Vortex Condensation in the Dual Chern-Simons Higgs Model

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The contribution of nontrivial vacuum (topological) excitations, more specifically vortex configurations of the self-dual Chern-Simons-Higgs model, to the functional partition function is considered. By using a duality transformation, we arrive at a representation of the partition function in terms of which explicit vortex degrees of freedom are coupled to a dual gauge field. By matching the obtained action to a field theory for the vortices, the physical properties of the model in the presence of vortex excitations are then studied. In terms of this field theory for vortices in the self-dual Chern-Simons Higgs model, we determine the location of the critical value for the Chern-Simons parameter below which vortex condensation can happen in the system. The effects of self-energy quantum corrections to the vortex field are also considered.

PACS numbers: 11.10.Kk, 11.15.Ex

I. INTRODUCTION

Gauge field theories in two spatial dimensions have long been recognized as important for understanding several physical phenomena that can be well approximated as planar ones, like high temperature superconductors and the fractional quantum Hall effect. In particular, Chern-Simons (CS) gauge theories have a special place in understanding these phenomena, not to mention the interest in their theoretical aspects on their own (for a review, see e.g. Ref. [1] and references therein). CS gauge theories exhibit a number of interesting properties. For example, it provides a mass term for the gauge field, while keeping renormalizability, without evoking spontaneous symmetry breaking. It can have the effect of statistical transmutation, attaching magnetic fluxes to a fermion or boson coupled to the gauge field and making them anyons [2].

Another important aspect regarding CS gauge theories, when coupled to symmetry broken scalar potentials, is the existence of both topological and nontopological vortex solutions [3]. These vortices are charged and anyon-like solutions that may be of relevance in explaining several phenomena in planar condensed matter systems, like in high temperature superconductors and the fractional quantum Hall effect, already mentioned above. Vortex solutions have been shown to exist in both a Maxwell-Chern-Simons-Higgs (MCSH) model (where both a kinetic Maxwell term and a CS term are present) as well in the Chern-Simons-Higgs (CSH) model (in the absence of a Maxwell term), where the vortices can have the property of self-duality, with the field equations reducing to first order differential equations [4, 5].

As far the physics of topological excitations like vortices are concerned, one important question is related to the possibility of condensation of these excitations in the system under appropriate conditions. Under these circumstances most of the physical properties of the system should be determined by those of the condensate. Condensation of topological excitations are believed to have relevance in the interpretation of many physical phenomena, like for instance in the confinement picture in dual formulations of gauge field theories [6]. We should also recall that there are many examples of physical systems in which phase transitions can be driven by topological excitations in quantum field theory as well as in condensed-matter physics [7], which makes the study of condensation of nontrivial vacuum excitations of relevance in different contexts.

It is known that the CSH model exhibits a phase transition between a vortex condensed phase and one in which the vortices are not condensed [8, 9]. In this work we consider the vortex condensation in the CSH model specialized to the case of the self-dual potential for the scalar field [4, 5], in which case vortices can be considered as noninteracting. This is used only for convenience, since then explicit expressions for the vortex energy and the dynamical mass for the vortices follow. Other generic potentials could well be used, provided that it does not depend on the phase of the scalar field \( \phi \) (so depends only on the product \( \phi \phi^* \)) and has a minimum at a non-zero value of \( \phi \). However, exact relations are not known for these more general potentials like for the self-dual one. Vortex condensation in CS theories has been considered before, in the context of the self-dual models, in Refs. [8, 9]. In particular, in Ref. [9], it was

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formally shown that vortices should condense below some critical value for the CS coupling parameter. However, no exact prediction for this value of the CS parameter was made. Also, being this condensation a transition point in the system, a determination of the order of this transition is also still lacking. These are important points that need to be addressed and that we will attempt to study in this paper, at least in part.

The existence of nontrivial solutions of the vortex type implies that, when performing the path integral over the fields in the partition function, there will be contributions coming from field configurations corresponding to these vortex solutions and, under appropriate conditions (depending on the values of the model parameters, for example), these solutions can dominate the partition function over constant vacuum configurations. This would then signal the possibility of condensation of these nontrivial vacuum excitations. Our strategy to study the vortex condensation problem in the CSH model is as follows. Vortex excitations are made explicit in the functional action by making use of a series of dual transformations for the original Lagrangian fields, obtaining an equivalent action, in which it becomes clear the vortex contributions. By properly matching our dual action to a field theory model, it is then possible to write it in terms of a vortex field $\psi$ coupled to a vectorial field. In this process of writing the action in terms of an explicit vortex field, a dynamical mass for the vortices is generated. From the dynamical generated mass for the vortices we can then infer, already at the tree-level, about the possibility of vortex condensation. In this case, there is a critical value for the CS parameter where the dynamical mass for the vortices vanishes and these become energetically favorable to condense. We also calculate the one-loop self-energy corrections for the introduced vortex field to check the stability of the condensation point when quantum corrections are included.

This paper is organized as follows. In Sec. II we introduce the CSH model and briefly review its usual vortex solutions. In Sec. III we derive the dual action, equivalent to the original action, but where the vortex excitation degrees of freedom can be made explicit. By matching this dual action to a field theory model, an appropriate vortex field can then be introduced. Working in the London approximation, where scalar field fluctuations are frozen, we obtain an explicit expression for the dynamical mass term for the vortex field that emerges naturally in the passage from the vortex coordinates to the vortex field. The point for vortex condensation is then derived from the dynamical generated mass term for the vortex field. In Sec. IV we compute the self-energy corrections for the vortex field in the dual model and investigate the change of the tree-level vortex condensation point when one-loop quantum corrections are considered. Our conclusions and final remarks are given in Sec. V.

II. THE CHERN-SIMONS-HIGGS MODEL AND ITS VORTEX SOLUTION

Let us consider the Abelian CSH model, defined in terms of a complex scalar field $\phi$ and Abelian gauge field $A_\mu$. The quantum partition function can be written as (throughout this work we work in the Euclidean space-time)

$$Z = \int DA_\mu D\phi D\phi^* \exp \{-S_E[A_\mu, \phi, \phi^*]\} ,$$

with Euclidean action given by (indices run from 1 to 3)

$$S_E[A_\mu, \phi, \phi^*] = \int d^3x \left[ -\frac{\theta}{4} \epsilon_{\mu\nu\gamma} A_\mu F_{\nu\gamma} + |D_\mu \phi|^2 + V(|\phi|) \right] ,$$

with $D_\mu \equiv \partial_\mu + ieA_\mu$ and $\theta$ is the CS parameter. $V(|\phi|)$ is a symmetry breaking polynomial potential, independent of the phase of the complex scalar field. The potential $V(|\phi|)$ is some potential with a nonvanishing symmetry breaking minimum for the scalar field $\phi$, $|\langle \phi \rangle| = \nu \neq 0$. For example, we can have the usual quartic symmetry-broken scalar potential,

$$V(|\phi|) = \frac{\lambda}{4} (|\phi|^2 - \nu^2)^2 ,$$

or the sixth-order self-dual potential [5],

$$V(|\phi|) = \frac{e^4}{\theta^2} (|\phi|^2 - \nu^2)^2 |\phi|^2 ,$$

with degenerate minima at $|\phi|^2 = \nu^2$ and $|\phi| = 0$. Note also that in $2 + 1$ dimensions, a sixth-order potential in the scalar field still gives a renormalizable theory. In the following we will consider the potential (2.4) that leads to the so-called dual vortex solutions [5].
The field equations corresponding to Eq. (2.2) are known to have finite energy field solutions corresponding to vortices, that carry an electric charge $Q$, given in terms of the magnetic flux $\Phi$ by

$$Q = \theta \int d^2 x F_{12} \equiv \theta \Phi,$$

and that they are also anyons, with a spin $j = Q\Phi/(4\pi)$ [2]. The field equations, though not having exact solutions, admit radially symmetric like solutions for a vortex, with multiplicity $n$, given by (using polar coordinates denoted here by $r, \chi$)

$$\phi_{\text{vortex}} = \varphi(r) \exp(i n \chi),$$
$$A_{\mu, \text{vortex}} = \frac{n}{e} A(r) \partial_\mu \chi,$$

where the functions $\varphi(r)$ and $A(r)$ vanish at the origin and have the asymptotic behavior $\varphi(r \to \infty) \to v$ and $A(r \to \infty) \to 1$. The functions $\varphi(r)$ and $A(r)$ are obtained (numerically) by solving the classical field equations. Then, from Eqs. (2.6) and (2.7), we see that at spatial infinity the scalar field $\phi$ goes to the vacuum $v$ and the gauge $A_{\mu}$ becomes a pure gauge. In this case, the flux $\Phi$ in Eq. (2.5) becomes $\Phi = 2\pi n/e$. Also, since the scalar field must be a single-valued quantity, Eq. (2.5) implies that, on the vortex, the phase $\chi$ must be singular. Therefore, the phase $\chi$ can be separated into two parts: in a regular part, $\chi_{\text{reg}}$, and in a singular one, $\chi_{\text{sing}}$, due to the vortex configuration.

The energy of the $n$-vortex excitations is determined by using the solutions (2.6) and (2.7) in the static energy action functional that is given by

$$E = \int d^2 x \left[ |D_i \phi|^2 - e^2 A_0^2 |\phi|^2 - \theta A_0 F_{12} + V(|\phi|) \right],$$

which, upon the use of the vortex solutions, gives [5]

$$E_{\text{vortex}} \geq 2\pi v^2 |n|,$$

III. THE DUAL-TRANSFORMED ACTION

We now describe the steps necessary to make explicit in the model action Eq. (2.2) the vortex degrees of freedom. We start by writing the complex scalar field in a polar-like parametrization form as $\phi = \rho \exp(i \chi)/\sqrt{2}$. From the discussion in the previous section, this implies that on the vortices, the phase $\chi$ is a multivalued function and $\chi$ in general can be expressed in terms of a regular (single valued) and a singular part as $\chi(x) = \chi_{\text{reg}}(x) + \chi_{\text{sing}}(x)$. The quantity

$$J^\mu = \frac{1}{2\pi} \epsilon^{\mu\nu\gamma} \partial_\nu \partial_\gamma \chi_{\text{sing}},$$

defines the vortex current [11].

The existence of the topological vortex solutions imply that, when performing the path integral over the fields in (2.1), there will be field configurations corresponding to these vortex solutions. We next make these solutions explicit in the functional action. From the modulus and phase parametrization for the complex scalar field, the partition function (2.1) becomes

$$Z = \int \mathcal{D}A_\mu \mathcal{D}\rho \mathcal{D}\chi \left( \prod_x \rho \right) \exp \left\{ - \int d^3 x \left[ \frac{1}{2} \rho^2 (\partial_\mu \chi + e A_\mu)^2 - i \frac{\theta}{2} \epsilon_{\mu\nu\gamma} A_\mu \partial_\nu A_\gamma + \frac{1}{2} (\partial_\mu \rho)^2 + V(\rho) \right] \right\}. $$

(3.2)
The functional integration over the regular phase $\chi_{\text{reg}}(x)$ dependent terms in the partition function is performed as follows,

\[
\int D\chi \exp \left[ - \int d^3x \frac{1}{2} \rho^2 (\partial_\mu \chi + eA_\mu)^2 \right] = \int D\chi_{\text{sing}} D\chi_{\text{reg}} DC_\mu \left( \prod_x \rho^{-3} \right) \exp \left\{ - \int d^3x \left[ \frac{1}{2\rho^2} C_\mu^2 - iC_\mu (\partial_\mu \chi_{\text{reg}}) - iC_\mu (\partial_\mu \chi_{\text{sing}} + eA_\mu) \right] \right\} = \int D\chi_{\text{sing}} \left( \prod_x \rho^{-3} \right) \mathcal{D}h_\mu \exp \left\{ - \int d^3x \left[ \frac{\kappa}{16\pi^2\rho^2} H_{\mu\nu}^2 - i\frac{e\kappa^{1/2}}{4\pi} \epsilon_{\mu\nu\gamma} h_\mu F_{\nu\gamma} - i\kappa^{1/2} J_\mu h_\mu \right] \right\}, \quad (3.3)
\]

where the functional integral over the field $C_\mu$, in the second line, was introduced in order to linearize the exponent term in the first line. The functional integral over $\chi_{\text{reg}}$ in the second line of Eq. (3.3) can now be easily done. It gives a constraint on the functional integral measure, $\delta(\partial_\mu C_\mu)$, which can be represented in a unique way by expressing the field $C_\mu$ in terms of a dual field, $C_\mu = \frac{\kappa^{1/2}}{2\pi} \epsilon_{\mu\nu\gamma} \partial_\nu h_\gamma \equiv \frac{\kappa^{1/2}}{4\pi} \epsilon_{\mu\nu\gamma} H_{\nu\gamma}$, where $\kappa$ is some arbitrary parameter with mass dimension and $H_{\mu\nu} = \partial_\mu h_\nu - \partial_\nu h_\mu$.

The functional integral over the original gauge field $A_\mu$ in Eq. (3.2) can also be immediately performed by using Eq. (3.3), from which we then obtain for Eq. (3.2) the result

\[
Z = \int D\chi_{\text{sing}} D\rho \left( \prod_x \rho^{-2} \right) \mathcal{D}h_\mu \exp (-S), \quad (3.4)
\]

where

\[
S = \int d^3x \left[ \frac{m^2}{16\pi^2\epsilon^2\rho^2} H_{\mu\nu}^2 - \frac{m}{2} iJ_\mu h_\mu + i\frac{2m^2}{8\pi^2\epsilon} \epsilon_{\mu\nu\gamma} h_\mu \partial_\nu h_\gamma \right.
\]
\[
\left. + \frac{1}{2} (\partial_\mu \rho)^2 + V(\rho) \right]. \quad (3.5)
\]

In Eq. (3.4) an overall field independent multiplicative factor was omitted and in Eq. (3.5) we have defined for convenience a new parameter $m = e\kappa^{1/2}$, with mass dimension. This arbitrary mass parameter $m$ in the final dual action is just a spurious constant that can be absorbed in a redefinition of the dual gauge field $h_\mu$ and none of our results will depend on it. Note also that the gauge invariance of the original model, $\delta A_\mu = \partial_\mu \Lambda(x)$, is now replaced in the dual action in Eq. (3.3) by $\delta h_\mu(x) = \partial_\mu \Lambda(x)$. Thus, gauge invariance, now in terms of the $h_\mu$ field, is still preserved.

The obtained expression for the partition function of the dual CSH model makes evident the contribution due to vortices in the path integral. In comparing Eqs. (3.4) and (3.5) with the result obtained by the authors of Ref. [1] for the CSH model, we note that their dual result, in the absence of external fields and currents included in there, is exactly equivalent to our expression. Another result that we observe in (3.5) is the characteristic dualization of the CS coefficient, where from the original action in (3.2) to the dual one Eq. (3.3), it is changed like $\theta \rightarrow -1/(4\pi^2 \theta)$. This dualization for the CS parameter has also been shown previously, like in Refs. [12] and [13] (in this last reference it was also shown a detailed derivation of the field functional derivations leading to this result). As explained in [11], this sign difference of the CS coefficient between the original action and the dual one is of relevance in interpreting the statistics of the vortices in the theory as anyons.

From Eq. (3.5) we see that the vortex degrees of freedom, represented by the vortex current (3.4), appear coupled with the new gauge field $h_\mu$. This is a non-vanishing quantity due to the singular nature of $\chi_{\text{sing}}$ of the Higgs field phase and, hence, this interaction term will contribute in the action, along with the worldline $x^\mu(\tau)$ swept by the vortex (if taken as pointlike particles). This can be made more explicit once we write the vortex current, for unity winding number vortex excitations ($n = 1$), corresponding to the energetically dominant configurations, in the form

\[
\mathcal{J}^\mu = \int dx^\mu(\tau) \delta^3 [x - x(\tau)], \quad (3.6)
\]

where $x^\mu(\tau)$ gives the vortex trajectory parametrized by $\tau$, $0 \leq \tau \leq 1$. Using (3.6), the action term in Eq. (3.5) corresponding to the interaction of the vortex with the gauge field becomes
where $g = 2m/e$ is the vortex strength (again for unit winding number vortices, which we are here considering). Eq. (3.7) is the analog of a classical action of a charged particle, with charge $q = g$ and null rest mass, interacting with a vectorial field $h_{\mu}$. Thus, technically, we can identify the vortex as the worldline of a dual charged particle, which, under second quantization, can be associated to a (local) complex vortex field $\psi$ (this is much the same as used in many condensed matter applications of duality for vortices, e.g. like in Refs. [13, 14]), where a phenomenological field theory for vortices in two-dimensional space was considered. Here, in order to write a field model for vortices we follow the approach adopted e.g. in Ref. [15] for matching the action part corresponding to the vortices degrees of freedom to a field theory model. First, let us consider the exponential term in Eq. (3.4) given by Eq. (3.5) and write it in terms of the vacuum expectation value for the scalar field, $v \equiv \rho_0/\sqrt{2}$.

$$S = S_{\text{vortex},0} + \int d^3x \left[ \frac{m^2}{16\pi^2e^2\rho_0^2} H_{\mu\nu}^2 - \frac{m}{e} J_{\mu} h_{\mu} + i \frac{m^2}{8\pi^2\theta} \epsilon_{\mu\nu\gamma} h_{\mu} \partial_{\nu} h_{\gamma} \right],$$

where

$$S_{\text{vortex},0} = \int d^3x \left[ \frac{m^2}{16\pi^2e^2} \left( \frac{1}{\rho^2} - \frac{1}{\rho_0^2} \right) H_{\mu\nu}^2 + \frac{1}{2} (\partial_{\mu} \rho)^2 + V(\rho) \right].$$

Note that in the approximation that the vortex is classical and fluctuations of $\rho$ can be neglected (this is usually called in the condensed matter problem as the London approximation [7]), which should be valid provided we remain deep inside the broken phase. This should be the case for instance when thermal effects are not important. We next note that Eq. (3.9) is only non-vanishing on the vortex core. Then, by properly connecting Eq. (3.9) in the dual transformed variables to its original field form in terms of the vortex solutions, we can recognize that Eq. (3.9) can be expressed back in the energy functional form for the vortices and it can then be written in a Nambu like form as [16, 17],

$$S_{\text{vortex},0} = E_{\text{vortex}} \int d\tau,$$

where $E_{\text{vortex}} = \pi \rho_0^2$, for unit-winding number self-dual vortices. Note that by associating Eq. (3.9) to the classical vortex energy and the explicit use of the vortex solutions, a definite scale is been introduced into the problem by the use of the characteristics of the classical vortex solutions. It is natural and consistent, therefore, to consider from this point onwards that we are effectively working with effective objects described by thick vortices. The natural scale being introduced by the use of the classical vortex solutions is the vortex radius, given in terms of the Higgs mass in the broken phase as $a \sim 1/m_\mu$.

Considering the full vortex contribution to the partition function and the action in the London approximation, then from Eq. (3.10) and the interaction term with the dual gauge field $h_{\mu}$, Eq. (3.7), we have that the total vortex contribution to the action is of the form

$$S_{\text{vortex}} = E_{\text{vortex}} \int d\tau + i \frac{2m}{e} \int \frac{dx^\mu}{d\tau} d\tau h_{\mu}.$$  \hspace{1cm} (3.11)

This vortex action term can be matched to a field theory model for vortices, described by a vortex field $\psi$ interacting with the gauge field $h_{\mu}$ and in such a way that the gauge symmetry of the action (3.5) remains preserved [12, 13, 19] (see also e.g. Ref. [20] for an explicit derivation concerning vortex-strings excitations in 3+1 dimensions).

$$S_{\text{vortex}} = \int d^3x \left[ \partial_{\mu} \psi + i \frac{2m}{e} h_{\mu} \psi \right]^2 + M^2 |\psi|^2],$$

with the additional gauge invariance for the complex vortex field, $\psi(x) \to \psi(x) \exp[-i2m\Lambda(x)/e]$. In the process of matching the functional integration over the vortex coordinates in the original functional partition function (3.4) to
the one in terms of the vortex field, a dynamical (entropy) mass term $M$ is induced. It is expressed in terms of the classical vortex energy in Eq. (3.11) and the characteristic length scale for classical vortices, $a$, as

$$M^2 = \frac{1}{a^2} \left( e^a E_{\text{vortex}} - 6 \right).$$

(3.13)

As mentioned above, $a$ is set as the characteristic coherence length in the vortex phase, which is given by the inverse of the scalar field Higgs mass in the broken phase, $a \sim 1/m_H$, where $m_H$ is determined depending on the scalar potential being used. For the self-dual potential Eq. (2.4) considered here, $m_H = e^2 \rho_0^2 / \theta$. Also, for self-dual vortices, in Eq. (3.13) we also have that $E_{\text{vortex}} = \pi \rho_0^2$. Note that by restricting the analysis to self-dual vortices, we do not need to consider e.g. interacting terms for the vortices, that can be a complicating matter to add to the field theory model Eq. (3.12).

In principle, interaction terms for the vortices can be constructed in general, for example by introducing in the vortex action a core energy term for the vortices (see e.g. Refs. [7, 21] where this is used) of the form $\sim \varepsilon J_{\mu}^2$, which still preserves the gauge symmetry for the total action term in the functional partition function. The coupling $\varepsilon$ could be for instance phenomenologically matched to some physical system of interest modeling two-dimensional systems inspired by a CS theory. In this work, however, we will not try to go that far, but this could be an interesting possible extension of the approach used here.

For the initial purposes set for this work, of confirming and determining the location of the critical point for vortex condensation formally demonstrated in Ref. [9], our result given by Eq. (3.13) already suffices. It follows from Eq. (3.13) that for model parameters for which $M^2$ vanishes and then beyond that becomes negative, points to the case where vortex excitations can condense, since a vortex condensate would be energetically favorable to form. For the self-dual potential considered, we obtain that Eq. (3.13) vanishes and change sign for CS parameters below a critical value $\theta_c$ given by

$$\theta_c \approx \frac{\ln 6}{\pi} e^2.$$

(3.14)

This result corroborates for instance the demonstration in Ref. [10] about the existence of a critical value for the CS parameter below which vortex condensation should exist. For potentials other than the self-dual one, this critical value for the Chern-Simons coefficient can be a complicated function of the model parameters, since we then need to know the complete expression for the vortex free energy. In the next section we investigate whether quantum corrections to the dynamical vortex mass can appreciably change the result given by Eq. (3.14).

IV. THE VORTEX FIELD SELF-ENERGY

The analysis of the stability of the result (3.14) towards quantum corrections can be made by means of the evaluation of the self-energy quantum contributions to the vortex field dynamical mass. From the action (3.3) and using the result (3.12) the action expressed in terms of the dual gauge field $h_\mu$ and the vortex field $\psi$ becomes

$$S = \int d^3x \left[ \frac{m^2}{16\pi^2 e^2 \rho_0^2} H_{\mu\nu}^2 + i \frac{m^2}{8\pi^2 \theta} \epsilon_{\mu\nu\gamma} h_\mu \partial_\nu h_\gamma + \left| \partial_\mu \psi + \frac{2m}{e} h_\mu \psi \right|^2 + M^2 |\psi|^2 + \frac{(\partial_\mu h_\mu)^2}{2\alpha} \right],$$

(4.1)

where in the above equation $(\partial_\mu h_\mu)^2/(2\alpha)$ is a gauge fixing term.

In the London approximation used to derive Eq. (4.1) the vortex field only couples to the dual gauge field $h_\mu$. The interaction vertices relevant for the calculation of the one-loop self-energy for the vortex field $\psi$ come then from the terms $i(2m/e)h_\mu [\psi \partial_\mu \psi^* - \psi^* \partial_\mu \psi]$ and $(2m/e)^2 h_\mu^2 |\psi|^2$. The propagators for $\psi$ and $h_\mu$ can be determined from the quadratic Lagrangian density in the form

$$\mathcal{L}_2 = \frac{1}{2} h_\mu D_{\mu\nu}^{-1} h_\nu + \psi D_{\psi\psi}^{-1} \psi^*,$$

(4.2)

where $D_{\mu\nu}^{-1}$, by redefining the $h_\mu$ field by a constant factor: $h_\mu \rightarrow (2m/e\rho_0/m) h_\mu$ (note that this corresponds just to fix $m^2$ to be $4\pi^2 e^2 \rho_0^2$), is given by
\[ D_{\mu \nu}^{-1} = \left( 1 - \frac{1}{\alpha} \right) \partial_\mu \partial_\nu - \partial^2 \delta_{\mu \nu} - \frac{e^2 \rho_0^2}{\theta} \epsilon_{\mu \gamma \nu} \partial_\gamma , \]  

(4.3)

while \( D_{\psi \psi}^{-1} \) is given by

\[ D_{\psi \psi}^{-1} = -\partial^2 + M^2 . \]  

(4.4)

The inverse of Eqs. (4.3) and (4.4) define the field propagators for \( h_\mu \) and \( \psi \), respectively. In particular, for the \( h_\mu \) field, we have that the propagator, in momentum space, is given by

\[ D_{\mu \nu}(k) = \frac{[(\alpha - 1)k^2 + \alpha e^4 \rho_0^2/\theta^2]}{k^4 (k^2 + e^4 \rho_0^4/\theta^2)} k_\mu k_\nu + \frac{(e^2 \rho_0^2/\theta) \epsilon_{\mu \nu \beta k_\beta}}{k^2 (k^2 + e^4 \rho_0^4/\theta^2)} \delta_{\mu \nu} + \frac{\delta_{\mu \nu}}{k^2 + e^4 \rho_0^4/\theta^2} , \]  

(4.5)

while for the vortex field it is just

\[ D_{\psi \psi}^{-1}(k) = \frac{1}{k^2 + M^2} . \]  

(4.6)

A convenient choice of gauge in Eq. (4.5) is the Landau gauge \( \alpha = 0 \). With this choice we have \( k_\nu D_{\mu \nu} = 0 \), which assures that all contributions coming from derivative vertex Feynman diagrams vanish. In the following we adopted the Landau gauge in the calculation of the vortex self-energy.

The diagrams contributing to the vortex self-energy, \( \Sigma \), at one-loop order are shown in Fig. 1.

\[
\Sigma(p) = 16\pi^2 \rho_0^2 \delta_{\mu \nu} \int_{\text{vort}} \frac{d^3k}{(2\pi)^3} D^{\mu \nu}(k) - 16\pi^2 \rho_0^2 \int_{\text{vort}} \frac{d^3k}{(2\pi)^3} (2\rho + k)_\mu D_{\psi \psi}^{-1}(p + k)(2\rho + k)_\nu D^{\mu \nu}(k) ,
\]  

(4.7)

where the index in the momentum integral, \( \int_{\text{vort}} \), is to remember that the momentum integral is to be evaluated considering the case of effective thick vortices, with characteristic scale set by the (inverse of the) Higgs mass \( m_H \). This then sets a momentum cutoff \( \Lambda \equiv 1/a = m_H \).

In terms of the self-energy \( \Sigma(p) \), we define an effective dynamical mass for the vortex as given by

\[ M_{\text{eff}}^2 = M^2 + \Sigma(M_{\text{eff}}) , \]  

(4.8)

where the self-energy is to be evaluated on-shell. Equation (4.8) is a gap equation that has to be evaluated for the effective mass \( M_{\text{eff}} \). The critical point for vortex condensation is then defined by the value of the Chern-Simons coefficient for which \( M_{\text{eff}}(\theta_c) = 0 \). The critical point \( \theta_c \) is then obtained from Eq. (4.8) as given by the solution of

\[
\begin{align*}
\Sigma(p) & = 0, \\
M_{\text{eff}}^2 & = M^2 + \Sigma(M_{\text{eff}}) .
\end{align*}
\]  

(4.9)
\[ [M^2 + \Sigma(0)] \bigg|_{\theta=\theta_c} = 0 \, . \] (4.9)

By explicitly using the field propagators in Eq. (4.7) and solving the gap equation at the critical point, we find that Eq. (4.9) is given by the following simple equation to be solved for \( \theta_c \),
\[
M^2(\theta_c) + \frac{16e^2 \rho_0^4}{\theta_c} \left( 1 - \frac{\pi}{4} \right) = 0 \, . \] (4.10)

Using the tree-level result for the dynamical vortex mass, Eq. (3.13), we can numerically solve Eq. (4.10) for \( \theta_c \) and obtain that
\[
\theta_c \simeq 0.825 \frac{\ln 6}{\pi} e^2 \, , \] (4.11)
which is about 17% smaller than the tree-level result, Eq. (3.14), derived in the previous section. Higher loop terms to the self-energy should lead to \( \mathcal{O}(e^3) \) and higher corrections to this result and, thus, they are not expected to change appreciably the leading order one-loop result for \( \theta_c \), at least for perturbatively small couplings.

V. CONCLUSIONS

In this work we have given the expression for the quantum partition function for vortices in the context of the CSH model. This is realized by obtaining the dualized version of the model, where the contribution of vortex excitations are made apparent in the action and also their coupling with the matter fields. The procedure explained in Sec. III to obtain the dual action allows us to take into account, in the functional path integral, the contribution of not only constant vacuum field fluctuations but also those nontrivial singular topological excitations that are known to exist in the original model.

By restricting the study of the obtained dual action in the London limit for the scalar Higgs field, \( \rho = \rho_0 \equiv \langle \rho \rangle \) taken as constant and considering the classical self-dual vortex solutions, the vortex degrees of freedom in the partition function action can be matched to a field theory model in terms of a vortex field with a dynamical mass for the vortices. From the expression of this dynamical vortex mass, we have shown that we can define and obtain the critical point for which vortices are energetically favorable to condense. This determines a specific critical value for the CS parameter, \( \theta_c \). For values of \( \theta < \theta_c \), vortex condensation is favorable already at the tree level. We have also considered the one-loop corrections to the dynamical vortex mass term and derived the self-energy contribution for the vortex field. We have shown that the critical point for vortex condensation slight decreases when the quantum corrections are included, but the prediction for vortex condensation still remains. Higher loop corrections are not expected to change in any appreciable way our predictions and results, at least for perturbative values for the gauge coupling constant.

It would be interesting to further investigate the vortex condensation problem in the quantum theory by possibly including interaction terms for the vortices (i.e. using scalar potentials other than the self-dual one), in which case the method developed here could be useful in modeling, phenomenologically, two-dimensional systems based on the CSH model, like in the study of condensed matter planar systems, which are of importance in the study of high-temperature cuprate superconductors and in the fractional quantum Hall effect. The inclusion of finite temperature effects, and then going beyond the London approximation considered in this work, would also be a natural extension to be done. We hope to pursue these and other problems related to this work in the future.

Acknowledgments

R.O.R. is partially supported by Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq-Brazil) and Fundação de Amparo à Pesquisa do Estado do Rio de Janeiro (FAPERJ).

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