Generalized Isotropic Berwald Manifolds

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Abstract

In this paper, we construct a new class of Finsler manifolds called generalized isotropic Berwald manifolds which is an extension of the class of isotropic Berwald manifolds. We prove that every generalized isotropic Berwald manifold is a generalized Douglas-Weyl manifold. On a compact generalized isotropic Berwald manifold, we show that the notions of stretch and Landsberg curvatures are equivalent. Then we prove that on these manifolds, a Finsler metric is R-quadratic if and only if it is a stretch metric with vanishing $\bar{E}$-curvature. Finally, we determine the flag curvature of generalized isotropic Berwald manifold with scalar flag curvature.

Keywords: Generalized Douglas-Weyl metric, Berwald metric.

1 Introduction

In Finsler geometry, there are several important non-Riemannian quantities. Let $(M, F)$ be a Finsler manifold. The second derivatives of $\frac{1}{2}F^2$ at $y \in T_xM_0$ is an inner product $g_y$ on $T_x M$. The third order derivatives of $\frac{1}{2}F^2$ at $y \in T_xM_0$ is a symmetric trilinear forms $C_y$ on $T_x M$. We call $g_y$ and $C_y$ the fundamental form and the Cartan torsion, respectively. The rate of change of the Cartan torsion along geodesics is the Landsberg curvature $L_y$ on $T_x M$ for any $y \in T_xM_0$. $F$ is said to be Landsbergian if $L = 0$.

For a Finsler metric $F = F(x, y)$, its geodesics curves are characterized by the system of differential equations $\ddot{c} + 2G^i(\dot{c}) = 0$, where the local functions $G^i = G^i(x, y)$ are called the spray coefficients and given by following

$$G^i = \frac{1}{4} g^{i j} \left\{ \frac{\partial^2[F^2]}{\partial x^k \partial y^j} y^k - \frac{\partial[F^2]}{\partial x^i} \right\}, \quad y \in T_x M.$$

$F$ is called a Berwald metric if $G^i = \frac{1}{2} \Gamma^i_{jk}(x) y^j y^k$ is quadratic in $y \in T_x M$ for any $x \in M$. In [8], it is proved that on a Berwald space, the parallel translation along any geodesic preserves the Minkowski functionals. Then Berwald spaces can be viewed as Finsler spaces modeled on a single Minkowski space.

Recently the various interesting special forms of Cartan, Berwald and Landsberg tensors have been obtained by some Finslerians. The Finsler spaces having such special forms have been called C-reducible, P-reducible, semi-C-reducible, isotropic Berwald curvature, isotropic mean Berwald curvature, and isotropic Landsberg curvature, etc [7][8][15][16][21][24].

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In [7], Shen-Chen by using the structure of Funk metric, introduce the notion of isotropic Berwald metrics. This motivates us to study special forms of Berwald curvature for other important special Finsler metrics.

Let \((M, F)\) be a two-dimensional Finsler manifold. We refer to the Berwald’s frame \((\ell^i, m^i)\) where \(\ell^i = y^i/F(y)\), \(m^i\) is the unit vector with \(\ell_i m^i = 0\) and \(\ell_i = g_{ij} \ell^j\). Then the Berwald curvature is given by

\[
B^i_{jkl} = F^{-1}(-2I_1 \ell^i + I_2 m^i) m_j m_k m_l,
\]

where \(I\) is 0-homogeneous function called the main scalar of \(F\) and \(I_2 = I_{12} + I_{112}\) (see page 689 in [1]). Since the Cartan tensor of \(F\) is given by \(C_{ijk} = F^{-1} I m_i m_j m_k\), then the Berwald curvature can be written as following

\[
B^i_{jkl} = -\frac{2I_1}{I} C_{jkl} \ell^i + \frac{I_2}{3F} \{h_{jk} h^i_l + h_{kl} h^i_j + h_{ij} h^i_k\}, \tag{1}
\]

where \(h_{ij} := m_i m_j\) is the angular metric.

Let \((M, F)\) be a Finsler manifold. Then \(F\) is said to be generalized isotropic Berwald metric if its Berwald curvature satisfies following

\[
B^i_{jkl} = \mu C_{jkl} \ell^i + \lambda (h_{ik}h_{jl} + h_{ij}h_{kl} + h_{kl}h_{ij}), \tag{2}
\]

where \(\mu = \mu(x, y)\) and \(\lambda = \lambda(x, y)\) are homogeneous functions of degrees 0 and -1 with respect to \(y\), respectively. Then \((M, F)\) is called a generalized isotropic Berwald manifold. It is remarkable that, if \(\mu = 2c\) and \(\lambda = cF^{-1}\), where \(c = c(x)\) is a scalar function on \(M\), then \(F\) reduces to a isotropic Berwald metric [22].

Then the class of generalized isotropic Berwald manifolds contains the class of isotropic Berwald manifolds, as a special case. By [2], it results that every Finsler surface has generalized isotropic Berwald curvature with \(\mu = -\frac{2I_1}{I}\) and \(\lambda = \frac{1}{2F}\).

**Example 1.** Consider the Funk metric on the unit ball \(B^n \subset \mathbb{R}^n\) defined by

\[
F(x, y) := \sqrt{|y|^2 - \langle |x|^2|y|^2 - \langle x, y \rangle^2\rangle + \langle x, y \rangle} \bigg/ \sqrt{1 - |x|^2}, \quad y \in T_x B^n = \mathbb{R}^n
\]

where \(||\) and \(<,>\) denote the Euclidean norm and inner product in \(\mathbb{R}^n\), respectively. \(F\) is a generalized isotropic Berwald metric with \(\mu = 1\) and \(\lambda = \frac{1}{2F}\).

The Douglas tensor is another non-Riemanian curvature which defined by

\[
D^i_{jkl} := \left(G^i - \frac{1}{n+1} \frac{\partial G^m}{\partial y^m} y^i y^m y^j y^k\right).
\]

For more details see [2][14]. A Finsler metric is called a generalized Douglas-Weyl (GDW) metric if the Douglas tensor satisfy in \(h^i_m D^i_{jkl} y^m = 0\) [12]. In [4], Bacsó-Papp show that this class of Finsler metrics is closed under projective transformation. In this paper, we prove the following.

**Theorem 1.1.** Every generalized isotropic Berwald metric is a generalized Douglas-Weyl metric.
As a generalization of Landsberg curvature, L. Berwald introduced a non-Riemannian curvature so-called stretch curvature and denoted by $\Sigma_y$ [6]. He showed that this tensor vanishes if and only if the length of a vector remains unchanged under the parallel displacement along an infinitesimal parallelogram. Therefore, we study complete generalized isotropic Berwald manifold with vanishing stretch curvature and prove the following.

**Theorem 1.2.** Let $(M, F)$ be a complete generalized isotropic Berwald manifold and $\mu$ be bounded function on $M$. Suppose that $F$ has vanishing stretch curvature. Then $F$ is a Landsberg metric.

The second variation of geodesics gives rise to a family of linear maps $R_y : T_xM \rightarrow T_xM$, at any point $y \in T_xM$, which is called the Riemann curvature in the direction $y$. A Finsler metric $F$ is said to be R-quadratic if the Riemannian curvature $R_y$ is quadratic in $y \in T_xM$ at each point $x \in M$. In the sense of Bácsó-Matsumoto, $F$ is R-quadratic if and only if the h-curvature of Berwald connection depends on position only [3][19][24]. Every Berwald metric and R-flat metric is R-quadratic metric. On the other hand, in [17] Shen find a new non-Riemannian quantity for Finsler metrics that is closely related to the $E$-curvature and call it $\bar{E}$-curvature. Recall that, the $E$-curvature is obtained from the mean Berwald curvature $E$, by the horizontal covariant differentiation along geodesics. In this paper, we study generalized isotropic Berwald manifolds with R-quadratic metrics and prove the following.

**Theorem 1.3.** Let $(M, F)$ be a generalized isotropic Berwald manifold of dimension $n > 2$. Then $F$ is R-quadratic if and only if it is a stretch metric with $\bar{E} = 0$.

For a Finsler manifold $(M, F)$, the flag curvature is a function $K(P, y)$ of tangent planes $P \subset T_xM$ and directions $y \in P$. Indeed the flag curvature in Finsler geometry is a natural extension of the sectional curvature in Riemannian geometry. $F$ is said to be of scalar flag curvature if the flag curvature $K(P, y) = K(x, y)$ is independent of flags $P$ associated with any fixed flagpole $y$. One of the important problems in Finsler geometry is to characterize Finsler manifolds of scalar flag curvature [14]. In this paper, we study generalized isotropic Berwald metrics of scalar curvature and partially determine the flag curvature. More precisely, we prove the following.

**Theorem 1.4.** Let $(M, F)$ be an $n$-dimensional generalized isotropic Berwald manifold of scalar flag curvature. Then the flag curvature $K = K(x, y)$ satisfies

$$\frac{n+1}{3} K_{y^k} + \left( K + \frac{\mu^2}{4} - \frac{\mu'}{2F} \right) I_k = 0,$$

where $\mu' := \mu_s y^s$ denotes the horizontal derivation of $\mu$ with respect to the Berwald connection.

There are many connections in Finsler geometry [13][20]. In this paper, we use the Berwald connection and the $h$- and $v$- covariant derivatives of a Finsler tensor field are denoted by “$|$” and “,” respectively.
2 Preliminaries

Let $M$ be a $n$-dimensional $C^\infty$ manifold. Denote by $T_x M$ the tangent space at $x \in M$, by $TM = \bigcup_{x \in M} T_x M$ the tangent bundle of $M$, and by $TM_0 = TM \setminus \{0\}$ the slit tangent bundle on $M$. A Finsler metric on $M$ is a function $F : TM \to [0, \infty)$ which has the following properties:

(i) $F$ is $C^\infty$ on $TM_0$;
(ii) $F$ is positively 1-homogeneous on the fibers of tangent bundle $TM$;
(iii) for each $F$, by $L = \mathcal{L}$ defined on $T_x M$ at $x \in M$, let $\mathcal{L}$ be a n-dimensional manifold. Given a Finsler manifold $(M, F)$, then a global vector field $G$ is induced by $F$ on $TM_0$, which in a standard coordinate $(x^i, y^j)$ for $TM_0$ is given by $G = y^j \frac{\partial}{\partial x^i} - 2F^j(x, y) \frac{\partial}{\partial y^i}$, where

$$ G^i := \frac{1}{4} y^j \left( [F^2]_{x^i} y^j - [F^2]_{x^j} y^i \right), \quad y \in T_x M. $$

The $G$ is called the spray associated to $(M, F)$. In local coordinates, a curve $c(t)$ is a geodesic if and only if its coordinates $(c^i(t))$ satisfy $\ddot{c}^i + 2G^i(\dot{c}) = 0$. 

For a tangent vector $y \in T_x M$, define $B_y : T_x M \otimes T_x M \otimes T_x M \to T_x M$ and $E_y : T_x M \otimes T_x M \to \mathbb{R}$ by $B_y(u, v, w) := B_{ijkl}(y)uwv^l \partial^i \partial j \partial k |_x$ and $E_y(u, v) := E_{jk}(y)w^j v^k$ where

$$B_{ijkl} := \frac{\partial^i G^j}{\partial y^i \partial y^j \partial y^k}, \quad E_{jk} := \frac{1}{2} B_{jk}.$$

The $B$ and $E$ are called the Berwald curvature and mean Berwald curvature, respectively. Then $F$ is called a Berwald metric and weakly Berwald metric if $B = 0$ and $E = 0$, respectively [17].

Define $D_y : T_x M \otimes T_x M \otimes T_x M \to T_x M$ by $D_y(u, v, w) := D_{ijkl}(y)uwv^l \partial^i \partial j \partial k |_x$ where

$$D_{ijkl} := B_{ijkl} - \frac{2}{n+1} \{E_{ijk} \delta^l_j + E_{jkl} \delta^l_i + E_{ijl} \delta^l_k \}.$$

We call $D := \{D_y\}_{y \in TM}$ the Douglas curvature. A Finsler metric with $D = 0$ is called a Douglas metric. The notion of Douglas metrics was proposed by Bácsó-Matsumoto as a generalization of Berwald metrics [2].

A Finsler metric is called a generalized Douglas-Weyl (GDW) metric if the Douglas tensor satisfy in

$$h^i_{\alpha} D^\alpha_{ijkl |\alpha} y^m = 0$$

In [3], Bácsó-Papp show that this class of Finsler metrics is closed under projective transformation. In [12], Najafi-Shen-Tayebi find the necessary and sufficient condition for a Randers metric to be a generalized Douglas-Weyl metric.

The Riemann curvature $R_y = R^i_{jk} dx^k \otimes \frac{\partial}{\partial x^i} |_x : T_x M \to T_x M$ is a family of linear maps on tangent spaces, defined by

$$R^i_{jk} = 2 \frac{\partial G^i}{\partial x^k} - \frac{\partial^2 G^i}{\partial x^j \partial y^k} + 2 G^i \frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^j} \frac{\partial G^i}{\partial y^k}.$$

The flag curvature in Finsler geometry is a natural extension of the sectional curvature in Riemannian geometry was first introduced by L. Berwald [5]. For a flag $P = \text{span}\{y, u\} \subset T_x M$ with flagpole $y$, the flag curvature $K = K(P, y)$ is defined by

$$K(P, y) := \frac{g_y(u, R_y(u))}{g_y(y, y)g_y(u, u) - g_y(y, u)^2}.$$  \hspace{1cm} (4)

When $F$ is Riemannian, $K = K(P)$ is independent of $y \in P$, and is the sectional curvature of $P$. We say that a Finsler metric $F$ is of scalar curvature if for any $y \in T_x M$, the flag curvature $K = K(x, y)$ is a scalar function on the slit tangent bundle $TM_0$. If $K$ is constant, then $F$ is said to be of constant flag curvature. A Finsler metric $F$ is called isotropic flag curvature, if $K = K(x)$.

A Finsler metric $F$ is said to be $R$-quadratic if $R_y$ is quadratic in $y \in T_x M$ at each point $x \in M$. Let

$$R^i_{jk}(x, y) := \frac{1}{3} \frac{\partial}{\partial y^i} \{ \frac{\partial R^i_{jk}}{\partial y^j} - \frac{\partial R^i_{kl}}{\partial y^k} \},$$

where $R^i_{jk}$ is the Riemann curvature of Berwald connection. Then we have $R^i_{jk} = R^i_{jk}(x, y) y^j y^k$. Therefore $R^i_{jk}$ is quadratic in $y \in T_x M$ if and only if $R^i_{jk}$ are functions of position alone. Indeed a Finsler metric is $R$-quadratic if and only if the $h$-curvature of Berwald connection depends on position only in the sense of Bácsó-Matsumoto [3].
3 Proof of Theorem 1.1

In this section, we are going to prove the Theorem 1.1. We need the following.

**Lemma 3.1.** ([12]) Let \((M, F)\) be a Finsler metric. Then \(F\) is a GDW-metric if and only if
\[
D_{ijkl}^i y^k = T_{ijkl} y^i,
\]
for some tensor \(T_{ijkl}\) on manifold \(M\).

**Proposition 3.1.** Let \(F\) be a non-Riemannian generalized isotropic Berwald metric. Then \(F\) is a Douglas metric if and only if it is a relatively isotropic Landsberg metric \(L + F^2 \lambda C = 0\).

**Proof.** By assumption, we have
\[
B_{ijkl}^i = \mu C_{ijkl}^i + \lambda (h_{ik}^j h_{jl} + h_{il}^j h_{jk}),
\]
Taking a trace of (6) yields
\[
E_{jk} = \frac{n}{2} + 1 \lambda h_{jk}.
\]
Thus
\[
B_{ijkl}^i = \mu C_{ijkl}^i + \frac{2}{n+1} (E_{jk} h_{ik}^j + E_{kl} h_{ij}^i + E_{lj} h_{ik}^j).
\]
Contracting (8) with \(y_i\) implies that
\[
\mu C_{ijkl} = -2F^{-1}L_{ijkl}.
\]
By taking (9) in (8) it follows that
\[
B_{ijkl}^i = -2F^{-1}L_{ijkl} + \frac{2}{n+1} (E_{jk} h_{ik}^j + E_{kl} h_{ij}^i + E_{lj} h_{ik}^j).
\]
On the other hand, we have
\[
h_{i,j,k} = 2C_{ijk} - F^{-2} (y_j h_{ik} + y_i h_{jk}),
\]
which implies that
\[
2E_{jk} = (n+1) \lambda h_{jk} + (n+1) \lambda \left\{ 2C_{ijkl} - F^{-2} (y_k h_{jl} + y_j h_{kl}) \right\}.
\]
The Douglas tensor is given by
\[
D_{ijkl}^i = B_{ijkl}^i - \frac{2}{n+1} \left\{ E_{jk} \delta_{il}^i + E_{kl} \delta_{ij}^i + E_{lj} \delta_{ik}^i + E_{il} \delta_{jk}^i \right\}.
\]
Putting (7), (10) and (12) in (13) yields
\[
D_{ijkl}^i = -2\left\{ F^{-2} L_{ijkl} + \lambda C_{ijkl} \right\} y^i - (\lambda y_i F^{-2} + \lambda y_i) h_{ijkl} y^i.
\]
For the Douglas curvature, we have \(D_{ijkl}^i = D_{ijlk}^i\). Then by (14), we have
\[
\lambda y_i F^{-2} + \lambda y_i = 0.
\]
From (13) and (15) we deduce that
\[
D_{ijkl}^i = -2\left\{ F^{-2} L_{ijkl} + \lambda C_{ijkl} \right\} y^i.
\]
By (13), it follows that \(F\) is a Douglas metric if and only if \(F^{-2} L_{ijkl} + \lambda C_{ijkl} = 0\). This completes the proof. \(\square\)
Corollary 3.1. Let \((M, F)\) be a non-Riemannian Finsler surface. Then \(F\) is a Douglas metric if and only if \(3I_1 + F I_2 = 0\).

Proof. As we explain in introduction, every two-dimensional Finsler manifolds are generalized isotropic Berwald manifolds. By \((9)\) and \((16)\) we get
\[
D^i_{jkl} = \{F^{-1} \mu - 2 \lambda\} C_{jkl} y^i.
\] (17)
Thus \(F\) is a Douglas metric if and only if \(\mu = 2 \lambda\). Since \(\mu = -\frac{3I_1}{F}\) and \(\lambda = \frac{1}{F^2}\), then we get the proof.

By \((9)\), we have the following.

Corollary 3.2. Let \((M, F)\) be a Finsler surface. Then \(F\) is a weakly Berwald metric if and only if \(I_2 = 0\).

Proof of Theorem 1.1: The Douglas tensor of \(F\) is given by
\[
D^i_{jkl} = -2\{F^{-2} L_{jkl} + \lambda C_{jkl}\} y^i.
\] (18)
Taking a horizontal derivation of \((18)\) implies that
\[
D^i_{jkl|s} y^s = -2\{F^{-2} L_{jkl|s} y^s + \lambda' C_{jkl} + \lambda L_{jkl}\} y^i.
\] (19)
where \(\lambda' = \lambda_{|m} y^n\). By Lemma 3.1, \(F\) is a GDW-metric with
\[
T_{jkl} = -2\{F^{-2} L_{jkl|s} y^s + \lambda' C_{jkl} + \lambda L_{jkl}\}.
\] (20)
This completes the proof.

4 Proof of Theorem 1.2

In this section, we study complete generalized isotropic Berwald manifold with vanishing stretch curvature.

Proof of Theorem 1.2: By definition
\[
\Sigma_{ijkl} := 2(L_{ijkl|l} - L_{ijjl|k}) = 0.
\] (21)
Contracting \((2)\) with \(y_i\) and using
\[
y_i B^i_{jkl} = -2L_{jkl}
\]
implies that
\[
L_{ijk} = -\frac{1}{2} \mu F C_{ijk}.
\] (22)
By \((21)\) and \((22)\), we get
\[
\mu_{jk} C_{ijl} - \mu_{lj} C_{ijk} = \mu(C_{ijkl|l} - C_{ijlj|k}).
\] (23)
Contracting \((23)\) with \(y^k\) and using \((22)\) yields
\[
(\mu' - \frac{1}{2}\mu^2 F)C_{ijk} = 0. \tag{24}
\]
If \(C_{ijk} = 0\), then \(F\) is a Riemannian metric, and thus it is a Landsberg metric. Suppose that \(F\) is a non-Riemannian metric. Then we have
\[
2\mu' = \mu^2 F. \tag{25}
\]
On a Finslerian geodesics, we have
\[
\mu' = \frac{d\mu}{dt}. \tag{26}
\]
Thus
\[
2\frac{d\mu}{dt} = \mu^2, \tag{27}
\]
which its general solution is
\[
\mu(t) = \frac{2\mu(0)}{2 - t\mu(0)}. \tag{28}
\]
If \(\mu(0) = 0\), then \(\mu(t) = 0\) and by \((22)\), we conclude that \(F\) is a Landsberg metric. Suppose that \(\mu(0) \neq 0\). Using \(||\mu|| < \infty\), and letting \(t \to +\infty\) or \(t \to -\infty\), implies that \(\mu = 0\). This complete the proof.

5 Proof of Theorem 1.3

To prove Theorem 1.3 we need the following.

Lemma 5.1. (8) For the Berwald connection, the following Bianchi identities hold:
\[
R^i_{jkl|m} + R^i_{jlm|k} + R^i_{jmk|l} = B^i_{jku}R^u_{lm} + B^i_{jlu}R^u_{mk} + B^i_{jmu}R^u_{kl}, \tag{29}
\]
\[
B^i_{jml|k} - B^i_{jkm|l} = R^i_{jkl,m} \tag{30}
\]
\[
B^i_{jkl,m} = B^i_{jkm,l} \tag{31}
\]
where \(R^i_{kl} := \ell^j R^i_{jkl}\).

Taking a trace of \((30)\) implies the following.

Lemma 5.2. (10) Let \(F\) be a R-quadratic Finsler metric. Then \(H = 0\).

Contracting \((30)\) with \(y_i\) yields
\[
y_iR^i_{jkl,m} = y_i B^i_{jml|k} - y_i B^i_{jkm|l} = (y_i B^i_{jml})_k - (y_i B^i_{jkm})_l = -2L_{jml|k} + 2L_{jkm|l} = \Sigma_{jkm\ell}. \tag{32}
\]
Thus we conclude the following.
Corollary 5.1. Every R-quadratic Finsler metric is a stretch metric.

Proof of Theorem 1.3. Let \((M,F)\) be a generalized isotropic Berwald manifold. Suppose that \(F\) is R-quadratic metric. By Corollary 5.1 it is sufficient to prove that \(\tilde{E} = 0\). By assumption, we have

\[
B^i_{jkl} = -2F^{-1}L_{jkl}\ell^i + \frac{2}{n+1}(E_{ijk}h^i_j + E_{kjl}h^i_j + E_{jkl}h^i_j).
\]

Then

\[
B^i_{jkl[s]} = -2F^{-1}L_{jkl[s]\ell^i + \frac{2}{n+1}(E_{jkl[s]h^i_j + E_{kjl[s]h^i_j + E_{jkl[s]h^i_j}).
\]

Replacing \(l\) and \(s\) in (39) yields

\[
B^i_{jks[l]} = -2F^{-1}L_{jks[l]\ell^i + \frac{2}{n+1}(E_{jks[l]h^i_j + E_{ks[l]h^i_j + E_{jks[l]h^i_j}).
\]

implies that

\[
B^i_{jkl[s]} - B^i_{jks[l]} = -2F^{-1}\left(L_{jkl[s] - L_{jks[l]\ell^i + \frac{2}{n+1}(E_{jkl[s]h^i_j - E_{jks[l]h^i_j}
\]

By (37) and Corollary 5.1 we have

\[
E_{jkl[s]h^i_j = (E_{jls} - E_{jsl})h^i_k + (E_{kl[s] - E_{ks[l]})h^i_j.
\]

Putting \(i = s\) in (37) and using \(h^i_s = n - 1\) and \(h^i_k = \delta^i_k - F^{-2}y^i_y\), we get

\[
(n - 2)E_{jkl} + F^{-2}H_{jkl} = (E_{jlk} + F^{-2}H_{jlk} - E_{jlk}) + (E_{klj} - F^{-2}H_{klj} - E_{klj}),
\]

or equivalently

\[
nE_{jkl} = E_{jlk} + E_{klj} - F^{-2}(H_{jlk} - H_{klj} + H_{jlk}).
\]

By Lemma 5.2, (38) reduces to following

\[
nE_{jkl} = E_{jlk} + E_{klj}.
\]

Permuting \(j, k, l\) in (39) leads to

\[
nE_{jkl} = E_{kjl} + E_{jkl} \quad \text{(40)}
\]

\[
nE_{kjl} = E_{jkl} + E_{jkl} \quad \text{(41)}
\]

implies that

\[
n(E_{jkl} + E_{klj}) = (n + 2)E_{jkl}.
\]

Putting (39) in (42) implies that

\[
E_{kjl} = E_{jkl}.
\]

This means that \(E_{jkl}\) is symmetric with respect to indices and (39) reduces to

\[
(n - 2)E_{jkl} = 0.
\]

Since \(n > 2\), thus \(\tilde{E} = 0\).

Conversely, let \(F\) be a stretch metric with \(\tilde{E} = 0\). Then by (30) and (35), we conclude that \(F\) is R-quadratic. This completes the proof. \(\square\)


6 Proof of Theorem 1.4

The following equation is hold

\[ L_{ijk|m}y^m + C_{ijk}R^m_k = -\frac{1}{3}(g_{im}R^m_{k,j} + g_{jm}R^m_{k,i}) - \frac{1}{6}(g_{im}R^m_{j,k} + g_{jm}R^m_{i,k}). \] (44)

For more details, see [11]. Contracting (44) with \( g^{ij} \) implies that

\[ J_{k|m}y^m + I_{m}R^m_k = -\frac{1}{3}\{2R^m_{k,m} + R^m_{m,k}\}. \] (45)

Proof of Theorem 1.4: Now we assume that \( F \) is of scalar curvature with flag curvature \( K = K(x, y) \). This is equivalent to the following identity:

\[ R_{i}^k = K F^2 h^i_k, \] (46)

where \( h^i_k := g^{ij}h_{jk} \). By (44), (45) and (46), we obtain

\[ L_{ijk|m}y^m = -\frac{1}{3} F^2 \{ K, h_{jk} + K, h_{ik} + K, h_{ij} + 3KC_{ijk} \} \]

and

\[ J_{k|m}y^m = -\frac{1}{3} F^2 \{ (n + 1)K, + 3KJ_k \}. \] (47)

By assumption, we have

\[ L_{ijkl} = -\frac{1}{2}\mu FC_{ijkl}. \] (48)

This yields

\[ J_i = -\frac{1}{2}\mu FI_i. \]

Since \( J_k = I_{k|m}y^m \), thus

\[ J_{i|m}y^m = -\frac{\mu'}{2} FI_i - \frac{\mu'}{2} FJ_i = \frac{1}{4}(\mu^2F - 2\mu')FI_i. \] (49)

It follows from (47) that

\[ \frac{n + 1}{3}K, i = (\mu' - \frac{\mu^2}{4} - K)I_i. \] (50)

Then we have (3).

Corollary 6.1. Let \((M, F)\) be a generalized isotropic Berwald manifold of dimension \( n \geq 3 \). Suppose that \( F \) is of scalar flag curvature \( K = K(x, y) \) such that \( 2\mu' - \{\mu^2 + 4K\}F = 0 \). Then \( F \) is of constant flag curvature.

Proof. By (50), we get \( K, j = 0 \) and then \( K = K(x) \). In this case, \( K = \text{constant} \) when \( n \geq 3 \) by the Schur theorem [4].

Finally, by (50) we can conclude the following.

Corollary 6.2. Let \((M, F)\) be a generalized isotropic Berwald manifold of isotropic flag curvature \( K = K(x) \) satisfies \( 2\mu' - \{\mu^2 + 4K\}F \neq 0 \). Then \( F \) reduces to a Riemannian metric.
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