RELAXATION DYNAMICS OF DISORDERED ISING MODEL. TWO-SITE CLUSTER APPROXIMATION

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Spin relaxation in a site-disordered Ising model within master equation approach is studied. The \( \bar{q}, \omega \)-dependent susceptibility of the model is calculated and investigated. Effects described by the two-site cluster approximation and lost by the mean field approximation are discussed. Comparison of obtained results with dielectric measurements in \( \text{Cs}(H_{1-x}D_x)_{2}PO_4 \) is presented.

1. Introduction

Being the model of critical phenomena in many different objects an Ising model compels an attention during many years. This paper investigate the kinetics of the model (within master equation \[1\] approach) having several goals. We consider the disordered (site chaos) Ising model in order to describe the dynamic properties of partially deuterrated ferroelectrics with hydrogen bonds. Besides the theory seems applicable to many other systems: alloys of magnets, diluted magnetic or ferroelectric systems and others. Within two-site cluster approximation (TCA) we calculate the dynamic structure factor to supplement our previous study \[2\] of correlation functions of the model.

2. The Hamiltonian. The approach

We consider the spin system with site chaos on the Bravais lattice which is described by the Ising Hamiltonian

\[
H = - \sum_i \kappa_i S_i - \frac{1}{2} \sum_{ij} K_{ij} S_i S_j .
\]

\( H \) describes the set of Ising spins with the pair exchange interaction \( K_{ij} \) in the site-dependent field \( \kappa_i \). The set of variables \( \{S_i\} (S_i = \pm 1; i = 1 \cdots N, N \) is a number of the sites\) represents a state of spin subsystem, and the set \( \{X_{i\alpha}\} \) describes the sort configuration (\( X_{i\alpha} = 1 \), if the site \( i \) is occupied by spin of the sort \( \alpha \), otherwise \( X_{i\alpha} = 0; \alpha = 1 \cdots \Omega, \Omega \) is a number of the sorts). The quantities \( \kappa_i, K_{ij} \) depend on sort configuration

\[
\kappa_i = \sum_\alpha \kappa_{i\alpha} X_{i\alpha}; \ K_{ij} = \sum_{\alpha\beta} K_{i\alpha,j\beta} X_{i\alpha} X_{j\beta}
\]
so the true parameters of the Hamiltonian (2.1) are \( \kappa_{i\alpha}, K_{i\alpha,j\beta} \). Here we suppose the pair interactions to be short-range

\[
K_{i\alpha,j\beta} = K_{\alpha\beta} \pi_{ij}; \quad K_{\alpha\beta} > 0; \quad \pi_{ij} = \begin{cases} 
1, & \text{if } j \in \pi_i \\
0, & \text{otherwise}
\end{cases} \tag{2.3}
\]

where \( \pi_i \) denotes the set of the nearest neighbours of the site \( i \) (the first coordination sphere).

From the very beginning we shall consider the quenched system (frozen sort configuration) and investigate its dynamics on the basis of Glauber’s master equation [1] for the distribution function \( \rho(\{S\}, t) \):

\[
\frac{d}{dt} \rho(\{S\}, t) = - \sum_i W_i(\cdots S_i \cdots) \rho(\cdots S_i \cdots, t) + \sum_i W_i(\cdots - S_i \cdots) \rho(\cdots - S_i \cdots, t) \tag{2.4}
\]

Following ref. [1] the probability per unit time that the \( i \)th spin flips from the value \( S_i \) to \( -S_i \) is assumed to be

\[
W_i(\cdots S_i \cdots) = \frac{1}{2\tau_i} (1 - S_i \tanh \beta \varepsilon_i) \tag{2.5}
\]

where

\[
\varepsilon_i = \kappa_i + \sum_j K_{ij} S_j \tag{2.6}
\]

is a local effective field acting on the \( i \)th spin. In this approach the dissipative properties of the spin subsystem is caused by its interaction with another (e.g. phonon) subsystems which play the role of a heat bath. The quantity \( \tau_i \) has the dimension of time and defines the time scale of relaxational phenomena. We assume the quantity \( \tau_i \) to be locally sort-dependent:

\[
\tau_i = \sum_{\alpha} \tau_{i\alpha} X_{i\alpha} \tag{2.7}
\]

Relations (2.4) and (2.5), (2.6) lead to the following kinetic equations for the average value of spin:

\[
D_{i,t} \langle S_i \rangle_{H,t} = \langle \tanh \beta \varepsilon_i \rangle_{H,t} \tag{2.8}
\]

where

\[
\langle \langle \cdots \rangle \rangle_{H,t} = Sp_s \rho(\{S\}, t) \langle \cdots \rangle; \quad D_{i,t} = 1 + \tau_i \frac{d}{dt} \tag{2.9}
\]

The system is disordered, therefore in order to calculate experimentally measured quantities one have to perform an averaging over the sort configurations. For example, the polarization of a partially deuterated H-bonded ferroelectric with the Hamiltonian (2.1) is defined as

\[
P(t) = \frac{1}{V} \sum_i \sum_{\alpha} \mu_{\alpha} \langle \langle S_{i\alpha} \rangle_{H,t} \rangle_x; \quad S_{i\alpha} = S_i X_{i\alpha} \tag{2.10}
\]

where \( V \) is a volume of the system, \( \mu_{\alpha} \) the dipole electric moment associated with the quasispin of sort \( \alpha \) (proton or deuteron), \( \langle \cdots \rangle_x \) denotes the
averaging over sort configurations with some density matrix \( \rho(\{X_{i\alpha}\}) \) that depends on conditions of system’s freezing. The two-site cluster approximation, which is used below, is sensitive only to the following moments of the distribution \( \rho(\{X_{i\alpha}\}) \)

\[
\langle X_{i\alpha} \rangle_x = c_\alpha, \quad \langle X_{i\alpha} X_{j\beta} \rangle_x = w_{\alpha\beta} (j \in \pi_i)
\]

(2.11)

where \( c_\alpha \) has the meaning of the sort \( \alpha \) spins’ concentration.

3. **Formulation of two-site cluster approximation**

The equation (2.8) is quite intractable. One can expand the \( \langle \tanh \beta \varepsilon_i \rangle_{H,t} \) in the right-hand side of (2.8) and find that this equation couples correlation functions (CFs) of the type \( \langle S_{i_1} \cdots S_{i_n} \rangle_{H,t} \) where the site indices belong to the first coordination sphere of the site \( i \) \((i_1 \cdots i_n \in \pi_i)\). The equation for \( \langle S_{i_1} \cdots S_{i_n} \rangle_{H,t} \) derived from (2.4) and (2.5) involves higher CFs, so the exact treatment of the Glauber’s equations has succeeded only in the case of one-dimensional system in zero field (ideal (one-sort) chain \([1, 3, 4]\), chain with single impurity \([5]\)). In other cases interpolative approximations are used which express higher CFs via lower, e.g. \( \langle \prod_j S_j \rangle_{H,t} \approx \prod_j \langle S_j \rangle_{H,t} \).

We suggest here the closure to the equation (2.8) in the spirit of a cluster approach. In a one-site approximation we replace contributions of all spins with effective fields

\[
\varepsilon_i \rightarrow \varepsilon_i^{[1]} = \bar{\kappa}_{i,t} + \sum_{r \in \pi_i} r \bar{\varphi}_{i,t} \equiv \bar{\kappa}_{i,t}
\]

(3.1)

where \( r \bar{\varphi}_{i\alpha} \) has the meaning of effective field acting on the spin \( i \) of sort \( \alpha \) from the nearest neighbour on the site \( r \). In a two-site approximation a contribution of one of the nearest neighbours is taken into account explicitly

\[
\varepsilon_i \rightarrow \varepsilon_i^{[2]} = \bar{\kappa}_{i,t} + \sum_{\substack{r \in \pi_i \\backslash 2 \r}} r \bar{\varphi}_{i,t} + K_{ij} S_j \equiv j \bar{\kappa}_{i,t} + K_{ij} S_j, (j \in \pi_i).
\]

(3.2)

In the case of ideal (one-sort) system no more assumptions are needed to obtain the closed set of equations for the average value of spin \( \langle S_i \rangle_{H,t} \)

\[
D_{i,t} \langle S_i \rangle_{H,t} = \langle \tanh \beta \varepsilon_i^{[1]} \rangle_{H,t}
\]

\[
D_{i,t} \langle S_i \rangle_{H,t} = \langle \tanh \beta \varepsilon_i^{[2]} \rangle_{H,t}.
\]

(3.3)

For the many-sort system it is necessary to point out a dependence of the fields \( r \varphi_{i\alpha} \) on the sort configuration. We shall believe the fields \( r \varphi_{i\alpha} \) to be different in the one-site approximations and the two-site one until averaging over sort configurations is carried out. For the sort averaging procedure both "one-site" and "two-site" fields will be assumed to be locally sort-dependent

\[
r \bar{\varphi}_{i\alpha,t} = \sum_{\alpha} X_{i\alpha} r \bar{\varphi}_{i\alpha,t}
\]

(3.4)

where \( r \bar{\varphi}_{i\alpha,t} \) is a c-number (it does not contain spin or sort operators).
After simple transformation the equation (3.3) can be written in the form

\[ D_{i,t}(S_i)_{H,t} = \tanh \beta \bar{\kappa}_{i,t} \]
\[ D_{i,t}(S_i)_{H,t} = L_{ij,t} + P_{ij,t}(S_j)_{H,t} \]
\[ D_{j,t}(S_j)_{H,t} = L_{ji,t} + P_{ji,t}(S_i)_{H,t}; \quad j \in \pi_i \]

where

\[ L_{ij,t} = \frac{1}{2} [\tanh \beta (j \bar{\kappa}_{it} + K_{ij}) + \tanh \beta (j \bar{\kappa}_{it} - K_{ij})] \]
\[ P_{ij,t} = \frac{1}{2} [\tanh \beta (j \bar{\kappa}_{it} + K_{ij}) - \tanh \beta (j \bar{\kappa}_{it} - K_{ij})] \]

The expressions (3.6) consist a system of nonlinear differential equations. To linearize this expressions we consider only linear response of the system on a small time-dependent changing of the external field. For this purpose we write the dynamic quantities in the following form:

\[ \kappa_{i,t} = \kappa_i + \delta \kappa_{i,t}; \quad \bar{\kappa}_{i,t} = \bar{\kappa}_i + \delta \bar{\kappa}_{i,t}; \]
\[ j \bar{\kappa}_{it} = j \bar{\kappa}_i + \delta j \bar{\kappa}_{i,t}; \quad \langle S_i \rangle_{H,t} = m^{(1)}_i + \delta m^{(1)}_i; \]
\[ L_{ij,t} = L_{ij} + L'_{ij} \delta j \bar{\kappa}_{i,t}; \quad P_{ij,t} = P_{ij} + P'_{ij} \delta j \bar{\kappa}_{i,t}; \]
\[ \tanh \beta \bar{\kappa}_{i,t} = \tanh \bar{\kappa}_i + (1 - \tanh^2 \bar{\kappa}_i) \delta \bar{\kappa}_{i,t} \]

Here \( \kappa_i, \bar{\kappa}_i, j \bar{\kappa}_i, m^{(1)}_i, L_{ij}, P_{ij} \) are the static (independent of time) components of corresponding quantities and \( \delta \kappa_{i,t}, \delta j \bar{\kappa}_{i,t}, \delta m^{(1)}_i, L'_{ij} \delta j \bar{\kappa}_{i,t}, P'_{ij} \delta j \bar{\kappa}_{i,t} \) represent linear in \( \delta \kappa_{i,t} \) deviations of these quantities from their static values.

\[ L_{ij} = \frac{1}{2} (\tanh \beta (j \bar{\kappa}_i + K_{ij}) + \tanh \beta (j \bar{\kappa}_i - K_{ij})) \]
\[ P_{ij} = \frac{1}{2} (\tanh \beta (j \bar{\kappa}_i + K_{ij}) - \tanh \beta (j \bar{\kappa}_i - K_{ij})) \]
\[ L'_{ij} = \frac{1}{2} \left( \cosh^{-2} \beta (j \bar{\kappa}_i + K_{ij}) + \cosh^{-2} \beta (j \bar{\kappa}_i - K_{ij}) \right) \]
\[ P'_{ij} = \frac{1}{2} \left( \cosh^{-2} \beta (j \bar{\kappa}_i + K_{ij}) - \cosh^{-2} \beta (j \bar{\kappa}_i - K_{ij}) \right) \]

Due to (3.7) two independent system of equations is obtained from (3.6). The static part:

\[ m^{(1)}_i = \tanh \beta \bar{\kappa}_i \]
\[ m^{(1)}_i = L_{ij} + P_{ij} m^{(1)}_j; \quad m^{(1)}_j = L_{ji} + P_{ji} m^{(1)}_i \]

can be reduced to the form

\[ m^{(1)}_i = \tanh \beta \bar{\kappa}_i = \frac{L_{ij} + P_{ij} L_{ji}}{1 - P_{ij} P_{ji}}. \]

The same equation is also obtained on the basis of cluster expansion of the free energy \[6\] and can be written in the form

\[ m^{(1)}_i = \langle S_i \rangle_{\rho_i} = \langle S_i \rangle_{\rho_{ij}} \]
where the one-site $\rho_i$ and intracluster $\rho_{ij}$ density matrices are defined as follows

$$
\rho_i = \frac{\exp(-\beta H_i)}{\text{Sp}_i \exp(-\beta H_i)}; \quad H_i = \bar{\kappa}_i S_i
$$

$$
\rho_{ij} = \frac{\exp(-\beta H_{ij})}{\text{Sp}_i \text{Sp}_j \exp(-\beta H_{ij})}; \quad H_{ij} = j \bar{\kappa}_i S_i + i \bar{\kappa}_j S_j + K_{ij} S_i S_j
$$

Multiplying (3.10) by $X_i^{\alpha}$ and averaging obtained relation over the sort configurations we get the equation for an average value of sort $\alpha$ spins $m_i^{(1)} \equiv \langle m_i^{(1)} X_i^{\alpha} \rangle_x$

$$
m_i^{(1)} = c_\alpha \tanh \bar{\kappa}_i^{\alpha}
$$

$$
m_i^{(1)} = \sum_\beta w_{\alpha\beta} \sinh \beta (j \bar{\kappa}_i^{\alpha} + i \bar{\kappa}_j^{\beta}) \cosh \beta (j \bar{\kappa}_i^{\alpha} - i \bar{\kappa}_j^{\beta}) + a_{\alpha\beta} \sinh \beta (j \bar{\kappa}_i^{\alpha} - i \bar{\kappa}_j^{\beta}) \cosh \beta (j \bar{\kappa}_i^{\alpha} + i \bar{\kappa}_j^{\beta})
$$

On assuming uniform field $\kappa_i^{\alpha} \to \kappa^{\alpha}$ (UF: $\bar{\varphi}_i^{\alpha} \to \bar{\varphi}^{\alpha}$, $j \bar{\kappa}_i^{\alpha} \to \kappa^{\alpha} + (z - 1) \bar{\varphi}_i^{\alpha}$, $\bar{\kappa}_i^{\alpha} \to \kappa^{\alpha} + z \bar{\varphi}_i^{\alpha}$, $z$ is a first coordination number) we obtain the known in corresponding static theories (TCA [2], first order of cluster variation method [7]) relation: (3.13) becomes the set of $2\Omega$ equations for the same number of quantities -- $\Omega$ order parameters $m_i^{(1)}$ and $\Omega$ effective fields $\bar{\varphi}^{\alpha}$.

The dynamic linearized part of (3.6) (here nonlinear on $\delta \kappa_{i,t}$ terms $L'_{ij}$, $\delta m_{i,t}^{(1)}$, $\delta i \bar{\kappa}_{j,t}$, $L'_{ji}$, $\delta m_{j,t}^{(1)}$, $\delta j \bar{\kappa}_{i,t}$ should be neglected) after time Fourier transformation and averaging over the sort configurations yields the Ornstein-Zernike-type equation for the dynamic structure factor $m_{i\alpha,j\beta}^{(2)}(\omega) = \frac{\delta m_{i\alpha,j\beta}^{(1)}(\omega)}{\delta \kappa_{i,j}(\omega)}$ of the model. Spatial Fourier transformation presents its solution in the form

$$
(\hat{m}^{(2)}(\vec{q}, \omega))^{-1} = (1 - z) (\hat{\phi}^{(2)})^{-1} + z (\hat{\phi}^{(20)} + \hat{\phi}^{(11)})^{-1}
$$

$$
+ (\pi_0 - \pi_\vec{q}) (\hat{\phi}^{(20)} (\hat{\phi}^{(11)})^{-1} \hat{\phi}^{(20)} - \hat{\phi}^{(11)})^{-1}
$$

where the matrix elements are

$$
(\hat{\phi}^{(2)}_{\alpha\beta})_{\alpha\beta}^{UF} = \delta_{\alpha\beta} c_\alpha \frac{1 - \tanh^2 \beta \bar{\kappa}_\alpha}{D_\alpha}
$$

$$
(\hat{\phi}^{(20)}_{\alpha\beta})_{\alpha\beta}^{UF} = \delta_{\alpha\beta} \sum_\gamma w_{\alpha\gamma} \frac{R_{\alpha\gamma} D_\gamma}{D_\alpha D_\gamma - P_{\alpha\gamma} P_{\gamma\alpha}}
$$

$$
(\hat{\phi}^{(11)}_{\alpha\beta})_{\alpha\beta}^{UF} = w_{\alpha\beta} \frac{R_{\beta\alpha} P_{\alpha\beta}}{D_\alpha D_\beta - P_{\alpha\beta} P_{\beta\alpha}}
$$

$$
(\hat{m}^{(2)}(\vec{q}, \omega))_{\alpha\beta} = \sum_j e^{i \vec{q} \vec{R}_j - \bar{R}_\alpha} m_{i\alpha,j\beta}^{(2)}(\omega)
$$

(3.15)
and
\[ P_{\alpha\beta} = \frac{1}{2} \left( \tanh \beta (\bar{\kappa}_{\alpha} + K_{\alpha\beta}) - \tanh \beta (\bar{\kappa'}_{\alpha} - K_{\alpha\beta}) \right) \] (3.16)
\[ R_{\alpha\beta} = 2a_{\alpha\beta} \left[ \left( 2a_{\alpha\beta} (1 + \cosh 2\beta \bar{\kappa'}_{\alpha} + \cosh 2\beta \bar{\kappa}_{\beta}) + (1 + a_{\alpha\beta}^2) (\cosh 2\beta \bar{\kappa'}_{\alpha} + \cosh 2\beta \bar{\kappa}_{\beta}) \right) \times \left( 1 + a_{\alpha\beta}^2 + 2a_{\alpha\beta} \cosh 2\beta \bar{\kappa}_{\alpha} \right) \right]^{-1} \times \left( \cosh \beta (\bar{\kappa'}_{\alpha} + \bar{\kappa}_{\beta}) + a_{\alpha\beta} \cosh \beta (\bar{\kappa'}_{\alpha} - \bar{\kappa}_{\beta}) \right)^{-2} \]

Factor \( \pi_q \) has the form
\[ \pi_q = \sum_{j \in \pi} e^{i\bar{\kappa}(R_j - R_i)}; \; \pi_0 = z \] (3.17)

4. Annealed model

In this case protons (deuterons) can go from one site (hydrogen bond) to another. Therefore the sort configuration changes in time. This motion should be described by Kawasaki master equation \[ ] [9]. But we assume that the sort configuration changes slowly and for the frequencies of magnetic (or ferroelectric) dispersion quasispins see sort configuration at rest. Therefore the formula (3.14) remains valid. If the temperature changes quite slowly, \( w_{\alpha\beta} \) is a temperature-dependent quantity which can be found by the static theory \[ ] [2].

5. Discussion. Numerical results.

The expression (3.14) is our target formula. Let us consider its partial cases. All expressions will be given for UF case.

1) Noninteracting system \( (K_{\alpha\beta} = 0 \forall \alpha, \beta) \) is a trivial case.
\[ m^{(1)}_{\alpha} = c_{\alpha} \tanh \beta \bar{\kappa}_{\alpha} \] (5.1)
\[ m^{(2)}_{\alpha\beta}(q, \omega) = \delta_{\alpha\beta} \frac{c_{\alpha}(1 - \tanh^2 \beta \bar{\kappa}_{\alpha})}{1 + i\omega \tau^0_{\alpha}} \] (5.2)

All the spins are independent, so the structure factor are independent of the wavelength. The relaxation times are equal to \( \tau^0_{\alpha} \).

2) One-sort (ideal) system. This case can be obtained in the limit \( c_1 = 1 \). The single order parameter \( m^{(1)}_1 \) and the effective field \( \varphi_1 \) satisfy the following equation
\[ m^{(1)}_1/c_1 \equiv \sigma^{(1)}_1 = \tanh \beta \bar{\kappa}_1 = \frac{\sinh 2\beta \bar{\kappa'}_1}{\cosh 2\beta \bar{\kappa'}_1 + a_{11}} \] (5.3)

where
\[ \bar{\kappa}_1 = \kappa_1 + z\varphi_1; \; \bar{\kappa'}_1 = \kappa_1 + z'\varphi_1; \; z' = z - 1; \; a = e^{-2K} \] (5.4)
The dynamic structure factor $m_{11}^{(2)}(\vec{q}, \omega)$ takes the form

$$m_{11}^{(2)}(\vec{q}, \omega)/c_1 \equiv \sigma_{11}^{(2)}(\vec{q}, \omega) = \frac{\sigma_{11}^{(2)}(\vec{q}, 0)}{1 + i\omega\tau(\vec{q})}, \quad (5.5)$$

Figure 1: Pure system on plane square lattice ($z = 4$). Static pair CF $m_{11}^{(2)}(\vec{q}, 0)$, inverse relaxation time $1/\tau(\vec{q})$, real and imaginary parts of dynamic pair CF $m_{11}^{(2)}(\vec{q}, \omega)$ at different wavevectors: $\pi\vec{q}$ equals 4 ($\vec{q} = 0$), 2, 0, $-2$, $-4$ (the last corresponds to the vertices of the first Brillouin zone).

where

$$\sigma_{11}^{(2)}(\vec{q}, 0) = \left(1 - \left(\sigma_{1}^{(1)}\right)^2\right) \frac{1 - P^2}{1 + \varepsilon' P^2 - P \pi_{\vec{q}}} \quad (5.6)$$
\[
P = \frac{1 - a^2}{1 + a^2 + 2a \cosh 2\beta \kappa'} - \frac{1}{1 - (\sigma_1^{(1)})^2},
\]

and \(\sigma_{11}^{(2)}(\vec{q},0)\) coincides with a spatial Fourier-transform of the static correlation function \(\langle S_i S_j \rangle_H\), which was founded with cluster expansion of the free energy within TCA in the ref. [3]. Result (5.6) is accurate in the one-dimensional case. Relaxation time

\[
\tau(\vec{q}) = \tau_0^0 \left(1 - \pi_1 \frac{P}{1 + z' P^2} \right)^{-1} T > T_c \tau_0^0 \left(1 - \pi_1 \frac{\tanh \beta K}{1 + z' \tanh^2 \beta K} \right)^{-1} \tag{5.7}
\]
is a decreasing function of \(\vec{q}\). Note, that known exact result [3] (zero external field, ideal one-dimensional system) is a partial case of the formula (5.5).

Figure 1 demonstrates the temperature and \(\vec{q}\)-dependence of the static structure factor \(m_{11}^{(2)}(\vec{q},0)\) and the relaxation time \(\tau(\vec{q})\) of the system on the plane square lattice (\(z = 4\)). Five lines represent five wavevectors: \(\pi_1 = 4, 2, 0, -2, -4\). In this figure (as well as in the formula (5.7)) one can see an interesting symmetry of inverse relaxation time: \(\tau_0^0 / \tau(\vec{q})\) line at \(\pi_1\) is mirror reflection of the line at \(-\pi_1\). The figure 1 shows also a behavior of real and imaginary parts of the dynamic structure factor (at the frequency \(\omega = 0.1 / \tau_0^0\)).

3. **Diluted system** (interacting spins (sort 1) are diluted by the noninteracting (nonmagnetic, nonferroelectric) impurities: \(K_{1\alpha} = 0\) for all \(\alpha \neq 1\)) is interesting due to percolation phenomena in such system: below some concentration of sort 1 spins there is no infinite cluster of interacting spins and ordered phase does not appear. The mean field approximation (MFA) fails to describe percolation phenomena [3]. At zero external field within MFA \(\sigma_1^{(1)}\) and pair CF per interacting spin \(\sigma_{11}^{(2)}\) depend on temperature, concentration and interaction strength only via \(T/T_c = T^{MFA}/T^{MFA}\) (\(T_c^{MFA} = z_1 c_1 K_{11}/k_B\)):

\[
\sigma_1^{(1)} = \sigma_1^{(1)}(T/T_c^{MFA}), \quad \sigma_{11}^{(2)} = \sigma_{11}^{(2)}(\vec{q},\omega,T/T_c^{MFA}).
\]

Therefore MFA predicts that \(\sigma_1^{(1)}\) reaches its saturation value \(\sigma_1^{(1)} = 1\) at \(T = 0\) at any concentration of interacting spins, and \(m_{11}^{(2)}(\vec{q},\omega)\) shows a single relaxation time behavior. In TCA \(T_c = \frac{2K_{11}}{k_B \ln \frac{z_1 c_1 K_{11}}{c_1 w_{11} - 1}}\),

\[
m_{11}^{(1)}(\vec{q}) = c_1 \tanh \beta \kappa_1 = w_{11} \frac{\sinh 2\beta \kappa'_1}{\cosh 2\beta \kappa'_1 + a_{11}} + (c_1 - w_{11}) \tanh \beta \kappa'_1 \tag{5.8}
\]

and this takes place for the case \(w_{1\alpha} = 0\) only, otherwise at zero temperature \(\sigma_1^{(1)} < 1\) (due to the presence of finite size clusters of interacting spins) and becomes zero if \(c_1 \geq z' w_{11}\) (below percolation point). In the case of complete chaos (\(w_{0\alpha} = c_0 c_\beta\)) TCA predicts percolation concentration \(c_p = (z - 1)^{-1}\).

We note that the Effective Field Approximation of Kaneyoshi et al. [8] (magnetization and static susceptibility) also reproduces this property of the diluted system.

Dynamic pair CF contains three terms

\[
m_{11}^{(2)}(\vec{q},\omega) = \frac{A_0}{1 + i\omega \tau_1^0} + \frac{A_+ (\vec{q})}{1 + i\omega \tau_+ (\vec{q})} + \frac{A_- (\vec{q})}{1 + i\omega \tau_- (\vec{q})} \tag{5.9}
\]
where

\[ A_0 = \frac{c_1 F_1^{(2)} w_{12} R_{12}}{z c_1 F_{1}^{(2)} - z' w_{12} R_{12}} \quad \text{as} \quad T \to T_c \quad \frac{c_1 w_{12}}{c_1 + z' w_{11}} \]

\[ A_{\pm} (\pi_{\vec{q}}) = \frac{B_{\pm} (\pi_{\vec{q}})}{1 - D_{\pm} (\pi_{\vec{q}})} \quad \tau_{\pm} (\vec{q}) = \frac{\tau_0}{1 - D_{\pm} (\pi_{\vec{q}})} \]  

(5.10)
Figure 2: Diluted system \((c_1 = 0.6, w_{11} = c_1^2)\) on plane square lattice (percolation concentration \(c_p = 1/3\)). Amplitudes \(A_+, A_-, A_0\), inverse relaxation times \(1/\tau_+(\vec{q})\), \(1/\tau_-(\vec{q})\), and dynamic pair CF \(m_{11}^{(2)}(\vec{q}, \omega)\) at different \(\vec{q} = \pi\vec{q} = 4, 2, 0, -2, -4\).
Figure 3: Diluted system. Temperature and frequency dependence of dynamic pair $C_{m}^{(2)}(\vec{q}, \omega)$ at $\vec{q} = 0$.

$$B_{\pm}(\pi_{\vec{q}}) = \pm \frac{c_{1} F_{1}^{(2)}}{PD_{\pm}(\pi_{\vec{q}}) \sqrt{\mathcal{D}(\pi_{\vec{q}})}} \left( \langle R_{1} \rangle^{2} D_{\pm}^{2}(\pi_{\vec{q}}) - (Pw_{12}R_{12})^{2} \right)$$

and

$$\langle R_{1} \rangle = w_{11}R_{11} + w_{12}R_{12}$$

$$D_{\pm}(\pi_{\vec{q}}) = P \frac{c_{1} F_{1}^{(2)} w_{11}R_{11}}{2\langle R_{1} \rangle} \pm \frac{\sqrt{\mathcal{D}(\pi_{\vec{q}})}}{\left( zc_{1} F_{1}^{(2)} - z'\langle R_{1} \rangle \right)}$$

$$\mathcal{D}(\pi_{\vec{q}}) = \left( c_{1} F_{1}^{(2)} w_{11}R_{11} \pi_{\vec{q}} \right)^{2}$$

$$+ 4\langle R_{1} \rangle \left( zc_{1} F_{1}^{(2)} - z'\langle R_{1} \rangle \right)w_{12}R_{12} \left( zc_{1} F_{1}^{(2)} - z'R_{12} \right)(5.11)$$
Figure 4: Spontaneous polarization $P$, inverse permittivity $1/\varepsilon$ and inverse relaxation time $1/\tau$ of the ferroelectric $Cs(H_{1-x}D_x)_2PO_4$ for different degrees of deuteration. Squares correspond to experimental data of [12], bold lines on the lower picture connect experimental points from the ref. [13]. Thick lines – theoretical results for annealed system with parameters given in table 1 and concentrations $x = 0, 0.18, 0.54, 0.88, 1$. 
Figure 5: Comparison of experiments and quenched model with assumption $w_{\alpha\beta} = c_\alpha c_\beta$ (complete chaos) (see caption to the previous figure). Thick lines – theoretical results for quenched system with parameters given in table 1 and concentrations $x = 0, 0.12, 0.47, 0.83, 1$. 
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Table 1: The model parameters describing experimental data on the pseudo-one-dimensional ferroelectric Cs(H$_1$-xD$_x$)$_2$PO$_4$: short-range coupling $K_{\alpha\beta}$; long-range interaction $J_{\alpha\beta}$; effective dipole electric moments $\mu_\alpha$ associated with proton or deuteron on hydrogen bond; parameters of master equation – relaxation times $\tau^0_\alpha$ of noninteracting protons or deuterons. $V_{\alpha\beta}$, $I_{\alpha\beta}$ are parameters describing nonexchange interactions between pseudospins [2]. Deuterons correspond to the pseudospins of sort 1.

| Parameter                                      | Quenched | Annealed |
|------------------------------------------------|----------|----------|
| $K_{11}/k_B$, K                                | 610      | 500      |
| $K_{12}/k_B$, K                                | 500      | 500      |
| $K_{22}/k_B$, K                                | 390      | 390      |
| $J_{11}/k_B$, K                                | 2.85     | 2.85     |
| $J_{12}/k_B$, K                                | 3.2      | 1.6      |
| $J_{22}/k_B$, K                                | 1.06     | 1.06     |
| $(\bar{V}_{11} + \bar{V}_{22} - 2\bar{V}_{12})/k_B$, K | -1000    | -1000    |
| $I_{12}/k_B$, K                                | -0       | -0       |
| $\mu_1$, e.s.u. x cm                           | $2.18 \times 10^{-18}$ |
| $\mu_2$, e.s.u. x cm                           | $1.87 \times 10^{-18}$ |
| $\tau^0_1$, s                                  | $3 \times 10^{-14}$ |
| $\tau^0_2$, s                                  | $5.5 \times 10^{-15}$ |

The functions $F_1^{(2)}$, $R_{12}$, $R_{11}$, $P$ are given by

$$F_1^{(2)} = c_1 (1 - (\sigma_1^{(1)})^2)$$
$$R_{12} = 1 - \tanh^2 \beta \kappa'_1$$
$$R_{11} = 4a \left( \frac{1 + a \cosh 2\beta \kappa'_1}{(\cosh 2\beta \kappa'_1 + a)(1 + a^2 + 2a \cosh 2\beta \kappa'_1)} \right)$$
$$P = \frac{1}{2} \left[ \tanh \beta (\kappa'_1 + K) - \tanh \beta (\kappa'_1 - K) \right]$$

(5.12)

The first term of (5.9) has the (5.2)-like behavior (relaxation time equals $\tau^0_1$), so we believe it to be a contribution of isolated spins (i.e. the spins, which all neighbours are noninteracting). The last two terms of (5.9) have the following symmetry

$$B_\pm (\pi \bar{q}) = B_x (\pi \bar{q}); \quad A_+ (-z) = A_- (z) = 0$$
$$D_\pm (\pi \bar{q}) = -D_\pm (\pi \bar{q}); \quad D_+ (\pi \bar{q}) > 0; \quad D_- (\pi \bar{q}) < 0$$

(5.13)

The relaxation time $\tau_-$ is always less then $\tau^0_1$ while $\tau_+$ is always greater then $\tau^0_1$ and slows down at the critical point ($\tau_- (0) \to \infty, \quad A_+ (\pi \bar{q}) \to \infty$).

Both $\tau_-$ and $\tau_+$ increase with the approaching to the center of the first Brillouin zone ($\bar{q} \to 0$). The amplitude $A_-$ is equal to 0 in the centre of the first Brillouin zone ($\pi \bar{q} = -z$) while $A_+$ vanishes at $\pi \bar{q} = -z$ (in the vertices of the first Brillouin zone).

In the case $w_1 = c_1$ ($w_{1\alpha} = 0 \forall \alpha \neq 1$, all interacting spins belong to one infinite cluster), $A_0$ is equal to zero, $A_-$ equals zero at $\pi (\bar{q}) > 0$ and $A_+$ is equal to zero at $\pi (\bar{q}) < 0$; $\sigma^{(1)}_1$ and pair CF are given by the formulae (5.3), (5.5).
Figure 2 shows the pair CF (5.9) of the diluted system on the plane square lattice \( (z = 4) \). In the figure 3 one can observe that presence of many relaxation times in dynamic pair CF of the diluted system can obviously be seen at small temperatures and frequencies. Abscissas of maxima of \(-\text{Im} m^{(2)}_{11}(\vec{0}, \omega)\) approximately equal inverse relaxation times: \( \omega_0 \approx 1/\tau_0^0 \), \( \omega_+ \approx 1/\tau_+^0 \). Specific temperature dependence of \( m^{(2)}_{11}(\vec{0}, \omega) \) is connected with nonmonotonous behavior of \( \tau_+^0(\vec{0}) \).

6. Experiments

One can see that TCA reflects essential properties of the diluted system. It seems to us that quality of the approximation is even higher for the system without noninteracting impurities \((K_{\alpha\beta} \neq 0)\) because such ”effective field” theories are more appropriate in the case of small fluctuations in a system considered. Obtained within TCA characteristics can be used in order to describe a variety of experimental data on partially deuterated ferroelectrics with hydrogen bonds and alloys of magnets. For example, good fit of TCA results for the ideal (one-sort) system to experimental data on the ferroelectric dispersion in pseudo-one-dimensional \((z = 2)\) ferroelectrics \( \text{CsH}_2\text{PO}_4 \) and its deuterated analog \( \text{CsD}_2\text{PO}_4 \) has been reached in the ref. \([10, 11]\). The many-sort theory developed here is applicable to partially deuterated compound \(- \text{Cs}(\text{H}_1-x\text{D}_x)_2\text{PO}_4 \). We obtain a good agreement of TCA results on annealed model and results of dielectric measurements \([12, 13]\) (figure 4). Fit of the quenched model (figure 5) to experimental data is not so good. The model parameters are shown in the table 1. More information about an actual type of the system can be given by measurement in the low-temperature region, where significant differences between behaviours of the annealed and quenched models exist, and by the wavevector-dependent experiments (e.g., neutron scattering).

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Here we consider a system with the Hamiltonian

\[
-\mathcal{H}/k_B T = \frac{1}{2} \sum_{i\alpha,j\beta} K_{i\alpha,j\beta} S_{i\alpha} S_{j\beta} + \frac{1}{2} \sum_{i\alpha,j\beta} J_{i\alpha,j\beta} S_{i\alpha} S_{j\beta} + \sum_{i\alpha} \Gamma_{i\alpha} S_{i\alpha}, \quad (A1)
\]

where \( K_{i\alpha,j\beta}, J_{i\alpha,j\beta} \) are pair interaction parameters, \( \Gamma_{i\alpha} \) external field acting on the spin of sort \( \alpha \) on the site \( i \). The interaction \( K_{i\alpha,j\beta} \) is same as in (2.1)-(2.3), but \( J_{i\alpha,j\beta} \) is assumed to be of long-range type, i.e. not restricted to the nearest neighbour sites. The Hamiltonian (A1) can be written as

\[
\mathcal{H} = MFA H - \frac{k_B T}{2} \sum_{i\alpha,j\beta} J_{i\alpha,j\beta} \Delta L_{(1)}^{m_{i\alpha}} \Delta L_{(1)}^{m_{j\beta}} \quad (A2)
\]

\[
MFA H = H + \frac{k_B T}{2} \sum_{i\alpha,j\beta} J_{i\alpha,j\beta} L_{(1)}^{m_{i\alpha}} L_{(1)}^{m_{j\beta}}
\]

\[
\Delta L_{(1)}^{m_{i\alpha}} = S_{i\alpha} - L_{(1)}^{m_{i\alpha}},
\]

where \( H \) is the Hamiltonian (2.1) in which the field \( \kappa \) is taken in the form

\[
\kappa_{i\alpha} = \Gamma_{i\alpha} + \sum_{j\beta} J_{i\alpha,j\beta} L_{(1)}^{m_{j\beta}}. \quad (A3)
\]

MFA assumes

\[
L_{(1)}^{m_{i\alpha}} = \langle \langle S_{i\alpha} \rangle_{\mathcal{H},t} \rangle_x \quad (A4)
\]

and ignores last term of (A2):

\[
\mathcal{H} \rightarrow MFA H = H + \frac{k_B T}{2} \sum_{i\alpha,j\beta} J_{i\alpha,j\beta} L_{(1)}^{m_{i\alpha}} L_{(1)}^{m_{j\beta}}. \quad (A5)
\]

One can see that formalism of the section 3 is applicable to the system with the Hamiltonian (A5). The equation of motion has the same form

\[
D_t \langle S_{i} \rangle_{\mathcal{H},t} = \langle \tanh \beta \varepsilon_{i} \rangle_{\mathcal{H},t}; \quad \varepsilon_{i} = \sum_{j} K_{ij} S_{j} + \kappa_{i}, \quad (A6)
\]
but now the field $\kappa_i$ includes molecular field (see (A3)) and thus couples the selfconsistency parameter $L m_{i\alpha}^{(1)}$. Then one can find that

$$L m_{i\alpha}^{(1)} = m_{i\alpha}^{(1)},$$  \hspace{1cm} \text{(A7)}$$

where $m_{i\alpha}^{(1)}$ is defined by the equation (3.13) and

$$L m_{i\alpha,j\beta}(\omega) \equiv \frac{\delta L m_{i\alpha}^{(1)}(\omega)}{\delta \Gamma_{j\beta}(\omega)} = \sum_{l\gamma} \frac{\delta m_{i\alpha}^{(1)}(\omega)}{\delta \kappa_{l\gamma}(\omega)} \frac{\delta \kappa_{l\gamma}(\omega)}{\delta \Gamma_{j\beta}(\omega)}$$

$$= \sum_{l\gamma} m_{i\alpha,l\gamma}(\omega) \left( \delta_{lj} \delta_{\gamma\beta} + \sum_{k\delta} J_{l\gamma,k\delta} L m_{k\delta,j\beta}^{(2)}(\omega) \right).$$  \hspace{1cm} \text{(A8)}$$

(A8) leads to the following expression for the structure factor

$$\left( L \hat{m}^{(2)}(\mathbf{q}, \omega) \right)^{-1} = \left( \hat{m}^{(2)}(\mathbf{q}, \omega) \right)^{-1} - \hat{J}(\mathbf{q}),$$  \hspace{1cm} \text{(A9)}$$

where $\hat{m}^{(2)}(\mathbf{q}, \omega)$ is defined by (3.14) and $\left( \hat{J}(\mathbf{q}) \right)_{\alpha\beta}$ is a Fourier-transform of $J_{i\alpha,j\beta}$. 
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