Learning a Single Neuron with Bias Using Gradient Descent

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Abstract

We theoretically study the fundamental problem of learning a single neuron with a bias term \((x \mapsto \sigma(\langle w, x \rangle + b))\) in the realizable setting with the ReLU activation, using gradient descent. Perhaps surprisingly, we show that this is a significantly different and more challenging problem than the bias-less case (which was the focus of previous works on single neurons), both in terms of the optimization geometry as well as the ability of gradient methods to succeed in some scenarios. We provide a detailed study of this problem, characterizing the critical points of the objective, demonstrating failure cases, and providing positive convergence guarantees under different sets of assumptions. To prove our results, we develop some tools which may be of independent interest, and improve previous results on learning single neurons.

1 Introduction

Learning a single ReLU neuron with gradient descent is a fundamental primitive in the theory of deep learning, and has been extensively studied in recent years. Indeed, in order to understand the success of gradient descent on complicated neural networks, it seems reasonable to expect a satisfying analysis of convergence on a single neuron. Although many previous works studied the problem of learning a single neuron with gradient descent, none of them considered this problem with an explicit bias term.

In this work, we study the common setting of learning a single neuron with respect to the squared loss, using gradient descent. We focus on the realizable setting, where the inputs are drawn from a distribution \(D\) on \(\mathbb{R}^{d+1}\), and are labeled by a single target neuron of the form \(x \mapsto \sigma(\langle v, x \rangle)\), where \(\sigma : \mathbb{R} \to \mathbb{R}\) is some non-linear activation function. To capture the bias term, we assume that the distribution \(D\) is such that its first \(d\) components are drawn from some distribution \(\tilde{D}\) on \(\mathbb{R}^d\), and the last component is a constant 1. Thus, the input \(x\) can be decomposed as \((\tilde{x}, 1)\) with \(\tilde{x} \sim \tilde{D}\), the vector \(v\) can be decomposed as \((\tilde{v}, b_v)\), where \(\tilde{v} \in \mathbb{R}^d\) and \(b_v \in \mathbb{R}\), and the target neuron computes a function of the form \(x \mapsto \sigma(\langle \tilde{v}, \tilde{x} \rangle + b_v)\). Similarly, we can define the learned neuron as \(x \mapsto \sigma(\langle \tilde{w}, \tilde{x} \rangle + b_w)\), where \(w = (\tilde{w}, b_w)\). Overall, we can write the objective function we wish to optimize as follows:

\[
F(w) := \mathbb{E}_{x \sim D} \left[ \frac{1}{2} \left( \sigma(w^\top x) - \sigma(v^\top x) \right)^2 \right] \quad (1)
\]

\[
= \mathbb{E}_{\tilde{x} \sim \tilde{D}} \left[ \frac{1}{2} \left( \sigma(\tilde{w}^\top \tilde{x} + b_w) - \sigma(\tilde{v}^\top \tilde{x} + b_v) \right)^2 \right]. \quad (2)
\]

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Throughout the paper we consider the commonly used ReLU activation function: $\sigma(x) = \max\{0, x\}$.

Although the problem of learning a single neuron is well studied (e.g. [18, 23, 5, 4, 11, 19, 14, 15]), none of the previous works considered the problem with an additional bias term. Moreover, previous works on learning a single neuron with gradient methods have certain assumptions on the input distribution $D$, which do not apply when dealing with a bias term (for example, a certain ”spread” in all directions, which does not apply when $D$ is supported on $\{1\}$ in the last coordinate).

Since neural networks with bias terms are the common practice, it is natural to ask how adding a bias term affects the optimization landscape and the convergence of gradient descent. Although one might conjecture that this is just a small modification to the problem, we in fact show that the effect of adding a bias term is very significant, both in terms of the optimization landscape and in terms of which gradient descent strategies can or cannot work. Our main contributions are as follows:

- We start in Section 3 with some negative results, which demonstrate how adding a bias term makes the problem more difficult. In particular, we show that with a bias term, gradient descent or gradient flow\(^1\) can sometimes fail with probability close to half over the initialization, even when the input distribution is uniform over a ball. In contrast, [23] show that without a bias term, for the same input distribution, gradient flow converges to the global minimum with probability 1.

- In Section 4 we give a full characterization of the critical points of the loss function. We show that adding a bias term changes the optimization landscape significantly: In previous works (cf. [23]) it has been shown that under mild assumptions on the input distribution, the only critical points are $w = \nu$ (i.e., the global minimum) and $w = 0$. We prove that when we have a bias term, the set of critical points has a positive measure, and that there is a cone of local minima where the loss function is flat.

- In Sections 5 and 6 we show that gradient descent converges to the global minimum at a linear rate, under some assumptions on the input distribution and on the initialization. We give two positive convergence results, where each result is under different assumptions, and thus the results complement each other. We also use different techniques for proving each of the results: The analysis in Section 6 follows from some geometric arguments and extends the technique from [23, 5]. The analysis in Section 5 introduces a novel technique, not used in previous works on learning a single neuron, and has a more algebraic nature. Moreover, that analysis implies that under mild assumptions, gradient descent with random initialization converges to the global minimum with probability $1 - e^{\Omega(d)}$.

- The best known result for learning a single neuron without bias using gradient descent for an input distribution that is not spherically symmetric, establishes convergence to the global minimum with probability close to $\frac{1}{2}$ over the random initialization [23, 5]. With our novel proof technique presented in Section 5 this result can be improved to probability at least $1 - e^{-\Omega(d)}$ (see Remark 5.7).

**Related work**

Although there are no previous works on learning a single neuron with an explicit bias term, there are many works that consider the problem of a single neuron under different settings and assumptions.

Several papers showed that the problem of learning a single neuron can be solved under minimal assumptions using algorithms which are not gradient-based (such as gradient descent or SGD). These algorithms include the Isotron proposed by [10] and the GLMtron proposed by [9]. The GLMtron algorithm is also

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\(^1\)I.e., gradient descent with infinitesimal step size.
analyzed in [3]. These algorithms allow learning a single neuron with bias. We note that these are non-
standard algorithms, whereas we focus on the standard gradient descent algorithm. An efficient algorithm
for learning a single neuron with error parameter $\epsilon = \Omega(1/\log(d))$ was also obtained in [6].

In [14] the authors study the empirical risk of the single neuron problem. However, their analysis does
not include the ReLU activation, or adding a bias term. A related analysis is also given in [15], where the
ReLU activation is not considered.

Several papers showed convergence guarantees for the single neuron problem with ReLU activation un-
der certain distributional assumptions, although none of these assumptions allows for a bias term. Notably,
[20, 18, 11, 1] showed convergence guarantees for gradient methods when the inputs have a standard Gauss-
ian distribution, without a bias term. [4] showed that under a certain subspace eigenvalue assumption a
single neuron can be learned with SGD, although this assumption does not allow adding a bias term. [23, 5]
use an assumption about the input distribution being sufficiently "spread" in all directions, which does not
allow for a bias term (since that requires an input distribution supported on $\{1\}$ in the last coordinate). [23]
showed a convergence result under the realizable setting, while [5] considered the agnostic and noisy set-
ings. In [19] convergence guarantees are given for the absolute value activation, and a specific distribution
which does not allow a bias term.

Less directly related, [21] studied the problem of implicit regularization in the single neuron setting.
In [22, 12] it is shown that approximating a single neuron using random features (or kernel methods) is
not tractable in high dimensions. We note that these results explicitly require that the single neuron which
is being approximated will have a bias term. Thus, our work complements these works by showing that
the problem of learning a single neuron with bias is also learnable using gradient descent (under certain
assumptions). Agnostically learning a single neuron with non gradient-based algorithms and hardness of a
agnostically learning a single neuron were studied in [3, 7, 8].

2 Preliminaries

Notations. We use bold-faced letters to denote vectors, e.g., $x = (x_1, \ldots, x_d)$. For $u \in \mathbb{R}^d$ we denote
by $\|u\|$ the Euclidean norm. We denote $\bar{u} = \frac{u}{\|u\|}$, namely, the unit vector in the direction of $u$. For
$1 \leq i \leq j \leq d$ we denote $u_{i:j} = (u_i, \ldots, u_j) \in \mathbb{R}^{j-i+1}$. We denote by $1(\cdot)$ the indicator function, for
example $1(t \geq 5)$ equals 1 if $t \geq 5$ and 0 otherwise. We denote by $U([-r, r])$ the uniform distribution over
the interval $[-r, r]$ in $\mathbb{R}$, and by $\mathcal{N}(0, \Sigma)$ the multivariate normal distribution with mean 0 and covariance
matrix $\Sigma$. Given two vectors $w, v$ we let $\theta(w, v) = \arccos \left( \frac{\langle w, v \rangle}{\|w\|\|v\|} \right) = \arccos(\langle \bar{w}, \bar{v} \rangle) \in [0, \pi]$. For a
vector $u \in \mathbb{R}^{d+1}$ we often denote by $\bar{u} \in \mathbb{R}^d$ the first $d$ components of $u$, and denote by $b_u \in \mathbb{R}$ its last
component.

Gradient methods. In this paper we focus on the following two standard gradient methods for optimizing
our objective $F(w)$ from Eq. (2):

- **Gradient descent:** We initialize at some $w_0 \in \mathbb{R}^{d+1}$, and set a fixed learning rate $\eta > 0$. At each iteration
  $t \geq 0$ we have: $w_{t+1} = w_t - \eta \nabla F(w_t)$.

- **Gradient Flow:** We initialize at some $w(0) \in \mathbb{R}^{d+1}$, and for every time $t \geq 0$, we set $w(t)$ to be the
  solution of the differential equation $\dot{w} = -\nabla F(w(t))$. This can be thought of as a continuous form of
  gradient descent, where the learning rate is infinitesimally small.
The gradient of the objective in Eq. (1) is:

\[ \nabla F(w) = \mathbb{E}_{x \sim D} \left[ \left( \sigma(w^\top x) - \sigma(v^\top x) \right) \cdot \sigma'(w^\top x) x \right]. \]  

(3)

Since \( \sigma \) is the ReLU function, it is differentiable everywhere except for 0. Practical implementations of gradient methods define \( \sigma'(0) \) to be some constant in \([0, 1]\). Following this convention, the gradient used by these methods still correspond to Eq. (3). We note that the exact value of \( \sigma'(0) \) has no effect on our results.

3 Negative results

In this section we demonstrate that adding bias to the problem of learning a single neuron with gradient descent can make the problem significantly harder.

First, on an intuitive level, previous results (e.g., [23, 5, 18, 4, 19]) considered assumptions on the input distribution, which require enough "spread" in all directions (for example, a strictly positive density in some neighborhood around the origin). Adding a bias term, even if the first \( d \) coordinates of the distribution satisfy a "spread" assumption, will give rise to a direction without "spread", since in this direction the distribution is concentrated on 1, hence the previous results do not apply.

Next, we show two negative results where the input distribution is uniform on a ball around the origin. We note that due to Theorem 6.4 from [23], we know that gradient flow on a single neuron without bias will converge to the global minimum with probability 1 over the random initialization. The only case where it will fail to converge is when \( w_0 \) is initialized in the exact direction \(-v\), which happens with probability 0 with standard random initializations.

3.1 Initialization in a flat region

If we initialize the bias term in the same manner as the other coordinates, then we can show that gradient descent will fail with probability close to half, even if the input distribution is uniform over a (certain) origin-centered ball:

**Theorem 3.1.** Suppose we initialize each coordinate of \( w_0 \) (including the bias) according to \( U([-1, 1]) \). Let \( \epsilon > 0 \) and let \( \tilde{D} \) be the uniform distribution supported on a ball around the origin in \( \mathbb{R}^d \) of radius \( \epsilon \). Then, \( w.p > 1/2 - \epsilon \sqrt{d} \), gradient descent on the objective in Eq. (2) satisfies \( w_t = w_0 \) for all \( t \) (namely, it gets stuck at its initial point \( w_0 \)).

Note that by Theorem 6.4 in [23], if there is no bias term in the objective, then gradient descent will converge to the global minimum w.p 1 using this random initialization scheme and this input distribution. The intuition for the proof is that with constant probability over the initialization, \( b_w \) is small enough so that \( \sigma(\tilde{w}^\top \tilde{x} + b_w) = 0 \) almost surely. If this happens, then the gradient will be 0 and gradient descent will never move. The full proof can be found in Appendix B.1. We note that Theorem 3.1 is applicable when \( \epsilon \) is sufficiently small, e.g. \( \epsilon \ll 1/\sqrt{d} \).

3.2 Targets with negative bias

Theorem 3.1 shows a difference between learning with and without the bias term. A main caveat of this example is the requirement that the bias is initialized in the same manner as the other parameters. Standard deep learning libraries (e.g. Pytorch [16]) often initialize the bias term to zero by default, while using random initialization schemes for the other parameters.
Alas, we now show that even if we initialize the bias term to be exactly zero, and the input distribution is uniform over an arbitrary origin-centered ball, we might fail to converge to the global minimum for certain target neurons:

**Theorem 3.2.** Let \( \tilde{D} \) be the uniform distribution on \( B = \{ \tilde{x} \in \mathbb{R}^d : \|\tilde{x}\| \leq r \} \) for some \( r > 0 \). Let \( v \in \mathbb{R}^{d+1} \) such that \( \tilde{v} = (1, 0, \ldots, 0)^\top \) and \( b_v = -(r - \frac{r}{2\rho}) \). Let \( w_0 \in \mathbb{R}^{d+1} \) such that \( b_{w_0} = 0 \) and \( \tilde{w}_0 \) is drawn from the uniform distribution on a sphere of radius \( \rho > 0 \). Then, with probability at least \( \frac{1}{2} - o_d(1) \) over the choice of \( w_0 \), gradient flow does not converge to the global minimum.

We prove the theorem in Appendix B.2. The intuition behind the proof is the following: The target neuron has a large negative bias, so that only a small (but positive) measure of input points are labelled as non-zero. By randomly initializing \( w \), with probability close to \( \frac{1}{2} \) there are no inputs that both \( v \) and \( w \) label positively. Since the gradient is affected only by inputs that \( v \) labels positively, then during the optimization process the gradient will be independent of the direction of \( v \), and \( w \) will not converge to the global minimum.

**Remark 3.3.** Theorem 3.2 shows that gradient flow is not guaranteed to converge to a global minimum when \( b_v \) is negative, instead it converges to a local minimum with a loss of \( F(0) \). However, the loss \( F(0) \) is determined by the input distribution. Take \( v = (1, 0, \ldots, 0, -(r - \frac{r}{2\rho}))^\top \) considered in the theorem. On one hand, for a uniform distribution on a ball of radius \( r \) as in the theorem we have:

\[
F(0) = \frac{1}{2} \mathbb{E}_x \left[ \left( \sigma(v^\top x) \right)^2 \right] = \frac{1}{2} \mathbb{E}_x \left[ 1(v^\top x \geq 0) \left(v^\top x\right)^2 \right]
\]

\[
= \frac{1}{2} \mathbb{E}_x \left[ 1 \left(x_1 \geq r - \frac{r}{2d^2}\right) \left(x_1 - \left(r - \frac{r}{2d^2}\right)\right)^2 \right]
\]

\[
\leq \frac{1}{2} \frac{r^2}{4d^2} \mathbb{P}_x \left(x_1 \geq r \left(1 - \frac{1}{2d^2}\right)\right) \leq r^2 e^{-\Omega(d)}.
\]

Thus, for any reasonable \( r \), a local minimum with loss \( F(0) \) is almost as good as the global minimum. On the other hand, take a distribution \( \tilde{D} \) with a support bounded in a ball of radius \( r \), such that half of its mass is uniformly distributed in \( A := \{ \tilde{x} \in \mathbb{R}^d : x_1 > r - \frac{r}{2d^2} \} \), and the other half is uniformly distributed in \( B \setminus A \). In this case, it is not hard to see that the same proof as in Theorem 3.2 works, and gradient flow will converge to a local minimum with loss \( F(0) = \Omega \left( \frac{r}{d^2} \right) \), which is arbitrarily large if \( r \) is large enough.

Although in the example given in Theorem 3.2 the objective at \( w = 0 \) is almost as good as the objective at \( w = v \), we emphasize that w.p almost \( \frac{1}{2} \) gradient flow cannot reach the global minimum even asymptotically. On the other hand, in the bias-less case by Theorem 6.4 in [23] gradient flow on the same input distribution will reach the global minimum w.p 1. Also note that the scale of the initialization of \( w_0 \) has no effect on the result.

### 4 Characterization of the critical points

In the previous section we have shown two examples where gradient methods on the problem of a single neuron with bias will either get stuck in a flat region, or converge to a local minimum. In this section we delve deeper into the examples presented in the previous section, and give a full characterization of the critical points of the objective. We will use the following assumption on the input distribution:
**Assumption 4.1.** The distribution \( \tilde{D} \) on \( \mathbb{R}^d \) has a density function \( p(\tilde{x}) \), and there are \( \beta, c > 0 \), such that \( \tilde{D} \) is supported on \( \{ \tilde{x} : \|\tilde{x}\| \leq c \} \), and for every \( \tilde{x} \) in the support we have \( p(\tilde{x}) \geq \beta \).

The assumption essentially states that the distribution over the first \( d \) coordinates (without the bias term) has enough ”spread” in all directions, and covers standard distributions such as uniform over a ball of radius \( c \). Other similar assumptions are made in previous works (e.g. \cite{23, 5}). We note that in \cite{23} it is shown that without any assumption on the distribution, it is impossible to ensure convergence, hence we must have some kind of assumption for this problem to be learnable with gradient methods. Under this assumption we can characterize the critical points of the objective.

**Theorem 4.2.** Consider the objective in Eq. (2) with \( v \neq 0 \), and assume that the distribution \( \tilde{D} \) on the first \( d \) coordinates satisfies Assumption 4.1. Then \( w \neq 0 \) is a critical point of \( F \) (i.e., is a root of Eq. (3)) if and only if it satisfies one of the following:

- \( w = v \), in which case \( w \) is a global minimum.
- \( w = (\tilde{w}, b_w) \) where \( \tilde{w} = 0 \) and \( b_w < 0 \).
- \( \tilde{w} \neq 0 \) and \( -b_w / \|\tilde{w}\| \geq c \).

In the latter two cases, \( F(w) = F(0) \). Hence, if \( 0 \) is not a global minimum, then \( w \) is not a global minimum.

We note that \( F(0) = \frac{1}{2} \mathbb{E}_x[\sigma(v^\top x)^2] \), so \( 0 \) is a global minimum only if the target neuron returns \( 0 \) with probability 1.

**Remark 4.3** (The case \( w = 0 \)). We intentionally avoided characterizing the point \( w = 0 \), since the objective is not differentiable there (this is the only point of non-differentiability), and the gradient there is determined by the value of the ReLU activation at \( 0 \). For \( \sigma'(0) = 0 \) the gradient at \( w = 0 \) is zero, and this is a non-differentiable saddle point. For \( \sigma'(0) = 1 \) (or any other positive value), the gradient at \( w = 0 \) is non-zero, and it will point at a direction which depends on the distribution. We note that in \cite{18} the authors define \( \sigma'(0) = 1 \), and use a symmetric distribution, in which case the gradient at \( w = 0 \) points exactly at the direction of the target \( v \). This is a crucial part of their convergence analysis.

We emphasize that with a bias term, there is a non-zero measure manifold of critical points (corresponding to the third bullet in the theorem). On the other hand, without a bias term the only critical point besides the global minimum (under mild assumptions on the input distribution) is at the origin \( w = 0 \) (cf. \cite{23}). The full proof is in Appendix C.

The assumption on the support of \( \tilde{D} \) is made for simplicity. It can be relaxed to having a distribution with exponentially bounded tail, e.g. standard Gaussian. In this case, some of the critical points will instead have a non-zero gradient which is exponentially small. We emphasize that when running optimization algorithms on finite-precision machines, which are used in practice, these ”almost” critical points behave essentially like critical points since the gradient is extremely small.

Revisiting the negative examples from Section 3, the first example (Theorem 3.1) shows that if we do not initialize the bias of \( w \) to zero, then there is a positive probability to initialize at a critical point which is not the global minimum. The second example (Theorem 3.2) shows that even if we initialize the bias of \( w \) to be zero, there is still a positive probability to converge to a critical point which is not the global minimum. Hence, in order to guarantee convergence we need to have more assumptions on either the input distribution, the target \( v \) or the initialization. In the next section, we show that adding such assumptions are indeed sufficient to get positive convergence guarantees.
5 Convergence for initialization with loss slightly better than trivial

In this section, we show that under some assumptions on the input distribution, if gradient descent is initialized such that $F(w_0) < F(0)$ then it is guaranteed to converge to the global minimum. In Subsection 5.2, we study under what conditions this is likely to occur with standard random initialization.

5.1 Convergence if $F(w_0) < F(0)$

To state our results, we need the following assumption:

Assumption 5.1.
1. The distribution $D$ is supported on $\{x \in \mathbb{R}^{d+1} : \|x\| \leq c\}$ for some $c \geq 1$.
2. The distribution $\tilde{D}$ over the first $d$ coordinates is bounded in all directions: there is $c' > 0$ such that for every $\tilde{u}$ with $\|\tilde{u}\| = 1$ and every $a \in \mathbb{R}$ and $b \geq 0$, we have $\Pr_{x \sim \tilde{D}}[\tilde{u}^\top x \in [a,a+b]] \leq b \cdot c'$.
3. We assume w.l.o.g. that $\|v\| = 1$ and $c' \geq 1$.

Assumption (3) helps simplifying some expressions in our convergence result, and is not necessary. Assumption (2) requires that the distribution is not too concentrated in a short interval. For example, if $\tilde{D}$ is spherically symmetric then the marginal density of the first (or any other) coordinate is bounded by $c'$. Note that we do not assume that $\tilde{D}$ is spherically symmetric.

Theorem 5.2. Under Assumption 5.1 we have the following. Let $\delta > 0$ and let $w_0 \in \mathbb{R}^{d+1}$ such that $F(w_0) \leq F(0) - \delta$. Let $\gamma = \frac{\delta^3}{8}\cdot\frac{1}{\|w_0\|^2\cdot\|w_0\|^2}$. Assume that gradient descent runs starting from $w_0$ with step size $\eta \leq \frac{\gamma}{\delta}$. Then, for every $t$ we have

$$\|w_t - v\|^2 \leq \|w_0 - v\|^2 (1 - \gamma\eta)^t.$$

The formal proof appears in Appendix D, but we provide the main ideas below. First, note that

$$\|w_{t+1} - v\|^2 = \|w_t - \eta\nabla F(w_t) - v\|^2 = \|w_t - v\|^2 - 2\eta \langle \nabla F(w_t), w_t - v \rangle + \eta^2 \|\nabla F(w_t)\|^2.$$

Hence, in order to show that $\|w_{t+1} - v\|^2 \leq \|w_t - v\|^2 (1 - \gamma\eta)$ we need to obtain an upper bound for $\|\nabla F(w_t)\|$ and a lower bound for $\langle \nabla F(w_t), w_t - v \rangle$. Achieving the lower bound for $\langle \nabla F(w_t), w_t - v \rangle$ is the challenging part, and we show that in order to establish such a bound it suffices to obtain a lower bound for $\Pr_x [w_t^\top x \geq 0, v^\top x \geq 0]$. We prove that if $F(w_t) \leq F(0) - \delta$ then $\Pr_x[w_t^\top x \geq 0, v^\top x \geq 0] \geq \frac{\delta}{c^2\|w_t\|}$. Hence, if $F(w_t)$ remains at most $F(0) - \delta$ for every $t$, then a lower bound for $\langle \nabla F(w_t), w_t - v \rangle$ can be achieved, which completes the proof. However, it is not obvious that $F(w_t)$ remains at most $F(0) - \delta$ throughout the training process. When running gradient descent on a smooth loss function we can choose a sufficiently small step size such that the loss decreases in each step, but here the function $F(w)$ is highly non-smooth around $w = 0$. That is, the Lipschitz constant of $\nabla F(w)$ is unbounded. We show that if $F(w_t) \leq F(0) - \delta$ then $w_t$ is sufficiently far from $0$, and hence the smoothness of $F$ around $w_t$ can be bounded, which allows us to choose a small step size that ensures that $F(w_{t+1}) \leq F(w_t) \leq F(0) - \delta$. Hence, it follows that $F(w_t)$ remains at most $F(0) - \delta$ for every $t$.

As an aside, recall that in Section 4 we showed that other than $w = v$ all critical points of $F(w)$ are in a flat region where $F(w) = F(0)$. Hence, the fact that $F(w_t)$ remains at most $F(0) - \delta$ for every $t$ implies...
that \(w_t\) does not reach the region of bad critical points, which explains the asymptotic convergence to the global minimum.

We also note that although we assume that the distribution has a bounded support, this assumption is mainly made for simplicity, and can be relaxed to have sub-Gaussian distributions with bounded moments. These distributions include, e.g. Gaussian distributions.

### 5.2 Convergence for Random Initialization

In Theorem 5.2 we showed that if \(F(w_0) < F(0)\) then gradient descent converges to the global minimum. We now show that under mild assumptions on the input distribution, a random initialization of \(F\) is such that gradient descent converges to the global minimum. For example, for \(\alpha = 2\), let \(\bar{w} = 0\) and \(\bar{v} = \frac{3\pi}{4}\). Suppose that \(-\frac{b_{\bar{w}}}{\|\bar{v}\|} \leq \alpha \cdot \frac{\sin\left(\frac{x}{2}\right)}{4}\). Then, \(F(w) \leq F(0) + \|w\|^2 \cdot \frac{c^2}{2} - \|w\| \cdot M < F(0)\).

We prove the theorem in Appendix D.1. The main idea is that since

\[
F(w) = \mathbb{E}_x \left[ \frac{1}{2} \left( \sigma(w^\top x) - \sigma(v^\top x) \right)^2 \right]
\]

\[
= F(0) + \frac{1}{2} \mathbb{E}_x \left[ (\sigma(w^\top x))^2 \right] - \mathbb{E}_x \left[ \sigma(w^\top x) \sigma(v^\top x) \right]
\]

\[
\leq F(0) + \|w\|^2 \cdot \frac{c^2}{2} - \|w\| \cdot \mathbb{E}_x \left[ \sigma(w^\top x) \sigma(v^\top x) \right],
\]

then it suffices to obtain a lower bound for \(\mathbb{E}_x \left[ \sigma(w^\top x) \sigma(v^\top x) \right]\). In the proof we show that such a bound can be achieved if the conditions of the theorem hold.

Suppose that \(w_0\) is such that \(\tilde{w}_0\) is drawn from a spherically symmetric distribution and \(b_{\tilde{w}_0} = 0\). By standard concentration of measure arguments, it holds w.p. at least \(1 - e^{\Omega(d)}\) that \(\theta(\tilde{w}_0, \bar{v}) \leq \frac{3\pi}{4}\) (where the notation \(\Omega(d)\) hides only numerical constants, namely, it does not depend on other parameters of the problem). Therefore, if \(\tilde{w}_0\) is drawn from the uniform distribution on a sphere of radius \(\rho < \frac{2M}{c^2}\), then the theorem implies that w.h.p. we have \(F(w_0) < F(0)\). For such initialization Theorem 5.2 implies that gradient descent converges to the global minimum. For example, for \(\rho = \frac{M}{c^2}\) we have w.h.p. that \(F(w_0) \leq F(0) + \frac{\delta^2 c^2}{2} - \rho M = F(0) - \frac{M^2}{2c^2}\), and thus Theorem 5.2 applies with \(\delta = \frac{M^2}{2c^2}\). Thus, we have the following corollary:

**Corollary 5.5.** Under Assumption 5.1 and Assumption 5.3 we have the following. Let \(M = \frac{\alpha^4 \delta \sin^3\left(\frac{x}{2}\right)}{256c}\), let \(\rho = \frac{M}{c^2}\), let \(\delta = \frac{M^2}{2c^2}\), and let \(\gamma = \frac{\delta^3}{312^2(\rho + 2)^2 c^2 e^c}\). Suppose that \(-\frac{b_{\tilde{w}}}{\|\bar{v}\|} \leq \alpha \cdot \frac{\sin\left(\frac{x}{2}\right)}{4}\), and \(w_0\) is such that \(b_{\tilde{w}_0} = 0\) and \(\tilde{w}_0\) is drawn from the uniform distribution on a sphere of radius \(\rho\). Consider gradient descent
with step size $\eta \leq \frac{\gamma}{4}$. Then, with probability at least $1 - e^{\Omega(d)}$ over the choice of $w_0$ we have for every $t$:

$$\|w_t - v\|^2 \leq \|w_0 - v\|^2 (1 - \gamma \eta)^t.$$  

We note that a similar result holds also if $\tilde{w}_0$ is drawn from a normal distribution $\mathcal{N}(0, \frac{\rho^2}{d} I)$.

**Remark 5.6** (The assumption on $b_v$). The assumption $-\frac{b_v}{\|v\|} \leq \alpha \cdot \sin\left(\frac{\pi}{4}\right)$ implies that the bias term $b_v$ may be either positive or negative, but in case it is negative then it cannot be too large. This assumption is indeed crucial for the proof, but for "well-behaved" distributions, if this assumption is not satisfied (for a large enough $\alpha$), then the loss at $F(0)$ is already good enough. For example, for a standard Gaussian distribution and for every $\epsilon > 0$, we can choose $\alpha$ large enough such that for any bias term (positive or negative) we either: (1) converge to the global minimum with a loss of zero, or; (2) converge to a local minimum with a loss of $F(0)$, which is smaller than $\epsilon$. Moreover, we can show that by choosing $\alpha$ appropriately, and using the example in Theorem 3.2, if $-\frac{b_v}{\|v\|} \geq 2\alpha$ then gradient flow will converge to a non-global minimum with loss of $F(0)$. This means that our bound on $\alpha$ is tight up to a constant factor. For a further discussion on the assumption on $b_v$, and how to choose $\alpha$ see Appendix E.

Previous papers have shown separation between random features (or kernel) methods and neural networks in terms of their approximation power (see [22, 12], and the discussion in [13]). These works show that under a standard Gaussian distribution, random features cannot even approximate a single ReLU neuron, unless the number of features is exponential in the input dimension. That analysis crucially relies on the single neuron having a non-zero bias term. In this work we complete the picture by showing that gradient descent can indeed find a near-optimal neuron with non-zero bias. Thus, we see there is indeed essentially a separation between what can be learned using random features and using gradient descent over neural networks.

**Remark 5.7** (Learning a neuron without bias). [23] studied the problem of learning a single ReLU neuron without bias using gradient descent on a single neuron without bias. For input distributions that are not spherically symmetric they showed that gradient descent with random initialization near zero converges to the global minimum w.p. at least $\frac{1}{2} - o_d(1)$. Their result is also under Assumption 5.3. An immediate corollary from the discussion above is that if we learn a single neuron without bias using gradient descent with random initialization on a single neuron with bias, then the algorithm converges to the global minimum w.p. at least $1 - e^{\Omega(d)}$. Moreover, our proof technique can be easily adapted to the setting of learning a single neuron without bias using gradient descent on a single neuron without bias, namely, the setting studied in [23]. It can be shown that in this setting gradient descent converges w.h.p to the global minimum. Thus, our technique allows us to improve the result of [23] from probability $\frac{1}{2} - o_d(1)$ to probability $1 - e^{\Omega(d)}$.

### 6 Convergence for spread and symmetric distributions

In this section we show that under a certain set of assumptions, different from the assumptions in Section 5, it is possible to show linear convergence of gradient descent to the global minimum. The assumptions we make for this theorem are as follows:

**Assumption 6.1.**

1. The target vector $v$ satisfies that $b_v \geq 0$ and $\|\tilde{v}\| = 1$.

2. The distribution $\tilde{D}$ over the first $d$ coordinates is spherically symmetric.
3. Assumption 5.3 holds, and denoting by $\tau := \frac{E_{\mathbf{x} \sim \bar{D}}[\|\mathbf{x}_1 \mathbf{x}_2\|]}{E_{\mathbf{x} \sim \bar{D}}[\|\mathbf{x}_1\|^2]}$, then $\alpha \geq 2.5\sqrt{2} \cdot \max \left\{ 1, \frac{1}{\sqrt{\tau}} \right\}$ where $\alpha$ is from Assumption 5.3.

4. Denote by $c := E_{\mathbf{x} \sim \bar{D}}[\|\mathbf{x}\|^4]$, then $c < \infty$.

We note that item (3) considers $x_1, x_2$, but due to the assumption on the symmetry of the distribution (item (2)), the assumption in item (3) holds for every $x_i, x_j$. Under these assumptions, we prove the following theorem:

**Theorem 6.2.** Assume we initialize $w_0$ such that $\|w_0 - \nu\|^2 < 1$, $b_{w_0} \geq 0$ and that Assumption 6.1 holds. Then, there is a universal constant $C$, such that using gradient descent on $F(w)$ with step size $\eta < C \cdot \frac{\beta}{\alpha^2} \min\{1, \tau\}$ yields that for every $t$ we have $\|w_t - \nu\|^2 \leq (1 - \eta\lambda)^t \|w_0 - \nu\|^2$, for $\lambda = C \cdot \frac{\beta}{\alpha^2}$.

This result has several advantages and disadvantages compared to those of the previous section. The main disadvantage is that the assumptions are generally more stringent: We focus only on positive target biases ($b_{\nu} \geq 0$) and spherically symmetric distributions $\bar{D}$. Also we require a certain technical assumption on the the distribution, as specified by $\tau$, which are satisfied for standard spherically symmetric distributions, but is a bit non-trivial\(^2\). Finally, the assumption on the initialization ($\|w_0 - \nu\|^2 < 1$ and $b_{w_0} \geq 0$) is much more restrictive (although see Remark 6.3 below). In contrast, the initialization assumption in the previous section holds with probability close to 1 with random initialization. On the positive side, the convergence rate does not depend on the initialization, i.e., here by initializing with any $w_0$ such that $\|w_0 - \nu\|^2 < 1$ and $b_{w_0} \geq 0$, we get a convergence rate that only depends on the input distribution. On the other hand, in Theorem 5.2, the convergence rate depends on the parameter $\delta$ which depends on the initialization. Also, the distribution is not necessarily bounded – we only require its fourth moment to be bounded.

**Remark 6.3** (Random initialization). For $b_{\nu} = 0$ the initialization assumption ($\|w_0 - \nu\|^2 < 1$) is satisfied with probability close to $1/2$ with standard initializations, see Lemma 5.1 from [23]). For $b_{\nu} > 0$, a similar argument applies if $b_w$ is initialized close enough to $b_{\nu}$.

The proof of the theorem is quite different from the proofs in Section 5, and is more geometrical in nature, extending previously used techniques from [23, 5]. It contains two major parts: The first part is an extension of the methods from [23] to the case of adding a bias term. Specifically, we show a lower bound on $\langle \nabla F(w), w - \nu \rangle$, which depends on both the angle between $w$ and $\nu$, and the bias terms $b_w$ and $b_{\nu}$ (see Theorem A.2). This result implies that for suitable values of $w$, gradient descent will decrease the distance from $\nu$. The second part of the proof is showing that throughout the optimization process, $w$ will stay in an area where we can apply the result above. Specifically, the intricate part is showing that the term $-\frac{b_w}{\|w\|^2}$ does not get too large. Note that due to Theorem 4.2, we know that keeping this term small means that $w$ stays away from the cone of bad critical points which are not the global minimum. The full proof can be found in Appendix F.

### 7 Discussion

In this work we studied the problem of learning a single neuron with a bias term using gradient descent. We showed several negative results, indicating that adding a bias term makes the problem more difficult than without a bias term. Next, we gave a characterization of the critical points of the problem under

\(^2\)For example, for standard Gaussian distribution, we have that $\tau = \frac{2}{3} \approx 0.63$, hence we can take $\alpha = 4.5$, and $\beta = O(1)$. Since the distribution $\bar{D}$ is symmetric, we present the assumption w.l.o.g with respect to the first 2 coordinates.
some assumptions on the input distribution, showing that there is a manifold of critical points which are not the global minimum. We proved two convergence results using different techniques and under different assumptions. Finally, we showed that under mild assumptions on the input distribution, reaching the global minimum can be achieved by standard random initialization.

We emphasize that previous works studying the problem of a single neuron either considered non-standard algorithms (e.g. Isotron), or required assumptions on the input distribution which do not allow a bias term. Hence, this is the first work we are aware of which gives positive and negative results on the problem of learning a single neuron with a bias term using gradient methods.

In this work we focused on the gradient descent algorithm. We believe that our results can also be extended to the commonly used SGD algorithm, using similar techniques to [23, 17], and leave it for future work. Another interesting future direction is analyzing other previously studied settings, but with the addition of a bias term. These settings can include convolutional networks, two layers neural networks, and agnostic learning of a single neuron.

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Appendices

A Auxiliary Results

In this appendix we extend several key results from [23] for the case of adding a bias term. Specifically, we extend Theorem 4.2 from [23] which shows that under mild assumptions on the distribution, the gradient of the loss points in a good direction which depends on the angle between the learned vector $w$ and the target $v$. We also bound the volume of a certain set in $\mathbb{R}^2$, which can be seen as an extension of Lemma B.1 from [23].

Lemma A.1. Let $P = \{y \in \mathbb{R}^2 : w^\top y > b, v^\top y > b, \|y\| \leq \alpha\}$ for $b \in \mathbb{R}$ and $w, v \in \mathbb{R}^2$ with $\|w\|, \|v\| = 1$ and $\theta(w, v) \leq \pi - \delta$ for $\delta \in [0, \pi]$. If $b < \alpha \sin\left(\frac{\delta}{2}\right)$ then $\text{Vol}(P) \geq \frac{(\alpha \sin\left(\frac{\delta}{2}\right) - b)^2}{4 \sin\left(\frac{\delta}{2}\right)}$.

Proof. The volume of $P$ is smallest when the angle is exactly $\pi - \delta$, thus we can lower bound the volume by assuming that $\theta(w, v) = \pi - \delta$. Next, we can rotate to coordinates to consider without loss of generality the volume of the set

$$P' = \{(y_1, y_2) \in \mathbb{R}^2 : \theta((y_1, y_2 - b'), e_2) \leq \delta/2, \|(y_1, y_2)\| \leq \alpha\},$$

where $b' = \frac{b}{\sin(\delta/2)}$ and $e_2 = (0, 1)$. Let $P'' = \{(x, y) \in \mathbb{R}^2 : x^2 + (y - b')^2 \leq (\alpha - b')^2\}$ be the disc of radius $\alpha - b'$ around the point $(0, b')$. It is enough to bound the volume of $P' \cap P''$. We define the rectangular sets:

$$P_1 = \left[\frac{\alpha - b'}{2} \sin\left(\frac{\delta}{4}\right), (\alpha - b') \sin\left(\frac{\delta}{4}\right)\right] \times \left[\frac{b'}{2} + \frac{\alpha - b'}{2} \cos\left(\frac{\delta}{4}\right), b' + (\alpha - b') \cos\left(\frac{\delta}{4}\right)\right]$$

$$P_2 = \left[-(\alpha - b') \sin\left(\frac{\delta}{4}\right), -\frac{\alpha - b'}{2} \sin\left(\frac{\delta}{4}\right)\right] \times \left[\frac{b'}{2} + \frac{\alpha - b'}{2} \cos\left(\frac{\delta}{4}\right), b' + (\alpha - b') \cos\left(\frac{\delta}{4}\right)\right]$$

See Figure 1 for an illustration. We have that $P_1, P_2 \subseteq P' \cap P''$. We will show it for $P_1$, the same argument also works for $P_2$. First, $P_1 \subseteq P''$ is immediate by the definition of the two sets. For $P'$, the straight line in the boundary of $P'$ is defined by $y_2 = b' + y_1 \frac{\cos\left(\frac{\delta}{2}\right)}{\sin\left(\frac{\delta}{2}\right)}$. It can be seen that each vertex of the rectangle $P_1$, is above this line. Moreover, the norm of each vertex of $P_1$ is at most $\alpha$. Hence all the vertices are inside $P'$, which means that $P_1 \subseteq P'$. In total we get:

$$\text{Vol}(P) \geq \text{Vol}(P' \cap P'') \geq \text{Vol}(P_1 \cup P_2)$$

$$= \frac{(\alpha - b')^2}{2} \sin\left(\frac{\delta}{4}\right) \cos\left(\frac{\delta}{4}\right)$$

$$= \frac{(\alpha \sin\left(\frac{\delta}{2}\right) - b)^2}{4 \sin\left(\frac{\delta}{2}\right)}$$

$\square$

Theorem A.2. Let $w, v \in \mathbb{R}^{d+1}$, denote by $\bar{w}, \bar{v}$ their first $d$ coordinates and by $b_w, b_v$ their last coordinate. Assume that $\theta(\bar{w}, \bar{v}) \leq \pi - \delta$ for some $\delta \in [0, \pi)$, and that the distribution $\mathcal{D}$ is such that its first $d$
coordinates satisfy Assumption 4.1 (1) from [23], and that its last coordinate is a constant 1. Denote 
\[ b' = \max\{-b_w/\|\tilde{w}\|, -b_v/\|\tilde{v}\|, 0\} \cdot \frac{1}{\sin(\frac{\delta}{2})} \], and assume that \( b' < \alpha \), then:

\[
\langle \nabla F(w), w - v \rangle \geq \frac{(\alpha - b')^4 \sin(\frac{\delta}{2})^3}{8} \cdot \min\left\{1, \frac{1}{4\alpha^2}\right\} \|w - v\|^2
\]

Proof. Let \( \tilde{x} \) be the first \( d \) coordinates of \( x \). We have that:

\[
\langle \nabla F(w), w - v \rangle = E_{x \sim D} \left[ \sigma'(w^\top x)(\sigma(w^\top x) - \sigma(v^\top x))(w^\top x - v^\top x) \right]
\]

\[
\geq E_{x \sim D} \left[ \mathbb{1}(w^\top x > 0, v^\top x > 0)(w^\top x - v^\top x)^2 \right]
\]

\[
= \|w - v\|^2 \cdot E_{x \sim D} \left[ \mathbb{1}(\tilde{w}^\top \tilde{x} > -b_w, \tilde{v}^\top \tilde{x} > -b_v)((w - v)^\top x)^2 \right]
\]

\[
\geq \|w - v\|^2 \cdot \inf_{u \in \text{span}\{w, v\}, \|u\| = 1} E_{x \sim D} \left[ \mathbb{1}(\tilde{u}^\top \tilde{x} > -b_w, \tilde{v}^\top \tilde{x} > -b_v)(u^\top x)^2 \right]
\]

Let \( b = \max\{-b_w/\|\tilde{w}\|, -b_v/\|\tilde{v}\|, 0\} \), then we can bound the above equation by:

\[
\|w - v\|^2 \cdot \inf_{u \in \text{span}\{w, v\}, \|u\| = 1} E_{x \sim D} \left[ \mathbb{1}(\tilde{w}^\top \tilde{x} > b, \tilde{v}^\top \tilde{x} > b)(u^\top x)^2 \right]
\]

\[
\geq \|w - v\|^2 \cdot \inf_{u \in \text{span}\{w, v\}, \|u\| = 1} E_{\tilde{x} \sim \tilde{D}} \left[ \mathbb{1}(\tilde{w}^\top \tilde{x} > b, \tilde{v}^\top \tilde{x} > b, \|\tilde{x}\| \leq \alpha)(\tilde{u}^\top \tilde{x} + b_u)^2 \right]
\]

(4)
Here $b_u$ is the bias term of $u$, $\tilde{u}$ are the first $d$ coordinates of $u$ and $\tilde{D}$ is the marginal distribution of $x$ on its first $d$ coordinates. Note that since the last coordinate represents the bias term, then the distribution on the last coordinate of $x$ is a constant 1. The condition that $\|u\| = 1$ (equivalently $\|u\|^2 = 1$) translates to $\|\tilde{u}\|^2 + b_u^2 = 1$.

Our goal is to bound the term inside the infimum. Note that the expression inside the distribution depends just on inner products of $\tilde{x}$ with $\tilde{w}$ or $\tilde{v}$, hence we can consider the marginal distribution $\tilde{D}_{\tilde{w},\tilde{v}}$ of $\tilde{x}$ on the 2-dimensional subspace spanned by $\tilde{w}$ and $\tilde{v}$ (with density function $p_{\tilde{w},\tilde{v}}$). Let $\tilde{w}$ and $\tilde{v}$ be the projections of $\tilde{w}$ and $\tilde{v}$ on that subspace. Let $P = \{ y \in \mathbb{R}^2 : \tilde{w}^Ty > b, \tilde{v}^Ty > b, \|y\| \leq \alpha \}$, then we can bound Eq. (4) with:

$$\inf_{u \in \mathbb{R}^2, b_u \in \mathbb{R} : \|u\|^2 + b_u^2 = 1} \int_{y \in P} (u^T y + b_u)^2 dy \geq \beta \|w - v\|^2 \inf_{u \in \mathbb{R}^2, b_u \in \mathbb{R} : \|u\|^2 + b_u^2 = 1} \int_{y \in P} (u^T y + b_u)^2 dy$$

Combining with Proposition A.3 finishes the proof

**Proposition A.3.** Let $P = \{ y \in \mathbb{R}^2 : \tilde{w}^Ty > b, \tilde{v}^Ty > b, \|y\| \leq \alpha \}$ for $b \in \mathbb{R}$ and $\tilde{w}, \tilde{v} \in \mathbb{R}^2$ with $\theta(\tilde{w}, \tilde{v}) \leq \pi - \delta$ for $\delta \in [0, \pi]$. Then

$$\inf_{u \in \mathbb{R}^2, b_u \in \mathbb{R} : \|u\|^2 + b_u^2 = 1} \int_{y \in P} (u^T y + b_u)^2 dy \geq \frac{(\alpha - b')^4 \sin^2 \left(\frac{\delta}{4}\right)}{8^4} \cdot \min \left\{ 1, \frac{1}{\alpha^2} \right\}$$

for $b' = \frac{b}{\sin(\delta/2)}$.

**Proof.** As in the proof of Lemma A.1, we consider the rectangular sets:

$$P_1 = \left[ \frac{(\alpha - b')}{2} \sin \left(\frac{\delta}{4}\right), (\alpha - b') \sin \left(\frac{\delta}{4}\right) \right] \times \left[ b' + \frac{(\alpha - b')}{2} \cos \left(\frac{\delta}{4}\right), b' + (\alpha - b') \cos \left(\frac{\delta}{4}\right) \right]$$

$$P_2 = \left[ - (\alpha - b') \sin \left(\frac{\delta}{4}\right), - (\alpha - b') \sin \left(\frac{\delta}{4}\right) \right] \times \left[ b' + \frac{(\alpha - b')}{2} \cos \left(\frac{\delta}{4}\right), b' + (\alpha - b') \cos \left(\frac{\delta}{4}\right) \right]$$

with $b' = \frac{b}{\sin(\delta/2)}$. Since we have $P_1 \cup P_2 \subseteq P$, and the function inside the integral is positive, we can lower bound the target integral by integrating only over $P_1 \cup P_2$. Now we have:

$$\inf_{u \in \mathbb{R}^2, b_u \in \mathbb{R} : \|u\|^2 + b_u^2 = 1} \int_{y \in P} (u^T y + b_u)^2 dy$$

$$\geq \inf_{u_1, u_2, b_u \in \mathbb{R} : u_1^2 + u_2^2 + b_u^2 = 1} \int_{y \in P_1 \cup P_2} (u_1 y_1 + u_2 y_2 + b_u)^2 dy$$

$$= \inf_{u_1, u_2, b_u \in \mathbb{R} : u_1^2 + u_2^2 + b_u^2 = 1} \int_{y \in P_1 \cup P_2} (u_1 y_1)^2 dy + \int_{y \in P_1 \cup P_2} (u_2 y_2 + b_u)^2 dy + \int_{y \in P_1 \cup P_2} 2 u_1 y_1 (u_2 y_2 + b_u) dy$$

$$= \inf_{u_1, u_2, b_u \in \mathbb{R} : u_1^2 + u_2^2 + b_u^2 = 1} \int_{y \in P_1 \cup P_2} (u_1 y_1)^2 dy + \int_{y \in P_1 \cup P_2} (u_2 y_2 + b_u)^2 dy$$
where in the last equality we used that $P_1 \cup P_2$ are symmetric around the $y_2$ axis, i.e. $(y_1, y_2) \in P_1 \cup P_2$ iff $(-y_1, y_2) \in P_1 \cup P_2$. By the condition that $u_1^2 + u_2^2 + b_a^2 = 1$ we know that either $u_1^2 \geq \frac{1}{2} \text{ or } u_2^2 + b_a^2 \geq \frac{1}{2}$. Using that both integrals above are positive, we can lower bound:

$$\inf_{u_1, u_2, b_a \in \mathbb{R} : u_1^2 + u_2^2 + b_a^2 = 1} \int_{y \in P_1 \cup P_2} (u_1 y_1)^2 dy + \int_{y \in P_1 \cup P_2} (u_2 y_2 + b_a)^2 dy \geq \min \left\{ \frac{1}{2} \int_{y \in P_1 \cup P_2} y_1^2 dy, \inf_{u_2, u_3 \in \mathbb{R} : u_2^2 + u_3^2 = \frac{1}{2}} \int_{y \in P_1 \cup P_2} (u_2 y_2 + u_3)^2 dy \right\}.$$ 

We will now lower bound both terms in the above equation. For the first term, note that for every $y \in P_1 \cup P_2$ we have that $|y_1| \geq \frac{(\alpha - b^2)}{2} \sin \left( \frac{\delta}{4} \right)$. Hence we have:

$$\frac{1}{2} \int_{y \in P_1 \cup P_2} y_1^2 dy \geq \frac{1}{2} \int_{y \in P_1 \cup P_2} (\alpha - b^2)^2 \sin \left( \frac{\delta}{4} \right)^2 dy \geq \frac{(\alpha - b^2)^2}{8} \sin \left( \frac{\delta}{4} \right)^2 \sin \left( \frac{\delta}{4} \right) \cos \left( \frac{\delta}{4} \right) \geq \frac{(\alpha - b^2)^4}{16 \sqrt{2} \sin \left( \frac{\delta}{4} \right)}.$$ 

(5)

where in the last inequality we used that $\delta \in [0, \pi]$, hence $\delta/4 \in [0, \pi/4]$.

For the second term we have:

$$\inf_{u_2, u_3 \in \mathbb{R} : u_2^2 + u_3^2 = \frac{1}{2}} \int_{y \in P_1 \cup P_2} (u_2 y_2 + u_3)^2 dy \geq \min \left\{ \int_{y \in P_1 \cup P_2} (u y_2 + \sqrt{\frac{1}{2} - u^2})^2 dy \right\} \frac{u \in \left[ -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right]}{u \in \left[ -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right]} \int_{y \in C} \left( u y_2 + \sqrt{\frac{1}{2} - u^2} \right)^2 dy_2.$$ 

(6)

The last equality is given by changing the order of integration into integral over $y_2$ and then over $y_1$, denoting the interval $C = \left[ b^2 + \frac{(\alpha - b^2)}{2} \cos \left( \frac{\delta}{4} \right), b^2 + (\alpha - b^2) \cos \left( \frac{\delta}{4} \right) \right]$, and noting that the term inside the integral does not depend on $y_1$.

Fix some $u \in \left[ -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right]$. If $u = 0$, then we can bound Eq. (6) by $\frac{(\alpha - b^2)^2}{4} \sin \left( \frac{\delta}{4} \right) \cos \left( \frac{\delta}{4} \right)$. Assume $u \neq 0$, we split into cases and bound the term inside the integral:

**Case I:** $\sqrt{\frac{1}{2} - u^2} \geq b^2 + \frac{3}{4} \cdot (\alpha - b^2) \cos \left( \frac{\delta}{4} \right)$. In this case, solving the inequality for $u$ we have

$$|u| \leq \sqrt{\frac{1}{2} - u^2} \geq \sqrt{\frac{1}{2} - \frac{1}{2 + 2 (b^2 + \frac{3}{4} (\alpha - b^2) \cos \left( \frac{\delta}{4} \right))^2}}.$$ 

Hence, we can also bound:

$$\sqrt{\frac{1}{2} - u^2} \geq \sqrt{\frac{1}{2} - \frac{1}{2 + 2 (b^2 + \frac{3}{4} (\alpha - b^2) \cos \left( \frac{\delta}{4} \right))^2}} = \sqrt{\frac{(b^2 + \frac{3}{4} (\alpha - b^2) \cos \left( \frac{\delta}{4} \right))^2}{2 + 2 (b^2 + \frac{3}{4} (\alpha - b^2) \cos \left( \frac{\delta}{4} \right))^2}}.$$ 

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In particular, for every $y_2 \in \left[ b' + \frac{(\alpha - b')}{2} \cos \left( \frac{\delta}{4} \right), b' + \frac{5(\alpha - b')}{8} \cos \left( \frac{\delta}{4} \right) \right]$ we get that:

$$
\left| uy_2 + \sqrt{\frac{1}{2} - u^2} \right| \geq \frac{(\alpha - b') \sin \left( \frac{\delta}{4} \right)}{8 \sqrt{2 + 2 (b' + (\alpha - b') \cos \left( \frac{\delta}{4} \right))}} \geq \frac{(\alpha - b')^2 \cos \left( \frac{\delta}{4} \right)}{8 \sqrt{2 + 2 (b' + (\alpha - b') \cos \left( \frac{\delta}{4} \right))}}
$$

Case II: $\sqrt{\frac{1}{2} - u^2} < b' + \frac{3}{4} \cdot (\alpha - b') \cos \left( \frac{\delta}{4} \right)$. Using the same reasoning as above, we get for every $y_2 \in \left[ b' + \frac{7(\alpha - b')}{8} \cos \left( \frac{\delta}{4} \right), b' + (\alpha - b') \cos \left( \frac{\delta}{4} \right) \right]$ that:

$$
\left| uy_2 + \sqrt{\frac{1}{2} - u^2} \right| \geq \frac{(\alpha - b') \sin \left( \frac{\delta}{4} \right)}{8 \sqrt{2 + 2 (b' + (\alpha - b') \cos \left( \frac{\delta}{4} \right))}} \geq \frac{(\alpha - b')^2 \cos \left( \frac{\delta}{4} \right)}{8 \sqrt{2 + 2 (b' + (\alpha - b') \cos \left( \frac{\delta}{4} \right))}}
$$

Combining the above cases with Eq. (6) we get that:

$$
\inf_{u_2, u_3 \in \mathbb{R}, u_2^2 + u_3^2 = \frac{1}{2}} \int_{y \in \mathcal{P}_1 \cup \mathcal{P}_2} (u_2 y_2 + u_3)^2 dy \geq (\alpha - b') \sin \left( \frac{\delta}{4} \right) \int_{y_2 \in \mathcal{C}} \frac{(\alpha - b')^2 \cos \left( \frac{\delta}{4} \right)}{8^2 (2 + 2 (b' + (\alpha - b') \cos \left( \frac{\delta}{4} \right)))^2} dy_2
$$

$$
\geq \frac{(\alpha - b')^4 \sin \left( \frac{\delta}{4} \right) \cos \left( \frac{\delta}{4} \right)}{2 \cdot 8^2 (2 + 2 (b' + (\alpha - b') \cos \left( \frac{\delta}{4} \right)))^2}
$$

$$
\geq \frac{\sin \left( \frac{\delta}{4} \right) \cos \left( \frac{\delta}{4} \right)}{2 \cdot 8^2 \sqrt{2} (2 + 2 (b' + (\alpha - b') \cos \left( \frac{\delta}{4} \right)))^2}
$$

$$
\geq \frac{(\alpha - b')^4 \sin \left( \frac{\delta}{4} \right) \cos \left( \frac{\delta}{4} \right)}{8^4} \cdot \min \left\{1, \frac{1}{\alpha^2} \right\}
$$

(7)

where in the second inequality we used that for $\delta \in [0, \pi]$ we have $\sin \left( \frac{\delta}{4} \right) \leq \cos \left( \frac{\delta}{4} \right)$, and in the last inequality we used that $b' \leq \alpha$, and $(\alpha - b') \cos \left( \frac{\delta}{4} \right) \leq \alpha$. Combining Eq. (5) with Eq. (7) finishes the proof.

\[ \square \]

### B Proofs from Section 3

#### B.1 Proof of Theorem 3.1

Let $\epsilon > 0$, for the input distribution, we consider the uniform distribution on the ball of radius $\epsilon$. Let $b_w$ be the last coordinate of $w$, and denote by $\tilde{w}, \tilde{x}$ the first $d$ coordinates of $w$ and $x$. Using the assumption on
the initialization of $w_0$ and on the boundness of the distribution $\tilde{D}$ we have:

$$|\langle \tilde{w}_0, \tilde{x} \rangle| \leq \|\tilde{w}_0\| \|\tilde{x}\| \leq \epsilon \sqrt{d}.$$ 

Since $b_{w_0}$ is also initialized with $U([-1, 1])$, w.p. $1/2 - \epsilon \sqrt{d}$ we have that $b_{w_0} < -\epsilon \sqrt{d}$. If this event happens, since the activation is ReLU we get that $\sigma'(\langle w_0, x \rangle) = 1(\langle \tilde{w}_0, \tilde{x} \rangle + b_{w_0} > 0) = 0$ for every $\tilde{x}$ in the support of the distribution. Using Eq. (3) we get that $\nabla F(w_0) = 0$, hence gradient flow will get stuck at its initial value.

### B.2 Proof of Theorem 3.2

**Lemma B.1.** Let $w \in \mathbb{R}^{d+1}$ such that $b_w = 0$, $\tilde{w}_1 < -\frac{4}{\sqrt{d}}$, and $\|\tilde{w}_{2:d}\| \leq 2\sqrt{d}$. Then,

$$\Pr_{x \sim D} \left[w^\top x \geq 0, v^\top x \geq 0 \right] = 0.$$

**Proof.** If $v^\top x \geq 0$ then $x_1 \geq r - \frac{r^2}{2d}$ and hence $x_1^2 \geq r^2 - \frac{r^2}{d}$. Since we also have $\|\tilde{x}\| \leq r$ then

$$\|\tilde{x}_{2:d}\|^2 = \|\tilde{x}\|^2 - x_1^2 \leq r^2 - \left(r^2 - \frac{r^2}{d}\right) = \frac{r^2}{d}.$$

Hence,

$$\Pr_{x \sim D} \left[w^\top x \geq 0, v^\top x \geq 0 \right] \leq \Pr_{x \sim D} \left[w^\top x \geq 0, x_1 \geq r - \frac{r}{2d^2}, \|\tilde{x}_{2:d}\| \leq \frac{r}{d}\right].$$

Since $b_w = 0$, $\|\tilde{w}_{2:d}\| \leq 2\sqrt{d}$ and $\tilde{w}_1 < -\frac{4}{\sqrt{d}}$, then for every $\tilde{x} \in B$ such that $x_1 \geq r - \frac{r}{2d^2} \geq \frac{r}{2}$ and $\|\tilde{x}_{2:d}\| \leq \frac{r}{d}$ we have

$$w^\top x = w^\top \tilde{x} = \tilde{w}_1 \tilde{x}_1 + \langle \tilde{w}_{2:d}, \tilde{x}_{2:d} \rangle < -\frac{4}{\sqrt{d}} \cdot \frac{r}{2} + 2\sqrt{d} \cdot \frac{r}{d} = 0.$$

Therefore, $\Pr_{x \sim D} \left[w^\top x \geq 0, v^\top x \geq 0 \right] = 0$. \hfill\(\Box\)

**Lemma B.2.** With probability $\frac{1}{2} - o_d(1)$ over the choice of $w_0$, we have

$$\Pr_{x \sim D} \left[w_0^\top x \geq 0, v^\top x \geq 0 \right] = 0.$$

**Proof.** Let $w \in \mathbb{R}^{d+1}$ such that $b_w = 0$ and $\tilde{w} \sim \mathcal{N}(0, I_d)$. Since $\tilde{w}_1$ has a standard normal distribution, then we have $\tilde{w}_1 < -\frac{4}{\sqrt{d}}$ with probability $\frac{1}{2} - o_d(1)$. Moreover, note that $\|\tilde{w}_{2:d}\|^2$ has a chi-square distribution and the probability of $\|\tilde{w}_{2:d}\|^2 \leq 4d$ is $1 - o_d(1)$. Hence, by Lemma B.1, with probability $\frac{1}{2} - o_d(1)$ over the choice of $w$, we have

$$\Pr_{x \sim D} \left[w^\top x \geq 0, v^\top x \geq 0 \right] = 0.$$

Therefore,

$$\Pr_{x \sim D} \left[\rho \frac{w^\top x}{\|w\|} \geq 0, v^\top x \geq 0 \right] = \Pr_{x \sim D} \left[w^\top x \geq 0, v^\top x \geq 0 \right] = 0.$$

Since $\rho \frac{w^\top x}{\|w\|}$ has the distribution of $w_0$, the lemma follows. \hfill\(\Box\)
Lemma B.3. Assume that \( w_0 \) satisfies \( \Pr_{x \sim \mathcal{D}} \left[ w_0^\top x \geq 0, v^\top x \geq 0 \right] = 0 \). Let \( \gamma > 0 \) and let \( w \in \mathbb{R}^{d+1} \) such that \( \bar{w} = \gamma \bar{w}_0 \), and \( b_w \leq 0 \). Then, \( \Pr_{x \sim \mathcal{D}} \left[ w^\top x \geq 0, v^\top x \geq 0 \right] = 0 \). Moreover, we have

- If \(-\frac{b_w}{\|w\|} < r\), then \( \frac{d\bar{w}}{dt} = -s\bar{w} \) for some \( s > 0 \), and \( \frac{db_w}{dt} < 0 \).
- If \(-\frac{b_w}{\|w\|} \geq r\), then \( \frac{d\bar{w}}{dt} = 0 \) and \( \frac{db_w}{dt} = 0 \).

**Proof.** For every \( x \) we have: If \( w^\top x = \gamma \bar{w}_0^\top \bar{x} + b_w \geq 0 \) then \( \gamma \bar{w}_0^\top \bar{x} \geq 0 \), and therefore \( w_0^\top x = \bar{w}_0^\top \bar{x} \geq 0 \). Thus

\[
\Pr_{x \sim \mathcal{D}} \left[ w^\top x \geq 0, v^\top x \geq 0 \right] \leq \Pr_{x \sim \mathcal{D}} \left[ w_0^\top x \geq 0, v^\top x \geq 0 \right] = 0. \tag{8}
\]

We have

\[
-\frac{d\bar{w}}{dt} = \nabla_{\bar{w}} F(w) = \mathbb{E}_x \left( \sigma(w^\top x) - \sigma(v^\top x) \right) \sigma'(w^\top x) \bar{x} = \mathbb{E}_x \left( \sigma(w^\top x) - \sigma(v^\top x) \right) 1(w^\top x \geq 0) \bar{x} + \mathbb{E}_x \left( \sigma(w^\top x) - \sigma(v^\top x) \right) 1(w^\top x \geq 0, v^\top x < 0) \bar{x} + \mathbb{E}_x \left( \sigma(w^\top x) - \sigma(v^\top x) \right) 1(w^\top x \geq 0, v^\top x \geq 0) \bar{x} \tag{Eq. (8)}
\]

If \( \bar{w} \) is spherically symmetric, then we obtain \( \nabla_{\bar{w}} F(w) = 0 \). Note that if \( \gamma \geq 0 \), i.e., \( \|\bar{w}\| > -b_w \), then \( \Pr_{\bar{x}} \left[ \bar{w}^\top \bar{x} > -b_w \right] > 0 \). Since \( \mathcal{D} \) is spherically symmetric, then we obtain \( \frac{d\bar{w}}{dt} = -s\bar{w} \) for some \( s > 0 \).

Next, we have

\[
-\frac{db_w}{dt} = \nabla_{b_w} F(w) = \mathbb{E}_x \left( \sigma(w^\top x) - \sigma(v^\top x) \right) \sigma'(w^\top x) \cdot 1 = \mathbb{E}_x \left( \sigma(w^\top x) - \sigma(v^\top x) \right) \cdot 1 \tag{Eq. (8)}
\]

If \( \frac{b_w}{\|w\|} \geq r \) then for every \( \bar{x} \in \mathcal{B} \) we have \( \bar{w}^\top \bar{x} \leq \|\bar{w}\| \geq -b_w \) and hence \( \frac{db_w}{dt} = 0 \). Otherwise, we have \( \frac{db_w}{dt} < 0 \).

**Proof of Theorem 3.2.** By Lemma B.2 \( w_0 \) satisfies \( \Pr_{x \sim \mathcal{D}} \left[ w_0^\top x \geq 0, v^\top x \geq 0 \right] = 0 \) w.p. at least \( \frac{1}{2} - o_d(1) \). Then, by Lemma B.3 we have for every \( t > 0 \) that \( \bar{w}_t = \gamma t \bar{w}_0 \) for some \( \gamma_t > 0 \), \( b_{w_t} < 0 \), and
Denote:

using Cauchy-Schwartz we have that

Thus, gradient flow does not converge to the global minimum $F(\mathbf{v}) = 0 < F(0)$.

C Proofs from Section 4

Proof of Theorem 4.2. The gradient of the objective is:

$$
\nabla F(\mathbf{w}) = \mathbb{E}_{\mathbf{x} \sim \mathcal{D}} \left[ \left( \sigma(\mathbf{w}^\top \mathbf{x}) - \sigma(\mathbf{v}^\top \mathbf{x}) \right) \cdot \sigma'(\mathbf{w}^\top \mathbf{x}) \mathbf{x} \right].
$$

We can rewrite it using that $\sigma$ is the ReLU activation, and separating the bias terms:

$$
\nabla F(\mathbf{w}) = \mathbb{E}_{\mathbf{x} \sim \mathcal{D}} \left[ \left( \sigma(\mathbf{w}^\top \mathbf{x} + b_w) - \sigma(\mathbf{v}^\top \mathbf{x} + b_v) \right) \cdot \mathbf{1}(\mathbf{w}^\top \mathbf{x} + b_w > 0) \mathbf{x} \right].
$$

First, notice that if $\mathbf{\tilde{w}} = 0$ and $b_w < 0$ then $\mathbf{1}(\mathbf{\tilde{w}}^\top \mathbf{\tilde{x}} + b_w > 0) = 0$ for all $\mathbf{\tilde{x}}$, hence $\nabla F(\mathbf{w}) = 0$. Second, using Cauchy-Schwartz we have that $|\langle \mathbf{\tilde{w}}, \mathbf{\tilde{x}} \rangle| \leq c \cdot \|\mathbf{\tilde{w}}\|$. Hence, for $\mathbf{w}$ with $\mathbf{\tilde{w}} \neq 0$ and $-\frac{b_w}{\|\mathbf{\tilde{w}}\|} \geq c$ we have that $\mathbf{1}(\mathbf{\tilde{w}}^\top \mathbf{\tilde{x}} + b_w > 0) = 0$ for all $\mathbf{\tilde{x}}$ in the support of the distribution, hence $\nabla F(\mathbf{w}) = 0$. Lastly, it is clear that for $\mathbf{w} = \mathbf{v}$ we have that $\nabla F(\mathbf{w}) = 0$. This shows that the points described in the statement of the proposition are indeed critical points. Next we will show that these are the only critical points.

Let $\mathbf{w} \in \mathbb{R}^{d+1}$ which is not a critical point defined above - i.e. either $\mathbf{\tilde{w}} = 0$ and $b_w > 0$, or $\mathbf{\tilde{w}} \neq 0$ and $-\frac{b_w}{\|\mathbf{\tilde{w}}\|} < c$. Then we have:

$$
\langle \nabla F(\mathbf{w}), \mathbf{w} - \mathbf{v} \rangle = \mathbb{E}_{\mathbf{x} \sim \mathcal{D}} \left[ \sigma'(\mathbf{w}^\top \mathbf{x})(\sigma(\mathbf{w}^\top \mathbf{x}) - \sigma(\mathbf{v}^\top \mathbf{x}))(\mathbf{w}^\top \mathbf{x} - \mathbf{v}^\top \mathbf{x}) \right]
$$

$$
= \mathbb{E}_{\mathbf{x} \sim \mathcal{D}} \left[ \mathbf{1}(\mathbf{w}^\top \mathbf{x} > 0, \mathbf{v}^\top \mathbf{x} > 0)(\sigma(\mathbf{w}^\top \mathbf{x}) - \sigma(\mathbf{v}^\top \mathbf{x}))(\mathbf{w}^\top \mathbf{x} - \mathbf{v}^\top \mathbf{x}) \right] +
$$

$$
+ \mathbb{E}_{\mathbf{x} \sim \mathcal{D}} \left[ \mathbf{1}(\mathbf{w}^\top \mathbf{x} > 0, \mathbf{v}^\top \mathbf{x} \leq 0)\sigma(\mathbf{w}^\top \mathbf{x})(\mathbf{w}^\top \mathbf{x} - \mathbf{v}^\top \mathbf{x}) \right] \geq \mathbb{E}_{\mathbf{x} \sim \mathcal{D}} \left[ \mathbf{1}(\mathbf{w}^\top \mathbf{x} > 0, \mathbf{v}^\top \mathbf{x} > 0)(\mathbf{w}^\top \mathbf{x} - \mathbf{v}^\top \mathbf{x})^2 \right] +
$$

$$
+ \mathbb{E}_{\mathbf{x} \sim \mathcal{D}} \left[ \mathbf{1}(\mathbf{w}^\top \mathbf{x} > 0, \mathbf{v}^\top \mathbf{x} \leq 0)(\mathbf{w}^\top \mathbf{x})^2 \right].
$$

Denote:

$$
A_1 := \{ \mathbf{\tilde{x}} \in \mathbb{R}^d : \mathbf{\tilde{w}}^\top \mathbf{\tilde{x}} > -b_w, \mathbf{\tilde{v}}^\top \mathbf{\tilde{x}} > -b_v, \|\mathbf{\tilde{x}}\| < c \}
$$

$$
A_2 := \{ \mathbf{\tilde{x}} \in \mathbb{R}^d : \mathbf{\tilde{w}}^\top \mathbf{\tilde{x}} > -b_w, \mathbf{\tilde{v}}^\top \mathbf{\tilde{x}} \leq -b_v, \|\mathbf{\tilde{x}}\| < c \}
$$

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Since $w$ is not a critical point as defined above, we know that the set $\{\tilde{x} \in \mathbb{R}^d : \tilde{w}^\top \tilde{x} > -b_w, \|\tilde{x}\| < c\}$ has a positive measure, hence either $A_1$ or $A_2$ have a positive measure. Assume w.l.o.g that $A_1$ have a positive measure, the other case is similar. Since both terms inside the expectations of Eq. (9) are positive, we can lower bound it with:

$$
\mathbb{E}_{x \sim D} \left[ \mathbb{1}(\tilde{w}^\top \tilde{x} > -b_w, \tilde{v}^\top \tilde{x} > -b_v)(w^\top x - v^\top x)^2 \right] = \|w - v\|^2 \mathbb{E}_{x \sim D} \left[ \mathbb{1}(\tilde{x} \in A_1)((\tilde{w} - \tilde{v})^\top x)^2 \right]
$$

(10)

Denote $u := \overline{w - v}$, and note that $w \neq v$, hence $\|u\| = 1$. Denote by $p(\tilde{x})$ the pdf of $\hat{D}$, then we can rewrite Eq. (10) as:

$$
\|w - v\|^2 \cdot \int_{\tilde{x} \in \mathbb{R}^d} \mathbb{1}(\tilde{x} \in A_1) \cdot (\tilde{u}^\top \tilde{x} + b_u)^2 p(\tilde{x}) d\tilde{x} = \|w - v\|^2 \cdot \int_{\tilde{x} \in A_1} (\tilde{u}^\top \tilde{x} + b_u)^2 p(\tilde{x}) d\tilde{x}
$$

(11)

Since the set $A_1$ has a positive measure, and the set $\{\tilde{x} : \tilde{u}^\top \tilde{x} + b_u = 0\}$ is of zero measure, there is a point $\tilde{x}_0$ such that $\tilde{u}^\top \tilde{x} + b_u \neq 0$. By continuity, there is a small enough neighborhood $A$ of $\tilde{x}_0$, such that $\tilde{u}^\top \tilde{x} + b_u \neq 0$ for every $\tilde{x} \in A$. Using Assumption 4.1 we can lower bound Eq. (11) by:

$$
|w - v|^2 \cdot \beta \int_{\tilde{x} \in A} (\tilde{u}^\top \tilde{x} + b_u)^2 d\tilde{x}
$$

where this integral is positive. This shows that $\langle \nabla F(w), w - v \rangle > 0$, which shows that $\nabla F(w) \neq 0$, hence $w$ is not a critical point.

\[ \square \]

## D Proofs from Section 5

The following lemmas are required in order to prove Theorem 5.2. First, we show that if $F(w) \leq F(0) - \delta$ then we can lower bound $\|w\|$ and $\Pr_x [w^\top x \geq 0, v^\top x \geq 0]$.

**Lemma D.1.** Let $\delta > 0$ and let $w \in \mathbb{R}^{d+1}$ such that $F(w) \leq F(0) - \delta$. Then

$$
\|w\| \geq \frac{\delta}{c^2},
$$

and

$$
\Pr_x [w^\top x \geq 0, v^\top x \geq 0] \geq \frac{\delta}{c^2 \|w\|}.
$$

**Proof.** We have

$$
F(0) - \delta \geq F(w) = \frac{1}{2} \mathbb{E}_x (\sigma(w^\top x) - \sigma(v^\top x))^2
$$

$$
= \frac{1}{2} \mathbb{E}_x (\sigma(w^\top x))^2 + \frac{1}{2} \mathbb{E}_x (\sigma(v^\top x))^2 - \mathbb{E}_x (\sigma(w^\top x)\sigma(v^\top x))
$$

$$
\geq F(0) - \mathbb{E}_x (\sigma(w^\top x)\sigma(v^\top x)).
$$

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\[ \delta \leq \mathbb{E}_x \sigma(w^T x) \sigma(v^T x) = \mathbb{E}_x \mathbb{1}(w^T x \geq 0, v^T x \geq 0) \cdot w^T x \cdot v^T x \]
\[ \leq \|w\|^2 \cdot \Pr_x \left[ w^T x \geq 0, v^T x \geq 0 \right]. \]

Thus,
\[ \|w\| \geq \frac{\delta}{c^2 \cdot \Pr_x \left[ w^T x \geq 0, v^T x \geq 0 \right]} \geq \frac{\delta}{c^2}, \]
and
\[ \Pr_x \left[ w^T x \geq 0, v^T x \geq 0 \right] \geq \frac{\delta}{c^2 \|w\|}. \]

Using the above lemma, we now show that if \( F(w) \leq F(0) - \delta \) then \( \|w - v\| \) decreases.

**Lemma D.2.** Let \( \delta > 0 \) and let \( B > 1 \). Let \( w \in \mathbb{R}^{d+1} \) such that \( F(w) \leq F(0) - \delta \) and \( \|w - v\| \leq B - 1 \). Let \( \gamma = \frac{\delta^3}{32c^3B^3} \) and let \( 0 < \eta \leq \frac{\gamma}{2}. \) Let \( w' = w - \eta \nabla F(w) \). Then,
\[ \|w' - v\|^2 \leq \|w - v\|^2 \cdot (1 - \gamma \eta) \leq (B - 1)^2. \]

**Proof.** We have
\[ \|w' - v\|^2 = \|w - \eta \nabla F(w) - v\|^2 = \|w - v\|^2 - 2\eta \langle \nabla F(w), w - v \rangle + \eta^2 \|\nabla F(w)\|^2. \]  \hspace{1cm} (12)

We first bound \( \|\nabla F(w)\|^2 \). By Jensen’s inequality and since \( \sigma \) is 1-Lipschitz, we have:
\[ \|\nabla F(w)\|^2 \leq \mathbb{E}_x \left[ \left( \sigma(w^T x) - \sigma(v^T x) \right)^2 \sigma'(w^T x) \|x\|^2 \right] \]
\[ \leq c^2 \mathbb{E}_x \left[ \left( \sigma(w^T x) - \sigma(v^T x) \right)^2 \right] \]
\[ \leq c^2 \mathbb{E}_x \left[ \left( w^T x - v^T x \right)^2 \right] \]
\[ = c^2 \mathbb{E}_x \left[ \left( w - v \right)^T x \right]^2 \]
\[ \leq c^4 \|w - v\|^2. \]  \hspace{1cm} (13)

Next, we bound \( \langle \nabla F(w), w - v \rangle \). Let \( u = \overline{w - v} \). We have
\[ \langle \nabla F(w), w - v \rangle = \mathbb{E}_x \left( \sigma(w^T x) - \sigma(v^T x) \right) \sigma'(w^T x)(w^T x - v^T x) \]
\[ = \mathbb{E}_x \left( w^T x - v^T x \right)^2 \mathbb{1}(w^T x \geq 0, v^T x \geq 0) + \]
\[ \mathbb{E}_x \left( w^T x \cdot (w^T x - v^T x) \right) \mathbb{1}(w^T x \geq 0, v^T x < 0) \]
\[ \geq \|w - v\|^2 \cdot \mathbb{E}_x \mathbb{1}(w^T x \geq 0, v^T x \geq 0)(u^T x)^2. \]
Let $\xi = \frac{\delta}{12B c^3c'}$. The above is at least

$$\|w - v\| \cdot \xi^2 \cdot \Pr_x \left[ w^T x \geq 0, v^T x \geq 0, (u^T x)^2 \geq \xi^2 \right]$$

$$= \|w - v\| \cdot \xi^2 \cdot \left( \Pr_x \left[ w^T x \geq 0, v^T x \geq 0 \right] - \Pr_x \left[ w^T x \geq 0, v^T x \geq 0, (u^T x)^2 < \xi^2 \right] \right).$$

By Lemma D.1, and since $\|w\| \leq \|w - v\| + \|v\| \leq B - 1 + 1 = B$, the above is at least

$$\|w - v\| \cdot \xi^2 \cdot \left( \frac{\delta}{c^2B} - \Pr_x \left[ w^T x \geq 0, v^T x \geq 0, |u^T x| < \xi \right] \right)$$

$$\geq \|w - v\| \cdot \xi^2 \cdot \left( \frac{\delta}{c^2B} - \Pr_x \left[ |u^T x| \leq \xi \right] \right)$$

$$= \|w - v\| \cdot \xi^2 \cdot \left( \frac{\delta}{c^2B} - \Pr_x \left[ \|u^T x + b_u\| \leq \xi \right] \right).$$

(14)

We now bound $\Pr_x \left[ \|u^T x + b_u\| \leq \xi \right]$. We denote $a = \|u\|$. If $\frac{1}{4c} \leq a \leq \frac{1}{4c}$, then since $\|u\| = 1$ we have $b_u \geq \sqrt{1 - \frac{1}{16c^2}} \geq \sqrt{1 - \frac{1}{16}} = \frac{\sqrt{15}}{4}$. Hence, for every $x$ with $\|x\| \leq c$ we have

$$\|u^T x + b_u\| \geq |b_u| - \|u^T \bar{x}\| \geq \frac{\sqrt{15}}{4} - ac \geq \frac{\sqrt{15}}{4} - \frac{1}{4} > \frac{1}{2}.$$  

Note that

$$\xi = \frac{\delta}{12B c^3c'} \leq \frac{F(0)}{12B c^3c'} = \frac{1}{12B c^3c'} \cdot \frac{1}{2} \cdot \|\sigma(v^T x)\|^2 \leq \frac{1}{12B c^3c'} \cdot \frac{1}{2} \cdot c^2 = \frac{1}{24B c c'} \leq \frac{1}{24},$$

where the last inequality is since $B, c, c' \geq 1$. Therefore, $|u^T \bar{x} + b_u| > \xi$. Thus,

$$\Pr_x \left[ \|u^T \bar{x} + b_u\| \leq \xi \right] = 0.$$

Assume now that $a \geq \frac{1}{4c}$. We have

$$\Pr_x \left[ \|u^T \bar{x} + b_u\| \leq \xi \right] = \Pr_x \left[ u^T \bar{x} \in [-\xi - b_u, \xi - b_u] \right] = \Pr_x \left[ u^T \bar{x} \in \left[ \frac{-\xi}{a} - \frac{b_u}{a}, \frac{\xi}{a} - \frac{b_u}{a} \right] \right]$$

$$\leq c' \cdot 2 \cdot \frac{\xi}{a}$$

$$\leq 8 c' \xi .$$

Combining the above with Eq. (14), we obtain

$$\langle \nabla F(w), w - v \rangle \geq \|w - v\|^2 \cdot \xi^2 \cdot \left( \frac{\delta}{c^2B} - 8 c' \xi \right)$$

$$= \|w - v\|^2 \cdot \frac{\delta^2}{122B^2 c^3 c'^2} \cdot \left( \frac{\delta}{c^2B} - 8 c' \cdot \frac{\delta}{12B c^3 c'} \right)$$

$$= \|w - v\|^2 \cdot \frac{\delta^2}{122B^2 c^3 c'^2} \cdot \left( \frac{\delta}{3c^2B} \right)$$

$$= \|w - v\|^2 \cdot \frac{\delta^3}{3 \cdot 122B^2 c^3 c'^2}.$$

(15)

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Combining Eq. (12), (13) and (15), and using $\gamma = \frac{\delta^3}{312^3 B \sqrt{c^2}}$, we have

$$\| \mathbf{w} - \mathbf{v} \|_2^2 \leq \| \mathbf{w} - \mathbf{v} \|^2 - 2\eta \| \mathbf{w} - \mathbf{v} \|^2 \cdot \gamma + \eta^2 c^4 \| \mathbf{w} - \mathbf{v} \|^2$$

$$= \| \mathbf{w} - \mathbf{v} \|^2 \cdot (1 - 2\eta \gamma + \eta^2 c^4) .$$

Since $\eta \leq \frac{\gamma}{c^4}$, we obtain

$$\| \mathbf{w}' - \mathbf{v} \|^2 \leq \| \mathbf{w} - \mathbf{v} \|^2 \cdot \left(1 - \frac{2\eta \gamma}{c^4} + \frac{\eta c^4}{c^4} \right)$$

$$= \| \mathbf{w} - \mathbf{v} \|^2 \cdot \left(1 - \eta \gamma \right) \leq \| \mathbf{w} - \mathbf{v} \|^2 \leq (B - 1)^2 .$$

□

Next, we show that $F(\mathbf{w})$ remains smaller than $F(\mathbf{0}) - \delta$ during the training. In the following two lemmas we obtain a bound for the smoothness of $F$ in the relevant region, and in the two lemmas that follow we use this bound to show that $F(\mathbf{w})$ indeed remains small.

**Lemma D.3.** Let $\mathbf{w} \in \mathbb{R}^{d+1}$ such that $F(\mathbf{w}) \leq F(\mathbf{0})$. Then, $\| \nabla F(\mathbf{w}) \| \leq c\sqrt{2F(\mathbf{0})}$.

**Proof.** By Jensen’s inequality, we have

$$\| \nabla F(\mathbf{w}) \|^2 \leq \mathbb{E}_{\mathbf{x}} \left( \sigma(\mathbf{w}^\top \mathbf{x}) - \sigma(\mathbf{v}^\top \mathbf{x}) \right)^2 \sigma'(\mathbf{w}^\top \mathbf{x}) \| \mathbf{x} \|^2$$

$$\leq c^2 \mathbb{E}_{\mathbf{x}} \left( \sigma(\mathbf{w}^\top \mathbf{x}) - \sigma(\mathbf{v}^\top \mathbf{x}) \right)^2$$

$$\leq c^2 2 F(\mathbf{w}) \leq 2c^2 F(\mathbf{0}) .$$

□

**Lemma D.4.** Let $M, B > 0$ and let $\mathbf{w}, \mathbf{w}' \in \mathbb{R}^{d+1}$ be such that for every $s \in [0, 1]$ we have $M \leq \| \mathbf{w} + s(\mathbf{w}' - \mathbf{w}) \| \leq B$. Then,

$$\| \nabla F(\mathbf{w}) - \nabla F(\mathbf{w}') \| \leq \| \mathbf{w} - \mathbf{w}' \| \cdot c^2 \left(1 + \frac{8(B+1)c^2}{M} \right) .$$

**Proof.** We assume w.l.o.g. that $\| \mathbf{w} - \mathbf{w}' \| \leq \frac{M}{2c}$. Indeed, let $0 = s_0 < \ldots < s_k = 1$ for some integer $k$, let $\mathbf{w}_i = \mathbf{w} + s_i(\mathbf{w}' - \mathbf{w})$, and assume that $\| \mathbf{w}_i - \mathbf{w}_{i+1} \| \leq \frac{M}{2c}$ for every $i$. If the claim holds for every pair $\mathbf{w}_i, \mathbf{w}_{i+1}$, then we have

$$\| \nabla F(\mathbf{w}) - \nabla F(\mathbf{w}') \| = \| \sum_{i=0}^{k-1} \nabla F(\mathbf{w}_i) - \nabla F(\mathbf{w}_{i+1}) \|$$

$$\leq \sum_{i=0}^{k-1} \| \nabla F(\mathbf{w}_i) - \nabla F(\mathbf{w}_{i+1}) \|$$

$$\leq \sum_{i=0}^{k-1} \| \mathbf{w}_i - \mathbf{w}_{i+1} \| \cdot c^2 \left(1 + \frac{8(B+1)c^2}{M} \right)$$

$$= c^2 \left(1 + \frac{8(B+1)c^2}{M} \right) \| \mathbf{w} - \mathbf{w}' \| .$$
We have
\[
\| \nabla F(w) - \nabla F(w') \|
\]
\[
= \| \mathbb{E}_x (\sigma(w^T x) - \sigma(v^T x)) \sigma'(w^T x) x - (\sigma(w'^T x) - \sigma(v^T x)) \sigma'(w'^T x) x \|
\]
\[
\leq \| \mathbb{E}_x 1(w^T x \geq 0, w'^T x \geq 0) (\nabla^T x - \sigma(v^T x) - w'^T x + \sigma(v^T x)) x \|
\]
\[
+ \| \mathbb{E}_x 1(w^T x \geq 0, w'^T x < 0) (\nabla^T x - \sigma(v^T x)) x \|
\]
\[
+ \| \mathbb{E}_x 1(w^T x < 0, w'^T x \geq 0) (\nabla^T x - \sigma(v^T x)) x \|
\].

By Jensen’s inequality and Cauchy-Shwartz, the above is at most
\[
\mathbb{E}_x 1(w^T x \geq 0, w'^T x \geq 0) \| w - w' \| \cdot \| x \| + \mathbb{E}_x 1(w^T x \geq 0, w'^T x < 0) (\| w \| \cdot \| x \| + \| v \| \cdot \| x \|) \cdot \| x \| + \mathbb{E}_x 1(w^T x < 0, w'^T x \geq 0) (\| w' \| \cdot \| x \| + \| v \| \cdot \| x \|) \cdot \| x \| .
\]

By our assumption we have \( \| x \| \leq c \) and \( \| w \|, \| w' \| \leq B \). Hence, the above is at most
\[
\| w - w' \| c^2 + \Pr_x \left[ w^T x \geq 0, w'^T x < 0 \right] c^2 (B + 1)
\]
\[
+ \Pr_x \left[ w^T x < 0, w'^T x \geq 0 \right] c^2 (B + 1).
\]

Now, we bound \( \Pr_x \left[ w^T x \geq 0, w'^T x < 0 \right] \). If \( w^T x \geq 0 \) and \( w'^T x < 0 \) then
\[
w^T x = w'^T x + (w - w')^T x < 0 + \| w - w' \| \cdot \| x \| \leq c \cdot \| w - w' \| .
\]

Hence, we only need to bound
\[
\Pr_x \left[ w^T x \in [0, c \cdot \| w - w' \|] \right] = \Pr_x \left[ \tilde{w}^T \tilde{x} + b_w \in [0, c \cdot \| w - w' \|] \right].
\]

We denote \( a = \| \tilde{w} \| \). If \( a \leq \frac{M}{4c} \), then since \( \| w \| \geq M \) we have \( |b_w| \geq \sqrt{M^2 - \left( \frac{M}{4c} \right)^2} = M \sqrt{1 - 1/(16c^2)} \).

Hence for every \( x \) we have
\[
| \tilde{w}^T \tilde{x} + b_w | \leq |b_w| - | \tilde{w}^T \tilde{x} | \geq |b_w| - ac \geq M \sqrt{1 - 1/(16c^2)} - \frac{M}{4} \geq M \sqrt{1 - 1/16} - \frac{M}{4}
\]
\[
= M \cdot \frac{\sqrt{15} - 1}{4} > M/2 \geq c \| w - w' \| .
\]

Thus, \( \Pr_x \left[ \tilde{w}^T \tilde{x} + b_w \in [0, c \cdot \| w - w' \|] \right] = 0. \)

Assume now that \( a \geq \frac{M}{4c} \). Hence, \( \frac{a}{c} \leq \frac{4c^2}{M} \). Therefore, we have
\[
\Pr_x \left[ \tilde{w}^T \tilde{x} + b_w \in [0, c \cdot \| w - w' \|] \right] = \Pr_x \left[ \tilde{w}^T \tilde{x} \in \left[ -\frac{b_w}{a}, -\frac{b_w}{a} + \frac{c}{a} \cdot \| w - w' \| \right] \right]
\]
\[
\leq \Pr_x \left[ \tilde{w}^T \tilde{x} \in \left[ -\frac{b_w}{a}, -\frac{b_w}{a} + \frac{4c^2}{M} \cdot \| w - w' \| \right] \right]
\]
\[
\leq \frac{c'}{M} \cdot \frac{4c^2}{M} \cdot \| w - w' \| .
\]

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Hence, \( \Pr_x \left[ w^\top x \geq 0, w'^\top x < 0 \right] \leq c' \cdot \frac{4c^2}{M} \cdot \| w - w' \|. \) By similar arguments, this inequality holds also for \( \Pr_x \left[ w^\top x < 0, w'^\top x \geq 0 \right] \). Plugging it into Eq. (16), we have

\[
\| \nabla F(w) - \nabla F(w') \| \leq \| w - w' \| \left( c^2 + 2 \cdot c^2 \cdot (B + 1) \cdot c' \cdot \frac{4c^2}{M} \right)
\]

\[
= \| w - w' \| \cdot c^2 \left( 1 + \frac{8(B + 1)c^2}{M} \right).
\]

\(\square\)

**Lemma D.5.** Let \( f : \mathbb{R}^d \to \mathbb{R} \) and let \( L > 0 \). Let \( x, y \in \mathbb{R}^d \) be such that for every \( s \in [0, 1] \) we have \( \| \nabla f(x + s(y - x)) - \nabla f(x) \| \leq Ls \| y - x \| \). Then,

\[
f(y) - f(x) \leq \nabla f(x)^\top (y - x) + \frac{L}{2} \| y - x \|^2.
\]

**Proof.** The proof follows a standard technique (cf. [2]). We represent \( f(y) - f(x) \) as an integral, apply Cauchy-Schwarz and then use the \( L \)-smoothness.

\[
f(y) - f(x) - \nabla f(x)^\top (y - x) = \int_0^1 \nabla f(x + s(y - x)) - \nabla f(x) \cdot (y - x) ds
\]

\[
\leq \int_0^1 \| \nabla f(x + s(y - x)) - \nabla f(x) \| \cdot \| y - x \| ds
\]

\[
\leq \int_0^1 Ls \| y - x \|^2 ds
\]

\[
= \frac{L}{2} \| y - x \|^2.
\]

Hence, we have

\[
f(y) - f(x) \leq \nabla f(x)^\top (y - x) + \frac{L}{2} \| y - x \|^2.
\]

\(\square\)

**Lemma D.6.** Let \( B, \delta > 0 \) and let \( L = c^2 \left( 1 + \frac{16(B + 1)c^4}{\delta} \right) \). Let \( w \in \mathbb{R}^{d+1} \) such that \( F(w) \leq F(0) - \delta \) and let \( w' = w - \eta \cdot \nabla F(w) \), where \( \eta \leq \min \left\{ \frac{\delta}{2c^3 \sqrt{2F(0)}}, \frac{\delta}{L} \right\} = \min \left\{ \frac{\delta}{2c^3 \sqrt{2F(0)}}, \frac{\delta}{\delta + 16(B + 1)c^2} \right\} \).

Assume that \( \| w \|, \| w' \| \leq B \). Then, we have \( F(w') - F(w) \leq -\eta \left( 1 - \frac{L}{2} \right) \| \nabla F(w) \|^2 \), and \( F(w') \leq F(w) \leq F(0) - \delta \).

**Proof.** Let \( M = \frac{\delta}{2c^2} \). By Lemmas D.1 and D.3, we have \( \| w \| \geq \frac{\delta}{c^2} \) and \( \| \nabla F(w) \| \leq c \sqrt{2F(0)} \). Hence for every \( \lambda \in [0, 1] \) we have

\[
\| w - \lambda \eta \nabla F(w) \| \geq \frac{\delta}{c^2} - \lambda \cdot c \sqrt{2F(0)} \geq \frac{\delta}{c^2} - \frac{\delta}{2c^3 \sqrt{2F(0)}} \cdot c \sqrt{2F(0)} = \frac{\delta}{2c^2} = M.
\]

Since \( \| w \|, \| w' \| \leq B \), we also have \( \| w - \lambda \eta \nabla F(w) \| \leq B \). By Lemma D.4, we have for every \( \lambda \in [0, 1] \) that

\[
\| \nabla F(w) - \nabla F(w - \lambda \eta \nabla F(w)) \| \leq \lambda \eta \| \nabla F(w) \| \cdot c^2 \left( 1 + \frac{8(B + 1)c^2}{M} \right).
\]
We have $L = c^2 \left(1 + \frac{16(B+1)c'c^2}{\delta}\right) = c^2 \left(1 + \frac{8(B+1)c'c^2}{M}\right)$. By Lemma D.5 we have

$$F(w - \eta \nabla F(w)) - F(w) \leq -\eta \|\nabla F(w)\|^2 + \frac{L}{2} \eta^2 \|\nabla F(w)\|^2.$$  

Since $\eta \leq \frac{1}{L}$, we also have $F(w - \eta \nabla F(w)) \leq F(w) \leq F(0) - \delta$. \hfill \Box

We are now ready to prove the theorem:

**Proof of Theorem 5.2.** Let $B = \|w_0\| + 2$. Assume that $\eta \leq \min \left\{ \frac{\delta}{2c^3 \sqrt{2F(0)}}, \frac{\delta}{\delta c^2 + 16(B+1)c'c^6}, \frac{\gamma}{c^4} \right\}$. We have $\|w_0 - v\| \leq \|w_0\| + 1 \leq B - 1$. By Lemmas D.2 and D.6, for every $t$ we have $\|w_t - v\| \leq B - 1$ (thus, $\|w_t\| \leq B$) and $F(w_t) \leq F(0) - \delta$. Moreover, by Lemma D.2, we have for every $t$ that $\|w_{t+1} - v\|^2 \leq \|w_t - v\|^2 \cdot (1 - \gamma \eta)$. Therefore, $\|w_t - v\|^2 \leq \|w_0 - v\|^2 (1 - \gamma \eta)^t$.

It remains to show that

$$\min \left\{ \frac{\delta}{2c^3 \sqrt{2F(0)}}, \frac{\delta}{\delta c^2 + 16(B+1)c'c^6}, \frac{\gamma}{c^4} \right\} = \frac{\gamma}{c^4}.$$  

Note that we have $\delta \leq F(0) = \frac{1}{2} \mathbb{E}_x \left(\sigma(w^\top x)\right)^2 \leq \frac{1}{2} \cdot c^2$. Thus

$$\frac{\gamma}{c^4} = \frac{\delta^3}{3 \cdot 12^2 B^3 c^2 c^6} \leq \frac{\delta}{3 \cdot 12^2 B^3 c^2 c^6} \cdot \frac{c^4}{4} = \frac{\delta}{12^3 B^3 c^8 c^6}.$$  

We have

$$\frac{\delta}{2c^3 \sqrt{2F(0)}} \geq \frac{\delta}{2c^4} \geq \frac{\gamma}{c^4},$$

where the last inequality is since $B, c, c' \geq 1$. Finally,

$$\frac{\delta}{\delta c^2 + 16(B+1)c'c^6} \geq \frac{c^3}{2} + 16(B+1)c'c^6 \geq \frac{\delta}{17(B+1)c'c^6} \geq \frac{\delta}{34Bc'c^6} \geq \frac{\gamma}{c^4}.$$ \hfill \Box

### D.1 Proofs from Subsection 5.2

**Proof of Theorem 5.4**

We have

$$F(w) = \frac{1}{2} \mathbb{E}_x \left(\sigma(w^\top x) - \sigma(v^\top x)\right)^2 = F(0) + \frac{1}{2} \mathbb{E}_x \left(\sigma(w^\top x)\right)^2 - \mathbb{E}_x \sigma(w^\top x) \sigma(v^\top x)\sigma(w^\top x).$$  

(17)
Let \( \xi = \frac{\alpha}{4v} \sin \left( \frac{\pi}{8} \right) \). We have

\[
\mathbb{E}_x \sigma(\hat{w}^\top x) \sigma(v^\top x) \geq \xi^2 \cdot \Pr_x \left[ \sigma(\hat{w}^\top x) \sigma(v^\top x) \geq \xi^2 \right] \\
\geq \xi^2 \cdot \Pr_x \left[ \hat{w}^\top x \geq 2\sqrt{c} \xi, v^\top x \geq \frac{\xi}{2\sqrt{c}} \right].
\]  

(18)

In the following two lemmas we bound \( \Pr_x \left[ \hat{w}^\top x \geq 2\sqrt{c} \xi, v^\top x \geq \frac{\xi}{2\sqrt{c}} \right]\).

**Lemma D.7.** If \( b_v \geq 0 \) then

\[
\Pr_x \left[ \hat{w}^\top x \geq 2\sqrt{c} \xi, v^\top x \geq \frac{\xi}{2\sqrt{c}} \right] \geq \frac{\beta (\alpha \sin \left( \frac{\pi}{8} \right) - 2\sqrt{c} \xi)^2}{4 \sin \left( \frac{\pi}{8} \right)}. 
\]

**Proof.** If \( \|\tilde{v}\| \geq \frac{1}{4c} \), then we have

\[
\Pr_x \left[ \hat{w}^\top x \geq 2\sqrt{c} \xi, v^\top x \geq \frac{\xi}{2\sqrt{c}} \right] \geq \Pr_x \left[ \hat{w}^\top x \geq 2\sqrt{c} \xi, \tilde{v}^\top \tilde{x} \geq \frac{\xi}{2\sqrt{c}} \right] \\
= \Pr_x \left[ \hat{w}^\top \tilde{x} \geq 2\sqrt{c} \xi, \tilde{v}^\top \tilde{x} \geq \frac{\xi}{2\sqrt{c} \|\tilde{v}\|} \right] \\
\geq \Pr_x \left[ \hat{w}^\top \tilde{x} \geq 2\sqrt{c} \xi, \tilde{v}^\top \tilde{x} \geq 2\sqrt{c} \xi \right] \\
\geq \frac{\beta (\alpha \sin \left( \frac{\pi}{8} \right) - 2\sqrt{c} \xi)^2}{4 \sin \left( \frac{\pi}{8} \right)},
\]

where the last inequality is due to Lemma A.1, since \( \theta(\hat{w}, \tilde{v}) \leq \frac{3\pi}{4} \).

If \( \|\tilde{v}\| \leq \frac{1}{4c} \), then

\[
b_v \geq \sqrt{1 - \frac{1}{16c^2}} \geq \sqrt{1 - \frac{1}{16}} = \frac{\sqrt{15}}{4} > \frac{3}{4},
\]

and hence

\[
v^\top x = \tilde{v}^\top \tilde{x} + b_v \geq -\frac{1}{4c} \cdot c + \frac{3}{4} = \frac{1}{2} \geq \xi \geq \frac{\xi}{2\sqrt{c}}.
\]

Therefore,

\[
\Pr_x \left[ \hat{w}^\top x \geq 2\sqrt{c} \xi, v^\top x \geq \frac{\xi}{2\sqrt{c}} \right] = \Pr_x \left[ \hat{w}^\top x \geq 2\sqrt{c} \xi \right] = \Pr_x \left[ \tilde{w}^\top \tilde{x} \geq 2\sqrt{c} \xi \right].
\]

For \( \tilde{u} \in \mathbb{R}^d \) such that \( \|\tilde{u}\| = 1 \) and \( \theta(\hat{w}, \tilde{u}) = \frac{3\pi}{4} \), Lemma A.1 implies that the above is at least

\[
\Pr_x \left[ \tilde{w}^\top \tilde{x} \geq 2\sqrt{c} \xi, \tilde{u}^\top \tilde{x} \geq 2\sqrt{c} \xi \right] \geq \frac{\beta (\alpha \sin \left( \frac{\pi}{8} \right) - 2\sqrt{c} \xi)^2}{4 \sin \left( \frac{\pi}{8} \right)}. 
\]

\[
\square
\]

**Lemma D.8.** If \( b_v < 0 \) and \( -\frac{b_v}{\|\tilde{v}\|} \leq \alpha \cdot \frac{\sin \left( \frac{\pi}{8} \right)}{4} \), then

\[
\Pr_x \left[ \hat{w}^\top x \geq 2\sqrt{c} \xi, v^\top x \geq \frac{\xi}{2\sqrt{c}} \right] \geq \frac{\beta (\alpha \sin \left( \frac{\pi}{8} \right) - 2\sqrt{c} \xi)^2}{4 \sin \left( \frac{\pi}{8} \right)}.
\]
Proof.

\[
\Pr_x \left[ \bar{w}^\top x \geq 2\sqrt{c} \xi, v^\top x \geq \frac{\xi}{2\sqrt{c}} \right] = \Pr_x \left[ \bar{w}^\top x \geq 2\sqrt{c} \xi, \bar{v}^\top \bar{x} \geq \frac{\xi}{2\sqrt{c}} - b_v \right] \\
= \Pr_x \left[ \bar{v}^\top \bar{x} \geq 2\sqrt{c} \xi, \bar{v}^\top \bar{x} \geq \frac{\xi}{2\sqrt{c}} \bar{v} - b_v \right]. \tag{19}
\]

Moreover, we have

\[
\left( \alpha \cdot \sin \left( \frac{\pi}{8} \right) \right)^2 \geq \left( \frac{b_v}{\|\bar{v}\|} \right)^2 = \frac{1 - \|\bar{v}\|^2}{\|\bar{v}\|^2} = \frac{1}{\|\bar{v}\|^2} - 1,
\]

and hence

\[
\|\bar{v}\|^2 \geq \frac{16}{\alpha^2 \sin^2 \left( \frac{\pi}{8} \right) + 16} \geq \frac{16}{\left( \alpha \sin \left( \frac{\pi}{8} \right) + 4 \right)^2} \geq \frac{16}{\left( c \cdot 1 + 4c \right)^2},
\]

where in the last inequality we used \( c \geq \alpha \) and \( c \geq 1 \). Thus,

\[
\|\bar{v}\| \geq \frac{4}{5c} \geq \frac{1}{2c}.
\]

Combining the above with Eq. (19), and using \(-\frac{b_v}{\|\bar{v}\|} \leq \alpha \cdot \sin \left( \frac{\pi}{8} \right) \), we have

\[
\Pr_x \left[ w^\top x \geq 2\sqrt{c} \xi, v^\top x \geq \frac{\xi}{2\sqrt{c}} \right] \geq \Pr_x \left[ \bar{w}^\top \bar{x} \geq 2\sqrt{c} \xi, \bar{v}^\top \bar{x} \geq \sqrt{c} \xi + \alpha \cdot \sin \left( \frac{\pi}{8} \right) \right] \\
= \Pr_x \left[ \bar{v}^\top \bar{x} \geq 2\sqrt{c} \xi, \bar{v}^\top \bar{x} \geq 2\sqrt{c} \xi \right] \\
\geq \frac{\beta \left( \alpha \sin \left( \frac{\pi}{8} \right) - 2\sqrt{c} \xi \right)^2}{4 \sin \left( \frac{\pi}{8} \right)},
\]

where the last inequality is due to Lemma A.1, since \( \theta(\bar{w}, \bar{v}) \leq \frac{3\pi}{4} \).

Combining Eq. (18) with Lemmas D.7 and D.8, we have

\[
\mathbb{E} \sigma(\bar{w}^\top x) \sigma(v^\top x) \geq \xi^2 \cdot \beta \left( \alpha \sin \left( \frac{\pi}{8} \right) - 2\sqrt{c} \xi \right)^2 = \frac{\alpha^2 \sin^2 \left( \frac{\pi}{8} \right)}{4 \sin \left( \frac{\pi}{8} \right)} \cdot \beta \left( \frac{\pi}{8} \sin \left( \frac{\pi}{8} \right) \right)^2 \\
= \frac{\alpha^4 \beta \sin^3 \left( \frac{\pi}{8} \right)}{256c} = M.
\]

Plugging the above into Eq. (17) we have

\[
F(w) \leq F(0) + \frac{\|w\|^2 c^2}{2} - \|w\| \cdot M.
\]

The above expression is smaller than \( F(0) \) if \( \|w\| < \frac{2M}{c^2} \).
E  Discussion on the Assumption on $b_v$

In Corollary 5.5 we had an assumption that $-\frac{b_v}{\|v\|} \leq \alpha \cdot \frac{\sin(\frac{\pi}{8})}{4}$. This implies that either the bias term $b_v$ is positive, or it is negative but not too large. Here we discuss why this assumption is crucial for the proof of the theorem, and what can we still say when this assumption does not hold.

In Theorem 3.2 we showed an example with $b_v < 0$ where gradient descent with random initialization does not converge w.h.p. to a global minimum even asymptotically. In the example from Theorem 3.2 we have $-\frac{b_v}{\|v\|} = r \left( 1 - \frac{1}{2d}\right)$, and the input distribution is uniform over a ball of radius $r$. In this case, we must choose $\alpha$ from Assumption 5.3 to be smaller than $r$ (otherwise $\beta = 0$ and hence $-\frac{b_v}{\|v\|} > \alpha \left( 1 - \frac{1}{2d}\right)$).

Therefore it does not satisfy the assumption $-\frac{b_v}{\|v\|} \leq \alpha \cdot \frac{\sin(\frac{\pi}{8})}{4}$ (already for $d > 1$). If we choose, e.g., $\alpha = \frac{r}{2}$, then the example from Theorem 3.2 satisfies $-\frac{b_v}{\|v\|} = \alpha \left( 2 - \frac{1}{d}\right) \leq 2\alpha$. It implies that our assumption on $-\frac{b_v}{\|v\|}$ is tight up to a constant factor, and is also crucial for the proof, since already for $-\frac{b_v}{\|v\|} = 2\alpha$ we have an example of convergence to a non-global minimum.

On the other hand, if $-\frac{b_v}{\|v\|} > \alpha \cdot \frac{\sin(\frac{\pi}{8})}{4}$ (i.e. the assumption does not hold) we can calculate the loss at zero:

$$F(0) = \frac{1}{2} \cdot \mathbb{E}_x \left[ \left( \sigma(\langle v^\top x \rangle) \right)^2 \right] = \frac{1}{2} \cdot \mathbb{E}_x \left[ 1 \left( \tilde{v}^\top \tilde{x} + b_v \geq 0 \right) \left( \tilde{v}^\top \tilde{x} + b_v \right)^2 \right]$$

$$= \frac{1}{2} \cdot \mathbb{E}_x \left[ 1 \left( \tilde{v}^\top \tilde{x} \geq -\frac{b_v}{\|v\|} \right) \|\tilde{v}\|^2 \left( \tilde{v}^\top \tilde{x} + b_v \right)^2 \right]$$

$$\leq \frac{\|\tilde{v}\|^2}{2} \cdot \mathbb{E}_x \left[ 1 \left( \tilde{v}^\top \tilde{x} \geq \alpha \cdot \frac{\sin(\frac{\pi}{8})}{4} \right) \left( \tilde{v}^\top \tilde{x} \right)^2 \right].$$

Let $\epsilon > 0$ be a small constant. Suppose that the distribution $\tilde{D}$ is spherically symmetric, and that $\alpha$ is large, such that the above expectation is smaller than $\epsilon$. For such $\alpha$, we either have $-\frac{b_v}{\|v\|} \leq \alpha \cdot \frac{\sin(\frac{\pi}{8})}{4}$, in which case gradient descent converges w.h.p. to the global minimum, or $-\frac{b_v}{\|v\|} > \alpha \cdot \frac{\sin(\frac{\pi}{8})}{4}$, in which case the loss at $w = 0$ is already almost as good as the global minimum. For standard Gaussian distribution, we can choose $\alpha$ to be a large enough constant that depends only on $\epsilon$ (independent of the input dimension), hence $\beta$ will also be independent of $d$. This means that for standard Gaussian distribution, for every constant $\epsilon > 0$ we can ensure either convergence to a global minimum, or the loss at $0$ is already $\epsilon$-optimal.

Note that in Remark 3.3 we have shown another distribution which is non-symmetric and depends on the target $v$, such that the loss $F'(0)$ is highly sub-optimal, but gradient flow converges to such a point with probability close to $\frac{1}{2}$.

F  Proofs from Section 6

Before proving Theorem 6.2, we first prove two auxiliary propositions which bounds certain areas for which the vector $w$ cannot reach during the optimization process. The first proposition shows that if the norm of $\bar{w}$ is small, and its bias is close to zero, then the bias must get larger. The second proposition shows that if the norm of $\bar{w}$ is small, and the bias is negative, then the norm of $\bar{w}$ must get larger.

3In Theorem 3.2 we have $\|v\| \neq 1$, but it still holds if we normalize $v$, namely, replace $v$ with $\frac{v}{\|v\|}$.
Proposition F.1. Assume that $\|\hat{w} - \bar{v}\|^2 \leq 1$, and that Assumption 6.1 holds. If $\|\hat{w}\| \leq 0.4$ and $b_w \in \left[0, \frac{\alpha \beta}{640}\right]$ then $(\nabla F(w))_{d+1} \leq -\frac{\alpha \beta}{640}$.

Proof. The $d + 1$ coordinate of the distribution $\mathcal{D}$ is a constant 1. We denote by $\tilde{\mathcal{D}}$ the first $d$ coordinates of the distribution $\mathcal{D}$. Hence, we can write:

$$
(\nabla F(w))_{d+1} = E_{x \sim \mathcal{D}} \left[ (\sigma(w^\top x) - \sigma(\bar{v}^\top x)) 1(w^\top x > 0) \right]
= E_{\tilde{x} \sim \tilde{\mathcal{D}}} \left[ (\sigma(\tilde{w}^\top \tilde{x} + b_w) - \sigma(\bar{v}^\top \tilde{x} + b_\bar{v})) 1(\tilde{w}^\top \tilde{x} > -b_w) \right]
= E_{\tilde{x} \sim \tilde{\mathcal{D}}} \left[ (\tilde{w}^\top \tilde{x} + b_w) \cdot 1(\tilde{w}^\top \tilde{x} > -b_w) \right] - E_{\tilde{x} \sim \tilde{\mathcal{D}}} \left[ (\bar{v}^\top \tilde{x} + b_\bar{v}) \cdot 1(\tilde{w}^\top \tilde{x} > -b_w, \bar{v}^\top \tilde{x} > -b_\bar{v}) \right]
$$

(20)

We will bound each term in Eq. (20) separately. Using the assumption that $\tilde{\mathcal{D}}$ is spherically symmetric, we can assume w.l.o.g that $\tilde{w} = \|\tilde{w}\| e_1$, the first unit vector. Hence we have that:

$$
E_{\tilde{x} \sim \tilde{\mathcal{D}}} \left[ (\tilde{w}^\top \tilde{x} + b_w) \cdot 1(\tilde{w}^\top \tilde{x} > -b_w) \right] = E_{\tilde{x} \sim \tilde{\mathcal{D}}} \left[ (\|\tilde{w}\| x_1 + b_w) \cdot 1( x_1 > \frac{b_w}{\|\tilde{w}\|}) \right] = \|\tilde{w}\| E_{\tilde{x} \sim \tilde{\mathcal{D}}} \left[ x_1 1( x_1 > \frac{b_w}{\|\tilde{w}\|}) \right] + b_w E_{\tilde{x} \sim \tilde{\mathcal{D}}} \left[ 1( x_1 > \frac{b_w}{\|\tilde{w}\|}) \right] \leq 0.4 \|\tilde{w}\| E_{\tilde{x} \sim \tilde{\mathcal{D}}} \left[ x_1 1( x_1 > 0) \right] + b_w.
$$

(21)

Here, (a) is since $\|\tilde{w}\| \leq 0.4$, and $E_{\tilde{x} \sim \tilde{\mathcal{D}}} \left[ 1( x_1 > \frac{b_w}{\|\tilde{w}\|}) \right] \leq 1$, (b) is since $b_w \geq 0$, hence

$$
E_{\tilde{x} \sim \tilde{\mathcal{D}}} \left[ x_1 1( 0 > x_1 > \frac{b_w}{\|\tilde{w}\|}) \right] \leq 0.
$$

For the second term of Eq. (20), we assumed that $\|\hat{w} - \bar{v}\|^2 \leq 1$, which shows that $\theta(\hat{w}, \bar{v}) \leq \frac{\pi}{2}$, and the term is largest when this angle is largest. Hence, to lower bound this term we can assume that $\theta(\hat{w}, \bar{v}) = \frac{\pi}{2}$, and since the distribution is spherically symmetric we can also assume w.l.o.g that $\bar{v} = e_2$, the second unit vector. Now we can bound:

$$
E_{\tilde{x} \sim \tilde{\mathcal{D}}} \left[ (\bar{v}^\top \tilde{x} + b_\bar{v}) \cdot 1(\bar{v}^\top \tilde{x} > -b_w, \bar{v}^\top \tilde{x} > -b_\bar{v}) \right] \geq E_{\tilde{x} \sim \tilde{\mathcal{D}}} \left[ (x_2 + b_\bar{v}) \cdot 1( x_1 > \frac{b_w}{\|\tilde{w}\|}, x_2 > -b_\bar{v}) \right] \geq \frac{1}{2} E_{\tilde{x} \sim \tilde{\mathcal{D}}} \left[ (x_2 + b_\bar{v}) \cdot 1( x_2 > -b_\bar{v}) \right] \geq \frac{1}{2} E_{\tilde{x} \sim \tilde{\mathcal{D}}} \left[ x_2 \cdot 1( x_2 > 0) \right] + \frac{1}{2} E_{\tilde{x} \sim \tilde{\mathcal{D}}} \left[ (x_2 + b_\bar{v}) \cdot 1( 0 > x_2 > -b_\bar{v}) \right] \geq \frac{1}{2} E_{\tilde{x} \sim \tilde{\mathcal{D}}} \left[ x_2 \cdot 1( x_2 > 0) \right] = \frac{1}{2} E_{\tilde{x} \sim \tilde{\mathcal{D}}} \left[ x_1 \cdot 1( x_1 > 0) \right],
$$

(22)

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where we used the assumption \( b_\nu \geq 0 \) and the symmetry of the distribution. Combining Eq. (21), Eq. (22) with Eq. (20) we get:

\[
(\nabla F(w))_{d+1} \leq b_w - 0.1 \mathbb{E}_{\tilde{D}}[x_1 \cdot 1(x_1 > 0)].
\]

Let \( \hat{D} \) be the marginal distribution of \( \tilde{D} \) on the plane spanned by \( e_1 \) and \( e_2 \), and denote by \( \hat{x} \) the projection of \( x \) on this plane. By Assumption 6.1(3) we have that the pdf of this distribution is at least \( \beta \) in a ball or radius \( \alpha \) around the origin. This way we can bound:

\[
\mathbb{E}_{\hat{x} \sim \hat{D}}[x_1 \cdot 1(x_1 > 0)] = \mathbb{E}_{\hat{x} \sim \hat{D}}[x_1 \cdot 1(x_1 > 0)] \\
\geq \frac{\alpha \beta}{2} \mathbb{P}(\alpha/2 < \|\hat{x}\| < \alpha, x_1 > \alpha/2) \\
\geq \frac{\alpha \beta}{2} \mathbb{P}(x_1 \in [\alpha/2, 3\alpha/4], x_2 \in [-\alpha/4, \alpha/4]) = \frac{\alpha^3 \beta}{32}.
\]

Combining the above, and using the assumption on \( b_w \) we get that:

\[
(\nabla F(w))_{d+1} \leq b_w - \frac{\alpha^3 \beta}{320} \leq - \frac{\alpha^3 \beta}{640}.
\]

\[
\square
\]

**Proposition F.2.** Assume that \( \|\tilde{w} - \tilde{v}\| < 1 \), and Assumption 6.1 holds. Denote by \( \tau = \frac{\mathbb{E}_{\tilde{x} \sim \tilde{D}}[x_1 x_2]}{\mathbb{E}_{\tilde{x} \sim \tilde{D}}[x_1^2]} \) where \( \tilde{D} \) is the projection of the distribution \( D \) on its first \( d \) coordinates. If \( \|\tilde{w}\| \leq \frac{\tau}{2} \) and \( b_w \leq 0 \) then \( \langle \nabla F(w)_{1:d}, \tilde{w} \rangle \leq 0 \).

**Proof.** Denote by \( \hat{D} \) the projection of the distribution \( D \) on its first \( d \) coordinates, we have that:

\[
\langle \nabla F(w)_{1:d}, \tilde{w} \rangle = \mathbb{E}_{\hat{x} \sim \hat{D}} \left[ (\sigma(w^\top x) - \sigma(\hat{v}^\top \hat{x})) \cdot 1(w^\top x > 0) \tilde{w}^\top \hat{x} \right] \\
= \mathbb{E}_{\hat{x} \sim \hat{D}} \left[ (\sigma(\tilde{w}^\top \hat{x} + b_w) - \sigma(\tilde{v}^\top \hat{x} + b_v)) \cdot 1(\tilde{w}^\top \hat{x} > -b_w) \tilde{w}^\top \hat{x} \right] \\
\leq \mathbb{E}_{\hat{x} \sim \hat{D}} \left[ \tilde{w}^\top \hat{x} + b_w - \sigma(\tilde{v}^\top \hat{x}) \cdot 1(\tilde{w}^\top \hat{x} > -b_w) \tilde{w}^\top \hat{x} \right].
\]

The inequality above is since \( b_\nu \geq 0 \). Recall that our goal is to prove that the above term is negative, hence we will divide it by \( \|\tilde{w}\| \). Also, since the distribution \( \hat{D} \) is symmetric we can assume w.l.o.g that \( \tilde{w} = \|\tilde{w}\|e_1 \). Hence, it is enough to prove that the following term is non-positive:

\[
\|\tilde{w}\| \mathbb{E}_{\tilde{x} \sim \tilde{D}} \left[ \|\tilde{w}\| x_1 + b_w - \sigma(\hat{v}^\top \hat{x}) \cdot 1(x_1 > -b_w/\|\tilde{w}\|) \right] x_1 \\
= \|\tilde{w}\| \mathbb{E}_{\tilde{x} \sim \tilde{D}} \left[ \|\tilde{w}\|^2 x_1 + b_w x_1 \cdot 1(x_1 > -b_w/\|\tilde{w}\|) \right] - \|\tilde{w}\| \mathbb{E}_{\tilde{x} \sim \tilde{D}} \left[ x_1 \tilde{v}^\top \hat{x} \cdot 1(x_1 > -b_w/\|\tilde{w}\|) \hat{v}^\top \hat{x} > 0 \right] \\
\leq \|\tilde{w}\| \mathbb{E}_{\tilde{x} \sim \tilde{D}} \left[ \|\tilde{w}\|^2 x_1 + b_w x_1 \cdot 1(x_1 > -b_w/\|\tilde{w}\|) \right] - \|\tilde{w}\| \mathbb{E}_{\tilde{x} \sim \tilde{D}} \left[ x_1 \tilde{v}^\top \hat{x} \cdot 1(x_1 > -b_w/\|\tilde{w}\|) \hat{v}^\top \hat{x} > 0 \right].
\]

We will first bound the second term above. Since the term only depend on inner products between \( \tilde{w}, \tilde{v} \) with \( \hat{x} \), we can consider the marginal distribution \( \hat{D} \), of \( D \) on the plane spanned by \( \tilde{w} \) and \( \tilde{v} \). Since \( \hat{D} \) is
symmetric we can assume w.l.o.g that \(D\) is spanned by the first two coordinates \(x_1\) and \(x_2\). Let \(\tilde{v}\) be the projection of \(v\) on this plane, then we can write \(\tilde{v} = (v_1, v_2)\) where \(v_1^2 + v_2^2 = 1\). Note that since the distribution \(D\) is symmetric, we have that \(\mathbb{E}[x_1^2] = \mathbb{E}[x_2^2]\). By Cauchy-Schwarz we have:

\[
|\text{cov}_D(x_1, x_2)| \leq \sqrt{\text{var}_D(x_1) \cdot \text{var}_D(x_2)} = \text{var}_D(x_1)
\]

Again, by symmetry of \(D\) we have that \(\mathbb{E}[x_1] = \mathbb{E}[x_2]\). Opening up the above terms we get that \(\mathbb{E}[x_1 \cdot x_2] \leq \mathbb{E}[x_2^2]\). Also, we assumed that \(\|\tilde{w} - v\| < 1\), then \(\theta(\tilde{v}, \tilde{w}) \leq \frac{\pi}{2}\) which means that \(v_1 \geq 0\). Hence, the second term of Eq. (24) is smallest when \(v = e_2\). In total, we can bound Eq. (24) by:

\[
\begin{align*}
&\|\tilde{w}\|\mathbb{E}_{\tilde{x} \sim D}\left[(\|\tilde{w}\| x_1 + b_w x_2) \cdot 1(x_1 > -b_w \|\tilde{w}\|)\right] - \|\tilde{w}\|\mathbb{E}_{\tilde{x} \sim D}\left[x_1 x_2 \cdot 1\left(x_1 > -b_w \|\tilde{w}\|, x_2 > 0\right)\right] \\
\leq &\|\tilde{w}\|\mathbb{E}_{\tilde{x} \sim D}\left[(\|\tilde{w}\| x_1 + b_w x_1 - \frac{1}{2} x_1 x_2) \cdot 1(x_1 > -b_w \|\tilde{w}\|)\right] \\
= &\|\tilde{w}\|\mathbb{E}_{\tilde{x} \sim D}\left[(\|\tilde{w}\| x_1 + b_w - \frac{1}{2} x_2) \cdot 1(x_1 > -b_w \|\tilde{w}\|)\right]
\end{align*}
\]

By our assumption, \(b_w \leq 0\). Both terms inside the expectation in Eq. (25) are largest when \(b_w = 0\). Hence, we can bound Eq. (25) by:

\[
\begin{align*}
&\|\tilde{w}\|^2 \mathbb{E}_{\tilde{x} \sim D}\left[(\|\tilde{w}\| x_1 - \frac{1}{2} x_2) \cdot 1(x_1 > 0)\right] \\
= &\frac{\|\tilde{w}\|^2}{2} \mathbb{E}_{\tilde{x} \sim D}\left[x_1^2\right] - \frac{\|\tilde{w}\|^2}{4} \mathbb{E}_{\tilde{x} \sim D}\left[x_1 x_2\right] \\
\leq &\frac{\|\tilde{w}\|^2}{2} \mathbb{E}_{\tilde{x} \sim D}\left[x_1^2\right] - \frac{\|\tilde{w}\|^4}{4} \mathbb{E}_{\tilde{x} \sim D}\left[x_1^2\right] = c_1 \left(\frac{\|\tilde{w}\|^2}{2} - \frac{\|\tilde{w}\|^4}{4}\right).
\end{align*}
\]

In particular, for \(\|\tilde{w}\| \leq \frac{\pi}{2}\), Eq. (26) non-positive.

We are now ready to prove the main theorem:

**Proof of Theorem 6.2.** Denote \(b_t = \max\{0, -\frac{b_{w_0}}{\|w_0\|}\}\). We will show by induction on the iterations of gradient descent that throughout the optimization process \(b_t < 2.4 \cdot \max\left\{1, \frac{1}{\sqrt{t}}\right\}\) and \(\theta(\tilde{w}_t, \tilde{v}) \leq \frac{\pi}{2}\) for every \(t \geq 0\).

By the assumption on the initialization we have that \(\|\tilde{w}_0 - \tilde{v}\|^2 \leq \|w_0 - v\|^2 < 1\), and also \(\|\tilde{v}\| = 1\), hence \(\theta(\tilde{w}_0, \tilde{v}) \leq \frac{\pi}{2}\). We also have that \(b_{w_0} \geq 0\), hence \(b_0 = 0\) this proves the case of \(t = 0\). Assume this is true for \(t\). We will bound the norm of the gradient of the objective using Jensen’s inequality:

\[
\begin{align*}
\|\nabla F(w)\|^2 &\leq \mathbb{E}_{x \sim D}\left[(\sigma(w^T x) - \sigma(v^T x))^2 1(w^T x > 0) x^T x\right] \\
&\leq \mathbb{E}_{x \sim D}\left[(w^T x - v^T x)^2 x^T x\right] \\
&\leq \|w - v\|^2 \mathbb{E}_{x \sim D}\left[\|x\|^4\right] = \|w - v\|^2 c.
\end{align*}
\]

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For the \((t + 1)\)-th iteration of gradient descent we have that:

\[
\|w_{t+1} - v\|^2 = \|w_t - \eta \nabla F(w_t) - v\|^2 \\
= \|w_t - v\|^2 - 2\eta \langle \nabla F(w_t), w_t - v \rangle + \eta^2 \|\nabla F(w_t)\|^2 \\
\leq \|w_t - v\|^2 - 2\eta \langle \nabla F(w_t), w_t - v \rangle + \eta^2 \|w_t - v\|^2.
\]

(28)

By Theorem A.2, and the induction assumption on \(\theta(\tilde{w}_t, \tilde{v})\) we get that there is a universal constant \(c_0\), such that \(\langle \nabla F(w_t), w_t - v \rangle \geq \frac{c_0 \beta}{\alpha w_{\tilde{t}}} \|w_t - v\|^2\). Using the induction assumption that \(b_t < 2.4 \cdot \max\left\{1, \frac{1}{\sqrt{\pi}}\right\}\) and Assumption 6.1(3) we can bound \((\alpha - \sqrt{2}b_t) \geq 0.1\). In total we get that \(\langle \nabla F(w_t), w_t - v \rangle \geq \frac{c_0 \beta}{10 \alpha^2} \|w_t - v\|^2\). By taking \(\eta \leq \frac{c_0 \beta}{10 \alpha^2}\) and combining with Eq. (28) we have that:

\[
\|w_{t+1} - v\|^2 < \|w_t - v\|^2.
\]

In particular, \(\|\tilde{w}_{t+1} - \tilde{v}\|^2 \leq \|w_{t+1} - v\|^2 < \|w_t - v\|^2 < 1\), which shows that \(\theta(\tilde{w}_{t+1}, \tilde{v}) \leq \frac{7}{2}\). and concludes the first part of the induction.

The bound for \(b_t\) is more intricate, for an illustration see Figure 2. Let \(t'\) be the first iteration for which \(\|\tilde{w}_{t'}\| \geq 0.4\). First assume that \(t \leq t'\), we will show that in this case \(b_t = 0\). Assume otherwise, and let \(t_0\) be the first iteration for which \(b_{t_0} > 0\). This means that \(b_{w_{t_0}} < 0\) and \(b_{w_{t_0-1}} \geq 0\). We have that:

\[
b_{w_{t_0}} = b_{w_{t_0-1}} - \eta \nabla F(w_{t_0})_{d=1}.
\]

If \(b_{w_{t_0-1}} \leq \frac{\alpha^2 \beta}{640}\), then by Proposition F.1 the last coordinate of the gradient is negative, hence \(b_{w_{t_0}} > b_{w_{t_0-1}} \geq 0\). Otherwise, assume that \(b_{w_{t_0-1}} > \frac{\alpha^2 \beta}{640}\). By Eq. (27): \(|\nabla F(w_{t_0})_{d=1}| \leq \|\nabla F(w)\| \leq \sqrt{\alpha}\). Hence, by taking \(\eta < \frac{\beta}{4 \alpha^2 \sqrt{\alpha}} \leq \frac{\alpha^2 \beta}{640 \sqrt{\alpha}}\), we get that \(b_{w_{t_0}} \geq 0\), which is a contradiction (note that by Assumption 6.1(3), we have \(\alpha \geq 1\)). We proved that if \(t \leq t'\) then \(b_{w_t} \geq 0\), which means that \(b_t = 0\).

Assume now that \(t > t'\). We will need the following calculation: Assume that \(\|\tilde{w}_t\| = \delta\). Then \(\|\tilde{w}_t - \tilde{v}\|^2 \geq (1 - \delta)^2\), and the minimum is achieved at \(\tilde{w} = \delta \tilde{v}\). Since we have:

\[
\|\tilde{w}_t - \tilde{v}\|^2 + (b_{w_t} - b_{v_t})^2 = \|w_t - v\|^2 \leq 1,
\]

we get that \((b_{w_t} - b_{v_t})^2 \leq 1 - (1 - \delta)^2 \leq 2\delta\). If we further assume that \(b_{w_t} \leq 0\), then \(b_{w_t}^2 \leq (b_{w_t} - b_{v_t})^2 \leq 2\delta\). Combining all the above, we get that if \(\|\tilde{w}_t\| = \delta\) then:

\[
b_t = \max\left\{0, - \frac{b_{w_t}}{\|\tilde{w}_t\|}\right\} \leq \sqrt{\frac{2}{7}} \delta.
\]

(29)

To show the bound on \(b_t\) we split into cases, depending on the norm of \(\tilde{w}_t\):

**Case I:** \(\frac{2\sqrt{7}}{7} < \|\tilde{w}_t\| \leq \frac{7}{2}\) and \(b_{w_t} \leq 0\). In this case we have:

\[
\|\tilde{w}_{t+1}\|^2 = \|\tilde{w}_t - \eta \nabla F(w_t)_{1:d}\|^2 \\
= \|\tilde{w}_t\|^2 - 2\eta \langle \tilde{w}_t, \nabla F(w_t)_{1:d} \rangle + \eta^2 \|\nabla F(w_t)_{1:d}\|^2 \\
\geq \|\tilde{w}_t\|^2 - 2\eta \langle \tilde{w}_t, \nabla F(w_t)_{1:d} \rangle.
\]

We can use Proposition F.2 to get that \(\langle \tilde{w}_t, \nabla F(w_t)_{1:d} \rangle \leq 0\), hence \(\|\tilde{w}_{t+1}\|^2 \geq \|\tilde{w}_t\|^2\). By Eq. (29) we get that \(b_{t+1} \leq \sqrt{\frac{5}{7}} \leq \frac{2\sqrt{3}}{\sqrt{7}}\).
Case II: \( \| \tilde{w}_t \| \geq \min \{ 0.4, \frac{x}{2} \} \). In this case, by choosing a step size \( \eta < \frac{1}{4\alpha} \min \{1, \tau \} \) we can bound

\[
\| \tilde{w}_{t+1} \| \geq \| \tilde{w}_t \|^2 - 2\eta (\tilde{w}_t, \nabla F(w_t)) 1:d \\
\geq \| \tilde{w}_t \|^2 - 2\eta \| \tilde{w}_t \| \| \nabla F(w_t) \| 1:d \\
\geq \| \tilde{w}_t \|^2 - 2\eta \| \tilde{w}_t \| \| \nabla F(w_t) \| \\
\geq \| \tilde{w}_t \|^2 - 2\eta \cdot 2c \geq \min \left\{ 0.39, \frac{2\tau}{5} \right\}.
\]

Again, by Eq. (29) we get that \( b_{t+1} \leq \max \left\{ \sqrt{\frac{\gamma_2}{\beta}}, \frac{\eta}{\sqrt{\gamma_2}} \right\} \leq 2.4 \cdot \max \left\{ 1, \frac{1}{\sqrt{\beta}} \right\} \). This concludes the induction.

**Case III:** \( \| \tilde{w}_t \| \leq \min \{ 0.4, \frac{2\tau}{5} \} \). We split into sub-cases depending on the previous iteration: (a) If \( b_{w_t} < 0 \), then by Case I the norm of \( \tilde{w} \) cannot get below \( \frac{2\tau}{5} \), hence this sub-case is not possible; (b) If \( b_{w_t} \geq 0 \) and \( \| \tilde{w}_{t-1} \| \leq \min \{ 0.4, \frac{2\tau}{5} \} \), then by the same reasoning in the case of \( t < t' \), \( b_{w_t} \) cannot get smaller than zero. Hence, we must have that \( b_{w_t+1} \geq 0 \); (c) If \( b_{w_t} > 0 \) and \( \| \tilde{w}_{t-1} \| \geq \min \{ 0.4, \frac{2\tau}{5} \} \) then the bound depend on whether \( \| \tilde{w}_{t-1} \| \) is larger than 0.4 or not. If \( \| \tilde{w}_{t-1} \| \leq 0.4 \), then using the same reasoning as the case of \( t < t' \) twice (both for the \( t - 1 \) and \( t \) iterations) we get that \( b_{t+1} \geq 0 \). If \( \| \tilde{w}_{t-1} \| > 0.4 \) and \( b_{w_t} \geq 0 \), then again this is the same case as in the case of \( t' < t \) (since \( \| \tilde{w}_t \| \leq 0.4 \). The last case is when \( \| \tilde{w}_{t-1} \| > 0.4 \) and \( b_{w_t} < 0 \), here using the same calculation as in Case II, we have that \( \| \tilde{w}_t \| \geq 0.39 \). Since \( \| \tilde{w}_t \| \leq \min \{ 0.4, \frac{2\tau}{5} \} \), using Proposition F.2, the norm of \( \tilde{w}_t \) can only grow, hence by the same reasoning as in Case I we can also bound \( b_{t+1} < 2.4 \cdot \max \left\{ 1, \frac{1}{\sqrt{\beta}} \right\} \).

Until now we have proven that throughout the entire optimization process we have that \( \theta(\tilde{w}_t, \tilde{v}) \leq \frac{\pi}{2} \) and \( b_t \leq 2.4 \cdot \max \left\{ 1, \frac{1}{\sqrt{\beta}} \right\} \). Let \( \delta = \pi - \theta(\tilde{w}, \tilde{v}) \), we now use Theorem A.2 and Eq. (27) to get that:

\[
\| w_{t+1} - v \|^2 = \| w_t - \eta \nabla F(w_t) - v \|^2 \\
= \| w_t - v \|^2 - 2\eta (\nabla F(w_t), w_t - v) + \eta^2 \| \nabla F(w_t) \|^2 \\
\leq \| w_t - v \|^2 - 2\eta \left( \frac{\alpha - \frac{b_t}{\sin(\frac{\pi}{2})}}{8\alpha^2} \right)^4 \beta \sin \left( \frac{\delta}{4} \right)^3 \| w_t - v \|^2 + \eta^2 c \| w_t - v \|^2 \\
\leq \| w_t - v \|^2 - \eta \left( \frac{\alpha - \frac{\sqrt{2}b_t}{\sin(\frac{\pi}{2})}}{8\alpha^2} \right)^4 \sin \left( \frac{\delta}{4} \right)^3 \| w_t - v \|^2 + \eta^2 c \| w_t - v \|^2 \\
\leq \| w_t - v \|^2 - \frac{\eta \bar{C} \beta}{\alpha^2} \| w_t - v \|^2 + \eta^2 c \| w_t - v \|^2 \\
\leq (1 - \eta \eta) \| w_t - v \|^2 \\
\leq \cdots \leq (1 - \eta \lambda)^t \| w_0 - v \|^2,
\]

which finished the proof.
Figure 2: A 2-d illustration of the optimization landscape. The $x$ axis represents $\|\tilde{w}\|$, and the $y$-axis represents $b_w$. In the figure, for simplicity, we assume that $b_v = 0$, and $\tau = 0.1$ which means that $\frac{\tau}{\tau} = 0.4$. The red circle represents the area with $\|w - v\| \leq 1$, throughout the optimization process $w_t$ stays in this circle. The black region represents the area where $b_t = -\frac{b_w}{\|w\|}$ can be potentially large, our goal is to show that $w_t$ stays out of this region. Case I shows that $w_t$ cannot cross the blue region. Case II shows that if $w_t$ is to the right of the black region, then $b_t$ is upper bounded. Case III shows that $w_t$ cannot cross the orange region (sub-cases (a) and (b)), and cannot cross from the green region directly to the black region (sub-case (c)).