THE HYPERBOLIC MODULI SPACE OF FLAT CONNECTIONS
AND THE ISOMORPHISM OF SYMPLECTIC MULTIPLICITY
SPACES

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Abstract. Let $G$ be a simple complex Lie group, $\mathfrak{g}$ be its Lie algebra, $K$ be a maximal compact form of $G$ and $\mathfrak{k}$ be a Lie algebra of $K$. We denote by $X \rightarrow \overline{X}$ the anti-involution of $\mathfrak{g}$ which singles out the compact form $\mathfrak{k}$.

Consider the space of flat $\mathfrak{g}$-valued connections on a Riemann sphere with three holes which satisfy the additional condition $A(z) = -A(\overline{z})$. We call the quotient of this space over the action of the gauge group $g(z) = g^{-1}(\overline{z})$ a hyperbolic moduli space of flat connections.

We prove that the following three symplectic spaces are isomorphic:
1. The hyperbolic moduli space of flat connections.
2. The symplectic multiplicity space obtained as symplectic quotient of the triple product of co-adjoint orbits of $K$.
3. The Poisson-Lie multiplicity space equal to the Poisson quotient of the triple product of dressing orbits of $K$.

1. Introduction

In this paper we investigate symplectic multiplicity spaces for compact simple Lie groups.

It has been recently observed by Jeffrey [8] that in the case when all three co-adjoint orbits belong to a small vicinity of $0 \in \mathfrak{k}^*$ symplectic multiplicity spaces coincide with the moduli spaces of $\mathfrak{k}$-valued flat connections on the sphere with three holes. It is also known that this result does not hold true for sufficiently big co-adjoint orbits.

Though we do not know how to extend the results of [8], we reformulate the problem for the hyperbolic moduli space of flat connections which is obtained as a symplectic quotient of the space of $\mathfrak{g}$-valued connections with additional condition $A(z) = -A(\overline{z})$ over the action of the gauge group $g(z) = g^{-1}(\overline{z})$. In this setting we establish the isomorphism of the symplectic multiplicity spaces to the hyperbolic moduli spaces for arbitrary co-adjoint orbits. This isomorphism is described explicitly.

The construction of the map between multiplicity spaces and moduli spaces is based on the simple observation that the holomorphic connection of the type

$$A(z) = \sum_i \frac{X_i}{z - z_i}dz$$

is flat. Here $X_i$ belong to given co-adjoint orbits. This construction has been suggested by Hitchin in [7].

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It is worth mentioning that both our main results (subsection 4.2 and 4.3) are inspired by the work of Fock and Rosly [4]. As a nice corollary of their description of the moduli spaces of flat connections we obtain an isomorphism of the hyperbolic moduli space and the Poisson-Lie multiplicity space corresponding to an arbitrary Poisson structure on the compact group $K$.

The paper is organized as follows. In Section 2 we define symplectic and Poisson-Lie multiplicity spaces. Section 3 includes a review of the Goldman and Fock-Rosly descriptions of the moduli space of flat connections. There we also define the hyperbolic moduli spaces for the sphere with three holes.

The isomorphisms between the multiplicity spaces and the hyperbolic moduli space are described in Section 4.

2. Symplectic Multiplicity Spaces

2.1. Multiplicity spaces for compact Lie groups. Let $G$ be a complex, connected and simply connected Lie group with Lie algebra $g$ and let $K$ be its maximal compact form. We denote by $k^*$ the dual space to the corresponding Lie algebra $k$. There exists a nondegenerate positively definite scalar product on $k^*$ (Killing form) which we denote by $<,>$. It induces the canonical isomorphism $I$ between $k^*$ and $k$:

\[ I : k \rightarrow k^*. \]

The group $K$ acts on $k^*$ by the coadjoint action $Ad^*$:

\[ Ad^*(k) = I \circ Ad(k) \circ I^{-1}. \]

The space $k^*$ carries a natural Poisson structure called Kirillov-Kostant Poisson structure which is linear and invariant under $Ad^*$-action.

Definition 1. Kirillov-Kostant bracket of two functions $\psi$ and $\psi'$ on $k^*$ is given by the following formula:

\[ \{ \psi, \psi' \}(P) = < P, [\nabla \psi, \nabla \psi'] >. \]

Here $P \in k^*$ and $[,]$ is the commutator in $k$ (we consider $\nabla \psi$ and $\nabla \psi'$ as elements of the Lie algebra $k$).

According to the general theory of Poisson manifolds [3, 16] the space $k^*$ splits into the set of symplectic leaves. Symplectic leaves coincide with orbits of the coadjoint action of $K$ on $k^*$. In this paper we consider only coadjoint orbits of the maximal dimension. Such orbits are in one to one correspondence with elements $H$ of the interior of the positive Weyl chamber $W_+$ in the Cartan subalgebra $\mathfrak{h} \subset k$:

\[ O_0(H) = \{ X \in k^* : X = Ad^*(k) I(H), k \in K, H \in W_+ \}. \]

The orbits of smaller dimension correspond to the elements $H$ on the boundary of the Weyl chamber.

The main objects of our interest in this paper are symplectic multiplicity spaces. In order to define them we consider a direct product of three coadjoint orbits $O_0(H_1) \times O_0(H_2) \times O_0(H_3)$. This is a symplectic space with the Poisson bracket equal to the sum of Kirillov-Kostant brackets on $O_0(H_1)$, $O_0(H_2)$ and $O_0(H_3)$. 
There is a diagonal action of the group $K$ on the triple product:

(6) $\mathcal{O}_0(H_1) \times \mathcal{O}_0(H_2) \times \mathcal{O}_0(H_3) \ni (X_1, X_2, X_3) \xrightarrow{\kappa} (\text{Ad}^*(k) X_1, \text{Ad}^*(k) X_2, \text{Ad}^*(k) X_3)$. 

This is a Hamiltonian action with the moment map:

(7) $\mu : \mathcal{O}_0(H_1) \times \mathcal{O}_0(H_2) \times \mathcal{O}_0(H_3) \to \mathfrak{t}^* :$

$\mu(X_1, X_2, X_3) = X_1 + X_2 + X_3.$

Symplectic multiplicity space is the symplectic quotient of the space $\mathcal{O}_0(H_1) \times \mathcal{O}_0(H_2) \times \mathcal{O}_0(H_3)$ with respect to the diagonal action (6) of $K$.

**Definition 2.** The symplectic multiplicity space $\mathcal{M}_0(H_1, H_2, H_3)$ is the quotient of the space $\mu^{-1}(0) \subset \mathcal{O}_0(H_1) \times \mathcal{O}_0(H_2) \times \mathcal{O}_0(H_3)$ over the diagonal action (6) of $K$:

(8) $\mathcal{M}_0(H_1, H_2, H_3) = \{(X_1, X_2, X_3) \in \mathcal{O}_0(H_1) \times \mathcal{O}_0(H_2) \times \mathcal{O}_0(H_3) : X_1 + X_2 + X_3 = 0\}/K.$

We always assume that the stabilizer of a point in $\mu^{-1}(0)$ is a discrete subgroup of $K$. Then 0 is said to be a regular value of the moment map and the quotient space $\mathcal{M}_0(H_1, H_2, H_3)$ is a smooth manifold. As a symplectic quotient it inherits symplectic structure from the parent space $\mathcal{O}_0(H_1) \times \mathcal{O}_0(H_2) \times \mathcal{O}_0(H_3)$.

Now we proceed to the definition of Poisson-Lie multiplicity spaces.

2.2. **Multiplicity spaces for Poisson-Lie groups.** The main idea of the theory of Poisson-Lie groups is to equip a Lie group with a special kind of Poisson structure.

**Definition 3.** A pair $(K, \mathcal{P})$ of a group $K$ and a Poisson bracket $\mathcal{P}$ on $K$ is called a Poisson-Lie group if the multiplication map $K \times K \to K$ is a Poisson map.

According to [11] the classes of isomorphic Poisson-Lie structures on a compact simple Lie group are parametrized by pairs $(t, u)$, where $t$ is a real number and $u$ is a real valued anti-symmetric (with respect to the Killing form) automorphism of the Cartan subalgebra $\mathfrak{h}$.

$u : \mathfrak{h} \to \mathfrak{h}$

(9) $<u(X), Y> = -<X, u(Y)> \ , \ \forall X, Y \in \mathfrak{h}.$

In order to describe this family of Poisson brackets we need some more notations.

Let $e_\alpha$ be the root generator of $\mathfrak{g}$ corresponding to a root $\alpha$ and $h_i$ be the generator of the Cartan subalgebra $\mathfrak{h}$ corresponding to a simple root $\alpha_i$. Define a pair of classical $r$-matrices (solutions to the classical Yang-Baxter equation [14]) $r_+^f(u), r_-^f(u) \in \mathfrak{g} \otimes \mathfrak{g}$, $\mathfrak{g}$ is considered as a Lie algebra over $\mathbb{R}$:

(10) $r_+^f(u) = \frac{1}{2} \sum_{j \in \Delta_+} i(h_j + iu(h_j)) \otimes h^j + \sum_{\alpha \in \Delta_+} (e_\alpha \otimes (i\alpha + i\alpha) + i\alpha \otimes (\alpha - \alpha)).$
where $\Delta_+$ is the set of positive roots, $\Delta_-$ is the set of simple roots, $P$ is the flip operation in the tensor product $g \otimes g$: $P(a \otimes b) = b \otimes a$. We shall often write $r^t_+$ and $r^t_-$ instead of $r^t_+(u)$ and $r^t_-(u)$. Let us remark that the $r$-matrices $r^t_\pm(u)$ satisfy the following reality condition:

$$r^t_+(u) = r^t_-(u).$$

Here $X \to \overline{X}$ is the anti-involution in $g$ which singles out the compact form $\mathfrak{k}$.

**Definition 4.** The Poisson-Lie bracket on the compact group $K$ corresponding to the pair of classical $r$-matrices $r^t_\pm(u)$ is defined as:

$$\{ \psi, \psi' \}_{(t,u)}(k) = \langle r^t_+(u), \nabla^L \psi(k) \otimes \nabla^L \psi'(k) \rangle - \langle r^t_+(u), \nabla^R \psi(k) \otimes \nabla^R \psi'(k) \rangle.$$

Here the bracket $\langle, \rangle$ is the canonical pairing between $\mathfrak{k}^* \otimes \mathfrak{k}^*$ and $\mathfrak{k} \otimes \mathfrak{k}$.

We always understand $\nabla^L$ and $\nabla^R$ as left and right derivations on $G$ considered as a real group with values in the dual space to the Lie algebra $g$. Let us note that the reality condition (12) implies reality of the Poisson tensor (13).

Besides the Poisson-Lie group $K$ we shall need the dual Poisson-Lie group $K^*$ and the Heisenberg double $D$, which provide the Poisson-Lie counterparts of the dual space $\mathfrak{k}^*$ and of the cotangent bundle $T^*K$.

For the Poisson-Lie structures on compact simple Poisson-Lie groups which we listed above, the dual group $K^*(u)$ and $D_t(u)$ can be described as follows. Let $N_+$ be the nilpotent subgroup of the group $G$ generated by positive roots and let $H$ be Cartan subgroup. The Borel subgroup $B_+$ is a semi-direct product of $H$ and $N_+$.

**Definition 5.** The Poisson-Lie group $K^*_t(u)$ is the following subgroup of $B_+$:

$$K^*_t(u) = \{ k^* \in B_+ : k^*_h = \exp(i(a + iu(a))) \text{ for some } a \in \mathfrak{h} \}.$$

Here $k^*_h$ is the image of the projection of $k^*$ into the Cartan subgroup of $G$:

$$k^* = k^*_h n , \quad n \in N_+.$$

The Poisson structure on the group $K^*_t(u)$ is given by the following formula:

$$\{ \psi, \psi' \}_{(t,u)}(k^*) = \langle r^t_+(u), \nabla^L \psi(k^*) \otimes \nabla^L \psi'(k^*) \rangle - \langle r^t_+(u), \nabla^R \psi(k^*) \otimes \nabla^R \psi'(k^*) \rangle.$$

Here we embed $K^*_t(u)$ into $G$ and understand $\nabla^L$ and $\nabla^R$ as left and right derivations on $G$.

**Definition 6.** The Heisenberg double $D_t(u)$ of the compact Poisson-Lie group $K$ is the group $G$ equipped with the following Poisson bracket:
\[(17) \quad \{\psi, \psi'\}_{(t,u)}(g) = \langle r^t_+(u), \nabla_L\psi(g) \otimes \nabla_L\psi'(g) \rangle + \langle r^t_-(u), \nabla_R\psi(g) \otimes \nabla_R\psi'(g) \rangle.\]

Let us make two remarks about the Heisenberg double:
1. The Heisenberg double is not a Poisson-Lie group.
2. The Poisson structure \((17)\) is nondegenerate.

The group \(K\) acts on \(D_t(u)\) by left and right multiplications. These are Poisson actions.

**Definition 7.** The action of a Poisson-Lie group \(K\) on a Poisson manifold \(M\) is called a Poisson action if the map \(K \times M \to M\) is a Poisson map.

Let us remark that if the Poisson bracket on the group \(K\) is nontrivial, the Poisson action does not preserve the Poisson bracket on \(M\).

In the case of Poisson-Lie action of the compact group \(K\) the standard notion of the moment map should be replaced by the notion of the moment map in the sense of Lu and Weinstein [13] (Poisson-Lie moment map). The target space of this new kind of moment map is the nonabelian Poisson-Lie group \(K^*\) instead of the abelian Poisson space \(\mathfrak{k}^*\). Let \(M\) be a Poisson manifold which carries a Poisson action of the Poisson-Lie group \(K\). Define a set of vector fields \(v_X\) on \(M\) for \(X \in \mathfrak{k}\) which represent the infinitesimal \(K\)-action.

**Definition 8.** A map \(m : M \to K^*\) is called a moment map in the sense of Lu and Weinstein if the following equation holds true:
\[(18) \quad v_X = \mathcal{P}(\cdot, X).
\]
Here \(\theta\) is the right-invariant Maurer-Cartan form on the group \(K^*\).

In order to describe the moment maps for left and right multiplication actions of \(K\) on \(D\) one can use the Iwasawa decomposition:
\[(19) \quad D = G = KK^* = K^*K,\]
\[g = \pi_L(g)\pi_R^*(g) = \pi_L^*(g)\pi_R(g).\]
So, we have defined right and left projections \(\pi_R, \pi_R^*\) and \(\pi_L, \pi_L^*\) from \(D\) to \(K\) and \(K^*\).

**Lemma 1.** The moment maps for the left and right multiplication actions of the group \(K\) on \(D\) are given by the projections \(\pi_R^*\) and \(\pi_L^*:\)
\[(20) \quad m^L(g) = \pi_L^*(g),\]
\[(21) \quad m^R(g) = (\pi_R(g))^{-1}.\]

Now we turn to the Poisson structure \((16)\) on \(K^*\). It is degenerate. The corresponding symplectic leaves coincide with orbits of the dressing action of \(K\) on \(K^*\) (these orbits are deformations of coadjoint orbits in \(\mathfrak{k}^*\)). One can define the dressing action by combining the left action of \(K\) on \(D\) with the right projection from \(D\) to
$K^*$. More explicitly, one can introduce maps $\rho : K \times K^* \to K$ and $\rho^* : K \times K^* \to K^*$ defined by the following equations:

\begin{align}
\rho(k, k^*) &= \pi_R(kk^*), \\
\rho^*(k, k^*) &= \pi_L^1(kk^*).
\end{align}

**Definition 9.** The result of the dressing action $AD^*$ of $k \in K$ on $k^* \in K^*$ is given by $\rho^*(k, k^*)$ in (22):

$$AD^*(k) k^* = \rho^*(k, k^*).$$

There exists a map from $k^*$ to $K^*$ which intertwines the actions $Ad^*$ and $AD^*$. In order to define it let us first introduce the following map:

$$f : K^* \to SK$$

where $SK$ is the following subspace in $G$:

$$SK = \{ \exp(iX), X \in \mathfrak{k} \} = \{ g \in G, \mathfrak{g} = g \}.$$

It is easy to prove that $f$ is invertible. So, one can define the map $e$ from $\mathfrak{k}^*$ to $K^*$:

$$e(X) = f^{-1}[\exp(2it I^{-1}(X))].$$

We remind that $I$ is the map from $\mathfrak{k}$ to $\mathfrak{k}^*$ defined by the Killing form.

It is easy to see that $e$ intertwines the actions of $K$ on $\mathfrak{k}^*$ and $K^*$:

$$AD^*(k) e(X) = e(Ad^*(k) X).$$

The map $e$ enables us to parametrize dressing orbits in $K^*$ by elements of the positive Weyl chamber in Cartan subalgebra $\mathfrak{h}$:

$$O_t(H) = \{ k^* = AD^*(k) e(I(H)), k \in K \}$$

In fact, $O_t(H)$ and $O_0(H)$ are isomorphic as symplectic spaces.

Now we can easily define the Poisson-Lie multiplicity spaces. Consider a product of three dressing orbits $O_t(H_1) \times O_t(H_2) \times O_t(H_3)$. Group $K$ acts on it in the following way

$$(k^*_1, k^*_2, k^*_3) \to (AD^*(k) k^*_1, AD^*(\rho(k, k^*_1)) k^*_2, AD^*(\rho(\rho(k, k^*_1), k^*_2)) k^*_3),$$

where $\rho$ is defined in equation (22). We refer to this action as to diagonal dressing action.

The action (29) is a Poisson one with the Poisson-Lie moment map $m(k^*_1, k^*_2, k^*_3) = k^*_1 k^*_2 k^*_3$. So one can apply the Poisson reduction (Poisson-Lie counterpart of the Hamiltonian reduction). The resulting Poisson quotients are Poisson-Lie multiplicity spaces.
Definition 10. Let $e_{K^*}$ be the unit element of the group $K^*_t(u)$. The Poisson-Lie multiplicity space $\mathcal{M}_t(H_1, H_2, H_3)$ is the quotient of the space $m^{-1}(e_{K^*})$ over the action (29) of $K$:

\begin{equation}
\mathcal{M}_t(H_1, H_2, H_3) = \{(k_1^*, k_2^*, k_3^*) \in \mathcal{O}_t(H_1) \times \mathcal{O}_t(H_2) \times \mathcal{O}_t(H_3) : k_1^*k_2^*k_3^* = e_{K^*}\}/K.
\end{equation}

The main aim of this text is to prove that there exists a global symplectic isomorphism between $\mathcal{M}_0(H_1, H_2, H_3)$ and $\mathcal{M}_t(H_1, H_2, H_3)$. The main tool in our proof is the moduli space of flat connections on the sphere with three holes and a certain real form of it.

3. The moduli space of flat connections on the sphere with three holes

3.1. Atiyah-Bott description. Let us consider a sphere with three holes. We denote it by $\Sigma$ and represent as a complex plain without two discs $D_1$ and $D_{-1}$ around $z = 1$ and $z = -1$:

\begin{equation}
\Sigma = \mathbb{C} \setminus (D_1 \cup D_{-1}).
\end{equation}

The third hole is at infinity.

Let $\mathcal{N}_\Sigma$ be the space of all connections on $\Sigma$ with values in the Lie algebra $\mathfrak{g}$. $\mathcal{N}_\Sigma$ is a Poisson manifold with Poisson bracket:

\begin{equation}
\{\psi, \psi'\} = \int_\Sigma < \frac{\delta \psi}{\delta A} \wedge \frac{\delta \psi'}{\delta A} >,
\end{equation}

where $\psi(A)$ and $\psi'(A)$ are functionals on $\mathcal{N}_\Sigma$. The Poisson bracket (32) is degenerate. In order to describe the corresponding symplectic leaves we fix conjugacy classes of the holonomies around holes. Let us choose contours $\Gamma_1, \Gamma_2$ and $\Gamma_3$ around the points 1, -1 and $\infty$ correspondingly. Define the space

\begin{equation}
\mathcal{N}(H_1, H_2, H_3) = \{A \in \mathcal{N}_\Sigma : \text{Hol}(A, \Gamma_j) \sim \exp(2itH_j)\},
\end{equation}

where $H_i$ belong to the positive Weyl chamber of the Cartan subalgebra $\mathfrak{h}$, $\text{Hol}(A, \Gamma_j) \sim \exp(2itH_j)$ means that $\text{Hol}(A, \Gamma_j)$ is conjugate to $\exp(2itH_j)$ as an element in $G$.

The spaces $\mathcal{N}(H_1, H_2, H_3)$ are the symplectic leaves of the Poisson bracket (32).

The bracket (32) is invariant with respect to the action of the gauge group $G_\Sigma$:

\begin{equation}
A \xrightarrow{g} gAg^{-1} - dgg^{-1}.
\end{equation}

Moreover, the gauge action is Hamiltonian, the corresponding moment map is given by the curvature:

\begin{equation}
F(A) = dA + A^2.
\end{equation}

The symplectic quotient of the space $\mathcal{N}(H_1, H_2, H_3)$ is the moduli space of flat connections on the sphere with three holes.
Definition 11. The moduli space of flat connections is a quotient of the space of flat connections over the gauge action of $G_{\Sigma}$:

\[ \mathcal{M}_{\Sigma}(H_1, H_2, H_3) = \{ A \in \mathfrak{h}(H_1, H_2, H_3) : F(A) = 0 \}/G_{\Sigma}. \]  

The space $\mathcal{M}_{\Sigma}(H_1, H_2, H_3)$ is a finite dimensional symplectic space with Poisson structure inherited from (32).

We shall need two alternative descriptions of the Poisson structure on the moduli space $\mathcal{M}_{\Sigma}(H_1, H_2, H_3)$. The first one was introduced by Goldman [6]. In order to describe the Goldman bracket let us consider the following set of functions on the space of flat connections on $\Sigma$. Denote by $\lambda$ some exact matrix representation of the group $G$. Then define

\[ \phi_{\Gamma} = \text{Tr}_{\lambda} \text{Hol}(A, \Gamma) \]  

for a closed contour $\Gamma$ on the surface. Occasionally we shall drop the sign $\lambda$ in the right hand side of this definition. The connection $A$ being flat, the function (37) depends exclusively on the homotopy class of $\Gamma$. Evidently, it is invariant with respect to the gauge transformations (34). So, it is a well-defined function on the moduli space of flat connections $\mathcal{M}_{\Sigma}(H_1, H_2, H_3)$ (or, more exactly, a pullback of such a function). In the case of $G = SU(n)$ Goldman [6] defined the Poisson bracket of two such functions $\phi_{\Gamma}$ and $\phi_{\Gamma'}$ as follows:

\[ \{ \phi_{\Gamma}, \phi_{\Gamma'} \} = \sum_{p \in \Gamma \cap \Gamma'} (-1)^{\nu(p)} (\phi_{\Gamma_p} - \frac{1}{n} \phi_{\Gamma} \phi_{\Gamma'}). \]  

Here $\nu(p)$ is the contribution into the intersection number of $\Gamma$ and $\Gamma'$ at the point $p$ and $\Gamma_p$ is the contour produced from the contours $\Gamma$ and $\Gamma'$ by the transformation in the vicinity of intersection point $p$ shown on Figure 1. We suppose that the contours intersect each other transversally and in finite number of points. This is not a restricting condition since only the homotopy class of a contour is important.

One can easily derive bracket (38) from (32).

More complicated functions on the moduli space can be defined by replacing countors by arbitrary graphs. The Poisson brackets of these functions similar to the Goldman’s brackets can be found in [2].

3.2. Fock-Rosly theorem. Another description of the Poisson bracket on the moduli space of flat connections was introduced by Fock and Rosly in [4]. They
used a notion of a ciliated fat graph which we are going to review now. Let us draw an oriented graph $L$ on the surface $\Sigma$ such that each face of the graph is homeomorphic either to a disk or to a disk with one hole. For instance, one can consider a triangulation of the surface. The graph $L$ has a cyclic order of ends of edges at each vertex which is inherited from the orientation of the surface. Such a graph is called a fat graph. It is convenient to introduce notations $s(l)$ and $t(l)$ for the starting and end points of the oriented edge $l$.

Let $\mathcal{N}_L$ be the space of graph connections. A graph connection is an assignment of an element $a_l$ of the group $G$ to each edge $l$ of the graph. Correspondingly, an element of the graph gauge group $G_L$ is an assignment of an element $g_\alpha \in G$ to each vertex $\alpha$ of the graph. The graph gauge group naturally acts on the space of graph connection:

$$a_l \rightarrow g_{t(l)} a_l g_{s(l)}^{-1}.$$  

(39)

We call a graph connection flat if its holonomy around each empty face is equal to identity in $G$ and the holonomy around the face containing the hole $\Gamma_i$ belongs to the same conjugacy class as $\exp(2itH_i)$. We denote the space of flat graph connections by $A_L(H_1, H_2, H_3)$. It is a standard fact that the quotient of the space of flat graph connections over the graph gauge group is equal to the moduli space of flat connections on $\Sigma$:

$$\mathcal{M}_\Sigma(H_1, H_2, H_3) = A_L(H_1, H_2, H_3)/G_L.$$  

(40)

Fock and Rosly [4] introduced a Poisson structure on the space $\mathcal{N}_L$. This structure depends on a linear order of the ends of edges at each vertex of the graph. One can fix the linear order by putting a small cilium at the vertex and choosing the edge just after the cilium to be the first in the enumeration (see Figure 2).

A fat graph with this additional structure is called a ciliated fat graph.

We need some notations. Let $N(L)$ be the set of all vertices of the graph $L$ and $E(L)$ be the set of ends of edges of $L$. We denote by $E(n)$ the subset of $E(L)$ which consists of the ends of edges incident to a given vertex $n \in N(L)$.

Let us introduce vector fields

$$X_\alpha = \begin{cases} \nabla_l^L & \text{for } \alpha \text{ being the target end of the edge } l \\ -\nabla_l^R & \text{for } \alpha \text{ being the starting end of the edge } l. \end{cases}$$  

(41)
Here $\nabla^L_l$ and $\nabla^R_l$ are left- and right-invariant derivations on the copy of the group $G$ assigned to an edge $l \in L$.

Now we are ready to write down the Poisson bracket of two functions $\psi$ and $\psi'$ on the space of graph connection. It is given by the following formula:

$$\{\psi, \psi'\} = \sum_{n \in N(L)} \left( \sum_{\alpha, \beta \in E(n)} \left< r_+^{t_\alpha}, X^{1\psi}_\alpha \wedge X^{2\psi'}_\beta \right> + \frac{1}{2} \sum_{\alpha \in E(n)} \left< r_+^{t_\alpha}, X^{1\psi}_\alpha \wedge X^{2\psi'}_\alpha \right> \right),$$

where $r_+^t$ is the classical $r$-matrix \([14]\). Here we use tensor notations $X^1 = X \otimes 1$ and $X^2 = 1 \otimes X$. We write $\alpha < \beta$ in the sense of the linear order in $E(n)$.

The main result of \([4]\) is described by the following theorem.

**Theorem 1.**

1. The Poisson bracket \((12)\) induces a Poisson bracket on the quotient space $N_L/G_L$.
2. The moduli space of flat connections is embedded into the quotient space $N_L/G_L$ as a symplectic leaf so that the induced Poisson bracket coincides with the Goldman bracket.

3.3. The hyperbolic moduli space. Let $\tau$ be a second order anti-automorphism (in the sense of complex structure) of the Riemann surface $\Sigma$:

$$\tau : \Sigma \rightarrow \Sigma, \quad \tau^2 = id.$$

Combining $\tau$ with the anti-involution of $g$ one can define an anti-automorphism $\sigma$ on the space of connections $N_\Sigma$:

$$\sigma(A)(z) = -A(\tau(z)).$$

One of the main objects of our interest in this text is the $\sigma$-invariant subspace in $N_\Sigma$:

$$N^\sigma_\Sigma = \{ A \in N_\Sigma : \sigma(A) = A \}.$$

The space $N^\sigma_\Sigma$ is acted on by the subgroup $G^\sigma_\Sigma$ of gauge group $G_\Sigma$:

$$G^\sigma_\Sigma = \{ g \in G_\Sigma : g(z) = g^{-1}(\tau(z)) \}.$$

If the surface $\Sigma$ has holes we demand that holonomies around the holes belong to some fixed conjugacy classes (see subsection 3.1). We assume that the map $\tau$ is an anti-automorphism of the Riemann surface with holes. We shall suppose that each particular hole is preserved by $\tau$. Choose the contours $\Gamma_i$ which are used to compute holonomies to be $\tau$-invariant. We can define the following subspace in the space $N^\sigma_\Sigma$:

$$N^\sigma_\Sigma(H_1, H_2, H_3) = \{ A \in N^\sigma_\Sigma : \text{Hol}(\Gamma_j, A) \sim \exp(2itH_j), H_j \in W_+ \}.$$

We also define the subspace of flat connections in $N^\sigma_\Sigma(H_1, H_2, H_3)$.
and the *hyperbolic* moduli space of flat connections.

**Definition 12.** The hyperbolic moduli space of flat connections is the quotient of the space \( \mathcal{A}_\Sigma^c(H_1, H_2, H_3) \) over the action of the gauge group \( G_\Sigma^c \).

\[
\mathcal{M}_\Sigma^c(H_1, H_2, H_3) = \mathcal{A}_\Sigma^c(H_1, H_2, H_3)/G_\Sigma^c.
\]

It is obvious that Poisson bracket (32) can be restricted to the space \( \mathcal{A}_\Sigma^c \) and so the hyperbolic moduli space is a symplectic quotient of \( \mathcal{A}_\Sigma^c \).

**Remark.** It is instructive to compare the hyperbolic moduli space of flat connections with the moduli space of flat connections with values in the compact form \( k \). In the latter case one considers the space of connections restricted by the condition

\[
A(z) = -A(z).
\]

The compact gauge group is defined as

\[
G_\Sigma^c = \{ g \in G_\Sigma : g(z) = g^{-1}(z) \}.
\]

Finally, the holonomies around holes are restricted to some conjugacy classes in the compact group \( K \):

\[
\mathcal{N}_\Sigma^c(H_1, H_2, H_3) = \{ A \in \mathcal{N}_\Sigma^c : \text{Hol}(\Gamma_j, A) \sim \exp(2tH_j), H_j \in W_+ \}.
\]

The eigenvalues of the holonomy matrices in this case are unitary numbers. For this reason one can call the moduli space \( \mathcal{M}_\Sigma^c(H_1, H_2, H_3) \) *elliptic* moduli space of flat connections. On the contrary, the eigenvalues of the holonomy matrices defined by (47) are positive real numbers. They define a set of hyperbolic elements of \( G \) and we call the corresponding moduli space the *hyperbolic* moduli space of flat connections.

Now we have to adjust the Fock and Rosly description to the hyperbolic moduli space. We choose a ciliated fat graph \( L \) to be invariant with respect to the anti-automorphism \( \tau \):

\[
\tau(L) = L.
\]

The analogue of the condition (42) for graph connections is

\[
a_l = (\bar{a}_{\tau(l)})^{-1},
\]

where \( \tau(l) \) is the image of the edge \( l \) under the action of \( \tau \).

Similarly, the condition for the graph gauge group is the following:

\[
g_\alpha = (\bar{g}_{\tau(\alpha)})^{-1}
\]

for any vertex \( \alpha \) and its image \( \tau(\alpha) \).

We leave it as an exercise for the reader to check that the Fock-Rosly bracket (42) can be restricted to graph connections obeying (54). In proving this fact it is important to note that the linear order at a vertex reverses under the action of \( \tau \) and to use the identity
Let $H$ the situation when the elements $H_1$, $H_2$, and $H_3$ are sufficiently small. The proof for the hyperbolic moduli space follows the same scheme.

The reduced space corresponding to the level $(z_1, z_2, z_3)$ of the moment map coincides with $M\Sigma (H_1, H_2, H_3)$.

4. Equivalence of Poisson-Lie and compact multiplicity spaces

A triple $(H_1, H_2, H_3)$ of elements of the positive Weyl chamber of $\mathfrak{h}$ defines three different symplectic spaces:

1. the symplectic multiplicity space $M_0(H_1, H_2, H_3)$,
2. the Poisson-Lie multiplicity space $M_t(H_1, H_2, H_3)$,
3. the hyperbolic moduli space of flat connections $M^\Sigma_t(H_1, H_2, H_3)$.

The goal of this paper is to compare these three manifolds as symplectic spaces.

4.1. Small values of $H_1$, $H_2$ and $H_3$. In this subsection we restrict ourselves to the situation when the elements $H_1$, $H_2$, and $H_3$ are sufficiently close to 0 in $\mathfrak{h}$.

Theorem 2. Let $t$ be a real number. There exists a neighborhood $U$ of 0 in $\mathfrak{h}$ such that

$$M_0(H_1, H_2, H_3) \approx M_t(H_1, H_2, H_3) \approx M^\Sigma_t(H_1, H_2, H_3)$$

if $H_1$, $H_2$, and $H_3$ belong to $U$.

Proof. In [3] it has been shown that $M_0(H_1, H_2, H_3)$ is isomorphic to the elliptic moduli space $M^\Sigma_t(H_1, H_2, H_3)$ if $H_1$, $H_2$, and $H_3$ are sufficiently small. The proof for the hyperbolic moduli space follows the same scheme.

Here we prove only the isomorphism of $M_0(H_1, H_2, H_3)$ and $M_t(H_1, H_2, H_3)$. The idea of the proof is very similar to the one in [3] (see Theorem 6.6). Consider the direct product of three copies of the Heisenberg double $D^3$. Define the action of the group $K^4$ on $D^3$ in the following fashion:

$$(d_1, d_2, d_3) \to (k_0^{-1}d_1k_1, k_0^{-1}d_2k_2, k_0^{-1}d_3k_3).$$

This is a Poisson action. The Poisson-Lie moment map for this action is given by

$$\mathfrak{m} : (d_1, d_2, d_3) \to (\mathfrak{m}^L(d_1)\mathfrak{m}^L(d_2)\mathfrak{m}^L(d_3), \mathfrak{m}^R(d_1), \mathfrak{m}^R(d_2), \mathfrak{m}^R(d_3)).$$

The reduced space corresponding to the level $(e_{K^4}, \exp(2itH_1), \exp(2itH_2), \exp(2itH_3))$ of the moment map coincides with $M_t(H_1, H_2, H_3)$.

It is known that for any Poisson action of a compact Poisson-Lie group $K$ on a symplectic manifold which possesses an equivariant moment map one can introduce a $K$-invariant symplectic structure on the same manifold such that the reduced spaces do not change [4]. Let us apply this construction to $D^3$ equipped with the $K^4$-action (59). Then by applying the theorem of Lerman and Sjamaar [10] (see
Proposition 2.5) one can prove that there exists a small neighborhood of the zero level set of the moment map in $D^3$ which is isomorphic to a small neighborhood of the zero section in $T^*K^3$. The corresponding symplectic quotients of $T^*K^3$ coincide with symplectic multiplicity spaces $\mathcal{M}_0(H_1, H_2, H_3)$.

Thus we conclude that

$$\mathcal{M}_t(H_1, H_2, H_3) \approx \mathcal{M}_0(H_1, H_2, H_3)$$

for $H_1$, $H_2$ and $H_3$ being sufficiently close to 0. \hfill \square

4.2. The Poisson map from the multiplicity space to the moduli space. Let us consider the map $\xi$ from the multiplicity space $\mathcal{M}_0(H_1, H_2, H_3)$ to the hyperbolic moduli space $\mathcal{M}_0^\sigma(H_1, H_2, H_3)$ given by the following formula:

$$\xi : \mathcal{M}_0(H_1, H_2, H_3) \to \mathcal{M}_0^\sigma(H_1, H_2, H_3)$$

$$\xi(X_1, X_2, X_3) = A(z)dz,$$

where

$$A(z) = \frac{X_1}{z-1} + \frac{X_2}{z+1}.$$ (63)

The map $\xi$ intertwines the diagonal coadjoint action of $K$ on the triple $(X_1, X_2, X_3)$ and the action of the constant gauge transformations on $A(z)$. The connection $A(z)$ is flat and it satisfies the condition $A(z) = -A(\overline{z})$. Thus, $\xi$ defines a map from the multiplicity space to the hyperbolic moduli space of flat connections.

**Theorem 3.** The map $\xi$ is a Poisson map.

Before proving the theorem let us prove the following lemma.

**Lemma 2.** The Poisson bracket on $\mathcal{M}_0^\sigma(H_1, H_2, H_3)$ induced from $\mathcal{M}_0(H_1, H_2, H_3)$ by the map $\xi$ is given by the following formula:

$$\{\psi, \psi'\} = \int_{\Sigma \times \Sigma} dz \, dw \left< \left[ \frac{\delta\psi}{\delta A(z)}, \frac{\delta\psi'}{\delta A(w)} \right], \frac{A(z) - A(w)}{z-w} \right>.$$ (64)

**Remark.** The Poisson bracket (64) is of the Sklyanin type [15]. In the connection with the moduli spaces it has been proposed in [4].

**Proof of Lemma.** The proof is a straightforward calculation using formula (4) for Poisson bracket on $\mathfrak{g}^*$ and formula (63) for $\xi$:

$$\{\psi, \psi'\} = \int_{\Sigma \times \Sigma} dz \, dw \left< \left[ \frac{\delta\psi}{\delta A(z)} \frac{\partial A(z)}{\partial X_1}, \frac{\delta\psi'}{\delta A(w)} \frac{\partial A(w)}{\partial X_1} \right], X_1 \right> +$$

$$+ \left< \left[ \frac{\delta\psi}{\delta A(z)} \frac{\partial A(z)}{\partial X_2}, \frac{\delta\psi'}{\delta A(w)} \frac{\partial A(w)}{\partial X_2} \right], X_2 \right> =$$
\[
\frac{\delta \psi}{\delta A(z)} \frac{\delta \psi'}{\delta A(w)} \left( \frac{X_1}{(z-1)(w-1)} \right) + \frac{\delta \psi}{\delta A(z)} \frac{\delta \psi'}{\delta A(w)} \left( \frac{X_2}{(z+1)(w+1)} \right) =
\]

\[
\mathcal{P} \left( \delta \psi \frac{\delta \psi'}{\delta A(z)} \frac{\delta \psi'}{\delta A(w)} \left( \frac{A(z) - A(w)}{z-w} \right) \right) = \int_{\Sigma \times \Sigma} dz \, dw < \left[ \frac{\delta \psi}{\delta A(z)} \frac{\delta \psi'}{\delta A(w)} \left( \frac{A(z) - A(w)}{z-w} \right) \right] >
\]

\[
\begin{align*}
\{ \phi_\Gamma, \phi_\Gamma' \} &= \int_{\Sigma \times \Sigma} dz \, dw < \left[ \frac{\delta \phi_\Gamma}{\delta A(z)} \frac{\delta \phi_\Gamma'}{\delta A(w)} \left( \frac{A(z) - A(w)}{z-w} \right) \right] > \\
&= \oint \oint dt \, ds \frac{\partial z(t)}{\partial t} \frac{\partial z(s)}{\partial s} \text{Tr}_{12} (\text{Hol}(A, \Gamma, t_0)^1 \text{Hol}(A, \Gamma', s_0)^1 \frac{C_{12}^1 A_{12}^2(z(t))}{z(t) - z(s)} + \text{Hol}(A, \Gamma, t_0)^1 \text{Hol}(A, \Gamma', s_0)^2 \frac{C_{12}^2 A_{12}^2(z(s))}{z(t) - z(s)} )
\end{align*}
\]
Then

\[
(75) \quad \oint \oint \frac{dtds}{\partial z} \left( \frac{\partial}{\partial z(t)} + \frac{\partial}{\partial z(s)} \right) Tr_{12} \left( C^{12} \text{Hol}(A, \Gamma, t)^1 \text{Hol}(A, \Gamma', s)^2 \right) \frac{z(t) - z(s)}{z(t) - z(s)} =
\]

\[
(76) \quad = \oint \oint \frac{dtds}{\partial z} \left( \frac{\partial}{\partial z(t)} + \frac{\partial}{\partial z(s)} \right) \left( \sum_n \eta_n(z(t)) \eta_n(z(s)) \right) Tr_{12} \left( C^{12} \text{Hol}(A, \Gamma, t)^1 \text{Hol}(A, \Gamma', s)^2 \right) +
\]

\[
+ \oint \oint \frac{dtds}{\partial z} \left( \frac{\partial}{\partial z(t)} + \frac{\partial}{\partial z(s)} \right) \left( 1 - \sum_n \eta_n(z(t)) \eta_n(z(s)) \right) Tr_{12} \left( C^{12} \text{Hol}(A, \Gamma, t)^1 \text{Hol}(A, \Gamma', s)^2 \right) \frac{z(t) - z(s)}{z(t) - z(s)} =
\]

\[
(77) \quad = J_1 + J_2
\]

Let us first consider the integral $J_2$. The integrand has no singularities. So one can integrate by parts:

\[
(78) \quad J_2 = \oint \oint \frac{dtds}{\partial z} \left( \frac{\partial}{\partial z(t)} + \frac{\partial}{\partial z(s)} \right) \left( \sum_n \eta_n(z(t)) \eta_n(z(s)) \right) Tr_{12} \left( C^{12} \text{Hol}(A, \Gamma, t)^1 \text{Hol}(A, \Gamma', s)^2 \right) \frac{z(t) - z(s)}{z(t) - z(s)} =
\]

\[
(79) \quad = \oint \oint \frac{dtds}{\partial z} \left( \frac{\partial}{\partial z(t)} + \frac{\partial}{\partial z(s)} \right) \left( 1 - \sum_n \eta_n(z(t)) \eta_n(z(s)) \right) \cdot \text{Tr}_{12} \left( C^{12} \text{Hol}(A, \Gamma, t)^1 \text{Hol}(A, \Gamma', s)^2 \right) \left( \frac{\partial}{\partial z(t)} + \frac{\partial}{\partial z(s)} \right) \frac{1}{z(t) - z(s)} = 0
\]

To calculate $J_1$ let us choose local coordinates $z_n$ in the vicinities $V_n$ in such a way that $z_n = t - is$. Depending on the intersection number of the contours at $p_n$ this can require the change of the orientation and so the change of the overall sign. Now the integral $J_1$ takes the following form:

\[
(80) \quad J_1 = \oint \oint \frac{dtds}{\partial z} \left( \frac{\partial}{\partial z(t)} + \frac{\partial}{\partial z(s)} \right) \left( \sum_n \eta_n(z(t)) \eta_n(z(s)) \right) Tr_{12} \left( C^{12} \text{Hol}(A, \Gamma, t)^1 \text{Hol}(A, \Gamma', s)^2 \right) \frac{z(t) - z(s)}{z(t) - z(s)} =
\]
\begin{align}
(81) & \quad = \sum_n (-1)^{\nu(p_n)} \int_{V_n} d\bar{z}_n dz_n \\
& \quad \cdot \frac{\partial}{\partial z_n} (\eta_n(\text{Re } z_n) \eta_n(\text{Im } z_n) \mathcal{H}_{12} (C^{12} \text{Hol}(A, \Gamma, \text{Re } z_n) \mathcal{H}_{12} (A, \Gamma', \text{Im } z_n)^2)) \\
& \quad = \sum_{p \in \Gamma \cap \Gamma'} (-1)^{\nu(p)} \mathcal{H}_{12} (C^{12} \text{Hol}(A, \Gamma, p) \mathcal{H}_{12} (A, \Gamma', p)^2).
\end{align}

For $K = SU(n)$ the right hand side can be rewritten as

\begin{align}
(82) & \quad = \sum_{p \in \Gamma \cap \Gamma'} (-1)^{\nu(p)} (\phi_{\Gamma_p} - \frac{1}{n} \phi_{\Gamma_1} \phi_{\Gamma_2}).
\end{align}

Here $\nu(p)$ is the intersection number at the point $p$ and the contour $\Gamma_p$ is described in section 3 (see Figure 1). Formula (83) coincides with the Goldman definition (38). So the map $\xi$ is a Poisson map.

Remark. Formula (82) is valid for any simple Lie algebra $\mathfrak{g}$. It always coincides with the Goldman bracket. In fact, one can repeat the same proof for an arbitrary function on the moduli space of flat connections \cite{4}. For the sake of simplicity we present only the calculation for the Goldman functions in the case of $K = SU(n)$.

4.3. The Poisson map from the moduli space to the Poisson-Lie multiplicity space. In this subsection we describe a Poisson map from the hyperbolic moduli space of flat connections on the sphere with three holes into the Poisson-Lie multiplicity space.

Let us draw a graph on the complex plain as shown at Figure 3. It is convenient to enumerate the edges of this graph as $\Gamma_i, \overline{\Gamma}_i, i = 1, 2, 3$ and the vertices as $P_i, i = 0, 1, 2, 3$. Given a flat connection satisfying the condition $A(z) = -A(\overline{z})$, one can define a triple $(g_1, g_2, g_3) \in G^3$ as

\begin{align}
(84) & \quad \mathcal{H} : A(z) \to \{g_i\}, \quad g_i = \text{Hol}(A, \Gamma_i)
\end{align}

The space $G^3$ is acted on by the group $K^4$ in the following fashion:

\begin{align}
(85) & \quad g_i \to k^{-1}_{i-1} g_i k_i.
\end{align}

The map $\mathcal{H}$ intertwines the action of the gauge group (34) and the action (85) understood as an action of the gauge group by means of the projection:

\begin{align}
(86) & \quad \pi_4 : G^3 \to K^4 \\
& \quad \pi_4 : g(z) \to \{g(P_i)\}.
\end{align}

So, the moduli space is mapped into the quotient $G^3/K^4$.

Let us split the projection $G^3 \to G^3/K^4$ into two pieces corresponding to the subgroups $K_0 \simeq K$ and $K_{123} \simeq K^3$ in $K^4$:

\begin{align}
(87) & \quad K_0 = \{(k, e_K, e_K, e_K) \in K^4\}, \\
& \quad K_{123} = \{(e_K, k_1, k_2, k_3) \in K^4\}.
\end{align}
As the group $G$ is the double for $K$, the quotient $G^3/K^3 \simeq (K^*)^3$. More explicitly:

$$ (g_1, g_2, g_3) \to (k_1^*, k_2^*, k_3^*) $$

(88)

$$ k_1^* = \pi_L^*(g_1), $$

$$ k_2^* = \pi_L^*(\pi_R(g_1)g_2), $$

$$ k_3^* = \pi_L^*(\pi_R(\pi_R(g_1)g_2)g_3). $$

Here $\pi_L^*$ and $\pi_R$ were defined in subsection 2.2. It is easy to see that $K_0$ acts on the quotient space $(K^*)^3$ by the diagonal dressing action (29). In this way we define the map

$$ \chi : \mathcal{M}_2^*(H_1, H_2, H_3) \to (K^*)^3/K. $$

(89)

Observe that the image of the map $\chi$ satisfies two additional conditions:

1. The connection $A(z)$ being flat, the product of $k_i^*$ is equal to identity:

$$ k_1^*k_2^*k_3^* = e_{K^*}, $$

(90)

2. As holonomies around the holes are restricted to certain conjugacy classes in $G$ then

$$ k_i^* \in O(H_i). $$

(91)

We conclude that $\chi$ actually maps the hyperbolic moduli space into the Poisson-Lie multiplicity space.

**Lemma 3.** The map $\chi : \mathcal{M}_2^*(H_1, H_2, H_3) \to \mathcal{M}_4(H_1, H_2, H_3)$ is a Poisson map.

**Proof.** Let us consider the space $G^3$ as a real manifold. The ciliated graph (Figure 3) defines a Fock-Rosly bracket on $G^3$. This bracket descends to $G^3/K^4$. One of symplectic leaves in $G^3/K^4$ coincides with the image of the subspace

$$ \mathcal{A} = \{(g_1, g_2, g_3) \in G^3, \ g_1g_2g_3 = e_G, \ g_j\overline{g_j} \sim \exp 2itH_j\} $$

(92)

and is isomorphic to the hyperbolic moduli space $\mathcal{M}_2^*(H_1, H_2, H_3)$.

On the other hand, one can check that the projection $G^3 \to (K^*)^3 \simeq G^3/K_{123}$ is a Poisson map if one takes the Fock-Rosly bracket on $G^3$ and the direct sum of
canonical brackets on three copies of $(K^*)^3$. By definition, the Poisson-Lie multiplicity space $\mathcal{M}_t(H_1, H_2, H_3)$ is a symplectic leaf in $(K^*)^3/K$ which is singled out by the following conditions:

\begin{align}
&k_1^*k_2^*k_3^* = e_{K^*}, \\
&k_i \in \mathcal{O}_i(H_i).
\end{align}

As $G^3/K^3 = (K^*)^3$ as Poisson spaces and conditions (92) coincide with conditions (93) and (94), we conclude that

\begin{align}
\mathcal{M}_\Sigma^e(H_1, H_2, H_3) &\approx \mathcal{M}_t(H_1, H_2, H_3).
\end{align}

Therefore we arrive at the following theorem.

**Theorem 4.** The hyperbolic moduli space $\mathcal{M}_\Sigma^e(H_1, H_2, H_3)$ is isomorphic to the Poisson-Lie multiplicity space $\mathcal{M}_t(H_1, H_2, H_3)$ as a symplectic space.

### 4.4. The main theorem.

Let us summarize all the information about the hyperbolic moduli space and the multiplicity spaces which we collected by now. We consider three families of smooth symplectic manifolds $\mathcal{M}_0(H_1, H_2, H_3)$, $\mathcal{M}_t(H_1, H_2, H_3)$ and $\mathcal{M}_\Sigma^e(H_1, H_2, H_3)$ and the following maps:

1. The Poisson map $\xi : \mathcal{M}_0(H_1, H_2, H_3) \to \mathcal{M}_\Sigma^e(H_1, H_2, H_3)$,
2. The symplectic isomorphism $\chi : \mathcal{M}_\Sigma^e(H_1, H_2, H_3) \to \mathcal{M}_t(H_1, H_2, H_3)$,
3. The symplectic isomorphisms between all three spaces for small values of $H_1$, $H_2$ and $H_3$. These isomorphisms are known to exist but are not known explicitly.

It is our goal to prove that the map $\xi$ is also a symplectic isomorphism.

**Lemma 4.** The map $\xi$ is a symplectic isomorphism.

**Proof.** Both spaces $\mathcal{M}_\Sigma^e(H_1, H_2, H_3)$ and $\mathcal{M}_0(H_1, H_2, H_3)$ have the same dimension equal to $(\dim K - 3 \text{ rank } K)$. Thus, the map $\xi$ is a Poisson map between symplectic spaces of the same dimension. Hence, it defines a covering of the space $\mathcal{M}_\Sigma^e(H_1, H_2, H_3)$ by the space $\mathcal{M}_0(H_1, H_2, H_3)$. We know that for small values of $H_1$, $H_2$ and $H_3$ these spaces are actually isomorphic as symplectic spaces. Both $\mathcal{M}_\Sigma^e(H_1, H_2, H_3)$ and $\mathcal{M}_0(H_1, H_2, H_3)$ depend continuously on $H_1$, $H_2$ and $H_3$. Therefore, the degree of the covering can not change. We conclude that the map $\xi$ is invertible.

This completes the proof of the main result of this paper.

**Theorem 5.** The following three symplectic spaces

1. the multiplicity space $\mathcal{M}_0(H_1, H_2, H_3)$,
2. the Poisson-Lie multiplicity space $\mathcal{M}_t(H_1, H_2, H_3)$,
3. the hyperbolic moduli space of flat connections on the sphere with three holes $\mathcal{M}_\Sigma^e(H_1, H_2, H_3)$

are isomorphic as symplectic manifolds.

Let us remark that both isomorphisms $\xi$ and $\chi$ which play the role in the last theorem are defined by explicit formulae (see previous subsections).
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