Topological Extensions of Noether Charge Algebras carried by D-\(p\)-branes

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Abstract

We derive the fully extended supersymmetry algebra carried by D-branes in a massless type IIA superspace vacuum. We find that the extended algebra contains not only topological charges that probe the presence of compact spacetime dimensions but also pieces that measure non-trivial configurations of the gauge field on the worldvolume of the brane. Furthermore there are terms that measure the coupling of the non-triviality of the worldvolume regarded as a U(1)-bundle of the gauge field to possible compact spacetime dimensions. In particular, the extended algebra carried by the D-2-brane can contain the charge of a Dirac monopole of the gauge field. In the course of this work we derive a set of generalized Gamma-matrix identities that include the ones presently known for the IIA case. – In the first part of the paper we give an introduction to the basic notions of Noether current algebras and charge algebras; furthermore we find a Theorem that describes in a general context how the presence of a gauge field on the worldvolume of an embedded object transforming under the symmetry group on the target space alters the algebra of the Noether charges, which otherwise would be the same as the algebra of the symmetry group. This is a phenomenon recently found by Sorokin and Townsend in the case of the M-5-brane, but here we show that it holds quite generally, and in particular also in the case of D-branes.
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Introduction

Topological extensions of the algebra of Noether currents and corresponding Noether charges have been studied in the past by a number of authors [1, 2]. In [1] the extensions of the algebra of Noether and modified Noether charges carried by supersymmetric extended objects have been examined; furthermore, it has been pointed out that the origin of these modifications is the Wess-Zumino term in the Lagrangian of the extended object. In [2] the algebra of the Noether supercharges of the M-5-brane was derived, and it was observed that not all central charges occurring in the superalgebra extension are entirely due to the Wess-Zumino term; it was shown that another contribution to the central charges originates in the presence of a gauge field potential on the worldvolume of the M-5-brane, which takes part in the action of the superPoincare group acting on the target space. In this work we shall prove a theorem explaining that this is a general feature for a whole class of Lagrangians containing a gauge potential on the worldvolume which is forced to transform under a Lie group that acts on the target space of the theory; in this case even the algebra of the (unmodified) Noether charges suffers modifications, which otherwise would close into the original algebra of the group that acts on the target space, possibly up to a sign, which is a consequence of whether the group acts from the left or from the right.

In this work we have performed an analysis of the extensions of the superalgebra of Noether and modified Noether charges carried by D-p-branes in a IIA superspace. In doing so, however, we have faced a number of difficulties which could not be illuminated by consulting the literature; in section 1.1 we therefore provide an introduction to the basic concepts of the algebra of Noether currents, Noether charges, and associated modified currents and charges, which might arise from a Lagrangian transforming as a semi-invariant under the action of the group. We show that for a left action the algebra of the associated Noether charges always closes to the original algebra, regardless of whether the Lagrangian is invariant under the transformation or not; we derive the action of the Noether charges on the currents and show that for a left action the Noether currents transform in the adjoint representation of the group. We then extend this analysis to the case of semi-invariant Lagrangians and examine carefully under which circumstances certain contributions to the Poisson brackets of the modified currents vanish or may be neglected; this question is not always fully addressed in the literature, and becomes even more non-trivial in the case of having a gauge field present on the worldvolume, since the gauge field degrees of freedom are subject to primary and secondary constraints (in Dirac's terminology). We analyze the constraint structure of a theory possessing such a gauge field on the world-volume; the results apply to D-branes and M-branes as well. We derive conditions under which the additional charges obtained so far are conserved and central. We finish the first section with showing how modifications of the Noether charge algebra arise from the presence of such a gauge field, even if the modifications of the charge algebra due to semi-invariant pieces in the Lagrangian are not yet taken into account.

In section 2 we apply these ideas to derive the extensions of the algebra
of modified Noether charges for D-p-brane Lagrangians in IIA superspace. We first derive a general form of these extensions applicable for the most general forms of gauge fields (NS-NS and RR) on the superspace; then we choose a particular background in putting all bosonic components of the gauge fields to zero, and taking into account that the remainder are subject to superspace constraints which allow to reconstruct the leading components of the RR gauge field strengths unambiguously. In doing so we must check whether the RR field strengths thus derived actually satisfy the appropriate Bianchi identities; we find that this question can be traced back to the validity of a set of generalized \( \Gamma \)-matrix identities; it is known that the first two members in this set are actually valid; as for the rest we derive a necessary condition using the technique that has been applied in similar circumstances previously, see \[3, 4\], and find that it is satisfied. The extended algebra thus derived contains topological charges that probe the existence of compact spacetime dimensions the brane is wrapping around; furthermore, which is a new feature here, we find that central charges show up that probe the non-triviality of the worldvolume regarded as a \( U(1) \)-bundle of the gauge field \( A_\mu \); we find that for the D-4-, 6- and 8-branes there exist central charges originating in the Wess-Zumino term that can be interpreted as probing the coupling of these non-trivial gauge-field configurations to compact dimensions in the spacetime; they are zero if either there are no compact dimensions, or the brane is not wrapping around them, or the \( U(1) \)-bundle is trivial, which requires the gauge field configuration to be trivial. As for the D-2-brane such a coupling of spacetime topology to the gauge field is only present in the central charges that stem from the fact that the gauge field transforms under supersymmetry; they have nothing to do with the Wess-Zumino term; for the special case of a D-2-brane given by \( \mathbb{R} \times S^2 \), where \( \mathbb{R} \) denotes the time dimension, we find that the algebra can contain the charge of a Dirac monopole of the gauge field; this result is very neat, so we present it here:

\[
\{Q_\alpha, Q_\beta\} = 2 \left( CT^m_{\alpha\beta}\right) P_m - 2i \left( CT_{11} \Gamma_m\right) \cdot Y^m - \\
- i \left( CT_{m_2m_1}\right) \cdot T^{m_1m_2} - 2i \left( CT_{11}\right) \cdot 4\pi g.
\]

Here \( Y^m \) is a central charge that couples the canonical gauge field momentum to compact dimensions in the spacetime allowing for 1-cycles in the brane wrapping around them; \( T^{m_1m_2} \) probes the presence of compact dimensions in spacetime the brane wraps around, i.e. allowing for 2-cycles wrapping around them, and \( g \) is the quantized charge of a Dirac monopole resulting from the gauge field.

1 Topological extensions of the algebra of Noether charges

1.1 Actions of a Lie group and associated Noether charges

1.1.1 Noether currents

Let \((x^\mu) = (t, \sigma^r), \mu = 0, \ldots, p; r = 1, \ldots, p\) denote coordinates on a \((p+1)\)-dimensional manifold ("worldvolume") \( W \). Here \( t \) refers to a "timelike" coordi-
nate, $\sigma^r$ refers to "spacelike" coordinates. Let $W(t)$ denote the hypersurfaces in $W$ with constant $t$. Let $L = L(\phi, \partial_\mu \phi)$ be a Lagrangian of a field multiplet $\phi = (\phi^1)$ defined on $W$, with unspecified dimension. The objects ($\phi^i$) are regarded to be coordinates on a target space $\Sigma$; at present we do not make any further assumptions on the precise nature of $\Sigma$. Let $G$ be a Lie group with generators $T_M \in \text{Lie}(G)$, where $\text{Lie}(G)$ is the Lie algebra of $G$; the generators $T_M$ act on $\phi^i$ according to $\phi^i \rightarrow \delta_M \phi^i = (\delta_M \phi^1)$; here $\delta_M \phi^i$ are the components of the vector field $\tilde{T_M}$ induced by the generator $T_M$ on $\Sigma$, i.e., the action of $e^{tT_M}$ defines a flow $(\phi, t) \rightarrow (e^{tT_M} \phi)$, which generates the vector field 

$$\frac{d}{dt} \left( e^{tT_M} \phi \right) \bigg|_{t=0} = \left( \tilde{T_M} \right)^i \bigg|_{\phi} \frac{\partial}{\partial \phi^i} = \delta_M \phi^i \frac{\partial}{\partial \phi^i} = \tilde{T_M}.$$  \hspace{1cm} (1)

For a right action the map $\text{Lie}(G) \ni X \mapsto \tilde{X}$, which sends an element of the Lie algebra of $G$ to an induced vector field on $\Sigma$, is a Lie algebra homomorphism into the set of all vector fields on $\Sigma$ endowed with the Lie bracket as multiplication:

$$[\tilde{X}, \tilde{Y}] = \left[ \tilde{X}, \tilde{Y} \right].$$  \hspace{1cm} (2)

For a left action this is true for the map $\text{Lie}(G) \ni X \mapsto -\tilde{X}$, since in this case

$$[\tilde{X}, \tilde{Y}] = - \left[ \tilde{X}, \tilde{Y} \right].$$  \hspace{1cm} (3)

Now denote the expression for the equations of motion for the fields $\phi^i$ by

$$\left( \text{eq, } L \right)_i := \frac{\partial L}{\partial \phi^i} - \partial_\mu \frac{\partial L}{\partial (\partial_\mu \phi^i)},$$  \hspace{1cm} (4)

then the action of the generator $T_M$ on $L$ takes the form

$$\delta_M L = \delta_M \phi^i \cdot \left( \text{eq, } L \right)_i + \partial_\mu j^\mu_M, \hspace{1cm} (5)$$

where

$$j^\mu_M = \delta_M \phi^i \cdot \frac{\partial L}{\partial (\partial_\mu \phi^i)}$$  \hspace{1cm} (6)

is the \textit{Noether current associated with $T_M$}. If $\text{sol}(L)$ denotes a solution to the \textit{equations of motion} $(\text{eq, } L)_i = 0$, we have

$$[\delta_M L = \partial_\mu j^\mu_M]_{\text{sol}(L)},$$  \hspace{1cm} (7)

i.e. on the solution $\text{sol}(L)$. Now assume that $L = L_0 + L_1$, where $L_0$ is invariant under $G$, $\delta_M L_0 = 0$. Then

$$\delta_M L_1 = \delta_M \phi^i \cdot (\text{eq, } L_0 + L_1)_i + \partial_\mu j^\mu_M, \hspace{1cm} (8)$$

where

$$j^\mu_M = \delta_M \phi^i \cdot \frac{\partial L_0}{\partial (\partial_\mu \phi^i)} + \delta_M \phi^i \cdot \frac{\partial L_1}{\partial (\partial_\mu \phi^i)} =: j^\mu_{0,M} + j^\mu_M. \hspace{1cm} (9)$$

5
Therefore,
\[ \delta_M \mathcal{L}_1 = \delta_M \phi^i \cdot (eq, \mathcal{L}_0 + \mathcal{L}_1)_i + \partial_\mu \left( j^\mu_{0,M} + J^\mu_M \right) , \tag{10} \]
and
\[ \left[ \delta_M \mathcal{L}_1 = \partial_\mu \left( j^\mu_{0,M} + J^\mu_M \right) \right]_{solv(\mathcal{L}_0+\mathcal{L}_1)} . \tag{11} \]
This is to be compared with
\[ 0 = \delta_M \phi^i \cdot (eq, \mathcal{L}_0)_i + \partial_\mu j^\mu_{0,M} , \tag{12} \]
and
\[ \left[ 0 = \partial_\mu j^\mu_{0,M} \right]_{solv(\mathcal{L}_0)} . \tag{13} \]
Since
\[ \left[ \partial_\mu j^\mu_{0,M} \right]_{solv(\mathcal{L}_0+\mathcal{L}_1)} \neq \left[ \partial_\mu j^\mu_{0,M} \right]_{solv(\mathcal{L}_0)} = 0 \]
in general, we see that \( j^\mu_{0,M} \) is no longer conserved in the presence of \( \mathcal{L}_1 \), although it is conserved on the critical trajectories of \( \mathcal{L}_0 \). Neither is the total current conserved,
\[ \left[ \delta_M \mathcal{L}_1 = \partial_\mu j^\mu_{0,M} \right]_{solv(\mathcal{L}_0+\mathcal{L}_1)} . \tag{14} \]
To proceed, we now specify the action of \( G \) on \( \mathcal{L}_1 \): We assume that, under the action of \( G \), \( \mathcal{L}_1 \) transforms as a total derivative \textbf{on- and off-shell}, i.e. without using the equations of motion. Then \( \mathcal{L}_1 \) is said to be \textit{semi-invariant} under the action of \( G \). This means that
\[ \delta_M \mathcal{L}_1 = \partial_\mu U^\mu_M , \tag{14} \]
for some functions \( U^\mu_M \) of the fields and its derivatives. This gives, using (10),
\[ 0 = \delta_M \phi^i \cdot (eq, \mathcal{L}_0 + \mathcal{L}_1)_i + \partial_\mu \left( j^\mu_{M} - U^\mu_M \right) , \tag{15} \]
and we see that the modified current
\[ \tilde{j}^\mu_M := j^\mu_{M} - U^\mu_M \tag{15} \]
is conserved on the critical trajectories of \( \mathcal{L}_0 + \mathcal{L}_1 \), i.e.
\[ \partial_\mu \tilde{j}^\mu_M = 0 \tag{16} \]
Note that in this case the \textbf{conserved current is no longer a Noether current}.

\[ \text{1.1.2 Algebra of Poisson brackets} \]
The Poisson brackets of the zero components \( j^0_M \) of the \textbf{total} Noether currents \( j^\mu_M \) associated with the action of \( T_M \) on some Lagrangian \( \mathcal{L} \) satisfy the Lie algebra of \( G \), possibly up to a sign, regardless of whether \( \mathcal{L} \) is invariant or not. This can be proven by introducing canonical momenta
\[ \Lambda_i := \frac{\partial \mathcal{L}}{\partial \dot{\phi}^i} , \]
so that
\[ j^0_M = \delta_M \phi^j \cdot \Lambda_i. \] (17)

Let \( C_{MN}^K \) denote the structure constants of the Lie algebra \( \text{Lie}(G) \) of \( G \), i.e.
\[ [T_M, T_N] = C_{MN}^K \cdot T_K, \]
where \([\cdot, \cdot]\) denotes a (graded) commutator. Working out the Poisson bracket we get
\[ \{ j^0_M (t, \sigma), j^0_N (t, \sigma') \}_{PB} = -\delta_M \phi^j \frac{\partial \delta_N \phi^j}{\partial \phi^i} \Lambda_i \delta (\sigma - \sigma') + \delta_N \phi^j \frac{\partial \delta_M \phi^j}{\partial \phi^i} \Lambda_i \delta (\sigma - \sigma'). \] (18)

where \( \{\cdot, \cdot\}_{PB} \) denotes a (graded) Poisson bracket with canonical variables \( \phi^i, \Lambda_i \). If we now use the results from (1) we find
\[ \{ j^0_M (t, \sigma), j^0_N (t, \sigma') \}_{PB} = \pm C_{MN}^K \cdot j^0_K (t, \sigma) \cdot \delta (\sigma - \sigma'), \] (19)
where ”+ / −” refers to a left / right action. If we define associated Noether charges
\[ Q_M (t) := \int_{W(t)} d^p \sigma \cdot j^0_M (t, \sigma) , \] (20)
then the once integrated version of (19) is
\[ \{ Q_M, j^0_N \}_{PB} = \pm j^0_K \cdot \text{ad}(T_M) N \] (21)
where \( \text{ad}(T) \) denotes the adjoint representation of the Lie algebra element \( T \). This now means that the total Noether currents span the adjoint representation of \( G \) in the case of a left action. Moreover, in this case a further integration of (21) yields back the algebra we have started with,
\[ \{ Q_M, Q_N \}_{PB} = \pm C_{MN}^K \cdot Q_K . \] (22)

We omit the subscript \( PB \) in what follows, and reintroduce it only when there is danger of confusion with an anticommutator.

Now we look at the situation when the Lagrangian contains a semi-invariant piece \( \mathcal{L}_1 \). In this case the conserved currents are \( \tilde{j}_M^\mu = j_M^\mu - U_M^\mu \), and their zero components have Poisson bracket relations
\[ \{ \tilde{j}_M^0 (t, \sigma), \tilde{j}_N^0 (t, \sigma') \} = \{ j_M^0, j_N^0 \} - \{ j_M^0, U_N^0 \} - \{ U_M^0, j_N^0 \} + \{ U_M^0, U_N^0 \}. \] (23)
The brackets \( \{ j_M^0, U_N^0 \} = \{ \delta_M \phi^i, U_N^0 \} \) are always unequal zero when \( U_N^0 \) is not a constant, since the presence of the canonical momenta amounts to derivatives with respect to the fields on \( U_N^0 \). However, the brackets \( \{ U_M^0, U_N^0 \} \)
are also non-vanishing in general; although in the Lagrangian description they contained only fields \( \dot{\phi}^j \) and their derivatives, the shift to the Hamiltonian (first order) picture amounts to inverting the relations

\[
\Lambda_i = \frac{\partial L}{\partial \dot{\phi}^i}(\phi, \partial_0 \phi, \partial_r \phi)
\]

for \( \partial_0 \dot{\phi}^i \), which gives \( \dot{\phi}^i = \Phi^i(\phi, \partial_t \phi, \Lambda) \), where \( r, s = 1, \ldots, p \) refers to the "spatial" coordinates on \( W \). Therefore,

\[
U_M^0(\phi, \partial_\mu \phi) \rightarrow U_M^0(\phi, \Phi(\phi, \partial_t \phi, \Lambda), \partial_s \phi) = \tilde{U}_M^0(\phi, \partial_t \phi, \Lambda)
\]

so that after Legendre transforming the \( U_M^0 \) do depend on the canonical momenta, which makes their mutual Poisson brackets in general non-vanishing. Note that the "hatted" \( \tilde{U}_M^0 \) is of course a different function of its arguments than \( U_M^0 \) which makes itself manifest when we are performing partial or functional derivatives, respectively. Therefore, in a Poisson bracket we are always dealing with \( \tilde{U}_M^0 \), outside a Poisson bracket we can replace \( \tilde{U}_M^0 \) by \( U_M^0 \), as we shall do in the following.

Now we subtract and add \( \pm C_{MN}^K \cdot U_R^0(t, \sigma) \cdot \delta (\sigma - \sigma') \) on the right hand side of (23), and use (19). This gives us

\[
\{ \bar{j}_M^0(t, \sigma), \bar{j}_N^0(t, \sigma') \} = \pm C_{MN}^K \cdot \bar{j}_K^0(t, \sigma) \cdot \delta (\sigma - \sigma) + S^0_{MN} + \{ \bar{U}_M^0(t, \sigma), \bar{U}_N^0(t, \sigma') \}
\]

with the "anomalous" piece

\[
S^0_{MN} + \{ \bar{U}_M^0(t, \sigma), \bar{U}_N^0(t, \sigma') \} = \{ \bar{j}_M^0(t, \sigma), \bar{U}_N^0(t, \sigma') \} - \{ \bar{U}_M^0(t, \sigma), \bar{j}_N^0(t, \sigma') \} = \pm C_{MN}^K \cdot U_R^0(t, \sigma) \cdot \delta (\sigma - \sigma') + \{ \bar{U}_M^0(t, \sigma), \bar{U}_N^0(t, \sigma') \}.
\]

Let us compute the brackets \( \{ \bar{j}_M^0, \bar{U}_N^0 \} \) for the special case that the action of \( G \) on covectors \( \Lambda_i \) is specified so as to make expressions like \( \dot{\phi}^i \Lambda_i \) transforming as scalars under the group operation; this means that \( \Lambda_i \) transform contragrediently to \( \phi^i \),

\[
\delta_M \Lambda_i = -\Lambda_j \frac{\partial \delta_M \phi^j}{\partial \phi^i}.
\]

To see that this specification leaves \( \dot{\phi}^i \Lambda_i \) invariant we apply \( \delta_M \),

\[
\delta_M (\dot{\phi}^i \Lambda_i) = \left( \delta_M \frac{d}{dt} \phi^i - \dot{\phi}^j \frac{\partial \delta_M \phi^j}{\partial \phi^i} \right) \Lambda_i;
\]

if we assume now, as usual, that \( \delta_M \) commutes with \( \partial_\mu \) the expression in the bracket vanishes. We find

\[
\{ \bar{j}_M^0(t, \sigma), \bar{U}_N^0(t, \sigma') \} = -\delta_M U^0_{MN} \cdot \delta (\sigma - \sigma) + \frac{\partial}{\partial \sigma'} \left[ \delta_M \phi^i \frac{\partial \bar{U}_N^0}{\partial \phi^i} \cdot \delta (\sigma' - \sigma) \right].
\]

(27)
Double integration of the second term over $W(t)$, $t = \text{const.}$, yields

$$
\int_{W(t)} d^p\sigma \cdot \frac{\partial}{\partial \sigma^i} \left[ \delta_M\phi^i \frac{\partial U_0^i}{\partial \sigma^j} \right] = \int_{\partial W(t)} d\mathcal{A}_{t}^{p-1} \cdot \delta_M\phi^i \frac{\partial U_0^i}{\partial \sigma^j} ,
$$

where $d\mathcal{A}_{t}^{p-1}$ is a $(p-1)$-dimensional area element. We must deal with this surface term appropriately. The manifold $W(t)$ can be infinitely extended in all spatial directions, or some of these spatial directions may be compact. To avoid bothering with the surface terms we assume from now on that the integrands of surface contributions vanish sufficiently strong at the boundary $\partial W(t)$, i.e. at points which lie at infinite values of the non-compact coordinates. As a special case this includes the possibility that $W(t)$ is closed, which implies that all spatial coordinates $\sigma^\mu$ are compact. Furthermore we assume that all expressions in a total derivative, such as on the left hand side of (28), are smooth and defined globally on $W$. (The emphasis on being globally defined is of course to prevent us from situations where Stokes’ theorem is not applicable, i.e. "surface terms cannot be integrated away"; this can be true for the topological current to be defined below). Under these circumstances all surface terms vanish, and we obtain for the current algebra

$$
\left\{ \tilde{j}^0_M (t, \sigma), \tilde{j}^0_N (t, \sigma') \right\} = \pm C_{MN}^K \tilde{j}^0_K (t, \sigma) \cdot \delta (\sigma - \sigma') + S_{0}^{MN} + \left\{ \tilde{U}_M^0 (t, \sigma), \tilde{U}_N^0 (t, \sigma') \right\} ,
$$

$$
S_{0}^{MN} (t, \sigma', \sigma) = \left[ \delta_M U_N^0 - \delta_N U_M^0 \pm C_{MN}^K \cdot U_K^0 \right] \cdot \delta (\sigma - \sigma') + \text{(total derivatives)} ,
$$

with the total derivatives from (27). Let us now define

$$
Q_M (t) := \int_{W(t)} d^p\sigma \cdot \tilde{j}^0_M (t, \sigma) ,
$$

$$
S_{MN}^\mu (t, \sigma) = \delta_M U_N^\mu - \delta_N U_M^\mu \pm C_{MN}^K \cdot U_K^\mu ,
$$

$$
Z_{MN} (t) := \int_{W(t)} d^p\sigma \cdot S_{0}^{MN} (t, \sigma) .
$$

Note that the charge $Q_M (t) = Q_M$ is no longer a Noether charge, since it is defined through the conserved current $\tilde{j}^0_M$ rather than the Noether current $j^0_M$. It is conserved, however, due to (16). We show now that $Z_{MN}$ is conserved as well.

### 1.1.3 Conservation of the new charges

To prove this, observe that $\delta_M$ commutes with $\partial_\mu$; therefore we can write

$$
\partial_\mu S_{MN}^\mu = \delta_M \partial_\mu U_N^\mu - \delta_N \partial_\mu U_M^\mu \pm C_{MN}^K \cdot \partial_\mu U_K^\mu = \left( \delta_M \delta_N - \delta_N \delta_M \pm C_{MN}^K \cdot \delta_K \right) \mathcal{L}_1 .
$$
If we work out the double variation we find that the last expression vanishes due to

\[ [\delta_M, \delta_N] \mathcal{L} = [\delta_M, \delta_N] \phi^i \cdot \mathcal{L}_{\phi^i} + \partial_\mu [\delta_M, \delta_N] \phi^i \cdot \mathcal{L}_{\partial_\mu \phi^i} \]

This can be seen yet in another way: On account of \([\partial_\mu, \delta_M] = 0\), \(\delta_N = \tilde{T}_N\) acts on coordinates \(\phi^i\) in the same way as it acts on \(\partial_\mu \phi^i\). Therefore we can replace the \(\delta's\) in the round bracket in (34) by vector fields \(\tilde{T}_N\), which yields

\[
\delta_M \delta_N - \delta_N \delta_M + C^K_{MN} \cdot \delta_K = [\tilde{T}_M, \tilde{T}_N] \pm C^K_{MN} \cdot \tilde{T}_K = \mp ([T_M, T_N] - C^K_{MN} \cdot T_K) = 0 ,
\]

according to the algebra of the generators \((T_M)\). What we have shown is

\[
\partial_\mu S^\mu_{MN} = 0 ,
\]

which is the local conservation law for the charge \(Z_{MN}(t)\) defined in (33).

Using the definitions (31, 33), we find on double integration of (29) (and on assumption that this integration is defined)

\[
\{Q_M, Q_N\}_{PB} = \pm C^K_{MN} \cdot Q_K + Z_{MN} + \int_{W(t)} d^p \sigma \, d^p \sigma' \cdot \left\{\widehat{U}_M^0 (t, \sigma), \widehat{U}_N^0 (t, \sigma')\right\} .
\]

We see that our original algebra has been extended by conserved charges \(Z_{MN}\); however, unless the Poisson brackets \(\left\{\widehat{U}_M^0, \widehat{U}_N^0\right\}\) vanish, this extension does not close to a new algebra!

### 1.2 Coset spaces of Lie groups as target spaces

#### 1.2.1 Closure of the algebra extension

In order to proceed further we now make more detailed assumptions about the structure of the target space and the geometric origin of the invariant and semi-invariant pieces in the Lagrangian. We assume that the target space is now the group \(G\) itself, with coordinates \(\phi^i\). More generally, we could have that \(G\) is a subgroup of a larger group \(\tilde{G}\), which contains yet another subgroup \(H : G, H \subset \tilde{G}\). Then \(\Sigma\) could be the coset space \(\tilde{G}/H\), and \(G\) would act on elements of \(\Sigma = \tilde{G}/H\) by left or right multiplication. This is the situation we shall consider later, where \(\tilde{G} = \text{superPoincare}\) in \(D = 10\) spacetime dimensions, \(G\) is the subgroup generated by \(\{P_m, Q_\alpha\}\), i.e. the generators of Poincare- and super-translations, and \(H\) is the subgroup \(SO(1,9)\). If the objects \(Q_\alpha\) build two 16-component spinors with opposite chirality, then the coset space \(\Sigma\) is type IIA superspace. However, for the purpose of illustrating of how topological currents emerge we shall in the following refrain from any graded groups, algebras, or whatsoever, and restrict ourselves to the simpler case of \(\Sigma = G\).

The fields \(\phi^i\) on \(W\) accomplish an embedding \(\text{emb} : W \to \Sigma\) of \(W\) into \(\Sigma\) by \(\text{emb}(x) = (\phi^1 (x), \ldots, \phi^{\dim G} (x))\). From now on we call \(W\) the "worldvolume", following standard conventions. If the hypersurfaces \(W (t)\) are closed then the
same holds for their images in $\Sigma$, since $\partial [embW(t)] = emb[\partial W(t)] = \emptyset$. In other words, the images $embW(t)$ are $p$-cycles in $\Sigma$ in this case. We assume that the previously made assumptions concerning surface terms in integrands still hold, and that those spatial dimensions of $W(t)$ which are not infinitely extended are closed. Furthermore we **assume** that the semi-invariant piece $L_1$ or Wess-Zumino (WZ) term, as it will be called in the sequel, is the result of the pull-back of a target space $(p+1)$-form $(WZ)$ to the worldvolume $W$; from now on, we write $L_1 = L_{WZ}$ for the semi-invariant piece. Its construction proceeds as follows:

Let $(\Pi^A)_{A=1,\ldots,\dim G}$ be left-invariant (LI) 1-forms on $\Sigma = G$; this means, that at every point in $\Sigma$ they span the cotangent space to $\Sigma$ at this point, and they are invariant under the action of the group,

$$\delta_M \Pi^A = \mathcal{L}_{\tilde{T}_M} \Pi^A = 0 \quad ,$$

where $\mathcal{L}_{\tilde{T}_M}$ denotes the Lie derivative with respect to the induced vector field $\tilde{T}_M$. The WZ-form $(WZ)$ on $\Sigma$ can be expanded in this basis,

$$(WZ) = \frac{1}{(p+1)!} \Pi^{A_1} \cdots \Pi^{A_{p+1}} \cdot (WZ)_{A_{p+1} \cdots A_1} (\phi) \quad ,$$

with pull-back

$$emb^* (WZ) = \frac{1}{(p+1)!} dx^\mu_1 \cdots dx^\mu_{p+1} \cdot \Pi^{A_1}_{,\mu_1} \cdots \Pi^{A_{p+1}}_{,\mu_{p+1}} \cdot (WZ)_{A_{p+1} \cdots A_1} (\phi) \quad ;$$

(38)

since $dx^\mu_1 \cdots dx^\mu_{p+1}$ is proportional to the canonical volume form $\omega_0$ with respect to the coordinates $(x^\nu)$ on $W$,

$$dx^\mu_1 \cdots dx^\mu_{p+1} = \epsilon^{\mu_1 \cdots \mu_{p+1}} \cdot \omega_0 \quad , \quad \omega_0 = dx^0 \cdots dx^p \quad ,$$

we find that $emb^* (WZ) = \omega_0 \cdot L_{WZ}$, where

$$L_{WZ} = \frac{1}{(p+1)!} \epsilon^{\mu_1 \cdots \mu_{p+1}} \cdot \Pi^{A_1}_{,\mu_1} \cdots \Pi^{A_{p+1}}_{,\mu_{p+1}} \cdot (WZ)_{A_{p+1} \cdots A_1} (\phi) \quad .$$

(39)

Note that we have tacitly used the superspace summation conventions on the indices $M_i$, which, of course, does not affect the validity of the results to be shown.

Semi-invariance of the WZ-term then implies that for every generator $T_M$ of $G$ there exists a $p$-form $\Delta_M$ on $\Sigma$ such that

$$\delta_M (WZ) = d\Delta_M \quad .$$

(40)

This implies that

$$\omega_0 \cdot \delta_M L_{WZ} = \delta_M [emb^* (WZ)] = emb^* \delta_M (WZ) = emb^* d\Delta_M =$$

$$= d (emb^* \Delta_M) \quad .$$

(41)
Expanding $\Delta_M$ in the LI-basis we can compute
\[ d(\text{emb}^*\Delta_M) = \omega_0 \cdot \frac{1}{p!} \epsilon^{\mu_1 \cdots \mu_{p+1}} \partial_{\mu} \left[ \Pi_{\mu_2}^A \cdots \Pi_{\mu_{p+1}}^A \cdot \Delta_M A_{p+1} \cdots A_2 \right] , \]
and comparison with (41) then yields
\[ \delta_M \mathcal{L}_{WZ} = \partial_\mu U^\mu_M , \]
\[ U^\mu_M = \frac{1}{p!} \epsilon^{\mu_2 \cdots \mu_{p+1}} \left[ \Pi_{\mu_2}^A \cdots \Pi_{\mu_{p+1}}^A \cdot \Delta_M A_{p+1} \cdots A_2 \right] . \]  
(42)

In particular, for $\mu = 0$ we obtain
\[ U^0_M = \frac{1}{p!} \epsilon^{0_2 \cdots \mu_{p+1}} \left[ \Pi_{\mu_2}^A \cdots \Pi_{\mu_{p+1}}^A \cdot \Delta_M A_{p+1} \cdots A_2 \right] , \]
from which it is seen that $U^0_M$ cannot contain $\Pi_0^A$, due to the antisymmetry of the $\epsilon$-tensor. Reexpanding the forms $\Pi_A$ in the coordinate basis $d\phi^M$ gives
\[ \Pi^A = \Pi_N^A d\phi^N , \quad \Pi_{\mu}^A = \Pi_N^{A,\phi^N} \]
from which we see that $U^0_M$ cannot contain $\phi_0^N = \dot{\phi}^N$ either. This point is crucial in light of our previous considerations, of course, since, if we now assume, that the equations
\[ \Lambda_M = \frac{\partial \mathcal{L}}{\partial \dot{\phi}^M} , \quad \text{for} \quad \mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{WZ} \]
ar invertible with respect to $\dot{\phi}^N$, then $\dot{\phi}^M = \Phi^M (\phi, \partial_r \phi, \Lambda)$ for $r = 1, \ldots, p$, and after performing the Legendre transformation $\left( \dot{\phi}^M, \dot{\phi}^N \right) \rightarrow \left( \phi^M, \Lambda_N \right)$ we have
\[ U^s_M = U^s_M \left( \phi, \Phi^M (\phi, \partial_r \phi, \Lambda), \partial_t \phi \right) = \widetilde{U}^s_M (\phi, \dot{\phi}, \Lambda) \quad ; \quad s = 1, \ldots, p ; \quad r, t \neq s , \]
but
\[ U^0_M = \widetilde{U}^0_M (\phi, \partial_r \phi) \quad ; \quad r = 1, \ldots, p . \]  
(43)
Therefore we now have Poisson brackets
\[ \left\{ J^0_M, \widetilde{U}^0_N \right\} = \delta_M^K \phi^K \cdot \left\{ \Lambda_K, \widetilde{U}^0_N \right\} , \]  
(44)
\[ \left\{ \widetilde{U}^0_M, \widetilde{U}^0_N \right\} = 0 . \]  
(45)

This point being clarified we omit the "hats" on $\widetilde{U}^s_M$ from now on, it being understood that it is the "hatted" version that appears in a Poisson bracket.

Referring to (36) we can now state that the algebra of the charges $Q_M$ closes to a linear combination of the $Q_M$ and the new charges $Z_{MN}$,
\[ \{ Q_M, Q_N \}_{PB} = \pm C^{K}_{MN} \cdot Q_K + Z_{MN} . \]  
(46)
Furthermore, due to (45), we have
\[ \left\{ S^0_{MN}, S^0_{M' N'} \right\} = 0 , \]  
(47)
and therefore
\[ \{ Z_{MN}, Z_{M'N'} \} = 0 \quad , \]
i.e. the mutual algebra of the new charges \( Z_{MN} \) also closes into the extension generated by \( (Q_M, Z_{MN}) \). But what about the algebra of \( \{ Q_K, Z_{MN} \}_{PB} \)? We now examine under which conditions this expression yields a linear combination of \( \{ Q_M, Z_{MN} \}_{PB} \).

### 1.2.2 Topological currents

Consider the object \( S_{MN}^\mu \) defined in (32),
\[ S_{MN}^\mu (t, \sigma) = \delta_M U_N^\mu - \delta_N U_M^\mu + C_{MN}^K \cdot U_K^\mu \quad . \]

Using the form of \( U_M^\mu \) given in (42) and taking into account that \( \delta_M \Pi_N^\mu = 0 \) we have
\[ S_{MN}^\mu = \frac{1}{p!} \epsilon^{\mu \mu_1 \cdots \mu_p} \Pi_{\mu_1}^A \cdots \Pi_{\mu_p}^A \times \]
\[ \times \left[ \delta_M \Delta_{NA_1 \cdots A_p} - \delta_N \Delta_{MA_1 \cdots A_p} \pm C_{MN}^K \cdot \Delta_{KA_1 \cdots A_p} \right] \quad . \]

For the sake of simplicity we define the expression
\[ \tilde{R}_{MNA_1 \cdots A_p} := \left[ \delta_M \Delta - \delta_N \Delta = C_{MN}^K \cdot \Delta_K \right]_{A_1 \cdots A_p} \quad , \]
so that
\[ S_{MN}^\mu = \frac{1}{p!} \epsilon^{\mu \mu_1 \cdots \mu_p} \Pi_{\mu_1}^A \cdots \Pi_{\mu_p}^A \cdot \tilde{R}_{MNA_1 \cdots A_p} \quad , \]
and rewrite this form in the coordinate basis \( (d\phi^N) \), \( \Phi^A = \Phi^A_N \Phi^N \), which yields
\[ S_{MN}^\mu = \frac{1}{p!} \epsilon^{\mu \mu_1 \cdots \mu_p} \Phi_{\mu_1}^A \cdots \Phi_{\mu_p}^A \cdot R_{MNN_1 \cdots N_p} \quad , \]
with the new components
\[ R_{MNN_1 \cdots N_p} = \Pi_{N_1}^A \cdots \Pi_{N_p}^A \cdot \tilde{R}_{MNA_1 \cdots A_p} \quad . \]

We note that \( R_{MNN_1 \cdots N_p} \) is a function of the fields \( \phi \) only. Appealing to (52) we now define the identically conserved \( \Phi \) topological currents
\[ j_T^{\mu_1 \cdots \mu_p} := \epsilon^{\mu_1 \cdots \mu_p} \Phi_{\mu_1}^M \cdots \Phi_{\mu_p}^M \quad , \]
and the topological charges
\[ T^{\mu_1 \cdots \mu_p} := \int_{W(t)} d^p \sigma \cdot j_T^{\mu_1 \cdots \mu_p} \quad , \]
which are conserved due to \( \partial_\mu j_T^{\mu_1 \cdots \mu_p} = 0 \). The topological charges \( T^{\mu_1 \cdots \mu_p} \) are invariant under the group action,
\[ \delta_K T^{\mu_1 \cdots \mu_p} = 0 \quad , \]
see \[ (58) \] below. We write \( \mu \) as
\[
S_{MN}^\mu = \frac{1}{p!} j_T^{\mu N_1 \cdots N_p} \cdot R_{MN N_p \cdots N_1} =: j_T^{\mu} \cdot R_{MN} , \tag{57}
\]
then the charges \( Z_{MN} \) take the form
\[
Z_{MN} = \int W(t) d^p \sigma \cdot S_{MN}^0 = \int W(t) d^p \sigma \ j_T^0 \cdot R_{MN} ; \tag{58}
\]
for constant \( R_{MN N_p \cdots N_1} \) this is
\[
Z_{MN} = T \cdot R_{MN} . \tag{59}
\]
Now we can turn to the bracket \( \{ Q_K, Z_{MN} \} \); a computation yields
\[
\{ Q_K, Z_{MN} \} = - \int W(t) d^p \sigma \cdot \delta_K \left[ j_T^0 \cdot R_{MN} \right] . \tag{60}
\]
It is clear that this can never close into an expression involving the charges \( Q_M \), since this would require the occurrence of \( j_M^0 = \delta_M^N \Lambda_N \) in the integrand, but the integrand contains no canonical momenta (recall that \( j_T^0 \) contains no time derivatives of fields, and \( R_{MN} \) contains no field derivatives at all). Hence, at best the left hand side can close into a linear combination of the new charges \( \{ Q_K, Z_{MN} \} \) is equivalent to demanding that
\[
\delta_K \left[ j_T^0 \cdot R_{MN} \right] = - \frac{1}{2} B_{KM N}^{M' N'} \cdot j_T^0 \cdot R_{M' N'} + \cdots , \tag{61}
\]
where \( \cdots \) denote possible surface terms, and where \( B_{KM N}^{M' N'} \) are constant; the factor \( \frac{1}{2} \) is due to the antisymmetry of \( R_{MN} \) in \( M \) and \( N \). \( \{ Q_K, Z_{MN} \} \) then reads
\[
\{ Q_K, Z_{MN} \} = - \frac{1}{2} B_{KM N}^{M' N'} \cdot Z_{M' N'} . \tag{62}
\]
\( \{ Q_K, Z_{MN} \} \) and \( \{ Q_K, Z_{MN} \} \) now tell us that the algebra of the conserved charges \( Q_K, Z_{MN} \) closes if and only if \( \{ Q_K, Z_{MN} \} \) holds.

### 1.2.3 When are the charges \( Z_{MN} \) central?

This can be read off from \( \{ Q_K, Z_{MN} \} \): The charges \( Z_{MN} \) are central, i.e. they commute with all other elements in the algebra, iff all coefficients \( B_{KM N}^{M' N'} \) vanish; according to \( \{ Q_K, Z_{MN} \} \) this is true iff
\[
\delta_K \left[ j_T^0 \cdot R_{MN} \right] = (\text{globally defined smooth surface term}) . \tag{63}
\]
Let us now examine
\[
\delta_K j_T^{0 M_1 \cdots M_p} = \sum_{k=1}^p \Theta_{\mu_k} \left[ \frac{1}{p!} \phi_{\mu_1}^{M_1} \cdots \phi_{\mu_p}^{M_p} \cdot \delta_K \phi^{M_k} \cdots \phi^{M_p}_{\mu_p} \right] . \tag{64}
\]
We take the point of view that the expression in square brackets is smooth and globally defined (since $\delta_K \phi^M$ amounts to a derivative of the field $\phi^M$ which can be smoothly continued over the whole of $W$) so that its integral over $W(t)$ indeed vanishes, on using Stokes’ theorem. This means that $\delta_K \left[ j^0_T \cdot R_{MN} \right]$ is a surface term, provided that $R_{MN}$ are constant. A sufficient condition for the charges $Z_{MN}$ to be central is therefore that

$$R_{MNp\cdots N_1} (\phi) = \text{const.} = R_{MNp\cdots N_1}, \quad (65)$$

where $R_{MNp\cdots N_1}$ are the components of $R_{MN}$ in the coordinate basis $(d\phi^N)$.

As an aside we remark that (64) implies that the topological charges $T_{M_1\cdots M_p}$ are invariant under the group action,

$$\delta_K T_{M_1\cdots M_p} = 0 \quad . \quad (66)$$

We now have (see (59)) $Z_{MN} = T \cdot R_{MN}$, and the non-vanishing brackets of our extended algebra then read

$$\{Q_M, Q_N\} = C^K_{MN} \cdot Q_K + T \cdot R_{MN} \quad ,$$

$$\{Q_K, T \cdot R_{MN}\} = \frac{1}{2} D^{M'N'}_{KMN} \cdot T \cdot R_{M'N'} \quad , \quad (67)$$

and the charges $T \cdot R_{MN}$ are all central.

### 1.3 Summary

At this point it is appropriate to summarize the results we have obtained so far in the form of three theorems.

#### 1.3.1 Theorem 1

The Noether currents satisfy the Poisson bracket algebra, possibly up to a sign,

$$\left\{ j^0_M (t, \sigma), j^0_N (t, \sigma') \right\}_{PB} = \pm C^K_{MN} \cdot j^0_K (t, \sigma) \cdot \delta (\sigma - \sigma') \quad , \quad (68)$$

regardless of whether the Lagrangian $\mathcal{L}$ is invariant or not. $\pm$ refers to a left/right action. The once integrated version is

$$\left\{ Q_M, j^0_N \right\}_{PB} = \pm j^0_K \cdot \text{ad} (T_M)^K_N \quad , \quad (69)$$

where $\text{ad} (T)$ denotes the adjoint representation of the Lie algebra element $T$. This implies that the total Noether currents span the adjoint representation of $G$ in the case of a left action.

Double integration of the current algebra yields the algebra of the generators of $G$, possibly up to a sign,

$$\{ Q_M, Q_N \}_{PB} = \pm C^K_{MN} \cdot Q_K \quad . \quad (70)$$
1.3.2 Theorem 2

Assume that the Lagrangian $L$ is semi-invariant under the action of $G$, i.e. $\delta_M L = \partial_M U^\mu_M$ for functions $U^\mu_M = U^\mu_M (\phi, \partial_\nu \phi)$ of the fields and its derivatives 
on-shell and off-shell; that the action of $G$ on canonical momenta $\Lambda_i = L_{\dot{\phi}^i}$ is defined by (26); and that surface integrals with smooth integrands may be neglected. Then

1. The modified currents
   $$\tilde{j}_M^\mu = j_M^\mu - U^\mu_M ,$$
   where $j_M^\mu$ are the Noether currents associated with $L$, are conserved,
   $$\partial^\mu \tilde{j}_M^\mu = 0 .$$

2. Double integration of the Poisson bracket algebra yields
   $$\{Q_M, Q_N\}_{PB} = \pm C_{MN}^K Q_K + Z_{MN} + \int_{W(t)} d^p \sigma \, d^p \sigma' \{ \hat{U}^0_M (t, \sigma), \hat{U}^0_N (t, \sigma') \} ,$$
   where
   $$Z_{MN} (t) := \int_{W(t)} d^p \sigma \cdot S_{MN}^0 (t, \sigma) ,$$
   and
   $$S_{MN}^\mu (t, \sigma) = \delta_M U^\mu_N - \delta_N U^\mu_M \pm C_{MN}^K \cdot U^\mu_K .$$
   Due to
   $$\partial^\mu S_{MN}^\mu = 0$$
   the ”charges” $Z_{MN}$ are conserved.

1.3.3 Theorem 3

Let those directions of the hypersurfaces $W (t)$ which are not infinitely extended be closed. Let the target space $\Sigma$ be the group $G$ itself; let the semi-invariant piece $L_1 = L_{WZ}$ in the Lagrangian be the pull-back of a target space $(p + 1)$-form to the worldvolume $W$, which transforms under $G$ according to $\delta_M L_{WZ} = \partial_M U^\mu_M$, with
   $$U^\mu_M = \frac{1}{p!} \epsilon^{\mu_2^\nu_1 \mu_p^1} \cdot \left[ \Pi^{M_2}_{\mu_2^1} \cdots \Pi^{M_{p+1}}_{\mu_{p+1}} \cdot \Delta_{M_{p+1}M_2} \right] ,$$
   where $\Delta_{M_{p+1}M_2}$ are the components of dim $G$ $p$-forms $\Delta_M$ in a left-invariant basis $\left( \Pi^M \right)$. Let the action of $G$ on canonical momenta $\Lambda_i = L_{\dot{\phi}^i}$ be defined according to (26). Then

1. The Poisson bracket algebra of the Noether charges $Q_M$ and the charges $Z_{MN}$ closes iff
   $$\delta_K \left[ j^0_T \mathbf{R}_{MN} \right] = -\frac{1}{2} B_{KMN}^{M'N'} j^0_T \mathbf{R}_{M'N'} + \cdots ,$$
where \( \cdots \) denote possible surface terms, \( B_{K'M''N} \) are constant, and where
\[
R_{MNNp\cdots N_1} = \Pi_{A_1}^A \cdots \Pi_{N_p}^A \left[ \delta_M \Delta_N - \delta_N \Delta_M \pm C_{MN} \cdot \Delta_K \right]_{A_p\cdots A_1}.
\]

The extended algebra then reads
\[
\{Q_{MN}, Q_{NP}\} = \pm C_{KM} \cdot Q_K + Z_{MN},
\]
\[
\{Q_K, Z_{MN}\} = \frac{1}{2} B_{K'M''N'} \cdot Z_{K'M''N'},
\]
\[
\{Z_{MN}, Z_{M'N'}\}_{PB} = 0.
\]

2. A sufficient condition for the charges \( Z_{MN} \) to be central is that
\[
R_{MNNp\cdots N_1}(\phi) = \text{const.} = R_{MNNp\cdots N_1}.
\]

3. The topological charges \( T_{M_1\cdots M_p} \) are invariant under the group action,
\[
\delta_K T_{M_1\cdots M_p} = 0.
\]

1.3.4 Corollary
If \( R_{MNNp\cdots N_1} = \text{const.} \), then all charges \( Z_{MN} \) are central, and are linear combinations of the topological charges \( T_{M_1\cdots M_p} \),
\[
Z_{MN} = T \cdot R_{MN}.
\]

1.4 Lagrangians including (Abelian) Gauge fields

1.4.1 Structure of the Lagrangian
Now let us study the case when the Lagrangian \( \mathcal{L} \) contains additional degrees of freedom in the form of an Abelian \((q-1)\)-form gauge potential \( A_{\mu_1\cdots\mu_{q-1}} \), \( q \leq p \), that is defined on the worldvolume. A priori, the group \( G \) acts on the target space \( \Sigma \) and there is no reason why \( A \) should be involved in the transformation of fields on \( \Sigma \), but that is what we now impose on \( A \), since it is the situation that occurs when the Lagrangian describes \( D-p \)-branes, which we want to study later. To this end, we assume that on the target space there exists a \( q \)-form potential \( B = \frac{1}{q!} \Pi_{Cq}^C \cdots \Pi_{C1}^C B_{C_1\cdots C_q} \), with an associated \((q+1)\)-form field strength \( H = dB \). The field strength \( H \) is taken to be invariant under the action of \( G \), i.e. \( \delta_M H = 0 \). This implies that locally
\[
\delta_M B = d\Delta_M,
\]
with dim \( G \) \((q-1)\)-forms
\[
\Delta_M = \frac{1}{(q-1)!} \Pi_{Aq}^A \cdots \Pi_{A2}^A \Delta_{MA_2\cdots A_q} = \frac{1}{(q-1)!} d\phi_{Aq}^A \cdots d\phi_{A2}^A \Delta_{MA_2\cdots A_q},
\]
where we have used a tilde to distinguish the components of \( \Delta_M \) with respect to the LI-basis \( (\Pi^A) \) from the components in the coordinate basis \( (d\phi^M) \),

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which we shall need later. $A_{\mu_1...\mu_{q-1}}$ are therefore $(p+1)$ additional degrees of freedom involved in the dynamics; it is assumed, however, that $A_{\mu_1...\mu_{q-1}}$ enters the Lagrangian only via the field strengths $F_{\mu_1...\mu_q} = q \cdot \partial_{[\mu_1} A_{\mu_2...\mu_q]}$. The Lagrangian again splits into an invariant piece $L_0$ and a semi-invariant piece $L_{WZ}$, where $L_0$ takes the form

$$L_0 = L_0 \left( \phi, \partial_{[\mu} \phi, F_{\mu_1...\mu_q]} \right) , \quad F_{\mu_1...\mu_q} = F_{\mu_1...\mu_q} - (emb^* B)_{\mu_1...\mu_q} ;$$  \hspace{1cm} (87)

in order to have $L_0$ invariant we impose the transformation behaviour

$$\delta_M A = emb^* \Delta_M , \quad (\delta_M A)_{\mu_2...\mu_q} = \phi^{A_q} \partial_{[\mu_1} \phi^{A_2} \Delta_M A_{\mu_2...\mu_q} (\phi)$$  \hspace{1cm} (88)

on $A$. Since

$$\delta_M \hat{F} = \delta_M [dA - emb^* B] = [d\delta_M A - emb^* d\Delta_M] = 0 ,$$  \hspace{1cm} (89)

this is sufficient to have an invariant $L_0$. As for the semi-invariant part $L_{WZ}$, we assume the following:

$$L_{WZ} = L_{WZ} \left( \phi, \partial_{[\mu} \phi, F_{\mu_1...\mu_q]} \right) ,$$  \hspace{1cm} (90)

with transformation behaviour $\delta_M L_{WZ} = \partial_{\mu} U^\mu_M$, where $U^\mu_M = U^\mu_M (\phi, \partial_{\mu} \phi, F_{\mu\nu})$, but

$$\partial U^\mu_M \partial_{\nu} \phi = 0 \quad \text{if} \quad \mu = \nu ; \quad \partial U^\mu_M \partial_{F_{\nu_1...\nu_q}} = 0 \quad \text{if} \quad \mu \in \{\nu_1, \ldots, \nu_q\} .$$  \hspace{1cm} (91)

Note the absence of a hat in the field $F$ in the definition of the field content of $U^\mu_M$.

Now we define canonical momenta

$$\Lambda_N = \frac{\partial L}{\partial \partial_0 \phi^N} , \quad \Lambda^{\nu_2...\nu_q} = \frac{\partial L}{\partial \partial_0 A_{\nu_2...\nu_q}} .$$  \hspace{1cm} (92)

1.4.2 Constraints on the gauge field degrees of freedom

The fact that we are dealing with a gauge field $A_{\mu_1...\mu_{q-1}}$ as dynamical degrees of freedom makes itself manifest in the form of constraints that are imposed on the dynamics [6]: Using the formula

$$\frac{\partial L}{\partial \partial_0 A_{\nu_1...\nu_q}} = \frac{1}{(q-1)!} \frac{\partial L}{\partial F_{\nu_1...\nu_q}}$$  \hspace{1cm} (93)

we see that, due to the antisymmetry of $F$, $L$ cannot contain $\partial_0 A_{\nu_2...\nu_q}$, whenever one of the $\nu_2, \ldots, \nu_q$ is zero; this implies that the canonical momenta

$$\Lambda^{\nu_2...\nu_q} = 0 \quad \text{for} \quad 0 \in \{\nu_2, \ldots, \nu_q\} ,$$  \hspace{1cm} (94)

i.e. they vanish identically. The number of independent constraints is $(p+1)\binom{q-2}{p}$. The second set of constraints follows from the equations of motion for $A_{\nu_2...\nu_q}$: They are given by

$$(eq)^{\nu_2...\nu_q} := \frac{\partial L}{\partial A_{\nu_2...\nu_q}} - \partial_0 \Lambda^{\nu_2...\nu_q} - \partial_0 \Lambda^{\nu_2...\nu_q} = 0 ,$$

18
where the sum over \( r \) ranges from 1 to \( p \). The first term on the RHS vanishes since \( \mathcal{L} \) contains no \( A_{\nu_2...\nu_q} \); the second one vanishes if we choose one of the \( \nu \)'s to be equal to zero, say \( \nu_2 \). On using (83) we have
\[
\frac{\partial \mathcal{L}}{\partial A_{\nu_0...\nu_q}} = -\frac{\partial \mathcal{L}}{\partial \partial_{\nu_0} A_{\nu_2...\nu_q}},
\]
so we get
\[
\frac{\partial A_{\nu_0...\nu_q}}{\partial A_{\nu_2...\nu_q}} = 0 , \ \ \ \nu_3, \ldots, \nu_q \ \ \ \text{arbitrary.} \tag{95}
\]
This yields a number of another \( \binom{p}{q-2} \) constraints.

These constraints are not on an equal footing, however; as can be seen from the above arguments, the first set (84) holds before any equations of motion are considered, and therefore amounts to a reduction of phase space to a submanifold of the original phase space of codimension \( \binom{p}{q-2} \); in Dirac’s terminology this is a set of primary constraints. The second set (85) comes into play only on-shell, i.e. on using equations of motion, and is called a set of secondary constraints. We shall not use Dirac’s machinery for handling these constraints here, but shall work with Poisson brackets instead; in this case, however, it is crucial to impose (84) \( \text{not before} \) all Poisson brackets have been worked out, otherwise we would obtain wrong results.

After these remarks let us now study the Poisson bracket algebra of the Noether currents. The Noether currents are
\[
J_M^\mu = \delta_M^K \phi^K \cdot \frac{\partial \mathcal{L}}{\partial \phi^K} + \frac{1}{(q-1)!} \delta_M A_{\nu_2...\nu_q} \cdot \frac{\partial \mathcal{L}}{\partial \partial_{\nu_0} A_{\nu_2...\nu_q}}, \tag{96}
\]
\[
J_M^0 = \delta_M^K \Lambda_K + \frac{1}{(q-1)!} \delta_M A_{\nu_2...\nu_q} \cdot \Lambda_{\nu_2...\nu_q}. \tag{97}
\]

### 1.4.3 Algebra of Noether currents

In working out brackets \( \{J_M^\mu, J_N^0\} \) we make use of the fact that \( \delta_M^K \phi^K \) is a function of the fields \( \phi \) only, therefore the brackets \( \{\delta_M^K \phi^K, \Lambda_{\nu_2...\nu_q}\} \) vanish; and that \( \delta_M A_{\nu_2...\nu_q} \) is a function of the fields \( \phi \) and their derivatives \( \partial_\mu \phi \) only, see (88), therefore the brackets \( \{\delta_M A_\nu, \Lambda_{\nu_2...\nu_q}\} \) vanish. The computation then yields
\[
\left\{ J_M^\mu \left( t, \sigma \right), J_N^0 \left( t, \sigma' \right) \right\} = \pm C_{MN}^K \cdot j_K^0 \cdot \delta \left( \sigma - \sigma' \right) + \frac{1}{(q-1)!} \left[ \mp C_{MN}^K \delta_K A_{\nu_2...\nu_q} \cdot \delta \left( \sigma - \sigma' \right) + \delta_M^K \Lambda_K, \delta_N A_{\nu_2...\nu_q} \right] \cdot \Lambda_{\nu_2...\nu_q}. \tag{98}
\]
This can be written as
\[
\left\{ J_M^\mu, J_N^0 \right\} = \pm C_{MN}^K \cdot j_K^0 \cdot \delta \left( \sigma - \sigma' \right) + \frac{1}{(q-1)!} \left[ \left( \mp \delta_M \delta_N + \delta_N \delta_M \mp C_{MN}^K \cdot \delta_K \right) A_{\nu_2...\nu_q} \right] \Lambda_{\nu_2...\nu_q} \cdot \delta \left( \sigma - \sigma' \right) \cdots , \tag{99}
\]
where \( \cdots \) denotes surface terms.

The first term is just what we have expected; \( \pm \) again refers to a left/right action. Since \( \delta M A = \text{emb}^* \Delta M \) we have
\[\left( \mp \delta_M \delta_N + \delta_N \delta_M \mp C_{MN}^K \cdot \delta_K \right) A = \text{emb}^* \left( \mp \delta_M \Delta_N + \delta_N \Delta_M \mp C_{MN}^K \Delta_K \right),\]
where the expression in the brackets

\[-\delta_M \Delta_N + \delta_N \Delta_M + C^K_{MN} \Delta_K =: S(\Delta)_{MN}\]

measures the deviation of the forms \(\Delta_M\) from transforming as a multiplet under the adjoint representation of the group \(G\); this is seen from

\[T_M \cdot \Delta_N = \mp \Delta_K \cdot \text{ad}(T_M)^K_N - S(\Delta)_{MN},\]

where the point denotes the action of the "abstract" generator \(T_M\) on the component \(\Delta_N\) according to \(T_M \cdot \Delta_N = [\delta_M, \Delta_N]\).

We now introduce the notation

\[emb^* S(\Delta)_{MN}^\nu_2...^\nu_q =: S(\Delta)_{MN}^\nu_2...^\nu_q,\]

(102)

then (99) reads

\[\{j_0^M, j_0^N\} = \pm C^K_{MN} \cdot j_0^K + S(\Delta)_{MN} \cdot \Lambda_{gauge},\]

(104)

Let us define

\[Q_M = \int_{W(t)} d^p\sigma \cdot j_0^M, \quad Y_{MN}(t) = \int_{W(t)} d^p\sigma \cdot S(\Delta)_{MN} \cdot \Lambda_{gauge},\]

(105)

then the once integrated version of (104) is

\[\{Q_M, j_0^N\} = \pm j_0^K \cdot \text{ad}(T_M)^K_N + S(\Delta)_{MN} \cdot \Lambda_{gauge},\]

(106)

which defines the action of the generator \(T_M\) on the Noether current \(j_0^K\). We see that due to the presence of the \(S\)-term on the right hand side the Noether currents now fail to transform as a multiplet in the adjoint representation, as was the case previously.

The twice integrated version is

\[\{Q_M, Q_N\} = \pm C^K_{MN} \cdot Q_K + Y_{MN} \cdot \Lambda_{gauge},\]

(107)

\(Q_M\) are conserved when the Lagrangian is invariant under \(G\); we need to check when \(Y_{MN}(t)\) are conserved. To this end we perform \(\frac{d}{dt}\) on \(Y_{MN}\) in (105) and assume, for the sake of convenience, that the hypersurfaces \(W(t)\) do not change shape as \(t\) varies; then the only contribution to \(\frac{dY_{MN}}{dt}\) comes from \(\frac{d}{dt}[S(\Delta)_{MN} \cdot \Lambda_{gauge}]\). A calculation then shows that a sufficient condition for the charge \(Y_{MN}(t)\) to be conserved is

\[S(\Delta)_{MNN_q...N_2}(\phi) = \text{const.} = S(\Delta)_{MNN_q...N_2}.\]

(108)

Under the same condition the charges \(Y_{MN}\) are seen to be central.

The complete algebra is then

\[\{Q_M, Q_N\} = \pm C^K_{MN} \cdot Q_K + Y_{MN}, \quad \{Q_K, Y_{MN}\} = \{Y_{MN}, Y_{M'N'}\} = 0.\]

(109)
1.4.4 Algebra of modified currents

At last then let us determine the general structure of the extended algebra of the charges associated with the modified \( \tilde{j}_M^\mu = j_M^\mu - U_M^\mu \), given that the WZ-term \( L_{WZ} \) behaves as in (90, 91). We again find that

\[
\{ \tilde{j}_M^0, \tilde{j}_N^0 \} = \{ j_M^0, j_N^0 \} - \{ U_M^0, j_N^0 \} - \{ j_M^0, U_N^0 \} ,
\]

with \( \{ j_M^0, j_N^0 \} \) given in (104). \( \{ j_M^0, U_N^0 \} \) can be determined using the properties of \( U_N^0 \) given in (91). Up to surface terms we then find

\[
\{ \tilde{j}_M^0, \tilde{j}_N^0 \} \approx \pm C_{KM}^N \cdot \delta (\sigma - \sigma') + \left[ \pm C_{KM}^N \cdot U_K^0 + S(\Delta)_{MN} \cdot \Lambda^{gauge} - \delta_N \phi^K \frac{\partial U_M^0}{\partial \phi^K} + \delta_M \phi^K \frac{\partial U_N^0}{\partial \phi^K} \right] \cdot \delta (\sigma - \sigma') .
\]

(110)

Analogous to (49) we define

\[
S(U)_{MN} = \delta_M \phi^K \cdot \frac{\partial U_N^0}{\partial \phi^K} - \delta_N \phi^K \cdot \frac{\partial U_M^0}{\partial \phi^K} \pm C_{KM}^N \cdot U_K^0
\]

(111)

and its integral

\[
Z_{MN} = \int_{W(t)} d^p \sigma \cdot S(U)_{MN} ,
\]

(112)

and \( Y_{MN} \) as the integral of \( S(\Delta)_{MN} \cdot \Lambda^{gauge} \) over \( W(t) \), according to (105). Then double integration of (111) yields

\[
\{ Q_M, Q_N \} = \pm C_{KM}^N \cdot Q_K + Y_{MN} + Z_{MN} .
\]

(113)

In the case of constant \( S(\Delta)_{MNN_Nq...N_2} \) (see (108)) we can write

\[
Y_{MN} = \frac{1}{(q-1)!} S(\Delta)_{MNN_Nq...N_2} \int_{W(t)} d^p \sigma \cdot \phi_{r_2}^{N_2} \cdots \phi_{r_q}^{N_q} \cdot \Lambda^{r_2...r_q} ,
\]

(114)

on the right hand side now there appear charges

\[
Y_{MN}^{N_2...N_q} := \int_{W(t)} d^p \sigma \cdot \phi_{r_2}^{N_2} \cdots \phi_{r_q}^{N_q} \cdot \Lambda^{r_2...r_q} ,
\]

(114)

and due to the first class constraints the summation in the integrand runs over ”spatial” indices \( r_2, \ldots, r_q \in \{1, \ldots, p\} \) only, so that finally

\[
Y_{MN} = \frac{1}{(q-1)!} S(\Delta)_{MNN_Nq...N_2} \cdot Y_{MN}^{N_2...N_q} ,
\]

and (113) becomes now

\[
\{ Q_M, Q_N \} = \pm C_{KM}^N \cdot Q_K + \frac{1}{(q-1)!} S(\Delta)_{MNN_Nq...N_2} \cdot Y_{MN}^{N_2...N_q} + Z_{MN} .
\]

(115)
Yet another expression for the above relations can be obtained \(\text{[2]}\) if we regard the worldvolume as a pseudo-Riemannian manifold with an (auxiliary) metric which is diagonal in the coordinate system \((t, \sigma)\),

\[-dt \otimes dt + \delta_{rs} \cdot d\sigma^r \otimes d\sigma^s\]

Then the restriction of this metric to the hypersurfaces \(W(t)\) is a Euclidean metric, and \(W(t)\) become Riemannian manifolds, on which we can introduce a Hodge star operator with respect to this metric. We need not distinguish between upper and lower indices here, so that \(\Lambda_{r_2 \ldots r_p}^{sq \ldots sp}\) for \(r_2, \ldots, r_q \in \{1, \ldots, p\}\) can be regarded as components of a \((q - 1)\)-form \(\Lambda^{gauge}\) on \(W(t)\). Its Hodge dual is then

\[(\ast \Lambda^{gauge})_{s_q \ldots s_p} = \frac{1}{(q - 1)!} \epsilon_{t_2 \ldots t_q s_q \ldots s_p} \cdot \Lambda^{t_2 \ldots t_q}, \quad (116)\]

where all indices are taken from the set \(\{1, \ldots, p\}\). On using

\[(\ast\ast \Lambda^{gauge}) = (-1)^{(q-1)(p-q+1)} \cdot \Lambda^{gauge} \quad (117)\]

we can write

\[d\sigma^1 \ldots d\sigma^p \cdot \phi^{N_2 \ldots r_q} \cdot \Lambda^{r_2 \ldots r_q} = \]

\[= \frac{1}{(p - q + 1)!} (\ast \Lambda^{gauge}) \cdot emb^* d\phi^{N_2} \ldots emb^* d\phi^{N_q} \quad (118)\]

in what follows we shall omit the "\(emb^*\)" for the sake of simplicity. Multiplication of \((118)\) by \(\frac{1}{(q-1)!} S(\Delta)_{M N N_q \ldots N_2}\) gives

\[d\sigma^1 \ldots d\sigma^p \cdot S(\Delta)_{M N} \cdot \Lambda^{gauge} = \]

\[= \frac{1}{(p - q + 1)!} (\ast \Lambda^{gauge}) \cdot S(\Delta)_{M N} \quad (119)\]

where \(S(\Delta)_{M N}\) now denotes the pullback of this form to \(W(t)\),

\[S(\Delta)_{M N} = emb^* \frac{1}{(q-1)!} d\phi^{N_2} \ldots d\phi^{N_q} \cdot S(\Delta)_{M N N_q \ldots N_2} \quad (120)\]

Thus we can rewrite \((118)\) as

\[Y_{MN}(t) = \frac{1}{(p - q + 1)!} \int_{W(t)} (\ast \Lambda^{gauge}) \cdot S(\Delta)_{M N} \quad (121)\]

We again summarize this section in the form of a theorem.

1.4.5 Theorem

Let an Abelian \((q - 1)\)-form gauge potential \(A_{\mu_2 \ldots \mu_q}, q \leq p\), be defined on the worldvolume. On the target space a \(q\)-form potential \(B\) transforms according to \(\delta_M B = d\Delta_M\) under \(G\). We impose a transformation behaviour \(\delta_M A = emb^* \Delta_M\) on \(A\). The Lagrangian splits into an invariant piece \(L_0\) and a semi-invariant piece \(L_{WZ}\), as described above. Then
1. The Poisson bracket algebra of the Noether currents is
\[
\{ j_0^M, j_0^N \} \approx \pm C_{KM} \cdot j_0^K + S(\Delta)_{MN} \cdot \Lambda^{gauge} \cdot \delta (\sigma - \sigma') ,
\] (122)
where \( S(\Delta)_{MN} = \text{emb}^* S(\Delta)_{MN \nu_2 \ldots \nu_q} \) and
\[
S(\Delta)_{MN} = -\delta_M \Delta_N + \delta_N \Delta_M \mp C_{MN} \Delta_K
\] (123)
measures the deviation of the forms \( \Delta_M \) from transforming as a multiplet under the adjoint representation of the group \( G \):
\[
T_M \cdot \Delta_N = \mp \Delta_K \cdot \text{ad}(T_M)^K_N - S(\Delta)_{MN} \cdot \Lambda \text{gauge} ,
\] (124)
The once integrated version of (122) is
\[
\{ Q_M, j_0^N \} = \pm j_0^K \cdot \text{ad}(T_M)^K_N + S(\Delta)_{MN} \cdot \Lambda \text{gauge} ,
\] (125)
where
\[
Y_{MN}(t) = \int_{W(t)} S(U)_{MN} \cdot \Lambda^{gauge} .
\] (127)
If \( S(\Delta)_{MN \nu_2 \ldots \nu_q} = \text{const.} \), then the charges \( Y_{MN} \) are conserved and central. (127) can be rewritten in the form
\[
Y_{MN}(t) = \frac{1}{(p - q + 1)!} \int_{W(t)} \left( \ast \Lambda^{gauge} \right) \cdot S(\Delta)_{MN} ,
\] (128)
with \( \ast \Lambda^{gauge} \) being the Hodge dual of the form \( \Lambda^{gauge} = \frac{1}{(q - 1)!} d\sigma^r_2 \ldots d\sigma^r_q \cdot \Lambda^{r_2 \ldots r_q} \) on the worldvolume.

2. The Poisson bracket algebra of the modified currents is
\[
\{ \tilde{j}_0^M, \tilde{j}_0^N \} \approx \pm C_{KM} \cdot \tilde{j}_0^K \cdot \delta (\sigma - \sigma') + [S(U)_{MN} + S(\Delta)_{MN} \cdot \Lambda^{gauge}] ,
\] (129)
with
\[
S(U)_{MN} = \delta_M \phi^K \cdot \frac{\partial \tilde{U}_0^K}{\partial \phi^K} - \delta_N \phi^K \cdot \frac{\partial \tilde{U}_0^K}{\partial \phi^K} \pm C_{MN} \cdot \tilde{U}_0^K
\] (130)
and its integral
\[
Z_{MN} = \int_{W(t)} d^p \sigma \cdot S(U)_{MN} .
\] (131)
Double integration of the current algebra yields the charge algebra
\[
\{ Q_M, Q_N \} = \pm C_{KM} \cdot Q_K + Y_{MN} + Z_{MN} .
\] (132)
In the case of constant \( S(\Delta)_{MN} \) this can be written as
\[
\{ Q_M, Q_N \} = C_{KM} \cdot Q_K + \frac{1}{(q - 1)!} S(\Delta)_{MN \nu_2 \ldots \nu_q} \cdot Y_{MN}^{N_2 \ldots N_q} + Z_{MN} .
\] (133)
2 Extended superalgebras carried by $D$-$p$-branes in IIA superspace

Now we apply the ideas we have developed in the previous sections to the case of $D$-$p$-branes in IIA superspace. This restricts $p$ to be even, $p = 0, 2, 4, 6, 8$. However, before doing so, we first discuss our superspace conventions, and our definitions of graded Poisson brackets. Then we first compute the algebra of Noether charges and of modified Noether charges resulting from a D-brane Lagrangian without specific assumptions on the background the brane propagates in, or on the specific form of the various gauge fields occurring in the Lagrangian. Then we recapitulate how supergravity determines the background in which the branes propagate, and the relation of superspace constraints with $\kappa$-symmetry of the branes. Then we study the Bianchi identities associated with a specific choice of background gauge fields in superspace; and only then we work out the explicit superalgebra extensions carried by D-branes in this particular $D = 10$ vacuum.

2.1 Conventions

2.1.1 Superspace conventions

The target space $\Sigma$ is now the coset space

$$\text{IIA-superMinkowski} = \text{IIA-superPoincare}/SO(1,9)$$

with coordinates $(X, \theta)$ that label the coset representative $e^{iX^P + \theta^Q}$. Adopting the convention that the complex conjugate of a product of two spinors reverses their order this implies that in an operator realization the coset representatives are mapped to unitary operators, provided that $P$ and $Q$ are hermitian. The assumption of IIA superspace means that we have two 16-component spinor generators of opposite chirality which transform under the two irreducible $(16 \times 16)$-dimensional spin representations of $SO(1,9)$; but effectively, this yields one non-chiral 32-component spinor, transforming under the direct sum of the two irreducible spin representations, which is just the representation of $SO(1,9)$ obtained from the 32-component $\Gamma$-matrices. The metric $\eta_{mn}$ on the target space is flat 10-dimensional "mostly plus" Minkowski metric. Spinor components occur with natural index up; an inner product between spinors is provided by the bilinear form $(\chi, \theta) \mapsto \chi^\alpha C_{\alpha\beta}\theta^\beta$, where $C$ is a charge conjugation matrix. In $D = 10$ and with the Minkowski metric as specified above we can choose a Majorana-Weyl representation for the spinors and the $\Gamma$-matrices, respectively, in which spinors have real Grassmann-odd components, the matrices $C\Gamma_m$ are real and symmetric, and $C$ is real and antisymmetric. In such a representation we can choose $C = \pm \Gamma_0$. Altogether we have 32 real fermion degrees of freedom a priori. Spinor indices are lowered and raised from the left with the charge conjugation matrix and its inverse, respectively; e.g., raising is accomplished with the inverse of $C$, the components of which are denoted by $C^\alpha{}^\beta$, by $\theta_\beta \mapsto \theta^\alpha = C^\alpha{}^\beta \theta_\beta$. By definition, $C^\alpha{}^\beta C^\beta{}^\gamma = \delta^\alpha{}^\gamma$. An expression like $\bar{\epsilon}\Gamma_m\theta$
therefore means
\[ \bar{e} \Gamma_m \theta = \epsilon^\alpha C_{\alpha\beta} (\Gamma_m)^\beta_\gamma \theta^\gamma \],
etc. Our supertranslation algebra is
\[ \{Q_{\alpha}, Q_{\beta}\} = 2 \Gamma^m_{\alpha\beta} \cdot P_m \]. (134)
The action of \( e^{iY \cdot P + \epsilon Q} \) on \((X, \theta)\) yields \((X', \theta')\), where \((X', \theta')\) is implicitly defined by
\[ e^{iY \cdot P + \epsilon Q} e^{iX \cdot P + \theta Q} = e^{iX' \cdot P + \theta' Q} \]; (135)
for infinitesimal \( \epsilon \) this yields \((X', \theta') = (X + Y + i\epsilon \Gamma \theta, \theta + \epsilon)\). From (135) it can be seen that this is a left action. The vector fields \( \tilde{T}_\alpha, i\tilde{T}_m \) induced by the generators \( Q_{\alpha}, iP_m \) on \( \Sigma \) are therefore
\[ \tilde{T}_\alpha = (i \Gamma m \theta)_{\alpha} \cdot \frac{\partial}{\partial X^m} + \frac{\partial}{\partial \theta^\alpha} =: \delta_\alpha \], (136)
\[ i\tilde{T}_m = \frac{\partial}{\partial X^m} =: \delta_m \]. (137)
By construction they are right-invariant vector fields. The corresponding left-invariant vector fields are obtained by replacing \( \theta \mapsto -\theta \) in (136, 137). Their duals are the left invariant 1-forms \( \Pi^M = (\Pi^m, \Pi^\alpha) \) on superspace, where
\[ \Pi^m = dX^m + id\bar{\theta} \Gamma^m \theta \], \( \Pi^\alpha = d\theta^\alpha \). (138)
From (137) we see that \( \delta_m \) is strictly speaking "\( i \times \) Poincare-translation with generator \( P_m \)."

The graded Lie-bracket of \( \tilde{T}_\alpha, \tilde{T}_\beta \) is \( \{\delta_\alpha, \delta_\beta\} = [\tilde{T}_\alpha, \tilde{T}_\beta]_{\text{graded Lie}} = \)
\[ = -2 \Gamma^m_{\alpha\beta} \left( -i \frac{\partial}{\partial X^m} \right) = -2 \Gamma^m_{\alpha\beta} \cdot \tilde{T}_m = 2i \Gamma^m_{\alpha\beta} \cdot \delta_m \], (139)
i.e. the algebra (134) is satisfied up to a sign, which is in accord with (3) in the first section, since the action of the supergroup on \( \Sigma \) is from the left, or equivalently, since the \( \tilde{T}_M \) are right-invariant.

Summation of superspace indices is defined according to
\[ \omega = \frac{1}{r!} dZ^{M_1} \ldots dZ^{M_r} \cdot \omega_{M_1 \ldots M_1} \],
where \( \omega \) is a superspace \( p \)-form. The forms on the worldvolume obey the usual summation conventions, however; e.g. for the pull-back of the above \( r \)-form to the worldvolume we write
\[ \text{emb}^* \omega = \frac{1}{r!} \partial_{\mu_1} Z^{M_1} \ldots \partial_{\mu_r} Z^{M_r} \cdot \omega_{M_1 \ldots M_1} \cdot dx^\mu_1 \ldots dx^\mu_r \].

Exterior derivative \( d \) is defined to act from the right on superspace forms as well as on worldvolume forms,
\[ d (\omega_\chi) = \omega d\chi + (-1)^q d\omega \cdot \chi \]
where \( \chi \) is a \( q \)-form.
2.1.2 Graded Poisson brackets

Given two (possibly graded) functionals \( F, G \) of the (possibly graded) time-dependent fields \( \phi^i(t, \sigma) \) and their canonical conjugate momenta \( \Lambda_i(t, \sigma) \) that are defined on a \( p \)-dimensional manifold \( S \) with coordinates \((\sigma^1, \ldots, \sigma^p)\), their Poisson bracket is defined by (see, e.g., [6])

\[
\{ F, G \}_{PB} = \int_S d^p \sigma \sum_i \left[ (-1)^{\phi^i} F \delta \frac{G}{\delta \phi^i(\sigma)} - G \delta \frac{F}{\delta \phi^i(\sigma)} \right] ;
\]

this can be expressed in terms of derivatives acting solely from the left by

\[
\{ F, G \}_{PB} = \int_S d^p \sigma \sum_i \left[ (-1)^{\phi^i} F \delta F \delta \phi^i(\sigma) - G \delta \Lambda_i(\sigma) \delta \phi^i(\sigma) \right]
\]

where \((-1)^{\phi^i} = 1\) iff both \( F \) and \( \phi^i \) are Grassmann-odd. With these definitions the following rules are satisfied:

1. Graded Antisymmetry,

\[ \{ F, G \} = -(-1)^{FG} \{ G, F \} \] .

2. Graded Leibnitz rule,

\[ \{ F, GH \} = \{ F, G \} H + (-1)^{FG} G \{ F, H \} \] .

3. Graded Jacobi identity,

\[ (-1)^{FH} \{ F, \{ G, H \} \} + (-1)^{GF} \{ G, \{ H, F \} \} + (-1)^{HG} \{ H, \{ F, G \} \} = 0 \] .

2.2 D-\( p \) brane Lagrangians

2.2.1 Structure of the Lagrangian

The kinetic supertranslation-invariant part \( \mathcal{L}_0 \) in the D-\( p \)-brane Lagrangian is given by

\[
\mathcal{L}_0 = \sqrt{-\det \left( g_{\mu\nu} + \tilde{F}_{\mu\nu} \right)} ,
\]

where \( g_{\mu\nu} = \Pi^m_\mu \Pi^o_\nu \eta_{nm} \) is the pull-back of the 10-dimensional "mostly plus" Minkowski metric \( \eta_{nm} \) to the worldvolume \( W \) of the D-\( p \)-brane using left-invariant (LI) 1-forms \( \Pi^A \), and

\[
\tilde{F}_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - \Pi^A_{\mu} \Pi^A_{\nu} \cdot B_{A2A1} \ ,
\]

where \( F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu \) are the components of the field strength \( F \) of the gauge potential \( A \) defined on the worldvolume, and \( B_{A2A1} \) are the components of the superspace 2-form potential \( B \) in the LI-basis whose leading component in a \( \theta \)-expansion is the NS-NS gauge potential. In the discussion below we shall
assume that its bosonic components are zero, but at present $B$ could be quite arbitrary. Under supertranslations $\delta_\alpha$ the field strength $H = dB$ is assumed to be invariant,

$$\delta_\alpha H = 0 \quad .$$

(147)

This implies that $B$ transforms locally as a differential,

$$\delta_\alpha B = d\Delta_\alpha \quad .$$

(148)

Therefore it transforms as a differential under Poincare translations as well: To see this compare

$$2i\Gamma^m_{\alpha\beta} \cdot \delta_m B = \{\delta_\alpha , \delta_\beta \} B = d(\delta_\alpha \Delta_\beta + \delta_\beta \Delta_\alpha) \quad ,$$

where in the first equation (139) has been used. If we multiply with another $\Gamma$-matrix and take the trace we find that

$$\delta_m B = d\Delta_m$$

(149)

with

$$\Delta_m = \frac{\Gamma^m_{\alpha\beta}}{i \cdot tr(132)} \delta_\alpha \Delta_\beta \quad .$$

(150)

Since $[\delta_m , \delta_n] = 0$ it follows from (149) that $d(\delta_m \Delta_n - \delta_n \Delta_m) = 0$, which implies, that locally

$$\delta_m \Delta_n - \delta_n \Delta_m = df_{mn}$$

(151)

for some function $f_{mn}$. This function need not be defined globally, however.

Although $A_\mu$ is a worldvolume field it is defined to transform under these translations according to

$$\delta_m A = emb^*\Delta_m \quad , \quad \delta_\alpha A = emb^*\Delta_\alpha \quad ,$$

(152)

where $emb : W \rightarrow \Sigma$ denotes the embedding of the worldvolume into the target space, and $emb^*$ denotes the associated pull-back. With this definition the quantity $\hat{F}$ is invariant under Poincare- and supertranslations, as explained in section 1.4. Since the same is true for $g_{\mu\nu}$ we see that therefore $\mathcal{L}_0$ is invariant as well.

### 2.2.2 Wess-Zumino term

The Wess-Zumino form $(WZ)$ in the D-$p$-brane Lagrangian is the $(p+1)$-form on the worldvolume

$$(WZ) = \sum_{n=0}^{[p+1]/2} \frac{1}{n!} emb^* C^{(p+1-2n)} \cdot \hat{F}^n \quad ,$$

(153)

where $C^{(r)}$ are the superspace potentials whose leading components in a $\theta$-expansion are the usual bosonic RR gauge potentials [7]; in the discussion below the bosonic components of its field strengths will be set to zero, which then amounts to the choice of a particular background, but here we make no specific
assumptions on the form of $C^{(r)}$. The pull-back of $(WZ)$ to the worldvolume gives the Wess-Zumino term $L_{WZ}$ in the Lagrangian,

$$
L_{WZ} = \sum_{n=0}^{\frac{p+1}{2}} \frac{\epsilon^{\lambda_1 \ldots \lambda_{p+1-2n} \nu_1 \ldots \nu_{2n}}}{(p + 1 - 2n)! \cdot 2^n \cdot n!} \left[ \text{emb}^* C^{(p+1-2n)} \right]_{\lambda_1 \ldots \lambda_{p+1-2n}} \hat{F}_{\nu_1 \nu_2} \cdots \hat{F}_{\nu_{2n-1} \nu_{2n}}.
$$

(154)

Since we are in IIA superspace we have actually $p = 2q$, $q = 0, \ldots, 4$.

If $\delta$ denotes either a supertranslation or a Poincare translation then invariance of $\hat{F}$ under either of these transformations implies that

$$
\delta L_{WZ} = \sum_{n=0}^{\frac{p+1}{2}} \frac{\epsilon^{\lambda_1 \ldots \lambda_{p+1-2n} \nu_1 \ldots \nu_{2n}}}{(p + 1 - 2n)! \cdot 2^n \cdot n!} \left[ \text{emb}^* \delta C^{(p+1-2n)} \right]_{\lambda_1 \ldots \lambda_{p+1-2n}} \hat{F}_{\nu_1 \nu_2} \cdots \hat{F}_{\nu_{2n-1} \nu_{2n}}.
$$

(155)

The field strengths associated with the RR-potentials $C^{(r)}$ are defined to be

$$
R^{(r+1)} = \left\{ \frac{dC^{(r)}}{dC^{(r)} - C^{(r-2)}H} ; \ r = 0, 1 \right\} ; \ r = 2, \ldots, 10
$$

(156)

they obey the Bianchi identities

$$
\left\{ \frac{dR^{(r+1)}}{dR^{(r+1)} - R^{(r-1)}H} ; \ r = 0, 1 \right\} ; \ r = 2, \ldots, 10.
$$

(157)

It is now assumed that the field strengths (156) are supertranslation invariant,

$$
\delta_\alpha R^{(r+1)} = 0 ; \ r = 0, \ldots, 10.
$$

(158)

Then it follows from

$$
\delta_m = \frac{\Gamma^{\alpha \beta}_m}{2i \cdot tr (1_{32})} \cdot \{\delta_\alpha, \delta_\beta\}
$$

(159)

that it is invariant under Poincare-translations $-i\delta_m$ as well. From (147, 157, 158) we can construct the general form of $\delta_\alpha C^{(r)}$ for both the IIA and IIB case recursively by starting with the lowest rank form $C^{(1)}$ or $C^{(0)}$, respectively. For the IIA case the result is that there exist superspace forms

$$
D^{(2r)}_\alpha ; \ r = 0, \ldots, 4 ; \ \alpha = 1, \ldots, 32
$$

(160)

such that

$$
\delta_\alpha C^{(2q+1)} = \sum_{k=0}^{q} dD^{(2q-2k)}_\alpha \cdot B^k_k .
$$

(161)

The superscript $(2r)$ in (160) refers to the fact that the index $\alpha$ does not take part in a summation, but labels one of 32 components of a spinor-valued $(2r)$-form $D^{(2r)} = \left( D^{(2r)}_\alpha \right)_{\alpha=1,\ldots,32}$. Using (159) we find that

$$
\delta_m C^{(2q+1)} = \sum_{k=0}^{q} dD^{(2q-2k)}_m \cdot B^k_k ,
$$

(162)
where

\[
D_m^{(0)} = \frac{\Gamma_{\alpha\beta}}{2i \times \text{tr} (1_{32})} \cdot \left[ \delta_\alpha D^{(0)}_\beta + \delta_\beta D^{(0)}_\alpha \right] = \frac{\Gamma_{\alpha\beta}}{i \times \text{tr} (1_{32})} \cdot \delta_\alpha D^{(0)}_\beta ,
\]

\[
D_m^{(2q)} = \frac{\Gamma_{\alpha\beta}}{2i \times \text{tr} (1_{32})} \cdot \left[ \delta_\alpha D^{(2q)}_\beta + \delta_\beta D^{(2q)}_\alpha + D^{(2q-2)}_\alpha \times d\Delta_\beta + D^{(2q-2)}_\beta \times d\Delta_\alpha \right] = \frac{\Gamma_{\alpha\beta}}{i \times \text{tr} (1_{32})} \cdot \left[ \delta_\alpha D^{(2q)}_\beta + D^{(2q-2)}_\alpha \times d\Delta_\beta \right].
\]

Now let us return to the supertranslation variation of \( C^{(r)} \) in (161). If we insert (161) in (155) we find that all terms involving \( B \) cancel,

\[
\delta_\alpha \mathcal{L}_{WZ} = \partial_\mu U_\mu^\alpha ,
\]

with

\[
U_\mu^\alpha = \sum_{n=0}^{q} \frac{\epsilon^{\mu\nu_1\nu_2\ldots\nu_{2q}+1-2n}}{(2q-2n)! \times 2^n \times n!} \left[ \text{emb}^{*} D^{(2q-2n)}_\alpha \right]_{\mu_2 \ldots \mu_{2q+1-2n}} F_{\nu_1\nu_2} \cdots F_{\nu_{2n-1}\nu_{2n}} .
\]

Similarly, application of (155) gives

\[
U_m^\mu = -i \sum_{n=0}^{q} \frac{\epsilon^{\mu\nu_1\nu_2\ldots\nu_{2q}+1-2n}}{(2q-2n)! \times 2^n \times n!} \left[ \text{emb}^{*} D^{(2q-2n)}_m \right]_{\mu_2 \ldots \mu_{2q+1-2n}} F_{\nu_1\nu_2} \cdots F_{\nu_{2n-1}\nu_{2n}} ,
\]

with the \( D_m \) given in (163) and (164). We observe that \( U_\alpha^0, U_m^0 \) contain neither of \( \dot{X}, \dot{\theta}, F_{0r} \).

### 2.2.3 Noether currents and Noether charges

We now want to compute the algebra of Noether currents and Noether charges resulting from these currents; as discussed in section 1.4 we may expect that due to the fact that the worldvolume gauge field \( A_\mu \) takes part in the supersymmetry transformations on the target space even the current algebra of the Noether currents fails to close in the ordinary form, but will be extended by central pieces. All the more this will be true for the algebra of the modified currents and charges, respectively.

Our degrees of freedom are now \((X^m, \theta^\alpha, A_\nu)\) with \( \nu = 0, \ldots, p \); the associated canonical conjugate momenta are \((\Lambda_m, \Lambda_\alpha, \Lambda^\nu)\), respectively. As discussed in section 1.4 we have a primary constraint \( \Lambda^0 = 0 \) which amounts to a reduction of phase space, and a secondary constraint \( \sum_{r=1}^{p} \delta_r \Lambda^r = 0 \), which holds only on-shell, i.e. on using the equations of motion. The zeroth components of the Noether currents associated with the generators \( Q_\alpha \) are

\[
J^{0}_\alpha = (i \Gamma^m \theta^\alpha) \cdot \Lambda_m + \Lambda_\alpha + (\text{emb}^{*} \Delta_\alpha) \cdot \Lambda^\nu ;
\]

as explained in section 1.4 the primary constraint \( \Lambda^0 = 0 \) must not be taken into account before all Poisson brackets have been worked out. The zeroth components of the Noether currents associated with the generators \( P_m \) are

\[
J^{0}_m = -i \Lambda_m - i (\text{emb}^{*} \Delta_m) \cdot \Lambda^\nu .
\]
Then the Poisson bracket of the currents \( \{ j^0_\alpha (t, \sigma), j^0_\beta (t, \sigma') \} \) is

\[
\{ j^0_\alpha (t, \sigma), j^0_\beta (t, \sigma') \} \approx -2i \Gamma^{m}_{\alpha \beta} \cdot \Lambda_m \cdot \delta (\sigma - \sigma') - \left[ \delta_\alpha (\text{emb}^* \Delta_\beta)_r + \delta_\beta (\text{emb}^* \Delta_\alpha)_r \right] \cdot \Lambda^r \cdot \delta (\sigma - \sigma') ,
\]

(170)

where “\( \approx \)” means “on using all constraints and equations of motion and on discarding surface terms”. If we insert (169) into the last equation we get

\[
\{ j^0_\alpha (t, \sigma), j^0_\beta (t, \sigma') \} \approx \left[ 2 \Gamma^{m}_{\alpha \beta} \cdot j^0_m + (\text{emb}^* S_{\alpha \beta} (\Delta))_r \cdot \Lambda^r \right] \cdot \delta (\sigma - \sigma') ,
\]

(171)

where we have used the primary constraint \( \Lambda^0 = 0 \) at last. \( S_{\alpha \beta} (\Delta) \) is given by

\[
S_{\alpha \beta} (\Delta) = 2i \Gamma^{m}_{\alpha \beta} \cdot \Delta_m - \delta_\alpha \Delta_\beta - \delta_\beta \Delta_\alpha .
\]

(172)

We see that the presence of the gauge field \( A_\mu \) taking part in the supersymmetry variation of the Lagrangian alters the form of the algebra even of the Noether currents, i.e. before taking into account the possible modifications of the Noether currents by terms originating in the Wess-Zumino term.

The once integrated version of (171) defines the action of the generator \( Q_\alpha \) on the current \( j^0_\beta \),

\[
\{ Q_\alpha, j^0_\beta \} = 2 \Gamma^{m}_{\alpha \beta} \cdot j^0_m + (\text{emb}^* S_{\alpha \beta} (\Delta))_r \cdot \Lambda^r .
\]

(173)

As explained in section 1.1 the presence of the Wess-Zumino term in the Lagrangian implies that the Noether charges \( Q_\alpha \) which are obtained by integrating the zero components \( j^0_\alpha \) over the hypersurface \( W(t) \) are no longer conserved; however, if the Wess-Zumino term \( \mathcal{L}_{\text{WZ}} \) and the NS-NS gauge potential are translational invariant, as will be the case below, the Noether charges \( P_m \) obtained by integrating \( j^0_m \) are still conserved.

The twice integrated version of (171) describes the modified algebra of the Noether charges,

\[
\{ Q_\alpha, Q_\beta \} = 2 \Gamma^{m}_{\alpha \beta} \cdot P_m + \int_{W(t)} d^p \sigma \sum_{r=1}^p (\text{emb}^* S_{\alpha \beta} (\Delta))_r \cdot \Lambda^r ;
\]

(174)

here \( (\text{emb}^* S_{\alpha \beta} (\Delta))_r = \partial_r Z^M \cdot S_{\alpha \beta M} \), when \( S_{\alpha \beta} \) is expanded in the coordinate basis \( (dZ^M) = (dX^m, d\theta^a) \). Now let us assume (see section 1.4) that \( S_{\alpha \beta M} \) are constants; we want to find extensions of the current and charge algebra by topological charges carried by the brane; but since it is only the bosonic coordinates \( X^m \) and the pull-back of their differentials to the worldvolume that describe the topology of the image \( \text{emb} W(t) \) of the brane in the spacetime \( \Sigma \) we need only consider the terms involving bosonic 1-forms \( dX^m \), i.e. \( \partial_r X^m \), in the above pull-back; therefore if we now define the charge

\[
Y^m = \int_{W(t)} d^p \sigma \sum_{r=1}^p \partial_r X^m \cdot \Lambda^r ,
\]

(175)
then the algebra of the Noether charges in (174) becomes

\[ \{ Q_\alpha, Q_\beta \} = 2 \Gamma^{m}_{\alpha\beta} \cdot P_m + S_{\alpha\beta m} \cdot Y^m \quad , \quad S_{\alpha\beta m} \text{ constant.} \] (176)

If we think of \( W(t) \) as being endowed with an auxiliary Euclidean metric which is diagonal in the coordinates \((\sigma^r)\) then we can introduce the Hodge dual of the 1-form \( \Lambda^{\text{gauge}} = \sum_{r=1}^{p} d\sigma^r \Lambda^r \), which is given by

\[ (*\Lambda^{\text{gauge}})_{s_2...s_p} = \epsilon_{r_2...r_p} \Lambda^r \] , (177)

and rewrite (175) as

\[ Y^m = \frac{1}{(p-1)!} \int_{W(t)} (*\Lambda^{\text{gauge}}) dX^m \] , (178)

where \( dX^m \) now denotes the pull-back \( \text{emb} \ast dX^m \), and exterior product of forms is understood in the integrand. Furthermore we note that \(*\Lambda^{\text{gauge}}\) is \textbf{closed} on the physical trajectories, since

\[ d * \Lambda^{\text{gauge}} = \partial_r \Lambda^r \cdot d\sigma^1 \cdots d\sigma^p \] . (179)

A similar computation now shows that

\[ \{ j^0_\alpha (t, \sigma), j^0_m (t, \sigma') \} \approx (\text{emb} \ast S_{\alpha m} (\Delta))_{r} \cdot \Lambda^r \cdot \delta (\sigma - \sigma') \] , (180)

with

\[ S_{\alpha m} (\Delta) = i (\delta_\alpha \Delta_m - \delta_m \Delta_\alpha) \] ; (181)

the factor of \( i \) comes from our parametrising of the coset elements of super-Minkowski space, see section 2.1.1.

Finally, we find

\[ \{ j^0_m (t, \sigma), j^0_n (t, \sigma') \} \approx (\text{emb} \ast S_{mn} (\Delta))_{r} \cdot \Lambda^r \cdot \delta (\sigma - \sigma') \] , (182)

where

\[ S_{mn} (\Delta) = \delta_m \Delta_n - \delta_n \Delta_m \] . (183)

Double integration of (182) using (151) then yields

\[ [P_m, P_n] = \int_{W(t)} d^p \sigma \cdot \partial_r (f_{mn} \Lambda^r) \] , (184)

where we have used the secondary constraint \( \partial_r \Lambda^r = 0 \) and the equations of motion. We see that the momenta can be \textbf{non-commuting} in the case that the functions \( f_{mn} \) are not globally defined; this could happen if some of the dimensions of \( W(t) \) are compact, and their images in the spacetime under the embedding describe a closed but non-contractible cycle.
2.2.4 Modified currents and charges

The modified currents are
\[
\tilde{j}_\alpha^0 = j_\alpha^0 - U_\alpha^0, \quad \tilde{j}_m^0 = j_m^0 - U_m^0.
\]
(185)

Their Poisson brackets are found to be
\[
\{\tilde{j}_\alpha^0, \tilde{j}_\beta^0\} \approx \left[2\Gamma_{\alpha\beta} \cdot \tilde{j}_m^0 + (emb*S_{\alpha\beta}(\Delta))_r \cdot \Lambda^r + S_{\alpha\beta}(U)\right] \cdot \delta(\sigma - \sigma')
\]
with
\[
S_{\alpha\beta}(U) = \delta_\alpha U_\beta^0 + \delta_\beta U_\alpha^0 + 2\Gamma_{\alpha\beta} \cdot U_n^0.
\]
(187)

\(S_{\alpha\beta}\) is given in (172). Furthermore,
\[
\{\tilde{j}_\alpha^0, \tilde{j}_m^0\} \approx \left[(emb*S_{\alpha m}(\Delta))_r \cdot \Lambda^r + S_{\alpha m}(U)\right] \cdot \delta(\sigma - \sigma')
\]
with \(S_{\alpha m}\) given in (181); and finally,
\[
\{\tilde{j}_m^0, \tilde{j}_n^0\} \approx \left[(emb*S_{mn}(\Delta))_r \cdot \Lambda^r + S_{mn}(U)\right] \cdot \delta(\sigma - \sigma')
\]
with \(S_{mn}\) from (183).

We can work out the expressions for \(S_{MN}(U)\) using (166, 167), which yields

\[
S_{\alpha\beta}(U) = \frac{\epsilon^{0\nu_1...\nu_{2q}}}{2q \cdot q!} emb^* \left[\delta_\alpha D_\beta^{(0)} + \delta_\beta D_\alpha^{(0)} + 2\Gamma_{\alpha\beta} \cdot D_n^{(0)}\right] \cdot F_{\nu_1 \nu_2} \cdots F_{\nu_{2q-1} \nu_{2q}} +
\]
\[
+ \sum_{k=0}^{q-1} \frac{\epsilon^{0\mu_2...\mu_{2q+1-2k} \nu_1...\nu_{2k}}}{(2q - 2k)! \cdot k!} \left\{emb^* \left[\delta_\alpha D_\beta^{(2q-2k)} + \delta_\beta D_\alpha^{(2q-2k)}\right] +
\right.
\]
\[
+ 2\Gamma_{\alpha\beta} \cdot D_n^{(2q-2k)} \right\}_{\mu_2...\mu_{2q+1-2k}} - (2q - 2k)(2q - 2k - 1) \cdot
\]
\[
\left[\left(emb^* D_\beta^{(2q-2k-2)}\right)_{\mu_2...\mu_{2q-1-2k}} \cdot \partial_{\mu_{2q-2k}} \left(emb^* \Delta_\alpha\right)_{\mu_{2q+1-2k}} +
\right.
\]
\[
+ \left(emb^* D_\alpha^{(2q-2k-2)}\right)_{\mu_2...\mu_{2q-1-2k}} \cdot \partial_{\mu_{2q-2k}} \left(emb^* \Delta_\beta\right)_{\mu_{2q+1-2k}} \right\} \cdot
\]
\[
F_{\nu_1 \nu_2} \cdots F_{\nu_{2q-1} \nu_{2q}}.
\]
(192)
of the underlying supergravity theory, i.e. RR-gauge fields.

conserved charges that act as sources for the various anti-symmetric gauge fields.

of type II superstring theories [8]. These theories have classical solutions which

Superstring constraints

In order to reduce the enormous field content of these superfields down to the

gauge fields one can derive field strengths with associated Bianchi identities.

identities the latter cease to be identities, but rather become equations the

consistency of which has to be examined separately. If the constraints are

properly chosen the equations so obtained are just the supergravity equations

2.3 D-p-branes in IIA supergravity backgrounds

2.3.1 Superspace constraints

D = 10 type II supergravity theories are the low-energy effective field theories

of type II superstring theories [3]. These theories have classical solutions which

describe extended objects called p-branes. The p-branes are solitons carrying

conserved charges that act as sources for the various anti-symmetric gauge fields

of the underlying supergravity theory, i.e. RR-gauge fields \(C^{(r)}\) and the NS-NS

2-form potential \(B\).

In superspace all ordinary components of RR and NS-NS gauge fields are

introduced as first components of their corresponding superfields. From the
gauge fields one can derive field strengths with associated Bianchi identities.

In order to reduce the enormous field content of these superfields down to the

on-shell content one introduces constraints on some of the components of the

superfield field strengths. When these constraints are inserted into the Bianchi

identities the latter cease to be identities, but rather become equations the

consistency of which has to be examined separately. If the constraints are

properly chosen the equations so obtained are just the supergravity equations

of motion.
D-branes arise from prescribing mixed (Neumann- and Dirichlet) boundary conditions on open strings in type II string theory. They are introduced as \((p+1)\)-dimensional hypersurfaces in spacetime where open strings are constrained to end on, but the ends are free to move on this submanifold. A spacelike section of a D-brane can be given a finite volume in a spacetime with compact dimensions by wrapping around topologically non-trivial cycles in the spacetime. In this case the supertranslation algebra of Noether charges or modified charges carried by the brane is extended by topological charges, which we derive below.

We want to consider D-branes in a flat IIA background. This condition requires the underlying supergravity theory to be massless, \(m = 0\), since it is known that \(D = 10\) Minkowski spacetime is not a solution to the field equations of massive IIA supergravity \([9]\). The massless theory allows a flat solution, however; its constraints, i.e. the constraints on the massless IIA supergravity background, can be obtained by dimensional reduction of the standard \(D = 11\) superspace constraints \([10]\); in particular, they imply the field equations of massless IIA supergravity. Moreover, we have the observation that, once the constraints on the NS-NS fields coupling to the kinetic (supersymmetry invariant) term in the D-brane action are given, the constraints on the RR-fields coupling to the brane via the Wess-Zumino term can be read off from \(\kappa\)-symmetry, see \([11]\); thus, consistent propagation of \(D\)-branes demands a background solving the equations of motion of the appropriate supergravity theory.

### 2.3.2 Superspace background and Bianchi identities

In the following we choose a massless flat background vacuum with Dilaton \(\phi = 0\), Dilatino \(D\phi = 0\), where \(D\) denotes a supercovariant derivative. Moreover, we assume that all bosonic components of the field strengths associated with the NS-NS fields and RR gauge fields, respectively, are zero; the non-bosonic components of these field strengths as well as the non-bosonic torsion components are uniquely determined by the superspace constraints, see \([7]\). The RR superfield potentials are usually collected in a formal sum \(C = \sum_{r=0}^{10} C^{(r)}\), where the ordinary RR gauge potentials are just the leading components of the \(C^{(r)}\) in a \(\theta\)-expansion; for the IIA case only the odd forms are relevant. Their field strengths are defined in \((156)\). The Bianchi identities associated with these field strengths are given in \((157)\). The Bianchi identities for the field strength \(H\) of the NS-NS field \(B\) is \(dH = 0\).

Let us now define a family of superspace forms \(K^{(p+2)}(S)\) by

\[
K^{(p+2)}(S) = \frac{i}{p!} \Gamma^{m_1} \cdots \Gamma^{m_{p+1}} \cdot d\bar{\theta} S \Gamma_{m_1 \cdots m_p} d\theta ,
\]

where \(p = 0, \ldots, 9\), \(S \in \{1_{32}, \Gamma_{11}\}\), and \(\Gamma_{m_1 \cdots m_p}\) is the usual antisymmetrised product of \(\Gamma\)-matrices. Then our choice of vacuum determines the field strengths.
to be \[\]
\begin{align*}
R^{(2)} &= K^{(2)} (\Gamma_{11}) , \\
R^{(4)} &= K^{(4)} (1) , \\
R^{(6)} &= K^{(6)} (\Gamma_{11}) , \\
R^{(8)} &= K^{(8)} (1) , \\
R^{(10)} &= K^{(10)} (\Gamma_{11}) ;
\end{align*}
Furthermore, the field strength \(H\) must take the form
\[H = -K^{(3)} (\Gamma_{11}) .\]
These field strengths are determined by superspace constraints; the Bianchi identities \([57]\) and the relation \(dH = 0\) are therefore identities no longer, and we must check whether they are actually satisfied.

2.3.3 Explicit form of \(B\)
Let us first consider the field strength \(H\) in \([197]\); the 3-form \(K^{(3)} (\Gamma_{11})\) has a potential
\[B = \left( -\Pi^m + \frac{i}{2} d\bar{\theta}\Gamma^m \theta \right) \cdot (i d\bar{\theta}\Gamma_{11} \Gamma_m \theta) ,\]
which yields \(H = dB = -K^{(3)} (\Gamma_{11})\) on account of the identity
\[d\bar{\theta}\Gamma^n \theta \cdot d\bar{\theta}\Gamma_{11} \Gamma_n d\theta + d\bar{\theta}\Gamma_{11} \Gamma_n \theta \cdot d\bar{\theta}\Gamma^n d\theta = 0 ;\]
this in turn is a consequence of the identity
\[\Gamma_{(\alpha\beta} (\Gamma_{11} \Gamma_n)_{\gamma\delta)} = 0 ,\]
which is known to hold in \(D = 10\). Therefore \([197]\) actually is a consistent choice for \(H\). Since \(H = -K^{(3)} (\Gamma_{11})\) is indeed supertranslation invariant we have \(\delta_\alpha B = d\Delta_\alpha\) for some 1-form \(\Delta_\alpha\); this can be computed to be
\[\Delta_\alpha = dX^m \cdot (i \Gamma_{11} \Gamma_m \theta)_{\alpha} - \frac{1}{6} [d\bar{\theta}\Gamma_m \theta \cdot (\bar{\theta}\Gamma_{11} \Gamma_m)_{\alpha} + d\bar{\theta}\Gamma_{11} \Gamma_m \theta \cdot (\bar{\theta}\Gamma_m)_{\alpha}] .\]
Moreover we note that
\[\delta_\alpha \Delta_\beta + \delta_\beta \Delta_\alpha = dX^m \cdot (2i \Gamma_{11} \Gamma_m)_{\alpha\beta} +
\]
\[+ \frac{1}{2} \cdot d \left[ (\Gamma_{11} \Gamma_m \theta)_{\alpha} \cdot (\Gamma^n \theta)_{\beta} + (\Gamma_{11} \Gamma_m \theta)_{\beta} \cdot (\Gamma^n \theta)_{\alpha} \right] .\]
Taking the trace with \(\Gamma^n_{\alpha\beta}\) of this expression yields zero: due to the tracelessness of products of \(\Gamma\)-matrices the first contribution vanishes, and the terms in the square bracket yield zero since
\[\bar{\theta}\Gamma_{11} \Gamma_m \Gamma_n \Gamma_m \theta = (2 - D) \bar{\theta}\Gamma_{11} \Gamma_n \theta = 0 ,\]
for \(C \Gamma_{11} \Gamma_n\) is symmetric, see Table 2. By \([50]\) this implies that
\[\Delta_m = 0 ,\]
\[ p = 2q \]

|   | 0 | 2 | 4 | 6 | 8 |
|---|---|---|---|---|---|
| \( S \) | \( \Gamma_{11} \) | \( \Gamma_{132} \) | \( \Gamma_{11} \) | \( \Gamma_{132} \) | \( \Gamma_{11} \) |

Table 1: Relation between \( p \) and \( S \).

as can be seen directly from (198), since \( B \) is translation invariant. From definitions (172, 181, 183) we now see that

\[
S_{\alpha\beta} (\Delta) = -dX^m \cdot (2i\Gamma_{11}\Gamma_m)_{\alpha\beta} + \cdots ,
\]

(204)

\[
S_{\alpha m} (\Delta) = S_{mn} (\Delta) = 0 ,
\]

(205)

where \( \cdots \) denotes terms that involve only fermionic 1-forms. Finally, note that \( d\Delta_m = 0 \).

2.3.4 Generalized \( \Gamma \)-matrix identities

Now let us turn attention to the RR field strengths. If we insert (196) into the Bianchi identities (157) we obtain

\[
dK^{(2)} (\Gamma_{11}) = 0 ,
\]

(206)

\[
dK^{(4)} (1) + K^{(2)} (\Gamma_{11}) K^{(3)} (\Gamma_{11}) = 0 ,
\]

\[
dK^{(6)} (\Gamma_{11}) + K^{(4)} (1) K^{(3)} (\Gamma_{11}) = 0 ,
\]

\[
dK^{(8)} (1) + K^{(6)} (\Gamma_{11}) K^{(3)} (\Gamma_{11}) = 0 ,
\]

\[
dK^{(10)} (\Gamma_{11}) + K^{(8)} (1) K^{(3)} (\Gamma_{11}) = 0 .
\]

(207)

Here (206) is trivially satisfied due to \( d(id\bar{\theta}\Gamma_{11}d\theta) = 0 \). The equations in (207) can be written as

\[
dK^{(2q+2)} (S) + K^{(2q)} (ST_{11}) K^{(3)} (\Gamma_{11}) = 0 ; \quad q = 1, 2, 3, 4 ,
\]

(208)

and \( p = 2q \) is related to \( S \) by Table 1. If we now use the explicit definitions of \( K^{(2q)} (S) \) as given in (195) we find that equations (207) are satisfied iff

\[
\Gamma^n_{(\alpha\beta} (ST_{nm1\ldots m2q-1})_{\gamma\delta)} + (2q - 1) \cdot (\Gamma_{11}\Gamma_{[m1})_{(\alpha\beta} (ST_{11}\Gamma_{m2\ldots m2q-1]}_{\gamma\delta)} = 0 .
\]

(209)

In the first term the symmetrisation involves spinor indices \( \alpha, \beta, \gamma, \delta \), but no covector indices \( n, m_1, \ldots, m_{2q-1} \), of course. In the second term we have a symmetrisation over \( \alpha, \beta, \gamma, \delta \), and independently, an antisymmetrisation over \( m_1, \ldots, m_{2q-1} \). (209) is a set of generalized \( \Gamma \)-matrix identities. We shall derive a necessary condition for them to hold, and show, that it is indeed satisfied. Before we do so, however, let us examine the special case of (209) when \( q = 1 \). In this case (209) becomes (see table 1 for the choice of \( S \))

\[
\Gamma^n_{(\alpha\beta} (\Gamma_{nm})_{\gamma\delta)} + (\Gamma_{11})_{(\alpha\beta} (\Gamma_{11}\Gamma_m)_{\gamma\delta)} = 0 .
\]

(210)

This is just the dimensional reduction to \( D = 10 \) of the \( D = 11 \) identity required for \( \kappa \)-symmetry of the \( D = 11 \) supermembrane \([10]\), and is known to hold in
where we have used the fact that if \( \Gamma_{\alpha\beta} \) is always antisymmetric, see Table 2. The last three contributions to (212) are:

\[
\begin{align*}
D = 11. & & \text{This means that the validity of at least the first equation in (207) is assured.}
\end{align*}
\]

To examine the validity of the other cases we reexpress (209) as

\[
\Gamma_{\alpha\beta}^{\gamma} (S \Gamma_{nm1...m_{2q-1}}^{\gamma})_{\gamma\delta} + (\Gamma_{11}\Gamma_{m1})_{\alpha\beta} (S \Gamma_{11}\Gamma_{m_{2}...m_{2q-1}}^{\gamma})_{\gamma\delta} + \\
+ \text{(cyc. } m_{1} \rightarrow m_{2} \rightarrow \cdots) + \cdots = 0 ,
\]

(211)

where "cyc." denotes a sum over all cyclic permutations of \( m_{i} \)-indices in the second term of the first line. Now we multiply (211) by \( \Gamma_{\alpha\beta}^{\gamma} \); this yields

\[
\begin{align*}
\text{tr} (\Gamma_{\alpha\beta}^{\gamma}) \cdot (C S \Gamma_{nm1...m_{2q-1}}^{\gamma}) + (C T_{\alpha\beta}^{\gamma}) \cdot \text{tr} (\Gamma_{1} S \Gamma_{nm1...m_{2q-1}}^{\gamma}) + \\
+ 4 (C T_{\alpha\beta}^{\gamma} \Gamma_{1} S \Gamma_{nm1...m_{2q-1}}^{\gamma})_{\text{sym}} + \\
+ \{ \text{tr} (\Gamma_{11}\Gamma_{m1}\Gamma_{1}) \cdot (C S \Gamma_{11}\Gamma_{m_{2}...m_{2q-1}}^{\gamma}) + (C T_{\alpha\beta}^{\gamma}) \cdot \text{tr} (\Gamma_{1} S \Gamma_{11}\Gamma_{m_{2}...m_{2q-1}}^{\gamma}) + \\
+ 4 (C T_{\alpha\beta}^{\gamma} \Gamma_{1} S \Gamma_{11}\Gamma_{m_{2}...m_{2q-1}}^{\gamma})_{\text{sym}} + (\text{cyc. } m_{1} \rightarrow m_{2} \rightarrow \cdots) + \cdots \} = 0 .
\end{align*}
\]

(212)

Here \((\text{sym})\) denotes the symmetric part of the matrix in brackets, i.e. \( M_{\text{sym}} = \frac{1}{2} (M + M^{T}) \). We list the contributions to (212):

\[
\begin{align*}
\text{tr} (\Gamma_{\alpha\beta}^{\gamma}) \cdot (C S \Gamma_{nm1...m_{2q-1}}^{\gamma}) = \text{tr} (132) \cdot (C S \Gamma_{lm1...m_{2q-1}}^{\gamma}) , \\
\text{tr} (\Gamma_{1} S \Gamma_{nm1...m_{2q-1}}^{\gamma}) = 0 \quad \text{for all } q = 1, \ldots, 4; S = S(q) , \text{ see Table (1)} , \\
(C T_{\alpha\beta}^{\gamma} \Gamma_{1} S \Gamma_{nm1...m_{2q-1}}^{\gamma})_{\text{sym}} = - (D - 2q - 1) \cdot (C S \Gamma_{lm1...m_{2q-1}}^{\gamma}) ,
\end{align*}
\]

(213)

where we have used the fact that if \((C S \Gamma_{lm1...m_{2q-1}}^{\gamma})\) is symmetric then \((C S \Gamma_{m_{2}...m_{2q-1}}^{\gamma})\) is always antisymmetric, see Table 2. The last three contributions to (213) are

\[
\text{tr} (\Gamma_{11}\Gamma_{m1}\Gamma_{1}) = 0 ,
\]

| \( p \) | \( S = 1_{32} \) | type | \( S = \Gamma_{11} \) | type |
|---|---|---|---|---|
| 0 | \( C \) | - | \( C \Gamma_{11} \) | + |
| 1 | \( C \Gamma_{m1} \) | + | \( C \Gamma_{11}\Gamma_{m1} \) | + |
| 2 | \( C \Gamma_{m1m_{2}} \) | + | \( C \Gamma_{11}\Gamma_{m1m_{2}} \) | - |
| 3 | \( C \Gamma_{m1,m_{3}} \) | - | \( C \Gamma_{11}\Gamma_{m1,m_{3}} \) | - |
| 4 | \( C \Gamma_{m1,m_{4}} \) | - | \( C \Gamma_{11}\Gamma_{m1,m_{4}} \) | + |
| 5 | \( C \Gamma_{m1,m_{5}} \) | + | \( C \Gamma_{11}\Gamma_{m1,m_{5}} \) | + |
| 6 | \( C \Gamma_{m1,m_{6}} \) | + | \( C \Gamma_{11}\Gamma_{m1,m_{6}} \) | - |
| 7 | \( C \Gamma_{m1,m_{7}} \) | - | \( C \Gamma_{11}\Gamma_{m1,m_{7}} \) | - |
| 8 | \( C \Gamma_{m1,m_{8}} \) | - | \( C \Gamma_{11}\Gamma_{m1,m_{8}} \) | + |
| 9 | \( C \Gamma_{m1,m_{9}} \) | + | \( C \Gamma_{11}\Gamma_{m1,m_{9}} \) | + |
| 10 | \( C \Gamma_{m1,m_{10}} \) | + | \( C \Gamma_{11}\Gamma_{m1,m_{10}} \) | - |
\[ tr \left( \Gamma_i \Sigma_{11} \Gamma_{m_2 \ldots m_{2q-1}} \right) = 0 \]
\[ (C \Sigma_{m_1} \Gamma_i \Gamma_{m_2 \ldots m_{2q-1}})_{\text{sym}} = - (C \Sigma_{l m_1 \ldots m_{2q-1}}) - (2q-1)(2q-3) \cdot \eta_{m_1[m_2} \cdot \eta_{||m_3} \left( C \Sigma_{m_4 \ldots m_{2q-1}} \right) . \quad (214) \]

Now we must perform the cyclic sum (cycl. \( m_1 \to m_2 \to \cdots \)) in (214). Since this is equal to \((2q-1) \times \) "antisymmetrisation of \((214)\) over \((m_1, \ldots, m_{2q-1})\" we see that the second contribution on the right hand side of (214) must vanish, since it involves antisymmetrisation over \(\eta_{m_1 m_2}\), and therefore the total contribution from this term is
\[ (2q-1) \cdot \left( C \Sigma_{[m_1} \Gamma_{||m_2 \ldots m_{2q-1}} \right)_{\text{sym}} = - (2q-1) \cdot \left( C \Sigma_{l m_1 \ldots m_{2q-1}} \right) . \quad (215) \]

Altogether, (212) leads to the condition
\[ [tr (1_{32}) - 4(D-2q-1) - 4(2q-1)] \cdot \left( C \Sigma_{l m_1 \ldots m_{2q-1}} \right) = 0 \quad ; \quad (216) \]

remarkably, the contributions involving \(q\) cancel each other in this equation, so we arrive at
\[ tr (1_{32}) - 4(D-2) = 0 \quad (217) \]
as a necessary condition for the \(\Gamma\)-matrix identities (209) to hold; but this is satisfied precisely in \(D=10\), independent of \(q\).

We do not know whether (217) is also sufficient to ensure (209); in the past, sufficiency of a similar condition to (217) to establish the well-known \(\Gamma\)-matrix identity \(\Gamma_{(a \delta)} (\Gamma_{\alpha \gamma})_{\gamma \delta} = 0\) in \(D=10\) could be established only via computer \([4]\). In the following we shall assume that (217) is sufficient and therefore (209) holds for all allowed values of \(q\); if this assumption should turn out to be wrong, then at least our analysis is valid for \(q=1\), since in this case the validity of (210) is known; our results then would be restricted to the D-2-brane in a IIA superspace.

2.3.5 Constructing the leading terms of \(C^{(r)}\)

Provided that (209) is valid we show that under these circumstances we can construct the potentials \(C^{(3)}, C^{(5)}, C^{(7)}, C^{(9)}\) recursively from \(C^{(1)}\). From (195, 196) we see that, up to a gauge transformation, we have
\[ C^{(1)} = i d \theta \Gamma_{11} \theta . \quad (218) \]

Now assuming that we have constructed \(C^{(2q-1)}\) we can use (198) to give
\[ dC^{(2q+1)} = K^{(2q+2)} (S) - C^{(2q-1)} K^{(3)} (\Gamma_{11}) , \quad (219) \]

where \(S\) is chosen according to Table [4]. A necessary and sufficient condition for the existence of a (local) \((2q+1)\)-form \(C^{(2q+1)}\) that satisfies (219) is that the differential of the right hand side of (213) vanishes; but since \(dC^{(2q-1)} = K^{(2q)} (S_{\Gamma_{11}}) - C^{(2q-3)} K^{(3)} (\Gamma_{11})\) by assumption, this is
\[ d \left[ K^{(2q+2)} (S) - C^{(2q-1)} K^{(3)} (\Gamma_{11}) \right] = dK^{(2q+2)} (S) + K^{(2q)} (S_{\Gamma_{11}}) K^{(3)} (\Gamma_{11}) , \quad (220) \]
where we have used the fact that \( H = -K^{(3)}(\Gamma_{11}) \) is a closed 3-form. But the right hand side of (221) are just the Bianchi identities (208), which are identically zero provided that (209) holds; the Bianchi identities are therefore integrability conditions for the forms \( C^{(2q+1)} \) in (219). The existence of \( C^{(r)} \) is therefore guaranteed at least for \( r = 1, 3 \).

We have solved (219) for \( C^{(3)} \) explicitly; the result is

\[
C^{(3)} = i\frac{1}{2} \Pi^m \Pi^n \cdot d\bar{\theta} \Gamma_{nm} \theta + \\
+ \frac{1}{2} \Pi^m \cdot [d\bar{\theta} \Gamma_n \theta \cdot d\bar{\theta} \Gamma_{nm} \theta - d\bar{\theta} \Gamma_{11} \theta \cdot d\bar{\theta} \Gamma_{11} \Gamma_m \theta] + \\
+ \frac{i}{6} d\bar{\theta} \Gamma_m \theta \cdot [d\bar{\theta} \Gamma_{11} \theta \cdot d\bar{\theta} \Gamma_{11} \Gamma_m \theta - d\bar{\theta} \Gamma^n \theta \cdot d\bar{\theta} \Gamma_{nm} \theta].
\] (221)

In proving that (221) is actually a solution to (219) for \( q = 1 \) one has to make use of the identities

\[
(Id 1) := d\bar{\theta} \Gamma^n \cdot d\bar{\theta} \Gamma_{11} \Gamma_n \theta + d\bar{\theta} \Gamma^n \theta \cdot d\bar{\theta} \Gamma_{11} \Gamma_n d\theta = 0
\] ,

(222)

and

\[
(Id 2)_m := d\bar{\theta} \Gamma^n \cdot d\bar{\theta} \Gamma_{nm} \theta + d\bar{\theta} \Gamma^n \theta \cdot d\bar{\theta} \Gamma_{nm} d\theta + \\
+ d\bar{\theta} \Gamma_{11} \cdot d\bar{\theta} \Gamma_{11} \Gamma_n \theta + d\bar{\theta} \Gamma_{11} \Gamma_n d\theta = 0
\] ,

(223)

where (222) is a consequence of (200), and (223) follows from (210). Then

\[
dC^{(3)} = K^{(4)}(1_{32}) - C^{(1)} K^{(3)} (\Gamma_{11}) + \\
+ \left( \frac{1}{2} \Pi^m - i\frac{1}{6} d\bar{\theta} \Gamma^m \theta \right) \cdot (Id 2)_m - \left( i\frac{3}{3} d\bar{\theta} \Gamma_{11} \theta \right) \cdot (Id 1)
\]

and (219) is fulfilled.

In principle we could apply the same procedure to construct the other potentials \( C^{(5)}, C^{(7)}, C^{(9)} \). But for the purpose we are pursuing here, namely the determination of the topological extensions of Noether algebras, we do not need to know the full expression for \( C^{(2q+1)} \); as mentioned earlier, these algebra extensions come into play when the D-\( p \)-brane wraps around compact dimensions in the spacetime; but the topology of this configuration is entirely determined by the bosonic coordinates \( X \) on the superspace, and the pull-back of the differentials \( dX^m \) to the worldvolume of the brane, respectively. In evaluating the anomalous contributions to the charge algebra as far as they origin in the WZ-term we therefore can restrict attention to those components of the \( C \)’s which have only bosonic indices. The strategy is as follows:

From section 2.2.4 we see that all we need are the components of the forms \( S_{MN}(U) \), \( M = (m, \alpha) \), carrying the maximum number of bosonic indices; since we shall work with the LI-basis now, this means that we need only consider terms involving the maximum number of bosonic basis-1-forms \( \Pi^m \); in the following we shall refer to such terms simply as “leading terms”; furthermore we shall call the number of bosonic indices in the leading term as the ”order” of the term. From (192)-(194) we see that \( S_{MN}(U) \) is composed of terms \( \delta_M D_N \),

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$D_M$ and $D_M d\Delta_N$. Since $\delta_M$ leaves the number of LI-1-forms invariant we see that in order to construct the leading terms of $S_{MN}(U)$ we need only construct the leading terms of $D_M$. Now let us look back at formula (161),

$$\delta_{\alpha} C^{(2q+1)} = \sum_{k=0}^{q} dD_{\alpha}^{(2q-2k)} \cdot \frac{B^k}{k!} \cdot . \quad (224)$$

From our choice of $B$ in (198) we see that $B$ contains only one bosonic 1-form $\Pi^m$; the order of the terms in the sum in (224) therefore decreases by 1 as $k$ increases by 1; this means that in order to construct the leading term of $dD_{\alpha}^{(2q)}$ we need only construct the leading term in $\delta_{\alpha} C^{(2q+1)}$; but this can be done using (219) recursively:

$$dC^{(2q+1)} = K^{(2q+2)} (S) - C^{(2q-1)} K^{(3)} (\Gamma_{11}) \cdot . \quad (225)$$

From (195) we see that the order of $K^{(3)} (\Gamma_{11})$ is one, and that of $K^{(2q+2)} (S)$ is $(2q)$; from (218) and (221) we deduce that the order of $C^{(2q-1)}$ is $(2q-2)$, therefore the first term on the right hand side of (225) is the leading term, and we must construct a $C^{(2q+1)}$ such that

$$dC^{(2q+1)} = K^{(2q+2)} (S) + \ldots .$$

Thus we find the leading term of $C^{(2q+1)}$ to be

$$C^{(2q+1)} = \frac{i}{(2q)!} \Pi^{m_{2q}} \ldots \Pi^{m_{1}} \cdot \bar{\theta} S \Gamma_{m_{1} \ldots m_{2q}} \theta \cdot , \quad (226)$$

with $S$ given in Table [1]. Therefore the leading term of $dD_{\alpha}^{(2q)}$ is

$$dD_{\alpha}^{(2q)} = - \frac{i}{(2q)!} \Pi^{m_{2q}} \ldots \Pi^{m_{1}} \cdot (\bar{\theta} S \Gamma_{m_{1} \ldots m_{2q}})_{\alpha} \cdot , \quad (227)$$

and, up to a differential, we have

$$D_{\alpha}^{(2q)} = - \frac{i}{(2q)!} \Pi^{m_{2q}} \ldots \Pi^{m_{1}} \cdot (\bar{\theta} S \Gamma_{m_{1} \ldots m_{2q}})_{\alpha} . \quad (228)$$

This gives

$$\delta_{\alpha} D^{(2q)}_{\beta} + \delta_{\beta} D^{(2q)}_{\alpha} = \frac{-2i}{(2q)!} \Pi^{m_{2q}} \ldots \Pi^{m_{1}} \cdot (S \Gamma_{m_{1} \ldots m_{2q}})_{\alpha \beta} = \quad (229)$$

$$= \frac{-2i}{(2q)!} dX^{m_{1}} \ldots dX^{m_{2q}} \cdot (S \cdot (2q) \Gamma_{m_{2q} \ldots m_{1}})_{\alpha \beta} + \ldots .$$

multiplying (229) with $\Gamma_{n}^{\alpha \beta}$ then yields a vanishing result due to the vanishing of the trace

$$tr (S \Gamma_{m_{1} \ldots m_{2q}} \Gamma_{n}) = 0$$

for all allowed values of $q$, $n$ and $S$. But since the order of the leading term of $D_{\alpha}^{(2q-2)} \cdot d\Delta_{\beta}$ is $(2q - 1)$, and the order of $\delta_{\alpha} D^{(2q)}_{\beta}$ is $(2q)$, as can be seen from (201) and (228), we infer from (164) that indeed

$$D_{m}^{(0)} = 0 \quad , \quad D_{m}^{(2q)} = 0 \quad ; \quad (230)$$
these equations will actually hold in a rigorous sense, not only to leading order; from \([213]\) and \([221]\) we see that at least \(C(1)\) and \(C(3)\) are strictly translation invariant, and this will be true for the others as well, since the higher rank potentials are constructed recursively from the lower rank ones. Furthermore, from \([228]\) we infer that \(\delta_m D_\alpha(2q) = 0\) for all \(q\).

2.3.6 Extended superalgebras for D-2q-branes

Now we can turn to evaluating the expressions \(S_{MN}(U)\) as given in \([192]\)–\([194]\). Since expressions involving \(d\Delta_M\) have leading order smaller than the leading order of \(\delta_\alpha D_\beta^{(2q-2k)}\), see \([204, 205]\), they can be omitted in the discussion. The final expression for \(S_{\alpha\beta}(U)\) is therefore

\[
S_{\alpha\beta}(U) = -2i \sum_{k=0}^{q} \left[ S(2q - 2k) \Gamma_{m_{2q} - 2k \ldots m_1} \right]_{\alpha\beta} \frac{\epsilon^{\mu_1 \ldots \mu_{2q-2k} \nu_1 \ldots \nu_{2k}}}{((2q - 2k)!)^2 2^k \cdot k!} \cdot \partial_{\mu_1} X^{m_1} \ldots \partial_{\mu_{2q-2k}} X^{m_{2q-2k}} \cdot F_{\nu_1 \nu_2} \ldots F_{\nu_{2k-1} \nu_{2k}}.
\]

Let us now write \(dX^m := emb^* dX^m\) for the sake of convenience; then we have

\[
dX^{m_1} \ldots dX^{m_{2q-2k}} \cdot \frac{(dA)^k}{k!} = \omega_0 \frac{\epsilon^{\mu_1 \ldots \mu_{2q-2k} \nu_1 \ldots \nu_{2k}}}{(2q - 2k)! 2^k \cdot k!} \cdot \partial_{\mu_1} X^{m_1} \ldots \partial_{\mu_{2q-2k}} X^{m_{2q-2k}} \cdot F_{\nu_1 \nu_2} \ldots F_{\nu_{2k-1} \nu_{2k}},
\]

where \(\omega_0 = \sigma^1 \ldots \sigma^{2q}\); therefore

\[
\omega_0 S_{\alpha\beta}(U) = -2i \sum_{k=0}^{q} \left[ S(2q - 2k) \Gamma_{m_{2q} - 2k \ldots m_1} \right]_{\alpha\beta} dX^{m_1} \ldots dX^{m_{2q-2k}} \cdot \frac{(dA)^k}{k!}.
\]

Furthermore, from \([193]\) and \([194]\) we infer that both \(S_{\alpha m}(U)\) and \(S_{m\alpha}(U)\) are zero.

Now we can collect everything together to write down the general structure of the modified charge algebra; we assume that double integration is defined, so that we get

\[
\{Q_\alpha, Q_\beta\} = 2n^{\alpha\beta} \cdot P_m - 2i (\Gamma_{11 \Gamma_m})_{\alpha\beta} \cdot Y^m - 2i \sum_{k=0}^{q} \left[ S(2q - 2k) \Gamma_{m_{2q} - 2k \ldots m_1} \right]_{\alpha\beta} Z^{m_1 \ldots m_{2q-2k}},
\]

with

\[
Y^m = \frac{1}{(2q - 1)!} \int_{W(t)} \left( *L^{\text{gauge}} \right) dX^m,
\]

which was defined in \([178]\), and

\[
Z^{m_1 \ldots m_{2q-2k}} = \int_{W(t)} dX^{m_1} \ldots dX^{m_{2q-2k}} \cdot \frac{(dA)^k}{k!}.
\]
\[
\int \frac{d^{2q} \sigma}{(2q - 2)! 2^k k!} \partial_\mu_1 X^{m_1} \ldots \partial_{2q-2k} X^{m_{2q-2k}} \cdot F_{\nu_1 \nu_2} \ldots F_{\nu_{2k-1} \nu_{2k}}.
\]

Moreover, the relation between \(q, S(2q)\) is given in Table 1. Note that the integrand of the charge \(Y^m\) is closed on the physical trajectories, see (179).

At last, from (188) and (190) we learn that
\[
[Q_\alpha, P_m] = 0, \quad [P_m, P_n] = 0.
\]

To avoid confusion we emphasize that in (233) the bracket \(\{Q_\alpha, Q_\beta\}\) denotes a graded Poisson-bracket between two Grassmann-odd quantities, but in (237) we have chosen a square bracket to denote the Poisson bracket between quantities of which at least one of them is Grassmann-even.

2.4 Interpretation of the central charges

Let us try to interpret the structure of the charges \(Z^{m_1 \ldots m_{2q-2k}}\) in (235). Let us fix \(q\) and first of all look at the extreme values of \(k\), i.e. \(k = 0\) and \(k = q\).

For \(k = 0\) we find
\[
Z^{m_1 \ldots m_{2q}} = \int_{W(t)} dX^{m_1} \ldots dX^{m_{2q}};
\]

from (54) we see that this is just the integral over the topological current \(j_{m_1 \ldots m_{2q}}\), i.e. the topological charge
\[
Z^{m_1 \ldots m_{2q}} = T^{m_1 \ldots m_{2q}}
\]
from (53). This charge will not be defined if the brane \(W(t)\) is infinitely extended in one of the spatial directions \(X^m\) occuring in \(T^{m_1 \ldots m_{2q}}\). On the other hand, if all spacetime directions occurring in \(T^{m_1 \ldots m_{2q}}\) are compact, but the brane is not wrapped around all of them then this charge will be zero. It will be non-zero only if the brane wraps around all these compact dimensions; consider, for example, a compact \(U(1)\)-factor in the spacetime, which may be taken as direction \(m = 1\), and a closed string that is wrapped around this \(n\) times [I] (the string is not a IIA brane, of course, but that does not affect the discussion here); then \(T^1\) is proportional to \(2n\pi\), where \(n\) is an integer. On the other hand, if the string is closed in a flat spacetime, then \(T^1 = 0\).

For \(k = q\) we find that
\[
Z = \frac{1}{2q!} \int_{W(t)} (dA)^q.
\]

This can be given a simple interpretation in the case of \(q = 1, p = 2q\), i.e. the D-2-brane: In this case
\[
Z = \int_{W(t)} dA.
\]
is just the flux of the field strength $F$ of the gauge potential through the brane $W(t)$. The worldvolume is now to be regarded as a $U(1)$-bundle $P(W, U(1))$. If the section $W(t)$ is infinitely extended then $Z$ will vanish provided that the gauge field vanishes sufficiently fast at infinity, and the bundle is trivial. If, however, $W(t)$ describes a $S^2$, say, then we have the possibility that the gauge potential is no longer defined globally on $S^2$; if two gauge patches are necessary to cover $S^2$ then the flux integral \((241)\) yields
\[
Z = 4\pi g, \quad 2g \in \mathbb{Z},
\]
where $g$ now is the charge of a Dirac monopole of the gauge field sitting "in the centre of $S^2$", and the quantization condition $2g \in \mathbb{Z}$ comes from the requirement that the transition function between the two gauge patches be unique, see for example \([12]\). It is not clear to us whether this interpretation extends to all possible values of $q$; we might conjecture that the $U(1)$-bundle can always be non-trivial, in which case similar arguments apply to \((240)\), since then we must cover $W(t)$ by more than one gauge patch, which should yield analogous results.

As for the values $1 \leq k \leq q$ we see that the currents $dX^{m_1} \cdots dX^{m_{2q-2k}}$ in \((235)\) now probe whether the brane has subcycles of dimension $(2q - 2k)$ embedded in it that wrap around $(2q - 2k)$ compact dimensions of the spacetime. Only in this case the charges $Z^{m_1 \cdots m_{2q-2k}}$ will be non-vanishing. Furthermore we see that the $U(1)$-bundle defined by the gauge field must be non-trivial in order to having a non-vanishing charge. To see this we can choose a static gauge $\sigma^\mu = X^\mu$, then from \((236)\) we have that
\[
Z^{m_1 \cdots m_{2q-2k}} = \int_{W(t)} d^2q \sigma \epsilon^{m_1 \cdots m_{2q-2k}\nu_1 \cdots \nu_{2k}} \frac{(2q - 2k)! \cdot k!}{(2q - 2k)!} \cdot \partial_{\nu_1} A_{\nu_2} \cdots \partial_{\nu_{2k-1}} A_{\nu_{2k}}. \tag{243}
\]
(This static gauge will be allowed at least on a certain coordinate patch on the worldvolume; in this case we have to sum over contributions from the different patches). We see that similar considerations concerning the non-triviality of the $U(1)$-bundle should apply here. In particular, \((243)\) will vanish if the bundle is trivial, since in this case the gauge potential $A_\mu$ is globally defined, and then \((233)\) yields a surface term. A tentative interpretation of the charges \((235)\) therefore would be that they measure the coupling of compact spacetime dimensions the brane or some directions of the brane wrap around to non-trivial gauge field configurations on the brane.

We have not found an easy interpretation for $Y^m$; from its structure we see that the $dX^m$-factor together with the fact that $*A_{\text{gauge}}$ is closed on the physical trajectories will make this charge non-vanishing only when the brane contains a 1-cycle wrapping around a compact spacetime dimension, e.g. a $S^1$-factor. This charge then describes the coupling of the canonical gauge field momentum to this particular topological configuration.

We finally present the modified charge algebra in the case of the D-2-brane with worldvolume $\mathbb{R} \times S^2$, since this allows for an easy interpretation, as we have seen above:
\[
\{Q_\alpha, Q_\beta\} = 2 (CT^m)_{\alpha\beta} \cdot P_m - 2i (CT_{11} \Gamma_m)_{\alpha\beta} \cdot Y^m -
\]
\[
- i (\Gamma_{m_2 m_1})_{\alpha\beta} \cdot T^{m_1 m_2} - 2i (\Gamma_{11})_{\alpha\beta} \cdot 4\pi g ,
\]
where \(T^{m_1 m_2}\) probes the presence of compact dimensions in spacetime the brane wraps around, and \(g\) is the quantized charge of a possible Dirac monopole resulting from the gauge field.

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