Renormalization theory of a two dimensional Bose gas: quantum critical point and quasi-condensed state

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To the memory of Kenneth Wilson,
for his inspiring and influential ideas

Abstract

We present a renormalization group construction of a weakly interacting Bose gas at zero temperature in the two-dimensional continuum, both in the quantum critical regime and in the presence of a condensate fraction. The construction is performed within a rigorous renormalization group scheme, borrowed from the methods of constructive field theory, which allows us to derive explicit bounds on all the orders of renormalized perturbation theory. Our scheme allows us to construct the theory of the quantum critical point completely, both in the ultraviolet and in the infrared regimes, thus extending previous heuristic approaches to this phase. For the condensate phase, we solve completely the ultraviolet problem and we investigate in detail the infrared region, up to length scales of the order $(\lambda^3 \rho_0)^{-1/2}$ (here $\lambda$ is the interaction strength and $\rho_0$ the condensate density), which is the largest length scale at which the problem is perturbative in nature. We exhibit violations to the formal Ward Identities, due to the momentum cutoff used to regularize the theory, which suggest that previous proposals about the existence of a non-perturbative non-trivial fixed point for the infrared flow should be reconsidered.
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1 Introduction and main results

The study of systems of non-relativistic bosons in three or less dimensions has been revived by the amazing experimental progresses of the last two decades. Starting from the mid nineties, it became possible to manipulate cold atoms in such a detailed and controlled way that Bose-Einstein Condensates (BEC) in optical traps and optical lattices were created, and the analogue of several quantum many body toy models were set up in actual experiments. It is becoming more and more possible to investigate crossover effects between different regimes and dimensionality. In this perspective, it is interesting to clarify and extend previous analyses about the low energy behavior of interacting Bose systems.

From a theoretical point of view, the two-dimensional case is particularly challenging: two dimensions appears to be a critical case both for the stability of the so-called free bosons fixed point, and for the stability of Bogoliubov theory [14], in the sense of the Renormalization Group (RG). This is the method for understanding from first principles the emergence of scaling laws in interacting many body systems at low or zero temperatures, as well as for computing thermodynamic and correlation functions. Applications to the Bose gas date back to the works of Beliaev (1958, [5]) and Hugenholtz–Pines (1959, [34]), whose ideas were further developed in Lee and Yang [35], Gavoret and Nozières [25], Nepomnyashchii and Nepomnyashchii [39] and Popov and Seredniakov [41].

All these works investigate the effects of corrections to Bogoliubov’s theory in the condensed phase, starting from the analysis of infrared divergences in perturbation theory (PT). They are based on diagrammatic techniques borrowed from Quantum Field Theory and on the summations of special classes of diagrams selected from the divergent PT, possibly by combining them with the use of Ward identities [25]. In this sense, they do not provide a systematic study of the problem, and the resulting indications of the stability of the Bogoliubov’s spectrum under perturbations are questionable.

The first systematic study of the whole PT in the presence of a non-zero condensate fraction in 3D, and the proof of its order by order convergence after proper resummations, is due to G. Benfatto in 1994 [7]. The method employed in his work is the Wilsonian RG, combined with the ideas of constructive RG, in the form developed by the roman school of Benfatto, Gallavotti et al since the late seventies [23] and already applied to a certain number of infrared problems in condensed matter systems [9, 8, 10, 13, 11, 28, 29, 31, 30, 37]. It is worth stressing that even though the methods used in [7] are based on the ideas of constructive field theory, the resulting bounds derived there are not enough for constructing the theory in a mathematically complete form: they are enough for deriving finite bounds at all orders in renormalized perturbation theory, growing like n! at the n-th order (“n! bounds”), but the possible Borel summability of the series remains an outstanding open problem. A program addressing this issue has been started by T. Balaban and collaborators (see [4] and references therein), but the solution is still far from being reached: the idea is to apply the Wilsonian RG in the form developed by Balaban in a series of works dedicated to the construction of theories with a broken
continuous symmetry [1, 2, 3], but the new technical problems arising in the case of a
complex gaussian measure (as the one appearing in the Bose gas) seem to be serious
obstacles to the solution.

The method and the ideas of Benfatto were later extended by Pistoleti et al [15, 40],
who implemented a systematic RG analysis of the Bose gas in $1 < d \leq 3$ dimensions,
by using dimensional regularization, and by implementing local Ward identities at all
orders of PT. The systematic use of local Ward identities allowed them to recover in a
conceptually simpler and more satisfactory way some of the cancellations determined by
Benfatto by inspection of PT. Moreover, they managed to extend the analysis down to
d = 2, modulo an assumption about the convergence of the flow of the particle-particle
effective interaction: this was assumed to flow towards an $O(1)$ infrared fixed point, as
suggested by a natural truncation of PT at the one loop level. A similar conclusion for the
infrared behavior of the zero temperature Bose gas was later recovered by Wetterich [48],
Dupuis [20, 17, 18], and Sinner et al [47], via the methods of the functional RG a’la
Wetterich, which involves a different regularization and truncation scheme, combined
with a numerical study of the resulting flow equations.

The main conclusion of [15, 40] (later confirmed by [48, 20, 17, 18, 47]), which we
will criticize below, is that the Bose-Einstein condensate is stable at zero temperature in
two dimensions, in the presence of weak repulsive interactions (and/or at low densities),
in agreement with Bogoliubov’s theory. This fact is a priori not obvious, since two
dimensions is critical for the existence of BEC; e.g., there is no condensation at any
finite temperature [33]. On top of that, [15, 40] find that the Bogoliubov’s spectrum is
stable (i.e., the speed of sound is finite), notwithstanding the fact that the flow of the
effective constants is anomalous, in the sense that it is different from what suggested by
naive power counting. In this respect, Ward identities play a crucial role in driving the
flow of the effective constants towards the “right direction”.

The zero-temperature condensed phase in two dimensions is not the only possible
critical phase for non-relativistic bosons: in fact, the condensed phase emerges in the
vicinity of a quantum (i.e., zero temperature) critical point controlled by the chemical
potential $\mu$, see [45]. The critical point separates a Mott insulator, “quantum disor-
dered”, phase [22, 45, 46] from the BEC; the critical theory at $\mu = 0$ has interesting
scaling properties, with logarithmic corrections in $d = 2$, which have been investigated
by RG methods in [21, 46]. By the same methods, Fisher and Hohenberg [21] also
computed the Kosterlitz-Thouless line $T = T(\mu)$ separating a quasi-condensed phase (in
the sense of algebraic decay of correlations) from a fully disordered one. More recently,
Dupuis and Rançøn [44, 43, 42] further extended the results of [21] by investigating the
critical behavior at the Kosterlitz-Thouless line by functional RG methods.

In the following we shall review and extend the Wilsonian RG approach to the
Bose gas in the two-dimensional continuum, in the formalism proposed by Benfatto
and Gallavotti. We shall prove renormalizability both of the critical theory and of the
condensed phase, both in the ultraviolet and in the infrared, and we shall develop a
theory valid at all orders in renormalized perturbation theory, with explicit bounds on
the generic order.
We shall first focus on the quantum critical point, which is very simple from the perspective of power counting: it defines a theory that is super-renormalizable in the ultraviolet and asymptotically free in the infrared; it describes the vacuum fluctuations of the theory, and it is in some sense the non-relativistic analogue of the dynamical vacuum of QED. Our results match with those of [21, 46], and extend them to all orders, in a framework that is the most promising for subsequent constructive approaches\(^1\).

Next, we shall move to the much more interesting condensed phase. The ultraviolet is super-renormalizable, in a similar sense as the critical theory. Things start to change at the energy scale where the kinetic energy becomes comparable with \(\lambda \rho_0\), where \(\rho_0\) is the condensate density, and \(\lambda\) the a-dimensional interaction strength: for smaller scales the theory becomes just renormalizable, with 8 running coupling constants, versus one free parameter (the chemical potential, which has to be fixed so to realize the right condensate density). Compared with previous works, we have an extra marginal coupling constant, associated with three-body interactions: this was previously overlooked, and its role is analyzed in our work for the first time.

Due to the presence of two relevant couplings, the flow of the effective constants is rapidly driven towards values of order 1, at which point the system leaves the perturbative regime and we stop the flow. The critical energy scale at which we interrupt the flow is of the order \(\lambda^2 \rho_0\) (corresponding to a length scale \((\lambda^3 \rho_0)^{-1/2}\)). For higher energies the theory is well defined at all orders and there we provide explicit bounds on the \(n\)-th order coefficients in the renormalized coupling for all the thermodynamic observables. For lower energies we cannot conclude anything from the RG analysis. The fact that the theory is well defined only in a limited range of length scales should be interpreted as an instance of the stability of the condensate order parameter up to length scales of the order \((\lambda^3 \rho_0)^{-1/2}\). Since we have no control on the behavior of the system on larger length scales, we cannot exclude the possibility that the condensate coherence length is finite in the ground state: this is the reason of the name “quasi-condensed state” in the title. However, we do not claim to have a proof of the absence of long-range order at zero temperature.

While we cannot investigate the region of energies smaller than \(\lambda^2 \rho_0\), we have a complete control of the theory at higher energy scales, including the effects of cutoffs and of the irrelevant terms. In particular, we can verify there the validity of the Ward Identities used extensively in [15, 40], as well as in [48, 20, 17, 18, 47], for extrapolating the flow to lower energy scales and for understanding the nature of the (putative) infrared fixed point. Quite remarkably, we prove that for energy scales intermediate between \(\lambda \rho_0\) and \(\lambda^2 \rho_0\) some of the Ward Identities derived in [15, 40] within a dimensional regularization scheme are explicitly violated at lowest order in the presence of a momentum regularization. We also identify the source of the violation in a correction term due to the cutoff function used for regularizing the theory. We stress that the violation appears at the one-loop level, in a region where perturbation theory is unambiguously valid, i.e., before

\(^1\)It would be very interesting to see whether the ideas of Balaban et al [4] could be applied to the actual construction of this theory, which is much easier than the condensed one, and still very relevant for the physics of the Mott transition.
we enter the non-perturbative regime. It may be related to the emergence of an anomaly term in the response functions. Therefore, the use of the Ward Identities in the deep infrared region by the authors of [15, 40] is questionable. We believe that our finding motivates a reconsideration of the nature and the very existence of a 2D condensate at zero temperature. It would be very interesting to investigate in an unbiased way the deep infrared region, i.e., the one at energy scales smaller than $\lambda^2 \rho_0$, via systematic non-perturbative simulations, possibly based on Borel resummations of the divergent series, which are missing so far. It would also be interesting to see whether it is possible to implement within our rigorous RG analysis the use of a different momentum regularization scheme that does not violate local gauge invariance at any finite scale, and to see whether in such a case the corrections to the local Ward Identities would vanish exactly or not\(^2\). We hope to come back to this issue in a future publication.

The rest of the paper is organized as follows: in Section 1.A we define the model and explain more precisely what we mean by quantum critical point or condensed state. In Section 2 we present the renormalization theory for the quantum critical point: first we reformulate the model in the form of a functional integral; then we describe the multiscale integration scheme for its computation, including the Gallavotti-Nicolò tree expansion and the proof of the basic $n!$ bounds; next, we modify the expansion by properly renormalizing and resumming the potentially divergent contributions; finally we study the flow of the effective constants. In Section 3 we present the renormalization theory of the condensed phase: as a starting point we manipulate the functional integral, so to put it into the form of a perturbed Bogoliubov’s theory; then we describe the multiscale integration of the resulting theory, first in the ultraviolet region, and then in the infrared; finally we study the flow equations in the infrared region, we discuss their compatibility with the Ward Identities and compare in a more technical way our findings with those of [15, 40]. Finally, in Section 4, we draw the conclusions. The Appendices collect a number of (very important!) technical aspects of the construction, among which: the one-loop beta function, the derivation of the global and local Ward Identities (including the correction terms due to the momentum cutoff), the verification of the Ward Identities at lowest order (including the proof of violation of the formal Ward Identities at lowest order).

1.A The model

We are interested in the study of the properties of a gas of non-relativistic bosons in a two-dimensional box $\Omega_L$ of side $L$ with periodic boundary conditions (to be eventually sent to infinity), interacting via a weak repulsive short range two-body potential. The corresponding grandcanonical Hamiltonian at chemical potential $\mu$ in second quantized form is

$$H_L = \sum_{\vec{k} \in D_L} (\vec{k}^2 - \mu) \hat{a}_{\vec{k}}^+ \hat{a}_{\vec{k}} + \frac{\lambda}{2L^2} \sum_{\vec{k},\vec{k}',\vec{p} \in D_L} \hat{v}(\vec{p}) \hat{a}_{\vec{k}+\vec{p}}^+ \hat{a}_{\vec{k}'}^+ \hat{a}_{\vec{k}'} \hat{a}_{\vec{k}} \equiv H_L^0 + V_L \ , \quad (1.1)$$

\(^2\)A possibility could be to suitably modify the infrared regulator used in [17, 19], so to make it locally gauge invariant at any finite scale.
where: \( \mathcal{D}_L = \{ \vec{k} = 2\pi \vec{n}/L, \ \vec{n} \in \mathbb{Z}^2 \} \); \( \hat{a}_k^\dagger \) and \( \hat{a}_k \) are creation and annihilation operators, satisfying the canonical commutation rules; \( \hat{v}(\vec{k}) = \int d\vec{x} e^{i\vec{k} \cdot \vec{x}} v(\vec{x}) \) is the Fourier transform of the two-body potential \( v(\vec{x}) \), which is assumed to be positive and positive definite (for simplicity); \( 0 < \lambda \ll 1 \) is the strength of the interaction. The Hamiltonian in (1.1) acts on the Fock space \( \mathcal{F}_L \) obtained as the direct sum of the \( N \)-particles Hilbert spaces of square-summable symmetric wave functions on the box \( \Omega_L \) of side \( L \). Our goal is to construct the thermal ground state in infinite volume, defined as the limit \( \beta \to \infty \) of the infinite volume grand canonical Gibbs state at inverse temperature \( \beta \) (i.e., we want to send \( L \to \infty \) first, and then \( \beta \to \infty \)). If needed, we allow \( \mu \) to depend upon \( L \) and \( \beta \), \( \mu = \mu_{\beta,L} \); the requirement is that the resulting ground state has total density \( \rho \) or total condensate density \( \rho_0 \); we shall be more specific in the following. Note that the units are chosen in such a way that \( \hbar = 2m = 1^3 \).

A basic thermodynamic quantity we are interested in is the specific ground state energy

\[
e_0(\rho) = - \lim_{\beta \to \infty} \lim_{L \to \infty} \frac{1}{\beta L^2} \log \text{Tr}_{\mathcal{F}_L} \text{e}^{-\beta(H_L - \mu_{\beta,L}N)} , \tag{1.2}
\]

with

\[
\mu = \lim_{\beta \to \infty} \lim_{L \to \infty} \mu_{\beta,L} = \partial_\rho e_0(\rho) , \tag{1.3}
\]

modulo exchange of limits issues, to be discussed more carefully below.

We are also interested in the \( 2n \)-points correlation functions in imaginary time, also known as Schwinger functions: given the “imaginary time” \( x_0 \in [0, \beta) \), and denoting by \( \vec{x} = (x_0, \vec{x}) \in [0, \beta) \times \Omega_L \) the space-time coordinates, we let

\[
S_{\sigma_1, ..., \sigma_{2n}}(x_1, ..., x_{2n}) = \lim_{\beta \to \infty} \lim_{L \to \infty} \frac{\text{Tr}_{\mathcal{F}_L} \left( \text{e}^{-\beta(H_L - \mu_{\beta,L}N)} \right) T \{ \hat{a}_{x_1}^{\sigma_1} \cdots \hat{a}_{x_{2n}}^{\sigma_{2n}} \} }{\text{Tr}_{\mathcal{F}_L} \text{e}^{-\beta(H_L - \mu_{\beta,L}N)}} , \tag{1.4}
\]

where, if

\[
a_\vec{x}^{\pm} = \frac{1}{L} \sum_{\vec{k} \in \mathcal{D}_L} \hat{a}_\vec{k}^{\pm} e^{\pm i\vec{k} \cdot \vec{x}} , \tag{1.5}
\]

then the time-evolved operator \( a_\vec{x}^{\pm} \) is

\[
a_\vec{x}^{\pm} = e^{(H - \mu_{\beta,L}N)x_0} a_\vec{x}^{\pm} e^{-(H - \mu_{\beta,L}N)x_0} \tag{1.6}
\]

Moreover, the operator \( T \) appearing in the r.h.s. of (1.4) is the time-ordering operator, which re-arranges times in decreasing chronological order:

\[
T \{ \hat{a}_{x_1}^{\sigma_1} \cdots \hat{a}_{x_{2n}}^{\sigma_{2n}} \} = \hat{a}_{x_{\pi(1)}}^{\sigma_{\pi(1)}} \cdots \hat{a}_{x_{\pi(2n)}}^{\sigma_{\pi(2n)}} , \tag{1.7}
\]

where \( \pi \) is a permutation of \( \{1, ..., n\} \) such that \( x_{\pi(1)} > \cdots > x_{\pi(n)} \). In particular, the two-points Schwinger function is often called the propagator, or interacting

\[^{3}\text{With this choice, the dimensions of the physical quantities speed (c), momentum (k), and energy (E) are respectively: } \sqrt{c} = |k| = |L|^{-1} \text{ and } |E| = |k_0| = |L|^{-2}. \]

\[^{4}\text{If some of the time coordinates are equal to each other, each group of operators with the same times is reordered in such a way that the creation operators are all on the left of the annihilation operators, within that group.} \]
propagator, to clearly distinguish it from the unperturbed one:

\[ S(x - y) := S_{-+}(x, y). \quad (1.8) \]

If \( \lambda = 0 \) and \( \mu_{\beta,L} \to 0^- \) as \( \beta, L \to \infty \), then the ground state has propagator

\[ S^0(x - y) = \rho_0 + \int_{\mathbb{R}^3} \frac{d|}{(2\pi)^3} \frac{e^{-i|k|(x-y)}}{1 - i|k| + |k|^2}, \quad (1.9) \]

where

\[ \rho_0 = \lim_{\beta \to \infty} \lim_{L \to \infty} \frac{1}{L^2 e^{-\beta \mu_{\beta,L} - 1}} \quad (1.10) \]

represents the average occupation number of the \( \vec{\kappa} = \vec{0} \) state per unit volume (condensate density). Note that the propagator tends polynomially to \( \rho_0 \), as \( |x| \to \infty \). In this sense the theory is critical, both if \( \rho_0 = 0 \) and \( \rho_0 > 0 \). Note also that in this non-interacting case, the higher points correlations can all be reduced to the propagator, via the Wick rule, which is valid simply because the Hamiltonian is quadratic in the creation/annihilation operators.

If \( \lambda > 0 \), the problem is not exactly solvable anymore. For small \( \lambda \), one can expand the thermodynamic and correlation functions in formal power series in \( \lambda \), and then try to give sense to it, at least order by order. The problem is not simple, since the perturbation theory is affected by divergences, both in the ultraviolet and infrared side. As we shall see, the non-trivial divergences are the infrared ones, which need to be treated differently, depending on whether \( \rho_0 = 0 \) or \( \rho_0 > 0 \).

The first case we shall treat is \( \rho_0 = 0 \), which is the quantum critical point, in the sense of Fisher et al. [22] and Sachdev [45]. In this context, we expand all thermodynamic and correlation functions by decomposing \( H_L \) as in the r.h.s. of (1.1), \( H_L = H_0^L + V_L \). The main result is that the formal perturbation theory in \( V_L \) with propagator (1.9) at \( \rho_0 = 0 \) is renormalizable at all orders. More than that: the theory is super-renormalizable in the ultraviolet and asymptotically free in the infrared, with the effective two-body interaction strength flowing to zero logarithmically as the infrared cutoff is removed. As a consequence, the dressed propagator has the same decay properties as the free one, modulo logarithmic corrections. Our method allows us to produce explicit estimates on the generic order of renormalized perturbation theory, growing like \( n! \) at the \( n \)-th order.

The second case is the condensate state, i.e., the analysis of the formal perturbation theory with propagator (1.9) at \( \rho_0 > 0 \). The reference quadratic theory, which the interacting system is supposedly a perturbation of, is the Bogoliubov’s Hamiltonian, which is obtained from (1.1) by replacing \( \hat{a}_0 \) by \( \sqrt{\rho_0} L \) and by keeping all the terms that are at most quadratic in the operators \( \hat{a}_k^\dagger \) with \( \vec{k} \neq \vec{0} \):

\[ H_{B,L} = C_0 L^2 + \sum_{\vec{k} \in \mathcal{D}_L \setminus \{0\}} \left[ f(\vec{k}) \hat{a}_k^\dagger \hat{a}_k + \frac{1}{2} g(\vec{k})(\hat{a}_k^\dagger \hat{a}_{-\vec{k}} + \hat{a}_{\vec{k}}^\dagger \hat{a}_{-\vec{k}\dagger}) \right], \quad (1.11) \]
with

\[ f(\vec{k}) = \vec{k}^2 - \mu_{B,L} + \lambda \rho_0 \hat{v}(\vec{0})(1 - \frac{1}{2\rho_0 L^2}) + \lambda \rho_0 \hat{v}(\vec{k}) , \quad g(\vec{k}) = \lambda \rho_0 \hat{v}(\vec{k}) , \quad (1.12) \]

\[ C_{0,L} = -\mu_{B,L} \rho_0 + \frac{\lambda}{2} \hat{v}(\vec{0}) \rho_0^2 (1 - \frac{1}{\rho_0 L^2}) . \quad (1.13) \]

Correspondingly, \(H_L\) can be rewritten identically as \(H_L = C_{0,L} L^2 + \tilde{H}_{B,L} + W_L\), with

\[ \tilde{H}_{B,L} = \sum_{\vec{k} \in \mathcal{D}_{L\setminus\{0\}}} \left[ F(\vec{k}) \hat{a}_{\vec{k}}^+ \hat{a}_{\vec{k}} + \frac{1}{2} g(\vec{k}) (\hat{a}_{\vec{k}}^+ \hat{a}_{-\vec{k}}^+ + \hat{a}_{\vec{k}}^+ \hat{a}_{-\vec{k}}) \right] , \quad F(\vec{k}) = \vec{k}^2 + \lambda \rho_0 \hat{v}(\vec{k}) . \quad (1.14) \]

Note that, up to the additive constant, \(\tilde{H}_{B,L}\) differs from \(H_{B,L}\) by a term \(-\mu_{\beta,L}^0 \sum_{\vec{k} \neq \vec{0}} \hat{a}_{\vec{k}}^+ \hat{a}_{\vec{k}}\), with \(\mu_{\beta,L}^0 = \mu_{\beta,L} - \lambda \rho_0 \hat{v}(\vec{0})(1 - \frac{1}{2\rho_0 L^2})\), which is thought as part of the perturbation, and acts as a counterterm, to be fixed in such a way that the condensate density is \(\rho_0\). Note also that the perturbation \(W_L\) is formally of smaller order than \(\tilde{H}_{B,L}\). The “naive” perturbation theory of \(H_L = H_L^0 + V_L\) in \(V_L\) with propagator (1.9) at \(\rho_0 > 0\) is formally equivalent to the perturbation theory of \(H_L = C_{0,L} L^2 + \tilde{H}_{B,L} + W_L\) in \(W_L\), with propagator of the form (in the \(\beta, L \to \infty\) limit):

\[ S^B(x - y) = \begin{pmatrix} S^B_{++}(x - y) & S^B_{+-}(x - y) \\ S^B_{-+}(x - y) & S^B_{--}(x - y) \end{pmatrix} = \rho_0 1 + \int_{\mathbb{R}^3} \frac{d\vec{k}}{(2\pi)^3} \frac{e^{-ik(x-y)}}{k_0^2 + F^2(\vec{k}) - g^2(\vec{k})} \begin{pmatrix} ik_0 + F(\vec{k}) & -g(\vec{k}) \\ -g(\vec{k}) & -ik_0 + F(\vec{k}) \end{pmatrix} , \quad (1.15) \]

The expansion with respect to this propagator is expected to have better convergence properties than the naive one, at least if we trust Bogoliubov’s theory. The very convergence of this modified perturbation theory, if valid, should be interpreted a posteriori as a confirmation of Bogoliubov’s picture at all orders.

For this case, our main result is that the modified perturbation theory around Bogoliubov’s Hamiltonian is renormalizable at all orders. More precisely, it is super-renormalizable in the ultraviolet, as for the \(\rho_0 = 0\) case, and just renormalizable in the infrared, with 8 running coupling constants and one free parameter (the chemical potential). After having fixed the chemical potential, we are left with 7 effective parameters, whose flows need to be controlled on the basis of the study of the beta function and of the use of Ward Identities. The presence of two marginal couplings rapidly drives the flow of the effective parameters out of the perturbative regime, from which point on we cannot conclude anything in a rigorous fashion. Still, at a heuristic level, one can try to guess what the system does in the deep infrared, non-perturbative, regime. A one-loop truncation to the beta function suggests that the (relevant) effective particle-particle interaction tends to an \(O(1)\) non-trivial fixed point. If this is assumed to be the case, one can then use the Ward Identities, which induce strong constraints on the flow of all the other effective parameters, to understand their behavior in the deep infrared. This is what is done in [15, 40], whose conclusion is that the infrared fixed point is consistent
at all orders. The fixed point has a behavior a’la Bogoliubov with a linear spectrum of excitations, notwithstanding the presence of certain anomalous terms. While, of course, we cannot rigorously prove or disprove this picture, we can (and we shall do so in the following) investigate the validity of the Ward Identities used in [15, 40] in the perturbative regime, i.e., before entering the deep infrared region. In that range of scales, the Ward Identities have an unambiguous meaning, and we can control all the terms involved in the identities at all orders, with explicit bounds on the coefficients at generic order. Our finding is that some Ward Identities, which play a crucial role in the analysis of [15, 40] are violated already at the one-loop level, due to the presence of momentum cutoffs, which were neglected in previous analyses (the authors of [15, 40] use dimensional regularization and extrapolation from three dimensions within a $3 - \varepsilon$ expansion). Therefore, we cannot confirm the consistency of the existence of a non-trivial fixed point in the deep infrared, and we think that the validity of a (dressed) Bogoliubov’s picture in 2D at zero temperature should be seriously reconsidered. We postpone further comments to Sections 3.C.4 - 3.C.5 below.

2 Renormalization Group theory of the Quantum Critical Point

2.A The functional integral representation

We start by analyzing the interacting theory at $\rho_0 = 0$ (quantum critical point). We focus on the perturbation theory for the partition function and the free energy. As well known, the perturbative expansion in $V_L$ with respect to the propagator (1.9) at $\rho_0 = 0$ can be expressed in the form of a coherent state path integral representation [38]: defining $\Lambda = [0, \beta] \times \Omega_L$, with $\Omega_L$ the square box of side $L$, if $Z_\Lambda = \text{Tr}\{e^{-\beta H_L}\}$ is the interacting partition function, and $Z^0_\Lambda = \text{Tr}\{e^{-\beta H^L_0}\}$ the non-interacting one,

$$\frac{Z_\Lambda}{Z^0_\Lambda} = \int P^0_\Lambda(d\varphi) e^{-V_\Lambda(\varphi)} .$$

(2.1)

Here:

1. $\varphi^+_x = (\varphi^-_x)^*$ is a complex classical field, labelled by the space-time point $x = (x_0, \vec{x}) \in \Lambda$.

2. $P^0_\Lambda(d\varphi)$ is a complex Gaussian measure with covariance

$$S^0_\Lambda(x - y) = \frac{1}{|\Lambda|} \sum_{k \in D_\Lambda} \frac{e^{-ik(x - y)}}{-ik_0 + |k|^2 - \mu^0_{\beta,L}} ,$$

(2.2)

where $D_\Lambda = 2\pi \beta^{-1}Z \times 2\pi L^{-1}Z^2$ and $\mu^0_{\beta,L} < 0$ goes to zero as $\beta, L \to \infty$ in such a way that (1.10) is zero, e.g., $\mu^0_{\beta,L} = -\kappa_0 \beta^{-1}$ and $\kappa_0 > 0$. If desired, the sum over
Let $S^0(x - y)$ be a scaling parameter (fixed once and for all, e.g., $\beta = 3$). Let $\gamma$ be a function of $t^2$ only, i.e., $\gamma(t) = \mathcal{K}(t^2)$ for some smooth $\mathcal{K}$, and we shall do so from now on.

$k_0$ can be performed exactly, and leads to

$$S^0_\Delta(x - y) = \frac{1}{L^2} \sum_{\bar{k} \in \mathcal{D}_L} e^{-i \bar{k}(\bar{x} - \bar{y}) - (x_0 - y_0)\varepsilon_0(\bar{k})} \left[ \frac{\vartheta(x_0 - y_0)}{1 - e^{-\beta \vartheta_0(\bar{k})}} + \frac{1 - \vartheta(x_0 - y_0)}{e^{\beta \vartheta_0(\bar{k})} - 1} \right],$$

(2.3)

where $\varepsilon_0(\bar{k}) = k^2 - \mu^2_{\beta,L}$ and $\vartheta(t)$ is the step function, equal to 1 for $t > 0$ and equal to 0 for $t \leq 0$. The function $S^0_\Delta(x - y)$ is defined a priori only for $|x_0 - y_0| < \beta$; if $-\beta < x_0 - y_0 < 0$, it satisfies $S^0_\Delta(x_0 - y_0, \bar{x} - \bar{y}) = S^0_\Delta(x_0 - y_0 + \bar{x}, \bar{y})$. Therefore, it can be naturally extended to the whole real axis by periodicity, and we shall indicate the resulting $\beta$-periodic function by the same symbol. Note that the limit $\beta, L \to \infty$ of $S^0_\Delta(x - y)$ gives (1.9) with $\rho_0 = 0$.

3. The interaction potential $V_\Delta(\varphi)$ is

$$V_\Delta(\varphi) = \frac{\lambda}{2} \int_{\mathbb{L}^2} d\bar{x} d\bar{y} |\varphi_\Delta|^2 w(x - y) |\varphi_\Delta|^2 - \nu_\Lambda \int_{\Lambda} d\bar{x} |\varphi_\Delta|^2$$

(2.4)

where $w(x - y) = \delta(x_0 - y_0)v(\bar{x} - \bar{y})$, and $\nu_\Lambda = \mu_{\beta,L} - \mu^2_{\beta,L}$ acts as a counterterm, to be fixed in such a way that the interacting propagator decays polynomially to zero at large distances.

We want to evaluate (2.1) via a Wilsonian RG analysis, in the form presented in [23, 9, 7]. The idea is to first introduce an ultraviolet cutoff, in order to make the number of degrees of freedom finite, and then integrate them slice by slice in momentum space. In the following we will try to describe the RG construction in a way as self-consistent as possible, and we refer the reader to the review papers [23, 9, 26] for further details on the methodology.

The ultraviolet cutoff is defined by the replacement of $S^0_\Delta(x - y)$ with a regularized version $S^0_{\Delta,N}(x - y)$ such that $\lim_{N \to \infty} S^0_{\Delta,N}(x - y) = S^0_\Delta(x - y)$ and defined as follows. Let $\gamma > 1$ be a scaling parameter (fixed once and for all, e.g., $\gamma = 2$). Let $\chi(t)$ be a smooth characteristic function on $\mathbb{R}$, such that $\chi(t) = 1$ for $|t| \leq 1$, and $\chi(t) = 0$ for $|t| \geq \gamma$. Then, if $x = (x_0, \bar{x})$ with $-\beta < x_0 < \beta$ and $\bar{x} \in \Omega_L$

$$S^0_{\Delta,N}(x) = \frac{1}{L^2} \sum_{\bar{k} \in \mathcal{D}_L} e^{-i \bar{k}(\bar{x} - \bar{y}) - x_0\varepsilon_0(\bar{k})} \left[ 1 - \chi \left( \gamma^{N + 1} d(x) \right) \right] \left[ \frac{\vartheta(x_0)}{1 - e^{-\beta \vartheta_0(\bar{k})}} + \frac{1 - \vartheta(x_0)}{e^{\beta \vartheta_0(\bar{k})} - 1} \right],$$

(2.5)

where $d(x) = \sqrt{||x_0||^2 + ||\bar{x}||^2}$, and $||| \cdot |||_\beta$ and $|| \cdot ||_L$ are the norms on the tori $\mathbb{R}/\beta \mathbb{Z}$ and $\mathbb{R}^2/L\mathbb{Z}^2$, respectively. The regularized propagator has by construction no singularity at small space-time distances. Note also that for any $\beta$ finite and $\mu^2_{\beta,L} = -\kappa_0 \beta^{-1}$ the

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For some of the bounds in the following, it is useful to assume that $\chi$ is a function of $t^2$ only, i.e., $\chi(t) = \mathcal{K}(t^2)$ for some smooth $\mathcal{K}$, and we shall do so from now on.
theory is automatically regular in the infrared, simply because the propagator decays exponentially on spatial scales larger than $O(\beta^{1/2})$. In order to construct the thermal ground state, we will show that the regularized theory is well defined, uniformly in $N$ and $\beta$, as $N, \beta \to \infty$. This is discussed in the next section.

2. B Multiscale decomposition, tree expansion and non-renormalized power counting

We consider the regularized partition function:

$$\Xi_{\Lambda,N} := \int P^0_{\Lambda,N}(d\varphi) e^{-V_{\Lambda}(\varphi)}.$$ (2.6)

where $P^0_{\Lambda,N}(d\varphi)$ is the gaussian integration with propagator $S^0_{\Lambda,N}(x-y)$. Our purpose is to compute $|\Lambda|^{-1} \log \Xi_{\Lambda,N}$ in an iterative fashion, and derive uniform bounds on the resulting expansion. Roughly speaking, we iteratively integrate the degrees of freedom supported on momenta of definite scale: $|k_0| + |k|^2 \sim \gamma^h$, starting from momenta of the same order as the ultraviolet cutoff, $h = N$, and then moving towards smaller and smaller scales. After having integrated the scales $N, N-1, \ldots, h+1$, the functional integral (2.6) is rewritten as an integral involving only the momenta smaller than $\gamma^h$, and the interaction is replaced by an “effective” one. This has the same qualitative structure as its “bare” counterpart, up to a redefinition of the interaction strength, and modulo “error terms”, called irrelevant in the RG jargon.

The iterative integration is based on the following multiscale decomposition of the propagator. We first rewrite the cutoff function in (2.5) as

$$1 - \chi(\gamma^{N+1}d(x)) = \sum_{h=h_{\beta}+1}^N u_h(x) + [1 - \chi(\gamma^{h_{\beta}+1}d(x))],$$ (2.7)

where $h_{\beta} := \lfloor \log_\gamma(\kappa_0\beta^{-1}) \rfloor$ and $u_h(x) = \chi(\gamma^h d(x)) - \chi(\gamma^{h+1} d(x))$. This induces the rewriting:

$$S^0_{\Lambda,N}(x) = \sum_{h=h_{\beta}+1}^N G_{\Lambda,h}(x) + R_{\Lambda,h_{\beta}}(x)$$ (2.8)

with

$$G_{\Lambda,h}(x) = \frac{1}{L^2} \sum_{\vec{k} \in \mathcal{D}_L} \frac{e^{-i\vec{k} \cdot (\vec{x} - \vec{y}) - x_0 \epsilon_0(\vec{k})} u_h(x) \left[ \frac{\vartheta(x_0)}{1 - e^{-\beta \epsilon_0(\vec{k})}} + \frac{1 - \vartheta(x_0)}{e^{\beta \epsilon_0(\vec{k})} - 1} \right].$$ (2.9)

and $R_{\Lambda,h_{\beta}}(x)$ defined by a similar expression, with $u_h(x)$ replaced by $1 - \chi(\gamma^{h_{\beta}+1}d(x))$. The decomposition is defined in such a way that the single-scale propagator $G_{\Lambda,h}$ decays exponentially on scale $\gamma^{-h}$ (see also (2.21) below), and similarly for $R_{\Lambda,h_{\beta}}(x)$. Correspondingly we rewrite the field $\varphi$ as a sum of independent fields $\varphi = \sum_{h=h_{\beta}}^N \varphi^{(h)}$, with
\( \varphi(h) \), \( h > h_\beta \), a gaussian field with propagator \( G_{\Lambda,h} \), and \( \varphi(h_\beta) \) a gaussian field with propagator \( R_{\Lambda,h_\beta} \), so that

\[
\Xi_{\Lambda,N} := \int \prod_{h=h_\beta}^N P_{\Lambda,h}(d\varphi(h)) e^{-V_{\Lambda}(\sum_{h=h_\beta}^N \varphi(h))},
\]  

(2.10)

with obvious notation. It is now clear how the iterative integration is performed: we first integrate the field \( \varphi(N) \), and rewrite

\[
\Xi_{\Lambda,N} := e^{-|\Lambda|F_{\Lambda,N}} \int \prod_{h=h_\beta}^{N-1} P_{\Lambda,h}(d\varphi(h)) e^{-V_{\Lambda}(\sum_{h=h_\beta}^{N-1} \varphi(h))},
\]

(2.11)

where \( \varphi(\leq N-1) = \sum_{h=h_\beta}^{N-1} \varphi(h) \), and \( V_{\Lambda}(N-1) \) is the effective potential on scale \( N-1 \), defined by

\[
|\Lambda|F_{\Lambda,N} + V_{\Lambda}(N-1)(\varphi) = -\log \int P_{\Lambda,N}(d\varphi(N)) e^{-V_{\Lambda}(\varphi+\varphi(N))}, \quad V_{\Lambda}(N-1)(0) = 0 .
\]

(2.12)

Next we integrate \( \varphi(N-1) \), thus defining \( F_{\Lambda,N-1} \) and \( V_{\Lambda}(N-2) \), and so on. At each step we define

\[
|\Lambda|F_{\Lambda,h} + V_{\Lambda}(h-1)(\varphi) = -\log \int P_{\Lambda,h}(d\varphi(h)) e^{-V_{\Lambda}(\varphi+\varphi(h))}, \quad V_{\Lambda}(h-1)(0) = 0
\]

(2.13)

and we proceed in this fashion until we reach the last scale \( h_\beta \). As a result, we get an expansion for the specific free energy in the form \( f_L(\beta) = \sum_{h=h_\beta}^N F_{\Lambda,h} \). Eventually, the specific ground state energy is obtained by taking the limit \( L \to \infty \) and then \( \beta \to \infty \) of this expression. The bounds on \( F_{\Lambda,h} \) are based on simple dimensional estimates on the propagator, and on a suitable resummation of the “divergent” terms, from which one can show that \( F_{\Lambda,h} \) is bounded uniformly in \( N, \beta, L \), for each fixed \( h \); on top of that, \( F_{\Lambda,h} \) reaches a well-defined limit as \( N, \beta, L \to \infty \) (to be denoted by \( F_h \)), and the limiting expression is absolutely summable in \( h \), both for \( h \to +\infty \) and for \( h \to -\infty \). In the following we illustrate in some detail the method for deriving these bounds. In order to keep the exposition as simple as possible, we shall perform the bounds on \( F_h \) directly in the \( \beta, L \to \infty \), leaving aside the issue of proving the uniform convergence of the finite \((\beta, L)\)-expressions to their limits\(^6\).

Using the inductive definition (2.13), we obtain the expansion for \( F_h \) and for the kernels of the effective potentials \( V^{(h)}(\varphi) = \lim_{|\Lambda| \to \infty} V_{\Lambda}^{(h)} \), where \( V^{(h)} \) has the form

\[
V^{(h)}(\varphi) = \sum_{m \geq 1} \int_{\mathbb{R}^{2m}} d\mathbf{x}_1 \cdots d\mathbf{y}_m W_2^{(h)}(\mathbf{x}_1, \ldots, \mathbf{x}_m; \mathbf{y}_1, \ldots, \mathbf{y}_m) \varphi_{\mathbf{x}_1}^+ \cdots \varphi_{\mathbf{x}_m}^+ \varphi_{\mathbf{y}_1}^- \cdots \varphi_{\mathbf{y}_m}^-
\]

(2.14)

\(^6\)A detailed discussion of the finite \( \beta \) effects for the ultraviolet integration is in [6]; a detailed discussion of the finite \((\beta, L)\) effects for the infrared integration is in [12].
with \( W_{2m}^{(h)} \) an integral kernel that is invariant under permutations of the \( x_i \)'s among themselves and/or of the \( y_i \)'s among themselves, and invariant under translations and rotations. In the following we will also need the “anchored” version of \( V^{(h)}(\varphi) \), to be denoted by \( \overline{V}_0^{(h)}(\varphi) \), which is the same as \( V^{(h)}(\varphi) \), up to the fact that one of the spacetime labels, say \( x_1 \), is fixed at an arbitrary position, say \( 0 \) (due to the permutation and translation invariance of the kernel, the resulting expression is independent in probability of the choice of the label that is fixed and of the localization point):

\[
\overline{V}_0^{(h)}(\varphi) = \sum_{m \geq 1} \int_{\mathbb{R}^{3(2m-1)}} dx_2 \cdots dy_m W_{2m}^{(h)}(0, x_2, \ldots, x_m; y_1, \ldots, y_m) \varphi_0^+ \varphi_{x_2}^+ \cdots \varphi_{x_m}^+ \varphi_{y_1}^- \cdots \varphi_{y_m}^-
\]

(2.15)

Let us illustrate how to derive the multiscale expansion for \( F_h \) and \( W_{2m}^{(h)} \). We first describe its “naive” version, which allows us to identify the potentially divergent terms, and suggests how to resum them in order to improve the convergence properties of the theory. The improved (resummed and renormalized) version will be described in the next sections.

Using (the limit \(|\Lambda| \to \infty\) of) (2.13), we get

\[
W_{2m}^{(h-1)}(x_1, \ldots, y_m) = \frac{1}{(m!)^2} \delta^2 \delta x_{1}^+ \cdots \delta x_m^+ \delta y_1^- \cdots \delta y_m^- \sum_{s \geq 1} \frac{1}{s!} \mathcal{E}_h^T(V^{(h)}(\varphi^{(h)}); s) \big|_{\varphi=0}
\]

(2.16)

and

\[
F_h = \sum_{s \geq 1} \frac{1}{s!} \mathcal{E}_h^T(\overline{V}_0^{(h)}(\varphi^{(h)}), V^{(h)}(\varphi^{(h)}); 1, s - 1).
\]

(2.17)

Here \( \mathcal{E}_h^T \) is the truncated expectation on scale \( h \), defined as

\[
\mathcal{E}_h^T(X_1(\varphi^{(h)}), \ldots, X_s(\varphi^{(h)}); n_1, \ldots, n_s) := \frac{\partial^{n_1 + \cdots + n_s}}{\partial \lambda_{1}^{n_1} \cdots \partial \lambda_{s}^{n_s}} \log \int P_h(d\varphi^{(h)}(\lambda_{1}, \ldots, \lambda_{s}) X_1(\varphi^{(h)}) + \cdots + \lambda_{s} X_s(\varphi^{(h)}) \big|_{\lambda_i = 0}
\]

(2.18)

where \( P_h \) is the gaussian measure with propagator \( G_h \), and \( G_h \) is the limit \( \beta, L \to \infty \) of (2.9) at \( h \) fixed:

\[
G_h(x) = \lim_{|\Lambda| \to \infty} G_{\Lambda,h}(x) = u_h(x) \vartheta(x_0) \int \frac{d^2 \vec{k}}{(2\pi)^2} e^{-i \vec{k} \cdot \vec{x} - x_0 \vec{k}} = u_h(x) \vartheta(x_0) e^{-|\vec{x}|^2/(4x_0)} / 4\pi x_0,
\]

(2.19)

which is scale covariant, in the sense that

\[
G_h(x_0, \vec{x}) = \gamma^h G_0(\gamma^h x_0, \gamma^{h/2} \vec{x})
\]

(2.20)

and, therefore, it satisfies the following bounds:

\[
||G_h||_{\infty} \leq A \gamma^h, \quad ||G_h||_1 \leq A \gamma^{-h},
\]

(2.21)
for a suitable $A > 0$. Moreover, due to the presence of $u_h(x)\vartheta(x_0)$ in (2.21),

$$G_h(x-y) \neq 0 \Rightarrow 0 < x_0 - y_0 < \gamma^{-h+1}, \quad (2.22)$$

which is a crucial property for the following analysis.

It is well known that the truncated expectation in (2.18) admits a natural graphical interpretation: if $X_i$ is graphically represented as a (non-local) vertex with external lines $\varphi^i$, (2.18) can be represented as the sum over the Feynman diagrams obtained by contracting in all possible connected ways the lines exiting from $n_1$ vertices of type $X_1$, $n_2$ of type $X_2$, etc, and $n_s$ of type $X_s$. Every contraction involves a pair of fields, one of type $\varphi^+$ and one of type $\varphi^-$, and corresponds to a propagator on scale $h$, as defined in (2.19).

Of course, (2.16) and (2.17) can be iterated so to re-express $V(h)$ in the r.h.s. in terms of $V(h+1)$, and so on, until we reach scale $N$. On scale $N$, we let $V^{(N)}(\varphi) = V(\varphi)$, where $V(\varphi)$ is the limit $|\Lambda| \to \infty$ of (2.4). We shall see that $\nu = \lim_{|\Lambda| \to \infty} \nu_\Lambda$ can be chosen equal to zero; therefore, in order not overwhelm the notation, we shall fix it directly equal to zero:

$$V^{(N)}(\varphi) = V(\varphi) = \frac{\lambda}{2} \int_{\mathbb{R}^6} dx dy |\varphi_x|^2 w(x-y) |\varphi_y|^2 \quad (2.23)$$

The final outcome is that the kernels of $V^{(h-1)}$ and the single scale contribution to the free energy, $F_h$, can be written as a sum over connected Feynman diagrams with lines on all possible scales between $h$ and $N$. The iteration of (2.16) induces a natural hierarchical organization of the scale labels of every Feynman diagram, which can be conveniently represented in terms of tree diagrams, first introduced by G. Gallavotti and F. Nicolò in [24]. Since then, the Gallavotti-Nicolò tree expansion has been described in detail in
several papers that make use of constructive renormalization group methods, see e.g. [23, 9, 26]. The main features of this expansion are described below.

(1) Let us consider the family of all unlabeled trees which can be constructed by joining a point $r$, the root, with an ordered set of $n \geq 1$ points, the endpoints of the tree, so that $r$ is not a branching point.

The unlabelled trees are partially ordered from the root to the endpoints in the natural way: nodes on the left are lower than those to their right; we shall use the symbol $<$ to denote this partial ordering; $n$ is called the order of the unlabelled tree. Two unlabelled trees are identified if they can be superposed by a suitable continuous deformation, so that the endpoints with the same index coincide.

We shall also consider the labelled trees (to be called simply trees in the following); they are defined by associating some labels with the unlabelled trees, as explained in the following items.

(2) We associate a label $h \leq N$ with the root, a label $N+1$ with the endpoints and we denote by $\tilde{T}^{(h)}_{N;n}$ the corresponding set of labelled trees with $n$ endpoints. Moreover, we introduce a family of vertical lines, labelled by an integer $h_v$, the scale label, taking values in $[h, N+1]$, see Fig. 1.

We call non trivial vertices of the tree its branch points; we call trivial vertices the points where the branches intersect the family of vertical lines. The set of the vertices will be the union of the endpoints and of trivial and non trivial vertices (see the dots in Fig. 1). Note that the root is not a vertex. Every vertex $v$ of a tree will be associated with its scale label $h_v$.

(3) There is only one vertex immediately following the root, called $v_0$ and with scale label equal to $h + 1$.

(4) Given a vertex $v$ of $\tau \in \tilde{T}^{(h)}_{N;n}$ that is not an endpoint, we can consider the subtrees of $\tau$ with root $v'$ (here $v'$ is the vertex immediately preceding $v$ on $\tau$), which correspond to the connected components of the restriction of $\tau$ to the vertices $w \geq v$. If a subtree with root $v'$ contains, besides the root, only $v$ and one endpoint on scale $h_v + 1$, it will be called a trivial subtree. Given a vertex $v$ we denote by $s_v$ the number of lines branching from $v$ (then $s_v = 1$ if $v$ is a trivial vertex).

(5) With each endpoint $v$ we associate $V^{(N)}(\phi)$, see (2.23), and a set $I_v$ of four field labels $f_1, \ldots, f_4$. Given $f \in I_v$, we let $x_f$ and $\sigma_f$ denote the space–time and the creation/annihilation label of the corresponding field; $\varphi^\sigma_{x_f}$, $\sigma_f = \pm$; due to the form of the interaction (2.23), the space–time labels of the four fields are equal two by two, and we indicate the two values by $x_1,v, x_2,v$. If $v$ is not an endpoint, we call $I_v$ the set of field labels associated with the endpoints following the vertex $v$. Moreover, we call cluster of $v$ (and indicate it by $x_v$) the family of space–time points associated with all the endpoints following $v$, if $v$ is not an endpoint, or $v$ itself, otherwise. In the following, we shall associate with every $\tau \in \tilde{T}^{(h)}_{N;n}$ several contributions, distinguished
by the choice of the fields that are contracted at every scale. In order to identify these contributions, we introduce a subset $P_v$ of $I_v$, whose interpretation is that of external fields of $v$. The $P_v$’s must satisfy various constraints: first of all, if $v$ is not an endpoint and $v_1, \ldots, v_{s_v}$ are the $s_v \geq 1$ vertices immediately following it, then $P_v \subseteq \cup_i P_{v_i}$; if $v$ is an endpoint, $P_v = I_v$. If $v$ is not an endpoint, we also denote by $Q_{v_i}$ the intersection of $P_v$ and $P_{v_i}$, so that $P_v = \cup_i Q_{v_i}$. The union of the subsets $P_{v_i} \setminus Q_{v_i}$ is, by definition, the set of the internal fields of $v$, and is non empty if $s_v > 1$.

In terms of these trees, the kernels of the effective potential can be written as (defining $\mathbf{x} = (x_1, \ldots, x_m)$ and $\mathbf{y} = (y_1, \ldots, y_m)$)

$$W_{2m}^{(h)}(\mathbf{x}; \mathbf{y}) = \sum_{n \geq 1} \sum_{\tau \in T_{N;n}^{(h)} \mathbb{P} \tau} \sum_{f \in I_{v_0} \setminus P_{v_0}} \int \prod_{f \in I_{v_0} \setminus P_{v_0}} d\mathbf{x}_f \ W^{(h)}(\tau, \mathbb{P}; \mathbf{x}_{v_0}) \quad (2.24)$$

where $\mathbb{P}_\tau$ is the family of all the choices of the sets $P_v$ compatible with the constraints illustrated above, and $\mathbb{P} = \{ P_v \}$ are elements of $\mathbb{P}_\tau$. The apex $(P_{v_0})$ on the sum indicates the constraint that we are not summing over $P_{v_0}$; rather, $P_{v_0}$ is a fixed set with $2m$ elements, $m$ of which have $\sigma$-label $+$ and space-time label $x_i$, while the remaining $m$ have $\sigma$-label $-$ and space-time label $y_i$. In particular, $\cup_{f \in P_{v_0}} (x_f) = (x, y)$. Similarly, $F_{h+1}$ can be written as

$$F_{h+1} = \sum_{n \geq 1} \sum_{\tau \in T_{N;n}^{(h)} \mathbb{P} \tau} \sum_{f \in P_{v_0}} \int \prod_{f \in P_{v_0} \setminus \{ f^* \}} d\mathbf{x}_f \ W^{(h)}(\tau, \mathbb{P}; \mathbf{x}_{v_0}) \quad (2.25)$$

where $f^*$ is an arbitrary field label: the reason why there is one integral less than the total number of space-time points comes from the fact that $F_h$ involves one anchored effective potential $\nabla_0^{(h)}(\varphi)$, see (2.17); the fact that $f^*$ is arbitrary comes from the translation invariance of the kernels. Moreover, the $*$ on the sum over $\mathbb{P}$ indicates the constraint that $P_{v_0} = \emptyset$ is fixed and the set of internal fields of $v_0$ is non empty. The contribution $W^{(h)}(\tau, \mathbb{P}; \mathbf{x}_{v_0})$ has the form:

$$W^{(h)}(\tau, \mathbb{P}; \mathbf{x}_{v_0}) = -(-\lambda/2)^n \left[ \prod_{v \text{ e.p.}} w(x_{1,v} - x_{2,v}) \right] \cdot \left[ \prod_{v \text{ not e.p.}} \frac{1}{s_v!} \mathcal{E}_{h_v}^T \left( \varphi(P_{v_1 \setminus Q_{v_1}}), \ldots, \varphi(P_{v_{s_v} \setminus Q_{v_{s_v}}}) \right) \right] \quad (2.26)$$

where the first product runs over the endpoints of $\tau$, the second over the vertices $v$ of $\tau$ that are not endpoints, for each of which we indicated by $v_1, \ldots, v_{s_v}$, the vertices immediately following $v$ on $\tau$. Moreover, $\varphi(P_v \setminus Q_v) := \prod_{f \in P_v \setminus Q_v} \varphi_{x_f}$. Note that the factors in (2.26) do not depend explicitly on $N$: therefore, the dependence of the effective potential on the ultraviolet cutoff is only due to the fact that the sum over the trees is restricted to $T_{N;n}^{(h)}$. 

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As anticipated above, the truncated expectations in the second line of (2.26) can be expanded in sums over connected Feynman diagrams, each of which is uniquely determined by the choice of the pairing among the $\varphi$ fields (each pairing is a grouping of the fields in pairs, each pair consisting of one field with $\sigma$-label equal to $+$, and one equal to $-$) and has a value equal to the product of the propagators associated with the paired fields. The choice of a (labelled) Feynman diagram is thus equivalent to the choice of a connected pairing per vertex; we indicate by $\tilde{\Gamma}(\tau, P)$ the set of connected labelled Feynman diagrams (or, equivalently, of sequences of connected pairings indexed by $v \in \tau$) compatible with the tree $\tau$ and the set of field labels $P$. Therefore, we can write:

$$W^{(h)}(\tau, P; \mathbf{x}_{v_0}) = \sum_{\mathcal{G} \in \tilde{\Gamma}(\tau, P)} \text{Val}(\mathcal{G})$$

where $\mathcal{G}_v$ is the pairing of the internal fields of $v$ associated with $\mathcal{G}$, and $\ell_\pm$ are the two contracted fields (with $\sigma$-labels $+$ and $-$, respectively) corresponding to the pair $\ell \in \mathcal{G}_v$. From a graphical point of view, the Feynman graph $\mathcal{G}$ can be depicted by drawing $n$ vertices as in Fig.2 (the exiting/entering solid half-lines represent fields of type $\varphi^+ / \varphi^-$ and the “wiggly” lines represent the interaction kernel $(\lambda/2) w(x - y)$), and then by joining the solid half-lines in pairs, in the way induced by the pairing $\{\mathcal{G}_v\}$ associated with $\mathcal{G}$; every contracted solid line $\ell$ has a well-defined direction (from $x_{\ell_+}$ to $x_{\ell_-}$), and it carries the scale label $h_v$, where $v$ is the vertex such that $\ell \in \mathcal{G}_v$: of course, it corresponds to the propagator $G_{h_v}(x_{\ell_+} - x_{\ell_-})$. Note that time increases along the lines in the natural direction (i.e., from $x_{\ell_+}$ to $x_{\ell_-}$), due to the time ordering condition (2.22). Let us also observe that every vertex $v$ corresponds to a set of propagators, those associated with the lines $\ell \in \mathcal{G}_v$. If we artificially associate a label $N$ with the wiggly lines, then the union of the solid lines in $\cup_{w \geq v} \mathcal{G}_w$, together with the wiggly lines associated with the endpoints following $v$ on $\tau$, form a maximal connected set of lines with scale labels $\geq h_v$, called a cluster; here “maximal” refers to the fact that any solid line exiting or entering the cluster has scale label strictly smaller than $h_v$, i.e., the cluster cannot be increased without decreasing its scale. See Fig.3 for an example of a labelled Feynman diagram, together with its cluster structure and its Gallavotti-Nicolò tree.

By using these explicit expressions and the dimensional bounds (2.21) on the prop-
Figure 3: An example of tree $\tau$ of order 3 with the corresponding cluster structure, where only the non trivial vertices are depicted. The cluster structure uniquely identifies a tree $\tau$ and viceversa. Down left an element $\mathcal{G}$ of the class of Feynman diagrams compatible with $\tau$.

agator, let us now derive a bound on (2.24) or, more precisely, on the $L_1$ norm of the kernel:

$$||W_{2m}^{(h)}|| = \int dx_2 \cdots dy_m |W_{2m}^{(h)}(0, x_2, \ldots, x_m; y_1, \ldots, y_m)|, \quad (2.28)$$

as well as on $|F_{h+1}|$. Note that the choice of not integrating over $x_1$ in (2.28) and of fixing its location at $x_1 = 0$ is irrelevant, due to the translation invariance of the kernel. In order to bound (2.28) and $|F_{h+1}|$ we proceed as follows: we use the representations (2.24)-(2.25), with $W^{(h)}(\tau, P; x_v)$ as in (2.27), and we get

$$||W_{2m}^{(h)}|| \leq \sum_{n \geq 1} (|\lambda|/2)^n \sum_{\tau \in \mathcal{T}_N^{(h)}, P \in \mathcal{P}_\tau} \sum_{G \in \Gamma(\tau, P)} \int dx_2 \cdots dy_m \prod_{f \in I_v \backslash P_v} df \cdot \left[ \prod_{v \text{ e.p.}} w(x_{1,v} - x_{2,v}) \right] \left[ \prod_{v \not\in \text{e.p.}} \frac{1}{s_v!} \prod_{\ell \in \mathcal{G}_v} |G_h(x_{\ell-} - x_{\ell+})| \right]. \quad (2.29)$$

$|F_{h+1}|$ is bounded by an analogous expression, with $P_v$ replaced by $\emptyset$ and the integration measure by $\prod_{f \in I_v \backslash \{f^*\}} df$ (and, in addition, the constraint on the sum over $P$ that the set of internal fields of $v_0$ is non empty). For each $G \in \Gamma(\tau, P)$, we arbitrarily choose a spanning tree $T_v \subseteq \mathcal{G}_v$ per vertex: $T_v$ consists of $s_v - 1$ lines connecting in a minimal way the $s_v$ clusters $v_1, \ldots, v_{s_v}$ immediately following $v$ on $\tau$. We rewrite

$$dx_2 \cdots dy_m \prod_{f \in I_v \backslash P_v} df = \left[ \prod_{v \not\in \text{e.p.}} \prod_{\ell \in T_v} d(x_{\ell-} - x_{\ell+}) \right] \left[ \prod_{v \text{ e.p.}} d(x_{1,v} - x_{2,v}) \right], \quad (2.30)$$

and similarly for $\prod_{f \in I_v \backslash \{f^*\}} df$. Next we bound all the propagators outside $\cup_v T_v$ by their $L_\infty$ norm, after which we are left with the product of the $L_1$ norms of the
propagators belonging to the spanning tree:

\[
\|W_{2m}^{(h)}\| \leq \sum_{n \geq 1} \sum_{\tau \in \tilde{T}_{N,n}^{(h)}} \sum_{(P_{v_0})} \sum_{g \in \tilde{\Gamma}(\tau, P)} |\lambda|^n 2^{-n} \left[ \int d\mathbf{x} \, w(|\mathbf{x}|) \right]^n \cdot \prod_{v \notin \text{e.p.}} \frac{1}{s_v!} \left( ||G_{h_v}||_{\infty} \right)^{\frac{\sum_{i=1}^{s_v} |P_{v_i}| - |P_v|}{2} - s_v + 1} (||G_{h_v}||_{1})^{s_v - 1} ,
\]

and similarly for \(|F_{h+1}|\). Inserting (2.21) into this equation we find:

\[
\|W_{2m}^{(h)}\| \leq \sum_{n \geq 1} \sum_{\tau \in \tilde{T}_{N,n}^{(h)}} \sum_{(P_{v_0})} \sum_{g \in \tilde{\Gamma}(\tau, P)} |\lambda|^n 2^{-n} \left[ \int d\mathbf{x} \, w(|\mathbf{x}|) \right]^n A^{2n-m} \cdot \prod_{v \notin \text{e.p.}} \frac{1}{s_v!} \gamma_{h_v} \left( \frac{\sum_{i=1}^{s_v} |P_{v_i}| - |P_v|}{2} - s_v + 1 \right) (4n - 2m)^{h_v - h_{v'}} \left( n_v - 1 \right) ,
\]

Next we rewrite every \(h_v\) appearing in the r.h.s. of this equation as \(h + (h_v - h)\) and we use the relations:

\[
\sum_{v \geq v_0} h \left( \sum_{i=1}^{s_v} |P_{v_i} - |P_v| \right) = h(4n - 2m) ,
\]

\[
\sum_{v \geq v_0} (h_v - h) \left( \sum_{i=1}^{s_v} |P_{v_i} - |P_v| \right) = \sum_{v \geq v_0} (h_v - h_{v'}) \left( 4n_v - |P_v| \right) ,
\]

\[
\sum_{v \geq v_0} h(s_v - 1) = h(n - 1) ,
\]

\[
\sum_{v \geq v_0} (h_v - h) (s_v - 1) = \sum_{v \geq v_0} (h_v - h_{v'}) (n_v - 1) ,
\]

where \(v'\) is the vertex immediately preceding \(v\) on \(\tau\) (so that \(h_v - h_{v'} = 1\)), and \(n_v\) is the number of endpoints following \(v\) on \(\tau\).

If we substitute these identities into (2.29), we obtain

\[
\|W_{2m}^{(h)}\| \leq \gamma^{h(2-m)} \sum_{n \geq 1} |\lambda|^n C^n A^{2n-m} \sum_{\tau \in \tilde{T}_{N,n}^{(h)}} \sum_{(P_{v_0})} \sum_{g \in \tilde{\Gamma}(\tau, P)} \prod_{v \notin \text{e.p.}} \frac{1}{s_v!} \gamma^{(h_v - h_{v'}) \left( 2 - |P_v| \right)}
\]

(2.37)
where \( C = 2^{-1} \int dx \, w(x) = 2^{-1} \int d\vec{x} \, v(\vec{x}) \). Now, the number of Feynman diagrams in \( \hat{\Gamma}(\tau, P) \) can be bounded by \( \prod_{v \not\in e.p.} (\sum_{i=1}^{s_v} |P_{v_i}| - |P_v|)! \leq C_1^n (n!)^2 \), while \( \prod_{v \not\in e.p.} s_v! \geq C_2^n n! \), so that

\[
||W_{2m}^{(h)}|| \leq \gamma^h (2-m) \sum_{n \geq 1} |\lambda|^n K^n n! \sum_{\tau \in \tau_n^{(h)}} \sum_{\mathcal{P} \in \mathcal{P}_\tau} \prod_{v \not\in e.p.} \gamma^{(h_v-h_{\omega_v})(2-\frac{|P_v|}{2})} \tag{2.38}
\]

with \( K = CAC_1C_2 \). Of course, \(|F_{h+1}| \) admits a completely analogous bound, with the only difference that \( P_{v_0} \) is replaced by \( \emptyset \) and, therefore, \( m = 0 \). A bound like (2.38) is often referred to as an \( n! \) bound on the perturbative expansion: if we could show that the last product \( \prod_{v \not\in e.p.} \gamma^{(h_v-h_{\omega_v})(2-\frac{|P_v|}{2})} \) is summable both over the field and the scale labels, then it would imply that the \( n \)-th order of perturbation theory would be finite and bounded explicitly by \((\text{const.})^n |\lambda|^n n! \) (compatible with the Borel summability of the theory).

Unfortunately, it is apparent that the subdiagrams with \( |P_v| = 2, 4 \) are not summable over the scale labels. This means that such subdiagrams need to be resummed. The iterative resummation procedure will be described in the next subsection, and it will be equivalent to a reorganization of the tree expansion, after which the resulting expansion will be finite at all orders, with \( n! \) bounds on the generic order.

### 2.C The renormalized expansion

In the previous subsection we saw that the only sub-diagrams leading to possible divergences in the multiscale expansion for the free energy and for the effective potential are those with \( |P_v| = 2, 4 \). An important fact that will be extensively used in the following is that, thanks to the time ordering condition (2.22), the values of most of these Feynman sub-diagrams are zero. More specifically, the value of any (sub)diagram with \( |P_v| = 0, 2 \) is zero, and the value of a subdiagram with \( |P_v| = 4 \) is different from zero only if it is a ladder graph, as the one in Fig.4 (this fact is well known both in the theoretical and in the mathematical physics literature, see e.g. [10, 6, 45]). A few examples of diagrams with 2 or 4 external legs and vanishing values are shown in Fig.5.

The reason why the diagrams with \( |P_v| = 0, 2 \), as well as the non-ladder diagrams with \( |P_v| = 4 \), are zero is very simple: they either contain a bosonic loop, i.e., a closed solid line, or they contain a wiggly line connecting two distinct points along the same open solid line, or both (see Fig.5). In any of these cases, the value of the diagram is proportional to the product of an order sequence of propagators \( G_{h_1}(x_1-x_2)G_{h_2}(x_2-x_3) \cdots G_{h_k}(x_k-x_{k+1}) \), with \( k \geq 1 \), times an interaction kernel \( w(x_1-x_{k+1}) = \delta(x_{1,0} - x_{k+1,0})v(\vec{x}_1 - \vec{x}_{k+1}) \): such a product is zero, simply because \( \delta(x_{1,0} - x_{k+1,0}) \) fixes the initial and final times in the sequence to be the same, while the time ordering condition (2.22) requires that \( x_{1,0} > x_{k+1,0} \) to order the sequence to be different from zero.

In conclusion, the only potentially dangerous subdiagrams, which make the bound (2.37) not uniformly summable in \( \{h_v\} \) and, therefore, require a resummation, are the
ladder diagrams as in Fig. 4. At the formal level, the resummation is very simple: for each \( h < N \), we just split the effective potential as 
\[
V(h)(\varphi) = LV(h)(\varphi) + RV(h)(\varphi),
\]
where
\[
\begin{align*}
L V(h)(\varphi) &= \int_{\mathbb{R}^{12}} dx_1 \cdots dy_2 L W_4(h)(x_1, x_2; y_1, y_2) \varphi^+_x \varphi^+_y \varphi^-_x \varphi^-_y, \\
&W_4(h)(x_1, x_2; y_1, y_2) = W_4(h)(x_1, x_2; y_1, y_2) \delta(x_1 - x_2) \delta(x_1 - y_1) \delta(x_1 - y_2).
\end{align*}
\]
if \( h < 0 \). In other words, \( L V(h)(\varphi) \) is either the sum of all the ladder subdiagrams with propagators carrying a scale label \( \geq h \), if \( h \geq 0 \), or its local part (in the sense of (2.40)), if \( h < 0 \). Moreover, \( RV(h)(\varphi) \) is the rest (the irrelevant part), i.e., the sum over all the field monomials of order \( \geq 6 \), plus possibly (if \( h < 0 \)) the non-local part of the ladder diagrams. \( L V(h)(\varphi) \) is thought of as the effective interaction on scale \( h \), and is treated in the same fashion as the interaction \( V(\varphi) \). The tree expansion is modified accordingly: the modified trees contributing to \( V(h) \) can now have endpoints on all scales between \( h + 2 \) and \( N + 1 \), where an endpoint on scale \( h_v \) represents an interaction vertex of type \( L V(h_{v-1})(\varphi) \), see Fig. 6; if such an endpoint has \( h_v \leq N \), then the vertex immediately preceding \( v \) on \( \tau \) is necessarily non trivial. An action of the operator \( R \) is associated with all the vertices \( v > v_0 \) that are not endpoints: if \( |P_v| > 4 \), such an action is trivial (i.e., it acts as the identity), while if \( |P_v| = 4 \) and \( h_v < 0 \), \( R \) extracts the non-local part
Figure 6: An example of renormalized tree belonging to $T_{N;n}^{(h)}$. Even if not reported explicitly in the picture, a label $\mathcal{R}$ is associated with each vertex different from $v_0$ and from the endpoints. The vertex $v_0$ is associated with an index $L$ or $R$. The endpoints at scale $h_v \in [h + 2, N]$ represent $-\mathcal{L}V^{(h_v-1)}(\varphi)$, while the endpoints at scale $N + 1$ correspond as before to $V(\varphi)$.

of the value of the subtree it acts on; if $|P_v| = 4$ and $h_v \geq 0$, the action of $\mathcal{R}$ kills the whole value of the subtree it acts on, i.e., it makes it vanish: therefore, we can freely decide that the modified trees are not allowed to have $|P_v| = 4$ on vertices such that $h_v \geq 0$, and we shall do so in the following.

We denote the family of the modified trees with $n$ endpoints by $T_{N;n}^{(h)}$. In terms of these new trees, the single-scale contribution to the kernels of the effective potential admit a representation that is very similar to (2.24)–(2.27): with some abuse of notation, we write it in the form:

$$W_{2m}(x; y) = \sum_{n \geq 1} \sum_{\tau \in T_{N;n}} \sum_{P \in P_{\tau}} \left( P_{v_0} \right) \int \prod_{f \in I_{v_0} \setminus \{f^*\}} dx_f \ W^{(h)}(\tau, P; x_{v_0}) \quad (2.41)$$

with

$$W^{(h)}(\tau, P; x_{v_0}) = \sum_{G \in \Gamma(\tau, P)} \text{Val}(G) , \quad (2.42)$$

$\Gamma(\tau, P)$ the set of connected labelled Feynman diagrams compatible with the renormal-
ized tree \( \tau \in \mathcal{T}_{N,n}^{(h)} \) and the set \( \mathcal{P} \), and

\[
\text{Val}(\mathcal{G}) = (-1)^{n+1} \prod_{v \text{ not e.p.}} \frac{\mathcal{R}_{v}}{s_{v}} \left[ \left( \prod_{\ell \in \mathcal{G}_{v}} G_{h_{\ell}}(x_{\ell_{-}} - x_{\ell_{+}}) \right) \right] \cdot \left( \prod_{v'^{>}v, h_{v'^{>}v} = h_{v} + 1} \mathcal{L}W_{4}^{(h_{v'} - 1)}(x_{1,v'}, x_{2,v'}; y_{1,v'}, y_{2,v'}) \right),
\] (2.43)

where \( \alpha_{v} = 0 \) if \( v = v_{0} \), and otherwise \( \alpha_{v} = 1 \). Moreover, it is understood that the operators \( \mathcal{R} \) act in the order induced by the tree ordering (i.e., starting from the endpoints and moving toward the root).

In the next subsection we will show that

\[
\left| \int d\mathbf{x}_{2} d\mathbf{y}_{1} d\mathbf{y}_{2} \mathcal{L}W_{4}^{(h)}(\mathbf{x}_{1}, \mathbf{x}_{2}; \mathbf{y}_{1}, \mathbf{y}_{2}) \right| \leq (\text{const.}) |\lambda|,
\] (2.44)

uniformly in \( h \) and \( N \). This is enough to show that the modified expansion is order by order convergent, uniformly in the ultraviolet cutoff, with \( n! \)-bounds on the \( n \)-th order of the free energy and of the kernels of the effective potential. Let us explain why this is the case. We start from (2.41)–(2.43) and proceed as in the proof of (2.38). As before, we need to estimate the integral over \( \mathbf{x}_{0} \setminus \mathbf{x}_{f} \) of the values of Feynman diagrams, and then sum over \( \mathcal{G} \in \Gamma(\tau, \mathcal{P}) \), over \( \mathcal{P} \) and over \( \tau \). The contributions from the Feynman diagrams in \( \Gamma(\tau, \mathcal{P}) \), with \( \mathcal{P} \) such that \( |P_{v}| \geq 6 \) for all vertices \( v \) of \( \tau \) that are not endpoints, are the easiest to estimate. In fact, the action of the operators \( \mathcal{R} \) in the formula (2.43) for the value of any such diagram is trivial, i.e., they act as the identity on all vertices. Therefore, the estimate goes over exactly as in the proof of (2.38), with the only difference that every endpoint \( v^{*} \) is associated with an integral like the one in the l.h.s. of (2.44), with \( h = h_{v^{*}} - 1 \), rather than with \( \frac{1}{2} \int d\mathbf{x}\mathcal{L}(\mathbf{x}) \): of course, this makes no qualitative difference because, thanks to (2.44), both integrals are bounded by \( (\text{const.}) |\lambda| \). In conclusion, the overall contribution from these diagrams on \( W_{2m}^{(h)}(\mathbf{x}, \mathbf{y}) \) (let us call it \( W_{2m}^{(h;1)}(\mathbf{x}, \mathbf{y}) \)) is bounded by an expression similar to (2.38), namely

\[
\| W_{2m}^{(h;1)} \| \leq \gamma^{(2-m)} \sum_{n \geq 1} |\lambda|^{n} \mathcal{K}^{n} n! \sum_{\tau \in \mathcal{T}_{N,m}^{(h)}} \sum_{\mathcal{P} \in \mathcal{P}_{\tau}} \gamma^{(h_{v} - h_{v'})(2 - |P_{v}|)} \prod_{v \text{ not e.p.}} \gamma^{(h_{v} - h_{v'})(2 - |P_{v}|)}
\] (2.45)

where the \( * (P_{\tau}) \) on the sum indicates the constraint that we are not summing over \( P_{v_{0}} \) and \( |P_{v_{0}}| = 2m \), and moreover \( |P_{v}| \geq 6 \) for all the vertices that are not endpoints. Since \( |P_{v}| \geq 6 \), the exponential factors \( \gamma^{(h_{v} - h_{v'})(2 - |P_{v}|)} \) in the r.h.s. of (2.45) are summable both over \( \{ h_{v} \} \) and over \( \{ P_{v} \} \), which leads to an \( n! \) bound of the form:

\[
\| W_{2m}^{(h;1)} \| \leq \gamma^{h(2-m)} \sum_{n \geq 1} |\lambda|^{n} (\mathcal{K}')^{n} n!,
\] (2.46)
as desired (here and below a bound like this, involving a non-summable series like the one in the r.h.s., should be interpreted as a bound valid order by order on the coefficients of the perturbative expansion of the l.h.s.). We are left with the contributions from trees and field labels such that of the perturbative expansion of the l.h.s.). We are left with the contributions from trees one in the r.h.s., should be interpreted as a bound valid order by order on the coefficients as desired (here and below a bound like this, involving a non-summable series like the one in the r.h.s.).

The emergence of this gain factor has been discussed several times in the literature, see e.g. [23, 9, 26]. Here we briefly explain the mechanism leading to it, in order to make our exposition as self-consistent as possible.

In order to explain the effect of \(R\), let us first concentrate on a vertex \(v\) that is not endpoint with \(|P_v| = 4\) and \(h_v < 0\), such that there are no vertices \(w > v\) with the same property. This means that the action of all the \(R\)'s associated with the vertices \(w > v\) that are not endpoints is equal to the identity. Therefore, the contribution from the subtree with root \(v'\), once integrated over \(\bar{x}_v := x_v \cup f \in P_v \{x_f\}\) is

\[
F_v(x_1, x_2; y_1, y_2) = (-1)^{n_v} \int d\bar{x}_v \prod_{w \not\in e.p.: \, w > v} \frac{1}{\omega_w} \left[ \prod_{\ell \in G_w} G_{h_w}(x_{\ell_-} - x_{\ell_+}) \right] \cdot \left( \prod_{v^* \in e.p.: \, v^* > v, \, h_{v^*} = h_w + 1} LW_{4}^{(h_w - 1)}(x_{1,v^*}, x_{2,v^*}; y_{1,v^*}, y_{2,v^*}) \right),
\]

where \(x_1, x_2\) (resp. \(y_1, y_2\)) are the elements of \(\cup f \in P_v \{x_f\}\) such that \(\sigma_f = +\) (resp. \(\sigma_f = -\)). Clearly, the \(L_1\) norm of this expression admits a bound analogous to \(\|W_{2m}^{(h; 1)}\|\), namely

\[
\int dx_2 dy_1 dy_2 |F_v(x_1, x_2; y_1, y_2)| \leq C_{m_v} |\lambda|^{n_v n_v}!.
\]

Similarly, we obtain

\[
\int dx_2 dy_1 dy_2 \left[ \sum_{j=0}^{2} |x_{2,j} - x_{1,j}|^{m_j} \right] |F_v(x_1, x_2; y_1, y_2)| \leq C_{m_0, m_1, m_2} \gamma^{-h_v(m_0 + \frac{1}{2}(m_1 + m_2))} |\lambda|^{n_v n_v}!,
\]

which will turn out to be useful below. Similar estimates are valid with \(x_{2,j} - x_{1,j}\)

\[\text{In order to prove (2.49), it is enough to rewrite each factor } x_{2,0} - x_{1,0}, \text{ etc., as a sum of differences } x_{\ell_-} - x_{\ell_+} \text{ along the spanning tree } \cup_{w > v} T_w \text{ (see the discussion preceding (2.31)), and to recognize that }\]

\[\text{each term } x_{\ell_-} - x_{\ell_+} \text{ goes together with the corresponding propagator } G_{h_w}(x_{\ell_-} - x_{\ell_+}); \text{ therefore, in the analogue of (2.31), the } L_1 \text{ norm of } G_{h_w}(x_{\ell_-} - x_{\ell_+}) \text{ is replaced by the one of } (x_{\ell_-} - x_{\ell_+}) G_{h_w}(x_{\ell_-} - x_{\ell_+}), \text{ which is (const.) } \gamma^{-h_v} \|G_{h_w}\| \leq (\text{const.}) \gamma^{-h_v} \|G_{h_w}\| 1. \]
replaced by $y_{1,j} - x_{1,j}$, etc. Let us now evaluate the action of $R$ on $F_v(x_1, x_2; y_1, y_2)$:

$$RF_v(x_1, x_2; y_1, y_2) = F_v(x_1, x_2; y_1, y_2) - \delta(x_1 - x_2)\delta(x_1 - y_1)\delta(x_1 - y_2)\int dx'_2 dy'_2 dy'_1 F_v(x_1, x'_2; y'_1, y'_2).$$

(2.50)

On top of that, in the expression (2.43), $RF_v(x_1, x_2; y_1, y_2)$ appears multiplied by the four propagators $G_{h_1}(x_{\ell_1, -} - x_1)$, $G_{h_2}(x_{\ell_2, -} - x_2)$, $G_{h'_1}(y_1 - x_{\ell'_1, +})$, $G_{h'_2}(y_2 - x_{\ell'_2, +})$ associated with the fields in $P_v$; here $h_1$ is the scale of the vertex $v_1$ such that $\ell_1 \in G_{v_1}$ and $x_{\ell_1, +} = x_1$, and similarly for the others. Note that $h_1, \ldots, h_4 < h_v$. Moreover, when we evaluate $\|W_{2m}^{(h)}\|$, we also need to integrate over three out of the four coordinates $x_1, \ldots, y_2$, and by translation invariance we can arbitrarily choose the one we are not integrating over, say $x_1$. In conclusion, we can isolate from the contribution under consideration to $\|W_{2m}^{(h)}\|$ the following expression:

$$\int dx_2 dy_1 dy_2 G_{h_1}(x_{\ell_1, -} - x_1) \cdots G_{h'_2}(y_2 - x_{\ell'_2, +})RF_v(x_1, x_2; y_1, y_2) = \int dx_2 dy_1 dy_2 \left[ G_{h_1}(x_{\ell_1, -} - x_1)G_{h_2}(x_{\ell_2, -} - x_2)G_{h'_1}(y_1 - x_{\ell'_1, +})G_{h'_2}(y_2 - x_{\ell'_2, +}) - 
\quad G_{h_1}(x_{\ell_1, -} - x_1)G_{h_2}(x_{\ell_2, -} - x_1)G_{h'_1}(y_1 - x_{\ell'_1, +})G_{h'_2}(y_2 - x_{\ell'_2, +}) \right] F_v(x_1, x_2; y_1, y_2).$$

(2.51)

Note that the expression in square brackets is the difference between the original product of propagators and a similar product where all the space-time points $x_1, x_2, y_1, y_2$ have been “localized” to $x_1$. Such expression can be equivalently rewritten as

$$\left[ G_{h_1}(x_{\ell_1, -} - x_1)\left( G_{h_2}(x_{\ell_2, -} - x_2) - G_{h_2}(x_{\ell_2, -} - x_1) \right)G_{h'_1}(y_1 - x_{\ell'_1, +})G_{h'_2}(y_2 - x_{\ell'_2, +}) + 
\quad G_{h_1}(x_{\ell_1, -} - x_1)G_{h_2}(x_{\ell_2, -} - x_1)\left( G_{h'_1}(y_1 - x_{\ell'_1, +}) - G_{h'_1}(x_1 - x_{\ell'_1, +}) \right)G_{h'_2}(y_2 - x_{\ell'_2, +}) + 
\quad + G_{h_1}(x_{\ell_1, -} - x_1)G_{h_2}(x_{\ell_2, -} - x_1)G_{h'_1}(x_1 - x_{\ell'_1, +})\left( G_{h'_2}(y_2 - x_{\ell'_2, +}) - G_{h'_2}(x_1 - x_{\ell'_2, +}) \right) \right]$$

and the differences in parentheses can be further rewritten in interpolated form as

$$\left( G_{h_2}(x_{\ell_2, -} - x_2) - G_{h_2}(x_{\ell_2, -} - x_1) \right) = -\int_0^1 ds (x_2 - x_1) \cdot \partial G_{h_1}(x_{\ell_2, -} - x_1)_{12(s)},$$

(2.52)

where $x_{12}(s) := x_1 + s(x_2 - x_1)$ and similar expressions are valid for the other two parentheses. Plugging this back into (2.51), we put the factor $x_{2,j} - x_{1,j}$ together with $F_v(x_1, x_2; y_1, y_2)$, from which we see that dimensionally it corresponds to a factor $\gamma^{-h_v}$ or $\gamma^{-h_v/2}$, depending on whether $j = 0$ or $j > 0$, see (2.49). Moreover,

---

*Actually, it could also happen that some of the fields in $P_v$ are also in $P_{v_0}$, in which case the corresponding propagators should be replaced by external fields. For simplicity, we exclude this possibility here.*
the derivative $\partial_j$ acting on $G_{h1}$ dimensionally corresponds to a factor $\gamma^{h_1}$ or $\gamma^{h_1/2}$, depending on whether $j = 0$ or $j > 0$, simply because $||\partial_0 G_{h1}||_1 \leq (\text{const.}) \gamma^h ||G_{h1}||_1$, $||\partial_1 G_{h1}||_1 \leq (\text{const.}) \gamma^{h/2} ||G_{h1}||_1$, etc. Putting things together, we see that the action of $\mathcal{R}$ is dimensionally bounded by a factor that is at least $\gamma^{-h_v + \max_i (h_i)}/2$, as desired. The argument explained here for an operator $\mathcal{R}$ acting on a vertex $v$ that is not followed by other vertices such that $\mathcal{R} \neq 1$, can then be iterated until the root is reached. In this way, each vertex that is renormalized non-trivially gains a factor $\gamma^{(h_v - h')/2}$, where $h' < h_v$. Note that in the iterative procedure described above it may happen that some of the derivatives coming from the interpolations act on the external fields. This means that the expansion (2.14) should be replaced by

$$V^{(h)}(\varphi) = \sum_{m \geq 1} \sum_{\alpha_1, \ldots, \alpha_m} \int_{\mathbb{R}^{6m}} dx_1 \cdots dx_m W_{2m; \alpha_1, \ldots, \alpha_m}^{(h)}(x_1, \ldots, x_m; y_1, \ldots, y_m), \hspace{1cm} \partial x_1, \ldots, \partial x_m \varphi_{x_i}, \partial y_1, \ldots, \partial y_m \varphi_{y_i} \hspace{1cm} (2.53)$$

where $\alpha_i = (\alpha_{i1}, \alpha_{i2}, \alpha_{i3})$, and $\partial x_i = \partial_{x_{i,0}} \partial_{x_{i,1}} \partial_{x_{i,2}}$, and similarly for $\partial y_i'. As a side remark, it can be shown that the summation over $\alpha_i, \alpha_i'$ can be restricted to the indices such that $\alpha_{i1}, \alpha_{i2} \in \{0, 1\}$, see [12, Section 3.3]. The result is

$$||W_{2m; \alpha_1, \ldots, \alpha_m}^{(h)}|| \leq \gamma^{(2-m-||\alpha^0|| - \frac{1}{2}||\alpha^1||)h} \left( \sum_{n \geq 1} |\lambda|^n \tilde{K}^n n! \sum_{\tau \in \mathcal{T}_{h_v,n}} \prod_{P \in \mathcal{P}} \gamma^{h_v - h_v' + \left(2 - \frac{|P_v|}{2} - z(P_v)\right)} \right), \hspace{1cm} (2.54)$$

where $||\alpha^0|| := \sum_i (\alpha_{i1}^0 + \alpha_{i2}^0)$, $||\tilde{\alpha}|| := \sum_i \sum_{j=1,2} (\alpha_{i1}^j + \alpha_{i2}^j)$, and

$$z(P_v) = \begin{cases} 1/2 & \text{if } |P_v| = 4, \\ 0 & \text{otherwise} \end{cases} \hspace{1cm} (2.55)$$

and we recall that $v'$ is the vertex immediately preceding $v$ on $\tau^0$. Note that now the renormalized scaling dimension $2 - |P_v|/2 - z(P_v)$ appearing at exponent in (2.54) is strictly negative, for all admissible $P_v$'s; therefore, the r.h.s. of (2.54) is summable both over $\{h_v\}$ and over $\{P_v\}$, and we get

$$||W_{2m; \alpha_1, \ldots, \alpha_m}^{(h)}|| \leq \gamma^{(2-m-||\alpha^0|| - \frac{1}{2}||\tilde{\alpha}||)h} \sum_{n \geq 1} |\lambda|^n \tilde{K}^n n!, \hspace{1cm} (2.56)$$

as desired.

---

* A complete proof of (2.54) requires the discussion of a few other technical points, including the change of variables from the original to the interpolated variables, as well as the possible accumulation of the derivatives coming from the interpolation procedure on a given propagator (which may a priori worsen the combinatorial factors in (2.54)). For these and other closely related issues we refer the reader to previous literature, see in particular [12, Section 3].

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2.D  The flow of the effective interaction

In this section we prove (2.44), by distinguishing the ultraviolet ($h \geq 0$) and infrared ($h < 0$) regimes.

The ultraviolet regime

We recall that if $h \geq 0$ then $\mathcal{L}W_4^{(h)}(x_1, x_2; y_1, y_2) := W_4^{(h)}(x_1, x_2; y_1, y_2)$ and that $W_4^{(h)}$ is the sum of all the ladder diagrams as in Fig.4 with scale labels larger than $h$ and lower than $N + 1$. In this particular case, $W_4^{(h)}$ admits an explicit expression in the following form:

$$W_4^{(h)}(x_1, y_1; x, y) = \frac{\lambda}{2} w(x - y) \delta(x - x_1) \delta(y - y_1) - \frac{1}{2} \sum_{n \geq 1} (-\lambda)^n \int dy_2 \cdots dy_n \delta(y - y_n) G[h+1,N](x_1 - x_2) \cdots G[h+1,N](x_{n-1} - x_n),$$

where $G[h+1,N](x) := \sum_{h<k\leq N} G_k(x)$. The replacement of the single scale propagator by $G[h+1,N](x)$ results from the sum over the scale labels of the special class of labelled diagrams we are looking at. Using the fact that $w(x) = \delta(x_0)v(\bar{x})$, we can immediately integrate out the $y_i,0$'s, so that

$$||W_4^{(h)}|| \leq \frac{1}{2} \sum_{n \geq 1} ||\lambda||^n ||v||_1 \cdot ||v||_\infty^{n-1} \cdot ||G[h+1,N]||_1^{n-1} \left[ \sup_{x_0} \int d\bar{x} G[h+1,N](x_0, \bar{x}) \right]^{n-1},$$

(2.58)

where $||v||_1 = \int d\bar{x} v(\bar{x})$ and $||v||_\infty = \sup_{\bar{x}} |v(\bar{x})|$. From (2.21) we see that

$$||G[h+1,N]||_1 \leq A'\gamma^{-h},$$

(2.59)

for a suitable $A' > 0$. Moreover, using (2.19), we also find

$$||G[h+1,N](x_0, \cdot)||_1 := \int d\bar{x} G[h+1,N](x_0, \bar{x}) \leq C,$$

(2.60)

uniformly in $x_0$, so that

$$||W_4^{(h)}|| \leq ||\lambda||^n ||v||_1 \sum_{n \geq 1} (K||\lambda||)^{n-1} \gamma^{-h(n-1)},$$

(2.61)

for $K = A'C$, which is the desired estimate for $h \geq 0$.

The idea behind the bound (2.61) is that, due to the structure of the ladder graphs, we can choose a “non usual” spanning tree which allows us to improve the naive dimensional estimate (2.38), whenever $h \geq 0$. According to the “usual” procedure for the derivation
of the dimensional bounds, i.e. the one presented after (2.29), the wiggly lines all belongs to the spanning tree; this means that we get a factor $||v||_1$ for each interaction $\omega(x-y)$, while the propagators not belonging to the spanning tree are bounded by their $L_\infty$ norm, see (2.31). From (2.58) it is apparent that in bounding the left side we followed a different procedure: only one of the wiggly lines was chosen to belong to the spanning tree, while the remaining $(n-1)$ interaction terms $v(\vec{x}_i - \vec{y}_i)$ were bounded by their $L_\infty$ norms. The connection among the remaining space-time points of the ladder is guaranteed by the propagators, see Fig.7. We conclude by noticing that the dependence of $W_4^{(h)}$ on the ultraviolet cutoff $N$ is exponentially small, thanks to the dimensional gain $\gamma^{-h}$ appearing in the previous bounds, see (2.61).

The infrared regime
If $h < 0$ the action of $L$ on $W_4^{(h)}$ is non trivial, see (2.40). In order to prove (2.44), we focus on the effective interaction

$$\lambda_h = \int dx_2 dy_1 dy_2 W_4^{(h)}(x_1, x_2; y_1, y_2) , \quad h \leq 0 , \quad (2.62)$$

defined in (2.40), and we note that the very definition of $W_4^{(h)}$ induces a recursive equation (“flow equation”) on $\lambda_h$:

$$\lambda_{h-1} = \lambda_h + \beta_h^*(\lambda_h, \ldots, \lambda_{-1}, \lambda) , \quad h \leq 0 , \quad (2.63)$$

\[ \beta_h^{(2)} = L_{v0} \frac{\lambda}{\lambda h} + R_{v0} \frac{\lambda h}{\lambda h + 1} \]

\[ = \frac{\lambda h}{h} + \frac{\lambda h + 1}{h + 1} \]

Figure 9: Labeled trees and Feynman diagrams representation of the second order contribution in \((\lambda h, \ldots, \lambda - 1, \lambda)\) to the beta function of \(\lambda_{h - 1}\), for \(h \leq -2\). The index \(L\) associated with \(v_0\) recalls the fact that \(\lambda_{h - 1}\) is the integral of \(L W_{h - 1}^{(4)}\).

where \(\beta_h^{(2)}\) is the local part (i.e., the integral over \(x_2, y_1, y_2\) of the) sum of the values of the diagrams associated with renormalized trees and field labels such that: the root is on scale \(h - 1\), \(|P_{v_0}| = 4\) and \(\bigcup v_i \neq P_{v_0}\), where \(v_1, \ldots, v_{s_{v_0}}\) are the vertices immediately following \(v_0\) on \(\tau\) (note that these conditions guarantee that the number of endpoints is \(\geq 2\)). Every endpoint \(v_i^+\) on a tree contributing to \(\beta_h^{(2)}\) is associated with an interaction

\[ \lambda_{h_i - 1} \int dx|\phi_x|^4 \quad \text{if } h_{v_i}^+ \leq 0, \quad (2.64) \]

or with

\[ \int dx_1 \cdots dy_2 W_{4}^{(h_i - 1)}(x_1, x_2; y_1, y_2) |\varphi_{x_1}^+ \varphi_{x_2}^+ \varphi_{y_1}^- \varphi_{y_2}^-|^2 \quad \text{if } h_{v_i}^+ > 0, \quad (2.65) \]

see Fig.8 for a graphical representation of the two vertices. In the latter equation \(W_{4}^{(h_i)}\) is the explicit function of \(\lambda\) in (2.57). Therefore, every tree contributing to \(\beta_h^{(2)}\) carries a natural dependence on \(\lambda_{h_i - 1}\), coming from the endpoints on scale \(\leq 0\), and on \(\lambda\), coming from the endpoints on scale \(> 0\): therefore, as indicated in (2.63), we think of \(\beta_h^{(2)}\) as a function of \(\lambda_h, \ldots, \lambda - 1, \lambda\). In the RG jargon, \(\beta_h^{(2)}\) is called the beta function for \(\lambda_h\).

In order to control the flow of \(\lambda_h\) with \(h \leq -1\), we explicitly compute the second order contribution in \((\lambda_h, \ldots, \lambda - 1, \lambda)\) to the beta function, see Fig.9. If \(h \leq -2\):

\[ \beta_h^{(2)}(\lambda_h, \ldots, \lambda - 1, \lambda) = -2 \int dx \, G_h(x)(\lambda_R^2 G_h(x) + 2\lambda_{h+1}^2 G_{h+1}(x)) \quad (2.66) \]
where we used the fact that $u_h(x)u_k(x) \equiv 0$, $\forall h, k : |h - k| > 1$. A similar formula is valid for $h = -1, 0$:

$$\beta_{-1}^{\lambda;2}(\lambda, \lambda) = -2 \int dx \ G_{-1}(x)(\lambda^2 - 1 G_{-1}(x)) + \frac{\lambda^2}{2} \tilde{G}_0(x), \tag{2.67}$$

$$\beta_0^{\lambda;2}(\lambda) = -2 \int dx \ G_0(x)(\frac{\lambda^2}{4} G_0(x) + \frac{\lambda^2}{2} \tilde{G}_1(x)), \tag{2.68}$$

where $\tilde{G}_h(x) := \int dy dz v(y) G_h(x_0, y - z) v(z - x)$, and we used the fact that

$$W_{h}^{(n)}(x_1, y_1; x, y) = \frac{\lambda}{2} w(x - y) \delta(x - x_1) \delta(y - y_1) + O(\lambda^2), \tag{2.69}$$

as it follows from (2.57)–(2.61). From the definition of $G_h$, (2.19), it is apparent that the second order beta function is strictly negative, $\forall h \leq 0$. Moreover, using the fact that $\lambda_h = \lambda_{h+1} + \frac{\beta_{h+1}}{2} = \lambda_{h+1} + O(\varepsilon_{h+1}^2)$, where $\varepsilon_h := \max_{k \geq h} \max\{|\lambda_k|, |\lambda|\}$, we see that in (2.66) we can replace $\lambda_{h+1}$ by $\lambda_h$, up to higher order corrections; moreover, using the scaling property (2.20), we find:

$$\beta_{h}^{\lambda;2}(\lambda_1, \ldots, \lambda, \lambda) = -\beta_2 \lambda_h^2 + O(\varepsilon_{h+1}^3), \tag{2.70}$$

with

$$\beta_2 := 2 \int dx G_0(x)(G_0(x) + 2G_1(x)) > 0. \tag{2.71}$$

In conclusion,

$$\lambda_{h-1} - \lambda_h = -\beta_2 \lambda_h^2 + O(\varepsilon_{h+1}^3), \tag{2.72}$$

with initial datum $\lambda_0 = \frac{1}{2}(1 + O(\lambda))$. Eq.(2.72) admits a solution that is bounded by $(\text{const.})|\lambda|$ uniformly in $h \leq 0$, and going to zero as $\sim |h|^{-1}$ as $h \to -\infty$. To see this, note that (2.72) is the finite-difference version of the equation $\dot{\lambda} = \beta_2 \lambda^2$, up to errors of the order $O(\lambda^3)$; since the latter ODE has solution $\lambda(h) = \lambda_0/(1 - \beta_2 h \lambda_0)$ for $h \leq 0$, it is easy to conclude that the solution to (2.72) has a qualitatively similar behavior. In the RG jargon, such behavior is referred to as asymptotic freedom.
3 Renormalization group theory of the Condensed State

3.A The functional integral representation for the perturbation theory around Bogoliubov Hamiltonian

From now on, we focus on the construction of the interacting condensed state, i.e. the interacting theory with propagator (1.9) at \( \rho_0 > 0 \), introduced at the end of Section 1.A. As discussed there, it is convenient to perform and analyze the perturbation theory of interest around the Bogoliubov Hamiltonian, which is supposedly the infrared fixed point for our theory (in the following we shall see that such an expectation must be reconsidered and corrected). With this purpose in mind, here we re-describe the perturbation theory around Bogoliubov’s Hamiltonian in the language of the functional integral.

We start from the representation (2.1) for the interacting partition function, with \( P^0_\Lambda(d\varphi) \) the gaussian measure with propagator (2.2) and \( \mu^0_{\beta,L} \) chosen in such a way that \( \lim_{\beta \to \infty} \lim_{L \to \infty} \beta L^2 (-\mu^0_{\beta,L}) = 1/\rho_0 \). We write the bosonic field \( \varphi^\pm \) as the sum of two independent fields: the first corresponding to its mean value, \( \xi^\pm := |\Lambda|^{-1} \int_\Lambda \varphi^\pm d\mathbf{x} \), whose interpretation is that of the amplitude of the condensate; the second representing the fluctuations around the condensate:

\[
\varphi^\pm := \xi^\pm + \psi^\pm, \quad \psi^\pm = \frac{1}{|\Lambda|} \sum_{\mathbf{k} \in D_\Lambda, \mathbf{k} \neq 0} e^{\pm i \mathbf{k} \cdot \mathbf{x}} \psi_k^\pm, \quad (3.1)
\]

where \( D_\Lambda \) was defined after (2.2). This decomposition induces a corresponding splitting of \( P^0_\Lambda(d\varphi) \) in the form of a product of a gaussian measure on \( \xi \) and a gaussian measure on \( \psi \). After this substitution the interaction potential takes the form

\[
V_\Lambda(\xi + \psi) = |\Lambda| \left( \frac{\lambda}{2} \hat{v}(\bar{0}) |\xi|^4 - \bar{\mu}_\Lambda^B |\xi|^2 \right) + V_\Lambda(\psi, \xi) \quad (3.2)
\]

with

\[
V_\Lambda(\psi, \xi) = \frac{\lambda}{2} \int_{\Lambda^2} dx dy \; w(x - y) \left[ (\xi^+)^2 \psi^-_x \psi^-_y + (\xi^-)^2 \psi^+_x \psi^+_y + 2 |\xi|^2 \psi^+_x \psi^-_y \right] - (\bar{\mu}_\Lambda^B - \lambda |\xi|^2 \hat{v}(\bar{0})) \int_{\Lambda} dx |\psi_x|^2 \\
+ \frac{\lambda}{2} \int_{\Lambda^2} dx dy \; |\psi_x|^2 w(x - y) |\psi_x|^2 + \lambda \int_{\Lambda^2} dx dy \left( \psi^+_x \xi^- + \xi^+ \psi^-_x \right) w(x - y) |\psi_x|^2 \\
+ \bar{\nu}_\Lambda \int_{\Lambda} dx (|\psi_x|^2 + |\xi|^2).
\]

Here \( \hat{v}(\bar{0}) = \int v(x) dx \) and \( \bar{\nu}_\Lambda = -\mu_{\beta,L} + \mu^0_{\beta,L} + \bar{\mu}^B_\Lambda \) must be chosen in order to fix the condensate density to \( \rho_0 \). In this language, the Bogoliubov approximation consists in neglecting the last two lines in (3.3). Rather than neglecting them, we combine the first two lines of (3.3) (which are quadratic in \( \psi \)) with the \( \psi \)-dependent part of \( P^0_\Lambda \); this defines a new reference gaussian measure \( P^B_{\xi,\Lambda}(d\psi) \), with modified propagator. The constant \( \bar{\mu}^B_\Lambda \) is another free parameter, which we can play with. The remaining terms
in (3.3) are treated perturbatively around the new reference measure. More explicitly, denoting by $V_A^B(\psi, \xi)$ the sum of terms on the last two lines of (3.3) we can rewrite (2.1) as:

$$
\frac{Z_A}{Z_A^0} = \int P_\Lambda(d\xi) e^{-|A|f_A^B(\xi)} \int P_{\xi,\Lambda}^B(d\psi) e^{-V_{\xi,\Lambda}(\psi)},
$$

where (if we define $\mu_A^B := \bar{\mu}_A^B + \mu_{0,\Lambda}^B$)

$$
P_\Lambda(d\xi) = \frac{d^2\xi}{N_0} e^{\frac{4}{3}(|\mu_A^B|^2 - \frac{1}{2} \Im(\xi)||\xi||^2)},
$$

Here $d^2\xi = d(\Re\xi)d(\Im\xi)$ and

$$
N_0 = \frac{\pi^{3/2}}{2} \sqrt{\frac{2}{\lambda\bar{\nu}(0)|A|}} e^{\frac{4}{3}(\mu_A^B)^2/(2\lambda\bar{\nu}(0))}.
$$

Moreover, the function $f_A^B(\xi)$ in the r.h.s of (3.4) arises from the ratio of the normalizations associated with the gaussian measures $P_\Lambda^0(d\psi)$ and $P_{\xi,\Lambda}^B(d\psi)$, and is given by (defining $F(\bar{k}) := |\bar{k}|^2 - \mu_A^B + \lambda\bar{\nu}(0)||\xi||^2 + \lambda\bar{\nu}(\bar{k})||\xi||^2$ and $\varepsilon'(\bar{k}) := \sqrt{F(\bar{k})^2 - (\lambda\bar{\nu}(\bar{k})||\xi||^2)^2}$)

$$
e^{-|A|f_A^B(\xi)} = \frac{N_0 Z_{\xi,\Lambda}^B e^{\varepsilon'(\bar{k})}}{Z_A^0 (-\mu_A^B)},
$$

where

$$
Z_{\xi,\Lambda}^B = \prod_{\bar{k} \in D_L} \frac{e^{\beta(F(\bar{k}) - \varepsilon'(\bar{k}))/2}}{1 - e^{-\beta\varepsilon'(\bar{k})}}
$$

and $D_L$ was defined after (1.1). In the rewritings above, $P_\Lambda(d\xi)$ is thought of as the “bare” measure associated with $\xi$, while $P_\Lambda(d\xi) e^{-|A|f_A^B(\xi)}$ is its dressed measure in the Bogoliubov’s approximation, and

$$
P_\Lambda(d\xi) e^{-|A|f_A(\xi)} := P_\Lambda(d\xi) e^{-|A|f_A^B(\xi)} \int P_{\xi,\Lambda}^B(d\psi) e^{-V_{\xi,\Lambda}(\psi)}
$$

is its fully dressed measure. It is apparent from its definition that $P_\Lambda(d\xi)$ tends to concentrate on the circle $||\xi|| = \mu_A^B/(\lambda\bar{\nu}(0))$ as $|A| \to \infty$. A similar property is valid for $P_\Lambda(d\xi)e^{-|A|f_A^B(\xi)}$. It is then natural to assume that also the fully dressed measure $P_\Lambda(d\xi)e^{-|A|f_A(\xi)}$ concentrates on a circle $||\xi|| = \rho_0$ as $|A| \to \infty$. The counterterm $\bar{\nu}_A$ must be fixed in such a way that $\rho_0$ corresponds to the actual condensate density. The consistency of this natural assumptions should be checked a posteriori of the construction of $f_\Lambda(\xi)$. By (gauge) symmetry, $f_\Lambda(\xi)$ is a radial function, i.e., it only depends upon $|\xi|$. Therefore, with no loss of generality, we can pick $\xi$ to be real.
From now on, motivated by the considerations above, we shall limit ourselves to describe the construction of \( f^\Lambda(\xi) \) in the case that \( \xi^+ = \xi^- = \sqrt{\rho_0} \), with \( \rho_0 \) fixed in such a way that \( f(\xi) = \lim_{|\Lambda| \to \infty} f^\Lambda(\xi) \) has a critical point at \( \rho_0 \). We will also make the convenient choice that \( \lim_{|\Lambda| \to \infty} \mu^B_\Lambda = \lambda \hat{v}(\vec{0}) \rho_0 \). A global control on \( f(\xi) \) is beyond the purpose of this paper. Note that the replacement of the fluctuating field \( \xi \) by the constant \( \sqrt{\rho_0} \) is the analogue of the well known \( c \)-number substitution for the creation and annihilation operators associated with the zero mode, which is rigorously known to be correct as far as the computation of the pressure is concerned \([36, 27]\); in this sense, our assumption of restricting our analysis to \( \xi^- = \sqrt{\rho_0} \) is not expected to be a serious limitation, and we expect that it leads to the correct result in the thermodynamic limit \( L \to \infty \). There are of course issues to be discussed about exchanging the limit \( L \to \infty \) with the replacement \( \xi \to \sqrt{\rho_0} \) (particularly as far as the computation of correlations are concerned), but also these issues are beyond the purpose of this paper.

After having fixed \( \xi^+ = \xi^- = \sqrt{\rho_0} \), the problem becomes that of constructing the theory associated with the functional integral

\[
\Xi_\Lambda = e^{-|\Lambda|(f^B_\Lambda(\sqrt{\rho_0}) + \bar{\nu}_\Lambda \rho_0)} \int P^B_\Lambda(d\psi) e^{-\bar{V}_\Lambda(\psi)}, \tag{3.10}
\]

where

\[
\bar{V}_\Lambda(\psi) = \frac{\lambda}{2} \int_{\Lambda^2} dxdy |\psi_x|^2 w(x - y) |\psi_y|^2 + \lambda \sqrt{\rho_0} \int_{\Lambda^2} dxdy (\psi^+_x + \psi^-_x) w(x - y) |\psi_y|^2 \\
+ \bar{\nu}_\Lambda \int_{\Lambda} dx |\psi_x|^2, \tag{3.11}
\]

see Fig.10. Moreover, \( P^B_\Lambda(d\psi) \) is the same as \( P^B_{\xi^\Lambda}(d\psi) \) above, with \( \xi \) replaced by \( \sqrt{\rho_0} \). We recall that \( \mu^B_\Lambda \) is chosen in such a way that \( \mu^B = \lim_{|\Lambda| \to \infty} \mu^B_\Lambda = \lambda \rho_0 \hat{v}(\vec{0}) \). In the
limit $\beta, L \to \infty$ the propagator associated with this modified measure is

$$g^B(x - y) = \left( \begin{array}{ccc} g^B_+(x - y) & g^B_- (x - y) \\ g^B_+(x - y) & g^B_- (x - y) \end{array} \right) =$$

$$= \int_{\mathbb{R}^3} \frac{dk}{(2\pi)^3} e^{-ik(x-y)} \begin{pmatrix} ik_0 + F(\vec{k}) & -\lambda \rho_0 \hat{\nu}(\vec{k}) \\ -\lambda \rho_0 \hat{\nu}(\vec{k}) & -ik_0 + F(\vec{k}) \end{pmatrix}$$

$$= \int_{\mathbb{R}^3} \frac{dk}{(2\pi)^3} e^{-ik(x-y)-|x_0-y_0|\varepsilon'(|\vec{k}|)} \begin{pmatrix} F(\vec{k}) + \sigma(x_0 - y_0)\varepsilon'(\vec{k}) & -\lambda \rho_0 \hat{\nu}(\vec{k}) \\ -\lambda \rho_0 \hat{\nu}(\vec{k}) & F(\vec{k}) + \sigma(y_0 - x_0)\varepsilon'(\vec{k}) \end{pmatrix},$$

where $F(\vec{k}) = |\vec{k}|^2 + \lambda \hat{\nu}(\vec{k})\rho_0$, $\varepsilon'(\vec{k}) = \sqrt{F(\vec{k})^2 - (\lambda \hat{\nu}(\vec{k})\rho_0)^2}$, and $\sigma(x_0 - y_0)$ is a sign function, equal to 1 if $x_0 - y_0 > 0$, equal to $-1$ if $x_0 - y_0 \leq 0$. In the following, and more precisely in the ultraviolet integration described in the following section, it will be convenient to think of $g^B$ as $g^B = \bar{g} + r$, where

$$\bar{g}(x - y) = \frac{e^{-|\vec{k}|^2/4|x_0-y_0|}}{4\pi|x_0-y_0|} \begin{pmatrix} \vartheta(x_0 - y_0) & 0 \\ 0 & \vartheta(y_0 - x_0) \end{pmatrix} \equiv \begin{pmatrix} G(x - y) & 0 \\ 0 & G(y - x) \end{pmatrix}$$

$$r(x - y) = -\int_{\mathbb{R}^3} \frac{dk}{(2\pi)^3} e^{-ik(x-y)} \frac{\lambda \rho_0 \hat{\nu}(\vec{k})}{|\vec{k}|^2 + (\varepsilon'(|\vec{k}|))^2} \begin{pmatrix} \frac{ik_0 + |\vec{k}|^2}{-ik_0 + |\vec{k}|^2} & 1 \\ 1 & \frac{-ik_0 + |\vec{k}|^2}{-ik_0 + |\vec{k}|^2} \end{pmatrix}.\tag{3.15}$$

In order to compute the free energy associated with the functional integral (3.10), we proceed as in Section 2, namely we regularize the theory by introducing an ultraviolet cutoff, to be eventually removed. The ultraviolet regularization is implemented by replacing the propagator $g^B_\Lambda$ by its cutoffed version $\sum_{h \leq N} g^B_{\Lambda,h}$, with $N$ a cutoff parameter to be eventually sent to $+\infty$. This replacement is the analogue of the replacement of $S^0_\Lambda$ in (2.3) by its regularized version (2.5) or, equivalently, (2.8). The limit $\beta, L \to \infty$ of the single scale propagator $g^B_{\Lambda,h}$ is analogous to (2.19): we write it as $g^B_h = \bar{g}_h + r_h$, where

$$\bar{g}_h(x - y) = \begin{pmatrix} G_h(x - y) & 0 \\ 0 & G_h(y - x) \end{pmatrix},$$

$$r_h(x - y) = -\int_{\mathbb{R}^3} \frac{dk}{(2\pi)^3} \tilde{f}_h(k)e^{-ik(x-y)} \frac{\lambda \rho_0 \hat{\nu}(\vec{k})}{|\vec{k}|^2 + (\varepsilon'(|\vec{k}|))^2} \begin{pmatrix} \frac{ik_0 + |\vec{k}|^2}{-ik_0 + |\vec{k}|^2} & 1 \\ 1 & \frac{-ik_0 + |\vec{k}|^2}{-ik_0 + |\vec{k}|^2} \end{pmatrix}\tag{3.17}$$

where $\tilde{f}_h(k) = \chi(\gamma^{-h}(k_0^2 + |\vec{k}|^4)^{1/2}) - \chi(\gamma^{-h+1}(k_0^2 + |\vec{k}|^4)^{1/2})$. It is implicit that all these regularizations have their natural counterpart at finite space-time volume $\Lambda$, which we will denote by $g^B_{\Lambda,h}, \bar{g}_{h,h}, r_{\Lambda,h}$.
The replacement of $g^B$ by its regularized version corresponds to the replacement of the partition function (3.10) by

$$\Xi_{A,N} = e^{-|A| f^B_A(\sqrt{\rho_0}) + \bar{\upsilon}_A \rho_0} \int \prod_{h \leq N} P^B_{A,h}(d\psi(h)) e^{-\hat{V}_A(\psi^{(\leq N)})}, \quad (3.18)$$

and the idea now is to compute the r.h.s. by iteratively integrating the degrees of freedom on scale $N$, $N - 1$, etc, thus rewriting the finite volume free energy as a series $f^B_A(\sqrt{\rho_0}) + \bar{\upsilon}_A \rho_0 + \sum_{h \leq N} F_{A,h}$, with $F_{A,h}$ representing the contribution from the single-scale integration $P^B_{A,h}(d\psi(h))$. Sending $N \to \infty$ corresponds to the ultraviolet limit, while the control of the series as $h \to -\infty$ corresponds to the infrared limit. As we will explain in detail below, the iterative computation proceeds smoothly (in a way very similar to the one described in Section 2) for all scales $h \geq \bar{h}$, where $\gamma^h = C_0 \lambda$, with $C_0$ a (finite but sufficiently large) positive constant. Note that, for $h \geq \bar{h}$, the term $|\tilde{k}|^4$ entering the definition of $\epsilon'(|\tilde{k}|) = \sqrt{|\tilde{k}|^4 + 2\lambda \rho_0 \hat{\upsilon}(\tilde{k})|\tilde{k}|^2}$ dominates the second term under the square root sign: $|\tilde{k}|^4 \geq (\text{const.}) \lambda \rho_0 \hat{\upsilon}(\tilde{k})|\tilde{k}|^2$; the opposite inequality is valid for lower scales. This implies that the Bogoliubov’s propagator satisfies different dimensional estimates in the two regimes. For $h \geq \bar{h}$, the theory looks like a small perturbation of the theory of the quantum critical point, see next section for a discussion of this regime. For $h < \bar{h}$ the multiscale integration procedure must be modified and it becomes much more involved (and interesting!), see Section 3.C below.

3. B The ultraviolet integration

In this subsection we describe how to integrate the degrees of freedom corresponding to the scales $\bar{h} \leq h \leq N$ (where $\bar{h}$ is fixed so that $\gamma^h = C_0 \lambda$) in the r.h.s. of (3.18). Proceeding in a way analogous to that described in Section 2.B, we construct the effective potential $V^{(\bar{h})}(\psi^{(\leq \bar{h})})$ via the analogue of (2.14) and (2.16), and the single-scale contribution to the free energy, $F_h$, via the analogue (2.17), with two main differences: (1) the symbol $E^B_h$ in (2.16) and (2.17) should now be re-interpreted as the truncated expectation with respect to $P^B_h(d\psi(h))$; (2) at the first step, $h = N$, the effective potential $V^{(N)}(\psi)$ should now be re-interpreted as being equal to $\hat{V}(\psi)$ in (3.11), rather than to $V(\psi)$ in (2.23). Note that neither the potential $\hat{V}(\psi)$ nor the propagator $g^B_h$ are now particle-number conserving: therefore, the effective potential on scale $h$ is not going to be particle-conserving either, and the analogue of (2.14) should now allow for a number $m_+$ of $\psi^+$ fields different in general from the number $m_-$ of $\psi^-$ fields. We shall denote by $W^{(h)}_{m_+, m_-}$ the kernel of the effective potential on scale $h$ with $m_+$ (resp. $m_-$) external legs of type $\psi^+$ (resp. $\psi^-$).

The outcome of the iterative construction can be expressed again in terms of a tree expansion, completely analogous to that described in Section 2.B, which implies the analogues of (2.24), (2.25) and (2.26). Once again, the only differences between the current formulas and those of Section 2.B are that both the potential on scale $N$ and the propagator are different. In particular, the trees contributing to $V^{(h)}$ can have endpoints
of different types, either of type 4, or $3^+$, or $3^-$, or 2, see Fig.10: we shall indicate by $n_4$ the number of endpoints of type 4, by $n_3^+$ the number of endpoints of type $3^+$, etc. Moreover, we let $n_3 = n_3^+ + n_3^-$. If $n = (n_2, n_3, n_4)$, we indicate by $T_{N,n}$ (resp. $\bar{T}_{N,n}$) the set of non-renormalized (resp. renormalized) trees contributing to $V^{(h)}$ with a specified number of endpoints of different types.

Regarding the new propagator $g_h^B = \bar{g}_h + r_h$, we note that while $\bar{g}_h$ is related in a trivial way with $G_h$ and, therefore, it is scale invariant (see (2.20)), this is not the case for $r_h$. Still, $r_h$ can be written as $r_h(x_0, x) = \int d\bar{y} v(x - \bar{y})\bar{r}_h(x_0, \bar{y})$, where $\bar{r}_h$ is defined by an expression similar to the r.h.s. (3.17), with $\bar{v}(\bar{k})$ replaced by 1 under the integral sign. Moreover, $\bar{r}_h$ is essentially scale invariant, i.e., $\bar{r}_h \simeq \gamma^{h}\bar{r}_0(\gamma^{h}x_0, \gamma^{h/2}x)$, up to corrections of relative size $\gamma^{h-h}$, which are negligible for $h \gg \bar{h}$. In particular, for all $h \geq \bar{h}$, $||r_h||_\infty \leq C\gamma^{h}\gamma^{h-h} \min\{1, \gamma^{-2}\}$ and $||r_h||_1 \leq C\gamma^{h}h^{h-h}$. Therefore, the same dimensional bounds (2.21) remain valid both for $g_h^B$ and for $\bar{g}_h$. On top of that, the rest $r_h$ is better behaved from a dimensional point of view, i.e., the $L_1$ (resp. $L_\infty$) norm of $r_h$ has an extra $\gamma^{h-h}$ (resp. $\gamma^{h-h} \min\{1, \gamma^{-h}\}$) as compared to the $L_1$ (resp. $L_\infty$) norm of $\bar{g}_h$. In the following, it will be useful to distinguish the contributions to the effective potential coming from the time-ordered, particle-conserving propagators $\bar{g}_h$ from those coming from the rest $r_h$. To this purpose, we write the analogue of (2.24) as

$$W^{(h)}_{(m_+, m_-)}(\bar{x}, y) = \sum_{n=0}^{\infty} \sum_{\tau \in \mathcal{T}^{(h)}_{N,n}} \sum_{P \in \mathcal{P}^e} \sum_{\bar{n} \in \mathcal{N}(P)} \int_{f \in \mathcal{L}_0 \setminus \mathcal{P}_0} d\bar{x}_f W^{(h)}(\tau, P, \bar{n}; \bar{x}_0)$$

(3.19)

where: (1) $n = (n_2, n_3, n_4) > 0$ means that $n_2, n_3, n_4$ are all non-negative, but not simultaneously zero; (2) $\bar{n} = (\bar{n}_r,v)_{v \in \tau}$ with $\bar{n}_r,v$ the number of propagators of type $r_h,v$ contained in $v$ [we say that a propagator is contained in $v$ if it associated with a pair of fields in $P_v \setminus Q_w$ for some $w > v$]; (3) $\mathcal{N}(P)$ denotes the set of values of $\bar{n}$ compatible with $P$. Of course, $F_{h+1}$ can be expanded in a similar way.

By proceeding exactly as in Section 2.B (in particular, by following the procedure described after (2.28)), we obtain the analogue of (2.38) (details are left to the reader):

$$||W^{(h)}_{(m_+, m_-)}|| \leq \gamma^{h(2-m/2)} \sum_{n=0}^{\infty} |\lambda|^{n_3+n_4} |\bar{p}|^{n_2} K^n (n_4 + n_3^+ - m_+ + m_-) \gamma^{-h(n_2+n_3/2)} \sum_{\tau \in \mathcal{T}^{(h)}_{N,n}} \sum_{P \in \mathcal{P}^e} \sum_{\bar{n} \in \mathcal{N}(P)} \gamma^{(\bar{h}-h)\bar{n}_r} \prod_{v \not \in \text{e.p.}}^{(\bar{h}_v-h_v)} d_v ,$$

(3.20)

where $n = n(n) = n_2 + n_3 + n_4$, $m = m_+ + m_-$, and the vertex dimension $d_v = d_v(n_v, \bar{n}_v, |P_v|)$ is given by

$$d_v = 2 - |P_v| - n_{2,v} - \frac{1}{2} n_{3,v} - \bar{n}_{r,v} .$$

(3.21)

Here and in the following $n_{q,v}$ denotes the number of endpoints with $q$ external legs following $v$ on $\tau$ and $n_q := n_{q,v}$. Similarly, $\bar{n}_r := \bar{n}_{r,v}$. Moreover, $n_v = (n_{2,v}, n_{3,v}, n_{4,v})$. The vacuum contribution, $F_{h+1}$, admits a similar bound.
The potentially divergent contributions to (3.20) come from the subdiagrams such that $d_v \geq 0$, namely by the subdiagrams such that:

(a) $|P_v| = 4$ with $(n_{2,v}, n_{3,v}, \tilde{n}_{r,v}) = (0, 0, 0)$;
(b) $|P_v| = 3$ with $(n_{2,v}, n_{3,v}, \tilde{n}_{r,v}) = (0, 1, 0)$;
(c) $|P_v| = 2$ with $(n_{2,v}, n_{3,v}, \tilde{n}_{r,v}) = (0, 0, 0), (1, 0, 0), (0, 2, 0), (0, 0, 1)$.

Note that the cases $|P_v| = 3$ with $(n_{2,v}, n_{3,v}, \tilde{n}_{r,v}) = (0, 0, 0)$, and $|P_v| = 2$ with $(n_{2,v}, n_{3,v}, \tilde{n}_{r,v}) = (0, 1, 0)$, which in principle would lead to $d_v > 0$, are impossible. In addition to this, since $g_v(x)$ preserves the time-ordering, some of the marginal or relevant sub-diagrams are identically zero, as it was the case in the theory of the quantum critical point. More specifically, it is easy to check that there exist no non-vanishing diagrams with $|P_v| = 2$ and $(n_{2,v}, n_{3,v}, \tilde{n}_{r,v}) = (0, 0, 0), (1, 0, 0)$, except the trivial one with $n_2 = 1$ and $n_3 = n_4 = 0$. Moreover, the only non-vanishing diagrams among the potentially divergent ones have a ladder structure, similar to the one we found in the theory of the quantum critical point. The list of the non-vanishing diagrams is shown in Fig. 11, 12 and 13.

Due to the presence of these relevant and marginal contributions, we need to define a proper localization procedure, and correspondingly to reorganize the expansion in a way similar to the one discussed in Sections 2.C and 2.D. In particular, at each iteration step, we define split $V^{(h)}(\psi) = \mathcal{L}V^{(h)}(\psi) + \mathcal{R}V^{(h)}(\psi)$ with

$$
\mathcal{L}V^{(h)}(\psi) = \int_{\mathbb{R}^{12}} dx_1 \cdots dy_2 \mathcal{L}W^{(h)}_{(2,2)}(x_1, x_2; y_1, y_2) \psi^+_{x_1} \psi^+_{x_2} \psi^-_{y_1} \psi^-_{y_2}
+ \int_{\mathbb{R}^6} dx_1 dx_2 dx_3 \left[ \mathcal{L}W^{(h)}_{(2,1)}(x_1, x_2; x_3) \psi^+_{x_1} \psi^+_{x_2} \psi^-_{x_3} + \mathcal{L}W^{(h)}_{(1,2)}(x_1; x_2, x_3) \psi^+_{x_1} \psi^-_{x_2} \psi^-_{x_3} \right]
+ \int_{\mathbb{R}^6} dx_1 dx_2 \left[ \mathcal{L}W^{(h)}_{(2,0)}(x_1, x_2) \psi^+_{x_1} \psi^+_{x_2} + \mathcal{L}W^{(h)}_{(1,1)}(x_1; x_2) \psi^+_{x_1} \psi^-_{x_2} + \mathcal{L}W^{(h)}_{(0,2)}(x_1, x_2) \psi^-_{x_1} \psi^-_{x_2} \right].
$$

The localized kernels $\mathcal{L}W^{(h)}_{(2,2)}, \mathcal{L}W^{(h)}_{(2,1)}$, etc, are defined differently, depending on whether

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10 The vanishing of the non-trivial diagrams with $|P_v| = 2$ and $(n_{2,v}, n_{3,v}, \tilde{n}_{r,v}) = (1, 0, 0)$ is valid if the 2-legged vertices are particle-conserving, i.e., if they are proportional to $\int dx |\psi(x)|^2 dx$, as in our case, see (3.11). However, we will see in a moment that non-particle-conserving 2-legged vertices, proportional to $\int dx (\psi^*_x \psi^*_x + h.c.)$, are generated by the iterative integration, as soon as we reach scales $\bar{h} \lesssim h \leq 0$. Therefore, a posteriori we will also have non-vanishing contributions with $|P_v| = 2$ and $(n_{2,v}, n_{3,v}, \tilde{n}_{r,v}) = (1, 0, 0)$, with the 2-legged vertex that is necessarily of type $\int dx (\psi^*_x \psi^*_x + h.c.)$. 

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Figure 12: Two legged ladder graphs of order $n$. In the first line the particle–conserving ladders with $(n_{2,v}, n_{3,v}, \tilde{n}_{r,v}) = (0, 2, 0)$. Two types of two legged graphs not–particle–conserving, missing in the initial potential (3.11), are generated by the iterative integration when $(n_{2,v}, n_{3,v}, \tilde{n}_{r,v}) = (0, 0, 1)$, as shown in the second line.

Figure 13: Two legged ladder graphs with $(n_{2,v}, n_{3,v}, \tilde{n}_{r,v}) = (1, 0, 0)$.

$h$ is larger or smaller than 0. In particular, if $h \geq 0$,

$$\mathcal{L}W^{(h)}_{(2,2)}(x_1, x_2; y_1, y_2) = W^{(h)}_{(2,2)}(x_1, x_2; y_1, y_2) \bigg|_{(0,0,0)},$$

(3.24)

where $[:|_{(0,0,0)}$ means that we are taking the contribution corresponding to $(n_{2,v}, n_{3,v}, \tilde{n}_{r,v}) = (0, 0, 0)$. Moreover, if $h < 0$,

$$\mathcal{L}W^{(h)}_{(2,2)}(x_1, x_2; y_1, y_2) = \tilde{\lambda}_h \delta(x_1 - x_2)\delta(x_1 - y_1)\delta(x_1 - y_2)$$

(3.25)

with

$$\tilde{\lambda}_h := \int dx_1'dy_1'dy_2' W^{(h)}_{(2,2)}(x_1, x_2; y_1', y_2') \bigg|_{(0,0,0)}.$$ 

(3.26)

In other words, $\mathcal{L}W^{(h)}_{(2,2)}$ is equal either to the sum of all the ladder sub-diagrams built out of 4-legged vertices and propagators of type $\bar{g}$ carrying a scale label $\geq h$, if $h \geq 0$, or to its local part (in the sense of (3.25)-(3.26)), if $h < 0$. This is the same definition that we had in the theory of the quantum critical point. The local parts of the 3- and 2-legged kernels are defined similarly. In particular, for $h < 0$, we introduce the running
Moreover, if an associated with one of the terms in (3.23) with \(v\) as in (3.28). In conclusion, after the resummation, we find, similarly to (2.54):

\[
\bar{\varphi}_h := \int dy W_{(2,0)}^{(h)}(x,y) \bigg|_{(0,0,1)} + W_{(2,0)}^{(h)}(x,y) \bigg|_{(1,0,0)}
\]

(3.27)

where in the second line the contribution corresponding to \((n_2,v,v_r) = (1,0,0)\) comes from graphs as in Fig.13 (i.e., the 2-legged vertex is necessarily of type \(\bar{z}_h\)). Moreover, by symmetry, the second line would be the same even if we changed the label \((2,0)\) to \((0,2)\). By proceeding in a way completely analogous to Section 2.D, we find that, for \((m_+,m_-) = (2,2),(2,1),(1,2),\)

\[
\int dx_2 \cdots dx_m \mathcal{L}W_{(m_+,m_-)}^{(h)}(x_1, \cdots, x_m) \leq (\text{const.}) \lambda . \quad (3.28)
\]

Moreover, if \((m_+,m_-) = (1,1),(2,0),(0,2)\) similar bounds are valid (see Appendix A):

\[
\bar{\varphi} - \int dx_2 \mathcal{L}W_{(1,1)}^{(h)}(x_1,x_2) \leq (\text{const.}) \lambda^2 \min\{\gamma^{-h}, -\delta\} , \quad (3.29)
\]

\[
\int dx_2 \mathcal{L}W_{(2,0)}^{(h)}(x_1,x_2) \leq (\text{const.}) \lambda^2 \min\{\gamma^{-h}, -\delta\} . \quad (3.30)
\]

After this resummation, \(F_{h+1}\) and \(W_{(m_+,m_-)}^{(h)}\) are expressed as sums over trees where all the vertices which are not endpoints have negative dimension. Every endpoint \(v^e\) is associated with one of the terms in (3.23) with \(h = h^e_+ - 1\), whose kernels are all bounded as in (3.28). In conclusion, after the resummation, we find, similarly to (2.54):

\[
|F_{h+1}| \leq \sum_{n > 0}^{\lambda^{h+n_2}} \bar{\varphi} \gamma^{-h(n_2+n_3/2)} \gamma^{h_+} \sum_{\tau \in T_{h^+}^{(h)}} \sum_{P \in T_{\lambda}^{(h)}} \sum_{r \in \Lambda(P)} \gamma^{(h_+ - h_r)} \min_{v \text{not e.p.}} \gamma^{(h_+ - h_v)} (d_v - z_v) \cdot \min\{1, \gamma^{-h_+}\} \sum_{n_2}^{n_3} \gamma^{h_+} \bar{\varphi}_h \min\{1, \gamma^{-h_+}\} \quad (3.31)
\]

where: (i) \(\bar{\varphi} := \sup_{h \geq 0} |\bar{\varphi}_h|\), which is smaller than \(|\bar{\varphi}| + (\text{const.}) \lambda^2 |\log \lambda|\), and \(\bar{\varphi}\) is chosen in such a way that

\[
\int dx_2 \mathcal{L}W_{(2,0)}^{(h)}(x_1,x_2) \leq \bar{\varphi} \gamma^{h_+} \min\{1, \gamma^{-h_+}\} \quad (3.32)
\]

i.e., using (3.30), we see that \(\bar{\varphi}\) is smaller than \((\text{const.}) |\log \lambda|\); (ii) the * on the sum indicates the constraints that \(P_{v_0} = 0\) is fixed and the set of internal fields of \(v_0\) is non empty; (iii) the symbol \(n_\nu\) (resp. \(n_z\)) denotes the number of endpoints of type \((m_+,m_-) = (1,1)\) (resp. \((m_+,m_-) = (2,0),(0,2)\), \(n_2 = n_\nu + n_z\) and \(n = n_2 + n_3 + n_4\); (iv) \(h^e_+\) is the highest among the scales of the endpoints of type \((2,0)\), if any, or of
the propagators of type \( r_h \), otherwise (note that the vacuum diagrams have at least two “non-particle-conserving” endpoints or propagators, i.e., endpoints of type \( \bar{z}_h \) or propagators of type \( r_h \); otherwise, their value vanishes). The factor \( \min\{1, \gamma^{-\bar{h} z_h}\} \) either comes from (3.32), or from the remark that \( \|r_h\|_\infty \leq C \gamma^{\bar{h} z_h} \min\{1, \gamma^{-\bar{h}}\} \) (if the tree has at least one propagator of type \( r_h \), in bounding the tree value via (2.31) we can decide to take the \( L_\infty \) norm of the propagator of type \( r_h \), which produces the desired gain factor). We also recall that \( T_{N,n}^{(h)} \) indicates the set of renormalized trees. Moreover, \( z_v \) is the dimensional gain induced by the renormalization procedure, which is a function of \( |P_v| \), \( n_v \) and \( \tilde{n}_v \). More precisely, \( z_v \) is equal to 1/2 in the cases listed in (3.22), and zero otherwise. Since the renormalized vertex dimension is always negative, the exponential factors \( \gamma^{(\delta_{v-h_v}(d_v-z_v))} \) in the r.h.s. of (3.31) are summable both over \( \{P_v\} \) and over \( \{P_v\} \). After these summations, we are led to an \( n! \) bound of the form \(([])_+ \) indicates the positive part):

\[
|F_{h+1}| \leq \gamma^{2h} \sum_{n>0} \lambda^{n_4+n_4} z_{n_4} \bar{r}^{n_{1}} \bar{K}^{n}(n_{4} + n_{4} - m_{4} - \frac{m_{4}}{2})! \gamma^{-h(n_{4}+n_{4}/2)} \gamma^{(h-h)(n_{z}+[2-n_{z}]_+)} \min\{1, \gamma^{-\bar{h}}\},
\]

for some \( \theta > 0 \). Eq.(3.33) is acceptable as long as \( h \geq \bar{h} \) and \( \bar{v} \gamma^{-\bar{h}} =: \delta \) is small enough, as we shall assume from now on. In a similar way, we derive an \( n! \) bound for the kernels of the (renormalized) effective potential, whose external legs can now be associated with the action of derivative operators, as in (2.53). Using a notation analogous to (2.54), we can write the result as:

\[
\|W_{(m_+,m_-)}^{(h)}(\alpha_1, \ldots, \alpha_m)\| \leq \gamma^{h(2-m_{4}/2)-\|\alpha\|_{\bar{a}}/2-\|\delta\|_{\bar{a}})} \sum_{n>0} \lambda^{n_4+n_4} z_{n_4} \bar{r}^{n_{1}} \bar{K}^{n}(n_{4} + n_{4} - m_{4} - \frac{m_{4}}{2})! \gamma^{-h(n_{4}+n_{4}/2)} \gamma^{(h-h)n_{z}},
\]

where the \( * \) on the sum recalls that the number of endpoints must be compatible with the number of external legs, namely \( n_{4} + n_{4}/2 \geq m_{4}/2 - 1 \).

In conclusion, we can use the iterative integration procedure above for all scales \( h \geq \bar{h} \). For smaller scales we need to modify the multiscale integration procedure, as described in the following sections. Note that at scale \( \bar{h} \) the bound (3.34) leads to the following estimate on the kernels with \( m \) external legs:

\[
\|W_{(m_+,m_-)}^{(h)}(\alpha_1, \ldots, \alpha_m)\| \leq \gamma^{\bar{h}(2-m_{4}/2)-\|\alpha\|_{\bar{a}}/2-\|\delta\|_{\bar{a}})} \sum_{n>0} \lambda^{n_4+n_4} z_{n_4} \delta^{n_{z}} \bar{K}^{n}(n_{4} + n_{4} - m_{4} - \frac{m_{4}}{2})!.
\]

Of course, since \( \gamma^{\bar{h}} \) is of the order \( \lambda \), the factor \( \gamma^{\bar{h}(2-m_{4}/2)-\|\alpha\|_{\bar{a}}/2-\|\delta\|_{\bar{a}})} \) could be partially simplified with \( \lambda^{n_{4}/2+n_4} \); however, for the subsequent bounds, it is conceptually more transparent to think of them as two separate factors.
3.C The infrared integration

In the previous section we discussed the integration of the ultraviolet scales, up to a scale \( \bar{h} \) such that \( \gamma^h \) is of the order \( \lambda \). We are now left with

\[
\Xi = e^{-[A(f^B(\sqrt{\Lambda})+\nu_0+\sum_{k>\bar{h}} F_k)]} \int P^B_{\leq \bar{h}}(d\psi(\leq \bar{h})) e^{-V^{(h)}(\psi(\leq \bar{h}))},
\]

where we dropped the labels \( \Lambda \) for simplicity, and \( P^B_{\leq \bar{h}}(d\psi(\leq \bar{h})) := \prod_{h<\bar{h}} P^B_h(d\psi^{(h)}) \). In order to perform the infrared integration we rewrite the propagator \( g(\leq \bar{h}) \) with the cutoff function \( g \), in order to perform the infrared integration we rewrite the propagator \( g(\leq \bar{h}) \) as \( g_{\leq \bar{h}} = \tilde{g}_h + g_{\bar{h}} \), where \( \tilde{g}_h = g^{(1)} + g^{(2)} \).

\[
\tilde{g}_h^{(1)}(x - y) = \int \frac{dk}{(2\pi)^3} e^{-ik(x-y)} \left( \frac{1}{-ik_0+|\vec{k}|^2} \right) \cdot \left[ 1 - \chi(\gamma^h(|x_0|^2 + |\vec{x}|^2)^{1/2}) - \chi(\gamma^\bar{h}(|x_0|^2 + |\vec{k}|^2)^{1/2}) \right],
\]

and

\[
\tilde{g}_h^{(2)}(x - y) = \int \frac{dk}{(2\pi)^3} \left( \frac{e^{-ik(x-y)}}{|\vec{k}^2|} \right) \left( \frac{ik_0 + F(\vec{k})}{-\lambda \rho_0 \hat{\nu}(\vec{k})} \right) \cdot \left[ \chi(\gamma^\bar{h}(|x_0|^2 + |\vec{k}|^2)^{1/2}) \right],
\]

\[
\| \vec{k} \|^2 := k_0^2 + 2\lambda \rho_0 \hat{\nu}(0)|\vec{k}|^2. \]

The propagator \( g_{\leq \bar{h}} \) is defined by an expression similar to (3.38), with the cutoff function \( \left[ \chi(\gamma^\bar{h}(|x_0|^2 + |\vec{k}|^2)^{1/2}) - \chi(\gamma^\bar{h}||k||) \right] \) replaced by \( \chi(\gamma^\bar{h}||k||) \) under the integral sign. In Appendix B we show that \( \tilde{g}_h \) admits qualitatively the same dimensional bound as \( g^\bar{h} \), namely

\[
|\partial_{\tilde{n}_0} \partial_{\tilde{x}^2} \tilde{g}_h(x)| \leq \frac{C_{N,n_0,\bar{n}} \bar{n}^{(1+n_0+|\vec{n}|)/2}}{1 + [\gamma^\bar{h}(|x_0|^2 + |\vec{x}|^2)^{1/2}]^N},
\]

for all \( N, n_0, n_1, n_2 \geq 0 \) (here \( \vec{n} = (n_1, n_2) \)) and suitable constants \( C_{N,n_0,\bar{n}} > 0 \). Therefore, we can rewrite the functional integral in the r.h.s. of (3.36) as

\[
\int P_{\leq \bar{h}}(d\psi(\leq \bar{h})) \int \tilde{P}_h(d\psi^{(h)}) e^{-V^{(h)}(\hat{\psi}(h)+\psi(\leq \bar{h}))},
\]

where \( \tilde{P}_h \) has propagator \( \tilde{g}_h \) and \( P_{\leq \bar{h}} \) has propagator \( g_{\leq \bar{h}} \). Next we integrate out the field \( \psi^{(h)} \) and we end up with a new effective potential \( V^{(h)}(h) \) admitting the same dimensional estimates as \( V^{(h)} \). The basic idea for integrating the lower scales is to rewrite \( \chi(\gamma^\bar{h}||k||) \) as \( \sum_{h<\bar{h}} f_h(k) \), with \( f_h(k) = \chi(\gamma^\bar{h}||k||) - \chi(\gamma^{h+1}||k||) \), into the definition of \( g_{\leq \bar{h}} \); such a rewriting induces a multiscale resolution of \( g_{\leq \bar{h}} \) in the form \( g_{\leq \bar{h}} = \sum_{h<\bar{h}} g^{(h)} \), which is used to integrate step by step the functional integral, in a way analogous to what we discussed so far. The outcome of this new multiscale integration can again
be expressed in terms of new trees, whose endpoints represent the effective interaction \( V^{(\bar{h})} \) on scale \( \bar{h} \). The point is that \( g^{(h)} \) satisfies new dimensional estimates, which are qualitatively different from (3.39). Therefore, the kernels of the effective potentials on scale \( h < \bar{h} \) satisfy new dimensional estimates, which force us to change the definition of localization and renormalization. In order to define the infrared integration and localization procedure in the most transparent way, it is convenient to re-express the infrared field \( \psi^{(\leq \bar{h})} \) in terms of its real and imaginary parts, as first suggested by G. Benfatto [7] and later used in [15, 40]:

\[
\psi^\pm_{x}^{(\leq \bar{h})} = \frac{1}{\sqrt{2}} \left( \psi^l_{x}^{(\leq \bar{h})} \pm i \psi^t_{x}^{(\leq \bar{h})} \right) .
\] (3.41)

These new fields have natural scaling properties, as we shall see in a moment, for the good reason that they represent the longitudinal and transverse components of \( \psi^{(\leq \bar{h})} \), with respect to the set of stationary points of \( f(\xi) \), see (3.20) and the following comments. In terms of the new “basis” \( \psi^l, \psi^t \), the propagator \( g^{\leq \bar{h}} \) takes the form

\[
g^{\leq \bar{h}}(x-y) = \int_{\mathbb{R}^3} \frac{dk}{(2\pi)^3} \frac{e^{-ik(x-y)}}{k^2 + (\epsilon'(k))^2} \left( |\bar{k}|^2 \frac{k_0}{-k_0} \right) \chi(\gamma^{-\bar{h}}||k||) ,
\] (3.42)

where the first row and column now correspond to the index \( l \), while the second row and column to the index \( t \).

### 3.C.1 Non-renormalized bounds

Let us briefly describe here the naive (i.e., non-renormalized) infrared multi-scale procedure that one would get by decomposing the propagator as suggested after (3.40). This digression will be helpful in order to compute the scaling dimensions and to define a proper localization procedure. We write \( g^{\leq \bar{h}}(x) = \sum_{h \leq \bar{h}} g^{(h)}(x) \), with \( g^{(h)}(x) \) that, in the basis \( \psi^l, \psi^t \), is given by an expression similar to (3.42), with \( \chi(\gamma^{-\bar{h}}||k||) \) replaced by \( f_{h}(k) \). The single-scale propagator has the following scaling property, which can be derived in a way analogous to the proof of (3.39) (see Appendix B): if \( \alpha, \alpha' \in \{l, t\} \)

\[
|\partial^{|n_0|}_x \partial^{|n|}_y g^{(h)}_{\alpha, \alpha'}(x)| \leq C_{N,n_0,n} \gamma^h(1+n_0+|n|) \gamma^{-\bar{h}}|n|/2 \frac{\gamma^{(h-\bar{h})}(\delta_{\alpha,l}+\delta_{\alpha',t})}{1 + \left[ \gamma^h(|x_0| + \gamma^{-\bar{h}/2}||\vec{x}||) \right]^N} \] (3.43)

for all \( N, n_0, n_1, n_2 \geq 0 \) and suitable constants \( C_{N,n_0,n} > 0 \). Moreover \( \vec{x}' := \vec{x}/\sqrt{\rho_0} \) which has the same physical dimensions as \( x_0 \). Eq.(3.43) makes apparent that the matrix elements of \( g^{(h)} \) in the basis \( l, t \) have well-defined scaling properties (while, of course, in the basis \( \pm \) they have not). Eq.(3.43) induces a dimensional estimate on the kernels of the effective potential, via the same procedure used in the previous sections. More precisely, we first rewrite the effective potential \( V^{(\bar{h})} \) in the basis \( l, t \), then we proceed as
in the derivation of (2.29), thus finding (details are left to the reader):

\[
\| W_{m_1,m_2;\alpha_1,...,\alpha_n} \| \leq \sum_{n>0} \sum_{\tau \in T_{\hbar,n}} K^n \sum_{P \in P_{\tau}} \left[ \prod_{v \text{ not e.p.}} \frac{1}{s_v!} \gamma^{h_{\tau}} \left[ \frac{1}{2} (\sum_{i=1}^{s_v} |P_{i,v}| - |P_i|) \right] \right] \left[ \prod_{v \text{ e.p.}} \gamma^{h_{\tau}} \left[ \frac{1}{2} |P_v| - q_i(P_v) - \frac{1}{2} q_i(P_v) \right] \right] \delta^{n_{3,v} + n_{4,v}} \delta^{n_{3,v} + n_{4,v}} (n_{3,v} + \frac{n_{3,v}}{2} - \frac{|P_{i,v}|}{2})!.
\]

(3.44)

where:

- \( T_{\hbar,n}^{(h)} \) is a family of trees with endpoints on scale \( \hbar \). Each endpoint represents one of the contributions to \( V^{(h)} \) generated by the ultraviolet integration described in the previous section; note that now there are infinitely many different types of endpoints, labelled by the number and types of external legs, and by their order: an endpoint \( v \) with \( n_{\text{ext}}^l \) (resp. \( n_{\text{ext}}^t \)) external legs of type \( l \) (resp. \( t \)) of order \( n_v = (n_{3,v}, n_{4,v}) \) is by definition the sum of the (ultraviolet) trees with the proper number of external legs and \( n_v \) endpoints of type (1, 1), etc., in the sense of Eqs. (3.31) and (3.35). The label \( n \) attached to \( T_{\hbar,n}^{(h)} \) refers to the total order of the tree, which is the sum of the orders of its endpoints.

- \( P_v \) is a set of indices labelling the fields “exiting” from the vertex \( v \), in the sense described in Section 2.B. We assume that \( P_v \) also carries the information about the type (either \( l \) or \( t \)) of the fields and the number of derivatives acting on them: given a field label \( f \in P_v \), we denote by \( \alpha(f) \in \{l,t\} \) its type, and by \( q_i(f) \) the number of derivatives \( \partial_i \) acting on it. Moreover, \( P_v^l = \{ f \in P_v : \alpha(f) = l \} \), \( P_v^t = \{ f \in P_v : \alpha(f) = t \} \), and \( q_i(P_v) \) is defined as the sum of the number of derivatives \( \alpha(f) \) acting on \( f \). Finally, \( q(f) \) is defined as a sum of the values of \( q_i(f) \), \( q(f) = \sum_{i=1}^{3} q_i(f) \), \( q(P_v) = \sum_{i=1}^{3} q_i(P_v) \) and \( q(P_v) = \sum_{i=1}^{3} q_i(P_v) \).

- The two dimensional factors in the second and third lines collect all the dimensional factors coming from the estimates of the propagators (see (3.43)) and the effect of the integrals over \( x \) along the lines of the spanning tree (see comments after (2.29)): roughly speaking every propagator on scale \( h \) obtained by contracting two fields \( f_1 \) and \( f_2 \) carries a dimensional factor

\[
\prod_{f \in \{f_1,f_2\}} \gamma^{h(\frac{1}{2} + q(f)) - \frac{1}{2} h q(f)} \gamma^{h-h \delta_{\alpha(f),l}}.
\]

(3.45)

and every integral carries a factor \( \gamma^{-3h + \hbar} \). Finally, the factor in the last line comes from the dimensional estimates of the endpoints, see (3.35).
Figure 14: A list of sub-diagrams with non-negative scaling dimensions (the scaling dimension is indicated under each diagram). Solid lines correspond to fields of type $t$, while dashed lines to fields of type $l$. The figure lists all possible relevant and marginal diagrams without derivatives acting on the external lines. The reader can easily reconstruct from (3.47) the other relevant and marginal couplings with $q(P_v) > 0$.

Using the analogues of (2.33)–(2.36) we find

$$||W_{m_1, m_2; \alpha_1, \ldots, \alpha_m}^{(h)}|| \leq \gamma^{h(-1 + m_1 + \frac{1}{2}||\alpha||)} \gamma^{h(3 - \frac{1}{2} m_2 - \frac{1}{2} m_1 - \frac{1}{2} ||\alpha^0|| - ||\alpha||)}$$

$$\sum_{n > 0} \sum_{\tau \in \tilde{T}_{h, n}^{(h)}} \sum_{P \in P_\tau} \sum_{\gamma \in \tilde{\Gamma}(\tau, P)} \left[ \prod_{v \text{ not } e.p.} \frac{1}{s_v} \right] \gamma^{(h_v - h_v')} d_v$$

$$\left[ \prod_{v \text{ e.p.}} \gamma^{(\bar{h} - h_v')} d_v \lambda^\frac{1}{2} n_{3,v} + n_{4,v} z^{n_{2,v}} \delta^{n_{4,v}} (n_{4,v} + \frac{n_{3,v}}{2} - |P_v|) \right]$$

(3.46)

where the scaling dimension $d_v$ is

$$d_v = 3 - \frac{3}{2} |P_v| - \frac{1}{2} |P_v^l| - q(P_v)$$

(3.47)

On the basis of (3.46)-(3.47), we see that there is a finite number of relevant and marginal sub-diagrams (once again, the relevant sub-diagrams are those with $d_v > 0$, while the marginal ones are those with $d_v = 0$). A (almost complete) list of the relevant and marginal terms is shown in Fig.14.

It should now be clear that the natural localization procedure needed for renormalizing the infrared theory requires the introduction of an $L$ operator acting non-trivially on all the kernels with $d_v \geq 0$, i.e., those in Fig.14 (plus the few others with derivatives acting on the external fields). At each step we iteratively “dress” the propagator by combining the marginal quadratic terms with the gaussian reference measure. This means that the renormalized single-scale propagator is not going to be the same $g^{(h)}$ introduced above, but rather a dressed version of it (i.e., a similar propagator, but defined in terms of a few renormalized parameters). This and other details are described in the next section.
3.C.2 Renormalized bounds

Motivated by the discussion in Section 3.C.1, we define a modified multiscale integration procedure of the degrees of freedom at scales \( h \leq \tilde{h} \), along the lines sketched at the end of the previous subsection. After the integration of the fields on scales \( \tilde{h}, \tilde{h} - 1, \ldots, h + 1 \), we rewrite (3.36) as

\[
\Xi = e^{-\Lambda(\int B(\sqrt{\rho_0}) + \int \rho_0 + \sum_{k \geq h} F_k)} \int P_{\leq h}(d\psi^{(\leq h)}) e^{-\psi^{(h)}(\psi^{(\leq h)})},
\]

where \( P_{\leq h}(d\psi^{(\leq h)}) \) has propagator (in the basis \( l, t \)):

\[
g^{(\leq h)}(x) = \int \frac{dk}{(2\pi)^2} \chi_h(k) e^{-ik \cdot x} \begin{pmatrix} \tilde{A}_h(k) |\vec{k}|^2 + \tilde{B}_h(k) k_0^2 & \tilde{E}_h(k) k_0 \\ -\tilde{E}_h(k) k_0 & \tilde{Z}_h(k) \end{pmatrix}
\]

with

\[
\mathcal{D}_h(k) = \tilde{Z}_h(k)(\tilde{C}_h(k)k_0^2 + \tilde{A}_h(k)|\vec{k}|^2), \quad \tilde{C}_h(k)\tilde{Z}_h(k) := \tilde{E}_h^2(k) + \tilde{B}_h(k)\tilde{Z}_h(k),
\]

and, defining \( A_h := \tilde{A}_h(0), B_h := \tilde{B}_h(0) \), etc, the cutoff function \( \chi_h(k) \) is:

\[
\chi_h(k) := \chi(\gamma^{-h}\|k\|_h), \quad \text{with} \quad \|k\|_h^2 := k_0^2 + (A_h/C_h)|\vec{k}|^2.
\]

Moreover,

\[
\psi^{(h)}(\psi) = \sum_{m, m_\ell \geq 0} \sum_{\alpha, \alpha'} \int dx_1 \cdots dx_{m_\ell} W_{m_\ell; m_\ell; \alpha, \alpha'}^{(h)}(x_1, \ldots, x_{m_\ell}; y_1, \ldots, y_{m_\ell}).
\]

with the \( * \) on the sum indicating the constraint that \( m := m_\ell + m_t > 0 \), and \( \alpha \) (resp. \( \alpha' \)) being a shorthand for \( (\alpha_1, \ldots, \alpha_{m_t}) \) (resp. \( (\alpha'_1, \ldots, \alpha'_{m_t}) \)). If \( h = \tilde{h} \), the effective potential \( \psi^{(h)} \) coincides with the function \( \tilde{V}^{(h)} \) introduced after (3.40). For what follows, it is also convenient to re-express (3.52) in momentum space:

\[
\psi^{(h)}(\psi) = \sum_{m, m_\ell \geq 0} \frac{dk_1}{(2\pi)^3} \cdots \frac{dk_m}{(2\pi)^3} \tilde{W}_{m_\ell; m_\ell; \alpha, \alpha'}^{(h)}(k_1, \ldots, k_m) (2\pi)^3 \delta(\sum_{i=1}^m k_i).
\]

where \( m = m_\ell + m_t \). Finally, \( F_h, \tilde{A}_h, \tilde{B}_h, \tilde{E}_h, \tilde{Z}_h \) and \( W_{m_\ell; m_\ell; \alpha, \alpha'}^{(h)} \) are defined recursively via the inductive construction described below.
In order to inductively prove (3.48), we split \( \mathcal{V}^{(h)} \) as \( \mathcal{L}\mathcal{V}^{(h)} + \mathcal{R}\mathcal{V}^{(h)} \), where \( \mathcal{R} = 1 - \mathcal{L} \) and \( \mathcal{L} \), the localization operator, is a linear operator on functions of the form (3.52), defined by its action on the kernels \( \hat{W}_{m, m_t} \) in the following way:

\[
\begin{align*}
\mathcal{L}\hat{W}_0^{(h)}(k_2, \ldots, k_6) & := \hat{W}_0^{(h)}(0, \ldots, 0) \\
\mathcal{L}\hat{W}_1^{(h)}(k_2, \ldots, k_5) & := \hat{W}_0^{(h)}(0, \ldots, 0) \\
\mathcal{L}\hat{W}_2^{(h)}(k_2, k_3, k_4) & := \hat{W}_0^{(h)}(0, 0, 0) + \sum_{i=2}^4 k_i \partial_{k_i} \hat{W}_0^{(h)}(0, 0, 0) \\
\mathcal{L}\hat{W}_3^{(h)}(k_2, k_3) & := \hat{W}_0^{(h)}(0, 0) + \sum_{i=2}^3 k_i \partial_{k_i} \hat{W}_0^{(h)}(0, 0) \\
\mathcal{L}\hat{W}_4^{(h)}(k_2, k_3, k_4) & := \hat{W}_0^{(h)}(0, 0, 0) + \sum_{i=2}^3 k_i \partial_{k_i} \hat{W}_0^{(h)}(0, 0, 0) \\
\end{align*}
\]

(3.54)

and \( \mathcal{L}\hat{W}_i^{(h)}(k_2, \ldots, k_6) := 0 \) otherwise. Of course, if desired, one could translate these definitions in real rather than momentum space, in which case they would take a form analogous to those given above, in the theory of the quantum critical point, or in the ultraviolet integration of the condensed phase. In momentum space, the definition of localization should be understood as follows. When we Taylor expand \( \hat{W}_{m, m_t}^{(h)} \) with respect to the momenta, the term of order \( n \) in the Taylor expansion has an improved scaling dimension, as compared to \( \hat{W}_{m, m_t}^{(h)} \) itself: more precisely, if \( \hat{W}_{m, m_t}^{(h)} \) has scaling dimension \( d(m_t, m_t) = 3 - \frac{3}{2} m_t - \frac{1}{2} m_t \), then the \( n \)-th order term in the Taylor expansion in \( k_t \) has dimension \( d(m_t, m_t) - n \). The reason is that each operator \( k_i \partial_{k_i} \) dimensionally corresponds to a scaling factor \( \gamma^{h-h'} \), where \( h' > h \): in fact, when \( \partial_{k_i} \) acts on \( \hat{W}_{m, m_t}^{(h)}(k_2, \ldots, k_m) \), which is a sum of Feynman diagrams, the derivative can act onto one of the propagators appearing in such diagrams, each of which has a scale label strictly larger than \( h \); from a dimensional point of view, a derivative \( \partial_{k_i} \) acting on a propagator \( \hat{g}^{(h')} \) on scale \( h' > h \) behaves dimensionally (up to \( \lambda \)-dependent factors) as a multiplication by \( \gamma^{-h'} \), see (3.43). Similarly, the factor \( k_i \) in \( k_i \partial_{k_i} \) can be thought of as being attached to one of the external legs, whose scale label is \( \leq h \), which means that it will be contracted in the multiscale integration process in the form of a propagator \( \hat{g}^{(\leq h)}(k) \): from a dimensional point of view, \( k_i \hat{g}^{(\leq h)}(k) \) behaves like \( \hat{g}^{(\leq h)}(k) \) times \( \gamma^{h} \). Therefore, the rationale behind the definition of \( \mathcal{L}\hat{W}_{m, m_t}^{(h)} \) is that we set it equal to its Taylor series in \( k_i \) truncated at order \( n \), where \( n \) is such that \( d(m_t, m_t) - n \geq 0 \) and \( d(m_t, m_t) - n - 1 < 0 \). In this way, the rest of the Taylor expansion, which is part of \( \mathcal{R}\mathcal{V}^{(h)} \), is irrelevant in the sense
that its scaling dimension is negative.

A priori, the definitions (3.54) produce as many running coupling constants as the number of terms appearing in the r.h.s. However, luckily enough, not all those terms are really there: many of them are vanishing by symmetry. More precisely, using the parity number of terms appearing in the r.h.s. However, luckily enough, not all those terms are really there: many of them are vanishing by symmetry. More precisely, using the parity number of terms appearing in the r.h.s. However, luckily enough, not all those terms are really there: many of them are vanishing by symmetry. More precisely, using the parity number of terms appearing in the r.h.s.

\[ \hat{W}^{(h)}_{0,5}(0, \cdots, 0) = 0, \quad \partial_k \hat{W}^{(h)}_{0,4}(0, 0, 0) = 0, \quad \hat{W}^{(h)}_{0,3}(0, 0) = 0, \quad \hat{W}^{(h)}_{0,2}(0) = \partial_k \hat{W}^{(h)}_{0,1}(0) = \partial_k \hat{W}^{(h)}_{1,1}(0) = 0. \]

Moreover

\[ \partial_{k_j} \hat{W}^{(h)}_{1,1}(0) = 0, \quad \text{if} \quad j = 1, 2 \]

\[ \partial_{k_i} \partial_{k_j} \hat{W}^{(h)}_{0,2}(0) = 0, \quad \text{if} \quad i \neq j, \]

and \( \partial_{k_3}^2 \hat{W}^{(h)}_{0,2}(0) = \partial_{k_3}^2 \hat{W}^{(h)}_{0,2}(0). \) In addition to these parity cancellations, we can also use the conservation of momentum and the permutation symmetry between the external fields to infer that

\[ \int \frac{dk_1 dk_2 dk_3}{(2\pi)^3} (k_2 \partial_{k_2} + k_3 \partial_{k_3}) \hat{W}^{(h)}_{0,3}(0, 0) \hat{\psi}_{k_1} \hat{\psi}_{k_2} \hat{\psi}_{k_3} \delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3) = 0. \]

We now let

\[ \gamma^{-\hat{h}} \lambda_{6,h} := \hat{W}^{(h)}_{0,6}(0, \cdots, 0), \quad \gamma^{h/2} \mu_h := \hat{W}^{(h)}_{1,2}(0, 0) \]

\[ \gamma^{h-\hat{h}} \lambda_h := \hat{W}^{(h)}_{0,4}(0, \cdots, 0), \quad \gamma^{2h-\hat{h}} \nu_h := \hat{W}^{(h)}_{0,2}(0). \]

The constants \( \lambda_{6,h}, \lambda_h, \mu_h, \nu_h \) are called the running coupling constants, and they are all real, as it follows from the reality properties of the propagator and of the interaction. The dimensional factors in the l.h.s. of the definitions (3.59) are all of the form \( \gamma^{(h-\hat{h})(3-\frac{3}{2}m_1-\frac{3}{2}m_1)} \gamma^{(2-\frac{3}{2}(m_1+m_2))} \) and are introduced in order to compensate a product of bad factors of the form \( \gamma^{(h-\hat{h})d_e} \) associated with the endpoints with \( d_e = 3 - \frac{3}{2}m_1 - \frac{3}{2}m_1 \geq 0, \) as those in the third line of (3.46); the choice of these scaling factors is justified a posteriori by the fact that in the dimensional estimate of the renormalized kernels (see (3.75) below) every marginal or relevant endpoint contributes with a factor \( \lambda_{6,h}, \lambda_h, \mu_h, \) or \( \nu_h, \) depending on its type, without any other extra bad dimensional factor. Using (3.35), we see that \( \lambda_{6,h} \) is of order \( \lambda^3, \lambda_h \) is of order \( \lambda, \mu_h \) is of order \( \lambda^{1/2}, \) and \( \nu_h \) is of order \( \delta. \)

We also define the wave function renormalization constants \( a_h, b_h, c_h \) and \( z_h \) as follows:

\[ a_h := \partial_{k_1}^2 \hat{W}^{(h)}_{0,2}(0) = \partial_{k_2}^2 \hat{W}^{(h)}_{0,2}(0), \]

\[ b_h := \partial_{k_3}^2 \hat{W}^{(h)}_{0,2}(0), \]

\[ c_h := -\partial_{k_3} \hat{W}^{(h)}_{1,1}(0), \]

\[ z_h := 2\hat{W}^{(h)}_{2,0}(0). \]
and remarkably also these constants are all \textit{real}. The quadratic part of \(\mathcal{L}\mathcal{V}^{(h)}\) associated with these constants is:

\[
\mathcal{L}_Q\mathcal{V}^{(h)}(\psi) = \frac{1}{2} \int \frac{dk}{(2\pi)^3} (\psi^J_{-k}, \psi^T_{-k}) \hat{M}_Q^{(h)}(k) \left( \frac{\psi^J_{k}}{\psi^T_{k}} \right) = \frac{1}{2} (\psi, M_Q^{(h)} \psi),
\]

which has exactly the same symmetry and reality structure as the exponent of the gaussian weight in the reference Bogoliubov's measure. Therefore, we can combine \(\mathcal{L}_Q\mathcal{V}^{(h)}\) with the gaussian measure at scale \(h\), by proceeding as follows. We let

\[
e^{-|\Lambda| t_h \tilde{P}_{\leq h}(d\psi^{(\leq h)})} := P_{\leq h}(d\psi^{(\leq h)}) e^{-\mathcal{L}_Q\mathcal{V}^{(h)}(\psi^{(\leq h)})}
\]

where \(t_h\) accounts for the change in the normalization of the two gaussian measures. The “dressed” measure \(P_{\leq h}(d\psi^{(\leq h)})\) has propagator

\[
g^{(\leq h)}(x) = \int \frac{dk}{(2\pi)^3} \chi_h(k) e^{-ik\cdot x} \left( \frac{\hat{A}_{h-1}(k) |\vec{k}|^2 + \hat{B}_{h-1}(k) k_0^2}{-\hat{E}_{h-1}(k) k_0} \frac{\hat{E}_{h-1}(k) k_0}{\hat{Z}_{h-1}(k)} \right)
\]

where

\[
\begin{align*}
\hat{A}_{h-1}(k) &= \hat{A}_h(k) + a_h \chi_h(k), & \hat{B}_{h-1}(k) &= \hat{B}_h(k) + b_h \chi_h(k), \\
\hat{E}_{h-1}(k) &= \hat{E}_h(k) + e_h \chi_h(k), & \hat{Z}_{h-1}(k) &= \hat{Z}_h(k) + z_h \chi_h(k),
\end{align*}
\]

and \(\mathcal{D}_{h-1}(k)\) is defined as in (3.50) with \(h\) replaced by \(h - 1\). The initial data at scale \(\tilde{h}\) for these renormalization constants are:

\[
\begin{align*}
A_{\tilde{h}}(k) &= A_{\tilde{h}} = 1 & B_{\tilde{h}}(k) &= B_{\tilde{h}} = 0 & E_{\tilde{h}}(k) &= E_{\tilde{h}} = 1 & Z_{\tilde{h}}(k) &= 2\lambda_0 \hat{\nu}(\vec{k}) + |\vec{k}|^2,
\end{align*}
\]

so that \(Z_{\tilde{h}} = 2\lambda_0 \hat{\nu}(\vec{0})\). Next we split the cutoff function in (3.67) as \(\chi_h(k) = \chi_{h-1}(k) + \hat{f}_h(k)\), where \(\chi_{h-1}(k)\) is defined as in (3.51), with \(h\) replaced by \(h - 1\), and we define

\[
\mathcal{L}_C\mathcal{V}^{(h)}(\psi) = \gamma^{-h} \lambda_6 \int d\chi (\psi^J_{\chi} \psi^T_{\chi})^6 + \gamma^{h/2} \mu_h \int d\chi (\psi^J_{\chi} \psi^T_{\chi})^2 + \gamma^{-h} \lambda_h \int d\chi (\psi^J_{\chi} \psi^T_{\chi})^4 + \gamma^{2h-h} \int d\chi (\psi^J_{\chi} \psi^T_{\chi})^2,
\]

the label \(C\) stands for \textquote{couplings} so that \(\mathcal{L}\mathcal{V}^{(h)}(\psi) = \mathcal{L}_Q\mathcal{V}^{(h)}(\psi) + \mathcal{L}_C\mathcal{V}^{(h)}(\psi)\). Then we rewrite (3.48) as

\[
\Xi = e^{-|\Lambda| (f^B(\sqrt{m}) + \hat{p}_0 + \sum_{k \geq h} F_k + t_h)} \int P_{\leq h-1}(d\psi^{(\leq h-1)}). \\
\cdot \int \tilde{P}_h(d\psi^{(h)}) e^{-\mathcal{L}_C\mathcal{V}^{(h)}(\psi^{(\leq h-1)} + \psi^{(h)}) - \mathcal{R}\mathcal{V}^{(h)}(\psi^{(\leq h-1)} + \psi^{(h)})}
\]

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where $P_{<h-1}(d\psi^{(\leq h-1)})$ has the same propagator as (3.49), with $h$ replaced by $h - 1$, while $\tilde{P}_h(d\psi^{(h)})$ has propagator $\tilde{\phi}^{(h)}(x)$ given by an expression analogous to the r.h.s. of (3.67), with $\chi_h(k)$ replaced by $f_h(k) = \chi_h(k) - \chi_{h-1}(k)$.

At this point, we integrate the field on scale $h$ and define:

$$\tilde{A} \tilde{F}_h + \mathcal{V}^{(h-1)}(\psi) := -\log \int \tilde{P}_h(d\psi^{(h)}) e^{-\mathcal{L}C\mathcal{V}^{(h)}(\psi^{(\leq h-1)} + \psi^{(h)}) - \mathcal{R}\mathcal{V}^{(h)}(\psi^{(\leq h-1)} + \psi^{(h)})}.$$  

(3.73)

By plugging (3.73) into (3.72), we reproduce our inductive assumption (3.48) at scale $h - 1$, with $F_{h-1} := t_h + \tilde{F}_h$.

The inductive integration procedure described above gives rise to a new family of renormalized trees, analogous to those described in Section 3.C.1. The renormalized trees contributing to $\mathcal{V}^{(h)}$ are now denoted by $\mathcal{T}^{(h)}_{h;i,n}$. The action of an operator $\mathcal{R}$ is associated with all the vertices $v > v_0$ that are not endpoints. The trees in $\mathcal{T}^{(h)}_{h;i,n}$ can have endpoints on all scales between $h + 2$ and $\bar{h} + 1$, and the endpoints can be either of type $\mathcal{L}$ (marginal or relevant) or of type $\mathcal{R}$ (irrelevant). The marginal or relevant endpoints can live on all scales between $h + 2$ and $\bar{h} + 1$, and they represent an interaction term of type $\mathcal{L}C\mathcal{V}^{(h_\nu-1)}(\psi)$. Depending on the specific monomial in $\mathcal{L}C\mathcal{V}^{(h_\nu-1)}$ that is associated with the endpoint, we shall say that the endpoint is of type $\lambda_6$, or $\mu$, or $\lambda$, or $\nu$, with obvious convention. An endpoint $v$ of type $\mathcal{L}$ is necessarily contracted on scale $h_\nu - 1$, i.e., $P_v \neq P_{v'}$, where $v'$ is the unique node of the tree preceding $v$ on $\tau$, with $h_{v'} = h_{\nu-1}$. Finally, the irrelevant endpoints are necessarily on scale $\bar{h} + 1$ and they can be contracted on any of the lower scales. They are associated with one of the contributions to $\mathcal{R}\mathcal{V}^{(h)}$, where $\mathcal{V}^{(h)} = \tilde{V}^{(h)}$ was defined after (3.40).

The renormalized single-scale propagator defined in (3.49) satisfies a modified dimensional bound, as compared to (3.43), depending on the renormalization constants: if $\alpha, \alpha' \in \{l, t\}$:

$$|\partial_{x_0}^{\mu_0} \partial_x^{\nu_0} \tilde{\phi}^{(h)}(x)| \leq C_{N,\mu_0, \nu_0}^{\gamma h(1 + n_0 + |\vec{n}|)} \gamma^{-h/2} \left( \gamma^{(h-k)(\delta_{\alpha,l} + \delta_{\alpha',l})} / \left( 1 + \left( \gamma^h \left( |x_0| + \sqrt{E_h} \right) \right)^N \right) \right).$$  

(3.74)

$$\cdot \frac{1}{A_h} \left( \gamma^h C_h Z_h \right)^{|\vec{n}|/2} \left( \frac{E_h}{\sqrt{C_h Z_h}} \right)^{(\delta_{\alpha,l} \delta_{\alpha',l} + \delta_{\alpha,t} \delta_{\alpha',t})} \left( \gamma \frac{C_h}{Z_h} \right)^{(\delta_{\alpha,l} + \delta_{\alpha',l})},$$
for all $N, n_0, n_1, n_2 \geq 0$ and suitable constants $C_{N, n_0, n_1} > 0$. In deriving (3.74) we assumed that $B_h \geq 0$ (a fact that will be proved below), so that $C_h \geq B_h$.

Using this dimensional estimate, and the fact that an $R$ operator acts on all the vertices $v > v_0$ of the tree that are not endpoints (so that all the scaling dimensions of such vertices are automatically negative), we find the (renormalized) analogue of (3.44) (details are left to the reader):

\[
\begin{align*}
\|W_{m_1, m_2; \alpha_1, \ldots, \alpha_m}(h)\| &\leq \gamma h(-1 + m_1 + \frac{1}{2} ||\alpha||) \gamma h(3 - \frac{1}{2} m_1 - \frac{1}{2} m_2 - ||\alpha||) \sum_{n > 0} \sum_{\tau \in T_{h, n}^{(h)}} K^n \sum_{P \in P_v} \sum_{\mathcal{G} \in \Gamma(\tau, P)} \left[ \prod_{v \text{ not } e.p.} \left( \frac{1}{s_{v'}}^{\gamma (h_{v'} - h_v)(d_{v'} - z_{v'})} U_v \right) \right] \left[ \prod_{v \text{ e.p.}} W_v \right],
\end{align*}
\]

(3.75)

where:

- the contribution $W_v$ associated with the endpoints is equal to:
  
  \( (1) \lambda_{h, h_{v-1}} \) (resp. $\lambda_{h, -1}$, or $\mu_{h, -1}$, or $\mu_{h_{v-1}}$) if the endpoint is marginal of type $\lambda_{h}$ (resp. relevant of type $\lambda$, or $\mu$, or $\nu$);

  \( (2) \gamma (h_{v'} - h_v)(d_{v'} - z_{v'}) \) if the endpoint is irrelevant; here $v'$ is the scale at which the endpoint is contracted, i.e., the scale of the first node preceding $v$ on $\tau$ such that $P_v \neq P_{v'}$.

- The function $U_v$ appearing in the product over the vertices that are not endpoints collects the constants depending on the renormalization constants $A_h, B_h, C_h$ etc., arising from the term in the second line of (3.74), and from the integrations along the spanning tree. It is defined as:

\[
U_v = A_{h_v}^{-\frac{1}{2} (\sum_{i=1}^{n_v} |P_{v_i}| - |P_v|)} \left( \gamma h C_{h_v} \right) \left( \frac{1}{A_{h_v}} \right) \left( \gamma h \right) \left( \frac{E_{h_v}}{C_{h_v} Z_{h_v}} \right) \left( \frac{n_v}{l_{h_v}} \right) \left( \frac{E_{h_v}}{A_{h_v}} \right) \left( \gamma h \right) \left( \frac{Z_{h_v}}{C_{h_v}} \right) \sum_{i=1}^{n_v} |P_{v_i}| - |P_v| \right),
\]

(3.76)

where $n_v$ is the number of propagators of type $(l, t)$ or $(t, l)$ internal to $v$, but not in any other cluster $u$ following $v$ on $\tau$ (i.e., it is the number of propagator of type $lt$ obtained by contracting two fields internal to $v$, in the sense of item (5) before (2.4)).

- The dimensional gain $z_v$ appearing at exponent in the factors $\gamma (h_{v} - h_{v'}) (d_{v} - z_{v'})$ is equal to $z_v = [d_v]$ (here $[\cdot]$ indicates the integer part plus 1), if $d_v \geq 0$, and $z_v = 0$ otherwise. By construction, $d_v - z_v \leq -1/2$ for all the vertices in the tree.

Since the renormalized scaling dimension $d_v - z_v$ in the r.h.s. of (3.75) is negative for all the vertices of the tree, we can sum over the scale labels and, by proceeding as in the previous sections, obtain an $n!$ bound analogous to (say) (3.33) for the renormalized kernels of the effective potential. An immediate consequence of the proof leading to (3.75) is that contributions from trees $\tau \in T_{h, n}^{(h)}$ with a vertex $v$ on scale $h_v = k > h$
admit an improved bound with respect to (3.75), with an extra factor $\gamma^{\theta(h-k)}$, for any $0 < \theta < 1/2$; this factor can be thought as a dimensional gain with respect to the “basic” dimensional bound in (3.75). This improved bound is usually referred to as the short memory property (i.e. long trees are exponentially suppressed); it is due to the fact that the renormalized scaling dimensions $(d_v - z_v)$ in (3.75) are all negative and smaller equal than $-1/2$, and can be obtained by taking a fraction of the factors $\gamma^{(h_v - h_{v'})(d_v - z_v)}$ associated with the branches of the tree $\tau$ on the path connecting the root with the vertex on scale $k$.

Of course, the bound makes sense as long as the factors $U_v$ and $W_v$ in (3.75) remain bounded, at least order by order in renormalized perturbation theory. Boundedness of the renormalization and running coupling constants is a non trivial fact, which can be analyzed in terms of the flow of such constants under the iterations of the multiscale expansion. As mentioned above, the function controlling the flow of the effective constants is called the beta function, and will be discussed in the next sections. It must be stressed that, due to the large number of renormalization and running coupling constants, a brute force study of the beta function is very hard, particularly if one wants to push the study to the whole infrared limit $h \to -\infty$. The hope would be to take advantage of a number of remarkable exact relations between the effective constants, which reduce the number of independent effective constants to be controlled. These exact relations, which can be thought of as cancellations in the beta function, follow from Ward Identities, see next section. Let us also anticipate the fact that in the presence of a momentum regularization, as the one we are using here, the Ward Identities are affected by finite correction terms, due to the cutoffs, which change them as compared with the formal expression one would get by neglecting the regularization effects. The implications of these correction (anomaly) terms are dramatic, since they do not allow to prove the exact vanishing of the “bad” terms in the beta function, i.e., of those terms that drive the effective constants to $+\infty$. These correction terms are already visible at the one-loop level, as discussed in the next sections.

3.C.3 The flow equations

The $n!$ bounds in (3.75) allow us to give a meaning at all orders to the renormalized theory, as long as the running coupling constants remain small, and the renormalization constants entering the dressed propagator are such that the factors $U_v$ in (3.76) remain of order 1 (as they are on scale $\bar{h}$). The iterative construction described above induces a flow equation for the running coupling and renormalization constants, of the form:

$$\lambda_{6,h-1} = \lambda_{6,h} + \beta^\lambda_h \lambda_{h-1} = \gamma(\lambda_{6,h} + \beta^\lambda_h) \quad (3.77)$$
$$\mu_{h-1} = \gamma^{1/2}(\mu_h + \beta^\mu_h) \quad \nu_{h-1} = \gamma^2(\nu_h + \beta^\nu_h) \quad (3.78)$$
$$A_{h-1} = A_h + \beta^A_h \quad B_{h-1} = B_h + \beta^B_h \quad (3.79)$$
$$E_{h-1} = E_h + \beta^E_h \quad Z_{h-1} = Z_h + \beta^Z_{h+1} \quad (3.80)$$

where the beta functions $\beta^\#_h$ are related to the kernels $W_{m_1,m_2,\alpha_1,...,\alpha_m}^{(h)}$ (see definitions from (3.59) to (3.63)) and, therefore, they are expressed by series in the running coupling...
and renormalization constants admitting the same bound (3.75). If under the evolution induced by the flow equations (3.77)–(3.80) we reach a scale at which one or more of the running coupling constants become of order one, then we stop the flow at that scale, which we denote by \( h^* \). Otherwise, i.e., if the running coupling constants remain small for all scales, we say that the theory is well defined in the infrared, in which case we set \( h^* = -\infty \).

Note that, as long as the theory makes sense (i.e., as long as the running coupling constants remain small), the flow equation is dominated by the first non-trivial truncation of the series defining the beta function. If the approximate flow obtained by such a truncation drives the constants towards smaller values, then we are in good shape; in such a situation, it is easy to show by a standard stability analysis that the complete flow stays close to its lowest order truncation. If, on the contrary, the first non-trivial truncation to the beta function drives the running coupling constants towards larger values, then \( h^* \) is finite, and we cannot conclude anything about the infrared behavior of the system at lower scales\(^{11}\). In fact, in such a case, if \( h < h^* \) the higher order contributions to the beta function tend to dominate, and we cannot conclude anything sensible from finite truncations to the beta function. Unfortunately, the present case belongs to the latter category. In fact, if we truncate the flow equations (3.77)–(3.80) at the lowest non-trivial order we find (see Appendix C):

\[
\lambda_{h-1} = \gamma \lambda_h - 2 \gamma \frac{1}{A_h C_h \gamma_h} \left( 18 \lambda_h^2 - 12 \lambda_h \mu_h^2 \frac{\gamma_h}{Z_h} + 2 \mu_h^4 \frac{\gamma_{2h}^2}{Z_{2h}} \right) \beta_0^{(2)}
\]  
\[+ 4 \gamma \frac{1}{A_h C_h \gamma_h} \left[ \left( -6 \lambda_h \mu_h^2 \frac{\gamma_h}{Z_h} + \mu_h^4 \frac{\gamma_{2h}^2}{Z_{2h}} \right) \beta_0^{(2,\chi)} + \mu_h^4 \frac{\gamma_{2h}^2}{Z_{2h}} \beta_0^{(3,\chi)} \right] \]  
(3.81)

\[
\mu_{h-1} = \gamma^{1/2} \mu_h - 2 \gamma^{1/2} \frac{1}{A_h C_h \gamma_h} \mu_h \left[ \left( 6 \lambda_h - 2 \mu_h^2 \frac{\gamma_h}{Z_h} \right) \beta_0^{(2)} + 2 \mu_h^2 \frac{\gamma_h}{Z_h} \beta_0^{(2,\chi)} \right] \]  
(3.82)

\[
\nu_{h-1} = \gamma^2 \nu_h + \gamma^2 \frac{1}{A_h} \left[ \left( 6 \lambda_h - 2 \mu_h^2 \frac{\gamma_h}{Z_h} \right) \beta_0^{(1)} - 2 \mu_h^2 \frac{\gamma_h}{Z_h} \beta_0^{(1,\chi)} \right] \]  
(3.83)

\(^{11}\)A special but important case realizes when the first non-trivial truncation of the beta function displays a cancellation, which makes the beta function zero at that order: in such a case we need to go to higher orders in order to see whether the beta function is truly zero or not. Remarkable examples where this happens are models of spinless fermions in one dimension [10, 26, 13], for which it is possible to prove that the beta function is zero at all orders, by making use of remarkable, very subtle, cancellations at all orders in perturbation theory, following from the Schwinger-Dyson equation combined with local Ward Identities, see [13].
where, denoting \( f_h(\rho) = \chi(\gamma^{-h}\rho) - \chi(\gamma^{-h+1}\rho) \),

\[
\beta_0^{(1)} = \frac{1}{2\pi^2} \int d\rho f_0(\rho), \\
\beta_0^{(2)} = \frac{1}{2\pi^2} \int \frac{d\rho}{\rho^2} \left( f_0(\rho) + 2f_0(\rho)f_1(\rho) \right), \tag{3.84}
\]

\[
\beta_0^{(1,\gamma)} = \frac{1}{2\pi^2} (1 - \gamma^{-1}) \int_1^\gamma d\rho \chi(\rho)(1 - \chi(\rho)), \tag{3.86}
\]

\[
\beta_0^{(2,\gamma)} = \frac{1}{2\pi^2} (\gamma - 1) \int_1^\gamma d\rho \chi(\rho)(1 - \chi(\rho))^2, \tag{3.87}
\]

\[
\beta_0^{(3,\gamma)} = \frac{1}{2\pi^2} (\gamma - 1) \int_1^\gamma d\rho \chi(\rho)(1 - \chi(\rho))^3. \tag{3.88}
\]

Moreover, the flow of \( \lambda_{6,h} \) has the form \( \lambda_{6,h-1} = \lambda_{6,h} + O(\lambda^3_0) \), while the flow of the renormalization constants reads:

\[
Z_{h-2} - Z_{h-1} = -2 \frac{1}{A_h C_h} \mu_h^2 \beta_0^{(2)} \tag{3.89}
\]

\[
E_{h-2} - E_{h-1} = -2 \frac{1}{A_h C_h} \mu_h^2 E_h Z_h \beta_0^{(2)} \tag{3.90}
\]

\[
B_{h-2} - B_{h-1} = 2 \frac{\mu_h^2}{A_h Z_h} \left( \frac{E_h^2}{C_h Z_h} \beta_0^{(2)} + \frac{1}{3} \beta_0^{(\gamma')} \right) \tag{3.91}
\]

\[
A_{h-2} - A_{h-1} = \frac{2}{3} \frac{\mu_h^2}{C_h Z_h} \beta_0^{(\gamma')} \tag{3.92}
\]

where

\[
\beta_0^{(\gamma')} = (\gamma - 1) \frac{1}{2\pi^2} \int_1^\gamma \frac{d\rho}{\rho} \left[ \rho(\chi'(\rho))^2 + 2\chi'(\rho)(1 - \chi(\rho)) \right]. \tag{3.93}
\]

The initial data are all fixed but \( \nu_{\bar{h}} \), which can be freely adjusted in order to control the flow of \( \nu_h \). In order to keep \( \nu_h \) small for all scales \( h \leq \bar{h} \), we invert the flow equation for \( \nu_h \) (see (3.78)) in the form

\[
\nu_h = \gamma^{2(k-h)}\nu_k - \sum_{k<k'\leq h} \gamma^{2(k'-h)}\beta_k^{(\nu')} \tag{3.94}
\]

and then impose \( \lim_{k \to \infty} \gamma^{2k}\nu_k = 0 \), so that

\[
\nu_h = -\sum_{k \leq h} \gamma^{2(k-h)}\beta_k^{(\nu')} \tag{3.95}
\]

which should be interpreted as a fixed point equation for the sequence \( \{\nu_k\}_{k \leq \bar{h}} \) (and, therefore, for \( \nu_{\bar{h}} \) itself, because the sequence \( \{\nu_k\}_{k \leq \bar{h}} \) is uniquely determined by the choice of \( \nu_{\bar{h}} \)). At lowest order (see (3.83)), the solution of (3.95) has the form

\[
\nu_h = -\sum_{k \leq h} \gamma^{2(k-h)} \frac{1}{A_k} \left[ \left( 6\lambda_k - 2\mu_k^2 \frac{\gamma}{Z_k} \right) \beta_0^{(1)} - 2\mu_k^2 \frac{\gamma}{Z_k} \beta_0^{(1,\gamma)} \right]. \tag{3.96}
\]
which tells us that $\nu_h$ is of the order $O(\lambda h)$. While the flow of $\nu_h$ can be controlled by properly fixing $\nu_\bar{h}$ (that is, by properly fixing $\bar{\nu}$), there is nothing we can do for controlling the flows of $\lambda h$ and $\mu h$: they are driven by the linear term, which induces an exponentially fast divergence in $\bar{h} - h$. The result is that (recall that $\lambda_{\bar{h}}$ is of order $\lambda$ and $\mu_{\bar{h}}$ is of order $\sqrt{\lambda}$):

$$
\lambda_h = \gamma^{\bar{h} - h} \lambda_{\bar{h}} (1 + O(\gamma^{\bar{h} - h} \lambda_{\bar{h}})) ,
\mu_h = \gamma^{(\bar{h} - h)/2} \mu_{\bar{h}} (1 + O(\gamma^{\bar{h} - h} \lambda_{\bar{h}})) ,
$$

which is valid as long as $\gamma^{\bar{h} - h} \lambda_{\bar{h}}$ is small enough, say smaller than a suitable constant $\varepsilon_0$ (independent of $\lambda$), i.e., up to a scale $h^*$ of the order $\log \gamma \lambda^2$. In the range of scales $h^* \leq h \leq \bar{h}$, the marginal constants $A_h, E_h$, and $Z_h$ remain close to their values at scale $\bar{h}$, that is they are changed at most by a factor $1 + O(\varepsilon_0)$. Similarly, $B_h$ grows from zero to a value of the order $O(\lambda^{-1} \varepsilon_0)$, so that $C_h$ remains close to its value at scale $\bar{h}$, up to a relative error of the order $O(\varepsilon_0)$. Finally, $\lambda_{\bar{h}, h}$ grows from its initial datum at scale $\bar{h}$, which is of order $\lambda^3$, to a value of the order $O(\varepsilon_0^{3/2})$.

For smaller scales we cannot conclude anything sensible from the renormalized perturbative expansion discussed here. In the literature [15, 40, 48, 20, 17, 18, 47] there are a few heuristic claims about the nature of the infrared theory, which are based on an extrapolation of the flow to the non-perturbative region $h \leq h^*$ and on the implementation of some remarkable identities and cancellations, known as Ward Identities [15, 40], which are supposedly “non-perturbative” in nature. Therefore, it has some interest to discuss here in a non-ambiguous way the predictions of these Ward Identities in the perturbative regime $h^* \leq h \leq \bar{h}$, where the Bogoliubov’s scaling is visible but the theory is still perturbative (and, therefore, our analysis at all orders not only makes sense, but it gives a precise meaning to the approximate schemes described in [15, 40] and in [48, 20, 17, 18, 47]).

The comparison of the predictions of the formal Ward Identities first derived in [15, 40] with our exact findings in the regime $h^* \leq h \leq \bar{h}$ is discussed in the next section. Quite interestingly, we find a violation of the predictions of one of the (local) Ward Identities at the level of the one-loop beta function. This violation, or anomaly, was completely overlooked in the literature so far. We also identify the source of the violation, in the form of correction due to the momentum cutoff, whose effect can be consistently treated in our multiscale integration procedure. The anomaly that we find forces us to reconsider the analysis of the infrared fixed point of the theory proposed in [15, 40], see Section 3.C.5 below for a discussion of this point.

### 3.C.4 Ward Identities and anomaly

The Ward Identities (WIs) are identities between correlation functions, which can be derived by a “change of coordinates” in the functional integral of the form $\psi_\pm^\pm \rightarrow e^{\pm i\alpha(x)} \psi_\pm^\pm$ (phase change, or gauge transformation). If $\alpha(x)$ is independent of $x$, then the corresponding WI is usually referred to as a global WI, while it is referred to as local, otherwise. In order to derive the WIs in a correct and non-ambiguous way, one first needs to define the quantities of interest (say, the partition function and the generating function
for correlations) in terms of a well-defined functional integral, that is a functional integral regularized by the presence of suitable ultraviolet and infrared cut-offs. Next, one performs the aforementioned change of variables in the regularized functional integral, thus deriving exact identities among regularized correlation functions. Finally, whenever possible, one removes the regularization and derives the limiting expression of the regularized WIs. In some cases, the extra correction terms due to the presence of the cutoff in the WIs do not vanish in the limit where we remove the cut-offs, in which case such terms are called anomalies. They are often interpreted as “quantum violations to the conservation laws”.

The WIs can be used to derive convenient identities among the beta functions of different running coupling constants, or to identify subtle cancellations in the beta function. In the present context, the use of WIs for controlling the flow of the running coupling or renormalization constants is due to [15, 40]. Using a dimensional regularization scheme and an ε expansion around dimension $d = 3$ (in the form $d = 3 - \varepsilon$) they derived a number of remarkable WIs among the running coupling and the renormalization constants, thus reducing the number of independent constants to just one$^{12}$. Moreover, they argued that the remaining constant (which can be chosen to be $\lambda_h$) reaches a non-trivial fixed point in the infrared. Once this is assumed, the flow of all the other constants is driven by the WIs and suggests that the infrared theory still displays a linear excitation spectrum a’la Bogoliubov, without anomalous dimensions in the physical response functions.

In this section we want to reconsider and criticize this picture, by explicitly showing the existence of a non-vanishing correction term in the identities suggested by the formal WIs (i.e., those in which the effects of cutoffs are a priori neglected). We will compare explicitly our findings only with those of [15, 40], but similar comments apply for [48, 20, 17, 18, 47].

Let us first focus on the global WIs relating among each other the running coupling and renormalization constants at scale $h$. As mentioned above, they can be obtained by performing a global phase transformation in an auxiliary functional integral, which we choose as follows:

$$e^{-|\Lambda|F_{h-1} - W^{(h-1)}(\phi)} = \int P_{\geq h}^B(d\psi)e^{-V(\psi + \phi)}.$$  (3.97)

Here $F_{h-1}$ is a normalization constant and $P_{\geq h}^B(d\psi)$ is a modified version of our reference Bogoliubov measure, with an extra infrared cutoff at scale $h$, see Appendix D for details. The local parts of the kernels of $W^{(h-1)}(\phi)$ are related in a simple way to the local parts of the kernels of $V^{(h-1)}(\phi)$, as proved in Appendix D.3, see (D.54). Therefore, by deriving $W^{(h-1)}(\phi)$ with respect to $\phi^t, \phi^t$ and then setting $\phi = 0$, we can get several useful identities among these local kernels, including the following (see (D.20) and (D.21) for

\footnote{12Actually, by taking into account the presence of the marginal coupling $\lambda_{6,h}$, neglected in [40], the number of independent couplings should be two, see comment 1 in the itemized list at the end of Section 3.C.5.}
\[ Z_{h-1} - 2\gamma^{2h-h} \nu_h - 2\sqrt{2\rho_0} \gamma^{h/2} \mu_h = \hat{W}^{(h)}_{1,1;J\lambda}(0,0) \]  
\[ \gamma^{h/2} \mu_h - 4\sqrt{2\rho_0} \gamma^{h-h} \lambda_h = \hat{W}^{(h)}_{0,3;J\lambda}(0,0) \]

The terms in the right sides, defined via (D.17) (see also (D.20)-(D.21) and (E.1)) are the correction terms due to the momentum cutoffs. They are due to the finite infrared cutoff on scale \( h \) appearing in the reference gaussian measure in (3.97). In principle, there could also be effects from the ultraviolet momentum cutoff on scale \( N \) (to be eventually removed) that we need to introduce in order to give a meaning to the right side of (3.97). However, the super-renormalizability of the ultraviolet theory proved in Section 3.B implies that the corrections due to a finite ultraviolet cutoff on scale \( N \) vanish exponentially as \( N \to \infty \). This can be proved by a simple modification of the analysis in Section 3.B, along the lines of e.g. [32, Appendix A.2], which we do not belabor here. If we neglect the right sides of (3.98)-(3.99), we obtain two formal WIs that coincide with those derived by Pistolesi et al. in a dimensional regularization scheme, see [40, Eq.(3.18)-(3.19)].

Note that by using dimensional regularization one neglects essentially by construction any anomaly term induced by the momentum cutoffs, and the resulting flow equations may have in general a different qualitative behavior. If our correction terms were dimensionally sub-dominant with respect to the left sides for \( h \ll \bar{h} \), then we could say that the formal WIs are asymptotically correct in the infrared regime. However, this is not the case: the one-loop computation shows that at lowest non trivial order (defining \( \beta^{(2,\lambda)}_0 \) and \( \beta^{(3,\lambda)}_0 \) as in (3.87)-(3.88))

\[ \hat{W}^{(h)}_{1,1;J\lambda}(0,0) = 4\frac{\mu_h^2}{A_h C_h} \frac{\gamma}{\gamma - 1} \beta^{(2,\lambda)}_0, \]
\[ \hat{W}^{(h)}_{0,3;J\lambda}(0,0) = \frac{1}{\sqrt{2\rho_0}} \frac{\mu_h^2}{A_h C_h} \frac{\gamma}{\gamma - 1} (6\beta^{(2,\lambda)}_0 - 4\beta^{(3,\lambda)}_0) \]

which are valid as long as \( h^* \leq h \leq \bar{h} \), up to higher order corrections in \( \lambda \) and in \( \gamma^{h-h} \), see Appendix E.1 for a proof. Therefore, strictly speaking, the formal global WIs are violated already at lowest order in perturbation theory. Nevertheless, it is apparent from the definitions of \( \beta^{(2,\lambda)}_0 \) and \( \beta^{(3,\lambda)}_0 \) that these terms vanish in the sharp cutoff limit, i.e., if we let the smooth cutoff function \( \chi(t) \) that enters all the definitions of our cutoffs tend to a step function that is equal to 1 for \( t < \gamma \) and equal to 0 for \( t > \gamma \). In this sense, the correction terms to the global WIs are “trivial” and we can conclude that the formal global WIs are asymptotically correct in the infrared regime, in the sharp cutoff limit.

The problem is more serious for the local WIs, which can be used to relate among each other the renormalization constants \( Z_h, E_h, A_h, B_h \). Two key local WIs are the

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\(^{13}\)Here by super-renormalizability of the ultraviolet theory we mean that all the interactions are effectively irrelevant: even those that are superficially marginal have dimensional gains in the ultraviolet that make them exponentially insensitive to the ultraviolet cutoff as \( N \to \infty \).
following:
\[
\frac{Z_{h-1}}{Z_h} \simeq E_{h-1} + \frac{1}{\sqrt{2\rho_0}} \left[ \partial_{\rho_0} \hat{W}^{(h)}_{1;0,J^\Delta}(0) - \partial_{\rho_0} \hat{W}^{(h)}_{1;0,J^{\beta T}}(0) \right],
\]
(3.102)

\[
\frac{E_{h-1} - 1}{Z_h} \simeq -B_{h-1} - \frac{1}{2\sqrt{2\rho_0}} \left[ \partial^2_{\rho_0} \hat{W}^{(h)}_{1;0,J^\Delta}(0) - \partial^2_{\rho_0} \hat{W}^{(h)}_{1;0,J^{\beta T}}(0) \right],
\]
(3.103)

where “\(\simeq\)” means “up to dimensionally negligible corrections”, i.e., up to errors of relative size \(\gamma^{\theta(h-h)}\) for some \(\theta > 0\). The two identities are an equivalent restatement of the equations (D.37) and (D.40) proved in Appendix D.2, and the terms in square brackets are the corrections due to the infrared cutoff in (3.97). They are the analogues of [40, Eq.(4.14)-(4.17)], which they reduce to by neglecting the correction terms. As for the global WIs, the identities (3.102)-(3.103) can be checked at lowest non-trivial order in the renormalized expansion, and the correction terms computed explicitly. This is done in Appendices E.2 and E.3. While the correction in (3.102) vanishes at lowest order (even though we see no reason why it should vanish exactly at all orders), the correction in (3.103) is non-trivial. At the one-loop level it is (see Appendix E.3)
\[
- \frac{1}{2\sqrt{2\rho_0}} \left[ \partial^2_{\rho_0} \hat{W}^{(h)}_{0,1,J^\Delta}(0) - \partial^2_{\rho_0} \hat{W}^{(h)}_{0,1,J^{\beta T}}(0) \right] = \frac{2}{3} \frac{B^2_0}{A_h Z_h} \beta^{(x)}(\chi) \frac{\gamma}{\gamma - 1},
\]
(3.104)

where \(\beta^{(x)}(\chi)\) is defined in (3.93). Quite surprisingly, not only the correction term does not vanish in the sharp cutoff limit, but in such a limit it gives a divergent contribution to the flow. In this sense, the correction strongly depends on the specific shape of the cutoff function; it can be checked that even its sign depends on the choice of \(\chi(t)\) and/or of the scaling parameter \(\gamma\) (e.g., take a smoothed version of the function that is 1 for \(t \leq 1\), = 0 for \(t \geq \gamma\), and linear in between; by varying \(\gamma\), the correction term passes from negative to positive values). This indicates that the cutoff affects substantially the computation of the infrared-regularized thermodynamic observables and that, therefore, the multi-scale scheme at hand may not be reliable at sufficiently low energies. Of course, the fact that the flow of \(B_h\) depends strongly on the (arbitrary) shape of the cutoff function does not mean that the thermodynamic observables are ill defined: it just means that the contribution from the energy scales below the cutoff is also dependent on the shape \(\chi\) and it is of comparable size with that from larger scales. It is hard to interpret the physical meaning of this phenomenon, but it may indicate that Bogoliubov theory (even if properly renormalized) is unstable at zero temperature in two dimensions, or the emergence of (possibly non-perturbative) anomaly terms in the response functions. The reader should compare this result with the three dimensional case [7], where the theory is asymptotically free in the infrared and, correspondingly, the correction terms in the local WIs are asymptotically vanishing in the infrared limit (see [16, Section 4.2] for a proof).

It should be stressed that in a Renormalization Group treatment of the theory based on dimensional regularization, no correction terms appear in the WIs and, therefore, they have no effect on the infrared flow of the coupling constants, irrespective of the dimensionality of the system. The mismatch between the predictions of dimensional
regularization and those based on a constructive scheme with momentum regularization, like ours, instills the doubt that dimensional regularization may not be a reliable method in the current context. In particular, it suggests that any extrapolation of the flow to the deep infrared (i.e., to scales lower than $h^*$) based on the use of local WIs is of doubtful validity. We comment more on this issue in the following section.

3.C.5 Some heuristic considerations on the nature of the infrared theory

As already mentioned, a Renormalization Group treatment of the two-dimensional condensate based on a dimensional regularization scheme and on extrapolation from $d = 3$ to lower dimensions was discussed in [40]. There the authors argue that at scales smaller than $h^*$ the beta function flow drives the running coupling and renormalization constants towards a non-trivial fixed point, which can be understood as follows. Neglecting the correction terms, and using systematically the WIs discussed above at lowest non-trivial order (in particular, using the replacements (E.4) which can be inductively justified at lowest order), the beta function for $\lambda$ can be rewritten as:

$$\lambda_{h-1} = \gamma \lambda_h - 4\gamma \frac{\lambda^2}{A_h C_h \gamma h} \beta_0^{(2)}$$ (3.105)

that, if taken (too) seriously, implies that the flow drives $\lambda_h$ towards the fixed point $\lambda^* = A^* C^* \gamma h / (4\beta_0^{(2)})$, provided that also $A_h$ and $C_h$ reach two fixed points, called $A^*$ and $C^*$. The formal WIs suggest that the fixed points $A^*$ and $C^*$ exist and are of order 1 and $\lambda^{-1}$, respectively. Therefore, the fixed point $\lambda^*$ is expected to be of order 1 with respect to $\lambda$ and, therefore, any analytical “proof” of its existence is inevitably heuristic (if $\lambda_h$ becomes of order 1, the use of the truncated equations is not justified). Of course one could hope that the $O(1)$ fixed point exists and its numerical value is sufficiently small, so that the truncation is a posteriori justified, but a possible proof of this fact would certainly require extensive numerical simulations in addition to analytical arguments. Still, it makes sense to assume that such a fixed point exists for $\lambda_h$ and, under this hypothesis, ask about the behavior of all the other running coupling and renormalization constants. The authors of [40] proposed a very nice and non-trivial argument for controlling the flow of all the other constants in terms of that of $\lambda_h$, via a smart combination of the WIs at disposal, discussed in [40, Section IV.B] and reproduced in the language of the current paper in [16, Chapters 3 and 4]. The use of WIs implies remarkable cancellations at all orders in the beta function: in particular they show that the contributions that potentially drive $A_h$ and $C_h$ to infinity are zero at all orders. The physical consequence of this fact is that the spectrum of excitation of the interacting theory remains linear and the physical response functions (e.g. density-density correlations) have the same qualitative behavior as predicted by Bogoliubov’s theory, notwithstanding a non trivial renormalization of the $l - l$ component of the propagator. The cancellation required for this argument to work can of course be checked explicitly at lowest order, as done in [40, Appendix B and C], see in particular [40, (C.6)-(C.8)], which should be compared with our Eqs.(3.89)-(3.91).
While very interesting and inspiring, this scheme seems to rely too heavily on the dimensional regularization scheme: as mentioned above, in our momentum cutoff scheme, which allows us to study two dimensions directly (rather than dimension $3 - \varepsilon$ with $\varepsilon$ a small parameter), the WIs are violated already at the one-loop level. In particular, the cancellations proved by the authors of [40] at all orders, are just false in a momentum cutoff scheme. Therefore, the key ingredient for the stability of the (putative) infrared fixed point is missing here, and this may indicate the emergence of non-perturbative anomaly terms, neglected in the analysis of [40], which may qualitatively change the nature of the ground state of the system. Of course, our analytical methods do not allow to investigate the existence of non-perturbative anomaly terms arising in the regime $h \leq h^*$. However, our finding calls for a reconsideration of the assumptions in [40] and for a non-perturbative numerical analysis of the model, which may help in understanding better the qualitative features of the ground state.

Let us conclude this section by mentioning that even if the cancellations proposed by [40] took place, there would still be a few extra issues to discuss in order to fully control the infrared theory, neglected in the analysis of [40]. These other issues, even if overlooked in [40], as well as in later works [48, 20, 17, 18, 47], can all be solved via closer inspection, as studied in detail in [16]:

1. In two dimensions there is the extra marginal coupling constant $\lambda_{6,h}$, which is absent, since irrelevant, in $3 - \varepsilon$ dimensions. Therefore, the hypothesis that $\lambda_h$ reaches a fixed point should be supplemented by the assumption that also $\lambda_{6,h}$ reaches one. The existence of a fixed point for the pair $(\lambda_h, \lambda_{6,h})$ is in fact compatible with a low-order truncation of the (coupled) flows of $\lambda_h, \lambda_{6,h}$, see [16, Section 3.5.2].

2. As proved in [40], the assumption that $\lambda_h, \lambda_{6,h}$ reach a fixed point implies that $Z_h$ and $\mu_h$ go to zero very fast in the infrared, like $\gamma^h$ and $\gamma^{h/2}$, respectively. The fast vanishing of $Z_h$ implies that the factors $U_v$ in (3.76) are dimensionally unbounded in general as $h \to -\infty$: this effectively changes the scaling dimensions of the operators involved, and requires the introduction of three more running coupling constants, whose flows can however be reduced again to that of $(\lambda_h, \lambda_{6,h})$ via the use of three novel global WIs. See [16, Section 3.1].

Of course, it is not worth entering the details of this discussion here, since the basic assumption for the study of the flow in the deep infrared region $h \leq h^*$, i.e., the validity of the local WIs for controlling the flow of the renormalization constant, is explicitly violated already at the one-loop level.

4 Conclusions

We presented a renormalization group construction of a weakly interacting two-dimensional Bose gas, both in the quantum critical regime (zero condensate and zero temperature) and in the presence of a condensate fraction. The construction is performed within
a rigorous renormalization group scheme, borrowed from the methods of constructive field theory, which allows us to derive explicit bounds on all the orders of renormalized perturbation theory. Contrary to other heuristic renormalization group approaches, our scheme allows us to evaluate and bound explicitly the effects of the irrelevant terms, without the need of neglecting them.

This scheme allows us to construct completely, at all orders, the theory of the quantum critical point, both in the ultraviolet and in the infrared. The theory turns out to be super-renormalizable in the ultraviolet (thanks to the finite range of the interaction potential between particles), and asymptotically free (marginally irrelevant) in the infrared, as expected on the basis of approximate renormalization schemes [21, 48, 44].

The application of the same scheme to the condensate phase is more subtle: while the ultraviolet regime corresponding to length scales smaller than the range of the potential $R_0$, and the crossover regime corresponding to length scales intermediate between $R_0$ and $(\lambda \rho_0)^{-1/2}$, are fully controllable, in the same fashion as the quantum critical point, the study of the infrared regime of larger length scales is much harder. In that regime there appear three relevant and five marginal effective constants; their flow is driven to larger values by the presence of two of the three relevant couplings and, therefore, we are forced to stop the flow at length scales of the order $(\lambda \rho_0)^{-1/2}$. For larger scales, non-perturbative approaches are required; they go beyond what can we do by our analytical methods and we cannot reach a definite conclusion about the nature of the infrared theory.

Still, interestingly enough, we explicitly exhibit violations to the formal Ward Identities in the third regime, the one corresponding to length scales between $(\lambda \rho_0)^{-1/2}$ and $(\lambda^3 \rho_0)^{-1/2}$. In this range the renormalized theory is perturbative and predictions based on low order truncations reliable. Therefore, the fact that some of the formal Ward Identities are violated at the one-loop level, as we prove, suggests the possibility that (non-perturbative) anomaly terms appear in the deep infrared regime. Certainly, it puts on shaky grounds their application to the deep infrared regime, which played a key role in previous proposals that the renormalization group flow reaches a non-perturbative infrared fixed point, characterized by a linear spectrum of excitations analogous to Bogoliubov’s one. We hope that future research, both from the analytical and numerical side, will help to clarify the nature of the infrared theory and to confirm or dismiss the Bogoliubov’s picture in two dimensional systems.

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$\beta_{\bar{z}}^h = \frac{\lambda}{h}$

Figure 16: Feynman diagrams representation for the first non trivial contribution to the beta function of $\bar{z}_{h-1}$, for $h \leq -1$.

### A Ultraviolet flow in the condensate phase

In this section we prove Eqs. (3.29)-(3.30). We denote by $\bar{z}_h$ and $\delta \nu_h$ the quantities defined via the second and third lines of (3.27), both for $h \geq 0$ and $\bar{h} \leq h < 0$. The graphs contributing to $\delta \nu_h$ are all and only ladder diagrams as in the first line of Fig. 12, for all $h \geq \bar{h}$; those contributing to $\bar{z}_h$ are all and only ladder diagrams as in the second line of Fig. 12, if $h \geq 0$, while they can be either as in the second line of Fig. 12 or as in Fig. 13, if $\bar{h} \leq h < 0$. For both couplings, we distinguish the regime $h \geq 0$ from $\bar{h} \leq h < 0$.

Let us first consider $\delta \bar{\nu}_h$. If $h \geq 0$ we proceed as in the “ultraviolet regime” subsection of Section 2.D, thus obtaining a bound on the $n$-th order diagrams of the same form as the $n$-th order term in the right side of (2.61). Since $n \geq 2$, we find that $|\delta \bar{\nu}_h| \leq C \lambda^2 \gamma^{-h}$ for all $h \geq 0$. For $h < 0$ we write the beta function equation for $\delta \bar{\nu}_h$, analogous to (2.63), with initial datum $\delta \bar{\nu}_h$ of the order $\lambda^2$: $\delta \bar{\nu}_{h-1} = \delta \bar{\nu}_h + \beta_{\bar{z}}^h$, with $\bar{h} \leq h \leq 0$. Using the (renormalized analogue of) (3.20), we see that $|\beta_{\bar{z}}^h| \leq C \lambda^2$, $\forall \bar{h} \leq h \leq 0$, which implies that $|\delta \bar{\nu}_h| \leq C \lambda^2 |h|$, $\forall \bar{h} \leq h \leq 0$, as desired.

We now apply the same strategy to the study of the flow of $\bar{z}_h$. If $h \geq 0$, we find that the diagrams of order $n$ are bounded by $\lambda(K\lambda)^{n-1} \gamma^{h-h}$, with $n \geq 1$; here the factor $\gamma^{h-h}$ is a dimensional estimate on the $L_\infty$ norm of the off diagonal propagator $r_h$. Therefore, $|\bar{z}_h| \leq (\text{const.}) \lambda \gamma^{h-h}$, for all $h \geq 0$. In particular, $\bar{z}_0$ is of order $\lambda^2$. From smaller scales, we study again the beta function equation $\bar{z}_{h-1} = \bar{z}_h + \beta_{\bar{z}}^h$, where $|\beta_{\bar{z}}^h| \leq C \lambda^2$, for all $\bar{h} \leq h \leq 0$. Therefore, as for $\delta \bar{\nu}_h$, we find $|\bar{z}_h| \leq C \lambda^2 |h|$, $\forall \bar{h} \leq h \leq 0$, which concludes the proof of (3.29)-(3.30).

### B Bounds on the propagators

In this section we prove the decay bound (3.39). We first focus on $\hat{\beta}_{\bar{h}}^{(1)}$. Of course, it is enough that we restrict to its $-+$ component, which we rewrite as the sum of two terms, whose definition is induced by the following rewriting of the cutoff function appearing under the integral sign in (3.37):

$$1 - \chi(\gamma^{\bar{h}}(k_0^2 + |\vec{x}|^4)^{1/2}) - \chi(\gamma^{-\bar{h}}(k_0^2 + |\vec{k}|^4)^{1/2}) \equiv (1 - h_{\bar{h}}(x))(1 - \tilde{\chi}_{\bar{h}}(k)) - h_{\bar{h}}(x)\tilde{\chi}_{\bar{h}}(k)$$  \hspace{1cm} (B.1)
where
\[ h_h(x) = \chi(\gamma^k_0 x^2 + |x|^4)^{1/2}, \quad \tilde{h}_h(k) = \chi(\gamma^{-2}k_0^2 + |k|^4)^{1/2}. \] (B.2)

The corresponding decomposition for the $-+$ component of $\tilde{g}_h^{(1)}$ is $[\tilde{g}_h^{(1)}(x)]_{-+} = g_h^{(1a)}(x) - g_h^{(1b)}(x)$. We now show that the two terms separately satisfy the same bound as (3.39).

The easiest term to treat is $g_h^{(1b)}(x)$, which we write as
\[ g_h^{(1b)}(x) = h_h(x) \int \frac{dk}{(2\pi)^3} e^{-ikx} \frac{\tilde{h}_h(k)}{-ik_0 + |k|^2}. \] (B.3)

Using the compact support properties of $\tilde{h}_h(k)$, it is immediate to check that
\[ |\tilde{g}_h^{(1b)}(x)| = \left| h_h(x) \int \frac{dk}{(2\pi)^3} e^{-ikx} \frac{\tilde{h}_h(k)}{-ik_0 + |k|^2} \right| \leq (\text{const.}) \gamma^h h_h(x), \] (B.4)

which implies (3.39) for $n_0 = |\vec{n}| = 0$. In order to estimate the derivatives of $\tilde{g}_h^{(1b)}(x)$, note that each derivative $\partial_{x_0}$ (resp. $\partial_{x_i}$ with $i = 1, 2$) acting on $h_h(x)$ produces a factor proportional to $\gamma^h$ (resp. $\gamma^{h/2}$), as desired. Moreover, each derivative $\partial_{x_0}$ (resp. $\partial_{x_i}$ with $i = 1, 2$) acting on the integral in the right side of (B.3) produces a factor $-ik_0$ (resp. $-ik_i$) under the integral sign, which is bounded proportionally to $\gamma^h$ (resp. $\gamma^{h/2}$), thanks to the compact support properties of $\tilde{h}_h$. This concludes the proof that $\tilde{g}_h^{(1b)}$ satisfies a bound like (3.39).

Let us now focus on $\tilde{g}_h^{(1a)}(x)$. In order to bound it, we rewrite the cutoff function $(1 - \tilde{h}_h(k))$ appearing in its definition as $\sum_{h > h_0} h_h(k)$, with $h_h(k)$ defined after (3.17) and note that, thanks to compact support properties of $h_h$,
\[ \left| \partial_{x_0} \partial_{x_0}^N \int \frac{d^3k}{(2\pi)^3} e^{-ikx} \frac{h_h(k)}{-ik_0 + |k|^2} \right| \leq C_{n_0, n} \gamma^h (1 + n_0 + |\vec{n}|^2/2). \] (B.5)

Moreover, integrating by parts and using again the compact support properties of $h_h$, we find
\[ |x_0|^N \left| \int \frac{d^3k}{(2\pi)^3} e^{-ikx} \frac{h_h(k)}{-ik_0 + |k|^2} \right| = \left| \int \frac{d^3k}{(2\pi)^3} e^{-ikx} \partial_{x_0}^N \frac{h_h(k)}{-ik_0 + |k|^2} \right| \leq C_N \gamma^h (1 - N) \] (B.6)

and, similarly,
\[ |\vec{x}|^N \left| \int \frac{d^3k}{(2\pi)^3} e^{-ikx} \frac{h_h(k)}{-ik_0 + |k|^2} \right| \leq C_N \gamma^h (1 - N/2). \] (B.7)

Combining the previous three equations we find
\[ \left| \partial_{x_0} \partial_{x_0}^N \int \frac{d^3k}{(2\pi)^3} e^{-ikx} \frac{h_h(k)}{-ik_0 + |k|^2} \right| \leq C_{N, n_0, n} \gamma^h (1 + n_0 + |\vec{n}|^2/2) \] (B.8)
which leads to
\[ \left| \partial_{x_0}^{n_0} \partial_{\vec{n}}^{(1a)} \tilde{g}_h^{(1a)}(\vec{x}) \right| \leq C_{N,n_0,\tilde{\pi}}(1 - \tilde{h}_h(\vec{x})) \sum_{h > h} \frac{\gamma^{h(1+n_0+|\vec{n}|/2)}}{1 + [\gamma^h(|x_0| + |\vec{x}|^2)]^N}. \] (B.9)

Now, on the support of \((1 - \tilde{h}_h(\vec{x}))\), the combination \(|x_0| + |\vec{x}|^2\) is larger than \((\text{const.})\gamma^h\)
and, therefore, for all \(N_1, N_2 \geq 0\) such that \(N_1 + N_2 = N\),
\[ \left| \partial_{x_0}^{n_0} \partial_{\vec{n}}^{(1a)} \tilde{g}_h^{(1a)}(\vec{x}) \right| \leq C_{N,n_0,\tilde{\pi}}(1 - \tilde{h}_h(\vec{x})) \sum_{h > h} \frac{\gamma^{(\tilde{h}-h)N_1 + h(1+n_0+|\vec{n}|/2)}}{1 + [\gamma^h(|x_0| + |\vec{x}|^2)]^{N_2}}, \] (B.10)
which is summable over \(h\) as soon as \(N_1 > n_0 + |\vec{n}|/2\). By picking such an \(N_1\) we obtain the desired bound on \(\tilde{g}_h^{(1a)}\).

The proof of the desired decay bound for \(\tilde{g}_h^{(2)}\) is completely analogous to the proof of (B.8) and, therefore, we do not give its details here. The same is true for the proof of (3.43), which we also leave to the reader.

C Lowest order computations

In this section we prove (3.81)–(3.83) and (3.89)–(3.92), i.e., we explicitly compute the beta functions for \(\lambda_h, \mu_h, \nu_h, Z_h, E_h, B_h\) and \(A_h\) at the lowest non-trivial order (which turns out to coincide with the one-loop computation). Rather than presenting the computations in the same order as we presented the formulas after (3.81), we proceed in order of increasing difficulty, starting from the easiest computation, which is the beta function of \(Z_h\), then moving to the beta function of \(\nu_h\), etc.

In the following, in the computation of the one-loop contributions to the beta function, we systematically perform a number of approximations, which either induce corrections of higher order than the one we are computing, or are dimensionally irrelevant for \(h \leq \tilde{h}\). The approximations we do are the following: (1) we systematically replace \(\gamma^{h+1} \lambda_{h+1}\) by \(\gamma^h \lambda_h\), \(\gamma^{h+1/2} \mu_{h+1}\) by \(\gamma^{h/2} \mu_h\), \(\gamma^{2(h+1)} \nu_{h+1}\) by \(\gamma^{2h} \nu_h\), \(A_{h+1}\) by \(A_h\), \(B_{h+1}\) by \(B_h\), \(E_{h+1}\) by \(E_h\), and \(Z_{h+1}\) by \(Z_h\), since these replacements induce an error that is of higher order in \(\lambda\); (2) similarly, we replace the renormalization functions \(\tilde{A}_{h-1}(k)\), etc., appearing in the definition of the propagator on scale \(h\) by \(A_h, \nu_h\), etc.; moreover, recalling the definition of \(\chi_{h}(k)\) in (3.51), we replace the support function \(\hat{f}_h(k) = \chi_{h}(k) - \chi_{h-1}(k)\) by \(\hat{f}_h(|k'|) := \chi(\gamma^{-h} |k'|) - \chi(\gamma^{-h+1} |k'|)\), where with some abuse of notation we let \(|k'| := (k_0, \sqrt{A_h/C_h} k) \equiv (k_0, k')\); note that also these replacements induce errors of higher order in \(\lambda\); (3) finally, we systematically neglect the terms coming from trees with at least one endpoint on scale \(\tilde{h}\), since these terms have relative size \(\gamma^{\theta(h-\lambda)}\) with \(0 < \theta < 1\) as compared to the main contributions to the beta function (see (3.75)).

C.1 Lowest order beta function for \(Z_h\)

The lowest order contribution to \(\beta_Z^Z\) (see (3.80) and (3.63)) is of order \(\lambda_h\) (or, equivalently, of order \(\mu_h^2\)) and can be represented graphically by diagrams of the form in Fig.17,
\[ \beta_h^Z = 2 \]

Figure 17: Leading order beta function for \( Z_h \). The graph represents the lowest order contribution to \( W^{(h)}(0) \), and the two solid lines (associated with two propagators of type \( g^{(h)}_{lt} \)) come with two labels \( h_1, h_2 \), which we need to sum over, with the constraints that \( \min\{h_1, h_2\} = h \) and \( |h_1 - h_2| \leq 1 \).

computed at zero external momentum, with the two propagators labelled by two scale labels \( h_1, h_2 \) such that \( \min\{h_1, h_2\} = h \). Note that by the compact support properties of the propagator \( \int_{-1}^{1} d(k) \rho \langle |k| \rangle \), the sum of the values of these diagrams is

\[
\beta_h^Z = -2 \int \frac{dk}{(2\pi)^3} \left[ \gamma^h \mu^2_h \left( \tilde{g}^{(h)}_{lt}(k) \right)^2 + 2\gamma^{h+1} \mu^2_{h+1} \tilde{g}^{(h)}_{lt}(k) \tilde{g}^{(h+1)}_{lt}(k) \right], \tag{C.1}
\]

where

\[
\tilde{g}^{(h)}_{lt}(k) = \frac{f_h(k)}{C_{h-1}(k) \left[ k^2 + (A_{h-1}(k)/C_{h-1}(k))|k|^2 \right]}, \tag{C.2}
\]

where \( \tilde{Z}_{h-1}(k)C_{h-1}(k) = \tilde{Z}_{h-1}(k)\tilde{B}_{h-1}(k) + \tilde{E}_{h-1}^2(k) \). Under the approximations explained above, this propagator can be replaced by

\[
\tilde{g}^{(h)}_{lt}(k') = f_h(|k'|) \frac{1}{C_h|k'|^2}. \tag{C.3}
\]

For future reference, let us write down the analogues of this equation, as far as the propagators with labels \( ll \) and \( lt \) are concerned:

\[
\tilde{g}^{(h)}_{lt}(k') = f_h(|k'|) \frac{B_h k_0^2 + C_h|\vec{k}'|^2}{Z_h C_h |k'|^2}, \quad \tilde{g}^{(h)}_{lt}(k) = f_h(|k|) \frac{E_h k_0}{Z_h C_h |k|^2}. \tag{C.4}
\]

Using these replacements, as well as the ones spelled above, we find that (C.1) is equal (up to higher order corrections) to

\[
\beta_h^Z = -2\gamma^h \mu^2_h \frac{1}{A_h C_h} \int \frac{dk}{(2\pi)^3} \left[ \frac{f_h^2(|k|) + 2f_h(|k|)f_{h+1}(|k|)}{|k|^4} \right], \tag{C.5}
\]

so that after rescaling and after the change of variables \( k_0 = \rho \cos \vartheta, |\vec{k}| = \rho \sin \vartheta \), with \( \rho \geq 0 \) and \( \vartheta \in [0, \pi] \), we get

\[
\beta_h^Z = -2 \frac{1}{A_h C_h} \mu^2_h \frac{1}{2\pi^2} \int \frac{d\rho}{\rho^2} \left[ f_h^2(\rho) + 2f_h(\rho)f_{h+1}(\rho) \right], \tag{C.6}
\]

which is the same as (3.89).
\[ \gamma^{2h-\bar{h}} \beta_h' = \begin{array}{c}
\end{array} + \begin{array}{c}
\end{array} + \begin{array}{c}
\end{array} \]

Figure 18: Leading order beta function for \( \nu_h \).

C.2 Lowest order beta function for \( \nu_h \)

The lowest order contributions to \( \beta_h' \) (see (3.78) and (3.59)) are of the order \( \lambda_h \) and can be represented graphically as in Fig.18. Once again, the propagators are associated with scale labels such that the minimum scale is equal to \( \bar{h} \). We denote by \( \beta_{h,1}^{\nu'} \) the contribution from the first diagram in Fig.18, and by \( \beta_{h,2}^{\nu'} \) those from the last two diagrams.

Proceeding as in the previous section we find that, up to higher order corrections,

\[ \gamma^{2h-\bar{h}} \beta_h^{\nu'(1)} = \frac{1}{A_h} \gamma^{h-\bar{h}} \lambda_h \int \frac{d^3k}{(2\pi)^3} f_h(|k|) = \frac{1}{A_h} \gamma^{2h-\bar{h}} \lambda_h \frac{1}{2\pi^2} \int d\rho f_0(\rho) . \quad (C.7) \]

In order to calculate \( \beta_{h,2}^{\nu'} \) we notice that the sum of the remaining two diagrams in Fig.18 involves the following combinations of propagators (after the usual replacements):

\[ \bar{g}_{tt}^{(h)}(k)\bar{g}_{ll}^{(h)}(k) + (\bar{g}_{tt}^{(h)}(k) + \bar{g}_{tt}^{(h)}(k)\bar{g}_{ll}^{(h)}(k) + 2\bar{g}_{tt}^{(h)}(k)\bar{g}_{tt}^{(h)}(k)) = \frac{2f_h(|k|)f_{h+1}(|k|)}{Z_h C_h |k|^2} + h.o. , \quad (C.8) \]

where \( h.o. \) in the last equation means “higher orders”. Using these identities we find, up to higher order corrections:

\[ \gamma^{2h-\bar{h}} \beta_h^{\nu'(2)} = -2\gamma^{2h} \mu_h^2 \frac{1}{Z_h} \frac{1}{A_h} \int \frac{d^3k}{(2\pi)^3} \frac{f_0^2(|k|) + 2f_0(|k|)f_1(|k|)}{|k|^2} \]

\[ = -2\gamma^{2h} \mu_h^2 \frac{1}{Z_h} \frac{1}{A_h} \frac{1}{2\pi^2} \int d\rho (f_0^2(\rho) + 2f_0(\rho)f_1(\rho)) . \quad (C.9) \]

Putting things together:

\[ \beta_h' = \left( 6\lambda_h - 2\mu_h^2 \frac{\gamma^h}{Z_h} \right) \frac{1}{A_h} \frac{1}{2\pi^2} \int d\rho f_0(\rho) \]

\[ - 2\mu_h^2 \frac{\gamma^h}{Z_h} \frac{1}{A_h} \frac{1}{2\pi^2} \int d\rho (f_0^2(\rho) + 2f_0(\rho)f_1(\rho) - f_0(\rho)) . \quad (C.11) \]
Using the definitions (3.84)–(3.88), this can be rewritten as

$$\beta_\mu' = \left( 6\lambda h - 2\mu^2 h \frac{\gamma}{Z_h} \right) \frac{1}{A_h} \beta^{(1)}_0 - 2\mu^2 h \frac{\gamma}{Z_h} \frac{1}{A_h} \beta^{(1,\chi)}_0,$$  

which implies (3.83).

## C.3 Lowest order beta function for $\mu_h$

The lowest order contributions to $\beta_\mu h$ are of the order $\mu h \lambda h$ and can be represented graphically as in Fig.19. Once again, the propagators are associated with scale labels such that the minimum scale is equal to $h$. We denote by $\beta^{(2)}_\mu h$ (resp. $\beta^{(3)}_\mu h$) the contribution to $\beta_\mu h$ coming from the first diagram (resp. second + third + fourth diagrams) in Fig.19. We find:

$$\gamma \beta^{(2)}_\mu = -12 \frac{1}{A_h C_h} \gamma \beta^{(2)}_0 \int \frac{d^3 k}{(2\pi)^3} \frac{f^2_h(|k|) + 2f_h(|k|)f_{h+1}(|k|)}{|k|^4},$$

and, using the analogues of (C.8)-(C.9),

$$\gamma \beta^{(3)}_\mu = 4 \frac{C_h}{A_h} \gamma \beta^{(3)}_0 \int \frac{d^3 k}{(2\pi)^3} \frac{1}{Z_h C_h^2 |k|^4} \left[ f^3_h(|k|) + 3f^2_h(|k|)f_{h+1}(|k|) + 3f_h(|k|)f^2_{h+1}(|k|) \right]$$

$$= 4 \frac{1}{A_h C_h} \gamma \beta^{(3)}_0 \frac{\mu^3}{Z_h^2} \int \frac{d \rho}{2\pi^2} \left[ f^3_0(\rho) + 3f^2_0(\rho)f_1(\rho) + 3f_0(\rho)f^2_1(\rho) \right].$$

In the last expression, we can rewrite

$$\frac{1}{2\pi^2} \int \frac{d \rho}{\rho^2} \left[ f^3_0(\rho) + 3f^2_0(\rho)f_1(\rho) + 3f_0(\rho)f^2_1(\rho) \right] = \beta^{(2)}_0 - \beta^{(2,\chi)}_0,$$

where $\beta^{(2,\chi)}_0$ is defined as in (3.87), so that, putting things together,

$$\beta^{(2)}_\mu = \frac{1}{A_h C_h \lambda h} \left[ \mu h \left( -12\lambda h + 4\mu^2 h \frac{\gamma}{Z_h} \right) \beta^{(2)}_0 - 4\mu^3 h \frac{\gamma}{Z_h} \beta^{(2,\chi)}_0 \right].$$

which implies (3.82)
Figure 20: Leading order beta function for $\lambda_h$.

C.4 Lowest order beta function for $\lambda_h$

The lowest order contributions to $\beta_h^\lambda$ are of the order $\lambda_h^2$ and can be represented graphically as in Fig.20. We denote with $\beta_h^{\lambda,(2)}$, $\beta_h^{\lambda,(3)}$ and $\beta_h^{\lambda,(4)}$ the contributions to $\beta_h^\lambda$ coming from the diagrams in Fig.20 with two, three and four end-points, respectively. Proceeding as in the previous sections, we find:

$$\gamma^\hbar \beta_h^{\lambda,(2)} = -36 \frac{1}{A_h C_h} \gamma^\hbar - 2 \lambda_h^2 \beta_0^{(2)}$$,  \hspace{1cm} \text{(C.17)}

$$\gamma^\hbar \beta_h^{\lambda,(3)} = 24 \frac{1}{A_h C_h} \gamma^\hbar \lambda_h \mu_h^2 \frac{\mu_h}{Z_h} (\beta_0^{(2)} - \beta_0^{(2,\chi)})$$,  \hspace{1cm} \text{(C.18)}

and

$$\gamma^\hbar \beta_h^{\lambda,(4)} = -4 \frac{1}{A_h C_h} \gamma^\hbar \frac{\mu_h^4}{Z_h^2} \frac{1}{2\pi^2} \int \frac{d\rho}{\rho \rho^2} T_4(\rho)$$,  \hspace{1cm} \text{(C.19)}

where

$$T_4(\rho) = \tilde{f}_0^4(\rho) + 4 f_0^2(\rho) f_1(\rho) + 6 f_0^2(\rho) f_1^2(\rho) + 4 f_0(\rho) f_1^3(\rho)$$.

Using the definitions (3.85)–(3.88), it can be checked that

$$\frac{1}{2\pi^2} \int \frac{d\rho}{\rho \rho^2} T_4(\rho) = \beta_0^{(2)} - \beta_0^{(2,\chi)} - \beta_0^{(3,\chi)}$$,  \hspace{1cm} \text{(C.21)}

so that, putting things together,

$$\beta_h^\lambda = \frac{1}{A_h C_h \gamma^\hbar} \left( -36 \lambda_h^2 + 24 \lambda_h \mu_h^2 \frac{\gamma^\hbar}{Z_h} - 4 \mu_h^4 \frac{\gamma^\hbar}{Z_h^2} \right) \beta_0^{(2)}$$

$$+ \frac{1}{A_h C_h \gamma^\hbar} \left[ -24 \lambda_h \mu_h^2 \gamma^\hbar \frac{\mu_h}{Z_h} + 4 \mu_h^4 \gamma^\hbar \frac{\mu_h}{Z_h^2} \right] \beta_0^{(2,\chi)} + 4 \mu_h^4 \gamma^\hbar \frac{\mu_h}{Z_h^2} \beta_0^{(3,\chi)}$$,  \hspace{1cm} \text{(C.22)}

which implies (3.81).
\[
\beta_h^E = \partial_{p_0} \left[ \cdot \right]_{p=0}
\]

Figure 21: Leading order beta function for \(E_h\). The graph is first computed at external momentum \(p = (p_0, \vec{0})\), then derived w.r.t. \(p_0\); after the action of the derivative, the external momentum is set to \(0\).

C.5 Lowest order beta function for \(E_h\)

The lowest order contribution to \(\beta_h^E\) (see (3.80) and (3.62)) is of order \(\lambda_h\) and is represented graphically in Fig.21. The value of the graph is (up to higher order corrections):

\[
\beta_h^E = \beta_{h,2}^E - \beta_{h,1}^E
\]

with

\[
\beta_{h,i}^E = 4 \frac{1}{A_h} \mu_h^2 \frac{E_h}{C_h Z_h} \int \frac{d^3 k}{(2\pi)^3} k_0 T_i(k) \frac{T_i(k + p)}{|k|^2} \partial_{p_0} \left[ T_i(k + p) \left| \frac{k}{|k|^2} \right|^2 \right]_{p=0}
\]

\[
= 2 \frac{1}{A_h} \mu_h^2 \frac{E_h}{C_h Z_h} \int \frac{d^3 k}{(2\pi)^3} k_0 \partial_{k_0} \left( \frac{T_i(k)}{|k|^2} \right)^2
\]

where \(T_2(k) := f_0(|k|) + f_1(|k|), T_1(k) := f_1(|k|)\). By integrating by parts:

\[
\beta_{h,i}^E = -2 \frac{1}{A_h} \mu_h^2 \frac{E_h}{C_h Z_h} \int \frac{d^3 k}{(2\pi)^3} \left( \frac{T_i(k)}{|k|^2} \right)^2,
\]

so that

\[
\beta_h^E = -2 \frac{1}{A_h} \frac{E_h}{C_h Z_h} \mu_h^2 \beta_0^{(2)},
\]

which gives (3.90).

C.6 Lowest order beta function for \(B_h\)

The lowest order contribution to \(\beta_h^B\) (see (3.79) and (3.61)) is of the order \(\lambda_h\) and is represented graphically in Fig.22. The sum of the values of the two graphs is (up to higher order corrections):

\[
\beta_h^B = \beta_{h,2}^B - \beta_{h,1}^B
\]

where, defining \(\alpha_h = \frac{E_h^2}{Z_h A_h}\), and letting \(T_1, T_2\) be the same as in the previous section,

\[
\beta_{h,i}^B = -2 \frac{\mu_h^2}{Z_h A_h} \int \frac{d^3 k}{(2\pi)^3} \partial_{p_0} \left[ \left( \frac{k^2 + \alpha_h k_0 p_0}{|k|^2 |k + p|^2} T_i(k + p) T_i(k) \right) \right]_{p=0}.
\]
\[ \beta^B_h = \partial^2_{p_0} \left[ \begin{array}{cc} & \\ \end{array} \right] \]  

Figure 22: Leading order beta function for \( B_h \).

By computing explicitly the derivative, and then integrating by parts, we find:

\[
\beta_{1,h}^B = -2 \frac{\mu_h^2}{Z_h A_h} \int \frac{d^3k}{(2\pi)^3} \left[ 2\alpha_h k_0 \frac{T_i(k)}{|k|^2} \partial_{k_0} \left( \frac{T_i(k)}{|k|^2} \right) + T_i(k) \partial^2_{k_0} \left( \frac{T_i(k)}{|k|^2} \right) \right] \quad \text{(C.29)}
\]

\[
\beta_{2,h}^B = -2 \frac{\mu_h^2}{Z_h A_h} \int \frac{d^3k}{(2\pi)^3} \left[ \alpha_h \left( \frac{T_i(k)}{|k|^2} \right)^2 + \partial_{k_0} T_i(k) \left( 2k_0 T_i(k) \frac{1}{|k|^4} - \frac{\partial_{k_0} T_i(k)}{|k|^2} \right) \right]. \quad \text{(C.30)}
\]

Let us now denote by \( \beta_{1,h}^{B,(1)} \) the contribution associated with the first term in square brackets, and by \( \beta_{1,h}^{B,(2)} \) the rest. We have:

\[
\beta_{1,h}^{B,(1)} = \beta_{2,h}^{B,(1)} - \beta_{1,h}^{B,(1)} = 2 \frac{\mu_h^2}{A_h Z_h Z_h C_h} \beta_0^{(2)}, \quad \text{(C.31)}
\]

which gives the first term in the r.h.s. of (3.91). Similarly, by passing to polar coordinates \( k_0 = \rho \cos \vartheta, |\vec{k}| = \rho \sin \vartheta \), with \( \vartheta \in [0, \pi] \), and by explicitly computing the integral over \( \vartheta \), we find:

\[
\beta_{1,h}^{B,(2)} = \beta_{2,h}^{B,(2)} - \beta_{1,h}^{B,(2)} = \frac{2}{3} \frac{\mu_h^2}{A_h C_h} (\gamma - 1) \frac{1}{2\pi^2} \int \frac{d\rho}{\rho} \left[ \rho (\chi'(\rho))^2 + 2\chi'(\rho) (1 - \chi(\rho)) \right], \quad \text{(C.32)}
\]

which gives the second term in the r.h.s. of (3.91).

C.7 Lowest order beta function for \( A_h \)

The lowest order contribution to \( \beta^A_h \) is represented graphically in a way similar to Fig.22, with \( \partial^2_{p_0} \) replaced by \( \partial^2_{p_1} \). Therefore, \( \beta^A_h = \beta_{1,h}^A - \beta_{1,h}^A \), with

\[
\beta_{1,h}^A = -2 \frac{\mu_h^2}{Z_h C_h} \int \frac{d^3k}{(2\pi)^3} T_i(k) \partial^2_{k_1} \left( \frac{T_i(k + p)}{|k + p|^2} \right) \bigg|_{p=0}. \quad \text{(C.33)}
\]

On the other hand, the integral in the r.h.s. is invariant under the exchange \( k_0 \leftrightarrow k_1 \) and, therefore, using the result of the previous section,

\[
\beta_{2,h}^A = \frac{2}{3} \frac{\mu_h^2}{C_h Z_h} (\gamma - 1) \frac{1}{2\pi^2} \int \frac{d\rho}{\rho} \left[ \rho (\chi'(\rho))^2 + 2\chi'(\rho) (1 - \chi(\rho)) \right], \quad \text{(C.34)}
\]

which proves (3.92).
D Ward identities

D.1 Derivation of the global Ward Identities

In this section we derive the Ward Identities associated with a phase transformation $\psi_x^\pm \to e^{\pm i\alpha(x)}\psi_x^\pm$, with $\alpha(x) \equiv \alpha$. Next, in the following section, we will show how to modify the computation in order to deal with a local phase transformation in which $\alpha(x)$ is a non-trivial function of $x$. We introduce a sequence of reference models, labelled by an integer $h$, defined in terms of a functional integral similar to (3.18), but with an extra infrared cutoff at a pre-fixed scale $h$ (that, for definiteness, we shall assume to be $\leq \bar{h}$):

$$e^{-|\Lambda|F_{h-1}-W^{(h-1)}(\phi)} = \int P^B_{\geq h}(d\psi)e^{-\bar{V}(\psi+\phi)}. \quad (D.1)$$

Here $P^B_{\geq h}(d\psi)$ is the complex gaussian measure with propagator given by the analogue of (3.13), modulo an extra infrared cutoff$^{14}$:

$$g^B_{\geq h}(x-y) = \int_{\mathbb{R}^3} \frac{dk}{(2\pi)^3} \tilde{\chi}_{\geq h}(k) \frac{e^{-ik(x-y)}}{k^2 + (\epsilon(k))^2} \begin{pmatrix} ik_0 + F(k) & -\lambda \rho \hat{\psi}(k) \\ -\lambda \rho \hat{\psi}(k) & -ik_0 + F(k) \end{pmatrix} \quad (D.2)$$

and $\tilde{\chi}_{\geq h}(k) = 1 - \chi_{h-1}(k)$, where $\chi_{h-1}(k)$ was defined in (3.51). Moreover, $\bar{V}(\psi)$ is given by (3.11) (we dropped the label $\Lambda$, for simplicity). The interest of the definition (D.1) is that the local parts of the kernels of $W^{(h-1)}$, with $h \leq \bar{h}$, are essentially$^{15}$ the same as the local part of the kernels of $V^{(h-1)} + \sum_{k=0}^{\bar{h}} L_q V(k)$, which we computed in Section 3.C.2 via a renormalized multiscale construction. Therefore, identities among the kernels of $W^{(h-1)}$ induce identities between the running coupling constants at scales $h \leq \bar{h}$, which are the relations we are interested in. See Section D.2 below for a detailed discussion about the connection between $W^{(h-1)}$ and $V^{(h-1)}$.

In order to obtain the desired identities among the kernels of $W^{(h-1)}(\phi)$ it is convenient to preliminarily manipulate the r.h.s. of (D.1). We recall that the Bogoliubov’s reference gaussian measure $P^B(d\psi)$ was obtained by first combining the negative exponential of the first two lines in (3.3) with the free gaussian measure $P^0$, see the discussion after (3.3), and then by performing the c-number substitution $\xi \to \sqrt{\bar{m}}$ spelled out after (3.20). A similar connection is valid, of course, between the cut-offed measures $P^B_{\geq h}(d\psi)$ and $P^0_{\geq h}(d\psi)$, where $P^0_{\geq h}(d\psi)$ is the free gaussian measure with propagator

$$\int \frac{dk}{(2\pi)^3} \tilde{\chi}_{\geq h}(k) \frac{e^{-ik(x-y)}}{-ik_0 + |\tilde{k}|^2}. \quad (D.3)$$

$^{14}$In this section, for simplicity, we formally write all the involved expressions in the limit $\beta, L \to \infty$, as we already did in the bulk of the paper. It is implicit in the discussion that all the involved quantities have a finite temperature/volume counterparts, which can be easily written down, but are slightly more cumbersome than their formal $\beta, L \to \infty$ limits (this is the only reason why we prefer not to write them explicitly).

$^{15}$We will see in Section D.2 below that the local parts of the kernels of $W^{(h-1)}$ coincide with those of $V^{(h-1)} + \sum_{k=0}^{\bar{h}} L_q V(k)$ up to a minor correction due to the “last integration step”, which is not visible at the level of the one-loop beta function, see (D.54) and the preceding discussion.
Therefore, we can write:
\[
P^B_{\geq h}(d\psi) = e^{-|\Lambda|\bar{f}_h} P^0_{\geq h}(d\psi) e^{-Q^B_h(\psi)},
\]
where, in the limit $\Lambda \not\to \mathbb{R}^3$ (and using the fact that the term in the second line of (3.3) vanishes in this limit),
\[
Q^B_{\geq h}(\psi) = \frac{\lambda}{2} \rho_0 \int_{\text{supp} \bar{\chi}_{\geq h}} \frac{dk}{(2\pi)^3} \bar{\psi}(\vec{k}) \bar{\chi}_{\geq h}^{-1}(k) (\hat{\psi}^+_k + \hat{\psi}^-_k)(\hat{\psi}^+_k + \hat{\psi}^-_k).
\]
Plugging (D.4) into (D.1), we find:
\[
e^{-|\Lambda|\bar{f}_h - \mathcal{V}_h(\phi)} = e^{-|\Lambda|\bar{f}_h} \int P^0_{\geq h}(d\psi) e^{-Q^B_{\geq h}(\psi) - \mathcal{V}(\psi + \phi)}.
\]
We now first perform the change of variables $\psi^\pm_x \to e^{\pm i\alpha} \tilde{\psi}_x^\pm$, and then we derive w.r.t. $\alpha$, so that:
\[
0 = \frac{\partial}{\partial \alpha} \int P^0_{\geq h}(d\psi) e^{-Q^B_{\geq h}(e^{i\alpha}\psi) - \mathcal{V}(e^{i\alpha}\psi + \phi)} \bigg|_{\alpha=0}.
\]
In order to compute the derivative explicitly, it is convenient to rewrite $\bar{V} = \bar{W}(\psi) + \bar{Q}(\psi)$ with
\[
\bar{W}(\psi) = \frac{\lambda}{2} \int dx \, dy \left[ |\psi_x + \sqrt{\rho_0}|^2 - \rho_0 \right] w(x-y) \left[ |\psi_y + \sqrt{\rho_0}|^2 - \rho_0 \right] + \bar{\nu} \int dx \left[ |\psi_x + \sqrt{\rho_0}|^2 - \rho_0 \right],
\]
\[
\bar{Q}(\psi) = -\frac{\lambda}{2} \rho_0 \int dx \, dy \left( \psi^+_x + \psi^-_x \right) w(x-y) \left( \psi^+_y + \psi^-_y \right) - \bar{\nu} \sqrt{\rho_0} \int dx \left( \psi^+_x + \psi^-_x \right),
\]
and then note that
\[
\frac{\partial}{\partial \alpha} \bar{W}(e^{i\alpha} \psi + \phi) \bigg|_{\alpha=0} = -i \int dx \left[ \frac{\delta \bar{W}(\psi + \phi)}{\delta \phi^+_x} (\phi^+_x + \sqrt{\rho_0}) - \frac{\delta \bar{W}(\psi + \phi)}{\delta \phi^-_x} (\phi^-_x + \sqrt{\rho_0}) \right],
\]
from which one gets (after an explicit computation of the contributions coming from $\bar{Q}$):
\[
\frac{\partial}{\partial \alpha} \bar{V}(e^{i\alpha} \psi + \phi) \bigg|_{\alpha=0} = -i \int dx \left[ \frac{\delta \bar{V}(\psi + \phi)}{\delta \phi^+_x} \cdot (\phi^+_x + \sqrt{\rho_0}) - \frac{\delta \bar{V}(\psi + \phi)}{\delta \phi^-_x} \cdot (\phi^-_x + \sqrt{\rho_0}) \right] - i \lambda \rho_0 \int dx \, dy \left( \psi^+_x + \phi^+_x - \psi^-_x - \phi^-_x \right) w(x-y) \left( \psi^+_y + \phi^+_y + \psi^-_y + \phi^-_y \right) - i \bar{\nu} \sqrt{\rho_0} \int dx \left( \psi^+_x + \phi^+_x - \psi^-_x - \phi^-_x \right).
\]
Note that the term in the last line is equal to \(-i\bar{\nu}\sqrt{\rho_0} \int dx (\phi_x^+ - \phi_x^-)\), simply because \(\psi\) has zero average, by construction. If we now use (D.12) into (D.7), after an explicit computation of the contribution coming from \(\hat{Q}_{\geq h}^B\) we find:

\[
\int P_{\geq h}^B(d\psi)e^{-\bar{V}(\psi+\phi)} \left\{ \int dx \left[ \frac{\delta\bar{V}(\psi+\phi)}{\delta \phi_x^+} (\phi_x^+ + \sqrt{\rho_0}) - \frac{\delta\bar{V}(\psi+\phi)}{\delta \phi_x^-} (\phi_x^- + \sqrt{\rho_0}) \right] + \right.
\]

\[
+ \lambda \rho_0 \int dx \, dy \left[ (\psi_x^+ + \phi_x^+ - \psi_x^- - \phi_x^-) w(x - y)(\psi_y^+ + \phi_y^+ + \psi_y^- + \phi_y^-) - (\psi_x^+ - \psi_x^-) w_{\geq h}(x - y)(\psi_y^+ + \psi_y^-) \right] + \bar{\nu}\sqrt{\rho_0} \int dx (\phi_x^+ - \phi_x^-) \bigg\} = 0 ,
\]

where \(w_{\geq h}(x)\) is the Fourier transform of \(\hat{\psi}(k)[\chi_{\geq h}(k)]^{-1}\). The equation (D.13) can be rewritten in terms of the fields \(\psi^t, \psi^t\) and \(\phi^t, \phi^t\) defined via (3.41), in terms of which we find:

\[
0 = \int P_{\geq h}^B(d\psi)e^{-\bar{V}(\psi+\phi)} \left\{ \int dx \left[ \frac{\delta\bar{V}(\psi+\phi)}{\delta \phi_x^t} \phi_x^t - \frac{\delta\bar{V}(\psi+\phi)}{\delta \phi_x^t} \left( \phi_x^t + \sqrt{2\rho_0} \right) \right] + \right.
\]

\[
+ 2\lambda \rho_0 \int dx \, dy \left[ (\phi_x^t + \psi_x^t) w(x - y)(\psi_y^t + \phi_y^t) - \psi_x^t w_{\geq h}(x - y) \psi_y^t \right] + \bar{\nu}\sqrt{2\rho_0} \int dx \phi_x^t \right\} .
\]

Defining \((\cdot, \cdot)_h^\phi = e^{iA[F_{h-1} + W^{(h-1)}]} \int P_{\geq h}^B(d\psi)e^{-\bar{V}(\psi+\phi)} [\cdot, \cdot]\), we can rewrite the last equation as

\[
0 = \int dx \left[ \frac{\delta W^{(h-1)}(\phi)}{\delta \phi_x^t} \phi_x^t - \frac{\delta W^{(h-1)}(\phi)}{\delta \phi_x^t} \left( \phi_x^t + \sqrt{2\rho_0} \right) \right] + \bar{\nu}\sqrt{2\rho_0} \int dx (\phi_x^t)_{h}^\phi + \right.
\]

\[
+ 2\lambda \rho_0 \int dx \, dy w(x - y)([(\phi_x^t + \psi_x^t)(\psi_y^t + \phi_y^t) - \psi_x^t \psi_y^t])_{h}^\phi - \int dx(\Delta_h(x))_{h}^\phi ,
\]

where in the second line \(\Delta_h(x)\) is the correction due to the presence of the cutoff:

\[
\Delta_h(x) := 2\lambda \rho_0 \int dy \psi_x^t(w_{\geq h}(x - y) - w(x - y)) \psi_y^t .
\]

Eq.(D.15) can be also rewritten in a convenient form by introducing the auxiliary functional \(W^{(h-1)}(\phi, \Phi^1, \Phi^2, J^{\Delta}):\)

\[
e^{-W^{(h-1)}(\phi, \Phi^1, \Phi^2, J^{\Delta})} = e^{iA[F_{h-1}]} \int P_{\geq h}^B(d\psi)e^{-\bar{V}(\psi+\phi)} - \int dx[F_x^1 F_x^1(\phi) + F_x^2 F_x^2(\psi, \phi) + J^{\Delta}_x \Delta_h(x)]
\]

(D.17)

with

\[
F_x^1(\phi) = \bar{\nu}\sqrt{2\rho_0} \phi_x^t , \quad F_x^2(\psi, \phi) = 2\lambda \rho_0 \int dy w(x - y)([(\phi_x^t + \psi_x^t)(\psi_y^t + \phi_y^t) - \psi_x^t \psi_y^t]) ,
\]

(D.18)
in terms of which (D.15) takes the form:

\[
0 = \int dx \left[ \frac{\delta \tilde{\mathcal{W}}^{(h-1)}(\phi, 0)}{\delta \phi_x^l} \phi_x^l - \frac{\delta \tilde{\mathcal{W}}^{(h-1)}(\phi, 0)}{\delta \phi_x^l} \left( \phi_x^l + \sqrt{2\rho_0} \right) \right] (D.19)
\]

and \((\phi, 0)\) is a shorthand for \((\phi, 0, 0, 0)\). At this point, we can obtain infinitely many identities among the kernels of \(\mathcal{W}^{(h-1)}\), known as global WIs, by further deriving (D.15) or (D.19) w.r.t. \(\phi^l, \phi^t\), and then taking \(\phi = 0\). The “formal” global WIs (discussed e.g. in [15, 40] in the framework of dimensional regularization) are those obtained by neglecting the effect of the cutoff, i.e., by neglecting \(\langle \Delta_h(x) \rangle_h^0\) in the second line of (D.15).

A few global WIs that we are interested in are those obtained by: (1) deriving w.r.t. \(\phi^l, \phi^t\), then integrating w.r.t. \(y\); (2) deriving w.r.t. \(\phi^l, \phi^t, \phi^t\), then integrating w.r.t. \(y, z\); (3) deriving w.r.t. \(\phi_x^l\) (of course in all these cases we put \(\phi = 0\) after the derivation). Their explicit expression is (neglecting for simplicity the issue of the “last integration scale” mentioned in footnote 10, a couple of pages above):

\[
Z_{h-1} - 2\gamma^{2h-\bar{h}}\nu_h - 2\sqrt{2\rho_0}\gamma^{h/2}\mu_h = \int dx dy \frac{\delta^3 \tilde{\mathcal{W}}^{(h)}(0)}{\delta J_0^x \delta \psi_x^l \delta \psi_y^t}, \quad (D.20)
\]

\[
\gamma^{h/2}\mu_h - 4\sqrt{2\rho_0}\gamma^{h-\bar{h}}\lambda_h = \frac{1}{3!} \int dx dy dz \frac{\delta^4 \tilde{\mathcal{W}}^{(h)}(0)}{\delta J_0^x \delta \psi_x^l \delta \psi_y^t \delta \psi_z^t}, \quad (D.21)
\]

\[
W_{i,0}^{(h)} + \sqrt{2\rho_0}(\nu - \tilde{W}_{0,2}^{(h)}(0)) = \int dx \frac{\delta^2 \tilde{\mathcal{W}}^{(h)}(0)}{\delta J_0^x \delta \psi_x^t}. \quad (D.22)
\]

The first two identities are clearly useful, because they relate the running coupling constants among each other. The last identity can be read as a renormalization condition, which fixes the chemical potential to the “right value”, and is known as the Hugenholtz-Pines identity. It is easy to check that the correction terms in these identities are “trivial”, in the sense that they vanish in the limit of sharp cutoff function, in which case we can drop all these terms at once.

### D.2 Local Ward Identities

The local Ward identities are derived in a way completely analogous to the global ones, with the only difference that the phase factor \(\alpha(x)\) appearing in the change of variables \(\psi^\pm \to e^{\pm i\alpha(x)}\psi^\pm\) is a non-trivial function of \(x\), and the derivatives w.r.t. \(\alpha\) performed in the previous section should be replaced by functional derivatives w.r.t. \(\alpha(x)\). When we derive (D.1) w.r.t. \(\alpha(x)\) we produce a number of terms that are essentially the same as those discussed in the previous section, except for the fact that there is an integration over \(x\) missing; e.g., the analogue of (D.11) is

\[
\frac{\delta}{\delta \alpha(x)} \tilde{\mathcal{W}}(e^{i\alpha} \psi + \phi) \bigg|_{\alpha(x) = 0} = -i \left( \frac{\delta \tilde{\mathcal{W}}(\psi + \phi)}{\delta \phi_x^l} (\phi_x^l + \sqrt{\rho_0}) - \frac{\delta \tilde{\mathcal{W}}(\psi + \phi)}{\delta \phi_x^l} (\phi_x^- + \sqrt{\rho_0}) \right), \quad (D.23)
\]
etc. In addition to these terms (i.e., the “local” analogues of those of the previous section), there is an extra contribution coming from the measure \( P^0_{\geq \hbar}(d\psi) \), which is not invariant under a local gauge transformation (while it was invariant under a global one). The gaussian weight entering the definition of \( P^0_{\geq \hbar}(d\psi) \) has the form

\[
\exp \left\{ - \int \frac{dk}{(2\pi)^3} \bar{x}_{\geq \hbar}(k) \psi^+_k (-ik_0 + |\vec{k}|^2) \psi^-_k \right\} =: \exp \left\{ - \int dx \psi^+_x (D_{\geq \hbar} \psi^-)_x \right\}
\]

(D.24)

where the pseudo-differential operator \( D_{\geq \hbar} \) is defined here for the first time. Taking the functional derivative of the expression in braces, we find, after some algebra:

\[
- \frac{\delta}{\delta \alpha(x)} \int d\chi' e^{i\alpha(x')} \psi^+_x \left( D_{\geq \hbar}(e^{-i\alpha(\cdot)} \psi^-) \right)_{x'} \bigg|_{\alpha(x)=0} = -i \left[ \partial_0(\psi^+_x \psi^-_x) + \Delta(\psi^+_x \psi^-_x) - 2\partial_x(\psi^+_x \partial_x \psi^-_x) \right] - \delta T_h(x),
\]

(D.25)

where \( \partial_0 = \partial_{x_0}, \partial = (\partial_{x_1}, \partial_{x_2}) \), and

\[
\delta T_h(x) = i \int \frac{dk\,dp}{(2\pi)^6} e^{ipx} \psi^+_k e^{ipx} C_h(k, p) \psi^-_k,
\]

(D.26)

with

\[
C_h(k, p) := (\bar{x}_{\geq \hbar}(k) - 1)(-ik_0 + |\vec{k}|^2) - (\bar{x}_{\geq \hbar}(k + p) - 1)(-i(k_0 + p_0) + |\vec{k} + \vec{p}|^2)
\]

(D.27)

The equations above can be rewritten in terms of the fields \( \psi^l, \psi^t \), in terms of which we find

\[
- \frac{\delta}{\delta \alpha(x)} \int d\chi' e^{i\alpha(x')} \psi^+_x \left( D_{\geq \hbar}(e^{-i\alpha(\cdot)} \psi^-) \right)_{x'} \bigg|_{\alpha(x)=0} = -i(\partial_0 j_{0,x} + \partial \bar{j}_x) - \delta T(x),
\]

(D.28)

with

\[
j_{0,x} = \frac{1}{2} \left( (\psi^l_x)^2 + (\psi^t_x)^2 \right),
\]

(D.29)

\[
\bar{j}_x = i\psi^t_x \partial \psi^t_x - i\psi^l_x \partial \psi^l_x
\]

(D.30)

and

\[
\delta T_h(x) = \int \frac{dk\,dp}{(2\pi)^6} e^{ipx} \left( \bar{x}_{\geq \hbar}(k) - 1 \right) \left[ k_0(\psi^l_{k-p} \psi^l_k + \psi^t_{k-p} \psi^t_k) + |\vec{k}|^2(\psi^t_{k-p} \psi^t_k - \psi^l_{k-p} \psi^l_k) \right].
\]

(D.31)

After having performed the functional derivative w.r.t. \( \alpha(x) \) and having re-expressed everything in terms of the \( \psi^l, \psi^t \) fields, we finally arrive at the analogue of (D.15), which
reads
\[ -i \langle \partial_0 j_{0,x} + \bar{\theta}_j \rangle_h^\phi = \frac{\delta W^{(h-1)}(\phi)}{\delta \phi_x} \phi_x^t - \frac{\delta W^{(h-1)}(\phi)}{\delta \phi_x^t} \left( \phi_x^t + \sqrt{2\rho_0} \right) + \nu \sqrt{2\rho_0} \langle \phi_x^t \rangle_h^\phi + 2\lambda \rho_0 \int dy \, w(x-y) \left[ \left( \phi_x^t + \psi_x^t \right) \left( \psi_x^t + \phi_x^t \right) - \psi_x^t \psi_x^t \right] \right]_h^0 - \left( \Delta_h(x) \right)_h^\phi + \langle \delta T_h(x) \rangle_h^\phi. \]

(D.32)

If desired, this equation can be put in a form similar to (D.19). It is enough to introduce the auxiliary potential
\[ e^{-W^{(h-1)}(\phi, J, \Phi, J^{\Delta, \delta T})} = e^{i\langle \Lambda \rangle_{F_{h-1}}} \int \mathcal{D}_h \psi \, e^{-V(\psi+\phi)-(J, J)-(\delta T, \delta T)} \]
where: \( J = (J_0, J_1, J_2) = (\bar{J}, \bar{\Phi} = (\Phi^1, \Phi^2), \text{and } (J, j) = \int dx (J_0 x_0 + \bar{J}_x, \bar{\Phi}_x, \bar{F}_x) = \sum_{i=1}^2 \int dx \bar{\Phi}_x F_x^i, \text{etc.} \) Using the auxiliary potential (D.33), Eq.(D.32) takes the form:
\[ -i \partial_0 \frac{\delta \bar{W}^{(h-1)}(\phi, 0)}{\delta j_{0,x}} - i \partial_0 \frac{\delta \bar{W}^{(h-1)}(\phi, 0)}{\delta \bar{j}_x} = \frac{\delta \bar{W}^{(h-1)}(\phi, 0)}{\delta \phi_x^t} \phi_x^t - \frac{\delta \bar{W}^{(h-1)}(\phi, 0)}{\delta \phi_x^t} \left( \phi_x^t + \sqrt{2\rho_0} \right) + \frac{\delta \bar{W}^{(h-1)}(\phi, 0)}{\delta \bar{j}_x^\Delta} + \frac{\delta \bar{W}^{(h-1)}(\phi, 0)}{\delta \bar{j}_x^{\delta T}}. \]

(D.34)

At this point, we can obtain infinitely many identities among the kernels of \( W^{(h-1)} \), known as local WIs, by further deriving this identity w.r.t. \( \phi^t \), and then taking \( \phi \equiv 0 \). The “formal” local WIs (discussed e.g. in [15, 40] in the framework of dimensional regularization) are those obtained by neglecting the effect of the cutoffs, i.e., by dropping the terms \( (\Delta_h(x))_h^\phi \) and \( (\delta T_h(x))_h^\phi \).

Two local WIs we are interested in are those obtained by deriving w.r.t. \( \phi^t \), or w.r.t. \( \phi \), and then taking the Fourier transform at \( p = (p_0, 0) \). Their explicit expression in momentum space is (neglecting for simplicity the issue of the “last integration scale”):
\[ p_0 \bar{W}^{(h)}_{1,0; 0_0}(p_0, 0) = -\sqrt{2\rho_0} \bar{W}^{(h)}_{1, 1, 0, \Delta}(p_0, 0) + \bar{W}^{(h)}_{1, 0, \Delta}(p_0, 0) - \bar{W}^{(h)}_{1, 0, J^{\delta T}}(p_0, 0), \]
\[ p_0 \bar{W}^{(h)}_{0, 1; 0_0}(p_0, 0) = 2\sqrt{2\rho_0} \bar{W}^{(h)}_{0, 2, 0}(p_0, 0) - \bar{W}^{(h)}_{1, 0}(p_0, 0) - \bar{W}^{(h)}_{0, 1; J^{\Delta}}(p_0, 0) - \bar{W}^{(h)}_{0, 1; J^{\delta T}}(p_0, 0), \]

(D.35)

(D.36)

where \( W^{(h)}_{1,0; 0_0}(x, y) \) is the kernel of \( J_0 x_0 \psi_y^t \) in \( W^{(h)}(\phi, J_0, 0) \), and \( \bar{W}^{(h)}_{1,0, J^{\Delta}}(p) = \int dx \, e^{i p x} \)
\[ W^{(h)}_{1,0, J^{\Delta}}(x, 0) \] (and similarly for \( W^{(h)}_{1,0, J^{\delta T}}(p) \), etc). Eqs.(D.35)-(D.36) are the analogues of [40, (3.11)-(3.12)]. If we divide (D.35) by \( p_0 \) and then take the limit \( p_0 \to 0 \) we find:
\[ Z_{h-1}^{J_0} = \sqrt{2\rho_0 (E_{h-1} - 1)} + p_0 \bar{W}^{(h)}_{1,0; J^{\Delta}}(0) - (\partial_{p_0} \bar{W}^{(h)}_{1,0; J^{\delta T}})(0), \]

(D.37)
where $Z_h^{(h)} := \hat{W}_{1,0;0}^{(h)}(0)$, and the name is justified by the fact that the Feynman diagram expansion for $Z_h^{(h)}$ has the same structure as that for $Z_h$. In particular, inspection of perturbation theory shows that

$$Z_h^{(h)} = \frac{Z_h - Z_h}{\lambda \hat{b}(0)} (1 + O(\gamma^{h-\tilde{h}})) \, ,$$  

(D.38)

where the error terms in parentheses come from the irrelevant terms on scale $\tilde{h}$. In a similar way, if we divide (D.36) by $p_0^2$ and then take the limit $p_0 \to 0$ we find, defining $E_{h-1}^{(h)} := -\partial_{p_0} \hat{W}_{0,1;0}^{(h)} (0)$:

$$-2E_{h-1}^{(h)} = 2 \sqrt{2p_0} B_{h-1} + \partial_{p_0}^2 \hat{W}_{0,1;J\Delta}^{(h)} (0) - \partial_{p_0}^2 \hat{W}_{0,1;J\delta T}^{(h)} (0) \, ,$$

(D.39)

where

$$E_{h}^{(h)} = \frac{E_h - 1}{\lambda \hat{b}(0)} \sqrt{2p_0} (1 + O(\gamma^{h-\tilde{h}})) \, .$$

(D.40)

Contrary to the correction terms of the global Ward Identities, the corrections in (D.37), (D.39) are not “trivial”, i.e., they do not vanish in the sharp cutoff limit. On the contrary, they give a finite (cutoff-dependent) contribution to the beta function, which shows up as an “anomaly” already at the level of the one-loop beta function, as discussed in Section 3.C.4.

D.3 Comparison between $\mathcal{W}^{(h-1)}$ and $\mathcal{V}^{(h-1)}$

In order to establish the exact relation between $\mathcal{W}^{(h-1)}$ and $\mathcal{V}^{(h-1)}$, we compute the r.h.s. of (D.1) via a multiscale integration analogous to the one used in the bulk of the paper for the computation of $\mathcal{V}^{(h-1)}$. The integration of the ultraviolet fields on scales $> \tilde{h}$, as well as the integration of the field $\tilde{\psi}\tilde{(h)}$ (see (3.40)), is identical to the one discussed in Section 3.B and at the beginning of Section 3.C, after which we can rewrite (D.1) as

$$e^{-|A|F_{h-1} - \mathcal{W}^{(h-1)}(\phi)} = e^{-|A|} \sum_{k \geq \tilde{h}} F_k \int P_{[h,\tilde{h}]}(d\psi) e^{-\mathcal{V}^{(h)}(\psi + \phi)} \, ,$$

(D.41)

where $P_{[h,\tilde{h}]}(d\psi)$ is the same as the gaussian measure $P_{\leq \tilde{h}}(d\psi)$ in (3.40), modulo the presence of an infrared cutoff on scale $h$ in the corresponding propagator (i.e., while the cutoff function appearing in the propagator of $P_{\leq \tilde{h}}(d\psi)$ is $\chi_{\tilde{h}}(k)$, the one in the propagator of $P_{[h,\tilde{h}]}(d\psi)$ is $\chi_{\tilde{h}}(k) - \chi_{h-1}(k)$). Moreover, using the same convention of Section 3.C.2, $\mathcal{V}^{(h)}$ coincides with the function $\tilde{\mathcal{V}}^{(h)}$ introduced after (3.40). At this point, we start dressing the gaussian measure in (D.41), in the same way as in Section 3.C.2. Let us describe the first integration step explicitly. Using the notations introduced in (3.64) and following equations, we rewrite the r.h.s. of (D.41) as

$$e^{-|A|} \sum_{k \geq \tilde{h}} F_k + L_Q \mathcal{V}^{(h)}(\phi) \int P_{[h,\tilde{h}]}(d\psi) e^{-L_Q \mathcal{V}^{(h)}(\psi) - \tilde{\mathcal{V}}^{(h)}(\psi + \phi) - \left(\psi + \phi, M_Q^{(h)}\phi\right)} \, ,$$

(D.42)
where $\tilde{\psi}(\tilde{h})$ is a shorthand for $L_{C}\psi(\tilde{h}) + R\psi(\tilde{h})$. We now combine $L_{Q}\psi(\tilde{h})(\psi)$ with the gaussian measure, as in (3.66), and then use the addition principle to rewrite the dressed measure as a product of a measure supported on scale $\tilde{h}$ and a measure supported on smaller scales, as in (3.72):

\[
(D.20) = e^{-|A|\sum_{k \geq h} F_{k} + t_{h}} + L_{Q}\psi(\tilde{h})(\phi) \int P_{[h,\tilde{h}-1]}(d\psi) \cdot \int \tilde{P}_{h}(d\psi(\tilde{h})) e^{-\tilde{\psi}(\tilde{h})(\psi(\tilde{h}) + \psi + \phi) - \left(\psi(\tilde{h}) + \psi + \phi, M_{Q}^{\tilde{h}}\phi\right)}.
\]

At this point, we integrate the field on scale $\tilde{h}$ and define:

\[
|A| \tilde{F}_{h} + \tilde{S}_{h}(\phi) + V(\tilde{h}-1)(\psi') + B(\tilde{h}-1)(\psi', \phi) = -\log \int \tilde{P}_{h}(d\psi(\tilde{h})) e^{-\tilde{\psi}(\tilde{h})(\psi(\tilde{h}) + \psi') - \left(\psi(\tilde{h}) + \psi', M_{Q}^{\tilde{h}}\phi\right)}.
\]

where, denoting by $\hat{\psi}_{k}$ the two-component column vector with components $\hat{\phi}_{k}^{L}$ and $\hat{\phi}_{k}^{T}$,

\[
\tilde{S}_{h}(\phi) = -\frac{1}{2} \int \frac{dk}{(2\pi)^{d}} \hat{\phi}_{-k}^{T} \hat{M}_{Q}^{\tilde{h}}(k) \hat{M}_{Q}^{\tilde{h}}(k) \hat{\phi}_{k} \equiv -\frac{1}{2} \left(\phi, M_{Q}^{\tilde{h}} T \hat{g}^{\tilde{h}}(k) M_{Q}^{\tilde{h}} \phi\right).
\]

We now set $F_{h-1} = t_{h} + \tilde{F}_{h}$ and $\tilde{S}_{h} = L_{Q}\psi(\tilde{h}) - \tilde{S}_{h}$, and then we iterate the same procedure. After the integration of the fields on scales $\geq k + 1$ we rewrite (D.1) as

\[
e^{-|A|\sum_{k' \geq k} F_{k'} + S(k)} \int P_{[h, k]}(d\psi) e^{-V(k)(\psi + \phi) - B(k)(\psi + \phi, \phi)},
\]

where one can inductively prove that, for $k < \tilde{h}$,

\[
S(k)(\phi) = \sum_{k' = k+1}^{\tilde{h}} \left[ L_{Q}\psi(k')(\phi) + \frac{1}{2} \left(\phi, Q(k'), T \hat{g}^{k'}(\phi)\right) - \frac{1}{2} \left(\phi, G^{(k'+1), T} M_{Q}^{k'} \psi^{(k'+1)}(\phi)\right)\right],
\]

\[
B(k)(\psi, \phi) = \left(\psi, Q(k+1)\phi\right) + \sum_{n \geq 1} \int d\mathbf{x}_{1} \cdots d\mathbf{x}_{n} \left[ \prod_{i=1}^{n} \left(\psi^{(k+1)}(\phi)\right)_{x_{i}} \right] \frac{\partial^{n}}{\partial \psi_{\mathbf{x}_{1}} \cdots \partial \psi_{\mathbf{x}_{n}}} \psi(\psi),
\]

and the vectorial nature of $\psi$ (i.e., the fact that $\psi$ has two components, labelled $l$ and $t$) is implicitly understood. Moreover, the functions $Q(k)$ and $G(k)$, $k \leq \tilde{h} + 1$, are defined by the iterative relations

\[
Q(k) = Q(k+1) + M_{Q}^{k} + M_{Q}^{k} G^{(k+1)}, \quad G(k) = G(k+1) + \hat{g}^{k}(Q(k)),
\]

with $Q^{(\tilde{h} + 1)} = G^{(\tilde{h} + 1)} = 0$. 79
The iteration goes on in the same fashion until we reach scale $h$, where a small difference from the previous scheme should be taken into account: in fact, by proceeding as described above, one finds that the dressed propagator on the last scale, rather than being equal to $\tilde{g}^{(h)}$, it is equal to

$$\tilde{g}^{(h)}(x) = \int \frac{dk}{(2\pi)^3} \tilde{f}_h(k) \frac{e^{-ikx}}{D_{h-1}(k)} \begin{pmatrix} \tilde{A}'_{h-1}(k) |\tilde{k}|^2 + \tilde{B}'_{h-1}(k) k_0^2 & \tilde{E}'_{h-1}(k) k_0 \\ -\tilde{E}'_{h-1}(k) k_0 & \tilde{Z}'_{h-1}(k) \end{pmatrix}$$

where $\tilde{f}(k) = \chi_h(k) - \chi_{h-1}(k)$, as usual, and

$$\tilde{A}'_{h-1}(k) = \tilde{A}_h(k) + a_h \tilde{f}_h(k), \quad \tilde{B}'_{h-1}(k) = \tilde{B}_h(k) + b_h \tilde{f}_h(k),$$
$$\tilde{E}'_{h-1}(k) = \tilde{E}_h(k) + e_h \tilde{f}_h(k), \quad \tilde{Z}'_{h-1}(k) = \tilde{Z}_h(k) + z_h \tilde{f}_h(k).$$

After the integration of the last scale we finally find

$$(D.1) = e^{-|A|(\sum_{k|h} F_k + \tilde{F}_{h-1}) + \tilde{S}^{(h-1)}(\phi) - \tilde{V}^{(h-1)}(\phi) - B^{(h-1)}(\phi,\phi)} \equiv e^{-|A|\bar{F}_{h-1} - \bar{W}^{(h-1)}(\phi)},$$

where the tildes on the functions at exponent recall the fact that these functions are defined in the same fashion as their analogues without tilde, with the only difference that the single-scale propagator on the last scale, $\tilde{g}^{(h)}$, wherever it enters the definition of these objects, should be replaced by $\tilde{g}^{(h)}$.

Eq.(D.53) provides us the desired relation between $\bar{W}^{(h-1)}$ and $\bar{V}^{(h-1)}$. It shows that the local parts of the kernels of $\bar{W}^{(h-1)}$ (in the sense of the values of their Fourier transforms at zero external momenta, as well as the zero momenta values of their derivatives w.r.t. $k$), are related in a very simple fashion with the corresponding local parts of the kernels of $\bar{V}^{(h-1)}$, simply because the local part of the kernels of $\bar{E}^{(h-1)}(\phi,\phi)$ is equal to $2 \sum_{k=h}^\bar{h} \mathcal{L} Q V^{(k)}(\phi)$, due to the compact support properties of $\tilde{g}^{(k)}$ and of $G^{(k)}$, and similarly the local part of $\bar{S}^{(h-1)}$ is equal to $\sum_{k=h}^\bar{h} \mathcal{L} Q V^{(k)}$. Therefore,

$$\mathcal{L} \bar{W}^{(h-1)}(\phi) = \mathcal{L} \tilde{V}^{(h-1)}(\phi) + \sum_{k=h}^\bar{h} \mathcal{L} Q V^{(k)},$$

which induces a simple, explicit, connection between the actual renormalization and running coupling constants of the model, introduced in Section 3.C.2, and $\mathcal{L} \bar{W}^{(h-1)}(\phi)$.

### E Verification of Ward Identities at lowest order

In this section, we verify the validity of some of the global and local Ward Identities among the running coupling and the renormalization constants, at lowest non trivial order in perturbation theory. In particular, we shall compute the effect of the infrared cutoff and discuss its role in the different identities worked out below.
$\hat{W}^{(h-1)}_{1,1;J^\Delta} = \text{Diagram A} + \text{Diagram B} + \text{Diagram C} + \text{Diagram D}$

Figure 23: Correction terms to the formal GWI relating $\mu_h$ and $Z_h$, see (D.20). The triangular vertex with the external zigzag line represents $J^\Delta \Delta_h(x)$, see (D.16) and (D.17).

E.1 Global Ward Identities

Let us verify the validity of (D.20) at lowest non trivial order in $\lambda$ and/or in $\gamma^{h-\bar{h}}$, in the region $h^* \leq h \leq \bar{h}$. If $W^{(h)}_{1,1;J^\Delta}(x, y; 0)$ is the kernel of $\int_0^\Delta \phi^i_x \phi^j_y$ in $\hat{W}^{(h)}(\phi, J^\Delta, 0)$, and

$$\hat{W}^{(h)}_{1,1;J^\Delta}(k, p) = \int dxdy e^{-ikx+i(k+p)y}W^{(h)}_{1,1;J^\Delta}(x, y; 0), \quad (E.1)$$

we can rewrite (D.20) as

$$Z_{h-1} - 2\gamma^{2h-\bar{h}}\nu_h - 2\sqrt{2\rho_0}\gamma^{h/2}\mu_h = \hat{W}^{(h)}_{1,1;J^\Delta}(0, 0) . \quad (E.2)$$

Using the beta function equations (3.77)-(3.80), the l.h.s. can be rewritten as

$$Z_{h-1} - 2\gamma^{\bar{h}}\nu_h - 2\sqrt{2\rho_0}\gamma^{h/2}\mu_h + \sum_{k=h+1}^{\bar{h}} \left( \beta_k^\gamma - 2\gamma^{2k-\bar{h}}\beta_k^\nu - 2\sqrt{2\rho_0}\gamma^{k/2}\beta_k^\mu \right) . \quad (E.3)$$

Now, using the explicit expressions of $Z_{h-1}, \nu_h, \mu_h$, one can check that the combination $Z_{h-1} - 2\gamma^{\bar{h}}\nu_h - 2\sqrt{2\rho_0}\gamma^{h/2}\mu_h$ is zero at lowest order (i.e., at the order $\lambda$, as well as at the order $\nu$) and, therefore, it is (at most) of the order $O(\lambda|\bar{h}|)$. Moreover, using the explicit expression of the beta functions for $Z_h, \nu_h$ and $\mu_h$, see (3.82), (3.83), (3.89), as well as the replacements

$$Z_h \to 2\sqrt{2\rho_0}\gamma^{h/2}\mu_h , \quad \lambda_h \to \frac{1}{4\sqrt{2\rho_0}}\gamma^{h-\bar{h}/2}\mu_h , \quad (E.4)$$

induced by the Ward Identities (D.20)-(D.21) (which, once inserted in the expressions (3.82), (3.83), (3.89) of the one-loop beta function, induce errors of higher order in $\lambda$ and/or in $\gamma^{h-\bar{h}}$, as one can prove inductively in $h$), we can rewrite, for $h^* \leq h \leq \bar{h}$,

$$\sum_{k=h+1}^{\bar{h}} \left( \beta_k^Z - 2\gamma^{2h-\bar{h}}\beta_k^\nu - 2\sqrt{2\rho_0}\gamma^{k/2}\beta_k^\mu \right) = \sum_{k=h+1}^{\bar{h}} \left( -\frac{2\mu^2}{A_k C_k} \beta_k^{(2)} \right) - \frac{2\gamma^h}{A_k C_k} \frac{1}{8\rho_0} \left( \beta_0^{(11)} - 2\beta_0^{(11, x)} \right) + \frac{2\mu^2}{A_k C_k} \left( \beta_0^{(2)} + 2\beta_0^{(2, x)} \right) . \quad (E.5)$$

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modulo higher order correction terms in $\lambda$ and/or in $\gamma^{h-h}$ (in deriving the expression of the first term in the second line we also used the fact that $C_k Z_k = 1$, up to higher order corrections). Now, the sum over $k$ of $-\frac{2\gamma k}{A_k C_k} \frac{1}{\delta_0'} (B_0^{(1)} - 2\beta_0^{(1,\gamma)})$ is of the order $\lambda^2$, which is subdominant with respect to the other two terms, whose sum over $k$ gives (using the fact that $\mu_k = \gamma^{(h-k)/2} \mu_{h}$, up to higher order corrections)

$$\gamma - 1 \frac{4\mu_{h+1}^2}{A_{h+1} C_{h+1}} \beta_0^{(2,\gamma)}.$$  

(E.6)

We now want to show that this expression is equal to $\hat{W}^{(h)}_{1,1;J \Delta}(0,0)$, modulo higher order corrections. In fact, the lowest order contribution to $\hat{W}^{(h)}_{1,1;J \Delta}(0,0)$ is given by the sum of the diagrams in Fig.23, with at least one propagator at scale $h$. After the same considerations made for the calculation of the beta function for $\mu_h$ (see Section C.3) we find that, at lowest order,

$$\hat{W}^{(h)}_{1,1;J \Delta}(0,0) = 8\lambda \rho_0 \hat{v}(\vec{0}) \frac{\mu_{h+1}^2}{Z_{h+1}} \frac{1}{A_{h+1} C_{h+1}} \times \int \frac{d^3k}{(2\pi)^3} \left( \sum_{k \geq 0} f_k(|k|) - 1 \right) \hat{f}_0(|k|) \frac{\hat{f}_0(|k|) + 3\hat{f}_0(|k|) \hat{f}_1(|k|) + 3\hat{f}_0(|k|) \hat{f}_2(|k|)}{|k|^4}.$$  

(E.7)

Note that, for $j \geq 0$,

$$\left( \sum_{k \geq 0} f_k(|k|) - 1 \right) \hat{f}_j(|k|) = u_0(|k|) \delta_{0,j}$$  

(E.8)

where

$$u_0(|k|) := \begin{cases} 1 - f_0(|k|), & \text{if } \gamma^{-1} \leq |k| \leq 0 \\ 0, & \text{otherwise}. \end{cases}$$  

(E.9)

Note that, if $\gamma^{-1} \leq |k| < 1$, then $\hat{f}_1(|k|) = 0$. Therefore,

$$\hat{W}^{(h)}_{1,1;J \Delta}(0,0) = 8\lambda \rho_0 \hat{v}(\vec{0}) \frac{\mu_{h+1}^2}{Z_{h+1}} \frac{1}{A_{h+1} C_{h+1}} \frac{1}{2\pi^2} \int_{\gamma^{-1}}^1 \frac{d\rho}{\rho^2} (1 - \hat{f}_0(\rho)) \hat{f}_0(\rho)$$

$$= 4 \mu_{h+1}^2 \frac{1}{A_{h+1} C_{h+1}} \frac{\gamma}{2\pi^2} \int_1^{\gamma} \frac{d\rho}{\rho^2} \lambda(\rho)(1 - \chi(\rho))^2$$

$$= 4 \mu_{h+1}^2 \frac{1}{A_{h+1} C_{h+1}} \frac{\gamma}{\gamma - 1} \beta_0^{(2,\gamma)},$$  

(E.10)

where we used that $Z_h = 2\lambda \rho_0 \hat{v}(\vec{0})$, up to higher order corrections.

A similar discussion can be repeated for proving the validity of (D.21), but we will not belabor the details here.
\[
\beta^E_h = \partial_{p_0} \left[ \begin{array}{c} \vdots \\ \vdots \\ \vdots \end{array} \right]_{p=0} \quad \beta^{Z_{J_0}}_h = \left[ \begin{array}{c} \vdots \\ \vdots \\ \vdots \end{array} \right]_{p=0} 
\]

\[
\partial_{p_0} \hat{W}^{(h)}_{1,0,J} (0) = \partial_{p_0} \left[ \begin{array}{c} \vdots \\ \vdots \\ \vdots \end{array} \right]_{p=0} \quad \partial_{p_0} \hat{W}^{(h-1)}_{1,0,J} (0) = \partial_{p_0} \left[ \begin{array}{c} \vdots \\ \vdots \\ \vdots \end{array} \right]_{p=0}
\]

Figure 24: Diagrams involved in the local WI relating \( E_h \) and \( Z_h \). On the first line the lowest order beta function for \( E_h \) and \( Z^{h}_{J_0} \). On the second line, the correction terms coming from the cutoffs. The triangular vertex with the external zigzag line represents \( J^\Delta_2 \Delta_h(x) \), see (D.16)-(D.17). The diamond vertex with wiggly external line represents \( J^{\delta T} \delta T_h(x) \), see (D.31)-(D.33).

### E.2 Local WI relating \( E_h \) and \( Z_h \)

In this section we prove the validity of the local WI (D.37), that is

\[
Z^{J_0}_{h-2} = \sqrt{2\rho_0} (E_{h-2} - 1) + \partial_{p_0} \hat{W}^{(h-1)}_{1,0,J} (0) - \partial_{p_0} \hat{W}^{(h-1)}_{1,0,J^{\delta T}} (0),
\]

(E.11)

at lowest non trivial order. Now, using (3.90), we find that at lowest order

\[
E_{h-2} - 1 = -2 \frac{1}{A_h C_h Z_h} \mu^2 \gamma - 1 \beta^{(2)}_0,
\]

(E.12)

while, using (D.38) and (3.89),

\[
Z^{J_0}_{h-2} = \sqrt{2\rho_0} \frac{Z_{h-2} - Z_h}{Z_h} = -2 \sqrt{2\rho_0} \frac{1}{A_h C_h Z_h} \mu^2 \gamma - 1 \beta^{(2)}_0,
\]

(E.13)

as can also be checked by an explicit calculation of the second diagram on the first line of Fig.24. Therefore, at lowest order \( Z^{J_0}_{h-2} = \sqrt{2\rho_0} (E_{h-2} - 1) \). In fact, it can be checked that the two correction terms in the r.h.s. cancel among each other: at lowest order their values are

\[
\partial_{p_0} \hat{W}^{(h-1)}_{1,0,J} (0) = -2 \gamma^3 \mu_h \frac{2\lambda \rho_0 \tilde{v} (0) E_h}{A_h C_h Z_h} \partial_{p_0} \int \frac{d^3k}{(2\pi)^3} \left( \hat{\chi}^{-1}_h (k) - 1 \right) \frac{f_h (|k + p|)}{|k + p|^2} \left| |k| \right| \left| |k| \right| \left| p = 0 \right|
\]

\[
\partial_{p_0} \hat{W}^{(h-1)}_{1,0,J^{\delta T}} (0) = -2 \gamma^3 \mu_h \frac{1}{A_h C_h} \partial_{p_0} \int \frac{d^3k}{(2\pi)^3} \left( \hat{\chi}^{-1}_h (k) - 1 \right) \frac{f_h (|k + p|)}{|k + p|^2} \left| |k| \right| \left| |k| \right| \left| p = 0 \right|
\]

(E.14)

which are the same, since \( E_h = 1 \) up to higher order corrections.
\[ \beta_h^B = \partial_{p_0}^2 \left[ \begin{array}{c} \scalebox{0.8}{\includegraphics{diagram1}} \\ \scalebox{0.8}{\includegraphics{diagram2}} \end{array} \right]_{p=0} \quad \beta_h^{E_h} = \partial_{p_0} \left[ \begin{array}{c} \scalebox{0.8}{\includegraphics{diagram3}} \end{array} \right]_{p=0} \]

\[ \partial_{p_0}^2 \hat{W}_{0,1;J\Delta}^{(0)}(0) = \partial_{p_0}^2 \left[ \begin{array}{c} \scalebox{0.8}{\includegraphics{diagram4}} \\ \scalebox{0.8}{\includegraphics{diagram5}} \end{array} \right]_{p=0} \]

\[ \partial_{p_0}^2 \hat{W}_{0,1;J\gamma}^{(0)}(0) = \partial_{p_0}^2 \left[ \begin{array}{c} \scalebox{0.8}{\includegraphics{diagram6}} \\ \scalebox{0.8}{\includegraphics{diagram7}} \end{array} \right]_{p=0} \]

Figure 25: Diagrams involved in the local WI relating \( B_h \) and \( E_h \). On the first line the lowest order beta function for \( B_h \) and \( E_h^{J_0} \). On the second and third lines, the correction terms coming from the cutoffs.

**E.3 Local WI relating \( B_h \) and \( E_h \)**

In this section we prove the validity of the local WI (D.39), that is

\[ -2E_{h-2}^{J_0} = 2\sqrt{2\rho_0}B_{h-2} + \partial_{p_0}^2 \hat{W}_{0,1;J\Delta}^{(h-1)}(0) - \partial_{p_0}^2 \hat{W}_{0,1;J\gamma}^{(h-1)}(0), \tag{E.15} \]

Let us discuss the various terms involved in this equation separately. Using (3.91) and the fact that \( \bar{B}_h = 0 \), we find

\[ B_{h-2} = 2\frac{\mu_h^2}{A_hZ_h} \gamma - 1 \left( \frac{E_h^2}{C_hZ_h} \rho_0 + \frac{1}{3} \beta_0^{(2)} \right). \tag{E.16} \]

Using (D.40) and (3.90) we find

\[ E_{h-2}^{J_0} = \frac{2}{\gamma - 1} \lambda \hat{v}(0) \sqrt{2\rho_0} \frac{1}{A_hC_h} \mu_h^2 \frac{E_h}{Z_h} \beta_0^{(2)}, \tag{E.17} \]

so that, recalling that at lowest order \( E_h = 1 \) and \( Z_h = 2\lambda \rho_0 \hat{v}(0) \),

\[ -2E_{h-2}^{J_0} - 2\sqrt{2\rho_0}B_{h-2} = -4\frac{\lambda \rho_0 \hat{v}(0)}{A_hZ_h} \gamma - 1 \beta_0^{(2)} \] \tag{E.18}

We now want to verify that the sum of the two correction terms in the r.h.s. of (E.15) gives exactly the same contribution.

The leading order contribution to the correction term \( \partial_{p_0}^2 \hat{W}_{0,1;J\Delta}^{(h-1)}(0) \) is represented on the second line of Fig.25. Using the same notations as in Section C.6, we find

\[ \partial_{p_0}^2 \hat{W}_{0,1;J\Delta}^{(h-1)}(0) = -4\lambda \rho_0 \hat{v}(0) \gamma^{-h/2} \mu_h \frac{1}{A_hZ_h} \int \frac{d^3k}{(2\pi)^3} \left( \frac{1}{\sum_{k>0} f_k(|k|)} - 1 \right) f_0(|k|) \cdot \partial_{p_0}^2 \left[ \left( 1 + \alpha_h \rho_0 \frac{k_0}{|k|^2} \right) f_0(|k + p|) \right]_{p=0}, \tag{E.19} \]
that is, using also the fact that at lowest order \(\alpha_h = 1\) and \(Z_h = 2\lambda \rho_0 \hat{\sigma}(\vec{0})\),
\[
\partial^2_{p_0} \hat{W}_{0,1;J^\Delta}(0) = -2 \frac{\gamma^{-h/2} \mu_h}{A_h} \int \frac{d^3 k}{(2\pi)^3} u_0(|k|) \left[ \frac{2k_0}{|k|^2} \partial k_0 \left( \frac{f_0(|k|)}{|k|^2} \right) + \partial^2 k_0 \left( \frac{f_0(|k|)}{|k|^2} \right) \right].
\]

We now compute the derivatives and pass to polar coordinates, thus finding
\[
\partial^2_{p_0} \hat{W}_{0,1;J^\Delta}(0) = -2 \frac{\gamma^{-h/2} \mu_h}{A_h} \int \frac{d^3 k}{(2\pi)^3} u_0(|k|) \left[ \frac{2f_0(|k|)k_0^2}{|k|^4} \right] \frac{f_0'(|k|)}{|k|^2} \left( k_0^2 - 2k_0^2 \right) + \frac{f_0''(|k|)}{|k|^4} \left( k_0^2 \right)
\]
\[
= \frac{\gamma^{-h/2} \mu_h}{A_h} \frac{1}{2\pi^2} \int d\rho u_0(\rho) \left[ \frac{4f_0(\rho)}{3\rho^2} - 2 \frac{f_0'(\rho)}{\rho} \right].
\]

The leading order contribution to the correction term \(\partial^2_{p_0} \hat{W}_{0,1;J^\Delta}(0)\) is represented on the third line of Fig.25, and is equal to
\[
\partial^2_{p_0} \hat{W}_{0,1;J^\Delta}(0) = -2 \frac{\gamma^{-h/2} \mu_h E_h}{A_h C_h Z_h} \frac{1}{2\pi^2} \int d\rho u_0(\rho) \left[ \frac{8f_0(\rho)}{3\rho^2} - \frac{4f_0'(\rho)}{3\rho} \right].
\]

Using the fact that at lowest order \(E_h/(C_h Z_h) = 1\), we find that the difference between
\((\ref{E.21})\) and \((\ref{E.22})\) is
\[
\partial^2_{p_0} \hat{W}_{0,1;J^\Delta}(0) - \partial^2_{p_0} \hat{W}_{0,1;J^\Delta}'(0) = \frac{\gamma^{-h/2} \mu_h}{A_h} \frac{1}{2\pi^2} \int d\rho u_0(\rho) \left[ \frac{4f_0(\rho)}{3\rho^2} + \frac{4f_0'(\rho)}{3\rho} - \frac{2f_0''(\rho)}{3\rho} \right].
\]

Using the first of \((\ref{E.4})\), we can rewrite this equation as
\[
\partial^2_{p_0} \hat{W}_{0,1;J^\Delta}'(0) - \partial^2_{p_0} \hat{W}_{0,1;J^\Delta}(0) = -4 \frac{\sqrt{2p_0 \mu_h^2}}{A_h 2\pi^2} \int d\rho u_0(\rho) \left[ \frac{2f_0(\rho)}{\rho^2} - \frac{2f_0'(\rho)}{\rho} + \frac{f_0''(\rho)}{\rho} \right]
\]
that, after an integration by parts, can be easily recognized to be the same as \((\ref{E.18})\).

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