A new class of group field theories for first order discrete quantum gravity

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Abstract

Group field theories, a generalization of matrix models for 2D gravity, represent a second quantization of both loop quantum gravity and simplicial quantum gravity. In this paper, we construct a new class of group field theory models, for any choice of spacetime dimension and signature, whose Feynman amplitudes are given by path integrals for clearly identified discrete gravity actions, in first order variables. In the three-dimensional case, the corresponding discrete action is that of first order Regge calculus for gravity (generalized to include higher order corrections), while in higher dimensions, they correspond to a discrete BF theory (again, generalized to higher order) with an imposed orientation restriction on hinge volumes, similar to that characterizing discrete gravity. This new class of group field theories may represent a concrete unifying framework for loop quantum gravity and simplicial quantum gravity approaches.

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1. Introduction

We present, in this paper, a new class of group field theory models for non-perturbative quantum gravity \cite{1–3}. GFTs can be understood as a second quantized formulation of both loop quantum gravity and simplicial quantum gravity, in which wavefunctions over the space of geometries are turned into classical fields first and then into operators. In fact, the quanta of the GFT field correspond to \(D\)-valent spin network vertices and, at the same time, to \((D - 1)\)-simplices. Since both loop quantum gravity and simplicial quantum gravity (in its quantum Regge calculus as well as in its dynamical triangulations formulation) are meant to be themselves a quantization of what is already a classical field theory, i.e. general relativity, a second quantization of the same
provides a formalism very similar in spirit to what was dubbed ‘third quantization’ of gravity [4]. This was a formal quantum field theory on superspace (the space of all 3-geometries on, say, $S^3$) which would have canonical quantum GR corresponding to its quantum 1-particle sector, and would produce a sum over topologies in perturbative Feynman expansion, obtained as interaction processes of quantum universes (the ‘particles’ of this formalism). Unfortunately the formidable mathematical and interpretation difficulties of the formalism in the continuum prevented any substantial development along these lines.

The group field theory formalism [1, 2, 5] greatly improves the situation, both at the mathematical and physical level, by turning to a discrete and local picture of superspace, in which what is propagated, created and annihilated are local chunks of a $(D-1)$-dimensional quantum space (again, spin network vertices or $(D-1)$-simplices). One feature of third quantization is retained, and even generalized, however: on the one hand these fundamental building blocks of quantum space can be combined at the kinematical level to represent, in principle, any spatial geometry and topology, and on the other hand their interactions still produce, in perturbative Feynman expansion, quantum spacetimes of any topology.

This step from a continuum to a discrete setting has another important consequence. If the quantum gravity Feynman diagrams in the third quantized formalism were smooth manifolds, and the continuum path integral for quantum gravity on the given manifold represented, by construction, the Feynman amplitude for each of them, in the discrete version of third quantization provided by group field theories the Feynman diagrams are given by combinatorial and un-embedded 2-complexes or, dually, by simplicial complexes, and the Feynman amplitudes have the interpretation of discrete, and well-defined path integrals (sum over discrete geometries) over each simplicial complex.

Each group field theory completely defines a possible dynamics for loop quantum gravity spin network states encoded in the GFT action as well as, implicitly, in its Feynman amplitudes, thus proposing an explicit solution (even if yet to be put to test) for one outstanding challenge of the LQG programme [6]. At the same time, it does so by defining a sum over histories that includes the crucial ingredients of both main simplicial quantum gravity approaches. As in quantum Regge calculus [7], it defines the dynamics of a quantum and classical simplicial geometry in terms of a sum over discrete geometric data (edge lengths, areas, etc; according to dimension) for a given base simplicial complex. In addition to this, it provides a possible way to encode all the continuum geometric degrees of freedom by a covariant procedure that is alternative to the infinite refinement procedure for a fixed incidence matrix, that has proven to be problematic at the quantum level, in quantum Regge calculus: a sum over inequivalent triangulations. In this it obviously agrees with the dynamical triangulations (DT) approach [8]; in fact, freezing the discrete geometric data attached to each complex (this can be done in various ways) turns the perturbative expansion of GFTs into a form very similar to the sum over triangulations weighted by a purely combinatorial factor that characterizes the DT approach; such formulation, in its causal restriction (corresponding to summing over a certain restricted class of Lorentzian triangulations) [8], has recently proven to be much more successful in recovering a continuum spacetime from the quantum theory. The links between group field theories and other approaches to quantum gravity are detailed in [1, 3], to which we refer.3

3 The crucial open issue that all these approaches face, at present, is the possibility of recovering a continuum description of spacetime, in the appropriate limit, and (a modified version of) Einstein’s general relativity as the effective dynamics in the same limit. Understanding the links between various approaches or even subsuming them within the GFT formalism may be an interesting and useful exercise in many respects, but it will remain futile if the problem of the emergence of continuum physics will not be solved or made easier by doing so. Our point of view on how, in fact, the GFT formalism, may in fact be crucial in order to tackle and solve this issue is discussed at length in [3].
It should be clear, however, that much more remains to be understood about these links and that only further work can confirm or refute the idea that group field theories really represent in concrete terms a unifying framework for all of them, as we have been suggesting. The focus of our present work is on the relation between GFTs and simplicial quantum gravity, and the aim is to make the correspondence between the two frameworks detailed and clear, with a precise matching between GFT Feynman amplitudes and simplicial quantum gravity sum over histories. In order to achieve this, we introduce and analyse in the present paper a new class of group field theories, characterized by Feynman amplitudes which have, in any dimension and in any signature, exactly the form of a simplicial gravity path integral. Its amplitudes, that is, are expressed as a (real) measure part times a phase factor, with phase clearly identified with a simplicial gravity action. In three dimensions, this will mean obtaining a path integral for three-dimensional simplicial gravity in first order formalism, corresponding to a first order Regge calculus action plus higher order \((f(R))-like\) corrections. In higher dimensions, we will obtain instead a path integral, augmented by a sum over triangulations of any topology, for what can be interpreted as topological BF theory with an additional orientation dependence, and, again, higher order (quantum) corrections to the action. This work can be understood as the further development of the line of research on the issue of causality in spin foam quantum gravity and GFT, and on the construction of a unified GFT framework for loop quantum gravity, spin foam models, quantum Regge calculus and dynamical triangulations, that started from an analysis of the issue of causality in spin foam models \([9]\), continued with the development of a refined technique for the construction of causal spin foam models, based on the particle analogy, in \([10]\), with the explicit construction of causal spin foam models for pure gravity and gravity coupled to matter in 3D \([11]\), and with the construction of a generalized GFT formalism \([12]\) based on the techniques and additional variables introduced in \([10]\). The present GFT construction is indeed, in a sense, a much improved and further developed version of the one in \([12]\), in a sense to be detailed in the following.

We will detail both the motivation, the basic ideas and the results of our work in the next section. In section 3 we present the general definition of the new class of GFTs, and the general structure of its Feynman amplitudes. Sections 4 and 5 report instead the detailed form of the amplitudes of these models in three and four dimensions, in both Riemannian and Lorentzian cases, and a discussion of their properties. We conclude with a summary of our results and an outlook on their relevance for further developments in this area.

\section{Motivation for the new models}

Let us now focus on the following questions: what types of amplitudes do we then expect or want the GFT Feynman diagrams to be assigned? How should they look, if they are indeed Feynman amplitudes for a field theory on a simplicial superspace?

\subsection{Causality, orientation dependence, third quantization and quantum (discrete) gravity path integrals}

Consider the simplest case of a 4D spacetime of topology \(\Sigma \times \mathbb{R}\), with compact \(\Sigma\). This spacetime has two boundaries, call them \(\Sigma_1\) and \(\Sigma_2\), to which we associate 3D spatial geometries \(h_1\) and \(h_2\), respectively. Assume now that we can uniquely associate (within a canonical quantum theory) a state \(|h_1\rangle\) with the geometry \(h_1\), and \(|h_2\rangle\) to the geometry \(h_2\). The basic idea underlying the ‘time-less’ characterization of the causal quantum gravity transition amplitudes \(|h_2\rangle|h_1\rangle\) is that, even if these cannot correspond to any time ordering, they do implement a ‘time-less ordering’ or, better, a causal ordering. This consists in the
requirement that $h_2$ lies in the causal future of $h_1$, which in turn can be formulated, when a canonical decomposition of the gravity variables is possible, as the requirement that the lapse function (which in a suitable gauge is equivalent to a proper time) between the two boundaries can only take positive values \[13, 14\]. Note that the formulation of this criterion does not require any use of coordinates. Note also that the above has a direct analogue in the definition of different Green functions for a relativistic particle \[15\], where it defines indeed the Feynman propagator. Moreover, this criterion can be generalized to the situation in which no canonical decomposition is available, for example, keeping the same boundaries and boundary states, for spacetimes of non-trivial topology. In such cases it can be formulated as the requirement that the amplitude is orientation dependent, i.e. that it turns into its complex conjugate if the spacetime orientation is reversed. If the dynamics is defined by a quantum gravity path integral, in metric variables, all these requirements are automatically encoded in the definition of the configuration space as the space of all metrics up to diffeomorphisms and of the quantum amplitude as the exponential of $i$ times the Einstein–Hilbert action (or some higher-derivatives extension), times a diffeo-invariant real measure:

$$\langle h_2 | h_1 \rangle = \int_D g | h_1, h_2 \rangle e^{i S_{\text{EH}}(g)}.$$  \(1\)

Indeed, this corresponds, in a canonical formulation to

$$\langle h_2 | h_1 \rangle = \int D\pi^i D\pi^j \left[ \exp \left( i \int_M d^3x \, dt \, (\pi^i \dot{h}_ij - N \mathcal{H} - N^i \mathcal{H}_i) \right) \right], \quad \text{(2)}$$

with the integration range $\left(0, +\infty\right)$ over the lapse function $N \left[13, 14\right]$. The above amplitude is complex, causal ($h_2$ is in the causal future of $h_1$) and orientation dependent (it turns into its own complex conjugate under a switch of spacetime orientation, as $i S_{\text{EH}}(g) \rightarrow -i S_{\text{EH}}(g)$ under this transformation, equivalent to switching positive to negative lapse). Moreover, it defines, at least formally, a Green function of the Hamiltonian constraint operator, the dynamical equation of motion of pure gravity, for trivial spacetime topology, not a solution of it, i.e. it satisfies $\mathcal{H} \langle h_2 | h_1 \rangle = \delta(h_1 - h_2)$. Notice that the same definition for the quantum gravity path integral results in an orientation-dependent transition amplitude also in the case of Riemannian quantum gravity, i.e. the quantization of Riemannian geometry, even though then the causality interpretation is not applicable to the same restriction on the lapse function.

In the formal third quantized framework, as in usual QFT, the path integral of the field theory itself provides, after field insertions, a definition of the causal transition amplitudes. Indeed, in usual QFT perturbative expansion, the causal ordering is reflected in the presence of the Feynman propagator in each individual particle line of a Feynman diagram. This propagator can be expressed as a sum over histories for the single particle it refers to, i.e. by a path integral weighted by the usual relativistic particle action \[15\]. If one does the same for all the particles involved in a given Feynman diagram, the whole Feynman amplitude can be put in the form of a path integral for a discrete system of particles, weighted by the exponential of the classical action, and with a constraint on their histories in position space, representing the particle interactions. The same happens in the formal third quantization setting for gravity \[4\]: each Feynman diagram, a discrete history of ‘universe interactions’, is weighted by an amplitude given by the path integral \(1\), the exponential of the gravity action, i.e. the geometric action for the particle universe for each line of propagation, plus appropriate joining conditions representing the interactions. In a GFT, in light of its proposed physical interpretation, we would expect a similar structure for the Feynman amplitudes. Therefore, if the GFT degrees of freedom and dynamics are to represent a discrete quantum geometry and its evolution, we would expect the amplitudes associated with its discrete (virtual) histories, the GFT Feynman amplitudes, to have the form of a path integral for discrete gravity, i.e. an exponential of
some classical discrete gravity action. This way, they would have the sought for properties of causality/orientation dependence and complexity on top of making the relation with classical and quantum discrete gravity manifest. Once more, it is the complex nature of the Feynman amplitudes and their having the form of an exponential of some simplicial gravity action, that would permit the interpretation of GFTs as third quantized theories of simplicial geometry and as providing a definition of discrete quantum gravity transition amplitudes.

This is not what happens in current GFT models. The Feynman amplitudes/spin foam models of all current GFTs are instead: real, a causal and orientation independent, so that they do not reflect the orientation of the underlying simplicial complex nor allow for the identification of any ordering between the boundary states. In this sense they define causal transition amplitudes, as discussed in detail in [9, 10].

We would then like to identify the true GFT analogue of the quantum gravity causal transition amplitudes, or, more precisely, we would like to construct group field theories for which, as in ordinary QFT, the Feynman expansion of \( n \)-point functions produces Feynman amplitudes given by the exponential of a discrete gravity action, i.e. with the causality restrictions implicitly, automatically, but also clearly implemented. This, for us, would be a clear sign that we are capturing the causal dynamics of discrete gravity correctly.

2.2. GFTs and simplicial quantum gravity

Another motivation for constructing this new class of GFTs is that they would bring simplicial quantum gravity approaches in much closer contact with the spin foam formalism for discrete gravity path integrals, and, via spin foams, with loop quantum gravity. Actually, there is hope that this new class of GFTs may represent the common unified framework in which both simplicial quantum gravity approaches, quantum Regge calculus and dynamical triangulations, as well as loop quantum gravity/spin foam one can be subsumed, for mutual benefit and further development of each. The general idea of GFTs as a common framework is discussed in [1, 3]. For this to be realized we need once more to construct GFT models with Feynman amplitudes given exactly by the exponential of a simplicial gravity action, times some appropriate measure. This is the step we take with the present work.

An interesting difference between usual simplicial quantum gravity and GFTs is that GFTs are based on a first order formulation of gravity and a group theoretic description of geometry. In other words they refer to a Palatini-like or BF-like formulation of gravity in terms of a D-bein field and a connection field. In a simplicial setting, therefore, we would expect to obtain a formulation of gravity in terms of some first order version of the Regge action, with variables being a discretized D-bein \( e_i \), associated with each link of the simplicial complex, or a discretized bivector field \( B_f \), associated with each \((D - 2)\)-face of the complex, in a BF-like formulation of gravity, plus a discretized connection variable, represented for example in terms of discrete parallel transports (group elements) of the same along dual links \( e^* \) of the simplicial complex, as in all current spin foam models [16]. This action would have a general form of the type:

\[
S = \sum_f V_f(e, B)\Theta_f(g_{e^*}) + \text{(higher order)},
\]

with \( V_f \) being the volume associated with the \((D - 2)\)-face \( f \), which is a function of either \( e \) variable or \( B \) variables, and \( \Theta \) being the corresponding deficit angle, i.e. the discretized curvature. Similar first order formulations of discrete gravity have been proposed and studied, e.g., in [17].
In dimension $D \geq 4$, one would have to add to (3) suitable (non-local) constraints on the discrete $B$ variables, if a BF-like formulation is the one sought for, ensuring their geometric interpretation. In absence of such constraints, in fact, we would just have a discrete version of classical BF theory, as it is clear from the fact that the variation of the action with respect to $B$ would produce the flatness condition $\Theta_f = 0$ for any face of the complex.

With the aim of reproducing the above type of classical action, plus the hope that this will lead to a more straightforward way of imposing the above constraint than in the usual spin foam procedure, and the main motivation of imposing the causality/orientation dependence condition, which is in fact a restriction on the integration range over the $B$ field, we are thus led to introduce additional variables, directly identifiable with the $B$ field of BF theory, into the usual GFT formalism, which is based on the connection variables only (group elements). We will indeed obtain, from our new GFTs, a third quantized version of discrete BF theory in any dimension and any signature, with an additional restriction on orientation automatically imposed, as well as incorporating what can be interpreted as quantum corrections to the above classical action (akin to higher derivative corrections to the Einstein–Hilbert action in effective approaches to continuum gravity). This means that we will obtain a third quantization of discrete first order gravity in 3D, and of an orientation-restricted BF theory in higher dimension.

### 2.3. New GFT variables and the relation between $B$ and $A$ fields in Lagrangian BF theory

In the usual spin foam models, such as the Ponzano–Regge model for BF theory, as well as in the GFT models that generate them, such as the Boulatov model, the variable that is interpreted as the discrete counterpart of the $B$ field of the original continuum BF theory is the representation label $J$ associated with each $(D - 2)$-face of the simplicial complex. This is first of all justified by the way it enters the expression for the spin foam amplitudes, after Peter–Weyl decomposition of the same. In the Riemannian 3D case, for example, one indeed gets, for each dual face [18]

$$
\int d^3 B_f \exp\{i \text{Tr} (B_f G_f )\} = \delta(G_f) = \sum_{J_f} d_{J_f} \chi^{J_f}(G_f).
$$

(4)

$B_f$ is the original discretized $B$ field, given by an $su(2)$ Lie algebra element, with which one starts from when deriving the spin foam amplitudes from a discrete Lagrangian path integral, but that does not appear in the corresponding GFT derivation, from which one just obtains the result of the $B_f$ integral above, i.e. the delta function over the group, forcing the flatness condition on the $SU(2)$ holonomy $G_f$. Starting from this delta function, by harmonic analysis on $SU(2)$ one gets the last expression in (4), which indeed resembles the starting expression but with a discrete replacement for the $B_f$ variables: the representation labels $J_f$. The same happens in the usual GFTs. Apart from the formal similarities, one physical rationale for the identification of the $J_f$ with a discretized $B$ field is the fact that it is conjugate to the connection variables, i.e. to the group elements $g_{e^r}$, in the sense of Fourier transforms, just as the $B$ field is canonically conjugate to the $A$ field in the Hamiltonian formulation of classical BF theory. This reasoning is of course sensible, and it is indeed supported by the respective role group representations and group elements play in loop quantum gravity, again following canonical analysis in the continuum, but it is also not fully conclusive. There are a few reasons for being dissatisfied with this interpretation, even if they are, admittedly, not at all conclusive either. One is that the $J_f$ corresponds more precisely to just one component of the original $B$ field, its
(discretized) absolute value, with the other components still missing any identification within
the formalism.

Our main concern with the identification of representation labels, and, before that, of
the generators of the Lie algebra, acting on connection group elements, with the discrete $B$
field comes from looking at the issue from a more general Lagrangian, rather than canonical
perspective (which is available only for trivial topology). Namely, we are looking for a group
field theory discrete realization of the path integral for a gravity theory in first order form,
which, as we have discussed, is likely to imply a restriction on the configurations of the $B$
field summed over, that would give a different result for the face amplitudes with respect
to (4), as it happens, for example, in the model of [11]. In such a path integral two sets
of variables are present, the geometric $B$ field and the connection, and the relation between
the two is one of the equations of motion of the theory (the one imposing metricity of the
connection) and thus should be imposed only by the dynamics of the theory, and not imposed
already at the kinematical level at the level of each history being summed over in the path
integral, as it appears to be done in current GFT models. We feel that imposing such a
condition already at the kinematical level results, in the usual spin foam models, in freezing
a part of the degrees of freedom of the theory. In particular it may be this restriction is
what is responsible for turning what should have been causal transition amplitudes into rather
awkward, from the GFT and third quantization perspective, a causal transition amplitudes,
imposing the canonical dynamical constraints even in situations, e.g. non-trivial spacetime
topologies, where a canonical interpretation is problematic and certainly not expected.

3. The new models

3.1. Basic idea behind the construction

BF theory in $D$ dimensions for a group $G$ with a Lie algebra $\mathfrak{g}$ is a topological field theory
defined by the action

$$S = \int_M \text{Tr}(B \wedge F(A)),$$

where $M$ is a $D$-dimensional manifold, $B$ can be thought of locally as a $\mathfrak{g}$-valued $(D-2)$-form
and $F$ is the curvature of the $G$-connection $A$, so it can also be thought of locally as a $\mathfrak{g}$-valued
2-form.

Let us now describe our main strategy and its rationale, illustrating it for simplicity in the
$D = 3$ case. The extension to different dimensions is straightforward and follows the same
type of arguments. It will be discussed in detail in the following.

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4 The situation, in this respect, has been ameliorated somewhat by the recent development of new spin foam models
for BF theory and gravity [19–23] based on coherent states. Here, the additional parameters labelling a coherent state
basis of vectors in each representation space $J_f$ are interpreted as the spin foam analogue of the missing components
of the $B$ field with modulus $J_f$. This is justified by the fact that the expectation value of a Lie algebra generator
in a coherent state is given by a (bi-)vector with modulus $J_f$ and components proportional to the coherent state
parameters. However, there are still several questions unanswered about the relation between a generic $B$ field and the
Lie algebra generators, the exact physical role played by coherent states, apart from their mathematical convenience,
etc. Moreover, since the above is a relation that concerns the expectation values of quantum operators, one may
suspect that it should be understood as a semi-classical one, holding only in some approximation of the dynamics
of the quantum theory. As we will discuss later on, we feel that our approach of introducing additional independent
variables playing in a straightforward sense the role of the discrete $B$ field, and whose relation with Lie algebra
generators for the group considered is governed by the dynamics of the theory, may help to clarify, with further work,
the role that coherent states play at the level of spin foam amplitudes.

5 Globally it is a section of the bundle associated with the principal $G$-bundle over which $A$ is defined via the adjoint
representation.
• We would like to introduce additional variables, corresponding to a discrete $B$ field associated with each 1-simplex in the simplicial complex, in the GFT perturbative expansion. This means that there should be one such variable for each argument of the GFT field.

• We would like the new variables to be identified with the generators of the Lie algebra of the relevant group. This implies that they must have the same number of components. The field should then be a complex function $\phi(g_1, B_1; g_2, B_2; g_3, B_3) : (G \times \mathbb{R}^3)^3 \to \mathbb{C}$ in this 3D example. The complexity of the field, together with symmetry under even permutation of the arguments, is needed to ensure orientability of the simplicial complexes arising in perturbative expansion.

• The identification should follow from some equation of motion of the theory, so to be part of the dynamics; at the same time, it should belong to the kinematical sector of the GFT, because we would like boundary states to satisfy it, at least partially, so to have a similar structure to that of loop quantum gravity spin network states.

• Such an equation of motion should then be of the type $B^i_a - J^i_a = 0$, where the indices $a = 1, 2, 3$ label the arguments of the field, while the indices $i$ are vector indices in $\mathbb{R}^3$, and $J^i_a$ are the generators of the $SU(2)$ or $SL(2, \mathbb{R})$ Lie algebra. However, the above equation is not invariant under group transformations, so we turn it into a covariant form, obtaining $B^2_a - J^2_a = 0$, for each argument of the field.

• The field being a function on the group, the generators of the corresponding Lie algebra act on the group arguments of the field as derivative operators, so that the above equation is actually implemented, in configuration space (with respect to $G$) as $B^2_a - \Box_a = 0$, where $\Box$ is the Laplace–Beltrami operator acting on the group manifold.

• After harmonic analysis, the $\Box_a$ is turned into the invariant Casimir of the group $G$, in a given representation $j_a$, acting as a multiplicative operator on the field now function on the same representation parameters $j_a$.

• $B^i_a$ act here as multiplicative operators; however, we can independently perform Fourier analysis on the $B$ variables as well, going to conjugate variables $X_a$, also in $\mathbb{R}^3$, and turn the quantity $B^2_a$ into a differential operator, a new Laplace–Beltrami operator acting on $\mathbb{R}^3$.

• By means of this extension of the group field theory formalism, we want also to reproduce proper simplicial gravity path integrals in perturbative expansion. Considering that, in the Regge formalism for discrete gravity, D-simplices are assumed to be flat and the whole dynamics of geometry comes from the boundary terms [7], we obtain a further motivation for restricting the modification of the GFT dynamics with respect to usual models to be confined to only the kinematical term in the GFT action.

• The interaction term is only modified by the extension in the number of variables as well as in a peculiar orientation dependence, in the variables $X$, Fourier conjugate to $B$, that is necessary to ensure the proper matching of $B$ variables across simplices, and encoded in the dependence on the complex structure of the field $\phi$, as we will see. As for the dependence on the group $G$, it maintains the same structure of the usual models describing BF theory.

In this way we obtain a new kinetic term given by a differential operator acting on the field, very similar to the usual Klein–Gordon operator of scalar field theory, but with a product structure coming from the independent action of one operator of the above type acting on each argument of the field: $K = \prod_a (\Box X_a - \Box g_a)$.

Note that there is almost nothing in the above choices that can select any specific dynamics of the geometric data ($B$ variables and group elements, say) at the level of the individual
Feynman diagram. The only dynamical ingredient above is the choice of a certain relation between them, but nothing seems to dictate, at the level of the GFT action, the individual dynamics of each set of variables. What we put in is then only (a) some complex structure resulting from the propagator representing the inverse of the chosen kinetic term, due to its singular nature as a differential operator, (b) the mentioned mutual relation between $B$'s and $g$'s, and (c) the combinatorics of the Feynman diagrams (dictated by the combinatorics of the variables in the action). It is only to be expected, then, that the simplicial action describing their dynamics at the level of each Feynman diagram, and appearing in the exponent of the phase part of the Feynman amplitudes (simplicial gravity path integral) will be pretty generic. The non-trivial tests will be to show (1) that this phase can be interpreted at all as a simplicial gravity action, because of the way the GFT variables will enter in it and (2) that this generalized action will reduce to the usual Regge action (in first order form) in appropriate, clearly identified and physically meaningful limits. Our proposed formalism passes these tests.

3.2. The new models: action and Feynman amplitudes

We now give the definition of the action for the new class of GFT models, for general dimension $D$ and general gauge group $G$.

Let $G$ be a semi-simple group (we will deal with the double covers of the rotation and the Lorentz groups in $D$ dimensions) and let $X$ be a space isomorphic, as a metric vector space, to the Lie algebra $g$ of $G$. The basic variable of the theory is a complex-valued field

$$\phi(g_1, g_2, \ldots, g_D; X_1, X_2, \ldots, X_D): G \times \cdots \times X \times \cdots X \to \mathbb{C},$$

where $D$ is the dimension of the model, which is the dimension of the generated simplicial complexes (we will concentrate on the three- and four-dimensional cases).

The field is interpreted as a $(D - 1)$-simplex, with the group and Lie algebra variables corresponding to its geometry. The group elements represent discrete parallel transports of a connection (the discrete analogue of the $A$ of BF theory) from the centre of the simplex to the boundaries, while the $X$ variables allow us the reconstruction of the volumes of the boundary $(D - 2)$-simplices, and are thus related to the $B$ field of BF theory $^6$.

The field is assumed to be invariant under even permutation of the labelling of its (pairs of) arguments $(g_i, X_i)$, and to turn into its own complex conjugate under change of this labelling by an odd permutation. In this way, the orientation of the corresponding $(D - 1)$-simplex is encoded in the complex structure of the field $^1$. $^{24}$

As in usual GFT models, geometric closure of the $(D - 2)$-simplices which form this $(D - 1)$-simplex translates into invariance of the field under the global symmetry $\phi(g_1 h, g_2 h, \ldots, g_D h; X_1, \ldots, X_D) = \phi(g_1, g_2, \ldots, g_D; X_1, \ldots, X_D)$ $^1$. We will impose this symmetry in the usual way by taking the field to be arbitrary and then projecting it onto the diagonal subspace, i.e. the field $\phi(g_i; X_i)$ is given by $\phi(g_i; X_i) = \int_G d h \phi(g_i h; X_i)$, where $\phi(g_i; X_i)$ is now arbitrary. Below, to reduce clutter, we will write the actions in terms of the $\phi$'s instead of the $\hat{\phi}$'s.

Also, we will denote both the field and its complex conjugate by $\phi^v$, with $v = \pm 1$ and $\phi^+ = \phi$ and $\phi^- = \phi^*$.

$^6$ As we shall see later on, it is the norms of the Fourier conjugate variables of the $X$'s, what we call below the $P$'s, that are to be interpreted as volumes, and are to be interpreted as the discrete analogue of the $B$ field of BF theory.
The model is defined by the following action

\[
S = \frac{1}{2} \int_{G^D} \left( \prod_{i=1}^{D} dg_i \right) \int_{X^D} \left( \prod_{i=1}^{D} dX_i \right) \phi^*(g_i; X_i) \left[ \prod_{i=1}^{D} \left( -\Box X_i + \Box_G - \frac{d}{24} m^2 \right) \right] \phi(g_i; X_i)
\]

\[
+ \frac{\lambda}{(D+1)!} \sum_{v_1, \ldots, v_{D+1}} \int_{G^{D+1}} \left( \prod_{i=1}^{D+1} dg_i \right) \int_{X^{D+1}} \left( \prod_{i \neq j} dX_{ij} \right) \times \prod_{i<j} \delta(g_{ij}g_{ji}^{-1}) \delta(v_i X_{ij} + v_j X_{ji}) \phi(v_i; X_{ij}) \cdots \phi(v_{D+1}; X_{D+1j}).
\]

(6)

\(\Box X\) and \(\Box_G\) are the Laplace–Beltrami operators on \(X\) and on \(G\) respectively, corresponding to the Killing form\(^7\) and the Cartan–Killing metric, and \(d\) is the dimension of \(G\) and \(X \cong g\).

As in usual GFTs, the combinatorics of arguments in the action is designed in such a way as to mimic the combinatorics of the \((D-2)\)-faces of a \(D\)-simplex in the interaction term, and the gluing of two \(D\)-simplices across their common boundary in the kinetic term.

The sum over \(v_i\) in the second term makes the action real. Interpreting the \(\phi\) as representing \((D-1)\)-simplices which are ‘incoming’ or ‘in the past boundary’, and the \(\phi^*\) as representing \((D-1)\)-simplices which are ‘outgoing’ or ‘in the future boundary’ with respect to the \(D\)-simplex corresponding to the GFT interaction vertex, we see that there are \(D+2\) possible vertices, corresponding to the cases in which \((D+1)\)–\(n\) ‘initial’ \((D-1)\)-simplices interact to give rise to \(n\) ‘final’ \((D-1)\)-simplices after the interaction has taken place. In turn these various terms correspond to the well-known \((D-1)\)-dimensional Pachner moves. As noticed above, the orientation of the \((D-1)\)-simplices, inducing a pre-order\([9, 10, 12]\) also on the set of \(D\)-simplices, and turning the resulting Feynman diagrams into directed graphs is encoded in the complex structure of the fields. For simplicity of presentation, we have chosen the weight the various interaction terms corresponding to different choices of \(v_i\)’s with the same coupling constant \(\lambda\); it is straightforward to relax this assumption defining coupling constants \(\lambda_{v_i}\), with \(\lambda_{v_i} = \lambda_{v_i}^*\) in order to ensure reality of the action, as was also done in \([12]\).

Let us remark that it is possible to choose a different vertex from the one above. One in which there is no dependence on the \(v\)’s, in the \(X\) variables, and this dependence is instead shifted to the \(P\) variables:

\[
\text{Vertex} = \prod_{i<j} \delta(g_{ij}g_{ji}^{-1}) \delta(X_{ij} - X_{ji}).
\]

Below, we will call the model given by (6) model A, while the one with this new vertex model B.

Note also that the kinetic operator is just a product of \(D\) copies of the Klein–Gordon one for a massive scalar field living in \(X \times G\), one for each pair of arguments of the field.

\(^7\) We would like to draw the reader’s attention to the fact that we are using opposite conventions for the Killing form and the Cartan–Killing metric. This means that the metric entering the definition of \(\Box_G\), is obtained by extending, using e.g. left-invariance, the negative of the metric used to define \(\Box_X\). The reason for this choice of conventions comes from the fact that if one uses the same sign for the \(\Box_G\) as the one used in the mathematical literature\([25, 26]\) for the \(\Box_G\), then one gets that a negative-definite operator in the case \(G\) is compact. So, for example, if \(G = SU(2)\), then the corresponding \(\Box_G\) would have been given (in the appropriate coordinates) by \(-\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}\right)\). On the other hand, with our conventions it is just the usual Laplacian on flat space.
Let us write the above action in ‘momentum’ space with respect to the \(X\) variables. We will denote the space dual to \(X\) as \(P\). Thus (6) is equal to

\[
S = \frac{1}{2(2\pi)^D} \int_{G^D} \left( \prod_{i=1}^{D} dg_i \right) \int_{P^D} \left( \prod_{i=1}^{D} dP_i \right) \phi^\ast(g_i; P_i) \left[ \prod_{i=1}^{D} \left( P_i^2 + \Box G_i - \frac{d^2}{24} m^2 \right) \right] \phi(g_i; P_i)
\]

\[+ \frac{\lambda}{(2\pi)^{D(D+1)}(D+1)!} \sum_{\nu_1, \ldots, \nu_D=1} \int_{G^D} \left( \prod_{i \neq j=1}^{D+1} dg_{ij} \right) \int_{P^{D+1}} \left( \prod_{i \neq j=1}^{D+1} dP_{ij} \right) \times \left[ \prod_{i<j} \delta(g_{ij} g_{ji}^{-1}) \delta(P_{ij} - P_{ji}) \right] \phi^{\nu_1}(g_1; P_1) \ldots \phi^{\nu_{D+1}}(g_{D+1}; P_{D+1}). \tag{7}
\]

\(P^2\) is the magnitude of \(P\) with respect to the Killing form. The kinetic term can be interpreted as the product of \(D\) Klein–Gordon operators on the group \(G\), and for a particle/field of (variable) mass square \(\frac{d^2}{24} - P_i^2\).

The written action is the one associated with model A (i.e. equation (6)). Note that the orientation dependence, i.e. the dependence of the vertex term on the \(\nu_i\)’s, is apparently lost in going to the \(P\) variables (of course, the vertex has this form in the \(P\) variables exactly because of the \(\nu\)-dependence in the \(X\) variables, thus this dependence is retained). In model B, instead, the vertex in the \((g, P)\) variables becomes

\[
\text{Vertex} = \left[ \prod_{i<j} \delta(g_{ij} g_{ji}^{-1}) \delta(P_{ij} - P_{ji}) \right] \phi^{\nu_1}(g_1; P_1) \ldots \phi^{\nu_{D+1}}(g_{D+1}; P_{D+1}).
\]

So the vertex of model B depends explicitly on the \(\nu\)’s in the \((g, P)\) variables, and not in the \(X\) variables. We will see that the Feynman amplitudes, when we use the \(P\) variables, thus in both the \((g, P)\) and \((J, P)\) representations, are the same for both models. The difference between them is apparent only when the \(X\) variables are invoked. Since, as we shall see later on, it is the \(P\) variables that have clear physical significance, and we are going to deal extensively only with the \((g, P)\) and \((J, P)\) representations, we shall not draw the distinction between the two versions of the model in what follows.

We can also perform the ‘Fourier transform’ with respect to the group variables. Expanding the field harmonically on the group and using its invariance under the global right shifts \([1, 5]\), we get

\[
\phi(g_i; P_i) = \sum_{J, \Lambda, \alpha_i} \phi^{J, \Lambda}_{\alpha_i}(P_i) \prod_{i=1}^{D} (D_{\alpha_i}^{\Lambda})(g_i)_{\hat{r}_i}^{J} \phi^{J, \Lambda}_{\alpha_i}.
\]

The \(J\)’s label the representations of the group \(G\). The index \(J\) can go over both discrete and continuous values in general as is the case for the Lorentz group. The \(D\)’s are the representation functions (the components of the representation matrices). \(t\) is an appropriate normalized intertwiner (between the representations labelled by \(J_1, \ldots, J_D\)), and \(\Lambda\) labels the different basis elements of the space of normalized intertwiners.

\[\text{Our convention for the Fourier transform is}
\]

\[
f(\bar{\rho}) = \int d^n \bar{x} \phi^{\bar{\rho} \bar{x}} f(\bar{\xi}), \quad f(\bar{\xi}) = \frac{1}{(2\pi)^n} \int d^n \bar{\rho} e^{-i\bar{\rho} \cdot \bar{\xi}} f(\bar{\rho}),
\]

where the vectors denote the coordinates in which the appropriate Killing form has a canonical form (diagonal matrix with \(\pm 1\) along the diagonal).
A very important property of the Laplace operator is that it is multiplicative on the representation functions (see [25] and references therein). More precisely, \( \Box \alpha (g) = C_J D^J (g) \) where \( C_J \) is the appropriate Casimir operator and the minus sign is used for compact groups while the plus sign for the noncompact ones.

Inserting the above into (7) we get

\[
S = \frac{1}{2(2\pi)^{D+1}} \sum_{P_D} \int_{P^0} \left( \prod_{i=1}^{D} dP_i \right) \phi^{\mu, \Lambda}_{\alpha^1} (P_1) \left( \prod_{j=1}^{D} \left( P_i^2 - \frac{d^2 m^2}{24} \right) \right) \phi^{\mu', \Lambda'}_{\alpha^2} (P_1) \\
+ \frac{\lambda}{(2\pi)^{D(D+1)}(D+1)!} \sum_{I_D, \alpha_I : I^{ij} \neq 0} \int_{P_{D(D+1)}} \left( \prod_{i=1}^{D+1} dP_{ij} \right) \phi^{\mu, \Lambda}_{\alpha^1} (P_{ij}) \ldots \phi^{\mu', \Lambda'}_{\alpha^2} (P_{ij}) \left( \prod_{i<j} \delta (P_{ij} - P_{ji}) \delta_{\alpha^1 i, \alpha^2 j} \right) .
\]

The interaction term is essentially the standard one, which is a product of fields whose arguments are contracted in the pattern of a \( D \)-simplex multiplied by the appropriate \( J \)-symbol. The only difference being that now it is not only the alphas and the \( J \)'s as well.

We quantize the theory now via the path-integral method. The partition function is given by

\[
Z = \int D\phi \, D\phi^* \, e^{iS[\phi, \phi^*]} .
\]

In lack of a better understanding of the quantum theory and of more powerful tools, we study the quantum dynamics of the theory in perturbative expansion around the vacuum, expanding the partition function in Feynman diagrams \( \Gamma \) in the usual way. We get

\[
Z = \sum_{\Gamma} \frac{\lambda^{|V_\Gamma|}}{\text{sym}(\Gamma)} Z_\Gamma ,
\]

where \( V_\Gamma \) is the number of vertices in the Feynman diagram \( \Gamma \), \( \text{sym}(\Gamma) \) is the symmetry factor of the diagram (order of automorphisms of the diagram/complex), and \( Z_\Gamma \) is the Feynman amplitude for the graph \( \Gamma \) obtained as is customary by taking the product of vertex functions and Feynman propagators, obtained by inverting the kinetic operator in the action. We then set out to extract vertex and propagator from our classical action. Let us begin with the vertex contribution. It is clear that the interaction term in the \((g, p)\) variables (equation (7)) is exactly

9 The reason there is this difference in the sign is because the Casimirs for the compact group are defined using the negative of the Killing form. So, for \( SU(2) \) the natural Casimir from the point of view of the Killing form would be \(-J_1^2 - J_2^2 - J_3^2 \) which is minus the usual one. The space of Casimirs of the rotation and the Lorentz groups in three dimensions is one dimensional, while in four dimensions it is two dimensional. In four dimensions, the Casimir that corresponds to the Laplace–Beltrami operator in the Riemannian case, where the representations of \( \text{Spin}(4) \) are labelled by a pair of spins \((J_1, J_2)\), is proportional to \( J_1 (J_1 + 1) + J_2 (J_2 + 1) \), while for the Lorentzian case, where representations of \( \text{SL}(2, \mathbb{C}) \) are labelled by an integer \( n \) and a real number \( \rho \), it is proportional (in our normalizations) to \( \rho^2 - n^2 + 2 \).

10 The easiest way to define this object is to use the diagram of a \( D \)-simplex. Thus take such a diagram, consisting of \((D+1)\) vertices and \((D+1)\) edges connecting them. Assign a representation label to each edge and choose at every vertex a normalized intertwiner (an invariant tensor) in the \( D \)-fold tensor product of the representations assigned to the edges meeting at this vertex. Now contract the tensor indices of the intertwiners in the pattern dictated by the edges of the \( D \)-simplex. The resulting object is what we call a \( J \)-symbol.
like the interaction terms in the usual GFTs for BF theory with the sole difference being the extra variables (which are contracted in exactly the same way as the group variables).

The vertex amplitude is then just the usual one, which is nothing but a product of delta functions connecting the group arguments in the $D$-simplex pattern, with the addition of extra delta functions connecting the $P$ variables paralleling the group ones. In other words, if we represent the vertex in the standard way\cite{1} we see that it consists of $(D + 1)$ bundles, of $D$-strands each, joined together in a pattern of a $D$-simplex (in the shaded area of figure 1).

Each strand represents a product of a delta function on the group with a delta function on the Lie algebra. The dark dots represent the arguments of the delta functions. Since we never have a situation when several strands meet at a point, it is obvious that there is no real interaction enforced by this vertex, at least not in usual local QFT sense, just a rerouting of the strands. It is in this sense that GFTs are sometimes referred to as ‘combinatorially non-local field theories’.

We now move on to the propagator. The easiest way to get it is to use the action written in terms of the $(J, P)$ variables, and the expansion of a delta function on a group in terms of the characters. We will not really need the explicit form of the character functions nor the precise values of the coefficients, rather just the fact that such an expansion is possible. For all the groups that we will consider in this work this is indeed the case\footnote{The groups we are interested in are the rotation and the Lorentz groups in $D$ dimensions, $SO(D)$ and $SO(1, D - 1)$, and their double covers. For these groups, the Plancherel expansion does exist and it has the above general form.}. We will use the following notation for this expansion

$$\delta(g) = \sum_J \Delta_J \chi_J(g), \quad (10)$$

where $\chi_J(g)$ is the character of the representation labelled by $J$, and as before the index $J$ can go over both discrete and continuous values (the sum standing for the usual sum or for the integral, respectively).

From this expansion and from the expression of the kinetic operator in (8) we can immediately read off the expression of the Feynman propagator\footnote{The integral over the $h$ follows from the invariance of the field under the right shifts. This integral will be important when we will compute the dual face amplitudes as the $h$’s are the only variables that will be left after all the ‘gluing’ integrals are done.}

$$D_F[g_i, h_i; P_i, Q_i] = \int_G dh \prod_{i=1}^D \left( \sum_J \frac{i\Delta_J}{p_i^2 + C_J - \frac{dm^2}{2} + i\epsilon} \chi_J(g_i h_i^{-1}) \delta(P_i - Q_i) \right). \quad (11)$$
The above expression, indeed, satisfies
\[
\prod_{i=1}^{D} \left( P_i^2 + \Box_i - \frac{d}{24} m^2 \right) D_F[g_i, h_i; P_i, Q_i] = i \int_G dh \prod_{i=1}^{D} \left( \delta(g_i h_i^{-1}) \delta(P_i - Q_i) \right). 
\] (12)

The fact that it is the Feynman propagator, as opposed to some other Green’s function, follows as usual from the \(i\epsilon\) prescription used in (11), as it is clear by recalling that the kinetic operator in (8), as noticed above, is essentially the Klein–Gordon operator in momentum variables.

To sum the series and obtain the propagator in the \((g, P)\) variables as opposed to the \((J, P)\) above, we take advantage of the fact that, once more, the kinetic term in the \((g, P)\) variables in (7) is just the product of Klein–Gordon ones on the group \(G\) with the mass given by \((P^2 - \frac{d}{24} m^2)\), and use the Feynman–Schwinger–DeWitt parametrization of the propagator [25, 27]. This parametrization relates the Klein–Gordon propagator of a massive scalar field on a space to the Schrödinger evolution kernel on that space, in a fictitious proper time parameter \(t\). For a short review of the properties of the Schrödinger kernel see the appendix, here we just give the result, which is given by
\[
D_F[g_i, h_i; P_i, Q_i] = \int_G dh \prod_{i=1}^{D} \left( K \left[ g_i h_i^{-1}, \left( P_i^2 - \frac{d}{24} m^2 \right) \right] \delta(P_i - Q_i) \right). 
\] (13)

where the \(K\) is the Schrödinger kernel in the mass representation.

As was mentioned above, the Feynman propagator for our theory in the \((g, P)\) variables is just (a product of \(D\) copies of) the Klein–Gordon propagator for a free particle on the group, with the mass equal to
\[
\mu = P^2 - \frac{d}{24} m^2.
\]

Analogously to the vertex above we represent the propagator by a bundle of \(D\) strands as is shown in figure 2.

Each strand represents a multiplicand from the right-hand side of equation (13), i.e. the \(i\)th strand is
\[
K \left[ g_i h_i^{-1}, \left( P_i^2 - \frac{d}{24} m^2 \right) \right] \delta(P_i - Q_i).
\]

The box across all the strands represents the common integral over \(h\). The dark dots at the ends of the strands represent the remaining arguments of the propagator \((g_i, P_i)\) on one side and \((h_i, Q_i)\) on the other. The reason why we have drawn the strands in the propagator differently from those in the vertex is that in distinction to the situation in the usual models where the strands represent the same thing (simple delta functions), this is not the case here, as a true propagation of degrees of freedom takes place between simplices, even though only a rerouting of the same occurs within each simplex.
Since we now have both the necessary ingredients, the vertex and the propagator, we proceed to construct explicitly the Feynman amplitudes. A Feynman diagram $\Gamma$ is obtained by gluing several vertices together using the propagators. If we pick one of the strands and follow it around the diagram, in the absence of external legs, as in the diagrams emerging from the perturbative expansion of the partition function, the strand closes back on itself. We can think of this loop as the boundary of a two-dimensional surface which we assume has the topology of a disk. The combinatorics of the vertex is such that if we take all these disks together they form the dual 2-complex $T^* \approx \Gamma$ of a simplicial $D$-complex $T$, the original disks being the 2-cells topologically dual to the $(D - 2)$-dimensional subsimplices in the simplicial complex (for details consult [1, 5]).

In the $(g, P)$ variables the Feynman graph amplitude factorizes per dual face (or, equivalently, per edge of the triangulation), i.e. the amplitude for a graph $\Gamma$ is a product of dual face amplitudes:

$$ Z_\Gamma = Z_{\Gamma'} = \int_{G^{T^*}} \left( \prod_{i=1}^{E^*} dh_i^* \right) \int_{P^{T^*}} \left( \prod_{i=1}^{F^*} dP_i \right) \prod_{f^* \in T^*} A_{f^*}(h_{e^*} \delta_{f^*}; P_{f^*}), \quad (14) $$

where, $E^*$ is the number of the dual edges in $T^*$ and $F^*$ is the number of the dual faces, and $A_{f^*}$ is the amplitude assigned to each dual face $f^*$. This amplitude depends on the group elements $h_{e^*}$ that are assigned to the dual edges $e^*$ on the boundary of the dual face $f^*$, and that result from the gauge symmetry of the field $\phi$ under $G$ (see [1]), and on a single $P$ variable associated with the whole dual face $f^*$ left after doing all the delta functions over intermediate momenta. More precisely, this amplitude is just a product of kernels with delta functions, integrated over the common group and momentum $P$ variables, and for a dual face with $N$ vertices (and thus $N$ links) it is given by

$$ A[h_1, \ldots, h_N; P] = \int_{G^{N}} \left( \prod_{i=1}^{N} dg_i g_i^{-1} \right) \prod_{i=1}^{N} K\left[ g_i h_i g_i^{-1}, \left( p^2 - \frac{d}{24} m^2 \right) \right] \times \left( \prod_{i=1}^{N} \delta(g_i^{-1} g_i^{-1}) \right), \quad (15) $$

where $g_0 = g_N$. The first multiplicand is just the propagators which are sitting on the dual edges, while the second multiplicand is the delta functions coming from the vertices\(^\text{13}\). See figure 3 for a diagrammatic representation of the above expression.

We can use the delta functions coming from the vertices to do the integrals over the $g$’s obtaining

$$ A[h_1, \ldots, h_N; P] = \int_{G^{N}} \left( \prod_{i=1}^{N} dg_i \right) \prod_{i=1}^{N} K\left[ g_i^{-1} h_i g_i^{-1}, \left( p^2 - \frac{d}{24} m^2 \right) \right]. $$

We would like to do the integrals over the remaining $g$’s and obtain something which depends only on the holonomy around the dual face, computed through the $h$ variables only, as in usual spin foam models and GFTs. However, the Schrödinger kernels in the mass representation in the $g$-variables do not compose in any simple way. To bypass this difficulty we use again the Feynman–Schwinger–DeWitt representation for the kernels in the previous equation:

$$ A[h_1, \ldots, h_N; P] = \int_{G^{N}} \left( \prod_{i=1}^{N} dg_i \right) \int_{R^{N}} \left( \prod_{i=1}^{N} dt_i \right) \prod_{i=1}^{N} e^{\delta(P^2 - \frac{d}{24} m^2) \theta(t_i)} K\left[ g_i h_i g_i^{-1}, t_i \right]. $$

\(^\text{13}\) We drop the infinite constant $\delta(0)$ which is a consequence of the translational symmetry in the $P$ variables, leaving the detailed treatment of this symmetry for future work.
Figure 3. This is a picture of all the variables appearing in (15) in the amplitude for a dual face. The black boxes represent the integrals over h's.

Now, since the kernels in the proper time representation do satisfy the composition identity (A.3) we can (after interchanging the order of integration) perform the group integrals obtaining

\[ A[h_1, \ldots, h_N; P] = \int_{\mathbb{R}^N} dt_1 \ldots dt_N \exp \left( i \left( P^2 - \frac{d}{24} m^2 \right) (t_1 + \cdots + t_N) \right) \theta(t_1) \ldots \theta(t_N) K \times [h_1 \ldots h_N, t_1 + \cdots + t_N]. \]

The product of the group elements in the kernel is exactly the holonomy around the dual face which we will denote by H. Thus \( A[h_1, \ldots, h_N; P] = A_N[H; P] \).

To do the integrals over the proper times we change variables

\[ A_N[H; P] = \int_{\mathbb{R}^N} dT dt_2 \ldots dt_N \exp \left( i \left( P^2 - \frac{d}{24} m^2 \right) T \right) \times \theta(T - t_2 - \cdots - t_N) \theta(t_2) \ldots \theta(t_N) K[H, T]. \]

The integrals over \( t_2, \ldots, t_N \) can now be performed as these variables appear only in the step functions giving

\[ A_N[H; P] = \frac{1}{(N-1)!} \int_{\mathbb{R}} dT \exp \left( i \left( P^2 - \frac{d}{24} m^2 \right) T \right) \theta(T) T^{N-1} K[H, T]. \] (16)

What we have shown above is that the dual face amplitude in the \((g, P)\) variables is the value at \( (P^2 - \frac{d}{24} m^2) \) of the Fourier transform of a monomial multiplied by the retarded Schrödinger kernel in the (proper) time \( T \). We will use this equation repeatedly in what follows. The explicit form of this object depends on the details of the group under consideration [26]. We will give the explicit formulae for the rotation and Lorentz groups in three and four dimensions in the next sections.

The above discussion gives the Feynman amplitude \( Z_T \) in terms of the \((g, P)\) variables. To make connection with the usual spin foam we want to write this amplitude in terms of the \((J, P)\) variables as well. This is done by returning to the general expression of the face amplitude (15), inserting the character expansion of the propagator (11) and using the fact that the characters satisfy\(^{14}\)

\[ \int_G dg_2 \Delta_J \chi_J(g_1 g_2^{-1}) \Delta_K \chi_K(g_2 g_3^{-1}) = \delta_{JK} \Delta_J \chi_J(g_1 g_3^{-1}). \]

\(^{14}\) As usual, the indices can go over discrete and continuous values. \( \delta_{JK} \) is the Kronecker delta in the discrete case and the Dirac delta in the continuous. The easiest way to see that this equation is true comes from seeing that it follows from the fact that the delta functions on the groups compose, i.e. that \( \int_G dg_2 \delta(g_1 g_2^{-1}) \delta(g_2 g_3^{-1}) = \delta(g_1 g_3^{-1}) \).
It is then easy to see that the dual face amplitude is given by\(^{15}\)

\[
A_N[H; P] = \sum_J \left[ \frac{i^N \Delta_J}{(P^2 - \frac{d}{2}\pm m^2 \mp C_J + i\epsilon)^N} \right] \chi_J(H),
\]

where \(N\) is again the number of dual edges (vertices) in the dual face \(f^*\). Going through the standard computations \(^{16}\) of group integrals, we can obtain from this formula the spin foam picture of our model. The amplitude of the dual 2-complex (the Feynman amplitude) obtained from our model is given by

\[
Z_T = \left( \prod_{f^* \in T} \sum_{J_{f^*}} \int_d P_{f^*} \right) \left( \prod_{f^* \in T} \left[ \frac{i^{N_{f^*}} \Delta_{J_{f^*}}}{(P^2_{f^*} - \frac{d}{2}\pm m^2 \mp C_{J_{f^*}} + i\epsilon)^{N_{f^*}}} \right] \right) \left( \prod_{v^* \in T} \{J\text{-symbol}\} \right).
\]

The sum goes over all labellings of the dual 2-complex by representations of \(G\), and \(J\text{-symbol}\) stands for the appropriate symbol coming from the representation theory of \(G\) (it is the 6-\(J\) symbol in three dimensions and 15-\(J\) symbol in four dimensions). Note that the Feynman amplitude in these variables is now factorized differently, as it is no longer just a product of amplitudes assigned to dual faces, but, as a result of the group integrations, there are contributions coming from the dual vertices.

It is easy to see that, in the spin foam representation, i.e. in momentum space, from the GFT perspective, the difference between the new models and the usual ones lies in the amplitudes assigned to the dual faces. These amplitudes are just products of the coefficients of the character expansion of the propagators above. However, albeit limited, this difference is crucial and has many consequences: (1) it makes the Feynman amplitudes complex; (2) it produces truly dynamical propagating quantum degrees of freedom, as the usual Feynman propagator of QFT does; (3) it selects as dominant contributions to the amplitudes the solutions of the kinematical QFT equations of motion, i.e. those for which \(P^2_{f^*} - \frac{d}{2}\pm m^2 \mp C_{J_{f^*}} = 0\), which, up to a constant \(\pm m^2\), implies the identification of the \(P\)’s with the Lie algebra generators for the group \(G\), which, as explained in the previous section, is what we want to mimic the structure of a BF path integral, given the identification (that we will confirm in detail in presenting the 3D and 4D models) of the \(P\) variables with the discrete analogue of the \(B\) field of BF theory.

Let us summarize what we have discussed so far. We have defined a new class of generalized GFT models in (6), (7), (8). We then analysed the Feynman rules of the theory. The vertex is easily seen to be almost the standard one. The propagator for the theory (which in a sense encodes most of the new features of the model) is obtained using the Schwinger–DeWitt parametrization. We have then constructed the Feynman amplitudes of the model in both the \((g, P)\) and the \((J, P)\) variables. Some general features of the new models are already apparent at this stage, such as the complexity of the amplitudes, the presence of propagating degrees of freedom at the quantum level, the relaxation at the quantum level of the relation between (the discrete analogue of) the \(B\) field and the generators of the Lie algebra of the group \(G\). We will now move on, and present in detail the model one obtains from this general definition in the 3D and 4D cases, in both Riemannian and Lorentzian settings. In doing so, the above features will become even clearer, as in particular it will become clearer the geometric interpretation of both the \(P\) and the \(g\) variables. Moreover, we will see that the Feynman amplitudes of the new models, in the \((g, P)\) variables, have indeed the form of path integrals for simplicial quantum

\(^{15}\) Note that this reconfirms equation (16) as if we take the given character expansion of the kernel, plug it into the right-hand side of (16) and evaluate the integral over \(T\), we obtain exactly the answer given in (17).
gravity of the form of a BF theory restricted to positive orientation. This extra condition is what makes the Feynman amplitudes we get not triangulation independent\footnote{This might seem surprising at first as we are dealing with 3D gravity and BF theory with no constraints imposed in higher dimensions, which are topological field theories. The reasons why one should not expect triangulation independence in our models is explained in [11], where a triangulation-dependent amplitude was written for a causal version of BF. In the end, the reason why we should not expect to obtain triangulation-independent amplitudes from our construction is that these are meant to describe, and indeed do describe, causally restricted BF theories, i.e. BF theories on which a causal restriction on the field configurations summed over is implemented. This restriction, heuristically speaking, turns the amplitudes from the typical BF delta functions over the flat connections into complex ones that are only peaked on such configurations, but admit quantum fluctuations beyond them. In other words, the causal models are not triangulation independent is because they are not on shell with respect to the flatness condition, and thus do not impose the Hamiltonian constraint on the quantum states of the theory, defining instead only a Green function of the same constraint. As a result, the causal amplitudes are not invariant under the canonical symmetries generated by the Hamiltonian constraint, while still being fully invariant under covariant (Lagrangian) diffeomorphisms [14], while this invariance is needed in the usual counting leading to the absence of local degrees of freedom in BF theory. The situation here is analogous to the case of a single relativistic particle. Classically, the momentum is constrained to be equal to the mass through the Hamiltonian constraint $\delta(p^2 - m^2)$. However, at the quantum level and for bulk (internal) configurations in the Feynman expansion of a multi-particle system, we no longer impose the mass-shell condition and the propagation of the particle is governed by the Feynman propagator $\frac{1}{p^2 - m^2 + i\epsilon}$, which is peaked on the classical configurations but includes other ones (not allowed classically).}.

3.3. New versus conventional models

Here we discuss the relation between the new and the usual models. There are two ways in which one re-obtains the more traditional GFTs and spin foam models for BF theory, as an appropriate restriction, from these generalized ones (the same was true for the models proposed in [12]).

• The conventional GFTs are obtained when we take the static-ultra-local limit [28] of the action (6), and for a specific choice of the mass parameter $m^2 = 1$ (which however does not play any role in the resulting amplitudes). The SUL limit is defined by the restriction of the theory obtained dropping the derivative contribution to the QFT action. In this limit, thus, one gets rid of the propagation in the theory by replacing the derivative terms in the kinetic term with delta functions:

$$\left[ \prod_{i=1}^{D} \left( -\Box X_i + \Box g_i - \frac{d}{24} m^2 \right) \delta(g_i, g'_i) \delta(\nu_i X_i + \nu'_i X'_i) \right] \to \left[ \prod_{i=1}^{D} \delta(g_i, g'_i) \delta(\nu_i X_i + \nu'_i X'_i) \right].$$

If we do this in (6) we will obtain essentially the usual GFT model but with the sole difference of having extra arguments which the field depends on.

How are the Feynman amplitudes affected by these extra variables? Since there is no coupling between the group and the $X$ (or $P$) variables, they are just propagated in parallel around the Feynman graph. The upshot of this is that the extra variables $X$ (or $P$) contribute just an overall (infinite) constant and thus do not affect the amplitudes, that reduce then to the usual spin foam models.

• Another way of looking at the relation between the new model and the conventional, which clarifies the fact the new model is the causal analogue of the usual ones, comes from considering the theory in the $(J, P)$ variables.

Take a single propagator and look at its character expansion (11). As is clear from this equation that the coefficients of the characters are just the usual Klein–Gordon propagators on a flat space $X$, whose dimensions are equal to the dimension of the group $G$ and which has a metric which is the Killing form. Also, it is clear that it is from here that the complexity (thus the causal nature, as we discussed) of the amplitudes comes. Using
Sohozki’s formula \(\frac{i}{\varepsilon_{\text{ren}}} = \pi \delta(x) + iP \left(\frac{i}{2}\right)\) and the reality of characters\(^{17}\) it follows that the real part of the propagator is given by

\[
D_{R}[g_{i}, h_{i}; P_{i}, Q_{i}] = \prod_{i=1}^{D} \left( \delta(P_{i} - Q_{i}) \int_{G} dh \left[ \sum_{J_{i}} \delta \left( P_{i}^{2} - \frac{d}{24} m^{2} \mp C_{J_{i}} \right) \chi_{J_{i}}(g_{i}, hh_{i}^{-1}) \right] \right).
\]  

(19)

Note that taking the real part of the propagator is the same as going on-shell with respect to the corresponding equation of motion, which is the classical relation between the \(P\) variables and the Lie algebra generators of the group \(G\), or, as we will confirm in the next sections and we have discussed in the previous, between the B and the A field of BF theory (metricity of the connection). If we now do the integrals over the \(P\) variables it is immediate that we just get the propagator and thus the whole spin foam amplitudes of the usual GFTs, as the delta functions integrate to 1.

4. New 3D GFT models

4.1. Riemannian 3D gravity

We now specialize the class of models considered above to the case \(D = 3\) and \(G = SU(2)\). The usual models (with the trivial kinetic term), for this choice of dimension and group, give Euclidean 3D BF theory, augmented by a sum over topologies, in perturbative expansion.

\(SU(2)\) is a compact group of rank 1, hence the kernel depends on a single periodic parameter. It is convenient to choose this parameter to be the ‘angle of rotation’ in the usual representation of \(SU(2)\). More precisely, if \(H \in G\) then \(H = e^{i \frac{n}{2} \sigma} \theta\) where \(\theta(H)\) is the angle of rotation, \(\vec{n} \in S^2\) is the axis of rotation and \(\vec{\sigma}\) are the Pauli matrices. The angle \(\theta(H)\) is a multivalued function of the group element. This should be clear as \(\theta(H)\) and \(\theta(H) + 4\pi n\) for \(n \in \mathbb{Z}\) correspond to the same group element. In other words, any choice of \(n\) in the expression \(\theta(H) + 4\pi n\) provides a possible definition of the angle characterizing the holonomy \(H\). What this means is that from a geometrical point of view, the angle of rotation is intrinsically an equivalence class of real numbers modulo addition of \(4\pi\). We will denote this equivalence class by \([\theta(H)] = \theta(H) \mod 4\pi\), and identify \([\theta(H)]\), i.e. \(\theta(H) + 4\pi n\) for any choice of \(n\), with the holonomy angle. However, since the equivalence class is not a number, to write any formula involving the angle of rotation, one should pick a representative of the equivalence class (i.e. choose a specific \(n\), for example \(n = 0\) thus restricting oneself to the \([0, 4\pi]\) range). This random choice does not matter if the function is automatically periodic when \(\theta \rightarrow \theta + 4\pi\) (e.g. the character function). However, when the functional expression one is dealing with is not periodic (the evolution kernel below), one needs to sum over all the equivalence classes (all possible \(n\) to obtain a function with the correct boundary conditions, i.e. a function on the group.

The explicit form of the evolution kernel on \(SU(2)\) in (proper) time \(T\) is given \([25]\) by the following formula

\[
K[H, T] = \frac{1}{(4\pi i T)^{\frac{1}{2}}} \sum_{n=-\infty}^{\infty} \left( \frac{\theta(H) + 4\pi n}{2 \sin \left( \frac{\theta(H)}{2} \right)} \right)^{i T} \left[ \frac{\theta(H) + 4\pi n}{2 T} (\theta(H) + 4\pi n)^{2} + \frac{i T}{8} \right].
\]  

(20)

\(^{17}\) Technically, this is not true for all groups. Fortunately the characters always enter into the delta function expansion (10) (from which the propagator is derived) in a symmetric way, such that the coefficients of characters which are complex conjugates of each other are real and equal.
Note the sum enforcing periodicity in $\theta \rightarrow \theta + 4\pi$. To avoid writing the sums which enforce periodicity in what follows, we adopt the following notation: whenever we have a sum which enforces periodicity of a certain function, i.e. whenever we have an expression of the form $\sum_{n=-\infty}^{\infty} f(\theta + 4\pi n)$, we will just write $f([\theta])$. The sum, which is required to convert an expression involving $[\theta]$ to a legitimate one involving just real numbers, will be kept implicit. This is perfectly reasonable from the geometric point of view as well, as it is exactly the entire equivalence class that has the meaning of an angle of rotation. This sum also has the meaning of a sum over all geodesics over the group (i.e. $S^3$) connecting the same two points [25].

Once more, we define the partition function of the model as a perturbative expansion in Feynman diagrams as in equation (9) and use the fact that the Feynman amplitudes factorize per dual face (14). Now, according to (16), to get the amplitude $A_f^\nu$ for a dual face with $N$ vertices in the $(g, P)$ variables, we should multiply the expression for the evolution kernel by $\frac{\theta(T)T^{N-1}}{(N-1)!}$ and then take the Fourier transform of the result, with respect to $T$, at the value $P^2 = \frac{m^2}{8}$. Thus

$$A_N[H, P] = \frac{1}{(4\pi i)^2(N-1)!} \sum_{n=-\infty}^{\infty} \left[ \frac{[\theta(H)]}{2 \sin\left(\frac{[\theta(H)]}{2}\right)} \right]_0^\infty \frac{dT}{T} T^{N-\frac{3}{2}} \times \exp \left[ \frac{i}{2T} [\theta(H)]^2 + iT \left( \frac{\tilde{p}^2}{2} - \frac{m^2 - 1}{8} \right) \right].$$

(21)

The integral can be evaluated explicitly [37] using the formula

$$\int_0^\infty \frac{dT}{T} T^{v-1} \exp \left[ \frac{ip}{2} \left( T + \frac{q^2}{T} \right) \right] = i^{v+1} \pi q^v H^{(1)}_v(qp),$$

(22)

where $H^{(1)}_v(z)$ is a Hankel function of the first kind of order $v$. The two coefficients $p$ and $q$ are complex numbers in general, but what is very important is that they should satisfy (Im$(p) > 0$ and Im$(pq^2) > 0$). It should be obvious that this should be the case as the integrals will simply not converge otherwise. Note that while the left hand side has $q^2$ in it, the right-hand side has $q$. The fact that we have to take a square root will be very important in the Lorentzian case.

For us

$$v = N - \frac{3}{2}, \quad p = \tilde{p}^2 - \frac{m^2 - 1}{4} \quad \text{and} \quad q^2 = \frac{[\theta(H)]^2}{\tilde{p}^2 - \frac{m^2 - 1}{4}}.$$  

(23)

It is clear that (Im$(p) > 0$ and Im$(pq^2) > 0$) imply that both the $\tilde{p}^2$ and the $[\theta(H)]$ should be complexified and given small positive imaginary parts. This complexification is nothing but the usual Feynman $\i$ prescription. The root of $q^2$ is defined in the usual way, by taking a cut along the negative real axis, letting $\sqrt{1} = 1$ and extending by continuity. As both the numerator and denominator have small phases (both are positive), their ratio also has a small phase. Thus the square root of $q^2$ is very close to the real axis and is very nearly equal to $\frac{[\theta(H)]}{|\tilde{p}^2 - \frac{m^2 - 1}{4}|}$.

18 The sum in the expression above converges in the sense of distributions [29].
19 Reconstructing an expression involving just real numbers from our notation is slightly ambiguous as one can insert the sum in several different ways in a formula which involves an equivalence class. To resolve this ambiguity we simply stipulate, and implicitly implement in the following, that the sum should be put on the left of the full expression. We would like to thank an anonymous referee for pointing out this ambiguity to us.
20 We are using the normalizations of [26]. The Killing form in our conventions is given by 2I, where I is the $3 \times 3$ identity matrix. As a consequence, since the metric on the dual to $X$ is given in terms of the inverse of this Killing form, $p^2 = \frac{1}{2} \tilde{p}^2$. 

20
Plugging (22) into (21) we get

\[
A_N[H, P] = \frac{i^{n-2}}{16\sqrt{\pi}(N-1)!} \left[ \frac{[\theta(H)]}{\sin(\frac{[\theta(H)]}{2})} \right] \left( \sqrt{\frac{2}{\sqrt{\theta^2 - m^2 - 1}} - \frac{1}{4}[\theta(H)]} \right)^{N-1} 
\]

\[
\times H_{\frac{N-2}{2}}^1 \left( \sqrt{\frac{2}{\sqrt{\theta^2 - m^2 - 1}} - \frac{1}{4}[\theta(H)]} \right). 
\]

The Hankel function of half-integer order can be given explicitly in terms of elementary functions via

\[
H_{\frac{N-2}{2}}^1(z) = \sqrt{\frac{2}{\pi z}} i^{-(N-1)} \sum_{K=0}^{N-2} (-1)^K \frac{(N + K - 2)!}{K!(N - K - 2)!} \frac{1}{(2iz)^K} e^{iz}. 
\]

Using this expression we get that the dual face amplitude has the form

\[
A_N[H, P] = \mu([\theta(H)], |\tilde{P}||, N) \exp(\frac{i}{4}\sqrt{\theta^2 - m^2 - 1}[\theta(H)]), 
\]

where \( \mu \) being given by

\[
\mu([\theta(H)], |\tilde{P}||, N) = \frac{-i\sqrt{2}}{16\pi(N-1)!} \left[ \frac{1}{\sin([\theta(H)])} \right] \left( \sqrt{\frac{2}{\sqrt{\theta^2 - m^2 - 1}} - \frac{1}{4}[\theta(H)]} \right)^{N-1} 
\]

\[
\times \sum_{K=0}^{N-2} (-1)^K \frac{(N + K - 2)!}{K!(N - K - 2)!} \frac{1}{(2i\sqrt{\theta^2 - m^2 - 1}[\theta(H)])^K} . 
\]

Above, we have given the amplitude for just one dual face, or recalling that in 3D a dual face is dual to an edge of the triangulation, it is the amplitude for a single edge. However, as was mentioned in the previous section, the amplitude of the dual complex in the \((g, P)\) variables is just the product of the dual-face amplitudes, or in the 3D context the product of edge amplitudes. Thus, we can easily write the amplitude for the whole triangulation (Feynman graph) \(Z_T\). It is

\[
Z_T = \int_{G^e} \left( \prod_{e \in T^*} dg_{e^*} \right) \int_{\mu^e} \left( \prod_{e \in T} d\tilde{P}_e \right) \mu(g_{e^*}, \tilde{P}_e^2, N_e) \exp(\frac{i}{4}\sum_{e} \sqrt{\tilde{P}_e^2 - \frac{m^2 - 1}{4}[\theta_e]}), 
\]

where the products go over all edges in the triangulation \(e \in T\) and all dual edges in the dual 2-complex \(e^* \in T^*\), and the factor \( \mu(g_{e^*}, \tilde{P}_e^2, T) \) is a product of all the \( \mu \)'s coming from each dual face i.e. \( \mu = \prod_{e \in T} \mu_e \) with \( \mu_e \) given by (26).

Now, consider the exponent in the above expression. We see immediately that it is just the Regge action for Euclidean 3D gravity

\[
S_{\text{Regge}} = \sum_{\text{edges}} L_e \theta_e, 
\]

in first order form, after identification of \( \sqrt{\tilde{P}_e^2 - \frac{m^2 - 1}{4}} \) with \( L_e \). Here the sum goes over all edges of the triangulation. \( L_e \) stands for the length of the edge \( e \) and \( \theta_e \) for the deficit angle, i.e. the discretized curvature, around the dual edge, which coincides with the angle of rotation \( [\theta_e(H_e)] \) that characterizes our holonomies \( H_e \) (again, equivalent to \( \theta(H) + 4\pi n \) for any choice of \( n \)).
This reconfirms and makes precise the interpretation for the new variables, the $P$'s, which was proposed in the introduction, as that they give the length of the edges to which they are associated, and thus as representing the discretized triad ($B$ field) associated with these edges, while the group elements are confirmed as a discretization of the Lorentz connection field $A$. Indeed, we obtain an expression for the simplicial gravity action of the same type as the ones in [17], and, as there, for which we use the identification of the length with the variables $\mu$, and, as there, with the edge lengths (hinge volumes) restricted to have a positive orientation. Note that this identification of the length with the variable $P$ becomes especially nice if we set $m^2 = 1$, as then it is the length of $\vec{P}_e$ directly, $|\vec{P}_e|$, which coincides with the length of the edge $L_e$. For this reason, as well as to simplify the formulae, we will adopt this choice for $m^2$ in the following discussion of the amplitude in the $(g, P)$ variables.

It is clear that the variation of the above action with respect to the edge lengths, or the variables $\vec{P}_e$, gives the classical equation $\theta_e = 0$, i.e. imposes flatness of the discrete geometry as the only classically allowed configuration, as we expect from 3D gravity. The variation with respect to the connection variables is more involved, and we would expect it to provide a discrete analogue of the continuum conditions enforcing metricity of the connection. We leave its analysis for future work.

The amplitude for the triangulation $Z_T$ is then just the partition function for discrete 3D Euclidean gravity, in first order form, with a measure factor $\mu(g_e, \vec{P}^2_e, N_e)$, as desired.

Let us now consider the measure factor $\mu(g_e, \vec{P}^2_e, N_e)$, in more detail. This factor is a complex number in general as should be evident from (26). Thus if we write

$$\mu(g_e, \vec{P}^2_e, N_e) = \left| \mu(g_e, \vec{P}^2_e, N_e) \right| e^{iS_e(g_e, \vec{P}^2_e, N_e)},$$

with

$$S_e(g_e, \vec{P}^2_e, N_e) = \sum_e \left[ -\frac{\pi}{2} + \arctan \left( \frac{\sum_{K=0}^{(N_e - 2K)} \frac{(-1)^K}{2^{2K}} \frac{(N_e - 2K - 3)!}{(N_e - 2K)! (N_e - 2K - 3)!} \frac{1}{2^{2K}} \left( \frac{\sum_{K=0}^{(N_e - 2K + 2)} \frac{(-1)^K}{2^{2K}} \frac{(N_e - 2K + 2)!}{(N_e - 2K)! (N_e - 2K + 2)!} \frac{1}{2^{2K}} \right) \right) \right],$$

we see that the full Feynman amplitude for the whole triangulation has the form

$$Z_T = \int_{G^e} \left( \prod_{e \in T} dg_e \right) \int_{\vec{P}^2_e} \left( \prod_{e \in T} d\vec{P}_e \right) \left| \mu(g_e, \vec{P}^2_e, N_e) \right| \times \exp \left( i \left[ S_{\text{Regge}}(g_e, \vec{P}^2_e) + S_e(g_e, \vec{P}^2_e, N_e) \right] \right).$$

The modulus of the quantum measure $\mu(g_e, \vec{P}^2_e, N_e)$, i.e. $\left| \mu(g_e, \vec{P}^2_e, N_e) \right|$ is then what should be considered as a proper quantum measure factor in our path integral, while the phase $e^{iS_e(g_e, \vec{P}^2_e, N_e)}$ gives what can be interpreted as quantum corrections to the Regge action (hence the subscript). We thus see that the amplitudes of our model, more precisely, have the form of a path integral (with an explicitly defined measure) of an extended first order Regge calculus, in which the Regge action is extended by (also explicitly computable) quantum corrections.

Let us then study in more detail these quantum corrections. We then study the explicit formula (28) for $S_e$, as well as the expression (26). Also, we focus on the dependence on the geometric data $P$ and $\theta$, neglecting constant factors, which give a constant contribution to the phase at every edge (equal to $-\frac{\pi}{2}$).

One of the most important properties of this part of expression (26) is that it depends on $[\theta_e]$ and $P_e$ solely through the combination $(|P_e|[[\theta_e]])$. This also implies that it can be expanded in (general, positive and negative) powers of the same combination $(|P_e|[[\theta_e]])$, weighted by factors that will necessarily be purely combinatorial, i.e. dependent on $N_e$ only.

Under the interpretation discussed above for the $P$ variables and for the $\theta$, a first possible interpretation of the powers of the expression $(L_e \theta_e)$ is that they represent the discrete
analogenes of higher order corrections to the Einstein–Hilbert action, given by powers of the Ricci scalar [30]. One could then expect the correspondence

$$\sum_c C_c (|\vec{P}_c||[\theta_c]|)^K \sim \int R^K(g) \text{Vol},$$

where $C_c$ is the mentioned combinatorial factor, Vol is the volume form and the aforementioned correspondence holds in the continuum approximation (in the sense of measures) [30].

However, the simplicial geometry of such higher powers of the Regge term is subtle (see again [30] for an extensive and detailed analysis). In particular, for the square power of the above expression, another plausible interpretation is provided by the square of the Riemann tensor, giving

$$\sum_c C_c (|\vec{P}_c||[\theta_c]|)^2 \sim \int R_{\mu\nu\rho\sigma}(g) R_{\mu\nu\rho\sigma}(g) \text{Vol}.$$ 

In general, in fact, higher order curvature terms, as traditionally defined in simplicial gravity, involve an additional geometric ingredient, a normalization of the hinge volumes, that gives them the correct dimensionless character. This is taken to be the contribution of the $D$-simplex volume associated with the specific hinge considered, $V_h$, giving a complete quadratic term of the form $(\frac{|\vec{P}_h||[\theta_h]|}{V_h})^2$. Its exact form could be argued, by universality arguments, to be most likely irrelevant for the continuum correspondence, but of course this is not at all obvious. With this choice of normalizing factor, one can indeed show that (the discrete analogue of) both $R^2$ and $(R_{\mu\nu\rho\sigma})^2$ agree when restricted to a single hinge. Therefore the difference between the two types of higher order terms depends exclusively on how different hinges are coupled, each being weighted individually by the quadratic expression above. The simplest choice of coupling $\sum_h V_h (\frac{|\vec{P}_h||[\theta_h]|}{V_h})^2$ gives then a contribution to the action corresponding to the square of the Riemann tensor. Other constructions are however possible for both the Riemann tensor itself and the quadratic terms that can be constructed from it [30]. Also, we are not aware of similar detailed analyses for higher powers, thus for curvature invariants beyond the quadratic order. In our model, the normalizing volume factor can be interpreted as being given by the Planck length to the appropriate power and multiplied by our purely combinatorial factor $C_c$, a function of $N_c$. Therefore a more complete interpretation scheme for the higher order corrections to the Regge action produced by our GFTs does involve a careful analysis of these combinatorial factors and in particular of the way they couple different hinges in the same $D$-simplex and beyond. This analysis will be performed and reported elsewhere.

From a more general perspective, however, these corrections to the Regge action, predicted by our model(s) share two main features: (1) they involve, as mentioned, both positive and negative powers of the curvature invariants, and (2) they depend on two independent sets of geometric variables, the (discrete analogues of) the D-bein and the connection fields. This implies, therefore, that the corrections to the bare Regge action produced by the model are of the general $f(R)$ type in the metric affine formalism [31].

We would like to emphasize once more that these corrections are not arbitrary, rather their form, including relative coefficients weighting them, and their behaviour in the various regimes of the theory are fully determined by the our choice of the original GFT action. This also means of course that one can modify the exact dependence on them of the simplicial action appearing in our Feynman amplitudes, by modifying the same GFT action, thus constructing different specific models within the general class of GFTs we have defined.

Let us analyse further the physics behind the corrections $S_c$. We are most interested in two approximations, both of which can be given a clear physical interpretation.
The first regime is when the lengths become large, i.e. when $|\vec{P}| \gg 1$ (remember that we are working in Planck units). Equivalently, this is the regime of large actions, in units of Planck’s constant, due to the way in which the edge lengths enter the discrete Regge action. This approximation can thus be considered as a ‘semi-classical approximation’ as it corresponds to the case where the relative size of the quantum fluctuations of the action (and of the edge lengths) is small. This is the analogue, for our models, of the asymptotics usually considered in the standard spin foams (the large-\(J\) asymptotic).

The second regime is approached when the edge lengths and discrete curvatures become small, and the triangulation becomes finer and finer, i.e. when $(|\vec{P}|/||\theta||) \to 0$ and $N \to \infty$. This can be thought of as the ‘continuum approximation’.

Let us first look at the behaviour of the measure and thus of the quantum corrections \(S_c\) at the heuristic level. Consider then the explicit expression for the (complex) measure in (26), and in particular to the part of it within curly brackets.

In the first case (large lengths $|\vec{P}|$) it is the first term in the sum in (26) that dominates, and since this term is real, it means the Regge action remains the dominant contribution to the phase of the path integral amplitude. We expect then the phase, including corrections, to be of the general form

$$S_{\text{Regge}} + O\left( \sum_e \frac{1}{|\vec{P}_e| \theta_e} \right),$$

thus with inverse powers of the curvature to play the role of quantum corrections to the Regge action, and the full Feynman amplitude (discrete gravity path integral) to be approximated by

$$Z_T \sim \int_{G^e} \left( \prod_{e' \in T^e} dg_{e'} \right) \left( \prod_{e \in T} d\vec{P}_e \right) \left\{ \prod_e \left[ \frac{-i\sqrt{2}}{16\pi (N_e - 1)!} \left[ \frac{\theta_e}{\sin(|\theta_e|)} \right] \left( \frac{1}{|\vec{P}_e||\theta_e|} \right)^{N_e - 1} \right] \right\} \times \left[ 1 + O\left( \frac{1}{|\vec{P}_e||\theta_e|} \right) \right] \exp\left( iS_{\text{Regge}}(g_{e'}, \vec{P}_e) + O(1/|\vec{P}_e||\theta_e|) \right).$$

In the second case (small edge lengths and very fine triangulation, i.e. high $N_e$) it is the last term in the sum in (26) that dominates. This term also contributes just a constant to $S_c$, (equal to $(N - 2)^2\frac{\pi}{2}$). We expect then the phase, including corrections, to be of the general form $S_{\text{Regge}} + O\left( \sum_e (|\vec{P}_e||\theta_e|)^2 \right)$, thus with positive powers of the curvature to play the role of quantum corrections to the Regge action, and the full Feynman amplitude (discrete gravity path integral) to be dominated by a term like

$$Z_T \sim \int_{G^e} \left( \prod_{e' \in T^e} dg_{e'} \right) \left( \prod_{e \in T} d\vec{P}_e \right) \left\{ \prod_e \left[ \frac{(C_N||\theta_e||)^{2(N_e - 1)}}{\sin(|\theta_e|)} \left( \frac{1}{|\vec{P}_e||\theta_e|} \right)^{2(N_e - 3)} \right] \right\} \times \exp\left( i\left( S_{\text{Regge}}(g_{e'}, \vec{P}_e^2) + O\left( \sum_e (|\vec{P}_e||\theta_e|)^2 \right) \right) \right).$$

We would like now to go beyond the naive heuristic considerations and analyse the form of the quantum measure, and of \(S_c\) in particular, in more detail.

This can be done with full confidence for the semi-classical approximation. The reason for this is (26), and thus the full Feynman amplitude, is regular at the limiting point $|P| \to \infty$ (it goes simply to zero), for a generic triangulation. Also, the proper analysis involves the asymptotic expansion of the Hankel function for large values of the argument, but, for half-integer order, this coincides with the expression (24) that we have used. This allows us to obtain full understanding of the way the phase behaves in the large length limit.
We can then use directly the expression (28) and, expanding the arctangent in powers of \( \frac{1}{\theta_e |\vec{P}_e|} \), we get that

\[
S_c(\sigma^e, \vec{P}_e^2, N_e) = \sum_e \left[ \frac{N_e}{2} \frac{1}{\theta_e |\vec{P}_e|} + o\left( \frac{1}{(\theta_e |\vec{P}_e|)^2} \right) \right].
\]

Of course, all the coefficients in the expansion can, in principle be computed within our model. As said, we can think of \( \sum_e C_e e^{\theta_e |\vec{P}_e|} \) as the inverse of the scalar curvature. Thus

\[
S_c \sim \int \left[ \frac{1}{R} + o\left( \frac{1}{R^2} \right) \right] \text{Vol}.
\]

Since the corrections are inverse in the curvature, they are of the infrared type, as it is intuitively to be expected as we are doing a large-scale approximation to our model. Thus we see that the new model predicts long-distance effects, at the simplicial level, of the same type as those predicted by effective \( f(R) \)-extended gravity models, and that have been found relevant in cosmological applications (most notably for modelling dark energy effects) [31].

The other case of interest (the ‘continuum’ limit) is much more involved to analyse, and the purely heuristic argument can be trusted as a limited indication of the relevant physics (it is intuitively obvious that in the small distance regime one gets quantum corrections of the ultraviolet type \( O(R^2) \)), but one that cannot be easily confirmed by a detailed analysis, at this point.

The reason for this is that, as is not difficult to see from (26), the Feynman amplitude has a badly singular point in \( (|\vec{P}_e| = 0) \): (1) it diverges in the limit like \( \frac{1}{|\vec{P}_e|^3} \); (2) the Hankel function has a branch point at 0, which poses extra problems one needs to deal with due to Stokes' phenomenon\(^{21} \), whose main consequence is, in this context, that the expression for the amplitude around this point depends heavily on how exactly the limit is taken, i.e. which path one takes in the complex domain to approach the singular point. Finally, the limit \( (|\vec{P}_e||\theta|) \rightarrow 0 \) by itself is not very physically meaningful. It acquires its importance when combined with the limit \( N \rightarrow \infty \). However, it is not difficult to see that the way the amplitude behaves is sensitive to the way these two limits are combined. Due to the above reasons we defer the detailed treatment of this regime of the model, as well as of the corresponding formulation of simplicial geometry for future work.

Finally, let us note that the fact that the amplitude diverges as \( \frac{1}{|\vec{P}_e|^3} \) is very appealing intuitively, as it implies that for larger triangulations it is the small values of \( |\vec{P}_e| \) that are the most relevant ones and that they become more and more dominant as we take larger and larger triangulations (this is because the higher powers are more divergent). Since we have interpreted the \( \vec{P} \) variables as giving the lengths of the edges of the triangulation, this looks exactly like the behaviour one would want in order to recover a good continuum limit: for a triangulation consisting of a large number of tetrahedra, the dominant histories are those for which the basic simplices are small, corresponding moreover to a singularity in the quantum amplitudes.

The new model is a causal one in the sense of [9, 10, 12] and it shares many features of the 3D model presented in [11]. Let us briefly recall the model proposed there. The action used in [11] is a discretized version of (5). The \( B \) field is replaced with a Lie algebra element \( \vec{P} = \vec{P} \cdot \vec{J} \) associated with every edge of the triangulation, and the connection \( A \) is substituted by its holonomy around the dual face \( H = \exp(\theta_{\vec{n}} \cdot \vec{J}) \). The discrete action is then given by

\[
S' = \sum_{e \in T} \vec{P} \cdot \vec{n} \sin(\theta_e).
\]

\(^{21}\) The phenomenon that the asymptotic expansion around a point of a nonanalyticity of a function depends on the sector chosen for the approach to the given point.
The model is quantized via the path integral method in the usual way, the only crucial difference being that the product $\vec{P}_e \cdot \vec{n}_e$ is restricted to be nonnegative. This is because, as was argued in [11], this corresponds to restricting the discretized ‘volume’ to be positive (by insertion of an appropriate step function in the BF partition function), and thus it represents the wanted implementation of the ‘causality’ restriction in quantum gravity transition amplitudes. Thus the partition function is given by

$$Z = \int_{G^*} \left( \prod_{e \in T} d\vec{g}_e \right) \int_{P^*} \left( \prod_{e \in T} d\vec{P}_e \right) \theta(\vec{P}_e \cdot \vec{n}_e) e^{i \vec{P}_e \cdot \vec{n}_e \sin(\theta_e)}.$$  \hfill (29)

The new model, which generates amplitudes given by (27), is causal in the same sense as (29) due to the simple fact that the $|\vec{P}_e|$ is always positive. Thus, keeping the interpretation of the $P$’s in mind, in our GFT model the integral over the discretized field is also restricted to be such that the hinge volumes are positive. This restriction results [11] in the causal analogues of usual spin foams in both the free and matter coupled cases.

There are several differences, however, between the model proposed here and the one proposed in [11], i.e. between (27) and (29).

- First, the discretizations used in the two cases are somewhat different. Although, both use the holonomies to represent the curvature and both average the $B$ field over an edge (and get a vector), the way these two objects enter into the discrete action is slightly different. Notably, the two variables are totally independent in the new model and interact simply through multiplicative coupling, at least before one uses the equations of motion resulting from the variation of the simplicial action. In the old model however, the variables mix more substantially: (a) there is extra coupling introduced by the dot product $\vec{P}_e \cdot \vec{n}_e$ ($\vec{n}_e$ is completely absent from the action in (27)); and (b) the domains of integration of the two variables are interdependent, due to the step function. With regards to both these points the new model is simpler than the old one. It is well possible, however, that one can get a 3D model, in the same new class of GFTs we are proposing, that is closer to the one in [11] by imposing additional (symmetry) conditions on the variables appearing in the GFT action.

- The measure factor $\mu(\vec{g}_e, \vec{P}^2_e, N_e)$ present in (27) is absent from (29). These are, as discussed above, corrections to the bare Regge action (and thus to the 3D BF action) that have been here deduced from first principles and not added in an ad hoc way (which of course could be done with (29)). Thus the new model is significantly richer than the old one. Also with respect to this point, we notice that there is still freedom left in choosing specific GFT actions within the general class of GFTs we introduced, and thus obtaining models with modified (and possibly simpler) path integral measures in the perturbative expansion.

- Due to the fact that the factor $\mu(\vec{g}_e, \vec{P}^2_e, N_e)$ depends on $N_e$, it should be clear that if we perform the integrals over the $P$’s in (27) we will get dual face amplitudes which depend on the number of vertices in each dual face, i.e. each dual face amplitude is a function of $N_e$. This however is not the case in the old model where the dual face amplitudes, which were computed explicitly in [11] were independent of this factor. The reason for this can be traced to the following fact. At the spin foam level, and in the construction of [11], the basic building block of the model was considered to be the dual face. At the GFT level, it is necessarily the wedge (i.e. the portion of the dual face contained within a D-simplex) from which everything else is constructed [32, 33]. The causal restriction advocated for in [11] was a dual-face one, and this is the reason for the independence of the resulting amplitudes from the number of wedges (vertices) making the dual face.
would expect that if the construction in [11] is repeated but with the causality restriction being imposed at the level of each wedge, one would obtain a model which is closer to the one reproduced here.

Finally, in [11] the causal restriction, although shown plausible, was implemented essentially by hand simply by inserting the step function into the partition function (29); therefore one could be left wondering about the possibility of different ways of implementing the same type of causal restriction. In the new model(s) we are proposing that such freedom is absent, at least for given choice of GFT action: the amplitudes are built, in a unique manner, from the same building block, the propagator, and it is exactly the propagator that has the information about causality, orientation dependence and the propagation of quantum degrees of freedom.

Consider now the model in the \((J, P)\) variables (i.e. equation (17) specified for \(D = 3\) and \(G = SU(2)\))

\[
Z_T = \left( \prod_{J < J^*} \int dJ^* \right) \prod_{J < J^*} \left[ \frac{i^{N^*} (2J^* + 1)}{(\frac{P^2_{J^*}}{2} - \frac{m^2}{8} - \frac{J^* (J^*+1)}{2} + i\epsilon)^{N^*}} \right] \prod_{\nu^* \in \Gamma^*} (6 - J^*),
\]

where of course now the representations are labelled by half-integers \(J_{J^*}\). Since it is the \(P\)'s that give the lengths, the interpretation of the \(J\) variables is not as straightforward as it is in the usual models. Looking at the expression above we can see that the \(J\)'s label the different poles of the dual face amplitude. Since the expression for the dual face amplitude is essentially a product of Feynman propagators we can think of the \(J\) variables as labelling the different semi-classical, on-shell values of \(|\vec{P}_e|\). The poles are

\[
\vec{P}^2 = J(J + 1) + \frac{m^2}{4}.
\]

If we make the same choice for \(m^2\) as before, i.e. if we set \(m^2 = 1\), then we see that \(\vec{P}^2 = \frac{(2J + 1)^2}{4} = \frac{\Delta J}{4}\).

Note that if we plug this back\(^{22}\) into (27) we see that it becomes

\[
Z_T = \sum_{J_{h_1 \ldots J_{f^*}} \in \Gamma^*} \int \left( \prod_{\nu^* \in \Gamma^*} dg_{\nu^*} \right) \mu(g^*, \Delta_{J^*}, N_e) \exp \left( i \sum_{\nu} \frac{\Delta J}{2} \theta_{\nu} \right),
\]

from which we see that the exponent is just the Regge action with the edge length restricted to be \(\frac{\Delta J}{4}\). This matches nicely with the expression obtained in [34] (see also [35]) for the eigenvalues of the length operator in 3D canonical quantum gravity.\(^{23}\)

Of course, this does not really mean that the ‘lengths’ are quantized in our model. This is because the ‘length’ information is given by \(|\vec{P}_e|\)'s and these are unconstrained, in general. Just like in the Feynman propagator for a scalar particle the momentum is not constrained, in the quantum theory, to the mass shell.

We can now obtain a pure spin foam expression for the Feynman amplitudes of our model, i.e. one involving only the representation variables. It is not difficult to perform the integrations

\(^{22}\) In other words, if we restrict the \(P\) variables to these discrete values by inserting \(\prod_{\nu} \sum \delta(|\vec{P}_\nu| - \frac{\Delta J}{2})\) into the path integral. We can heuristically interpret this restriction as imposing the connection metricity equation of motion (i.e. the equation obtained by varying the connection) into the path integral.

\(^{23}\) Apart from the factor of half, which is a consequence of the normalization we chose for the \(P\) field.
over $\vec{P}$'s in (30). The easiest way to do this is by using Cauchy’s formula\(^{24}\). The result of these integrations is given by

$$Z_T = \sum_{J_f, \ldots, J_{f'}} \left( \prod_{f' \in T^*} A_{f'}(J_{f'}, N_{f'}) \prod_{\nu' \in T^*} \{ 6 - \nu' \} \right),$$

where the dual face amplitude is given by

$$A_{f'}(J_{f'}, N_{f'}) = \frac{4\pi^2}{\Delta_{f'}^2} \left[ \frac{-2i}{\Delta_{f'}^2} \right]^{N_{f'}-1} \left[ \frac{(2N_{f'} - 2)!}{((N_{f'} - 1)!)^2} \right]^{N_{f'} - 2} \left[ \frac{2N_{f'} - 3}{2} \right].$$

Let us now try to extract some physical information on the model, and in particular how it depends on the combinatorics of the underlying triangulation, starting from this expression for the amplitudes.

Consider the regime of large $N_{f'} \gg 1$, i.e. consider the triangulations which are composed of many tetrahedra, which we have argued is one ingredient for approximating continuum physics in this setting. This should be combined with a small $|\vec{P}|$ approximation; however, having integrated out the $P$’s, we can only expect to read out from the amplitudes what the dominant configurations in the $J$ variables are. Using Stirling’s formula\(^{25}\)

$$\sqrt{2\pi n} \sim n^e \exp \left[ \frac{n}{e} \right].$$

we can easily see that the second multiplicand in (31) is asymptotic to $\frac{4^{N_{f'}-1}}{\sqrt{\pi (N_{f'}-1)}}$. Thus for large $N_{f'}$’s

$$A_{f'}(J_{f'}, N_{f'}) \sim \frac{1}{F(N_{f'})} \left[ \frac{8}{\Delta_{f'}^2} \right]^{N_{f'}-1},$$

where $F$ is a function of polynomial growth.

We conclude that the amplitude consisting of a large number of tetrahedra is dominated (as this is when $\frac{8}{\Delta_{f'}^2} > 1$) by the two lowest values of $J$’s, $J_{f'} = 0$, which can be thought of as the vacuum configuration, and $J_{f'} = \frac{1}{2}$, which is some sort of lowest excited state. So, if we interpret the values of $J$ as edge lengths, as in usual spin foam models, it is the shortest values that are the dominant ones for fine triangulations, as we would expect. In the limit of finer and finer triangulations (which, again, we would expect to lead to a continuum approximation of the discrete path integral), then, the partition function can be reasonably well approximated by a purely combinatorial sum, with amplitudes given by the above quantities evaluated at $J = 0$, i.e. for purely equilateral triangulation with edge lengths $L_e = l_P(2J_e + 1)|_{J_e=0} = l_P$.

In other words, in this regime, the model would effectively, and dynamically, reduce to a pure dynamical triangulations model\(^{8}\).

Consider the regime of large $J_{f'}$’s. Again, having integrated out the ‘true’ edge length variables $P$, we can heuristically interpret this regime as a large distance approximation. Looking again at the same expression (31), it is clear that it is the lowest values of $N_{f'}$’s that are most relevant in the limit. What this means is that if we look at the large length limit the most important Feynman diagrams are represented by the simplest triangulations, more precisely those with least number of vertices for each dual face. In other words, if one is interested only in large distance and semi-classical physics, then considering simple triangulations would suffice, as the GFT partition function, in perturbative expansion, is anyway dominated by such configurations.

\(^{24}\) This is done by changing to polar coordinates, extending the radial integral to go from $-\infty$ to $\infty$ and then closing the contour in the complex plane. By Jordan’s lemma, since $N_{f'} \geq 2$, the integral of the expression we have along a semicircle centred at the origin of radius $R$, goes to zero as $R \to \infty$. This allows us to add this bit to the integral closing the contour.

\(^{25}\) $e$ is the Euler number.
4.2. Lorentzian 3D gravity

We now move on to the case when \( D = 3 \) and \( G = SL(2, \mathbb{R}) \cong SU(1,1) \), i.e. \( G \) is the double cover of the Lorentz group in three dimensions. Thus, this model corresponds to the Lorentzian gravity in 3D.

\( SU(1,1) \) has two nonconjugate Cartan subgroups. This is easy to see as \( su(1, 1) \) can be obtained by complexifying two of the generators of \( su(2) \). Thus we would obtain a generator of rotation and two generators of boosts. The Cartan subgroups are thus the two subgroups generated by these different elements. One Cartan subgroup is just \( U(1) \) generated by the uncomplexified element, we will denote its conjugacy class by \( R \) (for rotation). The other Cartan subgroup is generated by one of the complexified elements and it is a noncompact group (isomorphic to \( \mathbb{R} \)) whose conjugacy class we will denote by \( B \) (for boost). The fact that they are Cartan subgroups means that any element of \( SU(1,1) \) is conjugate to either an element of \( R \) or of \( B \) apart from a set of elements of measure zero in the Haar measure.\(^{26}\) The conjugacy classes of the elements of \( R \) will be parametrized by a periodic parameter \( \theta \) (angle) for which we choose a normalization such that its period is \( 4\pi \). While the conjugacy classes of the elements of \( B \) will be parametrized by a real number \( \psi \) (the boost parameter, rapidity).

The explicit formula for the evolution kernel in proper time is given by the following formula [26]

\[
K[H, T] = \frac{1}{(4\pi iT)^2} \frac{[\theta(H)]}{2 \sin \left( \frac{\theta(H)}{4} \right)} \exp \left[ \frac{i}{2T} [\theta(H)]^2 + \frac{iT}{8} \right] \quad \text{when } H \in R
\]

\[
= \frac{1}{(4\pi iT)^2} \frac{\psi(H)}{2 \sin \left( \frac{\psi(H)}{4} \right)} \exp \left[ \frac{-i}{2T} \psi^2 + \frac{iT}{8} \right] \quad \text{when } H \in B,
\]

where we have used the same notation for the periodic parameter \( \theta \) as in the previous subsection.

Note that when the holonomy group element is a rotation then the \( SU(1,1) \) evolution kernel has exactly the same form as the \( SU(2) \) one (20). The crucial difference between the rotation and the boost cases is the different sign sitting in front of the parameters in the two cases: plus in the rotation case and minus in the boost case.

Once more we are interested in equations (9), (14). According to the general formula (16) we get the dual face amplitude \( A_f \), by multiplying the above expression for the kernel (33) by \( \frac{\mathrm{vol}(\mathbb{R}^n)}{\mathrm{vol}(\mathbb{R}^{n-1})} \) and evaluating its Fourier transform at \( P^2 - \frac{m^2}{8} \). Note that the Killing form (which enters into the definition of \( P^2 \)) now has signature \((++--)\). Thus there is one ‘timelike’ direction (the generator of the compact subgroup) and two ‘spacelike’ ones. Using the same normalizations as in the case of \( SU(2) \), we get \( P^2 = \frac{1}{4}(P_1^2 - P_2^2 - P_3^2) = \frac{1}{4}\bar{P}^2 \).

Now, consider the case when \( H \) is a rotation (\( H \in R \)). Then since the formula for the kernel (33) is exactly the same as the one we used in the \( SU(2) \) calculation (20) we can just write down the answer. Thus

\[
A_N[H, P] = \mu_R([\theta(H)], \sqrt{P^2}, N) \exp \left( i \sqrt{P^2 - \frac{m^2}{4}} [\theta(H)] \right)
\]

\[
\mu_R([\theta(H)], \sqrt{P^2}, N) = \frac{-i\sqrt{2}}{16\pi(N-1)!} \left[ \frac{1}{\sin([\theta(H)])} \left( \frac{[\theta(H)]}{\sqrt{P^2 - \frac{m^2}{4}}} \right) \right]^{N-1}
\]

\(^{26}\) At a technical level, let \( g \in SU(1,1) \). If \( |\mathrm{Tr}(g)| > 2 \) then \( g \in B \). If \( |\mathrm{Tr}(g)| < 2 \) then \( g \in R \). The set of elements which satisfy \( |\mathrm{Tr}(g)| = 2 \) is a set of measure zero.

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The formula analogous to (22) gives a Hankel function of the second kind.

\[ \times \left\{ \sum_{K=0}^{N-2} (-1)^K \frac{(N + K - 2)\!}{K!} \frac{1}{(2i\sqrt{P^2 - m^2\! - \!1}||\theta(H)||)} \right\}^K. \tag{35} \]

These are exactly the same formulae as before (25), (26), with the difference being that \( \tilde{P}^2 \) is calculated with the Minkowski metric and not the Euclidean one, and that this formula is not valid for arbitrary \( SU(1,1) \) element \( H \), rather only when \( HS \) is a ‘rotation’ \( (H \in R) \). Note the exponential factor in the amplitude. Restricting for the moment to the case when \( \tilde{P}^2 - \frac{m^2 - 1}{4} > 0 \), it should be clear that if we interpret, analogously to the Riemannian case, \( \tilde{P}^2 - \frac{m^2 - 1}{4} \) to be the square of the Minkowski length of the edge dual to the dual face under consideration, then the above exponent gives exactly the expected contribution to the Regge action coming from the edge under consideration. In order to simplify the following formulae and discussion we will set \( m^2 = 1 \). This of course also has the effect of making the length of \( \tilde{P}^2 \) to be directly the square of the edge length.

Now, the crucial difference between the Riemannian and the Lorentzian cases lies in the fact that \( \tilde{P}^2 \) can now go over both positive and negative values. As mentioned above the case when \( \tilde{P}^2 > 0 \) one just gets the exponent in the amplitude above becomes \( e^{i|\tilde{P}|||\theta(H)||} \). A simple oscillating phase.

When \( \tilde{P}^2 \) goes negative, clearly \( \sqrt{\tilde{P}^2} = \pm|\tilde{P}| \), and we have to choose a sign. As was mentioned in the previous section we know that \( \tilde{P}^2 \) should have a small positive imaginary part. This means that when \( \tilde{P}^2 \) goes from positive to negative values it does so above the origin in the complex. This means that \( \tilde{P}^2 \) has values above the cut we used to define the square root in the previous section. Thus we have to choose the ‘positive’ square root, i.e. \( \sqrt{\tilde{P}^2} = +|\tilde{P}| \). Plugging this into our exponent we see that it is equal to \( e^{-|\tilde{P}|||\theta(H)||} \).

Keeping in mind the interpretation of the \( P \) variables as that \( \tilde{P}^2 \) is the length of the corresponding edge of the triangulation, we see a very interesting phenomenon happening. The exponent as we said earlier coming from an edge contributes a summand towards the Regge action of the triangulation, with \( |\theta| \) being the deficit angle and \( |\tilde{P}| \) being the length of the relevant edge. Now, as long as the ‘length’ is positive, i.e. \( \tilde{P} \) is timelike we get an oscillating phase in the partition function. On the other hand when the \( \tilde{P} \) goes spacelike making the ‘length’ negative, we get an exponential suppression of the amplitude. Classically, in the Regge action when the edge of the triangulation is timelike, the curvature defect around it has to be a rotation (think of a massive point particle). We see that quantum mechanically this is not true. The edge corresponding to a rotational defect can be both timelike and spacelike, however the spacelike case is suppressed exponentially in the path integral. This is similar to the behaviour exhibited by the Feynman propagator of the relativistic point particle (which is not surprising as we have essentially the same mathematics here). The probability for the particle to propagate inside the lightcone is given by an oscillating phase. The particle can also leak outside the light cone (despite being relativistic). But, the probability of doing so is exponentially suppressed.

This of course is an intuitively satisfying feature of the model. However, the discussion above was limited to the case when the holonomy \( H \) around the dual face is a rotation (lies in \( R \)).

When \( H \) is a boost we can of course repeat the same calculation as before\(^{27}\). However, there is no real need to do this. Look at the two expressions in (33). Note that apart from the factor in front and a phase factor \( e^{i\xi} \), the case when \( g \in B \) is just the complex conjugate of the case when \( g \in R \), due to the difference in the sign in front of the \( \theta \) and \( \psi \). As, from the

\(^{27}\) The formula analogous to (22) gives a Hankel function of the second kind.
mathematical point of view, in order to get the dual face amplitude we are taking a Fourier transform, we can apply the general theorem that relates the Fourier transform of a function to the Fourier transform of its conjugate. Namely, if we denote the Fourier transform of a function \( f \) by \( \mathcal{F}(f)[k] \), then \( \mathcal{F}(f^*[−k]) \). The Fourier transform of a complex conjugate of a function is the complex conjugate of the Fourier transform evaluated at the negative of the argument. Using this we can immediately write down the dual face amplitude in the case when \( g \in B \). It is given by

\[
A_N[H,P] = \mu_B(\psi(H), \sqrt{\vec{P}^2}, N) \exp\left(-i\sqrt{-\vec{P}^2}\psi(H)\right)
\]

\[
\times \mu_B(\psi(H), \sqrt{\vec{P}^2}, N) = \frac{\sqrt{2}}{16\pi(N-1)!} \left[ \frac{1}{\sinh(\psi(H))} \left(\frac{\psi(H)}{\sqrt{-\vec{P}^2}}\right)^{(N-1)} \right]
\]

\[
\times \left\{ \sum_{K=0}^{N-2} (-1)^K \frac{(N + K - 2)!}{K!(N - K - 2)!} \frac{1}{(-2i\sqrt{-\vec{P}^2}\psi(H))^K} \right\}.
\]

The formula for \( \mu_B \) is obtained from \( \mu_R \) by letting \( \vec{P}^2 \rightarrow -\vec{P}^2 \), replacing \([\theta(H)]\) with \(\psi(H)\), switching the trigonometric sine for the hyperbolic one and finally taking the complex conjugate. Of course, by doing the whole calculation from scratch along the same lines as in the Riemannian case, one gets the same result.

Now we can easily see that the behaviour of the amplitude when \( H \) is a boost with respect to the different two possibilities of the sign of the \( \vec{P}^2 \) is opposite of that when \( H \) is a rotation, due to the minus sign in front of \( \vec{P}^2 \) in the formula above. In other words, when \( \vec{P}^2 \) is positive, i.e. \( \vec{P} \) is a spacelike vector, then we just have an oscillating phase. On the other hand when \( \vec{P}^2 \) goes negative, or equivalently, when \( \vec{P} \) becomes timelike, amplitude becomes a decaying exponent\(^{28}\). Again, this is in full agreement with expectations as classically the curvature defect around a spacelike edge is a boost.

Summarizing, if we put together all the dual face amplitudes and form the amplitude for the whole triangulation then what we get is

\[
Z_T = \int_{G^*} \left( \prod_{e \in T} dg_e^* \right) \int_{P^*} \left( \prod_{e \in T} d\vec{P}_e \right) \mu(g_e^*, \vec{P}_e^2, N_e) e^{S_{\text{Regge}}},
\]

where as before \( \mu \) is the quantum measure factor, being a product of \( \mu_R \)'s and \( \mu_B \)'s as appropriate, and \( S_{\text{Regge}} \) is given by

\[
S_{\text{Regge}} = \sum_e \alpha_e L_e \Theta_e.
\]

Here \( L_e \) stands for the absolute value of the length of the edge e (\( |\vec{P}_e| \)) and \( \Theta_e \) stands for the deficit parameter sitting at the edge e (an angle or a boost). Note that they are varied independently of each other showing that we have is first order theory. \( \alpha_e \) is a function of both \( L_e \) and \( \Theta_e \) and is given by the following table

|                | Rotation | Boost |
|----------------|----------|-------|
| Timelike       | +1       | +i    |
| Spacelike      | +i       | −1    |

\(^{28}\) Let us remark that we could have arrived at the same conclusions by being careful with the square root in the formula above. Since \( -\vec{P}^2 \) has to have a small positive imaginary part, \( \vec{P}^2 \) has a small negative imaginary part. Thus when we go from the positive values to the negative ones, we are doing so under the cut, thus choosing the 'negative' square root \( \sqrt{\vec{P}^2} = -i|\vec{P}| \).
Thus as we have said above when the variables are such that one is off-diagonal in this table (rotation-spacelike or boost-timelike) one gets exponential suppression of the amplitude. While when one is on the diagonal then one gets an oscillating phase. This means that the configurations that do not allow for a simultaneous classical geometric interpretation for both the discrete $B$ field and the discrete connection, i.e. those configurations that would be classically disallowed, are not forbidden but still exponentially suppressed. We would like to stress the fact that this causal behaviour is not put into the model by hand, but rather emerges naturally from its very definition as there were no arbitrary choices made anywhere in the construction (once the GFT action has been chosen).

Since the formulae in the Lorentzian case are so close to those in the Riemannian one we can easily carry over all the results from there. So, it is not difficult to see that (26) carries over without much change. In fact, there is no change when $H$ is a rotation apart from the definition of $P$. When $H$ is a boost the angle becomes a boost parameter, the trig sine goes to a hyperbolic one as well as a few sporadic minus signs. The conclusions deduced from the measure factor carry through without any change in the case in which we have an oscillatory contribution to the partition function (i.e. a complex exponential). The only difference being that when $H$ is a boost, all the phases go to their conjugates, which of course does not affect the qualitative behaviour.

When we are off-diagonal in the table, and we then have an exponential suppression, the integrand is, apart from an overall factor (a power of $i$), real. This is easiest to see from the fact that, as is evident from (24), the Hankel function for purely imaginary arguments is a (multiple of) real function. Thus strictly speaking one just has the measure factor in the path integral and no complex exponential (whose phase is to be interpreted to be the action). However, we find it far more clear, intuitively, and more insightful from the physical perspective to split again the integrand into a ‘measure factor’ and an exponent as we did above. Applying this philosophy to $\mu(g_e, P_e, N_e)$, we get corrections to the action $e^{-S_{\text{Regge}}}$ of the form $e^{-S_c}$, exactly in accordance with expectations, and in complete similarity with the results obtained in the other cases.

As before, in the large Minkowski length limit the quantum corrections $S_c$ coming from the phase of the factor $\mu$ are of the inverse scalar curvature type ($\sim \frac{1}{R}$), indicating the infrared corrections to the bare Regge action in the semi-classical limit.

Moreover, in all cases, we still get an amplitude that diverges like $\frac{1}{|\vec{P}|^{2N-3}}$ as $|\vec{P}| \rightarrow 0$, which means that when the number of tetrahedra in the triangulation increases, the shorter ‘lengths’ become more and more dominant ones, which is what one would expect if the model is to have a good continuum limit. The point $\vec{P}^2 = 0$ is a branch point of the amplitude which diverges there, thus requiring a much more detailed treatment deferred for future work.

Let us now move on to the Lorentzian analogue of the $(J, P)$ representation (30) for the quantum amplitudes. To do this note that $SU(1,1)$ has two types of representations [36]:

- Discrete ones labelled by a positive half integer $J$. The Casimir $C_J$ for these representations is negative and is equal to $C_J = -\frac{1}{2}J(J + 1)$. The constant $\Delta_J$ appearing in the character expansion of the delta function is $\Delta_J = 2J + 1$.

- Continuous ones labelled by a positive real number $\rho$. The Casimir for these representations is positive and is equal to $C_\rho = \frac{\rho^2}{2} + \frac{1}{8}$. The constant $\Delta_\rho$ is just $\Delta_\rho = 2\rho$.

Note that due to a difference in normalizations our Casimir is half the one in [36]. Also, we are not differentiating here, as it is not necessary nor very useful, between positive and negative discrete series of representations as they both enter in exactly the same way into the partition function (41). This duplicity of representations is responsible for the factor of 2 in front of the sum over the discrete representations.
If we plug these expressions into (18) (note that we have to pick the positive sign in front of the Casimir as SU(1,1) is noncompact) we get

\[
Z_{\Gamma} = \left( \prod_{f \in T^*} \left[ 2 \sum_{J_f} + \int_{0}^{\infty} d \rho_f \right] \int_{0}^{\infty} d^3 \vec{P}_{f^*} \right) \prod_{f \in T^*} A_{f^*}(J_{f^*}/\rho_{f^*}, P_{f^*}, N_{f^*}) \prod_{v \in T^*} \{6 - J\},
\]

where now we get two types of the dual face amplitude in the \((J, P)\) variables

\[
A_{f^*}[J_{f^*}, P_{f^*}, N_{f^*}] = \frac{i^{N_{f^*}} (2J_{f^*} + 1)}{\left( P_{f^*}^2 - \frac{m^2}{8} - J_{f^*}(J_{f^*} + 1) + i \epsilon \right)^{N_{f^*}}},
\]

when the representation is of discrete type, and

\[
A_{f^*}[\rho_{f^*}, P_{f^*}, N_{f^*}] = \frac{i^{N_{f^*}} (2\rho_{f^*})}{\left( \frac{P_{f^*}^2}{2} - \frac{\rho_{f^*}^2}{2} + \frac{1}{8} + i \epsilon \right)^{N_{f^*}}},
\]

when it is of the continuous type.

It is obvious from these two expressions that we get poles of two types, timelike and spacelike. What we mean by this is that there are two sets of poles, one when \(\vec{P}^2\) is positive, i.e. when \(\vec{P}\) is timelike; and one when it is negative, i.e. \(\vec{P}\) is spacelike. The first type of poles occurs when the representation labelling the dual face is of discrete type, as then we have in the denominator of (38) the following expression \((\vec{P}_{f^*}^2 - \frac{m^2}{8} - \frac{1}{2} J(J + 1))\), which vanishes when

\[
\vec{P}^2 = J(J + 1) + \frac{m^2}{4},
\]

which, if we set \(m^2 = 1\), gives \(\vec{P}^2 = \frac{\Delta_{f^*}^2}{4}\) as in the Riemannian case.

The other type of poles occurs when the relevant representation is of a continuous type as then we have in the denominator of the same equation the expression \((\frac{P_{f^*}^2}{2} - \frac{\rho_{f^*}^2}{2} + \frac{1}{8})\).

Which vanishes when

\[
\vec{P}^2 = -\rho^2 + \frac{m^2 - 1}{4},
\]

which, on setting \(m^2 = 1\), gives \(\vec{P}^2 = -\rho^2 = -\frac{\Delta_{f^*}^2}{4}\).

If we interpret these formulae as giving the semi-classical values of the ‘length’, we arrive at the intriguing fact that there are no preferred spacelike lengths as \(\rho\) is continuous and thus the ‘poles’ at the \(\Delta_{f^*}\)’s fill the line. In contrast, there are preferred timelike ‘lengths’, which are the discretely spaced \(\Delta_{f^*}\)’s.

Finally, by doing a ‘Wick rotation’ \((P_1, P_2, P_3) \to (P_1, iP_2, iP_3)\), we can perform the integrals over the \(\vec{P}_{f^*}\)’s along the lines of what was done in the Riemannian case. The asymptotic formula (32) goes through essentially unchanged, and we get

\[
A_{f^*}(J_{f^*}, N_{f^*}) \sim \frac{1}{F(N_{f^*})} \left[ \frac{8}{\Delta_{f^*}^2} \right]^{N_{f^*} - 1} \quad \text{or} \quad A_{f^*}(J_{f^*}, N_{f^*}) \sim \frac{1}{F(N_{f^*})} \left[ \frac{8}{\Delta_{f^*}^2} \right]^{N_{f^*} - 1}
\]

the only relevant difference being that the second factor (which dictates the behaviour of the asymptotic) is now either \(\left[ \frac{8}{\Delta_{f^*}^2} \right]^{N_{f^*} - 1}\) (as before) or alternatively equal to \(\left[ \frac{8}{\Delta_{f^*}^2} \right]^{N_{f^*} - 1}\). The conclusion is the same as before: for large \(N_{f^*}\)’s it is only the lowest \(J^\prime\)’s and \(\rho^\prime\)’s that contribute \((J^\prime < 1\) and \(\rho^\prime < \sqrt{5})\).

\[30\] Strictly speaking the function \(F\) here, when the representation is of continuous type, is different from the \(F\) in (32). However, the exact form of \(F\) is of no importance for us here as long as it is still a function of polynomial growth. We will continue to use this notation for this prefactor in 4D as well.
Note that we could have performed the mentioned `Wick rotation' anywhere in the above discussion. Most importantly, we could have done it in the triangulation amplitude (36). Since this amplitude is just a partition function for gravity, we thus see that there is a straightforward way of performing the `Wick rotation' in the gravity partition function coming from the new model which does not rely on the existence of any particular time slicing. We would like to point out however, that this `Wick rotation' (although very similar to the rotation in the squares of the edge lengths performed in causal dynamical triangulations [8]) is not the complete story, in the sense that it does not turn the action for Lorentzian gravity into one for Riemannian gravity, nor does it turn complex exponentials into real ones (thus quantum mechanical amplitudes into statistical weights). This is due to the fact that we have a first order theory with the $B$ and $A$ fields being totally independent. Thus, while we Wick rotate the $B$ field to a Euclidean one, we do not touch the connection. In this sense, the label `Wick rotation' is a slight abuse of language, as it really corresponds to some sort of partial or `half-performed Wick rotation', from a geometric perspective, hence the quotation marks. However, we find it very intriguing that even this partial transformation can be performed in such a natural way, and believe it can be a good starting point for a similarly natural, but this time complete definition of a geometric Wick rotation in simplicial quantum gravity.

Summarizing, we see that the Lorentzian case is not particularly different from the Riemannian one. There is essentially only one major, qualitative difference, which stems from the fact that the Lorentzian geometry is richer than the Euclidean one. Due to the first order nature of the theory, in the Lorentzian setting one gets additional, classically forbidden, histories, which have `mismatching' $B$ and $A$ fields. These histories are, as is customary in quantum mechanics, exponentially suppressed. As for the rest the same simplicial gravity path integral interpretation for the Feynman amplitudes of our GFT applies, and similar types of quantum corrections to the first order Regge action are identified.

5. New 4D GFT models

5.1. Riemannian BF theory

We now come to dealing with the four-dimensional case ($D = 4$). Our discussion in this subsection and the next will be rather brief as, if we stick to (causally restricted) BF theory (as opposed to gravity, in higher dimensions), there is little difference between the 3D and 4D cases. Our main aim in the present section is indeed to show explicitly that there are no qualitative new features added to the model by going to the fourth dimension, in neither the Riemannian nor the Lorentzian signatures, which shows how our proposed new class of GFTs behaves similarly in any dimension. As we shall see below, the four-dimensional models are essentially carbon copies of the three-dimensional ones.

The group that we are using for the Riemannian version of the 4D theory is the double cover of the rotation group in four dimensions $SO(4)$, which is just $Spin(4) \simeq SU(2) \times SU(2)$. The fact that the group is a direct product of two copies of the group we used for the 3D Riemannian case allows us to carry over easily essentially all the results we discussed in that case to the 4D setting. The reason for this is the fact that the Schrödinger kernel on $G_1 \times G_2$ is just the product of the kernels on $G_1$ and $G_2$. This in turn follows from the fact that the Laplacian on the direct product of two groups is just the sum of the two Laplacians $\Box_{G_1 \times G_2} = \Box_{G_1} + \Box_{G_2}$. This allows us to write down the kernel on $SU(2) \times SU(2)$ right away, essentially by squaring the expression given in (20)

$$K[H, T] = \frac{1}{(4\pi i T)^3} \frac{[\theta_1(H)] [\theta_2(H)]}{4 \sin \left(\frac{\theta_1(H)}{2}\right) \sin \left(\frac{\theta_2(H)}{2}\right)} \exp \left[ \frac{2T}{\theta_1^2 + \theta_2^2 + \frac{i \theta_1 \theta_2}{4}} \right].$$

(39)
We are of course using the same notation as before with respect to the periodic parameters \( \theta_1 \) and \( \theta_2 \). As before, we want to calculate the Feynman graph/triangulation amplitude \( Z_T = Z_{T'} \). Since this amplitude when written in terms of the \((g, P)\) variables factorizes per dual face, we concentrate on the amplitude for a single dual face.

Since the group is compact, its Killing form, in our conventions, is positive definite. Also, since the space \( P \) is isometric to the (dual of) \( \text{spin}(4) \cong \text{su}(2) \oplus \text{su}(2) \) we have \( P = P_1 \oplus P_2 \), with \( P_1 \cong P_2 \cong \text{su}^* (2) \). Thus, (with our normalizations) \( P^2 = \frac{1}{2} (\vec{P}_1^2 + \vec{P}_2^2) = \frac{1}{2} \vec{P}^2 \). Also, below we will denote the combination \( \sqrt{[\theta_1(H)]^2 + [\theta_2(H)]^2} \) as \( \theta (H) \). As in the 3D case, this (equivalence class of) parameter(s) has the geometric interpretation as the square distance between the origin and the point on the group manifold corresponding to the holonomy \( H \), measured along a geodesic.

Using formula (22), we get

\[
A_N[H, P] = \left( \frac{i^{N-1}}{(16\pi)^2 (N-1)!} \sin \left( \frac{\text{tan}(\theta_1(H))}{2} \right) \sin \left( \frac{\text{tan}(\theta_2(H))}{2} \right) \right) \left( \frac{[\Theta(H)]}{\sqrt{\vec{P}^2 - \frac{m^2}{4}}} \right)^{N-3} \times H^{(1)}_{N-3} \left( \sqrt{\vec{P}^2 - \frac{m^2 - 1}{2}}, [\Theta(H)] \right),
\]

with the same analytic continuation in the variables as in the 3D case. The Hankel function of integer order does not have an expression in terms of elementary functions analogous to (24). Instead it is given in terms of the following non-elementary integral\(^\text{31}\)

\[
H^{(1)}_{N-3}(z) = - \frac{2^{N-2} i z^{N-3}}{\Gamma (N - \frac{3}{2})} \sqrt{\pi} \int_0^\infty ds \frac{\cos (N - \frac{5}{2}) s}{\sin (2N - 5) s} \exp (-2 s \cot (s)) e^z.
\]

As we see, the formula above still furnishes a natural split of the amplitude into an exponential piece and a ‘measure’ piece. Thus the dual face amplitude is equal to

\[
A_N[H, P] = \mu ([\Theta(H)], \vec{P}, N) \exp \left( i \sqrt{\vec{P}^2 - \frac{m^2 - 1}{2}} [\Theta(H)] \right),
\]

with \( \mu \) given by

\[
\mu ([\Theta(H)], \vec{P}, N) = - \frac{i^{N-1} 2^{N-10}}{\pi^2 (N - 1)! \Gamma (N - \frac{5}{2})} \left( \frac{[\theta_1(H)] [\theta_2(H)]}{\sin \left( \frac{\text{tan}(\theta_1(H))}{2} \right) \sin \left( \frac{\text{tan}(\theta_2(H))}{2} \right)} \right) \left( \frac{[\Theta(H)]}{\sqrt{\vec{P}^2 - \frac{m^2}{4}}} \right)^{2(N-3)} \times \int_0^\infty ds \frac{\cos (N - \frac{5}{2}) s}{\sin (2N - 5) s} \exp \left( -2 s \cot (s) [\Theta(H)] \sqrt{\vec{P}^2 - \frac{m^2 - 1}{2}} \right).
\]

As before, we multiply together all the dual face amplitudes and obtain the amplitude for the Feynman diagram/triangulation

\[
Z_T = \int_{G^*} \left( \prod_{\epsilon \in T} dg_{\epsilon} \right) \int_{T'} \left( \prod_{\epsilon \in T} d\vec{P}_\epsilon \right) \mu (g_{\epsilon'}, \vec{P}^2_{\epsilon'}, N_{\epsilon'}) e^{iS_{\text{CBF}}},
\]

\(^\text{31}\) Strictly speaking this formula is valid only when the order of the Hankel function is greater than 0, i.e., when \( N \geq 3 \). However, there is a very simple relation between a Hankel function of a negative order with the one of a positive one, which is \( H^{(1)}_{N}(z) = e^{i\pi N} H^{(1)}_{-N}(z) \). This means that all we need to do when \( N = 2 \) (this is the only allowed value for \( N \) which is less than 3, since any dual face has at least two vertices) is multiply the given formula by a sign.
the products go over all the dual edges $e \ast$ of the dual complex $T \ast$ and over all the triangles $t$ in the triangulation $T$, the $\mu$ is the product of all the $\mu$'s coming from all the edges. The expression $S_{CBF}$ in the exponent is

$$S_{CBF} = \sum_{t} \sqrt{\vec{P}_{t}^2 - m^2 / 4} |\Theta_1|.$$  

Thus the Feynman amplitudes of the model are partition functions for an action of discretized BF theory type. Classically, the theory given by the action $S_{CBF}$ coincides with the one given by the usual BF action, as the equations of motion that they produce are the same (zero curvature)\(^{32}\). However, quantum mechanically, there is a significant difference between the two theories. The difference being that for the usual BF theory the integral over the $B$ field is unrestricted, with the integration producing the usual causal, real partition function. For the model given by $S_{CBF}$ the fact that the variable $\vec{P}$, which represents the discretized $B$ field, enters only through its length (which is always positive), means that what we have is the ‘causal’ analogue of the usual BF theory (hence the subscript).

It is tempting to call $\sqrt{\vec{P}_{t}^2 - m^2 / 4}$ the area of the triangle $t$ of the triangulation. However, this is untenable as the variable $\vec{P}$, being generic and non-simple (i.e. not itself a wedge product of 4-vectors), does not have an interpretation of defining the geometry of the triangle to which is associated, as one needs a simple bivector to do this. As before, the identification is cleanest if we set $m^2 = 1$ which we do in what follows to simplify the discussion and formulae. It is clear, however, that we are setting the stage for obtaining a proper causal spin foam model for 4D gravity, to be defined from the above by imposition of suitable simplicity constraints on the $P$ variables.

As before we write the $\mu$ in terms of magnitude and phase

$$\mu(g^e, \vec{P}_t^2, N_t) = |\mu(g^e, \vec{P}_t^2, N_t)| e^{iS_{CBF}(g^e, \vec{P}_t^2, N_t)}.$$  

Again, we interpret the $|\mu(g^e, \vec{P}_t^2, N_t)|$ as a quantum measure factor, while the phase $e^{iS_{CBF}}$ gives quantum corrections to the pure BF action $S_{CBF}$.

It is straightforward to extract the explicit expression for the phase from (40). It is given by,

$$S_{r}(g^e, \vec{P}_t^2, N_t) = \sum_{t} \left[ ((N + 1) \mod 4) \frac{\pi}{2} + \arctan \left\{ \frac{\int_{0}^{\pi/2} \cos^{N/2}(s) \sin^{N/2}(s) \left[ \sin^{N/2}(s) \cos^{N/2}(s) e^{-(N-4)s} \right]^{m^2 / 4}}{\int_{0}^{\pi/2} \cos^{N/2}(s) \sin^{N/2}(s) \left[ \sin^{N/2}(s) \cos^{N/2}(s) e^{-(N-4)s} \right]^{m^2 / 4}} \right\} \right].$$  

Although this expression looks totally different from the one we had in 3D (26) many of the features of the three-dimensional model carry through without any change. Most importantly, it still depends on the $\Theta_1$ and $|\vec{P}_t|$ solely through the combination $(|\vec{P}_t|/\Theta_1)$, which, at least when $\vec{P}_t$ is simple, can be interpreted to be the discrete analogue of the Ricci scalar $R$. Which means that the quantum corrections arising from the ‘measure’ $\mu$ are of the general form $f(R)$, just like in 3D.

It is possible to analyse asymptotically the expression for the phase above, along the lines done in 3D, and compute the exact coefficients and combinatorial factors weighting the

\(^{32}\) Notice also the similarity with the action appearing in the asymptotic (large-$J$) approximation of the Barrett–Crane spin foam vertex amplitude.
corrections to the Regge action.\footnote{Technically speaking, it is easier to use the asymptotic expansion of the Hankel function \cite{37}. This expansion looks very much like \eqref{24}, which, in the half-integer-order case, terminates and provides an explicit expression.} In the large ‘area’ asymptotic ($|\vec{P}| \to \infty$), the result is the same as before. One gets inverse scalar curvature corrections ($\int \left[ \frac{d}{\lambda} + O\left( \frac{1}{\lambda^2} \right) \right] \text{Vol}$) to the bare BF action, i.e. one gets infrared terms arising from the factor $\mu$ in the large distance and semi-classical regime.

Also, just as is the case in three dimensions, it is possible to see that the dual face amplitude goes like $\frac{1}{|\vec{P}|^{3}}$ when $|\vec{P}| \to 0$.\footnote{This simply follows from the fact that $H^{(1)}_{\nu}(z) \sim \frac{1}{z}$ when $z$ is close to zero.} As before, we would like to draw the reader’s attention to the fact that this type of behaviour is at least consistent with, if not suggestive of, the existence of the continuum limit.

Let us move on to the $J$ variables. Since our group is a product of two copies of $SU(2)$, its representation theory follows from that of the $SU(2)$. More precisely, each irrep of $SU(2) \times SU(2)$, is characterized by a pair of half-integers $(J_1, J_2)$. The dimension of such an irrep is $\Delta_{J_1, J_2} = (2J_1 + 1)(2J_2 + 1)$, finally the Casimir that concerns us\footnote{This is a somewhat technical point. The space of Casimirs for this group is two dimensional. It is spanned by, for example, the sum and the difference of the two Casimirs coming from each factor of $SU(2)$. However, among all these Casimirs there is one special which comes from the Killing form (often called, the scalar Casimir, in the literature). It is exactly this one that corresponds to the Laplace–Beltrami operator that we have and which is equal to this operator’s eigenvalues.} is just the sum of the two Casimirs coming from the two $SU(2)$ factors $\frac{1}{2}(J_1(J_1 + 1) + J_2(J_2 + 1))$.

The 4D case corresponding to equation \eqref{18} is

$$Z_{T^*} = \sum_{J_1, \ldots, J_{F}} \left( \prod_{f \in T^*} \left[ \int d^5 \vec{P}_f \right] \frac{i^{N_{J_f}}}{2^{J_{f^2} + 1}} \frac{(2J_{f_2} + 1)(2J_{f_1} + 1)}{\Delta_{J_f}^2 + \Delta_{J_f}^2} \right)^{N_{J_f} - 1} \times \prod_{v \in T^*} (15 - j),$$

from which we immediately obtain the semi-classical values of $|\vec{P}|$. They are

$$|\vec{P}| = \frac{\sqrt{(2J_1 + 1)^2 + (2J_2 + 1)^2 + m^2 - 1}}{2},$$

which as before have the nicest form when $m^2 = 1$.

Finally, it is not difficult to perform the integrals over the $P$ variables, and obtain the analogue of equation \eqref{32}. The result is

$$A_{t^*}(J_{f^*}, N_{f^*}) \sim \frac{1}{F(N_{f^*})} \left( \frac{8}{\Delta_{J_1}^2 + \Delta_{J_2}^2} \right)^{N_{f^*} - 1}.$$

As we see it is entirely analogous to the one before, with the crucial factor $\left[ \frac{4}{N_{f^*}^2} \right]^{N_{f^*} - 1}$, which determines the asymptotic being replaced by $\left[ \frac{8}{\Delta_{J_1}^2 + \Delta_{J_2}^2} \right]^{N_{f^*} - 1}$. The dominant contributions come from the representations for which $\Delta_{J_1}^2 + \Delta_{J_2}^2 \leq 8$. This is satisfied only when neither $J_1$ exceeds $\frac{1}{2}$. In other words, the two allowed values are those corresponding to the ‘vacuum’ and to the ‘lowest excited state’. Also, note that if we impose, by hand, the simplicity constraint at this level, in the way it is imposed in usual spin foam models, i.e. if we set $J_1 = J_2$ then there are exactly two dominant contributions: the vacuum $J_1 = J_2 = 0$, once more, and the configuration with $J_1 = J_2 = \frac{1}{2}$. Once again, one can think of this as an indication of a dynamical reduction of the model to a purely combinatorial one of the dynamical triangulations-type.
5.2. Lorentzian BF theory

Finally, let us consider the case when \( D = 4 \) and \( G = \text{SL}(2, \mathbb{C}) \). The technical difference between the (double cover of the) Lorentz group in four dimensions and the one in three is that in 4D the group \( \text{SL}(2, \mathbb{C}) \) has just one Cartan subalgebra. Thus, apart from a set of measure zero, all elements in the group are conjugate to the elements of the Cartan subgroup, which is the image of the Cartan subalgebra under the exponential map. As \( \mathfrak{sl}(2, \mathbb{C}) \) is spanned by three rotations and three boosts one can take the Cartan subalgebra spanned by a rotation and a boost along the same direction, i.e. one ‘compact’ and one ‘noncompact’ element. Thus the Schrödinger kernel will be parametrized by one periodic parameter \( \phi \) (with period \( 4\pi \)) and one non-periodic one \( \psi \). One can think of them as giving the angle of rotation and the boost parameter of the given group element. Intuitively, since \( \mathfrak{sl}(2, \mathbb{C}) \) can be thought of to be a complexification of \( \mathfrak{su}(2) \oplus \mathfrak{su}(2) \), we will see that what happens in the Lorentzian domain can be guessed by complexifying the results obtained in the Riemannian one. For example, the formula for the kernel on \( \text{SL}(2, \mathbb{C}) \) is effectively a complexification of that on \( \text{SU}(2) \times \text{SU}(2) \) given in (39)

\[
K[H, T] = \frac{1}{(4\pi i T)^3} \frac{[\theta(H)] \psi(H)}{4 \sin(\theta(H)) \sinh(\psi(H))} \exp \left[ \frac{i}{2T} (|\theta(H)|^2 - \psi^2(H)) + \frac{i T}{4} \right].
\]

As should be easy to see, the above expression is obtained by picking one of the \( \theta \)'s in (39) and analytically continuing it to \( i\psi \).

Again, we use formulae (9, 14, 16).

The Killing form on \( \mathfrak{sl}(2, \mathbb{C}) \) has signature \((+++---)\), thus in our normalization

\[
P^2 = \frac{1}{2} \tilde{P}^2 = \frac{1}{2} (P_1^2 + P_2^2 + P_3^2 - P_4^2 - P_5^2 - P_6^2).
\]

Now, there are two ways to do the required integral. Either we use the Hankel function (22) formula and plow ahead with the algebra, paying attention to how we approach the cut when we take the square root. Or we use similar arguments to what we used when we discussed the Lorentzian case in three dimensions, using the fact that mathematically we are just performing a 1D Fourier transform, which allows us to rely on the relation between the Fourier transforms of the function and its complex conjugate. Either way, the dual face amplitude is given by

\[
A_N[H, P] = \mu ([\Theta(H)], \tilde{P}, N) \exp(i\alpha([\Theta(H)], [\tilde{P}^2]));
\]

where \([\Theta(H)]^2\) is equal to \([\phi(H)]^2 - \psi^2(H)\) which is just the (square of the) length of the Cartan subalgebra element parametrizing the conjugacy classes, or equivalently it is the length of a geodesic on the group manifold joining the point given by the element \( H \) to the identity. The \( \alpha \) is given by the following table, which is a carbon copy of the one in 3D (37)

| Rotation | Boost |
|----------|-------|
| Timelike | +1    | +i   |
| Spacelike| +1    | -1   |

The columns are labelled by the two possible cases of the \( \Theta^2(H) \). The ‘rotation’ is when \( \Theta^2(H) \) is positive, as it is easy to see that the \( H \) is then conjugate to a rotation; while the ‘boost’ is when \( \Theta^2(H) \) is negative as this is when \( H \) is conjugate to a boost.

The rows, on the other hand, are labelled by the two possible cases of \( \tilde{P}^2 \). ‘timelike’ is when this vector has positive length and ‘spacelike’ when this vector has negative length.

Finally, the \( \mu \) is, apart from sporadic signs and factors of i, just the analytic continuation \((\theta \rightarrow i\psi)\) of the measure in the Riemannian case (40). For completeness we give the exact

\[36\] The set of elements whose trace is equal to 2.
formula here

\[
\mu([\Theta(H)], \vec{P}, N) = \frac{\pi i^N 2^{N-10} \alpha^{N-3} ([\Theta(H)], \vec{P}^2)}{\sin\left(\frac{\theta_1(H)[\theta_2(H)]}{2}\right) \sinh\left(\frac{\theta_1(H)[\theta_2(H)]}{2}\right)} \left[\Theta(H)\right]^{2(N-3)} \times \left\{ \int_0^\infty ds \frac{\cos^{N-\frac{3}{2}}(s) e^{\pi i s(N-\frac{3}{2})}}{\sin^{2N-3}(s)} \exp\left[-2\cot(s)\alpha([\Theta(H)], \vec{P}^2)\right] \right\}.
\]

It is straightforward to compute the Feynman graph amplitude \( A_T = A_{T^*} \) now. It is

\[
Z_{T^*} = \int_{Gr^*} \left( \prod_{v^* \in T^*} dg_{v^*} \right) \int_{P^*} \left( \prod_{e^* \in T} d\vec{P}_{e^*} \right) \mu(\rho^*, \vec{P}^2, N^*_{\alpha}) e^{i S_{CBF}},
\]

where as before \( \mu \) is a product of all the \( \mu \)'s coming from each dual face and \( S_{CBF} \) is given by

\[
S_{CBF} = \sum \alpha_t|\vec{P}_t|\theta_t|.
\]

where \( \alpha \) is given in the table above. Once again, we get a 'causal' BF action in our partition function, i.e. we get a theory whose classical equations of motion are just like those of the standard BF, while there are profound differences at the quantum level.

As was the case in 3D we get exponential suppression of the 'wrong' type of correlation between the variables. More precisely, had it not been for the fact that the variables \( P \) are in general not simple, we could have said that in the situation when the triangle corresponding to a holonomy given by a rotation is timelike or alternatively when it is spacelike when the holonomy is a boost, then this triangle contributes a phase to the partition function. On the other hand, if there is a mismatch between the \( \Theta \) and \( P \) (rotation-spacelike or boost-timelike), this triangle contributes an exponential suppression factor to the partition function. The behaviour of the model in the Lorentzian case is unaffected by dimension.

Also, since the 'measure' factor is effectively the same as in the Riemannian case, its phase depends on the deficit parameter \( \Theta \) and on the 'area' \( |\vec{P}| \) in the same way as the bare BF action does, i.e. the phase of \( \mu \) is a function of \( \alpha_t|\vec{P}_t|\theta_t \) (as well as the \( N^*_\alpha \)'s characterizing the triangulation). This fact is interpreted as before to mean that there are quantum corrections arising from the factor \( \mu \) of the general \( f(R) \) type.

The semi-classical analysis is exactly the same as in the Riemannian case and one sees that in the limit of large 'areas', we get inverse scalar curvature corrections to the bare BF action as before. Finally, the amplitude is as divergent as before in the neighbourhood of the \( \vec{P}^2 = 0 \). As in the previous section, we consider this fact to be a necessary condition for the existence of the continuum limit.

It is not difficult to write down the full Feynman amplitude in the \((J, P)\) variables. The relevant representations of \( SL(2, \mathbb{C}) \) are labelled by two parameters. A half integer \( J \) and a real positive parameter \( \rho \). The relevant Casimir and normalizations is equal to \( C_{J, \rho} = \frac{1}{4} - \frac{J^2 - \rho^2}{8} \).

The analogue of (18) is now

\[
Z_T = \left( \prod_{J^*} \left[ \sum_{j^*} \int_0^\infty d\rho_{j^*} \right] \int_P d^3 \vec{P}_{j^*} \right) \prod_{\rho_{j^*}} \left( \frac{P_{j^*}^2}{8} - m^2 + \frac{1}{4} - \frac{J_{j^*}^2 - \rho_{j^*}^2}{8} N_{j^*} \right) \prod_{v^*} \{15 - J\}.
\]

The poles in the expression (18) are obviously located at

\[
\vec{P}^2 = \frac{J^2 - \rho^2}{4} + \frac{m^2 - 1}{2}
\]

so these particular values are the preferred semi-classical 'areas'.

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Finally, as in 3D there exists a simple way of performing a (partial, as explained) Wick rotation in this model, by analytically continuing (some components of) the $P$ variables. This shows also that the existence of a good Wick rotation in our model is independent of the dimension. We can use this Wick rotation to perform the integrals over the $P$'s in the amplitude in the $(J, P)$ variables (equation (18)) and obtain the 4D Lorentzian analogue of (32), which is given by

$$A_{J^*, P^*}(J_{f^*}, \rho_{f^*}, N_{f^*}) \sim \frac{1}{F(N_{f^*})} \left[ \frac{8}{J_{f^*}^2 - \rho_{f^*}^2} \right]^{N_{f^*} - 1}.$$ 

The relevant difference from the Riemannian case is that the factor $\left[ \frac{8}{J_{\Delta}^2 - \Delta^2} \right]^{N-1}$, which controls the way the asymptotic behaves, gets replaced with $\left[ \frac{8}{J_{\Delta}^2 - \rho^2} \right]^{N-1}$. This means that the most dominant contributions are those which satisfy $|J^2 - \rho^2| < 8$. This does not of course force the $J$ and the $\rho$ to each be small (as was the case in 3D). However, it does force the Minkowski ‘length’ (or more appropriately area) to be small. If we restrict—by hand—to representations which are simple (i.e. those for which either $J$ or $\rho$ is zero), the expression above does force each of the parameters to be small, hinting again at a dynamical reduction to a dynamical triangulations-like sector.

5.3. Discussion: a new route from BF to gravity?

We have presented above a new GFT model for a BF-type formulation of quantum simplicial gravity, in four dimensions, in the spin foam formalism. The spin foam amplitudes (GFT Feynman amplitudes) have the form, modulo a quantum measure, of the exponential of a classical first order action based on two types of variables: a set of bivectors associated with 2-simplices of the simplicial complex and a set of Lorentz group elements representing parallel transports of a Lorentz connection. The action has a Regge calculus expression, augmented by higher order terms that can be interpreted as quantum corrections, that become negligible in the semi-classical limit. In this generalized simplicial gravity action, the ‘areas’ of the triangles are functions of the bivectors and the deficit angle associated, again, each triangle obtained from the holonomy of the same Lorentz connection, and thus a function of the corresponding group elements. The equations of motion following variation of the dominant contribution to the action, in the semi-classical (large distance) limit, restrict the holonomies to be flat, just as in ordinary BF theory, but the integration over the bivectors in the path integral does not treat on equal footing positive and negative orientations for the triangles, as BF theories do. The result is a complex amplitude, as said, and not a delta function over flat connections as in BF theory.

Now, in the new 4D BF-like model, thanks to the explicit presence of bivector variables and to their role in the discrete path integral clearly analogous to that played by the $B$ field in continuum formulations, the simplicity constraint is implemented in much more straightforwardly than in the usual approaches. This is achieved by constraining the integration over the bivector variables of our 4D model. The simplest way of doing so is to insert appropriate delta functions imposing the simplicity conditions on bivectors, and one has just to make sure that this is done consistently and as geometrically expected at the level of each Feynman diagram. Alternatively, and preferably, one should implement the constraints directly at the level of the new GFT action, and for doing so one has to identify clearly which constraints are needed in each 4-simplex (interaction term) and which refer to the gluing of 4-simplices (kinetic term), or whether one should instead constrain directly the field in both terms, as it is done for the other spin foam models [16]. Work on this is in progress. The
expected result of this new way of implementing the simplicity constraints, starting from our
4D model, is to obtain a constrained GFT whose Feynman amplitudes are given by a path
integral for an action that could be directly interpreted as the discretization of the Plebanski
action for classical gravity, i.e. of the form:
\[ S = \sum_f A_f(B_f) \theta_1(f(g_e^*) + \sum_v \lambda_v f(B_{f|v}), \]
where \( \lambda_v \) are Lagrange multipliers imposing the constraints \( f(B_{f|v}) \)
on the bivector variables \( B_{f|v} \) associated with each dual face (triangle) \( f \) incident to each dual vertex (4-simplex) \( v \) (we
have neglected here the quantum corrections to the first order Regge action coming from the
measure).

6. Conclusions and outlook

We have presented a new class of GFT models for the dynamics of quantum geometry, in any
spacetime dimension and signature. The construction was based on the extension of the GFT
formalism to include additional variables with the interpretation of a discrete counterpart of
the continuum \( B \) field in BF-like formulations of gravity. The Feynman amplitudes for the
new GFTs, i.e. the corresponding spin foam models, have exactly the form of true simplicial
gravity path integrals, with a clear-cut relation with discrete gravity actions, as opposed to
other known models in which the connection arises only in some asymptotic limit. In 3D
the new models are seen to provide a quantization of discrete quantum gravity in first order
(Palatini) form, in a local and discrete third quantized framework in which topology is allowed
to fluctuate. In 4D and higher, the new models have the form of a third quantized framework
for BF theory, but with an additional dependence of the amplitudes on the orientation of the
simplicial complex, of the type one would expect in a path integral quantization of first order
gravity.

The Lorentzian models also present a very nice interplay between the two sets of discrete
variables (\( B \) field and connection) which leads automatically to a suppression of all the
configurations which do not match the simultaneous geometric interpretation of both of them.

The GFT provides also a precise prescription for the quantum correction to the classical
Regge-like action (in first order form) that have to be included in the corresponding path
integral, in absence of further restrictions to the models. These additional terms in the action
become negligible in both the continuum limit (large number of simplices of small size) and
in the semi-classical limit (arbitrary number of simplices but large size of simplices, thus large
associated action), leaving only the Regge action to contribute to the path integral, as one
would expect. In the general case, and as soon as one goes beyond these limiting regimes, the
simplicial action provided by our GFT models turns into a generic \( f(R) \) extended action for
gravity. We feel that this has several interesting implications at the simplicial gravity level as
well as from a more phenomenological perspective, that deserve to be studied in more detail.

The way the large-\( P \) limit affects the amplitudes of the new models sheds new light, we
feel, on the usual large-\( j \) limit\(^{37} \) that brings usual spin foam models in closer relation with
simplicial gravity path integrals, by allowing an approximation of the vertex amplitudes with
the cosine of the Regge action. Indeed, our models suggest that this limit is a large distance
limit which is equivalent, at the discrete level, to a large action and thus semi-classical limit
(because of the way the Regge action, and its higher order corrections, depends on the hinge
volumes). As such, it has two effects: it kills any quantum interference between opposite
orientations for the hinges, in the usual models only, in which such opposite orientations are
treated on equal footing; it kills any short-distance effect such as \( R^n \) corrections to the action,

\(^{37}\) The large-\( P \) regime is related to the large \( j \) regime, as we have seen, in a `semi-classical approximation of the
amplitudes’ when these are expressed in the \((j, P)\) variables.
leaving only the Regge term as the leading contribution to the quantum amplitudes, with next
to leading order contributions being represented by inverse curvature terms $1/R^n$ which indeed modify the IR physics of the corresponding classical discrete gravity theory.

Appendix

The Schrödinger equation on the group is given by

$$i \frac{\partial \psi(g, t)}{\partial t} + \Box G \psi(g, t) = 0.$$  \hspace{1cm} (A.1)

The general solution to this equation is given by the aforementioned Schrödinger evolution
kernel $K[g_t, g_0, t]$ which gives the solution $\psi$ at time $t + t_0$ given the solution at time $t_0$,

$$\psi(g, t + t_0) = \int_G d g_0 \theta(t) K[g_t, g_0, t] \psi(g_0, t_0),$$

where $\theta(t)$ is the Heaviside step function. Many properties of $K$ immediately follow from
this equation and the fact that (A.1) is invariant under left and right shifts, notably symmetry,
composition and Green function property.

By the symmetry property of the kernel, we mean the fact that it is invariant under shifting
both arguments on the group on one hand and that it is a central function on the other. This
latter fact means that the kernel is expandable in characters, a feature we will use shortly. In
formulæ the kernel satisfies

$$K[g, h, t] = K[gh^{-1}, t] \quad \text{and} \quad K[g, t] = K[hgh^{-1}, t].$$  \hspace{1cm} (A.2)

The composition property of the kernel is the standard one:

$$\int_G d g_2 K[g_1, g_2, t] K[g_2, g_3, s] = K[g_1, g_3, t + s].$$  \hspace{1cm} (A.3)

This equation was used when we computed the Feynman amplitudes of our model.

Finally and most importantly the kernel satisfies the following two equations

$$i \frac{\partial K[g, h, t]}{\partial t} + \Box G K[g, h, t] = 0 \quad \text{and} \quad i \frac{\partial (\theta(t) K[g, h, t])}{\partial t} + \Box G (\theta(t) K[g, h, t]) = i \delta(t) \delta(gh^{-1}).$$

These coupled with the boundary condition $\lim_{t \to 0} K[g, h, t] = \delta(gh^{-1})$ mean that
$(\theta(t) K[g, h, t])$ is the retarded propagator for the Schrödinger equation. It is this last feature
that links the Schrödinger kernel with the Feynman propagator.

To see this link take the Fourier transform of the inhomogeneous equation with respect
to $t$, going to the conjugate variable $\mu$, which is the mass (square) of the particle in the
proper time parametrization of the Klein–Gordon propagator [25, 27], or the energy of the
same in the usual Schrödinger equation. If we denote by $K[g, h, \mu]$ the Fourier transform of
$(\theta(t) K[g, h, t])$ (i.e. what is called the mass representation of the Schrödinger kernel) then it
follows that

$$(\mu + \Box_G) K[g, h, \mu] = i \delta(gh^{-1}).$$

Comparing this to (12) we see immediately that $K$ satisfies essentially the same equation
as $D_F$. From this (13) follows trivially.
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