On the Edge Connectivity of Direct Products with Dense Graphs

Wei Wang · Zhidan Yan

Received: 26 April 2012 / Published online: 24 May 2012
© Springer 2012

Abstract Let $\kappa'(G)$ be the edge connectivity of $G$ and $G \times H$ the direct product of $G$ and $H$. Let $H$ be any graph with minimal degree $\delta(H) > |V(H)|/2$. We prove that for any graph $G$, $\kappa'(G \times H) = \min\{2\kappa'(G)|E(H)|, \delta(G)\delta(H)\}$. In addition, the structure of minimum edge cuts is described. As an application, we present a necessary and sufficient condition for $G \times K_n (n \geq 3)$ to be super edge connected.

Keywords Direct product graphs · Edge connectivity · Minimum edge cuts

Mathematics Subject Classification MSC 05C40

1 Introduction

All graphs considered in this paper are finite, undirected, loopless and without multiple edges. Let $G = (V(G), E(G))$ be a nontrivial graph. The edge connectivity $\kappa'(G)$ is the minimum number of edges whose removal disconnects $G$. A minimum disconnecting set of edges is necessarily an edge cut and is also called a minimum edge cut. The direct product $G \times H$ has vertex set $V(G) \times V(H)$. Two vertices $(x, u), (y, v)$ are adjacent when $xy \in E(G)$ and $uv \in E(H)$.

Weichsel observed half a century ago that the direct product of two nontrivial graphs $G$ and $H$ is connected if and only if both factors are connected and not both are bipartite graphs [13]. For a long time, this result was the only one that considered connectivity of direct product graphs. Recently, Brešar and Špacapan [2] obtained an upper bound and a lower bound on the edge connectivity of direct products. The exact value of edge connectivity of direct products has been given in special cases. Cao et al.

W. Wang · Z. Yan (✉)
College of Information Engineering, Tarim University, Alar 843300, China
e-mail: yanzhidan.math@gmail.com
obtained a formula for $\kappa'(G \times K_n)$ with $n \geq 3$ (see Corollary 1 below). Yang and Xu [15] determined the case when one factor is $K_2$. Based on this result, Ou [8] presented a sufficient condition for $G \times H$ to be super edge connected (a graph $G$ is super edge connected if every minimum edge cut is the set of all edges incident with a vertex in $G$). It is still open to determine the edge connectivity of $G \times H$ for general $G$ and $H$. This is quite opposite to the case of other three products, namely, the Cartesian product [6,14], the strong product [1,16] and the lexicographic product [15], where explicit formulae have been obtained in terms of invariants of factor graphs. We mention that some results on the (vertex) connectivity and super connectivity of direct products of graphs have been obtained recently, see [4,5,7,9–12].

In this paper, we investigate the case when one factor, say $H$, has minimum degree exceeding $|V(H)|/2$. Note that this condition implies that $H$ is a connected nonbipartite graph.

**Theorem 1** Let $H$ be a graph with $\delta(H) > |V(H)|/2$. Then for any graph $G$, $\kappa'(G \times H) = \min\{2\kappa'(G)|E(H)|, \delta(G)\delta(H)\}$.

The following corollary is a straightforward consequence of Theorem 1. It was first obtained by Cao et al. (see [3, Theorem 2.1]).

**Corollary 1** $\kappa'(G \times K_n) = \min\{n(n-1)\kappa'(G), (n-1)\delta(G)\}$ for $n \geq 3$.

Under the same restriction on $H$ as in Theorem 1, we also characterize the structure of all possible minimum edge cuts of $G \times H$. For $S_0 \subseteq E(G)$ we let $I_{G \times H}(S_0) = \{(x, u)(y, v), (x, v)(y, u) : xy \in S_0, uv \in E(H)\}$ and say that $I_{G \times H}(S_0)$ is induced by $S_0$. Note $|I_{G \times H}(S_0)| = 2|S_0||E(H)|$ and $G \times H - I_{G \times H}(S_0) = (G - S_0) \times H$.

**Theorem 2** Let $S$ be a minimum edge cut of $G \times H$, where $\delta(H) > |V(H)|/2$. Then either $S$ is induced by a minimum edge cut of $G$, or $S$ is the set of all edges incident with a vertex in $G \times H$, unless $G = K_2$ and $H = K_{2l-1} \lor lK_2$ for some $l$.

**Corollary 2** Let $n \geq 3$. Unless $G = K_2$ and $n = 3$, $G \times K_n$ is super edge connected if and only if $n\kappa'(G) > \delta(G)$.

## 2 Proof of the Main Results

For $x \in V(G)$, following [2], we let $xH = \{(x, u) : u \in V(H)\}$ and call it the $H$-fiber with respect to $x$. For $S \subseteq E(G \times H)$, define a new graph $G^*$ as follows:

(i) $V(G^*) = \{xH : x \in V(G)\}$, and

(ii) $E(G^*) = \{xH \gamma H : E_{G \times H - S}(xH, \gamma H) \neq \emptyset\}$, where $E_{G \times H - S}(xH, \gamma H)$ denotes the collection of all edges in $G \times H - S$ with one endvertex in $xH$ and the other in $\gamma H$.

**Lemma 1** If $G^*$ is disconnected, then either

(i) $|S| > 2\kappa'(G)|E(H)|$, or

(ii) $|S| = 2\kappa'(G)|E(H)|$ and $S$ is induced by a minimum edge cut of $G$. 

\(\textcircled{S}\) Springer