Fermions from photons: Bosonization of QED in 2+1 dimensions.

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Abstract

We perform the complete bosonization of 2+1 dimensional QED with one fermionic flavor in the Hamiltonian formalism. The Fermi operators are explicitly constructed in terms of the vector potential and the electric field. We carefully specify the regularization procedure involved in the definition of these operators, and calculate the fermionic bilinears and the energy-momentum tensor. The algebra of bilinears exhibits the Schwinger terms which also appear in perturbation theory. The bosonic Hamiltonian is a local, polynomial functional of $A_i$ and $E_i$, and we check explicitly the Lorentz invariance of the resulting bosonic theory. Our construction is conceptually very similar to Mandelstam’s construction in 1+1 dimensions, and is dissimilar from the recent bosonization attempts in 2+1 dimensions, which hinge crucially on the presence of a Chern-Simons term.

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1 Introduction

It is well known that, in 1+1 dimensions, fermion field operators can be constructed in terms of local bosonic fields. This bosonization procedure is of great theoretical interest in itself, since it provides a method of mutually mapping different local quantum field theories with a priori different Hilbert spaces. In the pioneering works of Coleman [1] and Mandelstam [2], the bosonization of a single Dirac field was performed. Subsequently it was realized, that this procedure is much more general. An extension was found to theories of several Fermi fields [3], and eventually non-Abelian bosonization was introduced by Witten [4]. Bosonization has proved to be very helpful in the analysis of different 1+1 dimensional models. It has been used in a variety of contexts: relativistic quantum field theory, condensed matter systems, string theory.

From the point of view of particle physics, bosonization techniques have found perhaps their most interesting applications in the realm of gauge theories. There they provide insights into the strong coupling regime of both Abelian [5], and non-Abelian [6] gauge theories, as well as better understanding of t’Hooft’s solution of large N $QCD_2$ [7]. Also in many other instances, in which 1+1 dimensional theories are used to model the behaviour of $QCD_4$, bosonization has proven helpful [8].

Clearly, the extension of bosonization to higher dimensions is highly desirable. It is reasonable to hope that, apart from its intrinsic theoretical interest, bosonization in higher dimensions will also find diverse applications. Although it is not likely that bosonized theories in 2+1 and 3+1 dimensions will be exactly solvable, they may be amenable to different analytical methods than their original fermionic formulations. A local bosonic formulation of gauge theories with fermions should also be very interesting from the lattice gauge theory point of view, since it would provide a convenient starting point for numerical simulations.

In the late 1970’s and early 1980’s several attempts, using different approaches, have been made to perform bosonization in more than 1+1 dimensions. None of them however has lead to a local bosonic theory. One approach was to perform an exact analog of the Jordan - Wigner transformation for lattice fermions [9]. The resulting bosonic theory is a $Z_2$ gauge theory, and the bosonic variables are not gauge invariant, and consequently nonlocal. The nature of the continuum limit of this bosonic model is also not known. In another approach [10], one divides a two (or three) dimensional space into one dimensional subspaces, and performs one - dimensional bosonization in each subspace. This procedure, however, does not give local expressions for some of the local fermionic bilinears. In particular, the fermion mass term is given by a complicated nonlocal operator. Related attempts, using tomografic projection [11], have produced similar results.

After the relative failure of these early approaches, it has been tacitly assumed until quite recently, that the extension of bosonization to higher dimensions is impossible. The feeling was that 1+1 dimensions is a very special case, since there is no spin in 1+1 dimensions, and therefore no ”real” difference between bosons and fermions.

The renewed interest in this problem was triggered by a possible connection between
the phenomenon of Fermi - Bose transmutation in 2+1 dimensions, and novel condensed matter systems, most notably high $T_c$ superconductors. Polyakov argued [12], that the addition of a Chern - Simons term [13] to the Lagrangian of QED$_3$ changes the statistics of charged excitations. His argument was elaborated further in many works [14], and the explicit realization of this Chern - Simons induced mechanism of Fermi - Bose transmutation in lattice theories has been given in [15], [16].

There is, however, an important conceptual difference between the Mandelstam construction in 1+1 dimensions, and the Chern - Simons bosonization in 2+1 dimensions. Let us recall the basic bosonization formulae in 1+1 dimensions [2]. The fermionic operator $\psi_\alpha$ is constructed in terms of a scalar bose field $\phi$, and its conjugate momentum $p$, as follows,

$$\psi_{1,2}(x) = \exp \left[ -\frac{2i}{\pi} \beta \int_{-\infty}^{x} dy p(y) + \frac{\beta}{2} \phi(x) \right]$$ (1)

which gives the following expressions for the bilinears,

$$\bar{\psi} \gamma_\mu \psi = -\frac{\beta}{2\pi} \epsilon_{\mu\nu} \partial_\nu \phi; \quad \bar{\psi} \psi \propto \cos \beta \phi$$ (2)

One important aspect of these formulae is that the fermion number current, which in the original theory is conserved due to the equations of motion, bosonizes into a topological current, which is conserved trivially. The U(1) charge, when expressed in terms of $\phi$, is a topological, rather than a Noether, charge. The bosonization is therefore achieved by a duality transformation: one constructs the fermionic operators $\psi$ which carry a global U(1) fermion number charge, in terms of a local bosonic field $\phi$, which is itself neutral.

The 2+1 dimensional construction of [15], [16] is very different in this respect. (See, however [17]). In that case the expression for the Fermi field is [18],

$$\psi(x) = \phi(x) \exp \left[ i \int d^2 y \theta(x - y) \rho(y) \right]$$ (3)

where $\theta(x)$ is the planar angle, and $\rho$ is the charge density operator. The fermi field $\psi$ and the bose field $\phi$ carry the same global quantum numbers. The element of duality is notably missing here. Moreover, in this picture, a free fermion is bosonized into a particle interacting with the vector potential with Chern - Simons action. But the associated bose field is not gauge invariant, and, after gauge fixing, becomes nonlocal. So one does not really construct fermionic operators in terms of local bosonic fields.

This 2+1 dimensional construction hinges crucially on the presence of the Chern - Simons term in the action. The Fermi - Bose transmutation occurs only for a fixed value of its coefficient. For other values of this coefficient this procedure gives nonlocal anyonic fields with fractional statistics. Another unsatisfactory feature of the procedure is, that it has been consistently implemented only for nonrelativistic fermions [15], [16]. When applied to continuum relativistically invariant theories [18], it suffers from regularization ambiguities, and it is not clear how to interpret the formal results. For example, since $[\rho(y), \phi(x)] = \phi(x)\delta^2(x - y)$, and the angle function is not defined at the origin, the
ordering on the right hand side of expression (3) is completely ambiguous, and formal manipulations using it are ill-defined. Furthermore, the covariant Dirac field has not yet been constructed in this framework, even formally.

In this paper we take a different approach to bosonization. Our aim is to construct a Dirac doublet of fermionic operators in 2+1 dimensions in terms of a local bosonic field. The main hypothesis on which we base this construction is the following. We assume that, in terms of the bose field, the fermion number charge must be topological, and thus the bosonized fermion number current must be trivially conserved. We are not aware of any theorem that states this, but this is the case in all the known examples when either exact bosonization has been performed (1+1 dimensions) or fermionic excitations are known to exist in the spectrum of a local bosonic theory in 2+1 or 3+1 dimensions. Another observation which indirectly supports this assumption is that fermionic fields are not local fields in the usual sense of the word. They satisfy local anticommutation, rather than commutation, relations. If these operators are to be constructed in terms of local bosonic variables, they must create a nonlocal configuration of the bosonic field. If the fermion number bosonizes into a topological charge of the bosonic theory, this will necessarily be true.

Given this assumption, the simplest system that suggests itself as a convenient object for bosonization is QED with one two-component Dirac fermion. The reason is that Maxwell’s equations,

\[ J^\nu = \frac{1}{e} \partial^\mu F^{\mu\nu} \quad (4) \]

imply that, if one takes as basic variables electric field \( E_i \) and the vector potential \( A_i \), the fermion number is trivially conserved, due to the antisymmetry of \( F_{\mu\nu} \). The fermion number charge is topological, since it is equal to the surface integral of \( E_i \) at spatial infinity. Since there are no global flavor symmetries in the model, it is the simplest of its kind.

In this paper we perform the analog of the Mandelstam construction in the one flavor QED_{3}. The paper is organized as follows. In Section 2 we explicitly construct the two-component Dirac spinor \( \psi_\alpha \) in the Hamiltonian formalism in terms of bosonic variables \( E_i \), and their conjugate momenta \( A_i \). Since the formal definition of \( \psi_\alpha \) suffers from the same kind of ambiguities as in the Chern - Simons case, we carefully specify the regularization procedure necessary to make it well-defined. In physical terms, the fermionic field is an operator which creates an electric charge \( e \) at the point \( x \), and also a pair consisting of a magnetic vortex and an antivortex of half integer strength \( \pi/e \), at infinitesimal separation.

In Section 3 we use the explicit form of the fermionic operators to calculate the fermionic bilinears \( \bar{\psi} \Gamma \psi \), and, in particular, show that the operators \( \psi_\alpha \) solve Gauss’ constraint. The calculation is performed in close analog with the 1+1 dimensional case, by expanding all quantities in inverse powers of the ultraviolet cutoff. As in 1+1 dimensions, higher order terms in this expansion contribute finite renormalization of the

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1 In this respect our approach is similar to [19], although in this reference the resulting bosonic theory is nonlocal.
coefficients of the lower order terms. As a result, Lorentz invariance has to be invoked in order to determine some coefficients in the expressions for the spatial components of the current. We then discuss a certain subtlety encountered in our calculation, and which is not present in 1+1 dimensions. This problem is due to the presence of the dimensional coupling $e$ in 2+1 dimensions, whereas the corresponding couplings in 1+1 dimensions are dimensionless. As we show, this restricts the validity of our bosonization procedure to scales lower than $\mu$, which is related to the UV cutoff $\Lambda$ by $\mu^3 = e^2 \Lambda^2$. For finite $e^2$ this is not a real restriction, since the scale $\mu$ becomes infinite in the infinite cutoff limit. However, this restriction prevents us from taking the limit $e^2 \to 0$ at finite $\Lambda$. This we believe, however, is not an artifact of our procedure, but rather a necessary consequence of the fact that the theory contains only one two-component fermion. The change of dynamics at scales higher than $\mu$ is necessary to avoid the fermion doubling problem. We calculate the algebra of the bilinears, and find that it reproduces the tree level algebra with the addition of Schwinger terms. These terms are also seen in perturbation theory at the one loop level.

In Section 4 we find the bosonized expression for the energy-momentum tensor. Again, as with the spatial components of the current, we use Lorentz invariance to determine several coefficients. It is shown that, with the right choice of these coefficients, the resulting theory is Lorentz-invariant in the limit $\Lambda \to \infty$, and one recovers Maxwell’s equations as its equations of motion.

Section 5 is devoted to the discussion, and some extensions of our results. In particular, we indicate how the well known regularization ambiguity of one flavor QED$_3$ appears in our framework. It is possible to construct a modified Fermi field $\chi_\alpha$, which solves the modified Gauss’ constraint $\chi^\dagger \chi = 1/e \partial_i E_i + en/2\pi B$. The coefficient $n$ must be an integer, to ensure the correct quantization of the fermion number charge. The new bosonized theory differs from the old one precisely by an addition of a Chern-Simons term to the Hamiltonian.

In Appendices we discuss the simple intuitive mechanism underlying the anticommutation relations of $\psi$, discuss the geometrical aspects of our construction, and give some details of the calculation of the bilinears.

## 2 The construction of Fermi operators

The 2+1 dimensional quantum electrodynamics is defined by the following Hamiltonian (we work in the timelike, or Weyl, gauge $A_0 = 0$),

$$H = \frac{1}{2} E^2 + \frac{1}{2} B^2 + \bar{\psi} \gamma^i (i \partial_i + e A_i) \psi + m \bar{\psi} \psi$$  \hspace{1cm} (5)

Together with the Gauss’ constraint,

$$\partial_i E_i = e \bar{\psi} \gamma^i \psi$$  \hspace{1cm} (6)
The two component Fermi field $\psi_\alpha$, $\alpha = 1, 2$, and the bosonic variables $E_i$, and $A_i$ satisfy the canonical (anti)commutation relations,

$$\{\bar{\psi}_\alpha(x), \psi_\beta(y)\} = \delta_{\alpha\beta}\delta^2(x - y); \quad [E^i(x), A^j(y)] = i\delta^{ij}\delta^2(x - y)$$

As is usually the case with gauge theories, the Hamiltonian eq.(5) acts on a large Hilbert space, which contains unphysical degrees of freedom. Those must be ultimately eliminated by solving the constraint eq.(6). One usually solves the constraint by expressing the longitudinal part of the electric field in terms of the matter fields $\psi$. In the context of our problem, however, we view eq.(6) as one of the bosonization equations. We will therefore retain both components of the electric field, and instead solve the Gauss’ constraint by constructing the doublet of anticommuting fermionic operators $\bar{\psi}_\alpha$ in terms of $E_i$, and $A_i$.\[\]

Substituting those back into the Hamiltonian eq.(5), we will obtain the completely bosonized form of the theory, defined on the physical Hilbert space of QED$_3$. Our bosonization procedure is therefore defined only for gauge invariant quantities. We will also calculate the fermionic bilinears, and find that they satisfy the tree level algebra, modified by Schwinger terms. The appearance of these Schwinger terms is also seen in perturbation theory at the one loop level.

To construct the operators $\psi_\alpha$, we must first fix the gauge freedom associated with the time-independent gauge transformations, generated by the Gauss’ constraint. We do this by considering $\psi$ in the Coulomb gauge. Those are the gauge - invariant operators,

$$\psi^{CG}_\alpha(x) = \psi_\alpha(x)\exp\left[ie\int d^2y e_i(y - x)A_i(y)\right]$$

where

$$e_i(x) = -\frac{1}{2\pi} \frac{x_i}{x^2}$$

is the electric field of a point charge. The exponential factor ensures the gauge invariance of the operator $\psi^{CG}$. In the following we will omit the superscript $CG$, and always understand that the fermionic operators we are constructing are defined by eq.(8). It is important to note, that the Hamiltonian eq.(5) can be rewritten in terms of these operators. In this formulation, the covariant derivative in eq.(5) contains only the transverse part of the vector potential $A_i$, $A^T_i$.

In addition to solving the constraint eq.(3), the fermionic operators $\psi_\alpha$ must satisfy the following conditions:\[\]

1. Carry unit electric charge, $[\psi_\alpha(x), \partial_i E_i(y)] = e\psi_\alpha(x)\delta^2(x - y)$;\[\]

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\[\]

Since the fermion field in QED$_3$ carries only one global quantum number, namely, the electric charge, it should be representable in terms of one bosonic field. This bosonic field will not be free of course, but will rather be strongly interacting. Our construction gives precisely this counting: the fields $E_i$ and their conjugate momenta describe a photon, and an additional scalar degree of freedom.

Bosonization of the massive Schwinger model in 1+1 dimensions can be formulated in precisely the same terms.
ii. Transform correctly under rotations. For convenience, we choose the basis of Dirac matrices in which the rotation generator is diagonal,

\[
\gamma^0 = \sigma^3; \quad \gamma^1 = i\sigma^2; \quad \gamma^2 = -i\sigma^1
\]  

(10)

Then,

\[
\psi_1 \rightarrow e^{i\phi/2}\psi_1; \quad \psi_2 \rightarrow e^{-i\phi/2}\psi_2
\]  

(11)

where \(\phi\) is the rotation angle;

iii. Fermionic bilinears must be local operators, \([\psi^\dagger_\alpha \psi_\beta(x), O(y)] = 0\) for \(x \neq y\), and for any local gauge invariant operator \(O(x)\).

To satisfy the first condition, we take the following ansatz,

\[
\psi_\alpha(x) = k\Lambda V_\alpha(x)\Phi(x)U_\alpha(x)
\]  

(12)

where,

\[
\Phi(x) = \exp\left[ie \int d^2y e_i(y-x)A_i(y)\right]
\]  

(13)

Here \(\Lambda\) is the ultraviolet cutoff, \(k\) is a finite dimensionless constant, which depends on the precise definition of the UV cutoff, and the operators \(V_\alpha\) and \(U_\alpha\) commute with the charge density operator. To ensure the anticommutativity of \(\psi\)'s, we take the following forms for \(U_\alpha\) and \(V_\alpha\),

\[
V_1(x) = -i\exp\left[\frac{i}{2e} \int d^2y(\theta(x-y) - \pi)\partial_iE_i(y)\right]; \quad V_2(x) = -iV_1^\dagger(x);
\]

\[
U_1(x) = \exp\left[-\frac{i}{2e} \int d^2y\theta(y-x)\partial_iE_i(y)\right]; \quad U_2(x) = U_1^\dagger(x)
\]

where \(\theta(x)\) is the polar angle at the point \(x\).

With these definitions we find,

\[
\begin{align*}
\psi_1(x)\psi_1^\dagger(y) &= \psi_1(y)\psi_1(x)e^{-i(\theta(y-x)-\theta(x-y))} = -\psi_1(y)\psi_1(x) \\
\psi_1(x)\psi_1^\dagger(y) &= \psi_1(y)\psi_1(x)e^{-i(\theta(y-x)-\theta(x-y))} = -\psi_1(y)\psi_1(x); \\
\psi_2(x)\psi_2^\dagger(y) &= \psi_2(y)\psi_2(x)e^{-i(\theta(y-x)-\theta(x-y))} = -\psi_2(y)\psi_2(x) \\
\psi_2(x)\psi_2^\dagger(y) &= \psi_2(y)\psi_2(x)e^{-i(\theta(y-x)-\theta(x-y))} = -\psi_2(y)\psi_2(x); \\
\psi_1(x)\psi_2^\dagger(y) &= \psi_1(y)\psi_1(x)e^{-i\pi} = -\psi_1(y)\psi_1(x) \\
\psi_1(x)\psi_2^\dagger(y) &= \psi_2(y)\psi_2(x)e^{-i\pi} = -\psi_2(y)\psi_2(x).
\end{align*}
\]  

(15)

The factor \(k\Lambda\) in eq.(12) ensures the correct dimensionality of the fermionic fields, and the correct normalization of the anticommutators at coincident points,

\[
\{\psi_1^\dagger(x), \psi_1(y)\} = \{\psi_2^\dagger(x), \psi_2(y)\} = \delta^2(x-y)
\]  

(16)

Here a comment is in order about the precise meaning of eqs.(12-15). The function \(\theta(x)\) is defined only modulo 2\(\pi\). On the other hand, the eigenvalues of the charge density
operator are quantized in units of $1/e$. As a consequence, the operators $U_\alpha$ and $V_\alpha$ (and thus $\psi_\alpha$) are double-valued. This, in fact, is to be expected on the general grounds. Under rotation by $2\pi$ the fermionic operators change sign, but the bosonic operators $A_i$ and $E_i$ are unchanged. Therefore $\psi$ cannot be a single valued function of $A_i$, $E_i$, and the coordinate $x$.[24]. Further geometrical details of the construction, and its interpretation along the lines of[24] are contained in the Appendix B.

These manipulations are, however, too formal, since the commutators of $V(x)$ and $U(x)$ with $\Phi(x)$ in eq.[12] are singular. The singularity is due to the fact that $\Phi(x)$ creates a charge at the point $x$, and consequently the commutator involves the ill-defined factor $e^{i\theta(0)/2}$. The expression for the fermionic operators needs regularization. The natural way to regularize it is by point splitting. We will therefore consider the following expressions,

$$
\psi_1^\eta(x) = V_1(x+\eta)\Phi(x)U_1(x-\eta); \quad \psi_2^\eta(x) = V_2(x+\eta)\Phi(x)U_2(x-\eta)
$$

The length of the regulator $\eta$ is taken to be proportional to the inverse of the UV cutoff $|\eta| \propto 1/\Lambda$. However, considering $\eta$ in any fixed direction breaks the rotational symmetry of the problem. To restore rotational invariance, we average over the direction of $\eta$. The averaging should be performed with an appropriate phase factor, in order to ensure the correct rotational properties of the operators. Under rotation by an angle $\phi$, $\theta$ transforms as follows, $\theta(x) \rightarrow \theta(x) + \phi$. We arrive therefore at,

$$
\psi_1 = \lim_{\Lambda \rightarrow \infty} \frac{k\Lambda}{2\pi} \int d\eta e^{-i\frac{\eta^2}{2\Lambda}} \psi_1^\eta(x), \quad \psi_2 = \lim_{\Lambda \rightarrow \infty} \frac{k\Lambda}{2\pi} \int d\eta e^{i\frac{\eta^2}{2\Lambda}} \psi_2^\eta(x)
$$

where the integral is over the angle of the vector $\eta$, and $\hat{\eta}$ is the unit vector parallel to $\eta$. It can be explicitly checked, that,

$$
\{\psi_\alpha(x),\psi_\beta(y)\} = 0, \quad |x-y| >> 1/\Lambda
$$

Therefore, in the limit $\Lambda \rightarrow \infty$, we regain the standard anticommutation relations.

However, we are still not done with the construction of $\psi$. The point that remains to be settled is the following. As defined up to now, the fermionic operators depend only on the longitudinal component of the electric field $E_i^L$. Any bilinear that one would calculate using these expressions would also depend only on $E_i^L$, and not on the transverse part $E_i^T$. However, $E_i^L$ is not a local field, and it is therefore impossible to obtain local bilinears with this definition of fermionic operators. (We have checked this by explicit calculation.) It turns out, that this problem can be remedied by modifying the expressions for $V(x)$ and $U(x)$, so that they become creation and annihilation operators of a magnetic vortex of half-integer strength,

$$
V_1(x) = -i \exp \left\{ \frac{i}{2e} \int d^2y \left[ (\theta(x-y) - \pi)\partial_i E_i(y) + 2\pi G^{(2)}(y-x)e_{ij}\partial_j E_j(y) \right] \right\}; \quad (20)
$$

$$
U_1(x) = \exp \left\{ -\frac{i}{2e} \int d^2y \left[ \theta(y-x)\partial_i E_i(y) + 2\pi G^{(2)}(y-x)e_{ij}\partial_j E_j(y) \right] \right\};
$$
Here \( G^{(2)}(x-y) = -\frac{1}{4\pi} \ln(\mu^2 x^2) \), \( x^2 = x^i x^i \), is the Green’s function of the two-dimensional Laplacian, with IR cutoff \( \mu \).

The physical meaning of the operators \( V_\alpha \) and \( U_\alpha \) is clear from the following commutation relations \[25\],

\[
[V_1(x), B(y)] = -\frac{\pi}{\epsilon} V_1(x) \delta^2(x-y); \quad [U_1(x), B(y)] = \frac{\pi}{\epsilon} U_1(x) \delta^2(x-y)
\] (21)

This modification does not change either the anticommutation relations or the rotational properties of \( \psi \). Moreover, now the point splitting procedure used in regularizing \( \psi \) has a very natural interpretation, since the vortex operators \( V \) and \( U \) are the dual variables of \( QED_3 \) \[26\]. In any lattice regularized version of the theory the charged field \( \Phi \) should live on the lattice sites, and the vortex fields should live on the sites of the dual lattice, and therefore be point split from \( \Phi \).

Equations (20), (18) and (17) are our final expressions for the fermionic operators in terms of the bose fields \( E_i \) and \( A_i \).

3 Calculation of bilinears

Our next step is to calculate fermionic bilinears which do not contain derivatives. As usual, those should be defined with the help of a point-splitting procedure. We use the following definition,

\[
J_\Gamma(x) = \bar{\psi}(x) \Gamma \psi(x) \equiv \frac{1}{8\pi} \int d\hat{\epsilon} e^{i\chi(\hat{\epsilon})} \left\{ [\psi^\dagger(x + \epsilon), \gamma^0 \Gamma \psi(x - \epsilon)] , e^{i\epsilon \int_{x-\epsilon}^{x+\epsilon} dx_i A_i^\Gamma} \right\}_{|\epsilon|,|\eta|\times1/\Lambda}
\]

(22)

It is implicitly understood in eq.(22) that the limit \( \Lambda \to \infty \) is taken at the end, after (independent) averaging over the directions of \( \epsilon \) and \( \eta \). The insertion of Wilson factor is appropriate for the definition of bilinears in a gauge theory. Since we are constructing the Coulomb gauge fermions, it is only the transverse part of the vector potential that appears in the Wilson factor. The phase \( e^{i\chi_\Gamma} \) is inserted to project onto the relevant irreducible representation of the 2D rotation group, while averaging over \( \hat{\epsilon} \). Thus, for \( \Gamma = \gamma_0 \) and \( \Gamma = 1 \) we have \( \chi(\epsilon) = 0 \), while for \( \Gamma = \gamma_+ = \gamma_1 + i\gamma_2 \), \( \chi(\epsilon) = \theta(\epsilon) \), and for \( \Gamma = \gamma_- = \gamma_1 - i\gamma_2 \), \( \chi(\epsilon) = -\theta(\epsilon) \). The ratio \( \eta/\epsilon \) is arbitrary, but the final results should not depend on it.

The calculation of bilinears proceeds in complete analogy to the 1+1 dimensional case. Namely, we expand the expression eq.(22) in powers of the inverse cutoff, and retain only the terms that do not vanish in the continuum limit \( \Lambda \to \infty \). This procedure has a certain caveat that one has to keep in mind. The operators which are multiplied by inverse powers of \( \Lambda \), and are formally small, may, in fact, give finite contribution in the continuum limit, if these operators have high enough dimensions. This problem, in fact, arises also in 1+1 dimensions, where the effect of the higher order terms is to
renormalize the coefficients of the lower dimensional operators in a way consistent with
the symmetries of the problem \[3\]. We will return to this point later, and argue that the
situation in our case is similar.

Let us begin by calculating the charge density and the mass operators. For that we
need \( \psi_i^1 \psi_1 \), and \( \psi_2^1 \psi_2 \),
\[
\psi_1^\dagger (e) \psi_1 (\varepsilon) = e^{-\frac{i}{2} \theta (\varepsilon - \varepsilon i) + \frac{i}{2} \int \psi (\xi, \eta) \psi (\xi, \eta) \psi (\xi, \eta) d^2 \xi d^2 \eta}
\]
where the c - number phase \( R_1 \) is given by,
\[
R_1 (\varepsilon, \xi, \eta) = \theta (\xi - 2 \varepsilon) - \theta (\xi + 2 \varepsilon) + \theta (\eta - 2 \varepsilon) - \theta (\eta + 2 \varepsilon)
\]
and the functions appearing in eq.(23) are defined as,
\[
f_i (y) \equiv \varepsilon_i (y + \varepsilon) - \varepsilon_i (y - \varepsilon);
\]
\[
a_1^y (y) \equiv \theta (y - \varepsilon + \xi) - \theta (\varepsilon + \xi - y) + \theta (\eta - \varepsilon - y) - \theta (y + \varepsilon + \eta)
\]
\[
\frac{1}{2 \pi} a_T^y (y) \equiv G (2) (y - \varepsilon + \xi) - G (2) (\varepsilon + \xi - y) + G (2) (\eta - \varepsilon - y) - G (2) (y + \varepsilon + \eta)
\]

We now have to expand the operatorial part in the exponential in eq.(23) in Taylor
series in \( \varepsilon_i, \xi_i, \) and \( \eta_i \). The only subtlety here is that the derivatives do not commute
when acting on \( \theta (x) \), and thus the order of derivatives has to be specified. Keeping in
mind the physical picture, however, it is clear that we must first expand all expressions
in powers of \( \eta_i \) and \( \xi_i \) at fixed \( \varepsilon_i \), and only afterwards expand in powers of \( \varepsilon \). The order
of taking the derivatives is therefore unambiguous. Expanding eq.(23) up to terms of the
second order, we obtain (for details see Appendix C),
\[
J_{11} \equiv \psi_1^\dagger \psi_1 = 2 \pi k^2 \Lambda^2 \left[ \frac{i}{\kappa} < \eta + \xi, \varepsilon_j > \partial_j \tilde{E}_i - < \varepsilon_i (\eta_j - \xi_j) > \{ A_i, \tilde{E}_j \} \right]
\]

Here \( \tilde{E}_i \equiv \varepsilon_i J E_j \), and the averaging is defined as follows,
\[
< \alpha > \equiv \frac{i}{8 \pi^2} \int d \varepsilon d \eta d \xi e^{-i [\theta (\varepsilon) - \theta (\xi)]} \text{Im} e^{\frac{i}{2} R_1 (\varepsilon, \xi, \eta)}
\]
Performing the averaging, and choosing the constant \( k \) appropriately, we get,
\[
\psi_1^\dagger \psi_1 = \frac{1}{2 \varepsilon} \partial_i E_i - A \cdot \tilde{E}
\]
The calculation of \( \psi_2^\adag \psi_2 \) proceeds analogously, and gives the parity conjugate of eq.(28),
\[
\psi_2^\dagger \psi_2 = \frac{1}{2 \varepsilon} \partial_i E_i + A \cdot \tilde{E}
\]
We obtain therefore for the charge density \( J_0 \) and the mass term \( J \),
\[
J_0 \equiv \psi^\adag \psi = \frac{1}{\varepsilon} \partial_i E_i
\]
The correction to these expressions are of the order $1/\Lambda^2$, and we neglect them.

We now turn to the calculation of the spatial components of the current. Following the same steps as in the previous derivation, we find,

\[
\psi_{2\xi}^\dagger(\epsilon)\psi_{\eta}(\epsilon) = -e^2(-\theta(\eta)-\theta(\xi) + \frac{1}{2}\mathcal{R}_2(\epsilon,\xi,\eta))e^{-i\pi} \int d^2y \frac{1}{2}\partial_j E_j(y) \]

\[
e^{i\epsilon} \int d^2y f_1(y) A_i(y) - \frac{i}{2} \int d^2y a_{ij}^2(y) \partial_i E_j(y)
\]

The phase $R_2$ is given by,

\[
R_2(\epsilon,\xi,\eta) = \theta(\xi - 2\epsilon) - \theta(\xi + 2\epsilon) - \theta(\eta - 2\epsilon) + \theta(\eta + 2\epsilon)
\]

and,

\[
a_{ij}^2(y) = \theta(y - \epsilon + \xi) - \theta(y + \xi - \eta) - \theta(\eta - y + \epsilon + \eta)\]

\[
\frac{1}{2\pi} a_{ij}^2(y) = G^{(2)}(y - \epsilon + \xi) - G^{(2)}(\epsilon + \xi - y) - G^{(2)}(\eta - \epsilon - y) + G^{(2)}(y + \epsilon + \eta)
\]

This again is to be expanded in powers of the inverse cutoff. After some algebra (see Appendix C), and keeping two terms in the expansion, we obtain,

\[
J_{21} = -k^2\Lambda^2 = 2 < \epsilon_i > e A_i - \\
- k^2\Lambda^2 \left[ \frac{e}{3} < \epsilon_i \epsilon_j \epsilon_k > \partial_i \partial_j A_k - \frac{4e^3}{3} < \epsilon_i \epsilon_j \epsilon_k > A_i A_j A_k \right] - 4\pi^2 k^2 \Lambda^2 \left[ \frac{1}{e} < \epsilon_i (\xi_j + \eta_j) (\xi_k + \eta_k) > A_i \tilde{E}_j \tilde{E}_k + \frac{i}{e^2} < \epsilon_i (\xi_j - \eta_j) (\xi_k + \eta_k) > \partial_i \tilde{E}_j \tilde{E}_k \right]
\]

where the averages are now defined as,

\[
< \alpha > = \frac{i}{8\pi^3} \int d\xi d\eta d\zeta \alpha e^{-i[\theta(\eta) + \theta(\xi) + \theta(\epsilon)]} \text{Re} \mathcal{R}_2(\epsilon,\xi,\eta)
\]

Already at this stage we see that the calculation of the spatial components of the current is more involved. The leading term is proportional to the UV cutoff. The next - to - leading term in the expansion is of order $1/\Lambda$, but its commutator with the leading term is finite. We therefore have to keep at least those next order terms which give a contribution to this commutator, even though they vanish in the naive continuum limit.

We thus have to come to grips here with the problem mentioned at the beginning of this section, namely, that discarding the formally small higher order terms in the above expansions is too naive. The simplest manifestation of this problem is the following. Explicitly evaluating the averages in eq.(35), and keeping only the leading terms, one obtains,

\[
J_i \equiv \bar{\psi} \gamma_i \psi = -e\kappa \Lambda A_i + \frac{1}{e\Lambda} \left[ \beta \tilde{E}_i (A \tilde{E}) + \gamma E^2 A_i \right]
\]

However, the constants $\kappa$, $\beta$ and $\gamma$ in the above formula depend on the ratio of the regulators $|\eta|/|\epsilon|$. On the other hand, it is clear that the final result may not depend
on this ratio. This means that the formally small terms that we discarded do not disappear completely, but renormalize the constants in eq. (37). The situation here is precisely analogous to the 1+1 dimensional case. There, the same naive expansion procedure gave an incorrect overall scale of $j_1$, which resulted in $j_0$ and $j_1$ not forming a Lorentz vector. In 1+1 dimensions one could, in principle, explicitly evaluate all the corrections coming from the higher order terms. Instead, the requirement of Lorentz invariance was sufficient to fix the scale of $j_1$. In the present case we use precisely the same method. In the next section, we will calculate the energy-momentum tensor, and require that the theory be Lorentz invariant in the continuum limit $\Lambda \to \infty$. We will also require that the spatial components of the current, together with the mass term, satisfy (up to possible Schwinger terms) the tree-level algebra.

Now it remains to convince ourselves, that the contribution of the higher order terms leads to finite renormalization of the coefficients in the expressions for the $J_i$'s. In 1+1 dimensions this is a trivial consequence of the fact that the coupling constant in, say the sine-Gordon theory, is dimensionless. Since the coupling constant $e$ in QED$_3$ is dimensionful, we cannot use similar arguments here. In fact, at first glance, it seems that some of the terms we have discarded are actually more important than the ones we have kept. The expansion in powers of $1/\Lambda$ will bring down terms of the same general form as in eq. (37), but multiplied by powers of the factors of the following three types,

$$\frac{\partial_i}{\Lambda}; \frac{eA_i}{\Lambda}; \frac{E_i}{e\Lambda}$$ (38)

The first type of factor is harmless, since, clearly, it can lead only to a finite renormalization of lower-dimensional terms. Since the scaling dimension of the vector potential in perturbation theory is 1/2, the second factor is small at any physical scale, and can be discarded. The third term, however, looks dangerous. The perturbative dimension of $E_i$ is 3/2, and this factor seems therefore to be of order $\Lambda^{1/2}$. If that were true, it would imply that the terms containing higher powers of this factor are more important than the lower order terms$^4$.

This conclusion is not correct, however, for the following, rather subtle, reason. It turns out that, apart from the UV cutoff $\Lambda$, the bosonized theory has an additional UV scale $\mu = (e^2\Lambda^2)^{1/3}$. The appearance of this scale can be seen as follows. The fermionic operator defined in eq. (18) is essentially a product of a vortex and an antivortex operator, separated by $\Lambda^{-1}$. The UV scaling behavior of the vortex operator in perturbation theory is known$^{25}$. At short distances it scales as an exponential,

$$<V(x)V^*(y)> \propto |x-y|^\alpha e^{c|x-y|^2/2}$$ (39)

where $c$ is a constant, and $\alpha/2$ is the scaling dimension of the electric field, $<E(x)E(y)> \propto \frac{1}{|x-y|^2}$. Assuming the perturbative scaling of the electric field at short distances, $\alpha = 3$,$^4$ Although in QED$_2$ the coupling constant has the dimension of mass, the electric field does not have a transverse component. Therefore the scaling dimension of $E/e$ is zero, and the problem does not arise.
we find that the fermion operator that we have constructed, at distances larger than \(1/\Lambda\), but still small relative to any physical distance scale, behaves, roughly, as,

\[
<\psi_\eta^\dagger(x)\psi_\xi(y)> \propto \left|\frac{c\eta^\dagger \xi}{e^2\Lambda^2|x-y|^3}\right|
\]  

(40)

The scale \(\mu\) is therefore clearly a crossover scale. At distances larger than,

\[
|x-y|^3 \propto \frac{1}{e^2\Lambda^2} \equiv \frac{1}{\mu^3}
\]  

(41)

the exponential in eq.(40) can be expanded in power series. In that case the fermion propagator scales as a power, which is consistent with its perturbative behavior.

At distances smaller than \(1/\mu\) the non-point-likeness of \(\psi\) becomes important. As a result the scaling behavior at these short distances in the bosonized theory must be different from the perturbative one. In fact, if the fermionic operator is still to scale with a power law for \(\mu \ll 1/x \ll \Lambda\), the leading UV behavior of the propagator of the electric field must be \(<E_i(x)E_i(y)> \propto \frac{e^2}{|x-y|^2}\). We will return to this point later, and see that this short distance behavior emerges naturally from the bosonic Hamiltonian.

Assuming for the moment that this is indeed the correct asymptotics, we find that our procedure of calculation of the bilinears is indeed self-consistent. The order of magnitude of the ”dangerous” correction factor is,

\[
\frac{E_i}{e\Lambda} \propto \frac{\mu^{3/2}}{e\Lambda} = O(1)
\]  

(42)

and the corrections due to the higher order terms in the expansion of the bilinears can only lead to a finite renormalization of the lower order terms.

We therefore take the current in the general form,

\[
J_i \equiv \bar{\psi}\gamma_i\psi = -e\kappa\Lambda A_i + \frac{1}{e\Lambda} \left[ \beta\tilde{E}_i(A\tilde{E}) + \gamma E^2 A_i \right]
\]  

(43)

Clearly, to make this expression well defined, we must normal order it with respect to the perturbative vacuum. This implies the subtraction of all terms with at least one Wick contracted pair of operators. It has been rigorously shown that, in 1+1 and 2+1 dimensions, Wick ordering with respect to the free measure of fields in an interacting theory always removes the most divergent parts of the correlation functions [27]. We assume that, in the present case, this is sufficient to subtract the divergent part of the operator in the square brackets in eq.(43).

To be more specific, we assume the following form of the UV asymptotics of the correlators of the bosonic fields, which is the most general one consistent with the UV scaling dimensions discussed above, rotational symmetry, and parity transformation properties,

\[
\lim_{x \rightarrow y} <A_i(x)A_j(y)> = r_1 \frac{\delta_{ij}}{|x-y|} + 2r_2 \frac{(x-y)_i(x-y)_j}{|x-y|^3};
\]  

(44)
\[
\lim_{x \to y} \frac{1}{e^2} < E_i(x) E_j(y) > = q_1 \frac{\delta_{ij}}{|x-y|^2} + 2q_2 \frac{(x-y)_i(x-y)_j}{|x-y|^4}; \\
\lim_{x \to y} < E_i(x) A_j(y) > = s_1 \frac{\delta_{ij}}{|x-y|^2} + 2s_2 \frac{(x-y)_i(x-y)_j}{|x-y|^4} \\
+ mp_1 \frac{\epsilon_{ij}}{|x-y|} + 2mp_2 \frac{(\tilde{x} - \tilde{y})_i(x-y)_j - (\tilde{x} - \tilde{y})_j(x-y)_i}{|x-y|^3}.
\]

The expectation values of the quadratic operators are then given by,

\[\frac{1}{e^2} < E^2 >= 2(q_1 + q_2)\Lambda^2; \quad < A^2 >= 2(r_1 + r_2)\Lambda \]  

\[< A \cdot E >= 2(s_1 + s_2)\Lambda^2; \quad < A \cdot \tilde{E} >= -2m(p_1 + p_2)\Lambda \]

\[< B^2 >= -(r_1 + 2r_2)\Lambda^3\]

The numerical coefficients are therefore subject to the following conditions,

\[q_1 + q_2 > 0; \quad r_1 + r_2 > 0; \quad r_1 + 2r_2 < 0; \quad s_1 + s_2 = 0 \]  

The first three conditions are obvious, while the last one should hold in a Lorentz invariant theory. It follows from the fact that \( E \cdot J \) is the zeroth component of a Lorentz vector, and therefore must have a vanishing expectation value.

In order to fix the coefficients in eq. (31), we first impose the condition that, together with other normal ordered bilinears, this current satisfies the tree-level current algebra, up to possible Schwinger terms, and terms that vanish in the continuum limit,

\[ [J_i(x), J_j(y)] = 2i\epsilon_{ij} J(x) \delta^2(x-y); \quad [J_i(x), J(y)] = -2i\epsilon_{ij} J_j(x) \delta^2(x-y) \]  

Evidently, with the mass bilinear given by eq. (31), the second equation holds without any corrections. Calculating the first commutator, we obtain,

\[ [J_i(x), J_j(y)] = i\epsilon_{ij} \{-2(q_1 + q_2)(p_1 + p_2)(\beta + 2\gamma)^2 + [-\kappa(3\beta + 2\gamma) + (q_1 + q_2)(\beta + 2\gamma)^2] : A \cdot \tilde{E} : \\
+ \frac{\beta + 2\gamma}{e^2\Lambda} [-m(p_1 + p_2)(\beta + 2\gamma) : E^2 : + \frac{\beta + \gamma}{\Lambda} : E^2 A \cdot \tilde{E} :] \} \]

This gives two conditions,

\[\beta + 2\gamma = 0; \quad \beta = \frac{2}{\kappa} \]  

The current is then determined as,

\[ J_i \equiv \bar{\psi} \gamma_i \psi = -e\kappa \Lambda A_i + \frac{1}{e\kappa \Lambda} \left[ 2\tilde{E}_i(A \tilde{E}) - E^2 A_i + 4(p_1 + p_2)m\Lambda \tilde{E}_i \right] \]  

The constant \( \kappa \) will be determined in the following section.
The fermionic bilinears are now expressed as local functions of $E_i$ and $A_i$. Their algebra can be calculated explicitly,

\[
[J_i(x), J_j(y)] = 2i\epsilon_{ij} J(x) \delta^2(x - y); \quad [J_i(x), J(y)] = -2i\epsilon_{ij} J_j(x) \delta^2(x - y) \quad (51)
\]

\[
[J_0(x), J_i(y)] = -i \left[ \kappa \Lambda \delta_{ij} + \frac{2}{e^2 \kappa \Lambda} (E_i E_j - 1/2E^2 \delta_{ij}) \right] \partial_j^x \delta^2(x - y);
\]

\[
[J(x), J_0(y)] = \frac{2i}{e} E_i(x) \epsilon_{ij} \partial_j^x \delta^2(x - y)
\]

The first two commutators are the canonical tree-level ones. The other two exhibit explicitly the Schwinger terms mentioned above. This, again, is a common feature of our procedure and the bosonization in 1+1 dimensions: the Schwinger terms, which in perturbation theory appear at the one loop level, appear in the bosonized theory in the tree level commutators.

### 4 The Hamiltonian and Lorentz invariance

We now calculate the energy-momentum tensor of the theory in terms of bosonic fields. The gauge invariant, symmetric energy-momentum tensor of QED$_3$ is given by,

\[
T_{\mu\nu} = T_{B\mu\nu} + T_{F\mu\nu}
\]

with the bosonic and fermionic parts,

\[
T_{B\mu\nu} = F^{\mu\lambda} F_{\lambda\nu} + \frac{1}{4} g^{\mu\nu} F^2
\]

\[
T_{F\mu\nu} = \frac{i}{4} \left( \bar{\psi} \gamma^\mu D^\nu \psi + \bar{\psi} \gamma^\nu D^\mu \psi - D^\nu \bar{\psi} \gamma^\mu \psi - D^\mu \bar{\psi} \gamma^\nu \psi \right)
\]

Here $D = \partial - ieA$ is the covariant derivative. In order to bosonize the fermionic part of the energy-momentum tensor, one has to calculate the bilinears in eq.(54). Their direct calculation in terms of the fermionic operators constructed in Section 2 gives the general form of the different terms which enter the Hamiltonian, but cannot establish their coefficients (because of the unknown finite renormalization due to the higher order terms).

In 1+1 dimensions there is an alternative way of calculating $T^{\mu\nu}$, using its representation in the Sugawara form. The energy-momentum tensor in a free fermionic theory can be written as a suitably regularized product of the currents [28]. In 1+1 dimensions it is,

\[
T_{F\mu\nu} = \left\{ J^\mu, J^\nu \right\} - g^{\mu\nu} J^\lambda J_\lambda + g^{\mu\nu} \frac{m}{2} \{ \bar{\psi}, \psi \}
\]

Thus the knowledge of the bosonized form of the currents suffices to obtain the bosonized energy-momentum tensor in this case.
In 2+1 dimensions one can still write a formal analog of eq.(55),

\[ T_{\mu \nu} = \frac{1}{\Lambda} \left[ \{ J^\mu, J^\nu \} - g^{\mu \nu} J^\lambda J_\lambda - \{ J^\mu, J^\nu \} : + g^{\mu \nu} : J^\lambda J_\lambda : \right] + g^{\mu \nu} \frac{m}{2} \{ \bar{\psi}, \psi \} \]  

(56)

Here : \( S : \) means normal ordering with respect to the perturbative vacuum. The normal ordered term is formally of order \( 1/\Lambda \), and vanishes in the continuum limit. However, being a composite operator of high dimension, it cannot be discarded a priori. The simplification in 1+1 dimensions is that the normal ordered part vanishes, due to the symmetries of the 1+1 dimensional Dirac matrices [28]. This does not happen in higher dimensions, and we cannot use the Sugawara construction directly.

We can still use the construction, however, as a guide to the form of \( T_{\mu \nu} \). The naive (i.e. neglecting the normal-ordered contributions) Sugawara form of \( T_{\mu \nu} \) contains precisely the same terms that one gets from the explicit calculation. We will therefore take an ansatz which is consistent with both approaches up to finite renormalizations, and determine the coefficients by requiring that Lorentz invariance is recovered in the continuum limit.

The general form for the Hamiltonian density is,

\[ T^{00} = \frac{1}{2} B^2 + \frac{1}{2} E^2 + \frac{a}{e^2 \Lambda} (\partial_i E_i)^2 + b \frac{e^2}{4} \Lambda A^2 + \frac{1}{\Lambda} \left[ c : (A \cdot \tilde{E})^2 : \right] + d : E^2 A^2 : + f A \cdot \tilde{E} \]  

(57)

If the theory is to be Lorentz invariant with standard transformation properties for \( E_i, B, J_0, \) and \( J_i \), Maxwell’s equations should be satisfied in the continuum limit. We therefore require,

\[ \dot{B}(x) = i[H, B(x)] = -\epsilon_{ij} \partial_j E_j + O(1/\Lambda) \]  

(58)

\[ \dot{E}_i = i[H, E_i] = -J_i + \epsilon_{ij} \partial_j B + O(1/\Lambda^2) \]  

(59)

Since \( J_i \) itself contains terms of order \( 1/\Lambda \), we demand that the second equation be satisfied to order \( O(1/\Lambda^2) \).

Calculating the commutators, we find the following conditions on the coefficients [5],

\[ f = 0, \quad b = \frac{\kappa}{2}, \quad c = -\frac{1}{\kappa}, \quad d = \frac{1}{2\kappa} \]  

(60)

This still leaves the coefficients \( \kappa \) and \( a \) undetermined. Our next step will be therefore to impose the complete Poincaré algebra.

A sufficient condition for Lorentz invariance of a theory invariant under spatial rotations and translations is the following commutation relation [23],

\[ -i[T^{00}(x), T^{00}(y)] = -(T^0_0(x) + T^0_0(y)) \partial^i \delta^2(x - y) \]  

(61)

\footnote{In fact, for these values of the coefficients the inhomogeneous Maxwell’s equation is obtained exactly, without corrections \( O(1/\Lambda^2) \). The correction term to the homogeneous equation is of the form, \( O(1/\Lambda) = +2/\kappa \Lambda e^2 \partial_k [(A^2 E^0) + (A \tilde{E})^2 - (M \kappa \Lambda/2) A^i] \). Normal-ordering the operator in the square brackets, we find that its divergent part vanishes. Thus, even for finite \( \Lambda \), this equation retains the form of a conservation law.}
It can be easily verified (by multiplying eq. (61) by linear functions of \( x \) and \( y \) and integrating) that eq. (61) implies the closure of the Poincaré algebra \( \{29\} \), with the following generators: the energy \( L_i = \int d^2 x [x_0 T_i^0 - x_i T_0^0] \), the momentum \( P_i = \int d^2 x T_i^0 \), and the angular momentum and boost generators, given, respectively by,

\[
L_i = \int d^2 x x^0 [T_0^0(x) - x_i T_i^0] \quad L = \int d^2 x \epsilon_{ij} x_i T_j^0
\]

(62)

It should also be verified that the angular momentum defined in eq. (62) does indeed generate rotations when acting on \( E_i \) and \( A_i \). Calculating the commutator in eq. (61), we find for the momentum density,

\[
T_i^0 = B \tilde{E}_i - 2a \kappa \partial_j E_i A_i
\]

(63)

\[
+ \partial_j \left[ \frac{4a(q_1 + q_2) + r_1 + 2r_2}{\kappa} \epsilon_{ij} A \cdot \tilde{E} - \frac{4a(q_1 + q_2) - r_1 - 2r_2}{\kappa} \delta_{ij} A \cdot E \right]
\]

\[
+ m \frac{p_1 + p_2}{2\kappa} \partial_j (A_i \tilde{A}_j + A_j \tilde{A}_i) + O(1/\Lambda)
\]

In arriving at this expression we have extracted finite parts of the formally vanishing operators using the operator product expansion (normal ordering) with the UV asymptotics of the propagators given in eq. (45). The relevant Wick contractions are,

\[
\lim_{x \to 0} \frac{1}{e^2} < \partial_i E_i(x) E_j(y) > O(y) = (q_1 + q_2) \Lambda^2 \partial_j O(y)
\]

(64)

\[
\lim_{x \to 0} \partial_i E_i(x) A_j(y) > O(y) = -\frac{1}{2} (p_1 + p_2) m \Lambda \epsilon_{ji} \partial_i O(y)
\]

\[
\lim_{x \to 0} \epsilon_{ki} \partial_k A_i(x) A_j(y) > O(y) = -\frac{1}{2} (r_1 + 2r_2) m \Lambda \epsilon_{ji} \partial_i O(y)
\]

\[
\lim_{x \to 0} \epsilon_{ki} \partial_k A_i(x) E_j(y) > O(y) = \frac{1}{2} (p_1 + p_2) m \Lambda \partial_j O(y)
\]

Requiring that the angular momentum generator has the standard "orbital" and "spin" parts,

\[
L \equiv \int d^2 x \epsilon_{ij} x_i T_j^0 = \int d^2 x \epsilon_{ij} x_i E_k \partial_j A_k - A \cdot \tilde{E}
\]

(65)

we obtain from eq. (63) the following relations,

\[
a = \frac{1}{2\kappa}, \quad \kappa = -\frac{2(q_1 + q_2)}{r_1 + 2r_2}
\]

(66)

or, using eq. (45),

\[
\kappa = \frac{\Lambda}{e^2} \frac{<E^2>}{<B^2>}
\]

(67)

The expression for the momentum density is also consistent with what is expected in a theory of a local vector field \( A_i \). The momentum operator \( P_i = \int d^2 x T_i^0 \) generates spatial translations by the usual rule,

\[
[A_i(x), P_j] = -i \partial_i A_j(x)
\]

(68)
We have now determined all the coefficients in the Hamiltonian density and the current in terms of $\langle E^2 \rangle$ and $\langle A \cdot \tilde{E} \rangle$. Our final expressions are (we record their non-normal-ordered forms),

\begin{align*}
T^{00} &= \frac{1}{2}B^2 + \frac{1}{2}E^2 + \frac{1}{2e^2\kappa \Lambda}(\partial_i E_i)^2 + \frac{e^2}{2}\kappa \Lambda A^2 + \frac{1}{2\kappa \Lambda} \left[ -2(A \cdot \tilde{E})^2 + E^2 A^2 \right] + MA \cdot \tilde{E}; \\
T^{0i} &= B\tilde{E}_i - (\partial_j E_j)A_i + \gamma \partial_i (A E) + \frac{M}{8} \partial_j (A_i \tilde{A}_j + A_j \tilde{A}_i) + O\left(\frac{1}{\Lambda}\right); \\
J_i &= -e\kappa \Lambda A_i + \frac{1}{e\kappa \Lambda} \left[ 2\tilde{E}_i A \cdot \tilde{E} - A_i E^2 \right] + \frac{M}{e} \tilde{E}_i
\end{align*}

with,

\begin{align*}
M &= -\frac{2}{\kappa \Lambda} \langle A \cdot \tilde{E} \rangle = \frac{4(p_1 + p_2)}{\kappa} m \\
\gamma &= -\frac{2}{\Lambda^3 \kappa} < B^2 >\kappa
\end{align*}

and $\kappa$ defined in eq.(67).

We want to stress here that, although eq.(69) contains a parameter $\kappa$, it does not define a one-parameter set of theories. The bosonized version of QED$_3$ corresponds to a unique choice of $\kappa$, which satisfies eq.(67). Unfortunately, since eq.(69) defines a strongly interacting theory, we cannot determine the numerical value of $\kappa$. This would involve the solution of the model (at least in the UV region) for arbitrary $\kappa$, calculating $\langle B^2 \rangle_\kappa$ and $\langle E^2 \rangle_\kappa$, and solving the selfconsistency equation (67).

We end this section with a comment. In arriving at the form of eq.(57), and subsequently eq.(69), we have truncated the series for the Hamiltonian density at the order $1/\Lambda$. This amounts to neglecting terms of the form $\frac{1}{e^2 \Lambda^3} E^4 A^2$. This is consistent with the asymptotic form of the correlation functions given in eq.(45), since the power counting based on it tells us that we have kept all the relevant terms. We can estimate the scale at which the irrelevant terms become important. Assuming perturbative scaling at low energies, a simple scaling argument shows that these terms are unimportant relative to the terms we kept in eq.(57) at energy scales smaller than $\mu = (e^2 \Lambda^3)^{1/3}$. This is again consistent with our previous estimate (see Section 3). At this scale all terms become equally important, and, once this happens, the scaling dimensions of fields will cease to be canonical. It is generally the case that the magnitudes of irrelevant and relevant terms in a Hamiltonian coincide at the UV cutoff scale. In our case this translates into the statement that, near the scale $\Lambda$, the electric field must have dimension 1. Although this argument does not constitute a proof of eq.(45) (and the proof cannot be given without solving the theory), it shows the selfconsistency of our assumptions.
5 Discussion

In this paper we have performed the analog of Mandelstam's construction in 2+1 dimensional QED$_3$, that is, we have constructed the two-component Dirac spinor field entirely in terms of the bosonic fields $E_i$, and their conjugate momenta $A_i$. The fermionic bilinears are local functions of the bosonic variables. They satisfy a current algebra which includes Schwinger terms. The bosonized theory is Lorentz invariant in the continuum limit. These aspects of our construction are very similar to what one encounters in 1+1 dimensions. There are, however, certain features which are conceptually different, and which reflect the differences between 1+1- and 2+1-dimensional physics. We now make several comments on these issues.

Firstly, the fermionic operators eq.(18) anticommute only at distances larger than the ultraviolet cutoff. In fact, the phase in the anticommutation relations contains a power tail of the form $(1/(\Lambda|x-y|)^s$, where $s$ is a number of order one. This is closely related to the fact (discussed in Sections 3 and 4), that the construction involves two UV scales $\Lambda$ and $\mu \propto (e^2\Lambda^2)^{1/3}$, and that the equivalence with continuum QED$_3$ holds only below the lower scale $\mu$. This also means that, if we were to discretize our bosonic theory, the fermionic operators would not have canonical local anticommutation relations. The lattice spacing $a$ should be identified with the smallest distance scale required in our regularization procedure, that is, $a \propto 1/\Lambda$. The Fermi fields would then anticommute only at distances larger than $1/\Lambda = a$. This is contrary to the assumptions made in the proof of the Nielsen-Ninomiya theorem [30], and provides a way in which the theory avoids the standard fermion-doubling problem. In fact, recall, that a Hamiltonian theory of one staggered lattice fermion field with local anticommutation relations in 2+1 dimensions becomes in the continuum limit a theory of four Fermi degrees of freedom. Our theory, however, has only two fermionic degrees of freedom in the continuum limit. This aspect is different from the 1+1 dimensional case, where a theory of one staggered lattice fermion leads in the continuum limit to a theory of one Dirac fermion. Accordingly, the anticommutation relations in 1+1 dimensions are canonical at all distances up to the cutoff.

Secondly, we comment on the explicit appearance of the UV cutoff in our final expressions, which is also a novel feature. The UV cutoff never appeared explicitly in 1+1 dimensional formulae, since there the theory of free fermions bosonized onto a theory of a free bosons. In higher dimensions there is however no reason to expect that the same will happen. On the contrary, one expects that fermions free in the UV will be represented by strongly interacting bosons. Indeed, the bosonic Hamiltonian eq.(69) contains, apart from quadratic terms, also a quartic interaction term with a coefficient of order $1/\Lambda$. Since this coefficient scales with the inverse power of the cutoff, by naive power counting, the term is irrelevant. However, this term is very important in determining the scaling dimensions of various operators. For example, if we were to omit it, the Hamiltonian would describe a (non-Lorentz-invariant) theory of two free bosonic degrees of freedom $A_i$, with scaling dimensions one. Both components of the electric field would then have dimension $3/2$. We know, however, that the divergence of electric field
(which is proportional to the fermion number density) must have scaling dimension 2. The formally irrelevant quartic interaction term is precisely responsible for the required change of the scaling.

It is also clear that this term cannot be treated in perturbation theory, since, even though it appears to be irrelevant, its effect is not small. The situation here is very similar to the one in some strongly interacting 2+1 - dimensional theories defined at finite fixed points of the coupling constant. The examples that recently have attracted much attention are the four - Fermi theories [31]. The four - Fermi interaction term is perturbatively irrelevant in 2+1 dimensions. However, if the coupling constant is defined at (or infinitesimally close to) its fixed point value, a renormalizable and nontrivial theory is obtained. The scaling dimensions of various operators at the fixed point are different from their values in the free theory. Since the coupling constant has the dimension of inverse mass, its fixed point value is always of the form \( g = \frac{k}{\Lambda} \), where \( \Lambda \) is the UV cutoff, and \( k \) is a pure number. The theory eq.(69) should be understood in the same sense. It is defined at the fixed point value of the coupling of the naively nonrenormalizable quartic interaction term, which is responsible for the correct scaling of the fields. In this spirit eq.(67), although it was not derived here from corresponding \( \beta \)- function, should be understood as a fixed point condition. The appearance of the positive powers of \( \Lambda \) in eq.(69) is also natural, once one realizes that the theory is interacting. Generically, in an interacting theory any coupling constant with positive dimension must scale as a power of the UV cutoff. This is the origin of the cutoff in the \( A^2 \) term in eq.(69). The appearance of the cutoff in the expression for the conserved current eq.(37) is then inevitable, since the form of the currents is determined by the Hamiltonian.

Unfortunately, the very fact that the bosonic theory we obtained is strongly interacting prevents us from analysing these questions quantitatively. Approximation schemes that were useful in analysing the four - Fermi interactions, such as the \( 1/N \) expansion, or the \( \epsilon \)- expansion, cannot be applied here, since an extension of the model either to large number of fermionic species, or to dimensions different from 2+1 is not available at the moment.

Even so, there are several directions in which our approach can be extended. The first, and most straightforward one, is to understand the regularization ambiguity of QED₃ [22]. The fermionic operators we have constructed solve the Gauss’ constraint. If one relaxes this condition, there are additional possibilities [23]. One natural modification is to substitute for \( E_i \) in eq.(21) the linear combination \( \Pi_i = E_i + \kappa e^2 \epsilon_\ij A_j \). This gives
\[
\chi^\dagger \chi = \frac{1}{e} \partial_i E_i + \kappa e B. \quad (\chi \text{ is the modified Fermi field}).
\]
However, the coefficient \( \kappa \) is not totally arbitrary in this case. It is crucial for the derivation that \( \int d^2 x \partial_i \Pi_i \) has quantized eigenvalues. Otherwise, fermionic operators will not be double valued, and the bilinears will not be local bosonic operators (see Appendix C). Since the magnetic flux in QED₃ is quantized in units of \( \frac{2\pi}{e} \), the coefficient \( \kappa \) must be given by \( \frac{n}{2} \), with \( n \) - an integer. This possible modification is the reflection of the well known regularization ambiguity in fermionic QED₃ [22], which leads to the appearance of the induced Chern - Simons term in the action or, in the Hamiltonian formalism, to the modification of the Gauss’ law constraint. This modification of fermionic operators will also modify the expressions
for the fermionic bilinears, and the energy - momentum tensor. The resulting bosonic theory can be obtained by the methods used in this paper.

Thirdly, although the theory we have derived is Lorentz invariant, it is not written explicitly in terms of Lorentz covariant fields. It would be desirable to make one more step, and find a formulation in terms of a different, covariant bosonic field. We believe that the most natural candidate for this is the magnetic vortex field. In $2+1$ dimensions it is a scalar field \[25\]. It has the additional virtue that in terms of it not only is the electric current trivially conserved, but also the electric charge has a meaning of a topological winding number, and is quantized classically \[32\]. The vortex operators appeared naturally in our construction of the Fermi fields. The counting of the number of degrees of freedom suggests that this additional transformation should be possible, since the complex vortex field is equivalent to two real bosonic fields. It would also be interesting to extend the construction to theories with more fermionic species. This generalization may be simpler in terms of the scalar vortex variables.

Finally, we note that the fermionic operators which we have constructed involve a product of vortex and antivortex operators, carrying opposite magnetic fluxes. Therefore the operators do not carry net magnetic flux, and thus are not analogous to $2+1$ - dimensional dyons. The mechanism of their anticommutativity is different from that discussed in \[23\], and subsequently extensively exploited in the analysis of Chern - Simons theories. In fact, the approach to the problem of bosonization presented here has a natural generalization to $3+1$ dimensions. The fermionic operators can be constructed quite easily in a form analogous to eq.(18). The anticommutation relations can be achieved by the mechanism described in the Appendix A. While in $2+1$ dimensions the fermionic operator creates a point charge and an infinitesimally close pair consisting of a magnetic vortex and an antivortex of half - integer strength, in $3+1$ dimensions it creates a point charge and an infinitesimally small vortex loop, again of half - integer strength. By averaging over the orientations of the loop, and by ensuring the correct transformation properties under the axial transformations, it is possible to construct the four component Dirac spinor, and calculate the bilinears. The work along these lines is currently in progress \[23\].

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6 Appendix A. A simple mechanism for anticommutation.

In this Appendix we give an intuitive explanation of the basic element in the construction of the anticommuting operators $\psi_\alpha$, and point out its similarity to the $1+1$ dimensional case. Recall that, in $1+1$ dimensional Abelian gauge theory, the fermionic operators can
be expressed in terms of the gauge field as follows,

$$\psi_{1,2}(x) = \exp\left[-ie \int_{-\infty}^{x} dy A(y)\right] \exp\left[\mp i \frac{\pi}{e} E(x)\right] \quad (73)$$

The two factors in eq.(73) have a very simple meaning. The first exponential factor creates a unit charge at the point $x$,

$$e^{-ie \int_{-\infty}^{x} dy A(y)} \frac{1}{e} \partial_i E_i = 1 + \delta(x - z) \quad (74)$$

The second exponential (up to a surface term) can be rewritten as,

$$V_\pm = \exp\left[\mp i \pi \int_{x}^{+\infty} d^2 y \frac{1}{e} \partial_i E_i(y)\right] \quad (75)$$

It therefore measures the total electric charge in the half space to the right of the point $x$. To see why two such operators anticommute, consider $\psi_\alpha(x) \psi_\beta(y)$. Since either $x > y$ or $y > x$, only one of these two operators creates the electric charge in the region where the other one is measuring it. When we change the order of the fermionic operators, the expression picks a phase according to $e^{A_i B} = e^B e^{[A_i B]}$. But always, only one of the $V'$s measures the charge. As a result the phase is always $\pm \pi$, and the operators anticommute.

It is clear therefore that in any number of dimensions one can achieve anticommutation relations (but not rotational invariance) from a construction of the following kind,

$$\psi_{\pm}(x) = \exp\left[i \int d^2 y e_i(x - y) A_i(y)\right] \exp\left[\pm i \pi \int_M d^2 y \frac{1}{e} \partial_i E_i(y)\right] \quad (76)$$

where $e_i(x)$ is the field of a point charge ($\partial_i e_i(x) = \delta^d(x)$), and the volume $M$ over which the integral is performed in the second factor is the half space “to the right of” the point $x$, defined for example by $y_1 > x_1$. The objects constructed in this way anticommute precisely for the same reason as in 1+1 dimensions.

Although 2+1 dimensional fermionic operators which we discussed in this paper where not represented in the form of eq.(70), the reason for their anticommutativity is essentially the same. Forgetting momentarily about the regularization, and the factor ordering (which are crucial for the calculation of bilinears, but not for the anticommutation properties of the operators at distant points), we can write eq.(17) as,

$$\psi_\alpha(x) \propto \exp\left[i \int d^2 y e_i(x - y) A_i(y)\right] \exp\left[\mp \frac{i}{2e} \int d^2 y \left(\theta(x - y) - \theta(y - x) - \pi\right) \partial_i E_i(y)\right] \quad (77)$$

For our present simplified discussion, we can think about the polar angle $\theta(x)$ as a function with a cut. The cut should begin at the origin, and go to spatial infinity. Let us choose the second axis as the direction of this line. Then, for $(x - y)_1 > 0$, we have $\theta(x - y) - \theta(y - x) = -\pi$, and for $(x - y)_1 < 0$ we have, $\theta(x - y) - \theta(y - x) = \pi$. The second phase factor in eq.(77) then becomes precisely the integral over the half space $(x - y)_1 > 0$ of the electric charge density $\frac{1}{e} \partial_i E_i$, with the coefficient $\pi$.

We therefore see, that the anticommutativity in our construction is achieved by precisely the same mechanism as in 1+1 dimensions.
7 Appendix B. The geometry of our construction.

In this appendix we discuss the geometry of our construction of the fermionic operators, and point out its connection to the formulation of Finkelstein and Rubinstein [24].

The function $\theta(x)$ is defined by the Cauchy - Riemann equation [15],

$$\epsilon_{ij}\partial_j G^{(2)}(x) = \frac{1}{2\pi}\partial_i \theta(x)$$  \hfill (78)

Here $G^{(2)}(x - y) = -\frac{1}{4\pi}\ln(\mu^2 x^2)$. The differential equation is solved by,

$$\theta(x) = \int_{C(M,x)} dy_i \epsilon_{ij} \frac{y_j}{y^2}$$  \hfill (79)

where the curve $C(M,x)$ starts at a base point $M$, and ends at the point $x$. It is convenient to take the base point $M$ to infinity. The function $\theta$ then depends only on the point $x$, and the first homotopy class of $C$, i.e., the number of times the curve $C$ winds around 0. A point on the universal cover can be parametrized by $x$, and the homotopy class of this curve. Therefore $\theta$ is a single valued function on the universal covering space [34]. Moreover, $e^{i\theta}$, and therefore also $\psi_\alpha(x)$, is a single valued function on the double cover of the plane $\mathbb{R}^2$.

We wish, however, to consider our theory as $QED_3$ on a plane, rather than on its cover. There is a natural mathematical framework - introduced by Finkelstein and Rubinstein - which, suitably generalized, accomodates this interpretation of our construction.

To set the scene, we will now discuss the relevant setup in some generality.

Let $X$ be the (time 0) configuration space of the theory. We will assume, that $X$ is a Banach space, noncompact in the natural topology. Let the set $\{\hat{\phi}_i\}_{i=1,\ldots,N}$ comprise of quantum fields, i.e., the operator - valued distributions on $X$. We shall assume that there is an action of the full Lorentz group on the fields $\hat{\phi}_i$, which are bosonic quantities. The Fock space over $X$, $\mathcal{F}^X$, is the field manifold of the theory. Let $Q$ denote the set of all quantum fields on $X$. $Q$ is explicitly realized as the $L^2$ space of the functional Gaussian measure, formally,

$$N \prod_{x \in X} \prod_i d\phi_i(x) \exp \left[-<\phi;K\phi>\right]$$  \hfill (80)

for the positive definite kernel $K$ defined by the quadratic part of the action. The integration is over $\mathcal{S}'(X)$, which is the space of Schwartz distributions on $X$.

Consider now a multivalued functional of the fields $\hat{\phi}$, say $\hat{\Psi}$. $\hat{\Psi}$ will in general depend not only on the fields themselves, but also on continuous functions $\{\theta_j\}_{j=1,\ldots,N}$, defined on the universal cover $\tilde{X}$ of the configuration space. We shall, however, elect to work not with the cover, but with $X$ itself, accordingly, we will take the domain of $\hat{\Psi}$ to be $Q \times \Pi^{-1}\{(C(\tilde{X}))^N\}$, where $\Pi^{-1}$ is the natural projection from the cover to the configuration space. In this approach, the “functions” $\theta$ are multivalued. We shall denote the domain of the functionals $\hat{\Psi}$ by $\mathcal{Q}$.
Now, let $F_\alpha$ be a flow on $\tilde{Q}$. Any such flow induces a continuous 1-parameter family of linear transformations on the functionals. We will take here, for definiteness, the flow to be induced by rotations about a fixed axis. Accordingly, we shall take $N = 1$, and $\theta_1 = \theta$, where $\theta$ is a polar variable parametrizing a 1-dimensional submanifold, perpendicular to the rotation axis. The linear transformation is then exhibited as,

$$\hat{\Psi}(\hat{\phi}, \theta) \rightarrow \hat{\Psi}(F_{\alpha}^{-1}\hat{\phi}F_\alpha, \theta + \alpha)$$

(81)

We shall take $\alpha \in [0, 2\pi]$, accordingly $F_0 = 1$, $F_{2\pi}$ is a rotation by $2\pi$, and $F_{2\pi}\hat{\phi} = \hat{\phi}$. However, $\hat{\Psi}$ is assumed to be multivalued, which implies, that $\pi_1(Q) \neq \emptyset$. Since $Q$ is a rather complicated infinite-dimensional set, we usually cannot compute this homotopy group directly. However, as we will show, we can relate this group to a much simpler homotopy group. To this end, consider a given topological sector of the theory, characterized by the asymptotic behavior of the operators,

$$\hat{\phi}_i(x) \rightarrow \phi_0$$

(82)

Denote the corresponding Fock space superselection sector by $\mathcal{F}^X(\phi_0)$. Now, consider a loop in $\tilde{Q}$. This can be viewed as a mapping of the unit cell $I^{dimX+1} \rightarrow \mathcal{F}^X(\phi_0)$, with values of the fields and functions on $\partial I^{dimX+1}$ specified. Hence,

$$\pi_1(Q) \approx \pi_{dimX+1}(\mathcal{F}^X(\phi_0))$$

(83)

as is easily seen by considering $\hat{\phi} \equiv \phi_0$.

It is now obvious, how this construction should proceed in our formulation of $QED_3$. We take $X = \mathbb{R}^2$, $\tilde{X} = \mathbb{C}/\mathbb{Z}_2$, and $\hat{\phi} = (E, A)$. Since we are working in the gauge-fixed formulation, the corresponding Gaussian measures are well-defined. We have explicitly constructed a pair of double-valued functionals $\hat{\Psi}$, which change their signs under a $2\pi$ rotation. We thus deduce, that $\pi_1(Q) \neq \emptyset$. This implies, by the isomorphism shown above, that $\pi_3(\mathcal{F}^X(\phi_0))$ does not vanish. We conjecture, that this result is tied to the fact, that while the operators $E_i$ have continuous spectra, the operator $\partial_i E_i$ has a purely discrete spectrum. This implies, that a “naive” picture of the resulting Fock space is certainly not correct, and this space has nontrivial topological characteristics.

8 Appendix C. Calculation of fermionic bilinears.

In this appendix we give details of the calculation of fermionic bilinears. We start with $\psi_1^\dagger \psi_1$,

$$\psi_1^\dagger(x)\psi_1(x) = \frac{1}{8\pi} \int d\epsilon \left\{ [\psi_1^\dagger - 1(x + \epsilon), \psi - 1(x - \epsilon)], e^{ie\int_{x-\epsilon}^{x+\epsilon} dx_i A_i^T} \right\}_{|\eta|,|\xi|,|\epsilon| \propto 1/\Lambda}$$

(84)

We have to represent the product of fermionic operators as a single exponential, expand it in powers of the regulators, and average over the directions of the regulators. Using the
Baker-Campbell-Hausdorff formula, $e^A e^B = e^{A + B + \frac{1}{2}[A,B]}$, which is valid when $[A, B]$ is a c-number, we find,

$$\psi_{1\xi}^\dagger(\epsilon)\psi_{1\eta}(-\epsilon) = e^{-\frac{i}{2}(\theta(\eta) - \theta(\xi) + \frac{1}{2}R_1(\epsilon, \xi, \eta))} e^{iF_1(A, E)} \tag{85}$$

$$\psi_{1\eta}(-\epsilon)\psi_{1\xi}^\dagger(\epsilon) = e^{-\frac{i}{2}(\theta(\eta) - \theta(\xi) - \frac{1}{2}R_1(\epsilon, \xi, \eta))} e^{iF_1(A, E)} \tag{86}$$

Here the c-number phase $R_1$ is given by,

$$R_1(\epsilon, \xi, \eta) = \theta(\xi - 2\epsilon) - \theta(\xi + 2\epsilon) + \theta(\eta - 2\epsilon) - \theta(\eta + 2\epsilon) \tag{87}$$

and the operatorial part is,

$$F_1(A, E) = e \int d^2 y f_i(y) A_i(y) + \frac{1}{2e} \int d^2 y a_1^A(y) \partial_i E_i - a_T^1(y) \epsilon_{ij} \partial_i E_j(y) \tag{88}$$

with,

$$f_i(y) \equiv e_i(y + \epsilon) - e_i(y - \epsilon); \tag{89}$$

$$a_1^A(y) \equiv \theta(y - \epsilon + \xi) - \theta(\epsilon + \xi - y) + \theta(\eta - \epsilon - y) - \theta(y + \epsilon + \eta);$$

$$\frac{1}{2\pi} a_T^1(y) \equiv G^{(2)}(y - \epsilon + \xi) - G^{(2)}(\epsilon + \xi - y) + G^{(2)}(\eta - \epsilon - y) - G^{(2)}(y + \epsilon + \eta)$$

We now expand the operatorial part in Taylor series. Since the derivatives do not commute when acting on $\theta(x)$, one has to specify their order. However, physically it is clear that we must first expand in $\eta$ and $\xi$ at fixed $\epsilon$, and therefore the order of derivatives is, in fact, unambiguously determined. We use the following identities,

$$\partial_i^x \int d^2 y \epsilon_j(y - x) A_j(y) = -A_i^L(x) \tag{90}$$

$$\partial_i^x \int d^2 y \theta(y - x) \partial_j E_j(y) = 2\pi E_i^L$$

$$\partial_i^x \int d^2 y G^2(y - x) \epsilon_{jk} \partial_j E_k(y) = \tilde{E}_i^T$$

where the superscripts $L$ and $T$ denote the longitudinal and transverse parts of the fields, respectively. Expanding the operatorial part up to the terms of second order, we obtain,

$$e^{iF_1(A, E)} = 1 + 2iec_i A_i^L - \frac{2\pi}{e} (\eta - \xi)_i \tilde{E}_i - \frac{2\pi}{e} (\eta + \xi)_i \epsilon_{ij} \partial_j \tilde{E}_i - 2\epsilon^2 \epsilon_{ij} A_i^L A_j^T \tag{91}$$

$$- 4\pi \epsilon_i (\eta - \xi)_j A_i^T \tilde{E}_j - \frac{2\pi^2}{\epsilon^2} (\eta - \xi)_i (\eta - \xi)_j \tilde{E}_i \tilde{E}_j$$

$$e^{ie \int^y dy A_i^T(y)} = 1 + 2iec_i A_i^T - 2\epsilon^2 \epsilon_{ij} A_i^T A_j^T$$

The phase $R_1$ depends on the ratio $|\eta|/|\epsilon|$. It turns out, however, that changing this ratio results in multiplying $\text{Im} e^{i\theta_1}$ by a constant. Since this is the only quantity that
appears in the present calculation, its variation can always be compensated by a suitable redefinition of \( k \) in eq.\( (18) \). We therefore evaluate \( R_1 \) for \(|\epsilon| > |\eta|\),

\[
R_1(\epsilon, \eta, \xi) = 2\pi \quad ; \quad \epsilon_{ij}\epsilon_{i}\eta_{j} > 0, \epsilon_{ij}\epsilon_{i}\xi_{j} > 0
\]

\[
R_1(\epsilon, \eta, \xi) = 0 \quad ; \quad \epsilon_{ij}\epsilon_{i}\eta_{j} > 0, \epsilon_{ij}\epsilon_{i}\xi_{j} < 0
\]

\[
R_1(\epsilon, \eta, \xi) = -2\pi \quad ; \quad \epsilon_{ij}\epsilon_{i}\eta_{j} < 0, \epsilon_{ij}\epsilon_{i}\xi_{j} < 0
\]

\[
R_1(\epsilon, \eta, \xi) = 0 \quad ; \quad \epsilon_{ij}\epsilon_{i}\eta_{j} < 0, \epsilon_{ij}\epsilon_{i}\xi_{j} > 0
\]

The next step is to calculate the averages, defined as,

\[
< \alpha > \equiv \frac{i}{8\pi^3} \int d\hat{\epsilon}d\hat{\eta}d\hat{\xi} e^{-i[\theta(\eta)-\theta(\xi)]} \text{Im} e^{iR_1(\epsilon,\xi,\eta)}
\]

We find,

\[
<1> = 0; \quad <\hat{\eta}_i \pm \hat{\xi}_i> = 0; \quad <(\hat{\eta}_i \pm \hat{\xi}_i)(\hat{\eta}_j \pm \hat{\xi}_j)> = 0; \quad <\hat{\epsilon}_i> = 0; \quad <\hat{\epsilon}_i\hat{\epsilon}_j> = 0
\]

\[
<\hat{\eta}_i - \hat{\xi}_i\hat{\epsilon}_j >= \frac{1}{4\pi}\delta_{ij}; \quad <\hat{\eta}_i + \hat{\xi}_i\hat{\epsilon}_j >= \frac{i}{4\pi}\epsilon_{ij}
\]

Assembling the terms, we obtain,

\[
\psi_1^\dagger \psi_1 = 2\pi k^2 \Lambda^2 \left[ -i <(\eta + \xi)_i\epsilon_j > \frac{1}{e}\partial_j \tilde{E}_i - <\epsilon_i(\eta_j - \xi_j)> \{A_i, \tilde{E}_j\} \right]
\]

\[
= \frac{k^2}{2} \Lambda^2 |\epsilon||\eta| \left[ \frac{1}{2}\partial_i E_i - 2A \cdot \tilde{E} \right]
\]

The calculation of \( \psi_2^\dagger \psi_2 \) proceeds in the same fashion. The only difference is that the electric field enters all expressions with the opposite sign, and the averaging is done with the complex conjugate phase factor,

\[
< \alpha > \equiv \frac{i}{8\pi^3} \int d\hat{\epsilon}d\hat{\eta}d\hat{\xi} e^{i[\theta(\eta)-\theta(\xi)]} \text{Im} e^{iR_1(\epsilon,\xi,\eta)}
\]

As a result the averages now become,

\[
<\hat{\eta}_i - \hat{\xi}_i\hat{\epsilon}_j >= \frac{1}{4\pi}\delta_{ij}; \quad <\hat{\eta}_i + \hat{\xi}_i\hat{\epsilon}_j >= -\frac{i}{4\pi}\epsilon_{ij}
\]

We therefore get,

\[
\psi_2^\dagger \psi_2 = \frac{k^2}{2} \Lambda^2 |\epsilon||\eta| \left[ \frac{1}{2}\partial_i E_i + 2A \cdot \tilde{E} \right]
\]

Choosing the length of the regulators in an appropriate way,

\[
|\epsilon||\eta| = k^{-2}\Lambda^{-2}
\]

we obtain,

\[
\psi^\dagger \psi = \frac{1}{e}\partial_i E_i, \quad \bar{\psi}\psi = -2A \cdot \tilde{E}
\]
We now turn to the calculation of spatial components of the current,
\[
J_-(x) = \psi_2^\dagger(x)\psi_1(x) \equiv \frac{1}{8\pi} \int d\epsilon e^{-i\theta(\epsilon)} \left\{ \left[ \psi_2^\dagger(x + \epsilon), \psi_1(x - \epsilon) \right], e^{i\epsilon \int_{x-\epsilon}^{x+\epsilon} dx_i A_i^T} \right\}[\eta,|\xi|,|\epsilon|]_{\epsilon \lambda 1/A}
\]  
(101)
Following analogous steps, we find,
\[
\psi_1^\dagger(\epsilon)\psi_2(\epsilon) = ic^{\frac{1}{2}}[-\theta(\eta) - \theta(\xi) + \frac{1}{2} R_2(\epsilon, \xi, \eta)] e^{-\frac{i}{
\rho \lambda} \int d^2y E_i(y) e^{iF_2(A, E)}
\]  
(102)
\[
\psi_2(\epsilon)\psi_1^\dagger(\epsilon) = -ie^{\frac{1}{2}}[-\theta(\eta) - \theta(\xi) - \frac{1}{2} R_2(\epsilon, \xi, \eta)] e^{-\frac{i}{
\rho \lambda} \int d^2y E_i(y) e^{iF_2(A, E)}
\]  
(103)

The operatorial phase is,
\[
F_2(A, E) = ie \int d^2y f_1(y) A_i(y) - \frac{i}{2} \int d^2y a_{1i}^2(y) \partial_i E_i + a_{1i}^2(y) \epsilon_{ij} \partial_j E_j(y)
\]  
(104)
and,
\[
a_{1i}^2(y) \equiv \theta(y - \epsilon + \xi) - \theta(\xi - \epsilon - y) - \theta(\eta - \epsilon - y) + \theta(\eta + \epsilon + \eta)
\]  
(105)
\[
a_{1i}^2(y) \equiv G^{(2)}(y - \epsilon + \xi) - G^{(2)}(\xi - \epsilon - y) - G^{(2)}(\eta - \epsilon - y) + G^{(2)}(\eta + \epsilon + \eta)
\]  
The \(c\) - number phase \(R_2\) is given by,
\[
R_2(\epsilon, \xi, \eta) = \theta(\xi - 2\epsilon) - \theta(\xi + 2\epsilon) - \theta(\eta - 2\epsilon) + \theta(\eta + 2\epsilon)
\]  
(106)
\[
R_2(\epsilon, \eta, \xi) = 0 ; \quad \epsilon_{ij} \epsilon_{i\xi} > 0, \quad \epsilon_{ij} \epsilon_{i\xi} > 0
\]  
(107)
\[
R_2(\epsilon, \eta, \xi) = -2\pi ; \quad \epsilon_{ij} \epsilon_{i\eta} > 0, \quad \epsilon_{ij} \epsilon_{i\xi} < 0
\]  
\[
R_2(\epsilon, \eta, \xi) = 0 ; \quad \epsilon_{ij} \epsilon_{i\eta} < 0, \quad \epsilon_{ij} \epsilon_{i\xi} < 0
\]  
\[
R_2(\epsilon, \eta, \xi) = 2\pi ; \quad \epsilon_{ij} \epsilon_{i\eta} < 0, \quad \epsilon_{ij} \epsilon_{i\xi} > 0
\]  
The Taylor series of \(F_2\) starts with a term of order zero,
\[
-\frac{1}{e} \int d^2y [\theta(y) - \theta(-y)] \partial_i E_i(y)
\]  
(108)
Again, using the Baker - Campbell - Hausdorff formula, we obtain,
\[
e^{iF_2} = -e^{iF_2} + \frac{1}{e} \int d^2y [\theta(y) - \theta(-y)] \partial_i E_i(y) e^{-\frac{1}{2} \int d^2y [\theta(y) - \theta(-y)] \partial_i E_i(y)}
\]  
(109)
The function \(\theta(y) - \theta(-y)\) is equal either to \(\pi\) or to \(-\pi\). The last operatorial factor in eq.(109) can therefore be written as,
\[
exp\left[-\frac{1}{e} \int d^2y [\theta(y) - \theta(-y)] \partial_i E_i(y)\right] = \exp[-i\pi(Q_+ - Q_-)]
\]  
(110)
where \(Q_+\) is the total electric charge in the regions of space where \(\theta(y) - \theta(-y) = \pi\), and \(Q_-\) the electric charge in the regions where \(\theta(y) - \theta(-y) = -\pi\). Multiplied by the factor \(\exp\{-i\pi Q\}\) in eqs. (102-103), the operatorial factor becomes,
\[
exp\{-i2\pi Q_+\}
\]  
(111)
Since the electric charge is quantized in sets which are sufficiently regular, so that their Newtonian capacity can be defined, this operator is, in fact, the identity. Note that the quantization of charge is crucial for the locality of \( J_i \), since otherwise the exponential in eq. (111) would be a complicated nonlocal operator. Now we can perform a straightforward Taylor expansion of the operatorial phase in eqs. (102-103), and average over the directions of the regulators. The nonvanishing contributions are,

\[
J_- = -k^2 \Lambda^2 \langle \epsilon_i > e A_i - \frac{4 e^3}{3} \langle \epsilon_i \epsilon_j \epsilon_k > A_i A_j A_k \rangle
\]

\[-4 \pi^2 k^2 \Lambda^2 \left[ \frac{1}{e} < \epsilon_i (\xi_j + \eta_j) (\xi_k + \eta_k) > A_i \tilde{E}_j \tilde{E}_k + \frac{i}{e^2} < \epsilon_i (\xi_j - \eta_j) (\xi_k + \eta_k) > \partial_i \tilde{E}_j \tilde{E}_k \right] \]

where the averages are now defined as,

\[
< \alpha > \equiv \frac{i}{8 \pi^3} \int d\hat{\epsilon} d\hat{\eta} d\hat{\xi} \hat{\alpha} e^{-i [\theta(\eta) + \theta(\xi) + \theta(\epsilon)]} \text{Re} [R_2(\epsilon, \xi, \eta)] \]

Calculation of \( J_+ \) proceeds in an analogous fashion. The expression for the current obtained in this way is,

\[
J_i \equiv \bar{\psi} \gamma_i \psi = eA k^2 \Lambda^2 |\epsilon| A_i + 4 \pi \Lambda^2 |\epsilon|^3 \left[ B (\partial_j \partial_j A_i + 2 \partial_i \partial_j A_j) + C e^2 A_i A^2 \right] + \frac{k^2}{e} \Lambda^2 |\epsilon| |\eta|^2 \left[ D \tilde{E}_i (A \tilde{E}) + F E^2 A_i \right]
\]

The coefficients \( A, B, C, D \) and \( F \) can be calculated explicitly. We do not give their values, however, since they are regularization - dependent. The final expression for \( J_i \) is determined in Sections 3 and 4 from the requirement of Lorentz invariance and tree - level current algebra.

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