Cotangent and tangent modules on quantum orbits

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Abstract

Let $k(S_2^q)$ be the “coordinate ring” of a quantum sphere. We introduce the cotangent module on the quantum sphere as a one-sided $k(S_2^q)$-module and show that there is no Yang-Baxter type operator converting it into a $k(S_2^q)$-bimodule which would be a flatly deformed object w.r.t. its classical counterpart. This implies non-flatness of any covariant differential calculus on the quantum sphere making use of the Leibniz rule. Also, we introduce the cotangent and tangent modules on generic quantum orbits and discuss some related problems of “braided geometry”.

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1 Introduction

Since the creation of super-theory it became clear that numerous aspects of commutative algebra and usual geometry could be generalized to the super-case. In particular, for any two $\mathbb{Z}_2$-graded one-sided (say, left) $A$-modules $M_1$ and $M_2$ over a super-commutative algebra $A$ their tensor product $M_1 \otimes_A M_2$ is well-defined.

In the latter 80’s it was recognized that many properties of (super-)commutative algebra and geometry could be further generalized onto objects related to a Yang-Baxter (YB) operator, i.e., a solution of the quantum YB equation

$$(S \otimes \text{id})(\text{id} \otimes S)(S \otimes \text{id}) = (\text{id} \otimes S)(S \otimes \text{id})(\text{id} \otimes S),$$

$S$ being an operator acting on $V^{\otimes 2}$ where $V$ is a vector space. If $S$ is an involutary YB operator ($S^2 = \text{id}$), the notion of an $S$-commutative algebra can be introduced in a natural way. If $A$ is such an algebra, the product $M_1 \otimes_A M_2$ of two one-sided $A$-modules can be introduced by means of the operator $S$ (under some natural conditions on it). Also, the operator $S$ plays the crucial role in a twisted or quantum (i.e. related to an operator
version of differential calculus. It can be used for ordering "quantum functions" and differentials mixed in virtue of the Leibniz rule.

Unfortunately, a straightforward application of this method to algebras related to a non-involuntary YB operator $S$ leads to non-flat deformations\(^1\).

As was shown in \([\text{Ar}], \text{[AAM]}\) the differential calculus on the quantum algebras $k_q(G), G = SL(n), SO(n), Sp(n)$ initiated in \([\text{W1}], \text{[W2]}\) and making use of the Leibniz rule gives rise to non-flat deformations of the classical differential algebras. However, this differential calculus plays the central role in all known attempts to introduce a quantum version of gauge theory related to Drinfeld-Jimbo quantum groups (cf., i.e. \([\text{BM}], \text{[HM]}\) and the references therein). From our viewpoint, non-flat deformations are somewhat pointless since in this case classical objects are not limits of their quantum counterparts.

Quantum orbits pertain to another type of algebras related to quantum groups. The simplest example of such an orbit is the quantum sphere $k(S^2_q)$ introduced in \([\text{P2}]\). A version of differential calculus on it was suggested in \([\text{P2}]\). However, as follows from our results, on the quantum sphere no flatly deformed $U_q(su(2))$-covariant differential calculus exists which makes use of the Leibniz rule. An explanation of this phenomenon, given in the present paper, consists in the following. It is possible to realize a flat deformation of the cotangent module\(^2\) considered as a one-sided $k(S^2)$-module. However, the flatness of deformation breaks down when one tries to deform the tangent vector bundle considered as a two-sided $k(S^2)$-module.

Thus, if we want deformed modules to be flatly deformed objects, the use of one-sided modules on quantum varieties\(^3\) are only relevant in general.

Note, that some one-sided modules which are $q$-analogues of line bundles on "quantum generic orbits" were constructed in \([\text{GS}]\) in the spirit of the Serre-Swan approach. Here we introduce the cotangent and tangent modules on the same orbits. Also, we discuss the problem of an operator meaning of the tangent modules, i.e. that of representing them by "braided vector fields" (note that in the case of the quantum sphere this problem was solved in \([\text{A}]\)).

In the sequel we prefer to use the quantum group (QG) $U_q(sl(n))$ instead of this $U_q(su(n))$ since we disregard any involution in quantum algebras in question. The basic field $k$ is assumed to be $\mathbb{R}$ or $\mathbb{C}$. Throughout the whole of the paper the parameter $q$ is assumed to be generic.

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\(^1\)Let us recall that a deformation $V_h$ of a vector space $V$ where $h$ is a formal parameter is called flat if $V_h/hV_h = V$ and $V_h$ is isomorphic to $V \otimes k[[h]]$ as $k[[h]]$-module (the tensor product is completed in the $h$-adic topology).

\(^2\)We use this term for the module corresponding to the tangent vector bundle in the framework of the Serre-Swan approach, cf. \([\text{Se}], \text{[Sw]}\). In a similar way we consider other classical and quantum modules. (All modules are assumed to be finitely generated.)

\(^3\)By abusing the language we call "quantum varieties" the corresponding "coordinate rings".
2 Cotangent $k(S^2_q)$-module

First, let us introduce the quantum sphere in a form appropriate for our goals. Let $V$ be a three dimensional vector space being a left $U_q(sl(2))$-module. The action of the quantum group $U_q(sl(2))$ can be extended to any tensor power of $V$ via the coproduct. Let us decompose the space $V^\otimes 2$ into a direct sum of irreducible $U_q(sl(2))$-modules:

$$V^\otimes 2 = V_0 \oplus V_1 \oplus V_2$$

(2.1)

where $V_i$ stands for the spin $i$ $U_q(sl(2))$-module. Let $v_0$ be a generator of the one-dimensional component $V_0$.

Let us introduce quantum sphere by

$$k(S^2_q) = T(V)/\{V_1, v_0 - c\}, \ c \in k, \ c \neq 0$$

(2.2)

where $T(V)$ stands for the free tensor algebra of the space $V$ and $\{I\}$ stands for the ideal generated by a family $I \subset T(V)$. This algebra is a particular (in some sense $q$-commutative) case of the Podles' sphere introduced in [P1]. (We disregard involution operators in this algebra. So, in fact we do not distinguish quantum sphere and quantum hyperboloid.)

Remark, that in definition (2.2) of the quantum sphere we do not use any coordinate form of the quantum sphere. In a similar non-coordinate way we introduce the cotangent module on it.

Let $V'$ be the space isomorphic to $V$ as $U_q(sl(2))$-modules but spanned by the differentials $dx, \ x \in V$. Let $V' \otimes k(S^2_q)$ be the free finitely generated right $k(S^2_q)$-module and its submodule $M_r$ generated by $(V' \otimes V)_0$. Thus, we have

$$M_r = \text{Im} \mu^{23}((V' \otimes V)_0 \otimes k(S^2_q)).$$

Hereafter $(V' \otimes V)_i$ stands for the spin $i$ component in the product $V' \otimes V$, $\mu$ stands for the product in the algebra $k(S^2_q)$, and $\mu^{ij}$ is the operator $\mu$ applied to the $i$-th and $j$-th factors. We call a right cotangent module on the quantum sphere the following quotient

$$T^*_r(S^2_q) = (V' \otimes k(S^2_q))/M_r.$$ 

We define a left cotangent module $T^*_l(S^2_q)$ on the quantum sphere in the same way as the quotient of the left $k(S^2_q)$-module $k(S^2_q) \otimes V'$ over its submodule

$$M_l = \mu^{12}(k(S^2_q) \otimes (V \otimes V'))_0.$$ 

(Hereafter we omit the symbol Im.)

These modules are quantum analogues (respectively, right and left ones) of the cotangent vector bundle over quantum sphere (or quantum hyperboloid) realized in the spirit of the Serre-Swan approach.

As was shown in [AG], the left cotangent module $T^*_l(S^2_q)$ is a flat deformation of its classical counterpart (in fact, the proof consists in showing that the classical and quantum objects are built from the same, respectively, $U(sl(2))$- and $U_q(sl(2))$-irreducible
components). By the same reason the right cotangent module $T_q^r(S_q^2)$ is a flatly deformed object.

Now, define a two-sided cotangent module (in the sequel called cotangent bimodule) on the quantum sphere. Let us set

$$T^*(S_q^2) = (k(S_q^2) \otimes V' \otimes k(S_q^2))/(M_l \otimes k(S_q^2) + k(S_q^2) \otimes M_r).$$

This $k(S_q^2)$-bimodule is much bigger than one-sided one even in the classical case ($q = 1$) because so far we do not have any rule for transposing the "quantum functions" and differentials (i.e., elements of $V'$). In what follows we omit the subscription $q$ if $q = 1$.

First, let us consider the classical case in details. In order to reduce this bimodule to the seize of the one-sided one we should define a commutation rule between elements of the algebra and those of $V'$. In the classical case it is always done by the flip. Namely, we set

$$T^*(S^2) = T^*(S^2)/\{a \otimes v - v \otimes a\} \quad a \in k(S^2), \ v \in V'. \quad (2.3)$$

It is not difficult to see that there exists a one-to-one correspondence between the one-sided (say, right) $k(S^2)$-module $T_q^r(S^2)$ and the bimodule $T^*(S^2)$. Indeed, modulo the denominator of $(2.3)$ any element of the $k(S^2)$-bimodule $k(S^2) \otimes V' \otimes k(S^2)$ can be reduced to an element of the right module $V' \otimes k(S^2)$. Thus, we have a map

$$\rho : k(S^2) \otimes V' \otimes k(S^2) \longrightarrow V' \otimes k(S^2).$$

The following inclusion is clear

$$\rho (M_l \otimes k(S^2) + k(S^2) \otimes M_r) \subset M_r.$$ 

This implies that the map $\rho$ sends the two-sided module $T^*(S^2)$ into $T_q^r(S^2)$. Moreover, it is isomorphism of linear spaces.

An analogous construction for algebras related to an involutary YB operator $S$ can be introduced in a similar way. In this case the denominator of formula $(2.3)$ should be replaced by $\{a \otimes v - S(a \otimes v)\}$. However, if we want to realize a similar approach for the algebra $k(S_q^2)$ there is an another approach for the $k(S_q^2)$ (or for other algebras related to non-involutary YB operators $S$) it is not clear what should be a proper analogue of the denominator in $(2.3)$.

Let $\overline{S} : V \otimes V' \longrightarrow V' \otimes V$ be any $U_q(sl(2))$-covariant invertible operator (called in the sequel a transposition). Let us replace the denominator in $(2.3)$ by $\{a \otimes v - \overline{S}(a \otimes v)\}$ (we also assume that $S^2$ in the numerator is replaced by $S_q^2$). The problem consists in finding all transpositions $\overline{S}$ such that the corresponding quotient denoted $T^*(S_q^2)$ would be a flat deformation of the quotient $(2.3)$.

It is evident that in order to give rise to a flatly deformed object a transposition $\overline{S}$ should preserve the ideal of the formula $(2.2)$ and take the submodule $M_l$ into $M_r$. Otherwise, by passing to the right $k(S_q^2)$-module $T^*_q(S^2)$ we would get some supplementary relations in it what would lead to a collapse of the final object. Essentially, this means that the map $\rho$ takes the $k(S_q^2)$-bimodule $T^*_q(S^2)$ onto some proper quotient of the right $k(S_q^2)$-module $T^*_q(S^2)$.

As we will see in the next section the only transpositions preserving the ideal in $(2.2)$ are $\pm S^{\pm 1}$ where $S$ is the YB operator coming from $U_q(sl(2))$. Then we will show that even these operators do not take the submodule $M_l$ into $M_r$. So, any transposition $\overline{S}$ leads to the collapse mentioned above.
3 Non-existence of a flat deformation

Now we represent the quantum sphere in a more explicit (coordinate) form. Let us fix the base \((u, v, w)\) in the space \(V\) with the following action of the QG \(U_q(sl(2))\)

\[
X.u = 0, \quad X.v = -(q + q^{-1})u, \quad X.w = v,
Y.u = -v, \quad Y.v = (q + q^{-1})w, \quad Y.w = 0,
H.u = 2u, \quad H.v = 0, \quad H.w = -2w.
\]

Hereafter \(X, H, Y\) are the standard generators of the QG \(U_q(sl(2))\) (cf. [CP]).

Note, that the QG \(U_q(sl(2))\) acts on the space \(V'\) in the same way (we should only replace the generators \((u, v, w)\) by \((du, dv, dw)\) in the formulae above).

Thus, the spaces \(V_i, i = 0, 1, 2\) being irreducible \(U_q(sl(2))-\)modules are as follows

\[
V_0 = \text{span}(v_0), \quad v_0 = (q^3 + q)uw + v^2 + (q + q^{-1})wu,
V_1 = \text{span}(q^2uv - vu, (q^3 + q)(uw - wu) + (1 - q^2)v^2, \ wv - q^2vw),
V_2 = \text{span}(u^2, \ uv + q^2vu, \ uw - qv^2 + q^3wu, \ vw + q^2vw, \ w^2)
\]

(the sign \(\otimes\) is systematically omitted).

It is well known that the YB operator \(S\) being restricted onto each component becomes scalar. Namely,

\[
S|_{V_0} = q^{-4} \text{id}, \quad S|_{V_1} = -q^{-2} \text{id}, \quad S|_{V_2} = q^2 \text{id}
\]

(cf. f.e. [C]). This implies that being applied to the product \(V \otimes V'\) the operator \(S\) acts as follows

\[
S(udu) = \alpha duu \quad (3.4)
S(q^2udv - vdu) = \beta (q^2dvw - dvu) \quad (3.5)
S((q^3 + q)udw + vdw + (q + q^{-1})wdv) = \gamma ((q^3 + q)duv + dvv + (q + q^{-1})dvw) \quad (3.6)
\]

with \(\alpha = q^2, \beta = -q^{-2}, \gamma = q^{-4}\) and similarly for other elements of each component.

Now consider an arbitrary invertible transposition \(\overline{S}: V \otimes V' \rightarrow V' \otimes V\) being a \(U_q(sl(2))-\)morphism (we do not require it to be a YB operator). It is given by the same formulae \((3.4)-(3.6)\) (and all their descendants) but with arbitrary non-trivial \(\alpha, \beta\) and \(\gamma\). By applying such an operator many times we can transform any element from \(V^\otimes k \otimes V'\) into that from \(V' \otimes V^\otimes k\). Let us note that the operator \(\overline{S} = S\) has the following remarkable property (this property is also valid for the operator \(S^{-1}\)).

**Proposition 1** We have

\[
S(\mu)^{12} = \mu^{23} S^{12} S^{23} \quad \text{and} \quad S(\mu)^{23} = \mu^{12} S^{23} S^{12}
\]

**Proof** The defining relations of the algebra \(k(S_q^2)\) are coordinated with the action of the QG \(U_q(sl(2))\) in the following sense

\[
X.\mu(a \otimes b) = \mu \Delta(X). (a \otimes b) \quad \forall X \in U_q(sl(2)), \ a, b \in k(S_q^2).
\]
Let $\mathcal{R}$ be the universal quantum R-matrix corresponding to the QG $U_q(sl(2))$. It satisfies the relations

$$\Delta^{12}\mathcal{R} = \mathcal{R}^{13}\mathcal{R}^{23}$$
and

$$\Delta^{23}\mathcal{R} = \mathcal{R}^{13}\mathcal{R}^{12}$$

(this means that the QG in question is quasitriangular).

Thus, for any $a, b, c \in k(S_q^2)$ we have (hereafter $\sigma$ is the usual flip)

$$S\mu^{12}(a \otimes b \otimes c) = \sigma\mathcal{R}\mu^{12}(a \otimes b \otimes c)$$

$$= \sigma(\mathcal{R}_1(\mu^{12}(a \otimes b)) \otimes \mathcal{R}_2 c)$$

$$= \sigma(\mu^{12}\Delta(\mathcal{R}_1)(a \otimes b) \otimes \mathcal{R}_2 c) = \sigma\mu^{12}(\Delta^{12}\mathcal{R})(a \otimes b \otimes c).$$

Here we use the Sweedler’s notation and apply the components of $\mathcal{R}$ to the elements $a, b, c$ w.r.t. the action $U_q(sl(2))$ on the algebra $k(S_q^2)$. Moreover, we use the relation $S = \sigma\mathcal{R}$.

The following chain of identities completes the proof of the first relation of proposition (the second one can be verified in a similar way)

$$\sigma\mu^{12}\Delta^{12}\mathcal{R} = \mu^{23}\sigma^{12}\sigma^{23}\Delta^{12}\mathcal{R}$$

$$= \mu^{23}\sigma^{12}\sigma^{23}\mathcal{R}^{13}\mathcal{R}^{23} = \mu^{23}\sigma^{12}\sigma^{23}\mathcal{R}^{13}\sigma^{23}\mathcal{R}^{23}$$

$$= \mu^{23}\sigma^{12}\sigma^{23}\mathcal{R}^{13}\sigma^{23}S^{23} = \mu^{23}\sigma^{12}\mathcal{R}^{12}S^{23}$$

$$= \mu^{23}\sigma^{12}S^{12}S^{23}.$$

It is evident that the above proposition is still valid if we replace the operator $S$ by $S^{-1}$ in the formulae above. However, if we replace the operator $S$ by any other transposition $\overline{S}$ the identities from this proposition become broken in virtue of the following.

**Proposition 2** The only operators $\overline{S}$ such that

$$\overline{S}(v_0 v') = v' v_0, \ \forall v' \in V'$$

and $\overline{S}(V_1 \otimes V') \subset V' \otimes V_1$ are $\pm S$ and $\pm S^{-1}$.

**Proof** In the sequel we represent an arbitrary transposition as

$$\overline{S} = xP_0 + yP_1 + zP_2, \ x, y, z \in k$$

where the operators

$$P_i : V \otimes V' \to (V' \otimes V)_i, \ i \in \{0, 1, 2\}$$

become the projectors $V'^{\otimes 2} \to V_i$ if we identify $V$ and $V'$. For $\overline{S} = S$ we have $x = \gamma = q^{-4}$,

$$y = \beta = -q^{-2}, \ z = \alpha = q^2.$$ In what follows we need the images of some elements under the action of the operators $P_i$:

$$P_0(vdv) = \alpha_1 \overline{v}_0, \ P_0(udw) = \alpha'_1 dv_0, \ P_2(udu) = duu,$$

$$P_2(vdv) = \beta'(duv + q^2dvu), P_2(vdu) = \overline{\beta}(duv + q^2dvu),$$

$\gamma = q^{-4}, \� = -q^{-2}, \α = q^2$. In what follows we need the images of some elements under the action of the operators $P_i$:
\[ P_2(vdv) = \gamma_1(duv - qdv + q^4 dwu), \]
\[ P_1(vdv) = \beta_1[(q^3 + q)(duw - dwu) + (1 - q^2)dv], \]
\[ P_1(vdu) = \xi(q^2 duv - dvu), \quad P_1(udv) = \alpha'(q^2 duv - dvu), \]
\[ dv_0 = (q^3 + q)udw + vdv + (q + q^{-1})wdu, \]
\[ d\overline{v}_0 = (q^3 + q)duw + (q + q^{-1})duv, \]
\[ P_{12}^{12}(ud\overline{v}_0) = \alpha'(q^2 duv - dvu)v + \beta_1'(q + q^{-1})[(q^3 + q)(duw - dwu) + (1 - q^2)dv], \]
\[ P_{22}^{12}(ud\overline{v}_0) = (q^3 + q)duuw + \beta'(duv + q^2 dvu)v + \beta_1'(q + q^{-1})(duw - qdvv + q^4 dwu)u, \]
\[ P_{02}^{12}(ud\overline{v}_0) = (q + q^{-1})\alpha'dv_0 u, \]

where

\[ \beta' = (1 + q^4)^{-1}, \quad \alpha' = q^2 \beta', \quad \xi = -\beta', \quad \overline{\beta} = q^2 \beta', \quad \gamma_1'' = q^4 \gamma_1', \]
\[ \beta_2 = \beta', \quad \alpha_2 = -\alpha', \quad \alpha'_2 = \beta', \quad \beta_2' = q^2 \alpha'_2, \quad \alpha''_1 = q^{-2} \alpha'_1, \]
\[ \alpha_1 = q^2(1 + q^2 + q^4)^{-1}, \quad \beta_1 = (1 - q^2)\beta', \quad \beta''_1 = -\beta'_1, \]
\[ \gamma_1 = -q(1 + q^2)^2(1 + q^2 + q^4)^{-1} \beta', \quad \gamma_1' = (1 + q^2 + q^4)^{-1} \beta', \]
\[ \alpha'_1 = q^3(1 + q^2)^{-1}(1 + q^2 + q^4)^{-1}, \quad \beta'_1 = q(1 + q^2)^{-1} \beta'. \]

On applying the transposition \( \overline{\mathcal{S}} \) to the element \( (q^2 uv - vu)du \) we get

\[ \overline{\mathcal{S}}((q^2 uv - vu)du) = \overline{\mathcal{S}}^{12\overline{23}}((q^2 uv - vu)du) = q^2 \overline{\mathcal{S}}^{12\overline{23}}(uvdu) - \overline{\mathcal{S}}^{12\overline{23}}(vudu). \]

By a straightforward but tedious computations with the use of the formulae above we get the following result for the coefficient at the element \( dv uu \) in the image above

\[ -q^4 \beta^2(\alpha^2 + (q^4 + q^{-4})\alpha\beta + \beta^2). \quad (3.8) \]

The condition \( \overline{\mathcal{S}}(V_1 \otimes V') \subset (V' \otimes V_1) \) implies

\[ \alpha^2 + (q^4 + q^{-4})\alpha\beta + \beta^2 = 0. \quad (3.9) \]

This equation has two solutions (in the projective sense)

\[ \alpha = -q^4 \beta \quad \text{and} \quad \alpha = -q^{-4} \beta. \quad (3.10) \]

Let us remark that the first (resp., second) solution is satisfied by the operator \( c\mathcal{S} \) (resp. \( c\mathcal{S}^{-1} \)) with an arbitrary \( c \neq 0 \). Then, the operator \( \overline{\mathcal{S}} \) in general can be represented as follows

\[ \overline{\mathcal{S}} = c\mathcal{S}^{\pm 1} + \delta P_0, \quad \delta \in k. \quad (3.11) \]
Now it remains to show that $\delta = 0$ and $c = \pm 1$. Let us do it for $S$ (the $S^{-1}$ case is analogous). From the above form of $\overline{S}$ we have

$$\overline{S}^{12}S^{23} = c^2S^{12}S^{23} + c\delta(S^{12}P_0^{23} + P_0^{12}S^{23}) + \delta^2P_0^{12}P_0^{23}. \quad (3.12)$$

Let us consider the image of the element $(q^2uv - vu)dv$ w.r.t. the action of the transposition $\overline{S}$. In this image we are interested in terms containing $duv_0$ or $dvwu$. Let us denote $I_1$ (resp. $I_2$) the coefficient at $duv_0$ (resp. $dvwu$) in $\overline{S}^{12}S^{23}((q^2uv - vu)dv)$. A straightforward computation shows that

$$I_1 = (q^3 + q)\alpha''I_0 + (q + q^{-1})\alpha_1\alpha'_1[q^6 - (q + q^{-1})\beta' + q^2\gamma'_1]c\delta,$$
$$I_2 = (q + q^{-1})I_0 + (q^3 + q)\alpha_1[\beta' + q^6\gamma'_1]c\delta$$
$$I_0 = (q^3 + q)\alpha_1\alpha'_1\delta^2 + \alpha_1[2q^{-3}(1 + q^2)\alpha'_1 + (\beta_1 - q^8(1 + q^2)\gamma'_1) - 1]c\delta. \quad (3.13)$$

Since the element $duv_0$ is that of the highest weight and the element $(q^2uv - vu)dv$ is not, the coefficient $I_1$ is trivial. Moreover, the condition $\overline{S}(V_1 \otimes V') \subset (V' \otimes V_1)$ implies that the coefficient $I_2$ is trivial. These two relations imply $c\delta = 0$.

Since $c \neq 0$ (unless the operator $\overline{S}$ is not invertible) we have $\delta = 0$.

Thus, if we admit the first condition of (3.10) the only transposition $\overline{S} = cS$ could preserve the defining ideal of $k(S_q^2)$. But in fact only factors $c = \pm 1$ are compatible with the centrality of the element $v_0$. This completes the proof.

Now we pass to showing that even the operator $S$ does not preserve the flatness of the deformation (for the operators $-S$ and $\pm S^{-1}$ the proof is analogous).

**Proposition 3** The image of $M_l$ w.r.t. the YB operator $S$ does not belong to $M_r$.

**Proof** By definition of the submodules $M_l$ and $M_r$, they consist respectively of the following elements

$$\mu^{12}(fdv_0) \quad \text{and} \quad \mu^{23}(d\overline{v}_0f), \ \forall f \in k(S_q^2). \quad (3.14)$$

Let us show that there exists an element $f \in k(S_q^2)$ such that $S(\mu^{12}(fdv_0)) \notin M_r$. Let $f = u$. We have

$$S(\mu^{12}(uv_0)) = \mu^{23}S^{12}S^{23}(uv_0)$$
$$= \mu^{23}(S^{23})^{-1}S^{12}S^{23}(uv_0)$$
$$= \mu^{23}(S^{23})^{-1}(d\overline{v}_0u).$$

By using

$$(S^{23})^{-1}(d\overline{v}_0.u) = \gamma^{-1}P_0^{23}(d\overline{v}_0u) + \beta^{-1}P_1^{23}(d\overline{v}_0u) + \alpha^{-1}P_2^{23}(d\overline{v}_0u), \quad (3.15)$$

we get

$$\mu^{23}(S^{23})^{-1}(d\overline{v}_0u) = \gamma^{-1}\mu^{23}(P_0^{23}(d\overline{v}_0u)) + \alpha^{-1}\mu^{23}(P_2^{23}(d\overline{v}_0u)). \quad (3.16)$$

We state that there is no element $g \in k(S_q^2)$ such that $\mu^{23}(d\overline{v}_0g)$ would be equal to the r.h.s. of (3.16). Indeed, it could be only an element of the form $g = \nu u$, $\nu \in k$. However, since $\alpha \neq \gamma$ and the both components in (3.10) are non-trivial we conclude that no appropriate factor $\nu$ exists. This completes the proof.
Remark 1 The statement of this proposition can be generalized to other quantum algebras related to non-involutive YB operators. The crucial property of the operator $S$ used in the proof is the following one: this operator has more than one distinct eigenvalues and the corresponding components do not vanish in the algebra $k(S_q^2)$.

However, for certain algebras and certain their two-sided modules the flatness of deformation is valid. Let us consider for instance, the quantum cone (it corresponds to the case $c = 0$). The module $T^*(S_q^2)$ defined as above is a flat deformation of its classical counterpart. This follows from the fact that in the corresponding quantum coordinate ring defined by $V_0 = 0, V_1 = 0$ the only component $V_2 \subset V^{\otimes 2}$ survives. This prevents us from the effect used in the proof above.

For the same reason, in quantum geometry dealing with non-involutive YB operators $S$ it is not convenient to use this operator (either any other transposition) in order to define a product $M_1 \otimes_A M_2$ of two one-sided $A$-modules assuming $A$ to be a quantum algebra looking like that $k(S_q^2)$. However, the product of two modules can be apparently defined as the quantum deformation of the product of their classical counterparts. So, the notation $M_1 \otimes_A M_2$ must be regarded in this restricted sense without any transposition of the elements of $A$ and those of $M_1$ (or $M_2$).

4 Generic quantum orbits and modules on them

The main purpose of this section is to generalize the construction of the cotangent module on quantum sphere to some other quantum orbits. Also, we define the tangent modules on these orbits and discuss the problem of equipping the tangent module with an action on the quantum coordinate ring in question. All constructions are done in the framework of one-sided modules over algebras in question. This allows us to hope that these modules are flatly deformed objects.

First of all, describe quantum orbits in question. Let us begin with evoking their quasiclassical counterparts (i.e. the corresponding Poisson structures).

As was shown in [DGS], on any orbit (of a semisimple element) $O \subset g^*$ where $g$ is a simple Lie algebra there exists a family of Poisson-Lie structures (for the compact form of the Lie algebra this family is labeled by the elements of $H^2(O)$). Moreover, in this family there exists a bracket which is compatible with the Kirillov-Kostant-Souriau one. A quantization of this particular Poisson bracket can be realized (at least in the $g = sl(n)$ case) in terms of the so-called reflection equation (RE) algebra. The resulting algebra can be described as an appropriate quotient of the RE algebra. Thus, we get an explicit realization of such an algebra in the spirit of algebraic geometry by means of a system of braided algebraic equations. (As for other Poisson-Lie structures they can be quantized by means of formal series in the spirit of deformation quantization. Their description in terms of so-called Hopf-Galois extension is also known, cf. [Sh], [HM].)

Let $g = sl(n)$ and $g_q$ be the same as vector space but equipped with a $U_q(sl(n))$-action which is a deformation of the adjoint action of $g$ onto itself. Let us extend this action to the space $g_q^{\otimes 2}$ by means of the coproduct in $U_q(sl(n))$ and decompose it into a direct sum

$$g_q^{\otimes 2} = I_+ \oplus I_- \quad (4.17)$$
of two $U_q(sl(n))$-invariant subspaces $I_+$ and $I_-$ so that the corresponding algebras
\[
\wedge_\pm = \wedge_\pm(g_q) = T(g_q)/\{I_\pm\}
\]
would be flat deformations of the symmetric $\wedge_+(g)$ and skewsymmetric $\wedge_-(g)$ algebras respectively. Since the space $g_q^{\otimes 2}$ is not multiplicity free (the component isomorphic to $g_q$ itself comes twice in the decomposition of $g_q^{\otimes 2}$ into a direct sum of irreducible $U_q(sl(n))$-modules) it is not evident that decomposition (4.17) exists.

Nevertheless, it does exist and can be constructed by means of the RE algebra mentioned above and of a $U_q(sl(n))$-covariant pairing. Let us recall that by the RE algebra one means the algebra generated by $n^2$ elements $l^i_j$, $1 \leq i, j \leq n$ subject to the relations
\[
SL_1 SL_1 - L_1 SL_1 S = 0
\]
where $L = (l^i_j)$ is the matrix with the entries $l^i_j$ and $L_1 = L \otimes \text{id}$.

It is known that this algebra has the center generated by the elements $C^p_q = \text{tr}_q L^p$, $p = 1, ..., n$ where $\text{tr}_q$ is the quantum analogue of the usual trace. Then the space $\text{span}(l^i_j)$ is a direct sum of a one-dimensional $U_q(sl(n))$-module generated by $\text{tr}_q L$ and a $n^2 - 1$-dimensional one which we identify with $g_q$. Then the space $I_-$ can be treated as the l.h.s. of (4.18) modulo the elements of the form
\[
l \otimes \text{tr}_q L, \ \text{tr}_q L \otimes l, \ l \in \text{span}(l^i_j).
\]

In virtue of [L] the algebra $\wedge_+$ is a flat deformation of its classical counterpart. (Let us note that the RE algebra and adjacent objects are also well defined for non-quasiclassical YB operators, cf. [GPS].)

We introduce the space $I_+$ as that orthogonal to $I_-$ w.r.t. the pairing
\[
(\ , \ ) : g_q^{\otimes 2} \otimes g_q^{\otimes 2} \rightarrow k, \ (\ , \ ) = <\ , \ >, >^{23}
\]
where $<\ , \ > : g_q^{\otimes 2} \rightarrow k$ is a (unique up to a factor) $U_q(sl(n))$-covariant pairing. Then following [D] we can state that the algebra $\wedge_-$ is a flatly deformed object as well.

Now, let us introduce ”the generic quantum orbits” by the following system of equations
\[
C^p_q - c^p = 0, \ c^p \in k, \ p = 1, ..., n.
\]

The constant $c^1$ is equal to 0 while the other constants are assumed to be generic. Let us note $k(M_q)$ the quotient of the RE algebra over the ideal $\{J\}$ generated by the l.h.s. elements of (4.20). This ”quantum coordinate ring” is a flat deformation of coordinate ring of a generic orbit in $sl(n)^\ast$.

As for q-analogues of other orbits of semisimple elements in $sl(n)^\ast$ the reader is referred to [DGK] where the case of the ”CP^n type orbits” was studied.

Now let us introduce quantum analogues of the cotangent module and its exterior powers on the orbits in question. (In the sequel all the modules are left.)

Consider the elements $d C^p_q$ looking like $d v_0$ of the previous section. This means that the differential $d$ is applied only to the last factor of the element $C^p_q$. Let us multiply the elements $d C^p_q$ by those of $k(M_q)$ from the left and the elements of $\wedge_-$ (in the sense of the algebra $\wedge_-$) from the right.
Now, consider the quotient of the left $k(M_q)$-module $k(M_q) \otimes \wedge^l$ over its submodule formed by the elements

$$f_p d C^p_q \wedge g_p, \quad f_p \in k(M_q), g_p \in \wedge^{l-1}$$

(hereafter $\wedge^l$ is the degree $l$ homogeneous component of the algebra $\wedge$). Conjecturally, this quotient is a flat deformation of the space of degree $l$ differential forms. A proof of this conjecture for the quantum sphere is given in [AC]. Also suggested in that paper was a $U_q(sl(n))$-covariant de Rham type complex which was a deformation of its classical counterpart without making use of the Leibniz rule.

However, the orbits in question are not multiplicity free any more and the problem of constructing a $U_q(sl(n))$-covariant complex which would be a flat deformation of the usual de Rham one becomes more delicate. Nevertheless, if we are only interested in $q$-analogues of 2-forms being generators of the cohomology ring on quantum orbits in question we can explicitly construct them in the following way (by analogy with the classical case).

It suffices to apply the $q$-cobracket to the last factor of each element $C^p_q$ and treat its image as an element of $\wedge^2$ (i.e. by realizing it as a sum of the summands $d x_i \wedge d x_j$).

By definition, the $q$-cobracket is the inverse (in some natural sense) of the $q$-Lie bracket whose construction was featured in [LS].

Let us call the above quotient module corresponding to the case $l = 1$ as the cotangent module and denote it as $T^\ast(M_q)$. Emphasize once more that this module is introduced as a one-sided module (namely, the left one but in the same way we can introduce the right one). Moreover, it is introduced explicitly by a system of equations. In a similar way we can realize the other modules defined above.

Now let us pass to defining the tangent module $T(M_q)$ on the orbits in question. In the classical case the tangent module $T(M)$ on a given regular affine variety $M$ has an operator realization by vector fields, i.e., there exists a map

$$T(M) \otimes k(M) \longrightarrow k(M) \quad (4.21)$$

which commutes with the $k(M)$-module structure product

$$k(M) \otimes T(M) \longrightarrow T(M).$$

Moreover, if $M$ is an orbit in $g^\ast$ there exists an embedding

$$g \hookrightarrow T(M)$$

such that map $(4.21)$ realizes a representation of Lie algebra $g$ by vector fields in the coordinate ring $k(M)$.

As for the tangent modules on quantum orbits $k(M_q)$ we define them by the same system as the cotangent ones. This is motivated by the fact that in the classical case the tangent and cotangent modules on orbits in question are isomorphic. So, conjecturally the deformation of the tangent module is flat.

However, in the quantum case there exists the problem of an operator meaning of the tangent module. For the case of the quantum sphere this problem was solved in [A]. Namely, it was shown that for the tangent quantum module $T(S^2_q)$ there exists a map

$$T(S^2_q) \otimes k(S^2_q) \longrightarrow k(S^2_q) \quad (4.22)$$
commuting with the module structure product
\[ k(S^2_q) \otimes T(S^2_q) \longrightarrow T(S^2_q). \]
Also in [A] an embedding was constructed of the form
\[ sl(2)_q \hookrightarrow T(S^2_q) \]
such that map (4.22) realized a representation of the \( q \)-Lie algebra \( sl(2)_q \) (this means that the relations between the generators of \( sl(2)_q \) in its enveloping algebra are preserved under map (4.22)).

We call braided vector fields the elements of \( T(S^2_q) \) realized as operators on \( k(S^2_q) \) via the map (4.22). However the problem of a similar treatment of the tangent modules on the quantum orbits in question is still open.

Let us complete the paper with the following remark. There exists a lot of articles devoted to different aspects of ”braided geometry”. However, often they do not consider the problem of flatness of quantum deformation. Nevertheless, a flat deformation is rather subtle phenomenon. Even if the flatness is fulfilled for a deformation of complexes related to a vector space, in general it disappears if one tries to restrict the differential algebras to a ”quantum variety”. We are sure that the approach making use of one-sided modules on quantum varieties developed here (as well as in [AG], [GS]) is more adequate for the needs of ”braided geometry” on quantum varieties since conjecturally it allows us to preserve the flatness of deformation (at the expense of the Leibniz rule).

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