Vanishing of the first continuous $L^2$-cohomology for II$_1$ factors

by Sorin Popa$^1$ and Stefaan Vaes$^2$

Abstract

We prove that the continuous version of the Connes-Shlyakhtenko first $L^2$-cohomology for II$_1$ factors, as proposed by A. Thom in [Th06], always vanishes.

In [CS03], A. Connes and D. Shlyakhtenko developed an $L^2$-cohomology theory for finite von Neumann algebras $M$, and more generally for weakly dense *-subalgebras $A \subset M$ of such von Neumann algebras. Then in [Th06], A. Thom provided an alternative, Hochschild-type characterization of the first such $L^2$-cohomology of $M$ as the quotient of the space of derivations $\delta : M \to \text{Aff}(M \otimes M^\text{op})$ by the space of inner derivations, where $\text{Aff}(M \otimes M^\text{op})$ denotes the *-algebra of operators affiliated with $M \otimes M^\text{op}$. Thom also proposed in [Th06] a continuous version of the first $L^2$-cohomology, by considering the (smaller) space of derivations $\delta$ that are continuous from $M$ with the operator norm to $\text{Aff}(M \otimes M^\text{op})$ with the topology of convergence in measure. He noted that in many cases (e.g., when $M$ has a Cartan subalgebra, or when $M$ is not prime), this cohomology vanishes, i.e. any continuous derivation of $M$ into $\text{Aff}(M \otimes M^\text{op})$ is inner.

Following up on this work, V. Alekseev and D. Kyed have shown in [AK11] that the first continuous $L^2$-cohomology also vanishes when $M$ has property (T), when $M$ is finitely generated with nontrivial fundamental group, or when $M$ has property Gamma. Recently, V. Alekseev proved in [Al13] that this is also the case for the free group factors $L(F_n)$.

In this article, we prove that in fact the first continuous $L^2$-cohomology vanishes for all finite von Neumann algebras. The starting point of our proof is a key calculation in the proof of [Al13, Proposition 3.1], which provides a concrete sequence of elements $y_n$ in the II$_1$ factor $M = L(F_3)$ of the free group $F_3$ with generators $a, b, c$, that tends to 0 in operator norm, but has the property that if a derivation $\delta : M \to \text{Aff}(M \otimes M^\text{op})$ satisfies $\delta(u_n) = u_n \otimes 1$ and $\delta(u_b) = \delta(u_c) = 0$, then $\delta(y_n)$ does not tend to 0 in measure. More precisely, the $y_n$’s in [Al13] are scalar multiples of words $w_n$ in $a, b, c$ with the property that $\delta(u_n)$ is a larger and larger sum of free independent Haar unitaries. In the case of an arbitrary II$_1$ factor $M$, we fix a hyperfinite II$_1$ factor $R \subset M$ with trivial relative commutant, and then use [Po92] to “simulate” (in distribution) $L(F_3)$ inside $M$, with $a$ any fixed unitary in $M$ and $b_m, c_m$ Haar unitaries in $R$ such that $a, b_m, c_m$ are asymptotically free. If now $\delta$ is a continuous derivation on $M$, then by subtracting an inner derivation, we may assume $\delta$ vanishes on $R$, thus on $b_m, c_m$. If $\delta(a) \neq 0$, and if we formally define $y_n$’s via the same formula as Alekseev’s, with $a, b_m, c_m$ in lieu of $a, b, c$, then a careful estimation of norms of $y_n$ and $\delta(y_n)$, which uses results in [HL99], shows that one still has $\|y_n\| \to 0$, while $\delta(y_n) \not\to 0$ in measure.

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1Mathematics Department, UCLA, CA 90095-1555 (United States), popa@math.ucla.edu
Supported in part by NSF Grant DMS-1101718

2KU Leuven, Department of Mathematics, Leuven (Belgium), stefaan.vaes@wis.kuleuven.be
Supported by Research Programme G.0639.11 of the Research Foundation – Flanders (FWO) and KU Leuven BOF research grant OT/13/079.
Let $M$ be a finite von Neumann algebra. We denote by $M^{\text{op}}$ the opposite von Neumann algebra and by $\text{Aff}(M \boxtimes M^{\text{op}})$ the $*$-algebra of operators affiliated with $M \boxtimes M^{\text{op}}$. A derivation $\delta : M \to \text{Aff}(M \boxtimes M^{\text{op}})$ is a linear map satisfying

$$\delta(ab) = (a \otimes 1)\delta(b) + (1 \otimes b^{\text{op}})\delta(a)$$

for all $a, b \in M$.

For every $\xi \in \text{Aff}(M \boxtimes M^{\text{op}})$, denote by $\partial \xi$ the inner derivation defined as

$$(\partial \xi)(a) = (a \otimes 1 - 1 \otimes a^{\text{op}})\xi$$

for all $a \in M$.

We equip $\text{Aff}(M \boxtimes M^{\text{op}})$ with the measure topology, i.e. the unique vector space topology with basic neighborhoods of 0 given by

$$B(\tau_1, \varepsilon) = \{ \xi \in \text{Aff}(M \boxtimes M^{\text{op}}) \mid \exists \text{ projection } p \in M \boxtimes M^{\text{op}} \text{ with } \tau_1(p) > 1 - \varepsilon, \|\xi p\| < \varepsilon \}$$

for all normal tracial states $\tau_1 : M \to \mathbb{C}$ and all $\varepsilon > 0$. If $\tau : M \to \mathbb{C}$ is a normal faithful tracial state, then $\{ B(\tau, \varepsilon) \mid \varepsilon > 0 \}$ is a family of basic neighborhoods of 0 and there is no need to vary the trace.

**Theorem 1.** Let $M$ be a finite von Neumann algebra. Every derivation $\delta : M \to \text{Aff}(M \boxtimes M^{\text{op}})$ that is continuous from the norm topology on $M$ to the measure topology on $\text{Aff}(M \boxtimes M^{\text{op}})$, is inner.

The following lemma is quite standard, but we include a detailed proof for completeness.

**Lemma 2.** It suffices to prove Theorem 1 for $I_1$ factors $M$ with separable predual.

**Proof.** We prove the lemma in different steps.

*Step 1.* It suffices to prove Theorem 1 for diffuse, countably decomposable $M$. Take a set $I = I_1 \sqcup I_2$ and an orthogonal family of projections $p_i \in \mathcal{Z}(M)$ with $\sum_{i \in I} p_i = 1$ and such that for all $i \in I_1$, we have that $M p_i$ is countably decomposable and diffuse, and such that for all $i \in I_2$, we have that $M p_i$ is a matrix algebra. Since the projections $p_i$ are orthogonal, the element

$$\xi = \sum_{i \in I} (p_i \otimes 1)\delta(p_i)$$

is well defined in $\text{Aff}(M \boxtimes M^{\text{op}})$. The following direct computation shows that

$$\delta(p_k) = (\partial \xi)(p_k)$$

for all $k \in I$. \hspace{1cm} (1)

Indeed,

$$\partial \xi)(p_k) = (p_k \otimes 1)\delta(p_k) - \sum_{i \in I} (p_i \otimes p_k^{\text{op}})\delta(p_i).$$

But, for all $i$ and $k$, we have

$$1 \otimes p_k^{\text{op}})\delta(p_i) = \delta(p_i p_k) - (p_i \otimes 1)\delta(p_k).$$

Multiplying with $p_i \otimes 1$ and summing over $i$, we find that

$$\sum_{i \in I} (p_i \otimes p_k^{\text{op}})\delta(p_i) = (p_k \otimes 1)\delta(p_k) - \delta(p_k).$$

In combination with (2), we find that (1) holds.
Replacing $\delta$ by $\delta - \partial \xi$, we may assume that $\delta(p_i) = 0$ for all $i \in I$. It follows that $\delta(Mp_i) \subset \text{Aff}(Mp_i \otimes (Mp_i)^{op})$ for every $i \in I$. We denote by $\delta_i$ the restriction of $\delta$ to $M_{p_i}$. By the assumption of step 1, $\delta_i$ is inner when $i \in I_1$. So for $i \in I_1$, we have that $\delta_i = \partial \xi_i$ for some $\xi_i \in \text{Aff}(M_{p_i} \otimes (M_{p_i})^{op})$. When $i \in I_2$, we have that $M_{p_i}$ is a matrix algebra and we can take a complete system of matrix units $e^i_{jk}$ for $M_{p_i}$. We then get that $\delta_i = \partial \xi_i$ for $i \in I_2$.

$$\xi_i = \sum_k (e^i_{k1} \otimes 1)\delta_i(e^i_{1k}) .$$

The vector $\xi = \sum_{i \in I} \xi_i$ is well defined in $\text{Aff}(M \otimes M^{op})$ and $\delta = \partial \xi$.

**Step 2.** It suffices to prove Theorem 1 when $M$ is diffuse and has separable predual. Using step 1, we may already assume that $M$ is diffuse and admits a faithful normal tracial state $\tau$ that we keep fixed. We start by proving the following three statements, using that $M$ admits the faithful trace $\tau$.

If $\xi \in \text{Aff}(M \otimes M^{op})$, there exists a von Neumann subalgebra $N \subset M$ with separable predual such that $\xi \in \text{Aff}(N \otimes N^{op})$. Take an increasing sequence of projections $p_n \in M \otimes M^{op}$ such that $\tau(1 - p_n) \to 0$ and $\xi \in M \otimes M^{op}$ for all $n$. In particular, $\xi p_n \in L^2(M \otimes M^{op}) = L^2(M) \otimes L^2(M^{op})$ and we can take separable Hilbert subspaces $H_n \subset L^2(M)$, $K_n \subset L^2(M^{op})$ such that $\xi p_n \in H_n \otimes K_n$. We then find countable subsets $V_n \subset M$ such that for every $n$, the vector $\xi p_n$ belongs to the $\| \cdot \|_2$-closed linear span of $\{a \otimes b^{op} \mid a, b \in V_n\}$. Defining $N$ as the von Neumann subalgebra of $M$ generated by all the sets $V_n$, our statement is proven.

Let $N_1 \subset M$ be a von Neumann subalgebra with separable predual. Then there exists a von Neumann subalgebra $N_2 \subset M$ with separable predual such that $N_1 \subset N_2$ and $\delta(N_1) \subset \text{Aff}(N_2 \otimes N_2^{op})$. Take a separable and weakly dense $C^*$-subalgebra $B_1 \subset N_1$. By the previous paragraph and because $\delta$ is norm-measure continuous, we can take $N_2 \subset M$ with separable predual such that $\delta(B_1) \subset \text{Aff}(N_2 \otimes N_2^{op})$. Replacing $N_2$ by the von Neumann algebra generated by $N_1$ and $N_2$, we may assume that $N_1 \subset N_2$. Since $\delta(B_1) \subset \text{Aff}(N_2 \otimes N_2^{op})$, it follows from [Th06, Lemma 4.2 and Theorem 4.3] that $\delta(N_1) \subset \text{Aff}(N_2 \otimes N_2^{op})$.

Let $N_3 \subset M$ be a von Neumann subalgebra with separable predual. Then there exists a von Neumann subalgebra $N \subset M$ with separable predual such that $N_1 \subset N$ and $\delta(N) \subset \text{Aff}(N \otimes N^{op})$. Using the previous paragraph, we inductively find an increasing sequence of von Neumann subalgebras $N_1 \subset N_2 \subset \cdots$ with separable predual such that $\delta(N_k) \subset \text{Aff}(N_k \otimes N_k^{op})$ for all $k$. We define $N$ as the von Neumann algebra generated by all the $N_k$. By construction, $N$ has separable predual and $\delta(N_k) \subset \text{Aff}(N \otimes N^{op})$ for all $k$. Again using [Th06, Lemma 4.2 and Theorem 4.3], it follows that $\delta(N) \subset \text{Aff}(N \otimes N^{op})$.

We can now conclude the proof of step 2. Since $M$ is diffuse, we can fix a diffuse abelian von Neumann subalgebra $A \subset M$ with separable predual. By [Th06, Theorem 6.4], we can replace $\delta$ by $\delta - \partial \xi$ for some $\xi \in \text{Aff}(M \otimes M^{op})$ and assume that $\delta(a) = 0$ for all $a \in A$. We prove that $\delta(x) = 0$ for all $x \in M$. Fix an arbitrary $x \in M$. Define $N_1$ as the von Neumann algebra generated by $A$ and $x$. Note that $N_1$ has a separable predual. By the previous paragraph, we can take a von Neumann subalgebra $N \subset M$ with separable predual such that $N_1 \subset N$ and $\delta(N) \subset \text{Aff}(N \otimes N^{op})$. By the initial assumption of step 2, the restriction of $\delta$ to $N$ is inner. So we can take a $\xi \in \text{Aff}(N \otimes N^{op})$ such that $\delta(y) = (\partial \xi)(y)$ for all $y \in N$. Since $A \subset N$ and $\delta(a) = 0$ for all $a \in A$, it follows that $(a \otimes 1)\xi = (1 \otimes a^{op})\xi$ for all $a \in A$. Since $A$ is diffuse, this implies that $\xi = 0$. Since $x \in N$, it then follows that $\delta(x) = 0$. This concludes the proof of step 2.

**Step 3.** Proof of the lemma: it suffices to prove Theorem 1 when $M$ is a $I_1$ factor with separable predual. Using step 2, we may already assume that $M$ is diffuse and has separable predual.
Let \( p_0 \in \mathcal{Z}(M) \) be the maximal projection such that \( \mathcal{Z}(M)p_0 \) is diffuse (possibly, \( p_0 = 0 \)). Let \( p_1, p_2, \ldots \) be the minimal projections in \( \mathcal{Z}(M)(1-p_0) \). Note that \( \sum_{n=0}^{\infty} p_n = 1 \). As in the proof of step 1, we may assume that \( \delta(Mp_n) \subset \text{Aff}(Mp_n \otimes (Mp_n)^\text{op}) \) for all \( n \). Denote by \( \delta_n \) the restriction of \( \delta \) to \( Mp_n \). Since \( Mp_0 \) has a diffuse center, it follows from [Th06, Theorem 6.4] that \( \delta_0 \) is inner. For all \( n \geq 1 \), we have that \( Mp_n \) is a II\(_1\) factor with separable predual. So by assumption, all \( \delta_n, n \geq 1 \), are inner. But then also \( \delta \) is inner. 

**Proof of Theorem 1.** Using Lemma 2 it suffices to take a II\(_1\) factor \( M \) with separable predual and a derivation \( \delta : M \to \text{Aff}(M \otimes M^\text{op}) \) that is continuous from the norm topology on \( M \) to the measure topology on \( \text{Aff}(M \otimes M^\text{op}) \). Denote by \( \tau \) the unique tracial state on \( M \). By [Po92, Corollary on p. 187], we can fix a copy of the hyperfinite II\(_1\) factor \( R \subset M \) such that \( R' \cap M = \mathbb{C}1 \). By [Th06, Theorem 6.4], we can replace \( \delta \) by \( \delta - \partial \xi \) and assume that \( \delta(\tau) = 0 \) for all \( \tau \in \mathcal{U}(R) \). We prove that \( \delta = 0 \). Fix a unitary \( u \in \mathcal{U}(M) \) with \( \tau(u) = 0 \). It suffices to prove that \( \delta(u) = 0 \).

Fix a free ultrafilter \( \omega \) on \( \mathbb{N} \) and consider the ultrapower \( M^\omega \). By [Po92, Corollary on p. 187], choose a unitary \( v \in R^2 \) such that the subalgebras \( v^kMv^{-k} \subset M^\omega, k \in \mathbb{Z}, \) are free. Fix a Haar unitary \( a \in \mathcal{U}(R) \), i.e. a unitary satisfying \( \tau(a^m) = 0 \) for all \( m \in \mathbb{Z} \setminus \{0\} \). Define, for \( k \geq 1 \), \( a_k = v^kav^{-k} \). It follows that \( a_k \in \mathcal{U}(R^\omega) \) are \(*\)-free Haar unitaries that are moreover free w.r.t. \( M \). Write \( a_k = (a_{k,n}) \) with \( a_{k,n} \in \mathcal{U}(R) \).

Similar to the definition of \( x_n \) in the proof of [AH13, Proposition 3.1], we consider for all large \( n \) and \( m \), the unitary

\[
w_{m,n} = a_{1,n}u a_{2,n}u \cdots a_{m,n}u
\]

and prove that either \( \delta(u) = 0 \) or \( \delta(w_{m,n}) \) is “very large almost everywhere”, contradicting the continuity of \( \delta \).

Since \( \delta(x) = 0 \) for all \( x \in R \), we get that

\[
\delta(w_{m,n}) = \left( \sum_{k=1}^{m} a_{1,n}u a_{2,n}u \cdots a_{k-1,n}u a_{k,n} \otimes (a_{k+1,n}u \cdots a_{m,n}u)^\text{op} \right) \delta(u),
\]

where, by convention, the first term in the sum is \( a_{1,n} \otimes (a_{2,n}u \cdots a_{m,n}u)^\text{op} \) and the last term is \( a_{1,n}u \cdots a_{m-1,n}u a_{m,n} \otimes 1 \).

Consider, in the ultrapower \( (M \otimes M^\text{op})^\omega \), the element

\[
T_m = \sum_{k=1}^{m} a_{1}u a_{2}u \cdots a_{k-1}u a_{k} \otimes (a_{k+1}u \cdots a_{m}u)^\text{op}.
\]

We claim that \( T_m \) is the sum of \( m \) \(*\)-free Haar unitaries. To prove this, it suffices to show that the first tensor factors \( a_{1}, a_{1}a_{2}, a_{1}a_{2}a_{3}, \ldots \) form a \(*\)-family of Haar unitaries. Since \( a_{1}, a_{2}, a_{3}, \ldots \) is a \(*\)-family of Haar unitaries that are \(*\)-free w.r.t. \( u \), also the Haar unitaries \( a_{1}, u a_{2}, u a_{3}, u a_{4}, \ldots \) are \(*\)-free. But then the conclusion follows by taking the product of the first \( k \) unitaries in this last sequence, again producing a \(*\)-family of Haar unitaries.

Since \( T_m \) is the sum of \( m \) \(*\)-free Haar unitaries, we get from [HL99, Example 5.5] an explicit formula for the spectral distribution of \( |T_m| \). It follows that \( |T_m| \) has the same distribution as \( 2\sqrt{m} - TS_m \), where \( S_m \) is a sequence of random variables satisfying \( 0 \leq S_m \leq 1 \) and converging in distribution to the normalized quarter circle law. Therefore, the spectral projections

\[
q_m = 1_{|\sqrt{m},+\infty)}(T_m T_m^*) = 1_{[m^{1/4},+\infty)}(|T_m|) = 1_{[m^{1/4}(4(m-1))^{-1/2},+\infty)}(S_m)
\]

satisfy \( \lim_m \tau(q_m) = 1 \). Write \( q_m = (q_{m,n}) \) where \( q_{m,n} \) are projections in \( M \otimes M^\text{op} \).
Fix an arbitrary $\varepsilon > 0$. Since $\delta$ is continuous, fix $\rho > 0$ such that $\delta(z) \in B(\tau, \varepsilon/2)$ whenever $z \in M$ and $\|z\| < \rho$. Take $m$ large enough such that $m^{-1/4} < \rho$ and $\tau(q_m) > 1 - \varepsilon$. For every $n$, the element $m^{-1/4}w_{m,n}$ has norm less than $\rho$. Therefore, $\delta(m^{-1/4}w_{m,n})$ belongs to $B(\tau, \varepsilon/2)$ and we find a projection $p_n \in M \overline{\otimes} M^{\text{op}}$ with

$$\tau(p_n) > 1 - \varepsilon/2 \quad \text{and} \quad \|\delta(m^{-1/4}w_{m,n}) p_n\| < \varepsilon/2.$$  

We also fix a projection $e_0 \in M \overline{\otimes} M^{\text{op}}$ with $\tau(e_0) > 1 - \varepsilon/2$ and such that $\delta(u)e_0 \in M \overline{\otimes} M^{\text{op}}$. We write $e_n = e_0 \wedge p_n$ and view $e = (e_n)$ as a projection in $(M \overline{\otimes} M^{\text{op}})^\omega$. By (3), we have in $(M \overline{\otimes} M^{\text{op}})^\omega$ the equality $(\delta(w_{m,n})e_0)_n = T_m(\delta(u)e_0)$, and therefore also that $(\delta(w_{m,n})e_n)_n = T_m(\delta(u)e_0) e$. We then find that

$$\varepsilon^2 > \lim_{n \to \omega} \|\delta(m^{-1/4}w_{m,n}) e_n\|^2 \geq \lim_{n \to \omega} \|\delta(m^{-1/4}w_{m,n}) e_n\|^2 \geq \tau(e \delta(u)e_0)^* m^{-1/2} T_m^* T_m (\delta(u)e_0) \geq \tau(e \delta(u)e_0)^* q_m (\delta(u)e_0) e.$$

Since $\tau(q_m) > 1 - \varepsilon$, we can fix $n$ such that

$$\|q_{m,n} \delta(u)e_n\|_2 < \varepsilon \quad \text{and} \quad \tau(q_{m,n}) > 1 - \varepsilon.$$

Since $\tau(e_n) > 1 - \varepsilon$, we have proven that for every $\varepsilon > 0$, there exist projections $p, q \in M \overline{\otimes} M^{\text{op}}$ such that $\tau(p) > 1 - \varepsilon$, $\tau(q) > 1 - \varepsilon$ and $\|q\delta(u)p\|_2 < \varepsilon$. This means that $\delta(u) = 0$. □

The proof of Theorem 1 gives no indication as to whether or not the Connes-Shlyakhtenko first $L^2$-cohomology vanishes as well. Note however that in order for a first $L^2$-cohomology theory to “work well” for II$_1$ factors $M$, the corresponding derivations should be uniquely determined by their values on a set of elements generating $M$ as a von Neumann algebra. In order for this to be the case, the derivations should normally satisfy some continuity property, even if that continuity is “very weak”. However, by combining Theorem 1 with the closed graph theorem, it follows that any derivation from $M$ into Aff$(M \overline{\otimes} M^{\text{op}})$ that satisfies some “reasonable” weak continuity property, must in fact be inner (see also Remark 4 hereafter):

**Corollary 3.** Let $M$ be a finite von Neumann algebra and write $E = \text{Aff}(M \overline{\otimes} M^{\text{op}})$. Assume that $\delta : M \to E$ is a derivation. If $\delta$ has a closed graph for the norm topology on $M$ and the measure topology on $\mathcal{E}$, then $\delta$ is inner.

This is in particular the case if $\delta$ is norm-$\mathcal{T}$-continuous w.r.t. any vector space topology $\mathcal{T}$ on $\mathcal{E}$ satisfying the following two properties:

- the inclusion $M \overline{\otimes} M^{\text{op}} \to \mathcal{E}$ is norm-$\mathcal{T}$-continuous;
- for every fixed $a \in M \overline{\otimes} M^{\text{op}}$, the map $\mathcal{E} \to \mathcal{E} : \xi \mapsto \xi a$ is $\mathcal{T}$-$\mathcal{T}$-continuous.

**Proof.** Take an orthogonal family of projections $p_i \in Z(M)$ such that $\sum_{i \in I} p_i = 1$ and every $M_{p_i}$ is countably decomposable. As in step 1 of Lemma 2 we may assume that $\delta(M_{p_i}) \subset \text{Aff}(M_{p_i} \overline{\otimes} (M_{p_i})^{\text{op}})$. If $\delta$ has closed graph, the restrictions $\delta_i$ of $\delta$ to $M_{p_i}$ still have closed graph. If all these restrictions $\delta_i$ are inner, also $\delta$ is inner. So to prove the first part of the corollary, we may assume that $M$ admits a faithful normal tracial state $\tau$ that we keep fixed. But then, the formula

$$d(\xi, \eta) = \inf \{ \varepsilon > 0 \mid \exists \text{ projection } p \in M \overline{\otimes} M^{\text{op}} : \tau(1 - p) < \varepsilon \text{ and } \|(\xi - \eta)p\| < \varepsilon \}$$

is a metric on $\text{Aff}(M \overline{\otimes} M^{\text{op}})$, and $\delta$ is inner with respect to this metric. □
defines a translation invariant, complete metric on $\text{Aff}(M \otimes M^{\text{op}})$ that induces the measure topology. So by [Ru91, 2.15], the closed graph theorem is valid and we find that $\delta$ is continuous, and hence inner by Theorem 1.

Assume now in general that $\delta$ is norm-$T$-continuous. To prove that $\delta$ has closed graph, assume that $\|x_n\| \to 0$ and $\delta(x_n) \to \xi$ in the measure topology. We have to show that $\xi = 0$. Fix a normal tracial state $\tau$ on $M$. Choose projections $p_n \in M \otimes M^{\text{op}}$ such that $\tau(p_n) > 1 - 2^{-n}$ and $\|\delta(x_n) - \xi\| p_n < 1/n$. Define the projections $q_k = \bigwedge_{n \geq k} p_n$ and note that $q_k$ is increasing and satisfies $\tau(q_k) \to 1$. For every fixed $k$, we get that $(\delta(x_n) - \xi) q_k$ converges to 0 in norm, as $n \to \infty$. By the first assumption on $T$, this convergence also holds in $T$. But also $\delta(x_n) \to 0$ in $T$ so that, by our second assumption on $T$, the sequence $\delta(x_n) q_k$ converges to 0 in $T$ as $n \to \infty$. We conclude that $\xi q_k = 0$ for all $k$. This implies that $\xi z_\tau = 0$ where $z_\tau \in Z(M)$ is the support projection of $\tau$. Since $\tau$ was arbitrary, it follows that $\xi = 0$.

**Remark 4.** We should point out that we have no concrete examples of vector topologies on $\text{Aff}(M \otimes M^{\text{op}})$ satisfying the conditions in the second part of Corollary 3 and that are strictly weaker than the measure topology (in fact, it is not even clear whether such a topology exists!). Let us also point out that there are other weak continuity properties of $\delta$ implying that $\delta$ has closed graph, thus following inner by the first part of Corollary 3. For instance, by using a similar argument as above, one can easily prove that this is the case when $\delta$ satisfies the following weak continuity property: whenever $x_n$ is a sequence in $M$ such that $\|x_n\| \to 0$, there exists a sequence of projections $p_n \in M \otimes M^{\text{op}}$ such that $p_n \to 1$ strongly, $\delta(x_n) p_n \in M \otimes M^{\text{op}}$ and $\delta(x_n) p_n \to 0$ $\sigma$-weakly.

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