ON THE DUFLO-SERGANOVA FUNCTOR FOR THE QUEER LIE SUPERALGEBRA

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Abstract. We study the Duflo-Serganova functor $DS_x$ for the queer Lie superalgebra $q_n$ and for all odd $x$ with $[x, x]$ semisimple. For the case when the rank of $x$ is 1 we give a formula for multiplicities in terms of the arch diagram attached to $\lambda$. Further, we prove that $DS_x(L)$ is semisimple if $L$ is a simple finite-dimensional module and $x$ is of rank 1 satisfying $x^2 = 0$.

1. Introduction

For a finite dimensional complex Lie superalgebra $g$ and an odd element $x$ satisfying $[x, x] = 0$, M. Duflo and V. Serganova defined a functor $DS_x : Rep(g) \rightarrow Rep(g_x)$ where $g_x := \text{Ker} \text{ad}(x)/\text{Im} \text{ad}(x)$. For $g = \mathfrak{gl}(m|n), \mathfrak{osp}(m|2n), p_n, q_n$ the algebra $g_x$ is isomorphic to $\mathfrak{gl}(m-s|n-s), \mathfrak{osp}(m-2s|2n-2s), p_{n-s}, q_{n-2s}$ respectively where $s$ is a non-negative number called rank of $x$.

1.1. Previous results. Let $g$ be a finite-dimensional Kac-Moody superalgebra and let $L := L_\lambda(g)$ be a simple finite-dimensional $g$-module of the highest weight $\lambda$. For a fixed $x$ of rank $r$ we denote $DS_x$ by $DS_r$ (we will only use this notation for Kac-Moody superalgebras). The following properties of the $DS$-functor were obtained in [20], [14], [12]:

(a) the multiplicities of irreducible constituents of $DS_1(L)$ are at most 2;
(b) the following are equivalent
- $L$ is typical;
- $DS_x(L) = 0$ for all non-zero $x$;
- $DS_1(L) = 0$;
(c) $DS_x(L)$ does not have subquotients differing by the parity change;
(d) $DS_x(L)$ is semisimple;
(e) $DS_s(L) \cong DS_1(\cdots (DS_1(L)\cdots))$.

The $p_n$-case was studied in [8]: in this case (a)–(c) hold and (d), (e) do not hold.

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Throughout this article, $\Pi$ stands for the parity change functor. We will use the notation $[N : L]$ for the “non-graded multiplicity” of a simple module $L$ in a finite length module $N$: this is equal to the usual multiplicity if $L \cong \Pi L$, and to the sum of multiplicites of $L$ and of $\Pi L$ if $L \not\cong \Pi L$.

In the “non-exceptional cases” $\mathfrak{gl}(m|n), \mathfrak{osp}(m|n)$ and $\mathfrak{p}_n$ the non-graded multiplicity $[\text{DS}_1(L_\lambda) : L_\mu(\nu)]$ is given in terms of so-called “arc diagrams”: this multiplicity is non-zero if and only if the arc diagram $\text{Arc}(\nu)$ can be obtained from the arc diagram $\text{Arc}(\lambda)$ by removing a maximal arc; this multiplicity is 1 for $\mathfrak{gl}(m|n)$, $\mathfrak{p}_n$ and is 1 or 2 for $\mathfrak{osp}(m|n)$.

Extension of the Duflo-Serganova functor. In this paper we study $\text{DS}_x(L)$ for the case when $L = L_q(\lambda)$ is a simple finite-dimensional $q_n$-module. For $q_n$ it is important to expand our use of the DS-functor to odd elements $x$ for which $[x, x]$ is a semisimple element of $\mathfrak{g}_0$; this is especially true because of the Cartan subalgebra admitting a nontrivial odd part.

Write $x^2 := \frac{1}{4}[x, x] \in \mathfrak{g}_0$. Recall that if $x \in \mathfrak{g}_\Sigma$ has that $x^2$ acts semisimply on a $\mathfrak{g}$-module $M$, we may define the Lie superalgebra

$$M_x := \frac{\ker(x : M^{x^2})}{\text{Im}(x : M^{x^2})},$$

where $M^{x^2}$ is the subspace of $M$ killed by $x^2$. In particular if $\text{ad}(x^2)$ acts semisimply, we obtain a Lie superalgebra $\mathfrak{g}_x$, and $M_x$ will naturally be a module over $\mathfrak{g}_x$. Thus we will be interested primarily in the space:

$$\mathfrak{g}_\Sigma^* = \{ x \in \mathfrak{g}_\Sigma \mid \text{ad}[x, x] \text{ is semisimple} \}.$$

Remark. When $\mathfrak{g}$ is a Kac-Moody Lie superalgebra, the usual definition of rank naturally extends to all elements in $\mathfrak{g}_\Sigma^*$, and for these superalgebras the description of $\text{DS}_x L(\lambda)$ remains independent of which element is chosen of a given rank, if $L(\lambda)$ is finite-dimensional. However their actions on the larger category of finite-dimensional $\mathfrak{g}$-modules differs significantly.

The case of $\mathfrak{g} = q_n$ and arc diagrams. On the other hand, for $\mathfrak{g} = q_n$ the behavior of the functor $\text{DS}_x$ on simple modules does not reduce to the case when $x^2 = 0$; in fact an element $x \in \mathfrak{g}_\Sigma^*$ has rank valued in $\frac{1}{2}\mathbb{N}$, whereas if $x^2 = 0$ we must have $\text{rank}(x) \in \mathbb{N}$. Further, there are infinitely many $G_0$-orbits on $\mathfrak{g}_\Sigma^*$ in contrast to the self-commuting cone which has only finitely many orbits. We as yet do not understand $\text{DS}_x$ on simple modules for an arbitrary $x \in \mathfrak{g}_\Sigma^*$; however we know what the possible simple constituents are, and in certain cases we know exactly how the functor behaves.

In order to study the simple constituents of $\text{DS}_x L(\lambda)$, we introduce arc diagrams that have similarities to those used for $\mathfrak{gl}, \mathfrak{osp}$, and $\mathfrak{p}$. In these arc diagrams there are full arcs, illustrated with solid lines, and half-arcs, illustrated with dotted lines. The half arcs are...
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exactly those emanating from 0. Each dominant weight \( \lambda \) has an associated arc diagram; for instance if \( \lambda = 7\epsilon_1 + 4\epsilon_2 + \epsilon_3 - \epsilon_6 - 7\epsilon_7 \) then \( \text{Arc}(\lambda) \) is given by:

In the above picture, the symbols \( \wedge \) lie at 0, and all other symbols lie at positions of \( \mathbb{N}_{>0} \).

Write \( P^+(\mathfrak{g}) \) for the dominant weights of a Lie superalgebra \( \mathfrak{g} \). The main theorem is as follows:

1.4. **Theorem.** Take \( \lambda \in P^+(\mathfrak{q}_n) \) and \( \nu \in P^+(\mathfrak{q}_{n-2s}) \) where \( s \in \frac{1}{2}\mathbb{N} \), and let \( x \in \mathfrak{g}_s^* \) be of rank \( s \).

(i) If \( [\text{DS}_x(L(\lambda)) : L(\nu)] \neq 0 \), then \( \text{Arc}(\nu) \) can be obtained from \( \text{Arc}(\lambda) \) by successively removing \( s \) maximal arcs.

(ii) The following are equivalent

- \( \text{smult}(\text{DS}_x L(\lambda), L(\nu)) \neq 0 \).
- \( [\text{DS}_x(L(\lambda)) : L(\nu)] = 1 \).
- \( \text{zero}(\lambda) - \text{zero}(\nu) = 2s \).

(iii) The indecomposable summands of \( \text{DS}_x(L(\lambda)) \) are isotypical.

Here \( \text{smult}(\text{DS}_x(L(\lambda)), L(\nu)) = 0 \) means that \( L(\nu) \) appears in \( \text{DS}_x L(\lambda) \) the same number of times as \( L(\nu) \); and if \( L(\nu) \cong L(\nu) \) then it further means that \( L(\nu) \) appears an even number of times in \( \text{DS}_x(L(\lambda)) \). Part (ii) is proven in [16], as well as in section 5.

Note that for a dominant \( \mathfrak{q}_n \)-weight \( \lambda \) there exists at most one dominant \( \mathfrak{q}_{n-2s} \)-weight \( \nu \) such that the sets of non-zero coordinates of \( \nu \) and of \( \lambda \) coincide; such \( \nu \) exists if and only if \( \lambda \) has at least \( 2s \) of zero coordinates.

1.5. **Case of rank** \( x \leq 1 \) **and** \( x = C_r \). For \( x \) with rank \( x \leq 1 \), we are able to precisely describe the multiplicities of the composition factors in a similar manner to the presentations of [20], [13], see Theorems 5.2, 5.3. Further, for each \( r \leq n \), we may consider the particular element

\[
C_r := \begin{bmatrix} 0 & B_r \\ B_r & 0 \end{bmatrix}, \quad \text{where} \quad B_r = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}.
\]

Then \( \text{DS}_{C_r} \) is completely described on simple modules, and has the special property that for a simple module \( L \), \( \text{DS}_{C_r}(L) \) is either again simple or is zero.

1.6. **Properties (a)-(e).** Let \( L \) be a simple finite-dimensional \( \mathfrak{q} \)-module. We discuss to what extent the properties from Section 1.1 extend to \( \mathfrak{g} = \mathfrak{q}_n \).

(a) \( \text{DS}_x(L) \) is either simple or zero if rank \( x = \frac{1}{2} \); the multiplicities of irreducible constituents of \( \text{DS}_x(L) \) are at most 2 if \( x \) is of rank 1;
(b) the following are equivalent
- $L$ is typical;
- $DS_x(L) = 0$ for all non-zero $x$;
- $DS_x(L) = 0$ for all $x$ of rank $\leq 1$.
(c) if the non-graded multiplicity of a simple $\mathfrak{g}_x$-module $L'$ in $DS_x(L)$ is not 1, then this non-graded multiplicity is even and $L'$ appears in $DS_x(L)$ the same number of times as $\Pi(L)$, see Theorem 1.4 (ii).

1.6.1. Remark on (b). If $DS_x(L(\lambda)) = 0$ for all $x$ of rank one, it does not imply that $L(\lambda)$ is typical; rather it implies that $\text{Arc}(\lambda)$ has at most one maximal arc, and if it exists it is a half-arc and zero($\lambda$) = 1. An example is given by the adjoint representation of $\mathfrak{psq}_3$. The condition that $DS_x(L(\lambda)) = 0$ for $x$ of rank one-half is equivalent to asking that $\text{Arc}(\lambda)$ has no half-arcs, in other words zero($\lambda$) = 0.

1.6.2. Property (d). Part (iii) of Theorem 1.4 states that

(d') $\text{Ext}^1(L', L'') = 0$ if $L'' \neq L'$, $\Pi L'$ are simple subquotients of $DS_x(L)$

holds in the $\mathfrak{q}_n$-case. If $\mathfrak{g}$ is a finite-dimensional Kac-Moody superalgebra, this property along with (c) implies (d) (i.e. the semisimplicity of $DS_x(L)$).

In the $\mathfrak{q}_n$-case we do not yet know if (d) holds in general; however (d') does imply the semisimplicity of $DS_x(L(\lambda))$ for $\lambda$ with no zero coordinates (in particular, for half-integral weights). Further, in Corollary 5.6 we prove that $DS_x(L)$ is semisimple for any simple finite-dimensional module $L$, when $x$ is of rank 1 with $x^2 = 0$.

The following example shows that semisimplicity does not hold for $\mathfrak{sq}_n$: for a dominant weight $\lambda$ of the form $\lambda = \sum_{i=1}^{p} \lambda_i \varepsilon_i + \sum_{i=p}^{n-2} \lambda_i \varepsilon_{i+2}$ with $\sum_{i=1}^{n-2} \lambda_i^{-1} = 0$ we have $L_{\mathfrak{q}_n}(\lambda) = L_{\mathfrak{sq}_n}(\lambda)$. The $\mathfrak{q}_{n-2}$-module $L_{\mathfrak{q}_{n-2}}(\sum_{i=1}^{n-2} \lambda_i \varepsilon_i)$ is a submodule of $DS_x(L_{\mathfrak{q}_n}(\lambda))$ for $x$ of rank 1 with $x^2 = 0$, and this module is a non-splitting extension of two simple $\mathfrak{sq}_{n-2}$-modules with the same highest weights.

1.6.3. Property (e). Such a property fails completely if consider arbitrary $x \in \mathfrak{g}_T^{sa}$; it is possible it will hold for those $x$ with $x^2 = 0$. However we have not computed $DS_x(L)$ for $x$ of rank greater than 1.

1.7. Grading. Given an element $x \in (\mathfrak{q}_n)_T$ with $x^2 = 0$, we may find a semisimple element $h \in (\mathfrak{q}_n)_T$ satisfying $[h, x] = x$. In this way we obtain an action of $h$ on $DS_x$ which commutes with $\mathfrak{g}_x$, and thus induces a grading on the $\mathfrak{g}_x$-module $DS_xV$ for a finite-dimensional $\mathfrak{q}_n$-module $V$ (details explained in Sections 3.5 and 7.1.2). Thus $h$ will act by a scalar on each composition factor of $DS_xV$. Further it allows us to view $DS_x$ as a functor from $\mathfrak{g}$-modules to $\mathfrak{g}_x \times \mathbb{C}(h)$-modules.

If $x$ is of rank 1 and $L$ is a finite-dimensional simple module, we have computed explicitly the eigenvalues of $h$ on the composition factors of $DS_xL$, and we obtain the following:
1.8. **Theorem.** If \( x \) is of rank 1 and \( L \) is a simple finite-dimensional \( \mathfrak{q}_n \)-module, then \( DS_x L \) is a semisimple, multiplicity-free \( \mathfrak{g}_x \times \mathbb{C}(h) \)-module.

We note that for a finite-dimensional Kac-Moody Lie superalgebra \( \mathfrak{g} \) we also always have such an element \( h \) which will acts on \( DS_x \) and commutes with \( \mathfrak{g}_x \), giving a grading. We expect that in these cases we also have that if \( L \) is a simple finite-dimensional \( \mathfrak{g} \)-module, then \( DS_x L \) will be multiplicity-free as a module over \( \mathfrak{g}_x \times \mathbb{C}(h) \). For \( \mathfrak{gl}(m|n) \) the grading on \( DS_1(L) \) was computed in [20].

1.9. **Projectivity.** In Corollary 5.3.2 we show that if \( L \) is a simple finite-dimensional module, then it is projective if and only if \( DS_x L = 0 \) for all \( x \) with rank \( x \leq 1 \) (in fact a slightly stronger statement is true). It would be interesting to understand whether this generalizes to all finite-dimensional modules.

1.10. **Methods.** The approach to computing composition factors and multiplicities of \( DS_x L(\lambda) \) is the same as in the \( \mathfrak{osp} \)-case (see [14]): using suitable translation functors the problem is reduced to the case of \( \mathfrak{q}_{2s} \), where \( x \) is of rank \( s \). For \( s \) of rank \( \leq 1 \), we may do the calculations on \( \mathfrak{q}_1 \) or \( \mathfrak{q}_2 \) where they are easily performed. The case of \( x = C_r \) is done by using the \( \mathfrak{g}_0 \)-invariance of \( x \) when \( n = r \).

The formula \((d')\) follows from Theorem 5.1 (i) and the following fact obtained in [13]: if \( \text{Ext}^1(L(\nu), L(\nu')) \neq 0 \) for some distinct dominant weights \( \nu, \nu' \), then the diagram of one for these weights is obtained from the diagram of the other one by moving a symbol \( \times \) along one of the arches (where \( \land \) is considered as a half of \( \times \)). The same reasoning works in the Kac-Moody case. Note that the previous proofs of \((d')\) in the Kac-Moody case were based on a stronger result: the existence bipartition of \( \text{Ext}^1 \)-graph compatible with the action of DS-functor. The latter result does not hold in the \( \mathfrak{q}_n \)-case.

Our proof of complete reducibility for \( DS_x(L) \) when \( x \) is of rank 1 with \( x^2 = 0 \) relies on the computation of the grading, and ultimately Theorem 1.8. The grading is computed by reducing to the case of \( \mathfrak{q}_2 \) as above, using the same techniques.

1.11. **Outlook for higher rank elements.** One of the central unanswered questions that we leave to future work is the computation of composition factors and multiplicities of \( DS_x(L(\lambda)) \) for \( x \) of rank bigger than 1. Unlike in the Kac-Moody cases, where one only has to look at those \( x \) with \([x, x] = 0 \) (see Remark 1.2.1), for \( \mathfrak{q}_n \) (as previously mentioned) we may obtain very different results for elements in \( \mathfrak{g}_{ss}^\times \) lying in distinct \( G_0 = GL(n) \)-orbits (see Section 3.1 for a description of these orbits).

The \( G_0 \)-orbits are parametrized by two pieces of data: first is the rank, a half-integer \( r \). Once we fix a rank \( r \), the orbits become parametrized by \( 2r \) unordered complex numbers, where an even number of them are 0 (they are given by the eigenvalues of the matrix \( B \) in [1,13]). This space is an open subvariety of \( \mathbb{C}^{2r}/S_{2r} \). It follows that the behavior of \( DS_x \) for \( x \) of rank \( r \) will be determined by values of symmetric functions on the \( 2r \) complex numbers; or more precisely, due to the canonical equality \( DS_{cx} = DS_x \) for all \( c \in \mathbb{C}^\times \),
the behavior of $DS_x$ should be determined by the zero sets of homogeneous symmetric functions on the $2r$ complex numbers; for an example see the computation of $DS_x(\text{psq}_n)$ below.

1.12. Kac-Wakimoto Conjecture and depth. It is of interest to understand to what extent a version of the (generalized) Kac-Wakimoto conjecture, proven in [26] for Kac-Moody Lie superalgebras, holds for $\mathfrak{q}_n$. Recall that the results of [26] show that in the Kac-Moody case we have $DS_x(L) \neq 0$ if $L$ is a simple finite-dimensional module of atypicality $r$ and rank $x \leq r$.

For $\mathfrak{g} = \mathfrak{q}_n$ such a result cannot hold (see Remark 1.6.1). Nevertheless one may ask, given $\lambda$ of atypicality $r$, does there exist $x$ of rank $r$ satisfying $DS_x(L(\lambda)) \neq 0$.

Another possible formulation is the following: given $\lambda$ of atypicality $r$, does there exist a chain of $DS$ functors, $DS_{x_1}, \ldots, DS_{x_k}$, such that $\sum \text{rank}(x_i) = r$ and $(DS_{x_k} \circ \cdots \circ DS_{x_1})L(\lambda) \neq 0$.

We give a positive answer to this question via the notion depth, introduced in Section 5.7 (note the definition differs slightly from the one in [11]). We show that, as in the Kac-Moody case, depth($N$) does not exceed the atypicality of $N$ and these numbers are equal if $N$ is a simple finite-dimensional module.

1.13. $DS_x$ for queer-type superalgebras. As an illustration of these ideas, we have computed the value of the functor $DS_x$ on all queer-type algebras, i.e. $\mathfrak{q}_n$, $\mathfrak{s}_n$, $\mathfrak{p}_n$, and $\mathfrak{psq}_n$. In each case we obtain a new superalgebra, and we compute its structure. If the rank of $x$ is less than $n/2$, then $DS_x(\mathfrak{q}_n) = \mathfrak{q}_{n-2r}$ with the similar formulae for $\mathfrak{s}_n$, $\mathfrak{p}_n$, and $\mathfrak{psq}_n$. For $x$ of rank $n/2$ we have $DS_x(\mathfrak{q}_n) = 0$, $DS_x(\mathfrak{s}_n) = \mathbb{C}$ and $DS_x(\mathfrak{p}_n) = \Pi\mathbb{C}$. For $\mathfrak{psq}_n$ we have the following interesting behavior: taking $x = \begin{pmatrix} 0 & B \\ B & 0 \end{pmatrix}$ of rank $n/2$ and letting $c_1, \ldots, c_n$ be the eigenvalues of $B$ we obtain

$$DS_x(\text{psq}_n) \cong \begin{cases} \mathbb{C}^{1|1} & \text{if } \sum_i c_i \ldots \hat{c_i} \ldots c_n = \sum_i c_i^3 \ldots \hat{c_i} \ldots c_n = 0 \\
\mathfrak{q}_1 & \text{if } \sum_i c_i \ldots \hat{c_i} \ldots c_n = 0 \neq \sum_i c_i^3 \ldots \hat{c_i} \ldots c_n \\
0 & \sum_i c_i \ldots \hat{c_i} \ldots c_n \neq 0 \end{cases}$$

where by $\mathbb{C}^{1|1}$ we mean the $(1|1)$-dimensional abelian Lie superalgebra (and $\widehat{\text{exclusion}}$ stands for the exclusion).

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\footnote{For $\mathfrak{q}_n$ there are several notions of atypicality, see for instance [1] and [11].}
1.15. **Index of frequently used notation.** Throughout the paper the ground field is \( \mathbb{C} \); \( \mathbb{N} \) stands for the set of non-negative integers. We will frequently used the following notation.

- \( g_{ss} \)
- \( T_{A,B} \)
- \( h, t, h_i, H_i, \text{zero}(\lambda), \text{nonzero}(\lambda) \)
- \( C\ell(\lambda), B_\lambda, C_\lambda, L(\lambda) \)
- \( \text{atyp, core, core-free} \)
- \( q(J), \Delta(J) \)
- \( x_{ss}, x_{nil} \)
- \( \text{howl} \)
- \( \text{depth} \)
- \( \text{smult} \)

2. **Preliminaries**

Throughout the paper the ground field is \( \mathbb{C} \); \( \mathbb{N} \) stands for the set of non-negative integers. We denote by \( \Pi \) the parity change functor. In Sections 2 – 6 we “identify” the modules \( N \) and \( \Pi(N) \) (where \( \Pi \) stands for the parity change functor). For a finite length module \( N \) and a simple module \( L \) we denote by \([N : L]\) the “non-graded multiplicity” (the number of simple subquotients in a Jordan-Hölder series of \( N \) which are isomorphic to \( L \) or to \( \Pi L \)). We say that a module \( N \) is **isotypical** if \( N \) is indecomposable and for each simple subquotient \( L, L' \) of \( N \) one has \( L' \cong L \) (up to the parity change).

We denote by \( \mathcal{F}\text{in}(\mathfrak{g}) \) the full subcategory of finite-dimensional modules which are semisimple over \( \mathfrak{g}_\mathbb{F} \).

2.1. **\( Q \)-type superalgebras.** In this paper \( \mathfrak{g} \) is the queer (\( Q \)-type) Lie superalgebra \( q_n \). Recall that \( q_n \) is a subalgebra of \( \mathfrak{gl}(n|n) \) consisting of the matrices with the block form

\[
T_{A,B} := \begin{pmatrix} A & B \\ B & A \end{pmatrix}
\]

One has \( \mathfrak{g}_\mathbb{F} = \mathfrak{gl}_n \). The group \( GL_n \) acts on \( \mathfrak{g} \) by the inner automorphisms; all triangular decompositions of \( q_n \) are \( GL_n \)-conjugated. We denote by \( t \) the Cartan subalgebra of \( \mathfrak{gl}_n \) spanned by the elements \( h_i = T_{E_{ii},0} \) for \( i = 1, \ldots, n \). Let \( \{\xi_i\}_{i=1}^n \subset t^* \) be the basis dual to \( \{h_i\}_{i=1}^n \). The algebra \( \mathfrak{h} := q_n^\mathbb{F} \) is a Cartan subalgebra of \( q_n^\mathbb{F} \); one has \( \mathfrak{h}_\mathbb{F} = t \). The elements \( H_i := T_{0,E_{ii}} \) form a basis of \( \mathfrak{h}_\mathbb{F} \); one has \( [H_i, H_j] = 2\delta_{ij}h_i \).

We fix a usual triangular decomposition: \( \mathfrak{g} = n^+ \oplus h \oplus n \), where \( \Delta^+ = \{\xi_i - \xi_j\}_{1 \leq i < j \leq n} \).

We write \( \lambda = \sum_{i=1}^{n} \lambda_i \varepsilon_i \in t^* \) as \( \lambda = (\lambda_1, \ldots, \lambda_n) \) and set

\[
\text{zero}(\lambda) := \#\{i \mid \lambda_i = 0\}, \quad \text{nonzero}(\lambda) := \#\{i \mid \lambda_i \neq 0\}.
\]
2.2. Modules $C_\lambda$ and $L(\lambda)$. The algebra $U(\mathfrak{h})$ can be naturally viewed as a Clifford algebra over the polynomial algebra $S(t)$ with the symmetric bilinear form

$$B : \mathfrak{h}_T \otimes \mathfrak{h}_T \to S(t) \quad \text{with} \quad B(H, H') = [H, H'].$$

For each $\lambda \in \mathfrak{t}^*$ the evaluation of $B$ gives the symmetric form $B_\lambda : (H, H') \mapsto \lambda([H, H'])$ which defines the Clifford algebra

$$\mathcal{C}(\lambda) := \mathcal{C}(\mathfrak{h}_T, B_\lambda) = U(\mathfrak{h})/U(\mathfrak{h})I(\lambda)$$

where $I(\lambda)$ stands for the kernel of the algebra homomorphism $S(t) \to \mathbb{C}$ induced by $\lambda$. Since $(H_i, H_j) = 2\delta_{ij} h_i$ one has

$$\text{rank } B_\lambda = \text{nonzero}(\lambda).$$

If $\text{zero}(\lambda) = 0$, then each finite-dimensional $\mathcal{C}(\lambda)$-module is projective.

The algebra $\mathcal{C}(\lambda)$ admits a unique simple module $C_\lambda$ (up to isomorphism and parity change). One has $\text{dim } C_\lambda = 2^{\frac{\text{rank } h_0 + 1}{2}}$ (and $\text{sdim } C_\lambda = 0$ for $\lambda \neq 0$).

We denote by $L_\varrho(\lambda)$ a simple $\mathfrak{g}$-module of the highest weight $\lambda \in \mathfrak{t}^*$; this module is a unique simple quotient of $\text{Ind}_{\mathfrak{h}}^\mathfrak{g} C_\lambda$, where $C_\lambda$ is viewed as a $\mathfrak{h} + \mathfrak{n}^+$-module with the zero action of $\mathfrak{n}^+$. One has $L_\varrho(\lambda) = C_\lambda$. We will often omit the index $\mathfrak{g}$ (resp., $\mathfrak{g}_x$) in the notation $L_\varrho(\lambda)$ (resp., $L_\varrho(\nu)$).

2.3. Core. The weight $\lambda$ is typical if $\lambda_i + \lambda_j \neq 0$ for all $i, j$; in particular we require that $\lambda_i \neq 0$ for all $i$. We define atyp $\lambda$ to be the number of nonzero disjoint pairs $\lambda_i + \lambda_j = 0$ for $i \neq j$, plus $\text{zero}(\lambda)/2$. For example, if $\lambda = (2, 1, 0, 0, 0, -2, -3)$, then atyp $\lambda = 5/2$.

Let $\text{core } \lambda$ be the set obtained from $\{\lambda_i\}^n_{i=1}$ by erasing all zeroes from $\lambda$ and all pairs $\lambda_i, \lambda_j$ with $i \neq j, \lambda_i + \lambda_j = 0$. For instance core($3, 2, 0, 0, 0, -2, -2$) = $\{3, -2\}$ and this weight has atypicality $5/2$. We say that $\lambda$ is core-free if core($\lambda$) = $\emptyset$.

We denote by $\chi_\lambda$ the central character of $L(\lambda)$. By [28], $\chi_\lambda = \chi_\nu$ if and only if core $\lambda$ = core $\nu$. In particular, the core-free weights are the weights with the central character equal to $\chi_0$. We set atyp $\chi_\lambda := \text{atyp } \lambda$.

2.4. Subalgebras $\mathfrak{q}(J)$. Set $I_n := \{1, \ldots, n\}$. For each $J \subset I_n$ we denote by $\mathfrak{h}(J)$ (resp., $\mathfrak{t}(J)$) the span of $H_i, h_i$ (resp., of $h_i$) with $i \in J$ and set

$$\Delta(J) := \{\varepsilon_i - \varepsilon_j | i, j \in J, i \neq j\}.$$ 

We denote by $\mathfrak{q}(J)$ the subalgebra spanned by $\mathfrak{h}(J)$ and $\sum_{\alpha \in \Delta(J)} \mathfrak{g}_\alpha$. Clearly, $\mathfrak{q}(J) \cong \mathfrak{q}_k$, where $k$ is the cardinality of $J$; we set $\mathfrak{h}_J := \mathfrak{h}(I_n \setminus J)$ and $\mathfrak{t}_J := (\mathfrak{h}_J \mathfrak{t})$ (the $\mathfrak{t}_J$ is spanned by $h_i$ with $i \in I_n \setminus J$). The triangular decomposition of $\mathfrak{g}$ induces a triangular decomposition of $\mathfrak{q}(J)$ with $\Delta(\mathfrak{q}(J))^+ = \{\varepsilon_i - \varepsilon_j | i < j, i, j \in J\}$. We will use analogous notation $\mathfrak{sq}(J), \mathfrak{pq}(J), \mathfrak{psq}(J)$. 
3. DS-functor in $\mathfrak{q}_n$-case

The DS-functor was introduced in [7]. We recall definitions and some results of [7,13] in the Appendix. A study of the DS-functor for $\mathfrak{q}_n$ when $x^2 = 0$ was initiated in [27], see also [11], Section 5 for the proofs. In this article we will consider arbitrary $x$ with $x^2$ semisimple; this means $x$ is of the form $x = T_0,B$, where $B^2$ is semisimple. If we write $B = B_{ss} + B_{nil}$ for the Jordan-Chevalley decomposition of $B$, we have $B_{nil}^2 = 0$ and $B_{ss}B_{nil} = 0$. Define rank $x = \text{rank } B_{nil} + \text{rank } B_{ss}/2$; in Proposition 3.2 we show that $\text{DS}_x(\mathfrak{g}_n) = \mathfrak{q}_{n-2\text{rank }x}$ as well as compute $\text{DS}_x(\mathfrak{g})$ for other $Q$-type superalgebras.

3.1. Representatives of $G_0$ on $\mathfrak{g}_s^{ss}$. Retain notation of [2,3]. Recall that the action of $G_0 = \text{GL}(n)$ on $\mathfrak{g}_i^{ss}$ is given by the adjoint action on $n \times n$-matrices. Therefore, finding the $G_0$-orbits on $\mathfrak{g}_i^{ss}$ is equivalent to finding the $\text{GL}(n)$ orbits on square-semisimple $n \times n$ matrices. We give below (non-unique) representatives of these orbits which are easy to work with for our purposes. Choose a subset $J := \{i_p\}_{p=1}^r \subseteq I_n$. Let $s \in \mathbb{N}$ with $s \leq r/2$, and for $p = 1, \ldots, s$ fix a non-zero odd element $x_p \in \mathfrak{g}_{i_2p - 1 - 2p}$. Further, let $c_{2s+1}, \ldots, c_r \in \mathbb{C}^\times$, and set

$$x = x_{nil} + x_{ss} := \sum_{p=1}^s x_p + \sum_{j=2s+1}^r c_j H_{ij}$$

Then $x \in \mathfrak{g}_i^{ss}$ and $x$ has rank $r/2$.

We will always choose $x \in \mathfrak{g}_i^{ss}$ of the above form and we will use the above identification of $\mathfrak{g}_x$ with $\mathfrak{g}(I_n \setminus J)$. Further we will say in this case that $x$ corresponds to the set $J$. In Section 3.4 we will show in what sense DS$_x$ is independent of the subset $J$.

The following lemma is an immediate consequence of Lemma 3.1.

3.1.1. Lemma. Let $x = x_{ss} + x_{nil}$, and write $\overline{x}_{ss}$ (resp., $\overline{x}_{nil}$) for the image of $x_{ss}$ (resp. $x_{nil}$) in $\mathfrak{g}_{x_{nil}}$ (resp., $\mathfrak{g}_{x_{ss}}$). Let $N$ a finite-dimensional $\mathfrak{g}$-module with semisimple action of $\mathfrak{g}_T$. Then

$$\dim \text{DS}_x(N) \leq \min \left( \dim \text{DS}_{\overline{x}_{ss}} \circ \text{DS}_{x_{nil}}(N), \dim \text{DS}_{\overline{x}_{nil}} \circ \text{DS}_{x_{ss}}(N) \right).$$

Further, if $x_{nil}$ is of rank $s$ and we write $x_{nil} = x_1 + \cdots + x_s$, where each $x_i$ is a rank 1 nilpotent operator with $[x_i, x_j] = [x_i, x_{ss}] = 0$, then for each $0 \leq r \leq s$ we have

$$\dim \text{DS}_x(N) \leq \dim \text{DS}_{\overline{x}_1} \cdots \text{DS}_{\overline{x}_r} \circ \text{DS}_{\overline{x}_{ss}} \circ \text{DS}_{\overline{x}_{r+1}} \cdots \circ \text{DS}_{\overline{x}_s}(N),$$

where $\overline{x}_i$ denotes the projection to the appropriate subquotient.

3.2. Proposition. Take $x$ of rank $r/2$ corresponding to $J \subseteq I_n$ as in [7,1].

(i) $\text{DS}_x(\mathfrak{q}_n)$ can be identified with $\mathfrak{q}(I_n \setminus J) \simeq \mathfrak{q}_{n-r}$;
(ii) $\text{DS}_x$ maps the standard $\mathfrak{q}_n$-module, $L(\varepsilon_1)$, to the standard $\mathfrak{q}_{n-r}$-module;
(iii) if $r < n$, then $\text{DS}_x(\text{sq}_n)$, $\text{DS}_x(\text{pq}_n)$, $\text{DS}_x(\text{psq}_n)$ can be identified with $\text{sq}(I_n \setminus J)$, $\text{pq}(I_n \setminus J)$, $\text{psq}(I_n \setminus J)$ respectively;
(iv) if $r = n$, then $\text{DS}_x(\text{sq}_n) \cong \mathbb{C}$, $\text{DS}_x(\text{pq}_n) \cong \Pi \mathbb{C}$;
(v) if $r = n > 1$, then $\text{DS}_x(\text{psq}_n)$ is a commutative $(1|1)$-dimensional Lie superalgebra if $x = x_{\text{nil}}$ or if $\sum c_i^{-1} = \sum c_i^{-3} = 0$, $\text{DS}_x(\text{psq}_n) \cong q_1$ if $\sum c_i^{-1} = 0 \neq \sum c_i^{-3}$ and $\text{DS}_x(\text{psq}_n) = 0$ if $\sum c_i^{-1} \neq 0$.

Proof. All formulae can be easily checked in the following cases: $x = x_{ss} = \sum_{p=1}^r c_i H_i$ or $x^2 = 0$ and rank $x = 1$ (in this case $J = \{i_1, i_2\}$). Now consider the general case. It is easy to see that $[x, q(I_n \setminus J)] = 0$ and that $[x, q_n] \cap q(I_n \setminus J) = 0$. Therefore $q(I_n \setminus J)$ with a subalgebra of $\text{DS}_x(q_n)$. Similarly, the standard $q(I_n \setminus J)$-module is a submodule of $\text{DS}_x(L(\varepsilon_1))$ and, if $r < n$, then $\text{sq}(I_n \setminus J)$, $\text{pq}(I_n \setminus J)$, $\text{psq}(I_n \setminus J)$ are subalgebras of $\text{DS}_x(\text{sq}_n)$, $\text{DS}_x(\text{pq}_n)$, $\text{DS}_x(\text{psq}_n)$ respectively.

Using Lemma 3.1.1 we obtain

$$
\dim \text{DS}_x(N) \leq \dim \text{DS}_{x_1} \circ \text{DS}_{x_2} \circ \ldots \circ \text{DS}_{x_n}(\text{DS}_{x_{ss}}(N))
$$

for any finite-dimensional module $N$. Using the cases $x = x_{ss}$ and rank $x = 1$ we get $\dim \text{DS}_x(q_n) \leq \dim q(I_n \setminus J)$ and the similar inequalities for other cases. This establishes (i), (ii), (iii).

Consider the remaining case rank $x = n/2$. By (i), $\text{DS}_x(q_n) = 0$. Using Hinich’s lemma we get (iv).

For (v) we have $J = \{1, \ldots, n\};$ thus $x$ takes the form

$$
x := \sum_{p=1}^s x_p + \sum_{j=2s+1}^n c_j H_j,
$$

with $c_j \in \mathbb{C}^\times$, $x_p \in \mathfrak{g}_{\varepsilon_{2p-1} - \varepsilon_{2p}}$. We set

$$
y := \sum_{p=1}^s y_p + \sum_{p=2s+1}^n c_i^{-1} H_i,
$$

where $y_p \in \mathfrak{g}_{\varepsilon_{2p-1} - \varepsilon_{2p}}$ is an odd element satisfying $[x_p, y_p] = 2h_{2p-1} + 2h_{2p}$. In particular, we have $[x, y] = 2T_{id,0}$.

Applying Hinich’s Lemma to the short exact sequence

$$
0 \rightarrow \mathbb{C} T_{id,0} \rightarrow \text{sq}_n \rightarrow \text{psq}_n \rightarrow 0
$$

and using $\text{DS}_x(\text{sq}_n) = \mathbb{C}$ we obtain a long exact sequence

$$
0 \rightarrow E \rightarrow \mathbb{C} \rightarrow \mathbb{C} \rightarrow \text{DS}_x(\text{psq}_n) \rightarrow \Pi E \rightarrow 0
$$

where $E := \mathbb{C} T_{id,0} \cap [x, \text{sq}_n]$. It is easy to see that $E = \mathbb{C} T_{id,0}$ if $y \in \text{sq}_n$ and $E = 0$ otherwise. This gives $\dim \text{DS}_x(\text{psq}_n) = (1|1)$ if $x = x_{\text{nil}}$ or if $\sum c_i^{-1} = 0$, and $\text{DS}_x(\text{psq}_n) = 0$ if $\sum c_i^{-1} \neq 0$.  

Consider the case when \( x = x_{nil} \) or \( \sum c_j^{-1} = 0 \), so that \( \text{dim } DS_x(\mathfrak{psq}_n) = (1|1) \). By \( \mathfrak{psq}_n \), \( DS_x(\mathfrak{psq}_n) \) is an ideal in \( DS_x(\mathfrak{psq}_n) \), so \( DS_x(\mathfrak{psq}_n) \) is either commutative or isomorphic to \( \mathfrak{q}_1 \). It is easy to see that \( \mathfrak{g} \) has a non-zero image in \( DS_x(\mathfrak{psq}_n) \); we denote this image by \( \mathfrak{g}_x \). Then \( DS_x(\mathfrak{psq}_n) \) is commutative if and only if \( \mathfrak{g}_x \) is an ideal in \( DS_x(\mathfrak{psq}_n) \).

If \( x = x_{nil} \), then \( y^2 = 0 \). In the remaining case \( \sum c_j^{-1} = 0 \) and \( y^2 = \sum c_j^{-1} h_j \). The elements in \([\mathfrak{sq}_n, x] \cap \mathfrak{t}\) are of the form \( \sum d_i h_i \), where \( d_{2p-1} = d_{2p} \) for \( p = 1, \ldots, s \) and \( \sum_j c_j^{-1} d_j = 0 \). Therefore \( y^2 \in \mathbb{C}T_{\text{ld}, 0} + [\mathfrak{sq}_n, x] \) is equivalent to the existence of \( d \in \mathbb{C} \) and \( z_j \in \mathbb{C} \) for \( j = 2s + 1, \ldots, n \) such that \( c_j z_j = c_j^{-2} + d \) for all \( j \) and \( \sum z_j = 0 \). Since \( \sum c_j^{-1} = 0 \), the last formula is equivalent to \( \sum c_j^{-3} = 0 \) This completes the proof. \( \square \)

3.3. Let \( x \in \mathfrak{g}_T^{ss} \), and write \( \mathbb{Z}(\mathfrak{g}) \) for the center of the enveloping algebra \( \mathcal{U}(\mathfrak{g}) \). Then we have

\[
\mathbb{Z}(\mathfrak{g}) \subseteq \mathcal{U}(\mathfrak{g})^x \rightarrow \mathcal{U}(\mathfrak{g}_x).
\]

Thus we have a natural map \( \mathbb{Z}(\mathfrak{g}) \rightarrow \mathbb{Z}(\mathfrak{g}_x) \), inducing a pullback map on central characters

\[
\eta^*_x : \text{Hom}(\mathbb{Z}(\mathfrak{g}_x), \mathbb{C}) \rightarrow \text{Hom}(\mathbb{Z}(\mathfrak{g}), \mathbb{C}).
\]

As in \( \mathfrak{gl}(m|n) \) and \( \mathfrak{osp}(m|n) \)-cases, the DS-functor respects preserves cores, in the following sense.

3.3.1. **Lemma.** Let \( x \in \mathfrak{g}_T^{ss} \) with \( r = \text{rank } x \). Then the map

\[
\eta_x : \mathbb{Z}(\mathfrak{g}_n) \rightarrow \mathbb{Z}(\mathfrak{g}_{n-2r})
\]

is surjective. Further, we have \( \eta^*_x(\chi_c) = \chi_c \).

**Proof.** See Theorem ([27], Thm. 6.3, [11], Cor. 5.8.1) for proofs in the case when \( x^2 = 0 \); the case of \( x \in \mathfrak{g}_T^{ss} \) is almost identical. \( \square \)

3.3.2. **Corollary.** If a \( \mathfrak{g} \)-module \( N \) has a central character \( \chi_\lambda \), then a \( \mathfrak{g}_x \)-module \( DS_x(N) \) has a central character \( \chi_\nu \), where \( \text{core}(\lambda) = \text{core}(\nu) \).

In particular, \( \text{atyp } \lambda = \text{atyp } \nu = \text{rank } x \) and \( DS_x(N) = 0 \) if \( \text{atyp } \lambda < \text{rank } x \).

3.4. **Independence of DSx from J ⊆ \{1, \ldots, n\}.** Retain notation of 3.1. Let \( I_s := \{n, n-1, \ldots, n-s+1\} \) and let \( x_s \in \mathfrak{g}_T^{ss} \) correspond to \( I_s \) as in Section 3.3.1. Let \( J = \{j_1 < \cdots < j_s\} \subseteq \{1, \ldots, n\} \) be an arbitrary subset of size \( s \) and let \( \sigma \in S_n \subseteq GL(n) \) denote the permutation with \( \sigma(n-s+1) = j_1, \ldots, \sigma(n) = j_s \), and set \( x := \sigma(x_s) \). The action of \( \sigma \) gives an isomorphism \( \sigma : \mathfrak{g}_{x_s} \rightarrow \mathfrak{g}_x \) and the commutative diagram

\[
\begin{array}{ccc}
\mathcal{F} \text{Fin}(\mathfrak{g}) & \xrightarrow{\phi} & \mathcal{F} \text{Fin}(\mathfrak{g}) \\
\downarrow DS_{x_s} & & \downarrow DS_x \\
\mathcal{F} \text{Fin}(\mathfrak{g}_{x_s}) & \xrightarrow{\phi_*} & \mathcal{F} \text{Fin}(\mathfrak{g}_x)
\end{array}
\]
where the functors $\phi$ and $\phi_x$ correspond to the shift of module structure along $\sigma : g \rightarrow g$ and $\sigma : g_{x_s} \rightarrow g_x$ respectively. If $N$ is a finite-dimensional $g$-module, the action of $g$ on $N$ induces an isomorphism $\phi(N) \cong N$ and a bijection between $\text{DS}_x(N)$ and $\text{DS}_y(N)$ which is compatible with the algebra isomorphism $g_x \xrightarrow{\sim} g_y$.

By Proposition \[3.2\] we have natural identifications $g_{x_s} \cong q_{n-s}$ and $g_x \cong q_{n-s}$ as the subalgebras of $q_n$ corresponding to the subsets $I_s$ and $J$. Under these identifications we have a commutative diagram

$$
\begin{array}{ccc}
g_{x_s} & \xrightarrow{\sigma} & g_x \\
\downarrow & & \downarrow \\
q_{n-s} & \xrightarrow{\sigma} & q_{n-s}
\end{array}
$$

Further, $\sigma$ takes the Cartan subalgebras and subalgebras of the respective copies of $q_n$ to one another. It follows that if $\nu \in P^+(q_{n-s})$ then $\phi_x(L_{g_{x_s}}(\nu)) \cong L_{g_x}(\nu)$.

It follows that the functor $\text{DS}_x : \mathcal{F}\text{in}(g) \rightarrow \mathcal{F}\text{in}(g_x)$ is independent of the choice of the set $J \subseteq I_n$ that $x$ corresponds to. (In the computations below our choice of $x$ will depend on the central character.)

### 3.4.1. Proposition

Let $M$ be a finite-dimensional $g$-module such that $\text{zero}(\nu) = 0$ for any weight $\nu$ of $M$. If $x \in (q_n)_{\text{ss}}^2$ with $x^2 \neq 0$, then $\text{DS}_x(M) = 0$.

**Proof.** As in Section \[3.1\] we may write $x = x_{ss} + x_{nil}$ with $x_{ss} \in h$ and $x_{ss}^2 = x^2$. Set $N := M^{ss}$. There exists $t \in t$ such that $[t, x_{nil}] = x_{nil}$. Since $x_{ss} \in h$ we have $[t, x_{ss}] = 0$. Since $\text{zero}(\nu) = 0$ for any weight $\nu$ of $M$, by \[2.2\] $M$ is a projective $h$-module, implying that $\text{DS}_x(M) = 0$ (since $x_{ss} \in h$), i.e. $M^{ss} = x_{ss}M$. Now the assertion follows from Lemma \[7.2.1\].

### 3.5. Gradings for elements with $x^2 = 0$

Fix $x \in q(J)$ with $x^2 = 0$ (i.e., $x = x_{nil}$). In this case there exists an element $h \in t \cap q(J)$ such that $[h, x] = x$. Thus, as is explained in Section \[7.1.2\] we obtain a grading on $M_x$ as a $g_x$-module for any finite-dimensional $g$-module $M$. Note that $[h, g_x] = 0$.

### 4. Dominant weights and arc diagrams

In this paper we study the action of $\text{DS}_x$ on finite-dimensional simple modules. We denote by $P^+(g)$ the set of dominant weights, i.e.

$$P^+(g) := \{ \lambda \in t^* \mid \dim L(\lambda) < \infty \}.$$

By \[22\], $\lambda \in P^+(g)$ if and only if $\lambda_i - \lambda_{i+1} \in \mathbb{N}_{\geq 0}$ and $\lambda_i = \lambda_{i+1}$ implies $\lambda_i = 0$. We call weight $\lambda$ integral (resp., half-integral) if $\lambda_i \in \mathbb{Z}$ (resp., $\lambda_i - \frac{1}{2} \in \mathbb{Z}$) for all $i$. If $\lambda \in P^+(g)$ is atypical, then $\lambda$ is either integral or half-integral. By \[3.3.2\] $\text{DS}_x(N) = 0$ for each typical module $N$ and $x \neq 0$. Without loss of generality we assume
\[ x \neq 0 \text{ and } \lambda \text{ is integral or half-integral.} \]

By Proposition 3.4.1 for half-integral weights it is only interesting to consider those \( x \) with \( x^2 = 0 \).

4.1. Weight diagrams. Weight diagrams were first defined in [4] for \( \mathfrak{gl}(m|n) \). The conventions on how to draw these weight diagrams differ; we follow essentially [18].

For \( \lambda = (\lambda_1, \ldots, \lambda_n) \) we construct the weight diagram \( \text{diag}(\lambda) \) as follows:

- for \( s \neq 0 \) we put \( > \) (resp., \( < \)) at the position \( s \) if there exists \( j \) with \( \lambda_j = s \) (resp., \( \lambda_j = -s \));
- we write \( \times \) if the position \( s \neq 0 \) contains \( > \) and \( < \);
- if zero(\( \lambda \)) \( \neq 0 \), we put \( \wedge^r \), where \( r = \text{zero}(\lambda) \);
- if \( \lambda \) is integral (resp., half-integral) we put the empty sign \( \circ \) at each non-occupied position with the coordinate in \( \mathbb{N} \) (resp., in \( \frac{1}{2} + \mathbb{N} \)).

If \( \lambda \) is integral, we draw the diagram from the zero position, and for \( \lambda \) half-integral we draw from the \( \frac{1}{2} \) position. For instance,

\[
\begin{align*}
(4, 1, -1, -3, -4) & \quad \circ \times \circ < \times \circ \circ \ldots \\
(5, 2, 0, 0, 0, 0, 0, -2, -3) & \quad \wedge^5 \circ \times < \circ > \circ \ldots \\
(\frac{5}{2}, \frac{1}{2}, -\frac{3}{2}, -\frac{2}{2}) & \quad \times \circ \circ \circ \ldots
\end{align*}
\]

where \( \ldots \) stands for the infinite sequence of the empty symbols \( \circ \).

We obtain a one-to-one correspondence between integral (resp., half-integral) dominant weights and the diagrams containing \( n \) symbols \( >, <, \wedge \) (where \( \times \) considered as the union of \( < \) and \( > \)), where each non-zero position contains exactly one of the symbols \( >, <, \times \) or \( \circ \) and the zero position contains \( \wedge^r \) or \( \circ \). Note that \( \text{atyp} \lambda \) is equal to the number of \( \times \)s in \( \text{diag}(\lambda) \) plus half the number of symbols \( \wedge \) at 0. We roughly think of the symbol \( \wedge \) as half of a symbol \( \times \).

For a weight diagram \( f \) denote by \( f(i) \) the symbols at the \( i \)th position.

4.1.1. Core diagrams. The symbols \( >, < \) are called core symbols.

A core diagram is a weight diagram which does not contain symbols \( \times \) and \( \wedge \). The core diagram of \( \lambda \) is obtained from the weight diagram of \( \lambda \) by erasing all \( \times \) and \( \wedge \) symbols.

We say that a \( \mathfrak{g} \)-central character \( \chi \) is dominant if there exists a finite-dimensional module with this central character. By above, the central characters are parametrized by \( \text{core}(\lambda) \), so the dominant central characters of atypicality \( k > 0 \) are parametrized by the core diagrams with \( n - 2k \) non-empty symbols. For instance, the central character of the maximal atypicality corresponds to the empty core diagram; for \( \mathfrak{q}_2 \) the diagrams of the weights with such central character are \( \wedge^2, \circ \times, \circ \circ \times \) and so on; for \( \mathfrak{q}_3 \) these diagrams are \( \wedge^3 \circ, \wedge \times, \wedge \circ \circ \times \) and so on.
For a core diagram $f$ we denote by $\chi(f)$ the corresponding central character.

4.1.2. Core-free diagrams. Note that $\lambda$ is core-free if and only if the weight diagram of $\lambda$ does not have core symbols. We assign to each diagram $f$ a core-free diagram $\text{howl}(f)$ which is obtained from $f$ by erasing all core symbols. For instance,

$$f = \wedge^4 \triangleright \bigstar \bigtriangledown \bigcirc \cdots \quad \text{core}(f) = \bigcirc \triangleright \bigstar \bigtriangledown \bigcirc \cdots \quad \text{howl}(f) = \wedge^4 \bigcirc \bigstar \bigtriangledown \bigcirc \cdots$$

$$g = \wedge^3 \bigstar \bigcirc \cdots \quad \text{core}(g) = \bigstar \bigcirc \bigstar \bigcirc \cdots \quad \text{howl}(g) = \wedge^3 \bigstar \bigcirc \cdots$$

For an atypical dominant weight $\lambda$ we denote by $\text{howl}(\lambda)$ the weight corresponding to the diagram $\text{howl}([\lambda])$.

4.2. Partial order. We will consider the standard partial order on $t^*$:

$$\lambda \succ \nu \quad \text{if} \quad \lambda - \nu \in \mathbb{N}\Delta^+.$$

4.2.1. Lemma. For atypical weights $\eta, \mu \in P^+(q_n)$ one has

$$\text{core}(\eta) = \text{core}(\mu) \; \& \; \eta > \mu \quad \Rightarrow \quad \text{howl}(\eta) \ngeq \text{howl}(\mu).$$

Proof. Let $f, g$ be weight diagrams of atypicality $p > 0$ with $\text{core} f = \text{core} g$, and let $a_1 \geq a_2 \geq \ldots \geq a_p$ (resp., $b_1 \geq b_2 \geq \ldots \geq b_p$) be the coordinates of the symbols $\bigstar$ in $f$ (resp., in $g$). We write $g \succ f$ if $b_j > a_j$ for some $j$ and $a_i = b_i$ for each $i < j$. For example

$$\wedge \bigstar \bigcirc \cdots \succ \wedge^3 \bigstar \bigcirc \bigstar \bigcirc \cdots.$$

It is easy to see that for atypical weights $\mu, \eta \in P^+(q_n)$ with $\text{core}(\mu) = \text{core}(\eta)$ one has

$$\eta > \mu \quad \Rightarrow \quad [\eta] \succ [\mu] \quad \text{and} \quad [\text{howl}(\eta)] \succ [\text{howl}(\mu)].$$

If $\eta > \mu$ and $\text{howl}(\eta) \leq \text{howl}(\mu)$, then $[\eta] \succ [\mu]$ and $[\text{howl}(\mu)] \nsubseteq [\text{howl}(\eta)]$, a contradiction. \hfill \Box

4.2.2. Remark. Note that

$$\text{core}(\eta) = \text{core}(\mu) \; \& \; \text{howl}(\eta) > \text{howl}(\mu) \quad \not\Rightarrow \quad \eta > \mu.$$

For example, for $\lambda := (4,1,0,0,-4)$, $\nu := (3,2,1,-2,-3)$ one has $\text{howl}(\lambda) = (3,0,0,-3)$ and $\text{howl}(\nu) = (2,1,-1,2)$. In this case $\lambda \not\succ \nu$ and $\text{howl}(\lambda) > \text{howl}(\nu)$.  

4.3. **Arc diagrams.** A *generalized arc diagram* is the following data:

1. a weight diagram $f$, where the symbols $\wedge$ at the zero position are drawn vertically;
2. a collection of non-intersecting arcs of two types:
   - a *full arc* $\text{arc}(a;b)$ connects the symbol $\times$ at the position $a \neq 0$ with the empty symbol at the position $b > a$; full arcs are depicted by solid arcs;
   - a *half arc* $\text{arc}(0;b)$ connects the symbol $\wedge$ at the zero position with an empty symbol at the position $b$; half arcs are depicted by dashed arcs.

An empty position is called *free* if it is not an end of an arc.

For $a \neq 0$, we call $\text{arc}(a;b)$ a *full arc supported at* $a$, and $\text{arc}(0;b)$ a *half arc supported at* 0. In this sense, when later on we talk about a number of arcs, two half arcs will make one arc. For example, a quantity of $3/2$ arcs consists either of a full arc with a half arc, or three half arcs.

A generalized arc diagram is called an *arc diagram* if each symbol $\times$ and $\wedge$ is the left end of exactly one arc and there are no free positions under the arcs. Each weight diagram $f$ admits a unique arc diagram which we denote by $\text{Arc}(f)$ (as in $\text{gl}(m|n)$ and $\text{osp}(m|n)$ case we construct the arcs successively starting from the rightmost symbol $\times$). We write $\text{Arc}(\nu)$ for $\text{Arc}(\text{diag}(\nu))$. We write $\text{Arc}(\nu) \subset \text{Arc}(\lambda)$ if each arc in $\text{Arc}(\nu)$ appears in $\text{Arc}(\lambda)$.

4.3.1. **Partial order.** We consider a partial order on the set of arcs by saying that one arc is smaller than another one if the first one is ”below” the second one, that is

$$\text{arc}(a;b) > \text{arc}(a';b') \text{ if and only if } a < a' < b < b'.$$

Since the arcs do not intersect, one has

$$\text{arc}(a;b) > \text{arc}(a';b') \iff a < a' < b,$$

and any two distinct arcs of the form $\text{arc}(0,b_1), \text{arc}(0,b_2)$ are comparable: either $\text{arc}(0,b_1) > \text{arc}(0,b_2)$ or $\text{arc}(0,b_1) < \text{arc}(0,b_2)$.

4.3.2. **Remarks.** Notice that a maximal arc can be “removed”: if we erase the symbol $\times$ (or $\wedge$) on the left end of the arc and the arc itself, we obtain another arc diagram (this does not hold if the arc is not maximal); if $\text{Arc}(\nu)$ is obtained from $\text{Arc}(\lambda)$ in this way, then $\text{Arc}(\nu) \subset \text{Arc}(\lambda)$.

Observe there is the natural one-to-one correspondence between $\text{Arc}(f)$ and $\text{Arc}(\text{howl}(f))$; this correspondence preserves the partial order on arcs and the type of arc (full or half).
4.3.3. Examples. The arc diagram for the weight \( \lambda = 7\varepsilon_1 + 4\varepsilon_2 + \varepsilon_3 - \varepsilon_6 - 7\varepsilon_7 \) looks as follows:

There are two maximal arcs: arc(0, 5) and arc(7, 8).

The arc diagram for \( \lambda = 5\frac{1}{2}\varepsilon_1 + \frac{1}{2}\varepsilon_2 - \frac{3}{2}\varepsilon_3 - \frac{5}{2}\varepsilon_4 - \frac{5}{2}\varepsilon_5 \) is given by:

In this case there is only one maximal arc: arc(\(1\frac{1}{2}, 9\frac{1}{2}\)).

5. Main results

In this section we formulate the main results, illustrate them by examples and give outlines of the proofs. In the following, for \( x \in \mathfrak{g}_{1}^{ss} \) of rank \( s \), \( \lambda \in P^+(\mathfrak{q}_n) \), and \( \nu \in P^+(\mathfrak{q}_{n-2s}) \), we write \( m_x(\lambda; \nu) := [\text{DS}_x(L_qn(\lambda)) : L_qn-2s(\nu)]. \)

5.1. Theorem. Take \( \lambda \in P^+(\mathfrak{q}_n) \) and \( \nu \in P^+(\mathfrak{q}_{n-2s}) \) where \( s \in \frac{1}{2}\mathbb{N} \), and let \( x \in \mathfrak{g}_{1}^{ss} \) be of rank \( s \).

(i) If \( m_x(\lambda; \nu) \neq 0 \), then \( \text{core}(\lambda) = \text{core}(\nu) \) and \( \text{Arc}(\nu) \) can be obtained from \( \text{Arc}(\lambda) \) by successively removing \( s \) maximal arcs.

(ii) We have \( m_x(\lambda, \nu) = 1 \) if and only if \( \text{zero}(\nu) = \text{zero}(\lambda) - 2s \). If \( \text{zero}(\nu) \neq \text{zero}(\lambda) - 2s \), then

\[
\text{smult}(\text{DS}_xL(\lambda), L(\nu)) = 0.
\]

(See Section 5.8 for the definition of \( \text{smult} \).)

(iii) Let \( x = T_{0,E11+...+Er} \). Then \( \text{DS}_x(L(\lambda)) = 0 \) if \( \text{zero}(\lambda) < r \), and if \( \text{zero}(\lambda) \geq r \) then \( \text{DS}_x(L(\lambda)) = L(\lambda') \), where \( \lambda' \) is obtained from \( \lambda \) by removing \( r \) zeroes; in other words \( \text{Arc}(\lambda') \) is obtained from \( \text{Arc}(\lambda) \) by removing \( r \) half arcs.

When \( x \) is of rank 1/2 its \( G_0 \)-orbit intersects \( \mathbb{C}(T_{0,E11}) \), and thus part (ii) of Theorem 5.1 covers this case. For the case when the rank of \( x \) is 1, we have the following more precise statements.

5.2. Theorem. Let \( x = T_{0,B} \in \mathfrak{g}_{1}^{ss} \) be of rank 1 and \( \lambda \) be a dominant integral weight.

(i) If \( \lambda \) is integral and \( \text{tr} B = 0 \) then \( m_x(\lambda; \nu) = 2 \) if \( \text{Arc}(\nu) \) can be obtained from \( \text{Arc}(\lambda) \) by removing one maximal full arc, \( m_x(\lambda; \nu) = 1 \) if \( \text{Arc}(\nu) \) can be obtained from \( \text{Arc}(\lambda) \) by removing two successive maximal half arcs and \( m_x(\lambda; \nu) = 0 \) in other cases.
(ii) If \( \lambda \) is integral and \( \text{tr} B \neq 0 \) then \( m_x(\lambda; \nu) = 1 \) if \( \text{Arc}(\nu) \) can be obtained from \( \text{Arc}(\lambda) \) by removing two successive maximal half arcs and \( m_x(\lambda; \nu) = 0 \) in other cases.

Recall that \( DS_x(L(\lambda)) = 0 \) if \( \lambda \) is a dominant half-integral weight and \( x^2 \neq 0 \), see Proposition 3.4.1.

5.3. **Theorem.** Let \( x \in g_{ss}^{ss} \) be of rank 1 with \( x^2 = 0 \). If \( \lambda \) is half-integral, then \( m_x(\lambda; \nu) = 2 \) if \( \text{Arc}(\nu) \) can be obtained from \( \text{Arc}(\lambda) \) by removing one maximal full arc and \( m_x(\lambda; \nu) = 0 \) in other cases.

5.3.1. **Remark.** If \( \text{Arc}(\nu) \) is obtained from \( \text{Arc}(\lambda) \) by removing \( r \) maximal arcs and \( s \) maximal half arcs, then \( \text{zero}(\nu) = \text{zero}(\lambda) - s \).

5.3.2. **Corollary.** For an atypical \( \lambda \in P^+(\mathfrak{q}_n) \) the following assertions are equivalent:

(i) \( DS_x(L(\lambda)) = 0 \) with \( x^2 = 0 \), rank \( x = 1 \);

(ii) \( \text{Arc}(\lambda) \) has a unique maximal arc and \( \text{zero}(\lambda) = 1 \);

(iii) \( DS_x(L(\lambda)) = 0 \) for any \( x \neq 0 \) with rank \( x \in \mathbb{Z} \).

5.3.3. **Corollary.** For every simple module \( L(\lambda) \) with atyp \( \lambda > 0 \), there exists a nonzero \( x \in g_{ss}^{ss} \) of rank \( \leq 1 \) such that \( DS_x L(\lambda) \neq 0 \).

**Proof.** Suppose that \( DS_x L(\lambda) = 0 \) for all nonzero \( x \in g_{ss}^{ss} \) of rank 1; then by Corollary 5.3.2 \( \text{zero}(\lambda) = 1 \). Therefore by Theorem 5.2, \( DS_C L(\lambda) \neq 0 \). \( \square \)

5.3.4. **Remark.** It is not hard to see that if the coordinates of \( \nu \) are obtained from the coordinates of \( \lambda \) by removal \( 2r \) zero coordinates, then \( m_x(\lambda; \nu) = 1 \) for \( x \) of rank \( r \) (see [11], Prop. 5.7.2 for slightly more general statement); in this case \( \text{Arc}(\nu) \) is obtained from \( \text{Arc}(\lambda) \) by removal of \( 2r \) half arcs.

5.4. **Corollary.** Let \( x \in g_{ss}^{ss} \).

(i) All indecomposable components of \( DS_x(L_{\mathfrak{q}_n}(\lambda)) \) are isotypical.

(ii) If \( \text{zero}(\lambda) = 0 \), then \( DS_x(L(\lambda)) \) is completely reducible.

**Proof.** In order to prove that each indecomposable component of \( DS_x(L(\lambda)) \) is isotypical, it is enough to verify that \( \text{Ext}^1(L(\nu), L(\mu)) = 0 \) if \( \mu \neq \nu \) and \( L(\mu), L(\nu) \) are subquotients of \( DS_x(L(\lambda)) \). Take \( \mu, \nu \) as above. By Theorem 5.1 all arcs in \( \text{Arc}(\mu) \), \( \text{Arc}(\nu) \) are also arcs in \( \text{Arc}(\lambda) \); in particular, if diag \( \lambda \) has \( \circ \) at some position, then both diagrams diag(\( \nu \)), diag(\( \mu \)) also have \( \circ \) at this position. Assume that \( \mu \not\geq \nu \) and that \( \text{Ext}^1(L(\nu), L(\mu)) \neq 0 \) or \( \text{Ext}^1(L(\mu), L(\nu)) \neq 0 \). By [12], Theorem A, this implies that diag(\( \nu \)) can be obtained from diag(\( \mu \)) by moving one symbol \( \times \) in diag(\( \mu \)) along the arc in \( \text{Arc}(\mu) \) or changing two symbols \( \wedge \) to a symbol \( \times \) which occupies a position connected to one of the symbols \( \wedge \) in
Arc(\(\mu\)). This means that there are two positions \(a < b\) which are connected by an arc in Arc(\(\mu\)) (\(\text{diag}(\mu)\) has \(\circ\) at the position \(b\)) and \(\text{diag}(\nu)\) has \(\times\) at the position \(b\). By above, the positions \(a, b\) are connected by an arc in Arc(\(\lambda\)), so \(\text{diag} \lambda\) has \(\circ\) at the position \(b\) and thus \(\text{diag}(\nu)\) has \(\circ\) at the position \(b\), a contradiction. This establishes (i).

If zero(\(\lambda\)) = 0 and \(L(\nu)\) is a subquotient of \(DS_x(L(\lambda))\), then zero(\(\nu\)) = 0; by Theorem 3.1 in [17] this gives \(\text{Ext}^1(L(\nu), L(\nu)) = \text{Ext}^1(L(\nu), \Pi L(\nu)) = 0\). Combining with (i) we deduce complete reducibility of \(DS_x(L(\lambda))\).

5.5. **A result on grading and semisimplicity.** Take \(x\) with \(x^2 = 0\) to be of rank 1.

5.5.1. **Notation.** Let \(\lambda \in P^+(q_n)\) be integral or half-integral. For \(k \in \frac{1}{2} \mathbb{N}\), we let

\[
g(\lambda, k) = \begin{cases} 
\ell(\lambda, k) + 1 & \text{if } \lambda \text{ integral} \\
\ell(\lambda, k) + 1/2 & \text{if } \lambda \text{ half-integral}
\end{cases}
\]

Here \(\ell(\lambda, k)\) denotes the number of nonzero free positions strictly to the left of \(k\) on the arc diagram of \(\lambda\) (see 4.3 for definition of a free position). For example, if \(\lambda\) is half-integral and has weight diagram \(\times \circ \times \times\) then \(g(\lambda, 7/2) = 3/2\) and \(g(\lambda, 1/2) = 1/2\), while if \(\lambda\) is integral with weight diagram \(\wedge^2 \times \circ \times\), then \(g(\lambda, 3) = g(\lambda, 1) = 1\).

5.5.2. **Theorem.** Let \(x\) be of rank 1 with \(x^2 = 0\), and let \(h\) be as in Section 3.5. Let \(\lambda \in P^+(q_n)\).

(i) If \(\mu \in P^+(q_{n-2})\) is obtained from \(\lambda\) removal of 2 zeros, then \(h\) acts trivially on \(L(\mu)\) as a submodule of \(DS_x L(\lambda)\).

(ii) If \(\mu \in P^+(q_{n-2})\) is obtained from \(\lambda\) by removal of a symbol \(\times\) at position \(k > 0\) in its arc diagram, then \(h\) acts with opposite eigenvalues \(\pm g(\lambda, k)\) on the two copies of \(L(\mu)\) in \(DS_x L(\lambda)\).

5.5.3. **Remark.** Theorems 5.2 and 5.5.2 imply that \(DS_x L(\lambda)\) is multiplicity-free as a module over \(g_x \times \mathbb{C}\langle h \rangle\). Thus we obtain as a simple corollary:

5.6. **Corollary.** For \(x\) of rank 1 with \(x^2 = 0\), \(DS_x L(\lambda)\) is semisimple.

**Proof.** This follows immediately from Corollary 3.4 and Theorem 3.5.2. □

5.6.1. **Example.** If \(\lambda = 7\epsilon_1 + 4\epsilon_2 + 2\epsilon_3 - 2\epsilon_5 - 4\epsilon_6 - 7\epsilon_7\), then the corresponding arc diagram looks as follows:

\[
\text{Arc diagram}
\]

As a \(g_x \times \mathbb{C}\langle h \rangle\)-module we have that

\[
DS_x L(\lambda) \cong L(\lambda_1)_1 \oplus \Pi L(\lambda_1)_{-1} \oplus L(\lambda_2)_1 \oplus \Pi L(\lambda_2)_{-1} \oplus L(\lambda_3)_2 \oplus \Pi L(\lambda_3)_{-2},
\]
where $\lambda_1 = 7\epsilon_1 + 4\epsilon_2 - 4\epsilon_4 - 7\epsilon_5$, $\lambda_2 = 7\epsilon_1 + 2\epsilon_2 - 2\epsilon_4 - 7\epsilon_5$, and $\lambda_3 = 4\epsilon_2 + 2\epsilon_3 - 2\epsilon_5 - 4\epsilon_5$. Here for a $g_x$-module $V$ and $t \in \mathbb{C}$ we write $V_t$ for the $g_x \times \mathbb{C}(h)$-module on which $h$ acts by $t$.

5.7. **Depth.** We set $X(g_r) := \{ x \in \mathfrak{g}_T^{ss} | \text{rank} \, x = r \}$. For a $g$-module $N$ we set

$$\tilde{X}(N) := \{ x \in \mathfrak{g}_T^{ss} \setminus \{0\} | \text{DS}_x(N) \neq 0 \},$$

and introduce $\text{depth}(N) \in \frac{1}{2} \mathbb{N} \cup \{\infty\}$ recursively by

$$\text{depth}(N) := \begin{cases} 0 & \text{if } \tilde{X}(N) = \emptyset \\ \max_{x \in \tilde{X}(N)} \left( \text{depth}(\text{DS}_x(N)) + \text{rank} \, x \right) & \text{if } \tilde{X}(N) \neq \emptyset. \end{cases}$$

By Corollary 3.3.2 one has $\text{depth}(N) \leq r$ if $N$ has a central character of atypicality $r$.

5.7.1. **Corollary.** For a finite-dimensional simple module $L = L(\lambda)$ one has

$$\text{depth}(L) = \text{atyp} \, \lambda.$$

**Proof.** By Corollary 3.3.2 one has $\text{depth}(L) \leq \text{atyp} \, \lambda$. Assume that $\text{atyp} \, \lambda > 0$. Combining Theorem 5.1 and Corollary 5.6 we deduce the existence of $x \neq 0$ (with $\text{rank} \, x \leq 1$) such that $\text{DS}_x(L)$ has a simple direct summand $L(\nu)$. By Corollary 3.3.2, $\text{atyp} \, \nu = \text{atyp} \, \lambda - \text{rank} \, x$. Using the induction on $\text{atyp} \, \lambda$ we obtain $\text{depth}(L) \geq \text{atyp} \, \lambda$. \qed

5.8. **Proof of Theorem 5.1 (ii).** Let us show that the following are equivalent

(a) $\text{smult}(\text{DS}_x L(\lambda), L(\nu)) \neq 0$;
(b) $[\text{DS}_x (L(\lambda)) : L(\nu)] = 1$;
(c) $\text{zero}(\lambda) - \text{zero}(\nu) = 2s$

where $\text{smult}$ (super multiplicity) of $L$ in $M$ is given by:

$$\text{smult}(M, L) := \begin{cases} [M : L] - [M : IL] & \text{if } L \not\cong IL \\ [M : L] \text{ mod } 2 & \text{if } L \cong IL \end{cases}$$

The implication (c) $\implies$ (b) can be easily established; for a proof in the case of $x^2 = 0$, see for example [11], Prop. 5.7.2. It remains to verify that the super multiplicity of $L(\nu)$ in $\text{DS}_x(L(\lambda))$ is zero except for the case when $\text{zero}(\lambda) - \text{zero}(\nu) = 2s$. 
5.8.1. **Corollary.** For $x$ of rank $r$ we have

$$[\text{Res}^q_{q_{n-2r}} L(\lambda)] = [\text{DS}_x(L(\lambda))],$$

Here $\lambda'$ is the dominant weight obtained from $\lambda$ by removing $2r$ zeroes.

*Proof.* Take $x = C_r$ and use Theorem 5.1 (iii). \hfill \Box

5.8.2. **Corollary.** Let $x$ be of rank $r$, and let $\lambda \in P^+(q_n)$, $\mu \in P^+(q_{n-2r})$. One has $\text{smult}(\text{DS}_x(L(\lambda)), L(\mu)) = \pm 1$ if $\mu$ is obtained from $\lambda$ by removing $2r$ zeroes and $\text{smult}(\text{DS}_x(L(\lambda)), L(\mu)) = 0$ otherwise.

*Proof.* In $\mathcal{K}_-(q_n)$ we have the following identity:

$$[M] = \sum_L \text{smult}(M, L)[L],$$

where $L$ runs over irreducible representations of $q_n$ up to parity. Now the statement follows from Corollary 5.8.1. \hfill \Box

This completes the proof Theorem 5.1 (ii).

5.9. **Examples.**

(i) For $\lambda = (4, 1, 0, -1, -4)$ we obtain the arc diagram:

```
\[ \xrightarrow{x} \xrightarrow{x} \xrightarrow{x} \xrightarrow{x} \]
```

We see that the diagram $\text{Arc}(\lambda)$ has three arcs $\text{arc}(0; 3)$, $\text{arc}(4; 5)$, $\text{arc}(1; 2)$;

and that the first two arcs are maximal, and the first is a half arc. If $x$ is of rank 1/2, then $\text{DS}_x(L(\lambda)) = L(\lambda')$ where $\lambda' = (4, 1, -1, -4)$. If $x$ is of rank 1 with $x = T_{0,B}$ and $\text{tr}(B) = 0$, then $\text{DS}_x(L(\lambda))$ has length two with subquotients $L(\nu), \Pi L(\nu)$ for $\nu = (1, 0, -1)$. If $x$ is of rank 1 with $x = T_{0,B}$ and $\text{tr}(B) \neq 0$, then $\text{DS}_x(L(\lambda)) = 0$.

(ii) By Corollary 5.3.2 for an atypical weight $\lambda \in P^+(q_3)$ one has $\text{DS}_x(L(\lambda)) = 0$ for any $x$ with $x^2 = 0$ if and only if $x = (1, 0, -1)$; for $q_4$ this holds if and only if $\lambda \in \{(2, 0, -1, 2), (2, 1, 0, -2), (a, 1, 0, -1), (1, 0, -1, -a)\}$ for $a \in \mathbb{N}_{>1}$.

(iii) A weight $\lambda \in P^+(q_5)$ of atypicality 2 satisfies $\text{DS}_x(L(\lambda)) = 0$ for any $x$ with $x^2 = 0$ if and only if $\lambda = (2, 1, 0, -1, -2)$ or $\lambda = (3, 1, 0, -1, -3)$. In these cases $\text{DS}_{1/2}(L(\lambda))$ is $L(\nu)$ for $\nu = (2, 1, -1, -2)$ and $\lambda = (3, 1, -1, -3)$ respectively.

(iv) Let $\lambda$ be one of the weights $(2, 1, 0, -1, -2), (3, 1, 0, -1, -3)$. By Lemma 3.1.1 if $x = x_{ss} + x_{nil} \in \mathfrak{g}_r^*$ satisfies $\text{DS}_x L(\lambda) \neq 0$, then $x_{nil} = 0$. 


5.10. **Shrinking.** Let \( f \) be a core-free diagram. Each minimal arc in \( \text{Arc}(f) \) takes the following form \( \text{arc}(a; a+1) \) for \( a \geq 0 \) such that \( f(a) = \times \) or \( \wedge \), and \( f(a+1) = \circ \).

If \( a \) supports a minimal arc we define the diagram \( \text{shr}_a(f) \) as follows:

- if \( a \neq 0 \), we produce \( \text{shr}_a(f) \) by “shrinking” the positions \( a \) and \( a+1 \); for instance,
  \[
  \text{shr}_3(\wedge^2 \circ \times \circ \times) = \wedge^2 \circ \times \times;
  \]
- if \( \text{arc}(0; 1) \in \text{Arc}(f) \) we produce \( \text{shr}_a(f) \) by a removal of a \( \wedge \) from the zero position and “shrinking” the position 1; for instance,
  \[
  \text{shr}_0(\wedge^3 \circ \times \circ \times) = \wedge^2 \times \circ \times.
  \]

Observe that \( \text{Arc}(\text{shr}_a(f)) \) is obtained from \( \text{Arc}(f) \) by “shrinking” the minimal arc supported at \( a \); the rest of the arcs remain “the same”, i.e. we have the natural injective map \( \text{Arc}(\text{shr}_a(f)) \to \text{Arc}(f) \) which preserves order between arcs.

The following proposition, which will be proven in Section 5, reduces the computation of \( \text{DS}_x(L(\lambda)) \) to the case \( g = q_s \), where \( \text{rank } x = s \).

5.11. **Proposition.** Let \( x \in g^s_T \) be of rank \( s \), and let \( \lambda \in P^+(q_n) \), \( \nu \in P^+(q_{n-2s}) \).

(i) If \( m_x(\lambda; \nu) \neq 0 \), then \( \text{core}(\lambda) = \text{core}(\nu) \) and \( \text{Arc}(\nu) \subset \text{Arc}(\lambda) \).

(ii) If \( \text{core}(\lambda) = \text{core}(\nu) \) and \( \text{Arc}(\nu) \subset \text{Arc}(\lambda) \), then

\[
(4) \quad m_x(\lambda; \nu) = \dim \text{DS}_x(L(\lambda')),
\]

where \( \text{Arc}(\lambda') \) is obtained from \( \text{Arc}(\text{howl}(\lambda)) \) by shrinking successively all arcs appearing in \( \text{Arc}(\text{howl}(\nu)) \) (starting from the minimal arcs).

5.11.1. **Example.** Suppose that rank \( x = 3 \). Then Proposition 5.11 gives

\[
[\text{DS}_x(L(\wedge^4 \circ \times \circ \times) : L(\wedge^2 \circ \circ \times)) = 0
\]

since \( \text{arc}(0, 2) \) lies in \( \text{Arc}(\wedge^2 \circ \circ \times) \) but does not lie in \( \text{Arc}(\wedge^4 \circ \times \circ \times) \). Another example: for rank \( x = 2 \) we have

\[
[\text{DS}_x(L(\wedge^2 > \circ \times \circ \circ \times) : L(\wedge^2 > \circ \circ)) = \dim \text{DS}_x(L(\circ \times \circ ))).
\]

The above formula is obtained as follows: in this case \( \text{howl}(\nu) \) has the diagram \( f = \wedge^2 \circ \times \) with minimal arcs \( \text{arc}(0, 1) \) and \( \text{arc}(2; 3) \); shrinking these arcs gives the diagram \( \text{shr}_0\text{shr}_2(f) = \wedge \), which has a unique arc \( \text{arc}(0; 1) \). The diagram of \( \text{howl}(\lambda) \) is \( g = \wedge^2 \circ \circ \circ \circ \times \times \) and

\[
\text{shr}_0\text{shr}_0\text{shr}_2(g) = \text{shr}_0\text{shr}_0(\wedge^2 \circ \circ \times \times) = \circ \times \times.
\]
5.12. **Reduction of Theorems 5.1, 5.2 and Theorem 5.5.2 to Proposition 5.11**

Proposition 5.11 implies Theorem 5.1 (i) and (ii) is proven in Section 5.8. In Theorems 5.2 and 5.3, \( x \) is of rank 1. For this case the weight \( \lambda' \) appearing in the right-hand side of (4) is a core-free weight of \( q_2 \), so Proposition 5.11 (ii) reduces the assertions to the case \( g=q_2 \), which can be easily computed explicitly: we have to compute \( DS_x(L(\lambda')) \) for a weight \( \lambda' \in P^+(q_2) \) of atypicality 2 and \( x \in (q_2)_{ss}^0 \) of rank 1. Recall that \( DS_x(L(\lambda')) = 0 \) if \( \lambda' \) is not integral and \( x^2 \neq 0 \). One has

\[
\dim DS_x(L(\lambda')) = \begin{cases} 1 & \text{if } \lambda' = 0 \text{ i.e. } \text{diag } \lambda' = \wedge^2 \\ 2 & \text{if } \lambda' \neq 0 \text{ i.e. } \text{diag } \lambda' = \circ \ldots \times \end{cases}
\]

if \( x^2 = 0 \) or if \( \lambda' \) is integral and \( x = T_{0,B} \) with \( \text{tr}(B) = 0 \). This establishes Theorems 5.2 (i) and 5.3. In the remaining case when \( \lambda' \) is integral and \( x = T_{0,B} \) with \( B^2 \) semisimple and \( \text{tr}(B) \neq 0 \) we have

\[
\dim DS_x(L(\lambda')) = \begin{cases} 1 & \text{if } \lambda' = 0 \text{ i.e. } \text{diag } \lambda' = \wedge^2 \\ 0 & \text{if } \lambda' \neq 0 \text{ i.e. } \text{diag } \lambda' = \circ \ldots \times \end{cases}
\]

This establishes Theorem 5.2 (ii). Finally, (iii) of Theorem 5.1 follows from the following lemma:

5.12.1. **Lemma.** *We have* \( DS_{C_n}(L(\lambda)) = 0 \) *for a non-zero weight* \( \lambda \in P^+(q_n) \).

*Proof.* Recall that \( DS_x(N) \) is an \( g^x \)-subquotient \( \text{Res}_g^x N \) which is annihilated by the ideal \([x,g] \cap g^x \). For \( x := C_n \) one has \( g^x = [x,g] = g_{0\pi} \), so \( DS_{C_n}(N) \) is a subquotient of \( N_{0\pi} \) if \( N \) is a finite-dimensional \( g \)-module. By [5], \( \text{Res}_n^g L(\lambda) \) contains no copies of the trivial module if \( L \) is not itself trivial; thus we must have \( DS_{C_n}(L(\lambda)) = 0 \). \( \square \)

5.12.2. **Reduction of Theorem Theorem 5.5.2.** In order to prove Theorem 5.5.2, we will follow the proof of Proposition 5.11 given in the next section while also keeping track of the action of \( h \). Namely, suppose that we are in the setup of Proposition 5.11 except that we assume that \( x \) is of rank 1 with \( x^2 = 0 \). Then \( h \) will act on any composition factor \( L(\nu) \) inside of \( DS_x L(\lambda) \) by a scalar; thus we may upgrade \( m_x(\lambda, \nu) \) to a vector space with an \( h \)-action that we write as \( M_x(\lambda, \nu) \), where we have \( \dim M_x(\lambda, \nu) = m_x(\lambda, \nu) \). Then in this case we will show that we obtain an isomorphism of \( h \)-vector spaces

\[(5) \quad M_x(\lambda, \nu) \cong DS_x(L(\lambda')).\]

The isomorphism (5) reduces the result to the \( q_2 \)-case, where we find that, for \( k \in \frac{1}{2} \mathbb{N} \), (up to parity)

\[DS_x L(k(\epsilon_1 - \epsilon_2)) = C_k \oplus \Pi C_{-k},\]

where \( C_k \) is the one-dimensional even vector space on which \( h \) acts by \( k \text{ Id} \).
6. Proof of Proposition 5.11

6.1. Translation functors and DS. Our main tools are translation functors described in \[1\]. Below we briefly recall the connection between the translation functor and DS functor, see \[14\], 7.1.

6.1.1. Notation. For a core diagram \(c\) we denote by \(\text{Irr}(\mathfrak{g})^c\) the set of isomorphism classes of finite-dimensional irreducible modules \(L(\lambda)\) with \(\text{core}(\lambda) = c\). Recall that all modules in \(\text{Irr}(\mathfrak{g})^c\) have the same central character \(\chi\). Let \(\mathcal{F}\text{in}(\mathfrak{g})^c\) be the full subcategory of \(\mathcal{F}\text{in}(\mathfrak{g})\) which corresponds to the central character \(\chi\) (i.e., \(N \in \mathcal{F}\text{in}(\mathfrak{g})^c\) if and only if \((z - \chi(z))^{\dim N}N = 0\) for each \(z \in \mathcal{Z}(\mathfrak{g})\)). Then \(\text{Irr}(\mathfrak{g})^c\) is the set of isomorphism classes of irreducible modules in \(\mathcal{F}\text{in}(\mathfrak{g})^c\).

Note that \(L(\varepsilon_1)\) is the standard module and \(L(-\varepsilon_n)\) is its dual. Let \(c, c'\) be core diagrams; we denote by \(T_{c'}^c\) (resp., \((T_{c'}^c)^*\)) the translation functor \(\mathcal{F}\text{in}(\mathfrak{g})^c \to \mathcal{F}\text{in}(\mathfrak{g})^{c'}\) which maps \(N\) to the projection of \(N \otimes L(\varepsilon_1)\) (resp., of \(N \otimes L(-\varepsilon_n)\)) to the subcategory \(\mathcal{F}\text{in}(\mathfrak{g})^{c'}\). The functors \(T_{c'}^c, (T_{c'}^c)^*\) are both left and right adjoint to each other.

6.1.2. It is easy to check that for any \(x \in \mathfrak{g}_x^s\) we have \(\text{DS}_x(L_{\mathfrak{g}}(\varepsilon_1)) = L_{\mathfrak{g}_x}(\varepsilon_1)\). Since DS commutes with tensor product and preserves cores (see 3.3.2) one has

\[
\text{DS}_x(T_{c'}^c(N)) \cong T_{c'}^c(\text{DS}_x(N))
\]

where \(c', c\) are core diagrams and \(T_{c'}^c\) stands for the functors \(\mathcal{F}\text{in}(\mathfrak{g})^c \to \mathcal{F}\text{in}(\mathfrak{g})^{c'}\) and \(\mathcal{F}\text{in}(\mathfrak{g}_x)^c \to \mathcal{F}\text{in}(\mathfrak{g}_x)^{c'}\).

Since the translation functors are exact, they induce morphisms of the Grothendieck ring. For any \(N \in \mathcal{F}\text{in}(\mathfrak{g})^c\) and \(L' \in \text{Irr}(\mathfrak{g}_x)^c\) we have

\[
[\text{DS}_x(T_{c'}^c(N)) : L'] = [T_{c'}^c(\text{DS}_x(N)) : L'] = \sum_{L_1 \in \text{Irr}(\mathfrak{g}_x)^c} [\text{DS}_x(N) : L_1][T_{c'}^c(L_1) : L'].
\]

(6)

Assume that \(L_1 \in \text{Irr}(\mathfrak{g}_x)^c\) and \(L' \in \text{Irr}(\mathfrak{g}_x)^c\) are such that for each \(L_2 \in \text{Irr}(\mathfrak{g}_x)^c\) with \(L_2 \neq L_1, \Pi L_1\) one has \([T_{c'}^c(L_2) : L'] = 0\). Then (6) gives

\[
[\text{DS}_x(T_{c'}^c(N)) : L'] = [T_{c'}^c(L_1) : L'] \cdot [\text{DS}_x(N) : L_1].
\]

(7)

6.2. Outline of the proof of Proposition 5.11. We will use the following notation

\[
\frac{\text{diag}(\lambda)}{\text{diag}(\nu)} := [\text{DS}_x(L(\lambda)) : L(\nu)].
\]

By 3.3.2 \(\mathfrak{g} \neq 0\) implies that \(\text{core}(f) = \text{core}(g)\). The proof of the remaining assertions of Proposition 5.11 follows the same steps as in \(\mathfrak{osp}\)-case. We briefly describe these steps below.

6.2.1. Reduction to the stable case. We call a weight diagram \( f \) stable if all symbols \( \times \) precede all core symbols, (notice that the symbols \( \wedge \) necessarily precede core symbols); we say that \( \eta \in P^+(g) \) is stable if \( \eta \) is typical or \( \text{diag}(\eta) \) is stable. For instance, \( (4, 1, 0, -1, -2) \) with the diagram \( \wedge \times < \circ > \) is stable and the weight \( \zeta = (2, 1, -2) \) with the diagram \( \circ > \times \) is not stable.

The general case can be reduced to the stable case with the help of translation functors described in [1]. For this step we will use translation functors which preserves howl: such functor transforms \( L(\eta) \) to \( L(\eta') \oplus \Pi L(\eta') \), where \( \text{diag}(\eta') \) is obtained from \( \text{diag}(\eta) \) by permuting two neighboring symbols at non-zero positions if exactly one of these symbols is a core symbol: for instance, from \( \zeta \) as above we can obtain \( \zeta' \) with the diagram \( \circ \times > \) which is stable. Note that howl \( (\eta) = \text{howl}(\eta') \). Using these functors we can transform any two simple modules \( L(\lambda), L(\nu) \) with core(\( \lambda \)) = core(\( \nu \)) to the modules \( L(\lambda_{st})^{\oplus r} \oplus \Pi L(\lambda_{st})^{\oplus r}, L(\nu_{st})^{\oplus r} \oplus \Pi L(\nu_{st})^{\oplus r} \), where \( \lambda_{st}, \nu_{st} \) are stable weights with

\[
\text{core}(\lambda_{st}) = \text{core}(\nu_{st}), \quad \text{howl}(\lambda) = \text{howl}(\lambda_{st}), \quad \text{howl}(\nu) = \text{howl}(\nu_{st})
\]

(the diagrams of \( \lambda_{st} \) and \( \nu_{st} \) are stable diagrams obtained from \( \text{diag}(\lambda) \), \( \text{diag}(\nu) \) by moving all core symbols from the non-zero position “far enough” to the right). For instance,

\[
\lambda = (6, 5, 1, 0, 0, -3, -5) \quad \nu = (6, 1, 0, 0, -3)
\]

\[
\text{diag}(\lambda) = \Lambda^2 > \circ < \circ \times > \quad \text{diag}(\nu) = \Lambda^2 > \circ < \circ \circ >
\]

\[
\text{diag}(\lambda_{st}) = \Lambda^2 \circ \circ \times \circ \circ >> \quad \text{diag}(\nu_{st}) = \Lambda^2 \circ \circ \circ \circ >>
\]

\[
\lambda_{st} = (8, 6, 3, 0, 0, -3, -7) \quad \nu_{st} = (8, 6, 0, 0, -7).
\]

Using (7) we will obtain the formula (8) which implies

\[
[DS_x(L(\lambda)) : L(\nu)] = [DS_x(L(\lambda_{st})) : L(\nu_{st})].
\]

This formula reduces the general case to the case when \( \lambda, \nu \) are stable.

6.2.2. Reduction to the core-free case. In [5] we will show that for a stable weight \( \lambda \) the computation of multiplicities in \( DS_x(L(\lambda)) \) can be reduced to the case when \( \lambda \) is core-free.

6.2.3. Shrinking in the core-free case. Using translation functors which do not preserve howl we will obtain useful “cancellation” formulae (9). Applying these formulae for core-free weights \( \lambda, \nu \) with the diagrams \( f, g \) we obtain

(i) if \( [DS_x(L(\lambda)) : L(\nu)] \neq 0 \), then each minimal arc in \( \text{Arc}(\nu) \) is a minimal arc in \( \text{Arc}(\lambda) \); in particular, if the operation \( \text{shr}_a \) (introduced in [5,10]) is defined for \( g \), then this operation is defined for \( f \);
(ii) if \( \text{shr}_a \) is defined for \( f \) and \( g \) for \( a \neq 0 \), then \( \frac{f}{g} = \frac{\text{shr}_a(f)}{\text{shr}_a(g)} \);
(iii) if \( [DS_x(L(\lambda)) : L(0)] \neq 0 \), then \( \text{Arc}(0) \subset \text{Arc}(\lambda) \) and \( \frac{f}{g} = \frac{\text{shr}_0(f)}{\text{shr}_0(g)} \).

Proposition [5,11] follows from these assertions by induction on \( \nu_1 \) (where \( \nu = (\nu_1, \ldots, \nu_{n-2r}) \)).
6.3. Useful translation functors. Below we recall several results of Lemma 4.3.8 in [1]. Let \( g \) be a weight diagram and \( c \) be the core diagram of \( g \). We retain notation of [6.1].

6.3.1. Translation functors preserving howl. Fix \( a \neq 0 \). For a weight diagram \( f \) we denote by \( \sigma(f) \) the diagram obtained by interchanging the symbols at the positions \( a \) and \( a + 1 \).

If \( c(a) \Rightarrow \) and \( c(a + 1) = \circ \) (i.e. \( c = \ldots > \circ \ldots \), \( \sigma(c) = \ldots \circ > \ldots \)), then
\[
T_c^{(g)}(L(g)) = L(\sigma(g)) \oplus \Pi L(\sigma(g)).
\]
The same formula holds if \( c(a) = \circ \), \( c(a + 1) \Leftarrow \). We depict these formulae as
\[
T_{\ldots \circ \ldots}(L(\ldots \circ \ldots)) = L(\ldots \circ \ldots) \oplus (L(\ldots \circ \ldots)) \oplus \Pi L(\ldots \circ \ldots)
\]
for \( \star \in \{\circ, \times\} \). The similar formulae hold for \( T^* \) if we interchange \( \Rightarrow \) and \( \Leftarrow \).

6.3.2. Translation functors reducing atypicality. Using the above notation we have
\[
\begin{align*}
T_{\ldots \circ \ldots}(L(\ldots \times \ldots)) &= T_{\ldots \circ \ldots}(L(\ldots \circ \ldots)) = 0 \\
T_{\ldots \circ \ldots}(L(\ldots \circ \ldots)) &= L(\ldots \circ \ldots) \oplus (L(\ldots \circ \ldots)) \oplus \Pi L(\ldots \circ \ldots)
\end{align*}
\]
where \( \star \in \{\circ, \times\} \), \( r \geq 0 \) and \( d = 1 \) if \( n - r \) is odd, \( d = 2 \) if \( n - r \) is even. For instance,
\[
T_{\circ \circ \circ \circ}(L(4, 3, 1, -1)) = L(4, 3, 2, -1) \oplus \Pi L(4, 3, 2, -1)
\]
since \( (4, 3, 1, -1) \) has the diagram \( \circ \times \circ \Rightarrow (4, 3, 2, -1) \) has the diagram \( \circ \Leftarrow \Rightarrow \Rightarrow \).

6.4. Cancellation formulae. We denote by \( f_- f_+ \) the diagram obtained by "gluing" the diagrams \( f_- \) and \( f_+ \) (where \( f_+ \) has exactly one symbol at each position) for instance,
\[
f_- = \land^4 \circ \quad f_+ = \times \quad f_- f_+ = \land^4 \circ \times.
\]

Recall that \( \frac{g}{\nu} \) stands for the multiplicity \( \text{DS}_\nu(L(\lambda)) : L(\nu) \), where \( \text{diag} \lambda = g \) and \( \text{diag} \nu = g' \). Applying (7) to the translation functors which appeared in 6.3.1 we obtain the following formulae
\[
(8) \quad \frac{f_- \circ f_+}{g_- \circ g_+} = \frac{f_- \times f_+}{g_- \times g_+} \quad \frac{f_- \times f_+}{g_- \times g_+} = \frac{f_- \circ f_+}{g_- \circ g_+},
\]
where the symbol \( \star \in \{\circ, \times\} \) occupies the same position in each diagram (for instance, the first formula gives \( \frac{\circ \times \circ \times \circ}{\circ \times \circ \times \circ} = \frac{\circ \times \circ \times \circ}{\circ \times \circ \times \circ} \)). Using (7) for the translation functors appearing in 6.3.2 we get for \( r, r' \geq 0 \)
\[
(9) \quad \frac{f_- \circ f_+}{g_- \circ g_+} = \frac{f_- \times f_+}{g_- \times g_+} = 0 \quad \frac{\land^r f_+}{g_- \circ g_+} = 0 \quad \frac{\land^{r+1} f_+}{g_- \circ g_+} = \frac{\land^{r+1} f_+}{g_- \circ g_+} = \frac{\land^{r+1} f_+}{g_- \circ g_+}.
\]
6.5. **Reduction to the core-free case.** Our next goal is to verify

\[
\frac{f}{g} = \frac{\text{howl}(f)}{\text{howl}(g)}
\]

if \(f, g\) are stable diagrams with \(\text{core}(f) = \text{core}(g) = c\). The formula is tautological if \(f, g\) are core-free, so we assume that this is not the case, i.e. \(c\) is non-empty.

6.5.1. **Notation for 6.5.** Let \(\lambda \in t^*, \nu \in t^*_x\) be stable weights with \(\text{core}(\lambda) = \text{core}(\nu) = c\).

Let \(u_+ (\text{resp., } u_-)\) be the number of symbols > (resp., <) of \(c\); we set

\[
u := u_- + u_+, \quad I' := \{u_+ + 1, \ldots, n - u_-\}, \quad I'' := I \setminus I'.
\]

The assumption on \(c\) implies \(u \neq 0\). Since DS “preserves the cores”, for \(x \in g_{\text{ss}}^1\) of rank \(s\) we have \(\text{DS}_x(\text{Fin}(g)^c) = 0\) if \(n - 2s < u\). Therefore we assume

\[0 < u \leq n - 2s.\]

Retain notation of 2.4 and introduce

\[g' := q(I'), \quad h' := h(I'), \quad h'' := h(I''), \quad t' := t(I'), \quad t'' := t(I'').\]

Note that \(g' \cong q_{n-u}, h'\) is a Cartan subalgebra of \(g'\) and \(h = h' \times h''\).

Consider the subalgebra \(g' + h = g' \times h'' \subset g\). We will use the notation \(L_{g' \times h''}(\mu), L_{g'}(\mu')\) for the corresponding simple modules over \(g' \times h''\) and \(g'\) respectively (where \(\mu \in t^*\) and \(\mu' \in (t')^*\)).

Let \(\text{wt}(c) \in (t'')^*\) be the weight “corresponding to the diagram \(c\)”: we take \(\langle \text{wt}(c), h_i \rangle\) (resp., \(\langle \text{wt}(f), h_{n+1-i} \rangle\)) equal to the coordinate of \(i\)th symbol > (resp., <) in \(c\) counted from the right. For example, for \(c = \Rightarrow \cdots \Rightarrow\) and \(n = 8\), \(t'\) (resp., \(t''\)) is spanned by \(h_3, \ldots, h_7\) (resp., \(h_1, h_2, h_8\)) and \(\text{wt}(c) = 5\varepsilon_1 + 3\varepsilon_2 - 4\varepsilon_8\).

6.5.2. **Remark.** For \(\mu \in P^+(q_n)\) with \(\text{core}(\mu) = c\) the following are equivalent

- \(\mu\) is stable;
- \(\mu|_t = \text{howl}(\mu)\);
- \(\mu|_{t'} = \text{wt}(c)\).

6.5.3. **Choice of \(x\).** We take \(x\) of rank \(s\) lying in \(g'\) (this can be done since \(2s \leq n - u\)). One has \(g'_x := \text{DS}_x(g') \cong q_{n-u-2s}\). We identify \(g_x\) (resp., \(g'_x\)) with the subalgebras of \(g = q_n\) as in 5.1 and identify \(\text{DS}_x(g' \times h'')\) with \(g'_x \times h'' \subset g_x\).
6.5.4. **Lemma.** \([\text{DS}_x(L(\lambda)) : L_{g^x}(\nu)] = [\text{DS}_x(L_{g^x \times h''}(\lambda)) : L_{g^x \times h''}(\nu)]\).

**Proof.** Choose \(z \in t'' \subset t\) in such a way that \(\{\varepsilon_i(z)\}_{i \in I''}\) are positive real numbers linearly independent over \(\mathbb{Q}\). We set
\[
a_0 := \text{wt}(c)(z).
\]
One has
\[
\mathfrak{g}^z = \mathfrak{g}' \times \mathfrak{h}'', \quad \lambda(z) = \nu(z) = a_0
\]
(the last formula follows from 6.5.2 and the stability of \(\lambda, \nu\)).

For a \(t''\)-module \(N\) we denote by \(\text{Spec}_z(N)\) the set of \(z\)-eigenvalues on \(N\), by \(N_a\) the \(a\)th eigenspace and view \(N_a\) as a module over \(\mathfrak{g}^z\).

It is easy to see that for any \(\mu \in t^*\) we have
\[
\begin{align*}
&(a) \text{ Spec}_z(L(\mu)) \subset \mu(z) - \mathbb{R}_{\geq 0}; \\
&(b) L(\mu)|_{\mu(z)} = L_{g' \times h''}(\mu) \quad (\text{this follows from the PBW-theorem}).
\end{align*}
\]
Set \(N := \text{DS}_x(L(\lambda))\). Since \(x \in \mathfrak{g}'\) we have \([z, x] = 0\) so \(N_a = \text{DS}_x(L(\lambda)_a)\); this gives
\[
\text{Spec}_z(N) \subset \lambda(z) - \mathbb{R}_{\geq 0}, \quad N_{a_0} = \text{DS}_x(L_{g' \times h''}(\lambda)).
\]
In particular,
\[
[\text{Res}_{g' \times h''}^g N : L_{g' \times h''}(\nu)] = [N_{a_0} : L_{g' \times h''}(\nu)] = [\text{DS}_x(L_{g' \times h''}(\lambda)) : L_{g' \times h''}(\nu)].
\]
On the other hand,
\[
[\text{Res}_{g' \times h''}^g N : L_{g' \times h''}(\nu)] = \sum_{\mu \in I_z} [N : L_{g^x}(\mu)] \cdot [L_{g^x}(\mu) : L_{g^x \times h''}(\nu)].
\]
Assume that \([N : L_{g^x}(\mu)] \cdot [L_{g^x}(\mu) : L_{g^x \times h''}(\nu)] \neq 0\). Then, by above, \(\nu(z) \leq \mu(z) \leq a_0\). Using (10) we get \(\mu(z) = \nu(z)\). Applying (b) we obtain \(\mu = \nu\) and \([L_{g^x}(\nu) : L_{g^x \times h''}(\nu)] = 1\). Hence (12) can be rewritten as
\[
[\text{Res}_{g' \times h''}^g N : L_{g' \times h''}(\nu)] = [N : L_{g^x}(\nu)] = [\text{DS}_x(L(\lambda)) : L_{g^x}(\nu)].
\]
Now the required assertion follows from (11). \(\square\)

6.5.5. **Lemma.** One has
\[
\text{Res}_{g' \times h''}^g L_{g' \times h''}(\lambda) = L_{g'}(\lambda')^{d_d}
\]
where \(\lambda' := \lambda|_{e'}\) and \(d := 2\lfloor \frac{d}{2} \rfloor\).

**Proof.** Retain notation of 2.2. Recall that \(B_\lambda\) is the symmetric form on \(b_T\) given by \((H, H') \mapsto \lambda([H, H'])\). We denote by \(B'_\lambda\) the restriction of \(B_\lambda\) to \(b'_T\) and view
\[
\mathcal{C}(\lambda') := \mathcal{C}(b'_T, B'_\lambda)
\]
as a subalgebra of $C\ell(\lambda)$. By 2.2, the highest weight space of $L_{g'\times h'}(\lambda)$ is $C_\lambda$ and the highest weight space of $L_{g'}(\lambda')$ is a simple $C\ell(\lambda')$-module which we denote by $E'_{\lambda'}$. It is easy to see that all simple subquotients of $\text{Res}^{g'\times h''}_{g'} L_\lambda(\lambda)$ are isomorphic to $L_{g'}(\lambda')$. Therefore it is enough to check that

$$\text{Res}^{C\ell(\lambda)}_{C\ell(\lambda')} C_\lambda = E'_{\lambda'} \oplus E''_{\lambda'}$$

for $d$ as above. By [16], Prop. 3.5.1 the semisimplicity of $\text{Res}^{C\ell(\lambda)}_{C\ell(\lambda')} C_\lambda$ is equivalent to the formula $\text{Ker} B'_{\lambda} = b_+ \cap \text{Ker} B_\lambda$. This formula holds since $\text{Ker} B_\lambda$ is spanned by $H_i$ with $\lambda_i = 0$, $b_+^{\prime}$ is spanned by $H_i$ with $i \in I'$, and $\text{Ker} B'_\lambda$ is spanned by $H_i$ with $\lambda_i = 0$ and $i \in I'$. Hence $\text{Res}^{C\ell(\lambda)}_{C\ell(\lambda')} C_\lambda$ is semisimple. By 2.2 $\dim C_\lambda = 2j$ where

$$j = \left\lfloor \frac{\text{rank} B_\lambda + 1}{2} \right\rfloor - \left\lfloor \frac{\text{rank} B'_\lambda + 1}{2} \right\rfloor = \left\lfloor \frac{\text{nonzero}(\lambda) + 1}{2} \right\rfloor - \left\lfloor \frac{\text{nonzero}(\lambda') + 1}{2} \right\rfloor.$$

By 6.5.2 $\lambda = \text{howl}(\lambda)$, so nonzero($\lambda'$) is even and nonzero($\lambda$) = nonzero($\lambda'$) + $u$. This gives $j = \left\lfloor \frac{\lambda + 1}{2} \right\rfloor$ and completes the proof. \(\square\)

6.5.6. Corollary. $L_g = \frac{\text{howl}(f)}{\text{howl}(g)}$

Proof. Notice that $t'_x := t' \cap t_x$ plays the same role for $g_x$ as $t'$ for $g$; therefore 6.5.5 gives

$$\text{Res}^{g'_{t'} \times h''}_{g'_{t}} L_{g'_{t'}}(\nu') = L_{g'_{t'}}(\nu') \oplus E$$

where $\nu' := \nu|_{t'_{\nu'}}$. Combining Lemmata 6.5.5, 6.5.4 and the above formula we get

$$[\text{DS}_{t}(L(\lambda)) : L(\nu)] = [\text{DS}_{t}(L_{g'}(\lambda')) : L_{g'_{t'}}(\nu')]$$

By 6.5.2 we have $\lambda|_{t_{\nu}} = \text{howl}(\lambda)$ and $\nu|_{t_{\nu}} = \text{howl}(\nu)$ as required. \(\square\)

6.6. The core-free case. It remains to verify (i)—(iii) in 6.2.3.

Thus let $x \in B_{t_{\nu}}^s$ be of rank $s$, and let $\lambda \in P^+(q_{m+2s}), \nu \in P^+(q_m)$ be core-free weights with $m \neq 0$.

6.6.1. Consider the case when $\nu \neq 0$. Then $\nu_1 > 0$. Write $\text{diag}(\nu) = g_- \times \circ$ (where $\circ$ is at the position $\nu_1$). By [12], if $[\text{DS}_{s}(L(\lambda)) : L(\nu)] \neq 0$, then $\text{diag}(\lambda) = f_- \times \circ f_+$ for some diagrams $f_-, f_+$ (where $\circ$ is at the position $\nu_1$). This gives (i) in 6.2.3. Combining (9), (8) and (6.5) we get

$$\frac{f_- \times \circ f_+}{g_- \times \circ} = \frac{f_- < \circ f_+}{g_- < \circ} = \frac{f_- f_+}{g_-}.$$

Note that both $\text{Arc}(f_- \times \circ f_+)$, $\text{Arc}(g_- \times \circ)$ contain a minimal $\text{arc}(\nu_1; \nu_1 + 1)$ and

$$f_- f_+ = \text{shr}_{\nu_1}(f_- \times \circ f_+), \quad g_- = \text{shr}_{\nu_1}(g_- \times \circ).$$

This establishes (ii) in 6.2.3.
6.6.2. The same argument shows that \([\text{DS}_x(L(\lambda)) : L_{q_m}(0)] \neq 0\) implies \(\text{diag } \lambda = \wedge^r \circ f_+\) for \(r \geq 1\) and that

\[
\frac{\wedge^r \circ f_+}{\wedge^m} = \frac{\wedge^{r-1} f_+}{\wedge^{m-1}}.
\]

Note that \(\wedge^m = \text{shr}_0(\wedge^{m+1})\) and \(\wedge^r f_+ = \text{shr}_0(\wedge^{r+1} \circ f_+).\)

From (13) we conclude that for \(m > 1\) one has

\[
[\text{DS}_x(L(\lambda)) : L_{q_m}(0)] \neq 0 \implies \text{diag } \lambda = \wedge^r \circ f_+
\]

and

\[
\frac{\wedge^r \circ f_+}{\wedge^m} = \frac{\wedge^{r-1} f_+}{\wedge^{m-1}}.
\]

This establishes 6.2.3 (iii) and completes the proof of Proposition 5.11. \(\square\)

6.7. Keeping track of the action of \(h\). We now explain how to prove the isomorphism (13) in the case when \(x^2 = 0\) and the rank of \(x\) is 1. In this case, we view \(\text{DS}_x\) as a functor from the category of \(\mathfrak{g}\)-modules to the category of \(\mathfrak{g}_x \times \mathbb{C}\langle h \rangle\)-modules. If \(V\) is semisimple over \(\mathfrak{g}_0\), which is the only case we consider, then \(\text{DS}_x\) will admit a semisimple action of \(h\). Given a \(\mathfrak{g}_x\)-module \(V\) and \(t \in \mathbb{C}\), we write \(V_t\) for the \(\mathfrak{g}_x \times \mathbb{C}\langle h \rangle\)-module with \(h\) acting by \(t\).

For the standard module \(L(\epsilon_1)\), the \(h\) action on \(\text{DS}_x L(\epsilon_1)\) is trivial, and thus for any pair of cores \(c, c'\) we have a natural isomorphism of \(\mathfrak{g}_x \times \mathbb{C}\langle h \rangle\)-modules

\[
\text{DS}_x(T_{c'}(N)) \cong T_{c'}(\text{DS}_x(N)).
\]

Notice that the central characters of \(\mathfrak{g}_x \times \mathbb{C}\langle h \rangle\) are parametrized by pairs \((c, t)\), where \(c\) is a core and \(t \in \mathbb{C}\). On the RHS of the above formula, \(T_{c'}\) denotes the translation functor between modules with central characters of the form \((c', t)\) to modules with central characters of the form \((c, t)\). In particular, \(T_{c'}\) takes modules with central character indexed by \((c', t)\) to modules with central character indexed by \((c, t)\).

In particular we obtain the following: for any \(N \in \mathcal{F}\text{Inn}(\mathfrak{g})^c\) and \(L' \in \text{Irr}(\mathfrak{g}_x)^c\) we have

\[
[\text{DS}_x(T_{c'}(N)) : L'_t] = [T_{c'}(\text{DS}_x(N)) : L'_t] = \sum_{L_1 \in \text{Irr}(\mathfrak{g}_x)} [\text{DS}_x(N) : (L_1)_t][T_{c'}(L_1) : L'_t].
\]

Using Equation (14), one may use the same operations with translation functors as is done in the general case above, and we see that the weight \(t\) of \(h\) will remain unaffected.

The other check that needs to be made is that for stable weights \(\lambda \in P^+(q_n), \mu \in P^+(q_n)\), and for \(t \in \mathbb{C}\), we have

\[
[\text{DS}_x L(\lambda) : L(\mu)_t] = [\text{DS}_x L(\text{howl}(\lambda)) : L(\text{howl}(\mu))_t].
\]
However following the argument of Lemma 6.5.4, we see that the main step is to take an eigenspace of a certain semisimple operator \( z \), which clearly commutes with \( h \) since they both lie in \( t \). Thus Lemma 6.5.4 becomes, in our case,

\[
[\text{DS}_x(L(\lambda)) : L_{g_x}(\nu)_t] = [\text{DS}_x(L_{g_x \times h^\lambda}(\lambda)) : L_{g_x \times h^\lambda}(\nu)_t].
\]

Now using the statement of Lemma 6.5.5 along with this, we obtain Equation (15).

From this, we may use the same algorithm implicitly described in Section 6.2.3 to finally obtain the isomorphism (5).

7. Appendix

The DS-functor was introduced in [7]; see also [15] for an expanded exposition. We will be using a slight extension of DS-functor which we define below.

7.1. Construction. Let \( g \) be a finite-dimensional Lie superalgebra. Define

\[
g^x_{ss} = \{ x \in g_t | \text{ad}[x, x] \text{ is semisimple} \}.
\]

For a \( g \)-module \( M \) and \( x \in g \) we set \( M^x := \text{Ker}_M x \). Now let \( x \in g^x_{ss} \) and write \( x^2 := \frac{1}{2}[x, x] \). For a \( g \)-module \( M \) on which \( x^2 \) acts semisimply we set

\[
M_x = \text{DS}_x(M) := \frac{\ker(x : M^{x^2})}{\text{Im}(x : M^{x^2})}.
\]

Then \( \text{DS}_x \) is a tensor functor; in particular \( g_x \) will be a Lie superalgebra, and \( M_x \) will have the natural structure of a \( g_x \)-module. Further there are canonical isomorphisms

\[
\text{DS}_x(\Pi(N)) \cong \Pi(\text{DS}_x(N)) \quad \text{and} \quad \text{DS}_x(M) \otimes \text{DS}_x(N) \cong \text{DS}_x(M \otimes N).
\]

In addition we have a canonical isomorphism of \( g_x \)-modules \( \text{DS}_x(N^*) \cong (\text{DS}_x(N))^* \).

Thus \( \text{DS}_x : M \mapsto \text{DS}_x(M) \) is a tensor functor from the category of \( g \)-modules on which \( x^2 \) acts semisimply to the category of \( g_x \)-modules.

7.1.1. We say that \( g_x \) can be identified with a certain subalgebra of \( g \) if

\[
g_x = g_x \ltimes ([x, g] \cap g^x)
\]

7.1.2. Grading on \( \text{DS}_x \). Let \( x \in g^x_{ss} \) with \( x^2 = 0 \), and write \( n(x) \) for the normalizer of \( x \) in \( g \). Then if \( M \) is a \( g \)-module, \( M_x \) will have the natural structure of a \( n(x) \)-module. The action map gives an exact sequence

\[
0 \rightarrow g^x \rightarrow n(x) \rightarrow \mathbb{C}\langle x \rangle,
\]

which defines a short exact sequence whenever there exists \( h \in g \) such that \([h, x] = 1\).
If $\mathfrak{g}$ is a Kac-Moody superalgebra or $\mathfrak{q}_n$, $\mathfrak{sq}_n$, such an element $h$ always exists, and further we may choose it such that under the embedding $\mathfrak{g}_x \subseteq \mathfrak{g}^x$, $h$ commutes with $\mathfrak{g}_x$. In this way we obtain naturally an action of $\mathfrak{g}_x \times \mathbb{C}(h)$ on $M_x$.

The action of $h$ gives rise to a grading on $M_x$ as a $\mathfrak{g}_x$-module according to the eigenvalues of $h$:

$$M_x = \bigoplus_{t \in \mathbb{C}} (M_x)_t.$$  

7.1.3. Hinich Lemma. Each short exact sequence of $\mathfrak{g}$-modules with semisimple action of $x^2$

$$0 \to M_1 \to N \to M_2 \to 0$$

induces a long exact sequence of $\mathfrak{g}_x$-modules

$$0 \to Y \to DS_x(M_1) \to DS_x(N) \to DS_x(M_2) \to \Pi(Y) \to 0,$$

where $Y$ is a some $\mathfrak{g}_x$-module; identifying $M_1$ with its image in $N$ we have

$$Y = (M_1^x \cap [x, N])/(M_1^x \cap [x, M_1]).$$

If $[x, x] = 0$ and we have an element $h$ as in Section 7.1.2, the morphism $DS_x(M_2) \to \Pi Y$ has weight 1 for the action of $h$, while all other maps commute with $h$.

7.1.4. DS and restriction map. Let $\mathcal{K}(\mathfrak{g})$ denote the Grothendieck ring of finite-dimensional $\mathfrak{g}$-modules and let $\mathcal{K}_-(\mathfrak{g})$ denote the the quotient of $\mathcal{K}(\mathfrak{g})$ by the relations $[N] = -[\Pi N]$, where $\Pi$ stands for the parity change functor. For a finite-dimensional $\mathfrak{g}$-module we denote by $[N]$ its image in $\mathcal{K}_-(\mathfrak{g})$. Although $DS_x$ is not an exact functor, by the Hinich lemma $DS_x$ defines ring homomorphisms on reduced Grothendieck rings $\mathcal{K}_-(\mathcal{F}\text{in}(\mathfrak{g})) \to \mathcal{K}_-(\mathcal{F}\text{in}(\mathfrak{g}^x))$ which coincides with the restriction map $[M] \mapsto [\text{Res}_{\mathfrak{g}_x}^\mathfrak{g} M]$, see [15]. The ring $\mathcal{K}_-(\mathcal{F}\text{in}(\mathfrak{g}_x))$ is a subring of $\mathcal{K}_-(\mathcal{F}\text{in}(\mathfrak{g}^x))$ and the image of $\mathcal{K}_-(\mathcal{F}\text{in}(\mathfrak{g})) \to \mathcal{K}_-(\mathcal{F}\text{in}(\mathfrak{g}^x))$ lies in this subring, see [15].

If $\mathfrak{g}_x$ is identified with a subalgebra of $\mathfrak{g}$ we have

$$[DS_x(M)] = [\text{Res}_{\mathfrak{g}_x}^\mathfrak{g} M].$$

7.1.5. Remark. The morphism $[M] \mapsto [DS_xM] = [\text{Res}_{\mathfrak{g}_x}^\mathfrak{g} M]$ is often denoted by $ds_x$ in many papers, including [15]. We chose to avoid this notation to emphasize the simplicity and universality of restriction, in favor of the more limited setting of the DS functor.

7.2. Case of commuting $x, y \in \mathfrak{g}^{ss}_x$. Fix $x, y \in \mathfrak{g}^{ss}_x$ and $h \in \mathfrak{g}$ such that

$$[x, y] = 0, \quad [h, x] = c_x x, \quad [h, y] = c_y y \quad \text{with} \quad c_x, c_y \in \mathbb{C}, \quad c_x \neq c_y.$$

Note that $x \in \mathfrak{g}^y$; we denote by $\mathfrak{f}$ the image of $x$ in $\mathfrak{g}^y$. 
7.2.1. Lemma. Let $N$ be a finite-dimensional $\mathfrak{g}$-module with a diagonal action of $h, x^2,$ and $y^2$. Assume that $(p|q) := \dim DS_{p,q}(N).$ Then $\dim DS_{x+y}(N) = (p-j|q-j)$ for some $j \geq 0$. Moreover, $\dim DS_{x-y} = (p|q)$ if $xN \cap yN = 0$.

Proof. By Lemma 3.1 of [29], $DS_{x+y} N \cong DS_{x+y} N^{x^2,y^2}$, so we may assume that $x^2 = y^2 = 0$. This reduces the statement to the case when $\mathfrak{g}$ is $(0|2)$-dimensional commutative Lie superalgebra and $N$ is a $\mathbb{C}$-graded $\mathfrak{g}$ module where $x, y$ have different degrees. In particular, for $v \in N$ the equality $xv = yv$ implies $xv = 0$.

The indecomposable finite-dimensional modules over the ring $\mathbb{F}[u,v]/(u^2, v^2)$ were classified in [25]. From this classification it follows the indecomposable summands of $N$ are, up to a parity change, from the following list: a 4-dimensional projective modules $M_4$ satisfying $DS_{x}(M_4) = DS_{y}(M_4) = 0$ and the “zigzag” modules $V_{s}^{\pm}$. Each zigzag module has a basis $\{v_i\}_{i=1}^{n}$ with $p(v_{i+1}) = t$; we depict each module by the diagram, where $xv_i = v_{i+1}$ and $yv_i = v_{i-1}$ is depicted as $v_i \rightarrow v_{i+1}$. We have

$$
V_{2n}^+ : \quad v_1 \rightarrow v_2 \leftarrow v_3 \rightarrow v_4 \leftarrow \ldots \rightarrow v_{2n} \\
V_{2n}^- : \quad v_1 \leftarrow v_2 \rightarrow v_3 \leftarrow v_4 \rightarrow \ldots \leftarrow v_{2n} \\
V_{2n-1}^+ : \quad v_1 \rightarrow v_2 \leftarrow v_3 \rightarrow v_4 \leftarrow \ldots \leftarrow v_{2n-1} \\
V_{2n-1}^- : \quad v_1 \leftarrow v_2 \rightarrow v_3 \leftarrow v_4 \rightarrow \ldots \rightarrow v_{2n-1}
$$

(The modules $V_1^+ \cong V_1^-$ are trivial). One sees that

$$
\dim DS_{p,q}(V_{2n}^\pm) = \dim DS_{p,q}(V_{2n+1}^\pm) = \dim DS_{x+y}(V_{2n+1}^\pm) = (1|0), \\
DS_{p}(V_{2n}^\pm) = DS_{p}(V_{2n}^\pm) = 0, \quad DS_{x+y}(V_{2n}^\pm) = 0
$$

and $\dim DS_{p,q}(V_{2n}) = (1|1)$ for $n > 1$. This gives $\dim DS_{x+y}(N) = (p-j|q-j)$ for some $j \geq 0$. If $xN \cap yN = 0$, then $N$ is a direct sum and the modules of the form $V_1^\pm, V_2^\pm, V_3^-$. By above, this gives $\dim DS_{x+y}(N) = DS_{p,q}(N).$ \hfill \Box

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