SUPERCONVERGENCE IN FREE PROBABILITY LIMIT THEOREMS FOR ARBITRARY TRIANGULAR ARRAYS

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Abstract. It is known that limit theorems for triangular arrays with identically distributed rows yields convergence of densities rather than just convergence in distribution. We show that this superconvergence result holds—at least at points at which the limit density is nonzero—even if the rows of the array are not identically distributed.

1. Introduction

The central limit theorem of free probability \cite{15} asserts that for a sequence of freely independent, identically distributed random variables $X_1, X_2, \ldots$ with zero expected value and unit variance, the variables
$$Y_n = \frac{X_1 + \cdots + X_n}{\sqrt{n}}$$
converge in distribution to the standard semicircular law with density
$$\frac{1}{2\pi} \sqrt{4 - t^2}$$
for $t \in [-2, 2]$. In case the variables $X_n$ are also bounded, it was observed in \cite{9} that the distribution of $Y_n$ is necessarily absolutely continuous for large $n$, and that the densities of these distributions (along with their derivatives of all orders) converge uniformly to the semicircular density on every interval $[a, b] \subset (-2, 2)$. Moreover, the distribution of $Y_n$ is supported on an interval $[c_n, d_n]$ such that $\lim_{n \to \infty} c_n = -2$ and $\lim_{n \to \infty} d_n = 2$. This phenomenon was called superconvergence in \cite{6}.

Chistyakov and Götze computed an asymptotic expansion of the free central limit theorem \cite{10} and the rate of convergence of the density \cite{9}. Various aspects of superconvergence were extended to more general limit processes for sums as well as for products of infinitesimal arrays of random variables, generally under the assumption that the rows of the arrays are formed of freely independent, identically distributed variables (see \cite{11, 7, 8, 13, 14}). The purpose of this note is to remove the identical distribution hypothesis on the arrays. The technique is closer to the one employed in \cite{6} than to the more recent developments and it essentially uses only the fact that the $R$-transforms of the variables in an infinitesimal array are defined on arbitrarily large Stolz angles. In particular, the subordination property of free convolution is not needed in this paper. The results do not even require that we work with a convolution of infinitesimal variables, but they only apply locally to the points that have a neighborhood where the limit distribution is absolutely

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continuous. We provide complete arguments for the additive case. The arguments for products are very similar and only the relevant differences are pointed out.

2. ADDITIVE CONVOLUTION AND SUPERCONVERGENCE

The distribution of a sum of freely independent selfadjoint random variables is the free additive convolution of the distributions of the summands. Because of this fact, we can forget about the variables themselves and focus instead on free convolutions. Suppose that $\mu$ is a Borel probability measure on $\mathbb{R}$, and define two analytic functions on $\mathbb{H} = \{ z \in \mathbb{C} : \Im z > 0 \}$ by

$$G_\mu(z) = \int_{\mathbb{R}} \frac{d\mu(t)}{z-t}, \quad F_\mu(z) = \frac{1}{G_\mu(z)}, \quad z \in \mathbb{H}. $$

The measure $\mu$ is uniquely determined by either of these functions, and the Stieltjes inversion formula shows that the value of the density of $\mu$ at almost every $x \in \mathbb{R}$ is equal to the boundary limit $-\pi^{-1} \lim_{y\to 0} \Im G_\mu(x+iy)$. The function $F_\mu$ satisfies

$$\lim_{z \to \infty} \frac{F_\mu(z)}{z} = 1,$$

where the limit is taken as $z = x+iy \to \infty$ such that $|x|/y$ remains bounded. Given $\alpha, \beta > 0$, define the Stolz angle at infinity by

$$\Gamma_{\alpha, \beta} = \{ x+iy \in \mathbb{C} : y \geq \max\{\beta, \alpha|x|\} \}.$$

Then (2.1) implies that for every $\alpha > 0$, there exist $\beta > 0$ and an analytic function $H_\mu : \Gamma_{\alpha, \beta} \to \mathbb{H}$ such that

$$F_\mu(H_\mu(z)) = z, \quad z \in \Gamma_{\alpha, \beta};$$

see [5]. Since $\Im F_\mu(z) \geq \Im z$ for $z \in \mathbb{H}$, it follows that the function

$$\varphi_\mu(z) = H_\mu(z) - z$$

satisfies

$$\Im \varphi_\mu(z) \leq 0, \quad z \in \Gamma_{\alpha, \beta}. $$

The function $\varphi_\mu$ may have analytic continuations to other regions of the form $\Gamma_{\alpha', \beta'}$ that still satisfy (2.2) but possibly $\Im(z + \varphi_\mu(z)) \leq 0$ for some $z \in \Gamma_{\alpha', \beta'}$, so the quantity $F_\mu(z + \varphi_\mu(z))$ is not defined. Any two such analytic continuations coincide on their common domain of definition, and we continue denoting by $\varphi_\mu$ the function obtained by assembling all of these possible analytic continuations to domains of the form $\Gamma_{\alpha', \beta'}$. Thus, if $\varphi_\mu$ is defined at a point $z = x + iy \in \mathbb{H}$, then it is also defined at all points $x' + iy'$ such that $y' \geq y$ and $|x'|/y' \leq |x|/y$. The equation

$$F_\mu(z + \varphi_\mu(z)) = z$$

persists for $z$ in every connected open set $U$ such that $U \subset \Gamma_{\alpha', \beta'}$ for some $\alpha', \beta' > 0$ and $z + \varphi_\mu(z) \in \mathbb{H}$ for every $z \in U$. The related identity

$$F_\mu(z) + \varphi_\mu(F_\mu(z)) = z$$

holds in every connected open set $V \subset \mathbb{H}$ with the property that $F_\mu(z)$ is in the domain of $\varphi_\mu$ for $z \in V$. The function $\varphi_\mu$ defined above is known as the Voiculescu transform of $\mu$ and it serves to linearize the free additive convolution of probability measures on $\mathbb{R}$; that is

$$\varphi_{\mu_1 \boxplus \mu_2} = \varphi_{\mu_1} + \varphi_{\mu_2}.$$
on the common domain of $\varphi_1$ and $\varphi_2$. We also recall that a measure $\mu$ is \( \mathbb{H} \)-infinitely divisible if and only if $\varphi_\mu$ is defined on the entire $\mathbb{H}$ and $\varphi_\mu(\mathbb{H}) \subset \mathbb{C}\setminus\mathbb{H}$. If $\mu$ is $\mathbb{H}$-infinitely divisible, the identity (2.3) holds for for every $z \in \mathbb{H}$ and thus $F_\mu$ maps $\mathbb{H}$ conformally onto a domain $\Omega_\mu \subset \mathbb{H}$. It was shown in [5] that

$$\Omega_\mu = \{ z \in \mathbb{H} : z + \varphi_\mu(z) \in \mathbb{H} \}.$$  

In fact, there exists a continuous function $f_\mu : \mathbb{R} \to [0, +\infty)$ such that

$$\Omega_\mu = \{ x + iy : x \in \mathbb{R}, y > f_\mu(x) \},$$

and the map $x \mapsto H_\mu(x + if_\mu(x))$ (extended by continuity when $f_\mu(x) = 0$) is a homeomorphism of $\mathbb{R}$ onto $\mathbb{R}$, see [11] or [7, Section 2]. Moreover, the function $f_\mu$ is real-analytic wherever it is nonzero. If $f_\mu > 0$ on an interval $[A, B]$ then $\mu$ has a real-analytic density on the interval $[a, b]$, where

$$a = H_\mu(A + if_\mu(A)), \quad b = H_\mu(B + if_\mu(B)),$$

and the density at a point $t = H_\mu(s + if_\mu(s))$, $s \in [A, B]$, is given by

$$-\frac{1}{\pi} \Im G_\mu(t) = -\frac{1}{\pi} \Im \frac{1}{s + if_\mu(s)} = \frac{1}{\pi} \frac{f_\mu(s)}{s^2 + f_\mu(s)^2}.$$  

Every interval $[a, b]$ on which $\mu$ has a positive density is of the form described above. The following fact is used in the proof of the main result.

**Lemma 2.1.** Suppose that $\mu$ is a $\mathbb{H}$-infinitely divisible measure on $\mathbb{R}$, and let $s \in \mathbb{R}$ be such that $f_\mu(s) > 0$. Then

$$\frac{\partial}{\partial t} \Im H_\mu(s + it) > 0 \text{ for } t \geq f_\mu(s).$$

**Proof.** When $\mu$ is a point mass, we have $\Im H_\mu(s + it) = t$, so the result is immediate. Suppose therefore that $\mu$ is not a point mass and write the function $\varphi_\mu$ in its Nevanlina representation

$$\varphi_\mu(z) = c + \int_{\mathbb{R}} \frac{1 + zx}{z - x} \, d\sigma(x),$$

where $c$ is a real constant and $\sigma$ is a nonzero, finite, positive Borel measure on $\mathbb{R}$. By the Cauchy-Riemann equations, it suffices to show that

$$\Re H_\mu'(z) > 0 \text{ for } z = s + it, \ t \geq f_\mu(s).$$

Easy calculations show that

$$H_\mu'(s + it) = 1 - \int_{\mathbb{R}} \frac{1 + x^2}{(s + it - x)^2} \, d\sigma(x),$$

and

$$\Im H_\mu(s + it) = t \left[ 1 - \int_{\mathbb{R}} \frac{1 + x^2}{|s + it - x|^2} \, d\sigma(x) \right].$$

The fact that $f_\mu(s) > 0$ means that

$$\int_{\mathbb{R}} \frac{1 + x^2}{|s + it - x|^2} \, d\sigma(x) = 1, \quad t = f_\mu(s),$$

and thus

$$\int_{\mathbb{R}} \frac{1 + x^2}{|s + it - x|^2} \, d\sigma(x) \leq 1, \quad t \geq f_\mu(s).$$
On the other hand, one has
\[
\Re \int_R \frac{1 + x^2}{(s + it - x)^2} d\sigma(x) < \left| \int_R \frac{1 + x^2}{(s + it - x)^2} d\sigma(x) \right| \leq \int_R \frac{1 + x^2}{|s + it - x|^2} d\sigma(x) \leq 1
\]
for \( t \geq f_\mu(s) \), and this implies the desired inequality \( \Re H'_\mu(z) > 0 \). \hfill \Box

Remark 2.2. In the above proof, we were able to use an explicit formula for the derivative. Calculations are more cumbersome for free multiplicative convolutions and therefore the following general fact will be useful. Suppose that \( \alpha, \beta > 0 \), and that \( f : (\alpha - \varepsilon, \alpha + \varepsilon) \to \mathbb{R} \) is a differentiable function. Set
\[
D = \{ x + iy : x \in (\alpha - \varepsilon, \alpha + \varepsilon) \text{ and } |y - f(x)| < \delta \},
\]
and let \( H : D \to \mathbb{C} \) be an analytic function such that
\[
\Im H(x + iy) > 0 \quad \text{for } y = f(x), \quad x \in (\alpha - \varepsilon, \alpha + \varepsilon).
\]
Then
\[
(2.4) \quad \frac{\partial}{\partial y} \Im H(x + iy) > 0 \quad \text{for } y = f(x), \quad x \in (\alpha - \varepsilon, \alpha + \varepsilon).
\]
To see this, we observe that the function \( H(x + if(x)) \) is necessarily real and increasing on \( (\alpha - \varepsilon, \alpha + \varepsilon) \), and thus
\[
0 \leq \frac{d}{dx} H(x + if(x)) = H'(x + if(x))(1 + if'(x)).
\]
Writing \( H'(x + if(x)) = a + ib \) with \( a, b \in \mathbb{R} \), we see that \( b = af'(x) \) and
\[
\frac{d}{dx} H(x + if(x)) = a - bf'(x) = a(1 + f'(x)^2).
\]
Since it is easily seen that \( H'(x + if(x)) \neq 0 \) (otherwise, the function \( H(x + if(x)) \) behaves locally as a power function, and \( H(x + if(x)) \) cannot be real on \( (\alpha - \varepsilon, \alpha + \varepsilon) \)), we conclude that \( a > 0 \), and \( (2.4) \) follows from the Cauchy–Riemann equations.

Weak convergence is easily described in terms of the functions \( F_\mu \) or \( \varphi_\mu \). If a sequence \( \{ \mu_n \}_{n \in \mathbb{N}} \) of Borel probability measures is tight, then there exist \( \alpha, \beta > 0 \) such that \( \Gamma_{\alpha,\beta} \) is contained in the domain of \( \varphi_{\mu_n} \) for every \( n \in \mathbb{N} \). Moreover, the weak convergence of \( \{ \mu_n \}_{n \in \mathbb{N}} \) to \( \mu \) is equivalent to the local uniform convergence of \( \varphi_{\mu_n} \) to \( \varphi_\mu \) on \( \Gamma_{\alpha,\beta} \), as well as to the local uniform convergence of \( F_{\mu_n} \) to \( F_\mu \) on \( \mathbb{H} \). See \cite{1} for the proofs of these results.

Suppose now that \( k_1, k_2, \ldots \in \mathbb{N} \), and that \( \{ \mu_{n,i} : n, i \in \mathbb{N}, i \leq k_n \} \) is an array of Borel probability measures on \( \mathbb{R} \). This array is said to be \textit{infinitesimal} if
\[
\lim_{n \to \infty} \min_{1 \leq i \leq k_n} \mu_{n,i} \left( (-\varepsilon, \varepsilon) \right) = 1
\]
for every \( \varepsilon > 0 \). It was observed in \cite{4} that, given an infinitesimal array as above and arbitrary \( \alpha, \beta > 0 \), there exists \( N \in \mathbb{N} \) such that \( \varphi_{\mu_{n,i}} \) is defined on \( \Gamma_{\alpha,\beta} \) for every \( n \geq N \) and \( i = 1, \ldots, k_n \). In particular, if we set
\[
\nu_n = \mu_{n,1} \boxplus \cdots \boxplus \mu_{n,k_n},
\]
\( \varphi_{\nu_n} \) is defined on \( \Gamma_{\alpha,\beta} \) for \( n \geq N \). Therefore the following result applies in particular to free additive convolutions of measures in an infinitesimal array and provides our extension of superconvergence to such an array.
Theorem 2.3. Let \( \{\nu_n\}_{n \in \mathbb{N}} \) be a sequence of Borel probability measures on \( \mathbb{R} \) that converges weakly to an \( \mathbb{R} \)-infinitely divisible measure \( \nu \). Suppose that for every \( \alpha, \beta > 0 \) there exists \( N \in \mathbb{N} \) such that \( \varphi_{\nu_n} \) is defined in \( \Gamma_{\alpha, \beta} \) for every \( n \geq N \). Let \( J \subset \mathbb{R} \) be a compact interval such that \( \nu \) is absolutely continuous and \( d\nu/dx > 0 \) in a neighborhood of \( J \). Then \( \nu_n \) is absolutely continuous on a neighborhood of \( J \) with a real-analytic density for sufficiently large \( n \), and the densities \( d\nu_n/dx \) converge uniformly on \( J \), along with all their derivatives, to \( d\nu/dx \) as \( n \to \infty \).

Proof. Since unit point masses are purely singular we may, and shall, assume that the support of \( \nu \) contains more than one point. It suffices to show that \( d\nu_n/dx \) converges locally uniformly to \( d\nu/dx \) on a set on which the latter density is strictly positive. Fix \( \alpha_0, \beta_0 > 0 \) such that \( \Gamma_{\alpha_0, \beta_0} \) is contained in the domain of \( \varphi_{\nu_n} \) for every \( n \in \mathbb{N} \) and \( \varphi_{\nu_n} \) converges locally uniformly to \( \varphi_{\nu} \) on \( \Gamma_{\alpha_0, \beta_0} \). As noted above, the function \( \varphi_{\nu} \) is defined on \( \mathbb{H} \). Fix a point \( x_0 \in \mathbb{R} \) such that \( \nu \) is absolutely continuous in a neighborhood of \( x_0 \) and \( (d\nu/dx)(x_0) > 0 \). It follows that there exists \( s_0 \in \mathbb{R} \) such that \( f_{\nu}(s_0) > 0 \) and \( x_0 = H_{\nu}(s_0 + if_{\nu}(s_0)) \). Given an arbitrary number \( \varepsilon > 0 \) with \( \varepsilon < f_{\nu}(s_0) \), Lemma 2.1 allows us to choose an interval \([A, B]\) containing \( s_0 \) such that

\[
Q = \{ s + it : s \in [A, B], |t - f_{\nu}(s_0)| \leq \varepsilon \}
\]

is contained in \( \mathbb{H} \), \( |f_{\nu}(s) - f_{\nu}(s_0)| < \varepsilon \) for \( s \in [A, B] \), and

\[
\frac{\partial}{\partial t} \exists H_{\nu}(s + it) > 0, \quad s + it \in Q.
\]

Choose also \( \delta > 0 \) such that the interior of the rectangle

\[
K = \left\{ s + it : s \in [A, B], \delta \leq t \leq \frac{1}{\delta} \right\}
\]

has nonempty intersection with \( \Gamma_{\alpha_0, \beta_0} \) and \( Q \subset K \). Finally, choose \( \alpha, \beta > 0 \) such that

\[
K \cup \Gamma_{\alpha_0, \beta_0} \subset \Gamma_{\alpha, \beta}.
\]

By hypothesis, there exists \( N \in \mathbb{N} \) be such that \( \varphi_{\nu_n} \) is defined on \( \Gamma_{\alpha, \beta} \) for every \( n \geq N \). The sequence of restrictions \( \{\varphi_{\nu_n}|_{\Gamma_{\alpha_0, \beta_0}}\}_{n \geq N} \) converges to \( \varphi_{\nu} \) uniformly in some closed disk contained in \( \Gamma_{\alpha_0, \beta_0} \), and since these functions take values in \( -\mathbb{H} \), the Vitali-Montel theorem implies that \( \varphi_{\nu_n} \) actually converges locally uniformly to \( \varphi_{\nu} \) on \( \Gamma_{\alpha, \beta} \), and thus \( H_{\nu_n} \) also converges locally uniformly to \( H_{\nu} \) on \( \Gamma_{\alpha, \beta} \). In particular, \( H_{\nu_n} \) converges uniformly to \( H_{\nu} \) on a neighborhood of \( K \) so, after replacing \( N \) by a larger value, we may assume that

\[
\exists H_{\nu_n}(s + it) > 0, \quad s + it \in Q, \quad n \geq N.
\]

Note also that the set

\[
K_+ = \{ s + it \in K : t \geq f_{\nu}(s_0) + \varepsilon \}
\]

is contained in \( \Omega_{\nu} \), while

\[
K_- = \{ s + it \in K : t \leq f_{\nu}(s_0) - \varepsilon \}
\]

is disjoint from \( \Omega_{\nu} \). Thus, possibly making \( N \) even larger, we may assume that

\[
\exists H_{\nu_n}(z) > 0 \text{ for } z \in K_+ \text{ and } \exists H_{\nu_n}(z) < 0 \text{ for } z \in K_-,
\]
provided that \( n \geq N \). Combining this with (2.25), we see that for every \( n \geq N \) and for every \( s \in [A, B] \) there exists a unique \( f_n(s) > 0 \) such that

\[
\exists H_{\nu_n}(s + it) > 0 \text{ for } t > f_n(s) \text{ and } \exists H_{\nu_n}(s + it) < 0 \text{ for } \delta < t < f_n(s).
\]

Of course, we have \( |f_n(s) - f_\nu(s)| < \varepsilon \). Since \( \varepsilon \) can be made arbitrarily small, we conclude that \( f_n \) converges to \( f_\nu \) uniformly on \([A, B]\).

We observe next that \( F_{\nu_n}(H_{\nu_n}(z)) = z \) for \( z \in \Gamma_{\infty, \delta_0} \), and our choice of \( K \), along with analytic continuation, show that

\[
F_{\nu_n}(H_{\nu_n}(s + it)) = s + it, \quad s + it \in Q, \quad t > f_n(s), \quad n \geq N.
\]

Now, the set \( Q \) is convex, and thus (2.25) implies that \( H_{\nu_n} \) is injective with an analytic inverse on a neighborhood of \( Q \), and that \( F_{\nu_n} \) coincides with the inverse of \( H_{\nu_n} \) on the set \( \{s + it \in Q : t > f_n(s)\} \). It follows that \( F_{\nu_n} \) has an analytic continuation to a neighborhood of \( H_{\nu_n}(Q) \). In particular, \( F_{\nu_n} \) has an analytic continuation across the segment

\[
\{H_{\nu_n}(s + if_n(s)) : s \in [A, B]\},
\]

and this segment tends to

\[
\{H_\nu(s + if_\nu(s)) : s \in [A, B]\}
\]

which is a neighborhood of \( x_0 \). This shows that \( \nu_n \) is absolutely continuous with a real-analytic density \( d\nu_n/dx \) in a neighborhood of \( x_0 \), and this density satisfies the formula

\[
\frac{d\nu_n}{dx}(H_{\nu_n}(s + if_n(s))) = \frac{1}{\pi} \frac{f_n(s)}{s^2 + f_n(s)^2}, \quad s \in [A, B].
\]

Finally, to show that the densities \( d\nu_n/dx \) converge uniformly to \( d\nu/dx \) in a neighborhood of \( x_0 \), it suffices to show that \( F_{\nu_n} \) converges uniformly to \( F_\nu \) in a neighborhood of \( x_0 \). This, as well as the convergence of the derivatives, follows from the formula

\[
F_{\nu_n}(z) = H_{\nu_n}^{-1}(z) = \frac{1}{2\pi i} \int_{\partial Q} \frac{\zeta H_{\nu_n}(\zeta)}{H_{\nu_n}(\zeta) - z} d\zeta
\]

that holds for \( z \in \mathbb{H} \) close to \( x_0 \) and for sufficiently large \( n \).

The following corollary formalizes the discussion preceding Theorem 2.3.

**Corollary 2.4.** Suppose that \( k_1, k_2, \cdots \in \mathbb{N} \), and that \( \{\mu_{n,i} : n, i \in \mathbb{N}, i \leq k_n\} \) is an infinitesimal array of Borel probability measures on \( \mathbb{R} \). Set

\[
\nu_n = \mu_{n,1} \boxplus \cdots \boxplus \mu_{n,k_n}.
\]

Assume that \( \nu_n \) converges to a \( \mathbb{H} \)-infinite divisible measure \( \nu \). Then for any compact interval \( J \subset \mathbb{R} \) such that \( \nu \) is absolutely continuous and \( d\nu/dx > 0 \) in a neighborhood of \( J \), \( \nu_n \) is absolutely continuous on a neighborhood of \( J \) with a real-analytic density for sufficiently large \( n \), and the densities \( d\nu_n/dx \) converge uniformly on \( J \) to \( d\nu/dx \) as \( n \to \infty \).
3. Multiplicative convolution on $\mathbb{R}_+$

Suppose that $\mu$ is a Borel probability measure on $\mathbb{R}_+ = [0, +\infty)$, other than the point mass $\delta_0$ at the origin. The analytic functions

$$
\psi_\mu(z) = \int_{\mathbb{R}_+} \frac{zt}{1-zt} \, d\mu(t), \quad \eta_\mu(z) = \frac{\psi_\mu(z)}{1+\psi_\mu(z)}, \quad z \in \mathbb{C}\setminus\mathbb{R}_+,
$$

are real-valued on $(-\infty, 0)$ and the measure $\mu$ is uniquely determined by either $\psi_\mu$ or $\eta_\mu$. Indeed, we have

$$
zG_\mu(z) = \frac{1}{1 - \eta_\mu(z)}, \quad z \in \mathbb{C}\setminus\mathbb{R}_+,
$$

(3.1)

to which the Stieltjes inversion can be applied to recover $\mu$. Moreover, $\psi_\mu$ and $\eta_\mu$ map $\mathbb{H}$ to itself and, in addition,

$$
\arg(\eta_\mu(z)) \geq \arg z, \quad z \in \mathbb{H},
$$

where ‘arg’ stands for the principal value of the argument, so $\arg z \in (0, \pi)$ for $z \in \mathbb{H}$, see [2]. It was shown in [3] that there exist an open set $V$, containing some interval of the form $(-a, 0)$, and an analytic function $\Sigma_\mu$ defined on $V$ such that

$$
\eta_\mu(z\Sigma_\mu(z)) = z, \quad z \in V.
$$

The related equation $\eta_\mu(z)\Sigma_\mu(\eta_\mu(z)) = z$ holds for $z$ in every connected open set $U \subset \mathbb{C}\setminus\mathbb{R}_-$ that intersects $\mathbb{R}_-$ and such that $\eta_\mu(z)$ belongs to the domain of $\Sigma_\mu$. The function $\Sigma_\mu$ serves an analogous role relative to multiplicative free convolution to that of $\varphi_\mu$ relative to additive free convolution, namely

$$
\Sigma_\mu \ast \nu(z) = \Sigma_\mu(z)\Sigma_\nu(z)
$$

for $z$ in every domain on which the three functions are defined. The measure $\mu$ is $\mathbb{H}$-infinitely divisible precisely when the function $\Sigma_\mu$ continues analytically to the entire domain $\mathbb{C}\setminus\mathbb{R}_+$ and this analytic continuation can be written as

$$
\Sigma_\mu(z) = \gamma \exp \left[ \int_{[0, +\infty]} \frac{1 + tz}{z - t} \, d\sigma(t) \right], \quad z \in \mathbb{C}\setminus\mathbb{R}_+,
$$

where $\gamma > 0$ and $\sigma$ is a finite, positive Borel measure on the one-point compatification $[0, +\infty]$ of $\mathbb{R}_+$ [4].

Suppose now that $\mu$ is $\mathbb{H}$-infinitely divisible. The equation

$$
\eta_\mu(z)\Sigma_\mu(\eta_\mu(z)) = z
$$

extends by analytic continuation to the entire slit plane $\mathbb{C}\setminus\mathbb{R}_+$. In particular, it shows that $\eta_\mu$ maps $\mathbb{C}\setminus\mathbb{R}_+$ conformally onto a domain $\Omega_\mu$ that is symmetric relative to the real line. In addition to the interval $(-\infty, 0]$, the boundary of $\Omega_\mu \cap \mathbb{H}$ consists of a curve of the form

$$
C = \{ re^{ih_\mu(r)} : r \in (0, +\infty) \},
$$

where $h_\mu : (0, +\infty) \to (0, \pi)$ is a continuous function, real analytic wherever it is strictly positive. In other words,

$$
\Omega_\mu \cap \mathbb{H} = \{ re^{i\theta} : r > 0, h_\mu(r) < \theta < \pi \}.
$$

Moreover, the function $\eta_\mu$ extends continuously and injectively to $\mathbb{H} \cup (0, +\infty)$ such that the range $\eta_\mu((0, +\infty))$ is exactly the curve $C$ and that the inversion relationships between $\eta_\mu$ and the map $\Phi(z) = z\Sigma_\mu(z)$ now extend to the boundary of the relevant domains. (See [5] Section2 for a review of these results concerning
boundary behavior, and the references therein for their origin.) Thus, there is at most one value \( t_0 > 0 \) such that \( \eta_{\mu}(t_0) = 1 \). Such a point exists precisely when \( h_\mu(1) = 0 \). Outside possibly the point \( 1/t_0 \), the measure \( \mu \) is absolutely continuous and its density is locally analytic wherever it is strictly positive. We give a short proof of this analyticity for future reference. Suppose that \( x_0 > 0 \) is a point where the density of \( \mu \) is positive. We write \( \frac{1}{x_0} = r_0 e^{i \theta_{\mu}(r_0)} \sum_{\mu}(re^{i \theta_{\mu}(r_0)}) \) for some \( r_0 > 0 \) such that \( h_\mu(r_0) > 0 \). The continuity of \( h_\mu \) yields \( h_\mu(r) > 0 \) for \( r \) near \( r_0 \). For such \( r \), the function \( \Phi(z) = z \sum_{\mu}(z) \) has a non-zero complex derivative at \( z = re^{i \theta_{\mu}(r)} \). This is because \( \Phi'(re^{i \theta_{\mu}(r)}) = 0 \) would imply that the positive number \( \Phi(re^{i \theta_{\mu}(r)}) \) has multiple preimages located on the curve \( C \), contradicting the fact that \( \Phi \) is injective on \( C \). Since the inversion equation

\[
\Phi(\eta_{\mu}(z)) = z
\]

holds for \( z \in \mathbb{H} \) close to \( 1/x_0 \), we conclude that \( \eta_{\mu} \) can be analytically continued to a neighborhood of \( 1/x_0 \) as the inverse of the function \( \Phi \). The desired analyticity now follows from \( \mu_{\alpha,\beta} \) and the Stieltjes inversion formula. (One can also write an implicit formula for the density, just as in the additive case, to see its analyticity directly; see for instance \( \mathbb{R} \).)

Suppose now that \( k_1 < k_2 < \cdots \) is a sequence of natural numbers and \( \{\mu_{n,i} : n, i \in \mathbb{N}, i \leq k_n\} \) is an infinitesimal array of probability measures on \( \mathbb{R}_+ \); that is,

\[
\lim_{n \to \infty} \min_{1 \leq i \leq k_n} \mu_{n,i}(\{1 - \epsilon, 1 + \epsilon\}) = 1
\]

for every \( \epsilon \in (0, 1) \). The analog of the domain \( \Gamma_{\alpha,\beta} \) in the context of multiplicative free convolution is the domain

\[
\Omega_{\rho,\theta} = \left\{ re^{it} : \rho < r < \frac{1}{\rho}, t \in (\theta, 2\pi - \theta) \right\},
\]

where \( \rho > 0 \) and \( \theta \in (0, \pi) \). By the results in \( \mathbb{R} \), given arbitrarily small \( \rho > 0 \) and \( \theta \in (0, \pi) \), all the functions \( \Sigma_{\mu_{n,i}} \) are defined in \( \Omega_{\rho,\theta} \) if \( n \) is sufficiently large. As shown in \( \mathbb{R} \), the weak convergence of probability measures on \( \mathbb{R}_+ \) can be translated into local uniform convergence of the corresponding \( \eta \)-functions, or into local uniform convergence of the corresponding \( \Sigma \)-functions on some \( \Omega_{\rho,\theta} \). The following result is analogous to Theorem \( \mathbb{R} \) and therefore it provides a superconvergence result for infinitesimal arrays.

**Theorem 3.1.** Let \( \{\nu_n\}_{n \in \mathbb{N}} \) be a sequence of Borel probability measures on \( \mathbb{R}_+ \) that converges weakly to an \( \mathbb{R} \)-infinitely divisible measure \( \nu \). Suppose that for every \( \rho > 0 \) and \( \theta \in (0, \pi) \) there exists \( N \in \mathbb{N} \) such that \( \Sigma_{\nu_n} \) is defined in \( \Omega_{\rho,\theta} \) for every \( n \geq N \). Let \( J \subset (0, +\infty) \) be a compact interval such that \( \nu \) is absolutely continuous and \( d\nu/dx > 0 \) in a neighborhood of \( J \). Then \( \nu_n \) is absolutely continuous with an analytic density on a neighborhood of \( J \) for \( n \) sufficiently large, and the densities \( d\nu_n/dx \) converge uniformly on \( J \), along with all their derivatives, to \( d\nu/dx \) as \( n \to \infty \).

**Sketch of proof.** To simplify notation, we set

\[
\Phi(z) = z \sum_{\nu}(z) \text{ and } \Phi_n(z) = z \sum_{\nu_n}(z), \quad n \in \mathbb{N}.
\]

Suppose that \( x_0 > 0 \) is a point where the density of \( \nu \) is positive. As seen above, we have

\[
\frac{1}{x_0} = \Phi(r_0 e^{i \theta_{\mu}(r_0)})
\]
for some \( r_0 > 0 \) such that \( h_\nu(r_0) > 0 \). The point \( r_0 e^{ih_\nu(r_0)} \) belongs to the domain \( \Omega_{\rho,\theta} \) provided that \( \rho \) and \( \theta \) are sufficiently small. Fix such values \( \rho \) and \( \theta \), and note that \( \Phi_n \) is defined on \( \Omega_{\rho,\theta} \) provided that \( n \) is sufficiently large. As seen in the earlier discussions, the essential point is to verify that, for \( z \in \mathbb{H} \) sufficiently close to \( 1/x_0 \), we have \( \eta_{\nu_n}(z) \in \Omega_{\rho,\theta} \) and the identity
\[
(3.2) \quad \Phi_n(\eta_{\nu_n}(z)) = z
\]
holds for \( n \) sufficiently large. The absolute continuity of \( \nu_n \) near \( x_0 \) would follow as a byproduct. Note that the identity \( (3.2) \) holds for \( z \in (-T,-1/T) \) if \( T \) is large enough, and it extends by analytic continuation to the largest interval of this form with the property that \( \eta_{\nu_n}((-T,-1/T)) \subset \Omega_{\rho,\theta} \).

We observe now that, by using the polar coordinates, the argument of Remark 2.2 (or alternatively, more explicit calculations from [12, Lemma 4.2]) yields
\[
\frac{\partial}{\partial \theta} \text{arg} (\Phi(r e^{i\theta})) > 0 \quad \text{for } r = r_0 \text{ and } \theta = h_\nu(r_0).
\]
So we may choose a neighborhood \( W \) of \( r_0 e^{ih_\nu(r_0)} \) and an integer \( N \) such that \( \Phi_n \) is defined on \( \Omega_{\rho,\theta} \) for \( n \geq N \), \( W \subset \Omega_{\rho,\theta} \cap \mathbb{H} \), and
\[
\frac{\partial}{\partial \theta} \text{arg} (\Phi_n(r e^{i\theta})) > 0 \quad \text{for } r e^{i\theta} \in W \text{ and } n \geq N.
\]
In particular, the complex derivative \( \Phi_n' \) does not vanish on \( W \), so that \( \Phi_n|_W \) has an analytic inverse. Then we choose \( \varepsilon > 0 \) so small that
1. \( |h_\nu(r) - h_\nu(r_0)| < \varepsilon \text{ for } |r - r_0| \leq \varepsilon \),
2. the compact set
   \[
   K = \{ re^{i\theta} : |r - r_0| \leq \varepsilon, |\theta - h_\nu(r_0)| \leq \varepsilon \}
   \]
is contained in \( W \);
3. \( \Phi(re^{ih_\nu(r_0)+i\varepsilon}) \in \mathbb{H} \) for \( |r - r_0| \leq \varepsilon \), and
4. \( \Phi(re^{ih_\nu(r_0)-i\varepsilon}) \in -\mathbb{H} \) for \( |r - r_0| \leq \varepsilon \).
Since \( \Phi_n \) converges to \( \Phi \) uniformly on \( K \), we can assume that properties (2) and (3) also hold for \( \Phi_n \) after making \( N \) bigger. It follows that there exists a unique function \( h_n : [r_0 - \varepsilon, r_0 + \varepsilon] \rightarrow (h_\nu(r_0) - \varepsilon, h_\nu(r_0) + \varepsilon) \) such that
\[
\Phi_n(re^{ih_n(r)}) \in (0, +\infty), \quad |r - r_0| \leq \varepsilon.
\]
The function \( \Phi_n \) is one-to-one on \( K \), and the interval
\[
\{ \Phi_n(re^{ih_n(r)}) : |r - r_0| \leq \varepsilon \}
\]
is a neighborhood of \( 1/x_0 \) for sufficiently large \( n \). We need to show that \( (3.2) \) holds in the set
\[
\{ \Phi_n(re^{i\theta}) : |r - r_0| \leq \varepsilon, h_n(r) < \theta \leq h_\nu(r) + \varepsilon \},
\]
provided that \( n \) is sufficiently large. This will follow by analytic continuation once we show that \( \Phi_n(K') \subset \mathbb{H} \), where
\[
K' = \{ re^{i\theta} : |r - r_0| \leq \varepsilon, \theta \in [h_\nu(r) + \varepsilon, \pi] \}.
\]
This last fact follows immediately because \( \Phi(K') \subset \mathbb{H} \) and \( \Phi_n \) converges uniformly on \( K' \) to \( \Phi \).

The following corollary formalizes the discussion preceding Theorem 3.1.
Corollary 3.2. Suppose that \( k_1, k_2, \cdots \in \mathbb{N} \), and that \( \{\mu_{n,i} : n, i \in \mathbb{N}, i \leq k_n\} \) is an infinitesimal array of Borel probability measures on \( \mathbb{R}_+ \). Set

\[ \nu_n = \mu_{n,1} \otimes \cdots \otimes \mu_{n,k_n}. \]

Assume that \( \nu_n \) converges to a \( \mathbb{E} \)-infinite divisible measure \( \nu \). Then for any compact interval \( J \subset (0, \infty) \) such that \( \nu \) is absolutely continuous and \( d\nu/dx > 0 \) in a neighborhood of \( J \), \( \nu_n \) is absolutely continuous on a neighborhood of \( J \) with a real-analytic density for sufficiently large \( n \), and the densities \( d\nu_n/dx \) converge uniformly on \( J \) to \( d\nu/dx \) as \( n \to \infty \).

### 4. Multiplicative convolution on \( T \)

Finally, we consider the superconvergence phenomenon on the unit circle \( T = \{e^{it} : \theta \in [0, 2\pi]\} \). If \( \mu \) is a Borel probability measure on \( T \), one sets again

\[ \psi_\mu(z) = \int_T \frac{z\zeta}{1 - z\zeta} d\mu(\zeta), \quad \eta_\mu(z) = \frac{\psi_\mu(z)}{1 + \psi_\mu(z)}, \]

but these functions are now defined on the unit disk \( D = \{z \in \mathbb{C} : |z| < 1\} \). The density of \( \mu \) relative to the normalized arclength measure \( m = d\theta/2\pi \) is given almost everywhere by

\[ \frac{d\mu}{dm}(\xi) = \lim_{r \uparrow 1} \frac{1 + \eta_\mu(r\xi)}{1 - \eta_\mu(r\xi)}, \quad \xi \in T. \]

We restrict our considerations to the case in which \( \int_T \zeta d\mu(\zeta) \neq 0 \) and we observe that this condition is satisfied for (all but finitely many) measures in an infinitesimal array. In terms of \( \psi \) and \( \eta \), this condition amounts to the requirement that \( \psi_\mu'(0) = \eta_\mu'(0) \neq 0 \). This implies the existence of an analytic function \( \Sigma_\mu \), defined in a disk \( \rho\mathbb{D} = \{z \in \mathbb{C} : |z| < \rho\} \) with \( \rho \in (0, 1) \) and with values in \( \mathbb{D} \), such that

\[ \eta_\mu(z\Sigma_\mu(z)) = z, \quad z \in \rho\mathbb{D}. \]

We denote by \( \rho_\mu \) the radius of convergence of the Taylor series of \( \Sigma_\mu \), so \( \Sigma_\mu \) is defined in \( \rho_\mu \mathbb{D} \). The free multiplicative convolution of measures on \( T \) satisfies the identity

\[ \Sigma_\eta \circ \Sigma_\mu(z) = \Sigma_\mu(z) \Sigma_\nu(z), \quad |z| < \min\{\rho_\mu, \rho_\nu\}. \]

Among the \( \mathbb{E} \)-infinitely divisible measures on \( T \), the normalized arclength measure \( m \) is the only one with zero first moment. The other \( \mathbb{E} \)-infinitely divisible measures \( \mu \) on \( T \) are characterized by the fact that \( \rho_\mu \geq 1 \) and \( \Phi_\mu(\mathbb{D}) \subset \mathbb{D} \), where \( \Phi_\mu(z) = z\Sigma_\mu(z) \). Clearly, if \( \mu \) is \( \mathbb{E} \)-infinitely divisible, the equation

\[ (4.1) \]

extends by analytic continuation to the entire disk \( \mathbb{D} \), showing that \( \eta_\mu \) maps \( \mathbb{D} \) conformally onto a domain \( \Omega_\mu \subset \mathbb{D} \). In fact, by the results in [12] (see also [8, Section 5] for a review), \( \eta_\mu \) extends to a homeomorphism of \( \overline{\mathbb{D}} \) onto \( \overline{\Omega_\mu} \). In particular, \( \eta_\mu \) maps \( T \) homeomorphically onto \( \partial \Omega_\mu \). The domain \( \Omega_\mu \) is starlike, in particular its boundary is a closed curve of the form

\[ \{R_\mu(\zeta) \zeta : \zeta \in T\}, \]

where \( R_\mu : T \to (0, 1] \) is a continuous function that is real-analytic at all points \( \zeta \) satisfying \( R_\mu(\zeta) < 1 \). Thus, one has

\[ \Omega_\mu = \{r\zeta : \zeta \in T, 0 \leq r < R_\mu(\zeta)\}. \]
For a \(\mathbb{E}\)-infinitely divisible measure \(\mu\), there is at most one point \(\zeta_0\) such that \(\eta_\mu(\zeta_0) = 1\), and this happens precisely when \(R_\mu(1) = 1\). Outside the point \(\zeta_0\), the measure \(\mu\) is absolutely continuous with a locally analytic density wherever this density is positive. This is a consequence of the identity (4.11) that holds for \(z \in \mathbb{D}\) close to a point \(\xi_0 \in \mathbb{T}\) such that \(|\eta_\mu(\xi_0)| < 1\) because (as we also discuss in the proof below) \(\Phi(\eta_\mu(\xi_0)) \neq 0\).

Suppose that \(k_1 < k_2 < \cdots\) is a sequence of natural numbers and \(\{\mu_{n,i} : n, i \in \mathbb{N}, i \leq k_n\}\) is an infinitesimal array of Borel probability measures on \(\mathbb{T}\), that is,

\[
\lim_{n \to \infty} \min_{1 \leq i \leq k_n} \mu_{n,i}(\{\zeta \in \mathbb{T} : |\zeta - 1| < \varepsilon\}) = 1
\]

for every \(\varepsilon > 0\). According to [3], given an arbitrary \(\rho \in (0, 1)\), we have \(\rho_{\mu_{n,i}} > \rho\) for all \(1 \leq i \leq k_n\) provided that \(n\) is sufficiently large.

The weak convergence of probability measures on \(\mathbb{T}\) can be translated into local uniform convergence of the \(\eta\)-functions. If all the measures under consideration satisfy \(\int_\mathbb{T} \zeta d\mu(\zeta) \neq 0\) it also translates into uniform convergence of \(\Phi\)-functions on \(\rho \mathbb{D}\) for some \(\rho > 0\).

We are now ready to state the circle version of the superconvergence result.

**Theorem 4.1.** Let \(\{\nu_n\}_{n \in \mathbb{N}}\) be a sequence of Borel probability measures on \(\mathbb{T}\) that converges weakly to an \(\mathbb{E}\)-infinitely divisible measure \(\nu\), where \(\int_\mathbb{T} \zeta d\nu(\zeta) \neq 0\). Suppose that \(\lim_{n \to \infty} \rho_{\nu_n} = 1\). Let \(J \subset \mathbb{T}\) be a compact arc such that \(\nu\) is absolutely continuous and its density is positive in a neighborhood of \(J\). Then \(\nu_n\) is absolutely continuous with a real-analytic density on a neighborhood of \(J\) for \(n\) sufficiently large, and the densities of \(\nu_n\) converge uniformly on \(J\), along with all their derivatives, to that of \(\nu\) as \(n \to \infty\).

**Sketch of proof.** We use the notation \(\Phi = \Phi_\nu\) and \(\Phi_n = \Phi_{\nu_n}\), \(n \in \mathbb{N}\). Suppose that \(\xi_0 \in \mathbb{T}\) is such that \(\eta_\nu(\xi_0) \in \mathbb{D}\). As in the proof of Theorem 3.1 we need to show that, for sufficiently large \(n\), we have \(\eta_{\nu_n}(z) \in \rho_{\nu_n} \mathbb{D}\) for \(z \in \mathbb{D}\) close to \(\xi_0\), and that \(\Phi_n(\eta_{\nu_n}(z)) = z\) for such \(z\). By hypothesis, we may assume that \(\rho_{\nu_n} > \rho > |\eta_\nu(\xi_0)|\) for some \(\rho \in (0, 1)\), and that \(|\Phi_n(z)| < 1\) for \(|z| < \rho\). It is easily seen, by using logarithmic polar coordinate, from Remark 2.2 (see also [12, Section 3] for a proof by explicit formulas) that

\[
\frac{\partial}{\partial r} \log |\Phi(re^{i\theta})| > 0
\]

at the point \(re^{i\theta_0} = \eta_\nu(\xi_0)\). Thus, we may also assume that

\[
\frac{\partial}{\partial r} \log |\Phi_n(re^{i\theta})| > 0
\]

for \(re^{i\theta}\) in a fixed neighborhood of \(\eta_\nu(\xi_0)\) and for large \(n\). It follows that there is a sequence \(\{R_n\}_{n \in \mathbb{N}}\) of functions defined in a neighborhood of \(e^{i\theta_0}\), with values in \((0, 1)\), such that

\[
|\Phi_n(r\zeta)|\begin{cases} < 1, & r < R_n(\zeta), \\ = 1, & r = R_n(\zeta), \\ > 1, & r > R_n(\zeta). \end{cases}
\]

Moreover, the functions \(R_n\) converge uniformly to \(R_\nu\) in a neighborhood of \(e^{i\theta_0}\). Then, we need to verify that

\[
\eta_{\nu_n}(\Phi_n(z)) = z
\]
for \( z = r\zeta \) provided that \( r < R_n(\zeta) \) and \( \zeta \) is close to \( e^{i\theta_0} \), say \( |\zeta - e^{i\theta_0}| \leq \varepsilon \). This statement follows by analytic continuation because \( \Phi_n \) converges uniformly to \( \Phi \) in the form of a compact set of the form

\[
\{ r\zeta : |\zeta - e^{i\theta_0}| \leq \varepsilon, r \leq R_n(e^{i\theta_0}) - \delta \},
\]

where \( \varepsilon, \delta > 0 \). Absolute continuity of \( \nu_n \) near \( 1/\xi_0 \), and the uniform convergence of the densities, follow as in the free additive case.

As in the additive case and the multiplicative case on \( \mathbb{R}_+ \), we have the following corollary that formalizes the discussion preceding Theorem 4.1.

Corollary 4.2. Suppose that \( k_1, k_2, \cdots \in \mathbb{N} \), and that \( \{ \mu_{n,i} : n, i \in \mathbb{N}, i \leq k_n \} \) is an infinitesimal array of Borel probability measures on \( \mathbb{T} \). Set

\[
\nu_n = \mu_{n,1} \boxtimes \cdots \boxtimes \nu_{n,k_n}.
\]

Assume that \( \nu_n \) converges to a \( \boxtimes \)-infinite divisible measure \( \nu \) such that \( \nu \neq m \). Then for any compact arc \( J \subset \mathbb{T} \) such that \( \nu \) is absolutely continuous and \( d\nu/dx > 0 \) in a neighborhood of \( J \), \( \nu_n \) is absolutely continuous on a neighborhood of \( J \) with a real-analytic density for sufficiently large \( n \), and the densities \( d\nu_n/dx \) converge uniformly on \( J \) to \( d\nu/dx \) as \( n \to \infty \).

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