FAST TRACK COMMUNICATION

Large deviations of the top eigenvalue of large Cauchy random matrices

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Abstract

We compute analytically the large deviation tails of the probability density function (pdf) of the top eigenvalue $\lambda_{\text{max}}$ in rotationally invariant and heavy-tailed Cauchy ensembles of $N \times N$ matrices for any Dyson index $\beta > 0$, where $\beta = 1, 2, 4$ correspond, respectively, to orthogonal, unitary and symplectic ensembles. Furthermore, we show that these large deviation tails flank a central non-Gaussian regime for $\lambda_{\text{max}} \sim \mathcal{O}(N)$ on both sides. By matching these tails with the central regime, we obtain the exact leading asymptotic behaviors for any $\beta$ of the pdf in the central regime, which generalizes the Tracy–Widom distribution known for Gaussian ensembles. Our analytical results are confirmed by numerical simulations.

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(Some figures may appear in colour only in the online journal)

1. Introduction

Characterizing the distribution of extreme eigenvalues of random matrices is of paramount interest in many applications. A cornerstone in this field is the discovery of the Tracy–Widom (TW) laws [1] for the typical fluctuations of the largest eigenvalue of Gaussian random matrices. Let us first recall a few basic definitions and results for the Gaussian ensembles. Consider an $N \times N$ matrix $H$ whose upper triangular entries are drawn at random from a standard Gaussian distribution in the real ($\beta = 1$), complex ($\beta = 2$) or quaternion ($\beta = 4$) domain. Symmetrizing, we end up with a symmetric, Hermitian or quaternionic self-dual random matrix (respectively) whose $N$ eigenvalues $\lambda_1, \ldots, \lambda_N$ are real random variables, characterized by the joint probability distribution function (pdf):

$$P_{\text{joint}}(\lambda_1, \ldots, \lambda_N) = BN(\beta) e^{-\frac{\beta}{2} \sum \lambda_i^2} \prod_{j<k} |\lambda_j - \lambda_k|^\beta,$$  

(1)
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where the normalization constant $B_N(\beta)$ is known from the celebrated Selberg’s integral. This joint law (1) allows one to interpret $\lambda_i$ as the positions of charged particles repelling each other via a 2d-Coulomb (logarithmic) potential: they are confined on a 1d line and each one is subject to a harmonic potential. Here we focus on the largest eigenvalue $\lambda_{\text{max}} = \max_1 \lambda_i \leq N \lambda_i$. What can we say about its statistical properties? This is a non-trivial question because, due to the all-to-all interaction term $\prod_{j<k} |\lambda_j - \lambda_k|^{\beta}$, the standard results about extreme value statistics of independent random variables [2] (namely the existence of just three universality classes: Gumbel, Fréchet or Weibull) are no longer applicable.

The average location of the largest eigenvalue is readily determined by the shape of the average density of the eigenvalues in the large $N$ limit. For a Gaussian random matrix of large size $N$, the average density of eigenvalues (normalized to unity) $\rho_N(\lambda) = \langle \frac{1}{N} \sum_i \delta(\lambda - \lambda_i) \rangle$ has a semi-circular shape for any $\beta > 0$ on the compact support $[-\sqrt{2N}, \sqrt{2N}]$ called the Wigner semi-circle

$$\rho_N(\lambda) \approx \frac{1}{\sqrt{N}} \hat{\rho}_W \left( \frac{\lambda}{\sqrt{N}} \right)$$

(2)

The average location of the largest eigenvalue is given for large $N$ by the upper edge of the density support:

$$\langle \lambda_{\text{max}} \rangle \approx \sqrt{2N}$$

(3)

However, the largest eigenvalue fluctuates from one realization of the matrix to another. What can be said about the full probability density of $\lambda_{\text{max}}$?

From the jpdf (1), it is easy to write the cumulative distribution of $\lambda_{\text{max}}$ as a multiple integral:

$$P(\lambda_{\text{max}} \leq w) = B_N(\beta) \prod_{i=1}^N \int_{-\infty}^w \, d\lambda_i \, P_{\text{joint}}(\lambda_1, \ldots, \lambda_N),$$

(4)

which can be interpreted as the partition function of a Coulomb gas in the presence of a hard wall in $w$. Carrying out this multiple integration is a non-trivial task. It turns out that it is necessary to deal separately with two natural scales for the fluctuations in the asymptotic limit. Typical (small) fluctuations scale as $\sim N^{-1/6}$, while atypical (large) fluctuations scale as $N^{1/2}$, and the two corresponding distributions are described by different functional forms:

- **Typical fluctuations.** Forrester [3] followed by Tracy and Widom [1] realized that in the large $N$ limit, the largest eigenvalue follows the law

$$\lambda_{\text{max}} \approx \sqrt{2N} + a_1 N^{-1/6} \chi_{\beta},$$

(5)

with $a_{1,2} = 1/\sqrt{2}$ and $a_3 = 2^{-7/6}$ and where the random variable $\chi_{\beta}$ has an $N$-independent distribution, $P(\chi_{\beta} \leq x) = F_{\beta} (x)$ called the TW distribution [1], which has highly asymmetric tails:

$$F_{\beta} (x) \sim \exp \left[ -\frac{\beta}{24} x^3 \right] \quad \text{as} \ x \to -\infty,$$

(6)

$$\sim \exp \left[ -\frac{2}{3} x^{3/2} \right] \quad \text{as} \ x \to \infty.$$  

(7)

These TW distributions also describe the top eigenvalue statistics of large real [4, 5] and complex [6] Gaussian Wishart matrices, which play an important role in the principal component analysis of large datasets. Amazingly, the same TW distributions have emerged in a number of *a priori* unrelated problems [7] such as the longest increasing subsequence of random permutations [8], directed polymers [6, 9] and growth models [10] in the
Kardar–Parisi–Zhang universality class in 1+1 dimensions, sequence alignment problems [11], mesoscopic fluctuations in quantum dots [12], height fluctuations of non-intersecting Brownian motions [13, 14], the number of minima of random energy landscapes [15] and also in finance [16]. Remarkably, the TW distributions associated with the Gaussian unitary and orthogonal ensembles have been recently observed in experiments on nematic liquid crystals [17].

- Atypical fluctuations. The TW laws do not account for atypically large fluctuations, e.g. of order $O(\sqrt{N})$ around the mean value $\sqrt{2N}$. Questions related to large deviations of extreme eigenvalues have recently emerged in cosmology [18] and disordered systems [19, 20], and in the assessment of the efficiency of data compression [21]. Recently, the large deviations of the largest eigenvalue of Wishart matrices have been measured in experiments involving coupled fiber lasers [22]. The probability of atypical large fluctuations, to leading order of the largest eigenvalue of Wishart matrices have been measured in experiments involving coupled fiber lasers [22].

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Given the apparent robustness of the predictions stemming from random matrices with Gaussian-independent entries, it is of paramount interest to investigate whether these TW distributions and their large deviation tails for $\lambda_{\text{max}}$ actually hold for a broader class of matrices. Results are available for (1) non-invariant ensembles, with independent and identically distributed (i.i.d.) entries, (2) invariant ensembles, and are summarized as follows.
(1) It is known that TW holds asymptotically for symmetric \( N \times N \) matrices with i.i.d.
entries of variance \( 1/N \), such that all moments are finite \[29\]. On the other hand, when
the distribution of i.i.d. entries decays as a power law, \( \sim |H_{ij}|^{-1-\mu} \), the case \( \mu = 4 \)
leads to a new class of limiting distribution. For \( \mu > 4 \) the TW distribution still holds
asymptotically, while for \( \mu < 4 \) the statistics of the largest eigenvalue is governed by the
Fréchet law \[16, 30–32\].

(2) For invariant ensembles, the gap probability can be generically written as a Fredholm
determinant (or Pfaffian) of universal kernels (see e.g. \[33\]). More specifically, the TW
law has been established for the classical Wishart ensembles in \[4–6\] and for the Jacobi
ensemble in \[34\]. Disordered ensembles of random matrices were studied in \[35\] where
continuous transitions between TW and other extreme value distributions were found. In
the context of Lévy stable ensembles, the largest eigenvalue distribution for the so-called
Lévy–Smirnov ensemble has been derived in \[36\].

The Cauchy ensemble

Here we address these challenging questions on top eigenvalue statistics beyond TW and
focus on yet another instance of random matrices which incorporates (i) highly non-Gaussian
statistics and (ii) correlated entries, while retaining rotational invariance. We consider the
Cauchy ensemble of the \( N \times N \) matrix \( H \), which might be symmetric (\( \beta = 1 \)), Hermitian
(\( \beta = 2 \)) or quaternionic (\( \beta = 4 \)). The weight associated with \( H \) is given by

\[ P(H) \propto |\text{det}(I_N + H^2)|^{-\beta(N-1)/2-1}, \tag{9} \]

where \( I_N \) denotes the \( N \times N \) identity matrix. Note that equation (9) is manifestly invariant
under the similarity transformation \( H \rightarrow UHU^{-1} \), with \( U \) being an orthogonal (\( \beta = 1 \)),
unitary (\( \beta = 2 \)) or symplectic (\( \beta = 4 \)) matrix. Due to this invariance, the jpdf of the \( N \) real
eigenvalues can be straightforwardly written as follows:

\[ P_{\text{joint}}(\lambda_1, \ldots, \lambda_N) \propto \prod_{i=1}^{N} \frac{1}{\left(1 + \lambda_i^2\right)^{(N-1)/2+1}} \prod_{j<k} |\lambda_j - \lambda_k|^\beta. \tag{10} \]

As in the Gaussian case, the above expression allows us to interpret \( \lambda_i \) as the positions of
charged particles (with, say, positive unit charge) repelling each other via the 2d-Coulomb
(logarithmic) interaction. Here they are confined on the real line and interact in addition with
a fixed particle, with charge \(-N + 2/\beta\), which is placed at the point of coordinate \((0, 1)\)
in the complex plane.

This ensemble has been studied in the literature in \[37\] in the context of mesoscopic
transport as a model of a quantum dot coupled to the outside world by non-ideal leads
containing \( N \) scattering channels. It is also one of the paradigmatic examples of application
of free probability theory in the context of random matrices \[38, 39\], besides having an
interesting connection with generalized circular ensembles \[40\]. A remarkable difference with
the standard Gaussian ensemble is that the density \( \rho_N(\lambda) \) is independent of \( N \), \( \rho_N(\lambda) = \rho^*(\lambda) \)
(for all values of \( N \)). Its support is unbounded and given, for any \( \beta \), by \[41, 42\]

\[ \rho^*(\lambda) = \frac{1}{\pi} \frac{1}{1 + \lambda^2}, \lambda \in (-\infty, \infty). \tag{11} \]

This heavy-tailed density does not have positive moments, and as a consequence, only a \textit{finite}
number of orthogonal polynomials do exist for this ensemble, in contrast with other classical
invariant ensembles.

In this paper, we study the statistics of the largest eigenvalue \( \lambda_{\text{max}} \) for the Cauchy ensemble
in equation (10) for any \( \beta \), while up to now only the case \( \beta = 2 \) was considered in \[43, 44\]—
and even in this case, the large deviations of \( \lambda_{\text{max}} \) have not been computed. To estimate the typical scale of \( \lambda_{\text{max}} \), denoted by \( \Lambda_{\text{max}} \), we note that for a general matrix model, it satisfies

\[
\int_{\Lambda_{\text{max}}}^{\infty} \rho_N(\lambda) d\lambda \approx 1/N,
\]

as the fraction of eigenvalues to the right of the maximum (including itself) is typically \( 1/N \).

Substituting \( \rho_N(\lambda) = \rho^*(\lambda) \), from (11), in (12), one obtains that \( \Lambda_{\text{max}} \sim \mathcal{O}(N) \). As we show below, this gives rise to three distinct regimes in the fluctuations of \( \lambda_{\text{max}} \) (as in equation (8) for Gaussian ensembles, where in that case \( \Lambda_{\text{max}} \sim \mathcal{O}(\sqrt{N}) \)).

2. Summary of new results

Before presenting the details of the calculations, it is useful first to summarize our main results. We show that the probability density function (pdf) of \( \lambda_{\text{max}} \), \( P(w, N) = \partial_w P(\lambda_{\text{max}} \leq w) \), displays three different regimes to leading order for large \( N \):

\[
P(w, N) \approx \begin{cases} 
\exp[-\beta N^2 \psi(w)] & w \ll N \\
N^{-1} f_\beta(w/N) & w \sim N \\
N \phi(w) & w \gg N,
\end{cases}
\]

where \( \psi(w) \) and \( \phi(w) \) are independent of \( \beta \) and are given by

\[
\psi(w) = \frac{1}{4} \left( \frac{1}{2} \log(w^2 + 1) - \log(w + \sqrt{w^2 + 1}) \right) + \frac{\log(2)}{4}.
\]

and

\[
\phi(w) = \frac{1}{\pi w^2}.
\]

The full expression of the scaling function \( f_\beta(x) \), describing typical fluctuations and playing the role analogous to the TW distributions in the Gaussian case (8), can be computed explicitly only for \( \beta = 2 \) [43, 44], where it can be expressed in terms of the solution of a Painlevé-V equation (see equations (48) and (49)). The function \( f_2(x) \) has its support over \([0, \infty)\). From the Painlevé-V equation, we can check explicitly that \( f_2(x) \) has the following asymptotic tails: \( f_2(x) \sim \exp(-1/8 x^2) \) as \( x \to 0 \) and \( f_2(x) \sim 1/\pi x^2 \) as \( x \to \infty \). While we were unable to compute \( f_\beta(x) \) analytically for arbitrary \( \beta \), we are however able to predict its precise tails for any \( \beta > 0 \)

\[
f_\beta(x) \sim \begin{cases} 
\exp\left(-\frac{\beta}{16 x^2}\right) & x \to 0 \\
\frac{1}{\pi x^2} & x \to \infty.
\end{cases}
\]

We achieve this by smoothly matching the central regime \( \lambda_{\text{max}} \sim \mathcal{O}(N) \) with the left (\( \lambda_{\text{max}} \ll N \)) and the right (\( \lambda_{\text{max}} \gg N \)) large deviation tails that we can compute for arbitrary \( \beta \).

By comparing the results for the Gaussian case in equation (8) and the Cauchy case in equation (13), we see that the behavior for the right tail is rather different in the two cases. Indeed, quite generally, one can show (see later) that far to the right of the central peak, the pdf \( P(w, N) \) can be simply expressed in terms of the average density of states as \( P(w, N) \approx N \rho_N(\lambda) \). In the Gaussian case, \( \rho_N(\lambda) \) has a bounded support in the \( N \to \infty \) limit (Wigner semi-circle). However, for finite but large \( N \), there is a non-trivial correction to

1 The symbol \( \approx \) stands for a logarithmic equivalence. For instance, the leftmost regime reads \( \lim_{N \to \infty} [-\log P(w, N)/\beta N^2] = \psi(w) \).
the density arising from the vicinity of the edge of the semi-circle [46], which leads to the non-trivial right large deviation behavior described by $\psi_r(z)$ in equation (8). In contrast, in the Cauchy case, the average density of states has an unbounded support and there does not seem to be any non-trivial edge corrections for finite $N$ as in the Gaussian case.

2.1. Right large deviation tail

Let us start by deriving the expression of the pdf $P(w, N)$ for the right tail, i.e. where $w \gg N$. For this purpose, we define, for each eigenvalue $\lambda_i$, a binary variable $\sigma_i$ such that $\sigma_i = 1$ if $\lambda_i \geq w$ and $\sigma_i = 0$ if $\lambda_i < w$. Then, the probability that the region $[w, \infty)$ is free of eigenvalues can be written as

$$P(\lambda_{\text{max}} \leq w) = \langle (1 - \sigma_1)(1 - \sigma_2) \cdots (1 - \sigma_N) \rangle,$$

where the average $\langle \cdot \rangle$ is over the joint pdf of the eigenvalues (10). Expanding the product in (17), one obtains

$$P(\lambda_{\text{max}} \leq w) = 1 - N \int_{\infty}^{\infty} \rho_N(\lambda) \, d\lambda + \frac{N^2}{2} \int_{\infty}^{\infty} d\lambda_1 \int_{\infty}^{\infty} d\lambda_2 \rho_N^2(\lambda_1, \lambda_2) + \cdots,$$

where $\rho_N^2(\lambda_1, \lambda_2)$ is the two-point correlation function, while the higher order terms omitted in (18) involve $k$-point correlation functions, with $k \geq 3$. Note that this expansion (18) for $\beta = 2$ (respectively, for $\beta = 1, 4$) corresponds to the expansion of the Fredholm determinant (respectively, Pfaffian) for the gap probability. When $w \to \infty$, keeping $N$ fixed, which corresponds to the extreme right tail, one can retain only the first two terms in (18) [46]. Hence, substituting $\rho_N(\lambda) = \rho^*(\lambda)$, from (11), into (18), one obtains

$$P(\lambda_{\text{max}} \leq w) \approx 1 - N \int_{\infty}^{\infty} \rho^*(\lambda) \, d\lambda.$$

Taking the derivative of equation (19) with respect to (w.r.t.) $w$ and using $\rho^*(\lambda) \sim 1/(\pi \lambda^2)$ for $\lambda \gg 1$ (11) yield the result announced in equation (13) for $w \gg N$ and any $\beta$, with $\phi(w) = 1/(\pi w^2)$ (15). Note that this implies that the tail of the pdf of $\lambda_{\text{max}}$ coincides, up to a factor $N$, with the one of the density $\rho_N(\lambda) = \rho^*(\lambda)$ (11).

2.2. Left large deviation tail

Let us now focus on the left tail $w \ll N$, which we tackle using a Coulomb gas technique. By definition,

$$P(\lambda_{\text{max}} \geq w) = \frac{\int_{(-\infty, w)^N} d\lambda_1 \cdots d\lambda_N P_{\text{joint}}(\lambda_1, \ldots, \lambda_N)}{\int_{(-\infty, \infty)^N} d\lambda_1 \cdots d\lambda_N P_{\text{joint}}(\lambda_1, \ldots, \lambda_N)}.$$

We can represent $P_{\text{joint}}(\lambda_1, \ldots, \lambda_N)$ in the Boltzmann form, $P_{\text{joint}}(\lambda_1, \ldots, \lambda_N) \propto \exp(-\beta/2)E[|\lambda|]$ where the energy function is given by

$$E[|\lambda|] = \left(N - 1 + \frac{2}{\beta}\right) \sum_{i=1}^{N} \ln (1 + \lambda_i^2) - \sum_{i \neq j} \ln |\lambda_i - \lambda_j|.$$

This thermodynamical analogy, originally due to Dyson [47], allows us to treat the system of eigenvalues as a 2d gas of charged particles (Coulomb gas) confined on the real line, in equilibrium under competing interactions at inverse temperature $\beta/2$. In the large $N$ limit, the Coulomb gas with $N$ discrete charges becomes a continuous gas which can be described by a continuum (normalized to unity) density function $\rho_w(x) = (1/N) \sum_i \delta(x - \lambda_i)$. Consequently, one can replace the original multiple integral in (20) by a functional integral over the
space of $\varrho_w(x)$. This procedure, originally introduced by Dyson [47] for the problem without a wall, was first used successfully for the Gaussian case with a wall in [19, 20] and subsequently was found useful in a number of different contexts (see, for example, [48–50] and references therein). Introducing the constrained density of eigenvalues $\varrho_w(x)$, one obtains

$$\mathbb{P}[\lambda_{\text{max}} \leq w] \propto \int \mathcal{D}[\varrho_w] \exp \left( -\frac{\beta N^2}{2} S_w[\varrho_w] \right),$$

(22)

where the action $S_w[\varrho_w]$ is given by

$$S_w[\varrho_w] = \int_{-\infty}^w \ln(1 + x^2) \varrho_w(x) \, dx - \int_{-\infty}^w \int_{-\infty}^w dx' \varrho_w(x) \varrho_w(x') \ln |x - x'|$$

$$+ C \left( \int_{-\infty}^w dx \varrho_w(x) - 1 \right),$$

(23)

where $C$ is a Lagrange multiplier enforcing the normalization of the density to 1. The cumulative distribution in equation (22) must be normalized, i.e. the proportionality constant in equation (22) is determined from the condition $\mathbb{P}[\lambda_{\text{max}} \leq w] \to 1$ as $w \to \infty$. The physical meaning of the action $S_w[\varrho_w]$ is readily understood as the free energy of the Coulomb gas, whose particles are constrained to lie to the left of a hard wall at $x = w$. The fact that the free energy is proportional to $N^2$ (and not just $N$) is a consequence of the strong all-to-all interactions among the particles: the free energy of this correlated gas is dominated by the energetic component $\sim O(N^2)$, while the entropic term [$\sim O(N)$] is subdominant in the large $N$ limit (recently, this entropic term has been computed explicitly for the Gaussian [51] and the Wishart–Laguerre ensembles [52]). For large $N$, the functional integral in equation (22) can be evaluated by the saddle point method. Clearly, the equilibrium configuration of the gas (the saddle point density) in the presence of the hard wall at $x = w$ will be determined by minimizing the action in (23). Following the result in the Gaussian case [19, 20], we can anticipate that the modified equilibrium (saddle point) density $\varrho_\star(x)$ in the presence of the wall will display an integrable divergence as $x \to w^-$. In terms of the equilibrium density $\varrho_\star(x)$, the probability of the largest eigenvalue reads from (22) (see also [53]):

$$\mathbb{P}[\lambda_{\text{max}} \leq w] \approx \exp[-\beta N^2 \psi(w)], \quad \psi(w) = \frac{1}{2} \left( S_w[\varrho_\star] - S_\infty[\varrho_\star] \right).$$

(24)

The additional term $S_\infty[\varrho_\star]$ comes from the normalization constant. In order to compute $\varrho_\star(x)$, we vary the action $\frac{\delta S_w[\varrho]}{\delta \varrho(x)} = 0$, yielding

$$\ln(1 + x^2) + C = 2 \int_{-\infty}^w dx' \varrho_\star(x') \ln |x - x'|, \quad x \in (-\infty, w].$$

(25)

Differentiating (25) w.r.t. $x$ and setting $x' = w - \tau$ and $x = w - z$ yields

$$\text{Pr} \int_0^\infty d\tau \frac{\partial \varrho_\star}{\partial \varrho_w}(w - \tau) = \frac{w - z}{1 + (w - z)^2},$$

(26)

where Pr stands for Cauchy’s principal part and the shifted density $\varrho_\star(\tau) = \varrho_\star(\tau) - \tau$ is introduced. The inversion of the half-Hilbert transform (26) is the main technical challenge. However, the task can be fully accomplished [54, 55], and the general solution of equation (26) of the Tricomi type can be written as follows:

$$\varrho_\star(\tau) = -\frac{1}{\pi^2 \sqrt{\tau}} \left\{ \text{Pr} \int_0^\infty ds \frac{1}{s - \tau} \frac{\sqrt{s(w - s)}}{1 + (w - s)^2} + B \right\}, \quad \tau \geq 0,$$

(27)
Figure 1. Plot of $\varrho_w^*(x) = \hat{\varrho}_w(w - x)$ in equation (28) for different values of $w$ (solid lines), compared to the results of our numerical simulations (symbols), for which $\beta = 2$ and $N = 100$.

where $B$ is a constant enforcing the normalization $\int_0^\infty d\tau \hat{\varrho}_w(\tau) = 1$. After straightforward manipulations, one finds that $B = 0$ and eventually

$$\hat{\varrho}_w(\tau) = \frac{1}{\pi \sqrt{2\pi}} \sqrt{k_w^2 + 1} \times \begin{cases} \frac{(k_w + w)^{1/2} - (w - \tau)(k_w - w)^{1/2}}{1 + (w - \tau)^2} & \frac{w}{w - \tau} \leq 0, \\ \frac{\left((\tau w - k_w^2)(w - \tau) + k_w(w - \tau)^2 + k_w\right)(k_w + w)^{1/2} + \tau (w - \tau)(k_w - w)^{1/2}}{k_w[(w - \tau)^2 + 1]} & \frac{w}{w - \tau} \geq 0, \end{cases}$$

where $k_w = \sqrt{w^2 + 1}$. One can check that $\int_0^\infty d\tau \hat{\varrho}_w(\tau) = 1$ and $\hat{\varrho}_w(w - \tau) \sim \frac{1}{\pi \sqrt{1 + \tau^2}}$ for $w \to \infty$, which yields the result announced above (11). In figure 1, we show a plot of $\varrho_w^*(x)$ for different values of $w$, and compare our exact analytical expression to numerical simulations (see below for more details about them).

2.2.1. Evaluation of the saddle point action. From the solution $\varrho_w^*(x)$ (depending parametrically on $w$), we can evaluate the action (23) as follows:

$$S_w[\varrho_w^*] = \int_{-\infty}^{w} \ln(1 + x^2)\varrho_w^*(x) \, dx - \int_{-\infty}^{w} \int_{-\infty}^{w} dx \, dx' \varrho_w^*(x)\varrho_w^*(x') \ln |x - x'|. \quad (29)$$

The double integral in (29) can be written in terms of a single integral, using the following trick. We multiply (25) by $\varrho_w^*(x)$ and integrate over $x$ to obtain eventually

$$\int_{-\infty}^{w} \int_{-\infty}^{w} dx \, dx' \varrho_w^*(x)\varrho_w^*(x') \ln |x - x'| = \frac{1}{2} \int_{-\infty}^{w} dx \varrho_w^*(x) \ln(1 + x^2) + \frac{C}{2}. \quad (30)$$

Therefore, the action reads

$$S_w[\varrho_w^*] = \frac{1}{2} \int_{-\infty}^{w} \ln(1 + x^2)\varrho_w^*(x) \, dx - \frac{C}{2}. \quad (31)$$
Given the different expression of \( \hat{\psi}(\tau) \) for \( w > 0 \) and \( w \leq 0 \) (28), these two cases have to be treated separately. We first consider the case \( w > 0 \) where the constant \( C \) can be determined by setting \( x = 0 \) in (25), yielding

\[
C = 2 \int_{-\infty}^{w} dx \ln |x| \psi^*_w(x).
\]  

(32)

Thus, eventually

\[
S_w[\psi^*_w] = \frac{1}{2} \int_{-\infty}^{w} \ln(1 + x^2) \psi^*_w(x) dx - \int_{-\infty}^{w} dx \ln |x| \psi^*_w(x).
\]  

(33)

Let us focus on the action \( S_w[\psi^*_w] \) in (33). Setting \( x = w - \tau \) and recalling that \( \hat{\psi}(\tau) = \psi^*_w(w - \tau) \), we can rewrite the action, for \( w > 0 \), as follows:

\[
S_w[\hat{\psi}_w] = \frac{1}{2} \int_{0}^{\infty} \ln(1 + (w - \tau)^2) \hat{\psi}_w(\tau) d\tau - \int_{0}^{\infty} d\tau \ln |w - \tau| \hat{\psi}_w(\tau),
\]  

(34)

where \( \hat{\psi}_w(\tau) \) is given in equation (28). After cumbersome manipulations, these integrals can be computed explicitly and we finally obtain a remarkably simple expression:

\[
S_w[\hat{\psi}_w] = \frac{1}{2} \left( \frac{1}{2} \log(w^2 + 1) - \log(w + \sqrt{w^2 + 1}) \right) + \frac{3 \log(2)}{2}.
\]  

(35)

In particular, it admits the large \( w \) expansion

\[
S_w[\hat{\psi}_w] = \log 2 + \frac{1}{8w^2} + \mathcal{O}(w^{-4}).
\]  

(36)

For \( w < 0 \), we start again with the expression in equation (31) but in this case, \( C \) cannot be evaluated by setting \( x = 0 \) in equation (25) as \( x = 0 \) is not in the support on \( \psi^*_w(x) \) (see equation (25)). One can however evaluate \( C \) by setting \( x = w \) in equation (25) to obtain

\[
C = 2 \int_{-\infty}^{w} \psi^*_w(x) \log |w - x| dx - \log (1 + w^2).
\]  

(37)

After some manipulations, one can finally show that, for \( w < 0 \), the expression of \( S_w[\hat{\psi}_w] \) is also given by equation (35). Therefore, as announced in equation (13), the left tail of the cumulative distribution of \( \lambda_{\text{max}} \) is given by

\[
\mathbb{P}[\lambda_{\text{max}} \leq w] \approx \exp(-\beta N^2 \psi(w))
\]

\[
\psi(w) = \frac{1}{2} (S_w[\hat{\psi}_w] - S_w[\hat{\psi}_w]) = \frac{1}{4} \left( \frac{1}{2} \log(w^2 + 1) - \log(w + \sqrt{w^2 + 1}) \right) + \frac{\log(2)}{4}.
\]  

(38)

The rate function \( \psi(w) \) has the following asymptotic tails:

\[
\psi(w) \approx 1/[16w^2] \quad \text{as } w \to +\infty,
\]  

(39)

\[
\approx \frac{1}{4} \ln(|w|/2) + \mathcal{O}(|w|^{-2}) \quad \text{as } w \to -\infty.
\]  

(40)

Substituting the asymptotic tail of \( \psi(w) \) as \( w \to -\infty \) into \( \mathbb{P}[\lambda_{\text{max}} \leq w] \approx \exp(-\beta N^2 \psi(w)) \) then predicts a power law tail of the distribution as \( w \to -\infty \): \( \mathbb{P}[\lambda_{\text{max}} \leq w] \sim |2/|w||^{\beta N^2/2} \).

2.2.2. Comparison with previous works for \( \beta = 2 \). For \( \beta = 2 \), it is interesting to compare the result (38) with previous works on the pdf of \( \lambda_{\text{max}}, P(w, N) \) [43, 44]. In [43], the authors derived a nonlinear differential equation for the function

\[
\sigma(w, N) = \left(1 + w^2\right) \frac{P(w, N)}{\int_{-\infty}^{w} P(z, N) dz} \equiv \left(1 + w^2\right) \frac{\partial}{\partial w} \ln \mathbb{P}[\lambda_{\text{max}} \leq w].
\]  

(41)
They found that, for arbitrary positive $N$, $\sigma(w, N) \equiv \sigma$ satisfies a nonlinear differential equation

$$
(1 + w^2)^2 (\sigma'')^2 + 4(1 + w^2)(\sigma')^3 - 8w\sigma(\sigma')^2 + 4\sigma^2\sigma' + 4N^2(\sigma')^2 = 0,
$$

(42)

with the asymptotic property

$$
\sigma(w, N) \to 0, \quad w \to \infty.
$$

(43)

Using the results of [45], it was shown in [43] that equation (42) can be transformed into a Painlevé-VI equation.

Let us now check if our result in equation (38), after setting $\beta = 2$, is compatible with the above differential equation in the large $N$ limit. Let us first evaluate $\sigma(w, N)$ defined in equation (41) by using our large $N$ prediction in equation (38). Evidently, for large $N$ but finite $w$, we anticipate that $\sigma(w)$ must be proportional to $N^2$, i.e.

$$
\sigma(w, N) \to N^2\hat{\sigma}(w),
$$

(44)

where the function $\hat{\sigma}(w)$ is independent of $N$ and our result predicts that

$$
\hat{\sigma}(w) = - (1 + w^2)\psi'(w) = \frac{1}{2}(\sqrt{1 + w^2} - w).
$$

(45)

On the other hand, substituting $\sigma(w, N) = N^2\hat{\sigma}(w)$ into the differential equation (42) and keeping only the leading order $O(N^6)$ terms, one finds that $\hat{\sigma}(w)$ must satisfy the following nonlinear differential equation:

$$
(1 + w^2)(\hat{\sigma}'')^2 - 2w\hat{\sigma}\hat{\sigma}' + \hat{\sigma}^2 + \hat{\sigma}' = 0.
$$

(46)

Hence, in order to be compatible, our predicted form $\hat{\sigma}(w) = \frac{1}{2}(\sqrt{1 + w^2} - w)$ must satisfy this nonlinear differential equation (46). Indeed, one can check quite easily that $\hat{\sigma}(w) = \frac{1}{2}(\sqrt{1 + w^2} - w)$ does satisfy equation (46), with the correct boundary condition (43): this thus provides a non-trivial check of our result for the rate function $\psi(w)$ (38).

2.3. The central part

We now come to the central part of the pdf of $\lambda_{max}$, where $\lambda_{max} \sim O(N)$. From standard scaling arguments, one expects, for $w \sim O(N)$

$$
P(w, N) = \frac{1}{N} f_\beta \left( \frac{w}{N} \right),
$$

(47)

for any value of $\beta$, as announced in (13). For $\beta = 2$, one can check that it is consistent with the large $N$ limit of (42) from which one obtains [44]

$$
f_2(x) = \frac{d}{dx} \exp \left[ \int_0^{1/x} \tau(z) \frac{dz}{z} \right], \quad x > 0,
$$

(48)

where $\tau(x) \equiv \tau$ satisfies a Painlevé-V equation on the interval $[0, +\infty)$

$$
x^2(\tau'')^2 + 4(\tau - x\tau') (\tau - \tau' (x + \tau')) = 0.
$$

(49)

Interestingly, the same equation (49) also appears in the expression of the pdf of the (scaled) spacing between consecutive eigenvalues in the bulk of the spectrum of GUE random matrices [56]. Hence, one expects that $f_2(x)$ can also be expressed in terms of a Fredholm determinant involving a kernel very similar to the sine kernel [42, 57]. This can be shown directly from equations (10) and (20), using orthogonal polynomials together with the asymptotic analysis performed in [58] to obtain [44]

$$
f_2(x) = \partial_x \det(\mathbb{I} - \Pi_x K \Pi_x), \quad K(y_1, y_2) = \frac{1}{\pi} \frac{\sin(1/y_1 - 1/y_2)}{y_1 - y_2},
$$

(50)
where \( I \) denotes the identity operator and the operator \( \Pi_x \) is the projector on the interval \([x, +\infty)\). Under the change of variable \( y_i = 1/(\pi x_i), i = 1, 2, \) and taking into account the corresponding change of the differential \( dy_i\), the kernel \( K \) in (50) translates to the famous sine kernel.

What are the asymptotic behaviors of \( f_2(x) \)? From equation (49), it is easy to see that \( \tau(x) \sim -x^2/4, as x \to \infty\). This yields the small argument behavior of \( f_2(x) \), from (48):

\[
\ln f_2(x) \sim -\frac{1}{8x^2}, x \to 0.
\]  

(51)

On the other hand, from equation (49), one obtains the small argument behavior of \( \tau(x) \sim -\lambda x \), when \( x \to 0 \), but the constant \( \lambda \) remains unspecified by this equation. Its determination can however be obtained by analyzing the Fredholm determinant in equation (50) which yields \( \tau(x) \sim -x/\pi, as x \to 0 \) [56], and finally

\[
f_2(x) \sim \frac{1}{\pi x^2}, x \to \infty.
\]

(52)

As we show now, these asymptotic behaviors for \( \beta = 2 \) in equations (51) and (52) can be obtained straightforwardly, and generalized to any value of \( \beta \), by exploiting the matching between the central part and the right and left tails of the distribution of \( \lambda_{\max} \). Indeed, if we study the distribution of \( \lambda_{\max} \) when \( \lambda_{\max} \) approaches its typical value \( \Lambda_{\max} \sim \mathcal{O}(N) \) from below, i.e. from the left tail, one expects from our result above (38) that, given \( \psi(w) \sim 1/(16w^2) \) for \( w \gg 1 \)

\[
P(\lambda_{\max} \leq w) \sim \exp\left[-N^2 \frac{\beta}{16w^2}\right], w \gg 1 & w \ll N.
\]

(53)

The above expression is obviously a function of the scaled variable \( x = w/N \) and coincides with the small argument behavior of the pdf \( f_2(x) \) when \( x = w/N \ll 1 \). This yields the asymptotic behavior announced in the introduction (16) and coincides, in particular, with the above result for \( \beta = 2 \) (51).

Similarly, if one studies the pdf of \( \lambda_{\max} \) when \( \lambda_{\max} \) approaches its typical value \( \Lambda_{\max} \sim \mathcal{O}(N) \) from above, i.e. from the right tail, one expects from our result in (13) that the pdf behaves like

\[
P(w, N) \sim \frac{N}{\pi w^2}, w \gg N
\]

(54)

which is obviously a function of the scaled variable \( w/N \times 1/N \). If one assumes that this behavior matches with the large argument behavior of the central part where \( P(w, N) \sim (1/N)f_\beta(w/N) \) when \( w \gg N \), one obtains that \( f_\beta(x) \sim 1/(\pi x^2) \) for large \( x \), as announced in (13).

3. Numerical simulations

We now present the results of our numerical simulations. The distribution of the eigenvalues (10) can be directly simulated by exploiting the thermodynamical analogy, interpreting the eigenvalues \( \lambda_i \) as a one-dimensional gas of charged particles with Coulomb interactions [42, 57]. Hence, the distribution of the maximum eigenvalue can be obtained by means of a Metropolis algorithm, the acceptance rate between two different configurations being \( e^{-\beta(\Delta E)} \), where the energy function \( E(\{\lambda_i\}) \) is given by equation (21). Within this numerical scheme, we have tested our analytical predictions in equations (13) and (16). In figure 2, we show a plot of \( NP(w, N) \) as a function of the scaled variable \( x = w/N \) for \( \beta = 2 \) and different values of \( N = 80, 160, 320 \). The good collapse of the different curves is in agreement with the scaling form predicted for the typical fluctuations in the central regime (13), (47).
The scaling relation $P(w, N) = N^{-1} f_2(w/N)$ for $\beta = 2$ is tested against numerical simulations varying $N$. Monte Carlo data are compared with the solution of equation (49) showing a perfect agreement. Inset: the asymptotic behavior of the tails (16) is verified.

Figure 3. Monte Carlo results for the Coulomb gas system. The logarithm of the probability density is shown in the case $N = 100$ for $\beta = 1$ in (a), $\beta = 2$ in (b) and $\beta = 4$ in (c). We compare it with our predictions for the left (38) and right (15) tails in a dashed line. For $\beta = 2$, we also show the results for $f_2(x)$ (48), in a solid line, which describes the typical fluctuations. Note that one expects that $\psi(w)$ has a support on the full real axis, including $w < 0$, where $P(w, N)$ is however extremely small and thus very hard to measure numerically for $N = 100$.

The solid line corresponds to a numerical evaluation of $f_2(x)$ in equation (48) obtained by solving numerically the equation for $\tau(x)$ in equation (49) (together with the asymptotic behavior $\tau(x) \sim -x/\pi$ for $x \to 0$); the agreement with the numerical data generated by the Metropolis algorithm is excellent. In the inset of figure 2, we see that our exact results in equation (16) describe very well the asymptotic behaviors of $f_2(x)$. In addition, in figure 3, we show a plot of $-\ln P(w, N)$ as a function of $w$ for $N$ fixed and large ($N = 100$) and for different values of $\beta = 1, 2$ and 4. This plot demonstrates the existence of the three distinct regimes as predicted in equation (13). It shows a very good quantitative agreement with the large deviation functions $\psi(w)$ (38) and $\phi(w)$ (15) which we predicted above. Note that $\psi(w)$...
has a support on the full real line, not only on the positive axis. However, for \( w < 0 \), \( P(w, N) \) is very small for \( N = 100 \) and is thus extremely hard to compute numerically (see figure 3). Simulations for smaller values of \( N \), not shown here, agree with our calculations for \( w < 0 \), although they suffer from strong finite \( N \) corrections.

4. Conclusions

In summary, we have investigated the fluctuations of the largest eigenvalue \( \lambda_{\text{max}} \) of \( N \times N \) random matrices belonging to rotationally invariant Cauchy ensembles. We have considered symmetric (\( \beta = 1 \)), Hermitian (\( \beta = 2 \)) and quaternionic (\( \beta = 4 \)) matrix ensembles. We have identified three different regimes for the pdf of \( \lambda_{\text{max}} \): (i) \( \lambda_{\text{max}} \ll N \) (left tail), (ii) \( \lambda_{\text{max}} \sim O(N) \) (central part) and (iii) \( \lambda_{\text{max}} \gg N \) (right tail). We have obtained exact results for the large deviation tails, both left and right and for any \( \beta \), which allows us to also obtain the leading asymptotic behaviors of the pdf in the central regime, which generalizes the TW distributions known for Gaussian ensembles. These exact results have been confirmed by thorough numerical simulations. The exact expression for the central part of the distribution, describing the typical fluctuations of \( \lambda_{\text{max}} \sim O(N) \), beyond the case \( \beta = 2 \), remains a challenging problem.

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