Stable hypersurfaces with constant scalar curvature in Euclidean spaces

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Abstract. We obtain some nonexistence results for complete noncompact stable hypersurfaces with nonnegative constant scalar curvature in Euclidean spaces. As a special case we prove that there is no complete noncompact strongly stable hypersurface $M$ in $\mathbb{R}^4$ with zero scalar curvature $S_2$, nonzero Gauss-Kronecker curvature and finite total curvature (i.e. $\int_M |A|^3 < +\infty$).

Key words: scalar curvature, stability, index, hypersurface.

1 Introduction

In this paper we study the complete noncompact stable hypersurfaces with constant scalar curvature in Euclidean spaces. It has been proved by Cheng and Yau [CY] that any complete noncompact hypersurfaces in the Euclidean space with constant scalar curvature and nonnegative sectional curvature must be a generalized cylinder. Note that the assumption of nonnegative sectional curvature is a strong condition for hypersurfaces in the Euclidean space with zero scalar curvature since the hypersurface has to be flat in this case. Let $M^n$ be a complete orientable Riemannian manifold and let $x : M^n \to \mathbb{R}^{n+1}$ be an isometric immersion into the Euclidean space $\mathbb{R}^{n+1}$ with constant scalar curvature. We can choose a a global unit normal vector field $N$ and the Riemannian connections $\nabla$ and $\tilde{\nabla}$ of $M$ and $\mathbb{R}^{n+1}$, respectively, are related by

$$\tilde{\nabla}_XY = \nabla_XY + \langle A(X), Y \rangle N,$$

where $A$ is the second fundamental form of the immersion, defined by

$$A(X) = -\tilde{\nabla}_X N.$$

Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of $A$. The $r$-mean curvature of the immersion in a point $p$ is defined by

$$H_r = \frac{1}{\binom{n}{r}} \sum_{1 \leq i_1 < \ldots < i_r} \lambda_{i_1} \ldots \lambda_{i_r} = \frac{1}{\binom{n}{r}} S_r,$$

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where $S_r$ is the $r$-symmetric function of the $\lambda_1, \ldots, \lambda_n$, $H_0 = 1$ and $H_r = 0$, for all $r \geq n + 1$. For $r = 1$, $H_1 = H$ is the mean curvature of the immersion, in the case $r = 2$, $H_2$ is the normalized scalar curvature and for $r = n$, $H_n$ is the Gauss-Kronecker curvature.

It is well-known that hypersurfaces with constant scalar curvature are critical points for a geometric variational problem, namely, that associated to the functional

$$A_1(M) = \int_M S_1 \, dM, \tag{1}$$

under compactly supported variations that preserves volume. Let $M$ be a hypersurface in the Euclidean space with constant scalar curvature. Following [AdCE], when the scalar curvature is zero, we say that a regular domain $D \subset M$ is **stable** if the critical point is such that $(\frac{d^2 A_1}{dt^2})_{t=0} \geq 0$, for all variations with compact support in $D$ and when the scalar curvature is nonzero, we say that a regular domain $D \subset M$ is **strongly stable** if the critical point is such that $(\frac{d^2 A_1}{dt^2})_{t=0} \geq 0$, for all variations with compact support in $D$. It is natural to study the global properties of hypersurfaces in the Euclidean space with constant scalar curvature. For example we have the following open question (see 4.3 in [AdCE]).

**Question 1.1** Is there any stable complete hypersurfaces $M$ in $\mathbb{R}^4$ with zero scalar curvature and nonzero the Gauss-Kronecker curvature?

We have a partial answer to the question 1.1.

**Theorem A.** (see Theorem 3.1) There is no complete noncompact stable hypersurface $M$ in $\mathbb{R}^{n+1}$ with zero scalar curvature $S_2$ and 3-mean curvature $S_3 \neq 0$ satisfying

$$\lim_{R \to +\infty} \frac{\int_{B_R} S_3^3}{R^2} = 0, \tag{2}$$

where $B_R$ is the geodesic ball in $M$.

When $S_2 = 0$, $S_1^2 = |A|^2$ we have

**Corollary B.** There is no complete noncompact stable hypersurface $M$ in $\mathbb{R}^4$ with zero scalar curvature $S_2$, nonzero Gauss-Kronecker curvature and finite total curvature (i.e. $\int_M |A|^3 < +\infty$).

We remark that Shen and Zhu (see [SZ]) proved that a complete stable minimal $n$-dimensional hypersurface in $\mathbb{R}^{n+1}$ with finite total curvature must be a hyperplane. The above Corollary can be seen as a similar result in dimension 3 for hypersurfaces with zero scalar curvature.

We also prove the following result for hypersurfaces with positive constant scalar curvature in Euclidean space.

**Theorem C.** (see Theorem 3.2) There is no complete immersed strongly stable hypersurface $M^n \to \mathbb{R}^{n+1}$, $n \geq 3$, with positive constant scalar curvature and polynomial growth of 1-volume, that is

$$\lim_{R \to \infty} \frac{\int_{B_R} S_1 \, dM}{R^n} < \infty,$$
where $B_R$ is a geodesic ball of radius $R$ of $M^n$.

As a consequence of the properties of a graph with constant scalar curvature, we have the following corollary:

**Corollary D.** (see Corollary 4.1) Any entire graph on $\mathbb{R}^n$ with nonnegative constant scalar curvature must have zero scalar curvature.

This can be compared with a result of Chern [Ch] which says any entire graph on $\mathbb{R}^n$ with constant mean curvature must be minimal. It has been known by a result of X. Cheng in [Che] (see also [ENR]) that any complete noncompact stable hypersurface in $\mathbb{R}^{n+1}$ with constant mean curvature must be minimal if $n < 5$. It is natural to ask that any complete noncompact stable hypersurface in $\mathbb{R}^{n+1}$ with nonnegative constant scalar curvature must have zero scalar curvature.

It should be remarked that Chern [Ch] proved that there is no entire graph on $\mathbb{R}^n$ with Ricci curvature less than a negative constant. We don’t know whether there exists an entire graph on $\mathbb{R}^n$ with constant negative scalar curvature.

The rest of this paper is organized as follows: we include some results and definitions which will be used in the proof of our theorems in Section 2. The proof of main results are given in Section 3 and Section 4 is an appendix in which we prove some stability properties for graphs with constant scalar curvature in the Euclidean space.

## 2 Some stability and index properties for hypersurfaces with $S_2 = \text{const.}$

We introduce the $r$’th Newton transformation, $P_r : T_p M \to T_p M$, which are defined inductively by

$$
P_0 = I, \quad P_r = S_r I - A \circ P_{r-1}, \quad r \geq 1.
$$

The following formulas are useful in the proof (see, [Re], Lemma 2.1).

$$
\text{trace}(P_r) = (n - r)S_r, \quad (3) \\
\text{trace}(A \circ P_r) = (r + 1)S_{r+1}, \quad (4) \\
\text{trace}(A^2 \circ P_r) = S_1 S_{r+1} - (r + 2)S_{r+2}. \quad (5)
$$

From [AdCC] we have the second variation formula for hypersurfaces in a space form of constant curvature $c$, $Q^{n+1}_c$, with constant 2-mean curvature:

$$
\frac{d^2 A_1}{dt^2} \bigg|_{t=0} = \int_D (P_1(\nabla f), \nabla f) dM - \int_D (S_1 S_2 - 3S_3 + c(n - 1)S_1) f^2 dM, \quad \forall f \in C_0^\infty(D). \quad (6)
$$

**Definition 2.1** When $S_2 = 0$ and $c = 0$, $M$ is stable if and only if

$$
\int_M \langle P_1(\nabla f), \nabla f \rangle dM \geq -3 \int_M S_3 f^2 dM, \quad (7)
$$
for any $f \in C_\infty^c(M)$. One can see that if $P_1 \equiv 0$, then $S_3 = 0$ and $M$ is stable. When $S_2 = \text{const.} \neq 0$, $M$ is stable if and only if
\[
\int_D \langle P_1(\nabla f), \nabla f \rangle dM \geq \int_D (S_1S_2 - 3S_3 + c(n - 1)S_1)f^2dM,
\]
for all $f \in C_\infty^c(M)$ and $\int_M fdM = 0$. We say that $M$ is strongly stable if and only if the above inequality holds for all $f \in C_\infty^c(M)$.

Similar to minimal hypersurface we can also define the index $I$ for hypersurfaces with constant scalar curvature. Given a relatively compact domain $\Omega \subset M$, we denote by $\text{Ind}^1(\Omega)$ the number of linearly independent normal deformations with support on $\Omega$ that decrease $A_1$. The index of the immersion are defined as
\[
\text{Ind}^1(M) := \sup \{\text{Ind}^1(\Omega) \mid \Omega \subset M, \ \Omega \text{ relatively compact}\}. \tag{8}
\]

$M$ is strongly stable if $\text{Ind}^1(M) = 0$. The following result has been known in [El].

**Lemma 2.1** Let $M^n \rightarrow \mathbb{Q}^{n+1}_c$ be a noncompact hypersurface with $S_2 = \text{const.} > 0$. If $M$ has finite index then there exist a compact set $K \subset M$ such that $M \setminus K$ is strongly stable.

For hypersurfaces with constant mean curvature, do Carmo and Zhou [dCZ] proved that

**Theorem 2.1** Let $x : M^n \rightarrow \mathbb{Q}^{n+1}_c$ be an isometric immersion with constant mean curvature $H$. Assume $M$ has subexponential volume growth and finite index. Then there exist a constant $R_0$ such that
\[
H \leq -\overline{\text{Ric}}_{M \setminus B_{R_0}}(N),
\]
where $N$ is a smooth normal vector field along $M$ and $\overline{\text{Ric}}(N)$ is the Ricci curvature of $M$ in the normal vector $N$.

The technique in [dCZ] was generalized by Elbert [El] to prove the following result:

**Theorem 2.2** Let $x : M^n \rightarrow \mathbb{Q}(c)^{n+1}$ be an isometric immersion with $S_2 = \text{constant} > 0$. Assume that $\text{Ind}^1M < \infty$ and that the 1-volume of $M$ is infinite and has polynomial growth. Then $c$ is negative and
\[
S_2 \leq -c.
\]

In particular, it implies that when $c = 0$ the hypersurfaces in the above theorem must have nonpositive scalar curvature.

## 3 Proof of the theorems

When $S_2 = 0$ we know that $|S_1|^2 = |A|^2$. Thus, if $S_3 \neq 0$, we have that $|A|^2 > 0$. Hence $S_1 \neq 0$ and we can choose an orientation such that $P_1$ is semi-positive definite. Since
\[
|\sqrt{P_1}A|^2 = \text{trace}(A^2 \circ P_1)
= -3S_3,
\]

then, when \( c = 0 \), \( M \) is stable if

\[
\int_M \langle P_1(\nabla f), \nabla f \rangle dM \geq \int_M |\sqrt{P_1} A|^2 f^2 dM,
\]

for any \( f \in C^\infty_c(M) \).

By Lemma 4.1 in \[AdCC\], when \( S_2 = 0 \), we know that \( |\nabla A|^2 - |\nabla S_1|^2 \geq 0 \). In the following lemma, we characterize the equality case in some special case.

**Lemma 3.1** Let \( M^n(n \geq 3) \) be a non-flat connected immersed 1-minimal hypersurface in \( \mathbb{R}^{n+1} \). If \( |\nabla A|^2 = |\nabla S_1|^2 \) holds on all nonvanishing point of \( |A| \) in \( M \), then each component of \( M \) with \( |A| \neq 0 \) must be a cylinder over a curve.

**Proof.** Choose a frame at \( p \) so that the second fundamental form is diagonalized. From the computations in \[SSY\], we have \( |A|^2 = \sum_i h^2_{ii} \), and

\[
\sum_{i,j,k} h^2_{ijk} - |\nabla |A||^2 = \left[ \sum_{i,j} \left( \sum_{s,t,k} h^2_{stik} - \sum_k (\sum_{i,j} h_{ij} h_{ijk})^2 \right) (\sum_{i,j} h^2_{ij})^{-1} \right]
\]

\[
= \frac{1}{2} \sum_{i,j,k,s,t} \left( h_{ij} h_{stk} - h_{st} h_{ijk} \right)^2 |A|^{-2}
\]

\[
= \frac{1}{2} \left[ \sum_{i,j,k,s} \left( h_{ii} h_{stk} - h_{st} h_{iik} \right)^2 + \sum_s \sum_{i,j} h^2_{ss} \left( \sum_k \sum_{i \neq j} h^2_{ijk} \right) \right] |A|^{-2}
\]

\[
= \frac{1}{2} \sum_{i,j,k,s} \left( h_{ii} h_{ssk} - h_{ss} h_{iik} \right)^2 + \sum_i \sum_{s,t} h^2_{ii} \left( \sum_k \sum_{s \neq t} h^2_{stk} \right) |A|^{-2}
\]

\[
+ \frac{1}{2} \sum_k \sum_{i \neq j} h^2_{ijk}
\]

\[
= \frac{1}{2} \left[ \sum_{i,j,k} \left( h_{ii} h_{ssk} - h_{ss} h_{iik} \right)^2 \right] |A|^{-2} + \sum_k \sum_{i \neq j} h^2_{ijk}
\]

\[
= \frac{1}{2} \left[ \sum_{i,j,k} \left( h_{ii} h_{ssk} - h_{ss} h_{iik} \right)^2 \right] |A|^{-2} + \sum_{i,j,k} h^2_{ijk} \geq 0.
\]

It is clear that the right hand side is nonnegative and is zero if and only if all terms on the right hand side vanish.

\[
\sum_{i,j,k} h^2_{ijk} - |\nabla |A||^2 \geq 0.
\]
Suppose \( x : M \to \mathbb{R}^{n+1} \) is the 1-minimal immersion. Since \( M \) is not a hyperplane, then \(|A|\) is a nonnegative continuous function which does not vanish identically. Let \( p \) be such a point such that \(|A|(p) > 0\). Then \(|A| > 0\) in a connected open set \( U \) containing \( p \). The equality in \( \text{(11)} \) implies

\[
\begin{align*}
  h_{ jj i} &= 0, \text{ for all } j \neq i, \\
  h_{ i j k} &= 0, \text{ for all } j \neq i, j \neq k, k \neq i \\
  h_{i i } h_{s s k} &= h_{s s} h_{i i k}, \text{ for all } i, s, k.
\end{align*}
\]

So we have \( h_{ jj i} = 0 \), for all \( j \neq i \), and from the last equation we claim that at most one \( i \) such that \( h_{i i i} \neq 0 \). Otherwise, without the loss of generality we assume \( h_{111} \neq 0 \), and \( h_{222} \neq 0 \), we have \( h_{11} h_{22 k} = h_{22} h_{11 k} \) for all \( k \). This implies \( h_{11} = h_{22} = 0 \) by choosing \( k = 1, 2 \). Using again the third formula we have \( h_{jj} h_{111} = h_{11} h_{jj1} \) for \( j = 3, \cdots, n \). Hence \( h_{jj} = 0 \) for all \( j = 3, \cdots, n \), which contradicts to \(|A| \neq 0\).

We now assume \( h_{111} \neq 0 \) by continuity we can also assume \( h_{11} \neq 0 \). From the last equation of above equation, we have \( h_{11} h_{s s 1} = h_{s s} h_{111} \) for \( s \neq 1 \). Hence \( h_{s s} = 0 \) for all \( s \neq 1 \). This implies that \( M \) is a cylinder over a curve.

\[ \square \]

We are now ready to prove

**Theorem 3.1** There is no complete noncompact stable hypersurfaces in \( \mathbb{R}^{n+1} \) with \( S_2 = 0 \) and \( S_3 \neq 0 \) satisfying

\[
\lim_{R \to +\infty} \frac{\int_{B_R} S_3^2}{R^2} = 0.
\]

**Proof.** Assume for the sake of contradiction that there were such a hypersurface \( M \). From Lemma 3.7 in [AdCC], we have

\[
L_1 S_1 = |\nabla A|^2 - |\nabla S_1|^2 + 3 S_1 S_3. \tag{12}
\]

Since for any \( \phi \in C_c^\infty(M) \),

\[
\int_M \langle P_1(\nabla(\phi S_1)), \nabla(\phi S_1) \rangle dM = \int_M \langle P_1((\nabla \phi) S_1 + \phi \nabla S_1), (\nabla \phi) S_1 + \phi \nabla S_1 \rangle dM
\]

\[
= \int_M \langle P_1(\nabla \phi), \nabla \phi \rangle S_1^2 dM + 2 \int_M \langle P_1(\nabla \phi), \nabla S_1 \rangle \phi S_1 dM
\]

\[
+ \int_M \phi^2 \langle P_1(\nabla S_1), \nabla S_1 \rangle dM,
\]

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then using (12) we have

\[
\int_M \phi^2 S_1(|\nabla A|^2 - |\nabla S_1|^2) dM = \int_M (L_1 S_1 - 3S_1 S_3) \phi^2 S_1 dM
\]

\[
= - \int_M \langle P_1(\nabla S_1), \nabla (\phi^2 S_1) \rangle dM - \int_M 3S_3 \phi^2 S_1^2 dM
\]

\[
= - \int_M \phi^2 \langle P_1(\nabla S_1), \nabla S_1 \rangle dM - 2 \int_M \langle P_1(\nabla S_1), \nabla \phi S_1 \rangle dM - \int_M 3S_3 \phi^2 S_1^2 dM
\]

\[
= - \int_M \langle P_1(\nabla (\phi S_1)), \nabla (\phi S_1) \rangle dM + \int_M \langle P_1(\nabla S_1), \nabla (\phi S_1) \rangle dM - \int_M 3S_3 \phi^2 S_1^2 dM
\]

\[
\leq \int_M \langle P_1(\nabla S_1), \nabla S_1 \rangle dM
\]

\[
\leq \int_M |\nabla \phi|^2 S_1^2 dM
\]

for any \( \phi \in C^\infty_c(M) \). Here we have used the stability inequality (7) in the fifth line and use the following consequence of (3) in the last inequality:

\[
S_1 |\nabla \phi|^2 \geq \langle P_1(\nabla S_1), \nabla \phi \rangle.
\] (13)

We can choose \( \phi \) as

\[
\phi(x) = \begin{cases} 
\frac{2R - r(x)}{R}, & \text{on } B_{2R} \setminus B_R; \\
1, & \text{on } B_R; \\
0, & \text{on } M \setminus B_{2R}.
\end{cases}
\]

Thus from the choice of \( \phi \) we have \( S_1(|\nabla A|^2 - |\nabla S_1|^2) \equiv 0 \). Therefore the ellipticity of \( L_1 \) implies \( L_1 S_1 = 3S_1 S_3 \). From Lemma 3.1, \( M \) must be a cylinder over a curve which contradicts \( S_3 \neq 0 \). The proof is complete.

The following Lemma is of some independent interest and we include here since its second part is useful in the proof of Theorem 3.2.

**Lemma 3.2** Let \( M \) be a complete immersed hypersurface in \( \mathbb{Q}^{n+1}_c \) with nonnegative constant scalar curvature \( S_2 > -\frac{n(n-1)c}{2} \) and \( S_1 \neq 0 \).

1) If \( M \) is strongly stable outside a compact subset, then either \( M \) has finite 1-volume, or

\[
\lim_{R \to +\infty} \frac{1}{R^2} \int_{B_R} S_1 = +\infty.
\]

2) If \( M \) is strongly stable, then

\[
\lim_{R \to +\infty} \frac{1}{R^2} \int_{B_R} S_1 = +\infty.
\]

In particular \( M \) has infinite 1-volume.
Proof. We can assume that there exists a geodesic ball $B_{R_0} \subset M$ such that $M \setminus B_{R_0}$ is strongly stable. That is,

$$
\int_M (S_1 S_2 - 3S_3 + c(n-1)S_1) f^2 dM \leq \int_M \langle P_1(\nabla f), \nabla f \rangle dM,
$$

for all $f \in C_c^\infty(M \setminus B_{R_0})$.

Now, since $S_2 \geq 0$, we have (see [AdCR], p. 392)

$$
H_1 H_2 \geq H_3,
$$

and

$$
H_1 \geq H_2^{1/2}.
$$

By using that $S_1 = nH_1$, $S_2 = \frac{n(n-1)}{2}H_2$ and $S_3 = \frac{n(n-1)(n-2)}{6}H_3$, it follows that

$$
\frac{(n-2)}{n} S_1 S_2 \geq 3S_3,
$$

that is,

$$
-3S_3 \geq -\frac{(n-2)}{n} S_1 S_2.
$$

(15)

We also have that

$$
\frac{S_1}{n} \geq \left( \frac{2S_2}{n(n-1)} \right)^{1/2},
$$

which implies

$$
S_1 \geq \left( \frac{2n}{n-1} \right)^{1/2} S_2^{1/2}.
$$

(16)

By using inequality (15) in (14), it follows that

$$
\int_M \left( S_1 S_2 - \frac{n-2}{n} S_1 S_2 + c(n-1)S_1 \right) f^2 dM \leq \int_M \langle P(\nabla f), \nabla f \rangle dM,
$$

that is,

$$
\int_M \left( S_2 + \frac{n(n-1)c}{2} \right) S_1 f^2 dM \leq \frac{n}{2} \int_M \langle P(\nabla f), \nabla f \rangle dM.
$$

By using (13), we obtain that

$$
\int_M S_1 |\nabla f|^2 dM \geq \int_M \langle P(\nabla f), \nabla f \rangle dM
$$

Therefore, there exists a constant $C > 0$ such that

$$
\int_M S_1 |\nabla f|^2 dM \geq C \int_M S_1 f^2 dM.
$$

(17)
1) When $M$ is strongly stable outside $B_{R_0}$. We can choose $f$ as

$$f(x) = \begin{cases} 
    r(x) - R_0, & \text{on } B_{R_0+1} \setminus B_{R_0}; \\
    1, & \text{on } B_{R+R_0+1} \setminus B_{R_0+1}; \\
    \frac{2R+R_0+1-r(x)}{R}, & \text{on } B_{2R+R_0+1} \setminus B_{R+R_0+1}; \\
    0, & \text{on } M \setminus B_{2R+R_0+1},
\end{cases}$$

where $r(x)$ is the distance function to a fixed point. Then

$$\frac{1}{R^2} \int_{B_{2R+R_0+1} \setminus B_{R+R_0+1}} S_1 dM + \int_{B_{R_0+1} \setminus B_R} S_1 dM \geq C \int_{B_{R+R_0+1} \setminus B_{R_0+1}} S_1 dM. $$

If the 1-volume is infinite, we can choose $R$ large such that

$$C \int_{B_{R+R_0+1} \setminus B_{R_0+1}} S_1 dM > \int_{B_{R_0+1} \setminus B_R} S_1 dM, $$

hence

$$\lim_{R \to +\infty} \frac{1}{R^2} \int_{B_{2R+R_0+1} \setminus B_{R+R_0+1}} S_1 dM = +\infty. $$

2) When $M$ is strongly stable we can choose a simpler test function $f$ as

$$f(x) = \begin{cases} 
    1, & \text{on } B_R; \\
    \frac{2R-r(x)}{R}, & \text{on } B_{2R} \setminus B_R; \\
    0, & \text{on } M \setminus B_{2R},
\end{cases}$$

which implies that when $S_1 \neq 0$,

$$\lim_{R \to +\infty} \frac{1}{R^2} \int_{B_{2R}} S_1 dM = +\infty. $$

The proof is complete.

**Theorem 3.2** There is no complete immersed strongly stable hypersurface $M^n \to \mathbb{R}^{n+1}$, $n \geq 3$, with positive constant scalar curvature and polynomial growth of 1-volume, that is

$$\lim_{R \to \infty} \frac{\int_{B_R} S_1 dM}{R^n} < \infty, $$

where $B_R$ is a geodesic ball of radius $R$ of $M^n$.

**Proof.** Suppose that $M$ is a complete immersed strongly stable hypersurface $M^n \to \mathbb{R}^{n+1}$, $n \geq 3$, with positive constant scalar curvature. From Theorem 2.2, it suffices to show that the 1-volume $\int_M S_1 dM$ is infinite which is the part (2) of Lemma 3.2.

□
4 Graphs with $S_2 = \text{const}$ in Euclidean space

In this section we include some stability properties and estimates for entire graphs on $\mathbb{R}^n$ which may be known to experts and not easy to find a reference. Using these facts we give the proof of Corollary 4.1. Let $M^n$ a hypersurface of $\mathbb{R}^{n+1}$ given by a graph of a function $u : \mathbb{R}^n \to \mathbb{R}$ of class $C^\infty(\mathbb{R}^n)$. For such hypersurfaces we have:

**Proposition 4.1** Let $M^n$ a graph of a function $u : \mathbb{R}^n \to \mathbb{R}$ of class $C^\infty(\mathbb{R}^n)$. Then

1. If $S_2 = 0$ and $S_1$ does not change sign on $M$, then $M^n$ is a stable hypersurface.

2. If $M$ has $S_2 = C > 0$, then $M^n$ is strongly stable.

**Proof.** Consider $f : M \to \mathbb{R}$ a $C^\infty$ function with compact support and let $W = \sqrt{1 + |\nabla u|^2}$. In order to calculate $\langle P_1(\nabla f), \nabla f \rangle$, write $g = fW$. Thus

$$
\langle P_1(\nabla f), \nabla f \rangle = \langle P_1(\nabla(gW)), \nabla(gW) \rangle = \langle P_1(g\nabla\left(\frac{1}{W}\right), g\nabla\left(\frac{1}{W}\right) + \frac{1}{W}\nabla g) 
$$

$$
= \langle gP_1(\nabla\frac{1}{W}) + \frac{1}{W}P_1(\nabla g), g\nabla(\frac{1}{W}) + \frac{1}{W}\nabla g \rangle 
$$

$$
= g^2\langle P_1(\nabla\frac{1}{W}), \nabla\frac{1}{W} \rangle + \frac{g}{W}\langle P_1(\nabla\frac{1}{W}), \nabla g \rangle + \frac{1}{W}\langle P_1(\nabla g), \nabla g \rangle. 
$$

By using that $P_1$ is selfadjoint, we have:

$$
\langle P_1(\nabla f), \nabla f \rangle = g^2\langle P_1(\nabla\frac{1}{W}), \nabla\frac{1}{W} \rangle + 2\frac{g}{W}\langle P_1(\nabla\frac{1}{W}), \nabla g \rangle + \frac{1}{W^2}\langle P_1(\nabla g), \nabla g \rangle. \tag{18}
$$

On the other hand, if $\{e_1, ..., e_n\}$ is a geodesic frame along $M$,

$$
\text{div}(fgP_1(\nabla\frac{1}{W})) = \sum_{i=1}^n\langle \nabla_{e_i}(fgP_1(\nabla\frac{1}{W})), e_i \rangle 
$$

$$
= \sum_{i=1}^n\{fg_iP_1(\nabla\frac{1}{W}) + f_iP_1(\nabla\frac{1}{W}) + f\nabla_{e_i}(P_1(\nabla\frac{1}{W})), e_i \} 
$$

$$
= \sum_{i=1}^n\{fg_iP_1(\nabla\frac{1}{W}), e_i \} + f_iP_1(\nabla\frac{1}{W}), e_i \} + f\nabla_{e_i}(P_1(\nabla\frac{1}{W})), e_i \} 
$$

Since $f = \frac{g}{W}$, we get

$$
f_i = g_i \frac{1}{W} + g\left(\frac{1}{W}\right)_i, 
$$
that is,

\[ gf_i = gg_i \frac{1}{W} + g^2 \left( \frac{1}{W} \right)_i \]

\[ = fg_i + g^2 \left( \frac{1}{W} \right)_i \]

Hence,

\[
\text{div}(fgP_1(\nabla \frac{1}{W})) = \sum_{i=1}^{n} \{fg_i \langle P_1(\nabla \frac{1}{W}), e_i \rangle + (fg_i + g^2 \left( \frac{1}{W} \right)_i) \langle P_1(\nabla \frac{1}{W}), e_i \rangle \} + fgL_1(\frac{1}{W})
\]

\[ = \sum_{i=1}^{n} \{2fg_i \langle P_1(\nabla \frac{1}{W}), e_i \rangle + g^2 \left( \frac{1}{W} \right)_i \langle P_1(\nabla \frac{1}{W}), e_i \rangle \} + fgL_1(\frac{1}{W})
\]

\[ = 2f \langle P_1(\nabla \frac{1}{W}), \nabla g \rangle + g^2 \langle P_1(\nabla \frac{1}{W}), \nabla(\frac{1}{W}) \rangle + fgL_1(\frac{1}{W})
\]

\[ = 2g \langle P_1(\nabla \frac{1}{W}), P_1(\nabla g) \rangle + g^2 \langle P_1(\nabla \frac{1}{W}), \nabla(\frac{1}{W}) \rangle + f^2WL_1(\frac{1}{W}).
\]

Thus,

\[ 2g \langle P_1(\nabla \frac{1}{W}), P_1(\nabla g) \rangle = \text{div}(fgP_1(\nabla \frac{1}{W})) - g^2 \langle P_1(\nabla \frac{1}{W}), \nabla(\frac{1}{W}) \rangle - f^2WL_1(\frac{1}{W}). \quad (19)
\]

Now, by using (19) into equation (18), we get

\[ \langle P_1(\nabla f), \nabla f \rangle = \text{div}(fgP_1(\nabla \frac{1}{W})) - f^2WL_1(\frac{1}{W}) + \frac{1}{W^2} \langle P_1(\nabla g), \nabla g \rangle.
\]

Now, the divergence theorem implies that

\[
\int_M \langle P_1(\nabla f), \nabla f \rangle dM = - \int_M f^2WL_1(\frac{1}{W})dM + \int_M \frac{1}{W^2} \langle P_1(\nabla g), \nabla g \rangle dM.
\]

Choose the orientation of \( M \) in such way that \( S_1 \geq 0 \). Since \( S_1^2 - |A|^2 = 2S_2 \geq 0 \), we obtain that \( S_1 \geq |A| \). Thus, \( \langle P_1(\nabla g), \nabla g \rangle = S_1|\nabla g|^2 - \langle A \nabla g, \nabla g \rangle \geq (S_1 - |A|)|\nabla g|^2 \geq 0 \), which implies that

\[
\int_M \langle P_1(\nabla f), \nabla f \rangle dM \geq - \int_M f^2WL_1(\frac{1}{W})dM.
\]

(20)

When \( S_2 \) is constant, we will use the following formula proved by Reilly (see [Re], Proposition C).

\[ L_1(\frac{1}{W}) = L_1(\langle N, e_{n+1} \rangle) + (S_1S_2 - 3S_3)\langle N, e_{n+1} \rangle = 0, \]

where \( N \) is the normal vector of \( M \) and \( e_{n+1} = (0, ..., 0, \pm 1) \), according to our choice of the orientation of \( M \).

Thus,

\[
\int_M \langle P_1(\nabla f), \nabla f \rangle dM \geq - \int_M f^2WL_1(\frac{1}{W})dM = 0
\]

for all function \( f \) with compact support. Hence \( M \) is stable if \( S_2 = 0 \) and strongly stable in the case \( S_2 \neq 0 \).
Remark 4.1 We would like to remark that the operator $L_1$ need not to be elliptic in the above proof.

Proposition 4.2 Let $M^n$ a graph of a function $u : \mathbb{R}^n \to \mathbb{R}$ of class $C^\infty(\mathbb{R}^n)$, with $S_1 \geq 0$. Let $B_R$ be a geodesic ball of radius $R$ in $M$. Then

$$\int_{B_{\theta R}} S_1 dM \leq \frac{C(n)}{1 - \theta} R^n,$$

where $C(n)$ and $\theta$ are constants, with $0 < \theta < 1$. In particular, $\int_M S_1 dM$ has polynomial growth.

Proof. Let $f : M \to \mathbb{R}$ be a be a function in $C^\infty_0(M)$, that is a smooth function with compact support. Observe that

$$\text{div} \left( f \frac{\nabla u}{W} \right) = f \text{div} \left( \frac{\nabla u}{W} \right) + \left\langle \nabla f, \frac{\nabla u}{W} \right\rangle,$$

where $W = \sqrt{1 + |\nabla u|^2}$. By using the fact that $S_1$ is given by $S_1 = \text{div} \left( \frac{\nabla u}{W} \right)$, we have that

$$\int_M f S_1 dM = \int_M f \text{div} \left( \frac{\nabla u}{W} \right) dM = - \int_M \left\langle \nabla f, \frac{\nabla u}{W} \right\rangle dM. \quad (21)$$

Now, choose a family of geodesic balls $B_R$ that exhausts $M$. Fix $\theta$, with $0 < \theta < 1$ and let $f : M \to \mathbb{R}$ be a continuous function that is one on $B_{\theta R}$, zero outside $B_R$ and linear on $B_R \setminus B_{\theta R}$. Therefore, from equation (21) we obtain

$$\int_{B_{\theta R}} S_1 dM \leq \int_{B_R} f S_1 dM \leq \int_{B_R} \left\langle \frac{\nabla u}{W}, \nabla f \right\rangle dM.$$

By using Cauchy-Schwarz inequality and the fact that $|\frac{\nabla u}{W}| \leq 1$, it follows that

$$\int_{B_{\theta R}} S_1 dM \leq \int_{B_R} |\nabla f| dM \leq \int_{B_r \setminus B_{\theta R}} \frac{1}{(1 - \theta) R} dM \leq \frac{1}{(1 - \theta) R} \text{vol}(B_R).$$

We observe that since $M$ is a graph, if $\Omega_R = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} | -R \leq x_{n+1} \leq R; \sqrt{x_1^2 + \ldots + x_n^2} \leq R\}$, then

$$\text{vol}(B_R) \leq \int_{\Omega_R} 1 dx_1 \ldots dx_{n+1} = C(n) R^{n+1}.$$

Hence,

$$\int_{B_{\theta R}} S_1 dM \leq \frac{1}{(1 - \theta) R} \text{vol}(B_R) = \frac{C(n)}{1 - \theta} R^n.$$

$\square$
We have the following Corollary of Theorem 3.2

**Corollary 4.1** Any entire graph on $\mathbb{R}^n$ with nonnegative constant scalar curvature must have zero scalar curvature.

**Proof.** Suppose by sake of contradiction that there exist a entire graph with $S_2 = \text{const} > 0$. Such graph is strongly stable and if $S_2 > 0$, we get that $S_1^2 = |A|^2 + 2S_2 > 0$, we obtain that $S_1$ does not change sign and we can choose the orientation in such way that $S_1 > 0$. Thus the graph has polynomial growth of the 1-volume. Thus we have a contradiction with Theorem 3.2. Thus it follows that $S_2 = 0$. $\square$

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