Bringing entanglement to the high temperature limit

Fernando Galve
IFISC (CSIC - UIB), Instituto de Física Interdisciplinar y Sistemas Complejos, Campus Universitat Illes Balears, E-07122 Palma de Mallorca, Spain

Leonardo A. Pachón
Departamento de Física, Universidad Nacional de Colombia, Bogotá D.C., Colombia.

David Zueco
Instituto de Ciencia de Materiales de Aragón y Departamento de Física de la Materia Condensada, CSIC-Universidad de Zaragoza, E-50012 Zaragoza, Spain.

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We show the existence of an entangled nonequilibrium state at very high temperatures when two linearly coupled harmonic oscillators are parametrically driven and dissipate into two independent heat baths. This result has a twofold meaning: first, it fundamentally shifts the classical-quantum border to temperatures as high as our experimental ability allows us, and second, it can help increase by at least one order of magnitude the temperature at which current experimental setups are operated.

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Introduction.— Since the establishment of quantum theory in last century there has been a long evolution on our concept of what is quantum and to what extent it is required to explain observations in nature. At the very beginning the reduction postulate was proposed, clearly separating between quantum microscopic entities and classical macroscopic measuring apparatuses. Since then macroscopic quantum phenomena such as superconductivity and coherent superposition in Bose–Einstein condensates [1], together with interference fringes of very massive molecules [2] have been observed. Recently a proposal to create superpositions of dielectric bodies, such as viruses up to micron size, inside a high finesse optical cavity has been given [3]. Hence the border between the classical and quantum worlds seems to be more diffuse and intriguing than we could have conceived one century ago.

Neither the usual transition criterion of \( \hbar / S_{\text{ch}} \to 0 \) (with \( S_{\text{ch}} \) the characteristic action of the system) is to be trusted, since this limit could be not completely continuous and strong deviations have been reported in the semiclassical regime [4]. In the more realistic situation the system interacts with the surrounding environment, dissipation restricts purely quantum phenomena to within the very low temperatures limit [5],

\[
k_B T / \hbar \omega \ll 1 ,
\]

where \( \hbar \omega \) denotes the typical energy scale of the system and \( k_B T \) the thermal energy. Above this limit, quantum correlations are inaccessible behind a ’mask’ of thermal fluctuations.

As a consequence, observing quantum phenomena implies the need for a very delicate pre-cooling process. But, is there any alternative to cooling for being quantum? In the present Letter, we defy the above classicality criterion and report the existence of a nonequilibrium entangled steady state for coupled harmonic oscillators at high temperatures, obtained through parametric driving. This result is quite fundamental, meaning that we might expect entanglement in hot highly nonequilibrium situations, as pointed out [6] for biological systems. Further, it could lighten the burden on quantum experiments requiring delicate pre-cooling setups. We note that though quantum coherence can play a role in biological processes at ambient temperature [7], demonstration of entanglement in hot highly nonequilibrium situations is there any alternative to cooling for being quantum? We show the existence of an entangled nonequilibrium state at very high temperatures when two linearly coupled harmonic oscillators are parametrically driven and dissipate into two independent heat baths. This result has a twofold meaning: first, it fundamentally shifts the classical-quantum border to temperatures as high as our experimental ability allows us, and second, it can help increase by at least one order of magnitude the temperature at which current experimental setups are operated.

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monic oscillators with linear coupling can be ascribed to a result, or semiclassical energy scales, etc., thus avoiding any high temperature, strong or weak damping, deeply quantum-mechanical effects. We can study the central system in any regime: low or high frequency, but not too low. With our analytical result we have been able to derive an exact analytic expression for the case without driving. The environmental influence enters via the spectral density of the Mathieu oscillator (See Appendices B and C). The opposite case corresponds to a model with common influence, which in the case of no driving [15] is only possible in very few cases [14, 18]. Here, we have been able to derive an exact analytic expression for the case without driving. The environmental influence enters via the spectral density $I(\omega) = \sum_j c_j^2/(2m_j\omega_j)\delta(\omega - \omega_j)$. Here, we assume for simplicity Ohmic noise $I(\omega) = m\gamma\omega$. It produces white noise in the classical limit $\gamma = 0$. With our analytical result we can study the central system in any regime: low or high temperature, strong or weak damping, deeply quantum or semiclassical energy scales, etc., thus avoiding any spurious approximative corrections or limitations. As a result, any system that can be considered as two harmonic oscillators with linear coupling can be ascribed exactly to our description.

Entanglement computation.— Linearity of the total Hamiltonian ensures that the state is always Gaussian, and thus its entanglement properties are fully characterized by the covariance matrix $\sigma_{i,j} = \langle \xi_i \xi_j + \xi_j \xi_i \rangle/2 - \langle \xi_i \rangle \langle \xi_j \rangle$ with $\xi = (Q_1, Q_2, P_1, P_2)$. An exact measure of entanglement is known for Gaussian states, the Logarithmic Negativity $E_N$, as explained in Appendix A. It is computed from the covariance matrix, which can be calculated from the propagator $J(X_f, t, X_i, 0)$ (see appendix B). In what follows we will exclusively use this measure.

Entanglement in the time independent case.— In contact with an environment, each particle is asymptotically forced into a thermal state with a temperature equal to that of the bath it is connected to. This state is reached independently on the initial condition of the oscillator, which in the case of no driving $c_1 = 0$ in [3] leads to the entanglement characteristics shown in figure 1. That is, any state will, after thermalization, fall into either the blue (entangled) part or the white (separable) part, depending only on the ratio $c_0/m_0\omega^2$ and the bath’s temperature $k_B T$. The entanglement region is restricted to the so called quantum limit $\hbar \omega < k_B T$, as expected from intuition, above such a temperature each oscillator has an independent description because the quantum state is separable.

![FIG. 1: Generation of an entangled nonequilibrium state with dissipative environments.](image)

**Entanglement creation by driving—** We sketch here a simple idea of how to produce an entangled nonequilibrium state at high temperatures. It may provide a huge leap in experimental requirements, while in addition it definitely removes temperature from the list of possible criteria for classicality, the latter being an important theoretical topic. The normal mode transformation for the oscillator Hamiltonian $H = \sum_{\alpha} p_{\alpha}^2/2m_\alpha + c_\alpha Q_\alpha^2/2$ where $Q_\alpha^2 = (Q_1 + Q_2)/\sqrt{2}$ and $P_\alpha^2 = (P_1 + P_2)/\sqrt{2}$ and $\omega_\alpha^2 = \omega + c(t)/m$. In the continuous variable setting, it is known that the maximally entangled state -a kind of reference state, which is close to a thermal state-is the Einstein, Podolsky, Rosen wavefunc-
It is just the infinite squeezing limit of the two-mode squeezed vacuum state, in which the indeterminacies of \( Q_+ \) and \( P_+ \) are under the standard quantum limit set by Heisenberg’s principle, while \( Q_- \) and \( P_- \) are above it (such that \( \Delta Q_+ / \Delta P_+ = \exp (\mp 2r) / \omega^2 \), with \( r \) the so-called squeezing parameter). The opposite situation is also valid. Thus generation of entanglement can be provided by squeezing of the normal modes, which in turn can be generated through parametric driving of their frequencies [20]. Both a time dependence in \( \omega \) or \( c \) will do, however the latter is better because it naturally provides the correct combination of squeezing between \( \pm \) modes. At the same time, the environment will try to destroy quantum coherence through equilibration to the thermal state. Thus we have two competing effects, whose balance will determine whether the steady state is entangled or not. In figure 1 we provide an example of normal mode squeezing in presence of the bath above the typical quantum limit \( k_B T / \hbar \omega = 10 > 1 \).

In figure 2 we summarize our results. Indeed, we find sets of parameters where entanglement is present at temperatures beyond the quantum limit, notice that in both figures \( k_B T > \hbar \omega \). Starting with a thermal state at the bath’s temperature, the system reaches after a certain time a nonequilibrium steady state with nonzero entanglement. We have chosen rather conservative couplings to the baths, as we will explain later, and still very high temperatures, \( k_B T \gg \hbar \omega \), can be reached.

It is a remarkable fact that while the system is forced into a highly nonequilibrium state, a steady state of entanglement is reached which is independent on the initial state of the system. To show this effect we plot in figure 3 (see inset) the time evolution of entanglement when the system starts with a two mode squeezed state and squeezing parameters \( r = 0, 0.5, 1 \), and compare it to the case of an initial thermal state with the same temperature as the bath.

*New 'phase diagram' for entanglement*— Parametric driving yields a new asymptotic behaviour which defines a new 'phase diagram', now dependent on four parameters: driving amplitude and frequency, temperature and the coupling to the bath. The driving frequency is overall chosen to be \( \omega_d = 2 \times 0.998 \omega \), and we also set \( c_0 = 0 \). While the optimal squeezing generation is obtained with a \( \omega_d \) dependent on \( \omega \) and \( c_1 \), the latter number seems to produce results nearly as good for different parameters, so it will be used unless otherwise stated. In figure 3 we see the points which delimit the border between presence(left)/absence(right) of entanglement, which is linear in temperature and driving amplitude and, as expected, the more isolated and driven the system is (low \( \gamma \) and high \( c_1 \)), the higher the temperature can be reached. In addition to the exact result, we have plotted a simple estimation of the border which we explain next.

We already mentioned that the entanglement production in this system can be viewed as a competition between the squeezing due to the driving and *mixing* because of the environment. The rate of squeezing can be obtained from the solutions to the nondissipative driven problem. They have the Mathieu form \( x(t) = \exp (i \mu_M t) \phi(t) \), where \( \phi(t) \) is a periodic function. If the Mathieu characteristic exponent \( \mu_M \) is real, they are stable, otherwise they are divergent which implies production of squeezing at a rate \( |\text{Im}(\mu_M)| \) (for every damped solution there is a divergent one) [21]. The rate of decoherence can be estimated from the diffusion coefficient \( D = \mu_M^2 / \omega_d \) (see Appendix B), yielding \( \gamma D \sim \gamma k_B T / \hbar \omega \) whenever \( k_B T > \hbar \omega \). Thus by comparison of both rates we obtain the new condition under which entanglement is present:

\[
\frac{k_B T}{\hbar \omega} \leq \frac{|\text{Im}(\mu_M)|}{\gamma},
\]

which is seen to be a rather impressive match to the exact evolution. The condition above should be compared with the *standard condition* [1]. In a nutshell the driving brings in a new quantum limit.

*Some examples*— We give next some actual examples of experiments which could profit from our strategy. However an additional comment is in order: the fact that squeezing grows approximately as \( |\text{Im}(\mu_M)| t \) also means that the energy and delocalization in space are increasing exponentially in time. Thus checking consistency with experimental size and energy considerations is a must.

![Figure 2: Time evolution of entanglement](image)
FIG. 3: 'Phase diagram’ of entanglement in the presence of parametric driving. We compare the condition [8] [lines] with the exact time evolution [dots] for different bath couplings $\gamma = 0.005\omega$ (blue triangles), $0.001\omega$ (green circles) and $0.0005\omega$ (red squares). Inset: time evolution for different initial conditions, namely a two mode squeezed vacuum state (dotted curves) with squeezing parameter $r = 0$ (red), $0.5$ (blue), $1$ (green), as compared to that of an initial thermal state (black). They all converge after some tens of periods. The parameters here are $\gamma = 0.001\omega$, $c_1 = 0.2\omega m^2$, $\omega_d = 2 \times 0.9908\omega$ and $k_B T / h\omega = 10$.

Take for example two Calcium ions, each confined in its own planar Penning traps [22]. A trap can be fabricated by nanolithography with a size of $d \sim 0.12\mu m$. If a voltage of $V = 10V$ is applied, the motional frequency is $\omega \approx 21GHz$ and thus we can interpret figure 1 as the temperature in Kelvin. A wire mediated capacitive coupling between traps allows to reduce the effective distance between ions and makes the coupling increase up to a reasonable level $c(t) = c_0 = 0.047m\omega_d^2$. If the frequencies are driven instead of the coupling (i.e. $\omega(t) = \omega_0 + \omega_1 \sin \omega_dt$), and assuming $\gamma = 0.0005\omega$, we still manage to get entanglement up to $\sim 50K$, while the delocalization of the oscillators is yet below the trap size, ensuring no confinement temperature leakage. To reach room temperature a very strong coupling would be required indeed, but our method allows the experimentalist to avoid building a sub-4K (liquid Helium) setup. We believe this to be a huge experimental step.

Another example is microwave superconducting cavities [23]. The coupling between two cavities can be modulated placing a superconducting qubit between them. The effective frequencies in these resonators are in the GHz regime, operating usually in the millikelvin range. The decoherence in these systems is $\gamma \approx 10^{-5}\omega$, or even less. However the coupling is weak, around 10MHz. In this case, due to the weak coupling, the parametric driving would enhance the amount entanglement that could be measured by nowadays technology [24].

Current experiments with nanomechanical resonators have these typical parameters: $\omega = 2\pi n = 2\pi \times 15MHz$, $m = 10^{-17}kg$, $c_1 \sim 10^{-4}m^2$, and a quality factor $Q \sim 20000$, which yields a damping $\gamma = 5 \times 10^{-5}\omega$ [23]. An entangled state can be observed at 2K. If the frequency can be increased a factor 10, it might reach the entangled regime in presence of liquid Helium.

In addition it is notable that the strong coupling regime has been reached between a massive mechanical microresonator and light [9]. Furthermore, a proposal for parametrically driving the coupling between a nanomechanical resonator and a superconducting electrical resonator has been given in [29]. Thus we might well foresee that these advances could be used to measure entanglement in yet unsuspected temperature regimes in the near future, while eliminating the need for complex and costly setups to cool objects to the quantum regime.

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[30] The phase diagram depends slightly on the dissipation strenght. For weak dissipation, as in our case, the equilibrium phase diagram is mostly independent on $\gamma$ [29]. In figure 1b we used $\gamma = 0.005\omega$.
[31] Notwithstanding each of the oscillators might be still regarded as quantum up to yet higher temperatures, we focus on entanglement since it underlies the very heart of the quantum weirdness.
Appendix A: Entanglement quantification

Entanglement can be easily quantified for a bipartite system of continuous variables in a Gaussian state. The logarithmic negativity [27] gives a characterization of the amount of entanglement which can be distilled into singlets. In the case of Gaussian continuous variable states, only the covariance matrix is needed. The covariance matrix \( \sigma \) is defined as

\[
\sigma_{ij} = \langle \xi_i \xi_j + \xi_j \xi_i \rangle / 2 - \langle \xi_i \rangle \langle \xi_j \rangle
\]

where \( \xi_i = Q_i, Q_2, P_1, P_2 \). The logarithmic negativity is defined as

\[
E_N = -\frac{1}{2} \sum_{i=1}^{4} \log_2[\text{Min}(1, 2|l_i|)]
\]

where \( l_i \) are the symplectic eigenvalues of the covariance matrix. They are simply the normal eigenvalues of the matrix \(-i \Sigma \sigma\), with \( \Sigma \) the symplectic matrix

\[
\sigma = \begin{pmatrix}
0 & 1_2 \\
-1_2 & 0
\end{pmatrix}
\]

and \( 1_2 \) is the \( 2 \times 2 \) identity matrix.

Whenever the logarithmic negativity of the system is zero, we have a separable state \( \rho_s = \sum_i p_i \rho_1^{(i)} \otimes \rho_2^{(i)} \), and each oscillator can be described independently. In continuous variable systems, the amount of entanglement is unbounded from above, having as a limiting case the maximally entangled EPR wavefunction with \( E_N \rightarrow \infty \).

Appendix B: Decoupling the total system in normal modes

The Hamiltonian of the total system reads

\[
H_s = \frac{p_1^2}{2m} + \frac{1}{2} m \omega^2 q_1^2 + \frac{p_2^2}{2m} + \frac{1}{2} m \omega_2^2 q_2^2 + c(t) q_1 q_2,
\]

\[
H_{1B} = \sum_{k=1}^{N} \frac{1}{2m_k} p_k^2 + \frac{1}{2} m_k \omega_k^2 \left( x_k - \frac{c_k q_1}{m_k \omega_k^2} \right)^2 + \sum_{k=1}^{N} \frac{1}{2m_k'} p_k'^2 + \frac{1}{2} m_k' \omega_k'^2 \left( x_k' - \frac{c_k' q_2}{m_k' \omega_k'^2} \right)^2,
\]

where \( c(t) = mc_0 + mc_1 \cos(\omega_0 t) \). Introducing the normal modes coordinates \( x_+ \) and \( x_- \) defined by

\[
q_1 = \frac{1}{\sqrt{2}} (x_+ + x_-), \quad p_1 = \frac{1}{\sqrt{2}} (p_+ + p_-),
\]

\[
q_2 = \frac{1}{\sqrt{2}} (x_+ - x_-), \quad p_2 = \frac{1}{\sqrt{2}} (p_+ - p_-),
\]

\( H_{1B} \) reads

\[
H_S = \frac{p_+^2}{2m} + \frac{1}{2} m \Omega_+^2(t) x_+^2 + \frac{p_-^2}{2m} + \frac{1}{2} m \Omega_-^2(t) x_-^2,
\]

where \( \Omega_{\pm}^2(t) = \omega^2 \pm c(t)/m \) and \( H_{1B} \)

\[
H_{1B} = \sum_{k=1}^{N} \frac{1}{2m_k} p_k^2 + \frac{1}{2} m_k \omega_k^2 x_k^2 - \frac{1}{\sqrt{2}} c_k x_k (x_+ + x_-) + \frac{c_k^2}{2 \sqrt{2 m_k \omega_k^2}} (x_+ + x_-)^2
\]

\[
+ \sum_{k=1}^{N} \frac{1}{2m_k'} p_k'^2 + \frac{1}{2} m_k' \omega_k'^2 x_k'^2 - \frac{1}{\sqrt{2}} c_k' x_k' (x_+ - x_-) + \frac{c_k'^2}{2 \sqrt{2 m_k' \omega_k'^2}} (x_+ - x_-)^2.
\]

These coordinates introduce a cross-term \( x_+ x_- \), which cancels out if

\[
\frac{c_k^2}{m_k \omega_k^2} = \frac{c_k'^2}{m_k' \omega_k'^2}.
\]
This requirement does not mean that the oscillators in the baths are identical but their modes distributions. In the continuous limit, it implies that the spectral distributions characterizing the baths, \( J_1(\omega) \) and \( J_2(\omega) \), are the same. In this case,

\[
H_{IB} = \sum_{k=1}^{N} \left\{ \frac{1}{2m_k} p_k^2 + \frac{1}{2} m_k \omega_k^2 x_k^2 + \frac{1}{2} m'_k \omega_k'^2 x'_k^2 + \frac{c_k^2}{\sqrt{2m_k\omega_k^2}} x_k^2 + \frac{c_k^2}{\sqrt{2m_k\omega_k'^2}} x_k'^2 - \left( \frac{1}{\sqrt{2}} c_k x_k + \frac{1}{\sqrt{2}} c'_k x'_k \right) x_+ - \left( \frac{1}{\sqrt{2}} c_k x_k - \frac{1}{\sqrt{2}} c'_k x'_k \right) x_- \right\} .
\]

(B8)

This expression suggests the introduction of new set of coordinates \( q_k \) and \( \Omega_k \) defined by

\[
q_k = \frac{1}{\Lambda_k \sqrt{2}} (c_k x_k + c'_k x'_k) , \quad \Omega_k = \frac{1}{\Lambda_k \sqrt{2}} (c_k x_k - c'_k x'_k) ,
\]

(B9)

which can be inverted

\[
x_k = \frac{1}{\sqrt{2} \Lambda_k} (\lambda_k q_k + \Lambda_k \Omega_k) , \quad x'_k = \frac{1}{\sqrt{2} \Lambda'_k} (\lambda_k q_k - \Lambda_k \Omega_k) ,
\]

(B10)

\[
p_k = \frac{1}{\sqrt{2} \Lambda_k} (\lambda_k p_k + \Lambda_k \Psi_k) , \quad p'_k = \frac{1}{\sqrt{2} \Lambda'_k} (\lambda_k p_k - \Lambda_k \Psi_k).
\]

(B11)

After substituting in \( H_{IB} \) and choosing \( m_k c_k^2 = m'_k c'_k^2 \) to eliminate a term proportional to \( p_k \Psi_k \), we have

\[
H_{IB} = \sum_{k=1}^{N} \left\{ \frac{\lambda_k^2}{2m_k c_k^2} p_k^2 + \frac{m_k \omega_k^2 \lambda_k^2}{2c_k^2} q_k^2 - \lambda_k q_k x_+ + \frac{\Lambda_k^2}{2m_k c_k^2} \Psi_k^2 + \frac{m_k \omega_k^2 \Lambda_k^2}{2c_k^2} \Omega_k^2 - \Lambda_k \Omega_k x_- \right\} .
\]

(B12)

To obtain a more standard version of the Hamiltonian, we could redefine \( m_k \to \lambda_k^2 / c_k^2 m_k \) and \( \omega_k^2 = c_k^2 \omega_k^2 / \lambda_k^2 \) and impose \( \lambda_k^2 = \Lambda_k^2 \) or just by choosing \( \lambda_k^2 = c_k^2 = \Lambda_k^2 \), so

\[
H_{IB} = \sum_{k=1}^{N} \left\{ \frac{1}{2m_k} p_k^2 + \frac{m_k \omega_k^2}{2} q_k^2 - c_k q_k x_+ + \frac{1}{2m_k} \Psi_k^2 + \frac{m_k \omega_k^2}{2} \Omega_k^2 - c_k \Omega_k x_- \right\} .
\]

(B13)

It means that we can conserve a small arbitrariness in the phase of the coupling by choosing different signs by \( \lambda_k \) and \( \Lambda_k \). However, for convenience we choose ‘+’ for both.

In summary, we have

\[
\mathcal{H} = \frac{p_+^2}{2m} + \frac{1}{2} m \Omega_+ x_+^2 + \frac{p_-^2}{2m} + \frac{1}{2} m \Omega_- x_-^2 + \sum_{k=1}^{N} \left\{ \frac{1}{2m_k} p_k^2 + \frac{m_k \omega_k^2}{2} q_k^2 - c_k q_k x_+ + \frac{1}{2m_k} \Psi_k^2 + \frac{m_k \omega_k^2}{2} \Omega_k^2 - c_k \Omega_k x_- \right\} ,
\]

(B14)

or

\[
\mathcal{H} = \frac{p_+^2}{2m} + \frac{1}{2} m \Omega_+ x_+^2 + \sum_{k=1}^{N} \left\{ \frac{1}{2m_k} p_k^2 + \frac{m_k \omega_k^2}{2} q_k^2 - c_k q_k x_+ \right\} \\
+ \frac{p_-^2}{2m} + \frac{1}{2} m \Omega_- x_-^2 + \sum_{k=1}^{N} \left\{ \frac{1}{2m_k} \Psi_k^2 + \frac{m_k \omega_k^2}{2} \Omega_k^2 - c_k \Omega_k x_- \right\} .
\]

(B15)

It is quite trivial, but we have derived an effective microscopic description of our initial assumption: normal modes coupled to identical but independent baths. Itworths to be mentioned that not only the baths have the same modes, \( \frac{c_k^2}{m_k \omega_k^2} = \frac{c'_k^2}{m_k \omega_k'^2} \), but also the coupling between the system and the bath is the same, \( \lambda_k = + c_k = \Lambda_k \).

In order to complete our program an important point is left, if we want that the propagating function factorize, \( J[x_+, x_-, x'_+, x'_-] = J[x_+, x'_+] J[x_-, x'_-] \), obtaining that each normal mode evolves actually in an independent way, we
have to verify that the product by pairs of the equilibrium density matrix of the baths modes remains uncorrelated in the new coordinates. In the current case, the transformation of coordinates reads

\[ x_k = \frac{1}{\sqrt{2}} (q_k + \Omega_k), \quad x'_k = \frac{1}{\sqrt{2}} (q_k - \Omega_k), \]  
\[ p_k = \frac{1}{\sqrt{2}} (p_k + \Psi_k), \quad p'_k = \frac{1}{\sqrt{2}} (p_k - \Psi_k), \]

then, the product of the equilibrium density matrix of the \( k \)-th mode of each bath reads

\[
\frac{1}{Z^k} \left( \frac{m_k \omega_k}{2\pi \hbar \sinh(\omega_k \hbar \beta)} \right)^{\frac{1}{2}} \exp \left[ -\frac{m_k \omega_k}{2\pi \hbar \sinh(\omega_k \hbar \beta)} \left( (x_{i,k}^2 + x_{i,k}^{'2}) \cosh(\omega_k \hbar \beta) - 2x_{i,k}x_{i,k}^{'} \right) \right] \\
\times \frac{1}{Z^k} \left( \frac{m_k' \omega_k'}{2\pi \hbar \sinh(\omega_k' \hbar \beta)} \right)^{\frac{1}{2}} \exp \left[ -\frac{m_k' \omega_k'}{2\pi \hbar \sinh(\omega_k' \hbar \beta)} \left( (x_{i,k}^2 + x_{i,k}^{'2}) \cosh(\omega_k \hbar \beta) - 2x_{i,k}x_{i,k}^{'} \right) \right] \\
\rightarrow \frac{1}{Z^k} \left( \frac{m_k \omega_k}{2\pi \hbar \sinh(\omega_k \hbar \beta)} \right)^{\frac{1}{2}} \exp \left[ -\frac{m_k \omega_k}{2\pi \hbar \sinh(\omega_k \hbar \beta)} \left( (\Omega_{i,k}^2 + \Omega_{i,k}^{'2}) \cosh(\omega_k \hbar \beta) - 2\Omega_{i,k} \Omega_{i,k}^{'} \right) \right].
\]

To obtain this desired result, we had to impose \( m_k = m_k' \) and \( \omega_k = \omega_k' \). So, it reduces our baths to be equal in detail, we mean, oscillator by oscillator. Only at this point we can affirm that the normal modes will evolve independently. This result for the bath modes can be interpret in geometrical terms as follows: the isopotential lines of two uncoupled identical harmonic are defined by circumferences, so they are invariant under any rotation, which imply that the dynamical quantities obey exactly the same motion equations. It is important to mention that the normal modes are coupled to the bath in different coordinates than the real modes, however the introduction of the normal modes for the bath leaves the Jacobian of the transformation equals to 1, so after the trace the will generate completely equivalent results.

**Appendix C: Propagating function for the density matrix**

In normal modes, the evolution of the density matrix is governed by,

\[
\rho(x_+, y_+, x_-, y_-, t) = \int dx_+ \int dy_+ \int dx_- \int dy_- J(x_+, y_+, x_-, y_-, t | x_+, y_+, x_-, y_-, 0) \rho(x_+, y_+, x_-, y_-, t),
\]

where \( J(x_+, y_+, x_-, y_-, t | x_+, y_+, x_-, y_-, 0) \) is the propagator of the reduced density matrix,

\[
J(x_+, y_+, x_-, y_-, t | x_+, y_+, x_-, y_-, 0) = \int Dx_+ \int Dy_+ \int Dx_- \int Dy_- \exp \left\{ \frac{i}{\hbar} S[x_+, x_-] - S[y_+, y_-] \right\} F[x_+, y_+, x_-, y_-],
\]

where \( S[x_+, x_-] \) is the classical action and \( F[x_+, y_+, x_-, y_-] \) the influence functional. \( Dx \) denotes an infinite product of measures in configuration space and implies a path integration over the paths \( x_+(t), y_+(t), x_-(t) \) and \( y_-(t) \) with endpoints \( x_+(0) = x_+ \), \( y(0) = y_+ \), \( x_-(0) = x_- \), \( y-(0) = y_- \), \( x_+(t) = x_+ \), \( y(t) = y_+ \), \( x_-(t) = x_- \) and \( y-(t) = y_- \). However, at this point we have decoupled our system and we are describing it by two different harmonic oscillators coupled to identical but independent baths. So,

\[
\rho(x_+, y_+, x_-, y_-, t) = \int dx_+ \int dy_+ \int dx_- \int dy_- J_+(x_+, y_+, x_-, y_-, t | x_+, y_+, x_-, y_-, 0) J_-(x_-, y_-, x_+, y_+, t | x_-, y_-, x_+, y_+, 0) \rho(x_+, y_+, x_-, y_-, t),
\]

with

\[
J_\pm(x_+, y_+, x_-, y_-, t | x_+, y_+, x_-, y_-, 0) = \int Dx_\pm \int Dy_\pm \exp \left\{ \frac{i}{\hbar} (S_\pm[x_\pm] - S_\pm[y_\pm]) \right\} F[x_\pm, y_\pm].
\]
For the case of a bath modeled by harmonic oscillators \cite{13}, the general result for $F[x_+, y_+]$ was derived by Caldeira and Leggett \cite{14} and it reads

$$F[x_+, y_+] = \exp \left\{ -\frac{i}{\hbar} \frac{m}{2} \left[ (x_+ + y_+ - y_f) \int_0^t ds \gamma(s) [x_+(s) - y(s)] + \int_0^t ds \int_0^s du \gamma(s - u) [\dot{x}_+(u) + \dot{y}(u)] [x_+(s) - y(s)] \right] \right\} \times \exp \left\{ -\frac{i}{\hbar} \int_0^t ds \int_0^s du \gamma(s - u) [\dot{y}_+(u) - \dot{y}(u)] [x_+(s) - y(s)] \right\},$$

(C5)

similar expressions stands for the $F[x_-, y_-]$ mode, $K(s)$ denotes the noise kernel

$$K(s) = \int_0^\infty \frac{d\omega}{\omega} \coth \left( \frac{\omega \hbar}{2 k_B T} \right) \cos(\omega s) I(\omega),$$

(C6)

wherein $k_B$ denotes the Boltzmann constant and $T$ the temperature of the bath. The friction kernel $\gamma(s)$ in terms of the spectral density reads

$$\gamma(s) = \frac{2}{m} \int_0^\infty \frac{d\omega}{\omega} \frac{I(\omega)}{\pi} \cos(\omega s), \quad \text{in Ohmic case } \gamma(s) = 2 \gamma_0 \delta(s).$$

(C7)

An identical expression stands for $F[x_-, y_-]$. Since path integrals in $J$ are quadratic, they can be done exactly to yield

$$J = \frac{1}{N_+(t) N_-(t)} \exp \left\{ \frac{i}{\hbar} \left( S_+ [x_+ - y_+ + x_+ (s; \varphi)] + S_- [x_+ - y_+ + x_- (s; \varphi)] \right) \right\} F[x_+, y_+] F[x_-+, y_-],$$

(C8)

being $N_\pm$ a normalization factor determined by the normalization of the propagator. To simplify further expressions, let's us to introduce the center of mass and difference variables, i.e.,

$$q_\pm = x_\pm - y_\pm, \quad Q_\pm = \frac{1}{2} (x_\pm - y_\pm),$$

(C9)

satisfying

$$\dot{q}_\pm(s) = -\gamma q_\pm(s) + \Omega_\pm^2(s; \varphi) q_\pm(s) = -2 q_{f,\pm} \gamma_0(t - s),$$

(C10)

$$\dot{Q}_\pm(s) = \gamma Q_\pm(s) + \Omega_\pm^2(s; \varphi) Q_\pm(s) = -2 Q_{f,\pm} \gamma_0(s).$$

(C11)

It is important to mention that solution to these equations will be valid only for $s > 0$ and it reads \cite{21}

$$q_\pm(s) = v_{1,\pm}(t, s; \varphi) q_{i,\pm} + v_{2,\pm}(t, s; \varphi) q_{f,\pm},$$

(C12)

$$Q_\pm(s) = u_{1,\pm}(t, s; \varphi) Q_{i,\pm} + u_{2,\pm}(t, s; \varphi) Q_{f,\pm}.$$  

(C13)

Since baths are defined by the same spectral density, then note that $\gamma$ is the same for $\pm$ cases. So we have that

$$J(x_+, y_+, x_-, y_-, t|x_+, y_+, x_-, y_-, 0) = \frac{1}{N(t)}$$

$$\times \exp \left[ -\frac{i}{\hbar} \left\{ \left[ b_{3,+}(t; \varphi) q_{+i} - b_{4,+}(t; \varphi) q_{+f} + b_{1,+(t; \varphi) q_{i,+,i} - b_{2,+(t; \varphi) q_{+f}} \right] \right\} \right] \times \exp \left[ -\frac{i}{\hbar} \left\{ \left[ b_{3,-}(t; \varphi) q_{-i} - b_{4,-}(t; \varphi) q_{-f} + b_{1,-(t; \varphi) q_{i,-,i} - b_{2,-(t; \varphi) q_{-f}} \right] \right\} \right]$$

$$\times \exp \left[ -\frac{i}{\hbar} \left\{ \left[ a_{11,+(t; \varphi) q^{2}_{+,i} + a_{12,+(t; \varphi) q_{+i} + a_{21,+(t; \varphi) q_{+i} + a_{22,+(t; \varphi) q^{2}_{+,f} \right] \right\} \right] \times \exp \left[ -\frac{i}{\hbar} \left\{ \left[ a_{11,-(t; \varphi) q^{2}_{-,i} + a_{12,-(t; \varphi) q_{-i} + a_{21,-(t; \varphi) q_{-i} + a_{22,-(t; \varphi) q^{2}_{-,f} \right] \right\} \right]$$

(C14)

where $N(t) = N_+(t) N_-(t)$,

$$a_{i,j,\pm} = \frac{1}{2} \int_0^t ds_1 \int_0^t ds_2 v_{i,\pm}(t, s_1; \varphi) v_{j,\pm}(t, s_2; \varphi) K(s_1 - s_2),$$

(C15)
and

\begin{align}
    b_{1,\pm}(t; \varphi) &= \dot{u}_{1,\pm}(t, 0; \varphi) + \gamma, & b_{2,\pm} &= \dot{u}_{1,\pm}(t, t; \varphi), \\
    b_{3,\pm}(t; \varphi) &= \dot{u}_{2,\pm}(t, 0; \varphi), & b_{4,\pm} &= \dot{u}_{2,\pm}(t, t; \varphi),
\end{align}

(C16)

(C17)

Using last definitions we can express \( N_\pm \) as \( N_\pm = \frac{2\pi\hbar}{b_{3,\pm}(t)} \). Next step is the derivation of the master equation. We based our calculation on the paper of Zerbe and Hángi [21] where the authors derived the exact quantum master equation for a single driven harmonic oscillator.

**Appendix D: Quantum Master Equation (QME)**

Quantum master equation for the normal modes of the initial system reads

\[
    i\hbar \frac{\partial}{\partial t} \rho(x_+, y_+, x_-, y_-) = \left[ -\frac{\hbar^2}{2m} \left( \frac{\partial^2}{\partial x_+^2} - \frac{\partial^2}{\partial y_+^2} \right) + \frac{m}{2} \Omega_+^2(t; \varphi)(x_+^2 - y_+^2) \right] \rho(x_+, y_+, x_-, y_-)
    + \left[ -\frac{\hbar^2}{2m} \left( \frac{\partial^2}{\partial x_-^2} - \frac{\partial^2}{\partial y_-^2} \right) + \frac{m}{2} \Omega_-^2(t; \varphi)(x_-^2 - y_-^2) \right] \rho(x_+, y_+, x_-, y_-)
    - i\hbar \gamma (x_+ - y_+ \left( \frac{\partial}{\partial x_+} - \frac{\partial}{\partial y_+} \right) \rho(x_+, y_+, x_-, y_-) + iD_{+, pp}(t, 0)(x_+^2 - y_+^2)\rho(x_+, y_+, x_-, y_-)
    - i\hbar \gamma (x_- - y_- \left( \frac{\partial}{\partial x_-} + \frac{\partial}{\partial y_-} \right) \rho(x_+, y_+, x_-, y_-) + iD_{-, pp}(t, 0)(x_-^2 - y_-^2)\rho(x_+, y_+, x_-, y_-)
    - \frac{\hbar}{m} [D_{+, xp}(t, 0) + D_{+, px}(t, 0) \left( \frac{\partial}{\partial x_+} + \frac{\partial}{\partial y_+} \right) \rho(x_+, y_+, x_-, y_-)
    - \frac{\hbar}{m} [D_{-, xp}(t, 0) + D_{-, px}(t, 0) \left( \frac{\partial}{\partial x_-} + \frac{\partial}{\partial y_-} \right) \rho(x_+, y_+, x_-, y_-),
\]

\( D_{\pm, pp}(t, 0) = 2 \left( b_{4,\pm} + \frac{b_{2,\pm}^2}{b_{2,\pm}} \right) a_{22,\pm} - a_{22,\pm} + 2 \frac{b_{2,\pm} b_{4,\pm}}{b_{2,\pm} b_{3,\pm}} - \frac{b_{4,\pm}}{b_{3,\pm}} a_{12,\pm},
\]

\( D_{\pm, px}(t, 0) = D_{\pm, xp}(t, 0) = \frac{1}{b_{3,\pm}} a_{12,\pm} + a_{22,\pm} + \frac{b_{2,\pm}}{b_{2,\pm} b_{3,\pm}} a_{12,\pm}.
\]

For small values of \( \hbar \), \( D_{\pm, px}(t, 0) \) and \( D_{\pm, pp}(t, 0) \) can be written as [28]

\[
    D_{\pm, pp}(t, 0) = \frac{m\gamma}{\beta} + \frac{2m^2\gamma\Lambda}{\beta} (\Omega_+^2(t) - \gamma^2),
\]

\[
    D_{\pm, px}(t, 0) = \frac{2m\gamma^2\Lambda}{\beta},
\]

where \( \Lambda = \hbar^2 \beta^2/24m \).

**Appendix E: Mean values and variances**

\[
    \langle f(x_\pm) \rangle = \int dQ_{f,\pm} f(Q_{f,\pm}) \rho(Q_{f,\pm}, q_{f,\pm}, 0, t)
\]

(E1)

The first moments read in terms of the initial values \( \langle x_\pm(t_0 = 0) \rangle = \langle (x_\pm, 0) \rangle \) and,

\[
    \langle x_\pm(t) \rangle = \langle f_{2,\pm}(t) - \frac{\gamma}{2} f_{1,\pm}(t) \rangle \langle (x_\pm, 0) \rangle + \frac{1}{m} f_{1,\pm}(t) \langle (p_\pm, 0) \rangle
\]

(E2)
\[ \langle p_\pm(t) \rangle = m \frac{d}{dt} \langle x_\pm(t) \rangle \]  
\[ = m [\dot{f}_{2,\pm}(t) - \frac{\gamma}{2} \dot{f}_{1,\pm}(t)] \langle (x_{\pm,0}) \rangle + \frac{1}{m} \dot{f}_{1,\pm}(t) \langle (p_{\pm,0}) \rangle \]  
\[ \tag{E3} \]  
\[ \text{The evolution of } \langle p_\pm(t) \rangle \text{ is discontinuous at } t_0 = 0, \text{ i.e., } \lim_{t \to 0^+} \langle p_\pm(t) \rangle = \langle p_{\pm,0} \rangle - m \gamma \langle x_{\pm,0} \rangle / 2 \text{ es in general not equal to } \langle p_{\pm,0} \rangle. \text{ This instantaneous jump of } \langle p_\pm(t) \rangle \text{ can be removed with an environmental cutoff } \omega_c, \text{ or a non-factorizing initial state } \langle \text{18} \rangle. \]  
\[ \text{The variances are obtained accordingly. They are given by} \]
\[ \sigma_{x_\pm x_\pm}(t) = \left( f_{2,\pm} - \frac{\gamma}{2} f_{1,\pm} \right)^2 \sigma_{x_\pm x_\pm}^0 + \frac{2}{m} f_{1,\pm} \left( f_{2,\pm} - \frac{\gamma}{2} f_{1,\pm} \right) \sigma_{x_\pm p_\pm}^0 + \frac{1}{m^2} f_{1,\pm}^2 \sigma_{p_\pm p_\pm}^0 + \frac{2h}{m} f_{1,\pm} a_{11,\pm}, \]  
\[ \tag{E5} \]  
\[ \sigma_{x_\pm p_\pm}(t) = m \left[ f_{2,\pm} f_{2,\pm} - \frac{\gamma}{2} \left( f_{1,\pm} f_{2,\pm} + \dot{f}_{1,\pm} f_{2,\pm} - \frac{\gamma}{2} f_{1,\pm} \dot{f}_{1,\pm} \right) \right] \sigma_{x_\pm x_\pm}^0 + \left( f_{1,\pm} f_{1,\pm} + \dot{f}_{1,\pm} f_{1,\pm} + \gamma \dot{f}_{1,\pm} f_{1,\pm} - \frac{\gamma}{2} \dot{f}_{1,\pm} \dot{f}_{1,\pm} \right) \sigma_{x_\pm p_\pm}^0 + \frac{1}{m^2} f_{1,\pm}^2 \sigma_{p_\pm p_\pm}^0 + 2h \left( f_{1,\pm} \dot{f}_{1,\pm} a_{11,\pm} + f_{1,\pm} a_{12,\pm} \right), \]  
\[ \tag{E6} \]  
\[ \sigma_{p_\pm p_\pm}(t) = m^2 \left( f_{2,\pm} - \frac{\gamma}{2} \dot{f}_{1,\pm} \right)^2 \sigma_{x_\pm x_\pm}^0 + 2m f_{1,\pm} \left( f_{2,\pm} - \frac{\gamma}{2} \dot{f}_{1,\pm} \right) \sigma_{x_\pm p_\pm}^0 + \frac{\dot{f}_{1,\pm}^2}{m^2} \sigma_{p_\pm p_\pm}^0 + 2m \left( f_{1,\pm}^2 a_{11,\pm} + 2 \dot{f}_{1,\pm} a_{12,\pm} + a_{22,\pm} \right), \]  
\[ \tag{E7} \]  
where we omitted the arguments of the functions \( a_{ij,\pm} \) and \( f_{i,\pm} \) for better lucidity. Here we note two missprints in \[\text{21}, \] one is the presence of a global factor \( \frac{1}{2} \) in the last term of \( \sigma_{x_\pm p_\pm} \) and the other is in the last term of \( \sigma_{x_\pm p_\pm} \), in \[\text{21} \] it reads \( 2 \hbar m \left( 2 \dot{f}_{1,\pm}^2 a_{11,\pm} + \dot{f}_{1,\pm} a_{12,\pm} + a_{22,\pm} \right) \). Due to the discontinuity at \( t = 0 \), variances at \( t = 0^+ \) jump to
\[ \sigma_{x_\pm x_\pm}(t_0^+) = \sigma_{x_\pm x_\pm}^0, \]  
\[ \sigma_{x_\pm p_\pm}(t_0^+) = -\gamma \sigma_{x_\pm x_\pm}^0 + \sigma_{x_\pm p_\pm}^0, \]  
\[ \sigma_{p_\pm p_\pm}(t_0^+) = \gamma^2 \sigma_{x_\pm x_\pm}^0 - 2\gamma \sigma_{x_\pm p_\pm}^0 + \sigma_{p_\pm p_\pm}^0, \]  
\[ \tag{E8,E9,E10} \]  
where \( t_0^+ \) means \( \lim t \to 0^+ \).