On second solutions of the sixth-order nonlinear mathematical model with measured derivatives

Elena Borodina\(^1\), Sergey Shabrov\(^2\), Faina Golovaneva\(^3\), Eleanora Kurkinskaya\(^4\)

\(^1\) Voronezh State University of Engineering Technologies, 394036, 19, Revolutsii Prospect, Voronezh, Russia
\(^2\) Department of Mathematical Analysis, 394018, 1, University Square, Voronezh State University, Voronezh, Russia
\(^3\) Department of Department of Partial Differential Equations and Probability Theory, Voronezh State University, Voronezh, Russia
\(^4\) Department of Mathematical Analysis, Voronezh State University, Voronezh, Russia

E-mail: eaborodina@inbox.ru; shabrov.s.a@math.vsu.ru; gfainav@mail.ru

Abstract.
In the present paper the sixth-order nonlinear mathematical model with nonsmooth solutions is studied. We consider a case, when the problem is guaranteed to have one solution and investigate the question on the presence of one more. Using the pointwise approach of Yu. V. Pokorny, which has shown its effectiveness in analyzing models of the second and fourth orders, sufficient conditions of the existence of the second solution for the sixth-order model with derivatives with respect to measure are obtained.

1. Introduction
In the present paper a mathematical model

\[
\begin{align*}
Lu &\equiv - (pu''_{xx})_{xx} + ru''_{xx} - (gu')_{x} + Q_{x} = f(x,u) \quad (x \in [0; \ell]), \\
&u(0) = u'(0) = u''(0) = 0, \\
&u(\ell) = u'(\ell) = u''(\ell) = 0,
\end{align*}
\]

(1)

that obviously has one known solution is studied. We investigate the question on the presence of one more solution. Without loss of generality, we will consider the known solution to be zero, since it is possible to make a functional change that makes this solution be equal to zero. Notice that a qualitative theory of equations with nonsmooth solutions began to develop rapidly after the publication in 1999 of the work of Yu. V. Pokorny [1]. The most profound results concerning this topic associated with monographs [2–5], works [6–21]. This efficiency is explained quite simply: when we apply derivatives with respect to the measures, the equation, in contrast to theory of generalized functions, becomes defined in each point and makes it possible to use qualitative methods for analyzing of solutions. Indeed, when using the theory of Schwartz-Sobolev distributions according difficult problems are emerged. The first problem is that only weak solvability can be established. Hence equations are not suitable for applications. The second problem, which has not yet been solved, arises when the generalized function multiply...
by the discontinuous one. The third problem is that equations in generalized functions are the equalities of two functionals defined on the space of basic functions. Hence it is extremely difficult to apply methods of qualitative analysis to such equations.

2. Preliminaries

We will consider a solution of (1) in the class \( E \) of twice continuously differentiable functions \( u(x) \) such as \( u''(x) = \mu \)-absolutely continuous on \([0; \ell]\); \( p_{zzxx}(x) \) is twice continuously differentiable; 
\[
(pu'_{xx})'' = \mu \text{-absolutely continuous on } [0; \ell].
\]

At the points \( \xi \) belonging to the set of discontinuity points of \( \mu(x) \), the equation in (1) is understood as the equality

\[
-\Delta \left( pu_{xx}'' \right)''(\xi) + \Delta \left( g_{xx} u_{xx}' \right)'(\xi) - \Delta(\mu u_{xx}' + Q_{xx})(\xi) + u(\xi) \Delta Q(\xi) = f(\xi, u(\xi)),
\]

where \( \Delta u(\xi) \) is a complete jump of the function \( u(x) \) at the point \( \xi \).

The equation from (1) is given almost everywhere (with respect to the measure \( \mu \)) on the special extension of the interval \([0; \ell]\). Let \( S(\mu) \) be a set of discontinuity points of the function \( \mu(x) \). We introduce on \( J_\mu = [0; \ell] \setminus S(\mu) \) the metric \( d(x,y) = |\mu(x) - \mu(y)| \). The resulting metric space \( (J_\mu; \mu) \) is not complete. The standard completion leads (up to isomorphism) to the set \([0; \ell]_S\), in which each point \( \xi \in S(\mu) \) is replaced with a pair of elements \( \xi - 0, \xi + 0 \), which were previously limit values. By inducing ordering from the original set, we obtain the inequalities \( x < \xi - 0 < \xi + 0 < y \) for all \( x, y \) for which the inequalities \( x < \xi < y \) hold in the initial segment.

The function \( v(\xi) \) at the points \( \xi - 0 \) and \( \xi + 0 \) of the set \([0; \ell]_S\) is defined by the limiting values. For a function defined in this way, we retain the previous notation. The function defined on this set becomes continuous with respect to the metric \( d(x,y) \).

The union of \([0; \ell]_S\) and \( S(\mu) \) gives us the set \([0; \ell]_\mu\), in which each point \( \xi \in S(\mu) \) is replaced by a triple of elements \( \{\xi - 0; \xi; \xi + 0\} \). We suppose that the equation is given on this set.

We assume that the functions \( p(x) \), \( r(x) \), \( g(x) \) and \( Q(x) \) are \( \mu \)-absolutely continuous on \([0; \ell]_S\); \( \min_{x \in [0; \ell]_S} p(x) > 0 \), \( Q(x) \) does not decrease, and \( f(x, u) \) satisfies the conditions of Carathéodory, i.e.,

- \( f(x, u) \) is defined and continuous in \( u \) for almost all \( x \) (with respect to the \( \mu \)-measure);
- the function \( f(x, u) \) is measurable in \( x \) for every \( u \);
- \( |f(x, u)| \leq m(x) \), where \( m(x) \) is a \( \mu \)-summable function on \([0; \ell]_S\).

We say that the homogeneous equation

\[
-\left( pu_{xx}'' \right)'' + (ru_{xx}')' - (gu_{xx}')' + Q_{xx}u = 0
\]

is non-oscillating on \([0; \ell]\) if any of its non-trivial solutions has at most five zeros with respect to multiplicities.

We denote by \( K \) the cone of non-negative functions on \([0; \ell]\).

**Lemma.** Let \( G(x,s) \) be the influence function [5] of the model

\[
Lu = F_{\sigma}'(s), \quad u(0) = u_x(0) = u_{xx}(0) = 0, \quad u(\ell) = u_x(\ell) = u_{xx}(\ell) = 0,
\]

moreover, (2) possesses the non-degeneracy property (the boundary value problem for \( F_{\sigma}'(x) = 0 \) has only a trivial solution); homogeneous equation

\[
Lu = 0
\]
does not oscillate on \([0; \ell]_\sigma\); \(G^*(x, s)\) is the influence function of the
\[
\begin{align*}
- \left( p u''_{xx\mu} \right)_{xx\sigma} &= F'_\sigma, \\
u(0) &= u'_x(0) = u''_{xx}(0) = 0, \\
u(\ell) &= u'_x(\ell) = u''_{xx}(\ell) = 0.
\end{align*}
\]

Then
\[m \cdot G^*(x, s) \leq G(x, s) \leq M \cdot G^*(x, s),\]
where \(m, M\) are finite positive constants belonging to \([0, \ell]\) for all \(x\) and \(s\).

Proof. Let us show that the non-oscillation of the homogeneous equation implies the positive invertibility of the boundary value problem.

Let \(F'_\sigma(x) \geq 0\) (\(\neq 0\)). Based on the Polya–Mamman representation (the proof is similar to [5]), we have the representation
\[
- \psi_6 \left( \psi_5 \left( \psi_4 \left( \psi_3 \left( \psi_2 \left( \psi_1 (\psi_0 u)_{x\mu} \right)_{x\sigma} \right)_{x\sigma} \right)_{x\sigma} \right)_{x\sigma} \right)_{x\sigma} = F'_\sigma,
\]
(here \(\psi_i(x)\) are separated from zero and can be differentiated with respect to the corresponding measure) from which it follows that
\[
\left( \psi_5 \left( \psi_4 \left( \psi_3 \left( \psi_2 \left( \psi_1 (\psi_0 u)_{x\mu} \right)_{x\sigma} \right)_{x\sigma} \right)_{x\sigma} \right)_{x\sigma} \right)_{x\sigma} \leq 0
\]
for all \(x\). The last inequality shows that the function \(\psi_5 \left( \psi_4 \left( \psi_3 \left( \psi_2 \left( \psi_1 (\psi_0 u)_{x\mu} \right)_{x\sigma} \right)_{x\sigma} \right)_{x\sigma} \right)_{x\sigma}(x)\) does not increase on the segment \([0; \ell]\). Therefore, it has at most one sign change. Since \(\psi_5(x) > 0\) we obtain that the function \(\psi_4 \left( \psi_3 \left( \psi_2 \left( \psi_1 (\psi_0 u)_{x\mu} \right)_{x\sigma} \right)_{x\sigma} \right)_{x\sigma}(x)\) has at most one sign change.

Continuing the reasoning, we obtain that the functions \(\psi_4 \left( \psi_3 \left( \psi_2 \left( \psi_1 (\psi_0 u)_{x\mu} \right)_{x\sigma} \right)_{x\sigma} \right)_{x\sigma}(x)\) and \(\left( \psi_3 \left( \psi_2 \left( \psi_1 (\psi_0 u)_{x\mu} \right)_{x\sigma} \right)_{x\sigma} \right)_{x\sigma}(x)\) have no more than two sign changes; functions \(\psi_3 \left( \psi_2 \left( \psi_1 (\psi_0 u)_{x\mu} \right)_{x\sigma} \right)_{x\sigma}(x)\) and \(\left( \psi_2 \left( \psi_1 (\psi_0 u)_{x\mu} \right)_{x\sigma} \right)_{x\sigma}(x)\) have no more than three sign changes; functions \(\psi_2 \left( \psi_1 (\psi_0 u)_{x\mu} \right)_{x\sigma}(x)\) and \(\left( \psi_1 (\psi_0 u)_{x\mu} \right)_{x\sigma}(x)\) have no more than four sign changes.

The boundary conditions imply the equalities \((\psi_1 (\psi_0 u)_{x\mu})_{x\sigma}(0) = (\psi_1 (\psi_0 u)_{x\mu})(\ell) = 0\). Therefore, the function \((\psi_1 (\psi_0 u)_{x\mu})_{x\sigma}(x)\) has at most two sign changes. Similarly, the function \((\psi_0 u)_{x\mu}(x)\) has no more than one sign change, functions \(\psi_0 u(x)\) and \(u(x)\) keep the sign on \([0; \ell]\). It remains to note that \(u(x)\) cannot have negative values.

Thus, both influence functions \(G(x, s)\) and \(G^*(x, s)\) are positive on \((0; \ell) \times (0; \ell)\).

To complete the proof, it remains to note that the relation \(\frac{G(x, s)}{G^*(x, s)}\) is bounded and separated from zero.

The inequalities
\[u_0(x)v_1(s) \leq G(x, s) \leq u_0(x)v_2(s)\]
implies that \(Lu = 0\) does not oscillate on \([0; \ell]_\sigma\), where
\[u_0(x) = \int_0^x (x - \tau)^2 d\mu(\tau) \cdot \int_x^\ell (\tau - x)^2 d\mu(\tau),\]
and \(v_1(s), v_2(s)\) is a \(\sigma\)-summable functions.
3. Main results

The following theorems are proved.

**Theorem 1.** Let the following conditions be satisfied:

1) the superposition operator generated by the function \( f(x, u_0(x)) \) acts continuously from \( C[0; \ell] \) to \( L_{p, \mu}[0; \ell] \) for some \( p \in (1; +\infty) \);
2) \( f(x, u) \geq 0 \) for all \( x \in [0; \ell] \) and \( u \geq 0 \);
3) equation \( Lu = 0 \) does not oscillate on \([0; \ell] \);
4) for some \( R > 0 \) and any \( \lambda \in (0; 1) \) the model

\[
\begin{align*}
\begin{cases}
Lu = \lambda f(x, u), \\
u(0) = u_x'(0) = u_{xx}''(0) = 0, \\
u(\ell) = u_x'(\ell) = u_{xx}''(\ell) = 0,
\end{cases}
\end{align*}
\]

has no solutions \( u(x) \) such that

\[
\sup_{x \in (0, \ell)} \frac{u(x)}{u_0(x)} \geq R,
\]

where as before \( u_0(x) = \int_0^x (x - \tau)^2 d\mu(\tau) \cdot \int_0^{\ell} (\tau - x)^2 d\mu(\tau) \).

Then problem (1) has at least one solution in \( K \).

Proof. Since \( Lu = 0 \) does not oscillate on \([0; \ell] \), the integral operator

\[
(Au)(x) = \int_0^\ell G(x, s) f(s, u(s)) \, d\sigma(s)
\]

transforms the cone \( K \) into a narrower set

\[
K(\hat{u}_0) = u(x \in C[0; \ell] | u(x) \geq \hat{u}_0(x) \| u \|_C, x \in [0; \ell]),
\]

where

\[
\hat{u}_0(x) = M \int_0^x (x - \tau)^2 d\mu(\tau) \cdot \int_x^{\ell} (\tau - x)^2 d\mu(\tau);
\]

moreover, it acts and is completely continuous on \( C[0; \ell] \). Moreover, any fixed point (8) is a solution to the differential model (1). Thus, the question of the existence of a fixed point in \( K \) for the operator \( A \) is restricted to \( K(\hat{u}_0) \).

The solvability (6) is equivalent to the solvability of the equation \( \lambda Au = u \) with the operator (8). If, in addition, it turns out that the last equation has a solution \( u_1 \in K(\hat{u}_0) \) satisfying the inequality \( \| u_1 \|_C \geq R_0 \) for some \( R_0 > 0 \), then according to the definition of \( K(\hat{u}_0) \) \( u_1(x) \geq R_0 \hat{u}_0(x) \) for all \( x \in [0; \ell] \); solution \( u_1(x) \) satisfies (7) with \( R = M \cdot R_0 \). Therefore, the condition of the theorem on the absence of such solutions means that the equation \( \lambda Au = u \) for \( \lambda \in (0, 1) \) has no solutions \( u_1(x) \in K(\hat{u}_0) \) such that \( \| u_1 \| \geq \hat{R} = \frac{R_0}{M} \). Consider on \( K(\hat{u}_0) \) the operator \( \hat{A} \):

\[
\hat{A}u = \begin{cases}
Au & \text{if } \| u \|_C \leq \hat{R}, \\
\hat{A}(\hat{R}\frac{u}{\| u \|_C}) & \text{if } \| u \|_C > \hat{R}.
\end{cases}
\]

(9)
The operator $\hat{A}$ is completely continuous on $K(\hat{u}_0)$ and transforms $K(\hat{u}_0)$ into a bounded part, that is, $\hat{A}$ leaves invariant the intersection of $K(\hat{u}_0)$ with a ball of some radius centered at the origin. And since this intersection is convex, bounded and closed, by virtue of the Schauder principle $\hat{A}$ has a fixed point $\hat{u}$ in $K(\hat{u}_0)$: $\hat{u} = \hat{A}\hat{u}$.

If we assume that $\|\hat{u}\|_C > \hat{R}$, then $\hat{u} = \hat{A}\hat{u} = A(\hat{R}\frac{u}{\|u\|_C})$. Hence, setting $\hat{v} = \hat{R}\frac{u}{\|u\|_C} - \hat{u}$, we have $\frac{\hat{R}}{\|u\|_C}A\hat{v} = \hat{v}$. In other words, problem (6) with $\lambda = \frac{\hat{R}}{\|u\|_C} < 1$ has a solution $\hat{v}$ satisfying the inequality $\|\hat{v}\|_C > \hat{R}$. Thus, the inequality $\|\hat{u}\|_C > \hat{R}$ is impossible. Therefore, $\|\hat{v}\|_C \leq \hat{R}$.

By virtue of the definition $\hat{A}$, we obtain $\hat{A}\hat{u} = \hat{u}$. The theorem is proved.

It is natural to check the conditions of the theorem for sufficiently large $R$. Therefore, we can specify its conditions in terms of the asymptotic properties of the function $f(x, u)$.

The function $f(x, u)$, which generates a superposition operator acting from $C[0, \ell]$ to some $L_{p, \sigma}[0; \ell]_\sigma$ ($p \in (1, \infty)$), is called asymptotically zero if

$$
\lim_{n \to \infty} \|\hat{f}(\cdot, u)\|_p^n = \lim_{n \to \infty} \int_0^\ell |\hat{f}(x, u)|^p \, d\sigma(x) = 0,
$$

$\hat{f}(x, u) = \sup_{v \geq u} \frac{|f(x, v)|}{v}$; $f(x, u)$ is called asymptotically linear if for some $q(x) \in L_{p, \sigma}[0; \ell]_\sigma$ the function $f(x, u)$ is asymptotically zero. In this case, we will write $q(x) = f'_\infty(x)$.

The simplest example of an asymptotically zero function is a function that is bounded on the entire plane. Less trivial: $f(x, u)$ is not bounded, but has slow, for example, logarithmic, growth at infinity.

**Theorem 2.** Let the following conditions be satisfied:

1) there is a majorant $f^\oplus(x, u)$ such that $0 \leq f(x, u) \leq f^\oplus(x, u)$ for all $x \in [0, \ell]$ and $u \geq 0$; $f^\oplus(x, u)$ is asymptotically linear;

2) equation $Lu = 0$ does not oscillate on $[0; \ell]_\mu$;

3) for $q(x) = f^\oplus_\infty(x)$ the spectral problem

$$
\begin{cases}
L u = \lambda q(x) u, \\
u(0) = u_x'(0) = u''_{xx}(0) = 0, \\
u(\ell) = u_x'(\ell) = u''_{xx}(\ell) = 0,
\end{cases}
$$

has no spectrum points in the unit circle.

Then model (10) is decidable in $K$.

Proof. Let us show that for a sufficiently large $R$ and any $\lambda \in (0, 1)$ the model

$$
\begin{cases}
L u = \lambda f(x, u), \\
u(0) = u_x'(0) = u''_{xx}(0) = 0, \\
u(\ell) = u_x'(\ell) = u''_{xx}(\ell) = 0,
\end{cases}
$$

has no solutions satisfying the inequality $u(x) \geq R\hat{u}_0(x)$. Suppose the opposite: for any $R_n \to \infty$ there exists a sequence of functions $u_n(x) \in K$ and numbers $\lambda_n \in (0, 1)$ such that $u_n(x)$ is a solution of the differential model (10) and satisfies the inequality $u_n(x) \geq R_n\hat{u}_0(x)$. The last inequality implies $\|u_n\|_C \to \infty$. The function $u_n(x)$ satisfies the equation

$$
u_n = \lambda_n Au_n,$$
where \( A = GF, G, \) and \( F \) are defined by

\[
(Gf)(x) = \int_0^\ell G(x, s)f(s)\,d\sigma(s),
\]

where

\[
(Fu)(x) = f(x, u(x)).
\]

Thus, by the first condition of the theorem

\[
u_k(x) = \lambda_k Au_k(x) \leq \int_0^\ell G(x, s)f_0^\oplus[s, u_k(s)]\,d\sigma(s), \tag{11}\]

and the functions \( f_0^\oplus(x, u_k(x)) \) belong to \( L_{p,\sigma}[0, \ell]_\sigma \). Inequality (11) can be rewritten as

\[
\frac{u_n(x)}{\|u_n\|_C} \leq \lambda_n \int_0^\ell G(x, s)q(s)\frac{u_n(s)}{\|u_n\|_C}\,d\sigma(s) + \lambda_n \int_0^\ell G(x, s)\frac{f_0^\oplus(s, u_n(s)) - q(s)u_n(s)}{\|u_n\|_C}\,d\sigma(s). \tag{12}\]

Let us show that the supremum on \((0, \ell)\) of the last term on the right-hand side of (12) tends to zero. Indeed, the asymptotic linearity of the function \( f_0^\oplus(x, u) \) implies that for any \( \epsilon > 0 \) there exists \( N = N(\epsilon) \) such that the inequality

\[
\left\| \frac{f_0^\oplus(x, u_0(x)v(x)) - q(x)u_0(x)v(x)}{u_0(x)} \right\|_p < \epsilon \tag{13}\]

holds for every \( v(x) \) satisfying the inequality \( v(x) \geq N(\epsilon) \). Since \( u_n \in K(\tilde{u}_0) \) and \( \|u_n\|_C \to \infty \) we obtain \( u_n(x) \geq \|u_n\|_C\tilde{u}_0(x) \), and for \( \|u_n\|_C \geq N \) we will have \( \|u_n\|_C \geq N\tilde{u}_0(x) \). Substituting \( \frac{u_n(x)}{\tilde{u}_0(x)} \) in (13) instead of \( v(x) \), we have

\[
\left\| \frac{f_0^\oplus(x, u_n(x)) - q(x)u_n(x)}{u_n} \right\|_p < \epsilon \|u_n\|_C.
\]

Since the integral operator with kernel \( G(x, s) \) is continuous (in fact, completely continuous) under the action from \( L_{p,\sigma}[0, \ell]_\sigma \) in \( C[0, \ell] \) we have

\[
\sup_{0 \leq x \leq \ell} \left| \int_0^\ell G(x, s)f_0^\oplus(s, u_n(s))\frac{f_0^\oplus(s, u_n(s)) - q(s)u_n(s)}{\|u_n\|_C}\,d\sigma(s) \right| \to 0
\]

for \( n \to \infty \), as required.

The integral operator

\[
(G_qf)(x) = \int_0^\ell G(x, s)q(s)f(s)\,d\sigma(s),
\]

acting from \( L_{p,\sigma}[0, \ell] \) to \( C[0, \ell] \) is completely continuous, therefore, the sequence \( w_n = G_q\left(\frac{u_n}{\|u_n\|_C}\right) \) is compact; sequence \( \{\lambda_n\} \subset [0, 1] \) is also compact. Let \( \{w_{n_k}(x)\} \) and \( \{\lambda_n\} \) be convergent sequences of sequences \( \{w_n(x)\} \) and \( \{\lambda_n\} \) respectively. We denote by \( w_0(x) \)
uniform \( \{w_{n_k}(x)\} \) and \( \lambda_0 \) is the limit of the numerical sequence \( \{\lambda_{n_k}\} \). From (12) it follows that \( \|w_0\|_C \neq 0 \) and \( \lambda_0 > 0 \).

By the condition of the theorem, the integral operator \( G_q \) is monotone. We obtain (we apply the operator \( G_q \) to both parts of (12))

\[
w_{n_k}(x) = G_q \left( \frac{u_{n_k}}{\|u_{n_k}\|_C} \right) \leq \lambda_{n_k} (G_q w_{n_k})(x) + \Omega_k(x),
\]

where

\[
\Omega_k(x) = \frac{\lambda_k}{\|u_{n_k}\|_C} \int_0^\ell G(\cdot, s) \left( f^{\hat{\circ}}(s, u_{n_k}(s)) - g(s) u_{n_k}(s) \right) d\sigma(s),
\]

wherein \( \Omega_k \to 0 \) at \( k \to \infty \). Passing to the limit as \( k \to \infty \) in equality (14), we will have \( w_0(x) \leq \lambda_0 (G_q w_0)(x) \), while \( w_0(x) \geq 0 \) and \( \lambda_0 \neq 0 \). It follows that the completely continuous operator \( G_q \) has an eigenvalue \( \lambda^* \in [0; \lambda_0) \subseteq [0; 1] \), to which the eigenfunction corresponds, and this contradicts the hypothesis of the theorem. The theorem is proved.

If in addition to the conditions of the theorem we require \( f(x, 0) \equiv 0 \), then problem (1) certainly has a trivial solution, and the application of the theorem does not provide any additional information.

**Theorem 3.** Let the following conditions be satisfied:

1) equation \( Lu = 0 \) does not oscillate on \([0; \ell]\);
2) the superposition operator generated by the function \( f(x, u) \) acts continuously from \( C[0, \ell] \) into some \( L_{p, \sigma}[0; \ell] \);
3) \( f(x, u) \geq 0 \) for all \( x \in [0, \ell] \) and \( u \geq 0 \);
4) \( f(x, 0) \equiv 0 \);
5) for some \( R > 0 \) and any \( \lambda \in (0, 1) \) model (6) has no solutions \( u(x) \) such that (7);
6) for some \( r_0 > 0 \) and some function \( h(x) \geq 0 \) other than identically zero such that \( h(x) \in L_{\infty}[0, \ell] \), for sufficiently small \( \lambda > 0 \) the model

\[
\begin{aligned}
L u &= f(x, u) + \lambda h(x), \\
u(0) &= u''(0) = u_x''(0) = 0, \\
u(\ell) &= u''(\ell) = u_x''(\ell) = 0,
\end{aligned}
\]

has no solutions for which the inequality

\[
\|\tilde{u}_0(x)\|_C \leq u(x) \leq r_0 \tilde{u}_0(x)
\]

holds for all \( x \) belonging to \([0; \ell]\).

Then mathematical model (1) has a nontrivial solution in \( K \).

Proof. Under the assumptions of the theorem, the operator \( A \), defined by equality (8), transforms \( K \) into \( K(\tilde{u}_0) \) and \( A\theta = \theta \). Let us consider a family of completely continuous (together with \( A \)) on \( K(\tilde{u}_0) \) operators

\[
A_r u = \begin{cases} 
Au + (r - \|u\|_C) h_0, & \text{if } \|u\|_C \leq r, \\
Au, & \text{if } \|u\|_C > r,
\end{cases}
\]

where \( r > 0 \) and

\[
h_0(x) = \int_0^\ell G(x, s) h(s) d\sigma(s).
\]
Arguments similar to those in Theorem 1 show that if a positive number $r$ is small enough, then for $\lambda > 1$ the equality $A_r u = \lambda u$ cannot hold on elements of large norm $K(\tilde{u}_0)$. Therefore, each operator $A_r$ for small $r > 0$ has a fixed point $u_r \in K(\tilde{u}_0)$: $A_r u_r = u_r$.

If the inequality $\|u_r\|_C < r$ holds, then by virtue of the definition of $A_r$ for $u_r$ the equality

$$u_r = Au_r + \lambda h$$

is true for $\lambda = r - \|u_r\|_C > 0$. Then the function $u_r(x)$ is a solution of mathematical model (15) for $\lambda > 0$, and for sufficiently small $\lambda$ satisfies inequalities (16), which due to conditions of the theorem are impossible. The resulting contradiction shows that the inequality $\|u_r\|_C < r$ cannot hold for all $r > 0$. Thus, for some $r > 0$ the inequality $\|u_r\|_C \geq r$ is true. Then, due to (17), this point will be fixed for the operator $A$, being a solution of model (1). The theorem is proved.

The next theorem shows how the conditions of the Theorem 2 can be verified in a neighborhood of zero.

**Theorem 4.** Let the following conditions be satisfied:

1. equation $Lu = 0$ does not oscillate on $[0; \ell]$,;
2. $f(x; 0) \equiv 0$;
3. there exists a function $f^\ominus(x, u)$ such that for some $r > 0$ the inequality

$$0 \leq f^\ominus(x, u) \leq f(x, u)$$

holds for all $x \in [0; \ell]$ and $u \in [0; r]$, and $f^\ominus(x, u)$ is monotone in $u$ for $u \in [0; r]$;
4. there exists a function $u_0(x) \in K \setminus \{0\}$ satisfying the conditions $u(0) = u'(0) = u''(0) = 0$ and $u(\ell) = u'(\ell) = u''(\ell) = 0$ and for sufficiently small $\lambda > 0$ the following inequality

$$\lambda(Lu_0)(x) \leq f^\ominus(x, \lambda u_0(x)), \quad (x \in [0; \ell])$$

holds. Then problem (1) has a nontrivial solution in $K$.

Proof. Let us show that for $h(x) = \frac{u_n(x)}{w_0(x)}$ all conditions of the Theorem 2 are satisfied. Assume that it is not. Then there are functions $u_n \in K$ and numbers $\lambda_n > 0$ such that $\|u_n\|_C \to 0$ and each function is a solution of the model

$$\begin{cases}
Lu = f(x, u) + \lambda_n h, \\
u(0) = u'(0) = u''(0) = 0, \\
u(\ell) = u'(\ell) = u''(\ell) = 0.
\end{cases}$$

It follows that the functions $v_n(x) = \frac{u_n(x)}{w_0(x)}$ satisfy the equalities

$$v_n = Av_n + \lambda_n h.$$ 

This implies that $v_n(x) \geq \lambda_n h(x)$ for all $x \in [0; \ell]$. Let $w_n = \inf_{0 \leq x < \ell} v_n(x, h(x))$. We have $w_n \geq \lambda_n > 0$ and $v_n(x) \geq w_n h(x)$. Therefore, $w_n \to 0$ for $n \to \infty$. From the fourth condition of the theorem (together with the equality $\lim_{n \to \infty} w_n = 0$) we find that the inequality $(A(w_n h))(x) \geq w_n h(x)$ is true for all $x \in [0; \ell]$ for sufficiently large $n$.

The integral operator

$$(A^\ominus v)(x) = \int_0^{\ell} G(x, s) f^\ominus(s, v(s)) d\sigma(s)$$

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acts in $C[0; \ell]$ and is monotone on $K(\tilde{u}_0)$ in a neighborhood of zero. From the third condition of the theorem and the uniform convergence of $u_n(x)$ to zero it follows that for large $n$ the inequality $Au_n \geq A^\ominus u_n$ is true. Further, due to the monotonicity of $A^\ominus$ the inequality $u_n(x) \geq \omega_n h(x)$ implies $A^\ominus u_n \geq A^\ominus(\omega_n h)$. Hence, from the equality $v_n = Av_n + \lambda_n h$ and the inequality $(A(\omega_n h))(x) \geq \omega_n h(x)$ we obtain the chain of inequalities

$$ u_n = Au_n + \lambda_n h \geq A^\ominus u_n + \lambda_n h \geq A^\ominus(\omega_n h) + \lambda_n h \geq (\omega_n + \lambda_n)h. $$

Thus,

$$ u_n(x) \geq (\omega_n + \lambda_n)h(x), $$

which contradicts the definition of the numbers $\omega_n$ (since $\lambda_n > 0$). The resulting contradiction proves the theorem.

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