Skew spectra of oriented bipartite graphs

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Abstract
A graph $G$ is said to have a parity-linked orientation $\phi$ if every even cycle $C_{2k}$ in
$G^\phi$ is evenly (resp. oddly) oriented whenever $k$ is even (resp. odd). In this paper,
this concept is used to provide an affirmative answer to the following conjecture of
D. Cui and Y. Hou [D. Cui, Y. Hou, On the skew spectra of Cartesian products
of graphs, The Electronic J. Combin. 20(2):#P19, 2013]: Let $G = G(X,Y)$ be
a bipartite graph. Call the $X \rightarrow Y$ orientation of $G$, the canonical orientation.
Let $\phi$ be any orientation of $G$ and let $Sp_S(G^\phi)$ and $Sp(G)$ denote respectively
the skew spectrum of $G^\phi$ and the spectrum of $G$. Then $Sp_S(G^\phi) = iSp(G)$ if and only
if $\phi$ is switching-equivalent to the canonical orientation of $G$. Using this result,
we determine the switch for a special family of oriented hypercubes $Q_d^\phi$, $d \geq 1$.
Moreover, we give an orientation of the Cartesian product of a bipartite graph and
a graph, and then determine the skew spectrum of the resulting oriented product
graph, which generalizes a result of Cui and Hou. Further this can be used to
construct new families of oriented graphs with maximum skew energy.

Keywords: oriented bipartite graphs; skew energy; skew spectrum; canonical ori-
tentation; parity-linked orientation; switching-equivalence

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1 Introduction

Let $G = (V, E)$ be a finite simple undirected graph of order $n$ with $V = \{v_1, v_2, \ldots, v_n\}$ as its vertex set and $E$ as its edge set. An orientation $\phi$ of $E$ results in the oriented graph $G^\phi = (V, \Gamma)$, where $\Gamma$ is the arc set of $G^\phi$. The adjacency matrix of $G$ is the $n \times n$ matrix $A = (a_{ij})$, where $a_{ij} = 1$ if $(v_i, v_j) \in E$ and $a_{ij} = 0$ otherwise. As the matrix $A$ is real and symmetric, all its eigenvalues are real. The spectrum of $G$, denoted by $Sp(G)$, is the spectrum of $A$. The energy $\mathcal{E}(G)$ of a graph $G$ of order $n$, introduced by Ivan Gutman [8] in 1978, is defined as the sum of the absolute values of its eigenvalues. The skew adjacency matrix of the oriented graph $G^\phi$ is the $n \times n$ matrix $S(G^\phi) = (s_{ij})$, where $s_{ij} = 1 = -s_{ji}$ whenever $(v_i, v_j) \in \Gamma(G^\phi)$ and $s_{ij} = 0$ otherwise. As the matrix $S(G^\phi)$ is real and skew symmetric, its eigenvalues are all pure imaginary. The skew spectrum of $G^\phi$ is the spectrum of $S(G^\phi)$. The concept of graph energy was recently generalized to oriented graphs as skew energy by Adiga, Balakrishnan and Wasin So in [1]. The skew energy $\mathcal{E}_s(G^\phi)$ of an oriented graph $G^\phi$ is defined as the sum of the absolute values of all the eigenvalues of $S(G^\phi)$. For the properties of the energy and spectrum of a graph, the reader may refer to [3, 9, 12], and for skew energy and skew spectrum of an oriented graph, to [1, 4, 10, 11, 13]. We follow [3] for standard graph theoretic notation.

By a cycle in $G^\phi$, we refer to not necessarily a directed cycle. An oriented even cycle is classified into two types based on its structure. An even cycle is defined to be a parity-linked orientation if it is evenly oriented whenever $k$ is even and oddly oriented whenever $k$ is odd. If every even cycle in $G^\phi$ is a parity-linked orientation, the orientation $\phi$ is defined to be a parity-linked orientation of $G$. (The parity-linked orientation is termed as uniform orientation in [6].)

Let $G = (X, Y)$ be a bipartite graph with bipartition $(X, Y)$. The canonical orientation of $G$ is that orientation which orients all the edges from one partite set to the other. It is immaterial if it is from $X$ to $Y$ or from $Y$ to $X$. Shader and So [13] have shown that for the canonical orientation $\sigma$ of $G(X, Y)$,

$$Sp_S(G^\sigma) = iSp(G).$$  \hspace{1cm} (1)

From this point onward, $\sigma$ stands for the canonical orientation with respect to a bipartite graph $G$ with a fixed bipartition $(X, Y)$.

Let $G^\phi$ be an oriented graph of order $n$. An even cycle $C_{2k}$ of length $2k$ in $G^\phi$ is said have a parity-linked orientation if it is evenly oriented whenever $k$ is even and oddly oriented whenever $k$ is odd. If every even cycle in $G^\phi$ has a parity-linked orientation, then the orientation $\phi$ is defined to be a parity-linked orientation of $G$. (The parity-linked orientation is termed as uniform orientation in [6].)
In [6], Cui and Hou have given a characterization of oriented graphs $G^\phi$ that satisfy Equation (1) by using the parity-linked orientation of graphs.

**Theorem 2 ([6]).** Suppose $G^\phi$ is an oriented bipartite graph with $G$ as its underlying graph. Then $Sp_S(G^\phi) = iSp(G)$ if and only if the orientation $\phi$ of $G$ is parity-linked.

Let $U$ be any proper subset of $V(G)$ of an oriented graph $G^\phi_1$ and let $\overline{U} = V(G) \setminus U$ be its complement. Reversing the orientations of all the arcs between $U$ and $\overline{U}$ results in another oriented graph $G^\phi_2$. This process is called the *switch* of $G^\phi_1$ with respect to $U$. The oriented graph got by two successive switches with respect to $U_1$ and $U_2$ is just the oriented graph obtained from $G$ by the switch with respect to the set $U_1 \Delta U_2$, the symmetric difference of $U_1$ and $U_2$.

Suppose $\phi_1$ and $\phi_2$ are two orientations of a graph $G$. Then $G^\phi_1$ and $G^\phi_2$ are said to be *switching-equivalent* if $G^\phi_2$ can be obtained from $G^\phi_1$ by a switch. It is clear that switching-equivalence among the set $\mathcal{G}$ of all orientations of a graph $G$ is indeed an equivalence relation on $\mathcal{G}$. The following result is proved in [1].

**Theorem 3 ([1]).** Let $\phi_1$ and $\phi_2$ be two orientations of a graph $G$. If $G^\phi_1$ and $G^\phi_2$ are switching-equivalent, then $Sp_S(G^\phi_1) = Sp_S(G^\phi_2)$.

We mention that the converse of Theorem 3 is not true for non-bipartite graphs.

**Example 4.** Consider the two orientations $\phi_1$ and $\phi_2$ of the cycle graph $C_5$ as given in Figure 1.

![Figure 1: Two orientations of the cycle graph $C_5$](image)

Since $C_5 \in \mathcal{G}$, the family of graphs without even cycles, by Theorem 1,

$$Sp_S(C_5^{\phi_1}) = Sp_S(C_5^{\phi_2}).$$

The oriented cycle $C_5^{\phi_1}$ has 5 arcs in one direction (clockwise) while $C_5^{\phi_2}$ has 4 arcs in the same direction for the given labeling. Any switch in $C_5^{\phi_1}$ will cause an even number of changes in the number of arcs in both the directions. Hence the 5 arcs in the clockwise direction can only become either 3 arcs or 1 arc in the clockwise direction after any switch but never 4 arcs in the clockwise direction. Therefore $\phi_1$ and $\phi_2$ are not switching-equivalent in $C_5$. 
In [6], Cui and Hou conjectured that for an oriented bipartite graph \( G^\phi \), \( Sp_S(G^\phi) = iSp(G) \) if and only if \( G^\phi \) is switching-equivalent to \( G^\sigma \), where \( \sigma \) is the canonical orientation of \( G \). In this paper, we settle the above conjecture in the affirmative and present it as the following theorem.

**Theorem 5** (Conjectured in [6]). Suppose \( \phi \) is an orientation of a bipartite graph \( G = G(X,Y) \). Then \( Sp_S(G^\phi) = iSp(G) \) if and only if \( G^\phi \) is switching-equivalent to \( G^\sigma \), where \( \sigma \) is the canonical orientation of \( G \).

**Proof.** Without loss of generality, we may assume that \( G \) is a connected graph.

**Sufficiency.** If \( G^\phi \) and \( G^\sigma \) are switching-equivalent, then by Theorem 3, \( Sp_S(G^\phi) = iSp(G) \) (where the second equality follows from (1)).

**Necessity.** We prove by induction on the number of edges \( m \) of the bipartite graph \( G \). The result is trivial for \( m = 1 \).

Assume that the result is true for all bipartite graphs with at most \( m - 1 \) (\( m \geq 2 \)) arcs. Let \( G \) be a bipartite graph with \( m \) edges and \( (X,Y) \) be the bipartition of the vertex set of \( G \). Suppose that \( \phi \) is an orientation of \( G \) such that \( Sp_S(G^\phi) = iSp(G) \). We have to prove that \( \phi \) is switching-equivalent to \( \sigma \). Let \( e \) be any edge of \( G \). By Theorem 2, \( \phi \) is a parity-linked orientation of \( G^\phi \) and hence of \( (G - e)^\phi \). Consequently, \( (G - e)^\phi_e \) has a parity-linked orientation, where \( \phi_e \) is the restriction of \( \phi \) to the graph \( G - e \). So again by Theorem 2,

\[
Sp_S((G - e)^\phi_e) = iSp(G - e).
\]

Consequently, by induction hypothesis, \( (G - e)^\phi_e \) is switching-equivalent to \( (G - e)^\sigma_e \), where \( \sigma_e \) is the restriction of \( \sigma \) to the graph \( G - e \).

Let \( \alpha \) be the switch that takes \( (G - e)^\phi_e \) to \( (G - e)^\sigma_e \) effected by the subset \( U \) of \( V(G - e) = V(G) \). We claim that \( \alpha \) takes \( \phi \) to \( \sigma \) in \( G \). If not, then the resulting oriented graph \( G^\phi \) will be of the following type: All the arcs of \( G - e \) will be oriented from one partite set (say, \( X \)) to the other (namely, \( Y \)) while the arc \( e \) will be oriented in the reverse direction, that is, from \( Y \) to \( X \) (See Figure 2).

![Figure 2: The oriented bipartite graph \( G^{\phi'} \) in Theorem 5](image)

Consider first the case when \( e \) is a cut edge of \( G \). The subgraph \( G - e \) will then consist of two components with vertex sets, say, \( S_1 \) and \( S_2 \). Now switch with respect to \( S_1 \). This will change the orientation of the only arc \( e \) and the resulting orientation is \( \sigma \). Consequently, \( \phi \) is switching-equivalent to \( \sigma \).
Note that the above argument also takes care of the case when $G$ is a tree since each edge of $G$ will then be a cut edge. Hence we now assume that $G$ contains an even cycle $C_{2k}$ containing the arc $e$ and complete the proof. But then any such $C_{2k}$ has $k - 1$ arcs in one direction and $k + 1$ arcs in the opposite direction (see Figure 3) thereby not admitting a parity-linked orientation. Hence this case can not arise. Consequently, $\phi$ is switching-equivalent to $\sigma$ in $G$. 

![Figure 3](https://example.com/figure3.png)

Figure 3: Cycle $C_{2k}$ for $k = 3, 4$ in $G^\phi$

Theorem 2 provides a nice characterization for an oriented bipartite graph $G^\phi$ to have the property that $Sp_S(G^\phi) = iSp(G)$. But it requires to check if every cycle in $G^\phi$ possesses a parity-linked orientation. A natural question is the following: Is it possible to reduce the number of checks to determine whether an oriented graph $G^\phi$ has a parity-linked orientation? Our next result provides an answer in this direction.

**Theorem 6.** Let $G$ be a bipartite graph and $\phi$ be an orientation of $G$. If $\phi$ induces a parity-linked orientation on every chordless (even) cycle of $G$, then $Sp_S(G^\phi) = iSp(G)$.

**Proof.** By virtue of Theorem 2, it suffices to show that if $\phi$ induces a parity-linked orientation on every chordless (even) cycle of $G$, then $\phi$ induces a parity-linked orientation on every cycle of $G$. If the result were not true, then there exists a cycle $C_{2\ell}$ in $G^\phi$ of least length $2\ell$ such that $\phi$ does not induce a parity-linked orientation on $C_{2\ell}$. This of course means that $C_{2\ell}$ is evenly (resp. oddly) oriented if $l$ is odd (resp. even). By hypothesis, $C_{2\ell}$ contains a chord $x_1 y_1$. Suppose that $C_{2\ell} = x_1 x_2 \ldots x_{\ell_1} y_1 y_2 \ldots y_{2\ell - \ell_1} x_1$ in clockwise direction. Consider the two cycles $C_1 = x_1 x_2 \ldots x_{\ell_1} y_1 x_1$ and $C_2 = x_1 y_1 y_2 \ldots y_{2\ell - \ell_1} x_1$ with respective lengths $\ell_1 + 1$ and $2\ell - \ell_1 + 1$ in clockwise direction. Note that $C_1$ and $C_2$ are also even ($G$ being bipartite). Suppose that $C_1$ and $C_2$ contain respectively $r_1$ and $r_2$ arcs in the clockwise direction. By the choice of $C_{2\ell}$, $C_1$ and $C_2$ possess the parity-linked orientation. Hence

$$\frac{\ell_1 + 1}{2} \equiv r_1 (\text{mod } 2) \quad \text{and} \quad \frac{2\ell - \ell_1 + 1}{2} \equiv r_2 (\text{mod } 2).$$
It follows that $\ell + 1 \equiv (r_1 + r_2)(\mod 2)$. Observe that if the arc corresponding to $x_1y_1$ is clockwise in $C_1$, then it must be anticlockwise in $C_2$ and vice versa. This of course means that $C_{2\ell}$ also admits the parity-linked orientation. This contradiction proves the result. \hfill \Box

Combining Theorem 2 and Theorem 6, we obtain immediately the following corollary.

**Corollary 7.** Let $G$ be a bipartite graph and $\phi$, an orientation of $G$. Then $Sp_S(G^\phi) = iSp(G)$ if and only if $\phi$ induces a parity-linked orientation on all the chordless cycles of $G$.

**Remark 8.** Let $C$ denote the set of all cycles of a bipartite graph $G$. A subset $\mathcal{S}$ of $C$ is called a generating set of $C$ if for any cycle $C$ of $C$ either $C \in \mathcal{S}$ or there is a sequence of cycles $C_1, C_2, \ldots, C_k$ in $\mathcal{S}$ such that $C = (\Delta C_1 \Delta C_2) \ldots \Delta C_k$ and for $2 \leq p \leq k - 1$, $((\Delta C_1 \Delta C_2) \Delta C_3) \ldots \Delta C_p$ are all cycles of $G$. With this notation, one can prove that for any oriented bipartite graph $G^\phi$, $Sp_S(G^\phi) = iSp(G)$ if and only if $\phi$ induces a parity-linked orientation for every cycle in a generating set $\mathcal{S}$ of $C$ in $G$. Actually, the set of all chordless cycles of a graph $G$ is a generating set of the set of all cycles of $G$.

### 3 Switching-equivalence in oriented hypercubes

We present below an illustration for Theorem 5. In [2], Amuradha and Balakrishnan have constructed an oriented hypercube $Q_d^\phi$ for which $Sp_S(Q_d^\phi) = iSp(Q_d)$, $d \geq 1$.

By Theorem 5, $\phi$ must be switching-equivalent to the canonical orientation $\sigma$ of $Q_d$. We now determine a switching set $U_d$ in $Q_d^\phi$ that takes $\phi$ to $\sigma$.

We first recall the algorithm given in [2] by means of which $Q_d^\phi$, $d \geq 1$, is constructed.

**Algorithm 9.** The hypercube $Q_d$, $d \geq 2$, can be constructed by taking two copies of $Q_{d-1}$ and making the corresponding vertices in the two copies adjacent. Let $V(Q_d) = \{(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_d) : \varepsilon_i = 0 \text{ or } 1\}$ be the vertex set of $Q_d$.

1. For $Q_1 = K_2$, $V(Q_1) = \{(0), (1)\}$. Set $(1, 0) \in \Gamma(Q_1^\phi)$.

2. Assume that for $i = 1, 2, \ldots, k(< d)$, the oriented hypercube $Q_k^\phi$ has been constructed. For $i = k + 1$, the oriented hypercube $Q_{k+1}^\phi$ is formed as follows:

   (a). Take two copies $C_0^{(k)}$ and $C_1^{(k)}$ of $Q_k^\phi$. Reverse the orientation of all the arcs in $C_1^{(k)}$.

   (b). For $j = 0, 1$, relabel the vertices of $C_j^{(k)}$ by adding $j$ as the first coordinate, that is, if $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k) \in Q_k$, then the vertex $(0, \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k) \in C_0^{(k)}$ and the vertex $(1, \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k) \in C_1^{(k)}$.

   (c). Let $(X_0, Y_0)$ be the bipartition of $V(Q_k)$ in $C_0^{(k)}$ such that the vertex labeled $(0, 0, \ldots, 0)$ is in $X_0$. Set the corresponding bipartition in $C_1^{(k)}$ as $(X_1, Y_1)$. (Note that the vertex labeled $(1, 0, 0, \ldots, 0) \in X_1$.) Consequently $X = X_0 \cup Y_1$ and $Y = X_1 \cup Y_0$ form the bipartition of $V(Q_{k+1})$. 

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(d). Add an edge between the vertices of $C_0^{(k)}$ and $C_1^{(k)}$ that differ in exactly the first coordinate. For each such edge, assign the orientation from $X_0$ to $X_1$ and from $Y_1$ to $Y_0$ (see Figure 4). This yields the oriented hypercube $Q_3^{\phi}$. (See Figure 5.)

3. If $k + 1 = d$, stop; else take $k \leftarrow k + 1$, return to Step 2. □

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**Figure 4:** Example for Step 2(d) in Algorithm 9

Let $(X, Y)$ be the bipartition of $V(Q_d^{\phi})$ for $d \geq 1$ such that the vertex $(0, 0, \ldots, 0)$ is
in $X$. It is then easy to observe from the construction of $Q^\phi_d$ that the indegree, $\deg^+ (u)$, of each vertex $u \in X$ is 1 while the outdegree $\deg^- (v)$ of each vertex $v \in Y$ is 1 in $Q^\phi_d$.

For each $d \geq 1$, we now define a set $U_d \subset V(Q^\phi_d)$ recursively as follows: For the oriented hypercube $Q^\phi_1$, set $U_1 = \{(0)\}$. For $k \geq 1$ ($k < d$), assume that the set $U_k$ has been determined. Form the set $U_{k+1}$ of the oriented hypercube $Q^\phi_{k+1}$ by taking, for each vertex $v = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k) \in U_k$ of the hypercube $Q^\phi_k$, the vertices $v_0 = (0, \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k)$ and $v_1 = (1, \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k)$. Note that in the oriented hypercube $Q^\phi_{k+1}$, if $U^0_k$ and $U^1_k$ are the two sets corresponding to $U_k$ in the two copies $C^{(k)}_0$ and $C^{(k)}_1$ of $Q^\phi_k$ then $U_{k+1} = U^0_k \cup U^1_k$ and $|U_{k+1}| = 2^k$.

We now show that for each $d \geq 1$, a switch with respect to the set $U_d$ in the oriented hypercube $Q^\phi_d$ results in the canonical orientation $\sigma$ of $Q_d$.

**Theorem 10.** Suppose $Q^\phi_d$ is the oriented hypercube obtained by Algorithm 9. Let $U_d \subset V(Q^\phi_d)$ be determined as above. Then a switch with respect to the set $U_d$, for $d = 1, 2, \ldots$, yields the canonical orientation $\sigma$ of $Q_d$.

**Proof.** Proof by induction on $d$. It is obvious for $d = 1, 2$. (For $d = 1, 2$, $U_1 = \{(0)\}$ and $U_2 = \{(0,0), (1,0)\}$.)

Suppose that a switch with respect to the set $U_k$ ($k < d$), yields the canonical orientation in the oriented hypercube $Q^\phi_k$. Consider the set $U_{k+1}$ of the oriented hypercube $Q^\phi_{k+1}$, clearly $Q^\phi_{k+1}$ consists of two copies $C^{(k)}_0$ and $C^{(k)}_1$ of $Q^\phi_k$. For $i = 0, 1$, let $U^i_k$ be the switch in the corresponding copy $C^{(k)}_i$. It is then easy to observe that

$$U_{k+1} = U^0_k \cup U^1_k.$$  

This shows that the copies $C^{(k)}_0$ and $C^{(k)}_1$ in $Q^\phi_{k+1}$ exhibit canonical orientation after the switch with respect to $U_{k+1}$. Further any arc between the two copies agrees with the canonical orientation (see Step 2(d) of Algorithm 9). Hence the switch with respect to $U_{k+1}$ results in $Q^\phi_{k+1}$. Applying induction, the result follows. 

\[\Box\]

### 4 The skew spectrum of $H \Box G$ with $H$ bipartite

In [1], Adiga et al. have shown that the skew energy of any oriented graph $G^\phi$ of order $n$, for which the underlying undirected graph $G$ is $k$-regular, is bounded above by $n \sqrt{k}$ and posed the following problem:

**Problem 11.** Which $k$-regular graphs $G$ on $n$ vertices have orientations $\phi$ with $\mathcal{E}_S(G^\phi) = n \sqrt{k}$, or equivalently, $S(G^\phi)^T S(G^\phi) = kI_n$?

In this section, we give an orientation of the Cartesian product $H \Box G$, where $H$ is bipartite, by extending the orientation of $P_m \Box G$ in [6], and we calculate its skew spectrum. As an application of this orientation, we construct new families of oriented graphs with maximum skew energy, which generalizes the construction in [6].
Let $H$ and $G$ be graphs with $p$ and $n$ vertices, respectively. Recall that the Cartesian product $H \Box G$ of $H$ and $G$ is the graph with vertex set $V(H) \times V(G)$ and the vertices $(u_1, v_1)$ and $(u_2, v_2)$ are adjacent in $H \Box G$ if and only if $u_1 = u_2$ and $v_1 v_2$ is an edge of $G$, or if $v_1 = v_2$ and $u_1 u_2$ is an edge of $H$. Assume that $\tau$ is any orientation of $H$ and $\phi$ is any orientation of $G$. There is a natural way to define the oriented Cartesian product $H^\tau \Box G^\phi$ of $H^\tau$ and $G^\phi$ whose underlying undirected graph is $H \Box G$. There is an arc from $(u_1, v_1)$ to $(u_2, v_2)$ if and only if $u_1 = u_2$ and $(v_1, v_2)$ is an arc of $G^\phi$, or if $v_1 = v_2$ and $(u_1, u_2)$ is an arc of $H^\tau$. The skew spectrum of $H^\tau \Box G^\phi$ has been determined in [6]. Some interesting results on the skew spectrum of the product $H^\tau \Box G^\phi$, where $H^\tau$ is an oriented hypercube are obtained in [2].

When $H$ is a bipartite graph with bipartition $X$ and $Y$, we modify the above definition of $H^\tau \Box G^\phi$ to obtain a new product graph $(H^\tau \Box G^\phi)^o$ with the following condition: If $u \in Y$ and $(v_1, v_2) \in (G^\phi)^o$, then we make $(u, v_2)(u, v_1)$ an arc of $H^\tau \Box G^\phi$ (instead of $(u, v_1)(u, v_2)$); the other arcs of $H^\tau \Box G^\phi$ remain unchanged.

**Theorem 12.** Let $H^\tau$ be an oriented bipartite graph of order $p$ and let the skew eigenvalues of $H^\tau$ be the nonzero complex numbers $\pm i\mu_1, \pm i\mu_2, \ldots, \pm i\mu_p$, and $p - 2r$ 0’s. Let $G^\phi$ be an oriented graph of order $n$ and let the skew eigenvalues of $G^\phi$ be the nonzero complex numbers $\pm i\lambda_1, \pm i\lambda_2, \ldots, \pm i\lambda_t$, and $n - 2t$ 0’s. Then the skew eigenvalues of the oriented graph $(H^\tau \Box G^\phi)^o$ are $\pm i\sqrt{\mu_j^2 + \lambda_k^2}$, $j = 1, \ldots, r$, $k = 1, \ldots, t$, each with multiplicity 2; $\pm i\mu_j, j = 1, \ldots, r$, each with multiplicity $n - 2t$; $\pm i\lambda_j, k = 1, \ldots, t$, each with multiplicity $p - 2r$ and 0 with multiplicity $(p - 2r)(n - 2t)$.

**Proof.** Let $H = H(X, Y)$ with $|X| = p_1$ and $|Y| = p_2$. With suitable labeling of the vertices of $H \Box G$, the skew adjacency matrix $S = S((H^\tau \Box G^\phi)^o)$ can be chosen as follows:

$$S = I_{p_1 + p_2} \otimes S(G^\phi) + S(H^\tau) \otimes I_n,$$

where $I_{p_1 + p_2} = I_p \otimes (a_{ij})$, $a_{ii} = 1$ if $1 \leq i \leq p_1$, $a_{ii} = -1$ if $p_1 + 1 \leq i \leq p$ and $a_{ij} = 0$ otherwise; $S(H^\tau)$ is the partitioned matrix

$$\begin{pmatrix} 0 & B \\ -BT & 0 \end{pmatrix},$$

where $B$ is a $p_1 \times p_2$ matrix. Further, $\otimes$ stands for the Kronecker product of two matrices [3].

We first determine the singular values of $S$. Note that the matrices $S$, $S(H^\tau)$ and $S(G^\phi)$ are all skew symmetric. By calculation, we have

$$SS^T = [I_p \otimes S(G^\phi) + S(H^\tau) \otimes I_n][I_p \otimes (-S(G^\phi)) + (-S(H^\tau)) \otimes I_n]$$

$$= -[I_p \otimes S^2(G^\phi) + S^2(H^\tau) \otimes I_n] + (I_p \otimes S(G^\phi))(S(H^\tau) \otimes I_n)$$

$$+ (S(H^\tau) \otimes I_n)(I_p \otimes S(G^\phi))].$$

Define $\omega_i = 1$ for $i = 1, 2, \ldots, p_1$ and $\omega_i = -1$ for $i = p_1 + 1, p_1 + 2, \ldots, p$. Denote $P^{(1)} = (I_{p_1 + p_2} \otimes S(G^\phi))(S(H^\tau) \otimes I_n)$ and $P^{(2)} = (S(H^\tau) \otimes I_n)(I_{p_1 + p_2} \otimes S(G^\phi))$. Note that $P^{(1)}$ and $P^{(2)}$ are both partitioned matrices each of order $p_1 \times p_2$ in which each entry is an $n \times n$ submatrix. The $(i, j)^{th}$ block in the matrix $P^{(1)} + P^{(2)}$ is given by

$$P^{(1)}_{ij} + P^{(2)}_{ij} = S(H^\tau)_{ij}S(G^\phi)(\omega_i + \omega_j).$$
For any \(1 \leq i, j \leq p\), if \(S(H^r)_{ij} = 0\), then \(P^{(1)}_{ij} + P^{(2)}_{ij} = 0\). Otherwise the vertices corresponding to \(i\) and \(j\) in \(H^r\) are in different parts of the bipartition. That is, \(1 \leq i \leq p_1\), \(p_1 + 1 \leq j \leq p\) or \(1 \leq j \leq p_1\), \(p_1 + 1 \leq i \leq p\). Then \(\omega_i + \omega_j = 0\). Thus it follows that \(P^{(1)} + P^{(2)} = 0\). Hence

\[
SS^T = -(I_{p_1 + p_2} \otimes S^2(G^\phi) + S^2(H^r) \otimes I_n).
\]

Therefore (cf. [3]), the eigenvalues of \(SS^T\) are \(\mu(H^r)^2 + \lambda(G^\phi)^2\), where \(\mu(H^r) \in Sp_S(H^r)\) and \(\lambda(G^\phi) \in Sp_S(G^\phi)\) and hence the eigenvalues of \(S\) are of the form \(\pm i\sqrt{\mu(H^r)^2 + \lambda(G^\phi)^2}\). Thus the skew spectrum of \((H^r \square G^\phi)^o\) is as given in the statement of the theorem. The proof is thus complete. \(\square\)

As an application of Theorem 12, we now construct a new family of oriented graphs with maximum skew energy.

**Theorem 13.** Let \(H^r\) be an oriented \(\ell\)-regular bipartite graph on \(p\) vertices with maximum skew energy \(E_S(H^r) = p\sqrt{\ell}\) and \(G^\phi\) be an oriented \(k\)-regular bipartite graph on \(n\) vertices with maximum skew energy \(E_S(G^\phi) = n\sqrt{k}\). Then the oriented graph \((H^r \square G^\phi)^o\) of \(H \square G\) has the maximum skew energy \(E_S((H^r \square G^\phi)^o) = np\sqrt{\ell} + k\).

**Proof.** Since \(H^r\) and \(G^\phi\) have maximum skew energy, \(S(H^r)S(H^r)^T = \ell I_p\) and \(S(G^\phi)S(G^\phi)^T = k I_n\). Then the skew eigenvalues of \(H^r\) are all \(\pm i\sqrt{\ell}\) and the skew eigenvalues of \(G^\phi\) are all \(\pm i\sqrt{k}\). By Theorem 12, all the skew eigenvalues of \((H^r \square G^\phi)^o\) are of the form \(\pm i\sqrt{\ell} + k\) and hence its skew energy is \(np\sqrt{\ell} + k\), the maximum possible skew energy that an \((\ell + k)\)-regular graph on \(np\) vertices can have. \(\square\)

An immediate corollary of Theorem 13 is the following result of Cui and Hou [6].

**Corollary 14.** Let \(G^\phi\) be an oriented \(k\)-regular graph on \(n\) vertices with maximum skew energy \(E_S(G^\phi) = n\sqrt{k}\). Then the oriented graph \((P_2 \square G^\phi)^o\) of \(P_2 \square G\) has maximum skew energy \(E_S((P_2 \square G^\phi)^o) = 2n\sqrt{k} + 1\).

Adiga et al. [1] showed that a 1-regular connected graph that has an orientation with maximum skew energy is \(K_2\); while a 2-regular connected graph has an orientation with maximum skew energy if and only if it is an oddly oriented cycle \(C_4\). Tian [14] proved that there exists a \(k\)-regular graph with \(n = 2^k\) vertices having an orientation \(\psi\) with maximum skew energy. Cui and Hou [6] constructed a \(k\)-regular graph of order \(n = 2^{k-1}\) having an orientation \(\varphi\) with maximum skew energy. The following examples provide new families of oriented graphs with fewer vertices that have maximum skew energy.

**Example 15.** Let \(G_1 = K_{4,4}\). For each \(r \geq 2\), set \(G_r = K_{4,4} \square G_{r-1}\). As there is an orientation of \(K_{4,4}\) with maximum skew energy 16 (see [5]), for each \(r \geq 1\), there exists an orientation of \(G_r\) that yields the maximum skew energy \(2^{3r}\sqrt{4r}\). This provides a family of \(4r\)-regular graphs of order \(n = 2^{3r}\) each having an orientation with skew energy \(2^{3r}\sqrt{4r}\), \(r \geq 1\).
**Example 16.** Let $G_1 = K_4$. For each $r \geq 2$, set $G_r = K_{4,4} \square G_{r-1}$. Since there exist orientations for $K_4$ with maximum skew energy $4\sqrt{3}$ (see [1, 7]), the skew energy of $G_r$, $r \geq 1$, is $2^{3r-1} \sqrt{4r - 1}$ and it is maximum. This provides a family of $4r - 1$-regular graphs of order $2^{3r-1}$ each having an orientation with maximum skew energy $2^{3r-1} \sqrt{4r - 1}$, $r \geq 1$.

**Example 17.** A new family of $4r - 2$-regular oriented graphs of order $2^{3r-1}$ with maximum skew energy $2^{3r-1} \sqrt{4r - 2}$, $r \geq 1$ is obtained when we set $G_1 = C_4$ in place of $K_4$ in Example 16.

**Example 18.** A new family of $4r - 3$-regular oriented graphs of order $2^{3r-2}$ with maximum skew energy $2^{3r-2} \sqrt{4r - 3}$, $r \geq 1$ is obtained when we set $G_1 = P_2$ in place of $K_4$ in Example 16.

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