The Sparse Hausdorff Moment Problem, with Application to Topic Models

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Abstract

We consider the problem of identifying, from its first $m$ noisy moments, a probability distribution on $[0, 1]$ of support $k < \infty$. This is equivalent to the problem of learning a distribution on $m$ observable binary random variables $X_1, X_2, \ldots, X_m$ that are iid conditional on a hidden random variable $U$ taking values in $\{1, 2, \ldots, k\}$. Our focus is on accomplishing this with $m = 2k$, which is the minimum $m$ for which verifying that the source is a $k$-mixture is possible (even with exact statistics). This problem, so simply stated, is quite useful: e.g., by a known reduction, any algorithm for it lifts to an algorithm for learning pure topic models.

In past work on this and also the more general mixture-of-products problem ($X_i$ independent conditional on $U$, but not necessarily iid), a barrier at $m^{O(k^2)}$ on the sample complexity and/or runtime of the algorithm was reached. We improve this substantially. We show it suffices to use a sample of size $\exp(k \log k)$ (with $m = 2k$). It is known that the sample complexity of any solution to the identification problem must be $\exp(\Omega(k))$. Stated in terms of the moment problem, it suffices to know the moments to additive accuracy $\exp(-k \log k)$. Our run-time for the moment problem is only $O(k^{2+\omega(1)})$ arithmetic operations.

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1 Introduction

Motivation. The Hausdorff moment problem is that of determining what moment sequences

\[ \mu_i := \int_0^1 \alpha^i \, dP(\alpha) \quad (i \geq 0) \]  

are possible for a probability distribution \( P \) supported on \([0, 1]\). For background on this classical topic, see [12, 27]. Associated with this is the computational problem of determining \( P \) from \((\mu_i)_i\) (or approximating it from finite, and possibly noisy, prefixes).

In this paper we are concerned with the sparse version of the computational problem, namely, the task of computing \( P \) in case it is assumed to have support of cardinality at most \( k < \infty \). This problem has the following equivalent interpretation, due to which we call it the “\( k \)-coin problem”: identify the parameters of a distribution on \( m \) observable binary random variables \( X_1, X_2, \ldots, X_m \) that are iid conditional on a hidden random variable \( U \) taking values in \( \{1, 2, \ldots, k\} \). Due to the symmetry among the variables (coins), the information available is precisely empirical estimates of moments \( 0, \ldots, m \) of the (shared) distribution of the variables \( X_i \). (Of course \( \mu_0 = 1 \) so there are \( m \) nontrivial statistics.)

Our focus is on accomplishing this with \( m = 2k \), which is the minimum \( m \) for which verifying that the source is a \( k \)-mixture is possible (even with exact statistics). We freely go back and forth in this paper between the formulation in terms of mixture models and the formulation in terms of the moment problem.

The problem of reconstructing \( P \), so simply stated, is quite useful:

(i) By a known reduction, any algorithm for this problem lifts to an algorithm for learning topic models. This will be discussed in Sec. 6.

(ii) This problem is a special case of the problem of identifying mixture models of \( k \) product distributions on binary variables, a problem on which there has been an impressive sequence of contributions in the last two decades, as we will discuss below. The best runtime for the product-distributions problem is however \( m^{O(k^2)} \). Until the present paper, the best runtime even for the iid case was no better. It seems likely that the true complexity of the general product case may be \( m^{O(k)} \). We show this to be true for the iid case, and this rules out some of the potential obstacles toward the conjecture for the general case.

(iii) Algorithms for identifying mixtures of product distributions are the simplest case of the yet-more-general problem of identifying distributions on “structural causal models” [23]. There has been little work in this direction, [2] being a notable exception. However, even that work has to make strong assumptions about the distributions of the variables \( X_i \), and in particular they cannot be binary. (Except for the case \( k = 2 \), but we are concerned here with complexity of the problem as a function of \( k \).) Source identification in causal graphical models is an important direction for future research, and it will matter a great deal whether the dependence on \( k \) (a measure of how much “confounding” there is in the model) scales as \( \exp(k) \) or as \( \exp(k^2) \).

The method. The algorithm we analyze is essentially that of Prony, 1795 [8]. The idea is to (a) characterize the coin biases (the support of \( P \)) as the roots of a polynomial whose coefficient vector is the kernel of the Hankel matrix; (b) use polynomial root-finding to determine the empirical coin biases; (c) reconstruct the mixture weights by polynomial interpolation.

Prior work on the sparse Hausdorff, i.e., \( k \)-coin mixture, problem. It has long been acknowledged in the numerical analysis literature (e.g., [12] §9.4, [15]) that the Prony method is sensitive to sample error (i.e., to errors in the moments). In fact, the instability is not limited to the Prony method; a lower bound ([25] Thm 6.1) for the problem (source identification of \( k \)-coin mixture models) shows that even for any \( c < \infty \), if \( c k \) (rather than just the minimum \( 2k \)) noisy moments are available, it remains the case that accurate source
Our result. Our main result is that using the Prony method, source identification can be performed for \( k \)-coin mixtures with sample complexity \( m^{O(k)} \leq \exp(k^2 \log k) \) (equivalently, the required moment accuracy is \( \exp(-k \log k) \), and runtime \( k^{2+o(1)} \). We have posted a working implementation of the algorithm on the following public Jupyter Notebook: Online notebook implementation\(^1\) (Tested in Chrome and Safari.)

The asymptotic notation hides a dependence on a separation parameter between the coin biases, which is necessary since the mixture weights are ill-defined in the limit of coinciding coin biases. For the reader interested in the key technical novelties of the paper we might point to the sequence of error-control lemmas in Section 7 and particularly to Lemma 22 which shows why a pseudo-kernel-vector of the empirical Hankel matrix will, as a polynomial, have roots close to those of the kernel of the model Hankel matrix.

This result also implies an improvement in identifying pure topic models, via the reductions in \([25,16]\). These reductions require solving \( k \) binary instances, and the required accuracy of the solution implies a post-reduction sample complexity of at least \( m^{O(k^2)} \) in both papers. Our results improve the post-reduction sample complexity to \( m^{O(k)} \) and the post-reduction runtime to \( O(k^{3+o(1)}) \). A more detailed comparison with previous work on topic models is given in Section 6.

Related work. The \( k \)-coin problem becomes easier when \( m \) is superlinear in \( k \), and trivial when \( m \) is \( \Omega(k^2 \log k) \). Therefore, we focus on the smallest \( m \) for which the problem is solvable, which is \( m = 2k - 1 \) if \( k \) is assumed, or \( m = 2k \) if \( k \) needs to be verified. As noted, the paper \([25]\) solved the problem (using \( m = 2k - 1 \)) in sample complexity \( s = m^{O(k^2)} \) and post-sampling runtime of \( \poly(k) \). By post-sampling runtime we mean the time complexity of the algorithm after the frequencies \( h_j, 0 \leq j \leq m \) (frequency that \( j \) of the conditionally-iid coins come up “heads”) have been collected. That paper also proves a lower bound of \( \exp(k) \) on the sample complexity needed to solve the problem. Subsequently, a different solution using near optimal sample complexity \( s = m^{O(k)} \), but much worse post-sampling runtime of \( m^{O(k^2)} \), was given in \([16]\).

In both papers mentioned above, the \( k \)-coin problem arises as the output of a reduction from the problem of identifying topic models, introduced in \([13,20]\). A (pure) \( k \)-topic model is simply analogous to the \( k \)-coin problem with highly multi-sided coins. There has been ample work on learning pure and mixed topic models, under various restrictive assumptions on the model, and also without restrictions \([3,1,25,16]\). The reductions of \([25,16]\) can be used in conjunction with our algorithm to reduce the sample complexity and post-sampling runtime required to solve the topic model problem. This is discussed in Section 6.

We also mention some generalizations of the \( k \)-coin problem that were considered in the literature. Most obvious is mixtures of \( k \) product distributions on \( \{0,1\}^m \). That is, the formulation is the same as ours except that \( X_1, \ldots, X_m \) are merely required to be independent, but not necessarily iid, conditional on the hidden variable \( U \). This problem has been the focus of considerable research in the past two decades \([14,10,7,5,9,6]\). Clearly, in this case a larger \( m \) is no longer purely helpful, since the number of degrees of freedom of the problem also goes up with \( m \). It should be noted, though, that the strongest results in this sequence, \([9]\) and \([6]\), do not address the problem of identifying the source model; rather, they learn a model which generates similar statistics. On the positive side, this task can sometimes be performed even under conditions where there is not enough information in the statistics for identification (i.e., when there are models with

\(^1\)https://colab.research.google.com/drive/1qK6V0YSjqd8LPxq8hhyYoap_U1lapt9yS?usp=sharing
near-enough statistics that are far apart in, say, transportation distance); but on the negative side, since these algorithms (as well as the algorithm in [16]) are forced to perform an exhaustive enumeration over a large grid of potential models, their computational efficiency does not much improve even when the statistics are known to sufficiently-good accuracy that only a very small-diameter (in transportation distance) set of models could generate them.

The distinction between the “identification” and “learning” goals was made already in [10], who solved the identification problem for mixtures of \( k = 2 \) product distributions on \( \{0, 1\}^m \). Similar results for somewhat more general models were achieved at a similar time in [7]. The best result to date [6] learns in time \( k^3 \cdot m^{O(k^2)} \), improving upon a previous result [9] of \( m^{O(k^3)} \). The same paper [6] shows a lower bound of \( m^{\Omega(\sqrt{k})} \) on the sample complexity of the task.

Beyond mixtures of product distributions, an even more complex but important class of source identification problems arises when the hidden variable (our “\( U \)” may be just one of several such variables, and when a known directed causal structure exists among the observed variables (the “\( X_i \)”)). This is a very broad field of investigation and we point only to [23, 24] for background, and to [2] for an example of how (with some additional assumptions on the distributions of the \( X_i \)) certain models can be handled.

### 2 Mixture Models and other Definitions

**Definition 1** (The \( k \)-coin model). A \( k \)-coin model \( \mathcal{M} = (\alpha, w) \) is a mixture of \( k \) Bernoulli variables with success probabilities \( \alpha_1, \ldots, \alpha_k \) with non-negative mixing weights \( w_1, \ldots, w_k \), respectively.

**Definition 2** (\( m \)-snapshots of a \( k \)-coin model). Given a \( k \)-coin model \( \mathcal{M} = (\alpha, w) \), an \( m \)-snapshot is a sample from the mixture of binomial distributions \( w_1 \text{ Binomial}(\alpha_1) + \ldots + w_k \text{ Binomial}(\alpha_k) \). (The binomial is a sufficient statistic for \( m \text{ rv’s} X_1, \ldots, X_m \) because they are iid given the selected coin.)

For a \( k \)-coin model, the moments defined in equation (1) can be written as follows where \( \delta_\alpha \) being the Dirac measure at \( \alpha \),

\[
\mathcal{P} = w_1 \delta_{\alpha_1} + \cdots + w_k \delta_{\alpha_k}, \quad \mu_i = \sum_{j=1}^{k} \alpha_j^i w_j.
\]

**Definition 3** (Separation for polynomials and mixtures). For a \( k \)-coin probability model \( \mathcal{M} = (\alpha, w) \), define the separation by \( \zeta(\mathcal{M}) = \min_{i \neq j} |\alpha_i - \alpha_j| \). For a degree \( k \) polynomial with roots \( \beta_1, \ldots, \beta_k \in \mathbb{C} \), define the root separation by \( \min_{i \neq j} |\beta_i - \beta_j| \).

**Definition 4.** The Vandermonde matrix \( V_\alpha \in \mathbb{R}^{k \times k} \) associated with a vector \( \alpha \in \mathbb{C}^k \) is given by

\[
V_\alpha = \begin{bmatrix}
1 & 1 & 1 & \cdots & 1 \\
\alpha_1 & \alpha_2 & \alpha_3 & \cdots & \alpha_k \\
\alpha_1^2 & \alpha_2^2 & \alpha_3^2 & \cdots & \alpha_k^2 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\alpha_1^{k-1} & \alpha_2^{k-1} & \alpha_3^{k-1} & \cdots & \alpha_k^{k-1}
\end{bmatrix}
\]

**Definition 5** (Hankel Matrix). The \((k + 1) \times (k + 1)\) Hankel matrix \( \mathcal{H}_{k+1} = \mathcal{H}_{k+1}(\mathcal{P}) \) is defined as:

\[
\mathcal{H}_{k+1} = \begin{bmatrix}
\mu_0 & \mu_1 & \mu_2 & \cdots & \mu_k \\
\mu_1 & \mu_2 & \mu_3 & \cdots & \mu_{k+1} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\mu_k & \mu_{k+1} & \mu_{k+2} & \cdots & \mu_{2k}
\end{bmatrix}
\]
Note that if $\mathcal{P}$ is supported on a set of cardinality $k$ (a.k.a. a $k$-coin distribution), then

$$\mathcal{H}_{k+1} = \sum_{j=1}^{k} w_j \alpha_j \alpha_j^T = V_\alpha \text{diag}(w_1, \ldots, w_k) V_\alpha^T$$

where $\alpha_j^T = (1, \alpha_j, \alpha_j^2, \alpha_j^3, \ldots, \alpha_j^k)$. This also shows that the Hankel matrix is positive semi-definite.

**Definition 6** (Polynomial associated with a vector). We associate to each vector $q \in \mathbb{R}^k$ a degree $k - 1$ polynomial $\hat{q}(x) = \sum_{j=0}^{k-1} q_j x^j$. (For this reason we use zero indexing for the vector.)

**Definition 7.** For a matrix $M$, let $\|M\|_2$ denote the $2 \to 2$ operator norm of $M$. Thus, $\|M\|_2 = \sigma_{\max}(M)$, the largest singular value of $M$.

**Definition 8.** For a Hermitian matrix $M$, let $\lambda_i(M)$ denote the $i$th smallest eigenvalue of $M$. In particular $\lambda_1(M)$ is the smallest eigenvalue of $M$.

**Definition 9** (Euclidean projection onto a closed convex set). For a closed convex set $S \subseteq \mathbb{R}^k$ and any point $x \notin S$, the Euclidean projection of $x$ onto $S$ is $\text{Proj}_S(x) := \arg\min_y \|y - x\|_2$. This projection is unique.

### 3 Properties of Hankel Matrices

**Lemma 10.** Let $\mathcal{P}$ be a probability measure on $[0, 1]$. Then,

1. $\mathcal{P}$ is supported on a set of cardinality at most $k$ iff $\mathcal{H}_{k+1}$ is singular.

2. If the support of $\mathcal{P}$ is a set $\{\alpha_1, \ldots, \alpha_k\} \subset [0, 1]$ then the kernel of $\mathcal{H}_{k+1}$ is spanned by the vector $q \in \mathbb{R}^{k+1}$ where $\hat{q}(z) = \prod_{i=1}^{k} (z - \alpha_i)$ is the unique monic polynomial with roots at the support of $\mathcal{P}$.

**Proof.** (Part 1.) By Equation (3), the rank of $\mathcal{H}_{k+1}$ for a $t$-coin distribution is at most $t$, and that implies that if $t \leq k$, then $\mathcal{H}_{k+1}$ is singular. So consider a distribution $\mathcal{P}$ on $[0, 1]$ that has positive mass at $k + 1$ points or more. Let $q \in \mathbb{R}^{k+1}$ be a non-zero vector. We have

$$q^T \mathcal{H}_{k+1} q = \int_0^1 \left( \sum_{j=0}^{k} q_j \alpha^j \right)^2 d\mathcal{P}(\alpha) = \int_0^1 \hat{q}^2(\alpha) d\mathcal{P}(\alpha).$$

There are at most $k$ points in $[0, 1]$ where the polynomial $\hat{q}$ evaluates to 0, and the total $\mathcal{P}$ measure of those points is less than 1. Thus, $q^T \mathcal{H}_{k+1} q > 0$, so $\mathcal{H}_{k+1}$ is positive definite.

(Part 2.) Since $\mathcal{H}_{k+1}$ is symmetric, its kernel is spanned by $q$ s.t. $q^T \mathcal{H}_{k+1} q = 0$. In order for the above integral to evaluate to zero over $\mathcal{P}$, we need that $\hat{q}^2(\alpha) = 0$ for each point $\alpha \in \text{supp}(\mathcal{P})$. As $\hat{q}$ is of degree $\leq k$, it is necessarily a scalar multiple of $\prod_{i=1}^{k} (z - \alpha_i)$. \(\square\)

**Lemma 11.** Let $\mathcal{P} = (\alpha, w)$ be a $k$-coin distribution with separation $\zeta$, and let $\mathcal{H}_k := \mathcal{H}_k(\mathcal{P})$. For every monic degree $k' \leq k - 1$ polynomial represented by $q \in \mathbb{R}^k$,

$$q^T \mathcal{H}_k q \geq \min \left( \frac{\zeta}{16} \right)^{2k-2} \cdot \|q\|_2^2.$$
\textbf{Proof.} Let \(\beta_1, \beta_2, \ldots, \beta_{k'}\) be the roots (possibly complex) of the polynomial \(q\), ordered so that \(|\beta_1| \geq |\beta_2| \geq \cdots \geq |\beta_{k'}|\). Since \(q\) is monic, we can write \(q(x) = \prod_{j=1}^{k'} (x - \beta_j)\). As the balls \(B(\alpha_i, \zeta/2), i = 1, 2, \ldots, k\), are disjoint, by the pigeonhole principle, there exists an \(i \in \{1, 2, \ldots, k\} \) such that \(B(\alpha_i, \zeta/2) \cap \{\beta_1, \beta_2, \ldots, \beta_{k'}\} = \emptyset\). The value of \(q\) at \(\alpha_i\) is

\[
\hat{q}(\alpha_i) = \prod_{j=1}^{k'} (\alpha_i - \beta_j).
\]

As \(\|q\|_1 \geq \|q\|_2\), there must be some \(\ell \in \{0, 1, 2, \ldots, k'\}\) such that \(|q_{\ell}| \geq \|q\|_2 \geq \|q\|_1\). Notice that \(|q_{\ell}| = e_{k' - \ell}(\beta_1, \beta_2, \ldots, \beta_{k'})\), where \(e_r\) is the \(r\)-th elementary symmetric polynomial over \(k'\) variables. \((e_0 = 1, e_1 = \sum \beta_i, e_2 = \sum_{i<j} \beta_i \beta_j \text{ etc.})\) So, \(e_{k' - \ell}\) is the sum over \((k' - \ell)\) \(\leq 2^{k'}\) monomials, hence \(|\beta_1 \beta_2 \cdots \beta_{k' - \ell}| \geq \frac{|q_{\ell}|}{(k' + 1)^{2^{k'} - r}}\). Eliminating from the product all the factors whose absolute value is below 2, we get that for some \(r \leq k' - \ell\), \(|\beta_1 \beta_2 \cdots \beta_r| \geq \frac{|q_{\ell}|}{(k' + 1)^{2^{k'} - r}}\). For \(j \in \{1, 2, \ldots, r\}\), since \(|\beta_j| \geq 2\) and \(\alpha_i \in [0, 1]\), it follows that \(|\alpha_i - \beta_j| \geq \frac{|\beta_j|}{2}\). Also, by the definition of \(i\) we have that \(|\alpha_i - \beta_j| > \zeta/2\) for all \(j \in \{1, 2, \ldots, k'\}\). Thus, we have that

\[
|\hat{q}(\alpha_i)| = \left( \prod_{j=1}^{r} |\alpha_i - \beta_j| \right) \left( \prod_{j=r+1}^{k'} |\alpha_i - \beta_j| \right) \geq \frac{|\beta_1 \beta_2 \cdots \beta_r|}{2^r} \left( \frac{\zeta}{2} \right)^{k' - r} \geq \frac{1}{k} \cdot \left( \frac{\zeta}{8} \right)^{k' - r} \|q\|_2.
\]

Therefore,

\[
q^T \mathcal{H}_k q = \sum_{j=1}^{k} w_j \cdot (\hat{q}(\alpha_j))^2 \geq w_{\min} \cdot (\hat{q}(\alpha_i))^2 > w_{\min} \cdot \frac{1}{k^2} \cdot \left( \frac{\zeta}{8} \right)^{2k'} \|q\|_2^2.
\]

Finally, for all \(k \geq 2\), we have \(\frac{1}{k^2} \cdot \left( \frac{\zeta}{8} \right)^{2k'} \geq \left( \frac{\zeta}{16} \right)^{2k'}\). \hfill \Box

\textbf{Corollary 12.} For a \(k\)-coin model \((\alpha, w), \lambda_2(\mathcal{H}_{k+1}) > w_{\min} \cdot \left( \frac{\zeta}{16} \right)^{2k} \).

\textbf{Proof.} By the Courant-Fischer-Weyl min-max principle, the smallest eigenvalue of \(\mathcal{H}_k\) is given by minimizing the Rayleigh-Ritz quotient. Let \(q \neq 0\) be a minimizer of \(\frac{q^T \mathcal{H}_k q}{q^T q}\). Let \(k'\) be greatest such that \(q_{k'} \neq 0\), and w.l.o.g. set \(q_{k'} = 1\). Then by Lemma 11

\[
\lambda_1(\mathcal{H}_k) = \min_{q \neq 0} \frac{q^T \mathcal{H}_k q}{q^T q} \geq w_{\min} \cdot \left( \frac{\zeta}{16} \right)^{2k - 2}.
\]

Notice that \(\mathcal{H}_k\) is a principal submatrix of \(\mathcal{H}_{k+1}\). Therefore, by the Cauchy interlacing theorem (Theorem 35), \(\lambda_2(\mathcal{H}_{k+1}) \geq \lambda_1(\mathcal{H}_k)\). \hfill \Box

\section{The Empirical Moments}

We bound the sampling error as follows. Sample \(s\) coins and let each of the random variables \(h_j, 0 \leq j \leq 2k\), be the fraction of coins which came up “heads” exactly \(j\) times. Then by the additive deviation bound known as Hoeffding’s inequality \([28]\), \(\Pr(|h_j - E(h_j)| \geq t) \leq 2 \exp(-2t^2 s)\). Thus
Lemma 13. If we use \( s > \frac{1}{22} \log(4k/\delta) \) samples then with probability at least \( 1 - \delta \): \( \forall j, |h_j - E(h_j)| < t \).

We can convert between the normalized histogram \( h \) and the standard moments of the distribution by using the observation (Lemma 1 in [26]) that for any \( t \in \mathbb{R} \),

\[
t^i = \sum_{j=i}^{n} \binom{j}{i} \times \binom{n}{j} t^j (1 - t)^{n-j}
\]

This gives us a linear transformation for converting from \( h \) to the vector \( \tilde{\mu} = (\tilde{\mu}_0, \ldots, \tilde{\mu}_{2k}) \). Define \( P_{as} \in \mathbb{R}^{(2k+1) \times (2k+1)} \), using zero-indexing by

\[
P_{as}_{ij} = \begin{cases} 
\binom{j}{i} & \text{if } j \geq i \\
0 & \text{otherwise};
\end{cases}
\]

then \( \tilde{\mu} = P_{as} h \).

Lemma 14. \( \|P_{as}\|_2 \leq 6^k \). Proof in Appendix A.

Now let \( \mu = (\mu_0, \ldots, \mu_{2k}) \) be the actual vector of moments of the distribution \( P \).

Lemma 15. For every \( \epsilon > 0 \), using \( s = 2^{O(k)} \cdot \frac{1}{\epsilon^2} \cdot \log(1/\delta) \) samples gives us estimated moments \( \tilde{\mu} = (\tilde{\mu}_0, \ldots, \tilde{\mu}_{2k}) \) satisfying \( \|\tilde{\mu} - \mu\|_\infty \leq \epsilon \) with probability at least \( 1 - \delta \).

**Proof.** Follows directly from Lemma 14 and Lemma 13.

Given an \( s \)-sample as above with empirical moments \( \tilde{\mu}_0, \tilde{\mu}_1, \ldots, \tilde{\mu}_{2k} \), denote by \( \tilde{H}_{k+1} \) the empirical Hankel matrix

\[
\tilde{H}_{k+1} = \begin{bmatrix}
\tilde{\mu}_0 & \tilde{\mu}_1 & \tilde{\mu}_2 & \cdots & \tilde{\mu}_k \\
\tilde{\mu}_1 & \tilde{\mu}_2 & \tilde{\mu}_3 & \cdots & \tilde{\mu}_{k+1} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\tilde{\mu}_k & \tilde{\mu}_{k+1} & \tilde{\mu}_{k+2} & \cdots & \tilde{\mu}_{2k}
\end{bmatrix}
\]

(4)

**Corollary 16.** For every \( \epsilon > 0 \), using \( s = 2^{O(k)} \cdot \frac{1}{\epsilon^2} \cdot \log(1/\delta) \) samples, we can obtain an empirical Hankel matrix satisfying \( \|\tilde{H}_{k+1} - H_{k+1}\|_2 \leq \epsilon \) with probability at least \( 1 - \delta \).

**Proof.** We have \( \|\tilde{H}_{k+1} - H_{k+1}\|_2 \leq \|\tilde{H}_{k+1} - H_{k+1}\|_F \leq (k + 1) \cdot \|\tilde{\mu} - \mu\|_\infty \). Now use Lemma 15 with \( \epsilon_k = \frac{\epsilon}{k+1} \).

### 5 Learning the Source

In this section, we define our learning algorithm, and we state and prove our main result and applications. The auxiliary lemmas are stated and proved in Section 7. The algorithm is specified given \( k \), lower bounds on the source parameters \( \zeta \) and \( w_{\min} \), the empirical histogram \( h \), and a parameter \( \gamma \) controlling the output accuracy. See Algorithm 1 for the full description of the algorithm (where the parameter for probability of success, \( 1 - \delta \), has been suppressed in favor of a constant “0.99”).
Algorithm 1 Algorithm LEARNCOINMIXTURE

1: procedure LEARNCOINMIXTURE($k, \zeta, w_{\text{min}}, h, \gamma$)
2: \hspace{1em} $\tilde{\mu} \leftarrow \text{Pas} h$
3: \hspace{1em} $\hat{H}_{k+1} \leftarrow \text{Hankel}(\tilde{\mu})$
4: \hspace{1em} $v \leftarrow \underset{\varepsilon_{\chi} \approx \text{approx}}{\arg \min} \{ v^T \hat{H}_{k+1} v : v^T v = 1 \}$ \hspace{1em} $\triangleright \varepsilon_1 = w_{\text{min}} \cdot 2^{-\gamma} \cdot (\zeta/16)^{2k}$
5: \hspace{1em} $\tilde{\beta}_1, \tilde{\beta}_2, \ldots, \tilde{\beta}_k \leftarrow \text{Proj}[0,1](\tilde{\hat{V}}_1), \ldots, \text{Proj}[0,1](\tilde{\hat{V}}_k)$
6: \hspace{1em} $\tilde{\alpha}_0, \tilde{\alpha}_1, \ldots, \tilde{\alpha}_k \leftarrow \text{Proj}[0,1](\tilde{\hat{V}}_1), \ldots, \text{Proj}[0,1](\tilde{\hat{V}}_k)$
7: \hspace{1em} $\tilde{\beta}_0 \leftarrow \text{RectifyWeights}(V_{\tilde{\alpha}}^{-1} \tilde{\mu})$ \hspace{1em} $\triangleright$ see Algorithm 2 on page 14
8: \hspace{1em} Output $\hat{M} = (\tilde{\alpha}, \tilde{\beta})$
9: end procedure

Theorem 17. Let $\mathcal{M} = (\alpha, w)$ be a $k$-coin model with separation $\zeta = \zeta(\mathcal{M})$. For any $\gamma \geq 1$, Algorithm 1 uses a histogram $h$ for a sample of $2k$-snapshots of size $s = w_{\text{min}}^{-2} \cdot 2^{O(k+\gamma)} \cdot \zeta^{-O(k)} \cdot \log \delta^{-1}$, and outputs a model $\hat{M} = (\tilde{\alpha}, \tilde{\beta})$ satisfying

$$\|\alpha - \tilde{\alpha}\|_\infty, \|w - \tilde{w}\|_\infty \leq 2^{-\gamma}$$

with probability at least $1 - \delta$. After sampling, Algorithm 1 computes the approximate model $\hat{M}$ using $O(k^2 \log k + k \log^2 k \cdot \log(\log \zeta^{-1} + \log w_{\text{min}}^{-1} + \gamma))$ arithmetic operations.

Proof. Throughout the proof, we make no attempt to optimize the absolute constants that are used. Let $u_1$ denote the unit vector spanning the kernel of $\hat{H}_{k+1}$, and let $v_1$ denote the eigenvector corresponding to the smallest eigenvalue of $\hat{H}_{k+1}$. Also, let $\varepsilon_0 > 0$ be a sufficiently small constant, to be determined later. The analysis of Algorithm 1 can be broken down into steps, each of which degrades the accuracy obtained in the initial sampling. The outline is as follows. The auxiliary claims and proofs appear mostly in Section 7.

1. We assume that $\|\hat{H}_{k+1} - \hat{H}_{k+1}\|_2 \leq w_{\text{min}} \cdot 2^{-\gamma} \cdot (\zeta/16)^{4k}$. This is guaranteed by Lemma 15 and Corollary 16 for a sample of size $s = w_{\text{min}}^{-2} \cdot 2^{O(k+\gamma)} \cdot \zeta^{-O(k)} \cdot \log \delta^{-1}$, with probability at least $1 - \delta$.

2. As $\|\hat{H}_{k+1} - \hat{H}_{k+1}\|_2 \leq w_{\text{min}} \cdot 2^{-\gamma} \cdot (\zeta/16)^{4k}$, by Lemma 19,

$$\|u_1 - v_1\|_2 < \sqrt{2(k+1)} \cdot 2^{-\gamma} \cdot (\zeta/16)^{2k} < \frac{1}{2} \cdot 2^{-\gamma} \cdot (\zeta/8)^{2k}.$$  

3. We use Lemma 20 with $\varepsilon = w_{\text{min}} \cdot 2^{-\gamma} \cdot (\zeta/16)^{2k}$, which satisfies the conditions of the lemma. We compute $v \in \mathbb{R}^{k+1}$ such that

$$\|v - v_1\|_2 \leq \varepsilon < \frac{1}{2} \cdot 2^{-\gamma} \cdot (\zeta/16)^{2k},$$

using $O(k^2 \log k + k \log^2 k \cdot \log(\log \zeta^{-1} + \log w_{\text{min}}^{-1} + \gamma))$ arithmetic operations.

4. As $\|u_1 - v_1\|_2, \|v - v_1\|_2 < \frac{1}{2} \cdot 2^{-\gamma} \cdot (\zeta/16)^{2k}$, we have that $\|u_1 - v\|_2 < 2^{-\gamma} \cdot (\zeta/16)^{2k}$. So, by Lemma 21,

$$\|q - r\|_\infty < 2^k \cdot \sqrt{k+1} \cdot 2^{-\gamma} \cdot (\zeta/16)^{2k} < 2^{-\gamma} \cdot (\zeta/8)^{2k},$$

where $q := u_1/|u_1|, r := v/|v|$.  

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5. As \( \|q - r\|_\infty < 2^{-\gamma} \cdot (\zeta/8)^{2k} \leq \frac{1}{2^{\alpha(k+1)}} \cdot 2^{-\gamma} \cdot (\zeta/2)^{2k-1} \), by Lemma 22 we have that
\[
d(\alpha, \beta) \leq \frac{1}{6(k + 1)} \cdot 2^{-\gamma} \cdot (\zeta/2)^k
\]
(where \( \alpha \) is the vector of roots of \( \hat{q} \) and \( \beta \) is the vector of roots of \( \hat{r} \) and \( d \) is the matching distance, defined in Lemma 22).

6. We use Corollary 24 with \( \rho = \varepsilon = \frac{1}{6(k + 1)} \cdot 2^{-\gamma} \cdot (\zeta/2)^k \), which satisfy the conditions of the corollary. Thus, we can compute biases \( \hat{\alpha}_1, \ldots, \hat{\alpha}_k \) satisfying
\[\|\hat{\alpha} - \alpha\|_\infty \leq \rho + \varepsilon \leq \frac{1}{3(k + 1)} \cdot 2^{-\gamma} \cdot (\zeta/2)^k,\]
using \( O(k \log^2 k \cdot (\log(\log \zeta^{-1} + \gamma) + \log^2 k)) \) arithmetic operations.

7. Finally, line 7 can be executed in the time its takes to invert the Vandermonde matrix \( V_{\hat{\alpha}} \) (i.e., \( O(k^2) \) arithmetic operations, for instance using Parker’s algorithm [21]; by Lemma 30 the procedure RECTIFYWEIGHTS takes \( O(k) \) operations). By Corollary 31, as \( \|\hat{\alpha} - \alpha\|_\infty, \|\hat{\mu} - \mu\|_\infty \leq \frac{1}{3(k + 1)} \cdot 2^{-\gamma} \cdot (\zeta/2)^k \) (the guarantee for \( \hat{\mu} \) is implied with plenty of room to spare by our assumption on the sample), we have \( \|\hat{w} - w\|_\infty \leq 2^{-\gamma}. \)

Notice that the proof actually gives a stronger guarantee for \( \|\hat{\alpha} - \alpha\|_\infty \), which is relative to \( \zeta/2^k \). We can get a relative guarantee \( \|\hat{w} - w\|_\infty \leq w_{\min} \cdot 2^{-\gamma} \) by increasing the sample size by a factor of \( w_{\min}^2 \).

**Corollary 18.** Let \( W(\mathcal{M}, \tilde{\mathcal{M}}) \) denote the Wasserstein distance between models \( \mathcal{M} \) and \( \tilde{\mathcal{M}} \) (viewed as metric measure spaces on \( [0, 1] \)). Then, \( W(\mathcal{M}, \tilde{\mathcal{M}}) \leq (k + 1) \cdot 2^{-\gamma} \) with probability at least 0.99. Proof in Appendix A.

### 6 Implications for Topic Models

Theorem 17 improves upon the upper bound of Theorem 5.1 in [25], which uses a sample of \( (2k - 1) \)-snapshots of size \( \max \left\{ (2/\zeta)^{O(k)}, (2^k k)^{O(k^2)} \right\} \) to achieve accuracy \( 2^{-\gamma} \) with high probability, using runtime of \( O(k^c) \) arithmetic operations, for a relatively large constant \( c \) (in particular, the algorithm solves a convex quadratic program whose representation uses \( k^3 \) bits). Theorem 17 also improves upon the upper bound of [16] [See Theorem 3.9 in the ArXiv version: https://arxiv.org/pdf/1504.02526.pdf]. That algorithm uses a sample size comparable to ours, but requires runtime \( (2^k k)^{O(k^2)} \) to achieve accuracy \( 2^{-\gamma} \) with high probability.

These improvements imply immediately a similar improvement for learning pure \( k \)-topic models, using known reductions from \( k \)-topic models to \( k \)-coin models. The reduction in Theorem 4.1 of [25] uses a sample of \( 1 \) and \( 2 \)-snapshots of size \( O(n \cdot \text{poly} (\log n, k, w_{\min}^{-1}, (\zeta^{-1}, 2^\gamma))) \), and runtime polynomial in the sample size, to reduce the problem to solving \( k \) instances of the \( k \)-coin problem with accuracy \( \min \left\{ (2^k k/w_{\min})^{-O(1)}, (2^k k)^{-O(k)} \right\} \). The reduction in [16] [See Theorem 6.1 in the ArXiv version.] uses a sample of \( 1 \) and \( 2 \)-snapshots of size \( \text{poly} (n, k, 2^\gamma) \), and runtime polynomial in the sample size, to reduce the problem to solving at most \( k \) instances of the \( k \)-coin problem with accuracy \( (2^k k)^{-O(k)} \). Notice that solving the \( k \)-coin outcome of either one of the two reductions using either one of the two previous algorithms requires a sample size of at least \( k^{O(k^3)} \) (on account of the required accuracy). Our algorithm enables a solution to the outcome of these reductions using a sample size of \( k^{O(k)} \) (and total runtime of \( O(k^{3+o(1)}) \)). We note that the accuracy in [25, 16] is stated in terms of Wasserstein distance, which is a weaker guarantee than the one we use here (see Corollary 18).
7 Analysis

In this section we prove the lemmas that are needed in the proof of Theorem 17. We have to cope with the fact that roots of polynomials (and even, generally, of polynomials with well-separated roots), are notoriously ill-conditioned in terms of the polynomial coefficients [29]. For this reason we will be developing bounds specifically adapted to our situation. We begin with an estimate on the accuracy of the recovered kernel of the Hankel matrix.

7.1 Approximating the kernel of \( \tilde{H}_{k+1} \)

**Lemma 19.** Let \( \mathcal{P} \) be any \( k \)-coin distribution with separation \( \zeta \). Then, for every \( \varepsilon < w_{\min} \cdot \left( \frac{\zeta}{16} \right)^{2k} \) the following holds. Suppose that \( \| \tilde{H}_{k+1} - H_{k+1} \|_2 \leq \varepsilon \). Let \( u_1 \) be the unit vector in the kernel of \( H_{k+1} \) and let \( v_1 \) be the unit eigenvector corresponding to \( \lambda_1(\tilde{H}_{k+1}) \) (chosen so that \( u_1^T v_1 \geq 0 \)). Then \( \| u_1 - v_1 \|_2 < \sqrt{2(k + 1)} \cdot \left( \frac{16}{\zeta} \right)^{2k} \cdot \frac{\varepsilon}{w_{\min}} \).

**Proof.** By Weyl’s inequality, we have that \( \lambda_1(\tilde{H}_{k+1}) \leq \varepsilon \). By Corollary 12, the eigengap \( \lambda_2(H_{k+1}) - \lambda_1(H_{k+1}) \) is at least

\[
\hat{w}_{\min} \left( \frac{\zeta}{16} \right)^{2k} - \hat{w}_{\min} \left( \frac{\zeta}{16} \right)^{2k} > \hat{w}_{\min} \left( \frac{\zeta}{16} \right)^{2k}.
\]

Now we can use Corollary 34 to obtain

\[
u_1^T v_1 = \| u_1^T v_1 \| \geq \left( 1 - \frac{\| H_{k+1} - \tilde{H}_{k+1} \|_F^2}{\lambda_2(H_{k+1}) - \lambda_1(H_{k+1})} \right)^{1/2} > 1 - \frac{(k + 1) \cdot \varepsilon^2}{w_{\min}^2 \left( \frac{\zeta}{16} \right)^{4k}}.
\]

Since \( \| u_1 - v_1 \|_2^2 = 2 - 2 u_1^T v_1 \) we get that

\[
\| u_1 - v_1 \|_2 < 2(k + 1) \cdot \left( \frac{16}{\zeta} \right)^{4k} \cdot \left( \frac{\varepsilon}{w_{\min}} \right)^2.
\]

Recall that \( (\lambda_1(\tilde{H}_{k+1}), v_1) \) is an eigenpair of \( \tilde{H}_{k+1} \). We need to compute a good approximation of \( v_1 \). This can be done using the following lemma. The result is implied by the algorithm of Pan and Chen (Theorem 1.2 of [18]). Extracting our lemma from the result in that paper is somewhat involved and we provide in Appendix A a brief outline of the argument (in particular, the parts that are not spelled out in that paper).

**Lemma 20.** For every \( \varepsilon \) such that \( 0 < \varepsilon \ll \min\{ \lambda_2(\tilde{H}_{k+1}) - \lambda_1(\tilde{H}_{k+1}), 1 \} \), we can compute a unit vector \( v \) satisfying \( \| v - v_1 \|_2 < \varepsilon \) using \( O \left( k^2 \log k + k \log^2 k \log \log(1/\varepsilon) \right) \) arithmetic operations.

7.2 The roots of the approximate kernel polynomial

We need to show that our computed eigenvector of the empirical Hankel matrix is close to the true eigenvector of the true Hankel matrix.
Lemma 21. Let $\mathcal{P}$ be any $k$-coin distribution with separation $\zeta$. Let $u_1$ be a unit vector in the kernel of $\mathcal{H}_{k+1}$. Let $v$ be a unit vector satisfying $\|u_1 - v\|_2 < \varepsilon$ for some $\varepsilon > 0$. Let $q = u_1/\|(u_1)_{k}\|$ and let $r = v/\|(u_1)_{k}\|$. Then $\|q - r\|_\infty < 2^k \sqrt{k+1} \cdot \varepsilon$.

Proof. Notice that $q$ and $r$ are well-defined, as $(u_1)_{k} \neq 0$ by the second part of Lemma 10. Now each of the coefficients of $q$ can be bounded by

$$|q_i| = |e_{k-i}(\alpha_1, \ldots, \alpha_k)| \leq \binom{k}{i} \varepsilon$$

where $e_r$ is the $r$-th elementary symmetric polynomial over $k$ variables. Now $\|q\|_2 \leq \sqrt{k+1} \cdot q_1 \leq 2^k \sqrt{k+1}$. Since $\|(u_1)_{k}\| \cdot q_2 = \|u_1\|_2 = 1$, we have $|(u_1)_{k}| \leq \frac{1}{2^k \sqrt{k+1}}$, and

$$\|q - r\|_\infty \leq \|q - r\|_2 \leq 2^k \sqrt{k+1} \cdot \|u_1 - v\|_2 < 2^k \sqrt{k+1} \cdot \varepsilon,$$

as stipulated.

We’re going to use the roots of the polynomial $\hat{r}$ as our guessed coin biases (after projecting the roots back to $[0, 1]$). We first need to show that the roots of $\hat{q}$ are well-behaved with respect to perturbations of $q$ so that when $q$ and $r$ are close the roots of $\hat{q}$ are close to the roots of $\hat{r}$.

Lemma 22. Let $q \in \mathbb{R}^{k+1}$ be the vector representing a degree-$k$ monic polynomial with roots $\alpha_1, \alpha_2, \ldots, \alpha_k$ contained in $[0, 1]$. Let $\zeta$ be the root separation for $\hat{q}$. Let $r \in \mathbb{R}^{k+1}$ represent another degree-$k$ polynomial. Let $\varepsilon \in \left(0, \frac{(\zeta/2)^k}{4k}\right)$. If $r$ satisfies $\|q - r\|_\infty \leq \varepsilon$, then the (possibly complex) roots $\beta_1, \beta_2, \ldots, \beta_k$ of $\hat{r}$ satisfy

$$d(\alpha, \beta) \leq \frac{4k \varepsilon}{(\zeta/2)^k - 1}$$

where $d(\alpha, \beta)$ is the optimal matching distance defined by

$$d(\alpha, \beta) := \min_{\sigma \in S_k} \max_{i} |\alpha_i - \beta_{\sigma(i)}| .$$

Proof. Fix any root $\alpha_i$ of $\hat{q}$, and consider the ball

$$B_i = B(\alpha_i, \frac{4k \varepsilon}{(\zeta/2)^k - 1})$$

in the complex plane. By assumption, $\frac{4k \varepsilon}{(\zeta/2)^k - 1} < \frac{\zeta}{2}$, so there are no other roots of $\hat{q}$, aside from $\alpha_i$, in $B_i$. Moreover, for any $x \in B_i$, and for any $j \neq i$, we have that $|x - \alpha_j| \geq \frac{\zeta}{2}$. Thus for every $x \in \partial B_i$, we have

$$|\hat{q}(x)| = \left| (x - \alpha_i) \prod_{j \neq i} (x - \alpha_j) \right| \geq \frac{4k \varepsilon}{(\zeta/2)^k - 1} \left( \frac{\zeta}{2} \right)^{k-1} = 4k \varepsilon.$$

On the other hand, we also have that $B_i \subset B(0, (2k - 1)/(2k - 2))$, as $\alpha_1, \ldots, \alpha_k \in [0, 1]$ and $\zeta \leq \frac{1}{x - 1}$. 

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Therefore, $|x| \leq \frac{2k-1}{2k-2}$, and thus

$$|\hat{q}(x) - \hat{r}(x)| = \left| \sum_{j=0}^{k} (q_j - r_j)x^j \right|$$

$$\leq \sum_{j=0}^{k} |q_j - r_j| \cdot |x|^j$$

$$\leq (k + 1) \cdot \left( \frac{2k - 1}{2k - 2} \right)^k \cdot \|q - r\|_\infty$$

$$\leq 4k \varepsilon.$$

By Rouché’s theorem (Theorem 37), we conclude that there is exactly one zero of $\hat{r}$ in $B_i$ and the matching distance bound follows immediately.

Our reconstructed coin biases will be denoted $\hat{\alpha}_1, \hat{\alpha}_2, \ldots, \hat{\alpha}_k$. We compute these biases by finding the roots of $\hat{\nu}$ (approximately), and then by projecting these roots onto the unit interval. To find the approximate roots we can use the following result of Pan.

**Theorem 23** (Pan’s Algorithm: Theorem 1.1 of [17]). Given a monic degree $k$ polynomial $\hat{p}$ with roots $\rho_1, \ldots, \rho_k \in B(0, 1)$ and an accuracy parameter $\gamma > 1$, we can compute approximate roots $\hat{\rho}_1, \ldots, \hat{\rho}_k$ satisfying $\|\rho - \hat{\rho}\|_\infty \leq 2^{-\gamma} \text{ in time } O(k \log^2 k \cdot (\log \gamma + \log^2 k))$.

**Corollary 24.** Let $q \in \mathbb{R}^{k+1}$ represent the polynomial $\hat{q}(z) = \prod_{i=1}^{k} (z - \alpha_i)$ where $\alpha_1, \ldots, \alpha_k \in [0, 1]$ are $\zeta$-separated, and let $r \in \mathbb{R}^{k+1}$ represent a polynomial of degree $k$ with roots $\beta_1, \ldots, \beta_k$ satisfying $d(\alpha, \beta) \leq \rho < \zeta/2$. For every $\varepsilon \in (0, \zeta/2 - \rho)$, we can reconstruct biases $\hat{\alpha}_1, \ldots, \hat{\alpha}_k$ satisfying $\|\hat{\alpha} - \alpha\|_\infty \leq \rho + \varepsilon$ using $O(k \log^2 k \cdot (\log \log(1/\varepsilon) + \log^2 k))$ arithmetic operations.

**Proof.** We’ll first find approximate the roots $\hat{\beta}_1, \ldots, \hat{\beta}_k$ of $\hat{r}$ using Theorem 23. Since the roots of $\hat{r}$ are in $B \left(0, \frac{2k - 1}{2k - 2} \right)$ instead of $B(0, 1)$, we’ll actually find the roots of $\hat{r} \left( \frac{2k - 2}{2k - 1} z \right)$ and then multiply by $\frac{2k - 2}{2k - 1}$ to get the roots of $\hat{r}$ up to accuracy $\varepsilon$ in time $O(k \log^2 k \cdot (\log \log(1/\varepsilon) + \log^2 k))$. (Notice that in order to get the desired accuracy we need to run Pan’s algorithm to get the rescaled roots to within distance $\frac{2k - 2}{2k - 1} \cdot \varepsilon$; this doesn’t matter for the purposes of runtime.)

Our output is $\hat{\alpha}_i := \text{Proj}_{[0,1]}(\hat{\beta}_i)$ for $i = 1, \ldots, k$, where we label the roots $\hat{\beta}_1, \ldots, \hat{\beta}_k$ by the permutation achieving the matching distance, i.e., the ordering of coordinates so that $\|\alpha - \beta\|_\infty = d(\alpha, \beta)$. Now

$$|\alpha_i - \hat{\alpha}_i| \leq \left| \alpha_i - \Re(\hat{\beta}_i) \right|$$

$$\leq |\alpha_i - \Re(\beta_i)| + \left| \Re(\beta_i) - \Re(\hat{\beta}_i) \right|$$

$$\leq |\alpha_i - \beta_i| + |\beta_i - \hat{\beta}_i|$$

$$\leq \rho + \varepsilon.$$

7.3 Recovering the mixture weights from the roots

Once we’ve recovered the parameters $\hat{\alpha}_1, \ldots, \hat{\alpha}_k$, we need to use those to recover mixture weights. This sequence of steps—first solving (approximately) for the roots, then for the mixture weights—is the essence of Prony’s method [8, 12] §9.4, [15]. In this section, we’ll show that this recovery can be done by solving a linear system without paying too great a price in terms of accuracy.
We’ll begin by stating results characterizing the condition number of a Vandermonde system under perturbations of a Vandermonde matrix that preserve the Vandermonde structure.

**Lemma 25** (Operator norm bound for a Vandermonde inverse; equation 3.2 in [4]). Let \( \alpha \in \mathbb{R}^k \) be entry-wise non-negative, and let \( q(z) = \prod_{i=1}^{k} (z - \alpha_i) \). Then

\[
\|V^{-1}_\alpha\|_\infty = \frac{|q(-1)|}{\min_i \{(1 + \alpha_i) |q'(\alpha_i)|\}}.
\]

**Claim 26.** For roots \( \alpha_1, \ldots, \alpha_j \) satisfying \(|\alpha_i - \alpha_j| \geq \zeta\), we have \( \|V^{-1}_\alpha\|_\infty \leq 2^k/\zeta^{k-1} \).

**Proof.** We apply Lemma 25 and observe that \( |q(-1)| \leq 2^k \) and \( q'(\alpha_i) \geq \zeta^{k-1} \).

We define the derivative matrix of the Vandermonde matrix by interpreting each entry as the evaluation of a polynomial at a point, \( [V_a]_{ij} = p_i(a_j) \), where \( p_i(t) = t^i \). Then \( [V_a']_{ij} = p'_i(a_j) = i\alpha_j^{i-1} \).

We’ll now define the condition number of the system,

\[
\text{cond}_\infty(a, b) := \lim_{\varepsilon \to 0} \sup_{\|\Delta a\|_\infty \leq \varepsilon, \|\Delta b\|_\infty \leq \varepsilon} \left\{ \frac{\|\Delta x\|_\infty}{\varepsilon} : V(a + \Delta a)(x + \Delta x) = b + \Delta b \right\}.
\]

(5)

We’ll utilize a bound from [4]. After instantiating the theorem with the parameters relevant to our problem, the bound is the following:

**Theorem 27** (Theorem 2.2 of [4]).

\[
\text{cond}_\infty(a, b) \leq \|V^{-1}_a\|_\infty + \|V^{-1}_aV'_a \text{diag}(x)\|_\infty.
\]

**Lemma 28.** Let \( \alpha \in [0,1]^k \), and let \( w \in \mathbb{R}^k \) be a probability distribution over \([k]\). Let \( \mu = V_\alpha w \). If \( \zeta \leq \min_{i \neq j} |\alpha_i - \alpha_j| \),

\[
\text{cond}_\infty(\alpha, \mu) \leq (k + 1)2^k/\zeta^{k-1}.
\]

**Proof.** We observe that

\[
\|V^{-1}_aV'_a \text{diag}(w)\|_\infty \leq \|V^{-1}_a\|_\infty \|V'_a \text{diag}(w)\|_\infty \\
\leq 2^k/\zeta^{k-1} \|V'_a \text{diag}(w)\|_\infty \\
= \frac{2^k/\zeta^{k-1}}{i \in [k-2]} \sum_{j=1}^{k} |\alpha_j^iw_j| \\
\leq k2^k/\zeta^{k-1}.
\]

Applying the bound of Theorem 27 gives the conclusion.

**Lemma 29.** Let \( \alpha \in [0,1]^k \) and let \( w \in \mathbb{R}^k \) be a probability distribution over \([k]\). Let \( \mu = V_\alpha w \), and \( \zeta \leq \min_{i \neq j} |\alpha_i - \alpha_j| \). Then \( w' := V_\alpha^{-1} \tilde{\mu} \) satisfies

\[
\|w' - w\|_\infty \leq \frac{(k + 1)2^k}{\zeta^{k-1}} \max \{\|\tilde{\alpha} - \alpha\|_\infty, \|\tilde{\mu} - \mu\|_\infty\}.
\]

**Proof.** This follows from Lemma 28 and the definition of the condition number.
Lemma 30. Given any weights \( w' \in \mathbb{R}^k \) satisfying \( \sum_{i=1}^{k} w'_i = 1 \), the procedure \( \text{RECTIFYWEIGHTS}(w') \) outputs in time \( O(k) \) a weight vector \( \tilde{w} \in [0,1]^k \) satisfying the following conditions

(i) \( \sum_{i=1}^{k} \tilde{w}_i = 1 \).

(ii) \( \| \tilde{w} - w \|_\infty \leq (k + 1) \| w' - w \|_\infty \).

Proof. Note that in \( 2 \), \( I^- \) denotes the indices of the negative weights, and \( I^+ \) the positive weights. \( W^- \) and \( W^+ \) denote the sums of the weights in the corresponding set of indices.

We’ll now analyze \( \tilde{w} \). First, note that we maintain property (i):

\[
\sum_{i=1}^{k} \tilde{w}_i = \sum_{i \in I^+} w'_i \left( 1 + \frac{W^-}{W^+} \right) = W^+ \left( 1 + \frac{W^-}{W^+} \right) = W^+ + W^- = 1
\]

Now we show that the weights are non-negative. Trivially, \( \tilde{w}_i \geq 0 \) for \( i \in I^- \). For \( i \in I^+ \),

\[
W^+ = 1 - W^-
= 1 + |W^-|
\geq |W^-|
\]

So \( w'_i (1 + \frac{W^-}{W^+}) \geq 0 \) if \( i \in I^+ \) as well.

We now prove (ii). We know that the true weights \( w \) lie in \([0,1]\), so increasing the negative weights to 0 only moves them closer to their true values. Thus, we have \( |\tilde{w}_i - w_i| \leq |w'_i - w_i| \) for all \( i \in I^- \). We observe that

\[
|W^-| \leq \| w' - w \|_1 \leq k \| w' - w \|_\infty
\]

and then that

\[
|\tilde{w}_i - w'_i| = \left( \frac{w'_i}{W^+} \right) W^- \leq k \| w' - w \|_\infty.
\]

It follows that \( \| \tilde{w} - w' \|_\infty \leq k \| w' - w \|_\infty \). Now we can apply the triangle inequality to get that

\[
\| \tilde{w} - w \|_\infty \leq \| \tilde{w} - w' \|_\infty + \| w' - w \|_\infty \leq (k + 1) \| w' - w \|_\infty.
\]

To see that the runtime is \( O(k) \) we observe we can compute \( I^- \) and \( I^+ \) in linear time and likewise for \( W^- \) and \( W^+ \). Each subsequent computation of \( \tilde{w}_i \) takes constant time. \( \square \)

Corollary 31. Letting \( \tilde{w} \in [0,1]^k \) be the output of \( \text{RECTIFYWEIGHTS}(w') \) where \( w' \) is as in Lemma 29, \( \| \tilde{w} - w \|_\infty \leq \frac{(k + 1)^2}{\zeta \gamma^k} \max \{ \| \tilde{\alpha} - \alpha \|_\infty, \| \tilde{\mu} - \mu \|_\infty \} \).

Proof. Notice that the first equation in the linear system defining \( w' \) is

\[
\sum_{i=1}^{k} w'_i = \mathbf{1}^T w' = \tilde{\mu}_0 = 1.
\]

Thus, \( w' \) satisfies the hypothesis of Lemma 30 and the conclusion follows. \( \square \)
Algorithm 2 Algorithm RECTIFYWEIGHTS

1: procedure RECTIFYWEIGHTS($w'$)
2:     \( I^- \leftarrow \{i \mid w'_i < 0\}, \quad I^+ \leftarrow \{i \mid w'_i \geq 0\} \)
3:     \( W^- \leftarrow \sum_{i \in I^-} w'_i, \quad W^+ \leftarrow \sum_{i \in I^+} w'_i \)
4:     for \( i = 1, \ldots, k \) do
5:         \( \tilde{w}_i \leftarrow \begin{cases} 0 & \text{if } i \in I^- \\ w'_i \left(1 + \frac{w^-}{w^+}\right) & \text{if } i \in I^+ \end{cases} \)
6:     end for
7:     Output \( \tilde{w} \)
8: end procedure

A Deferred Proofs

Proof of Lemma 14 We first observe that

\[
\begin{bmatrix}
(0) \ldots (0) \\
(0) \ldots (0) \\
\vdots \\
(0) \ldots (0)
\end{bmatrix}
= \begin{bmatrix}
(0) \ldots (0) \\
(0) \ldots (0) \\
\vdots \\
(0) \ldots (0)
\end{bmatrix}
\begin{bmatrix}
(0) \ldots (0) \\
(0) \ldots (0) \\
\vdots \\
(0) \ldots (0)
\end{bmatrix}
\]

which can be factored to obtain

\[
\begin{bmatrix}
(2k)^{-1} \\
0 \\
0 \\
0
\end{bmatrix}
\begin{bmatrix}
(2k)^{-1} \\
(2k)^{-1} \\
(2k)^{-1} \\
(2k)^{-1}
\end{bmatrix}
\begin{bmatrix}
(0) \ldots (0) \\
(0) \ldots (0) \\
\vdots \\
(0) \ldots (0)
\end{bmatrix}
\begin{bmatrix}
(0) \ldots (0) \\
(0) \ldots (0) \\
\vdots \\
(0) \ldots (0)
\end{bmatrix}
\]

Now

\[
\|
\text{diag}\left(\begin{array}{ccc}
(2k) \\
(2k) \\
\vdots \\
(2k)
\end{array}\right)
\right|_2 
\leq 1.
\]

The Frobenius norm of the latter matrix is

\[
\left(\sum_{j=0}^{2k} \sum_{i=0}^{j} \left(\frac{j}{i}\right)^2 \right)^{1/2} = \left(\sum_{j=0}^{2k} \left(\frac{2j}{j}\right) \right)^{1/2} \leq \left(2k \left(\frac{4k}{2k}\right) \right)^{1/2} \leq \left(2k \left(\frac{4k}{2k}\right) \right)^{1/2} \leq 6^k
\]

for \( k \geq 2 \). Using the sub-multiplicativity of the operator norm and the fact that the Frobenius norm upper bounds the operator norm, we get that \( \|\text{Pas}\| \leq 6^k \), as desired.

Proof of Corollary 18 Each \( \alpha_i \) can be matched to its corresponding \( \tilde{\alpha}_i \) up to weight \( \min\{w_i, \tilde{w}_i\} \). The
additional $|w_i - \tilde{w}_i|$ must move an additional distance of at most 1. This gives

$$W(\mathcal{M}, \tilde{\mathcal{M}}) \leq \sum_{i=1}^{k} |\alpha_i - \tilde{\alpha}_i| \cdot \min\{w_i, \tilde{w}_i\} + \sum_{i=1}^{k} |w_i - \tilde{w}_i|$$

$$\leq \sum_{i=1}^{k} 2^{-\gamma} \min\{w_i, \tilde{w}_i\} + \sum_{i=1}^{k} 2^{-\gamma}$$

$$\leq (k + 1) \cdot 2^{-\gamma},$$

using Theorem 17 and the fact that $\sum_{i=1}^{k} \min\{w_i, \tilde{w}_i\} \leq \sum_{i=1}^{k} w_i = 1.$

**Proof sketch of Lemma 20** We follow the outline in the papers by Pan, Chen, and Zheng [18, 19]. As $\widetilde{\mathcal{H}}_{k+1}$ is a Hankel matrix, a similarity transformation $A = T \mathcal{H}_{k+1} T^{-1}$, where $A$ is tridiagonal, can be computed in time $O(k^2 \log k)$. The characteristic polynomial $c_A(x)$ of $A$ can then be computed in time $O(k)$. Then, a root $\tilde{\lambda}$ that satisfies $|\tilde{\lambda} - \lambda_1(\mathcal{H}_{k+1})| < \varepsilon^2$ can be computed in time $O((k \log^2 k)(\log \log (1/\varepsilon) + \log^2 k))$ (see Theorem 23, note that $\|A\|_2 = \|\mathcal{H}_{k+1}\|_2$, thus it is trivially upper bounded by $(k + 1)^2$). Next, proceed to compute $v$ as follows. Pick an initial guess $v^{(0)}$ uniformly at random on the unit sphere (i.e., from the unit Haar measure on the sphere). We need $v^{(0)}$ to have $\|v^{(0)}\|_2 \geq 1$, which happens with constant probability. To boost the success probability to $1 - \delta$, we can repeat the entire process $O(\log(1/\delta))$ times. For constant $\delta$, this does not affect the asymptotic bound. We compute $v^{(1)}, v^{(2)}, \ldots$ using the inverse power iteration (see, for instance, Chapter 4 in [22]): Solve for $v^{(t)}$ the system of linear equations $(\tilde{\lambda}I - \mathcal{H}_{k+1}) v^{(t)} = v^{(t-1)}$, then set $v^{(t+1)} = \frac{v^{(t)}}{\|v^{(t)}\|_2}$. As $\mathcal{H}_{k+1}$ is a Hankel matrix, this can be done using $O(k^2)$ arithmetic operations. How many iterations are needed?—It is known that if $\lambda_1(\mathcal{H}_{k+1})$ is the unique eigenvalue of $\mathcal{H}_{k+1}$ that is closest to $\tilde{\lambda}$, and if $v^{(0)}_i \geq 0$, then $\tan \theta^{(t)} \leq 1$, where $\theta^{(t)}$ is the angle between $v_1$ and $v^{(t)}$, and $\rho = \frac{|\tilde{\lambda} - \lambda_1(\mathcal{H}_{k+1})|}{|\lambda_1 - \lambda_2(\mathcal{H}_{k+1})|}$, where $\lambda_2$ is the eigenvalue of $\mathcal{H}_{k+1}$ that is second-closest to $\tilde{\lambda}$. Notice that in our case $\rho = \frac{|\tilde{\lambda} - \lambda_1(\mathcal{H}_{k+1})|}{|\lambda_1 - \lambda_2(\mathcal{H}_{k+1})|} \leq \frac{\varepsilon^2}{1 - \varepsilon^2} < 2\varepsilon$. As $\tan \theta^{(0)} \leq \sqrt{k}$, after $t = O(\log(1/\delta))$ iterations, we have $\tan \theta^{(t)} < \varepsilon$. This implies that $\|v^{(t)} - v_1\|_2 < \varepsilon.$

**B Useful Theorems**

Consider two $n \times n$ Hermitian matrices $A, B$, with spectral decompositions $A = \sum_{i=1}^{n} \kappa_i u_i u_i^T$ and $B = \sum_{i=1}^{n} \lambda_i v_i v_i^T$, where the eigenvalues of both matrices are sorted in increasing order (i.e., $\kappa_1 \leq \kappa_2 \leq \cdots \leq \kappa_n$ and $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$). Also, let $P = B - A$ and let $\rho_1 \leq \rho_2 \leq \cdots \leq \rho_n$ be the eigenvalues of $P$ in increasing order.

**Theorem 32** (Weyl’s inequality). For every $i \in \{1, 2, \ldots, n\}$,

$$\kappa_i + \rho_1 \leq \lambda_i \leq \kappa_i + \rho_n.$$

**Theorem 33** (Davis-Kahan sin $\Theta$ theorem). Using the above definitions, let $i_0, i_1$ be integers such that $1 \leq i_0 \leq i_1 \leq n$, and let

$$g = \inf\{\|\kappa - \lambda\| : \kappa \in [\kappa_{i_0}, \kappa_{i_1}] \land \lambda \in (-\infty, \lambda_{i_0 - 1}] \cup [\lambda_{i_1 + 1}, +\infty)\},$$

where we define $\lambda_0 = -\infty$ and $\lambda_{n+1} = +\infty$. Then,

$$\|\sin \Theta(U, V)\|_F \leq \frac{\|P\|_F}{g},$$

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where \( U (V, \text{respectively}) \) is the \( n \times i_1 - i_0 + 1 \) matrix whose columns are \( u_{i_0}, \ldots, u_{i_1} \) (\( v_{i_0}, \ldots, v_{i_1} \), respectively), \( \Theta(U, V) \) is the \( i_1 - i_0 + 1 \times i_1 - i_0 + 1 \) diagonal matrix whose \( i \)-th diagonal entry is the \( i \)-th principal angle between the column spaces of \( U \) and \( V \), and \( \sin \Theta(U, V) \) is the diagonal matrix derived by applying the function \( \sin \) entrywise to \( \Theta(U, V) \). The same inequality holds if the Frobenius norm is replaced by any orthogonally invariant norm, e.g., an operator norm \( \| \cdot \|_{\text{op}} \).

**Corollary 34.** Using the same definitions,

\[
|u_1^T v_1| \geq \sqrt{1 - \frac{\| P \|^2}{|\kappa_1 - \lambda_2|^2}}.
\]

**Proof.** Take \( i_0 = i_1 = 1 \). By Theorem 33, \( |\sin \theta(u_1, v_1)| \leq \frac{\| P \|}{|\kappa_1 - \lambda_2|} \). The corollary follows as \( |u_1^T v_1| = |\cos \theta(u_1, v_1)| = \sqrt{1 - \sin^2 \theta(u_1, v_1)} \). \( \square \)

**Theorem 35** (Courant-Fischer-Weyl min-max principle). For every \( i = 1, 2, \ldots, n \),

\[
\lambda_i = \min_{U \subseteq \mathbb{R}^n} \left\{ \max_{x \in U} \left\{ \frac{x^T B x}{x^T x} : x \neq 0 \right\} : \dim(U) = i \right\}
= \max_{U \subseteq \mathbb{R}^n} \left\{ \min_{x \in U} \left\{ \frac{x^T B x}{x^T x} : x \neq 0 \right\} : \dim(U) = n - i + 1 \right\}.
\]

Let \( C \) be an \( m \times m \) Hermitian matrix with eigenvalues \( \nu_1 \leq \nu_2 \leq \cdots \leq \nu_m \), where \( m \leq n \).

**Theorem 36** (Cauchy’s interlacing theorem). If \( C = \Pi^* B \Pi \) for an orthogonal projection \( \Pi \), then for all \( i = 1, 2, \ldots, m \) it holds that \( \lambda_i \leq \nu_i \leq \lambda_{n-m+i} \).

**Theorem 37** (Rouché's theorem). Let \( f \) and \( g \) be two complex-valued functions that are holomorphic inside a region \( R \) with a closed simple contour \( \partial R \). If for every \( x \in \partial R \) we have that \( |g(x)| < |f(x)| \), then \( f \) and \( f + g \) have the same number of zeros inside \( R \), counting multiplicities.
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