A Vector Supersymmetry Killing IR Divergences in Non-Commutative Gauge Theories

Daniel N. Blaschke
Institute for Theoretical Physics, Vienna University of Technology
Wiedner Hauptstrasse 8-10, A-1040 Vienna (Austria)
E-mail: blaschke@hep.itp.tuwien.ac.at

Abstract. This is a report on the joint work with François Gieres, Stefan Hohenegger, Olivier Piguet and Manfred Schweda. We consider a non-commutative U(1) gauge theory with an extension which was originally proposed by A. A. Slavnov [3, 4] in order to get rid of UV/IR mixing problems. Here we show, that the improved IR behaviour of this model is mainly due to the appearance of a linear vector supersymmetry.

1. Introduction
We consider 3+1 dimensional $\theta$-deformed Minkowski space-time $M^4_{\theta}$, with the commutation relation

$$[\hat{x}^\mu, \hat{x}^\nu] = i \theta^{\mu\nu}$$

for the space-time coordinates (cf. [7, 8], see also [9] for a review). The non-commutativity parameter $\theta^{\mu\nu}$ is assumed to be constant and in order to avoid difficulties with time-ordering in the field theory, we choose the special case where $\theta^{0\mu} = 0$. In order to construct the perturbative field theory formulation, it is more convenient to use fields $A(x)$ (which are functions of ordinary commuting coordinates) instead of operator valued objects like $\hat{A}(\hat{x})$. One therefore defines the linear map $\hat{f}(\hat{x}) \mapsto S[f](x)$, called the “symbol” of the operator $\hat{f}$. One can then represent the original operator multiplication in terms of star products of symbols as

$$\hat{f} \hat{g} = S^{-1} \left[ S[f] \star S[g] \right].$$

In using the Weyl-ordered symbol (which corresponds to the Weyl-ordering prescription of the operators) one arrives at the following definition of the Weyl-Moyal $\star$-product ($S[f](x) \rightarrow A(x)$):

$$A_1(x) \star A_2(x) = e^{\frac{i}{2} \theta^{\mu\nu} \partial_\mu \partial_\nu} A_1(x) A_2(y) \bigg|_{x=y}.$$

It has the important property of invariance under cyclic permutations of the integral

$$\int d^4 x A_1(x) \star A_2(x) \star A_3(x) = \int d^4 x A_3(x) \star A_1(x) \star A_2(x).$$

© 2008 IOP Publishing Ltd
Moreover, bilinear terms are unaffected by the star product:

\[
\int d^4x A_1(x) \star A_2(x) = \int d^4x A_1(x) A_2(x). \tag{5}
\]

For a field theory this means that interaction vertices gain phases, whereas propagators remain unchanged. In constructing Feynman graphs one hence has to deal with so-called planar and non-planar diagrams [10]. While planar diagrams have the same ultraviolet divergences known from commutative field theory, the non-planar ones are finite due to phase factors. A simple example of an integral appearing in non-planar graphs is

\[
\int d^4k \frac{e^{ik\tilde{p}}}{k^2 + i\epsilon} \propto \frac{1}{\tilde{p}^2} \quad \text{with} \quad \tilde{p}^\mu = \theta^{\mu\nu} p_\nu.
\]

It is obvious, that the phases act as UV-regulators, but since this regulating effect can only take place for non-vanishing \(\tilde{p}\), a new infrared divergence appears as \(\tilde{p} \to 0\). This is the origin of the \textit{UV/IR mixing} problem [11, 12].

Here we would like to consider a non-commutative \(U(1)\) gauge theory action

\[
S = -\frac{1}{4} \int d^4xF_{\mu\nu} \star F^{\mu\nu}, \tag{6}
\]

where the field tensor

\[
F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu \star A_\nu]
\]

is endowed with a non-Abelian structure due to the star product. As shown by several authors [6, 14, 15], this action leads to an IR singular vacuum polarization, whose quadratic IR divergent term

\[
\Pi^{\mu\nu}_\text{IR}(k) = 2g^2 \frac{k_\mu k_\nu}{(k^2)^2} \quad \text{with} \quad \tilde{k}^\mu = \theta^{\mu\nu} k_\nu
\]

is gauge fixing independent. Obviously, graphs with this insertion are IR divergent.

2. The Slavnov Term

In order to get rid of IR divergences, Slavnov [3, 4] has proposed a modification of Yang-Mills theories, adding to the action a term

\[
\frac{1}{2} \int d^4x \lambda \star \theta^{\mu\nu} F_{\mu\nu}. \tag{9}
\]

In doing this, the constraint \(\theta^{\mu\nu} F_{\mu\nu} = 0\) is implemented which has the effect of making the gauge field propagator transversal with respect to \(\tilde{k}^\mu\), i.e. \(\tilde{k}^\mu \Delta_\mu^{AA}(k) = 0\). Hence, IR divergent Feynman graphs such as the one depicted in Figure 1 become finite.

There is, however, a catch: Even though \(\lambda\) is introduced as a Lagrange multiplier implementing a constraint, it becomes a dynamical field. Or to be more precise: One has additional Feynman rules, namely a \(\lambda\)-propagator, a mixed \(\lambda A\)-propagator and a \(\lambda AA\)-vertex, and hence numerous additional Feynman graphs. Since the additional propagators are not transversal with respect to \(\tilde{k}^\mu\), Slavnovs trick does not work for certain diagrams, i.e. the one depicted in Figure 2.

Let us now discuss special features of the gauge fixed action including the Slavnov term: In order to avoid unitarity problems [13] we choose the non-commutativity tensor spacelike, i.e.

\[
\theta^{ij} = \theta^{ij}, \quad i, j = 1, 2
\]
Furthermore, we choose the gauge fixing to be of an axial type \cite{16,17} with gauge fixing vector in the plane of the non-commutative coordinates:

\[ n^I = 0 \quad I = 0,3 \]

With these choices the Slavnov term, together with the gauge fixing terms, have the form of a 2-dimensional topological BF model (cf. \cite{1,5} and references therein):

\[ S_{\text{inv}} = \int d^4x \left( -\frac{1}{4} F_{\mu\nu} \star F^{\mu\nu} + \frac{\theta}{2} \lambda \star \epsilon^{ij} F_{ij} \right) \]

and

\[ S_{\text{gf}} = \int d^4x \left( B \star n^i A_i - \bar{c} \star n^i D_i c \right) \]

with the covariant derivative

\[ D_\mu c = \partial_\mu c - ig [A_\mu, c] . \]

3. Symmetries & Consequences

The action \( S = S_{\text{inv}} + S_{\text{gf}} \) is invariant under the BRST symmetry

\[ s A_\mu = D_\mu c, \quad s \bar{c} = B, \]
\[ s \lambda = -ig [\lambda, c], \quad sB = 0, \]
\[ sc = \frac{ig}{2} [c, c], \quad s^2 = 0, \]

where we have omitted the stars, and the commutators are considered to be graded by the ghost-number. Additionally, the gauge fixed action is also invariant under a (non-physical) linear vector supersymmetry (VSUSY), whose field transformations are

\[ \delta_i A_\mu = 0, \quad \delta_i c = A_i, \]
\[ \delta_i \tilde{c} = 0, \quad \delta_i B = \partial_i \tilde{c}, \]
\[ \delta_i \lambda = \frac{\epsilon_{ij}}{\theta} n^j \tilde{c}, \quad \delta^2 = 0. \]
Since the operator $\delta_i$ lowers the ghost-number by one unit, it represents an antiderivation (very much like the BRST operator $s$ which raises the ghost-number by one unit).

**Note:** Only the interplay of appropriate choices for $\theta^{\mu\nu}$ and $n^\mu$ lead to the existence of the VSUSY.

In contrast to the pure topological theories, we have an additional vectorial symmetry:

$\hat{\delta}_i A_J = -F_{ij}$,

$\hat{\delta}_i \lambda = -\frac{\epsilon_{ij}}{\theta} D_K F^{Kj}$,

$\hat{\delta}_i \Phi = 0$ for all other fields.  \hspace{1cm} (14)

This further symmetry is in fact a (non-linear) symmetry of the gauge invariant action. Its existence is due to the presence of the Yang-Mills part of the action which, in contrast to the BF-type part of the action, involves also $A_0$ and $A_3$. Notice that the algebra involving $s$, $\delta_i$, $\hat{\delta}_i$ and the $(x_1, x_2)$-plane translation generator $\partial_i$ closes on-shell (cf. [1]).

We shall now discuss the consequences of the linear VSUSY: The generating functional $Z^c$ of the connected Green functions is given by the Legendre transform of the generating functional $\Gamma$ of the one-particle irreducible Green functions. At the classical level (tree graph approximation) one has $\Gamma \sim S$, and hence the Ward identity describing the linear vector supersymmetry in terms of $Z^c$ in the tree graph approximation is given by

$$W_i Z^c = \int d^4 x \left\{ j_B \partial_i \frac{\delta Z^c}{\delta j_B} - j_c \frac{\delta Z^c}{\delta j_A} + \frac{\epsilon_{ij}}{\theta} n^j j_\lambda \frac{\delta Z^c}{\delta j_\lambda} \right\} = 0,$$

where $\{j_A^\mu, j_\lambda, j_B, j_c, j_\bar{c}\}$ are sources of $\{A_\mu, \lambda, B, c, \bar{c}\}$, respectively. Varying this expression with respect to $j_c$ and $j_\mu^A$ yields for the gauge field propagator:

$$\Delta_{A_i A_\mu} = 0$$ \hspace{1cm} (16)

In other words, as soon as one of its indices is either 1 or 2, the gauge field propagator is zero.

As the $\lambda AA$-vertex is proportional to $\theta_{ij}$, which here is non-vanishing only in the $(x_1, x_2)$-plane, relation (16) has the following important consequence for the Feynman graphs: The combination of gauge boson propagator and $\lambda AA$ vertex is zero (see Figure 3). Furthermore, it is impossible to construct a closed loop including a $\lambda AA$-vertex without having such a combination somewhere. *Hence, all loop graphs involving the $\lambda AA$-vertex vanish!*

In particular, dangerous vacuum polarization insertions as in Figure 2 vanish. This is the reason, why the model is free of the most dangerous, i.e. the quadratic, infrared singularities, as pointed out by Slavnov [4] for the special case of $\mu^\mu = (0, 1, 0, 0)$.
4. Generalization

In this section we would like to discuss the question whether we can show cancellation of IR singular Feynman graphs for a more general choice of $\theta^{\mu \nu}$ and $n^\mu$. The answer is yes, but we need to impose stronger Slavnov constraints. The initial Slavnov constraint was $\theta^{12}F_{12} + \theta^{13}F_{13} + \theta^{23}F_{23} = 0$. With “stronger” we mean that each term in the sum should vanish separately. Upon imposing these stronger conditions we may write for the action [2]:

$$S_{\text{inv}} = \int d^4x \left[ -\frac{1}{4}F_{\mu \nu}F^{\mu \nu} + \frac{1}{2}\epsilon^{ijk}F_{ij}F_{k}\lambda_k \right],$$

(17)

with $i, j, k \in \{1, 2, 3\}$. This action looks like a 3 dimensional BF model coupled to Maxwell theory. As in the pure BF-case, the action has two gauge symmetries

$$\delta g_1 A_\mu = D_\mu \Lambda, \quad \delta g_2 A_\mu = 0,$$

$$\delta g_1 \lambda_k = -ig [\lambda_k, \Lambda], \quad \delta g_2 \lambda_k = D_k \lambda'.$$

(18)

Similar to the previous model, we have an additional bosonic vector symmetry of the gauge invariant action:

$$\hat{\delta} i A_0 = -F_{i0}, \quad \hat{\delta} i \lambda_j = \epsilon_{ijk}D_0 F^{0k},$$

$$\hat{\delta} i A_i = 0.$$  

(19)

There is, however, a difference to the previous case: The additional vectorial symmetry is broken when fixing the second gauge symmetry $\delta g_2$.

If we consider a space-like axial gauge fixing of the form

$$S_{\text{gf}} = \int d^4x \left[ Bn^i A_i + d' n^i \lambda_i - \bar{c} n^i D_i \bar{c} - \bar{\phi} n^i D_i \phi \right],$$

(20)

the gauge fixed action is invariant under the linear VSUSY

$$\delta i c = A_i, \quad \delta i \lambda_j = -\epsilon_{ijk}n^k \bar{c},$$

$$\delta i B = \partial_i \bar{c}, \quad \delta i \Phi = 0, \quad \text{for all other fields},$$

(21)

in addition to the usual BRST invariance. The Ward identity describing the linear vector supersymmetry in terms of $Z^c$ at the classical level is given by

$$W_i Z^c = \int d^4x \left[ j_B \partial_i \frac{\delta Z^c}{\delta j^c_A} - j_c \frac{\delta Z^c}{\delta j^c_A} + \epsilon_{ijk}n^j \bar{c} \frac{\delta Z^c}{\delta j^c_i} \right] = 0.$$

(22)

Hence, the same arguments as before show the absence of IR singular graphs. However, the model exhibits numerous further symmetries which have been discussed in [2].

One should also note, that a generalization to higher dimensional models is possible. For example if $\lambda$ had $n$ indices the VSUSY would become

$$\delta i c = A_i, \quad \delta i \lambda_{j_1 \cdots j_n} = \epsilon_{i j_1 \cdots j_n} n^k \bar{c},$$

$$\delta i B = \partial_i \bar{c},$$

(23)

after appropriate redefinitions of Lagrange multipliers.

1 $d' = d - ig [\bar{\phi}, c]$ is the redefined multiplier field fixing the second gauge freedom $\delta g_2$. 

---

International Conference on Non-commutative Geometry and Physics IOP Publishing
Journal of Physics: Conference Series 103 (2008) 012009
doi:10.1088/1742-6596/103/1/012009
5. Conclusion and Outlook

Slavnov-extended Yang Mills theory can be shown to be free of the worst infrared singularities, if the Slavnov term is of BF-type. Furthermore, supersymmetry, in the form of VSUSY, seems to play a decisive role in theories which are not Poincaré supersymmetric. Open questions are:

- What is the role of VSUSY with respect to UV/IR mixing in topological NCGFT in general?
- What are the consequences of the additional symmetries appearing in these models?

Acknowledgments

The author would like to thank S. Hohenegger for helpful comments. D. N. Blaschke is a recipient of a DOC-fellowship of the Austrian Academy of Sciences at the Institute for Theoretical Physics at Vienna University of Technology.

References

[1] Blaschke D N, Gieres F, Piguet O and Schweda M 2006 *JHEP* **05** 059 (Preprint arXiv:hep-th/0604154).
[2] Blaschke D N and Hohenegger S 2007 Preprint arXiv:0705.3007.
[3] Slavnov A A 2003 *Phys. Lett.* B **565** 246 (Preprint arXiv:hep-th/0304141).
[4] Slavnov A A 2004 *Teor. Mat. Fiz.* **140** N3 388.
[5] Blaschke D N, Hohenegger S and Schweda M 2005 *JHEP* **11** 041 (Preprint arXiv:hep-th/0510100).
[6] Attems M, Blaschke D N, Ortner M, Schweda M, Stricker S and Weiretmayr M 2005 *JHEP* **07** 071 (Preprint arXiv:hep-th/0506117).
[7] Snyder H S 1947 *Phys. Rev.* **71** 38, **72** 68.
[8] Filk T 1996 *Phys. Lett.* B **376** 53.
[9] Douglas M R and Nekrasov N A *Rev. Mod. Phys.* **73** (2001) 977 (Preprint arXiv:hep-th/0106048).
[10] Minwalla S, Van Raamsdonk M and Seiberg N 2000 *JHEP* **02** 020 (Preprint arXiv:hep-th/9912072).
[11] Micu A and Sheikh Jabbari M M 2001 *JHEP* **01** 025 (Preprint arXiv:hep-th/0008057).
[12] Matusis A, Susskind L and Toumbs N 2000 *JHEP* **12** 002 (Preprint arXiv:hep-th/0002075).
[13] Seiberg N, Susskind L and Toumbs N 2000 *JHEP* **06** (2000) (Preprint arXiv:hep-th/0005015).
[14] Hayakawa M 2000 *Phys. Lett.* B **478** 394 (Preprint arXiv:hep-th/9912094).
[15] Ruiz F R 2001 *Phys. Lett.* B **502** 274 (Preprint arXiv:hep-th/0012171).
[16] Kummer W 1961 *Acta Phys. Austriaca* **14** 149, 1975 *Acta Phys. Austriaca* **41** 315.
[17] Boresch A, Emery S, Moritsch O, Schweda M, Sommer T and Zerrouki H 1998 *Applications of Noncovariant Gauges in the Algebraic Renormalization Procedure* (World Scientific).