Chiral supergravity and anomalies

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Abstract

Similarly as in the Ashtekar approach, the translational Chern–Simons term is, as a generating function, instrumental for a chiral reformulation of simple supergravity. After applying the algebraic Cartan relation between spin and torsion, the resulting canonical transformation induces not only decomposition of the gravitational fields into selfdual and antiselfdual modes, but also a splitting of the Rarita–Schwinger fields into their chiral parts in a natural way. In some detail, we also analyze the consequences for axial and chiral anomalies.

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1 Introduction

Recent developments by Ashtekar [1, 2] constitute considerable progress in the canonical formulation of general relativity and its relation to $SU(2)$ Yang–Mills theory. The key feature of Ashtekar’s formulation of general relativity

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is the introduction of a selfdual connection as one of the basic dynamical variables. This allows us to formulate general relativity in the familiar phase space of the Yang–Mills theories. It is indeed a theory of a connection and proves to be very useful to have the possibility of using loop variables (Wilson loops of the Ashtekar selfdual connection) in both the classical and the quantum descriptions of general relativity. This new formulation has generated not only a new approach to the problem of formulating quantum theory, but opened up a new avenue to study quantum gravity from a non-perturbative point of view. Since a great deal of nonperturbative effects in the Yang–Mills theories is known, it seems now feasible to carry over several techniques from quantum Yang–Mills theories to quantum gravity.

On the other hand, the operation of decomposing a field into positive and negative frequencies has no counterpart beyond linearized theory, while decomposing a field into selfdual and antiselfdual pieces makes sense for both non–linear gauge fields and general relativity. The role of chirality can be identified with the concept of selfduality. This is how, concretely, selfduality might be significant for quantizing either gauge theories or gravity.

In quantum theory one commonly chooses to work with positive frequency fields: positive helicity photons correspond to selfdual fields, whereas negative helicity ones to anti–selfdual fields. Alternatively, however, one could adopt only selfdual fields. The positive frequency fields would yield in this case the helicity +1 photons and the negative frequency fields the helicity −1 photons.

In spite of all the success of the Ashtekar formulation, the theory gets afflicted by the complex character of the variables, thus the issue of the reality conditions becomes compulsory. In the classical theory, general relativity is embedded in a larger complex theory.

As it is well known, supergravity suffers the same diseases as general relativity. Had we aimed at giving to the supergravity Lagrangian its chiral form, a similar analysis to the one given by Ashtekar should be performed. Jacobson advanced to some extent in this direction. Here we will turn, however, to the Clifford–algebra approach which yields the chiral decomposition from the more fundamental point of view of the generating function.

In this paper, we introduce a purely imaginary translational Chern–Simons term as a generating function which induces the chiral decomposition into selfdual and anti–selfdual parts in the bosonic sector as well as in the fermionic
sector of simple, i.e. \((\mathcal{N}=1)\) supergravity.

The chiral form of Einstein–Cartan theory of gravity and its coupling to Rarita–Schwinger field turn out to be very simple when expressed in the language of Clifford algebras. For instance, the coupling necessary to implement the local supersymmetry of the \((\mathcal{N}=1)\) supergravity action is a natural step in the Clifford formulation of the theory.

The crucial point is here to add suitable boundary terms of the Chern–Simons type to the action which allow for a chiral formulation; this induces in a natural way the introduction of an anti–selfdual or selfdual connection. In the Hamiltonian formulation such a boundary term acts as generating function for new variables, see [40, 33]. The structure of these boundary terms is fixed by the requirement of local Lorentz invariance and their relation to the two Bianchi identities.

The plan of the paper is as follows: In Sec. 2 we present the necessary requisites of the geometrical setting in terms of Clifford–algebra–valued forms. The generating function of the Chern–Simons type boundary terms is discussed in Sec. 3. In Sec. 4, we revisit the simple supergravity. In Sec. 5, chiral supergravity is generated by the ‘on shell’ boundary term. In Sec. 6, the chiral anomaly is investigated and some controverses are resolved. In Sec. 7 we explain the issue of the reality constraint in terms of a toy model and finally in Sec. 8 our results are contrasted with other approaches.

2 Riemann–Cartan spacetime and Clifford algebras

The Riemann–Cartan (RC) geometry \([22]\) will be described by means of a very concise formalism employing Clifford algebra–valued exterior differential forms. The Dirac matrices \(\gamma_\alpha\) obey

\[
\gamma_\alpha \gamma_\beta + \gamma_\beta \gamma_\alpha = 2 o_{\alpha \beta} \mathbf{1}_4 ,
\]

where \(\alpha, \beta, \cdots = \hat{0}, \hat{1}, \hat{2}, \hat{3}\) denote the (anholonomic) indices of the frame field \(e_\alpha\) which is assumed to be orthonormal. The signature of the Minkowski metric \(o_{\alpha \beta}\) of the frame bundle is \((+--\cdots\cdots)\). The 16 matrices \(\{\mathbf{1}_4, \gamma_\alpha, \sigma_{\alpha \beta} , \gamma_5, \gamma_5 \gamma_\alpha\}\) form a basis of a Clifford algebra in four dimensions, see \([10, 9]\). The totally antisymmetric product of Dirac matrices is the zero–form \(\gamma_5 := (i/4!) * (\gamma \wedge \)
\( \gamma \wedge \gamma \) and fulfills
\[
\gamma_5 = -i \gamma_0 \gamma_1 \gamma_2 \gamma_3, \quad \gamma_5 \gamma_5 = +1, \quad \{ \gamma_5, \gamma_\alpha \} = 0. \tag{2}
\]

With respect to the trace \( \text{Tr} \), the elements of the Clifford algebra are normalized by
\[
\text{Tr}(\gamma_\alpha \gamma_\beta) = 4 \delta_{\alpha\beta} \quad \text{and} \quad \text{Tr}(\sigma_{\alpha\beta} \sigma^{\gamma\delta}) = 8 \delta_{\alpha\gamma} \delta_{\beta\delta}, \tag{3}
\]
where \([\alpha \beta] = \frac{1}{2}(\alpha \beta - \beta \alpha)\) denotes the antisymmetrization of indices. Following the notation of Ref. [37, 38, 24], the constant \( \gamma_\alpha \) matrices can be converted into Clifford algebra–valued coframe one– or three–forms, respectively:
\[
\gamma := \gamma_\alpha \theta^\alpha, \quad \ast \gamma = \gamma^\alpha \eta_\alpha = \frac{i}{6} \gamma_5 \gamma \wedge \gamma \wedge \gamma. \tag{4}
\]

Here, we restrict ourselves to a topologically trivial frame bundle where \( \eta = (1/4!) \eta_{\alpha\beta\gamma\delta} \theta^\alpha \wedge \theta^\beta \wedge \theta^\gamma \wedge \theta^\delta \) is the volume four–form with the normalization \( \eta_{0123} = +1 \), \( \eta_\alpha := \epsilon_\alpha \eta = \ast \theta_\alpha \) is the coframe “density”, and \( \ast \) the Hodge dual. Moreover, in order to complete the \( \eta \)–basis for forms, we define \( \eta_{\alpha\beta} := \epsilon_\beta \eta_\alpha, \eta_{\alpha\beta\gamma} := \epsilon_\gamma \eta_{\alpha\beta}, \eta_{\alpha\beta\gamma\delta} := \epsilon_\delta \eta_{\alpha\beta\gamma} \). The contraction operator acting on \( p \)–forms from the left is defined as
\[
\tilde{\gamma} := \gamma^\beta e_\beta = \gamma^\beta e^i_\beta \partial_i \quad \text{with} \quad \tilde{\gamma} \gamma = 4 \cdot 1_4. \tag{5}
\]
It generalizes the usual Feynman “dagger” convention \( \mathcal{A}^\dagger := \gamma^\alpha e_\alpha \mathcal{A} \) for one–forms \( \mathcal{A} \).

The Lorentz generators \( \sigma_{\alpha\beta} := \frac{i}{2} (\gamma_\alpha \gamma_\beta - \gamma_\beta \gamma_\alpha) \) obey
\[
[\gamma_\alpha, \sigma_{\beta\gamma}] = 2i \big( o_{\alpha\beta} \gamma_\gamma - o_{\alpha\gamma} \gamma_\beta \big), \quad \{ \gamma_\alpha, \sigma_{\beta\gamma} \} = 2i \gamma_\alpha \gamma_\beta \gamma_\gamma = -2 \eta_{\alpha\beta\gamma\delta} \gamma_5 \gamma_\delta. \tag{6}
\]
Their Lie (or right) duals and their inverses are related via
\[
\sigma_{\alpha\beta}^* := \frac{1}{2} \sigma^{\gamma\delta} \eta_{\alpha\beta\gamma\delta} = i \gamma_5 \sigma_{\alpha\beta}, \quad \sigma_{\alpha\beta} = -\frac{1}{2} \sigma^*_{\mu\nu} \eta^{\mu\nu} \sigma_{\alpha\beta}. \tag{7}
\]
The associated two–forms are given by
\[
\sigma := \frac{1}{2} \sigma_{\alpha\beta} \theta^\alpha \wedge \theta^\beta = \frac{i}{2} \gamma \wedge \gamma, \quad \ast \sigma = \frac{1}{2} \sigma_{\alpha\beta} \eta^{\alpha\beta} =: \sigma^* = i \gamma_5 \sigma. \tag{8}
\]
Note that in orthonormal frames the Hodge dual $\ast$ and the Lie dual $\star$ are identical operations for $\sigma$. For Minkowski signature, the Hodge dual satisfies $\ast\ast = -1$, therefore often $i\ast$ is used in field theory in order to have an involutive duality operator. We will encounter also the self–or antiself dual combination

\[ \sigma_{\pm} := (\sigma \pm i\ast\sigma)/2 = \frac{1}{2}(1 \mp \gamma_5)\sigma, \quad \text{with} \quad i\ast\sigma_{\pm} = \pm\sigma_{\pm} \quad (9) \]

which at times is referred to as the Plebański two–form [52], cf. also Debever [16] and Brans [11] for related earlier constructions.

### 2.1 Clifford algebra-valued torsion and curvature

In terms of the Clifford algebra–valued connection $\Gamma := i\Gamma^{\alpha\beta}\sigma_{\alpha\beta}$, the $\mathcal{SO}(1,3) \cong SL(2,C)$–covariant exterior derivative $D = d + [\Gamma, \cdot]$ employs the algebra–valued form commutator $[\Psi, \Phi] := \Psi \wedge \Phi - (-1)^{pq}\Phi \wedge \Psi$. Differentiation of the basic variables leads to the Clifford algebra–valued torsion and curvature two–forms, respectively:

\[ \Theta := D\gamma = T^{\alpha}\gamma_{\alpha}, \quad \Omega := d\Gamma + \Gamma \wedge \Gamma = i\frac{4}{3}R^{\alpha\beta}\sigma_{\alpha\beta}. \quad (10) \]

In accordance with (7), its Lie dual is given by

\[ \Omega^\star := i\frac{8}{3}R_{\alpha\beta}\eta^{\alpha\beta\gamma\delta}\sigma_{\gamma\delta} = -\frac{1}{4}R^{\alpha\beta}\gamma_5\sigma_{\alpha\beta} = i\gamma_5\Omega. \quad (11) \]

The Ricci identity reads

\[ DD\Psi = [\Omega, \Psi]. \quad (12) \]

In this “Clifform” approach, the torsion two–form can be irreducibly decomposed into the trace part \( (2)\Theta \), the axial torsion \( (3)\Theta \), and the tensor torsion \( (1)\Theta \) as follows:

\[ (2)\Theta := \frac{1}{3}\gamma \wedge T = \frac{1}{12}\gamma \wedge \ast\text{Tr}(\ast\Theta \wedge \gamma), \quad (13) \]

\[ (3)\Theta := -\frac{1}{3}\ast[\gamma \wedge A] = \frac{1}{12}\gamma_5\text{Tr}(\gamma \wedge \Theta), \quad (14) \]

\[ (1)\Theta := \Theta - (2)\Theta - (3)\Theta. \quad (15) \]
The one–forms of the trace and axial vector torsion, respectively, are defined by

\[ T := \frac{1}{4} Tr (\dot{\gamma} \Theta) = e_\alpha T^\alpha, \quad A := \frac{1}{4} *Tr(\gamma \wedge \Theta) = *(\vartheta_\alpha \wedge T^\alpha). \quad (16) \]

The purely geometrical identity \[ \Theta \wedge \Theta \equiv (1)\Theta \wedge (1)\Theta + 2 (2)\Theta \wedge (3)\Theta, \quad (17) \]
vanishes for purely vector or axial torsion.

3 Chern–Simons type boundary terms in gravity

Quite generally, the Chern–Simons term

\[ C := Tr \left( A \wedge F - \frac{1}{3} A \wedge A \wedge A \right), \quad (18) \]

of a non–Abelian gauge theory yield boundary terms \( dC \) which turn out to be generating functions. In the Hamiltonian formulation, they induce a new pair of canonical variables, see for instance Mielke \[ 39, 40 \] in the case of the Ashtekar reformulation of GR.

In a Riemann–Cartan geometry, the Chern–Simons term for the Lorentz connection reads

\[ C_{RR} := -Tr \left( \Gamma \wedge \Omega - \frac{1}{3} \Gamma \wedge \Gamma \wedge \Gamma \right), \quad (19) \]

whereas the translational Chern–Simons term can be written as

\[ C_{TT} := \frac{1}{8\ell^2} Tr (\gamma \wedge \Theta) = \frac{1}{2\ell^2} \vartheta_\alpha \wedge T^\alpha. \quad (20) \]

A fundamental length \( \ell \) introduced here is necessary for dimensional reasons.

The corresponding boundary terms \( dC \) are distinguished by the fact that their variational derivatives

\[ \frac{\delta}{\delta \gamma} dC_{TT} = \frac{1}{\ell^2} (D\Theta - [\Omega, \gamma]) \quad \text{and} \quad \frac{\delta}{\delta \Gamma} dC_{RR} = -D\Omega, \quad (21) \]
lead us back to the first and second Bianchi identity
\[ D\Theta \equiv [\Omega, \gamma], \quad D\Omega \equiv 0, \] (22)
respectively \[23\].

The contracted Bianchi identities
\[ D[\gamma, \Theta] \equiv 2i[\sigma, \Omega], \quad D[\gamma, \Omega] \equiv [\Theta, \Omega], \quad D\mathcal{E} \equiv i[\Theta, \gamma_5\Omega], \] (23)
ensure the automatic conservation of the Einstein three–form
\[ \mathcal{E} := \mathcal{E}_\alpha\gamma^\alpha := \frac{1}{2}R^{\mu\nu} \wedge \eta_{\mu\nu\lambda}\gamma^\lambda = -i\gamma_5(\Omega \wedge \gamma + \gamma \wedge \Omega) = i[\gamma, \gamma_5\Omega] \] (24)
in Einstein’s GR with \(\Theta = 0\).

3.1 Chiral decomposition of the connection

The translational Chern–Simons term induces a chiral representation of the supergravity (Riemann–Cartan) curvature and, in the wake of this, two chiral pieces of the connection are projected out. The self– or antisel dual part of the Lorentz connection is defined according to
\[ \Gamma = \Gamma_+ + \Gamma_-, \quad \Gamma_{\pm}^{\alpha\beta} := \frac{1}{2}\Gamma^{\alpha\beta} \pm \frac{i}{2}\eta^{\alpha\beta\gamma\delta}\Gamma_{\gamma\delta}. \] (25)
This decomposition of the connection results also from the action of the corresponding projection operator \(P_\pm = \frac{1}{2}(1 \pm \gamma_5)\), the chirality operator:
\[ \Gamma_\pm = P_\pm\Gamma = \frac{i}{8}\Gamma^{\alpha\beta}(1 \mp \gamma_5)\sigma_{\alpha\beta} = \frac{i}{2}\Gamma^{\alpha\beta}(\sigma_{\alpha\beta} \pm i\sigma_{\alpha\beta}^*) = \frac{i}{4}\sigma_{\alpha\beta}^*(\frac{1}{2}\Gamma^{\alpha\beta} \pm \frac{i}{2}\eta^{\alpha\beta\gamma\delta}\Gamma_{\gamma\delta}). \] (26)
whereas the basis one–form partially anticommutes with the chiral projection
\[ \gamma_\pm := P_\pm\gamma = \frac{1}{2}(1 \mp \gamma_5)\gamma = \frac{1}{2}\gamma(1 \pm \gamma_5) = \gamma P_\pm. \] (27)

Note that the chirality operator \(P_\pm\) of our Clifford approach is the same as the one used in elementary particle physics.
Observe also that $\Gamma_-$ has tensorial transformation properties. Substituting the connection into the curvature, we find a corresponding decomposition of the curvature:

$$\Omega = \Omega_+ + \Omega_- , \quad \Omega_\pm := \Omega(\Gamma_\pm) .$$  \hspace{1cm} (28)

into selfdual and antselfdual pieces \cite{13}.

The connection dynamics looks in many respects simpler than usual geometrodynamics based on the metric, but the simplicity has been bought at a price that the connection $\Gamma_\pm$ is necessarily complex.

4 Simple ($\mathcal{N}=1$) supergravity

Supergravity \cite{51, 19} with one supersymmetry generator, i.e. $\mathcal{N}=1$, represents the simplest consistent coupling of a Rarita–Schwinger (RS) type spin-$\frac{3}{2}$ field to gravity.

The corresponding Hermitian Lagrangian four–form reads

$$L_{\text{Sugra}} = V_{\text{EC}} + L_{\text{RS}} ,$$ \hspace{1cm} (29)

$$L_{\text{RS}} = -\frac{1}{2} ( \overline{\Psi} \wedge \gamma_5 \gamma_5 \wedge D \Psi - \overline{D \Psi} \wedge \gamma_5 \gamma_5 \wedge \Psi ) ,$$ \hspace{1cm} (30)

where

$$V_{\text{EC}} = \frac{i}{2\ell^2} Tr (\Omega \wedge \gamma_5 \gamma_5 \wedge \sigma) = -\frac{1}{2\ell^2} Tr (\Omega \wedge \gamma_5 \sigma) .$$ \hspace{1cm} (31)

denotes the Einstein–Cartan (EC) Lagrangian.

It should be noted that the Rarita–Schwinger field $\Psi := \Psi_\alpha \theta^\alpha$ entering Eq. (30) is a Majorana spinor valued one-form. As it is well known, it satisfies the Majorana condition, i.e. $\Psi = C \overline{\Psi}^t$, where $\overline{\Psi} := \Psi^t C$, and the charge conjugation matrix \cite{34} given by $C = -i \gamma^0$ satisfies $C^t = C^{-1}$, $C^t = -C$, $C^{-1} \gamma^0 C = - (\gamma^0)^t$. Consequently,

$$\overline{\Psi} \wedge \Psi = 0 , \quad \overline{\Psi} \wedge \gamma_5 \gamma^0 \gamma_5 ^\alpha \Psi = 0 , \quad \overline{\Psi} \wedge \gamma_5 \Psi = 0 .$$ \hspace{1cm} (32)

Here we will use the real Majorana representation in which all $\gamma^\alpha$ are purely imaginary but the components of the gravitino vector–spinor are consequently all real, see \cite{31, 34}.
The covariant exterior derivatives
\[ D\Psi := d\Psi + \Gamma \wedge \Psi, \quad \overline{D}\Psi = d\overline{\Psi} + \overline{\Psi} \wedge \Gamma \]
(33)
of spinor–valued one–forms, induce minimal coupling, contrary to the tensorial component notation [53].

Energy–momentum and spin currents of the Rarita–Schwinger field turn out to be
\[ \Sigma_{\alpha} = \frac{1}{2} \left( \overline{\Psi} \wedge \gamma_5 \gamma_{\alpha} D\Psi + \overline{D\Psi} \wedge \gamma_5 \gamma_{\alpha} \Psi \right) \]
(34)
and
\[ \tau_{\alpha\beta} := \frac{\partial L_{RS}}{\partial \Gamma_{\alpha\beta}} = -\frac{i}{8} \overline{\Psi} \wedge \gamma_5 (\gamma_{\sigma_{\alpha\beta}} + \sigma_{\alpha\beta\gamma}) \wedge \Psi = \frac{i}{4} \eta_{\alpha\beta\gamma} \wedge \overline{\Psi} \wedge \gamma^\gamma \Psi , \]
(35)
respectively.

With our Clifford definition (10) for the torsion, we find for the Rarita–Schwinger equation, cf. [62],
\[ \gamma \wedge D\Psi - \frac{1}{2} \Theta \wedge \Psi = 0 . \]
(36)

For the coupled Einstein–Cartan–Rarita–Schwinger Lagrangian, the first field equation reads
\[ i \gamma_5 (\Omega \wedge \gamma + \gamma \wedge \Omega) = \frac{\ell^2}{2} \gamma^\alpha \left( \overline{\Psi} \wedge \gamma_5 \gamma_{\alpha} D\Psi + \overline{D\Psi} \wedge \gamma_5 \gamma_{\alpha} \Psi \right) , \]
(37)
whereas the second field equation, our “Cartan relation”, turns out to be:
\[ -\frac{1}{2} \eta_{\alpha\beta\gamma} \wedge T^\gamma = \frac{i}{4} \ell^2 \eta_{\alpha\beta\gamma} \wedge \overline{\Psi} \wedge \gamma^\gamma \Psi , \]
(38)
or, more simply written,
\[ \Theta = T^\alpha \gamma_\alpha = -\frac{i}{2} \ell^2 \overline{\Psi} \wedge \gamma^\alpha \Psi \gamma_{\alpha} . \]
(39)

For the explicit torsion term in (36), we find
\[ \Theta \wedge \Psi = -\frac{i}{2} \ell^2 \overline{\Psi} \wedge \gamma^\alpha \Psi \wedge \gamma_{\alpha} \Psi = 0 , \]
(40)
as can be easily shown by means of a commutation of the one–form $\gamma^\alpha \Psi$ among itself, i.e. a Fierz reordering \[51\]. Thus, in sharp contrast to the Dirac case, for the supergravity coupling of the Rarita–Schwinger field, torsion does not induce a nonlinear term in (36). This is because the RS equation amounts to four Dirac equation plus the constraint (40). Covariant exterior differentiation of the Rarita–Schwinger equation (36) yields:

\[ D \left( \gamma \wedge D \Psi - \frac{1}{2} \Theta \wedge \Psi \right) = \Theta \wedge D \Psi - \gamma \wedge DD \Psi - \frac{1}{2} \Theta \wedge D \Psi \]

\[ = \frac{1}{2} \Theta \wedge D \Psi - \frac{1}{2} (\Omega \wedge \gamma + \gamma \wedge \Omega) \wedge \Psi \, . \tag{41} \]

It is a remarkable fact of supergravity also in higher dimensions, cf. \[7\], that the integrability condition (41) for the fermionic fields is the bosonic equation. Indeed, by substitution of the gauge field equations (37) and (40) and a Fierz reordering, the integrability condition

\[ D \left( \gamma \wedge D \Psi - \frac{1}{2} \Theta \wedge \Psi \right) = D (\gamma \wedge D \Psi) = 0 \tag{42} \]

is satisfied without having used explicitly the RS field equation (36), but implicitly employed it in the energy–momentum current because of $L_{RS} \approx 0$.

### 4.1 SUSY transformation

The integrability condition (42) points to a gauge invariance of simple supergravity. The invariance of $L_{Sugra}$ under such local supersymmetric transformations (SUSY) \[17\] becomes now rather natural by using the Clifford algebra–valued coframe and connection. They read in our notation:

\[ \delta \Psi = 2D\epsilon \, , \]

\[ \delta \gamma = i\ell^2 \varepsilon^\alpha \gamma_\alpha \, , \]

\[ \delta \Gamma^{\alpha} = i\ell^2 \varepsilon_5 \gamma^\alpha D\Psi \, , \tag{43} \]

where $\delta \Gamma^{\alpha} := \frac{1}{2} \delta \Gamma^{\beta\gamma} \wedge \eta_{\alpha\beta\gamma}$ corresponds to the Lie dual of the variation of the connection, and $\epsilon^\alpha(x)$ is an anticommuting spinor–valued zero–form (Majorana field).
Quite generally, we find for the variation of the supergravity Lagrangian
\[
\delta L_{\text{Sugra}} = \delta \partial^\alpha \wedge \left[ \frac{1}{2 \ell^2} \eta_{\alpha \beta \gamma} \wedge R^{\beta \gamma} - \frac{1}{2} (\Psi_5 \gamma_\alpha \wedge D \Psi + \overline{D \Psi} \wedge \gamma_5 \gamma_\alpha \Psi) \right] \\
+ \frac{1}{2} \eta_{\alpha \beta \gamma} \wedge \left( -\frac{1}{\ell^2} T^\gamma + \frac{i}{2} \overline{\Psi} \wedge \gamma^\gamma \Psi \right) \wedge \delta \Gamma^{\alpha \beta} \\
+ \overline{\delta \Psi} \wedge \gamma_5 \left( i \gamma \wedge D \Psi - \frac{i}{2} \Theta \wedge \Psi \right) .
\]
(44)
Substituting (43), this becomes
\[
\delta L_{\text{Sugra}} = i \ell^2 \sigma^\alpha \Psi \wedge \left[ \frac{1}{2 \ell^2} \eta_{\alpha \beta \gamma} \wedge R^{\beta \gamma} - \frac{1}{2} (\Psi_5 \gamma_\alpha \wedge D \Psi + \overline{D \Psi} \wedge \gamma_5 \gamma_\alpha \Psi) \right] \\
+ i \gamma_5 \gamma_\alpha D \Psi \wedge \left( T^\alpha - \frac{i \ell^2}{2} \overline{\Psi} \wedge \gamma^\alpha \Psi \right) \\
+ 2 \overline{D \epsilon} \wedge \gamma_5 \left( i \gamma \wedge D \Psi - \frac{i}{2} \Theta \wedge \Psi \right) .
\]
(45)
By means of (41) and a Fierz reordering, this turns out to be a boundary term, which vanishes ‘on shell’:
\[
\delta L_{\text{Sugra}} = 2 d \left\{ \overline{\sigma} \wedge \gamma_5 \left( i \gamma \wedge D \Psi - \frac{i}{2} \Theta \wedge \Psi \right) \right\} \approx 0 .
\]
(46)
Therefore, we have explicitly shown that simple supergravity is locally supersymmetric; the vanishing of the covariant derivative of (36), provided that (17) and (39) are fulfilled, accounts for it. Simple supergravity leads to a generalization of the Dirac equation for the massless gravitino, which has the virtue of being free of the Velo–Zwanziger causality problems of other higher spin equations.

Now according to our results concerning the chiral Clifford algebra–valued formulation there is in the Lagrangian density \( V_{EC} \) a term \( \sim Tr(\Theta \wedge \Theta) \), quadratic in the torsion. This term vanishes for supergravity due to the following Fierz rearrangement similar to (41):
\[
Tr(\Theta \wedge \Theta) = -\frac{\ell^4}{4} Tr(\overline{\Psi} \wedge \gamma^\alpha \Psi \wedge \overline{\Psi} \wedge \gamma_\alpha \Psi) \\
= -\frac{i \ell^4}{4} Tr(\overline{\Psi} \wedge \overline{\Psi} \wedge \gamma^\alpha \Psi \wedge \gamma_\alpha \Psi) \\
= 0 .
\]
(47)
5 Chiral supergravity induced via a translational Chern–Simons term

In order to give the supergravity Lagrangian (29) its chiral form, an analysis similar to the one given in the previous section should be performed. Jacobson [29] advanced to some extent in this direction and there exists also a twistor formulation [14]; here we focus on the more fundamental point of view of the generating function [33]. To this end the Rarita–Schwinger spinor one–form is decomposed [31, 32] into left and right–handed pieces Ψ = Ψ_L + Ψ_R, where

\[ \Psi_L := \frac{1 - \gamma_5}{2} \Psi = P_- \Psi, \quad \Psi_R := \frac{1 + \gamma_5}{2} \Psi = P_+ \Psi, \]

with \( P_{\pm} \Psi = \Psi P_{\pm} \), \( D_{\pm} \Psi_{LR} := d \Psi_{LR} + \Gamma_{\pm} \wedge \Psi_{LR} \), and \( D_{\pm} \Psi_{LR} := d \Psi_{LR} + \Psi_{LR} \wedge \Gamma_{\pm} \).

In our elegant “Clifform” approach, we note that ‘on shell’, i.e. after using the Cartan relation (38), the translational Chern–Simons term (20) is given by

\[ C_{TT} \sim = \frac{i}{4} \bar{\Psi} \wedge \gamma \wedge \Psi = \frac{1}{4} J_5, \]

which is proportional to the axial current \( J_5 := i \bar{\Psi} \wedge \gamma \wedge \Psi \) of the Rarita–Schwinger field.

Similarly as in the Dirac case, the chiral version of the Rarita–Schwinger Lagrangian is obtained [13, 11] by adding the boundary term \( dC_{TT} \) with the imaginary unit:

\[ L_{RS_{\pm}} := \frac{1}{2} L_{RS} \pm idC_{TT} \]

\[ \quad = \pm \frac{1}{2} \left( D \Psi_{LR} \wedge \gamma \wedge \Psi + \bar{\Psi} \wedge P_\pm \gamma \wedge D \Psi \right). \]

Note that in the ‘on shell’ expression of the boundary term the torsion \( \Theta = D \gamma \) drops out due to (38). Acting with the chirality projector on the spinor one–forms as well as on the connection, the chiral Rarita–Schwinger Lagrangian reads explicitly

\[ L_{RS_{\pm}} = \pm \frac{1}{2} \left\{ \Psi_{R,L} \wedge \gamma \wedge D_{\pm} \psi_{R,L} + D_{\pm} \Psi_{L,R} \wedge \gamma \wedge \Psi_{L,R} \right\}, \]

12
where the positive (negative) sign goes once again with the first (second) of the indices appearing in the vector–spinor $\Psi_{L,R}$. According to this result, the chiral and standard actions for supergravity differ by an imaginary boundary term, once the equation of motion for the connection obtained from the chiral one is used. In full, the chiral supergravity theory can be related to simple supergravity through the identity

$$L_{\text{Chiral Sugra}} := \frac{1}{2} \left( \pm \right) V_{EC} + L_{\text{RS} \pm} = \frac{1}{2} \left( L_{\text{Sugra}} \mp i dC_{TT} \right) .$$

(52)

More precisely, the imaginary part of the chiral supergravity Lagrangian (52) is the boundary term

$$L_{\text{Chiral Sugra}} - \mathcal{L}_{\text{Chiral Sugra}} = \mp i dC_{TT} \cong i \left[ \pm \frac{i}{4} d \left( \bar{\Psi} \gamma \wedge \Psi \right) \right] ,$$

(53)

whereas the real part is the standard supergravity Lagrangian [29, 33].

5.1 SUSY invariance of the translational boundary term

Let us finally check, if the additional boundary term $dC_{TT}$ preserves local supersymmetry ‘on shell’: For the variation of the translational Chern–Simons term $C_{TT}$ we can use the identity $\delta (\theta^\alpha \wedge T_\alpha) = \delta \theta^\alpha \wedge 2T_\alpha - d(\theta^\alpha \wedge \delta \theta_\alpha)$. Under the usual assumption that $[\delta, d] = 0$, the variation of the corresponding boundary term yields

$$\delta dC_{TT} = \frac{1}{2\ell^2} \left[ d (\delta \theta^\alpha \wedge 2T_\alpha) - dd (\theta^\alpha \wedge \delta \theta_\alpha) \right] = \frac{1}{\ell^2} d (\delta \theta^\alpha \wedge T_\alpha) .$$

(54)

Thus, for the SUSY transformations (43) we find

$$\delta dC_{TT} = \frac{\ell^2}{2} d \left( \bar{\psi} \gamma^\alpha \psi \wedge T_\alpha \right) $$

$$\cong \frac{\ell^2}{2} d \left( \bar{\psi} \gamma^\alpha \psi \wedge \bar{\psi} \wedge \gamma_\alpha \psi \right) $$

$$= 0 .$$

(55)

In the last step we used the Cartan relation (39) and a Fierz rearrangement of two $\gamma^\alpha \psi$ one–forms. Consequently, the translational Chern–Simons term does not spoil the supersymmetry for our chiral formulation of supergravity.
Hence, the two theories are dynamically equivalent, but have canonically transformed variables in the Hamiltonian formulation.

Moreover, in the chirally projected form the SUSY transformations read

\[
\begin{align*}
\delta \Psi_{L,R} &= 2D_{\pm} \epsilon_{\pm}, \\
\delta \gamma_{\pm} &= i\ell^2 \epsilon_{\pm} \gamma^\alpha \Psi_{L,R} \gamma_\alpha, \\
\delta \Gamma^{\alpha}_{\pm} &= i\ell^2 \epsilon_{\pm} \gamma_5 \gamma^\alpha D_{\pm} \Psi_{L,R}.
\end{align*}
\]

For an extended model of chiral supergravity with complex tetrads, left- and right-handed SUSY transformations with \( \delta_{L,R} \Psi_{R,L} = 0 \) have tentatively adopted \[59\], in order to avoid inconsistencies. However, by adopting our Chern–Simons type boundary term \[41, 43\] for the imaginary part \[52\] of chiral supergravity the consistency problem can be solved also for \( (N=2) \), see \[60\].

## 6 Chiral anomaly

From the Dirac equation and its conjugate one can readily deduce for the axial current \( j_5 := \bar{\psi} \gamma_5 \gamma^\alpha \psi \) that the well-known “classical axial anomaly” \[30\]

\[ dj_5 = 2imP = 2im\bar{\psi} \gamma_5 \psi \tag{57} \]

for massive Dirac fields \( \psi \) holds also in a Riemann–Cartan background spacetime. If we restore chiral symmetry in the limit \( m \to 0 \), this leads to classical conservation law of the axial current for massless Weyl spinors, or since \( dj := d\bar{\psi} \gamma^\alpha \psi = 0 \), equivalently, for the chiral current \( j_{\pm} := \bar{\psi}(1 \pm \gamma_5) \gamma^\alpha \psi / 2 = \bar{\psi} P_{\pm} \gamma^\alpha \psi. \)

However, within the dynamical framework of the Einstein–Cartan–Dirac theory we also have

\[ dC_{TT} \simeq (1/4) dj_5 \to 0 \tag{58} \]

'on shell'. This is consistent with the fact that a Weyl spinor does not couple to torsion, because the remaining axial torsion \( A := \ast(\partial_\alpha \wedge T^\alpha) \) becomes a lightlike covector, i.e. \( A_\alpha A^\alpha \eta = A \wedge \ast A \simeq (\ell^4 / 4) j_5 \wedge j_5 = 0. \)

It is therefore worth mentioning at this point that the Ashetkar type dynamical equivalence of the chiral formulation holds at the classical level because \( dC_{TT} \simeq 0 'on shell', at least for massless spinors.
When quantum field theory is involved, other boundary terms may arise from the \textit{chiral anomalies} due to the non-conservation of the axial current, cf. [63, 27]. In our case, its calculation is much facilitated by regarding one term in the decomposed Dirac Lagrangian as an \textit{external} axial covector $A$ (without “internal” indices) coupled to the axial current $j_5$ of the Dirac field in an \textit{initially flat} spacetime. Then we can apply the result (11–225) of Itzykson and Zuber [28] for the axial anomaly. Accordingly, we find that only the term $dA \wedge dA$ arises in the axial anomaly [42], but \textit{not} the Nieh–Yan type term $d^* A \sim dC_{TT}$ as was recently claimed [13]. After switching on the curved spacetime of Riemannian geometry, we finally obtain for the axial anomaly

$$< dj_5 > = 2im < P > + \frac{1}{24\pi^2} \left[ Tr \left( \Omega^{(i)} \wedge \Omega^{(i)} \right) - \frac{1}{4} dA \wedge dA \right]. \quad (59)$$

This result, which can easily be transferred to the chiral current $j_{\pm}$, is based on diagrammatic techniques and the Pauli–Villars regularization scheme. It deviates sharply from the heat kernel method [47, 48, 49] which seems to lead to partially \textit{divergent} terms, and thus cannot be applied to our problem.

In the limit $m \to 0$, one can easily read off the corresponding chiral anomaly. For gravitational Rarita–Schwinger fields $\Psi$ within supergravity, the anomaly [21, 15] for the \textit{axial current} $J_5 := i\nabla \wedge \gamma \wedge \Psi$ is $-21 \times$ the anomaly for Dirac fields, whereas for the corresponding supersymmetric Yang–Mills anomaly one finds $3 \times$ the Dirac result.

| Spin | Gravitational | YM anomaly |
|------|--------------|------------|
| 1/2  | 1            | 1          |
| 3/2  | $-21$        | 3          |

Depending on the asymptotic helicity states, there occur contributions of topological origin of the Riemannian Pontrjagin or Euler type, respectively. Interesting enough, there is a Pontrjagin type contribution $dA \wedge dA$ from axial torsion in Riemann–Cartan spacetime. The role of spinors for the index theorem and in the 4D Donaldson invariants via Seiberg–Witten equation has recently been reviewed by Atiyah [5].
7 Reality conditions

The canonical variables introduced by Ashtekar cast the constraints of general relativity into polynomial form. This is a major achievement since the general relativity constraints when written in the usual ADM variables are highly non-polynomial and not even analytic in these variables and this has been one of the major difficulties in quantizing canonical gravity. The simplicity of the Ashtekar formulation gets afflicted by the complex character of his variables, the use of reality conditions is compulsory. The pursuit of the Ashtekar program was formulated as an alternative route for classical general relativity and its canonical quantization. In the classical theory, general relativity is considered to be embedded in a larger complex theory; the restriction to the usual Einstein theory is made imposing by hand some reality conditions. In quantum theory, one first ignores the reality conditions, solves the quantum constraints of the complex theory and the reality conditions finally surface as constraints upon the admissible inner products by requiring that the real classical observables become self-adjoint operators. The use of the reality conditions as a way of selecting an inner product has worked well in some models, but the same has not happened for the full gravity or supergravity so far.

7.1 One-dimensional harmonic oscillator analog

To illustrate the procedure of the last sections on a most elementary level, we take the one-dimensional harmonic oscillator of one Hertz as a toy model and show how the complex formulation of supergravity should work a la Dirac and why the Ashtekar formulation does not allow in a straightforward way the definition of an inner product needed to enforce the reality conditions.

7.1.1 Dirac factorization method

Let us recapitulate the factorization scheme introduced by Dirac and Bargmann for obtaining the eigenstates of the one-dimensional harmonic oscillator. Classically, its Lagrangian reads

\[ \mathcal{L} = \frac{1}{2} \dot{q}^2 - U(q) = \frac{1}{2} (\dot{q}^2 - q^2), \]  

(60)
where \( \cdot := \partial/\partial t \) denotes the time derivative. In terms of the canonical momentum \( p := \partial L/\partial \dot{q} = \dot{q} \) the Hamiltonian takes the form

\[
H = p\dot{q} - L = \frac{1}{2}(p^2 + q^2). \tag{61}
\]

In quantum mechanics (QM), the corresponding annihilation and creation operators \( a \) and \( a^\dagger \) satisfying \([a, a^\dagger] = 1\) are defined as follows:

\[
a := \frac{1}{\sqrt{2}} (q + i p), \quad a^\dagger := \frac{1}{\sqrt{2}} (q - i p). \tag{62}
\]

Here \( q \) is the generalized coordinate operator and \( p = (a - a^\dagger)/2i \) its canonical conjugate momentum. The dagger is used to denote the adjoint.

The Hamiltonian (61) converts into the operator

\[
H = a a^\dagger - \frac{1}{2} = a^\dagger a + \frac{1}{2} = \frac{1}{2} (a a^\dagger + a^\dagger a) \tag{63}
\]

and allows to write the Schrödinger equation \( \mathcal{H}\psi_E = E\psi_E \) in three different but completely equivalent ways.

As it is well known, the operator \( a \) annihilates the ground state \( \psi_0 = c_0 \exp(-q^2/2) \) with energy \( E_0 = 1/2 \) of the harmonic oscillator, i.e. \( a |\psi_0\rangle = 0 \).

The remaining states can be constructed by successive application of the creation operator \( a^\dagger \) to \( \psi_0 \), the energy increasing at each step by one, i.e.

\[
\psi_n = (n!)^{-1/2} (a^\dagger)^n \psi_0 = A_n H_n (q) \exp \left(-\frac{q^2}{2}\right). \tag{64}
\]

Here \( c_0 \) and \( A_n \) are normalization constants, \( H_n(q) \) the Hermite polynomials, and \( E_n = n + \frac{1}{2} \) the eigenvalues of the energy.

In this case there exist a well-defined inner product

\[
\langle \psi_m | \psi_n \rangle = (n! m!)^{-1/2} \langle \psi_0 | a^m a^\dagger^n | \psi_0 \rangle = (n! m!)^{-1/2} n! \delta_{mn} \langle \psi_0 | \psi_0 \rangle = \delta_{mn}, \tag{65}
\]

which allows real expectation values of self-adjoint operators like the kinetic energy operator:

\[
\langle \psi_n | \frac{p^2}{2} | \psi_n \rangle = -\frac{1}{4} \langle \psi_n | (a - a^\dagger)^2 | \psi_n \rangle = \frac{1}{4} \langle \psi_n | a a^\dagger + a^\dagger a | \psi_n \rangle = \frac{1}{2} \left(n + \frac{1}{2}\right) = \frac{1}{2} E_n. \tag{66}
\]
It is important to stress the fact that in the Dirac’s factorization method both variables, i.e. the generalized coordinate and its canonical conjugated momentum, are complex. Although this is a complex formulation of the harmonic oscillator, it leads to Hermitian operators having real eigenvalues. In contrast, in the Ashtekar formulation of the next section, only the canonical momenta become complex while the generalized coordinates, the triad, remain real, leading to non-Hermitian operators. Moreover, until now there does not exist a well defined inner product in order to obtain real eigenvalues.

7.2 Hybrid representation

In a ‘nutshell’, the Ashtekar approach to quantum gravity proceeds from a canonical transformation \((q, p) \rightarrow (\tilde{q}, \tilde{p})\) induced by \(\mathcal{C} = q^2/2\) as generating function. On the Lagrangian level of our toy model this is equivalently achieved by adding the boundary term \(i \mathcal{C}\), resulting in the complex Lagrangian

\[
\tilde{\mathcal{L}} = \mathcal{L} \pm i \mathcal{C} = \mathcal{L} \pm i q.
\]  

(67)

The corresponding complex momenta are \(\tilde{\mathcal{C}} := \dot{q} \pm i \partial \mathcal{C}/\partial q = p \pm iq\) \([= \pm i \sqrt{2a}(t)]\). This resembles the Bargmann representation of the harmonic oscillator, but only partially, because merely one choice of the complex momenta, for instance \(p - iq\), but not its conjugate will be employed in the Ashtekar formulation. In this ‘hybrid’ representation, the Hamiltonian \(\tilde{\mathcal{H}} := \dot{\tilde{q}} - \dot{\tilde{p}} = \frac{1}{2} \tilde{p}^2 - \dot{q}^2 = \mathcal{H}\) turns out to be holomorphic in \(\tilde{p}\).

In QM, where \(q\) and \(p\) become operators obeying \([q, p] = i\), the canonical transformation is achieved by the similarity transformation

\[
\begin{cases}
\tilde{q} = N q N^{-1} = q \\
\tilde{p} = N p N^{-1} = p \pm i \partial \mathcal{C}/\partial q = \tilde{p}
\end{cases}

\Rightarrow N = e^{\pm \mathcal{C}}
\]

(68)

The construction of the non-unitary operator \(N\) proceeds via \(e^{\mathcal{C}} p e^{-\mathcal{C}} = p + [\mathcal{C}, p] + \frac{1}{2!} [\mathcal{C}, [\mathcal{C}, p]] + \frac{1}{3!} [\mathcal{C}, [\mathcal{C}, [\mathcal{C}, p]]] + \cdots \) \(= p \pm i q\). Since \([q^2, p] = q[q, p] + [q, p]q = 2iq\), we necessarily recover \(\mathcal{C} = q^2/2\). Observe that in the Schrödinger representation the addition of the ‘complexifier’ \(i \mathcal{C}\) in the Lagrangian (67) induces a renormalization of the wave function

\[
\psi = N \tilde{\psi} = \exp \left( \pm q^2 \right) \tilde{\psi}, \quad \tilde{\psi}_n = A_n H_n (q).
\]  

(69)
In the $\tilde{p}$ representation the Schrödinger equation becomes simplified to a first–order differential equation, but at the price of modifying the inner product by a nonlocal measure \[20\]. With a Wick rotation in phase space one can go back to the usual creation and annihilation operators $a$ and $a^\dagger$ and retain an inner product for implementing the reality conditions, see Ref. \[57\].

The Hamiltonian formulation of gravity can be easily implemented in our scheme following the usual Yang–Mills or Poincaré gauge description, see Refs. \[40, 34, 6\]. In the Ashtekar formulation with complex variables, the tangential part of the basis one–forms, i.e. the “triad densities”, and the tangential part of the self– or antisel dual connection connection will become the generalized coordinates and momenta of the bosonic sector. In supergravity, the tangential Rarita–Schwinger field will be on par with the triads, see Ref. \[35\] for details. In the context of EC theory, Maluf \[36\] has argued that the reality condition leads to the vanishing of the spacelike torsion; but this would clash with the algebraic torsion relation \(39\) in supergravity.

There have been many attempts to solve the reality conditions issue, namely by considering “generalized Wick transform” \[57, 3\], by imposing the reality conditions, not by hand, but via Dirac constraints \[45\], or by using a modified form of the self–dual action that leads to the $SO(3)$ ADM formalism \(8\) without the appearence of difficult second–order constraints. However, there is a price to be paid, the Hamiltonian constraint is no longer a simple quadratic expression in both the densitized triad and the Ashtekar connection. This fact makes it more difficult to discuss all those issues that depend critically on having the theory formulated in terms of simple constraints; in particular, solving the constraints will be harder now. Therefore, the structure of the Hamiltonian constraint is at least as complicated as the one of the familiar ADM constraint \[26\].

One interesting point of discussion, suggested by the lack of success of all the attempts to retain Lorentzian general relativity by introducing simple modifications in the know actions, has to do with the obvious asymmetry between the formulations of gravity or supergravity in a real Ashtekar phase space for Lorentzian and Euclidean signatures. In fact it all boils down to the relative signs between the potential and kinetic terms in the scalar constraint. Consequently, the origin of the signature at the Lagrangian level is also rather obscure. The marked asymmetry between the real Hamiltonian formulation for different spacetime signatures strongly suggests that it would
differ very much from the usual self-dual action. This could have intriguing consequences in a perturbative setting because the ultraviolet behavior of the Euclidean and the Lorentzian theories could be very different. It is worthwhile to stress this point for the Hilbert–Einstein action and the so-called higher derivative theories, that differ in some terms quadratic in the curvatures. The first one is nonrenormalizable, whereas the second one is perturbatively renormalizable but nonunitary \cite{56}.

In full, the reality conditions issue deserves further investigation.

8 Discussion

As we have shown, there exists a natural way to translate ($\mathcal{N}=1$) supergravity theory into chiral form, by adding certain Chern–Simons boundary terms to the original action. This prescription works elegantly not only for the case of chiral simple ($\mathcal{N}=1$) supergravity, but also for the Einstein–Cartan–Dirac theory, see \cite{43} for details.

For the bosonic sector, the boundary term (20) coming from the translational Chern–Simons term (20) has the usual geometric form. In the case of the gravitino spinor field such boundary term is proportional to the axial current and arises naturally by using the algebraic Cartan relation as the second field equation. Since standard Rarita–Schwinger Lagrangian is zero ‘on shell’ it is not surprising that the same happens in their chiral form. Consistently, the correspondent boundary terms relating both forms of the above Lagrangian semi–classically turn out to be zero ‘on–shell’ again, i.e. $dC_{TT} \cong 0$, at least for massless spinor field. Consequently, the addition of a suitable Chern–Simons boundary terms seems to be a unifying principle in order to obtain the chiral formulation of fermions coupled to gravity.

Spinor formulations of general relativity have become very popular again. For instance, Nester and Tung \cite{46} invented an interesting quadratic spinor Lagrangian for general relativity. However, the variational principle seems to be incomplete, since the normalization $\nabla \Psi = 1$ of the Rarita–Schwinger type one–form $\Psi$ is not consistently included via a Lagrange multiplier two–form.

In previous papers of Tsuda et al. \cite{58, 59} on ($\mathcal{N}=1$) and ($\mathcal{N}=2$) supergravity, boundary terms are consistenly neglected thus completely missing the crucial role of the translational Chern–Simons term as the generating function for the chiral variables. In a recent work of Tung and Jacobson \cite{61},
our earlier differential form approach including a translational Chern–Simons term has been partially recovered in spinor notation without, however, using the unifying notion of Clifford algebras. For massive fermions in a gravitational field with angular momentum, there occur chirality transitions, see [12], which in extreme astrophysical situations could give way to sterile particles. A recent paper [18] seems to indicate that simple supergravity may not even be one–loop finite in the presence of boundaries.

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