Deterministic force-free resonant activation

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Abstract. The combined action of noise and deterministic force in dynamical systems can induce resonant effects. Here, we demonstrate a minimal, deterministic force-free setup allowing for the occurrence of resonant, noise-induced effects. We show that in the archetypal problem of escape from finite intervals driven by $\alpha$-stable noise with a periodically modulated stability index, depending on the initial direction of the modulation, resonant-activation-like or noise-enhanced-stability-like phenomena can be observed. Consequently, in comparison to traditional Lévy flights, Lévy flights with a time-dependent jump length exponent are capable of facilitating or slowing down the escape from finite intervals in an analogous way, such as the modulation of the potential in the resonant activation setup.

Keywords: stochastic particle dynamics, stochastic processes

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1. Introduction

The action of noise in dynamical systems results in the occurrence of so-called noise-induced effects [1], which demonstrate the constructive role of fluctuations. The combined action of stochastic and deterministic forces is responsible for the emergence of many counterintuitive effects. Among them, resonant activation (RA) [2, 3], stochastic resonance (SR) [4–6], the ratcheting effect [7, 8] and noise-enhanced stability [9–11] are the most well-known. In the RA phenomenon, the escape over a modulated potential barrier under the action of noise can be optimized, i.e., there exists such a parameter of the barrier modulating process for which the mean first passage time (MFPT) is minimal. The efficiency of dynamic RA [2], stochastic RA [3] as well as SR [12] relies on frequency or time scaling matching [13–16]. Typically, in RA, the potential barrier is dichotomously [3] or periodically [17] modulated. For Gaussian white noise, the height of the potential barrier is measured in units of $k_B T$; therefore instead of modulating the potential barrier, one can modulate the system temperature. Here, we extend this approach by studying the escape from finite intervals under the action of $\alpha$-stable noises. In contrast to the majority of earlier studies, we assume that the stability index $\alpha$ is no longer constant, but it is altered in time. Such a system is out of equilibrium; nevertheless, changes in $\alpha$ modify the width of the noise-induced displacement distribution, as measured by the interquantile distance. Similar to the properties of Lévy ratchets [18, 19], we expect that the action of modulated noise can produce resonant effects in simpler setups that are traditionally considered for the inspection of noise-induced effects. This hypothesis is based on the non-equilibrium properties of $\alpha$-stable Lévy-type noises.

Lévy noises are especially well suited for the description of out-of-equilibrium systems, as they allow for the occurrence of large fluctuations with a significantly larger probability than Gaussian white noise. Well-developed theory and desired mathematical properties, for example, self-similarity, infinite divisibility and generalized central limit theorem, ensure $\alpha$-stable noises are widely applied in various out-of-equilibrium models.

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and setups displaying anomalous fluctuations. Non-Gaussian, heavy-tailed fluctuations have been recorded in diverse experimental setups ranging from rotating flows [20], optical systems and materials [21, 22], physiological applications [23], disordered media [24], biological systems [25], financial time series [26–28], the dispersal patterns of humans and animals [29, 30], and laser cooling [31] to gaze dynamics [32] and search strategies [33, 34]. Lévy noise systems are also extensively studied theoretically [35–42]. Furthermore, possible applications of Lévy drivings in various systems, for example, population dynamics [43] and fluctuation detectors [44], have been suggested.

Models, assuming a variability of system parameters, have been explored in various contexts. For instance, Gaussian white noise with fluctuating temperature [45] can induce Lévy flights, which are described by the space fractional Smoluchowski–Fokker–Planck equation [40]. An appropriate fluctuation protocol in temperature [46] can transform the Boltzmann–Gibbs distribution into that following from the Tsallis statistics [47, 48]. In general, macroscopic fluctuations of the system parameters are studied within superstatistics [49]. Here, we assume that some of the system parameters evolve in time, but these changes are deterministic, such as in a generalization of the escape from the positive half-line [50, 51] to the time-dependent drift and diffusion coefficients [52]. Nevertheless, the studied system displays increased randomness because its stochastic properties are determined by the evolving parameter [53, 54]. Moreover, due to the time variability of the exponent characterizing the jump length distribution, the studied process is not of the Lévy type [55]. It is still characterized by independent increments, but contrary to traditional Lévy flights, they are no longer identically distributed.

The noise-driven escape from finite intervals is an archetypal problem studied within the theory of stochastic processes [56]. For Gaussian white noise, it is possible to find not only the MFPT [57], but also time-dependent densities [56]. Under the action of \( \alpha \)-stable noises, the MFPT is known [35, 36], but time-dependent densities can be constructed numerically only [58, 59] due to difficulties in the construction of the eigenvalues and eigenfunctions of fractional Laplacians [60–63]. From a microscopic point of view, these problems are produced by the discontinuity of trajectories of \( \alpha \)-stable motions. Consequently, in order to leave the domain of motion, a particle can escape via a single long jump [64–67] instead of a sequence of short jumps, which is the typical escape scenario under Gaussian white noise driving. The discontinuity of trajectories is also responsible for the failure of the method of images [68] and leapovers of Lévy flights [69–71].

In this manuscript, we extend the discussion on the overdamped, deterministic force-free kinetics driven by Lévy noises with time-dependent parameters. The studied process generalizes Lévy flights by assuming that the exponent characterizing the jump length distribution varies in time. Consequently, the studied process is characterized by independent but non-identically distributed increments. In the next section (section 2) we define the minimal setup allowing for the occurrence of RA. In section 3, we present the results of the extensive numerical (Monte Carlo) simulations. The manuscript is closed with a summary and conclusions (section 4).
2. Model

We begin our considerations with the overdamped motion described by the following Langevin equation

\[ \dot{x}(t) = \zeta(t), \]  

where \( \zeta(t) \) is a symmetric \( \alpha \)-stable noise \([42, 72]\). The \( \alpha \)-stable noise is the formal time derivative of the \( \alpha \)-stable process \( L(t) \) (see \([72]\)), which is the process with stationary, independent increments distributed according to the \( \alpha \)-stable distribution \([55]\). Values of the symmetric \( \alpha \)-stable motion \( L(t) \) are distributed according to the symmetric \( \alpha \)-stable distribution which is defined by the characteristic function

\[ \phi(k) = \langle \exp[ikL(t)] \rangle = \exp[-t\sigma^\alpha |k|^\alpha], \]

where \( \alpha \) \((0 < \alpha \leq 2)\) is the stability index, while \( \sigma \) \((\sigma > 0)\) is the scale parameter. The stability index \( \alpha \) determines the tails’ asymptotics, which for \( \alpha < 2 \) is of the power-law type, i.e., \( p(x) \propto |x|^{-(\alpha+1)} \). Consequently, the variance of \( \alpha \)-stable random variables with \( \alpha < 2 \) diverges. More precisely, for \( \alpha < 2 \), only fractional moments of order \( \nu < \alpha \) are finite, i.e., \( \langle |x|^\nu \rangle < \infty \). The Gaussian distribution \((\alpha = 2)\) is the only one \( \alpha \)-stable distribution with all moments finite. Furthermore, the noise-driven motion (see equation (1)) is restricted by two absorbing boundaries placed at \( \pm l \), i.e., the motion is performed within a bounded \([-l, l]\) domain as long as \( |x(t)| < l \). For the system described by equation (1), it is possible to define the MFPT \( T \)

\[ T = \langle \tau \rangle = \langle \min\{\tau : x(0) = 0 \land |x(\tau)| \geq l\} \rangle. \]

The exact formula for the MFPT \([35, 36]\) in the setup described above, i.e., for fixed noise parameters, reads

\[ T = \frac{1}{\Gamma(1+\alpha)} \ell^{\alpha}, \]

where \( \ell = l/\sigma \), which suggests that the natural variable for the studied problem is \( x/\sigma \). Such a transformation is equivalent to setting \( \sigma = 1 \) in equation (1). Using equation (4), it is possible to draw a phase-diagram showing domains where \( T(\alpha) \) is a decreasing \((T'(\alpha) < 0—\text{colored in blue})\) or increasing \((T'(\alpha) > 0—\text{colored in orange})\) function of the stability index \( \alpha \) (see the top panel of figure 1). For \( \ell \geq \exp(\frac{3}{2} - \gamma) \approx 2.51 \), where \( \gamma \) is the Euler–Mascheroni constant, the MFPT increases with the increase of \( \alpha \), while for \( \ell \leq \exp(-\gamma) \approx 0.56 \), the MFPT is the decreasing function of \( \alpha \). In the intermediate domain, \( \exp(-\gamma) < \ell < \exp(\frac{3}{2} - \gamma) \), the MFPT is a non-monotonous function of the stability index \( \alpha \). Moreover, there exists such a value of the stability index \( \alpha \), let us say \( \alpha_c \) \((0 \leq \alpha_c \leq 2)\), for which the MFPT attains maximum value. The bottom panel of figure 1 shows all three possible patterns of the MFPT curves. In particular, the blue dash-dotted line shows the results for \( \ell \leq \exp(-\gamma) \), i.e., \( \ell = 0.5 \), the black solid line depicts the results for \( \exp(-\gamma) < \ell < \exp(3/2 - \gamma) \), i.e., \( \ell = 1.5 \), and orange dashed for \( \ell \geq \exp(3/2 - \gamma) \), i.e., \( \ell = 3 \). These three lines represent: the decreasing, non-monotonous and increasing
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Figure 1. Phase diagram (a) showing domains where MFPT is the increasing ($T'(\alpha) > 0$) (orange) or decreasing ($T'(\alpha) < 0$) (blue) function of the stability index $\alpha$ and sample dependence of $T(\alpha)$ (b) corresponding to all (three) possible shapes of MFPT curves ($\ell = 0.5$—blue dash-dotted, $\ell = 1.5$—black solid and $\ell = 3$—orange dashed).

The existence of the domain of the non-monotonous dependence of MFPT on the stability index $\alpha$ can be intuitively explained. In order to escape from a narrow interval, i.e., small $\ell$, a sequence of short jumps is sufficient. Short jumps are controlled by the central part of the jump length distribution, which is a growing function of $\alpha$. Therefore, the smallest MFPT is recorded for $\alpha = 2$. A very different situation is observed for large $\ell$. In such a case, the most probable escape scenario is via a single long jump [66, 67]. Consequently, densities with heavier tails result in a faster escape and a minimal MFPT is recorded for $\alpha = 0$. Finally, there is an intermediate domain of $\ell$ where the competition between short and long jumps is observed in which MFPT is a non-monotonous function of $\alpha$. 

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Here, instead of a constant value of the stability index $\alpha$, we assume that the parameter $\alpha$ periodically changes in time

$$\alpha(t) = \bar{\alpha} + \frac{\Delta \alpha}{2} \sin(2\pi ft),$$

(5)

where

$$\bar{\alpha} = \frac{\alpha_{\text{min}} + \alpha_{\text{max}}}{2}$$

(6)

and

$$\Delta \alpha = \alpha_{\text{max}} - \alpha_{\text{min}}.$$  

(7)

Consequently, one can consider the studied process as Lévy flights with time-dependent exponent $\alpha$ characterizing the jump length distribution. The process defined by equation (1) with a fixed value of $\alpha$ is a classical example of the Lévy process [55] for which the MFPT is given by equation (4). In the case of time-dependent stability index $\alpha$, the studied process is no longer of the Lévy type as its increments are non-stationary.

A very similar situation is also observed for processes in which probability densities evolve according to distributed order fractional equations [73–76], scaled Brownian motion [77] or processes with a time-dependent diffusion coefficient. Nevertheless, the studied process is uniquely defined by equation (1) accompanied by equations defining the dependence of the stability index $\alpha$ on time (see equations (5)–(7)). Importantly, as will be verified below, the formula for the MFPT with fixed $\alpha$ can also be used to provide some predictions for the MFPT under action of time-dependent $\alpha$-stable noises. Such an approximation can be applied thanks to the fact that for the studied process, the escape scenario is similar to the escape scenario induced by a Lévy noise with a fixed $\alpha$.

The trajectory can still leave the domain of motion via a single long jump [64–67].

For the modulation given by equation (5), the mean value of the stability index $\alpha$ over the period of modulation, $T$, is equal to $\bar{\alpha}$. Despite the fact that, by construction, $\Delta \alpha > 0$ we use the $\Delta \alpha$ with positive or negative signs in order to indicate the initial dependence of $\alpha$ on time. Therefore, for $\Delta \alpha > 0$, we have $\dot{\alpha}(0) > 0$ while for $\Delta \alpha < 0$, there is $\dot{\alpha}(0) < 0$. In further studies, we use $\Delta \alpha = \pm 1$. Consequently, for $\Delta \alpha = 1$, $\alpha(t)$ initially increases, while for $\Delta \alpha = -1$, it decays. Despite the fact that $\alpha$ is no longer constant, equation (4) provides a qualitative explanation of the MFPT dependence on the frequency $f$. Equation (4) indicates whether MFPT is an increasing, decreasing or non-monotonous function of the stability index $\alpha$. The smaller MFPT corresponds to the situation where the first passage time density is narrower because the asymptotics of the first passage time density can be approximated by $p(\tau|\alpha) \sim \exp(-\frac{\tau}{T(\alpha)})$ (see [78]). Putting it differently, the width of the instantaneous first passage time density has qualitatively the same dependence on the stability index $\alpha$ as the MFPT $T(\alpha)$ (see equation (4)). When the first passage time density is narrower, individual escapes are (statistically) faster. If individual escapes, for a given value of $\alpha$, become faster, the escape kinetics is facilitated. Therefore, in the time-dependent case, $\alpha$ with smaller $T(\alpha)$, on average, speeds up the escapes, while $\alpha$ with the larger MFPT statistically slows down the escape kinetics. The overall efficiency of the escape kinetics is determined by the time scale associated with the modulation of $\alpha$ and the initial direction of changes.
in the value of the stability index. In the next section, we present the numerical results for MFPTs with periodically modulated stability index $\alpha$.

3. Results

The model of escape from finite intervals under the action of time-dependent $\alpha$-stable noises is studied by Monte Carlo methods. More precisely, using the Euler–Maruyama method [79, 80], we have generated an ensemble of trajectories following equation (1) with $\sigma = 1$ and $\alpha(t)$ varying according to equations (5)–(7). For more details, see the appendix, which presents the applied numerical algorithm [79–82]. Every trajectory was simulated until the first escape from the $[-l, l]$ interval, i.e., as long as $|x(t)| < l$.

From the set of collected first passage times $\tau$s the MFPT $T = \langle \tau \rangle$ was calculated (see equation (3)). The obtained results are presented in a series of figures showing MFPTs and other motion characteristics (figures 2–5), while, further, figures 6 and 7 show MFPTs for other sets of parameters. The studied system is characterized by two timescales. The first timescale is determined by the periodic modulation of the stability index $\alpha$, which is characterized by its period $T (T = 1/f)$. The second timescale is imposed by the escape kinetics and it is determined by the MFPT. Due to the modulation of $\alpha$, the stability index $\alpha$ is no longer constant during the motion. Different escapes are recorded at various values of instantaneous $\alpha$. In the dynamic regime, the recorded MFPTs are always between the minimum and maximum of MFPTs with fixed $\alpha$, i.e., $\min_{\alpha \in [\alpha_{\min}, \alpha_{\max}]} \{ T(\alpha) \} \leq T \leq \max_{\alpha \in [\alpha_{\min}, \alpha_{\max}]} \{ T(\alpha) \}$.

We begin our analysis with such values of $\alpha$, $\Delta \alpha$ and $\ell$ for which $T'(\alpha)$ does not change its sign with the change in the stability index $\alpha$. In other words, $T'(\alpha)$ is always smaller or larger than 0. In figure 2, the results corresponding to $T'(\alpha) < 0$ are depicted (see the blue domain in the top panel of figure 1). We have used $\alpha \in [\alpha_{\min}, \alpha_{\max}] = [0.5, 1.5]$, $\Delta \alpha = \pm 1$ and $\ell = 1$ resulting in $\pi = 1$ and $T'(\alpha) < 0$ for every
Figure 3. Histograms of instantaneous values of the stability index $\alpha$, $p(\alpha)$, at first passage time for $\alpha \in [0.5, 1.5]$ with $\ell = 1$. The different curves correspond to various values of the driving frequency $f$. (The black dots, red squares, and blue and orange triangles correspond to $f \in \{0.05, 0.1, 0.6, 3.1\}$ respectively).

Figure 4. Histograms of the last hitting points, $p(x_{\text{last}})$ for $\alpha \in [0.5, 1.5]$ with $\ell = 1$. The different curves correspond to various values of the frequency $f$. (The black dots, red squares, blue and orange triangles correspond to $f \in \{0.05, 0.1, 0.6, 3.2\}$ respectively.)

The recorded dependence of the MFPT on the frequency $f$ follows resonant-activation-like [3] ($\Delta \alpha = 1$) or noise-enhanced-stability-like [9] ($\Delta \alpha = -1$) patterns. More precisely, for $\Delta \alpha = 1$, $\alpha(t)$ initially grows. In subsequent moments, $\alpha$ becomes larger and $T(\alpha)$ smaller, as we are in the $T'(<0)$ domain. For $t < T/2$, with the increasing time, the chance of escape increases. As long as $T \ll T$, the modulation of $\alpha$ facilitates the escape kinetics. Therefore, there exists such a value of $f$ for which the MFPT attains, analogously, as in the RA, the minimal value. A further increase in $f$ makes the MFPT larger. For $\Delta \alpha = -1$, in comparison to $\Delta \alpha = 1$, the MFPT curve is inverted because the initial decay in $\alpha$ is associated with the decreasing chances of escape. Therefore, the MFPT curve displays a noise-enhanced-stability-like behavior; there exists such a value of the frequency $f_c$ for which the MFPT is maximal. In the
Figure 5. Survival probability $S(t)$ for $\alpha \in [0.5, 1.5]$, $\ell = 1$ with $\Delta \alpha = 1$. The various curves correspond to different values of $f$: $f = 0.2$ (black dashed) and $f = 3.1$ (red solid).

For the escape from the metastable potential, the MFPT is sensitive to the barrier configuration and depends on parameters characterizing a modulation protocol. In our setup, there is no deterministic force but the MFPT depends on the stability index $\alpha$ and its variation. The MFPT curve displays a single extreme if changes in $\alpha$ do not change the sign of $T'(\alpha)$. For $f = 0$, the MFPT is equal to $T(\alpha(0)) = T(\bar{\alpha})$ (see equation (4)) because the stability index $\alpha$ is constant and equal to its initial value. For $f_c$, the minimum (if $\text{sign}(\Delta \alpha) = -\text{sign}(T'(\alpha))$) or the maximum (if $\text{sign}(\Delta \alpha) = \text{sign}(T'(\alpha))$) of the MFPT is recorded.

The main quantity which characterizes the escape kinetics is the MFPT. Furthermore, the escape kinetics can be characterized by the instantaneous value of the stability index $\alpha$ at the moment of first escape, i.e., at the first passage time. Figure 3 shows the $p(\alpha)$ histograms for various values of the frequency $f$ corresponding to MFPTs depicted in figure 2. For $f = 0$, the $p(\alpha)$ density is given by the Dirac’s delta ($p(\alpha) = \delta(\alpha - \bar{\alpha})$), because all escapes take place with $\alpha = \bar{\alpha}$. For $f = 0.05$, there is a single maximum near the $\alpha = \bar{\alpha}$ and slowly decaying part towards $\alpha_{\text{max}} = 1.5$. With an increase in $f$, the height of the central peak at $\alpha = \bar{\alpha}$ decreases and the maximum at $\alpha = \alpha_{\text{max}}$ emerges.
For $f = 0.1$, the central peak does not decay completely; however, most particles exit with $\alpha \approx \alpha_{\text{max}}$ as it corresponds to the minimal MFPT. For small $f$, almost all escapes take place during the time when $\alpha$ has not managed to drop down below $\overline{\alpha}$. Therefore, for $\alpha < \overline{\alpha}$, the histogram vanishes, i.e., $p(\alpha) \equiv 0$. With the further increase in $f$, the non-zero probability $p(\alpha)$ for $\alpha < \overline{\alpha}$ emerges because a substantial fraction of escape events is recorded for small values of the stability index $\alpha$. For $f = 0.6$ and $f = 3.1$, most particles escape with extreme values of $\alpha$, i.e., $\alpha \approx \alpha_{\text{min}}$ or $\alpha \approx \alpha_{\text{max}}$; however, for $f = 0.6$, the $p(\alpha)$ still has discontinuity at $\alpha = \overline{\alpha}$. Finally, for large $f$, for example, $f = 3.1$, the $p(\alpha)$ density is almost symmetric along $\alpha = 1$. Nevertheless, for $f$ which is not large enough, $p(\alpha)$ densities are skewed into the direction of $\Delta \alpha$. The change in $\Delta \alpha$ from $\Delta \alpha = 1$ to $\Delta \alpha = -1$ reflects the $p(\alpha)$ density along the $\alpha = \overline{\alpha} = 1$ line.

The $p(\alpha)$ distribution with $f \to \infty$ approaches

$$p_\infty(\alpha) = \frac{2}{\pi \sqrt{\Delta \alpha^2 - 4(\alpha - \overline{\alpha})^2}},$$

which is of an analogous shape such as the $p(v)$ and $p(x)$ distributions in the Lévy walk scenario in the parabolic potential [83]. The density given by equation (8) is of

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the arcsine type and it is normalized on \( \alpha \in [\alpha_{\text{min}}, \alpha_{\text{max}}] = [\overline{\alpha} - \Delta \alpha/2, \overline{\alpha} + \Delta \alpha/2] \). If one knows the first passage time density \( p(\tau) \), using the transformation of variables, it is possible to obtain the \( p(\alpha) \) distribution. From the numerical simulation (results not shown), we see that for \( f \) large enough \( \tau \mod T \) is approximately uniform on the \([0, T]\) interval. Consequently, the argument of the sine function in equation (5) is uniformly distributed over the \([0, 2\pi]\) interval and \( p(\alpha) \) approaches the arcsine distribution \( p_\infty(\alpha) \) (see equation (8)). From equation (8), it is possible to calculate \( \langle T(\alpha) \rangle_{\text{arcsine}}, \) i.e.,

\[
\langle T(\alpha) \rangle_{\text{arcsine}} = \int_{\alpha_{\text{min}}}^{\alpha_{\text{max}}} p_\infty(\alpha) T(\alpha) d\alpha,
\]

which is marked with a blue dash-dotted line in figure 2. Moreover, using the distribution \( p(\alpha) \) at the first passage time, one can calculate

\[
\langle T(\alpha) \rangle_{p(\alpha)} = \int_{\alpha_{\text{min}}}^{\alpha_{\text{max}}} p(\alpha) T(\alpha) d\alpha.
\]

In the limit of \( f \to \infty \), one could expect that \( T \to \mathcal{T}(\langle \alpha \rangle) \), but actually one sees that \( T \) is closer to \( \langle T(\alpha) \rangle_{\text{arcsine}} \) (see equations (8) and (9)). Therefore, the asymptotic properties of escape from finite intervals induced by the \( \alpha \)-stable noise with the time-dependent stability index are very different from the asymptotic properties of RA \([17]\). In \([17]\), it has been shown that in the \( f \to \infty \) limit, the MFPT over a periodically modulated potential barrier is equal to the MFPT over the potential averaged over the period of modulation. Here, it is the other way round as \( \alpha \) averaged over \( p_\infty(\alpha) \) is equal to \( \overline{\alpha} \), which gives the \( f = 0 \) limit. Such an approximate limiting behavior for \( f \to \infty \) arises due to the general properties of the escapes induced by the \( \alpha \)-stable noise. Under the \( \alpha \)-stable driving, the most probable escape scenario is the escape via a single long jump. As can be seen from figure 4, a significant fraction of particles waits for an extreme jump and then escapes with a fixed, instantaneous, value of the stability index \( \alpha \). Therefore, a sequence of escapes with various values of \( \alpha \) is recorded. The level of agreement between \( \mathcal{T} \) and \( \langle T(\alpha) \rangle_{\text{arcsine}} \) depends on the system parameters. Interestingly, using the numerically estimated \( p(\alpha) \) distribution, it is possible to calculate \( \langle T(\alpha) \rangle_{p(\alpha)} \). In figure 2, these estimates are plotted with red solid (\( \Delta \alpha = -1 \)) and black solid (\( \Delta \alpha = 1 \)) curves. They nicely follow the results of computer simulations (points). The agreement is most likely coincidental because \( \langle T(\alpha) \rangle_{p(\alpha)} \) corresponds to the evolution with the fixed \( \alpha \) distributed according to \( p(\alpha) \). Consequently, the waiting scenario for the final jump is different from the actual one. Here, similar to distributed order fractional derivatives \([73, 75, 76]\), the stability index \( \alpha \) changes in time, but these changes are deterministic.

The distribution of the stability index \( \alpha \) at the first passage time \( p(\alpha) \) can be contrasted with the distribution \( p_i(\alpha) \) of instantaneous values of the stability index \( \alpha \) prior to the escape \( p_i(\alpha) = \langle \delta(\alpha - \alpha(t)) \rangle \). If the first passage time is long enough, such a distribution also tends to the arcsine distribution given by equation (8), but this time the convergence rate is much faster because every trajectory adds a whole ensemble of \( \alpha \). Therefore, in contrast to the \( p(\alpha) \) distribution, the arcsine distribution can be recorded for finite frequencies also. For instance, for the setup studied in figure 2, for \( f > 0.2 \), the \( p_i(\alpha) \) distribution is already indistinguishable from the arcsine distribution (see equation (8)). From figure 3, it is clearly visible that the value of the stability index
\(\alpha\) at the escape time follows a different pattern than the during-the-motion distribution of the instantaneous values of the stability index. Nevertheless, in both cases, i.e., for \(p(\alpha)\) and \(p_l(\alpha)\), the same asymptotic density is reached in the \(f \to \infty\) limit. One can conclude that the instantaneous value of the stability index \(\alpha\) at the first escape does not need to be the most probable value of the stability index which is recorded during the motion.

Figure 4 extends the examination of the properties of the escape scenarios performed in figure 2. It shows \(p(x_{\text{last}})\), i.e., histograms of the last visited points \((x_{\text{last}})\) before leaving the \([-1,1]\) interval. It clearly indicates that a significant fraction of particles escapes from the initial point, i.e., from \(x_{\text{last}} = 0\), while the majority of particles approach absorbing boundaries at \(\pm l\). The \(p(x_{\text{last}})\) distribution is symmetric along \(x_{\text{last}} = 0\), reflecting the symmetry of the noise. The observed distributions of the last hitting points are very similar to the one recorded for fixed \(\alpha\) in [84].

Figure 5 completes the exploration of the setup studied in figure 2 (\(\alpha \in [0.5,1.5]\), \(\ell = 1\) and \(\Delta \alpha = 1\)). It presents the survival probability \(S(t)\), i.e., the probability that at time \(t\), a particle is still in the \([-1,1]\) interval, for \(f = 0.2\) (black dashed line) and \(f = 3.1\) (red solid line). The survival probability is a decaying function of time because, with the increasing time, the chances of finding a particle in the domain of motion diminish. Furthermore, the survival probability displays a typical exponential trend which can be decorated by some bending due to the modulation in \(\alpha\). Such a bending is especially well visible for small values of frequencies, for example, \(f = 0.2\) (black dashed line).

In figure 6, the parameter \(\ell\) is set to \(\ell = 2.5\) consequently for all recorded values of \(\alpha\) \((\alpha \in [0.5, 1.5])\) \(\mathcal{T}'(\alpha) > 0\) (see the orange domain in the top panel of figure 1). Therefore, in comparison to \(\ell = 1\), the increase in \(\ell\) from 1 to 2.5 exchanges the monotonicity of the MFPT curves (see figures 2 and 6). The dashed line in figure 6 shows \(\mathcal{T}(\alpha)\) while the blue dash-dotted line \(\langle \mathcal{T}(\alpha) \rangle_{\text{arcsine}}\). The additional solid black and red curves show \(\langle \mathcal{T}(\alpha) \rangle_{\text{ap}}\). This time, the level of agreement between the results of the computer simulations and the \(\langle \mathcal{T}(\alpha) \rangle_{\text{ap}}\) approximation is worse than in figure 2. The decrease of agreement is produced by the dynamics prior to the last escape. The increase in \(\ell\) from 1 to 2.5 increases the MFPT with fixed \(\alpha\) (see equation (4)) by 2.5\(^n\) times. Consequently, the escape process is slower and the particles have more time to diffuse, making the integration of equation (4) over \(p(\alpha)\) not fully reliable. For \(\Delta \alpha = 1\), in the limit of \(f \to \infty\) the MFPT approaches \(\langle \mathcal{T}(\alpha) \rangle_{\text{arcsine}}\).

Finally, in figure 7, the model parameters are adjusted in such a way that \(\mathcal{T}'(\alpha)\) changes its sign during the periodic modulation of \(\alpha\). We have used two sets of parameters: (i) \(\ell = 1.5\), \(\alpha_{\text{min}} = 0.5\), \(\alpha_{\text{max}} = 1.5\) with \(\alpha(0) = 1\) (top panel) and (ii) \(\ell = 2\), \(\alpha_{\text{min}} = 1.0\), \(\alpha_{\text{max}} = 2.0\) with \(\alpha(0) = 1.5\) (bottom panel). In (i) \(\mathcal{T}'(0.973) \approx 0\) while in (ii) \(\mathcal{T}'(1.479) \approx 0\) and \(\alpha(0)\) in both cases lie in the domain where \(\mathcal{T}'(\alpha(0)) < 0\). Therefore, for \(\Delta \alpha = 1\), with the increasing \(\alpha\) the \(\mathcal{T}'(\alpha)\) changes its sign from negative to positive. Such an initial condition allows for the rapid decay of the MFPT with very slowly increasing \(\alpha\), i.e., for small \(f\). A further increase in the frequency is sufficient to move \(\alpha\) to the value for which \(\mathcal{T}'(\alpha)\) becomes positive, which in turn increases the MFPT. Therefore, in the situation when \(\mathcal{T}'(\alpha)\) changes its sign, in addition to the maximum of MFPT, there is a local, narrow minimum at a small \(f\) (\(f \approx 0.1\)). The change of sign in \(\Delta \alpha\) from \(\Delta \alpha = 1\) to \(\Delta \alpha = -1\), analogously as in figures 2 and 6, inverts the shape of the MFPT curves, i.e.,
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there is a narrow maximum at small $f$ and a wide minimum at larger $f$. Finally, in the limit of $f \to \infty$, $\mathcal{T}$ approaches $\langle \mathcal{T}(\alpha) \rangle_{\text{arcsine}}$, but for $\ell = 2$ (bottom panel of figure 7), the agreement is better. Furthermore, as can be seen from figure 7, the MFPT estimated using equation (10) significantly deviates from the numerically obtained values of the MFPTs. This discrepancy originates in the fact that, for the setup studied in figure 7, the $p(\alpha)$ density attains the arcsine $p_\infty(\alpha)$ shape already for finite $f$.

4. Summary and conclusions

Resonant activation is a phenomenon manifesting the constructive role of fluctuations. RA is a generic effect for barrier crossing events in the conformally modulated energy landscape [3, 17], i.e., for a fine-tuned rate of periodic or dichotomous modulation of the potential barrier, the minimal average escape time is observed.

Here, we have studied the properties of the minimal setup allowing for occurrence of RA. In comparison to the typical models, we have reduced the number of elements by eliminating the deterministic force. The model of Lévy noise-induced escape from the finite intervals is capable of revealing the phenomenon of RA if the stability parameter of the noise is periodically modulated. This model generalizes Lévy flights to the situation when jumps are still independent but are no longer identically distributed. The variability of the exponent characterizing the jump length distribution is responsible for the occurrence of noise-induced effects. Here, as in the RA phenomenon, there is such a value of the modulating frequency, for which the MFPT is minimal. The model can display not only the RA but also the noise-enhanced stability because the MFPT can be not only decreased but also increased. The model itself displays sensitivity to the initial direction in the modulation of the stability index $\alpha$, as it inverts the shape of the MFPT curves. At the same time, the survival probabilities follow exponential decay and the distribution of the instantaneous values of the stability index $\alpha$ at first passage time are skewed. The direction of asymmetry is determined by the initial monotonicity of modulation. Finally, nontrivial asymptotic behavior is recorded, i.e., in the high frequency limit, the recorded MFPT does not correspond to the MFPT with an average value of the stability index $\alpha$.

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Data availability

The data that support the findings of this study are available from the corresponding author (K C) upon reasonable request.

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Appendix. Numerical methods for equations driven by $\alpha$-stable noise

The problem of escape from a finite interval induced by $\alpha$-stable noises might be studied by Monte Carlo methods. The Langevin equation

$$\frac{dx}{dt} = \zeta(t) \quad (A.1)$$

can be discretized [80] into

$$x(t + \Delta t) \approx x(t) + \zeta^{(i)} \Delta t^{1/\alpha}, \quad (A.2)$$

where $\zeta^{(i)}$ is the sequence of independent and identically distributed random variables following the (symmetric) $\alpha$-stable density with fixed $\alpha$, i.e., the probability density with the characteristic function

$$\phi(k) = \exp[-\sigma^\alpha |k|^\alpha]. \quad (A.3)$$

Symmetric $\alpha$-stable random variables can be generated using the formula [81, 82]

$$\zeta = \sigma \frac{\sin(\alpha V)}{(\cos V)^{1/\alpha}} \left[\frac{\cos(V - \alpha V)}{W}\right]^{(1-\alpha)/\alpha}, \quad (A.4)$$

where $V$ and $W$ are independent random variables. $V$ is uniformly distributed on $(-\pi/2, \pi/2)$, while $W$ follows the exponential distribution with the unit mean. In case of escape from the finite intervals, trajectories are generated using equation (A.2) as long as condition $|x(t)| < l$ is fulfilled. Equations (A.1) and (A.2) can be generalized to situations where the stability index $\alpha$ is no longer constant, but it changes over time, i.e., $\alpha = \alpha(t)$

$$\frac{dx}{dt} = \zeta_{\alpha(t)}(t) \quad (A.5)$$

and

$$x(t + \Delta t) \approx x(t) + \zeta^{(i)}_{\alpha(t)} \times \Delta t^{1/\alpha(t)}, \quad (A.6)$$

where $\zeta^{(i)}_{\alpha(t)}$ is the sequence of independent (symmetric) $\alpha$-stable random variables with $\alpha(t) = \alpha(i \times \Delta t)$ (see equation (A.3)). Writing out equation (A.6) one has the recurrence

$$\begin{cases} x_0 = 0 \\ x_{n+1} = x_n + \zeta^{(i)}_{\alpha(t_i)} \times \Delta t^{1/\alpha(t_i)} \end{cases} \quad (A.7)$$

where $t_i = i \times \Delta t$ and $x_i = x(t_i) = x(i \times \Delta t)$. Therefore, by the construction, increments $x(t + \Delta t) - x(t)$ of the studied process are independent and they follow the symmetric $\alpha$-stable density with the time-dependent stability index $\alpha = \alpha(t)$, for example, see equations (5) and (A.3). Equations (A.2) and (A.6), accompanied by initial and boundary conditions, can be used to generate multiple trajectories. From an ensemble of trajectories, it is possible to obtain the required properties of non-stationary processes.
Deterministic force-free resonant activation defined by equations (A.1) or (A.5). Furthermore, in contrast to equation (A.1), the process defined by equation (A.5) is not of the Lévy type [55] as its increments are non-stationary. Finally, \( \alpha \)-stable random variables with \( \alpha < 2 \) are characterized by diverging variance, consequently some characteristics of the processes generated by equations (A.1) or (A.5) are not well defined. In particular, for such a process, it is not possible to calculate the (standard) autocorrelation.

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