REVERSE STEIN-WEISS, HARDY-LITTLEWOOD-SOBOLEV, HARDY, SOBOLEV
AND CAFFErellI-KOHN-NIRENBERG INEQUALITIES ON HOMOGENEOUS
GROUPS

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ABSTRACT. In this note we prove the reverse Stein-Weiss inequality on general homogeneous Lie
groups. The obtained results extend previously known inequalities. Special properties of homoge-
neous norms and the reverse integral Hardy inequality play key roles in our proofs. Also, we show
reverse Hardy, Hardy-Littlewood-Sobolev, Sobolev and Caffarelli-Kohn-Nirenberg inequali-
ties on homogeneous groups.

1. INTRODUCTION

In one of their pioneering work [24], Hardy and Littlewood considered the one dimensional
fractional integral operator on $(0, \infty)$ given by

$$T_\lambda u(x) = \int_0^\infty \frac{u(y)}{|x-y|^\lambda} dy, \quad 0 < \lambda < 1,$$

and proved the following theorem:

**Theorem 1.1.** Let $1 < p < q < \infty$ and $u \in L^p(0, \infty)$ with $\frac{1}{q} = \frac{1}{p} + \lambda - 1$. Then

$$\|T_\lambda u\|_{L^q(0, \infty)} \leq C \|u\|_{L^p(0, \infty)},$$

where $C$ is a positive constant independent of $u$.

The $N$-dimensional analogue of (1.1) can be written by the formula:

$$I_\lambda u(x) = \int_{\mathbb{R}^N} \frac{u(y)}{|x-y|^\lambda} dy, \quad 0 < \lambda < N.$$

The $N$-dimensional case of Theorem 1.1 was extended by Sobolev in [44]:

**Theorem 1.2.** Let $1 < p < q < \infty$, $u \in L^p(\mathbb{R}^N)$ with $\frac{1}{q} = \frac{1}{p} + \frac{\lambda}{N} - 1$. Then

$$\|I_\lambda u\|_{L^q(\mathbb{R}^N)} \leq C \|u\|_{L^p(\mathbb{R}^N)},$$

where $C$ is a positive constant independent of $u$.

Later, in [46] Stein and Weiss obtained the following two-weight extension of the Hardy-Littlewood-
Sobolev inequality, which is known as the Stein-Weiss inequality or weighted Hardy-Littlewood-
Sobolev inequality.

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equality, homogeneous Lie group.

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**Theorem 1.3.** Let $0 < \lambda < N$, $1 < p < \infty$, $\alpha < \frac{N(p-1)}{p}$, $\beta < \frac{N}{q}$, $\alpha + \beta \geq 0$ and $\frac{1}{q} = \frac{1}{p} + \frac{\lambda + \alpha + \beta}{N} - 1$. If $1 < p \leq q < \infty$, then

\begin{equation}
||x|^{-\beta} I_\lambda u||_{L^q(\mathbb{R}^N)} \leq C ||x|^{\alpha} u||_{L^p(\mathbb{R}^N)},
\end{equation}

where $C$ is a positive constant independent of $u$.

The Hardy-Littlewood-Sobolev inequality on Euclidean spaces and the regularity of fractional integrals was studied in [10], [12], [26] and [30]. On the Heisenberg group, Folland and Stein in [14] obtained the Hardy-Littlewood-Sobolev inequality and in [23] the authors also proved an analogue of the Stein-Weiss inequality. In [18] the authors studied the Stein-Weiss inequality on the Carnot groups. On homogeneous Lie groups, the Hardy-Littlewood-Sobolev and Stein-Weiss inequalities were obtained in [43] and [21]. In [8] the author proved the Stein-Weiss inequality on the Euclidean half-space.

The reverse Hardy-Littlewood-Sobolev inequality in the Euclidean space was obtained in the works [9], [11] and [27]. In [5] and [4], the authors obtained the reverse Stein-Weiss inequality on the Euclidean space and half-space, respectively. In this paper, we first show the reverse Stein-Weiss inequality on the homogeneous groups. In the proof we use special properties of homogeneous norms of the homogeneous Lie groups and reverse integral Hardy inequality, which are playing key roles in our calculations. Thus, in Theorem 2.5 we establish the reverse Stein-Weiss inequality on general homogeneous groups based on the reverse integral Hardy inequalities with one negative exponent. In particular, the obtained result recovers the previously known results of Abelian (Euclidean), Heisenberg, Carnot groups since the class of the homogeneous Lie groups contains those and since we can work with an arbitrary homogeneous quasi-norm. Note that in this direction systematic studies of different functional inequalities on general homogeneous (Lie) groups were initiated by the paper [37]. We refer to this and other papers by the authors (e.g. [39]) for further discussions.

We also note that the best constant in the Hardy-Littlewood-Sobolev inequality on the Heisenberg group is now known, see Frank and Lieb [16] (in the Euclidean case this was done earlier by Lieb in [25]). The expression for the best constant depends on the particular quasi-norm used and may change for a different choice of a quasi-norm.

A multidimensional version of one of the well-known inequalities of G.H. Hardy is

\begin{equation}
\left\| \frac{f}{|x|} \right\|_{L^p(\mathbb{R}^N)} \leq \frac{p}{N-p} \| \nabla f \|_{L^p(\mathbb{R}^N)}, \quad 1 < p < N,
\end{equation}

where $f \in C_0^\infty(\mathbb{R}^N)$, $\nabla$ is the Euclidean gradient and constant $\frac{p}{N-p}$ is sharp. The Hardy inequality was intensively studied. For example, the Hardy inequalities were considered on the Euclidean space in [19], [20], on the Heisenberg group in [6], [7], on stratified groups in [3], [31, 32, 33], [36], [42], for the vector fields in [34] and [38], on homogeneous groups in [37, 39]. In [37] (see e.g., [35]), authors showed the Hardy inequality with radial derivative on homogeneous groups in the following form:

**Theorem 1.4.** Let $G$ be a homogeneous group of homogeneous dimension $Q$. Let $| \cdot |$ be a homogeneous quasi-norm on $G$. Let $1 < p < Q$. Let $f \in C_0^\infty(G \setminus \{0\})$ be a complex-valued function. Then

\begin{equation}
\left\| \frac{f}{|x|} \right\|_{L^p(G)} \leq \frac{p}{Q-p} \| R f \|_{L^p(G)}, \quad 1 < p < Q,
\end{equation}

where $R$ is the radial derivative.
where $\mathcal{R} = \frac{d}{d|x|}$ is the radial derivative. The constant $\frac{p}{Q-p}$ is sharp.

In Abelian case ($\mathbb{R}^N, +$) with $Q = N$ and $| \cdot | = | \cdot |_E$ where $| \cdot |_E$ is the standard Euclidean distance, from (1.7) we have

$$\left\| \frac{f}{|x|} \right\|_{L^p(\mathbb{R}^N)} \leq \frac{p}{N-p} \left\| \frac{x}{|x|} \cdot \nabla f \right\|_{L^p(\mathbb{R}^N)}, \quad 1 < p < N,$$

where $\nabla$ is the standard Euclidean gradient. By using the Cauchy-Schwartz inequality from the inequality (1.8), we get (1.6). In this note, we show the reverse Hardy inequality on homogeneous general Lie groups.

We are also interested in Sobolev and Caffarelli-Kohn-Nirenberg inequalities, let us recall them briefly. In the classical work of Sobolev, he showed the following inequality:

$$\| u \|_{L^{p^*}(\mathbb{R}^N)} \leq C \| \nabla u \|_{L^p(\mathbb{R}^N)}, \quad 1 < p < N,$$

where $p^* = \frac{Np}{N-p}$. The Sobolev inequalities on stratified groups were obtained in [39, 40, 41]. In [41], on homogeneous groups, the authors showed $L^p$-Sobolev inequality in the following form:

**Theorem 1.5.** Let $\mathbb{G}$ be a homogeneous group of homogeneous dimension $Q$. Let $| \cdot |$ be a homogeneous quasi-norm on $\mathbb{G}$. Let $1 < p < Q$. Let $f \in C^\infty_0(\mathbb{G} \setminus \{0\})$ be a complex-valued function. Then

$$\| f \|_{L^p(\mathbb{G})} \leq \frac{p}{Q} \| \mathbb{E} f \|_{L^p(\mathbb{G})}, \quad 1 < p < \infty,$$

where $\mathbb{E} = \frac{d}{d|x|}$ is the Euler operator. The constant $\frac{p}{Q}$ is sharp.

In Abelian case ($\mathbb{R}^N, +$) with $Q = N$ and $| \cdot | = | \cdot |_E$ where $| \cdot |_E$ is the standard Euclidean distance, from (1.7) we have

$$\| f \|_{L^p(\mathbb{R}^N)} \leq \frac{p}{N-p} \| x \cdot \nabla f \|_{L^p(\mathbb{R}^N)}, \quad 1 < p < \infty,$$

where $\nabla$ is the standard Euclidean gradient. In the Euclidean case, this $L^p$-Sobolev inequality has been considered in [28].

In their fundamental work [2], L. Caffarelli, R. Kohn and L. Nirenberg established:

**Theorem 1.6.** Let $N \geq 1$, and let $l_1, l_2, l_3, a, b, d, \delta \in \mathbb{R}$ be such that $l_1, l_2 \geq 1, l_3 > 0, 0 \leq \delta \leq 1$, and

$$\frac{1}{l_1} + \frac{a}{N}, \quad \frac{1}{l_2} + \frac{b}{N}, \quad \frac{1}{l_3} + \frac{\delta d + (1-\delta)b}{N} > 0.$$

Then,

$$\| x^{\delta d + (1-\delta)b} u \|_{L^{l_3}(\mathbb{R}^N)} \leq C \| x^{a \nabla u} \|_{L^{l_1}(\mathbb{R}^N)} \| x^{b \nabla u} \|_{L^{l_2}(\mathbb{R}^N)}^\delta, \quad u \in C^\infty_c(\mathbb{R}^N),$$

if and only if

$$\frac{1}{l_3} + \frac{\delta d + (1-\delta)b}{N} = \delta \left( \frac{1}{l_1} + \frac{a - 1}{N} \right) + (1 - \delta) \left( \frac{1}{l_2} + \frac{b}{N} \right),$$

$$a - d \geq 0, \quad \text{if} \quad \delta > 0,$$

$$a - d \leq 1, \quad \text{if} \quad \delta > 0 \quad \text{and} \quad \frac{1}{l_3} + \frac{\delta d + (1-\delta)b}{N} = \frac{1}{l_1} + \frac{a - 1}{N}.$$
where \( C \) is a positive constant independent of \( u \).

On homogeneous groups, \( L^{p}\)-Caffarelli-Kohn-Nirenberg inequality was obtained in [29].

**Theorem 1.7.** Let \( G \) be a homogeneous group of homogeneous dimension \( Q \) and let \( \alpha, \beta \in \mathbb{R} \). Then for all complex-valued functions \( f \in C_0^\infty(G \setminus \{0\}) \), \( 1 < p < \infty \), and any homogeneous quasi-norm \( | \cdot | \) on \( G \) we have

\[
\frac{|Q-\gamma|}{p} \left\| \frac{f}{|x|^\gamma} \right\|_{L^p(G)}^p \leq \left\| \frac{1}{|x|^\alpha} \mathcal{R}f \right\|_{L^p(G)} \left\| \frac{f}{|x|^\beta} \right\|_{L^p(G)}^{p-1},
\]

where \( \gamma = \alpha + \beta + 1 \). If \( \gamma \neq Q \) then the constant \( \frac{|Q-\gamma|}{p} \) is sharp.

The Caffarelli-Kohn-Nirenberg inequalities, on stratified and homogeneous groups were obtained in [39, 40, 41]. On homogeneous groups in [29], a general result which was obtained in particular cases gives the Hardy inequality (1.7) and \( L^{p}\)-Sobolev inequality (1.10).

In this note, we show the reverse Hardy, Hardy-Littlewood-Sobolev, \( L^{p}\)-Sobolev, \( L^{p}\)-Caffarelli-Kohn-Nirenberg inequalities on homogeneous Lie groups. The main idea for proving such reverse inequalities was developed in [37]. In addition, to the best of our knowledge, such reverse inequalities above on a homogeneous group \( G \) are new even in the Euclidean case.

Summarising our main results of the present note, we prove the following facts:

- The reverse Stein-Weiss and Hardy-Littlewood-Sobolev inequalities on homogeneous groups;
- The reverse Hardy inequality on homogeneous groups;
- The reverse \( L^{p}\)-Sobolev inequality on homogeneous groups;
- The reverse \( L^{p}\)-Caffarelli-Kohn-Nirenberg inequality on homogeneous groups.

## 2. MAIN RESULTS

Let us recall that a Lie group (on \( \mathbb{R}^N \)) \( G \) with the dilation

\[
D_\lambda(x) := (\lambda^{v_1}x_1, \ldots, \lambda^{v_N}x_N), \quad v_1, \ldots, v_N > 0, \quad D_\lambda : \mathbb{R}^N \to \mathbb{R}^N,
\]

which is an automorphism of the group \( G \) for each \( \lambda > 0 \), is called a homogeneous (Lie) group. For simplicity, throughout this paper we use the notation \( \lambda x \) for the dilation \( D_\lambda \). The homogeneous dimension of the homogeneous group \( G \) is denoted by \( Q := v_1 + \ldots + v_N \). Also, in this note we denote a homogeneous quasi-norm on \( G \) by \( |x| \), which is a continuous non-negative function

\[
|G| \ni x \mapsto |x| \in [0, \infty),
\]

with the properties

i) \( |x| = |x|^{-1} \) for all \( x \in G \),
ii) \( |\lambda x| = \lambda |x| \) for all \( x \in G \) and \( \lambda > 0 \),
iii) \( |x| = 0 \) iff \( x = 0 \).

Moreover, the following polarisation formula on homogeneous Lie groups will be used in our proofs: there is a (unique) positive Borel measure \( \sigma \) on the unit quasi-sphere \( \mathcal{S} := \{ x \in G : |x| = 1 \} \), so that for every \( f \in L^1(G) \) we have

\[
\int_G f(x)dx = \int_0^\infty \int_{\mathcal{S}} f(ry)r^{Q-1}d\sigma(y)dr.
\]
The quasi-ball centred at \( x \in \mathbb{G} \) with radius \( R > 0 \) can be defined by

\[
B(x, R) := \{ y \in \mathbb{G} : |x^{-1}y| < R \}.
\]

We refer to [15] for the original appearance of such groups, and to [13, 35] for a recent comprehensive treatment.

Let us consider the integral operator

\[
I_{\lambda, \mu} u(x) = |x|^\lambda \ast u = \int_{\mathbb{G}} |y^{-1}x|^\lambda u(y)dy, \quad \lambda > 0,
\]

where \( \ast \) is the convolution. Let us recall briefly the reverse Hölder’s inequality.

**Theorem 2.1** ([1], Theorem 2.12, p. 27). Let \( p \in (0, 1) \), so that \( p' = \frac{p}{p-1} < 0 \). If non-negative functions satisfy \( f \in L^p(\mathbb{G}) \) and \( 0 < \int_{\mathbb{G}} g^{p'}(x)dx < +\infty \), we have

\[
\int_{\mathbb{G}} f(x)g(x)dx \geq \left( \int_{\mathbb{G}} f^p(x)dx \right)^{\frac{1}{p'}} \left( \int_{\mathbb{G}} g^{p'}(x)dx \right)^{\frac{1}{p}}.
\]

Let us also recall a well-known fact about quasi-norms.

**Proposition 2.2** ([13], Theorem 3.1.39). Let \( \mathbb{G} \) be a homogeneous Lie group. Then there exists a homogeneous quasi-norm on \( \mathbb{G} \) which is a norm, that is, a homogeneous quasi-norm \( | \cdot | \) which satisfies the triangle inequality

\[
|x y| \leq |x| + |y|, \quad \forall x, y \in \mathbb{G}.
\]

Furthermore, all homogeneous quasi-norms on \( \mathbb{G} \) are equivalent.

The next theorem is the reverse integral version of Hardy inequalities on general homogeneous groups that will be instrumental in our proof.

**Theorem 2.3** ([22]). Let \( \mathbb{G} \) be a homogeneous group of homogeneous dimension \( Q \). Assume that \( p \in (0, 1) \) and \( q < 0 \). Suppose that \( W, U \geq 0 \) are locally integrable functions on \( \mathbb{G} \). Then the inequality

\[
\left[ \int_{\mathbb{G}} \left( \int_{B(0,|x|)} f(y)dy \right)^q W(x)dx \right]^{\frac{1}{q}} \geq C_1(p, q) \left( \int_{\mathbb{G}} f^p(x)U(x)dx \right)^{\frac{1}{p}}
\]

holds for some \( C_1(p, q) > 0 \) and all non-negative measurable functions \( f \), if and only if

\[
0 < A_1 := \inf_{x \neq a} \left[ \left( \int_{G\setminus B(0,|x|)} W(y)dy \right)^{\frac{1}{q}} \left( \int_{B(0,|x|)} U^{1-p'}(y)dy \right)^{\frac{1}{p'}} \right].
\]

Moreover, the biggest constant \( C_1(p, q) \) in (2.6) has the following relation to \( A_1 \):

\[
A_1 \geq C_1(p, q) \geq \left( \frac{p'}{p' + q} \right)^{-\frac{1}{q}} \left( \frac{q}{p' + q} \right)^{\frac{1}{p'}} A_1.
\]

Also, inequality

\[
\left[ \int_{G\setminus B(0,|x|)} \left( \int_{B(0,|x|)} f(y)dy \right)^q W(x)dx \right]^{\frac{1}{q}} \geq C_2(p, q) \left( \int_{\mathbb{G}} f^p(x)U(x)dx \right)^{\frac{1}{p}}
\]
holds for some $C_2(p,q) > 0$ and all non-negative measurable functions $f$, if and only if

$$0 < A_2 := \inf_{x \neq a} \left[ \left( \int_{B(0,|x|)} W(y)dy \right)^\frac{1}{q} \left( \int_{G \setminus B(0,|x|)} U^{1-p'}(y)dy \right)^\frac{1}{p'} \right].$$

Moreover, the biggest constant $C_2(p,q)$ in (2.9) has the following relation to $A_2$:

$$A_2 \geq C_2(p,q) \geq \left( \frac{p'}{p' + q} \right)^{-\frac{1}{q}} \left( \frac{1}{p'} + \frac{q}{p'} \right)^{-\frac{1}{p'}} A_2. \tag{2.11}$$

**Remark 2.4.** In our sense, the negative exponent $q < 0$ of 0, we understand in the following form:

$$0^q = (+\infty)^{-q} = +\infty, \quad \text{and} \quad 0^{-q} = (+\infty)^q = 0.$$

Now we formulate the reverse Stein-Weiss inequality on $G$.

**Theorem 2.5.** Let $G$ be a homogeneous group of homogeneous dimension $Q \geq 1$ and let $| \cdot |$ be an arbitrary homogeneous quasi-norm on $G$. Let $\lambda > 0$, $p, q' \in (0, 1)$, $0 \leq \alpha < -\frac{Q}{q'}$, $0 \leq \beta < -\frac{Q}{q'}$, $rac{1}{q'} + \frac{1}{p} = \frac{\alpha + \beta + \lambda}{Q} + 2$, where $\frac{1}{p} + \frac{1}{q'} = 1$ and $\frac{1}{q} + \frac{1}{q'} = 1$. Then for all non-negative functions $f \in L^{q'}(G)$ and $h \in L^p(G)$ we have

$$\int_G \int_G |x|^\alpha |y^{-1}x|^\beta f(x)h(y)|y|^\beta dxdy \geq C \|f\|_{L^{q'}(G)} \|h\|_{L^p(G)}, \tag{2.12}$$

where $C$ is a positive constant independent of $f$ and $h$.

**Corollary 2.6.** By setting $\alpha = \beta = 0$ we get the reverse Hardy-Littlewood-Sobolev inequality on the homogeneous groups, in the following form:

$$\int_G \int_G |y^{-1}x|^\beta f(x)h(y)dxdy \geq C \|f\|_{L^{q'}(G)} \|h\|_{L^p(G)}, \tag{2.13}$$

for all non-negative functions $f \in L^{q'}(G)$ and $h \in L^p(G)$ with $\lambda > 0$, $p, q' \in (0, 1)$, $\frac{1}{q'} + \frac{1}{p} = \frac{1}{q} + 2$, where $\frac{1}{p} + \frac{1}{q'} = 1$ and $\frac{1}{q} + \frac{1}{q'} = 1$.

**Remark 2.7.** In the Abelian (Euclidean) case $G = (\mathbb{R}^N, +)$, hence $Q = N$ and $| \cdot |$ can be any homogeneous quasi-norm on $\mathbb{R}^N$, in particular with the usual Euclidean distance, i.e. $| \cdot | = \| \cdot \|_E$, this was investigated in [4].

**Proof of Theorem 2.5.** By using reverse Hölder’s inequality with $\frac{1}{q} + \frac{1}{q'} = 1$ (Theorem 2.1) in (2.12), we calculate,

$$\int_G \int_G |x|^\alpha f(x)|y^{-1}x|^\beta h(y)|y|^\beta dxdy = \int_G \left( \int_G |x|^\alpha |y^{-1}x|^\beta h(y)|y|^\beta dy \right) f(x)dx \geq \left( \int_G \left( \int_G |x|^\alpha |y^{-1}x|^\beta h(y)|y|^\beta dy \right)^q dx \right)^\frac{1}{q} \|f\|_{L^{q'}(G)}^q.$$

So for (2.12), it is enough to show that

$$\left( \int_G \left( \int_G |x|^\alpha |y^{-1}x|^\beta h(y)|y|^\beta dy \right)^q dx \right)^\frac{1}{q} \geq C \|h\|_{L^p(G)},$$
and by changing \( u(y) = h(y)|y|^{\beta} \), this is equivalent to

\[
\int_G \left( \int_G |x|^q |y^{-1}x|^2 u(y) dy \right)^{\frac{q}{q-1}} dx \leq C \| |y|^{-\beta} u \|_L^q(G).
\]

We have that

\[
\int_G |x|^q |y^{-1}x|^2 u(y) dy \geq \int_{B(0, \frac{1}{2})} |x|^q |y^{-1}x|^2 u(y) dy,
\]

then

\[
\left( \int_G |x|^q |y^{-1}x|^2 u(y) dy \right)^{\frac{q}{q-1}} \leq \left( \int_{B(0, \frac{1}{2})} |x|^q |y^{-1}x|^2 u(y) dy \right)^{\frac{q}{q-1}}.
\]

Therefore, we obtain

\[
(2.14) \quad \left( \int_G |x|^q \left( \int_G |y^{-1}x|^2 u(y) dy \right)^q dx \right)^{\frac{1}{q}} \geq \left( \int_G |x|^q \left( \int_{\Gamma \setminus B(0, 2|x|)} |y^{-1}x|^2 u(y) dy \right)^q dx \right)^{\frac{1}{q}} := I_1^q.
\]

Similarly with (2.14), we have

\[
(2.15) \quad \left( \int_G |x|^q \left( \int_G |y^{-1}x|^2 u(y) dy \right)^q dx \right)^{\frac{1}{q}} \geq \left( \int_G |x|^q \left( \int_{\Gamma \setminus B(0, 2|x|}) |y^{-1}x|^2 u(y) dy \right)^q dx \right)^{\frac{1}{q}} := I_2^q.
\]

By summarising above facts, from (2.14)-(2.15), we have

\[
(2.16) \quad \left( \int_G |x|^q \left( \int_G |y^{-1}x|^2 u(y) dy \right)^q dx \right)^{\frac{1}{q}} \geq \frac{I_1^q}{2} + \frac{I_2^q}{2}.
\]

From now on, in view of Proposition 2.2 we can assume that our quasi-norm is actually a norm.

**Step 1.** Let us consider \( I_1 \). By using Proposition 2.2 and the properties of the quasi-norm with \( |y| \leq \frac{|x|}{2} \), we get

\[
(2.17) \quad |x| = |x^{-1}| = |x^{-1}yy^{-1}| \leq |x^{-1}y| + |y^{-1}| = |y^{-1}x| + |y| \leq |y^{-1}x| + \frac{|x|}{2}.
\]

Then for any \( \lambda > 0 \), we have

\[
2^{-\lambda} |x|^{\lambda} \leq |y^{-1}x|^\lambda.
\]

It means,

\[
2^{-\lambda} \int_{B(0, \frac{1}{2})} |x|^\lambda u(y) dy \leq \int_{B(0, \frac{1}{2})} |y^{-1}x|^\lambda u(y) dy,
\]

so that

\[
\left( \int_{B(0, \frac{1}{2})} |y^{-1}x|^\lambda u(y) dy \right)^q \leq 2^{-\lambda q} \left( \int_{B(0, \frac{1}{2})} |x|^\lambda u(y) dy \right)^q.
\]
Therefore, we get

\[ I_1 = \int_G |x| \alpha \left( \int_{B(0, \frac{|x|}{2})} |y^{-1} x| \beta u(y) dy \right) dx \]

\[ \leq 2^{-\lambda q} \int_G |x|^{(\alpha + \lambda)q} \left( \int_{B(0, \frac{|x|}{2})} u(y) dy \right) q \, dx. \]

If condition (2.7) in Theorem 2.3 with \( W(x) = |x|^{Q+\lambda q} \) and \( U(y) = |y|^{-\beta p} \) in (2.6) is satisfied, then we have

\[ I_1 \leq 2^{-\lambda q} \int_G \left( \int_{B(0, \frac{|x|}{2})} u(y) dy \right)^q |x|^{(\alpha + \lambda)q} dx \leq C_1 \| |y|^{-\beta p} \|^q_{L^p(G)}. \]

Let us verify condition (2.7). By using assumption \( \beta < -\frac{Q}{p'} \), we obtain

\[ \frac{1}{p} + \frac{1}{q'} = \frac{\alpha + \beta + \lambda}{Q} + 2 < \frac{\alpha + \lambda}{Q} - \frac{1}{p'} + 2, \]

that is, \( \frac{Q+\lambda q}{Q_q} > 0 \), then \( Q + (\alpha + \lambda)q < 0 \) and by using the polar decomposition (2.2):

\[ \left( \int_{G \setminus B(0, |x|)} W(y) dy \right)^{\frac{1}{q}} = \left( \int_{G \setminus B(0, |x|)} |y|^{(\alpha + \lambda)q} dy \right)^{\frac{1}{q}} \]

\[ = \left( \int_{|x|}^{\infty} \int_{r}^{\infty} r^{Q-1} r^{(\alpha + \lambda)q} dr d\sigma(\omega) \right)^{\frac{1}{q}} \]

\[ = \left( \left| \mathcal{E} \right| \int_{|x|}^{\infty} r^{Q-1+(\alpha + \lambda)q} dr \right)^{\frac{1}{q}} \]

\[ = \left( -\frac{\left| \mathcal{E} \right|}{Q + (\alpha + \lambda)q} \left| x \right|^{Q+(\alpha + \lambda)q} \right)^{\frac{1}{q}} \]

\[ = \left( \frac{\left| \mathcal{E} \right|}{|Q + (\alpha + \lambda)q|} \right)^{\frac{1}{q}} \left| x \right|^{\frac{Q+(\alpha + \lambda)q}{q}}. \]

Since \( \beta < -\frac{Q}{p'} \), we have

\[ -\beta p(1 - p') + Q > -\beta p(1 - p') - \beta p' = 0. \]
So, \(-\beta p(1 - p') + Q > 0\). Then, let us consider

\[
\left( \int_{B(0,|x|)} U^{1-p'}(y)dy \right)^{\frac{1}{p'}} = \left( \int_{B(0,|x|)} |y|^{-\beta p(1-p')}dy \right)^{\frac{1}{p'}}
\]

\[
= \left( \int_{0}^{|x|} \int_{\mathbb{S}} r^{-\beta p(1-p')} r^{Q-1} dr d\sigma(\omega) \right)^{\frac{1}{p'}}
\]

\[
=(|\mathbb{S}| \int_{0}^{1} r^{-\beta p(1-p')+Q-1} dr )^{\frac{1}{p'}}
\]

\[
= \left( \frac{|\mathbb{S}|}{Q - \beta p(1-p')} \right)^{\frac{1}{p'}} |x|^{\frac{\beta p(1-p')+Q}{p'}}.
\]

Therefore, we have

\[
A_1 = \inf_{x \neq 0} \left( \int_{G \cap B(0,|x|)} W(y)dx \right)^{\frac{1}{q}} \left( \int_{B(0,|x|)} U^{1-p'}(y)dy \right)^{\frac{1}{p'}}
\]

\[
= \left( \frac{|\mathbb{S}|}{Q + (\alpha + \lambda)q} \right)^{\frac{1}{q}} \left( \frac{|\mathbb{S}|}{Q - \beta p(1-p')} \right)^{\frac{1}{p'}} \inf_{x \neq 0} |x|^{\frac{(\alpha + \lambda)q + Q}{q} + \frac{\beta p(1-p')+Q}{p'}}
\]

\[
= \left( \frac{|\mathbb{S}|}{Q + (\alpha + \lambda)q} \right)^{\frac{1}{q}} \left( \frac{|\mathbb{S}|}{Q - \beta p(1-p')} \right)^{\frac{1}{p'}} \inf_{x \neq 0} |x|^{Q \left( \frac{\alpha + \lambda}{q} + \frac{1}{q} + \frac{\beta p(1-p')+Q}{p'} \right)}
\]

\[
= \left( \frac{|\mathbb{S}|}{Q + (\alpha + \lambda)q} \right)^{\frac{1}{q}} \left( \frac{|\mathbb{S}|}{Q - \beta p(1-p')} \right)^{\frac{1}{p'}} \inf_{x \neq 0} |x|^{Q \left( \frac{2-\frac{1}{q} - \frac{1}{q} + \frac{\alpha + \lambda}{q}}{q} \right)}
\]

\[
= \left( \frac{|\mathbb{S}|}{Q + (\alpha + \lambda)q} \right)^{\frac{1}{q}} \left( \frac{|\mathbb{S}|}{Q - \beta p(1-p')} \right)^{\frac{1}{p'}} > 0.
\]

Then by using (2.6), we obtain

\[
I_1 \leq 2^{-\lambda q} \int_{G} |x|^{(\alpha + \lambda)q} \left( \int_{B(0,|x|/2)} u(y)dy \right)^{q} \leq 2^{-\lambda q} C_1 \|y\|^{-\beta u}\|u\|^q_{L^p(G)},
\]

so that

\[
I_1^{\frac{1}{q}} \geq 2^{-\lambda} C_1 \|y\|^{-\beta u}\|u\|_{L^p(G)} = 2^{-\lambda} C_1 \|h\|_{L^p(G)}.
\]

**Step 2.** As in the previous case \(I_1\), now we consider \(I_2\). From \(2|x| \leq |y|\), we calculate

\[
|y| = |y^{-1}| = |y^{-1}xx^{-1}| \leq |y^{-1}x| + |x| \leq |y^{-1}x| + \frac{|y|}{2},
\]

that is,

\[
\frac{|y|}{2} \leq |y^{-1}x|.
\]
Then, if condition (2.10) with $W(x) = |x|^a$ and $U(y) = |y|^{-(\beta + \lambda)p}$ is satisfied, then we have

$$I_2 = \int_G \left( \int_{G \setminus B(0,2|x|)} |x|^a |y|^{-1} |x|^{\lambda} u(y) dy \right)^q dx \leq 2^{-\frac{aq}{Q}} \int_G |x|^{aq} \left( \int_{G \setminus B(0,2|x|)} u(y) |y|^{\lambda} dy \right)^q dx \leq 2^{-\frac{aq}{Q}} \| |y|^{-\beta} u \|_{L^q(G)}^q.$$  

Now let us check condition (2.10). We have

$$\left( \int_{B(0,|x|)} W(y) dy \right)^{\frac{1}{q'}} = \left( \int_{B(0,|x|)} |y|^{aq} dy \right)^{\frac{1}{q'}} = \left( \int_0^{|x|} \int_{\mathbb{R}} r^{aq} r Q^{-1} dr d\sigma(\omega) \right)^{\frac{1}{q'}} = \left( \frac{|\mathbb{S}|}{Q + aq} \right)^{\frac{1}{q'}} |x|^{\frac{Qaq}{Q}}.$$  

where $Q + aq > 0$. From $\alpha \leq -\frac{Q}{q}$, we have

$$\frac{1}{q'} + \frac{1}{p} = \frac{\alpha + \beta + \lambda}{Q} + 2 < -\frac{1}{q} + \frac{\beta + \lambda}{Q} + 2 = \frac{\beta + \lambda}{Q} + 1 + \frac{1}{q'},$$  

then

(2.22) \quad $(\beta + \lambda)p' + Q < 0.$

By using this fact, we have

$$\left( \int_{G \setminus B(0,|x|)} U^{1-p'} (y) dy \right)^{\frac{1}{q'}} = \left( \int_{G \setminus B(0,|x|)} |y|^{-(\beta + \lambda)(1-p')p} dy \right)^{\frac{1}{q'}} = \left( \int_{|x|}^{\infty} \int_{\mathbb{R}} r Q^{-1} r^{-(\beta + \lambda)(1-p')p} dr d\sigma(\omega) \right)^{\frac{1}{q'}} = \left( \frac{|\mathbb{S}|}{Q - (\beta + \lambda)(1-p')p} \right)^{\frac{1}{q'}} |x|^{\frac{Q(\beta + \lambda)p'}{p'}}.$$  

(2.22) \quad $(\beta + \lambda)p' + Q < 0.$

By using this fact, we have

$$\left( \int_{G \setminus B(0,|x|)} U^{1-p'} (y) dy \right)^{\frac{1}{q'}} = \left( \int_{G \setminus B(0,|x|)} |y|^{-(\beta + \lambda)(1-p')p} dy \right)^{\frac{1}{q'}} = \left( \frac{|\mathbb{S}|}{Q - (\beta + \lambda)(1-p')p} \right)^{\frac{1}{q'}} |x|^{\frac{Q(\beta + \lambda)p'}{p'}}.$$  

(2.22) \quad $(\beta + \lambda)p' + Q < 0.$
Combining these facts we have

\[
A_2 = \inf_{x \neq a} \left( \int_{B(0, |x|)} W(y) \, dx \right)^{\frac{1}{q}} \left( \int_{G \setminus B(0, |x|)} U^{1-\mu} (y) \, dx \right)^{\frac{1}{\mu'}} \\
= \left( \frac{|G|}{Q + a_q} \right)^{\frac{1}{q}} \left( \frac{|G|}{Q + (\beta + \lambda) p'} \right)^{\frac{1}{\mu'}} \inf_{x \neq a} |x|^\frac{Q + a_q}{\mu'} + \frac{Q + (\beta + \lambda) p'}{\mu'} \\
= \left( \frac{|G|}{Q + a_q} \right)^{\frac{1}{q}} \left( \frac{|G|}{Q + (\beta + \lambda) p'} \right)^{\frac{1}{\mu'}} \inf_{x \neq a} |x|^\frac{Q + a_q + Q + (\beta + \lambda) p'}{\mu'} \\
= \left( \frac{|G|}{Q + a_q} \right)^{\frac{1}{q}} \left( \frac{|G|}{Q + (\beta + \lambda) p'} \right)^{\frac{1}{\mu'}} \inf_{x \neq a} |x|^O \left( \frac{Q + a + Q + \beta + \lambda}{\mu'} \right) \\
= \left( \frac{|G|}{Q + a_q} \right)^{\frac{1}{q}} \left( \frac{|G|}{Q + (\beta + \lambda) p'} \right)^{\frac{1}{\mu'}} \inf_{x \neq a} |x|^O \left( \frac{2^\frac{1}{q} - \frac{1}{q} + \frac{a + \beta + \lambda}{\mu'}}{\mu'} \right) \\
= \left( \frac{|G|}{Q + a_q} \right)^{\frac{1}{q}} \left( \frac{|G|}{Q + (\beta + \lambda) p'} \right)^{\frac{1}{\mu'}} > 0.
\]

Therefore, we have

\[
I_2 = \int_G \left( \int_{G \setminus B(0, |x|)} |x|^q u(y) |y|^{-\frac{1}{q}} x^\frac{1}{2} \, dy \right)^q \, dx \leq 2^{-\lambda q} C_2^q \| y |^{-\mu} u \|_{L^p (G)}^q.
\]

Then, we have

\[
I_2^{\frac{1}{q}} \geq 2^{-\lambda} C_2^q \| y |^{-\mu} u \|_{L^p (G)} = 2^{-\lambda} C_2^q \| h \|_{L^p (G)}.
\]

Finally, by using (2.21) and (2.24) in (2.16), we obtain

\[
\left( \int_G |x|^{aq} \left( \int_G |y|^{-\frac{1}{q}} x^\frac{1}{2} u(y) \, dy \right)^q \, dx \right)^{\frac{1}{q}} \geq \frac{I_2^{\frac{1}{q}} + I_2^{\frac{1}{2}}}{2} \\
\geq \frac{2^{-\lambda}(C_1 + C_2)}{2} \| y |^{-\mu} u \|_{L^p (G)} \\
= \frac{2^{-\lambda}(C_1 + C_2)}{2} \| y |^{-\mu} u \|_{L^p (G)} \\
= C_3 \| y |^{-\mu} u \|_{L^p (G)},
\]

where \( C_3 = \frac{2^{-\lambda}(C_1 + C_2)}{2} > 0. \)

Theorem 2.5 is proved.

Let us give improved reverse Stein-Weiss inequality.

**Theorem 2.8.** Let \( G \) be a homogeneous group of homogeneous dimension \( Q \geq 1 \) and let \( | \cdot | \) be an arbitrary homogeneous quasi-norm on \( G \). Let \( \lambda > 0, p, q' \in (0, 1) \) and \( \frac{1}{q} + \frac{1}{p'} = \frac{a + \beta + \lambda}{Q} + 2 \), where \( \frac{1}{p} + \frac{1}{p'} = 1, \frac{1}{q} + \frac{1}{q'} = 1 \). Then for all non-negative functions \( f \in L^{q'}(G) \) and \( h \in L^p(G) \), inequality
(2.12) holds, that is,

$$
\int_G \int_G |x|^a |y^{-1} x|^f f(x) h(y) |y|^\beta \, dx \, dy \geq C \|f\|_{L^p(G)} \|h\|_{L^p(G)},
$$

if one of the following conditions is satisfied:

(a) $0 \leq \alpha < -\frac{Q}{q}$,

(b) $0 \leq \beta < -\frac{Q}{p}$.

**Proof.** Let us prove (a). By using some notations from proof of Theorem 2.5 and (2.16), we obtain

$$
(2.26) \quad \left( \int_G |x|^q \left( \int_G |y^{-1} x|^f u(y) dy \right)^q dx \right)^{1/q} \geq I_2^{1/q},
$$

and by using Step 2 in the proof of Theorem 2.5 and from (2.24), we have $I_2^{1/q} \geq C \|y|^{-\beta} u\|_{L^p(G)}$, then we get

$$
(2.27) \quad \left( \int_G |x|^q \left( \int_G |y^{-1} x|^f u(y) dy \right)^q dx \right)^{1/q} \geq I_2^{1/q} \geq C \|y|^{-\beta} u\|_{L^p(G)}.
$$

Let us prove (b). By (2.16), we obtain

$$
(2.28) \quad \left( \int_G |x|^q \left( \int_G |y^{-1} x|^f u(y) dy \right)^q dx \right)^{1/q} \geq I_1^{1/q},
$$

and by using Step 1 in the proof of Theorem 2.5 and from (2.14), we have $I_1^{1/q} \geq C \|y|^{-\beta} u\|_{L^p(G)}$, then we get

$$
(2.29) \quad \left( \int_G |x|^q \left( \int_G |y^{-1} x|^f u(y) dy \right)^q dx \right)^{1/q} \geq I_1^{1/q} \geq C \|y|^{-\beta} u\|_{L^p(G)}.
$$

Let us give reverse Hardy, $L^p$-Sobolev and $L^p$-Caffarelli-Kohn-Nirenberg inequalities on $G$. Assume now that $f$ is a radially decreasing function, i.e., $R_f := \frac{d}{d|x|} f < 0$. Let us give the reverse Hardy inequality on homogeneous Lie groups, the reverse to Theorem 1.4.

**Theorem 2.9** (Reverse Hardy inequality). Let $G$ be a homogeneous Lie group with homogeneous dimension $Q \geq 1$. Assume that $p \in (0, 1)$. Then for any non-negative, real-valued and radially decreasing function $f \in C_0^\infty(G \setminus \{0\})$, we have

$$
(2.30) \quad \left\| \frac{f}{|x|} \right\|_{L^p(G)} \geq \frac{p}{Q-p} \|R_f\|_{L^p(G)},
$$
Proof. Let us denote $\mathcal{R}_1 = -\mathcal{R}$, so that we have $\mathcal{R}_1 f > 0$. By using polar decomposition (2.2), integration by parts and reverse Hölder’s inequality, we obtain
\[
\int_G f^p(x) \frac{dx}{|x|^p} = \int_0^\infty \int_{\mathbb{R}^n} f^p(ry) r^{Q-1} dr d\sigma(y)
\]
\[
= \frac{p}{Q-p} \int_G f^{p-1}(x)|x|^{-p} \mathcal{R} f(x) dx
\]
(2.31)
\[
= \frac{p}{Q-p} \int_G f^{p-1}(x)|x|^{-p} \mathcal{R}_1 f(x) dx
\]
\[
\geq \frac{p}{Q-p} \| f \|_{L^p(G)} \| \mathcal{R}_1 f \|_{L^p(G)}.
\]
This gives
\[
\| \frac{f}{|x|} \|_{L^p(G)} \geq \frac{p}{Q-p} \| \mathcal{R}_1 f \|_{L^p(G)},
\]
implying (2.30).

Let us define by $\mathcal{E} = |x| \mathcal{R}$ the Euler operator. Then we have the reverse $L^p$-Sobolev inequality, the reverse to Theorem 1.5.

**Theorem 2.10** (Reverse $L^p$-Sobolev inequality). Let $G$ be a homogeneous Lie group with homogeneous dimension $Q \geq 1$. Assume that $p \in (0, 1)$. Then for any non-negative, real-valued and radially decreasing function $f \in C_0^\infty(G \setminus \{0\}$), we have
\[
\| f \|_{L^p(G)} \geq \frac{p}{Q} \| \mathcal{E} f \|_{L^p(G)}.
\]

Proof. Let us denote $\mathcal{E}_1 = |x| \mathcal{R}_1$, so that $\mathcal{E}_1 f > 0$. By using polar decomposition (2.2), integration by parts and reverse Hölder’s inequality, we obtain
\[
\int_G f^p(x) dx = \int_0^\infty \int_{\mathbb{R}^n} f^p(ry) r^{Q-1} dr d\sigma(y)
\]
\[
= \frac{p}{Q} \int_G f^{p-1}(x)|x| \mathcal{R} f(x) dx
\]
(2.34)
\[
= \frac{p}{Q} \int_G f^{p-1}(x)|x| \mathcal{R}_1 f(x) dx
\]
\[
= \frac{p}{Q} \int_G f^{p-1}(x) \mathcal{E}_1 f(x) dx
\]
\[
\geq \frac{p}{Q} \| f \|_{L^p(G)} \| \mathcal{E}_1 f \|_{L^p(G)}.
\]
This gives
\[
\| f \|_{L^p(G)} \geq \frac{p}{Q} \| \mathcal{E}_1 f \|_{L^p(G)},
\]
implying (2.33).

Let us give the reverse $L^p$-Caffarelli-Kohn-Nirenberg inequality on $G$, that is, the reverse to Theorem 1.7.
**Theorem 2.11** (Reverse $L^p$-Caffarelli-Kohn-Nirenberg inequality). Let $\mathbb{G}$ be a homogeneous Lie group with homogeneous dimension $Q \geq 1$. Assume that $p \in (0, 1)$. Then for any nonnegative, real-valued and radially decreasing function $f \in C_0^\infty(\mathbb{G} \setminus \{0\})$, we have

$$(2.36) \quad \left\| \frac{f}{|x|^\beta} \right\|_{L^p(\mathbb{G})}^p \geq \frac{p}{Q - \gamma} \left\| \frac{Rf}{|x|^a} \right\|_{L^p(\mathbb{G})} \left\| \frac{f}{|x|^\beta} \right\|_{L^p(\mathbb{G})}^{p-1},$$

for all $\alpha, \beta \in \mathbb{R}$ and $\gamma = \alpha + \beta + 1$, such that $Q > \gamma$.

**Proof.** By using polar decomposition (2.2), integration by parts and reverse Hölder’s inequality, we obtain

$$\int_\mathbb{G} \frac{f^p(x)}{|x|^\gamma} \, dx = \int_0^\infty \int_{\mathbb{S}} \frac{f^p(ry)}{r^\gamma} r^{Q-1} \, dr \, d\sigma(y)$$

$$= -\frac{p}{Q - \gamma} \int_\mathbb{G} \frac{f^p(x)}{|x|^{r-1}} Rf(x) \, dx$$

$$= \frac{p}{Q - \gamma} \int_\mathbb{G} \frac{f^p(x)}{|x|^{a+\beta}} R_1f(x) \, dx$$

$$= \frac{p}{Q - \gamma} \int_\mathbb{G} \frac{f^p(x)}{|x|^\beta} \frac{R_1f}{|x|^a} \, dx$$

$$(2.37) \geq \frac{p}{Q - \gamma} \left\| \frac{f}{|x|^\beta} \right\|_{L^p(\mathbb{G})}^p \left\| \frac{R_1f}{|x|^a} \right\|_{L^p(\mathbb{G})}$$

$$= \frac{p}{Q - \gamma} \left\| \frac{f}{|x|^\beta} \right\|_{L^p(\mathbb{G})}^p \left\| \frac{R_1f}{|x|^a} \right\|_{L^p(\mathbb{G})}$$

This gives

$$(2.38) \quad \left\| \frac{f}{|x|^\beta} \right\|_{L^p(\mathbb{G})}^p \geq \frac{p}{Q - \gamma} \left\| \frac{R_1f}{|x|^a} \right\|_{L^p(\mathbb{G})} \left\| \frac{f}{|x|^\beta} \right\|_{L^p(\mathbb{G})}^{p-1},$$

which implies (2.36).

**Remark 2.12.** In (2.36), if we take $\gamma = p$ and $\alpha = 0$, then we have the reverse Hardy inequality. Also, if we take $\gamma = 0$ and $\beta = 0$, then we have the reverse $L^p$-Sobolev inequality.

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