Nonparametric Two-Sample Hypothesis Testing for Random Graphs with Negative and Repeated Eigenvalues

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Abstract

We propose a nonparametric two-sample test statistic for low-rank, conditionally independent edge random graphs whose edge probability matrices have negative eigenvalues and arbitrarily close eigenvalues. Our proposed test statistic involves using the maximum mean discrepancy applied to suitably rotated rows of a graph embedding, where the rotation is estimated using optimal transport. We show that our test statistic, appropriately scaled, is consistent for sufficiently dense graphs, and we study its convergence under different sparsity regimes. In addition, we provide empirical evidence suggesting that our novel alignment procedure can perform better than the naïve alignment in practice, where the naïve alignment assumes an eigengap.

1 Introduction

Network data arises naturally in several fields, including neuroscience (Bullmore and Sporns, 2009; Bullmore and Bassett, 2011; Vogelstein et al., 2013; Finn et al., 2015; Priebe et al., 2019; Arroyo-Relión et al., 2019) and social networks (Newman et al., 2002; Newman, 2006; Carrington et al., 2005) among others. With the introduction of network data in the various sciences, there is a need for developing corresponding statistical methodology and theory. Often one wishes to determine whether or not two graphs exhibit similar distributional properties for some notion of similarity between distributions on networks. Furthermore, as in classical statistics, one may wish to analyze graph data with only a few assumptions on the probability distributions. For example, for Euclidean data, given i.i.d. observations \( \{X_i\}_{i=1}^n \) and \( \{Y_j\}_{j=1}^m \in \mathbb{R}^d \) with cumulative distribution functions denoted \( F_X \) and \( F_Y \) respectively, a model-agnostic way to test whether \( F_X = F_Y \) is use nonparametric methods, such as Mood (1954); Anderson et al. (1994); Gretton et al. (2012); Székely and Rizzo (2013), and Chen and Friedman (2017).

For the two-sample test we consider, we take the perspective that a single network constitutes an observation as is often the case in the statistical network analysis literature. A number of works have studied hypothesis testing when the graphs are on the same vertex set (Tang et al., 2017a; Li and Li, 2018; Levin et al., 2019; Ghoshdastidar et al., 2020; Draves and Sussman, 2020), analogous to the “matched pairs” paradigm in Euclidean data. However, in many settings, there is not necessarily an a priori matching between vertices; for example, one could introduce and remove vertices without fundamentally altering the network structure (e.g. Cai and Li (2015)). In this paper we study a nonparametric two-sample test without assuming that the graphs have an alignment between the vertices.

We assume only that the two networks have conditionally independent edges and that the graphs’ edge-probability matrices are low rank (see Section 2.1 for a formal description). We consider a latent-space model (Hoff et al., 2002) introduced in Rubin-Delanchy et al. (2020), wherein each vertex has a latent vector in Euclidean space associated to it. The latent-space framework is specific enough to allow for a meaningful notion of similarity between graphs on different vertex sets, and it general enough to allow for arbitrary low-rank graphs. Low rank, conditionally edge-independent random graphs include a number of submodels...
including the stochastic blockmodel (Holland et al., 1983), the random dot product graph (Athreya et al., 2017), and finite-rank graphons (Lovász, 2012). In addition, other tests have been designed for fixed models and related problems, such as Lei (2016); Bickel and Sarkar (2016) and Fan et al. (2019). A more thorough discussion of related literature is in Section 3.3.

A major difficulty in allowing for negative eigenvalues in the graphs requires an understanding the relationship between the latent-space Euclidean geometry and the indefinite geometry induced by the negative eigenvalues. Nevertheless, we show that despite the underlying geometry, consistent testing can be performed given access only to the adjacency matrices. More specifically, we show that a test procedure based on a two-sample $U$-statistic with radial kernel $\kappa$ applied to rotated adjacency spectral embeddings yields a consistent test. Since we only assume that the graphs have a low-rank structure, our analysis includes random graphs whose edge-probability matrices have negative eigenvalues and a vanishing eigengap amongst the nonzero eigenvalues, and we conduct our study under different sparsity regimes. In particular, our proposed test procedure and its theoretical properties depend on a careful analysis of the interplay between indefinite orthogonal transformations, optimal transport, and convergence of degenerate $U$-statistics.

We further show that under the null hypothesis, for sufficiently dense graphs, the nondegenerate limiting distribution of our test statistic can be related to that of the $U$-statistic evaluated at suitably transformed latent vectors, and we provide additional results for sparser graphs. The convergence of our test statistic is analogous to that of Anderson et al. (1994); Gretton et al. (2012) in the Euclidean setting. Furthermore, our sparsity requirement is consistent with a number of works on network bootstraps for nondegenerate analogous to that of Anderson et al. (1994); Gretton et al. (2012) in the Euclidean setting. Furthermore, our sparsity requirement is consistent with a number of works on network bootstraps for nondegenerate $U$-statistics (Levin and Levina, 2019; Lunde and Sarkar, 2019; Zhang and Xia, 2020; Lin et al., 2020a,b). which occur as subgraph frequencies. An important aspect of our results is that our test statistic is a degenerate (two-sample) $U$-statistic (e.g. Serfling (1980)).

The paper is organized as follows. In the following subsection, we motivate the problem more thoroughly, and in Section 2, we give the relevant definitions and describe our setting. In Section 3 we state our main theoretical results for sparse, indefinite random graphs with negative and repeated eigenvalues and describe a modification to handle repeated eigenvalues. In Section 4 we show our results on simulated data, and in Section 5 we discuss our results. Section 6 contains the proofs of our main results.

**Notation:** We use capital letters to denote random vectors $X \in \mathbb{R}^d$, bold lowercase letters to denote fixed vectors, and bold capital letters for fixed or random matrices (which will be clear from context). The distribution of a random vector $X$ will be denoted by $F_X$, and for $X_1, \ldots, X_n$ i.i.d. some distribution $F_X$, we use $X$ to denote the $n \times d$ matrix with its rows the vectors $X_1, \ldots, X_n$. In many occasions, given $X_1, \ldots, X_n$ i.i.d. $F_X$, we let $X$ denote a realization from $F$ that is independent from $\{X_i\}_{i=1}^n$. We write $\| \cdot \|$ for the usual Euclidean norm on vectors and the spectral norm on matrices and $\| \cdot \|_F$ for the Frobenius norm. For a matrix $M$ we write its $\ell_2$ to $\ell_\infty$ operator norm via $\|M\|_{2,\infty} \equiv \max_i \|M_i\|$, where $M_i$ are the rows of $M$. We use $I_k$ to denote the $k \times k$ identity matrix. For a matrix $M$, the operator $\text{diag}(M)$ extracts its diagonal as a matrix, and for two matrices $M$ and $N$, the operator $\text{bdiag}(M, N)$ constructs the block-diagonal matrix $\begin{pmatrix} M & 0 \\ 0 & N \end{pmatrix}$. We write $f(n) = O(g(n))$ if there exists a constant $C$ such that $f(n) \leq Cg(n)$ for all $n$ sufficiently large, and $f(n) = \omega(g(n))$ if there exists a constant $c$ such that $cg(n) \leq f(n)$ for all $n$ sufficiently large. We also write $f(n) \gg g(n)$ if $g(n)/f(n) \to 0$ as $n \to \infty$.

### 1.1 Motivating Example

Suppose there are $n$ and $m$ vertices in two different graphs respectively, and suppose the vertices can be partitioned into three disjoint sets, called communities, where each vertex belongs to community $k$, $k \in \{1, 2, 3\}$, with probability $1/3$. Suppose further that for vertices in the same community, the probability of connection is $a$ and for vertices in different communities the probability of connection is $b$; such a model is referred to as the three-block balanced homogenous stochastic blockmodel in the literature. The matrix

$$B := \begin{pmatrix} a & b & b \\ b & a & b \\ b & b & a \end{pmatrix}$$
Figure 1: Comparisons of the naïve sign-flip alignment procedure (left) and the optimal transport alignment procedure (right) for two adjacency spectral embeddings for the stochastic blockmodel. On the left hand side, we see that visually the clusters do not lie on top of each other, and on the right hand side, the clusters appear to lie on top of each other.

has three eigenvalues; one always positive eigenvalue of \( a + 2b \) and a repeated eigenvalue \( a - b \), which is negative when \( b > a \). Let \( Z^{(1)} \in \{0, 1\}^{n \times 3} \) be the matrix such that \( Z^{(1)}_{ik} = 1 \) if vertex \( i \) belongs to community \( k \) and 0 otherwise, and similarly for \( Z^{(2)} \in \{0, 1\}^{m \times 3} \). Define

\[
P^{(1)} := Z^{(1)} B (Z^{(1)})^\top; \quad P^{(2)} := Z^{(2)} B (Z^{(2)})^\top.
\]

Now, consider the eigendecomposition of \( P^{(1)} \) and \( P^{(2)} \). In the setting that exactly \( 1/3 \) of the vertices belong to each community respectively, the eigenvalues are simply scaled by \( n/3 \) or \( m/3 \) for graphs one and two respectively. Furthermore, if one scales the eigenvectors by the square roots of the absolute values of these eigenvalues, by viewing each row as a point, one obtains three distinct points on \( \mathbb{R}^3 \) that remain constant in \( n \) and \( m \). For example, the first eigenvector scaled by the \( \sqrt{n/3} (a + 2b) \) yields the vector whose entries are all \( \sqrt{a+2b}/3 \).

Elementary computation shows that the second and third eigenvector correspond to the same eigenvalue \( n(a-b)/3 \) and correspond to a two-dimensional subspace. Hence, even though scaling the eigenvector by \( \sqrt{n(a-b)/3} \) (or \( \sqrt{m(a-b)/3} \)) yields a term that does not change as \( n \) and \( m \) increase, it is not defined uniquely because of the repeated eigenvalue. In fact, since the second and third eigenvectors correspond to any choice of basis for the subspace corresponding to the eigenvalue \( n(a-b)/3 \), one can arbitrarily rotate the second and third eigenvectors by any \( 2 \times 2 \) orthogonal transformation and still obtain eigenvectors.

Suppose one observes two graphs \( A^{(1)} \in \{0, 1\}^{n \times n} \) and \( A^{(2)} \in \{0, 1\}^{m \times m} \) with \( A^{(1)} \) independent of \( A^{(2)} \), where each \( A_{ij} \sim \text{Bernoulli}(P_{ij}) \) for \( i \leq j \), with \( A_{ij} = A_{ji} \) for \( j \leq i \) and similarly for \( A^{(2)} \). A common way to estimate the scaled eigenvectors of the matrices \( P^{(1)} \) and \( P^{(2)} \) given observed graphs \( A^{(1)} \) and \( A^{(2)} \) is the adjacency spectral embedding, which here is just the leading 3 eigenvectors scaled by the square roots of the absolute values of their eigenvalues (see Definition 4). Even if the adjacency spectral embedding is consistent, it will only be consistent up to some transformation that takes into account the nonidentifiability.
of the second and third eigenvectors.

As we will make clear in Section 2.1, when we refer to (in)equality in distribution we are referring to the fact that two stochastic blockmodels could be different in both the block assignment probabilities and the block probabilities matrix $B$. If one knew an a priori correspondence between the vertices, one could simply perform orthogonal Procrustes, which has a closed-form solution. Unfortunately, for two graphs of different sizes, there are many situations in which one need not have a correspondence between the graphs. Therefore, though the two graphs do not have an a priori alignment, under the null hypothesis that the two graphs have the same distribution there is a block-orthogonal matrix $\tilde{W}$ that will approximately align the supports of the empirical distributions of the rows of the adjacency spectral embeddings (see Proposition 2). Motivated by this problem and the literature on optimal transport, we show that estimating the orthogonal matrix by aligning the support of the empirical distributions suffices to obtain consistency; in particular, we propose minimizing the Orthogonal Wasserstein Distance, which outputs a block-orthogonal matrix $\hat{W}$ (see Section 3.2). This remedies the nonidentifiability of the second and third eigenvectors above.

For a generic eigenvector, if the corresponding eigenvalue has multiplicity one, then the only freedom in selecting the eigenvector is the choice of sign. In general, ignoring the orthogonal transformations would yield a test statistic that minimizes over all possible sign-flip combinations. In Figure 1, we plot the second and third dimensions of the adjacency spectral embeddings for adjacency matrices simulated from a stochastic blockmodel with connectivity matrix $B = \begin{pmatrix} .5 & .8 & .8 \\ .8 & .5 & .8 \\ .8 & .8 & .5 \end{pmatrix}$, and probability of community membership $\frac{1}{3}$ for each community. In the left figure, we plot the second and third dimensions of the adjacency spectral embeddings for two graphs on $n = 300$ vertices, where the first embedding is rotated using only the best sign flip. Here “best” corresponds to the minimum value of the test statistic. In the right figure, we plot the second and third dimensions of the two embeddings after using the optimal transport-based alignment we outline in Section 3.2.

From a purely visual standpoint, when the distributions for each graph are the same, the empirical distributions should lie approximately on top of one another; however, we see that sign flips fail to recover this correspondence. The left hand side shows visually how the second and third dimensions are not approximately aligned, and the right hand illustrates how the supports of the distributions approximately lie on top of one another, showing that estimating the rotation approximately recovers the implicit distributional correspondence. This figure demonstrates an important point for spectral methods in statistical network analysis: simply ignoring repeated eigenvalues could yield inconsistent testing. Section 4 contains further simulations and quantitative comparisons under more general model settings.

2 Preliminaries

We will now situate the hypothesis test described in the previous section in the general setting in which we will be performing our hypothesis test.

2.1 Setting

We use the latent position framework of the generalized random dot product graph proposed in Rubin-Delanchy et al. (2020) and closely related to that in Lei (2020b). First, we discuss the notion of a $(p,q)$ admissible distribution. In what follows, the matrix $I_{p,q}$ is defined as $I_{p,q} := \text{diag}(I_p, -I_q)$.

**Definition 1.** We say $F_X$ with support $\Omega \subset \mathbb{R}^d$ is a $(p,q)$ admissible distribution if for all $x, y \in \Omega$, $x^\top I_{p,q} y \in [0,1]$.

For a fixed $(p,q)$ admissible distribution, we consider the generalized random dot product graph as follows.

**Definition 2 (Rubin-Delanchy et al. (2020)).** We say a graph $A \in \{0,1\}^{n \times n}$ is a generalized random dot product graph on $n$ vertices with $(p,q)$-admissible distribution $F_X$, sparsity factor $\alpha_n$, and latent positions
$X$ if the matrix $A$ is symmetric, and the entries $A_{ij}$ are conditionally independent given $X$ and Bernoulli random variables with

$$\Pr(A_{ij} = 1|X) = \alpha_nX^\top_i I_{p,q}X_j$$

with $A_{ij} = A_{ji}$, and $X_1, \ldots, X_n \sim F_X$ are i.i.d. We write $(A, X) \sim \text{GRDPG}(F_X, n, \alpha_n)$.

The introduction of the matrix $I_{p,q}$ is to model large in magnitude negative eigenvalues of the adjacency matrix. The GRDPG model allows for arbitrary low-rank models, so is sufficiently agnostic to provide a meaningful setting for nonparametric inference. In a nonparametric setting, the parameters for the GRDPG model are simply the signature $(p, q)$, the sparsity parameter $\alpha_n$, and the distribution $F_X$ (which may or may not be parametric). For this work, we assume the signature $(p, q)$ is known. Practically speaking, this model is equivalent to assuming only that the probability generating matrix is low-rank, and there are several works showing that low-rank models can approximate full-rank models arbitrarily well (Tang et al., 2013; Xu, 2018; Udell and Townsend, 2019; Lei, 2020b; Rubin-Delanchy, 2020). Finally, in the setting $q = 0$, one recovers the random dot product graph (RDPG) model (Athreya et al., 2017), which assumes a low rank, positive semidefinite probability matrix.

One potential issue with the GRDPG model is that it exhibits nonidentifiability. To be explicit, suppose $Q$ is a matrix such that $Q I_{p,q} Q^\top = I_{p,q}$ (this is known as the indefinite orthogonal group $\mathcal{O}(p,q)$). Define the distribution $F_X := F_X \circ Q$, where $F_X \circ Q$ means that one generates $X_i \sim F_X$ and then left multiplies the vectors $X_i$ by $Q^\top$. Then the probabilities of each edge are fixed since

$$\Pr_{F_X}(A_{ij} = 1|X) = \alpha_n(Q^\top X_i)^\top I_{p,q}(Q^\top X_j)$$

$$= \alpha_nX_i^\top (Q I_{p,q} Q^\top)X_j$$

$$= \alpha_nX_i^\top I_{p,q}X_j$$

$$= \Pr_{F_X}(A_{ij} = 1|X).$$

Hence, the distribution of the graph remains unchanged if one transforms the support of $F_X$ by any indefinite orthogonal transformation $Q$. Therefore, any nonparametric test of equality of distribution must allow equality up to indefinite orthogonal transformations. This motivates the following definition.

**Definition 3.** Let $F_X$ and $F_Y$ be two $(p, q)$ admissible distributions. We say $F_X$ and $F_Y$ are equal up to indefinite orthogonal transformation if there exists a matrix $Q \in \mathcal{O}(p, q)$ such that

$$F_X = F_Y \circ Q.$$

In this case, we write $F_X \simeq F_Y$.

We note that in the RDPG model, $F_X \simeq F_Y$ is equivalent to saying the distributions are equivalent up to orthogonal transformation, since when $q = 0$ the nonidentifiability is of the form of orthogonal matrices.

We are now ready to formally describe our hypothesis test under the generalized random dot product graph framework. Suppose we observe two graph adjacency matrices $A^{(1)}$ and $A^{(2)}$ such that $(A^{(1)}, X) \sim \text{GRDPG}(F_X, n, \alpha_n)$ and $(A^{(2)}, Y) \sim \text{GRDPG}(F_Y, m, \beta_m)$ are mutually independent and have the same signature $(p, q)$. We consider the hypothesis test

$$H_0 : F_Y \simeq F_X$$

$$H_A : F_Y \not\simeq F_X.$$

Again, we assume throughout that $(p, q)$ is known and fixed in $n$ and $m$. In general, we do not assume $(\alpha_n, \beta_m)$ are known, but, for ease of exposition we shall first assume that they are known. They can be estimated consistently, so we will revisit these issues later (see Corollary 2).

**Remark 1** (Equivalence to Section 1.1). Although the above hypothesis test seems to suffer from a lack of identifiability, the nonidentifiability is primarily an artifact of working in the framework of the GRDPG model. Were we to reformulate the test in Section 1.1 in terms of the stochastic block model with generating matrices $B^{(1)}$ and $B^{(2)}$ and probability vectors $\pi^{(1)}$ and $\pi^{(2)}$, this test would just be determining whether both $B^{(1)} = B^{(2)}$ and $\pi^{(1)} = \pi^{(2)}$ up to permutation of the communities. To see the explicit equivalence,
one can transform any stochastic blockmodel with connectivity matrix $B$ into a GRDPG model by letting $B = V D V^\top$ and fixing $K$ vectors $\nu_1, \ldots, \nu_K$ as the rows $V|D|^{1/2}$, where $V$ is chosen arbitrarily if there are repeated eigenvalues. Then the GRDPG model in question is just a mixture of point masses corresponding to the entries of $\pi$, where the vectors are the rows of $V|D|^{1/2}$.

This test also allows one graph to be a submodel of the other. For example, this test includes the situation that $A^{(1)}$ comes from a stochastic blockmodel and $A^{(2)}$ comes from a model that may be broader, such as the mixed-membership stochastic blockmodel.

In practice, one observes only the graph adjacency matrix, and therefore must estimate the latent position matrix $X$. The statistical properties of the scaled eigendecomposition, referred to as the adjacency spectral embedding (ASE), are investigated in Rubin-Delanchy et al. (2020). The definition is given below.

**Definition 4** (Adjacency Spectral Embedding). Suppose $(A, X) \sim GRDPG (F_X, n, \alpha_n)$, and write the eigendecomposition of $A$ as $\sum_{i=1}^n \lambda_i u_i u_i^\top$, where the $\lambda_i$ are ordered by magnitude; $|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_n|$, and the $u_i$ are orthonormal. Form the $d \times d$ matrix $\Lambda_A$ by taking the $d$ largest (in magnitude) eigenvalues of $A$ sorted by positive and then negative components, and the $n \times d$ matrix $U_A$ with columns consisting of the eigenvectors associated to the eigenvalues in $\Lambda_A$. The adjacency spectral embedding of $A$ is the $n \times d$ matrix

$$\hat{X} := U_A |\Lambda_A|^{1/2},$$

where the operator $| \cdot |$ takes the absolute values of the entries.

### 2.2 A Kernel Estimator

To describe our test statistic, we must first define mean embedding of a distribution. Consider a symmetric positive-definite kernel $\kappa(\cdot, \cdot) : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ with associated reproducing kernel Hilbert space $\mathcal{H}$. The mean embedding of a distribution function $F$ with support $\Omega$ is defined via

$$\mu[F] := \int_{\Omega} \kappa(\cdot, x) dF(x).$$

A kernel $\kappa$ is called characteristic (Sriperumbudur et al., 2011) if the embedding $\mu[F]$ is injective, so that $F = G$ if and only if $\mu[F] = \mu[G]$. Examples of characteristic kernels include the Gaussian kernel $\kappa(x, y) = \exp\left(-\frac{1}{2\sigma^2} ||x - y||^2\right)$ and the Laplace kernel $\kappa(x, y) = \exp\left(-\frac{1}{\sigma} ||x - y||_1\right)$.

Since $\kappa$ is a function of two variables, given independent samples $X = \{X_i\}_{i=1}^n$ and $Y = \{Y_j\}_{j=1}^m$, we define the (two-sample) $U$-statistic

$$U_{n,m}(X, Y) := \frac{1}{n(n-1)} \sum_{j \neq i} \kappa(X_i, X_j) - \frac{2}{mn} \sum_{i=1}^n \sum_{k=1}^m \kappa(X_i, Y_k) + \frac{1}{m(m-1)} \sum_{l \neq k} \kappa(Y_l, Y_l).$$

In the asymptotic regime that $n$ and $m$ tend to infinity, under the assumption $\frac{m}{n+m} \to \rho \in (0, 1)$, Gretton et al. (2012) showed that

$$U_{n,m}(X, Y) \to \|\mu[F_X] - \mu[F_Y]\|_{\mathcal{H}}^2$$

almost surely, and, when $\kappa$ is characteristic, then $\|\mu[F_X] - \mu[F_Y]\|_{\mathcal{H}}^2 = 0$ if and only if $F_X = F_Y$. Moreover, they showed that $(n+m)U_{n,m}(X, Y)$ has a nondegenerate limiting distribution under the null hypothesis $F_X = F_Y$. The scaling by $(n+m)$ is due to the fact that the $U$-statistic is degenerate, where degeneracy of a $U$-statistic with kernel $h$ of two variables means that $E_{F_X}(h(X, \cdot))$ is constant. See, for example, Serfling (1980) for more details on the theory of degenerate $U$-statistics.

### 3 Hypothesis Testing With Negative and Repeated Eigenvalues

We now present a detailed asymptotic analysis of our two-sample test statistic. Given two graphs $A^{(1)}$ and $A^{(2)}$ on $n$ and $m$ vertices respectively our test statistic is defined via

$$U_{n,m}(\hat{X}, \hat{Y}) := \frac{1}{n(n-1)} \sum_{j \neq i} \kappa\left(\hat{X}_i, \hat{X}_j\right) - \frac{2}{mn} \sum_{i=1}^n \sum_{k=1}^m \kappa\left(\hat{X}_i, \hat{Y}_k\right) + \frac{1}{m(m-1)} \sum_{l \neq k} \kappa\left(\hat{Y}_l, \hat{Y}_l\right).$$
where $\hat{X}$ and $\hat{Y}$ are the adjacency spectral embeddings of $A^{(1)}$ and $A^{(2)}$ respectively. In what follows, all of our asymptotic results are stated as $n$ and $m$ tend to infinity. We will require some assumptions on the kernel $\kappa$.

**Assumption 1.** The kernel $\kappa$ is characteristic, radial, and twice continuously differentiable on $\mathbb{R}^d$.

The assumption that $\kappa$ is radial is so that our results can be expressed in terms of individual orthogonal matrices that may themselves be products of several orthogonal matrices. For example, we have the identity $U(XW_1, YW_2) = U(\hat{X}\hat{W}, \hat{Y})$, where $\hat{W} = W_1W_2^\top$. Our main results can be modified slightly to hold without this assumption at the penalty of introducing more orthogonal matrices. Differentiability is a relatively mild requirement, and the assumption of $\kappa$ being characteristic is satisfied by continuous kernels whose embeddings are dense in $\mathcal{H}$ equipped with the supremum norm, since the support of $F_X$ and $F_Y$ can be taken to be compact (see Theorem 3), and hence any universal kernel defined on $\mathbb{R}^d$ is characteristic for the problem we consider herein.

Since real-world graphs are sparse, we conduct a more thorough study of our test statistic under sparsity. First, we make assumptions on the sparsity for which our more general results hold. We implicitly assume that either $\alpha_n, \beta_n \rightarrow 0$ or that $\alpha_n \rightarrow 0$ and $\beta_n \rightarrow 0$, since if $\alpha_n$ or $\beta_n$ are converging to some constant greater than zero, one can just rescale the distribution $F_X$ or $F_Y$.

**Assumption 2a.** The sparsity parameters for the graphs satisfy

$$\min(n\alpha_n, m\beta_m) = \omega(\log^4(n)),$$

and

$$\frac{m\beta_m}{m\beta_m + n\alpha_n} \rightarrow \rho \in (0, 1).$$

If instead we have a slightly denser graph, we make the following assumption.

**Assumption 2b.** The sparsity parameters for the graphs satisfy for some $\eta > 0$,

$$\min(n\alpha_n, m\beta_m) = \omega(n^{1/2}\log^{1+\eta}(n)),$$

and

$$\frac{m}{m + n} \rightarrow \rho \in (0, 1).$$

In both asymptotic regimes, there are two competing factors: the first is in the approximation of the unperturbed $U$-statistic to the population value; that is, the $U$-statistic obtained given access to the latent vectors $X_1, \ldots, X_n$, and the second is in the approximation of the estimated $U$-statistic to the unperturbed $U$-statistic. In the first asymptotic regime, the primary difficulty stems from the approximation of the $U$-statistic to the maximum mean discrepancy between two appropriately defined distributions. The asymptotic regime in Assumption 2b up to the logarithmic term has been assumed in the literature (Tang et al., 2017c; Jones and Rubin-Delanchy, 2020) and is a common assumption in the theory of bootstrapped $U$-statistics for random graphs, particularly as they pertain to subgraph counts. See Levin and Levina (2019); Lunde and Sarkar (2019); Zhang and Xia (2020); Lin et al. (2020a), and Lin et al. (2020b) for details.

When considering negative eigenvalues, if one uses the GRDPG model framework, one necessarily has to contend with indefinite orthogonal transformations. From the equation $Q I_{p,q} Q^\top = I_{p,q}$ we have that $|\det(Q)| = 1$, and hence $Q$ is invertible and $Q^{-1} \in \mathcal{O}(p,q)$ as well. We also note that the set $\mathcal{O}(p,q)$ includes block-diagonal orthogonal matrices; i.e., if we have $W_p$ and $W_q$ for $p \times p$ and $q \times q$ orthogonal matrices, then

$$\begin{pmatrix} W_p & 0 \\ 0 & W_q \end{pmatrix} I_{p,q} \begin{pmatrix} W_p^\top & 0 \\ 0 & W_q^\top \end{pmatrix} = \begin{pmatrix} W_p W_p^\top & 0 \\ 0 & -W_q W_q^\top \end{pmatrix} = I_{p,q}.$$

We refer to the subgroup $\mathcal{O}(p,q) \cap \mathcal{O}(d)$ as the subgroup of block-orthogonal matrices. Note that $\|Q\| = 1$ for any block-orthogonal $Q$, whereas for any finite $M > 0$, there exists $Q \in \mathcal{O}(p,q) \setminus \mathcal{O}(d)$ with $\|Q\| > M$. 

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Therefore, allowing for negative eigenvalues involves studying matrices \( Q \in \mathbb{O}(p,q) \) that could be badly behaved (in a spectral norm sense). Nevertheless, using the limiting results in Agterberg et al. (2020), by subtly passing between indefinite and Euclidean geometry, we can show that when using the adjacency spectral embeddings, one does not even have to consider indefinite orthogonal matrices. Our first proposition shows that the \( U \)-statistic applied to the rows of the adjacency spectral embedding yields a consistent test. All proofs are deferred to Section 6.

**Proposition 1.** Let \( (A^{(1)}, X) \sim GRDPG(F_X, n, \alpha_n) \) and \( (A^{(2)}, Y) \sim GRDPG(F_Y, m, \beta_m) \) be independent. Suppose Assumption 2a or Assumption 2b is satisfied, and suppose further that \( \kappa \) satisfies Assumption 1. Set
\[
\Delta_X := \mathbb{E}(XX^\top),
\]
and similarly for \( \Delta_Y \). Suppose that both \( \Delta_X I_{p,q} \) and \( \Delta_Y I_{p,q} \) have distinct eigenvalues. Then
\[
U_{n,m}(\hat{X}/\alpha_n^{1/2}, \hat{Y}/\beta_m^{1/2}) \to \begin{cases} 
0 & F_X \simeq F_Y \\
c > 0 & F_X \not\simeq F_Y,
\end{cases}
\]
almost surely.

Our main requirements are the uniqueness of a certain indefinite orthogonal matrix from Agterberg et al. (2020), which is given by the assumption that \( \Delta_X I_{p,q} \) and \( \Delta_Y I_{p,q} \) have distinct eigenvalues which corresponds to distinct eigenvalues of \( P^{(1)} \) and \( P^{(2)} \). Without distinct eigenvalues, one is still able to obtain consistency up to a block orthogonal transformation.

**Proposition 2.** Let \( (A^{(1)}, X) \sim GRDPG(F_X, n, \alpha_n) \) and \( (A^{(2)}, Y) \sim GRDPG(F_Y, m, \beta_m) \) be independent. Suppose Assumption 2a or Assumption 2b is satisfied, and suppose further that \( \kappa \) satisfies Assumption 1. If \( F_X \simeq F_Y \), there exists a sequence of block orthogonal matrices \( \hat{W}_n \in \mathbb{O}(p,q) \cap \mathbb{O}(d) \) such that
\[
U_{n,m}(\hat{X}\hat{W}_n/\alpha_n^{1/2}, \hat{Y}/\beta_m^{1/2}) \to 0
\]
almost surely. However, if \( F_X \not\simeq F_Y \), then for any sequence of orthogonal matrices \( \hat{W}_n \in \mathbb{O}(p,q) \cap \mathbb{O}(d) \), there exists a constant \( C > 0 \) depending only on \( F_X \) and \( F_Y \) such that almost surely
\[
\liminf_{n,m} U_{n,m}(\hat{X}\hat{W}_n/\alpha_n^{1/2}, \hat{Y}/\beta_m^{1/2}) \geq C.
\]

We emphasize that these results are desirable since the matrices \( \hat{W}_n \) are block orthogonal matrices and not indefinite orthogonal matrices, and hence the only estimation required is the matrix \( \hat{W}_n \). Proposition 2 suggests that if one can estimate the matrices \( \hat{W}_n \) consistently, then we can devise a consistent test procedure through a permutation test and bootstrapping the test statistic distribution.

### 3.1 Main Results

Our main results include a more refined study of our test statistic. Define
\[
P^{(1)} := \alpha_n X I_{p,q} X^\top; \quad P^{(2)} := \beta_m Y I_{p,q} Y^\top,
\]
and let \( U_X A_X U_X^\top \) and \( U_Y A_Y U_Y^\top \) be their respective eigendecompositions, with \( A_X \) and \( A_Y \) arranged with the \( p \) positive eigenvalues first and \( q \) negative eigenvalues second. Define
\[
\hat{X} := U_X A_X^{1/2}; \quad \hat{Y} := U_Y A_Y^{1/2}.
\]
The matrices \( \hat{X} \) and \( \hat{Y} \) can be viewed as surrogates for the matrices \( \alpha_n^{1/2} X \) and \( \beta_m^{1/2} Y \) up to indefinite orthogonal transformation. In fact, the proof of Proposition 1 reveals that under the distinct eigenvalues assumption there exists a block-orthogonal matrix \( \hat{W}_n \) such that
\[
U_{n,m}(\hat{X}/\sqrt{\alpha_n}, \hat{Y}/\sqrt{\beta_m}) - U_{n,m}(\hat{X}\hat{W}_n/\alpha_n^{1/2}, \hat{Y}/\beta_m^{1/2}) \to 0
\]
and, furthermore,
\[
U_{n,m}(\hat{X}\hat{W}_n/\alpha_n^{1/2}, \hat{Y}/\beta_m^{1/2}) \to \|\mu[F_X \circ \hat{Q}_X^{-1}] - \mu[F_Y \circ \hat{Q}_Y^{-1}]\|^2_{\hat{Y}},
\]
where \( \hat{Q}_X \) and \( \hat{Q}_Y \) are indefinite orthogonal matrices defined only in terms of the distributions \( F_X, F_Y \) and the signature \((p,q)\). Therefore, we analyze the convergence of a scaled \( U \)-statistic.
Theorem 1. Let \((A^{(1)}, X) \sim GRDPG(F_X, n, \alpha_n)\) and \((A^{(2)}, Y) \sim GRDPG(F_Y, m, \beta_m)\) be independent, and suppose Assumption 2a is satisfied. Suppose \(\Delta_X I_{p,q}\) and \(\Delta_Y I_{p,q}\) have distinct eigenvalues, and let \(\kappa\) satisfy Assumption 1. Then, under the null hypothesis \(F_X \simeq F_Y\), there exists a sequence of block-orthogonal matrices \(W_n \in \mathbb{O}(d) \cap \mathbb{O}(p, q)\) such that

\[
(m \beta_m + n \alpha_n) \left( U_{n,m} \left( \frac{\hat{X}}{\sqrt{\alpha_n}}, \frac{\hat{Y}}{\sqrt{\beta_m}} \right) - U_{n,m} \left( \frac{\hat{X}W_n}{\sqrt{\alpha_n}}, \frac{\hat{Y}}{\sqrt{\beta_m}} \right) \right) \rightarrow 0
\]

almost surely. If instead \(F_X \not\simeq F_Y\),

\[
\frac{(m \beta_m + n \alpha_n)}{\log(n)} \left( U_{n,m} \left( \frac{\hat{X}}{\sqrt{\alpha_n}}, \frac{\hat{Y}}{\sqrt{\beta_m}} \right) - U_{n,m} \left( \frac{\hat{X}W_n}{\sqrt{\alpha_n}}, \frac{\hat{Y}}{\sqrt{\beta_m}} \right) \right) \rightarrow 0
\]

almost surely.

The hypotheses of the previous theorem include that \(\Delta I_{p,q}\) has distinct eigenvalues. Similar to Proposition 2, we can actually remove this assumption if we are willing to include additional orthogonal matrices.

Theorem 2. Consider the setting of Theorem 1, but suppose that \(\Delta I_{p,q}\) does not necessarily have distinct eigenvalues. Then, we have that under the null \(F_X \simeq F_Y\) there exist two sequences of block-orthogonal matrices \(\hat{W}_n\) and \(W_n\) such that

\[
(m \beta_m + n \alpha_n) \left( U_{n,m} \left( \frac{\hat{X}W_n}{\sqrt{\alpha_n}}, \frac{\hat{Y}}{\sqrt{\beta_m}} \right) - U_{n,m} \left( \frac{\hat{X}W_n}{\sqrt{\alpha_n}}, \frac{\hat{Y}}{\sqrt{\beta_m}} \right) \right) \rightarrow 0
\]

almost surely. If instead \(F_X \not\simeq F_Y\), then

\[
\frac{(m \beta_m + n \alpha_n)}{\log(n)} \left( U_{n,m} \left( \frac{\hat{X}W_n}{\sqrt{\alpha_n}}, \frac{\hat{Y}}{\sqrt{\beta_m}} \right) - U_{n,m} \left( \frac{\hat{X}W_n}{\sqrt{\alpha_n}}, \frac{\hat{Y}}{\sqrt{\beta_m}} \right) \right) \rightarrow 0
\]

almost surely.

In Theorem 2, the additional orthogonal matrix \(\hat{W}_n\) appears because without distinct eigenvalues assumption \(\hat{X}\) and \(\hat{Y}\) need to be simultaneously aligned to each other as well as to \(\hat{X}\) and \(\hat{Y}\). Again, a priori there are indefinite orthogonal transformations to contend with, but, as we show in Section 6, we can effectively bypass these transformations through careful analyses of their convergence. Similar to the proof of Proposition 1, the proof of Theorem 2 reveals that \(U_{n,m}(\hat{X}W_n/\alpha_n^{1/2}, \hat{Y}/\beta_m^{1/2})\) is converging to a quantity that depends only on population parameters; however, without the distinct eigenvalues assumption the matrices \(Q_X^{-1}\) and \(Q_Y^{-1}\) are no longer unique, so the convergence analysis and its statement require careful tabulation of additional block-orthogonal matrices.

If instead Assumption 2b holds, one can obtain a similar limiting result without including the sparsity in the scaling under the null hypothesis.

Corollary 1. Suppose the setting of Theorem 2, but suppose instead that Assumption 2b is satisfied. Under the null hypothesis, we have that

\[
(m + n) \left( U_{n,m} \left( \frac{\hat{X}W_n}{\sqrt{\alpha_n}}, \frac{\hat{Y}}{\sqrt{\beta_m}} \right) - U_{n,m} \left( \frac{\hat{X}W_n}{\sqrt{\alpha_n}}, \frac{\hat{Y}}{\sqrt{\beta_m}} \right) \right) \rightarrow 0
\]

almost surely, for the same sequences of orthogonal matrices \(\hat{W}_n\) and \(W_n\) as in Theorem 2.

Finally, we note that in general the sparsity factors \(\alpha_n\) and \(\beta_m\) are not known. If instead we wish to use the estimated sparsity factors, we have the following result.
Corollary 2. Assume that $E(X_1^T I_{p,q} X_2) = 1$ and that $\alpha_n, \beta_m \to 0$. Define

$$\hat{\alpha}_n := \frac{1}{m} \sum_{i<j} A_{ij}^{(1)}, \quad \hat{\beta}_m := \frac{1}{n} \sum_{i<j} A_{ij}^{(2)}.$$

Then the limiting results in Theorem 1, Theorem 2, and Corollary 1 all hold under their respective conditions with $\overline{X}/\alpha_n^{1/2}$ and $\overline{Y}/\beta_m^{1/2}$ replaced with $\hat{X}/\alpha_n^{1/2}$ and $\hat{Y}/\beta_m^{1/2}$ respectively and the almost sure convergence replaced with convergence in probability.

The condition $E(X_1 I_{p,q} X_2) = 1$ is used only for identifiability of $\alpha_n$ and $\beta_m$ when they need to be estimated. See e.g., Lunde and Sarkar (2019) for an identical condition in the setting of graphons.

3.1.1 Interpretation

There are several different alignment matrices that appear in order to show the convergence in Theorems 1 and 2. However, in our analysis we are able to show that only the indefinite orthogonal matrices that are simultaneously orthogonal have any effect on the limiting values. Given Propositions 1 and 2, the main results in Theorems 1 and 2 further detail that under the null hypothesis $F_X \simeq F_Y$ one can perform consistent testing given access to only the graphs $A^{(1)}$ and $A^{(2)}$. The results of Gretton et al. (2012) imply that $(m + n)U_{n,m}(X, Y)$ has a nondegenerate limiting distribution under the null hypothesis. For graphs with average degree growing faster than $n^{1/2} \log(n)$, Corollary 1 says that the same scaling occurs under the null hypothesis with $\overline{X}$ and $\overline{Y}$ as replacements for $X$ and $Y$. For almost surely dense graphs; i.e. graphs with $\alpha_n = \beta_m = 1$, Theorems 1 and 2 also provide a result under the alternative.

As mentioned in Section 3.1, under the distinct eigenvalues assumption, the proof of Proposition 1 reveals that

$$U_{n,m}(\hat{X}W_n/\sqrt{\alpha_n}, \hat{Y}/\sqrt{\beta_m}) \to \| \mu[F_X \circ \hat{Q}_X^{-1}] - \mu[F_Y \circ \hat{Q}_Y^{-1}] \|^2,$$

where $\hat{Q}_X^{-1}$ and $\hat{Q}_Y^{-1}$ are deterministic quantities depending only on $F_X$ and $F_Y$. A similar convergence happens without the distinct eigenvalues assumption but with additional block-orthogonal matrices. Lemma 2 shows that the rate of the approximation of $\hat{X}W_n/\sqrt{\alpha_n}$ and $\hat{Y}/\sqrt{\beta_m}$ to $X\hat{Q}_X^{-1}$ and $Y\hat{Q}_Y^{-1}$ is of order $\sqrt{\log(n)/n}$, which, in general, is not fast enough to guarantee the convergence of

$$(n + m) \left( U_{n,m}(\hat{X}W_n/\alpha_n^{1/2}, \hat{Y}/\beta_m^{1/2}) - U_{n,m}(X\hat{Q}_X^{-1}, Y\hat{Q}_Y^{-1}) \right)$$

to zero. However, Propositions 1 and 2 show that the $U$-statistic evaluated at $\hat{X}W_n$ and $\hat{Y}$ still tends to zero under the null hypothesis and to a constant under the alternative, which, together with Theorems 1 and 2, suggests that $(n + m)U_{n,m}(\hat{X}W_n/\alpha_n^{1/2}, \hat{Y}/\beta_m^{1/2})$ has a nondegenerate limiting distribution.

For testing purposes, the lack of distributional results is of no consequence, since if one can reliably estimate the orthogonal transformation $\hat{W}_n$ appearing in Theorem 2 (and Proposition 2) then one can perform consistent testing through a bootstrapped permutation test; see the following section. Furthermore, the limiting distribution for the maximum mean discrepancy between two distributions $F_X$ and $F_Y$ will not be independent of $F_X$ and $F_Y$ in general, so one may have to use a permutation test to approximate the null distribution anyways.

For sparser graphs, the almost sure convergence in Theorems 1 and 2 under the null hypothesis requires the scaling $m\beta_m + n\alpha_n$, which, if $n \gg m$ and $\alpha_n \gg \beta_m$ is slower than the convergence in Gretton et al. (2012) by a factor of $\alpha_n$. The reason for this stems primarily from the fact that for sparse graphs it is much more difficult to estimate $\overline{X}$ and $\overline{Y}$. The sparsity factor here brings down the effective sample size; one observes only $O(n\alpha_n)$ edges on average for sparse graphs instead of $O(n)$ edges for dense graphs. Therefore, though the scaling may not be optimal, Theorems 1 and 2 provide a more refined study of the test statistic, since Propositions 1 and 2 already imply consistent testing.

We also note that our results can be adapted to the conditional mixed-membership and degree-corrected stochastic blockmodel. Consider a deterministic sequence of matrices $P = P_n$, where $P$ has the structure

$$P = \alpha_n \Theta Z B Z^\top \Theta,$$
where \( Z \in [0,1]^{n \times K} \) is a membership matrix whose rows that sum to 1, and \( \Theta \in (0,1)^{n \times n} \) is a diagonal matrix of degree-correction parameters. For identifiability, assume \( \max_i \Theta_{ii} = 1 \). Let \( B = V D V^\top \) be the eigendecomposition of \( B \), and let \( p \) and \( q \) denote the number of positive and negative entries of \( D \) respectively. Define \( X = \Theta Z V |D|^{1/2} \); then \( P = \alpha_n X_{1:p,q} X^\top \). Though Theorem 2 is not immediately applicable as the rows of \( X \) are no longer drawn i.i.d., the proof can be modified as long as \( \Theta \) and \( Z \) both converge in the sense that the limit \( \frac{1}{n} Z^\top \Theta \theta Z \) exists. If \( B \) is full rank and does not change in \( n \), the notion of repeated eigenvalues then refers to the eigenvalues of the limit of \( \frac{1}{n} Z^\top \Theta \theta Z \), which depends on the particular sequence of \( Z \) and \( \Theta \) matrices.

### 3.2 Optimal Transport for Repeated Eigenvalues

We note that thus far, we have demonstrated that negative eigenvalues do not affect limiting results despite \textit{a priori} having to consider indefinite orthogonal transformations. Such a result is desirable, as one does not have to resort to numerical algorithms optimizing over \( O(p,q) \), which could be unstable due to the ill-conditioning inherent in indefinite orthogonal transformations. Furthermore, we have shown that any modification to our test need estimate only the matrix \( \hat{W}_n \) from Theorem 2. We now draw our attention to estimating \( \hat{W}_n \).

Let \( \hat{X} \) and \( \hat{Y} \) be the adjacency spectral embeddings of \( A^{(1)} \) and \( A^{(2)} \). One can view a collection of points as a distribution by assigning equal point mass to each point. Define \( \hat{F}_X \) as the empirical distribution for \( \hat{X} \) and define \( \hat{F}_Y \) as the empirical distribution for \( \hat{Y} \); that is, \( \hat{F}_X \) places point mass of \( \frac{1}{m} \) at each \( \hat{X}_i \), and \( \hat{F}_Y \) places point mass of \( \frac{1}{m} \) at each \( \hat{Y}_j \). Let \( d_2(\cdot,\cdot) \) denote the Wasserstein \( \ell^2 \) distance between two distributions; that is, given two distributions \( F \) and \( G \), we define

\[
d_2(F,G) := \inf_{\Gamma \in \Gamma_{F,G}} \mathbb{E}_{\Gamma \sim \Gamma_{F,G}} \|X - Y\|_2^{1/2},
\]

where \( \Gamma_{F,G} \) is the set of distributions whose marginals are \( F \) and \( G \). The set \( \Gamma_{F,G} \) is called the set of \textit{couplings} of \( F \) and \( G \). If \( F \) and \( G \) are empirical distributions on \( n \) and \( m \) points respectively, the couplings can be represented by matrices whose rows and columns sum to \( \frac{1}{m} \) and \( \frac{1}{n} \); these are the \textit{assignment matrices}.

In light of Theorem 2, we propose finding the orthogonal matrix \( \hat{W}_n \) that solves the problem

\[
\inf_{W \in O(d) \cap O(p,q)} d_2(\hat{F}_X^{a^{1/2}}, \hat{F}_Y^{b^{1/2}} \circ W),
\]

where \( \hat{F}_Y^{b^{1/2}} \circ W \) is the empirical distribution \( \hat{F}_Y^{b^{1/2}} \) transformed by an orthogonal matrix \( W \). The above distance is considered in both Lei (2020b) and Levin and Levina (2019) as the \textit{orthogonal Wasserstein distance}.

The problem in expression 1 is simultaneously an optimal transport problem in finding the minimum over couplings and a Procrustes problem in finding the minimum over orthogonal matrices. Define the matrix \( C_W \in \mathbb{R}^{n \times m} \) as the \textit{cost matrix with respect to} \( W \) by setting

\[
(C_W)_{ij} := \|X_i - WY_j\|^2.
\]

Then expression 1 can be written as

\[
\min_{W,\Pi} \langle \Pi, C_W \rangle
\]

where the inner product is the Frobenius (matrix) inner product, \( W \) is a block-orthogonal matrix, and \( \Pi \) satisfies \( \Pi 1 = \frac{1}{m} 1 \) and \( \Pi^\top 1 = \frac{1}{n} 1 \); that is, \( \Pi \) is an assignment matrix. We have the following proposition.

\textbf{Proposition 3.} Assume that \( \mathbb{E}(X_1^1 I_{p,q} X_2) = 1 \) and that \( \alpha_n, \beta_m \rightarrow 0 \). Suppose \( \hat{W}_n \) minimizes \( \langle \Pi, C_W \rangle \) over the block-orthogonal matrices. Suppose further that \( F_X \simeq F_Y \); that is, the null hypothesis holds. Then there exists constants \( c > 0 \) and \( C > 0 \) possibly depending on \( d \) such that with probability at least \( 1 - c(n^{-2} + m^{-2}) \),

\[
d_2(\hat{F}_X^{a^{1/2}}, \hat{F}_Y^{b^{1/2}} \circ \hat{W}_n) \leq C \left( \frac{\log^{1/d}(n)}{n^{1/d}} + \frac{\log^{1/d}(m)}{m^{1/d}} + \frac{\log(n)}{(n\alpha_n)^{1/2}} + \frac{\log(m)}{(m\beta_m)^{1/2}} \right).
\]
We also show that the orthogonal Wasserstein distance does not tend to zero under the alternative.

**Proposition 4.** Assume that \( \mathbb{E}(X_{1}^\top I_{p,q}X_{2}) = 1 \) and that \( \alpha_{n}, \beta_{m} \to 0 \). Suppose \( \tilde{W}_{n} \) minimizes \( \langle \Pi, C_{W} \rangle \) over the block-orthogonal matrices. Suppose that \( F_{X} \not\equiv F_{Y} \). Then there exists a constant \( C > 0 \) depending on \( F_{X} \) and \( F_{Y} \) such that

\[
\liminf_{n,m} d_{2}(\hat{F}_{X}/\sqrt{\alpha_{n}}, \hat{F}_{Y}/\sqrt{\beta_{m}} \circ \tilde{W}_{n}) \geq C
\]

almost surely.

Again, the assumption \( \mathbb{E}(X_{1}^\top I_{p,q}X_{2}) = 1 \) is for identifiability of \( \alpha_{n} \) and \( \beta_{m} \). If instead one assumes that \( \alpha_{n} = \beta_{m} = 1 \), the result still holds without the sparsity factors.

The above theory shows that given two adjacency matrices \( A^{(1)} \) and \( A^{(2)} \), calculating the adjacency spectral embeddings \( \hat{X} \) and \( \hat{Y} \), aligning them with an orthogonal matrix by solving Equation 2, and calculating the corresponding \( U \)-statistic, \( U_{n,m}(\hat{X}, \hat{Y}\tilde{W}_{n}) \) yields a consistent test statistic. From there, one can bootstrap the null distribution of \( U_{n,m} \) to get an approximate \( p \)-value. The procedure is summarized in Algorithm 1.

---

**Algorithm 1 Nonparametric Two-Graph Hypothesis Testing**

**Require:** \( A^{(1)} \in \mathbb{R}^{n \times n}, A^{(2)} \in \mathbb{R}^{m \times m} \)

1. Embed \( A^{(1)} \) and \( A^{(2)} \) into \( \mathbb{R}^{d} \) using the adjacency spectral embeddings, obtaining \( \hat{X} \) and \( \hat{Y} \) and sparsity estimates \( \hat{\alpha}^{1/2} \) and \( \hat{\beta}^{1/2} \).
2. Find \( \tilde{W}_{n} \) minimizing Equation 2 above using Algorithm 2;
3. Calculate the value of the \( U \)-statistic \( U_{n,m}(\hat{X}/\hat{\alpha}_{n}^{1/2}, \hat{Y}\tilde{W}_{n}/\hat{\beta}_{m}^{1/2}) \);
4. Bootstrap the \( U \)-statistic distribution assuming the null hypothesis;
5. Calculate the empirical probability of observing \( U_{n,m}(\hat{X}/\hat{\alpha}_{n}^{1/2}, \hat{Y}\tilde{W}_{n}/\hat{\beta}_{m}^{1/2}) \) under the bootstrapped null distribution.
6. **Return** Estimated \( p \)-value.

---

To solve for the matrix \( \tilde{W}_{n} \) in practice, we use the method proposed in Alvarez-Melis et al. (2019) tailored to our specific problem, in which the authors propose solving an entropy-regularized version of the problem which can be done efficiently. Define the auxiliary expression

\[
\inf_{\Pi, W} \langle \Pi, C_{W} \rangle + \varepsilon H(\Pi) \tag{3}
\]

where \( H(\Pi) \) is the entropy of the distribution given by \( \Pi \). For a fixed \( \varepsilon \), Equation 3 can be computed efficiently via the Sinkhorn algorithm (Cuturi, 2013). We then alternately minimize over \( W \) and \( \Pi \) to find the solution. Finally, given a fixed orthogonal matrix \( W \), we project \( W \) onto the block-orthogonal matrices. The (Frobenius) projection is given by the following proposition. We summarize the entire procedure in Algorithm 2.

**Proposition 5.** Let \( W \in \mathbb{O}(d) \). Then

\[
\inf_{R \in \mathbb{O}(d) \cap \mathbb{O}(p,q)} \| R - W \|_{F}
\]

is attained by taking the orthogonal components of the singular value decomposition of the top \( p \times p \) block of \( W \) and the bottom \( q \times q \) block of \( W \).

In essence, the algorithm alternates between solving for an orthogonal transformation given a fixed assignment matrix and solving for the assignment matrix given the orthogonal transformation.

### 3.2.1 Close Eigenvalues

Before moving on, we provide some intuition as to why estimating a rotation can be beneficial even when one does not have exactly repeated eigenvalues. We focus on the positive semidefinite case for convenience, though the analysis for the indefinite case is similar.
Suppose $\mathbb{E}(XX^\top)$ has $d$ distinct eigenvalues, and let $U_A$ and $U_P$ be the leading $d$ eigenvectors of $A$ and $P = \alpha_n XX^\top$. Let $W_*$ be defined via

$$W_* = \inf_{W \in \mathcal{O}(d)} \|U_A - U_P W\|_F,$$

which has a closed-form solution in terms of the left and right singular vectors of the matrix $U_A^\top U_P$. Since $\mathbb{E}(XX^\top)$ has distinct eigenvalues, without loss of generality assume that the columns of $U_A$ are chosen so that the inner product between the columns of $U_A$ and $U_P$ are positive. Then the sequence of matrices $W_*$ is converging to the identity, which also provides the uniqueness (up to sign) of the matrices $\hat{X}$ and $\hat{Y}$.

Define

$$\delta := \min_{1 \leq i \leq d} \left( \lambda_i(\mathbb{E}(XX^\top)) - \lambda_{i+1}(\mathbb{E}(XX^\top)) \right),$$

where $\lambda_{d+1} := -\infty$ by convention. It can be shown (see Appendix A) that

$$\|W_* - I\|_F = O\left(\frac{\log(n)}{n\alpha_\delta}\right),$$

where the big $O(\cdot)$ notation hides dependence on the dimension $d$. Hence, even though the right hand side tends to zero as $n\alpha_\delta \to \infty$, so the eigenvectors of $A$ and $P$ are well-aligned (up to sign), the rate of convergence of the orthogonal matrix depends on $n, \alpha_\delta$, and the corresponding eigengap.

In practice, one observes only the two graphs, and the eigenvalues must be estimated from the eigenvalues of $\hat{A}$. Therefore, even though the orthogonal matrix is converging to the identity, for any finite $n$, it may not be close if the eigengap is small relative to $\alpha$. So if one observes two graphs from the same model, but both $n$ and $m$ are small relative to $\delta$, then one may still need to estimate a rotation to align $\hat{X}$ and $\hat{Y}$, despite asymptotically having distinct eigenvalues.

### 3.3 Relation to Previous Results

There have been several tests proposed assuming that the graphs have the same set of vertices, such as Tang et al. (2017a); Ghoshdastidar et al. (2017); Li and Li (2018); Levin et al. (2019) and Draves and Sussman (2020). In Tang et al. (2017a); Levin and Levina (2019) and Draves and Sussman (2020), the authors work under the random dot product graph model, though they require that the expected degree grows as $\omega(n)$ (that is, the sparsity parameter is constant). In Li and Li (2018), working under the stochastic blockmodel, the authors are able to derive more explicit limiting results for their test statistic. In Ghoshdastidar et al. (2020), the authors allow for arbitrary distributions on two graphs, but again require that the graphs be on the same set of nodes. In contrast to all of these works, we do not assume that the two graphs are on the same set of vertices.

Our test statistic is based on a two-sample $U$-statistic using the rows of $\hat{X}$ and $\hat{Y}$. In Levin and Levina (2019), the authors consider bootstrapping nondegenerate $U$-statistics for random dot product graphs by estimating the latent positions. In addition, there have been a number of works on $U$-statistics for graphs.
in the more general graphon model (Lunde and Sarkar, 2019; Zhang and Xia, 2020; Lin et al., 2020a,b), but these involve bootstrapping moments of the underlying graphon, which can be computationally infeasible in practice. In this paper, we study a degenerate two-sample test statistic, which is not considered in any of these works.

Both Ghoshdastidar et al. (2017) and Tang et al. (2017b) consider a similar test as in this paper. In Ghoshdastidar et al. (2017), the authors introduce a formalism for two-sample testing under the assumption one observes only the adjacency matrices. Although our broad setting is similar to theirs, the model we study has more structural assumptions, allowing us to construct a test statistic using estimated latent positions. In addition, since our population test statistic is injective, we obtain a universally consistent test statistic using estimated latent position distributions under the single-graph setting, though in both of these works they proposed methodology are not simply trivial extensions of the results in Tang et al. (2017b).

The example of the random dot product graphs, the authors show that if \( \mathbf{X}, \mathbf{Y} \) are positive definite, our results include the random dot product graph as a special case, though the analysis and our results involve bootstrapping moments of the underlying graphon, which can be computationally infeasible in practice. In this paper, we study a degenerate two-sample test statistic, which is not considered in any of these works.

While at first glance the test statistic proposed in Tang et al. (2017b) is similar to our test statistic, analyzing the test statistic in a general low-rank setting involves substantial theoretical and methodological consideration with respect to indefinite orthogonal transformations, optimal transport, and sparsity. In addition, the assumption that \( \mathbb{E}(\mathbf{X}^\top \mathbf{X}) \) and \( \mathbb{E}(\mathbf{Y}^\top \mathbf{Y}) \) have repeated eigenvalues precludes testing in the case of the \( K \)-block balanced homogeneous stochastic blockmodel from Section 1.1 even if the \( \mathbf{B} \) matrix is positive definite. Our results include the random dot product graph as a special case, though the analysis and our proposed methodology are not simply trivial extensions of the results in Tang et al. (2017b).

We remark that Propositions 3 and 4 provide similar bounds to Theorem 5 of Levin and Levina (2019) and Theorem 4.4 of Lei (2020b), both of which consider convergence of empirical distributions to the corresponding latent position distribution under the single-graph setting, though in both of these works they require that the orthogonal matrix in the eigendecomposition of \( \mathbb{E}(\mathbf{X}^\top \mathbf{X}) \) is block diagonal. As a counterexample, consider the following GRDPG model. Let \( \mathbf{B} \in [0, 1]^{K \times K} \) be a symmetric connectivity matrix of rank \( K \), and let \( \mathbf{V} \mathbf{D} \mathbf{V}^\top \) be its eigendecomposition. Let \( Z_i \sim \text{Dirichlet}(\alpha) \) for some \( \alpha \in \mathbb{R}_+^K \), and define \( X_i = \mathbf{V} |\mathbf{D}|^{1/2} Z_i \), which is a valid GRDPG distribution. Then the matrix \( \mathbb{E}(\mathbf{X}^\top \mathbf{X}) \) has a block-orthogonal eigendecomposition if and only if \( |\mathbf{D}|^{1/2} \mathbb{E}(\mathbf{Z}^\top \mathbf{Z}) |\mathbf{D}|^{1/2} \mathbf{V}^\top \) does, where \( \mathbb{E}(\mathbf{Z}^\top \mathbf{Z}) \) is the second moment matrix for a Dirichlet random variable. If \( \alpha \) is the all ones vector, then

\[
\mathbf{V} |\mathbf{D}|^{1/2} \mathbb{E}(\mathbf{Z}^\top \mathbf{Z}) |\mathbf{D}|^{1/2} \mathbf{V}^\top = \frac{K}{K + 1} \mathbf{V} |\mathbf{D}| |\mathbf{V}^\top + \frac{1}{K(K + 1)} \mathbf{V} |\mathbf{D}|^{1/2} \mathbf{1} \mathbf{1}^\top |\mathbf{D}|^{1/2} \mathbf{V}^\top.
\]

The example \( \mathbf{B} = -0.1 \mathbf{I} + 0.21 \mathbf{1} \mathbf{1}^\top \) yields an orthogonal matrix that is not block-diagonal. Hence, even though assuming the eigendecomposition of \( \mathbb{E}(\mathbf{X}^\top \mathbf{X}) \) has a block diagonal structure is an attractive assumption amenable to theoretical analysis, this assumption can be violated by many different models.

Because of the prevalence of spectral methods in the literature, estimation of \( \hat{W}_n \) arises often in related inference tasks. For example, Zhang (2018) proposes solving a smooth function of the Laplace distance between distributions to estimate \( \hat{W}_n \), and Li and Li (2018), operating under the stochastic blockmodel,
consider estimating $\hat{W}_n$ by minimizing over the community memberships. Indeed, both methods are practically similar to ours, and may provide comparable results in practice, though we believe we are the first to apply it to nonparametric hypothesis testing and to provide asymptotic statistical guarantees under both the null and alternative hypotheses. Furthermore, Optimal Transport-Procrustes has been used to some success in the literature on natural language processing. Though our methodology is similar to Alvarez-Melis et al. (2019), other methods have been proposed for numerically solving the problem (e.g. Grave et al. (2019)).

Finally, we mention that though our algorithm is based on entropy-regularized Wasserstein distance, our results are stated in terms of the unregularized Wasserstein distance. While it may be possible to extend results on regularized optimal transport (e.g. Gangrade et al. (2019); Bigot et al. (2019)) to the mixed continuous and discrete setting implicitly required for our purposes, such an extension would require nontrivial analysis of the regularized Sinkhorn distance between $\hat{X}$ and $\hat{Y}$.

4 Simulations

Recall the motivating example in Section 1.1, given by the balanced homogeneous stochastic blockmodel, in which for vertices in the same community, the probability of an edge is $a$ and between communities the probability of an edge is $b$. When $b > a$ and the probability of belonging to a community is $1/K$ if there are $K$ communities, then this model has both repeated and negative eigenvalues. With the same model as in Section 1.1, where

$$B = \begin{pmatrix} .5 & .8 & .8 \\ .8 & .5 & .8 \\ .8 & .8 & .5 \end{pmatrix}$$

and the probability of community membership is $1/3$, we simulate 100 Monte Carlo iterations on $n = 300$ vertices from this stochastic blockmodel, and we calculate the value of our test statistic with both the naïvely rotated versions and the output of Algorithm 2.
Figure 3: Comparisons of the naïve sign-flip alignment procedure and the optimal transport alignment procedure for two adjacency spectral embeddings for the degree-corrected stochastic blockmodel. The left hand side shows the naïve alignment, and visually the clusters are not on top of each other, and the right hand side shows that using Algorithm 2 places the clusters approximately on top of each other.

Under the distinct eigenvalues assumption, choosing the sign of $\hat{X}$ to match those of $\hat{Y}$ suffices to give convergence as in Theorem 1; we dub this naïve alignment procedure the sign flips procedure. In Figure 2, we plot the density of the difference of our estimated test statistic using the Gaussian kernel with both the sign flips procedure and the estimated rotation from Algorithm 2. Since this is a finite sample simulation, we find the alignment $\hat{W}_n$ by running Algorithm 2 from multiple different initializations, and we take the value $\hat{W}$ that minimizes the test statistic, where $\hat{W}$ are the local minimums from Algorithm 2 for the estimated rotation and $\hat{W}$ are the sign matrices for $\hat{U}_{\text{sign flip}}$. We see that the density lies almost completely to the right of zero which suggests that the naïve estimate is nearly always larger than the estimated rotation. Moreover, there are some situations in which the difference is quite large, which demonstrates that the test statistic estimated using only sign flips need not necessarily converge to zero under the null hypothesis.

Under the null hypothesis, the test statistic should be tending to zero almost surely, and we see that the value of the test statistic evaluated using the estimated rotation is much more concentrated about zero. Moreover, a Wilcoxon test gives a $p$-value of less than 0.0001 for testing whether the estimated rotation test statistic is smaller than the sign-flips.

In Figure 3, using the same $B$ matrix as in the previous example, we also allow for independent degree correction parameters $\theta_i \sim .5 \times U(0, 1) + .5$, where $U(0, 1)$ denotes the Uniform distribution on $(0, 1)$. For two vertices $i$ and $j$ with communities $k$ and $l$ respectively, the probability of an edge is defined as $\theta_i \theta_j B_{kl}$. We plot the second and third dimensions of $\hat{X}$ and $\hat{Y}$ with both sign flips and the estimated orthogonal matrix output from Algorithm 2. By the independence of the degree-correction parameters, the matrix $E(XX^\top)I_{p,q}$ still has repeated eigenvalues, since it will be a scalar times the corresponding second moment matrix for a stochastic blockmodel. Here we use $n = m = 500$ to encourage the convergence of the second moment matrix. We see that visually the corresponding clusters lie on top of each other despite the added noise from the degree correction. Note that the clusters are “elongated” relative to the stochastic blockmodel in Figure 1; this is due to the fact that the latent position distribution for a degree-corrected stochastic blockmodel is supported on a ray, since the degree correction parameters change the magnitude of the latent positions but not the direction.
Figure 4: Density plot of the difference $\hat{U}_{\text{sign flips}} - \hat{U}_{\text{rotation}}$ for 100 Monte Carlo iterations of a degree-corrected stochastic blockmodel. A Wilcoxon test gives a $p$-value of < .0001 for testing whether the estimated rotation is better than the Sign Flips.

In Figure 4 we plot the density of the test statistic with and without the rotation, and a Wilcoxon test gives a $p$-value of less than .0001 for whether the rotated lies to the left of the naive alignment. We see that the distribution of the estimated rotation lies to the left of the distribution of sign flips.

4.1 Simulated Power Analysis

For the stochastic blockmodel in the previous section, the left hand side of Figure 5 gives estimated power curves for the following setup. First, we generate $A^{(1)}$ as a stochastic blockmodel with $B^{(1)}$ as before. We then generate $A^{(2)}$ independently from $A^{(1)}$ with probability matrix $B^{(1)} + \varepsilon I$ for $\varepsilon \in \{0, .1, .2\}$ in red, green, and blue respectively. Note that all these choices of $\varepsilon$ still yield a probability generating matrix with negative eigenvalues. We run 100 simulated permutation tests with 500 permutations and choose the critical value for $\alpha = .05$.

Similarly, the right hand side of Figure 5 shows estimated power curves for the following setup. First, we generate $A^{(1)}$ from a stochastic blockmodel with $B$ as before Then we generate $A^{(2)}$ as the degree-corrected stochastic blockmodel with degree-correction parameters chosen independently as $\beta \times U(0, 1) + (1 - \beta)$ for various choices of $\beta$. When $\beta = 0$ there are no degree-correction parameters, and the null hypothesis holds. As in the previous example, we run this simulation 100 times with 500 permutations per run. We consider $\beta \in \{0, .1, .2, .3\}$ in red, green, blue, and purple respectively. Larger values of $\beta$ can be understood qualitatively as moving further away from the null hypothesis. We see that under the alternative for larger values of $n$ the estimated power tends to 1, and the Type I error remains below .05 under the null hypothesis.

5 Discussion

We have shown that a test statistic defined by using the maximum mean discrepancy applied to the rows of the adjacency spectral embedding yields a consistent test in a natural asymptotic regime as the number of vertices tends to infinity. The methodology we propose shows that solving the optimal transport problem estimates the orthogonal matrix stemming from the eigenvalue multiplicity of the matrices $E(X X^\top)I_{p,q}$ and
\( \mathbb{E}(YY^\top) I_{p,q} \). While our optimization scheme alternates between points, we note that we have not proven that it yields a globally optimum solution in general, and many different initializations may be required to find the global minimizer. In addition, we note that using the resulting orthogonal transformation, if globally minimized, does not asymptotically affect power in the case of distinct eigenvalues, as the orthogonal transformation it is approximating is a sign matrix.

Our results show that the \( U \)-statistic associated to the reproducing kernel yields consistent testing under appropriate edge density; determining the exact nondegenerate limiting distribution is yet an open problem. The proof reveals that it will depend on the asymptotic distribution of the difference of indefinite orthogonal matrices \( \sqrt{n}(Q_X - \hat{Q}_X) \) in Section 6, but for practical purposes, this is irrelevant, as the resulting limit will not be independent of \( F_X \) and \( F_Y \) in general. While exact derivation of the limiting distribution is complicated, we note that our procedure yields a consistent test through a simple bootstrapping procedure. Our main results have demonstrated that only repeated eigenvalues (and not negative eigenvalues) require any modification to obtain consistency for two graph hypothesis testing.

As in Tang et al. (2017b), one can also extend our methodology to determine whether \( F_X \simeq F_Y \circ c \) for some constant \( c > 0 \), or for the setting \( F_X \circ \pi \simeq F_Y \circ \pi \), where \( \pi \) is the projection onto the sphere. Here \( F_X \simeq F_Y \circ c \) means that \( F_X \simeq F_{cY} \), and \( F_X \circ \pi \simeq F_Y \circ \pi \) means that \( F_{\pi(X)} \simeq F_{\pi(Y)} \). For \( c \) appropriately defined so that \( F_Y \circ c \) is a valid \((p,q)\)-admissible distribution, one can use the estimates

\[
\hat{s}_X = n^{-1/2}\| \hat{X} \|_F; \hat{s}_Y = m^{-1/2}\| \hat{Y} \|_F,
\]

and hence, by Lemma 3, these are consistent estimates of the parameters

\[
s_X = n^{-1/2}\| \hat{X} \|_F; s_Y = m^{-1/2}\| \hat{Y} \|_F.
\]

Similarly, one can project the estimates \( \hat{X} \) and \( \hat{Y} \) to the unit sphere to test whether \( F_Y \circ \pi \simeq F_Y \circ \pi \).

It remains an open question as to whether our results can be extended to graphons or other random graph models. In addition, while we have demonstrated that estimation of sparsity is sufficient to obtain consistency, graphs below the \( \sqrt{n} \) threshold may require additional analysis.
6 Proofs of Main Results

In this section, we first prove Proposition 1, Theorems 1 and 2 and Corollaries 1 and 2. We then prove Propositions 2, 3, 4, and 5. Finally, we prove the more technical results. Our proofs require careful tabulation of the various alignment matrices orthogonal matrices. We remark that all orthogonal matrices appearing in the following proofs are written with the letter \( W \) and all indefinite orthogonal matrices are written with the letter \( Q \), and we allow the constants implicit in the notation \( O(\cdot) \) to depend on \( d \) in an arbitrary manner. Finally, in our proofs, we will provide bounds with a constant \( C \) that may change from line to line.

Before proving Proposition 1, we include some important related results that we will require. First, Theorem 7 in Solanki et al. (2019) says that when we have a \((p,q)\)-admissible distribution, the support is bounded.

**Theorem 3** (Theorem 7 of Solanki et al. (2019)). Suppose \( F \) is a \((p,q)\)-admissible distribution; that is for all \( x,y \in \text{supp}(F) \), \( x^\top I_{p,q} y \in [0,1] \). Then \( \text{supp}(F) \) is bounded.

If needed, we can assume without loss of generality that the support \( \Omega \) is compact by extending it to its closure if necessary. We will also need an adaptation of Theorem 1 from Agterberg et al. (2020) for the two graph setting. The proof is straightforward and included in Section 6.5.

**Lemma 1.** Suppose \( F_X \simeq F_Y \), and that \( \Delta_X I_{p,q} \) and \( \Delta_Y I_{p,q} \) have distinct eigenvalues. Let \( Q_X \) be the matrix such that \( U_X |\Lambda|^{1/2} Q_X = X \) and similarly for \( Q_Y \) and \( Y \). Then there exists a fixed matrix \( \tilde{Q} \) such that both \( \|Q_X - \tilde{Q}\| \to 0 \) and \( \|Q_Y^{-1} - \tilde{Q}\| \to 0 \) almost surely, where \( T \in O(p,q) \) is the matrix such that \( F_Y = F_X \circ T \).

Finally, we need the following restatement of Theorem 5 of Rubin-Delanchy et al. (2020). Given Lemma 1 and the concentration results in this section (c.f. Lemma 2) the proof is straightforward by adapting the proof in Rubin-Delanchy et al. (2020) and is thus omitted. Note that \( Q_X^{-1} = I_{p,q} Q_X^\top I_{p,q} \) from the equation \( Q_X I_{p,q} Q_X^\top = I_{p,q} \), which will be useful in the sequel.

**Theorem 4.** Let \((A,X) \sim \text{GRDPG}(F_X, n, \alpha_n) \) for \( \alpha_n = \omega(\log^4(n)) \). Let \( Q_X \) be the matrix such that \( U_X |\Lambda|^{1/2} Q_X = X \). Then there exists an orthogonal matrix \( W_* \in O(d) \cap O(p,q) \) depending on \( n \) such that with probability at least \( 1 - n^{-2} \)

\[
\|\tilde{X} - \alpha_n^{1/2} X Q_X^{-1} W_*\|_{2,\infty} = O\left(\frac{\log(n)}{n^{1/2}}\right).
\]

Furthermore, as \( n \to \infty \), if \( \Delta I_{p,q} \) has distinct eigenvalues, then \( \|W_* - I\| = O\left((n\alpha_n)^{-1}\right) \) with probability at least \( 1 - n^{-2} \). In this case, we have the bound

\[
\|\tilde{X} - \alpha_n^{1/2} X Q_X^{-1}\|_{2,\infty} = O\left(\frac{\log(n)}{n^{1/2}}\right).
\]

**Proof of Proposition 1.** We first show that \( |U_{n,m}(\tilde{X}/\alpha_n^{1/2},\tilde{Y}/\beta_m^{1/2}) - U_{n,m}(XQ_X^{-1}W_X^*,YQ_Y^{-1}W_Y^*)| \to 0 \) almost surely. By continuity of \( \kappa \) and the fact that the supports of \( F_X \) and \( F_Y \) are bounded by Theorem 3, we have that

\[
|\kappa(\tilde{X}_i/\alpha_n^{1/2},\tilde{Y}_j/\beta_m^{1/2}) - \kappa((W_X^* Q_X^{-1})^\top X_i, (W_Y^* Q_Y^{-1})^\top Y_j)| \leq C||\alpha_n^{-1/2} \tilde{X} - X Q_X^{-1} W_X^*||_{2,\infty};
\]

\[
|\kappa(\tilde{Y}_i/\beta_m^{1/2},\tilde{Y}_j/\beta_m^{1/2}) - \kappa((W_Y^* Q_Y^{-1})^\top Y_i, (W_Y^* Q_Y^{-1})^\top Y_j)| \leq C||\beta_m^{-1/2} \tilde{Y} - Y Q_Y^{-1} W_Y^*||_{2,\infty};
\]

\[
|\kappa(\tilde{X}_i/\alpha_n^{1/2},\tilde{Y}_j/\beta_m^{1/2}) - \kappa((W_X^* Q_X^{-1})^\top X_i, (W_Y^* Q_Y^{-1})^\top Y_j)| \leq C\max\left(||\alpha_n^{-1/2} \tilde{X} - X Q_X^{-1} W_X^*||_{2,\infty}, ||\beta_m^{-1/2} \tilde{Y} - Y Q_Y^{-1} W_Y^*||_{2,\infty}\right).
\]

Each term tends to zero almost surely by Theorem 4. Hence,

\[
|U_{n,m}(\tilde{X}/\alpha_n^{1/2},\tilde{Y}/\beta_m^{1/2}) - U_{n,m}(XQ_X^{-1}W_X^*,YQ_Y^{-1}W_Y^*)| \to 0
\]
almost surely. Furthermore, we see that since \( \Delta_{p,q} \) has distinct eigenvalues, the matrices \( W_X^X \) and \( W_X^Y \) are converging to the identity as \( n \to \infty \).

Hence, it suffices to consider what the term \( U_{n,m}(XQ_X^{-1}, YQ_Y^{-1}) \) is converging to under the null and alternative respectively. Note that

\[
|U_{n,m}(XQ_X^{-1}, YQ_Y^{-1}) - U_{n,m}(X\tilde{Q}_X^{-1}, Y\tilde{Q}_Y^{-1})| \leq C \left( \|X(Q_X^{-1} - \tilde{Q}_X^{-1})\|_{2,\infty} + \|Y(Q_Y^{-1} - \tilde{Q}_Y^{-1})\|_{2,\infty} \right) 
\leq C \left( \|Q_X^{-1} - \tilde{Q}_X^{-1}\| + \|Q_Y^{-1} - \tilde{Q}_Y^{-1}\| \right),
\]

where we have implicitly used Theorem 3 and the fact that \( \kappa \) is twice continuously differentiable and hence Lipschitz on the closure of the support of \( F_X \circ \tilde{Q}^{-1} \). The right hand side tends to zero almost surely by Lemma 1 and Theorem 2 in Agterberg et al. (2020).

Hence, it suffices to analyze the convergence of \( U_{n,m}(X\tilde{Q}_X^{-1}, Y\tilde{Q}_Y^{-1}) \) under the null and alternative respectively. Note that \( \tilde{Q}_X^{-1} \) and \( \tilde{Q}_Y^{-1} \) are deterministic matrices. Under the null hypothesis, the matrix \( \tilde{Q}_Y^{-1} \) can be replaced with the limiting matrix \( T^{-1}\tilde{Q}^{-1} \) by Lemma 1. Therefore, under the null hypothesis, \( F_X \circ T = F_Y \), so \( \mu[F_X \circ \tilde{Q}^{-1}] = \mu[F_Y \circ T^{-1}\tilde{Q}^{-1}] \) since \( \kappa \) is assumed to be a characteristic kernel. By Gretton et al. (2012), as \( n, m \to \infty \) and \( n/(n + m) \to \rho \in (0, 1) \), we have that

\[
U_{n,m}(X\tilde{Q}_X^{-1}, Y\tilde{Q}_Y^{-1}) \to \|\mu[F_X \circ \tilde{Q}^{-1}] - \mu[F_Y \circ T^{-1}\tilde{Q}^{-1}]\|_{\mathcal{H}}^2 = 0. \tag{4}
\]

Hence, \( U_{n,m}(\hat{X}/\alpha_n^{1/2}, \hat{Y}/\beta_m^{1/2}) \) converges to zero under the null hypothesis.

Under the alternative hypothesis, the matrix \( T^{-1}\tilde{Q}^{-1} \) is replaced with a matrix \( \tilde{Q} \), and the term

\[
\|\mu[F_X \circ \tilde{Q}^{-1}] - \mu[F_Y \circ \tilde{Q}^{-1}]\|_{\mathcal{H}}^2 = c > 0
\]
or otherwise the null hypothesis would be true. Therefore, under the alternative, the term \( U_{n,m}(\hat{X}/\alpha_n^{1/2}, \hat{Y}/\beta_m^{1/2}) \) converges to some positive constant which completes the proof.

### 6.1 Proof of Theorems 1 and 2 and Corollaries 1 and 6

We will require a few supporting lemmas. The first is on the rate of approximation of the limiting matrix \( \tilde{Q} \) from Theorem 2 of Agterberg et al. (2020). We note that this improves on a bound of order \( (\log(n)/n)^{1/8} \) given in Solanki et al. (2019). The proof is in Section 6.5.

**Lemma 2.** Define \( Q_X \) as the matrix such that \( U_X|A_X|^{1/2} = \alpha_n^{1/2}XQ_X^{-1} \), and let \( \tilde{Q} \) be its corresponding limit. Then with probability at least \( 1 - n^{-2} \) that there exists an orthogonal matrix \( W_X \in \mathcal{O}(d) \cap \mathcal{O}(p,q) \) such that

\[
\|W_XQ_X - \tilde{Q}\| = O \left( \frac{\sqrt{\log(n)}}{\sqrt{n}} \right). \tag{5}
\]

Moreover, \( W_X \) has blocks corresponding to repeated eigenvalues of \( \Delta_{p,q} \). If \( \Delta_{p,q} \) has no repeated eigenvalues, then \( W_X \) is a sign matrix. In addition, the matrices \( Q_X \) and \( \tilde{Q} \) do not depend on the sparsity factor.

The next lemma is a technical result concerning the Frobenius norm concentration of \( \hat{X} \) to \( X \) and is needed to guarantee the existence of the specific matrices \( W_X^X \) and \( W_X^Y \), though it is also used in the proof of Lemma 4. We note that similar results were proven in Tang et al. (2017a) and Tang and Priebe (2018) in the setting of random dot product graphs, and in Athreya et al. (2020) in the setting of numerical linear algebra for random matrices. The asymptotic normality of a related Frobenius norm error for stochastic blockmodels was proven in Li and Li (2018). The proof is in Section 6.3. Throughout this section, recall that we define

\[
\tilde{X} := U_X|A_X|^{1/2}; \quad P^{(1)} = U_XA_XU_X^T = \alpha_nXI_{p,q}X^T
\]
with similar notation for \( \hat{Y} \) and \( P^{(2)} \). We therefore have the identity

\[
\hat{X} = \alpha_{n}^{1/2} X Q^{-1}_X.
\]

(6)

Note that \( \hat{X} \) and \( \hat{X} \) depend on the sparsity \( \alpha_n \), but the matrix \( X \) does not (since its rows are i.i.d. \( F_X \)), and that the matrix \( \hat{X} \) can be thought of as the adjacency spectral embedding of the probability generating matrix \( P = \alpha_n X I_{p,q} X^\top \). See also Table 2 below.

**Lemma 3.** Let \( (A, X) \sim GRDPG(F_X, n, \alpha_n) \) for some \( \alpha_n \) satisfying \( n\alpha_n \geq \omega(\log^4(n)) \). Asymptotically almost surely, for the sequence of block-orthogonal matrices \( W_* \) from Theorem 4, the matrix \( \hat{X} W_*^\top - \hat{X} \) admits the decomposition

\[
\hat{X} W_*^\top - \hat{X} = (A - P) U_X |A_X|^{-1/2} I_{p,q} + R
\]

where the matrix \( R \) satisfies

\[
\|R\|_F = O\left( \sqrt{\log(n) / n\alpha_n} \right)
\]

with high probability. Furthermore, also with high probability,

\[
\|\hat{X} - \hat{X} W_*\|_F^2 - C(\hat{X})^2 = O\left( \sqrt{\log(n) / n\alpha_n} \right),
\]

where

\[
C^2(X) := E\| (A - P) U |S|^{-1/2} \|_F^2.
\]

where the expectation is with respect to the randomness in \( A \). Finally, as \( n \to \infty \), we have that

\[
\|\hat{X} - \hat{X} W_*\|_F^2 \to \text{Tr} \left( \hat{Q}^{-1} \Delta^{-1} \Gamma \Delta^{-1} \hat{Q}^{-\top} \right).
\]

almost surely, where \( \Gamma \) is defined via

\[
\Gamma := \begin{cases} 
E[X X^\top (X^\top I_{p,q} \mu - X^\top I_{p,q} \Delta I_{p,q} X)] & \alpha_n \equiv 1 \\
E[X X^\top (X^\top I_{p,q} \mu)] & \alpha_n \to 0.
\end{cases}
\]

Finally, we present the following functional central limit theorem for the approximation of \( \hat{X} \) to \( \hat{X} \) under sparsity. The proof can be found in Section 6.4. The result is similar to Theorem 5 in Tang et al. (2017b), but requires a number of different technical considerations.

**Lemma 4.** Let \( (A, X) \sim GRDPG(F_X, n, \alpha_n) \) where \( n\alpha_n \gg \log^4(n) \). Suppose \( F : \mathbb{R}^d \to \mathbb{R} \) is a collection of twice continuously differentiable functions. Let \( \hat{X} := U_X |A_X|^{1/2} \), and let \( W_* \) be the matrix guaranteed by Lemma 3. Then, as \( n \to \infty \), the empirical process

\[
f \in F \mapsto \hat{g}_n f := \sqrt{\frac{\alpha_n}{n}} \sum_{i=1}^{n} \left[ f \left( \frac{W_* \hat{X}_i}{\sqrt{\alpha_n}} \right) - f \left( \frac{\hat{X}_i}{\sqrt{\alpha_n}} \right) \right] \to 0
\]

(7)

almost surely as \( n \to \infty \). Moreover, with probability at least \( 1 - O(n^{-2}) \),

\[
\sup_{f \in F} \left| \sqrt{\frac{\alpha_n}{n}} \sum_{i=1}^{n} \left[ f \left( \frac{W_* \hat{X}_i}{\sqrt{\alpha_n}} \right) - f \left( \frac{\hat{X}_i}{\sqrt{\alpha_n}} \right) \right] \right| = O\left( \sqrt{\frac{\log(n)}{n\alpha_n}} \right).
\]

(8)

In addition, the results hold with the replacement \( W_* \hat{X}_i \) and \( \hat{X}_i \) replaced with \( \hat{X}_i \) and \( W_*^\top \hat{X}_i \) respectively. Finally, if \( \sqrt{n\alpha_n} = \omega(n^{1/2} \log^{1+\eta}(n)) \) for some \( \eta > 0 \) then the result in Equation 7 still holds under the scaling \( \frac{1}{\sqrt{n}} \) instead of \( \sqrt{\alpha_n} / \sqrt{n} \), and in Equation (8) the right hand side bound is of the form \( O\left( \frac{\log^{1/2}(n)}{n^{1/2} \alpha_n} \right) \).
Notation | Definition
--- | ---
$Q_X, Q_Y \in \mathcal{O}(p, q)$ | The matrices such that $U_X|\Lambda_X|^{1/2}Q_X = \alpha_n^{1/2}X$ and $U_Y|\Lambda_Y|^{1/2}Q_Y = \beta_m^{1/2}Y$ (Equation 6)
$	ilde{Q}_X, \tilde{Q}_Y \in \mathcal{O}(p, q)$ | The limiting matrices for $Q_X$ and $Q_Y$ from Lemma 2 (Equation 6)
$W^*_X, W^*_Y \in \mathcal{O}(d) \cap \mathcal{O}(p, q)$ | The block-orthogonal matrices aligning $\hat{X}$ and $\tilde{X}$ (Equation 6) (Lemma 3 and Theorem 4)
$W_X, W_Y \in \mathcal{O}(d) \cap \mathcal{O}(p, q)$ | The block-orthogonal matrices aligning $Q_X$ and $\tilde{Q}_X$ from Lemma 2.

Table 1: Table of Notation

Table 2: Diagram of the alignment matrices and where they come from. Both $\tilde{Q}_X$ and $W_X$ come from Lemma 2, whereas the matrix $W^*_X$ comes from Lemma 3 (or Theorem 4).

Armed with these technical results, we are now ready to prove Theorems 1 and 2. To make the proof more straightforward, we have compiled the notation for all the alignment matrices in Table 1. Although the proof mirrors that in Tang et al. (2017b), the steps require careful tabulation of sparsity parameters, orthogonal transformations, and indefinite orthogonal transformations, all of which require novel technical analyses. Table 2 also shows how to find the various alignment matrices. Essentially, we have the approximations

$$\hat{X}W^*_XW_X \approx \hat{X}W_X = \alpha_n^{1/2}XQ_X^{-1}W_X \approx \alpha_n^{1/2}X\tilde{Q}_X^{-1},$$

using Lemmas 2 and 3. Similar approximations hold for $\hat{Y}$ and $\tilde{Y}$.

**Proof of Theorems 1 and 2.** Define the matrices $W^*_X$ and $W_X$ where $(W^*_X)^\top$ is the orthogonal matrix from Lemma 3, and $W_X$ is the matrix from Lemma 2. Define $W^*_Y$ and $W_Y$ similarly.
First, define $\hat{V}_{n,m} := V_{n,m} \left( \hat{X}_{X}, \hat{Y}_{Y} \right)$ via

$$
\hat{V}_{n,m} \left( \frac{\hat{X}_{X}}{\sqrt{\alpha_n}}, \frac{\hat{Y}_{Y}}{\sqrt{\beta_m}} \right) = \left\| \frac{1}{n} \sum_{i=1}^{n} \Phi \left( \frac{\hat{X}_{X,i}}{\sqrt{\alpha_n}} \right) - \frac{1}{m} \sum_{k=1}^{m} \Phi \left( \frac{\hat{Y}_{Y,k}}{\sqrt{\beta_m}} \right) \right\|_{H}^{2}
$$

$$
= \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{m} \kappa \left( \frac{\hat{X}_{X,i}}{\sqrt{\alpha_n}}, \frac{\hat{X}_{X,j}}{\sqrt{\alpha_n}} \right) - \frac{2}{mn} \sum_{i=1}^{n} \sum_{k=1}^{m} \kappa \left( \frac{\hat{X}_{X,i}}{\sqrt{\alpha_n}}, \frac{\hat{Y}_{Y,k}}{\sqrt{\beta_m}} \right) + \frac{1}{m^2} \sum_{k=1}^{m} \sum_{i=1}^{n} \kappa \left( \frac{\hat{Y}_{Y,k}}{\sqrt{\beta_m}}, \frac{\hat{Y}_{Y,i}}{\sqrt{\beta_m}} \right)
$$

and analogously for $\hat{V}_{n,m}$. We have the decomposition

$$(m \beta_m + n \alpha_n) \left( V_{n,m} \left( \frac{\hat{X}_{X}}{\sqrt{\alpha_n}}, \frac{\hat{Y}_{Y}}{\sqrt{\beta_m}} \right) - V_{n,m} \left( \frac{\hat{X}_{X}^X \hat{X}_{X}}{\sqrt{\alpha_n}}, \frac{\hat{Y}_{Y}^Y \hat{Y}_{Y}}{\sqrt{\beta_m}} \right) \right)
$$

$$
= (m \beta_m + n \alpha_n) \left( U_{n,m} \left( \frac{\hat{X}_{X}}{\sqrt{\alpha_n}}, \frac{\hat{Y}_{Y}}{\sqrt{\beta_m}} \right) - U_{n,m} \left( \frac{\hat{X}_{X}^X \hat{X}_{X}}{\sqrt{\alpha_n}}, \frac{\hat{Y}_{Y}^Y \hat{Y}_{Y}}{\sqrt{\beta_m}} \right) \right) + r_1 + r_2,
$$

where

$$
r_1 = \frac{(m \beta_m + n \alpha_n)}{n(n - 1)} \sum_{i=1}^{n} \left[ \kappa \left( \frac{\hat{X}_{X,i}}{\sqrt{\alpha_n}}, \frac{\hat{X}_{X,i}^X \hat{X}_{X,i}}{\sqrt{\alpha_n}} \right) - \kappa \left( \frac{\hat{X}_{X,i}^X \hat{X}_{X,i}}{\sqrt{\alpha_n}}, \frac{\hat{X}_{X,i}^X \hat{X}_{X,i}}{\sqrt{\alpha_n}} \right) \right]
$$

$$
+ \frac{(m \beta_m + n \alpha_n)}{n(n - 1)} \sum_{k=1}^{m} \left[ \kappa \left( \frac{\hat{Y}_{Y,k}}{\sqrt{\beta_m}}, \frac{\hat{Y}_{Y,k}^Y \hat{Y}_{Y,k}}{\sqrt{\beta_m}} \right) - \kappa \left( \frac{\hat{Y}_{Y,k}^Y \hat{Y}_{Y,k}}{\sqrt{\beta_m}}, \frac{\hat{Y}_{Y,k}^Y \hat{Y}_{Y,k}}{\sqrt{\beta_m}} \right) \right];
$$

$$
r_2 = \frac{(m \beta_m + n \alpha_n)}{n^2(n - 1)} \sum_{i=1}^{n} \sum_{j=1}^{n} \left[ \kappa \left( \frac{\hat{X}_{X,i}}{\sqrt{\alpha_n}}, \frac{\hat{X}_{X,j}^X \hat{X}_{X,j}}{\sqrt{\alpha_n}} \right) - \kappa \left( \frac{\hat{X}_{X,j}^X \hat{X}_{X,j}}{\sqrt{\alpha_n}}, \frac{\hat{X}_{X,j}^X \hat{X}_{X,j}}{\sqrt{\alpha_n}} \right) \right]
$$

$$
+ \frac{(m \beta_m + n \alpha_n)}{m^2(n - 1)} \sum_{k=1}^{m} \sum_{l=1}^{m} \left[ \kappa \left( \frac{\hat{Y}_{Y,k}}{\sqrt{\beta_m}}, \frac{\hat{Y}_{Y,l}^Y \hat{Y}_{Y,l}}{\sqrt{\beta_m}} \right) - \kappa \left( \frac{\hat{Y}_{Y,l}^Y \hat{Y}_{Y,l}}{\sqrt{\beta_m}}, \frac{\hat{Y}_{Y,l}^Y \hat{Y}_{Y,l}}{\sqrt{\beta_m}} \right) \right].
$$

By Theorem 3, $\hat{\Omega}$ is compact, so by the fact that $\kappa$ is twice continuously differentiable, $\kappa$ is Lipschitz on $Q^{X}^{-1} \hat{\Omega}$ since $Q^{X}^{-1} \hat{\Omega}$ is compact. In particular, for some positive $K$, we have that

$$
\|r_1\| \leq K \frac{m \beta_m + n \alpha_n}{n - 1} \max_{i} \left\| \frac{\hat{X}_{X,i}}{\sqrt{\alpha_n}} - \frac{(\hat{X}_{X}^X \hat{X}_{X,i})}{\sqrt{\alpha_n}} \right\|
$$

$$
+ K \frac{m \beta_m + n \alpha_n}{m - 1} \max_{i} \left\| \frac{\hat{Y}_{Y,i}}{\sqrt{\alpha_n}} - \frac{(\hat{Y}_{Y}^Y \hat{Y}_{Y,i})}{\sqrt{\alpha_n}} \right\|
$$

$$
\leq 2K \frac{m \beta_m + n \alpha_n}{n \alpha_n} \sqrt{\alpha_n} \log(n) \sqrt{m} + 2K \frac{m \beta_m + n \alpha_n}{m \beta_m \sqrt{\beta_m} \sqrt{m}} \log(m)
$$

$$
\leq C \left( \frac{\log(n)}{\sqrt{n}} + \frac{\log(m)}{\sqrt{m}} \right).
$$
by the assumption that \( \frac{m\beta_m + n\alpha_n}{m\beta_m + n\alpha_n} \to \rho \in (0, 1) \) and the \( 2, \infty \) bound in Theorem 4. By a similar argument,

\[
\|r_2\| \leq K \frac{m\beta_m + n\alpha_n}{(n-1)} \max_i \left\| \frac{W_X^T \tilde{X}_i}{\sqrt{\alpha_n}} - \frac{(W_s X W_X)^T \tilde{X}_i}{\sqrt{\alpha_n}} \right\| \\
+ K \frac{m\beta_m + n\alpha_n}{m-1} \max_i \left\| \frac{W_Y^T \tilde{Y}_i}{\sqrt{\alpha_n}} - \frac{(W_s Y W_Y)^T \tilde{Y}_i}{\sqrt{\alpha_n}} \right\|
\leq C \left( \frac{\log(n)}{\sqrt{n}} + \frac{\log(m)}{\sqrt{m}} \right).
\]

Both of these bounds are independent of \( \alpha_n \) and \( \beta_m \). Define

\[
\xi : = \frac{\sqrt{m\beta_m + n\alpha_n}}{n} \sum_{i=1}^n \kappa(W_X^T \tilde{X}_i/\sqrt{\alpha_n}, \cdot) - \frac{\sqrt{m\beta_m + n\alpha_n}}{m} \sum_{k=1}^m \kappa(W_Y^T \tilde{Y}_k/\sqrt{\beta_m}, \cdot)
\]

\[
\dot{\xi} : = \frac{\sqrt{m\beta_m + n\alpha_n}}{n} \sum_{i=1}^n \kappa((W_s X W_X)^T \tilde{X}_i/\sqrt{\alpha_n}, \cdot) - \frac{\sqrt{m\beta_m + n\alpha_n}}{m} \sum_{k=1}^m \kappa((W_s Y W_Y)^T \tilde{Y}_k/\sqrt{\beta_m}, \cdot).
\]

We now have that

\[
\left\| (m\beta_m + n\alpha_n) \left( V_{n,m} \left( \frac{\tilde{X} W X W_X}{\sqrt{\alpha_n}}, \frac{\tilde{Y} W Y W_Y}{\sqrt{\beta_m}} \right) - V_{n,m} \left( \frac{\tilde{X} W X}{\sqrt{\alpha_n}}, \frac{\tilde{Y} W Y}{\sqrt{\beta_m}} \right) \right) \right\|
= \left\| \| \xi \|_H^2 - \| \dot{\xi} \|_H^2 \right\|
\leq \| \xi - \dot{\xi} \|_H \left( 2\| \xi \|_H + \| \xi - \dot{\xi} \|_H \right).
\]

We have that

\[
\xi - \dot{\xi} = \sqrt{\frac{m\beta_m + n\alpha_n}{n}} \sum_{i=1}^n \frac{\kappa(W_X^T (W_s X)^T \tilde{X}_i/\sqrt{\alpha_n}, \cdot) - \kappa(W_X^T \tilde{X}_i/\sqrt{\alpha_n}, \cdot)}{\sqrt{n}}
- \sqrt{\frac{m\beta_m + n\alpha_n}{n}} \sum_{i=1}^n \frac{\kappa(W_Y^T (W_s Y)^T \tilde{Y}_i/\sqrt{\beta_m}, \cdot) - \kappa((W_s Y)^T \tilde{Y}_i/\sqrt{\beta_m}, \cdot)}{\sqrt{n}}
:= \zeta_X - \zeta_Y.
\]

We note that since \( \kappa \) is radial, we can disregard \( W_X^T \) and \( W_Y^T \). Moreover,

\[
\| \zeta_X \| = \left\| \sqrt{\frac{m\beta_m + n\alpha_n}{n}} \sum_{i=1}^n \frac{\kappa((W_X^T \tilde{X}_i/\sqrt{\alpha_n}, \cdot) - \kappa(\tilde{X}_i/\sqrt{\alpha_n}, \cdot)}{\sqrt{n}} \right\|
\leq \left\| \sqrt{\frac{m\beta_m + n\alpha_n}{n}} \sum_{i=1}^n \frac{\kappa((W_X^T \tilde{X}_i/\sqrt{\alpha_n}, \cdot) - \kappa(\tilde{X}_i/\sqrt{\alpha_n}, \cdot) (9) \|ight. \]

Since \( m\beta_m/(m\beta_m + n\alpha_n) \to \rho \in (0, 1) \) the term outside of the norm is of order \( O(1) \). Lemma 4 then implies that the term inside of the norm tends to zero, and the same argument holds for \( \zeta_Y \). In particular, we see that by Lemma 4,

\[
\| \xi - \dot{\xi} \|_H = O(\sqrt{\frac{\log(n)}{n\alpha_n}} + \sqrt{\frac{\log(m)}{m\beta_m}}).
\]
with probability at least \(1 - O(n^{-2} + m^{-2})\).

We now bound \(\|\xi\|_{\mathcal{M}}\) under the null and alternative respectively. Recall that \(\tilde{Q}_X\) is the limiting matrix guaranteed by Lemma 2. Note further that \(\tilde{X}/\sqrt{\alpha n} = XQ^{-1}_X\) so that \(\tilde{X}W_X/\sqrt{\alpha n} = XQ^{-1}_XW_X\), where \(W_X\) was the matrix such that \(Q_X - W_X\tilde{Q}_X\) is of order \(\sqrt{\log(n)/n}\) with probability at least \(1 - n^{-2}\). Hence, from the equation

\[ Q^{-1}_X = I_{p,q}Q^\top_XI_{p,q}, \]

we see that with probability at least \(1 - n^{-2}\) that

\[ \|Q^{-1}_XW_X - \tilde{Q}^{-1}_X\| = \|I_{p,q}Q^\top_XI_{p,q}W_X - I_{p,q}\tilde{Q}^\top_XI_{p,q}\| \]

\[ = \|Q^\top_XI_{p,q}W_X - \tilde{Q}^\top_XI_{p,q}\| \]

\[ = \|Q^\top_XW_X - \tilde{Q}_X\| \]

\[ = \|W_XQ_X - \tilde{Q}_X\| \]

\[ = \|Q_X - W_X\tilde{Q}_X\| \]

\[ = O\left(\sqrt{\log(n)}/\sqrt{n}\right). \]

Hence, we have that

\[ \xi = \frac{\sqrt{m\beta_m + n\alpha_n}}{n} \sum_{i=1}^{n} \kappa(W_X^\top\tilde{X}_i/\sqrt{\alpha m}, \cdot) - \frac{\sqrt{m\beta_m + n\alpha_n}}{m} \sum_{k=1}^{m} \kappa(W^\top_Y\tilde{Y}_k/\sqrt{\beta_m}, \cdot) \]

\[ = \frac{\sqrt{m\beta_m + n\alpha_n}}{\sqrt{n}} \sum_{i=1}^{n} \kappa((Q^{-1}_XW_X)^\top X_i, \cdot) - \frac{\sqrt{m\beta_m + n\alpha_n}}{\sqrt{m}} \sum_{k=1}^{m} \kappa((Q^{-1}_XW^\top_Y)^\top Y_{i,k}, \cdot) \]

\[ = \frac{\sqrt{m\beta_m + n\alpha_n}}{\sqrt{n}} \sum_{i=1}^{n} \kappa((Q^{-1}_XW_X)^\top X_i, \cdot) - \frac{\sqrt{m\beta_m + n\alpha_n}}{\sqrt{m}} \sum_{k=1}^{m} \kappa((Q^{-1}_XW^\top_Y)^\top Y_{i,k}, \cdot) \]

\[ + \frac{\sqrt{m\beta_m + n\alpha_n}}{\sqrt{n}} \sum_{i=1}^{n} \kappa((Q^{-1}_XW_X)^\top X_i, \cdot) - \frac{\sqrt{m\beta_m + n\alpha_n}}{\sqrt{m}} \sum_{k=1}^{m} \kappa((Q^{-1}_XW^\top_Y)^\top Y_{i,k}, \cdot) \]

\[ = \frac{\sqrt{m\beta_m + n\alpha_n}}{\sqrt{n}} \sum_{i=1}^{n} \kappa((Q^{-1}_XW_X)^\top X_i, \cdot) - \frac{\sqrt{m\beta_m + n\alpha_n}}{\sqrt{m}} \sum_{k=1}^{m} \kappa((Q^{-1}_XW^\top_Y)^\top Y_{i,k}, \cdot) \]

\[ + \frac{\sqrt{m\beta_m + n\alpha_n}}{\sqrt{n}} \sum_{i=1}^{n} \kappa((Q^{-1}_XW_X)^\top X_i, \cdot) - \frac{\sqrt{m\beta_m + n\alpha_n}}{\sqrt{m}} \sum_{k=1}^{m} \kappa((Q^{-1}_XW^\top_Y)^\top Y_{i,k}, \cdot) \]

\[ + \frac{\sqrt{m\beta_m + n\alpha_n}}{\sqrt{n}} \sum_{i=1}^{n} \kappa((Q^{-1}_XW_X)^\top X_i, \cdot) - \frac{\sqrt{m\beta_m + n\alpha_n}}{\sqrt{m}} \sum_{k=1}^{m} \kappa((Q^{-1}_XW^\top_Y)^\top Y_{i,k}, \cdot) \]

For the first two terms, by the Lipschitz property of \(\kappa\) and the fact that \(\tilde{\Omega}\) is bounded by Theorem 3, we have that

\[ \sqrt{m\beta_m + n\alpha_n} \sum_{i=1}^{n} \kappa((Q^{-1}_XW_X)^\top X_i, \cdot) \leq \sqrt{m\beta_m + n\alpha_n} \sum_{i=1}^{n} \kappa((Q^{-1}_XW_X)^\top X_i, \cdot) \]

\[ \leq \sqrt{m\beta_m + n\alpha_n} \sum_{i=1}^{n} \kappa((Q^{-1}_XW_X)^\top X_i, \cdot) \]

\[ \leq \|X\|_{2,\infty} \sqrt{m\beta_m + n\alpha_n} \|Q^{-1}_XW_X - \tilde{Q}^{-1}_X\| \]

\[ \leq C\sqrt{m\beta_m + n\alpha_n} \sqrt{\log(n)} \]

\[ = O\left(\sqrt{\alpha_n \log(n)}\right). \]
with probability at least \(1 - O(n^{-2})\) by Lemma 2. Hence the first term is of order
\[
O\left(\sqrt{\alpha_n \log(n)} + \sqrt{\beta_m \log(m)}\right)
\]
with probability at least \(1 - O(n^{-2} + m^{-2})\).

Now, define
\[
\psi_X := \sum_{i=1}^{n} \kappa(\tilde{Q}^{-1}_X X_i, \cdot) - \mu(F_X \circ \tilde{Q}^{-1}_X) / n
\]
\[
\psi_Y := \sum_{i=1}^{n} \kappa(\tilde{Q}^{-1}_Y Y_i, \cdot) - \mu(F_Y \circ \tilde{Q}^{-1}_Y) / m
\]

By Remark 1 in Schneider (2016), we see that
\[
P(\|\psi_X\|^2 > \varepsilon^2) \leq 2 \exp\left(-\frac{n \varepsilon^2}{64}\right)
\]
which in particular shows that \(\|\psi_X\| \leq C \sqrt{\log(n)}\sqrt{\alpha_n \log(n)} + \sqrt{\beta_m \log(m)}\) with probability at least \(1 - O(n^{-2})\), and similarly for \(\psi_Y\).

Thus far, we have shown with probability at least \(1 - O(n^{-2} + m^{-2})\) that
\[
\|\xi\|_H = O\left(\sqrt{\alpha_n \log(n)} + \sqrt{\beta_m \log(m)}\right) + \sqrt{\beta_m + \alpha_n} \left(\|\mu[F_X \circ \tilde{Q}^{-T}_X] - \mu[F_Y \circ \tilde{Q}^{-T}_Y]\|_H\right).
\]

Under the null hypothesis, the term in the parentheses is zero as \(\tilde{Q}_Y\) can be chosen to be \(T^{-1}\tilde{Q}^{-1}_X\) by Lemma 2 and the fact that \(\tilde{Q}_Y\) and \(\tilde{Q}_X\) do not depend on the sparsity factors by Lemma 2. Hence, we see that
\[
\|\xi - \hat{\xi}\|_H = O\left(\sqrt{\alpha_n \log(n)} + \sqrt{\beta_m \log(m)}\right) + \sqrt{\beta_m + \alpha_n} \left(\|\mu[F_X \circ \tilde{Q}^{-T}_X] - \mu[F_Y \circ \tilde{Q}^{-T}_Y]\|_H\right).
\]

Under the alternative the term \(\|\mu[F_X \circ \tilde{Q}^{-T}_X] - \mu[F_Y \circ \tilde{Q}^{-T}_Y]\|_H\) is not necessarily zero, so we have that
\[
\|\xi - \hat{\xi}\| \left(2\|\xi\| + \|\xi - \hat{\xi}\|\right) = O\left(\sqrt{\log(n)} n^{-1} + \sqrt{\log(m)} \beta_m^{-1}\right) \left(\sqrt{\alpha_n \log(n)} + \beta_m \log(m) + \sqrt{n \alpha_n + m \beta_m}\right).
\]

Hence, dividing by \(\log(n)\) yields the result.

We immediately derive the proof of Corollary 1.

**Proof of Corollary 1.** We will highlight where the previous proof changes. Examining the proof of Theorems 1 and 2, we see that we have (under the new scaling \((m + n)\)) the residual bounds
\[
r_1 = O\left(\frac{\log(n)}{\sqrt{n \alpha_n}} + \frac{\log(m)}{\sqrt{m \beta_m}}\right).
\]
and similarly for \(r_2\). Furthermore, by the final statement in Lemma 4, we have that
\[
\|\xi - \hat{\xi}\|_H \leq O\left(\frac{\sqrt{\log(n)}}{\sqrt{n \alpha_n}} + \frac{\sqrt{\log(m)}}{\sqrt{m \beta_m}}\right).
\]
Under the null hypothesis, through a similar analysis, we have that

$$\|\xi\|_H = O(\sqrt{\log(n) + \log(m)}).$$

Therefore, with probability $1 - O(n^{-2} + m^{-2})$, we have the bound

$$\|\xi - \hat{\xi}\| \leq O\left(\frac{\sqrt{\log(n)}}{\sqrt{n}\alpha_n} + \frac{\sqrt{\log(m)}}{\sqrt{m}\beta_m}\right)\left(\sqrt{\log(n)} + \sqrt{\log(m)}\right)$$

$$= O\left(\frac{\log(n)}{\sqrt{n}\alpha_n} + \frac{\log(m)}{\sqrt{m}\beta_m}\right)$$

since $m/(n + m) \to \rho \in (0, 1)$. Therefore, since $\min(\alpha_n, \beta_m) \geq n^{-1/2} \log^{1+\eta}(n)$ for some $\eta > 0$, the right hand side tends to zero.

In order to prove Corollary 2, we will need the following additional lemmas. The first is straightforward and included in Section 6.5 for completeness.

**Lemma 5.** When $E(X_1^T I_{p,q} X_2) = 1$, we have with probability at least $1 - O(n^{-2})$ that

$$\frac{1}{\sqrt{\alpha_n}} - \frac{1}{\sqrt{\alpha_n}} = O\left(\frac{\sqrt{\log(n)}}{n^{1/2}\alpha_n}\right).$$

The next lemma shows we can replace $\alpha_n$ with $\hat{\alpha}_n$ in the appropriate place and still maintain convergence in probability. The proof is in Section 6.3 and is simply a modification of the proof of Lemma 4.

**Lemma 6.** Under the setting of Lemma 4, the limiting result holds with $\hat{X}_i/\alpha_n^{1/2}$ replaced with $\hat{X}_i/\hat{\alpha}_n^{1/2}$ with the almost sure convergence replaced with convergence in probability.

**Proof of Corollary 2.** Now, the result follows by simply noting that the functional CLT still holds in probability, and hence the result holds with $\hat{X}/\alpha_n^{1/2}$ and $\hat{Y}/\beta_m^{1/2}$ replaced with $\hat{X}/\hat{\alpha}_n^{1/2}$ and $\hat{Y}/\hat{\beta}_m^{1/2}$.

### 6.2 Proofs of Propositions

In this section we prove Propositions 2, 3, and 5.

#### 6.2.1 Proof of Proposition 2

We need the following generalization of Lemma 1 to the repeated eigenvalues setting. The proof is also straightforward and postponed to Section 6.5.

**Lemma 7.** Let $Q_X$ be defined as above, and $Q_Y$ similarly. Then there exist deterministic matrices $\hat{Q}_X$ and $\hat{Q}_Y$ and sequences of orthogonal matrices $W_X$ and $W_Y$ such that $Q_X W_X - \hat{Q}_X \to 0$ and similarly for $Q_Y$ and $W_Y$. In particular, under the null hypothesis there exists some $T$ such that $F_X \circ T = F_Y$, in which case $\hat{Q}_Y$ can be chosen such that $\hat{Q}_Y = \hat{Q}_X T$.

**Proof of Proposition 2.** We will show the result holds for any two sequences of block-orthogonal matrices $W_n^1$ and $W_n^2$, since the result follows by taking $W_n := W_n^1(W_n^2)^\top$. First, by a similar argument to the proof of Proposition 1, we note that it suffices to prove the result with the replacement $X Q_X^{-1}$ and $Y Q_Y^{-1}$ instead of $\hat{X}/\alpha_n^{1/2}$ and $\hat{Y}/\beta_m^{1/2}$, since the $2 \to \infty$ result in Theorem 4 and the Lipschitz property of $\kappa$ shows that

$$\left|U_m(\hat{X} W_n^1 X_n/\alpha_n^{1/2}, \hat{Y} W_n^2 Y_n/\beta_m^{1/2}) - U_m(X Q_X^{-1} W_n^1, Y Q_Y^{-1} W_n^2)\right| \to 0$$

for any sequence of block-orthogonal matrices $W_n^1$ and $W_n^2$. 


We prove under the alternative first; that is, suppose $F_X \not\equiv F_Y$. Suppose that

$$\liminf U_{n,m}(XQ_X^{-1}W_n^1, YQ_Y^{-1}W_n^2) = 0$$

for some sequence of block-orthogonal matrices $W_n^1$, $W_n^2$. By passing to a convergent subsequence, we may assume the limit exists. Let $\tilde{Q}_X^{-1}$ and $\tilde{Q}_Y^{-1}$ be the limiting matrices given by Lemma 7, and similarly for the sequences of block orthogonal matrices $W^X$ and $W^Y$. We have that

$$|U_{n,m}(X\tilde{Q}_X^{-1}W_n^1, Y\tilde{Q}_Y^{-1}W_n^2)| = |U_{n,m}(X\tilde{Q}_X^{-1}W_X(W_X)^TW_n^1, Y\tilde{Q}_Y^{-1}W_Y(W_Y)^TW_n^2)|$$

$$\leq U_{n,m}(X\tilde{Q}_X^{-1}W_X(W_X)^TW_n^1, Y\tilde{Q}_Y^{-1}W_Y(W_Y)^TW_n^2)$$

$$- U_{n,m}(XQ_X^{-1}(W_X)^TW_n^1, YQ_Y^{-1}(W_Y)^TW_n^2)$$

$$+ |U_{n,m}(XQ_X^{-1}(W_X)^TW_n^1, YQ_Y^{-1}(W_Y)^TW_n^2)|. \quad (10)$$

We note that

$$\left| U_{n,m}(X\tilde{Q}_X^{-1}W_X(W_X)^TW_n^1, Y\tilde{Q}_Y^{-1}W_Y(W_Y)^TW_n^2) - U_{n,m}(XQ_X^{-1}(W_X)^TW_n^1, YQ_Y^{-1}(W_Y)^TW_n^2) \right|$$

tends to zero by Lemma 2. Therefore, we see that

$$\limsup |U_{n,m}(X\tilde{Q}_X^{-1}W_n^1, Y\tilde{Q}_Y^{-1}W_n^2)| \leq \liminf |U_{n,m}(XQ_X^{-1}(W_X)^TW_n^1, YQ_Y^{-1}(W_Y)^TW_n^2)|$$

where by assumption the right hand side is presumed to exist. Note that the only terms on the left hand side that are random are the matrices $X$ and $Y$ whose rows are drawn i.i.d. $F_X$ and $F_Y$ respectively. If the term on the right hand side tends to zero (which it does by assumption), then that implies that

$$\limsup |U_{n,m}(X\tilde{Q}_X^{-1}W_n^1, Y\tilde{Q}_Y^{-1}W_n^2)| \to 0.$$ 

This implies that

$$\|\mu[F_X \circ \tilde{Q}_X^{-1}W_n^1] - \mu[F_Y \circ \tilde{Q}_Y^{-1}W_n^2]\|_{\mathcal{H}}^2 \to 0,$$

by, e.g. Remark 1 in Schneider (2016) (as in the proof of Theorems 1 and 2). The only quantities that are changing in $n$ and $m$ are $W_n^1$ and $W_n^2$. Define the map

$$W \mapsto \|\mu[F_X \circ \tilde{Q}_X^{-1}] - \mu[F_Y \circ \tilde{Q}_Y^{-1}W]\|_{\mathcal{H}}^2. \quad (11)$$

By Lemma 6 in Gretton et al. (2012), we have that for any two distributions $X \sim F$ and $Y \sim G$ that

$$\|\mu[F] - \mu[G \circ W_1]\|_{\mathcal{H}}^2 - \|\mu[F] - \mu[G \circ W_2]\|_{\mathcal{H}}^2 = 2E_{F,G} \left[ \kappa(X, W_1^\top Y) - \kappa(X, W_2^\top Y) \right]$$

$$+ E_G \left[ \kappa(W_1^\top Y, W_2^\top Y') - \kappa(W_2^\top Y, W_2^\top Y') \right]$$

by the definition that $Y \sim G \circ W$ if $W^\top Y \sim G$. Hence, continuity of the map in (11) follows from continuity of $\kappa$. Therefore, by the assumption that $\kappa$ is radial, we see that since

$$\|\mu[F_X \circ \tilde{Q}_X^{-1}] - \mu[F_Y \circ \tilde{Q}_Y^{-1}W_n^2(W_n^2)^\top]\|_{\mathcal{H}}^2 \to 0,$$

we must have that some subsequence of $W_n^2(W_n^2)^\top$ is converging (since the set $\mathcal{O}(p, q) \cap \mathcal{O}(d)$ is compact). Let $\tilde{W}$ be this subsequential limit. Then this implies that

$$\|\mu[F_X \circ \tilde{Q}_X^{-1}] - \mu[F_Y \circ \tilde{Q}_Y^{-1}\tilde{W}]\|_{\mathcal{H}}^2 = 0.$$
However, under the alternative, since \( \kappa \) is characteristic, we have that \( \mu[F_X] \neq \mu[F_Y \circ T] \) for any \( T \in \mathcal{O}(p,q) \). But then the above equation is a contradiction. Furthermore, working backwards, we have the chain of inequalities

\[
\inf_{W \in \mathcal{O}(d) \cap \mathcal{O}(p,q)} \| \mu[F_X \circ \hat{Q}_X^{-1}] - \mu[F_Y \circ \hat{Q}_Y^{-1}W] \|_H^2 \leq \liminf_{n,m} \| U_{n,m}(X\hat{Q}_X^{-1}W_n^1, Y\hat{Q}_Y^{-1}W_m^2) \|_H^2 \\
\leq \limsup_{n,m} \| U_{n,m}(X\hat{Q}_X^{-1}W_n^1, Y\hat{Q}_Y^{-1}W_m^2) \|_H^2 \\
\leq \liminf_{n,m} \| U_{n,m}(XQ_X^{-1}(X^TW_n^1)^T, YQ_Y^{-1}(Y^TW_m^2)^T) \|_H^2,
\]

which shows that

\[
C := \inf_{W \in \mathcal{O}(d) \cap \mathcal{O}(p,q)} \| \mu[F_X \circ \hat{Q}_X^{-1}] - \mu[F_Y \circ \hat{Q}_Y^{-1}W] \|_H^2
\]

is a lower bound independent of the particular sequence \( W_n^1, W_m^2 \). This proves the second assertion.

Now, suppose the null hypothesis holds. Then, let \( \hat{Q}_Y = Q_XT \), where \( T \) is such that \( F_X \circ T = F_Y \). But then

\[
\left| U_{n,m}(X\hat{Q}_X^{-1}, Y\hat{Q}_Y^{-1}) \right| \leq \left| U_{n,m}(X\hat{Q}_X^{-1}, Y\hat{Q}_Y^{-1}) - U_{n,m}(XQ_X^{-1}(X^TW_n^1)^T, YQ_Y^{-1}(Y^TW_m^2)^T) \right| \\
+ \left| U_{n,m}(X\hat{Q}_X^{-1}, Y\hat{Q}_Y^{-1}) \right|.
\]

Again, the first term tends to zero by Lemma 2. We argue the second term tends to zero, and hence the result follows. We note that by Lemma 7, we have that \( \hat{Q}_Y = Q_XT \), where \( T \in \mathcal{O}(d) \cap \mathcal{O}(p,q) \). We have that \( |U_{n,m}(X\hat{Q}_X^{-1}, Y\hat{Q}_Y^{-1})| = |U_{n,m}(X\hat{Q}_X^{-1}, Y(1-T^{-1})\hat{Q}_X^{-1})| \), and since \( F_X \circ T = F_Y \), we also have that \( F_X \circ \hat{Q}_X^{-1} = F_Y \circ \hat{Q}_X^{-1} \), and hence the left hand side tends to zero.

\[\square\]

### 6.2.2 Proofs of Propositions 3 and 4

The proofs of these propositions are similar to Propositions 1 and 2 but require analysis of Wasserstein distances.

**Proof of Proposition 3.** Suppose first that the sparsity factors \( \alpha_n \) and \( \beta_m \) are known. By Lemmas 7 and 2, we have that under the null hypothesis there exist sequences of orthogonal matrices \( W_X \) and \( W_Y \) such that with probability at least \( 1 - n^{-2} - m^{-2} \)

\[
\|Q_X - W_X\hat{Q}\| = O \left( \frac{\sqrt{\log(n)}}{\sqrt{n}} \right); \quad \|Q_Y{T}^{-1} - W_Y\hat{Q}\| = O \left( \frac{\sqrt{\log(m)}}{\sqrt{m}} \right).
\]

(12)

Furthermore, by Theorem 4, there exists block-orthogonal \( W_X^* \) and \( W_Y^* \) (depending on \( n \) and \( m \)) such that with probability \( 1 - n^{-2} - m^{-2} \)

\[
\|X - \alpha_n^{1/2}XQ_X^{-1}W_X^*\|_{2 \rightarrow \infty} = O \left( \frac{\log(n)}{\sqrt{n}} \right); \quad \|Y - \beta_m^{1/2}YQ_Y^{-1}W_Y^*\|_{2 \rightarrow \infty} = O \left( \frac{\log(m)}{\sqrt{m}} \right).
\]

(13)

Define the event \( \mathcal{A} := \{(12) \text{ and } (13) \text{ hold}\} \). Note that \( \mathbb{P}(\mathcal{A}) \geq 1 - 2n^{-2} - 2m^{-2} \). Note that on \( \mathcal{A} \) we have that

\[
\|X - \alpha_n^{1/2}XQ_X^{-1}W_X^*\|_{2 \rightarrow \infty} = O \left( \frac{\log(n)}{\sqrt{n}} \right); \quad \|Y - \beta_m^{1/2}YQ_Y^{-1}W_Y^*\|_{2 \rightarrow \infty} = O \left( \frac{\log(m)}{\sqrt{m}} \right).
\]

Define \( \Gamma_{\hat{X},\hat{Y}} \) to be the set of couplings of \( \hat{F}_{\alpha_n^{1/2}} \) and \( \hat{F}_{\beta_m^{1/2}} \). We have that for any block-orthogonal \( W_n \), the minimizer \( \hat{W}_n \) satisfies

\[
d_2(\hat{F}_{\alpha_n^{1/2}}, \hat{F}_{\beta_m^{1/2}} \circ \hat{W}_n) \leq d_2(\hat{F}_{\alpha_n^{1/2}}, \hat{F}_{\beta_m^{1/2}} \circ W_n).
\]

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To show this tends to zero, we choose an appropriate block-orthogonal matrix $W_n$. Define

$$W_n := (W^X W_X)^\top (W_Y W_Y)^\top.$$  

Note that under the null hypothesis $F_X \simeq F_Y$ we have that $F_Y \circ \tilde{Q}^{-1} = F_Y \circ T^{-1} \circ Q^{-1}$ for some $T \in \mathbb{O}(p, q)$. Then, by the rotational invariance of the Euclidean norm, we have that,

$$d_2(\hat{F}_{X/\alpha_n^{1/2}} \circ W_n, \hat{F}_{Y/\beta_m^{1/2}}) = d_2(\hat{F}_{X/\alpha_n^{1/2}} \circ (W^X W_X)^\top, \hat{F}_{Y/\beta_m^{1/2}} \circ (W_Y W_Y)^\top)
\leq d_2(\hat{F}_{X/\alpha_n^{1/2}} \circ (W^X W_X)^\top, \hat{F}_X \circ \tilde{Q}^{-1})

+ d_2(\hat{F}_{Y/\beta_m^{1/2}} \circ (W_Y W_Y)^\top, \hat{F}_Y \circ T^{-1} \tilde{Q}^{-1})

+ d_2(\hat{F}_X \circ \tilde{Q}^{-1}, F_X \circ \tilde{Q}^{-1}) + d_2(\hat{F}_Y \circ T^{-1} \tilde{Q}^{-1}, F_Y \circ T^{-1} \tilde{Q}^{-1})$$

We show each term is small. For the first two terms, note that these are the empirical CDFs of the points with probability $2$ of Fournier and Guillin (2015) to see that with probability

$$\frac{1}{\sqrt{n}}\left\|\hat{X}/\sqrt{\alpha_n} - X/\tilde{Q}^{-1}\right\|^2_p$$

$$\leq \frac{1}{\sqrt{n}}\|\hat{X}/\sqrt{\alpha_n} - X/\tilde{Q}^{-1} W_X W_X\|_F$$

$$\leq O\left(\frac{\log(n)}{(n\alpha_n)^{1/2}}\right)$$

Similarly,

$$d_2(\hat{F}_{Y/\beta_m^{1/2}} \circ (W_Y W_Y), \hat{F}_Y \circ T^{-1} \tilde{Q}^{-1}) = O\left(\frac{\log(m)}{(m\beta_m)^{1/2}}\right).$$

Finally, we bound the final two terms. By Theorem 3, supp$(F)$ is bounded and hence so is any fixed invertible linear transformation of supp$(F)$. Hence, since $\|X\|_\infty \leq M$ almost surely, we can apply Theorem 2 of Fournier and Guillin (2015) to see that with probability $1 - n^{-2}$ that

$$d_2(\hat{F}_X \circ \tilde{Q}^{-1}, F_X \circ \tilde{Q}^{-1}) = O\left(\frac{\log^{1/d}(n)}{n^{1/d}}\right);$$

see also Levin and Levina (2019); Lei (2020a,b) for a related problem. Similarly,

$$d_2(\hat{F}_Y \circ T^{-1} \tilde{Q}^{-1}, F_Y \circ T^{-1} \tilde{Q}^{-1}) = O\left(\frac{\log^{1/d}(m)}{m^{1/d}}\right)$$

with probability $1 - m^{-2}$. Therefore, putting it all together, we see that with probability $1 - O(n^{-2} + m^{-2})$,

$$d_2(\hat{F}_{X/\alpha_n^{1/2}}, \hat{F}_{Y/\beta_m^{1/2}} \circ \tilde{W}) = O\left(\frac{\log^{1/d}(n)}{n^{1/d}} + \frac{\log^{1/d}(m)}{m^{1/d}} + \frac{\log(n)}{(n\alpha_n)^{1/2}} + \frac{\log(m)}{(m\beta_m)^{1/2}}\right).$$

Finally, if the sparsity is not known, we have that by Lemma 5, $
\frac{1}{\sqrt{\alpha_n}} = \frac{1}{\sqrt{\alpha_n}}\left(1 + O\left(\frac{\sqrt{\log(n)}}{n^{1/2}\alpha_n}\right)\right)$ with
probability at least $1 - O(n^{-2})$. Hence, we see that

$$
\| \hat{\mathbf{X}}(\hat{\alpha}_n^{-1/2} - \alpha_n^{-1/2})\|_{2,\infty} \leq \| \hat{\mathbf{X}}\|_{2,\infty} O\left( \frac{\log(n)}{n^{1/2}\alpha_n} \right)
= O\left( \frac{\log(n)}{n^{1/2}\alpha_n} \right) \left( \| \mathbf{X} - \sqrt{\alpha_n} \mathbf{Q}_\mathbf{X}^{-1} \mathbf{W}_\ast \|_{2,\infty} + \| \sqrt{\alpha_n} \mathbf{Q}_\mathbf{X}^{-1} \mathbf{W}_\ast \|_{2,\infty} \right)
= O\left( \frac{\log(n)}{n^{1/2}\alpha_n} \right) \left( \log(n) \frac{1}{\sqrt{n}} + O(\sqrt{\alpha_n}) \right)
= O\left( \frac{\log(n)}{n\alpha_n} \right) + O\left( \sqrt{\log\left( \frac{1}{\alpha_n} \right)} \right)
= O\left( \frac{\log(n)}{n\alpha_n} \right).
$$

Therefore, by analogous arguments as in the setting with the sparsity factors known, replacing $\hat{\alpha}_n^{-1/2}$ and $\beta_m^{-1/2}$ with $\alpha_n^{-1/2}$ and $\beta_m^{-1/2}$ is negligible compared to the $2,\infty$ bound. Hence, we have that with probability at least $1 - O(n^{-2} + m^{-2})$

$$
d_2(\hat{F}_{X/\alpha_n^{1/2}}, \hat{F}_{Y/\beta_m^{1/2}} \circ \hat{\mathbf{W}}_n) = O\left( \frac{\log^{1/d}(n)}{n^{1/d}} + \frac{\log^{1/d}(m)}{m^{1/d}} + \frac{\log(n)}{(n\alpha_n)^{1/2}} + \frac{\log(n)}{(m\beta_m)^{1/2}} \right).
$$

Proof of Proposition 4. Like the previous proof, we assume first that the sparsity factors $\alpha_n$ and $\beta_m$ are known. Let $\mathbf{W}_\mathbf{X}, \mathbf{W}_\mathbf{Y}, \mathbf{W}_\mathbf{X}^\dagger$, and $\mathbf{W}_\mathbf{Y}^\dagger$ be as in the proof of Proposition 3, and let $\mathbf{Q}_\mathbf{X}$ be the limit guaranteed by Lemma 2 and similarly for $\mathbf{Q}_\mathbf{Y}$. Note in this setting $\mathbf{Q}_\mathbf{X}$ is not necessarily equal to $\mathbf{Q}_\mathbf{Y}$. Let $\mathcal{A}$ be the same event as in the previous proof. By the reverse triangle inequality, we have that

$$
d_2(\hat{F}_{X/\alpha_n^{1/2}}, \hat{F}_{Y/\beta_m^{1/2}} \circ \hat{\mathbf{W}}_n) \geq -d_2(\hat{F}_{X/\alpha_n^{1/2}}, F_X \circ (\mathbf{Q}_\mathbf{X}^{-1}(\mathbf{W}_\mathbf{X} \mathbf{W}_\ast^\dagger)))
- d_2(\hat{F}_{Y/\beta_m^{1/2}} \circ \hat{\mathbf{W}}_n, F_Y \circ (\mathbf{Q}_\mathbf{Y}^{-1}(\mathbf{W}_\mathbf{Y} \mathbf{W}_\ast^\dagger) \circ \hat{\mathbf{W}}_n))
+ d_2(F_X \circ (\mathbf{Q}_\mathbf{X}^{-1}(\mathbf{W}_\mathbf{X} \mathbf{W}_\ast^\dagger)), F_Y \circ (\mathbf{Q}_\mathbf{Y}^{-1}(\mathbf{W}_\mathbf{Y} \mathbf{W}_\ast^\dagger) \circ \hat{\mathbf{W}}_n))
$$

From the proof of Proposition 3, with probability $1 - O(n^{-2})$ it holds that

$$
d_2(\hat{F}_{X/\alpha_n^{1/2}}, F_X \circ (\mathbf{Q}_\mathbf{X}^{-1}(\mathbf{W}_\mathbf{X} \mathbf{W}_\ast^\dagger))) = O\left( \frac{\log(n)}{n^{1/2}} + \frac{\log^{1/d}(n)}{n^{1/d}} \right),
$$
and analogously for the term depending on $F_Y$.

Consider the sequence $\mathbf{\bar{W}}_n := (\mathbf{W}_\mathbf{Y} \mathbf{W}_\mathbf{Y}^\dagger)^\top \mathbf{W}_n(\mathbf{W}_\mathbf{X} \mathbf{W}_\ast^\dagger)$. We note as a product of block-orthogonal matrices $\mathbf{W}_n$ is block-orthogonal and hence an element of $\mathcal{O}(p,q)$. Therefore, through the above argument and the invariance of the Frobenius norm to orthogonal transformations, we have that with probability $1 - O(n^{-2} + m^{-2})$ that

$$
d_2(\hat{F}_{X/\alpha_n^{1/2}}, \hat{F}_{Y/\beta_m^{1/2}} \circ \hat{\mathbf{W}}_n) \geq c_n - \varepsilon_n > 0,
$$
where $\varepsilon_n \to 0$, and

$$
c_n := d_2(F_X \circ \mathbf{Q}_\mathbf{X}^{-1}, F_Y \circ \mathbf{Q}_\mathbf{Y}^{-1} \circ \hat{\mathbf{W}}_n).
$$

Note that the only dependence on $n$ in the above comes from $\mathbf{\bar{W}}_n$. Furthermore, $\mathcal{O}(p,q) \cap \mathcal{O}(d)$ is a closed subgroup of $\mathcal{O}(d)$, and hence compact. Also, the mapping $\mathbf{\bar{W}} \mapsto d_2(F_X \circ \mathbf{Q}_\mathbf{X}^{-1}, F_Y \circ \mathbf{Q}_\mathbf{Y}^{-1} \circ \mathbf{\bar{W}})$
is continuous since for any fixed \( \mu_0 \in \mathbb{R}^d \), we have that

\[
\int \| \mu_0 - \hat{W}^T Y \|^2 dF(Y) = E\| \mu_0 - \hat{W}^T Y \|^2,
\]

is continuous. Hence, consider a convergent subsequence \( \hat{W}_{k_n} \) obtaining \( \liminf c_n \) and let \( \hat{W} \) be its associated subsequential limit. Hence, on the sequence of events \( A_n \),

\[
\liminf d_2(F_X \circ \hat{Q}^{-1}_X, F_Y \circ \hat{Q}^{-1}_Y \circ \hat{W}_{k_n}) = d_2(F_X \circ \hat{Q}^{-1}_X, F_Y \circ \hat{Q}^{-1}_Y \circ \hat{W}) \geq C
\]

for some constant \( C > 0 \) or else we would have \( F_X \simeq F_Y \) since \( \hat{W} \in \mathcal{O}(p,q) \). Therefore, by Borel-Cantelli, there exists a constant \( C > 0 \) such that

\[
\liminf d_2(F_{X/\alpha_{n/2}}, F_{Y/\beta_{n/2}} \circ \hat{W}_n) \geq C
\]

almost surely.

To replace the \( \alpha_n \) and \( \beta_m \) with their respective estimated counterparts, the same argument as in the proof of Proposition 3 goes through.

\( \square \)

6.2.3 Proof of Proposition 5

Proof of Proposition 5. Let \( R \) have block diagonal components \( R_p \) and \( R_q \), and let \( W_p \) and \( W_q \) be the top \( p \times p \) and bottom \( q \times q \) block of \( W \) respectively. Expanding out the Frobenius norm, we have that

\[
\arg\min_{R \in \mathcal{O}(d) \cap \mathcal{O}(p,q)} \| R - W \|_F = \arg\min_{R \in \mathcal{O}(d) \cap \mathcal{O}(p,q)} \| R - W \|_F^2,
\]

\[
= \arg\min_{R \in \mathcal{O}(d) \cap \mathcal{O}(p,q)} \text{Tr} \left( (R - W)^\top (R - W) \right)
\]

\[
= \arg\min_{R \in \mathcal{O}(d) \cap \mathcal{O}(p,q)} \text{Tr} \left( R^\top R - 2R^\top W + W^\top W \right)
\]

\[
= \arg\max_{R \in \mathcal{O}(d) \cap \mathcal{O}(p,q)} \text{Tr} (R^\top W)
\]

\[
= \arg\max_{R \in \mathcal{O}(d) \cap \mathcal{O}(p,q)} \text{Tr}(R_p^\top W_p) + \text{Tr}(R_q^\top W_q)
\]

since \( R^\top R = W^\top W = I \). Let \( W_p \) have singular value decomposition \( U_p \Sigma_p V_p^\top \) and similarly for \( W_q \). Then the maximum for each of the above is achieved by setting \( R_p = U_p V_p^\top \) and \( R_q = U_q V_q^\top \).

\( \square \)

6.3 Proof of the Frobenius Concentration (Lemma 3)

In this section we present the proof of Lemma 3. We will need the following Lemma, adapted from Lemma A.4 Athreya et al. (2020).

Lemma 8. Let \( A \) be a matrix whose entries are generated via \( A_{ij} \sim \text{Bernoulli}(p_{ij}) \) for \( i \leq j \), and let \( V = U_X |X|^{-1/2} \). Then with probability at least \( 1 - O(n^{-2}) \),

\[
\left| \| (A - P) V \|_F^2 - \mathbb{E}(\| (A - P) V \|_F^2) \right| = O\left( \sqrt{\frac{\log(n)}{n\alpha_n}} \right).
\]

Proof of Lemma 8. We follow the proof in Athreya et al. (2020), though we have a slightly different argument for the inclusion of the sparsity factor. Let \( A' \sim P \) be independent from \( A \). For \( 1 \leq r \leq s \leq n \) define the term

\[
Z_{rs} := \| (A^{(r,s)} - P)V \|_F^2,
\]
where the matrix $A^{(r,s)}$ agrees with $A$ in every entry except the $(r,s)$ and $(s,r)$-ones, where it equals $A'$. Defining $Z := \|(A - P)V\|^2_F$, we see that for $r \neq s$

$$Z - Z_{rs} = 2(A - A')_{rs} \left[\begin{array}{c}
((A - P)VV^T)_{rs} + ((A - P)VV^T)_{sr} + (A' - P)_{rs} \left((VV^T)_{ss} + (VV^T)_{rr}\right) \end{array}\right]$$

Furthermore, we have that

$$(Z - Z_{rs})^2 \leq \begin{cases} 
16 \left[\left((A - P)VV^T\right)_{rs}^2 + \left((A - P)VV^T\right)_{sr}^2 + (A' - P)_{rs}^2 \left((VV^T)_{ss} + (VV^T)_{rr}\right)^2\right] & r \neq s \\
8 \left[\left((A - P)VV^T\right)_{rr}^2 + (A' - P)_{rs}^2 \left((VV^T)_{ss} + (VV^T)_{rr}\right)^2\right] & r = s 
\end{cases}$$

Hence,

$$\sum_{r \leq s} (Z - Z_{rs})^2 \leq 16Z\|V\|^2_F + 8 \sum_r (VV^T)_{rr}^2 + 16 \sum_{r<s} (A' - P)^2_{rs} [(VV^T)_{ss}^2 + (VV^T)_{rr}^2]$$

$$= 16Z\|V\|^2_F + 8\|\text{diag}(VV^T)\|^2_F + 16 \sum_{s=1}^{n} \sum_{r=1}^{s-1} (A' - P)^2_{rs} [(VV^T)_{ss}^2 + (VV^T)_{rr}^2].$$

For the final term above, we have that

$$\mathbb{E}_{A'} \left[ \sum_{s=1}^{n} \sum_{r=1}^{s-1} (A' - P)^2_{rs} [(VV^T)_{ss}^2 + (VV^T)_{rr}^2] \right] = \sum_{s=1}^{n} \sum_{r=1}^{s-1} \mathbb{E}_{A'} (A' - P)^2_{rs} [(VV^T)_{ss}^2 + (VV^T)_{rr}^2] \leq 2n\alpha_n \|\text{diag}(VV^T)\|^2_F.$$

Moreover, from the definitions of $V$, we have that $\|\text{diag}(VV^T)\|^2_F \leq d\lambda_d^{-2} \leq Cd(n\alpha_n)^{-2}$, and $\|V\|^2_F = |\lambda_d|^{-1} \leq C(n\alpha_n)^{-1}$. Hence, we see that

$$\mathbb{E}_{A'} \sum_{r \leq s} (Z - Z_{rs})^2 \leq \frac{C_1}{n\alpha_n} Z + \frac{C_2d}{(n\alpha_n)^2} + \frac{C_3d}{n\alpha_n}.$$

Define $a := \frac{C_1}{n\alpha_n}$ and $b := \frac{C_2d}{(n\alpha_n)^2} + \frac{C_3d}{n\alpha_n}$. By Theorems 5 and 6 in Boucheron et al. (2003), we have that

$$\mathbb{P}(|Z - \mathbb{E}Z| > t) \leq 2 \exp \left(-\frac{t^2}{4a\mathbb{E}(Z) + 4b + 2at}\right)$$

$$\leq 2 \exp \left(-\frac{t^2n\alpha_n}{4C_1\mathbb{E}(Z) + 4C_2d(n\alpha_n)^{-1} + 4C_3d + 2C_1t}\right)$$

$$\leq 2 \exp \left(-\frac{t^2n\alpha_n}{C_1 + C_2t}\right)$$

for some constants $\tilde{C}_1$ and $\tilde{C}_2$ (depending on $d$) and for $n\alpha_n$ sufficiently large. Note that we implicitly used
that \(EZ = O(1)\), which can be seen from the fact that
\[
E(Z) = \mathbb{E}[(A - P)V]_F^2
\]
\[
= \sum_{i=1}^n \sum_{k=1}^d \mathbb{E} \left( \sum_j (A_{ij} - P_{ij}) V_{jk} \right)^2
\]
\[
= \sum_{i=1}^n \sum_{k=1}^d \sum_{j=1}^n V_{jk}^2 \mathbb{E}((A_{ij} - P_{ij}))^2 + \sum_{j \neq l} V_{jk} V_{lk} \mathbb{E}((A_{ij} - P_{ij})(A_{il} - P_{il}))
\]
\[
= \sum_{i=1}^n \sum_{k=1}^d \sum_{j=1}^n V_{jk}^2 P_{ij}(1 - P_{ij})
\]
\[
\leq n\alpha_n \sum_{k=1}^d \sum_{j=1}^n V_{jk}^2
\]
\[
\leq n\alpha_n \|U|\Lambda|^{-1/2}\|_F^2
\]
\[
\leq C
\]
for some constant \(C\) depending on \(d\) and \(\lambda d\). Hence, with the choice \(t = \tilde{C} \sqrt{\log(n)/n\alpha_n}\) for some constant \(\tilde{C}\) depending on \(d\), this is bounded above by \(2n^{-2}\).

We are now ready to prove Lemma 3.

**Proof of Lemma 3.** First, by the proof of Theorem 5 in Rubin-Delanchy et al. (2020), we note that there exists an orthogonal matrix \(W^* \in O(d) \cap O(p,q)\) (see equations 5 and 6 in Rubin-Delanchy et al. (2020)) such that
\[
\hat{U}|\hat{\Lambda}|^{1/2} - U|\Lambda|^{1/2}W_+ = (A - P)U|\Lambda|^{-1/2}W_+^T I_{p,q} + R,
\]
where the matrix \(R\) satisfies
\[
\|R\|_{2,\infty} = O\left(\frac{d^{1/2}\log^{1/2}(n)}{\sqrt{n(n\alpha_n)^{1/2}}}\right).
\]
Passing to the Frobenius norm, we see that
\[
\|R\|_F = O\left(\frac{d^{1/2}\log^{1/2}(n)}{(n\alpha_n)^{1/2}}\right).
\]
This proves the first claim. Hence,
\[
\|\hat{X} - XW_+^T\|_F = \|(A - P)U|\Lambda|^{-1/2}\|_F + O\left(\sqrt{\frac{d\log(n)}{n\alpha_n}}\right).
\]
We then can apply Lemma 8 to see that
\[
\mathbb{P} \left( \|(|A - P)V\|_F^2 - C(P)^2 \right) > C\sqrt{\log(n)/n\alpha_n} = O(n^{-2}),
\]
where \(V = U|\Lambda|^{-1/2}\) and \(C^2(P) = \mathbb{E}[(A - P)V]_F^2\). The rest of the proof is similar to Tang and Priebe (2018). By similar manipulations as those leading to Equation 18 in Rubin-Delanchy et al. (2020), we have that
\[
(A - P)U|\Lambda|^{-1/2}W_+^T I_{p,q} = \alpha_n^{-1/2}(A - P)X(X^TX)^{-1}I_{p,q}QX^{-1}.
\]
By Lemma 2, we have that there exists a sequence of block-orthogonal matrices such that $W_n^TQ_x^{-1} \rightarrow \tilde{Q}^{-1}$ almost surely. Hence, we have that

$$\|(A - P)\|\Lambda|^{-1/2}\|_{F}^2 = \frac{1}{\alpha_n} \|(A - P)X(X^TX)^{-1}I_{p,q}Q_x^{-1}\|_{F}^2$$

$$= \frac{1}{\alpha_n} \text{Tr}(Q_x^{-1}(X^TX)^{-1}X^TE(A - P)^2X(X^TX)^{-1}Q_x^{-1})$$

$$= \frac{1}{\alpha_n} \text{Tr}(W_n^TQ_x^{-1}(X^TX)^{-1}X^TE(A - P)^2X(X^TX)^{-1}Q_x^{-1}W_n)$$

$$= \text{Tr}(W_n^TQ_x^{-1}(n(X^TX)^{-1}) \left[ \frac{X^TE(A - P)^2X}{n^2\alpha_n} \right] (n(X^TX)^{-1})Q_x^{-1}W_n).$$

By the strong law of large numbers the term $X^TX/n \rightarrow \Delta$ almost surely, so $n(X^TX)^{-1} \rightarrow \Delta^{-1}$ almost surely by the continuous mapping theorem. In addition, we have that

$$\frac{X^TE(A - P)^2X}{n^2\alpha_n} = \frac{1}{n^2\alpha_n} \sum_{i=1}^{n} \sum_{k} X_iX_i^T (p_{ik}(1 - p_{ik}))$$

$$= \frac{1}{n^2\alpha_n} \sum_{i=1}^{n} \sum_{k} X_iX_i^T (X_i^T I_{p,q} X_k - \alpha_n X_i^T I_{p,q} X_k X_k^T I_{p,q} X_i).$$

As $n \rightarrow \infty$, this is tending to the matrix $\Gamma$, where $\Gamma$ is defined via

$$\Gamma := \begin{cases} 
E(XX^T (X^T I_{p,q} \mu - X^T I_{p,q} \Delta I_{p,q} X)) & \alpha \equiv 1 \\
E(XX^T (X^T I_{p,q} \mu)) & \alpha \rightarrow 0,
\end{cases}$$

where $\mu = E(X)$. Hence, putting it together, we have almost surely,

$$\|\tilde{X} - \hat{X} W_{*}^{T}\|_{F}^2 \rightarrow \text{Tr} \left( \tilde{Q}^{-1} \Delta^{-1} \Gamma \Delta^{-1} \tilde{Q}^{-1} \right).$$

\[ \square \]

6.4 Proof of the Functional CLT (Lemma 4) and Related Lemmas

In this section, we prove Lemma 4 and Lemma 6.

Proof of Lemma 4. We follow the proof by analogy to the proof Lemma 3 of Tang et al. (2017b), though we use the decomposition from Lemma 3. As $F$ is twice continuously differentiable, for $f \in F$ we Taylor expand to note that

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{(f(X_i) - f(W_{*}X_i))}{\sqrt{\alpha_n}} = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (\partial f)(X_i) \left( \frac{X_i - W_{*}X_i}{\sqrt{\alpha_n}} \right)$$

$$+ \frac{1}{2\sqrt{n}} \sum_{i} \left( X_i - W_{*}X_i \right)^T (\partial^2 f)(X_i^*) (X_i - W_{*}X_i) \frac{1}{\alpha_n},$$

for some $X_i^*$. The second order term is straightforwardly bounded by noting that by Theorem 3 $\bar{\Omega}$ is compact, and hence we can apply Lemma 3 to see that there exists some constant $C$ such that

$$\sup_{f \in F} \sum_{i=1}^{n} \frac{(X_i - W_{*}X_i)^T (\partial^2 f)(X_i^*) (X_i - W_{*}X_i)}{\sqrt{\alpha_n}} \leq \sup_{f \in F, X \in \bar{\Omega}} \frac{\|\partial^2 f(X)\|}{\sqrt{\alpha_n}} \|X_i - W_{*}X_i\|_{F}^2$$

$$\leq \frac{C}{\sqrt{\alpha_n}}.$$
which converges to zero almost surely.

We now bound the linear terms. Let \( \mathbf{M}(\partial f) = \mathbf{M}(\partial f; \tilde{X}_1, \ldots, \tilde{X}_n) \in \mathbb{R}^{n \times d} \) be the matrix whose rows are the vectors \( \partial f(\tilde{X}_i) \). Then

\[
\zeta(f) := \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (\partial f)(\tilde{X}_i)\top(\tilde{X}_i - \mathbf{W}_*\tilde{X}_i)
= \frac{1}{\sqrt{n}} \text{Tr}((\mathbf{X} - \mathbf{X}\mathbf{W}_*)[\mathbf{M}(\partial f)]\top)
= \frac{1}{\sqrt{n}} \text{Tr}((\mathbf{A} - \mathbf{P})\mathbf{U}|\mathbf{A}|^{-1/2}\mathbf{I}_{p,q})[\mathbf{M}(\partial f)]\top + \frac{1}{\sqrt{n}} \text{Tr}(\mathbf{R}[\mathbf{M}(\partial f)]\top),
\]

where \( \mathbf{R} \) is the residual matrix in 3. Recall the second term satisfies

\[
\frac{1}{\sqrt{n}} \text{Tr}(\mathbf{R}[\mathbf{M}(\partial f)]\top) = \frac{1}{\sqrt{n}} \langle \mathbf{R}, \mathbf{M}(\partial f) \rangle
\leq \sup_{f \in \mathcal{F}, \mathbf{X} \in \Omega} \frac{\sqrt{n}\|\partial f(\mathbf{X})\|}{\sqrt{n}} \|\mathbf{R}\|_F
\leq C\frac{\sqrt{n}}{\sqrt{n} \alpha n}
\leq C\frac{\sqrt{\log(n)}}{\sqrt{\alpha n}},
\]

where the penultimate inequality comes from the fact that \( \bar{\Omega} \) can be taken to be compact by Theorem 3, and since \( \mathcal{F} \) is twice-continuously differentiable, the gradient is Lipschitz on any fixed transformation of the support.

Now, we show that the final term converges to zero. The rest of the proof is largely the same as in Tung et al. (2017b). Define the set of derivatives of \( \partial \mathcal{F} := \{\partial f : f \in \mathcal{F}\} \). Let \( \|\partial f\|_\infty \) be the maximum Euclidean norm attained by \( f \) on \( \bar{\Omega} \). Note that by enlarging it if necessary, \( \bar{\Omega} \) can be taken to be compact and contain the \( \tilde{X}_i \)'s by the fact that \( \kappa \) is twice continuously differentiable on \( \mathbb{R}^d \) and by virtue of Theorems 4 and 3 and Lemma 2. Therefore the set of derivatives is totally bounded; define \( M := \sup_{\partial \mathcal{F}} \| \partial f \|_\infty \).

Then for any \( j \) there exists a finite subset \( S_j \) of covering functions such that for any \( g \in \partial \mathcal{F} \), we have that \( \| g - f_j \|_\infty \leq 2^{-j} M \). Define the mapping \( \mathcal{P}_j \) as the mapping that assigns the function \( g \in \partial \mathcal{F} \) to its closest function \( f_j \in S_j \). Then we have that

\[
\sup_f \left| \frac{1}{\sqrt{n}} \text{Tr}((\mathbf{A} - \mathbf{P})\mathbf{U}|\mathbf{A}|^{-1/2}\mathbf{I}_{p,q})[\mathbf{M}(\partial f)]\top \right|
= \sup_f \left| \frac{1}{\sqrt{n}} \sum_{i=1}^{\infty} \sum_{j=0}^{\infty} (\mathcal{P}\mathbf{A}\mathbf{U}(\mathbf{A})^{-1/2}\mathbf{I}_{p,q})[\mathbf{M}(\partial f)]\top \right|
\leq \sum_{j=0}^{\infty} \sup_f \left| \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (\mathcal{P}\mathbf{A}\mathbf{U}(\mathbf{A})^{-1/2}\mathbf{I}_{p,q})[\mathbf{M}(\partial f)]\top \right|
+ \frac{1}{\sqrt{n}} \sum_{i=1}^{n} |\mathcal{P}\mathbf{A}\mathbf{U}(\mathbf{A})^{-1/2}\mathbf{I}_{p,q})[\mathbf{M}(\partial f)]\top |
\]

We note that for fixed \( j \), defining the term \( \mathcal{Q}_j \) as the \( n \times d \) matrix whose rows are \( (\mathcal{P}\mathbf{A}\mathbf{U}(\mathbf{A})^{-1/2}\mathbf{I}_{p,q})[\mathbf{M}(\partial f)]\top \), we have that

\[
\frac{1}{\sqrt{n}} \sum_{i=1}^{n} |(\mathcal{P}\mathbf{A}\mathbf{U}(\mathbf{A})^{-1/2}\mathbf{I}_{p,q})[\mathbf{M}(\partial f)]\top |
= \frac{1}{\sqrt{n}} \left| \sum_{s=1}^{d} ((\mathcal{Q}_j)\top (\mathbf{A} - \mathbf{P})\mathbf{U}\mathbf{s} - \frac{\mathbb{I}_{s \geq p} + \mathbb{I}_{s \leq p}}{\sqrt{|\lambda_s|^{1/2}}} \right|
\]

We note that

\[
\| (\mathcal{Q}_j) \|_2 \leq \frac{3}{2} 2^{-j} M \sqrt{n}.
\]
Hence, for fixed $s \in \{1 \ldots, d\}$ this is a linear combination of mean-zero random variables. Therefore we have by Hoeffding’s inequality that

$$\mathbb{P}\left(\left(\mathcal{Q}_f^T\mathcal{Q}_f\right)_s \left(\left(\mathbf{A} - \mathbf{P}\right)\mathbf{U}_{s, \mathbb{I}_{s \geq p} + \mathbb{I}_{s \leq p}} \mathbf{U}_{s, \mathbb{I}_{s \geq p} + \mathbb{I}_{s \leq p}} - \mathbf{X}_{s, \mathbb{I}_{s \geq p} + \mathbb{I}_{s \leq p}}\right) > t\right) \leq 2 \exp\left(-\frac{t^2}{C^2 - 2d|\lambda_d|^{-1}}\right),$$

for some constant $C$ depending on $M$. Hence, by the union bound, we have that

$$\mathbb{P}\left[\frac{1}{\sqrt{n}} \sum_{s=1}^{d} \left(\left(\mathcal{Q}_f^T\mathcal{Q}_f\right)_s \left(\left(\mathbf{A} - \mathbf{P}\right)\mathbf{U}_{s, \mathbb{I}_{s \geq p} + \mathbb{I}_{s \leq p}} - \mathbf{X}_{s, \mathbb{I}_{s \geq p} + \mathbb{I}_{s \leq p}}\right) > dt\right) \right] \leq 2d \exp\left(-\frac{t^2}{C^2 - 2d|\lambda_d|^{-1}}\right),$$

provided that $t$ is chosen appropriately. By another union bound over the set $S_j$, we have that

$$\mathbb{P}\left[\sup_f \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left(\left(\mathcal{Q}_f^T\mathcal{Q}_f\right)_i \left(\left(\mathbf{A} - \mathbf{P}\right)\mathbf{U}|\mathbf{A}|^{-1/2}\mathbf{I}_{p,q}\right)_i > dt\right) \right] \leq 2d|S_j| \exp\left(-\frac{t^2}{C^2 - 2d|\lambda_d|^{-1}}\right).$$

We note that $|S_j| \leq (C2)^d$ by using the bound on the covering number (e.g. Lemma 2.5 in van de Geer (2009)). Following the steps from equation A.5 to A.6 in Tang et al. (2017b), by rearranging the equation above, we have that for any $t_j > 0$

$$\mathbb{P}\left[\sup_f \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left(\left(\mathcal{Q}_f^T\mathcal{Q}_f\right)_i \left(\left(\mathbf{A} - \mathbf{P}\right)\mathbf{U}|\mathbf{A}|^{-1/2}\mathbf{I}_{p,q}\right)_i > \eta_j \right) \right] \leq 2d \exp(-t_j^2),$$

where $\eta_j = d\sqrt{C2 - 2d\lambda_d^{-1}}(t_j^2 + \log|S_j+1|^2)$. Summing over $j$ and bounding the zeroth order term similarly, we have

$$\mathbb{P}\left[\sup_f \frac{1}{\sqrt{n}} \text{Tr}\left((\mathbf{A} - \mathbf{P})\mathbf{U}|\mathbf{A}|^{-1/2}\mathbf{I}_{p,q}\right)\mathbb{M}(\mathbf{\partial f})\mathbb{M}(\mathbf{\partial f})^T\right) \geq \sum_{j=0}^{\infty} \tilde{C}\eta_j \right] \leq 2d \sum_{j=0}^{\infty} \exp(-t_j^2).$$

Taking $t_j^2 = 2(\log(j) + \log(n))$, we have that

$$\mathbb{P}\left[\sup_f \frac{1}{\sqrt{n}} \text{Tr}\left((\mathbf{A} - \mathbf{P})\mathbf{U}|\mathbf{A}|^{-1/2}\mathbf{I}_{p,q}\right)\mathbb{M}(\mathbf{\partial f})\mathbb{M}(\mathbf{\partial f})^T\right] \geq d\lambda_d^{-1/2}(C_1\sqrt{\log(n) + C_2}) \leq \frac{2dC_3}{n^2},$$

for some constants $C_1$, $C_2$, and $C_3$. Hence, combining all this, we see the linear term satisfies

$$\sup_f |\zeta(f)| \leq C\frac{\sqrt{\log(n)}}{\sqrt{\alpha_n}}$$

with probability at least $1 - O(n^{-2})$. We note that the hidden constants and the constants in the bound depend on the diameter of the set $\partial\mathcal{F}$ and the dimension $d$.

Finally, if $\sqrt{\alpha_n} \geq \log^{1+\eta}(n)$, then the right hand side still tends to zero with an additional factor of $\sqrt{\alpha_n}$ in the denominator, which is the final assertion of Lemma 4. \hfill \square

**Proof of Lemma 6.** First, by Lemma 5, we have that $\sqrt{\alpha_n}/\sqrt{\alpha_n} \rightarrow 1$ in probability (and almost surely). In the proof of Lemma 4, we have shown that

$$\sup_f \frac{\sqrt{\alpha_n}}{\sqrt{n}} \sum_{i=1}^{n} \left(\mathbf{X}_i - \mathbf{W}^*\mathbf{X}_i\right) \frac{(\mathbf{X}_i - \mathbf{W}^*\mathbf{X}_i)}{\sqrt{\alpha_n}} \rightarrow 0;$$

$$\sup_f \frac{\sqrt{\alpha_n}}{2\sqrt{n}} \sum_{i} \left(\mathbf{X}_i - \mathbf{W}^*\mathbf{X}_i\right)^T \left(\mathbf{\partial^2 f}(\mathbf{X}_i^*)\right) \left(\mathbf{W}^*\mathbf{X}_i\right) \alpha_n \rightarrow 0.$$
almost surely. Therefore, we can replace \( \sqrt{\alpha_n} \) by \( \sqrt{\alpha_n} \) and apply Slutsky’s Theorem to conclude that

\[
\sup_{f} \frac{\sqrt{\alpha_n}}{\sqrt{n}} \left( \sum_{i=1}^{n} (\partial f)(X_i) \right) \left( \frac{X_i}{\sqrt{\alpha_n}} \right) - W_{\alpha_n}X_i \to 0;
\]

\[
\sup_{f} \frac{\sqrt{\alpha_n}}{2\sqrt{n}} \left( \sum_{i=1}^{n} (\partial f)(X_i) \right) \left( \frac{X_i}{\sqrt{\alpha_n}} \right) - W_{\alpha_n}X_i \to 0
\]

in probability.

6.5 Proofs of Auxiliary Lemmas

In this section we prove the additional technical lemmas; namely Lemmas 1, 7, 2, and 5.

6.5.1 Proofs of Lemmas 1, 7, and 2

This section contains various results associated to the approximation of the matrix \( Q_X \) to its limiting value.

**Proof of Lemma 1.** By Agterberg et al. (2020), we see that \( Q_X \xrightarrow{a.s.} \tilde{Q}_X \), where \( \tilde{Q} \) is a fixed indefinite orthogonal matrix, and similarly for \( Q_Y \), so that \( Q_Y \to \tilde{Q}' \) for some fixed matrix \( \tilde{Q}' \). Define the matrix \( \tilde{X} := YT^{-1} \). Note that the rows of \( \tilde{X} \) are distributed iid \( F_X \). Suppose that \( Q_{\tilde{X}} \) is the indefinite orthogonal matrix such that

\[
U_Y |S_Y|^{1/2} Q_{\tilde{X}} = \tilde{X}.
\]

Then we apply Theorem 1 from Agterberg et al. (2020) again, which implies that \( \|Q_{\tilde{X}} - Q_X\| \to 0 \) almost surely. However, noting \( \tilde{X} = YT^{-1} \) gives that \( Q_{\tilde{X}} = Q_Y T^{-1} \), which gives the result.

**Proof of Lemma 7.** The proof mostly is similar to that of Lemma 1, only instead we apply Corollary 2 from Agterberg et al. (2020). We have that there exists a sequence of block-orthogonal matrices \( W_X \) such that \( \|Q_X - W_X \tilde{Q}\| \to 0 \). Again, we are free to repeat the argument in the case \( F_X = F_Y \circ T \), only replacing \( Q_X \) with \( Q_Y T^{-1} \), to see that

\[
\|Q_Y T^{-1} - W_Y \tilde{Q}\| \to 0.
\]

Note that the (block)-orthogonal matrix above need not necessarily be the same due to nonidentifiability of the eigenvectors. Hence,

\[
W_Y^T Q_Y T^{-1} - W_X^T Q_X \to 0,
\]

which in particular implies that both terms are tending towards \( \tilde{Q} \) almost surely.

**Proof of Lemma 2.** Define \( \Delta = E(XX^T) \), and let \( \tilde{V} \) and \( V \) be the orthogonal matrices in the eigendecomposition of \( \left( X^T X \right)^{1/2} I_{p,q} \left( X^T X \right)^{1/2} \) and \( \Delta^{1/2} I_{p,q} \Delta^{1/2} \) respectively. Let \( \Lambda \) be the eigenvalues of \( XI_{p,q}X^T \) and let \( \tilde{\Lambda} \) be the eigenvalues of \( \Delta^{1/2} I_{p,q} \Delta^{1/2} \). We will first show that with probability at least \( 1 - n^{-2} \), that the following hold simultaneously:

\[
\|\Delta^{1/2} - \left( \frac{X^T X}{n} \right)^{1/2} \|^2 = O \left( \frac{\sqrt{\log(n)}}{\sqrt{n}} \right);
\]

(14)

\[
\| \left( \frac{\Lambda}{n} \right)^{-1/2} - \tilde{\Lambda}^{-1/2} \|^2 = O \left( \frac{\sqrt{\log(n)}}{\sqrt{n}} \right);
\]

(15)

\[
\|V - \tilde{V} W_n^T \| = O \left( \frac{\sqrt{\log(n)}}{\sqrt{n}} \right);
\]

(16)
where \( W_n \in \mathbb{O}(p, q) \cap \mathbb{O}(d) \) will be defined later.

First, by Theorem 6.2 in Higham (2008), for \( A \) and \( B \) positive-definite matrices, we have that

\[
\| A^{1/2} - B^{1/2} \| \leq \frac{1}{\lambda_{\min}(A)^{1/2} + \lambda_{\min}(B)^{1/2}} \| A - B \|.
\]  (17)

The result above is stated in Higham (2008) for matrices with distinct eigenvalues. However, if \( A \) and \( B \) do not have distinct eigenvalues, the result holds by adding small values of \( \varepsilon \) to each of the repeated eigenvalues, applying the result for the new slightly perturbed matrices, and then taking the limit as \( \varepsilon \to 0 \).

By the Law of Large Numbers, \( \frac{X_n^\top X_n}{n} \) is almost surely positive definite whenever \( \Delta \) is. Applying the above inequality to \( \Delta \) and \( \frac{X_n^\top X_n}{n} \), we see that

\[
\| \Delta^{1/2} - \left( \frac{X_n^\top X_n}{n} \right)^{1/2} \| \leq \frac{1}{\lambda_{\min}(\Delta)^{1/2} + \lambda_{\min}(\frac{X_n^\top X_n}{n})^{1/2}} \| \Delta - \frac{X_n^\top X_n}{n} \| \\
\leq \frac{1}{\lambda_{\min}(\Delta)^{1/2}} \| \Delta - \frac{X_n^\top X_n}{n} \|.
\]

By Theorem 6.5 in Wainwright (2019) (which applies to second moment matrices by shifting by the mean), we have that for some constants \( c_1, c_2 \) and \( c_3 \),

\[
\| \Delta - \frac{X_n^\top X_n}{n} \| \leq \| \Delta \| c_1 \left( \sqrt{\frac{d}{n}} + \frac{d}{n} \right) + \delta
\]

with probability at least \( 1 - c_3 \exp(-c_2 n \min(\delta, \delta^2)) \). Taking \( \delta = \sqrt{\frac{2 \log(n)}{c_3 n}} \), we see that

\[
\| \Delta - \frac{X_n^\top X_n}{n} \| = O \left( \| \Delta \| \sqrt{\frac{d}{n}} \sqrt{\log(n)} \right)
\]

with probability at least \( 1 - O(n^{-2}) \). Putting it all together and noting \( \| \Delta \| \) and \( d \) are constants in \( n \), we arrive at

\[
\| \Delta^{1/2} - \frac{X_n^\top X_n}{n}^{1/2} \| \leq O \left( \frac{\sqrt{\log(n)}}{\sqrt{n}} \right)
\]

which proves (14).

For (15), we note that \( \Delta \) and \( \tilde{\Delta} / n \) are the eigenvalues of the matrix \( \Delta^{1/2} I_{p,q} \Delta^{1/2} \) and \( \left( \frac{X_n^\top X_n}{n} \right)^{1/2} I_{p,q} \left( \frac{X_n^\top X_n}{n} \right)^{1/2} \) respectively. To see the latter, we note that \( X I_{p,q} X^\top \) has the same nonzero eigenvalues as \( (X_n^\top X_n)^{1/2} I_{p,q} (X_n^\top X_n)^{1/2} \) by similarity. By Weyl’s inequality, we have that

\[
|\lambda_i - \tilde{\lambda}_i| \leq \| \frac{X_n^\top X_n}{n} I_{p,q} \frac{X_n^\top X_n}{n}^{1/2} - \Delta^{1/2} I_{p,q} \Delta^{1/2} \| \\
\leq \| \frac{X_n^\top X_n}{n} I_{p,q} \Delta^{1/2} - \Delta^{1/2} I_{p,q} \Delta^{1/2} \| + \| \frac{X_n^\top X_n}{n} I_{p,q} \Delta^{1/2} - \frac{X_n^\top X_n}{n} I_{p,q} \frac{X_n^\top X_n}{n}^{1/2} \| \\
\leq \| \frac{X_n^\top X_n}{n} \| - \Delta^{1/2} \| \Delta^{1/2} \| + \| \frac{X_n^\top X_n}{n} \| - \| \frac{X_n^\top X_n}{n} \| \| \Delta^{1/2} - \frac{X_n^\top X_n}{n} \| \\
= O \left( \frac{\sqrt{\log(n)}}{\sqrt{n}} \right)
\]

by (14) and the fact that \( \frac{X_n^\top X_n}{n} \) can be bounded above by a constant. Now, define the function \( g(\lambda) := \frac{1}{|\lambda|^{1/2}} \). Since \( \Delta \) is full-rank and \( I_{p,q} \) is orthogonal, so is \( \Delta I_{p,q} \) and hence \( \Delta^{1/2} I_{p,q} \Delta^{1/2} \) by similarity. Therefore, there
exists an $\varepsilon > 0$ such that all eigenvalues of $\Delta^{1/2}I_{p,q}\Delta^{1/2}$ are outside the range $(-\varepsilon, \varepsilon)$, and hence so are those of $\left(\frac{X_n^T X_n}{n}\right)^{1/2}I_{p,q}\left(\frac{X_n^T X_n}{n}\right)^{1/2}$ for $n$ sufficiently large with probability at least $1 - n^{-2}$. The function $g(\lambda)$ is differentiable outside of $(-\varepsilon, \varepsilon)$, and hence by the Delta method applied to each eigenvalue individually, 

$$g(\lambda_i) - g(\tilde{\lambda}_i) = O\left(\frac{\sqrt{\log(n)}}{\sqrt{n}}\right),$$

for $n$ sufficiently large with probability $1 - O(n^{-2})$, where the hidden constant depends on $\alpha$. This proves (15).

For (16), we simply apply the Davis-Kahan Theorem to the eigenvectors associated to each eigenvalue. First, consider $i$ such that $\lambda_i$ is unique. By (15), for $n$ sufficiently large, the eigenvalues outside of $i$ are separated from each other and we can apply the Davis-Kahan Theorem. We see that for (16), we simply apply the Davis-Kahan Theorem to the eigenvectors associated to each eigenvalue.

We are now ready to prove the result in Equation 5. By Agterberg et al. (2020), we can write

$$Q_n = \left(\frac{1}{n}\right)^{-1/2}V^T \left(\frac{X_n^T X_n}{n}\right)^{1/2}$$

and

$$\tilde{Q} = \left(\frac{1}{n}\right)^{-1/2}\tilde{V}^T (\Delta)^{1/2}.$$

The result follows by using (14), (15), and (16) together and adding and subtracting terms and noting that by construction the orthogonal matrix from (16) commutes with $\left(\frac{1}{n}\right)^{-1/2}$.

We note that the matrix $Q_n$ is invariant to the sparsity factor, since the eigenvectors of $\alpha_nP = \alpha_n X_n I_{p,q} X_n^T$ are the same as those of $P$ and the eigenvalues are scaled by $\alpha_n$ so that if $D$ are the eigenvalues of $\alpha_nP$ then

$$U_n|D|^{1/2} = \sqrt{\alpha_n} U_n |A_n|^{1/2}$$

$$= \sqrt{\alpha_n} X_n Q_n^{-1},$$

showing that the matrix $Q_n$ depends only on the matrix $P$ and not the sparsity component $\alpha_n$.

### 6.5.2 Proof of Lemma 5

**Proof of Lemma 5.** First, for any fixed matrix $X$, we have that the $A_{ij}$’s are independent random variables, and recall that

$$\hat{\alpha}_n = \frac{1}{\binom{m}{2}} \sum_{i<j} A_{ij}.$$ 

Define $\theta_n := \frac{\alpha_n}{\binom{n}{2}} \sum_{i<j} P_{ij}$. Then $E(\hat{\alpha}_n|X) = \theta_n$. Therefore, we have that

$$P\left(|\hat{\alpha}_n - \alpha_n| > 2t\right) \leq P(|\hat{\alpha}_n - \theta_n| > t) + P(|\theta_n/\alpha_n - 1| > t/\alpha_n).$$
For the first term, we note that by applying Hoeffding’s inequality, we see that
\[
P\left(|\hat{\alpha}_n - \theta_n| \geq t\right) \leq 2 \exp\left(-\frac{n^2 t^2}{2}\right) \leq 2 \exp\left(-\frac{(n^2 + n) t^2}{2}\right) \leq 2 \exp\left(-\frac{n^2 t^2}{2}\right).
\]

For the second term, note that \(\frac{1}{(n/2)} \sum_{i<j} X_i^\top I_{p,q} X_j\) is a \(U\)-statistic with expected value 1. Hoeffding’s inequality for \(U\)-statistics (e.g. Example 2.23 in Wainwright (2019)) shows that
\[
P\left(|\frac{1}{n} \sum_{i<j} X_i^\top I_{p,q} X_j - 1| \geq t\right) \leq 2 \exp\left(-\frac{nt^2}{8}\right).
\]

Hence, we see that
\[
P\left(|\hat{\alpha}_n - \alpha_n| > 2t\right) \leq 2 \exp\left(-\frac{n^2 t^2}{2}\right) + 2 \exp\left(-\frac{nt^2}{8}\alpha_n^2\right).
\]

Set \(t = 4 \sqrt{\frac{\alpha_n \log(n)}{n}}\). Then recalling that for some \(C > 0\) \(1 \geq \alpha_n \geq C \log^4(n)/n\), we have
\[
P\left(|\hat{\alpha}_n - \alpha_n| > 8 \sqrt{\frac{\alpha_n \log(n)}{n}}\right) \leq 2 \exp\left(-8n\alpha_n \log(n)\right) + 2 \exp\left(-2\log(n)\alpha_n^{-1}\right) \leq 2 \exp\left(-8C \log^5(n)\right) + 2n^{-2} \leq 4n^{-2}.
\]

Now, define the event \(A := \{|\hat{\alpha}_n - \alpha_n| \leq 4 \sqrt{\alpha_n \log(n)/n}\}\. On A, since \(\alpha_n \geq \frac{C \log^4(n)}{n}\),
\[
\frac{|\hat{\alpha}_n - \alpha_n|}{\alpha_n} \leq \frac{4}{\log(1.5(n))},
\]

which is small for \(n\) sufficiently large. Hence, by Taylor expansion, we have that
\[
\frac{1}{\sqrt{\alpha_n}} = \frac{1}{\sqrt{\alpha_n + (\hat{\alpha}_n - \alpha_n)}} = \frac{1}{\sqrt{\alpha_n}} \left(1 + \frac{\hat{\alpha}_n - \alpha_n}{\alpha_n}\right)^{-1} = \frac{1}{\sqrt{\alpha_n}} \left(1 + \frac{1}{2} \frac{\hat{\alpha}_n - \alpha_n}{\alpha_n} + O\left(\frac{\hat{\alpha}_n - \alpha_n}{\alpha_n}\right)^2\right).
\]

By the previous observations, we have that this is equal to
\[
\frac{1}{\sqrt{\alpha_n}} \left(1 + O\left[\frac{\log(n)}{n\alpha_n}\right]\right).
\]

for \(n\) sufficiently large.

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### A More on the Discussion in Section 3.2.1

In this section, for $f$ and $g$ two functions of $n$, we write $f(n) \ll g(n)$ if $f(n)/g(n) \to 0$ as $n$ tends to infinity.
The eigenvalues of $\alpha_n XX^T$ are the same as those of $\alpha_n X^T X$, and as $n \to \infty$, the matrix $\frac{1}{n} X^T X$ is converging almost surely to $E(X X^T)$. Therefore, as $n \to \infty$,
\[
\frac{1}{n\alpha_n} (\lambda_i(P) - \lambda_{i+1}(P)) \to \delta_i,
\]
where $\delta_i = \lambda_i(E(X X^T)) - \lambda_{i+1}(E(X X^T))$. We have
\[
\|W_\ast - I\|_F \leq \|W_\ast - U_A^T U_P\|_F + \|U_A^T U_P - I\|_F.
\]
The first term can be bounded directly using the Davis-Kahan Theorem as
\[
\|W_\ast - U_A^T U_P\|_F = O\left(\frac{\|A - \mu\|}{\lambda_d(P)}\right)^2 = O((n\alpha_n)^{-1}).
\]
For the second term, following the analysis on Page 24 of Rubin-Delanchy et al. (2020), we have that for
\[
(U_A^T U_P)_{ij} = -\frac{\langle U_A \rangle_i^T (A - \mu)(U_P)_{ij}}{\lambda_j(P) - \lambda_i(A)}.
\]
By previous results on eigenvalue concentration (e.g. Eldridge et al. (2018); O’Rourke et al. (2018); Cape et al. (2017)), we have that
\[
|\lambda_i(A) - \lambda_i(P)| \leq C \log(n)
\]
with high probability. Hence, the above bound can be written as
\[
\frac{(U_A^T (A - \mu)(U_P)_{ij}}{\lambda_j(P) - \lambda_i(A)} = \frac{(U_A^T (A - \mu)(U_P)_{ij}}{\lambda_j(P) - \lambda_i(A) \pm C \log(n)}
\]
Moreover, by Koltchinskiy and Giné (2000), we have that $\frac{\lambda_j(P)}{n\alpha_n} - \lambda_i(E(X X^T)) = O_p(n^{-1/2})$. Hence, we can further simplify the bound to when $i = j + 1$
\[
\frac{(U_A^T (A - \mu)(U_P)_{ij}}{\lambda_j(P) - \lambda_i(A) \pm C \log(n)} = \frac{(U_A^T (A - \mu)(U_P)_{ij}}{\lambda_j(P) - \lambda_i(A) \pm O_p(\sqrt{n\alpha_n})}
\]
Expanding the numerator, we see that we can write this via
\[
\left(n\alpha_n \delta \pm O_p(\sqrt{n\alpha_n}) \pm O(\log(n))\right)^{-1}\left(U_A^T (A - \mu)(U_P)_{ij}\right)
\]
\[
\left(n\alpha_n \delta \pm O_p(\sqrt{n\alpha_n}) \pm O(\log(n))\right)^{-1}\left(U_A^T (U_P U_P^T)(A - \mu)(U_P)_{ij}\right)
\]
\[
+ \left(n\alpha_n \delta \pm O_p(\sqrt{n\alpha_n}) \pm O(\log(n))\right)^{-1}\left(U_A^T (I - U_P U_P^T)(A - \mu)(U_P)_{ij}\right).
\]
The first term $U_P^T (A - \mu)(U_P)_{ij}$ is a sum of independent random variables, so Hoeffding’s inequality reveals that it is of order at most $O(\log(n))$ with high probability. The second term can be bounded via the Davis-Kahan theorem as
\[
\left(U_A^T (I - U_P U_P^T)(A - \mu)(U_P)_{ij}\right) \leq \|(U_A^T (I - U_P U_P^T)||(|(A - \mu)||
\]
\[
\leq \frac{C\|A - \mu\|}{\lambda_d(P)} ||(A - \mu)||
\]
\[
= O(1).
\]
Putting it together, we arrive at

$$\|U_{\Lambda}^T U \mathbf{P} - \mathbf{I}\|_F = O\left(\frac{\log(n)}{n\alpha n^2}\right),$$

where \(\delta = \min_i \delta_i\), provided \(\sqrt{n\alpha_n} \ll n\alpha \delta\) and \(\log(n) \ll n\alpha \delta\).

From an asymptotic standpoint, the term \(\delta\) in the denominator makes little difference as it is a constant, and \(n\alpha_n \rightarrow \infty\). However, for finite \(n\), depending on \(\alpha_n\) and \(n\), even if \(n\alpha_n \gg \log^4(n)\), the constant may depend on \(\delta\). Therefore, for any fixed model, though the rate of convergence depends on \(n\alpha_n\), for all practical purposes it also depends on the eigengap \(\delta\).