Intersections of curves on surfaces and their applications to mapping class groups

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Abstract

We introduce an operation that measures the self intersections of paths on a surface. As applications, we give a criterion of the realizability of a generalized Dehn twist, and derive a geometric constraint on the image of the Johnson homomorphism.

1 Introduction

In study of the mapping class group of a surface, it is sometimes convenient and crucial to work with curves, i.e., loops or paths, on the surface. The mapping class group acts on the homotopy classes of curves, and as is illustrated in the classical theorem of Dehn-Nielsen, this action distinguishes elements of the mapping class group well. Moreover, since an element of the mapping class group is represented by a diffeomorphism, it preserves intersections of curves.

In this paper we introduce an operation that measures the self intersections of paths on a surface, and discuss its applications to the mapping class groups. Let $S$ be an oriented surface, and $*_{0}, *_{1} \in \partial S$ points on the boundary. We denote by $\Pi_{S}( *_{0}, *_{1})$ the set of homotopy classes of paths from $*_{0}$ to $*_{1}$, and by $\hat{\pi}'(S)$ the set of homotopy classes of non-trivial free loops on $S$. In §2 we introduce a $\mathbb{Q}$-linear map

$$
\mu: \mathbb{Q}\Pi S( *_{0}, *_{1}) \rightarrow \mathbb{Q}\Pi S( *_{0}, *_{1}) \otimes \mathbb{Q}\hat{\pi}'(S),
$$

by looking at the self intersections of a given path. This map is closely related to Turaev’s self intersection [18], and is actually a refinement of it.

One motivation to introduce $\mu$ comes from the Turaev cobracket on the Goldman-Turaev Lie bialgebra. The free vector space $\mathbb{Q}\hat{\pi}'(S)$ spanned by the set $\hat{\pi}'(S)$ is an involutive Lie bialgebra with respect to the Goldman bracket [5] and the Turaev cobracket [19]. In [2], we introduced a filtration on $\mathbb{Q}\hat{\pi}'(S)$ and showed that the Goldman bracket induces a Lie bracket on the completion $\hat{\mathbb{Q}}\hat{\pi}(S)$, which we called the completed Goldman Lie algebra. As we will see, the operation $\mu$ recovers the Turaev cobracket. Analyzing the behavior of $\mu$ under the conjunction of paths, we show that $\mu$ naturally extends to completions and the Turaev cobracket extends to a complete cobracket on $\hat{\mathbb{Q}}\hat{\pi}(S)$, thus we could call $\hat{\mathbb{Q}}\hat{\pi}(S)$ the completed Goldman-Turaev Lie bialgebra. Along the course, we find that there is a naturally defined bimodule of $\mathbb{Q}\hat{\pi}'(S)$. In [8] [9], we showed that $\mathbb{Q}\Pi S( *_{0}, *_{1})$ is a (left) $\mathbb{Q}\hat{\pi}'(S)$-module with respect to a structure map $\sigma: \mathbb{Q}\hat{\pi}'(S) \otimes \mathbb{Q}\Pi S( *_{0}, *_{1}) \rightarrow \mathbb{Q}\Pi S( *_{0}, *_{1})$. In fact, by investigating the properties of $\mu$ we arrive at the notion of a comodule of a Lie coalgebra, and that of a bimodule of a Lie bialgebra (see Appendix), and we show that $\mathbb{Q}\Pi S( *_{0}, *_{1})$ is a $\mathbb{Q}\hat{\pi}'(S)$-bimodule with respect to $\sigma$ and $\mu$. 
Since $\mu$ is defined in terms of the intersections of curves, it is automatically compatible with the action of the mapping class group. In §4 we give two applications of this fact to study of the mapping class group. The first one is an application to generalized Dehn twists \textsuperscript{9}, \textsuperscript{11}, \textsuperscript{13}, which are elements of a certain enlargement of the mapping class group. We can ask whether a generalized Dehn twist is realized by a diffeomorphism of the surface, and we give a criterion of the realizability of a generalized Dehn twist using $\mu$. This criterion is powerful enough so that we can extend results about a figure eight \textsuperscript{9}, \textsuperscript{11} to loops in wider classes. The second one is an application to the Johnson homomorphism, which is an embedding of the ‘smallest’ Torelli group (in the sense of Putman \textsuperscript{17}) into a pro-nilpotent group. Using the fact that a diffeomorphism preserves $\mu$, we derive a geometric constraint on the image of the Johnson homomorphism. That this constraint is non-trivial can be seen from examples of null-homologous, non-simple loops whose generalized Dehn twists are not realized by diffeomorphisms. But it is not clear how our obstruction is related to the known obstructions such as the Morita trace \textsuperscript{16}.

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**Contents**

1 Introduction \hfill 1

2 The Goldman-Turaev Lie bialgebra and its bimodule \hfill 3
   2.1 The Goldman-Turaev Lie bialgebra \hfill 3
   2.2 A $\mathbb{Q}\hat{\pi}'(S)$-bimodule \hfill 4
   2.3 The case of $\ast_0 = \ast_1$ \hfill 4

3 Completion of the Turaev cobracket \hfill 12
   3.1 Completion of the Goldman Lie algebra \hfill 12
   3.2 Intersection of paths \hfill 13
   3.3 Product formulas \hfill 14
   3.4 $\mu$ and the filtrations of $\mathbb{Q}\text{IIS}(\ast_0, \ast_1), \mathbb{Q}\hat{\pi}'(S)$ \hfill 16

4 Application to mapping class groups \hfill 19
   4.1 Generalized Dehn twists \hfill 19
   4.2 A criterion of the realizability \hfill 20
   4.3 New examples not realized by a diffeomorphism \hfill 21
   4.4 The Johnson homomorphisms \hfill 26
   4.5 A constraint on the Johnson image \hfill 27

A Lie bialgebras and their bimodules \hfill 29
   A.1 Lie bialgebras \hfill 29
   A.2 Lie comodules and bimodules \hfill 30
2 The Goldman-Turaev Lie bialgebra and its bimodule

Let $S$ be a connected oriented surface. We denote by $\hat{\pi}(S) = [S^1, S]$ the homotopy set of oriented free loops on $S$. In other words, $\hat{\pi}(S)$ is the set of conjugacy classes of $\pi_1(S)$. We denote by $| | : \pi_1(S) \to \hat{\pi}(S)$ the natural projection, and we also denote by $| | : Q\pi_1(S) \to Q\hat{\pi}(S)$ its $Q$-linear extension.

2.1 The Goldman-Turaev Lie bialgebra

We recall the Goldman-Turaev Lie bialgebra [5] [19].

Let $\alpha$ and $\beta$ be oriented immersed loops on $S$ such that their intersections consist of transverse double points. For each $p \in \alpha \cap \beta$, let $\alpha_p \beta_p \in \pi_1(S, p)$ be the loop first going along the loop $\alpha$ based at $p$, then going along $\beta$ based at $p$. Also, let $\varepsilon(p; \alpha, \beta) \in \{\pm 1\}$ be the local intersection number of $\alpha$ and $\beta$ at $p$. See Figure 1. The Goldman bracket of $\alpha$ and $\beta$ is defined as

$$[\alpha, \beta] := \sum_{p \in \alpha \cap \beta} \varepsilon(p; \alpha, \beta)|\alpha_p \beta_p| \in Q\hat{\pi}(S). \quad (2.1.1)$$

The free vector space $Q\hat{\pi}(S)$ spanned by the set $\hat{\pi}(S)$ equipped with this bracket is a Lie algebra. See [3]. Let $1 \in \hat{\pi}(S)$ be the class of a constant loop, then its linear span $Q1$ is an ideal of $Q\hat{\pi}(S)$. We denote by $Q\hat{\pi}'(S)$ the quotient Lie algebra $Q\hat{\pi}(S)/Q1$, and let $\varpi : Q\hat{\pi}(S) \to Q\hat{\pi}'(S)$ be the projection. We write $| |' := \varpi \circ | | : Q\pi_1(S) \to Q\hat{\pi}'(S)$.

Let $\alpha : S^1 \to S$ be an oriented immersed loop such that its self intersections consist of transverse double points. Set $D = D_\alpha := \{(t_1, t_2) \in S^1 \times S^1; t_1 \neq t_2, \alpha(t_1) = \alpha(t_2)\}$. For $(t_1, t_2) \in D$, let $\alpha_{t_1t_2}$ (resp. $\alpha_{t_2t_1}$) be the restriction of $\alpha$ to the interval $[t_1, t_2]$ (resp. $[t_2, t_1]$) \subset $S^1$ (they are indeed loops since $\alpha(t_1) = \alpha(t_2)$). Also, let $\dot{\alpha}(t_i) \in T_{\alpha(t_i)}S$ be the velocity vectors of $\alpha$ at $t_i$, and set $\varepsilon(\dot{\alpha}(t_1), \dot{\alpha}(t_2)) = +1$ if $(\dot{\alpha}(t_1), \dot{\alpha}(t_2))$ gives the orientation of $S$, and $\varepsilon(\dot{\alpha}(t_1), \dot{\alpha}(t_2)) = -1$ otherwise. The Turaev cobracket of $\alpha$ is defined as

$$\delta(\alpha) := \sum_{(t_1, t_2) \in D} \varepsilon(\dot{\alpha}(t_1), \dot{\alpha}(t_2))|\alpha_{t_1t_2}|' \otimes |\alpha_{t_2t_1}|' \in Q\hat{\pi}'(S) \otimes Q\hat{\pi}'(S). \quad (2.1.2)$$

This gives rise to a well-defined Lie cobracket $\delta : Q\hat{\pi}'(S) \to Q\hat{\pi}'(S) \otimes Q\hat{\pi}'(S)$ (note that $\delta(1) = 0$). Moreover, $Q\hat{\pi}'(S)$ is an involutive Lie bialgebra with respect to the Goldman bracket and the Turaev cobracket. See [19]. The involutivity is due to Chas [2]. We call $Q\hat{\pi}'(S)$ the Goldman-Turaev Lie bialgebra.

Figure 1: local intersection number

\[ \varepsilon(p; \alpha, \beta) = +1 \]

\[ \varepsilon(p; \alpha, \beta) = -1 \]
2.2 A $\mathbb{Q}\hat{\pi}'(S)$-bimodule

Hereafter we assume that the boundary of $S$ is not empty. Take distinct points $*_0, *_1 \in \partial S$, and let $I\!I\!S(*_0, *_1)$ be the homotopy set $\{([0,1],0,1), (S,*_0,*_1)\}$. We shall show that the free vector space $\mathbb{Q}\!I\!S(*_0, *_1)$ spanned by the set $I\!I\!S(*_0, *_1)$ has a structure of involutive right $\mathbb{Q}\hat{\pi}'(S)$-bimodule. For the definition of a bimodule, see Appendix. In §2.2 we treat the case $*_0 = *_1$.

A left $\mathbb{Q}\hat{\pi}'(S)$-module structure. Let $\alpha$ be an oriented immersed loop on $S$, and $\beta : [0,1] \to S$ an immersed path from $*_0$ to $*_1$ such that their intersections consist of transverse double points. Then the formula

$$\sigma(\alpha \otimes \beta) := \sum_{p \in \alpha \cap \beta} \varepsilon(p; \alpha, \beta) \beta_{*p_0} \alpha_p \beta_{*p_1} \in \mathbb{Q}\!I\!S(*_0, *_1)$$

(2.2.1)

gives rise to a well-defined $\mathbb{Q}$-linear map $\sigma : \mathbb{Q}\hat{\pi}'(S) \otimes \mathbb{Q}\!I\!S(*_0, *_1) \to \mathbb{Q}\!I\!S(*_0, *_1)$. Here, $\varepsilon(p; \alpha, \beta) \in \{\pm 1\}$ has the same meaning as before, and $\beta_{*p_0} \alpha_p \beta_{*p_1}$ means the path first going from $*_0$ to $p$ along $\beta$, then going along $\alpha$ based at $p$, and finally going from $p$ to $*_1$ along $\beta$. By the same proof as that of [8] Proposition 3.2.2, we see that $\mathbb{Q}\!I\!S(*_0, *_1)$ is a left $\mathbb{Q}\hat{\pi}'(S)$-module with respect to $\sigma$. See also [9] §4.

A right $\mathbb{Q}\hat{\pi}'(S)$-comodule structure. Let $\gamma : [0,1] \to S$ be an immersed path from $*_0$ to $*_1$ such that its self intersections consist of transverse double points. Let $\Gamma = \Gamma_\gamma \subset S$ be the set of double points of $\gamma$. For $p \in \Gamma$, we denote $\gamma^{-1}(p) = \{t^1_p, t^2_p\}$, so that $t^1_p < t^2_p$. Set

$$\mu(\gamma) := -\sum_{\gamma \in \Gamma} \varepsilon(\gamma(t^1_p), \gamma(t^2_p)) (\gamma(t^1_p) \cap \gamma(t^2_p))' \in \mathbb{Q}\!I\!S(*_0, *_1) \otimes \mathbb{Q}\hat{\pi}'(S).$$

(2.2.2)

Here $\varepsilon(\gamma(t^1_p), \gamma(t^2_p)) \in \{\pm 1\}$ has the same meaning as before, $\gamma(t^1_p) \cap \gamma(t^2_p)$ is the conjunction of the restrictions of $\gamma$ to $[0, t^1_p]$ and $[t^2_p, 1]$, and $\gamma(t^1_p) \cap \gamma(t^2_p)$ is the restriction of $\gamma$ to $[t^1_p, t^2_p]$.

Proposition 2.2.1. The formula (2.2.2) gives rise to a well-defined $\mathbb{Q}$-linear map

$$\mu : \mathbb{Q}\!I\!S(*_0, *_1) \to \mathbb{Q}\!I\!S(*_0, *_1) \otimes \mathbb{Q}\hat{\pi}'(S).$$

Moreover, $\mathbb{Q}\!I\!S(*_0, *_1)$ is a right $\mathbb{Q}\hat{\pi}'(S)$-comodule with respect to $\mu$.

Proof. Any immersions $\gamma$ and $\gamma'$ with $\gamma(0) = \gamma'(0) = *_0$ and $\gamma(1) = \gamma'(1) = *_1$, homotopic to each other relative to $\{0,1\}$, such that their self intersections consist of transverse double points, are related by a sequence of three local moves $(\omega 1)$, $(\omega 2)$, $(\omega 3)$, and an ambient isotopy of $S$. See Goldman [5] §5 and Figure 2. To prove that $\mu$ is well-defined, it is sufficient to verify that $\mu(\gamma) = \mu(\gamma')$ if $\gamma$ and $\gamma'$ are related by one of the three moves. Suppose $\gamma$ and $\gamma'$ are related by the move $(\omega 1)$. The contribution of the double point in the right picture of the move $(\omega 1)$ is zero, since the class of a null-homotopic loop is zero in $\mathbb{Q}\hat{\pi}'(S)$. Hence $\mu(\gamma) = \mu(\gamma')$.

Suppose $\gamma$ and $\gamma'$ are related by the move $(\omega 2)$. We may assume the left picture corresponds to $\gamma'$. Then $\gamma$ has two more double points than $\gamma'$. We write them by $p$ and $q$ so that $t^0_p < t^1_p$. As in Figure 3, there are two possibilities: $t^0_p < t^2_p$ or $t^0_p < t^3_p$, but in any case, $\gamma(t^0_p) \cap \gamma(t^2_p)$ is homotopic to $\gamma(t^0_p) \cap \gamma(t^3_p)$ relative to $\{0,1\}$, and $\varepsilon(\gamma(t^0_p), \gamma(t^2_p)) = -\varepsilon(\gamma(t^0_p), \gamma(t^3_p))$. Hence the contributions from $p$ and $q$ cancel and $\mu(\gamma) = \mu(\gamma')$.

Suppose $\gamma$ and $\gamma'$ are related by the move $(\omega 3)$. Similarly to the case of $(\omega 2)$, we see that a cancel happens and $\mu(\gamma) = \mu(\gamma')$. Typical cases are illustrated in Figure 4, where the contributions from $p$ (resp. $q$) and $p'$ (resp. $q'$) cancel.
Figure 2: local moves \((\omega_1), (\omega_2), \text{ and } (\omega_3)\)

\((\omega_1)\) birth-death of monogons

\((\omega_2)\) birth-death of bigons

\((\omega_3)\) jumping over a double point

Figure 3: invariance under the move \((\omega_2)\)
We next show that $\mathcal{QIIS}(\ast_0, \ast_1)$ is a right $\mathcal{Q}\pi'(S)$-comodule, i.e., \((1\mathcal{QIIS}(\ast_0, \ast_1) \otimes (1 - T))(\mu \otimes 1\mathcal{Q}\pi(S)) \circ \mu = (1\mathcal{QIIS}(\ast_0, \ast_1) \otimes \delta) \circ \mu\) (see Appendix). Let $\gamma : [0, 1] \to S$ be an immersed path from $\ast_0$ to $\ast_1$ such that its self intersections consist of transverse double points. To compute \((1\mathcal{QIIS}(\ast_0, \ast_1) \otimes (1 - T))(\mu \otimes 1\mathcal{Q}\pi(S))\mu(\gamma)\), we need to compute $\mu(\gamma_{t_1^p} \gamma_{t_2^p})$ where $p \in \Gamma$. The double points of $\gamma_{t_1^p} \gamma_{t_2^p}$ come from those of $\gamma$. Let $q$ be a double point of $\gamma_{t_1^p} \gamma_{t_2^p}$ and denote $\gamma^{-1}(q) = \{t_1^p, t_2^p\}$, so that $t_1^p < t_2^p$. There are three possibilities: (i) $t_2^q < t_1^p < t_2^p$, (ii) $t_2^q < t_2^p < t_1^p$, (iii) $t_2^q < t_1^p < t_2^p < t_2^q$. In cases (i) and (ii), the contribution to \((\mu \otimes 1\mathcal{Q}\pi(S))\mu(\gamma)\) from $(p, q)$ is

$$
e(\hat{\gamma}(t_2^q), \hat{\gamma}(t_1^p))\ne(\hat{\gamma}(t_1^p), \hat{\gamma}(t_2^q)) (\gamma_{t_1^p} \gamma_{t_2^q} \gamma_{t_2^p} \gamma_{t_1^p}) \otimes \gamma_{t_1^p}^\prime \otimes \gamma_{t_1^p}^\prime \otimes \gamma_{t_1^p}^\prime,$$

respectively. Here $\gamma_{t_1^p}^\prime$ means the restriction of $\gamma$ to $[t_1^p, t_2^p]$ and $\gamma_{t_2^q} \gamma_{t_1^p} \gamma_{t_2^p} \gamma_{t_1^p}$ means the conjunction. Therefore the contributions to \((\mu \otimes 1\mathcal{Q}\pi(S))\mu(\gamma)\) from $(p, q)$ of type (i) or (ii) are written as a linear combination of tensors of the form $u \otimes (v \otimes w \otimes v)$. Since \((1 - T)(v \otimes w \otimes v) = 0\), these contributions vanish on $\mathcal{QIIS}(\ast_0, \ast_1) \otimes \mathcal{Q}\pi'(S) \otimes \mathcal{Q}\pi'(S)$. Hence we only need to consider the contributions from (iii), and

\[
(1\mathcal{QIIS}(\ast_0, \ast_1) \otimes (1 - T))(\mu \otimes 1\mathcal{Q}\pi(S))\mu(\gamma) = \sum_{\ell_1^q < t_1^p < t_2^p < t_2^q} \ne(\hat{\gamma}(t_2^q), \hat{\gamma}(t_1^p))\ne(\hat{\gamma}(t_1^p), \hat{\gamma}(t_2^q))\ne(\hat{\gamma}(t_2^q), \hat{\gamma}(t_1^p))z(\gamma, p, q),
\]

(2.2.3)

where $z(\gamma, p, q) = (\gamma_{t_1^p} \gamma_{t_2^q}) \otimes ((\gamma_{t_1^p} \gamma_{t_2^q})^\prime \otimes \gamma_{t_1^p}^\prime \otimes \gamma_{t_1^p}^\prime \otimes \gamma_{t_1^p}^\prime)$. On the other hand, to compute \((1\mathcal{QIIS}(\ast_0, \ast_1) \otimes \delta)\mu(\gamma)\) we need to compute $\delta(\gamma_{t_1^p}^\prime)$ where $p \in \Gamma$. Each double point of the loop $\gamma_{t_1^p} \gamma_{t_2^q}$ comes from $q \in \Gamma$ such that $t_1^p < t_1^q < t_2^p < t_2^q$. Thus $\delta(\gamma_{t_1^p}^\prime)$
Figure 5: the formula does not work if a basepoint lies in $\text{Int}(S)$

is equal to

$$\sum_{p,q} \varepsilon(\dot{\gamma}(t_1^p), \dot{\gamma}(t_2^q)) \left( |\gamma_0 \delta_q \gamma_1^p \gamma_1^q \gamma_0^q \gamma_0^p \gamma_0^q | \otimes |\gamma_0 \delta_q \gamma_1^p \gamma_1^q \gamma_0^q \gamma_0^p \gamma_0^q | - |\gamma_0 \delta_q \gamma_1^p \gamma_1^q \gamma_0^q \gamma_0^p \gamma_0^q | \otimes |\gamma_0 \delta_q \gamma_1^p \gamma_1^q \gamma_0^q \gamma_0^p \gamma_0^q | \right) + \varepsilon(\dot{\gamma}(t_1^p), \dot{\gamma}(t_2^q)) w(\gamma, p, q),$$

(2.2.4)

where $w(\gamma, p, q) = (\gamma_0 \delta_q \gamma_1^p \gamma_0^q \gamma_1^p \gamma_0^q | \otimes |\gamma_0 \delta_q \gamma_1^p \gamma_1^q \gamma_0^q \gamma_0^p \gamma_0^q | - |\gamma_0 \delta_q \gamma_1^p \gamma_1^q \gamma_0^q \gamma_0^p \gamma_0^q | \otimes |\gamma_0 \delta_q \gamma_1^p \gamma_1^q \gamma_0^q \gamma_0^p \gamma_0^q | \right)$. Since $z(\gamma, p, q) = -w(\gamma, q, p)$, the right hand sides of (2.2.3) and (2.2.4) are equal. This completes the proof. \hfill \Box

Remark 2.2.2. We have taken $*_0$ and $*_1$ from $\partial S$ and assumed $*_0 \neq *_1$. If at least one of $*_0$ and $*_1$ lies in $\text{Int}(S)$, we need to consider another kind of local move illustrated in Figure 5. In this case the formula (2.2.2) does not work. For example, in Figure 5, the contribution from $p$ in the left picture is non-trivial. Also, the formula (2.2.2) needs to be fixed when $*_0 = *_1$. See [2.3]

We show that $\sigma$ and $\mu$ satisfy the compatibility (A.2.2) and the involutivity (A.2.3).

Proposition 2.2.3. The vector space $\mathbb{QILS}(*, *)$ is an involutive right $\mathbb{Q\hat{\sigma}(S)}$-bimodule with respect to $\sigma: \mathbb{Q\hat{\sigma}(S)} \otimes \mathbb{QILS}(*, *) \to \mathbb{QILS}(*, *)$ and $\mu: \mathbb{QILS}(*, *) \to \mathbb{QILS}(*, * \otimes \mathbb{Q\hat{\sigma}(S})$.

Proof. We first prove the involutivity. Let $\gamma: [0, 1] \to S$ be an immersed path from $*_0$ to $*_1$ such that its self intersections consist of transverse double points. Then

$$\overline{\sigma} \mu(\gamma) = \sum_{p \in \Gamma} \varepsilon(\dot{\gamma}(t_1^p), \dot{\gamma}(t_2^q)) \sigma \left( |\gamma_0 t_1^p \gamma_1^p \gamma_0^p | \otimes |\gamma_0 t_1^p \gamma_1^p \gamma_0^p | \right).$$

Let $q$ be an intersection of the loop $\gamma_0 t_1^p$ and the path $\gamma_0 t_1^p \gamma_1^p \gamma_0^p$ and we denote $\gamma^{-1}(q) = \{t_1^q, t_2^q\}$ so that $t_1^q < t_2^q$. There are two possibilities: (i) $t_1^q < t_1^p < t_2^q < t_2^p$; (ii) $t_1^q < t_1^p < t_2^p < t_2^p$. The contribution to $\overline{\sigma} \mu(\gamma)$ from $(p, q)$ are

$$\varepsilon(\dot{\gamma}(t_1^p), \dot{\gamma}(t_2^q)) \varepsilon(\dot{\gamma}(t_2^q), \dot{\gamma}(t_1^p)) \gamma_0 t_1^q \gamma_1^p \gamma_1^q \gamma_0^p \gamma_1^q \gamma_0^p$$

(2.2.5)
in case (i), and
\[ \varepsilon(\gamma(t^2_1), \gamma(t^2_0)) \varepsilon(\gamma(t^1_1), \gamma(t^1_0)) \gamma_{pq} \gamma_{pq} \gamma_{pq} \gamma_{pq} \gamma_{pq} \gamma_{pq} \]
(2.2.6)
in case (ii). If we interchange \( \alpha \) in case (i), and \( \gamma \) in case (ii). If we interchange \( p \) and \( q \) in (2.2.6), we get the minus of (2.2.5). Therefore the contributions from \((p, q)\) in case (i) and those in case (ii) cancel and \( \alpha \mu(\gamma) = 0 \).

We next show the compatibility. Let \( \alpha \) be an immersed loop on \( S \) and \( \gamma : [0, 1] \rightarrow S \) an immersed path from \( *_0 \) to \( *_1 \) such that their intersections and self intersections consist of transverse double points. The compatibility is equivalent to the following.

\[ \mu(\alpha \times \gamma) = \sigma(\alpha) \gamma(\gamma) - (\sigma \otimes 1_{Q^2(S)})(\gamma \otimes \delta(\alpha)). \]

(2.2.7)

Here, \( \sigma(\alpha) \mu(\gamma) = (\sigma \otimes 1_{Q^2(S)})(\alpha \otimes \mu(\gamma)) + (1_{Q^2(S)} \otimes \omega(\alpha)) \mu(\gamma) \). We compute the left hand side of (2.2.7). First of all, we have

\[ \mu(\alpha \times \gamma) = \sum_{\gamma \in \alpha \times \gamma} \varepsilon(p; \alpha, \gamma) \mu(\gamma_{\alpha p} \gamma_{p \gamma}). \]

Let \( q \) be a double point of \( \gamma_{\alpha p} \gamma_{p \gamma} \). There are three possibilities: (i) \( q \) comes from a double point of \( \alpha \), (ii) \( q \) comes from a double point of \( \gamma \), (iii) \( q \) comes from an intersection of \( \alpha \) and \( \gamma \), which is different from \( p \).

Suppose \( q \) comes from a double point of \( \alpha \). We denote \( \alpha^{-1}(p) = \{ t^0_1, t^0_2 \} \subset S^1 \), so that \( t^0_2, \alpha^{-1}(p), t^0_1 \) are arranged in this order according to the orientation of \( S^1 \). (Since \( \alpha \) is a simple point of \( \alpha \), the preimage \( \alpha^{-1}(p) \) consists of one point. For simplicity, we write \( \alpha^{-1}(p) \) for the unique point in the preimage.) The contribution to \( \mu(\gamma_{\alpha p} \gamma_{p \gamma}) \) from such \( q \) is

\[ -\varepsilon(\dot{\alpha}(t^0_1), \dot{\alpha}(t^0_2)) \gamma_{\alpha p} |/ \gamma_{p \gamma} \gamma_{p \gamma} \gamma_{p \gamma} \gamma_{p \gamma} \gamma_{p \gamma} \gamma_{p \gamma} \gamma_{p \gamma} \gamma_{p \gamma}. \]

(2.2.8)

Here, \( \alpha_{pq} \) (resp. \( \alpha_{qp} \)) means the restriction of \( \alpha \) to the interval \([\alpha^{-1}(p), t^1_1] \) (resp. \([t^1_2, \alpha^{-1}(p)] \)).

Thus the contributions to \( \mu(\alpha \times \gamma) \) from \((p, q)\) such that \( q \) is of type (i) is

\[ -\sum_{p \in \alpha \times \gamma} \sum_{(t^0_1, t^0_2)} \varepsilon(p; \alpha, \gamma) \varepsilon(\dot{\alpha}(t^0_1), \dot{\alpha}(t^0_2)) \gamma_{\alpha p} \gamma_{p \gamma} \gamma_{p \gamma} \gamma_{p \gamma} \gamma_{p \gamma} \gamma_{p \gamma} \gamma_{p \gamma} \gamma_{p \gamma}. \]

(2.2.8)

where the second sum is taken over ordered pairs \((t^0_1, t^0_2)\) such that \( \alpha(t^0_1) = \alpha(t^0_2) \) and \( \alpha^{-1}(p) \in [t^0_2, t^0_1] \). On the other hand, we have

\[ \delta(\alpha) = \sum_{(t^0_1, t^0_2)} \varepsilon(\dot{\alpha}(t^0_1), \dot{\alpha}(t^0_2)) |/ \gamma_{t^0_1} \gamma_{t^0_2} \gamma_{t^0_2} \gamma_{t^0_1} \gamma_{t^0_1} \gamma_{t^0_2} \gamma_{t^0_2} \gamma_{t^0_1}|, \]

where the sum is taken over ordered pairs \((t^0_1, t^0_2)\) such that \( \alpha(t^0_1) = \alpha(t^0_2) \), \( t^0_1 \neq t^0_2 \), and

\[ \sigma(\gamma_{\alpha p} \gamma_{p \gamma}) = \sum_p \varepsilon(p; \alpha, \gamma) \gamma_{\alpha p} \gamma_{p \gamma} \gamma_{p \gamma} \gamma_{p \gamma} \gamma_{p \gamma} \gamma_{p \gamma} \gamma_{p \gamma} \gamma_{p \gamma}, \]

where the sum is taken over \( p \in \alpha \cap \gamma \) such that \( \alpha^{-1}(p) \in [t^0_2, t^0_1] \). Therefore, (2.2.8) is equal to \(-\sigma \otimes 1_{Q^2(S)}(\gamma \otimes \delta(\alpha)). \)

Suppose \( q \) comes from a double point of \( \gamma \). We denote \( \gamma^{-1}(q) = \{ s^0_1, s^0_2 \} \), so that \( s^0_1, s^0_2 \) are arranged in this order according to the orientation of \( S^1 \). There are three possibilities: (i-a) \( \gamma^{-1}(p) < s^0_1 < s^0_2 \), (ii-b) \( s^0_1 < \gamma^{-1}(p) < s^0_2 \), (ii-c) \( s^0_1 < s^0_2 < \gamma^{-1}(p) \). The contributions to \( \mu(\alpha \times \gamma) \) from \((p, q)\) of type (ii-a) are

\[ \sum_{p \in \alpha \cap \gamma} \sum_{q \in \gamma^{-1}(p) \cap (s^0_1, s^0_2)} \varepsilon(p; \alpha, \gamma) \varepsilon(\gamma(s^0_1), \gamma(s^0_2)) \gamma_{\alpha p} \gamma_{p \gamma} \gamma_{p \gamma} \gamma_{p \gamma} \gamma_{p \gamma} \gamma_{p \gamma} \gamma_{p \gamma} \gamma_{p \gamma}. \]
Figure 6: the type (iii)

\[ \gamma^{-1}(p) < \gamma^{-1}(q) \]
and those from \((p, q)\) of type (ii-c) are

\[ \sum_{p \in \alpha \cap \gamma} \sum_{s_1^q < s_2^q < \gamma^{-1}(p)} \varepsilon(p; \alpha, \gamma) \varepsilon(q; s_1^q, s_2^q) \gamma_{s_1^q q} \gamma_{p q} \alpha_p \gamma_{p+1} \otimes \gamma_{s_2^q s_1^q}. \]

The sum of these two is equal to \((\sigma \otimes 1_\mathcal{Q}(S))(\alpha \otimes \mu(\gamma))\). The contributions to \(\mu(\sigma(\alpha \otimes \gamma))\) from \((p, q)\) of type (ii-b) are

\[ \sum_{p \in \alpha' \cap \gamma} \sum_{s_1^q < s_2^q < \gamma^{-1}(p)} \varepsilon(p; \alpha, \gamma) \varepsilon(q; s_1^q, s_2^q) \gamma_{s_1^q q} \gamma_{p q} \alpha_p \gamma_{p q}. \]

This is equal to \((1_\mathcal{QNS}(s_0, s_1) \otimes \text{ad}(\alpha))\mu(\gamma)\). Therefore, the contributions from \((p, q)\) such that \(q\) is of type (ii) is \((\sigma \otimes 1_\mathcal{Q}(S))(\alpha \otimes \mu(\gamma)) + (1_\mathcal{QNS}(s_0, s_1) \otimes \text{ad}(\alpha))\mu(\gamma) = \sigma(\alpha)\mu(\gamma)\).

Suppose \(q\) comes from an intersection of \(\alpha\) and \(\gamma\), which is different from \(p\). If \(\gamma^{-1}(p) < \gamma^{-1}(q)\), the contribution is

\[ -\varepsilon(p; \alpha, \gamma) \varepsilon(q; \alpha, \gamma) \gamma_{s_0 q} \alpha_{pq} \gamma_{q+1} \otimes \alpha_{pq} \gamma_{pq}. \quad (2.2.9) \]

and if \(\gamma^{-1}(q) < \gamma^{-1}(p)\), the contribution is

\[ -\varepsilon(p; \alpha, \gamma) \varepsilon(q; \alpha, \gamma) \gamma_{s_0 q} \alpha_{pq} \gamma_{p+1} \otimes \gamma_{pq} \alpha_{pq}. \quad (2.2.10) \]

See Figure 6. If we interchange \(p\) and \(q\) in (2.2.10), we get the minus of (2.2.9). Therefore the sum of the contributions from \((p, q)\) of type (iii) is zero. We have established the formula (2.2.7).

### 2.3 The case of \(s_0 = s_1\)

Take \(s \in \partial S\). We shall give a structure of an involutive right \(\mathcal{Q}_n'(S)\)-bimodule on the vector space \(\mathcal{Q}_n(S, \alpha) = \mathcal{Q}\mathcal{I}\mathcal{S}(\alpha, \beta)\). In fact, as we will see, there are two possibilities for the structure morphism of \(\mathcal{Q}_n'(S)\)-comodules: \(\mu_+, \mu_- : \mathcal{Q}_n(S, \alpha) \to \mathcal{Q}_n(S, \alpha) \otimes \mathcal{Q}_n'(S)\).

**Definition of \(\sigma\).** The \(\mathbb{Q}\)-linear map \(\sigma : \mathcal{Q}_n'(S) \otimes \mathcal{Q}_n(S, \alpha) \to \mathcal{Q}_n(S, \alpha)\) is defined by letting \(s_0 = s_1\) and applying the formula (2.2.1).

**Definition of \(\mu_+\) and \(\mu_-\).** We regard that the orientation of \(\partial S\) is induced from that of \(S\). Let \(\ell_+\) and \(\ell_-\) be embeddings from \([-1, 0]\) to \((\partial S, \alpha)\) such that the tangent vectors \(\ell_+'(0)\) and \(-\ell_-'(0)\) agree with the orientation of \(\partial S\). We denote \(\ell_+'(1) = s_1\) and \(\ell_-'(1) = s_0\). See Figure 7.
We have the isomorphisms $c_+ : \mathbb{Q}\pi_1(S,*) \to \mathbb{Q}\Pi S(\ast, \ast_+)$ and $c_- : \mathbb{Q}\pi_1(S,*) \to \mathbb{Q}\Pi S(\ast, \ast_-)$ given by $c_+(\gamma) = \gamma \ell_+$ and $c_-(\gamma) = \gamma \ell_-$. We define $\mu_+$ (resp. $\mu_-$) to be $(c_+^{-1} \otimes 1_{\mathbb{Q}\pi(S)}) \circ \mu \circ c_+$ (resp. $(c_-^{-1} \otimes 1_{\mathbb{Q}\pi(S)}) \circ \mu \circ c_-)$). Namely, we define $\mu_+$ so that the diagram

$$
\begin{array}{c}
\mathbb{Q}\pi_1(S,*) \xrightarrow{\mu_+} \mathbb{Q}\pi_1(S,*) \otimes \mathbb{Q}\hat{\pi}'(S) \\
\downarrow c_+ \\
\mathbb{Q}\Pi S(\ast, \ast_+) \xrightarrow{\mu} \mathbb{Q}\Pi S(\ast, \ast_+) \otimes \mathbb{Q}\hat{\pi}'(S)
\end{array}
$$

(2.3.1)

commutes, and define $\mu_-$ similarly.

We derive formulas for $\mu_+$ and $\mu_-$ similar to (2.2.2). Let $\gamma : [0, 1] \to S$ be an immersed path with $\gamma(0) = \gamma(1) = \ast$ such that its self intersections consist of transverse double points and the velocity vectors $\dot{\gamma}(0)$ and $\dot{\gamma}(1)$ are linearly independent. Let $\Gamma \subset \text{Int}(S)$ be the set of double points of $\gamma$ except $\ast$. There exists a small $\varepsilon > 0$ such that $\gamma^{-1}(\Gamma)$ is contained in $[0, 1 - \varepsilon]$. We can choose a representative of $c_+(\gamma)$ such that its restriction to $[0, 1 - \varepsilon]$ coincides with $\gamma|_{[0, 1 - \varepsilon]}$. Moreover, if $\varepsilon(\dot{\gamma}(0), \dot{\gamma}(1)) = +1$, i.e., $(\dot{\gamma}(0), \dot{\gamma}(1))$ gives the orientation of $S$, we can assume that the set of double points of the representative is $\Gamma$. If $\varepsilon(\dot{\gamma}(0), \dot{\gamma}(1)) = -1$, we can assume that the set of double points of the representative is the union of $\Gamma$ and an additional point, whose contribution to $\mu(c_+\gamma)$ is $\ell_+ \otimes |\gamma'|$. See Figure 8. Therefore we have

$$
\mu_+(\gamma) = \begin{cases} 
- \sum_{p \in \Gamma} \varepsilon(\dot{\gamma}(t_1^p), \dot{\gamma}(t_2^p))(\gamma_{0t_1^p}\gamma_{0t_2^p}) \otimes |\gamma_{t_1^p}^{t_2^p}'|', & \text{if } \varepsilon(\dot{\gamma}(0), \dot{\gamma}(1)) = +1 \\
1 \otimes |\gamma'| - \sum_{p \in \Gamma} \varepsilon(\dot{\gamma}(t_1^p), \dot{\gamma}(t_2^p))(\gamma_{0t_1^p}\gamma_{0t_2^p}) \otimes |\gamma_{t_1^p}^{t_2^p}'|', & \text{if } \varepsilon(\dot{\gamma}(0), \dot{\gamma}(1)) = -1.
\end{cases}
$$

(2.3.2)

Similarly, we have

$$
\mu_-(\gamma) = \begin{cases} 
-1 \otimes |\gamma'| - \sum_{p \in \Gamma} \varepsilon(\dot{\gamma}(t_1^p), \dot{\gamma}(t_2^p))(\gamma_{0t_1^p}\gamma_{0t_2^p}) \otimes |\gamma_{t_1^p}^{t_2^p}'|', & \text{if } \varepsilon(\dot{\gamma}(0), \dot{\gamma}(1)) = +1 \\
- \sum_{p \in \Gamma} \varepsilon(\dot{\gamma}(t_1^p), \dot{\gamma}(t_2^p))(\gamma_{0t_1^p}\gamma_{0t_2^p}) \otimes |\gamma_{t_1^p}^{t_2^p}'|', & \text{if } \varepsilon(\dot{\gamma}(0), \dot{\gamma}(1)) = -1.
\end{cases}
$$

(2.3.3)

In particular, we see that

$$
\mu_+(\gamma) - \mu_-(\gamma) = 1 \otimes |\gamma'|
$$

(2.3.4)

for any $\gamma \in \mathbb{Q}\pi_1(S, \ast)$.  

Figure 7: $\ell_+$ and $\ell_-$

$\includegraphics[width=0.5\textwidth]{figure7}$
Figure 8: computation of $\mu_+$

the case $\varepsilon(\gamma'(0), \gamma'(1)) = +1$

\[ \gamma \quad \Rightarrow \quad \Rightarrow \]  

the case $\varepsilon(\gamma'(0), \gamma'(1)) = -1$

\[ \gamma \quad \Rightarrow \quad \Rightarrow \]

Figure 9: changing the sign of $\varepsilon(\dot{\gamma}(0), \dot{\gamma}(1))$

Remark 2.3.1. Our construction is closely related to Turaev’s self intersection $\mu = \mu^T : \pi_1(S, *) \to \mathbb{Z}\pi_1(S, *)$ introduced in [18] §1.4. Actually, for any $\gamma \in \pi_1(S, *)$, we have

$$\mu^T(\gamma) \gamma = -(1_{\mathbb{Q}\pi_1(S, *)} \otimes \varepsilon) \mu_+ (\gamma). \quad (2.3.5)$$

Here $\varepsilon : \mathbb{Q}\hat{\pi}'(S) \to \mathbb{Q}$ is the $\mathbb{Q}$-linear map given by $\varepsilon(\alpha) = 1$ for $\alpha \in \hat{\pi}'(S)$.

Proposition 2.3.2. Both the pairs $(\sigma, \mu_+)$ and $(\sigma, \mu_-)$ define an involutive right $\mathbb{Q}\hat{\pi}'(S)$-bimodule structure on the vector space $\mathbb{Q}\pi_1(S, *)$.

Proof. By the commutativity of the diagram (2.3.1) and Proposition 2.2.1 it follows that $\mu_+$ defines a right $\mathbb{Q}\hat{\pi}'(S)$-comodule structure on $\mathbb{Q}\pi_1(S, *)$. Also since $c_+$ is compatible with $\sigma$, i.e., the action of $\mathbb{Q}\hat{\pi}'(S)$, Proposition 2.2.3 implies the compatibility and the involutivity of $(\sigma, \mu_+)$. The assertion for $(\sigma, \mu_-)$ follows similarly.

Another proof is obtained by the following observation: by applying a move illustrated in Figure 9 if necessary, any $\gamma \in \pi_1(S, *)$ is represented by an immersed path with $\varepsilon(\dot{\gamma}(0), \dot{\gamma}(1)) = +1$. Then $\mu_+(\gamma)$ is given by the first formula of (2.3.2). Applying the proof of Proposition 2.2.1 and 2.2.3 verbatim, we get the assertion for $(\sigma, \mu_+)$. \qed
Remark 2.3.3. More generally, for $t \in \mathbb{Q}$, consider the linear combination $\mu_t = ((1 + t)/2)\mu_+ + ((1-t)/2)\mu_-$ (so that $\mu_1 = \mu_+$ and $\mu_{-1} = \mu_-$. By (2.3.4) and $\mu_+(1) = \mu_-(1) = 0$, we see that $(\sigma, \mu_t)$ defines an involutive right $\widehat{\mathbb{Q}\pi'}(S)$-bimodule structure on $\widehat{\mathbb{Q}\pi_1}(S, \ast)$.

3 Conclusion of the Turaev cobracket

The vector space $\mathbb{Q}\widehat{\pi}(S)$ has a natural decreasing filtration and we can consider the completion $\widehat{\mathbb{Q}\pi}(S)$. As is shown in [9] §4, the Goldman bracket induces to a Lie bracket on $\widehat{\mathbb{Q}\pi}(S)$. In this section we show that $\mu$ is compatible with the filtrations of $\mathbb{Q}\Pi(S_{\ast_0}, \ast_1)$ and $\mathbb{Q}\pi'(S)$, and also show that the Turaev cobracket extends to a complete Lie cobracket

$$\delta: \mathbb{Q}\widehat{\pi}(S) \to \mathbb{Q}\widehat{\pi}(S) \otimes \mathbb{Q}\widehat{\pi}(S).$$

3.1 Completion of the Goldman Lie algebra

We make a few remarks on filtered vector spaces. Let $V = F_0V \supset F_1V \supset \cdots$ be a filtered $\mathbb{Q}$-vector space. The projective limit $\widehat{V} := \lim_{\to} V/F_nV$ is again a filtered $\mathbb{Q}$-vector space with the filter $F_n\widehat{V} := \text{Ker}(\widehat{V} \to V/F_nV)$. We say $V$ is complete if the natural map $V \to \widehat{V}$ is isomorphic. If $V$ and $W$ are filtered $\mathbb{Q}$-vector spaces, the tensor product $V \otimes W$ is naturally filtered by $F_n(V \otimes W) = \sum_{p+q=n} F_pV \otimes F_qW$. The complete tensor product $V \widehat{\otimes} W$ is defined as $V \widehat{\otimes} W := \widehat{V} \otimes \widehat{W} = \lim_{\to} V \otimes W/F_n(V \otimes W)$. Note that we have a natural isomorphism $\widehat{V} \widehat{\otimes} \widehat{W} \cong V \widehat{\otimes} W$.

**Definition 3.1.1.** A complete Lie algebra is a pair $(V, \nabla)$, where $V$ is a complete filtered $\mathbb{Q}$-vector space and $\nabla: V \widehat{\otimes} V \to V$ is a $\mathbb{Q}$-linear map continuous with respect to the topologies coming from the filtrations, and satisfies the skew condition $\nabla T = -\nabla: V \widehat{\otimes} V \to V$ and the Jacobi identity $\nabla(\nabla \otimes 1) = 0: V \widehat{\otimes} V \widehat{\otimes} V \to V$. Here, $T: V \widehat{\otimes} V \to V \widehat{\otimes} V$ is that induced from $T: V \otimes V \to V \otimes V$, etc. $\nabla$ is called a complete Lie bracket. Similarly, we define a complete Lie coalgebra, bialgebra, and a complete $V$-module, comodule, and bimodule.

Let $S$ be a connected oriented surface. Take some base point $\ast \in S$ and set

$$\mathbb{Q}\widehat{\pi}(S)(n) := |\mathbb{Q}1 + I\pi_1(S, \ast)^n| \subset \mathbb{Q}\widehat{\pi}(S), \text{ for } n \geq 0.$$ 

Here, $I\pi_1(S, \ast) := \text{Ker}(\mathbb{Q}\pi_1(S, \ast) \to \mathbb{Q}, \pi \ni x \mapsto 1)$ is the augmentation ideal of the group ring $\mathbb{Q}\pi_1(S, \ast)$. We regard $I\pi_1(S, \ast)^0 = \mathbb{Q}\pi_1(S, \ast)$. The space $\mathbb{Q}\pi(S)(n)$ is independent of the choice of $\ast$. Moreover, the Goldman bracket satisfies

$$[\mathbb{Q}\pi(S)(n_1), \mathbb{Q}\pi(S)(n_2)] \subset \mathbb{Q}\pi(S)(n_1 + n_2 - 2), \text{ for } n_1, n_2 \geq 1 \quad (3.1.1)$$

(see [9] §4.1). This implies that the Goldman bracket induces a complete Lie bracket on the projective limit

$$\widehat{\mathbb{Q}\pi}(S) := \lim_{\to} \mathbb{Q}\pi(S)/\mathbb{Q}\pi(S)(n).$$

We call $\widehat{\mathbb{Q}\pi}(S)$ the completed Goldman Lie algebra of $S$. We denote

$$\widehat{\mathbb{Q}\pi}(S)(n) := F_n\widehat{\mathbb{Q}\pi}(S) = \text{Ker}(\widehat{\mathbb{Q}\pi}(S) \to \mathbb{Q}\pi(S)/\mathbb{Q}\pi(S)(n)), \text{ for } n \geq 0.$$
For \( n \geq 0 \), let
\[
\hat{Q}_n'(S)(n) := \varpi(\hat{Q}_n(S)(n)) = |I_{\pi_1}(S, *)^n|' \subset Q_{\hat{n}}'(S).
\]
Since \(|1'| = 0\), \( Q_{\hat{n}}'(S)(0) = Q_{\hat{n}}'(S)(1) = Q_{\hat{n}}'(S) \). The natural map \( Q_{\hat{n}}'(S)/Q_{\hat{n}}'(S)(n) \to Q_{\hat{n}}'(S)/Q_{\hat{n}}'(S)(n) \) is a \( \mathbb{Q} \)-linear isomorphism for any \( n \). Hence \( \hat{Q}_n(S) \) is also written as
\[
\hat{Q}_n(S) = \lim_{n} Q_{\hat{n}}'(S)/Q_{\hat{n}}'(S)(n).
\]  
(3.1.2)

Let \( s_0, s_1 \in \partial S \) be as in \( \S 2.2 \). We make \( \Pi(S_0, s_1) \) filtered by taking some path \( \gamma \in \Pi(S_0, s_1) \) and setting
\[
F_n\Pi(S_0, s_1) := \gamma I_{\pi_1}(S, s_1)^n, \quad \text{for } n \geq 0.
\]  
(3.1.3)

Then this is independent of the choice of \( \gamma \) (see \[9\] Proposition 2.1.1). In particular we can consider the completion \( \hat{\Pi}(S_0, s_1) \). By \[9\] \( \S 4.1 \), we see that \( \sigma \) induces a \( \mathbb{Q} \)-linear map \( \hat{\Pi}(S, s_1) \otimes \hat{\Pi}(S_0, s_1) \to \hat{\Pi}(S_0, s_1) \), and the complete vector space \( \hat{\Pi}(S_0, s_1) \) has a structure of a complete right \( \hat{Q}_{\hat{n}}(S) \)-module. As a special case, the completed group ring \( \hat{\Pi}_1(S, s_1) := \lim_{n} \Pi_1(S, s_1)/(I_{\pi_1}(S, s_1))^n \), where \( s \in \partial S \), has a structure of a complete right \( \hat{Q}_{\hat{n}}(S) \)-module.

### 3.2 Intersection of paths

We would like to show \( \mu : \Pi(S_0, s_1) \to \Pi(S_0, s_1) \otimes \hat{Q}_n'(S) \) is compatible with the filtrations. For this purpose we introduce another map \( \kappa \), which measures the intersections of two paths.

Take four distinct points \( s_1, s_2, s_3, s_4 \in \partial S \). Let \( x, y : [0, 1] \to S \) be immersed paths such that \( x(0) = s_1, x(1) = s_2, y(0) = s_3, y(1) = s_4 \) and their intersections consist of transverse double points. Set
\[
\kappa(x, y) := -\sum_{p \in x \cap y} \varepsilon(p, x, y)(x_{\ast p}y_{\ast p}) \otimes (y_{\ast p}x_{\ast p}) \in \Pi(S_1, s_4) \otimes \Pi(S_3, s_2).
\]  
(3.2.1)

By an argument similar to the proof of Proposition 2.2.1, we see that (3.2.1) gives rise to a well-defined \( \mathbb{Q} \)-linear map
\[
\kappa : \Pi(S_1, s_2) \otimes \Pi(S_3, s_4) \to \Pi(S_1, s_1) \otimes \Pi(S_3, s_2).
\]

Next take three distinct points \( s, s_1, s_2 \in \partial S \). We shall introduce the degenerate versions of \( \kappa \), i.e., \( \mathbb{Q} \)-linear maps \( \kappa_+ \), \( \kappa_- : \Pi(S_1, s_1) \otimes \Pi(S_3, s_2) \to \Pi(S_1, s_1) \otimes \Pi(S_3, s_2) \). Let \( \ell_+ : ([0, 1], 0) \to (\partial S, s) \) be embedded paths as in \( \S 2.3 \). We assume that the images of \( \ell_+ \) or \( \ell_- \) do not contain \( s_1 \) and \( s_2 \). We have the isomorphisms \( c_+ : \Pi(S_1, s_1) \to \Pi(S_1, s_1) \) and \( c_- : \Pi(S_1, s_1) \to \Pi(S_1, s_1) \) given by \( c_+(\gamma) = \gamma \ell_+ \) and \( c_-(\gamma) = \gamma \ell_- \). We define \( \kappa_+ \) (resp. \( \kappa_- \)) to be \( (1_{\Pi(S_1, s_2)} \otimes c_+^{-1}) \circ \kappa \circ (c_+ \otimes 1_{\Pi(S_1, s_2)}) \) (resp. \( (1_{\Pi(S_1, s_2)} \otimes c_-^{-1}) \circ \kappa \circ (c_- \otimes 1_{\Pi(S_1, s_2)}) \)). Namely, we define \( \kappa_+ \) so that the diagram
\[
\begin{array}{ccc}
\Pi(S_1, s_1) \otimes \Pi(S_1, s_1) & \xrightarrow{\kappa} & \Pi(S_1, s_1) \otimes \Pi(S_1, s_1) \\
\downarrow_{c_+ \otimes 1_{\Pi(S_1, s_2)}} & & \downarrow_{1_{\Pi(S_1, s_2)}} \\
\Pi(S_1, s_1) \otimes \Pi(S_1, s_1) & \xrightarrow{\kappa} & \Pi(S_1, s_1) \otimes \Pi(S_1, s_1)
\end{array}
\]
commutes, and define $\kappa_-$ similarly.

Let $x, y : [0, 1] \to S$ be immersed paths with $x(0) = *_1$, $x(1) = y(0) = *$, $y(1) = *_2$, such that their intersections except * consist of transverse double points and the velocity vectors $\dot{x}(1)$ and $\dot{y}(0)$ are linearly independent. By a similar way to [23], we derive the following formulas.

\[
\kappa_+(x, y) = \begin{cases} 
-xy \otimes 1 - \sum_{p \in x \cap y \setminus \{\ast\}} \varepsilon(p; x, y)(x_{s_p} y_{p_2}) \otimes (y_{s_p} x_{p_2}), & \text{if } \varepsilon(\dot{x}(1), \dot{y}(0)) = +1 \\
- \sum_{p \in x \cap y \setminus \{\ast\}} \varepsilon(p; x, y)(x_{s_p} y_{p_2}) \otimes (y_{s_p} x_{p_2}), & \text{if } \varepsilon(\dot{x}(1), \dot{y}(0)) = -1.
\end{cases}
\]

\[
\kappa_-(x, y) = \begin{cases} 
- \sum_{p \in x \cap y \setminus \{\ast\}} \varepsilon(p; x, y)(x_{s_p} y_{p_2}) \otimes (y_{s_p} x_{p_2}), & \text{if } \varepsilon(\dot{x}(1), \dot{y}(0)) = +1 \\
x y \otimes 1 - \sum_{p \in x \cap y \setminus \{\ast\}} \varepsilon(p; x, y)(x_{s_p} y_{p_2}) \otimes (y_{s_p} x_{p_2}), & \text{if } \varepsilon(\dot{x}(1), \dot{y}(0)) = -1.
\end{cases}
\]

In particular, we have

\[
\kappa_+(x, y) - \kappa_-(x, y) = -xy \otimes 1
\]

for any $x \in \mathbb{Q} \Pi S(*_1, *)$ and $y \in \mathbb{Q} \Pi S(*, *_2)$.

\textbf{Remark 3.2.1.} To define $\kappa_-$, we have taken $*_i$ from $\partial S$. By the same reason as is explained in Remark 2.2.2, the formula \[\text{(3.2.1)}\] does not work if at least one of $*_i$ lies in $\text{Int}(S)$. Nevertheless, for four distinct points $*_1, *_2, *_3, *_4 \in S$, \[\text{(3.2.1)}\] defines a well-defined $\mathbb{Q}$-linear map $\kappa : \mathbb{Q} \Pi S(*_1, *_2) \otimes \mathbb{Q} \Pi S(*_3, *_4) \to \mathbb{Q} \Pi S(*_1, *_3) \otimes \mathbb{Q} \Pi S(*_2, *_4)$, where $S_{ij} = S \setminus \{*_i, *_j\}$.

\textbf{Remark 3.2.2.} The bilinear pairing $\kappa$ is closely related to Turaev's intersection $\lambda : \mathbb{Z} \pi_1(S, \ast) \otimes \mathbb{Z} \pi_1(S, \ast) \to \mathbb{Z} \pi_1(S, \ast)$ introduced in [IS] §1.4. Take distinct points $*, *' \in \partial S$. Letting $*_1 = *_2 = *$, $*_3 = *_4 = *'$ and applying \[\text{(3.2.1)}\], we get a $\mathbb{Q}$-linear map

\[
\kappa : \mathbb{Q} \pi_1(S, *) \otimes \mathbb{Q} \pi_1(S, *)' \to \mathbb{Q} \Pi S(*, *) \otimes \mathbb{Q} \Pi S(*', *)
\]

(using the same letter $\kappa$). Let $* \in \partial S$ be a base point of $S$ and $\ell_+ : ([0, 1], 0) \to (\partial S, *)$ an embedded path as in [23]. Then for $x, y \in \pi_1(S, *)$, we have

\[
\lambda(x, y) = -(c_+^{-1} \otimes \varepsilon)\kappa(1_{\mathbb{Q} \pi_1(S, *)} \otimes c_+)(x, y^{-1}).
\]

Here, $(1_{\mathbb{Q} \pi_1(S, *)} \otimes c_+) : \mathbb{Q} \pi_1(S, *) \otimes \mathbb{Q} \pi_1(S, *) \to \mathbb{Q} \pi_1(S, *) \otimes \mathbb{Q} \pi_1(S, *)$ is given by $x \otimes y \mapsto x \otimes \ell_+^{-1} y \ell_+$ for $x, y \in \pi_1(S, *)$, and $\varepsilon : \mathbb{Q} \Pi S(*)_+ \to \mathbb{Q}$ is the $\mathbb{Q}$-linear map given by $\varepsilon(\gamma) = 1$ for $\gamma \in \Pi S(*_+, *)$.

\subsection*{3.3 Product formulas}

\textbf{Lemma 3.3.1.} Let $*_1, *_2, *_3, *_4 \in \partial S$ be distinct four points. For any $x \in \mathbb{Q} \Pi S(*_1, *_2)$, $y \in \mathbb{Q} \Pi S(*_2, *_3)$, and $z \in \mathbb{Q} \Pi S(*_3, *_4)$, we have

\[
\kappa_+(xy, z) = \kappa(x, z)(1 \otimes y) + (x \otimes 1)\kappa_+(y, z) \in \mathbb{Q} \Pi S(*_1, *_4) \otimes \mathbb{Q} \pi_1(S, *_3)
\]

\[
\kappa_+(x, yz) = \kappa_+(x, y)(z \otimes 1) + (1 \otimes y)\kappa(x, z) \in \mathbb{Q} \Pi S(*_1, *_4) \otimes \mathbb{Q} \pi_1(S, *_2).
\]

14
Here, κ(x, z)(1 ⊗ y) means the image of κ(x, z) ⊗ y by the map QIPS(∗1, ∗4) ⊗ QIPS(∗3, ∗2) ⊗ QIPS(∗2, ∗3) → QIPS(∗1, ∗4) ⊗ QΠ1(S, ∗1), u ⊗ v ⊗ w → u ⊗ vw, etc. Similar formulas for κ− are obtained by replacing κ+ with κ− in the formulas above.

Proof. We only prove the formula κ+(xy, z) = κ(x, z)(1 ⊗ y) + (x ⊗ 1)κ+(y, z). The other formulas are proved similarly. Let x, y, z : [0, 1] → S be immersed paths with x(0) = ∗1, x(1) = y(0) = ∗2, y(1) = z(0) = ∗3, z(1) = ∗4, such that their intersections except ∗2 and ∗3 consist of transverse double points. Moreover, we assume that ε(˙x(1), ˙y(0)) = ε(˙y(1), ˙z(0)) = −1.

Applying the second formula of (3.2.2), we compute κ+(xy, z) as

\[ κ+(xy, z) = - \sum_{p ∈ (xy)∩\{∗3\}} ε(p; xy, z)(xy)_{∗1p}z_{∗3p}(xy)_{p∗3} \]
\[ = - \sum_{p ∈ x∩z} ε(p; x, z)(x_{∗1p}z_{∗3p}) ⊗ (z_{∗3p}x_{p∗3}y) \]
\[ - \sum_{p ∈ y∩z\setminus\{∗3\}} ε(p; y, z)(xy_{∗2p}z_{∗3p}) ⊗ (z_{∗3p}y_{p∗3}). \]

The first and the second terms are equal to κ(x, z)(1 ⊗ y) and (x ⊗ 1)κ+(y, z), respectively. This completes the proof. □

Corollary 3.3.2. Let n ≥ 2 and let ∗1, . . . , ∗n+2 ∈ ∂S be distinct (n + 2) points. For any x1, . . . , xn+1, xi ∈ QIPS(∗i, ∗i+1), we have

\[ κ+(x1 · · · xn, x_{n+1}) = \sum_{i=1}^{n} ((x1 · · · xi−1) ⊗ 1)κ+(xi, x_{n+1})(1 ⊗ (x_{i+1} · · · xn)) \]
\[ κ+(x1, x2 · · · xn) = \sum_{i=2}^{n+1} (1 ⊗ (x2 · · · xi−1))κ+(x1, xi)((x_{i+1} · · · xn+1) ⊗ 1). \]

Here, to simplify notations, for j − i ≥ 2, we write κ+(xi, xj) instead of κ(xi, xj). Similar formulas hold for κ−.

Proof. Note that κ+(x1, x_{n+2})(1 ⊗ (x_{i+1} · · · xn))(1 ⊗ x_{n+1}) = κ+(x1, x_{n+2})(1 ⊗ (x_{i+1} · · · xn)x_{n+1})), etc. By Lemma 3.3.1 and induction on n, we obtain the result. □

Lemma 3.3.3. Let ∗1, ∗2, ∗3 ∈ ∂S be distinct three points. For any x ∈ QIPS(∗1, ∗2) and y ∈ QIPS(∗2, ∗3), we have

\[ μ(xy) = μ(x)(y ⊗ 1) + (x ⊗ 1)μ(y) + (1 ⊗ (QIPS(∗1, ∗3) | )′)κ+(x, y) ∈ QIPS(∗1, ∗3) ⊗ QΠ(S). \]

Here, μ(x)(y ⊗ 1) means the image of μ(x) ⊗ y by the map QIPS(∗1, ∗2) ⊗ QΠ(S) ⊗ QIPS(∗2, ∗3) → QIPS(∗1, ∗3) ⊗ QΠ(S), u ⊗ v ⊗ w → uw ⊗ v, etc. Note that by (3.2.4) and | )′ = 0, we can replace κ+ with κ− in the formula above.

Proof. Let x, y : [0, 1] → S be immersed paths with x(0) = ∗1, x(1) = y(0) = ∗2, y(1) = ∗3, such that their intersections and self intersections consist of transverse double points. Moreover, we assume that ε(˙x(1), ˙y(0)) = −1. Let Γx and Γy be the set of double points
of $x$ and $y$, respectively. Then the set of double points of $xy$ is $\Gamma_x \cup \Gamma_y \cup (x \cap y \setminus \{*2\})$. We have

$$
\mu(xy) = - \sum_{p \in \Gamma_x} \varepsilon(x'(t_1'), x'(t_2'))(x_{\ast 1,p}x_{\ast 2,p}y) \otimes |x_{t_1'}y_{t_1'}|
- \sum_{p \in \Gamma_y} \varepsilon(y'(t_1'), y'(t_2'))(xy_{\ast 2,p}y_{\ast 1,p}) \otimes |y_{t_2'}x_{t_2'}|
- \sum_{p \in \partial \gamma \setminus \{*2\}} \varepsilon(x'(p), y'(p))(x_{\ast 1,p}y_{\ast 1,p}) \otimes |x_{\ast 2,p}y_{\ast 2,p}|.
$$

The first and the second terms are equal to $\mu(x)(y \otimes 1)$ and $(x \otimes 1)\mu(y)$, respectively. Since $|x_{\ast 2,p}y_{\ast 2,p}| = |y_{\ast 2,p}x_{\ast 2,p}|$, the third term is equal to $(1_{\mathcal{Q}\Pi\Sigma(*1, *3)} \otimes |\gamma|)\kappa_+(x, y)$. Hence $\mu(xy) = \mu(x)(y \otimes 1) + (x \otimes 1)\mu(y) + (1_{\mathcal{Q}\Pi\Sigma(*1, *3)} \otimes |\gamma|)\kappa_+(x, y)$.

By Corollary 3.3.2 and Lemma 3.3.3 and induction on $n$, we have the following.

**Corollary 3.3.4.** Let $n \geq 2$ and let $\ast_1, \ldots, \ast_{n+1} \in \partial S$ be distinct $(n + 1)$ points. For any $x_1, \ldots, x_n, x_i \in \mathcal{Q}\Pi\Sigma(\ast_i, \ast_{i+1})$, we have

$$
\mu(x_1 \cdots x_n) = \sum_{i=1}^{n} ((x_1 \cdots x_{i-1}) \otimes 1)\mu(x_i)((x_{i+1} \cdots x_n) \otimes 1)
+ \sum_{i < j} ((x_1 \cdots x_{i-1}) \otimes 1)K_{i,j}((x_{j+1} \cdots x_n) \otimes 1),
$$

where $K_{i,j} = (1_{\mathcal{Q}\Pi\Sigma(\ast_i, \ast_{j+1})} \otimes |\gamma|)(\kappa_+(x_i, x_j)(1 \otimes (x_{i+1} \cdots x_{j-1})))$.

### 3.4 $\mu$ and the filtrations of $\mathcal{Q}\Pi\Sigma(\ast_0, \ast_1)$, $\mathcal{Q}\hat{\pi}'(S)$

We assume that the boundary of $S$ is not empty. Take $\ast \in \partial S$.

**Lemma 3.4.1.** The following diagram is commutative:

$$
\begin{array}{ccc}
\mathcal{Q}\pi_1(S, \ast) & \xrightarrow{\mu_+} & \mathcal{Q}\pi_1(S, \ast) \otimes \mathcal{Q}\hat{\pi}'(S) \\
|\gamma| & \downarrow & (1 - T)(|\gamma| \otimes 1_{\mathcal{Q}\hat{\pi}'(S)}) \\
\mathcal{Q}\hat{\pi}'(S) & \xrightarrow{\delta} & \mathcal{Q}\hat{\pi}'(S) \otimes \mathcal{Q}\hat{\pi}'(S).
\end{array}
$$

*If we replace $\mu_+$ with $\mu_-$, the diagram still commutes.*

**Proof.** Let $\gamma: [0, 1] \to S$ be an immersed path with $\gamma(0) = \gamma(1)$ such that its self-intersections consist of transverse double points and $\varepsilon(\gamma(0), \gamma(1)) = +1$. By (2.3.2),

$$
\mu_+(\gamma) = - \sum_{p \in \Gamma} \varepsilon(\check{\gamma}(t_1'), \check{\gamma}(t_2'))(\gamma_{t_1'}^{\ast 1} \gamma_{t_2'}^{\ast 1}) \otimes |\gamma_{t_1'}^{\ast 1} |'.
$$

Using $|\gamma_{t_1'}^{\ast 1} \gamma_{t_2'}^{\ast 1}| = |\gamma_{t_1'}^{\ast 2} \gamma_{t_2'}^{\ast 2}|$, we obtain

$$
(1 - T)(|\gamma| \otimes 1_{\mathcal{Q}\hat{\pi}'(S)})\mu_+(\gamma)
= \sum_{p \in \Gamma} \varepsilon(\check{\gamma}(t_1'), \check{\gamma}(t_2'))(|\gamma_{t_1'}^{\ast 2} \gamma_{t_2'}^{\ast 2}|' \otimes |\gamma_{t_1'}^{\ast 1} \gamma_{t_2'}^{\ast 1}|' - |\gamma_{t_1'}^{\ast 1} \gamma_{t_2'}^{\ast 1}|' \otimes |\gamma_{t_1'}^{\ast 2} \gamma_{t_2'}^{\ast 2}|').
$$

This coincides with $\delta(|\gamma|)$. The proof for $\kappa_-$ is similar. \qed

16
Let \( \ell \) be the restriction of \((\ell_1 \cdots \ell_i)\) to the interval between \(*_i\) and \(*_{i+1}\). Take distinct \((\ell_1 \cdots \ell_i)\) we have (3.4.1). These computations imply that the second term of (3.4.1) lies in \(\pi_1(S, \ast) \otimes \partial\). See Figure 10.

Proposition 3.4.2. For \(n \geq 2\), we have
\[
\mu_+(\pi_1(S, \ast)^n) \subset \sum_{p+q=n-2} \pi_1(S, \ast)^p \otimes \hat{\pi}'(S)(q) \subset \pi_1(S, \ast) \otimes \hat{\pi}'(S).
\]
The same formula holds for \(\mu_-\).

Proof. We only consider \(\mu_+\). Take distinct \((n+1)\) points \(*_1 = \ast, \ast_2, \ldots, \ast_n, \ast_{n+1} = \ast_+\) on the segment \((0, 1) \subset \partial S\), so that they are arranged according to the orientation of \(\partial S\). Let \(\ell_i\) be the restriction of \(\ell_+\) to the interval between \(*_i\) and \(*_{i+1}\). See Figure 10.

Let \(\gamma_1, \gamma_2, \ldots, \gamma_n \in \pi_1(S, \ast)\). For \(1 \leq i \leq n\), set \(x_i := (\ell_1 \cdots \ell_{i-1})^{-1} \gamma_i(\ell_1 \cdots \ell_i) \in \text{QIIS}(\ast_i, \ast_{i+1})\). We have \(c_+(\gamma_1 \gamma_2 \cdots \gamma_n) = x_1 x_2 \cdots x_n\). By Corollary 3.3.4
\[
\mu_+(\gamma_1 \cdots \gamma_n) = (c_+^{-1} \otimes 1) \mu(x_1 \cdots x_n)
\]
where \(c_+^{-1} \otimes 1 = c_+^{-1} \otimes 1_{\text{QIIS}}\). For \(y \in \text{QIIS}(\ast_i, \ast_{i+1})\), we have
\[
c_+^{-1}(x_1 \cdots x_{i-1} y x_{i+1} \cdots x_n) = x_1 \cdots x_{i-1} x_{i+1} \cdots x_n (\ell_1 \cdots \ell_{i-1})^{-1}
\]
\[
\gamma_1 \cdots \gamma_{i-1} (\ell_1 \cdots \ell_{i-1}) y (\ell_1 \cdots \ell_{i})^{-1} \gamma_{i+1} \cdots \gamma_n \in \pi_1(S, \ast)^{n-1}.
\]
This implies that the first term of (3.4.1) lies in \(\pi_1(S, \ast)^{n-1} \otimes \hat{\pi}'(S)\). Next, for \(y \otimes z \in \text{QIIS}(\ast_i, \ast_{i+1}) \otimes \text{QIIS}(\ast_j, \ast_{j+1})\) we have
\[
(c_+^{-1} \otimes 1) (((x_1 \cdots x_{i-1}) \otimes 1) (1 \otimes z) ((y \otimes z)(1 \otimes (x_{i+1} \cdots x_{j-1}))) (x_{j+1} \cdots x_n) \otimes 1) i_{x_{i+1} \cdots x_{j-1}})
\]
As we have just seen, \(c_+^{-1}(x_1 \cdots x_{i-1} y x_{i+1} \cdots x_n) \in \pi_1(S, \ast)^{n+i-j-1}\). Also, since
\[
|z x_{i+1} \cdots x_{j-1}| \in \hat{\pi}'(S)(j-i-1)\). These computations imply that the second term of (3.4.1) lies in \(\sum_{p+q=n-2} \pi_1(S, \ast)^p \otimes \hat{\pi}'(S)(q)\). This completes the proof.

\[\Box\]
In particular, \( \mu_+ \) and \( \mu_- \) induce \( \mathbb{Q} \)-linear maps \( \mu_+: \hat{\mathbb{Q}}\pi_1(S,*) \rightarrow \mathbb{Q}\pi_1(S,*) \otimes \hat{\mathbb{Q}}\pi(S) \) which we denote by the same letters. Together with Lemma 3.4.1 and (3.1.2) in mind, we obtain

**Corollary 3.4.3.** The Turaev cobracket \( \delta: \mathbb{Q}\hat{\pi}'(S) \rightarrow \mathbb{Q}\hat{\pi}'(S) \otimes \mathbb{Q}\hat{\pi}(S) \) satisfies

\[
\delta(\mathbb{Q}\hat{\pi}'(S)(n)) \subset \sum_{p+q=n-2} \mathbb{Q}\hat{\pi}'(S)(p) \otimes \mathbb{Q}\hat{\pi}'(S)(q)
\]

for any \( n \geq 2 \).

As a consequence, \( \delta \) induces a \( \mathbb{Q} \)-linear map \( \delta: \mathbb{Q}\hat{\pi}(S) \rightarrow \mathbb{Q}\hat{\pi}(S) \otimes \mathbb{Q}\hat{\pi}(S) \). Together with the complete Lie bracket in 3.1.1, \( \mathbb{Q}\hat{\pi}(S) \) has a structure of an involutive complete Lie bialgebra, which we call the completed Goldman-Turaev Lie bialgebra. Moreover, the vector space \( \mathbb{Q}\hat{\pi}_1(S,*) \) has a structure of a complete involutive right \( \mathbb{Q}\hat{\pi}(S) \)-bimodule with respect to the structure maps \( \sigma \) and \( \mu_- \). This also holds for \( \mu_- \).

Take distinct points \( *_0, *_1 \in \partial S \) and recall from (3.1.3) the filtration \( F_n \hat{\Pi}(S)(*) = \gamma I\pi_1(S,*)^n \). Here \( \gamma \) is a path from \( *_0 \) to \( *_1 \).

**Proposition 3.4.4.** For \( n \geq 2 \), we have

\[
\mu(F_n \hat{\Pi}(S)(*)_0,*_1)) \subset \sum_{p+q=n-2} F_p \hat{\Pi}(S)(*)_0,*_1 \otimes \mathbb{Q}\hat{\pi}'(S)(q) \subset \hat{\Pi}(S)(*)_0,*_1 \otimes \mathbb{Q}\hat{\pi}'(S).
\]

**Proof.** Understanding * = \( *_1 \), let \( *_i \in \partial S \), 1 ≤ \( i \) ≤ \( n+1 \), \( \gamma_i \in I\pi_1(S,*) \) and \( x_i \in \hat{\Pi}(S,*)_i)+1 \), 1 ≤ \( i \) ≤ \( n \), be as in the proof of Proposition 3.4.2. Then we have \( \mu(\gamma_1 \cdots \gamma_n) = (c_+^{-1} \otimes 1)\mu(\gamma_1 x_1 \cdots x_n) \). From Lemma 3.3.2 it follows that

\[
\mu(\gamma_1 \cdots \gamma_n) = \mu(\gamma)(x_1 \cdots x_n) + (1 \otimes 1)\mu(x_1 \cdots x_n) + (1 \otimes 1)(Q\Pi(S*)(*)_0,*_1) \otimes \mathbb{Q}\hat{\pi}'(S)(q).
\]

It is clear \( (c_+^{-1} \otimes 1)\mu(\gamma)(x_1 \cdots x_n) \in F_n \hat{\Pi}(S)(*)_0,*_1 \otimes \mathbb{Q}\hat{\pi}'(S) \). By Proposition 3.4.2 we have \( (c_+^{-1} \otimes 1)(\gamma \otimes 1)\mu(x_1 \cdots x_n) \in \sum_{p+q=n-2} F_p \hat{\Pi}(S)(*)_0,*_1 \otimes \mathbb{Q}\hat{\pi}'(S)(q) \). From Corollary 3.3.2 we obtain \( (c_+^{-1} \otimes 1)(\gamma \otimes 1)(x_1 \cdots x_n) \in F_{n-1} \hat{\Pi}(S)(*)_0,*_1 \otimes \mathbb{Q}\hat{\pi}'(S)(n-i) \). This proves the proposition.

We define

\[
\hat{\Pi}(S)(*)_0,*_1 := \lim_{n \rightarrow \infty} \hat{\Pi}(S)(*)_0,*_1 \land F_n \hat{\Pi}(S)(*)_0,*_1,
\]

which is a \( \mathbb{Q}\hat{\pi}(S) \)-module by means of \( \sigma \). See 9.4.1. Proposition 3.4.4 implies \( \mu \) induces a \( \mathbb{Q} \)-linear map

\[
\mu: \hat{\Pi}(S)(*)_0,*_1 \rightarrow \hat{\Pi}(S)(*)_0,*_1 \otimes \hat{\mathbb{Q}}\pi(S),
\]

which makes \( \hat{\Pi}(S)(*)_0,*_1 \) a complete involutive right \( \mathbb{Q}\hat{\pi}(S) \)-bimodule. In 44 we will use this bimodule structure to prove that some generalized Dehn twists are not realized by diffeomorphisms.

### 4 Application to mapping class groups

In this section we discuss applications of our consideration of the (self) intersections of curves. In the first three subsections we study generalized Dehn twists, which was introduced in 9.11. In the last two subsections we study the Johnson homomorphisms following the treatments in 9.
4.1 Generalized Dehn twists

Generalized Dehn twists are associated with not necessarily simple loops on a surface, and are defined as elements of a certain enlargement of the mapping class group of the surface. We recall generalized Dehn twists following [9] §5. For another treatment, see [13].

Let $S$ be a compact connected oriented surface with non-empty boundary, or a surface obtained from such a surface by removing finitely many points in the interior. We denote by $\mathcal{M}(S, \partial S)$ the mapping class group of the pair $(S, \partial S)$, i.e., the group of orientation preserving diffeomorphisms of $S$ fixing $\partial S$ pointwise, modulo isotopies relative to $\partial S$. The group $\mathcal{M}(S, \partial S)$ naturally acts on each $\text{IIS}(p_0, p_1)$, $p_0, p_1 \in \partial S$.

Let $E \subset \partial S$ be a subset which contains at least one point of any connected component of $\partial S$. Then we can construct a small additive category $\hat{Q}(S, E)$, whose set of objects is $E$, and whose set of morphisms from $p_0 \in E$ to $p_1 \in E$ is $\hat{Q}\text{IIS}(p_0, p_1)$. As we mentioned in §3.1, $\hat{Q}\text{IIS}(p_0, p_1)$ is filtered and its completion $\hat{Q}\text{IIS}(p_0, p_1)$ is defined. Let $QC(S, E)$ be a small additive category whose set of objects is $E$, and whose set of morphisms from $p_0 \in E$ to $p_1 \in E$ is $\hat{Q}\text{IIS}(p_0, p_1)$. In [9], $QC(S, E)$ is called the completion of $QC(S, E)$.

The action of $\mathcal{M}(S, \partial S)$ on $\text{IIS}(p_0, p_1)$ naturally induces a $Q$-linear automorphism of $\hat{Q}\text{IIS}(p_0, p_1)$, as well as a $Q$-linear automorphism of $\hat{Q}\text{IIS}(p_0, p_1)$. In this way we obtain a group homomorphism of Dehn-Nielsen type

$$\hat{\text{DN}}: \mathcal{M}(S, \partial S) \to \text{Aut}(\hat{Q}(S, E)),$$

where $\text{Aut}(\hat{Q}(S, E))$ is the group of covariant functors from $\hat{Q}(S, E)$ to itself, which act on the set of objects as the identity, and act on each set of morphisms as $Q$-linear automorphisms. This group homomorphism is injective (see [9] Theorem 3.1.1).

Let $C \subset S \setminus \partial S$ be an unoriented loop. Take $q \in S$ and let $x \in \pi_1(S, q)$ be a based loop which is homotopic to $C$ as an unoriented loop. The quantity

$$L(C) := \frac{1}{2}(\log x)^2 \in \hat{\pi}(S)(2),$$

where $\mid \mid: \hat{Q}\pi_1(S, q) \to \hat{Q}\pi(S)$ is the map induced by $\mid \mid: Q\pi_1(S, q) \to Q\pi(S)$, is independent of the choice of $q$ and $x$.

A family of $Q$-linear homomorphisms $D = D^{(p_0, p_1)}: \hat{Q}\text{IIS}(p_0, p_1) \to \hat{Q}\text{IIS}(p_0, p_1)$, $p_0, p_1 \in E$, is called a derivation of $\hat{Q}(S, E)$, if it satisfies the Leibniz rule

$$D(uv) = (Du)v + u(Dv)$$

for any $p_0, p_1, p_2 \in E$, $u \in \hat{Q}\text{IIS}(p_0, p_1)$, and $v \in \hat{Q}\text{IIS}(p_1, p_2)$. The set of derivations of $\hat{Q}(S, E)$ naturally has a structure of a Lie algebra, which we denote by $\text{Der}(\hat{Q}(S, E))$. Then we obtain a Lie algebra homomorphism

$$\sigma: \hat{\pi}(S) \to \text{Der}(\hat{Q}(S, E)),$$

by collecting the structure morphisms $\sigma: \hat{\pi}(S) \otimes \hat{Q}\text{IIS}(p_0, p_1) \to \hat{Q}\text{IIS}(p_0, p_1)$, $p_0, p_1 \in E$ (see [9] §4.1). For $p_0, p_1 \in E$, the exponential of the derivation $\sigma(L(C)) \in \text{End}(\hat{Q}\text{IIS}(p_0, p_1))$ converges and we obtain an automorphism

$$\exp(\sigma(L(C))) \in \text{Aut}(\hat{Q}(S, E)),$$
which we call the generalized Dehn twist along $C$ ([9] Lemma 5.1.1, Definition 5.3.1). If $C$ is simple, then this is (the $\hat{\mathcal{DN}}$-image of) the usual right handed Dehn twist along $C$ ([9] Theorem 5.2.1).

**Remark 4.1.1.** Actually $\exp(\sigma(L(C)))$ lies in a subgroup $A(S,E) \subset \text{Aut}(\mathcal{QC}(S,E))$, which was introduced in [9] Definition 3.3.1.

### 4.2 A criterion of the realizability

A natural question is whether $\exp(\sigma(L(C)))$ is realized by a diffeomorphism, i.e., is in the $\hat{\mathcal{DN}}$-image. In [9] [11] we showed that if $C$ is a figure eight, then $\exp(\sigma(L(C)))$ is not in the $\hat{\mathcal{DN}}$-image. To extend this result for curves in wider classes, we consider the self intersections of curves.

Let $C \subset S \setminus \partial S$ be an unoriented free loop, and $N \subset S \setminus \partial S$ a connected compact subsurface which is a neighborhood of $C$, and not diffeomorphic to $D^2$. If the generalized Dehn twist $\exp(\sigma(L(C)))$ is the $\hat{\mathcal{DN}}$-image of a mapping class $\varphi \in \mathcal{M}(S,\partial S)$, the support of (a representative of) $\varphi$ is included in the subsurface $N$, by the localization theorem [9] Theorem 5.5.3.

Using the fact that $\mu$ maps simple paths to zero and a diffeomorphism preserves the simplicity of curves, together with cut and paste techniques developed in [9], we have the following.

**Proposition 4.2.1.** Suppose the inclusion homomorphism $\pi_1(N) \to \pi_1(S)$ is injective. Assume the generalized Dehn twist $\exp(\sigma(L(C)))$ is realized by a diffeomorphism. Then we have

$$\mu(\sigma(L(C))(\gamma)) = 0 \in \hat{\mathcal{PN}}(*_0,*_1) \otimes \hat{\mathcal{Q}}(N)$$

for any distinct points $*_0,*_1 \in \partial N$ and any simple path $\gamma \in \mathcal{PN}(*_0,*_1)$.

**Proof.** Let $\varphi \in \text{Diff}(S,\partial S)$ be a representative of $\exp(\sigma(L(C)))$. By the remark above, we may assume that the support of $\varphi$ is included in $N$. We denote by the same letter $\varphi$ the restriction of $\varphi$ to $N$, which we can regard as an element of the mapping class group $\mathcal{M}(N,\partial N)$. Also we regard $C$ as an unoriented free loop on $N$ and $L(C)$ as an element of $\hat{\mathcal{Q}}(N)$.

Let $\partial N = \bigsqcup_i \partial_i N$ be the decomposition into connected components. Then, by [9] Proposition 3.3.4, there exist some $a_i \in \mathbb{Q}$ such that

$$\varphi \exp(-\sigma(L(C))) = \exp\left(\sigma \left( \sum_i a_i L(\partial_i N) \right) \right) \in \text{Aut}(\hat{\mathcal{QC}(N,\partial N)})$$

(see also the proof of [9] Theorem 5.4.1). Since $C$ and $\partial_i N$ are disjoint, the derivations $L(C)$ and $L(\partial_i N)$ commute with each other. This implies $\varphi^n = \exp(n(\sigma(L(C)) + \sum_i a_i L(\partial_i N)))$ for any $n \in \mathbb{Z}$. Since $\varphi^n(\gamma)$ is a simple path, we have $\mu(\varphi^n(\gamma)) = 0$. Hence we obtain

$$\mu \left( \sigma \left( L(C) + \sum_i a_i L(\partial_i N) \right) \right)(\gamma) = 0.$$

On the other hand, by [9] Theorem 5.2.1, $\exp(\sigma(L(\partial_i N)))$ is realized by the Dehn twist along the simple closed curve $\partial_i N$. This implies $\mu(\sigma(L(\partial_i N))(\gamma)) = 0$. Hence we obtain $\mu(\sigma(L(C))(\gamma)) = 0$. This completes the proof. □
In the case $S$ is compact, i.e., has no punctures, we have another criterion for the realizability of generalized Dehn twists.

**Proposition 4.2.2.** Assume $S$ is compact, and let $C \subset S \setminus \partial S$ be an unoriented loop whose generalized Dehn twist $\exp(\sigma(L(C)))$ is realized by a diffeomorphism. Then we have

$$\delta L(C) = 0 \in \hat{Q}\hat{\pi}(S) \otimes \hat{Q}\hat{\pi}(S).$$

**Proof.** Take $*_{0}, *_{1} \in E$. Any orientation-preserving diffeomorphism $\varphi$ of $S$ fixing the boundary pointwise preserves the comodule structure map $\mu: \hat{Q}\hat{\Pi}\hat{S}(*_{0}, *_{1}) \rightarrow \hat{Q}\hat{\Pi}\hat{S}(*_{0}, *_{1}) \otimes \hat{Q}\hat{\pi}(S)$. Here we understand $\mu = \mu_{+}$ or $\mu_{-}$ in the case $*_{0} = *_{1}$. Hence, for any $n \in \mathbb{Z}$, we have

$$\mu \exp(n\sigma(L(C))) = \exp(n\sigma(L(C))) \mu,$$

and so

$$(\sigma(L(C)) \otimes 1 + 1 \otimes \sigma(L(C))) \mu = \mu \sigma(L(C)) : \hat{Q}\hat{\Pi}\hat{S}(*_{0}, *_{1}) \rightarrow \hat{Q}\hat{\Pi}\hat{S}(*_{0}, *_{1}) \otimes \hat{Q}\hat{\pi}(S).$$

From (A.2.2) this is equivalent to

$$((\pi \otimes 1_{\hat{Q}(S)}) (1_{\hat{Q}\hat{\Pi}\hat{S}(*_{0},*_{1})} \otimes \delta)(v \otimes L(C))) = 0 \in \hat{Q}\hat{\Pi}\hat{S}(*_{0},*_{1}) \otimes \hat{Q}\hat{\pi}(S)$$

for any $v \in \hat{Q}\hat{\Pi}\hat{S}(*_{0},*_{1})$. By [11] Theorem 6.2.1, the intersection of the kernels of the structure map $\sigma: \hat{Q}\hat{\pi}(S) \rightarrow \text{End}(\hat{Q}\hat{\Pi}\hat{S}(*_{0},*_{1}))$ for all $*_{0}, *_{1} \in E$, is zero. Hence we have $\delta L(C) = 0$. This proves the proposition.

In [11] the second-named author posed the following question.

**Question 4.2.3 ([11] Question 5.3.4).** Let $C$ be an unoriented loop on $\Sigma_{g,1}$, a surface of genus $g$ with one boundary component, and suppose the generalized Dehn twist along $C$ is realized by a diffeomorphism. Is $C$ homotopic to a power of a simple closed curve?

In view of Proposition 4.2.2 we come to the following conjecture.

**Conjecture 4.2.4.** Suppose an unoriented loop $C$ satisfies $\delta L(C) = 0$. Then $C$ would be homotopic to a power of a simple closed curve.

If the conjecture is true, then the question is also affirmative. But the conjecture looks like the question which was posed by Turaev [19] and whose counter-examples Chas gave in [2].

### 4.3 New examples not realized by a diffeomorphism

In this subsection we prove the following.

**Theorem 4.3.1.** Let $S$ and $E \subset \partial S$ be as in [14, 4] and $C \subset S \setminus \partial S$ an unoriented immersed loop whose self intersections consist of transverse double points. Assume $C$ has at least one self intersection and the inclusion homomorphism $\pi_{1}(C) \rightarrow \pi_{1}(S)$ is injective. Then the generalized Dehn twist $\exp(\sigma(L(C)))$ is not in the image of $\hat{D}\hat{N}: \mathcal{M}(S, \partial S) \rightarrow \text{Aut}(\hat{Q}\hat{C}(S, E))$. 


The rest of this subsection is devoted to the proof of Theorem 4.3.1.

Let $S$ be an oriented surface and $*_{0}, *_{1} \in \partial S$ distinct points. Using $\mu$ and the augmentation $\mathbb{Q}IIS(*_{0}, *_{1}) \to \mathbb{Q}, IIS(*_{0}, *_{1}) \ni x \mapsto 1$, we define a $\mathbb{Q}$-linear map $\hat{\mu}: \mathbb{Q}IIS(*_{0}, *_{1}) \to \mathbb{Q}\hat{\pi}'(S)$ as the composite

$$
\hat{\mu}: \mathbb{Q}IIS(*_{0}, *_{1}) \xrightarrow{\mu} \mathbb{Q}IIS(*_{0}, *_{1}) \otimes \mathbb{Q}\hat{\pi}'(S) \to \mathbb{Q} \otimes \mathbb{Q}\hat{\pi}'(S) = \mathbb{Q}\hat{\pi}'(S).
$$

By Proposition 3.4.4, $\hat{\mu}$ extends to a $\mathbb{Q}$-linear map $\hat{\mu}: \mathbb{Q}IIS(*_{0}, *_{1}) \to \mathbb{Q}\hat{\pi}(S)$. We denote by $\mathbb{Q}H_{1}(S; \mathbb{Z})$ the completed group ring of the integral first homology group $H_{1}(S; \mathbb{Z})$. There is a natural projection $\hat{\pi}(S) \to H_{1}(S; \mathbb{Z})$, which induces a $\mathbb{Q}$-linear map $\varpi: \mathbb{Q}\hat{\pi}(S) \to \mathbb{Q}H_{1}(S; \mathbb{Z})/\mathbb{Q}1$. Here $\mathbb{Q}1$ is the 1-dimensional subspace spanned by the identity element of $H_{1}(S; \mathbb{Z})$.

Let $C \subset S \setminus \partial S$ be an unoriented immersed loop such that its self intersections consist of transverse double points, and let $\gamma \in IIS(*_{0}, *_{1})$ be a simple path meeting $C$ transversally in a single point. In this situation, we shall compute the quantity $\varpi\hat{\mu}(\sigma(L(C))\gamma)$.

Let $c$ be a $\pi_{1}(S, *_{1})$-representative of $C$, as in Figure 11. Then we have

$$
\sigma(L(C))\gamma = \gamma \log c \in \mathbb{Q}IIS(*_{0}, *_{1}),
$$

since $\sigma(\{c^n\})\gamma = n\gamma c^n$ for $n \geq 0$. Now fix a parametrization $c: ([0, 1], \{0, 1\}) \to (S, *_{1})$. When $p \in S$ is a double point of $C$, we denote $c^{-1}(p) = \{t_{1}^{p}, t_{2}^{p}\}$ so that $t_{1}^{p} < t_{2}^{p}$. Set $x_{p} := c_{t_{1}^{p}}c_{t_{1}^{p}}$, and $y_{p} := c_{t_{2}^{p}}c_{t_{2}^{p}}$. By abuse of notation, we use the same letter $x_{p}$ and $y_{p}$ for the homology classes represented by these loops. Finally, let $h(x)$ be the formal power series defined by

$$
h(x) := \sum_{n=0}^{\infty} \frac{(-1)^{n}}{n + 2}(x - 1)^{n}.
$$

**Proposition 4.3.2.** Keep the notations as above. Then

$$
\varpi\hat{\mu}(\sigma(L(C))\gamma) = - \sum_{p} \varepsilon(c(t_{1}^{p}), c(t_{2}^{p}))(y_{p} + x_{p}(y_{p}^{2} - 1)h(c)) \in \mathbb{Q}H_{1}(S; \mathbb{Z})/\mathbb{Q}1.
$$

Here we write the product of the group ring $\mathbb{Q}H_{1}(S; \mathbb{Z})$ multiplicatively.

To prove this proposition, we need a lemma.
Lemma 4.3.3. In the polynomial ring $\mathbb{Q}[x]$, the following equalities hold.

1. For $n \geq 1$, 
   \[
   \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \sum_{j=0}^{k-1} x^j = (x - 1)^{n-1}.
   \]

2. For $n \geq 1$, set 
   \[
   f_n(x) := \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \sum_{j=0}^{k-1} (k-j)x^j.
   \]

   Then $f_1(x) = 1$ and $f_n(x) = x(x-1)^{n-2}$ for $n \geq 2$.

Proof. 1. Since $\sum_{j=0}^{k-1} x^j = (x^k - 1)/(x - 1)$, the left hand side is equal to 
   \[
   \frac{1}{x-1} \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} (x^k - 1) = \frac{1}{x-1} \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} x^k
   \]
   \[
   = \frac{1}{x-1} (x - 1)^n = (x - 1)^{n-1}.
   \]

2. The case $n = 1$ is clear. Let $n \geq 2$. By the first part, we compute 
   \[
   f_n(x) - (x - 1)^{n-1} = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \sum_{j=0}^{k-1} (k-1-j)x^j
   \]
   \[
   = \sum_{k=0}^{n} \left( \binom{n-1}{k-1} + \binom{n-1}{k} \right) (-1)^{n-k} \sum_{j=0}^{k-1} (k-1-j)x^j
   \]
   \[
   = \sum_{k=0}^{n-1} \binom{n-1}{k} (-1)^{n-k-1} \sum_{j=0}^{k} (k-j)x^j + \sum_{k=0}^{n} \binom{n-1}{k} (-1)^{n-k} \sum_{j=0}^{k-1} (k-1-j)x^j
   \]
   \[
   = 0.
   \]

Therefore $f_n(x) = (x - 1)^{n-1} + (x - 1)^{n-2} = x(x-1)^{n-2}$.

\[\square\]
Figure 12: a representative of $\gamma c^k$ ($k = 4$)

Figure 13: a picture near $p$ ($\varepsilon(\dot{c}(t_p^1), \dot{c}(t_p^2)) = 1$, $k = 4$)

points in the box $j^-$ ($1 \leq j \leq k - 1$) contribute as $+|x_p(y_px_p)^{j-1}'|$. If $\varepsilon(\dot{c}(t_p^1), \dot{c}(t_p^2)) = -1$, the contributions are the minus of the case of $\varepsilon(\dot{c}(t_p^1), \dot{c}(t_p^2)) = 1$. Therefore, we obtain

$$\hat{\mu}(\gamma c^k) = -\sum_p \varepsilon(\dot{c}(t_p^1), \dot{c}(t_p^2)) \left( \sum_{j=0}^{k-1} (k-j)|y_p(x_py_p)^j'| - \sum_{j=1}^{k-1} (k-j)|x_p(y_px_p)^{j-1}'| \right).$$

(4.3.1)

We next compute $\hat{\mu}(\gamma(c-1)^n)$ for $n \geq 1$. We claim that the contribution from $p$ to $\hat{\mu}(\gamma(c-1)^n)$ is $-\varepsilon(\dot{c}(t_p^1), \dot{c}(t_p^2))|y_p'|$ if $n = 1$, and

$$-\varepsilon(\dot{c}(t_p^1), \dot{c}(t_p^2)) \left( |y_p(x_py_p(x_px_p-1)^{n-2}' - |x_p(y_px_p-1)^{n-2}'| \right)$$

if $n \geq 2$. The case $n = 1$ is clear. If $n \geq 2$, by (4.3.1) the contribution is $-\varepsilon(\dot{c}(t_p^1), \dot{c}(t_p^2))$ times

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \left( \sum_{j=0}^{k-1} (k-j)|y_p(x_py_p)^j'| - \sum_{j=1}^{k-1} (k-j)|x_p(y_px_p)^{j-1}'| \right).$$

(4.3.2)
By
\[ \sum_{j=1}^{k-j} (k - j)|x_p(y_p x_p)^j-1|' = \sum_{j=0}^{k-1} (k - j)|x_p(y_p x_p)^j|' - \sum_{j=0}^{k-1} |x_p(y_p x_p)^j|' \]
and Lemma 1.3.1, 1.3.2 is equal to
\[ |y_p x_p y_p(x_p y_p - 1)^{-2}|' - |x_p y_p x_p(y_p x_p - 1)^{-2}|' + |x_p(y_p x_p - 1)^{-1}|' \]
\[ = |y_p x_p y_p(x_p y_p - 1)^{-2}|' - |x_p y_p x_p(y_p x_p - 1)^{-2}|'. \]
The claim is proved. Now we conclude
\[ \hat{\mu}(\gamma \log c) = -\sum_p \varepsilon(c(t_1^p), c(t_2^p)) \left(|y_p|' + |y_p x_p y_p h(x_p y_p)|' - |x_p h(y_p x_p)|' \right). \]
Applying \(\varpi\) and using \(x_p y_p = c = y_p x_p \in H_1(S; \mathbb{Z})\), we obtain the desired formula. This completes the proof. \(\square\)

**Proof of Theorem 4.3.1.** Assume the generalized Dehn twist \(\exp(\sigma(L(C)))\) is realized by a diffeomorphism. Let \(N\) be a closed regular neighborhood of \(C\). Take a simple point \(a \in S\) of \(C\) and let \(\gamma : ([0, 1], \{0, 1\}) \to (N, \partial N)\) be a simple path in \(N\) meeting \(C\) transversally only at \(a\). We denote \(\gamma(0) = *_0\) and \(\gamma(1) = *_1\). By Proposition 4.2.1, we have \(\mu(\sigma(L(C)) \gamma) = 0.\)

In particular, we have \(\varpi \hat{\mu}(\sigma(L(C)) \gamma) = 0 \in \mathbb{Q}H_1(N; \mathbb{Z})/\mathbb{Q}1.\)

We claim: 1) \(\{x_p\}_p \cup \{c\}\) constitute a \(\mathbb{Z}\)-basis of \(H_1(N; \mathbb{Z}) = H_1(C; \mathbb{Z})\), and 2) by an appropriate choice of \(a\), we can arrange that \(\sum_p \varepsilon(c(t_1^p), c(t_2^p)) \neq 0.\)

To prove the first claim, note that only the underlying 4-valent graph structure of \(C\), together with its (unoriented) parametrization matters. We proceed by induction on the number of double points of \(C\). If \(C\) is simple, the claim is trivial. Suppose \(C\) has at least one self intersection and let \(q\) be a double point of \(C\). Let \(f : \tilde{C} \to C\) be a resolution of \(q\). Namely, \(\tilde{C}\) is a 4-valent graph with a surjective map \(S^1 \to \tilde{C}\), such that the composition \(S^1 \to \tilde{C} \to C\) gives a parametrization of \(C\), \(f^{-1}(x)\) consist of a single point if \(x \neq q\), and \(f^{-1}(q)\) consist of two points, say \(q_+\) and \(q_-\).

By the excision isomorphism, we have \(H_1(C; \mathbb{Z}) = H_1(C; \{q\}; \mathbb{Z}) \cong H_1(\tilde{C}, \{q_+, q_-\}; \mathbb{Z})\).

Consider the homology exact sequence of the pair
\[ 0 \to H_1(\tilde{C}; \mathbb{Z}) \to H_1(C, \{q_+, q_-\}; \mathbb{Z}) \xrightarrow{\partial} H_0(\{q_+, q_-\}; \mathbb{Z}) \to 0. \]
Then the \(\partial\)-image of \(x_q \in H_1(C; \mathbb{Z}) = H_1(\tilde{C}, \{q_+, q_-\}; \mathbb{Z})\) is \(\pm(q_+ - q_-),\) which is a generator of \(H_0(\{q_+, q_-\}; \mathbb{Z}) \cong \mathbb{Z}\). By the inductive assumption, the lifts of \(\{x_p\}_{p \notin q} \cup \{c\}\) to \(\tilde{C}\) constitute a \(\mathbb{Z}\)-basis of \(H_1(\tilde{C}; \mathbb{Z})\). Therefore the lifts of \(\{x_p\}_p \cup \{c\}\) to \(C\) constitute a \(\mathbb{Z}\)-basis of \(H_1(C, \{q_+, q_-\}; \mathbb{Z})\), which completes the proof of the first claim.

To prove the second claim, let \(\ell\) be the component of the set of simple points of \(C\) containing \(a\), and \(\ell'\) a component next to \(\ell\). Take a simple point \(a' \in \ell'\) and let \(\gamma'\) be a simple path meeting \(C\) transversally only at \(a'\). We arrange that \(\varepsilon(c(a), \gamma(a)) = \varepsilon(c(a'), \gamma'(a'))\).

Let \(q\) be the double point of \(C\) between \(\ell\) and \(\ell'\). We compare \(\varepsilon(c(t_1^q), c(t_2^q))\)'s with respect to \(a\) and \(a'\). If \(p \neq q\), then they are the same. If \(p = q\), they are minus of each other. Hence the difference of the sums \(\sum_p \varepsilon(c(t_1^p), c(t_2^p))\) for \(a\) and \(a'\) is two, in particular at least one of them is not zero. This proves the second claim.
Now choose a such that $\sum_p \varepsilon(\hat{c}(t^p_1), \hat{c}(t^p_2)) \neq 0$. By the first claim, we can define a group homomorphism $\Phi: H_1(N; \mathbb{Z}) \to \langle t \rangle$ to an infinite cyclic group generated by $t$, by $\Phi(x_p) = 1$ and $\Phi(c) = t$. This group homomorphism induces a $\mathbb{Q}$-linear map $\Phi: \mathbb{Q}H_1(N; \mathbb{Z})/\mathbb{Q}1 \to \mathbb{Q}(\hat{t})/\mathbb{Q}1$. Since $x_py_p = c$, we have $\Phi(y_p) = t$. By Proposition 4.3.2,

$$\Phi(\omega \mu(\sigma(L(C))\gamma)) = -\left(\sum_p \varepsilon(\hat{c}(t^p_1), \hat{c}(t^p_2))\right) (t + (t^2 - 1)h(t)) \in \mathbb{Q}(\hat{t})/\mathbb{Q}1.$$  

Finally, we claim $t + (t^2 - 1)h(t) \neq 0$. To prove this, consider an algebra homomorphism from $\mathbb{Q}(\hat{t})$ to $\mathbb{Q}[[s]]$, the ring of formal power series in $s$, given by $t \mapsto 1 + s$. This is a filter-preserving isomorphism, and the image of $t + (t^2 - 1)h(t)$ is $1 + 2s$ (higher term), which is not zero in $\mathbb{Q}[[s]]/\mathbb{Q}1$. This shows $t + (t^2 - 1)h(t) \neq 0$, which contradicts to $\omega \mu(\sigma(L(C))\gamma) = 0$. This completes the proof.  

4.4 The Johnson homomorphisms

The higher Johnson homomorphisms on the higher Torelli groups for a once bordered surface are important tools to study the algebraic structure of the mapping class group. In [9], we gave a generalization of the classical construction of the Johnson homomorphisms to arbitrarily compact surfaces with non-empty boundaries. In this subsection we briefly recall this construction and connections to the classical theory.

Let $S$ be a compact connected oriented surface of with non-empty boundary, and $E \subset \partial S$ a subset such that each connected component of $\partial S$ has a unique point of $E$. We assume the genus of $S$ is positive. We define the Torelli group $\mathcal{I}(S, E)$ to be the kernel of the action of the mapping class group $\mathcal{M}(S, E)$ on the first homology group $H_1(S, E; \mathbb{Z})$, which is the smallest Torelli group in the sense of Putman [17]. On the other hand, we define

$$L^+(S, E) := \{u \in \widehat{\pi_1}(S)(3); \sigma(u)(\Delta(\ast_0, \ast_1)) = 0 \text{ for any } \ast_0, \ast_1 \in E\},$$

where $\Delta$ is the coproduct $\Delta = \Delta(\ast_0, \ast_1): \widehat{\Pi S}(\ast_0, \ast_1) \to \mathbb{Q}\Pi S(\ast_0, \ast_1) \otimes \mathbb{Q}\Pi S(\ast_0, \ast_1)$ given by $\Delta x := x \otimes x$ for any $x \in \Pi S(\ast_0, \ast_1)$, $\ast_0, \ast_1 \in E$.

Using the Hausdorff series, we can regard $L^+(S, E)$ as a pro-nilpotent group. In other words, using the injectivity of $\sigma$ ([9] Theorem 6.2.1) and the exponential map, we have a bijection $L^+(S, E) \xrightarrow{\Sigma} \exp(\sigma(L^+(S, E))) \subset \text{Aut}(\mathbb{Q}\mathcal{C}(S, E))$, which endows $L^+(S, E)$ with a group structure. In [9] §6.3 we showed the inclusion

$$\widehat{\text{DN}}(\mathcal{I}(S, E)) \subset \exp(\sigma(L^+(S, E))),$$

using a result of Putman [17] about generators of $\mathcal{I}(S, E)$ in the case the genus of $S$ is positive and our formula for Dehn twists [8]. Hence we obtain a unique injective group homomorphism

$$\tau: \mathcal{I}(S, E) \to L^+(S, E) \quad (4.4.1)$$

satisfying $\widehat{\text{DN}}_{\mathcal{I}(S, E)} = \exp \circ \sigma \circ \tau$.

Suppose $S = \Sigma_{g,1}$, a surface of genus $g$ with one boundary component. Then as we will briefly recall below (for details, see [9] §6.1), the Lie algebra $L^+(\Sigma_{g,1}, \{\ast\})$ is identified with (the completion of) the positive part of Kontsevich’s Lie, $\ell^+_g$ [10]. Preceding Kontsevich,
Morita [14] [15] introduced the Lie algebra $H_+ = \ell^+_g$ as a target of the higher Johnson homomorphisms.

Let $H := H_1(\Sigma_{g,1}; \mathbb{Q})$ be the first homology group of $\Sigma_{g,1}$, and consider $\widehat{T} := \prod_{m=0}^{\infty} H^{\otimes m}$, the completed tensor algebra generated by $H$. Via the interstion pairing $(\cdot, \cdot): H \times H \to \mathbb{Q}$, we identify $H$ and its dual $H^* = \text{Hom}(H, \mathbb{Q})$: $H \cong H^*$, $X \mapsto (Y \mapsto (Y \cdot X))$. Let $\omega \in H^{\otimes 2}$ be the two tensor corresponding to $-1_H \in \text{Hom}(H, H) = H^* \otimes H = H^{\otimes 2}$. By definition, the Lie algebra of symplectic derivations is $\mathfrak{a}^-_g = \text{Der}_\omega(\widehat{T})$, i.e., the Lie algebra of (continuous) derivations on the algebra $\widehat{T}$ annihilating $\omega$. The restriction

$$\mathfrak{a}^-_g \to \text{Hom}(H, \widehat{T}) = H^* \otimes \widehat{T} = H \otimes \widehat{T} = \prod_{m=1}^{\infty} H^{\otimes m}, \quad D \mapsto D|_H$$

is injective. In particular, the Lie algebra $\mathfrak{a}^-_g$ is naturally filtered. We say $D \in \mathfrak{a}^-_g$ is degree $n$ if $D \in \prod_{m \geq n+2} H^{\otimes m}$. Now the algebra $\widehat{T}$ has the complete coproduct $\Delta$ given by $\Delta(X) = X \otimes 1 + 1 \otimes X$, $X \in H$. Let $\mathfrak{l}^+_g$ be the Lie subalgebra of $\mathfrak{a}^-_g$ consisting of the derivations of positive degree and stabilizing the coproduct on $\widehat{T}$. The Lie algebra $\mathfrak{l}^+_g$ is an ideal of $(\text{the completion of})$ Kontsevich’s Lie $\ell_g$ [10].

In [9] Theorem 6.1.4 and 6.1.5, we showed that there exists a filter preserving isomorphism of Lie algebras

$$-\lambda_\theta: \widehat{\mathcal{Q}}^\pi(\Sigma_{g,1}) \xrightarrow{\sim} \mathfrak{a}^-_g,$$

inducing a filter preserving isomorphism of Lie algebras

$$-\lambda_\theta: L^+(\Sigma_{g,1}, \{\ast\}) \xrightarrow{\sim} \mathfrak{l}^+_g.$$

The map $\lambda_\theta$ depends on the choice of a so called symplectic expansion $\theta$, which is a map from $\pi_1(\Sigma_{g,1})$ to $\widehat{T}$ satisfying some conditions. As was mentioned in [9] §6.3, the composite $-\lambda_\theta \circ \tau$ is essentially the same as the Johnson map introduced by Kawazumi [17] and Massuyeau [12]. Its graded quotients with respect to a suitable filtration are the Johnson homomorphisms of all degrees introduced by Johnson [6] and improved by Morita [16]. Indeed, it is this context in which the Lie algebra $\ell_g$ was introduced by Morita [14] [15].

### 4.5 A constraint on the Johnson image

We show that the Turaev cobracket gives an obstruction of the surjectivity of $\tau$.

**Theorem 4.5.1.**

$$\delta \circ \tau = 0: \mathcal{I}(S, E) \xrightarrow{\tau} L^+(S, E) \subset \widehat{\mathcal{Q}}^\pi(S) \xrightarrow{\delta} \widehat{\mathcal{Q}}^\pi(S) \otimes \widehat{\mathcal{Q}}^\pi(S).$$

**Proof.** The proof is similar to that of Proposition 4.2.2. From the definition of $\tau$, for any $\varphi \in \mathcal{I}(S, E)$, there exists a unique $u \in L^+(S, E)$ such that $\varphi = \exp \sigma(u)$ on $\widehat{\mathcal{Q}}^\pi(\ast_0, \ast_1)$ for any $\ast_0$ and $\ast_1 \in E$. Then we have $\tau(\varphi) = u$ by definition. Let $\mu: \widehat{\mathcal{Q}}^\pi(\ast_0, \ast_1) \to \widehat{\mathcal{Q}}^\pi(\ast_0, \ast_1) \otimes \widehat{\mathcal{Q}}^\pi(S)$ be the structure map of the comodule $\widehat{\mathcal{Q}}^\pi(\ast_0, \ast_1)$. We understand $\mu = \mu_+$ or $\mu_-$ in the case $\ast_0 = \ast_1$. It is clear $\mu$ is preserved by $\varphi^n$ for any $n \in \mathbb{Z}$, namely, we have

$$(\exp \sigma(nu) \otimes \exp \sigma(nu)) \mu(v) = \mu(\exp \sigma(nu)(v))$$

27
Figure 14: the case \( g = 3, r = 2 \)

\[
\Sigma_{1,1}'s \quad (N, C) \quad \Sigma_{0,r+1}
\]

for any \( n \in \mathbb{Z} \) and \( v \in \widehat{\mathbb{P}} \mathbb{S}(\ast_0, \ast_1) \). Hence we have

\[
(\sigma(u) \hat{\otimes} 1 + 1 \hat{\otimes} \sigma(u))\mu(v) = \mu(\sigma(u)(v))
\]

which is equivalent to

\[
(\sigma \otimes 1)(v \hat{\otimes} \delta u) = 0 \in \widehat{\mathbb{P}} \mathbb{S}(\ast_0, \ast_1) \hat{\otimes} \widehat{\mathbb{P}} \mathbb{S}(S)
\]

for any \( \ast_0 \) and \( \ast_1 \in E \), from (A.2.2). Again by [9] Theorem 6.2.1, we conclude \( \delta u = 0 \). This proves the theorem.

This constraint is non-trivial if the genus of the surface \( S \) is greater than 1.

**Proposition 4.5.2.** If \( g \geq 2 \), we have \( \delta |_{L^+(S,E)} \neq 0 \).

**Proof.** We denote by \( \Sigma_{g,r} \) a connected oriented compact surface of genus \( g \) with \( r \) boundary components. Consider a spine \( C \) of the surface \( N := \Sigma_{0,g+1} \) as in Figure 14. If \( g \geq 2 \), \( C \) has a self-intersection. We cap each of the \( g \) boundaries the curve \( C \) surrounds by a surface diffeomorphic to \( \Sigma_{1,1} \) to obtain a compact surface \( S_0 \) diffeomorphic to \( \Sigma_{g,1} \), and glue \( \Sigma_{0,r+1} \) to the boundary of \( S_0 \) to get a compact surface \( S \) diffeomorphic to \( \Sigma_{g,r} \). See Figure 14. Choose one point in each boundary component of \( S \). We define \( E \) by the set of all these points. We may regard \( N \) as a regular neighborhood of \( C \).

Consider the invariant \( L(C) \in \widehat{\mathbb{P}} \mathbb{S}(S) \). As was proved in [9] Lemma 5.1.2, the action of \( L(C) \) stabilizes the coproduct \( \Delta \). Since \( [C] = 0 \in H_1(S; \mathbb{Q}) \), we have \( L(C) \in L^+(S,E) \). From the proof of Theorem 4.3.1 and the compatibility of the comodule structure map and the cobracket (A.2.2), we have \( \delta L(C) \neq 0 \in \widehat{\mathbb{P}} \mathbb{S}(N) \hat{\otimes} \widehat{\mathbb{P}} \mathbb{S}(N) \). As was proved in [9] Proposition 6.2.3, the inclusion homomorphism \( \widehat{\mathbb{P}} \mathbb{S}(N) \rightarrow \widehat{\mathbb{P}} \mathbb{S}(S_0) \) is injective. Since the inclusion homomorphism \( \pi_1(S_0) \rightarrow \pi_1(S) \) has a right inverse coming from capping all the boundaries except one by \( r - 1 \) discs, the inclusion homomorphism \( \widehat{\mathbb{P}} \mathbb{S}(S_0) \rightarrow \widehat{\mathbb{P}} \mathbb{S}(S) \) is injective. Hence we have \( \delta L(C) \neq 0 \in \widehat{\mathbb{P}} \mathbb{S}(S) \hat{\otimes} \widehat{\mathbb{P}} \mathbb{S}(S) \). This proves the proposition. □
From Theorem 4.5.1 the Zariski closure of the subgroup $\tau(\mathfrak{I}(S, E))$ is included in the closed Lie subalgebra $\operatorname{Ker}(\delta|_{L^+(S, E)})$. In view of this theorem we raise the following conjecture.

**Conjecture 4.5.3.** The Zariski closure of the subgroup $\tau(\mathfrak{I}(S, E))$ equals the closed Lie subalgebra $\operatorname{Ker}(\delta|_{L^+(S, E)})$.

$\tau(\mathfrak{I}(S, E)) = \operatorname{Ker}(\delta|_{L^+(S, E)})$.

By Turaev’s theorem [18], p.234, Corollary 2, $\mu$ captures the simplicity of a based loop on a surface. This conjecture is its analogue in the mapping class group. It is closely related to Conjecture 4.2.4. But it seems quite optimistic even in the simplest case $S = \Sigma_{g, 1}$. The cokernel of the Johnson homomorphisms in the case $S = \Sigma_{g, 1}$ is known to have plenty of $Sp$-irreducible components including the Morita trace [16]. For details, see [4] and references therein. But, unfortunately, we don’t know how our $\delta$ is related to these components, for at the moment we have no explicit description of $\delta^\theta$, the cobracket on $\mathfrak{a}_g^-$ which is induced by the isomorphism (4.4.2).

## A Lie bialgebras and their bimodules

For the sake of the reader we collect the definitions of a Lie bialgebra and its bimodules.

We work over the rationals $\mathbb{Q}$. Let $V$ be a $\mathbb{Q}$-vector space and let $T = T_V : V^{\otimes 2} \to V^{\otimes 2}$ and $N = N_V : V^{\otimes 3} \to V^{\otimes 3}$ be the linear maps defined by $T(X \otimes Y) = Y \otimes X$ and $N(X \otimes Y \otimes Z) = X \otimes Y \otimes Z + Y \otimes Z \otimes X + Z \otimes X \otimes Y$ for $X, Y, Z \in V$.

### A.1 Lie bialgebras

Let $\mathfrak{g}$ be a $\mathbb{Q}$-vector space equipped with $\mathbb{Q}$-linear maps $\nabla : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ and $\delta : \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$. Recall that $\mathfrak{g}$ is called a **Lie bialgebra** with respect to $\nabla$ and $\delta$, if

1. the pair $(\mathfrak{g}, \nabla)$ is a Lie algebra, i.e., $\nabla$ satisfies the skew condition and the Jacobi identity
   $$\nabla T = -\nabla : \mathfrak{g}^{\otimes 2} \to \mathfrak{g}, \quad \nabla(\nabla \otimes 1)N = 0 : \mathfrak{g}^{\otimes 3} \to \mathfrak{g},$$
2. the pair $(\mathfrak{g}, \delta)$ is a Lie coalgebra, i.e., $\delta$ satisfies the coskew condition and the coJacobi identity
   $$T\delta = -\delta : \mathfrak{g} \to \mathfrak{g}^{\otimes 2}, \quad N(\delta \otimes 1)\delta = 0 : \mathfrak{g} \to \mathfrak{g}^{\otimes 3},$$
3. the maps $\nabla$ and $\delta$ satisfy the compatibility
   $$\forall X, Y \in \mathfrak{g}, \quad \delta[X, Y] = \sigma(X)(\delta Y) - \sigma(Y)(\delta X).$$

Here we denote $[X, Y] := \nabla(X \otimes Y)$ and $\sigma(X)(Y \otimes Z) = [X, Y] \otimes Z + Y \otimes [X, Z]$ for $X, Y, Z \in \mathfrak{g}$. The map $\nabla$ is called the **bracket**, and the map $\delta$ is called the **cobracket**.

Moreover if the involutivity
$$\nabla \delta = 0 : \mathfrak{g} \to \mathfrak{g}$$
holds, we say $\mathfrak{g}$ is **involutive**.
A.2 Lie comodules and bimodules

Let \( g \) be a Lie algebra. Recall that a left \( g \)-module is a pair \((M, \sigma)\) where \( M \) is a \( \mathbb{Q} \)-vector space and \( \sigma \) is a \( \mathbb{Q} \)-linear map \( \sigma: g \otimes M \to M, X \otimes m \mapsto Xm, \) satisfying

\[
\forall X, Y \in g, \forall m \in M, \quad [X, Y]m = X(Ym) - Y(Xm).
\]

This condition is equivalent to the commutativity of the diagram

\[
\begin{array}{ccc}
g \otimes g \otimes M & \xrightarrow{(1 \otimes - \otimes 1_M)(1_T \otimes \sigma)} & g \otimes M \\
\n\n\n\n\n\downarrow \nabla \otimes 1_M & & \downarrow \sigma \\
\n\n\n\n\n\downarrow 1_M \otimes \nabla & & \downarrow M. \\
\n\n\n\n\n\end{array}
\]

If we define \( \sigma: M \otimes g \to M \) by \( \sigma(m \otimes X) := -\sigma(X \otimes m) = -Xm \) for \( m \in M \) and \( X \in g \), then the pair \((M, \sigma)\) is a right \( g \)-module, i.e., the following diagram commutes:

\[
\begin{array}{ccc}
M \otimes g \otimes g & \xrightarrow{(1_M \otimes (1-T))((1_M \circ \sigma) \otimes 1_g)} & M \otimes g \\
\n\n\n\n\n\downarrow 1_M \otimes \nabla & & \downarrow \sigma \\
\n\n\n\n\n\downarrow M \otimes g & & \downarrow M. \\
\n\n\n\n\n\end{array}
\]

Next let \((g, \delta)\) be a Lie coalgebra and \( M \) a \( \mathbb{Q} \)-vector space equipped with a \( \mathbb{Q} \)-linear map \( \mu: M \to M \otimes g \). We say the pair \((M, \mu)\) is a right \( g \)-comodule if the following diagram commutes:

\[
\begin{array}{ccc}
M & \xrightarrow{\mu} & M \otimes g \\
\n\n\n\n\n\downarrow \mu & & \downarrow 1_M \otimes \delta \\
\n\n\n\n\n\downarrow M \otimes g & & \downarrow M \otimes g. \\
\n\n\n\n\n\end{array}
\]

Similarly, we say a pair \((M, \overline{\sigma})\) is a left \( g \)-comodule if \( \overline{\sigma} \) is a \( \mathbb{Q} \)-linear map \( \overline{\sigma}: M \to g \otimes M \) and the following diagram commutes:

\[
\begin{array}{ccc}
M & \xrightarrow{\overline{\sigma}} & g \otimes M \\
\n\n\n\n\n\downarrow \overline{\sigma} & & \downarrow \delta \otimes 1_M \\
\n\n\n\n\n\downarrow g \otimes M & & \downarrow g \otimes g \otimes M. \\
\n\n\n\n\n\end{array}
\]

If we denote the switch map by \( T_{g,M}: g \otimes M \to M \otimes g, X \otimes m \mapsto m \otimes X \), then it is easy to see that \((M, \overline{\sigma})\) is a left \( g \)-comodule if and only if \((M, -T_{g,M} \overline{\sigma})\) is a right \( g \)-comodule.

Finally let \( g \) be a Lie bialgebra with \( \nabla \) the bracket and \( \delta \) the cobracket, \((M, \sigma)\) a left \( g \)-module, and \((M, \mu)\) a right \( g \)-comodule with the same underlying vector space \( M \). We define \( \overline{\sigma}: M \otimes g \to M \) by \( \overline{\sigma}(m \otimes X) := -Xm, \) as before. Then \((M, \overline{\sigma})\) is a right \( g \)-module.

We say the triple \((M, \sigma, \mu)\) is a \( g \)-bimodule if \( \sigma \) and \( \mu \) satisfy the compatibility

\[
\forall m \in M, \forall Y \in g, \quad \sigma(Y)\mu(m) - \mu(Ym) - (\overline{\sigma} \otimes 1_g)(1_M \otimes \delta)(m \otimes Y) = 0. \tag{A.2.2}
\]

Here \( \sigma(Y)\mu(m) = (\sigma \otimes 1_M)(Y \otimes \mu(m)) + (1_M \otimes \text{ad}(Y))\mu(m) \) and \( \text{ad}(Y)(Z) = [Y, Z] \) for \( Z \in g \). Then we also call the triple \((M, \sigma, \overline{\sigma})\) defined by \( \overline{\sigma} := -T_{g,M} \mu: M \to g \otimes M \) a left \( g \)-bimodule. Moreover, if \( g \) is involutive and the condition

\[
\overline{\sigma} \mu = 0: M \to M \tag{A.2.3}
\]

holds, we say \( M \) is involutive.
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