Weyl Anomaly Induced Current in Boundary Quantum Field Theories

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We show that when an external magnetic field parallel to the boundary is applied, Weyl anomaly gives rise to a new anomalous current in the vicinity of the boundary. The induced current is a magnetization current in origin: the movement of the virtual charges near the boundary give rise to a non-uniform magnetization of the vacuum and hence a magnetization current. Unlike other previous studied anomalous current phenomena such as the chiral magnetic effect or the chiral vortical effect, this induced current does not rely on the presence of a material system and can occur in vacuum. Similar to the Casimir effect, our discovered phenomena arises from the effect of the boundary on the quantum fluctuations of the vacuum. However this induced current is pure quantum mechanical and has no classical limit. We briefly comment on how this induced current may be observed experimentally.

\textit{Introduction} — Quantum anomaly induced current is an interesting phenomena. Much has been discussed in the literature \cite{1}. A number of such effects are known. The famous one is the chiral magnetic effect (CME) \cite{2–6} which refers to the generation of currents parallel to an external magnetic field $B$. The chiral vortical effect (CVE) \cite{7–10} refers to the generation of a current due to rotational motion in the charged fluid. The induced currents take the form

$$J_V = \sigma_{(B)V}B + \sigma_{(V)V}\omega, \quad J_A = \sigma_{(B)A}B + \sigma_{(V)A}\omega, \quad (1)$$

where $\sigma_{(B)V} = \frac{\epsilon\mu_A}{2\pi}$, $\sigma_{(B)A} = \frac{\epsilon\nu_A}{2\pi}$ are the chiral magnetic conductivities, $\sigma_{(V)V} = \frac{\nu\mu_A}{2\pi}$, $\sigma_{(V)A} = \frac{\nu^2 + \mu_A^2}{2\pi} + \frac{T^2}{\pi}$ are the chiral vortical conductivities, $\mu_A, \nu$ are the chemical potentials and $T$ is the temperature of the medium.

The chemical potential dependent current arises as a result of an imbalance in the left and right moving modes due to the axial anomaly, while the temperature dependent part comes from the gravitational anomaly \cite{11}. More recently, it has also been pointed out that anomalous current also occurs in a conformally flat gravitational spacetime due to Weyl anomaly \cite{12,13}. It should be noted that these anomalous current occurs only in a material system where the chemical potentials are non-vanishing, or in a curved spacetime. Since axial anomaly is an intrinsic property of Quantum Field Theory (QFT) which is present even in flat spacetime and in vacuum, it is natural to ask whether the phenomena of anomalous current may also occur in flat spacetime due to quantum fluctuation of the vacuum.

The Casimir effect is one of the most well known manifestation of the quantum fluctuation of electromagnetic vacuum in the presence of boundary \cite{14,16}. Recently the Casimir effects has been analyzed in full generality for arbitrary shape of boundary and for arbitrary spacetime metric, and new universal relations between the Casimir coefficients and the boundary central charge in a boundary conformal field theory have been discovered \cite{17}. The presence of boundary has also many other interesting physical consequences, e.g. renormalization group flows and critical phenomena \cite{18} or the topological insulator \cite{19} etc.

In this paper, we show that for a general class of boundary quantum field theory (BQFT) with $U(1)$ gauge symmetry, the quantum Weyl anomaly of the theory induces a new kind of induced current near the boundary. Consider a general BQFT defined on a four dimensional spacetime manifold $M$ with coordinates $x^\mu$, and has boundary $\partial M$ with coordinates $y^a$. The Weyl anomaly can be defined as the difference between the trace of renormalized stress tensor and the renormalized trace of stress tensor \cite{20,21}. We find it useful to introduce the following integrated Weyl anomaly

$$A = \int_M \sqrt{g} \left[ g^{\mu\nu} \langle T_{\mu\nu} \rangle - \langle g^{\mu\nu} T_{\mu\nu} \rangle \right]. \quad (2)$$

$A$ is equal to the variation of the effective action with respect to constant re-scaling of the metric\cite{22}. For simplicity, we focus on QFT which are covariant, gauge invariant, unitary and renormalizable, e.g. QED. By “renormalizable”, we mean, in the sense of perturbation theory, that all the coupling constants are of non-negative mass dimension. We also assume that the Weyl anomaly depends on only the positive powers of the coupling constants (including the mass $m$), so that it has a well-defined limit when we turn off the coupling constants. For this class of QFT, $A$ takes the following form \cite{20,23}

$$A = \int_M \sqrt{g} \left[ b_1 F_{\mu\nu} F^{\mu\nu} + O(R^2) \right] + \int_{\partial M} \sqrt{h} O(Rk). \quad (3)$$

Here $O(R^2)$ denotes terms constructed out of the bulk curvature tensor, including terms with positive powers of coupling constants; e.g. $R^2$, $R_{\mu\nu} R^{\mu\nu}$, $R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta}$, $\Box R$, $m^2 R$, $m^4$, $\cdots$, and $O(Rk)$
denotes the boundary Weyl anomaly [28, 29] that is constructed out of the boundary curvature tensor and the exterior curvature of the boundary. \( b_1 \) is the bulk central charge which govern the gauge field contribution to the Weyl anomaly [3]. For the normalization of the gauge field kinetic term \( S = -1/(4e^2) \int F^2 \), \( b_1 \) is related to the beta function as \( b_1 = -2/(2\pi^2) \) [30]. Below we show that for general BQFT as specified above, the expectation value of the induced current at a distance \( x \) very close to the boundary [31] is given by

\[
\langle J \rangle = \frac{e^2 c}{\hbar} 4 b_1 n \times B, \quad x \sim 0, \quad (4)
\]

where \( n \) is the inner normal to the boundary. The current (4) is a magnetization current \( J = \nabla \times M \) and corresponds to a quantum magnetization

\[
\langle M \rangle = \frac{e^2 c}{\hbar} 4 b_1 \log x B \quad (5)
\]

of the vacuum. It is remarkable that the anomalous current (4) and the vacuum magnetization (5) takes place even in flat spacetime and at zero temperature. These are pure quantum effect since it is inversely proportional the Planck constant and has no classical limit \( \hbar \to 0 \). The induced current is measured by quantum Hall conductance \( \sigma_H = e^2/\hbar \) which govern the quantum Hall effect. In fact the current (4) is in resemblance to the quantum Hall effect except that the current now is parallel to the boundary instead of perpendicular to the boundary as in the case of the standard Hall effect. One may therefore refer to (4) as an Anomalous Quantum Hall Effect [32].

**Physical Picture** — To understand the physical origin of the current (4) and the magnetization (5). Let us consider for simplicity the set up of a BQFT in flat spacetime with a flat boundary. Consider a point \( P \) at distance \( x \) from the boundary. We are interested in the amount of charges passing through \( P \) due to vacuum process of virtual particle creation and annihilation. Suppose there is a magnetic field normal to \( P \) (pointing out of the figure), the charged particles will move along circles due to the Lorentz force. If there is no boundary, the virtual particles created by quantum fluctuations at \( O' \) would annihilate at \( P \) after moving along the dotted circle. This gives rise to a transport of charges to the right. This is however precisely canceled by the movement of charges due to quantum fluctuation at the point \( O'' \). Summing over all possible locations of the source points, it is clear that there is no net transport of charges induced at \( P \). The situation is different when there is a boundary. In this case, those contribution from source points at \( x < 0 \) are missing. This leads to a net amount of charges moving to \(-y\) direction. In addition, vacuum pairs created at source point \( O''' \) could now reach \( P \) due to (virtual) reflection of the boundary. What exactly happens, perfect reflection or partial absorption, will depend on the boundary condition. But in any case there will be a net separation of charges and this contributes a transport of charges to the \(+y\) direction.

The current (4) can also be understood as a result of the magnetic response of the vacuum to the presence of boundary. As we noted already, quantum fluctuation of the vacuum leads to temporary creation of virtual pair of charged particles, which are then guided to move on circles in the presence of a magnetic field. As a result, tiny current loops are formed with the positively and negatively charged virtual particles contributing in the same way to the magnetic dipole moment. Summing all these contribution results in a total magnetization \( M \) of the vacuum. When there is no boundary, \( M \) is just an infinite constant that can be subtracted away by renormalization and the renormalized vacuum magnetization \( \langle M \rangle = 0 \) has no physical effect. When there is a boundary, it is clear that the renormalized \( \langle M \rangle \) is zero far away from the boundary, but become nontrivial near the boundary. This is very much like the Casimir effect. The magnetization (5) of the vacuum is a new effect and occurs only because of the presence of the boundary. Let us now turn to the rigorous QFT derivation.

**Rigorous Derivation** — We start with a proper analysis of the structure of the renormalized current \( J^\mu \) near the boundary. In general, for a BQFT, the renormalized current is generally singular near the boundary and the expectation value takes the asymptotic form near \( x \sim 0 \):

\[
\langle J^\mu \rangle = \frac{1}{x^3} J^{(3)}_\mu + \frac{1}{x^2} J^{(2)}_\mu + \frac{1}{x} J^{(1)}_\mu + \log x J^{(0)} + \cdots, \quad (6)
\]

where \( \cdots \) denotes terms regular at \( x = 0 \), and \( J^{(n)}_\mu \) depend only the background geometry, the background vector field strength and the type of fields under consideration. Hereafter we will drop the symbol ( ) for the expectation value. A similar expansion has been considered for the renormalized stress tensor [33]. We consider

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**FIG. 1.** Induced current from virtual pair creation in presence of boundary.
current that is conserved \((D_\mu J^\mu = 0)\) up to possibly an anomaly term. Since this term is finite, it is irrelevant to the divergent part of renormalized current \(\delta \mathcal{A}\). As a result, we obtain the gauge invariant solution

\[
J^{(3)}_\mu = 0, \quad J^{(2)}_\mu = 0, \quad J^{(1)}_\mu = \alpha_1 F_{\mu\nu} n^\nu + \alpha_2 D_\mu k + \alpha_3 D_\nu k_\mu + \alpha_4 \star F_{\mu\nu} n^\nu
\]

where \(F_{\mu\nu}, \star F_{\mu\nu}, n_\mu, D_\mu, k_\mu\) and \(h_{\mu\nu}\) are respectively the background field strength, Hodge dual of field strength, the normal vector, induced covariant derivative, extrinsic curvature and induced metric of the boundary. Note that in \(\delta \mathcal{A}\) we have re-expressed \(n^\mu R_{\mu\nu} h_\nu^\nu\) in terms of extrinsic curvatures by using the Gauss-Codazzi equation

\[
\frac{\delta}{\delta A_\mu} \mathcal{A} = \left( \int_M \sqrt{g} J^\mu (\delta A_\mu) \right) \log \frac{1}{\epsilon},
\]

where a regulator \(\epsilon \geq \epsilon\) to the boundary is introduced for the integral on the right hand side (RHS) of \(\delta \mathcal{A}\). The relation \(\delta \mathcal{A}\) identifies the boundary contribution of the current that is consistent with the conservation law and gauge invariance. We will now show that these current coefficients are indeed completely fixed by the central charges of the theory.

To establish this result, let us follow an observation of \([17]\) which allows one to relate the variation of \(\mathcal{A}\) with the asymptotic form of the stress tensor near the boundary. For the present case of current, we have the relation

\[
(\delta \mathcal{A})_{\partial M} = \left( \int_M \sqrt{g} J^\mu (\delta A_\mu) \right) \log \frac{1}{\epsilon},
\]

where a regulator \(\epsilon \geq \epsilon\) to the boundary is introduced for the integral on the right hand side (RHS) of \(\delta \mathcal{A}\). The relation \(\delta \mathcal{A}\) identifies the boundary contribution of the variation of the integrated anomaly \(\mathcal{A}\) under an arbitrary variation of the gauge field \(\delta A_\mu\) with the UV logarithmic divergent part of the integral involving the expectation value \(J^\mu\) of the renormalized \(U(1)\) current. The power of the relation \(\delta \mathcal{A}\) lies in the fact that the left hand side of \(\delta \mathcal{A}\) is a total variation and impose constraints on the RHS of \(\delta \mathcal{A}\) that are powerful enough to fix completely the asymptotic behavior of the current in terms of the Weyl anomaly of the theory. We refer the readers to the appendix for the derivation of this key relation \(\delta \mathcal{A}\).

Now let us use \(\delta \mathcal{A}\) to fix the coefficients. To proceed, let us consider the metric written in the Gauss normal coordinates \(ds^2 = dx^2 + (h_{ab} - 2x k_{ab} + x^2 q_{ab} + \cdots) dy^a dy^b\), where \(x \in [0, +\infty)\) and \(n_\mu = (1, 0, 0, 0)\) is the inward pointing normal vector. We also choose a gauge \(A_x = 0\) and expand the gauge field about the boundary: \(A_x = a_b + x A_x^{(1)} + \cdots\). Taking the variation of Weyl anomaly \(\delta \mathcal{A}\) with respect to the gauge field, we have \(\delta \mathcal{A}\mathcal{B} = 4b_1 \int_M \sqrt{g} F_{b}^n \delta a_b\). Next, we substitute \(\delta \mathcal{A}\), \(\delta \mathcal{A}\mathcal{B}\) into the RHS of \(\delta \mathcal{A}\), integrate over \(x\) and select the logarithmic divergent term, we obtain

\[
\left( \int_M \sqrt{g} J^\mu (\delta A_\mu) \right) \log \frac{1}{\epsilon} = \int_M \sqrt{h} \left( \alpha_1 F_{b}^n + \alpha_2 D^b k + \alpha_3 D_j k^j + \alpha_4 \star F_{b}^n \right) \delta a_b.
\]

As a result, we obtain for unitary QFT without the parity odd anomaly term \([23]\), \(\alpha_1 = 4b_1, \alpha_2 = \alpha_3 = \alpha_4 = 0\), and our main result for the expectation value of the current near the boundary:

\[
J_b = \frac{4b_1 F_{bn}}{x}, \quad x \sim 0,
\]

We emphasize that the current \(J_b\) does not involve any on-shell charged particle as we were considering the vacuum state and there is no Schwinger effect for magnetic field. Instead the induced current should be identified with a magnetization current as a result of the magnetization \(\mathcal{M}\) of the vacuum. This can be derived directly without first referring to the current \(J_b\) by using the magnetic coupling \(S = \int_M \sqrt{g} M \cdot B\) and the relation

\[
(\delta \mathcal{A})_{\partial M} = \left( \int_M \sqrt{g} M \cdot \delta B \right) \log \frac{1}{\epsilon},
\]

By considering a variation \(\delta B_z = \delta(x - \epsilon) \delta f(y, z)\) that is localized on the boundary \(\partial M\), one obtain \(\delta \mathcal{A}\).

The universal laws \(\delta \mathcal{A}\) and \(\delta \mathcal{A}\mathcal{B}\) hold for general BQFTs which are covariant, gauge invariant, unitary and renormalizable, or equivalently, for BQFTs whose Weyl anomaly is given by \(\delta \mathcal{A}\). Several comments are in order.

1. Since \(\delta \mathcal{A}\) and \(\delta \mathcal{A}\mathcal{B}\) depend only on the bulk central charge instead of boundary central charge, it is independent of the choices of boundary conditions. Thus the current is more universal than the renormalized stress tensor near the boundary which depends on boundary conditions \([17, 33–35]\). 2. The magnitude of the induced current is much larger than that of the stress tensor. To see this, let us recover the units in the formula. We have

\[
J_b = \frac{e^2 c b_1 F_{bn}}{\hbar} x, \quad T_{ab} = \hbar c d_1 h_{ab} x^2,
\]

where \(c\) is the charge, \(\hbar\) is the Planck constant, \(b_1, d_1\) are dimensionless constants and \(h_{ab}\) is the boundary metric. We have re-scaled \(F_{a\mu} \rightarrow e F_{a\mu}\) so that the field strength is related to electric field and magnetic field in the usual manner: \(E_i = c F_{i0}, B_i = \frac{1}{2} \varepsilon_{ijk} F_{jk}\). 3. Our result shows that constant magnetic field parallel to the boundary can induce a current \(J_b\). As we illustrated above, the boundary is crucial in realizing a separation of charges which result in the induced anomalous current and in the non-uniform magnetization for the vacuum. 4. We emphasize that our current is not due to on-shell movement of charges, but transport of virtual charges as a result of non-uniform vacuum magnetization. As such our current does not obey Ohm’s law and is not dissipative. It does not require an energy source to support it. 5. The result \(\delta \mathcal{A}\) is for a single boundary. For a real system with finite extent, e.g. a rectangular slab with two parallel boundaries, we will have current of the same form in each boundary components of the system. The total current is zero and satisfy the Bloch theorem \([30]\). 6. The relation \(\delta \mathcal{A}\) also implies an induced charge density \(\rho = \frac{e^2}{\hbar} x \cdot B\) near the boundary. Here \(E = E x\). 7. Our
results \([4]\) and \([5]\) were derived for the vacuum. In a material system, one need to take into account of the presence of charge carriers and non-vanishing conductivity of the media. The direct field theory analysis seems rather complicated. However due to the close relation with the Weyl anomaly, we expect that these results will continue to hold. In \([27]\) we use a holographic model to study the effect of conductivity, and we find that the current and the magnetization are not modified in the leading order of closeness to the boundary.

**Story of Free QFT** — Our general result \([9]\) is verified by free BQFT. For simplicity, let us consider complex scalar field with the action \(I = - \int_M \sqrt{|D^\mu \phi|} D_\mu \phi^* D^\mu \phi + E \phi^* \phi\), where \(D_\mu = \partial_\mu + iA_\mu\) are gauge invariant covariant derivatives and \(E\) are functions including only the coupling constants with non-negative mass dimension. For example, we can have \(E = m^2 + \lambda_0 R + \ldots\). However we exclude the terms like \(E = \lambda_1 F_{ij} F^{ij} + \lambda_2 R^2\) since they are non-renormalizable. In general, there are two kinds of boundary conditions for the scalar \([38]\): Dirichlet BC \((\phi|_{\partial M} = 0)\) and the Robin BC \((D_\mu \phi|_{\partial M} = 0)\). Here the function \(\psi\) defines a renormalizable theory, for example, \(\psi = 2\lambda_0 k + m f(y) + \cdots\). For a free complex scalar field theory, the expectation value of the current near the boundary has been derived in \([38]\) using heat kernel expansion. The result is \(J_b = -\frac{\partial_{\mu} \bar{\psi}}{24\pi^2 x}\) for both Dirichlet BC and Robin BC. The Weyl anomaly for the complex scalar field theory can be derived as the heat-kernel coefficient \(a_q\) \([39, 40]\). In this way, we get the Weyl anomaly \([3]\) with the central charge \(b_1 = -\frac{1}{24\pi^2}\). It is clear that the obtained current indeed satisfies our derived universal law \([9]\). From this simple example, we have learned two important facts. First, the near-boundary current is indeed independent of the choices of boundary conditions. Second, the universal law \([9]\) works for not just BCFT, but also for more general QFT. The only constraints we impose on the functions \(E, \psi\) are that they define a renormalizable theory. In particular, the theory need not be conformal invariant with \(E = \frac{1}{6} R, \psi = \frac{1}{2} k\).

**Finite Total Current** — Similar to the case of stress tensor \([17, 34, 41]\), there are boundary contributions to the current which make the total current finite. To see this, consider the gauge variation of finite part of the effective action. Due to gauge invariance, we obtain the conservation laws \(D_\mu J^\mu = 0\) in the bulk and \(D_\mu j^a = -J^a\) on the boundary. From the bulk current conservation and \([9]\), we get \(J_n = 4b_1 D_b F^{b n} \ln x + O(1)\). Substituting \(J_n\) into the boundary conservation law, we obtain the current boundary \(j_b = 4b_1 F_{bn} \ln \epsilon\). As a result, we have

\[
J_b = \frac{4b_1 F_{bn}}{x} + \delta(x; \partial M)4b_1 F_{bn} \ln \epsilon + O(1). \tag{12}
\]

where we have shifted the boundary from \(x = 0\) to a position \(x = \epsilon\). It is remarkable that the boundary current obtained from the conservation law automatically yields the total current \([12]\) which represent a finite flow of charge through any interval in the normal direction.

**On Experimental Observation** — Our current \([11]\) can be observed by measuring the magnetic response of the vacuum to external field in the presence of boundary. We have shown that the renormalized current and the quantum magnetization are independent of the choices of well-defined boundary conditions (BC). By ‘well-defined BC’, we means no current can flow out the boundary. The insensitivity of boundary conditions would decrease the difficulty in experiments. In reality since modes with sufficiently high frequencies would penetrate the boundary, this corresponds to an effective length cutoff and our formula \([11]\) will work well only for \(x > \epsilon\) with the cut off naturally being the lattice length \(a_{\text{lattice}}\) of the material in consideration. Consider, for simplicity, a constant magnetic field \(B\) and constant temperature \(T\) for the material. On the other hand, the formula \([11]\) applies only to the region close enough to the boundary such that \(x < x_{\text{max}} = \min\left(\frac{\hbar c}{k T}, \frac{\hbar c}{(eB)}\right)\), where \(m_{\text{eff}}\) is the effective mass of the charged particle. Taking \(T = 300\text{K}, m_{\text{eff}} = m_e\) to be the mass of electron and \(B = 0.01\text{T}\), we have \(x_{\text{max}} \sim 10^{-3}\text{m}, 10^{-13}\text{m}, 10^{-6}\text{m}\), which shows that the large mass of charged particle is the main obstruction to experimental observation of the phenomena. Thus one must try to decrease the effective mass in materials in order to satisfy \(\epsilon \approx x_{\text{max}}\). Fortunately, the availability of charge carriers with zero effective mass in graphene \([42]\) and Dirac or Weyl semimetals \([43]\) makes these systems a more promising setup for experimental observation of this induced current phenomena.

**Conclusions and Discussions** — In this letter, we show that for general four dimensional BQFTs which are gauge invariant, unitary and renormalizable, the renormalized current takes the universal form \([9]\) near the boundary. This covers fundamental theories such as QED, as well as many typical condensed matter systems of interests. The induced current is independent of the boundary conditions and the states of BQFT, and depends only on the beta function of the theory. Since the current is proportional to the quantum Hall conductance \(e^2/\hbar\), it is potentially large enough to be measured experimentally. It is interesting to perform experiment to observe this effect. It is also interesting to look for suitable implication of this effects for other physical systems such as astronomical objects or branes in string theory. Our discussions can be easily generalized to system with background non-Abelian gauge field and with spacetime dimensions other than four. See the appendix for the expectation value of current in dimensions other than four. We note however that only in four dimensions is the near boundary value of the current determined universally by the bulk central charge and is independent of boundary conditions.
ACKNOWLEDGEMENTS

We thank Bei-Lok Hu, Yan Liu, Jian-Xin Lu, Da-Wei Wang and Ling-Yan Hung for useful discussions and comments. This work is supported in part by NCTS and the grant MOST 105-2811-M-007-021 of the Ministry of Science and Technology of Taiwan. Rong-Xin Miao thank the funding of Sun Yat-Sen University.

Supplementary Information

1. The derivation of key formula

Consider a BQFT with a well defined effective action. The integrated Weyl anomaly $A$ defined by (2) can be obtained as the logarithmic UV divergent term of the effective action,

$$I = \cdots + A \log \left( \frac{1}{\epsilon} \right) + I_{\text{finite}},$$  \hspace{1cm} (13)

where $\cdots$ denotes terms which are UV divergent in powers of the UV cutoff $1/\epsilon$, and $I_{\text{finite}}$ is the renormalized, UV finite part of the effective action. To derive this result, let us consider a constant Weyl transformation $g_{\mu \nu} \rightarrow \exp(2\omega)g_{\mu \nu}$. Under this transformation, the UV cutoff transforms as $\epsilon \rightarrow \exp(\omega \epsilon)$ and the variation of effective action (13) becomes

$$\delta \omega I = \cdots + \omega (\delta A + \int_M \sqrt{g}(\partial^\mu g_{\mu \nu}) + O(\omega^2),$$ \hspace{1cm} (14)

where we have used $\delta \omega A = 0$ and $\delta \omega I_{\text{finite}} = \omega \int_M \sqrt{g}(\partial^\mu g_{\mu \nu}) + O(\omega^2)$. On the other hand, by definition we have

$$\frac{1}{2} \int_M \sqrt{g} \hat{T}^{\mu \nu} \delta g_{\mu \nu} = \omega \int_M \sqrt{g} \hat{T}^{\mu \nu} g_{\mu \nu} + O(\omega^2),$$ \hspace{1cm} (15)

where $\hat{T}^{\mu \nu}$ is the non-renormalized stress tensor. We use the hatted symbol (e.g. $\hat{T}_{\mu \nu}$) to denote non-renormalized quantity and un-hatted symbol (e.g. $T_{\mu \nu}$) to denote renormalized quantity. Separating $\hat{T}^{\mu \nu} g_{\mu \nu}$ into the renormalized UV finite part $\langle \hat{T}^{\mu \nu} g_{\mu \nu} \rangle$ and divergent part, we have

$$\delta \omega I = \cdots + \omega \int_M \sqrt{g} \langle \hat{T}^{\mu \nu} g_{\mu \nu} \rangle + O(\omega^2).$$ \hspace{1cm} (16)

Comparing the finite part of (14) and (16), we obtain the expression (2) for $A$ and hence our claim.

Now we are ready to prove the result (8) quoted in the main text of this letter. Inspired by [44, 45], let us regulate the effective action by excluding from its volume integration a small strip of geodesic distance $\epsilon$ from the boundary. Then there is no explicit boundary divergences in this form of the effective action, however there are boundary divergences implicit in the bulk effective action which is integrated up to distance $\epsilon$. The variation of effective action with respect to the vector is given by

$$\delta I = \int_{x \geq \epsilon} \sqrt{g} J^\mu \delta A_\mu,$$ \hspace{1cm} (17)

where $\hat{J}^\mu = \frac{\delta I}{\sqrt{g} A_\mu}$ is the non-renormalized bulk current. The renormalized bulk current is defined by the difference of the non-renormalized bulk current against a reference one [33]:

$$J^\mu = \hat{J}^\mu - J^\mu_0,$$ \hspace{1cm} (18)

where $J^\mu_0$ is the non-renormalized current defined for the same CFT without boundary. It is

$$\delta I_0 = \int_{x \geq \epsilon} \sqrt{g} J^\mu_0 \delta A_\mu,$$ \hspace{1cm} (19)

where $I_0$ is the effective action of the CFT with the boundary removed, hence the integration over the region $x \geq \epsilon$. Subtract (19) from (17) and focus on only the logarithmically divergent terms, we obtain our key formula

$$(\delta A)_{\partial M} = \left( \int_{x \geq \epsilon} \sqrt{g} J^\mu \delta A_\mu \right)_{\log(1/\epsilon)},$$ \hspace{1cm} (20)

where $(\delta A)_{\partial M}$ is the boundary terms in the variations of Weyl anomaly and $J^\mu$ is the renormalized bulk current. In the above derivations, we have used the fact that $\delta I$ and $\delta I_0$ have the same bulk variation of Weyl anomaly so that

$$(\delta A)_{\partial M} = (\delta I - \delta I_0)_{\log(1/\epsilon)}.$$ \hspace{1cm} (21)

We remark that we have considered global Weyl rescaling here and this is sufficient for the derivation of our results. For general local Weyl rescaling, the transformation properties of the effective action in the presence of boundaries can be analyzed in the line of [46] and we will leave it for future work.

2. Renormalized currents in $d \neq 4$

Consider BQFT in $d$-dimensional spacetime, flat for simplicity. In higher dimensions, it is expected that the renormalized current takes the following form

$$J_\mu = \frac{\alpha_1 F_{\mu n}}{x^{d-3}} + \frac{\alpha_2 \partial_\nu F_{\mu \nu}}{x^{d-4}} + \cdots, \hspace{1cm} x \sim 0,$$ \hspace{1cm} (22)

near the boundary. We claim that for $d > 4$, $\alpha_1$ depends on the boundary conditions in general. Let us take $d = 5$ as an example, where the Weyl anomaly has only boundary contributions

$$A = \int_{\partial M} \sqrt{h} [b_1 F_{ab} F^{a b} + b_2 F_{ab} F^{a b} + \cdots],$$ \hspace{1cm} (23)
Here $b_1, b_2$ are boundary central charges which depends on the choices of boundary conditions. By using our key formula \((2)\) together with $A_b = a_b - x F_{bn} + \cdots$ and the gauge choice $A_x = 0$, we obtain

$$\alpha_1 = -2b_1, \quad \alpha_2 = 4b_2,$$

(24)

which implies that the current \((22)\) depends on boundary conditions for $d = 5$. Note that the first relation in \((24)\) for $\alpha_1$ actually holds for general curve space. For the free complex scalar theory, the coefficient $\alpha_1$ for has been derived \([35]\):

$$\alpha_1 = \begin{cases} -\frac{2\Gamma(\frac{d}{2})}{(4\pi)^{d-1}}, & \text{Dirichlet BC} \\ -\frac{(d-2)\Gamma(\frac{d}{2}-1)}{(4\pi)^{d-1}}, & \text{Robin BC}. \end{cases}$$

(25)

As we have seen, for $d > 4$ the current takes different values for Dirichlet BC and Robin BC, which agrees with our result above.

For lower dimensions $d < 4$, similar analysis as \((4)\) – \((7)\) gives near a plane boundary the asymptotic current density

$$J_b = \begin{cases} \alpha \frac{e}{\hbar} F_{nb} x, & d = 2 \\ \alpha \frac{e}{\hbar} F_{nb}(\beta + n_R \ln x), & d = 3, \end{cases}$$

(26)

where $\alpha, \beta, n_R$ are constant parameters of order one. We emphasis, however, for lower dimensions $d < 4$, the current \((26)\) are not related to the Weyl anomaly. Hence the parameters in \((26)\) are not related to the central charge of the theory, but they are determined by the specific details of the theory. For example for free complex scalars, we have $n_R = 1$ for Robin BC and $n_R = 0$ for Dirichlet BC \([38]\).



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