = \{ W^n | 0 \leq n \leq N \} \) and notations

\[ \delta W^{n-\frac{1}{2}} := \frac{1}{k_n}(W^n - W^{n-1}), \quad W^{n-1/2} := \frac{1}{2}(W^n + W^{n-1}), \quad 1 \leq n \leq N, \]

and the piecewise constant approximation

\[ W(t) := \begin{cases} W^1, & t_0 < t < t_1, \\ W^{n-1/2}, & t_{n-1} < t < t_n, \quad n \geq 2. \end{cases} \quad (2.1) \]

Then, employing \( W \), we denote the following discrete fractional integral (see [5])

\[
I^{(\alpha)}W^{n-1/2} = \frac{1}{k_n} \int_{t_{n-1}}^{t_n} \left( \beta(t - \zeta)W(\zeta) \right) d\zeta dt = \sum_{s=2}^{n} \hat{w}_{ns} W^{s-1/2} k_s, 
\]

where we utilize the PI rule, besides,

\[
\hat{w}_{ns} = \frac{1}{k_n k_s} \int_{t_{n-1}}^{t_n} \int_{t_{s-1}}^{\min(t, t_s)} \beta(t - \zeta) d\zeta dt > 0. \quad (2.3)
\]

Then for \( n \geq 2 \), it holds that

\[
\hat{w}_{ns} = \frac{1}{k_n k_s} \int_{t_{n-1}}^{t_n} \int_{t_{s-1}}^{t_s} \beta(t - \zeta) d\zeta dt = \frac{\Gamma(\alpha+2)}{k_n k_s} \left[ (t_n - t_{s-1})^{\alpha+1} + (t_n - t_s)^{\alpha+1} - (t_{n-1} - t_{s-1})^{\alpha+1} - (t_{n-1} - t_s)^{\alpha+1} \right] 
\]

\[ k_n k_s \Gamma(\alpha+2) \]

and

\[
\hat{w}_{nn} = \frac{1}{k_n^2} \int_{t_{n-1}}^{t_n} \int_{t_{n-1}}^{t_n} \beta(t - \zeta) d\zeta dt = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+2)}, \quad n \geq 1. \quad (2.4)
\]

Additionally, regarding the source term \( f \), we give the approximation as follows

\[
f^{n-1/2} \approx \frac{1}{k_n} \int_{t_{n-1}}^{t_n} f(\cdot, t) dt, \quad n \geq 1, \quad (2.5)
\]

assumed to meet that

\[
\left\| f^{n-\frac{1}{2}} k_1 - \int_{t_0}^{t_1} f(\cdot, t) dt \right\| \leq C \int_{t_0}^{t_1} \| f(\cdot, t) \| dt, \quad (2.6)
\]

\[ n \geq 1. \]
and for \( n \geq 2 \),

\[
\left\| f^{n-1/2}k_n - \int_{t_{n-1}}^{t_n} f(\cdot, t) dt \right\| \leq Ck_n^2 \int_{t_{n-1}}^{t_n} \| f''(\cdot, t) \| dt. \tag{2.8}
\]

Here, for example, \( f^{n-1/2} = f(\cdot, t_{n-1/2}) \) or \( f^{n-1/2} = \frac{1}{2}[f(\cdot, t_{n-1}) + f(\cdot, t_n)] \) is also permissible.

Next, for eliminating the singular behaviour of the exact solution at \( t = 0 \) and obtaining accurate second order for time, the following hypotheses regarding non-uniform meshes [5] is presented by

\[
k_n \leq \tilde{C}_\gamma k \min \left\{ 1, t_n^{1-\frac{1}{\gamma}} \right\}, \quad n \geq 1, \quad \gamma \geq 1, \tag{2.9}
\]

from which, \( \tilde{C}_\gamma \) is independent of \( k \),

\[
t_1 = k_1 \geq \tilde{C}_\gamma k^3, \quad t_n \leq \tilde{C}_\gamma t_{n-1}, \quad n \geq 2, \tag{2.10}
\]

and a more rigid assumption about the temporal mesh, i.e.,

\[
0 \leq k_{n+1} - k_n \leq \tilde{C}_\gamma k^2 \min \left\{ 1, t_n^{1-2/\gamma} \right\}, \quad n \geq 2. \tag{2.11}
\]

Thus for \( t_0 \leq t \leq T \), the case satisfying above three hypotheses (2.9)-(2.11) is

\[
t_n = (nk)^\gamma, \quad 0 \leq n \leq N, \quad k = \frac{T^{1/\gamma}}{N}. \tag{2.12}
\]

2.2. Notations of spatial discretizations

First, denote nodes \( x_j = jh \ (0 \leq j \leq J) \) with \( h = \frac{L}{J} \) for the positive integer \( J \). Let \( \Omega_h = \{ x_j | 0 \leq j \leq J \} \). We define the grid function \( W = \{ W_j | 0 \leq j \leq J \} \). Then we give the following notations

\[
\Delta_x W_j = \frac{1}{2h}(W_{j+1} - W_{j-1}), \quad \delta_x W_j = \frac{1}{h}(W_{j+1} - W_j), \quad T_+ W_j = W_{j+1}, \quad T_- W_j = W_{j-1},
\]

\[
\Delta W_j = T_+ W_j - T_- W_j, \quad \Delta_x W_j = W_{j+1} - W_j, \quad \Delta_\xi W_j = W_j - W_{j-1},
\]

\[
\delta_x^2 W_j = \frac{1}{h^2}(\delta_x W_{j+1} + \delta_x W_{j-1}) - \delta_x W_j, \quad \nabla W_j = \frac{1}{h}(W_{j+1} - W_j + u_{j+1}).
\]

Then we present the following lemma.

Lemma 2.1. [13]. Assuming \( G(x) \in C_2^4([0, L]) \), we can get

\[
G''(x_j) = \delta_x^2 G_j - \frac{h^2}{6} \int_0^1 \left( G''''(x_j + \zeta h) + G''''(x_j - \zeta h) \right) (1 - \zeta)^3 d\zeta.
\]

Remark 1. Throughout this paper, \( C \) denotes a general positive number, which may be various in different situations, however, independent of the space-time step sizes \( k_n \) with \( 1 \leq n \leq N \) and \( h \).

3. Construction of second-order implicit difference scheme

Below, we can establish a fully discrete second-order implicit difference scheme for the problem (1.1)-(1.3).

Firstly, denote the following grid functions

\[
u^n = u(x, t_n), \quad 0 \leq n \leq N, \quad f^n = f(x, t_n), \quad 1 \leq n \leq N,
\]
\[ u^n_j = u(x_j, t_n), \quad 0 \leq n \leq N, \quad f^n_j = f(x_j, t_n), \quad 1 \leq n \leq N, \quad 0 \leq j \leq J, \]

and assume that

\[ c_0 := \max_{(x, t) \in [0, L] \times [0, T]} \{|u(x, t)|, \left| \frac{\partial u}{\partial t}(x, t) \right| \}. \]

Then we integrate (1.1) from \( t = t_{n-1} \) to \( t = t_n \) and multiply \( \frac{1}{k_n} \) to obtain

\[
\delta_t u^{n-\frac{1}{2}} + \frac{1}{k_n} \int_{t_{n-1}}^{t_n} uu_x dt = \frac{1}{k_n} \int_{t_{n-1}}^{t_n} \int_{t_0}^{t} \beta(t - \zeta)u_{xx}(x, \zeta) d\zeta dt \\
+ \frac{1}{k_n} \int_{t_{n-1}}^{t_n} f(x, t) dt, \quad x \in (0, L), \quad 1 \leq n \leq N, \tag{3.1}
\]

from which, we have

\[
\delta_t u^{n-\frac{1}{2}} + \frac{1}{k_n} \int_{t_{n-1}}^{t_n} uu_x dt = \frac{1}{k_n} \int_{t_{n-1}}^{t_n} \int_{t_0}^{t} \beta(t - \zeta)u_{xx}(x, \zeta) d\zeta dt \\
+ f^{n-1/2} + (R_1)^{n-1/2}, \quad x \in (0, L), \quad 1 \leq n \leq N, \tag{3.2}
\]

where \((R_1)^{n-1/2} = (R_{11})^{n-1/2} + (R_{12})^{n-1/2}\) with

\[
(R_{11})^{n-1/2} = \frac{1}{k_n} \int_{t_{n-1}}^{t_n} f(x, t) dt - f^{n-1/2}, \tag{3.3}
\]

and

\[
(R_{12})^{n-1/2} = \frac{1}{k_n} \int_{t_{n-1}}^{t_n} \int_{t_0}^{t} \beta(t - s) \left[ u_{xx}(x, s) - u_{xx}(x_1) \right] ds dt, \tag{3.4}
\]

where

\[
f^{n-1/2} = \frac{1}{2} \left( f(x, t_n) + f(x, t_{n-1}) \right). \tag{3.5}
\]

Then for second term in (3.2), we employ the right rectangle formula and the middle rectangle formula to get respectively

\[
\frac{1}{k_1} \int_{t_0}^{t_1} uu_x dt = u(x, t_1)u_x(x, t_1) + O(k_1), \tag{3.6}
\]

\[
\frac{1}{k_n} \int_{t_{n-1}}^{t_n} uu_x dt = u(x, t_{n-1/2})u_x(x, t_{n-1/2}) + O(k^2_n), \quad 2 \leq n \leq N. \tag{3.7}
\]

Next, we employ (3.2), (3.6) and (3.7) to obtain that

\[
\delta_t u^{n-\frac{1}{2}} + u(x, t_1)u_x(x, t_1) = I^{(\alpha)}u_{xx}^{1/2} + f^{1/2} + (R_1)^{1/2} + O(k_1), \quad x \in (0, L), \tag{3.8}
\]

and

\[
\delta_t u^{n-\frac{1}{2}} + u(x, t_{n-1/2})u_x(x, t_{n-1/2}) = I^{(\alpha)}u_{xx}^{n-1/2} + f^{n-1/2} \\
+ (R_1)^{n-1/2} + O(k^2_n), \quad x \in (0, L), \quad 2 \leq n \leq N. \tag{3.9}
\]

Then, we construct the fully discrete implicit difference scheme based on the spacial difference approximation.
For (3.8) and (3.9), we consider them at the point $x_j$, and use Lemma 2.1 to get

$$
\delta_t u_j + u(x_j, t_1)u_x(x_j, t_1) = I^{(\alpha)} \delta^2 u_j + f_j + (R_1)_{j} + O \left( \left( \frac{t^n - t^{n-1}}{k_1 \Gamma(\alpha + 1)} \right) h^2 \right) + O(k_1 + h^2), \quad 1 \leq j \leq J - 1, (3.10)
$$

and

$$
\delta_t u_j^n + u(x_j, t_{n-\frac{1}{2}})u_x(x_j, t_{n-\frac{1}{2}}) = I^{(\alpha)} \delta^2 u_j^n + f_j^n + (R_1)_{j} + O \left( \left( \frac{t^n - t^{n-1}}{k_n \Gamma(\alpha + 1)} \right) h^2 \right) + O(k_n^2 + h^2), \quad 1 \leq j \leq J - 1, \quad 2 \leq n \leq N, (3.11)
$$

from which, we utilize the Galerkin method based on piecewise linear test functions [4] to approximate the nonlinear convection term

$$
u(x_j, t_1)u_x(x_j, t_1) = \nabla u_j^1 \Delta_x u_j^1 + O(h^2), \quad (3.12)
$$

$$
u(x_j, t_{n-\frac{1}{2}})u_x(x_j, t_{n-\frac{1}{2}}) = \nabla u_j^n \Delta_x u_j^n + O(h^2), \quad 2 \leq n \leq N. \quad (3.13)
$$

Thus, we can get the following fully discrete implicit difference equations

$$
\delta_t u_j + \nabla u_j \Delta_x u_j = I^{(\alpha)} \delta^2 u_j + f_j + (R_1)_{j} + (R_2)_{j}, \quad 1 \leq j \leq J - 1, \quad (3.14)
$$

and

$$
\delta_t u_j^n + \nabla u_j^n \Delta_x u_j^n = I^{(\alpha)} \delta^2 u_j^n + f_j^n + (R_1)_{j} + (R_2)_{j}, \quad 1 \leq j \leq J - 1, \quad 2 \leq n \leq N, \quad (3.15)
$$

where

$$
|R_1| \leq C \left( k_1 + h^2 + \frac{t^n - t^{n-1}}{k_1 \Gamma(\alpha + 1)} h^2 \right),
$$

$$
|R_2| \leq C \left( k_n^2 + h^2 + \frac{t^n - t^{n-1}}{k_n \Gamma(\alpha + 1)} h^2 \right), \quad 2 \leq n \leq N, \quad (3.16)
$$

which subject to the following initial and boundary conditions

$$
u_j^0 = u_0(x_j), \quad 1 \leq j \leq J - 1, \quad (3.17)
$$

$$
u_j^n = u^n_j = 0, \quad 1 \leq n \leq N. \quad (3.18)
$$

Now, omitting $(R_1)_j$ and $(R_2)_j$ in (3.14)-(3.15) and replacing $u_j^0$ with its numerical approximation $U_j^0$, the fully discrete implicit difference scheme can be yielded by

$$
\delta_t U_j + \nabla U_j \Delta_x U_j = I^{(\alpha)} \delta^2 U_j + f_j^{1/2}, \quad 1 \leq j \leq J - 1, \quad (3.19)
$$

$$
\delta_t U_j^n + \nabla U_j^n \Delta_x U_j^n = I^{(\alpha)} \delta^2 U_j^n + f_j^n, \quad 1 \leq j \leq J - 1, \quad 2 \leq n \leq N, \quad (3.20)
$$

$$
U_j^0 = u_0(x_j), \quad 1 \leq j \leq J - 1, \quad (3.21)
$$

$$
U_j^n = U^n_j = 0, \quad 1 \leq n \leq N. \quad (3.22)
$$
4. Theoretical analysis

In this section, we give some notations and lemmas for our theoretical results, including the existence and uniqueness of numerical solutions, and the stability and convergence of implicit difference scheme.

Let $W_h = \{w|w = (w_0, w_1, \cdots, w_L), w_0 = w_L = 0\}$. For any $w, v \in W_h$, the discrete inner product and norms can be denoted via

$$\langle w, v \rangle := h \sum_{s=1}^{J-1} w_s v_s, \quad \|w\| := \sqrt{\langle w, w \rangle}, \quad \|w\|_\infty := \max_{0 \leq s \leq J} \{|w_s|\}. \quad (4.1)$$

**Lemma 4.1.** [4] For any $w, v \in W_h$, it holds that

$$\langle \delta^2 w, v \rangle = -h \sum_{s=0}^{J-1} \delta_s w_{s+1} \delta_s v_{s+1}.$$ 

**Lemma 4.2.** [4] For any $w, v \in W_h$, we have

$$\langle \Delta(wv), v \rangle = \frac{1}{2}(T_+ v \Delta w + T_- v \Delta w, v).$$ 

**Lemma 4.3.** For any $w, v \in W_h$, we can yield

(i) $\langle w \Delta v + \Delta(wv), v \rangle = 0$;

(ii) $\langle w \Delta w, v \rangle + \langle \Delta(wv), w \rangle = 0$;

(iii) $\langle w \Delta v, w \rangle + \langle \Delta(w)^2, v \rangle = 0$.

**Proof.** (i) From the definition of $\langle \cdot, \cdot \rangle$, we first yield that

$$\langle w \Delta v + \Delta(wv), v \rangle$$

$$= h \sum_{s=1}^{J-1} w_s (v_{s+1} - v_{s-1}) v_s + h \sum_{s=1}^{J-1} (w_{s+1} v_{s+1} - w_{s-1} v_{s-1}) v_s$$

$$= h \sum_{s=1}^{J-1} w_s v_{s+1} v_s - h \sum_{s=1}^{J-1} w_s v_{s-1} v_s + h \sum_{s=1}^{J-1} w_{s+1} v_{s+1} v_s - h \sum_{s=1}^{J-1} w_{s-1} v_{s-1} v_s$$

$$= h \sum_{s=2}^{J-1} w_{s-1} v_{s-1} v_s - h \sum_{s=1}^{J-1} w_{s-1} v_{s-1} v_s + h \sum_{s=2}^{J-1} w_s v_{s-1} v_s - h \sum_{s=1}^{J-1} w_s v_{s-1} v_s$$

$$= hwJ_{-1} v_{J-1} - hwJ_{-1} v_{J-1} = 0.$$ 

The proofs of (ii) and (iii) can be obtained analogously. Then we complete the proof. 

**Lemma 4.4.** For any $w, v \in W_h$, it holds that

$$\langle v \Delta(v - w) + (v - w) \Delta w + \Delta(v - w)(v + w), v - w \rangle$$

$$= \langle (v - w) \Delta w + \Delta(w(v - w)), v - w \rangle.$$ 


Proof. First, using Lemma 4.3, we get

$$
\langle v \Delta (v - w) + (v - w) \Delta w + \Delta (v - w)(v + w), v - w \rangle \\
= \langle (v \Delta (v - w) + (v - w) \Delta w + \Delta (v - w)(v + w), v - w \rangle \\
= \langle (v - w) \Delta (v - w) + \Delta (v - w)^2, v - w \rangle \\
= \langle (v - w) \Delta w + \Delta (w(v - w)), v - w \rangle + \langle w \Delta (v - w) + \Delta (w(v - w)), v - w \rangle \\
= A_1 + A_2.
$$

Then we only need to derive $A_2 = 0$. Employing Lemma 4.3 again, we yield

$$
A_2 = \langle w \Delta (v - w) + \Delta (w(v - w)), v - w \rangle \\
= \langle w \Delta (v - w) + \Delta (w(v - w)), v \rangle - \langle w \Delta (v - w) + \Delta (w(v - w)), w \rangle \\
= \langle w \Delta v - w \Delta w + \Delta (wv) - \Delta (w)^2, v \rangle - \langle w \Delta v - w \Delta w + \Delta (wv) - \Delta (w)^2, w \rangle \\
= \langle w \Delta v + \Delta (wv), v \rangle - \langle (w \Delta w, v) + \langle \Delta (wv), w \rangle \rangle - \langle (w \Delta v, w) + \langle \Delta (w)^2, v \rangle \rangle \\
= 0,
$$

which finishes the proof. \( \square \)

4.1. Existence of numerical solutions

Herein, we use the Leray-Schauder theorem [7] to derive the existence of numerical solutions for the second-order implicit difference scheme (3.19)-(3.22).

Theorem 4.1. Given $J, N \in \mathbb{Z}^+$ and $U^0 \in \mathbb{R}^{J-1}$, then second-order implicit difference equations (3.19)-(3.20) have the solution $U^n$ with $1 \leq n \leq N$.

Proof. With $U^0 \in \mathbb{R}^{J-1}$, we first utilize [4, Section 3] to show that (3.19) has a solution $U^1$. Then we need to apply the mathematical induction to show that, provided $U^s$ for $2 \leq s \leq n - 1$, the equation (3.20) for $U^n$ has a solution.

Below define the mapping $\mathcal{D}: \mathbb{R}^{J-1} \to \mathbb{R}^{J-1}$ by

$$
\mathcal{D}(v) := -\frac{k_n}{12h} (v \Delta v + \Delta (v)^2) + \frac{1}{2} k_n^2 \tilde{\omega}_{n1} \delta_x^2 v,
$$

then $U^n$ is a solution of equation (3.20) if and only if

$$
U^{n-1/2} = \mathcal{D}(U^{n-1/2}) + W,
$$

where

$$
W = U^{n-1} + \frac{1}{2} k_n k_1 \tilde{\omega}_{n1} \delta_x^2 U^1 + \frac{1}{2} k_n \sum_{s=2}^{n-1} k_s \tilde{\omega}_{ns} \delta_x^2 U^{s-1/2} + \frac{1}{2} k_n f^{n-1/2}.
$$

Therefore, we have to illustrate that the mapping $\mathcal{P}(\cdot) = \mathcal{D}(\cdot) + W$ has a fixed point. Next, we consider an open ball $\mathcal{B} = B(0, r)$ in $\mathbb{R}^{J-1}$ endowed with the norm $\| \cdot \|$ in (4.1). Assume that for $\lambda > 1$ and $U^{n-1/2}$ in the boundary of $\mathcal{B},$

$$
\lambda U^{n-1/2} = \mathcal{P}(U^{n-1/2}) = \mathcal{D}(U^{n-1/2}) + W. \quad (4.2)
$$

Then using Lemmas 4.1-4.3, we have

$$
\langle \mathcal{D}(U^{n-1/2}), U^{n-1/2} \rangle \leq 0.
$$

Now, we take the inner product of (4.2) with $U^{n-1/2}$ and apply the Cauchy-Schwarz inequality,
\[
\lambda \|U^{n-1/2}\|^2 \leq \langle W, U^{n-1/2} \rangle \leq \|W\| \|U^{n-1/2}\| \leq \frac{1}{2}(\|W\|^2 + \|U^{n-1/2}\|^2),
\]

thus,
\[
\lambda \leq \frac{1}{2} \left( \frac{\|W\|^2}{\|U^{n-1/2}\|^2} + 1 \right) \leq \frac{\|W\|^2}{2r^2} + \frac{1}{2}.
\]

As \( r \) large, the above inequality contradicts with the assumption \( \lambda > 1 \). Hence, (4.2) has no solution on \( \partial \mathcal{Q} \). Then employing the Leray-Schauder theorem (cf. [7], Thm. 6.3.3), we yield the existence of a fixed point of the mapping \( \mathcal{P} \) in the closure of \( \mathcal{Q} \).

4.2. Stability analysis

We below use the energy method to prove the stability of the second-order implicit difference scheme (3.19)-(3.22).

**Theorem 4.2.** Supposing that \( \{U^n\mid 1 \leq j \leq J-1, 1 \leq n \leq N\} \) is the solution of the second-order implicit difference scheme (3.19)-(3.22), for \( 1 \leq n \leq N \), we have
\[
\|U^n\| \leq \|U^0\| + 2 \sum_{l=1}^{N} k_l \|f^{l-1/2}\|.
\]

**Proof.** Taking the inner product of (3.19)-(3.20) with \( U^1 \) and \( U^{n-1/2} \), respectively, we get
\[
\langle \delta_t U^n, U^1 \rangle + \langle \nabla U^{n} \Delta_x U^1, U^1 \rangle = \langle I(\alpha) \delta_x^2 U^{1/2}, U^1 \rangle + \langle f^{1/2}, U^1 \rangle,
\]
and
\[
\langle \delta_t U^{n-1/2}, U^{n-1/2} \rangle + \langle \nabla U^{n-1/2} \Delta_x U^{n-1/2}, U^{n-1/2} \rangle = \langle I(\alpha) \delta_x^2 U^{n-1/2}, U^{n-1/2} \rangle
\]
\[
+ \langle f^{n-1/2}, U^{n-1/2} \rangle, \quad 2 \leq n \leq N,
\]
then, employing Lemmas 4.3 and 4.1, and for \( N \geq 1 \), we yield
\[
k_1 \langle \delta_t U^n, U^1 \rangle + \sum_{n=2}^{N} k_n \langle \delta_t U^{n-1/2}, U^{n-1/2} \rangle = k_1 \langle f^{1/2}, U^1 \rangle + \sum_{n=2}^{N} k_n \langle f^{n-1/2}, U^{n-1/2} \rangle
\]
\[
- \left( k_1 \langle I(\alpha) \delta_x U^{1/2}, \delta_x U^1 \rangle + \sum_{n=2}^{N} k_n \langle I(\alpha) \delta_x U^{n-1/2}, \delta_x U^{n-1/2} \rangle \right).
\]

Each term in (4.5) will be bounded one by one. At first, utilizing
\[
\langle \delta_t U^{n-1/2}, U^n \rangle = \frac{1}{2k_n}(U^n - U^{n-1}, U^n - U^{n-1} + U^n + U^{n-1}) \geq \frac{1}{2k_n}(\|U^n\|^2 - ||U^{n-1}||^2),
\]
we have
\[
k_1 \langle \delta_t U^n, U^1 \rangle + \sum_{n=2}^{N} k_n \langle \delta_t U^{n-1/2}, U^{n-1/2} \rangle \geq \frac{\|U^n\|^2 - ||U^{n-1}||^2}{2}.
\]

Secondly, from [5, pp. 483-485, (1.14)], we can obtain
\[
\left( k_1 \langle I(\alpha) \delta_x U^{1/2}, \delta_x U^1 \rangle + \sum_{n=2}^{N} k_n \langle I(\alpha) \delta_x U^{n-1/2}, \delta_x U^{n-1/2} \rangle \right) \geq 0.
\]
Then, by substituting (4.6) and (4.7) into (4.5), and utilizing the Cauchy-Schwarz inequality, we have

\[ ||U^N||^2 \leq ||U^0||^2 + 2k_1||f^0||||U^1|| + 2 \sum_{n=2}^{N} k_n ||f^{n-1/2}||||U^{n-1/2}||. \]

By choosing a suitable \( M \) such that \( ||u^M|| = \max_{0 \leq n \leq N} ||U^n|| \), we can yield

\[ ||U^M|| \leq ||U^0|| + 2 \sum_{n=1}^{M} k_n ||f^{n-1/2}|| \leq ||U^0|| + 2 \sum_{n=1}^{N} k_n ||f^{n-1/2}||. \] (4.8)

We finish the proof. \( \square \)

### 4.3. Convergence analysis

Here, we consider the convergence of the second-order implicit difference scheme (3.19)-(3.22). Denote

\[ e_j^n = U^n_j - U^n_j, \quad 0 \leq j \leq J, \quad 0 \leq n \leq N. \]

By subtracting (3.19)-(3.22) from (3.14)-(3.15), (3.17)-(3.18), respectively, we obtain the error equations as follows

\[ \delta_t e_j^1 - \nabla e_j^1 \Delta x e_j^1 = I(\alpha) \delta_x^2 e_j^{1/2} + \sum_{m=1}^{4} (R_m)_{j}^{1/2}, \quad 1 \leq j \leq J - 1, \] (4.9)

\[ \delta_t e_j^{n-1/2} - \nabla e_j^{n-1/2} \Delta x e_j^{n-1/2} = I(\alpha) \delta_x^2 e_j^{n-1/2} + \sum_{m=1}^{4} (R_m)_{j}^{n-1/2}, \] (4.10)

\[ e_j^0 = 0, \quad 0 \leq j \leq J, \quad e_j^n = e_j^{n} = 0, \quad 1 \leq n \leq N, \] (4.11)

from which,

\[ (R_3)_{j}^{1/2} = -\frac{1}{\Delta t} \left[ u_j^1 \Delta e_j^1 + \Delta (e_j^1 u_j^1) \right], \]
\[ (R_3)_{j}^{n-1/2} = -\frac{1}{\Delta t} \left[ u_j^{n-1/2} \Delta e_j^{n-1/2} + \Delta (e_j^{n-1/2} u_j^{n-1/2}) \right], \quad 2 \leq n \leq N, \]
\[ (R_4)_{j}^{1/2} = -\frac{1}{\Delta t} \left[ e_j^1 \Delta u_j^1 + \Delta (e_j^1 u_j^1) \right], \]
\[ (R_4)_{j}^{n-1/2} = -\frac{1}{\Delta t} \left[ e_j^{n-1/2} \Delta u_j^{n-1/2} + \Delta (e_j^{n-1/2} u_j^{n-1/2}) \right], \quad 2 \leq n \leq N. \]

In order to obtain the convergence, we present the following lemmas.

**Lemma 4.5.** If satisfying the conditions

\[ t||u''_{xx}(., t)|| + t^2||u'''_{xx}(., t)|| \leq \mathcal{M}t^{\sigma - 1}, \] (4.12)
\[ t||f'(., t)|| + t^2||f''(., t)|| \leq \mathcal{M}t^{\sigma - 1}, \quad \sigma > 0, \] (4.13)

then we have

\[ \sum_{n=1}^{N} k_n \left( \left\| (R_1)^{n-1/2} \right\| + \left\| (R_2)^{n-1/2} \right\| \right) \leq C \left( K_L + h^2 \right), \]
where
\[
K_L := \begin{cases} 
    k^{\gamma\sigma}, & \text{if } 1 \leq \gamma < \frac{2}{\sigma}, \\
    k^2 \log(t_N/t_1), & \text{if } \gamma = \frac{2}{\sigma}, \\
    k^2, & \text{if } \gamma > \frac{2}{\sigma}.
\end{cases}
\]

Proof. With assumptions (4.12) and (4.13), and from [5, Corollary 4.3], we can get
\[
\sum_{n=1}^{N} k_n \left\| (R_4)^{n-1/2} \right\| \leq \sum_{n=1}^{N} k_n \left\| (R_{11})^{n-1/2} \right\| + \sum_{n=1}^{N} k_n \left\| (R_{12})^{n-1/2} \right\|
\]
\[
\leq C_{\alpha,\gamma,\sigma,T,M} \times \begin{cases} 
    k^{\gamma\sigma}, & \text{if } 1 \leq \gamma < 2/\sigma, \\
    k^2 \log(t_N/t_1), & \text{if } \gamma = 2/\sigma, \\
    k^2, & \text{if } \gamma > 2/\sigma.
\end{cases}
\] (4.14)

In addition, using (3.16), we have
\[
\sum_{n=1}^{N} k_n \left\| (R_2)^{n-1/2} \right\| = k_1 \left\| (R_2)^{1/2} \right\| + \sum_{n=2}^{N} k_n \left\| (R_2)^{n-1/2} \right\|
\]
\[
\leq C_k \left( k_1 + h^2 + \left( \frac{t_0^n - t_0^0}{k_1 \Gamma(\alpha + 1)} \right) h^2 \right) + C \sum_{n=2}^{N} k_n \left( k_2^n + h^2 + \left( \frac{t_0^n - t_0^{n-1}}{k_n \Gamma(\alpha + 1)} \right) h^2 \right) \] (4.15)

Combining (4.14) and (4.15), we finish the proof. □

Lemma 4.6. If \( u_0 = u_J = 0 \) and \( e_0 = e_J = 0 \), it holds that
\[
\langle (R_3)^{1/2}, e^1 \rangle = 0, \quad \langle (R_3)^{n-1/2}, e^{n-1/2} \rangle = 0, \quad 2 \leq n \leq N.
\]

Proof. The proof can be finished by Lemma 4.3. □

Lemma 4.7. If \( u_0 = u_J = 0 \) and \( e_0 = e_J = 0 \), then we get
\[
\left\| (R_4)^{1/2}, e^1 \right\| \leq \frac{c_0}{2} \left\| e^1 \right\|^2, \quad \left\| (R_4)^{n-1/2}, e^{n-1/2} \right\| \leq \frac{c_0}{2} \left\| e^{n-1/2} \right\|^2, \quad 2 \leq n \leq N.
\]

Proof. From Lemma 4.2, we have
\[
\langle \Delta(e^m u^m), e^m \rangle = \frac{1}{2} \langle T_+ e^m \Delta_+ u^m + T_- e^m \Delta_- u^m, e^m \rangle.
\]

With conditions \( u_0 = u_J = 0, e_0 = e_J = 0 \) and using Cauchy-Schwarz inequality, we yield that
\[
\left\| (T_+ e^m \Delta_+ u^m, e^m) \right\| \leq \left\| \Delta_+ u^m \right\| \| T_+ e^m \|_{\infty} \left\| (T_+ e^m, e^m) \right\| \leq \left\| \Delta_+ u^m \right\| \sum_{s=1}^{J-1} h^m \| e^m \|^2 \leq \frac{1}{2} \| \Delta_+ u^m \| \left( \sum_{s=1}^{J-1} h^m \| e^m \|^2 \right) \leq \frac{1}{2} \left\| \Delta_+ u^m \right\| \left\| e^m \right\|^2.
\]
Similarly, it holds that
\[
|\langle T_m e^m \Delta u^m, e^m \rangle| \leq \|\Delta u^m\|_\infty \|e^m\|^2.
\]
Thus, we can get
\[
\left| - \langle e^m \Delta u^m + \Delta(e^m U^m), e^m \rangle \right| = \left| \langle e^m \Delta u^m, e^m \rangle \right| + \frac{1}{2} \left( \|\Delta u^m\|_\infty + \|\Delta u^m\|_\infty \right) \|e^m\|^2
\leq 3c_0 h \|e^m\|^2,
\]
which finishes the proof.

Theorem 4.3. Let \( \{u^n\}_{n=0}^N \in C^4_T(\{0, L\}) \) and \( \{U^n\}_{n=0}^N \) be the solutions of (1.1)-(1.3) and (3.19)-(3.21), respectively. Assume that \( f^{n-1/2} \) is selected to meet that (2.7) and (2.8) hold. If the exact solution and the forcing term satisfy
\[
t \|u''_{xx}(\cdot, t)\| + t^2 \|u'''_{xx}(\cdot, t)\| \leq M t^{\sigma-1},
\]
\[
t \|f'\(\cdot, t\)\| + t^2 \|f''\(\cdot, t\)\| \leq M t^{\sigma-1}, \quad \sigma > 0,
\]
respectively for \( t > 0 \), then for \( t_n \in [0, T] \) and \( \gamma \geq 1 \), it holds that
\[
\max_{1 \leq n \leq N} \|U^n - u^n\| \leq C(T) \left( h^2 + K_L \right).
\]
from which,
\[
K_L := \begin{cases} k^\gamma, & \text{if } 1 \leq \gamma < \frac{2}{3}, \\ k^2 \log(t_N/t_1), & \text{if } \gamma = \frac{2}{3}, \\ k^2, & \text{if } \gamma > \frac{2}{3}. \end{cases}
\]

Proof. First, taking the inner product of (3.19)-(3.20) with \( k_1 e^1 \) and \( k_n e^{n-1/2} \), respectively, summing up for \( n \) from 1 to \( N \), and utilizing Lemmas 4.3 and 4.1, we have
\[
k_1 \langle \delta_1 e^{1/2}, e^1 \rangle + \sum_{n=2}^{N} k_n \langle \delta_1 e^{n-1/2}, e^{n-1/2} \rangle
\]
\[
= k_1 \left( \sum_{n=1}^{4} (R_m)^{1/2}, e^1 \right) + \sum_{n=2}^{N} k_n \left( \sum_{m=1}^{4} (R_m)^{n-1/2}, e^{n-1/2} \right)
\]
\[
- \left( k_1 \langle I^{(\alpha)} \delta_x e^{1/2}, \delta_x e^1 \rangle + \sum_{n=2}^{N} k_n \langle I^{(\alpha)} \delta_x e^{n-1/2}, \delta_x e^{n-1/2} \rangle \right),
\]
which is the desired result.
from which, we use Lemma 4.6 and Lemma 4.7 to get

\[
k_1 \langle \delta \epsilon^1, e^1 \rangle + \sum_{n=2}^{N} k_n \langle \delta e^{n-\frac{1}{2}}, e^{n-\frac{1}{2}} \rangle \leq k_1 \left( \sum_{m=1}^{2} (R_m)^{\frac{1}{2}}, e^1 \right) + \sum_{n=2}^{N} k_n \left( \sum_{m=1}^{2} (R_m)^{-1/2}, e^{-1/2} \right) + \frac{c_0}{2} k_1 \parallel e^1 \parallel^2 + \frac{c_0}{2} \sum_{n=2}^{N} k_n \parallel e^{n-1/2} \parallel^2 \]

\[
- \left( k_1 \langle f^{(\alpha)} \delta_z e^{\frac{n}{2}}, \delta_z e^{1} \rangle + \sum_{n=2}^{N} k_n \langle I^{(\alpha)} \delta_x e^{n-\frac{1}{2}}, \delta_x e^{n-1/2} \rangle \right).
\]

Then using (4.6), (4.7) and the Cauchy-Schwarz inequality, we have

\[
\frac{1}{2} \left( \parallel e^n \parallel^2 - \parallel e^n \parallel^2 \right) \leq k_1 \parallel (R_1)^{\frac{1}{2}} \parallel \parallel e^1 \parallel + \sum_{n=2}^{N} k_n \parallel (R_1)^{n-\frac{1}{2}} \parallel \parallel e^{n-\frac{1}{2}} \parallel \]

\[
+ k_1 \parallel (R_2)^{\frac{1}{2}} \parallel \parallel e^1 \parallel + \sum_{n=2}^{N} k_n \parallel (R_2)^{n-\frac{1}{2}} \parallel \parallel e^{n-\frac{1}{2}} \parallel \]

\[
+ \frac{c_0}{2} k_1 \parallel e^1 \parallel^2 + \frac{c_0}{2} \sum_{n=2}^{N} k_n \parallel e^{n-\frac{1}{2}} \parallel^2.
\]

By taking an appropriate \( K \) such that \( \parallel e^K \parallel = \max_{0 \leq n \leq N} \parallel e^n \parallel \) and using (4.11), we have

\[
\parallel e^K \parallel^2 \leq 2 \sum_{n=1}^{K} k_n \parallel (R_1)^{n-\frac{1}{2}} \parallel \parallel e^K \parallel + 2 \sum_{n=1}^{K} k_n \parallel (R_2)^{n-\frac{1}{2}} \parallel \parallel e^K \parallel \]

\[
+ c_0 k_1 \parallel e^1 \parallel \parallel e^K \parallel + c_0 \sum_{n=2}^{K} k_n \parallel e^{n-\frac{1}{2}} \parallel \parallel e^K \parallel,
\]

therefore

\[
\parallel e^K \parallel \leq 2 \sum_{n=1}^{K} k_n \parallel (R_1)^{n-\frac{1}{2}} \parallel + 2 \sum_{n=1}^{K} k_n \parallel (R_2)^{n-\frac{1}{2}} \parallel \]

\[
+ c_0 k_1 \parallel e^1 \parallel + \frac{c_0}{2} \sum_{n=2}^{K} k_n \left( \parallel e^{n-1} \parallel + \parallel e^n \parallel \right).
\]

This naturally can obtain

\[
\left( 1 - \frac{c_0}{2} k_N \right) \parallel e^N \parallel \leq 2 \sum_{n=1}^{N} k_n \parallel (R_1)^{n-\frac{1}{2}} \parallel + 2 \sum_{n=1}^{N} k_n \parallel (R_2)^{n-\frac{1}{2}} \parallel \]

\[
+ \frac{3c_0}{2} \sum_{n=1}^{N-1} k_n \parallel e^n \parallel.
\]
When $k_N \leq \frac{1}{6h}$, we utilize the discrete Grönwall inequality to yield

$$\|e^N\| \leq C \exp \left( \sum_{n=1}^{N} k_n \right) \left( 2 \sum_{n=1}^{N} k_n \| (R_1)^{n-\frac{1}{2}} \| + 2 \sum_{n=1}^{N} k_n \| (R_2)^{n-\frac{1}{2}} \| \right),$$  \hspace{1cm} (4.24)$$

from which, we can complete the proof by Lemma 4.5. \hfill \square

### 4.4. Uniqueness of numerical solutions

Next, based on the hypothesis of Theorem 4.3, we will prove the uniqueness of numerical solutions for second-order implicit difference scheme (3.19)-(3.22).

**Theorem 4.4.** If $h$ is sufficiently small, $k = o(h^{\frac{1}{2}})$ and $\gamma = \frac{1}{2}$, then second-order implicit difference scheme (3.19)-(3.22) has a unique solution.

**Proof.** Let $U^n \in \mathbb{R}^{J-1}$ and $V^n \in \mathbb{R}^{J-1}$, $0 \leq n \leq N$, be the solutions of (3.19)-(3.22) which satisfy $U^0 = V^0$. From [4, p. 29], we can demonstrate similarly that (3.19) has a unique solution $U^1$. Then we suppose that $U^n = V^n$, $1 \leq m \leq n - 1$, $2 \leq n \leq N$.

Next, we only need to show $U^n = V^n$ for (3.20), with $2 \leq n \leq N$. Firstly, by analyzing (3.20), we get

$$\left( \delta t U_j^{n-\frac{1}{2}} - \delta t V_j^{n-\frac{1}{2}} \right) = \frac{1}{6h} \left( U_j^{n-\frac{3}{2}} \Delta U_j^{n-\frac{1}{2}} + \Delta (U_j^{n-\frac{1}{2}}) - V_j^{n-\frac{3}{2}} \Delta V_j^{n-\frac{1}{2}} + \Delta (V_j^{n-\frac{1}{2}}) \right)$$

$$= I^{(a)} \delta t U_j^{n-\frac{1}{2}} - V_j^{n-\frac{1}{2}} \right), \hspace{1cm} 1 \leq j \leq J-1. \hspace{1cm} (4.25)$$

Then, taking the inner product of (4.25) with $U^{n-\frac{1}{2}} - V^{n-\frac{1}{2}}$ and employing Lemma 4.1, we yield

$$\frac{1}{2k_n} \left( \| U^n - V^n \|^2 - \| U^{n-1} - V^{n-1} \|^2 \right) \leq -\frac{1}{6h} \left( U^{n-\frac{1}{2}} \Delta U^{n-\frac{1}{2}} + \Delta (U^{n-\frac{1}{2}}) - V^{n-\frac{1}{2}} \Delta V^{n-\frac{1}{2}} + \Delta (V^{n-\frac{1}{2}}) \right)$$

$$= \frac{1}{6h} \left( U^{n-\frac{1}{2}} \Delta U^{n-\frac{1}{2}} - V^{n-\frac{1}{2}} \right) \Delta V^{n-\frac{1}{2}} + \Delta (U^{n-\frac{1}{2}} - V^{n-\frac{1}{2}})(U^{n-\frac{1}{2}} + V^{n-\frac{1}{2}}), U^{n-\frac{1}{2}} - V^{n-\frac{1}{2}} \right)$$

and using Lemma 4.4, we obtain

$$\| U^n - V^n \|^2 \leq \| U^{n-1} - V^{n-1} \|^2$$

$$\frac{k_n}{3h} \left( (U^{n-\frac{1}{2}} - V^{n-\frac{1}{2}}) \Delta V^{n-\frac{1}{2}} + \Delta (V^{n-\frac{1}{2}}(U^{n-\frac{1}{2}} - V^{n-\frac{1}{2}})) \right) \leq \frac{k_n}{3h} \left( (U^{n-\frac{1}{2}} - V^{n-\frac{1}{2}}) \Delta V^{n-\frac{1}{2}} + \Delta (V^{n-\frac{1}{2}}(U^{n-\frac{1}{2}} - V^{n-\frac{1}{2}})) \right).$$

Using the Cauchy-Schwarz inequality, then we get

$$\Theta_n \equiv \left( (U^{n-\frac{1}{2}} - V^{n-\frac{1}{2}}) \Delta V^{n-\frac{1}{2}} + \Delta (V^{n-\frac{1}{2}}(U^{n-\frac{1}{2}} - V^{n-\frac{1}{2}})) \right)$$

$$\leq \left\| \Delta V^{n-\frac{1}{2}} \right\|_\infty \| U^{n-\frac{1}{2}} - V^{n-\frac{1}{2}} \| + \frac{1}{2} \left( \| \Delta V^{n-\frac{1}{2}} \|_\infty + \| \Delta V^{n-\frac{1}{2}} \|_\infty \right) \| U^{n-\frac{1}{2}} - V^{n-\frac{1}{2}} \|^2,$$  \hspace{1cm} (4.27)$$
from which, applying the triangle inequality, we obtain
\[ \left\| \Delta V^{n-\frac{1}{2}} \right\|_\infty = \max_{1 \leq j \leq J-1} \left\{ \left| V^{n-\frac{1}{2}}_{j+1} - V^{n-\frac{1}{2}}_{j-1} \right| \right\} \]
\[ \leq \max_{1 \leq j \leq J-1} \left\{ \left| V^{n-\frac{1}{2}}_{j+1} - v^{n-\frac{1}{2}}_{j+1} \right| + \left| v^{n-\frac{1}{2}}_{j+1} - v^{n-\frac{1}{2}}_{j-1} \right| + \left| v^{n-\frac{1}{2}}_{j-1} - V^{n-\frac{1}{2}}_{j-1} \right| \right\} \]
\[ \leq 2 \left\| V^{n-\frac{1}{2}} - v^{n-\frac{1}{2}} \right\|_\infty + Ch \leq 2h^{-\frac{1}{2}} \left\| V^{n-\frac{1}{2}} - v^{n-\frac{1}{2}} \right\|_\infty + Ch \]
\[ \leq h^{-\frac{1}{2}} \left\| V^n - v^n \right\| + Ch, \]
hence, we employ Theorem 4.3 with \( \gamma = \frac{2}{3} \), then
\[ \Theta^n \leq C \left[ h^{-\frac{1}{2}}(k^2 + h^2) + Ch \right] \left\| U^n - V^n \right\|^2. \tag{4.28} \]

By substituting (4.28) into (4.26), we can get
\[ \left\| U^n - V^n \right\|^2 \leq Ck/h \left[ h^{-\frac{1}{2}}(k^2 + h^2) + Ch \right] \left\| U^n - V^n \right\|^2, \quad 2 \leq n \leq N. \tag{4.29} \]

With (4.29), we obtain \( \left\| U^n - V^n \right\|^2 = 0 \) for \( k = o(h^{\frac{1}{2}}) \) as \( h \to 0 \). Within this assumption condition, the proof is finished. \( \square \)

5. Numerical experiment

In this section, we set \( L = T = 1 \). We use the second-order implicit difference scheme (3.19)-(3.22) to solve problem (1.1)-(1.3), based on an iterative algorithm [9, Algorithm 1], with MaxStep = 300 and eps = 1e-6. All experiments are carried out by MATLAB (R2014b) on a Windows 10 with CPU (2.20 GHz), RAM (8.0 GB).

Then define the following errors
\[ E_{CN}(N,J) := \left\| U^N - u^N \right\| \]
and the convergence orders
\[ Rate^t := \log_2 \left( \frac{E_{CN}(N,J)}{E_{CN}(2N,J)} \right), \quad Rate^x := \log_2 \left( \frac{E_{CN}(N,J)}{E_{CN}(N,2J)} \right), \]
respectively. In addition, we define
\[ f^{n-\frac{1}{2}} := \frac{1}{k_n} \int_{t_{n-1}}^{t_n} f(x,t)dt \tag{5.1} \]
to replace \( f^{n-1/2} = f(t_{n-1/2}) \).

**Example 1.** First, we give the exact solution
\[ u(x,t) = \sin \pi x - \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+2)} \sin 2\pi x, \tag{5.2} \]
hence, \( u_0(x) = \sin \pi x \) and
\[ f(x,t) = \frac{\pi^2 \Gamma(\alpha)}{\Gamma(\alpha+1)} \sin \pi x - \left( \frac{4\pi^2 \Gamma(\alpha+1)}{\Gamma(2\alpha+2)} + \frac{\Gamma(\alpha)}{\Gamma(\alpha+1)} \right) \sin 2\pi x \]
\[ + \left( \sin \pi x - \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+2)} \sin 2\pi x \right) \left( \pi \cos \pi x - \frac{2\pi \Gamma(\alpha+1)}{\Gamma(\alpha+2)} \cos 2\pi x \right). \tag{5.3} \]
Table 1: The \(L^2\) errors, temporal convergence orders and CPU time (seconds) for \(J = 1024\) and a uniform mesh (\(\gamma = 1\)) with \(f^{n-1/2} = \frac{1}{2}(f^n + f^{n-1})\).

| \(\alpha\) | \(N\) | \(\|E_{CN}\|\) | Rate | CPU(s) |
|---|---|---|---|---|
| 8  | 2.0116e-3 | * | 4.61 |
| 16 | 9.6258e-4 | 1.06 | 8.20 |
| 0.05 | 32 | 4.6360e-4 | 1.05 | 31.90 |
| 64 | 2.2437e-4 | 1.05 | 31.90 |
| 8  | 5.2489e-3 | * | 4.66 |
| 16 | 2.1570e-3 | 1.28 | 8.85 |
| 0.25 | 32 | 9.0520e-4 | 1.25 | 16.88 |
| 64 | 4.6360e-4 | 1.25 | 33.63 |
| 8  | 7.8050e-3 | * | 5.27 |
| 16 | 2.5050e-3 | 1.64 | 10.09 |
| 0.60 | 32 | 8.3659e-4 | 1.58 | 18.13 |
| 64 | 2.8073e-4 | 1.58 | 33.63 |
| 8  | 1.8670e-3 | * | 6.34 |
| 16 | 4.1218e-3 | 2.18 | 10.68 |
| 0.95 | 32 | 1.0774e-4 | 1.94 | 17.76 |
| 64 | 3.0643e-5 | 1.81 | 31.09 |

Therefore, for \(t > 0\), we have [5]

\[
\|u'_{xx}(\cdot,t)\| + t^2\|u''_{xx}(\cdot,t)\| \leq M t^{\sigma+1}, \quad \|f'(\cdot,t)\| + t^2\|f''(\cdot,t)\| \leq M t^{\sigma},
\]  

which implies that the regularity conditions (4.16)-(4.17) of Theorem 4.3 are satisfied with \(\sigma = \alpha + 1\). In Table 1, we show some results when taking \(f^{n-1/2} = \frac{1}{2}(f^n + f^{n-1})\). As predicted, the temporal convergence orders approximate \(k^{1+\alpha}\) under a uniform mesh, i.e., \(\gamma = 1\). However, from Table 2, the convergence orders can improve to \(k^2\) when \(\gamma = 2\alpha + 1\) (cf. Theorem 4.3).

Then, from Table 3, we illustrate the effectiveness of eliminating the errors caused by the approximation (2.6), employing (5.1) to replace \(f^{n-1/2} = f(\cdot, t_{n-1/2})\), which means that the singular behaviour of \(f\) on longer plays an important role, and Theorem 4.3 holds with \(\sigma = \alpha + 2\) (namely \(\gamma \geq \frac{2}{\sigma}\)), thus we predict the second-order convergence for time with \(\gamma = 1\). In addition, the second-order accuracy for time can be observed in Table 4, by fixing \(J = 1024\) and \(\gamma = \frac{2}{\sigma+1}\) when \(f^{n-1/2}\) is given by (5.1).

When \(f^{n-1/2}\) is given by (5.1) and fixing \(\alpha = 0.05, 0.35, 0.65, 0.95\), respectively, it can be observed clearly from Tables 5 and 6 that the implicit difference scheme is convergent to the order 2 for space as expected, which is in accordance with our theoretical analysis.

**Example 2.** Herein, we present the exact solution by

\[
\left(\frac{\pi^2 t^{2\alpha}}{\Gamma(2\alpha + 1)} + \frac{\tan^{-1} \left(\frac{\pi}{\alpha} \sqrt{2} \sin 2\pi x\right)}{2\Gamma(2\alpha + 1)}\right) \sin \pi x + \frac{\pi t^{2\alpha}}{2\Gamma(\alpha + 1)} \sin 2\pi x.
\]  

Herein, for \(t > 0\), we select \(f^{n-1/2}\) by (5.1) in order that the singular behaviour of \(f\) does not effect the convergence order (see [5]). Besides, we see that

\[
\|u'_{xx}(\cdot,t)\| + t^2\|u''_{xx}(\cdot,t)\| \leq M t^{\alpha},
\]  

which is more singular than the previous example. Therefore \(u_0(x) = 0\) and

\[
f(x, t) = \left(\frac{\pi^2 t^{2\alpha}}{\Gamma(2\alpha + 1)} + \frac{\tan^{-1} \left(\frac{\pi}{\alpha} \sqrt{2} \sin 2\pi x\right)}{2\Gamma(2\alpha + 1)}\right) \sin \pi x + \frac{\pi t^{2\alpha}}{2\Gamma(\alpha + 1)} \sin 2\pi x.
\]
Table 2: The $L^2$ errors, temporal convergence orders and CPU time (seconds) for $J = 1024$ and a graded mesh ($\gamma = \frac{2}{\alpha + 1}$) when $f_{n-1/2} = \frac{1}{2}(f_n + f_{n-1})$.

| $\alpha$ | N   | $E_{GN}$      | Rate$^c$ | CPU(s) |
|----------|-----|---------------|----------|--------|
| 8        |     | 4.8808e-4    | *        | 4.21   |
| 16       |     | 1.4323e-4    | 1.77     | 7.71   |
| 0.05     | 32  | 4.0144e-5    | 1.84     | 14.74  |
|          | 64  | 1.0342e-5    | 1.96     | 32.05  |
|          | 8   | 1.6004e-3    | *        | 4.55   |
|          | 16  | 4.4928e-4    | 1.83     | 8.30   |
| 0.25     | 32  | 1.2346e-4    | 1.86     | 16.16  |
|          | 64  | 3.3192e-5    | 1.90     | 33.07  |
|          | 8   | 4.3516e-3    | *        | 5.53   |
|          | 16  | 1.1749e-3    | 1.89     | 9.79   |
| 0.60     | 32  | 3.2463e-4    | 1.86     | 17.70  |
|          | 64  | 8.9770e-5    | 1.85     | 32.28  |
|          | 8   | 1.6900e-3    | *        | 8.32   |
|          | 16  | 3.6973e-4    | 2.19     | 10.29  |
| 0.95     | 32  | 9.6631e-5    | 1.94     | 16.67  |
|          | 64  | 2.7185e-5    | 1.83     | 32.56  |

Table 3: The $L^2$ errors, temporal convergence orders and CPU time (seconds) for $J = 1024$ and a uniform mesh ($\gamma = 1$) when $f_{n-1/2}$ is presented by (5.1).

| $\alpha$ | N   | $E_{GN}$      | Rate$^c$ | CPU(s) |
|----------|-----|---------------|----------|--------|
| 8        |     | 4.9645e-4    | *        | 4.38   |
| 16       |     | 1.2346e-4    | 2.01     | 10.50  |
| 0.25     | 32  | 2.9472e-5    | 2.07     | 17.53  |
|          | 64  | 6.0076e-6    | 2.29     | 37.04  |
|          | 8   | 1.1946e-3    | *        | 5.03   |
|          | 16  | 2.5727e-4    | 2.21     | 9.16   |
| 0.50     | 32  | 6.3106e-5    | 2.03     | 16.63  |
|          | 64  | 1.4618e-5    | 2.11     | 34.55  |
|          | 8   | 1.9949e-3    | *        | 5.83   |
|          | 16  | 4.9271e-4    | 2.02     | 10.19  |
| 0.75     | 32  | 1.1761e-4    | 2.07     | 16.74  |
|          | 64  | 2.7211e-5    | 2.11     | 31.59  |
|          | 8   | 1.8929e-3    | *        | 6.39   |
|          | 16  | 3.6419e-4    | 2.38     | 10.77  |
| 0.95     | 32  | 8.1789e-5    | 2.15     | 17.34  |
|          | 64  | 1.9272e-5    | 2.08     | 31.79  |
Table 4: The $L^2$ errors, temporal convergence orders and CPU time (seconds) for $J = 1024$ and a graded mesh ($\gamma = \frac{2}{\alpha+1}$) when $f^{n+1/2}$ is presented by (5.1).

| $\alpha$ | $N$  | $E_{CN}$         | Rate | CPU(s) |
|----------|------|------------------|------|--------|
|          | 8    | 2.4027e-3        | *    | 4.36   |
|          | 16   | 6.3361e-4        | 1.92 | 8.38   |
| 0.25     | 32   | 1.6421e-4        | 1.95 | 15.87  |
|          | 64   | 4.1219e-5        | 1.99 | 31.94  |
|          | 8    | 2.7254e-3        | *    | 4.81   |
|          | 16   | 6.7966e-4        | 2.00 | 8.89   |
| 0.50     | 32   | 1.7026e-4        | 2.00 | 16.37  |
|          | 64   | 4.1702e-5        | 2.03 | 32.92  |
|          | 8    | 2.3405e-3        | *    | 5.48   |
|          | 16   | 5.9713e-4        | 1.97 | 9.69   |
| 0.75     | 32   | 1.4693e-4        | 2.02 | 18.30  |
|          | 64   | 3.5482e-5        | 2.05 | 34.10  |
|          | 8    | 1.8367e-3        | *    | 6.20   |
|          | 16   | 3.7228e-4        | 2.30 | 11.53  |
| 0.95     | 32   | 8.8065e-5        | 2.08 | 16.67  |
|          | 64   | 2.1234e-5        | 2.05 | 30.99  |

Table 5: The $L^2$ errors, spatial convergence orders and CPU time (seconds) for $N = 256$ and a graded mesh ($\gamma = \frac{2}{\alpha+1}$) when $f^{n+1/2} = \frac{1}{2}(f^n + f^{n-1})$.

| $\alpha$ | $J$  | $E_{CN}$         | Rate | CPU(s) |
|----------|------|------------------|------|--------|
|          | 8    | 3.7100e-2        | *    | 0.31   |
|          | 16   | 9.0767e-3        | 2.03 | 0.32   |
| 0.05     | 32   | 2.2566e-3        | 2.01 | 0.35   |
|          | 64   | 5.6294e-4        | 2.00 | 0.50   |
|          | 8    | 3.2425e-2        | *    | 0.32   |
|          | 16   | 7.9333e-3        | 2.03 | 0.36   |
| 0.35     | 32   | 1.9727e-3        | 2.01 | 0.38   |
|          | 64   | 4.9245e-4        | 2.00 | 0.54   |
|          | 8    | 2.7412e-2        | *    | 0.32   |
|          | 16   | 6.7176e-3        | 2.03 | 0.35   |
| 0.65     | 32   | 1.6726e-3        | 2.01 | 0.38   |
|          | 64   | 4.1903e-4        | 2.00 | 0.56   |
|          | 8    | 2.7276e-2        | *    | 0.32   |
|          | 16   | 6.5935e-3        | 2.05 | 0.33   |
| 0.95     | 32   | 1.6360e-3        | 2.01 | 0.38   |
|          | 64   | 4.0899e-4        | 2.00 | 0.57   |
which illustrates that Theorem 4.3 holds when $\sigma = 1 + \alpha$. In Table 7, the results of a uniform mesh report expected temporal convergence orders $k^{1+\alpha}$. And then, with different $\alpha = 0.25, 0.50, 0.75, 0.95$, Table 8 shows $L^2$ errors, temporal convergence rates and CPU time for $J = 512$ and the graded mesh $(\gamma = \frac{2}{\alpha+1})$ when $f^{n+1/2}$ is given by (5.1), which shows the second-order convergence for time, as predicted.

Then for disparate $\alpha = 0.05, 0.35, 0.65, 0.95$, Table 9 shows the second-order accuracy for space when $N = 512$ and a uniform mesh. This point is also demonstrated in Table 10 with $\alpha = 0.01, 0.39, 0.69, 0.99$, respectively, when $N = 256$ and $\gamma = \frac{2}{\alpha+1}$. These all validate the theoretical results.

6. Concluding remarks

In this work, an implicit difference scheme has been constructed and analyzed for the nonlinear partial integro-differential equation with a weakly singular kernel. For compensating the singular behavior of the exact solution $u(\cdot, t)$ at $t = t_0$, the non-uniform meshes have been proposed. Then, the discrete energy method was used to derive the stability and convergence of the fully discrete implicit difference scheme. And the existence and uniqueness of numerical solutions were proved based on partly Leray-Schauder theorem. In order to compute proposed implicit difference scheme, an iterative algorithm was employed to obtain approximation solutions. Finally, numerical results have been given to confirm the predicted space-time convergence rates of our approach, which is consistent with the theory.

Conflict of interest: The authors declare that they have no conflict of interest.

References

[1] J. M. Burgers, A mathematical model illustrating the theory of turbulence, Adv. Appl. Mech., 1 (1948) 171-199.
[2] H. Chen, D. Xu, A second-order fully discrete difference scheme for a nonlinear partial integro-differential equation, J. Sys. Sci. Math. Sci., 28 (2008) 51-70. (in Chinese)
[3] M. Dehghan, Solution of a partial integro-differential equation arising from viscoelasticity, Int. J. Comput. Math., 83 (2006) 123-129.
[4] J. C. Lopez-Marcon, A difference scheme for a nonlinear partial integrodifferential equation, SIAM J. Numer. Anal., 27 (1990) 20-31.
Table 7: The $L^2$ errors, temporal convergence orders and CPU time (seconds) for $J = 512$ and a uniform mesh ($\gamma = 1$) when $f^{n-1/2}$ is presented by (5.1).

| $\alpha$ | $N$ | $E_{CN}$ | Rate | CPU(s) |
|----------|-----|----------|------|--------|
| 16       | 32  | 5.4117e-4 | *    | 1.37   |
| 0.25     | 64  | 2.0346e-4 | 1.41 | 2.57   |
| 128      | 3.4438e-5 | 1.24 | 6.25 |
| 16       | 2.6227e-3 | *    | 1.44 |
| 32       | 8.7778e-4 | 1.58 | 2.86 |
| 0.50     | 64  | 2.9870e-4 | 1.55 | 6.44   |
| 128      | 1.0389e-4 | 1.52 | 13.40 |
| 16       | 3.0356e-3 | *    | 1.59 |
| 32       | 9.2739e-4 | 1.71 | 2.83 |
| 0.75     | 64  | 2.7537e-4 | 1.75 | 5.71   |
| 128      | 8.1873e-5 | 1.75 | 12.78 |
| 16       | 1.7302e-3 | *    | 1.48 |
| 32       | 4.5492e-4 | 1.93 | 2.60 |
| 0.95     | 64  | 1.1830e-4 | 1.94 | 5.42   |
| 128      | 3.0877e-5 | 1.94 | 12.53 |

Table 8: The $L^2$ errors, temporal convergence orders and CPU time (seconds) for $J = 512$ and a graded mesh ($\gamma = \frac{2}{\alpha + 1}$) when $f^{n-1/2}$ is given by (5.1).

| $\alpha$ | $N$ | $E_{CN}$ | Rate | CPU(s) |
|----------|-----|----------|------|--------|
| 8        | 7.2543e-4 | *    | 0.74 |
| 16       | 1.6705e-4 | 2.12 | 1.35 |
| 0.25     | 32  | 4.0837e-5 | 2.03 | 2.50 |
| 64       | 1.1431e-5 | 1.84 | 5.46 |
| 8        | 3.5297e-3 | *    | 0.86 |
| 16       | 7.7871e-4 | 2.18 | 1.72 |
| 0.50     | 32  | 1.7826e-4 | 2.13 | 2.88 |
| 64       | 4.2639e-5 | 2.06 | 5.97 |
| 8        | 6.0540e-3 | *    | 0.81 |
| 16       | 1.6480e-3 | 1.88 | 1.53 |
| 0.75     | 32  | 4.2152e-4 | 1.97 | 2.68 |
| 64       | 1.0608e-4 | 1.99 | 5.25 |
| 8        | 5.8041e-3 | *    | 0.85 |
| 16       | 1.5193e-3 | 1.93 | 1.52 |
| 0.95     | 32  | 3.8612e-4 | 1.98 | 2.67 |
| 64       | 9.7184e-5 | 1.99 | 5.78 |
Table 9: The $L^2$ errors, spatial convergence orders and CPU time (seconds) for $N = 512$ and a uniform mesh ($\gamma = 1$) when $f^{n-1/2}$ is given by (5.1).

| $\alpha$ | $J$ | $E_{CN}$ | Rate | CPU(s) |
|----------|-----|----------|-------|--------|
| 0.05     | 8   | 9.7674e-3 | *     | 1.12   |
|          | 16  | 2.4272e-3 | 2.01  | 1.22   |
|          | 32  | 6.0618e-4 | 2.00  | 1.30   |
|          | 64  | 1.5179e-4 | 2.00  | 1.82   |
|          | 8   | 1.0983e-2 | *     | 1.06   |
|          | 16  | 2.7335e-3 | 2.01  | 1.10   |
| 0.35     | 32  | 6.8844e-4 | 1.99  | 1.19   |
|          | 64  | 1.7826e-4 | 1.95  | 1.86   |
|          | 8   | 1.1791e-2 | *     | 1.10   |
|          | 16  | 2.9364e-3 | 2.01  | 1.20   |
| 0.65     | 32  | 6.3968e-4 | 1.99  | 1.21   |
|          | 64  | 1.9159e-4 | 1.95  | 1.90   |
|          | 8   | 1.0631e-2 | *     | 1.07   |
|          | 16  | 2.6397e-3 | 2.01  | 1.11   |
| 0.95     | 32  | 6.5930e-4 | 2.00  | 1.21   |
|          | 64  | 1.6499e-4 | 2.00  | 1.73   |

Table 10: The $L^2$ errors, spatial convergence orders and CPU time (seconds) for $N = 256$ and a graded mesh ($\gamma = \frac{2}{\alpha+1}$) when $f^{n-1/2}$ is given by (5.1).

| $\alpha$ | $J$ | $E_{CN}$ | Rate | CPU(s) |
|----------|-----|----------|-------|--------|
| 0.01     | 4   | 3.9177e-2 | *     | 0.27   |
|          | 8   | 9.5538e-3 | 2.04  | 0.28   |
|          | 16  | 2.3740e-3 | 2.01  | 0.30   |
|          | 32  | 5.9263e-4 | 2.00  | 0.32   |
|          | 4   | 4.5697e-2 | *     | 0.28   |
|          | 8   | 1.1112e-2 | 2.04  | 0.29   |
| 0.39     | 16  | 2.7597e-3 | 2.01  | 0.30   |
|          | 32  | 6.8939e-4 | 2.00  | 0.34   |
|          | 4   | 4.8230e-2 | *     | 0.29   |
|          | 8   | 1.1778e-2 | 2.03  | 0.29   |
| 0.69     | 16  | 2.9308e-3 | 2.01  | 0.30   |
|          | 32  | 7.3527e-4 | 1.99  | 0.33   |
|          | 4   | 4.2250e-2 | *     | 0.29   |
|          | 8   | 1.0310e-2 | 2.03  | 0.29   |
| 0.99     | 16  | 2.5634e-3 | 2.01  | 0.30   |
|          | 32  | 6.3922e-4 | 2.00  | 0.34   |
[5] W. McLean, K. Mustapha, A second-order accurate numerical method for a fractional wave equation, Numer. Math., 105 (2007) 481-510.

[6] W. E. Olmstead, S. H. Davis, S. Rosenblat, W. L. Kath, Bifurcation with memory, SIAM J. Appl. Math., 46 (1986) 171-188.

[7] J. M. Ortega, W. C. Rheinboldt, Iterative solution of nonlinear equations in several variables, Academic Pr., 1970.

[8] I. Podlubny, Fractional Differential Equations, Academic Press, San Diego, 1999.

[9] W. Qiu, H. Chen, X. Zheng, An implicit difference scheme and algorithm implementation for the one-dimensional time-fractional Burgers equations, Math. Comput. Simul., 166 (2019) 298-314.

[10] J. M. Sanz-Serna, A Numerical method for a partial integro-differential equation, SIAM J. Numer. Anal., 25 (1988) 319-327.

[11] Q. Sheng, T. Tang, Optimal convergence of an Euler and finite difference method for nonlinear partial integro-differential equations, Math. Comput. Model., 21 (1995) 1-11.

[12] T. Tang, A finite difference scheme for partial integro-differential equations with a weakly singular kernel, Appl. Numer. Math., 11 (1993) 309-319.

[13] Y. Zhang, Z. Sun, H. Wu, Error estimates of Crank-Nicolson type difference schemes for the sub-diffusion equation, SIAM J. Numer. Anal., 49 (2011) 2302-2322.

[14] X. Zheng, H. Chen, W. Qiu, A Crank-Nicolson-type finite-difference scheme and its algorithm implementation for a nonlinear partial integro-differential equation arising from viscoelasticity, Comput. Appl. Math., 39 (2020) 1-23.