COFREE HOPF ALGEBRAS ON HOPF BIMODULE ALGEBRAS

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Abstract. We investigate a Hopf algebra structure on the cotensor coalgebra associated to a Hopf bimodule algebra which contains universal version of Clifford algebras and quantum groups as examples. It is shown to be the bosonization of the quantum quasi-shuffle algebra built on the space of its right coinvariants. The universal property and a Rota-Baxter algebra structure are established on this new algebra.

1. Introduction

The notion of a cotensor coalgebra was first introduced by Nichols ([16]) more than three decades ago. It is the dual version of tensor algebras built on bimodules over algebras. By using the universal property of cotensor coalgebras, Nichols constructed an algebra structure on the underlying space of the cotensor coalgebra. He investigated the subalgebra generated by elements of degree 0 and 1, which is called the Nichols algebra or quantum shuffle algebra nowadays. This sort of algebras lead to the quantum shuffle interpretation of quantum groups ([18]) and a series of works on the classification of finite dimensional pointed Hopf algebras over finite abelian groups of order prime to 210 (cf. [1], [2] and the references therein). These unexpected relations with other mathematical objects such as operads and logarithm conformal field theory evoke great interests on cotensor coalgebras.

Nevertheless, the study of this subject is just at the beginning. On one hand, the Nichols algebra itself admits various algebraic structures and many open questions. One may expect that new properties on these objects about this subject and connections with other algebraic or geometric objects will be discovered. On the other hand, we could generalize the Nichols algebra in a much bigger framework. For the second direction, the first step was advised by Rosso. He considered a Hopf bimodule $M$ with an associative product $m$. After imposing some compatibilities between the multiplication and the Hopf bimodule structure, he constructed a Hopf algebra structure on the cotensor coalgebra of this Hopf bimodule algebra $(M, m)$. We call this new algebra the cofree Hopf algebra on the Hopf bimodule algebra.

As in the case of quantum shuffle algebras, these Hopf algebras bring surprising applications to quantum groups. Recently, Rosso and the first author showed that the whole quantum group associated to a symmetrizable Kac-Moody Lie algebra can

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be realized as a natural quotient of the subalgebra of the cofree Hopf algebra built on some well-chosen Hopf bimodule algebras generated by elements of degree 0 and 1 (3). This provides a "global" construction of the whole quantum group, not just the positive part. We expect that this result could give some new insights on quantum groups. Moreover, Rosso's construction is entirely generalized in the framework of multi-brace cotensor Hopf algebras and the algebra structures on cotensor coalgebras are classified there in spirit of Loday and Ronco (14).

This new algebra built on a Hopf bimodule algebra \( M \) has a richer combinatorial structure and therefore is complicated. The explicit formula of this new multiplication involves all structures on \( M \) and our first task is to understand it. Analogue to the relation between cofree Hopf algebras on Hopf bimodules and quantum shuffle algebras (18) the quantum quasi-shuffle algebra \( 12 \) can be found inside of a cofree Hopf algebra on a Hopf bimodule algebra. More precisely, the new algebra is isomorphic, as a Hopf algebra, to the bosonization of quantum quasi-shuffle. These results should be known by Rosso, but he did not published any proof. Due to the importance of this algebra, it seems quite necessary to write down a complete and self-contained demonstration of this observation.

In this paper, after a careful analysis, we use a universal property showed in \( 12 \) to provide a detailed proof. Thanks to this isomorphism, we can concentrate our study on quantum quasi-shuffle algebras, which are the quantization of the classical quasi-shuffle algebras introduced by Newman and Radford (15), Hoffmann (7), Guo and Keigher (6) independently. They are constructed as a special case of quantum multi-brace algebras (11) and have some interesting properties and applications to other mathematical objects (cf. 9, 10, 12). We can extend the results about quantum quasi-shuffle algebras to the cofree Hopf algebras on Hopf bimodule algebras. These extensions are significant for the reason that the former ones are not real Hopf algebras but the latter ones are. For example, the second author constructed in \( 9 \) a Rota-Baxter algebra structure on quantum quasi-shuffle algebras, which can be extended to the bosonized Hopf algebra and then gives a Rota-Baxter algebra structure on the cofree Hopf algebra built on Hopf bimodule algebras.

We also establish some important tools for the further study of this new algebra, such as universal property. Finally, we provide examples of our new algebras, such as universal Clifford algebras and universal quantum groups. These examples demonstrate that the cofree Hopf algebras on Hopf bimodule algebras include many important algebras as special case. As we mentioned above that these new objects admit plenty of structures, this paper can be viewed as an opening of this subject.

This paper is organized as follows. We provide a detailed study of the algebra structure built on the cofree Hopf algebras on Hopf bimodule algebras in Section 2. In Section 3, a Rota-Baxter algebra structure on the new algebra, as well as a universal property, are established. Several examples including universal Clifford algebras and universal quantum groups are provided in Section 4.

**Notations.** In this paper, we denote by \( \mathbb{K} \) a ground field of characteristic 0. All vector spaces, algebras, coalgebras and tensor products we discuss are defined over \( \mathbb{K} \).
if not specified otherwise. For a vector space $V$, we denote by $T(V)$ the tensor vector space of $V$, by $\otimes$ the tensor product within $T(V)$, and by $\boxtimes$ the one between $T(V)$ and $T(V)$.

A braiding $\sigma$ on a vector space $V$ is an invertible linear map in $\text{End}(V \otimes V)$ satisfying the quantum Yang-Baxter equation on $V^{\otimes 3}$:

$$(\sigma \otimes \text{id}_V)(\text{id}_V \otimes \sigma)(\sigma \otimes \text{id}_V) = (\text{id}_V \otimes \sigma)(\sigma \otimes \text{id}_V)(\text{id}_V \otimes \sigma).$$

A braided vector space $(V, \sigma)$ is a vector space $V$ equipped with a braiding $\sigma$. For any $n \in \mathbb{N}$ and $1 \leq i \leq n - 1$, we denote by $\sigma_i$ the operator $\text{id}_V^{\otimes i-1} \otimes \sigma \otimes \text{id}_V^{\otimes n-i-1} \in \text{End}(V^{\otimes n})$. We denote by $\mathfrak{S}_n$ the symmetric group acting on the set $\{1, 2, \ldots, n\}$ and by $s_i$, $1 \leq i \leq n - 1$, the standard generators of $\mathfrak{S}_n$ permuting $i$ and $i + 1$. For any $w \in \mathfrak{S}_n$, we denote by $T_w^\sigma$ the corresponding lift of $w$ in the braid group $\mathfrak{B}_n$, defined as follows: if $w = s_{i_1} \cdots s_{i_l}$ is any reduced expression of $w$, then $T_w^\sigma = \sigma_{i_1} \cdots \sigma_{i_l}$. By Theorem 4.12 in [13], it is well-defined.

We define $\beta : T(V) \otimes T(V) \rightarrow T(V) \otimes T(V)$ by requiring that the restriction of $\beta$ on $V^{\otimes i} \otimes V^{\otimes j}$, denoted by $\beta_{ij}$, is $T_{\chi_{ij}}^\sigma$, where

$$\chi_{ij} = \begin{pmatrix} 1 & 2 & \cdots & i & i+1 & i+2 & \cdots & i+j \\ j+1 & j+2 & \cdots & j+i & 1 & 2 & \cdots & j \end{pmatrix} \in \mathfrak{S}_{i+j},$$

for any $i, j \geq 1$. For convenience, we denote by $\beta_{0i}$ and $\beta_{i0}$ the usual flip.

Let $(H, \Delta, \varepsilon, S)$ be a Hopf algebra. We denote $\Delta^{(0)} = \text{id}$, $\Delta^{(1)} = \Delta$, and $\Delta^{(n+1)} = (\Delta^{(n)} \otimes \text{id})\Delta$ recursively for $n \geq 1$. The coradical $\text{corad}(H)$ of $H$ is the sum of all simple subcoalgebras of $H$ (cf. [19]).

In the sequel, we adopt Sweedler’s notation for coproducts and comodule structure maps. For any $h \in H$,

$$\Delta(h) = \sum h_{(1)} \otimes h_{(2)}.$$ 

For a left $H$-comodule $(M, \delta_L)$ (resp. right $H$-comodule $(M, \delta_R)$) and any $m \in M$,

$$\delta_L(m) = \sum m_{(-1)} \otimes m_{(0)} \quad (\text{resp. } \delta_R(m) = \sum m_{(0)} \otimes m_{(1)}).$$

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In this section, we always assume $(H, \mu_H, \Delta_H, \varepsilon_H, S)$ is a Hopf algebra.

2.1. Algebra objects in $\mathcal{YD}^H$ and $\mathcal{M}_H^H$. We start by recalling the equivalence of $H$-Yetter-Drinfeld module categories and $H$-Hopf bimodule categories. Our basic references are [1], [18] and [19].

2.1.1. Yetter-Drinfeld module algebras.

Definition 2.1. A (left) $H$-Yetter-Drinfeld module is a vector space $V$ equipped simultaneously with a left $H$-module structure · and a left $H$-comodule structure such that whenever $h \in H$ and $v \in V$,

$$\sum h_{(1)}v_{(-1)} \otimes h_{(2)} \cdot v_{(0)} = \sum (h_{(1)} \cdot v)_{(-1)}h_{(2)} \otimes (h_{(1)} \cdot v)_{(0)}.$$
The category of $H$-Yetter-Drinfeld modules, denoted by $H^H\mathcal{YD}$, consists of the following data:

1. objects in $H^H\mathcal{YD}$ are $H$-Yetter-Drinfeld modules;
2. morphisms in $H^H\mathcal{YD}$ are linear maps which are both module and comodule morphisms.

Together with the usual tensor product, $H^H\mathcal{YD}$ is a tensor category. Moreover it is braided with the following commutativity constraint: for any $V, W \in H^H\mathcal{YD}$,

$$\sigma : V \otimes W \to W \otimes V, \quad v \otimes w \mapsto \sum v_{(0)} \cdot w \otimes v_{(1)}.$$

A unital algebra $(V, m, u)$ in $H^H\mathcal{YD}$ is a unital associative algebra such that the underlying space $V$ is an object of $H^H\mathcal{YD}$ and its multiplication $m$ and unit $u$ are morphisms in $H^H\mathcal{YD}$.

2.1.2. Hopf bimodule algebras.

**Definition 2.2.** An $H$-Hopf bimodule is a vector space $M$ equipped simultaneously with $H$-bimodule and $H$-bicomodule structures (with left coaction $\delta_L$ and right coaction $\delta_R$) such that both $\delta_L$ and $\delta_R$ are $H$-bimodule maps.

The category of $H$-Hopf bimodules, denoted by $H^H\mathcal{M}_H^H$, consists of the following data:

1. objects in $H^H\mathcal{M}_H^H$ are $H$-Hopf bimodules;
2. morphisms in $H^H\mathcal{M}_H^H$ are linear maps which are both $H$-bimodule and $H$-bicomodule morphisms.

Taking tensor product over $H$, $H^H\mathcal{M}_H^H$ is a tensor category: the bimodule structure is on two extremal terms and the bicomodule structure arises from the tensor product. In his seminal paper [20], Woronowicz showed that $H^H\mathcal{M}_H^H$ is braided.

As in the case of Yetter-Drinfeld modules, we can consider unital algebras in $H^H\mathcal{M}_H^H$. More precisely, we call $M \in H^H\mathcal{M}_H^H$ an $H$-Hopf bimodule algebra if $M$ is an associative algebra with multiplication

$$\mu : M \otimes M \to M \in H^H\mathcal{M}_H^H,$$

where the tensor product is the one in $H^H\mathcal{M}_H^H$.

2.1.3. Equivalence of categories.

**Theorem 2.3** ([18]). There exists an equivalence of braided tensor categories $H^H\mathcal{M}_H^H \sim H^H\mathcal{YD}$ sending an $H$-Hopf bimodule $M$ to the set of its right coinvariants $M^R = \{ m \in M | \delta_R(m) = m \otimes 1_H \}$.

We give some details of this equivalence. By Theorem 4.1.1 in [19], $M^R = P_R(M)$, where $P_R$ is the projection from $M$ onto $M^R$ given by $P_R(m) = \sum m_{(0)}S(m_{(1)})$ for
Then \( M \in \mathcal{M} \). A direct verification shows that \( M^R \) inherits a left \( \mathcal{H} \)-comodule structure from \((\mathcal{M}, \delta_L)\) and a left \( \mathcal{H} \)-module structure given by the adjoint action:

\[
h \cdot m = \sum h_{(1)} m S(h_{(2)}), \quad \text{for } h \in \mathcal{H} \text{ and } m \in M^R.
\]

With these structures, \((\mathcal{M}^R, \cdot, \delta_L)\) is an \( \mathcal{H} \)-Yetter-Drinfeld module.

Conversely, if \( V \) is an \( \mathcal{H} \)-Yetter-Drinfeld module, we denote \( \mathcal{M} = V \otimes \mathcal{H} \) and define: for any \( h, h' \) and \( v \in V \),

\[
h'(v \otimes h) = \sum h'_{(1)} \cdot v \otimes h'_{(2)} h, \quad (v \otimes h)h' = v \otimes hh',
\]

\[
\delta_L(v \otimes h) = \sum v_{(-1)} h_{(1)} \otimes (v_{(0)} \otimes h_{(2)}), \quad \delta_R(v \otimes h) = \sum (v \otimes h_{(1)}) \otimes h_{(2)}.
\]

Then \( \mathcal{M} \), equipped with these structures, is an \( \mathcal{H} \)-Hopf bimodule.

The algebras in \( \mathcal{M}^{\mathcal{H}} \mathcal{YD} \) and those in \( \mathcal{H}^\mathcal{M} \mathcal{H} \mathcal{YD} \) are corresponded under the category equivalence. For an \( \mathcal{H} \)-Hopf bimodule algebra \( \mathcal{M} \), \( M^R \) is naturally an \( \mathcal{H} \)-module object in \( \mathcal{H}^\mathcal{M} \mathcal{YD} \). In order to construct \( \mathcal{H} \)-Hopf bimodule algebras from algebras in \( \mathcal{H}^\mathcal{M} \mathcal{YD} \), we need the notion of smash product. Suppose that \( V \) is an algebra in \( \mathcal{H}^\mathcal{M} \mathcal{YD} \). The smash product \( V \# \mathcal{H} \) of \( V \) and \( \mathcal{H} \) is defined to be \( V \otimes \mathcal{H} \) as a vector space and

\[
(v \# h)(v' \# h') = \sum v(h_{(1)} \cdot v') \# h_{(2)} h', \quad v, v' \in V, h, h' \in \mathcal{H},
\]

where we use \( \# \) instead of \( \otimes \) to emphasize the smash product. One can prove that \( V \# \mathcal{H} \), together with the above \( \mathcal{H} \)-Hopf bimodule structure, is an \( \mathcal{H} \)-Hopf bimodule algebra.

**Example 2.4.** It is well-known that \( \mathcal{H} \) is a Yetter-Drinfeld module over itself with the following structures: for any \( x, h \in \mathcal{H} \),

\[
x \cdot h = \sum x_{(1)} h S(x_{(2)}), \quad \rho(h) = \sum h_{(1)} \otimes h_{(2)}.
\]

In addition, it is a unital algebra in \( \mathcal{H}^\mathcal{M} \mathcal{YD} \). Then \( \mathcal{H} \otimes \mathcal{H} \) is an \( \mathcal{H} \)-Hopf bimodule algebra with \( \mathcal{H} \)-Hopf bimodule structure defined before and algebra structure

\[
(h^1 \otimes h^2)(h^3 \otimes h^4) = \sum h^1 h_{(1)}^2 h^3 S(h_{(2)}^2) \otimes h_{(3)}^2 h^4.
\]

### 2.2. Cofree Hopf algebras on \( \mathcal{H} \)-Hopf bimodule algebras.

#### 2.2.1. Cotensor coalgebras.

Given two \( \mathcal{H} \)-Hopf bimodules \( \mathcal{M} \) and \( N \), we denote by \( \mathcal{M} \square \mathcal{H} N \) their cotensor product, i.e., the kernel of the map \( \text{id}_M \otimes \delta_L - \delta_R \otimes \text{id}_N : \mathcal{M} \otimes N \rightarrow \mathcal{M} \otimes \mathcal{H} \otimes N \). It is a subspace of \( \mathcal{M} \otimes N \). We endow \( \mathcal{M} \boxdot \mathcal{H} N \) with the following \( \mathcal{H} \)-Hopf bimodule structure:

\[
h(m \otimes n) = \sum h_{(1)} m \otimes h_{(2)} n, \quad (m \otimes n)h = \sum mh_{(1)} \otimes nh_{(2)},
\]

\[
\rho_L(m \otimes n) = \sum m_{(-1)} \otimes (m_{(0)} \otimes n), \quad \rho_R(m \otimes n) = \sum (m \otimes n_{(0)}) \otimes n_{(1)}.
\]

The cotensor product is associative, which allows one to define recursively \( \mathcal{M} \boxdot_n = \mathcal{H} \), \( \mathcal{M} \boxdot_{n+1} = \mathcal{M} \boxdot_n \mathcal{H} \mathcal{M} \) for \( n \geq 1 \). We denote \( T^c_{\mathcal{H}}(\mathcal{M}) = \bigoplus_{n \geq 0} \mathcal{M} \boxdot_n \mathcal{M} \).
and use $\sum m^1 \bigotimes \cdots \bigotimes m^n$ instead of $\sum m^1 \otimes \cdots \otimes m^n$ to indicate an element in $M^{\square H}$.

We define a linear map $\Delta : T^c_H(M) \rightarrow T^c_H(M) \bigotimes T^c_H(M)$ as follows: for any $h \in H$,

$$\Delta(h) = \Delta_H(h) = \sum h_{(1)} \bigotimes h_{(2)},$$

for any $m \in M$,

$$\Delta(m) = (\delta_L + \delta_R)(m) = \sum m_{(-1)} \bigotimes m_{(0)} + \sum m_{(0)} \bigotimes m_{(1)},$$

and for any $m^1 \bigotimes \cdots \bigotimes m^n \in M^{\square H}$ with $n \geq 2$,

$$\Delta(m^1 \bigotimes \cdots \bigotimes m^n) = \sum m^1_{(-1)} \bigotimes m^1_{(0)} \bigotimes m^2 \bigotimes \cdots \bigotimes m^n + \sum_{i=1}^{n-1} m^1 \bigotimes \cdots \bigotimes m^i \bigotimes m^{i+1} \bigotimes m \bigotimes \cdots \bigotimes m^n + \sum m^1 \bigotimes \cdots \bigotimes m^{n-1} \bigotimes m_{(0)} \bigotimes m_{(1)}.$$

We also define $\varepsilon : T^c_H(M) \rightarrow \mathbb{K}$ by requiring $\varepsilon|_H = \varepsilon_H$ and $\varepsilon|_{M^{\square H}} = 0$ for $n \geq 1$. Together with these maps, $T^c_H(M)$ is a graded coalgebra called the cotensor coalgebra over $H$ and $M$, which is characterized by the following universal property (cf. [16]).

**Proposition 2.5** ([16]). Let $(C, \Delta_C)$ be a coalgebra and $g : C \rightarrow H$ be a coalgebra map. We view $C$ as an $H$-bicomodule with left coaction $(g \otimes \text{id}_H)\Delta_C$ and right coaction $(\text{id}_C \otimes g)\Delta_C$ respectively. If $f : C \rightarrow M$ is an $H$-bicomodule map such that $f(\text{corad}(C)) = 0$, then there exists a unique coalgebra map $F : C \rightarrow T^c_H(M)$ making the following diagrams commute:

$$\begin{array}{ccc}
T^c_H(M) & \xrightarrow{F} & C \\
\pi \downarrow & & \downarrow \pi \\
H & \xrightarrow{g} & M \\
\downarrow p & & \downarrow f \\
M & & M
\end{array}$$

where $\pi$ and $p$ are the projections from $T^c_H(M)$ onto $H$ and $M$ respectively.

Explicitly, $F = g + \sum_{n\geq 1} f^\otimes n \circ \Delta^{(n-1)}_C$.

### 2.2.2. Algebra structure on $T^c_H(M)$

We will construct a bialgebra structure on $T^c_H(M)$ when $M$ admits moreover an algebra structure, generalizing constructions in [15] and [12]. We mention that this bialgebra appears recently as a special case in the framework of multi-brace cotensor Hopf algebras and is used to realize the whole quantum group ([3]). We would like to give a direct approach here.

Define

$$g = \mu_H(\pi \otimes \pi) : T^c_H(M) \bigotimes T^c_H(M) \rightarrow H,$$

and

$$f = \cdot_L(\pi \otimes p) + \cdot_R(p \otimes \pi) + \mu(p \otimes p) : T^c_H(M) \bigotimes T^c_H(M) \rightarrow M,$$

where $\cdot_L$ and $\cdot_R$ are the left and right module structure maps of $M$ respectively.
Endowed with the coproduct of the tensor product of two coalgebras, $T^c_H(M) \otimes T^c_H(M)$ is a graded coalgebra. Since both $\pi$ and $\mu_H$ are coalgebra maps, $g$ is a coalgebra map as well. For an analogue reason, $f$ is an $H$-bicomodule map.

Since the coalgebra $(T^c_H(M), \Delta)$ is graded, we have that $\text{corad} (T^c_H(M)) \subset \text{corad} (H) \subset H$. On the other hand, it is easy to show that for any two coalgebras $C$ and $D$ we have $\text{corad} (C \otimes D) \subset \text{corad} (C) \otimes \text{corad} (D)$, where $C \otimes D$ is endowed with the tensor coproduct (cf. [19]). So $\text{corad} (T^c_H(M) \otimes T^c_H(M)) \subset H \otimes H$, and consequently, $f (\text{corad} (T^c_H(M) \otimes T^c_H(M))) = 0$.

By the universal property, there exists a unique coalgebra map $F : T^c_H(M) \otimes T^c_H(M) \rightarrow T^c_H(M)$ such that $\pi F = g$ and $pF = f$. We want to show that $F$ is indeed an associative product. In order to prove the associativity of $F$, we consider

$$\pi F (F \otimes \text{id}_{T^c_H(M)}) , \pi F (\text{id}_{T^c_H(M)} \otimes F) : T^c_H(M) \otimes T^c_H(M) \otimes T^c_H(M) \rightarrow H,$$

and

$$pF (F \otimes \text{id}_{T^c_H(M)}) , pF (\text{id}_{T^c_H(M)} \otimes F) : T^c_H(M) \otimes T^c_H(M) \otimes T^c_H(M) \rightarrow M.$$ 

It is clear that $\pi F (F \otimes \text{id}_{T^c_H(M)})$ and $\pi F (\text{id}_{T^c_H(M)} \otimes F)$ are coalgebra maps, while $pF (F \otimes \text{id}_{T^c_H(M)})$ and $pF (\text{id}_{T^c_H(M)} \otimes F)$ are $H$-bimodule maps. If we could show that

$$\pi F (F \otimes \text{id}_{T^c_H(M)}) = \pi F (\text{id}_{T^c_H(M)} \otimes F), \text{ and } pF (F \otimes \text{id}_{T^c_H(M)}) = pF (\text{id}_{T^c_H(M)} \otimes F),$$

then the associativity of $F$ is a consequence of the universal property of $T^c_H(M)$.

The first identity arises from the associativity of $\mu_H$. Let us prove the second one:

$$pF (F \otimes \text{id}_{T^c_H(M)})$$

$$= \cdot_L (\mu_H (\pi \otimes \pi) \otimes p) + \cdot_R (f \otimes \pi) + \mu (f \otimes p)$$

$$= \cdot_L (\pi \otimes f) + \cdot_R (p \otimes \mu_H (\pi \otimes \pi)) + \mu (p \otimes f)$$

$$= pF (\text{id}_{T^c_H(M)} \otimes F).$$

We denote $x \ast y = F (x \otimes y)$ for any $x, y \in T^c_H(M)$. Then the following theorem is proved in the above argument.

**Theorem 2.6.** Let $M$ be an $H$-Hopf bimodule algebra. Then $(T^c_H(M), \ast, \Delta, \varepsilon)$ is a bialgebra.

**Remark 2.7.** We remind the reader that the $H$-Hopf bimodule algebra $M$ is not necessarily unital in the above construction.

2.3. **The structure of** $(T^c_H(M), \ast, \Delta, \varepsilon)$. The bialgebra $(T^c_H(M), \ast, \Delta, \varepsilon)$ is constructed by using the universal property in the last subsection. But the algebra structure is not clear from the abstract definition. We will show in this subsection that this product admits a combinatorial interpretation by using the quantum quasi-shuffle product ([12]).
2.3.1. Quantum quasi-shuffle algebras. For stating the construction of quantum quasi-shuffle algebras, we introduce some necessary notions first.

**Definition 2.8.** A quadruple \((A, m_A, \sigma)\) is called a braided algebra if \((A, m_A)\) is an associative algebra with a braiding \(\sigma\) on \(A\) satisfying the following conditions:

\[
(id_A \otimes m_A)\sigma_1 \sigma_2 = \sigma(m_A \otimes id_A), \quad (m_A \otimes id_A)\sigma_2 \sigma_1 = \sigma(id_A \otimes m_A).
\]

A braided algebra \((A, m_A, \sigma)\) is called unital if there exists \(1_A \in A\) such that \((A, m_A, 1_A)\) is a unital associative algebra satisfying

\[
\sigma(a \otimes 1_A) = 1_A \otimes a, \quad \sigma(1_A \otimes a) = a \otimes 1_A.
\]

A quadruple \((C, \Delta_C, \varepsilon_C, \sigma)\) is called a braided coalgebra if \((C, \Delta_C, \varepsilon_C)\) is a coalgebra with braiding \(\sigma\) on \(C\) satisfying the following conditions:

\[
\sigma_1 \sigma_2(\Delta_C \otimes id_C) = (id_C \otimes \Delta_C)\sigma, \quad \sigma_2 \sigma_1(id_C \otimes \Delta_C) = (\Delta_C \otimes id_C)\sigma,
\]

\[
(id_C \otimes \varepsilon_C)\sigma = \varepsilon_C \otimes id_C, \quad (\varepsilon_C \otimes id_C)\sigma = id_C \otimes \varepsilon_C.
\]

A sextuple \((B, m_B, 1_B, \Delta_B, \varepsilon_B, \sigma)\) is called a braided bialgebra if \((B, m_B, 1_B, \sigma)\) is a braided algebra and \((B, \Delta_B, \varepsilon_B, \sigma)\) is a braided coalgebra such that

\[
\begin{align*}
\Delta_B m_B &= (m_B \otimes m_B)(id_B \otimes \sigma \otimes id_B)(\Delta_B \otimes \Delta_B), \\
\Delta_B(1_B) &= 1_B \otimes 1_B.
\end{align*}
\]

We sometimes denote it by \((B, \sigma)\) for simplifying the notation.

Since \(h^nHYD\) is a braided tensor category, an algebra (resp. coalgebra) object in \(h^nHYD\) is naturally a braided algebra (resp. coalgebra) with respect to the commutativity constraint in \(h^nHYD\).

Let \(BB\) be the category of braided bialgebras consisting of the following data:

(i) objects in \(BB\) are braided bialgebras;

(ii) morphisms between two objects \((B, \sigma)\) and \((B', \tau)\) in \(BB\) are bialgebra morphisms \(f : B \to B'\) satisfying \(\tau(f \otimes f) = (f \otimes f)\sigma\).

For an object \((B, \sigma) \in BB, B \otimes B\) is a braided bialgebra with the product \((m_B \otimes m_B)(id_B \otimes \sigma \otimes id_B)\) and coproduct \((id_B \otimes \sigma \otimes id_B)(\Delta_B \otimes \Delta_B)\).

We recall the definition of quantum quasi-shuffle algebras in a general setting. Suppose that \((A, m, \sigma)\) is a braided algebra. The quantum quasi-shuffle product \(\Join_{\sigma}\) on \(T(V)\) is defined as follows \([12]\): for any \(\lambda \in \mathbb{K}\) and \(x \in T(A)\), the quantum quasi-shuffle product of \(\lambda x\) and \(x\) is just the scalar multiplication \(\lambda x\). For \(i, j \geq 2\) and any \(a_1, \ldots, a_i, b_1, \ldots, b_j \in A\), the product \(\Join_{\sigma}\) is defined recursively by:

\[
\begin{align*}
 a_1 \Join_{\sigma} b_1 &= a_1 \otimes b_1 + \sigma(a_1 \otimes b_1) + m(a_1 \otimes b_1), \\
 a_1 \Join_{\sigma} (b_1 \otimes \cdots \otimes b_j) &= \left( id_A^{(\otimes j+1)} + (id_A \Join_{\sigma(1,j-1)}(\beta_{1,1} \otimes id_A^{\otimes j-1}) + m \otimes id_A^{\otimes j-1}) (a_1 \otimes b_1 \otimes \cdots \otimes b_j), \\
 (a_1 \otimes \cdots \otimes a_i) \Join_{\sigma} b_1 &= \left( id_A \Join_{\sigma(i-1,1)} + \beta_{i,1} + (m \otimes id_A^{\otimes i-1})(id_A \otimes \beta_{i-1,1}) \right) (a_1 \otimes \cdots \otimes a_i \otimes b_1),
\end{align*}
\]
\[(a_1 \otimes \cdots \otimes a_i) \triangleright (b_1 \otimes \cdots \otimes b_j)\]
\[= a_1 \otimes ((a_2 \otimes \cdots \otimes a_i) \triangleright (b_1 \otimes \cdots \otimes b_j))\]
\[+ (\text{id}_A \otimes \triangleright_{(i,j-1)}) (\beta_{1,1} \otimes \text{id}_A^{\otimes i-1}) (a_1 \otimes \cdots \otimes a_i \otimes b_1 \otimes \cdots \otimes b_j)\]
\[+ (m \otimes \triangleright_{(i-1,j-1)}) (\text{id}_A \otimes \beta_{i-1,1} \otimes \text{id}_A^{\otimes j-1}) (a_1 \otimes \cdots \otimes a_i \otimes b_1 \otimes \cdots \otimes b_j),\]
where \(\triangleright_{(i,j)}\) denotes the restriction of \(\triangleright\) on \(A^{\otimes i} \otimes A^{\otimes j}\).

**Remark 2.9.** A new combinatorial description of the quantum quasi-shuffle product in the spirit of operads is discovered by the first author in [4] by lifting shuffles to the generalized virtual braid monoid.

It is shown in [12] that \((T(V), \triangleright)\) is a unital algebra, which is called the quantum quasi-shuffle algebra associated to the braided algebra \(A\) and denoted by \(T_{\sigma,m}(A)\). Moreover, equipped with the deconcatenation coproduct and the braiding \(\beta\), \(T_{\sigma,m}(A)\) is a braided bialgebra. It has a universal property ([12]) which will play an essential role in the later use. To state this property, we need the following notion.

**Definition 2.10.** Suppose \((C, \Delta_C, \varepsilon_C)\) is a coalgebra with a preferred group-like element \(1_C\). We denote \(\overline{\Delta_C}(x) = \Delta_C(x) - x \otimes 1_C - 1_C \otimes x\) for any \(x \in C\) and recursively
\[F_0 C = \mathbb{K}1_C,\]
\[F_r C = \{x \in C | \overline{\Delta_C}(x) \in F_{r-1} C \otimes F_{r-1} C\}, \text{ for } r \geq 1.\]
The coalgebra \(C\) is said to be **connected** if \(C = \bigcup_{r \geq 0} F_r C\).

We denote by \(\mathcal{CB}\) the full subcategory of \(\mathcal{BB}\) whose objects are braided bialgebras \((B, \sigma)\) such that both \(B\) and \(B \otimes B\) are connected.

Given a braided algebra \((A, m_A, \sigma)\), the quantum quasi-shuffle algebra \(T_{\sigma,m_A}(A)\) is uniquely determined by the following universal property.

**Proposition 2.11 ([12]).** Let \((B, \tau) \in \mathcal{CB}\) be a braided bialgebra with preferred group-like element \(1_B\). Let \(f : B \to A\) be a linear map such that \(m_A(f \otimes f) = fm_B\) on \(\ker \varepsilon_B \otimes \ker \varepsilon_B\), \(f(1_B) = 0\) and \((f \otimes f)\tau = \sigma(f \otimes f)\). Then there exists a unique braided bialgebra morphism \(\overline{f} : B \to T_{\sigma,m}(A)\) extending \(f\). Explicitly, \(\overline{f} = \varepsilon_B + \sum_{n \geq 1} f^{\otimes n} \Delta_B^{(n-1)}\).

### 2.3.2. Radford biproduct construction.
In order to analyze the algebra structure of \((T^*_H(M), *, \Delta)\), we need some results about bialgebras with a projection onto a Hopf algebra due to Radford ([17]).

Let \(K\) be a Hopf algebra and \((A, m_A, 1_A, \Delta_A, \varepsilon_A)\) be a bialgebra with two bialgebra maps \(i : K \to A\) and \(\pi : A \to K\) such that \(\pi i = \text{id}_K\). Set \(\Pi = \text{id}_A \ast (i \delta \pi)\), where \(\ast\) is the convolution product on \(\text{End}(A)\), and \(B = \Pi(A)\). We give a summary on the main results in [17].

1. Equipped with left action \(m_A(i \otimes \text{id}_A)\), right action \(m_A(\text{id}_A \otimes i)\), left coaction \((\pi \otimes \text{id}_A)\Delta_A\) and right coaction \((\text{id}_A \otimes \pi)\Delta_A\), \(A\) is a \(K\)-Hopf bimodule. The set of right coinvariants \(A^R\) is exactly \(B\).
(2) $B$ is a subalgebra of $A$. If we define $\Delta_B = (\Pi \otimes \Pi)\Delta_A$, then $(B, \Delta_B, \varepsilon_A|_B)$ is a coalgebra.

(3) Equipped with the standard $K$-Yetter-Drinfeld module induced from the $K$-Hopf bimodule structure, $(B, m_A|_B, 1_A, \Delta_B, \varepsilon_A|_B)$ is a braided bialgebra.

(4) The map $B \otimes K \to A$ given by $b \otimes h \mapsto bi(h)$ is a bialgebra isomorphism, where $B \otimes K$ is endowed with the smash product and smash coproduct.

2.3.3. The smash structure of $(T_H^c(M), \ast, \Delta)$. Note that both of the inclusion map $i : H \to T_H^c(M)$ into degree 0 and the projection map $\pi : T_H^c(M) \to H$ onto degree 0 are bialgebra maps such that $\pi i = \text{id}_H$. By applying the construction of Radford to $(T_H^c(M), \ast, 1_H, \Delta, \varepsilon), (T_H^c(M)^R, \ast)$ is a subalgebra of $(T_H^c(M), \ast)$, and $T_H^c(M)^R \# H$ is isomorphic, as a bialgebra, to $T_H^c(M)$ via the isomorphism $\xi$ given by $\xi(x \otimes h) = x \ast i(h)$ for any $x \in T_H^c(M)^R$ and $h \in H$. According to the preceding discussion, the investigation of $(T_H^c(M), \ast)$ can be restricted to the subalgebra $(T_H^c(M)^R, \ast)$.

We denote by $T(M^R)$ the tensor vector space on $M^R$.

**Proposition 2.12.** The vector spaces $T_H^c(M)^R$ and $T(M^R)$ are isomorphic as $H$-Yetter-Drinfeld modules.

**Proof.** Essentially, this is proved in [18]. For each $n \geq 1$, we define

$$\varphi : (M \square^H n)^R \to (M^R)^{\square n} \quad \text{and} \quad \psi : (M^R)^{\square n} \to (M \square^H n)^R$$

as follows: for any $\sum m_1 \square \cdots \square m_n \in M \square^H n$ and $v^1 \otimes \cdots \otimes v^n \in (M^R)^{\square n}$, $\varphi$ sends $\sum m_1 \square \cdots \square m_n$ to

$$\sum m_1^{(0)} S(m_1^{(1)}) \otimes \cdots \otimes m_n^{(n-1)} S(m_n^{(n-1)}) \otimes m_n^{(n)} S(m_n^{(n)}) ,$$

$\psi$ sends $v^1 \otimes \cdots \otimes v^n$ to

$$\sum v^1 v_{(-1)}^2 v_{(-2)}^3 \cdots v_{(-n+1)}^n \square \cdots \square v_{(0)}^{n-1} v_{(-1)}^n \square v_{(0)}^n .$$

Then it is routine to verify that both $\varphi$ and $\psi$ are well-defined $H$-Yetter-Drinfeld module morphisms satisfying $\varphi \circ \psi = \text{id}_{(M^R)^{\square n}}$ and $\psi \circ \varphi = \text{id}_{(M \square^H n)^R}$.

This proves the proposition since both sides are graded. \hfill $\Box$

We fix this isomorphism $\varphi : T_H^c(M)^R \to T(M^R)$ in the sequel.

Our next task is to show that the subalgebra $(T_H^c(M)^R, \ast)$ is isomorphic to the quantum quasi-shuffle algebra built on $M^R$.

**Theorem 2.13.** The map $\varphi : T_H^c(M)^R \to T_{\sigma,m}(M^R)$ is a morphism of braided bialgebras. As a consequence, $T_H^c(M)$ is isomorphic, as a Hopf algebra, to $T_{\sigma,m}(M^R) \# H$.

**Proof.** Together with the coproduct $\delta = (\Pi \otimes \Pi)\Delta$, $T_H^c(M)^R$ is a braided bialgebra, where $\Pi$ is the convolution product of $\text{id}_{T_H^c(M)}$ and $i S \pi$. Note that $T_H^c(M)^R$ and $T_H^c(M)^R \square T_H^c(M)^R$ are connected since

$$F_0 T_H^c(M)^R = \mathbb{K}1_H, \quad F_r T_H^c(M)^R = \bigoplus_{0 \leq i \leq r} \Pi(M^\square^H i), \quad r \geq 1$$

and
\[
\begin{align*}
F_0(T^c_H(M)^R \otimes T^c_H(M)^R) &= \mathbb{K}1_H \otimes 1_H, \\
F_r(T^c_H(M)^R \otimes T^c_H(M)^R) &= \bigoplus_{0 \leq i_1 + i_2 \leq r} \Pi(M^{\square^i H^{i_1}}) \otimes \Pi(M^{\square^i H^{i_2}}), \quad r \geq 1.
\end{align*}
\]

Define a linear map \( f : T^c_H(M)^R \to M^R \) by \( 0 \) on degree other than 1 and \( f = P_R \) on degree 1. It is obvious that \( f \) commutes with the braidings and is an algebra map on \( \ker \varepsilon \otimes \ker \varepsilon = T^c_H(M)^R \otimes T^c_H(M)^R \) with \( f(1_H) = 0 \). According to Proposition 2.10, it extends to \( \overline{f} : T^c_H(M)^R \to T_{\sigma,m}(M^R) \). Then it remains to show that \( \overline{f} = \varphi \) on each degree. This is direct since \( f \) concentrate on degree 1, the only non-zero term in the restriction of \( \overline{f} \) on \( (M^{\square^i H^j})^R \) is \( f^\otimes n \circ \Delta^{(n-1)} \), which is given by the action of \( P_R^\otimes n \).

3. Other structures and properties

3.1. Rota-Baxter algebra structure on \((T^c_H(M), \ast)\). We turn to establish a Rota-Baxter algebra structure on \((T^c_H(M), \ast)\). A basic reference is [5].

**Definition 3.1.** Given \( \lambda \in \mathbb{K} \), a pair \((R, P)\) is called a Rota-Baxter algebra of weight \( \lambda \) if \( R \) is a \( \mathbb{K} \)-algebra and \( P \) is a \( \mathbb{K} \)-linear endomorphism of \( R \) satisfying
\[
P(x)P(y) = P(xP(y)) + P(P(x)y) + \lambda P(xy), \forall x, y \in R.
\]
The map \( P \) is called a Rota-Baxter operator of weight \( \lambda \).

If \( P \) is a Rota-Baxter operator of weight 1, then \( \lambda P \) is a Rota-Baxter operator of weight \( \lambda \) (cf. [5]). So we can just focus on the case of weight 1.

Given a Rota-Baxter algebra, the following remarkable property enables one to provide new Rota-Baxter algebra structure on the underlying space with a different product from the original one and the Rota-Baxter operator.

**Lemma 3.2** ([5]). Let \((R, P)\) be a Rota-Baxter algebra of weight \( \lambda \). We define \( R_\otimes \) be the vector space \( R \) with the multiplication
\[
x \otimes y = xP(y) + P(x)y + \lambda xy, \quad \text{for } x, y \in R.
\]
Then \((R_\otimes, P)\) is again a Rota-Baxter algebra of weight \( \lambda \).

Now we recall a construction of Rota-Baxter algebras from quantum quasi-shuffle algebras (cf. [9]). Let \((A, m, \sigma)\) be a unital braided algebra. We consider the vector space \( A \otimes T(A) \). As vector spaces, \( A \otimes T(A) \cong T^+(A) = A \oplus A^{\otimes 2} \oplus \ldots \). We define an associative product
\[
\hat{\diamond}_\sigma = (m \otimes \hat{x}_\sigma)(\text{id}_A \otimes \beta \otimes \text{id}_{T(A)})
\]
on \( A \otimes T(A) \). We denote by \( \mathcal{A} \) the algebra \((A \otimes T(A), \hat{\diamond}_\sigma)\). We also define an endomorphism \( Q \) on \( A \otimes T(A) \) by \( Q(a \otimes x) = 1_A \otimes a \otimes x \). Then \((\mathcal{A}, Q)\) a Rota-Baxter algebra of weight 1.

We let \( T^+_{\sigma,m}(A) \) denote the sub-algebra of \( T_{\sigma,m}(A) \) containing elements of positive degrees.
Proposition 3.3. We have that $A_\bigodot$ is isomorphic to $T_{\sigma,m}^+(A)$ as an algebra. Therefore $(T_{\sigma,m}^+(A), P^+)$ is a Rota-Baxter algebra of weight 1 with $P^+(x) = 1_A \otimes x$ for $x \in T_{\sigma,m}^+(A)$.

Proof. We define a linear map $f : A \otimes T(A) \to T^+(A)$ by $f(a \otimes x) = a \otimes x$ for any $a \otimes x \in A \otimes T(A)$. It is apparent that $f$ is a bijection. Notice that for any $a, b \in A$, $x \in A^{\otimes i}$ and $y \in A^{\otimes j}$,

$$f((a \otimes x) \bigodot (b \otimes y)) = f(Q + (a \otimes x) \bigotimes (b \otimes y) + (a \otimes x) \bigotimes Q(b \otimes y) + (a \otimes x) \bigotimes (b \otimes y))$$

$$= f((1_A \otimes a \otimes x) \bigotimes (b \otimes y) + (a \otimes x) \bigotimes (1_A \otimes b \otimes y) + (a \otimes x) \bigotimes (b \otimes y))$$

$$= (m \bigotimes \mathfrak{m}(i+1,j))(\id_A \otimes \beta_{i+1,1} \otimes \id_A^{\otimes j})(a \otimes x \otimes 1_A \otimes b \otimes y)$$

$$+ (m \bigotimes \mathfrak{m}(i+1))(\id_A \otimes \beta_{1,1} \otimes \id_A^{\otimes i+1})(a \otimes x \otimes 1_A \otimes b \otimes y)$$

$$= (a \otimes x) \bigotimes (b \otimes y)$$

$$= f(a \otimes x) \bigotimes f(b \otimes y).$$

We assume that $A$ is a braided algebra with unit $1_A$ and define an endomorphism $P$ on $T_{\sigma,m}(A)$ as follows:

$$\begin{cases} P(\lambda) = \lambda 1_A, & \text{if } \lambda \in \mathbb{K}, \\ P(x) = P^+(x), & \text{if } x \in T^+(A). \end{cases}$$

Proposition 3.4. The pair $(T_{\sigma,m}(A), P)$ is a Rota-Baxter algebra of weight 1.

Proof. First of all, for any $\lambda, \nu \in \mathbb{K}$,

$$P(P(\lambda) \bigotimes \nu) + P(\lambda \bigotimes P(\nu)) + P(\lambda \nu) = \lambda \nu(1_A \otimes 1_A + 1_A \otimes 1_A + m(1_A \otimes 1_A)) = (\lambda 1_A) \bigotimes (\nu 1_A) = P(\lambda) \bigotimes P(\nu).$$

For any $x \in T^+(A)$,

$$P(P(\lambda) \bigotimes x) + P(\lambda \bigotimes P(x)) + P(\lambda \bigotimes x) = \lambda 1_A \otimes (1_A \bigotimes x) + \lambda 1_A \otimes 1_A \otimes x + \lambda 1_A \otimes x = (\lambda 1_A) \bigotimes (1_A \otimes x) = P(\lambda) \bigotimes P(x).$$
Lemma 3.5. Suppose that \( \beta(A) = \lambda \). Then

\[ P(P(x \# \lambda) + P(x \# P(\lambda)) + P(x \# \lambda) \]

\[ = \lambda 1_A \otimes x + \lambda 1_A \otimes (x \# 1_A) + \lambda 1_A \otimes x \]

\[ = (1_A \otimes x) \# (\lambda 1_A) = P(x) \# P(\lambda), \]

For any \( x, y \in T^+(A) \),

\[ P(P(x \# \lambda) y) + P(x \# P(y)) + P(x \# y) \]

\[ = 1_A \otimes ((1_A \otimes x) \# \lambda y) + 1_A \otimes (x \# (1_A \otimes y)) + 1_A \otimes (x \# y) \]

\[ = (1_A \otimes x) \# (1_A \otimes y) = P(x) \# P(y). \]

In the above computations, we used the fact that \( \beta(1_A \otimes x) = x \otimes 1_A \) and \( \beta(x \otimes 1_A) = 1_A \otimes x \) for any \( x \in T^+(A) \).

Finally,

\[ P(P(\lambda + x) \# (\nu + y)) + P((\lambda + x) \# \nu + y) + P((\lambda + x) \# (\nu + y)) \]

\[ = P(P(\lambda) \# \nu + \lambda \# \nu + P(x) \# \nu + P(x) \# \nu + P(x) \# \nu) \]

\[ + P(\lambda \# \nu + \lambda \# \nu + x \# \nu + x \# \nu + x \# \nu) \]

\[ = P(\lambda) \# \nu + P(\lambda) \# \nu + P(x) \# \nu + P(x) \# \nu \]

\[ = P(\lambda + x) \# \nu + P(\nu + \lambda). \]

We introduce the notion of Rota-Baxter algebras in \( H \rhd \mathcal{YD} \). A Rota-Baxter algebra \((R, P)\) is said to be in \( H \rhd \mathcal{YD} \) if \( R \) is an object in \( H \rhd \mathcal{YD} \) and \( P \) is a morphism in \( H \rhd \mathcal{YD} \). There exists a unique linear extension of \( P \) to the smash product \( \tilde{P} : R \# H \rightarrow R \# H \) given by \( \tilde{P}(a \# h) = P(a) \# h \).

**Lemma 3.5.** Suppose that \((V, P)\) is a Rota-Baxter operator of weight 1 in \( H \rhd \mathcal{YD} \). Then so is the pair \((V \# H, \tilde{P})\).

**Proof.** For any \( v, v' \in V \) and \( h, h' \in H \),

\[ \tilde{P}(\tilde{P}(v \# h)(v' \# h')) + \tilde{P}((v \# h)P(v' \# h')) + \tilde{P}((v \# h)(v' \# h')) \]

\[ = \tilde{P}((P(\nu) \# h)(v' \# h')) + \tilde{P}((v \# h)(P(v') \# h')) + \tilde{P}((v \# h)(v' \# h')) \]

\[ = \sum \left( P(v)(h_{(1)} \cdot v') \otimes h_{(2)} h' + P(h_{(1)} \cdot P(v')) \otimes h_{(2)} h' + P(v(h_{(1)} \cdot v')) \otimes h_{(2)} h' \right) \]

\[ = \sum \left( P(v)(h_{(1)} \cdot v') + v(h_{(1)} \cdot P(v')) + P(v(h_{(1)} \cdot v')) \right) \otimes h_{(2)} h' \]

\[ = \sum P(v)(h_{(1)} \cdot P(v')) \otimes h_{(2)} h' \]
with the help of these maps. As before, the left and right comodule structures of are given by the following composition
\[ P(T^c_H(M)) \xrightarrow{\phi \otimes \xi} T(M^R)^\#H \xrightarrow{\tilde{P}} T(M^R)^\#H(\phi \otimes \xi)^{-1} T^c_H(M), \]
where \( \phi = \varphi \otimes \text{id}_H \).

The following theorem is a corollary of Proposition 3.4 and Lemma 3.5.

**Theorem 3.6.** Suppose \( M \) is a unital \( H \)-Hopf bimodule algebra. Then the pair \((T^c_H(M), \ast, \tilde{P})\) is a Rota-Baxter algebra of weight 1.

**Remark 3.7.** By combining Lemma 3.2 and the above theorem, we can construct a new product on \( T^c_H(M) \) which brings \( T^c_H(M) \) another Rota-Baxter algebra structure with the same \( \tilde{P} \).

### 3.2. Universal property.

We study the universal property of \((T^c_H(M), \ast)\). For this reason, it should start by describing the suitable category in which \((T^c_H(M), \ast)\) is a free object.

Let \((B, \mu_B, \Delta_B, \varepsilon_B)\) be a bialgebra such that:

1. \( \text{corad}(B) \) is a subalgebra of \( B \);
2. there exist two bialgebra maps \( g : B \to H \) and \( k : H \to B \) with \( g \circ k = \text{id}_H \).

If this is the case, we can endow \( B \) with an \( H \)-Hopf bimodule algebra structure with the help of these maps. As before, the left and right comodule structures of \( B \) are given by \( \varrho_L = (g \otimes \text{id}_B)\Delta_B \) and \( \varrho_R = (\text{id}_B \otimes g)\Delta_B \) respectively. The \( H \)-bimodule structure is given by \( h \cdot b \cdot h' = k(h)bk(h') \) for \( h, h' \in H \) and \( b \in B \). Since both \( g \) and \( k \) are Hopf algebra maps with \( g \circ k = \text{id}_H \), easy verifications show that \( \varrho_L(h \cdot b \cdot h') = \Delta_B(h)\varrho_L(b)\Delta_B(h') \) and \( \varrho_R(h \cdot b \cdot h') = \Delta_B(h)\varrho_R(b)\Delta_B(h') \). Since \( \mu_B((b \cdot h) \otimes b') = bk(h)b' = \mu_B(b \otimes (h \cdot b')) \), \( \mu_B \) induces a multiplication \( B \otimes H B \to B \). In addition, by a direct verification, \( \mu_B \) is a morphism in \( H_M^H \).

On the other hand, the condition \( g \circ k = \text{id}_H \) implies \( B = \text{Im} k \oplus \text{Ker} g \) as vector space.

**Theorem 3.8 (Universal property of \( T^c_H(M) \)).** Under the assumptions above, for any \( H \)-bimodule algebra \( M \) and any morphism \( f : B \to M \) in \( H_M^H \) such that \( f \circ k = 0 \), \( f \) is an algebra map and \( f(\text{corad}(B)) = 0 \), there exists a unique bialgebra map \( F : B \to
$T^c_H(M)$ making the following diagrams commute:

\[
\begin{array}{ccc}
T^c_H(M) & \xrightarrow{F} & B \\
\pi & & \downarrow g \\
H & & M \\
\end{array}
\quad \begin{array}{ccc}
T^c_H(M) & \xrightarrow{F} & B \\
p & & \downarrow f \\
M & & \\
\end{array}
\]

where $\pi$ and $p$ are the projections from $T^c_H(M)$ onto $H$ and $M$ respectively.

**Proof.** By Proposition 2.5, there exists a unique coalgebra map $F : B \to T^c_H(M)$ making the above diagrams commute. We want to show that $F$ is an algebra map, i.e., $F(bb') = F(b)*F(b')$ whenever $b, b' \in B$. Define two maps $F_1, F_2 : B \otimes B \to T^c_H(M)$ by $F_1(b \otimes b') = F(bb')$ and $F_2(b \otimes b') = F(b)*F(b')$ respectively. Since $F_1 = F \mu_B$ and both $F$ and $\mu_B$ are coalgebra maps, $F_1$ is a coalgebra map. So is $F_2$ for a similar reason.

Consider the following commutative diagrams:

\[
\begin{array}{ccc}
T^c_H(M) & \xrightarrow{F_1} & B \otimes B \\
\pi & & \downarrow \pi F_1 \\
H & & M \\
\end{array}
\quad \begin{array}{ccc}
T^c_H(M) & \xrightarrow{F_1} & B \otimes B \\
p & & \downarrow p F_1 \\
M & & \\
\end{array}
\]

Since $\pi F_1 = \pi F \mu_B = g \mu_B$ is the composition of two coalgebra maps $g$ and $\mu_B$, $\pi F_1$ is a coalgebra map. Since $p F_1 = p F \mu_B = f \mu_B$ is the composition of two $H$-bicomodule morphisms $f$ and $\mu_B$, $p \circ F_2$ is a $H$-bicomodule morphism. Note that

\[
p F_1(\text{corad}(B \otimes B)) = f \mu_B(\text{corad}(B \otimes B)) 
\subset f \mu_B(\text{corad}(B) \otimes \text{corad}(B)) = f(\text{corad}(B)) = 0.
\]

Hence, $F_1$ is the coalgebra map induced by $\pi F_1$ and $p F_1$ from Proposition 2.5.

Now consider another two commutative diagrams:

\[
\begin{array}{ccc}
T^c_H(M) & \xrightarrow{F_2} & B \otimes B \\
\pi & & \downarrow \pi F_2 \\
H & & M \\
\end{array}
\quad \begin{array}{ccc}
T^c_H(M) & \xrightarrow{F_2} & B \otimes B \\
p & & \downarrow p F_2 \\
M & & \\
\end{array}
\]

We have $\pi F_2 = \pi \circ \rho \circ (F \otimes F) = \mu_H(\pi \otimes \pi)(F \otimes F) = \mu_H(g \otimes g)$, and $p F_2 = \rho_L(g \otimes f) + \rho_R(f \otimes g) + \mu(f \otimes f)$, where $\mu$ is the multiplication of $M$. Evidently, $\pi F_2$ is a coalgebra map and $p F_2$ is an $H$-bicomodule morphism. Note that $p F_2(\text{corad}(B \otimes B)) \subset p F_2(\text{corad}(B) \otimes \text{corad}(B)) = (\rho_L(g \otimes f) + \rho_R(f \otimes g) + \mu(f \otimes f))(\text{corad}(B) \otimes \text{corad}(B)) = 0$. Hence, $F_2$ is the coalgebra map induced by $\pi F_2$ and $p F_2$ from Proposition 2.5.

The last step is to show that $\pi F_1 = \pi F_2$ and $p F_1 = p F_2$. Observe that for any $b, b' \in B$,

\[
\pi F_1(b \otimes b') = g(bb') = g(b)g(b') = \pi F_2(b \otimes b').
\]
By the decomposition $B = \text{Im}k \oplus \text{Ker}g$, we write $b = k(h) + d$ and $b' = k(h') + d'$ for some $h, h' \in H$ and $d, d' \in \text{Ker}g$. Then

$$pF_3(b \otimes b') = h \cdot f(k(h')) + g(d) \cdot f(k(h')) + h \cdot f(d') + g(d) \cdot f(d')$$

$$+ f(k(h)) \cdot h' + f(d) \cdot h' + f(k(h)) \cdot g(d') + f(d) \cdot g(d')$$

$$+ f(k(h))f(k(h')) + f(d)f(k(h')) + f(k(h))f(d') + f(d)f(d')$$

$$= f(h \cdot d') + f(d \cdot h') + f(d)f(d')$$

$$= f(bb')$$

$$= pF_1(b \otimes b'),$$

where the second equality follows from the fact that $f$ is an $H$-bimodule map and $fk = 0$. This completes our proof. \(\square\)

4. Examples

In this section, we will provide some concrete examples of cofree Hopf algebras on Hopf bimodule algebras, including universal Clifford algebras and universal quantum groups.

4.1. Universal Clifford algebras. Let $V$ be a vector space, $Q : V \to \mathbb{K}$ be a quadratic form on $V$ and $\langle \cdot, \cdot \rangle$ be the associated symmetric bilinear form via polarization. The Clifford algebra $Cl(V, Q)$ on $V$ is the associated $\mathbb{K}$-algebra generated by $V$ with respect to the relations $uv + vu = \langle u, v \rangle$ for any $u, v \in V$.

Let $v_1, \ldots, v_n$ be a linear basis of $V$. The algebra $Cl(V, Q)$ has generators $v_1, \ldots, v_n$ and relations $v_iv_j + v_jv_i = \langle v_i, v_j \rangle$.1

Definition 4.1. The universal Clifford algebra $Cl(V)$ is generated as a $\mathbb{K}$-algebra by $\epsilon, v_i, \xi_{ij}$ for $1 \leq i \leq j \leq n$ and relations:

$$\epsilon^2 = 1, \quad \epsilon \xi_{ij} \epsilon^{-1} = \xi_{ij}, \quad \epsilon v_i \epsilon^{-1} = -v_i,$$

$$v_iv_j + v_jv_i = \xi_{ij}, \quad \xi_{ij} \xi_{kl} = \xi_{kl} \xi_{ij},$$

where $1 \leq i \leq j \leq n$ and $1 \leq k \leq l \leq n$.

Let $I$ be the ideal in $Cl(V)$ generated by $\xi_{ij} - Q(v_i, v_j).1$ for $1 \leq i \leq j \leq n$. Then the Clifford algebra $Cl(V, Q)$ is a subalgebra of the quotient algebra $Cl(V)/I$ generated by $v_i$ for $i = 1, \ldots, n$.

The universal Clifford algebra admits a unique Hopf algebra structure defined by:

$$\Delta(\epsilon) = \epsilon \otimes \epsilon, \quad \Delta(v_i) = \epsilon \otimes v_i + v_i \otimes 1, \quad \Delta(\xi_{ij}) = 1 \otimes \xi_{ij} + \xi_{ij} \otimes 1,$$

$$\varepsilon(\epsilon) = 1, \quad \varepsilon(v_i) = 0, \quad \varepsilon(\xi_{ij}) = 0.$$

Since

$$\Delta(\xi_{ij} - Q(v_i, v_j).1) = 1 \otimes (\xi_{ij} - Q(v_i, v_j).1) + (\xi_{ij} - Q(v_i, v_j).1) \otimes 1 + Q(v_i, v_j).1 \otimes 1$$
and $Q$ is symmetric, the ideal $I$ is a Hopf ideal if and only if $Q(v_i, v_j) = 0$ for any $i, j = 1, \ldots, n$. If this is the case, the quotient $Cl(V)/I$ is isomorphic to the exterior algebra $\wedge(V)$ on $V$.

We want to construct the universal Clifford algebra in the framework of cofree Hopf algebras on Hopf bimodule algebras.

Let $H = \mathbb{K}[\mathbb{Z}/2] = \mathbb{K}.1 \oplus \mathbb{K}.\varepsilon$ be the group algebra. It admits a Hopf algebra structure by letting 1 and $\varepsilon$ be group-like. Let $W = U \otimes H$ where $U$ is the vector space generated by $v_i$ and $\xi_{ij}$ for $1 \leq i \leq j \leq n$. It admits an $H$-Hopf bimodule algebra structure as follows:

1. $U$ admits a left $H$-module and comodule structure: for $1 \leq i \leq j \leq n$,
   $$\delta_L(v_i) = \varepsilon \otimes v_i, \quad \delta_L(\xi_{ij}) = 1 \otimes \xi_{ij}, \quad \varepsilon v_i = -v_i, \quad \varepsilon \xi_{ij} = \xi_{ij}.$$

2. Left $H$-module and comodule structures: for $u \in U$ and $h, h' \in H$,
   $$\delta_L(u \otimes h) = \sum u(-1)h(1) \otimes u(0) \otimes h(2), \quad h.(u \otimes h) = \sum h'(1).u \otimes h'(2)h.$$

3. Right $H$-module and comodule structures: for $u \in U$ and $h, h' \in H$,
   $$\delta_R(u \otimes h) = \sum u \otimes h(1) \otimes h(2), \quad (u \otimes h)h' = u \otimes hh'.$$

4. The multiplication $m : W \otimes W \to W$ is uniquely determined by: for $1 \leq i \leq j \leq n$,
   $$m(v_i \otimes v_j h') = \xi_{ij} h', \quad m(v_i \varepsilon \otimes v_j h') = -\xi_{ij} \varepsilon h',$$
   and on all other elements, $m$ gives zero. Notice that we omitted the tensor product inside of $W$ for simplification.

We let $T^c_H(W)$ denote the cofree Hopf algebra built on the $H$-Hopf bimodule algebra $W$ and $Q_H(W)$ its subalgebra generated by $H$ and $W$. The multiplication in $Q_H(W)$ is denoted by $\ast$.

**Lemma 4.2.** There exists a Hopf algebra morphism $\varphi : Cl(V) \to Q_H(W)$ defined by

$$v_i \mapsto v_i, \quad \varepsilon \mapsto \varepsilon, \quad \xi_{ij} \mapsto \xi_{ij}.$$

**Proof.** The only thing which needs a proof is that in $Q_H(W)$, the following identity holds: for any $1 \leq i \leq j \leq n$, $v_i \ast v_j + v_j \ast v_i = \xi_{ij}$. Indeed,

$$v_i \ast v_j = v_i \otimes v_j - v_j \otimes v_i + m(v_i \otimes v_j) = v_i \otimes v_j - v_j \otimes v_i + \xi_{ij},$$

$$v_j \ast v_i = v_j \otimes v_i - v_i \otimes v_j.$$

□

**Theorem 4.3.** $\varphi$ is an isomorphism of Hopf algebras.

**Proof.** Since $Q_H(W)$ is isomorphic to the quotient of $Cl(V)$ by the Hopf ideal generated by $\xi_{ij}$ for $1 \leq i \leq j \leq n$. This ideal is sent by $\varphi$ to a Hopf ideal $J$ in $Q_H(W)$ generated by $\xi_{ij}$ for $1 \leq i \leq j \leq n$. Then $\varphi$ passes to the quotient to give a Hopf algebra surjection $\varphi : \wedge(V) \to Q_H(W)/J$ which is moreover an isomorphism.
Therefore \( \ker \varphi \) is included in the subalgebra \( C \) of \( \text{Cl}(V) \) generated by \( \xi_{ij} \) for \( 1 \leq i \leq j \leq n \), which is a polynomial algebra. Since the restriction of the multiplication \( m \) on \( W \) to the subspace spanned by these \( \xi_{ij} \) is zero, the subalgebra of \( Q_H(W) \) generated by \( \xi_{ij} \) is a symmetric algebra. As a conclusion, the restriction of \( \varphi \) to \( C \) gives a Hopf algebra surjection

\[
\varphi|_C : C \rightarrow \mathbb{K}[\xi_{ij} | 1 \leq i \leq j \leq n],
\]

hence \( \varphi|_C \) is injective and \( \ker \varphi = \{0\} \).

### 4.2. Universal quantum groups.

We first recall the construction (with slight generalizations and modifications) in [3].

Let \( g \) be a Kac-Moody Lie algebra associated to an integral matrix \( C = (c_{ij})_{n \times n} \in M_n(\mathbb{Z}) \) and \( q \in \mathbb{K}^* \) not be 0 and 1. We let \( I \) denote the index set \( \{1, \cdots, n\} \).

Let \( H = \mathbb{K}[K_i^{\pm 1}, \cdots, K_n^{\pm 1}] \) be the group algebra of the additive group \( \mathbb{Z}^n \): it is a Hopf algebra, \( W \) be the vector space spanned by \( \{E_i, F_i, \xi_i | i \in I\} \) and \( M = W \otimes H \).

The following abuse of notation will be applied hereafter: for \( x \in W \) and \( K \in H \), we shall write \( xK \) for \( x \otimes K \) in \( M \) for short.

We define an \( H \)-Hopf bimodule algebra structure on \( M \):

1. The right module and comodule structures are defined as follows: for \( w \in W \) and \( h, h' \in H \),

\[
\delta_R(w \otimes h) = \sum w \otimes h^{(1)} \otimes h^{(2)}, \quad (w \otimes h)h' = w \otimes hh'.
\]

2. The left structures are given in the following way: on \( W \), for any \( i, j \in I \),

\[
K_i.E_j = q^{ci_{ij}}E_j, \quad K_i.F_j = q^{-ci_{ij}}F_j, \quad K_i.\xi_j = \xi_j;
\]

\[
\delta_L(E_i) = K_i \otimes E_i, \quad \delta_L(F_i) = K_i \otimes F_i, \quad \delta_L(\xi_j) = K_i^2 \otimes \xi_i,
\]

then the structure on \( M = W \otimes H \) is taken to be the one arising from the tensor product.

3. We define \( m : M \otimes M \rightarrow M \) by: for any \( K, K' \in H \), if \( \lambda \) is the constant such that \( K.F_j = \lambda F_j \),

\[
m(E_iK \otimes F_jK') = \delta_{ij}\lambda \xi_i KK';
\]

and on any other elements not of the above form, \( \alpha \) gives 0.

It is not difficult to show that \( M \), with these structures, is an \( H \)-Hopf bimodule algebra (for example, see [3]).

We let \( T_H^c(M) \) denote the cofree Hopf algebra built on the \( H \)-Hopf bimodule algebra \( M \) and \( Q_H(M) \) be its sub-Hopf algebra generated by \( H \) and \( M \) whose multiplication will be denoted by \(*\).

**Definition 4.4.** The Hopf algebra \( Q_H(M) \) is called the universal quantum group associated to the matrix \( C \).

Some specializations of this universal object are of great interests:
(1) Let $A$ be a symmetrizable Cartan matrix, $C = DA$ be its symmetrization with $D = \text{diag}(d_1, \cdots, d_n)$ and $\mathfrak{g}$ be the associated Kac-Moody Lie algebra. Suppose that $q \in \mathbb{K}^*$ is not a root of unity. Let $J$ be the ideal in $Q_H(M)$ generated by
\[
\left\{ \xi_i - \frac{K_i^2 - 1}{q_i - q_i^{-1}} \mid i \in I \right\}
\]
where $q_i = q^{d_i}$.

The ideal $J$ is in fact a Hopf ideal and the following theorem is proved in [3]:

**Theorem 4.5 ([3])**. There exists an isomorphism of Hopf algebras $U_q(\mathfrak{g}) \cong Q_H(M)/J$.

(2) We respect all situations in (1) but suppose that $q' = 1$ is a primitive odd root of unity for $l \geq d_i$ for any $i \in I$. Let $I$ be the Hopf ideal in $Q_H(M)$ generated by $J$ and $K_i^l - 1$ for $i \in I$. Using Theorem 15 in [18] and applying the proof of the theorem above in [3] one has the following corollary:

**Corollary 4.6.** Let $u_q(\mathfrak{g})$ be the small quantum group associated to $\mathfrak{g}$ (Frobenius kernel). There exists an isomorphism of Hopf algebra $u_q(\mathfrak{g}) \cong Q_H(M)/I$.

(3) If there is no restriction on the matrix $C \in M_n(\mathbb{Z})$ but $q$ is not a root of unity. Similar to the proof of Proposition 7 in [3], it can be shown that the ideal generated by $\xi_i - P(K_i)$ for $i \in I$ and $P_i \in \mathbb{K}[K_1, \cdots, K_n]$ is a Hopf ideal if and only if there exist $\lambda_i \in \mathbb{K}$ such that $P(K_i) = \lambda_i(K_i^2 - 1)$. We suppose moreover that $\lambda_i \neq 0$ for any $i \in I$, then without loss of generality, we can suppose $\lambda_i = 1$.

The discussion above allows us to define the quantum group $U_q(C)$ associated to a general integral matrix $C$ as the quotient of the universal quantum group $Q_H(M)$ by the Hopf ideal generated by $\xi_i - K_i^2 + 1$ for $i \in I$.

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