Structures of BiHom-Poisson algebras

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Abstract

This paper gives some constructions results and examples of BiHom-Poisson algebras. Next, BiHom-flexible algebras are defined and it is shown that admissible BiHom-Poisson algebras are BiHom-flexible. Furthermore, generalized derivations of BiHom-Poisson algebras are introduced and some their basic properties are given. Finally, BiHom-Poisson modules and several constructions of these notions are obtained.

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1 Introduction

A Poisson algebra \((P, \{\cdot, \cdot\}, \mu)\) consists of a commutative associative algebra \((A, \mu)\) together with a Lie structure \(\{\cdot, \cdot\}\), satisfying the Leibniz identity:

\[
\{\mu(x, y), z\} = \mu(\{x, z\}, y) + \mu(x, \{y, z\})
\]

These algebras firstly appeared in the work of Siméon-Denis Poisson two centuries ago when he was studying the three-body problem in celestial mechanics. Since then, Poisson algebras have shown to be connected to many areas of mathematics and physics. Indeed, in mathematics, Poisson algebras play a fundamental role in Poisson geometry [34], quantum groups [8, 11] and deformation of commutative associative algebras [13] whereas in physics, Poisson algebras represent a major part of deformation quantization [17], Hamiltonian mechanics [3] and topological field theories [33]. Poisson-like structures are also used in the study of vertex operator algebras [12].

Algebras where the identities defining the structure are twisted by a homomorphism are called Hom-algebras. Hom-type algebras appeared in the Physics literature of the 1990’s, when looking for quantum deformations of some algebras of vector fields, like Witt and Virasoro algebras [2, 24]. It was observed that algebras obtained by deforming certain Lie algebras no longer satisfied the Jacobi identity, but a modified version of it involving a homomorphism. An axiomatization of this type of algebras (called Hom-Lie algebras) was given in [21, 25].

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The associative counterpart of Hom-Lie algebras (called Hom-associative algebras) has been introduced in [31], where it was proved also that the commutator bracket defined by the multiplication in a Hom-associative algebra gives rise to a Hom-Lie algebra.

A BiHom-algebra is an algebra in such a way that the identities defining the structure are twisted by two homomorphisms $\alpha$ and $\beta$. This class of algebras was introduced from a categorical approach in [18] as an extension of the class of Hom-algebras. These algebraic structures include BiHom-associative algebras, BiHom-Jordan algebras, BiHom-alternative algebras and BiHom-Lie algebras. BiHom-Poisson algebras was first introduced in [29] and studied in [1] where in particular, the concept of module of BiHom-Poisson algebra is introduced and admissible BiHom-Poisson algebras, and only these BiHom-algebras, are shown to give rise to BiHom-Poisson algebras via polarization. More applications of BiHom-type can be found in [19, 20, 9, 10, 26, 28, 30].

The purpose of this paper is further to study BiHom-Poisson algebras. The paper is organized as follows. Section 2 contains some important basic notions and notations related to BiHom-algebras, BiHom-Poisson algebras and modules over BiHom-associative algebras. Section 3 presents some constructions results of BiHom-Poisson algebras, BiHom-flexible structures and admissible BiHom-Poisson algebras. In section 4 we give some basic properties concerning derivation algebras, quasiderivation algebras and generalized derivation algebras of Bihom-Poisson algebras. In section 5, we introduce and give some properties of BiHom-Poisson modules. Next, we prove that from a given BiHom-Poisson modules, a sequence of this kind of modules can be constructed and we then define the semi-direct product of BiHom-Poisson algebras. Finally, we define the first and second cohomology spaces of BiHom-Poisson algebras.

Throughout this paper $\mathbb{K}$ is an algebraically closed field of characteristic 0 and $A$ is a $\mathbb{K}$-vector space.

### 2 Preliminaries

This section contains necessary important basic notions and notations which will be used in next sections. For the map $\mu : A \otimes 2 \rightarrow A$, we will sometimes $\mu(a \otimes b)$ as $\mu(a, b)$ or $ab$ for $a, b \in A$ and if $V$ is another vector space, $\tau_1 : A \otimes V \rightarrow V \otimes A$ (resp. $\tau_2 : V \otimes A \rightarrow A \otimes V$) denote the twist isomorphism $\tau_1(a \otimes v) = v \otimes a$ (resp. $\tau_2(v \otimes a) = a \otimes v$).

**Definition 2.1.** A BiHom-module is a pair $(M, \alpha_M, \beta_M)$ consisting of a $\mathbb{K}$-module $M$ and a linear self-maps $\alpha_M, \beta_M : M \rightarrow M$ such that $\alpha_M \beta_M = \beta_M \alpha_M$. A morphism $f : (M, \alpha_M, \beta_M) \rightarrow (N, \alpha_N, \beta_N)$ of BiHom-modules is a linear map $f : M \rightarrow N$ such that $f \alpha_M = \alpha_N f$ and $f \beta_M = \beta_N f$.

**Definition 2.2.** [15] A BiHom-algebra is a quadruple $(A, \mu, \alpha, \beta)$ in which $(A, \alpha, \beta)$ is a BiHom-module, $\mu : A \otimes 2 \rightarrow A$ is a linear map. The BiHom-algebra $(A, \mu, \alpha, \beta)$ is said to be multiplicative if $\alpha \circ \mu = \mu \circ \alpha$ and $\beta \circ \mu = \mu \circ \beta$ (multiplicativity).

**Definition 2.3.** 1. A BiHom-algebra $(A, \mu, \alpha, \beta)$ is said to be BiHom-commutative if

$$\mu(\beta(x), \alpha(y)) = \mu(\beta(y), \alpha(x)), \forall x, y \in A$$
2. A BiHom-associative algebra \([15]\) is a multiplicative Bihom-algebra \((A, \mu, \alpha, \beta)\) satisfying the following BiHom-associativity condition:

\[
as_A(x, y, z) := \mu(\mu(x, y), \beta(z)) - \mu(\alpha(x), \mu(y, z)) = 0, \text{ for all } x, y, z \in A. \quad (2.1)\]

3. A BiHom-Lie algebra \([15]\) is a multiplicative Bihom-algebra \((A, \{\cdot, \cdot\}, \alpha, \beta)\) satisfying the BiHom-skew-symmetry and the BiHom-Jacobi identities i.e.

\[
\{\beta(x), \alpha(y)\} = -\{\beta(y), \alpha(x)\} \\
\bigcirc_{x,y,z} \{\beta^2(x), \{\beta(y), \alpha(z)\}\} = 0 \quad (2.2)
\]

where \(\bigcirc_{x,y,z}\) denotes the summation over the cyclic permutation on \(x, y, z\).

Clearly, a Hom-associative algebra \((A, \mu, \alpha)\) can be regarded as a BiHom-associative algebra \((A, \mu, \alpha, \alpha)\).

**Definition 2.4.** [29] A BiHom-Poisson algebra consists of a vector space \(A\), two bilinear maps \(\mu, \{\cdot, \cdot\}: A \otimes A \to A\), linear maps \(\alpha, \beta: A \to A\) such that

1. \((A, \mu, \alpha, \beta)\) is a BiHom-commutative BiHom-associative algebra,
2. \((A, \{\cdot, \cdot\}, \alpha, \beta)\) is a BiHom-Lie algebra,
3. the BiHom-Leibniz identity

\[
\{\alpha \beta(x), \mu(y, z)\} = \mu(\{\beta(x), y\}, \beta(z)) + \mu(\beta(y), \{\alpha(x), z\}) \quad (2.3)
\]

is satisfied for all \(x, y, z \in A\).

In a BiHom-Poisson algebra \((A, \{\cdot, \cdot\}, \mu, \alpha, \beta)\), the operations \(\mu\) and \(\{\cdot, \cdot\}\) are called the BiHom-associative product and the BiHom-Poisson bracket, respectively.

**Remark 2.5.** A non-BiHom-commutative BiHom-Poisson algebra is a BiHom-Poisson algebra without the BiHom-commutativity assumption [1]. These Bihom-algebras are called BiHom-Poisson algebras [29].

**Definition 2.6.** Let \((A, \{\cdot, \cdot\}, \mu, \alpha, \beta)\) a BiHom-Poisson algebra. A subspace \(H\) of \(A\) is called

1. A BiHom-subalgebra of \(A\) if
   \[\alpha(H) \subseteq H, \ \beta(H) \subseteq H, \ \mu(H, H) \subseteq H \text{ and } \{H, H\} \subseteq H.\]
2. A left-BiHom ideal of \(A\) if
   \[\alpha(H) \subseteq H, \ \beta(H) \subseteq H, \ \mu(A, H) \subseteq H \text{ and } \{A, H\} \subseteq H.\]
3. A right-BiHom ideal of \(A\) if
   \[\alpha(H) \subseteq H, \ \beta(H) \subseteq H, \ \mu(H, A) \subseteq H \text{ and } \{H, A\} \subseteq H.\]
4. A two sided BiHom-ideal if \(H\) is both a left and a right BiHom-ideal of \(A\).
Note that, if \( \alpha \) and \( \beta \) are bijective, then the notion of left-BiHom ideals is equivalent to the one of right-BiHom ideals.

**Definition 2.7.** Let \( (A, \{\cdot, \cdot\}, \mu, \alpha, \beta) \) be a BiHom-Poisson algebra. If \( Z(A) = \{x \in A | \{x, y\} = \mu(x, y) = 0, \ \forall y \in A\} \), then \( Z(A) \) is called the centralizer of \( A \).

**Definition 2.8.** Let \( (A, \{\cdot, \cdot\}, \mu, \alpha, \beta) \) a BiHom-Poisson algebra.

1. \( A \) is multiplicative if
   \[
   \alpha\{\cdot, \cdot\} = \{\cdot, \cdot\} \alpha \otimes 2, \quad \beta\{\cdot, \cdot\} = \{\cdot, \cdot\} \beta \otimes 2 \quad \text{and} \quad \alpha \mu = \mu \alpha \otimes 2, \quad \beta \mu = \mu \beta \otimes 2.
   \]

2. \( (A, \mu, \alpha, \beta) \) is said to be regular if \( \alpha \) and \( \beta \) are algebra automorphisms.

3. \( (A, \mu, \alpha, \beta) \) is said to be involutive if \( \alpha \) and \( \beta \) are two involutions, that is \( \alpha^2 = \beta^2 = \text{id} \).

4. Let \( (A', \{\cdot', \cdot'\}, \mu', \alpha', \beta') \) be another BiHom-Poisson algebra. A weak morphism \( f : A \to A' \) is a linear map such that
   \[
   \{\cdot, \cdot\} \circ f = \{\cdot', \cdot'\} \circ f \otimes 2 \quad \text{and} \quad f \mu = \mu' \circ f \otimes 2.
   \]

An morphism \( f : A \to A' \) is a weak morphism such that \( f \alpha = \alpha' \circ f \) and \( f \beta = \beta' \circ f \).

Note that a 5-tuple \( (A, \{\cdot, \cdot\}, \mu, \alpha, \beta) \) is multiplicative if and only if the twisting map \( \alpha, \beta : A \to A \) are morphisms.

Denote by \( \Gamma_f = \{x + f(x); \ x \in A\} \subset A \oplus A' \) the graph of a linear map \( f : A \to A' \).

**Definition 2.9.** Let \( (A, \mu, \alpha, \beta) \) be any BiHom-algebra.

1. A BiHom-module \( (V, \phi, \psi) \) is called an \( A \)-bimodule if it comes equipped with a left and a right structures maps on \( V \) that is morphisms \( \rho_l : (A \otimes V, \alpha \otimes \phi, \beta \otimes \psi) \to (V, \phi, \psi) \), \( a \otimes v \mapsto a.v \) and \( \rho_r : (V \otimes A, \phi \otimes \alpha, \psi \otimes \psi) \to (V, \phi, \psi) \), \( v \otimes a \mapsto v.a \) of BiHom-modules.

2. A morphism \( f : (V, \phi, \psi, \rho_l, \rho_r) \to (W, \phi', \psi', \rho'_l, \rho'_r) \) of \( A \)-bimodules is a morphism of the underlying BiHom-modules such that
   \[
   f \circ \rho_l = \rho'_l \circ (f \otimes \text{Id}_A) \quad \text{and} \quad f \circ \rho_r = \rho'_r \circ (f \otimes \text{Id}_A).
   \]

That yields the commutative diagrams
\[
\begin{array}{ccc}
A \otimes V & \xrightarrow{\rho_l} & V \\
\downarrow{\text{Id}_A \otimes f} & & \downarrow{f} \\
A \otimes W & \xrightarrow{\rho'_l} & W
\end{array}
\quad \quad \quad \quad \quad \quad \quad \\
\begin{array}{ccc}
V \otimes A & \xrightarrow{\rho_r} & V \\
\downarrow{f \otimes \text{Id}_A} & & \downarrow{f} \\
W \otimes A & \xrightarrow{\rho'_r} & W
\end{array}
\]

Now, let consider the following notions for BiHom-associative algebras.

**Definition 2.10.** Let \( (A, \mu, \alpha, \beta) \) be a BiHom-associative algebra, \( (L, \{\cdot, \cdot\}, \alpha, \beta) \) be a Hom-Lie algebra and \( (V, \phi, \psi) \) be a BiHom-module. Then
1. A left BiHom-associative $A$-module structure on $V$ [15] consists of a morphism $\rho_l : A \otimes V \rightarrow V$ of BiHom-modules, such that
\[ \rho_l \circ (\alpha \otimes \rho_l) = \rho_l \circ (\mu \otimes \psi). \] (2.4)
In terms of diagram, we have
\[
\begin{array}{c}
A \otimes A \otimes V \\
\mu \otimes \psi \downarrow \\
A \otimes V \\
\rho_l \downarrow \\
V \\
\end{array}
\]
2. A right BiHom-associative $A$-module structure on $V$ [15] consists of a morphism $\rho_r : V \otimes A \rightarrow V$ of BiHom-modules, such that
\[ \rho_r \circ (\phi \otimes \mu) = \rho_r \circ (\rho_r \otimes \beta). \] (2.5)
In terms of diagram, we have
\[
\begin{array}{c}
V \otimes A \otimes A \\
\phi \otimes \mu \downarrow \\
V \otimes A \\
\rho_r \downarrow \\
V \\
\end{array}
\]
3. A left BiHom-Lie $L$-module structure on $V$ [15] consists of a structure map $\rho : L \otimes V \rightarrow V$ such that
\[ \rho(\{\beta(x), y\}, \psi(v)) = \rho(\alpha \beta(x), \rho(y, v)) - \rho(\beta(y), \rho(\alpha(x), v)) \] (2.6)

3 BiHom-Poisson algebras

3.1 Constructions of BiHom-Poisson algebras

In this subsection, we provide some constructions results of BiHom-Poisson algebras.

**Proposition 3.1.** Let $(A, \{\cdot, \cdot\}, \mu, \alpha, \beta)$ be a BiHom-Poisson algebra and $I$ be a two-sided BiHom-ideal of $(A, \{\cdot, \cdot\}, \mu, \alpha, \beta)$. Then $(A/I, \{\cdot, \cdot\}, \mu, \alpha, \beta)$ is a BiHom-Poisson algebra where $[x, y] = \{x, y\}, \overline{\mu(x, y)} = \mu(x, y), \alpha(x) = \alpha(x)$ and $\overline{\beta(x)} = \beta(x)$, for all $x, y \in A/I$

**Proof.** We only prove item 1. of definition [2.4] item 2. and item 3. are being proved similarly.

For all $\overline{x}, \overline{y}, \overline{z} \in A/I$ we have
\[
as_{A/I}(\overline{x}, \overline{y}, \overline{z}) = \overline{\mu(\overline{x}, \overline{y}, \overline{z})} - \overline{\mu(\overline{x}, \overline{y}, \overline{z})} \]
\[
= \overline{\mu(\mu(x, y), \beta(z))} - \overline{\mu(\mu(x, y), \beta(z))} \]
\[
= \overline{\mu(\mu(x, y), \beta(z))} - \overline{\mu(\mu(x, y), \beta(z))} \]
\[
= \overline{\mu(\beta(x), \alpha(y))} \]
Then $(A/I, \overline{\mu}, \overline{\alpha}, \overline{\beta})$ is a BiHom-associative algebra.
\[
\overline{\mu(\beta(x), \alpha(y))} = \overline{\mu(\beta(x), \alpha(y))} \]
\[
= \overline{\mu(\beta(y), \alpha(x))} \] (by BiHom – commutativity of $A$).
\[
= \overline{\mu(\beta(x), \alpha(y))} \]
Then $(A/I, \overline{\mu}, \overline{\alpha}, \overline{\beta})$ is a BiHom-commutative algebra. \[\square\]
Proposition 3.3. \cite{[13]} Let \((A, \mu, \alpha, \beta)\) be a regular BiHom-associative algebra. Then \(A^- = (A, \{\cdot, \cdot\}, \mu, \alpha, \beta)\) is a regular non-BiHom-commutative BiHom-Poisson algebra, where \(\{\cdot, \cdot\} = \mu - \mu \circ (\alpha^{-1} \beta \otimes \alpha^{-1}) \circ \tau\), where for all \(x, y \in A\), \(\tau(x \otimes y) = y \otimes x\).

Example 3.3. Consider the 2-dimensional regular BiHom-associative algebras \((A, \mu, \alpha, \beta)\) with a basis \((e_1, e_2)\), (see \cite{[13]}) defined by

\[
\begin{align*}
\alpha(e_1) &= e_1, & \alpha(e_2) &= \frac{b(1-a)}{a} e_1 + ae_2, \\
\beta(e_1) &= e_1, & \beta(e_2) &= be_1 + (1-a)e_2, \\
\mu(e_1, e_1) &= e_1, & \mu(e_1, e_2) &= be_1 + (1-a)e_2, \\
\mu(e_2, e_1) &= \frac{b(1-a)}{a} e_1 + ae_2, & \mu(e_2, e_2) &= \frac{b}{a} e_2,
\end{align*}
\]

where \(a, b\) are parameters in \(\mathbb{K}\), with \(a \neq 0, 1\). Using the Proposition \cite{[13]} the 5-tuple \((A, \{\cdot, \cdot\}, \mu, \alpha, \beta)\) is a regular non-BiHom-commutative BiHom-Poisson algebra where, \(\{\cdot, \cdot\} = \mu - \mu \circ (\alpha^{-1} \beta \otimes \alpha^{-1}) \circ \tau\) and

\[
\begin{align*}
\alpha^{-1}(e_1) &= e_1, & \alpha^{-1}(e_2) &= \frac{b(a-1)}{a^2} e_1 + \frac{1}{a} e_2, \\
\beta^{-1}(e_1) &= e_1, & \beta^{-1}(e_2) &= \frac{b}{a+1} e_1 + \frac{1-a}{a} e_2.
\end{align*}
\]

Proposition 3.4. \cite{[20]} Let \((A, \{\cdot, \cdot\}, \mu, \alpha, \alpha, \beta, \beta)\) and \((B, \{\cdot, \cdot\}, \beta, \beta, \beta)\) be two BiHom-Poisson algebras. Then there exists a BiHom-Poisson algebra \((A \oplus B, \{\cdot, \cdot\}, \mu, \alpha = \alpha_A + \alpha_B, \beta = \beta_A + \beta_B)\), where the bilinear maps \(\{\cdot, \cdot\}, \mu : (A \oplus B)^{\times 2} \to (A \oplus B)\) are given by

\[
\mu(a_1 + b_1, a_2 + b_2) = \mu_A(a_1, a_2) + \mu_B(b_1, b_2),
\]

\[
\{a_1 + b_1, a_2 + b_2\} = \{a_1, a_2\}_A + \{b_1, b_2\}_B, \forall \ a_1, a_2 \in A, \forall \ b_1, b_2 \in B.
\]

and the linear maps \(\beta = \beta_B, \alpha = \alpha_A + \alpha_B : (A \oplus B) \to (A \oplus B)\) are given by

\[
\begin{align*}
(\alpha_A + \alpha_B)(a + b) &= \alpha_A(a) + \alpha_B(b), \\
(\beta_A + \beta_B)(a + b) &= \beta_A(a) + \beta_B(b), \ \forall \ (a, b) \in A \times B.
\end{align*}
\]

Proof. It is easy to see that \((A \oplus B, \mu, \alpha_A + \alpha_B, \beta_A + \beta_B)\) is a BiHom-associative and BiHom-commutative algebra and \((A \oplus B, \{\cdot, \cdot\}, \alpha_A + \alpha_B, \beta_A + \beta_B)\) is a BiHom-Lie algebra. Then \((A \oplus B, \{\cdot, \cdot\}, \mu, \alpha_A + \alpha_B, \beta_A + \beta_B)\) is a BiHom-Poisson algebra.

Proposition 3.5. Let \((A, \{\cdot, \cdot\}, \alpha, \beta)\) and \((B, \{\cdot, \cdot\}, \alpha, \beta)\) be two BiHom-Poisson algebras and \(\varphi : A \to B\) be a linear map. Then \(\varphi\) is a morphism from \((A, \{\cdot, \cdot\}, \alpha, \beta)\) to \((B, \{\cdot, \cdot\}, \alpha, \beta)\) if and only if its graph \(\Gamma_{\varphi}\) is a BiHom-subalgebra of \((A \oplus B, \{\cdot, \cdot\}, \mu, \alpha, \beta)\).

Proof. Let \(\varphi : (A, \mu, \alpha, \beta) \to (B, \mu, \alpha, \beta)\) be a morphism of BiHom-Poisson algebras. Then for all \(u, v \in A\),

\[
\mu(u + \varphi(u), v + \varphi(v)) = \mu_A(u, v) + \mu_B(\varphi(u), \varphi(v)) = \mu_A(u, v) + \varphi(\mu_A(u, v))
\]

\[
\{u + \varphi(u), v + \varphi(v)\} = \{u, v\}_A + \varphi(\{u, v\}_A)
\]

Thus the graph \(\Gamma_{\varphi}\) is closed under the operations \(\mu\) and \(\{\cdot, \cdot\}\). Furthermore since \(\varphi \circ \alpha_1 = \alpha_2 \circ \varphi\), we have \((\alpha_1 \oplus \alpha_2)(u, \varphi(u)) = (\alpha_1(u), \alpha_2 \circ \varphi(u)) = (\alpha_1(u), \varphi \circ \alpha_1(u))\). In the same way, we have \((\beta_1 \oplus \beta_2)(u, \varphi(u)) = (\beta_1(u), \beta_2 \circ \varphi(u)) = (\beta_1(u), \varphi \circ \beta_1(u))\), which implies that \(\Gamma_{\varphi}\) is closed under \(\alpha_1 \oplus \alpha_2\) and \(\beta_1 \oplus \beta_2\) Thus \(\Gamma_{\varphi}\) is a BiHom-subalgebra of \((A \oplus B, \{\cdot, \cdot\}, \mu, \alpha, \beta)\).
Conversely, if the graph $\Gamma_\varphi \subset A \oplus B$ is a BiHom-subalgebra of $(A \oplus B, \{\cdot,\cdot\}, \mu, \alpha, \beta)$ then we have
\[
\mu(u + \varphi(u), v + \varphi(v)) = (\mu_A(u, v) + \mu_B(\varphi(u), \varphi(v))) \in \Gamma_\varphi,
\]
\[
\{u + \varphi(u), v + \varphi(v)\} = \{u, v\}_A + \{\varphi(u), \varphi(v)\}_B \in \Gamma_\varphi
\]
which implies that
\[
\mu_B(\varphi(u), \varphi(v)) = \varphi(\mu_A(u, v)),
\]
\[
\{\varphi(u), \varphi(v)\}_B = \varphi(\{u, v\}_A).
\]
Furthermore, $(\alpha_1 + \alpha_2)(\Gamma_\varphi) \subset \Gamma_\varphi$, $(\beta_1 + \beta_2)(\Gamma_\varphi) \subset \Gamma_\varphi$ implies
\[
(\alpha_1 + \alpha_2)(u, \varphi(u)) = (\alpha_1(u), \alpha_2 \circ \varphi(u)) \in \Gamma_\varphi,
\]
\[
(\beta_1 + \beta_2)(u, \varphi(u)) = (\beta_1(u), \beta_2 \circ \varphi(u)) \in \Gamma_\varphi.
\]
which is equivalent to the condition $\alpha_2 \circ \varphi(u) = \varphi \circ \alpha_1(u)$, i.e. $\alpha_2 \circ \varphi = \varphi \circ \alpha_1$. Similarly, $\beta_2 \circ \varphi = \varphi \circ \beta_1$. Therefore, $\varphi$ is a morphism of BiHom-Poisson algebras. □

**Theorem 3.6.** Let $(A, \{\cdot,\cdot\}, \mu, \alpha, \beta)$ be a (non-BiHom-commutative) BiHom-Poisson algebra and $\alpha', \beta': A \to A$ be endomorphisms of $A$ such that any two of the maps $\alpha, \beta, \alpha', \beta'$ commute. Then
\[
A_{\alpha', \beta'} = (A, \{\cdot,\cdot\}_{\alpha', \beta'}, \mu_{\alpha', \beta'}, \alpha', \beta')
\]
is also a (non-BiHom-commutative) BiHom-Poisson algebra. Moreover suppose that $(B, \{\cdot,\cdot\}', \mu', \gamma, \delta)$ is another BiHom-Poisson algebra and $\gamma', \delta'$ be endomorphisms of $B$ such that any two of the maps $\gamma, \delta, \gamma', \delta'$ commute. If $f : (A, \{\cdot,\cdot\}, \mu, \alpha, \beta) \to (B, \{\cdot,\cdot\}', \mu', \gamma, \delta)$ is a morphism such that $f \circ \alpha' = \gamma' \circ f$ and $f \circ \beta' = \delta' \circ f$, then $f : A_{\alpha', \beta'} \to B_{\gamma', \delta'}$ is also a morphism.

**Proof.** Let give the proof in BiHom-commutativity case. We only prove item 1. of Definition 2.4 item 2. and item 3. can be proved similarly.

For all $x, y, z \in A$ we have
\[
as_{A_{\alpha', \beta'}}(x, y, z) = \mu_{\alpha', \beta'}(\mu_{\alpha', \beta'}(x, y), \beta'(z)) - \mu_{\alpha', \beta'}(\alpha'(x), \mu_{\alpha', \beta'}(y, z))
\]
\[
\quad = \mu(\alpha'(x), \beta'(y), \beta'(z)) - \mu(\alpha'(x), \mu(\alpha'(y), \beta'(z)))
\]
\[
\quad = \mu(\alpha'(x), \beta'(y), \beta'(z)) - \mu(\alpha'(x), \beta'(y), \beta'(z))
\]
\[
\quad = \mu_{A_{\alpha', \beta'}}(x, y, z) = 0 (\text{BiHom – associativity condition of } A).
\]

Then $(A, \mu_{\alpha', \beta'}, \alpha', \beta')$ is a BiHom-associative algebra.

Now, for all $x, y \in A$ we have:
\[
\mu_{\alpha', \beta'}(\beta'(x), \alpha'(y)) = \mu(\alpha'(x), \beta'(y), \alpha'(y))
\]
\[
\quad = \mu(\alpha'(x), \beta'(y), \alpha'(y))
\]
\[
\quad = \mu(\alpha'(x), \beta'(y), \alpha'(y)) (\text{BiHom – commutativity in } A).
\]

Then $(A, \mu_{\alpha', \beta'}, \alpha', \beta')$ is a BiHom-commutative algebra.

The second part is proved as follows: $\forall x, y \in A$
\[
f\{x, y\}_{\alpha', \beta'} = f\{\alpha'(x), \beta'(y)\}
\]
\[
\quad = \{f\alpha'(x), f\beta'(y)\}'
\]
\[
\quad = \{\gamma f(x), \delta f(y)\}'
\]
\[
\quad = \{f(x), f(y)\}_{\gamma', \delta'}.
\]
In the same way we have $f \mu_{\alpha',\beta'}(x,y) = \mu'_{\alpha',\beta'}(f(x), f(y))$.

This finishes the proof. \hfill \Box

Taking $\alpha' = \alpha^k$, $\beta' = \beta^l$, yields the following statement.

**Corollary 3.7.** Let $(A,\{\cdot,\cdot\}, \mu, \alpha, \beta)$ be a (non-BiHom-commutative) BiHom-Poisson algebra. Then

$$A_{\alpha^k,\beta^l} = (A,\{\cdot,\cdot\}_{\alpha^k,\beta^l} = \{\cdot,\cdot\} \circ (\alpha^k \otimes \beta^l), \mu_{\alpha^k,\beta^l} = \mu \circ (\alpha^k \otimes \beta^l), \alpha^{k+1}, \beta^{l+1})$$

is also a (non-BiHom-commutative) BiHom-Poisson algebra.

Taking $\alpha = \beta = id$, yields the following statement.

**Corollary 3.8.** Let $(A,\{\cdot,\cdot\}, \mu)$ be a (non-commutative) Poisson algebra and $\alpha, \beta: A \to A$ be two endomorphisms such that $\alpha \beta = \beta \alpha$. Then

$$A_{\alpha,\beta} = (A,\{\cdot,\cdot\}_{\alpha,\beta} = \{\cdot,\cdot\} \circ (\alpha \otimes \beta), \mu_{\alpha,\beta} = \mu \circ (\alpha \otimes \beta), \alpha, \beta)$$

is also a (non-BiHom-commutative) BiHom-Poisson algebra.

**Definition 3.9.** Let $(A,\{\cdot,\cdot\}, \mu)$ be a non-commutative Poisson algebra.

1. Given two commuting morphisms $\alpha, \beta: A \to A$, the triple $A'_{\alpha,\beta} = (A,\{\cdot,\cdot\}_{\alpha,\beta} = \{\cdot,\cdot\} \circ (\alpha \otimes \beta), \mu_{\alpha,\beta} = \mu \circ (\alpha \otimes \beta))$ is called the $(\alpha,\beta)$-twisting of $A$. A twisting of $A$ is a $(\alpha,\beta)$-twisting of $A$ for some morphisms $\alpha, \beta: A \to A$.

2. The $(\alpha,\beta)$-twisting $A'_{\alpha,\beta}$ of $A$ is called trivial if

$$\{\cdot,\cdot\}_{\alpha,\beta} = 0 = \mu_{\alpha,\beta}.$$

$A'_{\alpha,\beta}$ is called non-trivial if either $\{\cdot,\cdot\} \neq 0$ or $\mu_{\alpha,\beta} \neq 0$.

3. $A$ is called rigid if every twisting of $A$ is either trivial or isomorphic to $A$.

**Proposition 3.10.** Let $(A,\{\cdot,\cdot\}, \mu)$ be a non-commutative Poisson algebra. Suppose there exists two commuting morphisms $\alpha, \beta: A \to A$ such that either:

1. $\mu_{\alpha,\beta} = \mu \circ (\alpha \otimes \beta)$ is not BiHom-associative or

2. $\{\cdot,\cdot\}_{\alpha,\beta} = \{\cdot,\cdot\} \circ (\alpha \otimes \beta)$ does not satisfy the BiHom-Jacobi identity.

Then $A$ is not rigid.

**Proof.** The $(\alpha,\beta)$-twisting $A'_{\alpha,\beta}$ is non-trivial, since otherwise $\mu_{\alpha,\beta}$ would be BiHom-associative and $\{\cdot,\cdot\}_{\alpha,\beta}$ would satisfy the BiHom-Jacobi identity. For the same reason, the $(\alpha,\beta)$-twisting $A'_{\alpha,\beta}$ cannot be isomorphic to $A$. \hfill \Box

**Remark 3.11.** Let $(\mathfrak{g},[\cdot,\cdot])$ be a finite-dimensional Lie algebra, and let $(S(\mathfrak{g}), \mu)$ be its symmetric algebra. If $\{e_i\}_{i=1}^n$ is a basis of $\mathfrak{g}$, then $S(\mathfrak{g})$ is the polynomial algebra $\mathbb{K}[e_1, \ldots, e_n]$. Suppose the structure constants for $\mathfrak{g}$ are given by

$$[e_i,e_j] = \sum_{k=1}^n c_{ij}^k e_k.$$
Then the symmetric algebra \( S(\mathfrak{g}) \) becomes a Poisson algebra with the Poisson bracket

\[
\{F, G\} = \frac{1}{2} \sum_{i,j,k=1}^{n} c_{ij}^{k} e_{k} \left( \frac{\partial F}{\partial e_{i}} \frac{\partial G}{\partial e_{j}} - \frac{\partial F}{\partial e_{j}} \frac{\partial G}{\partial e_{i}} \right)
\] (3.7)

for \( F, G \in S(\mathfrak{g}) \). This Poisson algebra structure on \( S(\mathfrak{g}) \) is called the linear Poisson structure. Note that \( \{e_{i}, e_{j}\} = [e_{i}, e_{j}] \).

**Example 3.12 \((\mathfrak{sl}(2))\) is not rigid.** In this example, we show that the symmetric algebra \((S(\mathfrak{sl}(2)), \mu)\) on the Lie algebra \( \mathfrak{sl}(2) \), equipped with the linear Poisson structure (3.7), is not rigid in the sense of Definition 3.9.

The Lie algebra \( \mathfrak{sl}(2) \) has a basis \( \{e, f, h\} \), with respect to which the Lie bracket is given by

\[
[h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = h.
\]

To show that \( S(\mathfrak{sl}(2)) = \mathbb{K}\{e, f, h\} \) is not rigid, consider the Lie algebra morphisms \( \alpha, \beta: \mathfrak{sl}(2) \rightarrow \mathfrak{sl}(2) \) given by

\[
\alpha(e) = \lambda e, \quad \alpha(f) = \lambda^{-1}f, \quad \alpha(h) = h, \\
\beta(e) = \gamma e, \quad \beta(f) = \gamma^{-1}f, \quad \beta(h) = h.
\]

where \( \lambda, \gamma \in \mathbb{K} \) is a fixed scalar with \( \lambda, \gamma \neq 0, 1 \). Denote by \( \alpha, \beta: S(\mathfrak{sl}(2)) \rightarrow S(\mathfrak{sl}(2)) \) the extended maps, which is a Poisson algebra morphisms. By Proposition 3.10, the Poisson algebra \( S(\mathfrak{sl}(2)) \) is not rigid if \( \mu_{\alpha, \beta} = \mu \circ (\alpha \otimes \beta) \) is not BiHom-associative. We have

\[
\mu_{\alpha, \beta}(\mu_{\alpha, \beta}(e, h), h) - \mu_{\alpha, \beta}(e, \mu_{\alpha, \beta}(h, h)) = \alpha^{2}(e)\alpha\beta(h)\beta(h) - \alpha(e)\alpha\beta(h)\beta^{2}(h) = (\lambda^{2} - \lambda)eh^{2},
\]

which is not 0 in the symmetric algebra \( S(\mathfrak{sl}(2)) \) because \( \lambda \neq 0, 1 \). Therefore, \( \mu_{\alpha, \beta} \) is not BiHom-associative, and the linear Poisson structure on \( S(\mathfrak{sl}(2)) \) is not rigid. \[\square\]

### 3.2 BiHom-flexibles structures and admissible BiHom-Poisson algebras

In this subsection, we introduce BiHom-flexible algebras and prove that admissible BiHom-Poisson algebras are BiHom-flexible.

**Definition 3.1.** Let \((A, \mu, \alpha, \beta)\) be a BiHom-algebra. Then \(A\) is called BiHom-flexible algebra if for any \(x, y \in A\)

\[
\mu(\mu(\beta^{2}(x), \alpha\beta(y)), \beta\alpha^{2}(x)) = \mu(\alpha\beta^{2}(x), \mu(\alpha\beta(y), \alpha^{2}(x))) = 0.
\] (3.8)

**Remark 3.2.**

1. If \(\alpha = \beta = \text{Id}\), then \((A, \mu, \alpha, \beta)\) is reduced to a flexible algebra \((A, \mu)\).

2. If \((A, \mu, \alpha)\) is a Hom-flexible algebra [31], then \((A, \mu, \alpha, \alpha)\) is a BiHom-flexible algebra. Conversely, if \((A, \mu, \alpha, \beta)\) is a BiHom-flexible algebra such \(\alpha\) is injective and \(\alpha = \beta\) then, \((A, \mu, \alpha)\) is a Hom-flexible algebra.

**Lemma 3.3.** Let \(A = (A, \mu, \alpha, \beta)\) be a BiHom-algebra. The following assertions are equivalent

1. \(A\) is BiHom-flexible.

2. For any \(x, y \in A\), \(a_{A}(\beta^{2}(x), \alpha\beta(y), \alpha^{2}(x)) = 0\).
3. For any $x, y, z$

$$\alpha(x, y, z) = \alpha(x, y, z) + \alpha(y, z, x) + \alpha(z, x, y) = 0 \quad (3.10)$$

Proof. The equivalence of the first two assertions follows from the definition. The assertion

$$\alpha(x, y, z) = \alpha(x, y, z) + \alpha(y, z, x) + \alpha(z, x, y) = 0 \quad (3.10)$$

holds by definition and it is equivalent to $\alpha(x, y, z) + \alpha(y, z, x) + \alpha(z, x, y) = 0$ by linearity.

It is easy to prove the following

Proposition 3.13. Let $(A, \mu)$ be a flexible algebra, $\alpha, \beta : A \to A$ be two commuting morphisms. Then the Bihom-algebra $(A, \mu, \alpha, \beta)$ is BiHom-flexible.

Proof. Let $(A, \mu)$ be a flexible algebra, $\alpha$ and $\beta$ be morphisms of $(A, \mu)$. Then for all $x, y \in A$,

$$\mu_{\alpha, \beta}(\alpha(x, y)) \circ \beta(y, z) = \mu(\alpha(x, y), \beta(y, z))$$

By Lemma 3.3.

Hence the Bihom-algebra $(A, \mu, \alpha, \beta)$ is BiHom-flexible.

Corollary 3.4. Any BiHom-associative algebra is BiHom-flexible.

Proposition 3.5. A BiHom-algebra $A = (A, \mu, \alpha, \beta)$ is BiHom-flexible if and only if

$$\alpha(x, y, z) = \alpha(x, y, z) + \alpha(y, z, x) + \alpha(z, x, y) = 0 \quad (3.10)$$

where $\{\cdot, \cdot\} = \frac{1}{4}(\mu - \mu \circ (\alpha \beta \circ \alpha \beta^{-1}) \circ \tau)$ and $\circ = \frac{1}{2}(\mu + \mu \circ (\alpha \beta \circ \alpha \beta^{-1}) \circ \tau)$.

Proof. Since $\{\cdot, \cdot\} = \frac{1}{2}(\mu - \mu \circ (\alpha \beta \circ \alpha \beta^{-1}) \circ \tau)$ and $\circ = \frac{1}{2}(\mu + \mu \circ (\alpha \beta \circ \alpha \beta^{-1}) \circ \tau)$, by expansion in terms of $\mu$

$$\alpha(x, y, z) = \alpha(x, y, z) + \alpha(y, z, x) + \alpha(z, x, y) = 0 \quad (3.10)$$

Conversely, assume we have the condition $\{3.10\}$. By setting $x = z$, one gets $\alpha(x, y, z) = 0$. Therefore $A$ is BiHom-flexible.

Let’s give the notion of an admissible Bihom-Poisson algebras.

Definition 3.14. Let $(A, \mu, \alpha, \beta)$ be a BiHom-algebra. Then $A$ is called an admissible Bihom-Poisson algebra if satisfies

$$\alpha(x, y, z) = \alpha(x, y, z) + \alpha(y, z, x) + \alpha(z, x, y) = 0 \quad (3.10)$$

$$\alpha(x, y, z) = \alpha(x, y, z) + \alpha(y, z, x) + \alpha(z, x, y) = 0 \quad (3.10)$$

Let’s give the notion of an admissible Bihom-Poisson algebras.

$$\alpha(x, y, z) = \alpha(x, y, z) + \alpha(y, z, x) + \alpha(z, x, y) = 0 \quad (3.10)$$

$$\alpha(x, y, z) = \alpha(x, y, z) + \alpha(y, z, x) + \alpha(z, x, y) = 0 \quad (3.10)$$

Let’s give the notion of an admissible Bihom-Poisson algebras.
It is observed that if $\alpha$ and $\beta$ are inversible, then (3.11) is equivalent to
\[
as_A(x, y, z) = \frac{1}{3} \left\{ \mu(\mu(x, \alpha^{-1} \beta(z)), \alpha(y)) - \mu(\mu(\alpha^{-2} \beta^2(z), \alpha^{-1}(x)), \alpha(y)) \\
+ \mu(\mu(\alpha^{-1} \beta(y), \alpha^{-1} \beta(z)), \alpha^2 \beta^{-1}(x)) - \mu(\mu(\alpha^{-1} \beta(y), \alpha^{-1} \beta(x)), \beta(z)) \right\}.
\]

Proposition 3.15. Every admissible BiHom-Poisson algebra $(A, \mu, \alpha, \beta)$ is BiHom-flexible, i.e.,
\[
as_A(\beta^2(x), \alpha \beta(y), \alpha^2(z)) + as_A(\beta^2(z), \alpha \beta(y), \alpha^2(x)) = 0
\]
for all $x, y, z \in A$.

Proof. The required identity follows immediately from the defining identity (3.11), in which the right-hand side is anti-symmetric in $x$ and $z$.
\[
as_A(\beta^2(x), \alpha \beta(y), \alpha^2(z)) + as_A(\beta^2(z), \alpha \beta(y), \alpha^2(x))
= \frac{1}{3} \left\{ \mu(\mu(\beta^2(x), \alpha \beta(z)), \beta \alpha^2(y)) - \mu(\mu(\beta^2(z), \alpha \beta(x)), \beta \alpha^2(y)) \\
+ \mu(\mu(\beta^2(y), \alpha \beta(z)), \beta \alpha^2(x)) - \mu(\mu(\beta^2(y), \alpha \beta(x)), \beta \alpha^2(z)) \\
+ \mu(\mu(\beta^2(z), \alpha \beta(x)), \beta \alpha^2(y)) - \mu(\mu(\beta^2(x), \alpha \beta(z)), \beta \alpha^2(y)) \\
+ \mu(\mu(\beta^2(y), \alpha \beta(x)), \beta \alpha^2(z)) - \mu(\mu(\beta^2(y), \alpha \beta(z)), \beta \alpha^2(x)) \right\} = 0.
\]

Next we observe that in an admissible BiHom-Poisson algebra the cyclic sum of the BiHom-associator is trivial.

Proposition 3.16. Let $(A, \mu, \alpha, \beta)$ be an admissible BiHom-Poisson algebra. Then
\[
S_A(x, y, z) := as_A(\beta^2(x), \alpha \beta(y), \alpha^2(z)) + as_A(\beta^2(y), \alpha \beta(z), \alpha^2(x)) + as_A(\beta^2(z), \alpha \beta(x), \alpha^2(y)) = 0
\]
for all $x, y, z \in A$.

Proof. Using the defining identity (3.11), we have:
\[
as_A(\beta^2(x), \alpha \beta(y), \alpha^2(z)) = \frac{1}{3} \left\{ \mu(\mu(\beta^2(x), \alpha \beta(z)), \beta \alpha^2(y)) - \mu(\mu(\beta^2(z), \alpha \beta(x)), \beta \alpha^2(y)) \\
+ \mu(\mu(\beta^2(y), \alpha \beta(z)), \beta \alpha^2(x)) - \mu(\mu(\beta^2(y), \alpha \beta(x)), \beta \alpha^2(z)) \\
+ \mu(\mu(\beta^2(z), \alpha \beta(x)), \beta \alpha^2(y)) - \mu(\mu(\beta^2(x), \alpha \beta(z)), \beta \alpha^2(y)) \\
+ \mu(\mu(\beta^2(y), \alpha \beta(x)), \beta \alpha^2(z)) - \mu(\mu(\beta^2(y), \alpha \beta(z)), \beta \alpha^2(x)) \right\}
= -as_A(\beta^2(z), \alpha \beta(x), \alpha^2(y)) + as_A(\beta^2(x), \alpha \beta(z), \alpha^2(y))
= -as_A(\beta^2(z), \alpha \beta(x), \alpha^2(y)) - as_A(\beta^2(y), \alpha \beta(z), \alpha^2(x)) (by (3.13)).
\]

Therefore, we conclude that $S_A = 0$.

4 Derivations of BiHom-Poisson algebras

In this section, we introduce and study derivations, generalized derivations and quasiderivations of BiHom-Poisson algebras.
Definition 4.1. Let \((A, \cdot, \cdot, \mu, \alpha, \beta)\) be a BiHom-Poisson algebra. A linear map \(D : A \to A\) is called an \((\alpha^k, \beta^l)\)-derivation of \(A\) if it satisfies

1. \(D \circ \alpha = \alpha \circ D, \; D \circ \beta = \beta \circ D;\)
2. \(D(\{x, y\}) = \{\alpha^k \beta^l (x), D(y)\} + \{D(x), \alpha^k \beta^l (y)\};\)
3. \(D(\mu(x, y)) = \mu(\alpha^k \beta^l (x), D(y)) + \mu(D(x), \alpha^k \beta^l (y)),\)

for all \(x, y \in A\).

We denote by \(\text{Der}(A) := \bigoplus_{k \geq 0} \bigoplus_{l \geq 0} \text{Der}_{(\alpha^k, \beta^l)}(A)\), where \(\text{Der}_{(\alpha^k, \beta^l)}(A)\) is the set of all \((\alpha^k, \beta^l)\)-derivations of \(A\). Obviously, \(\text{Der}(A)\) is a subalgebra of \(\text{End}(A)\).

Lemma 4.2. Let \((A, \cdot, \cdot, \mu, \alpha, \beta)\) be a BiHom-Poisson algebra. We define a subspace \(W\) of \(\text{End}(A)\) by \(W = \{w \in \text{End}(A) \mid w \circ \alpha = \alpha \circ w \text{ and } w \circ \beta = \beta \circ w\}\) and \(\sigma_1, \sigma_2 : W \to W\) linear maps satisfying \(\sigma_1(w) = \alpha \circ w\) and \(\sigma_2(w) = \beta \circ w\). Then a quadruple \((W, [\cdot, \cdot], \sigma_1, \sigma_2)\), where the multiplication \([\cdot, \cdot] : W \times W \to W\) is defined for \(w_1, w_2 \in W\) by

\[
[w_1, w_2] = w_1 \circ w_2 - w_2 \circ w_1,
\]

is a BiHom-Lie algebra.

Proof. For any \(w_1, w_2, w_3 \in W, \; k_1, k_2 \in \mathbb{K}\), we have

\[
[w_1, w_1] = w_1 \circ w_1 - w_1 \circ w_1 = 0,
\]

\[
[k_1 w_1 + k_2 w_2, w_3] = (k_1 w_1 + k_2 w_2)w_3 - w_3 (k_1 w_1 + k_2 w_2)
\]

\[
= k_1 (w_1 w_3 - w_3 w_1) + k_2 (w_2 w_3 - w_3 w_2) = k_1 [w_1, w_3] + k_2 [w_2, w_3],
\]

\[
[\sigma_2(w_1), \sigma_1(w_2)] = [\beta(w_1), \alpha(w_2)] = \alpha \beta(w_1 w_2 - w_2 w_1) = -\alpha \beta(w_2 w_1 - w_1 w_2)
\]

\[
= -[\sigma_2(w_2), \sigma_1(w_1)]
\]

\[
[\sigma_2^2(w_1), \sigma_2(w_2), \sigma_1(w_3)] + [\sigma_2^2(w_3), \sigma_2(w_1), \sigma_1(w_2)]
\]

\[
+ [\sigma_2^2(w_2), \sigma_2(w_3), \sigma_1(w_1)]
\]

\[
= \beta^3 \alpha w_1 w_2 w_3 - \beta^3 \alpha w_1 w_3 w_2 - \beta^3 \alpha w_2 w_3 w_1 + \beta^3 \alpha w_3 w_1 w_2 + \beta^3 \alpha w_1 w_3 w_2
\]

\[
+ \beta^3 \alpha w_2 w_1 w_3 - \beta^3 \alpha w_2 w_1 w_3 - \beta^3 \alpha w_1 w_3 w_2 + \beta^3 \alpha w_1 w_3 w_2
\]

\[
+ \beta^3 \alpha w_3 w_1 w_2 - \beta^3 \alpha w_3 w_1 w_2 - \beta^3 \alpha w_1 w_3 w_3 + \beta^3 \alpha w_1 w_3 w_3
\]

\[
= 0.
\]

Then \((W, [\cdot, \cdot], \sigma_1, \sigma_2)\) is a BiHom-Lie algebra. \(\Box\)

Theorem 4.3. Let \((A, \cdot, \cdot, \mu, \alpha, \beta)\) be a BiHom-Poisson algebra. For any \(D \in \text{Der}_{(\alpha^k, \beta^l)}(A)\) and \(D' \in \text{Der}_{(\alpha^k', \beta^l')} (A)\), define their commutator \([D, D']\) as usual:

\[
[D, D'] = D \circ D' - D' \circ D.
\]

Then \([D, D'] \in \text{Der}_{(\alpha^k+k', \beta^l+l')} (A)\).
Proof. It is sufficient to prove $[\text{Der}_{(\alpha^k,\beta^l)}(A), \text{Der}_{(\alpha^{k'},\beta^{l'})}(A)] \subseteq \text{Der}_{(\alpha^{k+k'},\beta^{l+l'})}(A)$. It is easy to check that $[D, D'] \circ \alpha = \alpha \circ [D, D']$ and $[D, D'] \circ \beta = \beta \circ [D, D']$.

For any $x, y \in A$, we have

$$
[D, D'](\{x, y\}) = D \circ D'\{x, y\} - D' \circ D\{x, y\}
$$

$$
= D\{D'(x), \alpha^{k'} \beta^{l'}(y)\} + \{\alpha^{k'} \beta^{l'}(x), D'(y)\}
$$

$$
- D'\{D'(x), \alpha^k \beta^l(y)\} + \{\alpha^k \beta^l(x), D(y)\}
$$

$$
= D\{\{D'(x), \alpha^{k'} \beta^{l'}(y)\}\} + D\{\{\alpha^{k'} \beta^{l'}(x), D'(y)\}\}
$$

$$
- D'\{\{\alpha^{k'} \beta^{l'}(x), D(y)\}\} + \{\alpha^{k+k'} \beta^{l+l'}(y), \beta\circ D'(y)\}
$$

$$
- \{\alpha^{k+k'} \beta^{l+l'}(x), \beta\circ D(x)\} - \{\alpha^k \beta^l \circ D(Y), \beta \circ [D', D']\}
$$

Similarly, we can check that

$$
[D, D'/(\mu(x, y))\mu([D, D'](x), \alpha^{k+k'} \beta^{l+l'}(y))] = \mu(\alpha^{k+k'} \beta^{l+l'}(x), [D, D'](y)).
$$

It follows that $[D, D'] \in \text{Der}_{(\alpha^{k+k'},\beta^{l+l'})}(A)$. \qed

Definition 4.4. Let $(A, \cdot, \cdot, \mu, \alpha, \beta)$ be a BiHom-Poisson algebra. $D \in \text{End}(A)$ is said to be a generalized $(\alpha^k, \beta^l)$-derivation of $A$, if there exists two endomorphisms $D', D'' \in \text{End}(A)$ such that

1. $D \circ \alpha = \alpha \circ D$; $D \circ \beta = \beta \circ D$
2. $D' \circ \alpha = \alpha \circ D'$; $D' \circ \beta = \beta \circ D'$
3. $D'' \circ \alpha = \alpha \circ D''$; $D'' \circ \beta = \beta \circ D''$
4. $\{D(x), \alpha^k \beta^l(y)\} \{\alpha^k \beta^l(x), D'(y)\} = D''\{x, y\}$
5. $\mu(D(x), \alpha^k \beta^l(y)) + \mu(\alpha^k \beta^l(x), D'(y)) = D''(\mu(x, y))$

for all $x, y \in A$.

The set of generalized $(\alpha^k, \beta^l)$-derivations of $A$ is $\text{GDer}_{(\alpha^k, \beta^l)}(A)$ and we denote

$$
\text{GDer}(A) := \bigoplus_{k \geq 0, l \geq 0} \text{GDer}_{(\alpha^k, \beta^l)}(A).
$$

Definition 4.5. Let $(A, \cdot, \cdot, \mu, \alpha, \beta)$ be a BiHom-Poisson algebra. $D \in \text{End}(A)$ is said to be an $(\alpha^k, \beta^l)$-quasiderivation of $A$, if there exists endomorphisms $D', D'' \in \text{End}(A)$ such that

1. $D \circ \alpha = \alpha \circ D$; $D \circ \beta = \beta \circ D$
2. $D' \circ \alpha = \alpha \circ D'$; $D' \circ \beta = \beta \circ D'$
3. $D'' \circ \alpha = \alpha \circ D''$; $D'' \circ \beta = \beta \circ D''$
4. $\{D(x), \alpha^k \beta^l(y)\} \{\alpha^k \beta^l(x), D'(y)\} = D'(\{x, y\})$.\]
5. $\mu(D(x), \alpha^k \beta^l(y)) + \mu(\alpha^k \beta^l(x), D(y)) = D''(\mu(x, y)),$

for all $x, y \in A.$

We then define

$$QDer(A) := \bigoplus_{k \geq 0} \bigoplus_{l \geq 0} QDer(\alpha^k, \beta^l)(A).$$

**Definition 4.6.** Let $(A, \{\cdot, \cdot\}, \mu, \alpha, \beta)$ be a BiHom-Poisson algebra. A linear map $D : A \to A$ is called an $(\alpha^k, \beta^l)$-centroid of $A$ if it satisfies

1. $D \circ \alpha = \alpha \circ D, \quad D \circ \beta = \beta \circ D;$
2. $\{D(x), \alpha^k \beta^l(y)\} = \{\alpha^k \beta^l(x), D(y)\} = D(\{x, y\});$
3. $\mu(D(x), \alpha^k \beta^l(y)) = \mu(\alpha^k \beta^l(x), D(y)) = D(\mu(x, y)), \quad \forall x, y \in A.$

We set

$$C(A) := \bigoplus_{k \geq 0} \bigoplus_{l \geq 0} C(\alpha^k, \beta^l)(A).$$

**Definition 4.7.** The $(\alpha^k, \beta^l)$-quasicentroid of a BiHom-Poisson algebra $(A, \{\cdot, \cdot\}, \mu, \alpha, \beta)$ denoted by $QC(\alpha^k, \beta^l)(A)$ is the set of linear maps $D$ such that

1. $D \circ \alpha = \alpha \circ D, \quad D \circ \beta = \beta \circ D;$
2. $\{D(x), \alpha^k \beta^l(y)\} = \{\alpha^k \beta^l(x), D(y)\};$
3. $\mu(D(x), \alpha^k \beta^l(y)) = \mu(\alpha^k \beta^l(x), D(y)) = D(\mu(x, y)), \quad \forall x, y \in A.$

We set

$$QC(A) := \bigoplus_{k \geq 0} \bigoplus_{l \geq 0} QC(\alpha^k, \beta^l)(A).$$

**Remark 4.8.** Let $(A, \{\cdot, \cdot\}, \mu, \alpha, \beta)$ be a BiHom-Poisson algebra. Then $C(A) \subseteq QC(A).$

**Definition 4.9.** A linear map $D$ is called an $(\alpha^k, \beta^l)$-central derivation of $A$ if it satisfies

1. $D \circ \alpha = \alpha \circ D, \quad D \circ \beta = \beta \circ D;$
2. $\{D(x), \alpha^k \beta^l(y)\} = D(\{x, y\}) = 0;$
3. $\mu(D(x), \alpha^k \beta^l(y)) = D(\mu(x, y)) = 0, \quad \forall x, y \in A.$

The set of $(\alpha^k, \beta^l)$-central derivations is denoted by $ZDer(\alpha^k, \beta^l)(A)$ and we set

$$ZDer(A) := \bigoplus_{k \geq 0} \bigoplus_{l \geq 0} ZDer(\alpha^k, \beta^l)(A).$$

**Remark 4.10.** Let $(A, \{\cdot, \cdot\}, \mu, \alpha, \beta)$ be a BiHom-Poisson algebra. Then

$$ZDer(A) \subseteq Der(A) \subseteq QDer(A) \subseteq GDer(A) \subseteq End(A).$$

**Proposition 4.11.** Let $(A, \{\cdot, \cdot\}, \mu, \alpha, \beta)$ be a BiHom-Poisson algebra. Then the following statements hold:
1. $GDer(A)$, $QDer(A)$ and $C(A)$ are BiHom-subalgebras of $(W, [\cdot, \cdot], \sigma_1, \sigma_2)$;

2. $ZDer(A)$ is a BiHom-ideal of $Der(A)$.

Proof. We only prove that $GDer(A)$ is a subalgebra of $W$. The case of $QDer(L)$ and $C(L)$ is similar.

1. Suppose that $D_1 \in GDer_{(\alpha^k, \beta^l)}(A)$, $D_2 \in GDer_{(\alpha^{k'}, \beta^{l'})}(A)$. Then for any $x, y \in A$,

$$\{\sigma_1(D_1)(x), \alpha^{k+1} \beta^l(y)\} = \{\alpha_1 \circ D_1(x), \alpha^{k+1} \beta^l(y)\} = \alpha(\{D_1(x), \alpha^k \beta^l(y)\})$$

$$= \alpha(D_1''(\{x, y\}) - \{\alpha^k \beta^l(x), D_1'(y)\}) = \sigma_1(D_1')(\{x, y\}) - \{\alpha^{k+1} \beta^l(x), \sigma_1(D_1')(y)\}.$$ 

Since $\sigma_1(D_1''), \sigma_1(D_1') \in \text{End}(A)$, we have $\sigma_1(D_1) \in GDer_{(\alpha^{k+1}, \beta^{l+1})}(A)$,

$$\{\sigma_2(D_1)(x), \alpha^k \beta^{l+1}(y)\} = \{\beta \circ D_1(x), \alpha^k \beta^{l+1}(y)\} = \beta(\{D_1(x), \alpha^k \beta^l(y)\})$$

$$= \beta(D_1''(\{x, y\}) - \{\alpha^k \beta^l(x), D_1'(y)\}) = \sigma_2(D_1')(\{x, y\}) - \{\alpha^k \beta^{l+1}(x), \sigma_2(D_1')(y)\}.$$ 

Since $\sigma_2(D_1''), \sigma_2(D_1') \in \text{End}(A)$, we have $\sigma_2(D_1) \in GDer_{(\alpha^k, \beta^{l+1})}(A)$.

$$\{(D_1, D_2)(x), \alpha^{k+k'} \beta^{l+l'}(y)\}$$

$$= \{D_1 \circ D_2(x), \alpha^{k+k'} \beta^{l+l'}(y)\} - \{D_2 \circ D_1(x), \alpha^{k+k'} \beta^{l+l'}(y)\}$$

$$= D_1''(\{D_2(x), \alpha^{k+k'} \beta^{l+l'}(y)\}) - \{\alpha^k \beta^l(D_2(x)), D_1'(\alpha^{k+k'} \beta^{l+l'}(y))\} - D_2''(\{D_1(x), \alpha^k \beta^l(y)\}) + \{\alpha^k \beta^{l+l'}(D_1(x)), D_2'(\alpha^k \beta^l(y))\}$$

$$= D_1''(\{x, y\}) - \{\alpha^k \beta^{l+l'}(x), D_1''(y)\}) - \{\alpha^k \beta^{l+l'}(D_1(x), D_1'(\alpha^{k+k'} \beta^{l+l'}(y))\}$$

$$= D_1''(\{x, y\}) - \{\{\alpha^k \beta^l(x), D_1'(y)\}) - \{\alpha^k \beta^{l+l'}(D_1(x), D_1'(\alpha^{k+k'} \beta^{l+l'}(y))\}$$

$$= D_1''(\{x, y\}) - \{\alpha^k \beta^{l+l'}(D_1(x), D_1'(\alpha^{k+k'} \beta^{l+l'}(y))\}$$

Since $D_1''(D_2''(\{x, y\}) - \{\alpha^k \beta^{l+l'}(x), D_1''(y)\}) - \{\alpha^k \beta^{l+l'}(D_1(x), D_1'(\alpha^{k+k'} \beta^{l+l'}(y))\}$$

$$= \{\alpha^k \beta^l(D_2(x)), D_1'(\alpha^{k+k'} \beta^{l+l'}(y))\} - D_2''(\{x, y\}) + \{\alpha^k \beta^{l+l'}(D_1(x), D_1'(\alpha^{k+k'} \beta^{l+l'}(y))\}$$

$$= \{\alpha^k \beta^{l+l'}(D_1(x), D_1'(\alpha^{k+k'} \beta^{l+l'}(y))\}$$

Similarly, we have $\mu(D_1, D_2) \in GDer_{(\alpha^{k+k'}, \beta^{l+l'})}(A)$.

Therefore, $GDer(A)$ is a BiHom-subalgebra of $(W, [\cdot, \cdot], \sigma_1, \sigma_2)$.

2. Suppose that $D_1 \in ZDer_{(\alpha^k, \beta^l)}(A)$, $D_2 \in Der_{(\alpha^{k'}, \beta^{l'})}(A)$. Then for any $x, y \in A$,

$$\sigma_1(D_1)(\mu(x, y)) = \alpha \circ D_1(\mu(x, y)) = 0.$$ 

$$\sigma_1(D_1)(\mu(x, y)) = \alpha \circ D_1(\mu(x, y)) = \alpha(\mu(D_1(x), \alpha^k \beta^l(y))) = \mu(\sigma_1(D_1)(x), \alpha^{k+1} \beta^l(y)).$$ 

Similarly, $\sigma_1(D_1)(\{x, y\}) = \{\sigma_1(D_1)(x), \alpha^{k+1} \beta^l(y)\}$. 

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Lemma 4.12. Let $(A, \{\cdot, \cdot\}, \mu, \alpha, \beta)$ be a BiHom-Poisson algebra, then the following statements hold:

1. $[\text{Der}(A), C(A)] \subseteq C(A)$.
2. $[\text{QDer}(A), QC(A)] \subseteq QC(A)$.
3. $[QC(A), QC(A)] \subseteq \text{QDer}(A)$.
4. $C(A) \subseteq \text{QDer}(A)$.
5. $\text{QDer}(A) + QC(A) \subseteq \text{GDer}(A)$.
6. $C(A) \circ \text{Der}(A) \subseteq \text{Der}(A)$.

Proof. 1. Suppose that $D_1 \in \text{Der}_{(\alpha^i, \beta^i)}(A)$, $D_2 \in C_{(\alpha^j, \beta^j)}(A)$. Then for any $x, y \in A$, we have

$$[D_1, D_2](\{x, y\}) = D_1 \circ D_2(\{x, y\}) - D_2 \circ D_1(\{x, y\})$$

$$= D_1(\{D_2(x), \alpha^k \beta^l(y)\}) - D_2(\{D_1(x), \alpha^k \beta^l(y)\}) + \{\alpha^k \beta^l(D_1(x)), D_1(y)\} - \{\alpha^k \beta^l(D_2(x)), D_2(y)\}$$

$$= \{D_1(D_2(x)), \alpha^k \beta^l(D_1(y))\} + \{\alpha^k \beta^l(D_2(x)), D_1(\alpha^k \beta^l(y))\} - \{D_2(D_1(x)), \alpha^k \beta^l(y)\}$$

Similarly, we have $[D_1, D_2](\{x, y\}) = \{\alpha^k \beta^l(dy), [D_1, D_2](y)\}$. Hence, we have $\sigma_1(D_1) \in Z\text{Der}_{(\alpha^i+1, \beta^i)}(A)$. In the same way, we have, $\sigma_2(D_1) \in Z\text{Der}_{(\alpha^i, \beta^i+1)}(A)$. Next, we have

$$[D_1, D_2](\mu(x, y)) = D_1 \circ D_2(\mu(x, y)) - D_2 \circ D_1(\mu(x, y))$$

$$= D_1(\mu(D_2(x), \alpha^k \beta^l(y))) + \mu(\alpha^k \beta^l(D_2(x), D_2(y))) - D_2(\mu(D_1(x), \alpha^k \beta^l(y)))$$

$$\mu(\alpha^k \beta^l(D_1(x)), D_2(\alpha^k \beta^l(y)))$$

$$= \mu([D_1, D_2](x), \alpha^k \beta^l(y)).$$

Similarly, we have $[D_1, D_2](\{x, y\}) = \{D_1, D_2\}(\{x, y\}) = 0$. Therefore, $Z\text{Der}(A)$ is a BiHom-ideal of $\text{Der}(A)$. 

□
In the same way, we have \([D_1, D_2](\mu(x, y)) = \mu([D_1, D_2](x), \alpha^{k+k'}\beta^{l+l'}(y))\)
\[= \mu(\alpha^{k+k'}\beta^{l+l'}(x), [D_1, D_2](y)).\]
Hence, \([D_1, D_2] \in C_{(\alpha^{k+k'}, \beta^{l+l'})}(A)\). Therefore \([D_1, D_2] \in C(A)\).

2. Suppose that \(D_1 \in QDer_{(\alpha^k, \beta^l)}(A)\), \(D_2 \in QC_{(\alpha^{k'}, \beta^{l'})}(A)\). Then for any \(x, y \in A\), we have

\[
\{[D_1, D_2](x), \alpha^{k+k'}\beta^{l+l'}(y)\} = \{D_1 \circ D_2(x), \alpha^{k+k'}\beta^{l+l'}(y)\} - \{D_2 \circ D_1(x), \alpha^{k+k'}\beta^{l+l'}(y)\} = D_1(D_2(x), \alpha^k\beta^l(y)) - D_2(D_1(x), \alpha^{k'}\beta^{l'}(y)) - \{\alpha^k\beta^l(D_1(x)), D_2(\alpha^{k'}\beta^{l'}(y))\} = D_2(\alpha^k\beta^l(x), \alpha^{k'}\beta^{l'}(D_1(y)) D_2(D_1(y)) - \{\alpha^{k'}\beta^{l'}(x), D_1(x)\} \in QC(A)\).

Similarly, \(\mu([D_1, D_2](x), \alpha^{k+k'}\beta^{l+l'}(y))) = \mu(\alpha^{k+k'}\beta^{l+l'}(x), [D_1, D_2](y))\).
Hence, we have \([D_1, D_2] \in QC_{(\alpha^{k+k'}, \beta^{l+l'})}(A)\). So \([QDer(A), QC(A)] \subseteq QC(A)\).

3. Suppose that \(D_1 \in QC_{(\alpha^k, \beta^l)}(A)\), \(D_2 \in QC_{(\alpha^{k'}, \beta^{l'})}(A)\). Then for any \(x, y \in A\), we have

\[
\{[D_1, D_2](x), \alpha^{k+k'}\beta^{l+l'}(y)\} = \{D_1 \circ D_2(x), \alpha^{k+k'}\beta^{l+l'}(y)\} - \{D_2 \circ D_1(x), \alpha^{k+k'}\beta^{l+l'}(y)\} = \{D_2(\alpha^k\beta^l(x)), \alpha^{k'}\beta^{l'}(D_1(y)) D_2(D_1(y)) - \{\alpha^{k'}\beta^{l'}(x), D_1(x)\} \in QC(A)\).

i.e., \([D_1, D_2](x), \alpha^{k+k'}\beta^{l+l'}(y)\) + \(\alpha^{k+k'}\beta^{l+l'}(x)) = 0\). Similarly, \(\mu([D_1, D_2](x), \alpha^{k+k'}\beta^{l+l'}(y)) + \mu(\alpha^{k+k'}\beta^{l+l'}(x), [D_1, D_2]) = 0\). Hence, we have \([D_1, D_2] \in QDer_{(\alpha^{k+k'}, \beta^{l+l'})}(A)\), which implies that \([QDer(A), QC(A)] \subseteq QC(A)\).

4. Suppose that \(D \in C_{(\alpha^k, \beta^l)}(A)\). Then for any \(x, y \in A\), we have

\[D(\{x, y\}) = \{D(x), \alpha^k\beta^l(y)\} = \{\alpha^k\beta^l(x), D(y)\} = 2D(\{x, y\}).\]
Hence, we have

\[\{D(x), \alpha^k\beta^l(y)\} + \{\alpha^k\beta^l(x), D(y)\} = 2D(\{x, y\}).\]
Similarly, $\mu(D(x), \alpha^k \beta^l(y)) + \mu(\alpha^k \beta^l(x), D(y)) = 2D(\mu(x, y))$, which implies that $D \in Q\text{Der}_{(\alpha^k, \beta^l)}(A)$. So $C(A) \subseteq Q\text{Der}(A)$.

5. In fact. Let $D_1 \in Q\text{Der}_{(\alpha^k, \beta^l)}(A), D_2 \in QC_{(\alpha^k, \beta^l)}(A)$. Then there exist $D_1', D_1'' \in \text{End}(A)$, for any $x, y \in A$, we have

$$\{D_1(x), \alpha^k \beta^l(y)\} + \{\alpha^k \beta^l(x), D_1(y)\} = D_1'\{\{x, y\}\},$$

$$\mu(D_1(x), \alpha^k \beta^l(y)) + \mu(\alpha^k \beta^l(x), D_1(y)) = D_1''\mu(x, y).$$

Thus, for any $x, y \in A$, we have

$$\{(D_1 + D_2)(x), \alpha^k \beta^l(y)\} = \{D_1(x), \alpha^k \beta^l(y)\} + \{D_2(x), \alpha^k \beta^l(y)\}$$

$$= D_1'\{\{x, y\}\} - \{\alpha^k \beta^l(x), D_1(y)\} + \{\alpha^k \beta^l(x), D_2(y)\}$$

and

$$\mu((D_1 + D_2)(x), \alpha^k \beta^l(y)) = \mu(D_1(x), \alpha^k \beta^l(y)) + \mu(D_2(x), \alpha^k \beta^l(y))$$

$$= D_1''\mu(x, y) - \mu(\alpha^k \beta^l(x), D_1(y)) + \mu(\alpha^k \beta^l(x), D_2(y))$$

$$= D_1''\mu(x, y) - \mu(\alpha^k \beta^l(x), (D_1 - D_2)(y)),$$

Therefore, $D_1 + D_2 \in G\text{Der}_{(\alpha^k, \beta^l)}(A)$.

6. Suppose that $D_1 \in C_{(\alpha^k, \beta^l)}(A), D_2 \in \text{Der}_{(\alpha^{k'}, \beta^{l'})}(A)$. Then for any $x, y \in A$, we have

$$D_1 \circ D_2(\{x, y\})$$

$$= D_1(\{D_2(x), \alpha^{k'} \beta^{l'}(y)\} + \{\alpha^{k'} \beta^{l'}(x), D_2(y)\})$$

$$= \{D_1(D_2(x)), \alpha^{k+k'} \beta^{l+l'}(y)\} + \{\alpha^{k+k'} \beta^{l+l'}(x), D_1(D_2(y))\},$$

and

$$\mu(D_1 \circ D_2(x), \alpha^{k'} \beta^{l'}(y)) + \mu(\alpha^{k'} \beta^{l'}(x), D_2(y))$$

$$= \mu(D_1(D_2(x)), \alpha^{k+k'} \beta^{l+l'}(y)) + \mu(\alpha^{k+k'} \beta^{l+l'}(x), D_1(D_2(y))),$$

which implies that $D_1 \circ D_2 \in \text{Der}_{(\alpha^{k+k'}, \beta^{l+l'})}(A)$. So $C(A) \circ \text{Der}(A) \subseteq \text{Der}(A)$.

\[\square\]

**Theorem 4.13.** Let $(A, \{., .\}, \mu, \alpha, \beta)$ be a BiHom-Poisson algebra, $\alpha$ and $\beta$ surjections, then $[C(A), QC(A)] \subseteq \text{End}(A, Z(A))$. Moreover, if $Z(A) = \{0\}$, then $[C(A), QC(A)] = \{0\}$.

**Proof.** For any $D_1 \in C_{(\alpha, \beta)}(A), D_2 \in QC_{(\alpha', \beta')} (A)$ and $x, y \in A$, since $\alpha$ and $\beta$ are surjections, there exist $y' \in A$ such that $y = \alpha^{k+k'} \beta^{l+l'}(y')$, we have

$$\{[D_1, D_2](x), y\}$$

$$= \{D_1(D_2(x), \alpha^{k+k'} \beta^{l+l'}(y'))\}$$

$$= \{D_1D_2(x), \alpha^{k+k'} \beta^{l+l'}(y')\} - \{D_2D_1(x), \alpha^{k+k'} \beta^{l+l'}(y')\}$$

$$= D_1(\{D_2(x), \alpha^{k'} \beta^{l'}(y')\}) - \{\alpha^{k'} \beta^{l'}D_1(x), D_2\alpha^{k'} \beta^{l'}(y')\}$$

$$= D_1(\{D_2(x), \alpha^{k'} \beta^{l'}(y')\}) - D_1(\{D_2(x), \alpha^{k'} \beta^{l'}(y')\})$$

$$= 0,$$
\[ \mu([D_1, D_2](x), y) = \mu(D_1 D_2(x), \alpha^{k+k'} \beta^{l+l'} (y')) = \mu(D_1 D_2(x), \alpha^{k+k'} \beta^{l+l'} (y')) - \mu(D_2 D_1(x), \alpha^{k+k'} \beta^{l+l'} (y')) = D_1(\mu(D_2(x), \alpha^k \beta^l (y')) - \mu(\alpha^k \beta^l D_1(x), D_2 \alpha^k \beta^l (y')) = D_1(\mu(D_2(x), \alpha^k \beta^l (y')) - \mu(D_2(x), \alpha^k \beta^l (y'))) = 0. \]

So \([D_1, D_2](x) \in Z(A)\) and therefore \([C(A), QC(A)] \subseteq \text{End}(A, Z(A))\). Moreover, if \(Z(A) = \{0\}\), then it is easy to see that \([C(A), QC(A)] = \{0\}\).

## 5 BiHom-Poisson modules

First, let recall the following.

**Definition 5.1.** Let \((A, \{\cdot, \cdot\}_A, \mu_A)\) be a Poisson algebra. Then a left Poisson module structure on a left \(A\)-module over \(A\) is linear maps \(\mu_M, \{\cdot, \cdot\}_M : A \otimes M \rightarrow M\) such that

\[
\mu_M(a, \mu_M(b, m)) = \mu_M(\mu_A(a, b), m) \tag{5.1}
\]

\[
\{a, b\}_A, m\}_M = \{a, b, m\} M - \{b, a, m\}_M M \tag{5.2}
\]

\[
\{a, \mu_M(b, m)\}_M = \mu_M(\{a, b\}_A, m) + \mu_M(b, \{a, m\}_M) \tag{5.3}
\]

\[
\{\mu_A(a, b), m\}_M = \mu_M(a, \{b, m\}_M) + \mu_M(b, \{a, m\}_M) \tag{5.4}
\]

for any \(a, b \in A\) and \(m \in M\).

**Remark 5.2.** In [27], Poisson algebras are defined without the associativity assumption and then, left Poisson modules are defined without the identity \([5, 7]\). In a similar way, basing on the definition above, one can defined a right Poisson module.

**Definition 5.3.** Let \((A, \{\cdot, \cdot\}, \mu, \alpha, \beta)\) be a BiHom-Poisson algebra.

1. A left BiHom-Poisson \(A\)-module is a BiHom-module \((V, \phi, \psi)\) with structure maps \(\lambda : A \otimes V \rightarrow V\) and \(\rho : A \otimes V \rightarrow V\) such that the following equalities hold:

\[
\lambda(\alpha(x), \lambda(y, v)) = \lambda(\mu(x, y), \psi(v)) \tag{5.5}
\]

\[
\rho(\{\alpha(x), y\}, \psi(v)) = \rho(\alpha \beta(x), \rho(y, v)) - \rho(\beta(y), \rho(\alpha(x), v)) \tag{5.6}
\]

\[
\rho(\alpha \beta(x), \lambda(y, v)) = \lambda(\{\beta(x), y\}, \psi(v)) + \lambda(\beta(y), \rho(\alpha(x), v)) \tag{5.7}
\]

\[
\rho(\mu(\beta(x), y), \psi(v)) = \lambda(\alpha \beta(x), \rho(y, v)) + \lambda(\beta(y), \rho(\alpha(x), v)) \tag{5.8}
\]

2. A right BiHom-Poisson \(A\)-module is a BiHom-module \((V, \phi, \psi)\) with structure maps \(\wedge : V \otimes A \rightarrow V\) and \(\delta : V \otimes A \rightarrow V\) such that the following equalities hold:

\[
\wedge(\wedge(\psi, x), \beta(y)) = \wedge(\phi(v), \mu(x, y)) \tag{5.9}
\]

\[
\delta(\phi(v), \{x, \alpha(y)\}) = \delta(\phi(v), \{x, \alpha(y)\}) + \wedge(\delta(v, \beta(x)), \alpha(y)) \tag{5.10}
\]

\[
\delta(\wedge(v, x), \alpha(\beta(y))) = \delta(\phi(v), \{x, \alpha(y)\}) + \wedge(\delta(v, \beta(x)), \alpha(y)) \tag{5.11}
\]

\[
\delta(\phi(v), \mu(x, \alpha(y))) = \wedge(\delta(v, x), \alpha(\beta(y))) + \wedge(\delta(v, \beta(x)), \alpha(y)) \tag{5.12}
\]
Remark 5.4. 1. A left BiHom-Poisson $A$-module is a BiHom-module $(V, \phi, \psi)$ with structure maps $\lambda : A \otimes V \rightarrow V$ and $\rho : A \otimes V \rightarrow V$ such that $(V, \phi, \psi, \lambda)$ is a left BiHom-associative $A$-module, $(V, \phi, \psi, \rho)$ is a left BiHom-Lie $A$-module [12] and $(5.7)$ and $(5.8)$ hold. Similarly a right BiHom-Poisson $A$-module is a BiHom-module $(V, \phi, \psi)$ with structure maps $\land : V \otimes A \rightarrow V$ and $\delta : V \otimes A \rightarrow V$ such that $(V, \phi, \psi, \lambda)$ is a right BiHom-associative $A$-module, $(V, \phi, \psi, \delta)$ is a right BiHom-Lie $A$-module and $(5.11)$ and $(5.12)$ hold.

2. If $\alpha = \beta = \text{Id}$ and $\phi = \psi = \text{Id}$, we recover a left (respectively a right) Poisson module. Thus if $(A, \{\cdot, \cdot\}, \mu)$ is a Poisson algebra and $V$ is a left Poisson $A$-module in the usual sense, then $(V, \text{Id}_V, \text{Id}_V)$ is a left BiHom-Poisson $A$-module where $A = (A, \{\cdot, \cdot\}, \mu, \text{Id}_A, \text{Id}_A)$ is a BiHom-Poisson algebra.

The following results allow to give some examples of left BiHom-Poisson $A$-modules.

Proposition 5.5. Let $(A, \{\cdot, \cdot\}, \mu, \alpha, \beta)$ be a regular BiHom-Poisson algebra. Then $(A, \alpha, \beta)$ is a left BiHom-Poisson $A$-module where the structure maps are $\lambda(a, b) = \mu(a, b)$ and $\rho(a, b) = \{a, b\}$. More generally, if $B$ is a left BiHom-ideal of $(A, \{\cdot, \cdot\}, \mu, \alpha, \beta)$, then $(B, \alpha, \beta)$ is a left BiHom-Poisson $A$-module where the structure maps are $\lambda(a, x) = \mu(a, x)$, $\rho(x, a) = \{x, a\}$ for all $x \in B$ and $(a, b) \in A^{\times 2}$.

Proof. The fact that $\lambda$ and $\rho$ are structure maps follows from the multiplicativity of $\alpha$ and $\beta$ with respect to $\mu$ and $\{\cdot, \cdot\}$. Next, observe that from the BiHom-commutativity of $\mu$ and the BiHom-skew-symmetry of $\{\cdot, \cdot\}$ that $\mu(x, y) = \mu(x, y) = \mu(x, y)$ and $\{x, y\} = \{x, y\}$ for all $x, y \in A$. Now, pick $(x, y, v) \in A^{\times 3}$ then, we have by the BiHom-associativity

$$\lambda(\alpha(x), \lambda(y, v)) = \mu(\alpha(x), \mu(y, v)) = \mu(x, \beta(y), v) = \lambda(\mu(x, y), \beta(v))$$

Next, compute $(5.6)$ using the BiHom-Jacobi identity in the third line, as follows

$$\rho(\\{\beta(x), y\}, \beta(v)) = \{\beta(x), y\}, \beta(v)\} = \{\alpha^{-1}\beta(\beta(v)), \alpha\beta^{-1}\{\alpha^{-1}\beta(y), \alpha\beta^{-1}(\beta(x))\}\}$$

$$= \{\beta(\alpha^{-1}(v)), \beta(\alpha^{-1}(y)), \alpha(\beta^{-1}(x))\}\}$$

$$= \{\alpha^{-1}\beta(y), \{\beta(\alpha^{-1}(x)), \alpha^{-1}(v)\} + \{\beta(\alpha^{-1}(x)), \alpha^{-1}(y)\}\}$$

$$= \{\alpha\beta(\beta^{-1}(x)), \alpha\beta^{-1}(\{x, y\})\} - \{\alpha\beta(\beta^{-1}(y)), \alpha\beta^{-1}(\{x, v\})\}$$

Similarly, using $(2.3)$, we compute

$$\rho(\alpha(x), \lambda(y, v)) = \{\alpha\beta(\beta^{-1}(x)), \alpha\beta^{-1}(\{x, v\})\} - \{\alpha\beta(\beta^{-1}(y)), \alpha\beta^{-1}(\{x, v\})\}$$

Finally, we obtain $(5.8)$ as follows

$$\rho(\mu(\beta(x), y), \beta(v)) = \{\mu(\beta(x), y), \beta(v)\} = \{\alpha^{-1}\beta(y), \alpha\beta^{-1}\mu(\beta(x), y)\}$$

$$= \{\alpha\beta(\alpha^{-1}\beta(y)), \alpha\beta^{-1}(\mu(\beta(\alpha^{-1}\beta(v)), \alpha(x)\}, \beta(\alpha^{-1}(\alpha(y))\}\}$$

$$= \{\alpha\beta(\alpha^{-1}\beta(y)), \alpha\beta^{-1}(\mu(\beta(\alpha^{-1}\beta(v)), \alpha(x)\}, \beta(\alpha^{-1}(\alpha(y))\}\}$$

$$= \mu(\beta(\alpha(x)), \{\alpha\beta(\alpha^{-1}(x)), \alpha\beta^{-1}(\{x, v\})\} + \mu(\alpha(x), \{y, v\})$$

$$= \mu(\beta(\alpha(x)), \{\alpha\beta(\alpha^{-1}(x)), \alpha\beta^{-1}(\{x, v\})\} + \mu(\alpha(x), \{y, v\})$$

$$= \mu(\beta(\alpha(x)), \{\alpha\beta(\alpha^{-1}(x)), \alpha\beta^{-1}(\{x, v\})\} + \mu(\alpha(x), \{y, v\})$$

$$= \mu(\beta(\alpha(x)), \{\alpha\beta(\alpha^{-1}(x)), \alpha\beta^{-1}(\{x, v\})\} + \mu(\alpha(x), \{y, v\})$$

$$= \mu(\beta(\alpha(x)), \{\alpha\beta(\alpha^{-1}(x)), \alpha\beta^{-1}(\{x, v\})\} + \mu(\alpha(x), \{y, v\})$$

$$= \mu(\beta(\alpha(x)), \{\alpha\beta(\alpha^{-1}(x)), \alpha\beta^{-1}(\{x, v\})\} + \mu(\alpha(x), \{y, v\})$$
Hence \((A, \alpha, \beta)\) is a left BiHom-Poisson \(A\)-module. Similarly, we prove that more generally, any two-sided BiHom-ideal \((B, \alpha, \beta)\) of \((A, \{\cdot, \cdot\}, \mu, \alpha, \beta)\) is a left BiHom-Poisson \(A\)-module. \(\square\)

**Remark 5.6.** The analogous of Proposition 5.5 can be proved for right BiHom-Poisson algebras.

More generally, we prove:

**Proposition 5.7.** If \(f : (A, \{\cdot, \cdot\}, \mu_A, \alpha_A, \beta_A) \to (B, \{\cdot, \cdot\}, \mu_B, \alpha_B, \beta_B)\) is a morphism of BiHom-Poisson algebras and \(\alpha_B\) and \(\beta_B\) are invertible then, \((B, \alpha_B, \beta_B)\) becomes a left BiHom-Poisson \(A\)-module via \(f\), i.e., the structure maps are defined as \(\lambda(a, b) = \mu_B(f(a), b)\) and \(\rho(a, b) = \{f(a), b\}_B\) for all \((a, b) \in A \times B\).

**Proof.** The fact that \(\lambda\) and \(\rho\) are structure maps follows from the multiplicativity of \(\alpha_B\) and \(\beta_B\) with respect to \(\mu_B\) and \(\{\cdot, \cdot\}_B\). Next, observe that from the BiHom-commutativity of \(\mu_B\) and the BiHom-skew-symmetry of \(\alpha_B\) and \(\beta_B\) that \(\mu_B(b_1, b_2) = \mu_B(\alpha_B^{-1}(b_2), \alpha_B^{-1}(b_1))\) and \(\{b_1, b_2\}_B = -\{\alpha_B^{-1}B_B(b_2), \alpha_B(\beta_B^{-1}(b_1))\}_B\) for all \(b_1, b_2 \in B\). Now, pick \((x, y) \in A \times 2\) and \(v \in B\) then since \(f\) is a morphism of BiHom-algebras, we have by the BiHom-associativity in \(B\)

\[
\begin{align*}
\lambda(\alpha_A(x), \lambda(y, v)) &= \mu_B(f(\alpha_A(x)), \mu_B(f(y), v)) = \mu_B(\alpha_B(f(x)), \mu_B(f(y), v)) \\
&= \mu_B(f(\mu_B(x, y), \beta_B(v))) = \lambda(\mu_B(x, y), \beta_B(v))
\end{align*}
\]

Next, compute (5.6) using the BiHom-Jacobi identity in the third line, as follows

\[
\begin{align*}
\rho(\{\beta_A(x), y\}_A, \beta_B(v)) &= \{f(\beta_A(x), y\}_A, \beta_B(v)\}_B = \{\beta_B(f(x)), f(y))\}_B, \beta_B(v)\}_B \\
&= \{\alpha_B^{-1}B_B(\beta_B(v)), \alpha_B^{-1}B_B(\beta_B(\alpha_B^{-1}B_B(f(x))))\}_B \\
&= \{\beta_B^2(\alpha_B^{-1}(v)), \{\beta_B(\beta_B^{-1}(f(y)), \alpha_B(\alpha_B^{-1}(f(x))))\}_B \\
&= -\{\beta_B(\beta_B^{-1}(f(y)), \{\beta_B(\alpha_B\beta_B^{-1}(f(x)), \alpha_B(\alpha_B^{-1}(f(x))))\}_B \\
&- \{\beta_B(\alpha_B\beta_B^{-1}(f(x)), \{\beta_B(\alpha_B^{-1}(v)), \alpha_B(\beta_B^{-1}(f(y)))\}_B \\
&- \{\beta_B(f(x)), \{\alpha_B(\alpha_B^{-1}(f(y)))), \beta_B(\alpha_B^{-1}(f(x))))\}_B \\
&= \{\beta_B(f(x)), \{\alpha_B(\alpha_B^{-1}(f(y)))), \beta_B(\alpha_B^{-1}(f(x))))\}_B \\
&\quad + \{\alpha_B(\alpha_B^{-1}(f(x))), \beta_B(\alpha_B^{-1}(f(y))))\}_B \\
&= \{\beta_B(f(x)), \{\alpha_B(\alpha_B^{-1}(f(y)))), \beta_B(\alpha_B^{-1}(f(x))))\}_B \\
&\quad + \{\alpha_B(\alpha_B^{-1}(f(x))), \beta_B(\alpha_B^{-1}(f(y))))\}_B \\
&= -\rho(\beta(y), \rho(\alpha(x), v)) + \rho(\alpha(x), v))
\end{align*}
\]

Similarly, using (2.3) for \(B\) and \(f\) is a morphism, we compute

\[
\begin{align*}
\rho(\alpha_A\beta_A(x), \lambda(y, v)) &= \{f(\alpha_A\beta_A(x)), \mu_B(f(y), v))\}_B = \{\alpha_B(\beta_B(f(x)), \mu_B(f(y), v))\}_B \\
&= \mu_B(\{\beta_B(f(x)), f(y))\}, \beta_B(v))+ \mu_B(\beta_B(\alpha_B(f(x))), \{\alpha_B(\alpha_B^{-1}(v)), \beta_B(\alpha_B^{-1}(f(x))))\}_B \\
&= \mu_B(f(\{\beta_A(x), y\}_A, \beta_B(v)) + \mu_B(f(\beta_A(y)), \{f(\alpha_B(x)), \beta_B(v)) \\
&= \lambda(\{\beta_A(x), y\}_A, \beta_B(v)) + \lambda(\beta_A(y), \rho(\alpha_A(x), v)) \text{ which is (5.7)}
\end{align*}
\]
Finally, using \( f \) is a morphism we obtain \( [5.8] \) as follows

\[
\begin{align*}
\rho(\mu_A(\beta_A(x), y), \beta_B(v)) &= \{ f \mu_A(\beta_A(x), y), \beta_B(v) \}_B \\
&= -\{ \alpha_B^{-1} \beta_B(\alpha_B f(y)), \beta_B(\alpha_B f(x)) \}_B \\
&= -\{ \beta_B(\alpha_B (\beta_B^{-1} f(y))), \beta_B(\alpha_B (\beta_B^{-1} f(x))) \}_B \\
&= -\{ \beta_B(\alpha_B (\beta_B^{-1} f(y))) \}_B - \{ \beta_B(\alpha_B (\beta_B^{-1} f(x))) \}_B \\
&= -\mu_B(\beta_B(\alpha_B f(y)), \beta_B(\alpha_B f(x))) + \mu_B(\alpha_B(\beta_B f(y)), \alpha_B(\beta_B f(x))) \\
&= -\mu_B(\beta_B(\alpha_B f(y)), \beta_B(\alpha_B f(x))) + \mu_B(\alpha_B(\beta_B f(y)), \alpha_B(\beta_B f(x))) \\
&= \lambda(\beta_A(y), \rho(\alpha_A(x), y)) + \lambda(\alpha_A(\beta_A(x)), \rho(y, v))
\end{align*}
\]

Hence \( (B, \alpha_B, \beta_B) \) is a left BiHom-Poisson \( A \)-module. \( \square \)

Similarly, we can prove:

**Proposition 5.8.** If \( f : (A, \cdot, \cdot)_A, \mu_A, \alpha_A, \beta_A) \rightarrow (B, \cdot, \cdot)_B, \mu_B, \alpha_B, \beta_B) \) is a morphism of BiHom-Poisson algebras and \( \alpha_B \) and \( \beta_B \) are invertible then, \( (B, \alpha_B, \beta_B) \) becomes a right BiHom-Poisson \( A \)-module via \( f \), i.e., the structure maps are defined as \( \lambda(b, a) = \mu_B(b, f(a)) \) and \( \rho(b, a) = \{ b, f(a) \}_B \) for all \( (a, b) \in A \times B \).

As the case of BiHom-alternative and BiHom-Jordan algebras \([7]\), in order to give another example of left BiHom-Poisson modules, let us consider the following:

**Definition 5.9.** An abelian extension of BiHom-Poisson algebras is a short exact sequence of BiHom-Poisson algebras

\[
0 \rightarrow (V, \alpha_V, \beta_V) \xrightarrow{i} (A, \cdot, \cdot)_A, \mu_A, \alpha_A, \beta_A) \xrightarrow{\pi} (B, \cdot, \cdot)_B, \mu_B, \alpha_B, \beta_B) \rightarrow 0
\]

where \( (V, \alpha_V, \beta_V) \) is a trivial BiHom-Poisson algebra, \( i \) and \( \pi \) are morphisms of BiHom-algebras. Furthermore, if there exists a morphism \( s : (B, \cdot, \cdot)_B, \mu_B, \alpha_B, \beta_B) \rightarrow (A, \cdot, \cdot)_A, \mu_A, \alpha_A, \beta_A) \) such that \( \pi \circ s = \text{id}_B \) then the abelian extension is said to be split and \( s \) is called a section of \( \pi \).

**Example 5.10.** Given an abelian extension as in the previous definition, the BiHom-module \( (V, \alpha_V, \beta_V) \) inherits a structure of a left BiHom-Poisson \( B \)-module and the actions of the BiHom-algebra \( (B, \cdot, \cdot)_B, \mu_B, \alpha_B, \beta_B) \) on \( V \) are as follows. For any \( x \in B \), there exist \( \tilde{x} \in A \) such that \( x = \pi(\tilde{x}) \). Let \( x \) acts on \( v \in V \) by \( \lambda(x, v) := \mu_A(\tilde{x}, i(v)) \) and \( \rho(x, v) := \{ \tilde{x}, i(v) \}_A \). These are well-defined, as another lift \( \tilde{x}' \) of \( x \) is written \( \tilde{x}' = \tilde{x} + v' \) for some \( v' \in V \) and thus \( \lambda(x, v) = \mu_A(\tilde{x}, i(v)) = \mu_A(\tilde{x}', i(v)) \) and \( \rho(x, v) = \{ \tilde{x}, i(v) \}_A = \{ \tilde{x}', i(v) \}_A \) because \( V \) is trivial. The actions property follow from the BiHom-Poisson identities. In case these actions of \( B \) on \( V \) are trivial, one speaks of a central extension.

The next result allow to construct a sequence of left BiHom-Poisson modules from a given one.
Proposition 5.11. Let $(A, \{\cdot, \cdot\}, \mu, \alpha, \beta)$ be a BiHom-Poisson algebra and $V, \phi, \psi$ be a left BiHom-Poisson $A$-module with the structure maps $\lambda$ and $\rho$. Then for each $n, m \in \mathbb{N}$, the maps

$$\lambda^{(n,m)} = \lambda \circ (\alpha^n \beta^m \otimes \text{Id}_V)$$

$$\rho^{(n,m)} = \rho \circ (\alpha^n \beta^m \otimes \text{Id}_V)$$

(5.13) (5.14)

give the BiHom-module $(V, \phi, \psi)$ the structure of a left BiHom-Poisson $A$-module that we denote by $V^{(n,m)}_{\phi,\psi}$.

Proof. Since the structure map $\lambda$ is a morphism of BiHom-modules, we get:

$$\phi \lambda^{(n,m)} = \phi \lambda^{(n+1,m)} \otimes \phi \text{ (by (5.15))}$$

$$= \lambda \circ (\alpha^{n+1} \beta^m \otimes \phi) = \lambda \circ (\alpha^n \beta^m \otimes \text{Id}_V) \circ (\alpha \otimes \phi) = \lambda^{(n,m)} \circ (\alpha \otimes \phi)$$

Similarly, we get that $\psi \lambda^{(n,m)} = \lambda^{(n,m)} \circ (\beta \otimes \psi)$, $\phi \rho^{(n,m)} = \rho^{(n,m)} \circ (\phi \otimes \alpha)$ and $\psi \rho^{(n,m)} = \rho^{(n,m)} \circ (\psi \otimes \beta)$. Thus $\rho^{(n)}$ and $\rho^{(n)}$ are morphisms of BiHom-modules. First, pick $(x, y) \in A^{\times 2}$ and $v \in V$, then using (5.5) in the second line for $V_{\phi,\psi}$, we get

$$\lambda^{(n,m)}(\alpha(x), \lambda^{(n,m)}(y, v)) = \lambda(\alpha^{n+1} \beta^m(x), \lambda(\alpha^n \beta^m(y), v))$$

$$\lambda = \lambda(\mu(\alpha^n \beta^m(x), \alpha^n \beta^m(y)), \psi(v)) = \lambda^{(n,m)}(\mu(x, y), \psi(v))$$

Secondly, we compute

$$\rho^{(n,m)}(\{\beta(x), y\}, \psi(v)) = \rho(\{\alpha^n \beta^{m+1}(x), \beta(y)\}, \psi(v))$$

$$= \rho(\alpha \beta \alpha^n \beta^m(x), \rho(\alpha^n \beta^m(y), v)) - \rho(\beta \alpha^n \beta^m(y), \rho(\alpha \alpha^n \beta^m(x), v)) \text{ (by (5.6) in } V_{\phi,\psi})$$

$$= \lambda^{(n,m)}(\alpha \beta(x), \lambda^{(n,m)}(y, v)) - \lambda^{(n,m)}(\beta(y), \lambda^{(n,m)}(\alpha(y), v))$$

Next, we obtain

$$\rho^{(n,m)}(\beta \alpha(x), \lambda^{(n,m)}(y, v)) = \rho(\alpha \beta \alpha^n \beta^m(x), \lambda(\alpha^n \beta^m(y), v))$$

$$= \lambda(\beta \alpha^n \beta^m(x), \alpha^n \beta^m(y)), \psi(v)) + \lambda(\beta \alpha^n \beta^m(y), \rho(\alpha \alpha^n \beta^m(x), v)) \text{ (by (5.7) in } V_{\phi,\psi})$$

$$= \lambda^{(n,m)}(\{\beta(x), y\}, \psi(v)) + \lambda^{(n,m)}(\beta(y), \rho^{(n,m)}(\alpha(x), v))$$

Finally, we compute

$$\rho^{(n,m)}(\mu(\beta(x), y), \psi(v)) = \rho(\mu(\alpha \beta \alpha^n \beta^m(x), \alpha^n \beta^m(y)), \psi(v))$$

$$= \lambda(\alpha \beta \alpha^n \beta^m(x), \rho(\alpha^n \beta^m(y), v)) + \lambda(\alpha \beta \alpha^n \beta^m(y), \rho(\alpha \alpha^n \beta^m(x), v)) \text{ (by (5.7) in } V_{\phi,\psi})$$

$$= \lambda^{(n,m)}(\alpha \beta(x), \rho^{(n,m)}(\alpha(x), v)) + \lambda^{(n,m)}(\beta(y), \rho^{(n,m)}(\alpha(x), v)).$$

Hence, $V^{(n,m)}_{\phi,\psi}$ is a left BiHom-Poisson $A$-module. \[\square\]

Proposition 5.11 reads for the case of right BiHom-Poisson module as:

Proposition 5.12. Let $(A, \{\cdot, \cdot\}, \mu, \alpha, \beta)$ be a BiHom-Poisson algebra and $V, \phi, \psi$ be a right BiHom-Poisson $A$-module with the structure maps $\land$ and $\delta$. Then for each $n, m \in \mathbb{N}$, the maps

$$\land^{(n,m)} = \land \circ (\text{Id}_V \otimes \alpha^n \beta^m)$$

$$\delta^{(n,m)} = \delta \circ (\text{Id}_V \otimes \alpha^n \beta^m)$$

(5.15) (5.16)

give the BiHom-module $(V, \phi, \psi)$ the structure of a right BiHom-Poisson $A$-module that we denote by $V^{(n,m)}_{\phi,\psi}$.
Theorem 5.13. Let \((A, \{\cdot, \cdot\}, \mu)\) be a Poisson algebra, \(V\) be a left Poisson \(A\)-module with the structure maps \(\lambda, \rho\) and \(\alpha, \beta\) be endomorphisms of the Jordan algebra \(A\) and \(\phi, \psi\) be linear self-maps of \(V\) such that \(\phi \circ \lambda = \lambda \circ (\alpha \otimes \phi)\), \(\phi \circ \rho = \rho \circ (\alpha \otimes \phi)\), \(\psi \circ \lambda = \lambda \circ (\beta \otimes \psi)\) and \(\psi \circ \rho = \rho \circ (\beta \otimes \psi)\). Write \(A_{\alpha, \beta}\) for the BiHom-Poisson algebra \((A, \{\cdot, \cdot\}_{\alpha, \beta} = \{\cdot, \cdot\}(\alpha \otimes \beta), \mu_{\alpha, \beta} = \mu(\alpha \otimes \beta), \alpha, \beta)\) and \(V_{\phi, \psi}\) for the BiHom-module \((V, \phi, \psi)\). Then the maps:

\[
\tilde{\lambda} = \lambda \circ (\alpha \otimes \psi) \quad \text{and} \quad \tilde{\rho} = \rho \circ (\alpha \otimes \psi)
\]

(5.17)

give the BiHom-module \(V_{\phi, \psi}\) the structure of a left BiHom-Poisson \(A_{\alpha, \beta}\)-module.

Proof. It is clear that \(\tilde{\lambda}\) and \(\tilde{\rho}\) are morphisms of BiHom-modules. Next, pick \((x, y) \in A^{\times 2}\) and \(v \in V\), then using (5.5) for \(V_{\phi, \psi}\) in the second line we get:

\[
\tilde{\lambda}(\alpha(x), \tilde{\lambda}(y, v)) = \lambda(\alpha^2 \beta(x), \lambda(\alpha^2 \beta(y), \psi^2(v))) = \lambda(\mu(\alpha^2 \beta(x), \alpha^2 \beta(y)), \psi^2(v)) = \tilde{\lambda}(\mu_{\alpha, \beta}(x, y), \psi(v))
\]

Secondly, we compute

\[
\tilde{\rho}(\{\beta(x), y\}_{\alpha, \beta}, \psi(v)) = \rho(\{\alpha^2 \beta(x), \alpha^2 \beta(y)\}, \psi^2(v)) = \rho(\alpha^2 \beta^2(x), \rho(\alpha^2 \beta(y), \psi^2(v))) - \rho(\alpha^2 \beta^2(y), \rho(\alpha^2 \beta^2(x), \psi^2(v))) \quad \text{(by (5.6) in } V_{\phi, \psi})
\]

Next, we obtain

\[
\tilde{\rho}(\beta \lambda(x), \tilde{\lambda}(y, v)) = \rho(\alpha^2 \beta(x), \lambda(\alpha^2 \beta(y), \psi^2(v))) = \lambda(\alpha^2 \beta(x), \alpha^2 \beta(y)), \psi^2(v)) + \lambda(\alpha^2 \beta(y), \rho(\alpha^2 \beta(x), \psi^2(v))) \quad \text{(by (5.7) in } V_{\phi, \psi})
\]

Finally, we compute

\[
\tilde{\rho}(\mu_{\alpha, \beta}(\beta(x), y), \psi(v)) = \rho(\mu(\alpha^2 \beta^2(x), \alpha^2 \beta^2(y)), \psi^2(v)) = \lambda(\alpha^2 \beta^2(x), \rho(\alpha^2 \beta(y), \psi^2(v))) + \lambda(\alpha^2 \beta^2(y), \rho(\alpha^2 \beta^2(x), \psi^2(v))) \quad \text{(by (5.8) in } V_{\phi, \psi})
\]

Hence, \(V_{\phi, \psi}^{(n, m)}\) is a left BiHom-Poisson \(A\)-module. \(\square\)

Corollary 5.14. Let \((A, \{\cdot, \cdot\}, \mu)\) be a Poisson algebra, \(V\) be a left Poisson \(A\)-module with the structure maps \(\lambda\) and \(\rho\), \(\alpha, \beta\) be endomorphisms of the Poisson algebra \(A\) and \(\phi, \psi\) be linear self-maps of \(V\) such that \(\phi \circ \lambda = \lambda \circ (\alpha \otimes \phi)\), \(\phi \circ \rho = \rho \circ (\alpha \otimes \phi)\), \(\psi \circ \lambda = \lambda \circ (\beta \otimes \psi)\) and \(\psi \circ \rho = \rho \circ (\beta \otimes \psi)\). Write \(A_{\alpha, \beta}\) for the BiHom-Poisson algebra \((A, \{\cdot, \cdot\}_{\alpha, \beta} = \{\cdot, \cdot\}(\alpha \otimes \beta), \mu_{\alpha, \beta} = \mu(\alpha \otimes \beta), \alpha, \beta)\) and \(V_{\phi, \psi}\) for the BiHom-module \((V, \phi, \psi)\). Then the maps:

\[
\tilde{\lambda}^{(n, m)} = \lambda \circ (\alpha^{n+1} \beta^{m+1} \otimes \psi) \quad \text{and} \quad \tilde{\rho}^{(n, m)} = \rho \circ (\alpha^{n+1} \beta^{m+1} \otimes \psi)
\]

(5.18)

give the BiHom-module \(V_{\phi, \psi}\) the structure of a left BiHom-Poisson \(A_{\alpha, \beta}\)-module for all \(n, m \in \mathbb{N}\).

Proof. The proof follows from Proposition 5.11 and Theorem 5.13. \(\square\)
Similarly, we can prove the following result which is the analogous of Theorem 5.15 for right BiHom-Poisson modules.

**Theorem 5.15.** Let $(A, \{\cdot, \cdot\}, \mu)$ be a Poisson algebra, $V$ be a right Poisson $A$-module with the structure maps $\lambda, \rho$ and $\alpha, \beta$ be endomorphisms of the Jordan algebra $A$ and $\phi, \psi$ be linear self-maps of $V$ such that $\phi \lambda = \lambda \phi (\alpha \otimes \phi), \phi \rho = \rho (\alpha \otimes \phi), \psi \lambda = \lambda (\beta \otimes \psi)$ and $\psi \rho = \rho (\beta \otimes \psi)$. Write $A_{\alpha, \beta}$ for the BiHom-Poisson algebra $(A, \{\cdot, \cdot\}, \alpha, \beta, \mu_{\alpha, \beta}, \mu_{\alpha, \beta} = \mu_{\alpha \otimes \beta}, \beta, \beta)$ and $V_{\phi, \psi}$ for the BiHom-module $(V, \phi, \psi)$. Then the maps:

$$
\tilde{\lambda} = \wedge \circ (\phi \otimes \alpha \beta) \text{ and } \tilde{\delta} = \rho \circ (\phi \otimes \alpha \beta)
$$

(5.19)

give the BiHom-module $V_{\phi, \psi}$ the structure of a right BiHom-Poisson $A_{\alpha, \beta}$-module.

In the case of Poisson algebras, we can form semidirect products when given a left (or a right) module. Similarly, we have

**Theorem 5.16.** Let $(A, \{\cdot, \cdot\}, \mu, \alpha, \beta)$ be a BiHom-Poisson algebra and $(V, \phi, \psi)$ be a left $A$-module with the structure maps $\lambda$ and $\rho$. Then $(A \oplus V, \{\cdot, \cdot\}, \alpha, \beta) = (A, \{\cdot, \cdot\}, \mu, \alpha, \beta)$ is a BiHom-Poisson algebra where $\ast, \{\cdot, \cdot\} : (A \oplus V)^{\otimes 2} \to A \oplus V$, $(a+u) \ast (b+v) := \mu(a, b)+\lambda(a, v)+\lambda(\alpha^{-1} \beta(b), \psi^{-1}(u))$, $(a+u, b+v) := \{a, b\} + \rho(a, v) - \rho(\alpha^{-1} \beta(b), \psi^{-1}(u))$ and $\alpha, \beta : A \oplus V \to A \oplus V, \alpha(a+u) := \alpha(a) + \phi(u)$ and $\beta(a+u) := \beta(a) + \psi(u)$ called the semidirect product of the BiHom-Poisson $(A, \mu, \{\cdot, \cdot\}, \alpha, \beta)$ and $(V, \phi, \psi)$.

**Proof.** Clearly, $\alpha$ and $\beta$ are multiplicative with respect to $\ast$ and $\{\cdot, \cdot\}$. Next

$$
\tilde{\beta}(a+u) \ast \tilde{\alpha}(b+v) = (\beta(a) + \psi(u)) \ast (\alpha(b) + \phi(v))
$$

$$
= \mu(\beta(a), \alpha(b)) + \lambda(\beta(a), \phi(v)) + \lambda(\beta(b), \phi(u))
$$

$$
= \mu(\beta(b), \alpha(a)) + \lambda(\beta(b), \phi(u)) + \lambda(\beta(a), \phi(v)) \text{ (by the BiHom-commutativity of } \mu\text{)}
$$

$$
= \beta(b+u) \ast \alpha(u+v). \tag{5.20}
$$

Next, pick $(a, b, c) \in A^{\times 2}$ and $(u, v, w) \in V^{\times 2}$, then

$$
((a+u) \ast (b+v)) \ast \tilde{\beta}(c+w) = \left(\mu(a, b) + \lambda(a, v) + \lambda(\alpha^{-1} \beta(b), \psi^{-1}(u))\right) \ast (\beta(c) + \psi(w))
$$

$$
= \mu(\mu(a, b), \beta(c)) + \lambda(\mu(a, b), \psi(w)) + \lambda(\alpha^{-1} \beta^2(c), \beta^{-1} \alpha(a)), \psi^{-1}(v))
$$

$$
+ \lambda(\alpha^{-1} \beta^2(c), \lambda(b, \psi^{-1}(u))) \text{ (using } \lambda \text{ is a morphism)}
$$

$$
= \mu(\mu(a, b), \beta(c)) + \lambda(\mu(a, b), \psi(w)) + \lambda(\mu(a, \beta \alpha^{-1}(c)), \phi(v))
$$

$$
+ \lambda(\alpha^{-1} \beta(b), \lambda(b, \psi^{-1}(u))) \text{ (using BiHom-associativity)}
$$

$$
+ \lambda(\alpha^{-1} \beta(b), \alpha^{-1} \beta(c), \psi^{-1}(u)) \text{ (using BiHom-commutativity)} \tag{5.21}
$$

Similarly, we prove that

$$
\tilde{\alpha}(a+u) \ast \tilde{\beta}(b+v) \ast (c+w) = \mu(\alpha(a), \mu(b, c)) + \lambda(\mu(a, b), \psi(w)) + \lambda(\mu(a, \beta \alpha^{-1}(c)), \phi(v))
$$

$$
+ \lambda(\alpha^{-1} \beta(b), \alpha^{-1} \beta(c), \psi^{-1}(u))
$$

Hence $(A \oplus V, \ast, \tilde{\alpha}, \tilde{\beta})$ is a BiHom-commutative BiHom-associative algebra. Now, observe that $(A \oplus V, \{\cdot, \cdot\}, \tilde{\alpha}, \tilde{\beta})$ is a BiHom-Lie algebra (Proposition 4.9, [15]). Finally, let $(a, b, c) \in A^{\times 2}$.
and \((u, v, w) \in V^{\times 2}\). Then

\[
[\bar{\alpha} \beta(a + u), (b + v) \ast (c + w)] = [\alpha\beta(a) + \varphi(u), \mu(b, c) + \lambda(b, w) + \lambda(\alpha^{-1}\beta(), \psi^{-1}\phi(v))
\]

\[
= \{\alpha\beta(a), \mu(b, c)\} + \rho(\alpha\beta(u), \lambda(b, w)) + \rho(\alpha\beta(a), \lambda(\alpha^{-1}\beta(c), \psi^{-1}\phi(v)))
\]

\[
-\rho(\mu(\alpha\beta^{-1}(b), \beta\alpha^{-1}(c)), \psi^{-1}\phi^2(u)) = \mu(\{\beta(a), b\}, \beta(c))
\]

\[
+\mu(\beta(c), \{\alpha(a), c\}) + \lambda(\{\beta(a), b\}), \psi(w)) + \lambda(\beta(b), \rho(\alpha(a), w)) + \lambda(\{\beta(a), \alpha^{-1}\beta(c)\}, \phi(v))
\]

\[
+\lambda(\beta^2\alpha^{-1}(c), \rho(\alpha(a), \psi^{-1}\phi(v))) - \lambda(\beta(b), \rho(\beta\alpha^{-1}(c), \psi^{-1}\phi^2(u)))
\]

\[
-\lambda(\beta^2\alpha^{-1}(c), \rho(b, \psi^{-1}\phi^2(u))) \text{ (by } (2.3), (5.7), (5.8)\text{)}
\]

\[
= (\mu(\{\beta(a), b\}, \beta(c)) + \lambda(\{\beta(a), b\}), \psi(w)) + \lambda(\beta^2\alpha^{-1}(c), \rho(\alpha(a), \psi^{-1}\phi(v)))
\]

\[
-\lambda(\beta^2\alpha^{-1}(c), \rho(b, \psi^{-1}\phi^2(u))) + (\mu(\beta(c), \{\alpha(a), c\}) + \lambda(\beta(b), \rho(\alpha(a), w))
\]

\[
-\lambda(\beta(b), \rho(\beta\alpha^{-1}(c), \psi^{-1}\phi^2(u))) + \lambda(\{\beta(a), \alpha^{-1}\beta(c)\}, \phi(v))
\text{ (rearranging terms )}
\]

\[
= [\bar{\alpha}(a + u), (b + v)] \ast \beta(c + w) + \beta(b + v) \ast [\bar{\alpha}(a + u), (c + w)]
\]

Hence, the conclusion follows. \(\square\)

**Remark 5.17.** Consider the split null extension \(A \oplus V\) determined by the left BiHom-Poisson module \((V, \phi, \psi)\) for the BiHom-Poisson algebra \((A, \{\cdot, \cdot\}, \mu, \alpha, \beta)\) in the previous theorem. Write elements \(a + v\) of \(A \oplus V\) as \((a, v)\). Then there is an injective homomorphism of BiHom-modules \(i : V \to A \oplus V \text{ given by } i(v) = (0, v)\) and a surjective homomorphism of BiHom-modules \(\pi : A \oplus V \to A \text{ given by } \pi(a, v) = a\). Moreover, \(i(V)\) is a two-sided BiHom-ideal of \(A \oplus V\) such that \(A \oplus V/i(V) \cong A\). On the other hand, there is a morphism of BiHom-algebras \(\sigma : A \to A \oplus V \text{ given by } \sigma(a) = (a, 0)\) which is clearly a section of \(\pi\). Hence, we obtain the abelian split exact sequence of BiHom-Poisson algebras and \((V, \phi, \psi)\) is a left BiHom-Poisson module for \(A\) via \(\pi\).

**Definition 5.18.** Let \((A, \{\cdot, \cdot\}, \mu, \alpha, \beta)\) be a BiHom Poisson algebra. A skew-symmetric \(n\)-linear map \(f : A \times \cdots \times A \to A\) that is a derivation in each argument is called an \(n\)-BiHom-cochain, if it satisfies

\[
f(\alpha(x_1), \cdots, \alpha(x_n)) = \alpha \circ f(x_1, \cdots, x_n),
\]

\[
f(\beta(x_1), \cdots, \beta(x_n)) = \beta \circ f(x_1, \cdots, x_n).
\]

The set of \(n\)-Hom-cochains is denoted by \(C^n_{\alpha, \beta}(A, A)\), for \(n \geq 1\).

**Definition 5.19.** Let \((A, \{\cdot, \cdot\}, \mu, \alpha, \beta)\) be a regular BiHom Poisson algebra. For \(n = 1, 2\), the coboundary operator \(\delta^n : C^n_{\alpha, \beta}(A, A) \to C^{n+1}_{\alpha, \beta}(A, A)\) is defined as follows:

\[
\delta^1 f(x, y) = \{\alpha(x), f(y)\} - \{f(x), \alpha(y)\} - f(\{\alpha^{-1}\beta(x), y\})
\]

\[
\delta^2 f(x, y, z) = \{\alpha\beta(x), f(y, z)\} - \{\alpha\beta(y), f(x, z)\} + \{\alpha\beta(y), f(x, z)\}
\]

\[
- f(\{\alpha^{-1}\beta(x), y\}, \beta(z)) + f(\{\alpha^{-1}\beta(x), z\}, \beta(y)) - f(\{\alpha^{-1}\beta(y), z\}, \beta(x))
\]

**Lemma 5.20.** The coboundary operators \(\delta^i\) are well defined, for \(i = 1, 2\).

**Proof.** For any \(x, y, z \in A\) we have:

\[
\delta^1 f(\alpha(x), \alpha(y)) = \{\alpha^2(x), f\alpha(y)\} - \{f\alpha(x), \alpha^2(y)\} - f(\{\alpha^{-1}\beta(x), \alpha(y)\})
\]

\[
= \{\alpha^2(x), \alpha f(y)\} - \{\alpha f(x), \alpha^2(y)\} - f(\{\alpha\alpha^{-1}\beta(x), \alpha(y)\}) = \alpha \circ \delta^1 f(x, y)
\]
Theorem 5.21. With notations as above, we have
\[ \delta^2 \circ \delta^1 = 0. \]

Proof. Let \( f \in C^1_{\alpha, \beta}(A, A) \) and \( (x, y, z) \in A^3 \) then, we have:

\[
\begin{align*}
\delta^2 \circ \delta^1 f(x, y, z) &= \{ \alpha \beta(x), \delta^1 f(x, y, z) \} - \{ \alpha \beta(y), \delta^1 f(x, z) \} + \{ \alpha \beta(y), \delta^1 f(x, z) \} \\
- \delta^1 f(\{ \alpha^{-1} \beta(x), y \}, \beta(z)) &+ \delta^1 f(\{ \alpha^{-1} \beta(x), z \}, \beta(y)) - \delta^1 f(\{ \alpha^{-1} \beta(y), z \}, \beta(x)) \\
= &\{ \alpha \beta(x), \{ \alpha(y), f(z) \} \} - \{ \alpha \beta(x), \{ \alpha(z), f(y) \} \} - \{ \alpha \beta(x), f(\{ \alpha^{-1} \beta(y), z \}) \} \\
- \{ \alpha \beta(y), \{ \alpha(x), f(z) \} \} + \{ \alpha \beta(y), \{ \alpha(z), f(x) \} \} + \{ \alpha \beta(y), f(\{ \alpha^{-1} \beta(x), z \}) \} \\
+ \{ \alpha \beta(z), \{ \alpha(x), f(y) \} \} - \{ \alpha \beta(z), \{ \alpha(y), f(x) \} \} - \{ \alpha \beta(z), f(\{ \alpha^{-1} \beta(x), y \}) \} \\
- \{ \{ \beta(x), \alpha(y) \}, f(\{ \alpha^{-1} \beta(x), y \}) \} + \{ \{ \beta(x), \alpha(z) \}, f(\{ \alpha^{-1} \beta(x), y \}) \} + f(\{ \{ \alpha^{-2} \beta^2(x), \alpha^{-1} \beta(y) \}, \beta(z) \}) \\
+ \{ \{ \beta(x), \alpha(z) \}, f(\{ \alpha^{-1} \beta(x), z \}) \} - f(\{ \{ \alpha^{-2} \beta^2(x), \alpha^{-1} \beta(z) \}, \beta(y) \}) \\
- \{ \{ \beta(y), \alpha(z) \}, f(\{ \alpha^{-1} \beta(x), z \}) \} + f(\{ \{ \alpha^{-2} \beta^2(y), \alpha^{-1} \beta(z) \}, \beta(x) \}) \\
= &\{ \beta^2(\alpha^{-1}(x)), \{ \beta(\alpha^{-1}(y)), f(\{ \alpha^{-1} \beta(z) \}) \} \} + \{ \beta^2(\alpha^{-1}(x)), \{ \beta(\alpha^{-1}(y)), f(\{ \alpha^{-1} \beta(z) \}) \} \} \\
- \{ \{ \alpha \beta(x), f(\{ \alpha^{-1} \beta(y), z \}) \} \} - \{ \beta^2(\alpha^{-1}(x)), f(\{ \alpha^{-1} \beta(z) \}) \} \} + \{ \beta^2(\alpha^{-1}(x)), f(\{ \alpha^{-1} \beta(z) \}) \} \} \\
+ \{ \beta^2(\alpha^{-1}(x)), \{ \beta(\alpha^{-1}(z)), f(\{ \alpha^{-1} \beta(x) \}) \} \} + \{ \beta^2(\alpha^{-1}(z)), f(\{ \alpha^{-1} \beta(x) \}) \} \} \\
+ \{ \alpha \beta(z), f(\{ \alpha^{-1} \beta(x), y \}) \} - f(\{ \beta^2(\alpha^{-1}(z)), f(\{ \alpha^{-1} \beta(x) \}) \} \} \\
+ \{ \{ \beta^2(\alpha^{-1}(y)), \{ \beta(\alpha^{-1}(z)), f(\{ \alpha^{-1} \beta(x) \}) \} \} \\
- \{ \{ \delta^2(\alpha^{-1}(y)), \{ \beta(\alpha^{-1}(z)), f(\{ \alpha^{-1} \beta(x) \}) \} \} \} + \{ \beta^2(\alpha^{-1}(x)), f(\{ \alpha^{-1} \beta(z) \}) \} \} \\
+ \{ \alpha \beta(x), f(\{ \alpha^{-1} \beta(y), z \}) \} - f(\{ \beta^2(\alpha^{-1}(z)), f(\{ \alpha^{-1} \beta(x) \}) \} \} \\
( \text{since } \{ u, v \} = -\{ \alpha \beta^{-1}(v), \alpha \beta^{-1}(u) \} \forall u, v \in A) \\
= &0 \text{ (by the BiHom-Jaobi identity).}
\]

For \( n = 1, 2 \), the map \( f \in C^n_{\alpha, \beta}(A, A) \) is called an \( n \)-BiHom-cocycle \( \delta^n f = 0 \). We denote the subspace spanned by \( n \)-BiHom-cocycles by \( Z^n_{\alpha, \beta}(A, A) \) and \( B^n_{\alpha, \beta}(A, A) = \delta^{n-1} C^{n-1}_{\alpha, \beta}(A, A) \). Since \( \delta^2 \circ \delta^1 = 0 \), \( B^2_{\alpha, \beta}(A, A) \) is a subspace of \( Z^2_{\alpha, \beta}(A, A) \). Hence we can define a cohomology space \( H^2_{\alpha, \beta}(A, A) \) of as the factor space \( Z^2_{\alpha, \beta}(A, A)/B^2_{\alpha, \beta}(A, A) \).
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