Skew Brownian motion with dry friction: joint density approach

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Abstract

This note concerns the distribution of Skew Brownian motion with dry friction and its occupation time. These distributions were obtained in [2] by using the Laplace transform and joint characteristic functions. We provide an alternative approach, which is based on the use of the joint density for Skew Brownian motion, its last visit to the origin, its local and occupation times derived in [3].

Keywords: Skew Brownian motion, Caughey-Dienes process, local time, occupation time

1 Introduction

Let \( X_t = (X_t, t \geq 0) \) be a continuous time stochastic process defined as a solution of the following stochastic differential equation

\[
X_t = X_0 + \int_0^t m(X_s)ds + (2p - 1)L_t + W_t, \tag{1}
\]

where \( p \in (0, 1), \)

\[
m(x) = m_11_{\{x \geq 0\}} + m_21_{\{x < 0\}}, \quad x \in \mathbb{R}, \tag{2}
\]

for some constants \( m_1 \) and \( m_2, \)

\[
L_t = \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_0^t 1_{\{-\varepsilon \leq X_s \leq \varepsilon\}}ds \tag{3}
\]

is the local time of the process at zero, and \( (W_t, t \geq 0) \) is standard Brownian motion (BM). The process \( X_t \) is well known. If \( p = \frac{1}{2} \) and \( m_1 = m_2 = 0, \) then \( X_t \) is standard BM. If \( m_1 = m_2 = 0, \) then it is Skew Brownian motion (SBM) with parameter \( p \) (e.g. see the survey [4] and references therein). The distribution of the process \( X_t \) and its functionals (e.g. local and occupation times etc) is of great interest in applications and attracted the attention of many researchers. For example, the trivariate density of BM, its local and occupation time was obtained in [5] and

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applied to problems of stochastic control. The distribution of the process \( X_t \), its local and occupation time was obtained in [1] in the case \( m_1 = m_2 \) and used to explain results of some laboratory experiments for an advection-dispersion phenomenon. The process \( X_t \) with the piecewise linear drift (2) naturally appeared in [3] in the study of a two-valued local volatility model. The latter is a generalisation of the log-normal model for the underlying price on the case when the volatility of the price can take two different values. The joint density of the process, its local and occupation times, and the last visit to the origin was obtained in [3] in an exact analytical form. This result was applied in that paper to generalize the Black-Scholes formula for the option price on the case of the two-valued volatility.

In the case \( m_2 = -m_1 = m \) the process \( X_t \) is also known as the skew Caughey-Dienes process, or SBM with dry friction (e.g., see [2] and references therein). Densities of both the skew Caughey-Dienes process and its occupation time on the non-negative half-line were derived in [2] by using the Laplace transform and joint characteristic functions, which requires rather heavy computations. In this note we show how these distributions can be alternatively obtained by using the results of [3].

### 2 Results

#### 2.1 The distribution of the skew Caughey-Dienes process

In this section we derive the distribution of the skew Caughey-Dienes process. This process is the solution of equation (1) in the special case

\[
dX_t = -m \cdot \text{sgn}(X_t) + (2p - 1) dL_t + dW_t
\]

(4)

**Theorem 1.** Let \( X_t = (X_t, t \geq 0) \) be the solution of equation (1). If \( X_0 = 0 \), then given \( T > 0 \) the density of \( X_T \) is

\[
\phi_T(x) = \begin{cases} 
2p \left( e^{\frac{(mT+x)^2}{2T}} + \frac{m}{T}e^{-2mx} \left[ 1 + \text{Erf} \left( \frac{mT+x}{\sqrt{2T}} \right) \right] \right), & \text{if } x \geq 0, \\
2q \left( e^{\frac{-(mT+x)^2}{2T}} + \frac{m}{T}e^{2mx} \left[ 1 + \text{Erf} \left( \frac{mT+x}{\sqrt{2T}} \right) \right] \right), & \text{if } x < 0,
\end{cases}
\]

(5)

where \( q = 1 - p \) and \( \text{Erf}(z) \) is the standard error function.

**Proof.** Fix \( T > 0 \) and define the following quantities

\[
\tau = \max \{ t \in (0, T] : X_t = 0 \},
\]

(6)

\[
V = \int_{\tau_0}^{\tau} 1_{\{X_t \geq 0\}} dt,
\]

(7)

where \( \tau_0 = \min \{ t : X_t = 0 \} \), and the function

\[
\psi_{p,T}(t, v, x, l) = \begin{cases} 
2p \cdot h(v, lp)h(t - v, lq)h(T - t, x), & \text{if } x \geq 0, \\
2q \cdot h(v, lp)h(t - v, lq)h(T - t, x), & \text{if } x < 0,
\end{cases}
\]

(8)
for $0 \leq v \leq t \leq T, l \geq 0$, where
\[ h(s, y) = \frac{|y|}{\sqrt{2\pi s^3}} e^{-\frac{y^2}{2s}}, \quad y \in \mathbb{R}, s \in \mathbb{R}^+, \] (9)
is the density of the first passage time to zero of the standard BM starting at $y$. It was shown in [3, Theorem 2] that, if $X_0 = 0$, then the joint density of $(\tau, V, X_T, L_T)$ is given by
\[ \phi_T(t, v, x, l) = \psi_{p,T}(t, v, x, l)e^{-\frac{m_2^2+3(T-v)}{2}-l(m_1p-qm_2)+m(x)x}, \] (10)
where $\psi_{p,T}(t, v, z, l)$ is defined by equation (8). If $m_1 = -m_2 = -m$, then the function (10) simplifies as follows
\[ \phi_T(t, v, x, l) = \psi_{p,T}(t, v, x, l)e^{-m_2T+m(x)x+l}, \] (11)
for $0 \leq v \leq t \leq T, l \geq 0$. Using the convolution property of the hitting times of Brownian motion and the fact that $p + q = 1$, we have that
\[ \int_0^t h(v, lp)h(t-v, lq)dv = h(t, l), \]
which gives the joint density of $(\tau, X_T, L_T)$, namely,
\[ \phi_T(t, x, l) = \int_0^t \phi_T(t, v, x, l)dv = \begin{cases} 2p \cdot h(t, l)h(T-t, x)e^{-\frac{m_2^2}{2}-m(x)x+l}, & \text{if } x \geq 0, \\ 2q \cdot h(t, l)h(T-t, x)e^{-\frac{m_2^2}{2}+m(x)x+l}, & \text{if } x < 0, \end{cases} \] (12)
Using the convolution property of the hitting times again gives that
\[ \int_0^T h(t, l)h(T-t, x)dt = h(T, l + |x|). \]
Therefore, the joint density of $X_T$ and $L_T^{(0)}$ is as follows
\[ \phi_T(x, l) = \begin{cases} 2p \cdot h(T, l+x)e^{-\frac{m_2^2}{2}-m(x)x+l}, & \text{if } x \geq 0, \\ 2q \cdot h(T, l-x)e^{-\frac{m_2^2}{2}+m(x)x+l}, & \text{if } x < 0, \end{cases} \] (13)
It is left to integrate out the local time in order to obtain the density of $X_T$. Integration gives that
\[ \int_0^\infty \phi_T(x, l)dl = \phi_T(x), \]
where $\phi_T(x)$ is the function defined in (5), as claimed.
2.2 The distribution of the occupation time

Let \( X_t = (X_t, t \geq 0) \) be the solution of equation \( (1) \). Given \( T > 0 \) define

\[
U = \int_0^T 1_{\{X_t \geq 0\}} \, dt,
\]

i.e. \( U \) is the occupation time of the non-negative half-line during the time period \([0, T]\) (the occupation time). In [2] the density of the occupation time is expressed in term of a double integral of a rather complicated function. We show that this density can be obtained as an integral of a function of one variable, which is explicitly expressed in terms of the complementary error function \( \text{Erfc}(z) = 1 - \text{Erf}(z) \).

Note first that if \( X_0 = 0 \), then \( U = V + T - \tau \), if \( X_T \geq 0 \), and \( U = V \), if \( X_T < 0 \), where quantities \( \tau \) and \( V \) are defined in (6) and (7) respectively. Therefore, the joint density \( \varphi_T(t, u, x, l) \) of \((\tau, U, X_T, L_T)\) is (see equation (15) in [3])

\[
\varphi_T(t, u, x, l) = \begin{cases} 
2p \cdot h(u + t - T, lp) h(T - u, lq) h(T - t, x) e^{-\frac{u^2 T}{2} - x \cdot m + l \cdot m}, & \text{if } x \geq 0, \ l > 0, \ \text{and } t \leq T, \ T - t \leq u \leq T; \\
2q \cdot h(u, lp) h(t - u, lq) h(T - t, x) e^{-\frac{T^2}{2} + x \cdot m + l \cdot m}, & \text{if } x < 0, \ l > 0, \ \text{and } 0 \leq u \leq t \leq T.
\end{cases}
\]

(15)

Using that

\[
\int_{T-u}^T h(u + T + t, lp) h(T - t, x) \, dt = h(u, lp + x) \quad \text{for } x \geq 0,
\]

\[
\int_u^T h(t - u, lq) h(T - t, x) \, dt = h(T - u, lq + |x|) \quad \text{for } x < 0,
\]

we obtain the joint density \( \varphi_T(u, x, l) \) of \((U, X_T, L_T)\). Integrating over the variable \( x \) gives the joint density of the occupation time \( U \) and the local time \( L_T \), that is

\[
\varphi_T(u, l) = 2e^{-\frac{u^2 T}{2} + lm} (pF(u, l, p) + qF(T - u, l, q)),
\]

(16)

where

\[
F(y, l, c) = \frac{e^{-\frac{y^2 l}{2\pi y}}}{\sqrt{2\pi y}} = \frac{m}{2} \text{Erfc} \left[ \frac{lc + my}{\sqrt{2y}} \right] e^{lm c + \frac{m^2}{2}}.
\]

(17)

Thus, the density of the occupation time is given by the integral \( \varphi_T(u) = \int_0^\infty \varphi_T(u, l) \, dl \), where the function \( \varphi_T(u, l) \) is explicitly expressed in terms of the complementary error function, as claimed.
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