CURVED $\mathcal{O}$-OPERATOR SYSTEMS

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Abstract. We introduce the notion of curved $\mathcal{O}$-operator systems as a generalization of T. Brzeziński’s (curved) Rota-Baxter systems, and then investigate their relations with (tri)dendriform systems and pre-Lie algebras. Moreover, we consider the special cases of generalized Rota-Baxter algebras with respect to associative compatible pairs and then double curved Rota-Baxter systems.

1. Introduction

Rota-Baxter algebras were introduced in [14] in the context of differential operators on commutative Banach algebras and since [3] intensively studied in probability and combinatorics, and more recently in mathematical physics. One can refer to the book [8] for the detailed theory of Rota-Baxter algebras. Dendriform algebras were introduced by Loday in [10]. The motivation to introduce these algebraic structures with two generating operations comes from K-theory. It turned out later that they are connected to several areas in mathematics and physics, including Hopf algebras, homotopy Gerstenhaber algebra, operads, homology, combinatorics and quantum field theory where they occur in the theory of renormalization of Connes and Kreimer. Later the notion of tridendriform algebra were introduced by Loday and Ronco in their study of polytopes and Koszul duality, see [11]. The relations between Rota-Baxter algebras and (tri)dendriform algebras were discussed by Ebrahimi-Fard in [7]. Bai, Guo and Ni introduced the extended $\mathcal{O}$-operator generalizing the concept of $\mathcal{O}$-operators and studied the relations with the associative Yang-Baxter equations ([2]).

In 2016, in an attempt to develop and extend aforementioned connections between Rota-Baxter algebras, dendriform algebras and infinitesimal bialgebras, T. Brzeziński introduced the notion of a Rota-Baxter system in [4]. In particular it has been shown that to any Rota-Baxter system one can associate a dendriform algebra and, in fact, any dendriform algebra of a particular kind arises from a Rota-Baxter system. In [5], T. Brzeziński presented the curved version of Rota-Baxter system and investigated the relations with weak pseudotwistors, differential graded algebras and pre-Lie algebras. But they never considered the relations between curved Rota-Baxter systems and (tri)dendriform algebras. Curved Rota-Baxter systems generalize Rota-Baxter operators and algebras at least in a threefold way. First, when the curvature vanishes, the triple $(A, R, S)$ is a Rota-Baxter system and hence the choice of $S$ to be $R + \lambda id$ with $\lambda \in K$ to be $S + \lambda id$...
makes it into a Rota-Baxter algebra of weight $\lambda$. On the other hand, a Rota-Baxter algebra of weight $\lambda$ is obtained from $(A, R, S, \omega)$ by setting $R = S$ and $\omega(a \otimes b) = \lambda(ab)$, for all $a, b \in A$.

In this paper, we introduce the notion of curved $\mathcal{O}$-operator system, which generalizes both T. Brzeziński’s Rota-Baxter system in [4] and C. M. Bai, L. Guo and X. Ni’ $\mathcal{O}$-operator in [2], and study its properties. In Section 1, we recall some basic definitions. In Section 2, we introduce the notions of curved $\mathcal{O}$-operator system and provide their connections with associative algebras, (tri)dendriform systems and pre-Lie algebras. Section 3 is devoted to a class of special curved $\mathcal{O}$-operator systems, mainly generalized Rota-Baxter algebras with respect to associative compatible pairs and double curved Rota-Baxter systems.

Throughout this paper, $K$ will be a field, and all vector spaces, tensor products, and homomorphisms are over $K$. And we denote $\text{id}_M$ for the identity map from $M$ to $M$. Unless otherwise specified, algebras refers to associative algebras.

To begin with, we recall some useful definitions in [2, 4, 13].

**Definition 1.1.** A Rota-Baxter algebra of weight $\lambda$ is an algebra $A$ together with a linear map $R : A \rightarrow A$ such that
\[
R(a)R(b) = R(aR(b)) + R(R(a)b) + \lambda ab
\]
for all $a, b \in A$ and $\lambda \in K$ and we denote it by $(A, R)$. Such a linear operator is called a Rota-Baxter operator of weight $\lambda$ on $A$.

**Definition 1.2.** Let $A$ be an algebra and $V$ a vector space. Let $\triangleright : A \otimes V \rightarrow V$ and $\triangleleft : V \otimes A \rightarrow V$ be two linear maps. Then $(V, \triangleright, \triangleleft)$ is called an $A$-bimodule if
\[
a \triangleright (b \triangleright x) = (ab) \triangleright x, \quad (x \triangleright a) \triangleleft b = x \triangleleft (ab), \quad (a \triangleright x) \triangleleft b = a \triangleright (x \triangleleft b),
\]
where $a, b \in A$ and $x \in V$.

An $A$-bimodule map $f$ from $(V, \triangleright_V, \triangleleft_V)$ to $(W, \triangleright_W, \triangleleft_W)$ is a linear map $f : V \rightarrow W$ such that $f$ is a left $A$-module map from $(V, \triangleright_V)$ to $(W, \triangleright_W)$ and at the same time $f$ is a right $A$-module map $(V, \triangleleft_V)$ to $(W, \triangleleft_W)$.

**Definition 1.3.** A pre-Lie algebra (or left symmetric algebra) $A$ is a linear space with a binary operation $(x, y) \mapsto xy$ satisfying
\[
(xy)z - x(yz) = (yx)z - y(xz), \forall x, y, z \in A.
\]

**2. Curved $\mathcal{O}$-operator systems**

In this section, we introduce the notion of curved $\mathcal{O}$-operator system, which generalizes both T. Brzeziński’s Rota-Baxter system in [4] and C. M. Bai, L. Guo and X. Ni’ $\mathcal{O}$-operator in [2].

**Definition 2.1.** A system $(A, V, R, S, \omega)$ consisting of an algebra $A$, an $A$-bimodule $(V, \triangleright, \triangleleft)$ and three linear maps $R, S : V \rightarrow A$, $\omega : V \otimes V \rightarrow V$ is called a curved $\mathcal{O}$-operator system associated to $(V, \triangleright, \triangleleft)$ if, for all $x, y \in V$,
\[
R(x)R(y) = R(R(x) \triangleright y + x \triangleleft S(y) + \omega(x \otimes y)) = R(R(x) \triangleright y + x \triangleleft S(y)) + \omega_1(x \otimes y)
\]
and
\[ S(x)S(y) = S(R(x) \triangleright y + x \triangleleft S(y)) + \omega_1(x \otimes y) = S(R(x) \triangleright y + x \triangleleft S(y)) + \omega_2(x \otimes y), \tag{2.2} \]
where \( \omega_1 = R \circ \omega \) and \( \omega_2 = S \circ \omega \).

A morphism of curved \( \varnothing \)-operator system from \((A, V, R, S, \omega)\) to \((B, W, P, T, \nu)\) is a pair \((f, g)\), where \( f : A \rightarrow B \) is an algebra map and \( g : V \rightarrow W \) is an \( A \)-bimodule map such that \( f \circ R = P \circ g, f \circ T = T \circ g, f \circ S = P \circ g, f \circ S = T \circ g \) and \( g \circ \omega = \nu \circ (g \otimes g) \).

**Example 2.2.** (1) When \( \omega(x \otimes y) = 0 \), then curved \( \varnothing \)-operator system \((A, V, R, S, 0)\) is exactly the \( \varnothing \)-operator system \((A, V, R, S)\) associated to \((V, \triangleright, \triangleleft)\) in [12].

(2) Curved \( \varnothing \)-operator system \((A, A, R, S, \omega)\) associated to \((A, L, R)\) is exactly curved Rota-Baxter system \((A, R, S, \omega)\) in [3], where \( L \) and \( R \) denote the left and right multiplication, respectively.

(3) When \( R = S \) and \( \omega(x \otimes y) = \lambda xy \) in Example 2.2 (2), then we can get Rota-Baxter operator of weight \( \lambda \) in [3] [14].

(4) If \((V, \triangleright, \triangleleft)\) is an \( A \)-bimodule algebra and \( R = S \), then curved \( \varnothing \)-operator system \((A, V, R, R, \lambda \omega)\) is the \( \varnothing \)-operator of weight \( \lambda \) associated to \((V, \triangleright, \triangleleft)\) (see [4] Definition 2.7).

(5) Let \((V, \triangleright, \triangleleft) = (A, L, R)\), \( R = S \) and \( \omega_1 = \omega_2 = - \bullet \circ (R \otimes R) \), where \( \bullet \) denotes the multiplication of \( A \). Then curved \( \varnothing \)-operator system \((A, A, R, R, \omega)\) is so-called Reynolds algebra in [15].

(6) Let \( A \) be an algebra with unit \( 1_A \), \( R = S \) and \( \omega(a \otimes b) = -aR(1_A)b \). Then curved \( \varnothing \)-operator system \((A, A, R, R, \omega)\) is the TD-algebra in [9].

(7) Let \((V, \triangleright, \triangleleft) = (A, L, R)\), \( R = S \) and \( \omega(x \otimes y) = -R(x \bullet y) \). Then \( R \) is a Nijenhuis operator in [6].

Next, we give some concrete examples.

**Example 2.3.** Let \( A \) be the 2-dimensional associative algebra where the multiplication is defined, with respect to a basis \( \{e_1, e_2\} \), by
\[
e_1 \cdot e_1 = e_1, \quad e_1 \cdot e_2 = e_2, \quad e_2 \cdot e_1 = e_2, \quad e_2 \cdot e_2 = e_2.
\]
Consider the \( A \)-bimodule structure on a 1-dimensional vector space \( V \) generated by \( v_1 \), where the right and left actions are defined as
\[
e_1 \triangleright v_1 = v_1, \quad e_2 \triangleright v_1 = 0, \quad v_1 \triangleleft e_1 = 0, \quad v_1 \triangleleft e_2 = 0.
\]

The corresponding curved \( \varnothing \)-operator systems which are not \( \varnothing \)-operator systems are defined as
\[
\begin{align*}
\bullet \quad & R(v_1) = 0, & S(v_1) = pe_1, & w(v_1, v_1) = pv_1, \\
\bullet \quad & R(v_1) = 0, & S(v_1) = pe_2, & w(v_1, v_1) = pv_1, \\
\bullet \quad & R(v_1) = 0, & S(v_1) = pe_1 - e_2, & w(v_1, v_1) = pv_1, \\
\bullet \quad & R(v_1) = pe_2, & S(v_1) = 0, & w(v_1, v_1) = pv_1,
\end{align*}
\]
\[
\begin{align*}
R(v_1) &= pe_2, \quad S(v_1) = pe_2, \quad w(v_1, v_1) = pv_1, \\
R(v_1) &= pe_2, \quad S(v_1) = pe_1, \quad w(v_1, v_1) = pv_1, \\
R(v_1) &= pe_2, \quad S(v_1) = p(e_1 - e_2), \quad w(v_1, v_1) = pv_1,
\end{align*}
\]
where \( p \) is a parameter. Moreover, we have the following \( \mathcal{O} \)-operator systems \((w(e_1, e_1) = 0)\).

\[\begin{align*}
R(v_1) &= pe_1, \quad S(v_1) = 0, \\
R(v_1) &= pe_1, \quad S(v_1) = pe_2, \\
R(v_1) &= pe_1, \quad S(v_1) = pe_1, \\
R(v_1) &= pe_1, \quad S(v_1) = p(e_1 - e_2), \\
R(v_1) &= p(e_1 - e_2), \quad S(v_1) = 0, \\
R(v_1) &= p(e_1 - e_2), \quad S(v_1) = pe_2, \\
R(v_1) &= p(e_1 - e_2), \quad S(v_1) = pe_1, \\
R(v_1) &= p(e_1 - e_2), \quad S(v_1) = p(e_1 - e_2).
\end{align*}\]

Now, consider the \( A \)-bimodule structure on the same vector space and where the right and left actions are defined as

\[e_1 \triangleright v_1 = v_1, \quad e_2 \triangleright v_1 = v_1, \quad v_1 \triangleleft e_1 = v_1, \quad v_1 \triangleleft e_2 = v_1.\]

The corresponding curved \( \mathcal{O} \)-operator systems which are not \( \mathcal{O} \)-operator systems are defined as

\[\begin{align*}
R(v_1) &= 0, \quad S(v_1) = p(e_1 - e_2), \quad w(v_1, v_1) = pv_1, \\
R(v_1) &= pe_2, \quad S(v_1) = pe_2, \quad w(v_1, v_1) = -pv_1, \\
R(v_1) &= pe_2, \quad S(v_1) = pe_1, \quad w(v_1, v_1) = -pv_1, \\
R(v_1) &= pe_2, \quad S(v_1) = pe_2, \quad w(v_1, v_1) = -pv_1, \\
R(v_1) &= pe_1, \quad S(v_1) = pe_1, \quad w(v_1, v_1) = -pv_1, \\
R(v_1) &= p(e_1 - e_2), \quad S(v_1) = 0, \quad w(v_1, v_1) = pv_1, \\
R(v_1) &= p(e_1 - e_2), \quad S(v_1) = p(e_1 - e_2), \quad w(v_1, v_1) = pv_1,
\end{align*}\]
where \( p \) is a parameter.

Finally, we consider the \( A \)-bimodule structure on the 2-dimensional vector space generated by \( \{v_1, v_2\} \) and where the right and left actions are defined as

\[\begin{align*}
e_1 \triangleright v_1 = v_1, \quad e_1 \triangleright v_2 = v_2, \quad e_2 \triangleright v_1 = v_1, \quad e_2 \triangleright v_2 = v_2, \\
v_1 \triangleleft e_1 = v_1, \quad v_2 \triangleleft e_1 = v_2, \quad v_1 \triangleleft e_2 = v_1, \quad v_2 \triangleleft e_2 = v_2.
\end{align*}\]

We have a curved \( \mathcal{O} \)-operator system defined by

\[\begin{align*}
R(v_1) &= 0, \quad R(v_2) = p_1 e_1, \quad S(v_1) = p_2(e_1 - e_2), \quad S(v_2) = p_1 e_2, \\
w(v_1, v_1) &= p_2 v_1, \quad w(v_1, v_2) = -p_1 v_1, \quad w(v_2, v_1) = -p_1 v_1, \quad w(v_2, v_2) = -p_1 v_2,
\end{align*}\]
where \( p_1 \) and \( p_2 \) are parameters.
Definition 2.4. Let $A$ and $(V, \circ)$ be two algebras and $\triangleright : A \otimes V \to V$, $\triangleleft : V \otimes A \to V$, $R, S : V \to A$ be four linear maps. We call $(V, \circ, \triangleright, \triangleleft)$ an extended $A$-bimodule algebra if $(V, \triangleright, \triangleleft)$ is an $A$-module and the following conditions hold:

$$R(x) \triangleright (y \circ z) = (R(x) \triangleright y) \circ z, \quad (x \circ y) \triangleleft S(z) = x \circ (y \triangleleft S(z)), \quad x \circ (R(y) \triangleright z) = (x \triangleleft S(y)) \circ z,$$

(2.3)

where $x, y, z \in V$.

Remark 2.5. (1) When $R = S$, then extended $A$-bimodule algebra $(V, \circ, \triangleright, \triangleleft)$ turns to $A$-bimodule algebra introduced in [2] which can be seen as a special case of matched pair of two algebras in [1].

(2) If $(V, \circ, \triangleright, \triangleleft)$ is an $A$-bimodule algebra, then $(V, \circ, \triangleright, \triangleleft)$ is an extended $A$-bimodule algebra if and only if

$$x \circ ((R - S)(y) \triangleright z) = 0 \text{ or } (x \triangleleft (R - S)(y)) \circ z = 0,$$

(2.4)

where $x, y, z \in V$.

The notion of dendriform algebra was introduced by J. L. Loday [10]. In [12], we extend this notion to dendriform system as follows:

Definition 2.6. A dendriform system $(A, V, \prec, \succ, \preceq, \succeq)$ consists of two linear space $A, V$ and four linear maps $\prec : A \otimes V \to A$, $\succ : V \otimes A \to A$, $\preceq : V \otimes V \to V$ and $\succeq : V \otimes V \to V$ satisfying the following conditions:

$$\begin{align*}
(a \prec x) \prec y &= a \prec (x \succ y + x \preceq y), \quad (2.5) \\
x \succ (a \preceq y) &= (x \succ a) \preceq y, \quad (2.6) \\
x \succ (y \succeq a) &= (x \succ y + x \preceq y) \succeq a, \quad (2.7) \\
(x \preceq y) \preceq z &= x \preceq (y \succeq z + y \preceq z), \quad (2.8) \\
x \succeq (y \preceq z) &= (x \succeq y) \preceq z, \quad (2.9) \\
x \succeq (y \succeq z) &= (x \succeq y + x \preceq y) \succeq z, \quad (2.10)
\end{align*}$$

where $a \in A$ and $x, y, z \in V$.

Theorem 2.7. Let $A$ be an algebra, $(V, \circ, \triangleright, \triangleleft)$ be an extended $A$-bimodule algebra, $R, S : V \to A$ two linear maps. Define linear maps $\succ : V \otimes A \to A$, $\prec : A \otimes V \to A$ and $\triangleright, \triangleleft : V \otimes V \to V$ as follows:

$$x \triangleright a = R(x)a, \quad a \preceq x = aS(x), \quad x \triangleright y = R(x) \triangleright y, \quad x \triangleleft y = x \triangleleft S(y) + x \circ y,$$

(2.11)

where $x, y \in V$ and $a \in A$. If $(A, V, R, S, \circ)$ is a curved $O$-operator system associated to $(V, \triangleright, \triangleleft)$, then $(A, V, \prec, \succ, \preceq, \succeq)$ is a dendriform system.
Proof. For all \( a \in A \) and \( x, y, z \in V \), we can check Eqs. (2.5)-(2.10) as follows:

\[
(a \prec x) \prec y = (aS(x))S(y) = a(S(x)S(y))
\]

(2.5)

\[
= aS(R(x) \triangleright y + x \triangleleft S(y) + x \circ y)
\]

(2.6)

\[
= a \prec (R(x) \triangleright y + x \triangleleft S(y) + x \circ y) = a \prec (x \triangleright y + x \triangleleft y),
\]

so Eq. (2.5) holds. Similarly by Eq. (2.1) and the associativity of \( A \), one can prove Eqs. (2.6) and (2.7).

\[
(x \prec y) \prec z = (x \triangleleft S(y)) \triangleleft S(z) + (x \circ y) \triangleleft S(z) + (x \triangleleft S(y)) \circ z + (x \circ y) \circ z
\]

(2.8)

\[
x \triangleleft (S(y)S(z)) + x \circ (y \triangleleft S(z)) + x \circ (R(y) \triangleright z) + x \circ (y \circ z)
\]

(2.9)

\[
= x \prec (R(y) \triangleright z + y \triangleleft S(z) + y \circ z) + x \circ (R(y) \triangleright z + y \triangleleft S(z) + y \circ z)
\]

(2.10)

\[
= x \prec (R(y) \triangleright z + y \triangleleft S(z) + y \circ z)
\]

(2.11)

\[
= x \prec (y \triangleright z + y \triangleleft z),
\]

thus Eq. (2.8) is satisfied. And Eqs. (2.9) and (2.10) hold by Eqs. (1.2) and (2.1). \(\square\)

Remark 2.8. By the proof of Theorem 2.7, we have: Let \( A \) be an algebra, \((V, \circ, \triangleright, \triangleleft)\) be an \( A \)-bimodule algebra, \((A, V, R, S, \circ)\) a curved \( O \)-operator system associated to \((V, \triangleright, \triangleleft)\). Define linear maps \( \triangleright: V \otimes A \rightarrow A \), \( \triangleleft: A \otimes V \rightarrow A \), \( \triangleright: V \otimes V \rightarrow V \) and \( \triangleleft: V \otimes V \rightarrow V \) by Eq. (2.11). Then \((A, V, \triangleleft, \triangleright, \prec, \succ)\) is a dendriform system if and only if Eq. (2.4) holds.

Corollary 2.9. Let \( A \) be an algebra, \((V, \circ, \triangleright, \triangleleft)\) an \( A \)-bimodule algebra, \( R: V \rightarrow A \) a linear map and \( \lambda \in K \). Define linear maps \( \triangleright: V \otimes A \rightarrow A \), \( \triangleleft: A \otimes V \rightarrow A \), \( \triangleright: V \otimes V \rightarrow V \) and \( \triangleleft: V \otimes V \rightarrow V \) as follows:

\[
x \triangleright a = R(x)a, \ a \triangleleft x = aR(x), \ x \triangleright y = R(x) \triangleright y, \ x \triangleleft y = x \triangleleft R(y) + \lambda x \circ y,
\]

where \( x, y \in V \) and \( a \in A \). If \( R \) is an \( O \)-operator of weight \( \lambda \) associated to \((V, \triangleright, \triangleleft)\), then \((A, V, \triangleleft, \triangleright)\) is a dendriform system.

Proof. By Theorem 2.7 and Example 2.2 (4), we can finish the proof. \(\square\)

The following notion is a generalization of tridendriform algebra introduced in (12).

Definition 2.10. A **tridendriform system** is a seven tuples \((A, V, \triangleleft, \triangleright, \triangleleft, \triangleright, \cdot)\) consisting of two linear spaces \( A \) and \( V \) and five linear maps \( \triangleleft: A \otimes V \rightarrow A \), \( \triangleright: V \otimes A \rightarrow A \) and \( \triangleleft, \triangleright, \cdot: V \otimes V \rightarrow V \) satisfy the following conditions:

\[
(a \triangleleft x) \triangleleft y = a \triangleleft (x \triangleleft y + x \triangleright y + x \cdot y),
\]

(2.12)

\[
x \triangleright (a \triangleleft y) = (x \triangleright a) \triangleleft y,
\]

(2.13)

\[
x \triangleright (y \triangleright a) = (x \triangleright y + x \triangleright y + x \cdot y) \triangleright a,
\]

(2.14)

\[
(a \triangleleft y) \triangleleft z = x \triangleleft (y \triangleleft z + y \triangleright z + y \cdot z),
\]

(2.15)
Remark 2.12. Under the assumption of Remark 2.8, define linear maps
\[ x \succ (y \ll z) = (x \succ y) \ll z, \]
\[ x \succ (y \gg z) = (x \ll y + x \gg y + x \cdot y) \gg z, \]
\[ (x \ll y) \cdot z = x \cdot (y \gg z), \]
\[ (x \gg y) \cdot z = x \gg (y \cdot z), \]
\[ (x \cdot y) \ll z = x \cdot (y \ll z), \]
\[ (x \cdot y) \gg z = x \cdot (y \gg z), \]
where \( a \in A \) and \( x, y, z \in V \).

Theorem 2.11. Let \( A \) be an algebra, \((V, \circ, \triangleright, \ll)\) be an extended \( A \)-bimodule algebra, \( R, S : V \rightarrow A \) two linear maps. Define linear maps \( \triangleright : V \otimes A \rightarrow A, \ll : A \otimes V \rightarrow A \) and \( \gg, \ll, \cdot : V \otimes V \rightarrow V \) as follows:
\[ x \triangleright a = R(x)a, \ a \ll x = aS(x), \ x \gg y = R(x)\gg y, \ x \ll y = x \ll S(y), \ x \cdot y = x \circ y, \]
where \( x, y \in V \) and \( a \in A \). If \((A, V, R, S, \circ)\) is a curved \( O \)-operator system associated to \((V, \triangleright, \ll)\), then \((A, V, \ll, \gg, \ll, \cdot)\) is a tridendriform system.

Proof. For all \( a \in A \) and \( x, y \in V \), we only check Eq. (2.12) as follows:
\[ (a \ll x) \ll y = (aS(x))S(y) = a(S(x)S(y)) \]
\[ = aS(R(x)\gg y + x \ll S(y) + x \circ y) \]
\[ = a \ll (R(x)\gg y + x \ll S(y) + x \cdot y) \]
\[ = a \ll (x \gg y + x \ll y + x \cdot y). \]
One can prove that Eqs. (2.13)–(2.19) hold similarly. \( \square \)

Remark 2.12. Under the assumption of Remark 2.8, define linear maps \( \triangleright : V \otimes A \rightarrow A, \ll : A \otimes V \rightarrow A \) and \( \gg, \ll, \cdot : V \otimes V \rightarrow V \) by Eq. (2.20). Then \((A, V, \ll, \gg, \ll, \cdot)\) is a tridendriform system if and only if Eq. (2.21) holds.

Corollary 2.13. Under the assumption of Corollary 2.9, define linear maps \( \triangleright : V \otimes A \rightarrow A, \ll : A \otimes V \rightarrow A \) and \( \gg, \ll, \cdot : V \otimes V \rightarrow V \) as follows:
\[ x \triangleright a = R(x)a, \ a \ll x = aR(x), \ x \gg y = R(x)\gg y, \ x \ll y = x \ll R(y), \ x \cdot y = \lambda x \circ y, \]
where \( x, y \in V \) and \( a \in A \). If \( R \) is an \( O \)-operator of weight \( \lambda \) associated to \((V, \circ, \triangleright, \ll)\), then \((A, V, \ll, \gg, \ll, \cdot)\) is a tridendriform system.

Proof. By Theorem 2.11 and Example 2.2 (4), we can finish the proof. \( \square \)

Theorem 2.14. Let \( A \) be an algebra, \( R, S : V \rightarrow A \) two linear maps, \((V, \circ, \triangleright, \ll)\) be an extended \( A \)-bimodule algebra and \( \ast : V \otimes V \rightarrow V \) be a linear map defined for \( x, y \in V \) by
\[ x \ast y = R(x)\gg y + x \ll S(y) + x \circ y. \]
Then \((V, \ast)\) is an algebra if and only if
\[ (R(x)R(y) - R(x \ast y))\gg z = x \ll (S(y)S(z) - S(y \ast z)). \]
Proof. For all $x, y, z \in V$, we have

\[
(x \ast y) \ast z = R(x \ast y) \triangleright z + (R(x) \triangleright y) \triangleleft S(z) + (x \triangleleft S(y)) \triangleright S(z) + (x \circ y) \triangleleft S(z)
\]

\[+ (R(x) \triangleright y) \circ z + (x \triangleleft S(y)) \circ z + (x \circ y) \circ z\]

\[= R(x) \triangleright z + R(x) \triangleright (y \triangleleft S(z)) + x \triangleleft (S(y)S(z)) + x \circ (y \triangleleft S(z))\]

\[+ R(x) \triangleright (y \circ z) + x \circ (R(y) \triangleright z) + x \circ (y \circ z)\]

\[\quad \text{and}\]

\[
x \ast (y \ast z) = R(x)R(y) \triangleright z + R(x) \triangleright (y \triangleleft S(z)) + R(x) \triangleright (y \circ z) + x \triangleleft (S(y \ast z)) + x \circ (y \circ z).
\]

Therefore, we finish the proof. \qed

Corollary 2.15. Let $A$ be an algebra, $(V, \circ, \triangleright, \triangleleft)$ be an extended $A$-bimodule algebra. If $(A, V, R, S, \circ)$ is a curved $\mathcal{O}$-operator system associated to $(V, \triangleright, \triangleleft)$, then $(V, \ast)$ is an algebra, where $\ast$ is given by Eq.(2.21).

Proof. By Eqs.(2.1) and (2.2), we can get $R(x)R(y) = R(x \ast y)$ and $S(y)S(z) = S(y \circ z)$. Thus Eq.(2.22) holds. \qed

Remark 2.16. By Theorem 2.7 and [12, Proposition 3.7], we also obtain the conclusion in Corollary 2.15.

Proposition 2.17. Let $A$ be an algebra, $(V, \circ, \triangleright, \triangleleft)$ be an $A$-bimodule algebra, $R : V \to A$ a linear map. Then the operation given by

\[
x \ast_R y = R(x) \triangleright y + x \triangleleft R(y) + x \circ y
\]

is associative if and only if

\[
(R(x)R(y) - R(x \ast_R y)) \triangleright z = x \triangleleft (R(y)R(z) - R(y \ast_R z)),
\]

where $x, y, z \in V$.

Proof. Let $R = S$ in Theorem 2.14. \qed

Remark 2.18. Proposition 2.17 is exactly [2, Lemma 2.12] by replacing $x \circ y$ by $\lambda x \circ y$.

Definition 2.19. Let $Char K \neq 2$.

\[
\alpha = \frac{R + S}{2} \quad \text{and} \quad \beta = \frac{R - S}{2}
\]

is called the symmetrizer and antisymmetrizer of $R$ and $S$, respectively (see [Sec. 2.3][2]).

Then we have $R = \alpha + \beta$ and $S = \alpha - \beta$. 
Corollary 2.20. Let $A$ be an algebra, $(V, \circ, \triangledown, \triangleright)$ be an $A$-bimodule algebra, $R, S : V \rightarrow A$ two linear maps and $\alpha, \beta$ their symmetrizer and antisymmetrizer defined by Eq. (2.25). If $\beta$ is a balanced linear map of mass $k = 1$ (see [2, Definition 2.9]), namely
\[
\beta(x) \triangledown y = x \triangleleft \beta(y), \ \forall \ x, y \in V,
\] (2.26)
then $(V, \star)$, where $\star$ is given by Eq. (2.21), is an algebra if and only if $\alpha$ satisfies Eq. (2.24).

Proof. Since $\beta$ is balanced, for all $x, y \in V$,
\[
x \star y = R(x) \triangledown y + x \triangleleft S(y) + x \circ R(y) = \alpha(x) \triangledown y + x \triangleleft \alpha(y) + x \circ y = x \star_\alpha y.
\]
Then the conclusion follows from Proposition 2.17.

Proposition 2.21. Let $(A, V, R, S, \omega)$ be a curved $\mathcal{O}$-operator system associated to $(V, \triangledown, \triangleright)$. For all $x, y \in V$, define
\[
x \circ y = R(x) \triangledown y + x \triangleleft S(y).
\] (2.27)
Then $(V, \circ)$ is an algebra if and only if
\[
\omega_1(x \otimes y) \triangledown z = x \triangleleft \omega_2(y \otimes z).
\] (2.28)
Proof. By [12, Theorem 2.14] and Eqs. (2.1) and (2.2), we can get the conclusion.

Remark 2.22. (1) Eq. (2.28) is the form of Eq. (2.22) for curved $\mathcal{O}$-operator system.
(2) Let $\omega = 0$ in Proposition 2.21, we can get [12, Corollary 3.9].

Corollary 2.23. Let $A$ be an algebra, $(V, \circ, \triangledown, \triangleright)$ be an $A$-bimodule algebra and $R : V \rightarrow A$ an $\mathcal{O}$-operator of weight $\lambda$ associated to $(V, \circ, \triangledown, \triangleright)$. Then the operation given by
\[
x \circ y = R(x) \triangledown y + x \triangleleft R(y).
\] (2.29)
is associative if and only if
\[
\lambda R(x \circ y) \triangledown z = \lambda x \triangleleft R(y \circ z),
\] (2.30)
where $x, y, z \in V$.
Proof. Let $R = S$ and $\omega_i(x \otimes y) = \lambda R(x \circ y), i = 1, 2$ in Proposition 2.21.

Remark 2.24. Eq. (2.30) is exactly one of the two conditions in the Definition of balanced homomorphism (see [2, Eq. (21)]).

Proposition 2.25. Let $A$ be an algebra and $(V, \circ, \triangledown, \triangleright)$ be an extended $A$-bimodule algebra. If $(A, V, R, S, \circ)$ is a curved $\mathcal{O}$-operator system associated to $(V, \triangledown, \triangleright)$, then $(V, \blacklozenge)$ is a pre-Lie algebra, where $\blacklozenge$ is given by
\[
x \blacklozenge y = R(x) \triangledown y - y \triangleleft S(x) + x \circ y
\] (2.31)
for all $x, y \in V$. 
Proof. Since for all \( x, y, z \in V \),

\[
x \diamond (y \bowtie z) = R(x) \triangleright (R(y) \triangleright z) - (R(y) \triangleright z) \triangleleft S(z) + x \circ (R(y) \triangleright z) - R(x) \triangleright (z \triangleleft S(y)) + z \triangleleft (S(y) \bowtie x) - x \circ (z \triangleleft S(y)) + R(x) \triangleright (y \circ z) - (y \circ z) \triangleleft S(x) + x \circ (y \circ z)
\]

and

\[
(x \diamond y) \bowtie z = (R(x) \triangleright y) \bowtie z + (x \triangleleft S(y)) \bowtie z + (x \circ y) \bowtie z
\]

then

\[
x \diamond (y \bowtie z) - (x \diamond y) \bowtie z = R(x \triangleleft S(y)) \triangleright z - (R(y) \triangleright z) \triangleleft S(z) + x \circ (R(y) \triangleright z) - R(x) \triangleright (z \triangleleft S(y)) + z \triangleleft (S(y) \bowtie x) + y \circ (z \triangleleft S(y)) - x \circ (z \triangleleft S(y)) - (y \circ z) \triangleleft S(x) + z \triangleleft S(x \circ y) - x \circ (z \triangleleft S(y)) + R(y \triangleleft S(x)) \triangleright z + (y \triangleleft S(x)) \circ z + z \triangleleft S(x \circ y).
\]

Thus we have

\[
x \diamond (y \bowtie z) - (x \diamond y) \bowtie z - y \diamond (x \bowtie z) + (y \bowtie x) \bowtie z
\]
Therefore Eq. (1.3) holds for \((V, \bullet)\). This finishes the proof. 

\[
\begin{align*}
\mu(\mu \otimes \text{id}) &= \mu(\text{id} \otimes \mu), \\
\nu(\nu \otimes \text{id}) &= \nu(\text{id} \otimes \nu), \\
\nu(\mu \otimes \text{id}) &= \mu(\text{id} \otimes \nu) = \nu(\nu \otimes \text{id}) = \nu(\text{id} \otimes \mu).
\end{align*}
\]

3. Associative compatible pairs

In this section we consider a special case of curved \(O\)-operator systems.

**Definition 3.1.** Let \(V\) be a vector space and \(\mu, \nu : V \otimes V \rightarrow V\) two linear maps such that

\[
\mu(\mu \otimes \text{id}) = \mu(\text{id} \otimes \mu), \quad \nu(\nu \otimes \text{id}) = \nu(\text{id} \otimes \nu),
\]

Then we call \((\mu, \nu)\) an **associative compatible pair** on \(V\).

**Remark 3.2.** (1) If \((A, \mu)\) is an associative algebra, then \((\mu, \mu)\) is an associative compatible pair on \(A\).

(2) If for all \(x, y \in V\), we write \(\mu(x \otimes y) = xy\) and \(\nu(x \otimes y) = x \circ y\), then the above compatibility conditions can be rewritten as

\[
\begin{align*}
(xy)z &= x(yz), \\
(x \circ y) \circ z &= x \circ (y \circ z), \\
(xy) \circ z &= x(y \circ z) = (x \circ y)z = x \circ (yz),
\end{align*}
\]

where \(x, y, z \in V\).

(3) We note that \((V, \circ, L, R)\) is an extended \((V, \mu)\)-bimodule algebra.

**Definition 3.3.** Let \(V\) be a vector space and \(R : V \rightarrow V\), \(\mu, \nu : V \otimes V \rightarrow V\) be three linear maps such that

\[
R(x)R(y) = R(R(x)y + xR(y) + x \circ y),
\]

where \(x, y \in V\). Then we call \((V, \mu, \nu, R)\) a **generalized Rota-Baxter algebra**.

**Remark 3.4.** (1) If \((V, \mu)\) is an associative algebra and set \(x \circ y = \lambda \mu(x \circ y)\), then \((V, \mu, \nu, R)\) is a Rota-Baxter algebra of weight \(\lambda\).

(2) If \((V, \mu)\) is an associative algebra and write \(\omega(x \otimes y) = R(x \circ y)\), then the generalized Rota-Baxter algebra \((V, \mu, \nu, R)\) is exactly Brzeziński’s curved Rota-Baxter algebra (when \(R = S\) in [5] Definition 1.1]).

**Example 3.5.** Let \(A\) be the 2-dimensional associative algebra where the multiplication is defined, with respect to a basis \(\{e_1, e_2\}\), by

\[
e_1 \cdot e_1 = e_1, \quad e_1 \cdot e_2 = e_2, \quad e_2 \cdot e_1 = e_2, \quad e_2 \cdot e_2 = e_2.
\]

It forms an associative compatible pair with the algebra defined with respect to the same basis by

\[
e_1 \circ e_1 = (a - b)e_1 + be_2, \quad e_1 \circ e_2 = ae_2, \quad e_2 \circ e_1 = ae_2, \quad e_2 \circ e_2 = ae_2,
\]

where \(a\) and \(b\) are parameters.
The following linear maps $R$ together with the compatible pair provide a generalized Rota-Baxter algebra.

- $R(e_1) = (b - a)e_1$, $R(e_2) = 0$,
- $R(e_1) = -ae_1$, $R(e_2) = -ae_1$,
- $R(e_1) = -be_2$, $R(e_2) = -ae_2$,
- $R(e_1) = b(e_1 - e_2)$, $R(e_2) = a(e_1 - e_2)$,
- $R(e_1) = (b - a)e_1 - be_2$, $R(e_2) = -ae_2$,
- $R(e_1) = -ae_2$, $R(e_2) = -ae_2$,
- $R(e_1) = (b - a)e_1 - ae_2$, $R(e_2) = -ae_2$,
- $R(e_1) = a(e_1 - e_2)$, $R(e_2) = a(e_1 - e_2)$,
- $R(e_1) = (a - b)e_2$, $R(e_2) = 0$,
- $R(e_1) = (b - 2a)e_1 + (a - b)e_2$, $R(e_2) = -ae_1$,
- $R(e_1) = (b - a)e_1 + (a - b)e_2$, $R(e_2) = 0$.

Similar to the results in Section 2, we have:

**Proposition 3.6.** Let $(V, \mu, \nu, R)$ be a generalized Rota-Baxter algebra such that $(\mu, \nu)$ is an associative compatible pair on $V$.

1. Set $x \prec y = xR(y) + x \diamond y$, $x \succ y = R(x)y$.
Then $(V, \prec, \succ)$ is a dendriform algebra.

2. Set $x \circ y = R(x)y - yR(x) + x \diamond y$.
Then $(V, \circ)$ is a pre-Lie algebra.

3. Set $x \prec y = xR(y)$, $x \succ y = R(x)y$, $x \cdot y = x \diamond y$.
Then $(V, \prec, \succ, \cdot)$ is a tridendriform algebra.

4. Set $x \ast y = R(x)y + xR(y) + x \diamond y$.
Then $(V, \ast)$ is an associative algebra.

**Remark 3.7.** (1) If set $x \diamond y = \lambda xy$ in Proposition 3.6, then we can get the cases for the usual Rota-Baxter algebra of weight $\lambda$.

2. By Remark 3.4(2), we can transfer the results in [5] for Brzeziński’s curved Rota-Baxter algebras to the case of generalized Rota-Baxter algebras. For example, we have:

Let $(V, \mu, \nu, R)$ be a generalized Rota-Baxter algebra such that $(V, \mu)$ is an associative algebra. Set

$$x \ast y = R(x)y + xR(y).$$
Then \((V, \ast)\) is an associative algebra if and only if, for all \(x, y, z \in V\),
\[xR(y \circ z) = R(x \circ y)z.\]

In particular, if \((V, \mu)\) has an identity, then \((V, \ast)\) is an associative algebra if and only if there exists a central element \(\kappa \in V\) such that, for all \(x, y \in V\),
\[R(x \circ y) = \kappa xy.\]

Following the definition of generalized Rota-Baxter algebra, instead of \(R \circ \omega, S \circ \omega\) by \(\omega_1, \omega_2\), respectively, we can also consider the double curved Rota-Baxter system as follows:

**Definition 3.8.** A system \((A, R, S, \omega_1, \omega_2)\) consisting of an associative (but not necessarily unital) algebra \((A, \mu)\) and four linear maps \(R, S : A \rightarrow A, \omega_1, \omega_2 : A \otimes A \rightarrow A\) is called a **double curved Rota-Baxter system** if, for all \(a, b \in A\),
\[R(a)R(b) = R(R(a)b + aS(b)) + \omega_1(a \otimes b)\]  \hspace{1cm} (3.4)

and
\[S(a)S(b) = S(R(a)b + aS(b)) + \omega_2(a \otimes b).\]  \hspace{1cm} (3.5)

For double curved Rota-Baxter system \((A, R, S, \omega_1, \omega_2)\), we have:

**Corollary 3.9.** Let \((A, R, S, \omega_1, \omega_2)\) be a double curved Rota-Baxter system. Define
\[a \ast b = R(a)b + aS(b).\]  \hspace{1cm} (3.6)

Then \((A, \ast)\) is an associative algebra if and only if, for all \(a, b, c \in A\),
\[\omega_1(a \otimes b)c = \omega_2(b \otimes c).\]  \hspace{1cm} (3.7)

**Proof.** Let \(V = A\) in Proposition 2.21 \(\square\)

**Example 3.10.** We consider again the 2-dimensional associative algebra \(A\) where the multiplication is defined, with respect to a basis \(\{e_1, e_2\}\), by
\[e_1 \cdot e_1 = e_1, \quad e_1 \cdot e_2 = e_2, \quad e_2 \cdot e_1 = e_2, \quad e_2 \cdot e_2 = e_2.\]

We construct in the following a family of examples of associated double curved Rota-Baxter systems \((A, R, S, \omega_1, \omega_2)\). The maps \(R, S, \omega_1, \omega_2\) are defined with respect to the basis \(\{e_1, e_2\}\) by
\[R(e_1) = -c_{1,1}^1 e_2, \quad R(e_2) = -e_2,\]
\[S(e_1) = 0, \quad S(e_2) = (c_{1,1}^2 - c_{1,2}^2)e_2,\]
\[\omega_1(e_1, e_1) = c_{1,1}^2 (1 - c_{1,1}^2)e_2, \quad \omega_1(e_1, e_2) = (c_{1,2}^2 - c_{1,1}^2)e_2,\]
\[\omega_1(e_2, e_1) = b_{2,1}^1 e_1 + (1 - c_{1,1}^2 - b_{2,1}^1 c_{1,1}^2)e_2, \quad \omega_1(e_2, e_2) = b_{2,2}^1 e_1 + (c_{1,2}^2 - c_{1,1}^2 - b_{2,2}^1 c_{1,1}^2)e_2,\]
\[\omega_2(e_1, e_1) = c_{1,1}^1 e_1 + c_{1,1}^2 e_2, \quad \omega_2(e_1, e_2) = c_{1,2}^1 e_1 + c_{1,2}^2 e_2,\]
\[\omega_2(e_2, e_1) = e_1 + e_2, \quad \omega_2(e_2, e_2) = e_2,\]

where \(b_{i,j}^k, c_{i,j}^k\) are parameters.
Next we introduce the notion of double curved weak pseudotwistor:

**Definition 3.11.** Let \((A, \mu)\) be an associative algebra. A linear map \(T : A \otimes A \to A \otimes A\) is called a **double curved weak pseudotwistor** if there exist linear maps \(\mathcal{T} : A \otimes A \otimes A \to A \otimes A \otimes A\) and \(\omega_i : A \otimes A \to A, i = 1, 2\), rendering commutative the following diagrams:

\[
\begin{array}{ccc}
A \otimes A \otimes A & \xrightarrow{id \otimes \mu} & A \otimes A \\
\downarrow{id \otimes T} & & \downarrow{T} \\
A \otimes A & \xrightarrow{(id \otimes \mu) \circ \mathcal{T} - id \otimes \omega_2} & A \otimes A \\
\downarrow{(\mu \otimes id) \circ \mathcal{T} - \omega_1 \otimes id} & & \downarrow{T \otimes id} \\
A \otimes A \otimes A & & A \otimes A \otimes A
\end{array}
\]

(3.8)

and

\[
\begin{array}{ccc}
A \otimes A \otimes A & \xrightarrow{id \otimes \omega_2} & A \otimes A \\
\downarrow{\omega_1 \otimes id} & & \downarrow{\mu} \\
A \otimes A & \xrightarrow{\mu} & A \\
\end{array}
\]

(3.9)

The map \(\mathcal{T}\) is called a **weak companion of** \(T\) and \(\omega_i, i = 1, 2\) is called the **curvatures of** \(T\).

In what follows, we consider the relations between the product (3.6) and double curved weak pseudotwistor. Firstly, one can obtain:

**Proposition 3.12.** Let \(T : A \otimes A \to A \otimes A\) be a double curved weak pseudotwistor with weak companion \(\mathcal{T}\) and curvatures \(\omega_i, i = 1, 2\). Then \(\mu \circ T\) is an associative product on \(A\).

**Proof.** With the help of associativity of \(\mu\) one easily computes,

\[
\mu \circ T \circ (id \otimes \mu \circ T) = \mu \circ (id \otimes \mu) \circ \mathcal{T} - \mu \circ (id \otimes \omega_2) = \mu \circ (\mu \otimes id) \circ \mathcal{T} - \mu \circ (\omega_1 \otimes id) = \mu \circ T \circ (\mu \circ T \otimes id),
\]

finishing the proof. \(\square\)

Secondly, we can get:
**Proposition 3.13.** Let $(A, R, S, \omega_1, \omega_2)$ be a double curved Rota-Baxter system with the curvatures $\omega_i, i = 1, 2$ that satisfy Eq. (3.7), and define, for all $a, b, c \in A$,

$$T(a \otimes b) = R(a) \otimes b + a \otimes S(b),$$

and

$$T(a \otimes b \otimes c) = R(a) \otimes R(b) \otimes c + R(a) \otimes b \otimes S(c) + a \otimes S(b) \otimes S(c).$$

Then $T$ is a double curved weak pseudotwistor with weak companion $T$ and curvatures $\omega_i, i = 1, 2$.

**Proof.** The commutativity of diagram (3.8) is equivalent to Eq. (3.7). To check the commutativity of the left square in diagram (3.8), let us take any $a, b, c \in A$ and compute

$$T \circ (id \otimes \mu \circ T)(a \otimes b \otimes c) = T(a \otimes R(b)c + a \otimes bS(c))$$

$$= R(a) \otimes (R(b)c + bS(c)) + a \otimes S(R(b)c + bS(c))$$

$$= R(a) \otimes R(b)c + R(a) \otimes bS(c) + a \otimes S(b)S(c)$$

$$= -a \otimes \omega_2(b \otimes c)$$

$$= ((id \otimes \mu) \circ T - id \otimes \omega_2)(a \otimes b \otimes c).$$

$$T \circ (\mu \circ T \otimes id)(a \otimes b \otimes c) = T(R(a)b \otimes c + aS(b) \otimes c)$$

$$= R(R(a)b + aS(b)) \otimes c + (R(a)b + aS(b)) \otimes S(c)$$

$$= R(a)R(b) \otimes c + R(a)b \otimes S(c) + aS(b) \otimes S(c)$$

$$= -\omega_1(a \otimes b) \otimes c$$

$$= ((\mu \otimes id) \circ T - \omega_1 \circ id)(a \otimes b \otimes c).$$

So the commutativity of the right square in diagram (3.8) is checked. $\square$

For the relation between double curved Rota-Baxter system and pre-Lie algebra, we have:

**Corollary 3.14.** Let $(A, R, S, \omega_1, \omega_2)$ be a double curved Rota-Baxter system with the curvatures $\omega_i, i = 1, 2$. Then $A$ with operation $\circ$ defined by

$$a \circ b = R(a)b - bS(a),$$

is a pre-Lie algebra if and only if, for all $a, b \in A$,

$$\omega_1(a \otimes b) - \omega_2(b \otimes a)$$

is in the center of $A$.

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