ON THE STABILITY OF \( m \)-FOLD CICLES AND THE DYNAMICS OF GENERALIZED CURVE SHORTENING FLOWS

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Abstract. In this paper we study the (asymptotic and exponential) stability of the \( m \)-fold circle as a solution of the \( p \)-curve shortening flow (\( p \geq 1 \) an integer).

Key words: curve-shortening; symmetry; blow-up; exponential convergence.

AMS 2010 Mathematics subject classification: Primary 54C44; 35K55.

1. Introduction

Let

\[
x : \mathbb{S}^1 \times [0, T) \rightarrow \mathbb{R}^2
\]

be a family of smooth immersions of \( \mathbb{S}^1 \), the unit circle, into \( \mathbb{R}^2 \). In this paper we will say that \( x \) satisfies the \( p \)-curve shortening flow, \( p \geq 1 \), if \( x \) satisfies

\[
\frac{\partial x}{\partial t} = -\frac{1}{p} k^p N,
\]

where \( k \) is the curvature of the embedding and \( N \) is the normal vector pointing outwards the region bounded by \( x (\cdot, t) \).

Much is known about this family of flows. To select a few among many beautiful and fundamental works on the subject, we must mention the works of Gage and Hamilton ([7]), and of Ben Andrews ([2, 3]).

In this paper we will be concerned with the stability of \( m \)-fold circles as solutions to the \( p \)-curves shortening flow. As it is well known there are small perturbations of the 2-fold circle that do not behave asymptotically as a shrinking 2-fold circle. However, very recently, Wang in [11] showed the asymptotic stability of \( m \)-fold circles under certain small \( \frac{2\pi m}{n} \)-periodic perturbations as solutions to the curve shortening flow (i.e., for the case when \( p = 1 \)). In this note we will extend the work of Wang in two ways: we will show asymptotic stability results for the \( m \)-fold circle as a solution to the \( p \)-curve shortening flow for any positive integer, and we will provide sharp stabilization estimates for the curvature of solutions to the \( p \)-curve shortening flow that are appropriate small perturbations of an \( m \)-fold circle. Other interesting works, besides Wang’s, regarding stability of solutions to the curve shortening flow are the by now classical papers of Abresch and Langer ([1]), and of Epstein and Weinstein ([5]).
To study the stability of $m$-fold circles as solutions to (1) we will consider the Boundary Value Problem

\[
\begin{aligned}
\frac{\partial k}{\partial t} &= k^2 \left(p + k^p - 1 \right) \left( k^p - 2 + \frac{1}{p} \right) & \text{in } [0, \frac{2\pi}{m}] \times (0, T) \\
k(\theta, 0) &= \psi(\theta) & \text{on } [0, \frac{2\pi}{m}]
\end{aligned}
\]

with periodic boundary conditions, $\lambda > \sqrt{\frac{p+2}{p}}$, and $\psi$ a strictly positive function. As it is (2) has no immediate geometric interpretation. However, when $\lambda$ is an appropriate rational number, (2) is the evolution equation of the curvature of a curve being deformed via (1) under the assumption that the curve being deformed satisfies certain symmetries; more precisely, when $\lambda = \frac{n}{m}$, the study of equation (2) is equivalent to the study of (1) when the initial data is a perturbation of an $m$-fold circle under a $\frac{2\pi}{m}$-periodic perturbation.

The method we will use to prove our stability results was introduced in [4] (inspired by [9]) to study the blow-up behavior of certain nonlinear parabolic equations with periodic boundary conditions, but as the reader will notice, it can be also used to study the stability of certain blow-up profiles, and the regularity of solutions to (2) (as a byproduct of the method we will employ, it can be shown that, under certain conditions on the initial data, solutions to (2) are analytic). So we hope that the reader may find the method used in this paper of independent interest.

The organization of this paper is as follows: in Section 2 we present our main result, its application to the stability problem of $m$-fold circles, and its proof; in Section 3 we discuss the exponential stability of the constant steady solution of the normalized version of (2).

2. MAIN RESULT

Our results on the behavior of the $p$-curve shortening flow will follow as a consequence of a result on the behavior of solutions to (2), that we will promptly describe; but before we state our main result, let us set some definitions and notation. Given $f \in L^2 \left( [0, \frac{2\pi}{m}] \right)$, we write its Fourier expansion as,

\[
\sum_{n \in \mathbb{Z}} \hat{f}(n) e^{i\lambda n x},
\]

and define the family of seminorms,

\[
\|f\|_\beta = \max \left\{ \sup_{n \neq 0} |n|^\beta |\text{Re} \left( \hat{f}(n) \right)|, \sup_{n \neq 0} |n|^\beta |\text{Im} \left( \hat{f}(n) \right)| \right\}.
\]

As is customary, we define $C^l \left( [0, \frac{2\pi}{m}] \right)$ as the space of functions with continuous derivatives of order $l$, equipped with the norm

\[
\|f\|_{C^l \left( [0, \frac{2\pi}{m}] \right)} = \max_{j=0,1,2,\ldots,l} \sup_{\theta \in [0, \frac{2\pi}{m}]} \left| \frac{d^j f(\theta)}{d\theta^j} \right|.
\]

The main result of this paper is the following theorem:

**Theorem 2.1.** Let $\lambda > \sqrt{\frac{p+2}{p}}$. There exists a constant $c_{p,\lambda} > 0$ such that if

\[
\frac{\lambda}{2\pi} \int_0^{\frac{2\pi}{m}} \psi(\theta) \, d\theta \geq c_{p,\lambda} \|\psi\|_2,
\]

then...
then a solution to (3) with initial condition $\psi$ is analytic (in $\theta$) and satisfies
\[
\left\| k(t, \theta) - \hat{k}(0, \theta) \right\|_{C^l([0, 2\pi])} \leq E_{l, p, \lambda}(T - t)(\lambda^2 - \frac{p+2}{p})^{\frac{p}{p+1}},
\]
where $0 < T < \infty$ is the blow-up time of the solution to (3) with initial condition $\psi$.

Let us remark that the fact that a solution to (2) with initial condition $\psi > 0$ blows up in finite time is a consequence of the Maximum Principle for Parabolic Equations. Also, the reader should notice that if $c_{p, \lambda} > 0$ is large enough then there is no need to assume that $\psi$ is strictly positive, since it would be a consequence of (3): this is why even though we are assuming the positivity of $\psi$, this assumption does not appear explicitly in the statement of the theorem.

Next we give the promised geometric corollaries of our main result.

\subsection*{2.1. A stability result for $m$-fold circles.} Theorem 2.1 has as a corollary an asymptotic (nonlinear) stability result for small perturbations of $m$-fold circles. Indeed, if an $m$-fold circle is perturbed by a $\frac{2\pi m}{n}$-periodic function, then its curvature function satisfies equation (3) with $\lambda = \frac{n}{m}$, $n$ and $m$ mutually primes. Hence, if
\[
\frac{n}{m} > \sqrt{\frac{p+2}{p}},
\]
and if the perturbation of the $m$-fold circle by the $\frac{2\pi m}{n}$-periodic function is such that its curvature function satisfies the hypothesis of Theorem 2.1 then it will shrink asymptotically as an $m$-fold circle under the $p$-curve shortening flow.

Let us explain with more care. Adopting a similar notation as in [11], let $\gamma_m$ be the $m$-fold circle, i.e.,
\[
\gamma_m(\theta) = (\cos \theta, \sin \theta), \quad \theta \in [0, 2\pi m],
\]
and let $\varphi$ be a smooth function of the normal angle $(\theta)$ of $\gamma_m$ of period $\frac{2\pi m}{n}$ with $\frac{n}{m} > \sqrt{\frac{p+2}{p}}$. Consider the curve
\[
\gamma_{0\delta} = \gamma_m + \delta \varphi N,
\]
where $N$ is the outward unit normal to $\gamma_m$. It is assumed that $|\delta| > 0$ is small enough so that the curvature of $\gamma_{0\delta}$ is strictly positive. Then we have the following result:

**Theorem 2.2.** If $|\delta| > 0$ is small enough, then the solution to the $p$-curve shortening flow shrinks to a point asymptotically like an $m$-fold circle.

How small is $|\delta|$ in the previous theorem is dictated by the fact that the curvature of $\gamma_{0\delta}$ must satisfy the hypothesis of Theorem 2.1. The proof of this result is immediate from Theorem 2.1 and it extends the results of Wang ([11]), at least in the case when $\frac{n}{m} > \sqrt{\frac{p+2}{p}}$. Notice also that the condition $\frac{n}{m} > \sqrt{\frac{p+2}{p}}$ translates to $\frac{m}{n} > \sqrt{3}$ in the case of the curve shortening flow which is obviously weaker than the Abresch-Langer condition $\frac{n}{m} > \sqrt{2}$ (which is covered in Wang’s work).

We can be more quantitative in describing how a solution to the normalized $p$-curve shortening flow with initial condition $\gamma_{0\delta}$ approaches the solution given by...
an $m$-fold circle. To this end we consider the following normalized version of (2)

\[
\begin{align*}
\frac{\tilde{\theta}}{\partial \tau} &= \tilde{p} \tilde{k} + \frac{\tilde{p}^2}{\tilde{p}^2 - \tilde{p}} \frac{\tilde{p}^2}{\tilde{p}^2 - \tilde{p}} \\
\tilde{k}(\cdot, 0) &= \left(\frac{pT}{p+1}\right)^{1/p+1} \psi,
\end{align*}
\]

with periodic boundary conditions. This normalized equation is obtained from (2) by the rescaling and change of time parameter given by

\[
\tilde{k}(\theta, t) = \left(\frac{p}{p+1}\right)^{1/p+1} (T-t)^{1/p+1} k(\theta, t), \quad \tau = -\frac{1}{p+1} \log \left(1 - \frac{t}{T}\right).
\]

So we have:

**Corollary 2.1.** Let $\lambda = \frac{n}{m}$, $n$ and $m$ relatively primes, and $\lambda > \sqrt{\frac{p+2}{p}}$. Let $\gamma_{0\delta}$ be a $\frac{2\pi}{\lambda}$-periodic perturbation of the $m$-fold circle. There exists a $c_{p, \lambda} > 0$ such that if $|\delta|$ is small enough so that the curvature $\psi$ of $\gamma_{0\delta}$ satisfies

\[
\lambda \frac{2\pi}{\lambda} \int_0^{\frac{2\pi}{\lambda}} \psi(\theta) d\theta = \frac{\lambda}{2\pi} \int_0^{\frac{2\pi}{\lambda}} \psi(\theta) d\theta \geq c_{p, \lambda} \|\psi\|_2,
\]

then the curvature $\tilde{k}(\cdot, \tau)$ of the solution to the normalized $p$-curve shortening flow with initial condition $\left(\frac{p+1}{p}\right)^{1/p+1} \gamma_{0\delta}$ satisfies

\[
\left\|\tilde{k}(\theta, \tau) - \lambda \frac{2\pi}{\lambda} \int_0^{\frac{2\pi}{\lambda}} \tilde{k}(\theta, \tau) d\theta\right\|_{C^1[0, \frac{2\pi}{\lambda}]} \leq A_{t, p, \lambda} \exp \left(-\beta(p, l) \tau\right),
\]

where

\[
\beta(p, \lambda) = \left(\lambda^2 - \frac{p+2}{p}\right) p + 1 = \lambda^2 p - p - 1,
\]

and $A_{t, p, \lambda} > 0$ is a constant, which only depends on the initial condition, and also on $p$ and $\lambda$. Furthermore, the deformation towards a circle is through analytic curves.

The stabilization rate given by the previous corollary is optimal in the sense that

\[
\beta(\lambda, p) = \left(\lambda^2 - \frac{p+2}{p}\right) p + 1 = \lambda^2 p - p - 1,
\]

is the smallest eigenvalue of the linearization around $w \equiv 1$ of the nonlinear operator

\[
p^1 \tilde{k} + \frac{p^2}{\tilde{p}^2} + p^2 + \frac{1}{\tilde{p}} - pu, \quad \theta \in \left[0, \frac{2\pi}{\lambda}\right],
\]

which corresponds to the elliptic part of the parabolic equation obtained by normalizing (2) as is described in the introduction, so we cannot expect better stabilization rates for the derivatives of $k$. In Section 4 the last one of this paper, we also show that in the case of the normalized flow $k \to 1$ at the rate predicted by the linearization of (1): standard methods predict that this rate is at least $e^{-\beta(\lambda, p) \epsilon \tau}$, for any $\epsilon > 0$: the nonobvious part is then to dispose of the $\epsilon$. The reader is advised to consult the interesting work of Wiegner on the subject (see [12]).
2.2. Proof of Theorem 2.1. The proof of Theorem 2.1 will be given through a series of lemmas, but the strategy we will follow can be described very succinctly: we shall show that the Fourier coefficients of the solution to (2) minus its average decay uniformly to 0 in time at the appropriate rate, when the blow-up time is approached. Our main tool is the analysis of the infinite dimensional ODE system satisfied by these Fourier coefficients.

To start with the proof of Theorem 2.1, let us introduce some notation to make our writing a bit easier.

\[ \hat{u}^m(q_1, q_2, \ldots, q_m, t) = \hat{u}(q_1, t) \hat{u}(q_2, t) \ldots \hat{u}(q_m, t), \]

\[ H(p, q_1, q_2) = \frac{1}{p} - (p - 1) \lambda^2 q_1 q_2 - \lambda^2 q_1^2, \]

and

\[ q = (q_1, q_2, \ldots, q_{p+1}, q_{p+2}). \]

Let \( \mathcal{Z} \) be a finite subset of the integers which contains 0 (i.e., \( 0 \in \mathcal{Z} \)), and which is symmetric around 0 (i.e., if \( n \in \mathcal{Z} \) then \(-n \in \mathcal{Z}\)).

Let \( B_n = \{(b_1, \ldots, b_{p+2}) \in \mathbb{Z}^{p+2} : b_{p+2} = n - b_1 - b_2 - \cdots - b_{p+1}\} \), and define the set \( \mathcal{A}_n = \{b \in B_n : \text{there are } 1 \leq i < j \leq p+2 \text{ such that } b_i \neq 0 \text{ and } b_j \neq 0\} \).

Consider the following finite dimensional approximation of (2) (obtained from (2) after formally taking Fourier transform and then restricting the infinite dimensional ODE system only to those Fourier wave numbers contained in \( \mathcal{Z} \))

(8)

\[ \begin{cases} \frac{d}{dt} \hat{k}_Z(0, t) = \frac{1}{p} \hat{k}_Z(0, t)^{p+2} + \sum_{q \in \mathcal{A}_0 \cap \mathbb{Z}^{p+2}} H(p, q_1, q_2) \hat{k}_Z^{(p+2)}(q, t), \\ \frac{d}{dt} \hat{k}_Z(n, t) = \left( \frac{p+2}{p} - \lambda^2 n^2 \right) \hat{k}_Z(0, t)^{p+1} \hat{k}_Z(n, t) + \sum_{q \in \mathcal{A}_n \cap \mathbb{Z}^{p+2}} H(p, q_1, q_2) \hat{k}_Z^{(p+2)}(q, t), \text{ if } n \neq 0, n \in \mathcal{Z}, \end{cases} \]

with initial condition

(9) \[ \hat{\psi}_Z(n) = \hat{\psi}(n), \quad \text{if } n \in \mathcal{Z}. \]

Notice, and this will be important but not explicitly mentioned in our arguments, that the symmetry of \( \mathcal{Z} \) guarantees that

\[ \sum_{q \in \mathcal{A}_0 \cap \mathbb{Z}^{p+2}} H(p, q_1, q_2) \hat{k}_Z^{(p+2)}(q_1, \ldots, q_{p+2}, t) \]

is real valued.

Our first lemma gives an interesting estimate on the behavior of solutions to (8).

Lemma 2.1 (Trapping Lemma). There exists a constant \( c_{p, \lambda} > 0 \) independent of the choice of \( \mathcal{Z} \) such that if the initial datum \( \psi \) satisfies (3) then there exist a \( \gamma > 0 \) that depends on \( \psi \) such that the solution to (8) satisfies

\[ \left| \hat{k}_Z(n, t) \right| \leq \frac{c_{p, \lambda} \hat{\psi}(0) e^{-\gamma |n| t}}{|n|^2} \quad n \neq 0. \]
Proof. Following [4], first we fix a set $Z$ as described above. Define the function
\[ \hat{v}(n,t) := \hat{\varphi}(n,t) \hat{k}_Z(n,t), \]
where,
\[ \hat{\varphi}(n,t) = e^{\gamma |n|^t}, \]
with $\gamma$ small, and we let
\[ \Phi(q_1,\ldots,q_{p+2},t) = \frac{\hat{\varphi}(q_1 + \cdots + q_{p+2},t)}{\varphi(q_1,t) \cdots \varphi(q_{p+2},t)}. \]

Then we obtain the following system of ODEs,
\[
\begin{align*}
\frac{d}{dt} \hat{v}(0,t) & = \frac{1}{p} \hat{v}(0,t)^{p+2} + \\
& \quad + \sum_{q \in A_0 \cap Z^{p+2}} H(p,q_1,q_2) \Phi(q_1,t) \hat{v}(q_1,t) \hat{v}(q_2,t) \Phi(q_2,t),
\end{align*}
\]
(10)
\[
\frac{d}{dt} \hat{v}(n,t) = \left( \frac{p+2}{p} \right) \hat{v}(0,t)^{p+1} \left( |n| - \lambda^2 n^2 \right) \hat{v}(n,t) + \\
& \quad + \sum_{q \in A_n \cap Z^{p+2}} H(p,q_1,q_2) \Phi(q_1,t) \hat{v}(q_1,t) \hat{v}(q_2,t) \Phi(q_2,t),
\]
where $q = (q_1,\ldots,q_{p+2})$.

Now consider the set $\Omega_Z$ defined by,
\[ \Omega_Z = \left\{ w \in \mathbb{C}^Z : w(0) \geq c_{p,\lambda} \max_{n \in Z} \left\{ n^2 |Re(w(n))|, n^2 |Im(w(n))| \right\} \right\}. \]

To prove the lemma we must show that if $\psi_Z = \left( \hat{\psi}_Z(n) \right)_{n \in Z}$ belongs to $\Omega_Z$, so does
\[ v(\cdot,t) = \left( \hat{v}(n,t) \right)_{n \in Z}, \]
the solution to the ODE system (10) with initial condition $\psi_Z$, as long as it is defined ($v(\cdot,t)$, defined by the ODE system (10), is a trajectory in $\mathbb{C}^Z$, and what we want to show is that once a trajectory enters $\Omega_Z$, it never leaves). In order to do so, we must show that whenever $v$ belongs to $\Omega_Z$ up to time $t = \tau$, then
\[ \frac{dv}{dt}(\tau) = \left( \frac{d\hat{v}}{dt}(n,\tau) \right)_{n \in Z} \]
points towards the interior of $\Omega_Z$ (in this case, $\tau$ can very well be 0, and then we have $v = \psi_Z$, which by hypothesis belongs to $\Omega_Z$).

Proving that $\frac{dv}{dt}(\tau)$ points towards the interior of $\Omega_Z$ whenever $v(\tau)$ belongs to its boundary is a consequence of the fact that for $c_{p,\lambda}$ conveniently chosen the following inequalities hold (see Section 2 in [9])
\[
\frac{1}{p} \hat{v}(0,\tau)^{p+2} \geq \left| \sum_{q \in A_0 \cap Z^{p+2}} H(p,q_1,q_2) \Phi(q_1,\ldots,q_{p+2},\tau) \hat{v}(q_1,\ldots,q_{p+2},\tau) \right|
\]
(11)
So, in what follows we will show that for a good choice of $c_{p,\lambda}$, inequality (12) holds whenever $v$ belongs to the boundary of $\Omega_Z$. The same reasoning can then be applied to prove inequality (11) under the same circumstances, which would then prove the lemma.

Now, if $v$ belongs to the boundary of $\Omega_Z$, then it holds that

$$c_{p,\lambda} \max_{n \in \mathbb{Z}, n \neq 0} \left\{ \left| n \right|^2 |\text{Re} (\hat{v}(n,\tau))|, \left| n \right|^2 |\text{Im} (\hat{v}(n,\tau))| \right\} \leq \frac{\lambda}{2\pi} \int_0^{2\pi} v(\theta,\tau) \, d\theta,$$

and that there is an $n \in \mathbb{Z}$ such that

$$c_{p,\lambda} \|v\|_2 = c_{p,\lambda} \left| n \right|^2 |\text{Re} (\hat{v}(n,\tau))| = \hat{v}(0,\tau)$$

or the same, but for the imaginary part. Under these assumptions, if we write $M = \|v\|_2$, the righthand side of inequality (12) is bounded above by

$$f(c_{p,\lambda}) \lambda^2 M^{p+2},$$

where $f$ is a polynomial of degree at most $p$, whereas the lefthand side is bounded from below (in absolute value) by

$$c_{p,\lambda}^{p+1} \left( \lambda^2 - \frac{p+2}{pm^2} - \frac{\gamma}{k(0,\tau)^{p+1} |n|} \right) M^{p+2},$$

and hence, as

$$\lambda^2 - \frac{p+2}{pm^2} - \frac{\gamma}{k(0,\tau)^{p+1} |n|} > 0$$

(which can be achieved as long as $\lambda > \sqrt{\frac{p+2}{p}}$, choosing $\gamma > 0$ small enough), by taking $c_{p,\lambda} > 0$ large enough, the lemma follows. The reader should have noticed also, that a judicious choice of $c_{p,\lambda} > 0$ implies that $\hat{k}(0,\tau)$ is increasing (which is a consequence of inequality (11)), so the choice $\gamma > 0$ only depends on $\hat{\psi}(0)$.

Remark 2.1. The method of proof of Lemma 2.1 gives a way to estimate $c_{p,\lambda}$. Indeed, if $p = 1$ (the case of the curve shortening flow), we can take

$$c_{1,\lambda} = \frac{64\lambda^2}{\lambda^2 - 3}.$$

Although the Trapping Lemma is proven for finite dimensional approximations of (2), the estimate given is strong enough so it "passes to the limit", i.e., it holds for solutions to (2), provided that the initial condition satisfies the hypothesis of Theorem 2.1—the details are left to the reader. Hence, from now on our estimates are given for solutions to (2), and in consequence we drop the dependence on $\mathcal{Z}$.

Notice that the Trapping Lemma implies that solutions to (2) for initial conditions that satisfy the hypothesis of Theorem 2.1 are analytic in space. Also, from the Trapping Lemma, we can conclude the following useful estimate: for a solution
k to (2) with initial condition $\psi$, which satisfies the hypothesis of Theorem 2.1, we can find constants $C, \mu > 0$ such that the estimate

$$\left| \hat{k}(n, t) \right| \leq Ce^{-\mu |n|}, \quad \text{for} \quad t \geq \frac{T}{2},$$

holds. Obviously $C$ and $\mu$ may depend on the initial condition. The interested reader can compare this result with the work of Ferrari and Titi in [6], where they prove an analyticity results for certain semilinear parabolic equations in the $d$-torus.

Also, from the Trapping Lemma and the ODE satisfied by $\hat{k}(0, t)$ we obtain the following result on the blow-up behavior of $\hat{k}(0, t)$.

**Lemma 2.2** (Blow-up Lemma). Let $T > 0$ be the blow-up time of a solution to (2). Under the hypothesis of Theorem 2.1, for every $\eta > 0$ there exists a $t_0 > 0$ such that

$$\hat{k}(0, t) \geq \left( \frac{p}{p+1} \right)^{\frac{1}{p+1}} \frac{(1-\eta)^{\frac{1}{p+1}}}{(T-t)^{\frac{1}{p+1}}},$$

for all $t \in (t_0, T)$.

**Proof.** Using the equation satisfied by $\hat{k}(0, t)$, by the Trapping Lemma it can be shown that $\hat{k}(0, t)$ satisfies the differential inequality

$$\frac{d}{dt} \hat{k}(0, t) \leq \frac{1}{p} \hat{k}(0, t)^{p+2} + A\hat{k}(0, t)^p,$$

where $A$ is a constant. Notice also that the Trapping Lemma implies that $\hat{k}(0, t)$ blows up: since $k$ blows up, and $\hat{k}(n, t)$ is conveniently bounded for $n \neq 0$, $\hat{k}(0, t)$ must blow up. Hence, for every $\eta > 0$, there is a $t_0 > 0$ such that

$$\frac{d}{dt} \hat{k}(0, t) \leq \frac{1}{(1-\eta)^{\frac{1}{p}}} \hat{k}(0, t)^{p+2} \quad \text{for all} \quad t_0 > 0.$$

The Lemma follows from integrating this differential inequality. 

Now we can give a first estimate on the rate of decay of the Fourier wave numbers of solutions to (2).

**Lemma 2.3.** There exists $\epsilon_0 > 0$ which depends on $\lambda$ such that if $t > \frac{T}{2} > 0$ then there is a constant constant $b > 0$ such that for any $0 < \epsilon < \epsilon_0$, for $n \neq 0$, the following estimate holds,

$$\left| \hat{k}(n, t) \right| < be^{-\mu |n|} (T-t)^{\gamma} \quad \text{whenever} \quad t > \frac{T}{2}.$$

**Proof.** Notice that a solution to the infinite dimensional ODE system in Fourier space corresponding to equation (2) (whose finite dimensional approximations are described by (3)) can be written as,

$$\hat{k}(n, t) = \hat{k}(n, \tau) e^{-\left(\lambda^2 n^2 - \frac{\mu}{p+1}\right) \int_\tau^t} \hat{k}(0, s)^{p+1} ds + \int_\tau^t e^{-\left(\lambda^2 n^2 - \frac{\mu}{p+1}\right) s} \hat{k}(0, \sigma)^{p+1} d\sigma \times$$

$$\sum_{q \in A_n} H(p, q_1, q_2) \hat{k}^{*(p+2)}(q_1, \ldots, q_{p+2}, t) ds.$$
From the Trapping Lemma, we can estimate the nonlinear term in the previous expression as
\[ \left| \sum_{q \in A_n} H(p, q_1, q_2) \hat{k}^{(p+2)}(q_1, \ldots, q_{p+2}, s) \right| \leq Ce^{-\gamma|n|^s}. \]

By the Blow-up Lemma, given \( \eta > 0 \) there is a \( \delta > 0 \) such that if \( t > T - \delta > \frac{T}{2} \) then
\[
\left| \hat{k}(n, t) \right| \leq \left| \hat{k}(n, T - \delta) \right| \left( \frac{T - t}{\delta} \right)^{(1-\eta)\alpha(\lambda, n, p)} + (T - t)^{(1-\eta)\alpha(\lambda, n, p)} e^{-\mu|n|} \int_{T-\delta}^{T-t} \frac{1}{(T - s)^{(1-\eta)\alpha(\lambda, n, p)}} ds,
\]
where
\[ \alpha(\lambda, n, p) = \left( \lambda^2 n^2 - \frac{p + 2}{p} \right) \frac{p}{p + 1}. \]
With \( p \) fixed, by taking \( \eta = \frac{1}{2} \), since \( \alpha(\lambda, n, p) > 0 \), we obtain a bound,
\[
\left| \hat{k}(n, t) \right| \leq C \left( \left| \hat{k}(n, T - \delta) \right| (T - t)^{\frac{1}{2}} + e^{-\mu|n|} (T - t)^{\frac{3}{2}} \right),
\]
for any \( 0 < \epsilon < \min \left\{ \frac{1}{2} \alpha(\lambda, n, p), \frac{1}{2} \right\} \), which proves the lemma.

The previous lemma already implies our stability results. However, in order to obtain sharp estimates on the rates of uniformization of solutions to (2), and to finish the proof of Theorem 2.1, Lemma 2.2 does not suffice. In fact, we can improve a bit on Lemma 2.2. So we have:

**Lemma 2.4.** There exists a \( t_0 > 0 \) such that for all \( t \in (t_0, T) \) the following estimate holds
\[ \hat{k}(0, t) \geq \left( \frac{p}{p + 1} \right)^{\frac{1}{p+2}} \frac{1}{(T - t)^{\frac{p}{p + 1}}} \cdot \]

**Proof.** As in the proof of Lemma 2.2 the following differential inequality holds
\[ \frac{d}{dt} \hat{k}(0, t) \leq \frac{1}{p} \hat{k}(0, t)^{p+2} + A \hat{k}(0, t)^p, \]
for a constant \( A > 0 \) independent of \( t \). So using Lemma 2.2 we obtain the differential inequality
\[ \frac{1}{\hat{k}(0, t)^{p+2}} \frac{d}{dt} \hat{k}(0, t) \leq \frac{1}{p} + C (T - t)^{\frac{p}{p+1}}, \]
which after integration gives the desired inequality.

We are ready to improve the estimate on the decay of the Fourier coefficients of solutions to (2), i.e., the estimate provided by Lemma 2.3. In order to proceed, we use the previous lemma to estimate the integral
\[ I = \frac{p + 1}{p} \int_\tau^{t} \hat{k}(0, t)^{p+1} \]
from below. Indeed, from Lemma 2.3 since \( t \) and \( \tau \) are close to \( T \), using Taylor’s
Theorem, a simple computation shows that

\[
I \geq \int_{\tau}^{t} \frac{1}{T - s + (T - s)^{1 + \frac{1}{p + 1}}} \, ds = -\log \left( \frac{T - t}{T - \tau} \right) + O(1),
\]

and using this and (13) to estimate \( \hat{k}(n, t) \) from above, yields

\[
(14) \quad |\hat{k}(n, t)| \leq C |\hat{k}(n, T - \delta)| \left( \frac{T - t}{\delta} \right)^{\alpha(\lambda, n, p)} + \\
\quad C (T - t)^{\alpha(\lambda, n, p)} \int_{T - \delta}^{t} \left( \frac{1}{T - s} \right)^{\alpha(\lambda, n, p)} \times \\
\quad \sum_{q \in A_n} |H(p, q_1, q_2) \hat{k}^{(p+2)}(q_1, \ldots, q_{p+2}, s)| \, ds,
\]

where

\[
\alpha(\lambda, n, p) = \left( \lambda^2 n^2 - \frac{p + 2}{p} \right) \frac{p}{p + 1}.
\]

Let us now improve on the estimate given by Lemma 2.3. If we introduce the bound from Lemma 2.3 into (14), we get

\[
|\hat{k}(n, t)| \leq C e^{-\mu n} (T - t)^{\min\{\alpha(\lambda, n, p), 1 + 2e\}},
\]

with \( 0 < \mu' < \mu \), and from which we obtain the estimate

\[
|\hat{k}(n, t)| \leq C' e^{-\mu' n} (T - t)^{\min\{\alpha(\lambda, n, p), 3 + 2e\}},
\]

with \( 0 < \mu'' < \mu' \).

Finally, it should be clear that if we repeat this procedure a finite number of times, we arrive at

\[
(15) \quad |\hat{k}(n, t)| \leq D e^{-\nu n} (T - t)^{\alpha(\lambda, n, p)}, \quad n \neq 0,
\]

which in turn implies estimate (14), i.e., the conclusion of Theorem 2.1.

3. A FEW COMMENTS ON THE STABILIZATION RATE TOWARDS THE CONSTANT
STeady State Solution of the Normalized \( p \)-CURVE SHORTENING FLOW

In this section we study the exponential stability of the steady solution \( u \equiv 1 \) of the normalized \( p \)-curve shortening flow. Recall that the normalized \( p \)-curve shortening flow is obtained from the unnormalized \( p \)-curve shortening flow by the process described in section 2. To be more precise, we have that the normalized \( p \)-curve shortening flow is equivalent to the Boundary Value Problem

\[
(16) \quad \begin{cases}
\frac{\partial u}{\partial \tau} = pu^{1+\frac{1}{p}} \frac{\partial^2 u}{\partial \theta^2} + pu^{2+\frac{1}{p}} - pu & \text{in} \quad [0, \frac{2\pi}{\lambda}] \times (0, \infty) \\
u(\cdot, 0) = \left( \frac{pu}{p+1} \right)^{\frac{p}{p+1}} \psi,
\end{cases}
\]
with periodic boundary conditions. Recall that this normalized equation is obtained from (2) by the rescaling and change of time parameter given by

\[ u(\theta, t) = \left( \frac{p}{p+1} \right)^{\frac{1}{p+1}} (T-t)^{\frac{1}{p+1}} \hat{k}(\theta, t), \quad \tau = -\frac{1}{p+1} \log \left( 1 - \frac{t}{T} \right). \]

The reader should have noticed that Corollary 2.1 does not give a rate of convergence of the solution to the normalized flow towards the steady solution \( u \equiv 1 \). In order to provide rates of convergence towards the steady solution, we will analyze the behavior of \( \hat{k}(0, t) \) in the case of the unnormalized equation. First observe that the following lemma holds, and that its proof is an obvious modification of the proof given in Lemma 2.4. Again, we are under the hypothesis of Theorem 2.1.

**Lemma 3.1.** There is a \( t_0 > 0 \) such that if \( t \in (t_0, T) \), then we have the estimate

\[ \hat{k}(0, t) \leq \left( \frac{p}{p+1} \right)^{\frac{1}{p+1}} \frac{1}{(T-t) - (T-t)^{1+\frac{2}{p+1}}}^{\frac{1}{p+1}}. \]

If we let

\[ \hat{u}(0, t) = \left( \frac{p}{p+1} \right)^{\frac{1}{p+1}} (T-t)^{\frac{1}{p+1}} \hat{k}(0, t), \]

and then compute

\[ \hat{u}(0, t) - 1 \leq \frac{(T-t)^{\frac{1}{p+1}} - (T-t)^{1+\frac{2}{p+1}}}{(T-t) - (T-t)^{1+\frac{2}{p+1}}} \frac{1}{(T-t)^{\frac{1}{p+1}}}, \]

factoring out from the numerator \( (T-t)^{\frac{1}{p+1}} \) and applying Taylor’s Theorem we obtain

\[ \hat{u}(0, t) - 1 \leq \frac{(T-t)^{\frac{1}{p+1}} - (T-t)^{1+\frac{2}{p+1}}}{(T-t) - (T-t)^{1+\frac{2}{p+1}}} \frac{1}{(T-t)^{\frac{1}{p+1}}} \leq C (T-t)^{\frac{2}{p+1}}. \]

In the same way, by using Lemma 2.4 we obtain an estimate

\[ 1 - \hat{u}(0, t) \leq C (T-t)^{\frac{2}{p+1}}. \]

This implies that for the normalized flow (179) holds that

\[ \left| \frac{\lambda}{2\pi} \int_0^{2\pi} u(\theta, \tau) \, d\theta - 1 \right| \leq C \exp(-2\tau). \]

Using this estimate and Corollary 2.1 we obtain the estimate

\[ \| u(\theta, \tau) - 1 \|_{L^\infty([0, 2\pi])} \leq C \exp(-\omega \tau), \]

where

\[ \omega = \min \left\{ \lambda^2 p - p - 1, 2 \right\}. \]

Notice that

\[ \lambda^2 p - p - 1 \leq 2 \]

whenever

\[ \sqrt{\frac{p+2}{p}} < \lambda \leq \sqrt{\frac{p+3}{p}}, \]
so it is in this case that we obtain a rate of decay towards the steady state corresponding to the first eigenvalue of the elliptic part of the normalized $p$-curve shortening flow. However, if we use estimate (15) in the proof of Lemma 2.4, we can bound

$$\left| \sum_{\mathbf{q} \in \mathcal{A}_0} H(p, q_1, q_2) \hat{k}^{(p+2)}(\mathbf{q}, t) \right| \leq C(T-t)^{2\alpha(\lambda,p)} \hat{k}(0,t)^p$$

with

$$\alpha(\lambda,p) = \left( \lambda^2 - \frac{p+2}{p} \right) \frac{p}{p+1},$$

then from the ODE satisfied by $\hat{k}(0,t)$, we can obtain the improved bound from below

$$\hat{k}(0,t) \geq \left( \frac{p}{p+1} \right)^{\frac{1}{p+1}} \frac{1}{\left( T-t + (T-t)^{2\alpha(\lambda,p)} + \frac{p}{p+1} \right)^{\frac{1}{p+1}}},$$

and also the corresponding bound from above. Hence, proceeding as we just did, we arrive at an estimate

$$\left| \frac{\lambda}{2\pi} \int_0^{2\pi} u(\theta, \tau) \, d\theta - 1 \right| \leq C \exp\left( - (2\lambda^2 p - p - 1) \tau \right).$$

from which follows that (17) holds now for all $\lambda > \sqrt{\frac{p+2}{p}}$ with

$$\omega = \min \left\{ \lambda^2 p - p - 1, 2\lambda^2 p - p \right\} = \lambda^2 p - p - 1.$$

So we finish with the following result.

**Proposition 3.1.** Let $\lambda > \sqrt{\frac{p+2}{p}}$, let $\psi > 0$ and assume that it satisfies the hypothesis of Theorem 2.1. Then the solution $u(\theta, \tau)$ to (16) corresponding to the solution to (2) with $\psi$ as initial data, satisfies an estimate

$$\|u(\theta, \tau) - 1\|_{L^\infty[0,\frac{2\pi}{\lambda}]} \leq C \exp\left( - (\lambda^2 p - p - 1) \tau \right),$$

where $C > 0$ is a constant that depends on $\psi$, $p$ and $\lambda$.

**References**

[1] U. Abresch and J. Langer, The normalized curve shortening flow and homothetic solutions. J. Differential Geom. 23 (1986), no. 2, 175–196.
[2] B. Andrews, Evolving convex curves. Calc. Var. Partial Differential Equations 7 (1998), no. 4, 315–371.
[3] B. Andrews, Classification of limiting shapes for isotropic curve flows. J. Amer. Math. Soc. 16 (2003), no. 2, 443–459.
[4] J. Cortissoz, On the blow-up behavior of a nonlinear parabolic equation with periodic boundary conditions. Arch. Math. (Basel) 97 (2011), 69–78.
[5] C. L. Epstein and M. I. Weinstein, A stable manifold theorem for the curve shortening equation. Commun. Pure Appl. Math. 40 (1987), 119–139.
[6] A.B. Ferrari, E. S. Titi, Gevrey regularity for nonlinear parabolic equations. Comm. Partial Differential Equations 23 (1998), no.1–2, 1–16.
[7] M. Gage and R.S. Hamilton, The heat equation shrinking convex plane curves. J. Differential Geom. 23 (1986), no. 1, 69–96.
[8] R. L. Huang, Blow-up rates for the general curve shortening flow. J. Math. Anal. Appl. 383 (2011), no 2, 482–487.
[9] J. Mattingly and Ya. Sinai, An elementary proof of the existence and uniqueness theorem for the Navier-Stokes equations. Commun. Contemp. Math. 1 (1999), no. 4, 497–516.
[10] N. Sesum, Rate of convergence of the mean curvature flow. Comm. Pure Appl. Math. 61 (2008), no. 4, 464–485.
[11] X.–L. Wang, The stability of m-fold circles in the curve shortening problem. Manuscripta Math. 134 (2011), no. 3–4, 493–511.
[12] M. Wiegner, On the asymptotic behaviour of solutions of nonlinear parabolic equations. Math. Z. 188 (1984), no. 1, 3–22.
[13] M. Winkler, Blow-up of solutions to a degenerate parabolic equation not in divergence form. J. Differential Equations 192 (2003), no. 2, 445–474.

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