GENERATORS FOR FINITE SIMPLE MOUFGAN LOOPS

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ABSTRACT. Moufang loops are one of the best-known generalizations of groups. There is only one countable family of nonassociative finite simple Moufang loops, arising from the split octonion algebras. We prove that every member of this family is generated by three elements, using the classical results on generators of unimodular groups.

Keywords: finite simple Moufang loops, Paige loops, minimal generators, generators for unimodular groups, projective unimodular groups.

1. Finite Simple Moufang Loops

A Moufang loop is a loop satisfying the identity \( x(y(xz)) = ((xy)x)z \). Every element \( x \) of a Moufang loop has a (unique) two-sided inverse \( x^{-1} \); every two elements generate a group. Loops with the latter property are called diassociative \([10]\). The best-known Moufang loop is the multiplicative loop of nonzero elements in the standard 8-dimensional real octonion algebra \( \mathbb{O} \). Surely the best-known finite Moufang loop is the 240-element loop \( L \) of integral octonions of norm one \([4]\).

In 1956, L. Paige \([9]\) found one nonassociative finite simple Moufang loop for every finite field. Following E. Bannai and S. Song \([3]\), we denote this Paige loop constructed over \( F = GF(q) \) by \( M^*(q) \). Let us give the most brief description of \( M^*(q) \) now.

Consider the Zorn multiplication

\[
\begin{pmatrix}
    a & \alpha \\
    \beta & b
\end{pmatrix}
\begin{pmatrix}
    c & \gamma \\
    \delta & d
\end{pmatrix}
= \begin{pmatrix}
    ac + \alpha \cdot \delta & a\gamma + \alpha d - \beta \times \delta \\
    \beta c + b\delta + \alpha \times \gamma & \beta \cdot \gamma + bd
\end{pmatrix},
\]

where \( a, b, c, d \in F \), \( \alpha, \beta, \gamma, \delta \in F^3 \), and where \( \alpha \cdot \delta \) (resp. \( \alpha \times \delta \)) denotes the dot product (resp. cross product) of \( \alpha \) and \( \delta \). This is the same formula M. Zorn used to construct the split octonion algebra over \( F \). The loop \( M^*(q) \) consists of all matrices

\[
M = \begin{pmatrix}
    a & \alpha \\
    \beta & b
\end{pmatrix}
\]

with \( \det M = ab - \alpha \beta = 1 \) that are multiplied according to (1), and where \( M \) and \( -M \) are identified. The neutral element of \( M^*(q) \) is the identity matrix \( I \), and the inverse of \( M \) is

\[
M^{-1} = \begin{pmatrix}
    b & -\alpha \\
    -\beta & a
\end{pmatrix}.
\]

MSC: PRIMARY 20N05, SECONDARY 20F05.
In 1987, M. Liebeck \[8\] proved that there are no other nonassociative finite simple Moufang loops. The loop $M^*(2)$ is exceptional in the sense that it shows up in the real algebra $O$, too. Namely, $M^*(2)$ is isomorphic to the quotient of $L$ by its center $Z(L) = \{1, -1\}$ (see \[4\] once again).

Associative finite simple Moufang loops are finite simple groups. It is a remarkable fact that every finite simple group is 2-generated \[2\]; even more so, since no proof using only the simplicity is known. Instead, every family of finite simple groups must be investigated separately. Because of diassociativity, the nonassociative Paige loops cannot be 2-generated. It is reasonable to expect that a small number of generators will do. Indeed, it this paper we prove that:

**Theorem 1.1.** Every nonassociative finite simple Moufang loop is 3-generated.

Note that Theorem 1.1 was proved in \[11\] for all Paige loops $M^*(p)$, $p$ a prime. Thus the main task of this paper is to cover the general case. We also present a simple proof for the prime case, and offer at least two generating sets for every $M^*(q)$. The reader who wants to establish 1.1 as quickly as possible should focus on 2.2, 2.3, and Section 3.

2. Generators for $L_2(q)$

The crucial observation concerning Paige loops is that $M^*(q)$ contains several copies of $L_2(q) = PSL_2(q)$. Given the canonical basis $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$, $e_3 = (0, 0, 1)$ of $F^3$, let $\phi_i : L_2(q) \to M^*(q)$ be defined by

$$\phi_i \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b c_i \\ c e_i & d \end{pmatrix},$$

and let $G_i$ be the image of $L_2(q)$ under $\phi_i$. Since the multiplication in $G_i$ coincides with the usual matrix multiplication (all cross products involved in (1) vanish), $\phi_i$ is an isomorphism $L_2(q) \to G_i$.

This brings our attention to the classical results concerning generators for $L_2(q)$ and $SL_2(q)$. First of all, we have the Dickson Theorem:

**Theorem 2.1** (L. E. Dickson, 1900). If $q \neq 9$ is an odd prime power or $q = 2$, then $SL_2(q)$ is generated by

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix},$$

where $\lambda$ is a primitive element of $GF(q)$.

The proof can be found in \[6\], and more recently in \[7, pp. 44–55\]. The statement of the theorem usually does not mention $q = 2$, although it is apparently true for $q = 2$, since $SL_2(2) \cong S_3$ is generated by any two involutions, in particular by (2).

A. A. Albert and J. Thompson proved \[1, Lemma 8\] that for any primitive element $\lambda$ of $GF(q)$, $q > 2$, the group $SL_2(q)$ is generated by $B$, $-B$, and $C$. 

where

\[(3) \quad B = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 1 \\ -1 & \lambda \end{pmatrix}.\]

We therefore have:

**Proposition 2.2** (A. A. Albert, J. Thompson, 1959). Let \( q > 2 \) be a prime power. Then \( L_2(q) \) is generated by \( B \), where \( \lambda \) is a primitive element of \( GF(q) \).

The generators \( B \) are especially convenient for our purposes, because \( \phi_i(B) = B \) for every \( i, 1 \leq i \leq 3 \); but let us not get ahead of ourselves. It is practical to know some generators that do not involve a primitive element. For that matter, Coxeter and Moser argue in [5] that

**Lemma 2.3.** For every prime \( p \), the group \( L_2(p) \) is generated by

\[(4) \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.

3. Generators for \( M^*(q) \)

Our first result concerning \( M^*(q) \) has nothing to do with the generators for \( L_2(q) \). In its proof, we take advantage of the following lemma due to Paige:

**Lemma 3.1** (L. Paige, 1956). \( M^*(q) \) is generated by

\[(5) \quad M_\beta = \begin{pmatrix} 1 & 0 \\ \beta & 1 \end{pmatrix}, \quad M'_\beta = \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix},\]

where \( \beta \) runs over all nonzero vectors in \( F^3 \).

**Proof.** Combine Lemmas 4.2 and 4.3 of [5]. \( \square \)

**Proposition 3.2.** \( M^*(q) \) is generated by \( G_1 \cup G_2 \cup G_3 \).

**Proof.** Let \( Q \) be the subloop of \( M^*(q) \) generated by \( G_1 \cup G_2 \cup G_3 \). Thanks to Lemma 3.1, it suffices to prove that \( Q \) contains all elements \( M_\beta, M'_\beta \), defined in \( [5] \). We show simultaneously that \( M_\beta \in Q \) and \( M'_\beta \in Q \).

Let \( k \) denote the number of nonzero entries of \( \beta \). There is nothing to prove when \( k \leq 1 \). Suppose that \( k = 2 \). Without loss of generality, let \( \beta = (a, b, 0) \) for some \( a, b \in F^* = F \setminus \{0\} \). Verify that

\[
\begin{pmatrix} 1 & ac_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & be_2 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ -abc_3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & (a, b, 0) \\ 0 & 1 \end{pmatrix},
\]

and thus that \( M_\beta \in Q \). Similarly, \( M'_\beta \in Q \). We can therefore assume that \( Q \) contains all elements \( M_\beta, M'_\beta \) with \( k \leq 2 \).

Let \( k = 3, \beta = (a, b, c) \) for some \( a, b, c \in F^* \). As

\[
\begin{pmatrix} 1 & (a, b, 0) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & (0, 0, c) \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ -bc, ac, 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & (a, b, c) \\ 0 & 1 \end{pmatrix},
\]

\( M_\beta \) belongs to \( Q \). Symmetrically, \( M'_\beta \in Q \), and we are done. \( \square \)
In fact, $G_1 \cup G_2$ already generates $M^*(q)$. The role of the cross product is especially apparent in the next Proposition.

**Proposition 3.3.** The subgroup $G_3$ is contained in the subloop of $M^*(q)$ generated by $G_1 \cup G_2$. In particular, $M^*(q)$ is generated by $G_1 \cup G_2$.

**Proof.** As it turns out, all we need are these two equations:

\[
\begin{pmatrix}
1 & 0 \\
\lambda e_3 & 1
\end{pmatrix}
= -\begin{pmatrix}
0 & e_2 \\
-e_2 & 0
\end{pmatrix}
\begin{pmatrix}
1 & \lambda e_1 \\
-\lambda^{-1} e_1 & 0
\end{pmatrix}
\begin{pmatrix}
1 & e_2 \\
-e_2 & 0
\end{pmatrix}
\begin{pmatrix}
1 & \lambda e_1 \\
-\lambda^{-1} e_1 & 0
\end{pmatrix},
\]

\[
\begin{pmatrix}
0 & e_3 \\
-e_3 & 0
\end{pmatrix}
= \begin{pmatrix}
0 & e_1 \\
e_1 & 0
\end{pmatrix}
\begin{pmatrix}0 & -e_2 \\
e_2 & 0\end{pmatrix}.
\]

Note that the left hand sides of these equations are elements of $G_3$, whereas the right hand sides are products of elements of $G_1 \cup G_2$. When $q = 2$, we are done by Lemma 2.3. When $q > 2$, observe that

\[
\begin{pmatrix}
1 & 0 \\
\lambda & 1
\end{pmatrix}
\begin{pmatrix}0 & 1 \\
-1 & 0\end{pmatrix}
= \begin{pmatrix}0 & 1 \\
-1 & \lambda\end{pmatrix} = C.
\]

Since $B = \phi_i(B)$ for every $i$, $1 \leq i \leq 3$, we are done by Proposition 2.2. \hfill \square

Theorem 1.1 is now proved. When $q > 2$, $M^*(q)$ is generated by $\phi_1(C)$, $\phi_2(C)$ and $B = \phi_1(B) = \phi_2(B)$, by Propositions 2.2 and 3.3. When $q = 2$, we are done by the main result of [11], Theorem 2.1.

For the sake of completeness, allow us to present an alternative, simpler proof of [11, Theorem 2.1].

**Proposition 3.4.** [11, Theorem 2.1] Let $p$ be a prime. Then $M^*(p)$ is generated by

\[
U_1 = \begin{pmatrix}1 & e_1 \\
0 & 1\end{pmatrix}, \quad U_2 = \begin{pmatrix}1 & e_2 \\
0 & 1\end{pmatrix}, \quad X = \begin{pmatrix}0 & e_3 \\
e_3 & 1\end{pmatrix}.
\]

**Proof.** First check that

\[\begin{pmatrix}1 & 0 \\
1 & 1\end{pmatrix} = \begin{pmatrix}0 & 1 \\
-1 & 0\end{pmatrix}\begin{pmatrix}1 & 1 \\
0 & 1\end{pmatrix}^{-1}\begin{pmatrix}0 & 1 \\
-1 & 0\end{pmatrix}^{-1}.
\]

Combine (4) and (6) to see that $L_2(p)$ is generated by

\[
U = \begin{pmatrix}1 & 1 \\
0 & 1\end{pmatrix}, \quad V = \begin{pmatrix}0 & 1 \\
1 & 0\end{pmatrix}.
\]

Consequently, $M^*(p)$ is generated by $U_1 = \phi_1(U)$, $U_2 = \phi_2(U)$, $V_1 = \phi_1(V)$, and $V_2 = \phi_2(V)$. Now,

\[
V_2 = -(XU_1 \cdot XU_2) \cdot X^{-1} U_1, \quad V_1 = -U_1 U_2 \cdot (V_2 \cdot U_1 X),
\]

and we are through. \hfill \square
4. More Generating Sets

We would like to show how to obtain additional generating sets for $M^*(q)$. We take advantage of Proposition 3.3, Dickson’s Theorem, and of the fact that $SL_2(2^r)$ (for $r > 1$) is generated by

$$D_1 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \quad D_2 = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix},$$

where $\lambda$ is a primitive element of $GF(2^r)$. We leave the verification of (7) to the reader.

Since $\phi_i(D_2) = D_2$ for $i = 1, 2, 3$, we immediately see from Proposition 3.3 that $M^*(2^r)$ (for $r > 1$) is generated by $\phi_1(D_1)$, $\phi_2(D_1)$ and $D_2$.

**Proposition 4.1.** Let $q \neq 9$ be an odd prime power or $q = 2$. Then $M^*(q)$ is generated by

$$\begin{pmatrix} 1 & e_1 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & e_2 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & \lambda e_3 \\ -\lambda^{-1} e_3 & 1 \end{pmatrix},$$

where $\lambda$ is a primitive element of $GF(q)$.

**Proof.** Keeping Proposition 3.3 and Dickson’s Theorem in mind, we only need to obtain the elements

$$\begin{pmatrix} 1 & 0 \\ \lambda e_{i} & 1 \end{pmatrix},$$

for $i = 1, 2$. Straightforward computation reveals that

$$
\begin{pmatrix} 1 & 0 \\ \lambda e_{1} & 1 \end{pmatrix}^{-1} = -\begin{pmatrix} 0 & \lambda e_3 \\ -\lambda^{-1} e_3 & 1 \end{pmatrix}^2 \begin{pmatrix} 1 & e_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & \lambda e_3 \\ -\lambda^{-1} e_3 & 1 \end{pmatrix},
$$

$$
\begin{pmatrix} 1 & 0 \\ \lambda e_{2} & 1 \end{pmatrix}^{-1} = -\begin{pmatrix} 0 & \lambda e_3 \\ -\lambda^{-1} e_3 & 1 \end{pmatrix}^2 \begin{pmatrix} 1 & e_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & \lambda e_3 \\ -\lambda^{-1} e_3 & 1 \end{pmatrix}.
$$

Note that the expressions on the right hand side can be evaluated in any order. □

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