Novelty of anisotropic contraction phase in Bianchi-I spacetime in metric $f(R)$ cosmology

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Abstract

The present work is related to anisotropic cosmological evolution in metric $f(R)$ theory of gravity. The initial part of the paper develops the general cosmological dynamics of homogeneous anisotropic Bianchi-I spaces in $f(R)$ cosmology. The later part of the present paper shows that during anisotropic contraction of Bianchi-I spacetime the cosmological system has to choose between multiple anisotropic contraction branches. The novelty of the system lies in the fact that all the different branches of cosmological development corresponds to one particular form of the scale-factor appearing in the metric. The multiple branches correspond to different solutions of a first order non-linear differential equation, with time varying coefficients, which yields the dynamics of the anisotropy parameter. In quadratic gravity a pre-bounce power law contraction phase displays the complexity of the situation. We show that all the possible branches of cosmological development does not lead to isotropization where as there exists distinct cases which may lead to isotropization.

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1 Introduction

The issue related to stability of homogeneous and isotropic cosmological solutions with respect to small anisotropy has been studied intensely in theoretical cosmology [1], [2], [3], [4]. Behavior of small anisotropy has been studied in cosmological models, using general relativity (GR), in the contexts of inflation [5], [6], [7], [8], [9] and pre-bounce ekpyrotic contraction phase [10], [11], [12], [13]. In the context of inflation the ‘No-Hair’ conjecture asserts that any pre-existing anisotropy must asymptotically die out in an inflating universe. Wald has been able to prove the conjecture for all the Bianchi models except Bianchi-IX [14] which requires a large cosmological constant to isotropize the spacetime. In a contracting universe, provided the universe is dominated by a matter component mimicking an ultra stiff barotropic fluid, growth of small anisotropy is suppressed with respect to that of the Hubble parameter. In absence of any such fluid in a contracting phase, small anisotropy grows large and dominates over all other matter components. This leads to the Belinsky-Khalatnikov-Lifshitz (BKL) instability [15], foiling the bounce. Mathematically, it can be shown that in presence of a slowly rolling scalar field the isotropic de-Sitter solution is an attractor for an expanding universe and in presence of a fast rolling scalar field the isotropic power law solution (for the scale-factor) is an attractor for contracting universe. Therefore an inflationary scenario is usually realized by a slowly rolling scalar field and an ekpyrotic scenario is usually realized by a fast rolling scalar field [16], [17], [18], [19], [20].

In the present work we have analyzed the evolution of spacetime anisotropy in $f(R)$ gravity\(^1\) where the analysis becomes significantly more involved than that of models based on GR. Although previously there have been some attempts to generalize the no-hair theorem to incorporate higher order gravity theories [22], [23], [24], [25] and some applications of dynamical system analysis to understand anisotropic cosmology in higher order gravity [26], [27], [28], the previous attempts missed an important property of anisotropic cosmological dynamics related to multiple possibilities for anisotropic contraction. In the present work we illuminate our point using the homogeneous and anisotropic Bianchi-I type of spacetime. It is shown that there can be contraction phases in Bianchi-I metric where specifying an unique scale-factor for contraction does not always yield a unique cosmological development. This result is possible in $f(R)$

\(^1\)For a general understanding of modified gravity theories one can look at the review in Ref. [21].
cosmology and in GR one cannot have this property. This non-uniqueness of cosmological development corresponding to a specific scale-factor opens up a new problem as cosmological evolution becomes more complex conceptually and as consequence only simple $f(R)$ models can be semi-analytically solved. Any $f(R)$ model which is a higher order polynomial in $R$ compared to the quadratic $f(R)$ model requires a complete numerical solution for anisotropy growth. The present work focusses on two main issues. In the first half we develop the formalism for anisotropic cosmological dynamics in Bianchi-I spacetimes in full generality and and in the second half we show the potential of our formalism in quadratic $f(R)$ theory. To make our points clear we have chosen quadratic gravity as the model for gravitation and then we have worked with a specific power law scale-factor during the contraction phase of the universe. Both of the assumptions do have relevance as quadratic gravity does accommodate a cosmological bounce and power law dependence of scale-factors are the simplest, and many times physically relevant, cosmological solutions. In general the assumptions stated above may be simplistic but the results we present show the intrinsic nature of the complexities of anisotropy growth in Bianchi-I spacetimes and we expect all other models of $f(R)$ with more general time dependence of scale-factors will show qualitatively similar results as presented in this paper. Our work may be taken as an effective toy model which is used to crack a formidable problem in cosmological dynamics. To our understanding the above mentioned topics are discussed for the first time in full generality in the present paper.

The material in the paper is organized in the following way. The second section discusses about the basics of Bianchi-I spacetimes and sets the notations and conventions followed throughout the paper. In section 3 we present the general formalism of homogeneous and anisotropic cosmological dynamics in metric $f(R)$ cosmology. This part contains important results. In this section for the first time one comes across the complex nature of anisotropy development. In section 4 we present the results for quadratic $f(R)$ theory induced contraction phase where we have chosen the scale-factor during contraction to be like a power law function of time. The next section is the concluding section where we summarize the results obtained in the paper.
2 The anisotropic Bianchi-I metric and its properties

For our analysis, we have used the metric for Bianchi-I spacetime,

\[ ds^2 = -dt^2 + a_1^2(t)dx_1^2 + a_2^2(t)dx_2^2 + a_3^2(t)dx_3^2, \]

where \( a_1(t), \ a_2(t) \) and \( a_3(t) \) are the different scale factors, whose relative differences specify the amount of anisotropy in the evolving universe. Here the three scale factors along the three spatial directions are all different from each other. Existence of such anisotropic cosmological models in higher order gravity theories have been extensively studied in literature [29], [30], [31], [32]. In most of the earlier attempts the authors have tried to find out the nature of anisotropic spacetimes using various forms of anisotropic metric and using various forms of gravitational Lagrangians. In the present paper we show that the previous attempts have missed a vital ingredient in anisotropic expansion/contraction. The effect we discuss is clearly visible in Bianchi-I spacetime, but we think similar effects must also be present in the other anisotropic cosmological models.

In this paper we will assume the presence of a perfect hydrodynamic fluid, in the Bianchi type-I spacetime, whose energy-momentum tensor (EMT)is given by

\[ T_{\mu\nu} = (\rho + P)u_\mu u_\nu + Pg_{\mu\nu}, \]

where \( \rho \) and \( P \) are the energy density and pressure of the perfect fluid. The 4-velocity of the fluid element is given by \( u_\mu \), which being a time-like vector is normalized as

\[ u^\mu u_\mu = -1. \]

Although the spacetime metric is anisotropic the fluid which pervades the spacetime is assumed to be isotropic. In this paper we will assume a barotropic equation of state for the perfect fluid,

\[ P = \omega \rho, \]

where \( \omega \) is supposed to be a constant.

One can rewrite the form of the anisotropic metric, given in Eq. (1), in terms of the (geometric) average of the three scale factors. The geometric average of the scale factors is given as

\[ a(t) = [a_1(t) a_2(t) a_3(t)]^{1/3}. \]
One can express the three different scale factors in terms of the geometric average scale factor in the following way,

\[ a_i(t) = a(t)e^{\beta_i(t)}, \tag{6} \]

where \( i = 1, 2, 3 \), with the constraint

\[ \beta_1 + \beta_2 + \beta_3 = 0. \tag{7} \]

Using the above relations one can now rewrite the metric given in Eq. (1) as

\[ ds^2 = -dt^2 + a^2(t) \left[ e^{2\beta_1(t)} dx_1^2 + e^{2\beta_2(t)} dx_2^2 + e^{2\beta_3(t)} dx_3^2 \right]. \tag{8} \]

In this notation one can define the Hubble parameter, as an arithmetic average, and its time-derivative as

\[ H(t) \equiv \frac{1}{3} \left( \frac{\dot{a}_1}{a_1} + \frac{\dot{a}_2}{a_2} + \frac{\dot{a}_3}{a_3} \right) = \frac{\dot{a}}{a}, \tag{9} \]

\[ \dot{H}(t) \equiv \frac{1}{3} \frac{d}{dt} \left( \frac{\dot{a}_1}{a_1} + \frac{\dot{a}_2}{a_2} + \frac{\dot{a}_3}{a_3} \right) = \frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2}. \tag{10} \]

Mainly for the sake of brevity, henceforth in this article we will omit the word average (either geometric or arithmetic) before scale-factor or Hubble parameter. The components of the Einstein tensor, \( G^\mu_\nu = R^\mu_\nu - \frac{1}{2} R \delta^\mu_\nu \), for the above metric are as follows

\[ G^0_0 = -3H^2 + \frac{1}{2} \left( \dot{\beta}_1^2 + \dot{\beta}_2^2 + \dot{\beta}_3^2 \right), \tag{11} \]

\[ G^i_i = - \left[ 2\dot{H} + 3H^2 - 3H \dot{\beta}_i - \ddot{\beta}_i + \frac{1}{2} \left( \dot{\beta}_1^2 + \dot{\beta}_2^2 + \dot{\beta}_3^2 \right) \right]. \tag{12} \]

In the above equation no summation over \( i \) is intended. The Ricci scalar turns out to be

\[ R = 6(\dot{H} + 2H^2) + (\dot{\beta}_1^2 + \dot{\beta}_2^2 + \dot{\beta}_3^2). \tag{13} \]

In the next section we formulate the anisotropic cosmological dynamics guided by metric \( f(R) \) theory. The fact that the derivatives of the anisotropy parameters are themselves present in the expression of the Ricci scalar will make anisotropic cosmological dynamics much more involved (compared to the general relativistic case).
3 Formulation of anisotropic cosmological dynamics guided by metric $f(R)$ theory

The field equation in $f(R)$ gravity is

$$G^\mu_{\nu} = \frac{\kappa}{f'(R)} [T^\mu_{\nu} + T^\mu_{\nu(\text{curv})}],$$

(14)

where $G^\mu_{\nu}$ is the Einstein tensor and $T^\mu_{\nu(\text{curv})}$ is the energy momentum tensor due to curvature. Here $\kappa = 8\pi G$, where $G$ is the Newton’s gravitational constant and is related to the Planck mass $M_P$ via $G = 1/M_P^2$. In the present paper we will approximately use $M_P \approx 10^{19}$ GeV. The prime on top right hand side of any function represents the ordinary derivative of that function with respect to the Ricci scalar $R$. In particular

$$T^\mu_{\nu(\text{curv})} \equiv \frac{1}{\kappa} \left[ -\left(\frac{R f'(R) - f(R)}{2} + \square f'(R)\right) \delta^\mu_{\nu} + g^{\alpha\beta} D_\alpha D_\beta f'(R) \right]$$

(15)

where $D_\mu A_\nu$ is the covariant derivative of the covariant 4-vector $A_\nu$ and $\square \equiv g^{\alpha\beta} D_\alpha D_\beta$. The 0 – 0 component of the field equation in $f(R)$ theory in an anisotropic spacetime is then given as,

$$G^0_0 = -\frac{\kappa}{f'(R)} (\rho + \rho_{\text{curv}}),$$

(16)

where

$$\rho_{\text{curv}} = \frac{1}{\kappa} \left[ \frac{R f'(R) - f(R)}{2} - 3H \dot{f}'(R) \right].$$

(17)

The other three equations, specifying the $i - i$ components become,

$$G^i_j = \frac{\kappa}{f'(R)} (T^i_j + T^i_j(\text{curv})),$$

(18)

where $i, j = 1, 2, 3$. Here $T^i_j = P \delta^i_j$ stands for pressure of the perfect hydrodynamic fluid(s) whose EMT(s) has(have) the same form as given in Eq. [2]. The form of $T^i_j(\text{curv})$ is given as

$$\kappa T^i_j(\text{curv}) = -\left[ \frac{R f'(R) - f(R)}{2} - \dot{R} f''(R) - \ddot{R} f'''(R) - 2H \dot{f}'(R) \right] \delta^i_j - B^i_j \dot{R} f''(R),$$

(19)

where

$$B^i_j = \dot{\beta}_i \delta^i_j.$$

(20)
In terms of the above quantities one can now write,

\[ G^i_j = \frac{\kappa}{f'(R)} (P + P_{\text{curv}}) \delta^i_j - B^i_j \dot{R} f''(R), \tag{21} \]

where

\[ P_{\text{curv}} = \frac{\dot{R}^2 f''' + 2H \dot{R} f'' + \ddot{R} f'' - R f' - f}{2\kappa}. \tag{22} \]

In terms of the Hubble parameter and the anisotropy parameter, Eq. (16), can be written as

\[ H^2 = \frac{\kappa}{3f'(R)} (\rho + \rho_{\text{curv}}) + \frac{1}{6} \sum_{i=1}^{3} \beta_i^2, \tag{23} \]

while Eq. (21) becomes

\[ 2\dot{H} + 3H^2 - 3H \dot{\beta}_i - \ddot{\beta}_i + \frac{1}{2} \sum_{i=1}^{3} \beta_i^2 = -\frac{\kappa}{f'(R)} (P + P_{\text{curv}}) + \dot{\beta}_i \dot{R} \frac{f''(R)}{f'(R)}. \tag{24} \]

Using Eq. (7) after some simple manipulations on the above equation one obtains

\[ \ddot{\beta}_i + \left(3H + \frac{\dot{f}'(R)}{f'(R)}\right) \dot{\beta}_i = 0, \tag{25} \]

whose solution is

\[ \dot{\beta}_i(t) = \frac{b_i}{a^3(t) f'(R)}, \tag{26} \]

where \( b_i \) are numerical constants and Eq. (7) predicts that

\[ \sum_{i=1}^{3} b_i = 0. \tag{27} \]

It must be noted, the expression of the Ricci scalar in the present case as given in Eq. (13) itself contains the squared sum of the \( \dot{\beta}_i \), as a result of which the solution given in Eq. (26) becomes circular. In general relativity this circular definition does not appear as there \( f'(R) = 1 \).

Adding the three equations in Eq. (24) (for each value of the index \( i \)) one obtains,

\[ 2\dot{H} + 3H^2 = -\frac{\kappa}{f'(R)} (P + P_{\text{curv}}) - \frac{1}{2} \sum_{i=1}^{3} \beta_i^2. \tag{28} \]
If one uses Eq. (23) in the above equation then one gets $\dot{H}$ as

$$\dot{H} = -\frac{k}{2f'(R)} [(1 + \omega)\rho + (\rho_{\text{curv}} + P_{\text{curv}})] - \frac{1}{2} \sum_{i=1}^{3} \dot{\beta}_i^2,$$  \hspace{1cm} (29)

where $P = \omega \rho$ has been used. Unlike the cosmological dynamics of general theory of relativity, $f(R)$ cosmology depends upon $\ddot{R}$ and $\dot{R}$. Because of the circular nature of the dynamical evolution equation of $\dot{\beta}_i$ one has to take special care to evaluate $\ddot{R}$ and $\dot{R}$ in anisotropic $f(R)$ cosmology. In the following part of this subsection we discuss how one can evaluate the time derivatives of the Ricci scalar which appear in Eq. (23) and Eq. (29).

In the present case we define the anisotropy factor $x$ as

$$x^2(t) \equiv \sum_{i=1}^{3} \dot{\beta}_i^2(t).$$ \hspace{1cm} (30)

For an isotropic universe $x^2 = 0$, implying that all the $\beta_i$’s are constant in time. In such a case one can appropriately make (time-independent) coordinate rescaling in an appropriate way to make the spacetime look exactly like the Friedmann-Lemaitre-Robertson-Walker (FLRW) spacetime. In Bianchi-I spacetime the above definition of the anisotropy factor satisfies a differential equation:

$$\dot{x} + \left(3H + \frac{\dot{f}'(R)}{f'(R)}\right)x = 0,$$ \hspace{1cm} (31)

whose (nontrivial) solution must be like

$$x = \frac{b}{a^3(t)f'(R)},$$ \hspace{1cm} (32)

where $b$ is a real integration constant. The above equation contains the most important theoretical ingredient of the present paper. As a consequence of the above relation in $f(R)$ gravity, one cannot define an unique anisotropy dynamics. For any given $f(R)$ gravity, in general multiple time evolutions of the anisotropy factor $x$ is possible, each corresponding to a different equation of state for the barotropic fluid. If one writes the Ricci scalar as

$$R = 6(\dot{H} + 2H^2) + x^2,$$ \hspace{1cm} (33)

then its derivative can be written as

$$\dot{R} = 6(\ddot{H} + 4H\dot{H}) - 6Hx^2 - 2\frac{\dot{R}f''(R)}{f'(R)}x^2.$$ 

The above equation yields,

\[ \dot{R} = \frac{6(\ddot{H} + 4H\dot{H} - Hx^2)}{\left(1 + 2\frac{f''(R)}{f'(R)}x^2\right)} . \] (34)

Working out similarly one can write,

\[ \ddot{R} = \frac{6(\dddot{H} + 4H\ddot{H} + 4\dot{H}^2) - 2\left[(3\dddot{H} + \frac{i^2f'''}{f'} - \frac{i^2}{f''}) - 2\left(3H + \frac{\dot{f}}{f'}\right)^2\right]x^2}{\left(1 + 2x^2\frac{f''}{f'}\right)} . \] (35)

The above equations show that once we know the form of \( x \) in terms of the scale factor, we can write the values of \( R, \dot{R} \) and \( \ddot{R} \) in terms of the scale factor. The cosmological dynamics of anisotropic \( f(R) \) theory is encoded in Eq. (29), Eq. (32) and the energy-momentum conservation equation

\[ \dot{\rho} + 3H\rho(1 + \omega) = 0 . \] (36)

This set of three equations and the initial conditions specifying \( a, \dot{a}, \ddot{a} \) and initial \( \rho \) are enough to specify the anisotropic dynamics in \( f(R) \) cosmology. It is to be noted that the above initial values do not include an initial condition on the anisotropy factor \( x \). In the next subsection we will show that the above initial conditions are enough for the present case.

### 3.1 Evolution of the anisotropic factor \( x(t) \) in quadratic gravity

In this paper we will focus on quadratic gravity where

\[ f(R) = R + \alpha R^2 , \] (37)

where \( \alpha \) is a real number. Although this is a simple form of \( f(R) \) but it can be used to model cosmological inflation as well as cosmological bounce for positive and negative signs of the constant \( \alpha \) respectively. In this section we will determine the evolution of \( x(t) \) in quadratic gravity. The technique of evolution of \( x(t) \) in higher order gravity will be similar but much more involved. For higher order gravity the polynomial order of the algebraic equation yielding the roots of \( x(t) \)
may be five (or higher) and consequently there does not exist any general algebraic formalism yielding those roots.

From Eq. (32) one can easily verify that the algebraic equation specifying \( x(t) \) in quadratic gravity is a cubic equation of the form:

$$ x^3 + A_1 x + A_2 = 0, $$

where

$$ A_1 = 6(\dot{H} + 2H^2) + \frac{1}{2\alpha}, $$

$$ A_2 = -\frac{b}{2\alpha a^3}. $$

The discriminant, \( \Delta \), specifying the roots and their properties is given by

$$ \Delta = -4A_1^3 - 27A_2^2. $$

If \( \Delta > 0 \) there will be three distinct real roots, if \( \Delta < 0 \) then there will be one real root (and two complex roots) and if \( \Delta = 0 \) there can be repeated real roots. The roots of Eq. (38) are as follows:

$$ x = \left\{ \begin{array}{l}
\frac{(2/3)^{1/3}A_1}{(-9A_2 + \sqrt{-3\Delta})^{1/3}} + \frac{(-9A_2 + \sqrt{-3\Delta})^{1/3}}{2^{1/3}3^{2/3}} \cdot \\
\frac{(1 \pm i\sqrt{3})A_1}{(-9A_2 + \sqrt{-3\Delta})^{1/3}} \cdot \\
\frac{2^{2/3}3^{1/3}}{(-9A_2 + \sqrt{-3\Delta})^{1/3}}.
\end{array} \right. $$

The solutions of Eq. (38) are specified in terms of the scale factor and its first and second time derivatives. For the sake of completeness we also present the roots of the cubic equation using trigonometric functions as,

$$ x = \left\{ \begin{array}{l}
2\sqrt{-\frac{A_1}{3}} \cos \left[ \frac{1}{3} \tan^{-1} \left( \frac{\sqrt{3}A_1}{-9A_2} \right) \right], \\
-2\sqrt{-\frac{A_1}{3}} \cos \left[ \frac{1}{3} \left( \pi \mp \tan^{-1} \left( \frac{\sqrt{3}A_1}{-9A_2} \right) \right) \right].
\end{array} \right. $$

In the numerical calculations we will use the above form of the roots as they are less cumbersome to handle when all the roots are real.

The set of field equations specified at the end of the last section has to be solved in order to completely specify the anisotropic cosmological dynamics in quadratic gravity. One can replace \( x \) appearing in the field equation given in Eq. (29) by the appropriate expression from the set of the roots specified in Eq. (42) (or Eq. (43)) and solve the fourth order nonlinear differential equation for the scale factor \( a \). If hydrodynamic matter is present then in conjunction
with Eq. (29) the energy-momentum conservation condition in Eq. (36) also has to be taken into account. The initial conditions on the scale factor, its time derivatives (up to third order) and energy density must satisfy the constraint equation given in Eq. (23). From this discussion it is seen that one does not require a separate initial condition for $x$ to describe the anisotropic cosmological dynamics in $f(R)$ gravity. The existence of the cubic equation for $x$ shows that under certain circumstance the system may have three distinct real roots and the specific form of $x(t)$, actually realized in the dynamic development, has to be chosen in a nontrivial way.

Before we start the application of the theory described so far we want to specify a unique feature about the anisotropy factor $x$ in $f(R)$ gravity. From Eq. (32) it is seen that the anisotropy factor depends upon $a$, $\dot{a}$, $\ddot{a}$ and the integration constant $b$. Moreover from the form of the cubic equation followed by $x$ it can be easily seen that out of the three roots one tends to vanishes when $b \to 0$ (irrespective of the values of the scale-factor and its derivatives), where as the other two roots in general do not tend to zero when $b$ becomes arbitrarily small. If all the roots are real then the root which vanishes when $b$ vanishes plays an important role as in this case one can tune the value of the initial anisotropy by tuning the value of $b$. In particular if one wants to have a very small initial anisotropy, for any value of the scale-factor and its derivatives, then one must choose the root which tends to zero when $b$ tends to zero. When the system admits only one real root then this root always tends to zero when $b$ tends to zero. In GR, when one deals with anisotropic Bianchi Type-I cosmology, the equation followed by the anisotropy factor is $x(t) = b/a^3(t)$ and hence no such complications arise. One can always make the anisotropy factor in GR tend to zero by appropriately tuning the value of $b$.

4 Power law contraction

In this section we discuss anisotropic contraction phase, guided by quadratic gravity. This kind of a contraction phase may precede a cosmological bounce. In the present case we will assume the existence of hydrodynamic matter and $\alpha < 0$ as these conditions are required for a subsequent bounce [33]. In GR it is known that anisotropy suppression during contraction phase requires the presence of an ultrastiff matter component with $\omega(= P/\rho) > 1$. The presence of an
ultrastiff matter component can produce a slow contraction phase where preexisting anisotropy is suppressed\(^2\). If the anisotropy during the contraction phase is not suppressed adequately then the contraction may lead to instability. In the present case we will see that a power law contraction phase may suppress initial anisotropy in quadratic \(f(R)\) cosmology. Many other possibilities of anisotropy development exists in the present case and all those various possibilities will be pointed out in the later part of this section.

We assume that during the contracting phase \(t < 0\) and bounce occurs at \(t = 0\). The power law contraction phase does not include the bounce. During the contracting phase the scale-factor decreases as

\[
a(t) \propto (-t)^n , \text{ where } 0 < n < 1 ,
\]

and consequently

\[
H = \frac{n}{t}.
\]

From physical considerations one can choose \(\alpha = -10^{12}\) in Planck units \(\text{[33]}\). In this unit system the standard value of a quantity, expressed in mass/energy units, is obtained by multiplying the value of the physical quantity by a particular power of Plank mass \(M_P\) as done in Refs. \(\text{[33,34]}\). The specific power corresponds to the mass dimension of the physical quantity. In the present case the actual value of \(|\alpha|\) is \(10^{12}M_P^{-2}\). Eliminating \(\rho\) in Eq. (29) by using Eq. (23) we get,

\[
2\dot{H} + 3H^2(1 + \omega) + \frac{\kappa}{f'}(P_{\text{curv}} - \omega\rho_{\text{curv}}) + \frac{1}{2}x^2(1 - \omega) = 0 .
\]

(45)

From the above equation one can write the expression of the barotropic ratio as

\[
\omega = \frac{(4\dot{H} + 6H^2 + x^2)f' + 2\kappa P_{\text{curv}}}{(x^2 - 6H^2)f' + 2\kappa \rho_{\text{curv}}}.
\]

(46)

In the present case the above equation yields the equation of state for the barotropic matter when one specifies the particular nature of the scale-factor.

Determining the form of the time evolution of anisotropy factor reduces to finding the root(s) of Eq. (32). The root structure obviously depends upon the mathematical form of \(f(R)\). If \(f(R)\) is of the logarithmic or exponential form then the equation becomes a transcendental equation whose root(s) will specify the nature of anisotropic evolution of the universe. One can have various phases of anisotropy development during a cosmological evolution depending upon the roots of Eq. (32). In this paper we will particularly focus on the contracting

\(^2\)Sometimes this phase of slow contraction under the dominance of a ultrastiff matter is called the ekpyrotic phase \(\text{[10]}\).
phase of the universe leading to a cosmic bounce. We will not carry our analysis exactly towards the bounce time, rather we will investigate the nature of the development of anisotropy during a power law contraction during which the scale-factor is as given in Eq. (44). In this model the cosmological bounce takes place at \( t = 0 \). The power law contraction does not extend up to the bounce time, it ends before \( t = 0 \). We present the results for the popular quadratic \( f(R) \) model which actually accommodates a cosmological bounce [33], [35], [36], [37]. The main reason of choosing quadratic gravity is twofold. The primary reason is that quadratic gravity actually can accommodate a cosmic bounce and the secondary reason is that the complexity of the anisotropy development process can be handled in a semi-analytic way. The nature of the anisotropic contraction phase predicted in this model will give a glimpse of the interesting effects of \( f(R) \) models of anisotropic contraction in the Bianchi-I spacetime. The plot in Fig. 1 shows the nature of the roots at time \( t = -10^{10} \) in Planck units. In this paper we represent the units of dimensional quantities in Planck units. The time period of contraction is chosen in such a way that all the constraints as

\[ f'(R) > 0 , \quad \rho > 0 , \]

are maintained during this phase of contraction. As the power law contraction can never lead to a bounce the constraints compel us to terminate the power law contraction process some time before the bounce and in this paper we use the time interval \(-10^{10} \leq t \leq -10^{7}\). The scale-factor during this time is assumed to be \( a(t) = (-t/10^{10})^n \) such that \( a(t = -10^{10}) = 1 \). The nature of the roots show that below a certain \( b \) value and above a certain \( b \) value there is only one real root. Near \( b = 0 \) the system admits three real roots of \( x(t) \). We have verified that

Figure 1: Possible values of the anisotropy factor at \( t = -10^{10} \).
the nature of the root structure, as specified in Fig. 1, does feebly depend upon $n$ in the interval $0 \leq n \leq 1$. The plot in Fig. 1 shows three branches in three colors. The middle green branch smoothly matches to the blue branch above and the orange one below. The green branch specifies a root of Eq. (38) which is real near $b = 0$ and gives rise to small values of anisotropy factor $x_0 = x(t = -10^{10})$ initially. We will call this root as the third root and it corresponds to the third root listed in Eq. (43). The orange and blue regions specify the other roots which are large for regions near $b = 0$. We will call the root specified by the orange region and the blue region as the second and first root. As time evolves the nature of the plot in Fig. 1 changes but the general structure of the plot always remains qualitatively similar as the one plotted at the initial time.

The dynamics of anisotropy growth depends upon the parameters $b$ and $n$. We can specify the region in the $b - n$ plane which gives rise to decreasing anisotropy. The plot in Fig. 2 shows such a region in the $b - n$ plane. The plot is done at $t = -10^{10}$, the initial time, when the region is most constrained. In Fig. 3 we show how $x^2/H^2$ varies in time if one uses any value of $b, n$ in the shaded region in Fig. 2. In particular we have chosen $b = 10^{-12}$ and $n = 1/4$. As the

$^3$The root with the positive sign after $\pi$ in the pair of roots appearing in the second line of on the right hand side of Eq. (43)

$^4$The second and the first roots corresponds to the corresponding terms on the right hand side of Eq. (43)
anisotropy factor \( x(t) \) is a dimensional quantity, and always appears as a square in the field equations, we have plotted \( x^2/H^2 \) to make the growth dimensionless. More over \( x^2/H^2 \) specifies the anisotropy to (spacetime) contraction which is a better qualifier of the effect of anisotropy growth. In Fig. 4 we show the region in the \( b-n \) plane which gives rise to increasing anisotropy, the plot is specified at \( t = -10^7 \), the end-time as during that time the region is most constrained. The corresponding plot in Fig. 5 actually shows how \( x^2/H^2 \) increases with time when we use some value of \( b, n \) lying in the shaded region in Fig. 4. In this case we have used \( b = 10^{-12} \) and \( n = 2/5 \). In all the plots we have actually used the third root of Eq. (38). The plot of the third root of Eq. (38), at the initial time, with respect to \( b \), is portrayed by the green curve in Fig. 1. There can be other plots for the first and second roots of Eq. (38) which correspond to large initial anisotropy.

As mentioned earlier the multiple possibilities for anisotropic contraction in metric \( f(R) \) gravity corresponds to different possible equations of state for the barotropic perfect hydrodynamic fluid pervading the contracting universe. The barotropic ratio for the hydrodynamic matter in Bianchi-I universe, where anisotropy development is shown in Fig. 3 is plotted in Fig. 6. Similarly the barotropic ratio for the hydrodynamic matter, where anisotropy development is shown in Fig. 5 is plotted in Fig. 7. The difference between them is mainly that in one case \( \omega \) is always greater than unity while in the other case \( \omega \) is always
smaller than unity.

There are many possibilities of spacetime contraction in the quadratic gravity model in the case where the scale-factor is given as a power law. Even for a particular $n$ one can change $b$ to get a completely different contraction or one can keep both $n$ and $b$ same and can generate three different contractions. The later possibility shows the effects of multiple initial conditions on complex anisotropy generation in Bianchi-I spacetime. In Fig. 1 one can see that near $b = 0$ all the three roots coexist giving rise to three different forms of cosmological contraction for identical values of $b, n$. The three roots of Eq. (38) have three different forms of time evolution. There can also be some values of $b, n$ which does not give rise to a well defined contraction. Such a pair is $b = 10^{-7}$ and $n = 1/4$, where one has to work with the second root of Eq. (38) as that is the only real root at the initial time as shown in Fig. 1. It happens that this root does not remain real through out the time period of our interest and such contractions cannot take place in quadratic gravity. In Fig. 8 we show the development of anisotropy for two cases where $x_0$ (the initial anisotropy) corresponding to the two curves have identical values of $b$ and $n$ and the curves correspond to two different roots of Eq. (38).

To plot the curves presented in Fig. 8 we used $n = 2/5$ and $b = 10^{-12}$. For this pair of $n, b$ values all the three roots exist initially. The blue curve represents the time evolution of the first root while the green dashed curve represents the time evolution of the third root. If one wants to plot the anisotropy development for the second root, for the same $b, n$ pair, then it turns out that $f' < 0$ during contraction and consequently the second root evolves with gravitational instability. As we are only interested for cosmological dynamics which respects $f' > 0$ and $\rho > 0$ we omit cosmological dynamics starting from the second root from our present discussion. The plots clearly show that although the magnitude of $x^2/H^2$ for
Figure 8: Development of anisotropy for two cases having same $n, b$ values. Here $(10^{-11}) \frac{x^2}{H^2}$ for first root (blue curve) and $\frac{x^2}{H^2}$ for third root (green dashed curve) when $n = 2/5, b = 10^{-12}$.

Figure 9: Equation of state for first root (blue curve) and third root (green dashed) curve for the case where $n = 2/5, b = 10^{-12}$. For details regarding the plots look at the text below.

the first root is much greater than $x^2/H^2$ corresponding to the third root, the initial large anisotropy decreases as time proceeds while the relatively small initial anisotropy corresponding to the third root increases with time. The next curve shows that the two cases of anisotropy development requires the presence of two different kind of fluids. Large initial anisotropy reduces in time in a universe filled with a dark-energy kind of fluid where $\omega \sim -1$, where as small anisotropy grows in the presence of a fluid with positive equation of state. These curves show that in Bianchi-I spacetime, under power law contraction, there are multiple possibilities of cosmological evolution for same $n, b$ values. The above discussion shows that there can be completely different kind of cosmological dynamics in Bianchi-I spacetimes, during the power law contraction phase, where all the cosmological phases may correspond to the same initial time to start, the same $n$ and same $b$.

5 Conclusion

This paper presents the general results for anisotropic cosmological development in Bianchi-I model in metric $f(R)$ gravity. The initial part of the paper develops the formalism which can be used to track cosmological development in homogeneous and anisotropic Bianchi-I model. The formalism developed is dynamically complete and can predict the development of all the relevant cosmological and fluid parameters in cosmological time. The methods developed in this paper can be applied to expanding as well as contracting phase of the universe. As anisotropy development demands special attention in the contracting phase in
cosmological models based on GR our aim was to see how the problem translates into $f(R)$ cosmology. As anisotropy reduces in the expanding phase in GR it does not mean that this rule will be generally followed in $f(R)$ cosmology as the equation predicting anisotropy growth is non-linear in nature and may have surprises in store. Our preliminary calculations predicts that inflation in quadratic $f(R)$ cosmology, in Bianchi-I spacetime, indeed suppresses anisotropy. The results related to inflationary models in anisotropic spacetimes in $f(R)$ theory will be presented in a separate publication. In this article we tried to verify whether anisotropy subsides in the $f(R)$ theory driven contraction phase. The result we obtain is complex and opens up new areas of research.

The important result in this paper is related to anisotropy development, during the contraction phase in metric $f(R)$ gravity, which is guided by a non-linear first order ordinary differential equation with time varying coefficients. We have chosen quadratic $f(R)$ theory to illustrate our results as in this case most of the calculations can be done analytically, for any other higher order polynomial $f(R)$ one has to use numerical methods to determine the solutions of the differential equation predicting anisotropy dynamics. Our work shows the qualitative nature of the cosmological system undergoing anisotropic contraction and we expect qualitatively similar but quantitatively much more formidable results for other complicated forms of $f(R)$. Even in the case of quadratic gravity the various results coming out from our formalism is non-trivial. We have pointed out that even when we restrict the cosmological dynamics by enforcing conditions as $f' > 0$ and $\rho > 0$ there appears various regions in the $n, b$ plane which gives rise to different kind of anisotropy growth. For some possible cosmological evolutions we see that anisotropy reduces with time while in many cases anisotropy increases with time during the contraction phase. Our work definitely shows that there are regions in the $n, b$ plane which gives rise to cosmological dynamics which leads to suppression of initial anisotropy. It remains a formidable task to scan the whole relevant $n, b$ plane for the three possible dynamical growths in quadratic gravity. The effect of anisotropy in Bianchi-I type spacetimes in $f(R)$ cosmology makes the theory complex. One can also use the formalism presented here for the anisotropic gravitational collapse leading to structure formation and there one can get interesting and verifiable results.

Before we conclude we want to specify the novelty of our result. If one specifies a scale-factor for the cosmological dynamics and uses a specific initial time for the cosmological evolution then one expects that the resulting cosmological dynamics
is unique. In our work it is shown that even if one chooses a specific initial time and a particular scale-factor the cosmological time evolution of the system can be non-unique. There are multiple branches of dynamical evolution in Bianchi-I spacetimes in $f(R)$ cosmology even for a unique scale-factor in the metric.

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