The Pointwise Stabilities of Piecewise Linear Finite Element Method on Non-obtuse Tetrahedral Meshes of Nonconvex Polyhedra

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Abstract
Let \( \Omega \) be a Lipschitz polyhedral (can be nonconvex) domain in \( \mathbb{R}^3 \), and \( V_h \) denotes the finite element space of continuous piecewise linear polynomials. On non-obtuse quasi-uniform tetrahedral meshes, we prove that the finite element projection \( R_h u \) of \( u \in H^1(\Omega) \cap C(\bar{\Omega}) \) (with \( R_h u \) interpolating \( u \) at the boundary nodes) satisfies

\[
\| R_h u \|_{L^\infty(\Omega)} \leq C |\log h| \| u \|_{L^\infty(\Omega)}.
\]

If we further assume \( u \in W^{1,\infty}(\Omega) \), then

\[
\| R_h u \|_{W^{1,\infty}(\Omega)} \leq C |\log h| \| u \|_{W^{1,\infty}(\Omega)}.
\]

Keywords The stability in \( L^\infty \) and \( W^{1,\infty} \). Finite element method · Nonconvex polyhedra

Mathematics Subject Classification 65N30 · 65L12

1 Introduction

In this paper we consider the the Ritz projection \( R_h u \in V_{r,h} \) of \( u \in H^1(\Omega) \cap C(\bar{\Omega}) \) satisfying

\[
(\nabla R_h u, \nabla v_h)_{\Omega} = (\nabla u, \nabla v_h)_{\Omega}, \quad \forall v_h \in V^0_{r,h},
\]

where \( V_{r,h} \) is the finite element subspace of \( H^1(\Omega) \) composed of piecewise polynomials of degree \( r \ (r \geq 1) \), \( V^0_{r,h} = H^1_0(\Omega) \cap V_{r,h} \), and \( R_h u \) interpolates \( u \) at the boundary nodes on \( \partial \Omega \). In fact, \( R_h u \) is the finite element projection of \( u \) onto \( V_{r,h} \) for the model problem
\[ \Delta u = f \text{ in } \Omega, \]  
with Dirichlet boundary condition on \( \partial \Omega \).

Our motivation is to establish the stability in \( L^\infty(\Omega) \)

\[
\| R_h u \|_{L^\infty(\Omega)} \leq C \| u \|_{L^\infty(\Omega)},
\]  
(1.3a)

or

\[
\| R_h u \|_{L^\infty(\Omega)} \leq C |\log h| \| u \|_{L^\infty(\Omega)};
\]  
(1.3b)

and the stability in \( W^{1,\infty}(\Omega) \) (if \( u \in W^{1,\infty}(\Omega) \))

\[
\| R_h u \|_{W^{1,\infty}(\Omega)} \leq C \| u \|_{W^{1,\infty}(\Omega)},
\]  
(1.4a)

or

\[
\| R_h u \|_{W^{1,\infty}(\Omega)} \leq C |\log h| \| u \|_{W^{1,\infty}(\Omega)};
\]  
(1.4b)

There are a lot of important works for estimates (1.3) and (1.4). [9,16] are the first contributions for general quasi-uniform meshes. On convex polygonal domains, [9] considered piecewise linear (\( r = 1 \)) approximation while [16] treated the finite element approximation (for any \( r \geq 1 \)) to Neumann problem of (1.2). Schatz [13] proved (1.3) and (1.4) on polygonal (can be nonconvex) domains. When \( r = 1 \), the estimates provided in [13] are (1.3b) and (1.4b). Thus, estimates (1.3) and (1.4) are valid for most practical domains in \( \mathbb{R}^2 \).

On the contrast, in three dimensional space, all existing works [1,3,4,7,8,11,12,14,15] for estimates (1.3) and (1.4) are available on either domains with smooth boundary or convex polyhedral domains (Instead of explicit assumptions on domains, Brenner and Scott [1] needs \( \| w \|_{W^{2,p}(\Omega)} \leq C \| \Delta w \|_{L^p(\Omega)} \) for some \( p > 3 \) in three dimensional space, for any function \( W \) with zero trace on \( \partial \Omega \)).

In this paper, we prove that if the meshes are non-obtuse (all internal dihedral angles of all tetrahedral elements are less than or equal to \( \frac{\pi}{2} \), then estimates (1.3b) and (1.4b) hold for the finite element projection (1.1) with piecewise linear finite element space \( V_h = V_{1,h} \) (\( r = 1 \)).

In Sect. 2, we provide the main results and all assumptions. In Sect. 3, we show the proofs of our main results.

## 2 Main results

Let \( \Omega \) be a Lipschitz polyhedra (can be nonconvex) in \( \mathbb{R}^3 \). We denote by \( T_h \) quasi-uniform conforming tetrahedral meshes of \( \Omega \). We define \( V_h = H^1(\Omega) \cap P_1(T_h) \) and \( V^0_h = H^1_0(\Omega) \cap V_h \).

For any \( u \in H^1(\Omega) \), we introduce the Ritz projection \( R_h u \in V_h \) to satisfy

\[
(\nabla R_h u, \nabla v_h)_\Omega = (\nabla u, \nabla v_h)_\Omega, \quad \forall v_h \in V^0_h,
\]  
(2.1)

where \( R_h u \) interpolates \( u \) at the boundary nodes on \( \partial \Omega \). In fact, \( R_h u \) is the finite element projection of \( u \) onto \( V_h \), and (2.1) is exactly the finite element projection (1.1) with \( r = 1 \).

**Assumption 2.1** For any \( T \in T_h \), all internal dihedral angles of the tetrahedral element \( T \) are less than or equal to \( \frac{\pi}{2} \). \( T_h \) is called non-obtuse tetrahedral meshes of \( \Omega \).

**Assumption 2.2** The mesh \( T_h \) of \( \Omega \) can be extended to a larger convex domain \( \hat{\Omega} \) quasi-uniformly with \( \Omega \subseteq \hat{\Omega} \). We denote by \( \tilde{T}_h \) the extension of \( T_h \) on \( \hat{\Omega} \).

**Remark 2.1** We don’t require \( \tilde{T}_h \) introduced in Assumption 2.2 to be non-obtuse for all tetrahedral elements. Only elements \( T \in T_h \) need to be non-obtuse.
Theorem 2.2 If Assumptions (2.1) and (2.2) hold, then there is a positive constant $C$ such that for any $u \in H^1(\Omega) \cap C(\Omega)$,

$$\|R_h u\|_{L^\infty(\Omega)} \leq C |\log h| \|u\|_{L^\infty(\Omega)}.$$  

Theorem 2.3 If Assumptions (2.1) and (2.2) hold, then there is a positive constant $C$ such that for any $u \in W^{1,\infty}(\Omega)$,

$$\|R_h u\|_{W^{1,\infty}(\Omega)} \leq C |\log h| \|u\|_{W^{1,\infty}(\Omega)}.$$  

3 Analysis

Proof (Proof of Theorem 2.2) Since $u \in C(\Omega)$, we denote by $\tilde{u}$ the extension of $u$ to $\tilde{\Omega}$, such that $u \in C_0(\tilde{\Omega})$ and $\|\tilde{u}\|_{L^\infty(\tilde{\Omega})} = \|u\|_{L^\infty(\Omega)}$. The existence of $\tilde{u}$ satisfying the above two properties follows from the facts that $u \in C(\tilde{\Omega})$ and the Whitney type extension operator $E_0$ in Section 2.2 of Chapter 6 in [17] (see (8) and the proposition in Section 2.2 of Chapter 6 in [17]). We would like to emphasize that we don’t need $\tilde{u} \in H^1(\tilde{\Omega})$.

We define $V_h^0 = H_0^1(\Omega) \cap P_1(T_h)$. Let $\tilde{u}_h \in V_h^0$ satisfy

$$(\nabla \tilde{u}_h, \nabla \tilde{v}_h)_\Omega = \Sigma_{T \in T_h} (-(\tilde{u}, \Delta \tilde{v}_h)_T + \langle \tilde{u}, \nabla \tilde{v}_h \cdot n \rangle_{\partial T}), \quad \forall \tilde{v}_h \in V_h^0. \quad (3.1)$$

Here $n$ is the outward unit normal vector along $\partial T$ for any $T \in T_h$. For any $v_h \in V_h^0 = H_0^1(\Omega) \cap P_1(T_h)$, we denote by $\tilde{v}_h \in V_h^0$ the zero extension of $v_h$ to $\tilde{\Omega}$. By (3.1) and the definition of $V_h^0$, it is easy to see that

$$(\nabla \tilde{u}_h, \nabla v_h)_\Omega = (\nabla \tilde{u}_h, \nabla \tilde{v}_h)_\Omega$$

$$= \Sigma_{T \in T_h} (-(\tilde{u}, \Delta \tilde{v}_h)_T + \langle \tilde{u}, \nabla \tilde{v}_h \cdot n \rangle_{\partial T})$$

$$= \Sigma_{T \in T_h} (-(u, \Delta v_h)_T + \langle u, \nabla v_h \cdot n \rangle_{\partial T}) = (\nabla u, \nabla v_h)_\Omega. \quad (3.2)$$

The last equality holds since $u \in H^1(\Omega)$. On the other hand, since $\tilde{\Omega}$ is convex and $\tilde{u} \in C_0(\tilde{\Omega})$, (3.1) and [8, Theorem 12] imply that

$$\|\tilde{u}_h\|_{L^\infty(\tilde{\Omega})} \leq C |\log h| \|\tilde{u}\|_{L^\infty(\tilde{\Omega})} = C |\log h| \|u\|_{L^\infty(\Omega)}. \quad (3.3)$$

We notice that $R_h u \in V_h = H^1(\Omega) \cap P_1(T_h)$ satisfies

$$(\nabla R_h u, \nabla v_h)_\Omega = (\nabla u, \nabla v_h)_\Omega, \quad \forall v_h \in V_h^0 = H_0^1(\Omega) \cap V_h.$$  

Thus, by the above equation and (3.2), we have that $(R_h u - \tilde{u}_h) \mid_{\Omega} \in V_h$ and

$$\langle \nabla (R_h u - \tilde{u}_h), \nabla v_h \rangle_{\Omega} = 0, \quad \forall v_h \in V_h^0 = H_0^1(\Omega) \cap V_h.$$  

By Assumption (2.1) and [18, Theorem 3.2 and Lemma 5.1(iii)] (or by [2,5,6]), the above equation implies that

$$\|R_h u - \tilde{u}_h\|_{L^\infty(\Omega)} \leq \|R_h u - \tilde{u}_h\|_{L^\infty(\partial \Omega)} \leq \|u\|_{L^\infty(\partial \Omega)} + \|\tilde{u}_h\|_{L^\infty(\partial \Omega)}. \quad (3.4)$$
Thus, by (3.3) and (3.4), it is easy to see that
\[
\| R_h u \|_{L^\infty(\Omega)} \leq \| R_h u - \tilde{u}_h \|_{L^\infty(\Omega)} + \| \tilde{u}_h \|_{L^\infty(\Omega)} \\
\leq \| u \|_{L^\infty(\tilde{\Omega})} + \| \tilde{u}_h \|_{L^\infty(\tilde{\Omega})} + \| \tilde{u}_h \|_{L^\infty(\Omega)} \\
\leq \| u \|_{L^\infty(\Omega)} + 2 \| \tilde{u}_h \|_{L^\infty(\Omega)} \leq C | \log h | \| u \|_{L^\infty(\Omega)}.
\]

The proof is complete. \(\square\)

**Proof** (Proof of Theorem 2.3) We denote by \( I_h u \) the standard interpolation of \( u \) on \( V_h = H^1(\Omega) \cap P_1(T_h) \).

By applying Theorem 2.2 to \( u - I_h u \), we have
\[
\| R_h u - I_h u \|_{L^\infty(\Omega)} \leq C | \log h | \| u - I_h u \|_{L^\infty(\Omega)}.
\]

By inverse inequality and approximation properties of \( I_h \),
\[
\| R_h u \|_{W^{1,\infty}(\Omega)} \leq \| R_h u - I_h u \|_{W^{1,\infty}(\Omega)} + \| I_h u \|_{W^{1,\infty}(\Omega)} \\
\leq C h^{-1} \| R_h u - I_h u \|_{L^\infty(\Omega)} + C \| u \|_{W^{1,\infty}(\Omega)} \leq C \| u \|_{W^{1,\infty}(\Omega)}.
\]

The proof is complete. \(\square\)

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**Declarations**

**Conflict of interest** No conflict of interest exists.

**Code availability** Not applicable.

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