Vanishing Theorems for the Self-Dual N=2 String

Nathan Berkovits

Dept. de Matemática Física, Univ. de São Paulo
CP 20516, São Paulo, SP 01498, BRASIL

and

IMECC, Univ. de Campinas
CP 1170, Campinas, SP 13100, BRASIL

e-mail: nberkovi@snfma1.if.usp.br

It is proven that up to possible surface terms, the only non-vanishing momentum-dependent amplitudes for the self-dual N=2 string in $R^{2,2}$ are the tree-level two and three-point functions, and the only non-vanishing momentum-independent amplitudes are the one-loop partition function and the tree-level two and four-point functions. The calculations are performed using the topological prescription developed in an earlier paper with Vafa. As in supersymmetric non-renormalization theorems, the vanishing proof is based on a relationship between the zero-momentum dilaton and axion.

December 1994
1. Introduction

In an earlier paper with Vafa\cite{1}, a new topological prescription was described for calculating scattering amplitudes of N=2 strings, and was shown to be equivalent to the usual prescription. Because the topological prescription does not require N=2 superconformal ghosts or integration over U(1) moduli, calculations are considerably simplified. Furthermore, this new prescription contains no ambiguities associated with the locations of the N=2 picture-changing operators.

In the earlier paper \cite{1}, it was proven for the self-dual string in $R^{2,2}$ that all momentum-dependent amplitudes vanish up to surface terms, with the exception of the three-point function. However there exist indirect arguments that other amplitudes should also vanish. Since a $g$-loop $N$-point amplitude can be cut into an $(N + 2g)$-point tree amplitude, one expects by “unitarity” arguments \cite{3} that a loop amplitude should vanish when the corresponding tree amplitude vanishes (since there are two time directions, the term “unitarity” should not be taken too literally). It was also argued by Siegel\cite{4} that spacetime-supersymmetry and Lorentz-invariance imply the vanishing af all loop amplitudes (note, however, that explicit calculations find the one-loop partition function to be non-vanishing).

In this paper, it will be proven that up to possible surface terms, the only non-vanishing momentum-dependent scattering amplitudes for the self-dual string in $R^{2,2}$ are the tree-level two and three-point functions, and the only non-vanishing momentum-independent amplitudes are the one-loop partition function and the tree-level two and four-point functions. This result appears to contradict an earlier one-loop calculation of the three-point function\cite{5} and a two-loop calculation of the partition function\cite{6} which found non-vanishing amplitudes. A possible resolution of this paradox is that these amplitudes can be written as integrals of total derivatives. It would be interesting to verify this fact with explicit calculations of the surface-term contributions in the topological prescription. Note that these calculations are possible since the topological prescription does not contain total-derivative ambiguities.

The vanishing proof in this paper will use the fact that inserting a zero-momentum dilaton into a correlation function is related by picture-changing to inserting a zero-momentum axion. It is interesting that non-renormalization theorems for four-dimensional

\footnote{In an early version of the preprint, it was also claimed that certain momentum-independent amplitudes vanish. However the proof of this claim was incorrect since it ignored contractions between $\partial x^\mu$ and $\partial x'^\nu$. \cite{2}}

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supersymmetric strings also use the zero-momentum dilaton and axion, which are related to each other by spacetime-supersymmetry transformations. This suggests that the picture-changing operators, $\int G^+$ and $\int \tilde{G}^+$, can be thought of as twisted spacetime-supersymmetry generators, an idea that was first proposed in the earlier paper with Vafa.

For the self-dual string in $R^{2,2}$, the worldsheet fields are $X_{ab}$, the right-moving $\psi^+_a$ and $\tilde{\psi}_a^+$, and the left-moving $\bar{\psi}^+_a$ and $\bar{\psi}^-_a$. Note that the SO(2,2) vector index is expressed as two SU(1,1) spinor indices, $a$ and $\hat{a}$, which take the values 1 or 2. In this notation, $x_{a\hat{a}} = x_\mu \sigma^\mu_{a\hat{a}}$ where

$$\sigma^0_{a\hat{a}} = i\delta_{a\hat{a}}, \quad \sigma^1_{a\hat{a}} = i\sigma^x_{a\hat{a}}, \quad \sigma^2_{a\hat{a}} = i\sigma^y_{a\hat{a}}, \quad \sigma^3_{a\hat{a}} = \sigma^z_{a\hat{a}}.$$  

SU(1,1) indices can be raised and lowered using the epsilon tensor in two dimensions.

After twisting, the superconformal constraints are:

$$L = \frac{1}{4} \partial_2 x^{\hat{a}b} \partial_2 x^{ab} + \epsilon^{ab} \psi^-_a \partial_2 \psi^+_b,$$

$$G^+ = \psi^+_b \partial_2 x^b, \quad \tilde{G}^+ = \tilde{\psi}^+_b \partial_2 x^b, \quad G^- = \psi^-_b \partial_2 x^b, \quad \tilde{G}^- = \tilde{\psi}^-_b \partial_2 x^b,$$

$$J^{++} = \frac{1}{2} \epsilon^{ab} \psi^+_a \dot{\psi}^+_b, \quad J = \epsilon^{ab} \psi^+_a \psi^-_b, \quad J^{--} = \frac{1}{2} \epsilon^{ab} \dot{\psi}^-_a \psi^-_b,$$

$$\bar{L} = \frac{1}{4} \partial_2 x^{\hat{a}b} \partial_2 x^{\hat{a}b} + \epsilon^{ab} \tilde{\psi}^-_a \partial_2 \bar{\psi}^+_b,$$

$$\bar{G}^+ = \bar{\psi}^+_b \partial_2 x^b, \quad \bar{\tilde{G}}^+ = \bar{\psi}^+_b \partial_2 x^{\hat{a}b}, \quad \bar{G}^- = \bar{\psi}^-_b \partial_2 x^b, \quad \bar{\tilde{G}}^- = \bar{\psi}^-_b \partial_2 x^{\hat{a}b},$$

$$J^{++} = \frac{1}{2} \epsilon^{ab} \bar{\psi}^+_a \dot{\bar{\psi}}^+_b, \quad J = \epsilon^{ab} \bar{\psi}^+_a \bar{\psi}^-_b, \quad J^{--} = \frac{1}{2} \epsilon^{ab} \dot{\bar{\psi}}^-_a \bar{\psi}^-_b,$$

which form a right-moving and left-moving twisted N=4 superconformal algebra.

In the topological description of the N=2 string, the physical vertex operators contain U(1) charge (1,1), are annihilated by $\int G^+$, $\int \tilde{G}^+$, $\int \bar{G}^+$, $\int \bar{\tilde{G}}^+$, and can not be written as $\int G^+ G^+ \Omega$ or $\int \bar{G}^+ \bar{G}^+ \bar{\Omega}$ for any $\Omega$. They consist of the momentum-dependent state

$$G^+ \tilde{G}^+ V = k^{1a} k^{1\hat{b}} \psi^+_a \dot{\psi}^+_b e^{ik \cdot x}$$

and the momentum-independent states $\psi^+_a \bar{\psi}^{+\hat{a}}$. Note that since $k \cdot k = 0$, $k^{2\hat{a}}$ is proportional to $k^{1\hat{a}}$ so there is only one momentum-dependent physical state.

It will first be proven that except for the tree-level two and three-point functions, all momentum-dependent amplitudes must vanish up to surface terms. It will then be proven that except for the one-loop partition function and the tree-level two and four-point functions, all momentum-independent amplitudes must vanish up to surface terms.
2. Vanishing Theorem for Momentum-Dependent Amplitudes

When there are momentum-dependent vertex operators, it was shown in reference [1] that the $g$-loop $N$-point amplitude on a surface of instanton number $(2g - 2 + N, 2 - 2g - N)$ can be expressed as:

\[ A = \int_{M_{g,n}} G^{-}(\mu_{i})\bar{G}^{-}(\bar{\mu}_{i}) \prod_{r=1}^{N-1} \phi_{r}(z_{r}) \quad e^{ik_{N} \cdot x(z_{N})} \quad (2.1) \]

\[
det(I_m \tau) | \det \omega^{k}(v_{l})|^{2} \prod_{j=1}^{g} \tilde{G}^{+}(v_{j}) \tilde{\bar{G}}^{+}(\bar{v}_{j}) \]

where $\mu_{i}$ are the Beltrami differentials and $\phi_{r}$ is a physical vertex operator of U(1) charge (1,1) of the type described in the introduction. Note that $\tilde{G}^{+}$ and $\tilde{\bar{G}}^{+}$ have no poles, so the amplitude is independent of the locations of the $v_{j}$’s.

The only amplitude that can not be expressed in this form is the tree-level two-point function since $3g - 3 + N < 0$. In this case, the amplitude is given by

\[
< e^{ik_{1} \cdot x(z_{1})} e^{ik_{2} \cdot x(z_{2})} J^{++}(z_{3}) \bar{J}^{++}(\bar{z}_{3}) > .
\]

As was shown in reference [1], amplitudes on surfaces of other instanton numbers $(I_{R}, I_{L})$ are related to $A$ by

\[ A_{I_{R}, I_{L}} = h^{I_{R} - I_{L} - 4g + 4 - 2N} A \quad (2.2) \]

where $hk^{2\hat{a}}_{N} = k^{1\hat{a}}_{N}$ (note that $\hbar h = 1$). Therefore $A=0$ implies that $A_{I_{R}, I_{L}} = 0$ for surfaces of all instanton numbers.

The first step in proving that $A$ vanishes is to insert a zero-momentum dilaton vertex operator into the amplitude. This vertex operator is given by $\int_{\Omega} d^{2}z(\partial_{z}x^{\mu}\partial_{\bar{z}}x_{\mu})$ where the region of integration, $\Omega$, covers the whole surface with the exception of small discs surrounding $\mu_{i}, v_{j},$ and $z_{r}$. As was shown in appendix B of reference [10], this amplitude with the insertion is equal to

\[ A^{dilaton} = (gd - \sum_{r=1}^{N} k_{r} \cdot \frac{d}{d\hat{k}_{r}}) A \quad (2.3) \]

where $g$ is the genus, $d$ is the dimension ($d = 4$), and $d/d\hat{k}^{\mu}$ acts only on the $k$’s appearing in the exponential $e^{ik_{r} \cdot x}$ of the vertex operators, but not on the $k$’s appearing as factors in front of the exponential (for example, $d/d\hat{k}^{\mu}(k^{1\hat{a}}\psi_{\hat{a}}^{+})(k^{1\hat{a}}\bar{\psi}_{\hat{a}}^{+})e^{ik_{r} \cdot x} = ix_{\mu}(k^{1\hat{a}}\psi_{\hat{a}}^{+})(k^{1\hat{a}}\bar{\psi}_{\hat{a}}^{+})e^{ik_{r} \cdot x}$).
As discussed in reference [10], the term \( gd \) in equation (2.3) comes from the contraction of \( \partial z x^\mu \) with \( \partial \bar{z} x^\mu \) in the dilaton vertex operator. The term \( - \sum_{r=1}^N k_r \cdot d/dk_r \) comes from writing the dilaton vertex operator as the surface term

\[
\int d^2z \partial z (x_\mu \partial \bar{z} x^\mu) = \frac{1}{2} \int_C d\bar{z} \partial \bar{z} (x_\mu x^\mu)
\]

where the contour \( C \) surrounds the points \( \mu_i, v_j, \) and \( z_r \). Since the operators at \( \mu_i \) and \( v_j \) only involve derivatives of \( x^\mu \), their surface terms vanish. However at \( z_r \), the surface term contributes \(- (ik_r \cdot x) \phi_r = -k_r \cdot d/dk_r \phi_r \) where the \( d/dk_r \) acts only on the \( k_r \) in the exponential.

Because the \( \partial z x_{1\dot{a}} \)'s and \( \partial \bar{z} x_{1\dot{a}} \)'s in \( G^- \), \( \tilde{G}^+, \tilde{G}^- \), and \( G^+ \) cannot contract with each other, they can only be contracted with the \( x^\mu \)'s appearing in the exponentials of \( \phi_r \), which bring down factors of \( \hat{k}_r^\mu \). Therefore \( \sum_{r=1}^N k_r \cdot d/dk_r A = ((3g-3+N) + g + (3g-3+N) + g) A \). Since \( d = 4 \), equation (2.3) implies that

\[
A^\text{dilaton} = (6 - 4g - 2N) A. \tag{2.4}
\]

The next step in the proof is to show that \( A^\text{dilaton} = 0 \). This is done by writing the integrand of the dilaton vertex operator in the form

\[
\partial z x^\mu \partial \bar{z} x_\mu = \frac{\epsilon^{\dot{a}b}}{2} (\partial z x_{1\dot{a}} \partial \bar{z} x_{2\dot{b}} - \partial z x_{2\dot{a}} \partial \bar{z} x_{1\dot{b}})
= \frac{\epsilon^{\dot{a}b}}{2} (\tilde{G}^+ \psi^-_\dot{a} \partial \bar{z} x_{2\dot{b}} - \partial z x_{2\dot{a}} (\tilde{G}^+ \psi^-_\dot{b})). \tag{2.5}
\]

In the first term of equation (2.5), the \( \tilde{G}^+ \) can be pulled off of the \( \psi^-_\dot{a} \) to encircle the vertex operator \( V_N(z_N) = e^{ik_N \cdot x(z_N)} \), where possible surface terms are being ignored. Since \( \tilde{G}^+ V_N = -h(G^+ V_N) \) where \( k_\dot{a}^\dot{a} = h k_{1\dot{a}}^\dot{a} \), the \( G^+ \) can be pulled off of the \( \psi^-_\dot{a} \) to give

\[
\frac{\epsilon^{\dot{a}b}}{2} (-h G^+ \psi^-_\dot{a} \partial \bar{z} x_{2\dot{b}}) = -h \frac{\epsilon^{\dot{a}b}}{2} \partial z x_{2\dot{a}} \partial \bar{z} x_{1\dot{b}}. \tag{2.6}
\]

Similarly for the second term of equation (2.3), the \( \tilde{G}^+ \) can be pulled onto \( V_N \), replaced with \( -h \tilde{G}^+ \), and returned to encircle \( \tilde{\psi}^-_\dot{b} \). This gives

\[
- \frac{\epsilon^{\dot{a}b}}{2} \partial z x_{2\dot{a}} (-h \tilde{G}^+ \tilde{\psi}^-_\dot{b}) = \frac{\epsilon^{\dot{a}b}}{2} \partial z x_{2\dot{a}} \partial \bar{z} x_{2\dot{b}}
\]

which cancels the contribution in equation (2.6).

So we have proven that \( A^\text{dilaton} = 0 \), which implies that either \( (6-4g-2N) = 0 \) or \( A = 0 \). So besides the tree-level two-point function, the only possible non-zero amplitudes (up to surface terms) are when \( N = 1, g = 1 \) or \( N = 3, g = 0 \). But by momentum conservation, the one-point amplitude can not contain momentum dependence. So up to surface terms, the only non-vanishing momentum-dependent amplitudes for the self-dual string are the tree-level two and three-point functions.
3. Vanishing Theorem for Momentum-Independent Amplitudes

It will now be proven that up to surface terms, all momentum-independent amplitudes must vanish except for the one-loop partition function and the tree-level two and four-point functions. As was shown in reference \[\text{[II]}, \text{momentum-independent amplitudes can be written in the form}\]

\[
F(u_R, u_L) = \int_{M_{g,N}} \prod_{i=1}^{3g-4+N} \left( \hat{G}^- (\mu_i) \hat{G}^- (\bar{\mu}_i) \right) J^{--} (\mu_{3g-3+N}) J^{++} (\bar{\mu}_{3g-3+N})
\]

\[
\prod_{r=1}^{N} \phi_r (z_r) \det (Im \tau) \left| \det \omega^k (v_l) \right|^2 \prod_{j=1}^{g} \hat{G}^+ (v_j) \hat{G}^+ (\bar{v}_j)
\]

where

\[
\hat{G}^- = u^R_a \partial_z x^{a\bar{a}} \psi_{a}^- = u^R_1 \hat{G}^- - u^R_2 \hat{G}^-,
\]

\[
\hat{G}^+ = u^R_a \partial_z x^{a\bar{a}} \bar{\psi}_{a}^+ = u^R_1 \hat{G}^+ - u^R_2 \hat{G}^+,
\]

\[
\hat{G}^- = u^L_a \partial_{\bar{z}} x^{a\bar{a}} \bar{\psi}_{a}^- = u^L_1 \hat{G}^- - u^L_2 \hat{G}^-,
\]

\[
\hat{G}^+ = u^L_a \partial_{\bar{z}} x^{a\bar{a}} \bar{\psi}_{a}^+ = u^L_1 \hat{G}^+ - u^L_2 \hat{G}^+.
\]

\(u^R_a\) and \(u^L_a\) are SU(1,1) spinors which parameterize the choice of complex structure, \(\phi_r\) are momentum-independent vertex operators of the form \(\psi_{a}^+ \bar{\psi}_{a}^+\), and \(F(u_R, u_L)\) is a polynomial of degree \((4g - 4 + N, 4g - 4 + N)\) in \((u_R, u_L)\) whose \((8g - 7 + 2N)^2\) components give the scattering amplitude on a surface of instanton-number \((I_R, I_L)\) where \(-4g - 4 + N \leq I_R, I_L \leq 4g - 4 + N\).

The only amplitudes that can not be expressed in this form are the tree-level two-point function and the one-loop partition function. In these cases, the amplitudes are defined as \(< \phi_1 (z_1) \phi_2 (z_2) >\) and as \(\int_{M_{1}} (\int d^2 z J(z) \tilde{J}(\bar{z}))^2\).

Since \(F\) is invariant under the SU(1,1) subgroup of SO(2,2) Lorentz transformations which transform \(u^R_a\) and \(u^L_a\) but leave \(\psi_{a}^\pm\) and \(\bar{\psi}_{a}^\pm\) invariant, \(F\) must be proportional to \((e^{ab} u^R_a u^L_b)^{4g-4+N}\). It will now be shown that the proportionality constant vanishes up to surface terms unless \(N = 2 - 2g\) or \(N = 4 - 4g\).

The first step is to compute the effect of inserting the zero-momentum axion vertex operator

\[
V_{ab} = b^{ab} \int_{\Omega} d^2 z (\partial_z x_{a\bar{a}} \partial_{\bar{z}} x_{b\bar{a}})
\]

where \(b_{ab} = b_{ba}\) is the polarization of the axion, and the region of integration, \(\Omega\), covers the whole surface with the exception of small discs surrounding \(\mu_i\), \(v_j\), and \(z_r\). One can write the vertex operator as the surface term

\[
\frac{1}{2} b^{ab} \left( \int_{C} d\bar{z} (x_{a\bar{a}} \partial_{\bar{z}} x_{b\bar{a}}) - \int_{C} dz (x_{a\bar{a}} \partial_{z} x_{b\bar{a}}) \right)
\]
where the contour $C$ surrounds the points $\mu_i, v_i$, and $z_i$.

It is easy to check that this surface term transforms $\partial_z x_{ab} \to 2b_{ab}\partial_z x_{b}^b$ in $\hat{G}^-$ and $\hat{G}^+$, transforms $\partial_z x_{ab} \to -2b_{ab}\partial_z x_{b}^b$ in $\hat{G}^-$ and $\hat{G}^+$, and leaves $\phi_r$ invariant. The amplitude with the axion insertion therefore satisfies the following identity:

$$A^{\text{axion}} = 2b^a_{eb} \epsilon_{bc}(u^R_a \frac{d}{du^R_c} - u^L_a \frac{d}{du^L_c}) A. \quad (3.3)$$

So the amplitude with an insertion of $\int d^2z \partial_z x_{1a} \partial_z x_{1}^a$ satisfies the identity

$$A^{b_{11}} = 2(u^R_1 d/du^R_2 - u^L_1 d/du^L_2) A. \quad (3.4)$$

Furthermore, since

$$\int d^2z \partial_z x_{2a} \partial_z x_{1}^a =$$

$$- \int d^2z \partial_z x_{2a} \partial_z x_{1}^a =$$

$$\int d^2z(\partial_z x_{1a} \partial_z x_{2}^a + \partial_z x_{2a} \partial_z x_{1}^a), \quad (3.5)$$

the amplitude with an insertion of $\int d^2z \partial_z x_{2a} \partial_z x_{1}^a$ satisfies the identity

$$A^{dilaton+b_{12}} = -[gd + (u^R_1 d/du^R_2 - u^R_2 d/du^R_1 - u^L_1 d/du^L_2 + u^L_2 d/du^L_1)] A \quad (3.6)$$

where the term $gd$ comes from the dilaton contribution as was explained in the previous section.

But $\int d^2z \partial_z x_{1a} \partial_z x_{1}^a$ is related by picture-changing to an insertion of $\int d^2z \partial_z x_{2a} \partial_z x_{1}^a$. In other words, $\int d^2z \partial_z x_{1a} \partial_z x_{1}^a = \hat{G}^+ W$ and $\int d^2z \partial_z x_{2a} \partial_z x_{1}^a = \hat{G}^- W$ where $W = \psi^+_a \partial_z x_{1}^a$. As was shown in reference [1], this implies that up to surface terms,

$$\frac{d}{du^R_1} A^{b_{11}} = -\frac{d}{du^R_2} A^{dilaton+b_{21}}. \quad (3.7)$$

Using equations (3.4), (3.5), and (3.7), and the fact that $A$ is proportional to $(u^R_1 u^L_2 - u^R_2 u^L_1)^{4g-4+N}$, it is straightforward to show that $(2g - 2 + N)(4g - 4 + N)A = 0$. So the amplitude must vanish unless $N = 2 - 2g$ or $N = 4 - 4g$.

Therefore up to surface terms, the only non-vanishing momentum independent amplitudes are the one-loop partition function and the tree-level two and four-point functions. It would of course be interesting to check if these results are spoiled by surface-term contributions.

**Acknowledgements:** I would like to thank Denis Dalmazi, Hirosi Ooguri, Victor Rivelles, and Cumrun Vafa for useful conversations, and the Conselho Nacional de Pesquisa for financial support.
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