POINTS, WHOSE PEDAL TRIANGLES ARE SIMILAR TO THE GIVEN TRIANGLE

GEORGI GANCHEV¹, GYULBEYAZ AHMED² AND MARINELLA PETKOVA³

Abstract. We study the eleven points in the plane of a given triangle, whose pedal triangles are similar to the given one. We prove that the six points whose pedal triangles are positively oriented, lie on a single circle, while the five points, whose pedal triangles are negatively oriented, lie on a common straight line.

1. Preliminaries

First, following mainly [2] we give some notions and facts, which we use further in the paper (see also [3]).

1.1. The map $f$. Let $ABC$ be a triangle with sides $BC = a$, $CA = b$, $AB = c$, and angles $\angle A = \alpha$, $\angle B = \beta$, $\angle C = \gamma$, which we call the basic triangle. For an arbitrary point $M$ in the plane of $\triangle ABC$ we denote the segments $MA = x$, $MB = y$ and $MC = z$.

Any circle inversion $\varphi(M, r)$ with center $M$ and an arbitrary radius $r$ maps the points $ABC$ into the points $A_1B_1C_1$ (Figure 1), respectively, so that

$B_1C_1 = \frac{r^2}{xyz} ax, \quad C_1A_1 = \frac{r^2}{xyz} by, \quad A_1B_1 = \frac{r^2}{xyz} cz.$

Figure 1

The points $A_1, B_1, C_1$ are collinear if and only if $M \in k$.

Next we assume $M \notin k$ and denote the angles of $\triangle A_1B_1C_1$ by $(\alpha_1, \beta_1, \gamma_1)$. Then (1) means that any circle inversion $\varphi(M)$ with center $M$ determines one and the same triple of angles $(\alpha_1, \beta_1, \gamma_1)$ of a triangle. Hence, there arises a map

$f : M \rightarrow (\alpha_1, \beta_1, \gamma_1); \quad M \notin k, \quad (\alpha_1, \beta_1, \gamma_1) - a triple of angles of a triangle.$

First we note that $f(O) = (\alpha, \beta, \gamma)$.

For points, different from $O$, we shall prove the following statement.

2000 Mathematics Subject Classification. Primary 53A05, Secondary 53A10.

Key words and phrases. Brocard points, pedal triangles.
Proposition 1.1. Given a \( \triangle ABC \) and the circum-circle \( k \) of the triangle. If \( M \) and \( N \) are two inverse points with respect to the circle \( k \), then

\[
f(M) = f(N).
\]

Proof: Let \( \varphi(M, r) \) and \( \varphi'(N, r') \) be two circle inversions with centers \( M \) and \( N \), respectively. Denote by \( A_1B_1C_1 \) the image of the triple \( ABC \) under the inversion \( \varphi \) and by \( A'B'C' \) the image of the triple \( ABC \) under the inversion \( \varphi' \) (Figure 1).

If \( MA = x, MB = y, MC = z \) and \( NA = x', NB = y', NC = z' \), then we have

\[
\begin{align*}
B_1C_1 &= \frac{r^2}{xyz} ax, \quad C_1A_1 = \frac{r^2}{xyz} by, \quad A_1B_1 = \frac{r^2}{xyz} cz; \\
B'C' &= \frac{r^2}{x'y'z'} ax', \quad C'A' = \frac{r^2}{x'y'z'} by', \quad A'B' = \frac{r^2}{x'y'z'} cz'.
\end{align*}
\]  

(2)

On the other hand, since \( M \) and \( N \) are inverse points with respect to the circle \( k \), then \( k \) is Apollonius circle with basic points \( M, N \) and ratio \( \frac{d}{R} \), where \( OM = d \). Therefore

\[
\frac{x}{x'} = \frac{y}{y'} = \frac{z}{z'} = \frac{d}{R}.
\]

Then (2) and (3) imply that

\[
\frac{B_1C_1}{B'C'} = \frac{C_1A_1}{C'A'} = \frac{A_1B_1}{A'B'} = \frac{r^2}{xyz} \frac{x'y'z'}{R} d.
\]

Hence \( \triangle A_1B_1C_1 \sim \triangle A'B'C' \), which means that \( f(M) = f(N) \). \( \square \)

Proposition 1.2. Given the basic triangle \( ABC \) with angles \( (\alpha, \beta, \gamma) \) and an arbitrary triple of angles \( (\alpha_1, \beta_1, \gamma_1) \) of a triangle. Prove that:

(i) if \( (\alpha_1, \beta_1, \gamma_1) \neq (\alpha, \beta, \gamma) \), then there exist exactly two points \( M \) and \( N \) such that \( f(M) = f(N) = (\alpha_1, \beta_1, \gamma_1) \);

(ii) if \( (\alpha_1, \beta_1, \gamma_1) = (\alpha, \beta, \gamma) \), then only the center \( O \) satisfies the condition \( f(O) = (\alpha, \beta, \gamma) \).

Proof: (i) Let \( A_1B_1C_1 \) be a triangle with angles \( (\alpha_1, \beta_1, \gamma_1) \), respectively, and denote by \( B_1C_1 = a_1, \ C_1A_1 = b_1, \ A_1B_1 = c_1 \). If \( M \) is a solution to the equation \( f(X) = (\alpha_1, \beta_1, \gamma_1) \), then according to (1) we have

\[
\frac{ax}{a_1} = \frac{by}{b_1} = \frac{cz}{c_1}
\]

and consequently

\[
\frac{xy}{z} = \frac{by}{b_1} = \frac{cz}{c_1} \quad \text{and} \quad \frac{xy}{z} = \frac{ax}{a_1} = \frac{by}{b_1} = \frac{cz}{c_1}.
\]

Thus we obtained that any solution \( M \) to the equation \( f(X) = (\alpha_1, \beta_1, \gamma_1) \) is a common point of the three Apollonius circles \( k_1 \left( B, C; \lambda = \frac{b_1}{c_1} \frac{c}{b} \right), k_2 \left( C, A; \mu = \frac{c_1}{a_1} \frac{a}{c} \right) \) and \( k_3 \left( A, B; \nu = \frac{a_1}{b_1} \frac{b}{a} \right) \) (Figure 2).

Since \( (\alpha_1, \beta_1, \gamma_1) \neq (\alpha, \beta, \gamma) \) and \( \lambda \mu \nu = 1 \), at least one of the numbers \( \lambda, \mu, \nu \) is greater than 1 and at least one of them is less than 1. For definiteness let \( \lambda > 1 \) and \( \mu < 1 \). Therefore the point \( C \) is interior for the circles \( k_1 \) and \( k_2 \).
Let \((O_1, r_1)\) and \((O_2, r_2)\) be the corresponding centers and radii of \(k_1\) and \(k_2\), respectively. To prove that \(k_1\) and \(k_2\) intersect into two points, it is sufficient to show that \(|r_1 - r_2| < O_1O_2\), which is equivalent to

\[O_1O_2^2 - (r_1 - r_2)^2 > 0.\]

It is easy to find that

\[O_1O_2 = \frac{\mu^2}{1 - \mu^2} \overrightarrow{BC} - \frac{1}{\lambda^2 - 1} \overrightarrow{AC}, \quad r_1 = \frac{\lambda a}{\lambda^2 - 1}, \quad r_2 = \frac{\mu b}{1 - \mu^2}.\]

Then we get

\[O_1O_2^2 - (r_1 - r_2)^2 = \frac{1}{(\lambda^2 - 1)(1 - \mu^2)} [(c\mu)^2 - (b\lambda\mu - a)^2].\]

The equalities \(\lambda = \frac{b_1}{c_1} \frac{c}{b}\) and \(\mu = \frac{c_1}{a_1} \frac{a}{c}\) imply that

\[\left| \frac{b\lambda - a}{c} - \frac{a \mu}{cm} \right| = \frac{|b_1 - a_1|}{c_1} < 1.\]

Therefore \((c\mu)^2 - (b\lambda\mu - a)^2 > 0\) and the circles \(k_1, k_2\) intersect into two points, which we denote by \(M\) and \(N\).

The condition \(\lambda\mu\nu = 1\) implies that the third circle \(k_3\) also passes through \(M\) and \(N\).

(ii) If \(\alpha_1 = \alpha, \beta_1 = \beta, \gamma_1 = \gamma\), then \(\lambda = \mu = \nu = 1\) and \(k_1, k_2, k_3\) are the perpendicular bisectors of the corresponding sides of \(\triangle ABC\). Hence, in this case \(k_1, k_2, k_3\) have one common point \(O\) and the only solution to the equation \(f(X) = (\alpha_1, \beta_1, \gamma_1)\) is the point \(O\).

Now Proposition 1.1 and Proposition 1.2 imply the following

**Theorem 1.3.** Given the basic triangle \(ABC\) with angles \((\alpha, \beta, \gamma)\) and an arbitrary triple of angles \((\alpha_1, \beta_1, \gamma_1)\) of a triangle. Then:

(i) there exists exactly one point \(M\), interior for \(k\), such that \(f(M) = (\alpha_1, \beta_1, \gamma_1)\);

(ii) if \((\alpha_1, \beta_1, \gamma_1) \neq (\alpha, \beta, \gamma)\), then there exists exactly one point \(N\), exterior for \(k\), such that \(f(N) = (\alpha_1, \beta_1, \gamma_1)\). Furthermore \(N\) and \(M\) from (i) are inverse points with respect to the circum-circle of \(\triangle ABC\).\]
Next we shall prove that this correspondence is a realization of the map \( f \).

**Proposition 1.4.** Given the basic \( \triangle ABC \) with circum-circle \( k \) and a point \( M \), interior for the circle \( k \). If \((\alpha_1, \beta_1, \gamma_1)\) are the corresponding angles of the pedal triangle of \( M \) with respect to the basic triangle, then

\[
f(M) = (\alpha_1, \beta_1, \gamma_1).
\]

**Proof:** Since \( AM = x \) is a diameter of the circum-circle of \( \triangle AC_1B_1 \) (Figure 3), we find

\[
B_1C_1 = AM \sin \alpha = \frac{ax}{2R}.
\]

Similarly we find \( C_1A_1 = \frac{by}{2R} \) and \( A_1B_1 = \frac{cz}{2R} \). Thus we obtain

\[
\frac{ax}{B_1C_1} = \frac{by}{C_1A_1} = \frac{cz}{A_1B_1} = 2R.
\]

Now the assertion follows from (1) and the definition of \( f \). \( \square \)

![Figure 3](image-url)

Taking into account how a circle inversion transforms the circles in the plane, we have:

Let \( \varphi \) be a circle inversion with center \( M \) and \( \varphi(ABC) = A'B'C' \). Then:

- \( \triangle A'B'C' \) is positively oriented (has the orientation of \( \triangle ABC \)) if and only if \( M \) is inside the circle \( k \);
- \( \triangle A'B'C' \) is negatively oriented if and only if \( M \) is outside the circle \( k \);

The above statement and Proposition 2.1 imply the following:

Let \( \triangle A_1B_1C_1 \) be the pedal triangle of a point \( M \). Then:

- \( \triangle A_1B_1C_1 \) is positively oriented (has the orientation of \( \triangle ABC \)) if and only if \( M \) is inside the circle \( k \);
- \( \triangle A_1B_1C_1 \) is negatively oriented if and only if \( M \) is outside the circle \( k \);

A point \( M \) lies on \( k \), if and only if its pedal triangle is degenerate (which is the Simson’s theorem).

These considerations, Proposition 1.4 and Theorem 1.3 imply the following

**Theorem 1.5.** Given the basic triangle \( ABC \) with angles \((\alpha, \beta, \gamma)\) and an arbitrary triple of angles \((\alpha_1, \beta_1, \gamma_1)\) of a triangle. Then:

(i) there exists exactly one point \( M \), inside the circum-circle \( k \) of \( \triangle ABC \), such that \((\alpha_1, \beta_1, \gamma_1)\) are the angles of its pedal triangle. In this case the pedal triangle of \( M \) and the basic triangle have the same orientation.

(ii) if \((\alpha_1, \beta_1, \gamma_1) \neq (\alpha, \beta, \gamma)\), then there exists exactly one point \( N \), outside the circum-circle \( k \) of \( \triangle ABC \), such that \((\alpha_1, \beta_1, \gamma_1)\) are the angles of its pedal triangle. In this case the
pedal triangle of \(N\) and the basic triangle have the opposite orientations. Furthermore, \(N\) and \(M\) from (i) are inverse points with respect to the circum-circle of the basic triangle.

Remark 1.6. It follows from Theorem 1.5 that there does not exist a point \(N\), whose pedal triangle is similar to the basic triangle and negatively oriented.

The angles of \(\triangle ABC\) and \(\triangle A_1B_1C_1\) determine the angles \(\angle BMC, \angle CMA, \text{ and } \angle AMB\) in the following way (Figure 3):

\[
\angle AMB = \gamma + \gamma_1, \quad \angle CMA = \beta + \beta_1, \quad \angle BMC = \alpha + \alpha_1.
\]

We adopt the following convention: \(\angle BMC = \pi\) if and only if \(M\) lies on the side \(BC\); \(\angle BMC > \pi\) if and only if \(M\) is inside the circle \(k\), but \(M\) and \(A\) are from different sides of the line \(BC\). Then formulas (1) are valid for all points \(M\) inside the circle \(k\). Then, for any point inside the circle \(k\), we have the following simple criterion:

Let \(M\) be inside the circle \(k\). Then the angles of the pedal \(\triangle A_1B_1C_1\) are

\[
\alpha_1 = \angle BMC - \alpha, \quad \beta_1 = \angle CMA - \beta, \quad \gamma_1 = \angle AMB - \gamma.
\]

2. Points, whose pedal triangles are similar to the given one

Let \(k(O, R)\) be the circum-circle of the basic \(\triangle ABC\). In this section we study the points, whose pedal triangles are similar to the given \(\triangle ABC\).

2.1. Points, whose pedal triangles are positively oriented. Let \(\Omega_1\) be the first Brocard point for the basic \(\triangle ABC\) (Figure 4). According to the definition of \(\Omega_1\) we have \(\angle \Omega_1AB = \angle \Omega_1BC = \angle \Omega_1CA = \omega\), \(\omega\) being the Brocard angle.

If \(A_1B_1C_1\) is the pedal triangle of \(\Omega_1\), then it follows immediately that \(\angle B_1A_1C_1 = \alpha_1 = \beta, \quad \angle C_1B_1A_1 = \beta_1 = \gamma, \quad \angle A_1C_1B_1 = \gamma_1 = \beta\). Hence, the pedal triangle of \(\Omega_1\) is similar to \(\triangle BCA\) and positively oriented.

The second Brocard point \(\Omega_2\) is introduced by the conditions \(\angle \Omega_2AC = \angle \Omega_2CB = \angle \Omega_2BA = \omega\), which implies that the angles of the pedal triangle of \(\Omega_2\) are \((\alpha_1, \beta_1, \gamma_1) = (\gamma, \alpha, \beta)\). Therefore, the pedal triangle of \(\Omega_2\) is similar to \(\triangle CAB\) and positively oriented.

Now, let \(L_3\) be the point, whose pedal \(\triangle A_1B_1C_1\) is positively oriented and similar to \(\triangle BAC\). This means that \(L_3\) is inside the circle \(k\) and

\[
\angle BL_3C = \angle CL_3A = \alpha + \beta, \quad \angle AL_3B = 2\gamma.
\]

The last equality implies that \(L_3\) is a point on the arc \(AOB\) (Figure 5).
Further, we denote by $k'$ the circum-circle of $\triangle ABO$ ($k'$ is the line $AB$ if $\angle ACB = 90^\circ$) and by $EF$ the diameter of $k$, perpendicular to the side $AB$. If $K$ is the common point of $CL_3$ and $EF$, then the condition $\angle AL_3K = \angle BL_3K$ implies that $OK$ is a diameter of $k'$.

Let $\varphi(O, R)$ be the inversion with respect to the circle $k$. If $D$ is the midpoint of the side $AB$, then $\varphi(D) = K$ and the quadruple $DKEF$ is harmonic. Therefore, the circle $k$ is Apollonius circle with basic points $D, K$ passing through the point $C$. Hence, $CE(CF)$ is the interior (exterior) bisector of $\angle DCK$. Thus we obtained that $L_3$ is the common point of the arc $AOC$ and the ray of the symmedian to the side $AB$. Since $OK$ is a diameter of the circle $k'$, then

\begin{equation}
OL_3 \perp CK.
\end{equation}

The orthogonal projections of the circumcenter on the three symmedians of $\triangle ABC$ are considered from many other aspects (see e.g. [1], Theorems 120-121).

So, the six points on the circle $k$, whose pedal triangles are positively oriented and similar to the basic triangle, are as follows:

- the center $O$ of the circum-circle: $\triangle A_1B_1C_1 \sim \triangle ABC$;
- the first Brocard point $\Omega_1$: $\triangle A_1B_1C_1 \sim \triangle BCA$;
- the second Brocard point $\Omega_2$: $\triangle A_1B_1C_1 \sim \triangle CAB$;
- the orthogonal projection $L_1$ of $O$ on the ray of the symmedian to the side $BC$: $\triangle A_1B_1C_1 \sim \triangle ACB$;
- the orthogonal projection $L_2$ of $O$ on the ray of the symmedian to the side $CA$: $\triangle A_1B_1C_1 \sim \triangle CBA$;
- the orthogonal projection $L_3$ of $O$ on the ray of the symmedian to the side $AB$: $\triangle A_1B_1C_1 \sim \triangle BAC$;

Let us denote by $L$ the Lemoine point for the basic triangle. It is wellknown fact (e.g. [1],[4]) that the Brocard points $\Omega_1$ and $\Omega_2$ lie on the circle $k_0$ with diameter $OL$ so that $\angle \Omega_1OL = \angle \Omega_2OL = \omega$, where $\omega$ is the Brocard angle.

The circle $k_0$ is called the Brocard’s circle.

Now the following statement follows at once.
Theorem 2.1. The six points $O, \Omega_1, \Omega_2, L_1, L_2, L_3$, whose pedal triangles are positively oriented and similar to the basic triangle, lie on the Brocard’s circle.

Using the properties of the symmedians of a triangle, we find the following representations for the points $L_1, L_2, L_3$ in barycentric coordinates:

\[
\overrightarrow{OL_1} = \frac{(b^2 + c^2 - a^2) \overrightarrow{OA} + b^2 \overrightarrow{OB} + c^2 \overrightarrow{OC}}{2(b^2 + c^2) - a^2},
\]

\[
\overrightarrow{OL_2} = \frac{a^2 \overrightarrow{OA} + (c^2 + a^2 - b^2) \overrightarrow{OB} + c^2 \overrightarrow{OC}}{2(c^2 + a^2) - b^2},
\]

\[
\overrightarrow{OL_3} = \frac{a^2 \overrightarrow{OA} + b^2 \overrightarrow{OB} + (a^2 + b^2 - c^2) \overrightarrow{OC}}{2(a^2 + b^2) - c^2}.
\]

2.2. Points, whose pedal triangles are negatively oriented. In this subsection we study the points exterior for the circle $k$, whose pedal triangles are similar to the basic triangle. It follows from Theorem 1.5 that these points are the inverse points of $\Omega_1, \Omega_2, L_1, L_2, L_3$ with respect to $\varphi$ and we denote them by $\Omega'_1, \Omega'_2, L'_1, L'_2, L'_3$, respectively.

Applying Theorem 2.2 we obtain the following statement.

Theorem 2.2. The points $\Omega'_1, \Omega'_2, L'_1, L'_2, L'_3$, whose pedal triangles are negatively oriented and similar to the basic triangle, lie on a single straight line.

We denote this straight line $g = \varphi(k_0)$.

The position of the five points from Theorem 2.2 is given in Figure 6 under the assumption $a < c < b$.

First we study the point $L'_3$ (Figure 6). From one hand, $L'_3$ lies on the line $OL_3$. On the other hand, $L'_3$ lies on the line $AB = \varphi(k')$. Hence $L'_3 = AB \cap OL_3$.

We note that if $CA = CB$ (isosceles $\triangle ABC$), then $g \parallel AB$ and $L'_3$ is the point at infinity of the line $AB$.

Taking into account that $L'_3 = \varphi(L_3)$ and (2.1), we conclude that $CL'_3$ is tangent to $k$ at $C$. Therefore $CL'_3$ is the exterior simmedian through the vertex $C$ and $L'_3A : L'_3B = b^2 : a^2$. The last equality shows that the point $L'_3$ is the center of the Apollonius circle $k_3$ with basic points $A, B$, which contains $C$. This circle becomes the perpendicular bisector of the side $AB$ when $CA = CB$. 

![Figure 6](image-url)
Similarly we have: $L_1'$ is the center of the Apollonius circle $k_1$ with basic points $B, C$, containing $A$; $L_2'$ is the center of the Apollonius circle $k_2$ with basic points $C, A$, containing $B$. We call these Apollonius circles the basic Apollonius circles of $\triangle ABC$.

Further, we obtain the following statement.

**Corollary 2.3.** For any triangle the straight line, passing through the center of the circum-circle and the Lemoine point, is perpendicular to the axis, containing the centers of the three basic Apollonius circles of the triangle.

The above statement also shows that $\Omega_1\Omega_2$ is parallel to the axis $g$.

The points $\Omega_1'$ and $\Omega_2'$ are $O\Omega_1 \cap g$ and $O\Omega_2 \cap g$, respectively, and $L'$ is the midpoint of the segment $\Omega_1'\Omega_2'$. Then we can calculate the barycentric coordinates of the point $\Omega_1'$:

$$O\Omega_1' = \frac{a^2(a^2 - b^2) \overrightarrow{OA} + b^2(b^2 - c^2) \overrightarrow{OB} + c^2(c^2 - a^2) \overrightarrow{OC}}{a^4 + b^4 + c^4 - a^2b^2 - b^2c^2 - c^2a^2}.$$  

Replacing $(a, b, c)$ with $(b, c, a)$ in the last formula, we obtain

$$O\Omega_2' = \frac{a^2(a^2 - c^2) \overrightarrow{OA} + b^2(b^2 - a^2) \overrightarrow{OB} + c^2(c^2 - b^2) \overrightarrow{OC}}{a^4 + b^4 + c^4 - a^2b^2 - b^2c^2 - c^2a^2}.$$  

Since $L_3'$ is the midpoint of the segment $\Omega_1'\Omega_2'$, then

$$OL' = \frac{a^2(2a^2 - b^2 - c^2) \overrightarrow{OA} + b^2(2b^2 - c^2 - a^2) \overrightarrow{OB} + (2c^2 - a^2 - b^2) \overrightarrow{OC}}{2(a^4 + b^4 + c^4 - a^2b^2 - b^2c^2 - c^2a^2)}.$$  

**Remark 2.4.** Taking into account the characterization:

The first (second) Brocard point $\Omega_1$ ($\Omega_2$) is the only point, whose pedal triangle is positively oriented and similar to $\triangle BCA$ ($\triangle CAB$),

we can consider the points $\Omega_1'$ and $\Omega_2'$ as the exterior Brocard points of the given triangle.

From the above formulas we obtain immediately:

**Corollary 2.5.** The following statements are equivalent:

(i) $b = c$;
(ii) $\Omega_1 \equiv L_2$;
(iii) $\Omega_2 \equiv L_3$.

Further we obtain the following new characterization for the triangles, whose sides are proportional to their medians.

**Proposition 2.6.** Given a $\triangle ABC$ with circum-circle $k(O)$. The following conditions are equivalent:

(i) $a^2 + b^2 = 2c^2$;
(ii) the Lemoine point $L$ lies on the circum-circle of $\triangle ABO$;
(iii) the inverse image of the Lemoine point with respect to $k(O)$ lies on the side-line $AB$.

**References**

[1] J. Coolidge, A Treatise on the Geometry of the Circle and Sphere. New York: Chelsea, (1916).
[2] G. Ganchev, Etudes on the theme Inversion, Mathematica plus, (1994),4, 24-32. (in Bulgarian)
[3] G.Ganchev, N. Nikolov, Isogonal conjugacy and Fermat Problems. Matematicheskoe Prosveshchenie, III, 12 (2008), 185-194. (In Russian)
[4] R. Honsberger, Episodes in nineteenth and twentieth century Euclidean geometry. The Mathematical association of America, (1995).
1 Bulgarian Academy of Sciences, Institute of Mathematics and Informatics, Acad. G. Bonchev Str. bl. 8, 1113 Sofia, Bulgaria
E-mail address: 1 ganchev@math.bas.bg

2,3 High school "P. R. Slaveykov" Kardjali, Bulgaria