A Euclidean Fourier-analytic approach to vertical projections in the Heisenberg group

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Abstract
An improved almost everywhere lower bound is given for Hausdorff dimension under vertical projections in the first Heisenberg group, with respect to the Carnot-Carathéodory metric. This improves the known lower bound, and answers a question of Fässler and Hovila. The approach uses the Euclidean Fourier transform, Basset’s integral formula, and modified Bessel functions of the second kind.

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1 | INTRODUCTION

Let $H$ be the first Heisenberg group, identified with $\mathbb{C} \times \mathbb{R}$ and equipped with the group law

$$(z, t) \ast (\zeta, \tau) = (z + \zeta, t + \tau + 2\omega(z, \zeta)),$$

where $\omega(z, \zeta) = \text{Im}(z \zeta)$. The Carnot–Carathéodory metric on $H$ is bi-Lipschitz equivalent to the Korányi metric

$$d_H((z, t), (\zeta, \tau)) = \left\| (\zeta, \tau)^{-1} \ast (z, t) \right\|_H,$$

where

$$\| (z, t) \|_H = (|z|^4 + t^2)^{1/4};$$

see [4, pp. 18–19]. This work gives an improved almost everywhere lower bound for the Hausdorff dimension of sets under vertical projections in $H$, where the Hausdorff dimension $\dim A$ of a set $A \subseteq H$ is defined through the Korányi metric (equivalently the Carnot–Carathéodory metric).
The definition of the vertical projections will be summarized briefly here, but see [1] and [2] for more background.

For each $\theta \in [0, \pi)$, let

$$\mathbb{V}_\theta^\perp = \left\{ \left( \lambda_1 i e^{i \theta}, \lambda_2 \right) \in \mathbb{C} \times \mathbb{R} : \lambda_1, \lambda_2 \in \mathbb{R} \right\},$$

and

$$\mathbb{V}_\theta = \left\{ \left( \lambda e^{i \theta}, 0 \right) \in \mathbb{C} \times \mathbb{R} : \lambda \in \mathbb{R} \right\}.$$

Then each $(z, t) \in \mathbb{H}$ can be uniquely written as a product

$$(z, t) = P_{\mathbb{V}_\theta^\perp}(z, t) \ast P_{\mathbb{V}_\theta}(z, t)$$

of an element of $\mathbb{V}_\theta^\perp$ on the left, with an element of $\mathbb{V}_\theta$ on the right. For each $\theta \in [0, \pi)$, this defines the vertical projection $P_{\mathbb{V}_\theta^\perp}$ and the horizontal projection $P_{\mathbb{V}_\theta}$. A formula for $P_{\mathbb{V}_\theta^\perp}$ is

$$P_{\mathbb{V}_\theta^\perp}(z, t) = \left( \pi_{\mathbb{V}_\theta^\perp}(z), t + 2\omega \left( \pi_{\mathbb{V}_\theta}(z), z \right) \right),$$

where $\pi_{\mathbb{V}_\theta^\perp}$ is the orthogonal projection onto the line in $\mathbb{R}^2$ with direction $i e^{i \theta}$, and $\pi_{\mathbb{V}_\theta}$ is the orthogonal projection onto the line in $\mathbb{R}^2$ with direction $e^{i \theta}$.

In [1, Conjecture 1.5], it was conjectured that for any (presumably Borel or analytic) set $A \subseteq \mathbb{H}$, $\dim P_{\mathbb{V}_\theta^\perp}(A) \geq \min\{\dim A, 3\}$ for almost every $\theta \in [0, \pi)$, and that if $\dim A > 3$, then $P_{\mathbb{V}_\theta^\perp}(A)$ has positive area for almost every $\theta \in [0, \pi)$. This conjecture is known in the range $\dim A \leq 1$; see [1, Theorem 1.4]. In [6, 8], some improvements were made beyond the lower bound $\dim P_{\mathbb{V}_\theta^\perp}(A) \geq 1$ for sets with $\dim A > 2$. Question 4.2 from [6] asked whether any improvement over the lower bound of 1 was possible for sets of dimension between 1 and 2. The following theorem gives a positive answer.

**Theorem 1.1.** Let $A \subseteq \mathbb{H}$ be an analytic set with $\dim A > 1$. Then

$$\dim P_{\mathbb{V}_\theta^\perp}(A) \geq \min \left\{ \frac{1 + \dim A}{2}, 2 \right\},$$

for almost every $\theta \in [0, \pi)$.

This improves† the known lower bound in the range $1 < \dim A < 7/2$. If $\dim A \geq 7/2$, then the lower bound $\dim P_{\mathbb{V}_\theta^\perp}(A) \geq 2 \dim A - 5$ from [1] is better than Theorem 1.1 and holds for every $\theta \in [0, \pi)$. A special case of the lower bound in Theorem 1.1 was proved in [1, Theorem 7.10] for sets contained in a vertical subgroup.

The proof of Theorem 1.1 uses the Euclidean Fourier transform. An approach to Hausdorff dimension via the (non-Euclidean) group Fourier transform was developed by Román-García [12], who proved a group Fourier-analytic formula for the energy of a measure, via the group Fourier

† After this paper was written, a result of Fässler and Orponen appeared which is better than Theorem 1.1 for $\dim A > 7/3$ [7].
transform of the Korányi kernels $\| \cdot \|_{\beta}^{-s}$. Unlike the group Fourier transform case, the Euclidean Fourier transforms of the Korányi kernels seem to be unexplored. In Lemma 3.2, it is shown that if $s \in (1, 3)$, then

$$0 < \hat{f}_s \lesssim f_{3-s},$$

where $f_s(x, t) = (x^4 + t^2)^{-s/4}$ for $(x, t) \in \mathbb{R}^2$. This seems to be a partial analogue of the formula $\hat{k}_s = c_{n,s} k_{n-s}$ for the Riesz kernels $k_s(x) = |x|^{-s}$ on $\mathbb{R}^n$. Only the case $s \in (1, 2)$ of the inequality $\hat{f}_s \lesssim f_{3-s}$ is used in the proof of Theorem 1.1.

The (Euclidean) Fourier transform of a compactly supported finite Borel measure $\mu$ on $\mathbb{R}^n$ is a locally Lipschitz function, defined by

$$\hat{\mu}(\xi) = \int e^{-2\pi i \langle x, \xi \rangle} \, d\mu(x), \quad \xi \in \mathbb{R}^n.$$ 

Frostman’s lemma will be used throughout, which states that if $A$ is an analytic (Suslin) subset of a complete separable metric space $(X, d)$, then

$$\dim A = \sup\{ s \geq 0 : \exists \mu \in \mathcal{M}(A) \text{ with } I_s(\mu) < \infty \}$$

$$= \sup\{ s \geq 0 : \exists \mu \in \mathcal{M}(A) \text{ with } c_s(\mu) < \infty \},$$

where

$$I_s(\mu) = \int \int d(x, y)^{-s} \, d\mu(x) \, d\mu(y),$$

$$c_s(\mu) = \sup_{x \in X} \frac{\mu(B(x, r))}{r^s},$$

and $\mathcal{M}(A)$ is the set of nonzero finite Borel measures compactly supported on $A$. In particular, Frostman’s lemma holds for Borel sets, since Borel sets are analytic. For an introduction to energies $I_s(\mu)$ see [10, chapter 8], and see, for example, [3, Appendix B] for a proof of Frostman’s lemma in the general case. See [9] for the first application of energies to projections.

Section 2 contains most of the background on modified Bessel functions needed in Section 3, and Section 3 contains the proofs of the main results. Section 4 has some remarks and further questions.

## 2 Background on Modified Bessel Functions

Define the modified Bessel function of the first kind, of order $\nu$, by

$$I_\nu(z) = \sum_{n=0}^{\infty} \frac{(z/2)^{\nu+2n}}{n! \Gamma(n + \nu + 1)},$$

(2.1)

for all $z \in \mathbb{C} \setminus \{0\}$ when $\nu \in \mathbb{C} \setminus \mathbb{Z}$, and for all $z \in \mathbb{C}$ when $\nu \in \mathbb{Z}$. To make $I_\nu$ a single-valued function, the function $z^\nu$ is defined to be $e^{\nu \log z}$ where $-\pi < \arg z \leq \pi$, unless mentioned otherwise. By convention the sum in (2.1) starts at $-\nu$ when $\nu$ is a negative integer. Then $I_\nu$ is an entire function when $\nu \in \mathbb{Z}$, and is analytic on $\mathbb{C} \setminus (-\infty, 0]$ when $\nu \in \mathbb{C} \setminus \mathbb{Z}$. 


Define the modified Bessel function of the second kind, of order \( \nu \), by

\[
K_\nu(z) = \frac{\pi}{2 \sin(\nu \pi)} (I_{-\nu}(z) - I_\nu(z)),
\]

when \( \nu \in \mathbb{C} \setminus \mathbb{Z} \), and for any \( n \in \mathbb{Z} \), define

\[
K_n(z) = \lim_{\nu \to n} K_\nu(z).
\] (2.2)

For any \( \nu \in \mathbb{C} \), the domain of \( K_\nu \) is \( \mathbb{C} \setminus \{0\} \), and \( K_\nu \) is analytic on \( \mathbb{C} \setminus (-\infty, 0] \). For any fixed \( z \in \mathbb{C} \setminus \{0\} \), the limit in (2.2) exists since \( I_\nu(z) \) is an entire function of \( \nu \), and so the limit in (2.2) can be expressed as a difference of partial derivatives with respect to \( \nu \). For fixed nonzero \( z \), the function \( K_\nu(z) \) is continuous at \( \nu = n \), for any \( n \in \mathbb{Z} \). Finally, the definition implies that \( K_\nu = K_{-\nu} \) for all \( \nu \in \mathbb{C} \).

**Proposition 2.1** [14, p. 79]. For any \( \nu \in \mathbb{C} \) and \( z \in \mathbb{C} \setminus (-\infty, 0] \),

\[
\frac{d}{dz} [z^\nu K_\nu(z)] = -z^\nu K_{\nu-1}(z),
\]

and

\[
\frac{d}{dz} [z^{-\nu} K_\nu(z)] = -z^{-\nu} K_{\nu+1}(z).
\]

The third formula for \( K_\nu \) in the theorem below is known as Basset’s integral formula, and a proof of the theorem can be found in [14, p. 172]. A more direct proof is outlined in [15, p. 384], though the definition of \( K_\nu \) given in [15] has an extra factor of \( \cos(\pi \nu) \) compared to the (now) standard definition.

**Theorem 2.2** [14, p. 172]. If \( \Re \nu > -1/2 \) and \( \Re z > 0 \), then

\[
K_\nu(z) = \frac{(z/2)^\nu \sqrt{\pi}}{\Gamma\left(\nu + \frac{1}{2}\right)} \int_0^\infty e^{-z \cosh \phi} (\sinh \phi)^{2\nu} d\phi
\]

\[
= \frac{(z/2)^\nu \sqrt{\pi}}{\Gamma\left(\nu + \frac{1}{2}\right)} \int_1^\infty e^{-zt} (t^2 - 1)^{\nu - \frac{1}{2}} dt
\]

\[
= \frac{(2z)^\nu \Gamma\left(\nu + \frac{1}{2}\right)}{2 \sqrt{\pi}} \int_{-\infty}^\infty e^{-iu(z^2 + u^2)^{\nu - \frac{1}{2}}} du,
\]

where the last integral is an improper Riemann integral. In particular (due to either of the first two formulae), \( K_\nu(x) \) is strictly positive for all \( x > 0 \) and \( \nu \in \mathbb{R} \).

One corollary of the preceding theorem is the (known) limit for \( K_\nu \) stated in Corollary 2.3. This limit follows from taking the first term in the more general asymptotic series expansion for \( K_\nu \) (see, for example, [14, p. 202]), but the limit can also be calculated easily from Theorem 2.2.
Corollary 2.3. For any \( \nu \in \mathbb{C} \),

\[
\lim_{x \to +\infty} \frac{K_\nu(x)}{\left(\frac{\pi}{2x}\right)^{1/2} e^{-x}} = 1.
\]

Another (known) corollary of Theorem 2.2 is that the singularity of \( K_0 \) at the origin is of logarithmic type, which follows from the series for \( K_0 \) (see [14, p. 80]), but a short proof is included below.

Corollary 2.4. For all \( x > 0 \),

\[
K_0(x) \leq 1 + \log 2 + |\log x|.
\]

Proof. By Theorem 2.2,

\[
K_0(x) = \int_0^\infty e^{-x \cosh t} \, dt \leq \int_0^\infty e^{-(x/2)e^t} \, dt.
\]

The change of variables \( s = (x/2)e^t \) gives

\[
K_0(x) \leq \int_{x/2}^\infty \frac{e^{-s}}{s} \, ds \leq 1 + \log 2 + |\log x|.
\]

□

3 PROOF OF LEMMAS AND THE MAIN THEOREM

The following result of Tuck from [13] gives a sufficient condition for the positivity of Fourier transforms. The proof given here is essentially the same as the one from [13], but is included for completeness.

Lemma 3.1 [13]. Let \( u : (0, \infty) \to \mathbb{R} \) be a differentiable convex function which is not identically zero. Assume that

\[
\lim_{x \to +\infty} u(x) = \lim_{x \to +\infty} u'(x) = \lim_{x \to 0^+} x u(x) = 0.
\]

Then

\[
\int_0^\infty \cos(x \xi) u(x) \, dx > 0,
\]

for any \( \xi \in \mathbb{R} \) such that the two-sided improper Riemann integral in (3.1) converges.

Proof. That \( u \) is convex and differentiable implies that \( u' \) is weakly increasing on \((0, \infty)\), and therefore continuous by Darboux’s theorem. The condition \( \lim_{x \to +\infty} u'(x) = 0 \) implies that \( u'(x) \leq 0 \) for all \( x > 0 \), and the condition \( \lim_{x \to +\infty} u(x) = 0 \) then implies that \( u(x) \geq 0 \) for all \( x > 0 \). Let \( \xi \in \mathbb{R} \) be such that the integral in (3.1) converges. By scaling and by symmetry, it may be assumed that \( \xi = 1 \) (the case \( \xi = 0 \) is trivial). Then

\[
\int_0^\infty \cos(x) u(x) \, dx = - \int_0^\infty \sin(x) u'(x) \, dx,
\]
where the right-hand side is again a convergent two-sided improper Riemann integral. This follows by integrating by parts on $[a, b]$ with $0 < a < b < \infty$, and letting $a \to 0^+$ and $b \to \infty$, using the conditions $\lim_{x \to 0^+} x u(x) = 0$ and $\lim_{x \to +\infty} u(x) = 0$ to eliminate the boundary terms. It suffices to show that
\[
\int_0^{\infty} \sin(x) v(x) \, dx > 0,
\]
whenever $v$ is a weakly decreasing continuous function on $(0, \infty)$ with $\lim_{x \to +\infty} v(x) = 0$, such that $v$ is not identically zero and such that the above two-sided improper Riemann integral converges. By the assumption of convergence, the integral above can be written as
\[
\sum_{k=0}^{\infty} \int_{2k\pi}^{(2k+1)\pi} \sin(x) [v(x) - v(x + \pi)] \, dx.
\]
It suffices to find a single summand which is strictly positive. The function $v$ is non-constant since $v$ is not identically zero and $\lim_{x \to +\infty} v(x) = 0$. Since $v$ is weakly decreasing and non-constant, there must exist a non-negative integer $k_0$ such that
\[
v(2k_0 \pi) - v((2k_0 + 2)\pi) > 0,
\]
where the right limit is taken for $v(0)$ and allowed to be $+\infty$ if $k_0 = 0$. By continuity of $v$, it follows that there exists $x_0 \in (2k_0 \pi, (2k_0 + 1)\pi)$ such that
\[
v(x_0) - v(x_0 + \pi) > 0.
\]
Again by continuity of $v$, this yields
\[
\sum_{k=0}^{\infty} \int_{2k\pi}^{(2k+1)\pi} \sin(x) [v(x) - v(x + \pi)] \, dx \geq \int_{2k_0 \pi}^{(2k_0+1)\pi} \sin(x) [v(x) - v(x + \pi)] \, dx > 0.
\]
This finishes the proof. $\square$

The following lemma is an inequality for the (2-dimensional) Euclidean Fourier transforms of (2-dimensional) Korányi kernels.

**Lemma 3.2.** For $s \in (0, 3)$, let
\[
f_s(x, t) = \frac{1}{(x^4 + t^2)^{s/4}}, \quad (x, t) \in \mathbb{R}^2.
\]

Then $\phi \mapsto \int \phi f_s$ defines a tempered distribution $f_s \in S' (\mathbb{R}^2)$, and if $s \in (1, 3)$ then the Euclidean Fourier transform of $f_s$ is a locally integrable function which satisfies
\[
0 < \hat{f}_s(\xi_1, \xi_2) \leq C_s f_{3-s}(\xi_1, \xi_2), \tag{3.2}
\]
for some positive constant $C_s$ depending only on $s$. Moreover, if $s \in (1, 3)$ then
\[
\hat{f}_s(\xi_1, \xi_2) = \frac{2\pi^{s/4} |\xi_2|^s}{\Gamma(s/4)} \int_{\mathbb{R}} e^{-2\pi i x^2} |x|^1 \frac{1}{2^{s/2}} K_{s/2} \left(2\pi |\xi_2| x^2\right) \, dx. \tag{3.3}
\]
Before the proof, a brief heuristic explanation will be given for the appearance of $f_{3-s}$ in (3.2). If $f_s$ were a function in $L^1(\mathbb{R}^2)$, the lemma would follow by changing variables $(x, t) \mapsto (\lambda x, \lambda^2 t)$ in the integral

$$\hat{f}_s(\xi_1, \xi_2) = \int_{\mathbb{R}^2} e^{-2\pi i l((x, t), (\xi_1, \xi_2))} f_s(x, t) \, dx \, dt,$$

using the scaling $f_s(\lambda x, \lambda^2 t) = \lambda^{-s} f_s(x, t)$ for the particular choice $\lambda = (\xi_1^4 + \xi_2^2)^{-1/4}$, to get

$$\hat{f}_s(\xi_1, \xi_2) = f_{3-s}(\xi_1, \xi_2) \int_{\mathbb{R}^2} e^{-2\pi i \left( (x, t), \left( \frac{\xi_1}{(\xi_1^4 + \xi_2^2)^{1/4}}, \frac{\xi_2}{(\xi_1^4 + \xi_2^2)^{1/2}} \right) \right)} f_s(x, t) \, dx \, dt.$$

If $f_s$ were in $L^1(\mathbb{R}^2)$, the integral on the right-hand side above would be a continuous function of $(\eta_1, \eta_2) = (\xi_1/(\xi_1^4 + \xi_2^2)^{1/4}, \xi_2/(\xi_1^4 + \xi_2^2)^{1/2})$ on the compact set $\eta_1^4 + \eta_2^2 = 1$, and the inequality $|\hat{f}_s| \lesssim f_{3-s}$ would follow. Unfortunately, there does not seem to be an obvious way to interpret the integral above as a bounded function of $\eta$, since $f_s$ is not integrable over $\mathbb{R}^2$. A scaling property also explains the analogous formula $\hat{k}_s = c_{n, s} k_{n-s}$ for the Riesz kernel on $\mathbb{R}^n$, but in that case the rotation invariance of the Riesz kernel makes the formula an exact equality. The lack of symmetry between the variables in $f_s$ is the reason for integrating in the $t$-variable first in the proof below.

**Proof of Lemma 3.2.** The first part of the proof will establish (3.3). The idea is to apply Basset’s integral formula, but some analysis is necessary to justify interchanging the order of integration. The second part of the proof will use (3.3) to prove (3.2).

The assumption that $0 < s < 3$ implies that $f_s$ is bounded outside $B(0, 1)$ and locally integrable, and $f_s$ is therefore a tempered distribution. Assume now that $1 < s < 3$. Let $\psi$ be a smooth bump function on $\mathbb{R}$ such that $\psi = 1$ on $[-1, 1]$ and $\psi = 0$ outside $[-2, 2]$. Let $\phi \in S(\mathbb{R}^2)$ and for each $\epsilon > 0$ let $\phi_\epsilon(\xi_1, \xi_2) = (1 - \psi(\xi_2/\epsilon))\phi(\xi_1, \xi_2)$. Then $\langle f_s, \hat{\phi} - \hat{\phi}_\epsilon \rangle \to 0$ as $\epsilon \to 0$. To see this, write

$$f_s = f_{s, 1} + f_{s, 2} + f_{s, 3},$$

where

$$f_{s, 1} = f_s \chi_{\{(x, t) \in \mathbb{R}^2 : x^4 + t^2 \leq 1\}},$$

$$f_{s, 2} = f_s \chi_{\{(x, t) \in \mathbb{R}^2 : x^4 + t^2 > 1 \text{ and } x^2 \leq |t|\}},$$

and

$$f_{s, 3} = f_s \chi_{\{(x, t) \in \mathbb{R}^2 : x^4 + t^2 > 1 \text{ and } x^2 > |t|\}}.$$

Then $\langle f_{s, 1}, \hat{\phi} - \hat{\phi}_\epsilon \rangle \to 0$ as $\epsilon \to 0$, since $f_{s, 1} \in L^1(\mathbb{R}^2)$ and $\hat{\phi} - \hat{\phi}_\epsilon \to 0$ in $L^\infty(\mathbb{R}^2)$.

By an integration by parts in the $\xi_2$ variable,

$$\langle f_{s, 2}, \hat{\phi} - \hat{\phi}_\epsilon \rangle = \int_{\mathbb{R}^2} \frac{f_{s, 2}(x, t)}{2\pi i t} \int_{\mathbb{R}^2} e^{-2\pi i l((x, t), \xi)} \left[ (1 - \psi(\xi_2/\epsilon))\phi(\xi) + \psi(\xi_2/\epsilon) \partial_2 \phi(\xi) \right] d\xi \, dx \, dt.$$
The functions

\[ \int_{\mathbb{R}^2} e^{-2\pi i \langle (x,t), \xi \rangle} [e^{-1} \psi'(\xi_2/\varepsilon) \phi(\xi) + \psi(\xi_2/\varepsilon) \partial_2 \phi(\xi)] \, d\xi \]  

(3.4)

are uniformly bounded in \( L^\infty(\mathbb{R}^2) \), and converge to zero uniformly on compact subsets of \( \mathbb{R}^2 \), as \( \varepsilon \to 0 \). The convergence to zero of the second term in (3.4) follows from the dominated convergence theorem. Convergence to zero of the first term in (3.4) follows by writing

\[
\int_{\mathbb{R}^2} e^{-2\pi i \langle (x,t), \xi \rangle} e^{-1} \psi'(\xi_2/\varepsilon) \phi(\xi) \, d\xi
\]

\[
= \int_{\mathbb{R}^2} e^{-2\pi i \langle (x,t), (\xi_1, \varepsilon \eta) \rangle} \psi'(\eta) [\phi(\xi_1, \varepsilon \eta) - \phi(\xi_1, 0)] \, d\xi_1 \, d\eta
\]

\[
+ \int_{\mathbb{R}^2} [e^{-2\pi i \langle (x,t), (\xi_1, \varepsilon \eta) \rangle} - e^{-2\pi i \langle (x,t), (\xi_1, 0) \rangle}] \psi'(\eta) \phi(\xi_1, 0) \, d\xi_1 \, d\eta,
\]

and applying the dominated convergence theorem. The function \( f_{s,2}(x,t) \) is in \( L^1(\mathbb{R}^2) \) since \( s > 1 \). Hence, \( \langle f_{s,2}, \hat{\phi} - \hat{\phi}_\varepsilon \rangle \to 0 \) as \( \varepsilon \to 0 \).

For the third function, integrating by parts twice in the \( \xi_1 \) variable gives that \( \langle f_{s,3}, \hat{\phi} - \hat{\phi}_\varepsilon \rangle \to 0 \) as \( \varepsilon \to 0 \). This shows that \( \langle f_s, \hat{\phi} - \hat{\phi}_\varepsilon \rangle \to 0 \) as \( \varepsilon \to 0 \). Hence,

\[
\int_{\mathbb{R}^2} \frac{\hat{\phi}(x,t)}{(x^4 + t^2)^{s/4}} \, dx \, dt
\]

\[
= \lim_{\varepsilon \to 0} \int_{\mathbb{R}^2} \frac{\hat{\phi}_\varepsilon(x,t)}{(x^4 + t^2)^{s/4}} \, dx \, dt
\]

\[
= \lim_{\varepsilon \to 0} \lim_{M \to \infty} \lim_{N \to \infty} \int_{-M}^{M} \int_{\mathbb{R}^2} e^{-2\pi i x \xi_1} \phi(\xi) \int_{-N}^{N} \frac{e^{-2\pi i t \xi_2}}{(x^4 + t^2)^{s/4}} \, dt \, d\xi \, dx. \quad (3.5)
\]

For any \( x, \xi_2 \in \mathbb{R} \) both nonzero, and any \( s > 0 \), Theorem 2.2 gives

\[ \int_{-\infty}^{\infty} \frac{e^{-2\pi i t \xi_2}}{(x^4 + t^2)^{s/4}} \, dt = \frac{2\pi^{s/4}}{\Gamma(s/4)} |x|^{1-s/2} |\xi_2|^{x^2/4} K_{s-2/4} \left(2\pi x^2 |\xi_2| \right), \]

(3.6)

where the integral is an improper Riemann integral. By the second mean value theorem for integrals (or an integration by parts),

\[ \left| \int_{-\infty}^{\infty} \frac{e^{-2\pi i t \xi_2}}{(x^4 + t^2)^{s/4}} \, dt - \int_{-N}^{N} \frac{e^{-2\pi i t \xi_2}}{(x^4 + t^2)^{s/4}} \, dt \right| \leq \frac{1}{|\xi_2|^s (|x|^s + N^s/2)}, \quad (3.7) \]

for any \( N > 0 \). Hence, by three applications of the dominated convergence theorem,

(3.5) = \lim_{\varepsilon \to 0} \lim_{M \to \infty} \int_{-M}^{M} \int_{\mathbb{R}^2} e^{-2\pi i x \xi_1} \phi(\xi) \int_{-\infty}^{\infty} \frac{e^{-2\pi i t \xi_2}}{(x^4 + t^2)^{s/4}} \, dt \, d\xi \, dx

\[
= \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{-2\pi i x \xi_1} \phi(\xi) \int_{-\infty}^{\infty} \frac{e^{-2\pi i t \xi_2}}{(x^4 + t^2)^{s/4}} \, dt \, d\xi \, dx.
\]
The first application used (3.7) to get the dominating function
\[
|\phi_\varepsilon(\xi)| \left[ \frac{1}{|\xi_2|||x|| + 1} + |x|^{1-2} |\xi_2|^{\frac{s-2}{4} K_{\frac{s-2}{4}} (2\pi |x|^2 |\xi_2|)} \right],
\]
where \((x, \xi) \in [-M, M] \times \mathbb{R}^2\), while the second and third applications used the dominating function
\[
|\phi(\xi)||x|^{1-2} |\xi_2|^{\frac{s-2}{4} K_{\frac{s-2}{4}} (2\pi |x|^2 |\xi_2|)}, \quad (x, \xi) \in \mathbb{R}^3,
\]
which is integrable on \(\mathbb{R}^3\) since \(1 < s < 3\); by changing variables and considering the behavior of \(K_{\frac{s-2}{4}}\) for small and large arguments. More precisely, for \(s \neq 2\) the series definition of \(K_{\frac{s-2}{4}}\) gives the behavior of \(K_{\frac{s-2}{4}}\) for small arguments, while Corollary 2.4 controls the behavior of \(K_0\) for small arguments. Corollary 2.3 controls the asymptotic behavior of \(K_{\frac{s-2}{4}}\) for large arguments. By Fubini, (3.6), and a change of variables,
\[
\int_{\mathbb{R}^2} \hat{\phi}(x, t) \frac{dx dt}{(x^4 + t^2)^{s/4}} = \int_{\mathbb{R}^2} \phi(\xi) \left[ 2\pi^{s/4} |\xi_2|^{\frac{s-2}{4}} K_{\frac{s-2}{4}} (2\pi |x|^2 |\xi_2|) \right] e^{-2\pi i x \cdot \xi_1} \left| x \right|^{1-\frac{s}{2}} dx d\xi
\]
\[
= \frac{2\pi^{s/4}}{\Gamma(s/4)} \int_{\mathbb{R}^2} \phi(\xi) \left[ |\xi_2|^{\frac{s-3}{2}} \int e^{-2\pi i x \cdot \xi_2} \left| x \right|^{1-\frac{s}{2}} K_{\frac{s-2}{4}} (2\pi |x|^2) \right] d\xi.
\]
This proves (3.3). It remains to show that for \(\xi_2 \neq 0\),
\[
|\xi_2|^{\frac{s-3}{2}} \int_{-\infty}^{\infty} e^{-2\pi i x \cdot \xi_2} |x|^{1-\frac{s}{2}} K_{\frac{s-2}{4}} (2\pi |x|^2) \, dx \lesssim f_{3-s}(\xi_1, \xi_2),
\]
and that the left-hand side of (3.8) is strictly positive. The function
\[
-\frac{d}{dx} \left( |x|^{1-\frac{s}{2}} K_{\frac{s-2}{4}} (2\pi |x|^2) \right) = \left[ 4\pi |x|^{1-s} \right] \cdot \left[ |x|^{\frac{s}{2} \left( \frac{s+2}{4} \right)} K_{\frac{s+2}{4}} (2\pi x^2) \right]
\]
is decreasing on \((0, \infty)\) since it is a product of positive, decreasing functions; by Proposition 2.1, Theorem 2.2, and the assumption that \(s > 1\). More precisely, the derivative was calculated by substituting \(u = 2\pi x^2\), calculating the derivative with respect to \(u\) using the second identity in Proposition 2.1, and then applying the chain rule. This verifies the convexity condition in Lemma 3.1. The other conditions of Lemma 3.1 follow by considering the behavior of \(K_{\frac{s-2}{4}}\) for small and large arguments; using the series definition, Proposition 2.1, Corollary 2.3, Corollary 2.4, and the assumption that \(s < 3\). By Lemma 3.1, the left-hand side of (3.8) is strictly positive for \(s \in (1, 3)\), and therefore \(\hat{f}_3 > 0\).

If \(\xi_2^{1/2} \geq |\xi_1|\), the inequality in (3.8) is immediate since the integrand has \(L^1\) norm \(\lesssim 1\). This covers the case \(\xi_2^{1/2} \geq |\xi_1|\).
Henceforth, suppose that $|\xi_2|^{1/2} < |\xi_1|$. By symmetry and an integration by parts,

$$
\int_{-\infty}^{\infty} e^{-2\pi i x \frac{\xi_1}{|\xi_2|^{1/2}}} |x|^{1 - \frac{s}{2}} K_{\frac{s-2}{4}} \left(2\pi x^2\right) \, dx
= \frac{-|\xi_2|^{1/2}}{\pi \xi_1^{1/2}} \int_0^\infty \sin \left(2\pi x \frac{\xi_1}{|\xi_2|^{1/2}}\right) \frac{d}{dx} \left(|x|^{1 - \frac{s}{2}} K_{\frac{s-2}{4}} \left(2\pi x^2\right)\right) \, dx.
$$

(3.10)

More precisely, the fact that $|x|^{1 - \frac{s}{2}} K_{\frac{s-2}{4}} \left(2\pi x^2\right)$ is even means that only the cosine of the exponential in (3.10) contributes, and the interval of integration can be changed to $(0, \infty)$ resulting in a factor of 2. The integration by parts is first done on $(\delta, 1/\delta)$ for some $\delta > 0$, and then $\delta$ is taken to zero, with the boundary terms vanishing in the limit since $s < 3$ and due to the behavior of $K_{\frac{s-2}{4}}$ for small and large arguments (again using the series definition, Corollaries 2.3 and 2.4). By (3.9),

$$
\int_0^{\frac{|\xi_1|}{|\xi_2|^{1/2}}} \sin \left(2\pi x \frac{\xi_1}{|\xi_2|^{1/2}}\right) \frac{d}{dx} \left(|x|^{1 - \frac{s}{2}} K_{\frac{s-2}{4}} \left(2\pi x^2\right)\right) \, dx
\lesssim \frac{|\xi_1|}{|\xi_2|^{1/2}} \int_0^{\frac{|\xi_1|}{|\xi_2|^{1/2}}} |x|^{3 - \frac{s}{2}} K_{\frac{s+2}{4}} \left(2\pi x^2\right) \, dx
\lesssim \frac{|\xi_1|}{|\xi_2|^{1/2}} \int_0^{\frac{|\xi_1|}{|\xi_2|^{1/2}}} |x|^{2-s} \, dx
\lesssim \left(\frac{|\xi_2|^{1/2}}{|\xi_1|}\right)^{2-s}.
$$

(3.11)

It remains to bound the part of the integral over $x > \frac{|\xi_2|^{1/2}}{|\xi_1|}$. Since the right-hand side of (3.9) is decreasing, the second mean value theorem for integrals (or an integration by parts) can be applied to get

$$
\int_{\frac{|\xi_2|^{1/2}}{|\xi_1|}}^{\infty} \sin \left(2\pi x \frac{\xi_1}{|\xi_2|^{1/2}}\right) \frac{d}{dx} \left(|x|^{1 - \frac{s}{2}} K_{\frac{s-2}{4}} \left(2\pi x^2\right)\right) \, dx
\lesssim \left(\frac{|\xi_2|^{1/2}}{|\xi_1|}\right)^{3 - \frac{s}{2}} K_{\frac{s+2}{4}} \left(\frac{2\pi |\xi_2|}{|\xi_1|^2}\right)
\lesssim \left(\frac{|\xi_2|^{1/2}}{|\xi_1|}\right)^{2-s}.
$$

(3.12)

Substituting (3.11) and (3.12) back into (3.10) shows that (3.8) holds for all $s \in (1, 3)$. This finishes the proof. \qed
Let \( t : \mathbb{R}^2 \to \mathbb{R}^2 \) be the inverse map \((x, t) \mapsto (-x, -t)\), and let \( F^{-1} \) be the Euclidean inverse Fourier transform. The proof of the following lemma follows [11, p. 39].

**Lemma 3.3.** If \( s \in (1, 3) \) and \( \mu \) is a finite compactly supported Borel measure on \( \mathbb{R}^2 \), then

\[
\int f_s \, d(t_{\#} \mu * \mu) \leq \int_{\mathbb{R}^2} \hat{f}_s F^{-1}(t_{\#} \mu * \mu).
\]

**Proof.** Let \( \phi \) be a smooth, even, non-negative bump function on \( \mathbb{R}^2 \) with \( \int \phi = 1 \). For each \( \varepsilon > 0 \), let \( \phi_\varepsilon(x) = \varepsilon^{-2} \phi(x/\varepsilon) \), and let \( \mu_\varepsilon = \mu * \phi_\varepsilon \). Since \( \hat{f}_s \geq 0 \) (by Lemma 3.2), it may be assumed that \( \hat{f}_s F^{-1} (t_{\#} \mu * \mu) \) is absolutely integrable. By the dominated convergence theorem, a change of variables, and Fatou’s lemma,

\[
\int_{\mathbb{R}^2} \hat{f}_s F^{-1} (t_{\#} \mu * \mu) = \lim_{\varepsilon \to 0} \int_{\mathbb{R}^2} \hat{f}_s F^{-1} (t_{\#} \mu_\varepsilon * \mu_\varepsilon)
\]

\[
= \lim_{\varepsilon \to 0} \int f_s \, d(t_{\#} \mu_\varepsilon * \mu_\varepsilon)
\]

\[
= \lim_{\varepsilon \to 0} \int \int \int_{\mathbb{R}^2} f_s(x' - y') \phi_\varepsilon(x - x') \phi_\varepsilon(y - y') \, d x' \, d y' \, d \mu_\varepsilon(x) \, d \mu_\varepsilon(y)
\]

\[
\geq \int f_s \, d(t_{\#} \mu * \mu).
\]

\( \square \)

**Proof of Theorem 1.1.** By the scaling \((z, t) \mapsto (\lambda z, \lambda^2 t)\), it may be assumed that \( A \) is contained in the Korányi unit ball around the origin. It may also be assumed that

\[
\dim (A \setminus \{(0, t) \in \mathbb{C} \times \mathbb{R}\}) = \dim A,
\] (3.13)

since otherwise the theorem is immediate.

Let \( \alpha \) be such that \( 1 < \alpha < \min\{3, \dim A\} \), and suppose that \( 1 < s < (1 + \alpha)/2 \). By Frostman’s lemma, it suffices to prove that for any \( \varepsilon > 0 \), there exists a Borel probability measure \( \mu \) supported on \( A \), and a Borel set \( E \subseteq [0, \pi) \) with \( m([0, \pi) \setminus E) \leq \varepsilon \), such that

\[
\int_E \int_{\mathbb{H}} \int_{\mathbb{H}} d_{\mathbb{H}}((z, t), (\zeta, \tau))^{-\alpha} \, d\left(P_{\mathbb{H}}(z, t) \cdot P_{\mathbb{H}}(z, t)\right)(\zeta, \tau) \, d\vartheta < \infty.
\]

By (3.13), there is a number \( c = c(A, \alpha) > 0 \) such that for any \( \varepsilon > 0 \), there exists \( \partial_0 \in [0, \pi) \), and a Borel probability measure \( \mu \) on \( A \), supported in a Korányi ball of radius 1/2, with

\[
c_\alpha(\mu) = \sup_{(z,t) \in \mathbb{H}} \frac{\mu(B(\mathbb{H})(z, t),r))}{r^s} < \infty,
\] (3.14)
such that
\[
|z| > c, \quad (3.15)
\]
for all \((z, t) \in \text{supp } \mu\), and such that either
\[
|\arg z - \theta_0| \mod 2\pi < \epsilon^3 \quad \text{for all } (z, t) \in \text{supp } \mu, \quad (3.16)
\]
or
\[
|\arg z + \pi - \theta_0| \mod 2\pi < \epsilon^3 \quad \text{for all } (z, t) \in \text{supp } \mu. \quad (3.17)
\]

Let \(\epsilon > 0\) be given, assuming \(\epsilon < 1/100\) without loss of generality, let \(c, \theta_0\) and \(\mu\) be as described above, and let
\[
E = \{ \theta \in [0, \pi) : |\theta - \theta_0 - \pi/2| \mod \pi > \epsilon/2 \}, \quad (3.18)
\]
which satisfies \(m([0, \pi) \setminus E) \leq \epsilon\). By rotating \(\mathbb{V}_{\theta}^\perp\) to \(\mathbb{R}^2\) and applying Lemma 3.3,
\[
\int_E \int_{\mathbb{V}_{\theta}^\perp} \int_{\mathbb{V}_{\theta}^\perp} d\eta((z, t), (\zeta, \tau))^{-s} d\left(P_{\mathbb{V}_{\theta}^\perp \# \mu}(z, t) d\left(P_{\mathbb{V}_{\theta}^\perp \# \mu}(\zeta, \tau) d\theta
\]
\[
= \int_E \int_{\mathbb{V}_{\theta}^\perp} f_s(|z|, t) d\left(t_{\mu} P_{\mathbb{V}_{\theta}^\perp \# \mu} * P_{\mathbb{V}_{\theta}^\perp \# \mu}(z, t) d\theta
\]
\[
\leq \int_E \int_{\mathbb{R}^2} \hat{f}_s(r, \rho) \mathcal{F}^{-1}\left(t_{\mu} P_{\mathbb{V}_{\theta}^\perp \# \mu} * P_{\mathbb{V}_{\theta}^\perp \# \mu}\right)(rie^{i\theta}, \rho) dr d\rho d\theta, \quad (3.19)
\]
where \(\epsilon\) is the inverse map \((z, t) \mapsto (-z, -t)\), and the convolution above is Euclidean convolution (which equals Heisenberg convolution on vertical subgroups of \(\mathbb{H}^1\)). By Lemma 3.2,
\[
(3.19) \lesssim \int_{\mathbb{R}^2} (r^4 + \rho^2)^{(s-3)/4} \int_E \mathcal{F}^{-1}\left(t_{\mu} P_{\mathbb{V}_{\theta}^\perp \# \mu} * P_{\mathbb{V}_{\theta}^\perp \# \mu}\right)(rie^{i\theta}, \rho) d\theta dr d\rho,
\]
where \(\mathcal{F}^{-1}(t_{\#} P_{\mathbb{V}_{\theta}^\perp \# \mu} * P_{\mathbb{V}_{\theta}^\perp \# \mu})\) is non-negative by the convolution theorem.

Choose \(\delta > 0\) such that \(\delta < ((1 + \alpha)/2 - s)/100\). The part of the above integral over \((r^4 + \rho^2)^{1/4} \leq 100\) is finite since \(s > 0\), and since the inverse Fourier transform of a probability measure is bounded by 1 in sup norm. The remaining integral over \((r^4 + \rho^2)^{1/4} > 100\) can be dyadically partitioned into a sum over sets \(B_j\) where \((r^4 + \rho^2)^{1/4} \sim 2^j\) and \(j \geq 0\). For each \(j\), since \(\mathcal{F}^{-1}(t_{\#} P_{\mathbb{V}_{\theta}^\perp \# \mu} * P_{\mathbb{V}_{\theta}^\perp \# \mu})\) is non-negative, the factor \((r^4 + \rho^2)^{(s-3)/4}\) can be changed to \(2^{-j(3-s)}\), and then the sets \(B_j\) can be enlarged to
\[
A_j = \{(\rho, \theta, r) : \theta \in E, \quad |\rho| \leq 2^j, \quad |r| \leq 2^j\},
\]
without any need for modulus signs. By then using Fubini to swap the integral defining \(\mathcal{F}^{-1}(t_{\#} P_{\mathbb{V}_{\theta}^\perp \# \mu} * P_{\mathbb{V}_{\theta}^\perp \# \mu})\) with all of the others, it suffices to show that
\[
\int \int \int_{A_j} e^{2\pi i \left(\left(r e^{i\theta}, \rho\right) \cdot (z-\zeta, t-\tau + 2\omega(\pi_{\mathbb{V}_{\theta}}(z), z) - 2\omega(\pi_{\mathbb{V}_{\theta}}(\zeta), \zeta))\right)} d\rho d\theta dr \mu(\zeta, \tau) d\mu(z, t) \lesssim 2^{j(3-s-\delta)},
\]
for any \( j \geq 0 \), where
\[
A_j = \{ (\rho, \vartheta, r) : \vartheta \in E, \ |\rho| \leq 2^j, \ |r| \leq 2^j \}.
\]

Let \( j \geq 0 \) be given. Since \( \mu \) is a probability measure, it is enough to show that for any \( (z, t) \in \text{supp} \mu \),
\[
\left| \int_{A_j} e^{2\pi i \left( (rie^i \varphi, \left( z - \zeta, t - \tau + 2\omega \left( \pi V_\vartheta (z) \right)^\frac{1}{2} \right) - 2\omega \left( \pi V_\vartheta (\zeta) \right)^\frac{1}{2} \right) \right)} d\rho \ d\vartheta \ dr \ d\mu(\zeta, \tau) \right| \lesssim 2^{j(3-s-\delta)}.
\]

Let \( (z, t) \in \text{supp} \mu \) be given. A trivial upper bound for the inner integral is \( 2^{3j} \), so using \( \delta < (\alpha - s)/100 \) and the Frostman condition (3.14) on \( \mu \) gives
\[
\left| \int_{B_{\mathbb{H}}((z, t), 2^{-j})} \int_{A_j} e^{2\pi i \left( (rie^i \varphi, \left( z - \zeta, t - \tau + 2\omega \left( \pi V_\vartheta (z) \right)^\frac{1}{2} \right) - 2\omega \left( \pi V_\vartheta (\zeta) \right)^\frac{1}{2} \right) \right)} d\rho \ d\vartheta \ dr \ d\mu(\zeta, \tau) \right| \lesssim 2^{j(3-s-\delta)},
\]

Therefore, it suffices to show that
\[
\sum_{k=0}^{j} \left| \int_{B_{\mathbb{H}}((z, t), 2^{-k}) \setminus B_{\mathbb{H}}((z, t), 2^{-(k+1)})} \int_{A_j} e^{2\pi i \left( (rie^i \varphi, \left( z - \zeta, t - \tau + 2\omega \left( \pi V_\vartheta (z) \right)^\frac{1}{2} \right) - 2\omega \left( \pi V_\vartheta (\zeta) \right)^\frac{1}{2} \right) \right)} d\rho \ d\vartheta \ dr \ d\mu(\zeta, \tau) \right| \lesssim 2^{j(3-s-\delta)},
\]

(3.20)

Fix \( k \in \{0, \ldots, j\} \) and \( (\zeta, \tau) \in \text{supp} \mu \) with \( 2^{-(k+1)} \leq d_{\mathbb{H}}((z, t), (\zeta, \tau)) \leq 2^{-k} \). It will be shown that
\[
\left| \int_{A_j} e^{2\pi i \left( (rie^i \varphi, \left( z - \zeta, t - \tau + 2\omega \left( \pi V_\vartheta (z) \right)^\frac{1}{2} \right) - 2\omega \left( \pi V_\vartheta (\zeta) \right)^\frac{1}{2} \right) \right)} d\rho \ d\vartheta \ dr \ d\mu(\zeta, \tau) \right| \lesssim j \min \left\{ 2^{j+3k}, 2^{2j+k} \right\},
\]

(3.21)

which will be enough to prove (3.20). To see that (3.21) implies (3.20), assume that (3.21) holds. Then substituting (3.21) into (3.20) gives
\[
\sum_{k=0}^{j} \left| \int_{B_{\mathbb{H}}((z, t), 2^{-k}) \setminus B_{\mathbb{H}}((z, t), 2^{-(k+1)})} \int_{A_j} e^{2\pi i \left( (rie^i \varphi, \left( z - \zeta, t - \tau + 2\omega \left( \pi V_\vartheta (z) \right)^\frac{1}{2} \right) - 2\omega \left( \pi V_\vartheta (\zeta) \right)^\frac{1}{2} \right) \right)} d\rho \ d\vartheta \ dr \ d\mu(\zeta, \tau) \right|
\]
\[
\lesssim j \sum_{k \in \{0, j/2\}} 2^{j+3k-k\alpha} + j \sum_{k \in \{j/2, j\}} 2^{2j+k-k\alpha}
\]
\[
\lesssim j 2^{j\left( \frac{5-s}{2} \right)}
\]
\[
\lesssim 2^{j(3-s-\delta)},
\]
since $1 < \alpha < 3$ and $0 < \delta < (1+\alpha)/2 - s)/100$. This proves that (3.20) holds conditionally on (3.21), which (as explained previously) implies the theorem.

It remains to prove (3.21). If $|z - \zeta| \leq 2^{-2k}/100$, then $|t - \tau + 2\omega(z, \zeta)| \geq 2^{-2k}/10$, and hence

$$|t - \tau + 2\omega(\pi_{V_0}(z), z) - 2\omega(\pi_{V_0}(\zeta), \zeta)| \geq 2^{-2k},$$

for all $\theta \in [0, \pi)$, by the identity

$$\omega(\pi_{V_0}(z), z) - \omega(\pi_{V_0}(\zeta), \zeta) = \omega(z, \zeta) + \omega(\pi_{V_0}(z + \zeta), z - \zeta).$$

It follows that

$$\left| \int_{A_j} e^{2\pi i\left((rie^{i\theta}, \zeta - \tau + 2\omega(\pi_{V_0}(z), z) - 2\omega(\pi_{V_0}(\zeta), \zeta)\right))} \, d\rho \, d\theta \, dr \right| \leq 2^{j+2k}. \quad (3.22)$$

This implies (3.21) in this case, so it will henceforth be assumed that $|z - \zeta| > 2^{-2k}/100$. Let $p = (z - \zeta)/|z - \zeta|, q = (z + \zeta)/|z + \zeta|$ and let

$$E^{(1)} = \left\{ \theta \in E : \left| \left\langle p, ie^{i\theta} \right\rangle \right| < \epsilon^3 \right\}, \quad E^{(2)} = E \setminus E^{(1)}.$$

For $l \in \{1, 2\}$ let $A^{(l)}_j = \{ (\rho, \theta, r) \in A_j : \theta \in E^{(l)} \}$. Then

$$\left| \int_{A_j} e^{2\pi i\left((rie^{i\theta}, \zeta - \tau + 2\omega(\pi_{V_0}(z), z) - 2\omega(\pi_{V_0}(\zeta), \zeta)\right))} \, d\rho \, d\theta \, dr \right| \leq \int_{A_j^{(1)}} e^{2\pi i\left((rie^{i\theta}, \zeta - \tau + 2\omega(\pi_{V_0}(z), z) - 2\omega(\pi_{V_0}(\zeta), \zeta)\right))} \, d\rho \, d\theta \, dr \left| + \int_{A_j^{(2)}} e^{2\pi i\left((rie^{i\theta}, \zeta - \tau + 2\omega(\pi_{V_0}(z), z) - 2\omega(\pi_{V_0}(\zeta), \zeta)\right))} \, d\rho \, d\theta \, dr \right|. \quad (3.23)$$

Integrating the exponential above in the $r$-variable would result in a factor of $\langle ie^{i\theta}, z - \zeta \rangle^{-1}$, which is bounded on $A_j^{(2)}$ by definition of $E^{(2)}$. The integral over $A_j^{(1)}$ must be treated differently, since for $\theta \in E^{(1)}$ there is potentially no cancelation occurring in the $r$-integration, but this will be made up for by better cancelation in the $\rho$-integration (on average).

The second integral in the right-hand side of (3.23) satisfies

$$\left| \int_{A_j^{(2)}} e^{2\pi i\left((rie^{i\theta}, \zeta - \tau + 2\omega(\pi_{V_0}(z), z) - 2\omega(\pi_{V_0}(\zeta), \zeta)\right))} \, d\rho \, d\theta \, dr \right| \leq \int_{\theta \in E^{(2)}} \left| t - \tau + 2\omega(\pi_{V_0}(z), z) - 2\omega(\pi_{V_0}(\zeta), \zeta) \right| < 2^{-2j} \right\}$$

$$\left| \int_{\pi(A_j)} e^{2\pi i\left((rie^{i\theta}, \zeta - \tau + 2\omega(\pi_{V_0}(z), z) - 2\omega(\pi_{V_0}(\zeta), \zeta)\right))} \, d\rho \, dr \right| d\theta.$$
\[ + \sum_{l=-\infty}^{2j} \int_{\{\theta \in \mathbb{R}^2 : 2^{-(l+1)} < |l - \tau + 2\omega(\pi_{V_\theta}(z), z) - 2\omega(\pi_{V_\theta}(\zeta), \zeta)| < 2^{-l}\}} \left| \int_{\pi(A_j)} e^{2\pi i \left( r i e^{i\theta}, \rho, \left( z - \zeta, l - \tau + 2\omega(\pi_{V_\theta}(z), z) - 2\omega(\pi_{V_\theta}(\zeta), \zeta)\right) \right)} d\rho \, dr \right| d\theta, \quad (3.24) \]

where

\[ \pi(A_j) = \{(\rho, r) \in \mathbb{R}^2 : |ho| \leq 2^{2j}, \quad |r| \leq 2^j\} \]

is the projection of \( A_j \) onto the \((\rho, r)\)-plane. A similar calculation to [8, Lemma 2.3] (following [2, section 4] and [6, Lemma 3.5]) gives that the function

\[ F(\theta) = t - \tau + 2\omega(\pi_{V_\theta}(z), z) - 2\omega(\pi_{V_\theta}(\zeta), \zeta) = t - \tau + 2\omega(\pi_{V_\theta}(z - \zeta), z + \zeta) - 2\omega(z, \zeta) \]

satisfies

\[ 2^{4k} \lesssim |z - \zeta|^2 |z + \zeta|^2 = \left| \frac{F'(\theta)}{2} \right|^2 + \left| \frac{F''(\theta)}{4} \right|^2, \quad (3.26) \]

where the lower bound in (3.26) uses \(|z + \zeta| \gtrsim 1\), which follows from (3.15) and (3.16)–(3.17). By the mean value theorem, the equality in (3.26) implies that the set \( \{\theta \in [0, \pi) : |F'(\theta)| < (|z - \zeta| |z + \zeta|) / 100\} \) is a union of \( \lesssim 1 \) intervals; since each connected component has length \( \gtrsim 1 \). It follows from [5, Lemma 3.3] that for any \( \varepsilon > 0 \),

\[ m\left\{ \theta \in [0, \pi) : \left| t - \tau + 2\omega(\pi_{V_\theta}(z), z) - 2\omega(\pi_{V_\theta}(\zeta), \zeta) \right| < \varepsilon \right\} \lesssim \frac{\varepsilon^{1/2}}{2^{-k}}. \quad (3.27) \]

Hence,

\[ \int_{\{\theta \in \mathbb{R}^2 : 2^{-l} < |l - \tau + 2\omega(\pi_{V_\theta}(z), z) - 2\omega(\pi_{V_\theta}(\zeta), \zeta)| < 2^{-l}\}} \left| \int_{\pi(A_j)} e^{2\pi i \left( r i e^{i\theta}, \rho, \left( z - \zeta, l - \tau + 2\omega(\pi_{V_\theta}(z), z) - 2\omega(\pi_{V_\theta}(\zeta), \zeta)\right) \right)} d\rho \, dr \right| d\theta \quad \text{eq qarray} \]

\[ \lesssim \min \{ 2^{j+3k}, 2^{2j+k} \}. \]

This can be justified as follows. For each \( \theta \) in the domain of integration, the integral over the rectangle \( \pi(A_j) \) can be written as a product of two integrals, and each integral can be calculated directly. The modulus of the \( r \)-integral is \( \lesssim 2^{2k} \), and is also trivially \( \lesssim 2^j \). The modulus of the \( \rho \)-integral is \( \lesssim 2^j \) trivially. Integrating these two bounds and using (3.27) with \( \varepsilon = 2^{-2j} \) gives the bound of \( \min\{2^{j+3k}, 2^{2j+k}\} \). A similar argument gives that for any \( l \leq 2j \),
\[
\int \left\{ \theta \in E^{(2)} : 2^{-l+1} \leq |t - \tau + 2\omega(\pi_{V_\theta}(z), z) - 2\omega(\pi_{V_\theta}(\zeta), \zeta)| < 2^{-l} \right\}
\int_{\pi(A_j)} e^{2\pi i \left( (\tau, \rho), (z - \xi, -t + 2\omega(\pi_{V_\theta}(z), z) - 2\omega(\pi_{V_\theta}(\zeta), \zeta)) \right)} \, d\rho \, dr \, d\theta
\lesssim \min \left\{ 2^{3k + l/2}, 2^{j + k + l/2} \right\};
\]
the differences being that now the \(\rho\)-integral is \(\lesssim 2^l\), and the bound from (3.27) uses \(\varepsilon = 2^{-l}\). By combining these two bounds with (3.24) and summing the geometric series over \(l\),
\[
(3.24) \lesssim \min \left\{ 2^{j + 3k}, 2^{2j + k} \right\}. \tag{3.28}
\]
It remains to bound the first integral in the right-hand side of (3.23). By the assumptions on the support of \(\mu\) (from (3.16)–(3.17)),
\[
\min \left\{ |q - e^{i\theta_0}|, |q + e^{i\theta_0}| \right\} < \varepsilon^3,
\]
and thus for any \(\theta \in E\),
\[
\left| \left\langle q, e^{i\theta} \right\rangle \right| \geq 1 - \left| \left\langle q, ie^{i\theta} \right\rangle \right|
\geq 1 - \varepsilon^3 - \left| \left\langle e^{i\theta_0}, ie^{i\theta} \right\rangle \right|
= 1 - \varepsilon^3 - |\sin(\theta - \theta_0)|
\geq \varepsilon^2/10,
\]
by the definition of \(E\) (see (3.18)). The function \(F\) from (3.25) therefore satisfies
\[
|F'(\theta)| \geq 2|z - \xi| \cdot |z + \zeta| \left( \left| \left\langle p, e^{i\theta} \right\rangle \right| \left| \left\langle q, e^{i\theta} \right\rangle \right| - \left| \left\langle p, ie^{i\theta} \right\rangle \right| \cdot \left| \left\langle q, ie^{i\theta} \right\rangle \right| \right)
\geq 2|z - \xi| \cdot |z + \zeta| \left( \varepsilon^2 / 20 - \varepsilon^3 \right)
\gtrsim 2^{-2k}
\]
for any \(\theta \in E^{(1)}\). By the mean value theorem, it follows that for any \(\varepsilon > 0\),
\[
m \left\{ \theta \in E^{(1)} : |t - \tau + 2\omega(\pi_{V_\theta}(z), z) - 2\omega(\pi_{V_\theta}(\zeta), \zeta)| < \varepsilon \right\} \lesssim \frac{\varepsilon}{2^{-2k}}.
\]
Summing over dyadic numbers \(\varepsilon\) with \(2^{-2j} \leq \varepsilon \leq 1\) yields that
\[
\int_{A^{(1)}_j} e^{2\pi i \left( (\tau, \rho), (z - \xi, -t + 2\omega(\pi_{V_\theta}(z), z) - 2\omega(\pi_{V_\theta}(\zeta), \zeta)) \right)} \, d\rho \, d\theta \, dr \lesssim j 2^{j + 2k}. \tag{3.29}
\]
Combining (3.22), (3.23), (3.28), and (3.29) gives (3.21) for any \(k \in \{0, \ldots, j\}\). This proves (3.21), and as explained previously, this finishes the proof of the theorem.
4 | REMARKS AND FURTHER QUESTIONS

(1) Theorem 1.1 also holds if the Korányi metric on the domain is replaced by the (non-equivalent) parabolic metric $d((z, t), (\zeta, \tau)) = (|z - \zeta|^4 + |t - \tau|^2)^{1/4}$, with only minor changes to the proof (around (3.22)). It is not necessary to vary the metric on the codomain, since in $\mathbb{H}^1$ the two metrics are equivalent on any vertical subgroup.

(2) Whether the inequality $|\hat{f}_s| \lesssim f^{3-s}$ from Lemma 3.2 holds for $s \in (0, 1)$ is an open problem; the method of proof does not seem to extend to this range. The formula (3.3) for $\hat{f}_s$ is possibly just an artifact of the method.

(3) In [1, Example 7.11 and Remark 7.12], examples are given of sets $A$ of any dimension $\dim A \in [1, 2]$, for which the standard energy method cannot yield any improvement over the lower bound of $(1 + \dim A)/2$. However, the examples from [1] do not seem to preclude further improvement by a modified energy method such as that used in the proof of Theorem 1.1.

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