ON ALGEBRAIC FUSIONS OF ASSOCIATION SCHEMES

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ABSTRACT. We give a complete description of the irreducible representations of algebraic fusions of association schemes, in terms of the irreducible representations of a Schur cover of the corresponding group of algebraic automorphisms.

1. INTRODUCTION

In a recent paper [4], Hanaki investigates the relationship between the irreducible representations of association schemes and those of their fusion schemes. Special attention is paid to the case of algebraic fusions, for which Hanaki proves two main results (Theorem 4.3 and Theorem 4.5) that he calls Clifford type theorems. This article can be considered as a sequel to [4]. Our main result is a duality correspondence theorem that describes the irreducible representations of the algebraic fusion $A^G$ of $A$ in terms of the irreducible representations of a Schur cover $C(G)$ of the group $G$ of algebraic automorphisms of $A$. This kind of duality is a known phenomenon in representation theory, and in fact can be stated in a more general setting (see e.g. [2, Section 4.2]). Some particular manifestations of this duality are referred to in the literature as Schur-Weyl duality, Howe duality and the theta correspondence. Nevertheless, it seemed beneficial to include a detailed description and a proof in the case of association schemes, especially since our case requires the use of a Schur cover of the group $G$.

In section 2 we give all necessary preliminaries, and it is divided into two parts, the first deals with association schemes, and the second deals with projective representations of groups and the Schur multiplier. The following section 3 is the main section where we develop the needed machinery and prove the main duality correspondence theorem. In the final section 4 we give an application of our theory in the form of describing the irreducible representations of the exponentiation of an association scheme by the symmetric group $S_k$. As a byproduct we obtain the famous Schur-Weyl duality.

2. PRELIMINARIES

The goal of this section is to fix notation and state all the previous results that we will use. The first part of this section introduces all basic concepts and notation regarding association schemes. The notation related to the algebraic aspects of schemes largely coincides with Hanaki’s notation in [3, 4]. The second part is a short exposition of the theory of projective representations of finite groups and its relation to the Schur multiplier and Schur covers, loosely based on the first section of Wiegold’s [5].
2.1. **Association schemes.** Let $X$ be a set of $n$ elements, and let $\mathcal{R} = \{R_0, R_1, \ldots, R_d\}$ be a partition of $X \times X$. The pair

$$\mathcal{A} = (X, \mathcal{R})$$

is called an association scheme (or simply a scheme) if the following conditions hold:

**(AS1)** $R_0 = \{(x, x) \mid x \in X\},$

**(AS2)** for every $R_i \in \mathcal{R}$ we have $R_i^c = \{(y, x) \mid (x, y) \in R_i\} \in \mathcal{R},$

**(AS3)** for a pair $(x, y) \in R_k$, the number $|\{z \mid (x, z) \in R_i, (z, y) \in R_j\}|$ depends only on $i, j, k$ and not on the choice of the pair $(x, y) \in R_k$. This number is denoted $p_{ij}^k$, and called an intersection number.

The $R_i$’s are called the basic relations of $\mathcal{A}$. For every $0 \leq i \leq d$, let $A_i$ be the adjacency matrix of the relation $R_i$. Each $A_i$ is an $n \times n \{0, 1\}$-matrix indexed by the elements of $X$, where

$$(A_i)_{x,y} = 1 \iff (x, y) \in R_i.$$  

Now the above conditions can be formulated in terms of the adjacency matrices as follows

**(AS1’)** $A_0 = I, A_0 + A_1 + \cdots + A_d = J$, where $J$ is the all-ones matrix,

**(AS2’)** for every $0 \leq i \leq d$, there is $0 \leq i' \leq d$ such that $A_i^{i'} = A_{i'},$

**(AS3’)** $A_i \cdot A_j = \sum_{k=0}^d p_{ij}^k A_k.$

The first condition implies that the matrices $\{A_0, \ldots, A_d\}$ form a linearly independent set (as a vector space over $\mathbb{C}$), the third condition says that the linear span $\mathbb{C}\mathcal{A} = \text{Span}_\mathbb{C}(A_0, \ldots, A_d)$ is closed under matrix multiplication, and therefore forms an algebra over $\mathbb{C}$. The algebra $\mathbb{C}\mathcal{A}$ is called the adjacency algebra of $\mathcal{A}$, the $A_i$’s are called the basic matrices of $\mathcal{A}$.

2.1.1. **Representations and characters of association schemes.** It is a well-known fact (see e.g. [3]) that the adjacency algebra $\mathbb{C}\mathcal{A}$ is semisimple. A representation of $\mathcal{A}$ is just a linear representation of $\mathbb{C}\mathcal{A}$, i.e. a $\mathbb{C}$-algebra homomorphism

$$\rho : \mathbb{C}\mathcal{A} \longrightarrow \text{End}_\mathbb{C}(V) \cong M_k(\mathbb{C})$$

where $V$ is a left $\mathbb{C}\mathcal{A}$-module with $\dim_\mathbb{C} V = k$. For a representation $\rho$, the function:

$$\chi_\rho : \mathbb{C}\mathcal{A} \longrightarrow \mathbb{C}$$

defined by

$$\chi_\rho(m) = \text{tr}(\rho(m))$$

defined by

is called the character of $\rho$.

By semisimplicity, every representation of $\mathcal{A}$ is completely reducible and we denote by $\text{IRR}(\mathcal{A})$ the set of irreducible representations of $\mathcal{A}$, and by $\text{Irr}(\mathcal{A})$ the set of irreducible characters of $\mathcal{A}$. For a character $\chi$ of $\mathcal{A}$ let $V_\chi$ be a left $\mathbb{C}\mathcal{A}$-module affording $\chi$, and set:

$$(\chi, \chi')_{\mathcal{A}} = \dim_\mathbb{C} \text{Hom}_{\mathbb{C}\mathcal{A}}(V_\chi, V_{\chi'}).$$

By Schur’s lemma, if $\chi, \chi' \in \text{Irr}(\mathcal{A})$ then $(\chi, \chi')_{\mathcal{A}} = \delta_{\chi, \chi'}$. We will often abuse notation and use $\mathcal{A}$ instead of $\mathbb{C}\mathcal{A}$, but it should be clear from the context that we mean the adjacency algebra.
2.1.2. Automorphisms of association schemes. Let $A = (X, \mathcal{R})$ and $B = (Y, S)$ be schemes. A (combinatorial) isomorphism between $A$ and $B$ is a pair of bijections $\phi : X \to Y$ and $\psi : \mathcal{R} \to S$ that satisfy the condition that

$$(a, b) \in R \iff (\phi(a), \phi(b)) \in \psi(R).$$

A (combinatorial) isomorphism from $A$ to itself is called a (combinatorial) automorphism if $\psi$ is the identity map. All other (combinatorial) isomorphisms from $A$ to itself are called color automorphisms of $A$. We denote the group of (combinatorial) automorphisms of $A$ by $\text{Aut}(A)$, and the group of color automorphisms of $A$ by $\text{CAut}(A)$. It is well-known and easy to check that $\text{Aut}(A) \subseteq \text{CAut}(A)$.

An algebraic isomorphism between $A$ and $B$ is a mapping $\psi : \mathcal{R} \to S$ such that $p_{\psi(i)\psi(j)}^\psi = p_{ij}^k$ (here we slightly abused notation when writing $\psi(i)$ while meaning $\psi(R_i)$). The above mapping is defined on the basic relations and thus on the basic matrices of $A$, the name algebraic automorphism comes from the fact that the $\mathbb{C}$-linear extension of this mapping to $\mathbb{C}A$ is an algebra automorphism. Another well-known fact is that the quotient group $\text{CAut}(A) / \text{Aut}(A)$ embeds into $A\text{Aut}(A)$. Algebraic automorphisms which are not induced by color automorphisms are called proper algebraic automorphisms.

2.1.3. Fusion schemes. Let $A = (X, \mathcal{R})$ be a scheme. A scheme $B = (X, S)$ (with the same underlying set $X$) is called a fusion scheme of $A$ if every basic relation of $B$ is a union of some basic relations of $A$, that is, if for every $R \in \mathcal{R}$ there exists an $S \subseteq S$ such that $R \subseteq S$. We also say that $A$ is a fusion of $B$. By definition, it is clear that the adjacency algebra $\mathbb{C}B$ of the fusion scheme is a subalgebra of $\mathbb{C}A$.

If $G \leq A\text{Aut}(A)$ is a group of algebraic automorphisms then merging each orbit of the action of $G$ on the set $\mathcal{R} = \{R_0, \ldots, R_d\}$ of basic relations produces a fusion scheme $A^G$, called the algebraic fusion of $A$ with respect to $G$. Explicitly, $A^G = (X, S)$, where each $S \subseteq S$ is a relation formed by unifying the basic relations $R \in \mathcal{R}$ in an orbit.

2.1.4. Induction, restriction and Frobenius reciprocity. Let $B$ be a fusion scheme of $A$. For a left $A$-module $V$ we define the $B$-module $V \downarrow_B$ by restricting the action of $A$ on $V$ to $B$, the module $V \downarrow_B$ is called the restriction of $V$ to $B$. For a left $B$-module $W$ we define the $A$-module $W \uparrow_A = A \otimes_B W$, the module $W \uparrow_A$ is called the induction of $W$ to $A$. The following is Frobenius reciprocity for schemes and their fusions (see [4, Theorem 2.4]).

**Proposition 2.1.** Let $B$ be a fusion scheme of $A$. Let $\chi \in \text{Irr}(A)$ and $\eta \in \text{Irr}(B)$ then we have:

$$\langle \eta, \chi \downarrow_B \rangle_B = \langle \eta \uparrow_A, \chi \rangle_A.$$

2.2. Projective representations of groups and the Schur multiplier. Let $G$ be a finite group. A projective representation on a finite dimensional vector space, $V$, is a homomorphism

$$\varphi : G \to \text{PGL}(V).$$

Such a homomorphism is a collection of matrices, $\{\varphi(g) \mid g \in G\}$, that satisfies

$$\varphi(g)\varphi(h) = \lambda(g, h)\varphi(gh),$$

where $\lambda(g, h)$ is a non-zero scalar.
for all $g, h \in G$, and some scalar $\lambda(g, h) \in \mathbb{C}^\times$. The associativity of matrix multiplication implies

$$
(2.1) \quad \lambda(h, k)\lambda(g, hk) = \lambda(g, h)\lambda(gh, k),
$$

for all $g, h, k \in G$. Conversely, for any function $\kappa : G \times G \to \mathbb{C}^\times$ satisfying the condition (2.1) we can define a projective representation in the following manner: for $g \in G$ let $\overline{\psi}(g)$ be the $|G| \times |G|$-matrix with rows and columns indexed by the elements of $G$, having $(h, g^{-1}h)$-entry $\kappa(g, g^{-1}h)$. Suppose now that we had made a different choice for the set of automorphisms $\{\overline{\psi}(g) \mid g \in G\}$, and let $\kappa : G \times G \to \mathbb{C}^\times$ be its associated function, then $\overline{\varphi}(g) = \alpha(g)\overline{\psi}(g)$ for some scalar $\alpha(g) \in \mathbb{C}^\times$, and we obtain

$$
\overline{\varphi}(g)\overline{\varphi}(h) = \lambda(g, h)\overline{\varphi}(gh) = \lambda(g, h)\alpha(gh)\overline{\psi}(gh),
$$
on the other hand, we have

$$
\overline{\varphi}(g)\overline{\varphi}(h) = \alpha(g)\overline{\psi}(g)\overline{\psi}(h) = \alpha(g)\alpha(h)\kappa(g, h)\overline{\psi}(gh).
$$

Thus, for all $g, h \in G$ we have

$$
(2.2) \quad \lambda(g, h) = \frac{\alpha(g)\alpha(h)}{\alpha(gh)}\kappa(g, h).
$$

Two functions $\lambda, \kappa : G \times G \to \mathbb{C}^\times$ are equivalent if there exists a function $\alpha : G \to \mathbb{C}^\times$ such that (2.2) holds. The set of equivalence classes can be given a natural group structure by $[\lambda]\kappa = [\alpha]\kappa$, where $[\alpha]$ is just multiplication of functions.

The functions $G \times G \to \mathbb{C}^\times$ satisfying (2.1) are called 2-cocycles (with values in $\mathbb{C}^\times$), this set is denoted by $Z^2(G, \mathbb{C}^\times)$. Those 2-cocycles $\lambda \in Z^2(G, \mathbb{C}^\times)$ satisfying $\lambda(g, h) = \frac{\alpha(g)\alpha(h)}{\alpha(gh)}$ for some $\alpha : G \to \mathbb{C}^\times$ are called 2-coboundaries, and the set of 2-coboundaries is denoted $B^2(G, \mathbb{C}^\times)$. The above described group structure on equivalence classes of 2-cocycles is in fact just the factor group $Z^2(G, \mathbb{C}^\times) / B^2(G, \mathbb{C}^\times)$, it is denoted

$$
H^2(G, \mathbb{C}^\times) = Z^2(G, \mathbb{C}^\times) / B^2(G, \mathbb{C}^\times),
$$

and nowadays called the second cohomology group of $G$ with coefficients in $\mathbb{C}^\times$. In Schur’s language this is the ”Multiplierator” of $G$, we will call it the Schur multiplier of $G$, and denote it by $M(G)$.

The Schur multiplier arises naturally when one studies the central extensions of a group. A normal subgroup $N \trianglelefteq H$ is called central if $N$ is contained in the center of $H$. If $\rho : H \to GL_m(\mathbb{C})$ is an irreducible representation, then by Schur’s lemma $\rho(n) = \omega(n)I_m$ for each $n \in N$, where $\omega(n)$ is an $m$-th root of unity. Thus we have a projective representation of $K = H / N$:

$$
\overline{\varphi} : K \to PGL_m(\mathbb{C}) = GL_m(\mathbb{C}) / Z,
$$

where $Z$ is the center of $GL_m(\mathbb{C})$ (the scalar matrices), by defining

$$
\overline{\varphi}(hN) = \rho(h)Z.
$$

Schur proved the following converse: if $G$ is any finite group, there exists a group $H$ and a central subgroup $N \trianglelefteq H$ such that

$$
G \cong H / N,
$$

and every irreducible projective representation of $G$ is obtained from a linear representation of $H$ in the way described above. For a given $G$, the $H$ of smallest
order were called by Schur "Darstellungsgruppen", literally meaning "representation groups". Nowadays they are called covering groups, we will refer to them as Schur covers of $G$, and denote such a Schur cover by $C(G)$. Different Schur covers of $G$ can be non-isomorphic, but Schur proved that they must have isomorphic derived groups. Later, it was proved that different Schur covers of $G$ are isoclinic. On the other hand, the Schur multiplier $M(G)$ of $G$ is unique (up to canonical isomorphism).

3. ALGEBRAIC FUSIONS AND THEIR REPRESENTATIONS

Let $\mathcal{A} = (X, \mathcal{R})$ be an association scheme, and let $G \leq A\text{Aut}(\mathcal{A})$ be a finite group of algebraic automorphisms, and $\mathcal{A}^G$ the corresponding algebraic fusion. Since every $g \in G$ acts on $\mathbb{C}\mathcal{A}$ as an algebra automorphism, the group $G$ acts on $\text{IRR}(\mathcal{A})$ by

$$\pi^g(A_i) = \pi(A_{i g^{-1}})$$

for an irreducible representation $\pi \in \text{IRR}(\mathcal{A})$ and $g \in G$, and correspondingly on $\text{Irr}(\mathcal{A})$ by

$$\chi^g(A_i) = \chi(A_{i g^{-1}})$$

for an irreducible character $\chi \in \text{Irr}(\mathcal{A})$ and $g \in G$. Due to an abundance of actions we will use the notation $ga$ for the image of $a$ under the action of $g$, for $a \in \mathbb{C}\mathcal{A}$ and $g \in G$.

The following basic fact appears in [4, Lemma 4.1]:

**Lemma 3.1.** The adjacency algebra $\mathbb{C}\mathcal{A}^G$ is the algebra of fixed points

$$\mathbb{C}\mathcal{A}^G = \{ m \in \mathbb{C}\mathcal{A} \mid Gm = m \}.$$ 

Since $\mathcal{A}$ is a semisimple algebra, by the Wedderburn-Artin theorem, it admits a decomposition

$$\mathcal{A} \cong \bigoplus_j M_{n_j}(\mathbb{C}).$$

Each matrix algebra is the union of minimal left ideals isomorphic to an irreducible module of $\mathcal{A}$. Furthermore, each irreducible module is of dimension $n_j$, and the representation of $\mathcal{A}$ is just the projection on the respective component. In this description, an algebraic automorphism of $\mathcal{A}$ can either rearrange components of the same dimension or act on a component via conjugation, due to the Skolem-Noether theorem. If an algebraic automorphism fixes an irreducible module it also fixes (as a set) the corresponding block in the Wedderburn-Artin decomposition.

The following series of Lemmas shows that in order to obtain all irreducible representations of $\mathcal{A}^G$ it suffices to decompose the restrictions of a set of representatives of the orbits of $\text{IRR}(\mathcal{A})$ under the action of $G$. In fact, we show that $\text{Irr}(\mathcal{A}^G)$ is the disjoint union of the decompositions of such a set of representatives. First, we show that two irreducible characters from the same orbit restrict to the same character.

**Lemma 3.2.** Let $\mathcal{O} \subseteq \text{Irr}(\mathcal{A})$ be an orbit under the action of $G$, let $\chi, \chi' \in \mathcal{O}$ then

$$\chi \downarrow_{\mathcal{A}^G} = \chi' \downarrow_{\mathcal{A}^G}.$$
Proof. We show that $\chi \downarrow_{A^G} = \chi' \downarrow_{A^G}$ by showing that $(\eta, \chi \downarrow_{A^G})_{A^G} = (\eta, \chi' \downarrow_{A^G})_{A^G}$ for every $\eta \in \text{Irr}(A^G)$. By Frobenius reciprocity we have

$$(\eta, \chi \downarrow_{A^G})_{A^G} = (\eta \uparrow^A, \chi)_{A}$$

Now, by [4, Theorem 4.3] we have

$$(\eta \uparrow^A, \chi)_{A} = (\eta \uparrow^A, \chi^g)_{A}$$

for all $g \in G$. In particular, for the element $g \in G$ such that $\chi' = \chi^g$ we have

$$(\eta \uparrow^A, \chi)_{A} = (\eta \uparrow^A, \chi')_{A}.$$ 

Using Frobenius reciprocity again we obtain

$$(\eta, \chi \downarrow_{A^G})_{A^G} = (\eta, \chi' \downarrow_{A^G})_{A^G}.$$ 

□

Next, we show that if we have two irreducible characters from different orbits then the two corresponding sets of irreducible constituents in the algebraic fusion are disjoint.

Lemma 3.3. Let $O, O' \subseteq \text{Irr}(A)$ be two different orbits under the action of $G$. Let $\chi \in O$ and $\chi' \in O'$, then

$$(\chi \downarrow_{A^G}, \chi' \downarrow_{A^G})_{A^G} = 0.$$ 

Proof. Assume the contrary. This implies that there exists some $\eta \in \text{Irr}(A^G)$ such that $(\eta, \chi \downarrow_{A^G})_{A^G} \neq 0$ and $(\eta, \chi' \downarrow_{A^G})_{A^G} \neq 0$. By Frobenius reciprocity we have that

$$(\eta \uparrow^A, \chi)_{A} \neq 0$$

and

$$(\eta \uparrow^A, \chi')_{A} \neq 0.$$ 

But according to [4, Theorem 4.3], the induction of an irreducible character $\eta \in \text{Irr}(A^G)$ decomposes into a sum of characters from a single orbit under the action of $G$ on $\text{Irr}(A)$. This is a contradiction. □

Now we show that in order to obtain $\text{Irr}(A^G)$ it suffices to decompose all elements of $\text{Irr}(A)$:

Lemma 3.4. Every irreducible character $\eta \in \text{Irr}(A^G)$ appears in the decomposition of $\chi \downarrow_{A^G}$ for some $\chi \in \text{Irr}(A)$.

Proof. Assume by contradiction that $\eta \in \text{Irr}(A^G)$ is not a constituent of $\chi \downarrow_{A^G}$, for every $\chi \in \text{Irr}(A)$. Then

$$(\eta, \chi \downarrow_{A^G})_{A^G} = 0$$

for all $\chi \in \text{Irr}(A)$. By Frobenius reciprocity we have:

$$(\eta \uparrow^A, \chi)_{A} = 0$$

for all $\chi \in \text{Irr}(A)$, which is a contradiction. □

So we have:

Corollary 3.5. Let $\chi_0, \ldots, \chi_k \in \text{Irr}(A)$ be a set of representatives of the orbits of $\text{Irr}(A)$ under the action of $G$. Then the disjoint union of decompositions of $\chi_0 \downarrow_{A^G}, \ldots, \chi_k \downarrow_{A^G}$ is the set $\text{Irr}(A^G)$ of irreducible characters of the algebraic fusion $A^G$. 

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We now turn our focus to the main question of this paper: how do restrictions of irreducible characters of $A$ decompose in $A^G$? This question has a very nice answer in terms of (group) representations of a Schur cover $C(G)$ of $G$.

Let $\chi \in \Irr(A)$. Note that by [4, Theorem 4.5] there is a 1-1 correspondence between the decomposition of $\chi \downarrow_{A^G}$ and the decomposition of $\chi \downarrow_{A^G}$, therefore we may assume that $G$ fixes $\chi$.

**Proposition 3.6.** Let $G \leq \AAut(A)$ and let $\chi \in \Irr(A)$ be fixed by the action of $G$. Then $\chi$ induces a projective representation $\varphi$ of $G$ on $V_\chi$.

**Proof.** Let $(V_\chi, \pi_\chi) \in \IRR(A)$ be a representation affording $\chi$. Since $V_\chi$ is fixed by the action of $G$, we have that $V_\chi \cong V_{\chi^g}$ for every $g \in G$. Let

$$\varphi: G \rightarrow \GL(V_\chi)$$

be defined by

$$\varphi(g) = \varphi_g,$$

where $\varphi_g$ is the automorphism implementing $V_\chi \cong V_{\chi^g}$. Then we have, for every $a \in A$:

$$\varphi_g \pi_\chi(g^{-1}a) = \pi_\chi(a) \varphi_g.$$

Now for any $g, h \in G$, we have:

$$\varphi_g \varphi_h \pi_\chi(h^{-1}g^{-1}a) = \varphi_g \pi_\chi(g^{-1}a) \varphi_h = \pi_\chi(a) \varphi_g \varphi_h.$$

On the other hand:

$$\pi_\chi(h^{-1}g^{-1}a) = \pi_\chi((gh)^{-1}a) = \varphi_{gh}^{-1} \pi_\chi(a) \varphi_{gh}^{-1}.$$

This implies:

$$\varphi_g \varphi_h \pi_\chi(a) = \pi_\chi(a) \varphi_g \varphi_h \varphi_{gh}^{-1}.$$

Therefore there exists a scalar $\lambda(g, h)$, such that

$$\varphi_g \varphi_h = \lambda(g, h) \varphi_{gh}.$$

Hence we obtain a projective representation $\varphi$ of $G$ on $V_\chi$. $\square$

**Remark 3.7.** Note that in the Wedderburn-Artin decomposition, $G$ fixes the corresponding block. This implies that we can consider $G$ as acting on $M_n(\mathbb{C})$, where $\dim V_\chi = n$. The action is in fact conjugation by the $\varphi_g$.

**Corollary 3.8.** Let $G \leq \AAut(A)$, and let $\chi \in \Irr(A)$ be fixed by $G$, then

$$\varphi: C(G) \rightarrow \GL(V_\chi)$$

is a representation of the Schur cover $C(G)$ of $G$.

**Corollary 3.9.** If $M(G)$ is trivial, then we in fact get a representation of $G = C(G)$.

We are now ready to state and prove the duality correspondence theorem which describes the decomposition of $\chi \downarrow_{A^G}$ into irreducibles. We will denote by $\IRR(C(G))$ the set of irreducible representations of $C(G)$.

**Theorem 3.10.** Let $G \leq \AAut(A)$ and let $\chi \in \Irr(A)$ be fixed by the action of $G$. Let $(V_\chi, \pi_\chi)$ be a module affording $\chi$. Let $\varphi$ be the corresponding representation of $C(G)$ on $V_\chi$. Then there is a bijection:

$$\iota: \{\rho \in \IRR(C(G)) \mid \rho \subset \varphi\} \rightarrow \{\eta \in \IRR(A^G) \mid (\eta, \chi \downarrow_{A^G})_{A^G} \neq 0\}.$$
Furthermore, for $\eta = \iota(\rho)$ we have

$$(\eta, \chi \downarrow_{A^G})_{A^G} = \dim \rho,$$
$$\dim \eta = (\rho, \varphi)_{C(G)}.$$

**Proof.** Since the representation of $A$ on $V_\chi$ is realized by projecting onto the relevant block in the Wedderburn-Artin decomposition and $G$ fixes that block, we can identify the image $\pi_\chi(A)$ with $M_n(\mathbb{C})$. Furthermore, for each $a \in A^G$, we get:

$$\varphi^{-1}_g \pi_\chi(a) \varphi_g = \pi_\chi(a).$$

Let

$$C = \text{Span}_G (\varphi_g \mid g \in G),$$

be the subalgebra of $M_n(\mathbb{C})$ generated by the $\varphi_g$. Now since both the block and its complement are $G$-invariant, we can identify the image of $A^G$, with the algebra $C' = \{ m \in M_n(\mathbb{C}) \mid m \varphi_g = \varphi_g m \ \forall g \in G \}$, the centralizer ring of $C$. On the other hand, we can identify $C'$ with $\text{End}_{C(G)}(V_\chi)$, all linear endomorphisms of $V_\chi$, commuting with the action of $C(G)$.

For a subrepresentation $(U, \sigma)$ of $\varphi$, let $P$ be the projection on $U$. By the definition of a subrepresentation we have

$$P \varphi_g = \varphi_g P,$$

for every $g \in G$, hence $P \in A^G$ is an idempotent.

Let now $(W, \rho)$ be an irreducible constituent of $(V_\chi, \varphi)$ of multiplicity $m$. The isotypic component, $U(\rho)$, of $\varphi$, is the subspace spanned by all vectors satisfying

$$\varphi_g v = \rho_g v.$$

Note that we can write

$$V_\chi = \bigoplus_{\rho \in \text{IRR}(C(G))} U(\rho).$$

By Schur’s lemma, $\text{Hom}_{C(G)}(U(\rho), U(\tau)) = 0$, whenever $\rho \neq \tau$, hence:

$$C' = \text{End}_{C(G)}(V_\chi) = \bigoplus_{\rho} \text{End}_{C(G)}(U(\rho)).$$

In other words, each $U(\rho)$ is a $C'$-submodule of $V_\chi$.

In order to decompose this module into irreducibles, we note again that $a|_{U(\rho)} \in \text{End}_{C(G)}(U(\rho))$. Since $U(\rho) \cong W^\oplus m$, the action of $C(G)$ on $U(\rho)$ has the block matrix form:

$$
\begin{pmatrix}
\rho(g) & 0 & \cdots & 0 \\
0 & \rho(g) & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & \rho(g)
\end{pmatrix}.
$$

By Schur’s lemma $\text{End}_{C(G)}(W) = \mathbb{C}$, hence each $a \in C'$ has the following block matrix form on $U(\rho)$:

$$
\begin{pmatrix}
\alpha_{11} I & \alpha_{12} I & \cdots & \alpha_{1m} I \\
\alpha_{21} I & \alpha_{22} I & \cdots & \alpha_{2m} I \\
\vdots & \ddots & \ddots & \vdots \\
\alpha_{m1} I & \alpha_{m2} I & \cdots & \alpha_{mm} I
\end{pmatrix}.
$$
Here $I$ is the identity on $W$. In fact if we present $U(\rho) \cong \mathbb{C}^m \otimes \mathbb{C} W$, then $C(G)$ is acting by $I \otimes \rho$ and $\sigma$ is acting by $T_n \otimes I$, with $T_n \in M_m(\mathbb{C})$. Hence $U(\rho)$ as $C'$-module decomposes into $\dim W$ copies of the same irreducible $m$-dimensional $C'$-module. Explicitly, fix a basis $e_1, \ldots, e_r$ of $W$ and consider the subspaces of $U(\rho)$ defined by:

$$Q_j = \{v \otimes e_j \mid v \in \mathbb{C}^m\}.$$ 

Each $Q_j$ is clearly $C'$ invariant and irreducible. Furthermore $U(\rho) = \bigoplus_{j=1}^r Q_j$. \hfill \Box

### 4. Exponentiation of Association Schemes

Let $A = (X, R)$ be an association scheme. The $k$-th tensor power $A^\otimes k$ is the scheme with basic matrices $\{a_1 \otimes \cdots \otimes a_k \mid a_i \in R\}$. This scheme has an algebraic group of automorphisms isomorphic to $S_k \leq \text{Aut}(A^\otimes k)$ acting by:

$$\sigma(a_1 \otimes \cdots \otimes a_k) = a_{\sigma^{-1}(1)} \otimes \cdots \otimes a_{\sigma^{-1}(k)}.$$ 

**Definition 4.1.** The algebraic fusion of $A^\otimes k$ with respect to $S_k$ is called the exponentiation of $A$ by $S_k$ and is denoted by $A \uparrow S_k$.

It is well known that every irreducible representation of $A^\otimes k$ is of the form $V = V_1 \otimes \cdots \otimes V_k$, where $\{V_j, \pi_j\} \in \text{IRR}(A)$ and the action of $A^\otimes k$ is given by:

$$\pi(a_1 \otimes \cdots \otimes a_k)(v_1 \otimes \cdots \otimes v_k) = (\pi_1(a_1)v_1) \otimes \cdots \otimes (\pi_k(a_k)v_k).$$

The action of $S_k$ on the representations is then explicitly given by:

$$\pi^\sigma(a_1 \otimes \cdots \otimes a_k) = \pi(a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(k)}).$$

So the fixed points of the action of $S_k$ on $\text{IRR}(A^\otimes k)$ are precisely the representations of the form $(V^\otimes k, \pi^\otimes k)$, for $(V, \pi) \in \text{IRR}(A)$. By Corollary 3.8, we have a representation of a Schur cover $C(S_k)$, but we will see that in this case the associated cocycle is trivial, that is:

**Proposition 4.2.** Let $(V, \pi) \in \text{IRR}(A)$, then the projective representation of $S_k$ on $V^\otimes k$ is in fact a representation.

**Proof.** Let $\sigma \in S_k$ act on $V^\otimes k$ via:

$$\varphi_\sigma(v_1 \otimes \cdots \otimes v_k) = v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(k)}.$$ 

Then for $a_1 \otimes \cdots \otimes a_k \in A^\otimes k$:

$$\varphi_\sigma^{-1}(\pi(a_1) \otimes \cdots \otimes \pi(a_k))\varphi_\sigma(v_1 \otimes \cdots \otimes v_k) = \varphi_\sigma^{-1}(\pi(a_1)v_{\sigma^{-1}(1)} \otimes \cdots \otimes \pi(a_k)v_{\sigma^{-1}(k)})$$

$$= \pi(a_{\sigma(1)})v_1 \otimes \cdots \otimes \pi(a_{\sigma(k)})v_k.$$ 

The last expression is precisely $(\pi^\otimes k)^\sigma$. Now clearly $\varphi_\sigma \varphi_\tau = \varphi_{\sigma \tau}$ and hence it is a representation. \hfill \Box

To use the duality correspondence, we need to understand the decomposition of the representation $\varphi$ of $S_k$ on $V^\otimes k$. These are classical results that go back Young's construction of the famous Young diagrams and Young tableaux, and the following results by Frobenius, Schur and Weyl.
It is well known that for every partition \( \lambda \) of \( k \), there exists a unique irreducible representation of \( S_k \), that we shall denote by \( V_\lambda \), with \( \dim V_\lambda = m_\lambda \). By the duality correspondence (Theorem 3.10) we know that if
\[
V^\otimes k \cong \bigoplus_\lambda V_\lambda^\otimes m_\lambda
\]
is the irreducible decomposition of \( V^\otimes k \) as a representation of \( S_k \), then the irreducible decomposition of \( V^\otimes k \) as an \( \mathcal{A} \uparrow S_k \)-module is:
\[
V^\otimes k \cong \bigoplus_\lambda U_\lambda^\otimes m_\lambda,
\]
where \( U_\lambda = \iota(V_\lambda) \) and hence \( \dim U_\lambda = n_\lambda \).

Now let us describe the spaces \( U_\lambda \) in more detail based on [1, Lecture 6]. Let \( \lambda = (\lambda_1, \ldots, \lambda_d) \) be a partition of the integer \( k \). To the partition \( \lambda \) there is associated a Young diagram with \( \lambda_i \) boxes in the \( i \)-th row, aligned to the left. The conjugate partition \( \mu = (\mu_1, \ldots, \mu_l) \) to the partition \( \lambda \) is defined by interchanging the rows and columns of the Young diagram of \( \lambda \). For a given Young diagram we define the canonical Young tableau to be the numbering of the boxes in the diagram by the integers \( 1, \ldots, k \) in a left-to-right, up-to-down ordering. We can now define two subgroups of \( S_k \) by
\[
P_\lambda = \{ g \in S_k \mid g \text{ preserves each row} \},
\]
and
\[
Q_\lambda = \{ g \in S_k \mid g \text{ preserves each column} \}.
\]
To these two subgroups we associate the following elements in the group algebra \( \mathbb{C}S_k \):
\[
a_\lambda = \sum_{g \in P_\lambda} e_g \quad \text{and} \quad b_\lambda = \sum_{g \in Q_\lambda} \text{sgn}(g) e_g.
\]
Finally, we define the Young symmetrizer:
\[
c_\lambda = a_\lambda \cdot b_\lambda \in \mathbb{C}S_k.
\]

The Weyl module \( S_\lambda V \) is the image of the Young symmetrizer \( c_\lambda \) on \( V^\otimes k \):
\[
S_\lambda V = \text{Im} \left( c_\lambda|_{V^\otimes k} \right).
\]

By [1, Theorem 6.3] we have that \( U_\lambda = S_\lambda V \). The dimension of \( S_\lambda V \) is computed in a few nice and different ways in [1], maybe the most explicit one is given in [1, Theorem 6.3 (1)]:
\[
n_\lambda = \dim S_\lambda V = \prod_{1 \leq i < j \leq d} \frac{\lambda_i - \lambda_j + j - i}{j - i}.
\]
Here \( d = \dim V \), and \( \lambda \) is a partition of \( k \) into at most \( d \) parts of sizes \( \lambda_1 \geq \cdots \geq \lambda_d \geq 0 \).

All other irreducible representation of \( \mathcal{A}^\otimes k \) are of the form
\[
V_1^\otimes k_1 \otimes \cdots \otimes V_t^\otimes k_t
\]
with \( k_1 + \cdots + k_t = k \). The stabilizer of such a module is isomorphic to \( G = S_{k_1} \times \cdots \times S_{k_t} \) as a subgroup of \( S_k \). Similarly to Proposition 4.2, this is not merely a projective representation, but in fact a linear representation of \( G \). We know that for a direct product of groups the irreducible representations are precisely
the tensor products of the irreducible representations of each one of the constituents. Hence if $V_{\lambda_j}$ is an irreducible representation of $S_{k_j}$ corresponding to a partition $\lambda_j$ of $k_j$, then we have that as a representation of $G$:

$$V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_t} \cong \bigoplus_{\lambda_1, \ldots, \lambda_t} (V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_t})^{n_{\lambda_1} \cdots n_{\lambda_t}}.$$ 

Hence the irreducible decomposition of this module as an $\mathcal{A} \uparrow S_k$-module is:

$$V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_t} \cong \bigoplus_{\lambda_1, \ldots, \lambda_t} U_{\lambda_1, \ldots, \lambda_t}.$$ 

Here

$$m_{\lambda_1, \ldots, \lambda_t} = \prod_{j=1}^t \dim V_j \quad \text{and} \quad U_{\lambda_1, \ldots, \lambda_t} = S_{\lambda_1} V_1 \otimes \cdots \otimes S_{\lambda_t} V_t,$$

in particular

$$\dim U_{\lambda_1, \ldots, \lambda_t} = n_{\lambda_1} \cdots n_{\lambda_t}.$$ 

**Remark 4.3.** At this point it is easy to see the Schur-Weyl duality in any algebraic fusion with respect to the symmetric group. For example, in the case of an exponentiation $\mathcal{A} \uparrow S_k$, and a fixed irreducible representation $(V, \pi) \in \text{IRR}(\mathcal{A})$, the duality correspondence $\iota$ in Theorem 3.10 is precisely the Schur-Weyl duality.

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