Description of Unstable Systems in Relativistic Quantum Mechanics in the Lax-Phillips Theory*

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Abstract: We discuss some of the experimental motivation for the need for semigroup decay laws, and the quantum Lax-Phillips theory of scattering and unstable systems. In this framework, the decay of an unstable system is described by a semigroup. The spectrum of the generator of the semigroup corresponds to the singularities of the Lax-Phillips S-matrix. In the case of discrete (complex) spectrum of the generator of the semigroup, associated with resonances, the decay law is exactly exponential. The states corresponding to these resonances (eigenfunctions of the generator of the semigroup) lie in the Lax-Phillips Hilbert space, and therefore all physical properties of the resonant states can be computed. We show that the parametrized relativistic quantum theory is a natural setting for the realization of the Lax-Phillips theory.

1. Introduction

There has been considerable effort in recent years in the development of the theoretical framework of the scattering theory of Lax and Phillips\(^1\) for the description of quantum mechanical systems\(^2,3,4\). This work is motivated by the requirement that the decay law of a decaying system should be exactly exponential if the simple idea that a set of independent unstable systems consists of a population for which each element has a probability, say $\Gamma$, to decay, per unit time. The resulting exponential law ($\propto e^{-\Gamma t}$) corresponds to an exact semigroup evolution of the state in the underlying Hilbert space, defined as a family of bounded operators on that space satisfying

$$Z(t_1)Z(t_2) = Z(t_1 + t_2), \quad (1.1)$$

where $t_1, t_2 \geq 0$, and $Z(t)$ may have no inverse. If the decay of an unstable system is to be associated with an irreversible process, then its evolution necessarily has the property (1.1).\(^5\) The standard model of Wigner and Weisskopf\(^6\), based on the computation of the survival amplitude $A(t)$ as the scalar product

$$A(t) = \langle \psi, e^{-iHt} \psi \rangle \quad (1.2)$$

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where $\psi$ is the initial state of the unstable system and $H$ is the Hamiltonian for the full evolution, results in a good approximation to an exponential decay law for values of $t$ sufficiently large (Wigner and Weisskopf calculated an atomic linewidth in this way) but cannot result in a semigroup. When applied to a two-channel system, such as the decay of the $K^0$ meson, one easily sees that the poles of the resolvent for the Wigner-Weisskopf evolution of the two channel systems result in non-orthogonal residues that generate interference terms, which destroy the semigroup property, to accumulate in the calculation of predictions for regeneration experiments. The Yang-Wu parametrization of the $K^0$ decay processes, based on a Gamow type evolution generated by an effective 2x2 non-Hermitian matrix Hamiltonian, on the other hand, results in an evolution that is an exact semigroup. It appears that the phenomenological parametrization of refs. 9, which results in semigroup evolution, is indeed consistent to a high degree of accuracy with the experimental results on $K$-meson decay.

The quantum Lax-Phillips theory provides a framework for understanding the decay of an unstable system as an irreversible process. It appears, in fact, that this framework is categorical for the description of irreversible process.

The scattering theory of Lax and Phillips assumes the existence of a Hilbert space $\mathcal{H}$ of physical states in which there are two distinguished orthogonal subspaces $\mathcal{D}_+$ and $\mathcal{D}_-$ with the properties

$$
\begin{align*}
U(\tau) \mathcal{D}_+ & \subset \mathcal{D}_+ \quad \tau > 0 \\
U(\tau) \mathcal{D}_- & \subset \mathcal{D}_- \quad \tau < 0 \\
\bigcap_\tau U(\tau) \mathcal{D}_\pm & = \{0\} \\
\bigcup_\tau U(\tau) \mathcal{D}_\pm & = \mathcal{H},
\end{align*}
$$

i.e., the subspaces $\mathcal{D}_\pm$ are stable under the action of the full unitary dynamical evolution $U(\tau)$, a function of the physical laboratory time, for positive and negative times $\tau$ respectively; over all $\tau$, the evolution operator generates a dense set in $\mathcal{H}$ from either $\mathcal{D}_+$ or $\mathcal{D}_-$. We shall call $\mathcal{D}_+$ the outgoing subspace and $\mathcal{D}_-$ the incoming subspace with respect to the group $U(\tau)$.

A theorem of Sinai then assures that $\mathcal{H}$ can be represented as a family of Hilbert spaces obtained by foliating $\mathcal{H}$ along the real line, which we shall call $\{s\}$, in the form of a direct integral

$$
\mathcal{H} = \int_\oplus \mathcal{H}_s,
$$

where the set of auxiliary Hilbert spaces $\mathcal{H}_s$ are all isomorphic. Representing these spaces in terms of square-integrable functions, we define the norm in the direct integral space (we use Lesbesgue measure) as

$$
\|f\|^2 = \int_{-\infty}^{\infty} ds \|f_s\|_H^2,
$$

where $f \in \mathcal{H}$ represents $\mathcal{H}$ in terms of the $L^2$ function space $L^2(-\infty, \infty, H)$, and $f_s \in H$, the $L^2$ function space representing $\mathcal{H}_s$ for any $s$. We see that, in the framework of the
relativistic quantum theory\textsuperscript{15}, the evolution parameter $\tau$ corresponds to the invariant Stueckelberg evolution of the system, and the foliation asserted by Sinai\textsuperscript{14} is constructed within the measure space of the relativistic quantum theory Hilbert space at each $\tau$. The space $\mathcal{H}$ then corresponds to the usual (Stueckelberg) Hilbert space of states of a relativistic system; it therefore provides a natural setting for the realization of the Lax-Phillips theory.

The Sinai theorem furthermore asserts that there are representations for which the action of the full evolution group $U(\tau)$ on $L^2(-\infty, \infty, H)$ is translation by $\tau$ units. Given $D_{\pm}$ (the $L^2$ spaces representing $\mathcal{D}_{\pm}$), there is such a representation, called the \textit{incoming representation}\textsuperscript{1}, for which functions in $D_-$ have support in $L^2(-\infty, 0, H)$, and another called the \textit{outgoing representation}, for which functions in $D_+$ have support in $L^2(0, \infty, H)$.

Lax and Phillips\textsuperscript{1} show that there are unitary operators $W_{\pm}$, called wave operators, which map elements in $\mathcal{H}$, respectively, to these representations. They define an $S$-matrix,

$$ S = W_+ W_-^{-1} \quad (1.6) $$

which connects these representations; it is unitary, commutes with translations, and maps $L^2(-\infty, 0)$ into itself. The singularities of this $S$-matrix, in what we shall define as the \textit{spectral representation}, correspond to the spectrum of the generator of the exact semigroup characterizing the evolution of the unstable system.

With the assumptions stated above on the properties of the subspaces $D_+$ and $D_-$, Lax and Phillips\textsuperscript{1} prove that the family of operators

$$ Z(\tau) \equiv P_+ U(\tau) P_- \quad (\tau \geq 0), \quad (1.7) $$

where $P_{\pm}$ are projections into the orthogonal complements of $\mathcal{D}_{\pm}$, respectively, is a contractive, continuous, semigroup. This operator annihilates vectors in $\mathcal{D}_{\pm}$ and carries the space

$$ \mathcal{K} = \mathcal{H} \ominus \mathcal{D}_+ \ominus \mathcal{D}_- \quad (1.8) $$

into itself, with norm tending to zero for every element in $\mathcal{K}$.

Functions in the space $\mathcal{H}$, representing the elements of $\mathcal{H}$, depend on the variable $s$ as well as the variables of the auxiliary space $H$. In the nonrelativistic theory, the measure space of this Hilbert space of states is one dimension larger than that of a quantum theory represented in the auxiliary space alone. Identifying this additional variable with an \textit{observable} (in the sense of a quantum mechanical observable) time, we may understand this representation of a state in the nonrelativistic theory as a \textit{virtual history}. The collection of such histories forms a quantum ensemble; the absolute square of the wave function corresponds to the probability that the system would be found, as a result of measurement, at time $s$ in a particular configuration in the auxiliary space (in the state described by this wave function), i.e., an element of one of the virtual histories. This corresponds precisely to the interpretation of the relativistic wave functions of the Stueckelberg theory. The variable $s$, foliating the space according to such virtual histories, plays a conditional constraint role on the coordinatization of the spacetime configurations of the system.

\begin{footnote}
\textit{It follows from (1.7) and the stability of $\mathcal{D}_{\pm}$ that $Z(\tau) = P_{\mathcal{K}} U(\tau) P_{\mathcal{K}}$ as well.}
\end{footnote}
2. The Subspaces $D_{\pm}$, Representations, and the Lax-Phillips $S$-Matrix

The one-parameter unitary group $U(\tau)$ which acts on the Hilbert space $H$ is generated by the invariant operator $K$ which is the generator of dynamical evolution of the physical states in $H$; we assume that there exist wave operators $\Omega_{\pm}$ which intertwine this dynamical operator with an unperturbed dynamical operator $K_0$.\textsuperscript{16} We shall assume that $K_0$ has only absolutely continuous spectrum in $(-\infty, \infty)$.

We begin the development of the quantum Lax-Phillips theory with the construction of these representations. In this way, we shall construct explicitly the foliations described in Section 1.

The natural association of the time variable of the relativistic quantum theory with the foliation asserted by the theorem of Sinai\textsuperscript{14} does not correspond to the proper embedding of the relativistic quantum theory into the Lax-Phillips framework. It is, in fact, clear that for a many body system, one cannot single out a $t$ variable associated with a single particle; moreover, the time variable associated with the center of mass of a system\textsuperscript{15}, when it is well-defined, is not conjugate to the evolution operator ($i$ times its commutator with the evolution operator is the total $E/M$ of the system), and it is therefore not a candidate either. We solve this problem by constructing a foliation on the free translation representation of $K_0$. The free spectral representation of $K_0$ is defined by

$$f\langle \sigma | g \rangle = \sigma f\langle \sigma | g \rangle,$$

This equation corresponds to the Stueckelberg-Schrödinger equation\textsuperscript{15} in $\tau$-independent form, when $|g\rangle \to |x\rangle$; in this case, $K_0$ acting to the right becomes $1/2M$ times the d’Alembertian (or a sum of such operators for a many body system). The solution of the free Stueckelberg problem therefore provides the transformation function between the model representation, for which the spectral values of $x$ have their usual interpretation as observables in the laboratory, to the free spectral representation. Here, $|g\rangle$ is a general element of $H$ and $\beta$ corresponds to the variables (measure space) of the auxiliary space associated to each value of $\sigma$, which, with $\sigma$, comprise a complete spectral set. These constitute the complement in the measure space of the spectrum of $K_0$. We shall discuss the structure of this space in more detail elsewhere. The functions $f\langle \sigma | g \rangle$ may be thought of as a set of functions of the variables $\beta$ indexed on the variable $\sigma$ in a continuous sequence of auxiliary Hilbert spaces isomorphic to $H$.

We now proceed to define the incoming and outgoing subspaces $D_{\pm}$. To do this, we define the Fourier transform from representations according to the spectrum $\sigma$ to the foliation variable $s$ of (1.5), i.e.,

$$f\langle s | g \rangle = \int e^{i \sigma s} f\langle \sigma | g \rangle d\sigma.$$  

Clearly, $K_0$ acts as the generator of translations in this representation. We shall say that the set of functions $f\langle s | g \rangle$ are in the free translation representation.

Let us consider the sets of functions (dense) with support in $L^2(0, \infty)$ and in $L^2(-\infty, 0)$, and call these subspaces $D^\pm_0$. The Fourier transform back to the free spectral representation provides the two sets of Hardy class functions

$$f\langle \sigma | g_0^\pm \rangle = \int e^{-i \sigma s} f\langle s | g_0^\pm \rangle ds \in H^\pm,$$
for \( g_0^\pm \in D_0^\pm \).

We may now define the subspaces \( D_\pm \) in the Hilbert space of states \( \overline{H} \). To do this we first map these Hardy class functions in \( \overline{H} \) to \( \overline{H} \), i.e., we define the subspaces \( D_0^\pm \) by

\[
\int \sum_\beta \langle \sigma \beta | f \rangle | \sigma \beta \rangle_{D_0^\pm} \rangle d\sigma = D_0^\pm.
\]

(2.4)

As remarked above, we assume that there are wave operators which intertwine \( K_0 \) with the full evolution \( K \), i.e., that the limits

\[
\lim_{\tau \to \pm \infty} e^{iK\tau} e^{-iK_0\tau} = \Omega_\pm
\]

exist on a dense set in \( \overline{H} \).

The construction of \( D_\pm \) is then completed with the help of the wave operators. We define these subspaces by

\[
D_+ = \Omega_+ D_0^+
\]

\[
D_- = \Omega_- D_0^-.
\]

(2.6)

We remark that these subspaces are not produced by the same unitary map. This procedure is necessary to realize the Lax-Phillips structure non-trivially. If a single unitary map were used, then there would exist a transformation into the space of functions on \( L^2(-\infty, \infty, H) \) which has the property that all functions with support on the positive half-line represent elements of \( D_+ \), and all functions with support on the negative half-line represent elements of \( D_- \) in the same representation; the resulting Lax-Phillips S-matrix would then be trivial. The requirement that \( D_+ \) and \( D_- \) be orthogonal is not an immediate consequence of our construction; as we shall see, this result is associated with the analyticity of the operator which corresponds to the Lax-Phillips S-matrix.

In the following, we construct the Lax-Phillips S-matrix and the Lax-Phillips wave operators.

The wave operators defined by (2.5) intertwine \( K \) and \( K_0 \), i.e.,

\[
K\Omega_\pm = \Omega_\pm K_0.
\]

(2.7)

We may therefore construct the outgoing (incoming) spectral representations from the free spectral representation. Since

\[
K\Omega_\pm |\sigma \beta\rangle_f = \Omega_\pm K_0 |\sigma \beta\rangle_f
\]

\[
= \sigma \Omega_\pm |\sigma \beta\rangle_f,
\]

(2.8)

we may identify

\[
|\sigma \beta\rangle_{\text{out}} = \Omega_\pm |\sigma \beta\rangle_f.
\]

(2.9)

Let us now act on these functions with the Lax-Phillips S-matrix in the free spectral representation, and require the result to be the outgoing representer of \( g \):

\[
\langle \text{out} | \sigma \beta | g \rangle = \int d\sigma' \sum_{\beta'} f \langle \sigma \beta | \mathbf{S} | \sigma' \beta' \rangle_f \langle \sigma' \beta' | \Omega_{-1} g \rangle
\]

(2.10)
where $S$ is the Lax-Phillips $S$-operator (defined on $\mathcal{H}$). Transforming the kernel to the free translation representation with the help of (2.2), i.e.,

$$f\langle s\beta|S|s'\beta'\rangle_f = \frac{1}{(2\pi)^2} \int d\sigma d\sigma' e^{i\sigma s} e^{-i\sigma' s'} f\langle \sigma\beta|S|\sigma'\beta'\rangle_f,$$

(2.11)

we see that the relation (2.10) becomes, after using the Fourier transform in a similar way to transform the in and out spectral representations to the corresponding in and out translation representations,

$$\text{out}(s\beta|g) = f\langle s\beta|\Omega_+^{-1}g\rangle = \int ds' \sum_{\beta'} f\langle s\beta|S|s'\beta'\rangle_f f\langle s'\beta'|\Omega_-^{-1}g\rangle$$

$$= \int ds' \sum_{\beta'} f\langle s\beta|S|s'\beta'\rangle_f \text{in}(s'\beta'|g).$$

(2.12)

Hence the Lax-Phillips $S$-matrix is given by

$$S = \{f\langle s\beta|S|s'\beta'\rangle_f\},$$

(2.13)

in free translation representation. It follows from the intertwining property (2.7) that

$$f\langle \sigma\beta|S|\sigma'\beta'\rangle_f = \delta(\sigma - \sigma')S^{\beta\beta'}(\sigma).$$

(2.14)

3. Orthogonality of $\mathcal{D}_\pm$ and Analyticity of the $S$-Matrix.

The orthogonality of $\mathcal{D}_\pm$ follows from the analytic properties of the $S$-matrix. To display this analyticity property, we study the operator $S$ in the form (from (2.10); this operator coincides with the relativistic quantum mechanical $S$-matrix)

$$S = \Omega_+^{-1}\Omega_- = \lim_{\tau \to \infty} e^{iK_0\tau} e^{-2iK\tau} e^{iK_0\tau}. $$

(3.1)

It then follows in the standard way\textsuperscript{17} that

$$f\langle \sigma\beta|S|\sigma'\beta'\rangle_f = \delta(\sigma - \sigma')\{\delta^{\beta\beta'} - 2\pi i f\langle \sigma\beta|T(\sigma + i\epsilon)|\sigma\beta'\rangle_f\},$$

(3.2)

where

$$T(z) = V + VG(z)V = V + VG_0T(z).$$

(3.3)

We remark that, by this construction, we see that $S^{\beta\beta'}(\sigma)$ is analytic in the upper half plane in $\sigma$.

We have constructed the incoming and outgoing subspaces $\mathcal{D}_\pm$ in (2.6). It is essential for application of the Lax-Phillips theory that these subspaces be orthogonal, i.e., for every $f_+ \in \mathcal{D}_+$, $f_- \in \mathcal{D}_-$, that $(f_+, f_-) = 0$. If

$$f_+ = \Omega_+ f_0^+,$$

$$f_- = \Omega_- f_0^-,$$

(3.4)
mapped from functions in \( D_0^- \), we see that the orthogonality condition is

\[
(f_+, f_-) = (f_0^+, \Omega_+^{-1}\Omega_- f_0^-) = 0. \tag{3.5}
\]

As shown in (2.11), the \( S \)-matrix in free representation transforms the incoming to the outgoing representation; we may therefore write the scalar product in (3.5) as

\[
(f_+, f_-) = \sum_{\beta\beta'} \int ds ds' (f_0^+ | s\beta\rangle_{\text{out}} f(s|\sigma|s'\beta' \rangle f | s'\beta' \rangle_{\text{in}} (s'\beta'| f_0^-) \tag{3.6}
\]

Now, in the free translation representation, we have

\[
f(s|\sigma|s'\beta')_f = \int d\sigma d\sigma' e^{i\sigma s} e^{-i\sigma' s'} f(s|\sigma|\sigma'\beta')_f = \int d\sigma e^{i\sigma (s-s')} S_{\beta\beta'}^\sigma (s-s') \tag{3.7}
\]

The function \( S(\sigma)_{\beta\beta'} \) is analytic in the upper half plane; it may have a null co-space, but is otherwise regular. Its singularity lies in the lower half plane. To find a non-vanishing value for \( S_{\beta\beta'}^\sigma (s-s') \), we must close the contour in the lower half plane. This can only be done if \( s' > s \). For \( s' < s \), one must close in the upper half plane, and there \( S(\sigma) \) has no singularity, so the integral vanishes. Hence \( S_{\beta\beta'}^\sigma (s-s') \) takes \( D_+ \) to \( D_- \) in the incoming representation, and the subspaces \( D_+ \) and \( D_- \) are orthogonal.

4. Conclusions and Discussion

It was shown that a necessary condition for a non-trivial Lax-Phillips theory, for which the singularities of the \( S \)-matrix in the spectral variable constitute the spectrum of the generator of the semigroup, is that the evolution operator act as a smooth (operator-valued) integral kernel on the time axis in the free translation representation.\(^4\) We have shown in this paper that a pointwise (in spacetime \( x \)) dynamical evolution operator in what we have called the model representation, in which the Hamiltonian of a system and its spacetime variables appear with their usual laboratory interpretation, maps into a smooth, non-trivial kernel (through (2.1)) in the free translation representation, and therefore satisfies this necessary condition. The relativistic quantum theory therefore provides a natural framework for the Lax-Phillips theory.

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References
1. P.D. Lax and R.S. Phillips, *Scattering Theory*, Academic Press, N.Y. (1967).
2. C. Flesia and C. Piron, Helv. Phys. Acta 57, 697 (1984).
3. L.P. Horwitz and C. Piron, Helv. Phys. Acta 66, 694 (1993).
4. E. Eisenberg and L.P. Horwitz, in *Advances in Chemical Physics*, XCIX, p. 245, ed. I. Prigogine and S. Rice, John Wiley and Sons, N.Y. (1997).
5. C. Piron, *Foundations of Quantum Physics*, Benjamin/Cummings, Reading, Mass. (1976).
6. V.F. Weisskopf and E.P. Wigner, Zeits. f. Phys. 63, 54 (1930); 65, 18 (1930).
7. L.P. Horwitz, J.P. Marchand and J. LaVita, J. Math. Phys. 12, 2537 (1971); D. Williams, Comm. Math. Phys. 21, 314 (1971).
8. L.P. Horwitz and L. Mizrachi, Nuovo Cimento 21A, 625 (1974).
9. T.D. Lee, R. Oehme and C.N. Yang, Phys. Rev. 106, 340 (1957); T.T. Wu and C.N. Yang, Phys. Rev. Lett. 13, 380 (1964).
10. G. Gamow, Z. Phys. 51, 204 (1928).
11. B. Weinstein, *et al.*, *Results from the Neutral Kaon Program at Fermilab’s Meson Center Beamline, 1985-1997*, FERMILAB-Pub-97/087-E, published on behalf of the E731, E773 and E799 Collaborations, Fermi National Accelerator Laboratory, P.O. Box 500, Batavia, Illinois 60510.
12. S.R Wilkinson, C.F. Bharucha, M.C. Fischer, K.W. Madison, P.R. Morrow, Q. Niu, B. Sundaram, and M. Raizen, Nature 387, 575 (1997).
13. W. Baumgartel, Math. Nachr. 69, 107 (1975); L.P. Horwitz and I.M. Sigal, Helv. Phys. Acta 51, 685 (1978); G. Parravicini, V. Gorini and E.C.G. Sudarshan, J. Math. Phys. 21, 2208 (1980); A. Bohm, *Quantum Mechanics: Foundations and Applications*, Springer, Berlin (1986); A. Bohm, M. Gadella and G.B. Mainland, Am. J. Phys. 57, 1105 (1989); T. Bailey and W.C. Schieve, Nuovo Cimento 47A, 231 (1978).
14. I.P. Cornfield, S.V. Formin and Ya. G. Sinai, *Ergodic Theory*, Springer, Berlin (1982).
15. E.C.G. Stueckelberg, Helv. Phys. Acta 14, 372, 588(1941); 15, 23 (1942); L.P. Horwitz and C. Piron, Helv. Phys. Acta 48, 316 (1974); R.E. Collins and J.R. Fanchi, Nuovo Cimento 48A, 314 (1978); J.R. Fanchi, *Parameterized Relativistic Quantum Theory*, Kluwer, Norwell, Mass. (1993), and references therein.
16. L.P. Horwitz and A. Soffer, Helv. Phys. Acta 53, 112 (1980).
17. For example, J.R. Taylor, *Scattering Theory*, John Wiley and Sons, N.Y. (1972); R.J. Newton, *Scattering Theory of Particles and Waves*, McGraw Hill, N.Y. (1976).