We introduce *Density sketches (DS)*: a succinct online summary of the data distribution. DS can accurately estimate point-wise probability density. Interestingly, DS also provides a capability to sample unseen novel data from the underlying data distribution. Thus, analogous to popular generative models, DS allows us to succinctly replace the real-data in almost all machine learning pipelines with synthetic examples drawn from the same distribution as the original data. However, unlike generative models, which do not have any statistical guarantees, DS leads to theoretically sound asymptotically converging consistent estimators of the underlying density function. Density sketches also have many appealing properties making them ideal for large-scale distributed applications. DS construction is an online algorithm. The sketches are additive, i.e., the sum of two sketches is the sketch of the combined data. These properties allow data to be collected from distributed sources, compressed into a density sketch, efficiently transmitted in the sketch form to a central server, merged, and re-sampled into a synthetic database for modeling applications. Thus, density sketches can potentially revolutionize how we store, communicate, and distribute data.

1 Introduction

The capability to sample from a distribution is a prerequisite for the classical task of statistical estimation, with innumerable applications. The popular Monte Carlo Estimation [1] algorithm uses sampling as the primary tool to approximate statistical quantities of interest. Data-driven machine learning is yet another example, which is essentially a statistical estimation problem to estimate a model from data.

Often, we use a set of samples (popularly known as the data) to specify the distribution. Sampling from a distribution described in such a way requires estimating the underlying distribution. Popular methods to infer the distribution and sample from it belong to the following three categories: 1. Parametric density estimation [2] 2. Non-parametric estimation - Histograms and Kernel Density Estimators (KDE) [3] 3. Learning-based approaches such as Variational Auto Encoders (VAE), Generative Adversarial Networks (GANs), and related methods [4]. Generally, parametric estimation is not suitable to model most real data as it can lead to large unavoidable bias from the choice of the model [3]. Learning the distribution, e.g., via neural networks is one solution to this problem. Although learning-based methods have recently found remarkable success, they do not have any theoretical guarantees for the distribution of generated samples. Histograms and KDEs, on the other hand, are theoretically well understood. These statistical estimators of density are known to uniformly converge to the underlying true distribution *almost surely*. In this paper, we focus on such estimators, which have theoretical guarantees.

**Our contribution**: Histograms and KDE are the most popular non-parametric density estimation methods. However, they are known to scale poorly with the dimension and size of data. In this work, we propose *Density Sketches (DS)* - a
Density Sketch as compressed surrogate for Data We propose Density Sketches (DS) constructed from data as a compressed alternative to keeping the data itself. Specifically, for statistical estimation tasks, a sample from the underlying distribution is sufficient for the vast majority of applications. In this regard, the original data is not sacrosanct - any sample from the underlying distribution may be used. DS can be used to efficiently sample the required amount of synthetic data from the underlying distribution. With increasing amounts of data, storage and transfer are important practical concerns of growing importance. DS provides an advantageous trade-off for applications. Our experiments show that with more data, the accuracy of the density sketch improves, but the size is practically unaffected. Our mathematical results show that size is dependent on the variety of the data rather than the volume. These properties make density sketches an appealing alternative to data.

Data Privacy and Sketching Fine-grained data collected from many individuals is necessary for many downstream estimation and modeling tasks. However, this could compromise the privacy of an individual. Maintaining the privacy of individuals in released data is an increasingly important topic of study. Data Anonymization is an essential operation towards making data release private, although, with weak privacy guarantees. As Density Sketches do not store the exact data, it is a possible way to release anonymized data. Differential privacy [6] provides better privacy guarantees for the released data. With some modifications on the lines of [7], the Density Sketch, which has similar properties to count based sketches, can be made differentially private.

Data Collection on Edge and Mobile Devices A significant portion of the data today is generated on Edge and Mobile devices. Frequent transfer of data between these devices and a central server is unavoidable because of their limited storage capacity. Density Sketches, because of their small size, offers a natural alternative to storing and transferring data on such devices. Density Sketch, being a sketching data structure, possesses the mergeable property. It implies that the density sketch of complete data can be achieved by composing density sketches of parts of data. This makes it possible to process data on edge devices to make sketches and combine them at a central location to obtain final Density Sketch.

2 Background

2.1 Count Sketches

Count sketch [8, 9], along with its variants, is one of the most popular probabilistic data structure used for the heavy hitter problem. Informally, the heavy hitter problem is to identify top frequent items in a stream of data. This setting can be generalized where each item in data-stream is a key and value pair. Then we can state the problem formally as, given a stream of data of type \((a_t, c_t)\) where \(a_t\) is a key belonging to an extensive set, say \(U\), and \(c_t\) is the value associated with \(a_t\) at time step \(t\). The goal is to output top keys with the most value \(c(a)\), where \(c(a)\) is the sum of all values associated with key \(a\) in the stream.

Count sketch [8, 9], along with its variants, is one of the most popular probabilistic data structure used for the heavy hitter problem. Informally, the heavy hitter problem is to identify top frequent items in a stream of data. This setting can be generalized where each item in data-stream is a key and value pair. Then we can state the problem formally as, given a stream of data of type \((a_t, c_t)\) where \(a_t\) is a key belonging to an extensive set, say \(U\), and \(c_t\) is the value associated with \(a_t\) at time step \(t\). The goal is to output top keys with the most value \(c(a)\), where \(c(a)\) is the sum of all values associated with key \(a\) in the stream.

Figure 1: Countsketch, sketching and query

\(U\) can be very large for certain scenarios, for example, to keep track of top queried web pages, the key is web page address which can go up to 100 characters (800 bits). In this case \(|U| = 2^{800}\). This prohibits use of array of size \(|U|\). Dictionary to store all keys is prohibitive for its update time and cost of storing all keys. Count sketch offers a probabilistic solution in memory logarithmic in \(|U|\). There is a standard memory accuracy trade-off for count sketches. Let \(m\) be the number of distinct keys and \(C\) be the vector of counts indexed by each key. For count median sketch [9], the \((\epsilon, \delta)\) guarantee

\[ P(|c(a) - c(a^*)| > \epsilon ||C||_2) \leq \delta \]

is achieved using \(O\left(\frac{1}{\epsilon^2} (\log m + \log |U|)\right)\) space. [10]. As can be seen from the above equation, the approximation accuracy for a particular key depends on how it compares to the ||C||_2. Specifically, count sketch can give very good approximation for keys with highest values in a setting where most of other keys have very low values.

The procedure of sketching and querying in illustrated in figure[1] Count sketch, parameterised by K and R, uses K independent pairs of hash functions \((h_i : U \rightarrow [0, R], g_i : U \rightarrow \{-1, 1\})\), \(i \in [1, K]\) uniformly drawn from a family of universal hash functions and a 2D array of size \(K \times R\), say A. While processing each element (insert operation), say \((a, c)\), for all \(i \in [1, K]\), we update \(A[i, h_i(a)]\) by adding \(g_i(a)c\) to it. To query value \(c(a)\), we get K unbiased estimates from each array by querying \(A[i, h_i(a)]\) for each
2.2 Histograms and Kernel Density Estimation

Histograms and KDEs are popular methods to estimate the density of a distribution given some finite i.i.d sample of size, say n, drawn from the true density, say f(x).

Histogram divides the support (S ⊂ Rd) of the data into multiple partitions. It then uses the counts in every partition to predict the density, f_H(x), at a point x. Formally the density predicted at the point x ∈ S is given by

\[ f_H(x) = \frac{c(bin(x))}{n \cdot volume(bin(x))} \]

bin(x) identifies the partition in which x lies, c(.) counts the number of samples in that partition, and volume(.) measures the partition volume. Regular Histogram uses hypercube partitions aligned with data axes with a width, say h. As h increases, the estimate’s bias increases, and its variance decreases. Histograms suffer from bin-edge problems where a slight change in data across the bin’s edge can change predictions. One solution to bin-edge problem is Average Shifted Histogram (ASH) [3]. It uses same bin partitions but with shifted origins. Consider a one dimensional histogram with width h. ASH with m histograms has origins at \(0, \frac{h}{m}, \frac{2h}{m}, \ldots, \frac{(m-1)h}{m}\). The density estimate is then the average of the density estimates obtained from the different histograms. Asymptotically as m goes to ∞, ASH converges to a KDE with the triangle kernel.

Kernel Density is another smoother estimate of f(x) which resolves the bin-edge problem of histograms. For a given kernel function k(x, y) : Rd × Rd → R and data, say D, the KDE at point, say x, is defined as

\[ f_K(x) = \frac{1}{n} \cdot \frac{1}{\sum_{i \in [1, n], x_i \in D} k(x, x_i)} \]

Kernel functions, generally, are positive, symmetric, and integrate to 1. Gaussian, Epanechnikov, Uniform, etc. are some of the most widely used kernels. A smoothing parameter h also parameterizes kernel function. It determines its variance for Gaussian kernel, while for uniform and Epanechnikov kernels, it determines the size of the window around x where the function is non-zero. Again, as h increases, the bias increases, and variance decreases.

Histogram and KDEs cannot provide unbiased estimates of the true density [12]. Hence, Mean Square Error (MSE) is used to analyze them. Both of these estimators uniformly converge to underlying true distribution asymptotically. However, both suffer from the curse of dimensionality. To get a decent estimate of density in high dimensions, the number of samples needed is exponential in dimensions. Generally, for the Density Estimation task, dimensions of 4-50 are considered large enough [13].

2.3 Randomly Partitioned Histograms and Kernel Density Estimates

The density estimate using histogram of a randomly drawn partition is

\[ f(x) \propto \frac{1}{n} \sum_{i \in [1, n], x_i \in D} I(x_i \in bin(x)) \]

This estimate of the density has a expected value (over random partitions) which looks very similar to a KDE. The subscript p in Expectation is to make it explicit that expectation is over the random partitions.

\[ E_p(f(x)) \propto \frac{1}{n} \sum_{i \in [1, n], x_i \in D} P(x_i \in bin(x)) \]

The kernel of this KDE is the probability of collision between the query point x and the data point x_i. For example, randomly shifted regular histograms can approximate triangle kernel or asymptotic ASH. Of course, to get a reasonable estimate, we would need to make multiple random partitions and combine estimates obtained from them. Depending on the different partitioning schemes, we can obtain estimators for different kernels. In a recent paper [14], authors observe this connection. They show using different lsh functions, it is possible to approximate corresponding interesting kernels. It is useful to note here that if we use 11-lsh or 12-lsh functions [15], the partition is a grid of parallelepipeds of random shape.

2.4 Uniform Sampling from Convex polytopes

Uniform Sampling from Convex spaces is a well-studied problem [16, 17]. For general convex polytopes, this is achieved by finding a point inside the polytope using convex feasibility algorithms and then running an MCMC walk inside the polytope to generate a point with uniform probability. In the case of regular convex polytopes like hypercubes and parallelopiped, uniform sampling is much simpler. Sampling a data point at random in a d-dimensional hypercube of width 1 is equivalent to sampling d real values uniformly in the interval [0, 1]. For sampling within a d-dimensional parallelopiped, we first locate (d – 1)-dimensional hyperplane parallel to each face at a distance drawn uniformly from [0, h] where h is the width of parallelopiped in that direction. The sampled point is then the intersection of these (d – 1) dimensional hyperplanes.

3 Density Sketches

3.1 Notation

Data D consists of n i.i.d samples of dimension d drawn from true distribution f(x) : Rd → R. bin(x): ID of the partition in which point x falls. In case of regular histograms, RACE style partitioning, bin(x) : Rd → N^d and each bin can be identified with a unique tuple of d integers. For example, in regular histogram with width h, bin(x)_i = \lfloor x_i / h \rfloor. Shapes of partitions in the case of regular histograms or RACE style partitioning are regular, and hence simple algorithms can be used for sampling a point inside it. bin(x) and sampling algorithm for
some partitioning schemes is mentioned in Table 1.

| Partitioning Scheme | parameters | bin(x) ∈ N^d, x ∈ Rd | Sample s from bin_ad |
|---------------------|------------|-----------------------|---------------------|
| Regular Histogram   | width: h   | bin(x)_i = \left[ \frac{x_i}{h} \right] | r ∈ R^d, r_i ∼ Uniform[0, 1] |
| Aligned Histogram   | widths: h = (h_1, h_2, ..., h_d) | bin(x)_i = \left[ \frac{x_i}{h_i} \right] | s_i = h_i (bin_ad + r_i) |
| Random Partitions using d 1/2 lsh functions | W ∈ R^{d×s}, W = (w_1, w_2, ..., w_d) b : R^{s×1}, b = (b_1, b_2, ..., b_d) | width: h | bin(x)_i = \left[ \frac{x_i + w_d}{h} \right] |

Table 1: bin(x) for different partitioning schemes

where c(bin(x)) is the count of data points that lie in bin(x).

When using the sketch, instead of using actual c(bin(x)), we would use the estimate of c(bin(x)) from the count sketch. Let this estimate be \hat{c}(bin(x)). Then we can write the density predicted using countsketch as \hat{f}_C(x)

\[ \hat{f}_C(x) = \frac{\hat{c}(bin(x))}{n\cdot volume(bin(x))} \]

We know from countsketch literature, that \hat{c}(x) is closely distributed around c(x) and so we can expect \hat{f}_C(x) to be close to \hat{f}_H(x) and hence to f(x). Note that though \hat{f}_C(x) is a good estimate of density at a point x, the function \hat{f}_C(.) is not a density function as it does not integrate to 1.

Algorithm 1: Constructing density sketch of f(x)

Result: Density Sketch

f(x) : Rd → R : true distribution
x_1, x_2, ..., x_n ∼ f(x) : sample drawn from f(x)
bin(x) : Rd → N^d : bin to which x belongs
mem : CountSketch(R, K) : sketch with R-range, K-repetitions
heap : Heap(H): min-heap to store top H elements

for i ← 1 to n do
  bin_id = bin(x_i)
  mem.insert(bin_id, 1)
  c = mem.query(bin_id)
  heap.update(bin_id, c)
end

Algorithm 2: query \hat{f}_C(y), y ∈ Rd

Result: \hat{f}_C(y)
y ∈ Rd
bin_id = bin(y)
count = mem.query(bin_id)
return \frac{\text{count}}{\text{volume(bin_id)}}

3.4 \hat{f}_C(x): Estimate of density Function

In order to obtain a density function from the sketches, we have to normalize the function \hat{f}_C(x) over the support. We can write \hat{f}_C(x) as

\[ \hat{f}_C(x) \propto \hat{c}(x) \]

\[ \hat{f}_C(x) = \frac{\hat{c}(x)}{\int \hat{c}(x)dx} \]
with probability proportional to the count of elements in As is clear from the equations for \( \hat{\Sigma} \) This is essentially the reason why we needed a count sketch. The count sketch is a good enough representation for query-
in 1, we can maintain top \( H \) partitions efficiently with an

data points and discarding other partitions. As mentioned
in the first place. Here, we further approximate the distri-
bution by storing only top \( H \) partitions which contain most

exponential in dimension, sampling a bin proportional to
use exact counts and \( \hat{f}_C(x) \), is replaced by \( \hat{n} = \Sigma_{b \in \text{bins}} \hat{c}(b) \) to get a
density function. We can check that \( \hat{n} \) is an estimate of
n using estimate of count for each bin from the density

3.5 \( \hat{f}_S(x) \): Sampling from Density Sketches

The count sketch is a good enough representation for querying
the density at a point. However, it is not the best data
structure to efficiently generate samples. One naive way of
sampling from these sketches is to randomly select a point in
support of \( f(x) \) and then do a rejection sampling using estimate \( \hat{f}_C(x) \). However, given the enormous volume of
support in high dimensions, this method is bound to be
immensely inefficient. Another way is to choose a partition with
probability proportional to the count of elements in that
partition and then sample a random point from this
chosen partition. It is easy to check that the probability of
sampling a point \( x \) in this manner, precisely, is \( f_H(x) \) if we
use exact counts and \( \hat{f}_C(x) \) if we use approximate counts
from count sketch. However, given that number of bins is
exponential in dimension, sampling a bin proportional to its
counts requires prohibitive memory and computation.
This is essentially the reason why we needed a count sketch in
the first place. Here, we further approximate the distribu-
tion by storing only top \( H \) partitions which contain most
data points and discarding other partitions. As mentioned
in [1] we can maintain top \( H \) partitions efficiently with an
augmented heap. We then sample a partition present in
this heap with probability proportional to its count and
sample a random data point from this partition (Algorithm
3). The probability of sampling a data point whose bin is
not present augmented heap is then zero. The distribution
of this sampling algorithm is,

\[
\hat{f}_S(x) = \frac{\hat{c}(x)}{\text{volume}(\text{bin}(x))} 
\]

As is clear from the equations for \( \hat{f}_C(x) \) and \( \hat{f}_S(x) \), \( n = \Sigma_{b \in \text{bins}} c(b) \), is replaced by \( \hat{n} = \Sigma_{b \in \text{bins}} \hat{c}(b) \) to get a
density function. We can check that \( \hat{n} \) is an estimate of
n using estimate of count for each bin from the density

4 Analysis

Histogram and Kernel Density Estimators are well studied
non-parametric estimators of density. Both of these
estimators are shown to be capable of approximating a large
class of functions [3]. For example, with the condition of
Lipschitz Continuity on \( f \), we can prove that point wise
\( \text{MSE}(\hat{f}_H(x)) \) converges to 0 at a rate of \( O(n^{-2/3}) \).
Better results can be obtained for functions that have continuous
derivatives. In our analysis we make all such assumptions on
the lines of those made in [3], specifically, existence and boundedness of all function dependent terms that appear
in the theorems below. We refer reader to [3] for in
depth discussion on assumptions.

We restrict our analysis to convergence in probability for
all the estimators discussed in this paper which is standard
[3]. In this section we consider the regular histogram
partitioning scheme and show that our density estimates
\( \hat{f}_C(x) \) and sampling distribution \( \hat{f}_S(x) \) are approximations
of underlying distribution \( f(x) \) and converge to it. However,
similar analysis holds even for random partitioning schemes / KDE and is skipped here.
Mean integrated square error (MISE): MISE of an estimator of function is a widely used tool to analyse the performance of a density estimator.

\[ MISE = E \int (\hat{f}(x) - f(x))^2 dx \]

A density estimator, with MISE asymptotically tending to 0, is a consistent estimator of true density and converges to it in probability. We would use this tool to make statements about convergence of our estimators. By Fubini’s theorem, MISE is equal to IMSE (Integrated mean square error). Also, for any estimator \( \hat{c} \), \( MSE(\hat{c}) = Variance(\hat{c}) + Bias(\hat{c})^2 \). Hence we can write IMSE (and hence MISE) as sum of integrated variance (IV) and integrated square bias (ISB).

\[ IMSE(\hat{f}) = \int E((\hat{f}(x) - f(x))^2 dx = IV + ISB \]

The remaining section is organized as follows. We first state the main theorem in our paper which relates the Sampling probability of Density Sketches, \( \hat{f}_S(x) \), to the underlying true distribution. We then state various theorems that interrelate the different density function estimates used in this paper. The combination of all these theorems lead to our main theorem. The proofs of all the theorems can be found in Appendix A. We then briefly interpret different theorems in a small subsection.

**Theorem 1 (Main Theorem)** \( \hat{f}_S(x) \) to \( f(x) \). The probability density function of samplings, \( \hat{f}_S(x) \), using a Density Sketch over regular histogram of width \( h \), with parameters \( K, R, H \) created with \( n \) i.i.d samples from original density function \( f(x) \), has an IMSE :

\[ IMSE(\hat{f}_S(x)) \leq 12(1 - \text{ratio}_h)^2 + 3(1 + 2\epsilon)(\frac{1}{nh^d} + \frac{R(f)}{n} + o(\frac{1}{n})) + \frac{\#bins - 1}{KRnh^d} + 3(1 + 3\epsilon)\frac{h^2d}{4}R(\|\nabla f\|_2) + \frac{3\epsilon(1 + 2R(f) + h\sqrt{d}\int_{x \in S}(f(x)\|\nabla f\|_2))}{\#bins} \]

with probability \((1 - \delta)\), where \( \delta = \frac{\#bins}{e\#nR} \), \#bins is the number of non-empty bins in histogram, ratio\(_h\) is the estimated capture ratio as described in section 3.5.

The dependence of IMSE on properties of \( f(x) \), such as roughness, is standard [3] and cannot be avoided.

**Interpretation** The estimator \( \hat{f}_S(x) \) of \( f(x) \) is obtained by a series of approximations from \( f(x) \rightarrow \hat{f}_H(x) \rightarrow \hat{f}_C(x) \rightarrow \hat{f}_S(x) \rightarrow f_S(x) \). Hence in order to interpret this result, we break down the result above into multiple theorems enabling the reader to easily notice which step of approximations lead to what terms in the theorem above. We urge readers to read the interpretation of each theorem below to obtain a complete picture of the Main theorem.

**Theorem 2.** \( \hat{f}_H(x) \) to \( f(x) \) The IMSE for the estimator \( \hat{f}_H(x) \) using regular histogram with width \( h \) built over \( n \) i.i.d samples drawn from true distribution \( f(x) \), is

\[ IMSE(\hat{f}_H) \leq \frac{1}{nh^d} + \frac{R(f)}{n} + o(\frac{1}{n}) + \frac{h^2d}{4}R(\|\nabla f\|_2) \]

Specifically, its IV is \((\frac{1}{nh^d} + \frac{R(f)}{n} + o(\frac{1}{n}))\) and ISB is bounded by \((\frac{h^2d}{4}R(\|\nabla f\|_2))\) where \( R(\phi) \) is the roughness of the function \( \phi \) as defined as \( R(\phi) = \int_{x \in S} \phi(x)^2 dx \)

**Interpretation** : Theorem 2 applies to all the functions for which we can apply Taylor series expansion up to 2 terms and roughness terms \( R(f) \) and \( R(\|\nabla f\|_2) \) exist and are bounded. It is clear that if \( h \to 0 \) and \( nh^d \to \infty \), then \( IMSE \to 0 \), which implies that \( \hat{f}_H(x) \) converges to \( f(x) \) asymptotically. As \( nh^d \to \infty \), \( n \) should grow at a rate faster than reduction of \( h^d \). This is exactly the curse of dimensionality.

**Theorem 3.** \( \hat{f}_C(x) \) to \( f(x) \) The IMSE of estimator \( \hat{f}_C(x) \) obtained from the Density Sketch with parameters \( K, R, H \) using histogram of width \( h \) built over \( n \) i.i.d samples drawn from true distribution \( f(x) \) is

\[ IMSE(\hat{f}_C) = IMSE(\hat{f}_H) + \frac{\#bins - 1}{KRnh^d} \]

where \#bins is the number of non-zero bins in histogram. Specifically, its IV is \( IV(\hat{f}_C) = IV(\hat{f}_H) + \frac{\#bins - 1}{KRnh^d} \) and ISB is bounded by \( ISB(\hat{f}_C) = ISB(\hat{f}_H) \)

**Interpretation** : Note that though \( \hat{f}_C(x) \) is not a density estimator as \( \int \hat{f}_C(x) \neq 1 \), it is still useful to look at the IMSE for this function. It is clear from the above theorem, if \( \frac{\#bins}{KR} \) is bounded, then as \( h \to 0 \) and \( nh^d \to \infty \), IMSE(\( \hat{f}_C \)) tends to 0 and the \( \hat{f}_C(x) \) converges to \( f(x) \) simultaneously at all points in probability. As expected, using count sketches adds to the variance of the estimator while keeping the bias unchanged. The term \( \frac{\#bins - 1}{KRnh^d} \) gives the memory accuracy trade-off here, as it compares the number of keys being inserted into sketch versus the Density Sketch memory (=KR) excluding heap.

**Theorem 4.** \( \hat{f}_S(x) \) to \( f(x) \) The IMSE of estimator \( \hat{f}_S(x) \) obtained from the Density Sketch with parameters \( K, R, H \) using histogram of width \( h \) built over \( n \) i.i.d samples drawn from true distribution \( f(x) \) is

\[ |IMSE(\hat{f}_S) - IMSE(\hat{f}_C)| \leq \epsilon(N + 2M) \]

Specifically, \( |IV(\hat{f}_S) - IV(\hat{f}_C)| \leq 2\epsilon M \)

\( |ISB(\hat{f}_S) - ISB(\hat{f}_C)| \leq \epsilon N \)

where, \( N = (1 + ISB(\hat{f}_C)) \)

\[ M = IV(\hat{f}_C) + 2R(f) + (h^2d/4)R(\|\nabla f\|_2) + h\sqrt{d}\int_{x \in S}(f(x)\|\nabla f\|_2) \]

with probability \((1 - \delta)\), where \( \delta = \frac{\#bins}{e\#nR} \)
Table 2: Classification Datasets from [18] with dimension $< 500$, number of samples per class $> 100,000$

| Dataset          | Dimension | Samples | zipped size |
|------------------|-----------|---------|-------------|
| skin/nonskin     | 3         | 180K    | 670KB       |
| susy             | 18        | 4.5M    | 0.8GB       |
| higgs            | 28        | 10M     | 2.5GB       |
| webspam(unigram) | 254       | 300K    | 120MB       |

Figure 3: Visualization of samples drawn from Density Sketch

5 Experiments

5.1 Visualization of samples from density sketches

Figure 3(a) shows the samples drawn from the actual multi Gaussian distribution, whereas figure 3(b) shows the samples drawn from the Density Sketch(DS) built on the samples from the true distribution. As can be seen, the two samples are indistinguishable. Figure 3(c) shows some samples drawn from DS built over the "MNIST" dataset [18]. It can be seen that DS give sensible samples which can be identified as digits by the naked eye for different widths of partitions. In these experiments, we used l2-lsh random partitioning.

5.2 Evaluation of Samples on Classification Tasks

For most datasets, it is not possible to inspect samples visually. Hence we evaluate the quality of samples from DS by using them to train classification models.

Datasets: To perform a fair evaluation, we chose all datasets from the liblinear website [18], which satisfy the constraints of 1) data dimension less than 500 and 2) the number of samples per class greater than 100,000. Large Datasets is the main application domain for DS. The datasets such obtained are noted in the table 2. We use l2-lsh random partitions with 0.01 bandwidth.

Results: As can be seen, for "Higgs" dataset, the accuracy of the model achieved on original data of size 2.5GB, can be achieved by using a DS of size 50MB. So we get around 50x compression! We see similar results for datasets of "skin" (200x compression, 670KB) and "susy" (150x compression, 0.8GB) as well. The results show that DS is much more informative than Random Sample. The dimension of Webspam data is 254. The DS for this dataset is at a disadvantage due to Curse of Dimensionality : Number of datapoints required for good estimates is exponential in dimension. See Discussion 6 for qualitative discussion on this aspect. Due to these reasons the performance of DS is comparatively poor in "Webspam" dataset although it still beats Random Sampling in this experiment.

Other Baselines: We consider comparing against other sophisticated baselines. However, none are comparable to
DS in their purpose/utility. (1) Coresets for sampling would first require KDE estimation, which has a huge memory cost to construct the point set. Also, despite recent progress towards coresets in streaming setting [19], coresets remain challenging to implement for KDE problems [20]. (2) Algorithms like K-means clustering are inappropriate for streaming datasets and don’t have mergeability property of DS. An alternative approach is to select points based on importance sampling [20], geometric properties [21], and other sampling techniques [22]. However, recent experiments show that for many real-world datasets, random samples have a competitive performance with these approaches [14]. Dimensionality reduction via random projections is another way to reduce the size, however, given the relatively small dimension. It is unlikely to work well.

5.3 Estimation of statistical properties of dataset

We evaluate the DS on another task of evaluating different properties of the distribution on the same Datasets. Specifically, in this section, we show that samples drawn from the DS can be used effectively to estimate the Covariance Matrix of the underlying distribution. See Figure 4 for plots. The prediction from DS is superior to Random Sample except for "Webspam". Again, due to high dimensional data, DS do not perform that well. In this case it is worse than Random sample.

5.4 Manifestation of Theory in Experiments

As can be seen from theory, a lot of factors affect performance of DS such as $nh^d$, $ratio_h$ (#bins/KR) among others. In our experiments, we see that "Webspam" as higher $d$ and smaller $n$ as compared to, say, "higgs", hence the quality of histogram is bad for Webspam which translates to DS as well. Another factor that affects sampling distribution is capture ratio $ratio_c$, we observe that data is more scattered in "Webspam", hence is $ratio_c$ is worse than other datasets adversely affecting results.

6 Discussion: Curse of Dimensionality

In the context of density estimators, the curse of dimensionality implies (1) that the number of data points required to get decent estimates of density increases exponentially with dimension. (2) The number of bins in histograms is exponential. As Density Sketches are built over Histograms, they inherit this curse from Histograms. With increased data collection, the issue of the unavailability of large amounts of data is fast vanishing. We want to emphasize that Density Sketches’ advantages are best seen when data is humongous. Density Sketches can absorb tons of data and give better density estimates and samples without increasing their memory usage. Also, most real data in high dimensions is clustered or stays on a low-dimensional manifold. Density Sketches, throw away empty bins, and only store the histogram’s populated bins. Density Sketches can deal with the curse of dimensionality better than Histograms.

7 Conclusion

We introduce Density sketches (DS[1]) a sketch summarizing density from data samples. With very tiny size, streaming nature, sketch properties and capability to sample, DS has to potential to change the way we store, communicate and distribute data.

\[1\] We provide code for Density Sketches in Supplementary
References

[1] George Fishman. Monte Carlo: concepts, algorithms, and applications. Springer Science & Business Media, 2013.

[2] Jerome Friedman, Trevor Hastie, Robert Tibshirani, et al. The elements of statistical learning, volume 1. Springer series in statistics New York, 2001.

[3] David W Scott. Multivariate density estimation: theory, practice, and visualization. John Wiley & Sons, 2015.

[4] Ian J Goodfellow, Jean Pouget-Abadie, Mehdi Mirza, Bing Xu, David Warde-Farley, Sherjil Ozair, Aaron Courville, and Yoshua Bengio. Generative adversarial networks. arXiv preprint arXiv:1406.2661, 2014.

[5] Ian Goodfellow, Yoshua Bengio, Aaron Courville, and Yoshua Bengio. Deep learning, volume 1. MIT press Cambridge, 2016.

[6] Cynthia Dwork, Aaron Roth, et al. The algorithmic foundations of differential privacy. Foundations and Trends in Theoretical Computer Science, 9(3-4):211–407, 2014.

[7] Graham Cormode, Magda Procopiuc, Divesh Srivastava, and Thanh TL Tran. Differentially private publication of sparse data. arXiv preprint arXiv:1103.0825, 2011.

[8] Graham Cormode and M Muthukrishnan. Count-min sketch., 2009.

[9] Moses Charikar, Kevin Chen, and Martin Farach-Colton. Finding frequent items in data streams. In International Colloquium on Automata, Languages, and Programming, pages 693–703. Springer, 2002.

[10] Amit Chakrabati. https://www.cs.dartmouth.edu/~ac/Teach/data-streams-lecnotes.pdf, volume 1. 2020.

[11] DW Scott and SR Sain. Multi-dimensional density estimation handbook of statistics vol 23 data mining and computational statistics ed cr rao and ej wegman, 2004.

[12] Richard A Davis, Keh-Shin Lii, and Dimitris N Politis. Remarks on some nonparametric estimates of a density function. In Selected Works of Murray Rosenblatt, pages 95–100. Springer, 2011.

[13] Zhipeng Wang and David W Scott. Nonparametric density estimation for high-dimensional data—algorithms and applications. Wiley Interdisciplinary Reviews: Computational Statistics, 11(4):e1461, 2019.

[14] Benjamin Coleman and Anshumali Shrivastava. Sub-linear race sketches for approximate kernel density estimation on streaming data. In Proceedings of The Web Conference 2020, pages 1739–1749, 2020.

[15] Trevor Darrell, Piotr Indyk, and Gregory Shakhnarovich. Nearest-neighbor Methods in Learning and Vision: Theory and Practice. MIT Press, 2005.

[16] Claude JP Bélisle, H Edwin Romeijn, and Robert L Smith. Hit-and-run algorithms for generating multivariate distributions. Mathematics of Operations Research, 18(2):255–266, 1993.

[17] Yuansi Chen, Raaz Dwivedi, Martin J Wainwright, and Bin Yu. Vaidya walk: A sampling algorithm based on the volumetric barrier. In 2017 55th Annual Allerton Conference on Communication, Control, and Computing (Allerton), pages 1220–1227. IEEE, 2017.

[18] Chih-Chung Chang and Chih-Jen Lin. Libsvm : a library for support vector machines. In Software available at http://www.csie.ntu.edu.tw/ cjlin/libsvm. ACM Transactions on Intelligent Systems and Technology, 2011.

[19] Jeff M Phillips and Wai Ming Tai. Near-optimal coresets of kernel density estimates. Discrete & Computational Geometry, 63(4):867–887, 2020.

[20] Moses Charikar and Paris Siminelakis. Hashing-based-estimators for kernel density in high dimensions. In 2017 IEEE 58th Annual Symposium on Foundations of Computer Science (FOCS), pages 1032–1043. IEEE, 2017.

[21] Efren Cruz Cortes and Clayton Scott. Sparse approximation of a kernel mean. IEEE Transactions on Signal Processing, 65(5):1310–1323, 2016.

[22] Yutian Chen, Max Welling, and Alex Smola. Super-samples from kernel herding. arXiv preprint arXiv:1203.3472, 2012.

[23] Dimitris Achlioptas. Database-friendly random projections: Johnson-lindenstrauss with binary coins. Journal of computer and System Sciences, 66(4):671–687, 2003.
8 Appendix A

8.1 Theorem 2: \( F_H \) to \( f \)

While estimating true distribution \( f(x) : R^d \to R \), the integrated mean square error (IMSE) for the estimator \( \hat{f}_H(x) \) using regular histogram with width \( h \) and number of samples \( n \), is

\[
IMSE(\hat{f}_H) \leq \frac{1}{nh^d} + \frac{R(f)}{n} + o\left(\frac{1}{n}\right) + \frac{h^2d}{4}R(\|\nabla f\|_2)
\]

Specifically, its

\[
IV = \frac{1}{nh^d} + \frac{R(f)}{n} + o\left(\frac{1}{n}\right)
\]

and

\[
ISB \leq \frac{h^2d}{4}R(\|\nabla f\|_2)
\]

where \( R(\phi) \) is the roughness of the function \( \phi \) defined as \( R(\phi) = \int \phi(x)^2 dx \)

**Proof.** Let \( x \in S \). \( S \) is the support of the distribution. The estimator \( \hat{f}_H(x) \) is defined as, where \( V(x) \) is volume of bin in which \( x \) lies. Equivalently, we can also use \( V(b) \) to denote volume of bin \( b \). For standard histogram, \( V(x) = h^d \)

\[
\hat{f}_H(x) = \frac{1}{nV(bin(x))} \sum_{i=1}^{n} I(x_i \in bin(x))
\]

(1)

First let us consider the integrated variance.

\[
IV = \int_{x \in S} Var(\hat{f}_H(x)) dx = \sum_{b \in bins(S)} \int_{x \in b} Var(\hat{f}_H(x)) dx
\]

(2)

For a particular bin \( b \), the variance is constant at all values of \( x \). Also for a particular \( x \) in bin \( b \), we can write the following for \( Var(\hat{f}_H(x)) \) using independence of samples.

\[
Var(\hat{f}_H(x)) = \frac{1}{nV(bin(x))^2} Var(I(x_i \in bin(x))
\]

(3)

Also \( Var(I(x_i \in b)) = p_b(1 - p_b) \) where \( p_b \) is the probability of \( x_i \) lying in bin \( b \). That is, \( p_b = \int_{x \in b} f(x) dx \)

Using this in equation 2

\[
IV = \sum_{b \in bins(S)} V(b) \frac{1}{nV(bin(x))^2} p_b * (1 - p_b)
\]

(4)

Simplifying,

\[
IV = \sum_{b \in bins(S)} \frac{1}{nV(bin(x))} p_b * (1 - p_b)
\]

(5)

For standard histogram \( V(b) \) is same across bins,

\[
IV = \frac{1}{nV(bin)} (\Sigma_{b \in bins(S)} p_b - \Sigma_{b \in bins(S)} p_b^2) = \frac{1}{nV(bin)} (1 - \Sigma_{b \in bins(S)} p_b^2)
\]

(6)

Using mean value theorem, we can write, \( p_b = V(b)f(\xi_b) \) for some point \( \xi_b \in b \).

\[
\Sigma_{b \in bins(S)} p_b^2 = \Sigma_{b \in bins(S)} V(b)2f(\xi_b)^2 = V(b)\Sigma_{b \in bins(S)} V(b)f(\xi_b)^2
\]

(7)

Using Rienen Integral approximation , we can write the following as the bin size reduces,

\[
\Sigma_{b \in bins(S)} V(b)f(\xi_b)^2 = \int_{x \in S} f^2(x) dx + o(1)
\]

(8)

\( \int_{x \in S} f^2(x) dx \) is also known as the roughness of the function. Let us denote it using \( R(f) \). Hence

\[
IV = \frac{1}{nV(bin)} (1 - V(b)(R(f) + o(1)))
\]

(9)
\[
IV = \frac{1}{n V(b)} - \frac{R(f)}{n} + o\left(\frac{1}{n}\right) \tag{10}
\]

Putting \(V(b) = h^d\)

\[
IV = \frac{1}{n h^d} - \frac{R(f)}{n} + o\left(\frac{1}{n}\right) \tag{11}
\]

Keeping only the leading term in the above expression,

\[
IV = O\left(\frac{1}{n h^d}\right) \tag{12}
\]

Now let us look at the ISB for this estimator, \(ISB(\hat{f}_h(x))\)

\[
ISB(\hat{f}_h(x)) = \int_{x \in S} (E(\hat{f}_H(x) - f(x))^2 \, dx \tag{13}
\]

Let us look at the estimator,

\[
\hat{f}_H(x) = \frac{1}{V(bin(x))} \int_{t \in bin(x)} f(t) \, dt \tag{14}
\]

Just to make it clear, \(x \in R^d\), we will use it as a vector in the following. Using 2nd order multivariate taylor series expansion of this \(f(t)\) around \(x\), we get:

\[
f(t) = f(x) + (t - x, \nabla f(x)) + \frac{1}{2} (t - x)^T \mathcal{H}(f(x))(t - x) \tag{15}
\]

Here \(\mathcal{H}(f(t))\) is the hessian of \(f\) at \(t\). Without loss of generality let us look at the \(bin(x) = [0, h]^d\) that is the bin at the origin. Let us say it is \(bin_0\)

\[
\int_{t \in bin_0} f(t) \, dt = f(x) h^d + h^d ((\frac{h}{2} - x, \nabla f(x)) + O(h^{d+2}) \tag{16}
\]

where \(x^{(j)}\) is the \(j^{th}\) component of \(x\). Using eq 17 in eq 15, we get

\[
\hat{f}_H(x) = f(x) + ((\frac{h}{2} - x), \nabla f(x)) + O(h^2) \tag{17}
\]

Hence, just keeping the leading term , we have

\[
\text{Bias}(\hat{f}_H(x)) = ((\frac{h}{2} - x), \nabla f(x)) \tag{18}
\]

Now,

\[
\int_{x \in bin_0} \text{Bias}(\hat{f}_H(x))^2 \, dx = \int_{x \in bin_0} ((\frac{h}{2} - x), \nabla f(x))^2 \, dx \tag{19}
\]

Using cauchy's inequality, we get

\[
\int_{x \in bin_0} \text{Bias}(\hat{f}_H(x))^2 \, dx \leq \int_{x \in bin_0} \left(\frac{h}{2} - x\right) \|
abla f(x)\|^2 \, dx \tag{20}
\]

As \([h/2, h/2, ... h/2]\) is a mid point of the bin. The max norm of \(x - h/2\) can be \(h \sqrt{d}/2\)

\[
\int_{x \in bin_0} \text{Bias}(\hat{f}_H(x))^2 \, dx \leq \frac{h^2 d}{4} \int_{x \in bin_0} \|
abla f(x)\|^2 \, dx \tag{21}
\]

Now looking at ISB

\[
ISB = \Sigma_{b \in bins} \int_{x \in bin_0} \text{Bias}(\hat{f}_H(x))^2 \, dx \leq \frac{h^2 d}{4} \int_{x \in S} \|
abla f(x)\|^2 \, dx \tag{22}
\]

\[
ISB \leq \frac{h^2 d}{4} R(\|
abla f\|_{2}) \tag{23}
\]
8.2 Theorem 3: \( f_C \) to \( f_H \)

While estimating true distribution \( f(x) : R^d \rightarrow R \), the integrated mean square error (IMSE) for the estimator \( \hat{f}_C(x) \) using regular histogram with width \( h \) and number of samples \( n \) and countsketch with parameters \( (R: \text{range}, K: \text{repetitions}) \) and average-recovery, is

\[
\text{IMSE}(\hat{f}_C) = \text{IMSE}(\hat{f}_H) + \frac{\#\text{bins}}{KRnh^d}
\]

where \( n_{nzp} \) is the number of non-zero partitions. Specifically, its

\[
\text{IV}(\hat{f}_C) = \text{IV}(\hat{f}_H) + \frac{\#\text{bins} - 1}{KRnh^d}
\]

and

\[
\text{ISB}(\hat{f}_C) = \text{ISB}(\hat{f}_H)
\]

where \( n_{nzp} \) is the number of non-zero bins/partitions.

Proof. Consider a Countsketch with range = \( R \) and just one repetition. Let it be parameterized by the randomly drawn hash functions \( g : \text{bin} \rightarrow \{0, 1, 2, ..., R - 1\} \) and \( s : \text{bin} \rightarrow \{-1, +1\} \). The estimate of density at point \( x \) can then be written as

\[
\hat{f}_C(x) = \frac{1}{nV(\text{bin}(x))} \left( c(\text{bin}(x)) + \sum_{i=1}^{n} I(x_i \notin \text{bin}(x) \land g(\text{bin}(x_i)) == g(\text{bin}(x)))s(\text{bin}(x_i))s(\text{bin}(x)) \right)
\]

We can rewrite this as,

\[
\hat{f}_C(x) = \hat{f}_H(x) + \frac{1}{nV(\text{bin}(x))} \left( \sum_{i=1}^{n} I(x_i \notin \text{bin}(x) \land g(\text{bin}(x_i)) == g(\text{bin}(x)))s(\text{bin}(x_i))s(\text{bin}(x)) \right)
\]

where \( c(.) \) is count and \( V(.) \) is volume of the bins. As \( E(s(b)) = 0 \), it can be clearly seen that.

\[
E(\hat{f}_C(x)) = E(\hat{f}_H(x))
\]

Hence, it follows that

\[
\text{ISB}(\hat{f}_C(x)) = \text{ISB}(\hat{f}_H(x))
\]

It can be checked that each of the terms in the summation for right hand side of equation 26 including the terms in \( \hat{f}_H(x) \) are independent to each other , i.e. covariance between them is 0. Hence we can write the variance of our estimator as,

\[
\text{Var}(\hat{f}_C(x)) = \text{Var}(\hat{f}_H(x)) + \frac{1}{nV^2(\text{bin}(x))} \text{Var}(I(x_i \notin \text{bin}(x) \land g(\text{bin}(x_i)) == g(\text{bin}(x)))s(\text{bin}(x_i))s(\text{bin}(x)))
\]

\[
\text{Var}(\hat{f}_C(x)) = \text{Var}(\hat{f}_H(x)) + \frac{1}{nV^2(\text{bin}(x))} \left( 1 - p_{bin}(x) \right) \frac{1}{R}
\]

\[
\text{Var}(\hat{f}_C(x)) = \text{Var}(\hat{f}_H(x)) + \frac{1}{nV^2(\text{bin}(x))} \left( 1 - p_{bin}(x) \right) \frac{1}{R}
\]

Hence, IV is

\[
\text{IV}(\hat{f}_C(x)) = \text{IV}(\hat{f}_H(x)) + \int_{x \in S} \frac{1}{nV^2(\text{bin}(x))} \left( 1 - p_{bin}(x) \right) \frac{1}{R}
\]

\[
\text{IV}(\hat{f}_C(x)) = \text{IV}(\hat{f}_H(x)) + \sum_{b \in \text{bins}} \int_{x \in b} \frac{1}{nV^2(b)} \left( 1 - p_b \right) \frac{1}{R}
\]

\[
\text{IV}(\hat{f}_C(x)) = \text{IV}(\hat{f}_H(x)) + \sum_{b \in \text{bins}} \frac{1}{nV(b)} \left( 1 - p_b \right) \frac{1}{R}
\]

Assuming standard partitions. \( V(b) = h^d \) for all \( b \)

\[
\text{IV}(\hat{f}_C(x)) = \text{IV}(\hat{f}_H(x)) + \frac{1}{nh^d} \left( \#\text{bins} - 1 \right)
\]

With average recovery, with K repetitions, the analysis can be easily extended to get IV as

\[
\text{IV}(\hat{f}_C(x)) = \text{IV}(\hat{f}_H(x)) + \frac{1}{nh^d} \left( \#\text{bins} - 1 \right)
\]

The ISB remains same in this case.
8.3 Theorem 4: $f_{c}^{\ast}$ to $f_{C}$

While estimating true distribution $f(x): \mathbb{R}^{d} \rightarrow \mathbb{R}$, the integrated mean square error (IMSE) for the estimator $\hat{f}_{c}^{\ast}(x)$ using regular histogram with width h and number of samples n and countsketch with parameters (R:range, K:repetitions), is related to the estimator $f_{C}(x)$ as follows

$$IMSE(f_{C}(x)) - \epsilon(N + 2M) \leq IMSE(f_{c}^{\ast}(x)) \leq IMSE(f_{C}(x)) + \epsilon(N + 2M)$$

Specifically,

$$IV(f_{C}(x)) - 2\epsilon M \leq IV(f_{c}^{\ast}(x)) \leq IV(f_{C}(x)) + 2\epsilon M$$

and

$$ISB(f_{C}(x)) - \epsilon N \leq ISB(f_{c}^{\ast}(x)) \leq ISB(f_{C}(x)) + \epsilon N$$

where

$$M \leq IV(\hat{f}_{C}(x)) + 2(R(f) + \frac{h^2d}{4}R(\|\nabla f\|_2)) + h\sqrt{d}\int_{x \in S} (f(x)\|\nabla f\|_2)$$

$$N = (1 + ISB(\hat{f}_{C}(x)))$$

with probability $(1 - \delta)$ where $\delta = \frac{2\text{bins}}{\epsilon^{2}nR}$

**Proof.** Let us look at the estimator

$$f_{c}^{\ast}(x) = \frac{c(bin(x))}{V(bin(x))\Sigma_{b}c(b)} = f_{C}(x) \ast \frac{n}{\hat{n}}$$

(36)

where $\hat{n} = \Sigma_{b}c(b)$ and $n = \Sigma_{b}c(b)$ \hfill $\square$

**\hat{n} and its relation to n** Let us first analyse $\hat{n}$ and how it is related to n.

$$\hat{n} = \Sigma_{b}c(b) = \Sigma_{b}\Sigma_{i=1}^{n}I(x_{i} \in b) + I(x_{i} \notin b \land g(bin(x_{i})) = g(b))s(bin(x_{i}))s(b)$$

(37)

$$\hat{n} = \Sigma_{b}I(x_{i} \in b) + I(x_{i} \notin b \land g(bin(x_{i})) = g(b))s(bin(x_{i}))s(b)$$

(38)

Note that $E(\hat{n}) = n$. For variance, observe that most of the terms in the summation have covariance 0, except the terms $Cov(I(x_{i} \in b_{1}), I(x_{i} \in b_{2}))$ which are negatively correlated. Hence

$$Var(\hat{n}) = \Sigma_{b}Var(I(x_{i} \in b)) + Var(I(x_{i} \notin b \land g(bin(x_{i})) = g(b))s(bin(x_{i}))s(b)) + 2\Sigma_{i,b_{1},b_{2},b_{1} \neq b_{2}}Cov(I(x_{i} \in b_{1}), I(x_{i} \in b_{2}))$$

(39)

We know that

$$Var(I(x_{i} \in b)) = p_{b}(1 - p_{b})$$

$$Var(I(x_{i} \notin b \land g(bin(x_{i})) = g(b))s(bin(x_{i}))s(b)) = E(I(x_{i} \notin b \land g(bin(x_{i})) = g(b))2) = \frac{1 - p_{b}}{R}$$

$$Cov(I(x_{i} \in b_{1}), I(x_{i} \in b_{2})) = -p_{b_{1}}p_{b_{2}}$$

Hence, we pluggin in the values in previous equation ,

$$Var(\hat{n}) = n\Sigma_{b}p_{b}(1 - p_{b}) + n\Sigma_{b}\frac{1 - p_{b}}{R} - 2n\Sigma_{b_{1},b_{2},b_{1} \neq b_{2}}p_{b_{1}}p_{b_{2}}$$

(40)

$$Var(\hat{n}) = n(1 - \Sigma_{b}p_{b}^{2}) + n\Sigma_{b}\frac{1 - p_{b}}{R} - 2n\Sigma_{b_{1},b_{2}}p_{b_{1}}p_{b_{2}}$$

(41)

$$Var(\hat{n}) = n\{1 + \Sigma_{b}\frac{1 - p_{b}}{R} - (\Sigma_{b}p_{b}^{2}) - 2n\Sigma_{b_{1},b_{2}}p_{b_{1}}p_{b_{2}}\}$$

(42)

$$Var(\hat{n}) = n\{1 + \Sigma_{b}\frac{1 - p_{b}}{R} - (\Sigma_{b}p_{b})^{2}\}$$

(43)

$$Var(\hat{n}) = n\{\Sigma_{b}\frac{1 - p_{b}}{R}\}$$

(44)
\[ Var(\hat{n}) = \frac{n(#\text{bins} - 1)}{R} < \frac{n(#\text{bins})}{R} \] (45)

Using Chebyshev’s inequality, we have
\[ P(|\hat{n} - n| > \epsilon n) \leq \frac{Var(\hat{n})}{\epsilon^2 n^2} \] (46)
\[ P(|\hat{n} - n| > \epsilon n) \leq \frac{\#\text{bins}}{\epsilon^2 n R} \] (47)

Hence with probability \((1 - \delta)\), \(\delta = \frac{\#\text{bins}}{\epsilon^2 n R}, \) \(\hat{n}\) is within \(\epsilon\) multiplicative error.

**relation of pointwise Bias and ISB**

With probability \(1 - \delta\),
\[ \frac{\hat{f}_C(x)}{1 + \epsilon} \leq \hat{f}_C^*(x) \leq \frac{\hat{f}_C(x)}{1 - \epsilon} \] (48)

As expectations respect inequalities
\[ \frac{E(\hat{f}_C(x))}{1 + \epsilon} \leq E(\hat{f}_C^*(x)) \leq \frac{E(\hat{f}_C(x))}{1 - \epsilon} \] (49)
\[ \frac{E(\hat{f}_C(x)) - f(x)}{1 + \epsilon} \leq Bias(\hat{f}_C^*(x)) \leq \frac{E(\hat{f}_C(x))}{1 - \epsilon} - f(x) \] (50)
\[ \frac{Bias(\hat{f}_C(x)) - \epsilon f(x)}{1 + \epsilon} \leq Bias(\hat{f}_C^*(x)) \leq \frac{Bias(\hat{f}_C(x)) + \epsilon f(x)}{1 - \epsilon} \] (51)
\[ \frac{Bias(\hat{f}_C(x)) - \epsilon f(x)}{1 + \epsilon} \leq Bias(\hat{f}_C^*(x)) \leq \frac{Bias(\hat{f}_C(x)) + \epsilon f(x)}{1 - \epsilon} \] (52)

Integrating expressions again respects inequalities
\[ \frac{ISB(\hat{f}_C(x)) - \epsilon \int f(x)}{1 + \epsilon} \leq ISB(\hat{f}_C^*(x)) \leq \frac{ISB(\hat{f}_C(x)) + \epsilon \int f(x)}{1 - \epsilon} \] (53)
\[ \frac{ISB(\hat{f}_C(x))}{1 + \epsilon} \leq ISB(\hat{f}_C^*(x)) \leq \frac{ISB(\hat{f}_C(x)) + \epsilon}{1 - \epsilon} \] (54)

Using first order taylor expansion of \(\frac{1}{1+\epsilon}\) and ignore square terms
\[ (1 - \epsilon)ISB(\hat{f}_C(x)) - \epsilon \leq ISB(\hat{f}_C^*(x)) \leq (1 + \epsilon)ISB(\hat{f}_C(x)) + \epsilon \] (55)
\[ ISB(\hat{f}_C(x)) - \epsilon(1 + ISB(\hat{f}_C(x))) \leq ISB(\hat{f}_C^*(x)) \leq ISB(\hat{f}_C(x)) + \epsilon(1 + ISB(\hat{f}_C(x))) \] (56)

Hence,
\[ ISB(\hat{f}_C(x)) - \epsilon N \leq ISB(\hat{f}_C^*(x)) \leq ISB(\hat{f}_C(x)) + \epsilon N \] (57)

where
\[ N = (1 + ISB(\hat{f}_C(x))) \]
Point wise variance and IV Using the similar arguments
\[
\frac{E(\hat{f}_C^2(x))}{(1+\epsilon)^2} - \frac{E^2(\hat{f}_C(x))}{(1-\epsilon)^2} \leq \text{Var}(\hat{f}_C^*(x)) \leq \frac{E(\hat{f}_C^2(x))}{(1-\epsilon)^2} - \frac{E^2(\hat{f}_C(x))}{(1+\epsilon)^2}
\] (58)

Again making first order taylor expansions of denominator and ignoring square terms

\[
\text{Var}(\hat{f}_C(x)) - 2\epsilon(\text{Var}(\hat{f}_C(x))) \leq \text{Var}(\hat{f}_C^*(x)) \leq \text{Var}(\hat{f}_C(x)) + 2(\text{Var}(\hat{f}_C^2(x)) + E^2(\hat{f}_C(x)))
\] (59)

Since, \(\text{Var}(\hat{f}_C(x)) = E(\hat{f}_C^2(x)) - E^2(\hat{f}_C(x))\)

\[
\text{Var}(\hat{f}_C(x)) - 2\epsilon(\text{Var}(\hat{f}_C(x))) \leq \text{Var}(\hat{f}_C^*(x)) \leq \text{Var}(\hat{f}_C(x)) + 2\epsilon(\text{Var}(\hat{f}_C(x)) + 2E^2(\hat{f}_C(x)))
\] (60)

\[
\text{IV}(\hat{f}_C(x)) - 2\epsilon(\text{IV}(\hat{f}_C(x))) \leq \text{IV}(\hat{f}_C^*(x)) \leq \text{IV}(\hat{f}_C(x)) + 2\epsilon(\text{IV}(\hat{f}_C(x)) + 2\int_{x \in S} E^2(\hat{f}_C(x)))
\] (61)

Let us now figure out the \(\int_{x \in S} E^2(\hat{f}_C(x))\)

\[
\int_{x \in S} E^2(\hat{f}_C(x)) = \int_{x \in S} E^2(\hat{f}_H(x))
\] (62)

From equation 18, \(E(\hat{f}_H(x))^2 = f(x)^2 + ((\frac{h}{2} - x), \nabla f(x))^2 + 2f(x)((\frac{h}{2} - x), \nabla f(x))\)

\[
\int_{x \in S} E^2(\hat{f}_H(x)) \leq R(f) + \frac{h^2d}{4}R(||\nabla f||_2) + h\sqrt{d}\int_{x \in S} (f(x)||\nabla f||_2)
\] (63)

Hence,

\[
\text{IV}(\hat{f}_C(x)) - 2\epsilon M \leq \text{IV}(\hat{f}_C^*(x)) \leq \text{IV}(\hat{f}_C(x)) + 2\epsilon M
\] (64)

Where

\[
M \leq \text{IV}(\hat{f}_C(x)) + 2(R(f) + \frac{h^2d}{4}R(||\nabla f||_2) + h\sqrt{d}\int_{x \in S} (f(x)||\nabla f||_2))
\] (65)

8.4 Lemma 1

Estimators \(\hat{f}_S(x)\) and \(\hat{f}_C^*(x)\), obtained from the Density Sketch with parameters(R,K,H) using histogram of width h built over n i.i.d samples drawn from true distribution have a relation

\[
\int |\hat{f}_C^*(x) - \hat{f}_S(x)|dx = 2(1 - \text{ratio}_h)
\]

where \(\text{ratio}_h\) is the capture ratio as defined in section 3

\[
\int |\hat{f}_C^*(x) - \hat{f}_S(x)|dx = \Sigma_{b \in \text{bins}} \int_{x \in b} |\hat{f}_C^*(x) - \hat{f}_S(x)|dx
\] (66)

\[
\int |\hat{f}_C^*(x) - \hat{f}_S(x)|dx = \Sigma_{b \in \text{bins}(H)} \int_{x \in b} |\hat{f}_C^*(x) - \hat{f}_S(x)|dx + \Sigma_{b \notin \text{bins}(H)} \int_{x \in b} |\hat{f}_C^*(x) - \hat{f}_S(x)|dx
\] (67)

we know that for \(x \in b, b \notin \text{bins}(H), \hat{f}_S(x) = 0\). Hence,

\[
\int |\hat{f}_C^*(x) - \hat{f}_S(x)|dx = \Sigma_{b \in \text{bins}(H)} \int_{x \in b} |\hat{f}_C^*(x) - \hat{f}_S(x)|dx + \Sigma_{b \notin \text{bins}(H)} \int_{x \in b} \hat{f}_C^*(x)dx
\] (68)
\[ f_{\in b}(x) \, dx \] is the probability of a data point lying in that bucket according to \( \hat{f}_C(x) \)

\[
\int |\hat{f}_C^*(x) - \hat{f}_S(x)| \, dx = \Sigma_{b \in \text{bins}(H)} \int_{x \in b} |\hat{f}_C^*(x) - \hat{f}_S(x)| \, dx + \Sigma_{b \notin \text{bins}(H)} \frac{c_b}{n} \tag{69}
\]

For points \( x \in b, b \in \text{bins}(H) \), \( \hat{f}_C^*(x) * \hat{n} = \hat{f}_S(x) * \hat{n}_h \). Hence, \( \hat{f}_S(x) = \frac{\hat{n}}{\hat{n}_h} \hat{f}_C^*(x) \)

\[
\int |\hat{f}_C^*(x) - \hat{f}_S(x)| \, dx = \Sigma_{b \in \text{bins}(H)} \int_{x \in b} \hat{f}_C^*(x)(\frac{\hat{n}}{\hat{n}_h} - 1) \, dx + \Sigma_{b \notin \text{bins}(H)} \frac{c_b}{n} \tag{70}
\]

\[
\int |\hat{f}_C^*(x) - \hat{f}_S(x)| \, dx = \Sigma_{b \in \text{bins}(H)} \int_{x \in b} \hat{f}_C^*(x)(\frac{n}{n_h} - 1) \, dx + \Sigma_{b \notin \text{bins}(H)} \frac{c_b}{n} \tag{71}
\]

\[
\int |\hat{f}_C^*(x) - \hat{f}_S(x)| \, dx = (\frac{\hat{n}}{\hat{n}_h} - 1) \Sigma_{b \in \text{bins}(H)} \frac{c_b}{n} + \Sigma_{b \notin \text{bins}(H)} \frac{c_b}{n} \tag{72}
\]

\[
\int |\hat{f}_C^*(x) - \hat{f}_S(x)| \, dx = (\frac{n}{n_h} - 1) + \frac{\hat{n} - \hat{n}_h}{n} \tag{73}
\]

\[
\int |\hat{f}_C^*(x) - \hat{f}_S(x)| \, dx = (1 - \frac{n}{n_h}) + \frac{\hat{n} - \hat{n}_h}{n} \tag{74}
\]

\[
\int |\hat{f}_C^*(x) - \hat{f}_S(x)| \, dx = 2(1 - \frac{n}{n_h}) \tag{75}
\]

\[
\int |\hat{f}_C^*(x) - \hat{f}_S(x)| \, dx = 2(1 - \text{ratio}_h) \tag{76}
\]

### 8.5 Theorem 5

The IMSE of estimator \( \hat{f}_S(x) \) obtained from the Density Sketch with parameters (R,K,H) using histogram of width \( h \) built over \( n \) i.i.d samples drawn from true distribution \( f(x) \) is

\[
\text{IMSE}(\hat{f}_S(x)) \leq 12(1 - \text{ratio}_h)^2 + 3\text{IMSE}(\hat{f}_C^*(x))
\]

where \( \text{ratio}_h \) is the capture ratio as defined in

**Proof.** Giving a very loose relation between \( \hat{f}_S \) and \( f \). We can write

\[
\int (\hat{f}_S(x) - f(x))^2 \, dx = \int ((\hat{f}_S(x) - \hat{f}_C^*(x)) - (\hat{f}_C^*(x) - f(x)))^2 \, dx \tag{77}
\]

\[
\int (\hat{f}_S(x) - f(x))^2 \, dx \leq 3 \int (\hat{f}_S(x) - \hat{f}_C^*(x))^2 \, dx + 3 \int (\hat{f}_C^*(x) - f(x))^2 \, dx \tag{78}
\]

\[
\int (\hat{f}_S(x) - f(x))^2 \, dx \leq 3(\int (\hat{f}_S(x) - \hat{f}_C^*(x))^2 \, dx)^2 + 3 \int (\hat{f}_C^*(x) - f(x))^2 \, dx \tag{79}
\]

\[
\int (\hat{f}_S(x) - f(x))^2 \, dx \leq 12(1 - \text{ratio}_h)^2 + 3 \int (\hat{f}_C^*(x) - f(x))^2 \, dx \tag{80}
\]

\[
\text{IMSE} = MISE(\hat{f}_S(x)) \leq 12(1 - \text{ratio}_h)^2 + 3\text{IMSE}(\hat{f}_C^*(x)) \tag{81}
\]

\[ \square \]
9 Theorem 1 (Main Theorem) combines all other theorems

This theorem directly relates the distribution \( \hat{f}_S(x) \) to the true distribution, \( f(x) \)

\[
IMSE(\hat{f}_S(x)) \leq 12(1 - ratio_h)^2 + 3IMSE(\hat{f}_C(x))
\] (82)

\[
IMSE(\hat{f}_S(x)) \leq 12(1 - ratio_h)^2 + 3(IMSE(\hat{f}_C(x) + \epsilon(N + 2M)))
\] (83)

\[
IMSE(\hat{f}_S(x)) \leq 12(1 - ratio_h)^2 + 3(IMSE(\hat{f}_H) + \frac{\# \text{bins} - 1}{KRnh^d} + \epsilon(N + 2M))
\] (84)

\[
IMSE(\hat{f}_S(x)) \leq 12(1 - ratio_h)^2 + 3(\frac{1}{nh^d} + \frac{R(f)}{n}) + o(\frac{1}{n}) + \frac{h^2d}{4}R(\|\nabla f\|_2) + \frac{\# \text{bins} - 1}{KRnh^d} + \epsilon(N + 2M))
\] (85)

\[
N = (1 + ISB(\hat{f}_C))
\]

\[
N \leq 1 + \frac{h^2d}{4}R(\|\nabla f\|_2)
\]

\[
M \leq IV(\hat{f}_C) + 2R(f) + \frac{h^2d}{4}R(\|\nabla f\|_2) + h\sqrt{d}\int_{x \in S} (f(x)\|\nabla f\|_2)
\]

\[
M \leq IV(\hat{f}_H) + \frac{\# \text{bins} - 1}{KRnh^d} + 2R(f) + \frac{h^2d}{4}R(\|\nabla f\|_2) + h\sqrt{d}\int_{x \in S} (f(x)\|\nabla f\|_2)
\]

\[
M \leq \frac{1}{nh^d} + \frac{R(f)}{n} + o(\frac{1}{n}) + \frac{\# \text{bins} - 1}{KRnh^d} + 2R(f) + \frac{h^2d}{4}R(\|\nabla f\|_2) + h\sqrt{d}\int_{x \in S} (f(x)\|\nabla f\|_2)
\]

\[
IMSE(\hat{f}_S(x)) \leq 12(1 - ratio_h)^2 + 3(\frac{1}{nh^d} + \frac{R(f)}{n}) + o(\frac{1}{n}) + (1 + \epsilon)\frac{h^2d}{4}R(\|\nabla f\|_2) + \frac{\# \text{bins} - 1}{KRnh^d} + 2\epsilon M + \epsilon
\] (86)

\[
IMSE(\hat{f}_S(x)) \leq 12(1 - ratio_h)^2 + 3(1 + 2\epsilon)(\frac{1}{nh^d} + \frac{R(f)}{n}) + o(\frac{1}{n}) + \frac{\# \text{bins} - 1}{KRnh^d} +
\]

\[
3(1 + 3\epsilon)\frac{h^2d}{4}R(\|\nabla f\|_2) + 
\]

\[
3\epsilon(1 + 2R(f) + h\sqrt{d}\int_{x \in S} (f(x)\|\nabla f\|_2))
\] .

10 Other Base lines

**Coresets:** We considered a comparison with sophisticated data summaries such as coresets. Briefly, a coreset is a collection of (possibly weighted) points that can be used to estimate functions over the dataset. To use coresets to generate a synthetic dataset, we would need to estimate the KDE. Unfortunately, coresets for the KDE suffer from practical issues such as a large memory cost to construct the point set. Despite recent progress toward coresets in the streaming environment [19], coresets remain difficult to implement for real-world KDE problems [20].

**Clustering and Importance Sampling:** Another reasonable strategy is to represent the dataset as a collection of weighted cluster centers, which may be used to compute the KDE and sample synthetic points. Unfortunately, algorithms such as \( k \)-means clustering are inappropriate for large streaming datasets and do not have the same mergeability properties as our sketch. Furthermore, such techniques are unlikely to substantially improve over random sampling when the
samples is spread sufficiently well over the support of the distribution. An alternative approach is to select points from the dataset based on importance sampling [20], geometric properties [21], and other sampling techniques [22]. However, recent experiments show that for many real-world datasets, random samples have competitive performance when compared to point sets obtained via importance sampling and cluster-based approaches [14].

**Dimensionality Reduction:** One can also apply sketching algorithms to compress a dataset by reducing the dimension of each data point via feature hashing, random projections or similar methods [23]. However, this is unlikely to perform well in our evaluation since our datasets are already relatively low-dimensional. Such algorithms also fail to address the streaming setting, where $N$ can grow very large, because the size of the compressed representation is linear in $N$. Finally, most dimensionality reduction algorithms do not easily permit the generation of more synthetic data in the original metric space.