Gauge-invariant variables, WZW models and 
(2+1)-dimensional Yang-Mills theory

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Abstract
Recent progress in understanding (2+1)-dimensional Yang-Mills ($YM_{2+1}$) theory via the use of gauge-invariant variables is reviewed. Among other things, we discuss the vacuum wavefunction, an analytic calculation of the string tension and the propagator mass for gluons and its relation to the magnetic mass for $YM_{3+1}$ at nonzero temperature. (Talk given at the Workshop on Physical Variables in Gauge Theories, Dubna, Russia, September 1999.)

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1. Introduction

In this talk I shall discuss a Hamiltonian approach to Yang-Mills theory in two spatial dimensions ($YM_{2+1}$), where nonperturbative calculations can be carried out to the extent that results on mass gap and string tension can be compared with lattice simulations of the theory. The work I shall report on was developed over the last few years in collaboration with D. Karabali and Chanju Kim [1, 2, 3]. (Even though I shall concentrate on the pure $YM_{2+1}$, some of the results can be extended to the Yang-Mills-Chern-Simons theory [4]). Physical variables or gauge-invariant variables play a key role in our analysis, in keeping with the theme of this conference, and I hope this will be a nice example of how such variables can elucidate the nonperturbative structure of gauge theories.

Before entering into the details of our work, let me say a few words about the relevance of $YM_{2+1}$. Gauge theories (without matter) in (1+1) dimensions are rather trivial since there are no propagating degrees of freedom, although there may be global degrees of freedom on spaces of nontrivial topology. In (2+1) dimensions, gauge theories do have propagating degrees of freedom and being next in the order of complexity, it is possible that they provide a model simple enough to analyze mathematically and yet nontrivial enough to teach us some lessons about (3+1)-dimensional $YM$ theories. Another very important reason to study $YM_{2+1}$ is its relevance to magnetic screening in $YM_{3+1}$ at high temperature. Gauge theories at finite temperature have worse infrared problems than at zero temperature due to the divergent nature of the Bose distribution for low energy modes. A dynamically generated Debye-type screening mass will eliminate some of these, but we need a magnetic screening mass as well to have a perturbative expansion which is well defined in the infrared.

A simple way to see that a magnetic mass can be dynamically generated is as follows. In the imaginary time formalism, with Matsubara frequencies $\omega_n = 2\pi n T$, where $T$ is the temperature, the gauge fields have a mode expansion as $A_i(\vec{x},x^0) = \sum_n A_{i,n}(\vec{x}) \exp(i2\pi n Tx^0)$. At high temperatures and for modes of wavelength long compared to $1/T$, the modes with nonzero Matsubara frequencies are unimportant and the theory reduces to the theory of the $\omega_n = 0$ mode, viz., a three (Euclidean) dimensional Yang-Mills theory (or a (2+1)-dimensional theory in a Wick rotated version). Yang-Mills theories in three or (2+1) dimensions are expected to have a mass gap and this is effectively the magnetic mass of the (3+1)-dimensional theory at high temperature [5]. In order to incorporate this feature into $YM_{3+1}$ at nonzero temperatures, one needs a decomposition of the $YM_{3+1}$ Feynman integrals wherein the $YM_{2+1}$ pieces are isolated; in other words, one needs to identify the “slots” in the perturbative expansion of $YM_{3+1}$ where the $YM_{2+1}$ results can be inserted. There is a recent analysis along these lines by Reichenbach and Schulz [6].

Let me now start by recalling a couple of facts about $YM_{2+1}$. The coupling constant $e^2$ has the dimension of mass and it does not run as the four-dimensional coupling does.
The dimensionless expansion parameter of the theory is $k/e^2$ or $e^2/k$, where $k$ is a typical momentum. Thus modes of low momenta must be treated nonperturbatively, while modes of high momenta can be treated perturbatively. There is no simple dimensionless expansion parameter. YM$_{2+1}$ is perturbatively super-renormalizable, so the ultraviolet singularities are well under control.

2. The parametrization of the fields

Coming now to the details of our analysis, let us consider a gauge theory with group $G = SU(N)$ in the $A_0 = 0$ gauge. The gauge potential can be written as $A_i = -it^aA^a_i$, $i = 1, 2$, where $t^a$ are hermitian $N \times N$-matrices which form a basis of the Lie algebra of $SU(N)$ with $[t^a, t^b] = if^{abc}t^c$, $\text{Tr}(t^at^b) = \frac{1}{2}\delta^{ab}$. The spatial coordinates $x_1, x_2$ will be combined into the complex combinations $z = x_1 - ix_2$, $\bar{z} = x_1 + ix_2$ with the corresponding components for the potential $A \equiv A_z = \frac{1}{2}(A_1 + iA_2)$, $\bar{A} \equiv A_{\bar{z}} = \frac{1}{2}(A_1 - iA_2) = -(A_z)\dagger$. The starting point of our analysis is a change of variables given by

$$A_z = -\partial_z M M^{-1}, \quad A_{\bar{z}} = M^{\dagger -1}\partial_{\bar{z}} M$$

(1)

Here $M, M^\dagger$ are complex matrices in general, not unitary. If they are unitary, the potential is a pure gauge. The parametrization (1) is possible and is standard in many discussions of two-dimensional gauge fields. (There are also has similarities between (1) and the construction of gauge-invariant particle states as discussed by McMullan, Lavelle and Horan at this workshop [7].) A particular advantage of this parametrization is the way gauge transformations are realized. A gauge transformation $A_i \rightarrow A_i^{(g)} = g^{-1}A_i g + g^{-1}\partial_i g$, $g(x) \in SU(N)$ is obtained by the transformation $M \rightarrow M^{(g)} = gM$. The gauge-invariant degrees of freedom are parametrized by the hermitian matrix $H = M^\dagger M$. Physical state wavefunctions are functions of $H$.

In making a change of variables in a Hamiltonian formalism, there are two things we must do: 1) evaluate the volume measure (or Jacobian of the transformation) which determines the inner product of the wavefunctions and 2) rewrite the Hamiltonian as an operator involving the new variables. A consistency check would then be the self-adjointness of the Hamiltonian with the given inner product. We begin with the volume measure for the configuration space.

3. The functional measure and inner product

The YM Lagrangian in the $A_0 = 0$ gauge is given by

$$L = \int d^2x \left[ \frac{e^2}{2} \frac{\partial A^a_i}{\partial t} \frac{\partial A^a_i}{\partial t} - \frac{1}{2e^2} B^a B^a \right]$$

(2)

where $B^a = \frac{1}{2}\epsilon_{ij}(\partial_i A^a_j - \partial_j A^a_i + f^{abc} A^b_i A^c_j)$. By comparison of the kinetic term with the standard point-particle Lagrangian $L = \frac{1}{2}g_{\mu\nu}\dot{q}^\mu\dot{q}^\nu$, we see that the metric for the fields $A, \bar{A}$
is $ds_A^2 = \int d^2x \, \delta A_i^a \delta A_i^a$, with the corresponding volume $d\mu(A) = \prod_{x,a} dA^a(x)d\bar{A}^a(x)$, which is the standard Euclidean volume. From (1) we see that

$$\delta A = -D(\delta MM^{-1}) = - \left( \partial(\delta MM^{-1}) + [A, \delta MM^{-1}] \right)$$

$$\delta \bar{A} = \bar{D}(M^{\dagger -1} \delta M^{\dagger})$$

which gives

$$d\mu(A) = (\det D \bar{D}) \ d\mu(M, M^{\dagger})$$

where $d\mu(M, M^{\dagger})$ is the volume for the complex matrices $M, M^{\dagger}$, which is associated with the metric $ds^2_M = 8 \int \text{Tr}(\delta MM^{-1} M^{\dagger -1} \delta M^{\dagger})$. This is given by the highest order differential form $dV$ as $d\mu(M, M^{\dagger}) = \prod_x dV(M, M^{\dagger})$ where

$$dV(M, M^{\dagger}) \propto \epsilon_{a_1...a_n}(d\rho^{-1})_{a_1} \wedge ... \wedge (d\rho^{-1})_{a_n} \times \epsilon_{b_1...b_n}(M^{\dagger -1} dM^{\dagger})_{b_1} \wedge ... \wedge (M^{\dagger -1} dM^{\dagger})_{b_n}$$

where $n = \dim G = \dim SU(N) = N^2 - 1$. (There are some constant numerical factors which are irrelevant for our discussion.) The complex matrix $M$ can be written as $M = U\rho$, where $U$ is unitary and $\rho$ is hermitian. This is the matrix analogue of the modulus and phase decomposition for a complex number. Since gauge transformations act as $M \rightarrow M^{(g)} = gM$, we see that $U$ represents the gauge degrees of freedom and $\rho$ represents the gauge-invariant degrees of freedom on $M$. Substituting $M = U\rho$, (3) becomes

$$dV(M, M^{\dagger}) \propto \epsilon_{a_1...a_n}(d\rho^{-1} + \rho^{-1} d\rho)_{a_1} \wedge ... \wedge (d\rho^{-1} + \rho^{-1} d\rho)_{a_n} \times \epsilon_{b_1...b_n}(U^{-1} dU)_{b_1} \wedge ... \wedge (U^{-1} dU)_{b_n} \propto \epsilon_{a_1...a_n}(H^{-1} dH)_{a_1} \wedge ... \wedge (H^{-1} dH)_{a_n} d\mu(U)$$

Here $d\mu(U)$ is the standard group volume measure (the Haar measure) for $SU(N)$. Upon taking the product over all points, $d\mu(U)$ gives the volume of the entire gauge group (namely all $SU(N)$-valued functions) which we denote by $\text{vol}(G)$ and thus

$$d\mu(M, M^{\dagger}) = \prod_x dV(M, M^{\dagger}) \ \text{vol}(G) = d\mu(H) \ \text{vol}(G)$$

$$d\mu(H) = \prod_{x,a} \det r [d\varphi^a]$$

$$\det r \prod_a d\varphi^a = \epsilon_{a_1...a_n}(H^{-1} dH)_{a_1}...(H^{-1} dH)_{a_n}$$

We have parametrized $H$ in terms of the real parameters $\varphi^a$ and $H^{-1} dH = d\varphi^a r_{ak}(\varphi) t_k$. The volume element or the integration measure for the gauge-invariant configurations can
now be written as
\[ \frac{d\mu(A)}{vol(G_*)} = \frac{[dA_zd\bar{A}_z]}{vol(G_*)} \]
\[ = (\det D_zD_{\bar{z}}) \frac{d\mu(M,M^\dagger)}{vol(G_*)} = (\det D\bar{D})d\mu(H) \]  
(9)
where we have used (7). The problem is thus reduced to the calculation of the determinant of the two-dimensional operator $D\bar{D}$. This is well known [8]. The simplest way to evaluate this is to define $\Gamma = \log \det D\bar{D}$, which gives
\[ \frac{\delta\Gamma}{\delta A^a} = -i \text{Tr} \left[ \bar{D}^{-1}(x,y)T^a \right]_{y \to x} \]  
(10)
\[ (T^a)_{mn} = -if^a_{mn} \] are the generators of the Lie algebra in the adjoint representation. The coincident-point limit of $D^{-1}(x,y)$ is singular and needs regularization. With a gauge-invariant regulator, one finds
\[ \text{Tr} \left[ \bar{D}^{-1}_{\text{reg}}(x,y)T^a \right]_{y \to x} = \frac{2c_A}{\pi} \text{Tr} \left[ (A - M^\dagger\partial M^\dagger)T^a \right] \]  
(11)
where $c_A\delta^{ab} = f^{amn}f^{bmn}$; it is equal to $N$ for $SU(N)$. Using this result in (10) and integrating we get
\[ (\det D\bar{D}) = \left[ \frac{\det' \partial\bar{\theta}}{\int d^2x} \right]^{\text{dim}G} \text{exp} \left[ 2c_A \mathcal{S}(H) \right] \]  
(12)
\[ \mathcal{S}(H) \] is the Wess-Zumino-Witten (WZW) action for the hermitian matrix field $H$ given by
\[ \mathcal{S}(H) = \frac{1}{2\pi} \int \text{Tr}(\partial H\bar{\partial}H^{-1}) + \frac{i}{12\pi} \int e^{\mu\alpha\nu} \text{Tr}(H^{-1}\partial_\mu HH^{-1}\partial_\nu HH^{-1}\partial_\alpha H) \]  
(13)
We can now write the inner product for states $|1\rangle$ and $|2\rangle$, represented by the wavefunctions $\Psi_1$ and $\Psi_2$, as
\[ \langle 1|2 \rangle = \int d\mu(H)e^{2c_A \mathcal{S}(H)} \Psi_1^*\Psi_2 \]  
(14)
4. **Transforming the Hamiltonian**

The next step is the change of variables in the Hamiltonian. However, there is some further simplification we can do before taking up the Hamiltonian. We would expect the wavefunctions to be functionals of the matrix field $H$, but actually we can take them to be functionals of the current of the WZW model (13) given by $J = (c_A/\pi)\partial_z H H^{-1}$. First of all we notice that the Wilson loop operator can be constructed from $J$ alone as
\[ W(C) = \text{Tr}P e^{-\oint C(Adz + A\bar{d}z)} = \text{Tr}P e^{(\pi/c_A)\oint C J} \]  
(15)
Since the Wilson loop operator can provide a complete description of gauge-invariant observables, it is sufficient to take wavefunctions to be functions of $J$. There is also a conformal theory argument to show that it is sufficient to consider only functions of $J_{[1,2]}$.

This means that we can transform the Hamiltonian $H = T + V$ to express it in terms of $J$ and functional derivatives with respect to $J$. This is achieved by the chain rule of differentiation

$$ T \Psi = \frac{e^2}{2} \int E_i^a E_i^a \Psi $$

$$ = \frac{e^2}{2} \left[ \int_{x,u} \frac{\delta J^a(u)}{\delta J^a(x)} \delta \Psi + \int_{x,u,v} \frac{\delta J^a(u)}{\delta J^a(x)} \frac{\delta J^b(v)}{\delta J^b(x)} \Psi \right] $$

$$ V = \frac{1}{2e^2} \int B^a B^a $$ (16)

Regularization is important in calculating the coefficients of the two terms in $T$. Carrying this out we find

$$ T = m \left[ \int J^a(u) \frac{\delta}{\delta J^a(u)} + \int \Omega_{ab}(u,v) \frac{\delta}{\delta J^a(u)} \frac{\delta}{\delta J^b(v)} \right] $$ (17)

$$ V = \frac{\pi}{mcA} \int \bar{\partial} J^a(\vec{x}) \bar{\partial} J^a(\vec{x}) $$ (18)

where $m = e^2 c_A / 2\pi$ and

$$ \Omega_{ab}(u,v) = \frac{c_A \delta_{ab}}{\pi^2 (u-v)^2} - i \frac{\int_{ab} f^c(v)}{\pi (u-v)} $$ (19)

The first term in $T$ shows that every power of $J$ in the wavefunction gives a value $m$ to the energy, suggesting the existence of a mass gap. The calculation of this term involves exactly the same quantity as in (10) and with the same regulator leads to (17), i.e.,

$$ - \frac{e^2}{2} \int d^2 y \frac{\delta^2 J^a(x)}{\delta A^b(y) \delta A^b(y)} = \frac{e^2 c_A}{2\pi} M_{\alpha \beta} \text{Tr} \left[ T^m \hat{D}^{-1}(y,x) \right]_{y \to x} $$

$$ = m \ J^a(x) $$ (20)

Finally, (17, 18) (with regularizations taken account of) give a self-adjoint Hamiltonian which, as I mentioned before, is a nice consistency check.

5. An intuitive argument

The next step is to solve the Schrödinger equation, at least for the vacuum state and the low lying excited states. However, before taking this up, I shall give a short intuitive argument for the existence of a mass gap.
First of all notice that the total volume of the configuration space as given by (9) is finite, modulo regularization of the Laplacian $\bar{\partial}$, and can be written as

$$\int [dA_2 dA_3] = \left( \frac{\det(\partial \bar{\partial})}{\int d^2 x} \right)^{-\dim G} \text{vol}(G^*)$$

In contrast to this, the corresponding result for an Abelian theory (which has $c_A = 0$) is infinite. (If we make a mode decomposition of $H$ or $\varphi^a$ over the eigenmodes of the Laplacian $\partial \bar{\partial}$, the integration over the amplitude of each mode is finite for the nonabelian case because of the exponential; the divergence arises from the infinity of modes and can be regularized by truncation to a finite number of modes. For the Abelian case, the integration for each mode is divergent.) This result is encouraging as regards the question of the mass gap. One can go further and make a slightly better argument. The crucial ingredient is the measure of integration in the inner product (14). Writing $\Delta E$, $\Delta B$ for the root mean square fluctuations of the electric field $E$ and the magnetic field $B$, we have, from the canonical commutation rules $[E^a_i, A^b_j] = -i \delta_{ij} \delta^{ab}$, $\Delta E \Delta B \sim k$, where $k$ is the momentum variable. This gives an estimate for the energy

$$\mathcal{E} = \frac{1}{2} \left( \frac{e^2 k^2}{\Delta B^2} + \frac{\Delta B^2}{e^2} \right)$$

For low lying states, we minimize $\mathcal{E}$ with respect to $\Delta B^2$, $\Delta B^2_{\text{min}} \sim e^2 k$, giving $\mathcal{E} \sim k$. This corresponds to the standard photon. For the nonabelian theory, this is inadequate since $\langle \mathcal{H} \rangle$ involves the factor $e^{2c_A S(H)}$. In fact,

$$\langle \mathcal{H} \rangle \sim \int d\mu(H) e^{2c_A S(H)} \frac{1}{2}(e^2 E^2 + B^2/e^2)$$

In terms of $B$, the WZW action goes like $S(H) \approx -\frac{(c_A/\pi)^{1/2}}{2} \int B(1/k^2)B + ...$; we thus see that $B$ follows a Gaussian distribution of width $\Delta B^2 \approx \pi k^2/c_A$, for small values of $k$. This Gaussian dominates near small $k$ giving $\Delta B^2 \sim k^2(\pi/c_A)$. In other words, even though $\mathcal{E}$ is minimized around $\Delta B^2 \sim k$, probability is concentrated around $\Delta B^2 \sim k^2(\pi/c_A)$. For the expectation value of the energy, we then find $\mathcal{E} \sim e^2 c_A/2\pi + \mathcal{O}(k^2)$. Thus the kinetic term in combination with the measure factor $e^{2c_A S(H)}$ could lead to a mass gap of order $e^2 c_A$. The argument is not rigorous, but captures the essential physics as we shall see in a moment.

### 6. The vacuum wavefunction

Let us now consider the eigenstates of the theory. The vacuum wavefunction is presumably the simplest to calculate. Ignoring the potential term $V$ for the moment, since
$T$ involves derivatives, we see immediately that the ground state wavefunction for $T$ is $\Phi_0 = 1$. This may seem like a trivial statement, but the key point is that it is normalizable with the inner product (14); in fact, the normalization integral is (21). Starting with this, we can solve the Schrödinger equation taking $\Psi_0$ to be of the form $\exp(P)$, where $P$ is a perturbative series in the potential term $V$ (equivalent to a $1/m$-expansion). We then get

$$P = - \frac{\pi}{m^2 c_A} \text{Tr} \int : \bar{\partial} J \partial J :$$

$$- \left( \frac{\pi}{m^2 c_A} \right)^2 \text{Tr} \int [ : \bar{\partial} J (D \bar{\partial}) \partial J + \frac{1}{3} \bar{\partial} J [J, \bar{\partial}^2 J] : ]$$

$$- 2 \left( \frac{\pi}{m^2 c_A} \right)^3 \text{Tr} \int : \bar{\partial} J (D \bar{\partial})^2 \partial J + \frac{2}{9} [D \bar{\partial} J, \partial J] \bar{\partial}^2 J + \frac{8}{9} [D \bar{\partial}^2 J, J] \bar{\partial}^2 J$$

$$- \frac{1}{6} [J, \bar{\partial} J, [\bar{\partial} J, \bar{\partial} J]] - \frac{2}{9} [J, \bar{\partial} J, [J, \bar{\partial} J]] + O\left( \frac{1}{m^5} \right)$$

(24)

where $Dh = (c_A/\pi) \partial h - [J, h]$. The series is naturally grouped as terms with $2 J$’s, terms with $3 J$’s, etc. These terms can be summed up; for the $2J$-terms we find

$$\Psi_0 = \exp \left[ - \frac{1}{2 e^2} \int_{x,y} B_a(x) \left[ \frac{1}{m + \sqrt{m^2 - \nabla^2}} \right]_{x,y} B_a(y) + O(3J) \right]$$

(25)

The first term in (25) has the correct (perturbative) high momentum limit, viz.,

$$\Psi_0 \approx \exp \left[ - \frac{1}{2 e^2} \int_{x,y} B_a(x) \left[ \frac{1}{\sqrt{-\nabla^2}} \right]_{x,y} B_a(y) + O(3J) \right]$$

(26)

Thus although we started with the high $m$ (or low momentum) limit, the result (25) does match onto the perturbative limit. The higher terms are also small for the low momentum limit.

We can now use this result to calculate the expectation value of the Wilson loop operator; for the fundamental representation, it is given by

$$\langle W_F(C) \rangle = \text{constant} \exp \left[ - \sigma A_C \right]$$

$$\sqrt{\sigma} = e^2 \sqrt{N^2 - 1} / 8 \pi$$

(27)

where $A_C$ is the area of the loop $C$. $\sigma$ is the string tension. This is a prediction of our analysis starting from first principles with no adjustable parameters. Notice that the dependence on $e^2$ and $N$ is in agreement with large-$N$ expectations, with $\sigma$ depending
only on the combination $e^2N$ as $N \to \infty$. (The first correction to the large-$N$ limit is negative, viz., $-(e^2N)/2N^2\sqrt{8\pi}$ which may be interesting in the context of large-$N$ analyses.) Formula (27) gives the values $\sqrt{\sigma/e^2} = 0.345, 0.564, 0.772, 0.977$ for $N = 2, 3, 4, 5$.

There are estimates for $\sigma$ based on Monte Carlo simulations of lattice gauge theory. The most recent results for the gauge groups $SU(2), SU(3), SU(4)$ and $SU(5)$ are $\sqrt{\sigma/e^2} = 0.335, 0.553, 0.758, 0.966$ [11]. We see that our result agrees with the lattice result to within $\sim 3\%$.

One might wonder at this stage why the result is so good when we have not included the 3$J$- and higher terms in the wavefunction. This is basically because the string tension is determined by large area loops and for these, it is the long distance part of the wavefunction which contributes significantly. In this limit, the 3$J$- and higher terms in (25) are small compared to the quadratic term.

We have summed up the 3$J$-terms as well. Generally, one finds that $P$, when expressed in terms of the magnetic field, is nonlocal even in a $(1/m)$-expansion, contrary to what one might expect for a theory with a mass. This is essentially due to gauge invariance combined with our choice of $A_0 = 0$; it has recently been shown that a similar result holds for the Schwinger model [12].

7. **Magnetic mass**

I shall now briefly return to the magnetic mass. From the expression (17) we see immediately that for a wavefunction which is just $J^a$, we have the exact result $T J^a = m J^a$. When the potential term is added, $J^a$ is no longer an exact eigenstate; we find

$$(T + V) J^a = \sqrt{m^2 - \nabla^2} J^a + \cdots$$

(28)

showing how the mass value is corrected to the relativistic dispersion relation.

Now $J^a$ may be considered as the gauge-invariant definition of the gluon. This result thus suggests a dynamical propagator mass $m = e^2c_A/2\pi$ for the gluon. A different way to see this result is as follows. We can expand the matrix field $J$ in powers of $\varphi_a$ which parametrizes $H$, so that $J \simeq (c_A/\pi)\partial_t \varphi_a t_a$. This is like a perturbation expansion, but a resummed or improved version of it, where we expand the WZW action in $\exp(2c_A S(H))$ but not expand the exponential itself. The Hamiltonian can then be simplified as

$$H \simeq \frac{1}{2} \int_x \left[ -\frac{\delta^2}{\delta \varphi_a^2(x)} + \phi_a(x)(m^2 - \nabla^2)\phi_a(x) \right] + \cdots$$

(29)

where $\phi_a(k) = \sqrt{c_A k k/(2\pi m)} \varphi_a(k)$, in momentum space. In arriving at this expression we have expanded the currents and also absorbed the WZW-action part of the measure into the definition of the wavefunctions, i.e., the operator $e^{c_A S(H)}$ acts on $\tilde{\Psi} = e^{c_A S(H)}\Psi$. The above
equation shows that the propagating particles in the perturbative regime, where the power series expansion of the current is appropriate, have a mass $m = e^2 c_A / 2\pi$. This value can therefore be identified as the magnetic mass of the gluons as given by this nonperturbative analysis.

For $SU(2)$ our result is $m \approx 0.32e^2$. There have been two lattice calculations of the propagator mass of gluons for $SU(2)$; the values are $m \approx 0.35e^2$ and $0.46e^2$ [13, 14]. There is reasonable evidence for a ‘constituent gluon’ picture for glueballs in $YM_3$ from lattice analysis, so another suggestion has been to extract a constituent mass for a gluon and interpret it as the magnetic mass [13]. This gives a value $m \approx 0.31e^2 - 0.40e^2$. Considering the difficulties of a lattice estimate and the variance within these calculations, we cannot draw any definite conclusion in comparing with our analysis.

8. **Excited states**

Eventhough $J$ is useful as a description of the gluon, it is not a physical state. This is because of an ambiguity in our parametrization [4]. Notice that the matrices $M$ and $M\tilde{V}(\tilde{z})$ both give the same $A, \tilde{A}$, where $\tilde{V}(\tilde{z})$ only depends on $\tilde{z}$ and not $z$. Since we have the same potentials, physical results must be insensitive to this redundancy in the choice of $M$; in other words, physical wavefunctions must be invariant under $M \rightarrow M\tilde{V}(\tilde{z})$. $J$ is not invariant; we need at least two $J$’s to form an invariant combination. An example is

$$\Psi_2 = \int_{x,y} f(x, y) [\tilde{\partial} J_\alpha(x)(H(x, \tilde{y})H^{-1}(y, \tilde{y}))_{ab} \tilde{\partial} J_\beta(y)]$$  \hspace{1cm} (30)

This is not an eigenstate of the Hamiltonian. Since there are two $J$’s we should expect at least a mass of $2m$, but beyond that it is difficult to say anything very conclusive, see however [4].

This work was supported in part by a grant from the National Science Foundation.

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