A SYMMETRY ANALYSIS OF THE $\infty$-POLYPLACIAN

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ABSTRACT. In this work we use Lie group theoretic methods and the theory of prolonged group actions to study two fully nonlinear partial differential equations (PDEs). First we consider a third order PDE in two spatial dimensions that arises as the analogue of the Euler-Lagrange equations from a second order variational principle in $L^\infty$. The equation, known as the $\infty$-Polylaplacian, is a higher order generalisation of the $\infty$-Laplacian, also known as Aronsson’s equation. In studying this problem we consider a reduced equation whose relation to the $\infty$-Polylaplacian can be considered analogous to the relationship of the Eikonal to Aronsson’s equation. Solutions of the reduced equation are also solutions of the $\infty$-Polylaplacian. For the first time we study the Lie symmetries admitted by these two problems and use them to characterise and construct invariant solutions under the action of one dimensional symmetry subgroups.

1. Introduction

Near the end of the nineteenth century, Sophus Lie initiated a group theoretic approach in the study of differential equations. Lie’s work was greatly inspired by the ideas and work of Évariste Galois and Niels Abel for the study of algebraic equations, see [Yag88] for a historic account. His ideas on the study of invariance properties of differential equations under the action of continuous groups of transformations have had many implications in modern mathematics and physics. More recently, the method was popularised by Ovsiannikov [Ovs82] and Olver [Olv93]. A detailed exposition of the theory the interested reader should consult the books [BA08; Hyd00; Ste89] and the review paper [Oli10; Win93] and the references given therein. There are many modern generalisations of Lie’s classical approach. Examples of such generalisations are the nonclassical symmetries [BC69; CM94] and approximate symmetries [BGI89], to name a few. Group theoretic methods in the study of differential equations have been applied successfully to problems arising from geometry, general theory of relativity, gas dynamics, hydrodynamics and many more, see [Ibr93].

Calculus of variations is a well studied classical topic that has had profound impact in a wide range of topics such as geometry, mathematical physics, optimisation, etc. [GH96]. The use of Lie symmetries in problems arising from calculus of variations is quite broad and well established [Olv93]. Not only can one use symmetries of the Euler-Lagrange operator to conduct a symmetry reduction, as we will do here, but also symmetries of the underlying minimisation functional give

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rise to additional conservation laws satisfied by the solutions. This is Noether’s Theorem \cite{Noe71}, a classical result whose importance cannot be overstated.

Minimising the Dirichlet functional

\begin{equation}
\mathcal{J}[u; 2] = \int_{\Omega} |\nabla u|^2 \, dx,
\end{equation}

over a domain $\Omega \subset \mathbb{R}^n$, where $\nabla u$ denotes the vector of partial derivatives of $u$, is a classical problem naturally posed in a Hilbertian setting. However, one may choose to move away from the Hilbertian framework, in order to minimise the Dirichlet energy in a stronger norm. In particular, one may choose to minimise

\begin{equation}
\mathcal{J}[u; p] = \int_{\Omega} |\nabla u|^p \, dx,
\end{equation}

for some $p \in (1, \infty)$, or even examine the formal limiting case $p \to \infty$, thus considering variational problems in $L^\infty$. There are many difficulties typically encountered in this case and the study of the associated Euler-Lagrange equations obtained in this way are notoriously challenging \cite{Kat15}. Usually solutions are non-classical and special constructions need to be made to even be able to define the correct notion of *weak solutions*, that of *viscosity solutions* \cite{BDM89; Jen93}.

Taking the formal limit of the Euler-Lagrange equations of (1.2) as $p \to \infty$ it can be verified \cite{Aro65} that the respective Euler-Lagrange equations are quasilinear, second order, and given by

\begin{equation}
\Delta_\infty u = \sum_{i,j=1}^{n} u_{x_i} u_{x_j} u_{x_i x_j} = 0.
\end{equation}

This equation is called the $\infty$-Laplace problem or Aronsson’s equation, see also \cite{Pry17; KP16a} for a modern review of the derivation.

It is also clear that (1.3) can be rewritten as

\begin{equation}
\frac{1}{2} \sum_{i=1}^{n} u_{x_i} \left( \sum_{j=1}^{n} u_{x_j}^2 \right)_{x_i} = 0.
\end{equation}

and hence any solution of the Eikonal equation

\begin{equation}
\sum_{j=1}^{n} u_{x_j}^2 = c,
\end{equation}

is also a solution of (1.3). The converse to this however is not true, one can construct examples of solutions of (1.3) that do not solve the Eikonal equation for example, the well known Aronsson solution for $n = 2$

\begin{equation}
u(x, y) = |x|^{4/3} - |y|^{4/3}
\end{equation}

is not a solution of the Eikonal equation.

In this work we consider a problem arising as a higher order analogy to the $\infty$-Laplace problem coined the $\infty$-Polylaplacian. This is the associated Euler-Lagrange equation to a minimisation functional involving the Hessian of a function \cite{KP16b}. To that end, let $f$ be defined as

\begin{equation}
f[u] := \sum_{i,j=1}^{n} u_{x_i, x_j}^2,
\end{equation}
The ∞-Polylaplacian equation is the following PDE for $u$

$$
\Pi^2_\infty u := \sum_{i,j=1}^{n} f[u]_{x_i} f[u]_{x_j} u_{x_i x_j} = 0,
$$

which is a third order, fully nonlinear PDE. This is surprising as it is the formal limit of the $p$-Polylaplacian

$$
\sum_{i,j=1}^{n} \left( f[u]^{p/2-1} u_{x_i x_j} \right)_{x_i x_j} = 0
$$
as $p \to \infty$ which is a fourth order, quasilinear PDE. The application of such problems are quite broad. For example, equations of this form arise as a model for travelling waves in suspension bridges [LM90; GM10], the modelling of granular matter [Igb12], image processing [ETT15] and game theory [BEJ08].

Interestingly enough, the ∞-Polylaplacian has an analogous relationship to an equation that can be seen as a higher order generalisation of the Eikonal equation. Given the structure of the operator (1.8), it is clear that solutions to

$$
f[u] = c,
$$
are also solutions to the ∞-Polylaplacian. Again, the converse is not true and one can construct counterexamples such as

$$
|u(x, y)| = \frac{x^{12/5}}{y^{12/5}},
given in [KP16b] which solves (1.8) in an appropriate weak sense [KP16b; KM17].

In this work we will restrict our attention to the case $n = 2$. We use $x$ and $y$ for the independent variables and then (1.8) can be written explicitly as

$$
(f[u]_{xx})^2 u_{xx} + 2f[u]_{x} f[u]_{y} u_{xy} + (f[u]_{y})^2 u_{yy} = 0
$$

where now

$$
f[u] = u_{xx} + 2u_{xy} + u_{yy}.
$$

The aim of this paper is to study equations (1.10) and (1.12) from an algebraic point of view. Towards this end we obtain all the Lie symmetries for both problems and we classify all non-equivalent 1-dimensional Lie subalgebras. Then through reductions to ordinary differential equations (ODEs) we construct some invariant solutions. We also propose some conjectures for the symmetry structure of the ∞-Polylaplacian in $n$ dimensions. With this work we aim to progress the study of these strongly nonlinear problems and their solutions, i.e. the ∞-Polyharmonic functions [KP16b] and find potential minimizers for problems arising in the $L^\infty$ variational calculus.

The paper is organised as follows: In the following section we study the reduced ∞-Polylaplacian and consider some of its algebraic properties. In Section three we briefly introduce the basics of Lie symmetries of differential equations and we fix the notation. Moreover, we derive the infinitesimal invariance conditions for both the ∞-Polylaplacian and its reduced version. We solve the determining equations in Section four and thus obtain the Lie algebras of the full symmetry groups of both equations. We present some of the algebraic properties of the Lie algebras and exponentiate to obtain the corresponding Lie symmetries. In Section five we obtain all the inequivalent generators under the adjoint representation, i.e. the action of the symmetry group to its Lie algebra for both equations. We perform
the corresponding reductions to ODEs and we construct some invariant solutions. We conclude in section six with a summary and a discussion of our results.

2. The reduced \( \infty \)-Polylaplacian equation

In this section we introduce the reduced \( \infty \)-Polylaplacian equation and describe some of its properties and discuss some possible extensions of this work over algebraically closed fields of characteristic zero.

From the form of equations (1.12) it follows that the equation \( f[u] = c \), where \( c \) is constant, defines a submanifold in the space of solutions of equation (1.12).

2.1. Lemma. The real solutions of \( f[u] = c \) which are not first degree polynomials in \( x \) and \( y \) can always be mapped to the real solutions of

\[
(2.1) \quad f[u] = u_{xx}^2 + 2u_{xy}^2 + u_{yy}^2 = 1
\]

and vice versa.

Proof Let \( u \) be a real solution of \( f[u] = c \) then from the form of \( f[u] \) it follows that \( c \geq 0 \). The case \( c = 0 \) decomposes to the system of equations \( u_{xx} = u_{xy} = u_{yy} = 0 \) with the general solution \( u = \alpha x + \beta y + \gamma \). Assuming that \( c > 0 \) then the map \( \rho_c : u \mapsto c^{1/2}u \) is invertible and moreover

\[
f[\rho_c^{-1}(u)] = f[c^{-1/2}u] = c^{-1}f[u] = 1.
\]

Conversely, if \( u \) is a real solution of \( f[u] = 1 \) and \( c > 0 \) then

\[
f[\rho_c(u)] = f[c^{1/2}u] = cf[u] = c.
\]

We call equation (2.1) the reduced \( \infty \)-Polylaplacian. The Lemma suggests that if we are interested in real solutions of the \( \infty \)-Polylaplacian that lie on the submanifold \( f[u] = c \), then without any loss of generality we can consider only the \( c = 1 \) case. Alternatively, one can study the equation \( f[u] = c \) for any constant \( c \) over \( \mathbb{C} \)-valued functions. For example, let \( \mathcal{A} = \mathbb{C}[x,y] \) be the ring of polynomials in variables \( x \) and \( y \) with complex coefficients and with the usual gradation

\[
\mathcal{A} = \bigoplus_{k=0}^{\infty} \mathcal{A}_k, \quad \mathcal{A}_i \cdot \mathcal{A}_j \subset \mathcal{A}_{i+j}
\]

where \( \mathcal{A}_k \) is the homogeneous component of all polynomials in \( \mathcal{A} \) of degree \( k \). Then, for simplicity, we can search for solutions of \( f[u] = c \) in each subspace \( \mathcal{A}_k \). It is easy to verify that \( \mathcal{A}_0 \oplus \mathcal{A}_1 \subset \ker f \) and that \( f : \mathcal{A}_k \rightarrow \mathcal{A}_{2k-4} \) for all \( k \geq 2 \). Specifically, for \( k = 2 \) we have that \( f[\mathcal{A}_2] = \mathcal{A}_0 = \mathbb{C} \) and thus it follows that for any \( c \in \mathbb{C} \) the equation \( f[u] = c \) admits the solution \( u = \alpha x^2 + \beta xy + \gamma y^2 \) if and only if the parameters \( (\alpha, \beta, \gamma) \in \mathbb{C}^3 \) are elements of the \( 1 \)-parametric family of affine varieties \( V(4\alpha^2 + 2\beta^2 + 4\gamma^2 - c) \). For \( k > 2 \) and since \( f[\mathcal{A}_k] \subset \mathcal{A}_{2k-4} \) it follows that necessarily \( c = 0 \) and thus we only have to consider the equation \( f[u] = 0 \). Moreover, the parameter space associated to a solution \( u \in \mathcal{A}_k \) has dimension \( \dim \mathcal{A}_k = k + 1 \) while the image has dimension \( \dim \mathcal{A}_{2k-4} = 2k - 3 \). It follows that \( f \) maps

\[
u = \sum_{i+j=k} \alpha_{i,j}x^iy^j \mapsto f[u] = \sum_{m+n=2k-4} F_{m,n}x^my^n
\]

where \( \mathcal{A}_{2k-4} \) is the homogeneous component of all polynomials in \( \mathcal{A} \) of degree \( 2k-4 \).
where \( F_{m,n} \) are homogeneous quadratic polynomials of \( \alpha_{i,j} \) and thus \( u \) will satisfy \( f[u] = 0 \) if and only if \( \alpha = (\alpha_{0,0}, \alpha_{k-1,1}, \ldots, \alpha_{0,k}) \in \mathbb{C}^{k+1} \) is an element of the variety \( V(I) \), where \( I \) is the ideal generated by all \( F_{m,n} \). Effectively, to find solutions of \( f[u] = 0 \) in \( A_k \) one has to solve \( 2k - 3 \) quadratic equations for \( k + 1 \) variables (the parameter space) and thus for \( k \geq 5 \) the system is overdetermined. In principle these equations can be solved using Gröbner basis [CLO92] or numerical schemes. To investigate the existence and the form of solutions in \( A_k \) for all \( k \) is an open problem. More generally, the problem of characterising and classifying the solutions of \( f \) by the authors. In this paper we will not pursue these ideas any further, instead we will focus only on the Lie symmetries of the equation \( f[u] = 1 \).

3. Lie symmetries of differential equations

In this section we fix notation and briefly introduce Lie’s method of extended groups for differential equations. In addition we derive the determining equations for the symmetry group for both the \( \infty \)-Polylaplacian (1.12) and its reduction (2.1).

Let \( G \) be a 1–parameter group of diffeomorphisms \( T_\epsilon \) in \( \mathbb{R}^3 \) with coordinates \((x, y, u)\), i.e. \( G = \{ T_\epsilon : \epsilon \in \mathbb{R} \} \) and

\[
T_\epsilon : (x, y, u) \mapsto (\tilde{x}(x, y, u; \epsilon), \tilde{y}(x, y, u; \epsilon), \tilde{u}(x, y, u; \epsilon))
\]

such that \( T_{\epsilon_1} \circ T_{\epsilon_2} = T_{\epsilon_1 + \epsilon_2} \) and \( T_0 = 1 \). Assuming analyticity of the the group with respect to the parameter \( \epsilon \) we can expand in Taylor’s series and consider the corresponding infinitesimal transformation

\[
\xi = x + \epsilon X x, \quad \eta = y + \epsilon X y, \quad \tilde{u} = u + \epsilon X u
\]

where \( X \) is the generator of the group \( G \)

\[
X = \xi_t + \xi_x \frac{\partial}{\partial x} + \xi_y \frac{\partial}{\partial y} + \eta \frac{\partial}{\partial u}.
\]

The components of the generator \( X \) are given by

\[
\xi_t = \frac{\partial \tilde{x}}{\partial \epsilon} \bigg|_{\epsilon=0}, \quad \xi_x = \frac{\partial \tilde{y}}{\partial \epsilon} \bigg|_{\epsilon=0}, \quad \eta = \frac{\partial \tilde{u}}{\partial \epsilon} \bigg|_{\epsilon=0}.
\]

On the other hand, knowing the generator \( X \) (or equivalently the infinitesimal transformation) the full group \( G \) is given by the Lie series

\[
(3.1)
\]

\[
\tilde{x} = e^X x, \quad \tilde{y} = e^X y, \quad \tilde{u} = e^X u,
\]

where \( e^X = \sum_{k=0}^{\infty} \frac{1}{k!} X^k \), with \( X^k = XX^{k-1} \) and \( X^0 = 1 \). A smooth function \( F(x, y, u) \) is called an invariant of the transformation \( T_\epsilon \) if and only if

\[
F(\tilde{x}, \tilde{y}, \tilde{u}) = F(x, t, u).
\]

It is easy to prove that this condition is equivalent to the equation \( XF = 0 \). The surface (or the level set) \( F(x, t, u) = 0 \) admits \( T_\epsilon \) as a symmetry if and only if \( F(\tilde{x}, \tilde{y}, \tilde{u}) = 0 \) whenever \( F(x, t, u) = 0 \). Similarly, a PDE of the form \( \Phi(x, y, u, \partial^s u) = 0 \), where by \( \partial^s u \) we denote the set of partial derivative of \( u \) up to order \( s \), is invariant under the action of \( T_\epsilon \) if and only if \( \Phi(\tilde{x}, \tilde{y}, \tilde{u}, \partial^s \tilde{u}) = 0 \) when \( \Phi(x, t, u, \partial^s u) = 0 \). The infinitesimal symmetry criterion reads

\[
X^{(s)} \Phi = 0, \quad \text{mod} (\Phi = 0)
\]
where \( X^{(s)} \) is the infinitesimal generator of the group \( G^{(s)} \), the \( s \)-th prolongation of the group \( G \), acting on the space with variables \( x, y, u, \partial^s u \).

The \( s \)-th prolongation of the vector field \( X \) is given by

\[
X^{(s)} = X + \sum_{1 \leq l + m \leq s} \frac{\partial}{\partial u_{x...x y...y}^i} \zeta^{(l,m)}
\]

where

\[
\zeta_{(1,0)} = D_x \eta - u_x D_x \xi_1 - u_y D_x \xi_2, \quad \zeta_{(0,1)} = D_y \eta - u_x D_y \xi_1 - u_y D_y \xi_2
\]

and

\[
D_x = \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{xx} \frac{\partial}{\partial u_x} + \cdots
\]

denotes the operator of the total derivative with respect to \( x \) and similarly for \( D_y \).

The higher \( \zeta^{(l,m)} \) can be calculated recursively by the following formulas

\[
\zeta^{(l+1,m)} = (D_x \zeta^{(l,m)}) - u_{x...x y...y}^i D_x \xi_1 - u_{x...x y...y}^i D_x \xi_2
\]

and similarly for \( \zeta^{(l,m+1)} \), using the operator \( D_y \) instead of \( D_x \). For details see [Olv93; BA08; Hyd00].

In order to calculate the symmetries of equation (1.12) (resp. equation (2.1)) we have to use the third order prolongation of \( X \), namely \( X^{(3)} \) (resp. the second order prolonged vector field \( X^{(2)} \)). The calculations become extremely cumbersome too fast. This is why in order to calculate the related expressions we are using a Mathematica based algebraic package called SYM [DT04; DT06]. The original infinitesimal invariance condition of equation (1.12) reads

\[
(3.2) \quad X^{(3)} \Pi^2_{\infty} u = 0 \quad \text{whenever} \quad \Pi^2_{\infty} u = 0
\]

and decompose to a large overdetermined system of linear PDEs for \( \xi_1, \xi_2 \) and \( \eta \).

3.1. Proposition. The infinitesimal invariance condition (3.2) is equivalent to the following system of 16 equations:

\[
(3.3) \quad \xi_{iu} = \xi_{ixx} = \xi_{ixy} = \xi_{iyy} = 0, \quad i = 1, 2
\]

\[
(3.4) \quad \eta_{xx} = \eta_{xy} = \eta_{yy} = \eta_{xu} = \eta_{yu} = \eta_{uu} = 0,
\]

\[
(3.5) \quad \xi_{1y} + \xi_{2x} = \xi_{1x} - \xi_{2y} = 0.
\]

Similarly, the infinitesimal invariance condition for equation (2.1) is equivalent to the following system of 12 equations:

\[
(3.6) \quad \xi_{2xx} = \xi_{2u} = \xi_{1y} + \xi_{2x} = \xi_{1x} - \xi_{2y} = 0,
\]

\[
(3.7) \quad \eta_{xx} = \eta_{xy} = \eta_{yy} = \eta_{xu} - 2 \xi_{1x} = 0,
\]

\[
(3.8) \quad \xi_{1xx} = \xi_{1xy} = \xi_{1xu} = \xi_{1u} = 0.
\]

Proof The result follows using differential eliminations and ideas from differential algebra [Sch07; Rei90]. All the computations were implemented in Mathematica. □
4. Lie symmetries of the $\infty$-Polylaplacian

In this section we focus our attention at the systems of equations (3.3)-(3.5) and (3.6)-(3.8). These equations form an overdetermined system of linear partial differential equations and thus it is possible that the systems only admit the trivial solution $\xi_1 = \xi_2 = \eta = 0$. Obviously, this would imply that the only symmetry of equation (1.12) is the identity transformation. As we are going to show in the following propositions this is not the case. In this way we obtain the Lie algebra for the symmetry generators for both equations (1.12), (2.1) and thus, using the Lie series (3.1), derive the full groups of Lie point symmetries for both equations.

4.1. Proposition. The general solution of the determining equations (3.3)-(3.5) is given by

$$\begin{align*}
\xi_1 &= c_1 x + c_2 y + c_3, \\
\xi_2 &= -c_2 x + c_1 y + c_4, \\
\eta &= c_5 x + c_6 y + c_7 u + c_8,
\end{align*}$$

where $c_i$, $i = 1, \ldots, 8$ are arbitrary real constants.

Proof We notice that the system of equations (3.3)-(3.5) is in triangular form. The system of equations (3.3) imply that both $\xi_1$ and $\xi_2$ are independent of $u$ and polynomials of first degree in $x$ and $y$. Moreover, from (3.5) we obtain the form of $\xi_i$ given in (4.1). Finally, the system of equations (3.4) fully determine $\eta$ to be the general polynomial of degree one in $x$, $y$ and $u$.

It follows directly from Proposition (4.1) that the solution (4.1) defines an 8–dimensional Lie algebra of generators where the obvious basis is formed by the following vector fields

$$\begin{align*}
X_1 &= \frac{\partial}{\partial x}, \\
X_2 &= \frac{\partial}{\partial y}, \\
X_3 &= -y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y}, \\
X_4 &= x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}, \\
X_5 &= u \frac{\partial}{\partial u}, \\
X_6 &= x \frac{\partial}{\partial x}, \\
X_7 &= y \frac{\partial}{\partial y}, \\
X_8 &= u \frac{\partial}{\partial u}.
\end{align*}$$

Similarly, for equation (2.1) we prove the following Proposition:

4.2. Proposition. The general solution of the determining equations (3.6)-(3.8) is given by

$$\begin{align*}
\xi_1 &= c_1 x - c_2 y + c_3, \\
\xi_2 &= c_2 x + c_1 y + c_4, \\
\eta &= c_5 x + c_6 y + 2c_1 u + c_7,
\end{align*}$$

where $c_i$, $i = 1, \ldots, 7$ are arbitrary real constants.

Proof Using equations $\xi_{1xx} = \xi_{1u} = \xi_{2xx} = \xi_{2u} = 0$ and $\eta_{xx} = \eta_{uy} = \eta_{xuy} = 0$ we obtain that

$$\eta = A_0(u)x + B_0(u)y + C_0(u), \quad \xi_1 = A_1(y)x + B_1(y), \quad \xi_2 = A_2(y)x + B_2(y).$$

Equation $\xi_{1xy} = 0$ implies that $A_1(y) = c_1$ where $c_1$ is a constant. Similarly, equations $\xi_{1y} + \xi_{2x} = \xi_{1x} - \xi_{2y} = 0$ imply that $A_2 = c_2$, $B_1 = -c_2 y + c_3$ and $B_2 = c_1 y + c_4$. Finally, from equation $\eta_{u} - 2\xi_{1x} = 0$ we obtain that $A_0 = c_5$, $B_0 = c_6$ and $C_0(u) = 2c_1 u + c_7$, where all $c_i$ are arbitrary constants.

The Lie algebra of vector fields defined by the solution (4.4) is almost the same to the algebra spanned by the vector fields (4.2)-(4.3). It is a 7–dimensional Lie algebra spanned by the vector fields

$$\begin{align*}
Y_1 &= \frac{\partial}{\partial x}, \\
Y_2 &= \frac{\partial}{\partial y}, \\
Y_3 &= -y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y}, \\
Y_4 &= x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + 2u \frac{\partial}{\partial u}, \\
Y_5 &= x \frac{\partial}{\partial u}, \\
Y_6 &= y \frac{\partial}{\partial u}, \\
Y_7 &= \frac{\partial}{\partial u}.
\end{align*}$$
The reason for this symmetry breaking is because the reduced equation (2.1) is not homogeneous and hence the equation only admits the scaling that makes each individual term, \(u_{xx}, u_{xy}\) and \(u_{yy}\) invariant, i.e. the symmetry generated by \(Y_4\).

We denote the 8-dimensional real Lie algebra by \(\mathfrak{g}\) and the 7-dimensional real Lie algebra by \(\mathfrak{h}\), viz.

\[
\mathfrak{g} = \text{Span}\{X_i, \ i = 1, \ldots 8\}, \quad \mathfrak{h} = \text{Span}\{Y_i, \ i = 1, \ldots 7\}.
\]

It then follows that equation (1.12) admits the symmetry group generated by \(\mathfrak{g}\) while the symmetries of equation (2.1) are generated by \(\mathfrak{h}\). Moreover, since for both equations \(\xi_1u = \xi_2u = 0\), it follows that the symmetry transformations of both (1.12) and (2.1) are fibre preserving transformations.

Both Lie algebras \(\mathfrak{g}\) and \(\mathfrak{h}\) are solvable. Indeed, it is easy to see that for both algebras the derived series

\[
\mathfrak{g}^{(n)} = [[\mathfrak{g}^{(n-1)}, \mathfrak{g}^{(n-1)}]], \quad \mathfrak{g}^{(0)} = \mathfrak{g}
\]

terminate to the trivial Lie algebra \(\mathfrak{o} = \{0\}\) for a positive integer \(n\). As usual \([\cdot, \cdot]\) denotes the commutator of vector fields which is the Lie bracket of \(\mathfrak{g}\) and \(\mathfrak{h}\). The first derived algebra, which is an ideal of \(\mathfrak{g}\), is

\[
\mathfrak{g}^{(1)} = [\mathfrak{g}, \mathfrak{g}] = \text{Span}\{X_1, X_2, X_6, X_7, X_8\}
\]

and

\[
\mathfrak{h}^{(1)} = \text{Span}\{Y_1, Y_2, Y_5, Y_6, Y_7\}
\]

as can be verified by inspecting Table 1 and Table 2.

| \([X_i, X_j]\) | \(X_1\) | \(X_2\) | \(X_3\) | \(X_4\) | \(X_5\) | \(X_6\) | \(X_7\) | \(X_8\) |
|-----------------|--------|--------|--------|--------|--------|--------|--------|--------|
| \(X_1\)        | 0      | 0      | 1      | 0      | 0      | 0      | 0      | 0      |
| \(X_2\)        | 0      | 0      | -1     | 0      | 0      | 0      | 0      | 0      |
| \(X_3\)        | -1     | -1     | 0      | 0      | 0      | -1     | 0      | 0      |
| \(X_4\)        | -1     | -1     | 0      | 0      | 0      | 0      | 0      | 0      |
| \(X_5\)        | 0      | 0      | 0      | 0      | -1     | -1     | -1     | -1     |
| \(X_6\)        | -1     | -1     | 0      | 0      | 0      | 0      | 0      | 0      |
| \(X_7\)        | 0      | -1     | -1     | 0      | 0      | 0      | 0      | 0      |
| \(X_8\)        | 0      | 0      | 0      | 0      | 0      | 0      | 0      | 0      |

**Table 1.** Commutation relations of the Lie algebra \(\mathfrak{g}\).

Similarly, we have that

\[
\mathfrak{g}^{(2)} = \text{Span}\{X_8\}, \quad \mathfrak{h}^{(2)} = \text{Span}\{Y_7\}
\]

and thus \(\mathfrak{g}^{(3)} = \mathfrak{h}^{(3)} = \mathfrak{o}\).

| \([Y_i, Y_j]\) | \(Y_1\) | \(Y_2\) | \(Y_3\) | \(Y_4\) | \(Y_5\) | \(Y_6\) | \(Y_7\) |
|-----------------|--------|--------|--------|--------|--------|--------|--------|
| \(Y_1\)        | 0      | 0      | -1     | 0      | 0      | 0      | 0      |
| \(Y_2\)        | 0      | 0      | 1      | 0      | 0      | 0      | 0      |
| \(Y_3\)        | -1     | 1      | 0      | 0      | 0      | 0      | 0      |
| \(Y_4\)        | -1     | -1     | 0      | 0      | 0      | 0      | 0      |
| \(Y_5\)        | -1     | -1     | 0      | 0      | 0      | 0      | 0      |
| \(Y_6\)        | 0      | 0      | 0      | 0      | 0      | 0      | 0      |
| \(Y_7\)        | 0      | -2     | 0      | 0      | 0      | 2      | 0      |

**Table 2.** Commutation relations of the Lie algebra \(\mathfrak{h}\).
It is worth mentioning that the Lie subalgebra \( \text{Span}\{X_1, X_2, X_3\} \) is the Euclidean Lie algebra \( e(2) \) formed by the \textit{Killing vector fields} of \( \mathbb{R}^2 \). Thus equation (1.12) inherits the symmetries of the metric structure of \( \mathbb{R}^2 \) as it was also pointed out for the Aronsson’s equation (1.3) in [FF11].

The extension \( \mathfrak{g}_1 = \text{Span}\{X_1, X_2, X_3, X_4, X_5\} \) of the Lie algebra \( e(2) \) is still a Lie subalgebra of \( \mathfrak{g} \). Finally, as can be easily seen by the Table 1, the Lie subalgebra \( \mathfrak{g}_2 = \mathfrak{g}\setminus \mathfrak{g}_1 \) forms an \textit{ideal} of \( \mathfrak{g} \) and thus we have the following commutation relations

\[
[\mathfrak{g}_1, \mathfrak{g}_1] \subset \mathfrak{g}_1, \quad [\mathfrak{g}_2, \mathfrak{g}_2] \subset \mathfrak{g}_2, \quad [\mathfrak{g}_1, \mathfrak{g}_2] \subset \mathfrak{g}_2
\]

which means that the Lie algebra \( \mathfrak{g} \) can be decomposed as the semi-direct sum \( \mathfrak{g}_2 \oplus_s \mathfrak{g}_1 \). Similar results hold for the Lie algebra \( \mathfrak{h} \). Here \( \mathfrak{h}_1 = \text{Span}\{Y_1, Y_2, Y_3, Y_4\} \) and \( \mathfrak{h}_2 = \text{Span}\{Y_5, Y_6, Y_7\} \) and we can decompose \( \mathfrak{h} \) as the semi-direct product \( \mathfrak{h} = \mathfrak{h}_2 \oplus_s \mathfrak{h}_1 \).

4.3. \textbf{Theorem.} The full group of Lie point symmetries of equation (1.12) is given by:

\[
\begin{align*}
G_1 : (x, y, u) &\mapsto (\tilde{x}, \tilde{y}, \tilde{u}) = (x + \epsilon, y, u), \\
G_2 : (x, y, u) &\mapsto (\tilde{x}, \tilde{y}, \tilde{u}) = (x, y + \epsilon, u), \\
G_3 : (x, y, u) &\mapsto (\tilde{x}, \tilde{y}, \tilde{u}) = (x \cos \epsilon - y \sin \epsilon, x \sin \epsilon + y \cos \epsilon, u), \\
G_4 : (x, y, u) &\mapsto (\tilde{x}, \tilde{y}, \tilde{u}) = (e^\epsilon x, e^\epsilon y, u), \\
G_5 : (x, y, u) &\mapsto (\tilde{x}, \tilde{y}, \tilde{u}) = (x, y, e^\epsilon u), \\
G_6 : (x, y, u) &\mapsto (\tilde{x}, \tilde{y}, \tilde{u}) = (x, y, u + \epsilon x), \\
G_7 : (x, y, u) &\mapsto (\tilde{x}, \tilde{y}, \tilde{u}) = (x, y, u + \epsilon y), \\
G_8 : (x, y, u) &\mapsto (\tilde{x}, \tilde{y}, \tilde{u}) = (x, y, u + \epsilon).
\end{align*}
\]

\textbf{Proof} The one-parameter Lie groups generated by each of the vector fields of the Lie algebra \( \mathfrak{g} \) can be be constructed using the Lie series (3.1). With \( G_i \) denoting the 1-parametric family of diffeomorphisms \( \exp \epsilon X_i \), \( i = 1, \ldots, 8 \) we obtain the desired result. \qed

4.4. \textbf{Corollary.} If \( u = g(x, y) \) is a solution of equation (1.12) then the following:

\[
\begin{align*}
(4.8) \quad u &= g(x - \epsilon, y), \\
(4.9) \quad u &= g(x, y - \epsilon), \\
(4.10) \quad u &= g(x \cos \epsilon + y \sin \epsilon, -x \sin \epsilon + y \cos \epsilon), \\
(4.11) \quad u &= g(e^{-\epsilon} x, e^{-\epsilon} y), \\
(4.12) \quad u &= e^\epsilon g(x, y), \\
(4.13) \quad u &= g(x, y) + \epsilon x, \\
(4.14) \quad u &= g(x, y) + \epsilon y, \\
(4.15) \quad u &= g(x, y) + \epsilon
\end{align*}
\]

are solutions of (1.12) for all \( \epsilon \in \mathbb{R} \).

We obtain a similar symmetry structure for the reduced \( \infty \)-Polylaplacian equation (2.1) with the only difference being a breaking of the scaling symmetries, resulting in one fewer.
4.5. **Corollary.** The full group of Lie point symmetries of (2.1) is given by:

\[
\begin{align*}
H_1 &: (x, y, u) \mapsto (x + \epsilon, y, u), \\
H_2 &: (x, y, u) \mapsto (x + \epsilon, y, u), \\
H_3 &: (x, y, u) \mapsto (x \cos - y \sin, x \sin + y \cos, u), \\
H_4 &: (x, y, u) \mapsto (e^{\epsilon}x, e^{\epsilon}y, e^{2\epsilon}u), \\
H_5 &: (x, y, u) \mapsto (x, y, u + \epsilon x), \\
H_6 &: (x, y, u) \mapsto (x, y, u + \epsilon y), \\
H_7 &: (x, y, u) \mapsto (x, y, u + \epsilon).
\end{align*}
\]

This implies that if \( u = g(x, y) \) is a solution of (2.1) then so is (4.8), (4.9), (4.10), (4.13), (4.14) and (4.15) for all \( \epsilon \in \mathbb{R} \).

The scaling \( H_4 \) can be seen as the composition of the scaling transformations (4.11) and (4.12) making each differential monomial of the left hand side of equation (2.1) invariant under the action of \( H_4 \).

It is worth mentioning that equations (1.12) and (2.1) admit also discrete symmetries. In particular, both equations (1.12) and (2.1) are invariant under the permutation \( \sigma \) of the independent variables \( x \) and \( y \), as well as the \( x \)-reflection \( \rho : x \mapsto -x \). Obviously the \( y \)-reflection is also a symmetry of both equations and can be expressed as \( \sigma \circ \rho \circ \sigma \). These are also symmetries that the general \( \infty \)-Polylaplacian (1.8) admits in any dimension. Moreover, (2.1), while not scaling invariant in the \( u \)-direction, remains invariant under the reflection \( u \mapsto -u \).

The study of the Lie symmetry structure of the \( \infty \)-Polylaplacian for a general dimension and its corresponding reduction (2.1) is an open problem which is left for a future work. Nevertheless, we formulate the following conjecture: The Lie algebra of the symmetry generators of \( \infty \)-Polylaplacian equation (1.8) in \( n \) independent variables has dimension \( 3 + n(n + 3)/2 \) and it is spanned by the vector fields related to translations, scalings and gauge symmetries

\[
\begin{align*}
\frac{\partial}{\partial x_i}, \frac{\partial}{\partial u}, \ u \frac{\partial}{\partial u}, \ \sum_{j=1}^{n} x_j \frac{\partial}{\partial x_j}, \ x_i \frac{\partial}{\partial u}, \ i = 1, \ldots, n,
\end{align*}
\]

as well as rotation symmetries

\[
\begin{align*}
-x_i \frac{\partial}{\partial x_j} + x_j \frac{\partial}{\partial x_i}, \ 1 \leq i < j \leq n.
\end{align*}
\]

Similarly, the symmetry algebra of the reduced \( \infty \)-Polylaplacian equation (2.1) in \( n \) independent variables has dimension \( 2 + n(n + 3)/2 \) and is spanned by the same generators for translation, rotations and gauge symmetries but with a different scaling symmetry

\[
\begin{align*}
\sum_{j=1}^{n} x_j \frac{\partial}{\partial x_j} + 2u \frac{\partial}{\partial u}.
\end{align*}
\]
5. Invariant Solutions and Symmetry Reductions

In this section we are concerned with the symmetry reductions and group invariant solutions of equations (1.12) and (2.1). We construct solutions that are invariant under one dimensional subgroups, of their symmetry groups, acting non-trivially on the independent variables. This implies that we need to classify all the one dimensional Lie subalgebras of \( g \) and \( h \) into equivalence classes under the action of the corresponding symmetry group. The vector fields \( X_6, \ldots, X_8 \) of the algebra \( g \) generate gauge transformations and thus cannot be used to reduce the equation (1.12). Similarly, \( Y_5, \ldots, Y_7 \) cannot be used to reduce equation (2.1).

We first consider the reductions of equation (1.12) using one dimensional subalgebras of \( g \) spanned by \( X_i, i = 1, \ldots, 5 \). To classify all the one dimensional subalgebras of \( g_1 = \text{Span}\{X_1, \ldots, X_5\} \) we need to consider the action of the adjoint representation of the symmetry group of equation (1.12) on \( g_1 \). The adjoint representation of a Lie group to its algebra is a group action and is defined by conjugacy as follows

\[
\text{Ad}_{\exp X} (Y) = e^{\epsilon X} Y e^{-\epsilon X} = e^{\text{ad}_X(Y)} = Y + \epsilon \text{ad}_X(Y) + \frac{\epsilon^2}{2!} \text{ad}_X^2(Y) + \cdots,
\]

where \( X \) and \( Y \) are elements of the Lie algebra and \( \text{ad}_X(Y) = [X,Y] \). For the sake of completeness we present the adjoint representation of the symmetry group of (1.12) on its whole Lie algebra \( g \) in Table 3 and of the symmetry group of (2.1) to \( h \) in Table 4.

| Ad | \( X_1 \) | \( X_2 \) | \( X_3 \) | \( X_4 \) | \( X_5 \) | \( X_6 \) | \( X_7 \) | \( X_8 \) |
|----|---------|---------|---------|---------|---------|---------|---------|---------|
| 1  | \( x_1 \) | \( x_2 \) | \( x_3 \) | \( x_4 \) | \( x_5 \) | \( x_6 \) | \( x_7 \) | \( x_8 \) |
| 2  | \( x_2 \) | \( x_3 \) | \( x_4 \) | \( x_5 \) | \( x_6 \) | \( x_7 \) | \( x_8 \) | \( x_9 \) |
| 3  | \( x_4 \) | \( x_5 \) | \( x_6 \) | \( x_7 \) | \( x_8 \) | \( x_9 \) | \( x_{10} \) | \( x_{11} \) |

Table 3. The \( \text{Ad}_{\exp X_i} X_j \) is shown in the \((i,j)\) entry of the table.

| Ad | \( Y_1 \) | \( Y_2 \) | \( Y_3 \) | \( Y_4 \) | \( Y_5 \) | \( Y_6 \) | \( Y_7 \) |
|----|---------|---------|---------|---------|---------|---------|---------|
| 1  | \( y_1 \) | \( y_2 \) | \( y_3 \) | \( y_4 \) | \( y_5 \) | \( y_6 \) | \( y_7 \) |
| 2  | \( y_2 \) | \( y_3 \) | \( y_4 \) | \( y_5 \) | \( y_6 \) | \( y_7 \) | \( y_8 \) |
| 3  | \( y_4 \) | \( y_5 \) | \( y_6 \) | \( y_7 \) | \( y_8 \) | \( y_9 \) | \( y_{10} \) |
| 4  | \( y_5 \) | \( y_7 \) | \( y_8 \) | \( y_9 \) | \( y_{10} \) | \( y_{11} \) | \( y_{12} \) |
| 5  | \( y_6 \) | \( y_7 \) | \( y_8 \) | \( y_{10} \) | \( y_{11} \) | \( y_{12} \) | \( y_{13} \) |
| 6  | \( y_6 \) | \( y_7 \) | \( y_8 \) | \( y_{10} \) | \( y_{11} \) | \( y_{12} \) | \( y_{13} \) |
| 7  | \( y_7 \) | \( y_8 \) | \( y_{10} \) | \( y_{11} \) | \( y_{12} \) | \( y_{13} \) | \( y_{14} \) |

Table 4. The \( \text{Ad}_{\exp Y_i} Y_j \) is shown in the \((i,j)\) entry of the table.

5.1. Proposition. Any 1-dimensional subalgebra of \( g_1 \) is equivalent, under the adjoint representation, to one of the following cases:

\[(A1) \quad X_1, \quad (A5) \quad \alpha X_3 + X_5,\]
\[(A2) \quad X_3, \quad (A6) \quad \alpha X_4 + X_5,\]
\[(A3) \quad X_4, \quad (A7) \quad \gamma X_1 + \alpha X_3 + X_4,\]
\[(A4) \quad \gamma X_1 + X_5 \quad (A8) \quad \gamma X_1 + \alpha X_3 + \beta X_4 + X_5,\]

where \( \gamma \in \{0,1\} \) and \( \alpha, \beta \in \mathbb{R}\setminus\{0\} \).

Proof Starting with a general element of \( g_1 \) of the form

\[X = \alpha_1 X_1 + \alpha_2 X_2 + \alpha_3 X_3 + \alpha_4 X_4 + \alpha_5 X_5,\]
we can use all \( \text{Ad}_{\exp X} \) in order to simplify as much as possible and effectively classify all different 1-dimensional subalgebras of \( \mathfrak{g}_1 \). The adjoint action \( \text{Ad} \) induces an action on the coefficients \( \alpha_i \), i.e., on \( \mathbb{R}^5 \). It is easy to see that \( \alpha_3, \alpha_4 \) and \( \alpha_5 \) are invariants of the induced action. This implies that we can classify all inequivalent vector fields according to whether these invariants are zero or not. Moreover, we can rescale the vector field \( X \), use the permutation symmetry \( \sigma \) and the reflection symmetries \( \rho \) and \( \sigma \circ \rho \circ \sigma \) to identify some subcases and thus simplify our classification list. For example, in the case where \( \alpha_3 = \alpha_4 = \alpha_5 = 0 \) we act with \( \text{Ad}_{\exp \gamma X} \) to \( X \) and we obtain \( \gamma X = (\alpha_1 \cos(\epsilon) + \alpha_2 \sin(\epsilon))X_1 + (\alpha_2 \cos(\epsilon) - \alpha_1 \sin(\epsilon))X_2 \). Choosing \( \epsilon = \arctan(\alpha_2 \alpha_1^{-1}) \) and multiplying by a constant factor we obtain \( X_1 \). The other cases are obtained in a similar manner but the calculations are omitted for simplicity. The interested reader can find details and examples in [Olv93; Hyd00; Ovs82].

Similar consideration hold for the \( \mathfrak{h}_1 = \text{Span}\{Y_1, \ldots Y_4\} \):

5.2. Proposition. Any 1-dimensional subalgebra of \( \mathfrak{h}_1 \) is equivalent, under the adjoint representation, to one of the following cases:

\[
\begin{align*}
(B1) & \quad Y_1, \\
(B2) & \quad Y_3, \\
(B3) & \quad Y_4, \\
(B4) & \quad \gamma Y_1 + \alpha Y_3 + Y_4,
\end{align*}
\]

where \( \gamma \in \{0, 1\} \) and \( \alpha \in \mathbb{R} \setminus \{0\} \).

Proof The proof follows a similar argument to Proposition 5.1. Beginning with a general element of \( \mathfrak{h}_1 \) of the form

\[ Y = \beta_1 Y_1 + \beta_2 Y_2 + \beta_3 Y_3 + \beta_4 Y_4 \]

we classify all inequivalent cases. In this case the invariants of the induced action are \( \beta_3 \) and \( \beta_4 \).

5.3. Symmetry reductions. We proceed by first considering the symmetry reductions to ODEs and then continue constructing some solutions which are invariant under the symmetry transformations corresponding to the previous cases. We first consider the reductions and solutions of equation (1.12).

1. Solutions of (1.12) which are invariant under the symmetry generated by \( X_1 \) are of the form \( u = g(y) \). This implies that \( g \) is a solution of the trivial ODE \( g'''(y)^3 g''''(y)^2 = 0 \) and thus it follows that \( u = c_1 y^2 + c_2 y^2 + c_3 \) satisfies the \( \infty \)-Polylaplacian. Since equation (1.12) admits the permutation \( \sigma \) and also contains derivatives of at least second order it is easy to verify that the general quadratic polynomial in \( x \) and \( y \)

\[
(5.1) \quad u = \sum_{0 \leq i+j \leq 2} c_{ij} x^i y^j
\]

is also a solution.

2. Rotationally invariant solutions are of the form \( u = g(s) \) where \( s = x^2 + y^2 \). The reduced equation is the following ODE for \( f(s) \)

\[
(5.2) \quad (2s g_{ss} + g_s) \left[s(2s g_{ss} + g_s) g_{sss} + (3s g_{ss} + 2g_s) g_{ss} \right]^2 = 0.
\]

The factorisation of the reduced equation implies that

\[
(5.3) \quad u = \sqrt{x^2 + y^2},
\]
is a solution of equation (1.12), which we obtain by solving the linear equation

\[ 2sg_{ss} + g_s = 0 \]

and then changing to the original \( x, y \)-variables. Note that this solution is also the most general rotationally invariant solution of Aronsson’s equation (1.3) in two independent variables. However, equation (1.12) may admit more solutions of this type that correspond to the equation defined by the second factor in (5.2); corresponding to a third order nonlinear ODE.

3. The quantities \( u \) and \( s = xy^{-1} \) are algebraic invariants of the Lie group generated by \( X_3 \). We assume that \( u = g(s) \) and we obtain a reduced differential equation for \( g(s) \) which, similarly to the previous case can be decomposed to a product of two factors. The more complicated of these factors is too complicated to write explicitly, however, the simplest factor defines the differential equation

\[ (1 + s^2)g_{ss} + 2sg_s = 0 \]

which implies the solution

\[ u = \arctan \left( \frac{x}{y} \right), \tag{5.4} \]

of equation (1.12). Using the permutation symmetry of the independent variable it follows that \( \arctan (y/x) \) is also a solution. It can be easily verified that any linear combination of these two solutions is again a solution.

4. In the case of the generator \( \gamma X_1 + X_5 \) we have two subcases depending on the value of \( \gamma \). If \( \gamma = 0 \) the only invariant solution is the trivial solution \( u = 0 \). If \( \gamma = 1 \) we have two invariants of the corresponding Lie group, namely \( e^{-x}u \) and \( y \). This implies that the most general form of an invariant solution is \( u = e^x h(y) \). Substituting the ansatz for \( u \) in (1.12) we obtain the following equation for \( h(y) \)

\[ 4h^2 + 4h_y g_y + h_y h_2 = 0 \]

This equation is difficult to solve and it does not admit any obvious factorisations as in the previous cases.

5. In the case of the generator \( \alpha X_3 + X_5 \) the invariants are \( s = x^2 + y^2 \) and \( r = \arctan \left( \frac{z}{y} \right) + \alpha \ln (u) \). The most general solution invariant under the symmetry generated by \( \alpha X_3 + X_5 \) is of the form

\[ u = \exp \left( \frac{1}{\alpha} g(s) - \frac{1}{\alpha} \arctan \left( \frac{x}{y} \right) \right). \]

The resulting reduced ODE for \( g(s) \) is too complicated to handle or even write down. For specific values of \( \alpha \) there may be a possibility to simplify the expression and write down explicit solutions, however this has eluded us.

6. The quantities \( s = xy^{-1} \) and \( r = u x^{-\frac{1}{\alpha}} \) are invariants under the action of the Lie symmetry generated by \( \alpha X_3 + X_5 \). In the limit \( \alpha \to \infty \) we reduce to the generator \( X_3 \). For \( \alpha \neq 0 \), the most general invariant solution of (1.12) is of the form \( u = x^\frac{1}{\alpha} g(s) \). For a general \( \alpha \) the reduced ODE is difficult to handle. Nevertheless, for \( \alpha = 1/2 \) we obtain the following factorisation

\[ E_1[g]^3 E_2[g]^2 = 0 \]
where
\[ E_1[g] := s^2(1 + s^2)^2g_{ss} + 2s(1 + s^2)(2 + s^2)g_s + 2g \]
and
\[ E_2[g] := sg_{sss} + 6sg_{ss} + 6g. \]
Equations \( E_1[g] = 0 \) and \( E_2[g] = 0 \) are both linear and can be solved exactly. Solving the first equation we obtain the following solutions for (1.12)

\[ u = (x^2 + y^2) \left[ c_1 \cos \left( \sqrt{2} \arctan \left( \frac{x}{y} \right) \right) + c_2 \sin \left( \sqrt{2} \arctan \left( \frac{x}{y} \right) \right) \right] \]
for \( c_i \in \mathbb{R} \). The general solution of equation \( E_2[g] = 0 \) can also be find and implies the solution
\[ u = c_1x^2 + c_2xy + c_3y^2 \]
with \( c_i \in \mathbb{R} \). It is interesting to notice that while the scaling \( (x, y, u) \mapsto (e^{\alpha}x, e^{\alpha}y, e^\beta u) \) is a symmetry for every \( \alpha \), each of the differential monomials of equation (1.12) has the same weight, i.e.,
\[ f_{x}^2 u_{xx} \mapsto e^{(5-12\alpha)} f_{x}^2 u_{xx} \]
and similarly for the other terms. This observation implies that for the special value \( \alpha = 12/5 \) all three terms of equation (1.12) are individually invariant under the scaling symmetry. This observation further implies that
\[ \Pi_2^\infty x^r \sim x^{5r-12} \]
and because of symmetry the same will hold for \( y^r \). Putting all these together it is easy to verify that
\[ \Pi_2^\infty (ax^r + by^r) \sim a^5 x^{5r-12} + b^5 y^{5r-12} \]
from where we obtain, for \( r = 12/5 \), a scaling invariant solution of equation (1.12) if and only if \((a,b)\) satisfies
\[ a^5 + b^5 = 0. \]
The only real solution is given by \( b = -a \) and in this way we recover the solution
(5.6) \[ u = x^{12/5} - y^{12/5}, \]
which was first constructed in [KP16b]. The same arguments are valid in the case of the general \( \infty \)-Polylaplacian in \( n \) independent variables. In this case it is easy to see that
\[ u = c_1 x_1^{12/5} + \cdots + c_n x_n^{12/5} \]
is a solution if and only if \((c_1, \ldots, c_n)\) lies on the affine variety \( V(c_1^5 + \cdots + c_n^5) \). This solution is a an invariant solution under the scaling symmetry generated by the vector field
\[ X = \frac{\partial}{\partial x_1} + \cdots + \frac{\partial}{\partial x_n} + \frac{12}{5} \frac{\partial}{\partial u}. \]
Indeed, it can be written in the following form
\[ I_0 = c_1 I_{12/5} + \cdots + c_n I_{12/5} + c_n \]
where
\[ I_0 = ux_n^{-12/5}, \quad I_j = x_j x_n^{-1}, \quad j = 1, \ldots, n-1 \]
are invariants of the scaling symmetry generated by \( X \).
The ansatz can lead to the special solution
\[ u = \frac{(x^2 + y^2)^{2(1 + \alpha^2)^{-1}}}{\exp\left(\frac{\alpha}{1 + \alpha^2}\arctan\left(\frac{x}{y}\right)\right)} \]

are invariants under the action of the corresponding Lie group. The ansatz \( u = g(s) \)
leads to an ODE that is too complicated to write down. Nevertheless, the reduced ODE admits a factorisation where one of the factors is given by
\[ E_1[g] := (1 + \alpha^2)g_{ss} + \alpha g_s. \]

The linear ODE \( E_1[g] = 0 \) can be easily solved for every \( \alpha \) and implies the following solution of the \( \infty \)-Polylaplacian
\[ u = \frac{(x^2 + y^2)^{2(1 + \alpha^2)^{-1}}}{\exp\left(\frac{\alpha}{1 + \alpha^2}\arctan\left(\frac{x}{y}\right)\right)} \]

Similarly, when \( \gamma = 1 \) the invariants are \( u \) and
\[ z = -4\arctan\left(\frac{\alpha x + y}{1 + x - \alpha y}\right) + 2\alpha \ln\left[\alpha^2 + 2\alpha^2 x - 2\alpha^3 y + \alpha^2(1 + \alpha^2)(x^2 + y^2)\right] \]
and the reduced ODE for \( h(z) \) contains the following linear factor
\[ E_2[h] := 4(1 + \alpha^2)h_{zz} - \alpha h_z. \]

Solving the linear ODE \( E_2[h] = 0 \) for all \( \alpha \) we finally obtain the following solution
\[ u = \frac{[\alpha^2 + 2\alpha^2 x - 2\alpha^3 y + \alpha^2(1 + \alpha^2)(x^2 + y^2)]^{2(1 + \alpha^2)^{-1}}}{\exp\left(\frac{\alpha}{1 + \alpha^2}\arctan\left(\frac{\alpha x + y}{1 + x - \alpha y}\right)\right)} \]
of the \( \infty \)-Polylaplacian. It is interesting to notice that solution (5.7) can be seen
as the dominant part of solution (5.8) as \( \alpha \to \infty \).

Unfortunately the reductions that correspond to the generator \( A8 \) are too complicated to handle. We now focus on the reductions of the reduced \( \infty \)-Polylaplacian (2.1) that give additional information. Solutions of equation (2.1) that are invariant under translation in the \( x \)-direction are of the form \( u = g(y) \) where \( f \) satisfies
\[ 0 = g_{yy}^2 - 1 = (g_{yy} + 1)(g_{yy} - 1). \]
The solutions of these equations are just quadratic polynomials in \( y \) and thus add nothing new. Solutions invariant under rotations are of the form \( u = g(s) \) where \( s = x^2 + y^2 \) and \( g(s) \) satisfies the following ODE
\[ 16s^2g_{ss}^2 + 16s^2g_{sg}^2 + 8g_s^2 - 1 = 0. \]
The general solution of this ODE is not known, nevertheless a simple polynomial ansatz can lead to the special solution
\[ s = \frac{2\sqrt{2}}{2}. \]
The corresponding solution of the (2.1) and hence of (1.12) is contained in the family of polynomial solutions. Finally, the ansatz \( u = x^2g(s) \) where \( s = xy^{-1} \)
leads to solutions that are invariant under the Lie symmetry generated by \( Y_4 \). In
this case the reduced ODE for \( g \) is given by
\[ s^4(1 + s^2)^2g_{ss}^2 + 4s^2g_s^2(2 + 3s^2 + s^4)(g)g_{ss} + 2s^2(8 + 9s^2 + 2s^4)(g_s^2 + 16sg_{ss} + 4g_s^2 - 1 = 1. \]
As before a Laurent polynomial ansatz leads to the special solutions
\[ \frac{1}{\sqrt{2s}} \text{ and } \frac{1}{2s^2}. \]
The corresponding solutions of the $\infty$-Polylaplacian are contained in the polynomial family. Due to the complexity of the expressions we didn’t manage to obtain something meaningful in the final case ($B4$).

6. Conclusions

In this paper we studied the $\infty$-Polylaplacian and the reduced $\infty$-Polylaplacian in two dimensions from an algebraic point of view. For both equations we found the complete group of Lie point symmetries, we performed the corresponding symmetry reductions using all non-equivalent one dimensional subgroups and obtained special solutions (5.1), (5.3), (5.4), (5.5), (5.6), (5.7), (5.8) studying the corresponding reduced ODEs. We presented a conjecture on the full group of Lie point symmetries of the $\infty$-Polylaplacian and its reduced version in $n$-dimensions.

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References

[Aro65] G. Aronsson. “Minimization problems for the functional $\sup_x F(x, f(x), f'(x))$”. In: Ark. Mat. 6 (1965), 33–53 (1965).
[BGI89] V. A. Baikov, R. K. Gazizov, and N. K. Ibragimov. “Approximate symmetries”. In: Sbornik: Mathematics 64 (1989), pp. 427–441.
[BEJ08] E. N. Barron, L. C. Evans, and R. Jensen. “The infinity Laplacian, Aronsson’s equation and their generalizations”. In: Trans. Amer. Math. Soc. 360.1 (2008), pp. 77–101.
[BDM89] T. Bhattacharya, E. DiBenedetto, and J. Manfredi. “Limits as $p \to \infty$ of $\Delta_p u_p = f$ and related extremal problems”. In: Rend. Sem. Mat. Univ. Politec. Torino 47 (1989), pp. 15–68.
[BA08] G. Bluman and S. Anco. Symmetry and integration methods for differential equations. Vol. 154. Springer Science & Business Media, 2008.
[BC69] G. Bluman and J. Cole. “The general similarity solution of the heat equation”. In: Journal of Mathematics and Mechanics 18.11 (1969), pp. 1025–1042.
[CM94] P. A. Clarkson and E. L. Mansfield. “Algorithms for the nonclassical method of symmetry reductions”. In: SIAM Journal on Applied Mathematics 54.6 (1994), pp. 1693–1719.
[CLO92] D. Cox, J. Little, and D. O’Shea. Ideals, varieties, algorithms: An introduction to computational algebraic geometry and commutative algebra, UTM. Springer-Verlag, New York, 1992.
[DT06] S. Dimas and D. Tsoubelis. “A new Mathematica-based program for solving overdetermined systems of PDEs”. In: 8th International Mathematica Symposium (2006).
[DT04] S. Dimas and D. Tsoubelis. “SYM: A new symmetry-finding package for Mathematica”. In: Proceedings of the 10th international conference in modern group analysis (2004), pp. 64–70.
REFERENCES

[ETT15] A. E., M. T., and D. T. “On the $p$-Laplacian and $\infty$-Laplacian on Graphs with Applications in Image and Data Processing”. In: SIAM Journal on Imaging Sciences 8.4 (2015), pp. 2412–2451.

[FF11] I. L. Freire and A. C. Faleiros. “Lie point symmetries and some group invariant solutions of the quasilinear equation involving the infinity Laplacian”. In: Nonlinear Analysis: Theory, Methods & Applications 74.11 (2011), pp. 3478–3486.

[GH96] M. Giaquinta and S. Hildebrandt. Calculus of variations. I. Vol. 310. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. The Lagrangian formalism. Berlin: Springer-Verlag, 1996, pp. xxx+474.

[GM10] T. Gyulov and G. Morosanu. “On a class of boundary value problems involving the $p$-Biharmonic operator”. In: Journal of Mathematical Analysis and Applications 367.1 (2010), pp. 43–57.

[Hyd00] P. E. Hydon. Symmetry methods for differential equations: a beginner’s guide. Vol. 22. Cambridge University Press, 2000.

[Ibr93] N. H. Ibragimov. CRC Handbook of Lie group analysis of differential equations. Vol. 1-3. CRC press, 1993.

[Igb12] N. Igbida. “A Partial Integrodifferential Equation in Granular Matter and Its Connection with a Stochastic Model”. In: SIAM Journal on Mathematical Analysis 44.3 (2012), pp. 1950–1975.

[Jen93] R. Jensen. “Uniqueness of Lipschitz extensions: minimizing the sup norm of the gradient”. In: Arch. Rational Mech. Anal. 123.1 (1993), pp. 51–74.

[Kat15] N. Katzourakis. An introduction to viscosity solutions for fully nonlinear PDE with applications to calculus of variations in $L^\infty$. Springer Briefs in Mathematics. Springer, Cham, 2015, pp. xii+123.

[KM17] N. Katzourakis and R. Moser. “Existence, Uniqueness and Structure of Second Order absolute minimisers”. In: ArXiv preprint https://arxiv.org/abs/1701.03348. (2017).

[KP16a] N. Katzourakis and T. Pryer. “On the numerical approximation of $\infty$-biharmonic mappings”. In: Nonlinear Differential Equations and Applications NoDEA 23.6 (2016), p. 61.

[KP16b] N. Katzourakis and T. Pryer. “Second order $L^\infty$ variational problems and the $\infty$-polylaplacian”. In: ArXiv preprint https://arxiv.org/abs/1605.07880. (2016).

[LM90] A. Lazer and P. McKenna. “Large-amplitude periodic oscillations in suspension bridges: some new connections with nonlinear analysis”. In: Siam Review 32.4 (1990), pp. 537–578.

[Noe71] E. Noether. “Invariant variation problems”. In: Transport Theory Statist. Phys. 1.3 (1971). Translated from the German (Nachr. Akad. Wiss. Göttingen Math.-Phys. Kl. II 1918, 235–257), pp. 186–207.

[Oli10] F. Oliveri. “Lie symmetries of differential equations: classical results and recent contributions”. In: Symmetry 2.2 (2010), pp. 658–706.

[Olv93] P. J. Olver. Applications of Lie groups to differential equations. Second. Vol. 107. Graduate Texts in Mathematics. New York: Springer-Verlag, 1993, pp. xxviii+513.
REFERENCES

[Ovs82] L. Ovsiannikov. “Group Analysis of Differential Equations”. In: Academic Press, New York (1982).

[Pry17] T. Pryer. “On the finite element approximation of Infinity-Harmonic functions”. In: To appear in Proceedings A of the Royal Society of Edinburgh. (2017).

[Rei90] G. Reid. “A triangularization algorithm which determines the Lie symmetry algebra of any system of PDEs”. In: Journal of Physics A: Mathematical and General 23.17 (1990), p. L853.

[Sch07] F. Schwarz. Algorithmic Lie theory for solving ordinary differential equations. CRC Press, 2007.

[Ste89] H. Stephani. Differential equations: their solution using symmetries. Cambridge University Press, 1989.

[Win93] P. Winternitz. “Lie groups and solutions of nonlinear partial differential equations”. In: Nato Asi Series C Mathematical And Physical Sciences 409 (1993).

[Yag88] M Yaglom. “Felix Klein and Sophus Lie”. In: Evolution of the Idea of Symmetry in the Nineteenth Century, Birkhäuser, Boston etc (1988).

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