Abstract
In this paper we shall introduce a simple, effective numerical method for finding
differential operators for scalar and vector-valued functions on surfaces. The
key idea of our algorithm is to develop an intrinsic and unified way to compute
directly the partial derivatives of functions defined on triangular meshes which
are the discretization of regular surfaces under consideration. Most importantly,
the divergence theorem and conservation laws on triangular meshes are fulfilled.

Keywords: Gradient, Divergence, Laplace-Beltrami operators, LTL method, Conservation law.

1. Introduction

Numerical methods to compute partial differential operators on regular sur-
faces have always received great interest over last decades. However, they are
still not well-understood. For example, Conservation laws for diffusion equa-
tions are usually unsatisfied. Conservation Law is an important principle in
physics. Indeed, Conservational laws plays a key role in the study of partial dif-
ferential equations and have many applications in the linearization, integrability
and numerical analysis. The solution \( u(x, t) \) of diffusion equation
\[
\frac{\partial u}{\partial t} = \alpha \Delta_{\Sigma} u
\]
with \( \alpha > 0 \) on a regular surface \( \Sigma \) preserves the total energy. That is,
\[
\frac{\partial}{\partial t} \int_{\Sigma} u(x, t) dx = 0.
\]

To numerically simulate \( u(x, t) \), one discretizes the regular surface \( \Sigma \) to ob-
tain a triangular surface mesh \( S \) of \( \Sigma \), and considers the discrete solution \( u(x, t) \)
on \( S \). In this way, a fundamental problem arise. Usually, the solution \( u(x, t) \) on
\( S \) will not preserve the total energy. That is,
\[
\int_{\Sigma} u(x, t_i)
\]
will change as \( t_i \) increases.

The violation of the Conservation Law comes from the discretization of the Laplacian-Beltrami operator \( \Delta \Sigma \) on \( \Sigma \). In this paper, we shall try to handle this defect. Lai et al. [14] discussed this problem for regular curves in 2008.

Partial differential equations (PDEs) need to be solved intrinsically and numerically for data defined on 3D regular surfaces in many applications. For instance, such examples exist in fluid dynamic flows (Diewald, Preufer and Rumpf [7]), (Bertalmio, Cheng, Osher and Sapiro [2]), texture synthesis (Turk [18], Witkin and Kass [19]), vector field visualization (Diewald, Preufer and Rumpf [7]), weathering (Dorsey and Hanrahan [10]) and cell-biology (Ayton, McWhirter, McMurty and Voth [1]). Usually, regular surfaces are presented by triangular or polygonal forms. Partial differential equations are then solved on these triangular or polygonal meshes with data defined on them. The use of triangular or polygonal meshes is very popular in all areas dealing with 3D models. However, it has not yet been a widely accepted method to compute differential characteristics such as principal directions, curvatures and Laplacians (Chen and Wu [3, 4], Wu, Chen and Chi [20], Taubin [16]). In Wu, Chen and Chi [20], the authors proposed a new intrinsic simple algorithm, LTL method, to handle this difficulty. In this note, we shall use this new technique to approximate gradient, divergence and Laplace-Beltrami operators on surfaces.

In Osher and Sethian [15] and Bertalmio, Cheng, Osher and Sapiro [2] discussed a framework, the implicit surface algorithm, to solve Variational problems and PDE’s for scalar and vector-valued data defined on regular surfaces. Their key idea is to use, instead of a triangular or polygonal representation, an implicit representation. The surface under consideration is the zero-level set of a higher dimensional embedding function. Then they smoothly extend the original data on the surface to the 3D domain, adapt the PDE’s accordingly, and implement all the numerical computations on the fixed Cartesian grid corresponding to the embedding function. The advantage of their method is the use of the Cartesian grid instead of a triangular mesh for the numerical implementation.

The discretizations of the gradient, divergence and Laplace-Beltrami operators that we will discuss in this paper will have the following advantages:

**Intrinsicness**: we use the intrinsic geometric LTL method to define these operators.

**Conservation**: our Laplace-Beltrami operator will satisfy conservation laws on triangular meshes for diffusion equations.

**Convergence**: our gradient, divergence and Laplace-Beltrami operators will have the linear convergence rate locally and uniformly.

**Simplicity**: the numerical computations are also very easy to implement.

The rest of this paper is organized as follows. In section 2, we recall the gradient, divergence and Laplace-Beltrami operators defined on regular surfaces. In section three we propose our new discrete algorithm for these differential
operators on triangular meshes. We also discuss the convergence problem and conservation laws for these operators. Numerical simulations are presented in section 4.

2. The gradient, divergence and LB operators on regular surfaces

In order to describe the gradient, divergence and the LB operator on functions or vector fields in a regular surface \( \Sigma \) in the 3D Euclidean space \( \mathbb{R}^3 \), we consider a parameterization \( x: U \to \Sigma \) at a point \( p \), where \( U \) is an open subset of the 2D Euclidean space \( \mathbb{R}^2 \). We can choose, at each point \( q \) of \( x(U) \), a unit normal vector \( N(q) \). The map \( N: x(U) \to S^2 \) is the local Gauss map from an open subset of the regular surface \( \Sigma \) to the unit sphere \( S^2 \) in the 3D Euclidean space \( \mathbb{R}^3 \). The Gauss map \( N \) is differentiable. Denote the tangent space of \( \Sigma \) at the point \( p \) by \( T\Sigma_p = \{v \in \mathbb{R}^3 | v \perp N(p)\} \). The tangent space \( T\Sigma_p \) is a linear space spanned by \( \{x_u, x_v\} \) where \( u, v \) are coordinates for \( U \).

The gradient \( \nabla_\Sigma \) of a smooth function \( g \) on \( \Sigma \) can be computed from

\[
\nabla_\Sigma g = \frac{g_u G - g_v F}{EG - F^2} x_u + \frac{g_v E - g_u F}{EG - F^2} x_v
\]

where \( E, F, \) and \( G \) are the coefficients of the first fundamental form and

\[
g_u = \frac{\partial g(x(u, v))}{\partial u} \quad \text{and} \quad g_v = \frac{\partial g(x(u, v))}{\partial v}.
\]

See do Carmo[8] for the details.

Let \( X = Ax_u + Bx_v \) be a local vector field on \( \Sigma \). The divergence, \( \nabla_\Sigma \cdot X \), of \( X \) is defined as a function \( \nabla_\Sigma \cdot X: \Sigma \to \mathbb{R} \) given by the trace of the linear mapping \( Y(p) \to \nabla Y(p) X \) for \( p \in \Sigma \). A direct computation gives

\[
\nabla_\Sigma \cdot X = \frac{1}{\sqrt{EG - F^2}} \left( \frac{\partial}{\partial u}(A\sqrt{EG - F^2}) + \frac{\partial}{\partial v}(B\sqrt{EG - F^2}) \right)
\]

The LB operator \( \Delta_\Sigma \) acting on the function \( g \) is defined by the integral duality

\[
(\Delta_\Sigma g, \varphi) = -(\nabla_\Sigma g, \nabla_\Sigma \varphi)
\]

for all smooth function \( \varphi \) on \( \Sigma \). That is, \( \Delta_\Sigma g = \nabla_\Sigma \cdot \nabla_\Sigma g \). A direct computation yields the following local representation for the LB operator \( \Delta_\Sigma g \) on a smooth function \( g \):

\[
\Delta_\Sigma g = \frac{1}{\sqrt{EG - F^2}} \left[ \frac{\partial}{\partial u} \left( \frac{G}{\sqrt{EG - F^2}} \frac{\partial g}{\partial u} \right) - \frac{\partial}{\partial u} \left( \frac{G}{\sqrt{EG - F^2}} \frac{\partial g}{\partial v} \right) \right]
\]

\[
+ \frac{1}{\sqrt{EG - F^2}} \left[ \frac{\partial}{\partial v} \left( \frac{G}{\sqrt{EG - F^2}} \frac{\partial g}{\partial u} \right) - \frac{\partial}{\partial v} \left( \frac{G}{\sqrt{EG - F^2}} \frac{\partial g}{\partial v} \right) \right]
\]
3. Discrete gradient, divergence and LB operators

In this section we shall describe a simple and effective method to define the discrete gradient, divergence and LB operator on functions or vector fields on a triangular mesh. The primary ideas were developed in Chen, Chi and Wu [3, 20] where we try to estimate the discrete partial derivatives of functions on 2D scattered data points. Indeed, the method that we shall use to develop our algorithm is divided into two main steps: first we lift the 1-neighborhood points to the tangent space and obtain a local tangential polygon. Second, we use some geometric idea to lift functions or vectors to the tangent space. We call this a local tangential lifting (LTL) method. Then we present a new algorithm to compute their gradients in the 2D tangent space. This means that the LTL process allows use to reduce the 2D curved surface problem to the 2D Euclidean problem.

Consider a triangular surface mesh \( S = (V, F) \), where \( V \) is the list vertices and \( F \) is the list of triangles.

3.1. The local tangential lifting (LTL) method

To describe the local tangential lifting (LTL) method, we introduce the local tangential polygon at a vertex \( v \) of \( V \) as follows:

1. The normal vector \( N_A(v) \) at the vertex \( v \) in \( S \) is given by

\[
N_A(v) = \frac{\sum_{T \in T(v)} \omega_T N_T}{\sum_{T \in T(v)} \omega_T N_T} \tag{9}
\]

where \( N_T \) is the unit normal to a triangle face \( T \) and the centroid weight is given in [3] by

\[
\omega_T = \frac{1}{\sum_{T \in T(v)} \frac{1}{\|G_T - v\|^2}} \tag{10}
\]

Here, \( G_T \) is the centroid of the triangle face \( T \) determined by

\[
G_T = \frac{v + v_i + v_j}{3}. \tag{11}
\]

Note that the letter A in the notation \( N_A(v) \) stands for the word "Approximation".

2. The approximating tangent plane \( TS_A(v) \) of \( S \) at \( v \) is now determined by \( TS_A(v) = \{ w \in \mathbb{R}^3 | w \perp N_A(v) \} \).

3. The local tangential polygon \( P_A(v) \) of \( v \) in \( TS_A(v) \) is formed by the vertices \( \bar{v}_i \) which is the lifting vertex of \( v_i \) adjacent to \( v \) in \( V \).

\[
\bar{v}_i = (v_i - v) - < v_i - v, N_A(v) > N_A(v) \tag{12}
\]

as in figure [1]
4. We can choose an orthonormal basis $e_1, e_2$ for the tangent plane $T S_A(v)$ of $S$ at $v$ and obtain an orthonormal coordinates $(x, y)$ for vectors $w \in T S_A(v)$ by $w = x e_1 + y e_2$. We set $\bar{v}_i = x_i e_1 + y_i e_2$ with respect to the orthonormal basis $e_1, e_2$.

Next we explain how to lift locally a function defined on $V$ to the local tangential polygon $P_A(v)$. Consider a function $h$ on $V$. We will lift locally the function $h$ to a function of two variables, denoted by $\bar{h}$, on the vertices $\bar{v}_i$ in $P_A(v)$ by simply setting

$$\bar{h}(x_i, y_i) = h(v_i).$$

and $\bar{h}(\bar{0})$ where $\bar{0}$ is the origin of $T S_A(v)$. Then one can extend the function $\bar{h}$ to a piecewise linear function on the whole polygon $P_A(v)$ in a natural and obvious way.

### 3.2. A new discrete gradient algorithm

In this subsection we present a new discrete 2D algorithm for the gradients of functions on the 2D domains in the $x - y$ plane and also on triangular surface meshes. Given a $C^3$ function $f$ on a domain $\Omega$ in the $x - y$ plane with the origin $(0, 0) \in \Omega$, Taylor’s expansion for two variables $x$ and $y$ gives

$$f(x, y) = f(0, 0) + x f_x(0, 0) + y f_y(0, 0) + \frac{x^2}{2} f_{xx}(0, 0) + xy f_{xy}(0, 0) + \frac{y^2}{2} f_{yy}(0, 0) + O(r^3)$$

when $r = \sqrt{x^2 + y^2}$ is small.

Consider a family of neighboring points $(x_i, y_i) \in \Omega$, $i = 1, 2, \cdots, n$, of the origin $(0, 0)$. Take some constants $\alpha_i$, $i = 1, 2, \cdots, n$ with $\sum_{i=1}^n \alpha_i^2 = 1$. Then one has

$$\sum_{i=1}^n \alpha_i (f(x_i, y_i) - f(0, 0)) = \left(\sum_{i=1}^n \alpha_i x_i\right) f_x(0, 0) + \left(\sum_{i=1}^n \alpha_i y_i\right) f_y(0, 0) + \frac{1}{2} \left(\sum_{i=1}^n \alpha_i x_i^2\right) f_{xx}(0, 0) + \left(\sum_{i=1}^n \alpha_i x_i y_i\right) f_{xy}(0, 0) + \frac{1}{2} \left(\sum_{i=1}^n \alpha_i y_i^2\right) f_{yy}(0, 0) + O(r^3).$$

(15)

We choose the constants $\alpha_i$, $i = 1, 2, \cdots, n$ so that they satisfy the following equations:
Choose and fix an orthonormal basis \( e_1, e_2 \) for the tangent plane \( TS_A(v) \) of \( S \) at \( v \) and obtain an orthonormal coordinates \((x, y)\) for vectors \( w \in TS_A(v) \) by \( w = xe_1 + ye_2 \). We set \( \tilde{v}_i = x_i e_1 + y_i e_2 \) with respect to the orthonormal basis \( e_1, e_2 \). Consider a function \( h \) on \( V \). We will lift locally the function \( h \) to a
function of two variables, denoted by \( f \), on the vertices \( \bar{v}_i \) in \( P_A(v) \) by simply setting

\[
f(x_i, y_i) = h(v_i)
\]

and \( f(0, 0) = h(v) \) where \( (0, 0) \) is the origin of \( TS_A(v) \). In this way, we can define the approximating gradient by

\[
\nabla_A h(v) = \left( \frac{\sum_{i=1}^{n} \alpha_i x_i}{\sum_{i=1}^{n} \alpha_i}, \frac{\sum_{i=1}^{n} \alpha_i y_i}{\sum_{i=1}^{n} \alpha_i} \right)^{-1} \left( \frac{\sum_{i=1}^{n} \alpha_i (f(x_i, y_i) - f(0, 0))}{\sum_{i=1}^{n} \beta_i (f(x_i, y_i) - f(0, 0))} \right)
\]

(22)

where \( \alpha_i, \beta_i \) can be computed from Equations (16).

Since the approximating normal vector satisfies \( N_\Sigma(v) = N_A(v) + O(r^2) \), one can tell from Equations (16), (17), and (18) and obtain easily the following convergence theorem.

**Theorem 1. (Convergence Theorem 1)**

Under the notations as above, one has

\[
\nabla_\Sigma h(v) = \nabla_A h(v) + O(r^2).
\]

(23)

### 3.3. A new discrete divergence algorithm on triangular meshes

In this subsection, we shall use the divergence theorem to give a discrete approximation of the divergence of a vector field \( X \) defined on a triangular surface mesh \( S = (V, F) \). Consider a vertex \( v \in V \) and let \( v_j, j = 0, 1, \cdots, n \) be the neighboring vertices of \( v \) with \( v_0 = v_n \). These vertices \( v_j \) are labeled counterclockwise about the normal vector \( N_A(v) \). Let \( T_j \) be the triangle with vertices \( v, v_j \) and \( v_j+1 \). We define the outer normal vectors \( n(T_j, v_j) \) and \( n(T_j, v_{j+1}) \) of the triangle \( T_j \) at the vertex \( v_j \) and \( v_{j+1} \) respectively as follows. See figure 2.

Consider the lifting vectors \( w_1 \) and \( w_2 \) of the vectors \( v_{j+1} - v_j \) and \( v - v_j \) to the approximating tangent space \( TS_A(v) \) of \( S \) at the vertex \( v_j \):

\[
\begin{align*}
w_1 &= (v_{j+1} - v_j) - <v_{j+1} - v_j, N_A(v_j)> N_A(v_j) \\
w_2 &= (v - v_j) - <v - v_j, N_A(v_j)> N_A(v_j)
\end{align*}
\]

(24)

Now the outer normal vector \( n(T_j, v_j) \) of the triangle \( T_j \) at the vertex \( v_j \) can be defined as

\[
n(T_j, v_j) = \frac{<w_2, w_1> w_1 - \|w_1\|^2 w_2}{\|<w_2, w_1> w_1 - \|w_1\|^2 w_2\|}
\]

(25)

Similarly, we consider the lifting vectors \( w_3 \) and \( w_4 \) of the vectors \( v_j - v_{j+1} \) and \( v - v_{j+1} \) to the approximating tangent space \( TS_A(v_{j+1}) \) of \( S \) at the vertex \( v_{j+1} \):

\[
\begin{align*}
w_3 &= (v_{j+1} - v_j) - <v_{j+1} - v_j, N_A(v_{j+1})> N_A(v_{j+1}) \\
w_4 &= (v - v_{j+1}) - <v - v_{j+1}, N_A(v_{j+1})> N_A(v_{j+1})
\end{align*}
\]

(26)

Note that the outer normal vectors \( n(T_j, v_{j+1}) \) and \( n(T_{j+1}, v_{j+1}) \) are different.
Under these notations, we can now define the discrete divergence $\text{Div}_AX$ of a vector field on the triangular surface mesh $S$ by

$$
\text{Div}_AX(v) = \frac{1}{\sum_{k=0}^{n-1} |T_k|} \left[ \sum_{j=0}^{n-1} \frac{\|v_{j+1} - v_j\|}{6} (2 < X(v_j), n(T_j, v_j) > + 2 < X(v_{j+1}), n(T_j, v_{j+1}) > + < X(v_j), n(T_j, v_j) >) \right]
$$

(27)

where $|T_j|$ denotes the area of the triangle $T_j$.

We can extend the divergence $\text{Div}_AX$ to the whole mesh $S$ piecewise linearly in a natural way. Therefore, we have the following lemma.

**Lemma 1.** Let $X$ be a vector field on the triangular mesh $S$. The integration of the divergence $\text{Div}_AX$ over $S$ is

$$
\int_S \text{Div}_AX = \sum_{v_i \in V} \left[ \frac{1}{3} \text{Div}_AX(v_i) \sum_{j \in N(i)} |T_j| \right].
$$

(28)
Lemma 1 along with the definition of Div$_A X$ gives the following result.

**Theorem 2. (Discrete Conservation Law 1)**

Let $S = (V, F)$ be a triangular mesh without boundary and $X$ a vector field on $S$. We have

$$
\int_S \text{Div}_A X = 0.
$$

(29)

Let $\Sigma$ be a regular surface and $S = (V, F)$ a triangular surface mesh of $\Sigma$ with mesh size $r > 0$. If the mesh size $r$ is sufficiently small, we can find a unique geodesic $\gamma_{ij}$ joining two adjacent vertices $v_i$ and $v_j$. In this way, every triangle $T \in F$ has a corresponding geodesic triangle $\tilde{T}$ on the surface $\Sigma$ with the same vertices as $T$. See figure 3. Since the approximating normal vector satisfies $N_{\Sigma}(v) = N_A(v) + O(r^2)$, the outer normal vectors $n(\tilde{T}_j, v_j)$ and $n(\tilde{T}_j, v_{j+1})$ of the geodesic triangle $\tilde{T}_j$ in $\Sigma$ at the vertices $v_j$ and $v_{j+1}$ respectively also have the relations

$$
\begin{align*}
    n(\tilde{T}_j, v_j) &= n(T_j, v_j) + O(r^2) \\
    n(\tilde{T}_j, v_{j+1}) &= n(T_j, v_{j+1}) + O(r^2)
\end{align*}
$$

(30)

The main purpose of this section is to prove the following result.

**Theorem 3. (Convergence Theorem 2)**

Let $\Sigma$ be a regular surface and $S = (V, F)$ a triangular surface mesh of $\Sigma$ with mesh size $r > 0$. Consider a smooth vector field $X$ on $\Sigma$, one has, for sufficiently small $r > 0$, and $v \in V$,

$$
\text{Div}_\Sigma X(v) = \text{Div}_A X(v) + O(r)
$$

(31)
According to the Divergence Theorem on regular surfaces, one has
\[ \int_W \text{Div}_\Sigma X = \int_{\partial W} < X, \vec{n} > \] (32)
where the domain \( W \) is the union of the geodesic triangles \( \tilde{T}_j \) with vertices \( v, v_j, v_{j+1}, j = 0, 1, \ldots, n \) and \( \vec{n} \) the outer normal vector of \( W \). We denote and parametrize the geodesic edge \( E_j \) from \( v_j \) to \( v_{j+1} \) in \( \tilde{T}_j \) by \( E_j(t), t \in [0, 1] \) with \( \| E_j'(t) \| = L(E_j) \). Then (31) gives
\[ \int_W \text{Div}_\Sigma X = \int_{\partial W} < X, \vec{n} > = \sum_{j=0}^{n-1} L(E_j) \int_0^1 < X(t), \vec{n}(t) > dt \] (33)
where \( L(E_j) \) is the length of the geodesic edge \( E_j \). We can approximate the vectors \( X(t) \) and \( \vec{n}(t) \) by
\[
X(t) = (1-t)X(v_j) + tX(v_{j+1}) + O(r^2) \\
\vec{n}(t) = (1-t)n(T_j, v_j) + tn(T_j, v_{j+1}) + O(r^2) \] (34)
These relations follow from the following easy lemma from Calculus.

**Lemma 2.** Consider a smooth function or vector field \( g \) on \([0, a], a > 0 \) and a sufficiently small \( r > 0 \). Then one has, for \( t \in [0, 1] \),
\[ g(tr) = (1-t)g(0) + tg(r) + O(r^2) \] (35)
Equations (30) and (34) imply
\[ \vec{n}(t) = (1-t)n(T_j, v_j) + tn(T_j, v_{j+1}) + O(r^2) \] (36)
Note also that the length \( L(E_j) \) can be approximated by
\[ L(E_j) = \| v_{j+1} - v_j \|(1 + O(r^2)) \] (37)
Hence one obtains
\[
\sum_{j=0}^{n-1} L(E_j) \int_0^1 < X(t), \vec{n}(t) > dt \\
= \sum_{j=0}^{n-1} \left( \| v_{j+1} - v_j \|/6(1 + O(r^2)) \right) [2 < X(v_j), n(T_j, v_j) > \\
+ 2 < X(v_{j+1}), n(T_j, v_{j+1}) > + < X(v_j), n(T_j, v_{j+1}) > \\
+ < X(v_{j+1}), n(T_j, v_j) >]. \] (38)
On the other hand, we also have
\[ \int_W \text{Div}_\Sigma X = |W| (\text{Div}_\Sigma X(v) + O(r)) \]
\[ = \sum_{j=0}^{n-1} |T_j| (\text{Div}_\Sigma X(v) + O(r)) \]
\[ = \sum_{j=0}^{n-1} |T_j| (1 + O(r^2)) (\text{Div}_\Sigma X(v) + O(r)) \] (39)

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Therefore we yield
\[
\text{Div}_\Sigma X(v) = \frac{1}{\sum_{k=0}^{n-1} |T_k|} \left[ \sum_{j=0}^{n-1} \frac{\|v_{j+1}-v_j\|}{6} (2 < X(v_j), n(T_j, v_j) > + 2 < X(v_{j+1}, n(T_j, v_j) > + < X(v_j), n(T_j, v_{j+1}) > ) \right] + O(r)
\]
(40)
and this proves the main theorem (Theorem 3).

Using the results in subsection 3.2 and this subsection, we can approximate the Laplace-Beltrami operators on regular surface as follows. Consider a smooth function \( h \) on a regular surface \( \Sigma \) and \( S = (V, F) \) a triangular surface mesh of \( \Sigma \) with mesh size \( r > 0 \). One can use Equations (16)-(19) to define the approximating gradient \( \nabla_A h(v) \) at a vertex \( v \in V \). Then Equation (6) gives the approximating Laplace-Beltrami operator by
\[
\Delta_A h(v) = \text{Div}_A (\nabla_A h)(v)
\]
(41)

Theorems 1 and 3 then give

**Theorem 4. (Convergence Theorem 3)**

Let \( \Sigma \) be a regular surface and \( S = (V, F) \) a triangular surface mesh of \( \Sigma \) with mesh size \( r > 0 \). Consider a smooth function \( h \) on \( \Sigma \), one has, for sufficiently small \( r > 0 \), and \( v \in V \),
\[
\Delta_{\Sigma} h(v) = \Delta_A h(v) + O(r)
\]
(42)

We can extend \( \Delta_A h \) to the whole triangular surface mesh \( S \) piecewise linearly in a natural way. Then, the Discrete Conservation Law (Theorem 2) also holds for the Laplace-Beltrami operators.

**Theorem 5. (Discrete Conservation Law 2)**

Let \( S \) be a triangular surface mesh without boundary and \( h \) a function on \( S \). We have
\[
\int_S \Delta_A h = 0
\]
(43)

**Remark 1.** The error terms \( O(r), O(r^2) \) in Theorems 1, 3 can be shown to depend only on curvatures, injectivity radius \( \Sigma \) of \( \Sigma \), vector fields \( X \) and/or functions \( h \).

**Remark 2.** We also would like to point out that the methods discussed in this section also work in higher dimensions. Namely, we can also use these ideas to approximate the gradient, divergence and the Laplace-Beltrami operators for hypersurfaces in nD Euclidean spaces with \( n \geq 3 \). We will discuss these in another paper.

4. Numerical simulations

The Laplace-Beltrami operator on a regular surfaces plays an important role on PDEs. In this section, we shall estimate the Laplace-Beltrami operators on triangular meshes by our proposed method and shows some numerical simulations about several important PDEs on regular surfaces.
4.1. Comparisons of Laplacian estimations

We compare our proposed method, the level set method and some other discretization methods for estimating the Laplacian of random polynomial functions of degree less than 5 on a unit sphere and a torus in figures 4 and 5. Xu's method is a discretization method proposed in 2004. One can find the details about Xu's method and Level-set method in [15, 21]. We choose 10,000 random polynomial functions on these surfaces. The \(l_\infty\) and \(l_2\) errors are used for all vertices on the triangular mesh. From our simulations, all of these methods are convergent and comparable.

4.2. PDEs on surfaces

In this subsection, we show numerical solutions of some PDEs on surfaces via our proposed method. First, we consider the diffusion equation on a sphere

\[
  u_t = \Delta_\Sigma u
\]

with the initial condition

\[
  u_0(\theta, \eta) = \cos(\eta)
\]

where \((r, \theta, \eta)\) is the spherical coordinate of the unit sphere. We calculate

\[
  u(\phi, \theta, t) = \exp(-2t) \cos(\eta)
\]

as the exact solution of equation (44) with initial condition (45). We compute the Laplace-Beltrami operator \(\Delta_\Sigma\) in equation (44) by our proposed method.
Figure 6: The diffusion equation on a unit sphere

and compare our numerical solution of equation (44) with the exact solution (46). Figure 6 illustrates the numerical solutions of equation (44) at time $t=0$, 0.5, 10 and 9,000. The "fvals in $[a, b]$" in the figure 6 means the values $u(t)$ on the surface between $a$ and $b$, and the "l-infty error" means the $l_\infty$ error of our simulations. Figure 4.2 gives the $l_\infty$ error of our numerical solutions. Obviously, our numerical solution approaches the exact solution when the time is large enough. Furthermore, the integration of $u$, $\int_S u(t; p)dS$, is preserved at all time.

Next, we solve the fourth order diffusion equation,

$$u_t = -\Delta_S \Delta_S u$$

(47)

on the sphere with the initial condition

$$u_0(r, \theta, \eta) = \sin(3\theta)\sin(7\eta).$$

(48)

One can find the details about this equation in Greer’s paper[12]. In our example, the number of triangles on a triangular mesh is 4096. Figure displays the solution at $t = 0$, $t = 0.01$, $t = 0.5$ and $t = 5$. Obviously, our solution and Greer’s numerical solution[12] are comparable.

For our final example, we compute the Allen-Cahn equation,

$$u_t = \epsilon^2 \Delta_S u + u^3 - u,$$

(49)
Figure 7: The $l_{\infty}$ error of the diffusion equation on a unit sphere
Figure 8: Linear fourth order diffusion on a unit sphere.
with the initial condition
\[
f(\theta, \eta) = \begin{cases} 
1 & \sqrt{(\theta - \frac{\pi}{2})^2 + (\eta - \frac{\pi}{2})^2} \leq \frac{\pi}{5} \\
-1 & \text{otherwise}
\end{cases} \tag{50}
\]
on a torus,
\[
x(\theta, \eta) = \left( \frac{1}{2} \cos \eta + 1 \cos \theta, \frac{1}{2} \cos \eta + 1 \sin \theta, \frac{1}{2} \sin \eta \right). \tag{51}
\]

Figure 9 shows the results. Again, our results and Greer’s numerical solutions \cite{Greer} are equal, well.

5. Conclusion

Our proposed method is a new discretization method for estimating the divergence of a vector field on surfaces. The convergence ratio of our proposed
method is as good as the other well-known convergence methods for estimating
the Laplace-Beltrami operators. Almost all of other methods does not obey the
divergence theorem, however our proposed method does. That is, our proposed
method for estimating the Laplace-Beltrami operator on the heat equation have
the conservation property. In the near future, we shall use our proposed method
to improve more partial differential equations, such as the Navier-Stokes equa-
tion, on regular surfaces, triangular meshes and general manifolds of dimension
\( n \geq 3 \).

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