Holographic classification of topological insulators and its eightfold periodicity

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Received 6 June 2012, in final form 5 September 2012
Published 10 October 2012
Online at stacks.iop.org/JPhysA/45/435203

Abstract
Using generic properties of Clifford algebras in any spatial dimension, we explicitly classify Dirac Hamiltonians with zero modes protected by the discrete symmetries of time reversal, particle–hole symmetry and chirality. Assuming that the boundary states of topological insulators are Dirac fermions, we thereby holographically reproduce the periodic table of topological insulators found by Kitaev (2009 AIP Conf. Proc. 1134 22) and Ryu et al (2010 New J. Phys. 12 065010), without using topological invariants or K-theory. In addition, we find candidate $\mathbb{Z}_2$ topological insulators in classes AI, AII of dimensions 0,4 mod 8 and in classes C, D of dimensions 2,6 mod 8.

PACS numbers: 03.65.Vf, 03.65.Pm

1. Introduction

Topological insulators (TIs) are characterized by bulk band structures with special topological properties [1–7]. Namely, from the bulk wavefunctions in momentum space, one can construct a gauge field and the topological invariant is essentially a Chern number. These physical systems possess a bulk/boundary correspondence, in that they necessarily have protected gapless excitations on the $d = (d-1)$-dimensional surface. These surface modes are typically described by Dirac Hamiltonians. For example in the integer quantum Hall effect (QHE) in $d = 2$, the Chern number is the same as in the quantized Hall conductivity, and the edge states are chiral Dirac fermions [8, 9].

Kitaev [10] and Ryu et al [12] classified TIs in any spatial dimension according to the discrete symmetries of time reversal $T$, particle–hole symmetry $C$ and chirality $P$ and found five classes of TIs of any dimension. These classifications were based on the existence of topological invariants [12] or K-theory [10]. The bulk/boundary correspondence was pointed out in [11] for $d = 3$ spatial dimensions; using the classification of $d = d - 1 = 2$ dimensional Dirac Hamiltonians in [15], it was found that precisely 5 of the 13 Dirac classes had protected zero modes with the predicted discrete symmetries. In that analysis, it was crucial that the
classification in [15] contained three additional classes beyond the ten Altland–Zirnbauer (AZ) classes, since it was precisely these additional classes that corresponded to some of the TIs.

This ‘holographic’ classification of TIs, i.e. based on the existence of symmetry-protected zero modes on the boundary, is not necessarily equivalent to a classification based on topology or K-theory. Indeed, this issue was studied in [13] for $d = 2$, and six additional possible classes of TIs were found in addition to the predicted five. This motivated the present work, which presents a holographic classification of TIs in any dimension. We should emphasize from the beginning that we do not perform a complete classification of the most general Dirac Hamiltonian in any dimension, as was done in [13, 15] for $d = 2$, 1, since this would hardly be useful in higher dimensions. Rather, the goal is to begin with a minimal form of Dirac Hamiltonian, and then classify those which have protected zero modes. If one can thereby reproduce the periodic table of TIs as given in [10, 12], then this supports the validity of this holographic classification and further supports the $d = 2$ results in [13].

The remainder of the paper is organized as follows. In the following section we review the definitions of the ten AZ classes. In section 3, we formulate the classification problem in terms of Clifford algebras. The specific representation of the Clifford algebras we will use are presented in section 4. In section 5, we describe how to realize all ten AZ classes in any dimension. Section 6 contains our classification of protected zero modes; we reproduce the periodic table, and find one additional candidate for a TI in every even dimension. In section 7, we briefly describe why $d = 2$ is exceptional.

2. Discrete symmetries

The ten AZ classes of random Hamiltonians arise when one considers time-reversal symmetry (T), particle–hole symmetry (C) and parity or chirality (P) [14]. These discrete symmetries are defined to act as follows on a first-quantized Hamiltonian $\hat{H}$:

$$
T: \hat{H} \rightarrow T \hat{H} T^\dagger = \hat{H}
$$

$$
C: \hat{H} = C \hat{H} C^\dagger = -\hat{H}
$$

$$
P: \hat{H} \rightarrow P \hat{H} P^\dagger = -\hat{H}
$$

with $TT^\dagger = CC^\dagger = PP^\dagger = 1$, and $\hat{H}^T$ denotes the transpose of $\hat{H}$. In our classification, two Hamiltonians $\hat{H}$ and $\hat{H}'$ related by a unitary transformation $\hat{H}' = U \hat{H} U^\dagger$ are in the same class, since they have the same eigenvalues. For $C$ and $T$, this translates to $C \rightarrow C' = UCU^T$ and $T \rightarrow T' = UTU^T$. For $P$, the unitary transformation is $P \rightarrow P' = UPU^\dagger$. In the following, we will refer to these unitary transformations as gauge transformations.

For Hermitian Hamiltonians, $\hat{H}^T = \hat{H}^*$, thus, up to a sign, C and T symmetries are the same. We then focus on these symmetries involving the transpose: $T\hat{H}T^\dagger = \hat{H}$ and $C\hat{H}C^\dagger = -\hat{H}$. Taking the transpose of this relation, one finds that there are two consistent possibilities: $T^\dagger = \epsilon_1 T$ and $C^\dagger = \epsilon_2 C$, where $\epsilon_1, \epsilon_2 = \pm 1$, which are gauge-invariant relations. The various classes are thus distinguished by $\epsilon_1 = \pm 1, \emptyset$ and $\epsilon_2 = \pm 1, \emptyset$, where $\emptyset$ indicates that the Hamiltonian does not have the symmetry. (In some literature, $T$ and $C$ are chosen to be real, unitarity implies $T^2 = \epsilon_1$, $C^2 = \epsilon_2$, and this sign of the square characterizes the classes; however, this is not a gauge-invariant statement.) One obtains $9 = 3 \times 3$ classes just
by considering the three cases for $T$ and $C$. If the Hamiltonian has both $T$ and $C$ symmetries, then it automatically has a $P$ symmetry, with $P = TC^\dagger$ up to a phase. If there is neither $T$ nor $C$ symmetry, then there are two choices $P = \emptyset, 1$, and this gives the additional class AIII, leading to a total of 10. Their properties are shown in table 1. We also mention that one normally requires $P^2 = 1$. Below, we will require $T$ and $C$ to commute; thus $P^2 = T^2 C^\dagger^2 = \pm 1$. However, one has the freedom $P \to iP$ to restore $P^2 = 1$. In the following, in the cases with both $T$ and $C$ symmetries, we simply define $P = TC^\dagger$, up to a phase.

### Table 1. The ten AZ Hamiltonian classes. The $\pm$ signs refer to $T^T = \pm T$ and $C^T = \pm C$, whereas $\emptyset$ denotes the non-existence of the symmetry.

| AZ classes | $T$  | $C$  | $P$  |
|------------|------|------|------|
| A          | $\emptyset$ | $\emptyset$ | $\emptyset$ |
| AIII       | $\emptyset$ | $\emptyset$ | 1    |
| AI         | $\emptyset$ | $\emptyset$ | $-1$ |
| C          | $\emptyset$ | $\emptyset$ | $1$  |
| D          | $\emptyset$ | $\emptyset$ | $1$  |
| BDI        | $+1$ | $\emptyset$ | 1    |
| DIII       | $-1$ | $+1$ | 1    |
| CII        | $-1$ | $-1$ | 1    |
| CI         | $+1$ | $-1$ | 1    |

3. Formulation in terms of Clifford algebras

Let $d$ denote the spatial dimension and $\overrightarrow{a} = d - 1$ the dimension of the boundary. On the boundary, we assume the first quantized Dirac Hamiltonian of the form

$$\mathcal{H} = -i \sum_{a=1}^{\overrightarrow{a}} \gamma_a \frac{\partial}{\partial x_a} + M,$$

where $x_a$ are coordinates on the boundary and $\gamma_a, M$ are matrices. In the momentum space $k$, in order for the Hamiltonian to satisfy $\mathcal{H}^2 = k^2 + M^2$, and have a single particle energy spectrum $E = \pm \sqrt{k^2 + M^2}$, the $\gamma_a, a = 1, \ldots, \overrightarrow{a}$, must satisfy a Clifford algebra, and $M$ must anti-commute with all $\gamma_a$ in order for the cross terms in $\mathcal{H}^2$ to vanish:

$$\{\gamma_a, \gamma_b\} = 2 \delta_{ab}; \quad \{\gamma_a, M\} = 0, \forall a.$$

Thus up to rescaling of $M$, the set $\{\gamma_a, M\}$ forms a Clifford algebra. (The explicit form of $M$ will be given below, where in general it will be an element of a Clifford algebra times a constant or tensored with an additional space.)

The conditions for $P$, $T$ and $C$ symmetries are the following $\forall a$: :

$$P : \{P, \gamma_a\} = 0, \quad \{P, M\} = 0$$

$$T : T \gamma_a^T = -\gamma_a T, \quad TM^T = MT$$

$$C : C \gamma_a^T = \gamma_a C, \quad CM^T = -MC$$

The way these conditions are implemented is that one constructs $P$, $T$ and $C$ satisfying the first condition in each of the above cases, which is the most stringent, and then checks whether the second condition on $M$ is satisfied.
4. Clifford algebra representation

In this section, we describe an explicit representation of the Clifford algebra which we will utilize. A Clifford algebra is constructed from $N$ basis elements $\Gamma_a$, $a = 1, 2, \ldots, N$, satisfying the relations

$$\{\Gamma_a, \Gamma_b\} = 2\delta_{ab}. \quad (7)$$

We will refer to the algebra generated by linear combinations of products of the $\Gamma_a$ as the enveloping algebra of the Clifford algebra. (In the mathematics literature, this enveloping algebra is simply referred to as the Clifford algebra.) The degree of a monomial in the $\Gamma$ is the minimal number of factors subject to relations (7). Using the above relations of the basis elements on a 2-dimensional space in terms of an $n$-fold tensor product of Pauli matrices:

$$\begin{align*}
\Gamma_1 &= \sigma_x \otimes \sigma_x \otimes \sigma_x \otimes \cdots \otimes \sigma_x \\
\Gamma_2 &= \sigma_y \otimes \sigma_y \otimes \sigma_y \otimes \cdots \otimes \sigma_y \\
\Gamma_3 &= 1 \otimes \sigma_x \otimes \sigma_x \otimes \cdots \otimes \sigma_x \\
\Gamma_4 &= 1 \otimes \sigma_y \otimes \sigma_y \otimes \cdots \otimes \sigma_y \\
&\vdots \\
\Gamma_{2^n-1} &= 1 \otimes 1 \otimes \cdots \otimes 1 \otimes \sigma_x \\
\Gamma_{2^n} &= 1 \otimes 1 \otimes \cdots \otimes 1 \otimes \sigma_y \\
\Gamma_{2^n+1} &= \sigma_z \otimes \sigma_z \otimes \cdots \otimes \sigma_z \otimes \sigma_z,
\end{align*} \quad (8)$$

where $\sigma_x = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\sigma_y = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $\sigma_z = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Note that all $\Gamma_a$ are Hermitian and that $\Gamma_1\Gamma_2\cdots\Gamma_{2n+1}$ is proportional to the identity. The matrices $M_{ab} = [\Gamma_a, \Gamma_b]/4i$ comprise the Lie algebra for the irreducible spinor representation of $SO(2n+1)$. The $a$ index of $\Gamma_a$ transforms as the vector representation of $SO(2n+1)$.

Since the transpose is an anti-automorphism of the Clifford algebra, and an involution, then $A^T = \pm A$ for any monomial in the enveloping algebra. We will need

$$\begin{align*}
\Gamma_a^T &= -\Gamma_a \quad \text{if } a \neq 2n+1 \text{ is odd} \\
&= \Gamma_a \quad \text{if } a \text{ is even} \\
\Gamma_{2n+1}^T &= \Gamma_{2n+1}.
\end{align*} \quad (9)$$

In the following, the following elements of the Clifford algebra of degree $n$ and $n+1$ will play a central role in constructing the $T$ and $C$ symmetries:

$$G = \Gamma_1\Gamma_3\Gamma_5\cdots\Gamma_{2n-1}, \quad \tilde{G} = \Gamma_{2n+1}. \quad (10)$$

Using the transpose properties in equation (10) and the Clifford algebra relations, one can show that they satisfy

$$G^T = (-1)^{n(n+1)/2}G, \quad \tilde{G}^T = (-1)^{n(n+1)/2}\tilde{G}. \quad (11)$$

We will also need

$$\begin{align*}
GT_{2n+1} &= (-1)^n\Gamma_{2n+1}G, \quad \tilde{G}\Gamma_{2n+1} = (-1)^n\Gamma_{2n+1}\tilde{G} \\
\Gamma_{2n} &= (-1)^n\Gamma_{2n}G, \quad \tilde{G}\Gamma_{2n} = (-1)^{n+1}\Gamma_{2n}\tilde{G}.
\end{align*} \quad (12)$$

Finally, note that $GG \propto \Gamma_{2n+1}$, which we will also need.
5. Generic classification of Dirac fermions in any dimension

It is useful to first summarize the results of this section. Since the dimension of the above enveloping algebra of the Clifford algebra is $2^N = 2 \times 2^n \times 2^a$, any $2^n \times 2^a$ complex matrix can be expressed as an element of the enveloping algebra. Thus, the matrices $P, T$ and $C$ can be expressed in terms of products of the $\Gamma_a$ matrices. We find that in a given dimension $d$, there is a unique $T, C$ satisfying the stringent first condition in (5, 6), which is either $G$ or $\tilde{G}$, depending on the spatial dimension. Which class this symmetry belongs to is determined by the transpose relations (11). The eightfold periodicity arises from the even/odd properties of the powers in equation (11). Namely, $n(n − 1)/2$ is even for $n = 4m, 4m + 1$ and odd for $n = 4m + 2, 4m + 3$, where $m$ is an integer. On the other hand $n(n + 1)/2$ is even for $n = 4m, 4m + 3$ and odd for $n = 4m + 1, 4m + 2$. Finally, in order to obtain all the ten classes, one needs to tensor in an additional space, as will be explained. We need to distinguish between even and odd dimensions.

5.1. For $d$ odd

Let $d = 2n + 1$. Without loss of generality one can choose $\gamma_a = \Gamma_a$ for $a = 1, 2, …, 2n$, and $M = \Gamma_{2n+1}$, since other choices are related by unitary SO(d) rotations. The $P$ symmetry can be imposed with $P = \Gamma_{2n+1}$.

First consider $n$ odd. Then the unique $T$ that satisfies the first condition in equation (5) is $T = G$. When $d = 8m + 3$, i.e. $n = 4m + 1$, then $T^T = −T$. In order to obtain $T$ symmetry with the other sign in the transpose, one must tensor in an additional space. Let $\tilde{\tau}$ denote another set of Pauli matrices. Up to unitary transformations, the additional factor in $T$ is either $1$ or $i\tau_\xi$ [16], since they have opposite sign in the relation with their transpose. Thus, the other choice for $T$ is $T' = i\tau_\xi \otimes G$, satisfying $T'^T = T'$. On the other hand, when $n = 4m + 3$, i.e. $d = 8m + 7$, then $T^T = T$ and $T'^T = −T'$. The $C$ symmetry is similar. The solution to the first equation in (6) is $C = \tilde{G}$. If $d = 8m + 3$, then $C^T = C$, whereas if $d = 8m + 7$, then $C^T = −C$.

Again, in order to obtain the other sign in the transpose, one needs to consider $C' = i\tau_\xi \otimes \tilde{G}$.

Next consider $n$ even, i.e. $n = 4m$ or $4m + 2$, corresponding to $d = 8m + 1, 8m + 5$. The symmetries are realized with $T = \tilde{G}$ and $C = G$. For $d = 8m + 1$, $T^T = −T$ and $C^T = C$, whereas for $d = 8m + 5$, $T^T = T$ and $C^T = −C$.

These results can be summarized in table 2. For a particular dimension $d$ modulo 8, the table indicates the ‘primitive’ $T, C$ and the specific sign in their transpose. In each case, in

| $d \mod 8$ | $T$ | $T^T$ | $C$ | $C^T/C$ | $s_T$ | $s_C$ |
|------------|-----|-------|-----|---------|-------|-------|
| 0          | $\tilde{G}$ | +1    | $G$ | +1      | −1    | +1    |
| 1          | $\tilde{G}$ | +1    | $G$ | +1      | +1    | +1    |
| 2          | $G$   | −1    | $\tilde{G}$ | +1      | −1    | −1    |
| 3          | $G$   | −1    | $\tilde{G}$ | +1      | −1    | −1    |
| 4          | $G$   | −1    | $\tilde{G}$ | −1      | −1    | +1    |
| 5          | $G$   | −1    | $\tilde{G}$ | −1      | +1    | +1    |
| 6          | $G$   | +1    | $\tilde{G}$ | −1      | −1    | +1    |
| 7          | $G$   | +1    | $\tilde{G}$ | −1      | −1    | −1    |
order to obtain a representative with the opposite sign in the relation with their transpose, one must use $T'$, $C'$, which henceforth will always denote the primitive $T$, $C$ tensored with $i\gamma_i$.

5.2. For $d$ even

Let $d = 2n$. It turns out that one cannot construct a Clifford algebra on a space smaller than the $2^n$-dimensional space in equation (8). Thus we take $\gamma_a = \Gamma_a$ for $a = 1$ to $2n - 1$, and $M = M^T = \Gamma_{2n}$. The extra matrix $\Gamma_{2n+1}$ commutes with all the SO(2n) generators; thus, the $2^n$-dimensional space is irreducible, and in fact the direct sum of the two spinor representations of SO(d). The projectors onto these two representations are $p_\pm = (1 \pm \Gamma_{2n+1})/2$, and we will refer to the projected representations as being of left- or right-handed chirality. Again $P$ symmetry can be imposed with $P = \Gamma_{2n+1}$.

The construction of the $T$ and $C$ symmetries is similar to the odd $d$ case. For $d = 8m$ and $8m + 4$, $T = G$ and $C = G$, whereas for $d = 8m + 2, 8m + 6$, they are reversed, i.e. $T = G, C = \tilde{G}$. These results, and the information on their transposes, are also given in table 2. Note that for the classes with both $T$ and $C$ symmetries, $P = TC^\dagger$ is proportional to $\Gamma_{2n+1}$, consistent with our previous identification of $P$.

6. Classification of protected zero modes and topological insulators

In this section, we classify gapless theories that are protected by the symmetries, i.e. the theories where the mass $M$ has a symmetry-protected zero eigenvalue. The existence of this zero mode can arise in two ways: either $M$ is forced to be zero, or from the weaker condition $\det(M) = 0$; as explained below, the first way corresponds to a $\mathbb{Z}$ or $2\mathbb{Z}$ topological insulator, whereas the second is of type $\mathbb{Z}_2$.

6.1. AIII

The existence of TIs in class AIII in odd dimensions is easy to understand. Recall that $P$ symmetry is implemented with $\Gamma_{2n+1}$. This leaves no $\Gamma$-matrix to associate with $M$ which necessarily anti-commutes with $P$. Thus $M$ is not allowed in odd dimensions in class AIII.

6.2. Chiral classes in even dimensions

Recall that in $d = 2n$ even dimensions, there are two additional $\Gamma$, $\Gamma_{2n}$ and $\Gamma_{2n+1}$, beyond the $d - 1$ of them associated with the $\gamma_a$. This leads to the property of chirality. In $d = 2$ dimensions, chirality corresponds to left or right movers on the edge. More generally, we can define chiral states as follows. Define projectors $p_\pm = (1 \pm g)/2$ onto states of ‘left’ versus ‘right’ chirality, where $g$ is either $\Gamma_{2n}$ or $\Gamma_{2n+1}$. $M$ is then associated with the other unused $\Gamma$, e.g. if $g = \Gamma_{2n}$ then $M = \Gamma_{2n+1}$ and vice versa. A chiral theory is then defined as one with a spectrum that consists of only particles of right or left chirality.

It is easy to see that $M$ necessarily couples both chiralities. Using the fact that $M p_+ = p_- M$ and $p_+ p_- = 0$, one has $\langle \psi | M | \psi \rangle = \langle \psi | (p_+ + p_-) M (p_+ + p_-) | \psi \rangle = \langle \psi | L | M | \psi \rangle + \langle \psi | R | M | \psi \rangle$. Thus, a purely chiral theory has no possible mass term and is a candidate TI.

Chiral theories can have $T$ or $C$ symmetry, but not both, since $TC^\dagger$ is a $P$ symmetry, and theories with $P$ symmetry require both chiralities. This is evident from the fact that $TC^\dagger \propto GG \propto \Gamma_{2n+1}$, which we have above associated with $P$. Whether a chiral theory can have $T$ or $C$ symmetry depends on the dimension. Let $S$ stand for either $T$ or $C$. The invariance
of a chiral state under \( S \) requires \([S, p] = 0\), i.e. \([S, g] = 0\). On the other hand, if a mass is forbidden by the \( S \) symmetry, then this requires \([S, M] = 0\). One then sees from equation (12) that \( S \) then must be associated with \( \tilde{G} \). According to table 2, in dimensions \( d = 0, 4 \) this is a \( T \) symmetry, whereas in \( d = 2, 6 \) it is a \( C \) symmetry. This is consistent with the identifications made in [13] for \( d = 2 \), i.e. that left or right movers are invariant under \( C \) symmetry, whereas \( T \) symmetry exchanges them. Since \( M \) is forced to be zero, these are TIs of topological type \( Z \) (see the general discussion below). If the symmetry involves \( T' \) or \( C' \), then the space is doubled, and this should correspond to a TI of type \( 2Z \). Thus, for \( d = 0 \), there exist TIs in class \( A\)I of type \( Z \) and in class \( A\)I of type \( 2Z \). whereas for \( d = 4 \) chiral, TIs exist in class \( A\)I of type \( Z \) and in class \( A\)II of type \( 2Z \). It is a similar story for TIs in classes \( C \) and \( D \) in \( d = 2, 6 \); see table 4 below.

Finally, one may consider pure chiral theories with no \( T \) or \( C \), which are in class \( A \). For example, for \( d = 2 \), chiral states do not preserve \( T \) since \( T \) exchanges left and right movers. Thus any class that does not have \( T \) symmetry can be chiral, namely \( C \) and \( D \) as described in the last section, but also \( A \). Here, a mass term is not allowed simply because the theory is chiral, which should be distinguished from the above cases where the mass term is also prohibited by \( T \) or \( C \) symmetry. These are class \( A \) TIs of type \( Z \) in any even dimension. To summarize:

\[
\begin{align*}
d = 0 \text{ mod } 8. & \text{ Chiral TIs of type } Z \text{ in classes } A \text{ and } A\!I; \text{ and of type } 2Z \text{ in class } A\!II. \\
d = 2 \text{ mod } 8. & \text{ Chiral TIs of type } Z \text{ in classes } A \text{ and } D; \text{ and of type } 2Z \text{ in class } C. \\
d = 4 \text{ mod } 8. & \text{ Chiral TIs of type } Z \text{ in classes } A \text{ and } A\!II; \text{ and of type } 2Z \text{ in class } A. \\
d = 6 \text{ mod } 8. & \text{ Chiral TIs of type } Z \text{ in classes } A \text{ and } C; \text{ and of type } 2Z \text{ in class } C. \\
\end{align*}
\]

### 6.3. Non-chiral classes

As described in section 4, henceforth, for odd dimensions we fix \( M = \Gamma_{2n+1} \) and for even dimensions \( M = \Gamma_{2n} \). The latter is the natural choice since it is consistent with \( P = TC' \) as explained above. The general forms of \( T \) and \( C \) are \( T = \tau_i \otimes X_i \) and \( C = \tau_c \otimes X_c \), where \( X_{i,c} \) are either \( G \) or \( \tilde{G} \) according to table 2, and \( \tau_{i,c} = 1 \) or \( i \tau_i \). The ‘mass’ can be generally expressed as \( M = V \otimes \Gamma \), where \( \Gamma = \Gamma_{2n+1}, \Gamma_{2n} \) for \( d = 2n+1, 2n \), respectively. We will consider only the minimal dimensions of the space that \( V \) lives in, i.e. one- or two-dimensional. Let us define the signs \( s_{i,c} \) as follows: \( X_{i,c} \Gamma = s_{i,c} \Gamma X_{i,c} \). Then the constraints on \( V \) coming from \( T \) and \( C \), equations (5) and (6) are

\[
\tau_i V^T = s_i V \tau_i, \quad \tau_c V^T = -s_c V \tau_c.
\]

The signs \( s_{i,c} \) follow from equation (12) and are shown by dimension in table 2.

The symmetries constrain \( V \) according to equations (4)–(6). A protected zero mode arises in one of two ways. The symmetries can force \( V = 0 \), which in lower dimensions was associated with a \( Z \) topological invariant. If the space is doubled, i.e. \( \tau_{i,c} = i \tau_i \), then this indicates that the topological invariant is an even integer, i.e. of type \( 2Z \). The other possibility is that the symmetries lead to the condition \( \det V = 0 \) which implies \( V \) has a zero eigenvalue. As in \( d = 2, 3 \), this condition arises when a particular vector space has an odd dimension, and follows for example from \( V^T = -V \), which implies \( \det V = -\det V = 0 \); this even/odd aspect is associated with a \( \mathbb{Z}_2 \) topological insulator.

Regardless of dimension, given the allowed \( \tau_{i,c} \) and \( s_{i,c} \), one can identify the nine cases that have a protected zero mode and are listed in table 3. The two constraints one obtains besides \( V = 0 \) are

\[
V^T = -V \implies \det V = 0 \text{ if dim}(V) \text{ is odd}
\]

\[\text{(14)}\]
Table 3. The nine different ways a protected zero mode can arise, regardless of dimension.

| case | τ_t | τ_c | s_t | s_c | Constraints on V | type |
|------|-----|-----|-----|-----|------------------|------|
| 1    | 1   | 1   | 0   | 0   | Equation (14)    | $\mathbb{Z}_2$ |
| 2    | 1   | 1   | 0   | 0   | Equation (14)    | $\mathbb{Z}_2$ |
| 3    | 1   | 1   | 0   | 0   | Equation (14)    | $\mathbb{Z}_2$ |
| 4    | 1   | 1   | -1  | -1  | $V = 0$          | $\mathbb{Z}_2$ |
| 5    | 1   | 1   | 0   | 0   | $V = 0$          | $\mathbb{Z}_2$ |
| 6    | $i\tau_t$ | $i\tau_t$ | -1 | -1 | $V = 0$          | $2\mathbb{Z}_2$ |
| 7    | $i\tau_t$ | $i\tau_t$ | +1 | +1 | $V = 0$          | $2\mathbb{Z}_2$ |
| 8    | $i\tau_t$ | 1   | +1  | +1  | Equation (15)    | $\mathbb{Z}_2$ |
| 9    | 1   | 1   | $i\tau_t$ | -1 | -1 | Equation (15)    | $\mathbb{Z}_2$ |

Table 4. Periodic table of TIs based on the classification of symmetry-protected zero modes. All the chiral classes are of class A, the first listed in cases with two entries and those labeled $2\mathbb{Z}_2$ in even dimensions (indicated in blue online). The new candidate TIs are the second listed in the cases with two entries (red online).

| $d$ mod 8 | \( \text{AZ class} \) | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|-----------|-----------------|---|---|---|---|---|---|---|---|
| A         | $\mathbb{Z}$   | $\emptyset$ | $\mathbb{Z}$ | $\emptyset$ | $\mathbb{Z}$ | $\emptyset$ | $\mathbb{Z}$ | $\emptyset$ |
| AI        | $\emptyset$    | $\mathbb{Z}$ | $\emptyset$ | $\mathbb{Z}$ | $\emptyset$ | $\mathbb{Z}$ | $\emptyset$ |
| Al        | $\mathbb{Z}, \mathbb{Z}_2$ | $\emptyset$ | $\emptyset$ | $\mathbb{Z}$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ |
| BDI       | $\mathbb{Z}_2$ | $\mathbb{Z}$ | $\emptyset$ | $\mathbb{Z}$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ |
| D         | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}, \mathbb{Z}_2$ | $\emptyset$ | $\emptyset$ | $2\mathbb{Z}$ | $\emptyset$ |
| DIII      | $\mathbb{Z}$   | $\mathbb{Z}_2$ | $\mathbb{Z}, \mathbb{Z}_2$ | $\emptyset$ | $\emptyset$ | $2\mathbb{Z}$ | $\emptyset$ |
| AlII      | $2\mathbb{Z}$  | $\emptyset$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ |
| CII       | $\emptyset$   | $2\mathbb{Z}$ | $\emptyset$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ |
| C         | $\emptyset$   | $\mathbb{Z}$ | $\emptyset$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ |
| Cl        | $\emptyset$   | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ |

\[
V = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} \text{ with } a^T = -a \implies \det V = 0 \text{ if } \dim(a) = 1.
\] (15)

TIs can now be classified by dimension as follows. (i) For a given dimension \( d \), identify \( s_{t,c} \) from table 2. (ii) Identify which cases in table 3 apply for these values of \( s_{t,c} \). (iii) The transpose properties of \( T \) and \( C \) can be inferred from table 2, bearing in mind that if \( T \) or \( C \) is \( T' \) or \( C' \), then the sign is flipped. (iv) Identify the class using table 1. The results are as follows.

For \( d \) odd.

\( d = 1 \mod 8 \). The cases from table 3 that apply are 2, 5, 7 and 8. Examining their transpose properties in table 2, one sees that these correspond to TIs in classes D, BDI, CII and DIII, respectively.

\( d = 3 \mod 8 \). The cases from table 3 that apply are 1, 4, 6 and 9, corresponding to TIs in classes AI, DIII, CI and CII, respectively.

\( d = 5 \mod 8 \). The cases that apply are 2, 5, 7 and 8, corresponding to TIs in classes C, CII, BDI and CI, respectively.

\( d = 7 \mod 8 \). The cases that apply are 1, 4, 6 and 9, corresponding to TIs in classes AI, CI, DIII and BDI, respectively.
For \( d \) even.

In all even dimensions, the cases from table 3 that apply are 1, 2 and 3 and are thus all of type \( \mathbb{Z}_2 \). Which classes they belong to are again determined by the transpose properties in table 2 and comparing with table 1.

\( d = 0 \mod 8 \). Cases 1, 2 and 3 give TIs in classes AI, D and BDI, respectively, all of type \( \mathbb{Z}_2 \). The TI in the class AI of type \( \mathbb{Z}_2 \) is new, i.e. it was not in the original periodic table in [10, 12].

\( d = 2 \mod 8 \). Cases 1, 2 and 3 give TIs in classes AII, D and DIII, respectively, all of type \( \mathbb{Z}_2 \). The TI in the class D of type \( \mathbb{Z}_2 \) is new.

\( d = 4 \mod 8 \). Cases 1, 2 and 3 give TIs in classes AII, C and CII, respectively, all of type \( \mathbb{Z}_2 \).

\( d = 6 \mod 8 \). Cases 1, 2 and 3 give TIs in classes AI, C and CI, respectively, all of type \( \mathbb{Z}_2 \). The TI in class C of type \( \mathbb{Z}_2 \) is new.

7. The exceptionality of two dimensions

For \( d = 2 \), the above Hamiltonian is \( \mathcal{H} = -i\sigma_y \partial_x + V\sigma_z \), where \( M = V\sigma_z \). Under a unitary transformation \( \mathcal{H} \rightarrow U\mathcal{H}U^\dagger \), with \( U \) being a rotation in the \( \sigma \) space by 90° about the \( z \) axis followed by a 90° rotation about the \( x \) axis, the Hamiltonian is equivalent to

\[
\mathcal{H} = -i\sigma_x \partial_x + V\sigma_z
\]

which is of the form studied in [13], where in the latter \( V = V_0 \). The reason more classes of TIs were found in [13], namely 11, is the following. In the generic classification of the minimal Dirac Hamiltonians in section 5, \( T \) and \( C \) were unique, and ten classes were obtained. In two dimensions, the generic construction gives \( T = G = \sigma_y \) according to table 2. However, since \( T \) only has to anti-commute with \( \sigma_x \) for the Hamiltonian (16), \( T \) can be realized also as \( T = \sigma_y \). These are gauge inequivalent since their transpose properties are different. A comprehensive classification of the most general Dirac Hamiltonians yields a richer, more refined structure, wherein some of the \( AZ \) classes have two inequivalent representatives [13]. A total of 17 gauge-inequivalent classes were found, of which 11 had protected zero modes and conjectured to be TI.

8. Comments and conclusion

To summarize, we have shown how the periodic table of topological insulators (TIs) in all spatial dimensions can be understood in an alternative manner, namely by classifying symmetry-protected zero modes of Dirac Hamiltonians on the boundary. Our holographic approach makes no use of topological invariants nor K-theory, but is based only on generic properties of Clifford algebras in any dimension. Our analysis suggests an additional topological insulator of type \( \mathbb{Z}_2 \) in every even dimension.

We also commented on why two dimensions have even more possible TIs [13], essentially because there are two gauge-inequivalent ways of implementing time-reversal symmetry. Physical realizations of these new classes of TIs were proposed in [13]; in particular, it was suggested that one of them describes strained graphene.

We have not addressed in a precise way why our holographic approach gives one additional TI in every even dimension compared to other approaches. However, one possibility is suggested by the recent work [17], where in addition to \( T, C \) and \( P \) symmetries an additional spatial reflection symmetry is also considered, and five new classes of TI were found beyond
the standard five which coincide with those proposed from the holographic approach [13]. The latter work constructs bulk Hamiltonians for these new classes. In our work, we have assumed a Dirac Hamiltonian on the boundary, and this constraint perhaps implies some additional symmetries such as the reflection symmetry.

Acknowledgments

AL would like to thank Csaba Csaki, Eun-Ah Kim and S Ryu for discussions. This work is supported by the National Science Foundation under grant number NSF-PHY-0757868 and by the ‘Agence Nationale de la Recherche’ contract ANR-2010-BLANC-0414.

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