NON-CONVEX GEOMETRY OF NUMBERS
AND CONTINUED FRACTIONS

NICKOLAS ANDERSEN, WILLIAM DUKE, ZACH HACKING, AND AMY WOODALL

ABSTRACT. In recent work, the first two authors constructed a generalized continued fraction called the $p$-continued fraction, characterized by the property that its convergents (a subsequence of the regular convergents) are best approximations with respect to the $L^p$ norm, where $p \geq 1$. We extend this construction to the region $0 < p < 1$, where now the $L^p$ quasinorm is non-convex. We prove that the approximation coefficients of the $p$-continued fraction are bounded above by $\frac{1}{\sqrt{5}} + \varepsilon_p$, where $\varepsilon_p \to 0$ as $p \to 0$. In light of Hurwitz’s theorem, this upper bound is sharp, in the limit. We also measure the maximum number of consecutive regular convergents that are skipped by the $p$-continued fraction.

1. Introduction

A rational number $r/s$ with $r, s \in \mathbb{Z}$ and $s > 0$ is said to be a best approximation to an irrational real number $\alpha$ if, for all rationals $r'/s' \neq r/s$ with $0 < s' < s$, we have

$$|r - s\alpha| < |r' - s'\alpha|.$$ 

Lagrange [3] showed that each irrational $\alpha$ has infinitely many best approximations $\{p_n/q_n\}_{n=1}^{\infty}$ and that for each of them

$$q_n|p_n - q_n\alpha| < 1.$$ 

The best approximations are given explicitly by the convergents

$$\frac{p_n}{q_n} = b_0 + \frac{1}{b_1 + \frac{1}{b_2 + \cdots + \frac{1}{b_n}}}$$

of the regular continued fraction expansion of $\alpha$:

$$\alpha = b_0 + \frac{1}{b_1 + \frac{1}{b_2 + \cdots + \frac{1}{b_n}}} := b_0 + \frac{1}{b_1 + \frac{1}{b_2 + \cdots}}.$$ 

Classical results of Vahlen and Borel (Theorems 5A and 5B of [8, Ch. I]) state that among any successive pair of convergents, there is at least one that satisfies $q_n|p_n - q_n\alpha| < \frac{1}{2}$ and among any successive triple, there is at least one that satisfies the Hurwitz bound $q_n|p_n - q_n\alpha| < \frac{1}{\sqrt{5}}$.

The notion of best approximation has the following generalization. For $(x, y) \in \mathbb{R}^2$ and a fixed $0 < p < \infty$, let

$$F^{(p)}(x, y) = (|x|^p + |y|^p)^{\frac{1}{p}},$$

(1.1)
while \( F^{(\infty)}(x, y) = \max\{|x|, |y|\} \). If \( p \geq 1 \) then \( F^{(p)} \) gives a norm on \( \mathbb{R}^2 \) while if \( 0 < p < 1 \) it is a quasinorm in that it only satisfies the weakened triangle inequality

\[
F^{(p)}(x_1 + x_2, y_1 + y_2) \leq 2^{\frac{1}{p} - 1}(F^{(p)}(x_1, y_1) + F^{(p)}(x_2, y_2)).
\]

For a fixed \( p \), we say that \( r/s \) is an \( L^p \)-best approximation to \( \alpha \) if there is a \( t > 1 \) depending only on \( r/s \) so that for any rational \( r'/s' \neq r/s \)

\[
F^{(p)}_t(s, r - s\alpha) < F^{(p)}_t(s', r' - s'\alpha),
\]

where

\[
F^{(p)}_t(x, y) = F^{(p)}(t^{-1}x, ty).
\]

It is not difficult to show that \( r/s \) is an \( L^\infty \)-best approximation to \( \alpha \) if and only if it is a best approximation to \( \alpha \) in Lagrange’s sense (see Lemma 6.1 in [1]). Generalizing the case \( p = 1 \), which is due to Minkowski [4, 6], it is shown in [1] that for any fixed \( p \geq 1 \) the \( L^p \)-best approximations to \( \alpha \) are those rationals given by the convergents

\[
\frac{r_n}{s_n} = a_0 + \frac{\varepsilon_1}{a_1 + \frac{\varepsilon_2}{a_2 + \cdots \frac{\varepsilon_n}{a_n}}} \quad (\varepsilon_j = \pm 1)
\]

of a uniquely determined semi-regular continued fraction expansion of \( \alpha \), the \( p \)-continued fraction. Furthermore, each such best approximation satisfies

\[
F^{(p)}(s_n, r_n - s_n\alpha) < \Delta_p^{-\frac{1}{2}}
\]

where \( \Delta_p \) is the critical determinant of the unit ball \( B^{(p)} = \{(x, y) \in \mathbb{R}^2 : F^{(p)}(x, y) < 1\} \). The value of \( \Delta_p \) is given in [1, Section 4] (see also the references therein). It is increasing on \( 1 \leq p \leq \infty \) with \( \Delta_1 = \frac{1}{2}, \Delta_2 = \frac{\sqrt{3}}{2} \) and \( \Delta_\infty = 1 \). For any \( 1 \leq p < \infty \), the inequality between arithmetic and geometric means gives

\[
s_n|r_n - s_n\alpha| \leq \left(\frac{s_n^p + |r_n - s_n\alpha|^p}{2}\right)^{\frac{2}{p}} < 4^{-\frac{1}{p}} \Delta_p^{-1}
\]

(1.2)

The right-hand side of this inequality increases from \( \frac{1}{2} \) to 1 as \( p \) goes from 1 to \( \infty \). The convergents \( r_n/s_n \) of the 1-continued fraction, which is Minkowski’s diagonal continued fraction [5], thus satisfy

\[
s_n|r_n - s_n\alpha| < \frac{1}{2}
\]

In fact, Minkowski showed that these \( r_n/s_n \) coincide with those regular convergents \( p_m/q_m \) that satisfy \( q_m|p_m - q_m\alpha| < \frac{1}{2} \). For any \( p \geq 1 \) the sequence of convergents of the \( p \)-continued fraction give a subsequence of the regular convergents, but in general do not give all of those that satisfy \( q_m|p_m - q_m\alpha| < (4^{1/p}\Delta_p)^{-1} \). For details and references see [1].

In this paper we generalize these results to \( L^p \)-best approximations where \( p < 1 \). In view of the above, our prime motivation is to show that the right-hand side in the generalization of (1.2) for these approximations can be as close to \( \frac{1}{\sqrt{8}} \) as desired. This necessitates letting \( p \to 0 \). The associated \( p \)-continued fractions

\[
\alpha = a_0 + \frac{\varepsilon_1}{a_1 + \frac{\varepsilon_2}{a_2 + \cdots \frac{\varepsilon_n}{a_n}}}, \quad \gcd(\varepsilon_n, a_n, \varepsilon_{n+1}) = 1,
\]

(1.3)

that we construct have the property that for a fixed \( p < 1 \) the partial numerators \( \varepsilon_n \) are integers with \( |\varepsilon_n| \leq M_p \) where \( M_p \) depends only on \( p \). However, \( M_p \to \infty \) as \( p \to 0 \). Since now \( F^{(p)} \) is only a quasinorm, the ball \( B^{(p)} \) is not convex and we must apply some results of Mordell and Watson [7, 9] from the non-convex geometry of numbers (see also the book
of Gruber and Lekkerkerker [2] and the references therein). The theorem below summarizes the main properties of the \( p \)-continued fraction.

**Theorem 1.1.** For each \( \delta > 0 \) there exists a \( p = p_\delta \in (0, 1) \) such that for any irrational \( \alpha \) the \( p \)-continued fraction (1.3) of \( \alpha \) has the following properties.

1. The convergents are precisely the best approximations to \( \alpha \) with respect to \( F^{(p)} \).
2. Each convergent \( r_n/s_n \) satisfies \( s_n|r_n - s_n\alpha| < \frac{1}{\sqrt{5}} + \delta \).
3. There exists a constant \( M_p \) (depending only on \( p \)) such that \( |\varepsilon_n| \leq M_p \) for all \( n \).

For any \( p > 0 \) the \( p \)-convergents of \( \alpha \) form a subsequence of the regular convergents. Computation confirms that more regular convergents are skipped by the \( p \)-continued fraction than just those with \( s|r - s\alpha| \) large. We now describe the relationship between the \( p \)-convergents \( r_n/s_n \) and the regular convergents \( p_n/q_n \).

**Theorem 1.2.** Fix \( p \in (0, 1) \) and an irrational \( \alpha \in \mathbb{R} \). Let \( n \) be a positive integer and let \( m \) and \( \ell \) be such that \( r_n/s_n = p_m/q_m \) and \( r_{n+1}/s_{n+1} = p_{m+\ell}/q_{m+\ell} \). Then

\[
\ell \leq \log_\varphi \left( 2^{4/p-1} \sqrt{5} \frac{\Gamma(1 + \frac{1}{p})^2}{\Gamma(1 + \frac{2}{p})} \right),
\]

where \( \varphi = \frac{1}{2}(1 + \sqrt{5}) \).

We now show that as \( p \) tends to zero, arbitrarily many regular convergents can be skipped in the \( p \)-continued fraction. This is the case for every irrational for which the regular continued fraction is 1-periodic. See Table 1.1 for the case \( \alpha = \frac{1}{2}(1 + \sqrt{5}) \).

| \( p \) | First 10 convergents of the \( p \)-continued fraction |
|---|---|
| \infty | 2 3 5 8 13 21 34 55 89 144 |
| 0.5 | 2 5 8 13 21 34 55 89 144 233 |
| 0.4 | 5 8 13 21 34 55 89 144 233 377 |
| 0.3 | 8 13 21 34 55 89 144 233 377 610 |
| 0.25 | 13 21 34 55 89 144 233 377 610 987 |
| 0.2 | 21 34 55 89 144 233 377 610 987 1597 |
| 0.18 | 34 55 89 144 233 377 610 987 1597 2584 |
| 0.16 | 55 89 144 233 377 610 987 1597 2584 4181 |

Table 1.1. Selected values of \( p \) with the first several convergents of the \( p \)-continued fraction for \( \varphi = \frac{1}{2}(1 + \sqrt{5}) \).
**Theorem 1.3.** Let \( a \geq 1 \) be an integer and let \( \alpha = a + \frac{1}{a + \frac{1}{a + \frac{1}{a + \cdots}}} = \frac{1}{2}(a + \sqrt{a^2 + 4}) \). Then for each sufficiently large positive integer \( m \) there exists a \( p \in (0, 1) \) such that \( r_0/s_0 = p_{m}/q_m \).

The paper is organized as follows. In Section 2 we review Mordell’s result on the geometry of numbers for the quasinorms \( F^{(p)} \). Section 3 contains the algorithm that generates the \( p \)-continued fraction and the proof of Theorem 1.1. Sections 4 and 5 address the question of how many of the regular convergents can be skipped by the \( p \)-continued fraction; they contain the proofs of Theorems 1.2 and 1.3.

In Sections 2, 3, and 4, the value of \( p \in (0, 1) \) is fixed, so in those sections we will usually suppress the dependence on \( p \) from the notation. In Section 5, the value of \( p \) is allowed to vary; there, the notation will reflect the dependence on \( p \).

2. **Geometry of numbers for non-convex bodies**

Let \( F = F^{(p)} \), defined in (1.1). When \( p \geq 1 \), \( F \) defines a norm on \( \mathbb{R}^2 \), but when \( p \in (0, 1) \), \( F \) is not a norm because the triangle inequality does not hold. However, for such \( p \) we have the following quasi-triangle inequality, so \( F \) defines a quasinorm.

**Lemma 2.1.** Let \( p \in (0, 1) \). Then for any \( (x_1, y_1), (x_2, y_2) \in \mathbb{R}^2 \), we have

\[
F(x_1 + x_2, y_1 + y_2) \leq 2^{\frac{1}{p} - 1} (F(x_1, y_1) + F(x_2, y_2)).
\]

**Proof.** We start by proving that \( |a + b|^p \leq |a|^p + |b|^p \) for any \( a, b \in \mathbb{R} \). By homogeneity and the usual triangle inequality, it suffices to prove that \((1 + t)^p \leq 1 + t^p \) for \( t \geq 0 \), but this follows from the fact that the derivative of \( t \mapsto (1 + t)^p - t^p - 1 \) is nonpositive when \( p < 1 \).

Using that \( |a + b|^p \leq |a|^p + |b|^p \) we obtain

\[
F(x_1 + x_2, y_1 + y_2) \leq (F(x_1, y_1)^p + F(x_2, y_2)^p)^{\frac{1}{p}}.
\]

Now since \( t \mapsto t^{1/p} \) is convex we have \((a+b)^{1/p} \leq \frac{a^{1/p} + b^{1/p}}{2}\), from which it follows that

\[
(F(x_1, y_1)^p + F(x_2, y_2)^p)^{\frac{1}{p}} \leq 2^{\frac{1}{p} - 1} (F(x_1, y_1) + F(x_2, y_2)),
\]

as desired. \( \square \)

Our starting point for studying best approximations with respect to the quasinorm \( F \) is the following theorem of Mordell [7, Section 8], which we have updated to reflect later work of Watson [9]. If \( L \subseteq \mathbb{R}^2 \) is a full lattice, the determinant of \( L \) equals \( |\text{det } g| \), where \( g \in \text{GL}_2(\mathbb{R}) \) is any matrix whose rows form a \( \mathbb{Z} \)-basis for \( L \).

**Theorem 2.2 (Mordell and Watson).** Let \( p \in (0, 1) \) and let \( (a_p, b_p) \) be the unique solution to the equations

\[
(a + b)^p + (a - b)^p = a^p + b^p,
\]

\[
a^2 + b^2 = 2,
\]

with \( a_p > b_p \). Define \( c_p \) by

\[
c_p = 2^{\frac{1}{p}} F(a_p, b_p).
\]

Then every full lattice has a point \( (x, y) \neq (0, 0) \) satisfying

\[
F(x, y) \leq 2^{\frac{1}{p} - \frac{1}{2}} (\text{det } L)^{\frac{1}{2}} c_p.
\]

Furthermore, for \( p \in (0.3295 \ldots, 1) \) the inequality is sharp.
Note that the maximum value of the function \((a, b) \mapsto a^p + b^p\) subject to the constraint \(a^2 + b^2 = 2\) is 2, and this occurs only when \(a = b = 1\). Thus \(c_p < 1\) for all \(p < 1\). Also, clearly \(a_p\) and \(b_p\) both approach 1 as \(p\) tends to 1, so we have
\[
\lim_{p \to 1^-} c_p = 1. \tag{2.3}
\]

Applying Theorem 2.2 to the lattice \((0,t)\mathbb{Z} + (t^{-1}, -\alpha t)\mathbb{Z}\), we find that for every \(t \geq 1\) there exists a rational \(r/s\) with \(s > 0\) such that
\[
F_t(s, r - s\alpha) \leq 2^{\frac{1}{p}} - \frac{1}{2} c_p. \tag{2.4}
\]

Define
\[
\beta_p = 2^{\frac{2}{p} - 1} c_p^2. \tag{2.5}
\]

Following Mordell, if we let \(j_p = b_p/a_p < 1\) then
\[
a_p = \sqrt{\frac{2}{j_p^2 + 1}} \quad \text{and} \quad b_p = j_p \sqrt{\frac{2}{j_p^2 + 1}}
\]
and from this we find that
\[
\beta_p = \frac{(1 + j_p^2)^{2/p}}{1 + j_p^2}. \tag{2.6}
\]

Applying the inequality between arithmetic and geometric means and (2.4)–(2.6), we find that for each \(t \geq 1\) there exists a rational \(r/s\) with \(s > 0\) such that
\[
s |r - s\alpha| \leq 4^{\frac{1}{p}} [F_t(s, r - s\alpha)]^2 \leq 4^{\frac{1}{p}} \beta_p,
\]
where equality holds for the first inequality if and only if \(s = t^2 |r - s\alpha|\). Letting \(t \to \infty\) we find that
\[
s |r - s\alpha| \leq 4^{\frac{1}{p}} \beta_p \tag{2.7}
\]
for infinitely many rational approximations \(r/s\). The proposition below describes the values of \(4^{1/p} \beta_p\) for \(p \in (0, 1)\).

**Proposition 2.3.** Let \(p \in (0, 1)\). Then

1. \(\lim_{p \to 1^-} 4^{1/p} \beta_p = \frac{1}{2}\),
2. \(\lim_{p \to 0^+} 4^{1/p} \beta_p = \frac{1}{\sqrt{5}}\), and
3. the function \(p \mapsto 4^{1/p} \beta_p\) is increasing on \((0, 1)\).

In order to study the behavior of \(\beta_p\) it is apparent that we must understand the behavior of \(j_p\). The next lemma follows from Section 2 of [9]; its proof is elementary but quite tedious.

**Lemma 2.4.** The function \(p \mapsto j_p\) is differentiable and strictly increasing on \((0, 1)\) and
\[
\lim_{p \to 0^+} j_p = \frac{-1 + \sqrt{5}}{2}. \tag{2.8}
\]

Using Lemma 2.4 the proof of Proposition 2.3 is relatively straightforward.

**Proof of Proposition 2.3.** (1) By (2.5) and (2.3) we have
\[
\lim_{p \to 1^-} 4^{1/p} \beta_p = \lim_{p \to 1^-} \frac{1}{2} c_p^2 = \frac{1}{2}.
\]
Figure 2.1. A plot of $4^{-1/p} \beta_p$ for $p \in (0, 1)$, with the limiting values of $\frac{1}{2}$ and $\frac{1}{\sqrt{5}}$ shown as dashed lines.

(2) As $p$ tends to zero we have

$$4^{-\frac{1}{p}} \frac{(1 + x^p)^{2/p}}{1 + x^2} = \frac{1}{x^{-1} + x} + O(p)$$

uniformly for $x \in [\frac{1}{2}, 1]$. By (2.8) we compute

$$\lim_{p \to 0^+} 4^{-\frac{1}{p}} \beta_p = \lim_{p \to 0^+} 4^{-\frac{1}{p}} \frac{(1 + j_p^p)^{2/p}}{1 + j_p^2} = \frac{1}{2 (-1 + \sqrt{5})^{-1} + \frac{1}{2} (-1 + \sqrt{5})} = \frac{1}{\sqrt{5}}.$$

(3) We will prove that the derivative of the function $p \mapsto 4^{-1/p} \beta_p$ is positive. Since $\beta_p > 0$ it is enough to prove that $f'(p) > 0$, where

$$f(p) = \log(4^{-1/p} \beta_p) = -\frac{2 \log 2}{p} + \frac{2 \log(1 + j_p^p)}{p} - \log(1 + j_p^2).$$

A straightforward computation yields

$$f'(p) = \frac{2}{p^2} \left( \log 2 - \log(1 + j_p^p) - \frac{j_p^p \log(1/j_p^p)}{1 + j_p^p} \right) + 2j_p' \left( \frac{j_p^{p-1}}{1 + j_p^p} - \frac{j_p}{1 + j_p^2} \right).$$

By Lemma 2.4 we have $j_p' > 0$. Since $j_p \in (0, 1)$ and $p \in (0, 1)$ we have $j_p^{p-2} > 1$ so

$$1 + j_p^p < j_p^{-2} + j_p^p = j_p^{p-2}(1 + j_p^2),$$

which implies

$$\frac{j_p^{p-1}}{1 + j_p^p} - \frac{j_p}{1 + j_p^2} > 0.$$

Thus to show that $f'(p) > 0$ it suffices to prove that $g(j_p^p) < \log 2$, where

$$g(x) := \log(1 + x) + \frac{x \log(1/x)}{1 + x}.$$
Indeed, we have $g(x) < \log 2$ for any $x \in (0, 1)$ because
\[
g'(x) = \frac{\log(1/x)}{(1 + x)^2} > 0,
\]
and $\lim_{x \to 1^-} g(x) = \log 2$.

\section{The $p$-continued fraction for $p \in (0, 1)$}

In this section we prove Theorem 1.1. Fix $p \in (0, 1)$. We will need the following analogue of Minkowski’s first convex body theorem for the non-convex bodies
\[
\mathcal{B}_t(P) = \{P' \in \mathbb{R}^2 : F_t(P') < F_t(P)\},
\]
where $P \in \mathbb{R}^2$ and $t > 0$. This is essentially an immediate corollary of Mordell and Watson’s Theorem 2.2. We say that a lattice is admissible for a set $S$ if the only lattice point inside $S$ is the origin.

\begin{proposition}
Fix $p \in (0, 1)$, $t \geq 1$, and $P \in \mathbb{R}^2$. If $L$ is an admissible lattice for $\mathcal{B}_t(P)$ then
\[
\text{area} \mathcal{B}_t(P) \leq C_p \det L,
\]
where
\[
C_p = 2^{p+1} \frac{\Gamma(1 + \frac{1}{p})^2}{\Gamma(1 + \frac{2}{p})} c_p^2
\]
and $c_p$ is given in (2.1).
\end{proposition}

\begin{proof}
Let $R = F_t(P)$. By Theorem 2.2, there is a point $(x, y) \in L$ that satisfies (2.2). But this point is not in $\mathcal{B}_t(P)$ since $L$ is admissible for $\mathcal{B}_t(P)$. Thus
\[
R \leq F(x, y) \leq 2^{\frac{1}{p} - \frac{1}{2}} c_p (\det L)^{\frac{1}{2}}. \tag{3.1}
\]
The area of $\mathcal{B}_t(P)$ is given by
\[
\text{area} \mathcal{B}_t(P) = 4 \int_0^R (R^p - x^p)^{\frac{1}{p}} \, dx = 4 \int_0^1 (1 - x^p)^{\frac{1}{p}} \, dx = 4 \frac{\Gamma(1 + \frac{1}{p})^2}{\Gamma(1 + \frac{2}{p})}.
\]
From this computation and (3.1) we have
\[
\text{area} \mathcal{B}_t(P) \leq 2^{\frac{p}{2} + 1} \frac{\Gamma(1 + \frac{1}{p})^2}{\Gamma(1 + \frac{2}{p})} c_p^2 (\det L),
\]
as desired.
\end{proof}

We will generate the convergents of the $p$-continued fraction using a recursive algorithm, described in Lemma 3.4, that yields a sequence of points in the lattice
\[
L_\alpha = (0, 1)\mathbb{Z} + (1, -\alpha)\mathbb{Z},
\]
where a point $(s, r - s\alpha) \in L_\alpha$ corresponds to a rational $r/s$. Of course we can assume without loss of generality that $\alpha \in (-\frac{1}{2}, \frac{1}{2})$. The properties of the algorithm will show that the sequence of convergents is, in fact, convergent and is precisely the sequence of best approximations with respect to $F$, ordered by increasing denominator.

We will use this elementary lemma several times.
Lemma 3.2. Fix \( p \in (0, 1) \). If \( |x'| \leq |x| \) and \( |y'| \leq |y| \), then \( F(x', y') \leq F(x, y) \).

We also have the following property of \( B_t(P) \) as \( t \) varies. Roughly speaking, this lemma shows that in the first quadrant, \( B_t(P) \) contracts to the left of \( P \) and expands to the right of \( P \) as \( t \) increases.

Lemma 3.3. Fix \( p \in (0, 1) \) and \( P = (x_P, y_P) \in \mathbb{R}^2 \) with \( x_P \geq 0 \), and let \( t_0, t_1 \in \mathbb{R} \) with \( t_1 > t_0 > 0 \). Let \( x > 0 \) and for \( j \in \{0, 1\} \) define \( y_j > 0 \) by \( F_{t_j}(x, y_j) = F_{t_j}(P) \), when a solution to this equation exists. We have that

1. if \( x < x_P \), then \( y_0 > y_1 \), and
2. if \( x > x_P \), then \( y_0 < y_1 \).

Proof. From the implicit definition of \( y_j \), we find

\[
y_j = t_{j}^{-2} \left( x_{P}^{p} - x_{P}^{p} + \alpha_{2P} |y_{P}|^{p} \right)^{\frac{1}{p}}
\]

as an explicit formula for \( y_j \).

For algebraic ease, we consider the quantity \( y_0^p - y_1^p \). Since \( p > 0 \), this quantity will have the same sign as \( y_0 - y_1 \). Hence

\[
y_0^p - y_1^p = t_0^{-2p} (x_P^p - x_P^p + t_0^2p |y_P|^p) - t_1^{-2p} (x_P^p - x_P^p + t_1^2p |y_P|^p)
\]

\[
= t_0^{-2p} (x_P^p - x_P^p + |y_P|^p) - t_1^{-2p} (x_P^p - x_P^p) - |y_P|^p
\]

\[
= (t_0^{-2p} - t_1^{-2p}) (x_P^p - x_P^p).
\]

Note that the first term in the above expression is positive, since \( t_1 > t_0 \). If \( x < x_P \), then the second term in the above expression is positive, meaning \( y_0^p - y_1^p \) is positive and \( y_0 > y_1 \). If instead \( x > x_P \), the argument reverses and \( y_0 < y_1 \).

The lemma below describes the aforementioned recursive algorithm.

Lemma 3.4. Fix an irrational \( \alpha \in (-\frac{1}{2}, \frac{1}{2}) \) and set \( P_{-1} = (0, 1) \) and \( t_{-1} = 1 \). For each \( m \in \mathbb{Z} \) with \( m \geq 0 \), there exists a \( P_m = (x_m, y_m) \in L_{\alpha} \) and a \( t_m > 1 \) with the following properties:

1. \( x_m > x_{m-1} \) and \( |y_m| < |y_{m-1}| \),
2. \( L_{\alpha} \) is admissible for \( B_{t_m}(P_m) \),
3. \( F_{t_m}(P_{m-1}) = F_{t_m}(P_m) \), and
4. for any \( t \in (t_{m-1}, t_m) \), there is no \( P' = (x', y') \in L_{\alpha} \) different from \( P_{m-1} \) with \( x' > 0 \) and \( F_{t}(P') \leq F_{t}(P_{m-1}) \).

Proof. We construct the points \( P_m \) inductively. By the construction of \( L_{\alpha} \), the only points on the unit ball \( B = B_{t_{-1}}(P_{-1}) \) are \( \pm P_{-1} \). For any \( P = (x, y) \) with \( y \neq 0 \) we have

\[
F_{t}(P) = F(t^{-1}x, ty) \geq F(0, ty) = t \cdot |y|,
\]

so \( F_{t}(P) \) increases as \( t \) increases. By Proposition 3.1, there must be a maximum \( t \) for which \( L_{\alpha} \) is admissible for \( B_{t}(P_{-1}) \). Call this maximum \( t_0 \). Among the finitely many \( P' = (x', y') \in L_{\alpha} \) with \( F_{t_0}(P') = F_{t_0}(P_{-1}) \), there is a unique \( P' \) with maximal \( x' \) because \( \alpha \) is irrational. Let \( P_0 = P' \). Note that \( x_0 > x_{-1} \). We repeat this process starting with \( P_0 \), always choosing the new point with maximal \( x \)-coordinate. Let \( m \geq 0 \). We construct \( P_{m+1} \) from \( P_m \) by increasing \( t \), starting at \( t_m \), until \( L_{\alpha} \) is no longer admissible for \( B_{t}(P_m) \), and set \( t_{m+1} \) as the maximum. We call the point on the boundary of \( B_{t_{m+1}}(P_m) \) with maximal \( x \)-coordinate
Figure 3.1. The lattice $L_\alpha$ for $\alpha = 3 - e$ with $p = 0.5$. Dark lattice points correspond to the regular convergents. The points $P_0 = (1, -\alpha)$ and $P_1 = (7, 2 - 7\alpha)$ give best approximations for $\alpha$ relative to $F^{(p)}$.

We show that the sequence of points constructed this way satisfies properties (1)–(4) of the lemma.

(1) By Lemma 3.3 and our choice of $P_m$ with maximal $x$-coordinate we see that $P_m$ must be to the right of $P_{m-1}$, so $x_m > x_{m-1}$. Additionally, the boundary of the ball $B_{t_m}(P_m)$ is concave, so $|y_m| < |y_{m-1}|$.

(2)–(3) These follow from our choice of $t_m$ as the maximal $t$ for which $L_\alpha$ is admissible for $B_{t}(P_{m-1})$ and our choice of $P_m$ on the boundary of $B_{t_m}(P_{m-1})$.

(4) For any $t \in (t_{m-1}, t_m)$, the lattice $L_\alpha$ is admissible for $B_{t}(P_{m-1})$, so there are no points $P' \in L_\alpha$ with $F_t(P') < F_t(P_{m-1})$. If there were a point $P' \in L_\alpha$ for which $F_t(P') = F_t(P_{m-1})$ then this would contradict the definition of $t_m$. □

Figure 3.1 shows a few steps of the algorithm described above.

We claim that the sequence $\{t_m\}$ from Lemma 3.4 tends to infinity. Indeed, the area of the ball $B_t(P_m)$ is

$$\text{area } B_t(P_m) = F^2_t(P_m) \text{ area } B,$$

so we observe that

$$x_m t_m^{-1} = F(x_m t_m^{-1}, 0) \leq F(x_m t_m^{-1}, y_m t_m) = F_t(P_m) \leq C_p^2 \text{ area } B^{-\frac{1}{2}},$$

by Lemma 3.2 and Proposition 3.1. Since $C_p$ is fixed for any fixed $p$ and $x_m \to \infty$, we must have that $t_m \to \infty$.

To any sequence $\{r_n/s_n\}$ of rational numbers we can associate a generalized continued fraction with rational coefficients; that is, an expression of the form

$$a_0 + \frac{\varepsilon_1}{a_1 + \frac{\varepsilon_2}{a_2 + \frac{\varepsilon_3}{a_3 + \cdots}}} := a_0 + \frac{\varepsilon_1}{a_1 + \frac{\varepsilon_2}{a_2 + \frac{\varepsilon_3}{a_3 + \cdots}}}$$

with $a_j, \varepsilon_j \in \mathbb{Q} \setminus \{0\}$ and

$$\frac{r_n}{s_n} = a_0 + \frac{\varepsilon_1}{a_1 + \frac{\varepsilon_2}{a_2 + \cdots}} \frac{\varepsilon_n}{a_n}.$$
The $\varepsilon_j$ and $a_j$ are defined recursively in terms of $r_n$ and $s_n$, as we describe below. If the sequence $\{r_n/s_n\}$ converges (resp. diverges), we say that the continued fraction is convergent (resp. divergent). For any sequence $\{p_n\}$ of nonzero reals we have the transformation
\[
a_0 + \frac{\varepsilon_1}{a_1 + \frac{\varepsilon_2}{a_2 + \frac{\varepsilon_3}{a_3 + \cdots}}} = a_0 + \frac{p_1\varepsilon_1}{\rho_1 a_1 + \frac{p_2\varepsilon_2}{\rho_2 a_2 + \frac{p_3\varepsilon_3}{\rho_3 a_3 + \cdots}}} \tag{3.2}
\]
which preserves the convergents, as can be shown by induction. In particular, a continued fraction with rational coefficients can be easily transformed into one with integral coefficients (except possibly $a_0$) by clearing denominators. Of course, for a generic sequence, this process of clearing denominators will produce arbitrarily large values of $|\varepsilon_j|$ and $|a_j|$. 

Fix an irrational $\alpha \in (-\frac{1}{2}, \frac{1}{2})$ and let $P_m = (x_m, y_m)$ as in Lemma 3.4. For $m \geq 1$, let
\[
g_m = \begin{pmatrix} x_m & y_m \\ x_{m-1} & y_{m-1} \end{pmatrix}
\]
and set
\[
g_0 = \begin{pmatrix} x_0 & y_0 \\ 0 & x_0 \end{pmatrix}.
\]
Note that $\det g_m \in \mathbb{Z}$ for all $m$. Define $\tilde{a}_m$ and $\tilde{\varepsilon}_m$ by
\[
g_m = \begin{pmatrix} \tilde{a}_m & \tilde{\varepsilon}_m \\ 1 & 0 \end{pmatrix} g_{m-1} \tag{3.3}
\]
for $m \geq 1$.

We further define $s_m = x_m$ and $r_m = y_m + x_m \alpha$. Then,
\[
g_m = \begin{pmatrix} s_m & r_m - s_m \alpha \\ s_{m-1} & r_{m-1} - s_{m-1} \alpha \end{pmatrix}.
\]
We see that
\[
\begin{pmatrix} r_m & s_m \\ r_{m-1} - s_{m-1} \alpha & s_{m-1} \end{pmatrix} = g_m \begin{pmatrix} \alpha & 1 \\ 1 & 0 \end{pmatrix}.
\]
Substituting for $g_m$, we have that
\[
\begin{pmatrix} r_m & s_m \\ r_{m-1} & s_{m-1} \end{pmatrix} = \begin{pmatrix} \tilde{a}_m & \tilde{\varepsilon}_m \\ 1 & 0 \end{pmatrix} \begin{pmatrix} s_{m-1} & r_{m-1} - s_{m-1} \alpha \\ s_{m-2} & r_{m-2} - s_{m-2} \alpha \end{pmatrix} \begin{pmatrix} \alpha & 1 \\ 1 & 0 \end{pmatrix}.
\]
From this, we get the recurrence relations
\[
\begin{align*}
    r_m &= \tilde{a}_m r_{m-1} + \tilde{\varepsilon}_m r_{m-2}, & r_0 &= y_0 + x_0 \alpha, & r_{-1} &= x_0, \\
    s_m &= \tilde{a}_m s_{m-1} + \tilde{\varepsilon}_m s_{m-2}, & s_0 &= x_0, & s_{-1} &= 0.
\end{align*}
\]
Following the analogous proof from the theory of regular continued fractions, we obtain
\[
\frac{r_m}{s_m} = a_0 + \frac{\varepsilon_1}{a_1 + \frac{\varepsilon_2}{a_2 + \cdots}} = \frac{a_0}{\tilde{a}_m} + \frac{\tilde{\varepsilon}_1}{\tilde{a}_1 + \frac{\tilde{\varepsilon}_2}{\tilde{a}_2 + \cdots}} = \frac{a_0}{\tilde{a}_m} + \frac{\tilde{\varepsilon}_m}{\tilde{a}_m},
\]
where $a_0 = r_0/s_0$.

We prefer to have integer coefficients. To that end, we define
\[
a_m = (\det g_{m-1})\tilde{a}_m
\]
for $m \geq 1$. For $m \geq 2$, we define
\[
\varepsilon_m = (\det g_{m-2})(\det g_{m-1})\tilde{\varepsilon}_m
\]
and set \( \varepsilon_1 = (\det g_0) \tilde{\varepsilon}_1 \). We note that the denominators of \( \tilde{a}_m \) and \( \tilde{\varepsilon}_m \) are divisors of \( \det g_{m-1} \) by their construction in (3.3), so \( a_m \) and \( \varepsilon_m \) are always integral. We then have a generalized continued fraction for \( r_m/s_m \), with only \( a_0 \) not strictly integral. We note that this transformation is one of the form given in (3.2), which preserves the convergents to the continued fraction. Thus,

\[
\frac{r_m}{s_m} = a_0 + \frac{\varepsilon_1}{a_1} + \frac{\varepsilon_2}{a_2 + \ldots} \frac{\varepsilon_m}{a_m}.
\]

We are not guaranteed that \( \gcd(\varepsilon_m, a_m) = 1 \) or even that \( \gcd(\varepsilon_m, a_m, \varepsilon_{m+1}) = 1 \). However, this generalized continued fraction is unique up to transformation.

**Proof of Theorem 1.1.** Let \( \delta > 0 \). By Proposition 2.3, there exists a \( p = p_\delta \) such that

\[
4^{-\frac{1}{p}} \beta_p \leq \frac{1}{\sqrt{5}} + \delta.
\]

(1) The \( p \)-convergents are best approximations with respect to \( F \) because the associated lattice points satisfy part (4) of Lemma 3.4.

(2) Since every \( p \)-convergent is a best approximation with respect to \( F \), (2.7) is satisfied. Thus, every \( r_m/s_m \) satisfies

\[
s_m |r_m - s_m \alpha| < \frac{1}{\sqrt{5}} + \delta.
\]

(3) Since \( \tilde{\varepsilon}_m = -\det g_m/\det g_{m-1} \), we have for \( m \geq 2 \) the bound

\[
|\varepsilon_m| \leq |(\det g_{m-2})/(\det g_m)|.
\]

For \( m = 1 \), we see that \( \varepsilon_1 \leq |\det g_m| \). We therefore only need to prove an upper bound on \( |\det g_m| \) to obtain a bound for \( |\varepsilon_m| \). This is accomplished in the next proposition, which proves that the determinant is bounded and the bound depends only on \( p \). By applying a transformation as in (3.2), we could obtain an equivalent continued fraction where \( \gcd(\varepsilon_m, a_m, \varepsilon_{m+1}) = 1 \), but the values of each new \( |\varepsilon_m| \) would only be smaller. These new partial numerators would still be bounded by a constant dependent only on \( p \). \( \square \)

**Proposition 3.5.** Fix \( p \in (0, 1) \) and let \( \alpha \in (-\frac{1}{2}, \frac{1}{2}) \) be an irrational number. With \( g_m \) as above, for each \( m \geq 1 \) we have

\[
|\det g_m| \leq 2^{4/p - 1} \Gamma\left(1 + \frac{1}{p}\right)^2 \frac{\Gamma\left(1 + \frac{2}{p}\right)c^2}{\Gamma\left(1 + \frac{2}{p}\right)c^2}.
\]

**Proof.** Without loss of generality we may assume that \( y_{m-1} > 0 \). We divide into two cases depending on the sign of \( y_m \).

Suppose first that \( y_m > 0 \). Then the points \( P_m = (x_m, y_m) \) and \( P_{m-1} = (x_{m-1}, y_{m-1}) \) are in the first quadrant, with \( P_{m-1} \) to the left of \( P_m \). By Lemma 3.4 there exists a \( t > 0 \) such that \( F_t(P_m) = F_t(P_{m-1}) \). Let \( Q_1 = (t^{-1}x_m, ty_m) = (u_1, v_1) \) and \( Q_2 = (t^{-1}x_{m-1}, ty_{m-1}) = (u_2, v_2) \). Then \( Q_1 \) and \( Q_2 \) both lie on the boundary of \( RB \), where \( R = F_t(P_m) \) and \( B = \{ P \in \mathbb{R}^2 : F(P) < 1 \} \) is the unit ball. Furthermore,

\[
|\det g_m| = u_1v_2 - u_2v_1.
\]

We begin by computing the area of the region \( D \) bounded by the line segments \( OQ_1 \) and \( OQ_2 \) and the portion of the curve \( u^p + v^p = R^p \) in the first quadrant. Let \( \theta_1 = \arctan(v_1/u_1) \)
and $\theta_2 = \arctan(v_2/u_2)$ denote the angles that the line segments $OQ_1$ and $OQ_2$ make with the positive $x$-axis. In polar coordinates $(r, \theta)$ the region $D$ is described as

$$\theta_1 \leq \theta \leq \theta_2, \quad 0 \leq r \leq r(\theta) = \frac{R}{((\cos \theta)^p + (\sin \theta)^p)^{1/p}}.$$ 

Thus we have

$$\text{area}(D) = \int_{\theta_1}^{\theta_2} \int_0^{r(\theta)} r \, dr \, d\theta = \frac{R^2}{2} \int_{\theta_1}^{\theta_2} \frac{d\theta}{((\cos \theta)^p + (\sin \theta)^p)^{2/p}}.$$ 

Making the change of variable $u = \tan \theta$ we find that

$$\text{area}(D) = \frac{R^2}{2} \int_{v_1/u_1}^{v_2/u_2} \frac{du}{(1 + u^p)^{2/p}} = \frac{R^2}{2} \int_{v_1/u_1}^{v_2/u_2} \left( \frac{(1 + u)^2}{(1 + u^p)^{2/p}} \right) du = \frac{R^2}{2} \int_{v_1/u_1}^{v_2/u_2} \frac{u_1 v_2 - v_1 u_2}{(u_1 + v_1)(u_2 + v_2)}.$$ 

The minimum of the function $u \mapsto \frac{(1 + u)^2}{(1 + u^p)^{2/p}}$ is $2^{2-2/p}$; it occurs at $u = 1$. Thus we have

$$\text{area}(D) \geq 2^{1-2/p} R^2 \int_{v_1/u_1}^{v_2/u_2} \frac{du}{(1 + u)^2} = 2^{1-2/p} R^2 \frac{u_1 v_2 - v_1 u_2}{(u_1 + v_1)(u_2 + v_2)}.$$ 

The maximum value of $(u, v) \mapsto u + v$ subject to the constraint $u^p + v^p = R^p$ is $R$; it occurs when $(u, v) = (R, 0)$. Thus we have

$$\text{area}(D) \geq 2^{1-2/p} |\det g_m|.$$ 

There are two copies of $D$ inside $RB = B_1(Q_1)$, so we conclude by Proposition 3.1 that

$$|\det g_m| \leq 2^{2/p - 2} (2 \text{area}(D)) \leq 2^{2/p - 2} \text{area}(B_1(Q_1)) < 2^{4/p - 1} \frac{\Gamma(1 + \frac{1}{p})^2}{\Gamma(1 + \frac{2}{p})} c_p^2.$$ 

If $y_m < 0$, define $Q_1$ and $Q_2$ as before and define $Q'_1 = (u_1, -v_1)$. Let $D$ denote the region bounded by the line segments $OQ'_1$ and $OQ_2$ and the portion of the curve $u^p + v^p = R^p$ in the first quadrant. Then we have

$$\text{area}(D) \geq 2^{1-2/p} (u_1 v_2 + v_1 u_2).$$ 

Let $D'$ denote the triangle $OQ'_1 Q'_1$. Then

$$\text{area}(D') = u_1 |v_1| \geq u_2 |v_1| \geq 2^{2-2/p} u_2 |v_1| = -2^{2-2/p} u_2 v_1,$$

so

$$\text{area}(D \cup D') \geq 2^{1-2/p} (u_1 v_2 - v_1 u_2) = 2^{1-2/p} |\det g_m|.$$ 

The remainder of the proof proceeds as in the first case. \hfill \Box

4. Regular convergents skipped by the $p$-continued fraction

In this section we prove Theorem 1.2 and several lemmas to aid us in the next section, all of which concern the question of how many of the regular convergents of a given irrational $\alpha$ are skipped by the $p$-continued fraction.

We first list several well-known results about the regular continued fraction. Fix an irrational $\alpha$. For $p \in (0, 1)$, let $p_n/q_n$ denote the regular convergents of $\alpha$ and $r_n/s_n$ the $p$-convergents. First, $p_n$ and $q_n$ are defined recursively by

$$p_n = b_n p_{n-1} + p_{n-2}, \quad p_{-2} = 0, \quad p_{-1} = 1,$$

$$q_n = b_n q_{n-1} + q_{n-2}, \quad q_{-2} = 1, \quad q_{-1} = 0,$$
where \( b_n \) is the \( n \)-th regular partial quotient. Additionally, we have that for \( n \geq -1 \),
\[
q_np_{n-1} - p_nq_{n-1} = (-1)^n, \tag{4.1}
\]
and for \( n \geq 0 \),
\[
q_np_{n-2} - p_nq_{n-2} = (-1)^{n-1}b_n. \tag{4.2}
\]
These results and their proofs can be found in Section 3 of [8, Ch. I].

The number of consecutive regular convergents \( p_n/q_n \) that can be skipped by the \( p \)-continued fraction is closely related to the determinants that appear in the following proposition.

**Proposition 4.1.** Notation as above, let \( b_n \) denote the \( n \)-th regular partial quotient for \( \alpha \) and for \( \ell \geq 0 \) define
\[
D_\ell(n) = (-1)^{n-1}(q_{n-1}p_{n+\ell} - p_{n-1}q_{n+\ell}).
\]
Then \( D_0(n) = 1 \), \( D_1(n) = b_{n+1} \), and for \( \ell \geq 2 \), we have
\[
D_\ell(n) = b_{n+\ell}D_{\ell-1}(n) + D_{\ell-2}(n).
\]

**Proof.** For the preliminary cases \( \ell \in \{0, 1 \} \) we have \( D_0(n) = 1 \) and \( D_1(n) = b_{n+1} \) by (4.1) and (4.2). For \( \ell \geq 2 \), using the recursive definitions of \( p_{n+\ell} \) and \( q_{n+\ell} \) we have that
\[
D_\ell(n) = (-1)^{n-1}(q_{n-1}(b_{n+\ell}p_{n+\ell-1} + p_{n+\ell-1}) - p_{n-1}(b_{n+\ell}q_{n+\ell-1} + q_{n+\ell-1}))
\]
\[
= b_{n+\ell}(-1)^{n-1}(q_{n-1}p_{n+\ell-1} - p_{n-1}q_{n+\ell-1}) + (-1)^{n-1}(q_{n-1}p_{n+\ell-2} - p_{n-1}q_{n+\ell-2})
\]
\[
= b_{n+\ell}D_{\ell-1}(n) + D_{\ell-2}(n),
\]
as desired. \( \square \)

**Proof of Theorem 1.2.** By Proposition 4.1, using that the regular partial quotients of \( \alpha \) satisfy \( b_n \geq 1 \) for all \( n \), we have the rough lower bound
\[
D_\ell(n) \geq F_{\ell+1} \geq \frac{\varphi^\ell}{\sqrt{5}},
\]
where \( F_\ell \) is the \( \ell \)-th Fibonacci number and \( \varphi = \frac{1}{2}(1 + \sqrt{5}) \). On the other hand, \( D_\ell(n) = |\det g_m| \) for some \( m \), so by Proposition 3.5 we have \( D_\ell(n) \leq C_p \) where \( C_p \) is the expression on the right hand side in (3.4). Thus
\[
\ell \leq \log_\varphi(\sqrt{5}C_p). \quad \square
\]

To aid in proving Theorem 1.3, we state several lemmas regarding 1-periodic continued fractions. For the remainder of this section we assume that \( a \in \mathbb{Z} \) with \( a \geq 1 \) and that
\[
\alpha = a + \frac{1}{a+} \frac{1}{a+} \frac{1}{a+} \cdots = a + \frac{\sqrt{a^2 + 4}}{2}. \tag{4.3}
\]

**Lemma 4.2.** Suppose that \( \alpha \) satisfies (4.3) and define a sequence \( R_n \) by \( R_0 = 0 \), \( R_1 = 1 \), and
\[
R_n = aR_{n-1} + R_{n-2} \quad \text{for } n \geq 2.
\]
Then for all \( n \geq 0 \) we have
\[
R_n = \frac{\alpha^n - (-\alpha)^{-n}}{\alpha + \alpha^{-1}}.
\]
Proof. We can rewrite (4.4) using matrices, letting us solve for any $R_{n+1}$ and $R_n$ for $n \geq 1$ by writing
\[
\begin{pmatrix} R_{n+1} \\ R_n \end{pmatrix} = \begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}.
\]
The eigenvalues of $\begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix}$ are $\alpha$ and $-\alpha^{-1}$, where we have used (4.3) and $\alpha = a + \alpha^{-1}$. Diagonalizing, we obtain
\[
\begin{pmatrix} R_{n+1} \\ R_n \end{pmatrix} = \frac{1}{\alpha + \alpha^{-1}} \begin{pmatrix} 1 & 1 \\ \alpha^{-1} & -\alpha \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ 0 & -\alpha^{-1} \end{pmatrix} \begin{pmatrix} \alpha^{-1} & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}.
\]
Further simplification yields
\[
\begin{pmatrix} R_{n+1} \\ R_n \end{pmatrix} = \frac{1}{\alpha + \alpha^{-1}} \begin{pmatrix} \alpha^{n+1} - (-\alpha)^{-n-1} & \alpha^n - (-\alpha)^{-n} \\ \alpha^n - (-\alpha)^{-n} & \alpha^{n-1} - (-\alpha)^{-n+1} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}.
\]
This completes the proof. \qed

**Lemma 4.3.** If $\alpha$ satisfies (4.3) then $p_{n-1} = q_n$ for all $n \geq 1$.

**Proof.** Since $\alpha$ satisfies (4.3), every partial quotient is $a$. Then because $q_0 = 1$, $p_n$ and $q_n$ each satisfy with offset the recursive definition for $R_n$ given in Lemma 4.2. In particular, $p_n = R_{n+2}$ and $q_n = R_{n+1}$. Thus $p_{n-1} = R_{n+1} = q_n$. \qed

**Lemma 4.4.** Suppose that $\alpha$ satisfies (4.3) and let $R_n$ be as in Lemma 4.2. Then for any $n \geq 0$ we have
\[
|R_{n+1} - R_n \alpha| = \alpha^{-n}.
\]

**Proof.** By Lemma 4.2,
\[
|R_{n+1} - R_n \alpha| = \left| \frac{\alpha^{n+1} - (-\alpha)^{-n-1}}{\alpha + \alpha^{-1}} \right| = \frac{\alpha^n - (-\alpha)^{-n}}{\alpha + \alpha^{-1}} \left| \frac{1}{\alpha} \right| = \frac{\alpha^{-n} \left( \alpha^2 + 1 \right)}{\alpha + \alpha^{-1}} = \alpha^{-n},
\]
as desired. \qed

## 5. Proof of Theorem 1.3

Fix an $\alpha$ that satisfies (4.3). We define a curve and examine its intersection with the lattice points corresponding to the regular convergents of $\alpha$. Let $Q_n = (R_n, |R_{n+1} - R_n \alpha|)$ for any $n \geq 0$. This point is either the lattice point corresponding to a regular convergent or the reflection of said point across the $x$-axis because of Lemma 4.3 and our choice of $\alpha$. Fix an $m \in \mathbb{N}$. We define $C_m^{(p)}$ to be the portion of the boundary of the ball $B_t^{(p)}((0,1))$ in the first quadrant, where $t$ is such that $Q_m$ is on the boundary of $B_t^{(p)}((0,1))$. We can describe the curve $C_m^{(p)}$ algebraically by the equation
\[
t^p = (xt^{-1})^p + (yt)^p.
\]
Since $Q_m$ is on $C_m^{(p)}$, we can write
\[
t = \left( \frac{R_m^p}{1 - |R_{m+1} - R_m \alpha|^p} \right)^{\frac{1}{p}}.
\]
We claim there exists a unique $p = p(m)$ such that $Q_{m+1}$ is also on $C^{(p)}_m$. Equivalently,

$$
\left( \frac{R^p_m}{1 - |R_{m+1} - R_m\alpha|^p} \right)^{\frac{1}{p}} = t = \left( \frac{R^p_{m+1}}{1 - |R_{m+2} - R_{m+1}\alpha|^p} \right)^{\frac{1}{p}}.
$$

Simplification yields

$$
R^p_m - R^p_m |R_{m+2} - R_{m+1}\alpha|^p = R^p_{m+1} - R^p_{m+1} |R_{m+1} - R_m\alpha|^p,
$$

or, after Lemma 4.4,

$$
\left( \frac{R_{m+1}}{R_m} \right)^p = \frac{1 - \alpha^{-mp-p}}{1 - \alpha^{-mp}}.
$$

(5.3)

**Lemma 5.1.** If $m \geq 3$ then there exists a unique $p = p(m) \in (0, 1)$ satisfying (5.3).

**Proof.** Let $m \geq 3$ be an integer and define

$$
f(p) = \frac{1 - \alpha^{-mp-p}}{1 - \alpha^{-mp}}.
$$

We claim that $f(p)$ is strictly decreasing for $p \in (0, \infty)$. Assuming this for now, and using that $f(p) > 1$ we find that

$$
d \frac{df(p)}{dp} = \left( \frac{f'(p)}{pf(p)} - \log f(p) \right) \left[ f(p) \right]^{\frac{1}{p}} < 0.
$$

Since $\lim_{p \to 0^+} [f(p)]^{1/p} = \infty$ and $\lim_{p \to \infty} [f(p)]^{1/p} = 1$, we see that $p \mapsto [f(p)]^{1/p}$ is a bijection from $(0, \infty)$ to $(1, \infty)$. Thus there is a unique $p$ for which $[f(p)]^{1/p} = R_{m+1}/R_m$. We claim that this $p$ must be in $(0, 1)$. Indeed, when $p \geq 1$ we have

$$
[f(p)]^{\frac{1}{p}} \leq f(1) = \frac{1 - \alpha^{-m-1}}{1 - \alpha^{-m}} = 1 + \alpha^{-m} \frac{1 - \alpha^{-1}}{1 - \alpha^{-m}} < 1 + \frac{1}{\alpha^3}.
$$

On the other hand, by Lemma 4.4 we have

$$
\frac{R_{m+1}}{R_m} \geq \alpha - \frac{1}{\alpha^m R_m} \geq \alpha - \frac{1}{\alpha^3}.
$$

Since $x > 1 + 2/x^3$ for all $x > 1.6$ and since $\alpha > 1.618$, we see that $[f(p)]^{1/p} < R_{m+1}/R_m$, so the $p$ satisfying (5.3) must be in $(0, 1)$. Thus it remains to show that $f(p)$ is decreasing.

We have

$$
f'(p) = -\frac{m(1 - \alpha^{-p}) - \alpha^{-p}(1 - \alpha^{-mp})}{(1 - \alpha^{-m})^2} \alpha^{-mp} \log \alpha,
$$

so it is enough to show that

$$
m(1 - \alpha^{-p}) > \alpha^{-p}(1 - \alpha^{-mp}).
$$

Factoring the polynomial $1 - (\alpha^{-p})^m$ we find that

$$
\alpha^{-p} \frac{(1 - \alpha^{-mp})}{(1 - \alpha^{-p})} = \alpha^{-p} \left( 1 + \alpha^{-p} + \ldots + \alpha^{-p(m-1)} \right) < m.
$$

This completes the proof. □
Figure 5.1. The edge of the ball, $C_m^{(p)}$, for $p \approx 0.27$, with $\alpha = \varphi = \frac{1+\sqrt{5}}{2}$ and $m = 5$. The points $Q_{m+k}$ for $k \in \{-3, -2, -1, 0, 1, 2, 3\}$ are all visible.

We claim that for $p = p(m)$ as above with sufficiently large $m$, $L_\alpha$ is admissible for the ball $B_{t}^{(p)}((0, 1))$. This is equivalent to $Q_{m+k}$ being above $C_m^{(p)}$ for any $k \in \mathbb{Z} \setminus \{0, 1\}$ such that $k > -m$. By solving (5.1) for $y$, and comparing with the $y$-coordinate of $Q_{m+k}$, we find that we can write the condition of $Q_{m+k}$ being above $C_m^{(p)}$ as

$$|R_{m+k+1} - R_{m+k}\alpha| > (1 - R_{m+k}^{p}k^{-2p})^{\frac{1}{p}}.$$  

After some simplification and substituting the expression in (5.2) for $t$, we see that for a fixed $m$, $Q_{m+k}$ remains above $C_m^{(p)}$ if and only if

$$R_{m+k}^{p} - R_{m+k}^{p}|R_{m+1} - R_{m}\alpha|^p > R_{m}^{p} - R_{m}^{p}|R_{m+k+1} - R_{m+k}\alpha|^p.$$  

By Lemma 4.4, this can be simplified to

$$\left(\frac{R_{m+k}}{R_{m}}\right)^{p} > \frac{1 - \alpha^{-mp}}{1 - \alpha^{-mp}}.$$  

(5.4)

Lemma 5.2. For each integer $m \geq 2$, define $p = p(m) \in (0, 1)$ by (5.3). Then $p \sim \frac{\log 2}{m \log \alpha}$ as $m \to \infty$. Furthermore, for all sufficiently large $m$, for any $k \in \mathbb{Z} \setminus \{0, 1\}$ with $k > -m$, the inequality (5.4) holds.

Proof. Let $k \in \mathbb{Z} \setminus \{0\}$. Using Lemma 4.2 we have

$$\frac{R_{m+k}}{R_{m}} = \alpha^{k} \frac{1 - (-1)^{m+k}\alpha^{-2m-2k}}{1 - (-1)^{m}\alpha^{-2m}}$$

$$= \alpha^{k} \left( 1 + (-1)^{m}\alpha^{-2m} \frac{1 - (-1)^{k}\alpha^{-2k}}{1 - (-1)^{m}\alpha^{-2m}} \right)$$

$$= \alpha^{k} \left( 1 + O(\alpha^{-2m}) \right).$$


Thus we have

\[ \log \left( \frac{R_{m+k}}{R_m} \right) = k \log \alpha + O \left( \alpha^{-2m} \right). \]  

(5.5)

Making the ansatz

\[ p = \frac{p_0}{m} + \frac{p_1}{m^2} + \frac{p_2}{m^3} + O \left( \frac{1}{m^4} \right), \]

we find that

\[ \alpha^{-mp} = \alpha^{-p_0 - p_1 m^{-1} - p_2 m^{-2} + O(m^{-3})} \]

\[ = \alpha^{-p_0} - \frac{p_0}{m} \alpha^{-p_0} \log \alpha + \frac{p_0^2 \alpha^{-p_0} \log \alpha - 2p_0}{2m^2} + O \left( \frac{1}{m^3} \right), \]

and thus

\[ \log \left( \frac{1 - \alpha^{-mp - kp}}{1 - \alpha^{-mp}} \right) \]

\[ = \frac{kp_0 \log \alpha}{(\alpha^{p_0} - 1)m} - \frac{k \log \alpha ((p_0^2 + 2p_0 p_1) \alpha^{p_0} \log \alpha - 2p_0 (\alpha^{p_0} - 1))}{2(\alpha^{p_0} - 1)^2 m^2} + O \left( \frac{1}{m^3} \right). \]  

(5.6)

Hence, by the logarithm of (5.3), (5.5), and (5.6) with \( k = 1 \) we must have

\[ \frac{p_0 \log \alpha}{m} = \frac{p_0 \log \alpha}{(\alpha^{p_0} - 1)m} \]

and

\[ \frac{p_1 \log \alpha}{m^2} = -\frac{\log \alpha ((p_0^2 + 2p_0 p_1) \alpha^{p_0} \log \alpha - 2p_0 (\alpha^{p_0} - 1))}{2(\alpha^{p_0} - 1)^2 m^2}. \]

The first equation shows that \( \alpha^{p_0} - 1 = 1 \), which shows that \( p_0 = \frac{\log 2}{\log \alpha} \) and simplifies the second equation significantly. Thus

\[ p = \frac{\log 2}{m \log \alpha} \left( 1 - \frac{1}{2m} \right) + O \left( \frac{1}{m^3} \right). \]  

(5.7)

Now we assume that \( k \neq 1 \). Using (5.5), (5.6), and (5.7), the logarithm of the inequality (5.4) becomes

\[ \frac{k \log 2}{m} - \frac{k \log 2}{2m^2} + O \left( \frac{1}{m^3} \right) > \frac{k \log 2}{m} - \frac{k \log 2}{2m^2} \left( 2k \log 2 + 1 - 2 \log 2 \right) + O \left( \frac{1}{m^3} \right), \]

which, for sufficiently large \( m \), is true for any \( k \in \mathbb{Z} \setminus \{0, 1\} \). \( \square \)

For any sufficiently large \( m \) and with \( p = p(m) \), the first two lattice points that touch the boundary of \( B_t^{(p)}((0, 1)) \) as \( t \) increases from 1 must be the lattice points corresponding to \( Q_m \) and \( Q_{m+1} \) by the previous lemma, and furthermore the lattice must be admissible for this ball. This then gives us that \( r_0/s_0 = R_{m+2}/R_{m+1} = p_m/q_m \). \( \square \)
References

[1] Nickolas Andersen and William Duke. On a theorem of Davenport and Schmidt. *Acta Arith.*, 198(1):37–75, 2021.

[2] P. M. Gruber and C. G. Lekkerkerker. *Geometry of numbers*, volume 37 of *North-Holland Mathematical Library*. North-Holland Publishing Co., Amsterdam, second edition, 1987.

[3] J. L. Lagrange. *Additions aux éléments d’algèbre d’Euler*, volume VII of *Oeuvres*.

[4] Hermann Minkowski. Généralisation de la théorie des fractions continues. *Ann. Sci. École Norm. Sup. (3)*, 13:117–60, 1896.

[5] Hermann Minkowski. Ueber die Annäherung an eine reelle Grösse durch rationale Zahlen. *Math. Ann.*, 54(1-2):91–124, 1900.

[6] Hermann Minkowski. *Geometrie der Zahlen*. Bibliotheca Mathematica Teubneriana, Band 40. Johnson Reprint Corp., New York-London, 1968.

[7] L. J. Mordell. On the geometry of numbers in some non-convex regions. *Proc. London Math. Soc. (2)*, 48:339–390, 1945.

[8] Wolfgang M. Schmidt. *Diophantine approximation*, volume 785 of *Lecture Notes in Mathematics*. Springer, Berlin, 1980.

[9] G. N. Watson. A problem in elementary analysis proposed by Mordell. *Proc. London Math. Soc. (2)*, 48:391–400, 1945.

Brigham Young University Mathematics Department, Provo UT 84602
Email address: nick@math.byu.edu

UCLA Mathematics Department, Box 951555, Los Angeles, CA 90095-1555
Email address: wdduke@ucla.edu

Brigham Young University Mathematics Department, Provo UT 84602
Email address: zmhacking@gmail.com

Brigham Young University Mathematics Department, Provo UT 84602
Email address: amy.woodall713@gmail.com