Effect of Friction on Dense Suspension Flows of Hard Particles

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We use numerical simulations to study the effect of particle friction on suspension flows of non-Brownian hard particles. By systematically varying the microscopic friction coefficient $\mu$, and the viscous number $J$, we build a phase diagram that identifies three regimes of flow: Frictionless, Frictional Sliding, and Rolling. Using energy balance in flow, we predict relations between kinetic observables, confirmed by numerical simulations. For realistic friction coefficient and small viscous numbers (below $J \sim 10^{-3}$) we show that the dominating dissipative mechanism is sliding of frictional contacts, and we characterize asymptotic behaviors as jamming is approached. Outside this regime, our observations support that flow belongs to the universality class of frictionless particles. We discuss recent experiments in the context of our phase diagram.

I. INTRODUCTION

Non-Brownian dense suspensions are central to industrial processes including oil extraction and food processing, as well as natural phenomena such as landslides and slurries. Understanding how such out-of-equilibrium, crowded systems flow remains a great challenge, as such systems can show shear-thinning, shear-thickening, and shear-banding depending on both the microscopic nature of the interparticle forces, inertial effects, and the boundary conditions \cite{1–5}. To make progress in such complex systems, two limiting cases have received considerable attention. On the one hand, for \textit{infinitely hard frictional} particles, dimensional analysis alone implies that the pressure carried by the particles, $p$, and shear rate $\dot{\varepsilon}$ do not affect the flow independently, but only through the viscous number $J = \eta_0 \dot{\varepsilon}/p$ where $\eta_0$ is the solvent viscosity \cite{6}. In particular, the packing fraction $\phi$ and the macroscopic friction $\mu = \sigma/p$ must be functions of $J$, and are found empirically to follow constitutive laws of the type $\phi(J) = \phi_c - a_\phi J^{\gamma_\phi}$ and $\mu = \mu_c + a_\mu J^{\gamma_\mu}$ \cite{7}. These laws imply a quasi-Newtonian behavior at fixed $\phi$, with a viscosity $\eta/\eta_0 \propto (\phi_c - \phi)^{-\gamma}$ with $\gamma = 1/\gamma_\phi$. Currently these relations are phenomenological, with $\gamma = 1/\gamma_\phi \approx 1/\gamma_\mu \approx 2$. On the other hand, for \textit{frictionless particles} more precise numerical measurements were made \cite{3,10} and report values $\gamma = 1/\gamma_\phi \approx 1/\gamma_\mu \in [2.5, 2.8]$. A microscopic theory can rationalize these findings and predicts $\gamma = 2.85$ and $\gamma_\phi = 0.35$ \cite{11,12}. The central idea is that as the density increases, the network of contacts between particles becomes more and more constrained: there are fewer and fewer floppy modes along which particles can flow without overlapping \cite{13}. As the number of floppy modes vanishes, nondimensionalized velocity fluctuations diverge as $\ell \sim J^{-1/2}$ \cite{13} and become correlated on a length scale $\ell_c \sim J^{-0.15}$ \cite{11,14}. These results assume that dissipation stems only from viscous forces; the extent to which they apply to frictional particles is unknown.

Similar considerations apply to \textit{inertial} flow of hard particles: the rheology has a singular dependence on the dimensionless strain rate, the inertial number $I = J = \mu \rho \dot{\varepsilon}/\eta_0$. A microscopic theory can rationalize these findings and predicts $\gamma = 2.85$ and $\gamma_\phi = 0.35$ \cite{11,12}. The central idea is that as the density increases, the network of contacts between particles becomes more and more constrained: there are fewer and fewer floppy modes along which particles can flow without overlapping \cite{13}. As the number of floppy modes vanishes, nondimensionalized velocity fluctuations diverge as $\ell \sim J^{-1/2}$ \cite{13} and become correlated on a length scale $\ell_c \sim J^{-0.15}$ \cite{11,14}. These results assume that dissipation stems only from viscous forces; the extent to which they apply to frictional particles is unknown.

In this work, we characterize the role of friction in suspensions of hard particles using a combination of numerics and scaling arguments. We systematically vary the friction coefficient $\mu$ and viscous number $J$, and establish a transition from regimes where energy dissipation is
II. NUMERICAL PROTOCOL

We use the Discrete Element Method to model flow of granular materials with strictly overdamped dynamics; grains are stiff elastic disks with Coulomb friction, subject to drag forces modelling their interaction with the solvent. When grains overlap at a contact\( \alpha \), they experience a harmonic elastic force \( \vec{f}_\alpha \) with tangential and normal components \( f^T_\alpha \) and \( f^N_\alpha \); the spring constants have a ratio \( k_T = 0.5k_N \). Coulomb friction restricts the elastic forces to satisfy \( |f^T_\alpha| \leq \mu_p f^N_\alpha \); contacts that saturate this constraint are said to be sliding, while the rest are said to be rolling. Long-range hydrodynamic interactions are neglected, as such interactions are believed to be screened by the dense and disordered solid part of the suspension; viscous interactions between the particles and the viscous fluid are modeled by Stokes drag, both normal components \( \vec{f} \) and \( \vec{f} \) exerted on particle \( j \) by particle \( i \). The torque balance equation is \( 0 = \tau_i^v + \tau_i^s - \sum_j \tau_{ij} \). For drag forces, we consider a simple Stokes drag \( \vec{F}^v_i = -\eta_0 c_d (\vec{V}_i - \vec{V}_i^a) \) and \( \tau_i^v = -\frac{4}{3} \eta_0 c_d R_i^2 (\omega_i - \omega_i^a) \) where \( R_i \) is radius of particle \( i \), \( \vec{V}_i^a = \dot{\epsilon}_a \vec{x} \) is the affine velocity, and \( \omega_i^a = \dot{\epsilon}_a / 2 \). Here \( c_d = 3\pi/(1 - \phi_0) \), with \( \phi_0 = 0.76 \), is a constant chosen to scale the drag contributions \( J \) to experiments [2]. Its importance is discussed in the Discussion section below.

The Coulomb friction coefficient is varied from 0 to 10. For realistic grains, \( \mu_p \) is often reported in the range 0.2 to 0.7, however smaller values may be relevant for emulsions, foams, and coated surfaces, and larger values may model the effect of particle angularity [19].

III. ENERGY BALANCE

Energy conservation is a strong constraint on stationary flows, as we now show. The energy balance equation is obtained by contracting the force and torque balance equations along the velocity field \( (\vec{V}_i, \omega_i) \). The result is

\[
\mathcal{P}_{\text{ext}} = \mathcal{D}_{\text{visc}} + \mathcal{D}_{\text{cont}},
\]

where \( \mathcal{P}_{\text{ext}} = \sum_i \vec{F}^\text{ext}_i \cdot \vec{V}_i + \tau_i^\text{ext} \omega_i \) is the power injected through external forces, \( \mathcal{D}_{\text{visc}} \) is the power dissipated through viscosity, and \( \mathcal{D}_{\text{cont}} \) is the net power dissipated through contact forces. The viscous term is

\[
\mathcal{D}_{\text{visc}} = c_d\eta_0 [\sum_i (\vec{V}_i - \vec{V}_i^a) + \frac{4}{3} R_i^2 \omega_i \cdot (\omega_i - \omega_i^a)]
\]

\[
= c_d\eta_0 [\sum_i (\vec{V}_i - \vec{V}_i^a)^2 + \frac{4}{3} R_i^2 (\omega_i - \omega_i^a)^2],
\]

since non-affine and affine velocities are uncorrelated on average, i.e. \( \sum_i \vec{V}_i^a \cdot (\vec{V}_i - \vec{V}_i^a) = 0 \), and similarly for the
torques. The contact term is

\[ D_{\text{cont}} = -\sum_i \tilde{V}_i \cdot \sum_j \tilde{f}_{ij} - \sum_i \omega_i \sum_j \tau_{ij} \]

\[ = \sum_{ij} \tilde{u}_{ij} \cdot \tilde{f}_{ij}, \quad (3) \]

where \( \tilde{u}_{ij} = \tilde{V}_j - \tilde{V}_i + \tilde{n}_{ij} (R_j \omega_j + R_i \omega_i) \) is the relative velocity of particle \( j \) with respect to particle \( i \) at their mutual contact point; \( \tilde{n}_{ij} \) is a vector perpendicular to the contact normal \( \tilde{n}_{ij} \). The second equality in Eq. (3) can be established by direct substitution of the definition of \( \tilde{u}_{ij} \). The work done by contact forces has both normal and tangential components. The former is entirely elastic and vanishes on average, reflecting the fact that elastic forces are conservative. We are left with:

\[ D_{\text{cont}} = \sum_{ij \in C_S} \tilde{u}_{ij}^T \cdot \tilde{f}_{ij}^T + \sum_{ij \in C_R} \tilde{u}_{ij}^T \cdot \tilde{f}_{ij}^T \]

\[ \equiv D_{\text{slid}} + D_{\text{roll}}, \quad (4) \]

where \( C_S \) denotes the sliding contacts, \( C_R \) denotes the non-sliding (rolling) contacts. The term \( D_{\text{roll}} \) can be shown to vanish on average if the normal force in the contact was held constant, as it would correspond to the loading of a transverse spring whose mechanical energy must be bounded. However, it can be finite for contact force history that include sliding periods. We find empirically that for all parameters probed \( D_{\text{roll}} \lesssim D_{\text{slid}} \), as shown in Fig. 4 and \( D_{\text{roll}} \) will thus be neglected in scaling arguments below.

To investigate which source of dissipation dominates, we plot the ratio \( D_{\text{cont}}/D_{\text{visc}} \) in Fig. 2. Following our previous work [16], we can define regime boundaries at the viscous number at which \( D_{\text{cont}}/D_{\text{visc}} = 1 \). The result is plotted in Figure 1. We find three regimes, as discussed above. Note that the transition from Frictionless to Rolling is a crossover: throughout this transition area, dissipation is dominated by viscous drag, but the structure of the contact network depends strongly on \( \mu_p \), as discussed below.

### IV. SCALING ESTIMATES OF DISSIPATION

In steady state, energy input from the shear stress is \( \Omega \sigma \dot{\epsilon} \), where \( \Omega \) is the system volume and \( \sigma \) the shear stress. For large systems, additional contributions from fluctuations of the normal position of the wall are insignificant, thus

\[ P_{\text{ext}} = \Omega \sigma \dot{\epsilon}. \quad (5) \]

To estimate the dissipation rate from viscous dissipation, we consider the velocity scale \( \delta V \) defined as the square root of the time-averaged second moment of the non-affine velocity, \( \dot{\epsilon} \), i.e., \( \delta V = \langle (\tilde{V}_i - \tilde{V}_i^0)^2 \rangle^{1/2} \). We expect that the scale of angular velocity fluctuations is then \( \dot{\omega} \approx \delta V/(2D) \).

To define dimensionless dissipation per particle, normalizing by \( \Omega \rho \dot{\epsilon} \approx N \pi D^2 2 \rho \dot{\epsilon} / \phi \). Thus we let \( \bar{D}_{\text{visc}} \equiv D_{\text{visc}}/(\Omega \rho \dot{\epsilon}) \), so that

\[ \bar{D}_{\text{visc}} \approx \frac{28\phi c_d \eta_0 N \delta V^2}{3\pi D^2 \rho \dot{\epsilon}} \approx \frac{28\phi c_d L^2 J}{3\pi}, \quad (6) \]

FIG. 4. Ratio of sliding dissipation at rolling contacts to that at sliding contacts.
is the fraction of sliding contacts, \( \chi \) and show that in this regime it displays a weak scal-
approximately \( \propto \) we have approximately \( \sim \), suggesting flow inhomogeneity once one enters in the sliding frictional regime.

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V. MICROSCOPIC OBSERVABLES

The quantity \( \lim \) is the ratio of typical velocity fluctuations to the affine velocity scale, shown in Fig. 3. We find that for any \( \mu_p \), \( \lim \) increases and apparently diverges as \( J \) decreases. In the frictionless and rolling regimes, \( \lim \propto (J/\mu)^{-1/2} \), as follows from Eq. [8].

In the frictional sliding regime, the divergence is weaker: we have approximately \( \lim \propto (J/\mu)^{-1/3} \).

The typical sliding velocity \( u_T \) can differ from \( \delta V \) if flow is inhomogeneous. We find evidence for this in the frictional sliding regime, for which a power-law appears to be developing in the accessible range, with approximately \( \lim \propto J^{0.3} \), as shown in Fig. 5.

In the frictional sliding regime, an important variable is the fraction of sliding contacts, \( \chi \). We plot it in Fig. 5: and show that in this regime it displays a weak scaling: \( \chi \sim J^{0.1} \) for \( \mu_p = 0.3 \) and \( \chi \sim J^{0.13} \) for \( \mu_p = 0.6 \). Together with the behavior of \( \lim \) and \( \lim \propto J^{0.3} \), this is consistent with the constraint from energy balance: we have \( \lim / \mu = \lim J^{0.1+0.3-0.33-0.33} = J^{0.07} \mu^{-0.66} \). Since \( \mu \approx \mu_c \) in this regime, this is very close to a constant, as predicted by Eq. [9].

In the frictionless and rolling regimes, \( \chi \) depends strongly on \( \mu_p \). For \( \mu_p \gtrsim 1 \) we have \( \chi < 0.03 \), indicating that most contacts are rolling, and justifying the regime’s name. We thus find that when crossing from the frictionless to rolling regime, although the velocity fluctuations do not display a strong signal, and the rheology is similar as shown below, the contact network is restructuring from having all sliding contacts to having only rolling contacts.

VI. RHEOLOGY

We now turn to the constitutive relations for the volume fraction of solid material, \( \phi(J) \), and the stress ratio \( \mu(J) \), shown in Fig. 6. As expected, there is a large dependence on \( \mu_p \) in the values of \( \mu_c \) and \( \phi_c \): the stress ratio increases and the compaction decreases as \( \mu_p \) increases at constant \( J \). To distinguish flow regimes, we consider in Fig. 6: the quantity \( \phi_c - \phi \sim J^{\gamma_p} \). In the Rolling regime, we find \( \gamma_p \approx 0.3 \), close to the prediction of [11, 12] for flows dominated by viscosity. When \( \mu_p \) is lowered, but at large enough \( J \) to be in the transition from Rolling to Frictionless, we observe that the curves bend. A direct fit for \( \mu_p \rightarrow 0 \) would suggest \( \gamma_p \approx 0.5 \). However, previous analyses in the frictionless limit showed that corrections to scaling are large for frictionless particles, an effect that can be traced back to the smallness of the constant \( \mu_c \). If these corrections are taken into account, one finds asymptotic exponents \( \gamma_p \approx (0.36, 0.39) \) [13, 16, 20]. We thus suggest that within flow regimes dominated by viscosity, we have \( \gamma_p \approx (0.30, 0.39) \). In contrast, for intermediate \( \mu_p \), the curves display a kink close to the transition from viscous to frictional dissipation; this allows us to fit a much larger slope \( \gamma_p \approx 2/3 \) within the Frictional Sliding regime.

VII. PHASE BOUNDARIES

We have shown that for realistic \( \mu_p \), sliding friction eventually dominates dissipation as jamming is approached. Consistent with the change in dissipation mechanism, key observables differ in the frictional sliding regime: the velocity fluctuations show a weaker divergence with \( J \), and the fraction of sliding contacts displays scaling behavior. When \( \mu_p \lesssim 0.1 \), the crossover to frictional sliding can be understood theoretically: inside the frictionless regime, \( \lim \sim \sim (C_d J/\mu)^{-1/2} \) and \( \chi \approx 1 \), so that \( \lim \sim \sim 2 \mu_p (C_d J/\mu)^{-1/2} \). A crossover to the frictional sliding regime will occur when this quantity is \( O(\mu) \). Since \( \mu \approx \mu_c \) at small \( J \), we see that \( \lim \sim \mu_c \) at \( J_c \sim 4 \mu_p^2 / (C_d \mu_c) \). The scaling of \( J_c \) with \( \mu_p \), and the small prefactor, are both verified in Fig. 4.
FIG. 6. (a) Stress ratio $\mu$, (b) volume fraction $\phi$, and (c) $\phi_c - \phi$ as function of the viscous number $J$ at indicated particle friction coefficients. In (c), values of $\phi_c$ were determined by fitting curves in (b). In (c), dashed slope is 0.30, and dash-dotted slope is $2/3$.

VIII. TRANSITION TO INERTIAL FLOW

It is possible to combine the phase diagram of Fig. [1] with earlier results for strictly inertial flow [16] to estimate the transition from viscous to inertial rheology. In inertial dynamics, controlled by the inertial number $I = \dot{\varepsilon} D/\rho \sqrt{p/\rho}$, in addition to viscous dissipation and sliding friction, energy can be dissipated by grain inelasticity. The collisional dissipation rate, which we denote by $D_{\text{coll}}$, was previously estimated for hard particles in [11, 16]. Defining, as above, $\bar{D}_{\text{coll}} = D_{\text{coll}}/(\Omega p\dot{e})$, and considering $O(1)$ restitution coefficient, it reads [11, 16]

$$\bar{D}_{\text{coll}} \approx IL^2.$$  \hfill (10)

We assume that this contribution will add to the viscous and frictional contributions, thus modifying power balance from Eq.[11] to

$$\mu \approx (C_d J + I) L^2 + 2\mu_p \chi L_T.$$  \hfill (11)

Using the scaling behavior of $L$, $\chi$, and $L_T$ in the various regimes, one can determine from this relation all the possible transitions.

Consider, for example, the transition when inertia starts to dominate over viscous dissipation [11, 2]. From

FIG. 7. Sketch of phase diagram when inertia is present, in terms of $J$, $\mu_p$, and Stokes number $St = \mu^2/J$, where $I = \dot{\varepsilon} D/\rho \sqrt{p/\rho}$ is the inertial number. Below the colored dome, dissipation is dominated by frictional sliding, while above it is dominated either by viscous dissipation or grain inelasticity. Color corresponds to the value of rescaled confining pressure $\tilde{p} = (I/(C_d J))^2$ along the critical surface, distinguishing viscous regimes (yellow, $\tilde{p} < 1$) from inertial regimes (dark blue, $\tilde{p} \gg 1$).

FIG. 8. Sketch of phase diagram when inertia is present, in terms of $J$, $\mu_p$, and rescaled confining pressure $\tilde{p} = p/(D/(C_d \eta_p))^2$. Below the colored dome, dissipation is dominated by frictional sliding, while above it is dominated either by viscous dissipation or grain inelasticity. Color corresponds to the value of $\tilde{p}$ along the critical surface, distinguishing viscous regimes (yellow, $\tilde{p} < 1$) from inertial regimes (dark blue, $\tilde{p} \gg 1$).
Eq. 11 this will occur when $I > C_d J$. This predicts that the relevant dimensionless number is

$$\tilde{\phi} = \left( \frac{I}{C_d J} \right)^2 = \frac{\rho D^2}{C_d J \phi_c}, \quad (12)$$

which is nominally independent of strain rate, depending only on the pressure $p$, and fixed parameters. This is consistent with experimental measurements in [11], where the critical shear stress was found to change by less than a factor of 2 when the strain rate varies over a decade. This gives strong support to the microscopic expressions for $D_{\text{visc}}$ and $D_{\text{coll}}$.

On the relevance of friction, consider flow inside the frictional sliding regime, where dissipation is dominated by $D_{\text{slid}}$. Whether the rheology is viscous, with $\sigma \propto \dot{\epsilon}$, or inertial, with $\sigma \propto \dot{\epsilon}^2$, depends on the magnitude of the two subdominant dissipation rates, $D_{\text{visc}}$ and $D_{\text{coll}}$. From Eq. 11 we see that viscous dissipation will be larger provided $J > I/C_d$, which defines a criterion for irrelevance of inertial effects.

Finally, we can sketch the phase diagram when inertial effects are present, in particular the boundary between regions dominated by frictional sliding, and by other sources of dissipation. This boundary defines a function $J = f(\mu_p)$, which from Fig. 1 has an approximately inverted parabolic shape on logarithmic axes. It satisfies, in particular, $f(\mu_p) \sim \mu_p^2$ for $\mu_p \ll 1$. If regime change is controlled by dissipation, as we have argued here and in [10], then in the presence of inertia we need only replace $J$ by $J + I/C_d$. In particular, the frictional/frictionless regime boundary will be modified to $J + I/C_d = f(\mu_p)$. We can check this relation by comparing with the corresponding boundary of the frictional sliding found in [10], where there is a function $I = g(\mu_p)$. The implied relation $g(\mu_p) = C_d f(\mu_p)$ is consistent with the data for $\mu_p < 1$; for example in [10] it was found that $I \sim \mu_p^2$ at small $\mu_p$, consistent with $f(x) \sim x^2$. Also, the peak value of $I$ having a frictional regime is $I \approx 0.1$, which implies a peak value of $J$ having a frictional regime at $J \approx 0.1/C_d \approx 10^{-3}$, consistent with Fig. 1.

At fixed Stokes number $St=I^2/J$, the criterion becomes $f(\mu_p) = J + \sqrt{JSt/C_d}$, which can be solved for $J$. The resulting phase diagram is sketched in Fig. 7 for the range $10^{-2} < St < 10^3$. Below the colored dome, dissipation is dominated by frictional sliding. The color corresponds to the value of $\tilde{\phi}$; the yellow regions are viscous, and the blue regions are inertial.

In fact, the above analysis suggests that the phase diagram could more simply be plotted in terms of $\tilde{\phi}$ directly; this is done in Fig. 8. In this representation, the boundary between viscous and inertial regimes is a cut along fixed $\tilde{\phi} \approx 1$.

IX. COMPARISON WITH EXPERIMENTS

To quantitatively compare our findings with experiments, we need to ensure that our definition of $J$ and the associated normalization of drag forces is reasonably accurate. We have used a Stokes drag model, where the constant of proportionality is known, analytically, for a force exerted by the fluid in the absence of other particles. When particles are very dense, the drag force is renormalized by an Archimedes force: the fluid drag is enhanced because most of the volume is occupied by particles themselves, unlike in the case of one sphere in an infinite volume of fluid [22, 23]. This leads to the large constant $C_d \sim 10^2$, which translates to the small transition value $J_c$. By the definition $J = \eta \epsilon/p$, a change in normalization would correspond to a multiplicative shift in $J$. To verify that our normalization is appropriate, and to compare 2D to 3D, it is natural to measure distance from the transition, i.e., $J$, by comparing constant values of $(\phi_c - \phi)/\phi_c$. To test this procedure, in Fig. 9 we reproduce the data of Boyer et al. [7], along with the fitting form proposed therein, $\phi_c/\phi = 1 + J^{1/2}$. For $J \gtrsim 10^{-3}$, where most data are taken, the data are close to ours for $\mu_p \in (0.1, 0.6)$. The experiment used spheres of two materials: polymethyl methacrylate (PMMA) for $J \in (10^{-3}, 10^{-2})$ and polystyrene for $J \in (10^{-4}, 10^{-1})$, both expected to have a friction coefficient $O(1)$ in dry conditions. Although we cannot strongly discriminate between different $\mu_p$ in this comparison, we conclude that the normalization of $J$ is appropriate.

Our results thus support that experiments probing $J \geq 10^{-3}$ are not in the sliding regime, and that dissipation is dominated by viscous drag. In recent works [22, 23], a dimensionless local shear rate, equivalent to $\mathcal{L}$, was inferred by considering the rheological properties of suspensions immersed in non-Newtonian fluids. The analysis of [23] supports that $\mathcal{L} \sim J^{-1/2}$ in the range $10^{-3} < J < 10^{-1}$, as predicted from our phase diagram.

A few experiments, however, investigate dense flows

FIG. 9. $(\phi_c - \phi)/\phi_c$ as function of the viscous number $J$ at indicated particle friction coefficients, in comparison with data from Boyer et al. [7]. The solid line shows $\phi_c/\phi = 1 + J^{1/2}$, the fitting form proposed in [7].
closer to jamming. In [7], mentioned above, some data for \( \phi_c - \phi \) extend down to \( J \approx 10^{-5} \). These data are close to our simulation results for \( \mu_p = (0.3, 0.6) \). Assuming a \( \mu_p \) in this range, our phase diagram indicates that the experiment spans both frictional and viscous regimes. We expect a crossover from \( \gamma_0 \sim 0.35 \) (but affected by corrections to scaling, as discussed above) to \( \gamma_0 \sim 0.67 \). Since fits to \( \phi_c - \phi \) are marred by imprecision in the value of \( \phi_c \), especially at small \( J \), it would be interesting to focus experimentally on the range \( J < 10^{-3} \) and extract \( L \) in particular, to seek clear deviations from scaling laws associated with frictionless behavior.

X. DISCUSSION: RIGID CLUSTERS AND ELASTOPLASTICITY IN GRANULAR FLOWS?

In the frictional sliding regime, the mechanism of flow leading to \( L \sim J^{-1/3} \) and \( \gamma_T/L \sim J^{0.3} \) calls for an explanation, because it implies strong heterogeneity. In the frictionless regime, such heterogeneity is absent, consistent with theory that considers a single velocity scale [11]. One possibility is that frictional systems become overconstrained, and flow is elastoplastic, as in models of amorphous solids [24]. In this scenario, flow can be considered as a series of avalanches. In 2D, such avalanches are similar to slip lines of extension \( \xi \) that locally accumulate a strain of order unity. Such avalanches lead to approximately rigid-body motion to an extent \( \sim \xi \) perpendicular to the line of slip, so that the non-affine velocity during an avalanche is of order \( V_{na} \sim \epsilon D \xi \), i.e. \( L \sim \xi \). However, within the rigid blocks, the pairwise relative velocity of particles is negligible; thus the typical relative velocity \( V_r \) obtains its value only from the slip lines, \( V_r \sim V_{na}/\xi \). In this regime we therefore expect \( L_R/L \sim 1/L \), where \( L_R = V_r/(\epsilon D) \).

In support of this picture, approximately rigid-body motion has been suggested to occur in granular flows [25-28]. However, scaling relations from elastoplasticity concerning microscopic flow heterogeneity have not previously been tested. Defining \( V_r \equiv \langle (V_j - V_i)^2 \rangle^{1/2} \) to be consistent with the definition of velocity fluctuations, we plot \( L_R/L \) for \( \mu_p = 0 \) and \( \mu_p = 0.3 \) in Fig 10. When \( L \) is small, corresponding to large \( J \), particles move nearly affinely, but still nearby particles will have a relative velocity; hence we expect \( L_R/L > 1 \), as we observe both for \( \mu_p = 0 \) and \( \mu_p = 0.3 \). However, as \( L \) increases from unity, corresponding to denser flow, the curve for \( \mu_p = 0 \) asymptotes near unity, consistent with the theory of [11], while the curve for \( \mu_p = 0.3 \) continues to decrease, approximately as \( \sim 1/\sqrt{L} \) (dotted line). This provides further support for flow inhomogeneity in frictional flow, and moreover appears distinct from the naive elastoplastic prediction \( \sim 1/L \). Our observations thus do not suggest that a simple rigid block picture applies. Further investigations are clearly needed to clarify this point, central to a microscopic understanding of flow.

XI. CONCLUSION

We have provided a map that allows one to classify suspension flows of hard particles into different regimes. This map delimits a region where dissipation is dominated by viscous forces, and where exponents describing constitutive laws and velocity fluctuations appear identical to those of frictionless particles—these systems belong to the same universality class. In this region we believe that the floppy mode description of flows [11, 12] is accurate. This description is mean-field, in the sense that dissipation is assumed to occur rather homogeneously in space.

For \( J \gtrsim 10^{-3} \), the leading macroscopic effect of friction is to offset \( \mu_c \) and \( \phi_c \), while otherwise the rheology is hardly altered [29]. However, another regime, coined Frictional Sliding, can be identified sufficiently close to jamming. In this regime, friction dominates dissipation and asymptotic behaviors differ. Dissipation appears to be localized in space, as supported by the presence of multiple velocity scales, such as the non-affine velocity and the sliding velocity.

For realistic values of the friction coefficient we predict a transition, in an experimentally measurable regime, from a frictionless-type flow to a frictional sliding regime when the viscous number is decreased, or equivalently as the jamming point is approached.

Similar observations have been reported for inertial flows. On one hand, we showed that a similar phase diagram can be built in that case by comparing sliding and collisional dissipation [16]. On the other hand, strong kinetic heterogeneities in frictional systems near jamming have been reported [20], corresponding in our taxonomy to the Frictional Sliding regime. Building a description of this regime and its microscopic properties characterized here remains a challenge for the future, both for inertial and over-damped flows.

Finally, the viscous and inertial limits considered here and in Ref. [16] are two distinguished surfaces in a 3-dimensional phase diagram spanned by \( \mu_p \), \( J \), and the Stokes number \( I^2/J \). We sketched this diagram here, and showed that transitions from viscous to inertial dy-

![FIG. 10. Ratio of typical relative velocity to velocity fluctuations. Dashed line \( L_R/L \sim L^{-0.3} \) is a guide to the eye.](image-url)
dynamics occur at a constant stress level, consistent with experiments [1]. In future work, it would be valuable to map out the intervening regions in detail.

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