Exact solutions of the generalized $K(m, m)$ equations

Nikolay A. Kudryashov, Svetlana G. Prilipko

Department of Applied Mathematics, National Research Nuclear University MEPHI, 31 Kashirskoe Shosse, 115409 Moscow, Russian Federation

Abstract

Family of equations, which is the generalization of the $K(m, m)$ equation, is considered. Periodic wave solutions for the family of nonlinear equations are constructed.

1 Introduction

Seeking to understand the role of nonlinear dispersion in the formation of patterns in liquid drops in 1993 Rosenau and Hyman [1] introduced a family of fully nonlinear $K(m, n)$ equations and also presented solutions of the $K(2, 2)$ equation to illustrate the remarkable behavior of these equations. The $K(m, n)$ equations have the property that for certain $m$ and $n$ their solitary wave solutions have compact support. That is, they vanish identically outside a finite core region. These properties have a wide application in the fields of Physics and Mathematics, such as Nonlinear Optics, Geophysics, Fluid Dynamics and others. Later, this equation was studied by various scientists worldwide [2–14].

In this paper we construct periodic wave solutions for the following family of nonlinear partial differential equations

$$\frac{\partial u}{\partial t} + \sum_{k=0}^{N} \alpha_k \frac{\partial^{2k+1} u^m}{\partial x^{2k+1}} = 0, \quad N \geq 1, \quad m \neq 1, \quad \alpha_k \neq 0. \quad (1)$$

The equation (1) is of order $2N + 1$ and depends on $N + 2$ parameters denoted by $\alpha_0, \ldots, \alpha_N, m$. This family contains a number of well-known
generalizations of partial differential equations which were considered before [15–29].

This paper is organized as follows. In Section 2 we describe a method which enables one to construct periodic wave solutions for the concerned family of nonlinear partial differential equations. In Sections 3-6 we give several specific examples for some meanings of N.

2 Method applied

Applying traveling-wave variable
\[ u(x,t) = y(z) \quad z = x - C_0 t \] (2)
to Eq.(1) and integrating the results yield the following Nth-order equation
\[
\sum_{k=0}^{N} \alpha_k \frac{d^{2k} y^m}{dz^{2k}} - C_0 y = 0, \quad N \geq 1, \quad m \neq 1, \quad \alpha_k \neq 0.
\] (3)

The constant of integration is set to be zero. Substituting \(y(z) = F(z)^p\) into
\[
\alpha_n \frac{d^{2N} y^m}{dz^{2N}} - C_0 y = 0
\]
we have \(p = \frac{2N}{m-1}\). Note that Eq.(3) is an autonomous equation, and we can substitute \(z\) to \((z - z_0)\). We will take this fact into account in final solution, but we omit this substitution in our calculations. We search solutions of Eq.(3) in the form
\[
y(z) = (A_N)^{\frac{1}{m-1}} \cos^{\frac{2N}{m-1}} (B_N(z - z_0)).
\] (4)

There is a remarkable property of a function \(\cos(B_N z)\). First of all we have to show expansion terms of Eq.(3).

In the case \(k = 1\) we have the following expression
\[
\frac{d^2}{dz^2} \cos^{\frac{2m}{m-1}} (B_1 z) = -\frac{(2mB_1)^2 (m - 1)^2}{(m - 1)^2} \cos^{\frac{2m}{m-1}} (B_1 z) + \frac{2m (m + 1) B_1^2 (m - 1)^2}{(m - 1)^2} \cos^{\frac{2m}{m-1}} (B_1 z).
\] (5)
In the case \( k = 2 \) we obtain
\[
\frac{d^4}{dz^4} \cos^{2m-1} \left( B_2 z \right) = \frac{(4mB_2)^4}{(m-1)^4} \cos^{2m-1} \left( B_2 z \right) - 16mB_2^4(15m^3 + 11m^2 + 5m + 1) \cos^{2(m+1)} \left( B_2 z \right) + \frac{8mB_4^4(3m + 1)(m + 3)(m + 1)}{(m-1)^4} \cos^{2m-1} \left( B_2 z \right). 
\] (6)

In the case \( k = 3 \) we get
\[
\frac{d^6}{dz^6} \cos^{2m-1} \left( B_3 z \right) = -\frac{(6mB_3)^6}{(m-1)^6} \cos^{2m-1} \left( B_3 z \right) + 96mB_3^6(5m + 1)(7m^2 + m + 1)(19m^2 + 7m + 1) \cos^{2(m+1)} \left( B_3 z \right) - \frac{144mB_3^6(2m + 1)(5m + 1)(m + 1)(14m^2 + 8m + 5)}{(m-1)^6} \cos^{2(m+2)} \left( B_3 z \right) + \frac{72mB_3^6(5m + 1)(2m + 1)(m + 5)(m + 2)(m + 1)}{(m-1)^6} \cos^{2m-1} \left( B_3 z \right). 
\] (7)

In the general case \( k = N \) derivative takes the form
\[
\frac{d^{2N}}{dz^{2N}} \cos^{2m-1} \left( B_N z \right) = (-1)^N \frac{(2NmB_N)^{2N}}{(m-1)^{2N}} \cos^{2Nm} \left( B_N z \right) + \left( -1 \right)^{N+1} \frac{B_N^{2N} M_1^{2N}}{(m-1)^{2N}} \cos^{2(Nm-1)(m-1)} \left( B_N z \right) + \frac{B_N^{2N} M_2^{2N}}{(m-1)^{2N}} \cos^{2(Nm-2N)} \left( B_N z \right) + \frac{B_N^{2N} M_3^{2N}}{(m-1)^{2N}} \cos^{2(Nm-3N+1)} \left( B_N z \right) + \text{...}
\] (8)

where \( M_1^{2N}, \ldots, M_N^{2N} \) are polynomials of \( 2N \) power.
Substituting Eq. (8) into Eq. (3) we obtain the expression

\[ A_N \left( \alpha_0 - \frac{(2NmB_N)^2}{(m-1)^2} \alpha_1 + \frac{(2NmB_N)^4}{(m-1)^4} \alpha_2 - \cdots + (-1)^N \frac{(2NmB_N)^{2N}}{(m-1)^{2N}} \alpha_N \right) \cos^{\frac{2N}{m-1}}(B_N z) + \]

\[ + A_N \left( \frac{B_N^2 M_1^2}{(m-1)^2} \alpha_1 - \frac{B_N^4 M_1^4}{(m-1)^4} \alpha_2 - \cdots + (-1)^{N+1} \frac{B_N^{2N} M_1^{2N}}{(m-1)^{2N}} \alpha_N \right) \cos^{\frac{2(Nm-m+1)}{m-1}}(B_N z) + \]

\[ + A_N \left( \frac{B_N^2 M_2^2}{(m-1)^4} \alpha_2 - \frac{B_N^6 M_2^6}{(m-1)^6} \alpha_3 - \cdots + (-1)^N \frac{B_N^{2N} M_2^{2N}}{(m-1)^{2N}} \alpha_N \right) \cos^{\frac{2(Nm-2m+2)}{m-1}}(B_N z) + \]

\[ + A_N \left( \frac{B_N^{2(N-1)} M_{N-1}^{2(N-1)}}{(m-1)^{2(N-1)}} \alpha_{N-1} - \frac{B_N^{2N} M_{N-1}^{2N}}{(m-1)^{2N}} \alpha_N \right) \cos^{\frac{2(Nm-(N-1)m+N-1)}{m-1}}(B_N z) + \]

\[ + \left( \frac{B_N^{2N} M_{N-1}^{2N}}{(m-1)^{2N}} \alpha_N A_N - C_0 \right) \cos^{\frac{2N}{m-1}}(B_N z) = 0. \]

(9)

Equating coefficients at powers of \( \cos(B_N z) \) to zero yields an algebraic system. Solving this system we obtain the values of parameters \( A_N, B_N \) and correlations on the coefficients \( \alpha_0, \ldots, \alpha_N \).

3 Periodic wave solutions of the K(m,m) equation

The first member of the family (1) in the case \( N = 1 \) takes the form

\[ u_t + \alpha_0(u^m)_x + \alpha_1(u^m)_{3,x} = 0, \quad (u^m)_{3,x} = \frac{d^3 u^m}{dx^3}, \]

\[ m \neq 1, \quad \alpha_0, \alpha_1 \neq 0. \]

Taking the traveling wave ansatz (2) into account, we have the equation with respect to \( y(z) \)

\[ \alpha_1(y^m)_{2,z} + \alpha_0 y^m - C_0 y = 0, \quad (y^m)_{2,z} = \frac{d^2 y^m}{dz^2}. \]

(11)

Following the procedure suggested in the previous section we obtain the equation

\[ \left( \alpha_0 - \frac{(2mB_1)^2}{(m-1)^2} \alpha_1 \right) A_I \cos^{\frac{2m}{m-1}}(B_1 z) + \]

\[ + \left( \frac{2m(m+1)}{(m-1)^2} B_1^2 A_I \alpha_1 - C_0 \right) \cos^{\frac{2}{m-1}}(B_1 z) = 0. \]

(12)
Equating coefficients at powers of $\cos (B_1 z)$ to zero we get values of parameters $A_1, B_1$

$$A_1 = \frac{2C_0 m}{\alpha_0(m+1)}, \quad m \neq -1,$$

$$B_1 = \sqrt{\frac{\alpha_0}{\alpha_1}} \frac{(m-1)}{2m}, \quad m \neq 0.$$  

(13)

(14)

Solutions of Eq. (12) are the following

$$y(z) = \left( \frac{2C_0 m}{\alpha_0(m+1)} \right)^{\frac{1}{m+1}} \cos^{\frac{2}{m+1}} \left( \sqrt{\frac{\alpha_0}{\alpha_1}} \frac{(m-1)}{2m} (z - z_0) \right).$$  

(15)

In the case $\alpha_0 = 1, \alpha_1 = 1$ formula (15) is a solution of a well-known $K(m, n)$ equation with $n = m$ [1].

4 Periodic wave solutions of Eq. (3) in the case $N=2$

In this section we will construct exact solutions of the family (1) at $N = 2$. In this case Eq. (3) takes the form

$$u_t + \alpha_0 (u^m)_x + \alpha_1 (u^m)_{3,x} + \alpha_2 (u^m)_{5,x} = 0, \quad (u^m)_{5,x} = \frac{d^5 u^m}{dx^5}, \quad m \neq 1, \quad \alpha_0, \alpha_1, \alpha_2 \neq 0.$$  

(16)

Making the substitution (2) into (16) and integrating the result we obtain

$$\alpha_2 (y^m)_{4,z} + \alpha_1 (y^m)_{2,z} - C_0 y + \alpha_0 y^m = 0, \quad (y^m)_{4,z} = \frac{d^4 y^m}{dz^4}.$$  

(17)

Substitution of solution (11) into (17) allows us to get the following equation

$$A_2 \left( \alpha_0 - \frac{(4mB_2)^2}{(m-1)^2} \alpha_1 + \frac{(4mB_2)^4}{(m-1)^4} \alpha_2 \right) \cos^{\frac{4m}{m+1}} (B_2 z) +$$

$$+ A_2 \left( \frac{4m(3m+1)}{(m-1)^2} B_2^2 \alpha_1 - \frac{16m(15m^3 + 11m^2 + 5m + 1)}{(m-1)^4} B_2^4 \alpha_2 \right) \cos^{\frac{2(m+1)}{m+1}} (B_2 z) +$$

$$+ \left( \frac{8m(3m^3 + 13m^2 + 13m + 3)}{(m-1)^4} B_2^4 A_2 \alpha_2 - C_0 \right) \cos^{\frac{4}{m+1}} (B_2 z) = 0.$$  

(18)
Solving this equation we obtain

\[ A_2 = \frac{2^3 C_0 m(m + 1)}{\alpha_0(3m + 1)(m + 3)}, \quad m \neq -\frac{1}{3}, 3, \]  

\[ B_2 = \sqrt{\frac{\alpha_0 (m - 1)\sqrt{5m^2 + 2m + 1}}{4m(m + 1)}}, \quad m \neq 0, -1, \]  

\[ \alpha_2 = \frac{4(m + 1)^2 \alpha_1^2}{m(5m^2 + 2m + 1)^2 \alpha_0}. \]  

Formula (4) in the case \( N = 2 \) is the following

\[ y(z) = \left( \frac{2^3 C_0 m(m + 1)}{\alpha_0(3m + 1)(m + 3)} \right)^{\frac{1}{m-1}} \cos \frac{4}{m-1} \left( \sqrt{\frac{\alpha_0 (m - 1)\sqrt{5m^2 + 2m + 1}}{4m(m + 1)}} (z - z_0) \right). \]  

5 Periodic wave solutions of Eq. (3) in the case \( N = 3 \)

Formula (1) in the case \( N = 3 \) is

\[ u_t + \alpha_0(u^m)_x + \alpha_1(u^m)_{3,x} + \alpha_2(u^m)_{5,x} + \alpha_3(u^m)_{7,x} = 0, \]  

\[ (u^m)_{7,x} = \frac{d^7 u^m}{dx^7}, \quad m \neq 1, \quad \alpha_0, \alpha_1, \alpha_2, \alpha_3 \neq 0. \]  

Using traveling wave reduction (2), the following ordinary differential equation takes the form

\[ \alpha_3(y^m)_{6,z} + \alpha_2(y^m)_{4,z} + \alpha_1(y^m)_{2,z} + \alpha_0 y^m - C_0 y = 0, \quad (y^m)_{6,z} = \frac{d^6 y^m}{dz^6}. \]  

In the case \( N = 3 \) from Eq. (9) we obtain values of the parameters \( A_3, B_3 \)

\[ A_3 = \frac{2^3 C_0 (m + 2)(2m + 1)m}{\alpha_0(5m + 1)(m + 5)(m + 1)}, \quad m \neq -\frac{1}{5}, -1, -5, \]
\[ B_3 = \sqrt{\frac{\alpha_0 (m - 1)\sqrt{49m^4 + 92m^3 + 78m^2 + 20m + 4}}{6(2m + 1)(m + 2)m}}, \quad m \neq 0, -\frac{1}{2}, -2. \] (26)

Relations between the coefficients \( \alpha_0, \alpha_1, \alpha_2 \) and \( \alpha_3 \) are the following

\[ \alpha_2 = \frac{9(2m + 1)^2(m + 2)^2m^2(14m^2 + 8m + 5)\alpha_1^2}{(49m^4 + 92m^3 + 78m^2 + 20m + 4)^2\alpha_0}, \] (27)

\[ \alpha_3 = \frac{81m^4(2m + 1)^4(m + 2)^4\alpha_1^3}{(49m^4 + 92m^3 + 78m^2 + 20m + 4)^2\alpha_0^2}. \] (28)

Solutions of Eq.(24) take the form

\[ y(z) = (A_3)^\frac{1}{m+\tau} \cos^{\frac{6}{m+\tau}} (B_3(z - z_0)), \] (29)

where values of parameters \( A_3, B_3 \) are given by formulae (25) and (26).

In the case \( m = 3, \ C_0 = 1, \ z_0 = 0, \ \alpha_0 = 1 \) and \( \alpha_1 = 1 \) solution of Eq. (29) is presented in Fig. 1.

![Figure 1: Solution of Eq. (29) in the case \( N = 3, \ m = 3, \ C_0 = 1, \ z_0 = 0, \ \alpha_0 = 1, \ \alpha_1 = 1 \).](image)

Note that in the case \( N = C(2l + 1), \) where \( C \) is an arbitrary constant and \( l = 1, 2, \ldots, \) solutions (1) are alike solution given in Fig. 1.
6 Periodic wave solutions of Eq. (3) in the case \(N=4\)

Let us look for exact solutions of the family (1) in the case \(N=4\)

\[
\begin{align*}
  u_t + \alpha_0(u^m)_x + \alpha_1(u^m)_{3,x} + \alpha_2(u^m)_{5,x} + \alpha_3(u^m)_{7,x} + \alpha_4(u^m)_{9,x} &= 0, \\
  (u^m)_{9,x} &= \frac{d^9 u^m}{dx^9}, \quad m \neq 1, \quad \alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4 \neq 0.
\end{align*}
\] (30)

This equation possesses the traveling wave reduction (2) with \(y(z)\) satisfying the equation

\[
\begin{align*}
  \alpha_4(y^m)_{8,z} + \alpha_3(y^m)_{6,z} + \alpha_2(y^m)_{4,z} + \alpha_1(y^m)_{2,z} + \alpha_0 y^m - C_0 y &= 0, \\
  (y^m)_{8,z} &= \frac{d^8 y^m}{dz^8}.
\end{align*}
\] (31)

Following the procedure suggested in section 2 we obtain values of the parameters \(A_4, B_4\)

\[
A_4 = \frac{2^7 C_0 m(m + 3)(3m + 1)(m + 1)}{\alpha_0(3m + 5)(7m + 1)(m + 7)(5m + 3)}, \quad m \neq \frac{1}{7}, \frac{5}{3}, \frac{3}{5}, -7, (32)
\]

\[
B_4 = \sqrt{\frac{\alpha_0 (m - 1)\sqrt{205m^6 + 830m^5 + 1423m^4 + 1108m^3 + 443m^2 + 78m + 9}}{\alpha_1 8(3m + 1)(m + 3)(m + 1)m}}, \quad m \neq 0, -\frac{1}{3}, -1, -7, (33)
\]

and correlations on the coefficients

\[
\alpha_2 = \frac{4(m + 3)^2(3m + 1)^2(m + 1)^2m^2(273m^4 + 508m^3 + 518m^2 + 188m + 49)\alpha_1^2}{(205m^6 + 830m^5 + 1423m^4 + 1108m^3 + 443m^2 + 78m + 9)^2\alpha_0}, (34)
\]

\[
\alpha_3 = \frac{128(15m^2 + 10m + 7)(m + 3)^4(3m + 1)^4(m + 1)^4m^4\alpha_1^3}{(205m^6 + 830m^5 + 1423m^4 + 1108m^3 + 443m^2 + 78m + 9)^3\alpha_0^3}, (35)
\]

\[
\alpha_4 = \frac{1024(m + 3)^6(3m + 1)^6(m + 1)^6m^6\alpha_1^4}{(205m^6 + 830m^5 + 1423m^4 + 1108m^3 + 443m^2 + 78m + 9)^4\alpha_0^4}. (36)
\]
Eq. (4) takes the form

\[ y(z) = (A_4) \frac{1}{m-1} \cos^{\frac{8}{m-1}} (B_4(z - z_0)), \]

where \( A_4 \) and \( B_4 \) are determined by formulae (32) and (33).

In the case \( m = 2 \) solution (37) is given in Fig. 2.

Note, that the amplitude and period of the traveling wave solution are growing with the increasing of \( N \). The dependence of amplitude on \( N \) is illustrated in Fig. 3 in the case \( m = 2 \).

In the general case solution of the family of equations (1) takes the form
where parameters $A_N, B_N$ can be written as

\[
A_N = \frac{2^N C_0 \prod_{j=0}^{N-1} ((N - j)m + j)}{\alpha_0 \prod_{j=0}^{N-1} ((2(N - j) - 1)m + 2j + 1)},
\]

\[
B_N = \sqrt{\frac{\alpha_0}{\alpha_1}} \frac{(m - 1)P_N^{N-1}}{\prod_{j=0}^{N-1} ((N - j)m + j)},
\]

for $m \neq -\frac{2j+1}{2(N-j)-1}, \frac{j}{N-j}$ ($j = 0, 1, ..., N - 1$). $P_N^{N-1}$ are polynomials of $N - 1$ power. Relations for values of the coefficients $\alpha_k$ ($k = 0, 1, ..., N$) are found from (7).

7 Conclusion

Let us formulate shortly the results of this paper. We have studied the generalized $K(m, m)$ equations. Taking into consideration the traveling wave ansatz we have found the periodic wave solutions for a family of nonlinear partial differential equation (1). This family generalizes a well-known $K(m,n)$ equation in the case $n = m$. Formula for the amplitude of the traveling wave in the general case is given. Exact solutions for the cases $N = 1, 2, 3, 4$ of the family (1) are presented.

Acknowledgment

This work was supported by the Federal target programm ”Research and scientific-pedagogical personnel of innovation in Russia” of 2009-2011.

References

[1] Rosenau P. and Hyman J., Compactons : Solitons with Finite Wavelength, Phys. Rev. Lett. 1993;5:70.

[2] Rosenau P., Nonlinear Dispersion and Compact Structures, Phys. Rev. Lett. 1994;13:73.
[3] Rosenau P., On a class of nonlinear dispersive-dissipative interaction, Physica D 1998;123(1-4).

[4] Rosenau P., Compact and noncompact dispersive structure. Phys Lett A 2000;275(3):193-203.

[5] Tian L.X., Yin J.L., Stability of multi-compacton solutions and Backlund transformation in K(m,n,1). Chaos, Solitons and Fractals 2005;23(1):159-69.

[6] Zhu Y., Gao X.S., Exact special solitary solutions with compact support for the nonlinear dispersive K(m, n) equations, Chaos Solitons Fractals 2006;27(2):487-493.

[7] Zhu Y.G., Lu Z.S., New exact solitary-wave special solutions for the nonlinear dispersive K(m,n) equations. Chaos, Solitons and Fractals 2006;27(3):836-42.

[8] Odibat Z. M., Solitary solutions for the nonlinear dispersive K(m,n) equations with fractional time derivatives, Phys. Lett. A 2007;370:295-301.

[9] Xu L., Variational approach to solitons of nonlinear dispersive K(m,n) equations, Chaos, Solitons and Fractals 2008;37:137-143.

[10] Biswas A., 1-soliton solution of the K(m,n) equation with generalized evolution, Phys. Lett. A 2008;372(25):4601-4602.

[11] Wang D.-S., Lou S. Y., Prolongation structures and exact solutions of K(m,n) equations, J. Math. Phys. 2009;50:123513.

[12] M.S. Bruzon, M.L. Gandarias, Traveling wave solutions of the K(m,n) equations with generalized evolution term, AIP Conference Proceedings 2009;1168:244-247.

[13] Bruzon M.S., Gandarias M.L., Classical potential symmetries of the K(m,n) equation with generalized evolution term, Computers and Mathematics with Applications 2010;59(8):2536-2540.

[14] Biswas A., 1-soliton solution of the K(m,n) equation with generalized evolution and time-dependent damping and dispersion, Computers and Mathematics with Applications 2010;59:2536-2540.

[15] Kudryashov N.A., Exact soliton solutions of the generalized evolution equation of wave dynamics, J. Appl. Math. Mech. 1988;52(3):360-365.
[16] Kudryashov N.A., On types nonlinear nonintegrable differential equations with exact solutions, Phys. Lett. A 1991;155:269-275.

[17] Kudryashov N.A., Partial differential equations with solutions having movable first-order singularities, Physics Letters A 1992;169:237-242.

[18] Fu S., Liu S., Liu S., New exact solutions to the KdVBurgers-Kuramoto equation, Chaos Solitons Fract. 2005;23:609-616.

[19] Kudryashov N.A., Exact solitary waves of the Fisher equation, Phys. Lett. A 2005;342:99-106.

[20] Kudryashov N.A., Simplest equation method to look for exact solutions of nonlinear differential equations, Chaos Solitons Fract. 2005;24:1217-1231.

[21] Kudryashov N.A., Demina M.V., Polygons of differential equation for finding exact solutions, Chaos Solitons Fract. 2007;33:1480-1496.

[22] Kudryashov N.A., Loguinova N.B., Extended simplest equation method for nonlinear differential equations, Appl. Math. Comput. 2008;205(1):396-402.

[23] M. Qin, G. Fan, An effective method for finding special solutions of nonlinear differential equations with variable coefficients, Phys. Lett. A 2008;372:3240-3242.

[24] Kudryashov N.A., Solitary and periodic solutions of the generalized Kuramoto-Sivashinsky equation, Regul. Chaotic Dynam. 2008;13(3):234-238.

[25] Lu D., Hong B., Tian L., New solitary wave and periodic wave solutions for general types of KdV and KdVBurgers equations, Commun. Nonlinear Sci. Numer. Simul. 2009;14:77-84.

[26] Kudryashov N.A., Loguinova N.B., Be careful with Exp-function method, Commun. Nonlinear Sci. Numer. Simul. 2009;14:1881-1890.

[27] Kudryashov N.A., Seven common errors in finding exact solutions of nonlinear differential equations, Commun. Nonlinear Sci. Numer. Simul. 2009;14:3507-3529.

[28] Kudryashov N.A., Demina M.V., Traveling wave solutions of the generalized nonlinear evolution equations, Appl. Math. Comput. 2009;210:551-557.
[29] Kudryashov N.A., Meromorphic solutions of nonlinear ordinary differential equations, Commun. Nonlinear Sci. Numer. Simul. 2010;15:2778-2790.