Embedding regular black holes and black bounces in a cloud of strings

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I. INTRODUCTION

General relativity is a theory of gravitation where the gravitational interaction is a result of the curvature of the spacetime [1, 2]. Since it was proposed, this theory has been tested and proved to be effective in describing phenomena that were already known before its creation, such as the procession of the perihelion of mercury [3–5], as well as predicting new phenomena, such as the bending of light rays and gravitational waves [1]. These predictions have been experimentally proven, with the bending of light being tested in 1919 [6] and gravitational waves in 2015 [7]. The detection of gravitational waves represents a major advance in the study of our universe, allowing us to obtain information from astrophysical objects that are located very distant from our solar system.

Another very relevant prediction of general relativity is the black hole [8]. These astrophysical objects stand out due to their causal structure since from a certain region, known as the event horizon, no particle can escape from it [2]. Through the detection of gravitational waves, information was obtained from a vast number of black holes, some with small masses approaching 5 solar masses and some with large masses of up to more than 100 solar masses [9–11].

A structure that is often present in black hole solutions is the singularity [12]. These structures are characterized by the fact that geodesics are interrupted by them [13]. In some cases, the presence of singularities can be verified through the curvature invariants [13]; these are known as curvature singularities. What characterizes a black hole is the presence of an event horizon [2]. In this way, it is possible for solutions without singularities to exist, which are the regular black holes [14]. The first regular solution was proposed by Bardeen and interpreted as a solution of the Einstein equations in the presence of nonlinear electrodynamics by Ayon–Beato and Garcia [15, 16]. There are a lot of regular solutions and studies on properties of these solutions in the literature [17–35].

Another class of regular black holes was proposed by Simpson and Visser, solutions known as black bounces [36]. This solution distinguishes it from standard regular black holes through a modification in the black hole’s area, allowing the presence of a nonzero radius throat at $r = 0$. Initially, Simpson and Visser did not propose the content of material that could describe this solution. Recently, some works have appeared to show that solutions of black bounces can be obtained through the Einstein equations when there is a scalar field and nonlinear electrodynamics [37, 38]. In addition to the Simpson and Visser solution, there is a vast number of black bounces, where some of them have modifications in the area, and there are works focused on analyzing properties of these solutions [39–50].
In 1978, Letelier proposed a new type of black hole solution. In this case, we have a standard Schwarzschild black hole surrounded by a cloud of strings [51]. The cloud as a whole is a closed system, such that the stress-energy tensor is conserved [51]. Several works analyze the consequences of the presence of the string cloud in different situations, such as accretion, thermodynamics, and quasinormal modes, and there are several proposed solutions considering the presence of the cloud [52–61]. There is a solution where the authors consider a regular black hole surrounded by the cloud; however, due to the presence of the cloud, the solution is not regular anymore [62]. It would be interesting if there were regular solutions whose presence of the string cloud does not insert a singularity. This is one of the purposes of this work.

The structure of this work is organized as follows. In section II we presented the field equations to a cloud of strings, considering a general spacetime. In section III we presented the energy condition to a general spacetime, which will be needed in the following sections. In section IV we present the thermodynamics quantities that are necessary to study the thermodynamic equilibrium of a solution. The Bardeen solution surrounded by the cloud of strings is present in section V having its regularity and energy conditions analyzed. In section VI we propose a stress-energy tensor that allows us to obtain the Simpson–Visser solution in a cloud of string. We also analyzed the regularity of this solution and the energy conditions. Our conclusions are present in section VII.

We adopt the metric signature (+,−,−,−). Given the Levi-Civita connection, $\Gamma^\alpha_{\mu\nu} = \frac{1}{2} g^{\alpha\beta} \left( \partial_\mu g_{\nu\beta} + \partial_\nu g_{\mu\beta} - \partial_\beta g_{\mu\nu} \right)$, the Riemann tensor is defined as $R^\alpha_{\beta\mu\nu} = \partial_\mu \Gamma^\alpha_{\beta\nu} - \partial_\nu \Gamma^\alpha_{\beta\mu} + \Gamma^\sigma_{\beta\nu} \Gamma^\alpha_{\sigma\mu} - \Gamma^\sigma_{\beta\mu} \Gamma^\alpha_{\sigma\nu}$. We shall work in geometrodynamics units where $G = \hbar = c = 1$. 
II. SPACETIMES WITH CLOUD OF STRINGS

We are interested in obtaining solutions that describe spacetimes in the presence of a cloud of strings. In this way, we will analyze these solutions in the context of general relativity. The action that describes general relativity minimally coupled with matter and the cloud of strings is given by

\[ S = \int d^4x \sqrt{-g}R + S_M + S_{CS}, \]  

where \( S_M \) is the action that describes the matter sector, which will be specified later, and \( S_{CS} \) is the Nambu–Goto action used to describe stringlike objects, given by \[ S_{CS} = \int \sqrt{-\gamma} M d\lambda^0 d\lambda^1. \]

Here, \( M \) is a dimensionless constant that characterizes the string and \( \gamma \) is the determinant of \( \gamma_{AB} \), which is an induced metric on a submanifold given by \[ \gamma_{AB} = g_{\mu\nu} \frac{\partial x^\mu}{\partial \lambda^A} \frac{\partial x^\nu}{\partial \lambda^B}. \]

Just as a particle is associated with a world line, a string is associated with a world sheet, which is described by \( x^\mu(\lambda^A) \), where \( \lambda^0 \) and \( \lambda^1 \) are timelike and spacelike parameters.

It is possible to write the Nambu–Goto action as \[ S_{CS} = \int M \left( -\frac{1}{2} \Sigma^{\mu\nu} \Sigma_{\mu\nu} \right)^{1/2} d\lambda^0 d\lambda^1, \]

where \( \Sigma^{\mu\nu} \) is a bivector written as \[ \Sigma^{\mu\nu} = \epsilon^{AB} \frac{\partial x^\mu}{\partial \lambda^A} \frac{\partial x^\nu}{\partial \lambda^B}, \]

with \( \epsilon^{AB} \) being the Levi–Civita symbol, \( \epsilon^{01} = -\epsilon^{10} = 1 \).

Varying the action (1) with respect to the metric we find

\[ R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \kappa^2 T_{\mu\nu} = \kappa^2 T_{\mu\nu}^M + \kappa^2 T_{\mu\nu}^{CS}. \]

Here, \( T_{\mu\nu}^M \) and \( T_{\mu\nu}^{CS} \) are the stress-energy tensor of the matter sector and the stress-energy tensor of the cloud of string. We will only define the form of \( T_{\mu\nu}^M \) when considering specific cases. The form of \( T_{\mu\nu}^{CS} \) is \[ T_{\mu\nu}^{CS} = \frac{\rho \Sigma^{\alpha\beta} \Sigma_{\alpha\beta}}{8\pi \sqrt{-\gamma}}, \]

where \( \rho \) is the proper density of the cloud. The stress-energy tensor must obey the conservation law

\[ \nabla_\mu T_{\mu\nu}^{CS} = \nabla_\mu \left( \frac{\rho \Sigma^{\mu\alpha} \Sigma_{\alpha\nu}}{8\pi \sqrt{-\gamma}} \right) = \nabla_\mu \left( \rho \Sigma^{\mu\alpha} \frac{\Sigma_{\alpha\nu}}{8\pi \sqrt{-\gamma}} \right) + \rho \Sigma^{\mu\alpha} \nabla_\mu \left( \frac{\Sigma_{\alpha\nu}}{8\pi \sqrt{-\gamma}} \right) = 0. \]

Multiplying the equation above by \( \Sigma_{\alpha\beta} / (-\gamma)^{1/2} \) and using the identity

\[ \Sigma^{\alpha\beta} \nabla_\alpha \Sigma_{\beta\nu} = \frac{3}{2} \Sigma^{\alpha\beta} \partial_{[\alpha} \Sigma_{\beta\nu]} - \frac{1}{4} \nabla_{\nu} (\Sigma_{\nu\beta} \Sigma_{\alpha\beta}) \]

it is possible to obtain the following equations \[ \nabla_\mu (\rho \Sigma^{\mu\nu}) = \partial_\mu (\sqrt{-g} \rho \Sigma^{\mu\nu}) = 0, \]

\[ \Sigma^{\mu\nu} \nabla_\mu \left[ \frac{\Sigma_{\nu\beta}/(-\gamma)^{1/2}}{2} \right] = 0. \]
A general line element that describes a spherically symmetric and static spacetime is written as
\[
ds^2 = A(r)dt^2 - B(r)dr^2 - C(r)\left(d\theta^2 + \sin^2 \theta d\phi^2\right). \tag{12}\]
However, to regular black holes, in the context of the general relativity, we usually have \(A(r) = B(r)^{-1} = C(r) = r^2\). When considering black bounces we still have \(A(r) = B(r)^{-1}\) but \(C(r) \neq r^2\). Therefore, it is useful for us to use the “Buchdahl coordinates”, where the line element is
\[
ds^2 = f(r)dt^2 - f(r)^{-1}dr^2 - \mathcal{R}^2 \left(d\theta^2 + \sin^2 \theta d\phi^2\right), \tag{13}\]
where \(f(r)\) and \(\mathcal{R}\) are general functions of the radial coordinate.

Using the line element \(\mathcal{R}^0\), we can integrate \((10)\) and \((11)\), and it results in
\[
\Sigma^{01} = \sqrt{-\gamma} = \frac{a}{\rho \mathcal{R}^2}, \tag{14}\]
where \(a\) is an integration constant related with the string. The nonzero components of the stress-energy tensor are
\[
T^{CS0} = T^{CS1} = \frac{a}{8\pi \mathcal{R}^2}. \tag{15}\]

In order to obtain a positive energy density, \(a > 0\).

The nonzero components of the Riemann tensor are given by
\[
R^{01} = \frac{1}{2} f', \quad R^{02} = R^{03} = \frac{f'R'}{2\mathcal{R}}, \quad R^{12} = R^{13} = \frac{f'\mathcal{R}' + 2f\mathcal{R}''}{2\mathcal{R}}, \quad R^{23} = \frac{f\mathcal{R}'^2 - 1}{\mathcal{R}}. \tag{16}\]

The Kretschmann scalar is written as a sum of squares of the Riemann tensor components
\[
K = 4 \left(R^{01}\right)^2 + 4 \left(R^{02}\right)^2 + 4 \left(R^{03}\right)^2 + 4 \left(R^{12}\right)^2 + 4 \left(R^{13}\right)^2 + 4 \left(R^{23}\right)^2
   = 4 \left(R^{01}\right)^2 + 8 \left(R^{02}\right)^2 + 8 \left(R^{12}\right)^2 + 4 \left(R^{23}\right)^2. \tag{17}\]

If there is divergence in any of the components of the Riemann tensor, this divergence must also appear in the Kretschmann scalar \([13, 40]\). To the line element \(\mathcal{R}^0\), \((17)\) is given by
\[
K = \frac{\left(2\mathcal{R}f''\right)^2 + 2\left(\mathcal{R}f'R'\right)^2 + 2\mathcal{R}^2\left(f'R' + 2f\mathcal{R}''\right)^2 + 4(1 - f\mathcal{R}'^2)^2}{\mathcal{R}^4}. \tag{18}\]

If Eq. \((18)\) is singular in any point, the spacetime presents curvature singularities \([13, 40]\).

Now we need to obtain the form of \(f(r)\) and \(\mathcal{R}(r)\) but it is still necessary to define the form of \(S_M\).

### III. ENERGY CONDITIONS

In general, any models of \(f(r)\) and \(\mathcal{R}\) can be proposed. However, these models are not always physically appropriate. So, to know if the solutions make physical sense, we need to check the energy conditions.

To obtain the energy conditions, we need to identify the components of the stress-energy tensor. Where \(t\) is the timelike coordinate, \(f > 0\), we have
\[
T^\mu_{\nu} = \text{diag} [\rho, -p_r, -p_t, -p_t], \tag{19}\]
where \(\rho\) is the energy density, \(p_r\) is the radial pressure, and \(p_t\) is the tangential pressure. Considering the Einstein equations and Eq. \((13)\), we find
\[
\rho = \frac{\mathcal{R} \left(f'R' + 2f\mathcal{R}''\right) + f\mathcal{R}'^2 - 1}{\kappa^2 \mathcal{R}^2}, \tag{20}\n\]
\[
p_r = \frac{\mathcal{R}f'R' + f\mathcal{R}'^2 - 1}{\kappa^2 \mathcal{R}^2}, \tag{21}\n\]
\[
p_t = \frac{\mathcal{R}f'' + 2f'R' + 2f\mathcal{R}''}{2\kappa^2 \mathcal{R}}. \tag{22}\n\]
To regions where \( f(r) < 0 \), \( t \) is the spacelike coordinate, we must have
\[
T^\mu_\nu = \text{diag} \left[-p_r, \rho, -p_t, -p_t \right].
\] (23)

The fluid quantities, in this region, are given by
\[
\rho = -\frac{Rf'R'}{\kappa^2 R^2} + f R'^2 - 1,
\] (24)
\[
p_r = \frac{R(f'R' + 2 f R'') + f R'^2 - 1}{\kappa^2 R^2},
\] (25)
\[
p_t = \frac{Rf'' + 2 f'R' + 2 f R''}{2 \kappa^2 R}.
\] (26)

The energy conditions [63] are given by the inequalities
\[
NEC_{1,2} = WEC_{1,2} = SEC_{1,2} \iff \rho + p_r, t \geq 0,
\] (27)
\[
SEC_3 \iff \rho + p_r + 2p_t \geq 0,
\] (28)
\[
DEC_{1,2} \iff \rho - |p_r, t| \geq 0 \iff (\rho + p_r, t \geq 0) \text{ and } (\rho - p_r, t \geq 0),
\] (29)
\[
DEC_3 = WEC_3 \iff \rho \geq 0.
\] (30)

We see that \( DEC_{1,2} \iff ((NEC_{1,2}) \text{ and } (\rho - p_r, t \geq 0)) \), so we replace \( DEC_{1,2} \iff \rho - p_r, t \geq 0. \)

Inserting the results given in \([20]-[22]\), where the coordinate \( t \) is timelike, we have
\[
NEC_1 = WEC_1 = SEC_1 \iff -\frac{2 f R''}{\kappa^2 R} \geq 0,
\] (31)
\[
NEC_2 = WEC_2 = SEC_2 \iff \frac{R^2 f'' - 2 f (R R'' + \langle R' \rangle^2) + 2}{2 \kappa^2 R^2} \geq 0,
\] (32)
\[
SEC_3 \iff \frac{R f'' + 2 f'R'}{\kappa^2 R} \geq 0,
\] (33)
\[
DEC_1 \iff \frac{2 (1 - f'R'R' - f(\langle R' \rangle^2 - f R R''))}{\kappa^2 R^2} \geq 0,
\] (34)
\[
DEC_2 \iff \frac{-R^2 f'' + R(4 f'R' + 6 f R'') + 2 f R'^2 - 2}{2 \kappa^2 R^2} \geq 0,
\] (35)
\[
DEC_3 = WEC_3 \iff -\frac{R (f'R' + 2 f R'') + f(\langle R' \rangle^2 - 1)}{\kappa^2 R^2} \geq 0.
\] (36)

To \( f(r) < 0 \), \( t \) is the spacelike coordinate, and the energy conditions are
\[
NEC_1 = WEC_1 = SEC_1 \iff \frac{2 f R''}{\kappa^2 R} \geq 0,
\] (37)
\[
NEC_2 = WEC_2 = SEC_2 \iff \frac{R^2 f'' - 2 (\langle R' \rangle^2 f + 2 \langle R'' \rangle f + 2)}{2 \kappa^2 R^2} \geq 0,
\] (38)
\[
SEC_3 \iff \frac{R f'' + 2 f'R' + 4 R'' f}{\kappa^2 R} \geq 0,
\] (39)
\[
DEC_1 \iff \frac{2 (1 - f'R'R' - f(\langle R' \rangle^2 - f R R''))}{\kappa^2 R^2} \geq 0,
\] (40)
\[
DEC_2 \iff \frac{-R^2 f'' - 2 R'' f + 4 R R' f - 2 (\langle R' \rangle^2 f + 2}{2 \kappa^2 R^2} \geq 0,
\] (41)
\[
DEC_3 = WEC_3 \iff -\frac{R R' f' + (\langle R' \rangle^2 f - 1)}{\kappa^2 R^2} \geq 0.
\] (42)

**IV. THERMODYNAMICS**

The thermodynamics of a black hole can give us information about the stability of a solution. To verify this stability, we calculate the Hawking temperature, which is given through the surface gravity as
\[
T = \frac{k}{2 \pi} = \left| \frac{f'(r)}{4 \pi} \right|_{r=r_+},
\] (43)
where \( k \) is the surface gravity and \( r_+ \) is the radius of the event horizon. In addition to temperature, a black hole also has an entropy associated with it. This entropy is given by the Bekenstein relation

\[
S = \frac{A}{4\pi} = \pi R^2 |_{r=r_+}.
\]  

(44)

Once we have entropy and temperature, it is possible to calculate the heat capacity at constant charge through the relation

\[
C_q = T \frac{\partial S}{\partial T} |_{q}.
\]  

(45)

If the heat capacity is positive, the solution is thermodynamically stable.

V. BARDEEN SOLUTION WITH CLOUD OF STRING

In General Relativity, the Bardeen solution arises when the gravitational theory is coupled with nonlinear electrodynamics. The action associated with the nonlinear electrodynamics is given by

\[
S_M = \int d^4x \sqrt{-g} \mathcal{L}(F),
\]  

(46)

where \( \mathcal{L}(F) \) is the Lagrangian density, which is a general function of the electromagnetic scalar \( F = F_{\mu\nu} F^{\mu\nu} / 4 \). Here, \( F_{\mu\nu} \) is the Maxwell-Faraday tensor. To a magnetically charged solution, the Lagrangian density to the Bardeen case is [16]

\[
\mathcal{L}(F) = \frac{3}{2s q^2} \left( \frac{\sqrt{2q^2 F}}{1 + \sqrt{2q^2 F}} \right)^{5/2},
\]  

(47)

where \( s = |q| / 2m \), with \( q \) being the magnetic charge. The magnetic field and the electromagnetic scalar are given by

\[
F_{23} = q \sin \theta,
\]

(48)

\[
F = \frac{q^2}{2r^4}.
\]

(49)

The stress-energy tensor to a nonlinear electrodynamics is

\[
T_{\mu\nu} = \frac{1}{4\pi} \left[ g_{\mu\nu} \mathcal{L} - \mathcal{L}_F F^{\alpha}_{\mu} F_{\nu\alpha} \right],
\]  

(50)

with \( \mathcal{L}_F = \partial \mathcal{L} / \partial F \).

Using \([15]\) and \([50]\) with the Einstein equations, the equations of motion are

\[
\frac{a - \mathcal{R} (f' \mathcal{R}' + 2f \mathcal{R}'' - 2f \mathcal{R}^2 + 1)}{\mathcal{R}^2} = 0,
\]  

(51)

\[
\frac{a - \mathcal{R} f' \mathcal{R}' - f \mathcal{R}^2 - 2f \mathcal{R}^2 + 1}{\mathcal{R}^2} = 0,
\]  

(52)

\[
\frac{-f' \mathcal{R}' + f \mathcal{R}''}{\mathcal{R}} - \frac{f''}{2} - \frac{2q^2 \mathcal{L}'}{\mathcal{R}^4 F'} - 2\mathcal{L} = 0.
\]  

(53)

Subtracting Eq. \([51]\) from Eq. \([52]\) we find

\[
-\frac{2f \mathcal{R}''}{\mathcal{R}} = 0.
\]  

(54)

In general \( \mathcal{R}(r) \) is not infinity and \( f(r) \) is not zero, so that \( \mathcal{R}'' = 0 \), which means that

\[
\mathcal{R} = c_1 r + c_0.
\]

(55)
where \( c_1 \) and \( c_0 \) are integration constants. Imposing \( c_0 = 0 \) and \( c_1 = 1 \) we get that \( R = r \). Using Eq. (17) and Eq. (49), Eq. (51) becomes

\[
a \frac{f'}{r^{2}} - r f' + f - \frac{6m}{q^{2} + r^{2}} = 0. \tag{56}
\]

Solving this differential equation to \( f(r) \), we obtain

\[
f(r) = 1 - a - \frac{2mr^{2}}{(r^{2} + q^{2})^{3/2}}. \tag{57}
\]

where \( a \) is limited by \( 0 < a < 1 \). To \( a = 0 \) the Bardeen solution is recovered. To \( q = 0 \) the solution proposed by Letelier is recovered. The Kretschmann scalar associated with the solution Eq. (57) is

\[
K = 4 \left( \frac{a^{2}}{r^{4}} + \frac{4am}{r^{2}(q^{2} + r^{2})^{3/2}} + \frac{3m^{2}(-4q^{6}r^{2} + 47q^{4}r^{4} - 12q^{2}r^{6} + 8q^{8} + 4r^{8})}{(q^{2} + r^{2})^{7/2}} \right). \tag{58}
\]

Expanding the Kretschmann scalar to \( r \to 0 \) and \( r \to \infty \), we find

\[
K \approx \frac{4a^{2}}{r^{4}} + \frac{16am}{q^{2} r^{2}}, \text{ to } r \to 0, \tag{59}
\]

\[
K \approx \frac{4a^{2}}{r^{4}} + \frac{48m^{2}}{r^{6}}, \text{ to } r \to \infty. \tag{60}
\]

The singularity appears only at \( r = 0 \). If \( a = 0 \), the therm that diverges disappears, so that there is curvature singularity only to \( a \neq 0 \).

The energy conditions, where \( t \) is the timelike coordinate, are given by

\[
NEC_1 \iff 0, \quad NEC_2 \iff \frac{a}{\kappa^{2} r^{2}} + \frac{15mq^{2}r^{2}}{\kappa^{2}(q^{2} + r^{2})^{7/2}} \geq 0, \tag{61}
\]

\[
WEC_3 \iff \frac{6mq^{2}}{\kappa^{2}(q^{2} + r^{2})^{5/2}} + \frac{a}{\kappa^{2} r^{2}} \geq 0, \quad SEC_3 \iff \frac{6m(3q^{2}r^{2} - 2q^{4})}{\kappa^{2}(q^{2} + r^{2})^{7/2}} \geq 0, \tag{62}
\]

\[
DEC_1 \iff \frac{12mq^{2}}{\kappa^{2}(q^{2} + r^{2})^{5/2}} + \frac{2a}{\kappa^{2} r^{2}} \geq 0, \quad DEC_2 \iff -\frac{a}{\kappa^{2} r^{2}} + \frac{3mq^{2}(r^{2} - 4q^{2})}{\kappa^{2}(q^{2} + r^{2})^{7/2}} \geq 0. \tag{63}
\]

Where \( t \) is the spacelike coordinate, the energy conditions are

\[
NEC_1 \iff 0, \quad NEC_2 \iff \frac{a}{\kappa^{2} r^{2}} + \frac{15mq^{2}r^{2}}{\kappa^{2}(q^{2} + r^{2})^{7/2}} \geq 0, \tag{64}
\]

\[
WEC_3 \iff \frac{a}{\kappa^{2} r^{2}} + \frac{6mq^{2}}{\kappa^{2}(q^{2} + r^{2})^{5/2}} \geq 0, \quad SEC_3 \iff \frac{6m(3q^{2}r^{2} - 2q^{4})}{\kappa^{2}(q^{2} + r^{2})^{7/2}} \geq 0, \tag{65}
\]

\[
DEC_1 \iff \frac{2a}{\kappa^{2} r^{2}} + \frac{12mq^{2}}{\kappa^{2}(q^{2} + r^{2})^{5/2}} \geq 0, \quad DEC_2 \iff \frac{a}{\kappa^{2} r^{2}} + \frac{3mq^{2}(4q^{2} - r^{2})}{\kappa^{2}(q^{2} + r^{2})^{7/2}} \geq 0. \tag{66}
\]

We see that the null energy condition is satisfied, which means that the weak energy condition is also satisfied. The presence of the cloud of strings do not modify \( SEC_3 \), so the strong energy condition is violated inside the event horizon. The dominant energy condition is violated outside the event horizon.

From Eq. (54), the temperature is

\[
T = \frac{m \left( r_{+}^{3} - 2q^{2}r_{+} \right)}{2 \pi \left( q^{2} + r_{+}^{2} \right)^{3/2}}. \tag{67}
\]
Figure 1: Heat capacity at constant charge to the solution Eq. (57) with $q = 0.1$.

However, from the condition $f(r_+) = 0$, the black hole mass is given by

$$m = \frac{(1 - a) \left( q^2 + r_+^2 \right)^{3/2}}{2r_+^2}, \quad (68)$$

and the temperature becomes

$$T = \frac{(1 - a) \left( r_+^2 - 2q^2 \right)}{4\pi r_+ \left( q^2 + r_+^2 \right)}. \quad (69)$$

The entropy to this case is

$$S = \pi r_+^2. \quad (70)$$

Once we have the temperature and entropy, the heat capacity at constant charge is

$$C_q = \frac{2\pi r_+^2 \left( r_+^2 - 2q^2 \right) \left( q^2 + r_+^2 \right)}{7q^2r_+^4 + 2q^4 - r_+^4}. \quad (71)$$

The constant $a$ does not appear in Eq. (71); however, we need to remember that $r_+ = r_+(m, q, a)$. In Fig. 1, see how the heat capacity behaves as the mass increases. There is an intermediate value of mass where the solution is stable, $C_q > 0$. As the parameter $a$ increases, the range of mass where the solution is stable decreases.

VI. SIMPSON–VISSEY SOLUTION WITH CLOUD OF STRINGS

The standard Simpson–Visser solution represents a regular black hole that has a minimum area that is not zero. Depending on the parameters of the metric, the solution can also represent a wormhole. To obtain the Simpson–Visser solution in the presence of a cloud of string, let us consider the Einstein equations (6), where $T_{\mu\nu}^M$ is

$$T_{\mu\nu}^S = T_{\mu\nu}^{SV} + T_{\mu\nu}^{NMC}, \quad (72)$$
where $T_{\mu\nu}^{SV}$ is the stress-energy tensor associated with the Simpson–Visser solution, with the nonzero components given by \[36\]

\[
T_{00}^{SV} = -\frac{L^2 (\sqrt{L^2 + r^2} - 4m) (\sqrt{L^2 + r^2} - 2m)}{8\pi (L^2 + r^2)^3}, \quad T_{11}^{SV} = \frac{L^2}{8\pi (L^2 + r^2)^{3/2} (2m - \sqrt{L^2 + r^2})},
\]

\[
T_{22}^{SV} = \frac{L^2 (\sqrt{L^2 + r^2} - m)}{8\pi (L^2 + r^2)^{3/2}}, \quad T_{33}^{SV} = \frac{L^2 \sin^2 \theta (\sqrt{L^2 + r^2} - m)}{8\pi (L^2 + r^2)^{3/2}},
\]

and $T_{\mu\nu}^{NMC}$ is the stress-energy tensor that provides information about the nonminimal coupling between the Simpson–Visser solution and the cloud of string, which is given by

\[
T_{00}^{NMC} = -\frac{aL^2 ((a - 2)\sqrt{L^2 + r^2} + 6m)}{8\pi (L^2 + r^2)^{3/2}},
\]

\[
T_{11}^{NMC} = \frac{2aL^2 m}{8\pi (L^2 + r^2)^{3/2} (\sqrt{L^2 + r^2} - 2m)((a - 1)\sqrt{L^2 + r^2} + 2m)},
\]

\[
T_{22}^{NMC} = -\frac{aL^2}{8\pi (L^2 + r^2)^{3/2}}, \quad T_{33}^{CP} = -\frac{aL^2 \sin^2 \theta}{8\pi (L^2 + r^2)}.
\]

If $a = 0$ or $L = 0$ all components of $T_{\mu\nu}^{NMC}$ are zero.

To the Simpson–Visser solution we have $R = \sqrt{L^2 + r^2} \text{ [36]}. As the presence of a cloud of strings does not modify the area, we will consider the same $R$. With this modification in the area, the equations related to the string are

\[
\Sigma^{01} = \sqrt{-\gamma} = \frac{a}{\rho (r^2 + L^2)}.
\]

This result implies a modification in the stress-energy tensor to a cloud of string, whose nonzero components are given by

\[
T^{CS0}_{C1} = T^{CS1}_{0} = \frac{a}{8\pi (r^2 + L^2)}.
\]

Again, to guarantee the positivity of the energy density, we impose $a > 0$.

Using the stress-energy tensors \[73\], \[74\], \[75\]–\[77\], and \[79\], the field equations are given by

\[
-aL^2 ((a - 2)L^2 + (a - 2)r^2 + 6m\sqrt{L^2 + r^2}) - a(L^2 + r^2)(rf' - 1) + f(2L^2 + r^2)
\]

\[
\frac{(a - 1)(L^2 + r^2)^2 + 2m(L^2 + r^2)^{3/2}}{f(L^2 + r^2)^2} = 0,
\]

\[
L^2 \left(-\frac{2am}{\sqrt{L^2 + r^2} - 2m} - a + 1\right) + (1 - a)r^2 - r \left(L^2 + r^2\right) f' + \frac{fL^2(L^2 + r^2)^{1/2}}{2m - \sqrt{L^2 + r^2}} - fr^2 = 0,
\]

\[
-\frac{aL^2}{(L^2 + r^2)^2} - \frac{rf'(r)}{L^2 + r^2} - \frac{f''(r)}{2} - \frac{L^2 f(r)}{(L^2 + r^2)^2} - \frac{L^2 m}{(L^2 + r^2)^{3/2}} + \frac{L^2 (1 - a)}{(L^2 + r^2)^2} = 0.
\]

Integrating these equations, we find

\[
f(r) = 1 - a - \frac{2m}{\sqrt{L^2 + r^2}}.
\]

It is the metric coefficient that describes the Simpson–Visser solution immersed in a cloud of strings. If $L = 0$, the Letelier solution is recovered, and if $a = 0$, the Simpson–Visser is recovered. If $a = 1$, there is no event horizon, so we impose $0 < a < 1$. The Kretschmann scalar associated with this solution is

\[
K(r) = \frac{4(a^2r^6 + L^4 ((2a - 1)a + 3)r^2 + 9m^2) + L^2 (a(a + 2)r^4 - 12m^2r^2) + (2a - 2)a + 3)L^6 + 12m^2r^4}{(L^2 + r^2)^5}
\]

\[
+ \frac{16m ((a - 2)L^2r^2 + 2(a - 1)L^4 + ar^4)}{(L^2 + r^2)^{9/2}}.
\]
Expanding the Kretschmann scalar to \( r \to 0 \) and \( r \to \infty \), we find

\[
K \approx \frac{8}{L^4} \left( a + \frac{2m}{\sqrt{L^2}} - 1 \right)^2 + \frac{4m^2}{L^6} + \frac{4}{L^4}, \text{ to } r \to 0,
\]

\[
K \approx \frac{4a^2}{r^4} + \frac{16am}{r^5} - \frac{8(2a^2L^2 - aL^2 - 6m^2)}{r^6}, \text{ to } r \to \infty.
\]

Different from the case before, this solution is always regular. What allows for regularity is the modification in the area.

The entropy to this case is

\[
S = \pi \left( r_+^2 + L^2 \right).
\]

Once we have the temperature, entropy, and \( r_+ = \sqrt{(1-a)^2L^2 + 4m^2} \), the heat capacity at constant charge is

\[
C_q = -\frac{\pi \left( (a-1)^2L^2 + 2m^2 \right) \left( (a-1)^2L^2 + 4m^2 \right)}{(a-1)^4m^2}.
\]

In Fig. 2 we see that the heat capacity is always negative, \( C_q < 0 \), and there is no phase transition.
VII. CONCLUSION

In this work, we analyze the consequences of inserting regular black holes into a cloud of strings in the context of general relativity, studying the regularity, energy conditions, and thermodynamics of these solutions. For this, we first found the stress-energy tensor of a cloud of strings.

We consider the Einstein equations together with the stress-energy tensor associated with the Bardeen solution and the stress-energy tensor of the string cloud. By integrating the equations of motion, we were able to obtain the Bardeen solution surrounded by a kind of string atmosphere. This solution presents a problem when considering regularity. The presence of a cloud makes the solution singular, so the geodesics are interrupted at \( r = 0 \). The presence of the atmosphere changes the energy conditions but not significantly. For instance, if the parameter \( a \) does not appear in \( SEC_3 \), the strong energy condition is violated inside the event horizon, as it happens in the standard Bardeen solution. The null energy condition is satisfied and the energy density, \( WEC_3 \), is positive. We noticed that the presence of the cloud affects the heat capacity, decreasing the region where the thermodynamically stable phase exists, \( C_q > 0 \). Despite the modifications in thermodynamics, the solution presents two phase transitions, in the same way as for Bardeen, from unstable to stable and then from stable to unstable.

To obtain the Simpson–Visser solution surrounded by a cloud of strings, it was necessary to consider the stress-energy tensor for a cloud of strings, for the Simpson–Visser spacetime, and also to introduce a new stress-energy tensor that describes the interaction between the cloud and the black hole. The interaction term disappears if \( a = 0 \) or \( L = 0 \). The solution was obtained through the integration of the Einstein equations. To \( L = 0 \) the Letelier solution is recovered, and to \( a = 0 \) the usual Simpson–Visser solution is recovered. Different from the usual regular black hole, solutions that do not have their area modified, the Simpson–Visser solution is regular even when immersed in the cloud. The parameter \( a \) does not appear only in \( SEC_3 \), which means that the gravitational interaction is always attractive. To \( a \geq 1/2 \) the energy density is always positive but if \( a \geq 1 \) there is no horizon. So, if \( a = 1/2 \), the event horizon radius is \( r_H = \sqrt{16m^2 - L^2} \). Regardless of the value of \( L \), the energy density will be positive, so we can have a black bounce or a wormhole with positive energy density. The thermodynamics of this solution is not drastically affected by the presence of the cloud, since, regardless of the value of \( a \), the solution is always thermodynamically unstable, \( C_q < 0 \).

In future work, we hope to investigate more properties of these spacetimes, such as the behavior of particles/fields around them.
Acknowledgements

M.E.R. thanks Conselho Nacional de Desenvolvimento Científico e Tecnológico - CNPq, Brazil for partial financial support. This study was financed in part by the Coordenação de Aperfeiçoamento de Pessoal de Nível Superior - Brasil (CAPES) - Finance Code 001.

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