MORITA EMBEDDINGS FOR DUAL OPERATOR ALGEBRAS AND DUAL OPERATOR SPACES

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ABSTRACT. We define a relation $\subset_\Delta$ for dual operator algebras. We say that $B \subset_\Delta A$ if there exists a projection $p \in A$ such that $B$ and $pAp$ are Morita equivalent in our sense. We show that $\subset_\Delta$ is transitive, and we investigate the following question: If $A \subset_\Delta B$ and $B \subset_\Delta A$, then is it true that $A$ and $B$ are stably isomorphic? We propose an analogous relation $\subset_\Delta$ for dual operator spaces, and we present some properties of $\subset_\Delta$ in this case.

1. Introduction

An operator space $X$ is said to be a dual operator space if $X$ is completely isometrically isomorphic to the operator space dual $Y^*$ of an operator space $Y$. If, in addition, $X$ is an operator algebra, then we call it a dual operator algebra. For example, Von Neumann algebras and nest algebras are dual operator algebras. Blecher, Muhly and Paulsen introduced the notion of the Morita equivalence of non-self-adjoint operator algebras [4]. Subsequently, Blecher and Kashyap developed a parallel theory for dual operator algebras [1], [14]. At the same time, the author of the present article proposed a different notion of Morita equivalence for dual operator algebras, called $\Delta$-equivalence. Two unital dual operator algebras $A$ and $B$ are $\Delta$-equivalent if there exist faithful normal representations $\alpha : A \to \alpha(A)$, $\beta : B \to \beta(B)$ and a ternary ring of operators $M$ (i.e., a space satisfying $MM^*M \subseteq M$) such that $\alpha(A) = [M^*\beta(B)M]^{-w^*}$ and $\beta(B) = [M\alpha(A)M^*]^{-w^*}$ [9]. In this case, we write $A \sim_\Delta B$. An important property is that two algebras are $\Delta$-equivalent if and only if they are stably isomorphic, as was proved by Paulsen and the present author in [12]. Subsequently, Paulsen, Todorov and the present author defined a Morita-type equivalence $\sim_\Delta$ for dual operator spaces [13]. This equivalence also has the property of being equivalent with the notion of a stable isomorphism.

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In this paper, we define a weaker relation between dual operator algebras. We say that the dual operator algebra $B$ ∆-embeds into the dual operator algebra $A$ if there exists a projection $p \in A$ such that $B \sim_\Delta pAp$. In this case, we write $B \subset_\Delta A$. We investigate the relation $\subset_\Delta$ between unital dual operator algebras, and we prove that it is a transitive relation. In the case of von Neumann algebras, it is a partial order relation. This means that it has the additional property that if $A, B$ are von Neumann algebras and $A \subset_\Delta B, B \subset_\Delta A$, then $A \sim_\Delta B$. We present a counterexample to demonstrate that this does not always hold in the case of non-self-adjoint algebras. In Section 2, we also present a characterisation of the relation $\subset_\Delta$ in the terms of reflexive lattices.

In Section 3, we present an analogous theory defining the relation $\subset_\Delta$ for dual operator spaces. In this case, if $X, Y$ are dual operator spaces such that $Y \subset_\Delta X$, then there exist projections $p$ and $q$ such that $pX \subseteq X, Xq \subseteq X$ and $Y \sim_\Delta pXq$. We also define a weaker relation $\subset_{cb_\Delta}$. We say that $Y \subset_{cb_\Delta} X$ if there exist $w^*$-continuous completely bounded isomorphisms $\phi : X \rightarrow \phi(X), \psi : Y \rightarrow \psi(Y)$ such that $\phi(X) \subset_\Delta \psi(Y)$. We present a theorem describing $\subset_{cb_\Delta}$ in the terms of stable isomorphisms (Theorem 3.11), and we investigate the problem of whether $\subset_\Delta$ is a transitive relation for dual operator spaces (Theorem 3.14).

In the following, we briefly describe the notions used in this paper. We refer the reader to the books [3], [6], [7], [15] and [16] for further details. If $V$ is a linear space and $S \subseteq V$, then by $[S]$ we denote the linear span of $S$. If $H, K$ are Hilbert spaces, then we write $B(H, K)$ for the space of bounded operators from $H$ to $K$. We denote $B(H, H)$ as $B(H)$. If $L$ is a subset of $B(H)$, then we write $L'$ for the commutant of $L$, and $L''$ for $(L')'$. If $A$ is an operator algebra, then by $\Delta(A)$ we denote its diagonal $A \cap A^*$. A ternary ring of operators $M$, referred to as a TRO from this point, is a subspace of some $B(H, K)$ satisfying the following:

$$m_1, m_2, m_3 \in M \Rightarrow m_1m_2^*m_3 \in M.$$ 

It is well known that in the case that $M$ is norm closed, it is equal to $[MM^*M]^{-\Vert \cdot \Vert}$ If $X$ is a dual operator space and $I$ is a cardinal, then we write $M_I(X)$ for the set of $I \times I$ matrices whose finite submatrices have uniformly bounded norm. We underline that $M_I(X)$ is also a dual operator space, and it is completely isometrically and $w^*$-homeomorphically isomorphic with $X \otimes B(I^2(I))$. Here, $\otimes$ denotes the normal spatial tensor product. We say that two dual operator spaces $X$ and $Y$ are stably isomorphic if there exists a cardinal $I$ and a $w^*$-continuous completely isometric map from $M_I(X)$ onto $M_I(Y)$. If $\mathcal{L} \subseteq B(H)$ is a lattice, then we write $Alg(\mathcal{L})$ for the algebra
of operators \( x \in B(H) \) satisfying
\[
l^\perp x l = 0, \quad \forall \ l \in L.
\]
If \( A \subseteq B(H) \) is an algebra, then we write \( \text{Lat}(A) \) for the lattice of projections \( l \in B(H) \) satisfying
\[
l^\perp x l = 0, \quad \forall \ x \in A.
\]
A lattice \( L \) is called reflexive if
\[
L = \text{Lat}(\text{Alg}(L)).
\]
A reflexive algebra is an algebra of the form \( \text{Alg}(L) \), for some lattice \( L \). An important example of a class of reflexive lattices is given by nests. A nest \( N \subseteq B(H) \) is a totally ordered set of projections containing the zero and identity operators, which is closed under arbitrary suprema and infima. The corresponding algebra \( \text{Alg}(N) \) is called a nest algebra. If \( A \subseteq B(H) \) is a \( w^* \)-closed algebra and \( I \) is cardinal, then we write \( A^I \) for the algebra of operators \( x \in B(H \otimes l^2(I)) \) satisfying
\[
x((\xi_i)_{i \in I}) = (a(\xi_i))_{i \in I}, \quad \forall \ (\xi_i)_{i \in I} \in H \otimes l^2(I)
\]
for some \( a \in A \).

2. Morita embeddings for dual operator algebras

We consider the following known theorem concerning von Neumann algebras.

**Theorem 2.1.** Let \( A, B \) be von Neumann algebras. Then, the following are equivalent:
(i) There exist \( w^* \)-continuous, one-to-one, *-homomorphisms
\[
\alpha : A \to B(H), \quad \beta : B \to B(K),
\]
where \( H, K \) are Hilbert spaces such that the commutants \( \alpha(A)' \), \( \beta(B)' \) are *-isomorphic.
(ii) The algebras \( A, B \) are weakly Morita equivalent in the sense of Rieffel.
(iii) There exists a cardinal \( I \) and a *-isomorphism from \( M_I(A) \) onto \( M_I(B) \).

**Definition 2.1.** Let \( A, B \) be \( w^* \)-closed algebras acting on the Hilbert spaces \( H \) and \( K \), respectively. We call these weakly TRO-equivalent if there exists a TRO \( M \subseteq B(H, K) \) such that
\[
A = [M^*BM]^{-w^*}, \quad B = [MAM^*]^{-w^*}.
\]
In this case, we write \( A \sim_{\text{TRO}} B \).

The following defines our notion of weak Morita equivalence for dual operator algebras.
Definition 2.2. Let $A, B$ be dual operator algebras. We call these weakly $\Delta$-equivalent if there exist $w^*$-continuous completely isometric homomorphisms $\alpha$ and $\beta$, respectively, such that $\alpha(A) \sim_{TRO} \beta(B)$. In this case, we write $A \sim_{\Delta} B$.

The following theorem is a generalisation of Theorem 2.1 to the setting of unital dual operator algebras:

Theorem 2.2. Let $A, B$ be unital dual operator algebras. Then, the following statements are equivalent:

(i) There exist reflexive lattices $\mathcal{L}_1$ and $\mathcal{L}_2$, $w^*$-continuous completely isometric onto homomorphisms $\alpha : A \rightarrow \text{Alg}(\mathcal{L}_1)$, $\beta : B \rightarrow \text{Alg}(\mathcal{L}_2)$, and a $\ast$-isomorphism $\theta : \Delta(A)' = \mathcal{L}_1'' \rightarrow \Delta(B)' = \mathcal{L}_2''$ such that $\theta(\mathcal{L}_1) = \mathcal{L}_2$.

(ii) The algebras $A, B$ are weakly $\Delta$-equivalent.

(iii) There exists a cardinal $I$ and a $w^*$-continuous completely isometric homomorphism from $M_I(A)$ onto $M_I(B)$.

The previous theorem has been proved in various papers. In fact, if (i) holds, then by Theorem 3.3 in [8] $\text{Alg}(\mathcal{L}_1) \sim_{TRO} \text{Alg}(\mathcal{L}_2)$, and thus $A \sim_{\Delta} B$. Conversely, if (ii) holds, then by Theorems 2.7 and 2.8 in [10], by choosing a $w^*$-continuous completely isometric homomorphism $\alpha : A \rightarrow \alpha(A)$ with reflexive range, there exists a $w^*$-continuous completely isometric homomorphism $\beta : B \rightarrow \beta(B)$, also with a reflexive range, such that $\alpha(A) \sim_{TRO} \beta(B)$. Thus, by Theorem 3.3 in [8], (i) holds. The equivalence of (ii) and (iii) constitutes the main result of [12].

Remark 2.3. In the remainder of this section, if $A$ is a unital dual operator algebra and $p \in A$ is a projection, then $pA$ is also a dual operator algebra with unit $p$. If $A$ is a $w^*$-closed unital algebra acting on the Hilbert space $H$ and $p \in A$ is a projection, then we identify $pA$ with the algebra $pA|_{p(H)} \subseteq B(p(H))$.

2.1. TRO-embeddings for dual operator algebras.

Definition 2.3. Let $A, B$ be unital $w^*$-closed algebras acting on the Hilbert spaces $H$ and $K$, respectively. We say that $B$ weakly TRO-embeds into $A$ if there exists a projection $p \in A$ such that $B \sim_{TRO} pA$. In this case, we write $B \subset_{TRO} A$.

Remark 2.4. The above definition is equivalent to the following statements. Let $A, B$ be unital $w^*$-closed algebras acting on the Hilbert spaces $H$ and $K$, respectively. Then,

(i) The algebra $B$ weakly TRO-embeds into $A$ if and only if there exists a TRO $M \subseteq B(H, K)$ such that $B = [M^*AM]^{-w^*}, \ MBM^* \subseteq A$. 

(ii) The algebra $B$ weakly TRO-embeds into $A$ if and only if there exists a TRO $M \subseteq B(H,K)$ such that $B = [M^*AM]^{-w*}$, $MM^* \subseteq A$.

**Remark 2.5.** If $A$ is a unital $w^*$-closed algebra and $p \in A$ is a projection, then $pAp \subset_{TRO} A$. For the proof, we can take the linear span of the element $p$ as a TRO.

**Proposition 2.6.** Suppose that $A, B, C$ are unital $w^*$-closed algebras acting on the Hilbert spaces $H, K$, and $L$, respectively. If $C \subset_{TRO} B$ and $B \subset_{TRO} A$, then $C \subset_{TRO} A$.

**Proof.** We may assume that there exist projections $p \in \Delta(B), q \in \Delta(A)$ such that

$$C \sim_{TRO} pBp, \quad B \sim_{TRO} qAq.$$  

By Proposition 2.8 in [8], there exists a TRO $M$ such that

$$B = [MqAqM^*]^{-w*}, \quad qAq = [M^*BM]^{-w*},$$

$$\Delta(B) = [MM^*]^{-w*}, q\Delta(A)q = [M^*M]^{-w*}.$$

Define $N = pM$. Then, we have that

$$(pM)(pM)^*(pM) = pMM^*pM \subseteq pMM^*\Delta(B)M \subseteq p\Delta(B)M \subseteq pM.$$  

Thus, $N$ is a TRO. Then, we have that

$$pBp = [pMqAqM^*p]^{-w*} = [NqAqN^*]^{-w*} = [N(N^*NqAqN^*N)N^*]^{-w*},$$

and therefore

$$[N^*pBpN]^{-w*} = [N^*NqAqN^*N]^{-w*}.$$  

Thus,

$$pBp \sim_{TRO} [N^*NqAqN^*N]^{-w*}. $$

Therefore,

$$C \sim_{TRO} [N^*NqAqN^*N]^{-w*}. $$

We may assume that there exists a TRO $L$ such that

$$C = [LN^*NqAqN^*NL^*]^{-w*}, \quad [N^*NqAqN^*N]^{-w*} = [L^*CL]^{-w*}. $$

We make the following observations:

$$[Nq\Delta(A)qN^*]^{-w*} = [pMM^*MM^*p]^{-w*} = [pMM^*p]^{-w*} = [NN^*]^{-w*}. $$

Furthermore, $N^*N = M^*pM \subseteq [M^*M]^{-w*} = q\Delta(A)q$. Thus,

$$L^*L \subseteq \Delta([N^*NqAqN^*N]^{-w*}) \subseteq q\Delta(A)q.$$

Define $D = [LN^*Nq]^{-w*}$. We shall prove that $D$ is a TRO. We have that

$$(LN^*Nq)(qN^*NL^*)(LN^*Nq) = LN^*NqN^*(NL^*LN^*)Nq.$$  

By (2.3), it holds that

$$NL^*LN^* \subseteq Nq\Delta(A)qN^* \subseteq NN^*.$$
Thus,
\[ DD^*D \subseteq [LN^*NqN^*NN^*Nq]^{-w^*}. \]
By (2.2), we have that \( NqN^* \subseteq NN^* \). Thus,
\[ DD^*D \subseteq [LN^*NN^*NN^*Nq]^{-w^*} = D. \]
Thus, \( D \) is a TRO. By (2.1), we have that
\[ C = [DAD^*]^{-w^*}. \]
Furthermore,
\[ D^*D = qN^*NL^*LN^*Nq. \]
By (2.3), we have that
\[ D^*D \subseteq qN^*NN^*NN^*Nq \subseteq \Delta(A)qN^*Nq. \]
In addition, by (2.2) we have that
\[ D^*D \subseteq qN^*NN^*Nq \subseteq q\Delta(A)q \subseteq A. \]
Thus, by Remark 2.4 (ii), we have that \( C \subset_{TRO} A. \) \( \square \)

**Remark 2.7.** In light of the above proposition, one could expect that the relation \( \subset_{TRO} \) is a partial order relation in the class of unital \( w^* \)-closed operator algebras, if we identify those algebras that are TRO-equivalent. This means that \( \subset_{TRO} \) has the additional property that
\[ A \subset_{TRO} B, \quad B \subset_{TRO} A \Rightarrow A \sim_{TRO} B. \]
This is true in the case of von Neumann algebras, as we will show in Section 1.3. However, it fails in the case of non-self-adjoint algebras, as we prove in Section 1.4.

The following Lemma will be useful.

**Lemma 2.8.** Let \( B \) be a \( w^* \)-closed unital operator algebra acting on the Hilbert space \( H \), and let \( q \in B \) be a projection. If \( p \) is the projection onto \( \Delta(B)(q(H)) \), then \( p \) is a central projection for the algebra \( \Delta(B) \), and \( qBq \sim_{TRO} pBp \).

**Proof.** Clearly, \( p \) is a central projection for \( \Delta(B) \). We consider the TRO
\[ M = \Delta(B)q \subseteq B(q(H), p(H)). \]
We have that
\[ M(q(H)) = p(H), \quad M^*(p(H)) = q(H), \]
and
\[ M^*pBpM \subseteq qBq, \quad MqBqM^* \subseteq pBp. \]
Then, Proposition 2.1 in [8] implies that
\[ [M^*pBpM]^{-w^*} = qBq, \quad [MqBqM^*]^{-w^*} = pBp. \]
2.2. \(\Delta\)-embeddings for dual operator algebras.

**Definition 2.4.** Let \(A, B\) be dual operator algebras. We say that \(B\) weakly \(\Delta\)-embeds into \(A\) if there exist \(w^*\)-continuous completely isometric homomorphisms \(\alpha : A \to \alpha(A)\), \(\beta : B \to \beta(B)\) such that \(\beta(B) \subset_{TRO} \alpha(A)\). In this case, we write \(B \subset_\Delta A\).

The following theorem is a generalisation of Theorem 2.2.

**Theorem 2.9.** Let \(A, B\) be unital dual operator algebras. Then, the following are equivalent:

(i) There exist reflexive lattices \(\mathcal{L}_1, \mathcal{L}_2\) acting on the Hilbert spaces \(H\) and \(K\), respectively, \(w^*\)-continuous completely isometric onto homomorphisms \(\alpha : A \to \text{Alg}(\mathcal{L}_1), \beta : B \to \text{Alg}(\mathcal{L}_2)\),

and an onto \(w^*\)-continuous \(\ast\)-homomorphism

\[\theta : \Delta(A)' = \mathcal{L}_1'' \to \Delta(B)' = \mathcal{L}_2''\]

such that \(\theta(\mathcal{L}_1) = \mathcal{L}_2\).

(ii) \(B \subset_\Delta A\).

(iii) There exists a cardinal \(I\), a projection \(q \in A\), and a \(w^*\)-continuous completely isometric homomorphism from \(M_I(B)\) onto \(M_I(qAq)\).

**Proof.** The equivalence of (ii) and (iii) is a consequence of Definition 2.4 and Theorem 2.2.

\[(i) \Rightarrow (ii)\]

It suffices to prove that

\(\text{Alg}(\mathcal{L}_2) \subset_{TRO} \text{Alg}\mathcal{L}_1\).

Define the lattice

\[\mathcal{L} = \left\{ \begin{pmatrix} l & 0 \\ 0 & \theta(l) \end{pmatrix} : \ l \in \mathcal{L}_1 \right\}\]

and the spaces

\[U = \{ x : l^* x \theta(l) = 0 \ \forall \ l \in \mathcal{L}_1 \}, \ V = \{ y : \theta(l)^* y l = 0 \ \forall \ l \in \mathcal{L}_1 \}\]

We can easily prove that

\[\text{Alg}(\mathcal{L}) = \begin{pmatrix} \text{Alg}(\mathcal{L}_1) & U \\ V & \text{Alg}(\mathcal{L}_2) \end{pmatrix}\]

Because the map

\[\rho : \mathcal{L}_1'' \to \mathcal{L}'', \ a \to \begin{pmatrix} a & 0 \\ 0 & \theta(a) \end{pmatrix}\]
is a \(*\)-isomorphism such that \(\theta(\mathcal{L}_1) = \mathcal{L}\), from Theorem 3.3 in [8] we have that

\[
\text{Alg}(\mathcal{L}_1) \sim_{\text{TRO}} \text{Alg}(\mathcal{L}).
\]

Define the TRO

\[
M = (0 \ C).
\]

Now, observe that

\[
M \text{Alg}(\mathcal{L}) M^* = \text{Alg}(\mathcal{L}_2)
\]

and

\[
M^* M \subseteq \text{Alg}(\mathcal{L}).
\]

Thus,

\[
\text{Alg}(\mathcal{L}_2) \subset_{\text{TRO}} \text{Alg}.\mathcal{L}.
\]

Then, Proposition 2.6 implies that

\[
\text{Alg}(\mathcal{L}_2) \subset_{\text{TRO}} \text{Alg}(\mathcal{L}_1).
\]

\((iii) \Rightarrow (i)\)

Suppose that \(M_1(B)\) and \(M(qAq)\) are completely isometrically and \(w^*\)-homeomorphically isomorphic. Then, \(B \sim_{\Delta} qAq\). Every unital dual operator algebra has a \(w^*\)-completely isometric representation whose image is reflexive. Thus, we may assume that \(\alpha : A \rightarrow B(H)\) is a \(w^*\)-continuous completely isometric homomorphism such that \(\alpha(A) = \text{Alg}(\mathcal{L}_1)\) for a reflexive lattice \(\mathcal{L}_1\).

If \(p = \alpha(q)\), then

\[
\alpha(qAq) = \text{Alg}(\mathcal{L}_1|_p(H)).
\]

Theorem 2.7 in [10] implies that there exists a reflexive lattice \(\mathcal{L}_2\) and a \(w^*\)-continuous completely isometric homomorphism \(\beta : B \rightarrow \text{Alg}(\mathcal{L}_2)\) such that

\[
\text{Alg}(\mathcal{L}_2) \sim_{\text{TRO}} \text{Alg}(\mathcal{L}_1|_p(H)).
\]

Thus, by Theorem 3.3 in [8] there exists a \(*\)-isomorphism \(\rho : (\mathcal{L}_1|_p(H))^\prime\prime \rightarrow \mathcal{L}_2^\prime\prime\) such that \(\rho(\mathcal{L}_1|_p(H)) = \mathcal{L}_2\). If \(\tau : \mathcal{L}_1^\prime\prime \rightarrow \mathcal{L}_2^\prime\prime|_{p(H)} : x \rightarrow x|_{p(H)}\), then we write \(\theta = \rho \circ \tau\). This is the required map. \(\square\)

**Theorem 2.10.** Let \(A, B, C\) be unital dual operator algebras such that

\[
C \subset_{\Delta} B, \quad B \subset_{\Delta} A.
\]

Then, \(C \subset_{\Delta} A\).

**Proof.** We may assume that there exist projections \(p \in B, q \in A\) such that

\[
C \sim_{\text{TRO}} pBp, \quad \beta(B) \sim_{\text{TRO}} qAq,
\]

where \(\beta : B \rightarrow \beta(B)\) is a \(w^*\)-continuous completely isometric homomorphism. From Theorem 2.7 in [10], we know that for the representation
\( \beta|_{pBp} : pBp \to \beta(p)\beta(B)\beta(p) \) there exists a \( w^* \)-continuous completely isometric homomorphism \( \gamma : C \to \gamma(C) \) such that

\[
\gamma(C) \sim \text{TRO} \beta(p)\beta(B)\beta(p).
\]

By Proposition 2.6 we have that

\[
\beta(p)\beta(B)\beta(p) \subset \text{TRO} qAq.
\]

Because \( qAq \subset \text{TRO} A \), we have that \( \gamma(C) \subset \text{TRO} A \), and thus \( C \subset \Delta A \). □

**Remark 2.11.** In view of Theorem 2.10, one should expect that weak \( \Delta \)-embedding is a partial order relation in the class of unital dual operator algebras if we identify those unital dual operator algebras that are weakly \( \Delta \)-equivalent. Thus, one should expect that \( A \subset \Delta B \), \( B \subset \Delta A \Rightarrow A \sim \Delta B \). In Section 1.4 we shall see that this is not true. However, in the case of von Neumann algebras, this is indeed true. For further details, see Section 2.3 below.

**Example 2.12.** Let \( A = \text{Alg}(\mathcal{N}_1) \), \( B = \text{Alg}(\mathcal{N}_2) \), where \( \mathcal{N}_1 \) is a continuous nest, and \( \mathcal{N}_2 \) is a nest with at least one atom. We shall prove that it is impossible that \( B \subset \Delta A \).

**Proof.** Suppose on the contrary that \( B \subset \Delta A \). Thus, there exists a projection \( p \in \Delta(A) \) such that \( B \sim \Delta pA|_p \). Because \( B \) and \( pA|_p \) are nest algebras, it follows from Theorem 3.2 in [10] that \( B \sim \text{TRO} pA|_p \). Thus, by Theorem 3.3 in [8] there exists a homeomorphism \( \theta : \mathcal{N}_2 \to \mathcal{N}_1|_p \). This is impossible, because \( \mathcal{N}_2 \) contains an atom, and \( \mathcal{N}_1|_p \) is a continuous nest. □

### 2.3. The case of von Neumann algebras.

**Lemma 2.13.** Let \( A \) be a von Neumann algebra, and let \( p, q \) be central projections of \( A \) such that \( p \leq q \) and \( A \sim \text{TRO} A p \). Then, \( A \sim \text{TRO} A q \).

**Proof.** By Theorem 3.3 in [8], there exists a \(*\)-isomorphism \( \theta : A' \to A'p \). We need to prove that there exists a \(*\)-isomorphism \( \rho : A' \to A'q \). Suppose that

\[
e_0 = Id_A, e_1 = q, e_2 = p, e_n = \theta(e_{n-2}), n = 2, 3, ...
\]

Clearly \((e_n)_n\) is a decreasing sequence of central projections. Observe that

\[
e_0 = \left( \sum_{n=0}^{\infty} \bigoplus (e_{2n} - e_{2n+1}) \bigoplus (e_{2n+1} - e_{2n+2}) \right) \bigoplus \wedge_n e_n.
\]

Thus, the map \( \rho : A' \to A'q \) sending

\[
a = \left( \sum_{n=0}^{\infty} \bigoplus a(e_{2n} - e_{2n+1}) \bigoplus a(e_{2n+1} - e_{2n+2}) \right) \bigoplus (a \wedge_n e_n)
\]
To

\[
\rho(a) = \left( \sum_{n=0}^{\infty} \oplus \theta(a)(e_{2n+2} - e_{2n+3}) \oplus a(e_{2n+1} - e_{2n+2}) \right) \oplus (a \land e_n)
\]

is a \(*\)-isomorphism.

The above Lemma is based on the fact if \(p, q\) are central projections of the von Neumann algebra \(A\) such that \(p \leq q\) and \(A \cong Ap\), then \(A \cong Aq\), where \(\cong\) is the \(*\)-isomorphism. We acknowledge that this was known to the authors of [5] (see the proof of Lemma 6.2.3). In this Lemma, an alternative proof to ours was provided.

**Theorem 2.14.** The relation \(\subset_{\text{TRO}}\) is a partial order relation for von Neumann algebras, if we identify those von Neumann algebras that are TRO-equivalent.

**Proof.** Let \(A, B\) be von Neumann algebras. It suffices to prove the implication that

\[
A \subset_{\text{TRO}} B,\quad B \subset_{\text{TRO}} A \Rightarrow A \sim_{\text{TRO}} B.
\]

Let \(q_0 \in B, p_0 \in A\) be projections such that

\[
A \sim_{\text{TRO}} q_0 B q_0,\quad B \sim_{\text{TRO}} p_0 A p_0.
\]

By Lemma 2.8 there exist central projections \(q \in B, p \in A\) such that

\[
A \sim_{\text{TRO}} B q,\quad B \sim_{\text{TRO}} A p.
\]

Thus, there exist \(*\)-isomorphisms

\[
\theta : A' \to B' q,\quad \rho : B' \to A' p.
\]

We can easily see that there exists a central projection \(\hat{p} \in A\) such that \(\hat{p} \leq p\) and

\[
\rho(B' q) = A' \hat{p}.
\]

Therefore, we obtain a \(*\)-isomorphism from \(A'\) onto \(A' \hat{p}\). Because \(\hat{p} \leq p\) and \(p, \hat{p}\) are central, Lemma 2.13 implies that there exists a \(*\)-isomorphism from \(A'\) onto \(A' p\). Thus, \(A \sim_{\text{TRO}} A p\), which implies that \(A \sim_{\text{TRO}} B\). \(\square\)

**Theorem 2.15.** Let \(A, B\) be von Neumann algebras. Then, the following are equivalent:

(i) There exist \(*\)-isomorphisms \(\alpha : A \to \alpha(A), \beta : B \to \beta(B)\) and a \(w^*\)-continuous onto \(*\)-homomorphism \(\theta : \alpha(A)' \to \beta(B)'\).

(ii) \(B \subset_{\Delta} A\).

(iii) There exists a cardinal \(I\) and a \(w^*\)-continuous onto \(*\)-homomorphism \(\rho : M_I(A) \to M_I(B)\).
Proof. The equivalence of (i) and (ii) is a consequence of Theorem 2.9

\[(ii) \Rightarrow (iii)\]

Suppose that \(B \subset \Delta A\). By Theorem 2.9 we may assume that there exists a \(w^*\)-continuous onto \(*\)-homomorphism \(\theta : A' \to B'\). Suppose that \(A'|p^\perp = \text{Ker} \theta\) for a projection \(p\) in the centre of \(A\). We also assume that \(A \subseteq B(H)\). Then, the map

\[A'|p(H) \to B' : a|_p(H) \to \theta(a)\]

is a \(*\)-isomorphism. Because

\[(A|_p(H))' = A'|_p(H)\],

Theorem 2.1 implies that there exists a cardinal \(I\) and a \(w^*\)-continuous onto \(*\)-isomorphism

\[M_I(pAp) \to M_I(B)\].

\[(iii) \Rightarrow (ii)\]

Suppose that \(\rho : M_I(A) \to M_I(B)\) is a \(w^*\)-continuous onto \(*\)-homomorphism. Let \(p\) be a projection in the centre of \(M_I(A)\) such that

\[M_I(A)p^\perp = \text{Ker} \rho\].

Because \(Z(M_I(A)) = Z(A)'\), where \(Z(A)\) (resp. \(Z(M_I(A))\)) is the centre of \(A\) (resp. \(M_I(A)\)), we may assume that \(p = q'\) for \(q \in Z(A)\). Thus, the map

\[M_I(Aq) \to M_I(B) : (a_{i,j}q) \to \rho((a_{i,j}))\]

is a \(*\)-isomorphism. Then, Theorem 2.9 implies that \(B \subset \Delta A\). \(\square\)

**Theorem 2.16.** The weak \(\Delta\)-embedding is a partial order relation in the class of von Neumann algebras, if we identify those von Neumann algebras that are weakly Morita equivalent in the sense of Rieffel.

**Proof.** Claim: Let \(A\) be a von Neumann algebra, and let \(r \in A\) be a projection such that \(A \sim_{\Delta} rAr\). If \(q\) is a projection in \(A\) such that \(r \leq q\), then it also holds that \(A \sim_{\Delta} qAq\).

**Proof of the claim:** There exists a cardinal \(I\) such that the algebras \(M_I(A)\) and \(M_I(rAr)\) are \(*\)-isomorphic. We suppose that \(M = M_I(A)\), \(N = M_I(qAq)\). Then, we have that \(M \cong M_I(rAr) = r'I Nr'\) and \(N \cong q'I Mq'\). By Lemma 6.2.3 in \([5]\), the von Neumann algebras \(M\) and \(N\) are stably isomorphic. Thus, \(M \sim_{\Delta} N\). However,

\[A \sim_{\text{TRO}} M, \quad qAq \sim_{\text{TRO}} N\].

Therefore, \(A \sim_{\Delta} qAq\), and the proof of the claim is complete.

To prove the theorem, it suffices to prove that if \(A, B\) are von Neumann algebras such that \(A \subset \Delta B\), \(B \subset \Delta A\), then \(A \sim_{\Delta} B\). We may assume that
there exist projections $p \in B, q \in A$ and $w^*$-continuous completely isometric homomorphisms $\alpha : A \to \alpha(A), \beta : B \to \beta(B)$, such that
\[
\alpha(A) \sim_{TRO} pBp, \quad \beta(B) \sim_{TRO} qAq.
\]
For the representation $\beta|_{pBp} : pBp \to \beta(p)\beta(B)\beta(p)$, there exists a $w^*$-continuous one-to-one $*$-homomorphism $\gamma : \alpha(A) \to \gamma(\alpha(A))$ such that
\[
\gamma(\alpha(A)) \sim_{TRO} \beta(p)\beta(B)\beta(p).
\]
By Proposition 2.6, we have that $\beta(p)\beta(B)\beta(p) \subset_{TRO} qAq$.

Therefore, there exists a projection $r \leq q$ such that
\[
\gamma(\alpha(A)) \sim_{TRO} rAr \Rightarrow \gamma(\alpha(A)) \sim rAr.
\]
The claim implies that $A \sim qAq$. However, $qAq \sim B$. Thus $A \sim B$. Thus, the proof is complete. \qed

**Corollary 2.17.** Let $A, B$ be von Neumann algebras, $I, J$ be cardinals, and
\[
\theta : M_I(A) \to M_I(B), \quad \rho : M_J(B) \to M_J(A)
\]
be onto $w^*$-continuous homomorphisms. Then, $A \sim B$.

**Proof.** By Theorem 2.15, $A \subset_{_\Delta} B$ and $B \subset_{_\Delta} A$. The conclusion then follows from the above theorem. \qed

**Corollary 2.18.** Let $A, B$ be unital dual operator algebras such that $A \subset_{_\Delta} B$ and $B \subset_{_\Delta} A$. Then, $\Delta(A) \sim_{_\Delta} \Delta(B)$.

**Proof.** We can easily see that $\Delta(A) \subset_{_\Delta} \Delta(B)$ and $\Delta(B) \subset_{_\Delta} \Delta(A)$. Now, we can apply the above theorem. \qed

**Example 2.19.** Let $A$ be a factor, and $B$ be a unital dual operator algebra such that $B \subset_{_\Delta} A$. Then, $B$ is a von Neumann algebra, and $B \sim_{_\Delta} A$.

**Proof.** There exist a $*$-isomorphism $\alpha : A \to \alpha(A)$, a $w^*$-continuous completely isometric homomorphism $\beta : B \to \beta(B)$, and a TRO $M$ such that if $p$ is the projection onto $[MM^*]^{-w^*}$, then
\[
\beta(B) = [M^*\alpha(A)M]^{-w^*}, \quad p\alpha(A)p = [M\beta(B)M^*]^{-w^*}.
\]
Define $N = [\alpha(A)pM]^{-w^*}$. Because
\[
MM^*p\alpha(A) \subseteq p\alpha(A) \subseteq \alpha(A),
\]
it follows that
\[
pMM^*p \subseteq \alpha(A) \Rightarrow [\alpha(A)pMM^*p\alpha(A)]^{-w^*} = [NN^*]^{-w^*}
\]
is an ideal of \( \alpha(A) \). However, \( \alpha(A) \) is a factor, and thus \( \alpha(A) = [NN^*]^{-w^*} \).

On the other hand,

\[
[N^*N]^{-w^*} = [M^*p\alpha(A)\alpha(A)pM]^{-w^*} = [M^*\alpha(A)M]^{-w^*} = \beta(B).
\]

Thus, \( A \) and \( B \) are weakly Morita equivalent in the sense of Rieffel. However, in the case of von Neumann algebras, Rieffel’s Morita equivalence is the same as \( \Delta \)-equivalence. \( \square \)

2.4. A counterexample in non-self-adjoint operator algebras. Despite the situation for von Neumann algebras, we shall prove that if \( A, B \) are unital non-self-adjoint dual operator algebras, it does not always hold that the implication

\[
(2.4) \quad A \subset_{\Delta} B, \quad B \subset_{\Delta} A \Rightarrow A \sim_{\Delta} B.
\]

By Theorem 3.12 in [10], if \( A, B \) are nest algebras, then

\[
A \sim_{\Delta} B \iff A \sim_{TRO} B.
\]

Because for every nest algebra \( B \) and every projection \( p \in B \) the algebra \( pBp \) is a nest algebra, we can conclude that

\[
A \subset_{\Delta} B \iff A \subset_{TRO} B.
\]

Thus, in order to prove that (2.4) does not hold, it suffices to find nest algebras \( A \) and \( B \) such that \( A \subset_{TRO} B, \quad B \subset_{TRO} A \) and \( A \) is not TRO-equivalent to \( B \). Let \( m \) be the Lebesgue measure on the Borel sets of the interval \([0,1]\].

Suppose that \( \mathbb{Q} \) is the set of rationals, and \( Q^+ \) (resp. \( Q^- \)) is the projection onto \( l^2(\mathbb{Q} \cap [0, t]) \) (resp. \( l^2(\mathbb{Q} \cap [0, t]) \)). Furthermore, let \( N_t \) be the projection onto \( L^2([0, t], m) \) for \( 0 \leq t \leq 1 \). We define the nest

\[
\mathcal{N} = \{Q_t^+ \oplus N_t, Q_t^- \oplus N_t, \quad 0 \leq t \leq 1 \}.
\]

By \( A = Alg(\mathcal{N}) \), we denote the corresponding nest algebra acting on the Hilbert space

\[
H = l^2(\mathbb{Q} \cap [0,1]) \oplus L^2([0,1], m).
\]

The above nest appeared in Example 7.18 in [6]. Suppose that \( f(t) = \frac{1}{2}, \quad t \in [0,1] \), and define

\[
\mathcal{M} = \{Q_{f(t)}^+ \oplus N_{f(t)}, Q_{f(t)}^- \oplus N_{f(t)}, \quad 0 \leq t \leq 1 \}.
\]

We can define a unitary

\[
u_2 : L^2([0, 1]) \to L^2([0, \frac{1}{2}])
\]

such that \( u_2(\chi_\Omega) = \sqrt{2} \chi_{f(\Omega)} \) where \( \chi_\Omega \) is the characteristic function of the Borel set \( \Omega \). This unitary maps \( N_t \) onto \( N_{f(t)} \) in the sense that

\[
u_2 N_t u_2^* = N_{f(t)}, \quad 0 \leq t \leq 1.
\]
Furthermore, the map
\[\{Q^+_t, Q^-_t : 0 \leq t \leq 1\} \rightarrow \{Q^+_{f(t)}, Q^-_{f(t)} : 0 \leq t \leq 1\}\]
sending \(Q^+_t\) onto \(Q^+_{f(t)}\) for \(j = +, -\) is a nest isomorphism. Because these nests
are multiplicity free (they generate a maximal abelian self-adjoint algebra, referred to as an MASA from this point) and totally atomic, the above map extends as a *-isomorphism between the corresponding MASAs. Thus, there
exists a unitary
\[u_1 : l^2(\mathbb{Q} \cap [0, 1]) \rightarrow l^2(\mathbb{Q} \cap [0, 1/2])\]
such that
\[u_2Q^+_tu_2^* = Q^+_{f(t)}, \quad u_2Q^-tu_2^* = Q^-_{f(t)}, 0 \leq t \leq 1.\]
Therefore, the unitary \(u = u_1 \oplus u_2\) implies a unitary equivalence between \(N\) and \(M\).

Let \(s\) be the projection
\[s : l^2(\mathbb{Q} \cap [0, 1]) \rightarrow l^2(\mathbb{Q} \cap [0, 1/2])\]
and \(r\) be the projection
\[r : L^2([0, 1], m) \rightarrow L^2([0, 1/2], m).\]
If \(p = s \oplus r\), then \(p \in A\) and \(pAp = \text{Alg}(M)\). By the above arguments, \(A\) and \(pAp\) are unitarily equivalent, and thus they are TRO-equivalent. Suppose that \(q_0\) is the projection
\[q_0 = \begin{pmatrix} 0 & 0 \\ 0 & I_{L^2([0,1])} - r \end{pmatrix} \in A\]
and \(q = p + q_0\). Because \(p \leq q \leq Id_A\), we have that
\[pAp \subset_{TRO} qAq \subset_{TRO} A.\]
However, \(A \sim_{TRO} pAp\). This implies that \(A \subset_{TRO} qAq\). Thus, if (2.4) holds, then we should have that
\[A \sim_{TRO} qAq.\]
Suppose that \(L\) is the nest \(\text{Lat}(qAq)\). By Theorem 3.3 in [8], there exists a *

isomorphism \(\theta : \Delta(A)' \rightarrow (\Delta(A)|_{q(H)})'\)
such that \(\theta(N) = L\). However, the algebras \(\Delta(A), \Delta(A)|_{q(H)}\) are MASAs. Therefore, there exists a unitary \(w : q(H) \rightarrow H\) such that
\[\theta(x) = w^*xw, \quad \forall x \in \Delta(A) = \Delta(A)'.\]
We have that
\[A = wqAw^*.\]
We can easily see that $L = L_1 \cup L_2$, where

$$L_1 = \{ Q_i^+ \oplus N_i, 0 \leq t \leq \frac{1}{2} \}$$

and

$$L_2 = \{ Q_i^+ \oplus N_i, \frac{1}{2} \leq t \leq 1 \}.$$ 

Observe that $L_1 \leq L_0 \leq L_2$ for all $L_i \in L_i$, $i = 1, 2$ where $L_0 = Q_{t_0}^+ \oplus N_{t_0}$. If

$$M_0 = wL_0w^*, \ N_1 = wL_1w^*, \ N_2 = wL_2w^*,$$

then

$$M_1 \leq M_0 \leq M_2$$

for all $M_i \in N_i$, $i = 1, 2$. Suppose that $M_0 = Q_{t_0}^+ \oplus N_{t_0}$. Then,

$$N_2 = \{ Q_i^+ \oplus N_i, Q_i^- \oplus N_i, t \leq t \leq 1 \}.$$ 

If

$$\hat{L}_2 = \{ (Q_i^+ \oplus N_i) - L_0 : \frac{1}{2} \leq t \leq 1 \},$$

we can consider $\hat{L}_2$ to be a nest acting on $L^2([\frac{1}{2}, 1], m)$. Furthermore, if

$$\hat{N}_2 = \{ (Q_i^+ \oplus N_i) - M_0, (Q_i^- \oplus N_i) - M_0, t \leq t \leq 1 \},$$

then $\hat{L}_2$ and $\hat{N}_2$ are isomorphic nests. However, this is impossible, because $\hat{L}_2$ is a continuous nest and $\hat{N}_2$ is a nest with atoms. This contradiction shows that $A$ and $qAq$ are not TRO-equivalent.

**Remark 2.20.** Let $A$ and $q$ be as above. As we have seen, $qAq \subset A, A \subset qAq$ but $A$ and $qAq$ are not $\Delta$-equivalent. We can prove further that they are not Morita equivalent even in the sense of Blecher and Kashyap [11]. If they were, then by [11] $N$ and $L$ would be isomorphic as nests. However, we can see that this is impossible by applying the same arguments as above.

### 3. Morita Embeddings for Dual Operator Spaces

Definition 2.1 can be adapted to the setting of dual operator spaces as follows.

**Definition 3.1.** Let $H_1, H_2, K_1, K_2$ be Hilbert spaces, and let

$$X \subseteq B(H_1, H_2), Y \subseteq B(K_1, K_2)$$

be $w^*$-closed spaces. We call these weakly TRO-equivalent if there exist TROs $M_i \subseteq B(H_i, K_i), i = 1, 2$ such that

$$X = [M_2^*YM_1]^{-w^*}, \ Y = [M_2XM_1^*]^{-w^*}.$$ 

In this case, we write $X \sim_{TRO} Y$. 

Remark 3.1. If $W_1, W_2$ are Hilbert spaces and $Z$ is a subspace of $B(W_1, W_2)$, then we call it nondegenerate if $Z(W_1) = W_2$, $Z^*(W_2) = W_1$. If $H_1, H_2, K_1, K_2, X, Y$ are as in the above definition, $p_2$ (resp. $q_2$) is the projection onto $X(H_1)$ (resp. $Y(K_1)$), and $p_1$ (resp. $q_1$) is the projection onto $X^*(H_2)$ (resp. $Y^*(K_2)$), then the spaces $p_2X|_{p_1(H_1)}$, $q_2Y|_{q_1(K_1)}$ are nondegenerate, and also weakly TRO-equivalent. This can be concluded from Proposition 2.2 in [13].

The following defines our notion of weak Morita equivalence for dual operator spaces.

Definition 3.2. [13] Let $X, Y$ be dual operator spaces. We call these weakly $\Delta$-equivalent if there exist $w^*$-continuous completely isometric maps $\phi, \psi$, respectively, such that $\phi(X) \sim_{TRO} \psi(Y)$. In this case, we write $X \sim_{\Delta} Y$.

The following theorem constitutes the main result of [13].

Theorem 3.2. Let $X, Y$ be dual operator spaces. Then, the following are equivalent:

(i) $X \sim_{\Delta} Y$.

(ii) There exists a cardinal $I$ and a $w^*$-continuous completely isometric map from $M_I(X)$ onto $M_I(Y)$.

Remark 3.3. Throughout Section 3, we shall employ the following notation. If $H_1, H_2$ are Hilbert spaces, $X \subseteq B(H_1, H_2)$ is a $w^*$-closed subspace, and $q \in B(H_1)$, $p \in B(H_2)$ are projections such that $pX \subseteq X$, $Xq \subseteq X$, then by $pXq$ we denote the space $\{pxq : x \in X\} \subseteq B(H_1, H_2)$. This space is $w^*$-closed and completely isometrically and $w^*$-homeomorphically isomorphic with the space $pX|_{q(H_1)} \subseteq B(q(H_1), p(H_2))$.

3.1. TRO-embeddings for dual operator spaces.

Definition 3.3. Let $H_1, H_2, K_1, K_2$ be Hilbert spaces, and let $X \subseteq B(H_1, H_2)$, $Y \subseteq B(K_1, K_2)$ be $w^*$-closed spaces. We say that $Y$ weakly TRO embeds into $X$ if there exist TROs $M_1 \subseteq B(H_1, K_1)$ and $M_2 \subseteq B(H_2, K_2)$ such that $Y = [M_2XM_1^*]^{-w^*}$, $M_2YM_1 \subseteq X$ and $M_2^*M_2X \subseteq X$, $XM_1^*M_1 \subseteq X$. In this case, we write $Y \subset_{TRO} X$.

Remark 3.4. We can easily see that if $Y \subset_{TRO} X$, then there exist projections $p, q$ such that $pX \subseteq X$, $Xq \subseteq X$ and $Y \sim_{TRO} pXq$.

Examples 3.5. (i) If $X \sim_{TRO} Y$, then clearly $X \subset_{TRO} Y$ and $Y \subset_{TRO} X$.

(ii) If $K_i, W_i, i = 1, 2$ are Hilbert spaces, $Y \subseteq B(K_1, K_2)$, $Z \subseteq B(W_1, W_2)$ are $w^*$-closed spaces, and

$$X = Y \oplus Z \subseteq B(K_1 \oplus W_1, K_2 \oplus W_2),$$
then $Y \subset_{TRO} X$. For the proof, we apply the TROs $M_1 = (CI_{K_1}, 0)$, $M_2 = (CI_{K_2}, 0)$.

(iii) If $X \subseteq B(H_1, H_2)$ is a $w^*$-closed operator space and $p \in B(H_2), q \in B(H_1)$ are projections such that $pX \subseteq X$, $Xq \subseteq X$, then $pXq \subset_{TRO} X$.

(iv) A generalisation of $W^*$-modules over von Neumann algebras is given by the projectively $w^*$-rigged modules over unital dual operator algebras. See [2] for more details. Given a unital dual operator algebra $A$, a projectively $w^*$-rigged module over $A$ is a dual operator space $Z$ that is completely isometrically and $w^*$-homeomorphically isomorphic to a space $Y = [MA] - w^*$, where $M$ is a TRO satisfying $M^*M \subseteq A$. Observe that $Y = [MAC] - w^*$, $M^*CY \subseteq A$ and $M^*MA \subseteq A$. Thus, $Y \subset_{TRO} A$. Therefore, for every projectively $w^*$-rigged module $Z$ over a unital dual operator algebra $A$, we have that $Z \subset_{\Delta} A$. Here, $\subset_{\Delta}$ is the relation defined in Definition 3.4 below.

**Proposition 3.6.** Let $X, Y, Z$ be $w^*$-closed operator spaces. If $Z \subset_{TRO} Y$, $Y \subset_{TRO} X$, then there exist projections $p, q$ such that $pX \subseteq X$, $Xq \subseteq X$ and $Z \subset_{TRO} pXq$.

**Proof.** There exist projections $p, q, r, s$ such that 

$$pX \subseteq X, \quad Xq \subseteq X, \quad rY \subseteq Y, \quad Ys \subseteq Y$$

and TROs $M_i, N_i, i = 1, 2$ such that 

$$Y = [M_2pXqM_1^*] - w^*, \quad pXq = [M_2^*YM_1] - w^*,$$

$$Z = [N_2rYSN_1^*] - w^*, \quad rYS = [N_2^*ZN_1] - w^*.$$ 

We may assume that 

$$M_2p = M_2, \quad M_1q = M_1, \quad N_2r = N_2, \quad N_1s = N_1.$$ 

Suppose that $D_i$ is the $W^*$-algebra generated by the set 

$$\{M_iM_i^*\} \cup \{N_i^*N_i\}, i = 1, 2.$$ 

Define 

$$L_i = [N_iD_iM_i] - w^*, i = 1, 2.$$ 

Because $M_1M_1^* \subseteq D_1$, $N_1^*N_1 \subseteq D_1$, it follows that 

$$N_1D_1M_1M_1^*D_1N_1^*N_1D_1M_1 \subseteq N_1D_1M_1,$$

and thus 

$$L_1L_1^*L_1 \subseteq L_1.$$ 

Therefore, $L_1$, and similarly $L_2$, are TROs. Now, we have that 

$$[L_2pXqL_1^*] - w^* = [N_2D_2M_2pXqM_1^*D_1N_1^*] - w^* = [N_2D_2YD_1N_1^*] - w^*.$$
Because

\[ [M_2 M_2^* Y]^{\sim - w^*} = [Y M_1 M_1^* Y]^{\sim - w^*}, \]

we have that \( D_2 Y = Y = Y D_1 \). Thus,

\[ (3.1) \quad [L_2 p X q L_1^*]^{\sim - w^*} = [N_2 Y N_1^* Y]^{\sim - w^*} = [N_2 r Y s N_1^* Y]^{\sim - w^*} = Z. \]

Furthermore,

\[ L_2^* Z L_1 \subseteq [M_2^* D_2 N_2^* Z N_1 D_1]^{\sim - w^*} = [M_2^* D_2 r Y s D_1 M_1]^{\sim - w^*} \subseteq [M_2^* D_2 Y D_1 M_1]^{\sim - w^*} = [M_2^* Y M_1]^{\sim - w^*}. \]

Thus,

\[ (3.2) \quad L_2^* Z L_1 \subseteq p X q. \]

On the other hand,

\[ L_2^* L_2 p X q \subseteq [M_2^* D_2 N_2^* N_2 D_2 M_2]^{\sim - w^*} \subseteq [M_2^* D_2 M_2 p X q]^{\sim - w^*} = [M_2^* D_2 M_2^* Y M_1]^{\sim - w^*} \subseteq [M_2^* Y M_1]^{\sim - w^*}. \]

Thus, \( L_2^* L_2 p X q \subseteq p X q \), and similarly \( p X q L_1^* L_1 \subseteq p X q \). Therefore, the relations (3.1) and (3.2) imply that \( Z \subset_{TRO} p X q \). □

**Remark 3.7.** From the above proof, we isolate the fact that if \( Z \subset_{TRO} Y \) and \( Y \sim_{TRO} X \), then \( Z \subset_{TRO} X \).

### 3.2. \( \Delta \)-embeddings for dual operator spaces.

**Definition 3.4.** Let \( X, Y \) be dual operator spaces. We say that \( Y \) weakly \( \Delta \)-embeds into \( X \) if there exist \( w^* \)-continuous completely isometric maps

\[ \phi : X \to \phi(X), \quad \psi : Y \to \psi(Y) \]

such that \( \psi(Y) \subset_{TRO} \phi(X) \). In this case, we write \( Y \subset_{\Delta} X \).

**Definition 3.5.** Let \( X, Y \) be dual operator spaces. A map \( \phi : X \to Y \) that is one-to-one, \( w^* \)-continuous, and completely bounded with a completely bounded inverse is called a \( w^* \)-c.b. isomorphism, and the spaces \( X, Y \) are called \( w^* \)-c.b. isomorphic. Under the above assumptions, the map \( \phi^{-1} \) is also \( w^* \)-continuous.

**Definition 3.6.** Let \( X, Y \) be dual operator spaces. We call these c.b. \( \Delta \)-equivalent if there exist \( w^* \)-c.b. isomorphisms \( \phi : X \to \phi(X), \quad \psi : Y \to \psi(Y) \) such that \( \phi(X) \sim_{TRO} \psi(Y) \). In this case, we write \( X \sim_{\Delta} Y \).

**Definition 3.7.** Let \( X, Y \) be dual operator spaces. We say that \( Y \) c.b. \( \Delta \)-embeds into \( X \) if there exist \( w^* \)-c.b. isomorphisms \( \phi : X \to \phi(X), \psi : Y \to \psi(Y) \) such that \( \psi(Y) \subset_{TRO} \phi(X) \). In this case, we write \( Y \subset_{\Delta} X \).
Remark 3.8. Observe the following:

(i) \( X \sim_{\Delta} Y \Rightarrow X \sim_{cb\Delta} Y \)
(ii) \( X \subset_{\Delta} Y \Rightarrow X \subset_{cb\Delta} Y \)

In what follows, if \( X \) is a dual operator space, then \( M_l(X) \) (resp. \( M_r(X) \)) denotes the algebra of left (resp. right) multipliers of \( X \). In this case, \( A_l(X) = \Delta(M_l(X)) \), (resp. \( A_r(X) = \Delta(M_r(X)) \)) is a von Neumann algebra \([3]\).

Lemma 3.9. Suppose that \( Z, Y \) are \( w^* \)-closed operator spaces satisfying \( Z \sim_{TRO} Y \), \( H_1, H_2 \) are Hilbert spaces such that

\[
A_l(Y) \subseteq B(H_2), \quad A_r(Y) \subseteq B(H_1),
\]

and \( \psi : Y \rightarrow B(H_1, H_2) \) is a \( w^* \)-continuous complete isometry such that

\[
A_l(Y)\psi(Y)A_r(Y) \subseteq \psi(Y).
\]

Then, there exists a \( w^* \)-continuous complete isometry \( \zeta : Z \rightarrow \zeta(Z) \) such that \( \zeta(Z) \sim_{TRO} \psi(Y) \).

Proof. Assume that \( M_1, M_2 \) are TROs such that

\[
Z = [M_2^*YM_1^*]^{-w^*}, \quad Y = [M_2YM_1^*]^{-w^*}.
\]

By Remark 3.1, we may assume that \( Z \) and \( Y \) are nondegenerate spaces. We denote

\[
A = [M_2^*M_2]^{-w^*}, \quad B = [M_1^*M_1]^{-w^*}, \quad C = [M_2^*M_2]^{-w^*}, \quad D = [M_1^*M_1]^{-w^*}.
\]

The algebras

\[
\Omega(Z) = \begin{pmatrix} A & Z \\ 0 & B \end{pmatrix}, \quad \Omega(Y) = \begin{pmatrix} C & Y \\ 0 & D \end{pmatrix}
\]

are weakly TRO-equivalent as algebras. Indeed,

\[
\Omega(Z) = [M^*\Omega(Y)M]^{-w^*}, \quad \Omega(Y) = [M\Omega(Z)M^*]^{-w^*},
\]

where \( M \) is the TRO \( M_2 \oplus M_1 \). If \( c \in C \), then define

\[
\gamma(c) : \psi(Y) \rightarrow \psi(Y), \quad \gamma(c)\psi(y) = \psi(cy).
\]

We can easily see that \( \gamma(c) \in A_l(Y) \) and \( \|\gamma(c)\| \leq 1 \). Thus, \( \gamma : C \rightarrow A_l(Y) \) is a contractive homomorphism and hence a \(*\)-homomorphism. If \( \gamma(c) = 0 \), then \( cY = 0 \). Because \( Y \) is nondegenerate, we conclude that \( c = 0 \). Thus, \( \gamma \) is a one-to-one \(*\)-homomorphism. Similarly, there exists a one-to-one \(*\)-homomorphism \( \delta : D \rightarrow A_r(Y) \) such that \( \psi(y)\delta(d) = \psi(yd), \quad \forall y \). The map

\[
\pi : \Omega(Y) \rightarrow \pi(\Omega(Y)), \text{ given by}
\]

\[
\pi \left( \begin{pmatrix} c & y \\ 0 & d \end{pmatrix} \right) = \begin{pmatrix} \gamma(c) & \psi(y) \\ 0 & \delta(d) \end{pmatrix},
\]

is a \( w^* \)-continuous completely isometric homomorphism. This can be shown by applying 3.6.1 in \([3]\). By Theorem 2.7 in \([10]\), there exists a \( w^* \)-continuous
completely isometric homomorphism \( \rho : \Omega(Z) \to \rho(\Omega(Z)) \) and a TRO \( N \) such that
\[
\rho(\Omega(Z)) = [N^* \pi(\Omega(Y)) N]^{w*}, \quad \pi(\Omega(Y)) = [N \rho(\Omega(Z)) N^*]^{w*}.
\]
As in the discussion concerning the map \( \Phi \) below Theorem 2.5 in \[13\], the map \( \rho \) is given by
\[
\rho \left( \begin{pmatrix} a & z \\ 0 & b \end{pmatrix} \right) = \begin{pmatrix} \alpha(a) & \zeta(z) \\ 0 & \beta(b) \end{pmatrix},
\]
where \( \alpha : A \to \alpha(A) \), \( \zeta : Z \to \zeta(Z) \), \( \beta : B \to \beta(B) \) are completely isometric maps. By Lemma 2.8 in \[13\], the TRO \( N \) is of the form \( N = N_2 \oplus N_1 \).

\[
\zeta(Z) = [N_2^* \psi(Y) N_1]^{-w*}, \quad \psi(Y) = [N_2 \zeta(Z) N_1^*]^{-w*}.
\]

\[\square\]

**Lemma 3.10.** Let \( Z, \Omega, X \) be dual operator spaces. We assume that \( Z \sim_{TRO} \Omega \), and that \( \psi_0 : \Omega \to \psi_0(\Omega) \) is a \( w^* \)-continuous complete isometry such that
\[
\psi_0(\Omega) \subset_{TRO} X.
\]
Then, there exists a \( w^* \)-c.b. isomorphism \( \hat{\phi} : X \to \hat{\phi}(X) \) and a \( w^* \)-continuous complete isometry \( \zeta : Z \to \zeta(Z) \) such that \( \zeta(Z) \subset_{TRO} \hat{\phi}(X) \).
Thus, \( Z \subset_{\phi \Delta} X \).

**Proof.** Suppose that
\[
A_l(\Omega) \subseteq B(H_2), \quad A_r(\Omega) \subseteq B(H_1)
\]
and \( \psi : \Omega \to B(H_1, H_2) \) is a \( w^* \)-continuous complete isometry such that
\[
A_l(\Omega) \psi(\Omega) A_r(\Omega) \subseteq \psi(\Omega).
\]
By Lemma 3.9 there exists a \( w^* \)-continuous complete isometry \( \zeta : Z \to \zeta(Z) \) such that
\[
\zeta(Z) \sim_{TRO} \psi(\Omega).
\]
We assume that \( p, q \) are projections such that \( pX \subseteq X, Xq \subseteq X \) and
\[
\psi_0(\Omega) \sim_{TRO} pXq.
\]
Again by Lemma 3.9 there exists a \( w^* \)-continuous complete isometry \( \phi : pXq \to \phi(pXq) \) such that \( \psi(\Omega) \sim_{TRO} \phi(pXq) \). Define
\[
\hat{\phi}(x) = \begin{pmatrix} \phi(pxq) & 0 & 0 \\ 0 & p^\perp xq & 0 \\ 0 & 0 & xq^\perp \end{pmatrix},
\]
for all \( x \in X \). Observe that \( \hat{\phi} \) is a \( w^* \)-continuous completely bounded and one-to-one map. If \( \hat{\phi}^\infty \) is the \( \infty \times \infty \) amplification of \( \hat{\phi} \), then \( \hat{\phi}^\infty \) has a closed range. Thus, by the open mapping theorem, \( \hat{\phi}^\infty \) has a bounded inverse.
Therefore, $\hat{\phi}^{-1}$ is completely bounded. We have that $\zeta(Z) \sim_{TRO} \phi(pXq)$. Thus, there exist TROs $M_1, M_2$ such that 

$$
\zeta(Z) = [M_2\phi(pXq)M_1]^* \sim_{w^*} \phi(pXq) = [M_2^*\zeta(Z)M_1]^* \sim_{w^*}.
$$

Define the TROs 

$$
N_i = \begin{pmatrix} M_i & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & \phi(x)q^\perp \end{pmatrix}, \quad i = 1, 2.
$$

We can see that 

$$
[N_2\hat{\phi}(X)N_1^*]^{-w^*} = \zeta(Z), \quad N_2^*\zeta(Z)N_1 \subseteq \hat{\phi}(X)
$$

and 

$$
N_2^*N_2\hat{\phi}(X) \subseteq \hat{\phi}(X), \quad \hat{\phi}(X)N_1^*N_1 \subseteq \hat{\phi}(X).
$$

Thus, $\zeta(Z) \subset_{TRO} \hat{\phi}(X)$. □

**Theorem 3.11.** Let $X,Y$ be dual operator spaces. Then, the following are equivalent.

(i) $Y \subset_{cb} \Delta X$.

(ii) There exist $w^*$-c.b. isomorphisms $\psi : Y \to \psi(Y)$, $\phi : X \to \phi(X)$; projections $p,q$ such that $p\phi(X) \subseteq \phi(X)$, $\phi(X)q \subseteq \phi(X)$; a cardinal $I$; and a completely isometric $w^*$-continuous onto map 

$$
\pi : M_I(\psi(Y)) \to M_I(p\phi(X)q).
$$

**Proof.**

$(i) \Rightarrow (ii)$

By definition, there exist $w^*$-c.b. isomorphisms $\psi : Y \to \psi(Y)$, $\phi : X \to \phi(X)$ such that 

$$
\psi(Y) \subset_{TRO} \phi(X).
$$

There exist projections $p,q$ such that $p\phi(X) \subseteq \phi(X)$, $\phi(X)q \subseteq \phi(X)$ and $\psi(Y) \sim_{TRO} p\phi(X)q$. By Theorem 3.12 there exists a cardinal $I$ and a completely isometric $w^*$-continuous onto map 

$$
\pi : M_I(\psi(Y)) \to M_I(p\phi(X)q).
$$

$(ii) \Rightarrow (i)$

Define 

$$
\hat{\phi}(x) = \begin{pmatrix} p\phi(x)q & 0 & 0 \\ 0 & p^\perp\phi(x)q & 0 \\ 0 & 0 & \phi(x)q^\perp \end{pmatrix},
$$

for all $x \in X$. As in the proof of Lemma 3.10 we can see that $\hat{\phi}$ is a $w^*$-c.b. isomorphism. By Example 3.5 (ii), we have that 

$$
p\phi(X)q \subset_{TRO} \hat{\phi}(X).$$
By Theorem \[3.2\] it holds that \( \psi(Y) \sim_{\Delta} p\phi(X)q \). Thus, there exist completely isometric \( w^*\)-continuous maps

\[
\mu : \psi(Y) \to \mu(\psi(Y)), \quad \chi : p\phi(X)q \to \chi(p\phi(X)q)
\]
such that \( \mu(\psi(Y)) \sim_{\text{TRO}} \chi(p\phi(X)q) \). Now, apply Lemma \([3.10]\) for

\[
Z = \mu(\psi(Y)), \quad \Omega = \chi(p\phi(X)q), \quad \psi_0 = \chi^{-1} : \Omega \to p\phi(X)q.
\]
We have that

\[
\psi_0(\Omega) \subset_{\text{TRO}} \hat{\phi}(X).
\]
We conclude that

\[
\mu(\psi(Y)) \subset_{\text{cb}} \hat{\phi}(X) \implies Y \subset_{\text{cb}} X.
\]

Lemma 3.12. Suppose that \( Z \subseteq B(W_1, W_2) \), \( Y \subseteq B(K_1, K_2) \) are \( w^*\)-closed spaces such that \( Z \subset_{\text{TRO}} Y \). We also assume that \( p_2 \) is the projection onto \( Y(K_1) \), and \( p_1 \) is the projection onto \( Y^*(K_2) \). Thus, \( Y_0 = p_2Y[p_1(K_1)] \) is a nondegenerate space into \( B(p_1(K_1), p_2(K_2)) \). We shall prove that \( Z \subset_{\text{TRO}} Y_0 \).

*Proof.* By definition, there exist TROs \( N_i \subseteq B(W_i, K_i) \), \( i = 1, 2 \) such that

\[
Z = [N_2^*YN_1]^{*\Delta}, \quad N_2ZN_1^* \subseteq Y, \quad N_2N_2^*Y \subseteq Y, \quad YN_1N_1^* \subseteq Y.
\]
Define \( M_i = p_iN_i \subseteq B(W_i, p_i(K_i)) \), \( i = 1, 2 \). We have that

\[
p_2y = y \implies p_2m_1m_2y = m_1m_2y, \quad \forall y \in Y, \quad m_1, m_2 \in N_2.
\]
Thus,

\[
p_2N_2N_2^*p_2 = N_2N_2^*p_2 \implies p_2N_2N_2^*p_2 = p_2N_2N_2^*.
\]
The above relation implies that

\[
M_2M_2^*M_2 = p_2N_2N_2^*p_2N_2 = p_2N_2N_2^*N_2 \subseteq p_2N_2 = M_2.
\]
Therefore, \( M_2 \) is a TRO. Similarly, \( M_1 \) is also a TRO. Now, we have that

\[
[M_2^*Y_0M_1]^{-w^*} = [N_2^*p_2Y_0p_1N_1]^{-w^*} = [N_2^*YN_1]^{-w^*} = Z
\]
and

\[
M_2ZM_1^* = p_2N_2ZN_1^*p_1 \subseteq p_2Yp_1 = Y_0.
\]
Furthermore,

\[
M_2M_2^*Y_0 = p_2N_2N_2^*p_2Yp_1 = p_2N_2N_2^*Yp_1 \subseteq p_2Yp_1 = Y_0.
\]
Therefore, \( Y_0 \) is a nondegenerate subspace of \( B(p_1(K_1), p_2(K_2)) \), and \( Z \subset_{\text{TRO}} Y_0 \). \(\Box\)

The Lemma below is weaker than Lemma \([3.9]\)
Lemma 3.13. Suppose that $Z, Y$ are $w^*$-closed operator spaces satisfying $Z \subset_{TRO} Y$, $H_1, H_2$ are Hilbert spaces such that
\[ A_l(Y) \subseteq B(H_2), \quad A_r(Y) \subseteq B(H_1), \]
and $\psi : Y \rightarrow B(H_1, H_2)$ is a $w^*$-continuous complete isometry such that
\[ A_l(Y)\psi(Y)A_r(Y) \subseteq \psi(Y). \]
Then, there exists a $w^*$-continuous complete isometry $\zeta : Z \rightarrow \zeta(Z)$ such that $\zeta(Z) \subset_{TRO} \psi(Y)$.

Proof. Assume that $M_1, M_2$ are TROs such that
\[ Z = [M_2^*YM_1]^{-w^*}, \quad M_2ZM_1^* \subseteq Y, \quad M_2M_2^*Y \subseteq Y, \quad YM_1M_1^* \subseteq Y. \]
By Lemma 3.12, we may assume that $Y$ is nondegenerate. Suppose that $p$ is the identity of $[M_2^*M_2]^{-w^*}$ and $q$ is the identity of $[M_1M_1^*]^{-w^*}$. Then, we have that $pY \subseteq Y, Yq \subseteq Y$. We denote $A, B, C, D, \Omega(Z)$ as in Lemma 3.9 and
\[ \Omega(pYq) = \begin{pmatrix} C & pYq \\ 0 & D \end{pmatrix}. \]
If $c \in C$, then define map
\[ \psi(Y) \rightarrow \psi(Y) : \psi(y) \rightarrow \psi(cy). \]
Clearly, this map belongs to $A_l(Y)$, and thus there exists $\gamma(c) \in A_l(Y)$ satisfying
\[ \gamma(c)\psi(y) = \psi(cy) \forall y \in Y. \]
Note that we can define a $*$-homomorphism $\gamma : C \rightarrow A_l(Y)$. If $\gamma(c) = 0$, then $cy = 0$ for all $y \in Y$, and thus because $Y$ is nondegenerate, it follows that $c = 0$. Therefore, $\gamma$ is one-to-one. Similarly, there exists a one-to-one $*$-homomorphism
\[ \delta : D \rightarrow A_r(Y) \]
such that
\[ \psi(y)\delta(d) = \psi(yqd) \forall y \in Y. \]
We can conclude that there exist projections $\hat{p} \in A_l(Y), \hat{q} \in A_r(Y)$ such that
\[ \psi(py) = \hat{p}\psi(y), \quad \psi(yq) = \psi(y)\hat{q}, \quad \forall y \in Y. \]
The map $\pi : \Omega(pYq) \rightarrow \pi(\Omega(pYq))$, given by
\[ \pi \left( \begin{pmatrix} c & pyq \\ 0 & d \end{pmatrix} \right) = \begin{pmatrix} \gamma(c) & \psi(pyq) \\ 0 & \delta(d) \end{pmatrix}, \]
is a $w^*$-continuous completely isometric homomorphism. Because $\Omega(Z) \sim_{TRO} \Omega(pYq)$,
as in Lemma 3.9 we can find a $w^*$-continuous completely isometric map $\zeta : Z \to \zeta(Z)$ and TROs $N_1, N_2$ such that

$$\zeta(Z) = [N_2^*\psi(pYq)N_1^*]^{-w^*}, \quad \psi(pYq) = [N_2^*\zeta(Z)N_1]^{-w^*} \subseteq \psi(Y),$$

$$\gamma(C) = [N_2^*N_2]^{-w^*}, \quad \delta(D) = [N_1^*N_1]^{-w^*}.$$

Because

$$N_2^*N_2\psi(Y) \subseteq \gamma(C)\psi(Y) = \psi(CY) \subseteq \psi(Y),$$

and similarly $\psi(Y)N_1^*N_1 \subseteq \psi(Y)$, we have that $\zeta(Z) \subseteq \text{TRO} \psi(Y)$. \hfill \qed

**Theorem 3.14.** Let $X, Y, Z$ be dual operator spaces such that $Z \subseteq Y \subseteq X$. Then:

(i) There exist a $w^*$-continuous complete isometry $\chi : X \to \chi(X)$ and projections $p, q$ such that $p\chi(X) \subseteq \chi(X)$, $\chi(X)q \subseteq \chi(X)$ and $Z \subseteq p\chi(X)q$.

(ii) $Z \subseteq_{\text{cbo}} X$.

**Proof.** Suppose that $H_1, H_2$ are Hilbert spaces such that

$$A_t(Y) \subseteq B(H_2), \quad A_r(Y) \subseteq B(H_1)$$

and $\psi : Y \to B(H_1, H_2)$ is a $w^*$-continuous complete isometry such that

$$A_t(Y)\psi(Y)A_r(Y) \subseteq \psi(Y).$$

By Lemma 3.13 there exists a $w^*$-continuous complete isometry $\zeta : Z \to \zeta(Z)$ such that

$$\zeta(Z) \subseteq \text{TRO} \psi(Y).$$

Because $Y \subseteq X$, there exist a $w^*$-continuous complete isometry $\chi : X \to \chi(X)$ and projections $p, q$ such that $p\chi(X) \subseteq \chi(X)$, $\chi(X)q \subseteq \chi(X)$ and

$$Y \sim_{\Delta} p\chi(X)q.$$

By Lemma 3.9 there exists a $w^*$-continuous complete isometry $\phi : p\chi(X)q \to \phi(p\chi(X)q)$ such that

$$\psi(Y) \sim_{\text{TRO}} \phi(p\chi(X)q).$$

Thus, by Remark 3.7 it holds that

$$\zeta(Z) \subseteq \text{TRO} \phi(p\chi(X)q) \Rightarrow Z \subseteq p\chi(X)q.$$

There exist projections $r, s$ and TROs $M_1, M_2$ such that

$$r\phi(p\chi(X)q) \subseteq \phi(p\chi(X)q), \quad \phi(p\chi(X)q)s \subseteq \phi(p\chi(X)q)$$

and

$$\zeta(Z) = [M_2r\phi(p\chi(X)q)sM_1^*]^{-w^*}, \quad r\phi(p\chi(X)q)s = [M_2^*\zeta(Z)M_1]^{-w^*}.$$
As in the proof of Lemma 3.10, we can see that \( \hat{\phi} \) is a \( w^* \)-c.b. isomorphism from \( X \) onto \( \hat{\phi}(X) \). Define the TROs
\[
N_i = (M_i 0 0 0), \quad i = 1, 2.
\]
We can see that
\[
[N_2 \hat{\phi}(X)N_i^*]^{-w^*} = \zeta(Z), \quad N_2^* \zeta(Z)N_1 \subseteq \hat{\phi}(X)
\]
and
\[
N_2^* N_2 \hat{\phi}(X) \subseteq \hat{\phi}(X), \quad \hat{\phi}(X) N_1^* N_1 \subseteq \hat{\phi}(X).
\]
Thus,
\[
\zeta(Z) \subset_{TRO} \hat{\phi}(X) \Rightarrow Z \subset_{cb} X.
\]

**Example 3.15.** Let \( Y \) be a dual operator space, and let \( H \) be a Hilbert space such that \( Y \subset_{cb} B(H) \). Then, \( Y \sim_{\Delta} B(H) \).

**Proof.** There exist \( w^* \)-continuous completely isometric maps \( \psi : Y \rightarrow \psi(Y) \) and projections \( p, q \) such that \( p\phi(B(H)) \subseteq \phi(B(H)), \phi(B(H))q \subseteq \phi(B(H)) \) and
\[
\psi(Y) \sim_{TRO} p\phi(B(H))q.
\]
We define the map \( \alpha : B(H) \rightarrow B(H) \) given by \( \alpha(x) = \phi^{-1}(p\phi(x)) \). This is a multiplier of \( B(H) \), and also a projection. Because \( A_\Gamma(B(H)) = B(H) \), there exists a projection \( \hat{p} \in B(H) \) such that
\[
\phi^{-1}(p\phi(x)) = \hat{p}x \Rightarrow \phi(\hat{p}x) = p\phi(x) \quad \forall x \in B(H).
\]
Similarly, there exists a projection \( \hat{q} \in B(H) \) such that
\[
\phi(x\hat{q}) = \phi(x)q \quad \forall x \in B(H).
\]
We have that
\[
\phi^{-1}(p\phi(B(H))q) = \hat{p}B(H)\hat{q}.
\]
Because
\[
A_\Gamma(\hat{p}B(H)\hat{q}) = B(\hat{p}(H)), \quad A_\gamma(\hat{p}B(H)\hat{q}) = B(\hat{q}(H)),
\]
it follows from Lemma 3.9 that there exists a \( w^* \)-continuous completely isometric map \( \zeta : \psi(Y) \rightarrow \zeta(\psi(Y)) \) such that
\[
\zeta(\psi(Y)) \sim_{TRO} \hat{p}B(H)\hat{q}.
\]
Thus, there exist TROs \( M_1, M_2 \) such that
\[
\zeta(\psi(Y)) = [M_2 \hat{p}B(H)\hat{q}M_1^*]^{-w^*}.
\]
Define
\[
N_2^* = [M_2 \hat{p}B(H)]^{-w^*}, \quad N_1 = [B(H)\hat{q}M_1^*]^{-w^*}.
\]
Then, we have that 
\[ [N_2 N_2^* N_2]^{w^*} = [B(H) \hat{p} M_2^* M_2 \hat{p} B(H) \hat{p} M_2^*]^{w^*} = [B(H) p M_2^*]^{-w^*} = N_2. \]
Thus, \( N_2 \) is a TRO. Similarly, \( N_1 \) is a TRO. Now,
\[ \zeta(\psi(Y)) = [N_2^* B(H) N_1]^{-w^*} \]
and
\[ [N_2 B(H) N_1^*]^{-w^*} = [B(H) \hat{p} M_2^* B(H) M_1 \hat{q} B(H)]^{-w^*} = B(H). \]
Thus,
\[ \zeta(\psi(Y)) \sim_{TRO} B(H) \Rightarrow Y \sim_{\Delta} B(H). \]

Example 3.16. Let \( H \) be a Hilbert space, \( \Phi : B(H) \to B(H) \) be a \( w^* \)-continuous completely bounded idempotent map, and
\[ Y = \text{Ran} \Phi, \ Z = \text{Ran} \Phi^\perp. \]
Let \( \phi : B(H) \to Y \oplus Z \) be the map given by \( \phi(x) = \Phi(x) \oplus \Phi^\perp(x) \). We can easily prove that \( \phi \) is a \( w^* \)-c.b. isomorphism onto \( Y \oplus Z \). By Example 3.5 (ii), we have that \( Y \subset_{TRO} \phi(B(H)) \), and thus \( Y \subset_{\Delta} B(H) \).

Example 3.17. Let \( H \) be a Hilbert space, and let \( e, f \in B(H) \) be nontrivial projections. Define
\[ \Phi : B(H) \to B(H), \ \Phi(x) = e x f + e x f^\perp + e^\perp x f^\perp, \]
and denote \( Y = \text{Ran} \Phi \). By Example 3.16 we have that \( Y \subset_{\Delta} B(H) \). If \( Y \text{ weakly } \Delta \text{-embeds into } B(H) \), then by Example 3.15 we should have that \( Y \sim_{\Delta} B(H) \). However, this contradicts the fact that \( B(H) \) is a self-adjoint algebra and \( Y \) is a non-self-adjoint algebra. Thus, the relation \( Y \subset_{\Delta} X \) does not always imply that \( Y \subset_{\Delta} X \) holds.

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References

[1] D. P. Blecher and U. Kashyap, Morita equivalence of dual operator algebras, J. Pure Appl. Algebra, 212 (2008), 2401-2412
[2] D. P. Blecher and J. E. Kraus, On a generalization of \( W^* \)-modules, Banach Center Publ. 91, (2010), 77-86
[3] D. P. Blecher and C. Le Merdy, Operator Algebras and Their Modules—An Operator Space Approach, Oxford University Press, 2004
[4] D. P. Blecher, P. S. Muhly, and V. I. Paulsen, Categories of operator modules—Morita equivalence and projective modules, Memoirs of the A.M.S. 143 (2000) no. 681
[5] D. P. Blecher and V. Zarikian, The calculus of one sided M-ideals and multipliers in operator spaces, *Memoirs of the A.M.S.* 179 (842), 2003
[6] K. R. Davidson, *Nest Algebras*, Longman Scientific & Technical, Harlow, 1988
[7] E. Effros and Z.-J. Ruan, *Operator Spaces*, Oxford University Press, 2000
[8] G.K. Eleftherakis, TRO equivalent algebras, *Houston J. of Mathematics*, 38:1 (2012), 153-175
[9] G.K. Eleftherakis, A Morita type equivalence for dual operator algebras, *J. Pure Appl. Algebra*, 212:5 (2008), 1060-1071
[10] G.K. Eleftherakis, Morita type equivalences and reflexive algebras, *J. Operator Theory*, 64 (2010) no 1, 3-17
[11] G.K. Eleftherakis, Morita equivalence of nest algebras, *Math. Scand.*, 113 (2013), no 1, 83-107
[12] G.K. Eleftherakis, V. I. Paulsen, Stably isomorphic dual operator algebras, *Math. Ann.*, 341:1 (2008), 99-112
[13] G. K. Eleftherakis, V. I. Paulsen, and I. G. Todorov, Stable isomorphism of dual operator spaces, *J. Funct. Anal.* 258 (2010), 260–278
[14] U. Kashyap, A Morita theorem for dual operator algebras, *J. Funct. Analysis*, 256 (2009) 3545-3567
[15] V. I. Paulsen, *Completely Bounded Maps and Operator Algebras*, Cambridge University Press, 2002
[16] G. Pisier, *Introduction to Operator Space Theory*, Cambridge University Press, 2000

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