Dynamic concentration of the triangle-free process

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Abstract

The triangle-free process begins with an empty graph on $n$ vertices and iteratively adds edges chosen uniformly at random subject to the constraint that no triangle is formed. We determine the asymptotic number of edges in the maximal triangle-free graph at which the triangle-free process terminates. We also bound the independence number of this graph, which gives an improved lower bound on the Ramsey numbers $R(3,t)$: we show $R(3,t) > (1/4 - o(1))t^2/\log t$, which is within a $4 + o(1)$ factor of the best known upper bound. Our improvement on previous analyses of this process exploits the self-correcting nature of key statistics of the process. Furthermore, we determine which bounded size subgraphs are likely to appear in the maximal triangle-free graph produced by the triangle-free process: they are precisely those triangle-free graphs with 2-density at most 2.

1 Introduction

Constrained random graph processes provide both an interesting class of random graphs models and a natural source for constructions in graph theory. Although the dependencies introduced by the constraints make such processes difficult to analyse, the evidence to date suggests that they are particularly useful for producing graphs of interest for certain extremal problems. Here we consider the triangle-free random graph process, which is defined by sequentially adding edges, starting with the empty graph, chosen uniformly at random subject to the constraint that no triangle is formed. Formally, let $G(0)$ be the empty graph on $n$ vertices. At stage $i$ we have a graph $G(i)$; we denote its edge set by $E(i)$, and let $O(i)$ be the set of pairs $xy$ that are open, in that $G(i) \cup \{xy\}$ has no triangle. We obtain $G(i+1)$ from $G(i)$ by adding a uniformly random pair from $O(i)$.

This process was introduced by Bollobás and Erdős (see [9]), and first analysed by Erdős, Suen and Winkler [12], using a differential equations method introduced by Ruciński and Wormald [21] for the analysis of the constrained graph process known as the ‘d-process’. One motivation for their work was that their analysis of the triangle-free process led to the best lower bound on the Ramsey number $R(3,t)$ known at that time. The Ramsey number $R(s,t)$ is the least number $n$ such that any graph on $n$ vertices contains a complete graph with $s$ vertices or an independent set with $t$ vertices. In general, very little is known about these numbers, even approximately. The upper bound $R(3,t) = O(t^2/\log t)$

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was obtained by Ajtai, Komlós and Szemerédi [1], but for many years the best known lower bound, due to Erdős [11], was $\Omega(\frac{t^2}{\log^2 t})$. The order of magnitude was finally determined by Kim [15], who showed that $R(3, t) = \Omega(\frac{t^2}{\log t})$. He employed a semi-random construction that is loosely related to the triangle-free process, thus leaving open the question of whether the triangle-free process itself achieves this bound; this was conjectured by Spencer [23] and proved by Bohman [5]. There is now a large literature on the general $H$-free process, obtained by replacing ‘triangle’ by any fixed graph $H$ in the definition; see [8, 10, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31]. However, the theory is still very much in its early stages: we conjectured that our lower bound for $H$ strictly 2-balanced, given in [8], gives the correct order of magnitude for the length of the process, but so far this has only been proved for some special graphs.

In this paper we specialise to the triangle-free process, where we can now give an asymptotically optimal analysis. Our improvement on previous analyses of this process exploits the self-correcting nature of key statistics of the process. For a treatment of self-correction in a simpler context see [6].

The methods that we use to establish self-correction of the triangle-free process build on the ideas used recently by Bohman, Frieze and Lubetzky [7] for an analysis of the triangle-removal process. Furthermore, the results of this paper have also been obtained independently and simultaneously by Fiz Pontiveros, Griffiths and Morris; their proof also exploits self-correction, but is different to ours in some important ways.

Let $G$ be the maximal triangle-free graph at which the triangle-free process terminates.

**Theorem 1.1.** With high probability, every vertex of $G$ has degree $(1 + o(1)) \sqrt{\frac{2}{\log n}}$. Thus the number of edges in $G$ is \( \left( \frac{2}{2 \sqrt{2}} + o(1) \right) (\log n)^{1/2} n^{3/2} \) with high probability.

We also obtain the following bound on the size of any independent set in $G$.

**Theorem 1.2.** With high probability, $G$ has independence number at most $(1 + o(1)) \sqrt{2n \log n}$.

An immediate consequence is the following new lower bound on Ramsey numbers. The best known upper bound is $R(3, t) < (1 + o(1)) \frac{t^2}{\log t}$, due to Shearer [22].

**Theorem 1.3.** $R(3, t) > \left( \frac{2}{3} - o(1) \right) \frac{t^2}{\log t}$.

These results are predicted by a simple heuristic. The graph $G(i)$ that we get after $i$ steps of the triangle-free process should closely resemble the Erdős-Rényi random graph $G_{n,p}$ with $i = n^2 p/2$, with the exception that $G_{n,p}$ should have many triangles while $G(i)$ has none.

In addition to Theorems 1.1 and 1.2 we show that this heuristic extends to all small subgraph counts; in particular, we answer the question of which subgraphs appear in $G$. Suppose $H$ is a graph with at least 3 vertices. The 2-density of $H$ is $d_2(H) = \frac{|E_H| - 1}{|V_H| - 2}$. The maximum 2-density $m_2(H)$ of $H$ is the maximum of $d_2(H')$ over subgraphs $H'$ of $H$ with at least 3 vertices.

**Theorem 1.4.** Let $H$ be a triangle-free graph with at least 3 vertices.

(i) If $m_2(H) \leq 2$ then $\mathbb{P}(H \subseteq G) = 1 - o(1)$.

(ii) If $m_2(H) > 2$ then $\mathbb{P}(H \subseteq G) = o(1)$.
Thus, the small subgraphs that are likely to appear in $G$ are exactly the same as the triangle-free subgraphs that appear in the corresponding $G_{n,p}$.

Note that the lower bound on $R(3, t)$ given by the triangle-free process is non-constructive; for an explicit construction of a triangle-free graph on $\Theta(t^{3/2})$ vertices with independence number less than $t$ see Alon [2]. Alon, Ben-Shimon and Krivelevich [3] gave a construction that can be applied to $G$ to produce a regular Ramsey $R(3, t)$ graph, at the cost of a worse constant in the lower bound on $R(3, t)$.

The bulk of this paper is occupied with the analysis required for the lower bound in Theorem 1.1. To prove this, we in fact prove much more generally that we can ‘track’ several ensembles of ‘extension variables’ for most of the process; this is formalised as Theorem 2.2. The proof of Theorem 2.2 is outlined in the next section, then implemented over the four following sections. In Section 3 we present some coupling and union bound estimates that are needed throughout the paper, and also prove Theorem 1.4 assuming Theorem 2.2. In Sections 4, 5 and 6, we prove Theorem 2.2 via a self-correcting analysis of three ensembles of random variables. Section 7 is mostly occupied by the proof of Theorem 1.2; it also contains the proof of the upper bound in Theorem 1.1, which is similar and easier. We conclude with some brief remarks in Section 8.

2 Overview of lower bound

In this section we outline the proof of the lower bound in Theorem 1.1. We are guided throughout by the heuristic that $G(i)$ should resemble $G_{n,p}$ with $i = n^2 p / 2$. Before proceeding with the outline of the proof we mention a consequence of this heuristic that is central to the entire argument. We introduce a continuous time that scales as

$$t = n^{-3/2}.$$ 

Note that $p = 2tn^{-1/2}$. We define $Q(i)$ to be the number of open ordered pairs in $G(i)$. (So $Q(i) = 2|O(i)|$.) This variable is crucial to our understanding of the process. We have $Q(0) = n^2 - n$, and the process ends exactly when $Q(i) = 0$. How do we expect $Q(i)$ to evolve? If $G(i)$ resembles $G_{n,p}$ then for any pair $uv$ we should have

$$\Pr(uv \in O(i)) \approx (1 - p^2)^{n-2} \approx e^{-np^2} = e^{-4t^2}.$$ 

We set $q(t) = e^{-4t^2} n^2$ and expect to have

$$Q(i) \approx q(t)$$ 

for most of the evolution of the process. This is exactly what we prove.

2.1 Strategy

We use dynamic concentration inequalities for a carefully chosen ensemble of random variables associated with the process. We aim to show $V(i) \approx v(t)$ for all variables $V$ in the ensemble, for some smooth function $v(t)$, which we refer to as the scaling of $V$. Here $V(i)$ denotes the value of $V$ after $i$
steps of the process, and we scale time as \( t = in^{-3/2} \). For each \( V \) we define a tracking variable \( TV(i) \) and aim to show that \( DV(i) = V(i) - TV(i) \) satisfies \( |DV(i)| < e_{V}(t)v(t) \), for some error functions \( e_{V}(t) \). We use \( TV(i) \) rather than \( v(t) \) so that we can isolate variations in \( V \) from variations in other variables that have an impact on \( V \).

The improvement to earlier analysis of the process comes from ‘self-correction’, i.e. the mean-reverting properties of the system of variables. We take \( e_{V}(t) = f_{V}(t) + 2g_{V}(t) \), where we think of \( f_{V}(t) \) as the ‘main error term’ and \( g_{V}(t) \) as the ‘martingale deviation term’. We usually have \( g_{V} \ll f_{V} \), but there are some exceptions when \( t \) is small and hence \( f_{V}(t) \) is too small. We require \( g_{V}(t)v(t) \) to be ‘approximately non-increasing’ in \( t \), in that \( g_{V}(t)v(t') = O(g_{V}(t)v(t)) \) for all \( t' \geq t \).[1]

We define the critical window

\[
W_{V}(i) = [(f_{V}(t) + g_{V}(t))v(t), (f_{V}(t) + 2g_{V}(t))v(t)].
\]

We aim to prove the trend hypothesis: \( ZV(i) := |DV(i)| - e_{V}(t)v(t) \) is a supermartingale when \( |DV(i)| \in W_{V}(i) \).[2] The trend hypothesis will follow from the variation equation for \( e_{V}(t) \), which balances the changes in \( DV(i) \) and \( e_{V}(t)v(t) \). Since errors can transfer from one variable to another, each variation equation is a differential inequality that can involve many of the error functions.

We aim to track the process up to the time

\[
t_{\text{max}} = \frac{1}{2}(1/2 - \varepsilon) \log n.
\]

More precisely, we will define a stopping time \( I \) such that if \( I < i \) then every ‘good’ variable in our ensemble satisfies the required estimates at step \( i \) (the distinction between ‘good’ and ‘bad’ variables is discussed in Section 2.4). Setting \( i_{\text{max}} = t_{\text{max}}n^{3/2} \), it will suffice to show that \( I > i_{\text{max}} \) with high probability. If \( I \leq i_{\text{max}} \) then it will follow that there exists \( i^{*} = I \leq i_{\text{max}} \) and a ‘good’ variable \( V \) such that \( DV(i^{*}) \) is too large. In this situation \( DV(i) \) enters \( W_{V}(i') \) from below at some step \( i' < i^{*} \), stays in \( W_{V}(i) \) for \( i' \leq i \leq i^{*} \) then goes above \( W_{V}(i^{*}) \) at step \( i^{*} \). During this time \( ZV(i) \) is a supermartingale, with \( ZV(i') \leq -g_{V}(t)v(t') \) and \( ZV(i^{*}) \geq 0 \), so we have an increase of at least \( g_{V}(t)v(t') \) against the drift of the supermartingale. Then we use Freedman’s martingale inequality,[13] which is as follows.

**Lemma 2.1 (Freedman).** Suppose \( (X(i))_{i \geq 0} \) is a supermartingale with respect to the filtration \( \mathcal{F} = (\mathcal{F}_{i})_{i \geq 0} \). Suppose that \( X(i+1) - X(i) \leq B \) for all \( i \) and define \( V(j) = \sum_{i=1}^{j} \text{Var}(X(i) \mid \mathcal{F}_{i-1}) \). Then for any \( a, v > 0 \) we have

\[
\mathbb{P}(\exists i \text{ such that } X(i) \geq X(0) + a \text{ and } V(i) \leq v) \leq \exp\left(-\frac{a^{2}}{2(v + Ba)}\right).
\]

To apply Freedman’s inequality, we estimate

\[
\text{Var}_{V}(t) = \text{Var}(ZV(i) \mid \mathcal{F}_{i-1}) \quad \text{and} \quad N_{V}(t) = |ZV(i + 1) - ZV(i)|.
\]

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1. There will be one exceptional type of variable, the vertex degrees, for which this does not hold.
2. We generally work with the ‘upper critical window’, i.e. we consider \( |DV(i)| = DV(i) \). The situation where the difference variable is negative can be treated in exactly the same way with reversed signs. We also remark that we need to ‘freeze’ \( ZV(i) \) if \( V \) becomes ‘bad’, as explained below.
Since $g_V(t)v(t)$ is approximately non-increasing (unless $V$ is a vertex degree variable), it suffices to have
\[ g_V(t)^2v(t)^2 \geq \text{Var}_V(t)(n \log n)^3/2 \quad \text{and} \quad g_V(t)v(t) \geq N_V(t) \log n \]
to get the required bound, with subpolynomial failure probability. We refer to this as the \textit{boundedness hypothesis}.

The lower bound of Theorem \[1.1\] follows from an application of the union bound to the union of these events over all variables we consider and all points $i'$ when such a variable might enter the critical interval. Thus, after establishing the trend and boundedness hypotheses for all variables, we may conclude that $|D_V(i)| < e_V(t)v(t)$ for all $t \leq t_{max}$ with high probability. In other words, we can conclude $I > i_{max}$ with high probability.

2.2 Variables

All definitions are with respect to the graph $G(i)$. Sometimes we use a variable name to also denote the set that it counts, e.g. $Q(i)$ is the number of ordered open pairs, and also denotes the set of ordered open pairs. We usually omit $(i)$ and $(t)$ from our notation, e.g. $Q$ means $Q(i)$ and $q$ means $q(t)$. We use capital letters for variable names and the corresponding lower case letter for the scaling. We express scalings using the (approximate) edge density and open pair density; these are respectively

\[ p = 2tn^{-2} = 2tn^{-1/2} \quad \text{and} \quad \hat{q} = e^{-4t^2}. \]

The next most important variable in our analysis, after the variable $Q$ defined above, is the variable $Y_{uv}$ which, for a fixed pair of vertices $uv$, is the number of vertices $w$ such that $uw$ is an open pair and $vw$ is an edge. It is natural that $Y_{uv}$ should play an important role in this analysis, as when the pair $uv$ is added as an edge, the number of open edges that become closed is exactly $Y_{uv} + Y_{vu}$. The motivation for introducing the ensembles of variables defined below is as follows: control of the global variables is needed to get good control of $Q$, control of the stacking variables is needed to get good control of $Y_{uv}$, and controllable variables play a crucial role in our analysis of the stacking variables.

We note that the proof of the fact that we can track the controllable variables (up to the precision needed for our purposes) is relatively short. In a certain sense, our results on controllable variables can be viewed as a triangle-free process analog of the concentration on subgraph extensions that follows from Kim-Vu polynomial concentration \[10\]. (We remark in passing that a similar analog should hold for the triangle removal process, and the introduction of this idea would simplify the analysis of the triangle removal process recently given by Bohman, Frieze and Lubetzky \[7\].)

2.2.1 Global variables

We begin with the variable that we are most interested in understanding: the number of open pairs. We also include two other variables that will allow us to maintain precise control on the number of open pairs.

- $Q = 2|O(i)|$ is the number of ordered open pairs. The scaling is $q = \hat{q}n^2$. 

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• $R$ is the number of ordered triples with 3 open pairs. The scaling is $r = \hat{q}^3 n^3$.

• $S$ is the number of ordered triples $abc$ where $ab$ is an edge and $ac, bc$ are open pairs. The scaling is $s = \hat{p}\hat{q}^2 n^3 = 2t\hat{q}^2 n^{5/2}$.

We refer to $Q$, $R$ and $S$ as global variables.

### 2.2.2 Stacking variables

In order to understand the evolution of $Q$, $R$ and $S$, we introduce a large collection of stacking variables. The basic building blocks for the stacking variables are the following one-vertex extensions, which are defined for every ordered pair $uv$.

• $X_{uv}$ is the number of vertices $w$ such that $uw$ and $vw$ are open pairs. The scaling is $x = \hat{q}^2 n$.

• $Y_{uv}$ is the number of vertices $w$ such that $uw$ is an open pair and $vw$ is an edge. The scaling is $y = 2t\hat{q}n^{1/2}$.

We also need information about the degree and the open degree of a vertex.

• $X_u$ is the open degree of $u$, defined as the number of vertices $\omega$ such that $u\omega$ is open. The scaling is $x_1 = n\hat{q}$.

• $Y_u$ is the degree of $u$, defined as the number of vertices $\omega$ such that $u\omega$ is an edge. The scaling is $y_1 = 2tn^{1/2}$.

We refer to $Y_{uv}$ and $X_{uv}$ as codegree variables, to $Y_u$ as a degree variable, and to $X_u$ as an open degree variable.

A stacking variable is an iterated extension variable, in which each iteration adds an extension from the collection $X_y, Y_y, X_{xy}, Y_{xy}, Y_{yx}$, where $y$ is the vertex added in the previous extension, and $xy$ is an open pair in the previous extension. We will only track a subset of the collection of the stacking variables, the $M$-bounded stacking variables, which are defined below. Let $S$ be the set of sequences

$$\pi \in \{O, E, Y^O, X^O\} \times \{O, E, Y^I, Y^O, X^I, X^O\}^*$$

such that if $E$ occurs then it only does so as the last symbol of $\pi$. For any $\pi \in S$ and pair of vertices $uv$ (such that $uv \not\in E(i)$) we define $S^\pi_{uv}$ according to the following rules. Each element of $\pi$ indicates the next extension to be added and specifies an open pair in this extension to be the next rung in the stacking variable. We initiate by letting $uv$ be the active rung and letting $v$ be the last vertex. Suppose we have constructed $i - 1$ steps of our stacking variable and that we have an active rung $xy$ with last vertex $y$. If $\pi(i) = O$ then the next step is an $X_y$ extension with the single open pair in this extension the next rung. If $\pi(i) = E$ then the next step is an $Y_y$ extension and then there is no active rung: the variable terminates here.

Now suppose $\pi(i) \not\in \{O, E\}$; that is, suppose $\pi(i)$ indicates an $X$ or $Y$ extension on the active rung. The superscript indicates the direction of this extension. For $Y$ it determines whether we add $Y_{xy}$ or $Y_{yx}$, and the new open pair becomes the active rung. For $X$ it determines which of the two
new open pairs becomes the active rung. In both cases, a superscript of O (for ‘outer’) indicates that the new active rung is incident with the last vertex, \( y \), while a superscript of I (for ‘inner’) indicates that the next active rung is not incident with \( y \) (i.e. it is incident with \( x \)). We say that an open pair of \( \pi \) that is not a rung is a stringer open pair. Note that each occurrence of \( X \) in \( \pi \) gives one stringer open pair. Edges of \( \pi \) are called stringer edges. Note that any rung is a cutset of \( \pi \).

The simplest stacking variables are those of length 1, namely
\[
S_{X}^{O} = X, \quad S_{Y}^{O} = Y, \quad S_{O}^{O} = X, \quad S_{E}^{E} = Y
\]
and
\[
S_{X}^{O} = X, \quad S_{Y}^{O} = Y, \quad S_{O}^{O} = X, \quad S_{E}^{E} = Y
\]
The last example illustrates the general phenomenon that when \( \pi(1) \in \{O,E\} \) we obtain an extension based at the single vertex \( v \), which does not depend on \( u \). While we could denote this variable more simply by \( S_{\pi v} \), it is convenient to have a unified notation for stacking variables that allows the base of the extension to have one or two vertices.

Formally, we can view \( S_{\pi uv} \) as counting the number of injections \( \psi : \{\alpha_u, \alpha_v, \alpha_1, \ldots, \alpha_{|\pi|}\} \rightarrow [n] \) such that \( \psi(\alpha_u) = u \), \( \psi(\alpha_v) = v \) and \( \psi(\alpha_i) \) is a vertex that plays the role in the extension defined by \( \pi(i) \) for \( i = 1, \ldots, |\pi| \). Fix two rungs \( \alpha_x \alpha_y \) and \( \alpha_a \alpha_b \), where \( x < y < a < b \) and none of \( \pi(y + 1), \ldots, \pi(b) \) are \( O \) or \( E \). Let \( \pi[xy,ab] \) be the portion of \( S_{\pi uv} \) induced by the set of vertices \( \{\alpha_x, \ldots, \alpha_b\} \). We call this structure the triangular ladder of \( \pi \) cut off by \( xy \) and \( ab \). Note that any stacking variable is a concatenation of some number of triangular ladders and paths of open pairs, possibly ending with a pendant edge.

The concept of direction may be clarified by the following pictorial representation. We visualise \( \pi[xy,ab] \) as a horizontal strip of two rows, ‘top’ and ‘bottom’, where \( x \) and \( y \) are the leftmost elements of the two rows, and \( a \) and \( b \) are the rightmost two elements of the two rows (in some order). The remaining vertices are assigned so that rungs have one vertex in each row, whereas stringers lie within rows (this uniquely defines the assignment). Our convention for superscripts in \( X \) and \( Y \) corresponds to walking between the rows and describing whether the new vertex is added on the same row or the opposite row; thus \( I \) indicates that the new vertex is added to the same row, and \( O \) that it is added to the other row. Conversely, any such drawing determines a unique order \( v_1, \ldots, v_t \) of vertices, which we call the stacking order, from which we can reconstruct \( \pi \). Figure 1 illustrates a stacking variable corresponding to \( \pi = Y^O X^O X^O Y^O O Y^O Y^O X^I O O Y^I X^O O E \) (thick lines represent edges and thin lines represent open pairs).

To define the \( M \)-bounded stacking variables that play a key role in the proof, we need some additional terminology. Consider a triangular ladder \( \pi[ab,xy] \). We say that \( \alpha_i \) is a turning point in this ladder if the superscript of \( \pi(i + 1) \) is \( O \). Note that if \( \alpha_i \) is a turning point then it is in at least

Figure 1: The stacking variable corresponding to \( \pi = Y^O X^O X^O Y^O O Y^O Y^O X^I O O Y^I X^O O E \). Thick lines represent edges and thin lines represent open pairs.
two rungs. The open pairs containing $\alpha_i$ are $\alpha_i \alpha_i$ and $\alpha_j \alpha_i$ for $i + 1 \leq j \leq i^+$, for some $i^-$ and $i^+$, which are respectively the previous and next turning points (or non-existent if there are no such turning points). If $\alpha_i$ is in the top row (for example) then $\alpha_i$ and $\alpha_j$, $i + 1 \leq j \leq i^+$ are consecutive along the bottom row.

We define three weights associated with $\pi \in S$ as follows. The 1-weight $w_1(\pi)$ is the number of positions in $\pi$ that contain the symbols $O$ or $E$. The 2-weight $w_2(\pi)$ is the number of positions in $\pi$ that contain the symbols $XO$ or $YO$. The weight $w(\pi)$ is $w_1(\pi) + w_2(\pi)$. An $M$-fan in $\pi$ is a sequence of $M + 1$ consecutive positions of $\pi$ whose entries are in the set 

$$\{X^O, Y^O\} \times \{X^I, Y^I\}^{M-1} \times \{X^I, Y^I, X^O\}.$$

Note that an $M$-fan corresponds to a sequence of $M + 1$ vertices in the same row of a triangular ladder (flipping the position of the last element of the sequence if the last element is $X^O$). We say that a stacking variable $\pi$ is $M$-bounded if it satisfies the following conditions. (Note that the first of these is a condition on any $\pi \in S$ which we copy here for ease of reference.)

(i) If $E$ occurs in $\pi$ then it is the last symbol,

(ii) The sequences $OY^I$ and $OX^I$ do not appear in $\pi$ in any pair of positions other than the last two positions,

(iii) $w(\pi) \leq 2M$,

(iv) If $w(\pi) = 2M$ then the last symbol in $\pi$ is $O$, $E$, $X^O$ or $Y^O$, and

(v) $\pi$ does not contain an $M$-fan.

We let $S_M$ be the set of $M$-bounded stacking variables. Note that such a variable has length at most $2M^2 - M + 2 < 2M^2$.

2.2.3 Controllable variables

Finally, we formulate a very general condition under which we have some control on a variable. Suppose $\Gamma$ is a graph, $J$ is a spanning subgraph of $\Gamma$ and $A \subseteq V_\Gamma$. We refer to $(A, J, \Gamma)$ as an extension. Suppose that $\phi : A \rightarrow [n]$ is an injective mapping. We define the extension variables $X_{\phi, J, \Gamma}(i)$ to be the number of injective maps $f : V_\Gamma \rightarrow [n]$ such that

(i) $f$ restricts to $\phi$ on $A$,

(ii) $f(e) \in E(i)$ for every $e \in E_J$ not contained in $A$, and

(iii) $f(e) \in O(i)$ for every $e \in E_\Gamma \setminus E_J$ not contained in $A$.

We call $(J, \Gamma)$ the underlying graph pair of $X_{\phi, J, \Gamma}$. We introduce the abbreviations $V = X_{\phi, J, \Gamma}$,

$$n(V) = |V_\Gamma| - |A|, \quad e(V) = e_J - e_{J[A]}, \quad \text{and} \quad o(V) = (e_\Gamma - e_J) - (e_{\Gamma[A]} - e_{J[A]}).$$
The scaling is \( v = x_{A,J,G} = n^{n(V)}p^{e(V)}q^{o(V)} \). We expect \( V \approx v \), provided there is no subextension that is ‘sparse’, in that it has scaling much smaller than 1. Given \( A \subseteq B \subseteq B' \subseteq V_T \) we write
\[
S_B' = S_B'(J, \Gamma) = n|B|^{(e_J|B| - e_J|B'|)(e_I|B'| - e_I|B|) - (e_I|B| - e_I|B'|)}.
\]
For example, \( S_A'^V = v \). Note that if \( A \subseteq B \subseteq B' \subseteq V_T \) we have \( S_B^B' = S_B^B'' S_B^B' \). Note also that the letter ‘S’ is used for scalings and stacking variables, but there is no possibility for confusion, as the use is determined by the form of the superscript.

Let \( t' \geq 1 \). We say that \( V \) is controllable at time \( t' \) if \( J \neq \Gamma \) (i.e. at least one pair is open) and for \( 1 \leq t \leq t' \) and \( A \subseteq B \subseteq V_T \) we have
\[
S_B^B(J, \Gamma) \geq n^\delta,
\]
where \( \delta > 0 \) is a fixed global parameter that is sufficiently small given \( \varepsilon \). We note in passing that this condition is essentially identical to the condition needed to prove concentration of subgraphs counts in \( G_{n,p} \) using Kim-Vu polynomial concentration \([16]\). (See Lemma 3.1 below.) We say that \( V \) is controllable if it is controllable at time 1.

### 2.3 Tracking variables

Recall that each variable \( V \) has a tracking variable \( TV \) and we track the difference \( DV = V - TV \). We do this to isolate variations in \( V \) from other variations in \( G(i) \).

The tracking variables are defined as follows. For the global variables we take
\[
TQ = q, \quad TR = n^3 \cdot (Q/n^2)^3 = Q^3 n^{-3}, \quad TS = n^3 \cdot 2tn^{-1/2} \cdot (Q/n^2)^2 = 2tn^{-3/2}Q^2.
\]
If \( V \) is a one-vertex extension with \( a \) edges and \( b \) open pairs we take
\[
TV = n \cdot (2tn^{-1/2})^a \cdot (Q/n^2)^b.
\]
That is, we set \( TX_{uv} = Q^2 n^{-3} \) and \( TY_{uv} = 2tn^{-3/2}Q \) and \( TX_u = Qn^{-1} \).

For the stacking variable \( S^\pi_{uv} \) with \( |\pi| \geq 2 \) we have two cases, depending on the form of \( \pi \). If \( \pi(|\pi| - 1) \neq O \) or \( \pi(|\pi|) \in \{O, E\} \) we write \( \pi = \pi^- \cdot U \), where \( U \) is the last element of \( \pi \). Then
\[
TS^\pi_{uv} = S^\pi_{uv} TV.
\]

Now suppose \( \pi(|\pi| - 1) = O \) and \( \pi(|\pi|) \notin \{O, E\} \). We say that the open pair \( \alpha_{|\pi|-2} \alpha_{|\pi|-1} \) and the pair \( \alpha_{|\pi|-2} \alpha_{|\pi|-1} \) (which can be an edge or an open pair) are partner pairs. In this case we need to modify the tracking variable to take account of the rule concerning \( OY^I \) and \( OX^I \) in the definition of \( S_M \). We write \( \pi = \pi^- OU \), where \( U \) is again the last element of \( \pi \), \( \beta = \alpha_{|\pi|-2} \) and
\[
TS^\pi_{uv} = \begin{cases} 
\sum_{f \in S_{uv}^\pi \subseteq S_{uv}^\pi} X_{f(\beta)}^2 \cdot Qn^{-2} & \text{if } U \in \{X^I, X^O\} \\
\sum_{f \in S_{uv}^\pi \subseteq S_{uv}^\pi} X_{f(\beta)}^2 \cdot 2tn^{-1/2} & \text{if } U = Y^I \\
\sum_{f \in S_{uv}^\pi \subseteq S_{uv}^\pi} X_{f(\beta)} Y_{f(\beta)} \cdot Qn^{-2} & \text{if } U = Y^O,
\end{cases}
\]
where \( Y_b \) denotes the degree of the vertex \( b \).

The estimates that we prove for controllable variables are much weaker than the estimates for the variables discussed above. For a controllable variable \( V \) we simply take the tracking variable to be the scaling \( v \).  

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2.4 Error functions, stopping times, estimates, conventions, notation

We begin with the constant $M$. The value of $M$ depends on the length of time for which we want to track the process. Throughout the paper we fix a global parameter $\varepsilon > 0$, which can be taken arbitrarily small. We track the process until the time $t_{\text{max}}$ at which $\hat{q}(t_{\text{max}}) = n^{-1/2+\varepsilon}$. Thus

$$t_{\text{max}} = \frac{1}{2} \sqrt{(1/2 - \varepsilon) \log n}.$$  

We set $M = 3/\varepsilon$.

Now we turn to the error bounds for variables in our three ensembles of variables. We define parameters

$$e = \hat{q}^{-1/2} n^{-1/4} \quad \text{and} \quad L = \sqrt{\log n}.$$  

Our error functions take the form

$$e_V = f_V + 2g_V,$$

where

$$f_V = c_V \varphi_V \quad \text{and} \quad g_V = c_V \vartheta L^{-1} (1 + t^{-e(V)}) \varphi_V$$

and $\varphi_V$ is one of $e = \hat{q}^{-1/2} n^{-1/4}$, or $e^2 = \hat{q}^{-1} n^{-1/2}$ or $e^\delta$. We take $\varphi_V = e$ when $V$ is a stacking variable, $\varphi_V = e^2$ when $V$ is a global variable, and $\varphi_V = e^\delta$ when $V$ is a controllable general extension.

The function $\vartheta$ is introduced to deal with some technicalities that arise for small $t$. We let $K$ be a constant such that $K > M^6$ and let $\vartheta(t)$ be any increasing smooth function such that $\vartheta(t) = e^{Kt}$ for $0 \leq t \leq 1$, $\vartheta(t) \leq 2e^K$ for all $t$ and $\sup_{t \geq 0} |\vartheta''(t)|$ is finite. Note that the power of $t$ in $g_V$ is chosen so that the dominant term in $vg_V$ as $t \to 0$ does not contain a power of $t$. All of our variables except for vertex degrees have at least one open pair; thus $vg_V$ has a non-negative power of $\hat{q}$, so is approximately non-increasing ($\vartheta$ makes it increase by a constant factor).

The constants $c_V$ are all polylogarithmic. We specify them now in advance of the analysis, but we will keep the notation general so that it is clear how to choose the constants. For all controllable variables we take $c_V = 1$. For the global variables we take

$$c_R = L^{40}, \quad c_S = 2L^{40}, \quad c_Q = 4L^{40}.$$  

The constants for the stacking variables need to be chosen very carefully. The idea is that the constants should decrease as the length of $\pi$ increases, and there is a more substantial decrease for each occurrence of $O$ or $E$. For a stacking variable $V = S_{uv}^\pi$, we set

$$c_V = c_\pi = L^{15} g^{4M^2 - |\pi|-Mw_1(\pi)}.$$  

Now we define various stopping times associated with the process. Consider any variable $V$ in our ensembles, and write $V = X_{\phi,i,J}$ for some extension $(A, J, \Gamma)$. We say that $V$ is bad (at step $i$) if there is an edge $e$ in $G(i)$ such that $(J, \Gamma) + \phi^{-1}(e)$ contains a triangle or a path of length two joining the vertices of an open pair. If $V$ is not bad we say that $V$ is good. Note that if $\phi$ is bad then $V = 0$, so certainly it is not following its expected trajectory. For example, if $uv$ is an edge then $Y_{uv}$ is bad. We let $J_V$ be the smallest $i \geq i_V$ such that $V$ is bad, or $\infty$ if there is no such time. As indicated
earlier, we modify the definition above of the variable \( ZV(i) \) by defining \( ZV(i) = ZV(J_V - 1) \) for \( i \geq J_V \) to ensure that it trivially satisfies the supermartingale condition.

We only start tracking variables using differential equations at the point where we require a better bound than that given by coupling, in that the coupling bound does not rule out the variable entering its critical window. More precisely, we start tracking the variable \( V \) at the first value \( i = i_V \) such that \( i \geq n^{5/4} \) and \( g_V(t) \leq L^{-1} \). It follows that \( e_V = o(1) \) for any \( t > t_V = i_V n^{-3/2} \) for all variables in our ensembles.

Now we let \( I_V \) be the smallest \( i \) with \( i_V \leq i < J_V \) such that \( |DV(i)| > e_V(t) v(t) \), or \( \infty \) if there is no such time. Then we let \( I_1, I_2, I_3 \) respectively be the minimum of \( I_V \) over all variables \( V \) in the global, controllable, stacking ensemble. Finally, we let \( I = \min\{I_1, I_2, I_3\} \). Note that if \( I < i_{max} \) then there is some \( V \) such that \( I = I_V = i^* \leq i_{max}, \) i.e. \( \mathcal{D}V(i^*) \) is too large and \( V \) is good at step \( i^* \), as required for the strategy described above. We emphasize that, since we can restrict our attention to \( i < I \), we may assume that \( |DV(i)| \leq e_V(t) v(t) \) for all good variables \( V \) when verifying the trend and boundedness hypotheses.

We prove the following theorem over the following four sections. In the next section we estimate errors in variables for small \( t \) by coupling to the usual random graph. We then apply the strategy stated above to the global, stacking and controllable ensembles, respectively, in Sections 4, 5 and 6.

**Theorem 2.2.** With high probability \( I > i_{max} \).

Note that Theorem 1.1 follows immediately from Theorem 2.2. We will employ the following useful lemma extensively to estimate sums of products. The proof given here is due to Patrick Bennett.

**Lemma 2.3. (Product Lemma)** Suppose \((x_i)_{i \in I} \) and \((y_i)_{i \in I} \) are real numbers such that \( |x_i - x| \leq \delta \) and \( |y_i - y| < \epsilon \) for all \( i \in I \). Then we have

\[
\left| \sum_{i \in I} x_i y_i - \frac{1}{|I|} \left( \sum_{i \in I} x_i \right) \left( \sum_{i \in I} y_i \right) \right| \leq 2|I| \delta \epsilon
\]

**Proof.** The triangle inequality gives

\[
\left| \sum_{i \in I} (x_i - x)(y_i - y) \right| \leq |I| \delta \epsilon.
\]

Rearranging this inequality gives

\[
\sum_{i \in I} x_i y_i = x \sum_{i \in I} y_i + y \sum_{i \in I} x_i - |I| xy \pm |I| \delta \epsilon
\]

\[= \frac{1}{|I|} \left( \sum_{i \in I} x_i \right) \left( \sum_{i \in I} y_i \right) - |I| \left( \frac{1}{|I|} \sum_{i \in I} x_i - x \right) \left( \frac{1}{|I|} \sum_{i \in I} y_i - y \right) \pm |I| \delta \epsilon.\]

\[
\square
\]

We conclude this Section with some notation and conventions that are used throughout the paper. We use compact notation for one-step differences, writing \( \Delta_i(F) = F(i + 1) - F(i) \) for any sequence \( F(i) \) and \( \Delta_i(f) = f((t + 1)n^{3/2}) - f(tn^{3/2}) \) for any function \( f(t) \). We also use the following notational conventions:
• We use the ‘O-tilda’ notation $f = \widetilde{O}(g)$ to mean $|f| \leq (\log n)^A|g|$ for some absolute constant $A$.

• We use the abbreviation ‘whp’ for ‘with high probability’; all such statements will have sub-polynomial failure probability, which will justify us taking a polynomial number of them in union bounds.

• We denote the vertex set by $[n] = \{1, \ldots, n\}$.

3 Coupling and union bounds

In this section we gather two types of estimates that can be made without using dynamic concentration, namely coupling and union bounds.

3.1 Extension variables in $G(n, p)$

We start by considering extension variables in the Erdős-Rényi random graph $G(n, p)$. Suppose $J$ is a graph and $A \subseteq V_J$. We refer to $(A, J)$ as an extension. Given an injective map $\phi : A \to [n]$, we let $X_{\phi, J}$ be the number of injective maps $f : V_J \to [n]$ such that $f$ restricts to $\phi$ on $A$ and $f(e)$ is an edge of $G(n, p)$ for every $e \in J$. Given $A \subseteq B \subseteq B' \subseteq V_J$ we define the scaling $S_B^B = S_B^B(J) = n^{|B'| - |B|}p^{\langle J', B \rangle - \epsilon J(B)}$. Note that if $A \subseteq B \subseteq B' \subseteq B'' \subseteq V_J$ we have $S_B^{B''} = S_B^{B''} - S_B^{B''}$. We say that $(A, J)$ is strictly balanced with respect to $G(n, p)$ if $S_B^A < 1$ for all $A \subseteq B \subseteq V_J$. The extension series for $(A, J)$, denoted $(B_0, \ldots, B_d)$, is constructed by the following rule. We let $B_0 = A$. For $i \geq 0$, if $(B_i, J)$ is not strictly balanced then we choose $B_{i+1}$ to be a minimal set $C$ with $B_i \subseteq C \subseteq V_J$ that minimises $S_C^A$; otherwise we choose $B_{i+1} = V_J$, set $d = i + 1$ and terminate the construction.

To control extensions we quote the following result of Kim and Vu [16, Theorem 4.2.4] in a weakened form that suffices for our purposes.

Lemma 3.1. Suppose $(A, J)$ satisfies $S_A^B > n^\alpha$ for all $A \subseteq B \subseteq V_J$ in $G(n, p)$ for some absolute constant $\alpha > 0$. Then there is an absolute constant $\beta > 0$ such that whp $X_{\phi, J} = (1 \pm n^{-\beta})S^A_J$ for all injections $\phi : A \to [n]$.

We also require a weaker estimate that also applies to sparse extensions, as given by the following union bound lemma. We include a brief proof as it illustrates a method we will also use for similar estimates in the triangle-free process.

Lemma 3.2. If $(A, J)$ is strictly balanced then whp $X_{\phi, J} < L^4|V_J|\max\{S^{V_J}_A, 1\}$ for all injections $\phi : A \to [n]$.

Proof. First we note that for any fixed $f : V_J \to [n]$ restricting to $\phi$ on $A$ we have $P(f \in X_{\phi, J}) = p^{\langle J, \phi(J) \rangle}$. Next we estimate the probability that there are $s$ extensions in $X_{\phi, J}$ that are disjoint outside of $\phi(A)$. An upper bound is $s!(n^{V_J - |A|})^s \cdot (p^{|J - \phi(J)|})^s < (3s^{-1}S_A^{V_J})^s$, which is subpolynomial for $s = L^4\max\{S^{V_J}_A, 1\}$. Now we show by induction on $|V_J| - |A|$ that $X_{\phi, J} < L^4(|V_J| - |A|)\max\{S^{V_J}_A, 1\}$. The base case $|V_J| - |A| = 1$ holds by the bound on disjoint extensions. By strict balance and
the induction hypothesis, at most \(2^{V_j}L^4([V]-|A|-1) < L^4([V]-|A|)-1\) embeddings intersect any fixed embedding outside of \(\phi(A)\). So \(X_{\phi,jt} < L^4([V]-|A|)-1s < L^4([V]-|A|) \max\{S_A^V,1\}\).

For general extensions we obtain the following bound by applying the previous lemma to each step of the extension series, noting that \(S_B^{t+1} \geq 1\) for \(i \geq 1\).

**Lemma 3.3.** For any extension \((A,J)\) whp \(X_{\phi,j} < L^4[V_j] \max_{A \subseteq B \subseteq V_j} S_B^j\) for all injections \(\phi : A \rightarrow [n]\).

### 3.2 Coupling estimates

Our error functions are chosen to control the process for as long as possible, and the interesting range is \(t = \Omega(\log n)\). However, the powers of \(t\) cause the functions to be badly behaved for small \(t\). So we need alternative estimates for \(t = o(1)\), which we obtain by coupling to the usual Erdős-Rényi random graph process. We start tracking a variable \(V\) using differential equations at the first time \(t \geq n^{-1/4}\) where we require a proportional error that is \(o(1)\) (we will be more precise below). Note that we have to exclude very small \(t\) to obtain concentration, and we will see from the calculations below that \(t = n^{-1/4}\) is a natural starting point.

Let \(ER(n,j)\) denote the Erdős-Rényi random graph process with \(n\) vertices and \(j\) edges, where at each time step we choose the next edge to be a random pair that is not yet an edge. To obtain the triangle-free process we modify \(ER(n,j)\) by rejecting any pair that is closed, in that it forms a triangle with the edges already selected. After \(j\) steps the selected edges form the triangle-free process \(G(i)\) at \(i\) steps, where \(j - i\) edges were rejected. The number of rejected edges is bounded by the number of triangles in \(ER(n,j)\); call this \(T\). We can approximate \(ER(n,j)\) by the binomial model where edges are chosen independently with probability \(p_0 = 2j/n^2\). If \(p_0 > n^{-1+\alpha}\) for some \(\alpha > 0\) then by Lemma 3.1 we have \(T < 2p_0^2n^3 = O(j/n)^3\) whp. We consider \(j = \Theta(n^{5/4+c})\) with \(0 \leq c < 1/4\) so that \(T = \Theta(n^{-1/2+2c})j\). Then \(i = (1 + O(n^{-1/2+2c}))j\), so we can approximate \(G(i)\) by \(G(n,p_0)\) with \(p_0 = (1 + O(n^{-1/2+2c}))p\). Recall that we aim to prove \(V = v(1 \pm e_V)\) where \(e_V = f_V + 2g_V\) for all variables that we track. We do this by showing that the probability that \(DV = V - TV\) ever crosses the critical interval

\[ ((f_V + g_V)v, (f_V + 2g_V)v) \]

is very small. In order to circumvent technical issues with powers of \(t\) when \(t\) is small we begin the critical interval analysis at the smallest index \(i\) such that \(i \geq n^{5/4}\) and \(g_V(t) \leq L^{-1}\). Recall that we denote this step by \(i_V\). Let \(t_V = i_Vn^{-3/2}\) be the corresponding time. If the structure that is counted by \(V\) has no edges (i.e. features only open pairs) then we simply have \(i_V = n^{5/4}\) and \(t_V = n^{-1/4}\). The key point here is that the variable \(V\) must not be in (or beyond) the critical interval at the moment that we begin the critical interval analysis for that variable. We now use the coupling with the Erdős-Rényi process to establish this property for all variables that we track.

**Lemma 3.4.** With high probability every tracked variable \(V\) satisfies \(|DV(i_V)| \leq (f_V + g_V)\).

**Proof.** We begin with an arbitrary variable \(V\) with the property that the structure counted by \(V\) has no edge. We show that \(V(i_V) = V(n^{5/4}) = v(1 \pm g_V)\) with high probability. Let \(V = X_{\phi,jt}\), using the notation for extension variables from the previous section. Note that we take \(A = \emptyset\) for the
global variables $Q,R$. We have $v(n^{-1/4}) = n^{n/4} q^{o(V)} = n^{n/4}(1 - O(n^{-1/2}))$. Since $g_V = \omega(Ln^{-1/2})$, it suffices to show that $X_{\phi,J_\Gamma} = n^{n/4}(1 + O(n^{-1/2}))$. Note that $n^{n/4} - V$ is bounded by $n^{n/4}$ times the number of paths of length 2 in $ER(n,j)$; call this $P_2$. Now, by Chernoff bounds in $ER(n,j)$ whp every vertex has degree $p_0n \pm (p_0n)^{0.51}$, which we roughly estimate as $(1 + O(n^{-0.1}))p_0n$ when $p_0 = \Omega(n^{-3/4})$. Using this estimate on degrees we have $P_2 = (1 + o(1))(p_0n)^2 = (1 + o(1))4t^2n^2 = O(n^{3/2})$, as desired.

It remains to consider variables $V = X_{\phi,J_\Gamma}$ where $J \neq \emptyset$. We consider $S$ first, as this is the only such variable with $g_V(n^{-1/4}) \leq L^{-1}$. (Thus this is the only remaining variable with $i_V = n^{5/4}$. We can bound $S$ above by $2jn_i$, as any copy of the structure counted by $S$ determines an ordered edge and a vertex. These structures are not counted if the non-edge pairs are either closed or edges. Thus, we bound $S$ below by $2jn - 2P_2 - 2P_3$, where $P_3$ is the number of paths of length 3 in $G(n,p_0)$. By the vertex degree estimate we have $P_3 = (1 + o(1))(p_0n)^3$, so $S(n^{5/4}) = 2n^{9/4} + O(n^{7/4})$. This is well within the desired error bound.

For the remaining variables $V$ (i.e. stacking or controllable variables) we have $g_V(n^{-1/4}) \geq 1$. Fix such a variable $V = X_{\phi,J_\Gamma}$. We bound $V(i_V)$ above by $X_{\phi,J}$ in $G(n,p_0)$ and below by

$$X_{\phi,J} - \sum_{xy \in (V/2) \setminus J} X_{\phi,J_{xy}}$$

in $G(n,p_0)$, where for $xy \in \Gamma \setminus J$ we define $J_{xy}$ to be the graph obtained from $J$ by adding a new vertex $z$ adjacent to $x$ and $y$, and for $xy \in (V/2) \setminus \Gamma$ we define $J_{xy}$ to be the graph obtained from $J$ by adding $xy$ as an edge.

First suppose that $V$ is a stacking variable. Considering $V$ one vertex at a time and applying Lemma 3.1 to each step we obtain $X_{\phi,J} = v(1 + O(n^{-\beta}))$ for some $\beta > 0$. As $S_{B}^{V_{xy}}$ is smaller than $v$ by a polynomial factor for any $A \subseteq B \subseteq V$, we get the desired bound on $\sum_{xy} X_{\phi,J_{xy}}$ from an application of Lemma 3.3.

Now suppose $V$ is a controllable variable. To estimate $X_{\phi,J}$ at time $t_V$ we first note that since $L^{-1} = g_V(t_V) = \varphi L^{-1}(1 + t_V^{e(V)})e^\delta$, we have $t_V^{e(V)} > L^{-1}n^{-\delta/4}$. Since $t_V \leq 1$, for any $A \subseteq B \subseteq V$, we have

$$S_A^B(t_V) = t_V^{e_J(B) - e_J(A)} S_A^B(1) \geq t_V^{e(V)} n^\delta > L^{-1}n^{3\delta/4}.$$

So we can again apply Lemma 3.1 to conclude that $X_{\phi,J} = v(1 + O(n^{-\beta}))$ for some $\beta > 0$. It also follows from [1] that for any $A \subseteq B \subseteq V$, the scaling $S_{B}^{V_{J_{xy}}}$ is smaller than $v$ by a polynomial factor, and we again get the desired bound on $\sum_{xy} X_{\phi,J_{xy}}$ from an application of Lemma 3.3.

### 3.3 Union bounds

In this subsection we adapt the argument of Lemmas 3.2 and 3.3 to give a crude bound on general extension variables that holds throughout the triangle-free process. Along the way, we will prove Theorem 1.4 assuming Theorem 2.2. First we need some estimates for $Q(i)$ and $Y_{uv}(i)$ for any pair $uv$ that hold throughout the process.

**Lemma 3.5.** On $G_i$, whp $Q(i) = (1 \pm n^{-e/2})q(t)$ for $0 \leq t \leq t_{\max}$. Also, whp for any non-edge $uv$ we have $Y_{uv}(i) = (1 \pm n^{-e/4})y(t)$ for $n^{-0.4} \leq t \leq t_{\max}$. 


Proof. The bound for \( Q(i) \) holds on the good event \( G_t \) for \( t \geq n^{-1/4} \). For \( t \leq n^{-1/4} \) it holds because \( q(t) = (1 - O(n^{-1/2}))n^2 \), \( Q(i) \geq Q(n^{5/4}) = (1 - O(n^{-1/2}))n^2 \) and \( Q(i) \leq Q(0) = n^2 \).

The bound for \( Y_{uv}(i) \) follows from \( e_Y \leq n^{-\varepsilon'/4} \) for \( t \geq n^{-0.24} \) and \( \varepsilon \) small, so we can assume \( t \leq n^{-0.24} \). Thus we are in the case \( n^{1.1} \leq i \leq n^{1.26} \). For \( j = O(n^{1.26}) \), by Lemma 3.1 we estimate the number of triangles in \( ER(n,j) \) by \( T = O(j/n)^3 = O(n^{-0.48})j \). Thus we can couple \( G(i) \) to \( ER(n,j) \) where \( j = (1 + O(n^{-0.48}))i \). Writing \( d(v) \) for the degree of \( v \) in \( ER(n,j) \), by Chernoff bounds whp \( d(v) = 2j/n + O(j/n)^{2/3} = (1 \pm n^{-0.03})y \). This gives the required upper bound on \( Y_{uv}(i) \). We also have \( Y_{uv}(i) \geq d(v) - T(v) - P_3(uv) \), where in \( ER(n,j) \) we write \( T(v) \) for the number of triangles containing \( v \), and \( P_3(uv) \) for the number of paths of length 3 from \( u \) to \( v \). Each of \( T(v) \) and \( P_3(uv) \) have scaling \( n^2p^3 \leq n^{-1/4} \ll 1 \), so by Lemma 3.3 whp both are at most \( L^8 < n^{-0.03}y \), and we also have the required lower bound on \( Y_{uv}(i) \). \( \Box \)

We need some further notation and terminology for general extensions in the triangle-free process, which mirrors that used previously for extensions in the Erdős-Rényi process. We say that \( (A,J,\Gamma) \) is \emph{strictly balanced at time} \( t \) if \( S^V_B < 1 \) for all \( A \subseteq B \subseteq V_\Gamma \). The \emph{extension series} at time \( t \) for \( (A,J,\Gamma) \), denoted \( (B_0,\ldots,B_d) \), is constructed by the following rule. We let \( B_0 = A \). For \( i \geq 0 \), if \( (B_i,J,\Gamma) \) is not strictly balanced then we choose \( B_{i+1} \) to be a minimal set \( C \) with \( B_i \subseteq C \subseteq V_\Gamma \) that minimises \( S^V_{B_i} \); otherwise we choose \( B_{i+1} = V_\Gamma \), set \( d = i + 1 \) and terminate the construction.

In Lemma 3.8 we will give a general estimate for extension variables in the triangle-free process. First we illustrate the argument in the following lemma, which shows that sparse graph pairs do not appear; this is the main tool needed for the proof of Theorem 1.4. Here we take \( A = \emptyset \), write \( V_{J,\Gamma} = X_{\phi,J,\Gamma} \), where \( \phi \) is the unique map from \( \emptyset \) to \( [n] \), and \( v_{J,\Gamma} = S^V_{\emptyset} (J,\Gamma) \).

\textbf{Lemma 3.6.} Suppose \( v_{J,\Gamma}(i') < n^{-c} \) for some \( c > 0 \) and time \( i' \). Then the probability that \( G_i \) holds for \( i \leq i' \) but \( V_{J,\Gamma}(i') > 0 \) is at most \( 2n^{-c} \).

\textbf{Proof.} Suppose first that \( i' \leq L^{-1} \). Here we appeal to the coupling with the Erdős-Rényi random graph process. It suffices to estimate the probability that \( J \) appears in \( G_{n,j} \), where \( j = (1 + o(1))i' \). The expected number of copies of \( J \) is at most \( 2n^{-c} \), so the required bound follows from Markov’s inequality. Thus it suffices to consider \( i' \geq L^{-1} \).

To estimate \( P(V_{J,\Gamma}(i') > 0) \), we take a union bound of events, where we specify the injection \( f : V_\Gamma \rightarrow [n] \), and for \( e \in J \) we specify the selection step \( i_e \) at which the process selects the edge \( f(e) \). Fix some choice and let \( \mathcal{E} \) be the specified event. For each \( i \leq i' \) we estimate the probability that the selected edge is compatible with \( \mathcal{E} \). At a selection step \( i = i_e \) the selected edge is specified, so the probability is \( 2/Q(i_e) = (1 + o(1))2q(t_e)^{-1} \), where \( t_e = n^{-3/2}i_e \). For other \( i \), the required probability is \( 1 - N_i/Q \), where \( N_i \) is the number of ordered open pairs that cannot be selected at step \( i \) on \( \mathcal{E} \). If \( i \) is a selection step we write \( N_i = 0 \). Then we estimate

\[ P(\mathcal{E}) \leq \prod_{e \in J} (1 + o(1))2q(t_e)^{-1} \cdot \prod_{i=1}^{i'} (1 - N_i/Q). \]

Now we estimate \( N_i \) when \( i \) is not a selection step. For \( i < L^{-1}n^{3/2} \) we use the trivial estimate \( N_i \geq 0 \), so suppose \( i \geq L^{-1}n^{3/2} \). Suppose there are \( k_i \) choices of \( e \in J \) with \( i_e > i \). Then there are \( |\Gamma \setminus J| + k_i \) open pairs that must not become closed, namely the open pairs of \( f(\Gamma \setminus J) \) and
the $k_i$ pairs that have yet to be selected as edges. Note that the number of choices of $e_i$ that close more than one such open pair is $O(L^4) = o(y)$. By Lemma 3.5 all $Y$-variables are $(1 + o(1))y$, so we obtain $N_i = (1 + o(1))(\lvert \Gamma \setminus J \rvert + k_i) \cdot 4y$. Thus for $i \geq n^{1.1}$ we can write $1 - N_i/Q \leq (1 - A_i - B_i)$, where $A_i = (1 + o(1))(\lvert \Gamma \setminus J \rvert \cdot 8tn^{-3/2}$ and $B_i = (1 + o(1))k_i \cdot 8tn^{-3/2}$. This holds for all $i$ if we set $A_i = B_i = 0$ for $i < L^{-1}n^{3/2}$.

Using the estimate $\prod (1 - A_i - B_i) \leq \exp \{- \sum A_i - \sum B_i\}$ we see that the contribution to $\mathbb{P}(\mathcal{E})$ from the first factor is

$$\exp \left\{ - \sum_{i=1}^{i'} A_i \right\} = \exp \left\{ - (1 + o(1)) \sum_{i=L^{-1}n^{3/2}}^{i'} \lvert \Gamma \setminus J \rvert \cdot 8tn^{-3/2} \right\}$$

$$= \exp \left\{ - (1 + o(1))\lvert \Gamma \setminus J \rvert \sum_{i=L^{-1}n^{3/2}}^{i'} 8in^{-3} \right\}$$

$$= (1 + o(1)) \exp \{ - \lvert \Gamma \setminus J \rvert \cdot 4(i')^2n^{-3} \}$$

$$= (1 + o(1))e^{-4(i')^2|\Gamma \setminus J|} = (1 + o(1))\hat{q}(t')|\Gamma \setminus J|,$$

since $\sum_{i=1}^{L^{-1}n^{3/2}} in^{-3} < L^{-2} = o(1)$. From the second factor we obtain

$$\exp \left\{ - \sum_{i=1}^{i'} B_i \right\} = \exp \left\{ - (1 + o(1)) \sum_{i=L^{-1}n^{3/2}}^{i'} k_i \cdot 8tn^{-3/2} \right\}$$

$$= \exp \left\{ - (1 + o(1)) \sum_{i=L^{-1}n^{3/2}}^{i'} \sum_{e \in J} 8in^{-3} \right\}$$

$$= \prod_{e \in J} (1 + o(1))\hat{q}(t_e).$$

Therefore

$$\mathbb{P}(\mathcal{E}) \leq (1 + o(1)) \prod_{e \in J} 2n^{-2} \cdot \hat{q}(t')|\Gamma \setminus J|,$$

Summing over at most $n^{|V_t'|}$ choices for $f$ and $(i')^{\lvert J \rvert}$ choices for the selection steps, we estimate $\mathbb{P}(V_{t',H'}(i') > 0) < (1 + o(1))v(t') < 2n^{-\varepsilon}$. \hfill \Box

**Proof of Theorem 1.4.** Statement (i) is immediate from [8, Theorem 1.6(iii)]. For (ii), fix $H' \subseteq H$ with $d_2(H') > 2$. We can assume that the global parameter $\varepsilon$ satisfies $|E_{H'}|(1/2 - \varepsilon) > |V_{H'}| + \varepsilon$. Note that if $H \subseteq G$ then $V_{t',H'}(i_{\max}) > 0$ for some spanning subgraph $J$ of $H'$. We have

$$v_{t',H'}(t_{\max}) = n^{|V_{H'}|}p_{|E_J|}q(t_{\max})^{E_{H'}-|E_J|} = n^{|V_{H'}-|E_J|/2-(1/2-\varepsilon)(|E_{H'}|-|E_J|)} < n^{-\varepsilon}.$$ 

Thus the result follows from Theorem 2.2 and Lemma 3.6. \hfill \Box

**Lemma 3.7.** If $S_{V_t'}^B < y/L^7$ for all $A \subseteq B \subseteq V_t$ at some time $t'$ then whp

$$X_{\phi,i,t}(t') < L^{|V_t'|} \max_{A \subseteq B \subseteq V_t} S_{V_t'}^B$$

for all injections $\phi : A \to \lvert n \rvert$. 

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Proof. It suffices to consider \( t' \geq L^{-1/2} \), as for smaller \( t' \) we can simply appeal to the coupling with the Erdős-Rényi random graph process and apply Lemma 3.3.

The argument is similar to that of Lemma 3.6. We write \( s = L^4 \max_{A \subseteq B \subseteq V_t} S^V_B \) and estimate the probability that there is a set of \( s \) extensions \( \{f_1, \ldots, f_s\} \) in \( V(i') := X_{\phi,J,i'}(i') \) that are disjoint outside of \( \phi(A) \). We take a union bound of events in which we specify \( f_1, \ldots, f_s \), and for each \( 1 \leq j \leq s \) and \( e \in J \setminus J[A] \) we specify the selection step \( i_{j,e} \) at which the process selects the edge \( f_j(e) \). Fix some choice and let \( \mathcal{E} \) be the specified event. For each \( i \leq i' \) we estimate the probability that the selected edge is compatible with \( \mathcal{E} \). At a selection step \( i = i_{j,e} \) the selected edge is specified, so the probability is \( 2/\mathcal{Q}(i_{j,e}) = (1 + o(1))2q(t_{j,e})^{-1} \), where \( t_{j,e} = n^{-3/2}i_{j,e} \). For other \( i \), the required probability is \( 1 - N_i/Q \), where \( N_i \) is the number of ordered open pairs that cannot be selected at step \( i \) on \( \mathcal{E} \). If \( i \) is a selection step we write \( N_i = 0 \). Then we estimate

\[
\mathbb{P}(\mathcal{E}) \leq \prod_{j=1}^{s} \prod_{e \in J \setminus J[A]} (1 + o(1))2q(t_{j,e})^{-1} \cdot \prod_{i=1}^{i'} (1 - N_i/Q).
\]

Now we estimate \( N_i \) when \( i \) is not a selection step. For \( i < L^{-1/2}n^{3/2} \) we use the trivial estimate \( N_i \geq 0 \), so suppose \( i \geq L^{-1/2}n^{3/2} \). Suppose there are \( k_i \) choices of \((j,e)\) with \( i_{j,e} > i \). Then there are \( o(V)s + k_i \) open pairs that must not become closed, namely the \( o(V)s \) open pairs specified by \( f_1, \ldots, f_s \) and the \( k_i \) pairs that have yet to be selected as edges. Note that the number of choices of \( e_i \) that close more than one such open pair is \( O(s^2L^4) = o(syL^{-2}) \), since \( s < y(t')L^{-7} < yL^{-6.5} \).

By Lemma 3.5 all \( Y \)-variables are \((1 + o(1))y_s \), so we obtain \( N_i = (1 + o(1))(o(V)s + k_i) \cdot 4y \). Thus for \( i \geq L^{-1/2}n^{3/2} \) we can write \( 1 - N_i/Q \leq 1 - A_i - B_i \), where \( A_i = (1 + o(1))o(V)s \cdot 8tn^{-3/2} \) and \( B_i = (1 + o(1))k_i \cdot 8tn^{-3/2} \). This holds for all \( i \) if we set \( A_i = B_i = 0 \) for \( i < L^{-1/2}n^{3/2} \).

Similarly to the proof of Lemma 3.6, the contribution to \( \mathbb{P}(\mathcal{E}) \) from the first factor is

\[
\exp \left\{ - \sum_{i=L^{-1/2}n^{3/2}}^{i'} A_i \right\} \leq \left[ (1 + o(1))\hat{q}(t')^{o(V)} \right]^{s},
\]

and from the second factor is \( \prod_{j=1}^{s} \prod_{e \in J \setminus J[A]} (1 + o(1))\hat{q}(t_{j,e}) \). Therefore

\[
\mathbb{P}(\mathcal{E}) \leq \hat{q}(t')^{o(V)s} \prod_{j=1}^{s} \left( 1 + o(1) \prod_{e \in J \setminus J[A]} 2n^{-2} \right).
\]

Summing over at most \( s!n^n(V)^s \) choices for \( f_1, \ldots, f_s \) and \((t')^n(V)^s \) choices for the selection steps, the probability that such \( f_1, \ldots, f_s \) exist is at most \( s![(1 + o(1)v(t')]^s < (3s^{-1}v(t'))^s \), which is subpolynomial.

In the case when \((A,J,\Gamma)\) is strictly balanced at time \( t' \), the required bound on \( X_{\phi,J,i'}(i') \) follows by induction as in the proof of Lemma 3.2. The general case follows by applying this to each step of the extension series. \( \square \)

Lemma 3.8. Whp for any extension \((A,J,\Gamma)\) on at most \( M^3 \) vertices and all injections \( \phi : A \to [n] \) we have

\[
X_{\phi,J,i'}(i') < L^{4|V_t|} \max_{A \subseteq B \subseteq V_t} S^V_B.
\]
Proof. Let $A = B_0, \ldots, B_f = V_f$ be the extension series at time $t$. For each step $(B_i, J[B_{i+1}], \Gamma[B_{i+1}])$ there are two cases. If $S_{B_i}^{B_{i+1}}(t) < n^\delta$ then we can apply Lemma 3.7 to bound the number of ways to make the step. On the other hand, if $S_{B_i}^{B_{i+1}}(t) > n^\delta$ then this extension variable is controllable at time $t$, and we bound the number of copies by Lemma 5.1. □

4 Global Ensemble

In this section we prove that the global variables have the desired concentration, assuming that this is the case for all ensembles at earlier times. Recall that we track each variable $V$ relative to a tracking random variable $T_V$ to isolate variations in $V$ from variations in other variables that might have an impact on $V$. Recall that the global variables have scalings $q = \hat{q} n^2$, $r = \hat{q}^3 n^3$ and $s = p\hat{q}^2 n^3 = 2t^2 n^{5/2}$. We use the tracking variables

$$T_Q = q, \quad T_R = Q^3 n^{-3}, \quad \text{and} \quad T_S = 2tn^{-3/2}Q^2.$$  

(Note that the tracking variable for $Q$ is in fact a deterministic function.) We show that the difference random variables

$$DV = V - T_V$$

for $V \in \{Q, R, S\}$ are all small throughout the process. Recall that $I_1$ is the minimum of the stopping times $I_V$ over all variables $V$ in the global ensemble, so the following theorem says that there is a very small probability that we reach the universal stopping time $I$ before step $i_{\text{max}}$ because a global variable $V$ fails to satisfy the required bounds $|DV| \leq e_V v$. (Note that a global variable cannot be bad.)

Theorem 4.1. $\mathbb{P}(I = I_1 \text{ and } I_1 \leq i_{\text{max}}) = o(1)$.

We organise the proof of Theorem 4.1 into three subsections, in which we respectively verify the trend hypothesis, variation equations and boundedness hypothesis.

4.1 Trend Hypothesis

In this subsection, for each variable $V$ in the Global Ensemble, we give an upper bound on the one-step expected change in the difference variable, namely

$$\mathbb{E}[\Delta_i DV | F_i] = \mathbb{E}[DV(i+1) - DV(i) | F_i],$$

under the assumption that $V$ is in its upper critical window, i.e.

$$(f_V + g_V)v < DV < (f_V + 2g_V)v.$$  

We consider the effect of each open pair and edge in the structure counted by $V$ separately; the final expression is then obtained by linearity of expectation. When an open pair in a copy of the structure counted by $V$ is chosen or closed, we say that the copy is destroyed. We balance the change in $V$ due to destructions with the change in $T_V$ due to the change in $Q$. (The case $V = Q$ is handled differently as $Q$ is tracked relative to the deterministic function $q$.) Adding the edge $e_{i+1}$ can also
create new copies of the structure counted by \( V \) in which \( e_{i+1} \) plays the role of one of the edges in the structure; then we say that a copy of \( V \) is created. The change in \( V \) that comes from creations is balanced with the change in \( t \) in \( TV \).

To illustrate this mechanism consider \( V = Y_{uv} \), which we do not treat in this section but is a good example here. We write \( Y_{uv} = Y_{uv}^c - Y_{uv}^d \), where \( Y_{uv}^c \) counts the number of creations and \( Y_{uv}^d \) destructions up to time \( i \). Recall that \( TV_{uv} = 2tQn^{-3/2} \). We define \( \Delta_i(TY_{uv}^c) \) and \( \Delta_i(TY_{uv}^d) \) by writing

\[
\Delta_i(TY_{uv}) = TY_{uv}/(tn^{3/2}) - \frac{\Delta_i(Q)}{Q} TV_{uv} + O(t + t^{-1})yn^{-3}
\]

\[
= \Delta_i(TY_{uv}^c) - \Delta_i(TY_{uv}^d) + O(t + t^{-1})yn^{-3}.
\]

Now we can write

\[
\Delta_i(DY_{uv}) = \Delta_i(DY_{uv}^c) - \Delta_i(DY_{uv}^d) + O(t + t^{-1})yn^{-3},
\]

where \( \Delta_i(DY_{uv}^c) = \Delta_i(Y_{uv}^c) - \Delta_i(TY_{uv}^c) \) and \( \Delta_i(DY_{uv}^d) = \Delta_i(Y_{uv}^d) - \Delta_i(TY_{uv}^d) \). In this way we balance any change in \( Y_{uv} \) with a corresponding changes in \( TY_{uv} \).

In general, for a variable \( V \) with underlying graph pair \((J, \Gamma)\), we use the notation

\[
V = \sum_{e \in J} V^e - \sum_{e \in \Gamma \setminus J} V^e,
\]

where for edges \( e \in J \) we let \( V^e \) count the number of creations of \( V \) up to time \( i \) by the selection of \( e \), and for open pairs \( e \in \Gamma \) we let \( V^e \) count the number of destructions of \( V \) up to time \( i \) by the closure or selection of \( e \). (Actually, these are approximate expressions because a given copy of \( V \) may be counted more than once, but we will see that this gives a negligible error term.) We decompose the changes of the tracking variable as \( \Delta_i(TV) = \sum_{e \in J} \Delta_i(TV^e) - \sum_{e \in \Gamma \setminus J} \Delta_i(TV^e) \), where \( \Delta_i(TV^e) \) is the change in some term in \( TV \) that corresponds naturally to \( e \) (there is often an additional error term). We then use \( \Delta_i(TV^e) \) to balance \( \Delta_i(V^e) \) for each \( e \). Thus we write

\[
\Delta_i(DV) = \sum_{e \in J} \Delta_i(DV^e) - \sum_{e \in \Gamma \setminus J} \Delta_i(DV^e),
\]

where \( \Delta_i(DV^e) = \Delta_i(V^e) - \Delta_i(TV^e) \).

Now we proceed to compute the impact on \( \mathbb{E}[\Delta_i(DV) \mid F_i] \) for each open pair and edge in the structure counted by global variables \( V \). We begin with creations. The main point to note in these calculations is that each open pair gives a self-correction term of \(-8tf_\ell vn^{-3/2}\), which will later balance a corresponding \( 8tf_\ell vn^{-3/2} \) term from the change in \( e \).

### 4.1.1 Simple destructions

The variable \( Q \) has the property that there is a larger extension that describes the situation when an open pair \( ab \) in \( Q \) is closed. We call destructions of this form simple destructions. (We will see examples of this type again in Section 6 where we treat the stacking variables.) Note that each triple in \( S \) describes a way in which the addition of an edge closes a pair, so we have

\[
\mathbb{E}[\Delta_iQ^d \mid F_i] = 2 + 4S/Q.
\]
Then
\[
E[\Delta_i(DQ) | F_i] = E[\Delta_i(Q) - \Delta_i(q) | F_i]
\]
\[
= -(2 + 4S/Q) + 8tqn^{-3/2} \pm \tilde{O}(qn^{-3})
\]
\[
= -8tQn^{-3/2} \pm (8 + O(e_Q))e_{stn^{1/2}q} + 8tqn^{-3/2} \pm O(1)
\]
\[
\leq -(f_Q + g_Q - (1 + o(1))e_S)8tqn^{-3/2}.
\]

In the last line we used \(DQ = Q - q \geq (f_Q + g_Q)q\) when \(Q\) is in its upper critical window and \(te_{sqn^{-3/2}} \geq c_se^2qn^{-3/2} = c_S \gg 1\).

### 4.1.2 Product destructions

For destructions, it remains to consider those of \(R\) and \(S\). These are not simple destructions, as no variable in the ensemble counts ways in which these structures are destroyed by the closure of open pairs. Instead we will apply the Product Lemma from Section 2. Let \(V\) be \(R\) or \(S\) and let \(ab\) be an open pair in \(V\). For each open pair \(\alpha\beta\) in \(G(i)\) let \(V_{\alpha\beta}\) be the number of copies of the structure counted by \(V\) that use \(\alpha\) as \(a\) and \(\beta\) as \(b\). So \(R_{\alpha\beta} = X_{\alpha\beta}\) and \(S_{\alpha\beta} = Y_{\alpha\beta}\). We note that

\[
\sum_{\alpha\beta \in Q} Y_{\alpha\beta} = S \quad \text{and} \quad \sum_{\alpha\beta \in Q} X_{\alpha\beta} = R.
\]

So by the Product Lemma we can write

\[
E[\Delta_i(DV^d) | F_i] = E \left[ \Delta_i(V^d) - \frac{\Delta_i(Q)}{Q} TV | F_i \right]
\]
\[
= \sum_{\alpha\beta \in Q} 2Q^{-1}(Y_{\alpha\beta} + Y_{\beta\alpha})V_{\alpha\beta} - (2 + 4SQ^{-1})Q^{-1}TV
\]
\[
= 4SQ^{-2}V + O(e_Y e V_{\alpha\beta} v_{\alpha\beta}) - 4SQ^{-2}TV \pm O(v/q)
\]
\[
= (1 \pm (1 + o(1))e_S)8tn^{-3/2}DV \pm O(e_Y e V_{\alpha\beta})tvn^{-3/2} \pm O(v/q)
\]
\[
\geq [(1 + o(1))(f_V + g_V) - O(e_Y e V_{\alpha\beta}) - O(t^{-1}e^2)] 8tn^{-3/2}.
\]

This calculation holds for every open pair in the structures counted by \(R\) and \(S\), but we also need a ‘destruction fidelity’ term to correct for simultaneously closing the two open pairs in a single copy of the structure counted by \(V\). Note that this cannot occur for \(S\), as the configuration for closing the two open pairs in \(S\) would make a triangle with the edge in \(S\). For \(R\) we need to estimate the number of \(K_4\)'s in which two adjacent pairs are edges and the other four pairs are open. By Lemma 3.8 there are at most \(L^{16}n^4p^2\hat{q}^4 = O(L^{16}t^2\hat{q}^t)\) such configurations, so the correction to \(E[\Delta_i(DR^d) | F_i]\) is \(O(q^{-1} \cdot t^2L^{16}\hat{q}^t) = O(L^{16}t^2rn^{-2}) = o(g_R)tn^{-3/2}\), assuming (say)

\[c_R \geq L^{20}\]

### 4.1.3 Creations

Finally, we turn to creations, which among the global variables occur only for \(S\). Let \(V = S\) and let \(ab\) be the edge in \(V\). Recall that \(V^c(i)\) denotes the number of creations of \(V\) through \(i\) steps of the
process (which can only occur through the addition of an edge that plays the role of $ab$). Let $V^+$ be the structure given by replacing the edge $ab$ with an open pair, so $S^+ = R$. Note that

$$\mathcal{T}V^+ = \mathcal{T}V - \frac{Q}{2tn^{3/2}}$$

and

$$v^+ = v \cdot \frac{\hat{q}n^{1/2}}{2t}.$$

For each extension in $V^+(i)$ the probability that the edge $e_{i+1}$ falls in the appropriate position to result in a creation counted in $\Delta_i(V^c)$ is $2Q(i)$, so we can write $\mathbb{E}[\Delta_i(V^c) \mid \mathcal{F}_i] = 2Q^{-1}V^+$. There is no ‘creation fidelity’ term, as it is not possible for the added edge to close an open pair in $V$.

We match this change in $V$ with the change in $\mathcal{T}V$ that results from the one-step change in $t$, which is $\Delta_i(\mathcal{T}V^c) = \mathcal{T}V/(tn^{3/2})$. We have

$$\mathbb{E}[\Delta_i(\mathcal{D}V^c) \mid \mathcal{F}_i] = \mathbb{E}[\Delta_i(V^c) - t^{-1}n^{-3/2}\mathcal{T}V \mid \mathcal{F}_i]$$

$$= 2Q^{-1}(\mathcal{T}V^+ \pm v^+e_{V^+}) - t^{-1}n^{-3/2}\mathcal{T}V$$

$$= \pm (1 + o(1))t^{-1}e_Vvn^{-3/2}.$$

### 4.2 Variation Equations

Gathering together the relevant creation and destruction calculations from the previous subsection, under the assumption that the variable considered is in its upper critical window, we have

$$\mathbb{E}[\Delta_i(\mathcal{D}Q) \mid \mathcal{F}_i] \leq -(f_Q + g_Q - (1 + o(1))e_S)stqn^{-3/2},$$

$$\mathbb{E}[\Delta_i(\mathcal{D}R) \mid \mathcal{F}_i] \leq -(1 + o(1))\left[3(f_R + g_R) - O(e_Ye_X) - O(t^{-1}e^2)\right]s\rho,n^{-3/2},$$

$$\mathbb{E}[\Delta_i(\mathcal{D}S) \mid \mathcal{F}_i] \leq -(1 + o(1))\left[2(f_S + g_S) - \frac{eR}{\mathcal{S}^{\mathcal{S}}_{\mathcal{S}}} - O(e_Ye_Y) - O(t^{-1}e^2)\right]s\rho n^{-3/2}.$$

For each variable $V$ in the Global Ensemble we consider the sequence of random variables

$$ZV(i) = DV - ve_V.$$

Under the assumption that $V$ is in its upper critical window, we show that this sequence is a supermartingale, i.e. that

$$\mathbb{E}[\Delta_iZV \mid \mathcal{F}_i] = \mathbb{E}[\Delta_i(DV - ve_V) \mid \mathcal{F}_i] \leq 0.$$

Note that

$$\Delta_i(e_Vv) = e_V(t + n^{-3/2})v(t + n^{-3/2}) - e_V(t)v(t)$$

$$= (ev' + e_V)vn^{-3/2} + O(e'' + e'v'/v + evv'/v)vn^{-3}.$$

We estimate the last term by $O(t^2 + t^{-2})e_Vvn^{-3} = \tilde{O}(e_Vvn^{-5/2})$, using the fact that we restrict our attention to $t \geq n^{-1/4}$. We write

$$v'/v = t^{-1}e(V) - 8to(V),$$

where $V$ has $e(V)$ edges and $o(V)$ open pairs not in $A$. Recalling that $e_V = f_V + 2g_V$, we see that we can cancel the $8to(V)f_Vvn^{-3/2}$ term that occurs both in $\Delta_i(e_Vv)$ and in $\mathbb{E}[\Delta_iDV \mid \mathcal{F}_i]$: this is
the self-correction that is fundamental to the analysis. Thus we obtain

\[
\mathbb{E} \left[ \Delta_t ZQ \mid \mathcal{F}_t \right] \leq - \left( \frac{e'_Q}{8t} + o(f_Q) - (1 + o(1))(g_Q + e_S) \right) 8tqn^{-3/2},
\]

\[
\mathbb{E} \left[ \Delta_t ZR \mid \mathcal{F}_t \right] \leq - \left( \frac{e'_R}{8t} + o(f_R) - (1 + o(1))(3g_R + O(e_Y e_X)) - O(t^{-1}e^2) \right) 8trn^{-3/2},
\]

\[
\mathbb{E} \left[ \Delta_t ZS \mid \mathcal{F}_t \right] \leq - \left( \frac{e'_S}{8t} + \frac{e_S}{8t^2} + o(f_S) - (1 + o(1))(2gs + \frac{e'_R}{8t^2} + O(e_Y e_Y)) - O(t^{-1}e^2) \right) 8tsn^{-3/2}.
\]

We now show that the error functions that we have chosen grow quickly enough for each of these sequences to be supermartingales (i.e. the \( e'_V \) term will be dominant in each case). We stress that the \( t \ll 1 \) regime behaves a bit differently from the rest of the process in the estimates that follow. Now recall that our error functions have the form

\[
f_V = c_V e^2 \quad \text{and} \quad g_V = c_V \vartheta L^{-1} \left( 1 + t^{-e(V)} \right) e^2.
\]

Recall further that \( e = q^{-1/2} n^{-1/4} \) and note that the power of \( t \) in \( g_V \) is chosen so that the dominant term in \( v g_V \) as \( t \to 0 \) does not contain a power of \( t \). Since \( e'/e = 4t \) we have

\[
f'_V/f_V = 8t \quad \text{and} \quad g'_V/g_V = \vartheta'/\vartheta - e(V)t^{-1} \left( 1 + t^{e(V)} \right)^{-1} + 8t.
\]

Therefore, we have

\[
e'_V \geq 8tf_V + (\vartheta'/\vartheta - e(V)t^{-1} + 8t) \cdot 2g_V.
\]

We apply this observation to each global variable in turn.

For \( Q \) we have

\[
\mathbb{E} \left[ \Delta_t ZQ \mid \mathcal{F}_t \right] \leq -(1 + o(1)) \left[ (f_Q + \frac{(2\vartheta')}{8t^2} + 2g_Q) - (g_Q + e_S) \right] 8tqn^{-3/2}
\]

\[
\leq -(1 + o(1)) \left[ (f_Q - g_Q) + (\frac{\vartheta'}{4t^2}g_Q + g_Q - 2g_S) \right] 8tqn^{-3/2}.
\]

Then the sequence \( ZQ \) forms a supermartingale provided

\[
c_Q \geq 2c_S.
\]

(Note that \( \frac{\vartheta'}{4t^2}g_Q \) dominates the remaining terms for \( t \ll 1 \).)

Next consider \( R \), where we have

\[
\mathbb{E} \left[ \Delta_t ZR \mid \mathcal{F}_t \right] \leq -(1 + o(1)) \left[ f_R + \left( \frac{(2\vartheta')}{8t^2} + 2 \right) \cdot g_R - 3g_R - O(e_Y e_X) - O(t^{-1}e^2) \right] 8trn^{-3/2}
\]

\[
\leq -(1 + o(1)) \left[ f_R + \left( \frac{\vartheta'}{4t^2} - 1 \right) g_R - O(f_Y f_X) - O(g_Y g_X) \right] 8trn^{-3/2}.
\]

Then the sequence \( ZR \) forms a supermartingale provided

\[
c_R \geq Lc_Y e_X.
\]

For this implies that the \( g_R \vartheta'/\left( 4t \vartheta \right) \) term dominates for \( t \ll 1 \) and that the \( f_R \) term dominates otherwise.
The final global variable is $S$, where we have

$$
E[\Delta_i ZS \mid F_i] \leq -(1 + o(1)) \left[ (1 + \frac{1}{8r^2}) f_s + \left( \frac{2q}{8r} + 2 \right) \cdot g_s \right] 8tsn^{-3/2} + \left( 1 + o(1) \right) \left[ 2g_s + \frac{c_s}{8r^2} + O(e_Y e_Y) \right] + O(t^{-1}e^2) \right] 8tsn^{-3/2}
$$

Then the sequence $ZS$ forms a supermartingale provided

$$c_s \geq 2c_R \quad \text{and} \quad c_s \geq Lc_Y^2.$$  

For this implies that the $f_s/t^2$ term dominates for $t \ll 1$ and the $f_s$ term dominates otherwise.

### 4.3 Boundedness hypothesis

To apply Freedman’s inequality, for each $V$ in the Global Ensemble we estimate $Var_V = Var(\mathcal{ZV}(i) \mid F_{i-1})$ and $N_V = |\Delta_i \mathcal{ZV}|$; recall that it suffices to show

$$Var_V \leq \frac{(gVv)^2}{L^3n^{3/2}} \quad \text{and} \quad N_V \leq \frac{gVv}{L^2}.$$

For convenience we replace $\mathcal{ZV}$ by $\mathcal{DV}$ in our calculations, as this does not change $Var_V$ and only changes $N_V$ up to a constant factor. We can estimate the contribution to the variance from each pair in $V$ separately, using the simple observation that if random variables $A$ and $B$ each have variance at most $v$ then $A + B$ has variance at most $4v$.

Here we estimate $Var_V \leq N_V^2$, i.e. for the global variables we simply apply the Hoeffding-Azuma inequality. For $Q$ we have $gQq \geq cQ L^{-1} n^{3/2}$, so it suffices to show $Var_Q \leq cQ^2 L^{-5/2} n^{3/2}$ and $N_Q \leq cQ L^{-3/2} n^{3/2}$. The change in $\mathcal{DQ}$ when the process chooses the edge $e_{i+1} = uv$ is

$$\Delta_i \mathcal{DQ}^d = 2(Y_{uv} + Y_{vu} + 1) - (q(i+1) - q(i)) = O(ye_Y) = O(c_Y L n^{1/4}).$$

Then $N_Q = \tilde{O}(n^{1/4})$, and the required bounds hold easily.

For $R$ we have $g_Rr \geq c_R L^{-1} q^2 n^{5/2}$, so it suffices to show $Var_R \leq c_R^2 L^{-5} q^4 n^{7/2}$ and $N_R \leq c_R L^{-3} q^2 n^{5/2}$. On choosing $e_{i+1} = uv$, for each open pair in $R$ we have

$$\Delta_i DR^d = \Delta_i R^d - \frac{\Delta_i(Q)}{Q} TR = 2 \sum_{ab \in Y_{uv} \cup Y_{vu} \cup \{uv\}} (X_{ab} - Q^2 n^{-3}) = \tilde{O}(q^{5/2} n^{5/4}).$$

There is also a destruction fidelity correction for closing two open pairs in a single triple counted by $R$, which occurs for open triples $uab$ such that $va$ and $vb$ are edges, and similarly interchanging $u$ and $v$. By Lemma 3.8 the number of such choices for $ab$ is $\tilde{O}(nq^3)$, which is negligible in comparison with $\Delta_i DR^d$. The required bounds hold easily.

For $S$ we have $g_{SS} \geq c_S L^{-1} q n^2$, so it suffices to show $Var_S \leq c_S^2 L^{-5} q^2 n^{5/2}$ and $N_S \leq c_S L^{-3} q n^2$. On choosing $e_{i+1} = uv$, for both open pairs in $S$ we have

$$\Delta_i DS^d = \Delta_i S^d - \frac{\Delta_i(Q)}{Q} TS = 2 \sum_{ab \in Y_{uv} \cup Y_{vu} \cup \{uv\}} (Y_{ab} - 2tQn^{-3/2}) = O(y \cdot e_Y y) = \tilde{O}(q^{3/2} n^{3/4}).$$

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Also, for the edge in $S$ we have creation

$$\Delta_t \mathcal{D} S^c = \Delta_t S^c - \mathcal{T} S/(tn^{3/2}) = 2X_{ab} - Q^2 n^{-3} = O(e_X x) = \tilde{O}(\delta^{3/2}n^{3/4}).$$

The required bounds hold easily.

Having verified the trend and boundedness hypotheses, the proof of Theorem 4.1 is complete.

5 The Controllable Ensemble

In this subsection we formulate a general condition under which we can control the variable $V = X_{\phi,J,\Gamma}$ with accuracy $e^\delta$. Let $t' \geq 1$. Recall that $V$ is controllable at time $t'$ if $J \neq \Gamma$ and for any $1 \leq t \leq t'$ we have

$$S_A^B(J,\Gamma) \geq n^\delta \quad \text{for all} \quad A \subset B \subset V_r.$$  \hfill (2)

Note in particular that these bounds are required to hold for $v = S_A^V(J,\Gamma)$. Note also that if $V$ is controllable and $V^+$ is obtained from $V$ by changing an edge to an open pair then $V^+$ is controllable.

Recall that $I_2$ is the minimum of the stopping times $I_V$ over all variables $V$ in the controllable ensemble. The following theorem says that there is a very small probability that we reach the universal stopping time $I$ before step $i_{\max}$ because a good controllable variable $V$ fails to satisfy the required bound $|\mathcal{D}V| \leq e_V v$.

**Lemma 5.1.** \(P(I = I_2 \text{ and } I_2 \leq i_{\max}) = o(1).\)

It is convenient to consider the following modified extension variable. Given an injective map \(f : V_\Gamma \to [n]\), we say that a pair \(ab\) in \(f(V_\Gamma)\) is \(f\)-open if there is no vertex $c$ such that $ac$, $bc$ are edges and \(c \notin f(V_\Gamma)\); note that it is the last condition that distinguishes the definition from that of ‘open’.

Let \(V^* = X^*_{\phi,J,\Gamma}(i)\) be defined in the same way as \(X_{\phi,J,\Gamma}(i)\), except that pairs that are required to be open in \(X^*_{\phi,J,\Gamma}(i)\) are only required to be \(f\)-open in \(X^*_{\phi,J,\Gamma}(i)\). We will show that \(V = V^* \pm g_v v/2\) and \(V^* = (1 \pm e_V) v\), where \(e_V = e_V - g_v/2\).

First consider the approximation of $V$ by $V^*$. Fix $e \in (V_\Gamma) \setminus \Gamma$ with $e$ not contained in $A$. Let $J^e = J \cup \{e\}$ and $\Gamma^e = \Gamma \cup \{e\}$. We can estimate $|V - V^*|$ by the sum over all such $e$ of $X_{\phi,J^e,\Gamma^e}$. By Lemma 3.8 we have $X_{\phi,J^e,\Gamma^e} \leq L^4 |V_\Gamma| S_A^{V^*}(J^e,\Gamma^e)/S_A^B(J^e,\Gamma^e)$, where $B$ is chosen to minimise $S_A^B(J^e,\Gamma^e)$. Note that $S_A^{V^*}(J^e,\Gamma^e) = pv$ and $S_A^B(J^e,\Gamma^e) \geq pn^\delta$ by controllability. Thus $X_{\phi,J^e,\Gamma^e} \leq L^4 |V_\Gamma| v n^{-\delta}$, which is at most $g_v v/2$ for $n$ sufficiently large. Here we recall that $e_V = f_V + 2g_v$ has the form described in Subsection 2.4, namely

$$f_v = e^\delta \quad \text{and} \quad g_v = \partial L^{-1}(1 + t^{-e(V)}) e^\delta.$$

and note that $e^\delta > n^{-\delta/4}$.

To estimate $V^*$ we will apply Freedman’s inequality to $ZV^* = V^* - \mathcal{T}V^* - (f_V + 3g_v/2)v$, where \(\mathcal{T}V^* = n^{(V)} p^{(V)}(Qn^{-2})^{o(V)}\) (under the assumption that $\mathcal{D}V^* = V^* - \mathcal{T}V^*$ is in the upper critical window). We begin with the boundedness hypothesis because it requires a delicate argument that uses (2) in a crucial way.
5.1 Boundedness hypothesis

Given $g_V$ as defined above, it suffices to show

$$\text{Var}_e \ll \frac{(t - e(V))e^2}{L^5n^{3/2}} \quad \text{and} \quad N_e \ll \frac{t - e(V)e^2}{L^3}. \quad (3)$$

Note that the one-step change in $TV^* + (g_V + f_V)v$ is $O((t + t^{-1})vn^{-3/2}) = \widetilde{O}(vn^{-5/4})$, which is negligible in comparison with the required estimates. Therefore, it suffices to consider changes in $V^*$ rather than $ZV^*$.

First consider the one-step change due to closing some $e = \alpha\beta \in \Gamma \setminus J$. Assume for the moment that $|V_{\Gamma}| < M^3$. (The case $|V_{\Gamma}| = M^3$ is treated below). Let $(J', \Gamma')$ be obtained from $(J, \Gamma)$ by ‘gluing a $Y$-variable on $\alpha\beta$’ as follows. Let $\gamma$ be a new vertex, $V' = V_\Gamma \cup \{\gamma\}$, $J' = J \cup \{\beta\gamma\}$ and $\Gamma' = \Gamma \cup \{\alpha\gamma, \beta\gamma\}$. Note that this definition depends on the order of $\alpha$ and $\beta$. We analyse closures of $e$ by adding the edge corresponding to $\alpha\gamma$.

Let $A' = A \cup \{\alpha, \gamma\}$ and $S_m = \min_{A' \cup B \in \mathcal{V}'} S^B_A$, where all scalings are with respect to $(J', \Gamma')$. We can bound the probability $p_e$ that $X_{\phi,J,G}$ is affected by closing an edge corresponding to $e$ by $p_e < L^{4|V'|}S_m/q$, using Lemma 3.8 and the assumption $|V_{\Gamma}| < M^3$, as this event is contained in the event that the new edge is contained in an extension to the graph induced by the set $B_m$ that defines $S_m$. Furthermore, we can bound the magnitude of the change in $V^*$ by $N_e < L^{4|V'|}L^{4|V'|}S_{A'}^\gamma / S_m$, applying Lemma 3.8 twice. Then the one-step variance in $V^*$ satisfies

$$\text{Var}_e < p_e N_e^2 < L^{20|V'|}(S_m/q)(S_{A'}^\gamma / S_m)^2 = L^{20|V'|}y^2 v^2 / (qS_m).$$

(Note that we use $S_{A'}^\gamma = yS_A^\gamma = yv$.) Now recall that $S_m = S_{A'}^B = S_{Bm}^{Bm}S_{Bm}^{Bm}\gamma$. We have $S_{Bm}^{\gamma} \geq y$ by construction of $(J', \Gamma')$ and $S_{Bm}^\gamma \geq n^\delta$ since $V$ is controllable. Thus $S_m \geq n^\delta y$, and we have

$$\text{Var}_e = \widetilde{O}(n^{-\delta} v^2 n^{-3/2}) \quad \text{and} \quad N_e = \widetilde{O}(n^{-\delta} v),$$

which gives the required bounds [3], provided that $n$ is sufficiently large. (Note that $e^\delta > n^{-\delta/4}$.)

We now turn to the one-step change due to destruction in the case $|V_{\Gamma}| = M^3$. In this case we cannot necessarily apply Lemma 3.8 as the extension we get by ‘gluing a $Y$-variable on $\alpha\beta$’ has $M^3 + 1$ vertices. Here we modify the argument slightly, getting the same bounds as above. First note that the bound on $p_e$ can be used exactly as above. If $B_m$ has $M^3 + 1$ vertices then first take the extension to $B_m \setminus \{\gamma\}$ and then extend to $\gamma$; this gives the same estimate as above because $V$ is controllable. The bound on $N_e$ is slightly more delicate. Consider the last step in the extension series for the extension from $B_m$ to $V'$ in the argument above. Let this extension be $(C, J', \Gamma')$. If $\beta \in C$ then the vertex $\gamma$ can be dropped from $C$ and $V_{\Gamma}$ without influencing the extension. This gives an extension with at most $M^3$ vertices and Lemma 3.8 applies to give the same bound as above. So we can restrict our attention to the case $\beta \notin C$. Note the extension from $C$ to $C \cup \{\beta\}$ contains the edge $\beta\gamma$ and the open pair $\alpha\beta$. So $S_{C'}^\gamma \leq S_{C \cup \{\beta\}}^\gamma \leq y$. Now, if $S_{C'}^\gamma > y/L^7$ then we can apply Lemma 3.7 directly to get the desired bound. On the other hand, if $S_{C'}^\gamma < y/L^7$ then $S_{C \cup \{\beta\}}^\gamma > y/L^7$, and in this case we can bound the number of extensions from $C$ to $V'$ by first bounding the number of extension from $C$ to $C \cup \{\beta\}$ by $Y_{\alpha\gamma}$ and then bounding the number of extensions from $C \cup \{\beta\}$ to $V'$ by an application of Lemma 3.8 (using the fact that $\gamma$ would play no role in such an extension). In every case we sacrifice at most a polylogarithmic factor in the bound on $N_e$, and the estimates above hold.

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Now consider the one-step change due to creating a copy of this extension. Suppose that \( e \in J \). Let \( e = \alpha \beta \) and \( A' = A \cup \{\alpha, \beta\} \). Define \( S_m = \min_{A' \subseteq B \subseteq \Gamma} S_B^\Gamma \) where all scalings are with respect to \( (J \setminus e, \Gamma) \). Similarly to the case \( e \in J \setminus J \), we can estimate the probability \( p_e \) that \( V \) is affected by choosing a pair corresponding to \( e \) by \( p_e < L^4|V|^4 S_m/q \), and the magnitude \( N_e \) of the effect by \( N_e < L^4|V| L^4|V|^2 S^\Gamma_A / S_m \). Here we have

\[
\text{Var}_e < p_e N_e^2 < L^{20|V'|}(S_m/q)(S^\Gamma_A / S_m)^2 = L^{20|V'|}(\hat{q}/p)^2 v^2/(qS_m).
\]

Since \((J, \Gamma)\) is controllable we have \( S_m \geq n^\delta(\hat{q}/p) \), and we have

\[
\text{Var}_e < L^{20|V'|}n^{-\delta}v^2/(2tn^{3/2}) \quad \text{and} \quad N_e = O(n^{-\delta}v),
\]

which gives the required bound \( \text{(3)} \) for \( n \) sufficiently large, since we must have \( e(V) > 1 \) in this case.

We have verified the boundedness hypothesis, so Lemma 5.1 will follow from Freedman’s inequality, after we establish the trend hypothesis in the following subsection.

### 5.2 Trend hypothesis

Assume \( V = X_{\phi, J, \Gamma} \) is controllable and \( V^* = X_{\phi^*, J, \Gamma}^* \) is in the upper critical window: \((f_V + g_V)v < DV^* < (f_V + 3g_V/2)v \). We give an upper bound on \( \mathbb{E}[\Delta_iDV^* \mid F_i] \), i.e. the one-step expected change in the difference variable. To organise the computations we gather terms according to each \( e \in J \) and \( e \in \Gamma \setminus J \). For \( e \in J \) contribute positively (creation); those with \( e \in \Gamma \setminus J \) contribute negatively (destruction). We write \( \Delta_i(V^*) = V^*(i + 1) - V^*(i) = \sum_{e \in J} \Delta_i(V^e) - \sum_{e \in \Gamma \setminus J} \Delta_i(V^e) \) and \( \Delta_i(V^e) = \Delta_i^*(V^e) \pm F_i(V^e) \), where \( \Delta_i^*(V^e) \) is the main term and \( F_i(V^e) \) is the ‘fidelity’ error term.

For \( e \in J \) we take \( \Delta_i(V^e) = \Delta_i^*(V^e) \) equal to the number of \( f \in X^*_{\phi^*, J, \Gamma}(i) \) such that \( f(e) \) is added at step \( i + 1 \). We can take the ‘creation fidelity’ term as \( F_i(V^e) = 0 \) because we are tracking \( V^* \) rather than \( V \). For \( e \in \Gamma \setminus J \) we take \( \Delta_i^*(V^e) \) equal to the number of \( f \in X^*_{\phi^*, J, \Gamma}(i) \) such that \( f(e) \) is selected as \( e_{i+1} \) or closed at step \( i + 1 \). The ‘destruction fidelity’ term \( F_i(V^e) \) bounds the number of such \( f \) such that \( f(e_{i+1}) \) also closes \( f(e') \) for some \( e' \in \Gamma \setminus J \) with \( e' \neq e \).

Recall that we write \( \Delta_i(TV^e) = TV^e(i + 1) - TV^e(i) = \sum_{e \in J} \Delta_i(TV^e) + O(t^2 + t^{-2})n^{-3}v \), where \( \Delta_i(TV^e) \) is \( \frac{TV^e}{tn^{3/2}} \) for \( e \in J \) or \( \frac{2Q}{Q}TV^e \) for \( e \in \Gamma \setminus J \). We also write

\[
\Delta_i(DV^e) = DV^e(i + 1) - DV^e(i) = \sum_{e \in J} \pm \Delta_i(DV^e) + O(n^{-5/2}v),
\]

where \( \Delta_i(DV^e) = \Delta_i(V^e) - \Delta_i(TV^e) \pm F_i(V^e) \).

#### 5.2.1 Creation

Fix \( e \in J \) (if \( J \neq \emptyset \)) and write \( V^+ = X_{\phi^*, J \setminus e, \Gamma} \). Then

\[
\mathbb{E}[\Delta_i(DV^e) \mid F_i] = \mathbb{E}[\Delta_i(V^e) - \Delta_i(TV^e) \mid F_i] = 2Q^{-1}V^+ - TV/(tn^{3/2}) = 2Q^{-1}DV^+ \leq (1 + o(1))t^{-1}eV^+vn^{-3/2}.
\]
5.2.2 Destruction

Fix $e \in \Gamma \setminus J$. We have $\mathbb{E}[\Delta_i(V^e) \mid F_i] = 2Q^{-1} \sum_{f \in V} (1 + C_{f(e)} + L^4)$, where $L^4$ bounds the number of open pairs that close $f(e)$ and also $f(e')$ for some other $e' \in \Gamma \setminus J$ (i.e. this is the destruction fidelity term). Then

$$\mathbb{E}[\Delta_i(DV^e) \mid F_i] = 2Q^{-1} \sum_{f \in V} (1 + C_{f(e)} + L^4) - \mathbb{E} \left[ \frac{\Delta_i(Q)TV}{Q} \mid F_i \right]$$

$$= (1 \pm e_Y)8tn^{-3/2}V + O(L^4q^{-1}v) - (1 \pm O(e_S))8tn^{-3/2}TV$$

$$= 8tn^{-3/2}DV \pm 8tn^{-3/2}e_Y V \pm O(e_S8tn^{-3/2}v) \pm O(L^4q^{-1}v)$$

$$\geq [f_Y + g_Y - (1 + o(1))e_Y - O(e_S) - O(L^4t^{-1}e^2)] 8tn^{-3/2}.$$ 

Note that $f_Y, f_S = o(f_V)$ and $g_S, L^4t^{-1}e^2 = o(g_V)$. So we have the bound

$$\mathbb{E}[\Delta_i(DV^e) \mid F_i] \geq (1 + o(1))(f_Y + g_Y - 2g_Y)8tn^{-3/2}.$$ 

5.3 Variation Equation

As for the Global Ensemble,

$$\Delta_i(e_{V^e}) = e_{V^e}(t + n^{-3/2})v(t + n^{-3/2}) - e_{V^e}(t)v(t)$$

$$= (e_{V^e}v'/v + e_{V^e})vn^{-3/2} + O(e_{V^e}' + e_{V^e}v'/v + e_{V^e}v''/v)vn^{-3},$$

where we estimate the last term by $O(t^2 + t^{-2})e_{V^e}vn^{-3} = \bar{O}(e_{V^e}vn^{-5/2})$. Note that

$$e_{V^e}' = 4\delta tf_Y + (\vartheta'/\vartheta - e(V)t^{-1}(1 + t^e(V))^{-1} + 4\delta t) \cdot 3g_Y/2.$$ 

Since $c_Y = 1$ for all $V$ in the Controllable Ensemble, we have $e_{V^e} = (3/2)(g_Y - g_Y)$. Note further that $g_Y \leq 2t^e v_y$ (and there is no $V^e$ term if $e(V) = 0$). Applying these observations we have

$$\frac{\mathbb{E}[\Delta_i(ZV) \mid F_i]}{8tn^{-3/2}} \leq (1 \pm o(1))\frac{e(Y)}{8t^e} - e_{V^e} - (1 + o(1))o(V)(f_Y + g_Y - 2g_Y)$$

$$- \left[\frac{(e(Y))}{8t^e} - o(V)e_{V^e} + \frac{e_{V^e}}{8t^e}\right] + \bar{O}(e_{V^e}t^{-1}n^{-1} + \bar{O}(t^{-1}n^{-1}))$$

$$\leq \frac{e(Y)}{8t^e} \cdot \frac{3}{2} (g_{V^e} - g_Y) + \frac{o(V)}{2} g_Y^\prime - \frac{1}{2} \delta f_Y$$

$$- \left( \frac{\vartheta'}{8t^e} - \frac{e(Y)}{8t^e} + \frac{3}{2}\right) \cdot \frac{3g_Y}{2} + O(g_Y) + o(e_Y^2 / t^2) + o(e_Y)$$

$$\leq \frac{g_Y}{2} \left( o(V) - \frac{3d_Y}{2} + \frac{3e(Y)}{4t} - \frac{3g_Y}{8t^e} - \frac{1}{2} \delta f_Y + O(g_Y) + o(e_Y^2 / t^2) + o(e_Y) \right).$$

Then $\mathbb{E}[\Delta_i(ZV) \mid F_i] \leq 0$, as $- \frac{\vartheta'}{8t^e} g_Y$ dominates for $t \ll 1$, and $- \frac{1}{2} \delta f_Y$ dominates otherwise.

6 Stacking ensemble

In this section we establish control of all variables in the stacking ensemble. Recall that $I_S$ is the minimum of the stopping times $I_V$ over all variables $V$ in the stacking ensemble. The following theorem says that there is a very small probability that we reach the universal stopping time $I$ before step $i_{\max}$ because a good stacking variable $V$ fails to satisfy the required bound $|DV| \leq e_Y v$. \hfill \{27\}
Theorem 6.1. $\mathbb{P}(I = I_3 \text{ and } I_3 \leq i_{\text{max}}) = o(1)$.

Before proceeding with the trend and boundedness hypotheses and variation equations, we discuss the extensions that require appeals to the controllable ensemble.

6.1 Subextensions of stacking variables

This subsection concerns certain subextensions of stacking variables that will be needed in the trend hypothesis. In two very important cases we will appeal to the Controllable Ensemble for our estimates, and so we need to show that these extensions are indeed controllable. The two special structures that we need to consider are the following.

- An $h$-fan at the triple $A = abc$, also called a $\Phi_h$-extension at $A$, is any extension of the form $(A, J, \Gamma)$, where the base $A = abc$, $V_T$ has $h$ additional vertices $v_1, \ldots, v_h$, $bv_1 \ldots v_h c$ is a path of length $h + 1$ in $\Gamma$, and $av_i \in \Gamma \setminus J$ is open for $i \in [h]$. We emphasize that the pairs in the path $bv_1 \ldots v_h c$ can be either edges or open pairs.

- Let $(uv, J, \Gamma)$ be the extension corresponding to some stacking variable $\pi \in S_M$ such that $w(\pi) = 2M$. Note that if $\alpha_x\alpha_y$ is the last rung of $\pi$ then the extensions we get by stacking another copy of $Y$ on $\alpha_x\alpha_y$ are not in $S_M$. The backward extension $B_\pi$ is the extension $(A', J', \Gamma')$ with $A' = \{\alpha_u, \alpha_v, \alpha_x, \alpha_y\}$, $J' = J$ and $\Gamma' = \Gamma \setminus \alpha_x\alpha_y$.

We now show that if we set

$$h = M = 3/\varepsilon$$

then both of these extension variables are controllable at the time $t_{\text{max}}$ when $\hat{q}(t_{\text{max}}) = n^{-1/2+\varepsilon}$.

We begin with the $h$-fan. The minimum scaling of a $\Phi_h$-extension is $(\hat{q}n)^h p^{h+1} > n^{eh-1/2}(2t_{\text{max}})^{h+1}$, which is achieved when the path belongs entirely to $J$. Fix $B$ with $A \subseteq B \subseteq V$ that minimises $S_A^B = S_A^B(J, \Gamma)$. We need to show that $S_A^B \geq n^\delta$. We can assume that $B \neq V_T$. Then we can find $v_i$ in $B$ such that not both $v_{i-1}$ and $v_{i+1}$ are in $B$. (Here $v_0 = c$ and $v_{h+1} = b$.) But removing $v_i$ from $B$ reduces the scaling by at least $\gamma \geq n^{\varepsilon}$. Thus the minimum scaling is achieved by a single vertex extension of $A$, which is at least $\gamma > n^\delta$.

Now consider a backward extension $B_\pi$ with $w(\pi) = 2M$, and fix $B$ with $A' \subset B \subseteq V$. We calculate $S_{A'}^B$ going one vertex at a time. If $B$ does not contain some vertex $\alpha_i$ such that $\pi(i+1) = O$ or does not contain at least one vertex from each rung then by counting from the ends we see that $S_{A'}^B > (n^{\varepsilon}|B|-|A'|$. On the other hand, if $B$ indeed contains every $\alpha_i$ such that $\pi(i+1) = O$ and intersects every rung, then we claim that $|B| \geq M + 2$. To see this, first note that the number of vertices in $B$ in the interior of each triangular ladder must be at least half the number of turning points in the ladder rounded down. If $\pi$ has $i$ occurrences of the symbol $O$ then there are at most $i + 1$ triangular ladders and $\pi$ has $2M - 2 - i$ turning points that are on the interior of ladders. It follows that

$$|B \setminus A'| \geq i + \frac{2M - 2 - i}{2} - \frac{i + 1}{2} \geq M - 2.$$ 

With this observation in hand, simply counting in the stacking order we have

$$S_{A'}^B > (n^{\varepsilon}|B|-2)/(n^2\hat{q}) > n > n^\delta.$$
6.2 Trend hypothesis

Fix a stacking variable \( V = S_{uv}^\pi \). Assume \( V = X_{\alpha,I} \) is in the upper critical window: \( (f_V + g_V)v < DV < (f_V + 2g_V)v \). We give an upper bound on \( \mathbb{E}[\Delta_i DV | \mathcal{F}_i] \), i.e. the one-step expected change in the difference variable. To organise the computations we gather terms according to each edge \( e \) of the underlying graph pair \((J, \Gamma)\) of \( \pi \). Edges of \( \pi \) contribute positively (creation); open pairs of \( \pi \) contribute negatively (destruction). We write \( J \), \( \Gamma \), the underlying graph pair \((J, \Gamma)\) of \( \pi \). Edges of \( \pi \) contribute positively (creation); open pairs of \( \pi \) contribute negatively (destruction). We write

\[
\Delta_i(V) = V(i + 1) - V(i) = \sum_{e \in J} \Delta_i(V^e) - \sum_{e \in \Gamma \setminus J} \Delta_i(V^e),
\]

where each \( \Delta_i(V^e) \) includes a ‘fidelity’ error term that accounts for the selection of edges that impact a given extension in more than one way.

Consider the case that \( |\pi| > 1 \), and \( \pi(|\pi| - 1) \neq O \) or \( \pi(|\pi|) \in \{O, E\} \). We write \( \pi = \pi^U \), \( V^\pi = S_{uv}^{\pi^-} \) and recall that \( TV = V - T\mathcal{U} \). We say that a pair \( e \) is terminal if it belongs to \( U \), i.e. it contains the final vertex of \( V \); otherwise we say that \( e \) is internal. For ease of notation, if \( |\pi| = 1 \) then we write \( U = V \), and let \( V^\pi \) be the variable that is identically equal to 1. We write \( \Delta_i(TV) = \Delta_i(V^-)T\mathcal{U} + V^-\Delta_i(T\mathcal{U}) = \sum_{e \in J} \Delta_i(TV^e) - \sum_{e \in \Gamma \setminus J} \Delta_i(TV^e) \), where \( \Delta_i(TV^e) \) is defined as follows.

(i) If \( e \) is a terminal edge then \( \Delta_i(TV^e) = \frac{TV}{t_n^{\beta_\pi}} \),

(ii) If \( e \) is a terminal open pair then \( \Delta_i(TV^e) = \frac{\Delta_i(Q)}{Q}TV \),

(iii) If \( e \) is internal then \( \Delta_i(TV^e) = \Delta_i(V^-)T\mathcal{U} \).

For each of these terms we will also include a ‘fidelity’ error term: for terminal pairs the fidelity term is \( O(t_n^{-5/2}) \).

If \( \pi(|\pi| - 1) = O \) and \( \pi(|\pi|) \notin \{O, E\} \), we instead write \( \pi = \pi^-O \), \( V^- = S_{uv}^{\pi^-} \), \( \beta = \alpha_{|\pi|-2} \) and recall that

\[
TV = \begin{cases} 
\sum_{f \in V^-} X_{f(\beta)}^2 \cdot Qn^{-2} & \text{if } U \in \{X^I, X^O\} \\
\sum_{f \in V^-} X_{f(\beta)}^2 \cdot 2tn^{-1/2} & \text{if } U = Y^I \\
\sum_{f \in V^-} X_{f(\beta)} \cdot Qn^{-2} & \text{if } U = Y^O.
\end{cases}
\]

In this case we say that only \( \alpha_{|\pi|-1}\alpha_{|\pi|} \) is terminal; its treatment is exactly as (i) and (ii) above. However, recall that \( \alpha_{|\pi|-2}\alpha_{|\pi|-1} \) and \( \alpha_{|\pi|-2}\alpha_{|\pi|} \) are partner pairs. We write \( \beta = \alpha_{|\pi|-2} \) and extend the above definition of \( \Delta_i(TV^e) \) as follows.

(iv) If \( e \) is a partner edge then \( \Delta_i(TV^e) = \sum_{f \in V^-} \Delta_i(Y_{f(\beta)}) \cdot X_{f(\beta)} \cdot Qn^{-2} \),

(v) If \( e \) is a partner open pair then

\[
\Delta_i(TV^e) = \begin{cases} 
\sum_{f \in V^-} \Delta_i(X_{f(\beta)}^d) \cdot X_{f(\beta)} \cdot Qn^{-2} & \text{if } U \in \{X^I, X^O\} \\
\sum_{f \in V^-} \Delta_i(X_{f(\beta)}^d) \cdot X_{f(\beta)} \cdot 2tn^{-1/2} & \text{if } U = Y^I \\
\sum_{f \in V^-} \Delta_i(X_{f(\beta)}^d) \cdot Y_{f(\beta)} \cdot Qn^{-2} & \text{if } U = Y^O.
\end{cases}
\]
We emphasise that we do not consider partner pairs to be terminal, even though one of them uses the last vertex of $V$.

Recall that we view $S^n_u$ as counting the number of injections $\psi : \{\alpha_u, \alpha_v, \alpha_1, \ldots, \alpha_{|\pi|}\} \to [n]$ such that $\psi(\alpha_u) = u$, $\psi(\alpha_v) = v$ and $\psi(\alpha_i)$ is a vertex that plays the role in the extension defined by $\pi(i)$ for $i = 1, \ldots, |\pi|$. We will also use the following notation:

- For any $y \leq |\pi|$ we let $\pi|_y$ denote the prefix of $\pi$ of length $y$.
- If $\pi(|\pi|) \in \{X^I, X^O, Y^I, Y^O\}$ we let $\pi^o$ be obtained from $\pi$ by interchanging $I$ and $O$ in $\pi(|\pi|)$.

In the following subsections we use the following classification for a terminal open pair $e$.

(a) If $e$ is a rung and $\pi Y^I$ and $\pi Y^O$ both belong to $S_M$ we say that $e$ is \textit{simple}. If $e$ is a stringer and $\pi^o Y^I$ and $\pi^o Y^O$ both belong to $S_M$ we say that $e$ is \textit{simple}.

(b) If $w(\pi) = 2M$ and $e$ is the terminal rung we say that $e$ is \textit{outer}. If $w(\pi) = 2M - 1$, $\pi(|\pi|) = X^I$ and $e$ is the terminal stringer then we say that $e$ is \textit{outer}.

(c) If $e$ is not simple or outer we say that $e$ is a \textit{fan end} pair.

We note that outer pairs are not simple, as adding $Y^O$ to any $\pi'$ with $w(\pi') = 2M$ gives a variable not in $S_M$: we apply this with $\pi' = \pi$ when $e$ is the terminal rung or $\pi' = \pi^o$ when $e$ is the terminal stringer. We also note that a fan end pair $e$ is aptly named, as in this case $\pi$ must end with an $(M - 1)$-fan.

We now calculate the contribution from each type of edge and open pair. The key point here is that every open pair yields a self-correcting term of the form $(f_V + g_V)8tvn^{-3/2}$. Furthermore, the only open pairs that give significant error terms are terminal open pairs. Thus, for any stacking variable there are only a small number of terms that need to be handled with care when we consider the variation equations.

### 6.2.1 Internal non-partner destruction

Suppose that $e = \alpha_x \alpha_y$ with $x < y < |\pi|$ is internal but not partner. Set $\pi' = \pi|_y$ and let $W = S^\pi_{uv}$, $W^I = S^\pi_{uv} Y^I$ and $W^O = S^\pi_{uv} Y^O$. Note that $W$ is in $S_M$, but $W^I$ and $W^O$ may or may not be in $S_M$. In any case we have

$$W^I + W^O = \sum_{f \in W} Y_{f(xy)} + \sum_{f \in W} Y_{f(yx)} = (1 \pm e_Y)2Wy.$$

(The attentive reader might note that the first equality above is not technically correct: we should have replaced $Y_{f(xy)}$ by $|Y_{f(xy)} \setminus \text{Im}(f)|$. However, this gives an error term that is miniscule by comparison with the others, so we ignore it here, and in similar situations throughout this section.)

For each $f \in W$ let $F_{f,\pi} = X_{f,\nu,\nu'}$ count the number of forward extensions from $f$ to copies of $(J, \Gamma)$, in which $A = \{\alpha_u, \alpha_v, \ldots, \alpha_y\}$, $J' = J \setminus J[A]$ and $\Gamma' = \Gamma \setminus \Gamma[A]$. Note that this is simply another variable in this ensemble, namely $F_{f,\pi} = S^\pi_{f(e)}$, where $\pi' \circ \pi'' = \pi$. We define $F_{f,\pi}$ similarly (recall that $\pi = \pi^{-U}$).
Since $Δ_i(TV) = Δ_i(V^-)TU$, applying Lemma 2.3 we have

$$E[Δ_i(DV) \mid F_i] = E[Δ_i(V^e) - Δ_i(TV^e) \mid F_i]$$

$$= 2Q^{-1} \sum_{f \in W} (Y_{f(xy)} + Y_{f(yx)} ± L^4)(F_{f,π} - F_{f,π^-}TU)$$

$$= (1 ± O(εY^e)) \left( \frac{2V^o(W^o + W^I)}{QW} \right) - (1 ± O(εY^e)) \left( \frac{2V^- (W^o + W^I)}{QW} \right)TU ± L^4V^/Q$$

$$= 2(W^o + W^I)DV \overline{QW} ± O(\Delta V^o_{xy}V^o) ± O(εY^e)tvn^{-3/2}$$

$$≥ (1 + o(1))(f + g) - O(εY^e) - O(\Delta V^o_{xy}V^o)tvn^{-3/2}.$$  

(Note that using $ε^e$ is an overestimate for the error in the forward extensions, but we will incur such an error elsewhere, so we use the same formula here and throughout for simplicity.)

### 6.2.2 Partner destruction

Here we consider a partner open pair $e = α_xα_y$ with $x < y$. In this case $π(|π| - 1) = O$, $π(|π|) \notin \{O, E\}$, $x = |π| - 2$ and $y \in \{|π| - 1, |π|\}$. Note that if $y = |π|$ then it is isomorphic to the case $y = |π| - 1$, so we can restrict our attention to the latter. We write $π = π^-OU$, $V^- = S_{uv}^\pi$, $W = S_{uv}^O$, $W^o = S_{uv}^OY^O$, and $W^I = S_{uv}^OY^I$. Note that

$$W^I + W^o = \sum_{f \in W} Y_{f(xy)} + \sum_{f \in W} Y_{f(yx)} = (1 ± εY^-)2W_y.$$

For $f \in V^-$ we set

$$\hat{U}_f = \begin{cases} X_{f(α_x)}Qn^2 & \text{if } U \in \{X', X^O\} \\ X_{f(α_x)}2tn^{-1/2} & \text{if } U = Y^I \\ Y_{f(α_x)}Qn^2 & \text{if } U = Y^O. \end{cases}$$

We have

$$E[Δ_i(DV) \mid F_i] = E[Δ_i(V^e) - Δ_i(TV^e) \mid F_i]$$

$$= 2Q^{-1} \sum_{f \in V^-} \sum_{z \in X_{f(α_x)}} (Y_{f(α_x)}z + Y_{f(α_x)}z ± L^4)(U_{f(α_x)}z - \hat{U}_f)$$

$$= 2(W^o + W^I)(V^- TU) \overline{QW} ± O(εY^e)tvn^{-3/2} ± O(L^4v/q)$$

$$≥ (1 + o(1))(f + g) - O(εY^e) - O(\Delta V^o_{xy}V^o)tvn^{-3/2}.$$  

### 6.2.3 Simple destruction

Let $e = α_xα_y$ be a simple rung, i.e. the last rung of a ladder defined by $π$ with the property that $π^I$ and $π^O$ both belong to $S_M$. We have $E[Δ_i(V^e) \mid F_i] = 2Q^{-1} \sum_{f \in V}(1 + C_{f(e)} ± L^4)$, where
\(L^4\) bounds the number of open pairs that close \(f(e)\) and also \(f(e')\) for some other \(e' \in \Gamma \setminus J\) (we use Lemma 3.8). Write \(V^I = S^I_{uv}\) and \(V^O = S^O_{uv}\). Note that \(TV^I = TV^O = 2tqn^{-3/2}V\) and \(v^I = v^O = 2tqn^{1/2}v\). Since \(\Delta_i(\mathcal{T}V^e) = \frac{\Delta(V^e)}{Q}TV\) we have

\[
\begin{align*}
\mathbb{E}[\Delta_i(\mathcal{D}V^e) \mid F_i] &= \mathbb{E}[\Delta_i(V^e) - \Delta_i(\mathcal{T}V^e) \mid F_i] \\
&= 2Q^{-1}(V^I + V^O \pm L^2V) - (2 + 4SQ^{-1})TVQ^{-1} \\
&= 2Q^{-1}(TV^I + TV^O \pm v^Ie_{V^I} + v^Oe_{V^O}) - (8tn^{-3/2} \pm 4esq^{-2})TV \pm O(L^4v/q) \\
&= 8tn^{-3/2}DV \pm (1 + o(1))st(e_{V^I}/2 + e_{V^O}/2 + es)vn^{-3/2} \pm O(L^4v/q) \\
&\geq (1 + o(1))(fv + gv - v_{V^I}/2 - v_{V^O}/2 - es - O(L^4t^{-1}e^2))stvn^{-3/2}.
\end{align*}
\]

The same calculation applies if \(e\) is a simple stringer (using \(\pi^o\) in place of \(\pi\)).

6.2.4 Outer destruction

Let \(e = \alpha_x\alpha_y\) be an outer rung, i.e. \(e\) is terminal and \(w(\pi) = 2M\). For each \(ab \in Q\) let the variable \(X_{uvab}\) count the number of backward extensions \(B_x\) that map the last rung of \(S^o_{uv}\) to the open pair \(ab\). Recall that \(B_x\) is controllable and note that there can be no bad edge as \(M\) is assumed to be large, so we can estimate \(X_{uvab}\) using Lemma 5.1. Since \(\Delta_i(\mathcal{T}V^e) = \frac{\Delta(V^e)}{Q}TV\) we have

\[
\begin{align*}
\mathbb{E}[\Delta_i(\mathcal{D}V^e) \mid F_i] &= \mathbb{E}[\Delta_i(V^e) - \Delta_i(\mathcal{T}V^e) \mid F_i] \\
&= 2Q^{-1} \sum_{ab \in Q} X_{uvab}(Y_{ab} + Y_{ba} \pm L^4) - \frac{4S + 2Q}{Q^2}TV \\
&= 4Q^{-1}(1 + O(e_ye^\delta))Q^{-1}VS - 4SQ^{-2}TV \pm O(L^4v/q) \\
&\geq (1 + o(1))(fv + gv - O(e_ye^\delta) - O(L^4t^{-1}e^2))stvn^{-3/2}.
\end{align*}
\]

The same calculation applies if \(e\) is an outer stringer (using \(\pi^o\) in place of \(\pi\)).

6.2.5 Fan end destruction

For destruction, it remains to consider the case when \(e = \alpha_x\alpha_y\) is a fan end, i.e. \(\pi\) ends with an \((M - 1)\)-fan and \(e\) is the terminal rung. Let \(\pi^* = \pi \downarrow O\) and \(\pi^* = \pi \downarrow E\) be obtained from \(\pi\) by replacing the fan by an \(X_x\) or \(Y_x\) extension. Let \(V^x = S^x_{uv}\), \(V^* = S^*_{uv}\), \(V^* = S^*_{uv}\), \(V^x = S^x_{uv}\) and \(V^O = S^O_{uv}\). We write \(V = \sum_{f \in V^x} F_{f,\pi}\), where \(F_{f,\pi}\) denotes the forward extension, i.e. the \(\Phi_{M-1}\) extension from \(f(\alpha_x-1\alpha_x\alpha_y)\); recall that this is controllable, so we can estimate it using Lemma 5.1. Note that by Lemma 2.3 we have

\[
\begin{align*}
TV^xI &= \sum_{f \in V^x} X^2_{f(\alpha_x)} \cdot 2tn^{-1/2} = (1 + e_{X^1}^2)2tn^{-1/2}(V^*)^2/V^x, \\
TV^xO &= \sum_{f \in V^x} X_{f(\alpha_x)}Y_{f(\alpha_x)} \cdot Qn^{-2} = (1 + e_{X^1}e_{Y^1})Qn^{-2}V^xV^*/V^x.
\end{align*}
\]

This implies

\[
\begin{align*}
V^xI/QV^* &= (1 + (1 + o(1))(e_{V^*I} + e_{V^*I}))2tn^{-3/2}, \quad \text{and} \\
V^xO/QV^* &= (1 + (1 + o(1))(e_{V^*O} + e_{V^*O}))2tn^{-3/2}.
\end{align*}
\]

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Since $\Delta_i(TV^e) = \frac{\Delta_i(Q)}{Q}TV$ we have

$$
\mathbb{E}[\Delta_i(DV^e) \mid F_i] = \mathbb{E}[\Delta_i(V^e) - \Delta_i(TV^e) \mid F_i]
$$

$$
= 2Q^{-1} \sum_{f \in V^*} (Y_{f(ab)} + Y_{f(ba)} \pm L^4)F_{f,\pi} - \frac{4S}{Q^2}TV
$$

$$
= \frac{2(V^*I + V^*O)V}{QV^*} - \frac{4S}{Q^2}TV \pm O(\epsilon_Y e^\delta tvn^{-3/2}) \pm O(L^4v/q)
$$

$$
= (1 \pm \frac{1}{2}(1 + o(1))(e_{V^*} + e_{V^*} + e_{V^*} + e_{V^*})8tn^{-3/2}V - (1 + e_S)8tn^{-3/2}TV
$$

$$
\geq (fV + gV - (1 + o(1))(\frac{1}{2}(e_{V^*} + e_{V^*} + e_{V^*} + e_{V^*}) + e_S) - O(\epsilon_Y e^\delta) - O(L^4t^{-1}e^2))8tn^{-3/2}.
$$

### 6.2.6 Internal creation

Suppose that $e = \alpha_x\alpha_y$ with $x < y < |\pi|$ is an internal stringer edge of $\pi$. Let $V^+, V^-\pi$ be the variables obtained from $V$, $V^-$ when $e$ is replaced by an open pair. Then $TV^+ = V^+ \cdot TV$ and $v^+ = v \cdot \frac{\hat{q}n^{1/2}}{2t}$.

To estimate the creation fidelity term consider $e' \in \binom{V}{2} \setminus \Gamma$ and let $V'$ be obtained from $V^+$ by making $e'$ an edge. Consider the construction of $V'$ one vertex at a time in $\pi$-order. We see the same extensions as for $V^+$, except that at one step there is an extra edge incident to the new vertex. This reduces the scaling for this vertex by a factor of $p = 2tn^{-1/2}$. When we use Lemma 3.8 to bound the number of copies of $V'$, we do not get the full $p$ reduction if $v' < 1$. However, $V$ must have at least two new vertices in order for creation fidelity to be an issue, and therefore $v^+ \geq q^2q^{-1}$. Note that $v^+ e^2 \geq y^2 q^{-1} e^2 = 2tq^2n > 1$. So max$\{v', 1\} < v^+ e^2$, and by Lemma 3.8 we estimate $V' = O(L^4|V|e^2)v^+$.

Since $\Delta_i(TV^e) = \Delta_i(V^e)TU$ we have

$$
\mathbb{E}[\Delta_i(DV^e) \mid F_i] = \mathbb{E}[\Delta_i(V^e) - \Delta_i(TV^e) \mid F_i]
$$

$$
= 2Q^{-1}V^+(1 \pm O(L^4|V|e^2)) - 2Q^{-1}V^-(1 \pm O(L^4|V|e^2)) \cdot TV
$$

$$
= 2Q^{-1}(V^+ - TV^+) \pm O(L^4|V|e^2 q^{-1} v^+)
$$

$$
= (1 + O(e_Q))2q^{-1}DV^+ \pm O(L^4|V|e^2 q^{-1} v^+)
$$

$$
= \pm (1 + o(1))t^{-1}e_{V^+} \pm O(t^{-1}L^4|V|e^2)\] vn^{-3/2}.
$$

### 6.2.7 Partner creation

Suppose that $e = \alpha_x\alpha_y$ with $x < y = |\pi|$ is the partner edge of $\pi$. We can write $\pi = \pi^OY^O$. We estimate the creation fidelity term as for internal creation. Then

$$
\mathbb{E}[\Delta_i(DV^e) \mid F_i] = \mathbb{E}[\Delta_i(V^e) - \Delta_i(TV^e) \mid F_i]
$$

$$
= 2Q^{-1} \sum_{f \in S_{\alpha x}^e} \sum_{z \in X_{f(\alpha x)}} (X_{f(\alpha x)}z - X_{f(\alpha x)}Qn^{-2})
$$

$$
= 2Q^{-1}(V^+(1 \pm O(L^4|V|e^2)) - TV^+),
$$

$$
= (1 + O(e_Q))2q^{-1}DV^+ \pm O(L^4|V|e^2 q^{-1} v^+),
$$

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which gives the same expression as for internal creation.

6.2.8 Terminal creation

Finally, suppose that $e$ is the terminal edge of $\pi$. Then $\pi(|\pi|)$ is $E$, $Y^I$ or $Y^O$, and if $\pi(|\pi|) = Y^O$ then $\pi(|\pi| - 1) \neq O$ (otherwise $e$ would be partner). Let $V^+$ be the variable obtained by changing $e$ to an open pair, i.e. replacing $Y$ by $X$ in $U = \pi(|\pi|)$. Note that $TV^+ = TV \cdot \frac{Q}{2tn^{3/2}}$ and $v^+ = v \cdot \frac{2n^{1/2}}{2t}$. We estimate the creation fidelity term as for internal creation. Since $\Delta_i(DV^e) = TV^+/tn^{3/2}$ we have

$$E[\Delta_i(DV^e) | F_i] = \sum_{e \in \pi} \pm E[\Delta_i(V^e) - \Delta_i(TV^e) | F_i] = 2Q^{-1}V^+(1 \pm O(t^{-1}L^4t^{-1}e^2)) - TV/((tn^{3/2})$$

which gives the same expression as for internal creation.

6.3 Variation equation

Now we combine all the estimates in this section to show that $ZV = DV - eVv$ forms a supermartingale. Recall that $E[\Delta_i(DV) | F_i] = \sum_{e \in \pi} \pm E[\Delta_i(DV^e) | F_i]$, where the summand is positive for an edge and negative for an open pair. There are $e(V)$ edges, each giving a creation term of $(1 + o(1))(t^{-1}e_{V+} \pm O(t^{-1}L^4|V|e^2))v_{n-3/2}^-$. There are $o(V)$ open pairs, each giving a destruction term in which the main term is a self-correction term of $(1 + o(1))(fV + g_{V} \pm O(e_{V}^2) \pm O(L^4t^{-1}e^2))8tv_{n-3/2}^-tV^-$. There are additional error terms that come from terminal open pairs, which we denote by $e_{add}$; the form of this error term depends on the form of the terminal open pair. The constants $c\pi$ are carefully chosen to ensure that this term does not spoil the supermartingale condition.

- If the terminal open pair is outer then there is no additional error term at all, i.e. $e_{add} = 0$ in this case.
- If the terminal open pair is simple then the error term is

$$e_{add} = (1 + o(1))(e_{V^I}/2 + e_{V^O}/2 + e_S).$$

Note that

$$c_{V^O} = c_{V^I} = c_V/9.$$  

- If the terminal open pair is a fan end, i.e. the last rung in an $(M - 1)$-fan, then the error term is

$$e_{add} = (1 + o(1))(\frac{1}{2}(e_{V^*} + e_{V^*} + e_{V^*} + e_{V^*} + e_S).$$

Note that the sequences defining $V^*$ and $V^*$ each have $M - 1$ fewer symbols than $\pi$, but also have an additional ‘O’ or ‘E’. Thus

$$c_{V^*} = c_{V^*} = c_V/9 \text{ and } c_{V^*} = c_{V^*} = c_V/81.$$
Observe that $\pi$ can have two terminal open pairs (both simple, or one simple and one fan end), and we must add the estimates from both to get a worst case bound. Writing $e_{\text{add}} = f_{\text{add}} + 2g_{\text{add}}$ in the natural way, we have

$$f_{\text{add}} < \frac{10}{St} f_V \quad \text{and} \quad g_{\text{add}} < \begin{cases} \frac{10}{St} t^{-1} g_V & \text{if } t < 1 \\ \frac{10}{St} g_V & \text{if } t \geq 1. \end{cases}$$

We have

$$\mathbb{E}[\Delta_i(DV) \mid F_i] = \frac{e(V)}{St} (1 + o(1)) t^{-1} e_V + O(t^{-1} L^4 |V|^2) - (1 + o(1)) o(V) (f_V + g_V) + (1 + o(1)) e_{\text{add}} + O(e_Y e^\delta) + O(L^4 t^{-1} e^2).$$

As for general extensions we write

$$\Delta_i(e_V v) = e_V (t + n^{-3/2}) v(t + n^{-3/2}) - e_V (t) v(t) = (e_V v' / v + e'_V) vn^{-3/2} + O(t^2 + t^{-2}) e_V vn^{-3},$$

where we bound the last term by $O(e_V vn^{-5/2})$. Note that

$$e'_V \geq 4 t f_V + (v' / \theta - e(V) t^{-1} + 4t) \cdot 2g_V.$$

Since $V^+$ and $V$ have the same length and the same number of copies of $O$ and $E$, we have $e_V = e_{V^+}$ and so $e_{V^+} - e_V = 2(g_{V^+} - g_V)$. We also have $g_{V^+} \leq 2tg_V$ (and there is no $V^+$ term if $e(V) = 0$).

Thus

$$\mathbb{E}[\Delta_i(ZV) \mid F_i] \leq (1 + o(1)) \frac{e(V)}{St} (t^{-1} e_{V^+} + O(t^{-1} L^4 |V|^2)) - (1 + o(1)) o(V) (f_V + g_V - e_{\text{add}}) + O(e_Y e^\delta) + O(L^4 t^{-1} e^2) - ((\frac{e(V)}{St} - o(V)) e_V + \frac{e'_V}{St})
\leq \frac{e(V)}{St} \cdot 2(g_{V^+} - g_V) + o(V) g_V + (1 + o(1)) e_{\text{add}} - f_V / 2
- (\frac{\theta' / \theta}{St} - \frac{e(V)}{St} + \frac{1}{2}) \cdot 2g_V + O(e_Y e^\delta) + O(t^{-2} L^4 |V|^2) + o(e_V t^{-2}) + o(e_V)
\leq g_V \left[ o(V) + \frac{e(V)}{2St} - \frac{\theta' / \theta}{St} - 1 \right] + 2(1 + o(1)) g_{\text{add}} - \frac{f_V}{2} + (1 + o(1)) f_{\text{add}} + O(e_Y e^\delta) + O(t^{-2} L^4 |V|^2) + o(e_V t^{-2}) + o(e_V).$$

Since $-g_V \theta' / (4t \theta)$ dominates for $t \ll 1$ and $-f_V / 2$ dominates otherwise, we have $\mathbb{E}[\Delta_i(ZV) \mid F_i] \leq 0$.

Finally, note that the above calculation would have still been valid if we had replaced $e_V$ by $e_V / 3$; we will use this observation for open degree variables to control degree variables in the next subsection.

### 6.4 Boundedness hypothesis

We start by recalling we cannot apply our general strategy to vertex degree variables, as the function $g_V(t) v(t)$ is not approximately non-increasing, so we give a separate argument for these variables. Consider the degree variable $Y_u(i')$ for any vertex $u$ and step $i'$. For each $1 \leq i \leq i'$, the probability that we choose an edge incident to $u$ is

$$\frac{X_u(i)}{Q(i)} = \frac{(1 \pm e_{X_i} / 3) x_1}{(1 \pm e_q) q} = \left(1 \pm (1 + o(1)) e_{X_1} / 3\right) \frac{2}{n},$$

where

- $x_1 = \frac{X_u(i)}{Q(i)}
- e_{X_1} = \frac{1}{2} \cdot \text{degree of } u
- e_q = \frac{1}{2} \cdot \text{degree of } q
- X_u(i) = \text{number of edges incident to } u
- Q(i) = \text{total number of edges in graph}
- n = \text{number of vertices in graph}$

Using this, we can bound the expected change in the degree of $u$ as

$$\Delta u = \frac{1}{2} e_{X_1} / 3.$$
using the observation in the previous subsection that we can bound the relative error in $X_u(i)$ by $e_{X_1}/3$. Thus we can couple $Y_u(i')$ to a sum of independent indicator variables with mean

$$\sum_{i=1}^{i'} \left( 1 \pm (1 + o(1))e_{X_1}(t)/3 \right) 2/n = \left( 1 \pm (1 + o(1))e_{X_1}(t')/3 \right)y_1(t').$$

Since $f_{Y_1} = f_{X_1}$ and $g_{Y_1} = g_{X_1}(1+t^{-1})/2$ we have $e_{Y_1} > e_{X_1}/2$, so if we do not have $Y_u(i) = (1 \pm e_{Y_1})y_1$ then $Y_u(i)$ deviates from its mean by more than $e_{Y_1}y_1/4 > L^{13}n^{1/4}$. By Chernoff bounds, why this does not occur for any vertex $u$.

Now recall that for any variable $V$ other than the vertex degrees, it suffices to show $Var_V \leq (g_Vv)^2/(L^3n^{3/2})$ and $N_V \leq g_Vv/L^2$.

Fix $\pi \in S_M$, a non-edge $uv$, and let $V = S_{uv}^\pi$. Let $(J, \Gamma)$ be the underlying graph pair of $\pi$ and $A = uv$. Since $g_Vv \geq c_VL^{-1}(1 + t^{-e(V)})ev$ and $e^2n^{-3/2} = q^{-1}$, it suffices to show $Var_e \leq L^{-6}q^{-1}(c_V(1 + t^{-e(V)})v)^2$.

We analyse the contributions from each edge $e = xz \in \Gamma \setminus \Gamma[A]$ separately. There are two cases, according to whether $e$ is an open pair or an edge.

First suppose $x \in \Gamma \setminus J$ is an open pair (this can be either a rung or a stringer). Let $(J', \Gamma')$ be obtained from $(J, \Gamma)$ by ‘gluing a $Y$-variable on $x \in \Gamma$’ as follows. Let $\gamma$ be a new vertex, $V' = V_\Gamma \cup \{x\}$, $J' = J \cup \{x \gamma\}$ and $\Gamma' = \Gamma \cup \{x \gamma, x \gamma \gamma\}$. Note that this definition depends on the order of $\alpha_x$ and $\alpha_y$. We analyse closures of $e$ by adding the edge corresponding to $\alpha_x \gamma$. Let $A = \{\alpha_\nu, \alpha_v\}, A' = A \cup \{\alpha_x \gamma\}$ and $S_m = \min_{\Phi \subseteq B \subseteq V} S^B_A$, where all scalings are with respect to $(J', \Gamma')$. Let $B_m$ be the minimal set $B$ with $A' \subseteq B \subseteq V'$ that achieves $S^B_A = S_m$. By Lemma 3.8, we can bound the probability $p_e$ that $X_{\t, \Gamma}$ is affected by closing an ordered edge corresponding to $e$ by

$$p_e < L^{4|V'|}S_m/q,$$

as this event is contained in the event that the new edge is contained in an extension to $B_m$. (We calculate the impact of closure of $e$ on the one-step variance of $V$. For many edges we could get better bounds for inner open pairs by working instead with $D\delta V$, noting that $B_m$ is the same for $V$ and $V^{-}$, but this is not necessary for the bounds we prove here.) Furthermore, we can bound the magnitude in the change in $V$ by

$$N_e < L^{4|V'|}L^{4|V'|}S^{V'}_A/S_m.$$

Since $S^{V'}_A = yS^V_A = yv$, the one-step variance in $V$ satisfies

$$Var_e < p_eN_e < L^{20|V'|}(S_m/q)(S^{V'}_A/S_m)^2 = L^{20|V'|}y^2v^2/(qS_m).$$

We can bound $S_m$ from below by accounting for the scalings contributed by vertices in $\pi$ order. Each vertex contributes at least an additional $p\tilde{m} = y$ to the scaling. If $|B_m \setminus A| \geq 3$ it follows that $S_m > y^3 \geq L^{2|V'|}y^2$, which gives the desired bound on the one-step variance.

It remains to consider $|B_m \setminus A| \leq 2$. First consider the case $|B_m \setminus A| = 2$. We view the extension from $A$ to $B_m$ as a sequence of two one-vertex extensions with the extension to $\gamma$ taken last.
that each of these two steps is an extension in the stacking ensemble. If either of these steps is not a $Y_{ab}$-extension then we have $S_m > xy > L^{2|V'|}y^2$ (there cannot be any $Y_a$-extension as $B_m$ contains the open pair $\alpha_x\gamma$). This gives the desired bound on the one-step variance. So we may assume that both steps are $Y_{ab}$-extensions. Now we can use stacking variables to estimate $p_e$ and $N_e$, since $(A, B_m)$ induces the extension $S_w^{\pi_{(1)}}$, and $N_e = S_{\alpha_y\alpha_x}^{\pi'}$, where $\pi = \pi(1)\pi'$. Then we can use the better bounds $p_e < 2S_m/q$ and $N_e < 2S_{A}/S_m$ in the above calculation, obtaining $\text{Var}_e < 8y^2v^2/(qS_m)$. This suffices, as $S_m > y^2$ and $c_V \geq L^7$.

Finally, consider $|B_m \setminus A| = 1$. Here we have $B_m = \{\alpha_u, \alpha_v, \gamma\}, S_m = \hat{q}n$ and $p_e < 2S_m/q$. We bound $N_e$ by taking the extension from $B_m$ in two steps: first an extension from $B_m$ to a fan $\Phi_e$ and then a forward extension to the remainder of $S_{A}^{\pi}$. We can estimate the latter with a proportional error of $O(c_V)$. It remains to estimate the number of extensions to $\Phi_e$. If we consider any time $t$ for which $\Phi_e$ is controllable then we have $N_e < 2yv/S_m$, so again we can improve the above calculation to obtain $\text{Var}_e < 8y^2v^2/(qS_m)$. This suffices as $S_m = \hat{q}n \geq y^2/2$ and $c_V \geq L^7$. On the other hand, if the fan $\Phi_e$ is not controllable at time $t$ then either $\Phi_e$ is an extension to a single vertex or $\hat{q} < n^{\delta-1/4}$. If $\hat{q} < n^{\delta-1/4}$ we have $S_m = \hat{q}n > L^{2|V'|}y^2$, which suffices as above. If $\Phi_e$ is an extension to a single vertex we estimate $N_e \leq L^4\cdot vy/S_m$. This gives $\text{Var}_e < 8L^8v^2/q$, which suffices as $c_V \geq L^{15}$.

Now suppose that $e \in J$ is an edge and $e = \alpha_x\alpha_y$ where $x < y$. Let $A' = A \cup \{\alpha_x, \alpha_y\}$ and $S_m = \min_{A' \subseteq B \subseteq V} S_{A}^{B}$, where all scalings are with respect to $(J \setminus e, \Gamma)$. Let $B_m$ be the minimal set $B$ with $A' \subseteq B \subseteq V$ that achieves $S_{A}^{B} = S_m$. Now we have $p_e < L^{4|V'|}S_m/q$ and $N_e < L^{8|V'|}S_{A}/S_m$. It follows that the one-step variance in $V$ satisfies

$$\text{Var}_e < p_eN_e^2 < L^{2|V'|}(S_m/q)(S_A/S_m)^2 = L^{2|V'|}(\hat{q}p^{-1}v)^2/(qS_m).$$

Again we calculate the scaling $S_m$ one vertex at a time. Each vertex contributes at least an additional $pqn = y$ to the scaling, and $\alpha_y$ contributes at least $\hat{q}^2n = x$, since the edge $\alpha_x\alpha_y$ is switched to an open pair in $(J \setminus e, \Gamma)$. If $|B_m \setminus A| \geq 2$ we have $S_m \geq xy$, so

$$\text{Var}_e < y^{-1}L^{2|V'|}q^{-1}(t^{-1}v)^2,$$

which is sufficient, as $e \in J$ implies $e(V) \geq 1$. It remains to consider the case $|B_m \setminus A| = 1$, i.e. $B_m = A'$ and $e$ creates the first $Y$-extension of $\pi$ (using the fact that $V$ is not a vertex degree variable). Then $p_e \leq 2x/q$ and $N_e = S_{\alpha_y\alpha_x}^{\pi'} \leq 2v/y$, so $\text{Var}_e \leq 8q^{-1}(t^{-1}v)^2$, which suffices as $c_V \geq L^7$.

Thus we have verified the trend and boundedness hypotheses for stacking variables. By Freedman’s inequality, this completes the proof of Theorem 6.1, and so of the lower bound in Theorem 1.1.

**7 Independence number and upper bound**

After some establishing some preliminary facts in subsection 7.1 we will prove Theorem 1.2 in Section 7.2. Then we will apply a similar and easier argument in Section 7.3 to obtain the upper bound that completes the proof of Theorem 1.1.
7.1 Preliminaries

We start with an observation that will be used many times in this section to estimate the one-step variances due to destruction. Suppose $V$ is a variable, $e = uv$ is an open pair in $V$, and we want to estimate the one-step variance $\text{Var}_e$ due to destruction of $e$. Suppose that at most $N$ configurations in $V$ can be closed in any given step. Consider the bipartite graph $H$ with parts $(A, B)$, where $A = V$ is the set counted by $V$, $B = Q$ is the set of ordered open pairs, and $f \in A$ is adjacent to $b \in B$ if selecting $b$ as an edge closes $f(e)$. By assumption $d_H(b) \leq N$ for all $b \in B$. We also have $e(H) = 2 \sum_{f \in V} (Y_{f(uv)} + Y_{f(vu)}) = (1 + o(1))4yV$. Then

$$\text{Var}_e \leq Q^{-1} \sum_{b \in B} d_H(b)^2 \leq (1 + o(1))q^{-1} e(H)N = (1 + o(1))8tn^{-3/2}NV. \quad (5)$$

Next we prove some lemmas on counting open pairs. For any set $S$ let $Q_S(t)$ be the number of ordered open pairs in $S$ at time $t$. For any sets $A, B$ let $Q_{AB}(t)$ be the number of open pairs $ab$ with $a \in A$, $b \in B$ at time $t$.

**Lemma 7.1.** Whp for any set $S$ of size $s$ we have the following estimates.

(i) Suppose that $s \geq n^{1/4}$, $\psi \geq n^{-\varepsilon/5}$, $h \leq L^{-10}\psi^2 \hat{q}s$ and any vertex has at most $h$ neighbours in $S$. Then $Q_S = (1 \pm \psi)\hat{q}s^2$.

(ii) If $\psi \geq n^{-\varepsilon/5}$ and $s \geq L^{11} \psi^{-2} \sqrt{n}$ then $Q_S = (1 \pm \psi)\hat{q}s^2$.

(iii) If $s < L^{12} \sqrt{n}$ then $Q_S < L^{13} s \hat{q} \sqrt{n}$.

**Proof.** First consider statements (i) and (ii). We use critical window analysis for $t \geq n^{-0.4}$ to prove the bound $Q_S = (1 \pm e_O)\hat{q}s^2$, where $e_O = (1 + t/L)\psi/2$. This suffices as $e_O \leq \psi$. We use the window $[(1 + e_O - g_O)\hat{q}s^2, (1 + e_O)\hat{q}s^2]$, where $g_O = \psi/(40L^2)$. First we use coupling to the Erdős-Rényi process to show that $Q_S$ does not enter the critical window at $t = n^{-0.4}$. This follows from the trivial upper bound $Q_S \leq s^2$, and the lower bound $Q_S \geq s^2 - 5n^{0.2}s$, obtained by subtracting the number of paths of length 2 starting in $S$ in the random graph.

Next we establish the trend hypothesis that $\mathbb{E}[Q_S = Q_S - \hat{q}s^2 - e_O \hat{q}s^2]$ is a supermartingale while $Q_S$ is in its critical window. The expected change in $Q_S$ is

$$\mathbb{E}[\Delta_t Q_S | F_t] = -2Q^{-1} \sum_{ab \in Q_S} (Y_{ab} + Y_{ba}) = -8tn^{-3/2}(1 + O(e_Y))Q_S.$$

We also note that $\Delta_t(\hat{q}s^2) = (-8tn^{-3/2} + O(L^2n^{-3}))\hat{q}s^2$ and $\Delta_t(e_O \hat{q}s^2) = (1 + o(1))(L + t)^{-1} - 8t)n^{-3/2} e_O \hat{q}s^2$. Using $\frac{e_O}{\sqrt{L(t + t)}} \geq 2g_O$ and $e_Y \leq n^{-\varepsilon/4} = o(g_O)$ by Lemma 3.5, we have

$$\mathbb{E}[\Delta_t Q_S | F_t] \leq -(1 + o(1))8tn^{-3/2}\hat{q}s^2 \cdot (e_O - g_O - O(e_Y)) + (\frac{1}{\sqrt{L(t + t)}} - 1)e_O) \leq 0.$$

For the boundedness hypothesis, we take a union bound over $S$ and apply Freedman’s inequality. Write $N_O$ and $\text{Var}_O$ for the maximum one-step change and variance of $Q_S$. Since $g_O = \psi/(40L^2)$, it suffices to show that $N_O \leq L^{-10}\psi^2 \hat{q}s$, as by (5) this also implies $\text{Var}_O \leq L^{-4}n^{-3/2}s^{-1}(L^{-2}\psi \hat{q}s^2)^2$. This holds by our assumptions, using $N_O \leq h$ for (i) or $N_O = O(y)$ for (ii).
Lemma 7.4. Whp for any set $dD$ we can deduce a bound on the number of vertices of large degree in a given set. Let $\epsilon_O = (1 + t/L)L^{13}s\sqrt{n}/2$. Note that the bound is trivial for $t \leq 1$, as $s < L^{12}\sqrt{n}$ implies $Q_S \leq s^2 < \epsilon_O$. For $t \geq 1$ we use critical window analysis with the window $[\epsilon_O - g_O, \epsilon_O]$, where $g_O = \epsilon_O/(40L^2)$. When $Q_S$ is in the critical window we estimate $\mathbb{E}[\Delta_i Q_S \mid F_i] \leq -(1 + o(1))8tn^{-3/2}(\epsilon_O - g_O)$. We write $Q_S = Q_S - \epsilon_O$ and note that $\epsilon'_O = (1 + o(1))(t(L + t)^{-1} - 8t)\epsilon_O$. Thus we obtain the trend hypothesis

$$\mathbb{E}[\Delta_i ZQ_S \mid F_i] \leq -(1 + o(1))8tn^{-3/2} \cdot (\epsilon_O - g_O + \frac{1}{8t(L+t)} - 1)\epsilon_O) \leq 0.$$ 

For the boundedness hypothesis we use $N_O \leq 2y \leq L^{-9}s^{-1}g_O$, which implies $Var_O \leq L^{-6}s^{-1}g_O^2n^{-3/2}$, so we can apply Freedman's inequality. □

We also need a bipartite version of Lemma 7.1(i). The proof is essentially the same, so we omit it.

**Lemma 7.2.** Suppose $r, s \geq n^{1/4}$, $\psi \geq n^{-\epsilon/5}$ and $h \leq L^{-10}\psi^2\min\{r, s\}$. Then whp we have $Q_{RS} = (1 \pm \psi)qrs$ for any sets $R, S$ of respective sizes $r, s$ such that any vertex that has a neighbor in one of these sets has at most $h$ neighbors in the other.

Next we establish some density estimates. For a set $S$, let $E_S$ denote the number of edges of $G(t_{\max})$ in $S$.

**Lemma 7.3.** Whp for any set $S$ of size $s$

(i) if $s \geq L^{12}\sqrt{n}$ then $E_S < L^2n^{-1/2}s^2$,

(ii) if $s < L^{12}\sqrt{n}$ then $E_S < L^{15}s$.

**Proof.** For (i), we estimate the probability that some such $S$ spans $M := L^2n^{-1/2}s^2$ edges, taking a union bound over $S$ and the steps at which the edges are chosen, for which there are $n \choose m$ choices. For a specified step at time $t$, the probability of choosing an edge in $S$ is $Q_S(t)/Q(t) = (1 + o(1))s^2/n^2$, using Lemma 7.1(ii). Thus the failure probability $p_0$ satisfies

$$p_0 \leq \binom{n}{s} \binom{m}{M} ((1 + o(1))s^2/n^2)^M.$$ 

Then $s^{-1}\log p_0 \leq O(\log n) + (1 + o(1))Ms^{-1}\log \frac{en^2}{M^2} \leq -L^2n^{-1/2}s \leq -L^8$.

For (ii), we estimate the probability of choosing an edge in $S$ as $Q_S(t)/Q(t) < 2L^{13}sn^{-3/2}$ by Lemma 7.1(iii). Then

$$p_0 \leq \binom{n}{s} \left( \frac{m}{L^{15}s} \right) (2L^{13}sn^{-3/2})L^{15}s,$$

so $s^{-1}\log p_0 \leq O(\log n) + L^{15}\log \frac{2em}{L^2n^{3/2}} \leq -L^{15}$. □

We can deduce a bound on the number of vertices of large degree in a given set. Let $D_d(S)$ be the set of vertices that have degree at least $d$ in $S$.

**Lemma 7.4.** Whp for any set $S$ of size $s$

(i) if $s \geq L^{12}\sqrt{n}$ and $d > 8L^2n^{-1/2}s$ then $|D_d(S)| < 8L^2n^{-1/2}s^2/d$. 

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(ii) if \( s < L^{12} \sqrt{n} \) and \( d > 4L^{15} \) then \( |D_d(S)| < 4L^{15} s/d \).

Proof. For (i), suppose on the contrary that there is \( T \subseteq D_d(S) \) of size \( 8L^2 n^{-1/2} s^2 /d \). Then \( S \cup T \) is a set of size at most \( 2s \) that spans at least \( d|T|/2 > L^2 n^{-1/2} (2s)^2 \) edges, which contradicts Lemma 7.3(i). Similarly, for (ii), if there is \( T \subseteq D_d(S) \) of size \( 4L^{15} s/d \) then \( |S \cup T| \leq 2s \leq 2L^{12} \sqrt{n} \) and \( E_{S \cup T} \geq d|T|/2 > 2L^{15} s \geq L^{15} |S \cup T| \) edges, which contradicts Lemma 7.3. \( \square \)

Next we introduce absolute constants \( 0 < \alpha < \gamma < \beta \). We emphasize that these constants are chosen in the order listed; that is, \( \alpha \) is chosen to be sufficiently large, \( \gamma \) is chosen to be sufficiently large relative to \( \alpha \) and \( \beta \) is chosen to be sufficiently large relative to \( \alpha, \gamma \). (In an attempt to make clear the role these constants play, we do not substitute actual values. But for concreteness we note that they can take the values \( \alpha = 25, \gamma = 50, \beta = 600 \).) These constants will control the polylogarithmic factors in our arguments. When these factors are unimportant we will use ‘tilde’ notation: we write \( f(n) = \tilde{O}(g(n)) \) and \( g(n) = \tilde{\Omega}(f(n)) \) if \( f(n) \leq (\log n)^A g(n) \) for some absolute constant \( A \).

Given a vertex \( x \) and a set \( H \), let \( W_{xH} \) denote the number of ordered triples \( (a, b, c) \) of vertices such that \( ax \) is open, \( \{b, c\} \subseteq H \) and \( ab, ac \) are edges.

**Lemma 7.5.** Whp for every set \( H \) of size \( h \) with \( L^\alpha < h < L^\beta \) \( \sqrt{n} \) with the property that for any vertex \( a \)

(i) there are at most \( L^4 \) edges \( ab \) with \( b \in H \), and

(ii) there are at most \( 2x = 2q^2 n \) open pairs \( ab \) with \( b \in H \),

we have \( W_{xH} < L^{-\alpha} h q \sqrt{n} \) for all \( x \not\in H \).

Proof. We start with some definitions. We say that a vertex \( a \) is heavy with respect to \( H \) at time \( t \) if at least \( \hat{q} \sqrt{n} L^{-\gamma} \) pairs \( ab \) with \( b \in H \) are open. We say that \( a \) is obese with respect to \( H \) at time \( t \) if at least \( \hat{q} \sqrt{n} L^{\gamma} \) pairs \( ab \) with \( b \in H \) are open. For any obese vertex \( a \) we declare some subset of the open pairs \( ab \) with \( b \in H \) inactive so that the active open degree into \( H \) is \( \lfloor \hat{q} \sqrt{n} L^{\gamma} \rfloor \). We stress that the status of an open pair as active or inactive can change back and forth in the course of the process, but once a pair is chosen as an edge its status as active or inactive remains the same for the rest of the process. For \( j \in \{0, 1, 2\} \) let \( W_{xH}^j \) denote the number of ordered triples \( (a, b, c) \) of vertices such that \( ax \) is open, \( \{b, c\} \subseteq H \), the pairs \( ab \) and \( ac \) are both active, and their status depends on \( j \): if \( j = 0 \) then both are open, if \( j = 2 \) then both are edges, and if \( j = 1 \) then \( ab \) is open and \( ac \) is an edge. Thus \( W_{xH}^2 \) has the same definition as \( W_{xH} \), with the additional condition that \( ab \) and \( ac \) are active at the steps they are chosen as edges.

First we show that there is a negligible difference between \( W_{xH} \) and \( W_{xH}^2 \), so it suffices to bound the latter. Let \( O \) be the set of vertices that are obese with respect to \( H \). We claim that

\[ |O| < 2h L^{13-\gamma}. \]

For suppose not, and consider \( O' \subseteq O \) of size \( 2h L^{13-\gamma} \). Then \( |H \cup O'| < 2h \) and \( Q_{H \cup O} \geq 2L^{13} h \hat{q} \sqrt{n} \). This contradicts Lemma 7.1(iii), so \( |O| < h L^{13-\gamma} \). Applying Lemma 7.1(iii) again, we bound the number of open pairs in \( H \cup O \) by \( Q_{H \cup O} < 3h L^{13} \hat{q} \sqrt{n}/2 \). Thus the probability that we choose an edge between an obese vertex and \( H \) is at most \( 2h L^{13} n^{-3/2} \). By coupling with a binomial variable
and taking a union bound over $H$, whp for all $H$ we bound the number of edges in $H \cup O$ by $E_{H \cup O} < hL^{15}$. By assumption (i) this gives $W_{xH} - W_{xH}^2 < hL^{19}$, which is negligible by comparison with the desired bound on $W_{xH}$.

For the remainder of the proof, we use critical window analysis to prove the bounds

$$W_{xH}^0 < w_0 := L^{-a/2}h \hat{q} \sqrt{n}$$
$$W_{xH}^1 < w_1 := L^{-a/2}h \hat{q} \sqrt{n}$$
$$W_{xH}^2 < w_2 := L^{-a/2}h \hat{q} \sqrt{n}.$$

For $W_{xH}^j$ we use the critical windows $[e_j - g_j, e_j]$, where $e_j = (1 + t/L)w_j/2$ and $g_j = w_j/(40L^2)$. This suffices as $e_j < w_j$ for $j = 0, 1$ and $e_2 + hL^{19} < w_2$.

First we use coupling to the Erdős–Rényi process to show that the variables do not enter the critical windows for $n^{-1/4} \leq t \leq 1$. When $j = 0$ we use the trivial bound $W_{xH}^0 \leq nh^2 \ll w_0$ (since $\beta$ is large compared with $\alpha$). When $j = 1$ we bound $W_{xH}^1$ by picking $\{b, c\} \subseteq H$ then a neighbour of $c$. Since the random graph has degrees $(1 + o(1))pn = O(y)$ for $t \leq 1$ we obtain $W_{xH}^1 \leq O(y)h^2 \ll w_1$. When $j = 2$ we bound $W_{xH}^2$ by picking $\{b, c\} \subseteq H$ then a common neighbour, for which there are at most $O(L^2)$ choices in the Erdős–Rényi process, so $W_{xH}^2 = O(L^2h^2) \ll w_2$.

Next we prove the trend hypothesis, i.e. that $ZW_{xH}^j = W_{xH}^j - e_j$ is a supermartingale while $W_{xH}^j$ is in its critical window. We analyse the contributions to $\mathbb{E}[\Delta_i ZW_{xH}^j | F_i]$ separately according to each of the pairs $ax$, $ab$, $ac$.

Note that we cannot count the contribution of the destruction of $ab$ or $ac$ when $a$ is obese because we might only change the status of some inactive pair when such an active pair $ab$ is closed. Let $A_a$ denote the set of $b \in H$ such that $ab$ is open and active. The contribution to $\mathbb{E}[\Delta_i W_{xH}^0]$ due to the closure of pair $ab$ or $ac$ where $a$ is obese is at most

$$2Q^{-1} \sum_{a \in O} \sum_{b \in A_a} (Y_{ab} + Y_{ba}) |A_a| \leq 5yq^{-1} \cdot 2hL^{13-\gamma} \cdot (\hat{q}n^{1/2}L^{\gamma})^2 \ll 8tn^{-3/2}e_0L^{-2}.$$

Similarly, the contributions to $\mathbb{E}[\Delta_i W_{xH}^1]$ due to the closure of pair $ac$ where $a$ is obese is at most

$$2Q^{-1} \sum_{a \in O} \sum_{b \in A_a} (Y_{ab} + Y_{ba})L^4 \leq 5yq^{-1}L^4 \cdot 2hL^{13-\gamma} \cdot \hat{q}n^{1/2}L^{\gamma} \ll 8tn^{-3/2}e_1L^{-2}.$$

We will see shortly that these contributions are negligible. When we calculate the expected change due to destruction of $ab$ or $ac$ below we restrict our attention to vertices $a$ that are not obese.

In the calculation of the expected change in $ZW_{xH}^j = W_{xH}^j - e_j$ we write $\Delta_i(e_j) = (1 + o(1))e'_j n^{-3/2}$, noting that $e'_j \geq ((L+t)^{-1} - 24t)e_0$, $e'_1 \geq (t^{-1} - 16t)e_1$, $e'_2 \geq ((L+t)^{-1} - 8t)e_2$. For each open pair $\alpha\beta$ we have a destruction term of $2Q^{-1} \sum_{f \in W_{xH}^j} (Y_{f(\alpha \beta)} + Y_{f(\beta \alpha)}) \geq (1 + o(1))8tn^{-3/2}(e_j - g_j)$. This gives self-correction against a corresponding $8tn^{-3/2}e_j$ term in $\Delta_i(e_j)$. For each edge we have a creation term of $2Q^{-1}W_{xH}^{j-1} \leq (1 + o(1))2q^{-1}e_{j-1}$. Note that $2q^{-1}e_0 = t^{-1}n^{-3/2}e_1$ and $2q^{-1}e_1 = 2L^{-2}n^{-3/2}e_2$. Using $\frac{e_j}{8(\sqrt{L}+t)} \geq 4g_j$ we obtain

$$\mathbb{E}[\Delta_i ZW_{xH}^0 | F_i] \leq -(1 + o(1))8tn^{-3/2} \cdot (3(e_0 - g_0) + (\frac{1}{8(\sqrt{L}+t)} - 3)e_0) \leq 0.$$

$$\mathbb{E}[\Delta_i ZW_{xH}^1 | F_i] \leq -(1 + o(1))8tn^{-3/2} \cdot (2(e_1 - g_1) - \frac{1}{4q}e_1 + (\frac{1}{4q} - 2)e_1) \leq 0.$$

$$\mathbb{E}[\Delta_i ZW_{xH}^2 | F_i] \leq -(1 + o(1))8tn^{-3/2} \cdot (e_2 - g_2 - \frac{1}{4L^2}e_2 + (\frac{1}{8L^2} - 1)e_2) \leq 0.$$
It remains to show the boundedness hypothesis. Note that since we can restrict our attention to \( t \geq 1 \) the functions \( g_j \) are approximately decreasing. For each pair \( e \) in \( W^j_{xH} \) let \( N_e \) bound the one-step change and \( \text{Var}_e \) the one-step variance due to \( e \). To apply Freedman’s inequality, since \( g_j = w_j/(4L^2) \), it suffices to show \( N^*_e \leq w_j/(hL^5) \) and \( \text{Var}_e \leq w_j^2/(hL^8n^{3/2}) \), where \( N^*_e \) is the maximum positive one-step change in \( ZW^j_{xH} \). In some cases we will show the stronger statement \( N_e < w_j/(hL^{10}) \), which clearly gives the desired bound on \( N^*_e \), and implies the desired bound for \( \text{Var}_e \) by [5].

First we note that the required bounds for creation hold trivially as \( N_e \leq yL^\gamma \ll w_1/(hL^{10}) \) for \( W^1_{xH} \) and \( N_e \leq L^4 \ll w_2/(hL^{10}) \) for \( W^2_{xH} \).

For destruction we obtain negative changes in \( ZW^j_{xH} \), so we only need to bound \( \text{Var}_e \). First we introduce some additional definitions. Let \( T = T_{xH} \) be the set of heavy vertices \( a \) such that \( xa \) is open. Appealing to the control we maintain on \( X_{av} \), we have

\[
|T| < 2hx/(\hat{q}\sqrt{nL}^{-\gamma}) = 2hL^\gamma \hat{q}\sqrt{n}.
\]

Let \( U \) be the set of vertices \( z \) such that \( zx \) is open and \( z \) has at least \( \hat{q}\sqrt{nL}^{-\gamma} \) neighbors in \( T \). By Lemma 7.4 we have

\[
|U| < \begin{cases} 8hL^{4\gamma+15} & \text{if } 2hL^{-12}\hat{q} < 1, \\ 32h^2L^{5\gamma+2}\hat{q} & \text{otherwise}. \end{cases}
\]

Here we used \( \hat{q}\sqrt{nL}^{-\gamma} > 4L^{15} \) and \( \hat{q}\sqrt{nL}^{-\gamma} > 8L^2n^{-1/2} \cdot 2hL^\gamma \hat{q}\sqrt{n} \), which follows from the fact that we choose \( \beta \) to be large relative to \( \gamma \), to get the lower bounds on \( d \) required for Lemma 7.4.

Now consider destruction for the variables \( W^j_{xH} \) for \( j = 0, 1 \). We write \( \Delta_iW^j_{xH} = \Delta_iV_1 + \Delta_iV_2 \), where \( \Delta_iV_1 \) accounts for the change in \( V = W^j_{xH} \) that comes from the choice of an edge \( xx \) where \( z \in U \), and \( \Delta_iV_2 \) accounts for the rest. For \( \Delta_iV_2 \) the bound on \( \text{Var}_e \) will follow from the bound on \( N_e \) and [5]. The contribution to \( N_e \) from the closure of pairs \( ab \) or \( ac \) is clearly sufficiently small. Next consider the contribution from closure of pairs \( xa \) where \( a \) is not heavy. For \( j = 0 \) this is

\[
(2y)(\hat{q}\sqrt{nL}^{-\gamma})^2 < 2\hat{q}\sqrt{n}L^{1-2\gamma}x, \quad \text{which suffices as } w_0/(hL^{10}) = L^{-\alpha-12}\hat{q}\sqrt{n} \text{ is much larger as } \gamma \text{ is large relative to } \alpha.
\]

For \( j = 1 \), appealing to condition (i) in the statement of the Lemma, the contribution is at most \( (2y)(\hat{q}\sqrt{nL}^{-\gamma})L^4 \), which suffices as \( w_1/(hL^{10}) = L^{-\alpha-12}y\hat{q}\sqrt{n} \) and \( \gamma \) is large relative to \( \alpha \). Now consider the contribution from the closure of pairs \( xa \) where \( a \) is heavy (but we do not select \( xu \) with \( u \in U \)). Since we only consider active pairs and \( t \geq 1 \), this is at most

\[
\hat{q}\sqrt{nL}^{-\gamma} \cdot (\hat{q}\sqrt{nL}^{-\gamma})^2 = L^{-\gamma}x\hat{q}\sqrt{n} \ll w_0/(hL^{10}) \quad \text{for } j = 0, \quad \text{or } \hat{q}\sqrt{nL}^{-\gamma}(\hat{q}\sqrt{nL}^{-\gamma})L^4 \ll w_1/(hL^{10}) \quad \text{for } j = 1.
\]

Thus we have the required bound on \( N_e \) for \( \Delta_iV_2 \).

For \( j = 0, 1 \) it remains to bound \( \text{Var}_e \) for \( \Delta_iV_1 \). The probability that an edge \( xx \) with \( z \in U \) is chosen is at most \( 2|U|/q \), and the resulting change in \( W^j_{xH} \) is at most \( (2y)(\hat{q}\sqrt{nL}^{-\gamma})^2 \) for \( j = 0 \), or \( (2y)(\hat{q}\sqrt{nL}^{-\gamma})L^4 \) for \( j = 1 \). Suppose first that \( 2hL^{-12}\hat{q} < 1 \). Then for \( j = 0 \) we have

\[
\text{Var}_e \leq 16hL^{4\gamma+15}q^{-1}(2y)^2(\hat{q}\sqrt{nL}^{-\gamma})^4 = O(hq^5n), \quad \text{which suffices as } w_0^2/(hL^8n^{3/2}) = \tilde{O}(h^2L^{-12}) \quad \text{and, for } j = 1 \text{ we have } \text{Var}_e \leq 16hL^{4\gamma+15}q^{-1}(2y)^2(\hat{q}\sqrt{nL}^{-\gamma})^2L^8 = \tilde{O}(hLq^n), \quad \text{which suffices as } w_2^2/(hL^8n^{3/2}) = \tilde{O}(hq^4n^{1/2}) \quad \text{recalling that we restrict our attention to } t \geq 1.
\]

Suppose \( 2hL^{-12}\hat{q} \geq 1 \). Then for \( j = 0 \) we have

\[
\text{Var}_e \leq 64h^2L^{5\gamma+2}n^{-2}(2y)^2(\hat{q}\sqrt{nL}^{-\gamma})^4 < 256h^2L^{9\gamma+4}q^6n,
\]
and for \( j = 1 \) we have

\[
Var_e \leq 64h^2L^{5\gamma + 2}n^{-2}(2y)^2(\hat{q}\sqrt{nL\gamma})^2L^8 < 256h^2L^{7\gamma + 12}\hat{q}^4.
\]

As \( \beta \) is chosen to be large relative to \( \alpha, \gamma \) these bounds suffice.

It remains to bound \( Var_e \) for destruction of \( W_{xH}^2 \). Let \( W \) be the set of vertices that are open to \( x \) and have at least two neighbors in \( H \). By our bound on \( Y_e \) we have \( |W| < 2yh \). Let \( U' \) be the set of vertices that are open to \( x \) and have at least \( yL^{-\gamma} \) neighbors in \( W \). Note that by Lemma 7.4 we have

\[
|U'| < \begin{cases} 
8hL^{\gamma + 15} & \text{if } 2yh < L^{12}\sqrt{n} \\
32h^2yn^{-1/2}L^{\gamma + 2} & \text{otherwise.}
\end{cases}
\]

Here we used \( yL^{-\gamma} > 4L^{15} \) and \( yL^{-\gamma} > 8L^2n^{-1/2} \cdot 2yh \), which follows from the fact that \( \beta \) is chosen to be large relative to \( \gamma \), to get the lower bound on \( d \) required for Lemma 7.4. We write \( \Delta_iW_{xH}^2 = \Delta_iV_1 + \Delta_iV_2 \), where \( \Delta_iV_1 \) accounts for the change in \( W_{xH}^2 \) that comes from the choice of an edge \( xz \) where \( z \in U' \), and \( \Delta_iV_2 \) accounts for the rest. To bound \( Var_e \) for \( \Delta_iV_2 \), note that by condition (ii) in the statement of the Lemma we have \( N_e < yL^{8-\gamma} < w_2/(hL^{10}) \). For \( \Delta_iV_1 \), suppose first that \( 2yh < L^{12}\sqrt{n} \). Then \( Var_e < 8hL^{\gamma + 15}q^{-1}(2y)^2L^{16} = \tilde{O}(hyn^{-3/2}) \), which suffices as \( w_2^2/(hL^{8}n^{3/2}) = \tilde{\Omega}(hy^2n^{-3/2}) \). On the other hand, if \( 2yh \geq L^{12}\sqrt{n} \) then \( Var_e < 32h^2yn^{-1/2}L^{\gamma + 2}q^{-1}(2y)^2L^{16} < 128h^2n^{-1/2}L^{\gamma + 10}y^2n^{-3/2} \), which also suffices because \( \beta \) is chosen to be large relative to \( \alpha, \gamma \). \( \square \)

### 7.2 Proof of Theorem 1.2

Let \( \epsilon' = 3\epsilon \) and \( k = (1 + \epsilon')\sqrt{2n \log n} \). We show whp \( \alpha(G) < k \). Note that \( \alpha(G) \leq \alpha(G(t_{\text{max}})) \), so it suffices to bound \( \alpha(G(t_{\text{max}})) \). We need to estimate the probability that there is an independent set \( K \) of size \( k \). We will take a union bound over all such sets \( K \) together with information about how neighborhoods in \( G(t_{\text{max}}) \) intersect \( K \). We define a sequence of vertices \( x_1, \ldots, x_z \), where each \( x_\ell \) is chosen to maximise the number of neighbours in \( K \) that are not also neighbours of some \( x_j \) for \( j < \ell \). More precisely, the \( \ell \)th ‘hole’ is \( H_\ell = (N(x_\ell) \setminus \cup_{\ell'<\ell}N(x_{\ell'})) \cap K \), where \( x_\ell \) is chosen to maximise \( h_\ell = |H_\ell| \). We stop the sequence if there are no vertices that give more than \( L^{2\alpha} \) new neighbours in \( K \). Note that \( x_\ell \notin K \) for \( \ell \in [z] \), as \( K \) is independent. We say that a hole is ‘large’ if has size more than \( L^{-\beta}\sqrt{n} \). We let \( Z_A \) be the set of \( \ell \) such that \( H_\ell \) is large, \( Z_B = [z] \setminus Z_A \), \( A = \cup_{\ell \in Z_A} H_\ell \), \( B = \cup_{\ell \in Z_B} H_\ell \), \( C = K \setminus (A \cup B) \). For \( \ell \in Z_B \) we write \( H_\ell = \{v_{\ell j} : j \in [h_\ell]\} \), where \( x_\ell v_{\ell j} \) is selected at step \( i_{\ell j} \), and \( i_{\ell j} \) is increasing in \( j \). For \( \ell \in Z_A \) we specify the entire neighbourhood of \( x_\ell \) in \( G(t_{\text{max}}) \); we write \( d_\ell = |N(x_\ell)| \) and \( N(x_\ell) = \{v_{\ell j} : j \in [d_\ell]\} \), where \( x_\ell v_{\ell j} \) is selected at step \( i_{\ell j} \), and \( i_{\ell j} \) is increasing in \( j \). We will estimate \( \mathbb{P}(E) \), where \( E \) is the event that there is an independent set \( K \) with some fixed choices of \( z, x_\ell, h_\ell \) for \( \ell \in [z] \) and \( d_\ell \) for \( \ell \in Z_A \). We will refer to these choices of hole sizes, vertices with large neighborhoods in \( K \) and vertex degrees as the initial data that defines \( E \).

Note that by Lemma 7.4 we can assume

\[
|Z_A| < 8L^{16+\beta} \text{ and } z < 4L^{15-2\alpha}k.
\]

Also, using our bounds on vertex degree variables, for \( \ell \in Z_A, j \in [d_\ell] \) we can assume

\[
i_{\ell j} = jn/2 \pm n^{3/2-\epsilon/3} \text{ and } d_\ell = d \pm n^{1/2-\epsilon/3}.
\]
Now in addition to the initial data, we fix the independent set $K$ and the specific edges and appearance times $v_{t_j}, i_{t_j}$ for $\ell \in Z_A, j \in [d_\ell]$ and $v_{t_j}, i_{t_j}$ for $\ell \in Z_B, j \in [h_\ell]$. We let $\mathcal{E}_K$ be the event $K$ is independent and all the specified edges appear at the specified steps of the process. Note that $\mathcal{E}$ is a union of events of the form $\mathcal{E}_K$. To estimate the probability of the event $\mathcal{E}_K$, for each step $i$ we need to estimate the probability that the selected edge is compatible with $\mathcal{E}_K$, conditional on the history of the process. We say $i$ is a selection step if $i$ is one of $i_{t_j}$ for $\ell \in Z_A, j \in [d_\ell]$ or $\ell \in Z_B, j \in [h_\ell]$; then the selected edge is specified by $\mathcal{E}_K$, so the required probability is simply $2/Q = (1 + 2e_Q)q^{-2}$. For other $i$, the required probability is $1 - N_i/Q$, where $N_i$ is the number of ordered open pairs that cannot be selected at step $i$ when $\mathcal{E}_K$ occurs. If $i = i_{t_j}$ is a selection step write $N_i = 0$ and $t_{i_{t_j}} = t$. Then we estimate

$$\mathbb{P}(\mathcal{E}_K) \leq \prod_{\ell \in Z_A} \prod_{j=1}^{d_\ell} (1 + 2e_Q)q(t_{i_{t_j}})^{-1} \cdot \prod_{\ell \in Z_B} \prod_{j=1}^{h_\ell} (1 + 2e_Q)q(t_{i_{t_j}})^{-1} \cdot \prod_{i=1}^m (1 - N_i/Q). \quad (6)$$

To estimate $N_i$, we classify open pairs that cannot be selected at step $i$ as follows.

- Let $N_{iA_i}$ be the number of ordered open pairs of the form $v_{t_j}v_{t_{j'}}$ for some $\ell \in Z_A, j, j' \in [d_\ell]$.
- Let $N_{iA_o}$ be the number of ordered open pairs of the form $xy \ell$ or $yx \ell$ where $\ell \in Z_A$ and $y \notin K \cup \{x_1, \ldots, x_z\}$.
- Let $N_{iB}$ be the number of ordered open pairs $ab$ such that selecting $e_i = ab$ would close an open pair of the form $x_{t_j}v_{t_{j'}}$ for $\ell \in Z_B, j \in [h_\ell]$.
- Let $N_{iK}$ be the number of ordered open pairs in $K$ that are not contained within any hole.

We write $N_i \geq N_{iA_i} + N_{iA_o} + N_{iB} + N_{iK} - N_{iO}$, where $N_{iO}$ corrects for any overcounted pairs.

To estimate $N_{iA_i}$, for $\ell \in Z_A$ define $j_\ell = j(i)$ to be the value of $j \in [d_\ell]$ such that $i_{t_j} < i < i_{t(j+1)}$, i.e. $j_\ell$ edges have been selected at $x_\ell$. Let $S_\ell = \{v_{t_j}\}_{j=j_\ell + 1}^{d_\ell}$ and $s_\ell = |S_\ell| = d_\ell - j_\ell$. Note that the number of ordered open pairs $v_{t_j}v_{t_{j'}}$ with $j > j_\ell, j' \leq j_\ell$ is $\sum_{v \in S_\ell} 2y_{v_\ell} (1 + cy)q^2s_\ell$. Also, by the bound on codegrees, any vertex has at most $L^4$ neighbors in $S_\ell$. Then by Lemma 7.1(i) which $Q_{S_\ell} = (1 + n^{-\varepsilon/5})q^2s_\ell^2$ if $s_\ell > n^{1/4}$ and $\tilde{q}s_\ell \geq n^{2/5}L^{14}$. Since $\tilde{q} \geq n^{-1/2+\varepsilon}$ this holds for $s_\ell > n^{1/2-\varepsilon/2}$, so we can write $Q_{S_\ell} \geq (1 - n^{-\varepsilon/5})\tilde{q}s_\ell(s_\ell - n^{1/2-\varepsilon/2})$. The bound on co-degrees also implies that the number of open pairs that can be counted by more than one $\ell \in |Z_A|$ is at most $|Z_A|L^4 = O(1)$, which is negligible. Thus

$$N_{iA_i} \geq (1 - n^{-\varepsilon/5}) \sum_{\ell \in Z_A} (2y_{s_\ell} + \tilde{q}s_\ell(s_\ell - n^{1/2-\varepsilon/2})) = \sum_{\ell \in Z_A} (2y_{s_\ell} + \tilde{q}s_\ell^2) - \tilde{O}(\tilde{q}n^{1-\varepsilon/5}).$$

Using control on the open degree of vertices, we can also estimate $N_{iA_o}$ by

$$N_{iA_o} \geq 2|Z_A|\left(\tilde{q}n(1 - n^{-\varepsilon/4}) - k - z\right) \geq 2|Z_A|\tilde{q}n(1 - n^{-\varepsilon/5}) \quad (7)$$

To estimate $N_{iB}$, we say an open pair that closes $x_{t_j}v_{t_{j'}}$ for some $\ell \in Z_B, j \in [h_\ell]$ is outer if it contains $x_\ell$ or inner otherwise. We write $N_{iB} = N_{iBo} + N_{iBi}$, where an ordered pair counted by $N_{iB}$ is counted by $N_{iBi}$ if it is inner or $N_{iBo}$ if it is outer. Let $S_\ell = \{v_{t_j}\}_{j=1}^{h_\ell}$ be the set of $v_{t_j}$ with $j \in [h_\ell]$ such
that \( x_{\ell}v_{\ell j} \) is still open, and write \( s_{\ell} = |S_{\ell}| \). Each \( v_{\ell j} \) in \( S_{\ell} \) contributes \( 2Y_{v_{\ell j}x_{\ell}} = (1 \pm e_{Y})2y \) to \( N_{iB_{i}} \) and \( 2Y_{x_{\ell}v_{\ell j}} = (1 \pm e_{Y})2y \) to \( N_{iB_{o}} \); however, we need to account for open pairs that may be counted by more than one pair \( x_{\ell}v_{\ell j} \). For outer pairs, this may occur for \( x_{\ell}v_{\ell j} \) and \( x_{\ell'}v_{\ell'j} \) with \( \ell \in Z_{B} \) and \( j,j' \in S_{\ell} \). Assuming that \( s_{\ell} \geq L^{\alpha} \), the number of such overcounted pairs is \( W_{x_{\ell}s_{\ell}} < L^{-\alpha}s_{\ell}\sqrt{n} \) by Lemma 7.5. (Note that we apply the upper bound on the size of a hole \( H_{\ell} \) with \( \ell \in Z_{B} \) here.)

Summing over \( \ell \in Z_{B} \), using \( |Z_{B}| \leq z \leq 4L^{15-2\alpha}k \) and \( \sum_{\ell \in Z_{B}} s_{\ell} \leq k \) we obtain

\[
N_{iB_{o}} \geq (1 - e_{Y})2y \sum_{\ell \in Z_{B}} (s_{\ell} - L^{\alpha}) - \sum_{\ell \in Z_{B}} L^{-\alpha}s_{\ell}\sqrt{n} \geq 2y \sum_{\ell \in Z_{B}} s_{\ell} - L^{17-\alpha}k\sqrt{n}.
\]

Next note that an inner pair for \( x_{\ell}v_{\ell j} \) for some \( \ell \in Z_{B} \) with \( \ell' \neq \ell \). For this would require the edges \( x_{\ell}v_{\ell'j} \) and \( x_{\ell}v_{\ell j} \), which cannot both exist by the hole construction procedure. Thus there is no overcounting for inner open pairs, and we obtain \( N_{iB_{i}} \geq (1 - e_{Y})2y \sum_{\ell \in Z_{B}} s_{\ell} \). We finally note that inner open pairs have at least one vertex in \( K \) while outer open pairs have no vertex in \( K \) (since we may assume \( K \) is an independent set), thus the collections of open pairs counted by \( N_{iB_{i}} \) and \( N_{iB_{o}} \) do not intersect. In total

\[
N_{iB} \geq (1 - e_{Y})4y \sum_{\ell \in Z_{B}} s_{\ell} - L^{17-\alpha}k\sqrt{n} = 4y \sum_{\ell \in Z_{B}} s_{\ell} - O(L^{-3}\sqrt{n}), \tag{8}
\]

provided \( \alpha \) is sufficiently large.

Now we turn to an estimate for \( N_{iK} \). We begin by defining a partition of \( A \cup B \) that plays a key role in this discussion. We write

\[
h^{*} = h^{*}(i) = \min\{n^{2/5}, L^{-50}\sqrt{n}\},
\]

and let \( \ell^{*} \in [z + 1] \) be such that \( h_{\ell} \geq h^{*} \) for \( 1 \leq \ell < \ell^{*} \) and \( h_{\ell} < h^{*} \) for \( \ell^{*} \leq \ell \leq z \). Let \( J_{1} = J_{1}(i) = \cup_{\ell \leq \ell^{*}} H_{\ell} \) and \( J_{2} = J_{2}(i) = \cup_{\ell > \ell^{*}} H_{\ell} \). Note that any vertex has degree at most \( h^{*} \) in \( J_{2} \), by the hole construction procedure. Let \( N_{iK_{B}} \) be the set of ordered pairs counted by \( N_{iK} \) with first vertex in \( J_{2} \) and let \( N_{iK_{C}} \) be the set of ordered pairs counted by \( N_{iK} \) with first vertex in \( C \). By Lemma 7.1(i) whp \( Q_{J_{2}} = (1 + L^{-5})\bar{q}||J_{2}|^{2} \) if \( \bar{q}||J_{2}| \geq L^{20}h^{*} \), so we can write \( Q_{J_{2}} \geq (1 - L^{-5})\bar{q}||J_{2}|(|J_{2}| - L^{-30}\sqrt{n}) \). Let \( Q'_{J_{2}} \) be the number of ordered open pairs in \( J_{2} \) that are not contained within any hole. Then

\[
Q'_{J_{2}} \geq Q_{J_{2}} - h^{*}|J_{2}| \geq (1 - L^{-5})\bar{q}||J_{2}|(|J_{2}| - 2L^{-30}\sqrt{n}).
\]

Next let \( T \) be the set of vertices that have at least \( L^{20}h^{*} \) neighbours in \( J_{1} \). We can assume \( |T| < 4L^{-5}|J_{1}|/h^{*} < 6L^{-4}\sqrt{n}/h^{*} \) by Lemma 7.4. It follows that \( |N(T) \cap J_{2}| < 6L^{-4}\sqrt{n} \). Applying Lemma 7.2 with \( R = J_{1} \) and \( S = J_{2} = J_{2} \setminus N(T) \), whp \( Q_{J_{1},J_{2}} = (1 + L^{-5})\bar{q}|J_{1}||J_{2}| \) if \( \bar{q} \min\{|J_{1}|, |J_{2}| \} \geq L^{40}h^{*} \), so we can write

\[
Q_{J_{1},J_{2}} \geq (1 - L^{-5})\bar{q}(|J_{1}| - L^{-4}\sqrt{n})(|J_{2}| - 7L^{-4}\sqrt{n}).
\]

We can apply the same argument to open pairs between \( J_{2} \) and \( C \). Let \( T' \) be the set of vertices that have at least \( L^{20+2\alpha} \) neighbours in \( J_{2} \). We can assume \( |T'| < 4L^{-5-2\alpha}|J_{2}| < 6L^{-4-2\alpha}\sqrt{n} \) by Lemma 7.4. Since any vertex has at most \( L^{2\alpha} \) neighbours in \( C \), it follows that \( |N(T') \cap C| < 6L^{-4}\sqrt{n} \). Applying Lemma 7.2 with \( R = J_{2} \) and \( S = C' = C \setminus N(T') \), whp \( Q_{J_{2},C'} = (1 + L^{-5})\bar{q}|J_{2}||C'| \) if
\[ \hat{q} \min \{|J_2|, |C'|\} \geq L^{40+2\alpha}, \] so we can write \( Q_{J_2C'} \geq (1 - L^{-5})\hat{q}(|J_2| - L^{-4}\sqrt{n})(|C| - 7L^{-4}\sqrt{n}). \) In total we obtain

\[ N_{iKB} \geq Q'_{J_2} + Q_{J_1J_2} + Q_{J_2C'} \geq \hat{q}k|J_2| - O(L^{-3}\hat{q}n). \]

Also, applying Lemma 7.1 to \( C \) we have \( Q_C \geq (1 - L^{-5})|C|(|C| - L^{-4}\sqrt{n}), \) so

\[ N_{iKC} \geq Q_{KC,C'} + Q_C \geq \hat{q}k|C| - O(L^{-3}\hat{q}n). \]

(For \( Q_{K\setminus C,C'} \) we apply the same method as for \( Q_{J_2C'} \).) When \( \hat{q} < n^{-1/6} \) these two estimates will suffice: we have

\[ \hat{q} < n^{-1/6} \quad \Rightarrow \quad N_{iK} \geq N_{iKB} + N_{iKC} \geq \hat{q}k(\frac{|C|}{|J_2|}) - O(L^{-3}\hat{q}n) \]

\[ = \hat{q}k \left( |C| + \frac{z}{\ell+1} h_\ell \right) - O(L^{-3}\hat{q}n). \] (9)

When \( \hat{q} \geq n^{-1/6} \) we improve this estimate as follows. Let \( z' \) be such that \( h_\ell \geq n^{2/5} \) for \( \ell \leq z' \) and \( h_\ell < n^{2/5} \) otherwise. Note that \( z' \leq \ell' \). We write \( N_{iK} \geq \sum_{\ell=1}^{z'} N_{iKH_\ell} + N_{iKB} + N_{iKC} \), where an open ordered pair counted by \( N_{iK} \) is counted by \( N_{iKH_\ell} \), \( N_{iKB} \) or \( N_{iKC} \) according as its first vertex is in \( H_\ell \), \( B \setminus \cup_{\ell=1}^{z'} H_\ell \) or \( C \). For \( \ell \in [z'] \) we claim that

\[ \hat{q} \geq n^{-1/6} \quad \Rightarrow \quad N_{iKH_\ell} > (1 - L^{-5})\hat{q}h_\ell k/2. \] (10)

To see this, we first apply Lemma 7.2 for each \( \ell' \) such that \( h_{\ell'} \geq n^{1/5} \) to \( R = H_\ell \setminus N(x_{\ell'}) \) and \( S = H_\ell \setminus N(x_\ell) \). This is valid by the codegree bound, as each has size at least \( n^{1/5} - L^4 \), and any vertex with a neighbor in one set has at most \( L^4 < L^{-20}\hat{q}(n^{1/5} - L^4) \) neighbors in the other. Thus we obtain \( Q_{H_\ell H_{\ell'}} = (1 + L^{-5})\hat{q}h_\ell h_{\ell'} \). Then we apply Lemma 7.2 with \( R = H_\ell \) and \( S = K' = K \setminus \cup_{\ell' : h_{\ell'} \geq n^{1/5}} H_{\ell'} \), which is valid as any vertex with a neighbor in one set has at most \( n^{1/5} < L^{-20}\hat{q}n^{2/5} \) neighbors in the other (by the hole construction procedure). Thus we obtain \( Q_{H_\ell K'} = (1 - L^{-5})\hat{q}h_{\ell'}(|K'| - n^{2/5}) \). Since \( k - h_\ell - n^{2/5} > k/2 \) this proves (10). Using the estimates from above for \( N_{iKC} \) and \( N_{iKB} \) we have

\[ \hat{q} \geq n^{-1/6} \quad \Rightarrow \quad N_{iK} \geq \hat{q}k \left( |C| + \sum_{\ell=1}^{z'} h_{\ell'/2} + \sum_{\ell=\ell'+1}^{z} h_\ell \right) - O(L^{-3}\hat{q}n). \] (11)

To estimate the overcount \( N_{iO} \), first note that there is no overcounting between \( N_{iA_0} + N_{iB_0} \) and \( N_{iBi} + N_{iK} \); pairs counted by the former do not intersect \( K \) while pairs counted by the latter do intersect \( K \). There is also no overcounted pair between \( N_{iA_0} \) and \( N_{iB_0} \), as such a pair would have to contain \( x_\ell \) with \( \ell \in Z_B \) to lie in \( N_{iB_0} \) and no such pair lies in \( N_{iA_0} \). There is no overcounted pair between \( N_{iBi} \) and \( N_{iAi} \) as the hole construction procedure ensures that no vertex in a hole \( H_\ell \) with \( \ell \in Z_B \) is a neighbor of some vertex \( x_{\ell'} \) such that \( \ell' \in Z_A \). It follows from the co-degree bound that the overcount between \( N_{iAi} \) and \( N_{iK} \) is \( \tilde{O}(n^{1/2}) \). To bound the overcount between \( N_{iAi} \) and \( N_{iA_0} \), note that this is determined by naming vertices \( x_\ell, x_{\ell'} \) such that \( \ell, \ell' \in Z_A \) and a vertex \( b \) in the (final) neighborhood of \( x_\ell \). The number of choices for these is at most \( |Z_A|^2 d = \tilde{O}(n^{1/2}) \). Also, the overcount between \( N_{iAi} \) and \( N_{iB_0} \) is determined by naming a vertex \( b \in B \), a vertex \( x_\ell \) such that \( \ell \in Z_A \), and a vertex \( c \) that is in the (final) common neighborhood of \( x_\ell \) and \( b \). The number of
choices is bounded by $k|Z_A|L^4 = \tilde{O}(n^{1/2})$. This leaves the overcount between $N_{iK}$ and $N_{iBi}$, which is the most significant potential source of overcounting.

In order to bound the overcount between $N_{iK}$ and $N_{iBi}$ we introduce a definition. We say that a hole $H$ with $\ell \in Z_B$ is ‘black’ if $x_\ell$ has more than $L^{30}h_\ell$ neighbours in $K$. Let $BH$ be the set of vertices that belong to black holes, and let $XH$ be the corresponding set of vertices $x_\ell$. Note that Lemma 7.3(ii) implies $|BH| \leq L^{-1}k$. The contribution to $N_{iBi}$ of pairs that would close edges incident to black holes is at most $3y|BH| \leq 3L^{-11}yk \leq 3L^{-10}qkn^{1/2}$. Now consider overcounted pairs that would close pairs that are not incident to black holes. Such a pair has the form $v_{\ell j}v_{\ell j'}$ where $x_\ell v_{\ell j}$ is an edge, so $\ell' < \ell$ by the hole construction procedure. If $h_\ell > n^{2/5}$ then by the degree bound there are at most $n^{-2/5}k \cdot L^4 < n^{1/5}$ such edges $x_\ell v_{\ell j}$ with the fixed $x_\ell$. These are only counted in our estimate for $N_{iK}$ while $\hat{q} > n^{-1/6}$, so the overcount for such a hole is at most $h_\ell n^{1/5} < h_\ell \hat{q} n^{2/5}$. For holes $H$ such that $h_\ell < n^{2/5}$, recall that open pairs between $H_\ell$ and $H_{\ell'}$ are only counted in our estimate for $N_{iK}$ if $h_\ell < h^* \leq L^{-40} \hat{q} \sqrt{n}$. Since $H_\ell$ is not black, the number of choices for $v_{\ell j'}$ is at most $L^{30}h_\ell < L^{-10} \hat{q} \sqrt{n}$. The total contribution of such pairs to $N_{iB}$ is at most $L^{-10}h_\ell \hat{q} \sqrt{n}$. Thus

$$N_{iO} \leq 5L^{-10} \hat{q} n^{1/2}k = O(L^{-3} \hat{q} n). \quad (12)$$

Next we substitute $1 - N_i/Q \leq \exp(-(1 - 2eQ)q^{-1}(N_{iA} + N_{iAo} + N_{iB} + N_{iK} - N_{iO})$ in the estimate for $P(\mathcal{E}_K)$ given in [6] to obtain

$$-\log P(\mathcal{E}_K) \geq S_{Ai} - T_A + S_B - T_B + S_{A0} + S_K - S_O + (2 \log n - \log 2) \left( \sum_{\ell \in Z_A} d_{\ell} + |B| \right) - \tilde{O}(n^{1/2-\varepsilon}), \quad (13)$$

where

$$S_{\mu} = \sum_{i=1}^{m} N_{i\mu}q^{-1} \quad \text{for } \mu \in \{Ai, Ao, B, K, O\},$$

$$T_A = \sum_{\ell \in Z_A} \sum_{j=1}^{d_{\ell}} 4t_{\ell j}^2 \quad \text{and} \quad T_B = \sum_{\ell \in Z_B} \sum_{j=1}^{h_{\ell}} 4t_{\ell j}^2.$$ 

We proceed to show that the contributions from $S_{Ai} - T_A$ and $S_B - T_B$ are negligible. The remaining terms will be used to balance the number of events in our union bound calculation.

To estimate $S_{Ai}$, note that

$$\sum_{i=1}^{m} \sum_{\ell \in Z_A} 2g_{\ell i} q^{-1} = \sum_{\ell \in Z_A} \sum_{j=1}^{d_{\ell}} \sum_{i=\ell(j-1)}^{i_{\ell j}} 4tn^{-3/2}s_{\ell}$$

$$= \sum_{\ell \in Z_A} \sum_{j=1}^{d_{\ell}} \sum_{i=1}^{i_{\ell j}} 4tn^{-3}$$

$$= \sum_{\ell \in Z_A} \sum_{j=1}^{d_{\ell}} 2t_{\ell j}^2 - \sum_{\ell \in Z_A} \sum_{j=1}^{d_{\ell}} 2t_{\ell j}n^{-3/2} = \frac{T_A}{2} - \tilde{O}(n^{-1}).$$

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Using \( i_{\ell j} = jn/2 \pm n^{3/2-\varepsilon/3} \) and \( d_{\ell} = 2i_{\max}/n \pm n^{1/2-\varepsilon/3} \) we have

\[
\sum_{i=1}^{m} \sum_{\ell \in \mathcal{A}} \hat{q}_i s_i q_i^{-1} = \sum_{\ell \in \mathcal{A}} \sum_{j=i(i+1)} \sum_{i \in j} n^{-2}(d_{\ell} - j)^2 \\
\geq |\mathcal{A}| \sum_{j=1}^{2i_{\max}/n - 2n^{1/2-\varepsilon/3}} (2n)^{-1} j^2 \\
\geq |\mathcal{A}| \sum_{j=1}^{2i_{\max}/n} (2n)^{-1} j^2 - \tilde{O}(n^{1/2-\varepsilon/3}).
\]

Similarly, we have

\[
\sum_{\ell \in \mathcal{A}} \sum_{j=1}^{d_{\ell}} 2i_{\ell j}^2 < |\mathcal{A}| \sum_{j=1}^{2i_{\max}/n + n^{1/2-\varepsilon/3}} 2\left(jn^{-1/2}/2 + n^{-\varepsilon/3}\right)^2 \\
< |\mathcal{A}| \sum_{j=1}^{2i_{\max}/n} j^2(2n)^{-1} + \tilde{O}(n^{1/2-\varepsilon/3}).
\]

Putting everything together we have \( T_A - S_A \leq \tilde{O}(n^{1/2-\varepsilon/3}). \) Similarly, for \( S_B \) we have

\[
\sum_{i=1}^{m} \sum_{\ell \in \mathcal{B}} 4y_{i\ell} q_{i\ell}^{-1} \geq \sum_{\ell \in \mathcal{B}} \sum_{j=1}^{h_{\ell}} \sum_{i \in j} 8tn^{-3/2} s_{\ell j} \\
= \sum_{\ell \in \mathcal{B}} \sum_{j=1}^{h_{\ell}} \sum_{i \in j} 8tn^{-3} = T_B - \tilde{O}(n^{-1}).
\]

This gives \( T_B - S_B \leq O(L^{-2}n^{1/2}). \)

We are now ready for the union bound bound calculation. Recall that we have fixed the initial data that defines the event \( \mathcal{E} \); that is, we have specified \( z \), the collection of vertices \( x_1, x_2, \ldots, x_z \), the collection of hole size \( h_1, \ldots, h_z \) and the degrees of vertices \( x_{\ell} \) for \( \ell \in \mathcal{A} \). The number of choices for the data that defines an event \( \mathcal{E}_K \) is at most

\[
\left( \prod_{\ell \in \mathcal{A}} \left( \frac{n}{d_{\ell}} \right)^{d_{\ell} i_{\max}} \right) \left( \prod_{\ell \in \mathcal{B}} \left( \frac{n}{h_{\ell}} \right)^{h_{\ell} i_{\max}} \right) \left( \frac{n}{|C|} \right).
\]

(Note that we name the vertices in \( A \cup B \) by specifying the vertices in holes.) To estimate \( \mathbb{P}(\mathcal{E}) \) we apply (13) to each such choice of \( \mathcal{E}_K \), using \( d_{\ell} = 2i_{\max}/n \pm n^{1/2-\varepsilon/3} \) and \( (d_{\ell}) < \exp(\log \log n)h_{\ell} \) for \( \ell \in \mathcal{A} \), that \( T_A - S_A, T_B - S_B \) and \( S_O \) are at most \( O(L^{-2}n^{1/2}) \), and \( S_{AO} \geq 2|\mathcal{A}|i_{\max}/n - \tilde{O}(n^{1/2-\varepsilon/5}) \).
of the event that bound over every vertex \( x \) follows from Theorem 2.2, so it remains to show the upper bound. Let

\[
This proof is very similar to that of Theorem 1.2, but much simpler. The lower bound on degrees in 

\[ m \]

up to time \( t \). This gives

\[
A \cup C \text{ spans no edge in } G(t_{\max}),
\]

of the event that

1. \( A \) is the neighborhood of \( x \) in \( G(t_{\max}) \),

2. \( A \cup C \) spans no edge in \( G(t_{\max}) \), and
3. $vx$ is open in $G(t_{\text{max}})$ for all $v \in C$.

We view $C$ as vertices that might be added to the neighborhood of $v$ between time $t_{\text{max}}$ and the end of the process. We show that whp there is no triple $(x, A, C)$ with these properties.

Let $x, A, C$ with $|A| = d' = d(1 + n^{-\varepsilon/3})$ be fixed, where we write
\[ d = 2t_{\text{max}}n^{1/2} = \sqrt{(1/2 - \varepsilon)n \log n}. \]

We also specify an appearance time $i_j$ for every edge $xv_j$ where $A = \{v_1, \ldots, v_{d'}\}$. Let $\mathcal{F}$ be the event $A \cup C$ is an independent set in $G(t_{\text{max}})$, all pairs joining $x$ and $C$ are open in $G(t_{\text{max}})$ and all the specified edges appear at the specified steps of the process. In order to estimate the probability of the event $\mathcal{F}$, for each step $i$ we need to estimate the probability that the selected edge is compatible with this event, conditional on the history of the process. We say $i$ is a selection step if $i$ is one of $i_j$ for $j \in [d']$; then the selected edge is specified by $\mathcal{F}$, so the required probability is simply $2/Q = (1 \pm 2\varepsilon)/2 q^{-1}$. For other $i$, the required probability is $1 - N_i/Q$, where $N_i$ is the number of ordered open pairs that cannot be selected at step $i$ when $\mathcal{F}$ occurs. If $i = i_j$ is a selection step write $N_i = 0$. Then we estimate
\[ \mathbb{P}(\mathcal{F}) \leq \prod_{j=1}^{d'} (1 \pm 2\varepsilon) / 2 q(t_{ij})^{-1} \cdot \prod_{i=1}^{m} (1 - N_i/Q). \]

We write $N_i = N_{ia} + N_{ic}$, where $N_{ia}$ counts the ordered open pairs within $A$ and $N_{ic}$ counts those in $A \cup C$ with at least one vertex in $C$. Following the proof of Theorem 1.2 we have
\[ - \log \mathbb{P}(\mathcal{F}) \geq S_A - T_A + S_C + (2 \log n - \log 2) d' - O(n^{1/2}), \]
where $S_\mu = \sum_{i=1}^{m} N_{ia} q^{-1}$ for $\mu \in \{A, C\}$ and $T_A = \sum_{j=1}^{d'} 4t_{ij}^2$.

Following the argument in the previous section for estimating $S_{Ai} - T_A$ we have
\[ S_A - T_A = \tilde{O}(n^{1/2 - \varepsilon/3}). \]

In order to estimate $S_C$ we make the following crucial observation. If some vertex $u$ has degree at least $L^2 n^{\varepsilon} > 2\gamma(t_{\text{max}})$ in $C$ then at time $t_{\text{max}}$ we have $Y_{xu} > 2\gamma$. We can assume $xu$ is a non-edge as $x$ is open to $C$, so this contradicts our estimate on $Y$-variables. Thus no such vertex $u$ exists. While $\hat{q}k > L^{15} n^{\varepsilon}$, which holds up to time $(1 + o(1))t_{\text{max}}$, we can apply Lemmas 7.1 and 7.2 to obtain $Q_C \geq (1 - L^{-1}) \hat{q}|C|^2$ and $Q_{AC} \geq (1 - L^{-1}) \hat{q}|A||C|$. This gives
\[ S_C \geq (1 - o(1))(|C|^2 + 2|A| \cdot |C|) \frac{1}{2} \sqrt{\frac{1}{2} - \varepsilon} \frac{\log n}{n} = (1 - o(1)) \left( 1 - 2\varepsilon + \varepsilon' \sqrt{\frac{1}{2} - \varepsilon} \right) |C|^{1/2} \log n. \]

Now, taking the union over all possible choices of the data that specifies an event $\mathcal{F}$, namely the choices of $v, d', A, C$ and the collection of times at which the edges joining $v$ to $A$ appear, we see that the probability that a triple $(x, A, C)$ exists with the given conditions is at most
\[ \frac{n}{d'} \left( \frac{n}{|C|} \right)^{d'} \left( \frac{2}{n^2} \right)^{d'} \exp \left\{ -(1 - o(1)) \left( 1 - 2\varepsilon + \varepsilon' \sqrt{\frac{1}{2} - \varepsilon} \right) |C|^{1/2} \log n + O(n^{1/2}) \right\}. \]
where the sum is taken over $d'$ in the interval $(1 \pm n^{-\varepsilon/3})d$. Since $\varepsilon' = 5\varepsilon$, assuming $\varepsilon < 1/4$, this probability is at most

$$n \sum_{d'} \exp \left\{ -\varepsilon |C| \frac{1}{3} \log n \right\} .$$

Thus the required bound on degrees holds with high probability. 

\[
\square
\]

8 Concluding remarks

We have determined $R(3,t)$ to within a factor of $4 + o(1)$, so we should perhaps hazard a guess for its asymptotics: we are tempted to believe the construction rather than the bound, i.e. that $R(3, t) \sim t^2/4 \log t$. We should note that we only have an upper bound on the independence number of the graph $G$ produced by the triangle-free process. So, formally speaking, the triangle-free process could produce a graph that gives a better lower bound on $R(3, t)$. But we believe that this is not the case; that is, we conjecture that the bound on the independence number in Theorem 1.2 is asymptotically best possible.

Another natural direction for future research is to provide an asymptotically optimal analysis in greater generality for the $H$-free process. No doubt the technical challenges will be formidable, given the difficulties that arise in the case of triangles. But on an optimistic note, it is encouraging that one can build on two different proofs of this case.

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