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Similarity Solutions for the Complex Burgers’ Hierarchy

Abstract: A detailed analysis of the invariant point transformations for the first four partial differential equations which belong to the complex Burgers’ hierarchy is performed. Moreover, a detailed application of the reduction process through the Lie-point symmetries is presented while we construct the similarity solutions. We conclude that the differential equations of our consideration are reduced to first-order equations such as the Abel, Riccati, and to a linearisable second-order differential equation by using similarity transformations.

Keywords: Complex Burgers’ Equation; Integrability; Reduction of Order; Symmetries.

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1 Introduction

Burgers’ equation is the centre of attraction for decades for its diverse applications in different fields [1–4]. Its importance can be seen in the mathematical formulation for various subjects in applied mathematics [5–9]. In this paper we study the symmetries of the complex Burgers’ hierarchy [10]. The hierarchy is given by the formula

\[ u_t = t(L)P(iu_x e^{-i(\bar{u}-\bar{u})}), \]

where \( t(z) \) is an arbitrary entire function and the operators \( P \) and \( L \) are defined as [10]

\[ P(\beta(t, x)) = ie^{i(\bar{u}-\bar{u})} \beta(t, x) \quad \text{and} \quad L(\tau(t, x)) = i\tau x + u_x \tau(t, x). \]

The dependent variables, \( u \) and \( \tau \), are functions of \( t \) and \( x \) of complex type. The members of the hierarchy are obtained by setting \( t(L) = L^n \), where \( n = 0, 1, 2, 3 \ldots \). For \( n = 0 \), it leads to \( u_t = -u_x \). Subsequently, for higher values of \( n \), the other members are obtained eventually. For \( n = 1 \) the second member of the hierarchy is

\[ u_t = -u_x^2 - iu_{xx}. \]

This work focuses on the study of certain members of the complex Burgers’ hierarchy through Lie’s approach. Lie symmetry analysis is a powerful method for the study of nonlinear differential equations and there are many applications of Lie’s theory in different subjects of applied mathematics [11–14]. For instance for the classical Burgers’ equation (2) the symmetry analysis was performed in [15], while the symmetry analysis for the 2+1 Burgers’ equation is given in [9, 16].

The importance of Lie symmetries is that they provide us with differential invariants which can be used to reduce the order of differential equations, and to construct similarity solutions for the original equation [17]. Hence, in this work we study the algebraic properties for the members of the complex Burgers’ hierarchy and we compare the admitted Lie algebras and infer conclusions.

Furthermore, we derive the travelling-wave solution for each of the members by using the Lie invariants. Moreover, the Lie point symmetries are applied to determine the travelling-wave solutions.

2 The Members of the Complex Burgers’ Hierarchy

We study the algebraic properties of the first four equations of the complex Burgers’ hierarchy. The first member of the complex Burgers’ hierarchy is

\[ u_t + u_x = 0, \]

where \( L(\tau(t, x)) = i\tau x + u_x \tau(t, x). \)
where \( u(t, x) \) is a complex function. By substitution of \( u(t, x) = v(t, x) + iw(t, x) \), where \( v, w \) are real functions, from the latter equation there follows the system

\[
\begin{align*}
    v_t + v_x &= 0, \quad (4) \\
    w_t + w_x &= 0. \quad (5)
\end{align*}
\]

The second member of the complex Burgers’ hierarchy is

\[
\begin{align*}
    v_t &= -v_x^2 + w_x^2 + w_{xx}, \quad (6) \\
    w_t &= -2v_x w_x - v_{xx} \quad (7)
\end{align*}
\]

when it is reduced to its real and imaginary parts. In terms of real functions the second member of the Burgers’ hierarchy is well known to be linearisable by the Cole-Hopf transformation [18].

The third member of the complex Burgers’ hierarchy is written in terms of its components as

\[
\begin{align*}
    v_t &= -v_x^3 + 3v_x w_x^2 + 3w_x v_{xx} + 3v_x w_x v_{xx} + v_{xxx}, \quad (8) \\
    w_t &= -3v_x^2 w_x + w_x^3 - 3v_x v_{xx} + 3w_x w_{xx} + w_{xxx}. \quad (9)
\end{align*}
\]

The latter system is also called the complex Sharma–Tasso–Olver equation [19–21].

Finally the fourth member of the complex Burgers’ hierarchy is

\[
\begin{align*}
    u_t &= u_{x}^4 + 3u_{xx}^2 + 4u_{xxx} u_{xxxx} + i(-3u_{xx} u_{xxx} - 3u_{x}^2 u_{xxxx} + u_{xxxxx}), \quad (10)
\end{align*}
\]

for which the corresponding real and imaginary parts are

\[
\begin{align*}
    v_t &= v_x^4 - 6v_x^2 w_x^2 + w_x^4 + 3w_x v_{xx} + 6v_x w_x v_{xx} \\
    &\quad + 3v_{xx}^2 + 3v_x w_{xx} + 3v_x^2 w_{xx} - 3w_x^2 w_{xx} \\
    &\quad - 3w_{xx}^2 + 4v_x v_{xxx} - 4w_x w_{xxx} - w_{xxxxx}, \quad (11)
\end{align*}
\]

and

\[
\begin{align*}
    w_t &= 4v_x^3 w_x - 4v_x^2 w_x^3 - 4v_x v_x v_{xx} - 3v_x^2 v_{xx} \\
    &\quad + 3w_x^2 v_{xx} + 3w_x w_{xx} + 6v_x w_x w_{xx} \\
    &\quad + 6v_{xx} w_{xx} + 4w_x v_{xxx} + 4v_x w_{xxx} + v_{xxxxx}. \quad (12)
\end{align*}
\]

For the set of four differential equations above, we apply Lie’s theory and we determine the point transformations under which the partial differential equations are invariant.

### 3 Lie Symmetries and Differential Invariants

For the convenience of the reader we briefly discuss the theory of Lie symmetries of differential equations and the application of the differential invariants for the construction of similarity solutions.

Let \( \Phi \) be the map of an one-parameter point transformation such as

\[
\Phi(u^A(t, x)) = u'^A(t, x) \quad (13)
\]

with infinitesimal transformation (\( \varepsilon \) is the parameter of smallness)

\[
\begin{align*}
    t' &= t + \varepsilon x^1 \left( t, x, u^B \right) \quad (14) \\
    x' &= x + \varepsilon x^2 \left( t, x, u^B \right) \quad (15) \\
    y' &= u^A + \varepsilon \eta^A \left( t, x, u^B \right) \quad (16)
\end{align*}
\]

and generator

\[
X = \frac{\partial t'}{\partial \varepsilon} \frac{\partial}{\partial t} + \frac{\partial x'}{\partial \varepsilon} \frac{\partial}{\partial x} + \frac{\partial u'^A}{\partial \varepsilon} \frac{\partial}{\partial u^A} \quad (17)
\]

in which \( u^A(t, x) = (v(t, x), w(t, x)) \).

Consider now that \( u^A(t, x) \) is a solution of the partial differential equation \( \mathcal{H}(u^A, u^A_t, u^A_{xx}, \ldots) = 0 \). Then under the map \( \Phi \) defined by (13), function \( u'^A(t', x') \) is also a solution of the differential equation \( \mathcal{H}(u^A, u^A_t, u^A_{xx}, \ldots) = 0 \) if and only if \( \mathcal{H}(u^A, u^A_t, u^A_{xx}, \ldots) = 0 \), that is, the differential equation \( \mathcal{H}(u^A, u^A_t, u^A_{xx}, \ldots) = 0 \) is invariant under the action of the map, \( \Phi \).

If this property be true, then the generator, \( X \), of the infinitesimal transformation of the one-parameter point transformation, \( \Phi \), is a Lie (point) symmetry of the differential equation \( \mathcal{H}(u^A, u^A_t, u^A_{xx}, \ldots) = 0 \). Mathematically that is expressed as

\[
X^{[n]}(\mathcal{H}) = 0, \quad (18)
\]

or equivalently

\[
X^{[n]}(\mathcal{H}) = \psi \mathcal{H}, \quad \text{mod}(\mathcal{H}) = 0, \quad (19)
\]

where \( X^{[n]} \) denotes the \( n \)-prolongation/extension of the symmetry vector in the space of variables \( \{ t, x, u^A, u^A_t, u^A_{xx}, \ldots \} \). The symmetry condition (18) provides a set of differential equations the solution of which provides the generator of the infinitesimal transformation (17).
The importance of the existence of a Lie symmetry for a partial differential equation is that from the associated Lagrange’s system,

\[
\frac{dt}{\xi^1} = \frac{dx}{\xi^2} = \frac{du^A}{\eta^A},
\]

zeroth-order invariants, \(U^{[0]}(t, x, u^A)\), can be determined which can be used to reduce the number of the independent variables of the differential equation and lead to the construction of similarity solutions.

### 3.1 Lie Symmetries for the First Member of the Complex Burgers’ Hierarchy

We continue by presenting the Lie point symmetries for the set of equations (4) and (5). Specifically this system admits the infinite number of symmetries

\[
\Gamma = \xi^1(x, t, v, w)\partial_x + \left(c_c(-t + x, v, w) + \xi^1(x, t, v, w)\right)\partial_t
+ c_a(-t + x, v, w)\partial_v + c_b(v, w, -t + x)\partial_w.
\]

The system of differential equations (4) and (5) is well known to admit a travelling-wave solution. That family of solutions can be easily derived by considering that functions \(\xi^1(t, x, v, w)\) and \(c_c(x - t, v, w)\) are constants. That leads to the differential invariants

\[
s = x - ct, \quad f(s) = v(t, x), \quad g(s) = w(t, x),
\]

where now the system of differential equations, (4) and (5) is simplified to

\[
\frac{df}{ds} = 0, \quad \frac{dg}{ds} = 0
\]

with similarity solution \(f(s) = f_0, g = g_0\).

We continue with the determination of the Lie point symmetries for the system of differential equations, (6) and (7).

### 3.2 Lie Symmetries for the Second Member of the Complex Burgers’ Hierarchy

The system of differential equations, (6) and (7), admits the following generic symmetry vector

\[
\Gamma = \left(A_0 + A_1 t + A_2 t^2\right)\partial_t
+ \left(A_3 + A_4 t + \frac{(A_1 + 2A_2 t)x}{2}\right)\partial_x
+ \left(A_5 - \frac{A_2 t}{2} + e^{-w} \cos(v)a(t, x)
- e^{-w} b(t, x)\sin(v)\right)\partial_w
+ \left(A_6 + \frac{(2A_4 x + A_2 x^2)}{4} - e^{-w} \cos(v)b(t, x)
- e^{-w} a(t, x)\sin(v)\right)\partial_v,
\]

where \(A_0, A_1, A_2, A_3, A_4, A_5, A_6\) are arbitrary constants, \(a(t, x)\) and \(b(t, x)\) are functions which satisfy the linear constant coefficient \((1+1)\) evolution equations

\[
a_t - a_{xx} = 0, \quad b_t - b_{xx} = 0.
\]

From the latter it is clear that the system, (6) and (7) admits seven plus infinity Lie symmetry vectors. The seven vector fields corresponds to the seven arbitrary constants \(A_0, A_1, A_2, A_3, A_4, A_5, A_6\) and are

\[
\begin{align*}
\Gamma_{1a} &= \partial_t, \\
\Gamma_{2a} &= t\partial_t + \frac{x}{2}\partial_x, \\
\Gamma_{3a} &= t^2\partial_t + tx\partial_x + \frac{x^2}{4}\partial_v - \frac{t}{2}\partial_w, \\
\Gamma_{4a} &= \partial_v, \\
\Gamma_{5a} &= t\partial_x + \frac{x}{2}\partial_v, \\
\Gamma_{6a} &= \partial_w, \\
\Gamma_{7a} &= \partial_w.
\end{align*}
\]

The Lie brackets between the symmetries are

\[
\begin{align*}
[\Gamma_{1a}, \Gamma_{5a}] &= \Gamma_{1a} - \frac{\Gamma_{3a}}{2}, \\
[\Gamma_{1a}, \Gamma_{6a}] &= \Gamma_{1a} - \frac{\Gamma_{4a}}{2}, \\
[\Gamma_{1a}, \Gamma_{7a}] &= \Gamma_{1a} - \frac{\Gamma_{5a}}{2}, \\
[\Gamma_{2a}, \Gamma_{5a}] &= \frac{\Gamma_{2a}}{2}, \\
[\Gamma_{5a}, \Gamma_{7a}] &= \Gamma_{7a}.
\end{align*}
\]

from which we can infer that the symmetry vectors form the Lie algebra \(A_{3,5}^0 \oplus 2A_1\). Hence the admitted Lie algebra for the system (6) and (7) is

\[
A_{3,5}^0 \oplus 2A_1 \oplus 2A_{1\infty}.
\]
3.3 Lie Symmetries for the Third Member of the Complex Burgers’ Hierarchy

As far as concerns the Lie symmetries for the third member of the complex Burgers’ hierarchy, i.e. system (8) and (9), we find the generic symmetry
\[
\Gamma = (B_0 + B_1 t)\partial_t + \left( B_2 + \frac{B_1 x}{3} \right) \partial_x
\]
\[
+ \left( B_5 - \frac{B_4 \cos(2 \nu)}{2} - e^{-w} \left( -d(t, x) \cos(v) \right. \right.
\]
\[
+ c(t, x) \sin(v) \left. \right) + \frac{B_3 \sin(2 \nu)}{2} \right) \partial_w
\]
\[
+ \left( B_6 - e^{-w} \left( c(t, x) \cos(v) + d(t, x) \sin(v) \right. \right.
\]
\[
+ \frac{B_3 \cos(2 \nu)}{2} + B_4 \sin(2 \nu) \right) \partial_v,
\]
where \( B_{0-7} \) are arbitrary constants, \( c(t, x) \) and \( d(t, x) \) satisfy the linear evolution equations
\[
c_t - c_{xxx} = 0 \quad \text{and} \quad (25)
\]
\[
d_t - d_{xxx} = 0. \quad (26)
\]

From the general symmetry, we can write the seven vector fields which are
\[
\Gamma_{1b} = \partial_t,
\]
\[
\Gamma_{2b} = \partial_t + \frac{x}{3} \partial_x,
\]
\[
\Gamma_{3b} = \partial_x,
\]
\[
\Gamma_{4b} = \partial_w,
\]
\[
\Gamma_{5b} = \frac{\sin 2 \nu}{2} \partial \nu - \frac{\cos 2 \nu}{2} \partial w,
\]
\[
\Gamma_{6b} = \frac{\cos 2 \nu}{2} \partial \nu + \frac{\sin 2 \nu}{2} \partial w,
\]
\[
\Gamma_{7b} = \partial \nu
\]
for which the non-zero Lie brackets are
\[
[\Gamma_{1b}, \Gamma_{3b}] = \Gamma_{1b} \quad [\Gamma_{5b}, \Gamma_{6b}] = -\frac{\Gamma_{7b}}{2}
\]
\[
[\Gamma_{2b}, \Gamma_{3b}] = \frac{\Gamma_{1b}}{3} \quad [\Gamma_{5b}, \Gamma_{7b}] = -2 \Gamma_{6b}.
\]
Hence, the Lie point symmetries for the third member of the complex Burgers’ hierarchy form the \( A^{3,4}_3 \oplus 3A_1 \oplus 2A_{1\infty} \) Lie algebra.

3.4 Lie Symmetries for the Fourth Member of the Complex Burgers’ Hierarchy

Finally, for the fourth member of the complex Burgers’ hierarchy we find that the system, (11, 12), admits only four Lie symmetry vectors,
\[
\Gamma_{1c} = \partial_t, \quad \Gamma_{2c} = \partial_v,
\]
\[
\Gamma_{3c} = \partial_w, \quad \Gamma_{4c} = \partial_x
\]
which constitute the Lie algebra \( 4A_1 \) under the operation of taking the Lie bracket.

We continue our analysis with the application of the symmetry vectors to reduce the system of partial differential equations and to find possible similarity solutions. The reduction process is studied for the second, the third, and the fourth members of the hierarchy, and more specifically we focus on the travelling-wave solutions. The reason that we choose to perform the reduction by searching for travelling-wave solutions is because that it is the only common reduction among all the four members of the hierarchy that we studied. With such an analysis we are able to compare the travelling-wave solutions as we move to the higher-order members of the hierarchy.

4 Travelling-Wave Solutions

4.1 Reduction Process for the Second Member of the Complex Burgers’ Hierarchy

Consider the vector fields, \( \Gamma_{1a} + c \Gamma_{4a} \), which are symmetries of the system, (6) and (7), and \( c \) is a constant which, as we see below, corresponds to the “speed” of the travelling-wave solution. The similarity variables, i.e. Lie invariants are given in (21).

In the new variables (6) and (7) reduce to a system of two second-order ordinary differential equations, namely,
\[
g''(s) - f'^2 + g'^2 + cf'(s) = 0, \quad (27)
\]
\[
f''(s) + 2f'(s)g'(s) - cg'(s) = 0. \quad (28)
\]

This system of ordinary differential equations admits a 12-dimensional algebra comprising vector fields
\[
\Gamma_{1d} = \partial_s
\]
\[
\Gamma_{2d} = \left( \frac{\cos 2f \cos cs}{2} + \frac{\sin 2f \sin cs}{2} \right) \partial_f
\]
Figure 1: Qualitative evolution for the solution (32) for different values of the constant of integration constant, $F$. Left figures is for function $F(s)$ while right figure is for function $G(s)$. 

Easily the system, (27) and (28), is reduced to the following first-order equations

\[ G'(s) + cF(s) + G^2(s) - F^2(s) = 0 \]  
\[ F'(s) + 2F(s)G(s) - cG(s) = 0, \]

where $G(s) = g'(s)$ and $F(s) = f'(s)$. Easily the solution of the latter system can be written in closed form as

\[ F(s) = \frac{c}{2}, \quad G(s) = -\frac{c}{2} \tan \left( \frac{c}{2} (s - s_0) \right) \]  

or

\[ G(s) = -\frac{F'}{2F - c}, \]

\[ F(s) = \frac{c}{2} \frac{F_0(e^{-2ics} - F_1c)^2 - 16c^2 - 8cF_0e^{-ics}}{F_0(e^{-2ics} - F_1c)^2 - 16c^2}. \]

The corresponding behaviour of the functions $F(s)$ and $G(s)$, for various values of $F_1$, is plotted in Figure 1 in which we can observe the existence of wave solutions.

As it is evident from the figure, less turbulence prevails for $F_1$ at 0, as compared to other values. It is important
to mention that (29) with use of (30) can be written as a second-order differential equation,
\[
\left( F'' + \frac{1}{(2F - c)^3}F^2 \right) - F(F - c)(2F - c) = 0, \tag{33}
\]
which is invariant under the elements of the \textit{si}(2, R) Lie algebra. This means that the equation can be easily transformed to the Ermakov–Pinney equation. We continue with the third member of the hierarchy.

### 4.2 Reduction Process for the Third Member of the Complex Burgers’ Hierarchy

The travelling-wave solution for the system, (8) and (9), with respect to the similarity variables for \( \Gamma_{1b} + c \Gamma_{3b} \) is of form of (21). The reduced equations are
\[
f'''(s) + cf'(s) - f'^3 + 3f'(s)g'^2 + 3g(s)f''(s) + 3f'(s)g''(s) = 0 \tag{34}
\]
\[
g'''(s) + cg'(s) - 3f'^2g(s) + g'^3 - 3f'(s)f''(s) + 3g'(s)g''(s) = 0 \tag{35}
\]
This system is invariant under the action of a five-dimensional Lie algebra comprising the following vector fields
\[
\Gamma_{1f} = \partial_f
\]
\[
\Gamma_{2f} = \partial_g
\]
\[
\Gamma_{3f} = -\left( \frac{\sin \sqrt{cs}}{\sqrt{c}} \right) \partial_s - \cos \sqrt{cs} \partial_g
\]
\[
\Gamma_{4f} = -\left( \frac{\cos \sqrt{cs}}{\sqrt{c}} \right) \partial_s + \sin \sqrt{cs} \partial_g
\]
\[
\Gamma_{5f} = \partial_s
\]
and non-zero Lie brackets,
\[
[\Gamma_{3f}, \Gamma_{4f}] = -\frac{\Gamma_{5f}}{\sqrt{c}}, \quad [\Gamma_{3f}, \Gamma_{5f}] = -\sqrt{c} \Gamma_{4f},
\]
\[
[\Gamma_{4f}, \Gamma_{5f}] = \sqrt{c} \Gamma_{3f},
\]
that is, the vector fields, \( \Gamma_{lf} \), form the \( 2A_1 \oplus \mathfrak{s} \mathfrak{o}(2, 1) \) Lie algebra.

The application of the autonomous symmetry \( \partial_s \) in the system (34), (35) leads us also to the autonomous system of second-order differential equations,
\[
F'' - cF - F^3 + 3FG^2 + 3GF' + 3FG' = 0 \tag{36}
\]
\[
G'' - cG - 3F^2G + G^3 - 3FG' + 3GG' = 0 \tag{37}
\]
where, as above, \( G(s) = g'(s) \) and \( F(s) = f'(s) \). From the Lie symmetries \( \Gamma_{1f} \) and \( \Gamma_{3f} \) of the system (36) and (37) we can define the particular solution and
\[
F(s) = 0,
\]
\[
G(s) = \sqrt{c} \frac{\sin(\sqrt{cs}) - G_0 \cos(\sqrt{cs})}{G_0 \sin(\sqrt{cs}) + \cos(\sqrt{cs}) + G_1}. \tag{38}
\]
By using the symmetry \( \Gamma_{2f} \) we conclude that
\[
G(s) = 0, \quad F(s) = F_0 \sin(s, a_1) \tag{39}
\]
where \( \sin(s, a_1) \) is the Jacobi elliptic function and \( a_0(s) = \frac{\sqrt{a_1 + \sqrt{4c - a_1}}}{2a_1 + 2c - 1} \). In a similar way we can construct similarity solutions by using the combination of the Lie symmetries \( \Gamma_{1f} + \beta \Gamma_{2f} \).

### 4.3 Reduction Process for the Fourth Member of the Complex Burgers’ Hierarchy

The reduced equations with respect to the \( \Gamma_{1c} + c \Gamma_{ac} \), as mentioned above, are
\[
g'''(s) = f'(s)^4 - 6f'(s)^2g'(s)^2 + g'(s)^4 + 3g'(s)f'''(s)
\]
\[
+ 6f'(s)g'(s)f''(s) + 3f''(s)^2 + 3f'(s)g''(s)
\]
\[
+ 3f'(s)^2g''(s) - 3g'(s)^2g''(s) - 3g''(s)^2
\]
\[
+ 4f'(s)f'''(s) + 4g'(s)g'''(s) + cf'(s),
\]
\[
f'''(s) = 4f'(s)g'(s)^3 - 6f'(s)^3g'(s) + 3f'(s)f'''(s)
\]
\[
+ 3f'(s)^2f''(s) - 3g'(s)^2f''(s) - 3g'(s)g''(s)
\]
\[
- 6f'(s)g'(s)g''(s) - 6f'(s)g''(s) - 4g'(s)f'''(s)
\]
\[
+ 4f'(s)g'''(s) - cg'(s).
\]
The Lie point symmetries of the resulting system are
\[
\Gamma_{1j} = \partial_s, \quad \Gamma_{2j} = \partial_f, \quad \Gamma_{3j} = \partial_g.
\]
The application of the Lie symmetry, \( \partial_s \), leads us again to an autonomous third-order dynamical system with only two symmetries, the \( \Gamma_{2j} \) and \( \Gamma_{3j} \). From these symmetry vectors the only possible solution that we can
get is
\[ g(s) = g_0, \ f(s) = f_1 s + f_0. \]  \hspace{1cm} (40)

This is a particular real solution. However, it is not a travelling-wave solution. Hence in order to study the existence of solutions we should generalise the context of symmetries to the case of non-point symmetries or use other methods for solving nonlinear differential equations.

## 5 Conclusions

This work focussed on the study of the algebraic properties of the differential equations which belong to the complex Burgers’ hierarchy. More specifically, we studied the Lie point symmetries for the first four equations of the complex Burgers’ hierarchy. We found that the first member of the hierarchy is invariant under an infinite number of symmetries. The second member of the hierarchy is invariant under the group of transformations with generators the elements of the \( A_{3,5}^n \oplus 2A_1 \oplus 2A_{1\infty} \), which comprises seven plus two times infinity symmetries. The third member of the hierarchy is invariant under \( A_{3,4}^n \oplus 3A_1 \oplus 2A_{1\infty} \). On the other hand, the fourth member of the hierarchy is invariant under the four-dimensional Lie algebra \( 4A_1 \).

From the symmetry analysis it is clear that the differential equations which belong to the first three members of the complex Burgers’ hierarchy can be linearised because the infinite number of point symmetries exists (vice versa). However, we cannot reach a similar conclusion for the fourth member of the hierarchy, at least as far as concerns point transformations. However, it is well known that the Burgers’ hierarchy is linearised by the Cole-Hopf transformation [10].

As far as concerns regarding the number of admitted Lie point symmetries, someone may expect a common feature among the different members of the hierarchy. However, that is not true. We observe that as we proceed through the hierarchy, the number of symmetries decreases. The only common symmetries are the time and space translation, \( \{ \partial_t, \partial_x \} \), which of course exist because the differential equations are autonomous and homogeneous. The linear combination of these two symmetries forms the \( 2A_1 \) Lie algebra and provides the similarity variables for the travelling-wave solutions.

We applied these two symmetries for all the members of the hierarchy in our study and we reduced the systems of partial differential equations to systems of ordinary differential equations. For these systems we determined the Lie point symmetries and we proceeded with the further reduction. We conclude that the travelling-wave solutions can be determined explicitly by the use of Lie point symmetries only for the first, second, and third members of the complex Burgers’ hierarchy.

From our analysis it is clear that one should generalise the context of symmetries to non-point symmetries in order to study the higher members of the hierarchy and to determine the analytic and exact solutions. Such an analysis is a subject of a further study.

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