PURITY OF THE EMBEDDINGS OF OPERATOR SYSTEMS INTO THEIR C*- AND INJECTIVE ENVELOPES

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ABSTRACT. We study the issue of issue of purity (as a completely positive linear map) for identity maps on operator systems and for their completely isometric embeddings into their C*-envelopes and injective envelopes. Our most general result states that the canonical embedding of an operator system \( \mathcal{R} \) into its injective envelope \( I(\mathcal{R}) \) is pure if and only if the C*-envelope \( C^*_e(\mathcal{R}) \) of \( \mathcal{R} \) is a prime C*-algebra. To prove this, we also show that the identity map on any AW*-factor is a pure completely positive linear map.

For embeddings of operator systems \( \mathcal{R} \) into their C*-envelopes, the issue of purity is seemingly harder to describe in full generality, and so we focus here on operator systems arising from the generators of discrete groups. Two such operator systems of interest are denoted by \( S_n \) and \( \text{NC}(n) \). The former corresponds to the generators of the free group \( \mathbb{F}_n \), while the latter corresponds to the generators of the group \( \mathbb{Z}_2 \ast \cdots \ast \mathbb{Z}_2 \), the free product of \( n \) copies of \( \mathbb{Z}_2 \). The operator systems \( S_n \) and \( \text{NC}(n) \) are of interest in operator theory for their connections to the weak expectation property and C*-nuclearity, and for their universal properties. Specifically, \( S_n \) is the universal operator system for arbitrary \( n \)-tuples of contractions acting on a Hilbert space and \( \text{NC}(n) \) is the universal operator system for \( n \)-self-adjoint contractions. We show that the embedding of \( S_n \) into \( C^*_e(S) \) is pure for all \( n \geq 2 \) and the embedding of \( \text{NC}(n) \) into \( C^*_e(\text{NC}(n)) \) is pure for every \( n \geq 3 \).

The question of purity of the identity is quite subtle for operator system that are not C*-algebras, and we have results only for the operator systems \( S_n \) and \( \text{NC}(n) \).

Lastly, a previously unrecorded feature of pure completely positive linear maps is presented: every pure completely positive linear map on an operator system \( \mathcal{R} \) into an injective von Neumann algebra \( \mathcal{M} \) has a pure completely positive extension to any operator system \( \mathcal{T} \) that contains \( \mathcal{R} \) as an operator subsystem, thereby generalising a result of Arveson for the injective type I factor \( B(H) \).

1. INTRODUCTION

A face \( \mathcal{F} \) in a proper convex cone \( \mathcal{C} \) is a half-line face \([20]\) p. 182] if there exists an element \( \phi \in \mathcal{C} \) such that

\[
\mathcal{F} = \{ t\phi \mid t \in \mathbb{R}, t \geq 0 \}.
\]

Half-line faces of convex cones are the analogues, for cones, of the notion of extreme points of convex sets. Under topological conditions such as closedness, convex combinations of elements taken from various half-line faces of a convex cone \( \mathcal{C} \) completely determine the cone \( \mathcal{C} \) \([20]\) Theorem 18.5]. In particular, any generator

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φ of a half-line face of a convex cone C is an extreme point of any convex subset E ⊆ C that contains φ.

The purpose of this paper is to study such faces in the case where the cone C is the cone \(\mathcal{CP}(R, T)\) of completely positive linear maps from an operator system \(R\) into an operator system \(T\). The generators φ of such half-line faces of \(\mathcal{CP}(R, T)\) are said to be pure completely positive linear maps. Expressed differently, a completely positive linear map \(φ : R → T\) is irreducible if, for any completely positive linear maps \(θ, ω : R → T\) such that \(θ + ω = φ\), there necessarily exists a scalar \(s ∈ [0, 1]\) such that \(θ = sφ\) and \(ω = (1 − s)φ\).

Henceforth, \(\mathcal{CP}(R, T)\) will be denoted by \(\mathcal{CP}(R)\) when the domain \(R\) and codomain \(T\) are equal (as operator systems).

If \(R\) is a unital C∗-algebra \(A\) and if \(T = B(H)\), the von Neumann algebra of bounded linear operators on a Hilbert space \(H\), then there is a very satisfactory and readily applicable criterion discovered by Arveson: namely, \(φ\) is a pure element of \(\mathcal{CP}(A, B(H))\) if and only if, for any minimal Stinespring decomposition \(φ = w^*πw\) of \(φ\), the representation \(π\) is irreducible [Corollary 1.4.3]. Replacing the C∗-algebra \(A\) by an operator system \(R\) is somewhat more problematic, but there is, nevertheless, a geometric criterion for a unital completely positive (ucp) linear map \(φ : R → M_n(\mathbb{C})\), where \(M_n(\mathbb{C})\) is the C∗-algebra of \(n × n\) complex matrices, to be a pure element of \(\mathcal{CP}(R, M_n(\mathbb{C}))\): namely, \(φ\) is pure if and only if \(φ\) is a matrix extreme point in the compact free convex set \(S(R)\) of all matrix-valued ucp maps on \(R\) [7]. Consequently, the results of [1] and [7] indicate that arbitrary elements of \(\mathcal{CP}(A, B(H))\) and \(\mathcal{CP}(R, M_n(\mathbb{C}))\) can be viewed as operator (or matrix) convex combinations of pure completely positive linear maps. In this sense, then, pure completely positive linear maps determine all completely positive linear maps.

The most general case occurs when both \(R\) and \(T\) are arbitrary operator systems; however, in such cases the absence of any structure theory (such as a Stinespring decomposition) makes the determination of pure elements of \(\mathcal{CP}(R, T)\) very difficult.

Arveson’s criterion demonstrates that the identity map \(ι : B(H) → B(H)\) is a pure element of \(\mathcal{CP}(B(H))\). It is, therefore, natural to ask: for which operator systems \(R\) is the identity map \(ι : R → R\) a pure element of the cone \(\mathcal{CP}(R)\)? A closely related question involves embeddings: if \(R ⊆ T\) is an inclusion of operator systems, then is the canonical inclusion map \(ι : R → T\) a pure element of \(\mathcal{CP}(R, T)\)? In particular, for this second question, is the inclusion of an operator system \(R\) into its C∗-envelope \(C_ω^*(R)\) pure in the cone \(\mathcal{CP}(R, C_ω^*(R))\)?

Thus, in this paper we are concerned with the following two questions.

(Q1) For which operator systems \(R\) is the identity map \(ι : R → R\) pure in the cone \(\mathcal{CP}(R)\)?

(Q2) For which operator systems \(R\) is the embedding \(ι_e : R → C_ω^*(R)\) pure in the cone \(\mathcal{CP}(R, C_ω^*(R))\)?

An operator system \(J\) is injective if, for every operator system \(R\) and every operator system \(T\) containing \(R\) as an operator subsystem, each completely positive linear map \(φ : R → T\) has an extension to a completely positive linear map \(Φ : T → J\). Arveson’s Hahn-Banach Extension Theorem [Theorem 1.2.3] states that \(B(H)\) is an injective operator system for every Hilbert space \(H\). Hamana [12] provided crucial additional information about inclusions of operator systems into injective
operator systems, showing that every operator system $\mathcal{R}$ is an operator subsystem of some minimal injective operator system $I(\mathcal{R})$, which is called the injective envelope of $\mathcal{R}$. If we denote the canonical inclusion of $\mathcal{R}$ into $I(\mathcal{R})$ by $\iota_\mathcal{R}$, then this map is a unital completely positive order embedding of the operator system $\mathcal{R}$ into the operator system $I(\mathcal{R})$. The final question addressed in this paper is:

(Q3) For which operator systems $\mathcal{R}$ is the embedding $\iota_\mathcal{R} : \mathcal{R} \to I(\mathcal{R})$ pure in the cone $\mathcal{C}(\mathcal{R}, I(\mathcal{R}))$?

In this paper we provide a complete answer to question (Q3); we also answer question (Q2) for operator systems that arise from generators of discrete groups. Question (Q1), however, is difficult to answer in a generic manner. Therefore, we only address question (Q1) for two classes of finite-dimensional operator systems, denoted by $S_n$ and $\text{NC}(n)$. These two classes of operator systems are relevant for their universal properties, for their encoding of quantum correlations, and for their role in operator-algebraic questions concerning the weak expectation property and C*-nuclearity \cite{8,9,10,11,15}.

2. Operator Systems and Pure Completely Positive Linear Maps

Our general reference for operator system theory are the books of Paulsen \cite{19} and Effros and Ruan \cite{6}.

2.1. Operator systems and their C*- and injective envelopes. If $\mathcal{R}$ is a complex *-vector space, then the space $M_n(\mathcal{R})$ of $n \times n$ matrices over $\mathcal{R}$ is also a complex *-vector space in which the adjoint of an $n \times n$ matrix $[x_{ij}]_{i,j=1}^n$ of elements $x_{ij} \in \mathcal{R}$ is defined by $( [x_{ij}]_{i,j=1}^n)^* = [x_{ji}^*]_{i,j=1}^n$. A matrix ordering of a complex *-vector space $\mathcal{R}$ is a family $(\mathcal{C}_n)_{n \in \mathbb{N}}$ of subsets $\mathcal{C}_n$ of the real vector spaces $(M_n(\mathcal{R}))_{sa}$ of selfadjoint matrices over $\mathcal{R}$ such that, for all $n, m \in \mathbb{N}$, (i) $\mathcal{C}_n$ is a convex cone, (ii) $\mathcal{C}_n \cap (-\mathcal{C}_n) = \{0\}$, and (iii) $\alpha^* x \alpha \in \mathcal{C}_m$ for all $x \in \mathcal{C}_n$ and all complex $n \times m$ matrices $\alpha$.

An element $e_\mathcal{R} \in C_1$ is an Archimedean order unit for a matrix ordering $(\mathcal{C}_n)_{n \in \mathbb{N}}$ of a complex *-vector space $\mathcal{R}$ if, for every $n \in \mathbb{N}$, the condition $e_\mathcal{R}^{[n]} + eQ \in \mathcal{C}_n$ holds for every real $\varepsilon > 0$ only for $Q \in \mathcal{C}_n$. Here, $e_\mathcal{R}^{[n]} = e_\mathcal{R} \oplus \cdots \oplus e_\mathcal{R}$, the $n$-fold direct sum of $e_\mathcal{R}$. Note that if there exists an Archimedean order unit for a matrix ordering of $\mathcal{R}$, then there are infinitely many choices for this order unit.

Formally, an operator system is a triple $(\mathcal{R}, (\mathcal{C}_n)_{n \in \mathbb{N}}, e_\mathcal{R})$ consisting of a complex *-vector space $\mathcal{R}$, a matrix ordering $(\mathcal{C}_n)_{n \in \mathbb{N}}$ of $\mathcal{R}$, and a distinguished element $e_\mathcal{R}$ that serves as an Archimedean order unit for the matrix ordering $(\mathcal{C}_n)_{n \in \mathbb{N}}$ of $\mathcal{R}$. Unless it is necessary to make explicit reference to the matrix ordering $(\mathcal{C}_n)_{n \in \mathbb{N}}$ and/or the order unit $e_\mathcal{R}$, the triple $(\mathcal{R}, (\mathcal{C}_n)_{n \in \mathbb{N}}, e_\mathcal{R})$ will be denoted simply by $\mathcal{R}$. The matrix cones $\mathcal{C}_n$ of an operator system $\mathcal{R}$ are generally denoted by $M_n(\mathcal{R})_{+}$.

Every unital C*-algebra $\mathcal{A}$ is an operator system, where the matrix cones are given by $\mathcal{C}_n = M_n(\mathcal{A})_{+}$, the cone positive elements of the C*-algebra $M_n(\mathcal{A})$ and the Archimedean order unit is the multiplicative identity $1$ of $\mathcal{A}$, which is the canonical choice of Archimedean order unit for this matrix ordering of $\mathcal{A}$.

If $\mathcal{R}$ and $\mathcal{I}$ are operator systems, then a linear map $\phi : \mathcal{R} \to \mathcal{I}$ is completely positive if $\phi^{[n]} : M_n(\mathcal{R}) \to M_n(\mathcal{I})$ maps $M_n(\mathcal{R})_{+}$ into $M_n(\mathcal{I})_{+}$, for every $n \in \mathbb{N}$, where $\phi^{[n]}$ is defined by $\phi^{[n]}([x_{ij}]_{i,j=1}^n) = [\phi(x_{ij})]_{i,j=1}^n$. If, in addition, $\phi(e_\mathcal{R}) = e_\mathcal{I}$, then $\phi$ is said to be a unital completely positive linear map, or a ucp map. A
linear isomorphism \( \phi : \mathcal{R} \to \mathcal{I} \) in which both \( \phi \) and \( \phi^{-1} \) are completely positive is called a complete order isomorphism. An one-to-one completely positive linear map \( \phi : \mathcal{R} \to \mathcal{I} \) in which \( \phi(\mathcal{R}) \) is an operator subsystem of \( \mathcal{I} \) is called a complete order embedding of \( \mathcal{R} \) into \( \mathcal{I} \) if \( \phi \) is a complete order isomorphism when considered as a map of the operator system \( \mathcal{R} \) onto the operator system \( \phi(\mathcal{R}) \).

The notation \( \mathcal{R} \simeq \mathcal{I} \) indicates the existence of a unital complete order isomorphism \( \phi : \mathcal{R} \to \mathcal{I} \), although we shall also have need of complete order isomorphisms that are not unital.

The Choi–Effros Embedding Theorem \([12]\) states that every operator system \( \mathcal{R} \) is unitaly completely order isomorphic to an operator subsystem of \( \mathcal{B}(\mathcal{K}) \) for some Hilbert space \( \mathcal{K} \). Therefore, every operator system \( \mathcal{R} \) is capable of generating a C*-algebra. Of all such possibilities, our interest is with the minimal one, which is called the C*-envelope of \( \mathcal{R} \) and whose existence was first established by Hamana \([12]\).

**Definition 2.1.** A C*-envelope of an operator system \( \mathcal{R} \) is a pair \((\mathcal{A}, \tau_e)\) consisting of

1. a unital C*-algebra \( \mathcal{A} \), and
2. a unital complete order embedding \( \tau_e : \mathcal{R} \to \mathcal{A} \) such that \( \tau_e(\mathcal{R}) \) generates the C*-algebra \( \mathcal{A} \)

such that, for every unital complete order embedding \( \kappa : \mathcal{R} \to \mathcal{B} \) of \( \mathcal{R} \) into a unital C*-algebra \( \mathcal{B} \) for which \( \kappa(\mathcal{R}) \) generates \( \mathcal{B} \), there exists a unital *-homomorphism \( \pi : \mathcal{B} \to \mathcal{A} \) with \( \pi \circ \kappa = \tau_e \).

**Theorem 2.2.** (Hamana, \([12]\)) Every operator system \( \mathcal{R} \) admits a C*-envelope \((\mathcal{A}, \tau_e)\). Furthermore, if \((\tilde{\mathcal{A}}, \tilde{\tau}_e)\) is any other C*-envelope of \( \mathcal{R} \), then there exists a unital *-isomorphism \( \rho : \tilde{\mathcal{A}} \to \mathcal{A} \) such that \( \rho \circ \tilde{\tau}_e = \tau_e \).

Because, by Theorem 2.2, the C*-envelope of an operator system is unique up to isomorphism, we shall use the notation \( C^*_e(\mathcal{R}) \) to denote a (the) C*-envelope of \( \mathcal{R} \) and \( \tau_e \) to denote a (the) unital complete order embedding of \( \mathcal{R} \) into \( C^*_e(\mathcal{R}) \).

Recall that an operator system \( \mathcal{I} \) is injective if, for every operator system \( \mathcal{R} \) and every operator system \( \mathcal{J} \) that contains \( \mathcal{R} \) as an operator subsystem, every completely positive linear maps \( \phi : \mathcal{R} \to \mathcal{J} \) has an extension \( \Phi : \mathcal{I} \to \mathcal{J} \) such that \( \Phi \) is completely positive.

**Definition 2.3.** An injective envelope of an operator system \( \mathcal{R} \) is a pair \((\mathcal{I}, \iota_{ie})\) consisting of

1. an injective operator system \( \mathcal{I} \), and
2. a unital complete order embedding \( \iota_{ie} : \mathcal{R} \to \mathcal{I} \)

such that, for every inclusion \( \iota_{ie}(\mathcal{R}) \subseteq \mathcal{Q} \subseteq \mathcal{I} \) as operator subsystems in which \( \mathcal{Q} \) is injective, then necessarily \( \mathcal{Q} = \mathcal{I} \).

**Theorem 2.4.** (Hamana, \([12]\)) Every operator system \( \mathcal{R} \) admits an injective \((\mathcal{I}, \iota_{ie})\). Furthermore, if \((\tilde{\mathcal{I}}, \tilde{\iota}_e)\) is any other injective envelope of \( \mathcal{R} \), then there exists a unital complete order isomorphism \( \phi : \tilde{\mathcal{I}} \to \mathcal{I} \) such that \( \phi \circ \tilde{\iota}_e = \iota_{ie} \).

In light of the uniqueness of the injective envelope up to complete order isomorphism, an injective envelope of \( \mathcal{R} \) is denoted by \( I(\mathcal{R}) \).

By the Choi–Effros Embedding Theorem \([12]\), every operator system \( \mathcal{R} \) is completely order isomorphic to an operator subsystem of \( \mathcal{B}(\mathcal{K}) \) for some Hilbert space \( \mathcal{K} \). Thus, every operator system is an operator subsystem of an injective operator
system. However, Hamana [12] proved that if \( \mathcal{R} \) is an operator subsystem of \( \mathcal{B}(\mathcal{H}) \), then there exists a unital completely positive idempotent map \( \varepsilon : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}) \) such that the range of \( \varepsilon \) is an (the) injective of \( \mathcal{R} \). This operator system \( \mathcal{I}(\mathcal{R}) \) is unital completely order isomorphic to an injective C*-algebra \( \mathcal{B} \) via the Choi-Effros product: specifically, \( \mathcal{B} \) is given by the operator system \( \mathcal{I}(\mathcal{R}) \) whereby the product \( x \circ y \) of \( x, y \in \mathcal{I}(\mathcal{R}) \) is defined to be \( x \circ y = \varepsilon(xy) \), with \( xy \) denoting the product of \( x \) and \( y \) in \( \mathcal{B}(\mathcal{H}) \). Within \( \mathcal{B} \), the unital C*-algebra generated by \( \mathcal{R} \) is isomorphic to the C*-envelope \( \mathcal{C}_\mathcal{R}^*(\mathcal{R}) \) of \( \mathcal{R} \). These results are summarised in the theorem below, along with a crucial property known as rigidity.

**Theorem 2.5.** (Hamana, [12]) For every operator system \( \mathcal{R} \), the injective envelope \( \mathcal{I}(\mathcal{R}) \) of \( \mathcal{R} \) has the following properties:

1. (C*-envelope) there is an injective C*-algebra \( \mathcal{B} \) and a unital complete order isomorphism \( \phi : \mathcal{I}(\mathcal{R}) \to \mathcal{B} \) such that, if \( A \) denotes the C*-subalgebra of \( \mathcal{B} \) generated by \( \phi(I_\mathcal{R}) \), then \( (A, \phi \circ I_\mathcal{R}) \) is a C*-envelope of \( \mathcal{R} \);
2. (Rigidity) if \( \omega : \mathcal{I}(\mathcal{R}) \to \mathcal{I}(\mathcal{R}) \) is a completely positive linear map for which \( \omega \circ I_\mathcal{R} = I_\mathcal{R} \), then \( \omega \) is the identity map on \( \mathcal{I}(\mathcal{R}) \).

Theorem 2.5 indicates that the injective envelope of \( \mathcal{R} \), when viewed as a unital injective C*-algebra, contains a copy of the C*-envelope of \( \mathcal{R} \) as a unital C*-subalgebra.

### 2.2. Dual operator systems, minimal tensor products, and entanglement.

As every operator system is a normed vector space [4, p. 179], the dual space \( \mathcal{R}^d \) of an operator system \( \mathcal{R} \) is a Banach space. A matrix ordering of \( \mathcal{R}^d \) occurs when we declare an \( n \times n \) matrix \( \mathcal{G} = [\gamma_{ij}]_{i,j=1}^n \) of linear functionals on \( \mathcal{R} \) to be positive if the linear function \( \mathcal{G} : \mathcal{R} \to \mathcal{M}_n(\mathbb{C}) \) defined by

\[
\mathcal{G}(x) = [\gamma_{ij}(x)]_{i,j=1}^n, \text{ for } x \in \mathcal{R},
\]

is completely positive [4, Lemma 4.2]. While it is not true that this matrix ordering admits an Archimedean order unit for every operator system, the duals of finite-dimensional operator systems do possess an Archimedean order unit for this matrix ordering [4, Corollary 4.5]. Indeed, every faithful state \( \delta \) on \( \mathcal{R} \) is an Archimedean order unit [15 Lemma 2.5] for the matrix ordering of \( \mathcal{R}^d \). Thus, the choice of Archimedean order unit \( \delta \) for the dual \( \mathcal{R}^d \) of a finite-dimensional operator system \( \mathcal{R} \) is not canonical.

With regards to questions of purity, the use of an operator system dual can be useful in light of the following straightforward result.

**Proposition 2.6.** Suppose that \( \mathcal{R} \) and \( \mathcal{T} \) are finite-dimensional operator systems and that \( \phi : \mathcal{R} \to \mathcal{T} \) is a completely positive linear map. Then:

1. the linear adjoint \( \phi^d : \mathcal{T}^d \to \mathcal{R}^d \) is completely positive, and
2. \( \phi \) is pure in \( \mathcal{CP}(\mathcal{R}, \mathcal{T}) \) if and only if \( \phi^d \) is pure in \( \mathcal{CP}(\mathcal{T}^d, \mathcal{R}^d) \).

The algebraic tensor product \( \mathcal{R} \otimes \mathcal{T} \) of operator systems \( \mathcal{R} \) and \( \mathcal{T} \) is a complex *-vector space, and there are many possible matrix orderings that are induced by the matrix orderings of \( \mathcal{R} \) and \( \mathcal{T} \) that give \( \mathcal{R} \otimes \mathcal{T} \) the structure of an operator system [16]. In the case where \( \mathcal{R} = \mathcal{M}_k(\mathbb{C}) \) and \( \mathcal{T} = \mathcal{M}_m(\mathbb{C}) \), there is a unique operator system tensor product structure on \( \mathcal{R} \otimes \mathcal{T} \) and it is the one induced by considering \( \mathcal{M}_n(\mathcal{M}_k(\mathbb{C}) \otimes \mathcal{M}_m(\mathbb{C})) \) as a unital C*-algebra.
Definition 2.7. ([16, §4]) The minimal operator system tensor product \( \mathcal{R} \otimes_{\min} \mathcal{T} \) of operator systems \( \mathcal{R} \) and \( \mathcal{T} \) is the operator system whose matrix ordering is defined so that a matrix \( X = [x_{ij}] \in M_n(\mathcal{R} \otimes \mathcal{T}) \) is positive if
\[
[(\phi \otimes \psi)(x_{ij})]_{i,j=1}^n \in (M_n(M_k(\mathbb{C}) \otimes M_m(\mathbb{C})))_+,
\]
for all unital completely positive linear maps \( \phi : \mathcal{R} \to M_k(\mathbb{C}) \) and \( \psi : \mathcal{R} \to M_m(\mathbb{C}) \) and for all \( k, m \in \mathbb{N} \), and whose canonical Archimedean order unit is given by \( e_\mathcal{R} \otimes e_\mathcal{T} \).

The minimal operator system tensor product \( \mathcal{R} \otimes_{\min} \mathcal{T} \) of operator systems \( \mathcal{R} \) and \( \mathcal{T} \) may be realised by representing \( \mathcal{R} \) and \( \mathcal{T} \) as operator subsystems of \( \mathcal{B}(\mathcal{H}) \) and \( \mathcal{B}(\mathcal{K}) \), respectively, and then endowing the vector space \( \mathcal{R} \otimes \mathcal{T} \) of operators on the Hilbert space \( \mathcal{H} \otimes \mathcal{K} \) with the operator system structure induced by the operator system structure of \( \mathcal{B}(\mathcal{H} \otimes \mathcal{K}) \) [16, Theorem 4.4].

If \( V \) is a finite-dimensional vector space, then the tensor product \( V \otimes \mathbb{V}^d \) of \( V \) with its dual \( \mathbb{V}^d \) is linearly isomorphic to \( \mathcal{L}(V) \), the vector space of linear transformations on \( V \). If we apply this linear isomorphism to a finite-dimensional operator system \( \mathcal{R} \) and its operator system dual \( \mathcal{R}^d \), then the cone \( \mathcal{C}(\mathcal{R}) \) in \( \mathcal{L}(\mathcal{R}) \) of completely positive linear maps on \( \mathcal{R} \) determines a cone in \( \mathcal{R} \otimes \mathcal{R}^d \). This cone is in fact the positive cone of \( \mathcal{R} \otimes_{\min} \mathcal{R}^d \), where \( \otimes_{\min} \) denotes the minimal operator system tensor product [16, §4], [17, Lemma 8.5].

The canonical linear isomorphism between \( \mathcal{R} \otimes \mathcal{R}^d \) and \( \mathcal{L}(\mathcal{R}) \) is the one that maps elementary tensors \( x \otimes \psi \in \mathcal{R} \otimes \mathcal{R}^d \) to rank-1 linear transformations \( \tau \mapsto \psi(\tau)x \), for \( \tau \in \mathcal{R} \). Let \( \Gamma : \mathcal{L}(\mathcal{R}) \to \mathcal{R} \otimes \mathcal{R}^d \) be the inverse of this canonical linear isomorphism. Thus,
\[
\Gamma(\mathcal{C}(\mathcal{R})) = (\mathcal{R} \otimes_{\min} \mathcal{R}^d)_+,
\]
the cone of positive elements of the operator system \( \mathcal{R} \otimes_{\min} \mathcal{R}^d \). Thus, it is clear that \( \phi \in \mathcal{C}(\mathcal{R}) \) is pure if and only if \( \Gamma(\phi) \) is pure (i.e., only if \( \Gamma(\phi) \) generates a half-line face of the cone \( (\mathcal{R} \otimes_{\min} \mathcal{R}^d)_+ \)).

Definition 2.8. ([14]) If \( \mathcal{R} \) is a finite-dimensional operator system, then an element \( \xi \in \mathcal{R} \otimes \mathcal{R}^d \) is maximally entangled if there exist bases \( \{x_0, \ldots, x_m\} \) and \( \{\delta_0, \ldots, \delta_m\} \) of \( \mathcal{R} \) and \( \mathcal{R}^d \), respectively, such that
\begin{enumerate}
  \item \( \{x_0, \ldots, x_m\} \) and \( \{\delta_0, \ldots, \delta_m\} \) are dual bases (i.e., \( \delta_i(x_j) = 1 \) and \( \delta_i(x_j) = 0 \) if \( j \neq i \) for all \( i \) and \( j \)),
  \item \( x_0 = e_{\mathcal{R}} \) and \( \delta_0 = e_{\mathcal{R}^d} \), and
  \item \( \xi = \sum_{j=0}^m x_j \otimes \delta_j \).
\end{enumerate}

Proposition 2.9. Let \( \iota : \mathcal{R} \to \mathcal{R} \) be the identity map of a finite-dimensional operator system \( \mathcal{R} \). Then \( \Gamma(\iota) \) is the unique maximally entangled element of \( \mathcal{R} \otimes \mathcal{R}^d \).

Proof. Set \( x_0 = e_{\mathcal{R}} \) and let \( \delta_0 \) be a faithful state on \( \mathcal{R} \); thus, \( \delta_0 \) is a positive linear functional, \( \delta_0(x_0) = 1 \), and \( \delta_0 \) serves as an Archimedean order unit \( e_{\mathcal{R}^d} \) for \( \mathcal{R}^d \). Select a basis \( \{x_1, \ldots, x_m\} \) of \( \ker \delta_0 \) and a dual basis \( \{\delta_1, \ldots, \delta_m\} \) for the vector space \( \ker \delta_0 \); this dual basis can be realised by linear functionals on \( \mathcal{R} \) such that \( \{x_0, x_1, \ldots, x_m\} \) and \( \{\delta_0, \delta_1, \ldots, \delta_m\} \) are dual bases for \( \mathcal{R} \) and \( \mathcal{R}^d \). Set \( \xi = \sum_{j=0}^m x_j \otimes \delta_j \), which by definition is maximally entangled. Under the canonical isomorphism \( \mathcal{R} \otimes \mathcal{R}^d \to \mathcal{L}(\mathcal{R}) \), each \( x_i \otimes \delta_i \) maps to an operator on \( \mathcal{R} \) that annihilates every \( x_j \).
Corollary 2.10. The identity map \( \iota : \mathcal{R} \to \mathcal{R} \) of a finite-dimensional operator system \( \mathcal{R} \) is pure if and only if the maximally entangled element \( \Gamma(\iota) \in (\mathcal{R} \otimes_{\min} \mathcal{R}^d)_+ \) is pure.

2.3. Boundary representations. Boundary representations of the \( \mathbb{C}^* \)-algebra generated by an operator system have an important role in the proving our results on the purity of identity mappings.

Definition 2.11. If \( \mathcal{A} \) is a unital \( \mathbb{C}^* \)-algebra generated by an operator system \( \mathcal{R} \), then a representation \( \pi : \mathcal{A} \to \mathcal{B}(\mathcal{K}) \) of \( \mathcal{A} \) on some Hilbert space \( \mathcal{K} \) is a boundary representation for \( \mathcal{R} \) if (i) \( \pi \) is irreducible and (ii) \( \pi \) is the unique ucp extension to \( \mathcal{A} \) of the completely positive linear map \( \pi_{|\mathcal{R}} : \mathcal{R} \to \mathcal{B}(\mathcal{K}) \).

The first tool we shall use is the following one.

Proposition 2.12. If \( \mathcal{A} \) is a unital \( \mathbb{C}^* \)-algebra generated by an operator system \( \mathcal{R} \), and if \( \pi : \mathcal{A} \to \mathcal{B}(\mathcal{K}) \) is a boundary representation of \( \mathcal{R} \), then \( \pi_{|\mathcal{R}} \) is a pure element of \( \mathcal{C}^p(\mathcal{R}, \mathcal{B}(\mathcal{K})) \).

Proof. Let \( \phi = \pi_{|\mathcal{R}} \), where \( \pi : \mathcal{A} \to \mathcal{B}(\mathcal{K}) \) is a boundary representation of \( \mathcal{R} \). Suppose that \( \Theta, \Omega : \mathcal{R} \to \mathcal{B}(\mathcal{K}) \) are completely positive linear maps such that \( \Theta + \Omega = \phi \). By the Arveson Hahn-Banach Extension Theorem [1, Theorem 1.2.3], there are completely positive linear extensions \( \Theta \) and \( \Omega \) of \( \Theta \) and \( \Omega \) from \( \mathcal{R} \) to \( \mathcal{A} \). Thus, \( \Theta + \Omega \) is a completely positive extension of \( \phi \); hence, \( \pi = \Theta + \Omega \). Because \( \pi \) is irreducible, the Radon-Nikodým Theorem for completely positive linear maps [1, Theorem 1.4.2] implies that \( \Theta = s\pi \) and \( \Omega = t\pi \), for some \( s, t \in [0, 1] \). Hence, \( \Theta = \phi \) and \( \Omega = \phi \), proving that \( \phi \) is pure. \( \square \)

The crucial link between boundary representations and \( \mathbb{C}^* \)-envelopes, given by the following Choquet-type theorem, is the second tool from the theory of boundary representations that is required to prove our purity results.

Theorem 2.13. (Arverson-Davidson-Kennedy [2, 5]) Suppose that \( \mathcal{R} \) is an operator subsystem of a unital \( \mathbb{C}^* \)-algebra \( \mathcal{A} \) and that \( \mathcal{A} = \mathcal{C}^*(\mathcal{R}) \). Let \( \mathcal{S} \) denote the ideal of \( \mathcal{C}^*(\mathcal{R}) \) given by

\[ \mathcal{S} = \{ a \in \mathcal{C}^*(\mathcal{R}) \mid \pi(a) = 0 \text{ for every boundary representation } \pi \text{ of } \mathcal{R} \}, \]

and let \( q : \mathcal{C}^*(\mathcal{R}) \to \mathcal{C}^*(\mathcal{R}) \to \mathcal{C}^*(\mathcal{R})/\mathcal{S} \) denote the canonical quotient homomorphism. Then \( \mathcal{C}^*(\mathcal{R})/\mathcal{S}, q_{|\mathcal{R}} \) is a \( \mathbb{C}^* \)-envelope of \( \mathcal{R} \). That is, \( \mathcal{C}^*_v(\mathcal{R}) = \mathcal{C}^*(\mathcal{R})/\mathcal{S} \) and the ucp map \( \iota_c = q_{|\mathcal{R}} \) is a complete order embedding of \( \mathcal{R} \) into its \( \mathbb{C}^* \)-envelope \( \mathcal{C}^*(\mathcal{R})/\mathcal{S} \). Furthermore, for any matrix \( X \in M_n(\mathcal{R}) \), the norm of \( X \) is given by

\[ \|X\| = \max \left\{ \|\pi^{(n)}(X)\| \mid \pi \text{ is a boundary representation of } \mathcal{R} \right\}, \]

where \( \pi^{(n)} \) denotes the unital \( * \)-homomorphism on \( M_n(\mathcal{C}^*(\mathcal{R})) \) that maps every matrix \( [a_{ij}]_{i,j=1}^n \) of elements of \( \mathcal{C}^*(\mathcal{R}) \) to \( [\pi(a_{ij})]_{i,j=1}^n \).
2.4. Noncommutativity and the purity of ucp maps. The following simple observation will be useful in cases where the codomain of a unital completely positive linear map is an algebra.

**Proposition 2.14.** If a unital C*-algebra \( A \) has nontrivial centre and if \( \phi : R \to A \) a unital completely positive linear map, the \( \phi \) is not a pure element of the cone \( \mathcal{CP}(R, A) \).

**Proof.** By the hypothesis that the centre \( Z(A) \) of \( A \) is nontrivial, there exists a non-scalar positive element \( a \in Z(A) \) of norm \( \|a\| = 1 \). If \( \phi : R \to A \) is a unital completely positive linear map, then define \( \delta, \omega : R \to A \) by \( \delta(x) = a^{1/2} \phi(x)a^{1/2} \) and \( \omega(x) = (1 - a)^{1/2} \phi(x)(1 - a)^{1/2} \), for \( x \in R \). Observe that, for every \( x \in R \),

\[
\delta(x) + \omega(x) = a^{1/2}xa^{1/2} + (1 - a)^{1/2} \phi(x)(1 - a)^{1/2} = \phi(x)(a + (1 - a)) = \phi(x).
\]

However, as \( a \) is nonscalar, \( a = \delta(e_R) \neq \lambda \phi(e_R) \) for every \( \lambda \geq 0 \). Hence, \( \phi \) is not pure.

\[ \square \]

2.5. Extension of pure completely positive linear maps. In addition to proving that the von Neumann algebra \( \mathcal{B}(\mathcal{H}) \) is injective in his seminal paper [1], Arveson proved that if \( R \) is an operator subsystem of an operator system \( T \), then a pure completely positive linear map \( \phi : R \to \mathcal{B}(\mathcal{H}) \) extends to a pure completely positive linear map \( \Phi : T \to \mathcal{B}(\mathcal{H}) \). The following result shows that pure extensions occur for pure maps into arbitrary injective von Neumann.

**Proposition 2.15 (Pure Extensions).** If \( R \subseteq T \) is an inclusion of operator systems, and if \( \phi : R \to M \) is a pure completely positive linear map into an injective factor \( M \), then \( \phi \) extends to a pure completely positive linear map \( \Phi : T \to M \).

**Proof.** Let \( \delta = \|\phi\| \) and consider the set \( \mathcal{CP}_\delta(R, \mathcal{B}(\mathcal{H})) \) of all completely positive linear maps \( \psi : R \to M \) of norm \( \|\psi\| \leq \delta \). In the point-ultraweak topology, \( \mathcal{CP}_\delta(R, M) \) is a compact space [19, Theorem 7.4]. Consider the subset \( \mathcal{E} \) of \( \mathcal{CP}_\delta(T, M) \) consisting of all completely positive linear maps \( \Phi : T \to M \) such that \( \Phi|_R = \Phi \). By the injectivity of \( M \), the set \( \mathcal{E} \) is nonempty. It is also plainly convex. We now show that \( \mathcal{E} \) is compact in \( \mathcal{CP}_\delta(T, M) \).

Let \( \{\phi_\alpha\}_\alpha \) be a net in \( \mathcal{E} \). Because the norm of any completely positive map on an operator system is achieved at the order unit of the operator system, we have that \( \|\Phi\| = \delta \) for every \( \Phi \in \mathcal{E} \); hence, \( \mathcal{E} \subset \mathcal{CP}_\delta(T, M) \) and \( \{\phi_\alpha\}_\alpha \) is a net in \( \mathcal{CP}_\delta(T, M) \).

By the compactness of \( \mathcal{CP}_\delta(T, M) \), there exists a subnet \( \{\phi_{\alpha, i}\}_i \) of \( \{\phi_\alpha\}_\alpha \) and a \( \Phi \in \mathcal{CP}_\delta(T, M) \) such that \( \phi(x) \) is the limit, in the ultraweak topology of \( \mathcal{B}(\mathcal{H}) \), of the net \( \{\phi_{\alpha, i}(x)\}_i \) of operators \( \phi_{\alpha, i}(x) \in M \). Because \( M \) is closed in the ultraweak topology of \( M \), the operator \( \phi(x) \) must belong to \( M \) for every \( x \). Hence, the limiting map \( \Phi \) is an element of \( \mathcal{E} \). Thus, because every net in \( \mathcal{E} \) admits a convergent subnet, \( \mathcal{E} \) is compact in the subspace topology of \( \mathcal{CP}_\delta(T, M) \).

The point-ultraweak topology is a weak*-topology, and so the Krein-Milman Theorem applies to the compact convex set \( \mathcal{E} \), which yields an extreme point \( \Phi \) of \( \mathcal{E} \). Suppose that \( \Phi = \Theta + \Omega \), for some nonzero completely positive linear maps \( \Psi, \Omega : T \to M \). Thus, if \( \Theta = \Theta|_R \) and \( \omega = \Omega|_R \), then \( \Phi = \Theta + \omega \); hence, by the purity of \( \phi \), there are nonzero \( s, t \in [0, 1] \) such that \( \Theta = s\phi \) and \( \omega = t\phi \). Therefore, the completely positive linear maps \( s\Theta \) and \( t\Omega \) are completely positive extensions of \( \phi \), making them elements of \( \mathcal{E} \), and these maps satisfy \( s(\frac{1}{2}\Theta) + t(\frac{1}{2}\Omega) = \Phi \). Furthermore, as \( \phi(x) = s\phi(x) + t\phi(x) = (s + t)\phi(x) \) for every \( x \in R \), we have that \( s + t = 1 \). Hence, \( \Phi \) is a convex combination of \( \frac{1}{2}\Theta \) and \( \frac{1}{2}\Omega \). Since \( \Phi \) is an extreme
point of $C$, we obtain $\Phi = \frac{1}{2} \Theta = \frac{1}{4} \Omega$, which proves that the extension $\Phi$ of $\phi$ is pure.

A consequence of the proof of Proposition 2.15 is the following result.

**Proposition 2.16.** If $\mathcal{R} \subseteq \mathcal{J}$ is an inclusion of operator systems and if $\phi : \mathcal{R} \to \mathcal{J}$ is a pure completely positive linear map of $\mathcal{R}$ into an injective operator system $\mathcal{J}$, then every extreme point $\Phi$ of the convex set of extensions of $\phi$ to $\mathcal{J}$ is a pure completely positive linear map $\Phi : \mathcal{J} \to \mathcal{J}$.

### 2.6. Operator systems from discrete groups

Suppose that $u$ is a generating set for a discrete group $G$. Considered as a subset of the (full) group $C^*(G)$, each element of $u$ is unitary. Define $S(u) \subseteq C^*(G)$ by

$$S(u) = \operatorname{Span}\{1, u, u^* \mid u \in u\},$$

where $1$ denotes the multiplicative identity of the $C^*$-algebra $C^*(G)$. Thus, $S(u)$ is an operator system in $C^*(G)$.

**Definition 2.17.** Let $u_1, u_2, u_3, \ldots$ denote generators of the free group $F_\infty$. For each $n \in \mathbb{N}$, let

$$S_n = \operatorname{Span}\{1, u_k, u_k^* \mid k = 1, \ldots, n\} \subseteq C^*(F_n),$$

where $F_n$ is the free group generated by $u_1, \ldots, u_n$. The operator system $S_n$ is called the operator system of the free group $F_n$.

Up to unital complete order isomorphism, the definition of $S_n$ is independent of the choice of generators of $F_n$. Moreover, it is clear that $C^*(S_n) = C^*(F_n)$, for every $n \in \mathbb{N}$.

**Definition 2.18.** For each $n \in \mathbb{N}$ with $n \geq 2$, let $v_1, \ldots, v_n$ be generators of the group $*_1^n \mathbb{Z}_2$, the free product of $n$ copies of $\mathbb{Z}_2$, and define

$$\text{NC}(n) = \operatorname{Span}\{1, v_1, \ldots, v_n\} \subseteq C^*(*_1^n \mathbb{Z}_2).$$

(Note that in $C^*(*_1^n \mathbb{Z}_2)$ each $v_i$ is a selfadjoint unitary (i.e., $v_i^2 = 1$).) The operator system $\text{NC}(n)$ is called the operator system of the noncommutative $n$-cube.

The condition $n \geq 2$ in Definition 2.18 is present only to justify the use of the term “noncommutative cube”. One can also define $\text{NC}(1)$ in the obvious manner.

As with free groups, up to unital complete order isomorphism the definition of $\text{NC}(n)$ is independent of the choice of generators of $*_1^n \mathbb{Z}_2$, and $C^*(\text{NC}(n)) = C^*(*_1^n \mathbb{Z}_2)$.

The following result describes the $C^*$-envelope of operator systems arising from discrete groups.

**Theorem 2.19.** (A Proposition 3.2) If $u$ is a generating set for a discrete group $G$, then $(C^*(G), \iota)$ is a $C^*$-envelope for $S(u)$, where $\iota : S(u) \to C^*(G)$ is the canonical inclusion.

**Corollary 2.20.** $C^e(S_n) = C^*(F_n)$ and $C^e(\text{NC}(n)) = C^*(*_1^n \mathbb{Z}_2)$.

It is somewhat remarkable that the operator systems $S_n$ and $\text{NC}(n)$ are operator system quotients of operator systems of matrices $[8, 10]$. As a consequence, the dual operator systems $S_n^d$ and $\text{NC}(n)^d$ can be realised by operator systems of matrices, which is crucial for our analysis of the purity of identity maps on the operator systems $S_n$ and $\text{NC}(n)$.
Theorem 2.21. For every choice of order unit for each of the operator system duals $S_n^d$ and $NC(n)^d$,.

1. (10 Theorem 4.5) the operator system dual $S_n^d$ of $S_n$ is completely order isomorphic to the operator system $\mathcal{X}_n$ of $2n \times 2n$ matrices defined by

$$\mathcal{X}_n = \left\{ \sum_{k=1}^{n} \begin{bmatrix} \alpha & \beta_k \\ \gamma_k & \alpha \end{bmatrix} \mid \alpha, \beta_k, \gamma_k \in \mathbb{C}, k = 1, \ldots, n \right\},$$

and

2. (8 Proposition 6.1) the operator system dual $NC(n)^d$ of $NC(n)$ is completely order isomorphic to the operator system $\mathcal{Y}_n$ of $2n \times 2n$ matrices defined by

$$\mathcal{Y}_n = \left\{ \sum_{k=1}^{n} \begin{bmatrix} \alpha & \beta_k \\ \beta_k & \alpha \end{bmatrix} \mid \alpha, \beta_k \in \mathbb{C}, k = 1, \ldots, n \right\}.$$

To conclude, we make explicit note of the earlier-mentioned universal properties of $S_n$ and $NC(n)$.

Theorem 2.22. Let $y_1, \ldots, y_n \in B(H)$ be arbitrary contractions, for $n \in \mathbb{N}$, and let $x_1, \ldots, x_m \in B(H)$ be arbitrary selfadjoint contractions, for $m \in \mathbb{N}$ with $m \geq 2$. There exist unital completely positive linear maps $\phi : S_n \to B(H)$ and $\psi : NC(m) \to B(H)$ such that

$$\phi(u_j) = y_j, \text{ for every } j, \text{ and } \psi(v_k) = h_k, \text{ for every } k.$$

Theorem 2.22 is proved in [17 Proposition 9.7] and [8 Proposition 6.5].

3. Embedding Operator Systems into Their Injective Envelopes

The main result of this section, Theorem 3.2, is a determination of the purity of the embedding of an arbitrary operator system $\mathcal{R}$ into its injective envelope by way of the $C^*$-envelope of $\mathcal{R}$, thereby answering question (Q3).

If $\mathcal{B}$ is an injective $C^*$-algebra, then $\mathcal{B}$ is a monotone complete $C^*$-algebra and, hence, an AW*-algebra [21]. By the seminal work of Hamana [13, 21], every unital $C^*$-algebra $A$ has a regular monotone completion $\overline{A}$ such that $\overline{A}$ is a unital $C^*$-subalgebra of its injective envelope $I(A)$ (when $I(A)$ is considered in its guise as an injective $C^*$-algebra). Moreover, for any operator system $\mathcal{R}$, the following system of unital complete order embeddings holds:

$$\mathcal{R} \subseteq C^*_e(\mathcal{R}) \subseteq C^*_e(\mathcal{R}) \subseteq I(\mathcal{R}).$$

Lemma 3.1. If $\mathcal{B}$ is an AW*-factor, then the identity map $\iota : \mathcal{B} \to \mathcal{B}$ is a pure element of $\mathcal{CP}(\mathcal{B})$.

Proof. Every AW*-factor is a primitive $C^*$-algebra [22]; hence, without loss of generality we may assume that $\mathcal{B}$ is an irreducible $C^*$-algebra of operators acting on some Hilbert space $\mathcal{H}$. The identity map $\iota : \mathcal{B} \to \mathcal{B}$ is, therefore, an irreducible representation of $\mathcal{B}$ on $\mathcal{H}$; thus, by Arveson’s criterion [1 Corollary 1.4.3], $\iota$ is a pure element of $\mathcal{CP}(\mathcal{B}, \mathcal{B}(\mathcal{H}))$.

Suppose now that $\vartheta, \omega \in \mathcal{CP}(\mathcal{B})$ are such that $\vartheta + \omega = \iota$. Considering $\vartheta$ and $\omega$ as elements of $\mathcal{CP}(\mathcal{B}, \mathcal{B}(\mathcal{H}))$, the purity of $\iota$ in $\mathcal{CP}(\mathcal{B}, \mathcal{B}(\mathcal{H}))$ and the equation $\vartheta + \omega = \iota$ imply that $\vartheta = s\iota$ and $\omega = (1 - s)\iota$ for some $s \in [0, 1]$. Hence, it is also true that $\iota : \mathcal{B} \to \mathcal{B}$ is a pure element of $\mathcal{CP}(\mathcal{B})$. □
Theorem 3.2. The following statements are equivalent for the canonical unital complete order embedding $\iota_e : \mathcal{R} \to I(\mathcal{R})$:

1. $\iota_e$ is pure in the cone $\mathcal{CP}(\mathcal{R}, I(\mathcal{R}))$;
2. the $C^*$-algebra $C^*_e(\mathcal{R})$ is prime.

Proof. To prove that (1) implies (2), we shall prove that $\iota_e$ is not pure if $C^*_e(\mathcal{R})$ is not prime. To this end, let $A = C^*_e(\mathcal{R})$, which we assume to be nonprime. The regular monotone completion of any nonprime $C^*$-algebra has nontrivial centre [13 Theorem 7.1]. Thus, the ucp map $\iota_e : \mathcal{R} \to \overline{\mathcal{A}}$ is not pure in the cone $\mathcal{CP}(\mathcal{R}, \overline{\mathcal{A}})$, by Proposition 2.14. Hence, via the system of inclusions (1) above, $\iota_e$ is not pure in the cone $\mathcal{CP}(\mathcal{R}, I(\mathcal{R}))$.

Conversely, to prove that (2) implies (1), assume that $C^*_e(\mathcal{R})$ is a prime $C^*$-algebra. Thus, $I(C^*_e(\mathcal{R}))$ is (unital) completely order isomorphic to an injective AW*-factor [13 Theorem 7.1]. Suppose that $\vartheta, \omega : \mathcal{R} \to I(\mathcal{R})$ are completely positive linear maps such that $\vartheta + \omega = \iota_e$. By the injectivity of $I(\mathcal{R})$, there exists completely positive extensions $\Theta$ and $\Omega$ of $\vartheta$ and $\omega$, respectively, from $\mathcal{R}$ to $I(\mathcal{R})$. Thus, $\Theta + \Omega$ is a ucp map on $I(\mathcal{R})$ for which $|\Theta + \Omega| x = x$, for every $x \in \mathcal{R}$. By the rigidity property of the injective envelope [12 Lemma 3.6], it is necessarily the case that $(\Theta + \Omega) z = z$, for every $z \in I(\mathcal{R})$. However, as $I(\mathcal{R})$ is an AW*-factor, the identity map on $I(\mathcal{R})$ is pure, by Lemma 3.1; thus, $\Theta$ and $\Omega$ are scalar multiples of the identity map on $I(\mathcal{R})$, implying that $\vartheta$ and $\omega$ are scalar multiples of $\iota_e$. Hence, $\iota_e$ is pure in the cone $\mathcal{CP}(\mathcal{R}, I(\mathcal{R})).$ \hfill $\square$

Corollary 3.3. The embedding of $S_n$ into its injective envelope is pure for every $n \geq 2$.

Proof. The $C^*$-envelope of $S_n$ is $C^*(F_n)$, which is primitive (and, hence prime) for every $n \geq 2$ [3]. Thus, Theorem 3.2 yields the purity of the embedding $\iota_e$. \hfill $\square$

4. Embedding Discrete-Group Operator Systems into their $C^*$-Envelopes

This section answers question (Q2) for operator systems from discrete groups.

Lemma 4.1. Suppose that $\mathcal{R}$ is an operator subsystem of a unital $C^*$-algebra $A$ for which $A = C^*(\mathcal{R})$, and that $\mathcal{R}$ contains a set $u$ of unitary elements of $A$ that generate $A$ as a $C^*$-algebra. Suppose that $\mathcal{B}$ is a unital $C^*$-subalgebra of $\mathcal{B}(\mathcal{H})$, for some Hilbert space $\mathcal{H}$. If $\phi : \mathcal{R} \to \mathcal{B}$ is a ucp map for which $\phi(u)$ is unitary, for every $u \in u$, and if the commutant of $(\phi(u) | u \in u)$ in $\mathcal{B}(\mathcal{H})$ is 1-dimensional, then $\phi$ is a pure element of $\mathcal{CP}(\mathcal{R}, \mathcal{B})$.

Proof. The hypothesis that $\phi(u)$ is unitary, for every $u \in u$, implies that $\phi$ admits a unique ucp extension $\pi : C^*(\mathcal{R}) \to \mathcal{B}$ and that the extension $\pi$ is a homomorphism [17, Lemma 9.3]. Because the commutant of $(\phi(u) | u \in u)$ in $\mathcal{B}(\mathcal{H})$ is 1-dimensional, the $C^*$-algebra $\pi(C^*(\mathcal{R}))$ is irreducible; in other words, $\pi$ is an irreducible representation of $C^*(\mathcal{R})$.

Suppose that $\vartheta$ and $\omega$ are completely positive linear maps $\mathcal{R} \to \mathcal{B}$ such that $\phi = \vartheta + \omega$. In considering $\vartheta$ and $\omega$ as elements of $\mathcal{CP}(\mathcal{R}, \mathcal{B}(\mathcal{H}))$, Arveson’s Extension Theorem yields completely positive extensions $\Theta$ and $\Omega$ of $\vartheta$ and $\omega$ that map $C^*(\mathcal{R})$ into $\mathcal{B}(\mathcal{H})$. Hence, $\Theta + \Omega$ is one completely positive extension of $\phi$, considering $\phi$ as an element of $\mathcal{CP}(\mathcal{R}, \mathcal{B}(\mathcal{H}))$. However, because $\phi(u)$ is unitary in $\mathcal{B}(\mathcal{H})$ for every $u \in u$, $\phi$ has unique ucp extension from $\mathcal{R}$ to $C^*(\mathcal{R})$ into $\mathcal{B}(\mathcal{H})$ [15, Lemma 5.5]. Furthermore, because the extension is a homomorphism, the extension is necessarily $\pi$ and the range of the extension is an irreducible $C^*$-subalgebra.
of $B$. Hence, $\pi = \Theta + \Omega$. Because $\pi$ is irreducible, the Radon-Nikodým Theorem for completely positive linear maps [1] Theorem 1.4.2] states that $\Theta = s\pi$ and $\Omega = (1 - s)\pi$, for some $s \in [0, 1]$. Hence, $\theta = s\phi$ and $\omega = (1 - s)\phi$, thereby proving that $\phi$ is pure in $\mathcal{E}(B, B)$. $\square$

The following result applies in cases where $u$ need not necessarily be a finite set.

**Theorem 4.2.** If a discrete group $G$ is generated by a set $u$, and if $C^*(G)$ is a primitive $C^*$-algebra, then the embedding $\iota_e : S(u) \to C^*(G)$ is pure.

**Proof.** By assumption of the primitivity of $C^*(G)$, there exists a faithful irreducible representation of $C^*(G)$ on a separable Hilbert space $\mathcal{H}$. Thus, without loss of generality, we assume that $C^*(G)$ is a unital $C^*$-subalgebra of $B(\mathcal{H})$, that $S(u)$ is an operator subsystem of $C^*(G)$, and that $\iota_e$ is the inclusion map $\iota_e(x) = x$, for $x \in S(u)$. Clearly $\iota_e$ maps the unitary elements of $S(u)$ to unitary elements of $C^*(G)$; hence, by Lemma 4.1, $\iota_e$ is a pure element of $\mathcal{E}(S(u), C^*(G))$. $\square$

A necessary condition for a full group $C^*$-algebra $C^*(G)$ to be primitive is that $G$ be an infinite conjugacy class (i.c.c.) group. Therefore, the next theorem, which applies only to finitely-generated groups, requires somewhat less of the group $C^*$-algebra $C^*(G)$ and answers question (Q2) for the operator systems $S_n$ and $NC(n)$.

**Theorem 4.3.** Suppose that $G$ is a finitely-generated discrete group and $u$ is any finite set of generators of $G$. Then the following statements are equivalent:

1. the canonical embedding $\iota_e : S(u) \to C^*(G)$ is pure;
2. $G$ is an infinite conjugacy class (i.c.c.) group.

**Proof.** To prove that (1) implies (2), we show that if $G$ is not an i.c.c. group, then the embedding $\iota_e : S(u) \to C^*(G)$ is not pure. Because full group algebras have trivial centre only for i.c.c. groups (see, for example, [18 Proposition 2.1]), the ucp map $\iota_e : S(u) \to C^*(G)$ can never be pure if $G$ is not an i.c.c. group, by Proposition 2.14.

To prove that (2) implies (1), suppose that $u$ is given by $u = \{u_1, \ldots, u_n\}$, for some unitaries $u_j \in C^*(G)$. Note that the hypothesis that $G$ is an i.c.c group implies that the reduced group $C^*$-algebra $C^*_r(G)$ is prime [18 Proposition 2.3]. Because separable prime $C^*$-algebras are primitive, there exists a faithful irreducible representation $\pi : C^*_r(G) \to B(\mathcal{H})$, for some Hilbert space $\mathcal{H}$. Let $B = \pi(C^*_r(G))$, an irreducible $C^*$-subalgebra of $B(\mathcal{H})$. If $\rho$ denotes the left regular representation of the group $G$ (thereby implementing a homomorphism $C^*(G) \to C^*_r(G)$), then define a unital completely positive linear map $\psi : S(u) \to B(\mathcal{H})$ by

$$\psi(\alpha_1 + \sum_{j=1}^n \alpha_j u_j) = \alpha \pi(\rho(1)) + \sum_{j=1}^n \alpha_j \pi(\rho(u_j)).$$

Because $\psi(u)$ is unitary for every $u \in u$ and the commutant of $\{\psi(u) | u \in u\}$ in $B(\mathcal{H})$ is 1-dimensional, $\psi$ is a pure element of $\mathcal{E}(B, B)$, by Lemma 4.1.

If there exists completely positive linear maps $\theta, \omega : S(u) \to C^*(G)$ such that $\theta + \omega = \iota_e$, then

$$(\pi \circ \rho) \circ \theta + (\pi \circ \rho) \circ \omega = (\pi \circ \rho) \circ \iota_e = \psi.$$

Hence, the purity of $\psi$ in $\mathcal{E}(S(u), B)$ yields $(\pi \circ \rho) \circ \theta = s\psi = (\pi \circ \rho) \circ (st_e)$, for some $s \in [0, 1]$. Because $\psi$ is a linear map that sends a space of dimension $(n + 1)$
onto a space of the same dimension, \( \psi \) is injective; hence, \((\pi \circ \rho) \circ \theta(x) = (\pi \circ \rho) \circ (st_e)(x)\), for \( x \in S(u) \), only if \( \theta(x) = st_e(x) \). That is, \( \theta = st_e \) and \( \omega = (1 - s)t_e \), proving that \( t_e \) is pure.

**Corollary 4.4.** The embedding \( t_e : S_n \to C^*(\mathbb{F}_n) \) is pure for every \( n \geq 2 \).

**Proof.** For every \( n \geq 2 \), \( \mathbb{F}_n \) is an i.c.c. group. \( \square \)

**Corollary 4.5.** The embedding \( t_e : NC(n) \to C^*(\mathbb{Z}_2\ast) \) is pure for every \( n \geq 3 \).

**Proof.** If \( n \geq 3 \), then \( \mathbb{Z}_2\ast \) contains \( \mathbb{F}_2 \) as a subgroup (see, for example, [11] Lemma D.2]). Hence, \( \mathbb{Z}_2\ast \) is an i.c.c. group. \( \square \)

With respect to \( S_1 \) and \( NC(2) \), because \( C^*(\mathbb{Z}) \) is abelian and \( C^*(\mathbb{Z}_2 \ast \mathbb{Z}_2) \) has nontrivial centre, Proposition 2.14 shows that the embeddings of \( S_1 \) and \( NC(2) \) into their \( C^* \)-envelopes are not pure.

## 5. Identity Maps on Some Universal Operator Systems

In this section we answer question (Q1) for the universal operator systems \( S_n \) and \( NC(n) \), making crucial use of boundary representations and the fact that the operator system duals of theses operator systems can be represented as operator systems of matrices (Theorem 2.21).

**Theorem 5.1.** The identity map \( i_n : S_n \to S_n \) is pure for every \( n \geq 1 \).

**Proof.** By Theorem 2.6 it is enough to prove that the dual map \( t_n^d : S_n^d \to S_n^d \) (which is again an identity map) is pure. By Theorem 2.21 therefore, it is enough to verify that the identity map \( i_n : X_n \to X_n \) is pure in the cone \( \mathcal{C}(X_n, X_n) \), where

\[
X_n = \left\{ \frac{\alpha}{\beta_k} \right\}_{k=1}^{n} \mid \alpha, \beta_k, \gamma_k \in \mathbb{C}, k = 1, \ldots, n \right\}.
\]

The \( C^* \)-algebra generated by \( X_n \) is \( C^*(X_n) = \bigoplus_{k=1}^{n} \mathcal{M}_2(\mathbb{C}) \). The map \( \pi_k : C^*(X_n) \to \mathcal{M}_2(\mathbb{C}) \) that projects \( x \in X_n \) onto its \( k \)-th direct summand is an irreducible representation of \( C^*(X_n) \), for each \( k = 1, \ldots, n \). We prove below that each irreducible representation \( \pi_k \) of \( C^*(X_n) \) is a boundary representation of \( X_n \).

First note that, up to unitary equivalence, \( \pi_1, \ldots, \pi_n \) are all of the irreducible representations of \( C^*(X_n) \); thus, the boundary representations for \( X_n \) are among these elements. By Theorem 2.13 the norm of each \( x \in X_n \) is given by

\[
\|x\| = \max \{ \|\pi(x)\| \mid \pi \text{ is a boundary representation of } X_n \}.
\]

For each \( k = 1, \ldots, n \) let \( x_k \in X_n \) be the matrix whose \( k \)-th direct summand is \(
\begin{bmatrix}
0 & 1 \\
0 & 0
\end{bmatrix}
\)
and whose other direct summands are the \( 2 \times 2 \) zero matrix. Because

\[
\pi_k(x_k) = \begin{bmatrix}
0 & 1 \\
0 & 0
\end{bmatrix}
\]
for each \( k \) and \( \phi_j(x_k) = 0 \) when \( j \neq k \), the only way that equation (3) can hold for every \( k \) is if \( \pi_k \) is a boundary representation of \( X_n \) for every \( k \).

Now let \( \phi_k = \pi_k|X_n \); thus, \( \phi_k \) is a pure element of \( \mathcal{C}(X_n, \mathcal{M}_2(\mathbb{C})) \). Denote the canonical orthonormal basis vectors of \( \mathbb{C}^2 \) by \( e_1, e_2 \) and the canonical orthonormal basis vectors of \( \mathbb{C}^2n \) by \( f_\ell \), for \( \ell = 1, \ldots, 4n \). Denote by \( \nu_k : \mathbb{C}^2 \to \mathbb{C}^{2n} \) the isometry
that maps $e_1$ to $f_{2k-1}$ and $e_2$ to $f_{2k}$, for $k = 1, \ldots, n$. Thus, the ucp maps $\phi_k$ are given by $\nu_k^* i_n v_k$.

Suppose now that $\vartheta, \omega : X_n \to X_n$ are completely positive linear maps for which $i_n = \vartheta + \omega$. Then

$$\phi_k = \nu_k^* i_n v_k = \nu_k^* \vartheta v_k + \nu_k^* \omega v_k,$$

where $\nu_k^* \vartheta v_k, \nu_k^* \omega v_k \in C(P(X_n, M_2(\mathbb{C})))$. Observe that the projections $p_k = \nu_k \nu_k^* \in M_2(\mathbb{C})$, for $k = 1, \ldots, n$, are mutually orthogonal, sum to the identity, and commute with every element of $X_n$. Thus,

$$(4) \quad \vartheta = \sum_{k=1}^n p_k \vartheta p_k \text{ and } \omega = \sum_{k=1}^n p_k \omega p_k,$$

and

$$(5) \quad i_n = \sum_{k=1}^n p_k i_n p_k = \sum_{k=1}^n p_k \vartheta p_k + \sum_{k=1}^n p_k \omega p_k = \vartheta + \omega.$$

Furthermore, because each $\phi_k$ is a pure element of $C(P(X_n, M_2(\mathbb{C})))$, there are $s_k \in [0, 1]$ such that $\nu_k^* \vartheta v_k = s_k \phi_k$ and $\nu_k^* \omega v_k = (1 - s_k)\phi_k$ for $k = 1, \ldots, n$. Thus,

$$p_k \vartheta p_k = \nu_k (s_k \phi_k) \nu_k^* = s_k p_k i_n p_k$$

and

$$p_k \omega p_k = \nu_k ((1 - s_k)\phi_k) \nu_k^* = (1 - s_k) p_k i_n p_k$$

for every $k$. Therefore, equations (4) become

$$(\vartheta) = \sum_{k=1}^n s_k (p_k i_n p_k) \text{ and } (\omega) = \sum_{k=1}^n (1 - s_k) p_k i_n p_k.$$  

Evaluation of the expression above for $\vartheta$ at the identity $1 \in M_2(\mathbb{C})$ yields

$$\vartheta(1) = \sum_{k=1}^n s_k (p_k i_n (1)p_k) = \bigoplus_{k=1}^n \begin{bmatrix} s_k & 0 \\ 0 & s_k \end{bmatrix}.$$  

Because $\vartheta(1) \in X_n$, the diagonal of $\vartheta(1)$ is constant; hence, $s_1 = \cdots = s_n = s$ for some $s \in [0, 1]$. That is, $\vartheta = s i_n$ and $\omega = (1 - s) i_n$. \hfill \Box

**Theorem 5.2.** The identity map $i_n : NC(n) \to NC(n)$ is pure for every $n \geq 2$.

**Proof.** By Theorem 2.6, it is sufficient to confirm that the dual map $t_2^d : NC(n)^d \to NC(n)^d$ (which is again an identity map) is pure. By Theorem 2.21, therefore, it is enough to verify that the identity map $i_n : Y_n \to Y_n$ is pure in the cone $C(P(Y_n, Y_n))$,

where

$$(6) \quad y_n = \bigoplus_{k=1}^n \begin{bmatrix} \alpha \beta_k \\ \beta_k \alpha \end{bmatrix} | \alpha, \beta_k \in \mathbb{C}, k = 1, \ldots, n.$$  

Note that an element $y = \bigoplus_{k=1}^n \begin{bmatrix} \alpha \beta_k \\ \beta_k \alpha \end{bmatrix}$ is positive if and only $\beta_k \in \mathbb{R}$ and $\alpha \geq |\beta_k|$, for every $k = 1, \ldots, n$.  

The C*-algebra \( \mathcal{Y}_n \) generates is abelian. Hence, the matrices in \( \mathcal{Y}_n \) are commuting normal matrices and, therefore, admit a common spectral decomposition. That is, there is a unitary \( u \in \mathcal{M}_{2n}(\mathbb{C}) \) such that

\[
  u^* \left( \bigoplus_{k=1}^{n} \begin{bmatrix} \alpha & \beta_k \\ \beta_k & \alpha \end{bmatrix} \right) u = \begin{bmatrix} \alpha + \beta_1 & \alpha - \beta_1 & & \\ & \ddots & \ddots & \\ & & \alpha + \beta_n & \alpha - \beta_n \end{bmatrix}.
\]

Hence, \( \mathcal{Y}_n \) is unitarily equivalent to the operator subsystem \( \mathcal{Z}_n \) of the unital abelian C*-algebra \( \ell^\infty(2n) \) given by

\[
  \mathcal{Z}_n = \{ (\alpha + \beta_1, \alpha - \beta_1, \ldots, \alpha + \beta_n, \alpha - \beta_n) \mid \alpha, \beta_k \in \mathbb{C}, k = 1, \ldots, n \}.
\]

Observe that \( \ell^\infty(2n) \) is the unital C*-algebra generated by \( \mathcal{Z}_n \). Let \( \lambda_n : \mathcal{Z}_n \to \mathcal{Z}_n \) denote the identity map on \( \mathcal{Z}_n \). We aim to prove that \( \lambda_n \) is pure.

For each \( k \in \{1, \ldots, 2n\} \) let \( \pi_k : \ell^\infty(2n) \to \mathbb{C} \) denote the projection map onto the \( k \)-th coordinate. Thus, each \( \pi_k \) is an irreducible representation of \( \ell^\infty(2n) \) on \( \mathbb{C} \), and \( \mathcal{B} = \{\pi_1, \ldots, \pi_{2n}\} \) is the set of all irreducible representations of \( \ell^\infty(2n) \). If, for a fixed \( k \), the map \( \pi_k \) can be shown to be a boundary representation for \( \mathcal{Z}_n \), then it will follow that \( \varphi_k = \pi_k|_{\mathcal{Z}_n} \) the projection of \( \mathcal{Z}_n \) onto the \( k \)-th coordinate, is a pure state on \( \mathcal{Z}_n \).

To this end, let \( \Phi_k : \ell^\infty(2n) \to \mathbb{C} \) be any state extending \( \varphi_k \). Because the states on \( \ell^\infty(2n) \) are convex combinations of extremal states—which in this case are the irreducible representations \( \pi_1, \ldots, \pi_{2n} \)—we deduce that

\[
  \Phi_k = \sum_{j=1}^{2n} \lambda_j \pi_j,
\]

for some \( \lambda_1, \ldots, \lambda_{2n} \in [0, 1] \) for which \( \sum_{j=1}^{2n} \lambda_j = 1 \). The representation in equation (7) above depends on the choice of \( k \) (that is, the convex coefficients \( \lambda_j \) depend on the choice of \( k \)).

For notational simplicity, we consider the case of \( k = 1 \) first. Thus, we aim to show in equation (7)—assuming \( k = 1 \)—that \( \lambda_1 = 1 \) and \( \lambda_j = 0 \) for all \( j \neq 1 \). To this end, consider the element \( x \in \mathcal{Z}_n \) that is given by

\[
  x = (1, 0, 1, 0, \ldots, 1, 0).
\]

(We achieve \( x \) by selecting \( \alpha = \beta_j = \frac{1}{2} \) for every \( j = 1, \ldots, n \).) Thus, equation (7) yields

\[
  1 = \varphi_1(x) = \Phi_1(x) = \sum_{j=1}^{2n} \lambda_j \pi_j(x) = \sum_{\ell=1}^{n} \lambda_{2\ell-1}.
\]

Thus, from \( 1 = \sum_{j=1}^{2n} \lambda_j = \sum_{\ell=1}^{n} \lambda_{2\ell-1} \) we deduce that \( \lambda_{2\ell} = 0 \) for every \( \ell = 1, \ldots, n \). Now using

\[
  y = (1, 0, 0, 1, 0, \ldots, 0, 1) \in \mathcal{Z}_n,
\]
which is achieved by using \( \alpha = \beta_1 = \frac{1}{2} \) and \( \beta_1 = \frac{1}{2} \) for \( j = 2, \ldots, n \), we obtain
\[
1 = \varphi_1(y) = \Phi_1(y) = \sum_{j=1}^{2n} \lambda_j \pi_j(y) = \lambda_1 + \sum_{t=1}^{n} \lambda_{2t} = \lambda_1 + 0 = \lambda_1.
\]

Hence, \( \lambda_j = 0 \) for all \( j \neq 1 \) and \( \Phi_1 = \pi_1 \). The arguments for every other \( \Phi_k \) are handled similarly, juxtaposed according to whether \( k \) is even or odd. Thus, for each \( k = 1, \ldots, 2n \), \( \pi_k \) has the unique extension property, and so \( \pi_k \) is a boundary representation for \( \mathcal{Z}_n \). Hence, \( \varphi_k = \pi_k \) is a pure state on \( \mathcal{Z}_n \).

Now suppose that there are completely positive linear maps \( \partial, \omega : \mathcal{Z}_n \to \mathcal{Z}_n \) for which \( \lambda_n = \partial + \omega \). For each \( k \), let \( \varphi_k = \pi_k \circ \lambda_n \), \( \partial_k = \pi_k \circ \partial \), and \( \omega_k = \pi_k \circ \omega \), which project onto the \( k \)-th coordinates of \( x \), \( \partial(x) \), and \( \omega(x) \), respectively, for every \( x \in \mathcal{Z}_n \). The state \( \varphi_k \) was shown in the previous paragraph to be pure, and so \( \varphi_k = \partial_k + \omega_k \) implies that there \( \partial_k \) and \( \omega_k \) are nonnegative scalar multiples of \( \varphi_k \).

Select
\[
x = (\alpha + \beta_1, \alpha - \beta_1, \ldots, \alpha + \beta_n, \alpha - \beta_n) \in \mathcal{Z}_n.
\]

By the previous paragraph, each \( \partial_k \) is nonnegative scalar multiple of \( \varphi_k \). Hence, there are nonnegative \( s_1, \ldots, s_n, t_1, \ldots, t_n \in \mathbb{R} \) such that
\[
\partial(x) = (s_1(\alpha + \beta_1), t_1(\alpha - \beta_1), \ldots, s_n(\alpha + \beta_n), t_n(\alpha - \beta_n)).
\]

However, since \( \partial(x) \in \mathcal{Z}_n \), there exist \( \lambda, \mu_1, \ldots, \mu_n \in \mathbb{C} \) such that
\[
\partial(x) = (\lambda + \mu_1, \lambda - \mu_1, \ldots, \lambda + \mu_n, \lambda - \mu_n).
\]

Hence, we have the following system of \( 2n \) equations:
\[
\begin{align*}
  s_1(\alpha + \beta_1) &= \lambda + \mu_1 \\
  t_1(\alpha - \beta_1) &= \lambda - \mu_1 \\
  s_2(\alpha + \beta_2) &= \lambda + \mu_2 \\
  t_2(\alpha - \beta_2) &= \lambda - \mu_2 \\
  & \vdots \\
  s_n(\alpha + \beta_n) &= \lambda + \mu_n \\
  t_n(\alpha - \beta_n) &= \lambda - \mu_n.
\end{align*}
\]

Viewing the equations above as \( n \) pairs of equations, adding the first two equations in each pair leads to:
\[
(8) \quad (s_k + t_k)\alpha + (s_k - t_k)\beta_k = 2\lambda, \quad \text{for every } k = 1, \ldots, n.
\]

That is,
\[
(8) \quad (s_k + t_k)\alpha + (s_k - t_k)\beta_k = (s_j + t_j)\alpha + (s_j - t_j)\beta_j, \quad \text{for every } k, j = 1, \ldots, n.
\]

These equations above hold regardless of the choice of \( \alpha, \beta_1, \ldots, \beta_n \), and so it must be that \( s_k = t_k \) for each \( k \). Therefore, the \( n \) equations given in (8) simplify to
\[
(9) \quad 2s_k \alpha = 2\lambda, \quad \text{for every } k = 1, \ldots, n.
\]

Hence, \( s_1 = \cdots = s_n \). If \( s \in \mathbb{R} \) denotes this nonnegative real number, then \( \partial(x) = sx \), for every \( x \in \mathcal{Z}_n \). Thus, \( \partial = s \lambda_n \) and \( \omega = (1 - s) \lambda_n \).

Hence, \( \lambda_n \) is pure, implying that \( \lambda_n \) and \( t_n \) are pure.
We now give another example among many known examples showing that the restriction of a pure map to an operator subsystem need not be pure. What makes the example here of interest is that the restriction is made to an operator subsystem of just 1 dimension lower.

**Proposition 5.3.** There exist operator systems $\mathcal{R}$, $\mathcal{T}$, and $\mathcal{Q}$, and a unital completely positive linear map $\Phi : \mathcal{T} \to \mathcal{Q}$ such that

1. $\mathcal{R}$ is an operator subsystem of $\mathcal{T}$,
2. the linear dimension of the vector space $\mathcal{T}/\mathcal{R}$ is 1,
3. the map $\Phi$ is pure in $\mathcal{CP}(\mathcal{T}, \mathcal{Q})$, and
4. the restriction of $\Phi$ to $\mathcal{R}$ is not pure in $\mathcal{CP}(\mathcal{R}, \mathcal{Q})$.

**Proof.** Let $\mathcal{T} = \mathcal{Q} = \mathcal{X}_1$; that is,

$$\mathcal{T} = \mathcal{Q} = \left\{ \begin{bmatrix} \alpha & \beta_1 \\ \beta_2 & \alpha \end{bmatrix} \mid \alpha, \beta_1, \beta_2 \in \mathbb{C} \right\}.$$

Let

$$\mathcal{R} = \left\{ \begin{bmatrix} \alpha & \beta \\ \beta & \alpha \end{bmatrix} \mid \alpha, \beta \in \mathbb{C} \right\},$$

which is an operator subsystem of $\mathcal{T}$ of co-dimension 1.

Let $\Phi : \mathcal{T} \to \mathcal{Q}$ be the identity map. By (the proof of) Theorem 5.1, $\Phi$ is a pure element of $\mathcal{CP}(\mathcal{T}, \mathcal{Q})$. The completely positive linear maps $\vartheta, \omega : \mathcal{T} \to \mathcal{Q}$ given by

$$\vartheta \left( \begin{bmatrix} \alpha & \beta_1 \\ \beta_2 & \alpha \end{bmatrix} \right) = \begin{bmatrix} \frac{\alpha}{2} + \frac{\beta_1 + \beta_2}{4} \\ \frac{\alpha}{2} + \frac{\beta_1 + \beta_2}{4} \\ \frac{\alpha}{2} + \frac{\beta_1 + \beta_2}{4} \end{bmatrix},$$

and

$$\omega \left( \begin{bmatrix} \alpha & \beta_1 \\ \beta_2 & \alpha \end{bmatrix} \right) = \begin{bmatrix} \frac{\alpha}{2} - \left( \frac{\beta_1 + \beta_2}{4} \right) \\ \frac{-\alpha}{2} + \frac{\beta_1 + \beta_2}{4} \\ \frac{-\alpha}{2} + \frac{\beta_1 + \beta_2}{4} \end{bmatrix},$$

satisfy, when $\beta_1 = \beta_2 = \beta$,

$$\vartheta \left( \begin{bmatrix} \alpha & \beta \\ \beta & \alpha \end{bmatrix} \right) + \omega \left( \begin{bmatrix} \alpha & \beta \\ \beta & \alpha \end{bmatrix} \right) = \begin{bmatrix} \alpha & \beta \\ \beta & \alpha \end{bmatrix}.$$ 

That is, $\vartheta(x) + \omega(x) = \Phi|_{\mathcal{R}}(x)$, for every $x \in \mathcal{R}$. Hence, as neither $\vartheta$ nor $\omega$ is a scalar multiple of $\Phi|_{\mathcal{R}}$, the restriction of $\Phi$ to $\mathcal{R}$ is not pure in the cone $\mathcal{CP}(\mathcal{R}, \mathcal{Q})$. □

If $\mathcal{R}$ is any operator system, then $\mathcal{R} \otimes_{\min} \mathcal{X}_n$ is canonically an operator subsystem of $\mathcal{R} \otimes_{\min} \mathcal{M}_n(\mathbb{C}) = \mathcal{M}_n(\mathcal{R})$; thus, we may describe $\mathcal{R} \otimes_{\min} \mathcal{X}_n$ as the operator system

$$\mathcal{R} \otimes_{\min} \mathcal{X}_n = \left\{ \bigoplus_{k=1}^n \begin{bmatrix} r & a_k \\ b_k & r \end{bmatrix} \mid r, a_k, b_k \in \mathcal{R} \right\}.$$

Likewise,

$$\mathcal{R} \otimes_{\min} \mathcal{Y}_n = \left\{ \bigoplus_{k=1}^n \begin{bmatrix} r & c_k \\ c_k & r \end{bmatrix} \mid r, c_k \in \mathcal{R} \right\}.$$

As a final consequence of Theorems 5.1 and 5.2 and Corollary 2.10 we have the following application.
Proposition 5.4. If \( \{u_1, \ldots, u_n\} \) is the generating set of unitaries for \( C^*({\mathbb F}_n) \), for \( n \geq 1 \), and if \( \{v_1, \ldots, v_m\} \) is the generating set of selfadjoint unitaries for \( C^*({\mathbb F}_m) \), where \( m \geq 2 \), then
\[
\xi = \bigoplus_{k=1}^n \begin{bmatrix} 1 & u_k \\ u_k^* & 1 \end{bmatrix}
\]
is a pure element of the cone \( (S_n \otimes_{\min} X_n)_{+} \), and
\[
\xi' = \bigoplus_{j=1}^m \begin{bmatrix} 1 & v_j \\ v_j^* & 1 \end{bmatrix}
\]
is a pure element of the cone \( (NC(m) \otimes_{\min} Y_m)_{+} \).

Proof. With respect to the canonical basis \( \{1, u_1, u_1^*, \ldots, u_n, u_n^*\} \) of \( S_n \) and the canonical basis \( \{1, v_1, v_2, \ldots, v_m\} \) of \( NC(m) \), the matrix representation (see [8, Proposition 6.1], [10, Theorem 4.5]) of the dual basis elements of \( S^d_n \) and \( NC^d(m) \), where \( n \geq 1 \) and \( m \geq 2 \), are given by
\[
\{1, e_{[k]}^{[12]}, e_{[k]}^{[21]} \mid k = 1, \ldots, n\}
\]
and
\[
\{1, e_{[j]}^{[12]} + e_{[j]}^{[21]} \mid j = 2, \ldots, m, \}
\]
where \( e_{ij}^{[\ell]} \) denotes the matrix formed by a direct sum of \( 2 \times 2 \) matrices in which the \( \ell \)-th summand is the \( 2 \times 2 \) matrix unit \( e_{ij} \) and every other direct summand is 0. Thus, \( \xi \) and \( \xi' \) represent the maximally entangled elements of \( S_n \otimes S^d_n \) and \( NC(m) \otimes NC^d(m) \), respectively. By Corollary 2.10 the purity of \( \xi \) and \( \xi' \) follows from the purity of the identity maps on \( S_n \), for \( n \geq 1 \), and \( NC(m) \), for \( m \geq 2 \). \( \square \)

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