Mitigation of tipping point transitions by time-delay feedback control

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In stochastic multistable systems driven by the gradient of a potential, transitions between equilibria is possible because of noise. We study the ability of linear delay feedback control to mitigate these transitions, ensuring that the system stays near a desirable equilibrium. For small delays, we show that the control term has two effects: i) a stabilizing effect by deepening the potential well around the desirable equilibrium, and ii) a destabilizing effect by intensifying the noise by a factor of \((1 - \tau \alpha)^{-1/2}\), where \(\tau\) and \(\alpha\) denote the delay and the control gain, respectively. As a result, successful mitigation depends on the competition between these two factors. We also derive analytical results that elucidate the choice of the appropriate control gain and delay that ensure successful mitigations. These results eliminate the need for any Monte Carlo simulations of the stochastic differential equations, and therefore significantly reduce the computational cost of determining the suitable control parameters. We demonstrate the application of our results on two examples.

Many complex systems, such as the climate, stock markets, population of species, and the nervous system, have multiple stable equilibrium states. Near the tipping point of the system, small amounts of stochastic disturbances can lead to transitions between these equilibria. Typically, one of the equilibria is desirable and transitions away from it lead to catastrophic consequences. Here, we analyze the ability of a delay feedback control to mitigate transitions away from a desirable equilibrium.

I. INTRODUCTION

Tipping point transitions refer to abrupt changes in the state of a dynamical system caused by a rapid transition from an equilibrium state of the system to another equilibrium. Such transitions have been observed in many systems such as climate dynamics,\(^1\) epilepsy,\(^2\) population ecology,\(^3\) fluid dynamics,\(^4\) stock markets,\(^5\) and stock markets.\(^6\) Typically, one of the system equilibria is desirable and transitions away from it are catastrophic. As such, it is necessary to devise control methods that mitigate such transitions.

Tipping point transitions are usually studied in the context of bifurcation theory, where the system parameters change gradually. Under certain parameter variations, a bifurcation takes place whereby an equilibrium loses stability and the system evolves towards another stable equilibrium.

However, even before such bifurcations take place, the system can switch equilibria due to small amounts of stochastic disturbances. Such stochasticity-induced transitions are common in multistable systems, where several stable equilibria exist simultaneously. These transitions become more feasible near the tipping point of the system (i.e., close to the bifurcation).

Here, we focus on transitions in stochastic multistable systems. These systems are generally modeled by the stochastic differential equation (SDE),

\[
d\mathbf{X} = \mathbf{f}(\mathbf{X})dt + \mathbf{\sigma}(\mathbf{X})d\mathbf{W},
\]

where \(\mathbf{X}(t)\) denotes the state of the system at time \(t\), \(\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n\) is the driving force, \(\mathbf{W}(t)\) is the Brownian motion (or noise), and \(\mathbf{\sigma}(\mathbf{X})\) is the diffusion tensor determining the intensity of the noise. The system is multistable if, in the absence of noise, there are several stable fixed points.

Much attention has been paid to real-time prediction\(^14\)-\(^18\) and quantification\(^19\)-\(^21\) of transitions in system (1). Real-time prediction refers to early warning signs of the upcoming transitions that enable their predictions. Quantification, on the other hand, refers to estimating the probability of transitions in the aggregate, and the most likely paths such transitions may take. However, mitigation of tipping point transitions has received relatively little attention.

Here, we analyze the ability of linear time-delay feedback control in mitigating transitions in stochastic multistable system. We consider the special case of a system driven by the gradient of a potential \(V\), such that \(\mathbf{f}(\mathbf{x}) = -\nabla V(\mathbf{x})\). In the absence of noise \((\mathbf{\sigma} = 0)\), all minima of the potential \(V\) are stable fixed points. However, small amounts of noise cause rare transitions between these stable equilibria (see figure 1, for an example).

Delay feedback control has been used widely in deterministic systems to stabilize otherwise unstable fixed points or periodic orbits.\(^22\)-\(^25\) Recently, Suresh and Chandrasekar\(^26\) used delay feedback to mitigate extreme events in a deterministic system driven by a time-periodic force. However, little is known about the ability of delay feedback control to also mitigate critical transitions in stochastic multistable systems.

Here, we discover several properties of such a control strategy. The controlled system is a stochastic delay

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derivation of the equilibrium states of the system. A detailed description of this system is given in Section VI A.

differential analysis is extremely difficult due to the infinite-dimensional nature of the problem. For small delay, we derive an approximating SDE whose behavior mimics that of the controlled system. This approximation reveals several properties of the control. In particular, we show that the delay feedback has two simultaneous effects. First, the control term modifies the potential $V$ into an effective potential $V'$ which is deeper around the desirable equilibrium. As a result, this equilibrium becomes more stable, and transitions away from it become less likely.

At the same time, the delay effectively enhances the noise intensity by a factor of $1/\sqrt{1-\alpha \tau}$, where $\alpha$ is the control gain and $\tau$ is the delay. This intensified noise increases the probability of transitions away from the desirable equilibrium. The success of the delay feedback control depends on the interplay between these two competing factors. Therefore, the appropriate choice of the gain and the delay becomes crucial.

We derive analytical results that inform this choice. In particular, we derive the exact stationary density of the approximating SDE, and an upper bound for the probability of transitions away from the desirable equilibrium. These two analytical results elucidate the proper choice of the control parameters at a low computational cost, and eliminate any need for Monte Carlo simulations of the underlying SDDE (or its approximating SDE).

This paper is organized as follows. In Section II, we introduce the set-up of the problem. In Section III, we derive the approximating SDE of the controlled system. Stationary distribution of the approximating SDE is presented in Section IV and an upper bound for the probability of transitions is derived in Section V. In Section VI, we demonstrate our results with two numerical examples. Finally, Section VII contains our concluding remarks and future directions.

II. PRELIMINARIES AND SET-UP

In this section, we introduce the notation and set-up of the problem. In particular, we describe the uncontrolled system and its controlled counterpart, and outline their general properties.

A. Uncontrolled system

Consider a stochastic process $X(t)$ that is generated by the SDE,

$$dX = -\nabla V(X)dt + \sigma(X)dW, \quad X(0) = x^0,$$

(2)

where $V : \mathbb{R}^n \rightarrow \mathbb{R}$ is the potential function and $x^0$ denotes the initial condition of the stochastic process. Components of the vector $W(t) \in \mathbb{R}^m (m \geq n)$ are mutually independent, standard Wiener processes so that the increments $W_i(t_2) - W_i(t_1)$ are Gaussian, satisfying

$$\mathbb{E}[W_i(t_2) - W_i(t_1)] = 0,$$

(3a)

$$\mathbb{E}[(W_i(t_2) - W_i(t_1))(W_j(t_2) - W_j(t_1))] = (t_2 - t_1)\delta_{ij},$$

(3b)

for all $i, j \in \{1, 2, \ldots, m\}$ and $0 \leq t_1 \leq t_2$. Here, $\delta_{ij}$ denotes the Kronecker delta. The diffusion tensor $\sigma(x) \in \mathbb{R}^{n \times m}$ is a function of $x$. We assume that the potential $V$ and the tensor $\sigma$ are smooth enough to ensure the regularity of the SDE (2).

Consider the case where the potential $V$ has a finite number of disjoint local minima $x_1, x_2, \ldots, x_K \in \mathbb{R}^n$. In the absence of noise ($\sigma \equiv 0$), the local minima $x_k (k = 1, 2, \ldots, K)$ are all locally stable fixed points of the ODE (2). The presence of noise ($\sigma \neq 0$) allows for rare transitions between these fixed points. We assume that one of these equilibria is desirable and would like to suppress the transition of system trajectories to other equilibria. We denote the desirable fixed point by $x_a$, and note that $x_a = x_k$ for some $k \in \{1, 2, \ldots, K\}$.

Our goal is to mitigate the transitions away from the desirable fixed point $x_a$, or at least reduced the probability of such transitions.

B. Controlled system

In order to suppress transitions away from the desirable equilibrium $x_a$, we add a time-delay feedback control to

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1.png}
\caption{Example of abrupt transitions in a stochastic multistable system. Shown are a one-dimensional potential $V$ (bottom panel) and a trajectory $X(t)$ (top panel) of the corresponding SDE (1) with $f = -\nabla V$. The blue dashed lines mark the equilibrium states of the system. A detailed description of this system is given in Section VI A.
} \label{fig:fig1}
\end{figure}
system (2). The controlled system reads

$$d\mathbf{X} = -\nabla V(\mathbf{X})dt - u(t-\tau)dt + \sigma(\mathbf{X})dW,$$

where $\tau > 0$ is the delay and $u(t-\tau)$ is the control term. We denote the trajectories of the controlled system (4) by $\mathbf{X}(t)$ to distinguish them from the trajectories $\mathbf{X}(t)$ of the uncontrolled system (2). The delay $\tau$ is desirable since in practice the time required for state measurements, and consequent actuation of the control, always imposes a time lag.

Here we consider linear control terms of the form

$$u(t-\tau) = \alpha (\mathbf{X}(t-\tau) - \mathbf{x}_a),$$

where $\alpha > 0$ is the control gain and $\mathbf{x}_a$ is the desirable state. Note that with this control term, Eq. (4) is always time lag $\tau$, and $\tau$ is a reasonable approximation of the original SDDE, and is desirable since in practice the time required for state measurements, and consequent actuation of the control, always imposes a time lag.

The delay feedback control (5) has been used extensively for stabilizing fixed points and periodic orbits in deterministic systems\textsuperscript{22,23}. Here, we investigate their effectiveness in suppressing transitions away from the desirable state $\mathbf{x}_a$ in the stochastic context.

The delay term introduces significant difficulties in analysis of the controlled system. In particular, the phase space of an SDDE is infinite-dimensional, and furthermore, the Fokker–Planck equation for the density is an integro-differential equation\textsuperscript{29,30}. In order to circumvent these difficulties we derive an SDDE which approximates the SDDE (4) in the small delay limit $0 < \tau \ll 1$. The approximating SDDE is more amenable to mathematical analysis.

### III. APPROXIMATING SDE

To derive an approximate SDE for the SDDE (4), we expand the control term $\mathbf{X}(t-\tau)$ in $\tau$. This type of Taylor expansion for the delay term has been used routinely for delay differential equations\textsuperscript{26,31}, although its mathematical justification remains an open problem. Mazanov and Tognetti\textsuperscript{31} point out the potential pitfalls of this approximation, namely, that if too many terms in the Taylor series are kept, the resulting approximating equation may become unstable. However, Insperger\textsuperscript{32} argues with numerical examples that if $O(\tau^2)$ terms are negligible, and one only keeps the $O(\tau)$ terms, the resulting SDE is a reasonable approximation of the original SDDE, and preserves many of its stability properties.

Here, we apply this approximation. For short delays $0 < \tau \ll 1$, we consider the Taylor expansion

$$\mathbf{X}(t-\tau) = \mathbf{X}(t) - \tau \dot{\mathbf{X}}(t) + O(\tau^2).$$

Substituting this expression in Eq. (4) and neglecting $O(\tau^2)$ terms, we obtain

$$(1 - \alpha \tau) d\mathbf{X} = -\nabla V(\mathbf{X})dt - \alpha (\mathbf{X} - \mathbf{x}_a)dt + \sigma(\mathbf{X})dW.$$

We refer to Eq. (7) as the approximating SDE since it approximates the SDDE (4). We further assume that $\alpha \tau < 1$, since otherwise the desirable state $\mathbf{x}_a$ becomes unstable.

The following theorem shows that the approximating SDE can be transformed into an equivalent SDE by rescaling time. The rescaled equation has the form of a standard SDE.

**Theorem 1.** The trajectories of the approximating SDE (7) coincide with the trajectories of the rescaled approximating SDE,

$$d\mathbf{X} = -\nabla \tilde{V}(\mathbf{X})ds - \alpha (\mathbf{X} - \mathbf{x}_a) ds + \tilde{\sigma}(\mathbf{X})dW(s),$$

for the process $\mathbf{X}(s)$, where $s = t/(1 - \alpha \tau)$ is the rescaled time, $W(s)$ is a standard Wiener process, and $\tilde{\sigma}(\mathbf{X}) := \sigma(\mathbf{x})/\sqrt{1 - \alpha \tau}$.

**Proof.** See Appendix A. \hfill \qed

Defining the effective potential $\tilde{V}$,

$$\tilde{V}(\mathbf{x}) := V(\mathbf{x}) + \frac{\alpha}{2} |\mathbf{x} - \mathbf{x}_a|^2,$$

the rescaled approximating SDE (8) can be written more compactly as

$$d\mathbf{X} = -\nabla \tilde{V}(\mathbf{X})ds + \tilde{\sigma}(\mathbf{X})dW.$$  

Table I contains a list of relevant equations, their names, and their parameter range.

**Remark 1.** Two remarks about Theorem 1 are in order:

(a) The control term modifies the potential $V$ in the original SDE (2) to the effective potential (9). The term $\alpha |\mathbf{x} - \mathbf{x}_a|^2/2$ increases the potential gap around the desirable state $\mathbf{x}_a$, deepening the potential well. As a result, the control further stabilizes the state $\mathbf{x}_a$, and consequently transitions away from this state become harder.

(b) At the same time, the control intensifies the noise by a factor of $1/\sqrt{1 - \alpha \tau}$. In principle, this can facilitate transitions away from the desirable state $\mathbf{x}_a$.

Therefore, the delay control introduces two competing factors: One modifies the potential to further stabilize the desirable state $\mathbf{x}_a$, whereas the other increases the noise intensity. In the next two sections, we quantify the contribution from each factor and identify the regime where the stabilizing effect outperforms the effect of intensified noise.

We also note that the delay $\tau$ could in principle vanish, $\tau = 0$. In this case, the control gain $\alpha$ can be arbitrarily
TABLE I. List of equations. The controlled system is a stochastic delay differential equation, whereas the rest are stochastic differential equations without delay.

| Equation | Terminology | Parameters |
|----------|-------------|------------|
| $d\mathbf{X} = -\nabla V(\mathbf{X})dt + \sigma(\mathbf{X})dW$ | Uncontrolled system (SDE) | $\alpha, \tau > 0$ |
| $d\mathbf{X} = -\nabla V(\mathbf{X})dt$ | Controlled system (SDDE) | $\alpha, \tau > 0$ |
| $\alpha (\mathbf{X}(t - \tau) \cdot \mathbf{x}_0) dt + \sigma(\mathbf{X})dW$ | Approximating SDE | $0 < \alpha \tau < 1$ |
| $(1 - \alpha \tau) d\mathbf{X} = -\nabla V(\mathbf{X})dt$ | | |
| $\alpha (\mathbf{X} - \mathbf{x}_0) dt + \sigma(\mathbf{X})dW$ | Rescaled approximating SDE | $s = t/(1 - \alpha \tau), \tilde{\sigma} = \sigma/\sqrt{1 - \alpha \tau}$ |
| $d\mathbf{X} = -\nabla V(\mathbf{X})ds$ | | |
| $-\alpha (\mathbf{X} - \mathbf{x}_0) ds + \tilde{\sigma}(\mathbf{X})dW$ | | |

Note that for $\alpha = 0$, the Fokker–Planck equation (11) determines the evolution of the density $p$ corresponding to the uncontrolled system (2).

We are particularly interested in stationary densities $p_0$ which are invariant in time and describe the asymptotic evolution of the stochastic process.

**Definition 1.** A function $p_0 : \mathbb{R}^n \rightarrow \mathbb{R}^+$ is a stationary density of the rescaled approximating SDE (8), if $p_0$ satisfies

$$\nabla \cdot \left[ \nabla \tilde{V}(\mathbf{x})p_0 \right] + \frac{1}{2} \nabla \cdot \left[ \nabla \cdot (\tilde{\sigma}(\mathbf{x})\tilde{\sigma}(\mathbf{x})^\top) p_0 \right] = 0, \quad (13)$$

and $\int_{\mathbb{R}^n} p_0(\mathbf{x})d\mathbf{x} = 1$. In other words, $p_0$ satisfies the Fokker–Planck equation (11) and $\partial_t p_0 = 0$.

In the most general case of stochastic processes, the stationary distribution may not exist. Even when it does, it may not be unique. In our case, under reasonable assumptions, the existence and uniqueness of the equilibrium density is guaranteed. In addition, the stationary density has a known closed-form solution. For that, we need the following assumptions.

**A1** We assume that the diffusion tensor has the form $\sigma(\mathbf{x}) = \sigma \Sigma$ where $\sigma > 0$ and $\Sigma \in \mathbb{R}^{n \times n}$ is a semi-orthogonal matrix such that $\Sigma \Sigma^\top = I_n$ where $I_n$ is the $n \times n$ identity matrix.

**A2** We assume that $V \in C^2(\mathbb{R}^n; [a, \infty))$, for some finite $a \in \mathbb{R}$, and furthermore

$$\int_{\mathbb{R}^n} e^{-V(\mathbf{x})}d\mathbf{x} < \infty. \quad (14)$$
Under assumption (A1), the Fokker–Planck equation (13) simplifies to

\[ \partial_t p = \nabla \cdot \left[ \nabla \tilde{V}(x) p \right] + \tilde{\nu} \Delta p, \quad p(\cdot, 0) = f, \tag{15} \]

where \( \Delta \) denotes the Laplacian, and the diffusion constant is given by \( \tilde{\nu} := \tilde{\sigma}^2 / 2 \) where \( \tilde{\sigma} := \sigma / \sqrt{1 - \alpha \tau} \).

Assumption (A2) implies that the potential \( V \) is bounded from below, \( V(x) \geq a \) for all \( x \in \mathbb{R}^n \). We may assume, without loss of generality, that \( a \geq 0 \). This is permissible since the graph of \( V \) can be shifted upward, by an arbitrary positive constant, without altering the dynamics of (2).

Assuming (A1) and (A2), the following result holds.

**Theorem 2.** Assume (A1) and (A2). Then

1. The stochastic process \( \mathcal{X}(s) \), generated by the rescaled approximating SDE (8), has a unique stationary density.

2. The stationary density is given by

\[ p_0(x) = C \exp \left[ -\frac{\tilde{V}(x)}{\tilde{\nu}} \right], \tag{16} \]

where \( C \) is a normalizing factor defined by

\[ C = \left[ \int_{\mathbb{R}^n} \exp \left( -\frac{\tilde{V}(x)}{\tilde{\nu}} \right) dx \right]^{-1}, \tag{17} \]

so that \( \int_{\mathbb{R}^n} p_0(x) dx = 1 \).

3. The stationary density \( p_0 \) is a globally asymptotically attracting solution of the Fokker–Planck equation (11).

**Proof.** This is a well-known result. The proof can be found in, e.g., Refs. 34–36.

**Remark 2.** A few remarks on Theorem 2 are in order.

(a) Theorem 2 holds for the uncontrolled system by setting \( \alpha = 0 \), in which case \( \tilde{V} = V \), and \( \tilde{\nu} = \nu \) where \( \nu := \sigma^2 / 2 \).

(b) The fact that the stationary density (16) is globally attracting implies that all initial densities \( f(x) \) converge asymptotically to the unique stationary density \( p_0 \).

Therefore, the asymptotic state of the controlled system can be approximated by the stationary density (16). Evaluating this density only requires the knowledge of the effective potential \( \tilde{V} \). As a result, no numerical integration of the SDDE (4), or its rescaled approximating SDE (8), is required. In Section V, we use the stationary density \( p_0 \) to approximate the probability of transitions away from the desirable equilibrium \( x_a \).

**V. QUANTIFYING THE PROBABILITY OF TRANSITIONS**

In this section, we estimate the probability of transitions away from the desirable equilibrium \( x_a \) towards an undesirable (or ‘bad’) state \( x_b \in \mathbb{R}^n \). Although our results hold for any state \( x_b \) (not necessarily an equilibrium), we are particularly interested in the case where \( x_b \neq x_a \) is a local minimum of the potential \( V \).

In order to quantify the transitions away from the desirable equilibrium \( x_a \), we consider the ratio of the probability of \( \mathcal{X} \) being close to \( x_b \) to the probability of \( \mathcal{X} \) being close to \( x_a \). More precisely, we define the ratio

\[ r_c(\epsilon, x_a, x_b) := \frac{\mathbb{P}(|\mathcal{X} - x_b| \leq \epsilon)}{\mathbb{P}(|\mathcal{X} - x_a| \leq \epsilon)}, \tag{18} \]

where the subscript \( c \) denotes the ratio corresponding to the controlled system, and \( \epsilon > 0 \) is a small radius. The smaller the ratio \( r_c(\epsilon, x_a, x_b) \), the lower is the probability of transitions away from the desirable equilibrium \( x_a \) and towards the undesirable equilibrium \( x_b \).

If \( \mathcal{X} \) is a solution of the controlled SDDE (4), then the probabilities in Eq. (18) can be estimated from an ensemble of long term numerical simulations. However, if we work with the rescaled approximating SDE (8), such expensive numerical simulations are unnecessary. Note that the relevant regime here is the asymptotic steady state of the system after the initial transients have decayed. Therefore, the probabilities in Eq. (18) are taken with respect to the steady state probability density (16).

The following theorem establishes an upper bound for the ratio \( r_c \) if \( \mathcal{X} \) is given by the rescaled approximating SDE (8). Here, \( B_\epsilon(x_0) = \{ x \in \mathbb{R}^n : |x - x_0| \leq \epsilon \} \) denotes the closed ball of radius \( \epsilon \) centered at \( x_0 \).

**Theorem 3.** Assume (A1) and (A2), and let \( \epsilon > 0 \) be small enough such that \( B_\epsilon(x_a) \cap B_\epsilon(x_b) = \emptyset \). If the stochastic process \( \mathcal{X}(s) \) is generated by the rescaled approximating SDE (8), then

\[ r_c(\epsilon, x_a, x_b) \leq \exp \left[ \frac{\delta V(x_a, x_b, \epsilon)}{\tilde{\nu}} + \frac{\epsilon^2}{\beta^2} - \frac{\min_{x \in B_\epsilon(x_a)} |x - x_a|^2}{\beta^2} \right], \tag{19} \]

where

\[ \delta V(x_a, x_b) := \max_{x \in B_\epsilon(x_a)} V(x) - \min_{x \in B_\epsilon(x_b)} V(x), \tag{20a} \]

\[ \beta = \frac{\sigma}{\sqrt{\alpha(1 - \alpha \tau)}}, \quad \tilde{\nu} = \frac{\tilde{\sigma}^2}{2} = \frac{\sigma^2}{2(1 - \alpha \tau)}. \tag{20b} \]

**Proof.** See Appendix B.

Figure 2 depicts the quantities appearing in Theorem 3. Smaller upper bounds in Eq. (19) guarantee
lower probability of transition from the desirable equilibrium $x_a$ to the undesirable equilibrium $x_b$. There are two primary competing factors contributing to this upper bound. First, a larger distance between the desirable equilibrium $x_a$ and the undesirable equilibrium $x_b$ leads to a smaller upper bound. This distance is approximated by $\min_{x \in B_b(x_a)} |x - x_a|^2$, and the upper bound vanishes exponentially fast as this quantity grows.

A second contributing factor is the potential gap $\delta V(\epsilon, x_a, x_b)$. If $\max_{x \in B_b(x_a)} V(x) < \min_{x \in B_b(x_a)} V(x)$, then the potential gap also contributes to reducing the upper bound, hence reducing the probability of transitions. On the other hand, if $\max_{x \in B_b(x_a)} V(x) > \min_{x \in B_b(x_a)} V(x)$, then the potential gap increases the upper bound. In this case, the upper bound on the probability of transition depends on the competition between the distance between two equilibria $x_a$ and $x_b$, and the potential gap $\delta V$.

The following corollary shows more clearly the contributions of these two competing factors by taking the limit $\epsilon \to 0$.

**Corollary 1.** Assume (A1) and (A2). If the stochastic process $\mathbf{X}(s)$ is generated by the rescaled approximating SDE (8), then

$$\lim_{\epsilon \to 0^+} r_c(\epsilon; x_a, x_b) \leq \exp\left\{ -\frac{\alpha(1-\alpha \tau)}{\sigma^2} \left[ 2 \alpha (V(x_b) - V(x_a)) + |x_b - x_a|^2 \right] \right\}. \quad (21)$$

**Proof.** This follows from Theorem 3 by taking the limit $\epsilon \to 0^+$ in the inequality (19), and noting that $\bar{V} = \alpha^2 \beta^2 / 2$.

Therefore, in the limit $\epsilon \to 0^+$ and prescribed delay $\tau$ and gain $\alpha$, the upper bound is defined by the potential gap $V(x_b) - V(x_a)$ and the distance $|x_b - x_a|$. A larger distance between the equilibria always leads to a lower probability of transition. If $V(x_b) > V(x_a)$, the potential gap also contributes to reduce the probability of transitions. However, if $V(x_b) < V(x_a)$, then the upper bound depends on the competition between the potential difference and the distance between the equilibria. In particular, if $V(x_a) - V(x_b) \gg |x_b - x_a|^2$, then the control may not lead to a meaningful reduction in the transition probability.

Note that in a given problem, the potential gap $V(x_b) - V(x_a)$ and the distance $|x_b - x_a|$ are prescribed. Therefore, the only free parameters are the delay $\tau$ and the control gain $\alpha$. Assuming no delay, i.e. $\tau = 0$, the gain $\alpha$ can be arbitrary large and eventually all transitions are mitigated (since the upper bound vanishes as $\alpha \to \infty$). However, in the presence of delay $\tau > 0$, the gain $\alpha$ cannot be arbitrarily large since $\alpha \tau < 1$. In this case, the gain $\alpha$ needs to be chosen judiciously.

We emphasize that the upper bounds in Theorem 3 and Corollary 1 are valid for the stochastic process $\mathbf{X}(s)$ generated by the rescaled approximating SDE (8), and not the original controlled SDDE (4). Nonetheless, we show in the next section, that for short time delays $\tau$, these upper bounds are reliable approximations for the controlled system (4).

**VI. NUMERICAL RESULTS**

The purpose of this section is twofold. First, we demonstrate with two examples that the controlled system (4) does indeed mitigate the transitions away from the desirable equilibrium. Second, we demonstrate the validity of the rescaled approximating SDE (8) as an approximation of the controlled system when the delay $\tau$ is relatively small. Recall that mathematical analysis of the SDDE (4) is cumbersome and its numerical simulations are expensive. In contrast, there are many useful analytical results for the rescaled approximating SDE as presented in sections III, IV and V.

In a given problem, the potential $V$ and the desirable equilibrium $x_a$ are prescribed. Therefore, the only free parameters are the delay $\tau$ and the control gain $\alpha$. The delay $\tau$ is a consequence of the fact that the measurement of the state $\mathbf{X}$, and the consequent actuation of the control, introduce a delay in practice. In the following examples, we assume that this delay is made as small as possible so that the approximation (6) is valid.

For comparison, Monte Carlo simulations of the uncontrolled SDE (2) and the controlled SDDE (4) are carried out. This requires generating sample trajectories of the stochastic equations by numerical integration. These simulations are carried out by the predictor-corrector scheme of Cao et al. For completeness, we review this numerical scheme in Appendix C.
FIG. 3. The one-dimensional potential function $V(x) = Ax^2 + Bx^4$. We set $B = c^2$ and $A = -2B$. The potential for two parameter values $c = 1$ (blue) and $c = 1/\sqrt{2}$ (red) is shown.

A. One-dimensional potential

We first consider a one-dimensional example with the potential

$$V(x) = Ax^2 + Bx^4,$$

where $A < 0$ and $B > 0$. We assume $B = c^2$, for some constant $c$, and $A = -2B$. The depth of the potential well is controlled by the parameter $c = A/\sqrt{B}$ (see, e.g., Ref. 38). Figure 3 shows the potential $V$ for two choices of the parameter $c$.

The corresponding deterministic differential equation $\dot{x} = -V'(x)$ has three fixed points located at $x = 0, \pm \sqrt{-A/2B}$. The origin is an unstable fixed point while $\pm \sqrt{-A/2B}$ are stable. If the initial condition $x(0) = x^0$ satisfies $x^0 > 0$, then the trajectories of the deterministic system converge asymptotically to the stable fixed point $x = +\sqrt{-A/2B}$. Conversely, if $x^0 < 0$, the trajectories converge to the stable fixed point $x = -\sqrt{-A/2B}$. However, addition of noise allows for transitions between the two fixed points without the trajectory ever settling onto one of them.

In the following, we set $c = 1/\sqrt{2}$, $B = c^2$ and $A = -2B$. This implies that the fixed points are located at $x = 0, \pm 1$. The resulting uncontrolled SDE reads

$$dX = -(-2X + 2X^3)dt + \sigma dW,$$

where the noise intensity is $\sigma = 1/2$. Note that $V'(x) = -2x + 2x^3$. Figure 4(a) shows a sample trajectory of this SDE which switches randomly between the fixed points $x = \pm 1$. Figure 4(b) shows the resulting stationary distribution $p_0$ (blue circles), estimated from 200 direct numerical simulations. Each simulation is 1000 time units long with the sampling rate of 0.01, resulting in $2 \times 10^7$ samples. Figure 4(b) also shows the exact stationary density (16) with $\alpha = 0$ (solid red line) which is in agreement with direct numerical simulations.

Next we consider the controlled system. Without loss of generality, we assume that $x_a = +1$ is the desirable equilibrium and $x_b = -1$ is the undesirable equilibrium. In this case, the controlled SDDE reads

$$d\alpha = -(-2\alpha + 2\alpha^3) \, dt - \alpha(\alpha(t-\tau) - 1) \, dt + \sigma dW,$$

where the term $-\alpha(\alpha(t-\tau) - 1)$ is the time-delay feedback control. For $\alpha = 0$ we recover the uncontrolled system (23). Figure 4(c) shows a sample trajectory of the controlled system with $\alpha = 1$ and $\tau = 0.1$. Note that this trajectory has no transitions to the neighborhood of the undesirable equilibrium. Ensemble simulations exhibit a similar behavior. Figure 4(d) shows that probability density of $\alpha(t)$ from 200 simulations, each consisting of 1000 time units. In contrast to the uncontrolled system, the stationary density of the controlled system is unimodal with most of the density concentrated around the desirable equilibrium $x_a = +1$. As a result, transitions to the neighborhood of the undesirable equilibrium $x_b = -1$ are very unlikely.

Recall that in this experiment, we chose the control gain $\alpha = 1$ and the delay $\tau = 0.1$. How would one choose the values of these variables? The answer is not clear if we attempt to work with the controlled SDDE (4). However, if the delay is small, the results found in sections IV and V provide an inexpensive method to choose these parameters.

The delay must be chosen as small as possible. In practice, there is a lower bound on how small the delay may be, dictated by the amount of time it takes to measure the state $X$. Let us fix the delay at $\tau = 0.1$. Figure 5 shows the effective potential $\tilde{V}$ as $\alpha$ varies. For $\alpha = 0$, we recover the uncontrolled potential $V$ with two minima at $x_a = +1$ and $x_b = -1$. As $\alpha$ increases, the effective potential around the undesirable equilibrium $x_b = -1$ increases while the effective potential at the desirable equilibrium $x_a = +1$ remains unchanged. Eventually, for $\alpha$ large enough, the minimum around the undesirable equilibrium disappears and the effective potential has a unique minimum at the desirable equilibrium $x_a = +1$ (see, e.g., the curve corresponding to $\alpha = 1$).

Correspondingly, at $\alpha = 0$ (no control), the stationary density $p_0$ is bimodal with equal probability of visiting each of the equilibria $x_a = +1$ and $x_b = -1$. However, as $\alpha$ increases the density around the undesirable equilibrium $x_b = -1$ decreases. Eventually, for large enough $\alpha$ (e.g., $\alpha = 1$), the stationary density becomes unimodal concentrated around the desirable equilibrium $x_a = +1$.

These observations reveal that $\alpha = 1$ is a good choice for the control gain at delay $\tau = 0.1$. Note that since the effective potential $\tilde{V}$ and the stationary density $p_0$ are known analytically, this analysis is computationally inexpensive. In particular, it does not require the numerical integration of any SDEs or SDDEs.

Furthermore, for $\tau = 0.1$ and $\alpha = 1$, the upper bound in Corollary 1 is approximately $5.6 \times 10^{-7}$. This roughly means that the probability of the state being around the undesirable equilibrium $x_b = -1$ is $5.6 \times 10^{-7}$ times...
FIG. 4. (a) A sample trajectory of the uncontrolled system (23). The red dashed line marks the desirable equilibrium $x_a = +1$. (b) Stationary PDF of the uncontrolled system. The blue circles mark the PDF estimated from longterm direct numerical simulations of the uncontrolled system (23). The solid red curve is the exact stationary PDF (see Theorem 2). (c) A sample trajectory of the controlled system (24) with $\tau = 0.1$ and $\alpha = 1$. (d) The stationary PDF of the controlled system. The blue circles mark the PDF estimated from longterm direct numerical simulations of the controlled system (24). The solid red curve is the exact stationary PDF of the rescaled approximating SDE (see Theorem 2).

smaller than the state being around the desirable equilibrium $x_a = +1$.

There is no general recipe for choosing the delay $\tau$. As mentioned earlier, it must be as small as possible, so that the approximating SDE is a reasonable approximation of the controlled SDDE. It is still interesting to investigate whether the controlled system would mitigate transitions away from the desirable equilibrium for larger delays. Figure 6 shows how the stationary density varies as the delay increases. For $\tau = 0.2$, the stationary density is unimodal, centered at the desirable equilibrium with negligible probability of transitioning to the undesirable equilibrium. In addition, the exact stationary density of the approximating SDE is still a good approximation to the numerically estimated PDF.

The results are similar for $\tau = 0.4$. However, the exact PDF of the approximating SDE starts to deviate from the numerically estimated PDF of the SDDE. This is to be expected since, as the delay increases, the neglected $O(\tau^2)$ terms become significant. Even at $\tau = 2$, the stationary PDF is unimodal and the probability of transitions is negligible. Note that at $\tau = 2$, the approximating SDE is no longer valid since $\alpha \tau > 1$.

If we keep increasing the delay, eventually for $\tau > 5$ the control fails in the sense that the stationary PDF becomes bimodal again with non-negligible density around the undesirable equilibrium $x_b = -1$.

B. Two-dimensional potential

In this section, we present a two-dimensional example with the potential

$$V(x) = a_0x^4 - a_1x^2 + a_2y^2 - a_3x^2y - a_4(x - x_0)^2$$

where $x = (x, y)$. The parameters are $a_0 = 1$, $a_1 = 3/8$, $a_2 = 1/4$, $a_3 = \sqrt{4a_2(a_0 - 2a_1)} = 1/2$, $a_4 = 0.02$ and $x_0 = \sqrt{2a_1a_2/(4a_0a_2 - a_4^2)} = 1/2$. This potential has two minima at

$$x_a = (x_0, a_3x_0^2/2a_2), \quad x_b = (-0.5254, 0.2760),$$

and a saddle at $x_s = (0.0254, 0.0006)$ (see figure 7).

Again the minima $x_a$ and $x_b$ are both stable fixed points of the deterministic system $\dot{x} = -\nabla V(x)$. Addition of noise allows for rare transitions between the two equilibria. Our goal is to mitigate the transitions away from the desirable equilibrium $x_a$.
The function $V$ is designed so that the potential is deeper at the minimum $x_b$ (see figure 7(a)). As a result, the trajectories of the uncontrolled SDE (2) are more likely to be found around this undesirable equilibrium. This is confirmed by direct numerical simulations of the uncontrolled SDE (2) and estimating the stationary equilibrium as shown in figure 8(a). For reference, the exact stationary PDF of the uncontrolled system is shown in figure 8(b), which is in agreement with the numerically estimated PDF. In both cases the noise intensity is $\sigma = 0.15$.

Next we consider the controlled system with delay $\tau = 0.1$. As before, the delay is chosen to be small so that the approximating SDE is a reasonable approximation of the controlled SDDE. Gradually increasing the control gain $\alpha$, we find that $\alpha = 0.3$ is adequate for mitigating the transitions away from the desirable equilibrium $x_a$. This is shown in figure 7(b), which depicts the effective potential $\tilde{V}(x) = V(x) + \alpha|x - x_a|^2/2$ with $\alpha = 0.3$. This effective potential has a unique minimum at the point $x_a$.

Furthermore, for $\tau = 0.1$ and $\alpha = 0.3$, the upper bound in Corollary 1 is approximately $1.6 \times 10^{-6}$, indicating low probability of transitions towards the undesirable equilibrium. We emphasize again that determining the control gain $\alpha$ can be done merely by analyzing the function $\tilde{V}$, the stationary density $p_0$ and the upper bound (21), without any numerical simulations of the controlled SDDE or its approximating SDE.

Figure 8(c-d) compares the numerically estimated stationary PDF of the controlled system (SDDE) and the exact stationary PDF of its approximating SDE. The two PDF are in good agreement. Furthermore, the PDFs are unimodal with almost all the density concentrated around the desirable equilibrium $x_a$. As a result, transition to the undesirable equilibrium $x_b$ is very unlikely, and the mitigation has been successful.

VII. CONCLUSIONS

We have demonstrated the effectiveness of delay feedback control for mitigation of transitions in stochastic multistable systems driven by the gradient of a potential. In the absence of control, small amounts of noise can lead to rare transitions between the stable equilibria of the system. The delay feedback control mitigates these transitions by stabilizing an arbitrary equilibrium.

We added a delay feedback control term that turns the original stochastic differential equation (SDE) into a stochastic delay differential equation (SDDE). For small delay $\tau$, we showed that this SDDE can be approximated by an SDE. This approximation reveals two interesting features of the control: (i) the delay feedback modifies the potential $V$ resulting in an effective potential $\tilde{V}$ which is deeper at the location of the desirable equilibrium $x_a$. As a result, transitions away from this equilibrium become harder. (ii) At the same time, the delay feedback intensifies the effect of noise. In particular, the noise intensity of the uncontrolled system $\sigma$ becomes $\sqrt{1 - \alpha^2} \sigma$ in the controlled system, where $\alpha$ is the control gain.

Therefore, the effect of delay feedback is twofold. A stabilizing effect through deepening of the potential well and a destabilizing effect due to intensified noise. As a result, the delay $\tau$ and the gain $\alpha$ should be chosen appropriately in order to ensure that the stabilizing effect dominates the intensified noise.

We also derived exact results for the approximating SDE that elucidate the appropriate choice of the control delay and gain. For a given small delay $\tau$, an appropriate control gain $\alpha$ can be determined by analysis of the effective potential $\tilde{V}$ and the stationary density $p_0$. Since these two quantities are known analytically, the control gain can be determined at a low computational cost, without any need for generating sample trajectories of an SDE (or SDDE). Using these results, we also derived an upper bound for the probability of transitions away from the desirable equilibrium and towards an undesirable one. This upper bound (cf. Theorem 3) depends on the control gain $\alpha$, the delay $\tau$, the potential difference between the two equilibria (before applying the control) and the Euclidean distance between them.

Our work can be extended in several directions. The approximating SDE (and its rescaled version; Theorem 1) is valid for multiplicative noise, where the diffusion tensor $\sigma(x)$ depends on space. However, our analytical results concerning the stationary PDF and the upper bound for transition probabilities are limited to the case where $\sigma$...
is constant. It would be attractive to generalize these results to the case of multiplicative noise.

More importantly, it is desirable to design low-dimensional controllers of the form

$$u(t - \tau) = P(X(t - \tau) - x_a),$$

(27)

where $P$ is a projection onto a lower dimensional subspace. Such controllers would be more practical for mitigating transitions in high-dimensional systems, since they only act on a few degrees of freedom. Our preliminary numerical observations (not presented here) indicate that, if the projection is onto the least stable subspace of the fixed point $x_a$, then the mitigation is successful. However, our analytical results do not immediately extend to this low-dimensional control. The difficulty arises from the fact that $P \nabla V(x)$ or $\nabla V(Px)$ cannot necessarily be written as gradient functions. In other words, an effective potential $\tilde{V}$ does not necessarily exist such that

$$\nabla \tilde{V}(x) = P \nabla V(x)$$

or

$$\nabla \tilde{V}(x) = \nabla V(Px).$$

A counter example is provided by the function $V(x, y) = xy$ and $P$ being the orthogonal projection onto the $x$ coordinate.

Finally, analytical results for the general SDE (1), where $f(X)$ is not the gradient of a potential, would be welcome. At the moment, very little is known about the delay feedback control of such general stochastic systems. These open questions will be explored in forthcoming work.

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**Appendix A: Proof of Theorem 1**

Consider $t$ as a function of the rescaled time $s$ so that $t(s) = (1 - \alpha \tau)s$. Furthermore, we define a new stochastic process $\tilde{X}(s)$ such that $\tilde{X}(s) := X(t(s))$. Noting that...
Numerically Approximated Control Uncontrolled

Control

Note that the Wiener process \( t \) to the multidimensional case is straightforward. show that for a scalar process the variance of the increment is \( \sigma(\tilde{X}) \).

\[ \tilde{W}(s) \]

that

\[ P(\tilde{W}(0) = 0) = 1 \]

Clearly, the increments \( \tilde{W}(s_2) - \tilde{W}(s_1) \) are Gaussian and independent for all \( s_1 < s_2 \), and \( \mathbb{E}[\tilde{W}(s_2) - \tilde{W}(s_1)] = \mathbb{E}[\tilde{W}(t_2) - \tilde{W}(t_1)] = 0 \). It remains to show that the variance of the increment is \( s_2 - s_1 \). For simplicity, we show that for a scalar process \( \tilde{W}(s) \). The generalization to the multidimensional case is straightforward.

\[ \mathbb{E} \left[ \left| \tilde{W}(s_2) - \tilde{W}(s_1) \right|^2 \right] = \frac{\mathbb{E} \left[ \left| \tilde{W}(t(s_2)) - \tilde{W}(t(s_1)) \right|^2 \right]}{1 - \alpha \tau} = \frac{t(s_2) - t(s_1)}{1 - \alpha \tau} = s_2 - s_1. \]

This proves that \( \tilde{W}(s) \) is a standard Wiener process.

Substituting \( d\tilde{W}(t) = \sqrt{1 - \alpha \tau} d\tilde{W}(s) \) into Eq. (A1) and dividing by \( (1 - \alpha \tau) \), we obtain

\[ d\tilde{X}(s) = -\nabla V(\tilde{X}) ds - \alpha \left( \tilde{X} - x_0 \right) ds + \frac{\sigma(\tilde{X})}{\sqrt{1 - \alpha \tau}} d\tilde{W}(s). \]

Finally, defining \( \tilde{\sigma}(x) = \sigma(x)/\sqrt{1 - \alpha \tau} \) and omitting the tilde signs from \( \tilde{X} \) and \( \tilde{W} \), we obtain the desired Eq. (8).

Appendix B: Proof of Theorem 3

Recall that \( P(\tilde{X} \in \mathcal{S}) = \int_{\mathcal{S}} p_0(x) dx \), for any Lebesgue-measurable set \( \mathcal{S} \). Using Theorem 2, we have

\[ r_c = \frac{\int_{B_c(x_a)} e^{-\frac{V(x)}{\nu_0}} e^{-\frac{\nu_0|x-x_0|^2}{2}} dx}{\int_{B_c(x_a)} e^{-\frac{V(x)}{\nu_0}} e^{-\frac{\nu_0|x-x_0|^2}{2}} dx}. \]

Using the estimates,

\[ e^{-\frac{V(x)}{\nu_0}} \leq e^{-\frac{\min_{x \in B_c(x_a)} V(x)}{\nu_0}}, \quad \forall x \in B_c(x_a), \]

\[ e^{-\frac{V(x)}{\nu_0}} \geq e^{-\frac{\max_{x \in B_c(x_a)} V(x)}{\nu_0}}, \quad \forall x \in B_c(x_a), \]
where we used the fact that $\beta^2 = 2\nu/\alpha$. Next, we observe that
\[
e^{-\frac{|x-a|}{\alpha^2}} \leq e^{-\min_{x \in B_{c}(x_{0})} \frac{|x-a|}{\alpha^2}} \leq e^{-\frac{|x-a|}{\alpha^2}}, \quad \forall x \in B_{c}(x_{0})\]
\[
e^{-\frac{|x-a|}{\alpha^2}} \geq e^{-\min_{x \in B_{c}(x_{0})} \frac{|x-a|}{\alpha^2}} = e^{-\frac{x^2}{\alpha^2}}, \quad \forall x \in B_{c}(x_{0}).\]
As a result,
\[
r_{c} \leq \exp \left[ \frac{\delta V (\epsilon, x_{0}, x_{1})}{\nu} \int_{B_{c}(x_{0})} e^{-\frac{|x-a|}{\alpha^2}} \frac{dx}{\nu} \right] \left( \frac{\alpha \sigma}{\sqrt{1+\beta}} \right)^n \mbox{Vol} (B_{c}(x_{0})) \mbox{Vol} (B_{c}(x_{b})),
\]
where $\mbox{Vol} (B_{c}(x_{b})) = \frac{(\alpha \sigma)^n}{\sqrt{1+\beta}}$ denotes the volume of the ball $B_{c}(x_{b}) \subset \mathbb{R}^n$. This completes the proof.

Appendix C: Numerical integration of SDDEs

Numerical integration of SDDEs in Section VI are carried out by the predictor-corrector scheme developed by Cao et al.\(^{37}\). For completeness, we present their method here. This numerical scheme applies to SDDEs of the form
\[
dX(t) = f(X(t), X(t-\tau))dt + \sigma(X(t))dW(t),
\]
where $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$, $W(t) \in \mathbb{R}^m$ is a standard multidimensional Wiener process, and $\sigma(X) \in \mathbb{R}^{n \times m}$ is the diffusion matrix. For the smoothness requirements of the maps $f$ and $\sigma$ refer to Section 2 of Cao et al.\(^{37}\). In the context of our paper, we have
\[
f(X(t), X(t-\tau)) = -\nabla V (X(t)) - \alpha (X(t-\tau) - x_{0}).
\]

Consider a small time step $h > 0$, and let $X^{i}$ denote the numerical approximation of $X(ih)$, i.e., $X^{i} \approx X(ih)$. Furthermore, we assume that the delay $\tau$ is an integer multiple of the time step $h$, such that $\tau = jh$ for some $j \in \mathbb{N} \cup \{0\}$. We denote the elements of the vector $W(t)$ by $W_{l}(t)$, and write $W_{l}$ for $W_{l}(ih)$. Similarly, the columns of $\sigma$ are denoted by $\sigma_{l}$,
\[
\sigma = [\sigma_{1} | \sigma_{2} | \cdots | \sigma_{m}].
\]
In this set-up, the predictor-corrector method can be written as the two-step scheme,
\[
X^{i+1} = X^{i} + hf(X^{i}, X^{i-j}) + \sum_{l=1}^{m} \sigma_{l}(X^{i}) \Delta W_{l}^{i},
\]
where $\Delta W_{l}^{i}$ denote the increments of the Wiener process,
\[
\Delta W_{l}^{i} = W_{l}^{i+1} - W_{l}^{i}.
\]
If the maps $f$ and $\sigma$ are smooth enough, the above scheme has a global mean-square convergence rate of $O(h)$ (see Theorem 2.2 in Ref. 37). The numerical integrations in Section VI are carried out using the scheme (C4) with $h = 0.01$.

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