Bäcklund transformations for Gelfand–Dickey Flows, revisited

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We construct Bäcklund transformations (BT) for the Gelfand–Dickey hierarchy (GD−hierarchy) on the space of nth order differential operators on the line. Suppose \( L = \partial^n - \sum_{i=1}^{n-1} u_i \partial_i^{n-1} \) is a solution of the jth GDn flow. We prove the following results:

1. There exists a system \( \text{BT} \) of non-linear ordinary differential equations for \( h : \mathbb{R}^2 \to \mathbb{C} \) depending on \( u_1, \ldots, u_{n-1} \) in \( x \) and \( t \) variables such that \( \tilde{L} = (\partial + h)^{-1}L(\partial + h) \) is a solution of the jth GDn flow if and only if \( h \) is a solution of \( \text{BT} \) for some parameter \( k \). Moreover, coefficients of \( \tilde{L} \) are differential polynomials of \( u \) and \( h \). We say such \( \tilde{L} \) is obtained from a BT with parameter \( k \) from \( L \).

2. \( \text{BT} \) is solvable.

3. There exists a compatible linear system for \( \phi : \mathbb{R}^2 \to \mathbb{C} \) depending on a parameter \( k \), such that if \( \phi_1, \ldots, \phi_{n-1} \) are linearly independent solutions of this linear system then \( h := \ln W(\phi_1, \ldots, \phi_{n-1}) \) is a solution of \( \text{BT} \) and \( (\partial + h)^{-1}L(\partial + h) \) is a solution of the jth GDn flow, where \( W(\phi_1, \ldots, \phi_{n-1}) \) is the Wronskian. Moreover, these give all solutions of \( \text{BT} \).

4. We show that the BT for the GDn hierarchy constructed by M. Adler is our BT with parameter \( k = 0 \).

5. We construct a permutability formula for our BTs and infinitely many families of explicit rational solutions and soliton solutions.

Keywords: Backlund transformations; Gelfand–Dickey flow; permutability formula.

1. Introduction

The jth flow of the Gelfand–Dickey (GDn) hierarchy (cf. [1]) is the following evolution equation

\[
\frac{\partial L}{\partial t_j} = [(L^+)_{ij}, L]
\]

on \( \{L = \partial^n - \sum_{i=1}^{n-1} u_i \partial_i^{n-1} \mid u_i \in C^\infty(\mathbb{R}, \mathbb{C}), 1 \leq i \leq n-1\} \), where \( (L^+)_{ij} \) is the differential operator component of the pseudo-differential operator \( L^+ \), \( \partial = \partial_t \) and \( j \not\equiv 0 \pmod{n} \). Bäcklund transformations (BTs) for the KdV (i.e., GDn) hierarchy are well-known (cf. [2, 3]). Adler in [4] used the Miura transform to construct BTs for (1.1) as follows: If \( L = \partial^n - \sum_{i=1}^{n-1} u_i \partial_i^{n-1} = (\partial - q_n) \cdots (\partial - q_1) \) is a solution of (1.1)
such that \( q = \sum_{i=1}^{n-1} q_i e_i \) is a solution of the \( j \)th \( n \times n \) mKdV flow, then \( \tilde{L} := (\partial - q_{n-1}) \cdots (\partial - q_1)(\partial - q_n) \) is again a solution of \( (1.1) \). Note that the operators \( \tilde{L} \) and \( L \) in Adler’s Theorem are related by \( \tilde{L} = (\partial - q_n)^{-1} L(\partial - q_n) \).

Assume \( L = \partial^n - \sum_{i=1}^{n-1} u_i \partial^{i-1} \) is a solution of the \( j \)th GD hierarchy. We prove the following results in this paper:

1. There exist differential polynomials \( \xi_i(u, h) \) and \( \eta_{ij}(u, h) \) in \( x \) variable such that given any constant \( k \in \mathbb{C} \), the following system,

\[
(\text{BT})_{h,k} \begin{cases}
 h^{(n-1)} = \xi_i(u, h) - k, \\
 h_i = \eta_{ij}(u, h),
\end{cases}
\]

is solvable for \( h \).

2. \( \tilde{L} = \partial^n - \sum_{i=1}^{n-1} u_i \partial^{i-1} := (\partial_i + h)^{-1} L(\partial_i + h) \) is a solution of \( (1.1) \) if and only if \( h \) is a solution of \( (\text{BT})_{h,k} \) for some constant \( k \in \mathbb{C} \) (we call \( L \mapsto \tilde{L} \) a Bäcklund transform (BT) of \( L \) with parameter \( k \)). Moreover, there exist differential polynomials \( s_i(u, h) \) such that \( u_i = u_i + s_i(u, h) \) for \( 1 \leq i \leq n - 1 \).

3. We give a Permutability formula for our BTs.

4. There exist differential polynomials \( p_i(u, \lambda) \) for \( 1 \leq i \leq n \) such that the following linear system for \( \phi : \mathbb{R}^2 \to \mathbb{C} \),

\[
\begin{cases}
 L\phi = \lambda \phi, \\
 \phi_1 = \sum_{i=1}^{n} p_i(u, \lambda) \phi_x^{(i-1)},
\end{cases}
\]

is solvable for all parameter \( \lambda \in \mathbb{C} \). Moreover, let \( \phi_1, \ldots, \phi_{n-1} \) be linearly independent solutions of \( (1.3) \) with \( \lambda = k \), and \( W(\phi_1, \ldots, \phi_{n-1}) \) the Wronskian of \( \phi_1, \ldots, \phi_{n-1} \). Then \( h = (\ln W(\phi_1, \ldots, \phi_{n-1}))_x \) is a solution of \( (\text{BT})_{h,k} \) \( (1.2) \) and \( \tilde{L} = (\partial + h)^{-1} L(\partial + h) \) is a solution of \( (1.1) \). This gives a Darboux transform (DT) for the \( j \)th GD hierarchy. Moreover, this construction gives all solutions of \( (1.2) \).

5. We prove that Adler’s BT is our BT with parameter \( k = 0 \).

This paper is organized as follows: We give a brief review of various KdV-type hierarchies in Section 2, construct Bäcklund and Darboux transforms for the GD hierarchy in Section 3, give a Permutability formula for these BTs and a relation between BTs and scaling transform in Section 4. We explain the relation between our BT and Adler’s BT in the last section.

### 2. Various KdV type hierarchies

We need to use Bäcklund transformations for the \( n \times n \) KdV hierarchy and the equivalence between the \( n \times n \) KdV and the \( \tilde{A}^{(1)}_{n-1} \)-KdV hierarchies to construct BTs for the \( \tilde{A}^{(1)}_{n-1} \)-KdV hierarchy. We also need the Miura transform between the \( n \times n \) mKdV and the GD hierarchy. In this section, we review the constructions of these KdV-type hierarchies.

Let \( B_{n}^{+} \), \( B_{n}^{-} \), \( T_{n} \) and \( N_{n}^{+} \) denote the sub-algebras of upper triangular, lower triangular, diagonal and strictly upper triangular matrices in \( sl(n, \mathbb{C}) \), respectively, and \( N_{n}^{+} \) the subgroup of upper triangular
matrices in $SL(n, \mathbb{C})$ with 1 on the diagonal entries. Let

$$\mathcal{L} = \left\{ \xi(\lambda) = \sum_{i \leq m_0} \xi_i \lambda^i \in \mathfrak{sl}(n, \mathbb{C}), m_0 \text{ an integer} \right\},$$

$$\mathcal{L}_+ = \left\{ \xi(\lambda) = \sum_{i \geq 0} \xi_i \lambda^i \in \mathcal{L} \right\}, \quad \mathcal{L}_- = \left\{ \xi(\lambda) = \sum_{i < 0} \xi_i \lambda^i \in \mathcal{L} \right\}.$$  

Then $\mathcal{L}_+, \mathcal{L}_-$ are Lie subalgebras of $\mathcal{L}$ and $\mathcal{L} = \mathcal{L}_+ \oplus \mathcal{L}_-$ as a direct sum of linear subspaces. Given $\xi = \sum_i \xi_i \lambda^i \in \mathcal{L}$, we will use $\xi_{\pm}$ to denote the projection of $\xi$ onto $\mathcal{L}_\pm$ w.r.t. $\mathcal{L} = \mathcal{L}_+ \oplus \mathcal{L}_-$, i.e.,

$$\xi_+ = \sum_{i \geq 0} \xi_i \lambda^i, \quad \xi_- = \sum_{i < 0} \xi_i \lambda^i.$$  

### 2.1 A commuting hierarchy of flows on $C^\infty(\mathbb{R}, B_n^+)$

Let

$$J = e_{1n} \lambda + b, \quad b = \sum_{i=1}^{n-1} e_{i+1,i}.$$  

(2.1)

Given $y \in C^\infty(\mathbb{R}, B_n^+)$, a direct computation shows that (cf. [5]) there exists a unique $Q(y, \lambda) \in \mathcal{L}$ such that $Q(0, \lambda) = J$ and

$$\begin{align*}
\left\{ \partial_x + J + y, Q(y, \lambda) \right\} &= 0, \\
Q(y, \lambda) &\text{ is conjugate to } J.
\end{align*}$$  

(2.2)

Write

$$Q'(y, \lambda) = \sum_{i \in \mathbb{Z} \setminus \{0\}} Q_{ji}(y) \lambda^i.$$  

(2.3)

Then

$$y_j = [\partial_x + b + y, Q_{j0}(y)]$$  

(2.4)

is a flow on $C^\infty(\mathbb{R}, B_n^+)$ for each $j \not\equiv 0 \pmod{n}$. These flows commute.

The group $C^\infty(\mathbb{R}, N_n^+)$ acts on $C^\infty(\mathbb{R}, B_n^+)$ by gauge transformations, $\Delta(\partial_x + b + y) \Delta^{-1} = \partial_x + b + \Delta \ast y$, i.e.,

$$\Delta \ast y = \Delta(b + y) \Delta^{-1} - \Delta_x \Delta^{-1} - b.$$  

(2.5)

It is known that (2.4) is invariant under the gauge action of $C^\infty(\mathbb{R}, N_n^+)$. Hence if $\Sigma$ is a linear subspace of $B_n^+$ such that $C^\infty(\mathbb{R}, \Sigma)$ is a cross section of the $C^\infty(\mathbb{R}, N_n^+)$-action, then (2.4) induces a quotient flow.
on $C^\infty(\mathbb{R}, \Sigma)$ by projecting down along the orbits. The $\hat{A}_{n-1}^{(1)}$-KdV and the $n \times n$ KdV flows are induced quotient flows on two different cross sections.

2.2 The $\hat{A}_{n-1}^{(1)}$-KdV hierarchy [6]

Proposition 2.1 ([6]) Let

$$V_n = \bigoplus_{i=1}^{n-1} C e_i,$$

and $v \in C^\infty(\mathbb{R}, B^n)$. Then there exist a unique $N_n^+$-valued differential polynomial $\Delta$ and a differential polynomial $u$ in $v$ such that $u = \Delta \ast v$, i.e.,

$$\Delta(\partial_x + J + v) \Delta^{-1} = \partial_x + J + u. \quad (2.6)$$

In other words, $C^\infty(\mathbb{R}, V_n)$ is a cross section of the $C^\infty(\mathbb{R}, N_n^+)$-action. So (i) there exists a $N_n^+$-valued differential polynomial $\eta_j$ of $u$ such that $[\partial_x + b + u, Q_j(u) + \eta_j(u)]$ is in $C^\infty(\mathbb{R}, V_n)$, (ii) flow (2.4) induces a quotient flow on $C^\infty(\mathbb{R}, V_n)$,

$$ut_j = [\partial_x + b + u, Q_j(u) + \eta_j(u)] \quad (2.7)$$

which is called the $j$th $\hat{A}_{n-1}^{(1)}$-KdV flow. Moreover, $u : \mathbb{R}^2 \to V_n$ is a solution of (2.7) if and only if

$$[\partial_x + J + u, \partial_j + Z_j(u, \lambda)] = 0 \quad (2.8)$$

for all parameter $\lambda \in \mathbb{C}$, where

$$Z_j(u, \lambda) = (Q^j(u, \lambda))_+ + \eta_j(u). \quad (2.9)$$

Equation (2.8) is the Lax pair of the solution $u$ of the $j$th $\hat{A}_{n-1}^{(1)}$-KdV flow.

Definition 2.2 Let $\mathcal{O}$ be a connected open subset of $\mathbb{C}$. A map $E : \mathbb{R}^2 \times \mathcal{O} \to GL(n, \mathbb{C})$ is called a frame of the solution $u$ of the $j$th $\hat{A}_{n-1}^{(1)}$-KdV flow if $E(x, t, \lambda)$ is holomorphic for $\lambda \in \mathcal{O}$ and satisfies

$$E^{-1} E_x = J + u, \quad E^{-1} E_t = Z_j(u, \lambda).$$

Theorem 2.3 ([6]) $u = \sum_{i=1}^{n-1} u_i e_i$ is a solution of the $j$th $\hat{A}_{n-1}^{(1)}$-KdV flow (2.7) if and only if $L_u = \partial^n - \sum_{i=1}^{n-1} u_i \partial^{-i}$ is a solution of the $j$th GD flow.

Hence the GD and the $\hat{A}_{n-1}^{(1)}$-KdV hierarchies are the same, and (2.8) is a matrix valued Lax pair for the GD hierarchy.
2.3 The \( n \times n \) KdV hierarchy [7]

Let \( \alpha = e^{2\pi i} \), and

\[
\Lambda = \sum_{k=1}^{n-1} \frac{1 - \alpha^k}{1 - \alpha} e_{k,k+1}, \quad W_n = \bigoplus_{j=1}^{n-1} C \Lambda^j.
\]

It was proved in [5] that \( C^\infty(\mathbb{R}, W_n) \) is a cross section of the \( C^\infty(\mathbb{R}, N^+_n) \)-action on \( C^\infty(\mathbb{R}, B^+_n) \). Hence there is a \( N^+_n \)-valued differential polynomial \( \bar{\eta}_j(u) \) such that \( [\partial_s + b + \xi, Q_{j,0}(\xi) + \bar{\eta}_j(\xi)] \) is in \( C^\infty(\mathbb{R}, W_n) \), where \( Q_{j,0} \) is as defined in (2.3). The corresponding quotient flow on \( C^\infty(\mathbb{R}, W_n) \) is the \( jth n \times n \) KdV flow,

\[
\xi_j = [\partial_s + b + \xi, Q_{j,0}(\xi) + \bar{\eta}_j(\xi)]. \tag{2.10}
\]

Moreover, \( \xi : \mathbb{R}^2 \to W_n \) is a solution of (2.10) if and only if

\[
[\partial_s + J + \xi, \partial_q + (Q^j(\xi, \lambda)) + \bar{\eta}_j(u)] = 0
\]

for all parameter \( \lambda \in \mathbb{C} \), where \( Q(\xi, \lambda) \) is defined by (2.2). Frames for a solution \( \xi \) of (2.10) is defined similarly as frames for the \( \Lambda^{(1)}_{n-1} \)-KdV hierarchy.

2.4 The \( n \times n \) mKdV hierarchy [6]

Let

\[
\mathcal{W}_+= \left\{ \xi(\lambda) = \xi_0 + \sum_{i>0} \xi^i \lambda^i \in L \mid \xi_0 \in B^-_n \right\},
\]

\[
\mathcal{W}_- = \left\{ \xi(\lambda) = \xi_0 + \sum_{i<0} \xi^i \lambda^i \in L \mid \xi_0 \in N^+_n \right\}.
\]

Then \( \mathcal{W}_+, \mathcal{W}_- \) are sub-algebras of \( L \) and \( L = \mathcal{W}_+ \oplus \mathcal{W}_- \) as linear subspaces. Let \( \pi_{\mathcal{W}_0} \) denote the projection of \( \mathfrak{s}(n, \mathbb{C}) \) onto \( B^-_n \) with respect to \( \mathfrak{s}(n, \mathbb{C}) = B^-_n \oplus N^+_n \), and let \( \pi_{+} \) denote the projection of \( L \) onto \( \mathcal{W}_+ \) with respect to \( L = \mathcal{W}_+ \oplus \mathcal{W}_- \). It was proved in [6] that if \( q \in C^\infty(\mathbb{R}, \mathcal{T}_n) \), then \([\partial_s + b + q, \pi_{\mathcal{W}_0}(Q_{j,0}(q))]\) lies in \( C^\infty(\mathbb{R}, \mathcal{T}_n) \). The \( jth n \times n \) mKdV flow is the following flow on \( C^\infty(\mathbb{R}, \mathcal{T}_n) \),

\[
q_j = [\partial_s + b + q, \pi_{\mathcal{W}_0}(Q_{j,0}(q))]. \tag{2.11}
\]

Moreover, \( q : \mathbb{R}^2 \to \mathcal{T}_n \) is a solution of (2.11) if and only if \([\partial_s + J + q, \partial_q + \pi_+(Q^j(q, \lambda))] = 0\) for all \( \lambda \in \mathbb{C} \). It is known that this hierarchy is equivalent to the Kupershmidt–Wilson hierarchy constructed in [8].

**Definition 2.4** Let \( \Psi : C^\infty(\mathbb{R}, B^+_n) \to C^\infty(\mathbb{R}, V_n) \) and \( \Gamma : C^\infty(\mathbb{R}, B^+_n) \to C^\infty(\mathbb{R}, N^+_n) \) be the maps defined by

\[
\Psi(v) = u, \quad \Gamma(u) = \Delta,
\]

respectively, where \( v, u \) and \( \Delta \) are related by (2.6).
Theorem 2.6 ([5]) Let \( q = \sum_{i=1}^{m} q_i e_i u \) is a solution of the \( j \)th \( n \times n \) mKdV flow (2.11), then \( u = \sum_{i=1}^{m} u_i e_i m \) is a solution of the \( j \)th \( \hat{A}_{n-1}^{(1)} \)-KdV flow (2.7) (this is the Miura transform).

Let \( \Gamma \) and \( \Psi \) be the maps defined in Definition 2.4, \( \Delta x = \Gamma(q, t) \) and \( K \) a frame of the solution \( q \) of (2.11). Then \( u(t) = \Psi(q(t), t) \) is a solution of (2.7) and \( E = K \Delta^{-1} \) is a frame of \( u \).

Corollary 2.7 If \( u \) is a solution of the \( j \)th \( \hat{A}_{n-1}^{(1)} \)-KdV flow (2.7), then there exists a unique \( \Delta : \mathbb{R}^2 \rightarrow N_n^+ \) such that \( \xi(t) = \Delta(t) \Delta^{-1} u(t) \) is a solution of the \( n \times n \) KdV flow (2.10). Moreover, \( \Delta(t) = \Gamma(\xi(t), t) \), where \( \Gamma \) is the gauge action defined by (2.5).

3. Bäcklund transformation for \( \hat{A}_{n-1}^{(1)} \)-KdV

We use Bäcklund transformations for the \( n \times n \) KdV hierarchy constructed in [7] to construct BTs for the \( \hat{A}_{n-1}^{(1)} \)-KdV hierarchy.

Theorem 3.1 ([7]) Let \( F \) be a frame of a solution \( \xi \) of the \( j \)th \( n \times n \) KdV flow (2.10), and \( k \in \mathbb{C} \) and \( c_0 \in \mathbb{C}^{n+1} \) constants. Let \( \xi = (y_1, \ldots, y_n) := F(x, t, k)^{-1} c_0 \) and \( h = -\frac{h_{n+1}}{h_n} \). Then there exists \( Y = h_0 + \sum_{i=1}^{n} h_i \lambda_i \) such that \( \tilde{F} = F(J + Y)^{-1} \) is a frame of a new solution of (2.10). Moreover, \( (ke_1 + b + Y)(y) = 0 \).

Theorem 3.2 Let \( E \) be a frame of a solution \( u \) of the \( j \)th \( \hat{A}_{n-1}^{(1)} \)-KdV flow (2.7), and \( k \in \mathbb{C} \) and \( c_0 \in \mathbb{C}^{n+1} \) constants. Let \( \xi = (\xi_1, \ldots, \xi_n) := E(x, t, k)^{-1} c_0 \) and \( h = -\frac{h_{n+1}}{h_n} \). Then there exists \( f \) of the form \( J + h_1 + N \) for some \( N_n^+ \)-valued map \( N \) such that \( \tilde{E} = Ef^{-1} \) is a frame of a new solution of (2.7). Moreover, \( f(x, t, k)(\xi(x, t)) = 0 \).

Proof. By Corollary 2.7, there exists a unique \( \Delta \in \mathcal{C}^\infty(\mathbb{R}, N_n^+) \) such that \( \xi = \Delta^{-1} u \) is a solution of (2.10). It follows from Theorem 2.6 that \( F = E \Delta \) is a frame of the solution \( \xi \) of (2.10). Let \( y = (y_1, \ldots, y_n) = F^{-1}(x, t, k) c_0 \) and \( h = -\frac{h_{n+1}}{h_n} \). By Theorem 3.1, there exists \( Y = J + h_1 + \sum_{i=1}^{n} h_i \lambda_i \) such that \( \tilde{F} = F(J + Y)^{-1} \) is a frame of a solution \( \xi \) of (2.10). By Theorem 2.6 (c), the \((n - 1)\)th and \(n\)th rows of \( \Delta \) are \((0, \ldots, 1)\) and \((0, \ldots, 1)\), respectively. Note that

\[
(\xi(x, t) = E^{-1}(x, t, k)^{-1}(c_0) = \Delta(x) F^{-1}(x, t, k) c_0 = \Delta(x, t) y(x, t).
\]

So we have \( y_{n-1} = \xi_{n-1} \) and \( y_n = \xi_n \). Hence \( h = -\frac{\xi_{n-1}}{\xi_n} \).
Let $\tilde{\Delta}(\cdot, t) = \Gamma(\tilde{\xi}(\cdot, t))$. By Theorem 2.6, $E = \tilde{F} \tilde{\Delta}^{-1}$ is a frame of a solution $\tilde{u}$ of (2.7). Note that

\[ \tilde{E} = \tilde{F} \tilde{\Delta}^{-1} = \Gamma(J + Y)^{-1} \tilde{\Delta}^{-1} = E\Delta(J + Y)^{-1} \tilde{\Delta}^{-1}. \]

So $f := \tilde{\Delta}(J + Y)\Delta^{-1}$ is of the form $J + hL_n + N$ and $\tilde{E} = Ef^{-1}$. Since $f = \tilde{\Delta}(J + Y)\Delta^{-1}$, $(ke_{lm} + b + Y)y = 0$, and $\xi = \Delta y$, we have $f(x, t, k)\xi(x, t) = 0$. \hfill \square

The following lemma can be proved by a direct computation.

**Lemma 3.3** Let $u = \sum_{i=1}^{n-1} u_ie_{in} E \in C^\infty(\mathbb{R}, GL(n, \mathbb{C}))$ and $\phi$ the first column of $E$. Then $E_\xi = E(J + u)$ if and only if $E = (\phi, \phi_1, \ldots, \phi_{n-1}^{(n)})$ and $\phi_{n}^{(n)} = \sum_{i=1}^{n-1} u_i\phi_{i}^{(n-1)} + \lambda\phi$, where $J = e_{i\xi} + b$ is defined by (2.1).

We call $y(h)$ a differential polynomial of order $k$ if $y$ is a polynomial of $h, h_1, \ldots, h_{\xi}^{(k)}$. Next we write down the entries of the map $f$ in Theorem 3.2 in terms of $u$ and $h$.

**Theorem 3.4** Let $E$ and $\tilde{E}$ be frames of solutions $u = \sum_{i=1}^{n-1} u_ie_{in}$ and $\tilde{u} = \sum_{i=1}^{n-1} \tilde{u}_ie_{in}$ of (2.7), respectively. Suppose $\tilde{E} = E(J + hL_n + N)^{-1}$ for some smooth $h : \mathbb{R}^2 \to \mathbb{C}$ and $N = (N_{ij}) : \mathbb{R}^2 \to N_n^\gamma$. Then we have:

(i) There exist differential polynomials $s_i(u, h)$ of order $(n - i)$ in $h$ such that

\[ \tilde{u}_i = u_i + s_i(u, h), \quad 1 \leq i \leq n - 1. \]  

(ii) Let $C_{ij} = \frac{\partial}{\partial u_j(u)}$. Then entries of $N$ are

\[
\begin{align*}
N_{ij} &= C_{j-1,i-1}h_{\xi}^{(j-1)}, & 1 \leq i < j < n, \\
N_{in} &= u_i + s_i(u, h) + C_{n-1,i-1}h_{\xi}^{(n-1)}, & 1 \leq i \leq n - 1.
\end{align*}
\]

(iii) $h$ satisfies

\[
\begin{align*}
h_{\xi}^{(n)} &= r_n(u, h), \\
h_i &= \eta_{nj}(u, h),
\end{align*}
\]

for some differential polynomials $r_n(u, h)$ of order $(n - 1)$ in $h$ and $\eta_{nj}(u, h)$ of order $j$ in $h$.

**Proof.** From $E^{-1}E_\xi = J + u$, $\tilde{E}^{-1}\tilde{E}_\xi = J + \tilde{u}$ and $E = \tilde{E}f$, we have

\[ f_\xi - f(J + u) + (J + \tilde{u})f = 0. \]

Compare the $n$th entry of (3.4) to get $u_{n-1} = h_{\xi} + \tilde{u}_{n-1} + (n - 1)h_{\xi}$, i.e.,

\[ \tilde{u}_{n-1} = u_{n-1} - nh_{\xi}. \]
Compare the \((n - 1, n)\)-entry of \((3.4)\) to get

\[
\tilde{u}_{n-2} = u_{n-2} - (u_{n-1})_x = \frac{(n - 3)n}{2} h_{xx} + nh_x.
\]

Use induction and compare the \((n - i, n)\)th entry of \((3.4)\) for \(1 \leq i \leq n - 1\) to see that \(\tilde{u}_{n-i} = u_{n-i} + s_{n-i}(u, h)\) for some differential polynomial \(s_{n-i}(u, h)\) of order \(i\) in \(h\). This proves (i).

Note that \(f = \lambda h + N\) implies the first column of \(f\) is \((h, 1, 0, \ldots, 0)^t\). Let \(\phi\) and \(\tilde{\phi}\) denote the first column of \(E\) and \(\tilde{E}\), respectively. By assumption, \(E = \tilde{E}\). So we have \(\phi = \tilde{\phi} + h\phi\). Lemma 3.3 implies that \(\tilde{\phi}^{(n)} = \lambda \phi + \sum_{i=1}^{n-1} \tilde{u}_i \phi^{(i-1)}\). Use \(\phi = \tilde{\phi} + h\phi\) to compute \(\phi^{(n)}\) and use \((3.1)\) to get \((3.2)\). This proves (ii).

The \(1\)th entry of the constant term of the left-hand side of \((3.4)\) (as a polynomial in \(\lambda\)) is \(h^{(n)} - r_{n}(u, h)\) for some differential polynomial \(r_{n}(u, h)\) of order \((n - 1)\) in \(h\). This proves \(h^{(n)} = r_{n}(u, h)\). Since \(E\) and \(\tilde{E}\) are frames of \(u\) and \(\tilde{u}\), we have \(E^{-1}E_t = Z_j(u, \lambda)\) and \(\tilde{E}^{-1}\tilde{E}_t = Z_j(\tilde{u}, \lambda)\), where \(Z_j\) is as in \((2.9)\). But \(E = \tilde{E}\) implies that

\[
f_t = fZ_j(u, \lambda) - Z_j(\tilde{u}, \lambda)f.
\]

Compare the \(11\)th entry of the constant term of \((3.5)\) to see that \(h_t = \eta_{n,j}(u, h)\) for some differential polynomial \(\eta_{n,j}(u, h)\) of order \(n\). This proves \((ii)\).

**Definition 3.5** Given \(u = \sum_{j=1}^{n-1} u_j e_j + h \in C^\infty(\mathbb{R}, \mathbb{C})\), let \(f_{u,h} = J + hI_n + N\), where \(N = (N_{ij})\) is as in \((3.2)\).

**Theorem 3.6** Let \(E\) and \(\tilde{E}\) be frames of solutions \(u\) and \(\tilde{u}\) of \((2.7)\), respectively. Suppose \(\tilde{E} = E_{f_{u,h}}^{-1}\) for some \(h\). Then there exist a constant \(k \in \mathbb{C}\) and a differential polynomial \(\xi_{u}(u, h)\) of order \((n - 2)\) in \(h\) such that

\[
\begin{align*}
\det(f_{u,h}(x, \lambda)) &= (-1)^{n-1}(\lambda - k), \\
r_{n}(u, h) &= (\xi_{u}(u, h))_x, \\
(BT)_{u,k} &= h^{(n-1)} = \xi_{u}(u, h) - k, \\
h_t &= \eta_{n,j}(u, h),
\end{align*}
\]

where \(r_{n}(u, h)\) and \(\eta_{n,j}(u, h)\) are as in \((3.3)\).

**Proof.** Since \(\text{tr}(J + u) = \text{tr}(J + \tilde{u}) = \text{tr}(Z(u, \lambda)) = \text{tr}(Z_j(\tilde{u}, \lambda)) = 0\), and \(d(\ln(\det(E))) = \text{tr}(E^{-1}dE)\), we see that both \(E(x, t, \lambda)\) and \(\tilde{E}(x, t, \lambda)\) are independent of \(x\) and \(t\). Hence \(\det(f_{u,h})\) is independent of \(x, t\). But \(\det(f_{u,h})\) is a degree one polynomial in \(\lambda\). So there exists a constant \(k \in \mathbb{C}\) such that \(\det(f_{u,h}(x, t, \lambda)) = (-1)^{n-1}(\lambda - k)\). This proves \((3.6)\). It follows from \((3.2)\) that we have \(\det(f_{u,h}) = (-1)^{n-1}(\lambda + h^{(n-1)} - \xi_{u}(u, h))\) for some differential polynomial \(\xi_{u}(u, h)\) of order \((n - 2)\). This proves

\[
h^{(n-1)} - \xi_{u}(u, h) = -k.
\]

By Theorem 3.4, \(h\) satisfies the second equation of \((3.8)\).
It remains to prove (3.7). Set \( f = f_{u,h} \). The proof of Theorem 3.2(ii) implies that

\[
\text{tr}(J) = \text{tr}(J + u) = \text{tr}(J + \tilde{u})f = (h^{(n)} - r_n(u,h))e_{1n}.
\]  

(3.9)

Let \( w = h^{(n)} - r_n(u,h) \). Use \( \ln \det(y) \) to obtain

\[
\ln \det(f) = \ln(f^{-1}f) = \ln((J + u) - f^{-1}(J + \tilde{u})f + f^{-1}we_{1n}).
\]

Since \( \text{tr}(J + u) = \text{tr}(J + \tilde{u}) = 0 \),

\[
\ln \det(f) = \ln(f^{-1}we_{1n}) = wf^{(n)},
\]

where \( f^{(n)} \) is the \((n1)\)th entry of the inverse of \( f \). By definition of \( f \), we see that \( f^{(n)} \) is equal to \((-1)^{n-1}(\det f)^{-1} \). This proves (3.7). \( \square \)

As a consequence of Theorems 3.2, 3.4 and 3.6 we obtain a DT for the \( \hat{A}_{n-1}^{(1)} \)-KdV flows.

THEOREM 3.7 Let \( E, u, k, c_0, \xi = (\xi_1, \ldots, \xi_n)' \) be as in Theorem 3.2. If \( \xi_n \neq 0 \), then

(i) \( h = -\frac{\xi_{n-1}}{\xi_n} \) is a solution of (BT)\(_{u,k} \), i.e., (3.8),

(ii) \( \tilde{E} = Ef_{u,h}^{-1} \) is a frame of a new solution \( \tilde{u} = \sum_{i=1}^{n-1} \tilde{u}_i e_{in} \) of (2.7), where \( \tilde{u}_i \)'s are given by (3.1),

(iii) \( f_{u,h}(x, t, k) \xi(x, t) = 0 \) for all \( x, t \).

THEOREM 3.8 Let \( u \) be a solution of (2.7), and \( k \in \mathbb{C} \) a constant. Then

(i) all solutions of (BT)\(_{u,k} \) can be obtained by Theorem 3.7.

(ii) (BT)\(_{u,k} \) is solvable.

Proof. Let \( E \) be the frame of a solution \( u \) of (2.7) such that \( E(0, 0, \lambda) = I_n \) for all \( \lambda \in \mathbb{C} \). Let \( \mathbf{e} = (c_1, \ldots, c_{n-1}, 1) \) and \( v = (v_1, \ldots, v_n) := E^{-1}(\cdot, \cdot, k)(\mathbf{e}) \). Then \( v_n \neq 0 \) in an open neighbourhood of \((0, 0)\) in \( \mathbb{R}^2 \). It also follows from \( E^{-1}E_\lambda = J + u \) that

\[
v_\lambda = -(e_{1n}k + b + u)v.
\]  

(3.10)

Let \( h = -v_{n-1}/v_n \), and \( \Phi : \mathbb{C}^{n-1} \to \mathbb{C}^{n-1} \) the map defined by

\[
\Phi(\mathbf{e}) = (h(0, 0), h_1(0, 0), \ldots, h_{n-2}(0, 0)).
\]

We claim that \( \Phi \) is onto. Let \( \phi = \frac{1}{v_n} \), then \( h = \phi_{n-1} \). Equation (3.10) implies that

\[
\left\{\begin{array}{l}
\phi_\lambda = -(ke_{1n} + b + u + h_\lambda)\phi, \\
\phi(0, 0) = -\mathbf{e}.
\end{array}\right.
\]
Compare the entries of both sides, we have

\[(\phi_1)_x = -\phi_1 \phi_{n-1} + k + u_1,\]
\[(\phi_i)_x = -\phi_{i-1} - \phi_i \phi_{n-1} + u_i, \quad 2 \leq i \leq n-1.\]

This shows that given any \((c_1, \ldots, c_{n-1})\) there exists a solution \(h\) of (3.8) satisfying

\[h(0, 0) = -c_{n-1}, \quad h_x(0, 0) = c_{n-2} + u_{n-1}(0, 0) - c_{n-1}^2, \quad \cdots\]

This proves (i). Since we have constructed solutions with all initial data, system (3.8) with parameter \(k\) is solvable. \(\square\)

As a consequence of Theorems 3.6, 3.7 and 3.8 that we have the following.

**Corollary 3.9** If \(u = \sum_{i=1}^{n-1} u_i e_i\) is a solution of (2.7) and \(k \in \mathbb{C}\) a constant, then \((\text{BT})_{h,k}\) is solvable for \(h\). Moreover,

1. if \(h\) is a solution of \((\text{BT})_{h,k}\), then \(\tilde{u} = \sum_{i=1}^{n-1} \tilde{u}_i e_i\) defined by \(\tilde{u}_i = u_i + s_i(u, h)\) as in (3.1) is a solution of (2.7),
2. if \(E\) is a frame of \(u\), then \(E f_{u,h}^{-1}\) is a frame of \(\tilde{u}\) for \(\lambda \neq k\) and \(\det(f_{u,h}(x, t, \lambda)) = (-1)^{n-1}(\lambda - k)\).

We use \(h^\sharp u\) to denote the new solution \(\tilde{u}\) constructed in the above theorem and call the map \(u \mapsto h^\sharp u\) a Bäcklund transformation with parameter \(k\) as in Section 1.

**Example 3.10 (BT for the second \(A_2^{(1)}\)-KdV flow)** Use the algorithm given in Section 2 to see that the second \(A_2^{(1)}\)-KdV flow is

\[
\begin{cases}
(u_1)_t = (u_1)_{xx} - \frac{2}{3} (u_2)_{xxx} + \frac{2}{3} u_2 (u_2)_x, \\
(u_2)_t = -(u_2)_{xx} + 2(u_1)_x,
\end{cases}
\tag{3.11}
\]

Take \(t\) derivative on both sides of the second equation of (3.11) and use the first equation to see that \(u_2\) satisfies the Boussinesq equation:

\[
(u_2)_tt = -\frac{1}{3} (u_2)_{x(4)} + \frac{4}{3} (u_2)_{x(2)}^2 + \frac{4}{3} u_2 (u_2)_{xx}.
\tag{3.12}
\]

We use the proof of Theorem 3.4 to see that system (3.8) for a solution \(u = u_1 e_{13} + u_2 e_{23}\) of (3.11) and \(f_{u,h}\) are

\[
\begin{cases}
h_x = -u_1 + (u_2)_x + hu_2 - 3hh_x - h^3 - k, \\
h_t = \frac{2}{3} (u_2)_x - h_x - 2hh_x, \\
f_{u,h}(x, t, \lambda) = e_{13} \lambda + \begin{pmatrix}
h & h & u_1 - (u_2)_x + h_x + 3hh_x \\
1 & h & u_2 - h_x \\
0 & 1 & h
\end{pmatrix}.
\end{cases}
\]
The new solution $\tilde{u}$ is given by

$$\tilde{u}_1 = u_1 - (u_2)_x + 3hh_x, \quad \tilde{u}_2 = u_2 - 3h_x.$$  \hspace{1cm} (3.13)

**Example 3.11** (1-soliton solutions of the $j$th $\hat{A}_{n-1}$-KdV flow)

Set $\alpha = e^{2\pi i}$, and $C = (e^{i(1-j)}e^{i-j})$, $D(\rho) = \text{diag}(1, \rho, \ldots, \rho^{n-1})$ and $\sigma = e_n + \sum_{i=1}^{n-1} e_{i+j}$. We apply Theorem 3.7 to $u = 0$ with frame $E(x, t, \lambda) = \exp(xJ + tJ^j)$, $k = \rho^n$ and $c_0 = D(\rho)^{-1}\sigma C^{-1}p_0$, where $p_0 = (1, 1, 0, \ldots, 0)$. A direct computation implies that the solution $h$ of (3.8) obtained by Theorem 3.7 is

$$h = \frac{\rho}{2} \left( (1 - \alpha) \tanh \left( \frac{1}{2} ((1 - \alpha) \rho x + (1 - \alpha^j) \rho^j t) \right) - (1 + \alpha) \right).$$ \hspace{1cm} (3.14)

If $\rho = (1 + \bar{\alpha})\mu$ for some $\mu \in \mathbb{R}$, then $h$ defined by (3.14) becomes

$$h = -\mu \sin \left( \frac{2\pi}{n} \right) \tan \theta - \left( 1 + \cos \left( \frac{2\pi}{n} \right) \right) \mu,$$ \hspace{1cm} (3.15)

is a real solution, where $\theta = \mu \sin \left( \frac{2\pi}{n} \right) x + \sum_{i=0}^{n-1} C_j \mu^i \sin \left( \frac{2i\pi}{n} \right) t$. In particular, when $j = 2$, (3.15) becomes

$$h = -\mu \sin \left( \frac{2\pi}{n} \right) \tan \left( \mu \sin \left( \frac{2\pi}{n} \right) x + 2 \left( 1 + \cos \left( \frac{2\pi}{n} \right) \right) \mu t \right) - \left( 1 + \cos \left( \frac{2\pi}{n} \right) \right) \mu.$$ \hspace{1cm} (3.16)

When $n = 2$ and $j = 2$. The above computation gives

(i) Let $\mu = -\frac{\alpha}{2} (1 - \alpha)$. Then $h$ defined by (3.14) becomes

$$h = \sqrt{3} \mu \tanh(\sqrt{3}\mu(x - 2\mu t)) + \mu t.$$ \hspace{1cm} (3.17)

Substitute (3.17) to (3.13) with $u = 0$ to see that

$$\begin{cases} u_1 = 9\mu^3 \text{sech}^2(\sqrt{3}\mu(x - 2\mu t))(\sqrt{3} \tanh(\sqrt{3}\mu(x - 2\mu t)) + i), \\ u_2 = -9\mu^2 \text{sech}^3(\sqrt{3}\mu(x - 2\mu t)) \end{cases}$$

is a complex solution of (3.11) and $u_2$ is the solution of (3.12) obtained in [9].

(ii) $h$ given by (3.16) is

$$h = -\frac{\sqrt{3}}{2} \mu \tan \left( \frac{\sqrt{3}}{2} \mu (x + \mu t) \right) - \frac{\mu}{2}.$$ 

Let $\nu = \frac{\mu}{2}$. Then

$$h_{\nu} = -\sqrt{3} \nu \tan(\sqrt{3}(x + 2\nu t)) - \mu.$$
3.12 Suppose Theorem c new frame again, we need to construct a frame that is holomorphic at $\lambda = k$. To apply BT to the new solution with the same parameter $k$ again, we need to construct a frame that is holomorphic at $\lambda = k$. This can be done by multiplying the new frame $Ef^{-1}$ on the left by some $C(\lambda)$ independent of $x, t$.

**Theorem 3.12** Suppose $E(x, t, \lambda)$ is a frame of a solution $u$ of (2.7) that is holomorphic for $\lambda$ in an open subset $\mathcal{O}$ of $\mathbb{C}$. Let $k \in \mathcal{O}$ be a constant, and $h$ the solution of (3.8) constructed from $E(x, t, k)$ and $e = (c_1, \ldots, c_n)^T \in \mathbb{C}^n \setminus 0$ as in Theorem 3.7. Then

(i) $\tilde{E} = Ef_{h}^{-1}$ is a frame of the new solution $h^2 u$ and $\tilde{E}$ is holomorphic for all $\lambda \in \mathcal{O} \setminus \{k\}$.

(ii) $\tilde{E}(x, t, \lambda) = C(\lambda)E(x, t, \lambda)_{f_{h}}^{-1}(x, t, \lambda)^{-1}$ is a frame for $h^2 u$ that is holomorphic for all $\lambda \in \mathcal{O}$, where $C(\lambda) = e_{1n}(\lambda - k) + b + \sum_{i=1}^{n-1} c_i e_{1n}$. $k$

**Proof.** By Corollary 3.9, $h$ is a solution of (3.3), $h^2 u$ is a solution of (2.7), and $Ef_{h}^{-1}$ is a frame of $h^2 u$. By definition, $f_{h}(x, t, \lambda)$ is holomorphic for all $\lambda \in \mathcal{O}$. Since $\det(f_{h}(x, t, \lambda)) = (-1)^{n-1}(\lambda - k)$, if $\lambda \neq k$ then $Ef_{h}^{-1}$ is holomorphic at $\lambda$.

Let $f = f_{h}$, and $f^T$ the transpose of the cofactor matrix of $f$, i.e., the $ij$th entry is $(-1)^{i+j}\det(M_{ij})$ where $M_{ij}$ is the $(n-1) \times (n-1)$ matrix obtained from $f$ by crossing out the $i$th row and $j$th column. Then $ff^T = (-1)^{n-1}(\lambda - k)I_n$. So we have

$$\text{Im}(f^T(x, t, k)) \subset \text{Ker}(f(x, t, k)) = \mathbb{C}E^{-1}(x, t, k)(e). \quad (3.18)$$

Let

$$F(x, t, \lambda) = C(\lambda)E(x, t, \lambda)f_{h}^{-1}(x, t, \lambda)$$

$$= (-1)^{n-1} \frac{1}{\lambda - k} C(\lambda)E(x, t, \lambda)f^T(x, t, \lambda).$$

Then $F(x, t, \lambda)$ is holomorphic for $\lambda \in \mathcal{O} \setminus \{k\}$ and has a possible simple pole at $\lambda = k$. We claim that the residue of $F(x, t, \lambda)$ at $\lambda = k$ is zero. The residue of $F(x, t, \lambda)$ at $\lambda = k$ is equal to $(-1)^{n-1}C(k)E(x, t, k)f^T(x, t, k)$. By (3.18), it is equal to $\phi(x, t)C(k)(e)$ for some function $\phi$. A direct computation implies that $C(k)(e) = 0$. This proves the claim and $F$ is holomorphic at $\lambda = k$. \qed

The above theorem will be used to construct Bäcklund transformations for the n-d central affine curve flows in [10].
4. Permutability and Scaling Transform

In this section, we (i) give a Permutability formula for Bäcklund transformations, (ii) prove that the conjugation of a Bäcklund transformation with parameter \( k = 1 \) by the \( r \)-scaling transform gives a BT with parameter \( k = r^{-n} \) for the \( \tilde{A}_{n-1}^{(1)} \)-KdV hierarchy.

**Theorem 4.1 (Permutability for BT)**

Let \( u \) be a solution of (2.7), \( k_1, k_2 \in \mathbb{C} \) constants, \( h_i \) solutions of (BT) \( _{u,k_i}^{(3.8)} \), and \( h_i^\sharp u \) the solution of (2.7) construct from \( u \) and \( h_i \) for \( i = 1, 2 \). Suppose \( h_1 \neq h_2 \). Set

\[
\begin{align*}
\tilde{h}_1 &= h_1 + \frac{(h_1 - h_2)}{h_1 - h_2}, \\
\tilde{h}_2 &= h_2 + \frac{(h_1 - h_2)}{h_1 - h_2}.
\end{align*}
\]

Then

(i) \( \tilde{h}_1 \) is a solution of (BT) \( _{h_2^\sharp u,k_1}^{(h_2^\sharp u)} \) and \( \tilde{h}_2 \) is a solution of (BT) \( _{h_1^\sharp u,k_2}^{(h_1^\sharp u)} \),

(ii) \( \tilde{h}_1^\sharp(h_2^\sharp u) = \tilde{h}_2^\sharp(h_1^\sharp u) \).

**Proof.** Let \( E(x, t, \lambda) \) be a frame of \( u \). By Theorem 3.8, there exist constant vectors \( v_1^0, v_2^0 \in \mathbb{R}^n \) that give \( h_1, h_2 \), i.e.,

\[
\begin{align*}
v_1 &= (v_{1,1}, \ldots, v_{1,n})^T = E(\cdot,\cdot,k_1)^{-1}v_1^0, \\
v_2 &= (v_{2,1}, \ldots, v_{2,n})^T = E(\cdot,\cdot,k_2)^{-1}v_2^0, \\
h_i &= -\frac{v_{i,n-1}}{v_{i,n}}, \quad i = 1, 2.
\end{align*}
\]

From (3.10),

\[
(v_i)_x = -E^{-1}E_i(x, t, k_i)v_i = -(\tilde{J} + u)|_{\lambda=h_i} v_i, \quad i = 1, 2.
\]

In particular, we have

\[
\begin{align*}
(v_{i,n-1})_x &= -v_{i,n-2} - u_{n-1}v_{i,n}, \\
(v_{i,n})_x &= -v_{i,n-1}, \quad i = 1, 2.
\end{align*}
\]

Therefore,

\[
(h_i)_x = \frac{v_{i,n-2}}{v_{i,n}} - h_i^2 + u_{n-1}, \quad i = 1, 2. \tag{4.1}
\]

By Theorem 3.7, \( E_i(x, t, \lambda) = E(x, t, \lambda)^f_{u,h_i}^{-1} \) is a frame for \( h_i^\sharp u \) for \( i = 1, 2 \), respectively. Let

\[
\tilde{v}_1 = E_2^{-1}(x, t, k_1)v_1^0, \quad \tilde{v}_2 = E_1(x, t, k_2)^{-1}v_2^0.
\]
Now we compute
\[ \tilde{h}_i = -\frac{\tilde{v}_{i,n-1}}{\tilde{v}_{i,n}}. \]

From
\[ \tilde{v}_1 = E(x, t, k_1)v_1^0 = f_{u, b_2}(k_1)E(x, t, k_1)^{-1}v_1^0 = f_{u, b_2}(k_1)v_1, \]
we get
\[
\begin{cases}
\tilde{v}_{1,n-1} = v_{1,n-2} + h_2 v_{1,n-1} + (u_{n-1} - (h_2)_z)v_{1,n}, \\
\tilde{v}_{1,n} = v_{1,n-1} + h_2 v_{1,n}.
\end{cases}
\]

Together with (4.1), we have
\[ \tilde{h}_1 = -h_2 + \frac{v_{1,n-2} v_{1,n} - v_{2,n-2} v_{1,n}}{v_{2,n-1} v_{1,n} - v_{1,n-1} v_{2,n}} = h_1 + \frac{(h_1 - h_2)_z}{h_1 - h_2}. \]

Similarly, \[ \tilde{h}_2 = h_2 + \frac{(h_1 - h_2)_z}{h_1 - h_2}. \]

A direct computation implies \[ f_{u_2, b_2}f_{u, b_2} = f_{u_1, b_1}f_{u, b_1}. \] Hence \[ \tilde{h}_2 \tilde{z}(h_1 \tilde{u}) = \tilde{h}_1 \tilde{z}(h_2 \tilde{u}). \]

Next we review the scaling transform of the \( \hat{A}_{n-1}^{(1)} \)-KdV hierarchy.

**Proposition 4.2** ([11]) Let \( u = \sum_{i=1}^{n-1} u_i e_m \) be a solution of the \( j \)th \( \hat{A}_{n-1}^{(1)} \)-KdV flow, and \( r \in \mathbb{R} \setminus \{0\} \). Let \( D(r) = \text{diag}(1, r, \ldots, r^{n-1}) \), and
\[
(r \cdot u)(x, t) = r^{n+1-j} u_i(rx, rt), \quad 1 \leq i \leq n - 1.
\]

Then

(i) \( (r \cdot u) = \sum_{i=1}^{n-1} (r \cdot u_i)e_m \) is a solution of the \( j \)th \( \hat{A}_{n-1}^{(1)} \)-KdV flow,

(ii) if \( E(x, t, \lambda) \) is a frame of \( u \) then
\[ \tilde{E}(x, t, \lambda) := D(r)^{-1}E(rx, rt, r^{-n}\lambda)D(r) \]
is a frame of \( r \cdot u \).

So the multiplicative group \( \mathbb{R} \setminus \{0\} \) acts on the space of solutions of the \( j \)th \( \hat{A}_{n-1}^{(1)} \)-KdV flow. The following theorem proves that the conjugation of BT with parameter \( k = 1 \) by a scaling transform gives BT with arbitrary non-zero real parameter.

**Theorem 4.3** Let \( u \) be a solution of the \( j \)th \( \hat{A}_{n-1}^{(1)} \)-KdV flow, \( r \in \mathbb{R} \setminus \{0\} \) and \( \tilde{h} \) a solution of (BT)\(_{r, u, 1} \). Then
Proof. Let $E$ be the frame of $u$ with $E(0, 0, \lambda) = I_n$. It follows from $d(\ln \det(E)) = \text{tr}(E^{-1} dE) = 0$ that

$$\det E(x, t, \lambda) = 1. \tag{4.2}$$

From Proposition 4.2,

$$F(x, t, \lambda) = D(r)^{-1} E(x, r^t, r^{-\eta} \lambda) D(r)$$

is a frame for $\tilde{u} = r \cdot u$, where $D(r) = \text{diag}(1, r, \ldots, r^{n-1})$. It follows from Corollary 3.9 that

$$F_1(x, t, \lambda) = F(x, t, \lambda) f_{u,h}^{-1}(x, t, \lambda)$$

is a frame for $h\tilde{u}$. Apply the scaling transform given by $r^{-1}$ to $h\tilde{u}$, we have

$$F_2(x, t, \lambda) = D(r) F_1(r^{-1} x, r^{-1} t, r^n \lambda) D(r)^{-1} = E(x, t, \lambda) D(r) f_{u,h}^{-1}(r^{-1} x, r^{-1} t, r^n \lambda) D(r)^{-1}, \tag{4.3}$$

is a frame for $\hat{u} = r^{-1} \cdot (h\tilde{u}(r \cdot u))$. A direct computation implies that

$$D(r) f_{u,h}(r^{-1} x, r^{-1} t, r^n \lambda) D(r)^{-1} = r(e_{1n} \lambda + b + r^{-1} h(r^{-1} x, r^{-1} t) I_n + r^{-1} D(r) N D(r)^{-1}),$$

where $f_{u,h} = e_{1n} \lambda + b + h I_n + N$ and $N$ is strictly upper triangular.

So we have proved $F_2(x, t, \lambda) = r^{-1} E(x, t, \lambda) g^{-1}(x, t, \lambda)$ is a frame of $\hat{u} = r^{-1} \cdot (h\tilde{u}(r \cdot u))$, where

$$g(x, t, \lambda) = e_{1n} \lambda + b + \hat{h}(x, t) I_n + r^{-1} D(r) N(r^{-1} x, r^{-1} t) D(r)^{-1}.$$

Note that $r^{-1} D(r) N(r^{-1} x, r^{-1} t) D(r)^{-1} \in N^+_n$ and

$$\hat{E}(x, t, \lambda) = r F_2(x, t, \lambda) = E(x, t, \lambda) g^{-1}(x, t, \lambda) \tag{4.4}$$

is a frame of $\hat{u}$. So $g = f_{u,h}$.

By Theorem 3.6, we have $\det(f_{u,h}) = (-1)^n (\lambda - 1)$. It follows from (4.2), (4.3) and (4.4) that

$$\det(f_{u,h}^{-1}) = \det(\hat{E}(x, t, \lambda)) = r^n \det(F_2(x, t, \lambda)) = r^n \det(f_{u,h}(r^{-1} x, r^{-1} t, r^n \lambda)) = (-1)^{n-1} \frac{r^n}{r^n \lambda - 1} = \frac{(-1)^{n-1}}{\lambda - r^{-n}}.$$

Hence $\det(f_{u,h}) = (-1)^{n-1} (\lambda - r^{-n})$. \qed
Remark 4.4 The known explicit rational solutions (cf. [2]) and $m$-soliton solutions for (2.7) (cf. [1]) can also be constructed by applying BTs.

(i) We apply Theorem 3.7 to the frame $E(x, t, \lambda) = \exp(xJ + tJ')$ of the trivial solution $u = 0$ of (2.7) with $k = 0$ to obtain explicit rational solutions $\tilde{u}$. An explicit frame for $\tilde{u}$ that is holomorphic at $\lambda = 0$ is given in Theorem 3.12(ii). So we can apply Theorem 3.7 again to obtain another family of explicit rational solutions. Repeat this process to produce infinitely many families of explicit rational solutions.

(ii) We use Theorem 3.7 with parameter $k \neq 0$ to construct 1-soliton solutions, then use the Permutability Theorem repeatedly to construct explicit $k$-soliton solutions.

5. Relation between Adler’s BT and our BT

In this section, we prove results (2), (4) and (5) stated in Section 1.

Theorem 5.1 Let $L = \partial^n - \sum_{i=1}^{n-1} u_i \partial^{i-1}$ be a solution of the $j$th GD$_n$ flow (1.1), and $h \in C^\infty(\mathbb{R}^2, \mathbb{C})$. Then the following statements are equivalent:

(a) $\tilde{L} = (\partial + h)^{-1}L(\partial + h)$ is a solution of the $j$th GD$_n$ flow (1.1),

(b) $h$ is a solution of (BT)$_{n,k}$ (3.8) for some constant $k \in \mathbb{C}$.

Proof. We first prove (a) implies (b). Suppose $\tilde{L} = (\partial + h)^{-1}L(\partial + h)$ is a solution of (1.1). Write $\tilde{L} = \partial^n - \sum_{i=1}^{n-1} \tilde{u}_i \partial^{i-1}$. By Theorem 2.3, both $u = \sum_{i=1}^{n-1} u_i e_{in}$ and $\tilde{u} = \sum_{i=1}^{n-1} \tilde{u}_i e_{in}$ are solutions of (2.7).

Let $\tilde{E}(x, t, \lambda)$ be a frame of the solution $\tilde{u}$, and $\tilde{\phi} = (\tilde{\phi}_1, \ldots, \tilde{\phi}_n)'$ the first column of $\tilde{E}$. It follows from Lemma 3.3 that $\{\tilde{\phi}_1, \ldots, \tilde{\phi}_n\}$ form a basis of solutions of $\tilde{L} - \lambda = 0$. Since $L - \lambda = (\partial + h)(L - \lambda)(\partial + h)^{-1}$, $\phi_1 = (\partial + h)\phi_1$ is a solution of $L - \lambda = 0$ for $1 \leq i \leq n$. Let $E = ((\phi_i)_{i=1}^n)$ and $f = E^{-1}E$. So $E = Ef$. A direct computation implies that the first column of $f$ is $(h, 1, \ldots, 0)'$, $f$ is of the form $J + hI_n + N$ for some $N_{ij}^+$ valued map $N$, and $\text{det}(f(x, t, \lambda))$ is of degree one in $\lambda$. Since $E$ is a frame for $\tilde{u}$, $\text{det}(\tilde{E})$ is independent of $x, t$, and $\text{det}(E) = \text{det}(\tilde{E}) = c(\lambda)$, and $c(\lambda) \neq 0$ for all $\lambda \in \mathbb{C}$. So $\text{det}(E(x, t, \lambda)) \neq 0$ for generic $(x, t, \lambda)$, i.e., $\{\phi_1, \ldots, \phi_n\}$ is a basis of $L - \lambda = 0$ for generic $\lambda$. Hence $E$ is a frame of $u$. It follows from Theorem 3.4 that we have $f = f_{\lambda,k}$ and $E = Ef_{\lambda,k}^{-1}$. By Theorem 3.6, there is a constant $k \in \mathbb{C}$ such that $h$ is a solution of (BT)$_{n,k}$ (3.8).

Next we prove (b) implies (a). Suppose $h$ is a solution of (BT)$_{n,k}$ (3.8). By Corollary 3.9 we have

(i) $\tilde{u} = h^* u$ is a solution of (2.7),

(ii) if $\tilde{E}$ is a frame of $\tilde{u}$, then $E = \tilde{E}f_{\lambda,k}$ is a frame for $u$ all $\lambda \neq \lambda$.

(iii) the $i$th entries of $\phi_1, E$ and $\phi_1 E$ are related by $\phi_i = (\partial + h)\phi_i$.

Assume $\lambda \neq k$. Lemma 3.3 implies that $\{\phi_1, \ldots, \phi_n\}$ is a basis of $L - \lambda = 0$. This is also a basis of $(\partial + h)(\tilde{L} - \lambda)(\partial + \tilde{h})^{-1} = 0$, where $\tilde{L} = \partial^n - \sum_{i=1}^{n-1} \tilde{u}_i \partial^{i-1}$. It follows from Lemma 3.3 that $L - \lambda = (\partial + h)(L - \lambda)(\partial + h)^{-1}$. Hence $L = (\partial + h)L(\partial + h)^{-1}$. Or equivalently, $\tilde{L} = (\partial + h)^{-1} L(\partial + h)$. □

Corollary 5.2 Let $L = \partial^n - \sum_{i=1}^{n-1} u_i \partial^{i-1}$ be a solution of the $j$th GD$_n$ flow (1.1), $u = \sum_{i=1}^{n-1} u_i e_{in}$ and $k \in \mathbb{C}$ a constant. Let $p_{i1}(u, \lambda)$ denote the $i$th entry of $Z_j(u, k)$, where $Z_j(u, \lambda)$ is defined by (2.9) for the
Theorem 5.4 Let $\hat{A}_{n-1}^{(1)}$-KdV flow. Then

$$
\begin{cases}
(L - k)\phi = 0, \\
\phi_i = \sum_{j=1}^n p_i(u,k)\phi_j^{(i-1)},
\end{cases}
$$

(5.1)

is solvable for $\phi : \mathbb{R}^2 \to \mathbb{C}$. Moreover, if $\phi_1, \ldots, \phi_{n-1}$ are linearly independent solutions of (5.1), then $h = (\ln(W(\phi_1, \ldots, \phi_{n-1})))_t$ is a solution of (BT)$_{a,h}$ (3.8) and $\hat{L} = (\partial + h)^{-1}L(\partial + h)$ is a solution of (1.1), where $W(f_1, \ldots, f_{n-1}) = \det((f_j)_k^{(i-1)})$ is the Wronskian.

Proof. By Theorem 2.3, $u$ is a solution of (2.7). Let $g = (\phi, \phi_1, \ldots, \phi_{n-1}^{(n-1)})$. Then system (5.1) is equivalent to

$$
\begin{align*}
g^{-1}g_t &= ke_{i,n} + b + u, \\
g^{-1}g_i &= Z_i(u,k).
\end{align*}
$$

(5.2)

This is the Lax pair of $u$ with $\lambda = k$, so it is solvable. If $\phi$ is a solution of (5.1), then $g$ satisfies (5.2). Let $(v_1, \ldots, v_n)^t = g^{-1}e_n$. It follows from Theorem 3.7 that $h = -\frac{\log v_n}{v_n}$ is a solution of (3.8) with parameter $k$. Use Cramer’s rule to get the formula of $h$ in terms of Wronskians. □

For example, when $n = 3$, equation (5.1) is

$$
\begin{align*}
\phi_{xxx} - u_2\phi_x - u_1\phi &= k\phi, \\
\phi_i &= -\frac{i}{2}u_2\phi + \phi_{xi}.
\end{align*}
$$

Recall that Adler’s BT is:

THEOREM 5.3 ([2]) If $L = \partial^n - \sum_{i=1}^{n-1} u_i \partial^{i-1} = (\partial - q_n) \cdots (\partial - q_1)$ is a solution of the $j$th GD$_n$ flow (1.1) and $q = \sum_{i=1}^n q_ie_i$ is a solution of the $j$th $n \times n$ mKdV flow, then $\hat{L} = (\partial - q_{n-1}) \cdots (\partial - q_1)(\partial - q_n)$ is again a solution of (1.1).

Note that $\hat{L}$ and $L$ in Adler’s Theorem satisfies the condition $\hat{L} = (\partial - q_n)^{-1}L(\partial - q_n)$. So by Theorem 5.1, $-q_n$ is a solution of (1.2) for some $k \in \mathbb{C}$. Next we show that $k = 0$.

THEOREM 5.4 Let $L, q_1, \ldots, q_n$ be as in Theorem 5.3. Then $-q_n$ is a solution (BT)$_{a,0}$. In other words, Adler’s BT is our BT with parameter $k = 0$.

Proof. Let $K$ denote the parallel frame for the Lax pair of the solution $q$ of the $j$th $n \times n$ mKdV flow at $\lambda = 0$ with $K(0,0) = I_n$, i.e., $K$ is the solution of

$$
\begin{align*}
K^{-1}K_x &= b + q, \\
K^{-1}K_i &= \pi_{ln}(Q_{\lambda,0}(q)),
\end{align*}
$$

where $Q_{\lambda,0}(q)$ is defined in Section 2, $b = \sum_{i=1}^{n-1} e_{i+1}$, and $\pi_{ln}$ is the projection of sl$(n, \mathbb{C})$ onto $\mathfrak{b}_n^-$ along $\mathfrak{n}_n^+$. Since both $b + q$ and $\pi_{ln}(Q_{\lambda,0}(q))$ are lower triangular and $K(0,0) = I_n$, we see that $K(x,t) \in \mathfrak{b}_n^-$.
i.e. lower triangular, for all \( x, t \). Let \( \phi = (\phi_1, \ldots, \phi_n)' \) denote the first column of \( K \). It follows from \( Kx = K(b + q) \) that the \( j \)th column of \( K \) is

\[
(\partial - q_{j-1}) \cdots (\partial - q_1) \phi,
\]

and \( L\phi = (\partial - q_n) \cdots (\partial - q_1) \phi = 0 \). Since \( K \) is lower triangular, we obtain

\[
(\partial - q_i) \cdots (\partial - q_1) \phi_i = 0, \quad (5.3)
\]

for \( 1 \leq i \leq n - 1 \).

Let \( \Delta(\cdot,t) := \Gamma(q(\cdot,t)) \), where \( \Gamma \) is given in Definition 2.4. By Theorem 2.5, \( g = K\Delta^{-1} \) is a frame of the Lax pair of the solution \( u \) of (2.7) at \( \lambda = 0 \). Note that \( \Delta \) is in \( N_n^+ \), so the first column of \( g \) is also \( \phi \). Use \( g^{-1}x = b + u \) to obtain that \( g = (\phi, \phi_1, \ldots, \phi_{n-1})' \).

Let

\[
R := (\partial - q_{n-1}) \cdots (\partial - q_1) = \partial^{n-1} - \sum_{i=1}^{n-1} \xi_i \partial^{i-1}.
\]

By (5.3), \( R\phi_j = 0 \) for all \( 1 \leq j \leq n - 1 \). So we have

\[
\sum_{i=1}^{n-1} \xi_i \phi_j^{(i-1)} = \phi_j^{(n-1)}, \quad 1 \leq j \leq n - 1.
\]

By Cramer’s rule,

\[
\xi_{n-1} = \frac{\det(\eta, \ldots, \eta_j^{(n-4)}, \eta_j^{(n-3)}, \eta_j^{(n-1)})}{\det(\eta, \eta_1, \ldots, \eta_j^{(n-2)})}, \quad (5.4)
\]

where \( \eta = (\phi_1, \ldots, \phi_{n-1})' \). Recall that \( g = (\phi, \ldots, \phi_{n-1})' \). Let \( g^{ij} \) denote the \( ij \)th entry of \( g^{-1} \). Then the numerator and denominator of the right-hand side of (5.4) are \( -\det(g)g^{n-1,n} \) and \( \det(g)g^{nn} \), respectively.

So \( \xi_{n-1} = -\frac{g^{n-1,n}}{g^{nn}} \). By Theorem 3.7, \( \xi_{n-1} \) is a solution of (1.2) with \( k = 0 \). Compare the coefficient of \( \partial^{n-1} \)

of \( L = (\partial - q_n)R \) to obtain \( q_n = -\xi_{n-1} \). Hence \( -q_n \) is a solution of (1.2) with \( k = 0 \).

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