SYMMETRIC CRYSTALS FOR $\mathfrak{gl}_\infty$

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Abstract. In the preceding paper, we formulated a conjecture on the relations between certain classes of irreducible representations of affine Hecke algebras of type B and symmetric crystals for $\mathfrak{gl}_\infty$. In the present paper, we prove the existence of the symmetric crystal and the global basis for $\mathfrak{gl}_\infty$.

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1. Introduction

Lascoux-Leclerc-Thibon ([LLT]) conjectured the relations between the representations of Hecke algebras of type $A$ and the crystal bases of the affine Lie algebras of type $A$. Then, S. Ariki ([A]) observed that it should be understood in the setting of affine Hecke algebras and proved the LLT conjecture in a more general framework. Recently, we presented the
notation of symmetric crystals and conjectured that certain classes of irreducible representations of the affine Hecke algebras of type $B$ are described by symmetric crystals for $\mathfrak{gl}_\infty$ ([EK]).

The purpose of the present paper is to prove the existence of symmetric crystals in the case of $\mathfrak{gl}_\infty$.

Let us recall the Lascoux-Leclerc-Thibon-Ariki theory. Let $H_n^A$ be the affine Hecke algebra of type $A$ of degree $n$. Let $K_n^A$ be the Grothendieck group of the abelian category of finite-dimensional $H_n^A$-modules, and $K^A = \oplus_{n \geq 0} K_n^A$. Then it has a structure of Hopf algebra by the restriction and the induction. The set $I = \mathbb{C}^*$ may be regarded as a Dynkin diagram with $I$ as the set of vertices and with edges between $a \in I$ and $ap_i^2$. Here $p_i$ is the parameter of the affine Hecke algebra usually denoted by $q$. Let $\mathfrak{g}_I$ be the associated Lie algebra, and $\mathfrak{g}_I^-$ the unipotent Lie subalgebra. Let $U_I$ be the group associated to $\mathfrak{g}_I^-$. Hence $\mathfrak{g}_I$ is isomorphic to a direct sum of copies of $A^{(1)}_\ell$ if $p_i$ is a primitive $\ell$-th root of unity and to a direct sum of copies of $\mathfrak{gl}_\infty$ if $p_i$ has an infinite order. Then $\mathbb{C} \otimes K^A$ is isomorphic to the algebra $\mathcal{O}(U_I)$ of regular functions on $U_I$. Let $U_q(\mathfrak{g}_I)$ be the associated quantized enveloping algebra. Then $U_q^-(\mathfrak{g}_I)$ has an upper global basis $\{G^{up}(b)\}_{b \in B(\infty)}$. By specializing $\bigoplus \mathbb{C}[q, q^{-1}]G^{up}(b)$ at $q = 1$, we obtain $\mathcal{O}(U_I)$. Then the LLTA-theory says that the elements associated to irreducible $H^A$-modules corresponds to the image of the upper global basis.

In [EK], we gave analogous conjectures for affine Hecke algebras of type $B$. In the type $B$ case, we have to replace $U_q^-(\mathfrak{g}_I)$ and its upper global basis with symmetric crystals (see §23). It is roughly stated as follows. Let $H_n^B$ be the affine Hecke algebra of type $B$ of degree $n$. Let $K_n^B$ be the Grothendieck group of the abelian category of finite-dimensional modules over $H_n^B$, and $K^B = \oplus_{n \geq 0} K_n^B$. Then $K^B$ has a structure of a Hopf bimodule over $K^A$. The group $U_I$ has the anti-involution $\theta$ induced by the involution $a \mapsto a^{-1}$ of $I = \mathbb{C}^*$. Let $U_I^\theta$ be the $\theta$-fixed point set of $U_I$. Then $\mathcal{O}(U_I^\theta)$ is a quotient ring of $\mathcal{O}(U_I)$. The action of $\mathcal{O}(U_I) \simeq \mathbb{C} \otimes K^A$ on $\mathbb{C} \otimes K^B$, in fact, descends to the action of $\mathcal{O}(U_I^\theta)$.

We introduce $V_\theta(\lambda)$ (see §23), a kind of the $q$-analogue of $\mathcal{O}(U_I^\theta)$. The conjecture in [EK] is then:

(i) $V_\theta(\lambda)$ has a crystal basis and a global basis.

(ii) $K^B$ is isomorphic to a specialization of $V_\theta(\lambda)$ at $q = 1$ as an $\mathcal{O}(U_I)$-module, and the irreducible representations correspond to the upper global basis of $V_\theta(\lambda)$ at $q = 1$.

Remark. In [KM], Miemietz and the second author gave an analogous conjecture for the affine Hecke algebras of type $D$.

In the present paper, we prove that $V_\theta(\lambda)$ has a crystal basis and a global basis for $\mathfrak{g} = \mathfrak{gl}_\infty$ and $\lambda = 0$.

More precisely, let $I = \mathbb{Z}_{odd}$ be the set of odd integers. Let $\alpha_i (i \in I)$ be the simple roots with

$$(\alpha_i, \alpha_j) = \begin{cases} 2 & \text{if } i = j, \\ -1 & \text{if } i = j \pm 2, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\theta$ be the involution of $I$ given by $\theta(i) = -i$. Let $\mathcal{B}_\theta(\mathfrak{gl}_\infty)$ be the algebra over $K := \mathbb{Q}(q)$ generated by $E_i$, $F_i$, and invertible elements $T_i$ ($i \in I$) satisfying the following defining relations:

(i) the $T_i$’s commute with each other,

(ii) $T_{\theta(i)} = T_i$ for any $i$,
(iii) $T_i E_i T_i^{-1} = q^{(\alpha_i + \alpha_\theta(i), \alpha_i)} E_i$ and $T_i F_i T_i^{-1} = q^{(\alpha_i + \alpha_\theta(i), -\alpha_i)} F_i$ for $i, j \in I$,
(iv) $E_i F_j = q^{-(\alpha_i, \alpha_j)} F_j E_i + (\delta_{i,j} + \delta_{\theta(i), j}) T_i$ for $i, j \in I$,
(v) the $E_i$'s and the $F_i$'s satisfy the Serre relations (see Definition 2.1 (4)).

Then there exists a unique irreducible $B_\theta(\mathfrak{gl}_\infty)$-module $V_\theta(0)$ with a generator $\phi$ satisfying $E_i \phi = 0$ and $T_i \phi = \phi$ (Proposition 2.11). We define the endomorphisms $\tilde{E}_i$ and $\tilde{F}_i$ of $V_\theta(0)$ by

$$\tilde{E}_i a = \sum_{n \geq 1} F_i^{(n-1)} a_n, \quad \tilde{F}_i a = \sum_{n \geq 0} F_i^{(n+1)} a_n,$$

when writing

$$a = \sum_{n \geq 0} F_i^{(n)} a_n \quad \text{with} \quad E_i a_n = 0.$$

Here $F_i^{(n)} = F_i^n / [n]!$ is the divided power. Let $A_0$ be the ring of functions $a \in K$ which do not have a pole at $q = 0$. Let $L_\theta(0)$ be the $A_0$-submodule of $V_\theta(0)$ generated by the elements $\tilde{F}_i \cdots \tilde{F}_i \phi$ ($\ell \geq 0, i_1, \ldots, i_\ell \in I$). Let $B_\theta(0)$ be the subset of $L_\theta(0)/qL_\theta(0)$ consisting of the $\tilde{F}_i \cdots \tilde{F}_i \phi$'s. In this paper, we prove the following theorem.

**Theorem (Theorem 4.15).**
(i) $\tilde{F}_i L_\theta(0) \subset L_\theta(0)$ and $\tilde{E}_i L_\theta(0) \subset L_\theta(0)$,
(ii) $B_\theta(0)$ is a basis of $L_\theta(0)/qL_\theta(0)$,
(iii) $\tilde{F}_i B_\theta(0) \subset B_\theta(0)$, and $\tilde{E}_i B_\theta(0) \subset B_\theta(0) \cup \{0\}$,
(iv) $\tilde{F}_i \tilde{E}_i (b) = b$ for any $b \in B_\theta(0)$ with respect to $\tilde{E}_i b \neq 0$, and $\tilde{E}_i \tilde{F}_i (b) = b$ for any $b \in B_\theta(0)$.

Let $-\bar{\phi}$ be the bar operator of $V_\theta(0)$. Namely, $-$ is a unique endomorphism of $V_\theta(0)$ such that $\bar{\phi} = \phi$, $\bar{a v} = \bar{a} v$ and $\bar{F}_i \bar{v} = \tilde{F}_i v$ for $a \in K$ and $v \in V_\theta(0)$. Here $\bar{a} q = a (q^{-1})$.

Then we prove the existence of global basis:

**Theorem (Theorem 5.5).** Let $V_\theta(0)_A$ be the smallest submodule of $V_\theta(0)$ over $A := \mathbb{Q}[q, q^{-1}]$ such that it contains $\phi$ and is stable by the $F_i^{(n)}$'s.

(i) For any $b \in B_\theta(0)$, there exists a unique $G^\text{low}_\theta(b) \in V_\theta(0)_A \cap L_\theta(0)$ such that $\overline{\bar{G}}^\text{low}_\theta(b) = G^\text{low}_\theta(b)$ and $b = G^\text{low}_\theta(b) \mod qL_\theta(0)$,
(ii) $L_\theta(0) = \bigoplus_{b \in B_\theta(0)} A_0 G^\text{low}_\theta(b)$, $V_\theta(0)_A = \bigoplus_{b \in B_\theta(0)} A G^\text{low}_\theta(b) \text{ and } V_\theta(0) = \bigoplus_{b \in B_\theta(0)} KG^\text{low}_\theta(b)$.

We call $G^\text{low}_\theta(b)$ the lower global basis. The $B_\theta(\mathfrak{gl}_\infty)$-module $V_\theta(0)$ has a unique symmetric bilinear form $(\ast, \ast)$ such that $(\phi, \phi) = 1$ and $E_i$ and $F_i$ are transpose to each other. The dual basis to $\{G^\text{low}_\theta(b)\}_{b \in B_\theta(0)}$ with respect to $(\ast, \ast)$ is called an upper global basis.

Let us explain the strategy of our proof of these theorems. We first construct a PBW type basis $\{P_\theta(m)\}_m$ of $V_\theta(0)$ parametrized by the $\theta$-restricted multisegments $m$. Then, we explicitly calculate the actions of $E_i$ and $F_i$ in terms of the PBW basis $\{P_\theta(m)\}_m$. Then, we prove that the PBW basis gives a crystal basis by the estimation of the coefficients of these actions. For this we use a criterion for crystal bases (Theorem 4.8).

## 2. General definitions and conjectures

### 2.1. Quantized universal enveloping algebras and its reduced $q$-analogues

We shall recall the quantized universal enveloping algebra $U_q(\mathfrak{g})$. Let $I$ be an index set (for simple roots), and $Q$ the free $\mathbb{Z}$-module with a basis $\{\alpha_i\}_{i \in I}$. Let $(\ast, \ast): Q \times Q \rightarrow \mathbb{Z}$ be a symmetric bilinear form such that $(\alpha_i, \alpha_i)/2 \in \mathbb{Z}_{\geq 0}$ for any $i$ and $(\alpha_i^\vee, \alpha_j) \in \mathbb{Z}_{\leq 0}$ for $i \neq j$.
where $\alpha^\gamma_i := 2\alpha_i/(\alpha_i, \alpha_i)$. Let $q$ be an indeterminate and set $\mathbf{K} := \mathbb{Q}(q)$. We define its subrings $\mathbf{A}_0$, $\mathbf{A}_\infty$ and $\mathbf{A}$ as follows.

$$
\begin{align*}
\mathbf{A}_0 &= \{ f \in \mathbf{K} \mid f \text{ is regular at } q = 0 \}, \\
\mathbf{A}_\infty &= \{ f \in \mathbf{K} \mid f \text{ is regular at } q = \infty \}, \\
\mathbf{A} &= \mathbb{Q}[q, q^{-1}].
\end{align*}
$$

**Definition 2.1.** The quantized universal enveloping algebra $U_q(\mathfrak{g})$ is the $\mathbf{K}$-algebra generated by elements $e_i, f_i$ and invertible elements $t_i$ ($i \in I$) with the following defining relations.

1. The $t_i$’s commute with each other.
2. $t_je_i t_j^{-1} = q^{(\alpha_i, \alpha_j)} e_i$ and $t_j f_i t_j^{-1} = q^{-(\alpha_j, \alpha_i)} f_i$ for any $i, j \in I$.
3. $[e_i, f_j] = \delta_{ij} t_i - t_i^{-1} \frac{q_i - q_j}{q_i - q_j^{-1}}$ for $i, j \in I$. Here $q_i := q^{(\alpha_i, \alpha_i)}/2$.
4. (Serre relation) For $i \neq j$,

$$
\sum_{k=0}^b (-1)^k e_i^{(k)} e_j e_i^{(b-k)} = 0, \quad \sum_{k=0}^b (-1)^k f_i^{(k)} f_j f_i^{(b-k)} = 0.
$$

Here $b = 1 - (\alpha_i^\gamma, \alpha_j)$ and

$$
e_i^{(k)} = e_i^k/[k]!, \quad f_i^{(k)} = f_i^k/[k]!, \quad [k]_i = (q_i^k - q_i^{-k})/(q_i - q_i^{-1}), \quad [k]_i! = [1][2] \cdots [k]_i.
$$

Let us denote by $U_q^-(\mathfrak{g})$ (resp. $U_q^+(\mathfrak{g})$) the $\mathbf{K}$-subalgebra of $U_q(\mathfrak{g})$ generated by the $f_i$’s (resp. the $e_i$’s).

Let $e_i^\prime$ and $e_i^\ast$ be the operators on $U_q^-(\mathfrak{g})$ defined by

$$
[e_i, a] = \frac{(e_i a)t_i - t_i^{-1} e_i a}{q_i - q_i^{-1}} \quad (a \in U_q^-(\mathfrak{g})).
$$

These operators satisfy the following formulas similar to derivations:

$$
e_i^\prime(ab) &= e_i^\prime(a)b + (\text{Ad}(t_i)a)e_i^\prime b, \\
e_i^\ast(ab) &= ae_i^\ast b + (e_i^\ast a)(\text{Ad}(t_i)b).
$$

The algebra $U_q^-(\mathfrak{g})$ has a unique symmetric bilinear form $(\cdot, \cdot)$ such that $(1, 1) = 1$ and

$$(e_i^\prime a, b) = (a, f_i b) \quad \text{for any } a, b \in U_q^-(\mathfrak{g}).$$

It is non-degenerate and satisfies $(e_i^\ast a, b) = (a, b f_i)$. The left multiplication of $f_j, e_i^\prime$ and $e_i^\ast$ have the commutation relations

$$
e_i^\prime f_j = q^{-(\alpha_i, \alpha_j)} f_j e_i^\prime + \delta_{ij}, \quad e_i^\ast f_j = f_j e_i^\ast + \delta_{ij} \text{Ad}(t_i),$$

and both the $e_i^\prime$’s and the $e_i^\ast$’s satisfy the Serre relations.

**Definition 2.2.** The reduced $q$-analogue $\mathcal{B}(\mathfrak{g})$ of $\mathfrak{g}$ is the $\mathbf{K}$-algebra generated by $e_i^\prime$ and $f_i$.

2.2. Review on crystal bases and global bases. Since $e_i^\prime$ and $f_i$ satisfy the $q$-boson relation, any element $a \in U_q^-(\mathfrak{g})$ can be uniquely written as

$$
a = \sum_{n \geq 0} f_i^{(n)} a_n \quad \text{with } e_i^\prime a_n = 0.
$$

Here $f_i^{(n)} = \frac{f_i^n}{[n]_i!}$.
Definition 2.3. We define the modified root operators $\tilde{e}_i$ and $\tilde{f}_i$ on $U_q^{-}(g)$ by
$$\tilde{e}_i a = \sum_{n \geq 1} f_i^{(n-1)} a_n, \quad \tilde{f}_i a = \sum_{n \geq 0} f_i^{(n+1)} a_n.$$ 

Theorem 2.4 ([K]). We define
$$L(\infty) = \sum_{\ell \geq 0, i_1, \ldots, i_\ell \in I} A_{i_1} f_{i_1} \cdots f_{i_\ell} \cdot 1 \subset U_q^{-}(g),$$
$$B(\infty) = \left\{ \tilde{f}_{i_1} \cdots \tilde{f}_{i_\ell} \cdot 1 \mod qL(\infty) \mid \ell \geq 0, i_1, \ldots, i_\ell \in I \right\} \subset L(\infty)/qL(\infty).$$
Then we have
(i) $\tilde{e}_i L(\infty) \subset L(\infty)$ and $\tilde{f}_i L(\infty) \subset L(\infty)$,
(ii) $B(\infty)$ is a basis of $L(\infty)/qL(\infty)$,
(iii) $\tilde{f}_i B(\infty) \subset B(\infty)$ and $\tilde{e}_i B(\infty) \subset B(\infty) \cup \{0\}$.
We call $(L(\infty), B(\infty))$ the crystal basis of $U_q^{-}(g)$.

Let $-\theta$ be the automorphism of $K$ sending $q$ to $q^{-1}$. Then $A_0$ coincides with $A_\infty$.
Let $V$ be a vector space over $K$, $L_0$ an $A_0$-submodule of $V$, $L_\infty$ an $A_\infty$-submodule, and $V_A$ an $A$-submodule. Set $E := L_0 \cap L_\infty \cap V_A$.

Definition 2.5 ([K]). We say that $(L_0, L_\infty, V_A)$ is balanced if each of $L_0$, $L_\infty$ and $V_A$ generates $V$ as a $K$-vector space, and if one of the following equivalent conditions is satisfied.
(i) $E \to L_0/qL_0$ is an isomorphism,
(ii) $E \to L_\infty/q^{-1}L_\infty$ is an isomorphism,
(iii) $(L_0 \cap V_A) \oplus (q^{-1}L_\infty \cap V_A) \to V_A$ is an isomorphism,
(iv) $A_0 \otimes Q E \to L_0$, $A_\infty \otimes Q E \to L_\infty$, $A \otimes Q E \to V_A$ and $K \otimes Q E \to V$ are isomorphisms.

Let $-\theta$ be the ring automorphism of $U_q(g)$ sending $q, t_i, e_i, f_i$ to $q^{-1}, t_i^{-1}, e_i, f_i$.
Let $U_q(g)_A$ be the $A$-subalgebra of $U_q(g)$ generated by $e_i^{(n)}$, $f_i^{(n)}$ and $t_i$. Similarly we define $U_q^{-}(g)_A$.

Theorem 2.6. $(L(\infty), L(\infty)^-, U_q^{-}(g)_A)$ is balanced.

Let
$$G^{\text{low}} : L(\infty)/qL(\infty) \to E := L(\infty) \cap L(\infty)^- \cap U_q^{-}(g)_A$$
be the inverse of $E \cap L(\infty)/qL(\infty)$. Then $\{G^{\text{low}}(b) \mid b \in B(\infty)\}$ forms a basis of $U_q^{-}(g)$. We call it a (lower) global basis. It is first introduced by G. Lusztig ([L]) under the name of “canonical basis” for the A, D, E cases.

Definition 2.7. Let
$$\{G^{\text{up}}(b) \mid b \in B(\infty)\}$$
be the dual basis of $\{G^{\text{low}}(b) \mid b \in B(\infty)\}$ with respect to the inner product $(\cdot, \cdot)$. We call it the upper global basis of $U_q^{-}(g)$.

2.3. Symmetric crystals. Let $\theta$ be an automorphism of $I$ such that $\theta^2 = \text{id}$ and $(\alpha_\theta(i), \alpha_\theta(j)) = (\alpha_i, \alpha_j)$. Hence it extends to an automorphism of the root lattice $Q$ by $\theta(\alpha_i) = \alpha_\theta(i)$, and induces an automorphism of $U_q(g)$.

Definition 2.8. Let $B_\theta(g)$ be the $K$-algebra generated by $E_i, F_i,$ and invertible elements $T_i$ $(i \in I)$ satisfying the following defining relations:
(i) the $T_i$’s commute with each other,
Lemma 2.9. Identifying $U_q^{-}(\mathfrak{g})$ with the subalgebra of $B_\theta(\mathfrak{g})$ generated by the $F_i$'s, we have

\begin{align}
(2.2) & \quad T_i a = (\text{Ad}(t_i t_{\theta(i)})) a T_i, \\
(2.3) & \quad E_i a = (\text{Ad}(t_i)) a E_i + e_i a + (\text{Ad}(t_i)(e_{\theta(i)} a)) T_i
\end{align}

for $a \in U_q^{-}(\mathfrak{g})$.

\textbf{Proof.} The first relation is obvious. In order to prove the second, it is enough to show that if $a$ satisfies \((2.2)\), then $f_j a$ satisfies \((2.3)\). We have

\begin{align*}
E_i(f_j a) &= (q^{-\alpha_i,\alpha_j}) f_j E_i + \delta_{i,j} + \delta_{\theta(i),j} T_i a \\
&= q^{-\alpha_i,\alpha_j} f_j (\text{Ad}(t_i)) a E_i + e_i a + (\text{Ad}(t_i)(e_{\theta(i)} a)) T_i \\
& \quad + \delta_{i,j} a + \delta_{\theta(i),j} (\text{Ad}(t_i t_{\theta(i)})) a T_i \\
&= ((\text{Ad}(t_i)(f_j a)) E_i + e_i f_j a + (\text{Ad}(t_i)(e_{\theta(i)}(f_j a))) T_i.
\end{align*}

Q.E.D.

The following lemma can be proved in a standard manner and we omit the proof.

Lemma 2.10. Let $K[T_i^{\pm}; i \in I]$ be the commutative $K$-algebra generated by invertible elements $T_i$ ($i \in I$) with the defining relation $T_{\theta(i)} = T_i$. Then the map $U_q^{-}(\mathfrak{g}) \otimes K[T_i^{\pm}; i \in I] \to B_\theta(\mathfrak{g})$ induced by the multiplication is bijective.

Let $\lambda \in P_+ := \{\lambda \in \text{Hom}(Q, \mathbb{Q}) \mid \langle \alpha_i^\vee, \lambda \rangle \in \mathbb{Z}_{\geq 0} \text{ for any } i \in I\}$ be a dominant integral weight such that $\theta(\lambda) = \lambda$.

Proposition 2.11. (i) There exists a $B_\theta(\mathfrak{g})$-module $V_\theta(\lambda)$ generated by a non-zero vector $\phi_\lambda$ such that

(a) $E_i \phi_\lambda = 0$ for any $i \in I$,
(b) $T_i \phi_\lambda = q^{\alpha_i,\lambda} \phi_\lambda$ for any $i \in I$,
(c) $\{u \in V_\theta(\lambda) \mid E_i u = 0 \text{ for any } i \in I\} = K \phi_\lambda$.

Moreover such a $V_\theta(\lambda)$ is irreducible and unique up to an isomorphism.

(ii) there exists a unique symmetric bilinear form $(\cdot, \cdot)$ on $V_\theta(\lambda)$ such that $(\phi_\lambda, \phi_\lambda) = 1$ and $(E_i u, v) = (u, F_i v)$ for any $i \in I$ and $u, v \in V_\theta(\lambda)$, and it is non-degenerate.

Remark 2.12. Set $P_\theta = \{\mu \in P \mid \theta(\mu) = \mu\}$. Then $V_\theta(\lambda)$ has a weight decomposition

$$V_\theta(\lambda) = \bigoplus_{\mu \in P_\theta} V_\theta(\lambda)_\mu,$$

where $V_\theta(\lambda)_\mu = \{u \in V_\theta(\lambda) \mid T_i u = q^{\alpha_i, \mu} u\}$. We say that an element $u$ of $V_\theta(\lambda)$ has a $\theta$-weight $\mu$ and write $\text{wt}_\theta(u) = \mu$ if $u \in V_\theta(\lambda)_\mu$. We have $\text{wt}_\theta(E_i u) = \text{wt}_\theta(u) + (\alpha_i + \alpha_{\theta(i)})$ and $\text{wt}_\theta(F_i u) = \text{wt}_\theta(u) - (\alpha_i + \alpha_{\theta(i)})$.

In order to prove Proposition 2.11 we shall construct two $B_\theta(\mathfrak{g})$-modules.
Lemma 2.13. Let $U_q^{-}(\mathfrak{g})\phi'_\lambda$ be a free $U_q^{-}(\mathfrak{g})$-module with a generator $\phi'_\lambda$. Then the following action gives a structure of a $B_\theta(\mathfrak{g})$-module on $U_q^{-}(\mathfrak{g})\phi'_\lambda$:

\[
\begin{align*}
T_i(a\phi'_\lambda) &= q^{(\alpha_i,\lambda)}(\text{Ad}(t_i t_{\theta(i)})a)\phi'_\lambda, \\
E_i(a\phi'_\lambda) &= (e_i' a + q^{(\alpha_i,\lambda)}\text{Ad}(t_i) (e_i'_{\theta(i)} a)) \phi'_\lambda, \\
F_i(a\phi'_\lambda) &= (f_i a) \phi'_\lambda,
\end{align*}
\]

(2.4) for any $i \in I$ and $a \in U_q^{-}(\mathfrak{g})$.

Moreover $B_\theta(\mathfrak{g})/ \sum_{i \in I} (B_\theta(\mathfrak{g})E_i + B_\theta(\mathfrak{g})(T_i - q^{(\alpha_i,\lambda)})) \to U_q^{-}(\mathfrak{g})\phi'_\lambda$ is an isomorphism.

Proof. We can easily check the defining relations in Definition 2.8 except the Serre relations for the $E_i$'s. For $i \neq j \in I$, set $S = \sum_{n=0}^{\infty} (-1)^n E_i^n E_j (b-n)$ where $b = 1 - \langle h_i, \alpha_j \rangle$. It is enough to show that the action of $S$ on $U_q^{-}(\mathfrak{g})\phi'_\lambda$ is equal to 0. We can check easily that $SF_k = q^{- (\alpha_i,\alpha_j)} F_k S$. Since $S\phi'_\lambda = 0$, we have $SU_q^{-}(\mathfrak{g})\phi'_\lambda = 0$.

Hence $U_q^{-}(\mathfrak{g})\phi'_\lambda$ has a $B_\theta(\mathfrak{g})$-module structure.

The last statement is obvious. Q.E.D.

Lemma 2.14. Let $U_q^{-}(\mathfrak{g})\phi''_\lambda$ be a free $U_q^{-}(\mathfrak{g})$-module with a generator $\phi''_\lambda$. Then the following action gives a structure of a $B_\theta(\mathfrak{g})$-module on $U_q^{-}(\mathfrak{g})\phi''_\lambda$:

\[
\begin{align*}
T_i(a\phi''_\lambda) &= q^{(\alpha_i,\lambda)}(\text{Ad}(t_i t_{\theta(i)})a)\phi''_\lambda, \\
E_i(a\phi''_\lambda) &= (e_i a)\phi''_\lambda, \\
F_i(a\phi''_\lambda) &= (f_i a + q^{(\alpha_i,\lambda)}(\text{Ad}(t_i) a)) \phi''_\lambda,
\end{align*}
\]

(2.5) for any $i \in I$ and $a \in U_q^{-}(\mathfrak{g})$.

Moreover, there exists a non-degenerate bilinear form $\langle \cdot, \cdot \rangle: U_q^{-}(\mathfrak{g})\phi'_\lambda \times U_q^{-}(\mathfrak{g})\phi''_\lambda \to \mathbb{K}$ such that $\langle Fu,v \rangle = \langle u, E_iv \rangle$, $\langle E_iu, v \rangle = \langle u, F_iv \rangle$, $\langle T_iu, v \rangle = \langle u, T_iv \rangle$ for $u \in U_q^{-}(\mathfrak{g})\phi'_\lambda$ and $v \in U_q^{-}(\mathfrak{g})\phi''_\lambda$, and $\langle \phi'_\lambda, \phi'_\lambda \rangle = 1$.

Proof. There exists a unique symmetric bilinear form $(\cdot, \cdot)$ on $U_q^{-}(\mathfrak{g})$ such that $(1,1) = 1$ and $f_i$ and $e_i'$ are transpose to each other. Let us define $(\cdot, \cdot): U_q^{-}(\mathfrak{g})\phi'_\lambda \times U_q^{-}(\mathfrak{g})\phi''_\lambda \to \mathbb{K}$ by $\langle a\phi'_\lambda, b\phi''_\lambda \rangle = (a, b)$ for $a \in U_q^{-}(\mathfrak{g})$ and $b \in U_q^{-}(\mathfrak{g})$. Then we can easily check $\langle F_iu, v \rangle = \langle u, E_i v \rangle$, $\langle T_iu, v \rangle = \langle u, T_i v \rangle$. Since $e_i'$ is transpose to the right multiplication of $f_i$, we have $\langle E_iu, v \rangle = \langle u, F_i v \rangle$. Hence the action of $E_i, F_i, T_i$ on $U_q^{-}(\mathfrak{g})\phi'_\lambda$ and $U_q^{-}(\mathfrak{g})\phi''_\lambda$ satisfy the defining relations in Definition 2.8.

Q.E.D.

Proof of Proposition 2.11. Since $E_i\phi''_\lambda = 0$ and $\phi''_\lambda$ has a $\theta$-weight $\lambda$, there exists a unique $B_\theta(\mathfrak{g})$-linear morphism $\psi: U_q^{-}(\mathfrak{g})\phi'_\lambda \to U_q^{-}(\mathfrak{g})\phi''_\lambda$ sending $\phi'_\lambda$ to $\phi''_\lambda$. Let $V_\theta(\lambda)$ be its image $\psi(U_q^{-}(\mathfrak{g})\phi'_\lambda)$.

(i)(c) follows from $\{ u \in U_q^{-}(\mathfrak{g}) \mid e_i' u = 0 \text{ for any } i \} = \mathbb{K}$ applying to $U_q^{-}(\mathfrak{g})\phi'_\lambda \supset V_\theta(\lambda)$. The other properties (a), (b) are obvious. Let us show that $V_\theta(\lambda)$ is irreducible. Let $S$ be a non-zero $B_\theta(\mathfrak{g})$-submodule. Then $S$ contains a non-zero vector $v$ such that $E_i v = 0$ for any $i$. Then (c) implies that $v$ is a constant multiple of $\phi'_\lambda$. Hence $S = V_\theta(\lambda)$.

Let us prove (ii). For $u, u' \in U_q^{-}(\mathfrak{g})\phi'_\lambda$, set $\langle (u, u') \rangle = \langle u, \psi(u') \rangle$. Then it is a bilinear form on $U_q^{-}(\mathfrak{g})\phi'_\lambda$ which satisfies

\[
\langle (\phi'_\lambda, \phi'_\lambda) \rangle = 1, \quad \langle (F_iu, u') \rangle = \langle (u, E_iu') \rangle, \quad \langle (E_iu, u') \rangle = \langle (u, F_iu') \rangle, \quad \langle (T_iu, u') \rangle = \langle (u, T_iu') \rangle.
\]

(2.6) It is easy to see that a bilinear form which satisfies (2.6) is unique. Since $\langle (u, u') \rangle$ also satisfies (2.6), $\langle (u, u') \rangle$ is a symmetric bilinear form on $U_q^{-}(\mathfrak{g})\phi'_\lambda$. Since $\psi(u') = 0$ implies $\langle (u, u') \rangle = 0$, $\langle (u, u') \rangle$ induces a symmetric bilinear form on $V_\theta(\lambda)$. Since $(\cdot, \cdot)$ is non-degenerate on $U_q^{-}(\mathfrak{g})$, $(\cdot, \cdot)$ is a non-degenerate symmetric bilinear form on $V_\theta(\lambda)$.

Q.E.D.
Lemma 2.15. There exists a unique endomorphism $-\varphi$ of $V_{\theta}(\lambda)$ such that $\varphi_{\lambda} = \phi_{\lambda}$ and $\varphi_{\lambda} = \lambda_{\varphi}$ for any $a \in K$ and $v \in V_{\theta}(\lambda)$.

Proof. The uniqueness is obvious.

Let $\xi$ be an anti-involution of $U^{-}_{q}(g)$ such that $\xi(q) = q^{-1}$ and $\xi(f_{i}) = f_{i}(q)$. Let $\tilde{\rho}$ be an element of $Q \otimes P$ such that $(\tilde{\rho}, \alpha_{i}) = (\alpha_{i}, \alpha_{\theta(i)})/2$. Define $c(\mu) = ((\mu + \tilde{\rho}, \theta(\mu + \tilde{\rho}))/(\alpha_{i}, \alpha_{\theta(i)}) + (\lambda, \mu))$ for $\mu \in P$. Then it satisfies

$$c(\mu) - c(\mu - \alpha_{i}) = (\lambda + \mu, \alpha_{\theta(i)}).$$

Then we define the endomorphism $\Phi$ of $U^{-}_{q}(g)\phi'_{\lambda}$ by $\Phi(a\phi'_{\lambda}) = q^{-c(\mu)}\xi(a)\phi'_{\lambda}$ for $a \in U^{-}_{q}(g)$. Let us show that

$$\Phi(F_{i}(a\phi'_{\lambda})) = F_{i}\Phi(a\phi'_{\lambda}) \quad \text{for any } a \in U^{-}_{q}(g).$$

For $a \in U^{-}_{q}(g)_{\mu}$, we have

$$\Phi(F_{i}(a\phi'_{\lambda})) = \Phi(f_{i}a + q^{(\alpha_{i}, \lambda + \mu)}a_{\theta(i)})\phi'_{\lambda}$$
$$= q^{-c(\mu)}\xi(a)f_{i}\phi'_{\lambda} + q^{(\alpha_{i}, \lambda + \mu) - c(\mu - \alpha_{i})}f_{i}\xi(a)\phi'_{\lambda}.$$ 

On the other hand, we have

$$F_{i}\Phi(a\phi'_{\lambda}) = F_{i}(q^{-c(\mu)}\xi(a)\phi'_{\lambda})$$
$$= q^{-c(\mu)}(f_{i}\xi(a) + q^{(\alpha_{i}, \lambda + \theta(\mu))}\xi(a)f_{i}(\theta(\mu)))\phi'_{\lambda}.$$ 

Therefore we obtain (2.7).

Hence $\Phi$ induces the desired endomorphism of $V_{\theta}(0) \subset U^{-}_{q}(g)\phi'_{\lambda}$. Q.E.D.

Hereafter we assume further that

there is no $i \in I$ such that $\theta(i) = i$.

We conjecture that $V_{\theta}(\lambda)$ has a crystal basis. This means the following. Since $E_{i}$ and $F_{i}$ satisfy the $q$-boson relation, any $u \in V_{\theta}(\lambda)$ can be uniquely written as $u = \sum_{n \geq 0} F_{i}(n) u_{n}$ with $E_{i}u_{n} = 0$. We define the modified root operators $\tilde{E}_{i}$ and $\tilde{F}_{i}$ by:

$$\tilde{E}_{i}(u) = \sum_{n \geq 1} F_{i}(n-1) u_{n} \quad \text{and} \quad \tilde{F}_{i}(u) = \sum_{n \geq 0} F_{i}(n+1) u_{n}.$$ 

Let $L_{\theta}(\lambda)$ be the $A_{\theta}$-submodule of $V_{\theta}(\lambda)$ generated by $\tilde{F}_{i_{1}} \cdots \tilde{F}_{i_{\ell}} \phi_{\lambda}$ ($\ell \geq 0$ and $i_{1}, \ldots, i_{\ell} \in I$), and let $B_{\theta}(\lambda)$ be the subset

$$\{ \tilde{F}_{i_{1}} \cdots \tilde{F}_{i_{\ell}} \phi_{\lambda} \mod qL_{\theta}(\lambda) \mid \ell \geq 0, i_{1}, \ldots, i_{\ell} \in I \}$$

of $L_{\theta}(\lambda)/qL_{\theta}(\lambda)$.

Conjecture 2.16. Let $\lambda$ be a dominant integral weight such that $\theta(\lambda) = \lambda$. Then we have

1. $\tilde{F}_{i} L_{\theta}(\lambda) \subset L_{\theta}(\lambda)$ and $\tilde{E}_{i} L_{\theta}(\lambda) \subset L_{\theta}(\lambda)$,
2. $B_{\theta}(\lambda)$ is a basis of $L_{\theta}(\lambda)/qL_{\theta}(\lambda)$,
3. $\tilde{F}_{i} B_{\theta}(\lambda) \subset B_{\theta}(\lambda)$ and $\tilde{E}_{i} B_{\theta}(\lambda) \subset B_{\theta}(\lambda) \cup \{0\}$,
4. $\tilde{F}_{i} \tilde{E}_{i}(b) = b$ for any $b \in B_{\theta}(\lambda)$ such that $\tilde{E}_{i} b \neq 0$, and $\tilde{E}_{i} \tilde{F}_{i}(b) = b$ for any $b \in B_{\theta}(\lambda)$.

As in [K], we have

Lemma 2.17. Assume Conjecture 2.16. Then we have

1. $L_{\theta}(\lambda) = \{ v \in V_{\theta}(\lambda) \mid (L_{\theta}(\lambda), v) \subset A_{\theta} \}$,
(ii) Let $(\cdot, \cdot)_0$ be the $\mathbb{Q}$-valued symmetric bilinear form on $L_\theta(\lambda)/qL_\theta(\lambda)$ induced by 
$(\cdot, \cdot)$. Then $B_\theta(\lambda)$ is an orthonormal basis with respect to $(\cdot, \cdot)_0$.

Moreover we conjecture that $V_\theta(\lambda)$ has a global crystal basis. Namely we have

**Conjecture 2.18.** $(L_\theta(\lambda), L_\theta(\lambda)^-, V_\theta(\lambda)^{\text{low}})$ is balanced. Here $V_\theta(\lambda)^{\text{low}} := U_q^{-}(\mathfrak{g}) A \phi_\lambda$.

Its dual version is as follows.

Let us denote by $V_\theta(\lambda)^{\text{up}}_A$ the dual space $\{ v \in V_\theta(\lambda) \mid (V_\theta(\lambda)^{\text{low}}_A, v) \subset A \}$. Then Conjecture 2.18 is equivalent to the following conjecture.

**Conjecture 2.19.** $(L_\theta(\lambda), c(L_\theta(\lambda)), V_\theta(\lambda)^{\text{up}}_A)$ is balanced.

Here $c$ is a unique endomorphism of $V_\theta(\lambda)$ such that $c(\phi_\lambda) = \phi_\lambda$ and $c(\alpha v) = \bar{a} c(v)$, $c(E_i v) = E_i c(v)$ for any $a \in K$ and $v \in V_\theta(\lambda)$. We have $(c(v'), v) = (v', \bar{v})$ for any $v, v' \in V_\theta(\lambda)$.

Note that $V_\theta(\lambda)^{\text{up}}_A$ is the largest $A$-submodule of $V_\theta(\lambda)$ such that $M$ is invariant by the $E_i(n)$'s and $M \cap K \phi_\lambda = A \phi_\lambda$.

By Conjecture 2.19, $L_\theta(\lambda) \cap c(L_\theta(\lambda)) \cap V_\theta(0)^{\text{up}}_A \to L_\theta(\lambda)/qL_\theta(\lambda)$ is an isomorphism. Let $G_\theta^{\text{up}}$ be its inverse. Then $\{ G_\theta^{\text{up}}(b) \}_{b \in B_\theta(\lambda)}$ is a basis of $V_\theta(\lambda)$, which we call the upper global basis of $V_\theta(\lambda)$. Note that $\{ G_\theta^{\text{up}}(b) \}_{b \in B_\theta(\lambda)}$ is the dual basis to $\{ G_\theta^{\text{low}}(b) \}_{b \in B_\theta(\lambda)}$ with respect to the inner product of $V_\theta(\lambda)$.

We shall prove these conjectures in the case $\mathfrak{g} = \mathfrak{gl}_\infty$ and $\lambda = 0$.

### 3. PBW basis of $V_\theta(0)$ for $\mathfrak{g} = \mathfrak{gl}_\infty$

#### 3.1. Review on the PBW basis.

In the sequel, we set $I = \mathbb{Z}_{\text{odd}}$ and $$(\alpha_i, \alpha_j) = \begin{cases} 2 & \text{for } i = j, \\ -1 & \text{for } j = i \pm 2, \\ 0 & \text{otherwise}, \end{cases}$$
and we consider the corresponding quantum group $U_q(\mathfrak{gl}_\infty)$. In this case, we have $q_i = q$. We write $[n]$ and $[n]!$ for $[n]_q$ and $[n]_q!$ for short.

We can parametrize the crystal basis $B(\infty)$ by the multisegments. We shall recall this parametrization and the PBW basis.

**Definition 3.1.** For $i, j \in I$ such that $i \leq j$, we define a segment $\langle i, j \rangle$ as the interval $[i, j] \subset I := \mathbb{Z}_{\text{odd}}$. A multisegment is a formal finite sum of segments:

$$m = \sum_{i \leq j} m_{ij} \langle i, j \rangle$$

with $m_{ij} \in \mathbb{Z}_{\geq 0}$. We call $m_{ij}$ the multiplicity of a segment $\langle i, j \rangle$. If $m_{ij} > 0$, we sometimes say that $\langle i, j \rangle$ appears in $m$. We sometimes write $m_{i,j}(m)$ for $m_{ij}$. We write sometimes $\langle i \rangle$ for \langle i, i \rangle. We denote by $M$ the set of multisegments. We denote by $\emptyset$ the zero element (or the empty multisegment) of $M$.

**Definition 3.2.** For two segments $\langle i_1, j_1 \rangle$ and $\langle i_2, j_2 \rangle$, we define the ordering $\geq_{PBW}$ by the following:

$$\langle i_1, j_1 \rangle \geq_{PBW} \langle i_2, j_2 \rangle \iff \begin{cases} j_1 > j_2 \\ \text{or } j_1 = j_2 \text{ and } i_1 \geq i_2. \end{cases}$$

We call this ordering the PBW-ordering.

**Definition 3.3.** For a multisegment $m$, we define the element $P(m) \in U_q^{-}(\mathfrak{gl}_\infty)$ as follows.
(1) For a segment $\langle i, j \rangle$, we define the element $\langle i, j \rangle \in U_q^{-}(\mathfrak{gl}_\infty)$ inductively by

$$\langle i, i \rangle = f_i, \quad \langle i, j \rangle = \langle i, j - 2 \rangle \langle j, j \rangle - q(j, j) \langle i, j - 2 \rangle \quad \text{for } i < j.$$ 

(2) For a multisegment $\mathbf{m} = \sum_{i \leq j} m_{ij} \langle i, j \rangle$, we define

$$P(\mathbf{m}) = \prod \langle i, j \rangle^{(m_{ij})}.$$ 

Here the product $\prod$ is taken over segments appearing in $\mathbf{m}$ from large to small with respect to the PBW-ordering. The element $\langle i, j \rangle^{(m_{ij})}$ is the divided power of $\langle i, j \rangle$ i.e.

$$\langle i, j \rangle^{(n)} = \begin{cases} \frac{1}{[n]} \langle i, j \rangle^n & \text{for } n > 0, \\ 1 & \text{for } n = 0, \\ 0 & \text{for } n < 0. \end{cases}$$

Hence the weight of $P(\mathbf{m})$ is equal to $\text{wt}(\mathbf{m}) := - \sum_{i \leq k \leq j} m_{i,j} \alpha_k$: $t_i P(\mathbf{m}) t_i^{-1} = q^{(\alpha_i, \text{wt}(\mathbf{m}))} P(\mathbf{m})$.

**Theorem 3.4 ([L]).** The set of elements $\{P(\mathbf{m}) \mid \mathbf{m} \in \mathcal{M}\}$ is a $\mathbf{K}$-basis of $U_q^{-}(\mathfrak{gl}_\infty)$. Moreover this is an $\mathbf{A}$-basis of $U_q^{-}(\mathfrak{gl}_\infty)_A$. We call this basis the PBW basis of $U_q^{-}(\mathfrak{gl}_\infty)$.

**Definition 3.5.** For two segments $\langle i_1, j_1 \rangle$ and $\langle i_2, j_2 \rangle$, we define the ordering $\geq_{\text{cry}}$ by the following:

$$\langle i_1, j_1 \rangle \geq_{\text{cry}} \langle i_2, j_2 \rangle \iff \begin{cases} j_1 > j_2 \\ \text{or} \\ j_1 = j_2 \text{ and } i_1 \leq i_2. \end{cases}$$

We call this ordering the crystal ordering.

**Example 3.6.** The crystal ordering is different from the PBW-ordering. For example, we have $\langle -1, 1 \rangle >_{\text{cry}} \langle 1, 1 \rangle >_{\text{cry}} \langle -1, -1 \rangle$, while we have $\langle 1, 1 \rangle >_{\text{PBW}} \langle -1, 1 \rangle >_{\text{PBW}} \langle -1, -1 \rangle$.

**Definition 3.7.** We define the crystal structure on $\mathcal{M}$ as follows: for $\mathbf{m} = \sum m_{i,j} \langle i, j \rangle \in \mathcal{M}$ and $i \in I$, set $A_k^{(i)}(\mathbf{m}) = \sum_{k' \geq k} (m_{i,k'} - m_{i+2,k'+2})$ for $k \geq i$. Define $\varepsilon_i(\mathbf{m})$ as max $\{A_k^{(i)}(\mathbf{m}) \mid k \geq i \} \geq 0$.

(i) If $\varepsilon_i(\mathbf{m}) = 0$, then define $\tilde{e}_i(\mathbf{m}) = 0$. If $\varepsilon_i(\mathbf{m}) > 0$, let $k_e$ be the largest $k \geq i$ such that $\varepsilon_i(\mathbf{m}) = A_k^{(i)}(\mathbf{m})$ and define $\tilde{e}_i(\mathbf{m}) = \mathbf{m} - \langle i, k_e \rangle + \delta_{k_e \neq i} \langle i + 2, k_e \rangle$.

(ii) Let $k_f$ be the smallest $k \geq i$ such that $\varepsilon_i(\mathbf{m}) = A_k^{(i)}(\mathbf{m})$ and define $\tilde{f}_i(\mathbf{m}) = \mathbf{m} - \delta_{k_f \neq i} \langle i + 2, k_f \rangle + \langle i, k_f \rangle$.

**Remark 3.8.** For $i \in I$, the actions of the operators $\tilde{e}_i$ and $\tilde{f}_i$ on $\mathbf{m} \in \mathcal{M}$ are also described by the following algorithm:

Step 1. Arrange the segments in $\mathbf{m}$ in the crystal ordering.

Step 2. For each segment $\langle i, j \rangle$, write $-$, and for each segment $\langle i + 2, j \rangle$, write $+$. 

Step 3. In the resulting sequence of $+$ and $-$, delete a subsequence of the form $++$ and keep on deleting until no such subsequence remains.

Then we obtain a sequence of the form $- - \cdots - + + \cdots +$.

(1) $\varepsilon_i(\mathbf{m})$ is the total number of $-$ in the resulting sequence.

(2) $\tilde{f}_i(\mathbf{m})$ is given as follows:
(a) if the leftmost + corresponds to a segment \langle i + 2, j \rangle, then replace it with \langle i, j \rangle,
(b) if no + exists, add a segment \langle i, i \rangle to m.

(3) \tilde{e}_i(m) is given as follows:
(a) if the rightmost − corresponds to a segment \langle i, j \rangle, then replace it with \langle i + 2, j \rangle,
(b) if no − exists, then \tilde{e}_i(m) = 0.

Let us introduce a linear ordering on the set \mathcal{M} of multisegments, lexicographic with respect to the crystal ordering on the set of segments.

**Definition 3.9.** For m = \sum_{i \leq j} m_{i,j} \langle i, j \rangle \in \mathcal{M} and and m' = \sum_{i \leq j} m'_{i,j} \langle i, j \rangle \in \mathcal{M}, we define m' < \text{cry } m if there exist i_0 \leq j_0 such that m'_{i_0,j_0} < m_{i_0,j_0}, m'_{i,j_0} = m_{i,j_0} for i < i_0, and m'_{i,j} = m_{i,j} for j > j_0 and i \leq j.

**Theorem 3.10.** (i) \(L(\infty) = \bigoplus_{m \in \mathcal{M}} A_0 P(m)\).
(ii) \(B(\infty) = \{P(m) \mod qL(\infty) \mid m \in \mathcal{M}\}\).
(iii) We have
\[
\tilde{e}_i P(m) \equiv P(\tilde{e}_i(m)) \mod qL(\infty),
\tilde{f}_i P(m) \equiv P(\tilde{f}_i(m)) \mod qL(\infty).
\]
Note that \(\tilde{e}_i\) and \(\tilde{f}_i\) in the left-hand-side is the modified root operators.
(iv) We have
\[
\overline{P(m)} \in P(m) + \sum_{m' < \text{cry } m} A P(m').
\]

Therefore we can index the crystal basis by multisegments. By this theorem we can easily see by a standard argument that \((L(\infty), L(\infty)^-, U_q^- (\mathfrak{g})_A)\) is balanced, and there exists a unique \(G^{\text{low}}(m) \in L(\infty) \cap U_q^- (\mathfrak{g})_A\) such that \(G^{\text{low}}(m)^- = G^{\text{low}}(m)\) and \(G^{\text{low}}(m) \equiv P(m) \mod qL(\infty)\). The basis \(\{G^{\text{low}}(m)\}_{m \in \mathcal{M}}\) is a lower global basis.

**3.2. \(\theta\)-restricted multisegments.** We consider the Dynkin diagram involution \(\theta\) of \(I\) defined by \(\theta(i) = -i\) for \(i \in I = \mathbb{Z}_{\text{odd}}\).

![Dynkin diagram](image)

We shall prove in this case Conjectures 2.16 and 2.18 for \(\lambda = 0\) (Theorems 4.15 and 5.5).

We set
\[
\tilde{V}_\theta(0) := B_\theta(\mathfrak{gl}_\infty) / \sum_{i \in I} (B_\theta(\mathfrak{gl}_\infty) E_i + B_\theta(\mathfrak{gl}_\infty) (T_i - 1) + B_\theta(\mathfrak{gl}_\infty) (F_i - F_{\theta(i)}))
\]
\[
\simeq U_q^- (\mathfrak{gl}_\infty) / \sum_i U_q^- (\mathfrak{gl}_\infty) (f_i - f_{\theta(i)}).
\]

Let \(\tilde{\phi}\) be the generator of \(\tilde{V}_\theta(0)\) corresponding to \(1 \in B_\theta(\mathfrak{gl}_\infty)\). Since \(F_i \phi''_0 = (f_i + f_{\theta(i)}) \phi''_0 = F_{\theta(i)} \phi''_0\), we have an epimorphism of \(B_\theta(\mathfrak{gl}_\infty)\)-modules
\[
(3.1) \quad \tilde{V}_\theta(0) \twoheadrightarrow V_\theta(0).
\]

We shall see later that it is in fact an isomorphism (see Theorem 4.15).
Definition 3.11. If a multisegment $\mathbf{m}$ has the form
\[ m = \sum_{-j \leq i \leq j} m_{ij} \langle i, j \rangle, \]
we call $\mathbf{m}$ a $\theta$-restricted multisegment. We denote by $\mathcal{M}_\theta$ the set of $\theta$-restricted multisegments.

Definition 3.12. For a $\theta$-restricted segment $\langle i, j \rangle$, we define its modified divided power by
\[ \langle i, j \rangle^{[m]} = \begin{cases} \langle i, j \rangle^{(m)} & (i \neq j), \\ \frac{1}{m!} \langle i, j \rangle^m & (i = j). \end{cases} \]

We understand that $\langle i, j \rangle^{[m]}$ is equal to 1 for $m = 0$ and vanishes for $m < 0$.

Definition 3.13. For $\mathbf{m} \in \mathcal{M}_\theta$, we define the element $P_\theta(\mathbf{m}) \in U_q^{-}(\mathfrak{gl}_\infty) \subset B_\theta(\mathfrak{gl}_\infty)$ by
\[ P_\theta(\mathbf{m}) = \prod_{\langle i, j \rangle \in \mathbf{m}} \langle i, j \rangle^{[m_{ij}]} \]
Here the product $\prod$ is taken over the segments appearing in $\mathbf{m}$ from large to small with respect to the PBW-ordering.

If an element $\mathbf{m}$ of the free abelian group generated by $\langle i, j \rangle$ does not belong to $\mathcal{M}_\theta$, we understand $P_\theta(\mathbf{m}) = 0$.

We will prove later that $\{P_\theta(\mathbf{m}) \phi \}_{\mathbf{m} \in \mathcal{M}_\theta}$ is a basis of $V_\theta(0)$ (see Theorem 4.15). Here and hereafter, we write $\phi$ instead of $\phi_0 \in V_\theta(0)$.

3.3. Commutation relations of $\langle i, j \rangle$. In the sequel, we regard $U_q^{-}(\mathfrak{gl}_\infty)$ as a subalgebra of $B_\theta(\mathfrak{gl}_\infty)$ by $f_i \mapsto F_i$.

We shall give formulas to express products of segments by a PBW basis.

Proposition 3.14. For $i, j, k, l \in I$, we have
\begin{enumerate}[1.]
\item $\langle i, j \rangle \langle k, \ell \rangle = \langle k, \ell \rangle \langle i, j \rangle$ for $i \leq j$, $k \leq \ell$ and $j < k - 2$,
\item $\langle i, j \rangle \langle j + 2, k \rangle = \langle i, k \rangle + q \langle j + 2, k \rangle \langle i, j \rangle$ for $i \leq j < k$,
\item $\langle j, k \rangle \langle i, \ell \rangle = \langle i, \ell \rangle \langle j, k \rangle$ for $i < j \leq k < \ell$,
\item $\langle i, k \rangle \langle i, k \rangle = q^{-1} \langle i, k \rangle \langle i, k \rangle$ for $i < j \leq k$,
\item $\langle i, j \rangle \langle i, k \rangle = q^{-1} \langle i, k \rangle \langle i, j \rangle$ for $i < j < k$,
\item $\langle i, k \rangle \langle i, \ell \rangle = \langle i, \ell \rangle \langle i, k \rangle + (q^{-1} - q) \langle i, \ell \rangle \langle i, k \rangle$ for $i < j < k < \ell$.
\end{enumerate}

Proof. (1) is obvious. We prove (2) by the induction on $k - j$. If $k - j = 2$, it is trivial by the definition. If $j < k - 2$, then $\langle k \rangle$ and $\langle i, j \rangle$ commute. Thus, we have
\[ \langle i, j \rangle \langle j + 2, k \rangle = \langle i, j \rangle ((\langle j + 2, k - 2 \rangle \langle k \rangle) - q \langle k \rangle \langle j + 2, k - 2 \rangle) \]
\[ = (\langle i, k - 2 \rangle + q \langle j + 2, k - 2 \rangle \langle i, j \rangle) \langle k \rangle - q \langle k \rangle \langle i, j \rangle \langle j + 2, k - 2 \rangle \]
\[ = \langle i, k - 2 \rangle \langle k \rangle + q \langle j + 2, k - 2 \rangle \langle i, j \rangle \]
\[ - q \langle k \rangle (\langle i, k - 2 \rangle + q \langle j + 2, k - 2 \rangle \langle i, j \rangle) \]
\[ = \langle i, k \rangle + \langle j + 2, k \rangle \langle i, j \rangle. \]

In order to prove the other relations, we first show the following special cases.

Lemma 3.15. We have for any $j \in I$
\begin{enumerate}[a.]
\item $\langle j - 2, j \rangle \langle j \rangle = q^{-1} \langle j \rangle \langle j - 2, j \rangle$ and $\langle j \rangle \langle j, j + 2 \rangle = q^{-1} \langle j, j + 2 \rangle \langle j \rangle$,
\end{enumerate}
(b) \( \langle j, j + 2 \rangle \langle j - 2, j + 2 \rangle = \langle j - 2, j + 2 \rangle \langle j \rangle \langle j \rangle \).
(c) \( \langle j - 2, j \rangle \langle j, j + 2 \rangle = \langle j, j + 2 \rangle \langle j - 2, j \rangle + (q^{-1} - q) \langle j - 2, j + 2 \rangle \langle j \rangle \).

**Proof.** The first equality in (a) follows from
\[
\langle j - 2, j \rangle \langle j \rangle - q^{-1} \langle j \rangle \langle j - 2, j \rangle = (f_{j-2}f_j - qf_jf_{j-2})f_j - q^{-1}f_j(f_{j-2}f_j - qf_jf_{j-2})
\]
\[
= f_{j-2}f_j^2 - (q + q^{-1})f_jf_{j-2}f_j + f_j^2f_{j-2} = 0.
\]

We can similarly prove the second.
Let us show (b) and (c). We have, by (a)
\[
\langle j - 2, j \rangle \langle j, j + 2 \rangle = \langle j + 2 \rangle \langle j - 2 \rangle (\langle j \rangle \langle j \rangle - q \langle j \rangle \langle j - 2 \rangle) \langle j \rangle
\]
\[
= q^{-1} \langle j \rangle \langle j - 2, j \rangle \langle j, j + 2 \rangle - q \langle j \rangle \langle j - 2, j \rangle + q \langle j \rangle \langle j - 2, j \rangle \langle j \rangle
\]
\[
(3.2)
\]
\[
= \langle j \rangle \langle j - 2, j \rangle \langle j - 2, j \rangle + q \langle j \rangle \langle j - 2, j \rangle \langle j \rangle - q \langle j \rangle \langle j - 2, j \rangle \langle j \rangle
\]
\[
= \langle j, j + 2 \rangle \langle j - 2, j \rangle + q^{-1} \langle j \rangle \langle j - 2, j \rangle - q \langle j \rangle \langle j - 2, j \rangle \langle j \rangle.
\]

Similarly, we have
\[
\langle j - 2, j \rangle \langle j, j + 2 \rangle = (\langle j - 2 \rangle \langle j \rangle - q \langle j \rangle \langle j - 2 \rangle) \langle j \rangle \langle j + 2 \rangle \langle j \rangle
\]
\[
(3.3)
\]
\[
= q^{-1} \langle j \rangle \langle j - 2, j \rangle \langle j, j + 2 \rangle - q \langle j \rangle \langle j - 2, j \rangle + q \langle j \rangle \langle j - 2, j \rangle \langle j \rangle
\]
\[
= (\langle j \rangle \langle j + 2 \rangle \langle j \rangle - q \langle j \rangle \langle j - 2 \rangle) \langle j - 2, j \rangle + q^{-1} \langle j \rangle \langle j - 2, j \rangle - q \langle j \rangle \langle j - 2, j \rangle \langle j \rangle
\]
\[
= (j, j + 2) \langle j - 2, j \rangle + q^{-1} \langle j \rangle \langle j - 2, j \rangle - q \langle j \rangle \langle j - 2, j \rangle \langle j \rangle.
\]

Then, (3.2) and (3.3) imply (b) and (c).

We shall resume the proof of Proposition 3.14. By Lemma 3.15 (b), \( \langle i, k \rangle \) commutes with \( \langle j \rangle \) for \( i < j < k \). Thus we obtain (3).

We shall show (4) by the induction on \( k - j \). Suppose \( k - j = 0 \). The case \( i = k - 2 \) is nothing but Lemma 3.13 (a).

If \( i < k - 2 \), then
\[
\langle i, k \rangle \langle k \rangle = \langle i, k - 4 \rangle \langle k - 2, k \rangle \langle k \rangle - q \langle k - 2, k \rangle \langle i, k - 4 \rangle \langle k \rangle
\]
\[
= q^{-1} \langle k \rangle \langle i, k - 4 \rangle \langle k - 2, k \rangle - \langle k \rangle \langle k - 2, k \rangle \langle i, k - 4 \rangle \langle k \rangle = q^{-1} \langle k \rangle \langle i, k \rangle.
\]

Suppose \( k - j > 0 \). By using the induction hypothesis and (3), we have
\[
\langle i, k \rangle \langle j, k \rangle = \langle i, k \rangle \langle j + 2, k \rangle - q \langle i, k \rangle \langle j + 2, k \rangle \langle j \rangle
\]
\[
= \langle j \rangle \langle i, k \rangle \langle j + 2, k \rangle - (j + 2, k) \langle i, k \rangle \langle j \rangle
\]
\[
= q^{-1} \langle j \rangle \langle j + 2, k \rangle \langle i, k \rangle - q \langle j + 2, k \rangle \langle i, k \rangle = q^{-1} \langle j \rangle \langle k \rangle \langle i, k \rangle.
\]

Similarly we can prove (5).
Let us prove (6). We have
\[
\langle i, k \rangle \langle j, \ell \rangle = (\langle i, j - 2 \rangle \langle j, k \rangle - q \langle j, k \rangle \langle i, j - 2 \rangle) \langle j, \ell \rangle
\]
\[
= q^{-1} \langle i, j - 2 \rangle \langle j, \ell \rangle \langle j, k \rangle - q \langle j, k \rangle \langle i, j - 2 \rangle \langle j, \ell \rangle \langle i, \ell \rangle \langle j \rangle \langle i, j - 2 \rangle
\]
\[
= q^{-1} (\langle i, \ell \rangle + q \langle j, \ell \rangle \langle i, j - 2 \rangle) \langle j, k \rangle - q \langle i, \ell \rangle \langle j, k \rangle - q \langle j, \ell \rangle \langle j, k \rangle \langle i, j - 2 \rangle
\]
\[
= \langle j, \ell \rangle \langle i, k \rangle + (q^{-1} - q) \langle i, \ell \rangle \langle j, k \rangle.
\]
Lemma 3.16. (i) For $1 \leq i \leq j$, we have $\langle -j, -i \rangle \sim = \langle i, j \rangle \sim$.

(ii) For $1 \leq i < j$, we have $\langle -j, i \rangle \sim = q^{-1}(\langle -i, j \rangle \sim)$.

Proof. (i) If $i = j$, it is obvious. By the induction on $j - i$, we have

\[
\langle -j, -i \rangle \sim = (\langle -j, -i - 2 \rangle \langle -i \rangle - q(\langle -i \rangle \langle -j, -i - 2 \rangle) \sim \phi
= (\langle -j, -i - 2 \rangle \langle i \rangle - q(\langle i \rangle + 2, j) \sim \phi
= (\langle i \rangle \langle -j, -i - 2 \rangle - q(i + 2, j) \langle -i \rangle \sim
= (\langle i \rangle \langle i + 2, j \rangle - q(i + 2, j) \langle i \rangle) \sim \phi = \langle i, j \rangle \sim.
\]

(ii) By (i), we have

\[
\langle -j, i \rangle \sim = (\langle -j, -1 \rangle \langle 1, i \rangle - q(\langle 1, i \rangle \langle -j, -1 \rangle) \sim \phi
= (\langle -j, -1 \rangle \langle -i, -1 \rangle - q(\langle 1, i \rangle \langle 1, j \rangle) \sim \phi
= (q^{-1}(\langle -i, -1 \rangle \langle -j, -1 \rangle - \langle 1, j \rangle \langle 1, i \rangle) \sim \phi
= (q^{-1}(\langle -i, -1 \rangle \langle 1, j \rangle - \langle 1, j \rangle \langle -i, -1 \rangle) \sim = q^{-1}(\langle -i, j \rangle \sim.
\]

Q.E.D.

Proposition 3.17. (i) For a multisegment $m = \sum_{i \leq j} m_{i,j} \langle i, j \rangle$, we have

\[
\text{Ad}(t_k)P(m) = q^{\sum_(m_{i,k} - m_{i,l}) + \sum_j(m_{k+2,j} - m_{k,j})} P(m).
\]

(ii)

\[
e_k'(i, j)^{(n)} = \begin{cases} 
q^{1-n} \langle i \rangle^{(n-1)} & \text{if } k = i = j, \\
(1 - q^2)q^{1-n} \langle i + 2, j \rangle \langle i, j \rangle^{(n-1)} & \text{if } k = i < j, \\
0 & \text{otherwise,}
\end{cases}
\]

\[
e_k^*(i, j)^{(n)} = \begin{cases} 
q^{1-n} \langle i \rangle^{(n-1)} & \text{if } i = j = k, \\
(1 - q^2)q^{1-n} \langle i, j \rangle^{(n-1)} \langle i - 2 \rangle & \text{if } i < j = k, \\
0 & \text{otherwise.}
\end{cases}
\]

Proof. (i) is obvious. Let us show (ii). It is obvious that $e_k'(i, j)^{(n)} = 0$ unless $i \leq k \leq j$.

It is known ([K]) that we have $e_k(k)^{(n)} = q^{1-n} \langle k \rangle^{(n-1)}$. We shall prove $e_k'(k, j)^{(n)} = (1 - q^2)q^{1-n} \langle k + 2, j \rangle \langle k, j \rangle^{(n-1)}$ for $k < j$ by the induction on $n$. By (2.1), we have

\[
e_k'(k, j) = e_k'((k) \langle k + 2, j \rangle - q \langle k + 2, j \rangle \langle k \rangle)
= \langle k + 2, j \rangle - q^2 \langle k + 2, j \rangle = (1 - q^2) \langle k + 2, j \rangle.
\]

For $n \geq 1$, by the induction hypothesis and Proposition 3.14 ([H]), we get

\[
[n]e_k'(k, j)^{(n)} = e_k(k, j)^{(n-1)} \langle k, j \rangle^{(n-1)}
= (1 - q^2) \langle k + 2, j \rangle \langle k, j \rangle^{(n-1)} + q^{-1} \langle k, j \rangle \cdot (1 - q^2)q^{2-n} \langle k + 2, j \rangle \langle k, j \rangle^{(n-2)}
= (1 - q^2) \{ \langle k + 2, j \rangle \langle k, j \rangle^{(n-1)} + q^{-1} \langle k, j \rangle \langle k + 2, j \rangle \langle k, j \rangle^{(n-2)} \}
= (1 - q^2)(1 + q^{-n}[n - 1]) \langle k + 2, j \rangle \langle k, j \rangle^{(n-1)}
= (1 - q^2)q^{1-n} \langle n \rangle \langle k + 2, j \rangle \langle k, j \rangle^{(n-1)}.
\]
Finally we show $e'_k(i, j) = 0$ if $k \neq i$. We may assume $i < k \leq j$. If $i < k < j$, we have
\[
e'_k(i, j) = e'_k((i, k - 2)(k, j) - q(k, j)\langle i, k - 2 \rangle) \\
= q(i, k - 2)e'_k(k, j) - q(e'_k(k, j))\langle i, k - 2 \rangle \\
= q(1 - q^2)\langle i, k - 2 \rangle(k + 2, j) - q(1 - q^2)\langle k + 2, j \rangle\langle i, k - 2 \rangle \\
= 0.
\]
The case $k = j$ is similarly proved. The proof for $e''_k$ is similar.

Q.E.D.

### 3.4. Actions of divided powers

**Lemma 3.18.** Let $a$, $b$ be non-negative integers, and let $k \in I_{>0} := \{k \in I \mid k > 0\}$.

1. For $\ell > k$, we have
\[
\langle -k \rangle\langle -k + 2, \ell \rangle^{(a)}\langle -k, \ell \rangle^{(b)} = [b + 1]\langle -k + 2, \ell \rangle^{(a-1)}\langle -k, \ell \rangle^{(b+1)} + q^{a-b}\langle -k + 2, \ell \rangle^{(a)}\langle -k, \ell \rangle^{(b)}\langle -k \rangle.
\]
2. We have
\[
\langle -k \rangle\langle -k + 2, k \rangle^{(a)}\langle -k, k \rangle^{[b]} = [2b + 2]\langle -k + 2, k \rangle^{(a-1)}\langle -k, k \rangle^{[b+1]} + q^{a-b}\langle -k + 2, k \rangle^{(a)}\langle -k, k \rangle^{[b]}\langle -k \rangle.
\]
3. For $k > 1$, we have
\[
\langle -k \rangle\langle -k + 2, k - 2 \rangle^{[a]} = (q^a + q^{-a})^{-1}\langle -k + 2, k - 2 \rangle^{[a-1]}\langle -k, k - 2 \rangle + q^a\langle -k + 2, k - 2 \rangle^{[a]}\langle -k \rangle.
\]
4. If $\ell \leq k - 2$, we have
\[
\langle \ell, k - 2 \rangle^{(a)}\langle k \rangle = \langle \ell, k \rangle\langle \ell, k - 2 \rangle^{(a-1)} + q^{a}\langle k \rangle\langle \ell, k - 2 \rangle^{(a)}.
\]
5. For $k > 1$, we have
\[
\langle -k + 2, k - 2 \rangle^{[a]}\langle k \rangle = (q^a + q^{-a})^{-1}\langle -k + 2, k \rangle\langle -k + 2, k - 2 \rangle^{[a-1]} + q^a\langle k \rangle\langle -k + 2, k - 2 \rangle^{[a]}.
\]

**Proof.** We show (1) by the induction on $a$. If $a = 0$, it is trivial. For $a > 0$, we have
\[
[a]\langle -k \rangle\langle -k + 2, \ell \rangle^{(a)}\langle -k, \ell \rangle^{(b)} \\
= (\langle -k, \ell \rangle + q\langle -k + 2, \ell \rangle\langle -k \rangle)\langle -k + 2, \ell \rangle^{(a-1)}\langle -k, \ell \rangle^{(b)} \\
= [b + 1]q^{1-a}\langle -k + 2, \ell \rangle^{(a-1)}\langle -k, \ell \rangle^{(b+1)} + q\langle -k + 2, \ell \rangle\{[b + 1]\langle -k + 2, \ell \rangle^{(a-2)}\langle -k, \ell \rangle^{(b+1)} + q^{a-b-1}\langle -k + 2, \ell \rangle^{(a-1)}\langle -k, \ell \rangle^{(b)}\langle -k \rangle\} \\
= [b + 1](q^{1-a} + q[a - 1])\langle -k + 2, \ell \rangle^{(a-1)}\langle -k, \ell \rangle^{(b+1)} + q^{a-b}[a]\langle -k + 2, \ell \rangle^{(a)}\langle -k, \ell \rangle^{(b)}\langle -k \rangle.
\]
Since $q^{1-a} + q[a - 1] = [a]$, the induction proceeds.

The proof of (2) is similar by using $\langle -k, k \rangle^{[b]} = [2b]\langle -k, k \rangle^{[b-1]}\langle -k \rangle$. 
We prove (3) by the induction on $a$. The case $a = 0$ is trivial. For $a > 0$, we have

$$
[2a](-k)^(-k + 2, k - 2)^{[a]} = ([(-k, k - 2) + q(-k + 2, k - 2)(-k)](-k + 2, k - 2)^{[a-1]}
= q^{1-a}(-k + 2, k - 2)^{[a-1]}(-k, k - 2)
+ q(-k + 2, k - 2)\left\{((q^{a-1} + q^{1-a})^{-1}(-k + 2, k - 2)^{[a-2]}(-k, k - 2)
+ q^{a-1}(-k + 2, k - 2)^{[a-1]}\right\}
= (q^{1-a} + \frac{q[2a - 1]}{q^a - 1 + q^{1-a}})(-k + 2, k - 2)^{[a-1]}(-k, k - 2) + q^a[2a](-k + 2, k - 2)^{[a]}(-k)
= (q^a + q^{-a})^{-1}[2a](-k + 2, k - 2)^{[a-1]}(-k, k - 2) + q^a[2a](-k + 2, k - 2)^{[a]}(-k).
$$

Similarly, we can prove (4) and (5) by the induction on $a$. Q.E.D.

**Lemma 3.19.** For $k > 1$ and $a, b, c, d \geq 0$, set

$$(a, b, c, d) = \langle k \rangle^{(a)}(-k + 2, k)^{(b)}(-k, k)^{(c)}(-k + 2, k - 2)^{(d)}\tilde{\phi}.$$

Then, we have

$$
\langle -k \rangle(a, b, c, d) = [2c + 2](a, b - 1, c + 1, d)
+ [b + 1]q^{b-2c}(a, b + 1, c, d - 1)
+ [a + 1]q^{2d-2c}(a + 1, b, c, d).
$$

**Proof.** We shall show first

$$
\langle -k \rangle\langle -k + 2, k - 2 \rangle^{[(d)]}\tilde{\phi}
= \langle -k + 2, k \rangle\langle -k + 2, k - 2 \rangle^{[(d-1)]} + q^{2d} \langle -k + 2, k - 2 \rangle^{[(d)]}\tilde{\phi}.
$$

By Lemma 3.18 (3), we have

$$
\langle -k \rangle\langle -k + 2, k - 2 \rangle^{[(d)]}\tilde{\phi}
= \langle (q^d + q^{-d})^{-1}\langle -k + 2, k - 2 \rangle^{[(d-1)]}(-k, k - 2)
+ q^d\langle -k + 2, k - 2 \rangle^{[(d)]}(-k)\rangle\tilde{\phi}.
$$

By Lemma 3.16 and Lemma 3.18 (5), it is equal to

$$
\langle (q^d + q^{-d})^{-1}q^{-d}\langle -k + 2, k \rangle\langle -k + 2, k - 2 \rangle^{[(d-1)]}
+ q^d\langle -k + 2, k \rangle\langle -k + 2, k - 2 \rangle^{[(d-1)]} + q^d\langle -k + 2, k \rangle\langle -k + 2, k - 2 \rangle^{[(d)]}\rangle\tilde{\phi}.
$$

Thus we obtain (3.5). Applying Lemma 3.18 (2), we have

$$
\langle -k \rangle(a, b, c, d) = \langle k \rangle^{(a)}\left\{[2c + 2]\langle -k + 2, k \rangle^{(b-1)}(-k, k)^{(c+1)}
+ q^{b-c}\langle -k + 2, k \rangle^{(b)}(-k, k)^{(c)}\langle -k \rangle(-k + 2, k - 2)^{(d)}\tilde{\phi}\right\}
= [2c + 2](a, b - 1, c + 1, d) + q^{b-c}\langle k \rangle^{(a)}(-k + 2, k)^{(b)}(-k, k)^{(c)}
\times\langle -k + 2, k \rangle\langle -k + 2, k - 2 \rangle^{[(d-1)]} + q^{2d}\langle -k + 2, k - 2 \rangle^{[(d)]}\tilde{\phi}
= [2c + 2](a, b - 1, c + 1, d) + q^{b-2c}[b + 1](a, b + 1, c, d - 1)
+ q^{b-c} + 2d-c-b[a + 1](a + 1, b, c, d).
$$

Hence we have (3.4). Q.E.D.
Lemma 3.18 (2) implies

\[ \langle -1 \rangle^{(a)} \langle -1, 1 \rangle^{[m]\,\sim} = \sum_{s=0}^{\lfloor a/2 \rfloor} \left( \prod_{\nu=1}^{\lfloor a/2 \rfloor} \frac{[2m + 2\nu]}{[2\nu]} \right) q^{-2(a-s)m + \frac{(a-2s)(a-2s-1)}{2}} \langle 1 \rangle^{(a-2s)} \langle -1, 1 \rangle^{[m+s]\,\sim}. \]

(2) For \( k > 1 \), we have

\[ \langle -k \rangle^{(n)} \langle -k + 2, k - 2 \rangle^{[a]\,\sim} = \sum_{u=0}^{n} \sum_{i+j+2t=m, j+t=u} q^{2ai + i(t-1) - i(t+u)} \langle k \rangle^{(i)} \langle -k + 2, k \rangle^{(j)} \langle -k, k \rangle^{[t]} \langle -k + 2, k - 2 \rangle^{[a-u]\,\sim}. \]

(3) If \( \ell > k \), we have

\[ \langle k \rangle^{(n)} \langle k + 2, \ell \rangle^{(a)} = \sum_{s=0}^{n} q^{(n-s)(a-s)} \langle k + 2, \ell \rangle^{(a-s)} \langle k, \ell \rangle^{(s)} \langle k \rangle^{(n-s)}. \]

**Proof.** We prove (1) by the induction on \( a \). The case \( a = 0 \) is trivial. Assume \( a > 0 \). Then, Lemma 3.18 (2) implies

\[ \langle -1 \rangle^{(1)} \langle -1, 1 \rangle^{[m]\,\sim} = \left( [2m + 2] \langle 1 \rangle^{(n-1)} \langle -1, 1 \rangle^{[m+1]} + q^n \langle 1 \rangle^{(n)} \langle -1, 1 \rangle^{[m]} \langle -1 \rangle \right) \phi \]

\[ = \left( [2m + 2] \langle 1 \rangle^{(n-1)} \langle -1, 1 \rangle^{[m+1]} + q^n \langle 1 \rangle^{(n)} \langle -1, 1 \rangle^{[m]} \langle 1 \rangle \right) \phi \]

\[ = \left( [2m + 2] \langle 1 \rangle^{(n-1)} \langle -1, 1 \rangle^{[m+1]} + q^n \langle 1 \rangle^{(n)} \langle n+1 \rangle \langle -1, 1 \rangle^{[m]} \right) \phi. \]

Put

\[ c_s = \left( \prod_{\nu=1}^{\lfloor a/2 \rfloor} \frac{[2m + 2\nu]}{[2\nu]} \right) q^{-2(a-s)m + \frac{(a-2s)(a-2s-1)}{2}}. \]

Then we have

\[ \langle -1 \rangle^{(a+1)} \langle -1, 1 \rangle^{[m]\,\sim} = \langle -1 \rangle^{(a)} \langle -1, 1 \rangle^{[m]\,\sim} \]

\[ = \langle -1 \rangle^{\lfloor a/2 \rfloor} c_s \langle -1, 1 \rangle^{[m+s]\,\sim} \]

\[ = \sum_{s=0}^{\lfloor a/2 \rfloor} c_s \left( [2(m + s + 1)] \langle 1 \rangle^{(a-2s-1)} \langle -1, 1 \rangle^{[m+s+1]} + q^{a-2s-2(m+s)} \right) \langle a - 2s + 1 \rangle^{(a-2s+1)} \langle -1, 1 \rangle^{[m+s]} \rangle \phi. \]

In the right-hand-side, the coefficients of \( \langle 1 \rangle^{a+1-2r} \langle -1, 1 \rangle^{[m+r]\,\sim} \) are

\[ [2(m + r)] c_{r-1} + q^{a-2m-4r} \langle a - 2r + 1 \rangle c_r \]

\[ = \prod_{\nu=1}^{r} \frac{[2m + 2\nu]}{[2\nu]} q^{-2(a-r+1)m + \frac{(a-2r)(a-2r+1)}{2}} \left( [2r] q^{a-2r+1} + [a - 2r + 1] q^{-2r} \right) \]

\[ = [a + 1] \prod_{\nu=1}^{r} \frac{[2m + 2\nu]}{[2\nu]} q^{-2(a-r+1)m + \frac{(a-2r)(a-2r+1)}{2}}. \]

Hence we obtain (1).

We prove (2) by the induction on \( n \). We use the following notation for short:

\[ (i, j, t, a) := \langle k \rangle^{(i)} \langle -k + 2, k \rangle^{(j)} \langle -k, k \rangle^{[t]} \langle -k + 2, k - 2 \rangle^{[a]} \phi. \]
Then Lemma 3.19 implies that
\[
\langle -k \rangle(i, j, t, a) = [2t + 2](i, j - 1, t + 1, a) \\
+ [j + 1]q^{j-2t}(i, j + 1, t, a - 1) \\
+ [i + 1]q^{2a-2t}(i + 1, j, t, a).
\]
Hence, by assuming (2) for \( n \), we have
\[
[n + 1] \langle -k \rangle^{(n+1)}(-k + 2, k - 2)[a] \tilde{\phi} = \langle -k \rangle(-k) \langle -k \rangle^{(n)}(-k + 2, k - 2)[a] \tilde{\phi}
\]
\[
= \sum_{u=0}^{n} \sum_{i+j+2t=n,j+t=u} \left\{ [2t + 2]q^{2ai + (j+1)2 - i(t+u)}(i, j - 1, t + 1, a - u) \\
+ [j + 1]q^{2ai + (j-1)2 - i(t+u) + j-2t}(i, j + 1, t, a - u - 1) \\
+ [i + 1]q^{2ai + (j-1)2 - i(t+u) + 2a-2u-2t}(i + 1, j, t, a - u) \right\}.
\]
Then in the right hand side, the coefficients of \((i', j', t', a - u')\) satisfying \( i' + j' + 2t' = n + 1, j' + t' = u' \) are
\[
[2t']q^{2ai' + (j'+1)2 - i'(t'+1+u')} + [j']q^{2ai' + (j'-1)2 - i'(t'+u') + j'-1-2t'} \\
+ [i']q^{j'(j'-1)/2 - i'(t'+u')} \left( [2t']q^{j'+i'} + [j']q^{j'-2t'} + [i']q^{-(t'+u')} \right) \\
= q^{2ai' + (j'-1)/2 - i'(t'+u')} [n + 1].
\]
We can prove (3) similarly as above. Q.E.D.

3.5. Actions of \( E_k, F_k \) on the PBW basis. For a \( \theta \)-restricted multisegment \( m \), we set
\[
\tilde{P}_\theta(m) = P_\theta(m) \tilde{\phi}.
\]
We understand \( \tilde{P}_\theta(m) = 0 \) if \( m \) is not a multisegment.

**Theorem 3.21.** For \( k \in I_{>0} \) and a \( \theta \)-restricted multisegment \( m = \sum_{-j \leq i \leq m} \langle i, j \rangle \), we have
\[
\tilde{P}_\theta(m)
\]
\[
= \sum_{\ell \geq k} [m_{-k, \ell} + 1]q_{\ell}^{m_{-k+2, \ell} - m_{-k, \ell}} \tilde{P}_\theta(m - \langle -k + 2, \ell \rangle + \langle -k, \ell \rangle) \\
+ q_{\ell}^{m_{-k+2, \ell} - m_{-k, \ell}} [2m_{-k, k} + 1] \tilde{P}_\theta(m - \langle -k + 2, k \rangle + \langle -k, k \rangle) \\
+ \sum_{k \geq k} q_{\ell}^{m_{-k+2, k} - m_{-k, k} + m_{-k+2, k} - 2m_{-k, k} + 1} \tilde{P}_\theta(m - \delta_{k \neq 1} \langle -k + 2, k - 2 \rangle + \langle -k + 2, k \rangle) \\
+ \sum_{-k+2 < i < k} q_{\ell}^{i} \sum_{k \geq k} q_{\ell}^{m_{-k+2, k} - m_{-k, k} + 2m_{-k+2, k} - 2m_{-k, k} + 1} \tilde{P}_\theta(m - \delta_{i<k} \langle i, k - 2 \rangle + \langle i, k \rangle).
\]

**Proof.** We divide \( m \) into four parts, \( m = m_1 + m_2 + m_3 + \delta_{k \neq 1} \langle m_{-k+2, k-2} - (k + 2, k - 2) \rangle \), where \( m_1 = \sum_{j \geq k} m_{i,j}(i,j) \), \( m_2 = \sum_{j \leq k} m_{i,j}(i,j) \), \( m_3 = \sum_{-k+2 < i < j < k} m_{i,j}(i,j) \). Then Proposition 3.14 implies
\[
\tilde{P}_\theta(m) = P_\theta(m_1)P_\theta(m_2)P_\theta(m_3)(\langle -k + 2, k - 2 \rangle [m_{-k+2, k-2}] \tilde{\phi}).
\]
If \( k = 1 \), we understand \( \langle -k + 2, k - 2 \rangle^{[n]} = 1 \). By Lemma 3.18 (1), we have

\[
\langle -k \rangle P_\theta(m_1) = \sum_{\ell > k} \sum_{q > \ell} \ell(m_{-k + 2}, \ell - m_{-k}, \ell) [m_{-k, \ell} + 1] P_\theta(m_1 - \langle -k + 2, \ell \rangle + \langle -k, \ell \rangle)
+ q \sum_{\ell > k} \ell(m_{-k + 2}, \ell - m_{-k}, \ell) P_\theta(m_1) \langle -k \rangle,
\]

and Lemma 3.18 (2) implies

\[
\langle -k \rangle P_\theta(m_2) = [2m_{-k, k} + 2] P_\theta(m_2 - \langle -k + 2, k \rangle + \langle -k, k \rangle)
+ q \sum_{m_{-k + 2}, k - m_{-k}, k} P_\theta(m_2) \langle -k \rangle.
\]

Since we have \( \langle -k \rangle P_\theta(m_3) = P_\theta(m_3) \langle -k \rangle \), we obtain

\[
\langle -k \rangle \tilde{P}_\theta(m) = \sum_{\ell > k} \sum_{q > \ell} \ell(m_{-k + 2}, \ell - m_{-k}, \ell) [m_{-k, \ell} + 1] \tilde{P}_\theta(m - \langle -k + 2, \ell \rangle + \langle -k, \ell \rangle)
+ q \sum_{\ell > k} \ell(m_{-k + 2}, \ell - m_{-k}, \ell) [2m_{-k, k} + 2] \tilde{P}_\theta(m - \langle -k + 2, k \rangle + \langle -k, k \rangle)
+ q \sum_{\ell > k} \ell(m_{-k + 2}, \ell - m_{-k}, \ell) P_\theta(m_1 + m_2 + m_3) \langle -k \rangle - \langle -k + 2, k - 2 \rangle^{[m_{-k + 2}, k - 2]} \varphi.
\]

By (3.5), we have

\[
\langle -k \rangle \langle -k + 2, k - 2 \rangle^{[m_{-k + 2}, k - 2]} \varphi = \langle -k + 2, k \rangle \langle -k + 2, k - 2 \rangle^{[m_{-k + 2}, k - 2]} \varphi
+ \delta_{k \neq 1} q^{2m_{-k + 2}, k - 2} \langle k \rangle \langle -k + 2, k - 2 \rangle^{[m_{-k + 2}, k - 2]} \varphi.
\]

Hence the last term in (3.6) is equal to

\[
q \sum_{\ell > k} \sum_{q > \ell} \ell(m_{-k + 2}, \ell - m_{-k}, \ell) - m_{-k, k} [m_{-k + 2, k} + 1] \tilde{P}_\theta(m - \delta_{k \neq 1} \langle -k + 2, k - 2 \rangle + \langle -k + 2, k \rangle)
+ \delta_{k \neq 1} q \sum_{\ell > k} \ell(m_{-k + 2}, \ell - m_{-k}, \ell) + 2m_{-k + 2, k - 2} P_\theta(m_1 + m_2 + m_3) \langle k \rangle \langle -k + 2, k - 2 \rangle^{[m_{-k + 2}, k - 2]} \varphi.
\]

For \( k \neq 1 \), Lemma 3.18 (4) implies

\[
P_\theta(m_3) \langle k \rangle = \sum_{-k + 2 < i \leq k} q^{\sum_{-k + 2 < j < i} m_{j, k} - 2} \langle i, k \rangle P_\theta(m_3 - \delta_{i < k} \langle i, k - 2 \rangle),
\]

and Proposition 3.14 implies

\[
P_\theta(m_2) \langle i, k \rangle = q^{-\sum_{i < i} m_{i, k}} [m_{i, k} + 1] P_\theta(m_2 + \langle i, k \rangle).
\]

Hence we obtain

\[
P_\theta(m_1) P_\theta(m_2) P_\theta(m_3) \langle k \rangle \langle -k + 2, k - 2 \rangle^{[m_{-k + 2}, k - 2]} \varphi
= \sum_{-k + 2 < i \leq k} q^{\sum_{-k + 2 < j < i} m_{j, k} - 2} \sum_{\sum_{k < i} m_{i, k} [m_{i, k} + 1]} \tilde{P}_\theta(m - \delta_{i < k} \langle i, k - 2 \rangle + \langle i, k \rangle).
\]

Thus we obtain the desired result. Q.E.D.
Theorem 3.22. For \( k \in \mathbb{Z}_0 \) and a \( \theta \)-restricted multisegment \( \mathbf{m} = \sum_{i,j} m_{i,j} \langle i, j \rangle \), we have

\[
E_{-k} \tilde{P}_\theta (\mathbf{m}) = (1 - q^2) \sum_{\ell \geq k} q 1^\mathbf{m} \langle m_{-k+2, \ell} \rangle [m_{-k+2, \ell} + 1] \tilde{P}_\theta (\mathbf{m} - \langle -k, \ell \rangle + \langle -k + 2, \ell \rangle) + \delta_{k \neq 1} (1 - q^2) \sum_{\ell \geq k} q 1^\mathbf{m} \langle m_{-k+2, \ell} \rangle [m_{-k+2, \ell} + 1] \tilde{P}_\theta (\mathbf{m} - \langle -k, k \rangle + \langle -k + 2, k \rangle) + \delta_{k \neq 1} (1 - q^2) \sum_{\ell \geq k} q 1^\mathbf{m} \langle m_{-k+2, \ell} \rangle [m_{-k+2, \ell} + 1] \tilde{P}_\theta (\mathbf{m} - \langle -k, k \rangle + \langle -k + 2, k \rangle)
\]

\[+ \sum_{-k < i < k - 2} (m_{i, k - 2} + 1) \tilde{P}_\theta (\mathbf{m} - \langle i, k \rangle + \langle i, k - 2 \rangle) \]

\[+ \delta_{k \neq 1} (1 - q^2) \sum_{\ell \geq k} q 1^\mathbf{m} \langle m_{-k+2, \ell} \rangle [m_{-k+2, \ell} + 1] \tilde{P}_\theta (\mathbf{m} - \langle -k + 2, k \rangle + \langle -k + 2, k - 2 \rangle)
\]

\[+ \delta_{k \neq 1} (1 - q^2) \sum_{\ell \geq k} q 1^\mathbf{m} \langle m_{-k+2, \ell} \rangle [m_{-k+2, \ell} + 1] \tilde{P}_\theta (\mathbf{m} - \langle -k + 2, k \rangle + \langle -k + 2, k - 2 \rangle)
\]

\[+ \delta_{k \neq 1} (1 - q^2) \sum_{\ell \geq k} q 1^\mathbf{m} \langle m_{-k+2, \ell} \rangle [m_{-k+2, \ell} + 1] \tilde{P}_\theta (\mathbf{m} - \langle -k + 2, k \rangle + \langle -k + 2, k - 2 \rangle)
\]

\[+ \delta_{k \neq 1} (1 - q^2) \sum_{\ell \geq k} q 1^\mathbf{m} \langle m_{-k+2, \ell} \rangle [m_{-k+2, \ell} + 1] \tilde{P}_\theta (\mathbf{m} - \langle -k + 2, k \rangle + \langle -k + 2, k - 2 \rangle)
\]

\[+ \delta_{k \neq 1} (1 - q^2) \sum_{\ell \geq k} q 1^\mathbf{m} \langle m_{-k+2, \ell} \rangle [m_{-k+2, \ell} + 1] \tilde{P}_\theta (\mathbf{m} - \langle -k + 2, k \rangle + \langle -k + 2, k - 2 \rangle)
\]

\[+ \delta_{k \neq 1} (1 - q^2) \sum_{\ell \geq k} q 1^\mathbf{m} \langle m_{-k+2, \ell} \rangle [m_{-k+2, \ell} + 1] \tilde{P}_\theta (\mathbf{m} - \langle -k + 2, k \rangle + \langle -k + 2, k - 2 \rangle)
\]

\[+ \delta_{k \neq 1} (1 - q^2) \sum_{\ell \geq k} q 1^\mathbf{m} \langle m_{-k+2, \ell} \rangle [m_{-k+2, \ell} + 1] \tilde{P}_\theta (\mathbf{m} - \langle -k + 2, k \rangle + \langle -k + 2, k - 2 \rangle)
\]

\[+ \delta_{k \neq 1} (1 - q^2) \sum_{\ell \geq k} q 1^\mathbf{m} \langle m_{-k+2, \ell} \rangle [m_{-k+2, \ell} + 1] \tilde{P}_\theta (\mathbf{m} - \langle -k + 2, k \rangle + \langle -k + 2, k - 2 \rangle)
\]

\[+ \delta_{k \neq 1} (1 - q^2) \sum_{\ell \geq k} q 1^\mathbf{m} \langle m_{-k+2, \ell} \rangle [m_{-k+2, \ell} + 1] \tilde{P}_\theta (\mathbf{m} - \langle -k + 2, k \rangle + \langle -k + 2, k - 2 \rangle)
\]

Proof. We shall divide \( \mathbf{m} \) into \( \mathbf{m} = \mathbf{m}_1 + \mathbf{m}_2 + \mathbf{m}_3 \) where \( \mathbf{m}_1 = \sum_{i,j > k} m_{i,j} \langle i, j \rangle \) and \( \mathbf{m}_2 = \sum_{i < k} m_{i,k} \langle i, k \rangle \) and \( \mathbf{m}_3 = \sum_{i,j < k} m_{i,j} \langle i, j \rangle \). By (2.3) and Proposition 3.17 we have

\[
E_{-k} \tilde{P}_\theta (\mathbf{m}) = \left( (e'_k P_\theta (\mathbf{m}_1)) P_\theta (\mathbf{m}_2 + \mathbf{m}_3) + (\text{Ad}(t_{-k}) P_\theta (\mathbf{m}_1))(e'_k P_\theta (\mathbf{m}_2 + \mathbf{m}_3)) + \text{Ad}(t_{-k}) \{ P_\theta (\mathbf{m}_1)(e'_k P_\theta (\mathbf{m}_2)) P_\theta (\mathbf{m}_3) \} \right) \tilde{P}_\theta (\mathbf{m} - \langle k \rangle).
\]

By Proposition 3.17 the first term is

\[
(e'_k P_\theta (\mathbf{m}_1)) P_\theta (\mathbf{m}_2 + \mathbf{m}_3) = (1 - q^2) \sum_{\ell > k} q 1^\mathbf{m} \langle m_{-k+2, \ell} \rangle [m_{-k+2, \ell} + 1] \tilde{P}_\theta (\mathbf{m} - \langle -k, \ell \rangle + \langle -k + 2, \ell \rangle).
\]

The second term is

\[
(\text{Ad}(t_{-k}) P_\theta (\mathbf{m}_1))(e'_k P_\theta (\mathbf{m}_2 + \mathbf{m}_3)) = q \sum_{\ell > k} (m_{-k+2, \ell} - m_{-k,k}) \left[ \frac{m_{-k,k}}{2m_{-k,k}} \right] \frac{m_{-k+2,k} + 1}{2m_{-k,k}} (1 - q^2) q^{m_{-k,k} + m_{-k+2,k}} \tilde{P}_\theta (\mathbf{m} - \langle -k, k \rangle + \langle -k + 2, k \rangle).
\]

Let us calculate the last part of (3.7). We have

\[
\text{Ad}(t_{-k}) \left( P_\theta (\mathbf{m}_1)(e'_k P_\theta (\mathbf{m}_2)) P_\theta (\mathbf{m}_3) \right) = q^{m_{-k+2, \ell} - m_{-k,k}} \sum_{\ell > k} m_{i,k} - 2 \delta_{k = 1} P_\theta (\mathbf{m}_1)(e'_k P_\theta (\mathbf{m}_2)) P_\theta (\mathbf{m}_3).
\]

We have

\[
e'_k P_\theta (\mathbf{m}_2) = q^{1-m_{i,k}} \sum_{\ell < k} m_{i,k} P_\theta (\mathbf{m}_2 - \langle k \rangle)
\]

\[+ (1 - q^2) \sum_{-k < i < k} q^{1-m_{i,k}} \sum_{\ell < i} m_{i,k} P_\theta (\mathbf{m}_2 - \langle k \rangle) \langle i, k - 2 \rangle
\]

\[+ \frac{m_{-k,k}}{2m_{-k,k}} (1 - q^2) q^{m_{-k,k}} P_\theta (\mathbf{m}_2 - \langle -k, k \rangle) \langle -k, k - 2 \rangle.
\]
For $-k < i < k$, we have

$$
\langle i, k - 2 \rangle P_\theta(m_3) = q ^ { - \sum_{i' > i} m_{i',k-2} } [(1 + \delta_{i = -k + 2})(m_{i,k-2} + 1)] P_\theta(m_3 + \langle i, k - 2 \rangle).
$$

By Lemma 3.16 we have

$$
\langle -k, k - 2 \rangle P_\theta(m_3) \tilde{\phi} = q ^ { - \sum_{i \leq k} m_{i,k-2} } P_\theta(m_3) \langle -k, k - 2 \rangle \tilde{\phi} = q ^ { - \sum_{-k \leq i \leq -k + 2} m_{i,k-2} - \delta_{k \neq 1} } P_\theta(m_3) \langle -k + 2, k \rangle \tilde{\phi} = q ^ { -m_{-k+2,k-2} - \sum_{-k \leq i \leq -k + 2} m_{i,k-2} - \delta_{k \neq 1} } \langle -k + 2, k \rangle P_\theta(m_3) \tilde{\phi}.
$$

Hence we obtain

$$
P_\theta(m_1) (e_i P_\theta(m_2)) P_\theta(m_3) \tilde{\phi} = q ^ { - \sum_{i \leq k} m_{i,k-2} } P_\theta(m - \langle k \rangle) + (1 - q^2) \sum_{-k+2 < i \leq -k-2} q ^ { 1 - \sum_{i' \leq i} m_{i',k-2} - \sum_{i' > i} m_{i',k-2} } \times [m_{i,k-2} + 1] \tilde{P}_\theta(m - \langle i \rangle, k) + \langle i, k - 2 \rangle)
$$

$$
+ (1 - q^2) \delta_{k \neq 1} q ^ { -m_{-k,k} - m_{-k+2,k-2} - \sum_{-k \leq i \leq -k + 2} m_{i,k-2} } \times [2(m_{-k+2,k-2} + 1)] \tilde{P}_\theta(m - \langle -k + 2, k \rangle + \langle -k + 2, k - 2 \rangle)
$$

$$
+ (1 - q^2) q ^ { 2(1 - m_{-k,k}) - m_{-k+2,k-2} - \sum_{-k \leq i \leq -k + 2} m_{i,k-2} - \delta_{k \neq 1} } \frac{[m_{-k+2,k} + 1][m_{-k,k}]}{[2m_{-k,k}]} P(m - \langle -k \rangle) + \langle -k + 2, k \rangle).
$$

Hence the coefficient of $\tilde{P}_\theta(m - \langle k \rangle)$ in $E_{-k} \tilde{P}_\theta(m)$ is

$$
\sum_{\ell \geq k} \sum_{i \leq k} \sum_{i \leq k} m_{i,k-2} - \delta_{i+1} + 1 \sum_{i \leq k} m_{i,k-2} = q ^ { \sum_{\ell \geq k} m_{-k+2,\ell - m_{-k,k}} } 2m_{-k,k} - \sum_{-k \leq i \leq -k + 2} m_{i,k-2} - \delta_{k \neq 1} (1 - m_{-k,k} + m_{-k+2,k-2} - \sum_{-k \leq i \leq -k + 2} m_{i,k-2} - m_{i,k}).
$$

The coefficient of $\tilde{P}_\theta(m - \langle -k, k \rangle + \langle -k + 2, k \rangle)$ in $E_{-k} \tilde{P}_\theta(m)$ is

$$
(1 - q^2) q ^ { \sum_{\ell \geq k} m_{-k+2,\ell - m_{-k,k}} + \sum_{i \leq k} m_{i,k-2} - \delta_{i+1} + 2(1 - m_{-k,k}) - m_{-k+2,k-2} - \sum_{-k \leq i \leq -k + 2} m_{i,k-2} - \delta_{k \neq 1} } \times \frac{[m_{-k+2,k} + 1][m_{-k,k}]}{[2m_{-k,k}]}
$$

$$
= (1 - q^2) q ^ { \sum_{\ell \geq k} m_{-k+2,\ell - m_{-k,k}} + \sum_{i \geq k} m_{-k+2,\ell - m_{-k,k}} } \frac{[m_{-k,k}][m_{-k+2,k} + 1]}{[2m_{-k,k}]} (1 + q ^ { -2m_{-k,k} } )
$$

$$
= (1 - q^2) q ^ { -m_{-k,k} + \sum_{\ell \geq k} m_{-k+2,\ell - m_{-k,k}} + \sum_{i \geq k} m_{-k+2,\ell - m_{-k,k}} } [m_{-k+2,k} + 1]
$$

$$
= (1 - q^2) q ^ { -m_{-k,k} + 2m_{-k,k} + \sum_{\ell \geq k} m_{-k+2,\ell - m_{-k,k}} } [m_{-k+2,k} + 1],
$$

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For $-k + 2 < i \leq k - 2$, the coefficient of $\bar{P}_\theta(m - (i, k) + (i, k - 2))$ in $E_{-k}\bar{P}_\theta(m)$ is

$$(1 - q^2) q^{\ell} \sum_{i \leq k - 2} m_{i,k-2}^\ell - \delta_{k=1}^{11} \sum_{i' \leq i} m_{i',k} - \sum_{i' > i} m_{i',k} - \sum_{k+2 \leq i'} m_{i',k} - \sum_{m_{i,k-2}^\ell} [m_{i,k-2} + 1]$$

$$(1 - q^2) q^{\ell} \sum_{i \leq k} (m_{-k+2,\ell} - m_{-k,\ell}) + 2m_{-k+2,k-2} + 2m_{-k,k} - \sum_{k+2 \leq i'} m_{i',k} - \sum_{m_{i,k-2}^\ell} [m_{i,k-2} + 1].$$

Finally, for $k \neq 1$, the coefficient of $\bar{P}_\theta(m - (-k + 2, k) + (-k + 2, k - 2))$ in $E_{-k}\bar{P}_\theta(m)$ is

$$(1 - q^2) q^{\ell} \sum_{i \leq k} (m_{-k+2,\ell} - m_{-k,\ell}) + 2m_{-k+2,k-2} + 2m_{-k,k} - \sum_{k+2 \leq i'} m_{i',k} - \sum_{m_{i,k-2}^\ell} [2(m_{-k+2,k-2} + 1)].$$

Q.E.D.

**Theorem 3.23.** For $k > 0$ and $m \in \mathcal{M}_\theta$, we have

$$E_k\bar{P}_\theta(m) = \sum_{\ell > k} (1 - q^2) q^{\ell} \left( \sum_{i \leq k} m_{i,k} + 1 \right) \bar{P}_\theta(m - (k, \ell) + (k + 2, \ell)) + q^{\ell + \delta_{\ell \neq k}} \bar{P}_\theta(m - (k, \ell) + (k, \ell)).$$

**Proof.** The first follows from $e_{-k}^\ast P_\theta(m) = 0$ and Proposition 3.17, and the second follows from Proposition 3.20

Q.E.D.

4. **Crystal basis of $V_\theta(0)$**

4.1. **Crystal structure on $\mathcal{M}_\theta$.** We shall define the crystal structure on $\mathcal{M}_\theta$.

**Definition 4.1.** Suppose $k > 0$. For a $\theta$-restricted multisegment $m = \sum_{-j \leq i \leq j} m_{i,j}(i, j)$, we set

$$\varepsilon_{-k}(m) = \max \left\{ A_j^{(-k)}(m) \mid j \geq -k + 2 \right\},$$

where

$$A_j^{(-k)}(m) = \sum_{\ell \geq j} (m_{-k,\ell} - m_{-k+2,\ell+2}) \text{ for } j > k,$$

$$A_k^{(-k)}(m) = \sum_{\ell > k} (m_{-k,\ell} - m_{-k+2,\ell}) + 2m_{-k,k} + \delta(m_{-k+2,k} \text{ is odd}),$$

$$A_j^{(-k)}(m) = \sum_{\ell > k} (m_{-k,\ell} - m_{-k+2,\ell+2}) + 2m_{-k,k} - 2m_{-k+2,k-2} + \sum_{-k-2 \leq i \leq j+2} m_{i,k} - \sum_{-k-2 < i \leq j} m_{i,k-2} \text{ for } -k + 2 \leq j \leq k - 2.$$

(i) Let $n_f$ be the smallest $\ell \geq -k + 2$, with respect to the ordering $\cdots > k + 2 > k > -k + 2 > \cdots > k - 2$, such that $\varepsilon_{-k}(m) = A_{n_f}^{(-k)}(m)$. We define

$$\bar{F}_{-k}(m) = \begin{cases} m - (-k + 2, n_f) + (-k, n_f) & \text{if } n_f > k, \\
 - (-k + 2, k) + (-k, k) & \text{if } n_f = k \text{ and } m_{-k+2,k} \text{ is odd}, \\
 m - \delta_{n_f \neq k-2}(-k + 2, k - 2) + (-k + 2, k) & \text{if } n_f = k \text{ and } m_{-k+2,k} \text{ is even}, \end{cases}$$

$$m - \delta_{n_f \neq k-2}(n_f + 2, k - 2) + (n_f + 2, k) \text{ if } -k + 2 \leq n_f \leq k - 2.$$
Then we obtain a sequence of the form $\cdot \cdot \cdot$ $\cdot$

$$\tilde{e}(3)$$

For $0 < k \in I$, the actions of $\tilde{E}_-k$ and $\tilde{F}_-k$ on $m \in \mathcal{M}_\theta$ are described by the following algorithm.

Step 1. Arrange segments in $m$ of the form $\langle -k, j \rangle \ (j > k), \langle -k + 2, j \rangle \ (j > k), \langle i, k \rangle \ (i \leq k)$, $\langle -k + 2, i \rangle \ (k < i \leq k)$, $\langle -k + 2, k \rangle$ in the order

$$\cdots, \langle -k, k + 2 \rangle, \langle -k + 2, k + 2 \rangle, \langle -k, k \rangle, \langle -k + 2, k \rangle, \langle -k + 2, k - 2 \rangle, \langle -k + 4, k \rangle, \langle -k + 4, k - 2 \rangle, \cdots, \langle k - 2, k \rangle, \langle k - 2, k - 2 \rangle, \langle k \rangle.$$

Step 2. Write signatures for each segment contained in $m$ by the following rules.

(i) If a segment is not $\langle -k + 2, k \rangle$, then
- For $\langle -k, k \rangle$, write $-\cdot$,
- For $\langle -k, j \rangle$ with $j > k$, write $-$,
- For $\langle -k + 2, k - 2 \rangle$ with $k > 1$, write $++$,
- For $\langle -k + 2, j \rangle$ with $j > k$, write $+$,
- For $\langle j, k \rangle$ with $-k + 2 < j \leq k$, write $-$,
- For $\langle j, k - 2 \rangle$ with $-k + 2 < j \leq k - 2$, write $+$,
- Otherwise, write no signature.

(ii) For segments $m_{-k+2,k}\langle -k + 2, k \rangle$, if $m_{-k+2,k}$ is even, then write no signature, and if $m_{-k+2,k}$ is odd, then write $-\cdot$.

Step 3. In the resulting sequence of $+$ and $-$, delete a subsequence of the form $++$ and keep on deleting until no such subsequence remains.

Then we obtain a sequence of the form $-\cdot \cdots - \cdot + \cdot + \cdot +$.

(1) $\varepsilon_{-k}(m)$ is the total number of $-$ in the resulting sequence.

(2) $\tilde{F}_{-k}(m)$ is given as follows:

(i) if the leftmost $+$ corresponds to a segment $\langle -k + 2, j \rangle$ for $j > k$, then replace it with $\langle -k, j \rangle$,
(ii) if the leftmost $+$ corresponds to a segment $\langle j, k - 2 \rangle$ for $-k + 2 \leq j \leq k - 2$, then replace it with $\langle j, k \rangle$,
(iii) if the leftmost $+$ corresponds to segment $m_{-k+2,k}\langle -k + 2, k \rangle$, then replace one of the segments with $\langle -k, k \rangle$,
(iv) if no $+$ exists, add a segment $\langle k, k \rangle$ to $m$.

(3) $\tilde{E}_{-k}(m)$ is given as follows:

(i) if the rightmost $-$ corresponds to a segment $\langle -k, j \rangle$ for $j \geq k$, then replace it with $\langle -k + 2, j \rangle$,
(ii) if the rightmost $-$ corresponds to a segment $\langle j, k \rangle$ for $-k + 2 < j < k$, then replace it with $\langle j, k - 2 \rangle$,
(iii) if the rightmost $-$ corresponds to segments $m_{-k+2,k}\langle -k + 2, k \rangle$, then replace one of the segments with $\langle -k + 2, k - 2 \rangle$,
(iv) if the rightmost $-$ corresponds to a segment $\langle k, k \rangle$ for $k > 1$, then delete it,
(v) if no $-$ exists, then $\tilde{E}_{-k}(m) = 0$.  

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Example 4.3. (1) We shall write \(\{a, b\}\) for \(a(-1, 1) + b(1)\). The following diagram is the part of the crystal graph of \(B_\theta(0)\) that concerns only the 1-arrows and the \((-1)\)-arrows.

Especially the part of \((-1)\)-arrows is the following diagram.

\[
\begin{align*}
\{0, 2n\} &\xrightarrow{-1} \{0, 2n + 1\} &\xrightarrow{-1} \{1, 2n\} &\xrightarrow{-1} \{1, 2n + 1\} &\xrightarrow{-1} \{2, 2n\} &\cdots \\
\end{align*}
\]

(2) The following diagram is the part of the crystal graph of \(B_\theta(0)\) that concerns only the \((-1)\)-arrows and the \((-3)\)-arrows. This diagram is, as a graph, isomorphic to the crystal graph of \(A_2\).

(3) Here is the part of the crystal graph of \(B_\theta(0)\) that concerns only the \(n\)-arrows and the \((-n)\)-arrows for an odd integer \(n \geq 3\):

\[
\begin{align*}
\phi \xrightarrow{n} \langle n \rangle &\xrightarrow{n} 2\langle n \rangle &\xrightarrow{n} 3\langle n \rangle &\xrightarrow{n} \cdots \\
\end{align*}
\]
Lemma 4.4. For \(k \in I_{>0}\), the data \(\tilde{E}_{-k}, \tilde{F}_{-k}, \varepsilon_{-k}\) define a crystal structure on \(\mathcal{M}_\theta\), namely we have

(i) \(\tilde{F}_{-k} \mathcal{M}_\theta \subset \mathcal{M}_\theta\) and \(\tilde{E}_{-k} \mathcal{M}_\theta \subset \mathcal{M}_\theta \sqcup \{0\}\),

(ii) \(\tilde{F}_{-k} \tilde{E}_{-k}(m) = m\) if \(\tilde{E}_{-k}(m) \neq 0\), and \(\tilde{E}_{-k} \circ \tilde{F}_{-k} = \text{id}\),

(iii) \(\varepsilon_{-k}(m) = \max \{n \geq 0 \mid \tilde{E}_n^k(m) \neq 0\}\) for any \(m \in \mathcal{M}_\theta\).

Proof. We shall first show that, for \(m = \sum_{-j \leq i < j} m_{i,j}(i,j) \in \mathcal{M}_\theta\), \(\tilde{F}_{-k}(m) = \theta\)-restricted, \(\tilde{E}_{-k}\tilde{F}_{-k}(m) = m\) and \(\varepsilon_{-k}\tilde{F}_{-k}(m) = \varepsilon_{-k}(m) + 1\). Let \(A_j := A_j^{(-k)}(m) (j \geq -k + 2)\) and let \(n_f\) be as in Definition 4.11. Set \(m' = \tilde{F}_{-k}m\). Let \(A'_j = A_j^{(-k)}(m')\) and let \(n'_c\) be \(n_e\) for \(m'\).

(i) Assume \(n_f > k\). Since \(A_{n_f} > A_{n_f-2} = A_{n_f} + m_{-k,n_f-2} - m_{-k+2,n_f}\), we have \(m_{-k,n_f-2} < m_{-k+2,n_f}\). Hence \(m' = m - \langle -k + 2, n_f \rangle + \langle -k, k_f \rangle\) is \(\theta\)-restricted. Then we have

\[
A'_j = \begin{cases} 
A_j & \text{if } j > n_f, \\
A_j + 1 & \text{if } j = n_f, \\
A_j + 2 & \text{if } j < n_f.
\end{cases}
\]

Hence \(\varepsilon_{-k}(m') = A_{n_f} + 1 = \varepsilon_{-k}(m) + 1\) and \(n'_c = n_f\), which implies \(m = \tilde{E}_{-k}(m')\).

(ii) Assume \(n_f = k\).

(a) If \(m_{-k+2,k}\) is odd, then \(m' = m - \langle -k + 2, k \rangle + \langle -k, k \rangle\) is \(\theta\)-restricted. We have

\[
A'_j = \begin{cases} 
A_j & \text{if } j > k, \\
A_j + 1 & \text{if } j = k, \\
A_j + 2 & \text{if } j < k.
\end{cases}
\]

Hence \(\varepsilon_{-k}(m') = \varepsilon_{-k}(m) + 1\) and \(n'_c = k\), which implies \(m = \tilde{E}_{-k}(m')\).

(b) Assume that \(m_{-k+2,k}\) is even. If \(k \neq 1\), then \(A_k > A_{-k+2} = A_k - 2m_{-k+2,k-2}\), and hence \(m_{-k+2,k-2} > 0\). Therefore \(m' = m - \delta_{k\neq1}(-k + 2, k - 2) + \langle -k + 2, k \rangle\) is \(\theta\)-restricted. We have

\[
A'_j = \begin{cases} 
A_j & \text{if } j > k, \\
A_j + 1 & \text{if } j = k, \\
A_j + 2 & \text{if } j < k.
\end{cases}
\]

Hence \(\varepsilon_{-k}(m') = \varepsilon_{-k}(m) + 1\) and \(n'_c = k\), which implies \(m = \tilde{E}_{-k}(m')\).

(iii) Assume \(-k + 2 \leq n_f < k - 2\). Since \(A_{n_f} > A_{n_f+2} = A_{n_f} + m_{n_f+4,k} - m_{n_f+2,k-2}\), we have \(m_{n_f+2,k-2} > m_{n_f+4,k}\). Hence \(m' = m - \langle n_f + 2, k - 2 \rangle + \langle n_f + 2, k \rangle\) is \(\theta\)-restricted.

Then we have

\[
A'_j = \begin{cases} 
A_j & \text{if } j > n_f, \\
A_j + 1 & \text{if } j = n_f, \\
A_j + 2 & \text{if } j < n_f.
\end{cases}
\]

(Here the ordering is as in Definition 4.11 (i).) Hence \(\varepsilon_{-k}(m') = \varepsilon_{-k}(m) + 1\) and \(n'_c = n_f\), which implies \(m = \tilde{E}_{-k}m'\).

(iv) Assume \(n_f = k - 2\). It is obvious that \(m' = m + \langle k \rangle\) is \(\theta\)-restricted. We have

\[
A'_j = \begin{cases} 
A_j & \text{if } j \neq n_f, \\
A_j + 1 & \text{if } j = n_f.
\end{cases}
\]

Hence \(\varepsilon_{-k}(m') = \varepsilon_{-k}(m) + 1\) and \(n'_c = n_f\), which implies \(m = \tilde{E}_{-k}(m')\).
Similarly, we can prove that if $\varepsilon_{-k}(m) > 0$, then $\tilde{E}_{-k}(m)$ is $\theta$-restricted and $\tilde{F}_{-k}\tilde{E}_{-k}(m) = m$. Hence we obtain the desired results. Q.E.D.

**Definition 4.5.** For $k \in I_{>0}$, we define $\tilde{F}_k$, $\tilde{E}_k$ and $\varepsilon_k$ by the same rule as in Definition 3.7 for $\tilde{f}_k$, $\tilde{e}_k$ and $\varepsilon_k$.

Since it is well-known that it gives a crystal structure on $\mathcal{M}$, we obtain the following result.

**Theorem 4.6.** By $\tilde{F}_k$, $\tilde{E}_k$, $\varepsilon_k$ ($k \in I$), $\mathcal{M}_\theta$ is a crystal, namely, we have

(i) $\tilde{F}_i\mathcal{M}_\theta \subset \mathcal{M}_\theta$ and $\tilde{E}_k\mathcal{M}_\theta \subset \mathcal{M}_\theta \cup \{0\}$,

(ii) $\tilde{F}_k\tilde{E}_k(m) = m$ if $\tilde{E}_k(m) \neq 0$, and $\tilde{E}_k \circ \tilde{F}_k = \text{id}$,

(iii) $\varepsilon_k(m) = \max \left\{ n \geq 0 \mid \tilde{E}_k^n(m) \neq 0 \right\}$ for any $m \in \mathcal{M}_\theta$.

The crystal $\mathcal{M}_\theta$ has a unique highest weight vector.

**Lemma 4.7.** If $m \in \mathcal{M}_\theta$ satisfies that $\varepsilon_k(m) = 0$ for any $k \in I$, then $m = \emptyset$. Here $\emptyset$ is the empty multisegment. In particular, for any $m \in \mathcal{M}_\theta$, there exist $\ell \geq 0$ and $i_1, \ldots, i_\ell \in I$ such that $m = \tilde{F}_{i_1} \cdots \tilde{F}_{i_\ell}\emptyset$.

**Proof.** Assume $m \neq \emptyset$. Let $k$ be the largest $k$ such that $m_{k,j} \neq 0$ for some $j$. Then take the largest $j$ such that $m_{k,j} \neq 0$. Then $j \geq |k|$. Moreover, we have $m_{k+2,\ell} = 0$ for any $\ell$, and $m_{k,\ell} = 0$ for any $\ell > j$. Hence we have

$$A^{(k)}_j(m) = \begin{cases} 2m_{k,j} & \text{if } k = -j, \\ m_{k,j} & \text{otherwise.} \end{cases}$$

Hence $\varepsilon_k(m) \geq A^{(k)}_j(m) > 0$. Q.E.D.

4.2. **A criterion for crystals.** We shall give a criterion for a basis to be a crystal basis. Although we treat the case for modules over $B(\mathfrak{g})$ in this paper, similar results hold also for $U_q(\mathfrak{g})$.

Let $K[e, f]$ be the ring generated by $e$ and $f$ with the defining relation $ef = q^{-2}fe + 1$. We define the divided power by $f^{(n)} = f^n/[n]!$.

Let $P$ be a free $\mathbb{Z}$-module, and let $\alpha$ be a non-zero element of $P$.

Let $M$ be a $K[e, f]$-module. Assume that $M$ has a weight decomposition $M = \bigoplus_{\xi \in P} M_\xi$, and $eM_\lambda \subset M_{\lambda+\alpha}$ and $fM_\lambda \subset M_{\lambda-\alpha}$.

Assume the following finiteness conditions:

$$\text{(4.1) for any } \lambda \in P, \dim M_\lambda < \infty \text{ and } M_{\lambda+n\alpha} = 0 \text{ for } n \gg 0.$$  

Hence for any $u \in M$, we can write $u = \sum_{n \geq 0} f^{(n)}u_n$ with $eu_n = 0$. We define endomorphisms $\tilde{e}$ and $\tilde{f}$ of $M$ by

$$\tilde{e}u = \sum_{n \geq 1} f^{(n-1)}u_n,$$

$$\tilde{f}u = \sum_{n \geq 0} f^{(n+1)}u_n.$$  

Let $B$ be a crystal with weight decomposition by $P$. In this paper, we consider only the following type of crystals. We have $\text{wt}: B \to P$, $\tilde{f}: B \to B$, $\tilde{e}: B \to B \cup \{0\}$, $\varepsilon: B \to \mathbb{Z}_{\geq 0}$ satisfying the following properties, where $B_\lambda := \text{wt}^{-1}(\lambda)$:

(i) $\tilde{f}B_\lambda \subset B_{\lambda-\alpha}$ and $\tilde{e}B_\lambda \subset B_{\lambda+\alpha} \cup \{0\}$ for any $\lambda \in P$.  

Assume that a decomposition \( u \subset \mathbb{L} \).

\[ (4.9) \]

\( \tilde{e} \lambda \)

(ii) \( \tilde{e} \)

\[ (4.10) \]

\( \tilde{e} \in \mathbb{B}, \)

\[ (4.11) \]

\( \tilde{e} \circ \tilde{f} = \text{id}_B, \)

(iii) for any \( \lambda \in \mathbb{P}, B_\lambda \) is a finite set and \( B_{\lambda + \alpha} = \emptyset \) for \( n \gg 0 \),

(iv) \( \varepsilon(b) = \max \{ n \geq 0 \mid \tilde{e}^n b \neq 0 \} \) for any \( b \in \mathbb{B} \).

Set \( \text{ord}(a) = \sup \{ n \in \mathbb{Z} \mid a \in q^n \mathbb{A}_0 \} \) for \( a \in \mathbb{K} \). We understand \( \text{ord}(0) = \infty \).

Let \( \{ C(b) \}_{b \in \mathbb{B}} \) be a system of generators of \( \mathbb{M} \) with \( C(b) \in M_{\text{wt}(b)} \): \( \mathbb{M} = \sum_{b \in \mathbb{B}} \mathbb{K} C(b) \).

Let \( \xi \) be a map from \( \mathbb{B} \) to an ordered set. Let \( c : \mathbb{Z} \to \mathbb{R}, f : \mathbb{Z} \to \mathbb{R} \) and \( e : \mathbb{Z} \to \mathbb{R} \).

Assume that a decomposition \( \mathbb{B} = \mathbb{B}' \cup \mathbb{B}'' \) is given.

Assume that we have expressions:

\[ (4.2) \]

\[ eC(b) = \sum_{b' \in \mathbb{B}} E_{b,b'} C(b'), \]

\[ (4.3) \]

\[ fC(b) = \sum_{b' \in \mathbb{B}} F_{b,b'} C(b'). \]

Now consider the following conditions for these data, where \( l = \varepsilon(b) \) and \( l' = \varepsilon(b') \):

(4.4) \( c(0) = 0 \), and \( c(n) > 0 \) for \( n \neq 0 \),

(4.5) \( c(n) \leq n + c(m + n) + e(m) \) for \( n \geq 0 \),

(4.6) \( c(n) \leq c(m + n) + f(m) \) for \( n \leq 0 \),

(4.7) \( c(n) + f(n) > 0 \) for \( n > 0 \),

(4.8) \( c(n) + e(n) > 0 \) for \( n > 0 \),

(4.9) \( \text{ord}(F_{b,b'}) \geq -\ell + f(\ell + 1 - \ell') \),

(4.10) \( \text{ord}(E_{b,b'}) \geq 1 - \ell + e(\ell - 1 - \ell') \),

(4.11) \( F_{b,\tilde{e}b} \in q^{-\ell}(1 + q \mathbb{A}_0) \),

(4.12) \( E_{b,\tilde{e}b} \in q^{1-\ell}(1 + q \mathbb{A}_0) \) if \( \ell > 0 \),

(4.13) \( \text{ord}(F_{b,b'}) > -\ell + f(\ell + 1 - \ell') \) if \( b' \neq \tilde{f}b, \xi(\tilde{f}b) \neq \xi(b') \),

(4.14) \( \text{ord}(F_{b,b'}) > -\ell + f(\ell + 1 - \ell') \) if \( \tilde{f}b \in \mathbb{B}', b' \neq \tilde{f}b \) and \( \ell \leq \ell' - 1 \),

(4.15) \( \text{ord}(E_{b,b'}) > 1 - \ell + e(\ell - 1 - \ell') \) if \( b \in \mathbb{B}'', b' \neq \tilde{e}b \) and \( \ell \leq \ell' + 1 \).

**Theorem 4.8.** Assume the conditions (4.4)–(4.15). Set \( L = \sum_{b \in \mathbb{B}} A_0 C(b) \). Then we have \( \tilde{e}L \subset L \) and \( \tilde{f}L \subset L \). Moreover we have

\[ \tilde{e}C(b) \equiv C(\tilde{e}b) \mod qL \quad \text{and} \quad \tilde{f}C(b) \equiv C(\tilde{f}b) \mod qL \quad \text{for any} \quad b \in \mathbb{B}. \]

Here we understand \( C(0) = 0 \).

We shall divide the proof into several steps.

Write

\[ C(b) = \sum_{n \geq 0} f^{(n)} C_n(b) \quad \text{with} \quad eC_n(b) = 0. \]

Set

\[ L_0 = \sum_{b \in \mathbb{B}, n \geq 0} A_0 f^{(n)} C_0(b). \]

Set for \( u \in \mathbb{M}, \text{ord}(u) = \sup \{ n \in \mathbb{Z} \mid u \in q^n L_0 \} \). If \( u = 0 \) we set \( \text{ord}(u) = \infty \), and if \( u \not\in \bigcup_{n \in \mathbb{Z}} q^n L_0 \), then \( \text{ord}(u) = -\infty \).

We shall use the following two recursion formulas (4.16) and (4.17).
We have
\[ eC(b) = \sum_{n \geq 1} q^{1-n} f^{(n-1)} C_n(b) \]
\[ = \sum_{n \geq 0} E_{b,b'} f^{(n)} C_n(b'). \]

Hence we have
\[ C_n(b) = \sum_{b' \in B_{\lambda+\alpha}} q^{n-1} E_{b,b'} C_{n-1}(b') \quad \text{for } n > 0 \text{ and } b \in B_\lambda. \]

If \( \ell := \varepsilon(b) > 0 \), then we have
\[ fC(\tilde{b}) = \sum_{b' \in B, n \geq 0} F_{\tilde{b},b'} f^{(n)} C_n(b') \]
\[ = \sum_{n \geq 0} [n + 1] f^{(n+1)} C_n(\tilde{b}). \]

Hence, we have by (4.11)
\[ \delta_{n \neq 0}[n] C_n(\tilde{b}) = \sum_{b'} F_{\tilde{b},b'} C_n(b') \in q^{1-\ell}(1 + qA_0) C_n(b) + \sum_{b' \neq b} F_{\tilde{b},b'} C_n(b'). \]

Therefore we obtain
\[ C_n(b) \in \delta_{n \neq 0}(1 + qA_0) q^{\ell-n} C_n(b') + \sum_{b' \neq b} q^{\ell-1} A_0 F_{\tilde{b},b'} C_n(b') \quad \text{if } \ell > 0. \]

**Lemma 4.9.** \( \text{ord}(C_n(b)) \geq c(n - \ell) \) for any \( n \in \mathbb{Z}_{\geq 0} \) and \( b \in B, \) where \( \ell := \varepsilon(b). \)

**Proof.** For \( \lambda \in P, \) we shall show the assertion for \( b \in B_\lambda \) by the induction on \( \sup \{ n \in \mathbb{Z} \mid M_{\lambda+\alpha} \neq 0 \}. \)

Hence we may assume
\[ \text{ord}(C_n(b)) \geq c(n - \ell) \quad \text{for any } n \in \mathbb{Z}_{\geq 0} \text{ and } b \in B_{\lambda+\alpha}. \]

(i) Let us first show \( C_n(b) \in KL_0. \)

Since it is trivial for \( n = 0, \) assume that \( n > 0. \) Since \( C_{n-1}(b') \in KL_0 \) for \( b' \in B_{\lambda+\alpha} \) by the induction assumption (4.18), we have \( C_n(b) \in KL_0 \) by (4.16).

(ii) Let us show that \( \text{ord}(C_n(b)) \geq c(n - \ell) \) for \( n \geq \ell. \)

If \( n = 0, \) then \( \ell = 0 \) and the assertion is trivial by (4.4). Hence we may assume that \( n > 0. \)

We shall use (4.16). For \( b' \in B_{\lambda+\alpha}, \) we have
\[ \text{ord}(C_{n-1}(b')) \geq c(n - 1 - \ell') \quad \text{where } \ell' = \varepsilon(b') \]
by the induction hypothesis (4.18). On the other hand, \( \text{ord}(E_{b,b'}) \geq 1 - \ell + c(\ell - 1 - \ell') \)
by (4.10). Hence,
\[ \text{ord}(q^{n-1} E_{b,b'} C_{n-1}(b')) \geq (n - 1) + (1 - \ell + c(\ell - 1 - \ell')) + c(n - 1 - \ell') \]
\[ = (n - \ell) + c(\ell - 1 - \ell') + c((n - \ell) + (\ell - 1 - \ell')) \]
\[ \geq c(n - \ell) \]
by (4.5).

(iii) In the general case, let us set \( r = \min \{ \text{ord}(C_n(b)) - c(n - \varepsilon(b)) \mid b \in B_\lambda, n \geq 0 \} \in \mathbb{R} \cup \{ \infty \}. \)

Assuming \( r < 0, \) we shall prove
\[ \text{ord}(C_n(b)) > c(n - \ell) + r \quad \text{for any } b \in B_\lambda, \]

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which leads a contradiction.

By the induction on $\xi(b)$, we may assume that
\begin{equation}
(4.19) \quad \text{if } \xi(b') < \xi(b), \text{ then } \text{ord}(C_n(b')) > c(n - \ell') + r \text{ where } \ell' := \varepsilon(b').
\end{equation}

By (ii), we may assume that $n < \ell$. Hence $\bar{e}b \in B$. By the induction hypothesis (4.18), we have $\text{ord}(q^{e-n}C_{n-1}(\bar{e}b)) \geq \ell - n + c((n - 1) - (\ell - 1)) \geq c(n - \ell) > c(n - \ell) + r$. By (4.17), it is enough to show
\[\text{ord}(q^{\ell-1}F_{\bar{e}b,b}C_n(b')) > c(n - \ell) + r \quad \text{for } b' \neq b.\]
We shall divide its proof into two cases.

(a) $\xi(b') < \xi(b)$.

In this case, (4.19) implies $\text{ord}(C_n(b')) > c(n - \ell') + r$. Hence
\[
\text{ord}(q^{\ell-1}F_{\bar{e}b,b}C_n(b')) > (\ell - 1) + (1 - \ell + f(\ell - \ell')) + c(n - \ell') + r
\]
by (4.9) and (4.6).

(b) Case $\xi(b') \leq \xi(b)$.

In this case, $\text{ord}(F_{\bar{e}b,b'}) > 1 - \ell + f(\ell - \ell')$ by (4.13), and $\text{ord}(C_n(b')) \geq c(n - \ell') + r$. Hence,
\[
\text{ord}(q^{\ell-1}F_{\bar{e}b,b}C_n(b')) > (\ell - 1) + (1 - \ell + f(\ell - \ell')) + c(n - \ell') + r
\]
\[
= f(\ell - \ell') + c((n - \ell) + (\ell - \ell')) + r \geq c(n - \ell) + r
\]
Q.E.D.

Lemma 4.10. $\text{ord}(C_\ell(b) - C_{\ell-1}(\bar{e}b)) > 0$ for $\ell := \varepsilon(b) > 0$.

Proof. We divide the proof into two cases: $b \in B'$ and $b \in B''$.

(i) $b \in B'$.

By (4.17), it is enough to show
\[\text{ord}(q^{\ell-1}F_{\bar{e}b,b}C_\ell(b')) > 0 \quad \text{for } b' \neq b.\]

(a) Case $\ell > \ell' := \varepsilon(b')$.

We have
\[
\text{ord}(q^{\ell-1}F_{\bar{e}b,b}C_\ell(b')) \geq (\ell - 1) + (1 - \ell + f(\ell - \ell')) + c(\ell - \ell') > 0
\]
by (4.7).

(b) Case $\ell \leq \ell'$.

We have $\text{ord}(F_{\bar{e}b,b'}) > 1 - \ell + f(\ell - \ell')$ by (4.14). Hence
\[
\text{ord}(q^{\ell-1}F_{\bar{e}b,b}C_\ell(b')) > (\ell - 1) + (1 - \ell + f(\ell - \ell')) + c(\ell - \ell') \geq 0
\]
by (4.6) with $n = 0$.

(ii) Case $b \in B''$.

We use (4.16). By (4.12), it is enough to show that
\[\text{ord}(q^{\ell-1}E_{b,b'}C_{\ell-1}(b')) > 0 \quad \text{for } b' \neq \bar{e}b.\]

(a) Case $\ell - 1 > \ell'$.

\[\text{ord}(q^{\ell-1}E_{b,b'}C_{\ell-1}(b')) \geq e(\ell - 1 - \ell') + c(\ell - 1 - \ell') > 0 \quad \text{by (4.10) and (4.8)}.\]

(b) Case $\ell - 1 \leq \ell'$.

\[\text{ord}(E_{b,b'}) > 1 - \ell + e(\ell - 1 - \ell') \quad \text{by (4.15), and } \text{ord}(q^{\ell-1}E_{b,b'}C_{\ell-1}(b')) > e(\ell - 1 - \ell') + c(\ell - 1 - \ell') \geq 0 \quad \text{by (4.5) with } n = 0.\]

Q.E.D.
Hence we have
\begin{align*}
C_n(b) & \equiv 0 \mod qL_0 \quad \text{for } n \neq \ell := \varepsilon(b), \\
C_\ell(b) & \equiv C_0(e\ell b) \mod qL_0, \\
C(b) & \equiv f^{(j)}C_{\ell}(b) \mod qL_0, \\
\tilde{f}C(b) & \equiv C(\tilde{f}b) \mod qL_0, \\
\varepsilon C(b) & \equiv C(\varepsilon b) \mod qL_0,
\end{align*}

\[ L_0 := \sum_{b \in B, n \geq 0} A_0 f^{(n)} C_0(b) = \sum_{b \in B} A_0 C(b). \]

Indeed, the last equality follows from the fact that \( \{C(b)\}_{b \in B} \) generates \( L_0/qL_0 \).

Thus we have completed the proof of Theorem 4.8.

The following is the special case where \( B' = B'' = B \) and \( \varepsilon(b) = \varepsilon(b) \).

**Corollary 4.11.** Assume \((4.4)-(4.12)\) and
\begin{align*}
(4.20) & \quad \text{ord}(F_{b'\ell'}) > -\ell + f(1 + \ell - \ell') \quad \text{if } \ell < \ell' \text{ and } b' \neq \tilde{f}b, \\
(4.21) & \quad \text{ord}(E_{b'\ell'}) > 1 - \ell + e(\ell - 1 - \ell') \quad \text{if } \ell \leq \ell' + 1 \text{ and } b' \neq \tilde{e}b.
\end{align*}

Then the assertions of Theorem 4.8 hold.

4.3. **Estimates of the order of coefficients.** By applying Theorem 4.8 we shall show that \( \{P_\theta(m)\phi\}_{m \in \mathcal{M}_\theta} \) is a crystal basis of \( V_\theta(0) \) and its crystal structure coincides with the one given in §4.1.

We define \( c, f, e : \mathbb{Z} \to \mathbb{Q} \) by \( c(n) = |n/2| \) and \( f(n) = e(n) = n/2 \). Then the conditions \((4.4)-(4.8)\) are obvious. Set \( \xi(m) = (-1)^{m-k+2, k} m_{-k,k} \) and
\begin{align*}
B'' & = \{ m \in \mathcal{M}_\theta \mid -k + 2 < n_e(m) < k \} \cup \{ m \in \mathcal{M}_\theta \mid m_{-k+2, k}(m) \text{ is odd} \}, \\
B' & = \mathcal{M}_\theta \setminus B''.
\end{align*}

Here \( n_e(m) \) is \( n_e \) given in Definition 4.1 (ii). If \( \varepsilon_{-k}(m) = 0 \), then we understand \( n_e(m) = \infty \).

We define \( F_{m,m'}^{-k} \) and \( E_{m,m'}^{-k} \) by the coefficients of the following expansion:
\begin{align*}
F_{-k}P_\theta(m)\tilde{\phi} & = \sum_{m'} F_{m,m'}^{-k} P_\theta(m')\tilde{\phi}, \\
E_{-k}P_\theta(m)\tilde{\phi} & = \sum_{m'} E_{m,m'}^{-k} P_\theta(m')\tilde{\phi},
\end{align*}

as given in Theorems 3.21 and 3.22. Put \( \ell = \varepsilon_{-k}(m) \) and \( \ell' = \varepsilon_{-k}(m') \).

**Proposition 4.12.** The conditions \((4.9), (4.11), (4.13) \text{ and } (4.14)\) hold, namely, we have
\begin{enumerate}
\item[(a)] if \( m' = \tilde{F}_{-k}(m) \), then \( F_{m,m'}^{-k} \in q^{-\ell}(1 + qA_0) \),
\item[(b)] if \( m' \neq \tilde{F}_{-k}(m) \), then \( \text{ord}(F_{m,m'}^{-k}) \geq -\ell + f(\ell + 1 - \ell') = -(\ell + \ell' - 1)/2 \),
\item[(c)] if \( m' \neq \tilde{F}_{-k}(m) \) and \( \text{ord}(F_{m,m'}^{-k}) = -(\ell + \ell' - 1)/2 \), then the following two conditions hold:
\begin{enumerate}
\item \( \xi(\tilde{F}_{-k}(m)) > \xi(m') \),
\item \( \ell \geq \ell' \) or \( \tilde{F}_{-k}(m) \in B'' \).
\end{enumerate}
\end{enumerate}

**Proof.** We shall write \( A_j \) for \( A_{-k}^{-j}(m) \). Let \( n_f \) be as in Definition 4.1 (i). Note that \( F_{m,\tilde{F}_{-k}(m)}^{-k} \neq 0 \).
If $F_{m,m'}^{-k} \neq 0$, we have the following four cases. We shall use $[n] \in q^{1-n}(1 + qA_0)$ for $n > 0$.

**Case 1.** $m' = m - \langle -k + 2, n \rangle + \langle -k, n \rangle$ for $n > k$.

In this case, we have

$$F_{m,m'}^{-k} = [m_{-k,n} + 1]q^{\sum_{j > n}(m_{-k+2,j} - m_{-k,j})} \in q^{-A_k}(1 + qA_0)$$

and

$$\ell = \max\{A_j (j > -k + 2)\},$$

$$\ell' = \max\{A_j (j > n), A_n + 1, A_j + 2 (j < n)\}.$$ 

If $m' = \tilde{F}_{-k}(m)$, then $\ell = A_n$ and we obtain (a). Assume $m' \neq \tilde{F}_{-k}(m)$. Since $A_n \leq \ell, \ell' - 1$, we have $\text{ord}(F_{m,m'}^{-k}) = -A_n \geq -(\ell + \ell' - 1)/2$. Hence we obtain (b). If $\text{ord}(F_{m,m'}^{-k}) = -(\ell + \ell' - 1)/2$, then we have $A_n = \ell = \ell' - 1$. Since $A_j + 2 \leq \ell' = A_n + 1$ for $j < n$, we have $n_f = n$ and $m' = \tilde{F}_{-k}(m)$, which is a contradiction.

**Case 2.** $m' = m - \langle -k + 2, k \rangle + \langle -k, k \rangle$.

In this case we have

$$F_{m,m'}^{-k} = [2m_{-k,k} + 2]q^{\sum_{j > k}(m_{-k+2,j} - m_{-k,j})} \in q^{-A_k-\delta(m_{-k+2,k} \text{ is even})}(1 + qA_0).$$

(i) Assume that $m_{-k+2,k}$ is odd. We have $F_{m,m'}^{-k} \in q^{-A_k}(1 + qA_0)$ and

$$\ell' = \max\{A_j (j > k), A_k + 1, A_j + 2 (j < k)\}.$$ 

If $m' = \tilde{F}_{-k}(m)$, then $\ell = A_k$ and (a) holds. Assume that $m' \neq \tilde{F}_{-k}(m)$. We have $A_k \leq \ell, \ell' - 1$ and hence $\text{ord}(F_{m,m'}^{-k}) = -A_k \geq -(\ell + \ell' - 1)/2$. If $\text{ord}(F_{m,m'}^{-k}) = -(\ell + \ell' - 1)/2$, then $A_k = \ell = \ell' - 1$, and we have $m' = \tilde{F}_{-k}(m)$, which is a contradiction.

(ii) Assume that $m_{-k+2,k}$ is even. Then $m' \neq \tilde{F}_{-k}(m)$, $F_{m,m'}^{-k} \in q^{-A_k-1}(1 + qA_0)$ and

$$\ell' = \max\{A_j (j > k), A_k + 3, A_j + 2 (j < k)\}.$$ 

We have $A_k \leq \ell, \ell' - 3$ and hence $\text{ord}(F_{m,m'}^{-k}) = -A_k - 1 \geq -(\ell + \ell' - 1)/2$. Hence (b) holds. Let us show (c). Assume $m' \neq \tilde{F}_{-k}(m)$, and $\text{ord}(F_{m,m'}^{-k}) = -(\ell + \ell' - 1)/2$. Then we have $A_k = \ell = \ell' - 3$. Hence $n_f \leq k$ and we have either $\tilde{F}_{-k}(m) = m - \delta_{i \neq k}\langle i, k - 2 \rangle + \langle i, k \rangle$ with $-k + 2 < i \leq k$ or $\tilde{F}_{-k}(m) = m - \delta_{k \neq 1}\langle -k + 2, k - 2 \rangle + \langle -k + 2, k \rangle$.

Hence we have $\xi(\tilde{F}_{-k}(m)) = \pm m_{-k,k} > -m_{-k,k} - 1 = \xi(m')$. Hence we obtain (c) (1).

(1) Assume $\tilde{F}_{-k}(m) = m - \delta_{i \neq k}\langle i, k - 2 \rangle + \langle i, k \rangle$ with $-k + 2 < i \leq k$. Then $k \neq 1$ and $\tilde{F}_{-k}(\tilde{F}_{-k}(m)) = \tilde{F}_{-k}(m) - \langle i, k \rangle + \delta_{i \neq k}\langle i, k - 2 \rangle$. Hence $n_c(\tilde{F}_{-k}(m)) = i - 2 < k$.

Hence $\tilde{F}_{-k}(m) \in B''$. Therefore we obtain (c) (2).

(2) Assume $\tilde{F}_{-k}(m) = m - \delta_{k \neq 1}\langle -k + 2, k - 2 \rangle + \langle -k + 2, k \rangle$. Then $m_{-k+2,k}(\tilde{F}_{-k}(m)) = \delta_{-k+2,k} + 1$ is odd. Hence $\tilde{F}_{-k}(m) \in B''$.

**Case 3.** $m' = m - \delta_{k \neq 1}\langle -k + 2, k - 2 \rangle + \langle -k + 2, k \rangle$. In this case, we have

$$F_{m,m'}^{-k} = [m_{-k+2,k} + 1]q^{\sum_{j > k}(m_{-k+2,j} - m_{-k,j}) + m_{-k+2,k} - 2m_{-k,k}} \in q^{-A_k+\delta(m_{-k+2,k} \text{ is odd})}(1 + qA_0).$$

(i) If $m_{-k+2,k}$ is odd, then $m' \neq \tilde{F}_{-k}(m)$, $F_{m,m'}^{-k} \in q^{-A_k+1}(1 + qA_0)$, and

$$\ell' = \max\{A_j (j > k), A_k - 1, A_j + 2 (j < k)\}.$$ 

We have $A_k \leq \ell, \ell' + 1$ and hence $\text{ord}(F_{m,m'}^{-k}) = -A_k - 1 \geq -(\ell + \ell' - 1)/2$. If $\text{ord}(F_{m,m'}^{-k}) = -(\ell + \ell' - 1)/2$, then $A_k = \ell = \ell' + 1$, and $n_f = k$. Hence we obtain
(c) (2), and \( \tilde{F}_{-k}(m) = m - (-k + 2, k) + (-k, k) \). Hence \( \xi(\tilde{F}_{-k}(m)) = m_{-k,k} + 1 > m_{-k,k} = \xi(m) \). Hence we obtain (c) (1).

(ii) If \( m_{-k+2,k} \) is even, then \( F_{m,m'}^{-k} \in q^{-A_k}(1 + qA_0) \) and

\[
\ell' = \max\{A_j \ (j > k), A_k + 1, A_j + 2 \ (j < k)\}.
\]

If \( m' = \tilde{F}_{-k}(m) \), then \( \ell = A_k \) and (a) is satisfied. Assume \( m' \neq \tilde{F}_{-k}(m) \). We have \( A_k \leq \ell, \ell' - 1 \) and hence ord\( (F_{m,m'}^{-k}) = A_k \geq - (\ell + \ell' - 1)/2 \). If ord\( (F_{m,m'}^{-k}) = -(\ell + \ell' - 1)/2 \), then \( A_k = \ell = \ell' - 1 \), and hence \( m' = \tilde{F}_{-k}(m) \), which is a contradiction.

**Case 4.** \( m' = m - \delta_{i \neq k}(i, k - 2) + \langle i, k \rangle \) for \( -k + 2 < i \leq k \). We have

\[
F_{m,m'}^{-k} = [m_{i,k} + 1]q^{\sum_j \langle i, j \rangle + 2m_{-k+2,k-2}-2m_{-k,k}+\sum_{-k+2 \leq j < i}(m_{j,k-2} - m_{j,k})} \\
\in q^{-A_{i-2}}(1 + qA_0),
\]

and

\[
\ell' = \max\{A_j \ (j \geq k), A_j \ (j < i - 2), A_{i-2} + 1, A_j + 2 \ (i - 2 < j \leq k - 2)\}.
\]

If \( m' = \tilde{F}_{-k}(m) \), then \( \ell = A_{i-2} \) and (a) holds. Assume \( m' \neq \tilde{F}_{-k}(m) \). Since \( A_{i-2} \leq \ell, \ell' - 1 \), we have ord\( (F_{m,m'}^{-k}) = A_{i-2} \geq - (\ell + \ell' - 1)/2 \). Hence we obtain (b). If ord\( (F_{m,m'}^{-k}) = -(\ell + \ell' - 1)/2 \), then we have \( A_{i-2} = \ell = \ell' - 1 \). Hence \( m' = \tilde{F}_{-k}(m) \), which is a contradiction.

Q.E.D.

**Proposition 4.13.** Suppose \( k > 0 \). The conditions (4.10), (4.12), and (4.15) hold, namely, we have

(a) if \( m' = \tilde{E}_{-k}(m) \), then \( E_{m,m'}^{-k} \in q^{1-\ell}(1 + qA_0) \),

(b) if \( m' \neq \tilde{E}_{-k}(m) \), then ord\( (E_{m,m'}^{-k}) \geq 1 - \ell + e(\ell - 1 - \ell') = - (\ell + \ell' - 1)/2 \),

(c) if \( m' \neq \tilde{E}_{-k}(m) \), \( \ell \leq \ell' + 1 \) and ord\( (E_{m,m'}^{-k}) = -(\ell + \ell' - 1)/2 \), then \( b \notin B'' \).

**Proof.** The proof is similar to the one of the above proposition.

We shall write \( A_j \) for \( A_j^{-k}(m) \). Let \( n_e \) be as in Definition 4.1 (ii).

Note that \( E_{m,m'}^{-k} \neq 0 \) if \( \tilde{E}_{-k}(m) \neq 0 \). If \( E_{m,m'}^{-k} \neq 0 \), we have the following five cases.

**Case 1.** \( m' = m - \langle -k, n \rangle + \langle -k + 2, n \rangle \) for \( n > k \).

In this case, we have

\[
E_{m,m'}^{-k} = (1 - q^2)[m_{k+2,n} + 1]q^{1+\sum_j \langle m_{-k+2,j} - m_{-k,j} \rangle} \in q^{-A_n}(1 + qA_0)
\]

and

\[
\ell = \max\{A_j \ (j \geq -k + 2)\},
\]

\[
\ell' = \max\{A_j \ (j > n), A_{n} - 1, A_j + 2 \ (j < n)\}.
\]

If \( m' = \tilde{E}_{-k}(m) \), then \( \ell = A_n \) and we obtain (a). Assume \( m' \neq \tilde{E}_{-k}(m) \). Since \( A_n \leq \ell, \ell' + 1 \), we have ord\( (E_{m,m'}^{-k}) = 1 - A_n \geq - (\ell + \ell' - 1)/2 \). Hence we obtain (b). If ord\( (E_{m,m'}^{-k}) = -(\ell + \ell' - 1)/2 \), then we have \( A_n = \ell = \ell' + 1 \). Since \( A_j \leq \ell' = A_n - 1 \) for \( j > n \), we have \( n_e = n \) and \( m' = \tilde{E}_{-k}(m) \), which is a contradiction.

**Case 2.** \( m' = m - \langle -k, k \rangle + \langle -k + 2, k \rangle \).

In this case we have

\[
E_{m,m'}^{-k} = (1 - q^2)[m_{-k+2,k} + 1]q^{1+\sum_j \langle m_{-k+2,j} - m_{-k,j} \rangle + m_{-k+2,k} - 2m_{-k,k}} \\
\in q^{-A_k + \delta(m_{-k+2,k} \text{ is odd})}(1 + qA_0).
\]
(i) Assume that \( m_{-k+2,k} \) is odd. Then \( m' \neq \tilde{E}_{-k}(m) \), \( E_{m,m'}^{-k} \in q^{2-A_k(1+qA_0)} \) and

\[
\ell' = \max\{A_j (j > k), A_k - 3, A_j - 2 (j < k)\}.
\]

We have \( A_k \leq \ell, \ell' + 3 \) and hence \( \text{ord}(E_{m,m'}^{-k}) = 2 - A_k \geq -(\ell + \ell' + 1)/2 \). Hence (b) holds. If \( \text{ord}(E_{m,m'}^{-k}) = -(\ell + \ell' - 1)/2 \), then \( A_k = \ell = \ell' + 3 \). Hence \( \ell > \ell' + 1 \) and (c) holds.

(ii) Assume that \( m_{-k+2,k} \) is even. Then \( E_{m,m'}^{-k} \in q^{1-A_k(1+qA_0)} \) and

\[
\ell' = \max\{A_j (j > k), A_k - 1, A_j - 2 (j < k)\}.
\]

If \( m' = \tilde{E}_{-k}(m) \), then \( \ell = A_k \), and we obtain (a). Assume \( m' \neq \tilde{E}_{-k}(m) \). We have \( A_k \leq \ell, \ell' + 1 \) and hence \( \text{ord}(E_{m,m'}^{-k}) = 1 - A_k \geq -(\ell + \ell' - 1)/2 \). If \( \text{ord}(E_{m,m'}^{-k}) = -(\ell + \ell' - 1)/2 \), then \( A_k = \ell = \ell' + 1 \) and \( n_e = k \). Hence \( m' = \tilde{E}_{-k}(m) \), which is a contradiction.

Case 3. \( m' = m - (-k + 2, k) + \delta_{k\neq1}(-k + 2, k - 2) \). If \( k \neq 1 \), we have

\[
E_{m,m'}^{-k} = (1 - q^2)(2(m_{-k+2,k-2} + 1))q^{1+\sum_{j > k}(m_{-k+2,j} - m_{-k,j}) + 2m_{-k+2,k-2} - 2m_{-k,k}} \in q^{1-A_k+\delta(m_{-k+2,k} is odd)(1+qA_0)}.
\]

If \( k = 1 \), we have

\[
E_{m,m'}^{-k} = q^{\sum_{j > k}(m_{-k+2,j} - m_{-k,j}) - 2m_{-k,k}} = q^{1-A_k+\delta(m_{-k+2,k} is odd)}.
\]

In the both cases, we have

\[
E_{m,m'}^{-k} \in q^{1-A_k+\delta(m_{-k+2,k} is odd)(1+qA_0)}.
\]

(i) If \( m_{-k+2,k} \) is odd, then \( E_{m,m'}^{-k} \in q^{1-A_k(1+qA_0)} \) and

\[
\ell' = \max\{A_j (j > k), A_k - 1, A_j - 2 (j < k)\}.
\]

If \( m' = \tilde{E}_{-k}(m) \), then \( \ell = A_k \) and (a) is satisfied. We have \( A_k \leq \ell, \ell' + 1 \) and hence \( \text{ord}(E_{m,m'}^{-k}) = 1 - A_k \geq -(\ell + \ell' - 1)/2 \). Assume \( m' \neq \tilde{E}_{-k}(m) \). If \( \text{ord}(E_{m,m'}^{-k}) = -(\ell + \ell' - 1)/2 \), then \( A_k = \ell = \ell' + 1 \), and \( n_e = k \). Hence \( m' = \tilde{E}_{-k}(m) \), which is a contradiction.

(ii) If \( m_{-k+2,k} \) is even, then \( m' \neq \tilde{E}_{-k}(m) \), \( E_{m,m'}^{-k} \in q^{1-A_k(1+qA_0)} \), and

\[
\ell' = \max\{A_j (j > k), A_k + 1, A_j - 2 (j < k)\}.
\]

We have \( A_k \leq \ell, \ell' - 1 \) and hence \( \text{ord}(E_{m,m'}^{-k}) = -A_k \geq -(\ell + \ell' - 1)/2 \). Hence we obtain (b). If \( \text{ord}(E_{m,m'}^{-k}) = -(\ell + \ell' - 1)/2 \), then \( A_k = \ell = \ell' - 1 \). Hence \( n_e(m) \geq k \) and \( m_{-k+2,k}(m) \) is even. Hence \( m \notin B'' \).

Case 4. \( m' = m - (i,k) + (i,k-2) \) for \( -k + 2 < i \leq k - 2 \).

We have

\[
E_{m,m'}^{-k} = (1 - q^2)[m_{i,k-2} + 1]q^{1+\sum_{j > k}(m_{i,k-2,j} - m_{i,j}) + 2m_{i,k-2,k-2} - 2m_{-k,k} + \sum_{k+2 < j < i}(m_{j,k-2} - m_{-j,k})} \in q^{1-A_{i-2}(1+qA_0)},
\]

and

\[
\ell' = \max\{A_j (j \geq k), A_j (j < i - 2), A_{i-2} - 1, A_j - 2 (i \leq j \leq k - 2)\}.
\]

If \( m' = \tilde{E}_{-k}(m) \), then \( \ell = A_{i-2} \) and (a) holds. Assume \( m' \neq \tilde{E}_{-k}(m) \). Since \( A_{i-2} \leq \ell, \ell' + 1 \), we have \( \text{ord}(E_{m,m'}^{-k}) = 1 - A_{i-2} \geq -(\ell + \ell' - 1)/2 \). Hence we obtain (b). If \( \text{ord}(E_{m,m'}^{-k}) = -(\ell + \ell' - 1)/2 \), then we have \( A_{i-2} = \ell = \ell' + 1 \). Hence \( m' = \tilde{E}_{-k}(m) \), which is a contradiction.
Case 5. \( k \neq 1 \) and \( m' = m - \langle k \rangle \). In this case,
\[
E^{-k}_{m,m'} = q^{\sum_{j > k}(m_{-k,j} - m_{-k,k}) - 2m_{-k,k} + 2m_{-k,k-2} + \sum_{k-2 < l < k}(m_{l,k-2} - m_{l,k})} q^{1-A_{k-2}(1 + qA_0)},
\]
and
\[
\ell' = \max\{A_j (j \neq k - 2), A_{k-2} - 1\}.
\]
If \( m' = \widetilde{E}_{-k}(m) \), then \( \ell = A_{k-2} \) and (a) holds. Assume \( m' \neq \widetilde{E}_{-k}(m) \). Since \( A_{k-2} \leq \ell, \ell + 1 \), we have \( \text{ord}(E^{-k}_{m,m'}) = 1 - A_{k-2} \geq -(\ell + \ell' - 1)/2 \). Hence we obtain (b). If \( \text{ord}(E^{-k}_{m,m'}) = -(\ell + \ell' - 1)/2 \), then we have \( A_{k-2} = \ell = \ell' + 1 \). Hence \( m' = \widetilde{E}_{-k}(m) \), which is a contradiction.

Proposition 4.14. Let \( k \in I_{>0} \). Then the conditions in Corollary 4.11 holds for \( \widetilde{E}_k, \widetilde{F}_k \) and \( \epsilon_k \), with the same functions \( c, e, f \).

Since the proof is similar to and simpler than the one of the preceding two propositions, we omit the proof.

As a corollary we have the following result. We write \( \phi \) for the generator \( \phi_0 \) of \( V_\theta(0) \) for short.

Theorem 4.15. (i) The morphism
\[
\widetilde{V}_\theta(0) := U_q^-(g)/\sum_{k \in I} U_q^-(g)(f_k - f_{-k}) \to V_\theta(0)
\]
is an isomorphism.

(ii) \( \{P_\theta(m)\}_m \subset M_\theta \) is a basis of the \( K \)-vector space \( V_\theta(0) \).

(iii) Set
\[
L_\theta(0) := \sum_{\ell \geq 0, i_1, \ldots, i_\ell \in I} A_0 \tilde{F}_{i_1} \cdots \tilde{F}_{i_\ell} \phi \subset V_\theta(0),
\]
\[
B_\theta(0) = \left\{ \tilde{F}_{i_1} \cdots \tilde{F}_{i_\ell} \phi \mod qL_\theta(0) \mid \ell \geq 0, i_1, \ldots, i_\ell \in I \right\}.
\]
Then, \( B_\theta(0) \) is a basis of \( L_\theta(0)/qL_\theta(0) \) and \( (L_\theta(0), B_\theta(0)) \) is a crystal basis of \( V_\theta(0) \), and the crystal structure coincides with the one of \( M_\theta \).

(iv) More precisely, we have
(a) \( L_\theta(0) = \bigoplus_{m \in M_\theta} A_0 P_\theta(m) \phi \),
(b) \( B_\theta(0) = \{P_\theta(m) \phi \mod qL_\theta(0) \mid m \in M_\theta \} \),
(c) for any \( k \in I \) and \( m \in M_\theta \), we have
1. \( \tilde{F}_k P_\theta(m) \phi \equiv P_\theta(\tilde{F}_k(m)) \phi \mod qL_\theta(0) \),
2. \( \tilde{E}_k P_\theta(m) \phi \equiv P_\theta(\tilde{E}_k(m)) \phi \mod qL_\theta(0) \), where we understand \( P_\theta(0) = 0 \),
3. \( \tilde{E}_n P_\theta(m) \phi \in qL_\theta(0) \) if and only if \( n > \epsilon_k(m) \).

Proof. Let us recall that \( P_\theta(m) \phi \in V_\theta(0) \) is the image of \( \tilde{P}_\theta(m) \in \tilde{V}_\theta(0) \). By Theorem 3221 \( \{\tilde{P}_\theta(m)\}_m \subset M_\theta \) generates \( \tilde{V}_\theta(0) \). Let us set \( \tilde{L} = \sum_{m \in M_\theta} A_0 \tilde{P}_\theta(m) \subset \tilde{V}_\theta(0) \). Then Theorem 4.8 implies that \( \tilde{F}_k \tilde{P}_\theta(m) \equiv \tilde{P}_\theta(\tilde{F}_k(m)) \mod q\tilde{L} \) and \( \tilde{E}_k \tilde{P}_\theta(m) \equiv \tilde{P}_\theta(\tilde{E}_k(m)) \mod q\tilde{L} \). Hence the similar results hold for \( L_0 := \sum_{m \in M_\theta} A_0 P_\theta(m) \phi \subset V_\theta(0) \) and \( P_\theta(m) \phi \).

Let us show that
\[
(A) \quad \{P_\theta(m) \phi \mod qL_0 \}_m \text{ is linearly independent in } L_0/qL_0,
\]
by the induction of the θ-weight (see Remark 2.12). Assume that we have a linear relation
\[ \sum_{m \in S} a_m P_\theta(m) \phi \equiv 0 \mod qL_0 \]
for a finite subset \( S \) and \( a_m \in \mathbb{Q} \setminus \{0\} \). We may assume that all \( m \) in \( S \) have the same θ-weight. Take \( m_0 \in S \). If \( m_0 \) is the empty multisegment \( \emptyset \), then \( S = \{ \emptyset \} \) and \( P_\theta(m_0) \phi = \phi \) is non-zero, which is a contradiction. Otherwise, there exists \( k \) such that \( \varepsilon_k(m_0) > 0 \) by Lemma 4.7. Applying \( \tilde{E}_k \), we have \( \sum_{m \in S} a_m \tilde{E}_k P_\theta(m) \phi \equiv \sum_{m \in S, \tilde{E}_k(m) \neq 0} a_m P_\theta(\tilde{E}_k(m)) \phi \equiv 0 \mod qL_0 \). Since \( \tilde{E}_k(m) \) \( (\tilde{E}_k(m) \neq 0) \) are mutually distinct, we have \( a_{m_0} = 0 \) by the induction hypothesis. It is a contradiction.

Thus we have proved (A). Hence \( \{ P_\theta(m) \phi \}_{m \in M_\theta} \) is a basis of \( V_\theta(0) \), which implies that \( \{ \tilde{P}_\theta(m) \}_{m \in M_\theta} \) is a basis of \( \tilde{V}_\theta(0) \). Thus we obtain (i) and (ii).

Let us show (iv) (a). Since \( \tilde{F}_i \cdots \tilde{F}_t \phi \equiv P_\theta(\tilde{F}_i \cdots \tilde{F}_t \emptyset) \mod qL_0 \), we have \( L_\theta(0) \subset L_\theta(0) + qL_0 \). Hence Nakayama’s lemma implies \( L_0 = L_\theta(0) \). The other statements are now obvious.

Q.E.D.

5. Global basis of \( V_\theta(0) \)

5.1. Integral form of \( V_\theta(0) \). In this section, we shall prove that \( V_\theta(0) \) has a lower global basis. In order to see this, we shall first prove that \( \{ P_\theta(m) \phi \}_{m \in M_\theta} \) is a basis of the \( A \)-module \( V_\theta(0)_A \). Recall that \( A = \mathbb{Q}[q, q^{-1}] \), and \( V_\theta(0)_A = U_q(\mathfrak{g} \mathfrak{l}_\infty)_A \phi \).

Lemma 5.1. \( V_\theta(0)_A = \bigoplus_{m \in M_\theta} A P_\theta(m) \phi \).

Proof. It is clear that \( \bigoplus_{m \in M_\theta} A P_\theta(m) \phi \) is stable by the actions of \( F^{(n)}_k \) by Proposition 3.20. Hence we obtain \( V_\theta(0)_A \subset \bigoplus_{m \in M_\theta} A P_\theta(m) \phi \).

We shall prove \( P_\theta(m) \phi \in U_q^{-}(\mathfrak{g} \mathfrak{l}_\infty)_A \phi \). It is well-known that \( \langle i, j \rangle^{(m)} \) is contained in \( U_q^{-}(\mathfrak{g} \mathfrak{l}_\infty)_A \), which is also seen by Proposition 3.20 (3). We divide \( m \) as \( m = m_1 + m_2 \), where \( m_1 = \sum_{-j < i \leq j} m_{ij} \langle i, j \rangle \) and \( m_2 = \sum_{k > 0} m_k \langle -k, k \rangle \). Then \( P_\theta(m) = P(m_1) P_\theta(m_2) \) and \( P(m_1) \in U_q^{-}(\mathfrak{g} \mathfrak{l}_\infty)_A \). Hence we may assume from the beginning that \( m = \sum_{0 \leq k \leq a} m_k \langle -k, k \rangle \).

We shall show that \( P_\theta(m) \phi \in V_\theta(0)_A \) by the induction on \( a \).

Assume \( a > 1 \). Set \( m' = \sum_{0 < k \leq a - 4} m_k \langle -k, k \rangle \) and \( v = P_\theta(m') \phi \). Then \( \langle -a + 2, a - 2 \rangle^{[m]} v \in V_\theta(0)_A \) for any \( m \) by the induction hypothesis.

We shall show that \( \langle -a, a \rangle^{[a]} \langle -a + 2, a - 2 \rangle^{[m]} v \) is contained in \( V_\theta(0)_A \) by the induction on \( n \). Since \( P_\theta(m') \phi \) commutes with \( \langle a \rangle \), \( \langle -a \rangle \), \( \langle -a + 2, a - 2 \rangle \), \( \langle -a + 2, a \rangle \) and \( \langle -a, a \rangle \), Proposition 3.20 (2) implies
\[
\langle -a \rangle^{[2n]} \langle -a + 2, a - 2 \rangle^{[n+m]} v
= \sum_{i+j+2t = 2n, j+t = u} q^{2(n+m)i+j(j-1)/2-i(t+u)} \langle a \rangle^{(i)} \langle -a + 2, a \rangle^{(j)} \langle -a, a \rangle^{(t)} \langle -a + 2, -2 \rangle^{[n+m-u]} v,
\]
which is contained in \( V_\theta(0)_A \). Since \( \langle a \rangle^{(i)} \langle -a + 2, a \rangle^{(j)} \langle -a, a \rangle^{(t)} \langle -a + 2, a - 2 \rangle^{[n+m-u]} v \) is contained in \( V_\theta(0)_A \) if \( (i, j, t, u) \neq (0, 0, n, n) \) by the induction hypothesis on \( n \), \( \langle -a, a \rangle^{[a]} \langle -a + 2, a - 2 \rangle^{[m]} v \) is contained in \( V_\theta(0)_A \).

If \( a = 1 \), we similarly prove \( P_\theta(m) \phi \in V_\theta(0)_A \) using Proposition 3.20 (1) instead of (2).

Q.E.D.

5.2. Conjugate of the PBW basis. We will prove that the bar involution is upper triangular with respect to the PBW basis \( \{ P_\theta(m) \}_{m \in M_\theta} \).

First we shall prove Theorem 3.10 (4).

For \( a, b \in \mathcal{M} \) such that \( a \leq b \), we denote by \( \mathcal{M}_{[a,b]} \) (resp. \( \mathcal{M}_{\leq b} \)) the set of \( m \in \mathcal{M} \) of the form \( m = \sum_{a \leq i \leq j \leq b} m_{ij} \langle i, j \rangle \) (resp. \( m = \sum_{i \leq j \leq b} m_{ij} \langle i, j \rangle \)). Similarly we define \( \{ \mathcal{M}_\theta \}_{i \leq b} \).
For a multisegment $m \in \mathcal{M}_{<b}$, we divide $m$ into $m = m_b + m_{<b}$, where $m_b = \sum_{i \leq b} m_{i,j}(i,b)$ and $m_{<b} = \sum_{i < j < b} m_{i,j}(i,j)$.

**Lemma 5.2.** For $n \geq 0$ and $a, b \in I$ such that $a \leq b$, we have

$$\langle a, b \rangle^{(n)} \in \langle a, b \rangle^{(n)} + \sum_{m < n(a,b)} K P(m).$$

**Proof.** We shall first show

$$\langle a, b \rangle \in \langle a, b \rangle + \sum_{a+2 < k \leq b} \langle k; b \rangle U_q^-(g) \langle a \rangle$$

by the induction on $b - a$. If $a = b$, it is trivial. If $a < b$, we have

$$\langle a, b \rangle = \langle a \rangle \langle a+2, b \rangle - q^{-1} \langle a+2, b \rangle \langle a \rangle$$

$$\in \langle a \rangle \left( \langle a+2, b \rangle + \sum_{a+2 < k \leq b} \langle k; b \rangle U_q^-(g) \right) - q^{-1} \left( \langle a+2, b \rangle + \sum_{a+2 < k \leq b} \langle k; b \rangle U_q^-(g) \right) \langle a \rangle$$

$$\subset \langle a, b \rangle + (q - q^{-1}) \langle a+2, b \rangle \langle a \rangle + \sum_{a+2 < k \leq b} \left( \langle k; b \rangle \langle a \rangle U_q^-(g) + \langle k; b \rangle U_q^-(g) \right).$$

Hence we obtain (5.1). We shall show the lemma by the induction on $n$. We may assume $n > 0$ and

$$\langle a, b \rangle^{n-1} \in \langle a, b \rangle^{n-1} + \sum_{m < (n-1)(a,b)} K P(m).$$

Hence we have

$$\langle a, b \rangle^n = \langle a, b \rangle^{(n-1)} \langle a, b \rangle^n + \sum_{a < k \leq b} \langle k, b \rangle U_q^-(g) + \sum_{m < (n-1)(a,b)} K \langle a, b \rangle P(m).$$

For $a < k \leq b$ and $m \in \mathcal{M}$ such that $\text{wt}(m) = \text{wt}(n(a,b)) - \text{wt}((k, b))$, we have $m \in \mathcal{M}_{[a,b]}$ and $m_0 = \sum_{a \leq i \leq b} m_{i,b}(i, b)$ with $\sum_i m_{i,b} = n - 1$. In particular, $m_{a,b} \leq n - 1$. Hence $\langle k, b \rangle P(m) \in K P(m + (k, b))$ and $m + \langle k, b \rangle < n(a,b)$.

If $m < (n-1)(a,b)$, then $\langle a, b \rangle P(m) \in K P((a,b) + m)$ and $\langle a, b \rangle + m < n(a,b)$. Q.E.D.

**Proposition 5.3.** For $m \in \mathcal{M}$,

$$\overline{P(m)} \in P(m) + \sum_{n < m} K P(n).$$

**Proof.** Put $m = \sum_{i \leq j < b} m_{i,j}(i,j)$ and divide $m = m_b + m_{<b}$. We prove the claim by the induction on $b$ and the number of segments in $m_b$. Suppose $m_b = m(a,b) + m_1$ with $m = m_{a,b} > 0$, where $m_1 = \sum_{a < i \leq b} m_{i,b}(i,b)$.

(i) Let us first show that

$$\overline{P(m_b)} \in P(m_b) + \sum_{m' < m_b} K P(m').$$
We have \( \overline{P(m_b)} = \overline{P(m_1)} \cdot \langle a, b \rangle^{(m)} \). Since \( \overline{P(m_1)} \in P(m_1) + \sum_{m' < m_1} K \overline{P(m')} \) by the induction hypothesis, and \( \langle a, b \rangle^{(m)} \in \overline{P(m_1)} + \sum_{m'' < m} K \overline{P(m'' \langle a, b \rangle)} \), we have

\[
\overline{P(m_b)} \in P(m_b) + \sum_{m' < m_1, \ m' \in M_{[a+2, b]}} K \overline{P(m')} \langle a, b \rangle^{(m)} + \sum_{m'' < m} K \overline{P(m')} \overline{P(m'')}.
\]

If \( m'_1 < m_1 \) and \( m'_1 \in M_{[a+2, b]} \), then \( P((m'_1)_{<b}) \) and \( \langle a, b \rangle^{(m)} \) commute. Hence we obtain \( (5.2) \).

If \( m'_1 \leq m_1, m'_1 \in M_{[a+2, b]} \) and \( m'' \leq m \), then we can write \( m''_1 = j \langle a, b \rangle + m_2 \) with \( j < m \) and \( m_2 \in M_{[a+2, b]} \). Hence we have

\[
P(m'_1)P(m'') = K P((m'_1)_b) P(j \langle a, b \rangle) P((m'_1)_{<b}) P(m_2) P(m''_{<b}).
\]

Since \( (m'_1)_{<b}, m_2 \in M_{[a+2, b]} \) we have \( P((m'_1)_{<b}) P(m_2) P(m''_{<b}) \in \sum_{n_b \in M_{[a+2, b]}} K P(n) \). Hence we have \( P(m'_1)P(m'') \in \sum_{n_b \in M_{[a+2, b]}} K P((m'_1)_b + j \langle a, b \rangle + n) \) and \( (m'_1)_b + j \langle a, b \rangle + n < m_b \). Hence we obtain \( (5.2) \).

(ii) By the induction hypothesis, \( \overline{P(m_{<b})} \in P(m_{<b}) + \sum_{m'' < m_{<b}} \overline{P(m'')} \). Since \( \overline{P(m)} = \overline{P(m_b)} \overline{P(m_{<b})} \), \( (5.2) \) implies that

\[
\overline{P(m)} \in P(m) + \sum_{m' < m_b, m'' \in M_{<b}} K \overline{P(m')} \overline{P(m'')} + \sum_{m'' < m_{<b}} K \overline{P(m_b)} \overline{P(m')}.
\]

For \( m' < m_b \) and \( m'' \in M_{<b} \), we have

\[
P(m')P(m'') = P(m'_b)P(m'_b)P(m''_{<b}) \subseteq \sum_{n_b \in M_{<b}, n_b = m'_b} K P(n) \subset \sum_{n < m} K P(n).
\]

For \( m'' \leq m_{<b} \), we have \( P(m_b)P(m'') = P(m_b + m'') \) and \( m_b + m'' < m \). Thus we obtain the desired result.

Q.E.D.

Proposition 5.4. For \( m \in M_\theta \), we have

\[
\overline{P_\theta(m)} \phi = P_\theta(m) \phi + \sum_{m'\in M_\theta, m' < m} K P_\theta(m') \phi.
\]

Proof. First note that

\[
P(m) \phi = \sum_{n \in M_\theta \leq m} K P_\theta(n) \phi \quad \text{for any } b \in I_{>0} \text{ and } m \in M_{[-b, b]},
\]

by the weight consideration.

For \( m \in M_\theta \), \( P_\theta(m) \) and \( P(m) \) are equal up to a multiple of bar-invariant scalar. Thus we have

\[
\overline{P_\theta(m)} = \overline{P(m)} + \sum_{m' \in M, m' < m} K P(m')
\]

by Proposition 5.3. Hence it is enough to show that

\[
P(m') \phi = \sum_{n \in M_\theta, n < m} K P_\theta(n) \phi
\]

(5.4)
for \( m' \in \mathcal{M} \) such that \( m' < m \) and \( \text{wt}(m') = \text{wt}(m) \). Put \( m = \sum_{i \leq j \leq b} m_{i,j} \delta(i, j) \) and write \( m = m_b + m_{<b} \). We prove (5.3) by the induction on \( b \). By the assumption on \( m' \), we have \( m' \in \mathcal{M}_{[-b,b]} \) and \( m'_{<b} \leq m_b \). Thus \( m'_b \in \mathcal{M}_0 \). Hence \( K\Phi(m'_b) = K\Phi(m'_b)P(m'_{<b})\phi \).

If \( m'_b = m_b \), then \( m'_{<b} = m_{<b} \), and the induction hypothesis implies \( P(m'_{<b})\phi \in \sum_{m \in \mathcal{M}_0, n < m_{<b}} K\Phi(n)\phi \). Since \( \theta_P(m'_b) = \theta_P(m'_b + n) \) and \( m'_b + n < m \), we obtain (5.3).

If \( m'_b < m_b \), write \( m' = m'_b + m'_{<b} \). Set \( s = m_{-b,b} - m'_{-b,b} \geq 0 \). Since \( \text{wt}(m') = \text{wt}(m) \), we have \( \sum_{j < b} m'_{j,b} = s \). If \( s = 0 \), then \( m'_{<b} \in \mathcal{M}_{[-b,b-2]} \), and \( P(m'_{<b})\phi \in \sum_{m \in \mathcal{M}_0, n < b} K\Phi(n)\phi \) by (5.3). Then (5.3) follows from (5.4).

Assume \( s > 0 \). Since \( m'_{<b} \in \mathcal{M}_{[-b,b]} \), we have \( P(m'_b)\phi \in \sum_{m \in \mathcal{M}_0, n < b} K\Phi(n)\phi \) by (5.3). We may assume \((1 + \theta) m'_{<b} = (1 + \theta) m_{<b} \) (see Remark 2.12). Hence, we have \( s = 2m_{-b,b}(n) + \sum_{i < b} m_{i,b}(n) \). In particular, \( m_{-b,b}(n) \leq s/2 \). We have \( m'_b + n \in \mathcal{M}_0 \) and \( P(m'_b)P(n)\phi \in P(m'_b + n)\phi \). Since \( m_{-b,b}(m'_b + n) \leq (m_{-b,b} - s) + s/2 < m_{-b,b} \), we have \( m'_b + n < m \). Hence we obtain (5.3).

5.3. Existence of a global basis. As a consequence of the preceding subsections, we obtain the following theorem.

**Theorem 5.5.** (i) \((L_0(0), L_0(0)^-, V_0(0)_A)\) is balanced.

(ii) For any \( m \in \mathcal{M}_0 \), there exists a unique \( G^\text{low}(m) \in L_0(0) \cap V_0(0)_A \) such that \( G^\text{low}(m) = G^\text{low}_m \) and \( G^\text{low}_m \equiv P(m)\phi \mod qL_0(0) \).

(iii) \( G^\text{low}_m \in P_0(0)\phi + \sum_{n < m} q^a P_0(n)\phi \) for any \( m \in \mathcal{M}_0 \).

(iv) \( \{G^\text{low}_m \}_{m \in \mathcal{M}_0} \) is a basis of the \( A \)-module \( V_0(0)_A \), the \( A_0 \)-module \( L_0(0) \) and the \( K \)-vector space \( V_0(0) \).

**Proof.** We have already seen that \( \overline{P_0(m)} \phi = \sum_{m' \leq m} c_{m,m'} P_0(m') \phi \) for \( c_{m,m'} \in \mathbb{A} \) with \( c_{m,m'} = 1 \). Let us denote by \( C \) the matrix \((c_{m,m'})_{m,m' \in \mathcal{M}_0} \). Then \( C^2 = \text{id} \) and it is well-known that there is a matrix \( A = (a_{m,m'})_{m,m' \in \mathcal{M}_0} \) such that \( \overline{AC} = A \), \( a_{m,m'} = 0 \) unless \( m' \leq m \), \( a_{m,m} = 1 \) and \( a_{m,m'} \in \mathbb{Q}[q] \) for \( m' < m \). Set \( \overline{G^\text{low}_m} = \sum_{m' \leq m} a_{m,m'} P_0(m') \phi \). Then we have \( \overline{G^\text{low}_m} = G^\text{low}_m \) and \( G^\text{low}_m \equiv P_0(m) \phi \mod qL_0(0) \). Since \( G^\text{low}_m \) is a basis of \( V_0(0)_A \), we obtain the desired results.

Q.E.D.

Errata to “Symmetric crystals and affine Hecke algebras of type B, Proc. Japan Acad., 82, no. 8, 2006, 131–136”:

(i) In Conjecture 3.8, \( \lambda = \Lambda_{p_0} + \Lambda_{p_0^{-1}} \) should be read as \( \lambda = \sum A_{a} \Lambda_{a} \), where \( A = I \cap \{p_0, p_0^{-1}, -p_0, -p_0^{-1}\} \). We thank S. Ariki who informed us that the original conjecture is false.

(ii) In the two diagrams of \( B_0(\lambda) \) at the end of § 2, \( \lambda \) should be 0.

(iii) Throughout the paper, \( A^{(1)}_k \) should be read as \( A^{(1)}_{k-1} \).

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