HOMOLOGICAL PROPERTIES OF A CERTAIN NONCOMMUTATIVE DEL PEZZO SURFACE

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Abstract. Recently, de Thanhoffer de Volcsey and Van den Bergh showed that Grothendieck groups of “noncommutative Del Pezzo surfaces” with an exceptional sequence of length 4 are isomorphic to one of three types, the third one not coming from a commutative Del Pezzo surface. In this paper, we adapt the theory of noncommutative P^1-bundles as appearing in the work of Van den Bergh and Nyman to produce a sheaf Z-algebra whose associated Proj has an exceptional sequence of length 4 for which the Gram matrix is of this third type. We show that this noncommutative scheme is noetherian and describe its local structure through the use of our generalized preprojective algebras (3).

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1. INTRODUCTION AND OVERVIEW

In the paper, [2], de Thanhoffer de Volcsey and Van den Bergh provide a numerical classification of possibly noncommutative Del Pezzo surfaces with an exceptional sequence of length 4. More precisely they consider a lattice Λ with a nondegenerate bilinear form ⟨−, −⟩ and consider the following sets of conditions:

• there is an s ∈ Aut(Λ) such that ⟨x, sy⟩ = ⟨y, x⟩ for x, y ∈ Λ
• (s − 1) is nilpotent
• rk(s − 1) = 2
Theorem 1.1. Let $\Lambda$ satisfy the above conditions. Then $\Lambda$ is isomorphic to $\mathbb{Z}^4$ where the matrix of the bilinear form is one of the following standard types:

$$
\begin{array}{cccc}
1 & 2 & 2 & 4 \\
0 & 1 & 0 & 2 \\
0 & 0 & 1 & 2 \\
0 & 0 & 0 & 1 \\
\end{array} \quad \text{and} \quad 
\begin{array}{cccc}
1 & 2 & 3 & 5 \\
0 & 1 & 1 & 3 \\
0 & 0 & 1 & 2 \\
0 & 0 & 0 & 1 \\
\end{array}
$$

The first two types correspond to the Grothendieck groups of the Del Pezzo surfaces $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{P}_1$ respectively. Moreover, it is an easy exercise to show that the third type in fact cannot correspond to a Del Pezzo surface. The goal of this paper is to construct a noncommutative analogue of a Del Pezzo surface equipped with an exceptional collection which forms a basis for the Euler form for which the Gram matrix is of the third type.

The entries of the matrix seem to suggest that we should find a 'noncommutative scheme' equipped with 2 'maps' to $\mathbb{P}^1$. We consider a construction which is an adaptation of Van den Bergh's theory of noncommutative $\mathbb{P}^1$-bundles. In $[9]$, he considers a symmetric sheaf $\mathbb{Z}$-algebras $\mathbb{S}(E)$ constructed from a locally free bimodule $E$ of rank $(2, 2)$ and shows that $\mathbb{S}(E)$ can be regarded as the noncommutative analogue of a $\mathbb{P}^1$-bundle. In this paper we will apply the same construction where the bimodule $E$ is of rank $(4, 1)$ instead as suggested by the entries $(1, 3)$ and $(2, 4)$ and construct a noncommutative analogue of a Del Pezzo surface equipped with an exceptional collection which forms a basis for the Euler form for which the Gram matrix is of the third type.

In the next section, we describe their local behaviour. We prove that there exists a cover such that over each open subset, they can be regarded as a generalized pre-projective algebra as introduced in the paper $[5]$. An immediate application of this result is that the category of graded $\mathcal{A}$-modules is a locally noetherian Grothendieck category (Theorems 3.1, 3.7).

In the final section, we introduce pullback functors $\Pi^*_m : \text{QCoh}(X_n) \rightarrow \text{Gr}(\mathbb{S}(E))$ and adapt the results of $[5]$ to our setting to prove a formula which computes the Ext groups of sheaves pulled back from $X$ or $Y$:

**Theorem.** (See [4, 11]) Let $E \in \text{bimod}(X, Y)$ be locally free of rank $(4, 1)$. Let $\mathcal{F}$ and $\mathcal{G}$ be locally free sheaves on $X_n$ respectively $X_m$ for $m, n \in \mathbb{Z}$ such that $m \geq n - 1$. Then

$$
\text{Ext}^i_{\text{Proj}(A)}(\Pi^*_m \mathcal{F}, \Pi^*_n \mathcal{G}) \cong \text{Ext}^i_{X_m}(\mathcal{F}, \mathcal{G} \otimes \mathbb{S}(E)_{n,m})
$$

for all $i \geq 0$.

This culminates in the construction of the desired noncommutative Del Pezzo surface:

**Theorem.** (See [4, 11]) Let $E$ be the $\mathbb{P}^1$-bimodule $f(\mathcal{O}_{\mathbb{P}^1})_{ld}$ and $\mathbb{S}(E)$ be the associated symmetric sheaf $\mathbb{Z}$-algebra. Then

$$
\Pi^*_1(\mathcal{O}_{\mathbb{P}^1}), \Pi^*_1(\mathcal{O}_{\mathbb{P}^1}(1)), \Pi^*_0(\mathcal{O}_{\mathbb{P}^1}), \Pi^*_0(\mathcal{O}_{\mathbb{P}^1}(1))
$$
is an exceptional sequence of graded $\mathcal{S}(\mathcal{E})$-modules for which the Gram matrix of the Euler form is given by
\[
\begin{pmatrix}
1 & 2 & 1 & 5 \\
0 & 1 & 0 & 4 \\
0 & 0 & 1 & 2 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

\section*{Acknowledgements}
Both authors wish to thank their supervisor, Michel Van den Bergh for suggesting the construction and many helpful discussions along the way. They would also like to explicitly mention that most of section 4 is a straightforward adaptation of the beautiful techniques developed by Adam Nyman in \cite{6, 7}.

\section{Symmetric Sheaf $\mathbb{Z}$-Algebras}

\subsection{Definitions and construction}
Sheaf-bimodules were defined in \cite{9} as follows:

\begin{definition}
Let $X \rightarrow S$ and $Y \rightarrow S$ be $S$-schemes. A coherent $X - Y$ bimodule $\mathcal{E}$ is a coherent $\mathcal{O}_{X \times_S Y}$-module such that the support of $\mathcal{E}$ is finite over $X$ and $Y$. We denote the corresponding abelian category by $\text{bimod}_S(X - Y)$. More generally an $X - Y$-bimodule is a quasi-coherent $\mathcal{O}_{X \times_S Y}$-module which is a filtered direct limit of objects in $\text{bimod}_S(X - Y)$. The abelian category of $X - Y$-bimodules is denoted $\text{BiMod}_S(X - Y)$. Finally, a bimodule $\mathcal{E}$ is called locally free if $\pi_{X^*}(\mathcal{E})$ and $\pi_{Y^*}(\mathcal{E})$ are locally free. If moreover $\pi_{X^*}(\mathcal{E})$ and $\pi_{Y^*}(\mathcal{E})$ have finite rank $m$ and $n$ respectively, then $\mathcal{E}$ is said to have rank $(m, n)$.

The tensor product of $\mathcal{O}_{X \times S Y \times S Z}$-modules induces a tensor product $\text{BiMod}_S(X - Y) \otimes \text{BiMod}_S(Y - Z) \rightarrow \text{BiMod}_S(X - Z)$ given by
\[
\mathcal{E} \otimes \mathcal{F} := \pi_{X \times Z^*}(\pi_{X \times Y}^* (\mathcal{E}) \otimes_{X \times Y} \pi_{Y \times Z}^* (\mathcal{F})).
\]
Moreover for each $\mathcal{E} \in \text{BiMod}_S(X - Y)$ there is a right exact functor:
\[
- \otimes \mathcal{E} : \text{QCoh}(X) \rightarrow \text{QCoh}(Y) : \mathcal{M} \otimes \mathcal{E} := \pi_{Y^*}(\pi_{X^*}(\mathcal{M}) \otimes_{X \times Y} \mathcal{E})
\]
which is exact if and only if $\mathcal{E}$ is locally free. By \cite{9} Lemma 3.1.1. this functor determines $\mathcal{E}$ uniquely.

\begin{definition}
Let $W$ be an $S$-scheme with finite $S$-maps $u : W \rightarrow X$, $v : W \rightarrow Y$. If $\mathcal{U} \in \text{QCoh}(W)$, then we denote $(u, v)_*\mathcal{U} \in \text{BiMod}_S(X - Y)$ as $u_*\mathcal{U}$. One easily checks:
\[
- \otimes u_*\mathcal{U} = v_*(u^*(-) \otimes_W \mathcal{U})
\]
Any bimodule isomorphic to one of the form $u_*\mathcal{U} \cong_{Id} (u_*\mathcal{U})_{Id}$ is called central.

\begin{definition}
Let $(X_i \rightarrow S)_{i \in \mathbb{Z}}$ be a collection of $S$-schemes. A sheaf $\mathbb{Z}$-algebra $\mathcal{A}$, is a collection of $X_i - X_j$-bimodules $\mathcal{A}_{ij}$ together with maps $\mathcal{A}_{ij} \otimes \mathcal{A}_{jk} \rightarrow \mathcal{A}_{ik}$ satisfying the usual associativity and unit properties (see \cite{9}). An $\mathcal{A}$-module is a sequence of $X_i$-modules $\mathcal{M}_i$ together with maps $\mathcal{M}_i \otimes \mathcal{A}_{ij} \rightarrow \mathcal{M}_j$ again satisfying the obvious axioms. The associated category is denoted $\text{Gr}(\mathcal{A})$. An $\mathcal{A}$-module is right bounded if $\mathcal{M}_i = 0$ for $i >> 0$. An $\mathcal{A}$-module is called torsion if it is a filtered colimit of right bounded modules. Let $\text{Tors}(\mathcal{A})$ be the subcategory
of $\text{Gr}(A)$ consisting of torsion modules. Then $\text{Tors}(A)$ is Serre and the corresponding quotient category is denoted by $\Proj(A)$. We have a projection functor $p : \text{Gr}(A) \rightarrow \Proj(A)$ with right adjoint $\omega$ (see [8]).

We shall need the following form of duality of bimodules:

**Lemma 2.4.** Let $E \in \text{bimod}_S(X - Y)$ be locally free. Then there is a unique object $E^* \in \text{bimod}_S(Y - X)$ such that

$$- \otimes_Y E^* : \text{QCoh}(Y) \rightarrow \text{QCoh}(X)$$

is the right adjoint of $- \otimes_X E$, i.e. for $M \in \text{QCoh}(X)$ and $N \in \text{QCoh}(Y)$:

$$\text{Hom}_Y(M \otimes E, N) \cong \text{Hom}_X(M, N \otimes E^*)$$

**Proof.** see [9, Section 3].

**Remark 2.5.** If $E = uUv$, then $E^*$ is given by $v\text{Hom}_W(U, v!O_Y)u$.

The dual notion leads to a an object $^*E$ such that

$$\text{Hom}_X(N \otimes ^*E, M) \cong \text{Hom}_Y(N', M \otimes E)$$

and Yoneda’s lemma proves that $E = (^*E^*) = (^*E)^*$. Repeated application of duals leads to the following notation:

$$E^{*n} = \begin{cases} E^* \ldots E^* & n \geq 0 \\ \cdots \otimes E & n < 0 \end{cases}$$

In this context it will be convenient to invoke the following notation:

(1) $X_n = X$ if $n$ is even and $Y$ if $n$ is odd.

There are unit and counit morphisms:

(2) $i_n : O_{X_n} \rightarrow E^{*n} \otimes E^{*n+1}$

$$j_n : E^{*n} \otimes E^{*n-1} \rightarrow O_{X_n}$$

Our next ingredient is that of a nondegenerate bimodule.

**Definition 2.6.** For $E \in \text{bimod}(X - Y)$ and $F \in \text{bimod}(Y - Z)$, a bimodule $Q \subset E \otimes_Y F$ is invertible if there is a $Q^{-1} \subset \text{bimod}(Z - X)$ such that $Q \otimes Q^{-1} \cong O_X$ and $Q^{-1} \otimes Q \cong O_Z$. If moreover the following canonical composition

$$E^* \otimes_X Q \rightarrow E^* \otimes_X E \otimes_Y F \rightarrow F$$

is in fact an isomorphism, then $Q$ is said to be nondegenerate.

We can now state the definition of a symmetric sheaf $Z$-algebra.

**Definition 2.7.** Let $(X_i \rightarrow S)_{i \in \mathbb{Z}}$ be a sequence of $S$-schemes and let $E_i$ be locally free $X_i - X_{i+1}$-bimodules. Then the tensor sheaf $Z$-algebra $T(\{E_i\})$ is the sheaf $Z$-algebra generated by the $\{E_i\}$. More precisely

$$T(\{E_i\})_{m,n} = \begin{cases} 0 & n < m \\ \text{id}(O_{X_n}) & n = m \\ E_m \otimes \ldots \otimes E_{n-1} & n > m \end{cases}$$

\[\text{The notation } Q\text{Gr}(A) \text{ is standard as well.}\]
If moreover for each \( i \) we are given a nondegenerate \( X_i \)-\( Y_i \)-bimodule \( Q_i \subset E_i \otimes E_{i+1} \), then the symmetric sheaf \( \mathcal{Z} \)-algebra \( S(\{E_i\}, \{Q_i\}) \) is the quotient of \( T(\{E_i\}) \) by the relations \((Q_i)\). I.e. \( S(\{E_i\}, \{Q_i\})_{m,n} \) is defined as

\[
\begin{cases}
T(\{E_i\})_{m,n} & n \leq m + 1 \\
T(\{E_i\})_{m,n}/[Q_m \otimes \ldots + (E_m \otimes Q_{m+1} \otimes \ldots) + \ldots + (\ldots \otimes Q_{n-2})] & n \geq m + 2
\end{cases}
\]

Given an \( X-Y \)-bimodule \( E \), the so-called standard tensor sheaf \( \mathcal{Z} \)-algebra \( T(E) \) and standard symmetric sheaf \( \mathcal{Z} \)-algebra \( S(E) \) are constructed as above by taking \( X_n \) as in (1) and \( E_n, Q_n \) as follows:

\[
E_n = E^* \\
Q_n = i_n (O_{X_n}) \subset E^* \otimes E^{*+1}
\]

There is a useful operation called twisting:

**Theorem 2.8.** Let \((X_i \rightarrow S)_i\) and \((Y_i \rightarrow S)_i\) be \( S \)-schemes and \( A \) a sheaf \( \mathcal{Z} \)-algebra.

Given a collection of invertible \( X_i-Y_i \)-bimodules \((T_i)_i\), one can construct a sheaf \( \mathcal{Z} \)-algebra \( B \) by

\[
B_{ij} := T_{i}^{-1} \otimes A_{ij} \otimes T_{i}
\]

called the twist of \( A \) by \((T_i)_i\).

There is an equivalence of categories given by the functor

\[
T : \text{Gr}(A) \cong \text{Gr}(B) : M_i \rightarrow M_i \otimes T_i
\]

Finally, every sheaf \( \mathcal{Z} \)-algebra can be obtained from a standard one by a twist.

**Proof.** This is proven in section 4.1 of [9] \( \square \)

The above theorem allows us to make the following definition:

**Definition 2.9.** A symmetric sheaf \( \mathcal{Z} \)-algebra is commutative if it is the twist of a standard symmetric sheaf \( \mathcal{Z} \)-algebra whose underlying bimodule is central.

2.2. The Rank \((2,2)\) Case. The following lemma (which was already announced but not proven in [9]) shows that commutative symmetric sheaf \( \mathcal{Z} \)-algebras of rank \((2,2)\) are essentially commutative. We shall use the notation \( \text{Sym}_{X \times X}(id V_{Id}) \)

to denote the sheaf-\( \mathcal{Z} \)-algebra whose \((i,j)\)-component is \( \text{Sym}_{X \times X}(id V_{Id})_{j-i} \) and considered its associated category of graded modules as in Definition 2.3.

**Lemma 2.10.** Let \( V \) be a locally free \( X \)-module of rank 2. There is an equivalence of the form

\[
\text{Gr}(S(id V_{Id})) \xrightarrow{T} \text{Gr}( \text{Sym}_{X \times X}(id V_{Id}) ) \xrightarrow{i} \text{Gr}( \text{Sym}_X(V) )
\]

where \( T \) is given by twisting through \( (\wedge^2 V)^{\frac{1}{2}} \) \( i \in \mathbb{Z} \).

**Proof.** The second equivalence in the composition follows from the fact

\[
M_i \otimes \text{Sym}_{X \times X}(id V_{Id})_{j-i} = M_i \otimes id (\text{Sym}_X(V)_{j-i})_{id} = M_i \otimes X \text{Sym}_X(V)_{j-i}
\]
implying that both multiplications coincide. We thus only need to exhibit the first equivalence.
Let \( \mathcal{E} = \text{id}_d \mathcal{V}_d \). Using the explicit expression for the dual, we obtain
\[
\mathcal{E}^* = \text{id}_d \text{Hom}(\mathcal{V}, \text{id}_d \mathcal{O}_d)_{\text{id}} = \text{id}_d(\mathcal{V}^*)_d
\]
In particular \( \mathcal{E}^{2n} = \mathcal{E} = \text{id}_d(\mathcal{V})_{\text{id}} \) and \( \mathcal{E}^{2n+1} = \mathcal{E}^* = \text{id}_d(\mathcal{V}^*)_{\text{id}} \) hold for all \( n \).
Recall from [4, II ex. 5.16.b] that the pairing \( \mathcal{V} \otimes \mathcal{V} \to \Lambda^2 \mathcal{V} \) is perfect, implying there is an isomorphism
\[
(4) \quad \mathcal{V}^* \otimes (\Lambda^2 \mathcal{V}) \xrightarrow{\cong} \mathcal{V}
\]
Let \( (T_i)_i = (\Lambda^2 \mathcal{V})|_{\frac{2}{3}} \). It follows that \( T(\mathcal{E}) \) is obtained from a twist of the classical tensor \( \mathbb{Z} \)-algebra \( T_X(\mathcal{V}) \) and by Theorem 2.8 there is an equivalence
\[
\text{Gr}(T(\mathcal{E})) \to \text{Gr}(T_X(\mathcal{V})): (\mathcal{M}_i)_{i} \mapsto (\mathcal{M}_i \otimes (\Lambda^2 \mathcal{V})|_{\frac{2}{3}})_{i}
\]
specifically in each component:
\[
(5) \quad T(\mathcal{E})_{m,n} \cong \text{id}_d \left( (\Lambda^2 \mathcal{V})|_{\frac{2}{3}} \otimes T_X(\mathcal{V})_{n-m} \otimes (\Lambda^2 \mathcal{V})^{-1}|_{\frac{2}{3}} \right)_{\text{id}}
\]
We now claim that the twisting in (5) induces a twisting
\[
\mathcal{S}(\mathcal{E})_{m,n} \cong \text{id}_d \left( (\Lambda^2 \mathcal{V})|_{\frac{2}{3}} \otimes \text{Sym}_X(\mathcal{V})_{n-m} \otimes (\Lambda^2 \mathcal{V})^{-1}|_{\frac{2}{3}} \right)_{\text{id}}
\]
and hence an equivalence of categories:
\[
(6) \quad \text{Gr}(\mathcal{S}(\mathcal{E})) \to \text{Gr}(\text{Sym}_X(\mathcal{V})): (\mathcal{M}_i)_{i} \mapsto \bigoplus_i \mathcal{M}_i \otimes (\Lambda^2 \mathcal{V})|_{\frac{2}{3}}
\]
So we are left with proving the claim. For this we must understand what happens under [4] to the relations that define \( \mathcal{S}(\mathcal{E}) \) as a quotient of \( T(\mathcal{E}) \).
As the relations are generated in degree 2 it suffices to consider \( \mathcal{S}(\mathcal{E})_{m,m+2} \otimes T(\mathcal{E})_{m,m+2} \otimes (\Lambda^2 \mathcal{V})_{\text{id}} \). This is the quotient of \( T(\mathcal{E})_{m,m+2} \otimes (\Lambda^2 \mathcal{V})_{\text{id}} \cong \text{id}_d(T_X(\mathcal{V}))_{\text{id}} = \text{id}_d(V \otimes \mathcal{V})_{\text{id}} \) by the relation \( i_{\text{id}}(\mathcal{O}_X)_{\text{id}} \otimes \text{id}_d(\Lambda^2 \mathcal{V})_{\text{id}} \subset \text{id}_d(V \otimes \mathcal{V}^* \otimes \Lambda^2 \mathcal{V})_{\text{id}} \cong \text{id}_d(V \otimes \mathcal{V})_{\text{id}} \).
We have to check that this relation is exactly the one that defines \( \text{Sym}_X(\mathcal{V}) \) as a quotient of \( T_X(\mathcal{V}) \). The latter relation is defined locally, so it suffices to check on a trivializing open subset \( U \) for \( \mathcal{V} \). If \( \mathcal{V}|_U \cong \mathcal{O}_X|_U \), \( u \otimes \mathcal{O}_X|_U \) then \( i_{\text{id}}(\mathcal{O}_X)_{\text{id}} \) is locally given by \( u \otimes u^* + v \otimes v^* \). One checks that the isomorphism [4] maps \( u^* \otimes (u \wedge v) \) to \( v \) and \( v^* \otimes (u \wedge v) \) to \( -u \), the induced relation in \( \mathcal{V} \otimes \mathcal{V} \) is locally given by \( u \otimes v - v \otimes u \), the defining relation of \( \text{Sym}_X(\mathcal{V}) \).

\textbf{Corollary 2.11.} With the assumptions from the previous theorem we have an induced equivalence:
\[
\Phi: \text{Proj}(\mathcal{S}(\mathcal{V}d)) \xrightarrow{\cong} \text{Proj}(\text{Sym}_X(\mathcal{V})) \xrightarrow{\cong} \text{QCoh}(\mathbb{P}_X(\mathcal{V}))
\]

\textbf{Proof.} The equivalence given in (6) obviously maps torsion modules onto torsion modules, hence it factors through \( \text{Proj}(\mathcal{S}(\mathcal{V}d)) \xrightarrow{\cong} \text{Proj}(\text{Sym}_X(\mathcal{V})) \).
The second equivalence is given by the following pair of functors
\[
\begin{array}{ccc}
\text{Proj}(\text{Sym}_X(V)) & \xrightarrow{\sim} & \text{Qcoh}(\mathbb{P}_X(V)) \\
p \circ \Gamma_* := p \left[ \oplus_i \pi_* ((-)i) \right] & & \\
\end{array}
\]
\[p \circ \Gamma_* := p \left[ \oplus_i \pi_* ((-)i) \right] \]

Where \( \pi \) is the projection \( \mathbb{P}_X(V) \to X \).

2.3. **Truncation Functors and periodicity.** Any sheaf \( Z \)-algebra is endowed with a sequence of truncation functors as follows: let \((X_i \to S)_i\) be \( S \)-schemes and \( A \) a sheaf \( Z \)-algebra. Then for each \( n \in \mathbb{Z} \), consider the functor
\[
\text{Gr}(A) \xrightarrow{(-)_n} \text{Qcoh}(X_n)
\]

We shall need the following easy result on these functors:

**Lemma 2.12.** Let \( e_n A \) be the right \( A \)-module \((A_{nm})_m\). There is an adjoint pair \(- \otimes e_n A \dashv (-)_n\)

**Proof.** The proof of this is standard and left to the reader \( \square \)

If \( A \) is a symmetric \( Z \)-algebra in standard form, then there is a 2-periodic behaviour among these functors in the following way:

**Proposition 2.13.** Let \( A \) be a symmetric sheaf \( Z \)-algebra. Then there is an autoequivalence \( \alpha \) on \( \text{Gr}(A) \) inducing a commuting diagram for each \( n \)
\[
\begin{array}{ccc}
\text{Gr}(A) & \xrightarrow{(-)_n} & \text{Qcoh}(X_n) \\
\alpha & & \downarrow \oplus \omega_{X_n/S} \\
\text{Gr}(A) & \xrightarrow{(-)_{n+2}} & \text{Qcoh}(X_n) \\
\end{array}
\]

**Proof.** By Theorem 2.8 \( A \) is Morita equivalent to a symmetric sheaf \( Z \)-algebra \( S(E) \) in standard form with \( E \in \text{bimod}(X - Y) \). Moreover by [9, 4.1.7], we have
\[
E \cong \omega^{-1}_{X/S} \otimes E \otimes \omega_{Y/S}
\]

Hence the twist by \( \omega_{X_i/S} \in \mathbb{Z} \) yields an equivalence
\[
\mathcal{T} : \text{Gr}(S(E)) \xrightarrow{\cong} \text{Gr}(\omega^{-1} \otimes S(E) \otimes \omega) \xrightarrow{\cong} \text{Gr}(S(E^\star))
\]

Where we used the short-hand notation
\[
(\omega^{-1} \otimes S(E) \otimes \omega)_{m,n} = \omega^{-1}_{X_{m,n}/S} \otimes S(E)_{m,n} \otimes \omega_{X_{n}/S}
\]

Next, the construction of a standard symmetric sheaf \( Z \)-algebra implies that there is an equivalence \( \Psi : \text{Gr}(S(E)(2)) \to \text{Gr}(S(E^\star)) \). We now simply define
\[
\alpha := (-2) \circ \Psi^{-1} \circ \mathcal{T} : \text{Gr}(S(E)) \to \text{Gr}(S(E^\star)) \to \text{Gr}(S(E)(2)) \to \text{Gr}(S(E))
\]

\( \square \)

In the case of a sheaf \( Z \)-algebra which is commutative (see [29]) of rank \((2,2)\), the \( 0 \)th truncation functor coincides with the pushforward functor in the following sense:
Theorem 2.14. Let $\mathbb{S}(\mathcal{V})$ be a commutative symmetric sheaf $\mathbb{Z}$-algebra of rank $(2,2)$ and let $\Phi : \text{Proj}(\mathbb{S}(\mathcal{V})) \to \text{QCoh}(\mathbb{P}(\mathcal{V}))$ be the equivalence provided by Corollary 2.14. Then the following diagram commutes

\[
\begin{array}{ccc}
\text{Gr}(\mathbb{S}(\mathcal{V})) & \xrightarrow{\omega} & \text{QCoh}(X) \\
\downarrow_{\Phi} & & \downarrow_{\pi_*} \\
\text{Proj}(\mathbb{S}(\mathcal{V})) & & \text{QCoh}(\mathbb{P}(\mathcal{V}))
\end{array}
\]

Proof. Let $Z := \mathbb{P}(\mathcal{V})$ and $\mathcal{A} := \mathbb{S}(\mathcal{V})$. The explicit isomorphism we need to exhibit is

\[\pi_* \left( \oplus_i (-) \otimes T_i \right) \cong \left( \omega(-) \right)_0\]

Now by Lemma 2.12 and the definition of $\omega$, the functor $(\omega(-))_0$ is right adjoint to $p((-) \otimes e_0\mathcal{A})$. Another formal computation using Corollary 2.11 shows that $\pi_* \left( \oplus_i (-) \otimes T_i \right)$ is right adjoint to $T^{-1} \left[ (p \circ \Gamma_*)(\pi^*(-)) \right] = p \left[ (\pi_*(\pi^*(-)) \otimes T_i^{-1})_i \right]$, which by the projection formula, simplifies to $p \left( ((-) \otimes \pi_* \mathcal{O}_Z(i) \otimes T_i^{-1})_i \right)$. The unicity of adjoint functors thus reduces the claim to showing the isomorphism

\[(7) \quad ((-) \otimes \pi_* \mathcal{O}_Z(i) \otimes T_i)_i \cong (-) \otimes e_0\mathcal{A}\]

Since $Z$ is a projective bundle over $X$, we have $\pi_* \left( \mathcal{O}_Z(i) \right) = \pi_* \left( \mathbb{Sym}_X(V)_i \right)$ and since $\mathcal{V}$ has rank 2, by [1] Proposition II.7.11.a, $\pi_* \left( \mathbb{Sym}_X(V)_i \right) = \mathbb{Sym}_X(V)_i$. Now, by the choice of $T_i$, we have $\mathbb{Sym}_X(V)_i = \mathcal{A}_i \otimes T_i$. (7) thus becomes

\[((-) \otimes \pi_* \mathcal{O}_Z(i) \otimes T_i^{-1})_i = ((-) \otimes \mathcal{A}_i \otimes T_i \otimes T_i^{-1})_i = ((-) \otimes \mathcal{A}_i)_i = (-) \otimes e_0\mathcal{A}\]

proving the claim.

We also have 1-periodicity for the truncation functors in this case.

Proposition 2.15. Let $\mathcal{V}$ be a locally free sheaf of rank 2 on $X$ and $\mathbb{S}(\mathcal{V})$ the associated symmetric sheaf $\mathbb{Z}$-algebra. Then there is an equivalence $\beta$ and for each $n$, a line bundle $\mathcal{L}_n$ on $X$ making the diagram

\[
\begin{array}{ccc}
\text{Gr}(\mathbb{S}(\mathcal{V})) & \xrightarrow{(-)_n} & \text{QCoh}(X) \\
\downarrow_{\beta} & & \downarrow_{- \otimes \mathcal{L}_n} \\
\text{Gr}(\mathbb{S}(\mathcal{V})) & \xrightarrow{(-)_{n+1}} & \text{QCoh}(X)
\end{array}
\]

commute.

Proof. By Lemma 2.10 there is a sequence of $X - X$-bimodules $T_i$ such that the following is an equivalence of categories

\[\text{Gr}(\mathbb{S}(\mathcal{V})) \to \text{Gr}(\mathbb{Sym}_X(V)) : (\mathcal{M}_i)_i \mapsto \bigoplus_i \mathcal{M}_i \otimes T_i\]

Let $(-)$ denote the inverse shift functor on $\text{Gr}(\mathbb{Sym}_X(V))$, i.e. $(\mathcal{M}(-))_i = \mathcal{M}_{i-1}$.
and define $\beta$ as being the autoequivalence making the diagram

\[
\begin{array}{ccc}
\text{Gr}(S(\text{Id}_V \text{Id})) & \xrightarrow{\tau} & \text{Sym}_X(V) \\
\downarrow \beta & & \downarrow (-1) \\
\text{Gr}(S(\text{Id}_V \text{Id})) & \xrightarrow{\tau} & \text{Sym}_X(V)
\end{array}
\]

commute. Since we clearly have $(-)^{n+1} \circ (-1) = (-)^n$, we get the required result by choosing the line bundle $L_n := T_n \otimes T_{n+1}$ with $T_n$ as in the proof of Lemma 2.10. □

**Remark 2.16.** the previous result of 1-periodicity clearly implies 2-periodicity after repeated application in the sense that

$$(-)^{n+2} \circ \beta^2 = (\mathcal{L}_{n+1} \otimes \mathcal{L}_n) \otimes (-)^n$$

hence one can wonder whether this coincides with Proposition 2.13. An explicit computation shows that this is not the case in general. Indeed, from the explicit form of $\mathcal{T}$ in Proposition 2.13 and $\beta$ in Proposition 2.15, we obtain

$$\mathcal{L}_n = \left(\bigwedge^2 V\right)^\left\lceil \frac{n}{2} \right\rceil \otimes \left(\bigwedge^2 V\right)^-\left\lfloor \frac{n+1}{2} \right\rfloor$$

and $\mathcal{L}_{n+1} \otimes \mathcal{L}_n = \left(\bigwedge^2 V\right)^{-1}$, which obviously does not coincide with $\omega_{X/S}$ in general.

### 3. Noetherianity of $\text{Gr}(S(\mathcal{E}))$

In this section we prove one of the main results of this paper:

**Theorem 3.1.** Let $X$ and $Y$ be smooth varieties and $\mathcal{E} \in \text{bimod}_S(X - Y)$ be locally free of rank $(4,1)$. Then the category $\text{Gr}(S(\mathcal{E}))$ is locally noetherian.

Throughout this section we will always assume that $X, Y$ and $\mathcal{E} \in \text{bimod}_S(X - Y)$ satisfy the conditions in the above theorem. The next lemma shows that under these assumptions, the bimodule $\mathcal{E}$ can written is a convenient form using a line bundle on $Y$ and a finite map $f$.

**Lemma 3.2.** Let $X$ and $Y$ be smooth varieties over an algebraically closed field $k$ and let $\mathcal{E} \in \text{bimod}_k(X, Y)$ be locally free of rank $(4,1)$. Then there is a line bundle $\mathcal{L}$ on $Y$ and a finite surjective morphism $f : Y \to X$ of degree 4 such that $\mathcal{E} \cong f(\mathcal{L})_{\text{Id}}$.

**Proof.** We consider the following diagram:

\[
\begin{array}{ccc}
\text{Supp}(\mathcal{E}) & \xrightarrow{\iota} & X \times Y \\
g \downarrow & & \downarrow h \\
X & \xrightarrow{\pi_X} & Y \\
\pi_Y \downarrow & & \downarrow \\
& X & \\
\end{array}
\]
By definition 2.1, $g$ and $h$ are finite and in particular they are closed. As $g_*(\mathcal{E})$ and $h_*(\mathcal{E})$ are locally free, this immediately implies that $g$ and $h$ must be surjective. The construction of $f$ relies on the fact that $h$ is in fact an isomorphism. The main step in proving this is to show that $\text{Supp}(\mathcal{E})$ is irreducible.

Let $V_1, \ldots, V_n$ be the irreducible components of $\text{Supp}(\mathcal{E})$ ordered such that $\dim(V_i) \geq \dim(V_{i+1})$ and let $d := \dim(Y)$. The surjectivity of $h$ implies that $\dim(V_1) \geq d$ and the finiteness implies $\dim(V_1) = d$. Let $m$ be the largest integer such that $\dim(V_m) = d$. We first prove $m = 1$ and then conclude the proof by showing $m = n$ as well.

By way of contradiction assume $m > 1$. As $h$ is finite, $\dim(h(V_1)) = \dim(V_1) = d$ and hence $h|_{V_i}$ is surjective for $i = 1, \ldots, m$. Then there is an open subset $U$ of $Y$ such that $h^{-1}(U) = \bigsqcup_{i=1}^m (h^{-1}(U) \cap V_i)$. This $U$ is obtained by removing the images of the lower dimensional irreducible components and the intersections of $V_1, \ldots, V_m$. I.e.

$$ Y \setminus U := \bigcup_{1 \leq i < j \leq m} h(V_i \cap V_j) \cup \bigcup_{i > m} h(V_i) $$

($Y \setminus U$ is closed as a finite union of closed subsets. Moreover all these closed sets have dimension strictly smaller than $d$ hence $U$ is non-empty and $h^{-1}(U) \cap V_i \neq \emptyset$ for $i = 1, \ldots, m$.) As $\mathcal{E}$ is coherent on $\text{Supp}(\mathcal{E})$, it is locally free on $\text{Supp}(\mathcal{E}) \setminus W$ where $W$ is a finite union of closed subsets of dimension strictly smaller than $d$. Moreover $\mathcal{E}$ has constant rank on the components of $\text{Supp}(\mathcal{E}) \setminus W$. Hence by reducing $U$ even more, we may assume that

$$ h^{-1}(U) = \bigsqcup_{i=1}^m (h^{-1}(U) \cap V_i) $$

with for each $i$: $\mathcal{E}|_{h^{-1}(U) \cap V_i}$ locally free of some constant rank $r_i$. Now for each $i$: $h_i : h^{-1}(U) \cap V_i \rightarrow U$ is a finite, surjective morphism of varieties of some degree $\delta_i$. Hence it is flat and by the above $h_{i,*}(\mathcal{E}|_{h^{-1}(U) \cap V_i})$ is a locally free sheaf of rank $r_i \cdot \delta_i > 0$. (iii) implies that

$$ (h_*\mathcal{E})|_U = \bigoplus_{i=1}^m h_{i,*}(\mathcal{E}|_{h^{-1}(U) \cap V_i}) $$

is a locally free sheaf of rank $\sum_{i=1}^m r_i \cdot \delta_i$. By assumption this rank is 1, which is only possible if $m = 1$ and $r_1 = \delta_1 = 1$.

Next we show $n = m = 1$. By way of contradiction assume $n > 1$. Then similar to (iii) we define an open subset $U' \subset Y$ by

$$ Y \setminus U' := \bigcup_{1 \leq i < j \leq n} h(V_i \cap V_j) $$

As for all $i < j : \dim(V_i \cap V_j) < \dim(V_2)$ we must have $h^{-1}(U') \cap V_2 \neq \emptyset$. On the other hand, as $\mathcal{E}|_{V_2}$ is coherent, $\mathcal{E}|_{h^{-1}(U') \cap V_2}$ is nonzero and hence $h_{2,*}(\mathcal{E}|_{h^{-1}(U') \cap V_2})$ is a nonzero torsion sheaf on $Y$. 

However as before we have

\[(h_\ast \mathcal{E})|_{U'} = \bigoplus h_{i_\ast} (\mathcal{E}|_{h^{-1}(U') \cap V_i})\]

leading to a contradiction as the left hand side is a line bundle and thus torsion free, whereas the right hand side has a nonzero torsion summand. Hence we have shown \(m = n = 1\), such that \(\text{Supp}(\mathcal{E})\) is irreducible. In particular \(\text{Supp}(\mathcal{E}) \rightarrow Y\) is a surjective, finite morphism of some degree \(\delta > 0\) and \(h_\ast (\mathcal{O}_{\text{Supp}(\mathcal{E})})\) is a locally free \(\mathcal{O}_Y\)-module of rank \(\delta\). Locally on \(h^{-1}(U)\) this sheaf has rank \(\delta_1 = 1\), hence \(\delta = 1\) and \(h\) is an isomorphism. One now easily checks that choosing \(f := g \circ h^{-1}\) and \(\mathcal{L} := h_\ast (\mathcal{E})\) gives an isomorphism of \(X - Y\)-bimodules \(\mathcal{E} \cong f(\mathcal{L})|_{U'}\).

\[\square\]

Remark 3.3. Throughout the text we shall assume that \(\mathcal{E}\) is given in the above form, i.e. \(\mathcal{E} = f(\mathcal{L})|_{U'}\).

### 3.1. Restricting to an open subset.

The first step in the proof of Theorem 3.1 is showing that there is an appropriate notion of restricting \(S(\mathcal{E})\) to an open subset of \(X\) and that the statement of Theorem 3.1 can be reduced to an open cover of \(X\) in this sense.

Throughout this section we will use the following notation:

If \(U \subset X\) is an open subset, then we define \(U^n \subset X^n\) as follows:

\[U^n = \begin{cases} U & \text{if } n \text{ is even} \\ f^{-1}(U) & \text{if } n \text{ is odd} \end{cases}\]

By construction \(U^n\) is an open subset of \(X_n\) and it is an affine open subset whenever \(U\) is because \(f\) is a finite morphism. For a bimodule \(F \in \text{bimod}(X_n - X_{n+1})\), a sheaf \(\mathbb{Z}\)-algebra \(A\) and an \(A\)-module \(M\) we will use the notation \(|_U\) to denote the restriction to the corresponding open subset. I.e.

\[
\begin{align*}
F|_U &:= F|_{U^n \times U^{n+1}} \\
(A|_U)|_{m,n} &:= (A_{m,n})|_U = (A_{m,n})|_{U^m \times U^n} \\
(M|_U)|_n &:= (M_n)|_U^n
\end{align*}
\]

To ensure that the restrictions of \(A\) to an open subset in turn has the structure of a sheaf \(\mathbb{Z}\)-algebra, we need the following technical condition:

**Lemma 3.4.** Let \(A\) be a sheaf \(\mathbb{Z}\)-algebra and \(U \subset X\) an open subset such that for each \(m,n\): \(\text{Supp}((A_{m,n})|_{U^m \times X_n}) \subset U^m \times U^n\) and \(\text{Supp}((A_{m,n})|_{X_n \times U^n}) \subset U^m \times U^n\). Then

i) \(A|_U\) has an induced algebra structure.

ii) Restriction of modules to \(U\) defines a functor \(|_U : \text{Gr}(A) \rightarrow \text{Gr}(A|_U)\)

**Proof.**

i) We must show that for all \(l,m,n \in \mathbb{Z}\) there are multiplication morphisms \(A_{l,m}|_{U^l \times A_{m,n}}|_{U^m} \rightarrow A_{l,n}|_U\) induced by the morphisms \(A_{l,m} \otimes A_{m,n} \rightarrow A_{l,n}\). The latter induces a morphism of \(U^l - U^n\)-bimodules:

\[(A_{l,m} \otimes A_{m,n})|_{U^l} \rightarrow A_{l,n}|_U\]
Now the claim follows from the following chain of isomorphisms:
\[
(A_{i,m} \otimes A_{m,n})_{|U} = (\pi_{X_1,X_n} (A_{i,m} \otimes X_1 \times X_n \pi_{X_1,X_n} (A_{m,n})))_{|U} \\
= \pi_{U_1,U^n} \left( (\pi_{X_1,X_n} (A_{i,m} \otimes X_1 \times X_n \pi_{X_1,X_n} (A_{m,n})))_{|U} \right) \\
= \pi_{U_1,U^n} \left( \pi_{X_1,X_n} (A_{i,m}) \otimes X_1 \times X_n \pi_{X_1,X_n} (A_{m,n}))_{|U} \right) \\
= \pi_{U_1,U^n} \left( \pi_{X_1,X_n} (A_{i,m} \otimes X_1 \times X_n \pi_{X_1,X_n} (A_{m,n})))_{|U} \right) \\
= \pi_{U_1,U^n} \left( \pi_{U_1,U^n} (A_{i,m} \otimes X_1 \times X_n \pi_{X_1,X_n} (A_{m,n})))_{|U} \right) \\
= \pi_{U_1,U^n} \left( \pi_{U_1,U^n} (A_{i,m} \otimes X_1 \times X_n \pi_{X_1,X_n} (A_{m,n})))_{|U} \right) \\
= A_{i,m}|_U \otimes A_{m,n}|_U
\]

Where $\pi_{U_1,U^n}$ and $\pi_{U_1,U^n}$ are the projections $\pi_{U_1,U^n} : U_1 \times X_1 \times X_n \to U_1 \times X_1$ and $\pi_{U_1,U^n} : U_2 \times X_2 \times X_n \to U_2 \times X_2$, with similar definitions for $\pi_{X_1,U^n}$ and $\pi_{X_1,U^n}$.

The first equality is the definition of tensor product of bimodules
\[
\text{bimod}(X_1 - X_2) \times \text{bimod}(X_2 - X_2) \to \text{bimod}(X_1 - X_1)
\]
The second equality follows from the commutation of pushforward and restriction of sheaves. The third equality follows from the commutation of tensor product of sheaves and restriction. The fourth equality follows from the commutation of pullback and restriction of sheaves. The fifth equality follows the assumption of the lemma. The last equality is the definition of multiplication
\[
\text{bimod}(U_1 - U_2) \times \text{bimod}(U_2 - U_2) \to \text{bimod}(U_1 - U_1)
\]

ii) This essentially reduces to showing $(\mathcal{M}_i \otimes \mathcal{A}_{i,j})_{|U} = (\mathcal{M}_{|U})_i \otimes (\mathcal{A}_{|U})_{i,j}$ which is completely similar to i).

As an immediate corollary we have

**Corollary 3.5.** For any $U \subset X$,

i) $\mathcal{S}(\mathcal{E})_{|U}$ has an algebra structure induced by $\mathcal{S}(\mathcal{E})$

ii) There is a functor $|U : \text{Gr}(\mathcal{S}(\mathcal{E})) \to \text{Gr}(\mathcal{S}(\mathcal{E})_{|U})$

iii) There is an isomorphism of symmetric sheaf $\mathbb{Z}$-algebras: $\mathcal{S}(\mathcal{E})_{|U} \cong \mathcal{S}(\mathcal{E}_{|U})$

**Proof.** i+ii) As $\mathcal{E}$ is given by $f(\mathcal{L})_{|id}$, the conditions in Lemma 3.4 are obviously satisfied if $\mathcal{A} = \mathcal{S}(\mathcal{E})$. For iii) We first show that for all $n \in \mathbb{N}$ there is a natural isomorphism

\[
(\mathcal{E}^*|U)^n = (\mathcal{E}|U)^*n
\]

By Remark 2.3 we see by induction that for each $i \geq 0$ there is a line bundle $\mathcal{L}_i$ such that

\[
\mathcal{E}^{*2i} = f(\mathcal{L}_i)_{|id} \\
\mathcal{E}^{*2i+1} = f(\mathcal{L}_i)_{|f}
\]

where $\mathcal{L}_0 = \mathcal{L}$. The explicit form of the dual (11) shows that it suffices to exhibit isomorphisms

\[
f(\text{Hom}(\mathcal{L}_i, f^! \mathcal{O}_X))_{|U}|_{id} \cong f|_U \left( \text{Hom}_{f^{-1}(U)}(\mathcal{L}_i)_{|f^{-1}(U), (f|_U)^! \mathcal{O}_U} \right)_{id_U}
\]
Note that by construction, in degree 2, we have $M_{\text{left}} = \text{component act on the left}$ and only the component act on the left. This implies that $\eta_{\text{left}}$ induces an isomorphism

$$i_n(\mathcal{O}_{U^n})|_{U_{ij}} \cong i_n(\mathcal{O}_{X^n})|_{U_{ij}}$$

and we can extend $\eta_{\text{left}}$ to an isomorphism

$$\mathbb{S}(\mathcal{E})|_{U_{ij}} \cong \mathbb{S}(\mathcal{E}|_{U_{ij}})$$

**Lemma 3.6.** Let $\bigcup U_i$ be a finite open cover for $X$. Moreover assume that $A$ is a sheaf $\mathbb{Z}$-algebra such that the conditions in Lemma 3.4 are satisfied for all $U_i$, then

$$\forall l : M|_{U_i} \in \text{Gr}(A|_{U_i}) \text{ is noetherian } \Rightarrow M \in \text{Gr}(A) \text{ is noetherian}$$

**Proof.** Suppose we are given an ascending chain of subobjects of $M^n \subset M$ in $\text{Gr}(A)$ such that the restriction of this chain to any of the $U_i$ stabilizes. As there are only finitely many $U_i$, there is an $N \in \mathbb{N}$ such that for all $n \geq N$ and for all $l$: $(M^n)|_{U_i} = (M^{n+1})|_{U_i}$. By a degree-wise application of the gluing axiom, the graded modules $M^n$ and $M^{n+1}$ must coincide. \hfill $\square$

### 3.2. covering by relative Frobenius pairs.

**Lemma 3.6** shows that proving that a given set of generators is in fact a set of noetherian generators can be done locally. In this subsection we construct an open cover $X = \bigcup U_i$ for which the categories $\text{Gr}(\mathcal{E}|_{U_i})$ can explicitly described (see 3.13). For this cover, the sections satisfy a relative version of the Frobenius property as introduced in the paper [3], whose definition and properties we recall below:

**Definition 3.7.** We say that $S/R$ is relative Frobenius of rank $n$ if:

- $S$ is a free $R$-module of rank $n$.
- $\text{Hom}_R(S, R)$ is isomorphic to $S$ as $S$-module.

**Remark 3.8.** It is clear that if $R$ is a field, then $S/R$ being relative Frobenius coincides with $S$ being a finite dimensional Frobenius algebra in the classical sense.

We shall need the following notation: for a relatively Frobenius pair, let $M := R \otimes S$. This $R - S$-bimodule can be considered a $R \oplus S$ bimodule by letting only the $R$-component act on the left and only the $S$-component on the right. Similarly, we let $N := S \otimes R$ and consider it an $R \oplus S$-bimodule by letting only the component $S$ act on the left and only the component $R$ act on the right. We now define

$$T(R, S) := T_{R \otimes S}(M \oplus N)$$

Note that by construction, in degree 2, we have $M \otimes_{R \otimes S} M = N \otimes_{R \otimes S} N = 0$, hence

$$T(R, S)_2 = (M_{R \otimes S} N) \oplus (N \otimes_{R \otimes S} M) = (R \otimes S S_R) \oplus (S \otimes S S_R)$$

The algebra we will be concerned in will be a quotient of $T(R, S)$ as follows: let $\lambda$ be a generator of $\text{Hom}_R(S, R)$ as an $S$-module. The $R$-bilinear form $(a, b) := \lambda(ab)$ is clearly nondegenerate and hence we can find dual $R$-bases $(e_i), (f_j)$ satisfying

$$\lambda(e_i f_j) = \delta_{ij}$$

However as restriction to open affine subsets commutes with $f_*, \mathcal{H}om_Y$ and $f^!$, this isomorphism is immediate.
Definition 3.9. For a relative Frobenius pair $S/R$, the generalized preprojective algebra $\Pi_R(S)$ is given by

$$T(R, S)/\text{(rels)}$$

where the relations are in degree 2 given by

$$1 \otimes 1 \in R \otimes_S S_R$$

$$\sum_i e_i \otimes f_i \in s \otimes_R S_S$$

Remark 3.10. If $S$ is the ring $R^{\oplus n}$. Then $\Pi_R(S)$ is isomorphic to the preprojective algebra over $R$ associated to the quiver with one central vertex and $n$ outgoing arrows. (See [3 Lemma 1.5])

We shall use the following result from [3]:

Theorem 3.11. Let $S/R$ be relative Frobenius of rank 4 and assume $R$ is noetherian, then $\Pi_R(S)$ is noetherian as well.

Throughout, we shall make use of the following lemma, well-known to experts:

Lemma 3.12. Let $\mathcal{L}$ be a line bundle on $Y$ and $p \in X$. Then there is an open subset $U \subset X$ containing $p$, such that $\mathcal{L}|_{f^{-1}(U)} \cong \mathcal{O}_{f^{-1}(U)}$.

Proof. We can reduce to the case where $X = \text{Spec}(R)$, $Y = \text{Spec}(S)$ are affine schemes where $S$ is finitely generated over $R$ and $\mathcal{L} = \mathcal{L}'$ for some invertible $S$-module $L$. Let $p$ be the prime ideal in $\text{Spec}(R)$ corresponding to $f(p) \in X$, then $S_p := S \otimes_R p$ is a semilocal ring, hence every finitely generated projective of constant rank is free and in particular the Picard group is trivial. Consequently, there exists an $l \in L$ such that

$$S_p \xrightarrow{l} L_p$$

is an isomorphism.

Now consider the morphism $S \xrightarrow{l} L$ with kernel $K$ and cokernel $C$. Then there is an exact sequence

$$(12) \quad 0 \rightarrow K \rightarrow S \xrightarrow{l} L \rightarrow C \rightarrow 0$$

$K$ is a finitely generated $R$-submodule of $S$ by the noetherianity of $R$. $L$ is finitely generated over $R$, being an invertible $S$-module. It follows that $C$ is finitely generated over $R$ as a quotient of $L$.

Now let $\alpha_1, \ldots, \alpha_n$ be a set of generators for $K$, then as $K \otimes_R p = 0$ there exist elements $x_1, \ldots, x_n \in R \setminus p$ such that $\alpha_1 x_1 = \ldots = \alpha_n x_n = 0$. Set $x := x_1 \cdots x_n \in R \setminus p$, then $x \neq 0$ for all $x \in K$. Similarly there is a $x' \in R \setminus p$ such that $\beta \cdot x' = 0$ for all $\beta \in C$. Now define $z = x \cdot x'$, then $K \otimes_R z = C \otimes_R z = 0$ implying that $\cdot l$ defines an isomorphism

$$S \otimes_R z \xrightarrow{\cong} L \otimes_R z$$

$U = \text{Spec}(R_z)$ then is the desired open subset. \qed

We can now prove the main result of this subsection

Theorem 3.13. Write $\mathcal{E} = f(\mathcal{L})_{|X}$ as in lemma 3.12. There is a finite cover $X = \bigcup_i U_i$ of affine open subsets $U_i = \text{Spec}(R_i)$ such that:

i) $\mathcal{L}|_{f^{-1}(U_i)}$ is a trivial $\mathcal{O}_{f^{-1}(U_i)}$-module

ii) $\omega_Y|_{f^{-1}(U_i)}$ is a trivial $\mathcal{O}_{f^{-1}(U_i)}$-module
iii) \( \omega_X|_{U_i} \) is a trivial \( \mathcal{O}_{U_i} \)-module

iv) \( f^{-1}(U_i) = \text{Spec}(S_i) \) where \( S_i/R_i \) is relative Frobenius of rank 4.

Proof. We first note the following two facts:

- Let \( \text{Spec}(R) \) be an affine open subset on which i), ii), iii) or iv) holds. Then the same statement holds for any standard open \( \text{Spec}(R_f) \subset \text{Spec}(R) \). This is obvious for i), ii) and iii). For iv) it follows from [3, Lemma 3.1].
- Let \( \text{Spec}(R) \) and \( \text{Spec}(R') \) be affine open subsets of \( X \), then their intersection is covered by affine open subsets of the form \( \text{Spec}(R_f) = \text{Spec}(R'_f) \).

By these two facts it suffices to find affine open covers for i), ii), iii) and iv) separately. For i) and ii) such a cover exists by Lemma 3.12 and the fact that \( \omega_Y \) is a line bundle on the smooth variety \( Y \). The existence for a cover satisfying iii) is immediate from the fact that \( \omega_X \) is a line bundle. Hence the proof reduces to finding a cover satisfying iv).

As \( f : Y \to X \) is a finite, by Lemma 3.12 \( f^!\omega_X \) is completely determined by \( f_* \) and we have an isomorphism of \( f_* \mathcal{O}_Y \)-modules

(13) \[ f_* (f^!\omega_X) := \text{Hom}_X(f_*\mathcal{O}_Y, \omega_X) \cong f_* \omega_Y \]

As moreover \( f \) is also surjective and flat, there is a cover \( X = \bigcup U_i \) with \( U_i = \text{Spec}(R_i) \) and \( f^{-1}(U_i) = \text{Spec}(S_i) \) where \( S_i \) is a free \( R_i \)-module of rank 4 for each \( i \). By the previous arguments we can assume that ii) and iii) are also satisfied on this cover. In this case, replacing \( f \) by its restriction \( f^{-1}(U_i) \to U_i \), (13) reads

\[ f_* (f^!\mathcal{O}_{U_i}) := \text{Hom}_{U_i}(f^!\mathcal{O}_{f^{-1}(U_i)}, \mathcal{O}_{U_i}) \cong f_* \mathcal{O}_{f^{-1}(U_i)} \]

and taking sections yields the required isomorphism of \( S_i \)-modules:

\[ \text{Hom}_{R_i}(S_i/R_i) \cong S_i \]

3.3. A local description of \( S(\mathcal{E}) \). In this section, we shall show that for affine schemes satisfying the conditions of 3.3 it is possible to describe \( S(\mathcal{E}) \) using generalized preprojective algebras from 3.9. We shall assume that \( X \) and \( Y \) are smooth affine varieties over some base field \( k \), say \( X = \text{Spec}(R) \) and \( Y = \text{Spec}(S) \) such that \( S/R \) is relative Frobenius of rank 4 and \( \omega_X \cong \mathcal{O}_X \), \( \omega_Y \cong \mathcal{L} \cong \mathcal{O}_Y \). We introduce some auxiliary notations: for the be the symmetric sheaf-\( \mathbb{Z} \)-algebra over \( X \) and \( Y \) in standard form \( S(\mathcal{E}) \), there is a \( \mathbb{Z} \)-algebra over \( k \), \( \Gamma(A) \) defined by

\[ \Gamma(A)_{m,n} := \Gamma(X_m \times X_n, \mathcal{A}_{m,n}) \]

since each component \( \Gamma(A)_{m,n} \) is an \( R \otimes S \) or \( S \otimes R \) bimodule depending on the indices, \( \Gamma(A) \) is in fact a \( \mathbb{Z} \)-algebra over the ring \( R \otimes S \) as in the discussion following remark 3.8. The classical equivalence between quasi-coherent modules and global sections can easily be adapted to our setting to obtain an equivalence:

\[ \Gamma : \text{Gr}(A) \overset{\cong}{\longrightarrow} \text{Gr}( \Gamma(A)) : \{ \mathcal{M}_n \}_{n \in \mathbb{Z}} \mapsto \{ \Gamma(X_n, \mathcal{M}_n) \}_{n \in \mathbb{Z}} \]

The following is an immediate consequence of the assumptions of this section:

Lemma 3.14. The \( \mathbb{Z} \)-algebra \( \Gamma(S(\mathcal{E})) \) is \( 2 \)-periodic in the sense that

\[ \Gamma(S(\mathcal{E}))_{m,n} \cong \Gamma(S(\mathcal{E}))_{m+2,n+2} \]

Proof. By 2.13 there are isomorphisms \( S(\mathcal{E})_{i+2,j+2} \cong \omega_{i+1}^{-1} \otimes S(\mathcal{E}) \otimes \omega_j \). By the assumptions in the beginning of this sections, both canonical bundles are trivial, implying that \( S(\mathcal{E})_{m,n} \cong S(\mathcal{E})_{m+2,n+2} \). The result follows after applying \( \Gamma(-) \).
Using the methods in appendix A (Lemma A.3), the 2-periodic \(\mathbb{Z}\)-algebra \(\Gamma(S(\mathcal{E}))\) gives rise to a graded algebra \(\Gamma(S(\mathcal{E}))\).

**Lemma 3.15.** Let \(X = \text{Spec}(R)\) and \(Y = \text{Spec}(S)\) be affine schemes such that \(S/R\) is relative Frobenius of rank 4. Let \(f : Y \to X\) be the induced morphism and \(\mathcal{E} = f(O_Y)_{Id}\). Then \(\Gamma(S(\mathcal{E})) \cong \Pi_R(S)\).

**Proof.** Consider the quotient map \(\mathcal{T}(\mathcal{E}) \to S(\mathcal{E})\)

Taking global sections in each component \(\Gamma(X_m \times X_n, (-)_{m,n})\) yields a surjection \(\Gamma(\mathcal{T}(\mathcal{E})) \to \Gamma(S(\mathcal{E}))\).

because \(X_m \times X_n\) is affine.

Since the functor \((-)\) preserves surjectivity (see Proposition A.6), we obtain a map

\[\pi : \Gamma(\mathcal{T}(\mathcal{E})) \to \Gamma(S(\mathcal{E})).\]

We first show that there is a canonical isomorphism of \(R \oplus S\)-modules

\[\Gamma(\mathcal{T}(\mathcal{E})) \cong T(R, S)\]

For this (as \(\Gamma(S(\mathcal{E}))\) is clearly generated in degrees 0 and 1) it suffices to show the following three facts

- \(\Gamma(\mathcal{T}(\mathcal{E}))_0 \cong T(R, S)_0 = R \oplus S\) as rings
- \(\Gamma(\mathcal{T}(\mathcal{E}))_1 \cong T(R, S)_1 \cong R S_S \oplus S R_S\) as \(R \oplus S\) modules
- the multiplication map yields isomorphisms

\[\Gamma(\mathcal{T}(\mathcal{E}))_1 \otimes \Gamma(\mathcal{T}(\mathcal{E}))_n \xrightarrow{\cong} \Gamma(\mathcal{T}(\mathcal{E}))_{n+1}\]

For the first item, we compute:

\[\Gamma(\mathcal{T}(\mathcal{E}))_0 = \begin{pmatrix} \Gamma(\mathcal{T}(\mathcal{E}))(0,0) & 0 \\ 0 & \Gamma(\mathcal{T}(\mathcal{E}))(1,1) \end{pmatrix} = \begin{pmatrix} \Gamma(X \times X, Id(O_X)_{Id}) & 0 \\ 0 & \Gamma(Y \times Y, Id(O_Y)_{Id}) \end{pmatrix}\]

next, we have

\[\Gamma(X \times X, Id(O_X)_{Id}) = \text{Hom}(O_{X \times X}, \Delta_*(O_X)) = \text{Hom}(\Delta^*(O_{X \times X}), O_X) = \text{Hom}(O_X, O_X) \cong R\]

And similarly \(\Gamma(Y \times Y, Id(O_Y)_{Id}) \cong S\) such that

\[\Gamma(\mathcal{T}(\mathcal{E}))_0 \cong \begin{pmatrix} R & 0 \\ 0 & S \end{pmatrix} \cong R \oplus S\]
In a completely similar fashion, we check the second condition:

\[
\begin{pmatrix}
0 & \Gamma(T(E))_{0,1} \\
\Gamma(T(E))_{1,2} & 0 \\
\end{pmatrix} = 
\begin{pmatrix}
0 & \Gamma(Y \times X, E^*) \\
\Gamma(Y \times X, E) & 0 \\
\end{pmatrix} = 
\begin{pmatrix}
0 & \Gamma(Y \times X, f(O_Y)Id) \\
\Gamma(Y \times X, Id(O_Y)) & 0 \\
\end{pmatrix} 
\]

\[
\cong 
\begin{pmatrix}
0 & R S \oplus s R S \ R S \\
s S R & 0 \\
\end{pmatrix} \cong R S \oplus s R S 
\]

To check the final condition, we have the isomorphisms

\[
T(E)_{i,i+1} \otimes T(E)_{i+1,i+n+1} \longrightarrow T(E)_{i,i+n+1} 
\]

We now apply the \(\Gamma(X_i \times X_{i+n+1}, \cdot)\). Note that as all schemes are affine, the tensor product and \(\Gamma(\cdot)\) commute, resulting in an isomorphism

\[
\Gamma(T(E))_{i,i+1} \otimes \Gamma(T(E))_{i+1,i+n+1} \longrightarrow \Gamma(T(E))_{i,i+n+1} 
\]

and finally, application of the functor \((\cdot)\) yields the required isomorphism

\[
\Gamma(T(E))_{i,i+n} \otimes \Gamma(T(E))_{n} \cong \Gamma(T(E))_{n+1} 
\]

proving the isomorphism [14]. Finally, we prove that the relations defining \(\Pi_R S\) coincide with the kernel of \(\pi\), i.e. there is a commutative diagram:

\[
\begin{array}{ccc}
\Gamma(T(E)) & \xrightarrow{\pi} & \Gamma(S(E)) \\
\cong & & \cong \\
T(R,S) & \xrightarrow{\varphi} & \Pi_R(S) \\
\end{array} 
\]

By the isomorphisms in the previous step and the construction of \(\Gamma\), there are isomorphisms:

\[
\zeta_0 : \text{Hom}_{X \times X}(id(O_X)Id, E \otimes E^*) \xrightarrow{\cong} \text{Hom}_R(R, R S \otimes S R) \\
\zeta_1 : \text{Hom}_{Y \times Y}(id(O_Y)Id, E^* \otimes E) \xrightarrow{\cong} \text{Hom}_S(S, S R \otimes R S) 
\]

\(S(E)\) is defined as a quotient of \(T(E)\) by the relations given by the unit morphisms \(i_0 \in \text{Hom}_{X \times X}(id(O_X)Id, E \otimes E^*), i_1 \in \text{Hom}_{Y \times Y}(id(O_Y)Id, E^* \otimes E)\) as in [2]. Similarly \(\Pi_R(S)\) is defined as a quotient of \(T_R(S)\) by elements \(\eta_0 \in \text{Hom}_R(R, R S \otimes S R), \eta_1 \in \text{Hom}_S(S, S R \otimes R S)\). Hence we must prove \(\zeta_0(i_0) = \eta_0\) and \(\zeta_1(i_1) = \eta_1\). First note that there is a commutative diagram of isomorphisms

\[
\begin{array}{ccc}
\text{Hom}_{X \times Y}(E, E) & \longrightarrow & \text{Hom}_{X \times X}(id(O_X)Id, E \otimes E^*) \\
\downarrow & & \downarrow \zeta_0 \\
\text{Hom}_{R \otimes S}(R S \otimes R S) & \xrightarrow{\varphi_0} & \text{Hom}_R(R, R S \otimes S R) \\
\end{array} 
\]
where \( \phi_0 \) is given by the adjunction \(- \otimes_R S_S \dashv - \otimes_S S_R = (-)_R\). Hence 
\( \xi_0(i_0) = \phi_0(I\text{Id}_{S_S}) : 1_R \mapsto 1_S \otimes 1_S \), which coincides with \( \eta_0 \). Similarly the existence of the dual bases \((e_i)_i, (f_j)_j\) implies there is an adjunction 
\(- \otimes_S S_R = (-)_R \dashv - \otimes_R S_S \) given by 
\[ \varphi_1 : \text{Hom}_R(M \otimes S_S R, N) \to \text{Hom}_S(M, N \otimes_R S_S) : \psi \mapsto \left( \psi' : m \mapsto \sum_i \psi(me_i) \otimes f_i \right) \]
Where we used Lemma 3.16 to see that the morphisms in the image of \( \varphi_1 \) indeed have an \( S \)-module structure. A commutative diagram as above shows that 
\( \xi_1(i_1) = \varphi_1(I\text{Id}_{S_S}) : 1_S \mapsto \sum_i e_i \otimes f_i \) which coincides with \( \eta_1 \). □

**Lemma 3.16.** \( \sum_i e_i \otimes f_i \) is central in the \( S \)-bimodule \( S \otimes_R S \). I.e. for all \( a \in S \) we have 
\[ \sum_i ae_i \otimes f_i = \sum_i e_i \otimes fa \]

**Proof.** It is sufficient to prove that for all \( j, k \) we have 
\[ \sum_i \lambda(ae_if_j)\lambda(f_i e_k) = \sum_i \lambda(e_i f_j)\lambda(f_i ae_k) \]
which is clear since both sides are equal to \( \lambda(ae_k f_j) \). □

3.4. **Proof of Theorem 3.1.** As \( X \) and \( Y \) are noetherian we know that \( \text{QCoh}(X) \) and \( \text{QCoh}(Y) \) are locally noetherian categories and hence there exist collections of noetherian generating objects for these categories, say \( \mathcal{N}_X := \{N^X_i\}_{i \in I} \) and \( \mathcal{N}_Y := \{N^Y_j\}_{j \in J} \). For each \( n \in \mathbb{Z} \) we define \( N^n \) in \( \text{QCoh}(X_n) \) as:
\[ N^n = \begin{cases} N^X_i & \text{if } n \text{ is even} \\ N^Y_j & \text{if } n \text{ is odd} \end{cases} \]
We shall prove that the collection 
\[ \{N \otimes e_n S(E) \mid n \in \mathbb{Z}, N \in N^n\} \]
forms a set of noetherian generators for \( \text{Gr}(S(E)) \). Not that the collection is easily seen to generate as for each \( M \in \text{Gr}(A) \) there is a surjective morphism 
\[ \oplus_{n \in \mathbb{Z}} M_n \otimes e_n A \to M \]
and for each \( n \in \mathbb{Z} \) there is a surjective morphism 
\[ \oplus_{\alpha} (N^{n}_{\alpha})^{m_{\alpha}} \to M_n \]
where \( N^{n}_{\alpha} \in N^n \). Hence we only need to show that the elements of (15) are noetherian objects in \( \text{Gr}(S(E)) \). By Lemma 3.6 and Corollary 3.5 this can be checked locally for any open cover \( X = \bigcup_i U_i \). By Theorem 3.13 we may hence assume that \( X = \text{Spec}(R) \) and \( Y = \text{Spec}(S) \) are affine schemes such that 
\[ \begin{align*} 
\text{i)} & \ L \cong \mathcal{O}_Y \cong \omega_Y \\
\text{ii)} & \ \omega_X \cong \mathcal{O}_X \\
\text{iii)} & \ S/R \text{ is relative Frobenius of rank } 4.
\end{align*} \]
With these assumptions there are functors

$$\begin{align*}
\text{Gr}(S(\mathcal{E})) & \cong \text{Gr}(\Gamma(S(\mathcal{E}))) \\
\text{Prop. A.6} & \\
\text{Lemma 3.15}
\end{align*}$$

Let $$F : \text{Gr}(S(\mathcal{E})) \to \text{Gr}(\Pi_R(S))$$ be the composition. Then the above diagram shows that $$F$$ is an exact embedding of categories. Hence $$N \otimes e_n S(\mathcal{E})$$ is a noetherian object in $$\text{Gr}(S(\mathcal{E}))$$ if $$F(N \otimes e_n S(\mathcal{E}))$$ is a noetherian object in $$\text{Gr}(\Pi_R(S))$$. On the other hand, as $$N$$ is noetherian in $$\text{QCoh}(X_n)$$ there is an $$m \in \mathbb{N}$$ and an surjection $$\mathcal{O}_{X_n} \to N$$ giving rise to a surjection

$$F(\mathcal{O}_{X_n} \otimes e_n S(\mathcal{E})) \to F(N \otimes e_n S(\mathcal{E}))$$

Hence it suffices to show that $$F(\mathcal{O}_{X_n} \otimes e_n S(\mathcal{E}))$$ is a Noetherian object in $$\Pi_R(S)$$. This is however obvious as

$$F(\mathcal{O}_{X_n} \otimes e_n S(\mathcal{E})) = \begin{cases} R \cdot \Pi_R(S)(-n) & \text{if } n \text{ is even} \\ S \cdot \Pi_R(S)(-n) & \text{if } n \text{ is odd} \end{cases}$$

As both $$R \cdot \Pi_R(S)$$ and $$S \cdot \Pi_R(S)$$ are direct summands of $$\Pi_R(S)$$, which is a noetherian ring by Theorem 3.11, we have proven the theorem. \(\square\)

Since the proof of Theorem 3.1 exhibits an explicit set of generators, we can also prove the following:

**Theorem 3.17.** The category $$\text{Gr}(S(\mathcal{E}))$$ is Grothendieck.

**Proof.** Let $$(\mathcal{M}_i, f_{ij})$$ be a direct system of graded $$S(\mathcal{E})$$-modules. In each degree $$d$$, we obtain a direct system of quasicoherent $$X_d$$-modules $$(\mathcal{M}_i^d, f^d_{ij})$$. Since $$\text{QCoh}(X_n)$$ is Grothendieck, we can form the direct limit in each degree to obtain a sequence of $$X_n$$-modules $$\mathcal{L}_n = \lim_i (\mathcal{M}_i^n, f^n_{ij})$$. If we fix a couple $$(n, m)$$, the universality of the direct limit naturally defines a map

$$S(\mathcal{E})_{n,m} \otimes \mathcal{L}_n = S(\mathcal{E})_{n,m} \otimes \lim_i (X_i^n, f^n_{ij}) \to \lim_i (X_i^m, f^m_{ij}) = \mathcal{L}_m$$

showing that $$\mathcal{L}$$ is in fact a graded $$S(\mathcal{E})$$-module. The fact that $$\mathcal{L}$$ is a direct limit and that the formation of $$\mathcal{L}$$ is exact is an easy consequence of the construction. Finally a collection of generators for $$\text{Gr}(S(\mathcal{E}))$$ is given by [15]. \(\square\)

### 4. Homological Properties of Sheaf $$\mathbb{Z}$$-Algebras

This section is dedicated to adapting the results in [9] and [5] to obtain a formula to compute certain Ext-groups. Throughout $$X$$ and $$Y$$ will denote smooth curves over an algebraically closed field $$k$$ of characteristic zero. Let $$\mathcal{A}$$ be a sheaf $$\mathbb{Z}$$ algebra over $$(X_i \to \text{Spec}(k))$$. To keep the geometric intuition (as in 2.14) we denote the truncation functors $$(\omega(-))_m : \text{Proj}(\mathcal{A}) \to \text{QCoh}(X_m)$$ by $$\Pi_{m*}$$. The left adjoints,
which are given explicitly by \( p(\cdot) \otimes e_m A \), are in turn denoted by \( \Pi_n^m \). We shall use the notations \( X_n \) and \( Q_n \) as in (3).

If \( E \in \text{bimod}(X - X) \) is locally free of rank \((2,2)\) and \( A = S(E) \), \( [5] \) computes the Euler characteristics \( \langle \Pi_m^m F, \Pi_n^m G \rangle \) for two locally free sheaves \( F \) and \( G \) on \( X \). In this section, we develop the machinery to perform an analogous calculation in the case of a bimodule \( E \in \text{bimod}_S(X - Y) \) of rank \((4,1)\). As mentioned in the introduction and motivated by Proposition 2.13 our main goal is to understand Euler characteristics of the form \( \langle \Pi_n^m F, \Pi_n^m G \rangle \) with \( n - m = 1, 0 \) or \(-1\). The goal of this section will be to prove the following, slightly stronger theorem:

**Theorem 4.1.** Let \( E \in \text{bimod}(X, Y) \) be locally free of rank \((4,1)\). Let \( F \) and \( G \) be locally free sheaves on \( X_n \) respectively \( m, n \in \mathbb{Z} \) such that \( m \geq n - 1 \). Then

\[
\text{Ext}^i_{\text{Proj}(A)}(\Pi_m^m F, \Pi_n^m G) \cong \text{Ext}^i_{X_n}(F, G \otimes S(E)_{n,m})
\]

for all \( i \geq 0 \).

From this we immediately have the following corollaries:

**Corollary 4.2.** With the above assumptions, the Euler characteristics of the pulled back sheaves on \( \text{Proj}(S(E)) \) are given by

\[
\langle \Pi_m^m F, \Pi_n^m G \rangle = \langle F, G \otimes S(E)_{n,m} \rangle
\]

**Corollary 4.3.** Let \( \{F_1, \ldots, F_n\} \) and \( \{G_1, \ldots, G_b\} \) be exceptional sequences of locally free sheaves on \( X_n \) and \( X_{n+1} \) respectively. Then \( \Pi_n^m F_1, \ldots, \Pi_n^m F_b, \Pi_n^m G_1, \ldots, \Pi_n^m G_b \) is an exceptional sequence on \( \text{Proj}(A) \).

As an immediate application of Corollary 4.2 we can construct a noncommutative Del-Pezzo surface with the desired Gram matrix (see introduction):

**Corollary 4.4.** Let \( E \) be the following \( \mathbb{P}^1 \)-bimodule \( f(O_{\mathbb{P}^1})_{id} \) and \( S(E) \) be the associated symmetric sheaf \( \mathbb{Z} \)-algebra. Then

\[
\Pi_1^0(O_{\mathbb{P}^1}), \Pi_1^1(O_{\mathbb{P}^1}(1)), \Pi_0^0(O_{\mathbb{P}^1}), \Pi_0^1(O_{\mathbb{P}^1}(1))
\]

is an exceptional sequence of graded \( S(E) \)-modules for which the Gram matrix of the Euler form is given by

\[
\begin{pmatrix}
1 & 2 & 1 & 5 \\
0 & 1 & 0 & 4 \\
0 & 0 & 1 & 2 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

Throughout this section \( E \) will be a locally free \( X - Y \)-bimodule of rank \((4,1)\) and we let \( A := S(E) \) denote the associated symmetric sheaf \( \mathbb{Z} \)-algebra in standard form. The proof of Theorem 4.1 is based on a chain of lemmas and the following technical result of which the proof will be the subject of appendix B:

**Theorem 4.5.** There is an exact sequence (in \( \text{bimod}(O_{X_n} - A) \), see [9] Section 3.2.) for the definition of this category)

\[
0 \to Q_m \otimes e_m A \to E^{*m} \otimes e_{m+1} A \to e_m A \to \text{id}(O_{X_m})_{id} \to 0
\]

**Proof.** By the nature of the relations this sequence is known to be right exact. The proof of the left exactness is based on point modules and is given in Appendix B. \( \square \)
As an immediate corollary of this theorem and its proof we find:

**Corollary 4.6.** for each \( i, j \in \mathbb{Z} \), the bimodule \( A_{ij} \) is locally free on the left and
on the right. Its rank is equal to

\[
\text{rk}(A)_{m,n} := \begin{cases} 
(n - m + 1, n - m + 1) & m \equiv n \text{ mod } 2 \\
\frac{n - m + 1}{2}, 2(n - m + 1) & m \text{ odd, } n \text{ even} \\
2(n - m + 1), \frac{n - m + 1}{2} & m \text{ even, } n \text{ odd}
\end{cases}
\]

An immediate application of this result is the following lemma that will be very
convenient in the rest of our discussion.

**Lemma 4.7.** For each \( m \in \mathbb{Z} \), the functor \( \Pi_m^* : \text{QCoh}(X_m) \rightarrow \text{Gr}(A) \) is an exact
functor.

**Proof.** For each \( n \geq m \), \( A_{m,n} \) is locally free by Corollary 4.6, hence the functor
\(- \otimes A_{m,n} : \text{QCoh}(X_m) \rightarrow \text{QCoh}(X_n) \) is exact. As taking direct limits in \( \text{QCoh}(X) \)
and \( \text{QCoh}(Y) \) is exact, the result follows. \( \square \)

**Lemma 4.8.** There is a natural isomorphism for all \( F \in \text{QCoh}(X_m) \) and \( C \in \text{D}^+(\text{Gr}(A)) \):

\[
\text{RHom}_{\text{Proj}(A)}(\Pi_m^* F, C) \cong \text{RHom}_{X_m}(F, R\Pi_m^* C)
\]

**Proof.** This follows from lemma [5, Lemma 4.2] using a spectral sequence argument, see for
example [5, Lemma 4.2]. \( \square \)

We are especially interested in the case where \( C = \Pi_n^* G \) for a locally free sheaf \( G \)
on \( X_n \). Hence we need to understand complexes of the form \( R\Pi_m^*(\Pi_n^* G) \). The strategy
for computing its homology is as follows: by Lemma 4.10 it suffices to understand
the derived functors of \( \tau \). These in terms follow from the derived functors of an
internal Hom-functor \( \text{Hom} \) (Lemma 4.12).

**Lemma 4.9.** We have the following facts for the derived functors of the torsion
functor \( \tau : \text{Gr}(A) \rightarrow \text{Tors}(A) \):

i) for \( i \geq 1 \), there is an isomorphism of functors
\[
R^{i+1} \tau \cong (R^i \omega) \circ p
\]

ii) For each \( \mathcal{M} \in \text{Gr}(A) \) there is an exact sequence:
\[
0 \rightarrow \tau(\mathcal{M}) \rightarrow \mathcal{M} \rightarrow \omega(p(\mathcal{M})) \rightarrow R^1 \tau(\mathcal{M}) \rightarrow 0
\]

**Proof.** Since \( \text{Gr}(A) \) is locally noetherian, by [7, Lemma 2.12], any essential extension
of a torsion module remains a torsion module. In particular, the category
\( \text{Tors}(A) \) is closed under injective enveloppes, the result now follows from [8, Theo-
rem 2.14.15]. \( \square \)

**Lemma 4.10.** For \( i \geq 1 \), there is an isomorphism
\[
R^i \Pi_m^*(\Pi_n^* V) \cong R^{i+1} \tau(V \otimes e_n A)_m
\]

**Proof.** As the functors \( p \) and \( (-)_m \) are exact there is a functorial isomorphism
\[
R^i (\Pi_m^*(p(-)) \cong R^i \omega(p(-))_m
\]

Combining this isomorphism with the one in Lemma 4.9 we obtain for each \( i \geq 1 : \)
\[
R^i \Pi_m^*(\Pi_n^* V) := R^i \Pi_m^*(p(V \otimes e_n A)) \cong R^i \omega(p(V \otimes e_n A))_m \cong R^{i+1} \tau(V \otimes e_n A)_m
\]
The following is based on [6, Section 3.2]:

Let $\text{BiMod}(A - A)$ denote the category whose objects are of the form 

$$ \{ B_{m,n} \in \text{BiMod}(X_m - X_n) \}_{m,n} $$

such that the left and right multiplications $A_l,m \otimes B_{m,n} \to B_{l,n} \text{ resp } B_{m,n} \otimes A_{n,l} \to B_{m,l}$

are compatible in the obvious sense. We denote by $\mathcal{B}$ the subcategory for which all $B_{m,n}$ are coherent and locally free. There are an Hom-functors

$$ \mathcal{H}\text{om} : \mathcal{B}^{\text{op}} \times \text{Gr}(A) \to \text{Gr}(A) $$

$$ \mathcal{H}\text{om} : \text{BiMod}(O_{X_n} - A) \times \text{Gr}(A) \to \text{QCoh}(X_n) $$

satisfying the following properties:

**Proposition 4.11.**

i) $\mathcal{H}\text{om}(B, M)_m = \mathcal{H}\text{om}(e_m \otimes B, M)$ for all $B \in \mathcal{B}$ and $M \in \text{Gr}(A)$

ii) $\mathcal{H}\text{om} : \mathcal{B}^{\text{op}} \times \text{Gr}(A) \to \text{Gr}(A)$ is a bifunctor, left exact in both its arguments

iii) $\mathcal{H}\text{om} : \text{BiMod}(O_{X_n} - A) \times \text{Gr}(A) \to \text{QCoh}(X_n)$ is a bifunctor, left exact in both its arguments

iv) $\mathcal{H}\text{om}(Q \otimes e_m A, M) \cong M_m \otimes Q^*$ for all $M \in \text{Gr}(A)$ and $Q \in \text{coh}(X_m)$ locally free

**Proof.** i) This follows immediately by checking the exact definitions in [6, Section 3.2]

ii) [6, Proposition 3.11, Theorem 3.16(1)]

iii) [6, Theorem 3.16(3)]

iv) [6, Theorem 3.16(4)]

By ii. and iii. in the above proposition it makes sense to define the right derived functors $\mathcal{E}\text{xt}^i$ and $\mathcal{E}\text{xt}^i$ for all $i \geq 0$. Moreover we use the notation $A_{\geq l}$ to denote the object in $\mathcal{B}$ given by

$$(A_{\geq l})_{m,n} = \begin{cases} A_{m,n} & \text{if } n - m \geq l \\ 0 & \text{else} \end{cases}$$

and $A_0 := A/A_{>1}$. Then we have the following relation between the derived functors of $\tau$ and the $\mathcal{E}\text{xt}^i$:

**Lemma 4.12.** $R^i\tau(-) \cong \lim_{l \to \infty} \mathcal{E}\text{xt}^i_{\text{Gr}(A)}(A/A_{\geq l}, -)$

**Proof.** By [7, Proposition 3.19], we have an isomorphism of functors

$$ \tau \cong \lim_{l \to \infty} \mathcal{H}\text{om}_{\text{Gr}(A)}(A/A_{\geq l}, -) $$

Since $\text{Gr}(A)$ is a Grothendieck category, the direct limit of an exact sequence remains exact and the isomorphism descends to an isomorphism

$$ R\tau \cong \lim_{l \to \infty} R\mathcal{H}\text{om}_{\text{Gr}(A)}(A/A_{\geq l}, -) $$

and taking homology yields

$$ R^i\tau(-) \cong \lim_{l \to \infty} \mathcal{E}\text{xt}^i_{\text{Gr}(A)}(A/A_{\geq l}, -) \qed $$
Lemma 4.13. Let \( B \in \mathcal{B} \) be concentrated in degree \( l \geq 0 \) (i.e. \( B_{m,n} = 0 \) whenever \( m+l \neq n \)) and \( V \) a locally free sheaf. Then for \( n-l-1 \leq m \) and for all \( i \geq 0 \):

\[
\mathcal{E}xt^i(B, V \otimes e_n A)_m = 0
\]

Proof. Using a classical \( \delta \)-functor argument, one sees that Proposition 4.11(iv) gives rise to an isomorphism

\[
\mathcal{E}xt^i(B, V \otimes e_n A)_m \cong \mathcal{E}xt^i(A_0, V \otimes e_n A)_{m+l} \otimes B^*_{m,m+l}
\]

which easily reduces the proof to the case \( B = A_0 \) for which \( l = 0 \).

By Proposition 4.11(iv) we see that the exact sequence from Theorem 4.5 is a resolution of \( e_m A_0 = t_d (O_{X_m})_{t_d} \) by \( \mathcal{H}om(\_ , V \otimes e_n A) \)-acyclic sheaves. In particular we can calculate \( \mathcal{E}xt^i(A_0, V \otimes e_n A)_m = \mathcal{E}xt^i(e_m A_0, V \otimes e_n A) \) by taking homology of the complex

\[
0 \rightarrow \mathcal{H}om(e_m A_0, V \otimes e_n A) \xrightarrow{d_0} \mathcal{H}om(\mathcal{E}^m \otimes e_{m+1} A, V \otimes e_n A)
\]

\[
\xrightarrow{d_1} \mathcal{H}om(Q_m \otimes e_{m+2} A, V \otimes e_n A) \rightarrow 0
\]

again using Proposition 4.11(iv), this complex becomes

(18) \[
0 \rightarrow V \otimes A_{m} \xrightarrow{d_0} V \otimes A_{n,m+1} \otimes \mathcal{E}^{m+1} \xrightarrow{d_1} V \otimes A_{n,m+2} \otimes Q^*_m \rightarrow 0
\]

Hence we have

- \( \mathcal{E}xt^0(e_m A_0, V \otimes e_n A) = \ker(d_0) \)
- \( \mathcal{E}xt^1(e_m A_0, V \otimes e_n A) = \ker(d_1) / \text{im}(d_0) \)
- \( \mathcal{E}xt^2(e_m A_0, V \otimes e_n A) = \text{coker}(d_1) \)
- \( \mathcal{E}xt^i(e_m A_0, V \otimes e_n A) = 0 \) for all \( i \geq 3 \)

We defer to the proof of \([6]\) Theorem 4.4] for showing that \( \ker(d_0) = 0 \) and \( \ker(d_1) = \text{im}(d_0) \). To show that \( d_1 \) is surjective, recall that each stalk of the structure sheaf of a smooth curve is a PID. In particular the stalk of \( \text{im}(d_1) \) at any point \( p \) is a free \( O_p \)-module as a submodule of \( (V \otimes A_{n,m+2} \otimes Q^*)_p \). Hence \( \text{im}(d_1) \) is a locally free subsheaf of \( V \otimes A_{n,m+2} \otimes Q^* \) and to show that \( d_1 \) is surjective, it suffices to show that the rank of \( \text{im}(d_1) \) equals the rank of \( V \otimes A_{n,m+2} \otimes Q^* \). By exactness of (18) at the first and middle term we know the rank of \( \text{im}(d_1) \) is given by:

\[
\text{rk}(\text{im}(d_1)) = \text{rk} (V \otimes A_{n,m+1} \otimes \mathcal{E}^{m+1}) - \text{rk}(\ker(d_1))
\]

\[
= \text{rk} (V \otimes A_{n,m+1} \otimes \mathcal{E}^{m+1}) - \text{rk}(\text{im}(d_0))
\]

\[
= \text{rk} (V \otimes A_{n,m+1} \otimes \mathcal{E}^{m+1}) - \text{rk} (V \otimes A_{n,m})
\]

Hence we define

\[
d_{n,m} := \text{rk} (V \otimes A_{n,m+2} \otimes Q^*) - \text{rk} (V \otimes A_{n,m+1} \otimes \mathcal{E}^{m+1}) + \text{rk} (V \otimes A_{n,m})
\]

and show \( d_{n,m} = 0 \) for all \( n, m \). As \( d_{n,m} \) is obviously linear in \( \text{rk}(V) \), so we can reduce to the case that \( V = L \) is a line bundle. We can then use Corollary 4.6 to compute the ranks whenever \( n \leq m \):
Proof. of Theorem 4.1

\begin{align*}
\text{In the case where } n = m + 1 \text{ (which by the assumption of the theorem is the only case with } n \geq m) \text{ we have}
\text{rk}(\text{im}(d_1)) &= \text{rk}(L \otimes A_{m+1,m+1} \otimes E^{*m+1}) - \text{rk}(L \otimes A_{m+1,m}) \\
&= \text{rk}(L \otimes E^{*m+1}) \\
&= \text{rk}(L \otimes A_{m+1,m+2} \otimes Q^*_m)
\end{align*}

again showing surjectivity of \(d_1\).

\section*{Lemma 4.14.} \(\text{Ext}^i(A/A_{\geq l}, V \otimes e_nA)_m = 0 \text{ for } m \geq n - 1 \text{ and } i \geq 0\)

\textbf{Proof.} Consider the short exact sequence

\[0 \rightarrow A_{\geq l}/A_{\geq l+1} \rightarrow A/A_{\geq l+1} \rightarrow A/A_{\geq l} \rightarrow 0\]

Applying \(\text{Hom}(\cdot, V \otimes e_nA)\) gives rise to a long exact sequence for each \(m \geq n - 1\)

\[\ldots \rightarrow \text{Ext}^i(A_{\geq l}/A_{\geq l+1}, V \otimes e_nA)_m \rightarrow \text{Ext}^i(A/A_{\geq l+1}, V \otimes e_nA)_m \rightarrow \text{Ext}^{i+1}(A_{\geq l}/A_{\geq l+1}, V \otimes e_nA)_m \rightarrow \ldots\]

As \(m \geq n - 1\) it follows from Lemma 4.13 that for each \(i \geq 0\) we have an exact sequence

\[0 \rightarrow \text{Ext}^i(A/A_{\geq l+1}, V \otimes e_nA)_m \rightarrow \text{Ext}^i(A/A_{\geq l}, V \otimes e_nA)_m \rightarrow 0\]

Hence

\[\text{Ext}^i(A/A_{\geq l}, V \otimes e_nA)_m \cong \text{Ext}^i(A/A_{\geq 0}, V \otimes e_nA)_m = \text{Ext}^i(0, V \otimes e_nA)_m = 0. \]

\section*{Proof. of Theorem 4.7}

Take \(m, n \in \mathbb{Z}\) with \(m \geq n - 1\). Let \(F\) be locally free on \(X_m\) and \(G\) locally free on \(X_n\), then by Corollary 4.8

\[\text{Ext}^i_{\text{proj}(A)}(\Pi_m F, \Pi_n^* G) = h^i \left(\text{RHom}_{\text{proj}(A)}(\Pi_m F, \Pi_n^* G)\right) \cong h^i \left(\text{RHom}_{X_m}(F, R \Pi_m, \Pi_n^* G)\right)\]

Now for \(i \geq 1\) we have

\[\text{R}^i \Pi_m, \Pi_n^* G \cong \text{R}^{i+1} \tau(G \otimes e_nA)_m \cong \lim_{l \rightarrow \infty} \text{Ext}^{i+1}(A/A_{\geq l}, G \otimes e_nA)_m = 0\]

by Lemmas 4.10, 4.12 and 4.14 respectively.

In particular the complex \(\text{RHom}_{m}, \Pi_n^* G\) is quasi-isomorphic to the complex that is
equal to $\Pi_m \Pi_n^* G$ concentrated in position zero. Finally we can conclude by noticing that $\Pi_m \Pi_n^* G = (\omega p(G \otimes e_n A))_m$ and by Lemma 4.9 there is an exact sequence

$$0 = \tau(G \otimes e_n A)_m \longrightarrow G \otimes A_{n,m} \xrightarrow{\cong} \omega(p(G \otimes e_n A))_m \longrightarrow R^1 \tau(G \otimes e_n A)_m = 0$$

where the first term equals zero because $G \otimes e_n A$ is torsion free and the last term is zero because $R^1 \tau(G \otimes e_n A)_m \cong \lim_{l \to \infty} \text{Ext}^1(A/A_{\geq l}, G \otimes e_n A)_m = 0$.

Hence we can conclude that for $m \geq n - 1$ we have

$$\text{Ext}_{\text{proj}(\mathcal{A})}^i(\Pi_m \mathcal{F}, \Pi_n^* G) \cong h^i(\text{R Hom}_{\mathcal{X}_m}(\mathcal{F}, R\Pi_m \Pi_n^* G))$$

$$\cong h^i(\text{R Hom}_{\mathcal{X}_m}(\mathcal{F}, G \otimes A_{n,m}))$$

$$= \text{Ext}_{\mathcal{X}_m}^i(\mathcal{F}, G \otimes A_{n,m}) \quad \square$$
In this section, we show how a periodic \( \mathbb{Z} \)-algebra \( A \) gives rise to a graded algebra \( \overline{A} \) such that \( \text{Gr}(A) \) is a direct summand of the category \( \text{Gr}(\overline{A}) \). We shall consider a slightly more general version of \( \mathbb{Z} \)-algebra to fit our needs.

**Definition A.1.** Let \((R_i)_{i \in \mathbb{Z}}\) be a collection of commutative \( k \)-algebras. A *bimodule \( \mathbb{Z} \)-algebra* is a collection of \( R_i - R_j \)-bimodules \( A_{ij} \) together with multiplication maps
\[
A_{ij} \otimes_{R_j} A_{jl} \rightarrow A_{il}
\]
and \( R_i \)-linear unit maps
\[
R_i \rightarrow A_{ii}
\]
satisfying the usual \( \mathbb{Z} \)-algebra axioms.

To ease notation we shall omit the word *bimodule* whenever there is no confusion.

We briefly recall what we mean by periodicity of a \( \mathbb{Z} \)-algebra.

**Definition A.2.** Let \( A \) be a \( \mathbb{Z} \)-algebra over \((R_i)_{i \in \mathbb{Z}}\) and \( d > 0 \) an integer. Assume that for each \( i \), we have \( R_{i+d} = R_i \). We say \( A \) is *\( d \)-periodic* if there is an isomorphism of \( \mathbb{Z} \)-algebras \( \phi: A \sim \rightarrow A(d) \). I.e. there is a collection of \( R_i - R_j \)-bimodule isomorphisms \( \{ \phi_{ij}: A_{ij} \sim \rightarrow A_{i+d,j+d} \}_{i,j} \) compatible with the multiplication and unit maps.

Let \( A \) be \( d \)-periodic and let \( R := \bigoplus_{i=0}^{d-1} R_i \). We shall construct a graded \( R \)-algebra \( \overline{A} \) as follows: let \( A_n \) be a \( d \times d \) matrix with entries
\[
A_{ij} = \begin{cases} A_{i,i+n} & \text{if } j - i \equiv n \pmod{d} \\ 0 & \text{else} \end{cases}
\]
(Where we use the convention that the numbering of rows and columns of the matrix starts at 0 instead of 1.)

By way of example, we have
\[
\begin{pmatrix}
0 & A_{0,1} & 0 & \ldots & 0 \\
0 & 0 & A_{1,2} & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & A_{d-2,d-1} \\
A_{d-1,d} & 0 & 0 & \ldots & 0 
\end{pmatrix}
\]

Each \( A_n \) is naturally a left (resp. right) \( R \)-module by letting a \( d \)-tuple \( (r_0, \ldots, r_{d-1}) \) act as a diagonal matrix \( D \) with entries \( D_{ii} := r_i \) on the left (resp. right). Moreover, there is a canonical multiplication map
\[
\overline{A}_n \otimes_R \overline{A}_m \rightarrow \overline{A}_{n+m}
\]
given by the ordinary matrix multiplication and applying the periodicity isomorphisms \( \phi_{ij} \) whenever necessary. The \( (R_i)_{i \in \mathbb{Z}} \)-linearity of the \( \mathbb{Z} \)-algebra multiplication implies that the above maps are indeed \( R \)-bilinear.

**Lemma A.3.** Suppose \( A \) is \( d \)-periodic, then the above maps define a graded \( R \)-algebra structure on the \( R \)-module \( \overline{A} := \bigoplus_{i \in \mathbb{Z}} \overline{A}_i \).

**Proof.** The reader checks that the compatibility of the periodicity isomorphisms with the \( \mathbb{Z} \)-algebra multiplication maps implies that the multiplication is associative. The multiplication is distributive by construction and the algebra has a unit.
given by

\[ 1 = \begin{pmatrix} e_0 & 0 & \ldots & 0 \\ 0 & e_1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & e_{d-1} \end{pmatrix} \in \mathcal{A}_0 \]

where \( e_i \) is the unit in \( A_{ii} \). \( \square \)

There is a convenient description of graded right \( \mathcal{A} \)-modules as follows: let \( M \in \text{Gr}(\mathcal{A}) \). Then \( M = \oplus_{i \in \mathbb{Z}} M_i \). Moreover, each \( R \)-module \( M_i \) in turn has a direct sum decomposition given by \( M_i = \oplus_{j=0}^{d-1} M_i e_j \). We define \( M^1_i := M_i e_j \).

The \( \mathcal{A} \)-module structure has a nice description in these terms. For a matrix \( \pi \in \mathcal{A}_m \), \( e_j \pi \) only has one nonzero entry at position \((j, j+m)\). It follows from the right \( R \)-structure on \( A_m \) that \( e_j \pi = \pi e_{j+m} \) (where we consider \( j+m \) mod \( d \)). Thus the right action of \( \mathcal{A}_m \) on \( M^1_i \) becomes a map of the form \( M^1_i \otimes A_{j,j+m} \rightarrow M^1_{i+j+m} \) or equivalently for \( l = j+m \),

\[ M^1_i \otimes A_{j,l} \rightarrow M^1_{i+l-j} \]

we now have:

**Lemma A.4.** Suppose \( A \) is \( d \)-periodic and let \( C \) be the category defined as follows:

1. an object is a collection of \( R \)-modules \((M^1_i)_{i \in \mathbb{Z}, 0 \leq j \leq d-1}\), such that \( M^1_i \) is an \( R_j \)-module together with multiplication maps

\[ \mu^M_{i,j,l} : M^1_i \otimes A_{j,l} \rightarrow M^1_{i+l-j} \]

for each \( i, j, l \) (where \( l \) and \( i+l-j \) should be interpreted modulo \( d \)) satisfying the obvious compatibility condition for multiplication and unit.

2. a morphism is a collection \( f_{i,j} \) of \( R_j \)-linear maps \( M^1_i \rightarrow N^1_i \) such that

\[ f_{i+l-j,l} \circ \mu^M_{i,j,l} = \mu^N_{i,j,l} \circ (f_{i,j} \otimes A_{j,l}) \]

Then there is a canonical isomorphism of categories \( C \cong \text{Gr}(\mathcal{A}) \).

**Proof.** The above discussion shows that the assignment \( M \rightarrow (M_i e_j)_{i \in \mathbb{Z}, 0 \leq i \leq n-1} \) is well defined and essentially surjective. A morphism of graded modules \( f : M \rightarrow N \) will satisfy \( f(M_i e_j) \subset N_i e_j \) and we can define \( f_{i,j} \) as the restriction to these submodules. The \( A \)-linearity guarantees that \( (f_{i,j})_{i,j} \) indeed defines a morphism in \( C \) and since \( \oplus M_i e_j = M \) it is clear that this assignment is faithful. Since any collection of maps \( f_{i,j} \) satisfying the above compatibility with the multiplication will sum up to an \( \mathcal{A} \)-linear map, the assignment is also full. \( \square \)

**Lemma A.5.** There exists a decomposition

\[ C = C_0 \oplus \ldots \oplus C_{d-1} \]

where \( C_n \) is the full subcategory of \( C \) whose objects are collections of \( R \)-modules \((M^1_i)_{i \in \mathbb{Z}, 0 \leq j \leq d-1}\) where \( M^1_i = 0 \) unless \( j - i \equiv n \) (mod \( d \)).

**Proof.** This follows immediately from the construction of \( C \) and the fact that \( j - i = l - (l + i - j) \). Hence, if \((M^1_i)_{ij}\) is a non-zero object in \( C_n \), then so is \((M^1_{i+l-j})_{ij}\) for all \( l \). \( \square \)
Proposition A.6. There is an exact embedding of categories
\[
\overline{(-)} : \text{Gr}(A) \to \text{Gr}(A)
\]
moreover the essential image is a direct summand of \(\text{Gr}(A)\).

Proof. Let \(M\) be an \(A\)-module with multiplication maps \(\mu_{i,m} : M_i \otimes_R A_m \to M_{i+m}\) and let \(C\) be as above. We define an object \(\overline{M}\) in \(C\) by
\[
\overline{M}_i = \begin{cases} M_i & \text{if } j \equiv i \mod d \\ 0 & \text{else} \end{cases}
\]
where the multiplication is given by
\[
\overline{\mu}_{i,j,l} = \begin{cases} \mu_{i,l-j} & \text{if } j \equiv i \mod d \\ 0 & \text{else} \end{cases}
\]
This assignment clearly defines an exact embedding \(\text{Gr}(A) \to \overline{C} \to C\), finishing the proof by Lemmas A.4 and A.5. \(\square\)

Appendix B. Point Modules and proof of Theorem 4.5

In this section we will assume \(A = \mathbb{S}(E)\) is a symmetric sheaf \(\mathbb{Z}\)-algebra in standard form with \(E \in \text{bimod}(X - Y)\) locally free of rank \((4, 1)\), in particular \(E = f(L)\text{Id}\) as in Lemma 3.2. As before we assume \(X, Y\) are smooth curves over an algebraically closed field \(k\) and let \(\alpha : X \to \text{Spec}(k), \beta : Y \to \text{Spec}(k)\) be the structure morphisms. As always we will write \((X_n, \alpha_n) = \begin{cases} (X, \alpha) & \text{if } n \text{ is even} \\ (Y, \beta) & \text{if } n \text{ is odd} \end{cases}\)
We say \(P_n \in \text{coh}(X_n)\) is locally free over \(k\) of rank \(l\) if \(\alpha_n^* P_n\) is free of rank \(l\).

A module \(P \in \text{Gr}(A)\) is said to be generated in degree \(m\) if \(P_n = 0\) for all \(n < m\) and \(P_m \otimes A_{m,n} \to P_n\) is surjective for all \(n \geq m\). As \(A\) is generated in degree one as an algebra, we have surjectivity of \(P_{n_1} \otimes A_{n_1,n_2} \to P_{n_2}\) for all \(n_2 \geq n_1 \geq m\) by the following commuting diagram
\[
\begin{array}{ccc}
P_m \otimes A_{m,n_1} \otimes A_{n_1,n_2} & \to & P_{n_1} \otimes A_{n_1,n_2} \\
| & & | \\
P_m \otimes A_{m,n_2} & \to & P_{n_2}
\end{array}
\]

Remark B.1. An obvious example of a module generated in degree \(m\) is \(e_mA\).

The above diagram implies that the maps \(A_{m,n} \otimes e_n A \to e_mA\) are surjective for all \(m \geq n\).

An \(m\)-shifted point-module over \(A\) is defined in 9 as an object \(P \in \text{Gr}(A)\) such that \(P\) is generated in degree \(m\) and for which \(P_n\) is locally free of rank one over \(k\) for all \(n \geq m\). As the next Lemma shows, this definition is not desirable in the current situation.

Lemma B.2. Let \(i \in \mathbb{Z}\) and \(P \in \text{Gr}(A)\) generated in degree \(2i\) such that \(P_{2i}\) and \(P_{2i+1}\) are locally free of rank one over \(k\). Then \(P_n = 0\) for all \(n \geq 2i + 2\).
Proof. Recall that the following composition
\[ P_{2i} \rightarrow P_{2i} \otimes \mathcal{E}^{*2i} \otimes \mathcal{E}^{*2i+1} \rightarrow P_{2i+1} \otimes \mathcal{E}^{*2i+1} \rightarrow P_{2i+2} \]
must be zero as it represents the action of $Q_{2i}$. By [9, Lemma 4.3.2.] this composition equals
\[ P_{2i} \xrightarrow{\varphi_{2i}} P_{2i+1} \otimes \mathcal{E}^{*2i+1} \xrightarrow{\varphi_{2i+1}} P_{2i+2} \]
where $\varphi_{2i}^*$ is obtained by adjointness from $\varphi_{2i} : P_{2i} \otimes \mathcal{E}^{*2i} \rightarrow P_{2i+1}$. Remark that as $P_{2i}$ and $P_{2i+1} \otimes \mathcal{E}^{*2i+1}$ are locally free of rank one over $k$ we have that either $\varphi_{2i}^*$ is an isomorphism or it is zero. Similarly $\varphi_{2i+1}$ is monic or zero. Hence the only way the composition can be zero is when $\varphi_{2i}^* = 0$ or $\varphi_{2i+1} = 0$. The first option cannot happen as $\varphi_{2i} \neq 0$ (because $P$ was generated in degree $2i$ and $P_{2i+1} \neq 0$). Hence we must have $\varphi_{2i+1} = 0$. However $\varphi_{2i+1}$ must be surjective (again because $P$ was generated in degree $2i$), hence $P_{2i+2} = 0$. Using surjectivity of $P_{2i+2} \otimes \mathcal{A}_{2i+2,n} \rightarrow P_n$ for all $n \geq 2i + 2$ the result follows. □

The following definition will make more sense:

**Definition B.3.** A shifted point module is an object $P \in \text{Gr}(\mathcal{A})$ which is generated in degree $2i$ for some integer $i$ and such that for all $n \geq 2i$, $P_n$ is locally free over $k$ of rank one if $n$ is even and rank two if $n$ is odd. We will often use the short hand notation $\dim_k(P_n) = \text{length}_{\text{Spec}(k)}(\alpha_n((P))$ whenever the latter is finite. So we could say $P$ is a shifted point module if is generated in degree $2i$ and:

\[
\dim_k(P_n) = \begin{cases} 
0 & \text{if } n < 2i \\
1 & \text{if } n \geq 2i \text{ is even} \\
2 & \text{if } n > 2i \text{ is odd}
\end{cases}
\]

The following Lemma shows that this new definition of point modules behaves way better than the naive one:

**Lemma B.4.** Let $P \in \text{Gr}(\mathcal{A})$ be a graded module and $i \in \mathbb{Z}$ such that:
\begin{itemize}
  \item $P$ is generated in degree $2i$
  \item $\dim_k(P_{2i}) = 1$
  \item $\dim_k(P_{2i+1}) = 2$
\end{itemize}
Then for all $n \geq 2i + 2$ fixed, we have
\[ \dim_k(P_n) \leq \begin{cases} 
1 & \text{if } n \text{ is even} \\
2 & \text{if } n \text{ is odd}
\end{cases} \]

Moreover if equality holds in (19), then $P_n$ is defined up to unique isomorphism by the data $\varphi_{2i} : P_{2i} \otimes \mathcal{E}^{*2i} \rightarrow P_{2i+1}$.

If on the other hand (19) is a strict inequality, then $P_l = 0$ for all $l > n$.

**Proof.** We prove all facts by induction on $n$. So suppose (19) and the subsequent claims hold for $n = 2i, \ldots, m$. We distinguish several cases depending on whether the inequalities are in fact equalities or not.

**Case 1: Equality holds in (19) for $n = 2i, \ldots, m$.**

The following composition is zero:
\[ P_{m-1} \xrightarrow{\varphi_{m-1}} P_m \otimes \mathcal{E}^{*m} \xrightarrow{\varphi_m} P_{m+1} \]
$\varphi_m$ is surjective, hence one can easily check that (19) holds if we can prove $\varphi_{m-1}^*$ is injective. Moreover if the equality holds for $\dim_k(P_{m+1})$, then $P_{m+1} \cong \text{coker}(\varphi_{m-1}^*)$
and is hence defined up to unique isomorphism.

**Case 1a: m is odd**

\( \dim_k(P_{m-1}) = 1 \) hence it suffices to prove \( \varphi_{m-1}^* \neq 0 \) and this holds because \( \varphi_{m-1} \neq 0 \)

**Case 1b: m is even**

If \( \varphi_{m-1} \) were not injective, then there is a \( W \subset P_{m-1} \), \( \dim_k(W) = 1 \) such that the composition

\[
W \hookrightarrow P_{m-1} \xrightarrow{\varphi_{m-1}^*} P_m \otimes \mathcal{E}^*m
\]

is zero. This implies that there is an \( W \in \text{Gr}(A) \) given by \( W_{m-1} = W \) and \( W_l = 0 \) for \( l \neq m - 1 \). By construction there is an embedding \( \chi : W \hookrightarrow P_{m-2} \) and let \( C = \text{coker}(\chi) \). Then \( C \) is generated in degree \( m - 2 \) (which is even!) and \( \deg_k(C_{m-2}) = \deg_k(C_{m-1}) = \deg_k(C_m) = 1 \) contradicting Lemma B.2.

**Case 2: There is an integer \( n \in \{2i+2, \ldots, m\} \) such that there is a strict inequality for \( \dim_k(P_n) \) in (19)**

Let \( n_0 \) be the smallest such \( n \). We have to show \( P_l = 0 \) for all \( l > n_0 \).

Assume that \( P_{n_0} = 0 \), then \( P_l = 0 \) by surjectivity of \( P_{n_0} \otimes A_{n_0, l} \to P_l \).

The only nontrivial case is when \( n_0 \) is odd and \( \dim_k(P_{n_0}) = 1 \). In this case \( \dim_k(P_{n_0-1}) = 1 \) as well and the result follows from Lemma B.2.

**Remark B.5.** The above Lemma also shows that any data \( \varphi_{2i} : P_{2i} \otimes \mathcal{E}^{*2i} \to P_{2i+1} \) with \( \dim_k(P_{2i}) = 1 \) and \( \dim_k(P_{2i+1}) = 2 \) can be extended to a shifted point module which is unique up to unique isomorphism.

From now on we use the following short hand notation:

\[
L_{n,p} := \mathcal{O}_p \otimes e_{n,A}
\]

where \( p \) is any point on \( X_n \).

**Proof. of Theorem 4.3**

Exactness of the sequence (17) can be checked for each degree \( n \) separately:

\[
(20) \quad 0 \to Q_m \otimes A_{m+2,n} \to \mathcal{E}^{*m} \otimes A_{m+1,n} \to A_{m,n} \to 0
\]

As all terms in this sequence are elements of bimod\((X_m - X_n)\), applying \( \pi_{m,*} \) gives a sequence of coherent sheaves on \( X_m \):

\[
(21) \quad 0 \to \pi_{m,*}(Q_m \otimes A_{m+2,n}) \to \pi_{m,*}(\mathcal{E}^{*m} \otimes A_{m+1,n}) \to \pi_{m,*}(A_{m,n}) \to 0
\]

and (21) is exact if and only if (20) is. The structure of the relations on \( A \) implies that (17) and hence also (20) and (21) are right exact. Now for any point \( p \in X_m \) the following will be right exact as well:

\[
(22) \quad 0 \to \mathcal{O}_p \otimes \pi_{m,*}(Q_m \otimes A_{m+2,n}) \to \mathcal{O}_p \otimes \pi_{m,*}(\mathcal{E}^{*m} \otimes A_{m+1,n}) \to \ldots \to \mathcal{O}_p \otimes \pi_{m,*}(A_{m,n}) \to 0
\]

and as all terms in (22) are locally free over \( k \), its left exactness can be checked numerically. If for all terms in (22) \( \dim_k \) does not depend on the point \( p_m \), then the terms in (21) are locally free and exactness of (21) follows from exactness of (22).

Hence in order to prove the Lemma we show that the terms in (22) have the “correct” constant length (see (27)). From this it follows that (20) is exact and its terms are locally free on the left. The locally freeness on the right then follows from (9).
Proposition 3.1.6].

So we are left with finding the length of the objects in $\mathcal{O}_p \otimes \pi_{m,*}(uU_v)$. Any object in $\text{bimod}(X_m - X_n)$ is of the form $uU_v$ for finite maps $u$ and $v$. As taking the direct image through a finite morphism does not change the length of sheaves, we have for such a bimodule:

$$\dim_k(\mathcal{O}_p \otimes \pi_{m,*}(uU_v)) = \dim_k(\mathcal{O}_p \otimes u_\mathcal{U})$$

$$= \dim_k(u^*(\mathcal{O}_p) \otimes \mathcal{U})$$

$$= \dim_k(u^*(\mathcal{O}_p) \otimes \mathcal{U})$$

$$= \dim_k(\mathcal{O}_p \otimes uU_v)$$

Hence the length of the terms in (22) can be calculated from

$$\dim_k(\mathcal{O}_p \otimes Q_m \otimes A_{m+2,n} \rightarrow \mathcal{O}_p \otimes E^{*m} \otimes A_{m+1,n} \rightarrow \mathcal{O}_p \otimes A_{m,n} \rightarrow \mathcal{O}_p) = 0.$$  

Now $\dim_k(\mathcal{O}_p \otimes Q_m) = 1$, hence there is a point $\tilde{p} \in X_{m+2}$ such that $\mathcal{O}_p \otimes Q_m = \mathcal{O}_{\tilde{p}}$. Similarly, in the case where $m = 2i - 1$ there must be a point $\tilde{q} \in X_{2i+1}$ such that $\mathcal{O}_p \otimes E_{2i-1} = \mathcal{O}_q$. In the case where $m = 2i$, we have $\dim_k(\mathcal{O}_p \otimes E_{2i}) = 4$, hence there must be points $\tilde{q} \in X_{2i+1}$, $a = 1, \ldots, 4$ such that $\mathcal{O}_p \otimes E_{2i}$ is an extension of the $\mathcal{O}_{\tilde{q}}$. We denote the corresponding extension of the $L_{2i+1, \tilde{q}}$ by $M_{2i+1, p}$. The sequence (23) now gives rise to the following right exact sequences:

$$L_{2i+1, \tilde{p}} \rightarrow L_{2i, q} \rightarrow L_{2i-1, p} \rightarrow 0$$

(24)

$$L_{2i+2, \tilde{p}} \rightarrow M_{2i+1, p} \rightarrow L_{2i, p} \rightarrow 0$$

(25)

Finally there also is a right exact sequence:

$$L_{2i+1, p'} \rightarrow L_{2i-1, p} \rightarrow P_p \rightarrow 0$$

(26)

where the morphism $L_{2i+1, p'} \rightarrow L_{2i-1, p}$ comes from the fact that $\dim_k(\mathcal{O}_p \otimes A_{2i-2,2i-1}) = 3 > 0$ such that there is a $p' \in X_{2i+1}$ with a nonzero morphism $\mathcal{O}_{p'} \rightarrow \mathcal{O}_p \otimes A_{2i-2,2i+1}$. $P_p$ is defined as the cokernel of this morphism.

We now prove the following by induction on $j$ (simultaneously for all $p$ and all $i$):

$$\dim_k((P_p)_{2j+2i}) = 1$$

$$\dim_k((P_p)_{2i+2j+1}) = 2$$

$$\dim_k((L_{2i,p})_{2i+2j}) = 2j + 1$$

$$\dim_k((L_{2i,p})_{2i+2j+1}) = 4j + 4$$

$$\dim_k((L_{2i-1,p})_{2i+2j}) = j + 1$$

$$\dim_k((L_{2i-1,p})_{2i+2j+1}) = 2j + 3$$

(27)

By construction these facts are known to hold for $j = 0$. So by induction we now suppose they hold for $j = 0, \ldots, l$, for all points and for all $i \in \mathbb{Z}$. We prove that these facts then also hold for $j = l + 1$.  


By (25) we can see:
\[
\dim_k((L_{2i,p})_{2i+2+2}) \geq \dim_k((M_{2i+1,p})_{2i+2+2}) - \dim_k((L_{2i+2,\tilde{p}})_{2i+2+2})
\]
\[
= \sum_{a=1}^{4} \dim_k((L_{2i+2,\tilde{q}_{2i+1}})_{2i+2+2}) - \dim_k((L_{2i+2,\tilde{p}})_{2i+2+2})
\]
\[
= 4 \cdot (l + 1) - (2l + 1)
\]
\[
= 2l + 3
\]

Where the last equality follows from the induction hypothesis. Similarly we find a lower bound
\[
\dim_k((L_{2i+1,p})_{2i+2+2}) \geq 4l + 4.
\]
This can be written schematically as:
\[
(28)
\]
Where the numbers on the right of a module denote \( \dim_k((-)_x) \) for \( x = 2i - 1, \ldots, 2i + 2l + 3 \). And an underlined number implies a lower bound for \( \dim_k \). Similarly we will write \( N \) to denote that \( N \) is an upperbound for a certain \( \dim_k \).

Now consider the module \( P_{p, \geq 2i+2l} \). This is generated in degree \( 2i + 2l \) because \( P_p \) is a quotient of \( L_{2i-1,p} \). Moreover \( \dim_k((P_p)_{2i+2l}) = 1 \) and \( \dim_k((P_p)_{2i+2l+1}) = 2 \), so Lemma [B.4] implies \( \dim_k((P_p)_{2i+2l+2}) \leq 1 \) and \( \dim_k((P_p)_{2i+2l+3}) \leq 2 \). Together with the right exact sequence (26) this gives us the following upper bounds:
\[
(29)
\]
Combining the bounds found in (28) and (29) and using (24) we have found
\[
(30)
\]
Right exactness of (24) implies that the bounds in (30) are in fact equalities, because for example we find the upper bound
\[
\dim_k((L_{2i,q})_{2i+2l+2}) \leq \dim_k((L_{2i-1,p})_{2i+2l+2}) + \dim_k((L_{2i+1,\tilde{p}})_{2i+2l+2})
\]
\[
\leq l + 2 + l + 1
\]
\[
= 2l + 3
\]
which equals the already known lower bound for \( \dim_k((L_{2i},q)_{2i+2}) \). Hence we have found the exact value for \( \dim_k((L_{2i+1},q)) \). A priori the above right exact sequence only gives this exact value for the points \( q \in X_{2i} \) for which there is a \( p \in X_{2i-1} \) such that \( O_p \otimes E^{*2i-1} = O_q \). But as \( E^{*2i-1} \) is of the form \( Id(L_{i-1}) \) as in (12) we have \( q = f(p) \) and surjectivity of \( f \) implies that \( q \) runs through all points of \( X_{2i-1} \). For \( L_{2i-1} \) no such problems arise.

Hence we have proven (27) for all \( i, j \in \mathbb{Z} \) and for all points \( p \). As these values do not depend on \( p \) we have that the terms in (24) are locally free on the left (and hence also on the right). Filling in these values for (22) we find that the sequences must be exact.

\[ \square \]

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