Folded Strings in Curved Spacetime

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Abstract

Two dimensional classical string theory is solved in any curved spacetime. The complete spacetime required to describe the classical string motions turns out to be larger than the global space required by complete particle geodesics. The solutions are fully classified by their behavior in the asymptotically flat region of spacetime. When the curvature is smooth, the string solutions are deformed folded string solutions as compared to flat spacetime folded strings that were known for 19 years. However, surprising new stringy behavior becomes evident at curvature singularities such as black holes. The global properties of the classical string theory require that the “bare singularity region” of the black hole be included along with the usual black hole spacetime. The mathematical structure needed to describe the solutions include a recursion relation that is analogous to the transfer matrix of lattice theories. This encodes lattice-like properties on the worldsheet on the one hand and the geometry of spacetime on the other hand. A case is made for the presence of folded strings in the quantum theory of non-critical strings for $d \geq 2$.

I. INTRODUCTION

The original physical motivations for studying string theory were: (1) understanding unification of forces including quantum gravity, and (2) understanding the Standard Model. In recent years it has become increasingly evident that these goals should be re-examined in the presence of curved 4D space-time string backgrounds. The gauge symmetries and spectrum of quark + lepton families, which are the main ingredients of the Standard Model, were probably fixed during the stringy early times in the evolution of the Universe. At such times 4D space-time was curved. Since curvature contributes to the central charge, duality and other topological aspects of String Theory, it is likely that the predictions of String Theory in curved 4D spacetime are quite different than the flat 4D approach. Therefore, String Theory in curved space-time must be better understood in order to discuss its connections to low energy physics.

One should consider all kinds of curved backgrounds, not only the traditional cosmological backgrounds, since the passage from curved space-time to flat space-time may involve various phase transitions, including inflation of a small region of the original curved universe to today’s large universe that is essentially homogeneous and flat. Despite the curvature during the early universe, one can identify the states that play a role in low energy physics purely on the basis of symmetries: The gauge bosons, and chiral families of quarks and leptons in a small region of the 4D early curved universe would become the “massless” particles observed in today’s inflated 4D flat universe. Therefore, one can search for the “right” curved space model on the basis of its symmetries and “low energy spectrum”. Such a scenario (including inflation, and tracking the low energy states) is technically little understood although logically and intuitively it is highly plausible. The possibility of such a scenario suggests that curved space-time string theory deserves intensive study.

In addition, the issues surrounding gravitational singularities should be answered in the context of curved space-time string theory, as it is the only known theory of quantum gravity.

With these questions in mind, we have been pursuing a program of building and analyzing exactly solvable models of string theory in curved spacetime. The main tool introduced in [1] [2] is the G/H gauged WZW model based on non-compact groups, such that the coset contains a single time coordinate. The geometrical properties of such models started to be understood when the $SL(2, R)/R$ case with $k = 9/4$ [3] was interpreted as a string moving in a black hole background [3]. Since then a lot of progress was made on the construction of exact conformal field theory models for bosonic, supersymmetric and heterotic strings in curved spacetime. Some exact results on the geometrical properties of the models (to all orders in $\alpha'$) were derived ( [4] [10]). In such models various questions can be investigated. In particular, the “low energy spectrum” can be computed group theoretically by using the unitary representations of the relevant non-compact group [1] [11].

More recently, it became apparent that the solution of the classical equations of such models would shed more light on their physical interpretation. This is relevant to fundamental questions of singularities in gravitational physics, as well as stringy questions about the early uni-
verse and its influence on the low energy spectrum of quarks and leptons. The principles of the full classical stringy solution for any gauged WZW model were obtained in general terms in [12]. The specialization to particle geodesics was given explicitly in [4]. A more detailed exploration of the general 2D classical string theory in any curved spacetime (i.e. not only WZW models) was done in [13]. In 2D the only non-trivial stringy solutions turn out to be necessarily folded strings, and therefore folded strings are the only path toward analyzing stringy phenomena due to the wave nature of physics. Folded strings fall into this category, especially in the area of string-QCD relations. Therefore our work touches on two aspects of string theory: (i) strings in curved space-time and (ii) folded strings.

The main discussion of the classical folded string solutions in 2D is given in sections 2-6. In sections 7,8 a case is made for the presence of folded strings in the quantum theory of non-critical strings and in higher dimensions, by clarifying the reasons for their absence in the standard methods of conformal field theory and other approaches. Comments and conclusions appear in section 9.

II. CLASSICAL SOLUTION

In [13] the complete set of solutions of two dimensional classical string theory were constructed for any 2D curved spacetime. The classical action is given by \( \int d^2\sigma G_{\mu\nu}(x) \partial_\mu x^\nu \partial_\nu x^\nu \). In 2D \( B_{\mu\nu}(x) \) can be eliminated since it produces a total derivative in the action, and in the classical theory the dilaton is absent. The most general metric can always be transformed into the conformal form \( G_{\mu\nu} = \eta_{\mu\nu}G(x) \). Then the most general 2D classical string equations of motion \( \partial_\mu \partial_\nu x^\mu + \Gamma^\sigma_{\mu\nu} \partial_\mu x^\nu \partial_\nu x^\lambda = 0 \), and conformal (Virasoro) constraints \( \partial_\pm x^\mu \partial_{\pm} x^\nu G_{\mu\nu}(x) = 0 \) (vanishing stress tensor) take the form

\[
\begin{align*}
\partial_+(G \partial_- u) + \partial_-(G \partial_+ u) &= \frac{\partial G}{\partial x^\pm}(\partial_+ u \partial_- v + \partial_+ v \partial_- u) \\
\partial_+(G \partial_- v) + \partial_-(G \partial_+ v) &= \frac{\partial G}{\partial x^\pm}(\partial_+ u \partial_- v + \partial_+ v \partial_- u) \\
\partial_+ u \partial_- v &= 0 = -\partial_- u \partial_- v ,
\end{align*}
\]

where we have used the target space lightcone coordinates \( u(\sigma^+, \sigma^-) = \frac{1}{\sqrt{2}}(x^0 + x^1) \), \( v(\sigma^+, \sigma^-) = \frac{1}{\sqrt{2}}(x^0 - x^1) \), and the world sheet lightcone coordinates \( \sigma^\pm = (\tau \pm \sigma)/\sqrt{2} \), \( \partial_\pm = (\partial_\tau \pm \partial_\sigma)/\sqrt{2} \).

Since a typical string state is massive, one should expect that the string will follow on the average the trajectory of a massive particle. Therefore, to understand the average behavior of the string geodesic it is useful to first consider the solution for the geodesic of a massive particle. The particle geodesic equations follow from the above ones by dimensional reduction. That is, by dropping the \( \sigma \) dependence, \( \partial_\pm \rightarrow \partial_\tau \), these equations reduce to the point particle geodesic equations. For particles, the last line in (1) imposes the condition for a null geodesic, which is too restrictive for our purpose. If this condition is modified to

\[
G \dot{u} \dot{v} = \frac{m^2}{2}
\]

then (1) becomes the equations for a timelike geodesic for a massive particle with mass \( m \). The zero mass limit may also be considered at the end. We will provide the explicit solutions to the particle as well as the string equations. As discovered in [13] there are additional stringy phenomena due to the wave nature that cannot be seen in the particle solution, and therefore it is useful to contrast the string solutions with the particle solutions.

In flat space-time the solutions are given in terms of arbitrary left-moving and right-moving functions \( x^\mu_L(\sigma^+) \), \( x^\mu_R(\sigma^-) \)

\[
x^\mu(\tau, \sigma) = x^\mu_L(\sigma^+) + x^\mu_R(\sigma^-).
\]

As shown by BBHP in several gauges [14] [15], the constraints are also satisfied provided

\[1\] Although the original BBHP solutions were for open
\[ u = u_0 + \frac{1}{2} \left[ (\sigma^+ + f(\sigma^+)) + (\sigma^- + g(\sigma^-)) \right] \]
\[ v = v_0 + \frac{1}{2} \left[ (\sigma^+ - f(\sigma^+)) + (\sigma^- + g(\sigma^-)) \right] \]

where \( f(\sigma^+) \) and \( g(\sigma^-) \) are any two periodic functions, \( f(\sigma^+) = f(\sigma^+ + \sqrt{2}) \), \( g(\sigma^-) = g(\sigma^- + \sqrt{2}) \), with slopes \( f'(\sigma^+) = \pm 1 \) and \( g'(\sigma^-) = \pm 1 \). The slopes can change discontinuously any number of times at arbitrary locations \( \sigma^+_1, \sigma^-_1 \) within the basic intervals \(-1/\sqrt{2} \leq \sigma^+ \leq 1/\sqrt{2} \) (and then repeated periodically), but the functions \( f, g \) are continuous at these points. The discontinuities in the slopes are allowed since the equations of motion are first order in either \( \partial_u \) or \( \partial_v \). The number of times the slope changes in the basic interval corresponds to the number of folds for left movers and right movers respectively.

The simplest BBHP solution is the so-called yo-yo solution given by \( f = |\sigma^+|_{\text{per}} \) and \( g = |\sigma^-|_{\text{per}} \) which are the periodically repeated absolute value. These solutions describe folded strings, with the folds oscillating against each other, and moving at the speed of light. Examples are plotted in figures 1.2.

In Fig. 1 one sees the yo-yo solution with equal periods for \( |\sigma^+|_{\text{per}} \) and \( |\sigma^-|_{\text{per}} \). Fig. 2 is generated by taking the period of \( |\sigma^-|_{\text{per}} \) to be half of that of \( |\sigma^+|_{\text{per}} \). In Fig. 2 the string has two folds plus an additional critical point moving at the speed of light that becomes a fold for part of the motion.

As discovered in [3], the complete set of classical solutions in curved spacetime are classified by their behavior in the asymptotically flat region of spacetime \( G(u, v) \rightarrow 1 \), where they tend to the complete set folded string solutions of BBHP given in (3) as boundary conditions.

The curved space solutions are given in the form of a map from the world sheet to target spacetime. As a mathematical convenience the world sheet is divided into lattice-like patches, with a map associated with each patch. The world-sheet lattice structure is determined by the sign patterns of \((f', g') = (\pm, \pm) \) inherent in the BBHP solutions, thus the lattice is dictated by the boundary conditions in the asymptotically flat region of spacetime \( G(u, v) \rightarrow 1 \). We emphasize that the lattice is on the world-sheet, not in curved spacetime, and it is only a mathematical tool to keep track of patches. In each patch of the lattice one set of signs holds, hence there are 4 types of patches called \( A, B, C, D \). For each such patch there is a solution of the equations of motion that is valid within the patch. The forms of the solutions in a group of neighboring \( A, B, C, D \) patches labelled by an integer \( k \) are (see eq.(13) for an example of a pattern of patches)

\[ A : \quad u = U_k(\sigma^+), \quad v = U_k(\sigma^-) \]
\[ B : \quad u = U_k(\sigma^-), \quad v = V_k(\sigma^+) \]
\[ C : \quad u = u_k, \quad v = W[\alpha_k(\sigma^+) + \beta_k(\sigma^-)], u_k \]
\[ D : \quad u = W[\alpha_k(\sigma^-) + \beta_k(\sigma^+), v_k], \quad v = v_k, \]

where the functions \( U_k(\sigma^+), V_k(\sigma^-), \alpha_k(\sigma^+), \beta_k(\sigma^-) \) and the constants \( u_k, v_k \) are given by a recursion relation whose form depends on the metric \( G \). It is easy to verify that, independently of the recursion relation, the forms listed in (3) solve the differential equations for any \( U_k(\sigma^+), V_k(\sigma^-), \alpha_k(\sigma^+), \beta_k(\sigma^-) \) provided the functions \( W_k(\sigma^+), \sigma^-), \tilde{W}_k(\sigma^+, \sigma^-) \) are defined by inverting the following relations

\[ F(u_k, W_k) \equiv \int_{W_k}^{W_k} du' G(u_k, v') = \alpha_k(\sigma^+) + \beta_k(\sigma^-), \]
\[ \tilde{F}(\tilde{W}_k, v_k) \equiv \int_{W_k}^{W_k} du' G(u', v_k) = \tilde{\alpha}_k(\sigma^-) + \tilde{\beta}_k(\sigma^+). \]

This is the complete set of solutions. So, for a given metric \( G(u, v) \) there exist the functions \( F(u, v) \) and \( \tilde{F}(u, v) \) such that their partial derivatives reproduce the metric

\[ \frac{\partial F(u, v)}{\partial v} = G(u, v) = \frac{\partial \tilde{F}(u, v)}{\partial u}. \]

and for each metric \( G \) we have the relations

\[ F(u_0, v) = \alpha + \beta \Leftrightarrow v = W(\alpha + \beta, u_0), \]
\[ \tilde{F}(u, v_0) = \tilde{\alpha} + \tilde{\beta} \Leftrightarrow u = W(\tilde{\alpha} + \tilde{\beta}, v_0), \]

that help define the solutions \( C, D \) in terms of the arbitrary functions \( \alpha(\sigma^+), \beta(\sigma^-) \). Consider the following three cases as illustrations

1. Flat metric \( ds^2 = du dv : \)
   \[ F = u_0 + v = \alpha + \beta, \]
   \[ quad \Rightarrow v = W(\alpha + \beta, u_0), \]
   \[ \tilde{F} = u + v_0 = \tilde{\alpha} + \tilde{\beta}, \]
   \[ \Rightarrow u = W(\tilde{\alpha} + \tilde{\beta}, v_0). \]

2. This set of solutions were noticed independently in [12] and [13], but the authors of [13] thought there are no stringy solutions. They did not realize that the validity of these solutions is limited to patches of the worldsheet, and they assumed that the stringy solutions discussed in [13] and here are gauged away by using the remaining conformal invariance.
2. SL(2,R)/R black hole metric $ds^2 = (1-uv)^{-1} du dv$:

$$F = -u_0^{-1} ln(1-u_0 v) = \alpha + \beta$$
$$v = W = u_0^{-1} \{ 1 - \exp[-u_0(\alpha + \beta)] \},$$
$$\bar{F} = -v_0^{-1} ln(1-u_0 v) = \bar{\alpha} + \beta$$
$$u = W = v_0^{-1} \{ 1 - \exp[-v_0(\bar{\alpha} + \beta)] \}.$$  

(11)

3. Cosmological (de Sitter) metric $ds^2 = dt^2 - e^{2Ht} du^2 = \frac{4}{|F(u,v)|^2}(u + v)^{-2} du dv$:

$$F = -(u_0 + v)^{-1} = \alpha + \beta$$
$$v = W = -(\alpha + \beta)^{-1} - u_0,$$
$$\bar{F} = -(u + v_0)^{-1} = \bar{\alpha} + \beta$$
$$u = W = -(\bar{\alpha} + \beta)^{-1} - v_0.$$  

(12)

The reader can verify that the flat spacetime BBHP solutions given in (10) take the 4 forms of (11) in regions of $\sigma^\pm$ where the 4 types of sign patterns $(f', g') = (\pm 1, \pm 1)$ hold. The form (10) looks more general than the BBHP solution because there still are boundary conditions and gauge freedom that will be fixed later. Then it agrees precisely with (11) as seen below. Intuitively we expect that in the presence of small curvature the physical character of the solution is similar to the folded string motions illustrated in Figs.1.2. As the curvature increases smoothly, except for the deformations due to curvature, the motions must also be similar. The question is what happens when there are curvature singularities?

At the boundaries of each patch continuity conditions must be imposed. This produces a recursion relation that describes the motion of the string as the proper time $\tau$ increases (see below). The recursion relation, which is analogous to a “transfer matrix” of lattice theories, connects the maps in different patches into a single continuous map from the worldsheet to spacetime. Thus, the functions $U_{k}(\sigma^+, \alpha_k(\sigma^+), \beta_k(\sigma^+))$, $V_{k}(\sigma^+, \alpha_k(\sigma^+), \beta_k(\sigma^+))$ in the various patches get related to each other. This “transfer matrix” encodes the properties of the world sheet lattice on the one hand and the geometry of spacetime on the other hand. Thus, lattices on the world-sheet plus geometry in space-time lead to “transfer matrices”. This seems to be a rich area of mathematical physics to explore in more detail in the future. Here we will derive the transfer matrix for one such lattice.

Recall that the lattice is dictated by the nature of the solution (11) in the asymptotically flat region of target spacetime. As an example we consider the simplest yo-yo solution as a boundary condition. This defines the sign patterns according to the slopes of the periodic functions $\sigma^+_{per}$ and $\sigma^-_{per}$, and the following lattice emerges from the periodic behavior of these functions. The world sheet is labelled by $\sigma$ horizontally and by $\tau$ vertically. Increasing values of $k$ correspond to increasing values of $\tau$. Periodicity in $\sigma$ is imposed, hence the world sheet is a cylinder. It is sliced by equally spaced 45° lines that form a light-cone lattice in $\sigma^\pm$. The crosses in the diagram represent the corners of the cells on the world sheet.

The transfer matrix for this “yo-yo lattice” was discussed in [13] for the SL(2,R)/R metric. Here we first give a compact general formula for any metric $G(u,v)$ and then specialize it to the examples of flat spacetime, the SL(2,R)/R black hole space-time, and the cosmological de Sitter space-time.

III. GENERAL SPACETIME

The continuity at the corners that join the $A$, $B$ cells is automatically insured by the use of the same functions $U_k(z), V_k(z)$ to describe the $A$, $B$ solutions, but with different arguments $z = \sigma^\pm$ that alternate between neighboring cells. Continuity at the boundaries between $A$, $B$ cells and $C$, $D$ cells requires

$$U_{k+1}(-1/\sqrt{2}) = U_k(1/\sqrt{2}) = u_k,$$
$$V_{k+1}(-1/\sqrt{2}) = V_k(1/\sqrt{2}) = v_k.$$  

(14)

where the $(u_k, v_k)$ are constants. Similarly, by taking into account the relations (10) at these boundaries one can construct the functions $W_k(\sigma^+, \sigma^-), \bar{W}_k(\sigma^+, \sigma^-)$ for the $C$, $D$ cells in terms of the functions $U_k(\sigma^\pm), V_k(\sigma^\pm)$

$$W_k = W[(F(u_k, V_k(\sigma^+)) + F(u_k, V_k(\sigma^-)) - F(u_k, v_{k-1})], u_k]$$
$$\bar{W}_k = W[(F(u_k(\sigma^+), v_k) + F(U_k(\sigma^-), v_k) - F(u_k-1, v_k)], v_k].$$  

(15)

Evaluating these at the lower (i.e. past) boundaries of the $C$, $D$ cells, using $V_k(-1/\sqrt{2}) = v_{k-1}, U_k(-1/\sqrt{2}) = w_{k-1}$

4It is also possible to take different functions in the $A$, $B$ patches and only require continuity at the boundaries. However, this freedom has no physical meaning since it can be changed by a conformal gauge transformation. Indeed, we have already used part of the remaining conformal invariance in choosing the forms in (11) to make the functions in neighboring $A$, $B$ patches the same. The same reasoning applies also to the $C$, $D$ patches.
\[ W(F(u_k, V_k(z)), u_k) = V_k(z), \quad (16) \]

and similarly for \( U_k(z) \). At the upper (i.e. future) boundaries of the \( C, D \) cells the boundary matching gives a recursion relation

\[ V_{k+1}(z) = W[(F(u_k, V_k(z)) + F(u_k, v_k) - F(u_k, v_{k-1})), u_k], \]
\[ U_{k+1}(z) = W[(F(U_k(z), v_k) + F(u_k, v_k) - F(u_{k-1}, v_k)), v_k]. \quad (17) \]

where \( z = \sigma \pm \). This recursion may be viewed as a transfer operation in proper time \( \tau \to \tau + 2 \), for any \( \sigma \), and is quite analogous to the concept of the "transfer matrix" in lattice theories. The recursion leads to the solution of all the \( U_k(\sigma \pm), V_k(\sigma \pm) \) in terms of \( U_0(z), V_0(z) \), that describe initial conditions at \( \tau = 0 \).

By evaluating the recursion relation (17) at the boundaries of each cell \( z = \pm 1/\sqrt{2} \) and using the values (14) at the boundaries, one finds a recursion relation for the constants \( (u_k, v_k) \)

\[ v_{k+1} = W \left[ \left( 2F(u_k, v_k) - F(u_k, v_{k-1}) \right), u_k \right], \]
\[ u_{k+1} = W \left[ \left( 2F(u_k, v_k) - F(u_k, v_{k-1}) \right), v_k \right]. \quad (18) \]

The solution of this recursion relation requires 4 initial constants \( u_0, v_0, u_{-1}, v_{-1} \)

\[ U_0(-1/\sqrt{2}) = u_{-1}, \quad U_0(1/\sqrt{2}) = u_0, \]
\[ V_0(-1/\sqrt{2}) = v_{-1}, \quad V_0(1/\sqrt{2}) = v_0. \quad (19) \]

Therefore, the positions \( (u_k, v_k) \) are fully determined in curved space-time in terms of 4 initial constants. The counting of initial parameters is right: since there are just two dynamical folds, specifying their initial positions and velocities corresponds to just 4 parameters.

The constants \( (u_k, v_k) \) are sufficient to describe the physical motion of the folds (or end points), as well as the whole string, as follows. Consider the diagram of eq. (13). At any \( \tau \) the trajectories of the folds are parametrized by the vertical lines that pass through \( \sigma = 0, 2 \) on the world sheet (and their periodic repetitions at \( \sigma = 4\ell, 4\ell + 2 \)). Likewise, vertical lines that pass through the crossings located at \( \sigma = 1, 3 \) (and their periodic repetitions at \( \sigma = 4\ell + 1, 4\ell + 3 \)) parametrize the trajectory of the midpoint between the folds. The center of mass of the string coincides with these midpoints. As \( \sigma \) increases one can read off the space-time trajectories of the center of mass and of the folds by moving upward along the vertical lines in the diagram. For example, consider the \( \sigma = 0 \) fold: during \( 2k - 2 \leq \tau \leq 2k \) it remains at constant \( u = u_{k-1} \) while the value of \( v = W_{k-1} \) increases from \( v = v_{k-2} \) to \( v = v_{k} \). Between \( 2k \leq \tau \leq 2k + 2 \) it remains at constant \( v = v_{k} \) while the value of \( u = W_{k} \) increases from \( u = u_{k-1} \) to \( u = u_{k+1} \), etc. In a similar way the trajectory of the second fold and of the center of mass are read off directly from the diagram in eq. (13). The space-time trajectories of these points are plotted in a \((u, v)\) plot in Fig.3.

The detailed motion of the intermediate points of the string at any \( \sigma \) are described by the functions \( U_k, V_k, W_k \) as indicated on the diagram (13) and mapped on Fig.3. The space-time trajectories of folds or end points that are the images of \( \sigma = 0, 2 \) are physical and cannot depend on conformal reparametrizations. Indeed, as seen from the above solution there is no freedom in the choice of the constants \( (u_k, v_k) \) except for the initial values (13). On the other hand, the motion of the rest of the string is gauge dependent at intermediate points \( \sigma \) (because of reparametrizations), and therefore it depends on the choice of \( U_0(z), V_0(z) \) that have remained unspecified. However, once the motion of the end points is plotted, it is clear from Fig.3 that the shape of the minimal surface is already determined without needing the details of the gauge dependent motion of the intermediate points.

The remaining conformal invariance may be used to fix the form of these functions in the initial cell (although this is not necessary). For the yo-yo solution the initial functions \( U_0(z), V_0(z) \) need not contain more than 4 constants that are related to the initial positions and velocities of the two folds. Therefore, the simplest gauge fixed form is

\[ U_0(z) = \frac{1}{2} (u_0 + u_{-1}) + \frac{1}{\sqrt{2}} (u_0 - u_{-1}) z_{\text{per}}, \]
\[ V_0(z) = \frac{1}{2} (v_0 + v_{-1}) + \frac{1}{\sqrt{2}} (v_0 - v_{-1}) z_{\text{per}}, \quad (20) \]

where \( z_{\text{per}} \) is the linear function \( z_{\text{per}} = z \) in the interval \( -1/\sqrt{2} \leq z \leq 1/\sqrt{2} \) and then repeated periodically. However, any other periodic function with the same 4 boundary constants will produce the same physical motion for the folds.

The recursion (13) is the fundamental physical relation that fully determines the motion of the yo-yo string in curved space-time. We called it the "transfer matrix" in the example of the black hole worked out in ref. [13]. It was found that it has certain invariances that are valid everywhere in target space-time, including near singularities. The invariance is related to a lattice version of the fundamental action \( A = \int d^2 \sigma G_{\mu \nu} \partial_\mu x^\nu \partial_\nu x^\nu \) that represents the minimal surface swept by the string. The lattice version of the minimal surface is expressed in terms of the constants \( (u_k, v_k) \) and its value for one period turns out to be a constant of motion. Explicit expressions for this lattice action will be given for specific metrics in the following sections. For every metric \( G \) one can find a lattice version of the action \( A \) that is an invariant under the recursion (13). The invariance is valid even in the vicinity of singularities in space-time (i.e., when \( G(u_k, v_k) \) grows) and helps in the understanding of new stringy phenomena. For example, it was found that classical strings can tunnel to regions of space-time (such as the bare singularity region of a black hole) that are forbidden to particle geodesics. Such a surprising motion of a string may be thought of as the analog of the diffraction of light around
corners, that is possible for classical waves, but is impossible for particle trajectories.

In this section we constructed the yo-yo solution in any curved space-time given by \( G \). In a similar way one may consider more complicated solutions with many folds. The general boundary condition near \( G \to 1 \) given by (4), with any number folds, defines a pattern of \( A, B, C, D \) on the world sheet that corresponds to the regions of \((\sigma^+, \sigma^-)\) that have definite signs of \( f', g' \) for some choice of \( f, g \). The pattern must be periodic horizontally, with a period of \( \sigma \to \sigma + 4 \), to insure periodicity. This generalizes the lattice in the diagram of (13). By virtue of the BBHP construction, any of these generalized patterns is guaranteed to correspond to strings that propagate forward in time. Then there remains to carry out the matching of the functions at the boundaries. This would give generalizations of the recursion relations and transfer matrices discussed above. It seems that this is a very rich area for mathematical physics, since one may explore relations between geometries defined by metrics \( G \), lattices, and transfer matrices. It is clear that the general behavior of the minimal surface that emerges from this procedure has to be quite similar to the one in flat space-time (which is already given by the choice of \( f, g \)), except for the deformations due to curvature and singularities. Moreover, it seems that the main physical stringy features related to the curvature and/or singularity structure of space-time may already be extracted from the yo-yo solution that has only two folds.

We now apply the general yo-yo results to several specific metrics and construct explicitly the corresponding “transfer matrices”, their invariants, and the corresponding string solutions.

**IV. FLAT SPACE-TIME**

The functions \( F, W \) corresponding to the flat space-time metric \( G = 1 \) are given in (11). Using them in the general formulas (14-19) we obtain the explicit recursion relations

\[
\begin{align*}
\tilde{W}_k &= U_k(\sigma^+) + U_k(\sigma^-) - u_{k-1}, \\
U_{k+1}(z) &= U_k(z) + u_k - u_{k-1} \\
u_{k+1} &= 2u_k - u_{k-1}
\end{align*}
\]

They are solved by

\[
\begin{align*}
u_k &= u_0 + k(u_0 - u_{-1}) \\
U_k(z) &= U_0(z) + k(u_0 - u_{-1}) \\
\tilde{W}_k &= U_0(\sigma^+) + U_0(\sigma^-) + (k + 1)(u_0 - u_{-1}) - u_0
\end{align*}
\]

where \( U_0(-1/\sqrt{2}) = u_{-1}, U_0(1/\sqrt{2}) = u_0, \) and the function \( U_0(z) \) is arbitrary. The solutions for \( V_k(z), W_k, v_k \) are obtained from the above by replacing \( U \to V \) and \( u \to v \). If \( U_0(z), V_0(z) \) are gauge fixed as in (20), then this solution takes the convenient form of the BBHP yo-yo string in (1) with \( f = |\sigma^+|_{\text{per}}, g = |\sigma^-|_{\text{per}} \). The present form is a generalization that permits other gauge choices. The motion of the end points, as plotted in Fig.1 is gauge independent, but the motion of the interior points of the string depends on the gauge choice, as expected.

Define a lattice version of the surface element \( dA = d^2 \sigma \left( \partial_+ u \partial_- v + \partial_- u \partial_+ v \right) \) swept by the string during \( 2k \leq \tau \leq 2k + 2 \). The area of one rectangle in Fig.1 is

\[
dA_k = (u_k - u_{k-1})(v_k - v_{k-1}).
\]

From the world sheet point of view this covers the image of one \( A \) or \( B \) cell. The image of a \( C \) or \( D \) cell has zero area in target space-time since they are mapped to the edges of the rectangle (see footnote 6 for a related point). Consider the transformation (21) as a transfer matrix that takes the system forward in time. Under this transformation \( dA_k \) is an invariant since \( dA_{k+1} = dA_k \). This is seen by rewriting (21) in the form \( U_{k+1}(z) - u_k = U_k(z) - u_{k-1}, \) etc.. Therefore, we may say that the “transfer matrix” for flat space-time given by (21) leaves invariant the “lattice action density” given by (13). This concept generalizes to curved space-time, as seen below.

**V. BLACK HOLE SPACE-TIME**

The case of the SL(2,R)/R two dimensional black hole metric \( ds^2 = (1 - uv)^{-1}du dv \) was already discussed in (13), but here we will show how the results of (13) follow from the general formulas, and also give the additional recursion relations for \( U_k, \tilde{V}_k, W_k \) at general \( k \) and general gauge that were not provided in (13).

The solution for the geodesic of a massive particle was given in our previous work (4). Here we rewrite it in a more convenient form

\[
\begin{align*}
u &= e^7 \sqrt{\gamma^2 + m^2} \left[ \frac{u_0 \cosh(\gamma \tau)}{\left( \gamma \sqrt{\gamma^2 + m^2} - u_0 \right) \sinh(\gamma \tau)} \right] \\
v &= e^{-7} \sqrt{\gamma^2 + m^2} \left[ \frac{v_0 \cosh(\gamma \tau)}{\left( \gamma \sqrt{\gamma^2 + m^2} + v_0 \right) \sinh(\gamma \tau)} \right]
\end{align*}
\]

where \( u_0, v_0, \dot{u}_0, \dot{v}_0 \) are initial velocities and momenta, \( m \) is the mass of the particle, and \( \gamma \) is a convenient parameter

\[
\gamma = \sqrt{(u_0 v_0 + \dot{u}_0 \dot{v}_0)^2 - 4u_0 \dot{v}_0},
\]

\[
m^2 = \frac{\dot{u}_0 \dot{v}_0}{(1 - u_0 v_0)}.
\]

In the zero mass limit either \( \dot{u}_0 = 0 \) or \( \dot{v}_0 = 0 \), and then the solution reduces to a light-like geodesic for which either \( u \) or \( v \) remain constant respectively at all times.

The singularity is at \( u(\tau)v(\tau) = 1 \). To see when the particle hits the singularity we compute this quantity
\[
\frac{uv - 1}{u_0v_0 - 1} = \left[ \cosh \gamma \tau + \frac{(\dot{u}_0\dot{v}_0 + \dot{u}_0v_0)}{\sqrt{(u_0\dot{v}_0 + u_0v_0)^2 - 4u_0v_0}} \right]^2
\]

(26)

In the massless limit this expression becomes

\[
\frac{u(\tau)v(\tau) - 1}{u_0v_0 - 1} = \exp \left( (u_0\dot{v}_0 + \dot{u}_0v_0) \tau \right), \quad \dot{u}_0\dot{v}_0 = 0. \quad (27)
\]

It is evident that the sign of \( uv - 1 \) cannot change as \( \tau \) changes, therefore the particle must remain in either the black hole region \( uv < 1 \) (can cross the horizon at \( u = 0 \) or \( v = 0 \)), or in the bare singularity region \( uv > 1 \). The boundary \( uv = 1 \) acts like an impenetrable wall from either side. This last feature is different for the string solution. In contrast to the point particle, the string will tunnel through the wall!! This surprising effect was discovered in [13].

It was evident from the work of [13] that, except for the tunneling type phenomena, the string follows more or less the geodesic of the massive particle. Therefore, it is useful to clarify the properties of the geodesics of the point particle, because they depend on the initial particle location as well as its velocity.

If the particle starts out in the “bare singularity” region, \( u_0v_0 > 1 \) (future or past regions), the mass formula in (25) requires \( \dot{u}_0\dot{v}_0 < 0 \) and \( \gamma \) is real. Then the motion is governed by hyperbolic functions, and (26) never vanishes. Therefore, a massive particle, or the string, cannot hit the singularity. In the massless limit, according to (27), the light-like geodesic will hit the bare singularity only if it starts out with initial conditions that give \( u_0\dot{v}_0 + \dot{u}_0v_0 < 0 \), but in any case it reaches the singularity only at infinite proper time \( \tau = \infty \). Therefore, the “bare singularity” region of the \( SL(2,\mathbb{R})/\mathbb{R} \) black hole is not a singularity that can be reached by physical signals in a finite amount of proper time. In this sense it is not really a singularity.

If the particle starts initially in the black hole region \( u_0v_0 < 1 \), either inside or outside the horizon, its trajectory has wildly different behavior depending on its velocity. There are two critical ratios of the velocities at which \( \gamma = 0 \).

(i) If the velocities lie in the range

\[
\left( \frac{1 - \sqrt{1 - u_0v_0}}{u_0v_0} \right)^2 < \frac{\dot{u}}{\dot{v}} < \left( \frac{1 + \sqrt{1 - u_0v_0}}{u_0v_0} \right)^2. \quad (28)
\]

then \( \gamma \) is imaginary and (24) vanishes periodically. The massive particle goes through the horizon and hits the black hole singularity at a finite value of \( \tau \). There it moves smoothly to a second sheet of the \((u, v)\) space-time, but still with \( uv < 1 \). It continues its journey toward the second branch of the singularity and hits it, moving on to a third sheet of space-time (or back to the first sheet, according to interpretation). The journey continues endlessly from singularity to singularity, always moving smoothly to another sheet, and always remaining in the region \( uv < 1 \). This behavior is similar to the behavior of geodesics in the many worlds of the Reissner-Nordtrom black hole.

(ii) If the velocities lie in the range

\[
\frac{\dot{v}}{\dot{u}} > \left( \frac{1 + \sqrt{1 - u_0v_0}}{u_0} \right)^2 \quad (29)
\]

then \( \gamma \) is real, the motion is hyperbolic, and (26) vanishes only once. Therefore, the particle hits the black hole at a finite \( \tau \) only once, and moves to a second sheet where it remains for the rest of time.

(iii) If the velocities lie in the range

\[
\frac{\dot{v}}{\dot{u}} < \left( \frac{1 - \sqrt{1 - u_0v_0}}{u_0} \right)^2 \quad (30)
\]

then \( \gamma \) is real, the motion is hyperbolic, but (26) never vanishes. Therefore, the particle never hits the black hole.

The string geodesics given below follow, on the average, the behavior of the massive particle geodesics above. But, because of the oscillatory motion we find new phenomena in the vicinity of the black hole. When the string approaches the black hole from the \( uv < 1 \) region, and hits the singularity, it behaves differently than the particle: it fully penetrates the wall to the \( uv > 1 \) region, but then it snaps back into the \( uv < 1 \) region, and then follows more or less the particle trajectory in the second sheet, etc. (see the solution below and the plots in Figs.4,5).

To construct the string solution we use the general formulas of the previous sections. The functions \( F, W \) corresponding to the flat space-time metric \( G = (1 - uv)^{-1} \) are given in [16]. Using them in the general formulas (44-45) we obtain the explicit recursion relations

\[
\tilde{W}_k(\sigma^+, \sigma^-) = \frac{1}{v_k} \left[ 1 - \frac{(1-U_k(\sigma^+)v_k)(1-U_k(\sigma^-)v_k)}{1-u_k^{-1}v_k} \right],
\]

\[
U_{k+1}(z) = \frac{1-u_kv_k}{1-u_k^{-1}v_k} \left[ U_k(z) + \frac{u_k-u_k^{-1}}{1-u_kv_k} \right],
\]

\[
u_{k+1} = \frac{2u_ku_k^{-1} - u_k^2v_k}{1-u_k^{-1}v_k}, \quad (31)
\]

5 In the present case the worlds are pasted to each other just at \( uv = 1 \) along the singularity. When the metric is modified by quantum corrections [13] a gap develops so that the singularity becomes unreachable while the geodesics move from one world to the next.
and similarly $W_k, V_k, v_k$ are obtained from the above by interchanging $U \leftrightarrow V$ and $u \leftrightarrow v$. This agrees with the results of [13]. Note that for $u, v \to 0$ or $\infty$ the metric approaches the flat metric (see footnote #2). In both of these limits the formulas in (31) approach the flat ones in (21).

Just as the flat case, we define a lattice version of the area element in curved space-time. The “lattice area” swept by the string for one of the rectangles in Fig.4,5 is defined as

$$dA_k = \frac{(u_k - u_{k-1})(v_k - v_{k-1})}{1 - \frac{1}{4}(u_k + u_{k-1})(v_k + v_{k-1})}. \quad (32)$$

As in the flat case, this is a lattice version of the target space area of the image of a $A$ or $B$ cell on the world sheet, while the area of the image of a $C$ or $D$ cell is zero. This expression is invariant under the “transfer matrix” [21], i.e. $dA_k = dA_{k+1}$. The invariance of this expression everywhere, including in the vicinity of the black hole singularity, is helpful in understanding the reason for the tunnelling to the bare singularity region. Namely, since the string must move in a way that conserves this minimal area, and must have a continuous trajectory, it cannot avoid the tunnelling for generic initial conditions set by an observer (see Fig.5).

By feeding the recursion relations to a computer, the trajectories of the folds are plotted in Fig.4,5. A physical discussion of the string falling into a black hole was given in [13]. The most surprising effect was the tunnelling of the string into the bare singularity region which is not possible for particles (Fig.5). As suggested before, this is analogous to the diffraction of classical light waves that is possible for waves but not for particles.

Therefore, we must conclude that a complete black hole classical spacetime must include the “bare singularity” region which is a region dual to the exterior of the horizon. We have shown that the classical motion of a string is incomplete without this region. Of course, we are also aware that the quantum conformal field theory also has duality properties that require the inclusion of all dual regions on an equal footing. All this seems to indicate that the problems surrounding gravitational singularities, including the information paradox problem, probably could not be fully resolved or even correctly described without the inclusion of spacetime regions that seemed inaccessible in traditional approaches.

VI. COSMOLOGICAL SPACE-TIME

Consider the cosmological space-time corresponding to a Friedman - Robertson - Walker (FRW) universe in 4D

$$ds^2 = dt^2 - R^2(t) \left( \frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right). \quad (33)$$

where $k = -1, 0, 1$ are related to the classification of cosmological space-times as “open, flat, closed” respectively. For a string moving purely along the radial direction $d\theta = d\phi = 0$ one concentrates on the 2D metric

$$ds^2 = dt^2 - R^2(t) \frac{dr^2}{1 - kr^2}. \quad (34)$$

It is convenient to change variables

$$\sqrt{k} r = \sin(\sqrt{k} X), \quad T = \int \frac{dt}{R(t)}, \quad \sqrt{k} = i, 0, 1$$

$$u = \frac{1}{\sqrt{k}}(T + X), \quad v = \frac{1}{\sqrt{k}}(T - X), \quad (35)$$

so that the line element takes the conformal form

$$ds^2 = R^2(\ddot{T}^2 - dX^2) = G(u, v) \, du dv,$$

$$G(u, v) = 2R^2(t) \, dT$$

Once written in terms of $(u, v)$ the complete manifold is usually obtained by analytic continuation to all values of these variables. Then one may apply the general formulas of the previous sections to obtain the classical motion of strings.

As an example consider the de Sitter universe for which the expansion factor of the universe is given by

$$|R(t)| = e^{Ht} \quad (37)$$

where $H = \dot{R}/R$ is the Hubble constant, and

$$ds^2 = \frac{4 \, du \, dv}{H^2(u - v)^2}. \quad (38)$$

This 2D space can be embedded in 3D as the surface of a hyperboloid described by (see Fig.6)

$$x_0 = \frac{u - H^{-2}}{u + v}, \quad x_1 = \frac{u + H^{-2}}{u + v}, \quad x_2 = \frac{1}{H} \frac{u - v}{u + v} \quad (39)$$

Then the metric in (38) takes the flat form

$$ds^2 = dx_0^2 - dx_1^2 - dx_2^2. \quad (40)$$

First consider the geodesic equations for a massive particle of mass $m$. They can be solved exactly as a function of proper time $\tau$

$$u(\tau) = c + \frac{\sinh(Hm\tau) - \sinh(Hm\tau_0)}{H \sinh(Hm(\tau + \tau_0))}, \quad v(\tau) = -c - \frac{\sinh(Hm\tau) + \sinh(Hm\tau_0)}{H \sinh(Hm(\tau + \tau_0))}$$

$$R(\tau) = \frac{\sinh(Hm(\tau + \tau_0))}{\sinh(Hm\tau_0)} = -\frac{\sqrt{2}}{H} \frac{1}{u + v} \quad (41)$$

where $c, \tau_0$ are constants, and $R(\tau = 0) = 1$ has been chosen for simplicity. The geodesic for the massive particle is best pictured on the surface of the hyperboloid $x_1^2 + x_2^2 = x_0^2 + H^{-2}$. Inserting the solution in (39) one sees that $x_0(\tau)$ increases monotonically and lies in the range $-\infty < x_0(\tau) < \infty$. The geodesic extends from a point on the infinitely large circle at $x_0 = -\infty$ to a
point on the infinitely large circle at $x_0 = \infty$. It is a line that spirals less than or equal to one time on this surface. Define the angle $\tan \theta = x_2/x_1$. If the mass is zero, the maximum spiralling angle $\Delta \theta = \theta(\infty) - \theta(-\infty)$ is exactly $2\pi$, but for the massive particle the angle is less than $2\pi$.

Thus, on the average, we must expect the string center of mass to spiral less than $2\pi$. Of course, the overall string performs the yo-yo oscillations of Fig.3 and sweeps a minimal surface on the hyperboloid, that is similar to the one in flat space-time except for deformations due to curvature.

The explicit solution that describes this motion is obtained by applying our general procedure that yields the transfer matrix

$$\begin{align*}
\hat{W}_k &= \left[\frac{1}{u_k(z) + v_k} +\right. \\
1 &\left.\frac{1}{u_k(z) + v_k} \right]^{-1} - v_k
\end{align*}$$

$$U_{k+1}(z) = \frac{1}{u_k(z) + v_k} -\frac{1}{u_k(z) + v_k} - v_k \quad u_{k+1} = \frac{2}{u_k + v_k} -\frac{1}{u_k} - v_k$$

(42)

Similar formulas hold for $W_k, V_k, v_k$ respectively. We define a discrete version of the minimal area for rectangle $k$ by

$$dA_k = \frac{4}{H^2} \frac{(u_k - u_{k-1})(v_k - v_{k-1})}{(u_k + v_k)(u_k + v_k)}.$$ (43)

The transfer matrix leaves invariant this discrete minimal area, i.e. $dA_k+1 = dA_k$. This is easily proven by rewriting the transfer matrix in the form

$$\begin{align*}
\frac{U_{k+1}(z) - u_k}{U_{k+1}(z) + v_k} (u_k + v_k) &= \frac{u_k(z) - u_{k-1}}{U_k(z) + v_k} (u_k(z) + v_k) \\
\frac{V_{k+1}(z) - v_k}{v_k + V_k(z)} (u_k + v_k) &= \frac{V_k(z) - v_k}{v_k + V_k(z)} (u_k(z) + v_k) \\
\end{align*}$$

(44)

By feeding the recursion relation to a computer, and plotting the trajectories of the folds, the minimal surface is constructed and seen to have the properties described above, as depicted in Fig.6.

**VII. QUANTUM FOLDED STRING**

Given the fact that the string in 2D is quite non-trivial classically, we expect that there is a consistent quantization procedure that includes the non-trivial folded states. Therefore we should try to make a case for folded strings in the quantum theory.

As pointed out many times in our past work, folded 2D-string states are present in the $d = 2$ and $c \leq 25$ sector of the quantum theory in flat as well as curved spacetime. In simple string models, when it has been possible to compute the spectrum, their norm is positive and is proportional to $(c-26)$. Only if $d = 2$ and $c = 26$ simultaneously (e.g. $d = 2$ flat space-time with linear dilaton such that $c = 26$) the folded string states become zero norm states and then the special discrete momentum states survive as the only stringy states. A simple model in which these properties may be easily seen is the covariant quantization of the 2D string theory, in which the physical states are identified as the subset that satisfies the Virasoro constraints, i.e. $L_0 - \frac{d-2}{2} = L_{n \geq 1} = 0$ applied on states. For example, it has been known for a long time that the $d \leq 25$ sector of the flat string theory has non-trivial positive norm states (including for $d = 2$) that satisfy the Virasoro constraints and that there are no ghosts. A similar covariant quantization can be carried out for the 2D black hole string by using the Kac-Moody current algebra formulation, and relaxing the $c = 26$ condition (i.e. $k < 9/4$) to include the folded strings.

Why $c = 26$? There are several approaches to the quantization of strings that converge on the requirement of $c = 26$. These include the light-cone gauge, the Polyakov path integral and the BRST quantization. However, they each involve certain steps that seem to inadvertently exclude the $c < 26$ string. We can point out that

(i) The usual light-cone approach throws away the folded states from the beginning by assuming a uniform momentum density $P^+(\tau, \sigma) = p^+$, a statement that is not true for the BBHP solutions even in flat spacetime.

(ii) The Polyakov approach assumes a certain measure for the path integral, thus locking into a definition of a quantum theory. A different measure that takes into account folds can be considered as in [24] mentioned below.

(iii) The BRST approach requires $Q_{BRST}^2 = 0$ as an operator. This is a stronger requirement than imposing the Virasoro constraints only in the physical subspace $<\text{phys}|L_0 - \alpha_0|\text{phys}> = 0$. An analogous statement would be $<\text{phys}|Q_{BRST}^2|\text{phys}> = 0$, which does not lead to $c = 26$. Actually, the fact that there exists a consistent covariant quantization of the *flat free string* in $d < 26$ is already proof that the $Q_{BRST}^2 = 0$ approach is too strong.

The critical $c = 26$ string is certainly consistent and well understood. The success of its methods have created a prejudice against other possibilities. However, as argued above, it appears that a more general quantization of string theory for $c < 26$, that would include classical features such as the folded string states, is possible. This is already known to be true for the free flat string. What would also be interesting is to find the correct formulation for interacting folded strings. The path integral approach discussed in [24] seems to be promising, and it may be possible to make faster progress by reformulating it in the conformal gauge and relating it to our classical solutions. Note that the definition of fold in ref. [24] does
not take into account that the map from the world sheet to spacetime may be many to one (i.e. a region mapped to a segment, as is the case for our solutions). This feature may be important in the formulation of folds and their interactions in the path integral approach. In particular, the description of folds in the conformal gauge, as in our papers, may eventually prove to be a more convenient mathematical formulation.

VIII. HIGHER DIMENSIONS

Folded strings exist in higher dimensions as well. One can display the general solution in flat space-time in the temporal gauge

\[ x^0 = p^0 \tau, \quad x(x, \sigma) = x_L(\sigma^+) + x_R(\sigma^-), \]

\[ (\partial_+ x_L)^2 = p_0^2 = (\partial_- x_R)^2 \]

where \( f(\sigma^+), \ g(\sigma^-) \) are arbitrary periodic vectors in \( d - 2 \) dimensions, \( \text{which could be discontinuous, and } \epsilon_L(\sigma^+), \epsilon_R(\sigma^-) \) take the values \( \pm 1 \) in patches of the corresponding variables such that the sign patterns repeat periodically (as in the 2D string). When \( f, g \) are both zero the solution reduces to the 2 dimensional BBHP case. In general, the presence of discontinuous \( \epsilon_L, \epsilon_R \), and the discontinuities in \( f(\sigma^+), \ g(\sigma^-) \) gives a larger set of solutions, which include strings that are partially or fully folded. Discontinuities are allowed since the differential equations are first order in the derivatives \( \partial_+ \) and \( \partial_- \). Such solutions are usually missed in the lightcone gauge even in the flat classical theory (therefore, the lightcone “gauge” is not really a gauge, except for the case of \( c = 26 \)).

The curved space-time analogs of such solutions in higher dimensions are presently under investigation. For other solutions of classical strings theory in higher dimensional curved spacetime see \[21\].

IX. COMMENTS AND CONCLUSIONS

We have solved generally the classical 2D string theory in any curved space-time. All stringy solutions correspond to folded strings. All solutions tend to the BBHP solutions (as boundary conditions) in the asymptotically flat region of the curved space-time. Therefore, the BBHP solutions of eq.\[4\] serve to classify all the solutions for any curved space-time. In fact, the sign patterns of the BBHP solutions provide the method for dividing the world-sheet into patches, thus defining the lattices associated with the \( A, B, C, D \) solutions. The matching of boundaries for these functions gives the general solution in curved space-time in the form of a “transfer matrix”. Thus, lattices on the world-sheet plus geometry in space-time lead to transfer matrices. This seems to be a rich area to explore in more detail.

The general physical motion of the string is: oscillations around a center of masss that follows on the average a geodesic of a massive particle, consistent with intuition. The oscillations are deformed by curvature as compared to the BBHP solutions in flat spacetime, but they maintain the same general character as long as the curvature is smooth. However, new stringy behavior becomes evident in the vicinity of singularities where new phenomena, such as tunneling (similar to diffraction), take place. There is also the continuation of the motion into new worlds, in a finite amount of proper time, that the string as well as the massive particle geodesics do (but not the massless particle! - see above). Because of the tunelling and the new worlds, the global space of the \( SL(2, R)/R \) black hole is not just the usual black hole space, \( uv < 1 \). Rather, it must include also the \( uv > 1 \) “bare singularity” region even for the classical description of strings (actually this region is not really singular, as argued above). We conjecture that the inclusion of the bare singularity region is a more general requirement than the \( SL(2, R)/R \) case for the correct description of string motion. Of course, by duality, the quantum theory must include all the regions.

Folded strings are also of interest in a string-QCD relation. Gluons are expected to behave just like the folds, since only at the location of a gluon the color flux tube can fold, and move at the speed of light. Some recent discussion on this point can be found in \[16\] \[22\].

We suspect that the inclusion of the quantum states corresponding to folded strings may lead to a consistent quantum theory in less than 26 dimensions. As already emphasized in an paper, the free string is per-
fectly consistent as a quantum theory for $c < 26$, including the folded states. The interacting quantum string with folds remains as an open possibility.

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X. FIGURE CAPTIONS

FIG. 1. The 2-fold solution in flat spacetime.

FIG. 2. The 3-fold solution obtained with different left-right periods.

FIG. 3. The two fold solution in curved spacetime.

FIG. 4. String falling into black hole.

FIG. 5. Bare singularity region is needed in classical string theory.

FIG. 6. String motion in de Sitter spacetime.
Fig. 1. Minimal surface of flat string with 2 critical points that move at 45 degrees. The paths of different points along the string are marked with different symbols.

Fig. 2. Minimal surface of flat string with 3 critical points that move at 45 degrees. The paths of different points along the string are marked with different symbols.

Fig. 3. Minimal area in curved spacetime. The sizes of the rectangles change depending on the curvature.

Fig. 4. Ingoing string on 1st sheet meets black hole, moves out to 2nd sheet.

Fig. 5. String minimal area tunnels to forbidden region beyond black hole. Arrows along trajectories of midpoint.

Fig. 6. String moving in de Sitter space.