Supersymmetric Hyperbolic $\sigma$-models and Decay of Correlations in Two Dimensions

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December 11, 2019

Abstract

In this paper we study a family of nonlinear $\sigma$-models in which the target space is the supermanifold $\mathbb{H}^{2|2N}$. These models generalize Zirnbauer’s $\mathbb{H}^{2|2}$ nonlinear $\sigma$-model [Zir91] which has a number of special features for which we find analogs in the general case. For example, by supersymmetric localization, the partition function of the $\mathbb{H}^{2|2}$ model is a constant independent of the coupling constants. Here we show that for the $\mathbb{H}^{2|2N}$ model, the partition function is a multivariate polynomial of degree $N-1$, increasing in each variable. We use these facts to provide estimates on the Fourier and Laplace transforms of the ‘$t$-field’ when these models are specialized to $\mathbb{Z}_2$. From the bounds, we conclude the $t$-field exhibits polynomial decay of correlations and has fluctuations which are at least those of a massless free field.

1 Introduction

We begin by introducing one of the main objects of study in this paper. Let $\Lambda$ be a finite subset in $\mathbb{Z}^d$ and for each $j \in \Lambda$, $t_j$ is a real variable. The $\Lambda$-tuple of numbers $\mathbf{t} := (t_j)_{j \in \Lambda}$ will be referred to as a (spin)-configuration and the collection of all such tuples $\mathbb{R}^\Lambda$ will be referred to as the sample (or configuration) space. For any two points $j_1, j_2 \in \Lambda$, $|j_1 - j_2|$ will denote the Euclidian distance on the lattice.

On $\mathbb{R}^\Lambda$, we define two functions $F$ and $M$ defined by the parameters $J_{jj'}, \varepsilon_j \geq 0$:

$$F_{\Lambda,J}(\nabla \mathbf{t}) = \sum_{(jj') \in \Lambda} J_{jj'}(\cosh(t_j - t_{j'}) - 1),$$

$$M_{\varepsilon}^{\Lambda}(\mathbf{t}) = \sum_{j \in \Lambda} \varepsilon_j (\cosh t_j - 1),$$

where we denoted by $(jj')$ the nearest neighbor pairs $|j - j'| = 1$. Next we define the operator $D_{\Lambda}^{\varepsilon}(\mathbf{t})$ by

$$(D_{\Lambda}^{\varepsilon})_{jj'} = 0 \quad |j' - j| > 1$$

$$(D_{\Lambda}^{\varepsilon})_{jj} = -J_{jj} \quad |j' - j| = 1$$

$$(D_{\Lambda}^{\varepsilon})_{jj'} = +[\sum_{j' \sim j} J_{jj'} + V_j + \varepsilon_j e^{-t_j}] \quad j' = j,$$

where

$$V_j = \sum_{j' \sim j} J_{jj'} [e^{t_j - t_{j'}} - 1].$$

It is worth noting that one can rewrite this operator as $e^{-t} \circ \Delta_{J,\varepsilon}(t) \circ e^{-t}$ where $e^{-t}$ is the diagonal operator in the position basis $(e^{-t})_{ij} = \delta_{ij} e^{-t_j}$ and where $\Delta_{J,\varepsilon}(t)$ is the sum of the weighted graph Laplacian with conductances $J_{jj'} e^{t_j + t_{j'}}$ and mass term (killing rates) $\varepsilon_j e^{-t_j}$. We will mostly consider the case $\varepsilon_0 = 1$ and $\varepsilon_j = 0$ otherwise, but the general setup is also of interest.

*supported by Israel Science Foundation grant number 1692/17
Given these objects, we propose a general study of the following Gibbs measures:

\[ d\mu_{\Lambda,a,J,\varepsilon}(t) = Z^{-1}_{\Lambda,a,J,\varepsilon} \prod_{j \in \Lambda} \frac{dt_j}{(2\pi)^{1/2}} e^{-F_{\Lambda}(\nabla t)} e^{-M_{\Lambda}^t(t)} \times \left| \det D_{\Lambda}(t) \right|^a, \]

where \( dt_j \) is the Lebesgue measure. The partition function \( Z_{\Lambda,a,J,\varepsilon} \) is defined to normalize the integral to be a probability measure. Note that if \( \varepsilon_j \equiv 0 \) then the model is not defined since a) \( \det D_{\Lambda}(t) \) vanishes identically and b) there is the non-compact symmetry \( t_j \mapsto t_j + x \). Thus taking \( \varepsilon_j \neq 0 \) for some \( j \) seems necessary for this measure to be normalizable. Later, we will see that, nevertheless, one can consider the \( \varepsilon \to 0 \) limit of the normalized measures. For future reference we let \( \langle \cdot \rangle_{\Lambda,a,J,\varepsilon} = \langle \cdot \rangle_{J,\varepsilon} \) denoting the corresponding Gibbs state (with \( a, \Lambda \) fixed). Our main interest is in the behavior of the correlation functions

\[ \langle e^{z[t_k-t_\ell]} \rangle_{\Lambda,a,J,\varepsilon} \quad z \in \mathbb{C}, \]

\[ \langle G_{\ell k} \rangle_{\Lambda,a,J,\varepsilon} \quad \text{where} \ G = D^{-1}, \]

as the volume \( \Lambda \uparrow \mathbb{Z}^d \).

This model is (the \( \ell \)-marginal distribution of) the titular Hyperbolic Non-linear \( \sigma \)-model. The reason for this name is that when \( a \in \mathbb{Z} \cup \mathbb{Z} + 1/2 \), the measure \( d\mu_{\Lambda,a,J,\varepsilon}(t) \) is a marginal distribution of the natural Gibbs measure on the space of maps from \( \Lambda \) into some Hyperbolic space/superspace. The study we are proposing may therefore be understood recalling the relation between vector valued Potts models and the more generally defined Random Cluster models. That said, our results will only hold for \( a \in \mathbb{N} + 1/2 \) and not \( a \in \mathbb{N} \) for reasons which should become clear as the exposition proceeds.

We will argue below that this family of models has a rich structure worthy of study. Our goal is to expose some basic facts about these models and to indicate interesting directions for further inquiry. Of most interest to us is the case when the coupling constants \( J_{jj'} = \beta I_{jj'} \), where \( I \) is the incidence matrix for \( \Lambda \), but we have good reasons to keep the formulation general. In this case \( \beta > 0 \) is a parameter that can be interpreted as an inverse temperature or, in the specific case \( a = 1/2 \) as a measure of the amount of disorder in a system [DSZ10].

1.1. Overview and Motivation. The consideration of Gibbs measures whose Gibbs weight is of the form \( e^{-F_{\Lambda}(\nabla t)} e^{-M_{\Lambda}^t(t)} \) is by now a familiar object of study. In particular if \( a = 0 \) the model falls into the category of \( \nabla \phi \) models with log-concave interactions. If \( a \neq 0 \) the det \( D \) term brings an interesting nonlocal modification into the definition of the Gibbs measure. In general, when \( a \leq 0 \) the function \( t \mapsto e^{-F_{\Lambda}(\nabla t)} e^{-M_{\Lambda}^t(t)} \times \left| \det D_{\Lambda}(t) \right|^a \) is log-concave, so that there are off-the-shelf techniques, principally the Brascamp-Lieb inequality and its relatives, which give upper bounds on fluctuations of the \( t \) field. Based on this, one in general expects that the \( t \)-field behaves at large length-scales like a Gaussian field. This is not the end of the story however, since given a choice of pinning \( \varepsilon_j \), it is important to understand the behavior of mean values \( \langle t_k - t_\ell \rangle_{\Lambda,a,J,\varepsilon} \) and to provide lower bounds on fluctuations, especially when the points \( k, \ell \) are far from the vertices being pinned. To our knowledge this has been carried out fully only in the case \( a = 0 \) by R. Bauerschmidt [Bau], see also [SZ04].

Our interest in this paper is generally the models with \( a > 0 \). The new feature here is that, if \( a > 0 \), the factor \( \left| \det D_{\Lambda}(t) \right|^a \) is log-convex and competes with the more conventional term \( e^{-F_{\Lambda}(\nabla t)} e^{-M_{\Lambda}^t(t)} \). The significance of this is that it raises the possibility of a phase transition as e.g. \( \beta \) varies (in the case \( J_{jj'} = \beta I_{jj'} \)). The specific case \( a = 1/2 \) is fairly well understood, but this is the only case which has been looked at in detail. If \( a = 1/2 \), it was long ago observed that this model is a certain marginal distribution of a \( \mathbb{H}^{2|2} \) non linear \( \sigma \)-model (we will explain this in more detail below) [Zir91,DS10,DSZ10]. We refer the reader to those papers for a fuller discussion of this model’s physical significance. Let us highlight some of their results. In [DS10], the authors show that on any (infinite) graph with uniformly bounded degree, there is \( \beta_0 \) depending only on the degree of the graph so that \( \langle G_{\ell k} \rangle_{J,\varepsilon} \) decays exponentially fast provided \( J_{jj'} \leq \beta_0 \) and under mild conditions on the \( \varepsilon_j \)'s. On the other hand, in [DSZ10], the authors show that on the graph \( \mathbb{Z}^d \) and provided \( J_{jj'} = \beta I_{jj'} \), there is \( \beta_0 \) such that for all \( \beta > \beta_0 \), and if \( \varepsilon_j = h \) independent of \( j \), then for all \( x, y \in \Lambda \)

\[ \langle \cosh(t_x - t_y) \rangle_{J,\varepsilon}, \langle \cosh(t_x) \rangle_{J,\varepsilon} \leq 3 \]
independent of $\Lambda, h$ (technically, here the proof requires periodic boundary conditions). Although this statement does not quite address the behavior of $\langle G_{xy}\rangle_{t,\epsilon}$, it suggests (and the authors morally show) that the walk induce by the (random) operator $\Delta J, \epsilon(t)$ is diffusive (whereas the previous exponential decay result implies positive recurrence). Thus the two results combine to show that on $\mathbb{Z}^d$, the model has a phase transition as $\beta$ varies.

Besides being a beautiful result within the field of statistical mechanics, the case $a = 1/2$ also gained visibility due to a conjectural connection to Edge Reinforced Random Walk (ERRW) and to the Vertex Reinforced Jump Process (VRJP), probabilistic models introduced and studied in a different context [CD86, DV01]. This connection was made explicit by Sabot and Tarrès [ST15], and the connection was then used, along with the ideas in [DS10] to prove positive recurrence under strong reinforcement. This result was simultaneously proved by a different, less computational method in [ACK14]. The reader should also note the excellent earlier work [MR09], which proves recurrence on dilute versions of $\mathbb{Z}^2$.

Since [ST15, ACK14], a number of related papers appeared, in particular [STZ17, SZ19, Sab19, BHS] which demonstrate recurrence/recurrent-like signatures for ERRW and VRJP on $\mathbb{Z}^2$. In particular, the present work was inspired by the insightful paper [BHS]. The main observation of that paper can be loosely summarized as saying that the generator of the VRJP coincides with the adjoint, with respect to the Gibbs measure, of the differential operator generating global hyperbolic symmetry. In principle their work suggests that the VRJP is relevant for all $a \in \mathbb{Z} \cup \mathbb{Z} + 1/2$, not just when $a = 1/2$. While this turns out to be true for $a \leq 0$ (in this case $d\mu_{\Lambda,a,\epsilon}(t)$ provides a nontrivial stationary mixing measure for the VRJP), for $a \in \mathbb{N} + 1/2$ the correspondence is only formal since the coefficients of the generator for the VRJP become Grassmann-valued [Bau]. It remains to be seen whether probabilistic sense can be made of this last remark.

There is still one major open problem remaining after [DS10, DSZ10]: the conventional wisdom, based on physical reasoning due to Polyakov (see Part 4 of the lecture notes [Ton] for a nice account of this) suggests that on $\mathbb{Z}^2$, $\langle G_{jk}\rangle_{t,\epsilon}$, with, say, $J_{jj'} = \beta I_{jj'}$ and $\epsilon_0 = 1, \epsilon_j = 0$ otherwise, decays exponentially in $|\ell - k|$. While the rate of decay should depend rather nontrivially on $\beta$, the fact that there is exponential decay should not. This is a special case of the longstanding open problem demonstrating mass generation for 2 dimensional nonlinear $\sigma$-models. This provided our initial interest: proving this result for $a = 1/2$ seemed out of reach, but we had hopes that, by considering other values of $a$, in particular $a = 3/2, 5/2, 7/2, \ldots$ and an alternative description to be explained below, one might make progress. We now believe these hopes were naive. Nevertheless we compile evidence below that the study of the model at odd half integers is, in itself, an incredibly interesting enterprise.

1.2. Summary of Results and Discussion. Let us now detail our findings. Our modest initial goal was to investigate whether, with $a > 0$, one could apply harmonic deformation techniques (e.g. the Mermin-Wagner theorem or McBryan-Spencer method) to say anything about the behavior of $e^{[t_k - t_0]}$ in large volumes $\Lambda$ and for $k, \ell$ far apart from one another. There are a few reasons to start with this question. For one, as already mentioned, the problem of mass generation in $d = 2$ remains a goal. While harmonic deformation techniques have no hope of addressing the question directly, they allow us to familiarize ourselves with the objects of study while also providing a good chance that new results may be derived.

For a second reason, in the case $a = 1/2$, these techniques have recently been applied in independent works by G. Kozma and R. Peled [KP19] and [Sab19] to obtain a crucial a priori estimate for irreducibility of the VRJP in $2d$. Kozma and Peled apply the harmonic deformation technique configuration-wise. In order for it to succeed, they require an a priori statement to the effect that the locations where $\cosh(t_j - t_j')$ is large, for nearest neighbors $(jj')$, are very sparse. To show this latter fact, they use the dual description of $d\mu_{\Lambda,a=1/2,J,\epsilon}(t)$ as the mixing measure for the VRJP. On the other hand Sabot deals with this issue in a slick but indirect way, relying instead on SUSY through the fact that the partition function $Z_{\Lambda,a=1/2,J,\epsilon} \equiv 1$ for all $J_{jj'}$ (the reason this is important will become clear below).

Let us now state our first result, the raison d’être for the paper.

**Theorem 1.1** (Bounds of Fourier Transforms). *Fix $a \in \mathbb{N} + 1/2$. Let $J_{jj'} = \beta I_{jj'}$, and choose*
$\varepsilon_0 = 1, \varepsilon_j = 0$ otherwise. For all $\Lambda$ sufficiently large, we have the bounds

\[
\left| \left\langle e^{ik(t_m-t)} \right\rangle_{\Lambda, a, J, \varepsilon_0} \right| \leq \exp \left( -\frac{k^2}{\beta + 2a} \left( \log(1 + ||m-\ell||_2) - \log \left( 1 + \frac{4|k|}{(\beta + 2a)} \right) \right) \right),
\]

\[
\left| \left\langle e^{ik(t_m-t)} \right\rangle_{\Lambda, a, J, \varepsilon_0} \right| \leq \exp \left( -\frac{|k| - \beta - a}{2} \left( \log(1 + ||m-\ell||_2) \right) \right).
\]

This gives non-concentration of the $t$-field analogous to a massless Gaussian free field.

Combining the main inequality appearing in [Sab19] with the main ingredient in our proof of Theorem 1.1 (which we will broadly outline in the subsequent two paragraphs) we can generalize the Sabot method to models for all $a \in \mathbb{N} + 1/2$ (in case $a = 3/2$ this bound appears in [BCHS19] as well).

**Theorem 1.2** (Bounds on Laplace Transforms). Let $a \in \mathbb{N} + 1/2$, and $0 < p < 2a$ be fixed. On $\mathbb{Z}^2$ with $\beta_{ij} = \beta_{1_{i-j}}$ there is $c(\beta, p) > 0$ such that

\[
\left\langle e^{p(t_0-t)} \right\rangle_{\Lambda, a, J, \varepsilon_0} \leq |\text{dist}(v, 0)|^{-c(\beta, p)}.
\] (1.7)

As mentioned above, the reason for the restriction $a \in \mathbb{N} + 1/2$ should be viewed through an analogy between Potts and random cluster models. While the Random Cluster Models make sense for any value of $q > 0$, it is only for integer values of $q$ that one is able to rewrite the model as a spin system. Having both descriptions allows powerful tools to be applied in the integer case which are otherwise unavailable, e.g. Reflection Positivity. So too, as we (partially) explain in Appendix A and use from Section 3 onwards, when $a \in \mathbb{N} + 1/2$ the measure $d\mu_{\Lambda, a, J, \varepsilon_0}(t)$ maybe realized as the marginal distribution, in horospherical coordinates, of a spin model over $\Lambda$ in which the spins take values in, respectively, a hyperbolic space $\mathbb{H}_a^{-2a}$ if $a \leq 0$ or in a hyperbolic superspace if $a > 0$ (either $\mathbb{H}_a^{1/2N}$ if $2a$ is an even integer) or $\mathbb{H}_a^{2/2a+1}$ (if $2a$ is an odd integer)). The details of this connection can be found in Appendix A and also in [DSZ10].

There are two ingredients that we make use of in proving these theorem. In each case, the first is an $a$ priori estimate. In the case of the Fourier transform, we use a technique with roots in the classic paper of McBryan and Spencer [MS77]. It amounts to bounding the Fourier transform of the distribution for the variable $t_k - t_\ell$ by shifting contours. For hyperbolic non-linear $\sigma$-models, more so than their compact counterparts (the subject of the original paper [MS77]), this step is essentially the method used to compute the characteristic function of a Gaussian variable. This technique and an associated bound on the Fourier transform of the distribution for $t_k - t_\ell$ is presented in the following section Section 2.

In the case of the Laplace transform, for this first step we instead apply a slight generalization of a recent bound of C. Sabot [Sab19]. We state this estimate at the beginning of Section 6 and refer the reader to [Sab19] for the original proof when $a = 1/2$, or to [BCHS19] for the general case. In both cases, the upshot will be to derive both Theorems 1.1 and 1.2, we need to show that if we consider the partition function $Z_{\Lambda, a, J, \varepsilon}$ as a function of the single coupling $J_{jj'}$ for some fixed $j, j'$ holding the remaining variables fixed, $F(J_{jj'}) := Z_{\Lambda, a, J, \varepsilon}$, then $F$ is necessarily increasing in $J_{jj'}$.

Thus, the heart of our paper is the following remarkable collection of facts. We state them in the general context of hyperbolic SUSY $\sigma$-models on a finite weighted graph $(G, J)$. The notation should be self-evident.

**Theorem 1.3.** Fix $a \in \mathbb{N} + 1/2$, let $G = (V, E)$ be a finite graph and choose $\varepsilon, J$ to be non-negative masses and couplings. Let us fix a nearest neighbor pair $j, j' \in E$ and view

\[
J_{jj'} \mapsto Z_{G, a, J, \varepsilon}
\]
as a function from $\mathbb{R}_+$ to $\mathbb{R}_+$. Then

- $Z_{G, a, J, \varepsilon}$ is a polynomial of degree $n = a - 1/2$ in $J_{jj'}$.
- For all $k \leq \lfloor n/2 \rfloor$ and for $k = n$, $\partial^k_{J_{jj'}} Z_{G, a, J, \varepsilon} > 0$. 

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• Specializing $\varepsilon$ to be supported at one vertex $v$ and $0 \leq \varepsilon_v \leq 1$ then $\partial^k_{j,j'} Z_{\Lambda,a,J,\varepsilon} \geq 0$ for all $k \leq n$.

In particular $Z_{\Lambda,a,J,\varepsilon}$ is increasing in each $J_{j,j'}$.

We strongly suspect that, in the third conclusion, the restriction to sufficiently small one point pinning is an artifact of our proof, and we would very much like to remove this hypothesis.

Recall that $\partial^k_{j,j'} Z_{\Lambda,a,J,\varepsilon}$ are proportional to the moments of the interaction between spins at $j,j'$. We will prove non-negativity of polynomial coefficients by bounding these moments. In particular, $Z_{\Lambda,a,J,\varepsilon}$ is increasing if the first moment of the interaction is positive for any positive choice of $J$’s. Positivity of this first moment turns out to be substantially easier to demonstrate than positivity in the general case. We have opted to go the extra mile and prove positivity of all moments (up to $a - 1/2$) for reasons to be discussed after Theorem 1.4.

Let discuss the context of this result with respect to the previously mentioned special case, when $a = 1/2$. In that case the model has the fundamental property that $Z_{J,\varepsilon} = 1$ for any choice of non-negative $J,\varepsilon$. This and other basic identities play a crucial role in [DS10, DSZ10, Sab19]. As explained in an appendix of [DSZ10], this fundamental identity follows from what physicists refer to as ‘Supersymmetric Localization’. The same technique goes by the name the Duistermaat-Heckman Theorem [DH82] in the mathematics community. For us, the consequence of this technology is that, after disintegrating $d\mu_{\Lambda,a,J,\varepsilon}(t)$ into the full $\mathbb{H}^{2|N}$ spin model, we may rewrite the system as the ‘Gibbs state’ of a $2(N - 1)$ component pure Grassmann field if $a \in \mathbb{N} + 1/2$. That $Z_{\Lambda,a,J,\varepsilon}$ is a polynomial in $J_{j,j'}$ is then manifest. Where we have to work is to show the second statement, positivity of the polynomial’s coefficients.

There are a few hints that such a positivity might be true. For one it is possible to compute by hand what is going on when $a = 3/2,5/2$. In these cases we initially managed to show the positivity provided the $J_{j,j'}$’s are uniformly large. However, what truly convinced us that it must be true is the following theorem, which combines the observation that localization reduces the $\mathbb{H}^{2|N}$ model to a purely Grassmann variable field theory, along with a beautiful algebraic paper [CSS07] which coincidentally connects the latter field theory with unrooted spanning forests (the reader may also consult the forthcoming [BCHS19] for an abbreviated account of this development). To state the result, we need to introduce this last object. Given a finite graph $G$, let $\mathcal{F}(G)$ denote the collection of unrooted spanning forest on $G$ and, for $F \in \mathcal{F}(G)$, let

$$W(F) = \left( \prod_{ij \in F} J_{ij} \right) \prod_{T \in F} [1 + \sum_{i \in T} \varepsilon_i]$$

**Theorem 1.4.** Let $a = 3/2$ and let $G$ be fixed and finite. For any $\varepsilon_i,J_{ij} \in \mathbb{R}$,

$$Z_{G,a,J,\varepsilon} = 2 \sum_{F \in \mathcal{F}(G)} W(F)$$

where $\prod_{T \in F}$ denotes the product over components (trees) of $F$.

As the reader may anticipate, this correspondence extends also to many types of correlation functions, for example $(G_{ik})_{G,a,J,\varepsilon}$ is equal to the probability $\ell$ and $k$ are connected in the probability measure determined by the weights $W(F)$. Thus, our hyperbolic nonlinear $\sigma$-model with $a$ set to $3/2$ provides a continuous representation of a natural class of probability measures on unrooted spanning forests. Together with Bauerschmidt, Helmuth and Swan, we explore this connection in some detail [BCHS19], proving in particular that in two dimensions, there are no infinite trees in a thermodynamic limit for any $\beta$ (the thermodynamic limit should be unique, but we do not have the technology to prove that).

Theorem 1.4 raises the question as to whether, given that the models with $a = 1/2$ and $a = 3/2$ have ‘dual representations’ in the discrete probability world, there are such representations for all $a \in \mathbb{N} + 1/2$. We have found such a representation when $a = 5/2$, but as yet have not uncovered a general mechanism producing such representation. For this reason we regard the positivity expressed in Theorem 1.3 as an important contribution above and beyond the fact that the partition functions increase coordinate-wise.
It provides a consistency check for this question, since if such a correspondence did exist, Theorem 1.3 would follow for free.

The plan of the remainder of the paper is as follows. Postponing the introduction of Grassmann variables as long as possible, in the next section we present a bound on the characteristic function of the random variable $t_k - t_e$. This computation holds for any $a \in \mathbb{R}$ and brings us to the fundamental problem of controlling partition function ratios $Z_{J,j'}^{\varepsilon,t}$. When coordinate-wise we have $J_{j,j'}^{\varepsilon,t} \leq J_{j,j'}$. In Section 3 we finally reveal the Grassmann variable representation of the hyperbolic $\sigma$ models of interest, summarizing just what we need to continue on to prove Theorem 1.3. For convenience we provide a more detailed exposition in Appendix A and we also refer the reader to the Appendices of [DSZ10] on which our discussion is based. In Section 4.2.1 we develop some important identities available in the Grassmann representation. For a bit of background on the origin of these identities, the reader may consult Appendix A.3 and Section 7 of [CSS07]. These identities form the basis for the computations which follow in Section 4 that demonstrate positivity of the polynomial coefficients. In Sections 5 and 6 we then complete the proofs of Theorems 1.1 and 1.2.

Acknowledgements. I thank Roland Bauerschmidt for explaining to me the results in [BHS], which motivated the present work. I also thank Bauerschmidt, Tyler Helmuth and Andrew Swan for numerous discussions on related topics throughout the preparation of this manuscript.

2 A general Fourier Transform upper bound for the $t$-field.

In this section $d = 2$ and $\Lambda \subset \mathbb{Z}^2$. Abusing notation slightly, we let $\varepsilon_0$ denote the pinning vector $\varepsilon_0 = 1$ and $\varepsilon_i = 0$ in subscripts where a pinning vector may appear, e.g. $Z_{\Lambda, a J, \varepsilon_0}$. Let $a \in \mathbb{R}$ and $J_{j,j'} > 0$ be fixed. Following Mc Bryan and Spencer, we consider the effect of translating the integration variables $t_x$ into the complex plane, $t_x \mapsto t_x + i \rho_x$. Given $\rho : \Lambda \rightarrow \mathbb{R}$, and a collection of coupling constants $J_{j,j'}$, let $J_{j,j'}(\rho) = J_{j,j'} \cos(\rho_j - \rho_{j'})$. Also, let

$$A := \{ \rho : \rho_0 = 0 \text{ and } \cos(\rho_j - \rho_{j'}) > 0 \text{ for all } (j,j') \}.$$ 

We begin with a general estimate:

**Lemma 2.1.** If $\rho \in A$,

$$\left| \left\langle e^{ik[t_m - t_l]} \right\rangle_{\Lambda, a J, \varepsilon_0} \right| \leq \frac{Z_{\Lambda, a J}(\rho) \varepsilon_0}{Z_{\Lambda, a J, \varepsilon_0}} \prod_{(j,j')} \cos(\rho_j - \rho_{j'})^{-\max(a,0)} e^{\sum_{(j,j') \in \Lambda} J_{j,j'} [1 - \cos(\rho_j - \rho_{j'})]} e^{-k|\rho_m - \rho_e|}.$$ 

**Proof.** We want shift the integration variables $t_x$ into the complex plane, $t_x \mapsto t_x + i \rho_x$. This need to be done carefully so as not to lose control of the Gibbs-Boltzman factor through the shift. Our solution to this issue is to make a series of small amplitude shifts and to verify that we maintain control on Gibbs-Boltzman factor after each shift.

To this end, consider the integral

$$I(\rho) := \int e^{ik[t_m - t_l] - k|\rho_m - \rho_e|} \prod_{j \in \Lambda} \frac{dt_j}{2\pi} e^{-\sum_{(j,j') \in \Lambda} J_{j,j'} [\cosh(t_j - t_{j'} + i[\rho_j - \rho_{j'}]) - 1]} e^{-M(\alpha)} \rho_j |\rho_j| [\det D_{\Lambda, J}^\varepsilon(t + i \rho)]^\alpha.$$ 

We begin by arguing that the integrand is in $L^1$ if $\rho_0 = 0$ and $\cos(\rho_j - \rho_{j'}) > 0$ for all $(j,j')$. Under this assumption, let $\alpha = \min_{(j,j')} \cos(\rho_j - \rho_{j'}) > 0$. First, for general complex-valued $\rho$, let $\rho_j(\Im(t)) = J_{j,j'} \cos(\Im(t_j - t_{j'}))$ and observe that

$$\left| e^{-F_{\Lambda, J}(\rho)} \right| \leq C(\Lambda) e^{-F_{\Lambda, J}(\Im(t))}.$$ 

Also, by the assumption on $\rho$, $J_{j,j'}(\Im(t)) \geq J_{j,j'} \alpha > 0$. Next the Matrix Tree Theorem implies that we have the point-wise bound

$$|\det D_{\Lambda, J}^\varepsilon(t)| \leq [\det D_{\Lambda, J}^\varepsilon(\Im(t))]$$
Combining these estimates implies the integrand appearing in the definition of \( I(\rho) \) is in \( L^1 \) and that
\[
|I(\rho)| \leq C(\Lambda)e^{k|\rho_m - \rho_2|} \int_{\rho \in \Lambda} \prod_{j \in \Lambda} \frac{dt_j}{2\pi} e^{-\sum_{(j,j') \in \Lambda} J_{jj'} |\cos(t_j - t_{j'}) - 1|} e^{-\lambda_j(t) \left( \det D_{\lambda,j}(t) \right)^a} < \infty.
\]

Next we wish to argue that \( I(\rho) \) is locally constant in \( \rho \) on \( \mathcal{A} \), and hence constant over all of \( \mathcal{A} \) by continuity and connectedness. For \( \rho \in \mathcal{A} \), let \( \alpha = \min_{(j,j')} \cos(\rho_j - \rho_{j'}) \) once again. To this end, we claim that there is \( \epsilon = \epsilon(\alpha) > 0 \) such that if \( \psi \) is chosen with \( \|\psi\|_\infty \leq \epsilon \) then \( I(\rho) = I(\rho + \psi) \).

To see this, consider the collection of contours (in \( \mathbb{C} \))
\[
\Gamma_K(j) = [-K + i\rho_j, K + i\rho_j] \cup [K + i\rho_j, K + i(\rho_j + \psi_j)] \\
\cup [-K + i(\rho_j + \psi_j), K + i(\rho_j + \psi_j)] \cup [-K + i\rho_j, -K + i(\rho_j + \psi_j)].
\]

By Green’s theorem,
\[
\oint_{\times_{j \in \Lambda} \Gamma_K(j)} dt_j e^{ik[t_m - t_l]} e^{-F_{\Lambda,j}(\nabla t)} e^{-\lambda_j(t)} \left( \det D_{\lambda,j}(t) \right)^a = 0,
\]
where we used that \( a > 0 \) so that no contribution is picked up from 0’s of \( \det D_{\lambda,j}(t) \).

To estimate the contribution to this multivariate contour integral from \( \{ \mathbb{L} : |\Re(t_y)| = K \text{ for some } y \} \), observe that by the remarks above if \( y \) is fixed with \( |\Re(t_y)| = K \) then we have
\[
\int_{\times_{j \notin \mathbb{L}} \Gamma_K(j)} \prod_t dt_j e^{ik[t_m - t_l]} e^{-F_{\Lambda,j}(\nabla t)} e^{-\lambda_j(t)} \left( \det D_{\lambda,j}(t) \right)^a \leq C(\Lambda) \int_{\times_{j \notin \mathbb{L}} \Gamma_K(j)} \prod_t dt_j e^{-F_{\Lambda,j}(\Re(\nabla t)) - \lambda_j(\Re(t))} \left( \det D_{\lambda,j}(\Re(t)) \right)^a.
\]

Now for \( |\Re(t_y)| = K \) and \( t_j \in \Gamma_K(j) \) otherwise,
\[
e^{-F_{\Lambda,j}(\Re(\nabla t)) - \lambda_j(\Re(t))} \leq e^{-\min_{j,j'} J_{jj'} \alpha (|\cos(K/|\Lambda|) - 1|)}
\]
due to the pinning at 0. Since we also have (via the matrix tree theorem again)
\[
\left| \left( \det D_{\lambda,j}(\Re(t)) \right) \right| \leq C(\Lambda)e^{k|\Lambda|K}
\]
the RHS of (2.2) is further bounded by
\[
C_1(\Lambda)e^{k|\Lambda|K} \left[ K \right] |\rho|_\infty |\Lambda|e^{-\min_{j,j'} J_{jj'} \alpha (|\cos(K/|\Lambda|) - 1|)}.
\]

Thus the contributions to the contour integral from \( \{ \mathbb{L} : |\Re(t_y)| = K \text{ for some } y \} \) tend to 0 as \( K \) tends to \( \infty \).

Applying the dominated convergence theorem, we see that we can shift each \( t_j \) integral from \( \mathbb{R} + i\rho_j \) to \( \mathbb{R} + i(\rho_j + \psi_j) \), provided \( \|\psi\|_\infty \leq \epsilon \) and \( \rho \in \mathcal{A} \). We obtain
\[
I(\rho) = I(\rho + \psi)
\]
in this case and then by connectedness that \( I(\rho) = I(0) \) throughout \( \mathcal{A} \). To finish the lemma we need to estimate \( I(\rho) \).

We have
\[
\left| e^{ik[t_m - t_l]} \right|_{\Lambda,a,J,\xi_0} = \frac{|I(\rho)|}{Z_{\Lambda,a,J,\xi_0}} \leq e^{-k|\rho_m - \rho_2| - \sum_{(j,j') \in \Lambda} J_{jj'} |\cos(\rho_j - \rho_{j'}) - 1|} \times \frac{1}{Z_{\Lambda,a,J,\xi_0}} \int_{\rho \in \Lambda} \prod_{\rho \in \Lambda} \frac{dt_j}{2\pi} e^{-\sum_{j,j' \in \Lambda} J_{jj'} |\cos(t_j - t_{j'}) - 1|} e^{-\lambda_j(t) \left( \det D_{\lambda,j}(t) \right)^a}. \tag{2.3}
\]
We observe that we can rewrite the integral on the RHS as

\[ I = \frac{Z_{J^{(0)},\varepsilon_0}}{Z_{J,J^{(0)}}} \left\langle \left( \frac{\det D_{J^{(0)},j}(t)}{\det D_{J,J^{(0)}}(t)} \right)^a \right\rangle_{\Lambda,\alpha,J^{(0)},\varepsilon_0} \]  

\[ (2.4) \]

Using the Matrix Tree Theorem again we have the point-wise bound (valid for any $a$)

\[ \left( \frac{\det D_{J^{(0)},j}(t)}{\det D_{J,J^{(0)}}(t)} \right)^a \leq \prod_{(jj') \in \Lambda} \cos(\rho_j - \rho'_j)^{-\max(a,0)} \]

and this finishes the proof. \hfill \Box

### 3 Grassmann Representation for $a \in \mathbb{N} + 1/2$. 

In the next four sections, we work on a general finite connected weighted graph $G = (V, E, J)$ where the edge-weights $J_{j'j} > 0$. In defining the pinning vector $\varepsilon_0$, choose some distinguished vertex $0 \in V$ to be viewed as the origin. We extend the definition of $d_{H,t}^{\alpha} \frac{\mu_{J^{(0)}}}{2}$ and the associated Gibbs state in the obvious way. In order to control Fourier/Laplace transforms of the $t$-field, we next need to control $Z_{G,a,J^{(0)},\varepsilon_0}^{\alpha} \frac{\mu_{J^{(0)}}}{2}$ assuming component-wise domination $J_{j'j}^{(0)} \leq J_{j'j}$.

In this section, our goal here is to provide the reader with a minimum of necessary notation and identities to proceed with the proof Theorem 1.3. More details may be found in Appendix A or in Appendix C of [DSZ10]. The result being quoted relies on supersymmetric localization. Given $n$, let $(\psi_i^0, \overline{\psi}_i^0)_{\ell=1,i \in \Lambda}$ be a system of generators of the Grassmann algebra $G_{\Lambda}$ with $2n$ variables per site. Let

\[ \bar{\psi}_i \cdot \psi_i = \sum_{\ell=1}^{n} \bar{\psi}_i^{\ell} \psi_i^{\ell}, \quad \sigma_i = \sqrt{1 + 2 \bar{\psi}_i \cdot \psi_i}, \quad D\mu_0(\bar{\psi}_i, \psi_i) = \prod_{\ell} \partial_{\bar{\psi}_i^{\ell}} \partial_{\psi_i^{\ell}} \sigma_i^{-1}, \]

\[ D_0(\bar{\psi}, \psi) = \prod_{i \in V, \ell} \partial_{\bar{\psi}_i^{\ell}} \partial_{\psi_i^{\ell}}, \quad D(\bar{\psi}, \psi) = \prod_{i \in V} D\mu_0(\bar{\psi}_i, \psi_i), \]

and define the Grassmann action

\[ S_{J,\varepsilon} = \sum_{(jj')} J_{j'j} \left\{ \sigma_j \sigma_j' - \bar{\psi}_j \cdot \psi_j' - \bar{\psi}_j' \cdot \psi_j - 1 \right\} + \sum_i \varepsilon_i [\sigma_i - 1]. \]

**Superintegral sign convention:** Note that our sign convention was chosen so that

\[ \int D\mu_0(\bar{\psi}_i, \psi_i) \cdot \sigma_i \exp(\lambda \bar{\psi}_i \cdot \psi_i) = (1 - \lambda)^n \]

so as to conform with the convention chosen in [DSZ10]. This convention clashes with the choice in another key paper [CSS07].

**Notation:** We now introduce the partition function and Gibbs state for the Grassmann field. We have

\[ Z_{G,a,J,\varepsilon}^{f} := \int D(\bar{\psi}, \psi) e^{-S_{J,\varepsilon}}, \]

\[ \langle \cdot \rangle_{G,a,J,\varepsilon}^{f} := \frac{\int D(\bar{\psi}, \psi) \cdot e^{-S_{J,\varepsilon}}}{Z_{G,a,J,\varepsilon}^{f}}. \]

In the rest of this paper, we refer to this model as that $\mathbb{H}^{0|2n}$ $\sigma$-model. To distinguish between Gibbs states involving the $t$-field, or the full $\mathbb{H}^{2|2n+1}$-valued spin (discussed in Appendix A), and expressions involving the purely Grassmann field, note the superscript $f$ appearing in these definitions. This convention will remain enforced throughout the rest of the paper.
Lemma 3.1. Let $\varepsilon$ be a nonzero pinning vector. Let $a \in \mathbb{N} \cup \{0\} + 1/2$ and $2n = 2a - 1$. Then we have a ‘duality’ between the $(t_i)_{i \in \Lambda}$ variables and the $(\psi_i^t, \bar{\psi}_i^t)_{t=1,i \in \Lambda}$. At the level of partition functions, this duality takes the form

$$Z_{G,a,J,\varepsilon} = Z_{G,a,J,\varepsilon}^f$$

When $n = 0, a = 1/2$ the identity reads

$$Z_{G,a,J,\varepsilon} = 1$$

It is worth remarking that it is not obvious from its definition that $Z_{G,a,J,\varepsilon}^f > 0$. The previous lemma confirms this, and therefore demonstrates how the connection between the two sets of variables can be used in both directions, even if one does not have an explicit probabilistic interpretation of the Grassmann variable Gibbs state.

If $a > 1/2$ one obtains further identities for correlation functions. We now state the most general one we will need in this paper. We introduce some shorthand notations to be used here and in the remainder of the text.

$$\tau_{ij}^f = -[\bar{\psi}_i^f - \bar{\psi}_j^f][\psi_i^f - \psi_j^f], \quad \eta_{ij}^f = -[\bar{\psi}_i^f - \bar{\psi}_j^f] \psi_i^f,$$

$$\pi_i^f = -\bar{\psi}_i^f \psi_i^f$$

Observe that

$$[\bar{\psi}_i^m - \bar{\psi}_j^m]^2 = [\psi_i^m - \psi_j^m]^2 = [\pi_i^m]^2 = 0.$$  

Furthermore, recall the (t-dependent) Green kernel $G_{ij}$ defined in the introduction and let

$$G_{ij}^{(1)} := [G_{ii} + G_{jj} - 2G_{ij}], \quad G_{ij}^{(2)} := G_{ij}^{(1)} G_{00} - [G_{i0} - G_{j0}]^2 \quad G_{ij}^{(3)} = [G_{i0} - G_{j0}].$$

Note, of course that $G_{ij}^{(1)}, G_{ij}^{(2)} \geq 0$ point-wise in the t-field.

Lemma 3.2. Fix $a, n$ so that $2a - 1 = 2n$ and suppose $J, M \in \mathbb{N}$ satisfy $J + M \leq n$. For any $A \subset [J + M]$ and denoting $|A \cap [J]| = a, |A| = b$,

$$\left< \prod_{\ell=1}^J \tau_{ij}^f \times \prod_{j \in A} \pi_j^m \times \prod_{M+J+1}^{n} \eta_{ij}^m \right>_{G_{i},n,J,\varepsilon_{0}}^f = \left< G_{ij}^{(1)} G_{ij}^{(2)} G_{00}^{b-a} G_{ij}^{(3)} n-(M+J) \right>_{G,a,J,\varepsilon_{0}}.$$  

Note in particular that if $n - (M + J)$ is even then this expression is non-negative.

4 Proof of Theorem 1.3

From now on, we fix a graph $G$ and $n$ and suppress them from the notation. To prove Theorem 1.3, the key idea is to pass back and forth between the two descriptions of the partition function and Gibbs state - in terms of the $t$ field on one hand and in terms of the pure Grassmann field on the other. We first encounter this idea in arguing that $Z_{G,a,J,\varepsilon}$ must be a multivariate polynomial.

4.1. Upper Triangulating the Grassmann variables. We want to find a coordinate system of $2n$-component Grassmann variables in which

$$-\frac{1}{2}(n_i - n_j, n_i - n_j) = 1 - \sigma_i \sigma_j + \bar{\psi}_i \psi_j + \bar{\psi}_j \psi_i$$

can be expressed in terms of expressions like $[(\bar{\psi}_j^f - \bar{\psi}_j^f)(\psi_j^f - \psi_j^f)]$ and $[\bar{\psi}_j^f \psi_j^f \bar{\psi}_j^f \psi_j^f]$. If we have such a coordinate system, then due to the fact that for any single component $t$

$$[(\bar{\psi}_j^f - \bar{\psi}_j^f)(\psi_j^f - \psi_j^f)]^2 = [\bar{\psi}_j^f \psi_j^f \bar{\psi}_j^f \psi_j^f]^2 = 0,$$
we could conclude
\[
\frac{1}{2}(n_i - n_j, n_i - n_j)^{n+1} = 0.
\]

Applied to \(e^{-S_{J,\epsilon}}\), we then conclude \(Z_{G,a,J,\epsilon}\) must be a multivariate polynomial of degree at most \(n\) in each variable.

Recall
\[
\pi_i = -\bar{\psi}_i \cdot \psi_i, \quad \pi_i^j = -\bar{\psi}_i^j \psi_i^j,
\]
and introduce the notation
\[
\sigma_i(J) = \sqrt{1 - 2 \sum_{\ell \leq j} \pi_i^j}.
\]
We have
\[
1 - \sigma_i\sigma_j = 1 - \sigma_i(n-1)\sigma_j(n-1) - \pi_i^j \sigma_j(n-1) - \pi_j^i \sigma_i(n-1) - \frac{\pi_i^n \pi_j^n}{\sigma_i(n-1)\sigma_j(n-1)}.
\]

We can now change variables, setting
\[
\sigma_i(n-1)\bar{\phi}_i^n = \bar{\psi}_i^n, \quad \sigma_i(n-1)\phi_i^n = \psi_i^n, \quad \phi_i^\ell = \bar{\psi}_i^\ell, \quad \phi_i^\ell = \psi_i^\ell \text{ for } \ell < n.
\]

Let \(\nu_i^n = -\bar{\phi}_i^n \phi_i^n\). Then \(-\frac{1}{2}(n_i - n_j, n_i - n_j)\) transforms into
\[
-\sigma_i(n-1)\sigma_j(n-1) \left\{ [\bar{\phi}_i^n - \bar{\phi}_j^n][\phi_i^n - \phi_j^n] + \nu_i^n \nu_j^n \right\} + 1 - \sigma_i(n-1)\sigma_j(n-1) + \sum_{\ell=1}^{n-1} \bar{\phi}_i^\ell \cdot \phi_i^\ell
\]
and the apriori integration form \(D\mu_0\) transforms to
\[
\prod_{i} \prod_{\ell} \partial_{\bar{\phi}_i^\ell} \partial_{\phi_i^\ell} \circ \sigma_i(n-1)^{-3} [1 - 2\nu_i^n]^{-1/2}.
\]

The point here is that Term I above is, (ignoring the fact that the coefficient \(\sigma_i(n-1)\sigma_j(n-1)\) is not a positive real number), algebraically of exactly the same form as the coupling in an \(\mathbb{H}^{n+1}\) model, while Term II is of the same form as that for an \(\mathbb{H}^{n+1}(n-1)\) model, with the caveat that there is extra mass due to the change in Jacobian \(\sigma_i(n-1)^{-3}\). By analogy with the language of probability theory, in the new variables the last component of spin is, ‘conditional’ on the first \(n-1\) components, generating a uniform spanning forest with ‘edge weights’ \(J_{ij}\sigma_i(n-1)\sigma_j(n-1)\).

By induction, we can choose a system of generators \(\bar{\alpha}, \alpha\) so that if we set \(\mu_i^k = -\bar{\alpha}_i^k \alpha_i^k\) the action \(S_{J,\epsilon}\) transforms into
\[
S_{J,\epsilon} \rightarrow \hat{S}_{J,\epsilon} = \sum_{(ij)} \sum_{\ell} \Gamma_i^\ell \left\{ [\bar{\alpha}_i^\ell - \bar{\alpha}_j^\ell][\alpha_i^\ell - \alpha_j^\ell] + \mu_i^\ell \mu_j^\ell \right\} + \sum \epsilon_i \Gamma_i^N
\]
where
\[
\Gamma_i^\ell = \prod_{k<\ell} [1 - \mu_i^k].
\]

Meanwhile, the integration form \(D\mu_0\) transforms to
\[
\prod_{i} \prod_{\ell} \partial_{\bar{\alpha}_i^\ell} \partial_{\alpha_i^\ell} \circ \frac{1}{\prod_{\ell=1}^{n} [1 - 2\mu_i^\ell]^{2(n-\ell)+1}}
\]

**Lemma 4.1.** For any \(n \geq 1\) the partition function \(Z_{J,\epsilon}^f\) is a multivariate polynomial of degree \(n\) in the coupling constants \(J_{ij}\). Since \(Z_{J,\epsilon}^f = Z_{J,\epsilon}\), the same holds for \(Z_{J,\epsilon}\).

**Proof.** In the new variables \(\bar{\alpha}, \alpha\), for any component \(\ell\),
\[
\left\{ [\bar{\alpha}_i^\ell - \bar{\alpha}_j^\ell][\alpha_i^\ell - \alpha_j^\ell] + \mu_i^\ell \mu_j^\ell \right\}^2 = 0
\]
The lemma follows immediately from the pigeon hole principle. \(\square\)

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4.2. Proof of Theorem 1.3, Positivity of the coefficients. Now we turn to the positivity of \( \partial_{J,\varepsilon}^k Z_{J,\varepsilon} \). Though part of the theorem is stated for general pinning fields \( \varepsilon \), for simplicity of exposition we will restrict attention to the case when this field is supported at one vertex, which we denote by 0 and denote the field by \( \varepsilon_0 \).

\[
0 < Z_{J,\varepsilon} = Z_{J,\varepsilon}^f \quad \text{and} \quad \frac{\partial_{J,\varepsilon}^k Z_{J,\varepsilon}^f}{Z_{J,\varepsilon}^f} = \left\langle \left\langle -(n_i - n_j)^2 \right\rangle \right\rangle_{J,\varepsilon}^f.
\]

we shall rather estimate \( \left\langle \left\langle -(n_i - n_j)^2 \right\rangle \right\rangle_{J,\varepsilon}^f \). Before proceeding to the main line of proof, we are going to introduce a few calculational tools.

4.2.1. Ward Identities for hyperboloids in the superspace. We will restrict attention to the case when this field is supported at one vertex, which we denote by 0. Since

\[
v_1 \cdot v_j := -\sigma_i \sigma_j + (\bar{\psi}_i \psi_j - \psi_i \bar{\psi}_j).
\]

So that \( n_i^2 = -1 \). This point of view leads naturally to the objects we now introduce.

Let

\[
Q_\pm^f = \sum_{i \in V} \sigma_i \partial_i^f,
\]

where \( \partial_i^f = \partial_{\psi_i^f} \) and \( \bar{\partial}_i = \partial_{\bar{\psi}_i^0} \). We note the identities

\[
Q_+^f \psi_i^f = Q_-^f \bar{\psi}_i^0 = \sigma_i, \quad Q_{\pm}^f [n_i \cdot n_j] = 0.
\]

For calculation purposes, we work with the unnormalized, unpinned state

\[
\left[ J \right]_{J,0}^f := \int D(\bar{\psi}, \psi) \cdot e^{-S_{J,0}}.
\]

Since \( \{ \sigma_i \sigma_j - \bar{\psi}_i \cdot \psi_j - \bar{\psi}_j \cdot \psi_i \} = -n_i \cdot n_j \) and the bare state is

\[
\mathcal{D}(\bar{\psi}, \psi) = \prod_{i,\ell} \partial_{\bar{\psi}_i^0} \partial_{\psi_i^f} \sigma_i^{-1}.
\]

As explained at (A.26), the state \( \left[ J \right]_{J,0}^f \) is invariant with respect to the infinitesimal symmetries \( Q_{\pm}^f \), and this fact implies a number of useful relations. The basic one is an integration-by-parts formula:

**Lemma 4.2.** If we write \( \sigma_i = Q_{\pm}^f \), then for any \( F \in \mathcal{G}_\Lambda \),

\[
\left[ F(\bar{\psi}, \psi) \sigma_i \right]_{J,0}^f = \left[ \bar{\psi}_i^f \partial_i^f F(\bar{\psi}, \psi) \right]_{J,0}^f = \left[ \psi_i^f \partial_i^f F(\bar{\psi}, \psi) \right]_{J,0}^f.
\]

Further discussion of (4.7) appears in Appendix A.3. The next lemma records a special case of this identity which will be useful below.

**Lemma 4.3** (Expansion Identity). Fix \( L \leq n - 1 \) and suppose \( F \) is an even element of the Grassmann algebra \( \mathcal{G}_\Lambda \) depending on the components \( (\bar{\psi}_i^f, \psi_i^f)_{i \leq L} \). Then

\[
\left[ F_0 e^{-\varepsilon_0 \sigma_0} \right]_{J,0}^f = \varepsilon_0 \left[ F_0 \pi^L \sigma_0^{-1} e^{-\varepsilon_0 \sigma_0} \right]_{J,0}^f - (m - 1) \left[ F_0 \pi^L \sigma_0^{-2} e^{-\varepsilon_0 \sigma_0} \right]_{J,0}^f.
\]
Proof. Since $F$ depends only on $(\tilde{\psi}_i^f, \psi_j^f)_{\ell \leq L}$ Equation (4.7) implies
\[
\left[ F\sigma_0 e^{-\varepsilon_0 \sigma_0} \right]^f_{J,0} = \varepsilon_0 \left[ F_{\sigma_0}^{\varepsilon_0} + 1 e^{-\varepsilon_0 \sigma_0} \right]^f_{J,0} \quad (4.9)
\]

Theorem 4.4 (Restatement of Theorem 1.3 for $\varepsilon_0 \geq 0$ in Grassmann variables). Fix $\varepsilon_0 > 0$. For all $J,k$ such that either $k \leq \lfloor n/2 \rfloor$ or $k = n$
\[
\langle [-(n_i - n_j)^2]^k \rangle^f_{J,\varepsilon_0} \geq 0.
\]
In addition, if $\varepsilon_0 \leq 1$ then
\[
\langle [-(n_i - n_j)^2]^k \rangle^f_{J,\varepsilon_0} \geq 0.
\]
for all $k$.

Proof. Let us consider first the case $k = n$, which we prove indirectly: As a function of $J_{ij}$ alone, $Z_{J,\varepsilon_0}$ is a polynomial of degree $n$ by Lemma 4.1 and we can express this via the formula
\[
\frac{Z_{J,\varepsilon_0}}{Z_{J,J_{ij}=0,\varepsilon_0}} = \sum_{j=0}^{n} \frac{j^k}{k!} \langle [-(n_i - n_j)^2]^k \rangle^f_{J,J_{ij}=0,\varepsilon_0}.
\]
Since the left hand side is positive for all $J_{ij} \geq 0$ we conclude that
\[
\langle [-(n_i - n_j)^2]^n \rangle^f_{J,J_{ij}=0,\varepsilon_0} \geq 0.
\]
By Section 4.1 $[-(n_i - n_j)^2]^{n+1} = 0$,
\[
\langle [-(n_i - n_j)^2]^n \rangle^f_{J,J_{ij}=0,\varepsilon_0} = \frac{Z_{J,\varepsilon_0}}{Z_{J,J_{ij}=0,\varepsilon_0}} \langle [-(n_i - n_j)^2]^n \rangle^f_{J,\varepsilon_0}
\]
so the claim holds for $k = n$ and all $J_{ij} \geq 0$.

For $k < n$ we need to get our hands dirty. It will turn out that, for $k \leq n/2$, we can provide a proof which is relatively clean and easy to check (in particular it applies in the main current application of interest, $\partial_{J_{ij}} \log Z_{J,\varepsilon_0} = \langle [-(n_i - n_j)^2]^k \rangle^f_{J,\varepsilon_0}$). For $n/2 < k < n$ we currently have no better method than to express $\langle [-(n_i - n_j)^2]^k \rangle^f_{J,\varepsilon_0}$ as the expectation of a polynomial in $G_{\sigma_0,0}, G_{\sigma_0}^{(2)}, G_{\sigma_0}^{(2)}, G_{\sigma_0}^{(3)}$ and to argue the the coefficients of the polynomial are positive. This requires a fair amount of computational endurance.

Let us start the proof for $k < n$. Using the symmetry of the state with respect to the Grassmann components and then (4.7) with $\sigma_i - \sigma_j = Q^\perp [\tilde{\psi}_i^f - \tilde{\psi}_j^f]$,
\[
\left[ [-(n_i - n_j)^2]^k e^{-\varepsilon_0 \sigma_0} \right]^f_{J,0} = \left[ 2n\tau_{ij}^1 + (\sigma_i - \sigma_j)^2 [-(n_i - n_j)^2]^{k-1} e^{-\varepsilon_0 \sigma_0} \right]^f_{J,0}.
\]
\[
= \left[ (2n - 1)\tau_{ij}^1 - \varepsilon_0 \eta_{ij}^1 (\sigma_i - \sigma_j) [-(n_i - n_j)^2]^{k-1} e^{-\varepsilon_0 \sigma_0} \right]^f_{J,0}. \quad (4.10)
\]

As a first aside, it is worth noting here that if the pinning $\varepsilon_0 = 0$ the second term in the second equality vanishes. We can then iterate this identity using (4.7) with $\sigma_i - \sigma_j = Q^\perp [\tilde{\psi}_i^f - \tilde{\psi}_j^f]$ for $2 \leq \ell \leq k$ successively in the analogous place where we had $\sigma_i - \sigma_j = Q^\perp [\tilde{\psi}_i^f - \tilde{\psi}_j^f]$. We obtain, if $\varepsilon_0 = 0$,
\[
\left[ [-(n_i - n_j)^2]^k \right]^f_{J,0} = \left[ 2(n - 1) + 1 \right] \cdots \left[ 2(n - k) + 1 \right] \left[ \prod_{\ell=1}^{k} \tau_{ij}^\ell \right]^f_{J,0}.
\]
valid for all $k \leq n$. It is tempting to use the duality (3.1) to conclude  
\[
\left\langle \prod_{\ell=1}^{k} \tau_{ij}^{\ell} \right\rangle_{J,0}^{f} \quad \mu = \mu \left\langle \left[ G_{ij}^{(1)} \right]^{k} \right\rangle_{J,0} \geq 0,
\]
but unfortunately the RHS is not well defined. The problem is that the $t$-field has a noncompact translational symmetry and requires a pinning term to make sense as a probability measure. Still, it provides motivation to push the computation further. Returning to (4.10), we use $\sigma_i - \sigma_j = Q^2 \bar{\psi}_i^2 - \bar{\psi}_j^2$ to obtain two summands  
\[
\left\langle -\left( n_i - n_j \right) e^{-\varepsilon_0 \sigma_0} \right\rangle_{J,0}^{f} = (2n - 1) \left\langle -\left( n_i - n_j \right)^2 e^{-\varepsilon_0 \sigma_0} \right\rangle_{J,0}^{f}
\]
\[
+ \varepsilon_0^2 \left\langle -\left( n_i - n_j \right)^2 \left[ \bar{\psi}_i^4 - \bar{\psi}_j^4 \right] \right\rangle_{J,0}^{f}.
\]
(4.11)
We want to continue iteratively, using that $\bar{\psi}_i^m - \bar{\psi}_j^m = 0$ and expanding $\sigma_i - \sigma_j$ using $Q^\ell$ until we generate $n$ distinct components between the $\tau_{ij}$’s and the $\bar{\psi}_i$’s. To handle the combinatorics of this expansion let, for $p,t \in \mathbb{N}$  
\[
I_n(p,t) = \{(i_1, \ldots, i_p) : t \leq \ell \leq n, i_\ell > i_{\ell+1}, i_s - i_{s+1} = 1 \text{ is odd, } n - i_1 \text{ is even}\},
\]
(4.12)
\[
A_n(p,t) = \sum_{I \in I_n(p,t)} [2i_1 - 1] \cdots [2i_p - 1],
\]
(4.13)
\[
C_n(0) = 1 \text{ and } C_n(p) = A_n(p,1) \text{ otherwise. In particular, if } p > nC_n(p) = 0
\]
(4.14)
(4.15)
When $n$ and there is no danger of confusion we will suppress the subscript $n$.

With these notations, and since the components of the Grassmann field are exchangeable, after iterating the previous computation using the available components $\ell = 3, 4, \ldots, n$ we have  
\[
\left\langle -\left( n_i - n_j \right)^2 e^{-\varepsilon_0 \sigma_0} \right\rangle_{J,0}^{f} = \sum_{p=2k-n}^{k} A(p, n + p - 2k) d_0^{2(k-p)} \left\langle \prod_{\ell=1}^{p} \eta_{ij}^{m} \epsilon_{ij}^{2} \right\rangle_{J,0}^{f}
\]
\[
+ \sum_{p=0}^{2k-n-1} C(p) d_0^{n-p} \left\langle \prod_{\ell=1}^{p} \eta_{ij}^{m} \epsilon_{ij}^{2} \right\rangle_{J,0}^{f}.
\]
(4.16)
Of course if $2k - n - 1 < 0$, the second sum should be interpreted as 0. We pause now to observe an intermediate result. In case $k \leq \lceil n/2 \rceil$ the second term in Equation (4.16) vanishes. We claim the first term is manifestly positive. To see this observe that if we absorb $e^{-\varepsilon_0 \sigma_0}$ into the action and pass back to horospherical coordinates, then we may express fermionic expectations for polynomials in the $\tau$’s, $\eta$’s and $\pi_\ell$’s in terms of polynomials in the $G_{ij}^{(1)}$, $G_{ij}^{(2)}$, $G_{00}$, and $G_{ij}^{(3)}$, see Lemma 3.2. We therefore obtain the following immediate corollary:

**Proposition 4.5.** For any $k$, all summands in the first term on the RHS of Equation (4.16) are positive. In particular if $k \leq \lceil n/2 \rceil$,  
\[
\left\langle -\left( n_i - n_j \right)^2 \right\rangle_{J,0}^{f} = \sum_{0 \leq p,q \leq 2k, 2p + 2q = 2k, p + 2q \leq n} A(p, n + p - 2k) d_0^{2(k-p)} \left\langle G_{ij}^{(1)p} G_{ij}^{(3)2q} \right\rangle_{J,0}^{f} \geq 0.
\]
In particular this shows that $Z_{J,0}^{f} = Z_{J,0}$ is increasing in the coupling constants.
Remark 4.6. It is worth stating how these formulas modify if the pinning field is supported at more than one point. The only change is that $\varepsilon_0 \eta_{ij}$ gets replaced by

$$
\kappa_{ij}(\varepsilon) := -[\psi_i^\ell - \psi_j^\ell]! \sum_j \varepsilon_j^i \psi_j^i
$$

leading to a replacement of $\varepsilon_0 G^{(3)}_{ij}$ by

$$
G^{(3)}_{ij}(\varepsilon) := \sum_{j'} \varepsilon_{j'}[G_{ij'} - G_{jj'}].
$$

From now on, we may assume $[n/2] < k < n$. Combining this with the positivity argument given for the $n$'th moment, we may also assume $n \geq 4$. From this point forward, the proof becomes more computational. For summands in the second term on the RHS of Equation (4.16), we continue by using $\sigma_i - \sigma_j = Q^\ell [\psi_i^\ell - \psi_j^\ell]$ for $\ell \geq p + 1$. Letting $\phi(j) = \sigma_0(j) - \varepsilon_0 \pi_0^j$,

$$
\left[ \prod_{\ell=1}^p \tau_{ij}^\ell \prod_{m=1}^r \eta_{ij}^{m+p}[\sigma_i - \sigma_j]^{2k-2p-r} e^{-\varepsilon_0 \sigma_0} \right] f_{J,0} \longrightarrow \left[ \prod_{\ell=1}^{p+1} \tau_{ij}^\ell \prod_{m=2}^{r} \eta_{ij}^{m+p}[\sigma_i - \sigma_j]^{2k-2p-r-1} \phi(p+1)e^{-\varepsilon_0 \sigma_0} \right] f_{J,0}
$$

where we used nilpotency of the $\psi_i^\ell$'s and $\psi_j^\ell$'s to reduce $\sigma_0$ to $\sigma_0(p + 1)$. Continuing in this way we arrive at

$$
\left[ \prod_{\ell=1}^p \tau_{ij}^\ell \prod_{m=1}^r \eta_{ij}^{m+p}[\sigma_i - \sigma_j]^{2k-2p-r} e^{-\varepsilon_0 \sigma_0} \right] f_{J,0} = \left[ \prod_{\ell=1}^{2k-n} \tau_{ij}^\ell \prod_{m=2k-n+1}^{n} \eta_{ij}^{2k-n} \phi(s)e^{-\varepsilon_0 \sigma_0} \right] f_{J,0}
$$

Plugging this expression into (4.16) and reorganizing terms

$$
\left[ (n_i - n_j)^2 p e^{-\varepsilon_0 \sigma_0} \right] f_{J,0} = \sum_{\substack{0 \leq p, q, \; 2p + 2q = 2k, \\ p + 2q \leq n}} A(p, n - p - 2q) e^{2p} \eta_{ij}^{2k-n} \phi(s) e^{-\varepsilon_0 \sigma_0} f_{J,0}
$$

Using an integration by parts in horospherical coordinates, we may estimate Term I from below by the expectation of a polynomial (with positive coefficients) in $G_{00}, G^{(1)}_{ij}, G^{(2)}_{ij}$.

Lemma 4.7. We have, for $k \leq n - 1$,

$$
\frac{I}{Z_{J,\varepsilon_0}} \geq \varepsilon_0 A(k, n - k) \cdot \sum_{\ell=0}^k D_n(\ell; k) \left( C_{00}^{n-k} G_{ij}^{(1)x} G_{ij}^{(2)k-x} \right)_{J,\varepsilon_0},
$$

where

$$
D_n(\ell; k) \geq \begin{cases} 
\begin{array}{ll}
(n_k - 1)! k^2 \ell^{n-k-2} & \text{if } k \leq n - 2, \\
(n_k - 2)! & \text{if } k = n - 1
\end{array}
\end{cases}
$$

Likewise, we may evaluate Term II in a compact, implicit manner using integration by parts in horospherical coordinates. To describe the output denote

$$
P_{L,m}(v; a, b) = D^L_1 \cdot D^m_2 \cdot 1.
$$

Note that $P_{L,m}$ is a rational function in $v$ and polynomial in $a, b$ of degree $L + m$. 


Lemma 4.8. We have
\[
\frac{II}{Z_{J,\varepsilon_0}} = \varepsilon_0 (n-k) \sum_{\ell=0}^{2k-n-1} C(p) \left| \langle G_{ij}^{(3)} \rangle^{2(n-k)} \mathcal{P}_{p,2k-n-p}(1; G_{ij}^{(1)}, G_{ij}^{(2)}) \rangle \beta_{J,\varepsilon_0} \right|
\]
(4.21)
and if \(0 \leq \varepsilon_0 \leq 1\)
\[
\frac{II}{Z_{J,\varepsilon_0}} \leq \frac{I}{Z_{J,\varepsilon_0}}
\]
(4.22)

The proofs of Lemmas 4.7 and 4.8 are presented in the next two subsections. Since \(k \leq n-1\), Lemma 4.7 and Equation (4.21) together show that by taking \(\varepsilon_0\) small relative to \(n\), Term II can be bounded by Term I. To get a quantitative estimate of the dependence of \(\varepsilon_0\) on \(n\), the main difficulty is to carefully understand \(\mathcal{P}_{p,2k-n-p}(1; a, b)\). This is undertaken in Section 4.4.1.

4.3. Proof of Lemma 4.7.

Proof. Using Equation (4.8) we have the following identities:

**Identity 1:** If \(F\) depends on at most the first \(n-1\) components \((\tilde{\psi}^f_i, \psi^f_i)_{i \in V, \ell \leq n-1}\) then
\[
\left\langle F \right\rangle^f_{J,\varepsilon_0} = \varepsilon_0 \left\langle F \delta_0^{n-1} \right\rangle_{J,\varepsilon_0} + \sum_{j=1}^{n} \left( \frac{F \pi_j^f}{\sigma_0(j-1)} \right)^f_{J,\varepsilon_0}.
\]
(4.23)

This follows by applying Equation (4.8) to \(\left\langle F \delta_0^{n} \right\rangle_{J,\varepsilon_0} \) with \(\sigma_0 = Q^n\).

**Identity 2:** If \(F\) depends on, at most, the first \(n-2\) components \((\tilde{\psi}^f_i, \psi^f_i)_{i \in V, \ell \leq n-2}\)
\[
\left\langle F \right\rangle^f_{J,\varepsilon_0} = \varepsilon_0 \left\langle F \delta_0^{n-1} \right\rangle_{J,\varepsilon_0} + \left\langle F \pi_j^f \right\rangle_{J,\varepsilon_0} + \sum_{j=1}^{n-1} \left( \frac{F \pi_j^f}{\sigma_0} \right)^f_{J,\varepsilon_0}.
\]
(4.24)

This follows similarly by applying Equation (4.8) to \(\left\langle F \delta_0^{n-1} \right\rangle_{J,\varepsilon_0} \) with
\[
\sigma_0 + \pi_j^n = Q^n \cdot \left[ \tilde{\psi}_j^n \delta_0 \right] = Q^n \cdot \left[ \tilde{\psi}_j^n \cdot Q_j^{-1} \cdot \tilde{\psi}_j^{n-1} \right].
\]

To prove the lemma, we distinguish two cases: Either \([n/2] \leq k \leq n-2\) or \(k = n-1\). In both cases, we shall start by applying these identities with \(F = \prod_{\ell=1}^{k} \tau_{ij}^f\). Note that we may ignore the remaining contributions to Term I since by Proposition 4.5 they are positive. Note also that, using the identity
\[
\frac{1}{\sigma_0(j)} = E[e^X \sum_{\ell \leq j} \pi_0^\ell] \quad (X \sim N(0,1)),
\]
and the horospherical integration, each summand on the RHS of these Identities 1, 2 is non-negative (if this is too brief, the reader may look to the case \(k = n-1\) below for more detail).

We take up the case \(k \leq n-2\) first. Since \(\left\langle \prod_{\ell=1}^{k} \tau_{ij}^f \pi_0^{n-k} \right\rangle_{J,\varepsilon_0} \geq 0\),
\[
\left\langle \prod_{\ell=1}^{k} \tau_{ij}^f \right\rangle_{J,\varepsilon_0} \geq 2 \sum_{j=1}^{n-1} \left\langle \prod_{\ell=1}^{k} \tau_{ij}^f \pi_j^f \right\rangle_{J,\varepsilon_0}.
\]
(4.25)

We repeatedly apply Identity 2 to each term on the RHS until we reach an expression which depends on \(n-k-1\) of the last \(n-k\) components. We obtain (using the exchangeability of the components)
\[
\left\langle \prod_{\ell=1}^{k} \tau_{ij}^f \right\rangle_{J,\varepsilon_0} \geq \sum_{J \leq k} b_J(k) \left( \prod_{j=1}^{k} \tau_{ij}^f \prod_{m=k-J+1}^{n-1} \pi_j^m \right)_{J,\varepsilon_0}
\]
(4.25)
with
\[
b_J(k) = \frac{(J + n - k - 2)! \cdot k!}{2 \cdot (k - J)!}.
\]

Once \( n - k - 1 \) of the last \( n - k \) components appear, we switch protocols and use Identity 1 to each summand on the RHS. We obtain, since the remarks on positivity remain in force for

\[
F := \prod_{\ell=1}^{k} \tau_{ij}^{\ell} \prod_{m=k-J+1}^{n-1} \pi_{0}^{m}
\]
as well,

\[
\left\langle \prod_{\ell=1}^{k} \tau_{ij}^{\ell} \prod_{m=k-J}^{n-1} \pi_{0}^{m} \right\rangle_{J,0}^{f} \geq \sum_{k-J \geq l} \frac{(k-J)!}{l!} \left\langle \prod_{\ell=1}^{k} \tau_{ij}^{\ell} \prod_{m=l+1}^{n} \pi_{0}^{m} \right\rangle_{J,0}^{f}.
\]

Plugging this into the RHS of Equation (4.25), we have

\[
\left\langle \prod_{\ell=1}^{k} \tau_{ij}^{\ell} \right\rangle_{J,0}^{f} \geq \sum_{l \leq k} D_{n}(l; k) \left\langle \prod_{\ell=1}^{k} \tau_{ij}^{\ell} \prod_{m=k-l}^{n} \pi_{0}^{m} \right\rangle_{J,0}^{f}
\]

where

\[
D_{n}(l; k) = \frac{(n-k-1)! \cdot k!}{l!} \sum_{J \leq k-l} \binom{J+n-k-2}{J} 2^{J+n-k-2}.
\]

Using Lemma 3.2, we have

\[
\left\langle \prod_{\ell=1}^{k} \tau_{ij}^{\ell} \prod_{m=k-l}^{n} \pi_{0}^{m} \right\rangle_{J,0}^{f} = \left\langle [G_{00}]^{n-k} G_{ij}^{(1)} \pi_{0}^{m} \right\rangle_{J,0}^{f}
\]

Finally we have an appropriate lower bound on \( D_{n}(l; k) \) simply by taking the term with largest index;

\[
D_{n}(l; k) \geq \frac{(n-k-1)! \cdot k! 2^{n-l-2}}{l!} \binom{n-l-2}{k-l}.
\]

Turning to the case \( k = n - 1 \), we insert the identity

\[
\frac{1}{\sigma_{0}(j)} = \mathbb{E}[e^{X^{2} \sum_{\ell \leq j} \pi_{0}^{\ell}}] \quad (X \overset{d}{=} N(0,1)),
\]
to obtain

\[
\left\langle \prod_{\ell=1}^{n-1} \tau_{ij}^{\ell} \right\rangle_{J,0}^{f} = \left\langle \pi_{0}^{n} \prod_{\ell=1}^{n-1} \tau_{ij}^{\ell} \right\rangle_{J,0}^{f} + \sum_{j=1}^{n-1} \mathbb{E} \left[ \left\langle \prod_{\ell=1}^{n-1} \tau_{ij}^{\ell} \cdot \pi_{0}^{m} \cdot e^{X^{2} \sum_{\ell \leq j-l} \pi_{0}^{\ell}} \right\rangle_{J,0}^{f} \right].
\]

By nilpotency of the \( \pi_{0}^{\ell} \)'s,

\[
e^{X^{2} \sum_{\ell \leq j} \pi_{0}^{\ell}} = \sum_{J \leq j} \frac{X^{2J}}{J!} \left( \sum_{\ell \leq j} \pi_{0}^{\ell} \right)^{J}.
\]

Collecting terms according to the number of \( \pi_{0}^{\ell} \)’s appearing,

\[
\left\langle \prod_{\ell=1}^{n-1} \tau_{ij}^{\ell} \right\rangle_{J,0}^{f} = \left\langle \pi_{0}^{n} \prod_{\ell=1}^{n-1} \tau_{ij}^{\ell} \right\rangle_{J,0}^{f} + \sum_{m=0}^{n-1} a_{m} \left\langle \prod_{\ell=1}^{n-1} \tau_{ij}^{\ell} \prod_{\ell'=1}^{m} \pi_{0}^{\ell'} \right\rangle_{J,0}^{f}
\]

where an empty product is treated as 1 and

\[
a_{m,n-1} = \frac{[2(m-1)]!!}{(m-1)!} \sum_{m \leq j \leq n-1} \frac{(j-1)!}{(j-m)!}.
\]
This identity can be recursed for each summand such that $\pi_0^r$ does not appear. We obtain
\[
\left\langle \prod_{t=1}^{n-1} \tau_{ij}^t \right\rangle_{J, \xi_0}^f = \sum_{m=0}^{n-1} D_n(n - m - 1; n - 1) \left\langle \pi_0^{n-1} \prod_{t=1}^{m} \prod_{t=1}^{n-1} \pi_0^t \right\rangle_{J, \xi_0}^f.
\]

Here, with $j_0 := 0$ and, for a given tuple of positive integers $j_1 \ldots j_i$, denoting $J_i = \sum_{t=0}^{i} j_t$,
\[
D_n(n - m - 1; n - 1) = \sum_{s \geq 0} \sum_{\sum_{i=1}^{s} j_i = n} \prod_{i=0}^{s-1} a_{j_{i+1}, n-J_i-1}.
\]

We wish to estimate $D_n(n - m - 1; n - 1)$ from below. The simplest option is to take the term $a_{m,n-1}$ to obtain
\[
D_n(n - m - 1; n - 1) \geq \frac{2^{m-1} \Gamma(m + 1/2)(n - 2)!}{\Gamma(m + 1)\Gamma(1/2)(n - m)!}.
\]

4.4. Proof of Lemma 4.8. The starting point is the formula for Term II of Equation (4.19), which we recall here for the readers convenience:
\[
\frac{\Pi}{Z_{J, \xi_0}} = \left\langle \prod_{t=1}^{2k-n} \tau_{ij}^t \times \prod_{m=2k-n+1}^{n} \eta_{ij}^m \times \left\{ \sum_{p=0}^{2k-n-1} C(p) \varepsilon_0^{m-p} \prod_{m'=p+1}^{2k-n} \phi(m') \right\} \right\rangle_{J, \xi_0}^f.
\]

To derive identity (4.21), on the RHS of (4.26) we pass back to the horospherical representation and perform a fermionic Gaussian integration. In each summand, we first integrate out the components $\ell \in \{2k-n+1, \ldots, n\}$. This yields a factor $[G_{ij}^{(3)}]_{2(n-k)}$. Next we integrate the remaining components starting with $\ell = 2k-n$ first and then proceeding backwards to $\ell = 1$. As a result of the component by component integration we have, after integrating out the components $2k-n, \ldots, s+1$, a recursively defined expression depending on the components $\ell = 1, \ldots, s$. The expression factors as a product of two terms: an explicit factor $T_s := \prod_{m=p+1}^{s} \phi(m)$ independent of the $s$th component and another factor $Q_s$ which collects the dependence on the $s$th component explicitly and through $\sigma(s)$. We compute that
\[
Q_s = \begin{cases} [G_{ij}^{(3)}]_{2(n-k)} \prod_{m=p+1}^{s} \phi(m) \times [D_2^{2k-n-s} \cdot 1]_{v=\sigma(s)} & \text{if } s \geq p + 1, \\
[G_{ij}^{(3)}]_{2(n-k)} D_1^{p-s} D_2^{2k-n-p} \cdot 1]_{v=\sigma(s)} & \text{if } s \leq p,
\end{cases}
\]

which is to be integrated over the $t$-field and the remaining Gaussian components - there is 1 real component and $s+1$ fermionic components. The additional components are due to the inversion of SUSY localization. Note that the additional Gaussian components do not appear in the delocalized versions of $T_s, Q_s$. The first part of Lemma 4.8 thus follows.

4.4.1. Computing the $\mathcal{P}$’s. For the second part of Lemma 4.8, let us consider
\[
Q(v, a, b) := \sum_{p=0}^{2k-n-1} C(p) \mathcal{P}_{p,2k-n-p}(v; a, b).
\]
as remarked before, this is a polynomial in $a, b$. Let $r = a/b$. Then setting $u = \sqrt{rv}$, we have
\[
D_2 = \varepsilon_0 b \sqrt{r} \frac{u}{u - \theta_u - \varepsilon_0} \cdot u,
\]
\[
D_1 = \frac{br}{u} [u - \theta_u].
\]

This leads to
\[
D_2^m \cdot 1 = [\varepsilon_0 b \sqrt{r}]^m \left\{ H_{m+1}(u - \frac{\varepsilon_0}{\sqrt{r}}) + \frac{\varepsilon_0}{\sqrt{r}} H_m(u - \frac{\varepsilon_0}{\sqrt{r}}) \right\}.
\]

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where $H_r$ are the Hermite polynomials. Therefore

$$Q(v; a, b) := \sum_{p=0}^{2k-n-1} C(p)[\varepsilon_0 b\sqrt{r}]^{2k-n-p}[br]^p D_1 \cdot \frac{1}{u} \left\{ H_{2k-n-p+1}(u - \frac{\varepsilon_0}{\sqrt{r}}) + \frac{\varepsilon_0}{\sqrt{r}} H_{2k-n-p}(u - \frac{\varepsilon_0}{\sqrt{r}}) \right\}$$

Recall that

$$H_m(z + y) = \sum_{j=0}^{m} \binom{m}{j} z^j H_{m-j}(y)$$

so

$$Q(1; a, b) :=
\begin{align*}
b^{2k-n} \sum_{p=0}^{2k-n-1} \sum_{j=0}^{2k-n-p+1} C(p)[\varepsilon_0 2k-n-p]^{2k-n-p+j-1} \binom{2k-n-p+1}{j} 2^p \left( \frac{j-1}{2} \right)_p H_{2k-n-p+1-j}(\frac{\varepsilon_0}{\sqrt{r}}) \\
- b^{2k-n} \sum_{p=0}^{2k-n-1} \sum_{j=0}^{2k-n-p} C(p)[\varepsilon_0 2k-n-p+1]^{2k-n-p+j-2} \binom{2k-n-p}{j} 2^p \left( \frac{j-1}{2} \right)_p H_{2k-n-p-j}(\frac{\varepsilon_0}{\sqrt{r}})
\end{align*}$$

where $(x)_j$ is the descending factorial and if $p = 0$, we set $\left( \frac{j-1}{2} \right)_p = 1$. Now we collect terms according to powers in $r$. Expanding the Hermite polynomials using the variable $m$, and denoting $t_p = 2k-n-p$,

$$Q(1; a, b) := b^{2k-n} \sum_{p=0}^{2k-n-1} \sum_{j=0}^{t_p+1} \sum_{m=0}^{t_p+1-j} (-1)^m c_{m; t_p+1-j} C_n(p)[\varepsilon_0 t_p+m]^{t_p+1-m} \binom{t_p+1}{j} 2^p \left( \frac{j-1}{2} \right)_p \\
- b^{2k-n} \sum_{p=0}^{2k-n-1} \sum_{j=0}^{t_p-j} \sum_{m=0}^{t_p-j} (-1)^m c_{m; t_p-j} C_n(p)[\varepsilon_0 t_p+m+1]^{t_p-j-m} \binom{t_p}{j} 2^p \left( \frac{j-1}{2} \right)_p$$

where

$$c_{w;r} = \begin{cases} 0 & \text{if } r-w \text{ is odd or } w < 0, \\ \frac{r!}{w!(r-w)!} & \text{else.} \end{cases}$$

Recalling that $Q(1; a, b)$ must be a polynomial of homogenous degree $2k-n$, the only terms which ultimately contribute satisfy the conditions

$$0 \leq t_p + j - 1 - m \leq 2k-n, \quad t_p + j - 1 - m \in 2\mathbb{Z} \text{ from the first sum},$$

$$0 \leq t_p + j - 2 - m \leq 2k-n, \quad t_p + j - 2 - m \in 2\mathbb{Z} \text{ from the second sum}.$$

So

$$Q(1; a, b) = \sum_{\ell=0}^{2k-n-1} d_n(\ell; k) a^{\ell} b^{2k-n-\ell}$$

where

$$d_n(\ell; k) = C_n(2k-n-\ell)[\varepsilon_0 2k-n-\ell]^{2k-n-\ell} \left( \frac{\ell}{2} \right)_{2k-n-\ell} + \sum_{p=0}^{2k-n-1} \sum_{j=0}^{t_p} (-1)^{t_p+j-1} C_n(p)[\varepsilon_0 t_p+j+1-2\ell]^{2t_p+j+1-2\ell} \frac{1}{2} \binom{t_p+1}{j} \left\{ c_{t_p+j-2-2\ell; t_p-j} \binom{t_p+1}{j} + c_{t_p+j-1-2\ell; t_p+1-j} \binom{t_p+1}{j} \right\}.$$

The proof of Lemma 4.8 is then concluded by verifying the following estimates.

**Lemma 4.9.** For all $n \geq 4$ and all $[n/2] \leq k \leq n - 1$, $0 \leq \ell \leq 2k-n-1$ and all $|\varepsilon_0| \leq 1$

$$A(k, n-k) D_{n-k+\ell}(k) \geq |d_\ell(k)|$$
To prove this lemma, we first must understand the growth of the $C_n(p)$’s.

**Lemma 4.10.** Let $n \geq 4$ be fixed. For all $0 \leq p \leq n - 3$,

$$C_n(p) \geq [2(n - p) - 1]C_n(p - 1).$$

and

$$C_n(n) = C_n(n - 1) \geq 2C_n(n - 2)$$

and

$$C_n(n - 2) \geq \frac{15}{7}C_n(n - 3)$$

**Proof.** The conditions on the $p$-tuples in $I_n(p, 1)$ imply that

$$C_n(p) = (2n - 1)C_{n-1}(p - 1) + C_{n-2}(p)$$

By induction, we see immediately that $C_n(p)$ increases in $p$. To prove the lemma we induct on hypothesis that the stated bound holds for $n' < n$ and all $p$ and also for $n$ and $p' < p$. Recall that $C_n(0) = 1$. $C_n(1) \geq 2n - 1$, so the bound clearly holds with $p = 1$.

Before proceeding to verify the induction step, we need to make some observations: First, by definition of the sets $I_n(p)$,

$$C_n(n) = C_n(n - 1) = A_n(1) = A_n(3)$$

$$C_n(n - 2) = C_n(n) \sum_{j=2}^{n} [2j - 1]^{-1}[2j - 3]^{-1} \leq \frac{1}{2}C_n(n)$$

since

$$\sum_{j=2}^{n} [2j - 1]^{-1}[2j - 3]^{-1} = \frac{1}{2} - \frac{1}{2(2n - 1)}.$$  

Second, by a simple induction $C_n(p) \geq 2C_n(p - 1)$ for all $n \geq 2$ and $p \leq n - 3$. To perform this induction, we check by hand that $C_2(2) = C_2(1) = 3, C_3(3) = C_3(2) = 15, C_3(1) = 6$.

Also by definition of $I_n(p)$

$$C_n(n - 2) - C_n(n - 3) = \frac{4C_n(n)}{3 \cdot 5}$$

so that we have

$$C_n(n - 2) = \frac{1}{1 - \frac{4}{15}}C_n(n - 3).$$

which implies $C_n(n - 2) = \frac{15}{11}C_n(n - 3)$ for all $n \geq 3$.

We are now ready to verify the induction step of in the proof of the main lemma. We induct on hypothesis that the stated bound holds for $n' < n$ and all $p$ and also for $n$ and $p' < p$. Then we have

$$C_n(p) = (2n - 1)C_{n-1}(p - 1) + C_{n-2}(p)$$

$$\geq (2n - 1)[2(n - p) - 1]C_{n-1}(p - 2) + [2(n - p - 2) - 1]C_{n-2}(p - 1)$$

$$=[2(n - p) - 1]C_{n}(p - 1) + [2(n - p) - 1][C_{n-2}(p) - C_{n-2}(p - 1)] - 2C_{n-2}(p)$$

where the inequality follows from the induction hypothesis. By the a priori estimate,

$$C_{n-2}(p) \geq 2C_{n-2}(p - 1)$$

Now since $p \leq n - 3$, $n - p \geq 3$ so $2(n - p) - 1 \geq 5$. Also, by the a priori estimate, $C_{n-2}(p) \geq 2C_{n-2}(p - 1)$. It follows that the last two terms on the RHS of (4.32) sum to something non-negative. Thus the induction step is proved. \[\square\]
Proof of Lemma 4.9. Let

\[ b_n(\ell, k) = C_n(2k - n - \ell)\varepsilon_0^\ell 2^{2k-n-\ell} \left( \frac{\ell}{2} \right)_{2k-n-\ell}. \]

Recall the formula for the Hermite coefficients \( c_{r,s} \) which provide also constraints on the summands of \( d_r(k) \) to be non-zero. By the triangle inequality, the assumption \(|\varepsilon_0| \leq 1 \) and Lemma 4.10

\[ |d_n(\ell; k) - b_n(\ell, k)| \leq \sum_{p=0}^{2k-n-\ell-1} \sum_{j=|2\ell+1-t_p|\forall 0}^{\ell+1} 2^{2p-n+4} C_n(n-3)\Gamma(\frac{5}{2}) |(j-1)/2|^p |(t_p+1)| \]

\[ \frac{C_n(n-3)\Gamma(\frac{5}{2}) |(j-1)/2|^p |(t_p+1)|}{(t_p+j-2-2\ell)!(\ell+1)!\Gamma(n-p-1/2)}. \]

With foresight on what our final comparison will be, we also bound

\[ \frac{1}{(\ell+1-j)!j!} \leq 2^{\ell+1} \frac{1}{(\ell+1)!}. \]

to obtain

\[ |d_n(\ell; k) - b_n(\ell, k)| \leq \sum_{p=0}^{2k-n-\ell-1} \sum_{j=|2\ell+1-t_p|\forall 0}^{\ell+1} 2^{2p+\ell-n+5} C_n(n-3)\Gamma(\frac{5}{2}) |(j-1)/2|^p |(t_p+1)| \]

\[ \frac{C_n(n-3)\Gamma(\frac{5}{2}) |(j-1)/2|^p |(t_p+1)|}{(t_p+j-1-2\ell)!(\ell+1)!\Gamma(n-p-1/2)}. \]

To estimate the RHS of this inequality, observe that for \( j \) fixed

\[ \frac{1}{(t_p+j-2-2\ell)!}, \quad \frac{(t_p+1)!}{\Gamma(n-p-1/2)} \]

are both increasing in \( p \). Denoting \( M_\ell = |(\ell/2)_{2k-n-\ell} |\Gamma((\ell+1)/2)_{2k-n-\ell} | \) we also have

\[ |(j-1)/2|^p |\leq 4M_\ell \]

for \( j \leq \ell+1 \) and \( p \leq 2k-n+j-1-2\ell \). Exchanging the sums over \( j \) and \( p \), these bounds imply

\[ |d_n(\ell; k) - b_n(\ell, k)| \leq \sum_{j=0}^{\ell+1} 2^{4k-3n+2j-3\ell+7} C_n(n-3)\Gamma(\frac{5}{2}) M_\ell (2\ell+1-j)! \]

\[ \frac{(2\ell+1-j)!}{(\ell+1)!\Gamma(2(n-k)+2\ell-j+1/2)}. \]

Since

\[ \frac{(2\ell+1-j)!}{\Gamma(2(n-k)+2\ell-j+1/2)} \]

is increasing in \( j \), the RHS of (4.4.1) can be bounded by

\[ |d_n(\ell; k) - b_n(\ell, k)| \leq 2^{4k-3n-\ell+7} C_n(n-3)\Gamma(\frac{5}{2}) M_\ell \]

\[ \frac{(\ell+1)!\Gamma(2(n-k)+\ell-1/2)}. \]

Finally, by Lemma 4.10,

\[ |b_n(\ell, k)| \leq 2^{-2(n-k)+3-\ell} \frac{C_n(n-3)M_\ell\Gamma(\frac{5}{2})}{\Gamma(2(n-k)+\ell-1/2)}. \]

Let us now check the statement of the Lemma. For \( k \leq n-2 \) we can write

\[ D_\ell(k)A(k, n-\ell) \geq \frac{(n-k-1)!\cdot k! 2^{k+n-\ell-2}}{\ell!} \left( \frac{n-\ell-2}{k-\ell} \right) \frac{\Gamma(n+1/2)}{\Gamma(n-k-1/2)} \]

where we used Lemma 4.7 and

\[ A(k, n-k) = \frac{2^k\Gamma(n+1/2)}{\Gamma(n-k-1/2)}. \]
We thus have

\[ \frac{|b_n(\ell; k)|}{D_\ell(k)A(k, n - k)} \leq 2^{k-3n+5} \ \frac{\Gamma\left(\frac{5}{2}\right) C_n(n - 3) \Gamma(n - k - 1/2)}{\Gamma(n - k) \cdot k! \cdot (n-\ell-2) \Gamma(2n - k + \ell - 1/2)} \]

and

\[ \frac{|d_n(\ell; k) - b_n(\ell, k)|}{D_\ell(k)A(k, n - k)} \leq 2^{3(k-n)-n+9} \ \frac{\Gamma\left(\frac{5}{2}\right) C_n(n - 3) \Gamma(n - k - 1/2)}{\Gamma(n - k) \cdot k! \cdot (n-\ell-2) \Gamma(2n - k + \ell - 1/2)}. \]

We combine the two previous estimates with the fact that

\[ 2^{2-n} C_n(n - 3) \leq \frac{\Gamma(n + 1/2)}{\Gamma(1/2)} \]

to obtain

\[ \frac{|d_n(\ell; k)|}{D_\ell(k)A(k, n - k)} \leq \frac{\Gamma\left(\frac{5}{2}\right)}{\Gamma(n - k) \cdot k! \cdot (n-\ell-2) \Gamma(2n - k + \ell - 1/2)} \left\{ 2^{k-2n+2} + \frac{2^{3(k-n)+6}}{\ell + 1} \right\}. \]

Continuing with the RHS, we have

\[
\frac{\Gamma\left(\frac{5}{2}\right)}{\Gamma(n - k) \cdot k! \cdot (n-\ell-2) \Gamma(2n - k + \ell - 1/2)} \left\{ 2^{k-2n+2} + \frac{2^{3(k-n)+6}}{\ell + 1} \right\} \leq \frac{(2k - n)!}{5 \cdot k!} \left\{ 2^{-n} + 1 \right\} \leq 1 \quad (4.33)
\]

where in the last inequality we used the fact that \( n \geq 4 \) and \( \lfloor n/2 \rfloor \leq k \leq n - 2 \). This verifies the Lemma in case \( k \leq n - 2 \).

If \( k = n - 1 \), the estimate proceeds in the same way except that we replace the lower bound on \( D_\ell(k) \) by

\[ D_\ell(n - 1) \geq \frac{2^{n-\ell-2} \Gamma(n - \ell - 1/2)(n - 2)!}{\Gamma(n - \ell) \Gamma(1/2) \ell!}. \]

The estimate gets slightly tighter. We find

\[ |d_n(\ell; n - 1) - b_n(\ell, n - 1)| \leq 2^{n-\ell+1} \frac{C_n(n) \Gamma\left(\frac{5}{2}\right) M_\ell}{(\ell + 1) \Gamma(2 + \ell - 1/2)} \]

and

\[ |b_n(\ell, n - 1)| \leq 2^{1-\ell} \frac{C_n(n) M_\ell \Gamma\left(\frac{5}{2}\right)}{2 + \ell - 1/2}. \]

We end up with

\[ \frac{|d_n(\ell; n - 1)|}{D_\ell(n - 1)A(n, 1)} \leq \frac{3}{2} \frac{\sqrt{n - \ell} M_\ell}{(n - 2)! \sqrt{\ell + 1/2}} \left\{ 2^{2-n} + \frac{2^3}{\ell + 1} \right\} \]

\[ \leq \frac{3}{8} \frac{\sqrt{2n}}{(n - 2)(n - 3)} \left\{ 2^{2-n} + 2^3 \right\} \]

For \( n \geq 6 \) the last expression is bounded by 1. For \( n \in \{4, 5\} \) and \( k = n - 1 \), one must unfortunately return to (4.27) and compute explicitly. We find

\[ |d_4(0; 3)| \leq 5/2 + 2^5; \quad |d_4(1; 3)| = 0; \]
\[ A(4, 1) = 7 \cdot 5 \cdot 3; \]
\[ |d_5(0; 4)| \leq 3^3 + 5^3 \cdot 4; \quad |d_5(1; 4)| \leq 2^5; \quad |d_5(2; 4)| = 0; \]
\[ A(5, 1) = 3^3 \cdot 7 \cdot 5. \]

Since \( D_n \ell; (n - 1) > 1 \) in the nonzero cases, these estimates complete the Lemma in the final two cases.
5 Proof of Theorem 1.1

In the setting of Theorem 1.1, \( J_{j,j'} = \beta I_{j,j'} \), \( \rho_j \) is any function which is \( j = 0 \) at 0 and \( \min_{(j,j')} \cos(\rho_j - \rho_{j'}) > 0 \). We will shortly specify a particular choice by optimizing an upper bound on the Fourier transform of \( t_m - t_\ell \). By Lemma 2.1 and Theorem 1.3, we have

\[
\left| \left\langle e^{ik(t_m-t_\ell)} \right\rangle_{\Lambda,a,J,\varepsilon_0} \right| \leq \prod_{(j,j')} \cos(\rho_j - \rho_{j'})^{-a} e^{\sum_{(j,j') \in \Lambda} (J_{j,j'}[1-\cos(\rho_j - \rho_{j'})]) e^{-k[\rho_m - \rho_\ell]}}.
\]

Using the bound \( 1 - x^2/2 \leq \cos(x) \),

\[
\left| \left\langle e^{ik(t_m-t_\ell)} \right\rangle_{\Lambda,a,J,\varepsilon_0} \right| \leq \prod_{(j,j')} \cos(\rho_j - \rho_{j'})^{-a} e^{\sum_{(j,j') \in \Lambda} (\beta(\rho_j - \rho_{j'})^2/2 e^{-k[\rho_m - \rho_\ell]}}.
\]

Since \( \cos(x) \geq 1 - x^2/2 \) and using the bound \( \frac{1}{1-u} \leq e^{bu} \) if \( b > 1 \) and \( 0 \leq u < \frac{b-1}{b} \), we have

\[
\left| \left\langle e^{ik(t_m-t_\ell)} \right\rangle_{\Lambda,a,J,\varepsilon_0} \right| \leq e^{\sum_{(j,j') \in \Lambda} [\beta(\rho_j - \rho_{j'})^2/2 e^{-k[\rho_m - \rho_\ell]}}.
\]

if we restrict attention to \( \rho' \)'s so that \( \|\nabla \rho\|_\infty < \sqrt{\frac{b-1}{b}} \).

We would like to optimize over \( \rho \), except that for \( k \) large one of these optimizers - approximately a solution to

\[-\Delta_\Lambda \cdot \rho = \frac{k}{\beta + a} [\delta_m - \delta_\ell]\]

- may not satisfy \( \cos(\nabla \rho) > 0 \) or \( \|\nabla \rho\|_\infty < \sqrt{\frac{b-1}{b}} \).

To get around this issue, we cut the expected optimizer off. Let \( c = \frac{1}{2} \sqrt{\frac{b-1}{b}} \) and let

\[B_x = \{j : \frac{|k|}{(\beta + ba)} < c\|x - j\|_2 \} \]

Given \( k \), define

\[\phi_j(k; x) = -\frac{k}{\beta + ba} \log(1 + \|x - j\|_2),\]

\[\psi_j(k; x) = \begin{cases} 
\phi_j(k; x) \text{ if } j \notin B_x, \\
-\frac{k}{\beta + ba} \log(1 + \frac{|k|}{c(\beta + ba)}) \text{ otherwise}, 
\end{cases}\]

\[\rho_j(k) = (\psi_j(k; m) - \psi_0(k; m) - (\psi_j(k; \ell) - \psi_0(k; \ell)).\]

Then

\[\|\nabla \rho\|_\infty \leq \sqrt{\frac{b-1}{b}}\]

and, for this choice of \( \rho \) we have

\[k[\rho_m - \rho_\ell] = \frac{2k^2}{\beta + ba} \left[ \log(1 + \|m - \ell\|_2) - \log \left( 1 + \frac{2|k|\sqrt{b}}{\sqrt{b - 1}(\beta + ba)} \right) \right],\]

\[\sum_{(j,j') \in \Lambda} [\beta + ba](\rho_j - \rho_{j'})^2/2 \leq \frac{1}{2} k[\rho_m - \rho_\ell],\]

so that

\[
\left| \left\langle e^{ik(t_m-t_\ell)} \right\rangle_{\Lambda,a,J,\varepsilon_0} \right| \leq \exp \left( -\frac{k^2}{\beta + ba} \left[ \log(1 + \|m - \ell\|_2) - \log \left( 1 + \frac{2|k|\sqrt{b}}{\sqrt{b - 1}(\beta + ba)} \right) \right] \right).
\]
for any \( b > 1 \). We take \( b = 2 \) to obtain one of the bounds claimed in the statement of the theorem.

To obtain the second bound, we simply scale the approximate optimizer, setting

\[
\rho_j = -\frac{k}{4|k|} \log(1 + \|m - j\|_2)/(1 + \|\ell - j\|_2) + \frac{k}{4|k|} \log(1 + \|m\|_2)/(1 + \|\ell\|_2).
\]

Then \( \|\nabla \rho\|_\infty \leq \frac{1}{2} \) and, reasoning as above to deal with the cosine terms, we have

\[
\left| \left( e^{ik(t_m - t_v)} \right)_{\Lambda, a, J, \epsilon, 0} \right| \leq \exp \left( -\frac{|k| - \beta - a}{2} \log(1 + \|m - \ell\|_2) \right).
\]

6 Bounds on Laplace transforms and the proof of Theorem 1.2

To begin, let us record an analogue of Lemma 2.1 for Laplace transforms. As discussed earlier, this bound appears in [Sab19] in the special case \( a = 1/2 \), but holds \textit{mutatis mutandis} for general \( a \), see [BCHS19] for a proof.

**Lemma 6.1.** Fix \( 0 < s < 1 \). Let \( \rho : \Lambda \rightarrow \mathbb{R} \) be given so that \( \rho_0 = 0 \) and \( \rho_v = 1 \). Choose \( q > 1 \) so that \( s + 1/q = 1 \) and choose \( \gamma > 0 \) so that \( q^2 \gamma \|\nabla \rho\|_\infty \leq 1/2 \). Then

\[
\langle e^{2a t_v} \rangle_{\Lambda, \beta, a, \epsilon, 0} \leq e^{-2a s \gamma} e^{\beta \sum_{j} q^2 \gamma^2 (\rho_j - \rho_v) Z_{\beta} (j, j) \prod (\beta_{j'})^{\psi_j}}
\]

where

\[
\beta_{j'} = \beta(1 - 2q^2 \gamma^2 (\rho_j - \rho_v)^2).
\]

Properly speaking, the proof of this lemma requires an \textit{a priori} identity to the effect that the \( 2a \)th moment of \( e^{t_v} \) is 1. The latter identity was already proved for the equivalent of one point pinning in [Sab19, BCHS19]. However, we feel it is worth pointing out that if \( a \in \mathbb{N} + 1/2 \) then the identity is \textit{algebraic} and therefore applies to any choice of pinning:

**Lemma 6.2.** For any finite weighted graph \((G, J)\) and any pinning \( \epsilon \),

\[
\langle e^{2a t_v} \rangle_{G, a, \epsilon} = 1.
\]

**Proof.** Since \( z_v + x_v = e^{t_v} \),

\[
\langle e^{2a t_v} \rangle_{J, \epsilon} = \sum_{j=0}^{2a} \binom{2a}{j} \langle z_v^{j} x_v^{2a-j} \rangle_{J, \epsilon} = \sum_{j=0, j \text{ odd}}^{2a} \binom{2a}{j} \langle z_v^{j} x_v^{2a-j} \rangle_{J, \epsilon}.
\]

The restriction to \( j \) odd on the RHS is due to the fact that \( \langle z_v^{j} x_v^{2a-j} \rangle_{J, \epsilon} = 0 \) if \( 2a - j \) is odd. Using \( a - j/2 \) distinct supersymmetry transformations,

\[
\langle z_v^{j} x_v^{2a-j} \rangle_{J, \epsilon} = \frac{\Gamma(2a - j + 1)}{2^{a-j/2} \Gamma(a - j/2 + 1)} \left\langle z_v^{\frac{a-j}{2}} \prod_{\ell=1}^{a-j/2} [-\xi_{\ell} \psi_{\ell}] \right\rangle_{J, \epsilon} = \frac{\Gamma(2a - j + 1)}{2^{a-j/2} \Gamma(a - j/2 + 1)} \left\langle \sigma_{\ell}^{f} \prod_{\ell=1}^{a-j/2} [-\bar{\psi}_{\ell} \psi_{\ell}] \right\rangle_{J, \epsilon}.
\]

Now we note that if we define

\[
\bar{\sigma}_{\ell} = 1 + 2 \sum_{\ell=a-j/2}^{a-1/2} \bar{\psi}_{\ell} \psi_{\ell} =: 1 - 2a_{\ell}
\]

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the nilpotency of the Grassmann variables implies
\[
\sigma_v^{a-j/2} \prod_{\ell=1}^{a-j/2} [-\bar{\psi}_v^\ell \psi_v^\ell] = \tilde{\sigma}_v(j)^{a-j/2} \prod_{\ell=1}^{a-j/2} [-\bar{\psi}_v^{\ell+(j-1)/2} \psi_v^{\ell+(j-1)/2}].
\]
Expanding \(\sigma_v(j)^{j/2}\), we have
\[
\sigma_v(j)^{j/2} = \sum_{k=0}^{(j-1)/2} \frac{(j/2) \cdots (j/2 - k)(-2)^k \alpha_j^k}{k!}
\]
By exchangeability, after we insert this expression,
\[
\langle e^{2a\sigma_v(v)} \rangle_{J,\varepsilon} = \sum_{L} d_L \left( \prod_{\ell=1}^{L} [-\bar{\psi}_v^\ell \psi_v^\ell] \right)^f_{J,\varepsilon},
\]
where
\[
d_L = 2^{-L} \sum_{k,j: j \text{ odd}, j \leq 2a, k \leq (j-1)/2, a-j/2+k=L} \frac{(2a)!}{j!(a-j/2)!} \frac{(j/2) \cdots (j/2 - k)(-1)^k 2^{2k}}{k!} (j-1)/2 \cdots [(j-1)/2 - k]
\]
\[
= 2^{-L} \sum_{k=0}^{L} \frac{(2a)! (-1)^k}{4(L-k)! k! (2a_0 - L)!} = 0.
\]
The claim now follows.

**Proof of Theorem 1.2.** Given \(\rho\) as in Lemma 6.1, as long as \(\tilde{\beta}_{jj'} \geq 1/2\), Theorem 1.3 implies
\[
\langle e^{s_2\sigma_v(v-t_0)} \rangle_{J,\beta,a,\varepsilon} \leq e^{-2as\gamma + (\beta+1)q^2\gamma^2E(\rho)}
\]
where
\[
E(\rho) = \sum_{(j,j')} (\rho_j - \rho_{j'})^2
\]
is the Dirichlet energy of \(\rho\).

Now choose \(\rho_j\) to be the function which is 0 at 0, 1 at \(v\) and discrete harmonic on \(\Lambda\setminus\{0,v\}\). As is well known, if \(\Lambda\) is large enough (for \(y\) fixed but large), we can find \(c_0\) so that
\[
\|\nabla \rho\|_{\infty} \leq \frac{c_0}{\log|v|}
\]
and the constants \(c_0\) and \(c_1\) are independent of \(\Lambda\) as \(\Lambda \uparrow \mathbb{Z}^2\), provided the surface-to-volume ratio \(\frac{|\partial \Lambda|}{|\Lambda|}\) tends to 0. To finish, given \(s\), we optimize over \(\gamma\) subject to the hypothesis of Lemma 6.1 is satisfied. Thus we set
\[
\gamma := \frac{as \log|v|}{c_1(\beta + 1)q^2}
\]
where \(c_1 = \max\left(\frac{2as}{(\beta+1)}, 1\right)\). With this choice \(q^2\gamma \|\nabla \rho\|_{\infty} \leq 1/2\) and
\[
-2as\gamma + (\beta+1)q^2\gamma^2E(\rho) \leq -as\gamma = -c_2 \log|v|.
\]
where \(c_2 = \frac{(as)^2}{c_1(\beta+1)q^2}\). The theorem thus holds.
A  SUSY for $a \in \mathbb{N} + 1/2$

In this appendix we sketch the development of the $\mathbb{H}^{2|2N}$ nonlinear $\sigma$-model from which the measure $d\mu_{G,a,J,\varepsilon}(t)$, defined at (1.4) is derived. Let us first introduce the $\mathbb{H}^{2|2N}$ $\sigma$-models. We consider the general case where the model is defined over a finite graph $G = (V,E)$.

For each $j \in V$ we introduce a supervector $u_j \in \mathbb{R}^{3|2N}$,

$$u_j = (z_j, x_j, y_j, \xi_j, \eta_j).$$

(A.1)

The variables $\xi_j, \eta_j$ are $N$-tuples of odd generators for a Grassmann algebra over $j$. Subscript indices denote locations in $G$ whereas superscript indices will denote internal components of the Grassmann vectors: $\xi_i^{(t)}$ is the $\ell$’th component of $\xi_i$. We then define a Minkowski signature bilinear form on $\mathbb{R}^{3|2N}$ by

$$(u,u') = -zz' + xx' + yy' + \xi \cdot \eta' - \eta \cdot \xi'$$

where

$$\xi \cdot \eta' = \sum_{\ell=1}^{N} \xi^{(t)} \eta'^{(t)}.$$  

(A.3)

To define the $\mathbb{H}^{2|2N}$ $\sigma$-model, the $u_j$’s are constrained to satisfy the quadratic equation

$$(u_j, u_j) = -1.$$  

(A.4)

Note that the solutions to this equation form a two-sheeted hyperboloid in $\mathbb{R}^{3|2N}$. If, for each $j$, we restrict attention to spins lying in the sheet with positive square root, we arrive at $\mathbb{H}^{2|2N}$. This is a super manifold. Geometrically, it is an infinitesimal extension by Grassmann variables of the $d = 2$ hyperbolic plane, parametrized by 2 real variables $x_j, y_j$ and $2N$ Grassman variables $\xi_j, \eta_j$.

On the product space $(\mathbb{H}^{2|2N})^{|A|}$ we introduce a ‘measure’ (more accurately, a Berezin superintegration form)

$$D\mu_a = \prod_{j \in V} dx dy \prod_{j \in V, \ell} \partial_{\xi_j} \partial_{\eta_j} (z_j)^{-1/2}.$$  

(A.5)

Note that $z_j^2 = 1 + x_j^2 + y_j^2 + 2\xi_j \cdot \eta_j$ - it is an element of the Grassmann algebra, not a real variable. The Gibbs state is then proportional to the Grassmann integration form $D\mu_a e^{-A_{J,\varepsilon}}$ with

$$A_{J,\varepsilon} = \frac{1}{2} \sum_{jj' \in E} J_{jj'} (u_j - u_{j'}, u_j - u_{j'}) + \sum_{i \in V} \varepsilon_i (z_i - 1)$$

(A.6)

The coupling constants $J_{jj'} > 0$ if $jj'$ are nearest neighbors in $G$ and $J_{jj'} = 0$ otherwise.

A.1. Horospherical Coordinates and $d\mu_{G,a,J,\varepsilon}(t)$. To connect the $\mathbb{H}^{2|2N}$ model to $d\mu_{G,a,J,\varepsilon}(t)$ with $a = N - 1/2$ we need to introduce a change of variables called horospherical (or Iwasawa) coordinates. The nex coordinates are $(t, s, \tilde{\theta}, \theta)$ defined by

$$x = \sinh t - e^t \left( \frac{1}{2} s + \sum_{i=1}^{N} \tilde{\theta}^i \theta^i \right), \
y = e^t s \text{ for } i \geq 2, \\xi^i = e^t \tilde{\theta}^i, \\eta^i = e^t \theta^i,$$

(A.7)

where $t \in \mathbb{R}$ and $s \in \mathbb{R}$ range over the real numbers. In the Poincaré disc and given a point $p$, $t$ represents the signed distance of the horocycle tangent to 1 from 0 and containing $p$ whereas $s$ represents the (normalized) location of $p$ on the horocycle). In these coordinates, the expression for the action becomes

$$A_{J,\varepsilon} = \sum_{ij} J_{ij} (S_{ij} - 1) + \sum_{k \in A} \varepsilon_k (z_k - 1),$$

(A.8)
where \((jj')\) are NN pairs and

\[
S_{jj'} = B_{jj'} + (\bar{\theta}_j - \bar{\theta}_{j'}) \cdot (\theta_j - \theta_{j'}) e^{t_j + t_{j'}} ,
\]

(A.9)

\[
B_{jj'} = \cosh(t_j - t_{j'}) + \frac{1}{2}(s_j - s'_{j'}) \cdot (s_j - s_{j'}) e^{t_j + t_{j'}} ,
\]

(A.10)

\[
z_j = \cosh t_j + (\frac{1}{2}s_j \cdot s_j + \bar{\theta}_j \cdot \theta_j) e^{t_j} .
\]

(A.11)

By applying Berezin’s transformation formula [BKNK87] for changing variables in a (super-)integral, one finds that

\[
D\mu_V = \prod_{j \in \Lambda} e^{-t_j(2N-1)} dt_j ds_j \prod_\ell \partial_{\bar{\theta}_j} \partial_{\theta_j}
\]

(A.12)

For any function \(f\) of the field variables \(\{t_j, s_j, \bar{\theta}_j, \theta_j\}_{j \in \Lambda}\) we now define its expectation as

\[
\langle f \rangle_{G,J,\epsilon} = \int D\mu_A e^{-A_{J,\epsilon}} f \int D\mu_V e^{-A_{J,\epsilon}},
\]

whenever the numerator integral exists.

There is no notational conflict with earlier definitions due to the following. Observe the beautiful feature that all coordinates, besides the \(t\) coordinate, are Gaussian in this representation. Thus if \(f\) depends on the \(t_j\)’s alone, we may easily integrate the other variables out. We find \((a = N - 1/2)\)

\[
\int D\mu_A e^{-A_{J,\epsilon}} = Z_{G,a,J,\epsilon} \langle f \rangle_{G,a,J,\epsilon} = \int \frac{f d\mu_A^{J,\epsilon}(t)}{Z_{G,a,J,\epsilon}} = \langle f \rangle_{G,a,J,\epsilon} .
\]

A.2. SUSY Localization. Section 2 highlighted the importance of controlling partition function ratios

\[
R_{J',J} := \frac{Z_{G,a,J',\epsilon}}{Z_{G,a,J,\epsilon}}
\]

for two choices of coupling constants \(J, J'\). For the \(\sigma\)-models taking values in \(\mathbb{H}^{2|2}\), this ratio is 1 by SUSY localization [DSZ10]. On \(\mathbb{H}^{2|2N}\) this is not true, but the localization argument is still extremely useful.

We now sketch the localization computation from [DSZ10] as it applies to the \(\mathbb{H}^{2|2N}\) models (the reader may consult that paper for the missing details). The computation is most easily explained by first considering the special case that \(G\) is just a single vertex. Let \(H\) be the quadratic polynomial

\[
H = x^2 + y^2 + 2\xi \cdot \eta.
\]

Let us isolate one pair of Grassmann components \((\xi^1, \eta^1)\). With respect to this pair, let \(q\) be the distinguished first-order differential operator defined by

\[
q = x\partial_{\eta^1(1)} - y\partial_{\xi^1(1)} + \xi^1\partial_x + \eta^1\partial_y .
\]

(A.14)

Note that \(q\) annihilates \(H\).

In this notation, the \textit{a priori} superintegration form is

\[
D\mu = dx dy \prod_\ell \partial_{\xi^1} \partial_{\eta^1} \circ (1 + H)^{-1/2} .
\]

**Lemma A.1.** The Berezin superintegration form \(D\mu\) is \(q\)-invariant, i.e.,

\[
\int D\mu \ q \cdot f = 0
\]

for any compactly supported smooth superfunction \(f\).

**Corollary A.2.** Suppose \(q \cdot f = 0\). Then for any \(\tau > 0\),

\[
\int D\mu \ f = \int D\mu e^{-\tau H} \ f.
\]
Every superfunction $f$ can be expanded over the Grassmann variables $(\xi^I, \eta^J)_{I \geq 2}$ as

$$ f = \sum_{I, J \subseteq \{2, \ldots, N\}} f_{I, J}(x, y, \xi^I, \eta^J) \xi^I \eta^J $$

where $\xi^I = \prod_{\ell \in I} x^{\ell}$ and similarly for $\eta$. Let $f_0$ be the superfunction obtained by setting $x = y = \xi^{(1)} = \eta^{(1)} = 0$,

$$ f_0 = \sum_{I, J \subseteq \{2, \ldots, N\}} f_{I, J}(0) \xi^I \eta^J, $$

so that the coefficient functions are evaluated as $f_{I, J}(0)$. Thus superfunction is a constant coefficient polynomial in $(\xi^I, \eta^J)_{I \geq 2}$. Let $n = N - 1$ and let $\bar{\psi}, \bar{\psi}$ denote the $n$-tuples defined by the last $n$ components of $\xi, \eta$, so $\bar{\psi}^I = \xi^{I + 1}$, $\bar{\psi}^I = \eta^{I + 1}$ for $I \in \{1, \ldots, n\}$ and let $\sigma = (1 + 2 \bar{\psi} \cdot \bar{\psi})^{1/2}$. The variables $\sigma, \bar{\psi}, \bar{\psi}$ live in a degenerate hyperboloid $\mathbb{H}^{0,2n} \subset \mathbb{R}^{1,2n}$.

$$ D\mu_0 = \prod_{\ell} \partial_{\psi^\ell} \partial_{\psi^\ell} \circ \sigma^{-1}. $$

**Lemma A.3** (Localization from $\mathbb{H}^{2,2N}$ to $\mathbb{H}^{0,2(N-1)}$). Let $f$ be a smooth superfunction which satisfies the invariance condition $Qf = 0$ and decreases sufficiently fast at infinity in order for the integral $\int D\mu f$ to exist. Then

$$ \int D\mu f = \int D\mu_0 f_0. $$

We now generalize this last lemma to $\mathbb{H}^{2,2N}$ for general finite graphs. Let $n_j = (\sigma_j, \bar{\psi}_j, \bar{\psi}_j')$ and set

$$ S_{J, \varepsilon} = \frac{1}{2} \sum_{(j, j') \in E} J_{j, j'} (n_j - n_j', n_j - n_j') + \varepsilon_i (\sigma_i - 1) \quad (A.15) $$

where the bilinear form is

$$ (n, n') = -\sigma \bar{\psi} \cdot \bar{\psi}' + \bar{\psi} \cdot \bar{\psi}. $$

Let $\langle \psi \rangle_{G, n, J, \varepsilon}^f$ denote the corresponding Gibbs state, in particular

$$ Z_{G, n, J, \varepsilon}^f := \int \prod_{j \in V} D\mu_0(\bar{\psi}_j, \bar{\psi}_j) e^{-S_{G, n, J, \varepsilon}}. $$

Let $q_V = \sum_{j \in V} q_j$

**Lemma A.4.** Let $N \in \mathbb{N}$, $\alpha = N + 1/2$, $n = N - 1$. Then the $\mathbb{H}^{2,2N}$ $\sigma$-model is equivalent to a purely fermionic $\mathbb{H}^{0,2n}$ $\sigma$-model in the sense that for any function $F(z, (\xi^I)_{I = 2}^N, (\eta^I)_{I = 2}^N)$ which decays sufficiently fast and such that $q_V \cdot F = 0$

$$ \langle F(z, (\xi^I)_{I = 2}^N, (\eta^I)_{I = 2}^N) \rangle_{G, n, J, \varepsilon}^f = \langle F(\sigma, 0, \bar{\psi}, \bar{\psi}) \rangle_{G, n, J, \varepsilon}^f. \quad (A.16) $$

In particular

$$ Z_{G, a, J, \varepsilon} = Z_{G, n, J, \varepsilon}^f. $$

**A.3. Residual SUSY After Localization; Connection with [CSS07].** The action $S_{J, \varepsilon}$ can be interpreted as a nonlinear $\sigma$-model with respect to the target space $\mathbb{R}^{1,2n}$, with even coordinate $\sigma$ and odd generators $(\bar{\psi}^I, \psi^I)_{I = 1}^n$. There are two natural quadratic forms we can put on this space:

$$ -\sigma^2 + 2 \bar{\psi} \cdot \psi \quad \text{Lorentzian}, \quad (A.17) $$

$$ \sigma^2 + 2 \bar{\psi} \cdot \psi \quad \text{Euclidean}. \quad (A.18) $$

The fermionic model on the RHS of (??) is a nonlinear $\sigma$-model on one sheet of the degenerate hyperboloid $\sigma^2 - 2 \bar{\psi} \cdot \psi = 1$ whereas in [CSS07] the spins are interpreted as taking values in the 'upper hemisphere'
of the degenerate sphere $\sigma^2 + 2\psi \cdot \psi = 1$ provides some intuition in [CSS07]. These two degenerate superspaces are the same via a change of fermionic coordinates. As such, the discussion of Section 7 in [CSS07] provides useful insight into our situation. Let us recapitulate and generalize that discussion for the sake of completeness.

We begin by introducing, at each vertex $i \in V$, a superfield $v_i := (\sigma_i, \psi_i, \bar{\psi}_i)$ consisting of a single bosonic variable $\sigma_i$ and $2n$ Grassmann variables $(\psi_i^\ell, \bar{\psi}_i^\ell)_{\ell=1}^n$. We equip $\mathbb{R}^{1|2n}$ with the Lorentzian scalar product

$$(v_i, v_j) := -\sigma_i \sigma_j + (\bar{\psi}_i \psi_j - \bar{\psi}_j \psi_i),$$

(A.19)

There are two types of symmetries which preserve this bilinear form. The first type is a symplectic linear transformation mapping the Grassmann variables into themselves and fixing the bosonic component. That is, if $M$ is an invertible $2n$-by-$2n$ matrix preserving the bilinear form induced by

$$J := \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

and if $u_i = (\sigma_i, M [\psi_i, \bar{\psi}_i])$, then $(u_i, u_j) = (v_i, v_j)$.

The second type of transformation is supersymmetric, mixing $\sigma$ with the $\psi, \bar{\psi}$'s. There are, in the general case, $n$ noncommuting SUSY transformations transformations, parametrized by fermionic (Grassmann-odd) global parameters $(\epsilon^\ell, \bar{\epsilon}^\ell)_{\ell=1}^n$:

$$\delta \sigma_i = (\epsilon^\ell \psi_i^\ell + \bar{\psi}_i^\ell \epsilon^\ell)$$

(A.20)

$$\delta \psi_i^\ell = \epsilon^\ell \sigma_i$$

(A.21)

$$\delta \bar{\psi}_i^\ell = \bar{\epsilon}^\ell \sigma_i$$

(A.22)

To check that these transformations leave eq. (A.19) invariant, we compute

$$\delta (v_i \cdot v_j) = (\delta \sigma_i) \sigma_j + \sigma_i (\delta \sigma_j) + [(\delta \bar{\psi}_i) \cdot \psi_j + \bar{\psi}_i \cdot (\delta \psi_j) - (\delta \psi_i) \cdot \bar{\psi}_j - \psi_i \cdot (\delta \bar{\psi}_j)]$$

(A.23)

$$= -(\epsilon^\ell \psi_i^\ell + \bar{\psi}_i^\ell \epsilon^\ell) \sigma_j - (\epsilon^\ell \psi_j^\ell + \bar{\psi}_j^\ell \epsilon^\ell) \sigma_i$$

$$+ [\epsilon^\ell \psi_j^\ell \sigma_i + \bar{\psi}_j^\ell \epsilon^\ell \sigma_j - \epsilon^\ell \psi_j^\ell \sigma_j - \psi_j^\ell \epsilon^\ell \sigma_j]$$

(A.24)

$$= 0.$$

(A.25)

Now let us consider a $\sigma$-model in which the superfields $v_i$ are constrained to lie on the upper sheet of the hyperboloid $\mathbb{R}^{1|2n}$, $\sigma_i^2 - 2\psi_i \cdot \bar{\psi}_i = 1$ This constraint is solved by writing

$$\sigma_i = \pm (1 + 2\bar{\psi}_i \cdot \psi_i)^{1/2},$$

so that $\sigma_i$ is an even invertible element of the Grassmann algebra. We take only the $+$ sign in (A.3) and denote the corresponding unit vector by $n_i$.

The $\mathfrak{sp}(2n)$ transformations continue to act as in (??) while the SUSY transformations act via

$$\delta \psi_i^\ell = \epsilon^\ell \sigma_i$$

$$\delta \bar{\psi}_i^\ell = \bar{\epsilon}^\ell \sigma_i$$

These transformations leave invariant the scalar product $n_i \cdot n_j$, and the corresponding generators $Q^\ell_\pm$ are defined as

$$Q^\ell_+ = \sum_{i \in V} \sigma_i \delta_i^\ell$$

$$Q^\ell_- = \sum_{i \in V} \sigma_i \bar{\delta}_i$$

where $\delta_i^\ell = \partial_{\psi_i^\ell}$ and $\bar{\delta}_i^\ell = \partial_{\bar{\psi}_i^\ell}$

$$Q^\ell_+ \psi_i^\ell = Q^\ell_- \bar{\psi}_i^\ell = \sigma_i, \quad Q^\ell_+ [n_i, n_j] = 0.$$

(A.26)
From this identity, $Q^\pm_\pm S J,\epsilon = Q^\pm_\pm e^{-S J,\epsilon} = 0$. Note also that \textit{a priori} integration form $\mathcal{D}_0(\bar{\psi},\psi)$ is invariant with respect to the $Q^\pm_\pm$'s:

$$\int \mathcal{D}_0(\bar{\psi},\psi) Q^\pm_\pm F(\psi,\bar{\psi}) = 0.$$ 

From these facts, (4.7) follows immediately.

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