Boutet de Monvel operators on Lie manifolds with boundary

Karsten Bohlen

Abstract

We introduce and study a general pseudodifferential calculus for boundary value problems on a class of non-compact manifolds with boundary (so-called Lie manifolds with boundary). This is accomplished by constructing a suitable generalization of the Boutet de Monvel calculus for boundary value problems. The data consists of a compact manifold with corners $M$ that is endowed with a Lie structure of vector fields $\mathcal{V}$, a so-called Lie manifold. The manifold $M$ is split into two equal parts $X_+$ and $X_-$ which intersect in an embedded hypersurface $Y \subset X_{\pm}$. Our goal is to describe a transmission Boutet de Monvel calculus for boundary value problems compatible with the structure of Lie manifolds. Starting with the example of $b$-vector fields, we show that there are two groupoids integrating the Lie structures on $M$ and on $Y$, respectively. These two groupoids form a bibundle (or a groupoid correspondence) and, under some mild assumptions, these groupoids are Morita equivalent. With the help of the bibundle structure and canonically defined manifolds with corners, which are blow-ups in particular cases, we define a class of Boutet de Monvel type operators. We then define the representation homomorphism for these operators and show closedness under composition with the help of a representation theorem. Finally, we consider appropriate Fredholm conditions and construct the parametrices for elliptic operators in the calculus.

Keywords: Boutet de Monvel’s calculus, groupoid, Lie manifold.

\textit{email:} bohlen.karsten@math.uni-hannover.de
1. Introduction

In this work we will enlarge the groupoid pseudodifferential calculus introduced in [30] and develop a general notion of a pseudodifferential calculus for boundary value problems in the framework of Lie groupoids. The most natural approach seems to be along the lines of the Boutet de Monvel calculus. Boutet de Monvel’s calculus (e.g. [1]) was established in 1971. This calculus provides a convenient and general framework to study the classical boundary value problems. At the same time parametrices are contained in the calculus and it is closed under composition of elements.

1.1. Overview

In our case, consider the following data: a Lie manifold \((X,\mathcal{V})\) with boundary \(Y\), which is an embedded, transversal hypersurface \(Y \subset X\) in the compact manifold with corners \(X\) and which is a Lie submanifold of \(X\) (cf. \([3], [2]\)). The Lie structure \(\mathcal{V} \subset \Gamma(TX)\) is a Lie algebra of smooth vector fields such that \(\mathcal{V}\) is a subset of the Lie algebra \(\mathcal{V}_0\) of all vector fields tangent to the boundary strata and a finitely generated projective \(C^\infty(X)\)-module. From \(X\) we define the double \(M = 2X\) at the hypersurface \(Y\) which is a Lie manifold \((M,2\mathcal{V})\). The corresponding Lie structure \(2\mathcal{V}\) on \(M\) is such that \(\mathcal{V} = \{V|_X : V \in 2\mathcal{V}\}\). Transversality of \(Y\) in relation to \(M\) means that for each given hyperface \(F \subset M\) we have

\[
T_xM = T_xF + T_xY, \ x \in Y \cap F. \tag{1}
\]

Introduce the following notation for interior and boundary: by \(\partial M\) we mean the union of all hyperfaces of the manifold with corners \(M\),

\[
M_0 := M \setminus \partial M, \ Y_0 := Y \cap M_0, \ X_0 := X \cap M_0 \text{ and } \partial Y := Y \cap \partial M.
\]

For an open hyperface \(F\) in \(M\) we denote by \(\overline{F}\) the closure in \(M\). Denote by \(\partial_{\text{reg}}F = \partial_{\text{reg}}\overline{F} = \overline{F} \cap Y\) the regular boundary of \(F\). The hypersurface \(Y\) is endowed with a Lie structure as in \([2]\):

\[
\mathcal{W} = \{V|_Y : V \in 2\mathcal{V}, V|_Y \text{ tangent to } Y\}. \tag{2}
\]

We make the following assumptions: i) The hypersurface \(Y\) is embedded in \(M\) in such a way that the boundary faces of \(Y\) are in bijective correspondence with the boundary faces of \(M\). This means the map \(\mathcal{F}(M) \ni F \mapsto F \cap Y \in \mathcal{F}(Y)\) should be a bijection, where \(\mathcal{F}(M), \mathcal{F}(Y)\) denotes the boundary faces of \(M\) and \(Y\) respectively. ii) Secondly, we assume that for the given Lie structure \(2\mathcal{V}\) there is an integrating Lie groupoid \(G\) which is Hausdorff, amenable and has the local triviality property: \(G_F \cong F \times F \times G\) for any open face \(F\) of \(M\) (where \(G\) is an isotropy Lie group). Also \(G_{M_0} \cong M_0 \times M_0\) is the pair groupoid on the interior and \(A_{|M_0} \cong TM_0\) the tangent bundle on the interior.

Note that the first assumption poses no loss of generality in the consideration of boundary value problems: We can always consider a small tubular neighborhood of \(Y\) and restrict the groupoid accordingly. Away from such a neighborhood there are no boundary value problems to be considered. The second condition is needed in the parametrix construction at the end of this work. In the main body of the paper we give examples of Lie structures for which it is known that they have integrating Lie groupoids that are Hausdorff, amenable and locally trivial. From now on we fix a Lie manifold with boundary with the notation given above and fulfilling the previous assumptions.

The goal is to construct a Boutet de Monvel calculus for general pseudodifferential boundary value problems adapted to this data. We model our construction on the pseudodifferential calculus on a Lie manifold as described in \([3]\). The authors define a representation of pseudodifferential operators on a Lie groupoid, i.e. a \(*\)-homomorphism from equivariant operators acting on the
groupoid fibers to operators acting on the original manifold with corners. Since the pseudodifferential calculus on a Lie groupoid is closed under composition (cf. 30), a representation theorem automatically yields closedness under composition for the corresponding pseudodifferential calculus on the given Lie manifold. In order to construct a groupoid calculus for boundary value problems we first consider the Lie algebroid $A_{2V} \to M$ such that $\Gamma(A_{2V}) \cong 2V$ using the Serre Swan theorem. We fix a Lie groupoid $\mathcal{G} \rightrightarrows M$ fulfilling our assumptions such that $\mathcal{A}(\mathcal{G}) \cong A_{2V}$. On the boundary Lie structure on $Y$ we also obtain a groupoid $\mathcal{G}_\partial = \mathcal{G}_Y \rightrightarrows Y$ with an associated Lie algebroid $\pi_\partial: \mathcal{A}_\partial \to Y$. We define a boundary structure in the following way: We introduce $\mathcal{X} = \mathcal{G}_Y = r^{-1}(Y)$, which is a longitudinaly smooth space and can be viewed as a desingularization of $Y \times M$ with regard to the diagonal $\Delta_Y$, suitably embedded as manifolds with corners, as well as $\mathcal{X}^t = \mathcal{G}_Y = s^{-1}(Y)$ of $M \times Y$ with regard to $\Delta_Y$. There is a canonical diffeomorphism $f: \mathcal{X} \to \mathcal{X}^t$. Additionally, the spaces $\mathcal{X}$ and $\mathcal{X}^t$ implement a bibundle correspondence between $\mathcal{G}$ and $\mathcal{G}_\partial$ as defined by Hilsum and Skandalis, 16. Since the hypersurface $Y$ divides the double $M = 2X$ we denote by $X := X_+$ the right half and by $X_-$ the left half. These halves have corresponding Lie structures and hence corresponding groupoids $\mathcal{G}_0 \rightrightarrows X_+$. On the symbols of pseudodifferential operators from the groupoid calculus we impose a fiberwise transmission property with regard to the subgroupoids $\mathcal{G}^+, \mathcal{G}^- \subset \mathcal{G}$. The compatibility requirements we will state particularly imply that $\mathcal{G}_0^+, \mathcal{G}_0^-$ have fiberwise boundaries consisting of the fibers $\mathcal{X}_x$ for $x \in X_+$. The Boutet de Monvel operators are defined and adapted to data given by the boundary structure, i.e. the tuple $(\mathcal{G}, \mathcal{G}_\partial, \mathcal{G}^+, \mathcal{X}, \mathcal{X}^t, f)$. The boundary structure depends on the initial Lie structure and integrability properties of the corresponding Lie algebroids.

1.2. Results

The Boutet de Monvel calculus adapted to this data is denoted by $\mathcal{B}^{0,0}(\mathcal{G}^+, \mathcal{G}_\partial)$ (of order 0 and type 0) and consists of matrices of operators, which are equivariant families of operators, such that

$$\mathcal{B}^{0,0}(\mathcal{G}^+, \mathcal{G}_\partial) \subset \text{End} \left( \begin{array}{c} C^\infty_c(\mathcal{G}^+) \\ \oplus \\ C^\infty_c(\mathcal{G}_\partial) \end{array} \right).$$

The first objective of this work is the proof of the following result.

**Theorem 1.1.** Given a Lie manifold $(X, \mathcal{V})$ with embedded hypersurface $Y \subset X$ yielding a Lie manifold $X$ with boundary $Y$ such that $M = 2X$, the double. Then for a pair of associated groupoids $\mathcal{G} \rightrightarrows M$, $\mathcal{G}_\partial \rightrightarrows Y$ adapted to a boundary structure the equivariant transmission Boutet de Monvel calculus is closed under composition. This means that given the order $m \in \mathbb{Z}$ we have

$$\mathcal{B}^{m,0}(\mathcal{G}^+, \mathcal{G}_\partial) \cdot \mathcal{B}^{0,0}(\mathcal{G}^+, \mathcal{G}_\partial) \subseteq \mathcal{B}^{m,0}(\mathcal{G}^+, \mathcal{G}_\partial).$$

In the next step we describe a vector-representation of our algebra. Just as in the case of a pseudodifferential operator on a groupoid there is a homomorphism which maps $\mathcal{B}^{0,0}(\mathcal{G}^+, \mathcal{G}_\partial)$ to an algebra $\mathcal{B}^{0,0}_V(X, Y)$. The first algebra on the left consists of equivariant families on a suitable boundary structure. This algebra on the right hand side is defined to consist of matrices of pseudodifferential, trace, potential and singular Green operators. These operators are extensions from the usual operator calculus on the interior manifold with boundary $(X_0, Y_0)$. Hence we want to define a homomorphism $\varrho_{BM}$ of algebras from

$$\text{End} \left( \begin{array}{c} C^\infty_c(\mathcal{G}^+) \\ \oplus \\ C^\infty_c(\mathcal{G}_\partial) \end{array} \right) \supset \mathcal{B}^{0,0}(\mathcal{G}^+, \mathcal{G}_\partial) \to \mathcal{B}^{0,0}_V(X, Y) \subset \text{End} \left( \begin{array}{c} C^\infty(X) \\ \oplus \\ C^\infty(Y) \end{array} \right).$$

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The indicial symbol $\text{Lopatinski ellipticity}$ defined as the restriction to a singular hyperface $F$. It is a non-trivial task to prove that in certain particular cases $\varrho_{BM}$ furnishes an isomorphism between these two algebras. Furthermore, as can already be shown by simply viewing the special case of pseudodifferential operators, it is not true in general. Instead we prove an analog of a result due to Ammann, Lauter and Nistor (cf. [3]).

**Theorem 1.2.** Given the vector representation $\varrho_{BM}$ and a boundary structure we have the following isomorphism

$$
\varrho_{BM} \left( B^{m,0}(G^+, \mathcal{G}_\partial) \right) \cong B^{m,0}_V(X,Y).
$$

A priori, the inverse of an invertible Boutet de Monvel operator will not be contained in our calculus due to the definition via compactly supported distributional kernels. We define a completion $\overline{B}^0_\infty V(X,Y)$ of the residual Boutet de Monvel operators with regard to the family of norms of operators $L \left( \begin{array}{cc} H^\ell_V(X) & H^\ell_V(X) \\ \oplus & \oplus \end{array} \right)$ on Sobolev spaces, cf. [4]. Define the completed algebra of Boutet de Monvel operators as

$$
\overline{B}^{0,0}_V(X,Y) = B^{0,0}_V(X,Y) + \overline{B}^\infty_\infty V(X,Y).
$$

The resulting algebra contains inverses and has favorable algebraic properties, e.g. it is spectrally invariant, cf. section [5]. We have a parametrix construction after defining a notion of Shapiro-Lopatinski ellipticity. The indicial symbol $R_F$ of an operator $A$ on $X$ is an operator $R_F(A)$ defined as the restriction to a singular hyperface $F \subset X$ (see [3]). Note that if $F$ intersects the boundary $Y$ non-trivially we obtain in this way a non-trivial Boutet de Monvel operator $R_F(A)$ defined on the Lie manifold $F$ with boundary $F \cap Y$.

We say that an operator $A \in \overline{B}^{0,0}_V(X,Y)$ is $V$-elliptic if the principal symbol and the principal boundary symbol of $A$ are both pointwise invertible. If a $V$-elliptic operator $A$ has pointwise invertible indicial symbols $R_F(A)$ for each singular hyperface $F$ we call $A$ elliptic. Then we prove the following result.

**Theorem 1.3.** i) Let $A \in \overline{B}^{0,0}_V(X,Y)$ be $V$-elliptic. There is a parametrix $B \in \overline{B}^{0,0}_V(X,Y)$ of $A$, in the sense

$$
I - AB \in \overline{B}^\infty_\infty V(X,Y), \quad I - BA \in \overline{B}^\infty_\infty V(X,Y).
$$

ii) Let $A \in \overline{B}^{0,0}_V(X,Y)$ be elliptic. There is a parametrix $B \in \overline{B}^{0,0}_V(X,Y)$ of $A$ up to compact operators

$$
I - AB \in \mathcal{K} \left( \begin{array}{c} L^2_0(X) \\ \oplus \end{array} \right), \quad I - BA \in \mathcal{K} \left( \begin{array}{c} L^2_0(X) \\ \oplus \end{array} \right) \left( \begin{array}{c} L^2_0(Y) \\ \oplus \end{array} \right) \left( \begin{array}{c} L^2_0(Y) \\ \oplus \end{array} \right).
$$

An application of groupoids to boundary value problems can be found in [8]. We also refer to recent work of Debord and Skandalis where the Boutet de Monvel algebra is being studied using deformation groupoids [11]. It would be interesting to explore applications of the calculus to cases of manifolds with piecewise smooth boundaries previously considered in the literature. One
example are Lipschitz boundaries, cf. [19], [26], [25]. On the other hand potential applications to the $\bar{\partial}$-problem could be explored, see [12].

The paper is organized as follows. In Section 2 we recall the definition of Boutet de Monvel’s calculus in the standard case and study an example for our construction. In Section 3 we briefly summarize the definitions and notation for Lie groupoids, groupoid actions and Lie algebroids. There we also introduce operators defined via their Schwartz kernels and discuss reduced reduced kernels. Section 4 is concerned with the notion of boundary structure. We motivate the definition by considering known examples of Lie structures with corresponding integrating Lie groupoids. We prove that under our assumptions such a boundary structure or tuple exists. In Section 5 we introduce the extended operators of Boutet de Monvel type which are special instances of the operators defined in Section 3. Then we show how to compose these operators in Section 6. Section 7 is concerned with the definition of the Boutet de Monvel calculus with regard to a given boundary structure. We prove the closedness under composition and the representation theorem. In Section 8 we describe several properties of the calculus, including order reductions and continuity on Sobolev spaces. Section 9 is concerned with the construction of parametrices in the Lie calculus.

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2. Boutet de Monvel’s calculus

2.1. Properties

The calculus of Boutet de Monvel contains the classical boundary value problems as well as their inverse if it exists and parametrices. This calculus was developed in 1971, see [7]. Let for the moment $X$ be a smooth compact manifold with boundary denoted $\partial X = Y$. Let $P$ be a pseudodifferential operator on a smooth neighborhood $M$ of the manifold with boundary. We denote by $P_+ = \chi^+ P \chi^0$ the operator obtained from $P$ via extension by zero $\chi^0$ of functions defined on $X$ to functions on $M$ and restriction $\chi^+$ to $X$. The pseudodifferential operator $P$ should fulfill the transmission property, i.e. $P_+$ should map functions smooth up to the boundary to functions again smooth up to the boundary ([14], p. 23, (1.2.6)). Furthermore, $G: C^\infty(X) \to C^\infty(X)$ is a singular Green operator ([14], p. 30), $K: C^\infty(\partial X) \to C^\infty(X)$ is a potential operator ([14], p. 29) and $T: C^\infty(X) \to C^\infty(\partial X)$ is a trace operator ([14], p. 27). Additionally, $S$ is a pseudodifferential operator on the boundary. Elements of the calculus consist of matrices of operators

$$A = \begin{pmatrix} P_+ + G & K \\ T & S \end{pmatrix} : \bigoplus_{E_1} C^\infty(X,E_1) \to \bigoplus_{E_2} C^\infty(X,E_2) \in B^{m,d}(X,\partial X) \quad (3)$$

The calculus has the following notable properties.

- If the bundles match, i.e. $E_1 = E_2 = E$, $F_1 = F_2 = F$ the calculus is closed under composition.
• If $F_1 = 0$, $G = 0$ and $K$, $S$ are not present, the classical BVP’s are contained in the calculus, e.g. Dirichlet problem.

• If $F_2 = 0$ and $T$, $S$ are not present, it contains inverses of classical BVP’s if they exist.

It is non-trivial to establish the closedness under composition, which means that two Boutet de Monvel operators composed are again of this type, compare the standard references [14], [33].

2.2. An elliptic boundary value problem

In this section we will study a Shapiro Lopatinski elliptic boundary value problem adapted to a Lie manifold with boundary. Specifically, we consider the appropriate generalization of the problem discussed in Section 2 of [20]. We therefore fix the data in the introduction, a Lie manifold $(X, V)$ with boundary $(Y, W)$ and double $(M, 2V)$. Let smooth hermitian vector bundles $\tilde{E}, \tilde{F} \to M$, restrictions $E, F \to X$ as well as $F_j \to M$ for $j = 1, \cdots, L$ be given. We fix the differential operators on the double Lie manifold $B_j \in \text{Diff}_{2V}^m(M; \tilde{E}, \tilde{F})$ for $j = 1, \cdots, L$ and $P \in \text{Diff}_{2V}^m(M, \tilde{E}, \tilde{F})$. Set $T = (\gamma_Y \circ (B_1)_+, \cdots, \gamma_Y \circ (B_L)_t)$ for the trace operator and let $d := \max_{j=1}^L m_j + 1$.

We consider the following boundary value problem

\[
P_u u = f \text{ in } X,
Tu = g \text{ on } Y.
\]

The problem $A = \begin{pmatrix} P_+ \\ T \end{pmatrix}$ in particular yields a continuous linear operator on the appropriate Sobolev spaces

\[
A = \begin{pmatrix} P_+ \\ T \end{pmatrix} : H^s_V(X, E) \to H^{s-m}(X, F) \oplus \bigoplus_{j=1}^L H^{s-m_j - \frac{1}{2}}(Y, F_j|_Y).
\]

**Theorem 2.1.** Assume the boundary value problem $A = \begin{pmatrix} P_+ \\ T \end{pmatrix}$ is Shapiro-Lopatinski elliptic (in the sense of Definition [9.4 in section 9], then for $s > \max\{m, d\} - \frac{1}{2}$

\[
A = \begin{pmatrix} P_+ \\ T \end{pmatrix} : H^s_V(X, E) \to H^{s-m}(X, F) \oplus \bigoplus_{j=1}^L H^{s-m_j - \frac{1}{2}}(Y, F_j|_Y).
\]

is Fredholm.

There is a parametrix $P = (Q_+ + G \ K_1 \cdots K_L)$ of $A$ in the completed Lie calculus $\overline{B}_{W, \max\{0, d-m\}}(X, Y)$ (cf. section 8).

Here $G$ is a singular Green operator of order $-m$ and $K_j$ are potential operators of order $-m_j - \frac{1}{2}$.

**Proof.** We fix the pseudodifferential order reductions (cf. section 8, [4, 29])

\[
R_j : H^s_W(Y, F_j|_Y) \approx H^{s-m_j}(Y, F_j|_Y)
\]

in the completed pseudodifferential calculus $\overline{W}^m_W(Y)$, $\mu_j = m - m_j - \frac{1}{2}$, $j = 1, \cdots, L$.

Set

\[
R := \begin{pmatrix} R_1 & 0 & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & R_L \end{pmatrix}
\]
and consider for $s > \max\{m, d\} - \frac{1}{2}$ the operator

$$
\mathcal{B} = \begin{pmatrix} 1 & 0 \\ 0 & R \end{pmatrix} \begin{pmatrix} P^+ \\ T \end{pmatrix} : H^s_Y(X, E) \to \bigoplus_{J_+} H^{s-m}_W(Y, J_+)
$$

where $J_+ = \bigoplus_{j=1}^d F_j y$.

We obtain an elliptic pseudodifferential boundary value problem contained in the completed calculus $\mathcal{B} \in \mathcal{B}^{m,d}(X, Y)$. The ellipticity follows by the multiplicativity of the principal and principal boundary symbol in Lemma 9.2. By Theorem 1.3 there is a parametrix $\mathcal{P} \in \mathcal{B}^{-m, (d-m)+}(X, Y)$ up to residual terms. Hence $\mathcal{P} = \mathcal{P} \begin{pmatrix} 1 & 0 \\ 0 & R \end{pmatrix}$ yields a parametrix of $\mathcal{A}$. The Fredholm property of $\mathcal{P}$ follows by another application of Theorem 1.3, part (b).

3. Groupoids, actions and algebroids

3.1. Lie groupoids

**Definition 3.1.** Groupoids are small categories in which every morphism is invertible.

First we will introduce and fix the notation for the rest of this paper. Then we will give the definition of a Lie groupoid. For more details on groupoids we refer the reader for example to the book [31].

**Notation 3.2.** A groupoid will be denoted $\mathcal{G} \rightrightarrows \mathcal{G}^{(0)}$. We denote by $\mathcal{G}^{(1)}$ the set of morphisms and by $\mathcal{G}^{(0)}$ the set of objects. By a common abuse of notation we write $\mathcal{G}$ for $\mathcal{G}^{(1)}$. We have the range / source maps $r, s : \mathcal{G} \to \mathcal{G}^{(0)}$ such that $\gamma \in \mathcal{G}$ is

$$
\gamma : s(\gamma) \to r(\gamma).
$$

The set of composable arrows is given as pullback

$$
\mathcal{G}^{(2)} := \mathcal{G}^{(1)} \times_{\mathcal{G}^{(0)}} \mathcal{G}^{(1)} = \{ (\gamma, \eta) \in \mathcal{G} \times \mathcal{G} : r(\eta) = s(\gamma) \}.
$$

Also denote the inversion $i : \mathcal{G} \to \mathcal{G}$, $\gamma \mapsto \gamma^{-1}$

and unit map

$$
u : \mathcal{G}^{(0)} \to \mathcal{G}, \, x \mapsto u(x) = \text{id}_x \in \mathcal{G}.
$$

Multiplication is denoted by $m : \mathcal{G}^{(2)} \to \mathcal{G}$, $(\gamma, \eta) \mapsto \gamma \cdot \eta$.

We also set $\mathcal{G}^x := s^{-1}(x), \mathcal{G}^x := r^{-1}(x), \mathcal{G}^x_x := \mathcal{G}^x \cap \mathcal{G}^x$ for the $r$ and $s$ fibers and their intersection $\mathcal{G}^x_x$. The latter is easily checked to be a group for each $x \in \mathcal{G}^{(0)}$.

**Axioms:** One can summarize the maps in a sequence

$$
\mathcal{G}^{(2)} \xrightarrow{m} \mathcal{G} \xrightarrow{i} \mathcal{G} \xrightarrow{r,s} \mathcal{G}^{(0)} \xrightarrow{u} \mathcal{G}.
$$

With the above notation we can give an alternative way of defining groupoids axiomatically as follows.

(i) $(s \circ u)|_{\mathcal{G}^{(0)}} = (r \circ u)|_{\mathcal{G}^{(0)}} = \text{id}_{\mathcal{G}^{(0)}}$. 

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For each $\gamma \in G$

$(u \circ r)(\gamma) \cdot \gamma = \gamma, \quad \gamma \cdot (u \circ s)(\gamma) = \gamma.

(iii) $s \circ i = r, \quad r \circ i = s.$

(iv) For $(\gamma, \eta) \in G^{(2)}$ we have

$r(\gamma \cdot \eta) = r(\gamma), \quad s(\gamma \cdot \eta) = s(\eta).$

(v) For $(\gamma_1, \gamma_2), (\gamma_2, \gamma_3) \in G^{(2)}$ we have

$(\gamma_1 \cdot \gamma_2) \cdot \gamma_3 = \gamma_1 \cdot (\gamma_2 \cdot \gamma_3).$

(vi) For each $\gamma \in G$ we have

$\gamma^{-1} \cdot \gamma = \operatorname{id}_{s(\gamma)}, \quad \gamma \cdot \gamma^{-1} = \operatorname{id}_{r(\gamma)}.$

**Definition 3.3.** The 7-tuple $(G^{(0)}, G^{(1)}, r, s, m, u, i)$ defines a *Lie groupoid* if and only if $G \Rightarrow G^{(0)}$ is a groupoid, $M := G^{(0)}, \quad G^{(1)}$ are $C^\infty$-manifolds (with corners), all the maps are $C^\infty$ and $s$ is a submersion.

**Remark 3.4.** We notice that $r$ is automatically a submersion due to the axiom (iii). Hence the pullback

$G^{(2)} \xrightarrow{p_2} G \xrightarrow{s} M$

exists in the $C^\infty$-category if $G$ is $C^\infty$ and thus $G^{(2)}$ is a smooth manifold as well.

### 3.2. Groupoid actions

Given a Lie groupoid $G \Rightarrow M$ we introduce spaces $X$ which are fibred over $G^{(0)}$ and such that $G$ acts on $X$. This notion as well as some of the notation is adapted from the paper [32].

**Definition 3.5.** Let $(X, q)$ be a $G$-space, i.e. $q: X \to M$ is a smooth map and $X$ is a smooth manifold. Set $X \ast G := X \times_M G = \{(x, \gamma) \in X \times G : q(x) = r(\gamma)\}$ to be the *composable elements*. We say that $G$ acts on $X$ from the right if the following conditions hold:

i) For each $(x, \gamma) \in X \ast G$

$q(x \cdot \gamma) = s(\gamma).$

ii) For each $(x, \gamma) \in X \ast G$ and $(\gamma, \eta) \in G^{(2)}$

$x \cdot (\gamma \cdot \eta) = (x \cdot \gamma) \cdot \eta.$

iii) For each $(x, \gamma) \in X \ast G$ we have

$(x \cdot \gamma) \cdot \gamma^{-1} = x.$

A left action of $G$ on a $G$-space $(X, p)$ is a right-action in the opposite category $G^{\text{op}}$.

**Remark 3.6.** Note that for any $\mathcal{H}$-space $(X, p)$ which is additionally *fibered* meaning $p$ is a surjective submersion then the pullback $\mathcal{H} \ast X$ exists in the $C^\infty$-category if $\mathcal{H}, \ X$ are $C^\infty$. Analogously for a fibered $G$-space.
Consider the following actions of two Lie groupoids $\mathcal{G}$, $\mathcal{H}$:

\[
\begin{array}{ccc}
\mathcal{H} & \circlearrowleft & \mathcal{X} \\
\downarrow p & & \downarrow q \\
\mathcal{H}^{(0)} & \rightarrow & \mathcal{G}^{(0)} \\
\end{array}
\]

We can define a so-called left Haar system on $\mathcal{X}$ induced by the action of $\mathcal{H}$ and analogously a right Haar system induced by the action of $\mathcal{G}$. This enables us to define left- and right-operators coming from the actions.

Let $\{\lambda_x\}_{x \in \mathcal{G}^{(0)}}$ be a Haar system induced on $\mathcal{X}$ by the right action of $\mathcal{G}$, see also [32], p. 6. This is a family of measures such that

- The support is $\text{supp } \lambda_x = \mathcal{X}_x$ for each $x \in \mathcal{G}^{(0)}$.
- The map $\mathcal{G}^{(0)} \ni x \mapsto \int_{\mathcal{X}_x} f \, d\lambda_x$
  is $C^\infty$.
- We have the invariance condition
  \[
  \int_{\mathcal{X}_{r(\gamma)}} f(z \cdot \gamma) \, d\lambda_{r(\gamma)}(z) = \int_{\mathcal{X}_{s(\gamma)}} f(w) \, d\lambda_{s(\gamma)}(w).
  \]  

Fix the right-multiplication for given $\gamma \in \mathcal{G}$

\[ r_\gamma: \mathcal{X}_{r(\gamma)} \rightarrow \mathcal{X}_{s(\gamma)}, \quad z \mapsto z \cdot \gamma. \]

This is a diffeomorphism.

The induced operators acting on $C^\infty$-functions are given by

\[ R_\gamma: C^\infty_c(\mathcal{X}_{s(\gamma)}) \rightarrow C^\infty_c(\mathcal{X}_{r(\gamma)}), \quad (R_\gamma f)(z) := f(z \cdot \gamma), \quad z \in \mathcal{X}. \]

These operators $R_\gamma$ yield $(\text{anti-})$homomorphisms since $(R_\gamma)^{-1} = R_{\gamma^{-1}}$ is the inverse and $R_{\gamma \cdot \eta} = R_\eta \circ R_\gamma$, $(\gamma, \eta) \in \mathcal{G}^{(2)}$.

**Definition 3.7.** i) A continuous linear operator $T: C^\infty_c(\mathcal{G}) \rightarrow C^\infty_c(\mathcal{X})$ is called a right $\mathcal{X}$-operator if and only if $T = (T_x)_{x \in \mathcal{G}^{(0)}}$ is a family of continuous linear operators $T_x: C^\infty_c(\mathcal{G}_x) \rightarrow C^\infty_c(\mathcal{X}_x)$ such that

\[
R_{\gamma^{-1}} T_{r(\gamma)} R_\gamma = T_{s(\gamma)}, \quad \gamma \in \mathcal{G}.
\]

This can be expressed alternatively by requiring the following diagram to commute for each $\gamma \in \mathcal{G}$

\[
\begin{array}{ccc}
C^\infty_c(\mathcal{G}_{s(\gamma)}) & \xrightarrow{T_{s(\gamma)}} & C^\infty_c(\mathcal{X}_{s(\gamma)}) \\
\downarrow R_{\gamma^{-1}} & & \downarrow R_\gamma \\
C^\infty_c(\mathcal{G}_{r(\gamma)}) & \xrightarrow{T_{r(\gamma)}} & C^\infty_c(\mathcal{X}_{r(\gamma)}).
\end{array}
\]
ii) By analogy \( \tilde{T} : C^\infty_c(\mathcal{X}) \to C^\infty_c(\mathcal{H}) \) is a \textit{left} \( \mathcal{X} \)-\textit{operator} if and only if \( \tilde{T} = (\tilde{T}^y)_{y \in \mathcal{H}}(0) \) is a family of continuous linear operators \( \tilde{T}^y : C^\infty_c(\mathcal{X}^y) \to C^\infty_c(\mathcal{H}^y) \) such that the diagram

\[
\begin{array}{ccc}
C^\infty_c(\mathcal{X}^s(\gamma)) & \xrightarrow{\tilde{T}^s(\gamma)} & C^\infty_c(\mathcal{H}_s(\gamma)) \\
\text{L}_\gamma & & \text{L}_{s-1} \\
\downarrow & & \downarrow \\
C^\infty_c(\mathcal{X}^r(\gamma)) & \xrightarrow{\tilde{T}^r(\gamma)} & C^\infty_c(\mathcal{H}_r(\gamma))
\end{array}
\]

commutes for each \( \gamma \in \mathcal{H} \) where \( L_\gamma \) denotes in this case the corresponding left multiplication.

The next Proposition tells us that the family of Schwartz kernels \( (k_x)_{x \in \mathcal{G}}(0) \) for a given \( \mathcal{X} \)-operator can be replaced by a so-called reduced kernel. This is not unlike the situation for groupoids and the pseudodifferential calculus where reduced kernels are used extensively (cf. [30]).

**Proposition 3.8.** Given a right-\( \mathcal{X} \)-operator \( T : C^\infty_c(\mathcal{G}) \to C^\infty_c(\mathcal{X}) \). Then for \( u \in C^\infty_c(\mathcal{G}) \), \( z \in \mathcal{X} \)

\[
(Tu)(z) = \int_{\mathcal{G}_{q(z)}} k_T(z \cdot \gamma^{-1})u(\gamma) \, d\mu_q(z)(\gamma)
\]

with \( k_T(z \cdot \gamma^{-1}) := k_r(\gamma)(z, \gamma) \) depending only on \( z \cdot \gamma^{-1} \in \mathcal{X} \) for each \( (z, \gamma^{-1}) \in \mathcal{X} \ast \mathcal{G} \).

**Proof.** First we can write for \( z \in \mathcal{X} \)

\[
(R_{\gamma^{-1}}T_r(\gamma)R_\gamma)u(z) = (T_r(\gamma)R_\gamma u)(z \cdot \gamma^{-1}) = \int_{\mathcal{G}_r(\gamma)} k_r(\gamma)(z \cdot \gamma^{-1}, \eta)u(\eta \gamma) \, d\mu_r(\gamma)(\eta)
\]

\[
= \int_{\mathcal{G}_s(\gamma)} k_r(\gamma)(z \cdot \gamma^{-1}, \bar{\eta} \cdot \gamma^{-1})u(\bar{\eta}) \, d\mu_s(\gamma)(\bar{\eta})
\]

via the substitution \( \bar{\eta} := \eta \gamma \) and invariance of Haar system. By use of (25) we see that the last integral equals

\[
(T_{s(\gamma)}u)(z) = \int_{\mathcal{G}_{s(\gamma)}} k_{s(\gamma)}(z, \eta)u(\eta) \, d\mu_{s(\gamma)}(\eta).
\]

This implies the following identity by the uniqueness of the Schwartz kernel for \( T_x \) for each \( x \in M \)

\[
\forall \gamma \in \mathcal{G} \quad k_{s(\gamma)}(z, \bar{\eta}) = k_r(\gamma)(z \cdot \gamma^{-1}, \bar{\eta} \cdot \gamma^{-1}). \quad (*)
\]

To see that \( k_T \) is well-defined assume \( \beta = z \cdot \gamma^{-1} = \bar{\bar{z}} \cdot \bar{\gamma}^{-1} \) and \( \delta = \gamma^{-1} \cdot \bar{\gamma} \), then

\[
k_{s(\gamma)}(\bar{\bar{z}}, \bar{\gamma}) = k_{s(\delta)}(\bar{\bar{z}}, \bar{\gamma}) = k_{r(\delta)}(\bar{\bar{z}} \delta^{-1}, \bar{\gamma} \delta^{-1}) = k_{s(\gamma)}(\beta \gamma \delta^{-1}, \bar{\gamma} \delta^{-1}) = k_{s(\gamma)}(z, \gamma).
\]

This completes the proof.

\[\square\]

3.3. Lie algebroids

The aim of this section is to give a definition of Lie algebroids and subalgebroids. We restrict ourselves to the bare minimum needed in the following text of the paper. For a more detailed exposition the reader may consult e.g. [27].
Definition 3.9.  

• A Lie algebroid is a tuple $(E, \varrho)$. Here $\pi: E \to M$ is a vector bundle over a manifold $M$ and $\varrho: E \to TM$ is a vector bundle map such that

$$\varrho \circ [V, W]_{\Gamma(E)} = \varrho \circ [V, \varrho W]_{\Gamma(TM)}$$

and

$$[V, fW]_{\Gamma(E)} = f[V, W]_{\Gamma(E)} + \varrho(V)(f)W, \; f \in C^\infty(M), \; V, W \in \Gamma(E).$$

• Given two Lie algebroids $(A, \varrho)$ and $(\tilde{A}, \tilde{\varrho})$ over the same manifold $M$. Then a Lie algebroid morphism is a map $\varphi: A \to \tilde{A}$ making the following diagram commute:

$$\begin{array}{ccc}
TM & \xrightarrow{\varrho} & TM \\
\downarrow{\varphi} & & \downarrow{\tilde{\varphi}} \\
A & \xrightarrow{\pi} & \tilde{A} \\
\downarrow{\tilde{\pi}} & & \downarrow{\pi} \\
M & & M
\end{array}$$

and such that $\varphi$ preserves Lie bracket: $\varphi[V, W]_{\Gamma(A)} = [\varrho(V), \varrho(W)]_{\Gamma(TM)}$.

We briefly summarize some relevant facts about the construction of Lie algebroids.

• For any given Lie groupoid $G$ we obtain an associated algebroid $A(G)$ in a covariantly functorial way. Define $T^sG := \ker(ds)$ the $s$-vertical tangent bundle as a sub-bundle of $TG$. Denote by $\Gamma(T^sG)$ the smooth sections and define $\Gamma_R(T^sG)$ as the sections $V$ such that

$$V(\eta_\gamma) = (R_\gamma)_* V_\eta \text{ for } (\eta, \gamma) \in G^{(2)}.$$ 

We then define the Lie-algebroid associated to $G$ via the pullback

$$\begin{array}{ccc}
A(G) & \xrightarrow{u^*} & T^sG \\
\downarrow{\pi_{|_M}} & & \downarrow{\pi_{|_M}} \\
M & \xrightarrow{u} & G
\end{array}$$

In other words $A(G) := \{(V, x)|ds(V) = 0, \; u(x) = 1_x = \pi(V)\}$.

• There is a canonical isomorphism of Lie-algebras $\Gamma_R(T^sG) \cong \Gamma(A(G))$. The set of smooth sections $\Gamma(A(G))$ is a $C^\infty(M)$-module with the module operation $f \cdot V = (f \circ r) \cdot V$ with $f \in C^\infty(M)$.

• Let $A(G)$ be given as above and define $\varrho: A(G) \to TM$ by $\varrho := dr \circ u^*$. Then $(A(G), \varrho)$ so defined furnishes a Lie algebroid.

• A Lie algebroid is said to be integrable if we can find an associated (with connected $s$-fibers) Lie groupoid. Not every Lie algebroid is integrable.
4. Boundary structure

With the given Lie manifold with boundary we want to associate a so-called boundary structure. We define a boundary structure and we show that for particular examples of Lie structures there is a boundary structure. What is necessary in the general case is a certain assumption on the groupoids $\mathcal{G}$, $\mathcal{G}_0$, namely they ought to define a bibundle structure which we are going to specify. The boundary structure is in fact a good analogy for blow-ups of the corners which are the intersections of $Y$ with the (singular) boundary at infinity of $M$. These blow-ups are in our setup canonically defined in terms of $\mathcal{G}$ and $\mathcal{G}_0$, the groupoids integrating $\mathcal{A}$ and $\mathcal{A}_0$.

Recall that a bibundle correspondence between two Lie groupoids $\mathcal{G}$ and $\mathcal{H}$ implemented by $\mathcal{X}$ is a left $\mathcal{H}$-space $(\mathcal{X}, p)$ and a right $\mathcal{G}$-space $(\mathcal{X}, q)$ with free and proper right action of $\mathcal{G}$ on $\mathcal{X}$ such that there is a homeomorphism $\mathcal{X}/\mathcal{G} \simto \mathcal{H}(0)$ induced by $p$ and the actions of $\mathcal{G}$ and $\mathcal{H}$ commute, see also [16].

In the following we give the axioms necessary to define a boundary structure.

**Definition 4.1.** A boundary structure is defined as a tuple $(\mathcal{G}, \mathcal{G}_0, \mathcal{G}^\pm, \mathcal{X}, \mathcal{X}^t, f)$ consisting of a Lie groupoid $\mathcal{G} \rightrightarrows M$ and two manifolds (possibly with corners) $\mathcal{X}$, $\mathcal{X}^t$ which are diffeomorphic via a flip diffeomorphism $f$ and subgroupoids $\mathcal{G}^\pm \rightrightarrows X^\pm$ of $\mathcal{G}$.

We impose the following axioms on this data:

i) $A(\mathcal{G}) \cong A_Y$, $A(\mathcal{G}_0) \cong A_\partial$ as well as $A(\mathcal{G}^\pm) \cong A_Y$ as Lie algebroids.

tt) $\mathcal{X}$ is a right $\mathcal{G}$- and a left $\mathcal{G}_\partial$-space and $\mathcal{X}^t$ is a left $\mathcal{G}$- and a right $\mathcal{G}_\partial$-space and these actions implement a bibundle correspondence between $\mathcal{G}$ and $\mathcal{G}_\partial$. The charge maps of the actions $p: \mathcal{X} \to Y$, $q: \mathcal{X} \to M$ and $p^t: \mathcal{X}^t \to M$, $q^t: \mathcal{X}^t \to Y$ are such that $p$ and $q^t$ are surjective submersions.

iii) Restricted to the interior we have

$$\mathcal{X}_{|Y_0 \times M_0} = p^{-1}(M_0) \cap q^{-1}(Y_0) \cong Y_0 \times M_0,$$

$$\mathcal{X}^t_{|M_0 \times Y_0} = (p^t)^{-1}(M_0) \cap (q^t)^{-1}(Y_0) \cong M_0 \times Y_0.$$

iv) The fibers of $\mathcal{G}^\pm$ are the interiors of smooth manifolds with boundary, namely:

$$\partial_{reg} \mathcal{G}^+_x = \mathcal{X}_x, \quad x \in X_+$$

$$\partial_{reg} \mathcal{G}^-_x = \mathcal{X}_x, \quad x \in X_-.$$

**Example 4.2.** i) Consider a compact manifold $X$ with boundary $\partial X = Y$ and interior $\tilde{X} := X \setminus Y$. Then we also fix the double $M = 2X$. In this (trivial) case the spaces are given by $\mathcal{X} := Y \times M$, $\mathcal{X}^t = M \times Y$ with the flip $f(x', y) = (y, x')$, $(x', y) \in Y \times M$. We have here the pair groupoids $\mathcal{G} = M \times M$, $\mathcal{G}_0 = Y \times Y$ as well as $\mathcal{G}^+ = \tilde{X}_+ \times \tilde{X}_+$, $\mathcal{G}^- = \tilde{X}_- \times \tilde{X}_-$. Then $p, q$ are just the projections $\pi_1: Y \times M \to Y$, $\pi_2: Y \times M \to M$.

ii) We consider the Lie structure $\mathcal{V}_b := \{ V \in \Gamma^\infty(TM) : V \text{ tangent to } F_i, \quad 1 \leq i \leq N \}$, cf. [23]. The Lie algebroid $\mathcal{A} \to M$ is the $b$-tangent bundle such that $\Gamma(\mathcal{A}) \cong \mathcal{V}_b$. Following Monthubert [28], we find a Lie groupoid $\mathcal{G}_b(M)$ integrating $\mathcal{A}$ which is $s$-connected, Hausdorff and amenable: We start with the set

$$\Gamma_b(M) = \{(x, y, \lambda) \in M \times M \times (\mathbb{R}_+)^N : \rho_i(x) = \lambda \rho_i(y), \quad 1 \leq i \leq N\}$$

endowed with the structure $(x, y, \lambda) \circ (y, z, \mu) = (x, z, \lambda \cdot \mu)$, $(x, y, \lambda)^{-1} = (y, x, \lambda^{-1})$ and

$$r(x, y, \lambda) = x, \quad s(x, y, \lambda) = y, \quad u(x) = (x, x, 1).$$

Here multiplication $\lambda \cdot \mu$ and inversion $\lambda^{-1}$ are componentwise.

We then define the $b$-groupoid $\mathcal{G}_b(M)$ as the $s$-connected component (the union of the connected components of the $s$-fibers of $\Gamma_b(M)$), i.e. $\mathcal{G}_b(M) := C_s \Gamma_b(M)$. 

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iii) Fix the Lie structure $V_{c_l}$ of generalized cusp vector fields for $l \geq 2$ given by the local generators in a tubular neighborhood of a boundary hyperface: $\{x^i_1\partial_{x_1}, \partial_{x_2}, \ldots, \partial_{x_n}\}$. Let us recall the construction of the associated Lie groupoid $G_l(M)$, the so-called generalized cusp groupoid, given in [21] for the benefit of the reader. We set

$$\Gamma_l(M) := \{(x, y, \mu) \in M \times M \times (\mathbb{R}_+)^N : \mu_i \rho_i(x)^i \rho_i(y)^i = \rho_i(x)^i - \rho_i(y)^i\}$$

with structure $r(x, y, \lambda) = x$, $s(x, y, \lambda) = y$, $u(x) = (x, x, 0)$ and $(x, y, \lambda)(y, z, \mu) = (x, z, \lambda + \mu)$.

We then define $G_l(M)$ as the $s$-connected component of $\Gamma_l(M)$.

iv) The following example of the fibered cusp calculus is from Mazzeo and Melrose [24] and we use the formulation and notation for manifolds with fibered corners as given in [10]. We briefly recall the definition of the associated groupoid and refer to loc. cit. for the details. See also [13] for the precise geometric construction of the Lie groupoid for a different type of fibered cusp Lie structure. Write $\pi = (\pi_1, \ldots, \pi_N)$, where $\pi_i : F_i \to B_i$ are fibrations; $B_i$ is the base, which is a compact manifold with corners. Define the Lie structure

$$V_\pi := \{V \in \mathcal{V}_b : V|_{F_i} \text{ tangent to the fibers } \pi_i : F_i \to B_i, \ V\rho_i \in \rho^2 C^\infty(M)\}.$$

Then $V_\pi$ is a finitely generated $C^\infty(M)$-module and a Lie sub-algebra of $\Gamma^\infty(TM)$. The corresponding groupoid is amenable [10], Lemma 4.6; as a set it is defined as

$$G_\pi(M) := (M_0 \times M_0) \cup \left( \bigcup_{i=1}^N (F_i \times \pi_i T^\pi B_i \times \pi_i F_i) \times \mathbb{R} \right),$$

where $T^\pi B_i$ denotes the algebroid of $B_i$.

**Theorem 4.3.** For the Lie structure $\mathcal{V} \in \{\mathcal{V}_b, \mathcal{V}_{c_l}, \mathcal{V}_r\}$, $l \geq 2$ there is a boundary structure.

**Proof.** The claim is proven for the case $\mathcal{V} = \mathcal{V}_b$. For the other cases the argument goes along the same lines, so we omit it. We have the fixed boundary defining functions for the hyperfaces of $M$ and denote this family by $(p_j)_{j \in I}$. On $Y$ there are the boundary defining functions (relative to $Y$), denoted by $(q_j)_{j \in I}$. These are the boundary defining functions of the faces from the intersections of $Y$ with the strata of $M$. We set $\mathcal{X} := r^{-1}(Y) = G(M)_M^Y$ and $\mathcal{X}^t := s^{-1}(Y) = G(M)_Y^M$.

Consider now the topology of $\mathcal{X}$ and $\mathcal{X}^t$. It is defined in local charts by the rule

$$Y_0 \times M_0 \ni (x'_n, y_n) \to (x', y, \lambda = (\lambda_j)_{j \in I}) := q_j(y_n) \to \lambda_j, \ n \to \infty, \ i, j \in I, \ x'_n \to x', \ y_n \to y.$$

The right action of $G(M)$ on $\mathcal{X}$ is given by right composition and the left action of $G(Y)$ is given by left composition. On the interior we only have the pair groupoids. This yields the trivial actions

$$Y_0 \times Y_0 \leftarrow Y_0 \times M_0 \cdots \leftarrow M_0 \times M_0$$

$$Y_0 \leftarrow Y_0 \times Y_0 \leftarrow Y_0 \times M_0 \cdots$$

These actions can be extended continuously to the closure of $Y_0 \times M_0$ in $G$ and also of $M_0 \times Y_0$, and we obtain from the definition of the topology that

$$\mathcal{X} = Y_0 \times M_0^G, \ \mathcal{X}^t = M_0 \times Y_0^G.$$

The continued actions are defined

$$G(Y) \leftarrow \mathcal{X} \leftarrow G(M)$$

$$Y \leftarrow Y_0 \mathcal{X} \leftarrow M$$

Theorem 4.3
Axiom i) holds because of [28] where it was shown that the given $b$-groupoids integrate the Lie structure of $b$-vector fields. It is immediate to check that the actions commute and by definition the charge map $q$ induced by the source map of $G(M)$ is a surjective submersion. The action of $G(M)$ on its space of units is free and proper, hence the action of $G(M)$ by right composition on $\mathcal{X}$ is free and proper. Here properness means that the map $\mathcal{X} \times G(M) \to \mathcal{X} \times \mathcal{X}$, $(z, \gamma) \mapsto (z \cdot \gamma, z)$ is a homeomorphism onto its image. Let $z \gamma = z$ then $q(z) = r(z) = r(\gamma)$ which implies composability and $s(\gamma) = q(z \gamma) = q(z)$. Hence $\gamma = \text{id}_{q(z)}$ which verifies that the action is free. The same can be proven for the left action under our assumption, but we do not need this fact.

Finally, we need to check that we have a diffeomorphism $\mathcal{X} / G(M) \cong Y$ induced by the charge map $p$. We first check this for the groupoids $\Gamma(M)$ and $\Gamma(Y)$, then we take the $s$-connected components which proves the assertion for the groupoid $G(M)$ and $G(Y)$. We have to show that $p(z) = p(w)$ for $z, w \in \mathcal{X}$ if and only if there is a necessarily unique $\eta \in G(M)$ such that $w = z \cdot \eta$. Let $z = (x', y, (\lambda_i)_{i \in I})$, $w = (x', \tilde{y}, (\mu_i)_{i \in I})$ and set $\eta = (y, \tilde{y}, (\frac{\tilde{y}_i}{y_i})_{i \in I})$. By definition of the topology of the groupoids fix the sequences $(x'_n, y_n)$ such that $\frac{q_i(x'_n)}{p_i(y_n)} \to \lambda_j$, $j \in I$, $n \to \infty$ and $(x'_n, y_n)$ such that $\frac{q_i(x'_n)}{p_i(y_n)} \to \mu_j$, $j \in I$, $n \to \infty$.

Then $\eta \in \Gamma(M)$ since

$$
\frac{p_i(y_n)}{p_i(\tilde{y}_n)} = \frac{q_i(x'_n)}{p_i(\tilde{y}_n)} \left( \frac{q_i(x'_n)}{p_i(y_n)} \right)^{-1} \to \frac{\mu_i}{\lambda_i}, \quad i \in I, \quad n \to \infty.
$$

This concludes the proof of the isomorphism $\mathcal{X} / G(M) \cong Y$. Hence we obtain a bibundle correspondence and Axiom ii) is verified. Part iii) follows from the definition of the actions we just gave. Also note that the flip diffeomorphism $f : \mathcal{X} \cong \mathcal{X}^t$ is defined by

$$
f : (x', y, (\lambda_i)_{i \in I}) \mapsto \left( y, x', \left( \frac{1}{\lambda_i} \right)_{i \in I} \right).
$$

It remains to verify condition iv). For this we define $G^\pm := G(X \pm)$ and prove that this groupoid has the required property. Thus we want to show that

$$
\partial_{reg} G^\pm_x = \mathcal{X}_x, \quad x \in X.
$$

The boundary is possibly empty (for $x$ not incident to the hypersurface $Y$). We have to distinguish two cases: $x$ in the interior and $x$ on the boundary of $M$. The groupoid fiber $G^\pm_x$ for $x$ in the interior $M_0$ trivializes to the pair groupoids and this case is thus immediate. We need to consider the case of a point on the boundary of $M$. Assume that $x \in F$ for some open face $F$ of $M$ which is incident to $Y$ (i.e. shares a hyperface with $Y$). By the local triviality property of groupoids (see [28]) we have $G^\pm_x \cong F \times \mathbb{R}^*_+$. The same follows by definition for $\mathcal{X}$, i.e.

$$
\mathcal{X}_x \cong F_{ij} \times \mathbb{R}^*_+.
$$

where $F_{ij}$ denotes the face of $F$ such that $F_{ij} = \overline{F} \cap Y$. Via the definition of the Lie manifold with boundary (cf. [2]) we obtain that the component $F \times \mathbb{R}^*_+$ is the interior of a manifold with boundary. In particular we see that

$$
\partial_{reg}(F \times \mathbb{R}^*_+) = \partial_{reg}(\overline{F} \times \mathbb{R}^*_+) \cong F_{ij} \times \mathbb{R}^*_+.
$$
In summary, we obtain
\[ \partial_{\text{reg}} G_x^+ \cong \begin{cases} 
Y_0 & \text{for } x \in X_0 \\
F_{ij} \times \mathbb{R}^*_+ & \text{for } x \text{ incident to some } F \\
\emptyset & \text{otherwise} 
\end{cases} \cong \mathcal{X}_x. \]

Hence condition iv) holds.

**Remark 4.4.** On the given Lie manifold with boundary we can always restrict the groupoid \( G \) integrating the Lie structure \( 2V \) to a small tubular neighborhood \( Y \subset \mathcal{U} \subset M \) such that the faces of \( Y \) are in bijective correspondence with the faces of \( \mathcal{U} \) as specified in the introduction. Then it is not hard to verify that there is a bibundle correspondence between \( G^+_Y = G_\partial \) and \( G \) implemented by \( \mathcal{X} = r^{-1}(Y), \mathcal{X}^t = s^{-1}(Y) \). Condition iii) follows since \( G_{M_0} \cong M_0 \times M_0 \) by assumption. Also setting \( G^\pm = G^+_X \) we see that condition i) holds. By the previous proof we also obtain that condition iv) holds whenever the groupoid \( G \) is locally trivial. Therefore there is in fact a boundary structure for any Lie manifold with boundary for which there is an integrating groupoid with the local triviality property.

5. Operators on groupoids

The next goal is to define potential, trace and singular Green operators on the groupoid level. These operators should be equivariant families of operators on the fibers, similar to the case of pseudodifferential operators on groupoids. The singular Green, trace and potential operators are ordinarily defined so as to act like pseudodifferential operators in the cotangent direction and as convolution operators in the normal direction. This somewhat complicated behaviour is difficult to realize in the groupoid setting. We start from a different but equivalent definition. Our approach is inspired by the ordinary case of a smooth, compact manifold with boundary, cf. [14]. Here the trace, potential and singular Green operators are extended to the double of the manifold and can be understood as conormal distributions with rapid decay along the normal direction. In our general setting we would therefore like to consider conormal distributions on \( Y \times M \) and \( M \times Y \) as well as \( M \times M \). Since we are working in the setting of manifolds with corners we will desingularize these manifolds using Lie groupoids. This is where the previously introduced notion of a boundary structure enters. For the cases of \( M \times M \) and \( Y \times Y \) this is realized through the groupoids \( G \) and \( G_\partial \) respectively and the pseudodifferential operators on groupoids. We introduce additional blowups \( \mathcal{X} \) and \( \mathcal{X}^t \) with good properties (fibered over the manifolds \( Y \) and \( M \)) with regard to \( G \) and \( G_\partial \). Then we define the trace, potential and singular Green operators as distributions on these spaces and \( G \) conormal to the diagonal \( \Delta_Y \).

5.1. Actions

**From now on we fix:** A boundary structure \((G, G_\partial, G^\pm, \mathcal{X}, \mathcal{X}^t, f)\) adapted to our Lie manifold \((X, V)\) with boundary \( Y \) and its double \((M, 2V)\). We then fix the groupoid actions which are summarized in the following picture. In the first column the dotted arrows indicate morphisms in the category of Lie groupoids (correspondences) which are implemented by the actions in the second column.

\[
\begin{array}{cccccc}
G_\partial & \longrightarrow & G & \longrightarrow & G_\partial \\
\circ & \quad & \circ & \quad & \circ \\
p & \quad & q & \quad & p^t \\
\mathcal{X} & \longrightarrow & \mathcal{X}^t \\
q & \quad & q^t \\
Y & \quad & M & \quad & Y \\
\end{array}
\]
Fix also Haar systems on the groupoids and fibered spaces as follows.

\[ G : \{ \mu_x \}_{x \in M}, \ \mathcal{X} : \{ \lambda_x \}_{x \in M}, \]

\[ G_{\partial} : \{ \mu_y \}_{y \in Y}, \ \mathcal{X}^t : \{ \lambda^t_x \}_{x \in M}. \]

In each case the system is a (left / right)-Haar system if the corresponding action is a (left / right)-action.

5.2. Local charts

In order to define the operators on groupoids and actions as given in the last section we have to introduce the local charts. The charts are given by diffeomorphisms which preserve the \( s \)-fibers, see also [32], p. 3.

Fix the dimensions \( n = \dim M = \dim M_0, \ n - 1 = \dim Y = \dim Y_0. \)

- A chart of \( G \) is an open subset \( \Omega \subset G \) which is diffeomorphic to two open subsets of \( G^{(0)} \times \mathbb{R}^n \). Choose two open subsets \( V_s \times W_s \) and \( V_r \times W_r \). Then choose two diffeomorphisms \( \psi_s : \Omega \to V_s \times W_s \) and \( \psi_r : \Omega \to V_r \times W_r \). Additionally, we require that these diffeomorphisms are fiber-preserving in the sense that \( s(\psi_s(x, w)) = x \) for \( (x, w) \in V_s \times W_s \) and \( r(\psi_r(x, w)) = x \) for \( (x, w) \in V_r \times W_r \). Hence we have the following commuting diagrams:

\[
\begin{array}{ccc}
V_r & \overset{r(\Omega)}{\longleftarrow} & \Omega \\
& \searrow & \downarrow \\
& \quad & s(\Omega) \\
\downarrow & & \downarrow \\
V_s & \overset{s(\Omega)}{\longleftarrow} & \Omega
\end{array}
\]

- Similarly, the charts for \( \mathcal{X} \) are given by the sets of the form \( \tilde{\Omega} = \Omega \cap \mathcal{X} \) for charts \( \Omega \) of \( G \) fitting into the following commuting diagrams:

\[
\begin{array}{ccc}
V_q \times W_q & \overset{\tilde{\Omega}}{\longrightarrow} & V_{p^t} \times W_{p^t} \\
W_q \overset{q(\tilde{\Omega})}{\longleftarrow} & \downarrow & \rightarrow \\
& \quad & \downarrow \\
& \quad & p^t(\tilde{\Omega}) \\
& & \rightarrow \\
& & V_{p^t}
\end{array}
\]

- Analogously, \( \tilde{\Omega} \subset \mathcal{X}^t \) are charts with the actions reversed and hence in this case we have the commuting diagrams:

\[
\begin{array}{ccc}
V_p \times W_p & \overset{\tilde{\Omega}}{\longrightarrow} & V_{q^t} \times W_{q^t} \\
W_p \overset{p(\tilde{\Omega})}{\longleftarrow} & \downarrow & \rightarrow \\
& \quad & \downarrow \\
& \quad & q^t(\tilde{\Omega}) \\
& & \rightarrow \\
& & V_{q^t}
\end{array}
\]

**Definition 5.1.** i) A family \( T = (T_x)_{x \in M} \) of operators \( T_x : C^\infty_c(G_x) \to C^\infty_c(X_x) \) is a differentiable family of trace type iff the following holds. Given any chart \( \Omega \subset G \) with fiber preserving diffeomorphism, \( s(\Omega) \sim \Omega \times W \) for some \( W \subset \mathbb{R}^n \) open. Moreover for \( \tilde{\Omega} := \Omega \cap \mathcal{X} \) such that \( \tilde{W} := W \cap \mathbb{R}^{n-1} \) we have a fiber-preserving diffeomorphism \( q(\Omega) \sim \tilde{\Omega} \times \tilde{W} \), and for each \( \varphi \in C^\infty_c(\Omega), \ \tilde{\varphi} \in C^\infty_c(\tilde{\Omega}) \) the operator \( \tilde{\varphi}T\varphi \) has a Schwartz kernel

\[
k \in I^m(s(\Omega) \times \tilde{W} \times W, \Delta_{\tilde{W}}) \approx C^\infty(s(\Omega)) \hat{\otimes} I^m(\tilde{W} \times W, \Delta_{\tilde{W}}).
\]
The operator \( \tilde{\varphi} T_x \varphi \) for each \( x \in s(\Omega) \) corresponds to the Schwartz kernel \( k_x \) via the diffeomorphisms \( X_x \cap \Omega \cong \tilde{W} \) and \( G_x \cap \Omega \cong W \).

ii) Analogously, we define a family \( K = (K_x)_{x \in M} \) of operators \( K : C^\infty_c(X_x^t) \to C^\infty_c(G_x) \) with the charts reversed. This is called differentiable family of potential type.

iii) A differentiable family of singular Green type \( (G_x)_{x \in M} \) is a family of operators \( G_x : C^\infty_c(G_x) \to C^\infty_c(G_x) \) defined as follows. Given any chart \( \Omega \subset G \) with fiber preserving diffeomorphism \( s(\Omega) \sim \Omega \times W \) for some \( W \subset \mathbb{R}^n \) open and \( \tilde{W} = W \cap \mathbb{R}^{n-1} \). Then for each \( \varphi \in C^\infty_c(\Omega) \) the operator \( \varphi G_x \varphi \) has a Schwartz kernel

\[
k \in I^n(s(\Omega) \times W \times \Delta_{\tilde{W}}) \cong C^\infty(\Omega) \otimes I^n(W \times \Delta_{\tilde{W}}).
\]

Furthermore, \( \varphi G_x \varphi \) for each \( x \in s(\Omega) \) corresponds to the Schwartz kernel \( k_x \) via the diffeomorphism \( G_x \cap \Omega \cong W \).

Fix the following operations

\[
\begin{align*}
\mu_G : G \times G &\to G, (\gamma, \eta) \mapsto \gamma \eta^{-1}, \\
\mu : X \times G &\to X, (z, \gamma) \mapsto z \cdot \gamma^{-1}, \\
\mu^t : G \times X^t &\to X, (\gamma, z) \mapsto \gamma^{-1} \cdot z
\end{align*}
\]

whenever defined.

A trace type family \( T \) has a family of Schwartz kernels \( (k^T_x)_{x \in M} \). Define the support of \( T \) as

\[
\text{supp}(T) = \bigcup_{x \in M} \text{supp}(k^T_x).
\]

The reduced support of \( T \) is written

\[
\text{supp}_\mu(T) = \mu(\text{supp}(T)).
\]

The analogous definitions for potential type operators \( K \) and Green type operators \( G \) are given by

\[
\text{supp}_{\mu^t}(K) = \mu^t(\text{supp}(K)), \quad \text{supp}_{\mu^t}(G) = \mu^t(\text{supp}(G)).
\]

**Definition 5.2.**

- An extended trace operator is a differentiable family \( T = (T_x)_{x \in M} \) of trace type which is a right \( X^t \)-operator (see Definition 3.7, p. 117) such that the reduced support of \( T \) is a compact subset of \( X \).

- An extended potential operator is a differentiable family \( K = (K_x)_{x \in M} \) of potential type which is a left \( X^t \)-operator such that the reduced support of \( K \) is a compact subset of \( X \).

- An extended singular Green operator is a differentiable family \( G = (G_x)_{x \in M} \) of singular Green type which is equivariant and such that the reduced support of \( G \) is a compact subset of \( G \).

**Remark 5.3.**

i) Since we also have a right action of \( G \) on \( X \) and \( X \) is diffeomorphic (via \( f \)) to \( X^t \), we obtain that being a left \( X^t \)-operator is equivalent to the equivariance condition with regard to the right action of \( G \) on \( X \) given in equation (5) on p. 9. Hence a potential operator is also a right operator with regard to \( X \) in this sense, which furnishes by the proof of Prop. 3.8 a reduced kernel for extended potential operators.

ii) Note that we obtain the reduced kernels for pseudodifferential operators on \( G \) and extended singular Green operators with an argument completely analogous to the proof of Prop. 3.8.
Remark 5.6. see [30], p. 24.

Schwartz kernel theorem it can be proven that the spaces $\Psi$ of extended trace operators.

Proof. We give a proof of conormality for the case $\mathcal{K} \times \mathcal{G}$ of extended singular Green operators of order $\mathcal{M}$. For the pseudodifferential operators on $\mathcal{K} \times \mathcal{G}$ has a reduced kernel $k_T^\ast$ being a compactly supported distribution on $\mathcal{G}$ conormal to $\Delta_Y$.

Proposition 5.4. i) Given an extended trace operator $T$ the reduced kernel $k_T$ (see Proposition 3.3, p. 10) is a compactly supported distribution on $\mathcal{X}$ conormal to $\Delta_Y$.

ii) Analogously an extended potential operator $K$ has reduced kernel $k_K$ a compactly supported distribution on $\mathcal{X}^t$ conormal to $\Delta_Y$. Furthermore, $K$ is the adjoint of an extended trace operator.

iii) An extended singular Green operator $G$ has a reduced kernel $k_G$ being a compactly supported distribution on $\mathcal{G}$ conormal to $\Delta_Y$.

Proof. We give a proof of conormality for the case $\mathcal{K} \times \mathcal{G}$ of extended trace operators.

Given a family of Schwartz kernels for $(k_T^x)_{x \in M}$ contained in $\Gamma^m(\mathcal{X}_x \times \mathcal{G}_x, \mathcal{X}_x)$ for each $x \in M$. Rewrite this as

$$k_T^x = \mu^*(k_T)|_{\mathcal{X}_x \times \mathcal{G}_x}, \quad \mathcal{X}_x \subset \mathcal{G}_x \text{ (transversal)}.$$ 

Here $\mu$ is the map $\mathcal{K} \times \mathcal{G} \ni (z, \gamma) \mapsto z \cdot \gamma^{-1} \in \mathcal{K}$ and

$$\langle \mu^*(k_T), f \rangle = \left< k_T(z), \int_{w=\gamma} f(w, \gamma) \right>.$$ 

Then we need to show that: $\text{singsupp}(k_T) \subset Y \cong \Delta_Y$.

To this end let $z \in \mathcal{X} \setminus \Delta_Y$ and $\varphi \in C^\infty_c(\mathcal{X})$ a cutoff function such that $\varphi$ is equal to 1 in a neighborhood of $\Delta_Y$ and equal to 0 in a neighborhood containing $z$. Then

$$\mu^*((1 - \varphi)k_T) = (1 - \varphi \circ \mu)\mu^*(k_T)$$

restricted to $\mathcal{X}_x \times \mathcal{G}_x$ yields $(1 - \varphi \circ \mu)k_T^x$ and this is $C^\infty$ because $\text{singsupp}(k_T^x) \subset \Delta_x \cong \mathcal{X}_x \subset \mathcal{X}_x \times \mathcal{G}_x$ by definition. Hence $(1 - \varphi \circ \mu)\mu^*(k_T)$ is $C^\infty$, but this implies that $(1 - \varphi)k_T$ is smooth as well. This proves conormality.

Finally, we show that a trace operator is the adjoint of a potential operator and vice versa. Let $T = (T_x)_{x \in M}$ be an extended trace operator and let $(k_T^x)_{x \in M}$ be the corresponding family of Schwartz kernels. The adjoint $T^* = (T^*_x)_{x \in M}$ is given by $T^*_x : C^\infty_c(\mathcal{X}_x) \to C^\infty_c(\mathcal{G}_x)$ such that for $u \in C^\infty_c(\mathcal{X}_x)$ we have

$$(T^*_x u)(\gamma) = \int_{\mathcal{X}_x(\gamma)} \overline{k_T^x(z, \gamma)} u(z) \, d\lambda_s(\gamma)(z).$$

Define the family of operators $K = (K_x)_{x \in M}$ by $K = T^*$ and $k_K^x(\gamma, z) := \overline{k_T^x(z, \gamma)}$. We obtain a family $(k_K^x)_{x \in M}$ of distributions on $\mathcal{G}_x \times \mathcal{X}_x$ conormal to $\Delta_x \cong \mathcal{X}_x$ for each $x \in M$. In addition $K$ is equivariant with regard to the right action of $\mathcal{G}$ which is by remark 5.3 equivalent to being a left $\mathcal{X}^t$-operator. Hence $K$ is an extended potential operator. The same argument shows that the adjoint of an extended potential operator is an extended trace operator.

Notation 5.5. We fix the notation for the reduced kernels and denote by $\Gamma^m_c(\mathcal{X}, \Delta_Y)$ the space of reduced kernels of extended trace operators of order $m$, by $\Gamma^m_c(\mathcal{X}^t, \Delta_Y)$ the reduced kernels of extended potential operators of order $m$ and by $\Gamma^m_c(\mathcal{G}, \Delta_Y)$ the space of reduced kernels of singular Green operators of order $m$. For the pseudodifferential operators on $\mathcal{G}$ of order $m$ we use the notation $\Psi^m(\mathcal{G})$ for the space of operators and $\Gamma^m_c(\mathcal{G}, \Delta_M)$ for the reduced kernels. With the Schwartz kernel theorem it can be proven that the spaces $\Psi^m(\mathcal{G})$ and $\Gamma^m_c(\mathcal{G}, \Delta_M)$ are isomorphic, see [30], p. 24.

Remark 5.6. We will also use the notation

$$\Psi^{m,0}(\mathcal{G}, \mathcal{G}_0) := J_{tr} \circ \Gamma^m_c(\mathcal{X}, \Delta_Y), \quad K^{m,0}(\mathcal{G}, \mathcal{G}_0) := J_{pot} \circ \Gamma^m_c(\mathcal{X}^t, \Delta_Y),$$

$$G^{m,0}(\mathcal{G}, \mathcal{G}_0) := J_{gr} \circ \Gamma^m_c(\mathcal{G}, \Delta_Y)$$

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for these classes of extended trace, potential and singular Green operators, respectively. The \( \mathcal{J} \) in each case are the appropriate isomorphisms from the Schwartz kernel theorem. Hence the operators defined previously act as follows. The mapping

\[
\mathcal{J}_{\text{tr}}: I^m_c(\mathcal{X}, \Delta_Y) \rightarrow \mathcal{F}^{m,0}(\mathcal{G}, \mathcal{G}_\theta) \subset \text{Hom}(C_c^\infty(\mathcal{G}), C_c^\infty(\mathcal{X}'))
\]

is for \( z \in \mathcal{X}, \ k_T \in I^m_c(\mathcal{X}, \Delta_Y) \) given by

\[
(\mathcal{J}_{\text{tr}}(k_T)u)(z) = \int_{\mathcal{G}_q(z)} k_T(z \cdot \gamma^{-1})u(\gamma) \, d\mu_q(\gamma).
\]

Analogously for the potential operators we have

\[
\mathcal{J}_{\text{pot}}: I^m_c(\mathcal{X}^t, \Delta_Y) \rightarrow \mathcal{K}^{m,0}(\mathcal{G}, \mathcal{G}_\theta) \subset \text{Hom}(C_c^\infty(\mathcal{X}^t), C_c^\infty(\mathcal{G}))
\]

which for \( \gamma \in \mathcal{G}, \ k_K \in I^m_c(\mathcal{X}^t, \Delta_Y) \) is given by

\[
(\mathcal{J}_{\text{pot}}(k_K)u)(\gamma) = \int_{\mathcal{X}^t_{\gamma(\gamma)}} k_K(\gamma^{-1} \cdot z)u(z) \, d\lambda^t_{\gamma(\gamma)}(z).
\]

Lastly, for the singular Green operators

\[
\mathcal{J}_{\text{gr}}: I^m_c(\mathcal{G}, \Delta_Y) \rightarrow \mathcal{G}^{m,0}(\mathcal{G}, \mathcal{G}_\theta) \subset \text{Hom}(C_c^\infty(\mathcal{G}), C_c^\infty(\mathcal{G}))
\]

we have for \( \gamma \in \mathcal{G}, \ k_G \in I^m_c(\mathcal{G}, \Delta_Y) \)

\[
(\mathcal{J}_{\text{gr}}(k_G)u)(\gamma) = \int_{\mathcal{G}_{s(\gamma)}} k_G(\gamma \eta^{-1})u(\eta) \, d\mu_s(\gamma)(\eta).
\]

With any fibered space, longitudinally smooth via an action of a nice enough groupoid one can associate an equivariant calculus of pseudodifferential operators. We want to define such a calculus on \( \mathcal{X} \) and \( \mathcal{X}^t \). The following definition can in somewhat greater generality also be found in [32].

**Definition 5.7.** A family of pseudodifferential operators of order \( m \) on \( \mathcal{X} \) is defined as \( S = (S_x)_{x \in M} \) such that

i) each \( S_x: C^\infty(\mathcal{X}_x) \rightarrow C^\infty(\mathcal{X}_x) \) is contained in \( \Psi^m(\mathcal{X}_x) \).

ii) For each chart of \( \mathcal{X} \) given by \( \Omega \sim q(\Omega) \times W \) there is a smooth function \( a: q(\Omega) \rightarrow S^m(T^*W) \) such that for each \( x \in q(\Omega) \) we have

\[
S_x|_{\Omega \cap \mathcal{X}_x} = a_x(y, D_y)
\]

via identifying \( \Omega \cap \mathcal{X}_x \) with \( W \). Here \( a_x(y, \xi) = a(x)(y, \xi) \). We denote by \( \Psi^m(\mathcal{X}) \) the set of pseudodifferential families on \( \mathcal{X} \).

This leads immediately to a definition of equivariant pseudodifferential operators on \( \mathcal{X} \) and \( \mathcal{X}^t \).

**Definition 5.8.** The space of equivariant pseudodifferential operators \( \Psi^\bullet(\mathcal{X})^\mathcal{G} \) on \( \mathcal{X} \) consists of elements \( S = (S_x)_{x \in M} \) of \( \Psi^\bullet(\mathcal{X}) \) such that the following equivariance condition holds

\[
R_{\gamma^{-1}} S_{r(\gamma)} R_\gamma = S_{s(\gamma)}, \ \gamma \in \mathcal{G}.
\]

By analogy we define the equivariant pseudodifferential operators \( \mathcal{G}^\bullet \Psi^\bullet(\mathcal{X}^t) \) on \( \mathcal{X}^t \) coming from the left action of \( \mathcal{G} \). The equivariance condition in this case is given as in Definition 3.7 ii) on p. 10.
The operators defined here are in each case families parametrized over the double $M$. We have to clarify what role the pseudodifferential operators defined on $\mathcal{G}_\theta$ play.

**Proposition 5.9.** We have the following exact sequence

$$C^\infty_Y(M)\Psi^*(\mathcal{X})^G \longrightarrow \Psi^*(\mathcal{X})^G \longrightarrow R^G Y \longrightarrow \Psi^*(\mathcal{G}_\theta)$$

where $R^G Y$ is a well-defined restriction of families $(S_x)_{x \in M} \mapsto (S_y)_{y \in Y}$. Here $C^\infty_Y(M)$ are the smooth functions on $M$ that vanish on $Y$.

**Proof.** First note that $\mathcal{G}_\theta$ acts (from the left and the right) on itself. Extend this action to the set of families $(S_y)_{y \in Y}$ with $S_y \in \Psi^*(\mathcal{G}_\theta)_y$. Invariance under this action is just the usual equivariance condition for pseudodifferential operators. Together with the uniform support condition we therefore recover the class of pseudodifferential operators, denoted $\Psi^*(\mathcal{G}_\theta)$.

The exactness of the sequence

$$C^\infty_Y(M)\Psi^*(\mathcal{X}) \longrightarrow \Psi^*(\mathcal{X}) \longrightarrow R^G Y \longrightarrow \Psi^*(\mathcal{G}_\theta)$$

for the restriction operator $R_Y$ defined by $R_Y((S_x)_{x \in M}) = (S_y)_{y \in Y}$ is immediate. Here note that

$$\mathcal{X}_Y = q^{-1}(Y) = r^{-1}(Y) \cap s^{-1}(Y) = G^c_Y = \mathcal{G}_\theta$$

by assumption.

Note also that by the previous remarks $\Psi^*(\mathcal{X}_Y)^G_Y \cong \Psi^*(\mathcal{G}_\theta)$. This furnishes the exact sequence of equivariant pseudodifferential operators with a well-defined restriction map $R^G_Y$

$$C^\infty_Y(M)\Psi^*(\mathcal{X})^G \longrightarrow \Psi^*(\mathcal{X})^G \longrightarrow R^G Y \longrightarrow \Psi^*(\mathcal{G}_\theta) \cong \Psi^*(\mathcal{G}_\theta).$$

\[ \square \]

6. **Compositions**

In order to prove the main Theorem we first establish a Lemma about compositions of conormal distributions.

**Lemma 6.1.** The classes of extended Boutet de Monvel operators are closed under compositions induced by groupoid actions and convolution. More precisely we have the following compositions:

\[
\begin{align*}
*: I^{m_1}(\mathcal{X}, \Delta_Y) \times I^{m_2}(\mathcal{X}, \Delta_Y) &\rightarrow I^{m_1+m_2}_c(\mathcal{G}, \Delta_Y), \\
*: I^{m_1}_c(\mathcal{X}, \Delta_Y) \times I^{m_2}(\mathcal{X}, \Delta_Y) &\rightarrow \Psi^{m_1+m_2}(\mathcal{X}, \Delta_Y), \\
*: \Psi^{m_1}(\mathcal{X})^G \times I^{m_2}(\mathcal{X}, \Delta_Y) &\rightarrow I^{m_1+m_2}_c(\mathcal{X}, \Delta_Y), \\
*: \Psi^{m_1}(\mathcal{G}) \times I^{m_2}(\mathcal{G}, \Delta_Y) &\rightarrow I^{m_1+m_2}_c(\mathcal{G}, \Delta_Y), \\
*: I^{m_1}(\mathcal{G}, \Delta_Y) \times \Psi^{m_2}(\mathcal{G}) &\rightarrow I^{m_1+m_2}_c(\mathcal{G}, \Delta_Y), \\
*: I^{m_1}(\mathcal{G}, \Delta_Y) \times I^{m_2}(\mathcal{X}, \Delta_Y) &\rightarrow I^{m_1+m_2}_c(\mathcal{X}, \Delta_Y), \\
*: I^{m_1}(\mathcal{X}, \Delta_Y) \times \Psi^{m_2}(\mathcal{G}) &\rightarrow I^{m_1+m_2}(\mathcal{X}, \Delta_Y), \\
*: \Psi^{m_1}(\mathcal{G}) \times I^{m_2}(\mathcal{X}, \Delta_Y) &\rightarrow I^{m_1+m_2}(\mathcal{X}, \Delta_Y), \\
*: I^{m_1}(\mathcal{X}, \Delta_Y) \times \Psi^{m_2}(\mathcal{X}) &\rightarrow I^{m_1+m_2}(\mathcal{X}, \Delta_Y), \\
*: I^{m_1}(\mathcal{X}, \Delta_Y) \times I^{m_2}(\mathcal{X}, \Delta_Y) &\rightarrow I^{m_1+m_2}(\mathcal{X}, \Delta_Y). 
\end{align*}
\]
Proof. We have the equivalences \((12) \Leftrightarrow (13), (14) \Leftrightarrow (8)\) and \((15) \Leftrightarrow (11)\) by Prop. 5.4. ii).
Since the argument in each case goes along the same lines we only treat the first 3 cases of compositions exemplarily.

i) We consider first the case of the composition \((9)\). Consider a family of extended trace operators \(T = (T_x)_{x \in M}\) and extended potential operators \(K = (K_x)_{x \in M}\). Denote the corresponding family of Schwartz kernels by \(k_x^1 \in \mathcal{I}_{m_1}(\mathcal{X}_x \times \mathcal{G}_x, \mathcal{X}_x)\) as well as \(k_x^2 \in \mathcal{I}_{m_2}(\mathcal{G}_x \times \mathcal{X}_x^t, \mathcal{X}_x^t)\) for \(x \in M\).

We make the following computation involving an interchange of integration we still have to justify via a reduction to local coordinates. Let \(\gamma \in \mathcal{G}_x\) then

\[
(K_x \cdot T_x)u(\gamma) = \int_{\mathcal{X}_x^t} k_x^2(\gamma, z)(T_x u)(z) \, d\lambda_x^t(z)
= \int_{\mathcal{X}_x^t} \int_{\mathcal{G}_x} k_x^2(\gamma, z) k_x^1 T(z, \eta) u(\eta) \, d\mu_x(\eta) \, d\lambda_x^t(z)
= \int_{\mathcal{G}_x} k_x^2T(\gamma, \eta) u(\eta) \, d\mu_x(\eta).
\]

The kernels of the composition \(k_x^{K \cdot T}\) would therefore take the form

\[
k_x^{K \cdot T}(\gamma, \eta) = \int_{\mathcal{X}_x^t} k_x^2(\gamma, z) k_x^1 T(z, \eta) \, d\lambda_x^t(z).
\]

This corresponds to the convolution of reduced kernels \(k_k \ast k_T\) which is immediately defined from the actions.

First we check the support condition of the composed operator. The reduced support is compact via the inclusion

\[
supp_{\mu}(K \cdot T) \subset \mu_G(supp_{\lambda}(K) \times supp_{\lambda'}(T)).
\]

Here the inversion of elements in the spaces \(\mathcal{X}\) and \(\mathcal{X}'\) is performed inside the groupoid \(\mathcal{G}\) where it is always defined.

Fix the projections

\[
p_1 : \mathcal{X} \times \mathcal{G} \rightarrow \mathcal{X}, \ p_2 : \mathcal{X} \times \mathcal{G} \rightarrow \mathcal{G}.
\]

Then by the uniform support condition the family \(T = (T_x : x \in M)\) is in particular properly supported. This means for compact sets \(K_1 \subset \mathcal{G}, \ K_2 \subset \mathcal{X}\) we have that

\[
p_i^{-1}(K_i) \cap supp(k_T) \subset \mathcal{X} \times \mathcal{G}, \ i = 1, 2
\]
is compact. We make use of this property for the following argument.

Next we check the smoothness property of compositions. Let \(f \in C_c^{\infty}(\mathcal{G})\) be given, we will show that \(Tf \in C_c^{\infty}(\mathcal{X})\). Assume that \(T = (T_x : x \in M)\) has a Schwartz kernel \(k_T\) contained in \(I^{-\infty}(\mathcal{X}, \Delta_Y) = \bigcap_m I^m(\mathcal{X}, \Delta_Y)\). Then \(k_T\) is \(C^{\infty}\) on the closed subset \(\{(\gamma, \eta) : q(\gamma) = s(\eta)\}\) of \(\mathcal{X} \times \mathcal{G}\) and via fiber preserving diffeomorphisms we obtain a \(C^{\infty}\)-atlas. The function \(Tf\) yields a smooth function because we integrate the kernels \(k_x^T\) which are smooth functions. Hence we can interchange integration and differentiation. Therefore \(Tf \in C_c^{\infty}(\mathcal{X})\) for \(k_T \in I^{-\infty}(\mathcal{X}, \Delta_Y)\).

Consider a general extended trace operator \(T\). Let \((\gamma, z) \in \mathcal{G}_x \times \mathcal{X}_x\) and \(\Omega \subset \mathcal{G}\) be a chart with fiber preserving diffeomorphism \(\Omega \sim s(\Omega) \times W\). We can assume that \(W \subset \mathbb{R}^n\) is convex, open, \(0 \in W\) and that \(\Omega\) is a neighborhood of \(\gamma\) such that \((x, 0)\) gets mapped to \(\gamma\) via the diffeomorphism. We also set \(\tilde{\Omega} = \Omega \cap \mathcal{X} \subset \mathcal{X}\) and \(\tilde{W} = W \cap \mathbb{R}^{n-1}\) with a fiber preserving diffeomorphism \(\tilde{\Omega} \sim q(\tilde{\Omega}) \times \tilde{W}\) (recall the fact that \(q = s|_{\mathcal{G}_x}\) by assumption). By the previous remarks the family \(T\) is properly supported, which implies in particular that each \(T_x\) is properly supported for \(x \in M\). Hence we obtain that the kernels \(k_x^T\) of \(T\) satisfy the support estimate

\[
p_1^{-1}\left(\frac{q(\tilde{\Omega}) \times \frac{W}{2}}{2}\right) \cap p_2^{-1}\left(\frac{s(\Omega) \times \frac{W}{2}}{2}\right) \cap \bigcup_x supp(k_x^T) \subset \left(\frac{q(\tilde{\Omega}) \times \frac{3W}{2}}{2}\right) \times \left(\frac{s(\Omega) \times \frac{3W}{2}}{2}\right)
\]
therefore the fact that \( T f \in C^\infty_c(\mathcal{X}) \) reduces to a computation in local coordinates. Similar reasoning applies to potential operators. Using the same argument as above we deduce that \( K f \in C^\infty_c(\mathcal{G}) \) for \( f \in C^\infty_c(\mathcal{X}) \) if \( K \) is smoothing. For a general \( K \) we note that each \( K_x \) is properly supported for \( x \in M \) and hence we obtain that the kernels \( k_x^K \) of \( K \) satisfy the support estimate

\[
p_1^{-1} \left( q(\bar{\Omega}) \times \frac{\bar{W}}{2} \right) \cap p_2^{-1} \left( s(\Omega) \times \frac{W}{2} \right) \subset \left( q(\bar{\Omega}) \times \frac{3\bar{W}}{2} \right) \times \left( s(\Omega) \times \frac{3W}{2} \right).
\]

The smoothness of \( K f \) reduces to a computation in local coordinates. Consider the general composition \( K \cdot T \) for \( T \) an arbitrary extended trace operator and \( K \) an arbitrary extended potential operator. Then make the suitable support estimates as above to show that \( K_x \cdot T_x \) are compositions of smooth families of conormal distributions which act on the sets \( W \subset \mathbb{R}^n, \bar{W} \subset \mathbb{R}^{n-1} \). It then follows from a general theorem of Hörmander about compositions of conormal distributions, see [18], Thm. 25.2.3, p. 21 that the composition of \( K \) and \( T \) is smoothing. For a general \( K \) supported since \( K \) is equivariant with regard to the right action of the groupoid \( G \), we calculate

\[
(\gamma \cdot T)_{\gamma^{-1}} = (K \cdot T)_{\gamma^{-1}} = K_{\gamma^{-1}} T_{\gamma^{-1}}.
\]

Since \( K \) is equivariant with regard to the right action of \( \mathcal{G} \) on \( \mathcal{X} \) by Remark 5.3, \( i \), the equivariance condition for \( K \) reads

\[
R_{\gamma^{-1}} K_{\gamma} = K_{\gamma} R_{\gamma^{-1}}, \quad \forall \gamma \in \mathcal{G}.
\]

For \( \gamma \in \mathcal{G} \) we calculate

\[
R_{\gamma^{-1}} (K \cdot T)_{\gamma} = R_{\gamma^{-1}} (K_{\gamma} \cdot T_{\gamma}) = R_{\gamma^{-1}} K_{\gamma} R_{\gamma} = R_{\gamma^{-1}} K_{\gamma} R_{\gamma} T_{\gamma} = K_{\gamma} (R_{\gamma}^{-1}) R_{\gamma} T_{\gamma} = K_{\gamma} T_{\gamma}.
\]

and hence \( K \cdot T \) has the required equivariance property with regard to the right action of the groupoid \( \mathcal{G} \). We have thus verified all the properties of an extended Green operator.

\( ii \) Consider the next composition \( T \cdot K : C^\infty_c(\mathcal{X}) \to C^\infty_c(\mathcal{X}) \) which is again for \( z \in \mathcal{X} \) and \( u \in C^\infty_c(\mathcal{X}) \) given by

\[
(T_x \cdot K_x)u(z) = \int_{\mathcal{G}_q(z)} k_x^T(z, \gamma) (K_x u)(\gamma) \, d\mu_x(\gamma)
\]

\[
= \int_{\mathcal{G}_x} \int_{\mathcal{Y}_z^i} k_x^T(z, \gamma) k_x^K(\gamma, w) u(w) \, d\lambda_x^i(w) \, d\mu_x(\gamma)
\]

\[
= \int_{\mathcal{Y}_z^i} k_x^{T \cdot K}(z, w) u(w) \, d\lambda_x^i(w).
\]

The kernel \( k_x^{T \cdot K} \) is written

\[
k_x^{T \cdot K}(z, w) = \int_{\mathcal{G}_x} k_x^T(z, \gamma) k_x^K(\gamma, w) \, d\mu_x(\gamma).
\]
Now we can argue again analogously to \( i \) that the composition has the right support condition and via a reduction to local charts the formal computation can be made precise. We therefore obtain a family of kernels \( k^T_{x,K} \in \Gamma_{m_1+m_2}(X_x^t \times \mathcal{X}_x, \Delta_{X_x}) \).

**iii)** The third composition \( S \cdot T : C^\infty_c(G) \rightarrow C^\infty_c(\mathcal{X}) \) gives a family of extended trace operators. We obtain for \( z \in \mathcal{X}, u \in C^\infty_c(G) \)

\[
(S_x \cdot T_x)(z) = \int_{X_x} k^S_x(z,w)(T_x u)(w) \, d\lambda_x(w)
\]

\[
= \int_{X_x} \int_{G_x} k^S_x(z,w)k^T_x(w,\gamma)u(\gamma) \, d\mu_x(\gamma) \, d\lambda_x(w)
\]

\[
= \int_{G_x} k^{S\cdot T}_x(z,\gamma)u(\gamma) \, d\mu_x(\gamma).
\]

We obtain the kernel

\[
k^{S\cdot T}_x(z,\gamma) = \int_{X_x} k^S_x(z,w)k^T_x(w,\gamma) \, d\lambda_x(w).
\]

We proceed by making the analogous argument as in \( i \), \( ii \). The right support condition holds on \( \Psi(\mathcal{X})^0 \) e.g. via the identification from \([5.9]\) The rest of the reasoning then yields a family of kernels \( k^{S\cdot T}_x \in \Gamma_{m_1+m_2}(X_x \times G_x, \Delta_{X_x}) \) with the correct support condition. \( \square \)

7. The calculus

7.1. Quantization

Let us denote by \( B^m_{\text{prop}}(M_0, Y_0) \) the properly supported extended Boutet de Monvel operators of order \( m \), defined on the interior. In this section we introduce an algebra of extended operators \( B^m_{\text{prop}}(M,Y) \) on the double Lie manifold \( M \). This is defined by extending the distributional kernels in \( B^m_{\text{prop}}(M_0, Y_0) \) to take the Lie structure into account. We also fix the actions with corresponding notation from \([5.1]\). Introduce the singular normal bundles for the inclusions \( \Delta_Y \hookrightarrow \mathcal{X}, \Delta_Y \hookrightarrow \mathcal{G} \) as well as \( \Delta_Y \hookrightarrow \mathcal{G} \):

\[
N^X \Delta_Y \rightarrow Y, \quad N^{X'} \Delta_Y \rightarrow Y, \quad N^\mathcal{G} \Delta_Y \rightarrow Y.
\]

Restricted to the interior we have by axiom \( iii \) in Def. \([4.1]\) the isomorphisms

\[
N^X \Delta_Y|_{Y_0} \cong N_{Y_0}^0 \times M_0 \Delta_{Y_0}, \quad N^{X'} \Delta_Y|_{Y_0} \cong N_{M_0}^0 \times Y_0 \Delta_{Y_0}, \quad N^\mathcal{G} \Delta_Y|_{Y_0} \cong N_{M_0}^0 \times M_0 \Delta_{Y_0}.
\] (16)

Here we denote by \( N_{Y_0}^0 \times M_0 \Delta_{Y_0} \) the normal bundle to the inclusion \( \Delta_{Y_0} \hookrightarrow Y_0 \times M_0 \) and the same for the others. It is not hard to see that \( N^X \Delta_Y \) can be identified with \( \mathcal{A}|_Y \) which is isomorphic to \( A_M \cong \mathcal{N} \).

**Remark 7.1.** \( i \) On the singular normal bundles we define the Hörmander symbols spaces \( S^m(N^X \Delta_y^\ast) \subset C^\infty(N^X \Delta_y^\ast) \) such that for \( U \subset Y \) open with

\[
N^X \Delta_y^\ast|_U \cong U \times \mathbb{R}^{n-1} \times \mathbb{R}, \quad K \subset U \text{ compact}.
\]

We have the estimates for \( t \in S^m(N^X \Delta_y^\ast) \)

\[
|D^\alpha \xi^\delta t(x',\xi)| \leq C_{K,\alpha,\beta}(\xi)^{|m-|\beta|}, \quad (x,\xi) \in K \times \mathbb{R}^{n-1} \times \mathbb{R}
\]

for each \( \alpha \in \mathbb{N}_0^{n-1}, \beta \in \mathbb{N}_0^n \). Note that we have by Hörmander’s results a correspondence between the spaces of symbols on the normal bundle to a smooth manifold and conormal distributions on the space (at least in the smooth case, cf. \([17]\), Thm 18.2.11):

\[
I^L(\mathcal{X}, \Delta_Y)/I^{-\infty}(\mathcal{X}, \Delta_Y) \cong S^m(N^X \Delta_y^\ast)/S^{-\infty}(N^X \Delta_y^\ast)
\]
where $L$ is the obligatory correction of order

$$m = L - \frac{1}{4}\dim \mathcal{X} + \frac{1}{2}\dim \Delta_Y.$$  

We will ignore this order convention in the following discussions. Note that our earlier definition of smooth families of operators defined as conormal distributions suggests immediately a quantization which we state next. We can require additionally the (local) rapid decay property stated earlier, then we use the notation $S^m_N(\mathcal{N}^X \Delta_Y^*)$ for these symbol spaces. Analogously the spaces

$$S^m_N(\mathcal{N}^X \Delta_Y^*) \subset C^\infty(\mathcal{N}^X \Delta_Y^*), \quad S^m_N(\mathcal{N}^G \Delta_Y^*) \subset C^\infty(\mathcal{N}^G \Delta_Y^*).$$

ii) A second definition we will need is that of conormal distributions on the normal bundles themselves. First given the normal and conormal bundles

$$\pi: \mathcal{N}^X \Delta_Y \to Y, \quad \pi: \mathcal{N}^G \Delta_Y^* \to Y,$$

(and analogously for $\mathcal{N}^X$, $\mathcal{N}^G$) define the fiberwise Fourier transform $F_\pi: S(\mathcal{N}^X \Delta_Y) \to S(\mathcal{N}^X \Delta_Y^*)$

$$F_\pi(\varphi)(\xi) := \int_{\pi(\zeta)=\pi(\xi)} e^{-i\langle \xi, \zeta \rangle} \varphi(\zeta) \, d\zeta.$$  

The inverse is given by duality

$$F_\pi^{-1}(\varphi)(\zeta) = \int_{\pi(\zeta)=\pi(\xi)} e^{i\langle \xi, \zeta \rangle} \varphi(\zeta) \, d\xi, \quad \varphi \in S(\mathcal{N}^X \Delta_Y^*).$$

Here we use the notation $S(\mathcal{N}^X \Delta_Y)$, $S(\mathcal{N}^X \Delta_Y^*)$ for the spaces of rapidly decreasing functions on the normal and conormal bundle respectively, see also [35], chapter 1.5. Then the spaces of conormal distributions are defined as:

$$I^m(\mathcal{N}^X \Delta_Y, \Delta_Y) := F_\pi^{-1} S^m(\mathcal{N}^X \Delta_Y^*)$$

and $I^m(\mathcal{N}^X \Delta_Y, Y)$, $I^m(\mathcal{N}^G \Delta_Y, Y)$ analogously.

For the definition of the quantization rule we need some further notation. Let $0 < r \leq r_0$ where $r_0$ is the (positive) injectivity radius of $M$.

- First for the case of trace operators. We set

$$\mathcal{N}^X \Delta_Y)_r = \{ v \in \mathcal{N}^X \Delta_Y : \|v\| < r \}$$

as well as

$$I^m_{(r)}(\mathcal{N}^X \Delta_Y, \Delta_Y) = I^m((\mathcal{N}^X \Delta_Y)_r, \Delta_Y).$$

Fix the restriction

$$R: I^m_{(r)}(\mathcal{N}^X \Delta_Y, \Delta_Y) \to I^m_{(r)}(N^{Y_0 \times M_0} \Delta_{Y_0}, \Delta_{Y_0}).$$

We denote by $\Psi$ the normal fibration of the inclusion $\Delta_{Y_0} \to Y_0 \times M_0$ such that $\Psi$ is the local diffeomorphism mapping an open neighborhood of the zero section $\mathcal{O}_{Y_0} \subset V \subset N^{Y_0 \times M_0} \Delta_{Y_0}$ onto an open neighborhood $\Delta_{Y_0} \subset U \subset Y_0 \times M_0$ (cf. [35], Thm. 4.1.1). Then we have the induced map on conormal distributions

$$\Psi_*: I^m_{(r)}(N^{Y_0 \times M_0} \Delta_{Y_0}, \Delta_{Y_0}) \to I^m(Y_0 \times M_0, \Delta_{Y_0}).$$

Also let $\chi \in C^\infty_c(\mathcal{N}^X \Delta_Y)$ be a cutoff function which acts by multiplication

$$I^m(\mathcal{N}^X \Delta_Y, \Delta_Y) \to I^m_{(r)}(\mathcal{N}^X \Delta_Y, \Delta_Y).$$
• For potential operators we use the analogous notation:
\[ \mathcal{R}^t, \Psi^t, \mathcal{F}^t, \chi^t. \]

• Finally, in the singular Green case we have the induced normal fibration
\[ \Phi_*: I^m(N^\Delta_{M_0 \times M_0} \Delta_{Y_0}, \Delta_{Y_0}) \to I^m(M_0 \times M_0, \Delta_{Y_0}). \]
and the fiberwise Fourier transform
\[ \mathcal{F}_f^\Delta: I^m(N^\Delta M, \Delta_M) \to S^m(N^\Delta \Delta_Y^*). \]
The restriction and cutoff is denoted by
\[ \mathcal{R}^\Delta: I^m(N^\Delta M, \Delta_M) \to I^m(N^{M_0 \times M_0} \Delta_{Y_0}, \Delta_{Y_0}) \]
and
\[ \mathcal{F}_f: I^m(N^\Delta M, \Delta_M) \to I^m(N^\Delta Y, \Delta_Y). \]

Definition 7.2 (Quantization). i) Define
\[ q_{T, \chi}: S^m(N^\Delta X^* \Delta^*_Y) \to \mathcal{F}^{m,0}(M,Y) \]
such that for \( t \in S^m(N^\Delta X^* \Delta^*_Y) \) we have
\[ q_{T, \chi}(t) = J_{tr} \circ q_{\Psi, \chi}(t) \]
where
\[ q_{\Psi, \chi}(t) = \Psi_* (\mathcal{R}(\chi \mathcal{F}^{-1}_f(t))). \]

ii) Define
\[ q_{K, \chi^t}: S^m(N^\Delta X^t \Delta^*_Y) \to \mathcal{K}^m(M,Y) \]
such that for \( k \in S^m(N^\Delta X^t \Delta^*_Y) \) we have
\[ q_{K, \chi^t}(k) = J_{pot} \circ q_{\Psi, \chi^t}(k). \]

iii) Define
\[ q_{G, \varphi}: S^m(N^\Delta Y^* \Delta^*_Y) \to \mathcal{G}^{m,0}(M,Y) \]
such that for \( g \in S^m(N^\Delta Y^* \Delta^*_Y) \) we have
\[ q_{G, \varphi}(g) = J_{pot} \circ q_{\Phi, \varphi}(g). \]

Proposition 7.3. The fibrations \( q_{\Psi, \chi}, q_{\Psi, \chi^t} \) and \( q_{\Phi, \varphi} \) define properly supported Schwartz kernels.

Proof. Consider exemplarily the trace operators. Since \( \chi \mathcal{R}(t) \) is properly supported we find that \( q_{T, \chi}(t) \) defines a properly supported operator. It is clear from the definition that \( q_{T, \chi}(t): C^\infty_c(M_0) \to C^\infty_c(Y_0) \) has the Schwartz kernel \( q_{\Psi, \chi}(t) \).

The following is easy to check.

Proposition 7.4. The quantizations \( q_{T, \chi}, q_{K, \chi}, q_{G, \chi} \) are in each case independent of the choice of cutoff functions up to smoothing errors.
From the compactness of $M$ and $Y$ we can associate to each vector field in $2V$ respectively $W$ a global flow

$$2V \ni V \mapsto \Phi_V : \mathbb{R} \times M \to M, \quad W \ni W \mapsto \Psi_W : \mathbb{R} \times Y \to Y.$$ 

Then consider the diffeomorphisms evaluated at time $t = 1$

$$\Phi(1, -) : M \to M \text{ and } \Psi(1, -) : Y \to Y$$

and fix the corresponding group actions on functions which we denote by

$$2V \ni V \mapsto \varphi_V : C^\infty(M) \to C^\infty(M), \quad W \ni W \mapsto \psi_W : C^\infty(Y) \to C^\infty(Y).$$

The upshot of this is a definition of the suitable smoothing terms for our calculus which we state next.

**Definition 7.5.** i) The class of $V$-trace operators is defined as

$$\mathcal{F}^{m, 0}_{2V}(M, Y) := \mathcal{F}^{m, 0}(M, Y) + \mathcal{F}^{-\infty, 0}_{2V}(M, Y).$$

Here $\mathcal{F}^{m, 0}(M, Y)$ consists of the extended operators from the previous definition. The residual class is defined as follows

$$\mathcal{F}^{-\infty, 0}_{2V}(M, Y) := \text{span}\{q_{\chi, T}(t)\varphi_{V_1} \cdots \varphi_{V_k} : V_j \in 2V, \; \chi \in C^\infty_c(A_\partial), \; t \in S_N^{-\infty}(\mathcal{N}^X \Delta_Y^*)\}.$$

ii) The class of $V$-potential operators is defined in the same fashion

$$\mathcal{K}^{m, 0}_{2V}(M, Y) := \mathcal{K}^{m, 0}(M, Y) + \mathcal{K}^{-\infty}_{2V}(M, Y)$$

with residual class

$$\mathcal{K}^{-\infty}_{2V}(M, Y) := \text{span}\{q_{\chi, K}(k)\psi_{W_1} \cdots \psi_{W_k} : W_j \in W, \; \chi \in C^\infty_c(A_\partial), \; k \in S_N^{-\infty}(\mathcal{N}^X \Delta_Y^*)\}.$$

iii) Lastly, the class of $V$-singular Green operators is defined as

$$\mathcal{G}^{m, 0}_{2V}(M, Y) := \mathcal{G}^{m, 0}(M, Y) + \mathcal{G}^{-\infty}_{2V}(M, Y)$$

with residual class

$$\mathcal{G}^{-\infty, 0}_{2V}(M, Y) := \text{span}\{q_{\chi, G}(g)\psi_{V_1} \cdots \psi_{V_k} : V_j \in 2V, \; \chi \in C^\infty_c(A_\partial), \; g \in S_N^{-\infty}(\mathcal{N}^G \Delta_Y^*)\}.$$

iv) Then the calculus $\mathcal{B}^{m, 0}_{2V}(M, Y)$ of extended operators consists of matrices of the form

$$A = \begin{pmatrix} P + G & K \\ T & S \end{pmatrix}$$

for $P \in \Psi^m_{2V}(M), S \in \Psi^m_W(Y)$ and $G$ an extended singular Green operator, $K$ extended potential and $T$ extended trace operator.
7.2. Composition

The restriction $\chi^+$ to the interior $\tilde{X}_0 := X_0 \setminus Y_0$ and the extension by zero operator $\chi^0$ are given on the manifold level by

$$L^2(M_0) \xrightarrow{\chi^+} L^2(\tilde{X}_0)$$

with $\chi^+\chi^0 = \text{id}_{L^2(\tilde{X}_0)}$ and $\chi^0\chi^+$ being a projection onto a subspace of $L^2(M_0)$.

On the groupoid level we use the same symbols since it will be clear from context which is meant. So we define the operators

$$L^2(\mathcal{G}) \xrightarrow{\chi^+} L^2(\mathcal{G}^+)$$

with $\chi^+\chi^0 = \text{id}_{L^2(\mathcal{G}^+)}$ and $\chi^0\chi^+$ being a projection onto a subspace of $L^2(\mathcal{G})$.

**Definition 7.6.** The operator $P \in \Psi^m(\mathcal{G})$ has the transmission property if the symbol $a \in S^m_{tr}(\mathcal{A}^*)$. Here the class of Hörmander symbols $a \in S^m_{tr}(\mathcal{A}^*)$ consists of families $a = (a_x)_{x \in \mathcal{M}}$ such that each symbol $a_x$ has the transmission property with regard to $\mathcal{X}_x \subset \mathcal{G}_x$. In particular the operators $(\chi^+ P \chi^0)_x$ map functions smooth up to the boundary $\mathcal{X}_x$ to functions which have the same property.

**Example 7.7.**

- Notice first that if $x \in M_0$ is an interior point we have that $\mathcal{G}_x \cong M_0$ and we recover the transmission property on the interior manifold $X_0$ with boundary $Y_0$.

- In our trivial case $\mathcal{G} = M \times M$ and $M = 2X$, $X$ a compact manifold with boundary $\partial X = Y$ we recover the transmission property meaning $\Psi^m_{tr}(M) \cong \Psi^m_{tr}(\mathcal{G})$.

**Notation 7.8.** The operation of truncation itself is given as a linear operator.

i) On the groupoid calculus this operator is given by

$$\text{End} \left( \frac{C^\infty_c(\mathcal{G})}{C^\infty_c(\mathcal{X})} \right) \supset B^{m,0}(\mathcal{G}, \mathcal{X}) \ni A = \begin{pmatrix} P + G & K \\ T & S \end{pmatrix} \mapsto \tilde{\mathcal{C}}(A) = \begin{pmatrix} \chi^+(P + G)\chi^0 & \chi^+K \\ T\chi^0 & S \end{pmatrix} \in \text{End} \left( \frac{C^\infty_c(\mathcal{G}^+)}{C^\infty_c(\mathcal{X})} \right).$$

ii) On the extended calculus we define

$$\text{End} \left( \frac{C^\infty_c(M_0)}{C^\infty_c(Y_0)} \right) \supset B^{m,0}_{2\mathcal{V}}(M, Y) \ni A = \begin{pmatrix} P + G & K \\ T & S \end{pmatrix} \mapsto \mathcal{C}(A) = \begin{pmatrix} \chi^+(P + G)\chi^0 & \chi^+K \\ T\chi^0 & S \end{pmatrix} \in \text{End} \left( \frac{C^\infty_c(X_0)}{C^\infty_c(Y_0)} \right).$$

**Definition 7.9.**

i) The Boutet de Monvel calculus on the boundary structure is defined as the set of operators for $m \leq 0$

$$B^{m,0}(\mathcal{G}^+, \mathcal{X}) := \tilde{\mathcal{C}} \circ B^{m,0}(\mathcal{G}, \mathcal{X}).$$

ii) The class of Boutet de Monvel operators on the Lie manifold with boundary is for $m \leq 0$ defined as

$$B^{m,0}_{2\mathcal{V}}(X, Y) := \mathcal{C} \circ B^{m,0}_{2\mathcal{V}}(M, Y).$$

The vector representation $\tilde{\mathcal{C}}_{BM}$ is defined by

$$A \begin{pmatrix} \varphi \circ r \\ \psi \circ r \end{pmatrix} = \left( \tilde{\mathcal{C}}_{BM}(A) \begin{pmatrix} \varphi \\ \psi \end{pmatrix} \right) \circ \begin{pmatrix} r \\ r \end{pmatrix}$$

for $A \in B^{m,0}(\mathcal{G}^+, \mathcal{X})$ and $\varphi \in C^\infty_c(X_0)$, $\psi \in C^\infty_c(Y_0)$.

**Proof** (of Theorem 7.1). Let $A, B \in B^{0,0}(\mathcal{G}^+, \mathcal{X})$, then $A = (A_x)_{x \in \mathcal{X}}$, $B = (B_x)_{x \in \mathcal{X}}$ are equivariant families of properly supported Boutet de Monvel operators the fibers $(\mathcal{G}^+_x, \mathcal{X}_x)$ which are smooth manifolds with boundary. The composition $A \cdot B = (A_x \cdot B_x)_{x \in \mathcal{X}}$ is on the fibers of the groupoids and the spaces $\mathcal{X}$ and $\mathcal{X}^\ell$. Using the support estimates in the proof of Lemma 6.1 the calculation can be performed in local charts by reduction to composition of properly supported operators of Boutet de Monvel type on smooth manifolds. \qed
7.3. Representation theorem

**Theorem 7.10.** Given a $\mathcal{V}$-boundary structure the previously defined vector representation $\varrho_{BM}$ furnishes the isomorphism

$$\varrho_{BM} \circ \mathcal{B}_{m,0}^0(\mathcal{G}, \mathcal{X}) \cong \mathcal{B}_{2V}^m(\mathcal{M}, \mathcal{Y}).$$

**Proof.** The proof is similar to the proof of Theorem 3.2. in [3]. For further details we also refer to [6, Theorem 7.9].

**Proof (of Thm. 1.2).** From the right action of the groupoid $\mathcal{G}$ on $\mathcal{X}$ we obtain the induced action of the subgroupoid $\mathcal{G}^+ \subset \mathcal{G}$. The right action $\mathcal{X}|_{\mathcal{G}^+}$ defines the equivariance of the families in the calculus $\mathcal{B}_{m,0}^0(\mathcal{G}^+, \mathcal{X})$. The range and source maps in $\mathcal{G}^+$ are the restrictions of the range and source maps in $\mathcal{G}$, hence the representation $\tilde{\varrho}_{BM}$ is well-defined. Thereby we obtain that the truncation maps are linear, equivariant maps and the following diagram commutes as linear operators

$$\begin{array}{ccc}
\mathcal{B}_{m,0}^0(\mathcal{X}, \mathcal{Y}) & \xleftarrow{\tilde{\varrho}_{BM}} & \mathcal{B}_{m,0}^0(\mathcal{G}^+, \mathcal{X}) \\
\mathcal{C} & \quad \downarrow \quad & \mathcal{C} \\
\mathcal{B}_{2V}^m(\mathcal{M}, \mathcal{Y}) & \xleftarrow{\varrho_{BM}} & \mathcal{B}_{m,0}^0(\mathcal{G}, \mathcal{X}).
\end{array}$$

Since $\mathcal{C}, \tilde{\mathcal{C}}$ and $\varrho_{BM}$ are surjective (by Theorem 7.10) we obtain the surjectivity of $\tilde{\varrho}_{BM}$ as follows. Let $\tilde{B} \in \mathcal{B}_{m,0}^0(\mathcal{G}^+, \mathcal{X})$ then by surjectivity of $\varrho_{BM}$ and $\mathcal{C}$ we lift this to an element $\tilde{B} \in \mathcal{B}_{m,0}^0(\mathcal{G}, \mathcal{X})$. Then $A := \tilde{\mathcal{C}}(\tilde{B})$ is the required preimage. By commutativity we have

$$\tilde{\varrho}_{BM}(A) = (\tilde{\varrho}_{BM} \circ \tilde{\mathcal{C}})(\tilde{B}) = (\mathcal{C} \circ \varrho_{BM})(\tilde{B}) = B.$$ 

Hence in this case $\tilde{\varrho}_{BM}$ is surjective. It is also immediate that it is a well-defined homomorphism of algebras. This yields the closedness under composition. 

**Vector bundles**

Up until now we have only considered scalar operators of Boutet de Monvel type. It does only require minor modifications to consider operators acting on smooth sections of smooth vector bundles, see also [30]. To this effect let $E_1, E_2 \to X$ be smooth vector bundles on $X$ and $J_\pm \to Y$ smooth vector bundles on $Y$. We can pull back these bundles to $\mathcal{G}^+$ via $\tilde{E}_i := r^*E_i \to \mathcal{G}^+$. Similarly, the actions allow us to pull back the bundles $J_\pm$ to $\mathcal{X}$ and obtain $\tilde{J}_\pm \to \mathcal{X}$.

It is not difficult to modify our construction for operators acting on the smooth sections such that $A \in \mathcal{B}_{V}^{m,0}(\mathcal{X}, \mathcal{Y}; E_1, E_2, J_\pm)$ is a continuous linear operator

$$\begin{array}{ccc}
C^\infty(X, E_1) & \oplus & C^\infty(X, E_2) \\
A: & \oplus & \oplus \\
C^\infty(Y, J_+) & \oplus & C^\infty(Y, J_-)
\end{array}$$

Similarly, $A \in \mathcal{B}^{0,0}(\mathcal{G}^+, \mathcal{X}; \tilde{E}_1, \tilde{E}_2)$ is a continuous linear operator

$$\begin{array}{ccc}
C_c^\infty(\mathcal{G}^+, \tilde{E}_1) & \oplus & C_c^\infty(\mathcal{G}^+, \tilde{E}_2) \\
A: & \oplus & \oplus \\
C_c^\infty(\mathcal{X}, \tilde{J}_+) & \oplus & C_c^\infty(\mathcal{X}, \tilde{J}_-)
\end{array}$$

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8. Properties of the calculus

The following sections mark the start of the analysis of the represented side of our calculus (the algebra $\mathcal{B}^{0,0}_Y(X,Y)$). In order to limit the size of the paper and preserve its readability most proofs are referred to the comprehensive literature on Boutet de Monvel’s calculus, where these constructions have already been given and which need only be adapted to our setup in a straightforward manner.

Set $r(\xi) := (1 + ||\xi||^2)^{\frac{1}{2}}$, then close to the boundary (in a fixed tubular neighborhood) we define the symbol

$$r_-^m(x, \xi', \xi_n) = \left( \varphi \left( \frac{\xi_n}{C(\xi')} \right) \langle \xi' \rangle^m - i \xi_n \right)^m$$

for $m \in \mathbb{R}$, some constant $C > 0$ and $\varphi \in S(\mathbb{R})$ such that $\varphi(0) = 1$, $\text{supp} \varphi^{-1} \varphi \subset \mathbb{R}_-$. Then $r_-^m(\xi)$ is an elliptic symbol and has the transmission property (cf. [13], Proposition 1.3).

We construct a global classical symbol $a \in \mathcal{S}^1_\mathcal{cl}(A^*)$ as follows. Fix a normal cover $\bigcup_{i=1}^\infty U_i = M$ of $M$ with local trivializations $\Psi_i : \mathcal{A}U_i \cong U_i \times \mathbb{R}^n$.

- If $U_i \cap Y = \emptyset$ (an interior chart) set $\Psi_i a(\xi) = r(\xi)$.
- If $U_i \cap Y \neq \emptyset$ (a boundary chart) set $\Psi_i a(\xi) = r_-^m(\xi)$.

**Definition 8.1.** Let $a \in \mathcal{S}^1_\mathcal{cl}(A^*)$ be the order reducing symbol as defined above and set $R_- := [q(a)^{\frac{1}{2}}]^{-1}$ for $k$ chosen sufficiently large as in the proof of [14, Corollary 4.3].

We can use the same argument as in [13, Proposition 1.7] to prove the following result.

**Proposition 8.2.** The operator $R_+ : \chi^+ R_-^m \chi^0$ extends to a linear isomorphism

$$R_+ : H^m_\mathcal{V}(X) \cong H^{m-\delta}_\mathcal{V}(X)$$

for each $s \in \mathbb{R}$.

Denote by $\gamma_Y : \mathcal{C}_c^\infty(M_0) \rightarrow \mathcal{C}_c^\infty(Y_0)$ the restriction operator $f \mapsto f|_{Y_0}$. Then $\gamma_Y : H^m_\mathcal{V}(M) \rightarrow H^m_\mathcal{V}(Y)$ is continuous for each $m \in \mathbb{R}$, see [2, Theorem 4.7]. We also fix the operator $\partial \in \text{Diff}^\infty_\mathcal{V}(M)$ which is supported in $[0, \epsilon) \times Y$ and near $Y$ coincides with $\partial_t$ for $t \in (-\epsilon, \epsilon)$.

**Lemma 8.3.** For $j \in \mathbb{N}_0$ with $s > j - \frac{1}{2}$ we obtain

$$\partial^j_+ : H^s_\mathcal{V}(X) \rightarrow H^{s-j}_\mathcal{V}(X)$$

is continuous.

**Proof.** The extension by zero operator $\chi^0 : H^s_\mathcal{V}(X) \rightarrow H^s_\mathcal{V}(M)$ is a priori continuous for $\frac{1}{2} > s > -\frac{1}{2}$. Since any differential operator has the transmission property we obtain that $\partial_+$ is continuous for $s > j - \frac{1}{2}$ by [13, Theorem 1.14].

**Definition 8.4.** Set for a given $f \in \mathcal{C}_c^\infty(X)$ with support in $Y(\epsilon)$

$$(\gamma_j f)(x') := \lim_{t \to 0} (\partial_{x_n}^j f)(x', t), \quad (x', x_n) \in Y(\epsilon).$$

Given $u \in \mathcal{C}_c^\infty(Y_0)$ denote by $u \otimes \delta_Y \in \mathcal{D}'(M_0)$ the distribution given by

$$\langle u \otimes \delta_Y, v \rangle = \langle u, v(-, 0) \rangle = \int_{Y_0} u(x') v_{|Y_0}(x') \, dv(x'), \quad v \in \mathcal{C}_c^\infty(M_0).$$
Proposition 8.5. For $s > j + \frac{1}{2}$ we have

$$\gamma_j : H^s_v(X) \to H^s_{W^{-\frac{j}{2}}}(Y)$$

is continuous.

Proof. The differential operator

$$\partial^j : H^s_v(M) \to H^{s-j}_v(M)$$

is continuous. Denote by $e_s : H^s_v(X) \to H^s_{2v}(M)$ a continuous extension operator depending on $s$. By the previous remarks

$$\gamma_Y : H^{s-j}_{2v}(M) \to H^{s-j-\frac{1}{2}}_{2v}(M)$$

is continuous provided $s > j - \frac{1}{2}$. In the fixed tubular neighborhood we can write $\gamma_j = \gamma_Y \circ \partial^j \circ e_s$. This proves the claim by the continuity of $\gamma_Y$, $\partial^j$ and $e_s$.

We are now in a position to define the operators $\mathcal{B}^{m,d}_v(X,Y)$ of arbitrary order $m \in \mathbb{R}$ and type $d \in \mathbb{N}_0$.

Definition 8.6. i) A trace operator of order $m \in \mathbb{R}$ and type $d \in \mathbb{N}_0$ is defined as $T \in \mathcal{T}^{m,d}_v(X,Y)$ if there is a sequence $T_j \in \mathcal{T}^{m-j,0}_v(X,Y)$ such that

$$T = \sum_{j=0}^{d} T_j \partial^j_+.$$

A residual trace operator of type $d \in \mathbb{N}_0$ is defined in the same way for a sequence $T_j \in \mathcal{T}^{-\infty,0}_v(X,Y), 0 \leq j \leq d$.

ii) A singular Green operator of order $m \in \mathbb{R}$ and type $d \in \mathbb{N}_0$ is defined as $G \in \mathcal{G}^{m,d}_v(X,Y)$ if there is a sequence $G_j \in \mathcal{G}^{m-j,0}_v(X,Y)$ such that

$$G = \sum_{j=0}^{d} G_j \partial^j_+.$$

A residual singular Green operator of type $d \in \mathbb{N}_0$ is defined in the same way for a sequence $G_j \in \mathcal{G}^{-\infty,0}_v(X,Y), 0 \leq j \leq d$.

Remark 8.7. In [33], p. 149 we have the following standard representations for elements in Boutet de Monvel’s calculus. A trace operator $T \in \mathcal{T}^{m,0}_v(X,Y)$ takes the form $T = \gamma_Y \circ Q_+$ for a $Q \in \Psi^{m,0}_{2v,\text{tr}}(M)$ and a potential operator $K \in \mathcal{K}^{m}_v(X,Y)$ takes the form $K = \tilde{Q}_+(- \otimes \delta_Y)$ for a $\tilde{Q} \in \Psi^{m,0}_{2v,\text{tr}}(M)$. We should remark that in our framework we can prove these properties in the same way. By an elementary but tedious computation we can show directly that the formal adjoint of a trace operator is a potential operator also in the order $m > 0$ cases.

Theorem 8.8. i) A trace operator $T \in \mathcal{T}^{m,d}_v(X,Y)$ extends to a continuous linear operator

$$T : H^s_v(X) \to H^{s-m-\frac{1}{2}}_v(W^{-\frac{j}{2}}(Y))$$

for $s > d - \frac{1}{2}$.

ii) A potential operator $K \in \mathcal{K}^{m}_v(X,Y)$ extends to a continuous linear operator

$$K : H^{s-\frac{1}{2}}_v(W^{-\frac{j}{2}}(Y)) \to H^s_v(X)$$

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iii) A singular Green operator \( G \in \mathcal{G}_{V}^{m,d}(X,Y) \) extends to a continuous linear operator 
\[ G: H_{V}^{s}(X) \to H_{V}^{s-m}(X) \]
for \( s > d - \frac{1}{2} \).

**Proof.** i) Let \( T \in \mathcal{D}_{V}^{m,d}(X,Y) \) and decompose \( T \) as follows
\[ T = T_{0} + T_{d} \]
where \( T \in \mathcal{D}_{V}^{m,0}(X,Y) \) and \( T_{d} \) can be written
\[ T_{d} = \sum_{j=0}^{d-1} S_{j} \gamma_{j}, \quad S_{j} \in \Psi_{W}^{m-j}(Y) \]
(see e.g. [34] for the proof).

Hence the continuity follows from Proposition 8.5.

ii) This follows because \( K \) is the formal adjoint of a trace operator \( T \).

iii) By the same proof as in [34], Lemma 2.2.14 we can write
\[ G = \sum_{j=0}^{d-1} K_{j} \gamma_{j} + G_{0}, \quad K_{j} \in \mathcal{K}_{V}^{m-j,0}(X,Y). \]

The first term on the right hand side is continuous by Proposition 8.5 and ii). The second term has the following series representation by [33], Lemma 7, p. 149
\[ G_{0} = \sum_{j=1}^{\infty} \lambda_{j} \hat{K}_{j} \hat{T}_{j} \]
for a sequence of potential operators \( \hat{K}_{j} \) and trace operators \( \hat{T}_{j} \). Hence the continuity follows by i) and ii).

In the sequel we will study the corresponding Guillemin completion for Boutet de Monvel’s calculus. The completion has favorable algebraic properties. Most importantly, inverses (if they exist) are contained in the calculus. Define the completion \( \mathcal{B}_{V}^{-\infty,0}(X,Y) \) of the residual Boutet de Monvel operators with regard to the family of norms of operators \( \mathcal{L} \left( \begin{array}{cc} H_{V}^{s}(X) & H_{V}^{s}(X) \\ \oplus & \oplus \\ H_{W}^{s}(Y) & H_{W}^{s}(Y) \end{array} \right) \) on Sobolev spaces.

**Definition 8.9.** Let
\[ \mathcal{B}_{V}^{m,d}(X,Y) := \mathcal{B}_{V}^{m,d}(X,Y) + \mathcal{B}_{V}^{-\infty,d}(X,Y) \]
be the completed calculus of Boutet de Monvel operators.

**Proposition 8.10.** Let \( R_{m}^{+} := \chi^{+} R_{m}^{m} \chi^{0} \) with \( R_{m}^{m} \in \Psi_{tr,V}^{m}(M) \) and \( \tilde{R}_{m}^{m} \in \Psi_{W}^{m}(Y) \) order reductions in the pseudodifferential calculi. Define \( \Lambda^{m} := \text{diag}(R_{m}^{+}, \tilde{R}_{m}^{m}) \), then \( \Lambda^{m} \in \mathcal{B}_{V}^{m,0}(X,Y) \).

**Proposition 8.11.** Let \( m \leq 0 \), then \( A \in \mathcal{B}_{V}^{m,0}(X,Y) \) implies that \( A^{*} \in \mathcal{B}_{V}^{m,0}(X,Y) \) where
\[ A^{*} = \begin{pmatrix} \chi^{+} \text{op}(a)^{*} \chi^{0} + G^{*} & T^{*} \\ K^{*} & S^{*} \end{pmatrix}. \]
Proof. We have \([\chi^+ \operatorname{op}(a) \chi^0]^* = \chi^+ \operatorname{op}(a)^* \chi^0\) immediately because \(m \leq 0\) and \(L^2\)-continuity. The adjoint of a trace operator is a potential operator and vice versa and the adjoint of a singular Green operator is a again a singular Green operator by definition. Finally, \(S^* \in \Psi^m_{\emptyset}(Y)\) since the Lie calculus is closed under adjoints, see \([3, \text{Corollary 3.3}]\).

For the following Theorem we use a common trick based on order reductions, see for example \([5, \text{Theorem 6.3.6}]\).

**Theorem 8.12.** Let \(A \in B_{V}^{m,d}(X,Y)\) with \(d \leq m_+\).

i) With \(m \leq 0\) and \(d = 0\) it follows that

\[
A \cdot A^* \in B_{V}^{2m,0}(X,Y).
\]

ii) For \(m > 0\) it follows that

\[
(A \cdot \Lambda^{-m}) \cdot (A \cdot \Lambda^{-m})^* \in B_{V}^{0,0}(X,Y).
\]

iii) If \(A \in \overline{B}_{V}^{m,d}(X,Y)\) is invertible for some \(s \geq m\) as an operator

\[
A: \begin{array}{c}
\oplus \quad \oplus \\
H_{V}^{s-m}(X,E) \quad H_{V}^{s-m}(X,E)
\end{array} \quad \begin{array}{c}
\oplus \quad \oplus \\
H_{W}^{s-m}(Y) \quad H_{W}^{s-m}(Y)
\end{array}
\]

then the inverse \(A^{-1} \in \overline{B}_{V}^{-m,\max\{s-m,0\}}(X,Y)\).

Proof. The properties i) and ii) follow immediately from Proposition [8.11]. In order to prove iii) we consider the pseudodifferential order reductions

\[
R_1^{-s}: L_{V}^{2}(X,E) \oplus H_{W}^{s}(Y,F) \quad \sim \quad H_{V}^{s}(X,E) \oplus H_{W}^{s}(Y,F)
\]

and

\[
R_2^{-s-m}: H_{V}^{s-m}(X,E) \oplus H_{W}^{s-m}(Y,F) \quad \sim \quad L_{V}^{2}(X,E) \oplus H_{W}^{s}(Y,F).
\]

of order \(-s\) and \(s - m\) respectively.

Then \(B := R_2^{-s-m}AR_1^{-s}\) is contained in \(B_{V}^{0,0}(X,Y)\) and is invertible. Therefore the assertion reduces to the order \((0,0)\) case. We apply the \(\Psi^\ast\)-property of the \((0,0)\)-algebra which is proved analogously to \([4, \text{Theorems 6.1-6.2}]\) and \([29, \text{Lemma 4.8}]\).

**9. Parametrix**

In this section we will introduce the principal and principal boundary symbol of an operator in our calculus. We will define the notion of ellipticity and show that a parametrix exists under the previously stated conditions on the calculus. A major technical problem is that in the Lie calculus already inverses of invertible operators are not necessarily contained. This makes a parametrix construction difficult and we state here a version of such a result. There are at least two approaches to overcome the problem of inverses: (1) using a larger calculus of pseudodifferential operators (with asymptotics) or (2) completing the algebra of pseudodifferential operators (non-canonically) such that inverses are contained. In the last section we outlined the second approach. We will henceforth consider the completed Boutet de Monvel calculus as defined in Section 8.

Fix the smooth, hermitian vector bundles \(E_1, E_2 \to X, J_\pm \to Y\) and recall the notation for the boundary algebroid and its co-bundle \(\pi_B: A_B \to Y, \pi_{B_\pm}: A_B^\pm \to Y\).

We define the principal symbol and principal boundary symbol on \(B^{m,0}(G^+, X; \tilde{E}_1, \tilde{E}_2, \tilde{J}_\pm)\) for \(m \leq 0\).
• Set $T^qX := \ker dq$ for the vertical tangent bundle over $X$.
• Let $A = (A_x)_{x \in X} \in \mathcal{B}^{m,0}(G^+,X)$ be a $C^\infty$-family of Boutet de Monvel operators.
• Then for each $x \in X$ we have a Boutet de Monvel operator $A_x \in \mathcal{B}^{m,0}_{prop}(G^+_x,X_x)$.
• Hence the principal symbol $\sigma(A_x)$ and the principal boundary symbol $\sigma_\partial(A_x)$ are defined invariantly on $T^*G^+_x$ and $T^*X_x$ respectively.
• By right-invariance of the family $A$ these symbols descend to a principal symbol $\sigma(A)$ and principal boundary symbol $\sigma_\partial(A)$ defined invariantly on $\mathcal{A}^+_+$ and $(T^qX)^*$ respectively.
• Introduce the indicial symbol $\mathcal{R}_F$ for a face $F \in \mathcal{F}_1(X)$ as the restriction $\mathcal{R}_F((A_x)_{x \in X}) = (A_y)_{y \in F}$.

By noting that $T^q|_Y X = \mathcal{A}_\partial$ we make the following definition for principal and principal boundary symbol on the represented algebra $\mathcal{B}^{m,0}_\nu(X,Y)$ for $m \leq 0$.

**Definition 9.1.** To an element $A \in \mathcal{B}^{m,0}_\nu(X,Y;E_1,E_2,J_{\pm})$ we associate the two principal symbols.

i) The **principal boundary symbol** $\sigma^\nu_\partial(A)$ is defined as

$$\sigma^\nu_\partial(A) := \sigma_{\partial|Y}(A).$$

This yields a section of the infinite dimensional bundle

$$C^\infty(\mathcal{A}_\partial^+, \text{Hom}(\pi_0E_{1|Y} \otimes \mathcal{S}^\nu, \pi_0E_{2|Y} \otimes \mathcal{S}^\nu)).$$

Here $\mathcal{S}^\nu \to \mathcal{A}_\partial^+$ is a bundle with fiber $S(K_{\pm})$ on the inward pointing normal direction.

In particular the restriction of $\sigma^\nu_\partial$ to the interior $(X_0, Y_0)$ agrees with the principal boundary symbol on the interior.

ii) The **principal symbol** $\sigma^\nu(A)$ which is the principal symbol of the pseudodifferential operator in the upper left corner of the matrix $A$. This yields a section in

$$C^\infty(\mathcal{A}_\partial^+, \text{Hom}(E_1, E_2)).$$

iii) Define by $\Sigma^{m,0}_\nu(X,Y;E_1,E_2,J_{\pm})$ the space consisting of pairs of principal symbols $(a,a_\partial)$. These are homogeneous or $\kappa$-homogenous sections of the bundles $\mathcal{A}_\partial^+,$ $\mathcal{A}_\partial^*$, respectively, with canonical compatibility condition.

**Lemma 9.2.** Given $A, B \in \mathcal{B}^{0,0}_\nu(X,Y;E_1,E_2,J_{\pm})$ we have

$$\sigma^\nu(A \cdot B) = \sigma^\nu(A) \cdot \sigma^\nu(B), \quad \sigma^\nu_\partial(A \cdot B) = \sigma^\nu_\partial(A) \cdot \sigma^\nu_\partial(B).$$

**Proof.** From the assumption that the groupoid $G$ is Hausdorff we obtain by the same argument as in [39, p.11] that the vector representation furnishes an isomorphism

$$\mathcal{B}^{0,0}_\nu(X,Y;E_1,E_2,J_{\pm}) \cong \mathcal{B}^{0,0}(G^+,X;\tilde{E}_1,\tilde{E}_2,\tilde{J}_{\pm}).$$

Since the principal and principal boundary symbol are each defined invariantly on the bundles $\mathcal{A}_\partial^+, \mathcal{A}_\partial^*$ the computation reduces to the equivariant families of Boutet de Monvel operators. Hence given $A = (A_x)_{x \in X}$, $B = (B_x)_{x \in X}$ in $\mathcal{B}^{0,0}(G^+,X;\tilde{E}_1,\tilde{E}_2,\tilde{J}_{\pm})$ we have

$$\sigma_\partial(A \cdot B) = \sigma_\partial((A_x \cdot B_x)_{x \in X}) = (\sigma_\partial(A_x \cdot B_x))_{x \in X}$$

$$= (\sigma_\partial(A_x) \cdot \sigma_\partial(B_x))_{x \in X} = \sigma_\partial(A) \cdot \sigma_\partial(B).$$

In the same way we obtain multiplicativity of the principal symbol. \qed
Since we will in the following only be concerned with represented operators we will simply write \( \sigma \) and \( \sigma_\partial \) for \( \sigma^V \) and \( \sigma^V_\partial \).

For the following result and proof in the standard case see e. g. [33].

**Theorem 9.3.** The following sequence is exact

\[
\mathcal{B}_V^{-1,0}(X, Y; E_1, E_2, J_\pm) \longrightarrow \mathcal{B}_V^{0,0}(X, Y; E_1, E_2, J_\pm) \longrightarrow \Sigma_\nu^0(X, Y; E_1, E_2, J_\pm).
\]

**Proof.** i) The same exact sequence holds for the interior calculus. Since the principal symbols are extensions of the interior we immediately obtain that \( \ker \sigma \oplus \ker \sigma_\partial = \ker \cap \ker \sigma_\partial = \mathcal{B}_V^{-1,0}(X, Y) \).

ii) To prove surjectivity let \( (a, a_\partial) \in \Sigma_\nu^0 \).

Since we also have an analogous exact sequence for the class of pseudodifferential operators \( \Psi_{\nu,2}^m \) it suffices to find singular Green, trace and potential operators in the preimage.

In a fixed small tubular neighborhood \( U \cong Y(\epsilon) \) trivialize the singular normal bundle \( \mathcal{N} \). Let \( \{U_i\} \) be a normal covering of \( Y \) (assumed finite by compactness of \( Y \)) and let \( \{\varphi_i\} \) be a subordinate partition of unity. We also fix a boundary defining function \( \rho_\nu: M \to \mathbb{R} \) such that \( \{\rho_\nu = 0\} = Y \).

This is defined by the tubular neighborhood theorem for Lie manifolds (cf. [2]). Hence for the diffeomorphism of tubular neighborhoods \( \nu: Y \times (-\epsilon, \epsilon) \to U \subset M \) we have

\[
(\rho_\nu \circ \nu)(x', x_n) = x_n, \quad x' \in Y, \quad x_n \in (-\epsilon, \epsilon).
\]

Let \( \varphi \in C^\infty(\mathbb{R}_+) \) be a cutoff function such that \( \varphi(x_n) = 0 \) close to 0. Hence we can locally on each trivialization \( \mathcal{N}_{U_i}^+ \cong U_i \times \mathbb{R}_+ \) construct singular Green, potential and trace operators. Via our partition of unity we obtain corresponding global symbols.

This furnishes a global right inverse morphism \( \text{Op}_\nu: \Sigma_\nu^0 \to \mathcal{B}_V^{0,0} \).

The notion of ellipticity we introduce here is the usual condition of Shapiro-Lopatinski type. It is sufficient to obtain a parametrix. We obtain Fredholm operators if we additionally request the invertibility of the indicial symbols.

**Definition 9.4.** An operator \( A \in \mathcal{B}_V^{m,0}(X, Y) \) is called

i) \( \mathcal{V} \)-elliptic if \( (a^m \oplus \sigma_\partial)(A) \) is pointwise bijective, 

ii) elliptic if \( A \) is \( \mathcal{V} \)-elliptic and for each \( F \in \mathcal{F}_1(X) \) the indicial symbol \( \mathcal{R}_F(A) \) is invertible.

**Remark 9.5.** The elements of the symbol algebra \( \Sigma_\nu \) can be written more simply in terms of the action in the normal direction. Let us define this action for a given symbol \( p \in S^m_\nu(A^*) \). We set

\[
\text{op}_n^+ p(x', \xi') = p(x', 0, \xi', D_n)_+
\]

then in local trivializations of the bundles \( E_1, E_2 \) and \( S^\nu \) we have

\[
\text{op}_n^+ p(x', \xi'): S(\mathbb{R}_+) \otimes \mathbb{C}^k \to S(\mathbb{R}_+) \otimes \mathbb{C}^k
\]

where \( k \) is the fibre dimension of \( E_1, E_2 \).

We will alternatively consider these maps as operators acting on \( L^2(\mathbb{R}_+) \), the closure of \( S(\mathbb{R}_+) \).

In [33] the action in the normal direction is defined for the boundary symbols which act more generally as *Wiener-Hopf operators* on the spaces \( H^\pm \) and their respective \( L^2 \)-closures. We will not need this formulation in the present work.

Denote by \( p: S^*A_\partial \to Y \) the canonical projection of the sphere bundle.

Let \( A \) be \( \mathcal{V} \)-elliptic and consider the principal symbol \( \sigma_P \) of the pseudodifferential operator \( P \) in the upper left corner. Since the infinite dimensional bundles are trivial the action in the normal direction

\[
\text{op}_n^+ \sigma_P: p^*E_1 \otimes S^\nu \to p^*E_2 \otimes S^\nu
\]
yields a Fredholm family, parametrized over $S^* A_0$ which preserves the index, cf. [33], Prop. 5, p.95.

We can express the principal boundary symbol (with analogously defined actions in the normal direction for Green, potential and trace operators)

$$\sigma_\partial(A) = \begin{pmatrix} \text{op}_n^+ p + \text{op}_n g & \text{op}_n k \\ \text{op}_n t & s \end{pmatrix}$$

as an element of $C^\infty(S^* A_0, \text{Hom}(p^*E_1 Y \otimes S^V, p^*E_2 Y \otimes S^V)$.

Proof (of Thm. [1.3]. a) First we have to show that the symbol algebra (for the completed calculus) $\Sigma_Y$ is closed under inverse. Specifically, let $A \in B^0_v(X, Y)$ be $\mathcal{V}$-elliptic and invertible. Denote by $P$ the pseudodifferential operator in the upper left corner of $A$. By $\mathcal{V}$-ellipticity of $A$ we fix a $Q \in \mathcal{G}_v(M)$ such that $\sigma(P^{-1}) = \sigma(Q)$. Set $Q = \text{op}(q)$ for a symbol $q \in S^0(A^*)$. We want to show that there is an elliptic boundary symbol of the form

$$\sigma_\partial(B) = \begin{pmatrix} \text{op}_n^+ q + \text{op}_n g & \text{op}_n k \\ \text{op}_n t & s \end{pmatrix} : p^*E_1 \otimes S^V \oplus p^*J_+ \rightarrow p^*E_2 \otimes S^V$$

such that $\sigma_\partial(B) = \sigma_\partial(A)^{-1}$.

For this we can adapt an approach to the problem the idea of which goes back to Boutet de Monvel, [7] (the index bundle, p.35) and which can be found in detail in the reference [33]. We divide the argument into three parts.

1) We have

$$\dim \ker \text{op}_n^+ \sigma_P \leq \text{const, dim coker op}_n^+ \sigma_P \leq \text{const}$$

with constants independent of the parameter $\varrho \in S^* A_0$. Then there is a finite dimensional trivial subbundle $W$ in $p^* E \otimes S^V$ such that

$$(\im \sigma_P)_{(x', \xi')} + W_{(x', \xi')} = (p^*E_2 \otimes S^V)_{(x', \xi')}$$

for each $(x', \xi') \in S^* A_0$. We obtain the index element

$$\text{ind}_{S^* A_0} \text{op}_n^+ \sigma_P \in K(S^* A_0)$$

which depends on the homotopy class of the Fredholm family. Let $(\sigma^{(t)})_{t \in [0,1]}$ be a homotopy of elliptic symbols in $S^0(A^*)$, then

$$\text{ind}_{S^* A_0} \text{op}_n^+ \sigma^{(0)} = \text{ind}_{S^* A_0} \text{op}_n^+ \sigma^{(1)}.$$  

Hence for $\sigma_P$ we obtain

$$\text{ind}_{S^* A_0} \text{op}_n^+ \sigma_P = [p^*J_+] - [p^*J_-].$$

By the same argument as in [33], Prop. 11 on page 199 it follows that

$$\text{ind}_{S^* A_0} \text{op}_n^+ p \in p^* K(Y).$$  

Claim: By (17) there is a Green symbol $g_0$ and bundles $\tilde{J}_+, \tilde{J}_- \rightarrow Y$ such that

$$\ker_{S^* A_0}(\text{op}_n^+ \sigma_P + \text{op}_n g_0) \cong p^* \tilde{J}_+,$$

$$\text{coker}_{S^* A_0}(\text{op}_n^+ \sigma_P + \text{op}_n g_0) \cong p^* \tilde{J}_-.$$  

For the proof see [33], p. 201.
II) The construction of the boundary symbol is a parameter dependent construction of $C^\infty$-boundary symbol (an isomorphism)

$$
b_\partial = \left( \begin{array}{cc}
op_n^+ q + \op_n g_1 & \op_n k_1 \\
op_n t_1 & s_1 \end{array} \right) : \begin{array}{c}
p^* E_1 \otimes S^V \\
p^* E_2 \otimes S^V \\
p^* J_+ \\
p^* J_- \end{array} \rightarrow \begin{array}{c} 
\oplus \\
\oplus \\
p^* J_+ \\
p^* J_- \end{array}.
$$

Since $\op_n^+ q$ is a parametrix of $\op_n^+ p$ in the sense of Fredholm families we have by part I)

$$\text{ind}_{S^* A_0} \op_n^+ q = [p^* J_+] - [p^* J_-].$$

In order to find an isomorphism of the form $b_\partial$ it suffices to find a Green symbol $g_1$ with

$$\text{ker}_{S^* A_0} (\op_n^+ q + \op_n g_1) \cong p^* J_+,$$
$$\text{coker}_{S^* A_0} (\op_n^+ q + \op_n g_1) \cong p^* J_-.$$  

We know by the claim in part I) that there is a Green symbol $g_0$ with

$$\tilde{J}_0^+ = \text{ker}_{S^* A_0} (\op_n^+ q + \op_n g_0) \cong p^* J_0^+,$$
$$\tilde{J}_0^- = \text{coker}_{S^* A_0} (\op_n^+ q + \op_n g_0) \cong p^* C^N$$

for suitable vector bundles $J_0^+ \rightarrow Y$ and $N$ such that

$$[p^* J_0^+] - [C^N] = [p^* J_+] - [p^* J_-]$$

by Lemma 5, p. 203 [33].

For $N$ sufficiently large there are decompositions

$$\tilde{J}_0^+ = \tilde{V} \oplus \tilde{J}_+, \quad \tilde{J}_0^- = \tilde{W} \oplus \tilde{J}_-,$$
$$J_0^+ \cong V \oplus J_+, \quad C^N \cong W \oplus J_-$$

for vector bundles $V, W \rightarrow Y$ with isomorphisms

$$\tilde{V} \cong p^* V, \quad \tilde{J}_+ \cong p^* J_+, \quad \tilde{W} \cong p^* W, \quad \tilde{J}_- \cong p^* J_-$$

where $\op_n^+ q + \op_n g_0$ induces an isomorphism $\beta: \tilde{W} \rightarrow \tilde{V}$. There is a Green symbol $g_2$ so that the isomorphism $\beta$ is induced by $- \op_n g_2$ and that $\op_n g_2$ vanishes on the complement of $\tilde{W}$ in $p^* E \otimes S^V$. Then $g_1 = g_0 + g_2$ is a Green symbol with the desired property.

III) The inverse of $a_\partial = \sigma_\partial(A)$ can be calculated separately for each $\varrho \in S^* A_\partial$ (cf. 2.1.2.4, Prop. 6, p. 110). What is left to show is the smoothness of the inverse symbol depending on $\varrho \in S^* A_\partial$. For $b_\partial$ the smoothness is clear. Additionally, the composition of smooth symbols yields smooth symbols. Thus $c_\partial = a_\partial b_\partial$ is in $C^\infty(S^* A_\partial)$. Hence if we can show that $c_\partial^{-1}$ is $C^\infty$ we obtain that $a_\partial^{-1} = b_\partial c_\partial^{-1}$ is $C^\infty$. This is a lengthy but elementary calculation for which we refer to Prop. 6, p. 204-206 in [33].

The parametrix is obtained as follows.

i) From the exact sequence given in Theorem 9.33 there is a $B \in B^{0,0}_Y (X, Y)$ such that

$$(\sigma \oplus \sigma_\partial)(B) = (\sigma \oplus \sigma_\partial)(A)^{-1}.$$ 

Then by the multiplicativity of the principal symbol and the principal boundary symbol it follows that

$$(\sigma \oplus \sigma_\partial)(I - AB) = 0.$$
Applying the exact sequence once more we have that $R := I - AB \in B^{-1,0}_V(X,Y)$. Hence $B$ is a right parametrix of $A$ of order 1. Setting $B_k = B(I + R + \cdots + R^{k-1})$ we obtain

$$AB_k = (I - R)(I + R + \cdots + R^{k-1}) = I - R^k$$

with $R^k \in B^{-k,0}_V(X,Y)$. Thus $B_k$ is a right parametrix of $A$ of order $k$.

**ii)** Let $(B_k)_{k \in \mathbb{N}_0}$ be a sequence of right parametrices of $A$ of orders $k \in \mathbb{N}_0$. By asymptotic completeness we can find $B \sim \sum_i B_i$ such that $AB - I \in B^{-\infty,0}_V(X,Y)$. Hence we have found a right parametrix up to residual terms.

**iii)** Fix a right parametrix up to residual terms $B_1$ of $A$. Analogously to **i)** and **ii)** we can find a left parametrix $B_2$ of $A$ such that

$$I - AB_1 = R_1 \in B^{-\infty,0}_V(X,Y) \text{ and } I - B_2 A = R_2 \in B^{-\infty,0}_V(X,Y).$$

Rewrite the operator $AB_2 B_1$ as follows

$$AB_2 B_1 = AB_1(I - R_1) = A(I - R_2)B_1 = AB_1 - AR_2 B_1 = (I - R_1) - AR_2 B_1$$

hence $AB_2 B_1 + AR_2 B_1 = I - R_1$ so that

$$(AB_2 A + AR_2)B_1 = I - R_1.$$

The left hand side of the previous equation equals

$$AB_2(I - R_1) + AR_2 B_1$$

and note that this is $\equiv AB_2 \mod B^{-\infty,0}_V(X,Y)$.

Hence $B_2$ is also a right parametrix equal up to residual terms to $I - R_1$. Analogously one proves that $B_1$ is also a left parametrix equal up to residual terms to $I - R_2$. It follows that $B_1 \equiv B_2 \mod B^{-\infty,0}_V(X,Y)$ and therefore there is an operator $B \in B^{-\infty,0}_V(X,Y)$ equal to $B_1$ mod $B^{-\infty,0}_V(X,Y)$ and $B_2$ mod $B^{-\infty}_V(X,Y)$ such that

$$I - AB \in B^{-\infty,0}_V(X,Y), \text{ and } I - BA \in B^{-\infty,0}_V(X,Y).$$

**b)** Consider the pullback algebra $\Sigma_{BM}$ which is defined as the restricted direct sum of $\Sigma_V$ and the indicial algebras $\oplus_{F \in F_1(X)} B^{0,0}_{V(F)}(F, F_{reg})$. We can summarize the situation in the following pullback diagram.

We will generalize the Fredholm conditions given in the special case of the pseudodifferential operators on Lie manifolds, see e.g. [29].

**Claim:** The following sequence is exact

$$\mathcal{K}(L^2_V(X)) \oplus \mathcal{K}(L^2_V(Y)) \longrightarrow B^{0,0}_V(X,Y) \longrightarrow \sigma_{\oplus \sigma_{\partial}}(\oplus_F R_F) \longrightarrow \Sigma_{BM}.$$
The surjectivity is immediate. Let $A \in \mathcal{B}_V^{0,0}(X,Y)$, then we prove that

$$
\begin{align*}
L^2_V(X) & \oplus L^2_V(Y) \\
A: & \rightarrow \oplus \\
L^2_W(Y) & \oplus L^2_W(Y)
\end{align*}
$$

is compact if and only if $\sigma(A) = 0$, $\sigma_B(A) = 0$ and $R_F(A) = 0$ for each $F \in \mathcal{F}_1(X)$. Assume that $\sigma \oplus \sigma_B \oplus (\oplus_F R_F)(A) = 0$. From the short exact sequence in Theorem \[\text{it} \] it follows that $A \in B_\mathcal{V}^{-1,0}(X,Y)$. Let $(\rho_F)_{F \in \mathcal{F}_1(X)}$ be the collection of boundary defining functions of the faces at infinity of $X$. Setting

$$
\rho := \prod_F \rho_F
$$

we have that $A = \rho B$ for some $B \in B_\mathcal{V}^{-1,0}(X,Y)$. Hence

$$
\begin{align*}
L^2_V(X) & \rho H^1_V(X) \\
A: & \rightarrow \oplus \\
L^2_W(Y) & H^1_W(Y)
\end{align*}
$$

is continuous.

By the generalized Kondrachov’s theorem, see [2, Theorem 3.6], we obtain that $\rho H^1_V(X) \hookrightarrow L^2_V(X)$ is compact. Also $H^1_W(Y) \hookrightarrow L^2_W(Y) = H^0_W(Y)$ is compact. We obtain that

$$
\begin{align*}
L^2_V(X) & L^2_V(X) \\
A: & \rightarrow \oplus \\
L^2_W(Y) & L^2_W(Y)
\end{align*}
$$

is compact.

For the other direction assume that $A$ is compact. Then $\sigma(A) = 0$ and $\sigma_B(A) = 0$. Assume for contradiction that there is a face $F \in \mathcal{F}_1(X)$ such that $R_F(A) \neq 0$ and fix such an $F$. By the Hausdorff and amenability property of $G$ we can identify bijectively $B_\mathcal{V}^{0,0}(X,Y) \cong B^{0,0}(G^+,X)$ via the vector representation, see [29, p.11]. The restriction $R_F$ is defined on the level of families by $(A_x)_{x \in X} \mapsto (A_y)_{y \in F}$ and by the Hausdorff property of the groupoid $G$ we have that $X \ni x \mapsto \left\| A_x \left( \frac{\varphi_x}{\psi_x} \right) \right\| \Pg$ a continuous function for each $\varphi \in C^\infty_c(G^+), \psi \in C^\infty_c(X)$. Let $x \in F$ and $\varphi \in C^\infty_c(G^+_x), \psi \in C^\infty_c(X_x)$ be given such that $A_x \left( \frac{\varphi}{\psi} \right) \neq 0$ and let $\tilde{\varphi}, \tilde{\psi}$ be extensions to functions in $C^\infty_c(G^+_x)$ and $C^\infty_c(X_x)$ respectively. Then we have $\tilde{\varphi}_y = \varphi|_{G_y}$, $\tilde{\psi}_y = \psi|_{X_y}$ for $y \in X_0$. These restrictions identify as functions $\tilde{\varphi}_y \in C^\infty_c(X_0)$ and $\tilde{\psi}_y \in C^\infty_c(Y_0)$ via the canonical isometries $G^+_y \cong X_0$, $X'_y \cong Y_0$. The supports of $\tilde{\varphi}_y$, $\tilde{\psi}_y$ increase via $y \rightarrow x$ for $y \in X_0$. This implies weak convergence in Sobolev spaces $\tilde{\varphi}_y \rightharpoonup 0$, $\tilde{\psi}_y \rightharpoonup 0$. By continuity it follows

$$
A_y \left( \frac{\varphi_y}{\psi_y} \right) \rightarrow A_x \left( \frac{\varphi_x}{\psi_x} \right) \neq 0.
$$

This yields a contradiction to the compactness of $A$. Hence the proof of the claim is finished.

Set $\mathcal{H} := L^2_V(X) \oplus L^2_W(Y)$ and denote by $\mathcal{C}(\mathcal{H}) := \mathcal{L}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ the Calkin algebra with quotient map $q_C: \mathcal{L}(\mathcal{H}) \rightarrow \mathcal{C}(\mathcal{H})$. We want to show that $A$ is elliptic if and only if $A$ is Fredholm on the Hilbert space $\mathcal{H}$. By amenability property of the groupoid we know that the vector representation furnishes an injective $*$-representation (cf. [22, Theorem 4.4])

$$
\pi: B_\mathcal{V}^{0,0}(X,Y) \rightarrow \mathcal{L}(\mathcal{H}).
$$
An operator $A \in \mathcal{L}(\mathcal{H})$ is Fredholm if and only if $q_C(A) \in C(\mathcal{H})$ is invertible. Apply the standard exact sequence for the indicial symbol for order $-1$ (note that by invariance of each singular hyperface $F$ the indicial symbol is surjective)

$$\ker \mathcal{R}_F^{-1} \longrightarrow B^1_v(X, Y) \xrightarrow{\mathcal{R}_F^{-1}} B^1_v(F, F_{reg}) =: \mathcal{B}^1_v.$$

Consider the diagram

$$\begin{align*}
\oplus_F \ker \mathcal{R}_F^{-1} & \xrightarrow{\pi} \mathcal{K}(\mathcal{H}) \xrightarrow{q_C} \mathcal{L}(\mathcal{H}) \xrightarrow{\sigma_{\partial}} \mathcal{C}(\mathcal{H}) \\
B^1_v & \xrightarrow{\sigma_{\partial}} \Sigma_v \\
\oplus_F \mathcal{B}^1_v & \xrightarrow{\sigma_{\partial}} \Sigma_{BM}
\end{align*}$$

One can check the equality ($A$ being SL-elliptic, see also [22, Theorem 9])

$$\sigma_{ess}(\pi(A)) = \bigcup_{F \in \mathcal{F}_1(X)} \sigma(\mathcal{R}_F(A)) \cup \bigcup_{\xi \in S^* A} \text{spec}(\sigma(A)(\xi)) \cup \bigcup_{\xi' \in S^* A_{\partial}} \text{spec}(\sigma_{\partial}(A)(\xi'))$$

where $\sigma_{ess}$ denotes the essential spectrum and $\text{spec}(\sigma_{\partial}(A)(\xi'))$ denotes the spectrum of the matrix defined by the operator valued symbol $\sigma_{\partial}(A)$ at the point $\xi'$. Hence $\pi$ induces an injective $\ast$-homomorphism

$$B^0_v(X,Y)/\mathcal{K}(\mathcal{H}) \to C(\mathcal{H}).$$

Thus $q_C(A)$ is invertible in $C(\mathcal{H})$ if and only if $(\mathcal{R}_F \oplus \sigma \oplus \sigma_{\partial})(A)$ is pointwise invertible for each $F \in \mathcal{F}_1(X)$.

[1] B. Ammann, C. Carvalho, V. Nistor, Regularity for Eigenfunctions of Schrödinger Operators, Letters in Math. Physics, Vol. 101, Issue 1, pp. 49-84.

[2] B. Ammann, A. Ionescu, V. Nistor, Sobolev spaces on Lie manifolds and regularity for polyhedral domains, Doc. Math. 11, 161-206 (2006).

[3] B. Ammann, R. Lauter, V. Nistor, Pseudodifferential operators on manifolds with a Lie structure at infinity, Ann. of Math. 165, 717-747 (2007).

[4] B. Ammann, R. Lauter, V. Nistor, A. Vasy, Complex powers and non-compact manifolds, Comm. Part. Diff. Eq. 29, no. 5/6 671-705 (2004).

[5] U. Battisti, Zeta functions of pseudodifferential operators and Fourier integral operators on manifolds with boundary, PhD Thesis, Hannover 2015.

[6] K. Bohlen, Boutet de Monvel’s calculus via groupoid actions, PhD Thesis, Hannover 2015.

[7] Louis Boutet de Monvel, Boundary problems for pseudo-differential operators, Acta Math., 126(1-2):11-51, 1971.

[8] C. Carvalho, Y. Qiao, Layer potentials $C^*$-algebras of domains with conical points, Cent. Eur. J. Math. 11 (2013), no. 1, 27-54.

[9] M. Crainic, R. L. Ferandes, Integrability of Lie brackets, Ann. of Math., 157 (2003), 575-620.

[10] C. Debord, J.-M. Lescure, F. Rochon, Pseudodifferential operators on manifolds with fibred corners, arXiv:1112.4575.
[11] C. Debord, G. Skandalis, *Adiabatic groupoid, crossed product by $\mathcal{R}_+^*$ and Pseudodifferential calculus*, Adv. in Math. 257 (2014), 66-91.

[12] C. Fefferman, *On Kohn’s microlocalization of $\overline{\partial}$ problems*, Modern methods in complex analysis (Princeton, NJ, 1992), 119–133, Ann. of Math. Stud., 137, Princeton Univ. Press, Princeton, NJ, 1995.

[13] L. Guillaume, *Géométrie non-commutative et calcul pseudodifférentiel sur les variétés à coins fibrés*, Ph.D. thesis, Université Paul Sabatier Toulouse 3, 2012.

[14] G. Grubb, *Functional Calculus of Pseudodifferential Boundary Problems*, Birkhäuser; 2nd edition, 1986.

[15] G. Harutjunjan, B.-W. Schulze, *Reduction of orders in boundary value problems without transmission property*, J. Math. Soc. Japan, Vol. 56, Number 1 (2004), 65-85.

[16] M. Høegh-Krohn, G. Skandalis, *Morphismes $K$-orientés d’espaces de feuilles et fonctorialité en théorie de Kasparov (d’après une conjecture d’A. Connes)*, Ann. Sci. École Norm. Sup. 20 (4) (1987) 325-390.

[17] L. Hörmander, *The Analysis of Linear Partial Differential Operators III*, Springer-Verlag, Berlin Heidelberg, 1985.

[18] L. Hörmander, *The Analysis of Linear Partial Differential Operators IV*, Springer-Verlag, Berlin Heidelberg, 1985.

[19] D. Jerison, C. E. Kenig, *The inhomogeneous Dirichlet Problem in Lipschitz Domains*, Journal of Functional Analysis vol. 130 (1995), 161-219.

[20] T. Krainer, *Elliptic Boundary Problems on Manifolds with Polycylindrical Ends*, Journal of Functional Analysis vol. 244 issue 2 March 15, 2007. p. 351-386.

[21] R. Lauter, B. Monthubert, V. Nistor, *Pseudodifferential Analysis on Continuous Family Groupoids*, Documenta Math. 5 (2000), 625-655.

[22] R. Lauter, V. Nistor, *Analysis of geometric operators on open manifolds: A groupoid approach*, Progress in Mathematics, Vol. 198, 2001, pp 181-229.

[23] R. Melrose, *The Atiyah-Patodi-Singer index theorem*, A. K. Peters, Ltd., Boston, Mass., 1993.

[24] R. Mazzeo, R. Melrose, *Pseudodifferential operators on manifolds with fibred boundary*, Asian Journal of Mathematics 2 No. 4 (1999) pp. 833-866.

[25] D. Mitrea, M. Mitrea, M. Taylor, *Layer Potential, the Hodge Laplacian and Global Boundary Problems in Nonsmooth Riemannian Manifolds*, Vol. 150, Number 713, Memoirs of the AMS, Providence, RI, 2001.

[26] M. Mitrea, M. Taylor, *Boundary layer methods for Lipschitz domains in Riemannian manifolds*, J. Funct. Anal. 163 (1999), 181-251.

[27] I. Moerdijk, J. Mrčun, *Introduction to Foliations and Lie Groupoids*, Cambridge Univ. Press, 2003.

[28] B. Monthubert, *Pseudodifferential calculus on manifolds with corners and groupoids*, Proc. A. Math. Soc., Vol. 127, 10, 2871-2881.

[29] V. Nistor, *Pseudodifferential operators on non-compact manifolds and analysis on polyhedral domains*, Proceedings of the Workshop on Spectral Geometry of Manifolds with Boundary and Decomposition of Manifolds, Roskilde University, 307–328, Contemporary Mathematics, AMS, Rhode Island, 2005.

[30] V. Nistor, A. Weinstein, P. Xu, *Pseudodifferential Operators on Differential Groupoids*, Pacific J. Math. 189 (1999), 117-152.
[31] A. L. T. Paterson, *Groupoids, Inverse Semigroups, and their Operator Algebras*, Progress in Mathematics Vol. 170, Birkhäuser, 1999.

[32] A. L. T. Paterson, *The equivariant analytic index for proper groupoid actions*, K-theory, 32(2004), 198-230.

[33] S. Rempel, B.-W. Schulze, *Index theory of elliptic boundary problems*, Math. Lehrbücher Monogr. II Abt. Math. Monogr., vol. 55, Akademie-Verlag, Berlin, 1982.

[34] E. Schrohe, B.-W. Schulze, *Boundary Value Problems in Boutet de Monvel’s Algebra for Manifolds with Conical Singularities I*, Advances in Partial Differential Equations 1, pp. 97-209.

[35] S. R. Simanca, *Pseudo-differential Operators*, Pitman Research Notes in Mathematics 236, 1990.