On Maximal Robust Positively Invariant Sets in Constrained Nonlinear Systems

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Abstract

We study the maximal robust positively invariant set for state-constrained continuous-time nonlinear systems subjected to a bounded disturbance. We show that this set is closed and that its boundary consists of two complementary parts, one of which we name the invariance barrier, which may be constructed using the maximum principle. We demonstrate on various examples that this set is nonconvex and nonsmooth in general, even for linear systems.

Key words: invariant sets, constraint satisfaction problems, nonlinear systems

1 Introduction

Set invariance is a fundamental concept in control theory due to its well-known relationship with stability, see for example [1, 2]. This paper focuses on the maximal robust positively invariant set (MRPI) of a continuous-time nonlinear system subjected to state constraints and a bounded disturbance term. Roughly speaking, a system’s MRPI is the largest set contained in the constrained state-space such that any system trajectory that initiates in this set remains in it for all future time, regardless of the disturbance realisation.

Robust invariant sets are useful as analysis tools in themselves, but have also been utilised in a number of theoretical investigations and applications: they have been shown to play an important role in some investigations of feasibility of robust predictive control schemes, see the works [3–6]; they play a central role in the design of some obstacle-avoiding path-planning methodologies, see [7–9]; they play a role in the design of reference governors, see [10, 11]; and they may act as the terminal constraint set to guarantee stability of MPC approaches, see for example [12, 13].

The majority of the literature that studies the MRPI focusses on constructing the set by utilising various algorithms that involve the iterative computation and intersection of one-step predecessor sets\footnote{Given a discrete-time system and a subset of the state-space, \( S \), by the one-step predecessor set we mean the set of all states such that for any disturbance input the subsequent state is contained in \( S \). These sets go under various names in the literature, and the reader is referred to the references mentioned in the paragraph.} of discrete-time systems. The computation of these predecessor sets is hard, in general, and so these algorithms are often only used in the setting of discrete-time linear systems subjected to polyhedral or ellipsoidal state and disturbance constraints. Under this setting under- or over-approximations of the MRPI may be effectively computed, and in some cases these algorithms may compute the set exactly. Works that study the MRPI along these lines include [3, 14–18]. We emphasise that in this work we study the MRPI in the context of constrained continuous-time nonlinear systems.

Other sets that are closely related to the MRPI, but should not be confused with it, include: minimal robust positively invariant sets, see for example [3, 19]; control invariant sets, closely related to viability kernels and admissible sets, see for example [2, 20–23]; capture basins, also called regions of attraction, see for example [24, 25]; and backwards reachable sets that appear in the context of differential games, see for example [26]. Each of these sets have a large body of literature, and we have only mentioned some of their important references. Moreover, researchers have considered many problems associated with robust invariant sets, including: the computation of...
robust invariant sets for nonlinear systems, see [27, 28], or the derivation of conditions under which a given set is robustly invariant, see [29]; and the computation of feedbacks along with their associated robust invariant sets for linear systems, see [30, 31]. We emphasise that the focus of the present paper is the study of the MRPI and that we do not cover any of the previously mentioned problems.

In this paper we adapt results from the recent theory of barriers in constrained nonlinear systems, see [22–36], where the focus is on characterising the admissible set, to the current setting of the MRPI. In our treatment we show that the results from the paper [22] adapt in an intuitive way, but that the proofs of the results are by no means easy adaptations.

Considering a constrained nonlinear system, under certain assumptions, we show that the MRPI is closed, and that its boundary consists of two complementary parts. One part is contained in the boundary of the constrained state-space. The other, which we call the invariance barrier, is made up of special trajectories of the system that, along with their associated disturbance realisations, satisfy the necessary conditions of the Pontryagin maximum principle. We show that if the MRPI is stationary, then these curves can be found through backwards integration from points that satisfy an ultimate tangentiality condition on the boundary of the constraint set. Through examples we show that, in general, the MRPI is nonconvex and nonsmooth, even for linear systems.

The outline of the paper is as follows. In Section 2 we present the constrained system under study, along with the assumptions we impose throughout the paper. In Section 3 we show that the MRPI is closed, and in Section 4 we characterise its boundary. Section 5 is dedicated to the ultimate tangentiality condition, which is satisfied at the intersection of the invariance barrier and the boundary of the constrained state-space. Section 6 presents our main result, which says that trajectories along the invariance barrier satisfy the maximum principle. We show a number of examples in Section 7, where we also discuss some interesting observations, and conclude the paper with Section 8.

## 2 Constrained System Formulation

We consider the following nonlinear system subjected to state and input constraints:

\[ \dot{x}(t) = f(x(t), d(t)), \]

\[ x(t_0) = x_0, \]

\[ d \in D, \]

\[ g_i(x(t)) \leq 0, \forall t \in [t_0, \infty], \ i = 1, 2, \ldots, p, \]

where \( x(t) \in \mathbb{R}^n \) is the state and \( d(t) \in \mathbb{R}^m \) is a disturbance input. We make the same assumptions as those made in [22]:

(A1) The space \( D \) is the set of all Lebesgue measurable functions that map the interval \([t_0, \infty]\) to a set \( D \subset \mathbb{R}^n \), which is compact and convex.

(A2) The function \( f \) is \( C^2 \) with respect to \( d \in D \), and for every \( d \) in an open subset containing \( D \), the function \( f \) is \( C^2 \) with respect to \( x \in \mathbb{R}^n \).

(A3) There exists a constant \( 0 < c < +\infty \) such that the following inequality holds true:

\[ \sup_{d \in D} |x^T f(x, d)| \leq c(1 + \|x\|^2), \quad \text{for all } x \in \mathbb{R}^n. \]

(A4) The set \( f(x, D) \equiv \{f(x, d) : d \in D\} \) is convex for all \( x \in \mathbb{R}^n \).

(A5) For every \( i = 1, 2, \ldots, p \), the function \( g_i \) is \( C^2 \) with respect to \( x \in \mathbb{R}^n \), and the set \( \{x : g_i(x) = 0\} \) defines a manifold.

The assumptions (A2) and (A3) are required to guarantee equicontinuity of a sequence of integral curves of the system restricted to a finite interval, and along with assumption (A4) are used to prove the important compactness result stated in Proposition 1.

By \( x^{(d,x_0,t_0)} \) we will refer to the solution of (1) with the initial condition \( x_0 \in \mathbb{R}^n \) at time \( t_0 \in \mathbb{R} \) and a disturbance realisation \( d \in D \). If the initial time is clear from context we will use the notation \( x^{(d,x_0)} \), and if, in addition, the initial condition is clear we will use the notation \( x^{(d)} \). We note that under (A2) above, this solution exists and is unique. By \( x^{(d,x_0,t_0)}(t), x^{(d,x_0)}(t) \) and \( x^{(d)}(t) \), with \( t \in [t_0, \infty] \), we will refer to the solution at time \( t \).

Given two disturbance realisations, \( d_1 \in D \) and \( d_2 \in D \), along with a time instant \( \tau \in [t_0, \infty] \), the concatenated disturbance given by \( d_3(t) = \begin{cases} d_1(t) \text{ for } t \in [t_0, \tau] \\ d_2(t) \text{ for } t \in [\tau, \infty] \end{cases} \) also satisfies \( d_3 \in D \). We denote this concatenation by \( d_3 = d_1 \mathbin{xor} d_2 \). Let \( g(x) = (g_1(x), g_2(x), \ldots, g_p(x))^T \).

By \( g(x) \neq 0 \) (resp. \( g(x) < 0 \)) we mean that \( g_i(x) \leq 0 \) (resp. \( g_i(x) < 0 \)) for all \( i = 1, 2, \ldots, p \). By \( g(x) \neq 0 \), we mean that \( g_i(x) = 0 \) for at least one \( i \). By \( \llbracket d \rrbracket \) we refer to the set \( \{i \in \{1, 2, \ldots, p\} : g_i(x) = 0\} \). We introduce the following sets in order to lighten our notation:

\[ G \triangleq \{x : g(x) \leq 0\}, \]

\[ G_\ominus \triangleq \{x : g(x) < 0\}, \]

\[ G_0 \triangleq \{x : g(x) = 0\}. \]

The notation \( L_f g(x, d) \) denotes the Lie derivative of a differentiable function \( g \) with respect to the vector field \( f(\cdot, d) \) at the point \( x \). If \( S \) is a set, then \( \text{int}(S) \) denotes its interior and \( \text{cl}(S) \) denotes its closure. If \( S_1 \) and \( S_2 \) are sets, then \( S_1 \subset S_2 \) indicates that \( S_1 \) is a subset of \( S_2 \), that is, if \( s_1 \in S_1 \), then \( s_1 \in S_2 \).
Theorem compactness result, for which a proof may be found in [22, Appx. A], will be used in the sequel.

**Proposition 1** Assume that (A1)-(A4) hold. Let \( \mathcal{X}(x_0) \) denote the set of all integral curves initiating from \( x_0 \in \mathbb{R}^n \), satisfying (1)-(3). Given a compact set \( X_0 \) of \( \mathbb{R}^n \), the set \( \mathcal{X} = \bigcup_{x_0 \in X_0} \mathcal{X}(x_0) \) is compact with respect to the topology of uniform convergence on \( C^0([0, T], \mathbb{R}^n) \) for all \( T \geq 0 \), where \( C^0([0, T], \mathbb{R}^n) \) denotes the set of continuous functions that map \([0, T]\) to \( \mathbb{R}^n \). In other words, from every sequence \( \{x^{(d_k, x_k)}\}_{k \in \mathbb{N}} \subset \mathcal{X} \) one can extract a uniformly convergent subsequence on every finite interval \([0, T]\), whose limit \( \xi \) is an absolutely continuous integral curve on \([0, \infty] \), belonging to \( \mathcal{X} \).

### 3 Closedness of the MRPI

In this section we show that the MPRI is closed. First we recall some notions from the literature on invariant sets, see for example [1-3].

**Definition 1** A set \( \Omega \subset \mathbb{R}^n \) is said to be a robust positively invariant set (RPI) of the system (1)-(3) provided that \( x^{(d_0, x_0, t_0)}(t) \in \Omega \) for all \( t \in [t_0, \infty] \), for all \( x_0 \in \Omega \) and for all \( d \in \mathcal{D} \).

**Definition 2** We denote by \( \mathcal{M} \) the maximal robust positively invariant set (MRPI) of the system (1)-(4) contained in \( G \). In other words, \( \mathcal{M} \) is the union of all robust positively invariant sets that are subsets of \( G \).

Next, we introduce an equivalent description of \( \mathcal{M} \) which will make it easier to study and construct.

**Proposition 2** An equivalent definition of \( \mathcal{M} \) for system (1)-(4) is given by:

\[
\mathcal{R} = \{ x_0 : x^{(d, x_0, t_0)}(t) \in G, \forall t \in [t_0, \infty], \forall d \in \mathcal{D} \}.
\]

In other words, \( \mathcal{M} = \mathcal{R} \).

This observation was made in [18], where it was noted that the result is not difficult to see. Nevertheless, we provide a proof for the current setting of continuous-time nonlinear systems.

**PROOF.** First, we argue by contradiction that the set \( \mathcal{R} \) is an RPI of system (1)-(3). To that end, suppose \( \mathcal{R} \) is not. Then there exists an \( x_1 \in \mathcal{R} \), a \( d_1 \in \mathcal{D} \), and a \( t_1 \in \{t_0, \infty\} \) such that \( x^{(d_1, x_1, t_0)}(t_1) \notin \mathcal{R} \). Let \( x_2 = x^{(d_1, x_1, t_0)}(t_1) \). Because \( x_2 \notin \mathcal{R} \), there exists \( d_2 \in \mathcal{D} \) and a \( t_2 \in \{t_1, \infty\} \) such that \( g_i(x^{(d_2, x_2, t_1)}(t_2)) > 0 \) for some \( i \in \{1, 2, \ldots, p\} \). Now consider the concatenated disturbance \( d_3 = d_1 \triangleright t_1 \times d_2 \). We have that \( g_i(x^{(d_3, x_1, t_0)}(t)) > 0 \) for some \( i \in \{1, 2, \ldots, p\} \), contradicting the fact that \( x_1 \in \mathcal{R} \). Therefore, \( \mathcal{R} \) is an RPI of system (1)-(3). Clearly, \( \mathcal{R} \subset G \), and along with the fact that \( R \) is an RPI, we conclude that \( \mathcal{R} \subset \mathcal{M} \). For any \( \hat{x} \in \mathcal{M} \) we have, by definition, \( x^{(d, \hat{x}, t_0)}(t) \in \mathcal{M} \subset G \) for all \( t \in [t_0, \infty] \), for all \( d \in \mathcal{D} \), implying that \( \hat{x} \in \mathcal{R} \), thus \( \mathcal{M} \subset \mathcal{R} \). We can thus conclude that \( \mathcal{M} = \mathcal{R} \).

**Proposition 3** Under the assumptions (A1)-(A5), the set \( \mathcal{M} \) is closed.

**PROOF.** The proof follows directly from the compactness result of Proposition 1, similar to its counterpart, [22, Prop. 4.1]. Consider an arbitrary disturbance realisation \( d \in \mathcal{D} \), along with any sequence of initial states \( \{x_k\}_{k \in \mathbb{N}} \), with \( x_k \in \mathcal{M} \) for all \( k \), converging to a point \( \hat{x} \in \mathbb{R}^n \). From the equivalent definition of \( \mathcal{M} \), as in (5), we have \( g_i(x^{(d, x_k, t_0)}(t)) \leq 0 \) for all \( t \in [t_0, \infty] \), for all \( k \in \mathbb{N} \). From Proposition 1 there exists a subsequence, which we denote by \( x^{(d, x_k, t_0)} \) (keeping the same index, \( k \)), that converges to an integral curve of the system (1)-(3), which we label \( x^{(d, \hat{x}, t_0)} \). From the continuity of \( g_i \), see (A5), we have \( g_i(x^{(d, \hat{x}, t_0)}(t)) \leq 0 \) for all \( t \in [t_0, \infty] \). Because our choice of \( d \) was arbitrary, we can conclude that \( g_i(x^{(d, \hat{x}, t_0)}(t)) \leq 0 \) for all \( t \in [t_0, \infty] \) for all \( d \in \mathcal{D} \). Thus \( \hat{x} \in \mathcal{M} \), concluding the proof.

### 4 The boundary of \( \mathcal{M} \)

Proposition 3 says that \( \mathcal{M} \) is closed. We let \( \partial \mathcal{M} \) denote its boundary, and introduce the following notation to refer to the two complementary parts of \( \partial \mathcal{M} \):

\[
[\partial \mathcal{M}]_0 \triangleq \partial \mathcal{M} \cap G_0, \\
[\partial \mathcal{M}]_- \triangleq \partial \mathcal{M} \cap G_-.
\]

**Proposition 4** The set \( [\partial \mathcal{M}]_0 \) is contained in the set of points \( z \in G_0 \) that satisfy \( \max_{d \in \mathcal{D}} \max_{t \in [z]} L_f g_i(z, d) \leq 0 \).  

**PROOF.** Consider any point \( z \in [\partial \mathcal{M}]_0 \) and assume that there exists a \( d \in \mathcal{D} \) such that \( \max_{d \in \mathcal{D}} \max_{t \in [z]} L_f g_i(z, d) > 0 \). Then there exists an open interval \( [t_0 - \epsilon, t_0 + \epsilon] \), with \( \epsilon > 0 \) and sufficiently small, such that the resulting integral curve \( x^{(d, z, t_0)} \) violates a constraint at some \( t \in [t_0 - \epsilon, t_0 + \epsilon] \). In other words, \( g_i(x^{(d, z, t_0)}(t)) > 0 \) for some \( i = 1, 2, \ldots, p \). But his would contradict the fact that \( z \in [\partial \mathcal{M}]_0 \subset \mathcal{M} \).

#### 4.1 The invariance barrier

We now turn our attention to the set \( [\partial \mathcal{M}]_- \), which we name the invariance barrier. But first, we need to introduce the property of stationarity, which will be important in the sequel.
Consider the following set, where $T < \infty$:

$$M_T = \{x_0 : x^{(d,x_0,t_0)}(t) \in G, \forall t \in [t_0, T], \forall d \in \mathcal{D}\}.$$  

If we refer to $M$ as defined in Proposition 2, then clearly $M \subset M_{T_1} \subset M_{T_2} \subset G$, for $0 \leq T_2 \leq T_1 < \infty$. Note that the sets $M_T$ are not necessarily robustly invariant. However, when this is the case, we have $M_{T} \subset M$, and thus $M = M_T$.\footnote{We note that a notion of “finite-time robust robustness” is not useful because it is redundant: if $\Omega_T$ is a set for which $x^{(d,x_0,t_0)}(t) \in \Omega_T$ for all $t \in [t_0, T]$, for all $x_0 \in \Omega_T$, for all $d \in \mathcal{D}$, then $\Omega_T$ is an RPI.}

**Definition 3** The set $M$ is said to be stationary with horizon $T < \infty$, provided that $M = M_T$.

In the context of discrete-time systems, a set satisfying this property is said to be finitely-determined, see [18].

**Proposition 5** Assume that (A1)-(A5) hold and that $M$ is stationary with horizon $T < \infty$. Consider a point $\bar{x} \in [\partial M]_\infty$. Then there exists a $\bar{d} \in \mathcal{D}$ such that the corresponding integral curve runs along $[\partial M]_\infty$ and intersects $G_0$ in finite time. In other words, there exists a $\bar{d} \in \mathcal{D}$ and a $\bar{t} \in [0, T]$ such that $x^{(\bar{d}, \bar{x}, t_0)}(t) \in [\partial M]_\infty$ for all $t \in [t_0, \bar{t}]$, and $x^{(\bar{d}, \bar{x}, t_0)}(\bar{t}) \in G_0$.

**PROOF.** Consider a sequence $\{x_k\}_{k \in \mathbb{N}}$, with $x_k \in M^C$ for all $k$, converging to a point $\bar{x} \in [\partial M]_\infty$. For every $x_k$ in this sequence, due to the stationarity of $M$ and the continuity of $g$, there exists a $k_d \in \mathcal{D}$, a $\bar{t}_k \in [0, T]$ and an $i_k \in \{1, 2, \ldots, p\}$ such that $g_{i_k}(x^{(\bar{d}, \bar{x}, t_0)}(t_k)) = 0$. We can select a subsequence from the sequence of curves $\{x^{(\bar{d}, \bar{x}, t_0)}\}$ such that the corresponding sequence of crossing-times, $\{t_k\}_{k \in \mathbb{N}}$ (where, by abuse of notation, we re-use $k$ to refer to this subsequence), is monotonically increasing. Note that the sequence $\{t_k\}$ is bounded above by some $\bar{t} \leq T$. Moreover, due to the compactness result from Proposition 1, we can select a further subsequence from $\{x^{(\bar{d}, \bar{x}, t_0)}\}$ that uniformly converges on the interval $[t_0, T]$ to an integral curve belonging to the system (1)-(3). Thus, we have constructed a sequence of integral curves, $\{x^{(\bar{d}, \bar{x}, t_0)}\}$ (where again we use the index $k$), that uniformly converges to the curve $x^{(\bar{d}, \bar{x}, t_0)}$ with $\bar{d} \in \mathcal{D}$, $\bar{x} \in [\partial M]_\infty$ and such that $g_{i_k}(x^{(\bar{d}, \bar{x}, t_0)}(t)) = 0$ for some $\bar{t} \leq T$ and some $i \in \{1, 2, \ldots, p\}$. The curve $x^{(\bar{d}, \bar{x}, t_0)}$ cannot intersect the interior of $M$ for any $t \in [t_0, \bar{t}]$, otherwise it will not be the uniform limit of a sequence of integral curves, entirely contained in $M^C$.

**Remark 1** If $M$ is not stationary then an integral curve initiating on $[\partial M]_\infty$ may remain in $[\partial M]_\infty$ for all future time. As a simple example, consider the one-dimensional system $\dot{x}(t) = -x(t) + d(t)$, with the constraints $x \in [-2, 2]$ and $|d(t)| \leq 1$. Then, clearly $M = [-1, 1]$, and for any $\bar{x} \in [\partial M]_\infty = \{-1\} \cup \{1\}$ there does not exist a disturbance realisation such that the resulting integral curve remains in $[\partial M]_\infty$ and eventually intersects $G_0$.

**Proposition 5 says that if $M$ is stationary, then** there exist integral curves of the system that run along its invariance barrier and eventually intersect $G_0$. The next proposition characterises this intersection: it says that it happens in a tangential manner.

**Proposition 6** Assume that (A1)-(A5) hold, and that $M$ is stationary with $T < \infty$. Consider a point $\bar{x} \in [\partial M]_\infty$ along with a disturbance realisation, $\bar{d} \in \mathcal{D}$, as in Proposition 5, such that the resulting integral curve runs along $[\partial M]_\infty$ and intersects $G_0$ at $t$. Let $\bar{z} \triangleq x^{(\bar{d}, \bar{x}, t_0)}(\bar{t}) \in G_0$. Then, the following holds:

$$\max_{d \in \mathcal{D}} \max_{i \in I(z)} L f g_i(z, d) = 0.$$  

The proof of Proposition 6 is much simpler than its admissible set counterpart, Proposition 6.1 of [22]. In particular, in the present context of invariant sets, we do not need to introduce a needle perturbation of the disturbance realisation $\bar{d}$ in order to prove the result.

**PROOF.** First we prove that $\bar{d}(\bar{t})$ maximises the maximum Lie derivative of all active constraints at the point $\bar{z}$. That is, we show:

$$\max_{i \in I(z)} L f g_i(z, \bar{d}(\bar{t})) = \max_{d \in \mathcal{D}} \max_{i \in I(z)} L f g_i(z, d). \quad (6)$$  

This fact will be important in the proof of our main result, Theorem 1.

Note that the mapping $t \mapsto g_i(x^{(\bar{d}, \bar{x}, t_0)}(t))$ is nondecreasing for all $i \in I(z)$ over an interval $[\bar{t} - \eta, \bar{t}]$ with $\eta > 0$ and sufficiently small, which implies that $\max_{i \in I(z)} L f g_i(z, \bar{d}(\bar{t})) > 0$. Suppose that there exists a $\bar{d} \in D$ such that $\max_{i \in I(z)} L f g_i(z, \bar{d}) > \max_{i \in I(z)} L f g_i(z, \bar{d}(\bar{t}))$. This would imply that $\max_{i \in I(z)} L f g_i(z, \bar{d}) > 0$, contradicting the fact that $z \in [\partial M]_\infty$, from Proposition 4. Thus, we have $\max_{i \in I(z)} L f g_i(z, d) \leq \max_{i \in I(z)} L f g_i(z, \bar{d}(\bar{t}))$ for all $d \in D$, which is the statement in (6). We have established that $0 \leq \max_{i \in I(z)} L f g_i(z, \bar{d}(\bar{t})) = \max_{i \in I(z)} L f g_i(z, \bar{d}(\bar{t}))$.  

\[ \]
max_{d \in D} \max_{i \in \{z\}} L_f g_i(z,d) \leq 0$, which completes the proof.

6 Construction of the invariance barrier

At this point we introduce the following set, see for example [37, Ch. 4], but in the context of our setting where the input is considered a disturbance:

**Definition 4** The reachable set at time $t \in \mathbb{R}$ from a point $\bar{x} \in \mathbb{R}^n$ is given by:

$$X_t(\bar{x}) \triangleq \{ x^{(d,\bar{x},t_0)}(t) : d \in D \}.$$

That is, it is the set of all states reachable at time $t$ as a consequence of an admissible disturbance realisation.

It is well-known, under the assumptions made in this work, that the reachable set is compact and continuously varying with time, see [37, Ch. 4].

**Proposition 7** Assume that (A1)-(A5) hold, and that $\mathcal{M}$ is stationary with $T < \infty$. Consider a point $\bar{x} \in [\partial \mathcal{M}]_-$ along with a disturbance realisation, $d \in D$, as in Proposition 5, such that the resulting integral curve runs along $[\partial \mathcal{M}]_-$ and intersects $G_0$ at $t \in [t_0, \infty)$. Let $z \triangleq x^{(d,\bar{x},t_0)}(\bar{t}) \in G_0$. Then, $x^{(d,\bar{x},t_0)}(t) \in \partial X_t(\bar{x})$ for all $t \in [t_0, \bar{t}]$.

PROOF. By definition, we have that $x^{(d,\bar{x},t_0)}(t) \in X_t(\bar{x})$ for all $t \in [t_0, \bar{t}]$. Moreover, because $x^{(d,\bar{x},t_0)}(t) \in [\partial \mathcal{M}]_-$ for all $t \in [t_0, \bar{t}]$, we have:

$$X_t(\bar{x}) \cap [\partial \mathcal{M}]_- \neq \emptyset \quad \forall t \in [t_0, \bar{t}]. \quad (7)$$

Suppose that there exists a $t_1 \in [t_0, \bar{t}]$ such that int$(X_{t_1}(\bar{x})) \cap [\partial \mathcal{M}]_- = \emptyset$. Then there would exist a point $x_1 \in [\partial \mathcal{M}]_-$ along with a neighbourhood of this point, labelled $\mathcal{N}(x_1)$, such that $\mathcal{N}(x_1) \subset$ int$(X_{t_1}(\bar{x}))$ and $\mathcal{N}(x_1) \cap M^C = \emptyset$. But this would imply that there exists a $d_1 \in D$ such that $x^{(d_1,\bar{x},t_0)}(t_1) \in M^C$, contradicting the fact that $\mathcal{M}$ is robustly positively invariant. We can conclude that int$(X_{t}(\bar{x})) \cap [\partial \mathcal{M}]_- = \emptyset$ for all $t \in [t_0, \bar{t}]$, and along with (7) conclude that $x^{(d,\bar{x},t_0)}(t) \in \partial X_t(\bar{x})$ for all $t \in [t_0, \bar{t}]$.

We have gathered enough facts about the MRPI to present our main result.

**Theorem 1** Assume that (A1)-(A5) hold, and that $\mathcal{M}$ is stationary with $T < \infty$. Every integral curve $x^{(d)}$ on $[\partial \mathcal{M}]_- \cap \text{cl}(\text{int}(\mathcal{M}))$ and the corresponding disturbance realisation, $d \in D$, as in Proposition 5, satisfy the following necessary conditions.

There exists a nonzero absolutely continuous maximal solution $\lambda^d$ to the adjoint equation:

$$\dot{\lambda}^d(t) = - \left( \frac{\partial f}{\partial x}(x^{(d)}(t), d(t)) \right)^T \lambda^d(t),$$

$$\lambda^d(\bar{t}) = (Dg_i(z))T,$$

(8)

where $\bar{t}$ denotes the time at which $x^{(d)}$ intersects $G_0$, $z \triangleq x^{(d)}(\bar{t})$, and $Dg_i(z) \triangleq \max_{i \in \{z\}} L_f g_i(z,d(\bar{t}))$, such that

$$\max_{d \in D} \{ \lambda^d(t)^T f(x^{(d)}(t), d) \} = \lambda^d(t)^T f(x^{(d)}(t), d(t)) = 0,$$

for almost every $t \in [t_0, \bar{t}]$. Moreover, at $\bar{t}$ the ultimate tangentiality condition holds:

$$\max_{d \in D} \max_{i \in \{z\}} L_f g_i(z, d) = \max_{i \in \{z\}} L_f g_i(z, d(\bar{t})) = L_f g_i(z, d(\bar{t})) = 0. \quad (10)$$

PROOF. By Proposition 7 the integral curve running along $[\partial \mathcal{M}]_-$ satisfies $x^{(d)}(t) \in \partial X_t(\bar{x})$ for all $t \in [t_0, \bar{t}]$. Therefore, from Theorem 2 of the appendix, there exists a solution, labelled $\lambda^d$, to the adjoint equation, (A2), such that the Hamiltonian is maximised for almost every $t \in [t_0, \bar{t}]$ as in (A3). We still need to show that with the final condition $\lambda^d(\bar{t}) = (Dg_i(z))T$ we obtain a solution to the adjoint equation such that the constant on the right-hand side of (A3) is zero. From the appendix, $\lambda^d(t)$ is the outward normal of a hyperplane that contains the elementary perturbation cone, $\mathcal{K}_t$, for every $t \in [t_0, \bar{t}]$. Moreover, we have:

$$\lambda^d(t)^T v(t) = \lambda^d(\bar{t})^T v(\bar{t}) \leq 0 \quad \forall t \in [t_0, \bar{t}],$$

(11)

where $v(t)$ is any elementary perturbation vector in $\mathcal{K}_t$. Let the arbitrary $v(t)$ be associated with the perturbation data $(d, \tau, \bar{t})$. Then, after dividing by $\tau$, we have

$$\lambda^d(\bar{t})^T \left[ f(x^{(d)}(\bar{t}), d) - f(x^{(d)}(\bar{t}), d(\bar{t})) \right] \leq 0 \quad \forall d \in D.$$

Recall from the proof of Proposition 6 that:

$$\max_{i \in \{z\}} L_f g_i(z, d(\bar{t})) = \max_{d \in D} \max_{i \in \{z\}} L_f g_i(z, d) = 0, \quad (12)$$

or equivalently,

$$\max_{i \in \{z\}} Dg_i(x^{(d)}(\bar{t})) \left[ f(x^{(d)}(\bar{t}), d) - f(x^{(d)}(\bar{t}), d(\bar{t})) \right] \leq 0,$$
for all \( d \in D \), from where we deduce that \( \lambda^d(t) = D_{g_1}(z)^T \). From (11) and (12) we also deduce that the constant on the right-hand side of (A.3) is zero. The ultimate tangentiality condition was proved in Proposition 6.

7 Examples

7.1 Mass-spring-damper and comparison with admissible sets

Consider the following mass-spring-damper system, as in [22], described by:

\[
\begin{align*}
\dot{x}_1 &= \left( \begin{array}{cc} 0 & 1 \\ -\frac{k}{m} - \frac{b}{m} & 0 \end{array} \right)x_1 + \left( \begin{array}{c} 0 \\ \frac{1}{m} \end{array} \right)d(t), \\
|d(t)| &\leq 1, \quad x_1(t) - 1 \leq 0, \quad \forall t \in [t_0, \infty[
\end{align*}
\]

where \( m \) is the mass, \( x_1(t) \) is the mass’s displacement, \( b \) the damping coefficient, \( k \) the spring coefficient and \( d(t) \) a force acting on the mass. With \( g(x) \triangleq x_1 - 1 \) we identify the sets \( G = \{ x \in \mathbb{R}^2 : x_1 \leq 1 \} \) and \( G_0 = \{ x \in G : x_1 = 1 \} \). Let \( z \triangleq (z_1, z_2)^T \) denote the point at which \( \partial [(\partial M)_{-}] \) intersects \( G_0 \), as in Theorem 1. Invoking the ultimate tangentiality condition, as in (10), we obtain:

\[
\max_{|d| \leq 1} \{ Dg(z)f(z, d) \} = \max_{|d| \leq 1} \{ z_2 \} = z_2 = 0.
\]

Thus, we identify the point \( z = (1, 0)^T \) as a point of ultimate tangentiality. From the Hamiltonian maximisation condition, as in (9), we identify the disturbance realisation associated with \( [\partial M]_{-} \) by considering \( \max_{|d| \leq 1} \{ \lambda_1(x_2) + \lambda_2(-\frac{k}{m}x_1 - \frac{b}{m}x_2 + \frac{1}{m}d) \} \), from where we obtain \( \ddot{d}(t) = \text{sgn}(\lambda_2(t)) \). The adjoint system is:

\[
\lambda_1 \dot{z}_1 = \lambda_2 \dot{z}_2 - \lambda_1 + \frac{b}{m}\lambda_2.
\]

We integrate backwards using \( \dot{d} \) from the point \( z \) to obtain the invariance barrier, \( [\partial A]_{-} \), as in Figure 1 for the parameters \( m = 1, k = 2 \) and \( b = 2 \).

7.1.1 Comparison with admissible sets

We now briefly discuss the relationship between robust positively invariant sets and admissible sets. The admissible set is defined as follows: \( A \triangleq \{ x_0 : \exists d \in D, \text{ s.t. } x^{(d, x_0, t_0)}(t) \in G \forall t \geq t_0 \} \), and, for the system described by (1)-(4), it is clear that \( \mathcal{M} \subset A \). The set \( A \) is interesting in the context where the input \( d \) is not interpreted as a disturbance, but as a control. The analysis in the paper [22], under the assumptions (A1)-(A5), showed that the set \( [\partial A]_{-} \triangleq \partial A \cap G_{-} \), which is called the barrier, may be constructed via a minimum-like principle analogous to the approach involving the maximum principle, as stated in Theorem 1 of the current paper. Figure 1 shows the barrier, \( [\partial A]_{-} \), along with the invariance barrier, \( [\partial M]_{-} \), for the linear spring example.

The name barrier comes from the fact that \( [\partial A]_{-} \) possess the semi-permeability property: any integral curve that crosses the barrier after having initiated in the interior of \( A \), cannot re-enter the interior of \( A \), before first violating a constraint. Similarly, we have chosen to name the set \( [\partial M]_{-} \) the invariance barrier because it also possesses a semi-permeability-like property: any integral curve that crosses the invariance barrier, having initiated in the interior of \( A \), can never leave the set \( M \).

7.2 Constrained double integrator

This example emphasises the fact that the conditions in Theorem 1 are necessary, and that parts of the obtained integral curves may need to be ignored when constructing the set \( M \). Consider the double integrator \( \dot{x}_1 = x_2, \quad \dot{x}_2 = d, \) with \( d \in [-0.5, -0.25] \) and \( g_1(x) \triangleq -x_1^2 - x_2^2 + 1 \). Invoking the ultimate tangentiality condition we get:

\[
\max_{d \in [-0.25, -0.5]} (-2x_1, -2x_2)(x_2, d)^T = 0,
\]

from where we identify four points of ultimate tangentiality: \((1, 0), (-1, 0), (0.5, \sqrt{1 - 0.5^2}) \) and \((0.25, -\sqrt{1 - 0.25^2}) \). The adjoint satisfies \( \lambda_1 = 0, \lambda_2 = -\lambda_1 \) with \( \lambda(t) = (-2x_1(t), -2x_2(t)) \). From the Hamiltonian maximisation condition we identify:

\[
\left\{ \begin{array}{ll}
\ddot{d}(t) = -0.25, & \text{for } \lambda_2(t) \geq 0 \\
\ddot{d}(t) = -0.5, & \text{for } \lambda_2(t) < 0.
\end{array} \right.
\]

We obtain the four candidate invariance barrier trajectories as in Figure 2. The trajectory ending at \((1, 0)\) vio-
lates the constraint and should clearly be ignored. Either
the curve ending at \((0.5, \sqrt{1 - 0.5^2})\), or the curve ending
at \((0.25, -\sqrt{1 - 0.25^2})\) needs to ignored, as both cannot
form the boundary of \(\mathcal{M}\). It may be verified that for all
points on the circle satisfying \(x_1 > 0.25\) and \(x_2 < 0\) there
exists a disturbance satisfying \(-0.5 \leq d \leq -0.25\) such
that \(Dg(x, d) > 0\). Thus, this part of the circle cannot
be part of \([\partial \mathcal{M}]_0\). We conclude that the curve ending
at \((0.5, \sqrt{1 - 0.5^2})\) needs to be ignored, and obtain the
system’s MRPI as in Figure 3.

We now add another state constraint, \(g_2(x) \triangleq x_1 -
3\). We identify the point of ultimate tangentiality
\((2.5, -0.5)\) and find the candidate invariance barrier trajec-
tory with the same \(\bar{d}\) as before. This curve intersects
the curve that ends on the circle at \((0.25, -\sqrt{1 - 0.25^2})\).
We need to ignore the parts of both curves that extend,
backwards in time, beyond this intersection point, be-
cause they are contained in parts of the state space for
which either \(g_1\) or \(g_2\) may be violated by an admissi-
ble disturbance realisation, and so we end up with the
set \(\mathcal{M}\) as in Figure 4. In the setting of admissible sets,
further prolongation of candidate barrier curves beyond
intersection points, called stopping points [33], always
need to be ignored. We conjecture that in the context
of MRPIs, all intersection points of candidate invariance
barrier curves are also stopping points.

7.3 Pendulum

The next example, taken from [18], considers a linearised
model of a pendulum actuated by a torque, described
by the linear differential equations \(\dot{\theta}_1(t) = \theta_2(t), \dot{\theta}_2(t) =
\theta_1(t) + \tau(t) + d(t)\), with \(\theta_1(t)\) the pendulum’s angle, \(\tau(t)\)
the applied torque and \(d(t)\) a disturbance torque. It is
desired that the actuator not saturate during the sys-
tem’s operation, and to this end the constraint \(|\tau(t)| \leq 2
is imposed. It is assumed that the disturbance is in the
bounded interval \(D \triangleq [d_{\min}, d_{\max}] = [-0.1, 0.1]\). In [18]
the authors specified a linear feedback law, \(\tau(\theta, w) =
-k_1\theta_1(t) - k_2\theta_2(t) + (k_1 - 1)w\), where \(\theta \triangleq (\theta_1, \theta_2)^T\), \(k_1\)
and \(k_2\) are design parameters, and \(w\) is a constant in-
put set-point that determines the equilibrium of the an-
gle \(\theta_1(t)\). Our goal is to find the system’s MRPI, which
would form a subset of the state space wherein it would
be guaranteed that the constraint on \(\tau\) would be satisfied.

To that end, we define \(g_1(\theta, w) \triangleq \tau(\theta, w) - 2\) and \(g_2(\theta) \triangleq
-\tau(\theta, w) - 2\). We invoke the ultimate tangentiality con-

![Fig. 3. The MRPI for the constrained double integrator ex-
ample.](image)

![Fig. 4. The MRPI for the constrained double integrator ex-
ample with an additional linear constraint.](image)
dition for \(g_1\) to get:

\[
0 = \max_{d \in D} \{Dg_1(\theta)f(\theta,d)\} = -k_1 \theta_2 - k_2 \theta_1 + k_1 k_2 \theta_1 + k_2^2 \theta_2 - k_1 k_2 w + k_2 w - k_2 d_{\min}.
\]

We substitute for \(\theta_2\), and use the parameters \(k_1 = 6.25, k_2 = 2.5\) (as was specified in [18]) along with \(w = -0.3\), to obtain the ultimate tangentiality point \(z \approx (-0.3190, -0.6324)\). A similar analysis for \(g_2\), identifies the point \(z \approx (-0.2810, 0.8724)\). As before, we obtain the invariance barrier curves by integrating backwards from these points using \(d\) obtained from the condition (9). The resulting MRPI is shown in Figure 5. A similar analysis for \(w = 0\) and \(w = 0.3\) produces the MRPIs also shown in Figure 5.

We now provide an analysis for the true (nonlinear) dynamics of the system, given by \(\ddot{\theta}(t) = -g \sin(\theta(t)) + \frac{1}{m} \tau(t) + d(t)\), where \(g\) is the gravitational constant, \(l\) the length of the pendulum and \(m\) the mass. We impose the same state and torque constraints as before and, using the parameters \(g = 9.81, l = 1, m = 1, k_1 = 6.25, k_2 = 2.5\) and \(w = -0.3\), again identify two points of ultimate tangentiality, \(z \approx (-0.1044, -1.1690)\) and \(z \approx (-0.0919, 0.3998)\). Integrating backwards we obtain the set \(\mathcal{M}\) as in Figure 6, where we have ignored the candidate invariance barrier trajectory ending at \((-0.1044, -1.1690)\). We can see that the true MRPI is much smaller when we consider the nonlinear dynamics.

The maximum principle. We used this fact to construct the MRPI for a number of examples, illuminating some interesting properties.

We adapted results on admissible sets, where the focus is on the existence of control functions such that constraints are always satisfied, to the current setting, where the focus is on guaranteeing the satisfaction of constraints for all possible disturbance realisations. An interesting extension of these ideas could be to the setting of differential games, where one could be interested in the existence of controls such that state and input constraints are always satisfied regardless of the disturbance realisation.

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A Tangent perturbation cone and the Maximum Principle

In this appendix we present well-known results that are used in some of the proofs of the paper. The content here is mainly taken from [37, Ch.4] and [38].
Consider the integral curve \( x(\tilde{d},\tilde{x},t_0) \), initiating at \( \tilde{x} \) at time \( t_0 \in \mathbb{R} \), associated with the disturbance realisation \( \tilde{d} \in D \), and defined on the compact interval \([t_0, \bar{t}]\). Let \( \pi \triangleq \{d_1, \tau, l\} \) denote the perturbation data with \( d_1 \in D \), \( \tau \) a Lebesgue point of \( \tilde{d} \) and \( l \geq 0 \), and form the following needle perturbation:

\[
d_x(t, \epsilon) = \begin{cases} 
    d_1, & \text{for } t \in [\tau - l, \tau], \\
    d(t), & \text{elsewhere on } [t_0, \bar{t}].
\end{cases}
\]

For \( \epsilon > 0 \) small enough, \( d_\pi \) is an admissible disturbance realisation. Define the elementary perturbation vector by \( v_\pi(t) \triangleq \left[ f(x(\tilde{d},\tilde{x},t_0)(\tau), d_1) - f(x(\tilde{d},\tilde{x},t_0)(\tau), \tilde{d}(\tau)) \right] l \).

This is a tangent vector at the point \( x(\tilde{d},\tilde{x},t_0)(\tau) \), whose parallel displacement along the curve \( x(\tilde{d},\tilde{x},t_0) \) is described by the variational equation:

\[
\dot{v}_\pi(t) = \left( \frac{\partial f}{\partial x}(x(\tilde{d},\tilde{x},t_0)(t), \tilde{d}(t)) \right) v_\pi(t).
\]

The resulting perturbed integral curve satisfies:

\[
x_{\tilde{d},\tilde{x}}(t_0, t) = x(\tilde{d},\tilde{x},t_0)(t) + c v_\pi(t) + o(\epsilon) \quad \forall t \in [\tau, \bar{t}],
\]

where \( o(\epsilon) \) is a function such that \( \lim_{\epsilon \to 0} \frac{o(\epsilon)}{\epsilon} = 0 \). Let \( \lambda(t) \) denote the normal of a hyperplane, \( \Pi(t) \), containing the point \( x(\tilde{d},\tilde{x},t_0)(\tau) \), such that \( \lambda(t) T v \) vanishes for any tangent vector \( v \in \Pi(t) \). Then, the solution to the adjoint equation:

\[
\dot{\lambda}(t) = - \left( \frac{\partial f}{\partial x}(x(\tilde{d},\tilde{x},t_0)(t), \tilde{d}(t)) \right)^T \lambda(t)
\]

describes the parallel displacement of the hyperplane \( \Pi(t) \) along the curve \( x(\tilde{d},\tilde{x},t_0) \). It may be verified that:

\[
\frac{d}{dt} \left[ \lambda(t)^T v_\pi(t) \right] = 0. \tag{A.1}
\]

**Definition 5** The tangent perturbation cone, \( \mathcal{K}_t \), at time \( t \) satisfying \( t_0 \leq t \leq \bar{t} \) is the smallest closed convex cone in the tangent space at \( x(\tilde{d},\tilde{x},t_0)(t) \) that contains all parallel displacements of all elementary perturbation vectors from all Lebesgue times \( \tau \) on \([t_0, \bar{t}]\).

It can be shown, see [37, Ch.4], that there exists a hyperplane, with outward normal specified by \( \lambda^d(t) \), such that \( \lambda^d(t)^T \lambda(t) \leq 0 \) for any elementary perturbation vector \( v(t) \in \mathcal{K}_t \). From (A.1) we can then deduce that:

\[
\lambda^d(t)^T v(t) \leq 0 \quad \forall t \in [t_0, \bar{t}],
\]

where \( v(t) \) is any elementary perturbation vector contained in \( \mathcal{K}_t \).

**Theorem 2 (Maximum Principle [37])** Consider the system (1)-(3). Let \( \tilde{d} \in D \) be a disturbance realisation such that \( x(\tilde{d},x_0,t_0)(t_1) \in \partial X_{\Pi}(x_0) \) for some \( t_1 > t_0 \). Then, there exists a non-zero absolutely continuous maximal solution \( \lambda^d(t) \) to the adjoint equation:

\[
\dot{\lambda}(t) = - \left( \frac{\partial f}{\partial x}(x(\tilde{d},x_0,t_0)(t), \tilde{d}(t)) \right)^T \lambda^d(t), \tag{A.2}
\]

such that

\[
\max_{\tilde{d} \in D} \{ \lambda^d(t)^T f(x(\tilde{d},x_0,t_0)(t), \tilde{d}) \} = \lambda^d(t)^T f(x(\tilde{d},x_0,t_0)(t), \tilde{d}(t)) \text{ is constant} \tag{A.3}
\]

for almost every \( t \in [t_0, t_1] \).

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