The precontraction group of the field of logarithmic transseries $T_{\log}$

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Abstract

As a first step to understand the theory of the structure $T_{\log}$ of logarithmic transseries as an ordered valued logarithmic field, we focus on the map $\chi$ induced by the logarithm of $T_{\log}$ in its value group $\Gamma_{\log}$ and study the theory of the precontraction group $(\Gamma_{\log}, \chi)$. Particularly, we show that this theory is model complete and complete, and we characterize all definable subsets of the discrete set $\chi(\Gamma_{\log})$.

Key words: Precontraction group, centripetal, logarithmic transseries.

1 Introduction

In [3][10], Franz-Victor Kuhlmann and Salma Kuhlmann showed that in a non-archimedean exponential field the exponential induces a map, called contraction, on the value group of the field with respect to its natural valuation. Specifically, if log denotes the inverse of the exponential map and $v$ the natural valuation of the ordered field, then for $a > 0$ and $v(a) < 0$ they defined the contraction map $\chi$ as $\chi(v(a)) = v(\log(a))$, $\chi(-v(a)) = -\chi(v(a))$ and $\chi(0) = 0$. Under this definition, the authors studied in [7][9] the first order theory of the value group of an exponential field endowed with such contraction map and showed that this theory is complete, decidable, admits quantifier elimination and is weakly o-minimal.

We recall that an exponential field is an ordered field equipped with an order preserving group isomorphism from the additive group of the field onto the multiplicative group of positive elements. The transseries field $T$ is an important non-archimedean exponential field introduced by Ecalle in [4] and by Dahn and Gring in [3], and widely studied as a valued differential field in [1] by Matthias Aschenbrenner, Lou van den Dries and Joris van der Hoeven. Particularly, the last authors show that the contraction map associated to the exponential map of $T$ is definable in the asymptotic couple of $T$, that is the structure of the value group of $T$ endowed with a function induced by the differential map.

In similar way, in [5] Allen Geheret shows that for the valued differential field of logarithmic transseries $T_{\log}$, a special subfield of $T$ defined in [1] and whose elements, informally speaking, are formal series which do not involve exponentiation, there is a precontraction map (i.e a non-surjective contraction map) definable in the asymptotic couple of $T_{log}$.

Following the classical strategy used in model theory to study the theory of a valued field by first studying the theory of its value group and of its residual field, as a first step to understand the theory of $T_{\log}$ as an ordered valued logarithmic field, i.e an ordered valued field equipped with an order preserving group morphism from the multiplicative group of positive elements of the field in the additive group. We study in this paper the model theory of its associated precontraction group, that is the structure given by its value group $\Gamma_{\log}$ endowed with a function $\chi$ induced by the logarithm map.

Base on the ideas used in [4][9] to study the theory of the contraction groups and those used in [5] to study the theory of the asymptotic couple of $T_{\log}$, we study the first order theory of the couple $(\Gamma_{\log}, \chi)$ as a precontraction group. We notice that although the map $\chi$ is not surjective, the image of $\Gamma_{\log}^{<0}$ by $\chi$ is a discrete set cofinal in $\Gamma_{\log}^{<0}$ and using this fact we prove that the theory of the precontraction group $(\Gamma_{\log}, \chi)$ is model complete and complete and we study the definable subsets of the image of $\Gamma_{\log}^{<0}$ by $\chi$.

The structure of the paper is as follows. In section 2, we recall some preliminary notions and notations about ordered abelian groups and valued abelian groups and we present a short description of
$T_{\log}$ and its value group. In section 3, we include some definitions and results about precontraction groups. In section 4, we define the language $L_{pdg}$ of ordered groups together a symbol function for the contraction map and a constant symbol, and study the $L_{pdg}$-theory $T_{pdg}$ of centripetal precontraction discrete groups. Particularly, we prove that the theory $T_{pdg}$ is model complete and complete. Next, we expand the language $L_{pdg}$ to ensure that the natural expansion of the theory $T_{pdg}$ has quantifier elimination and use it to characterize all definable subsets of the image of the group by the precontraction map. Finally, we study the simple extensions of models of $T_{pdg}$.

For the general notions and facts about model theory, we refer the reader to [2, 6] or [1, Appendix B].

2 Preliminaries

Throughout, $m$ and $n$ range over $\mathbb{N} = \{0, 1, 2, \ldots\}$, the set of natural numbers.

Ordered sets

By an ordered set $S$ we mean a set $S$ equipped with a distinguished total order relation $\leq$. If $B$ is a subset of $S$ we see $B$ as an ordered subset of $S$ ordered by the induced ordering and we define the set

$$S^{\geq B} = \{a \in S : a \geq b \text{ for all } b \in B\}.$$ 

In similar way we define $S^{>B}$, $S^{\leq B}$ and $S^{<B}$. Particularly, if $B = \{b\}$, then we set $S^{\geq b} = S^{\geq B}$.

We say that a subset $B$ of $S$ is convex in $S$ if for all $a, c \in B$ and $b \in S$ such that $a \leq b \leq c$ we have $b \in B$, and we define the convex hull of $B$ in $S$ as

$$\text{conv}(B) = \{b \in S : a \leq b \leq c \text{ for some } a, b \in B\}.$$ 

Moreover, we say that $B \subseteq S$ is a lower cut in $S$ if for all $b \in B$ and $a \in A$, $a < b$ implies $a \in B$.

Finally, we define intervals in $S$ as usual and for $\infty \notin S$, we define the set $S_\infty = S \cup \{\infty\}$ and extend the order of $S$ to $S_\infty$ setting $a < \infty$ for all $a \in S$.

Ordered abelian groups

An ordered abelian group $\Gamma$, written additively, is an abelian group with an ordering such that for all $a, b, c \in \Gamma$ if $a < b$ then $a + c < b + c$. For $a \in \Gamma$ we set $|a| = \max\{a, -a\}$ and define the archimedean class of $a$ in $\Gamma$ as

$$[a] = \{b \in \Gamma : |a| \leq n|b| \text{ and } |b| \leq n|a| \text{ for some } n \geq 1\}.$$ 

Thus, we say that $a$ is archimedean equivalent to $b$ in $\Gamma$ if $b \in [a]$. Moreover, the set $[\Gamma]$ of all archimedean classes become in an ordered set putting

$$[a] \leq [b] \iff |a| \leq n|b| \text{ for some } n \geq 1.$$ 

Moreover, we have

$$[a] < [b] \iff n|a| \leq |b| \text{ for all } n \geq 1.$$ 

Valued abelian groups

Let $\Gamma$ be an abelian group and $S$ be an ordered set. A valuation on $\Gamma$ is a surjective map $v : \Gamma \to S_\infty$ such that for all $a, b \in \Gamma$ the following conditions are satisfied:

1. $v(a) = \infty \iff a = 0$.
2. $v(-a) = v(a)$.
3. $v(a + b) \geq \min\{v(a), v(b)\}$.
A valued abelian group is a structure conformed by an abelian group $\Gamma$, and ordered set $S$ and a valuation $v$ on $\Gamma$.

For example, for an ordered abelian group $\Gamma$ if we put $S = [\Gamma]$ and equip $S$ with the reversed ordering of $[\Gamma]$, then the map $v : \Gamma \to S$ defined as $v(a) = [a]$ is a valuation on $\Gamma$. We call this valuation the natural valuation of $\Gamma$.

The field of logarithmic transseries $T_{\log}$

The field $T_{\log}$ of logarithmic transseries is a special subfield of the field $T$ of transseries (see [1] for a definition of $T$), in which each element is a formal series with real coefficients and monomials of the form $\ell_0^\alpha \ell_1^1 \cdots \ell_n^n$, with $\ell_0 = x$, $\ell_{n+1} = \log(\ell_n)$ for $n > 0$ and $r_0, \ldots, r_n \in \mathbb{R}$.

Formally, we can construct $T_{\log}$ as follows: First, for each $n$ we set $L_n$ as the formal multiplicative group given by

$$L_n = \{ \ell_0^{r_0} \ell_1^{r_1} \cdots \ell_n^{r_n} : r_0, r_1, \ldots, r_n \in \mathbb{R} \},$$

and ordered by the relation $\ell_0^{r_0} \ell_1^{r_1} \cdots \ell_n^{r_n} > 1$ if and only if the exponents $r_0, r_1, \ldots, r_n$ are not all zero, and $r_i > 0$ for the least $i$ with $r_i \neq 0$.

Next, for each $n$, we define the Hahn field $\mathbb{R}[[L_n]]$ of well based series with real coefficients and monomials in $L_n$. We mean the field of all functions $f : L_n \to \mathbb{R}$ (written as formal sums $f = \sum_{m \in L_n} f_m m$) such that $\text{supp}(f) := \{ m \in L_n : f_m \neq 0 \}$ has no strictly increasing infinite sequences.

Finally, since $L_n$ is an ordered subgroup of $L_{n+1}$ for $m \leq n$, the ordered group inclusions

$$L_0 \subseteq L_1 \subseteq \ldots \subseteq L = \bigcup_n L_n,$$

induce field inclusions

$$\mathbb{R}[[x^R]] = \mathbb{R}[[L_0]] \subseteq \mathbb{R}[[L_1]] \subseteq \ldots$$

and we define

$$T_{\log} := \bigcup_n \mathbb{R}[[L_n]].$$

It follows that $T_{\log}$ is an ordered subfield of $T$ and $\mathbb{R}[[L]] \cap T = T_{\log}$. Moreover, as each group $L_n$ is divisible, the fields $\mathbb{R}[[L_n]]$ and $T_{\log}$ are real closed.

Let $\Gamma_{\log}$ be the ordered $\mathbb{R}$-vector space $\bigoplus_{n>0} \mathbb{R} \ell_n$, where $\alpha = \sum_{i=0}^{n} r_i \ell_{i+1} > 0$ if $r_k > 0$ for the least $k$ in $\{0, 1, \ldots, n\}$ such that $r_k \neq 0$. We define a convex valuation $v$ of $T_{\log}$ as the unique map

$$v : T_{\log} \to \Gamma_{\log} \cup \{ \infty \}$$

such that

1. $v(\ell_0^{r_0} \ell_1^{r_1} \cdots \ell_n^{r_n}) = -r_0 \ell_1 - r_1 \ell_2 - \cdots - r_n \ell_{n+1}$,
2. $v(f) = v(\mathfrak{a}(f))$ for all $f \in T_{\log}^{\neq 0}$, where $\mathfrak{a}(f) := \text{max}(\text{supp}(f))$ is the dominant monomial of $f$.
3. $v(0) = \infty$.

Thus, $T_{\log}$ becomes an ordered valued field with valuation ring $O_{\log} = \mathbb{R} \oplus O_{\log}$, maximal ideal

$$O_{\log} = \{ f \in T_{\log} : v(f) > 0 \},$$

value group $\Gamma_{\log}$, and residue field $\mathbb{R}$.

Now, since each positive element $f \in T_{\log}$ can be decomposed as $f = \mathfrak{a}(f) \cdot f_{\mathfrak{a}(f)} \cdot (1 + \epsilon)$ where $\mathfrak{a}(f) \in L$, $f_{\mathfrak{a}(f)} \in \mathbb{R}^{>0}$ is the leading coefficient of $f$ and $\epsilon \in O_{\log}$ (see [2]), we may define the logarithm of $f$ as

$$\log(f) = \log_{\text{pre}}(\mathfrak{a}(f)) + \log_{\text{G}}(f_{\mathfrak{a}(f)}) + \log_{\text{R}}(1 + \epsilon),$$
where $\log_{\mathbb{R}}$ is the logarithm in $\mathbb{R}$, $L_\mu$ is the logarithm on 1-units given by

$$L_\mu(1 + \epsilon) = \sum_{i>0} (-1)^{i+1} \epsilon^i$$

and $L_{\text{pre}}$ is the logarithmic section defined as $L_{\text{pre}}(\ell^0 \ell^1 \cdots \ell^n) = r_0 \ell_1 + \cdots + r_n \ell_{n+1}$.

Under this definition we see that the map log is an ordered embedding from the multiplicative group $T^>_{\log}$ into the additive group $T_{\log}$, such that

$$\log(T^>_{\log}) = \Gamma_{\log} \oplus \mathbb{R} \oplus \mathcal{O}_{\log}$$

is an $\mathbb{R}$-vector subspace of $T_{\log}$.

Additionally, the valuation and the logarithm are related by the following property, which is known as Growth Axiom (see [10]): for all $f \in T^>_{\log}$ with $v(f) < 0$ we have that $v(\log(f)) > v(f)$, which implies

$$f > \log(f^n) = n \log(f) \text{ for all } n \in \mathbb{N}.$$  

Moreover, the map log induce an extra structure in the value group $\Gamma_{\log}$ given by the map

$$\chi : \Gamma_{\log} \rightarrow \Gamma_{\log}$$

defined as

$$\chi(\alpha) = \begin{cases} \chi'(\alpha), & \text{if } \alpha < 0, \\ 0, & \text{if } \alpha = 0, \\ -\chi'(-\alpha), & \text{if } \alpha > 0. \end{cases}$$

where $\chi' : \Gamma_{\text{log}}^{-} \rightarrow \Gamma_{\text{log}}^{-}$ is given by $\chi'(\alpha) = v(\log(f))$ with $f \in T^>_{\log}$, and $\alpha = v(f) < 0$. We see that $\chi$ is well defined, since for $f, g \in T^>_{\log}$ with $v(f) = v(g) < 0$ there is a positive unit $h$ in $\mathcal{O}_{\log}$ such that $f = gh$. Thus, $\log(f) = \log(g) + \log(h)$ and

$$v(\log(f)) \geq \min\{v(\log(g)), v(\log(h))\}.$$  

By definition of $\chi$ we have $v(\log(h)) \geq 0$ and $v(\log(g)) < 0$, and then

$$v(\log(f)) = \min\{v(\log(g)), v(\log(h))\} = v(\log(g)).$$

### 3 Precontraction groups

The notion of contraction map was used in [9] to study the structure of the value group of an exponential field and the theory of contraction groups was studied in [7][8]. We list here some useful definitions and results of those papers. Specifically:

**Definition 1.** Given a totally ordered abelian group $\Gamma$ and a map $\chi : \Gamma \rightarrow \Gamma$, the pair $(\Gamma, \chi)$ is called a precontraction group and $\chi$ is called a precontraction map if it satisfies for all $a, b \in \Gamma$ the following axioms:

1. $\chi(a) = 0 \iff a = 0,$
2. $a \leq b \rightarrow \chi(a) \leq \chi(b),$
3. $\chi(-a) = -\chi(a),$
4. if $a$ is archimedean equivalent to $b$ and $\text{sign}(a) = \text{sign}(b)$, then $\chi(a) = \chi(b).$

If in addition $\chi$ is surjective then $\chi$ is called a contraction map and $(\Gamma, \chi)$ is called a contraction group. Moreover, $(\Gamma, \chi)$ will be called centripetal if $\forall a \in \Gamma^0(|a| > |\chi(a)|)$ and divisible if $\Gamma$ is divisible.
Example 1. The map $\chi$ defined in the value group $\Gamma_{\log}$ of $T_{\log}$ is a precontraction map. Moreover, since the ordered valued logarithmic field $T_{\log}$ satisfies the Growth Axiom, then in fact $(\Gamma_{\log}, \chi)$ is a centripetal precontraction group.

Proof. We already see that $\chi$ is well defined. Now, let $v(f)$ be archimedean equivalent to $v(g)$ with $f, g \in T_{\log}^0$ and $v(f) \leq v(g) < 0$, then there is a natural number $n$ such that $nv(g) = v(g^n) \leq v(f)$. By convexity of $v$ we obtain that $g^n \geq f \geq g$, and then $\log(g^n) = n\log(g) \geq \log(f) \geq \log(g)$. Thus $v(\log(g)) = v(\log(f))$ and $\chi(v(f)) = \chi(v(g))$.

Finally, if $v(f) < 0$ with $f \in T_{\log}^0$, then by Growth Axiom we have $v(f) < v(\log(f)) = \chi(v(f))$. Thus, by definition of $\chi$ we conclude that $|v(a)| > |\chi(a)|$ for all $a \in \Gamma_{\neq 0}$, i.e. $(\Gamma_{\log}, \chi)$ is centripetal.

From the axioms we have some useful consequences:

Lemma 2. Let $(\Gamma, \chi)$ be a precontraction group and $a, b \in \Gamma$.

(1) Axiom (4) is equivalent to the single statement $\chi(2a) = \chi(a)$.

(2) $\chi(\Gamma^{<0}) \subseteq \Gamma^{<0}$ and $\chi(\Gamma^{>0}) = -\chi(\Gamma^{>0})$.

(3) $\chi(a + b) \geq \min\{\chi(a), \chi(b)\}$.

(4) If $\chi(a) < \chi(b) < 0$ then $\chi(a - b) = \chi(a)$.

(5) If $0 < \chi(a) < \chi(b)$ then $\chi(b - a) = \chi(b)$.

(6) Let $b > 0 > a$. If $\chi(|a|) > \chi(|b|)$ then $\chi(a - b) = \chi(a)$, and if $\chi(|b|) > \chi(|a|)$ then $\chi(b - a) = \chi(b)$.

Proof. (1) We just have to show that the statement $\forall a \in \Gamma \chi(2a) = \chi(a)$ implies axiom (4). First, by axiom (2) we can observe that if $\chi(2a) = \chi(a)$ then $\chi(na) = \chi(a)$ for all $n \in \mathbb{N}$ . Now, if $a$ is archimedean equivalent to $b$ and $\text{sign}(a) = \text{sign}(b)$ then there is a natural number $n$ such that $n|a| \leq |b|$ and $n|b| \geq |a|$, so $\chi(a) = \chi(na) \geq \chi(b) = \chi(ab) \geq \chi(a)$ and thus $\chi(a) = \chi(b)$.

(2) If $a < 0$ then by axioms 1 and 2 we have $\chi(a) < 0$ and by axiom (3) we have $\chi(-a) = -\chi(a) > 0$.

(3) Without loss of generality we can assume that $a < b$. Then

$$a + a < a + b < b + b$$

and

$$\chi(2a) \leq \chi(a + b) \leq \chi(2b).$$

Since $\chi(2x) = \chi(x)$ for all $x \in \Gamma$, then

$$\chi(a) \leq \chi(a + b) \leq \chi(b),$$

so $\chi(a + b) \geq \min\{\chi(a), \chi(b)\}$.

(4) Since $\chi(a) < \chi(b) < 0$ then $a < b < 0$ and $a - b > a$. Thus $\chi(a - b) \geq \chi(a)$. On the other hand, as $\chi(b) > \chi(a)$ and by item (3)

$$\chi(a) = \chi((a - b) + b) \geq \min\{\chi(a - b), \chi(b)\},$$

then $\chi(a) \geq \chi(a - b)$. Thus, $\chi(a - b) = \chi(a)$.

Items (5) and (6) follow of item (4).

Working with the natural valuation of $\Gamma$, for example we have the following immediate properties:
Lemma 3. Let \((\Gamma, \chi)\) be a precontraction group. Then

1. For all \(a, b \in \Gamma\), if \(v(a) \leq v(b)\) then \(|\chi(a)| \geq |\chi(b)|\).
2. For all \(a, b \in \Gamma\), if \(v(a - b) > v(a)\) then \(\chi(a) = \chi(b)\).
3. For all \(a_1, a_2, ..., a_n \in \Gamma\), if \(v(a_k) < v(a_i)\) for all \(i \neq k\) then \(\chi(\sum_{i=1}^{n} a_i) = \chi(a_k)\).
4. \((\Gamma, \chi)\) is a centripetal precontraction group if and only if \(v(\chi(a)) > v(a)\) for all \(a \in \Gamma^{\neq 0}\).

Now, the main result about contraction groups proved in [7, 8] is the following:

Theorem 4. In the language of ordered groups expanded by a unary function symbol for the contraction map, the theory of nontrivial divisible centripetal contraction groups is complete, decidable, admits quantifier elimination and is weakly o-minimal\(^1\) and it is the model completion of the theory of centripetal precontraction groups.

4 The theory \(T_{pdg}\)

A key feature of the centripetal precontraction group \((\Gamma_{\log}, \chi)\) is that the image of \(\Gamma_{\log}^{<0}\) by \(\chi\) is a discrete set with first element and where the immediate successor of an element \(a \in \chi(\Gamma_{\log}^{<0})\) is \(\chi(a)\). Thus, to capture this property we introduce the following definition:

Definition 5. Let \(L_{pdg} = \{+,-,0,\cdot,\chi,c\}\), be the language of ordered groups augmented by a unary function symbol \(\chi\) and a constant symbol \(c\). We say that a nontrivial centripetal precontraction group \((\Gamma, \chi)\) is a model of the \(L_{pdg}\)-theory \(T_{pdg}\) if:

1. \(\chi(\Gamma^{<0})\) has a least element \(c\),
2. \(\chi : \chi(\Gamma^{<0}) \to \chi(\Gamma^{<0})^{\geq c}\) is a bijection,
3. \(\forall a, b \in \chi(\Gamma^{<0})\) if \(a < b\) then \(a < \chi(a) \leq b\)
4. \(\Gamma\) is a divisible ordered group.

From the above definition, we can see that each substructure \(S\) of a model of \(T_{pdg}\) is a centripetal precontraction group where \(\chi(S^{<0})\) has a least element and \(\chi(a)\) is the immediate successor of \(a\) for each \(a \in \chi(S^{<0})\).

Example 2. (1) Clearly, \((\Gamma_{\log}, \chi)\) is a model of \(T_{pdg}\). Moreover,

\[\chi(\Gamma_{\log}^{<0}) = \{-\ell_2, -\ell_3, \ldots\},\]

where \(\chi(-\ell_k) = -\ell_{k-1}\) and \(-\ell_k < -\ell_{k+1}\).

(2) Let \(\oplus_1 \mathbb{Q}e_i\) be a vector space over \(\mathbb{Q}\) with ordered basis \(\{e_i\}\). Under the usual lexicographic order, i.e.,

\[\sum a_i e_i > 0\text{ iff } a_k > 0\text{ for the least }k\text{ such that }a_k \neq 0,\]

\(\oplus_1 \mathbb{Q}e_i\) becomes an ordered abelian group and if we define the function \(\chi_\mathbb{Q} : \oplus_1 \mathbb{Q}e_i \to \oplus_1 \mathbb{Q}e_i\) as

\[\chi_\mathbb{Q}(\sum a_i e_i) = \text{sign}(a_k) e_{k+1}\]

with \(k\) the minimal index such that \(a_k \neq 0\), then \((\oplus_1 \mathbb{Q}e_i, \chi_\mathbb{Q})\) is a model of \(T_{pdg}\).\(^1\)

\(^1\)A theory in which an order is given or definable is called weakly o-minimal if in every model of this theory, each definable subset is a finite union of convex subsets. Moreover, if each one of such convex subsets is an interval, then we say that the theory is o-minimal.
In addition to the properties listed in lemmas 2 and 3, we can observe that if \((\Gamma, \chi)\) is a model of \(T_{pdg}\), then the discrete set \(\chi(\Gamma^{<0})\) is cofinal in \(\Gamma^{<0}\) since for all \(a \in \Gamma^{<0}\) we have \(a < \chi(a)\). Now, although in the models of \(T_{pdg}\) the map \(\chi\) is not surjective and we can not proceed as in [7] to prove the model completeness of \(T_{pdg}\), here we will use the properties of the discrete set \(\chi(\Gamma^{<0})\) to do that.

### 4.1 Some algebraic properties of models of \(T_{pdg}\)

First, we can observe the following:

**Lemma 6.** If \((\Gamma, \chi)\) is a model of \(T_{pdg}\), then \(\Gamma\) is a vector space over \(\mathbb{Q}\) and \(\chi(\Gamma^{<0})\) is a linearly independent subset of \(\Gamma\).

**Proof.** Since \(\Gamma\) is a divisible ordered group it follows that \(\Gamma\) is a vector space over \(\mathbb{Q}\). Moreover, given \(q_1, q_2, ..., q_n \in \mathbb{Q}\) with \(q_1 \neq 0\) and \(a_1, a_2, ..., a_n \in \chi(\Gamma^{<0})\) with \(a_1 < a_2 < ... < a_n\) then

\[ \chi(a_1) < \chi(a_2) < ... < \chi(a_n), \]

and if \(\alpha = \sum_{i=1}^{n} q_i a_i\) then by lemma 2, we have that \(\chi(\alpha) = \chi(a_1)\) whenever \(q_1 > 0\) and \(\chi(\alpha) = -\chi(a_1)\) whenever \(q_1 < 0\). Thus \(\alpha \neq 0\). \(\square\)

Regarding the construction of new precontraction groups we have the following:

**Lemma 7.** Let \((\Gamma, \chi)\) be a centripetal precontraction group and \(\Delta \subseteq \Gamma\) be a nonempty convex subgroup such that if \(\chi(x) \in \Delta\) then \(x \in \Delta\) for \(x \in \Gamma\). Then:

1. There is a unique order \(\leq^\prime\) in \(\Gamma/\Delta\) has a such that \(\Gamma/\Delta\) is an ordered abelian group in which if \(a \leq b\) then \(\overline{a} \leq^\prime \overline{b}\) for \(a, b \in \Gamma\).
2. The map \(\chi' : \Gamma/\Delta \to \Gamma/\Delta\) given by \(\chi'(\overline{a}) = \overline{\chi(a)}\) is well defined and makes \((\Gamma/\Delta, \chi')\) a centripetal precontraction group.

**Proof.** Since the (1) is a general property of ordered abelian groups, it is enough to put \(\overline{a} > 0\) if and only if \(a > \Delta\). First we show that \(\chi'\) is well defined. To do that we prove that if \(a - b \in \Delta\) then \(\chi(a) - \chi(b) \in \Delta\). If \(\chi(a) = \chi(b)\) then clearly \(\chi(a) - \chi(b) = 0 \in \Delta\). Now, if \(\chi(a) \neq \chi(b)\) then we have the following cases:

- \(\chi(a) > \chi(b) > 0\). Thus, \(a - b > 0\) and by centripetal property we have \(a - b > \chi(a - b) > 0\). Moreover, \(\chi(a - b) = \chi(a)\). So, \(a - b > \chi(a) > \chi(b) > 0\) which implies \(\chi(a), \chi(b) \in \Delta\) and then \(\chi(a) - \chi(b)\).

- \(\chi(a) < \chi(b) < 0\). Similar to the previous case.

- \(a < 0 < b\). Thus \(\chi(a) < 0 < \chi(b) < b\). \(a - b < 0\) and \(a - b < \chi(a - b) < 0\). Moreover, \(a - b < \chi(a) - \chi(b) < 0\) and then \(\chi(a) - \chi(b) \in \Delta\).

Now, since \(\chi(x) \in \Delta\) implies that \(x \in \Delta\), then we can prove that \((\Gamma/\Delta, \chi')\) is a centripetal precontraction group. \(\square\)

### 4.2 Embedding lemmas

Let \((\Gamma, \chi), (\Gamma', \chi')\) be precontraction groups. We say that \((\Gamma', \chi')\) is an extension of \((\Gamma, \chi)\) if \(\Gamma'\) is an extension of \(\Gamma\) as valued groups \(\chi(\Gamma) = \chi'(\Gamma') \cap \Gamma\) and \(\chi'(a) = \chi(a)\) for \(a \in \Gamma\). Moreover, we say that

\[ \phi : (\Gamma, \chi) \to (\Gamma', \chi') \]
is an embedding of precontraction groups if $\phi: \Gamma \to \Gamma'$ is an embedding of ordered abelian groups such that

$$\phi(\chi(a)) = \chi'(\phi(a))$$

for all $a \in \Gamma$.

From this definition it follows that if $(\Gamma, \chi)$ and $(\Gamma', \chi')$ are models of $T_{pdg}$, then we have the following possibilities: First we can have $\chi(\Gamma^{<0}) = \chi'(\Gamma'^{<0})$, which is always true if $[\Gamma] = [\Gamma']$ and some times when $[\Gamma] \neq [\Gamma']$. Secondly, we can have $\chi(\Gamma^{<0}) \neq \chi'(\Gamma'^{<0})$, and here we have again two possibilities: either there is $b \in \chi'(\Gamma'^{<0})$ such that $b > \chi(\Gamma^{<0})$ or there is a nonempty lower cut $G$ in $\chi(\Gamma^{<0})$ and $b \in \chi'(\Gamma'^{<0}) \setminus \chi(\Gamma^{<0})$ such that $\chi(G) \subseteq G$ and $G < b < g^G$.

**Definition 8.** From now on, we call $G \subseteq \chi(\Gamma^{<0})$ a special cut if $G$ is a lower cut in $\chi(\Gamma^{<0})$ such that $\chi(G) \subseteq G$ and we denote by $\text{scut}(\chi(\Gamma^{<0}))$ the collection of all special cuts of $\chi(\Gamma^{<0})$.

Based on Gehret’s work about the theory of the asymptotic couple of $T_{\log}$ in [9], in the following we present some embedding lemmas which deal with the above cases and that will be used to prove the model completeness of $T_{pdg}$.

**Case 1.** $(\Gamma, \chi) \subseteq (\Gamma', \chi')$ with $[\Gamma] = [\Gamma']$.

From [7, lemma 3.6] we have the following result:

**Lemma 9.** Let $(\Gamma, \chi)$ be a centripetal precontraction group. Then for each extension $(\Gamma', <)$ of $(\Gamma, <)$ such that $[\Gamma] = [\Gamma']$, $\chi$ extends in a unique way to a centripetal precontraction $\chi'$ on $\Gamma'$ and we have $\chi(\Gamma') = \chi(\Gamma)$. Particularly, if $\mathbb{Q} \mathbb{G} = \mathbb{Q} \otimes \mathbb{Z}$, $\Gamma$ is the divisible hull of $\Gamma$ then $\chi(\mathbb{Q} \mathbb{G}) = \chi(\Gamma)$, since every element in $\mathbb{Q} \mathbb{G}$ is archimedean equivalent to some element of $\Gamma$.

Using the quantifier elimination of the theory of divisible ordered abelian groups we have:

**Lemma 10.** Let $(\Gamma, \chi), (\Gamma', \chi')$ and $(\Gamma^*, \chi^*)$ be models of $T_{pdg}$, such that $(\Gamma, \chi) \subseteq (\Gamma', \chi')$, $[\Gamma] = [\Gamma']$, $(\Gamma^*, \chi^*)$ is $k$-saturated for some $k > \text{card}(\Gamma')$, and $\phi: (\Gamma, \chi) \to (\Gamma^*, \chi^*)$ is an embedding, then there is an embedding $\phi': (\Gamma', \chi') \to (\Gamma^*, \chi^*)$ which extends $\phi$.

**Proof.** Since $\Gamma$, $\Gamma'$ and $\Gamma^*$ are divisible ordered abelian groups and such theory has quantifier elimination, then by saturation of $(\Gamma^*, \chi^*)$ there is an embedding $\phi': \Gamma' \to \Gamma^*$ that extends the embedding $\phi: \Gamma \to \Gamma^*$. Moreover, if $b \in \Gamma'$, because $[\Gamma] = [\Gamma']$, there is $a \in \Gamma$ such that $[a] = [b]$ and $\text{sign}(a) = \text{sign}(b)$.

Thus, $\chi'(b) = \chi'(a)$ and then

$$\phi'(\chi'(b)) = \phi'(\chi'(a)) = \phi'(\chi(a)),$$

but as $[\phi'(b)] = [\phi(a)]$ in $[\Gamma']$, then $\chi^*(\phi'(b)) = \chi^*(\phi(a))$. Finally, since $\phi$ is an embedding of centripetal divisible precontraction groups, then

$$\chi^*(\phi'(a)) = \phi'(\chi(a)) = \phi'(\chi'(b)).$$

**Case 2.** $(\Gamma, \chi) \subseteq (\Gamma', \chi')$ with $[\Gamma] \neq [\Gamma']$ and $\chi(\Gamma^{<0}) = \chi'(\Gamma'^{<0})$.

From [7, lemma 3.3] we know that:

**Lemma 11.** Let $(\Gamma, \chi) \subseteq (\Gamma', \chi')$ be precontraction groups. Let $a \in \Gamma'$ such that $[a] \notin [\Gamma]$ and $\chi'(a) = b \in \Gamma$. Then $(\Gamma + \mathbb{Z}a, \chi_a)$ is a precontraction group with $\chi_a(\Gamma + \mathbb{Z}a) = \chi(\Gamma) \cup \{b, -b\} < G$.

Moreover, the extension of $\chi$ from $(\Gamma, \chi)$ to $\Gamma + \mathbb{Z}a$ is uniquely determined by the assignment $\chi'(a) = b$.

If $\Gamma$ is divisible, $\Gamma + \mathbb{Q}a$ is the divisible hull of $\Gamma + \mathbb{Z}a$. Thus, by lemmas [9] and [11], we have $[\Gamma + \mathbb{Q}a] = [\Gamma + \mathbb{Z}a]$ and the image under $\chi_a$ coincide. From this we have the following lemma:
Lemma 12. Let $(\Gamma, \chi) \subseteq (\Gamma', \chi')$ be models of $T_{pdg}$ with $\chi(\Gamma^<0) = \chi'(\Gamma'^<0)$, $a \in \Gamma^<0$ such that $[a] \notin [\Gamma]$ and $C$ is the lower cut in $[\Gamma]$ defined by $[a]$. Then there is a model $(\Delta, \chi_\Delta)$ of $T_{pdg}$ such that:

1. $(\Gamma, \chi) \subseteq (\Delta, \chi_\Delta) \subseteq (\Gamma', \chi')$ with $[a] \in [\Gamma_\Delta]$, and
2. for any embedding $\phi$ of $(\Gamma, \chi)$ into a model $(\Gamma^*, \chi^*)$ of $T_{pdg}$ and each $a' \in \Gamma^*<0$ with $[a'] \notin [\phi(\Gamma)]$ which realize the cut $\{\phi(x) : x \in C\}$, there is a unique embedding $\phi' : (\Delta, \chi_\Delta) \rightarrow (\Gamma^*, \chi^*)$ that extends $\phi$ with $\phi'(a) = a'$.

Proof. Let $a \in \Gamma^<0$ with $[a] \notin [\Gamma]$ and $b = \chi'(a) \in \Gamma$. Define $(\Delta, \chi_\Delta) = (\Gamma + Qa, \chi_a')$ where $\chi_a'$ is the restriction of $\chi'$ to $\Gamma + Qa$. As $\chi(\Gamma) = \chi'(\Gamma)$ then $\chi_\Delta(\Delta) = \chi(\Gamma)$, so $\chi_\Delta(\Delta)$ is a centripetal divisible precontraction group.

Case 3. $(\Gamma, \chi) \subseteq (\Gamma', \chi')$ with $\chi(\Gamma^<0) \neq \chi'(\Gamma'^<0)$.

As we saw above if $(\Gamma, \chi) \subseteq (\Gamma', \chi')$ are model of $T_{pdg}$ and $\chi(\Gamma^<0) \neq \chi'(\Gamma'^<0)$, we have two cases. First, we can have that there is $b \in \chi'(\Gamma'^<0)$ such that $b > \chi(\Gamma^<0)$. So we want to extend $(\Gamma, \chi)$ to a model $(\Delta, \chi_\Delta)$ of $T_{pdg}$ in which $b \in \chi_\Delta(\Delta^<0)$. To do that, we can observe that if $b > \chi(\Gamma^<0)$, then $\chi^k(b) > \chi(\Gamma^<0)$ for any integer $k$. Thus, $\chi^k(b)$ is the lower cut in $\chi(\Gamma^<0)$ which extends $(\Gamma, \chi_\Delta)$.

Lemma 13. Let $(\Gamma, \chi) \subseteq (\Gamma', \chi')$ be divisible centripetal precontraction groups and $(b_n)_{n \geq 0}$ a family in $\chi(\Gamma^<0)$ such that $b_{n+1} = \chi(b_n)$ and $b_0 > \chi(\Gamma)$ for all $n \geq 0$, then there is a divisible centripetal precontraction group $(\Gamma'', \chi'')$ such that:

1. $(\Gamma, \chi) \subseteq (\Gamma'', \chi'') \subseteq (\Gamma', \chi')$ with $b_n \in \chi''(\Gamma^<0)$ for $n > 1$, and
2. for any embedding $\phi$ of $(\Gamma, \chi)$ into a divisible centripetal precontraction group $(\Gamma^*, \chi^*)$ and any family $(b'_n)_{n \geq 0}$ in $\chi^*(\Gamma'^<0)$ such that $b'_{n+1} = b'_{n+1}$ and $b_n > \chi(\Gamma^<0)$ for $n \geq 0$, there is a unique embedding $\phi' : (\Gamma'', \chi'') \rightarrow (\Gamma^*, \chi^*)$ which extends $\phi$ and such that $\phi'(b_n) = b'_n$ for all $n$.

Proof. Let $(\Gamma_0, \chi_0) \subseteq (\Gamma', \chi')$ be the family given by $\Gamma_0 = \Gamma$, $\Gamma_{n+1} = \Gamma_n + Qb_n$ and $\chi_n$ the restriction of $\chi'$ to $\Gamma_n$. By lemma 12, $(\Gamma_n, \chi_n)$ is a divisible precontraction group for each $n$ and since $(\Gamma_n, \chi_n) \subseteq (\Gamma_{n+1}, \chi_{n+1})$ and $\chi'(b_n) = b_{n+1}$, $(\Gamma_n, \chi_n) \subseteq (\Gamma', \chi')$ is a divisible centripetal precontraction group which extends $(\Gamma, \chi)$. Now, by induction if we assume that $\phi_n : (\Gamma_n, \chi_n) \rightarrow (\Gamma^*, \chi^*)$ is an embedding such that $\phi_n(b_n) = b'_n$ for $i \in \{0, 1, ..., b_{n-1}\}$, then by lemma 12 there is a unique embedding

$$\phi_{n+1} : (\Gamma_{n+1}, \chi_{n+1}) \rightarrow (\Gamma^*, \chi^*)$$

which extends $\phi_n$ and such that $\phi_{n+1}(b_n) = b'_n$. Thus, there is a unique embedding

$$\phi' = \bigcup_{i \geq 0} \phi_i : (\Gamma'', \chi'') \rightarrow (\Gamma^*, \chi^*)$$

which satisfies the required properties.

Now, we use the above lemma to include the predecessors of the element $b_0$ of the family:

Lemma 14. Let $(\Gamma, \chi) \subseteq (\Gamma', \chi')$ be divisible centripetal precontraction groups and $(b_k)_{k \in \mathbb{Z}}$ a family in $\chi(\Gamma^<0)$ such that $b_{k+1} = \chi(b_k)$ and $b_k > \chi(\Gamma)$ for all $k \in \mathbb{Z}$, then there is a divisible centripetal precontraction group $(\Delta, \chi_\Delta)$ such that:

1. $(\Gamma, \chi) \subseteq (\Delta, \chi_\Delta) \subseteq (\Gamma', \chi')$ with $b_k \in \chi_\Delta(\Delta^<0)$ for all $k \in \mathbb{Z}$, and
2. for any embedding $\phi$ of $(\Gamma, \chi)$ into a divisible centripetal precontraction group $(\Gamma^*, \chi^*)$ and any family $(b'_k)_{k \in \mathbb{Z}}$ in $\chi^*(\Gamma'^<0)$ such that $b'_{k+1} = \chi(b'_k)$ and $b_k > \chi(\Gamma^<0)$ for $k \in \mathbb{Z}$, there is a unique embedding $\phi' : (\Delta, \chi_\Delta) \rightarrow (\Gamma^*, \chi^*)$ which extends $\phi$ and such that $\phi'(b_k) = b'_k$ for all $k$. 

Proof. First for each \( n \in \mathbb{N} \) we define the family \( (a_i^n)_{i \geq 0} \) where \( a_i^n = b_{-n+i} \) for \( i \geq 0 \). Clearly, we have that \( a_i^{n+1} = \chi'(a_i^n) \) and \( a_i^{n+1} = a_i^n \). Now, using the lemma \([13]\) for each family \((a_i^n)_{i \geq 0}\) we obtain a divisible centripetal precontraction group \((\Gamma''_n, \chi''_n)\) such that \( a_i^{n+1} = \chi''_n(\Gamma''_n) \) and \( \chi''_n(a_i^n) = a_i^{n+1} \) and a unique embedding \( \psi_n : (\Gamma''_n, \chi''_n) \to (\Gamma''_{n+1}, \chi''_{n+1}) \) such that \( \psi_n(a_i^n) = a_i^{n+1} \). Thus we obtain the increasing chain

\[
(\Gamma, \chi) \subset (\Gamma''_0, \chi''_0) \subset (\Gamma'', \chi'') \subset \ldots
\]

and we define \((\Delta, \chi) = \bigcup_{n \geq 0}(\Gamma''_n, \chi''_n)\).

Now, if \( \phi : (\Gamma, \chi) \to \chi'(\Gamma^{<\infty}) \) is an embedding with \((b_k')_{k \in \mathbb{Z}} \) a family in \( \chi'(\Gamma^{<\infty}) \) such that \( b_{k+1}' = \chi'(b_k') \) and \( b_k > \phi(\chi(\Gamma^{<\infty})) \) for some \( k \in \mathbb{Z} \), then by lemma \([13]\) there is a unique embedding

\[
\phi_n : (\Gamma''_n, \chi''_n) \to (\Gamma'', \chi'')
\]

that extends \( \phi \) and such that \( \phi_n(a_i^n) = \phi_n(b_{-n+i}) = b'_{-n+i} \). Moreover, \( \phi_n \subseteq \phi_{n+1} \) because

\[
\phi_{n+1}(a_i^{n+1}) = \phi_n(\chi''_{n+1}(a_i^{n+1})) = b'_{-(n+1)+i+1} = b'_{-n+i} = \phi_n(a_i^n).
\]

Thus we have that \( \phi' = \cup \phi_n \) is the unique embedding from \((\Delta, \chi)\) into \((\Gamma'', \chi'')\) that extends \( \phi \) and such that \( \phi'(b_k) = b_k' \) for all \( k \in \mathbb{Z} \).

On the other hand, if \((\Gamma, \chi) \subseteq (\Gamma', \chi')\) are models of \( T_{pdg} \), \( \chi(\Gamma^{<\infty}) \neq \chi'(\Gamma^{<\infty}) \) and there is a nonempty special cut \( G \in \chi(\Gamma^{<\infty}) \) and \( b \in \chi(\Gamma^{<\infty}) \setminus \chi(\Gamma^{<\infty}) \) such that \( G < b < G^{>G} \), then there is a family \((b_k)_{k \in \mathbb{Z}} \) in \( \chi'(\Gamma^{<\infty}) \) such that \( G < b_k < G^{>G} \) and \( b_{k+1} = \chi'(b_k) \). So, in order to extend \((\Gamma, \chi)\) to a model \((\Delta, \chi)\) of \( T_{pdg} \) in which \( b \in \chi(\Delta^{<\infty}) \) we have to add a copy of \( b \) between some specific elements of \( \chi(\Gamma) \).

Lemma 15. Let \((\Gamma, \chi) \subseteq (\Gamma', \chi')\) be divisible centripetal precontraction groups, \( G \) be a nonempty special cut in \( \chi(\Gamma^{<\infty}) \) and \((b_k)_{k \in \mathbb{Z}} \) be a family in \( \chi'(\Gamma^{<\infty}) \) such that \( G < b_k < G^{>G} \) with \( b_{k+1} = \chi'(b_k) \), then there is a divisible centripetal precontraction group \((\Sigma, \chi)\) such that:

1. \((\Gamma, \chi) \subset (\Sigma, \chi) \subseteq (\Gamma', \chi')\) with \( b_k \in \chi(\Sigma^{<\infty}) \) for all \( k \in \mathbb{Z} \), and
2. for any embedding \( \phi \) of \((\Gamma, \chi)\) into a divisible centripetal precontraction group \((\Gamma'', \chi'')\) and any family \((b'_k)_{k \in \mathbb{Z}} \) in \( \chi'(\Gamma''^{<\infty}) \) such that \( b'_{k+1} = \chi'(b'_k) \) and \( \phi(G) < b'_k < \phi(G^{>G}) \), there is a unique embedding \( \phi' : (\Sigma, \chi) \to (\Gamma'', \chi'') \) which extends \( \phi \) and such that \( \phi'(b_k) = b'_k \) for all \( k \).

Proof. It is enough to take \( \Delta_G = G + \oplus_{k \in \mathbb{Z}} Qb_k \) and \( \chi_G \) the restriction of \( \chi' \) to \( \Delta_G \). Under the hypothesis of the above lemma, for any element \( a \in \Gamma^{<\infty} \setminus G \) we have \( b_k < a < 0 \) for all \( k \in \mathbb{Z} \), and by item \( 2 \) of lemma \([2]\) we obtain

\[
\chi'(b_k - a) = b_k.
\]

Thus, taking \( a_k = b_k - a \) we can define \( \Delta_G = G + \oplus_{k \in \mathbb{Z}} Qa_k \) and \( \chi_G \) the restriction of \( \chi' \) to \( \Delta_G \), with \( b_k \in \chi_G(\Sigma^{<\infty}) \).

4.3 Model completeness of \( T_{pdg} \)

To prove the model completeness of \( T_{pdg} \) we use the following result (see \([11]\) Corollary B.10.4.):

Lemma 16. The following are equivalent:

1. \( \Sigma \) is model complete;
2. for all models \( M, N \) of \( \Sigma \) with \( M \subseteq N \) and every elementary extension \( M^* \) of \( M \) that is \( k \)-saturated for some \( k > \text{card}(N) \), there is an embedding \( N \to M^* \) that extends the natural inclusion \( M \to M^* \).
Remark 1. Let $M, N, M^*$ be models of $\Sigma$ where $M \subseteq N$, $M \preceq M^*$ and $M^*$ is $k$-saturated for some $k > \text{card}(N)$. If we want to show that $\Sigma$ is model complete, by the last lemma and Zorn’s lemma, it is enough to show that there is a substructure $K$ of $N$ that properly contains $M$, is model of $\Sigma$ and embeds over $M$ in $M^*$.

Under such observation, the model completeness of $T_{pdg}$ is a consequence of the following theorem:

**Theorem 17.** Let $(\Gamma, \chi), (\Gamma', \chi')$ and $(\Gamma^*, \chi^*)$ be models of $T_{pdg}$, such that $(\Gamma, \chi) \subseteq (\Gamma', \chi')$ and $(\Gamma^*, \chi^*)$ is a $k$-saturated elementary extension of $(\Gamma, \chi)$, with $k > \text{card}(\Gamma')$. Then there is a submodel $(\Delta, \chi_\Delta)$ of $(\Gamma', \chi')$ which properly extends $(\Gamma, \chi)$ such that $(\Delta, \chi_\Delta)$ embeds over $(\Gamma, \chi)$ in $(\Gamma^*, \chi^*)$.

**Proof.** We call $\phi$ the embedding of $(\Gamma, \chi)$ into $(\Gamma^*, \chi^*)$ and just consider the following cases:

1. $[\Gamma] = [\Gamma']$: By lemma 10 it is enough to take $(\Delta, \chi_\Delta) = (\Gamma', \chi')$.

2. $[\Gamma] \neq [\Gamma']$ and $\chi(\Gamma) = \chi'(\Gamma')$: By hypothesis there is an element $a \in \Gamma^{<0}$ such that $[a] \notin [\Gamma]$, and by lemma 12 there is a model $(\Delta, \chi_\Delta) \subseteq (\Gamma', \chi')$ of $T_{pdg}$ that properly extends $(\Gamma, \chi)$. By saturation we can extend the embedding $\phi : (\Gamma, \chi) \rightarrow (\Gamma', \chi')$ to an embedding $\phi' : (\Delta, \chi_\Delta) \rightarrow (\Gamma^*, \chi^*)$.

3. $\chi(\Gamma^{<0}) \neq \chi'(\Gamma^{<0})$ and there is $b \in \chi(\Gamma^{<0})$ such that $b > \chi(\Gamma^{<0})$: We define the family $(b_k)_{k \in \mathbb{Z}}$ of $\chi(\Gamma^{<0})$ by $b_0 = b$, $b_{k+1} = \chi'(b_k)$ for $k > 0$ and $b_{k-1}$ as the unique element of $\chi'(\Gamma^{<0})$ such that $\chi(b_{k-1}) = b_k$ for $k > 0$, then in $\chi(\Gamma^{<0})$ there is a model $(\Delta, \chi_\Delta) \subseteq (\Gamma', \chi')$ of $T_{pdg}$ which properly extends $(\Gamma, \chi)$ and such that $b_k \in \chi_\Delta(\Delta^{<0})$.

Using the saturation of $(\Gamma^*, \chi^*)$ we can find a family $(b'_k)_{k \in \mathbb{Z}}$ in $(\Gamma^*, \chi^*)$ such that $b'_k > \phi(\chi(\Gamma^{<0}))$ for all $k \in \mathbb{Z}$ and $b_{k+1} = \chi^*(b_k)$. Thus, again by lemma 14 there is a unique embedding $\phi' : (\Delta, \chi_\Delta) \rightarrow (\Gamma^*, \chi^*)$

that extends $\phi$ and such that $\phi'(b_k) = b'_k$.

4. There is $b \in \chi'(\Gamma^{<0}) \setminus \chi(\Gamma)$ such that $b$ realize a special cut in $\chi(\Gamma^{<0})$: We define the set

$$G = \{ a \in \chi(\Gamma^{<0}) : a < b \}.$$

Since the models of $T_{pdg}$ are centripetal precontraction groups then we have that $\chi(G) \subseteq G$ and by axioms (3) and (4) there is a family $(b_k)_{k \in \mathbb{Z}}$ in $\chi'(\Gamma^{<0})$ such that $G \preceq b_k < \Gamma^{>G}$, $b_0 = b$, $b_{k+1} = \chi'(b_k)$ for $k > 0$ and $b_{k-1}$ is the unique element of $\chi'(\Gamma^{<0})$ such that $\chi(b_{k-1}) = b_k$ for $k > 0$ then in lemma 15 there is a model $(\Delta, \chi_\Delta) = (\Gamma, \chi_\Delta)$ of $T_{pdg}$ which properly extends $(\Gamma, \chi)$ and such that $b_k \in \chi_\Delta(\Delta^{<0})$.

By saturation there is a family $(b'_k)_{k \in \mathbb{Z}}$ in $(\Gamma^*, \chi^*)$ such that $\phi(G) < b'_k < \phi(\Gamma^{>G})$, $b_{k+1} = \chi^*(b_k)$ for all $k \in \mathbb{Z}$, and again by lemma 15 there is a unique embedding $\phi' : (\Delta, \chi_\Delta) \rightarrow (\Gamma^*, \chi^*)$ that extends $\phi$ and such that $\phi'(b_k) = b'_k$.

\[\square\]

**Corollary 18.** $T_{pdg}$ is model complete.

Now, we can observe that the model $(\oplus_i Qe_i, \chi_G)$ of $T_{pdg}$ defined in the first example of section 4 embeds in any model $(\Gamma, \chi')$ of $T_{pdg}$, since we can take any element $b \in \chi'(\Gamma^{<0})$, define the family $(b_n)_{n > 0}$ such that $b_1 = b$ and $b_{n+1} = \chi'(b_n)$, and identify the element $-e_n$ with the element $b_n$ for all $n \geq 1$. Thus we obtain that $T_{pdg}$ has a prime model and:

**Corollary 19.** $T_{pdg}$ is complete.
4.4 Quantifier elimination of $T_{pdg}$

Expanding the language $L_{pdg}$ to $L_{pdg^*} = L_{pdg} \cup \{ \infty, \chi^{-1}, \delta_1, \delta_2, \delta_3, \ldots \}$, where $\infty$ is a constant symbol, $\chi^{-1}$ and $\delta_n$ for $n > 0$ are unary function symbols, each model $(\Gamma, \chi)$ of $T_{pdg}$ can be seen as a $L_{pdg^*}$-structure with underlying set $\Gamma_\infty = \Gamma \cup \{ \infty \}$ in which:

- $\infty$ is such that $\infty \notin \Gamma$, $\infty + \infty = \chi(\infty) = -\infty = \infty$ and for all $x \in \Gamma$ we have $x + \infty = \infty$, and
- we interpret $\delta_n$ as division by $n$ and $\chi^{-1}$ as a function from $\Gamma_\infty$ to $\Gamma_\infty$ such that its restriction

$$\chi^{-1} : \chi(\Gamma_\infty)^{\geq c} \to \chi(\Gamma_\infty)$$

is the inverse of $\chi : \chi(\Gamma_\infty) \to \chi(\Gamma_\infty)^{\geq c}$, $\chi^{-1}(0) = 0, \chi^{-1}(c) = \infty$ and $\chi^{-1}(a) = \infty$ for all $a$ in $\Gamma_\infty \setminus \chi(\Gamma)$.

Thus, we define the theory $T_{pdg^*}$ as the $L_{pdg^*}$-theory whose models are the expansion of models of $T_{pdg}$.

Now, we observe that each $L_{pdg^*}$-substructure of a model of $T_{pdg^*}$ has a $T_{pdg^*}$-closure in the following sense:

**Lemma 20.** Let $(\Gamma, \chi)$ be a model of $T_{pdg}$ and $(\Gamma_0, \chi_0)$ be a $L_{pdg^*}$-substructure of $(\Gamma, \chi)$. There is a model $(\Gamma', \chi')$ of $T_{pdg^*}$ such that

1. $(\Gamma', \chi') \subseteq (\Gamma, \chi)$, and
2. $(\Gamma', \chi')$ can be embedded over $(\Gamma_0, \chi_0)$ into every model of $T_{pdg^*}$ which extends $(\Gamma_0, \chi_0)$.

**Proof.** If there is $a \in \Gamma_0$ such that $\chi(a) = c$, then in fact $(\Gamma_0, \chi_0)$ is a model of $T_{pdg^*}$ and we finish. Otherwise, there is $a \in \Gamma_\infty$ such that $\chi(a) = c$, so we define $\Gamma'$ as the divisible ordered abelian group generated by $\Gamma_0 \cup \{ a \}$, and $\chi' = \chi|_{\Gamma'}$. Thus, $(\Gamma', \chi')$ is a model of $T_{pdg^*}$.

Finally, given any model $(\Gamma', \chi')$ of $T_{pdg^*}$ which extends $(\Gamma_0, \chi_0)$, there is $b \in \Gamma'$ such that $\chi(b) = c$. We see that $a$ and $b$ have the same type over $\Gamma_0$. Thus, we define the embedding $\phi : (\Gamma', \chi') \to (\Gamma^*, \chi^*)$ as $\phi(\Gamma_0) = \Gamma_0$ and $\phi(a) = b$.

As a consequence of this lemma and mimicking the proof of the theorem [17] but considering $L_{pdg^*}$-structures instead of $L_{pdg}$-structures, we can prove that the $L_{pdg^*}$-theory $T_{pdg^*}$ has quantifier elimination.

4.5 Definable subsets of $\chi(\Gamma_\infty)$

In this section we mimic the study made by Gehret in [2] about some definable sets in the asymptotic couple of $\mathbb{T}_{\log}$ and show that given a model $(\Gamma, \chi)$ of $T_{pdg}$, each definable subset of $\chi(\Gamma_\infty)$ is a finite union of intervals in $\chi(\Gamma_\infty)$ and singletons. Specifically, to prove such result we will use a special kind of functions called $\chi$-functions.

Now, for any element $a \in \chi(\Gamma_\infty)$ and integer $k < 0$ we put $\chi^k(x) = (\chi^{-1})^{-k}(x)$ and $\chi^0(x) = x$.

**Definition 21.** We say that a function $G : \chi(\Gamma_\infty) \to \Gamma$ is a $\chi$-function if it is constant or

$$G(x) = \sum_{i=1}^{n} q_i \chi^{k_i}(x) - \alpha$$

for some $n > 0$, $k_1 < k_2 < \ldots < k_n$ in $\mathbb{Z}$, $q_1, \ldots, q_n \in \mathbb{Q}_{\neq 0}$ and $\alpha \in \Gamma$.

---

2 The notion of $\chi$-function used here was inspired in the notion of $\chi$-polynomial defined in [8].

3 The notion of $\chi$-function used here was inspired in the notion of $\chi$-polynomial defined in [S].
Since for each $k \in \mathbb{Z}^{<0}$, the $\chi$-function $\chi^k(x)$ has image $\infty$ for $x < \chi^{-k}(c)$ with $x \in \chi(\Gamma^{<0})$, and it is injective and strictly increasing in $\chi(\Gamma^{<0}) = \{x \in \chi(\Gamma^{<0}) : x \geq \chi^{-k}(c)\}$, then if for any $\chi$-function

$$G(x) = \sum_{i=1}^{n} q_i \chi^{k_i}(x) + \alpha$$

we define

$$\text{Dom}_G = \begin{cases} \chi(\Gamma^{<0})_{k_1} & \text{if } k_1 < 0 \\ \chi(\Gamma^{<0}) & \text{if } k_1 \geq 0 \end{cases}$$

then we have:

**Lemma 22.** Let $G : \chi(\Gamma^{<0}) \to \Gamma$ be the $\chi$-function given by $G(x) = \sum_{i=1}^{n} q_i \chi^{k_i}(x) + \alpha$, then

1. $G(a) = \infty$ for any $a \in \chi(\Gamma^{<0}) \setminus \text{Dom}_G$.
2. $G$ is injective on $\text{Dom}_G$.
3. If $q_1 > 0$ then $G(x)$ is strictly increasing on $\text{Dom}_G$, and if $q_1 < 0$ then $G(x)$ is strictly decreasing on $\text{Dom}_G$.

**Proof.**

1. If $k_1 < 0$ then $\chi^{k_1}(a) = \infty$ for all $a < \chi^{-k_n}(c)$. Now, if $k_n > 0$, then the proof is immediate.

2. If $x \in \text{Dom}_G$ then $\chi^{k_1}(x) < \chi^{k_2}(x) < \ldots < \chi^{k_n}(x)$. So, if $y, x \in \text{Dom}_G \subseteq \chi(\Gamma^{<0})$ with $y \neq x$, then $\chi^{k_i}(y) \neq \chi^{k_i}(x)$ for all $1 \leq i \leq n$, and by lemma 6 we have that $G(x) \neq G(y)$.

3. If $a, b \in \text{Dom}_G$ with $a < b$, then $[a] < [b]$, $\chi(a) < \chi(b)$ and by lemma 2 $\chi(a-b) = \chi(a)$. Thus, for all $i, j \in \mathbb{Z}$ with $i < j$ we have that

$$[\chi^i(b) - \chi^i(a)] > [\chi^j(b) - \chi^j(a)]$$

and then

$$[\chi^{k_1}(b) - \chi^{k_1}(a)] > [\chi^{k_2}(b) - \chi^{k_2}(a)] > \ldots > [\chi^{k_n}(b) - \chi^{k_n}(a)].$$

So, since $\chi^{k_1}(b) > \chi^{k_1}(a)$, we have that $G(b) - G(a) = \sum_{i=1}^{n} q_i (\chi^{k_i}(b) - \chi^{k_i}(a)) > 0$ if and only if $q_1 > 0$.

Since by lemma 6 we know that $\chi(\Gamma^{<0})$ is a linearly independent subset of $\Gamma$ as $\mathbb{Q}$-vector space, then depending on the constant value $\alpha$ we observe how many images has the restriction of the $\chi$-function

$$G(x) = \sum_{i=1}^{n} q_i \chi^{k_i}(x) - \alpha$$

to $\text{Dom}_G$ in $\chi(\Gamma^{<0})$:

**Lemma 23.** Given the $\chi$-function $G(x) = \sum_{i=1}^{n} q_i \chi^{K_i}(x) + \alpha$ then we have one of the following possibilities:

1. $\alpha = 0$, $n = 1$, $q_1 = 1$ and $G(\chi(\Gamma^{<0})) \subseteq \chi(\Gamma^{<0})$, or
2. $\text{card}(G(\text{Dom}_G) \cap \chi(\Gamma^{<0})) \leq 2$.

**Proof.** Considering the element $\alpha$ we have two main cases: $\alpha$ does not belongs to $\text{span}_\mathbb{Q} \chi(\Gamma^{<0})$ or $\alpha$ belong to $\text{span}_\mathbb{Q} \chi(\Gamma^{<0})$. In the first case, $G(x) \notin \chi(\Gamma^{<0})$ for all $x \in \text{Dom}_G$. In the second case we can assume that for some natural $m > 0$ there are $r_1, r_2, \ldots, r_m \in \mathbb{Q}$ and $a_1, a_2, \ldots, a_m \in \chi(\Gamma^{<0})$ with $a_1 < a_2 < \ldots < a_n$ such that $\alpha = r_1 a_1 + r_2 a_2 + \ldots + r_m a_m$. Clearly, if $x \in \text{Dom}_G$ then $G(x) \in \chi(\Gamma^{<0})$ if and only if $G(x) = \chi^{k_h}(x)$ for some $1 \leq h \leq n$ or $G(x) = a_s$ for some $1 \leq s \leq m$, which is possible only if all components except one of $G(x)$ are canceled. We analyze the possible cases:
Clearly if $G(x)$ and $H(x)$ are two $\chi$-functions then $G(x) + H(x)$, $G(x) - H(x)$ and $\delta_n(G(x))$ for all $n > 0$ are again $\chi$-functions. Thus

**Lemma 24.** The set of $\chi$-functions is closed under $+, -, \delta_n$.

On the other hand, although the composition $\chi(G(x))$ of $\chi$ and a $\chi$-function $G(x)$ is not necessarily a $\chi$-function, we can prove that $\chi(G(x))$ is given piecewise by $\chi$-functions (Lemma 25), which means that there are $a_1, a_2, ..., a_n \in \chi(\Gamma^0) \cup \{0\}$ for any element $\theta$ such that $|a_1 < a_2 < \ldots < a_n| = 0$ such that for any $i \in \{1, 2, ..., n-1\}$ there is a unique \(\lambda \in G(\delta_n(G(x)))\) of the form $\lambda = \sum_{i=1}^{n} q_i \chi^{k_i}(x)$ for all $x \in G(G(x))$ for all $x \in G(\delta_n(G(x)))$.

To prove this, we first observe that by Lemma 14 for any element $\theta = \sum_{i=1}^{n} q_i a_i$ of $\Gamma$ where $q_1, q_2, ..., q_n \in \mathbb{Q}^\neq 0$ and $a_1, a_2, ..., a_n \in \chi(\Gamma^0)$ with $a_1 < a_2 < \ldots < a_n$, we have that $\chi(\theta) = \chi(a_1)$ if $q_i > 0$ and $\chi(\theta) = -\chi(a_1)$ if $q_i < 0$. Thus we have:

**Lemma 25.** Let $G(x)$ be a $\chi$-function. Then $\chi(G(x))$ is given piecewise by $\chi$-functions.

**Proof.** If $G(x)$ is constant, then $\chi(G(x))$ is also a constant, which means that $\chi(G(x))$ is a $\chi$-function. And if $G(x) = \sum_{i=1}^{n} q_i \chi^{k_i}(x) + \alpha$ then clearly, for all $x \in \chi(\Gamma^0) \setminus \text{Dom}_G$ we have $\chi(G(x)) = \chi(\alpha) + \alpha$ which is constant. So, from now on $G(x)$ will be a $\chi$-function of the form $G(x) = \sum_{i=1}^{n} q_i \chi^{k_i}(x) + \alpha$ and we will focus on the values of $G$ on $\text{Dom}_G$.

If $\alpha = 0$ by the above lemma $\chi(G(x)) = \text{sign}(q_i)\chi^{k_i}(x)$ for all $x \in \text{Dom}_G$. Putting now $\alpha \neq 0$ and $\theta(x) = \sum_{i=1}^{n} q_i \chi^{k_i}(x)$ we have $\chi(G(x)) = \chi(\theta(x) + \alpha)$.

Without loss of generality we can assume $q_1 > 0$. Thus $\chi(\theta(x)) = \chi(\chi^{k_1}(x))$ for all $x \in \text{Dom}_G$, and there is a unique $x_0 \in \chi(\Gamma^0)$ such that $|\chi(\alpha)| = |x_0|$. Thus we have two possibilities:

1. $|\chi(\theta(x))| \neq |x_0|$ for all $x \in \text{Dom}_G$. If $\chi(\alpha) = x_0$ then either $x_0 < \chi(\theta(x))$ for all $x \in \text{Dom}_G$ and $\text{Dom}_G(x) = x_0$ for all $x \in \text{Dom}_G$, or there is a unique $x_1 \in \text{Dom}_G$ such that $\chi(\theta(x_1)) < x_0 < \chi(\theta(x_1))$ and $\chi(G(x)) = \chi(\chi^{k_1}(x))$ for all $x \in \text{Dom}_G$.

Now, if $\chi(\alpha) = -x_0$ then $\chi(G(x)) = \chi(\chi^{k_1}(x))$ for all $x \in \text{Dom}_G$. 
(2) There is a unique \( x_1 \in \text{Dom}_G \) such that \( |\chi(\theta(x_1))] = |x_0| \). We can see that \( \chi(G(x)) \) has the same behavior for all \( x \neq x_1 \) that in the previous case. However, if \( x = x_1 \) then we have the following cases: If \( \chi(\alpha) = x_0 \) then \( \chi(G(x)) = x_0 \), but if \( \chi(\alpha) = -x_0 \) then: Let \( \alpha = \alpha + q_1 \chi^{k_1}(x) \). If \( \chi(\alpha) = \chi(\chi^{k_1}(x)) \) then \( \chi(G(x)) = \chi(\chi^{k_1}(x)) \). In other case, we compare \( |\chi(\chi^{k_1}(x))| \) with \( |\chi(\alpha)| \). If \( |\chi(\chi^{k_1}(x))| \neq |\chi(\alpha)| \) then the value of \( G(x) \) is determined by the min\{sign\((q_2)\chi(\chi^{k_2}(x)), \chi(\alpha)\}\}. If \( |\chi(\chi^{k_2}(x))| = |\chi(\alpha)| \) then we have two cases, if \( \text{sign}(q_2)\chi(\chi^{k_2}(x)) = \chi(\alpha) \) then \( \chi(G(x)) = \text{sign}(q_2)\chi(\chi^{k_2}(x)) \), but if not, then we define \( \alpha_2 = \alpha + q_2 \chi^{k_2}(x) \) and repeat the analysis done for \( \alpha_1 \). This process is finite because in the possible last step we analyze \( \alpha_n = \alpha_{n-1} + q_n \chi^{k_n}(x) \).

In conclusion, for each \( \chi \)-function \( G(x) = \sum_{i=1}^n q_i \chi^{k_i}(x) + \alpha \), \( \chi(G(x)) \) is given piecewise by \( \chi \)-functions.

From lemmas 23, 24 and 25 we obtain:

**Proposition 26.** Let \( t(x) : \Gamma \rightarrow \Gamma \) be an \( L_{pdg^*} \)-term and \( G : \chi(\Gamma^{<0}) \rightarrow \Gamma \) the restriction of \( t \) to \( \chi(\Gamma^{<0}) \). Then \( G \) is given piecewise by \( \chi \)-functions.

**Proof.** The proof follows from lemmas 23, 24 and 25 doing induction on the complexity of the \( L_{pdg^*} \)-terms.

As a consequence of this proposition and the quantifier elimination in \( T_{pdg^*} \), we have:

**Corollary 27.** Every definable \( A \subseteq \chi(\Gamma^{<0}) \) is a finite union of intervals in \( \chi(\Gamma^{<0}) \) and singletons.

**Remark 2.** For each model \( (\Gamma, \chi) \) of \( T_{pdg^*} \), the definable set \( \chi(\Gamma^{<0}) \) is infinite and discrete, so \( (\Gamma, \chi) \) is not weakly \( \alpha \)-minimal.

Now, if we expand the language \( L_{pdg^*} \) by a new constant symbol \( d \) and define the theory \( T_{pdg^*} \) as

\[ T_{pdg^*} \cup \{ \chi(d) = c \} \]

then \( T_{pdg^*} \) has quantifier elimination and a universal axiomatization. Thus, from proposition 26 we have the following:

**Theorem 28.** Let \( G : \chi(\Gamma^{<0}) \rightarrow \Gamma \) be a definable function. Then \( G \) is given piecewise by \( \chi \)-functions.

**Proof.** Since \( T_{pdg^*} \) has quantifier elimination and has a universal axiomatization, then by corollary B.11.15 of [1] there are \( L_{pdg^*} \)-terms \( t_1(x), t_2(x), \ldots, t_n(x) \) such that \( G(x) = t_k(x) \) for \( x \in \chi(\Gamma^0) \) and some \( k \in \{1, 2, \ldots, n\} \). Thus, by proposition 26 the restriction of \( G(x) \) to

\[ \text{Dom}_k = \{ x \in \chi(\Gamma^{<0}) : G(x) = t_k(x) \} \subseteq \chi(\Gamma^0), \]

is given piecewise by \( \chi \)-functions.

### 4.6 Simple extensions

Let \( \mathbb{M} = (M, \chi_M) \) be a monster model of \( T_{pdg^*} \) and \( (\Gamma, \chi) \) a small submodel of \( \mathbb{M} \). In this section we show that each simple extension \( \Gamma(a) \) for \( a \in M \setminus \Gamma \) of \( \Gamma \) is isomorphic to a specific extension of \( \Gamma \) obtained utilizing the extensions given in lemmas 14 and 15.

To do that, first we will combine the lemmas 14 and 15 to define extensions of \( \Gamma \) which are built including many copies of \( \mathbb{Z} \) in a specific and ordered way. Specifically, if \( \text{scut}^{op}(\chi(\Gamma^{<0})) \) denote the linear ordered set of the elements \( G \subseteq \chi(\Gamma^{<0}) \) such that \( \chi(\Gamma^{<0}) \setminus G \) is a special cut of \( \chi(\Gamma^{<0}) \) and where \( G_1 \leq G_2 \) in \( \text{scut}^{op}(\chi(\Gamma^{<0})) \) if and only if \( G_2 \subseteq G_1 \), then given an ordinal \( \delta \) and an increasing
function $f : \delta \to \text{scut}^{op}(\chi(\Gamma^{<\alpha})) \setminus \{\chi(\Gamma^{<\alpha})\}$, for each $f(\alpha)$ with $\alpha < \delta$ we want to include a specific copy of $Z$ between $\chi(\Gamma^{<\alpha}) \setminus f(\alpha)$ and $f(\alpha)$. Moreover, if $\delta = \beta + 1$, it may happen that $f(\beta) = \emptyset$, which means that we have to include a copy of $Z$ at the end of $\chi(\Gamma^{<\alpha})$.

**Lemma 29.** Let $\delta$ be an ordinal. Given an increasing function $f : \delta \to \text{scut}^{op}(\chi(\Gamma^{<\alpha})) \setminus \{\chi(\Gamma^{<\alpha})\}$, there is a model $(\Gamma_f, \chi_f)$ of $T_{pdlg}$ and a family $(b_{k, \rho})_{k \in \mathbb{Z}, \rho < \delta}$ in $\chi(\Gamma^{<\alpha})$ such that:

1. $(\Gamma, \chi) \subset (\Gamma_f, \chi_f)$,
2. $\Gamma^{<f(\rho)} < b_{k, \rho} < f(\rho)$ and $\chi_f(b_{k, \rho}) = b_{k+1, \rho}$ for all $k \in \mathbb{Z}$, and $\rho < \delta$,
3. $b_{k_1, \rho_1} < b_{k_2, \rho_2}$ for all $k_1, k_2 \in \mathbb{Z}$ and $\rho_1 < \rho_2 < \delta$, and
4. for any embedding $\phi$ of $(\Gamma, \chi)$ into a model $(\Gamma^*, \chi^*)$ of $T_{pdlg}$ and any family $(b^n_{k, \rho})_{n \in \mathbb{Z}, k < \delta}$ in $\chi(\Gamma^{<\alpha})$ such that $\phi(\Gamma^{<f(\alpha)}) < b_{k, \rho} < \phi(f(\rho))$ and $\chi^*(b^n_{k, \rho}) = b^n_{k+1, \rho}$ for all $k \in \mathbb{Z}$, and $\rho < \delta$, and $b^n_{k_1, \rho_1} < b^n_{k_2, \rho_2}$ for all $k_1, k_2 \in \mathbb{Z}$ and $\rho_1 < \rho_2 < \delta$, there is a unique embedding $\phi'$ from $(\Gamma_f, \chi_f)$ into $(\Gamma^*, \chi^*)$ which extends $\phi$ and such that $\phi'(b_{k, \rho}) = b_{k, \rho}$ for all $k \in \mathbb{Z}$ and $\rho < \delta$.

**Proof.** The proof is by induction on $\delta$ and we only have to observe that for the successor step, if $\delta = \beta + 1$, then by inductive hypothesis there is an extension $(\Gamma_f, \chi_f, b_{k, \rho})$ of $(\Gamma, \chi)$ corresponding to $f|\beta : \beta \to \text{scut}^{op}(\chi(\Gamma^{<\alpha})) \setminus \{\chi(\Gamma^{<\alpha})\}$ and $f(\beta) \in \text{scut}^{op}(\chi(\Gamma^{<\alpha}))$.

Now, to study the simple extension $\Gamma(a)$ of $\Gamma$ with $a \in M \setminus \Gamma$, we consider first if $(\Gamma \oplus Qa)^{<\alpha}$ is closed under $\chi$ and to do that we use the set

$$\Delta_\Gamma = \chi((\Gamma + Qa^{<\alpha})^{<\alpha}) = \{\chi(x + qa) : q \in Q^{<\alpha}, x \in \Gamma \text{ and } x + qa < 0\}.$$

Specifically, we have the following results:

**Lemma 30.**

1. For all $x \in M^{<\alpha}$ and $y \in \Delta_\Gamma$ with $x < y$, $x \in \Delta_\Gamma$ if and only if $x \in \chi(M^{<\alpha})$.
2. For all $x \in \chi(M^{<\alpha})$ and $y \in \Delta_\Gamma \cap \chi(M^{<\alpha})$, if $x < y$ then $x \in \Delta_\Gamma \cap \chi(M^{<\alpha})$.
3. $\text{card}(\Delta_\Gamma \setminus \chi(M^{<\alpha})) < 1$.
4. If $\Delta_\Gamma \setminus \chi(M^{<\alpha}) = \{b\}$ with $b \in \chi(M^{<\alpha})$, then $b$ realize the cut

$$(\Delta_\Gamma \cap \chi(M^{<\alpha}) \setminus \chi(M^{<\alpha})) \setminus \Delta_\Gamma)$$

in $\chi(M^{<\alpha})$.

**Proof.** (1) Let $y = \chi(b + qa)$ for some $b \in \Gamma$ and $q \in Q^{<\alpha}$. If $x \in \Delta_\Gamma$, then $x = \chi(d + ra)$ for some $d \in \Gamma$ and $r \in Q^{<\alpha}$. Without loss of generality we can assume that $q, r > 0$. Thus, $x = \chi(d + r\frac{q}{r}a)$ and since $x < y < 0$ then

$$x = \chi(d + r\frac{q}{r}a) = \chi(d + r\frac{q}{r}a - (b + qa)) = \chi(d + b - b) \in \chi(M^{<\alpha}).$$

On the other hand, if $x \in \chi(M^{<\alpha})$ then $x = \chi(d)$ for some $d \in \Gamma^{<\alpha}$. Thus

$$x = \chi(d) = \chi(d + (b + qa)) = \chi((d + b) + qa) \in \Delta_\Gamma.$$

If $q < 0$ or $r < 0$, the demonstration is similar.
As a consequence of the above, we have two possibilities \( \chi(\Delta_T) \subseteq \Delta_T \) or \( \chi(\Delta_T) \setminus \Delta_T \neq \emptyset \). Hence it follows that:

**Corollary 31.** Exactly one of the following is true:

1. There is a nonempty special cut \( B \) in \( \chi(\Gamma) \) such that \( \Delta_T = B \).
2. There is \( b \in \chi(\Gamma) \) such that \( \Delta_T = (\chi(\Gamma)) \leq b \subseteq \chi(\Gamma) \).
3. There is a nonempty special cut \( B \) in \( \chi(\Gamma) \) and \( b \in \chi(M(\Gamma)) \setminus \chi(\Gamma) \) such that \( B < b \), \( b < (\chi(\Gamma) \setminus B) \) and \( \Delta_T = B \cup \{b\} \).

As a particular case, if \( \Delta_T \subseteq \chi(\Gamma) \) then the ordered divisible abelian subgroup \( \Gamma \oplus Qa \) of \( M \) is closed under \( \chi \) and \( \Gamma(a) = (\Gamma \oplus Qa, \chi) \). In general we have the following:

**Theorem 32.** If \( a \in M \setminus \Gamma \), then \( \Gamma(a) \) is isomorphic over \( \Gamma \) to one of the following:

1. \( \Gamma \) for some increasing function \( f : n \to \text{scut}(\Gamma) \setminus \{\chi(\Gamma)\} \) and some natural \( n \).
2. \( \Gamma \oplus Qa \) for some increasing function \( f : n \to \text{scut}(\Gamma) \setminus \{\chi(\Gamma)\} \) and some natural \( n \).
3. \( \Gamma \oplus Qa \) for some increasing function \( f : \omega \to \text{scut}(\Gamma) \setminus \{\chi(\Gamma)\} \).

**Proof.** The main idea of the proof is to construct by induction a chain \( 0 \subseteq \Gamma_0 \subseteq \Gamma_1 \subseteq \Gamma_2 \subseteq \ldots \subseteq \Gamma(a) \) of models of \( T_{pdg} \), in the model \( M \), each one isomorphic to \( \Gamma_f \) for some increasing function \( f : : n \to \text{scut}(\Gamma) \setminus \{\chi(\Gamma)\} \).

To do that, we put first \( \Gamma_0 = \Gamma \). Clearly, \( \Gamma_0 \) is isomorphic to \( \Gamma_f \) for \( f : 0 \to \text{scut}(\Gamma) \setminus \{\chi(\Gamma)\} \).

Assume we have built \( \Gamma_n \), with \( n \in \mathbb{N} \) and \( \Gamma_n \cong \Gamma_f \) for some increasing \( f : n \to \text{scut}(\Gamma) \setminus \{\chi(\Gamma)\} \):

Then we have two possibilities:

1. \( \Gamma_n = \Gamma(a) \), and then \( \Gamma(a) \cong \Gamma_f \).
2. \( a \notin \Gamma_n \). Thus we consider the set \( \Delta_{\Gamma_n} \) for \( \Gamma_n \), and we have other two cases:
   - \( \Delta_{\Gamma_n} \subseteq \chi(\Gamma_n) \). Thus, we put \( \Gamma_{n+1} = \Gamma_n \oplus Qa \). So, \( \Gamma(a) \cong \Gamma_f = \Gamma_{n+1} \) and \( \Gamma_f \cong \Gamma_f \oplus Qa \).
   - \( \chi(\Delta_{\Gamma_n}) \setminus \Delta_T \). Here, \( \Delta_{\Gamma_n} = B \cup \{b\} \) for some special cut \( B \subseteq \chi(\Gamma_n) \) and \( b \in \chi(M(\Gamma_n)) \setminus \chi(\Gamma) \) with \( B < b < (\chi(\Gamma) \setminus B) \). Thus, we define \( \Gamma_{n+1} \) as the model of \( T_{pdg} \) given by Lemma 15 by including the copy of \( Z \) corresponding to \( b \). Thus, there is \( g : n+1 \to \text{scut}(\Gamma) \setminus \{\chi(\Gamma)\} \) such that \( \Gamma_{n+1} \cong \Gamma_f \).

Now, if \( \Gamma(a) = \Gamma_n \) for some \( n \) we have finish. Otherwise, we put \( \Gamma(a) = \bigcup_n F_n \oplus Qa \). By construction, \( \Gamma(a) \cong \Gamma_f \oplus Qa \) for some increasing \( f : \omega \to \text{scut}(\Gamma) \setminus \{\chi(\Gamma)\} \).

**Example 3.** Let \( (\oplus_i \mathbb{Q}e_i, \chi \mathbb{Q}) \subseteq (\Gamma_{\log}, \chi) \) be the model of \( T_{pdg} \) considered in the first example of section 2. Let \( r \in \mathbb{R}^{<0} \) and \( a = re_m \in \Gamma_{\log} \oplus \oplus_i \mathbb{Q}e_i \), for some \( m \). Since for each \( b + qa \) in \( (\Gamma_{\log} + \mathbb{Q}^{\neq 0}a)^{<0} \) the entry \( m \) never is 0, then

\[
\Delta_{\Gamma_{\log}} = \chi(\Gamma_{\log} + \mathbb{Q}^{\neq 0}a)^{<0} = \{-e_i : 2 \leq i \leq m\} \subseteq \chi(\Gamma_{\log}^{<0}).
\]

Hence, \( (\oplus_i \mathbb{Q}e_i, \chi \mathbb{Q})(a) = (\oplus_i \mathbb{Q}e_i \oplus \mathbb{Q}a, \chi') \subseteq (\Gamma_{\log}, \chi) \), where \( \chi' \) is the restriction of \( \chi \) to \( \oplus_i \mathbb{Q}e_i \oplus \mathbb{Q} \).
(2) Let $(\Gamma, \chi)$ be a model of $T_{pdg}$ and $(\Gamma_f, \chi_f)$ be a fixed extension of $(\Gamma, \chi)$ for some increasing function 

$$f : n \to \text{scut}^{op}(\Gamma) \setminus \{\chi(\Gamma^{<0})\}$$

with $n \geq 1$. Let’s take one element $a_j \in (\text{span}_{\mathbb{Q}}(b_{k,j}))_{k \in \mathbb{Z}}^{\neq 0}$ for each $j < n$, where $(b_{k,j})_{k \in \mathbb{Z}}$ are the elements of the $j$-th copy of $\mathbb{Z}$ added to $\Gamma$ in $\Gamma_f$. Given $c \in \Gamma$ we define the element

$$a = c + \sum_{j=0}^{n-1} a_j \in \Gamma_f.$$ 

Thus, $\Gamma(a) = \Gamma_f$.

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