The comparison of the Hamiltonians of the noncommutative isotropic harmonic oscillator and Landau problem in symmetric and two Landau gauges are evaluated analytically. The Hamiltonian of a noncommutative isotropic harmonic oscillator is found by using Bopp’s shift in commutative coordinate space. The result shows that the two systems are isomorphic up to the similar values of \( n_r \) and \( m_l \) and \( qB = eB > 0 \) for both gauge choices. However, there is an additional requirement for Landau gauge where the noncommutative oscillator has to lose one spatial degree of freedom. It also needs to be parametrized by a factor \( \zeta \) for their Hamiltonians to be consistent with each other. The wavefunctions and probability density functions are then plotted and the behaviour that emerges is explained. Finally, the effects of noncommutativity or magnetic field on the eigenstates and their probability distributions of the isomorphic system are shown.

\textit{Keywords:} Landau levels, noncommutative isotropic harmonic oscillator

\section{Introduction}

Recently, noncommutative field theories and their derivatives have been a subject of intense research. We refer to [1, 2, 3] as reading materials to provide some historical background
and reviews on the subject over the past few years. Since its establishment, it has invaded various domains of theoretical physics and has effectively continued to evolve and meet the mathematical requirements of various situations. This includes string theory (e.g [4, 5, 6]), Yang - Mills theory whereby some of its forms is a noncommutative gauge theory (e.g [7, 8, 9]) and condensed matter theory most notably, quantum Hall effect (e.g [10]).

A simple insight on the role of noncommutativity in field theory can be obtained by studying the one particle sector, which prompted an interest in the study of noncommutative quantum mechanics. The deformation of space due to noncommutativity can be expressed by the commutation relations of Hermitian operators as shall be seen later in Section 3. Thus, several authors have solved many related problems for example, hydrogen atom[11], central potential[12], Aharonov - Bohm effect[13], Ahoronov - Casher effect[14], Klein–Gordon oscillators[15], the Landau problem[16], and the list goes on.

In [17], the relationship between the magnetic field and noncommutativity parameter has been established ($\theta = \frac{4\hbar}{qB}$) for the noncommutative quantum mechanics and the usual Landau problem to be equivalent theories in the lowest Landau levels. A further generalization of this relation was then made in [18] in the context of relativistic quantum mechanics. In this paper, we would like to argue that the isomorphism between one of the problems of noncommutative quantum mechanics i.e, isotropic harmonic oscillator and Landau problem can also exist in higher-order Landau levels in fact, all Landau levels provided that their specific conditions to be discussed are satisfied.

The paper is organized as follows. Section 2 is devoted to the energy spectra and wavefunctions of Landau problem in symmetric and Landau gauge by solving the Schrödinger equation analytically. Section 3 focuses on finding the solution to the eigenvalue problem of a noncommutative isotropic harmonic oscillator by virtue of comparison of its Hamiltonian and that of Landau problem. Section 4 attempts to explore the behaviour of the wavefunctions and the probability densities plot of the isomorphic system as different quantum numbers are shown. In section 5, the effects of the magnetic field or noncommutativity are demonstrated on the ground-state eigenfunctions and their probability distributions of the isomorphic system. In the final section, we state our conclusion.

## 2 Landau levels

In the presence of a magnetic field $\hat{B}$, the canonical momentum $\hat{p}$ is shifted by the magnetic vector potential $\hat{A}$ to give a Hamiltonian of the form

$$\hat{H} = \frac{\left(\hat{p} - q\hat{A}\right)^2}{2\mu},$$

(2.1)

where $q$ and $\mu$ are the charge and the mass of the particle of interest, and $\hat{p} - q\hat{A}$ is the kinetic momentum operator. Hereafter, we adopt $q = -e$ which denotes the charge of an electron.
Note that in the presence of a magnetic field, $\hat{p} - q\hat{A}$ represents the true momentum of the particle rather than $\hat{p}$ and is the result of minimal coupling rule [19]. In this problem, we consider an electron to be freely moving in a two-dimensional system in a uniform magnetic field, $\hat{B} = B\hat{z}$. For such a magnetic field, there are two common gauge choices which will be discussed in greater detail shortly.

### 2.1 Symmetric gauge

In symmetric gauge, the vector potential can be written as

$$\hat{A}_x = -\frac{1}{2}B\hat{y}, \quad \hat{A}_y = \frac{1}{2}B\hat{x}. \tag{2.2}$$

Then by substituting (2.2) into (2.1),

$$\hat{H} = \frac{1}{2\mu} \left[ \left( \hat{p}_x + \frac{qB}{2}\hat{y} \right)^2 + \left( \hat{p}_y - \frac{qB}{2}\hat{x} \right)^2 \right]. \tag{2.3}$$

Note that we define the z-axis such that $qB > 0$. For the case of the electron where $q < 0$, it means that the magnetic field is along the negative z-axis. Alternatively, $qB > 0$ is also true for the proton where $q > 0$ when the magnetic field is along the positive z-axis. Hence we will now adopt $qB = eB$ instead, where $eB$ is also greater than 0. After a few algebraic steps,

$$\hat{H} = \frac{1}{2\mu} \left( \hat{p}_x^2 + \hat{p}_y^2 \right) + \frac{e^2B^2}{8\mu} (\hat{x}^2 + \hat{y}^2) - \frac{eB}{2\mu} (\hat{x}\hat{p}_y - \hat{y}\hat{p}_x), \tag{2.4}$$

where

$$\hat{L}_z \equiv \hat{x}\hat{p}_y - \hat{y}\hat{p}_x, \tag{2.5}$$

is the z-component of the orbital angular momentum operator. Let

$$\omega_c = \frac{eB}{\mu}, \tag{2.6}$$

which represents the cyclotron frequency in SI units.

However in some works in literature such as [21, 24], the Hamiltonian can also be written as

$$\hat{H} = \frac{1}{2\mu} \left[ \left( \hat{p}_x + \frac{qB}{2c}\hat{y} \right)^2 + \left( \hat{p}_y - \frac{qB}{2c}\hat{x} \right)^2 \right], \tag{2.7}$$

which leads to

$$\hat{H} = \frac{1}{2\mu} \left( \hat{p}_x^2 + \hat{p}_y^2 \right) + \frac{e^2B^2}{8\mu c^2} (\hat{x}^2 + \hat{y}^2) - \frac{eB}{2\mu c} (\hat{x}\hat{p}_y - \hat{y}\hat{p}_x). \tag{2.8}$$
As a result,
\[ \omega_c = \frac{eB}{\mu c}, \] (2.9)
which still represents the cyclotron frequency but in Gaussian units as the Lorentz force
differs by a factor of \(1/c\) where \(c\) is the speed of light. Both are valid representations of
the Hamiltonian for the system. In this work, the Hamiltonian which considers the SI units of
the cyclotron frequency will be considered.

Returning to our initial problem of calculating the eigenstates and eigenvalues of (2.4),
by applying a few change of variables and orthogonality relation, the solution to the time-
independent Schrödinger equation is
\[
\Psi_{n_r,m_l}(r,\varphi) = \frac{1}{\sqrt{2\pi}} \left( \frac{eB}{\hbar} \right)^{\frac{3}{2}} \sqrt{n_r!} \frac{(eB_r^2)^{|m_l|}}{(2\hbar^2)^{|m_l|}} \exp \left( -\frac{eB}{4\hbar} \right) \frac{(eB_r^2)}{2\hbar^2} e^{im_l\varphi}.
\] (2.10)
where \(n_r\) and \(m_l\) are radial and angular momentum quantum numbers respectively [25] and
is in agreement with [26]. As for the eigenvalues, they are given as
\[
E_{n_r,m_l} = (2n_r + |m_l| + 1) \frac{\hbar \omega_c}{2} - m_l \frac{\hbar \omega_c}{2},
\] (2.11)
\[
= (2n_r + |m_l| + 1) \frac{\hbar eB}{2\mu} - m_l \frac{eB}{2\mu}.
\] (2.12)
The first term mimics the energy eigenvalues of the standard two-dimensional harmonic oscil-
lator (with the frequency \(\frac{\omega_c}{2}\) instead). The second term comes from the coupled momentum-
gauge term that is manifested in the Hamiltonian after expanding the kinetic momentum
operator. This term represents the interaction between the canonical momenta and the
magnetic field, and what distinguishes this problem from a standard planar isotropic oscillator [26].

Now, the interesting question that could be pondered is, what if we choose to use \(qB = -eB\) instead? At first, this may look trivial but a careful calculation reveals that the resulting
energy levels are actually different from the case that we obtain previously. In fact, it can be
shown that
\[
E_{n_r,m_l} = (2n_r + |m_l| + 1) \frac{\hbar \omega_c}{2} + m_l \frac{\hbar \omega_c}{2},
\] (2.13)
\[
= (2n_r + |m_l| + 1) \frac{\hbar eB}{2\mu} + m_l \frac{eB}{2\mu},
\] (2.14)
while the eigenfunction maintains the same form as in (2.10). Notice that the sign of the
second term is different than before which implies that, the mere assumption that we make
on the product, \(qB\) can actually change the pattern of degeneracy. We refer to [26] for a
detailed derivation of the eigenstates and eigenenergies obtained when \(qB = -eB\).
2.2 Landau gauge

As for the Landau gauge, it can be written in two different ways;

First Landau gauge : \( \hat{A}_x = -B\hat{y}, \hat{A}_y = 0 \),

Second Landau gauge : \( \hat{A}_x = 0, \hat{A}_y = B\hat{x} \).

From here thereon, we are only going to focus our attention on the first Landau gauge, as the discussion that follows will also be applied to the second Landau gauge in a similar manner to the first Landau gauge but with a minute difference as shall be pointed out later. Then by substituting (2.15) into (2.1)

\[
\hat{H} = \frac{1}{2\mu} \left[ (\hat{p}_x + qB\hat{y})^2 + (\hat{p}_y)^2 \right],
\]

Just like in the symmetric gauge, we will define the z-axis such that \( qB > 0 \) and hence, \( qB = eB \). After a few algebraic steps,

\[
\hat{H} = \frac{1}{2\mu} \left( \hat{p}_x^2 + \hat{p}_y^2 \right) + \frac{e^2B^2}{2\mu} \hat{y}^2 + \frac{eB}{\mu} \hat{y}\hat{p}_x,
\]

The derivation of the solution of the Schrödinger equation involving the Hamiltonian in (2.18) is very well known as it has the mathematical structure of a shifted harmonic oscillator [27] and thus, will not be repeated here. It is

\[
\Psi_{n_y,k_x}(x, y) = \frac{1}{\sqrt{2^n_y n_y!}} \left( \frac{eB}{\pi \hbar} \right)^{\frac{1}{4}} \exp \left[ -\frac{eB}{2\hbar} \left( y + \frac{\hbar k_x}{eB} \right)^2 \right] H_n \left( \frac{eB}{\hbar} \left( y + \frac{\hbar k_x}{eB} \right) \right) e^{i k_x x},
\]

where \( H_n \) is the Hermite polynomial. This method of solution has introduced two quantum numbers; \( k_x, \) a real number, and \( n_y, \) a non-negative integer. This latter quantum number counts the number of nodes of the probability density. On the other hand, the eigenvalues are

\[
E_{n_y,k_x} = (2n_y + 1) \hbar \frac{\omega_c}{2},
\]

Note the lack of dependence on the quantum number \( k_x \). Consequently, the energies are infinitely degenerate for every non-negative integer \( n_y \). The energy eigenvalues of the second Landau gauge will also take the form of (2.20). In regard to the energy eigenstates, we need to change the sign of the shifted term of the position of the harmonic oscillator to have \( -\frac{\hbar k_y}{eB} \) with quantum numbers, \( k_y \) and \( n_x \). In contrast to our result, in [26], an assumption of \( qB = -eB \) is analysed instead. The only difference of our results with those in [26] is, we need to change the sign of the spatial shift of the eigenfunction of the oscillator in both Landau gauges while having the same energy.
3 Noncommutative isotropic harmonic oscillator

The Hamiltonian of a particle of mass \( m \) which oscillates with an angular frequency \( \omega \) under the influence of an isotropic harmonic oscillator potential in the noncommutative space is formulated as

\[
\hat{H} = \frac{1}{2m} \left( \hat{p}_1^2 + \hat{p}_2^2 \right) + \frac{1}{2} m \omega^2 \left( \hat{x}_1^2 + \hat{x}_2^2 \right),
\]

(3.1)

There are various formulations of quantum mechanics on noncommutative Moyal phase spaces that can be used to deal with related problems such as the above. Among them include canonical, path-integral, Weyl-Wigner, and systematic formulations (see [28]). We will utilize canonical formulation in our work. Then, by applying Bopp’s shift transformation, the non-commuting coordinates can be expressed in terms of commuting coordinates in the following form

\[
\hat{x}_i = \hat{x}_i - \frac{\Theta}{2\hbar} \hat{p}_j, \\
\hat{p}_i = \hat{p}_i,
\]

(3.2)

We can see from the above that \( \hat{p}_i \) maps onto itself. In the sequel, we often take \( \Theta_{ij} = \theta \varepsilon_{ij} \) where \( \Theta \) is the constant, frame-dependent parameters and \( \varepsilon_{ij} \) is the Levi-Civita symbol [29] and they are connected as written as follows

\[
\varepsilon_{ij} = \begin{cases} 1 & i < j \\ 0 & i = j \\ -1 & i > j \end{cases}
\]

(3.3)

The commutators involving position and momentum in (3.2) are \([\hat{x}_i, \hat{x}_j] = i\hbar \delta_{ij} \) and \([\hat{x}_i, \hat{p}_j] = i\hbar \delta_{ij} \). Due to the transformation, the new variables will satisfy the usual commutation relations below

\[
[\hat{x}_i, \hat{x}_j] = 0, \\
[\hat{x}_i, \hat{p}_j] = i\hbar \delta_{ij}, \\
[\hat{p}_i, \hat{p}_j] = 0.
\]

(3.4)

With the new variables defined in (3.2), the Hamiltonian in the ordinary space will then be

\[
\hat{H} = \frac{1}{2m} \left[ \hat{p}_1^2 + \hat{p}_2^2 \right] + \frac{1}{2} m \omega^2 \left[ \left( \hat{x}_1 - \frac{\theta}{2\hbar} \hat{p}_2 \right)^2 + \left( \hat{x}_2 + \frac{\theta}{2\hbar} \hat{p}_1 \right)^2 \right].
\]

(3.5)

After a few algebraic steps,

\[
\hat{H} = \left( \frac{1}{2m} + \frac{m \omega^2 \theta^2}{8 \hbar^2} \right) \left( \hat{p}_1^2 + \hat{p}_2^2 \right) + \frac{1}{2} m \omega^2 \left( \hat{x}_1^2 + \hat{x}_2^2 \right) - \frac{\theta}{2\hbar} m \omega^2 \left( \hat{x}_1 \hat{p}_2 - \hat{x}_2 \hat{p}_1 \right).
\]

(3.6)
Let \( \hat{x}_1 = \hat{x} \) and \( \hat{x}_2 = \hat{y} \) then,

\[
\hat{H} = \left( \frac{1}{2m} + \frac{m \omega^2 \theta^2}{8\hbar^2} \right) (\hat{p}_x^2 + \hat{p}_y^2) + \frac{1}{2} m \omega^2 (\hat{x}^2 + \hat{y}^2) - \frac{\theta}{2\hbar} m \omega^2 (\hat{x}\hat{p}_y - \hat{y}\hat{p}_x),
\]

where

\[
\hat{L}_z \equiv \hat{x}_1\hat{p}_2 - \hat{x}_2\hat{p}_1,
\]

is the z-component of the orbital angular momentum operator. It is obvious that the third term will vanish once we set \( \theta = 0 \) and is analogous to that corresponding to the Landau problem on \( \mathbb{R}^2 \) [24]. Now, let

\[
\gamma = \frac{\theta}{2\hbar} M \Omega^2.
\]

Then, rearrange (3.7) by introducing the effective mass and frequency so that the first two terms have the form of the Hamiltonian of an isotropic harmonic oscillator as follows

\[
\hat{H} = \frac{1}{2M} (\hat{p}_x^2 + \hat{p}_y^2) + \frac{1}{2} M \Omega^2 (\hat{x}^2 + \hat{y}^2) - \gamma \hat{L}_z,
\]

where

\[
\frac{1}{2M} \equiv \frac{1}{2m} + \frac{m \omega^2 \theta^2}{8\hbar^2},
\]

\[
M = \frac{m}{1 + \frac{m^2 \omega^2 \theta^2}{4\hbar^2}}.
\]

As a result

\[
M \Omega^2 \equiv m \omega^2,
\]

\[
\Omega^2 = m \omega^2 \left( 1 + \frac{m^2 \omega^2 \theta^2}{4\hbar^2} \right),
\]

\[
\Omega = \omega \sqrt{1 + \frac{m^2 \omega^2 \theta^2}{4\hbar^2}}.
\]

where the definition of the parameters above are also used in [22, 23]. Here comes the main question in this paper, can we somehow find the energy eigenvalues and eigenstates of the noncommutative oscillator from the mathematical framework of Landau problem? This shall be explored in the following subsections.

### 3.1 Symmetric gauge

From Section 2, we have determined the energy spectra and wavefunctions of Landau levels. The question now is, what are the conditions to be obeyed in order for Landau levels and
noncommutative oscillator to be equivalent? We can naively compare the Hamiltonian of the two systems, (3.10) and (2.4) then try to infer from there. Thus we have,

\[
\frac{m}{1 + \frac{m^2 \omega^2 q^2}{4\hbar^2}} = \mu,
\]

\[
\frac{1}{2} m \omega^2 = \frac{e^2 B^2}{8\mu},
\]

\[
\frac{\theta}{2\hbar} m \omega^2 = \frac{eB}{2\mu}.
\]

(3.13)

Realize that, the set of equations that we have from the comparison of the coefficients of the Hamiltonians needs to be true simultaneously. In other words, (3.13) altogether has to be consistent but, that is not the case here. For the two systems to be isomorphic, we need to invoke some mathematical scheme by redefining \(M\) and \(\Omega\) in other terms instead of \(m\) and \(\omega\). This is allowable as \(M\) and \(\Omega\) are respectively the parameters that characterize the effective mass and frequency of the oscillator in a commutative space. With reference to [17] which is generalized for any noncommutative quantum mechanical problem in central field, we are going to let the effective mass and frequency to be

\[
M = \frac{2\hbar^2}{\theta^2},
\]

(3.14)

and

\[
\Omega = \frac{\theta}{\hbar},
\]

(3.15)

which are \(\theta\)-dependent. Due to (3.14) and (3.15), the resulting Hamiltonian of the noncommutative oscillator is the following,

\[
\hat{H} = \frac{\theta^2}{4\hbar^2} (\hat{p}_x^2 + \hat{p}_y^2) + (\hat{x}^2 + \hat{y}^2) - \frac{\theta}{\hbar} \hat{L}_z.
\]

(3.16)

Realize that by imposing the oscillator to have the new effective mass and frequency in terms of \(\theta\) alone, the Hamiltonian (3.16) lack of parameters permits an opportunity of introducing a factor to make the two systems consistent. By multiplying each of the coefficients of the Hamiltonian by a factor \(\zeta\) to be determined, then

\[
\hat{H} = \frac{\zeta \theta^2}{4\hbar^2} (\hat{p}_x^2 + \hat{p}_y^2) + \zeta (\hat{x}^2 + \hat{y}^2) - \frac{\zeta \theta}{\hbar} \hat{L}_z.
\]

(3.17)

Now, by comparing the newly-defined Hamiltonian of the noncommutative oscillator and the Landau levels, hence

\[
\frac{\zeta \theta^2}{4\hbar^2} = \frac{1}{2\mu},
\]
\[ \zeta = \frac{e^2 B^2}{8 \mu}, \]
\[ \zeta \frac{\theta}{\hbar} = \frac{e B}{2 \mu}. \]  

(3.18)

Unlike the comparison from (3.13), a close inspection reveals that (3.18) is altogether consistent and in accordance with [17]. As a result, together with factor \( \zeta \), the effective mass and frequency of the oscillator will then be equal to those of Landau levels as shown below

\[ \frac{1}{2M} = \frac{\zeta \theta^2}{4 \hbar^2}, \]
\[ M = \frac{2 \hbar^2}{\zeta \theta^2}, \]  

(3.19)

\[ \frac{1}{2} M \Omega^2 = \zeta, \]
\[ \Omega = \frac{\zeta \theta}{\hbar}, \]  

(3.20)

where \( \zeta = \frac{e^2 B^2}{8 \mu} \) is a controlling factor. By calculation, we can evaluate that when Landau levels have \( \mu = m_e = 9.109 \times 10^{-31} \) and \( \omega_c = \frac{e B}{\mu} = 2.110 \times 10^{12} \), then the oscillator would have \( M = 4.620 \times 10^{-37} \) and \( \Omega = 2.081 \times 10^{18} \) due to (3.14) and (3.15). Since we have established the relationship above, we can always examine a different mass and frequency of either of the systems and be able to find the same parameters of the other system studied for them to be isomorphic.

Let us contemplate the effect of a constant \( B \) and \( \theta \) on their respective systems. In the quantum Hall effect experiments, the magnetic fields are typically about \( B = 12T \) [17, 30, 31]. Using the second and third equations in (3.18),
\[ \theta = \frac{4 \hbar}{e B} = 2.195 \times 10^{-16}, \]  

(3.21)

in m\(^2\) or it can also be written as \( \theta = 0.2195 \times 10^{-11} \) in cm\(^2\) which is in conformance with [17]. For this value of \( \theta \), one cannot distinguish between noncommutative quantum mechanics (in this case, it would be the noncommutative isotropic harmonic oscillator) and the usual Landau problem.

Since we have determined the value of \( \theta \), we should also compute the value of the controlling factor \( \zeta \), that is
\[ \zeta = \frac{e^2 B^2}{8 \mu} = 5.071 \times 10^{-7}, \]  

(3.22)

which implies that just like \( \theta \), there is a dependence on the charge of the particle and the magnetic field which acts on it in the Landau problem with an additional dependence on the mass of the particle.
Hence, since their Hamiltonians are now similar, analysing Landau levels is tantamount to analysing the noncommutative oscillator. Let us revisit the energy eigenvalues of Landau levels in symmetric gauge for \( qB = eB > 0 \). They are

\[
E_{n_r, m_l} = (2n_r + |m_l| + 1)\hbar \frac{eB}{2\mu} - m_l\hbar \frac{eB}{2\mu}.
\] (3.23)

For the noncommutative oscillator, we can express (3.23) in terms of \( \theta \) and \( \zeta \) as follows

\[
E_{n_r, m_l} = (2n_r + |m_l| + 1)\zeta - m_l\zeta \theta.
\] (3.24)

Therefore, we conclude that the noncommutative oscillator and Landau levels are isomorphic provided that the pair of quantum numbers \( n_r \) and \( m_l \) of both systems have the same values. Besides, the Hamiltonian of the noncommutative oscillator has to be parametrized by the controlling factor \( \zeta \) so that it is consistent with the Hamiltonian of Landau levels in symmetric gauge. In regards to the wavefunctions of Landau levels, we recall that

\[
\Psi_{n_r, m_l}(r, \varphi) = \frac{1}{\sqrt{2\pi}} \left( \frac{eB}{\hbar} \right)^{\frac{1}{2}} \sqrt{\frac{n_r!}{(n_r + |m_l|)!}} \left( \frac{eB}{2\hbar} \right)^{|m_l|} \frac{1}{2} r^{|m_l|} \exp \left( -\frac{eB}{4\hbar} r^2 \right) L_{n_r}^{[m_l]} \left( \frac{eB}{2\hbar} r^2 \right) e^{im_l \varphi}.
\] (3.25)

For the noncommutative oscillator, again by expressing (3.25) in terms of \( \theta \) and \( \zeta \), one obtains

\[
\Psi_{n_r, m_l}(r, \varphi) = \frac{1}{\sqrt{2\pi}} \left( \frac{4}{\theta} \right)^{\frac{1}{2}} \sqrt{\frac{n_r!}{(n_r + |m_l|)!}} \left( \frac{2}{\theta} \right)^{|m_l|} \frac{1}{2} r^{|m_l|} \exp \left( -\frac{1}{\theta} r^2 \right) L_{n_r}^{[m_l]} \left( \frac{2}{\theta} r^2 \right) e^{im_l \varphi}.
\] (3.26)

With the above results, we can find the analytical spectra of the noncommutative oscillator in the context of Landau problem in a symmetric gauge. However, the results obtained are by no means representing the general spectra of the solutions of the noncommutative oscillator but rather specific to the case when \( M \) and \( \Omega \) are \( \theta \)-dependent designated in (3.19) and (3.20). We refer to [32, 33, 34, 35] as reading materials that deliver more elaborate research on the spectra of the noncommutative oscillator using path integral formulation.

The same question as raised before can still be asked here. What if we impose the following condition, \( qB = -eB \)? We will encounter a situation where the comparison cannot be made as

\[
-\frac{\theta}{2\hbar} = \frac{eB}{2\mu};
\] (3.27)

which is clearly false as the left-hand side is negative while the right-hand side is positive. This is the result when comparing the third term of their Hamiltonians. One might argue
that why do we not account for this with a new $\zeta$. The problem is, the $\zeta -$ term is also present in the first and second terms of the Hamiltonian of the noncommutative oscillator. If we change them, it will generate a false statement like in (3.27). Hence, we can say that this is also a condition for isomorphism. We want to emphasize again that the results of the energy spectra and wavefunctions of the noncommutative oscillator that we obtain here are applied only when $M$ and $\Omega$ are $\theta -$dependent which is the limitation of this formalism.

### 3.2 Landau gauge

As for the two Landau gauges, we cannot directly compare the coefficients of their Hamiltonians as their variables or operators are clearly different from one another. Nonetheless, an assumption can be made for each to account for this. For the first Landau gauge, assume that the position operator, $\hat{x}$ to be the zero operator, $\hat{0}$. The same supposition goes to the second Landau gauge as well but with $\hat{y}$. Hence, the correlation can then be made as follows,

$$\frac{m}{1 + \frac{m^2 \omega^2 \theta^2}{4 \hbar^2}} = \mu,$$

$$\frac{1}{2} m \omega^2 = \frac{e^2 B^2}{2 \mu},$$

$$\frac{\theta}{2 \hbar} m \omega^2 = \frac{eB}{\mu}. \quad (3.28)$$

Notice that the set of equations in (3.28) is reminiscent to the one in symmetric gauge and therefore, we will not repeat the discussion here. However we will show how their Hamiltonians take the form as portrayed below,

Table 1: Comparison between the Hamiltonian of Landau gauges and noncommutative oscillator

| Hamiltonian of Landau gauge | Hamiltonian of noncommutative oscillator |
|-----------------------------|------------------------------------------|
| $\frac{1}{2\mu} (\hat{p}_{x}^2 + \hat{p}_{y}^2) + \frac{e^2 B^2}{2\mu} \hat{y}^2 + \frac{e B}{\mu} \hat{y} \hat{p}_x$ | $\frac{\zeta \theta^2}{4\hbar^2} (\hat{p}_{x}^2 + \hat{p}_{y}^2) + \zeta (\hat{x}^2 + \hat{y}^2) - \frac{\zeta \theta}{\hbar} \hat{L}_z$ |
| $\frac{1}{2\mu} (\hat{x}^2 + \hat{y}^2) + \frac{e^2 B^2}{2\mu} \hat{x}^2 - \frac{e B}{\mu} \hat{x} \hat{p}_y$ | $\frac{\zeta \theta^2}{4\hbar^2} (\hat{p}_{x}^2 + \hat{p}_{y}^2) + \zeta (\hat{x}^2 + \hat{y}^2) - \frac{\zeta \theta}{\hbar} \hat{L}_z$ |

After solving the time-independent Schrödinger equation, the energy eigenvalues of the noncommutative oscillator in the context of the first Landau gauge are

$$E_{n_y} = (2n_y + 1) \zeta \theta \quad (3.29)$$
and the energy eigenstate is

\[
\Psi^{(k_0)}_{n_y}(y) = \frac{1}{\sqrt{2^{n_y}n_y!}} \left( \frac{4}{\pi \theta} \right)^{\frac{1}{4}} \exp \left[ -\frac{2}{\theta} \left( y + \frac{\theta}{4}k_0 \right)^2 \right] H_n \left[ \frac{4}{\theta} \left( y + \frac{\theta}{4}k_0 \right) \right]
\]

The important thing to note here is, the system is more restrictive compared to the case of symmetric gauge since we have to lose a spatial degree of freedom. As a result, the system is nondegenerate for every quantum number, \( n_y \). The real number \( k_x \) will be a constant that we will denote as \( k_0 \), since the system has lost the degree of freedom associated with position \( x \). This is equivalent to the statement that the position has localized to a fixed value. Consequently, it implies that it is impossible to determine \( k_0 \) as it is correlated with the momentum \( p_x \) of the particle due to the uncertainty principle. We will treat \( k_0 \) as a parameter instead of a quantum number and its effect on the system will be analysed in Section 4.2. If we use the second Landau gauge, the energy eigenvalues resemble the ones obtained before and, the energy eigenfunction will have a negative shift in position. Otherwise, as before, it will not work if we assume \( qB < 0 \).

4 Effect of quantum numbers on the isomorphic system

Since the conditions under which these theories are isomorphic have been laid out in the previous section, we will plot the wavefunctions and probability density functions for the first few lower quantum numbers to observe their effects on the isomorphic system. Let us substitute \( \theta = 2.195 \times 10^{-16} \) that is defined from the previous section into (3.26) for symmetric gauge and (3.30) for the first Landau gauge. Since their eigenstates are identical (under certain assumptions) as proven in the previous section, plotting the wavefunction of a noncommutative oscillator is tantamount to that corresponding to Landau levels. Hence, plotting a noncommutative oscillator alone is sufficient to avoid redundancy in the upcoming figures and discussion.

4.1 Symmetric gauge

By using Mathematica, the 3D plot of one of the ground-state wavefunctions and its probability density of the isomorphic system when both quantum numbers are zero are shown as follows,
In Figure 1, since $n_r$ and $m_l$ are both zero, the imaginary part of the wavefunction naturally vanishes and so are other related terms in (3.26) leaving the wavefunction to have a structure of a $e^{-r^2}$ function. The same argument goes to Figure 2 as well except that it follows the structure of a $e^{-2r^2}$ which explains its slightly different feature. Yet, the characteristic shape for both follows the two-dimensional Gaussian function. Due to its distribution, it is most likely to find the particle at the origin. Generally, the wavefunction bears no physical interpretation while the probability density function does as its own name implies, by giving us the likelihood of finding an electron (or some other system) at some given point in space based on the Born rule. Hence, in the upcoming discussion, the wavefunction (in this paper we will focus on its real part) will only serve as a visualisation while the probability distribution is subjected to further analysis. The ground-state energy is $1.112 \times 10^{-22} J$ or $6.943 \times 10^{-4} eV$. 

Figure 1: Ground-state, $\Psi_{0,0}(r, \varphi)$ of the isomorphic system with respect to symmetric gauge

Figure 2: Ground-state, $|\Psi_{0,0}(r, \varphi)|^2$ of the isomorphic system with respect to symmetric gauge
4.1.1 Effect of $n_r$ on $\Psi_{n_r,0}$

The exact magnitude of energy for an increasing value of the radial quantum number $n_r$ can be seen as follows.

Table 2: Ordered pair of quantum numbers and its corresponding energy of the isomorphic system in the units of $J$ and $eV$

| $(n_r, m_l)$ | Energy (in $J$)   | Energy (in $eV$) |
|-------------|-------------------|------------------|
| (1, 0)      | $3.340 \times 10^{-22}$ | $2.085 \times 10^{-3}$ |
| (2, 0)      | $5.566 \times 10^{-22}$ | $3.475 \times 10^{-3}$ |
| (3, 0)      | $7.793 \times 10^{-22}$ | $4.864 \times 10^{-3}$ |

Figure 3 shows that in a fixed domain, the number of concentric rings formed that can be seen on the nodes and antinodes of the wavefunctions increases as $n_r$ increases. This is actually reasonable because when we observe Figure 4, those antinodes will turn to nodes and it is in accordance with what $n_r$ actually represents. It is a positive integer that corresponds to the number of nodes in the wave function moving radially out from the center. Just like $|\Psi_{0,0}|^2$, the likelihood of finding the particle at the origin is also the highest. Though as $m_l$ increases, so does the number of nodes causing the probability distribution to be smeared out to the existing nodes. The more the number of nodes, the lower their resulting amplitudes.
(a) First excited state, $\Psi_{1,0}(r, \varphi)$
(b) Second excited state, $\Psi_{2,0}(r, \varphi)$
(c) Third excited state, $\Psi_{3,0}(r, \varphi)$

Figure 3: Effect of $n_r$ on $\Psi_{n_r,0}(r, \varphi)$ of the isomorphic system with respect to symmetric gauge
4.1.2 Effect of $m_l$ on $Re[\Psi_{0,m_l}]$

For an increasing value of the angular momentum quantum number $m_l$, the exact value of the energy levels will be evaluated by using the parameters mentioned before.
Table 3: Ordered pair of quantum numbers and its corresponding energy of the isomorphic system in the units of $J$ and $eV$

| $(n_r, m_l)$ | Energy (in $J$) | Energy (in $eV$) |
|-------------|----------------|------------------|
| (0, 3)      | $1.112 \times 10^{-22}$ | $6.943 \times 10^{-4}$ |
| (0, 2)      | $1.112 \times 10^{-22}$ | $6.943 \times 10^{-4}$ |
| (0, 1)      | $1.112 \times 10^{-22}$ | $6.943 \times 10^{-4}$ |
| (0, −1)     | $3.340 \times 10^{-22}$ | $2.085 \times 10^{-3}$ |
| (0, −2)     | $5.566 \times 10^{-22}$ | $3.475 \times 10^{-3}$ |
| (0, −3)     | $7.793 \times 10^{-22}$ | $4.864 \times 10^{-3}$ |

Notice that unlike Figures 1 until 4, there are two orders of state to which each of the plots in Figure 5 and 6 belongs. This comes from the rotational symmetry associated with states with opposite signs of $m_l$ and thus, producing identical geometrical shapes. When $n_r = 0$, all states with $m_l \geq 0$ stay at the ground level. The remaining states with $m_l < 0$ will mimic the behaviour of $n_r$ in Section 4.1.1 as documented in Table 3. Figure 5 displays the real part of the wavefunction and we can see that the number of local minima and maxima designated by the troughs and valleys formed increases as $|m_l|$ increases. In addition, Figure 6 indicates that just like Figure 4, the distribution is radially symmetric and the probability of finding a particle farther away from the origin is greater as $|m_l|$ increases since the radius at which the maximum amplitude is located increases. Unlike in the previous subsection, we only plot the real part of the wavefunctions but not as a whole. This is because, when $m_l \neq 0$, we cannot plot the real and imaginary part of the wavefunctions on the same set of axes.
(a) Ground-state, $\text{Re}[\Psi_{0,1}(r, \varphi)]$ and first excited state, $\text{Re}[\Psi_{0,-1}(r, \varphi)]$

(b) Ground-state, $\text{Re}[\Psi_{0,2}(r, \varphi)]$ and second excited state, $\text{Re}[\Psi_{0,-2}(r, \varphi)]$

(c) Ground-state, $\text{Re}[\Psi_{0,3}(r, \varphi)]$ and third excited state, $\text{Re}[\Psi_{0,-3}(r, \varphi)]$

Figure 5: Effect of $m_l$ on $\text{Re}[\Psi_{0,m_l}(r, \varphi)]$ on the isomorphic system with respect to symmetric gauge
(a) Ground-state, $|\Psi_{0,1}(r, \varphi)|^2$ and first excited state, $|\Psi_{0,1}(r, \varphi)|^2$

(b) Ground-state, $|\Psi_{0,2}(r, \varphi)|^2$ and second excited state, $|\Psi_{0,-2}(r, \varphi)|^2$

(c) Ground-state, $|\Psi_{0,3}(r, \varphi)|^2$ and third excited state, $|\Psi_{0,-3}(r, \varphi)|^2$

Figure 6: Effect of $m_l$ on $|\Psi_{0,m_l}(r, \varphi)|^2$ on the isomorphic system with respect to symmetric gauge

4.1.3 Effect of both $n_r$ and $m_l$ on $\text{Re}[\Psi_{n_r,m_l}]$

When both $n_r$ and $m_l$ are nonzero, for the sake of demonstration, Table 4 depict the energy levels of both systems.
Table 4: Ordered pair of quantum numbers and its corresponding energy of the noncommutative oscillator in the units of $J$ and $eV$

| $(n_r, m_l)$ | Energy (in $J$) | Energy (in $eV$) |
|-------------|-----------------|-----------------|
| (1, 1)      | $3.340 \times 10^{-22}$ | $2.085 \times 10^{-3}$ |
| (2, 1)      | $5.566 \times 10^{-22}$ | $3.475 \times 10^{-3}$ |
| (3, 1)      | $7.793 \times 10^{-22}$ | $4.864 \times 10^{-3}$ |
| (1, 2)      | $3.340 \times 10^{-22}$ | $2.085 \times 10^{-3}$ |
| (2, 2)      | $5.566 \times 10^{-22}$ | $3.475 \times 10^{-3}$ |
| (3, 2)      | $7.793 \times 10^{-22}$ | $4.864 \times 10^{-3}$ |
| (1, 3)      | $3.340 \times 10^{-22}$ | $2.085 \times 10^{-3}$ |
| (2, 3)      | $5.566 \times 10^{-22}$ | $3.475 \times 10^{-3}$ |
| (3, 3)      | $7.793 \times 10^{-22}$ | $4.864 \times 10^{-3}$ |
| (1, -1)     | $5.566 \times 10^{-22}$ | $3.475 \times 10^{-3}$ |
| (2, -1)     | $7.793 \times 10^{-22}$ | $4.864 \times 10^{-3}$ |
| (3, -1)     | $1.002 \times 10^{-21}$ | $6.254 \times 10^{-3}$ |
| (1, -2)     | $7.793 \times 10^{-22}$ | $4.864 \times 10^{-3}$ |
| (2, -2)     | $1.002 \times 10^{-21}$ | $6.254 \times 10^{-3}$ |
| (3, -2)     | $1.225 \times 10^{-21}$ | $7.644 \times 10^{-3}$ |
| (1, -3)     | $1.002 \times 10^{-21}$ | $6.254 \times 10^{-3}$ |
| (2, -3)     | $1.225 \times 10^{-21}$ | $7.644 \times 10^{-3}$ |
| (3, -3)     | $1.447 \times 10^{-21}$ | $9.034 \times 10^{-3}$ |

Figure 7 shows the complex features of some of the arbitrarily chosen states with specific values of $n_r$ and $m_l$ respectively to illustrate its combined effect (due to them being nonzero) on the real part of the wavefunctions. Let us divert our attention to the probability distribution plot. Realize that the distribution of Figure 8c seems to be spread out farther than the origin while maintaining the same structure compared to Figure 8a. This is not surprising as Figure 8c has a higher $|m_l|$ by an addend of 1. As for when comparing Figure 8a and 8b, the latter has an additional node which is also anticipated as it has a higher value of $n_r$ by an addend of 1. By combining these two features, we will have Figure 8d. This behaviour can also be extended to the higher-order state of the isomorphic system. We do not specify the order of state for each plot as the ideas to determine it, are straightforward and be extended from Section 4.1.1 and Section 4.1.2.
Figure 7: Effect of both $n_r$ and $m_l$ on $\text{Re}[\Psi_{n_r, m_l}(r, \varphi)]$ on the isomorphic system with respect to symmetric gauge.
Figure 8: Effect of both $n_r$ and $m_l$ on $|\Psi_{n_r,m_l}(r,\varphi)|^2$ on the isomorphic system with respect to symmetric gauge

4.2 Landau gauge

For this subsection, we study the effect of the quantum number $n_y$ and a parameter $k_0$ on the wavefunctions and probability distributions of the isomorphic system in the context of the first Landau gauge. Even though the functions are two-dimensional in nature as they only depend on the position $y$, we decide to let our $k_0$ to be the variable of our functions to study their collective effects (with $n_y$) on the same set of axes. This will not create any issue on the interpretation of the plot as the effect of $k_0$ is trivial as shall be seen later. Recall that $k_0$ is also unpredictable and hence, we will not provide a table to show some exact value of energies.

The contour plot of the aforementioned functions are provided in Figure 9 and 10 below,
Figure 9: Effect of $n$ on $\Psi^{(k_0)}_{n_y}(y)$ of the isomorphic system with respect to the first Landau gauge and varying parameter $k_0$
From Figures 9 and 10, we can see that the parameter $k_0$ only plays a role in shifting the harmonic oscillator to the right of the origin as $k_0$ increases. From the mathematical form of the wavefunction, we can think of the system as a shifted harmonic oscillator where the behaviour of such function is well understood. By right, the quantum number $n_y$ reflects the number of nodes and antinodes of the wavefunction and only the number of nodes for the probability density curve. However, we can clearly see that only two nodes are present even when $n_y$ is greater than 1 as shown in 10. This is due to the factor used in the wavefunction which makes the nodes look invisible. Besides, the amplitude of the two highest peaks designated by the brightest strips on the plot also increases as $n_y$ increases which implies that the amplitude of the other seemingly invisible peaks in between should decrease and thus, making it even harder to observe. This behaviour is also displayed in higher-order states.
5 Effect of $B$ or $\theta$ on the isomorphic system

In this section, the effect of the magnetic field or noncommutativity on the isomorphic system will be presented. We will also use the parameters that have been defined in the previous section for plotting purposes.

5.1 Symmetric gauge

Let us consider the following random values of the magnetic field; 10$T$, 15$T$ and 20$T$. From Figures 11 and 12, we only plot the ground-state wavefunctions and their respective probability densities i.e, $\Psi_{0,0}$ and $|\Psi_{0,0}|^2$ as the effect can naturally be extended to higher-order states. We observe that the radius at which the distribution begins to flatten out decreases as $B$ increases or $\theta$ decreases. This is due to the radial symmetry of the wavefunction. To compensate for the reduction in radius, the amplitude has to increase to maintain the norm to be one.
Figure 11: Effect of $B$ or $\theta$ on the function $\Psi_{0,0}(r, \varphi)$ with respect to symmetric
5.2 Landau gauge

This time we also assign the same values of the magnetic field (10\( T \), 15\( T \) and 15\( T \)). We only plot the ground-state wavefunctions and probability distributions just like before. Figures 13 and 14 below show that the deviation of the harmonic oscillator due to parameter \( k_0 \) from the origin to be lesser as the magnetic field increases or noncommutativity decreases. This is expected as these two parameters are present in the shifted position term in (3.30). The same behaviour is also present in higher-order states.
Figure 13: Effect of $B$ or $\theta$ on the function $\Psi_{n_y}^{(k_0)}(y)$ of the isomorphic system with respect to the first Landau gauge and varying parameter $k_0$. 

(a) $\Psi_0^{(k_0)}(y)$ at $B = 5T$

(b) $\Psi_1^{(k_0)}(y)$ at $B = 10T$

(c) $\Psi_2^{(k_0)}(y)$ at $B = 15T$
Figure 14: Effect of $B$ or $\theta$ on the probability density, $|\Psi^{(k_0)}_n(y)|^2$ of the isomorphic system with respect to the first Landau gauge and varying parameter $k_0$

6 Conclusions

In this work, we propose a mathematical formalism as an approach to solve noncommutative harmonic oscillator problems within the framework of the Landau problem. We argue that the equivalence of noncommutative quantum mechanics in the central field and the Landau problem as portrayed by [12] for the lowest Landau level can be extended to all Landau levels in the case of a harmonic oscillator. We find that the noncommutative isotropic harmonic oscillator and Landau levels can be considered as an isomorphic system in the context of both Landau and symmetric gauges by satisfying a number of conditions. In the context of symmetric gauge, they are isomorphic up to the similar values of $n_r$ and $m_l$ for $qB = eB > 0$. The noncommutative oscillator also has to be parametrized by a factor $\zeta$ to make the Hamiltonians consistent with each other. As for Landau gauge, the requirements are the
same but with an addition that, the noncommutative oscillator has to lose one spatial degree
of freedom.

With this research, the problems involving the analytical spectrum of the noncommutative
harmonic oscillator can now be understood within the framework of the Landau problem.
This framework provides a simpler and more generalized approach to the noncommutative
oscillator problems since the Landau problems are very well understood and it also applies
to all energy levels. It is also of major interest to study if this relation still holds when a
gauge-invariant Landau problem with some potential is considered. One can also imagine
the possibility of isomorphism if the Hamiltonian is energy-dependent.

Acknowledgments

The authors wish to acknowledge the financial support provided through the IPM-UPM
Grant, Project No. 9645700 Universiti Putra Malaysia.

References

[1] Douglas, M., & Nekrasov, N. (2001). Noncommutative field theory. Reviews of Modern
Physics, 73(4), 977–1029. doi:10.1103/revmodphys.73.977

[2] Rivelles, V. O. (2011). A Review of Noncommutative Field Theories. Journal of Physics:
Conference Series, 287, 012012. doi:10.1088/1742-6596/287/1/012012

[3] Szabo, R. (2003). Quantum field theory on noncommutative spaces. Physics Reports,
378(4), 207–299. doi:10.1016/s0370-1573(03)00059-0

[4] Connes, A., Douglas, M. R., & Schwarz, A. (1998). Noncommutative geometry and
Matrix theory. Journal of High Energy Physics, 1998(02), 003–003. doi:10.1088/1126-
6708/1998/02/003

[5] Douglas, M. R., & Hull, C. (1998). D-branes and the noncommutative torus. Journal
of High Energy Physics, 1998(02), 008–008. doi:10.1088/1126-6708/1998/02/008

[6] Seiberg, N., & Witten, E. (1999). String theory and noncommutative geometry. Journal
of High Energy Physics, 1999(09), 032–032. doi:10.1088/1126-6708/1999/09/032

[7] Eguchi, T., & Nakayama, R. (1983). Simplification of quenching procedure for large N
spin models. Physics Letters B, 122(1), 59–62. doi:10.1016/0370-2693(83)91168-1

[8] González-Arroyo, A., & Okawa, M. (1983). Twisted-Eguchi-Kawai model: A re-
duced model for large-N lattice gauge theory. Physical Review D, 27(10), 2397–2411.
doi:10.1103/physrevd.27.2397
[9] González-Arroyo, A., & Korthals Altes, C. P. (1983). Reduced model for large N continuum field theories. Physics Letters B, 131(4-6), 396–398. doi:10.1016/0370-2693(83)90526-9

[10] Bellissard, J., van Elst, A., & Schulz - Baldes, H. (1994). The noncommutative geometry of the quantum Hall effect. Journal of Mathematical Physics, 35(10), 5373–5451. doi:10.1063/1.530758

[11] Chaichian, M., Sheikh-Jabbari, M. M., & Tureanu, A. (2001). Hydrogen Atom Spectrum and the Lamb Shift in Noncommutative QED. Physical Review Letters, 86(13), 2716–2719. doi:10.1103/physrevlett.86.2716

[12] Gamboa, J., Loewe, M., & Rojas, J. C. (2001). Noncommutative quantum mechanics. Physical Review D, 64(6). doi:10.1103/physrevd.64.067901

[13] Li, K., & Dulat, S. (2006). The Aharonov–Bohm effect in noncommutative quantum mechanics. The European Physical Journal C, 46(3), 825–828. doi:10.1140/epjc/s2006-02538-2

[14] Mirza, B., Narimani, R., & Zarei, M. (2006). Aharonov–Casher effect for spin-1 particles in a non-commutative space. The European Physical Journal C, 48(2), 641–645. doi:10.1140/epjc/s10052-006-0047-z

[15] Mirza, B., & Mohadesi, M. (2004). The Klein-Gordon and the Dirac Oscillators in a Noncommutative Space. Communications in Theoretical Physics, 42(5), 664–668. doi:10.1088/0253-6102/42/5/664

[16] Dulat, S., & Kang, L. (2008). Landau problem in noncommutative quantum mechanics. Chinese Physics C, 32(2), 92–95. doi:10.1088/1674-1137/32/2/003

[17] Gamboa, J., Mendez, F., Loewe, M., & Rojas, J. C. (2001). The Landau Problem and Noncommutative Quantum Mechanics. Modern Physics Letters A, 16(32), 2075–2078. doi:10.1142/s0217732301005345

[18] Mirza, B., Narimani, R., & Zare, S. (2011). Relativistic Oscillators in a Noncommutative Space and in a Magnetic Field. Communications in Theoretical Physics, 55(3), 405–409. doi:10.1088/0253-6102/55/3/06

[19] Chiao, Raymond. (2011). On the origin of the minimal coupling rule, and on the possibility of observing a classical, ”Aharonov-Bohm-like” angular momentum. arXiv:1104.4493v1 [physics.class-ph]

[20] Desai, B. R. (2013). Quantum mechanics with basic field theory. Cambridge: Cambridge University Press. (pp. 208)
[21] Kang, L., Xiao-Hua, C., & Dong-Yan, W. (2006). *Heisenberg algebra for noncommutative Landau problem*. Chinese Physics, 15(10), 2236–2239. doi:10.1088/1009-1963/15/10/008

[22] Kijanka, A., & Kosiński, P. (2004). *Noncommutative isotropic harmonic oscillator*. Physical Review D, 70(12). doi:10.1103/physrevd.70.127702

[23] Muthukumar, B., & Mitra, P. (2002). *Noncommutative oscillators and the commutative limit*. Physical Review D, 66(2). doi:10.1103/physrevd.66.027701

[24] Jellal, Ahmed & Houca, Rachid. (2006). *Noncommutative Description of Quantum Spin Hall Effect*. International Journal of Geometric Methods in Modern Physics - INT J GEOM METHODS MOD PHYS. 06. doi:10.1142/S0219887809003539.

[25] Sullivan, D. M. (2012). *Quantum mechanics for electrical engineers*. Hoboken, NJ: IEEE Press. (pp. 240)

[26] Bhuiyan, Asadullah & Marsiglio, Frank. (2020). *Landau levels, Edge States, and Gauge Choice in 2D Quantum Dots*. arXiv:2001.03860v2 [cond-mat.mes-hall]

[27] D. J. Griffiths and D. F. Schroeter, *Introduction to Quantum Mechanics, Third Edition* (Cambridge University Press, Cambridge, 2018).

[28] Gouba, L. (2016). *A comparative review of four formulations of noncommutative quantum mechanics*. International Journal of Modern Physics A, 31(19), 1630025. doi:10.1142/s0217751x16300258

[29] Chung, W. S. (2016). *Generalized Uncertainty Relation in the Non-commutative Quantum Mechanics*. International Journal of Theoretical Physics, 55(6), 2989–3000. doi:10.1007/s10773-016-2931-0

[30] Klitzing, K. v., Dorda, G., & Pepper, M. (1980). *New Method for High-Accuracy Determination of the Fine-Structure Constant Based on Quantized Hall Resistance*. Physical Review Letters, 45(6), 494–497. doi:10.1103/physrevlett.45.494

[31] Tsui, D. C., Stormer, H. L., & Gossard, A. C. (1982). *Two-Dimensional Magneto-transport in the Extreme Quantum Limit*. Physical Review Letters, 48(22), 1559–1562. doi:10.1103/physrevlett.45.1559

[32] Jing, J., Zhao, S.-H., Chen, J.-F., & Long, Z.-W. (2008). *On the spectra of noncommutative 2D harmonic oscillator*. The European Physical Journal C, 54(4), 685–690. doi:10.1140/epjc/s10052-008-0560-3

[33] Benchika, A., & Chetouani, L. (2013). *Energy-dependent Potential and Normalization of Wave Function*. Modern Physics Letters A, 28(18), 1350079. doi:10.1142/s021773231350079x
[34] Benchikha, A., & Merad, M. (2017). *Energy-dependent harmonic oscillator in noncommutative space: A path integral approach*. International Journal of Modern Physics A, 32(32), 1750194. doi:10.1142/s0217751x17501949

[35] Benchikha, A., Merad, M., & Birkandan, T. (2017). *Energy-dependent harmonic oscillator in noncommutative space*. Modern Physics Letters A, 32(20), 1750106. doi:10.1142/s0217732317501061