INFINITESIMAL DEFORMATIONS OF PARABOLIC CONNECTIONS AND PARABOLIC OPERS

INDRANIL BISWAS, SORIN DUMITRESCU, SEBASTIAN HELLER, AND CHRISTIAN PAULY

Abstract. We compute the infinitesimal deformations of quadruples of the form 
\((X, S, E_\ast, D)\),
where \((X, S)\) is a compact Riemann surface with \(n\) marked points, \(E_\ast\) is a parabolic vector
bundle on \(X\) with parabolic structure over \(S\), and \(D\) is a parabolic connection on \(E_\ast\). Using
it we compute the infinitesimal deformations of \((X, S, D)\), where \(D\) is a parabolic \(\text{SL}(r, \mathbb{C})\)–oper on \((X, S)\). It is shown that the monodromy map, from the moduli space of triples
\((X, S, D)\), where \(D\) is a parabolic \(\text{SL}(r, \mathbb{C})\)–oper on \((X, S)\), to the \(\text{SL}(r, \mathbb{C})\)–character variety of \(X\) \(\setminus S\), is an immersion.

CONTENTS

1. Introduction 1
2. Holomorphic connections and the Atiyah bundle 3
3. Infinitesimal deformations and isomonodromy 10
4. Infinitesimal deformations of parabolic opers 17
5. Monodromy of parabolic opers 27
Appendix A. Parabolic opers 32
References 39

1. Introduction

Opers were introduced by Beilinson and Drinfeld [BD1, BD2]. They have turned out
to be very important in diverse topics, for example in geometric Langlands correspondence,
nonabelian Hodge theory, some branches of mathematics physics, differential equations et
cetera; see [BF], [DFK+], [FT], [FG1], [FG2], [CS], [Fr1], [Fr2], [KSZ], [MR], [BSY] and
references therein.

Parabolic vector bundles were introduced by Mehta and Seshadri in [MS]. In [BDP],
\(\text{SL}(r, \mathbb{C})\)–opers in the set-up of parabolic vector bundles were introduced. The aim here is
to further investigate the \(\text{SL}(r, \mathbb{C})\)–opers in the context of parabolic vector bundles.

2010 Mathematics Subject Classification. 14H60, 33C80, 53C07.
Key words and phrases. Parabolic oper, logarithmic Atiyah bundle, isomonodromy, monodromy map.
Being inspired by the interesting work [Sa], we study the infinitesimal deformations, and the monodromy map, of the parabolic \( \text{SL}(r, \mathbb{C}) \)-opers. Computation of the infinitesimal deformations of parabolic opers entails computation of the infinitesimal deformations of quadruples of the form \((X, S, E_*, D)\), where \((X, S)\) is a compact Riemann surface with \(n\) marked points, \(E_*\) is a parabolic vector bundle on \(X\) with parabolic structure over \(S\), and \(D\) is a parabolic connection on \(E_*\).

We introduce the parabolic analog of the Atiyah bundle and the Atiyah exact sequence. Given a parabolic vector bundle \(E_*\) on \((X, S)\), its Atiyah bundle \(\text{At}(E_*)\) fits in the short exact sequence of holomorphic vector bundles

\[
0 \rightarrow \text{End}^p(E_*) \rightarrow \text{At}(E_*) \xrightarrow{\sigma} TX \otimes \mathcal{O}_X(-S) \rightarrow 0,
\]

where \(\text{End}^p(E_*)\) denotes the sheaf of quasiparabolic flag preserving endomorphisms of \(E_*\) (see (2.2) for the quasiparabolic flags); the sequence in (1.1) is the Atiyah exact sequence in the set-up of parabolic bundles (see Definition 2.3). We show the following:

1. A holomorphic splitting of (1.1) produces a logarithmic connection on the holomorphic vector bundle \(E_\star\) underlying \(E_*\) such that the residues preserve the quasiparabolic flags of \(E_*\), and
2. a parabolic connection on \(E_*\) is a holomorphic splitting of (1.1) such that the eigenvalues of the residues are given by the parabolic weights (see (2.3) for the parabolic weights).

(See Lemma 2.4 and Corollary 2.5)

We prove the following (see Lemma 3.1):

The infinitesimal deformations of the triple \((X, S, E_*)\) are parametrized by \(H^1(X, \text{At}(E_*))\).

Now let \(D\) be a connection on the parabolic vector bundle \(E_\star\) over \((X, S)\). We assume that the local monodromy of \(D\) around each point of \(S\) is semisimple (meaning diagonalizable). Let

\[
\mathcal{D}_0 : \text{End}^p(E_\star) \rightarrow \text{End}^n(E_\star) \otimes K_X
\]

be the corresponding logarithmic connection on \(\text{End}^p(E_\star)\), where \(\text{End}^n(E_\star) \subset \text{End}^p(E_\star)\) is the sheaf of endomorphisms nilpotent with respect to the quasiparabolic flags of \(E_*\). We show that this operator \(\mathcal{D}_0\) extends to a holomorphic differential operator

\[
\mathcal{D} : \text{At}(E_\star) \rightarrow \text{End}^n(E_\star) \otimes K_X \otimes \mathcal{O}_X(S)
\]

(see (2.24)). Let \(\mathcal{B}_\bullet\) denote the following two-term complex of sheaves on \(X\)

\[
\mathcal{B}_\bullet : \mathcal{B}_0 = \text{At}(E_\star) \xrightarrow{\mathcal{D}} \mathcal{B}_1 = \text{End}^n(E_\star) \otimes K_X \otimes \mathcal{O}_X(S),
\]

where \(\mathcal{B}_i\) is at the \(i\)-th position.

We prove the following (see Lemma 3.4):

The infinitesimal deformations of the quadruple \((X, S, E_\star, D)\) are parametrized by the first hypercohomology \(H^1(\mathcal{B}_\bullet)\).
Now assume that $D$ is a parabolic $\text{SL}(r, \mathbb{C})$–oper (the definition of a parabolic $\text{SL}(r, \mathbb{C})$–oper is recalled in Section 4). Since $D$ is a parabolic $\text{SL}(r, \mathbb{C})$–oper, the local monodromy of $D$ around each point of $S$ is semisimple (see Lemma 4.2). In (4.7) and (4.9) we construct the holomorphic vector bundles $\text{At}_X(r)$ and $\text{ad}^n_1 \left( \text{Sym}^{r-1}(E_\ast) \right)$ respectively on $X$. Using $D$ we construct a differential operator $D_B : \text{At}_X(r) \rightarrow \text{ad}^n_1 \left( \text{Sym}^{r-1}(E_\ast) \right) \otimes K_X \otimes \mathcal{O}_X(S)$ (see (4.13)). Let $C_\bullet$ denote the following two-term complex of sheaves on $X$

$$C_0 : C_0 = \text{At}_X(r) \xrightarrow{D_B} C_1 = \text{ad}^n_1 \left( \text{Sym}^{r-1}(E_\ast) \right) \otimes K_X \otimes \mathcal{O}_X(S),$$

where $C_i$ is at the $i$-th position.

We prove the following (see Theorem 4.5):

**The space of all infinitesimal deformation of the triple $(X, S, D)$, where $D$ is a parabolic $\text{SL}(r, \mathbb{C})$–oper on $(X, S)$, is given by the hypercohomology $H^1(C_\bullet)$.**

A reformulation of the above result is proved in Corollary 4.6.

In Theorem 5.1 we prove the following:

**The monodromy map from the moduli space of triples $(X, S, D)$, where $D$ is a parabolic $\text{SL}(r, \mathbb{C})$–oper on $(X, S)$, to the $\text{SL}(r, \mathbb{C})$–character variety for $X \setminus S$ is an immersion.**

The isomonodromy condition defines a holomorphic foliation on the moduli space of quadruples $(X, S, E_\ast, D)$, where $D$ is a connection on the parabolic vector bundle $E_\ast$ over $(X, S)$. The proof of Theorem 5.1 involves computing this foliation. This computation is carried out in Lemma 3.5.

In the appendix we give an alternative definition of a parabolic $\text{SL}(r, \mathbb{C})$–oper in terms of $\mathbb{R}$-filtered sheaves as introduced and studied by Maruyama and Yokogawa. This definition is conceptually closer to the definition of an ordinary $\text{SL}(r, \mathbb{C})$–oper and clarifies the one given in [BDP].

### 2. Holomorphic connections and the Atiyah bundle

#### 2.1. Atiyah bundle for parabolic bundles.

Let $X$ be a compact connected Riemann surface. Fix a finite subset of $n$ distinct points

$$S := \{x_1, \ldots, x_n\} \subset X. \quad (2.1)$$

The reduced effective divisor $x_1 + \ldots + x_n$ on $X$ will also be denoted by $S$.

A quasiparabolic structure $x_1 + \ldots + x_n$ on $X$ will also be denoted by $S$.

A quasiparabolic structure on a holomorphic vector bundle $E$ on $X$ is a filtration of subspaces of the fiber $E_{x_i}$ of $E$ over $x_i$

$$E_{x_i} = E_{i,1} \supset E_{i,2} \supset \cdots \supset E_{i,l_i} \supset E_{i,l_i+1} = 0 \quad (2.2)$$
for every $1 \leq i \leq n$. A parabolic structure on $E$ is a quasiparabolic structure as above together with a string of real numbers

$$0 \leq \alpha_{i,1} < \alpha_{i,2} < \cdots < \alpha_{i,l_i} < 1$$

(2.3)

for every $1 \leq i \leq n$. The above number $\alpha_{i,j}$ is called the parabolic weight of the subspace $E_{i,j}$ in (2.2). The divisor $S$ is known as the parabolic divisor. (See [MS], [MY], [Bh].)

A parabolic vector bundle is a holomorphic vector bundle $E$ with a parabolic structure $(\{E_{i,j}\}, \{\alpha_{i,j}\})$. For notational convenience, $(E, (\{E_{i,j}\}, \{\alpha_{i,j}\}))$ will be denoted by $E^\ast$.

**Assumption 2.1.** Throughout we will work with the assumption that all the parabolic weights $\alpha_{i,j}$ are rational numbers. We assume that each $x_i$ has at least one nonzero parabolic weight. We also assume that $3(\text{genus}(X) - 1) + n > 0$.

For any $1 \leq i \leq n$ and $1 \leq j \leq l_i + 1$, let $\mathcal{E}_{i,j} \longrightarrow X$ be the holomorphic vector bundle defined by the following short exact sequence of coherent analytic sheaves on $X$:

$$0 \longrightarrow \mathcal{E}_{i,j} \longrightarrow E \longrightarrow E_{x_i}/E_{i,j} \longrightarrow 0$$

(2.4)

(see (2.2)). Since $E_{x_i}/E_{i,j}$ is supported on $x_i$, the subsheaf $\mathcal{E}_{i,j}$ of $E$ actually coincides with $E$ over the open subset $X \setminus \{x_i\} \subset X$.

The space of all holomorphic sections, over an open subset $U \subset X$, of a holomorphic vector bundle $V \longrightarrow X$ is denoted by $\Gamma(U, V)$.

Let

$$\text{End}^P(E^\ast) \subset \text{End}(E)$$

(2.5)

be the coherent analytic subsheaf such that for any open subset $U \subset X$, the subspace

$$\Gamma(U, \text{End}^P(E^\ast)) \subset \Gamma(U, \text{End}(E))$$

consists of all $\mathcal{O}_U$–linear homomorphisms $s : E|_U \longrightarrow E|_U$ satisfying the condition that

$$s(\mathcal{E}_{i,j}|_U) \subset \mathcal{E}_{i,j}|_U$$

for all $x_i \in U$ and all $1 \leq j \leq l_i$, where $\mathcal{E}_{i,j}$ is defined in (2.4).

Let

$$\text{End}^n(E^\ast) \subset \text{End}^P(E^\ast)$$

(2.6)

be the coherent analytic subsheaf consists of all $\mathcal{O}_U$–linear homomorphisms $s : E|_U \longrightarrow E|_U$ satisfying the condition that for any open subset $U \subset X$,

$$s(\mathcal{E}_{i,j}|_U) \subset \mathcal{E}_{i,j+1}|_U$$

for all $x_i \in U$ and all $1 \leq j \leq l_i$.

**Remark 2.2.** It is customary to define $\text{End}^P(E^\ast)$ as the subsheaf of $\text{End}(E)$ that preserves the subspace $E_{i,j} \subset E_{x_i}$ for all $x_i \in U$ and all $1 \leq j \leq l_i$. Similarly, $\text{End}^n(E^\ast)$ is defined to be the subsheaf of $\text{End}^P(E^\ast)$ that takes any $E_{i,j}$ to $E_{i,j+1}$. While the definitions in (2.5) and (2.6) are equivalent to these, we will see that the definitions in (2.5) and (2.6) are more useful for our purpose.
Using the pairing \((\text{End}(E) \otimes \mathcal{O}(S))^2 \longrightarrow \mathcal{O}(2S)\) defined by trace
\[
A \otimes B \longmapsto \text{trace}(AB),
\]
we have
\[
\text{End}^p(E_*)^* = \text{End}^n(E_*) \otimes \mathcal{O}(S).
\]

For any integer \(k \geq 0\), let \(\text{Diff}^k(E, E)\) be the holomorphic vector bundle on \(X\) given by the sheaf of all holomorphic differential operators, of order at most \(k\), from \(E\) to itself. We have the following short exact sequence of holomorphic vector bundles on \(X\)
\[
0 \longrightarrow \text{Diff}^0(E, E) = \text{End}_{\mathcal{O}_X}(E) \longrightarrow \text{Diff}^1(E, E) \longrightarrow \text{End}_{\mathcal{O}_X}(E \otimes \mathcal{O}(-S)) \longrightarrow 0,
\]
where \(TX\) is the holomorphic tangent bundle of \(X\) and \(\sigma_0\) is the symbol map on the first order differential operators. Let
\[
\text{Diff}^1_P(E, E) \subset \text{Diff}^1(E, E)
\]
be the coherent analytic subsheaf consists of all differential operators \(D_U : E|_U \longrightarrow E|_U\), where \(U \subset X\) is any open subset, satisfying the condition that for any \(s \in \Gamma(U, E_{i,j})\),
\[
D_U(s) \in \Gamma(U, E_{i,j})
\]
for all \(x_i \in U\) and all \(1 \leq j \leq l_i\), where \(E_{i,j}\) is constructed in (2.4).

Comparing the definitions given in (2.8) and (2.5) we conclude that
\[
\text{Diff}^1_P(E, E) \cap \text{End}_{\mathcal{O}_X}(E) = \text{End}^p(E_*) ;
\]
the above intersection takes place inside \(\text{Diff}^1(E, E)\) (see (2.7) and (2.8)). Consider the subsheaf
\[
TX \otimes \mathcal{O}_X(-S) \subset TX = \text{Id}_E \otimes TX \subset \text{End}(E) \otimes TX,
\]
where \(S\) is the divisor in (2.1). Define
\[
\text{At}(E_*):= \text{Diff}^1_P(E, E) \cap \sigma_0^{-1}(TX \otimes \mathcal{O}_X(-S)) \subset \text{Diff}^1(E, E),
\]
where \(\sigma_0\) is the projection in (2.7) and \(\text{Diff}^1_P(E, E) \subset \text{Diff}^1(E, E)\) is the subsheaf in (2.8).

Now from (2.7) and (2.9) we get the following short exact sequence of holomorphic vector bundles on \(X\)
\[
0 \longrightarrow \text{End}^p(E_*) \longrightarrow \text{At}(E_*) \longrightarrow TX \otimes \mathcal{O}_X(-S) \longrightarrow 0,
\]
where \(\sigma\) is the restriction of \(\sigma_0\) to \(\text{At}(E_*) \subset \text{Diff}^1(E, E)\). It is straightforward to check that the homomorphism \(\sigma\) in (2.12) is surjective; indeed, this follows immediately from the fact that \(TX \otimes \mathcal{O}_X(-S) \subset \sigma_0(\text{Diff}^1_P(E, E))\).

**Definition 2.3.** The vector bundle \(\text{At}(E_*)\) in (2.11) will be called the *Atiyah bundle* for the parabolic bundle \(E_\ast\), and the sequence in (2.12) will be called the *Atiyah exact sequence* for the parabolic bundle \(E_\ast\).

When \(S\) is the zero divisor (meaning \(n = 0\) in (2.1)), then \(\text{At}(E_*)\) is the usual Atiyah bundle \(\text{At}(E)\) for \(E\), and (2.12) is the usual Atiyah exact sequence for \(E\). (See [At].)
2.2. Holomorphic connections on a parabolic bundle. The holomorphic cotangent bundle of \( X \) will be denoted by \( K_X \).

Let \( V \) be a holomorphic vector bundle on \( X \). A *logarithmic connection* on \( V \) singular over \( S \) is a holomorphic differential operator of order one

\[
D : V \longrightarrow V \otimes K_X \otimes O_X(S)
\]

satisfying the Leibniz identity, which says that

\[
D(fs) = fD(s) + s \otimes df
\]  

(2.13)

for any locally defined holomorphic function \( f \) on \( X \) and any locally defined holomorphic section \( s \) of \( V \).

We note that any logarithmic connection on \( V \) is flat because \( \Omega^{2,0}_X = 0 \).

Take a point \( x_i \in S \). The fiber of \( K_X \otimes O_X(S) \) over \( x_i \) is identified with \( \mathbb{C} \) by the Poincaré adjunction formula [GH, p. 146]. To explain this isomorphism

\[
(K_X \otimes O_X(S))_{x_i} \sim \mathbb{C},
\]  

(2.14)

let \( z \) be a holomorphic coordinate function on \( X \) defined on an analytic open neighborhood of \( x_i \) such that \( z(x_i) = 0 \). Then we have the isomorphism \( \mathbb{C} \longrightarrow (K_X \otimes O_X(S))_{x_i} \) that sends any \( c \in \mathbb{C} \) to \( c \cdot \frac{dz}{z} \in (K_X \otimes O_X(S))_{x_i} \). It is straightforward to check that this map \( \mathbb{C} \longrightarrow (K_X \otimes O_X(S))_{x_i} \) is actually independent of the choice of the above holomorphic coordinate function \( z \).

Let \( D_V : V \longrightarrow V \otimes K_X \otimes O_X(S) \) be a logarithmic connection on \( V \). Using the Leibniz identity in (2.13) it is straightforward to deduce that the composition of homomorphisms

\[
V \xrightarrow{D_V} V \otimes K_X \otimes O_X(S) \longrightarrow (V \otimes K_X \otimes O_X(S))_{x_i} \sim V_{x_i}
\]  

(2.15)

is \( O_X \)–linear; the above isomorphism \( (V \otimes K_X \otimes O_X(S))_{x_i} \sim V_{x_i} \) is given by the isomorphism in (2.14). Therefore, the composition of homomorphisms in (2.15) produces a \( \mathbb{C} \)–linear homomorphism

\[
\text{Res}(D_V, x_i) : V_{x_i} \longrightarrow V_{x_i},
\]  

(2.16)

which is called the *residue* of \( D_V \) at \( x_i \); see [De]. If \( \lambda_1, \ldots, \lambda_r \) are the generalized eigenvalues of \( \text{Res}(D_V, x_i) \) with multiplicity, where \( r = \text{rank}(V) \), then the generalized eigenvalues of the local monodromy of \( D \) around \( x_i \) are

\[
\exp(-2\pi \sqrt{-1}\lambda_1), \exp(-2\pi \sqrt{-1}\lambda_1), \ldots, \exp(-2\pi \sqrt{-1}\lambda_r)
\]  

(2.17)

[De].

We now recall another description of the logarithmic connections on \( V \). Consider the short exact sequence in (2.7)

\[
0 \longrightarrow \text{End}(V) \longrightarrow \text{Diff}^1(V, V) \xrightarrow{\tilde{\sigma}_V} TX \otimes \text{End}(V) \longrightarrow 0
\]  

(2.18)

for \( V \). Define

\[
\text{At}(V, S) := \tilde{\sigma}_V^{-1}(TX \otimes O_X(-S)) \subset \text{Diff}^1(V, V),
\]
where \( TX \otimes O_X(-S) \subset TX \otimes \text{End}(V) \) is the subbundle defined as in (2.10) for \( V \). So from (2.18) we have the short exact sequence of holomorphic vector bundles

\[
0 \rightarrow \text{End}(V) \rightarrow \text{At}(V, S) \xrightarrow{\sigma_V} TX \otimes O_X(-S) \rightarrow 0,
\]

(2.19)

where \( \sigma_V \) is the restriction of the projection \( \hat{\sigma}_V \) in (2.18) to the subsheaf \( \text{At}(V, S) \subset \text{Diff}^1(V, V) \). A logarithmic connection on \( V \) is a holomorphic splitting of the short exact sequence in (2.19); in other words, a logarithmic connection on \( V \) is a holomorphic homomorphism of vector bundles

\[
h : TX \otimes O_X(-S) \rightarrow \text{At}(V, S)
\]
such that \( \sigma_V \circ h = \text{Id}_{TX \otimes O_X(-S)} \), where \( \sigma_V \) is the homomorphism in (2.19).

Take a parabolic vector bundle \( E^* = (E, \{E_{i,j}\}, \{\alpha_{i,j}\}) \); see (2.2), (2.3).

A connection on \( E^* \) is a logarithmic connection \( D \) on \( E \), singular over \( S \), such that

1. \( \text{Res}(D, x_i)(E_{i,j}) \subset E_{i,j} \) for all \( 1 \leq j \leq l_i, 1 \leq i \leq n \) (see (2.2)), and
2. the endomorphism of \( E_{i,j}/E_{i,j+1} \) induced by \( \text{Res}(D, x_i) \) coincides with multiplication by the parabolic weight \( \alpha_{i,j} \) for all \( 1 \leq j \leq l_i, 1 \leq i \leq n \).

(See [BL, Section 2.2].) A necessary and sufficient condition for \( E^* \) to admit a connection is given in [BL]. The condition in question says that \( E^* \) admits a connection if and only if the parabolic degree of every direct summand of \( E^* \) is zero [BL, p. 594, Theorem 1.1].

A holomorphic splitting of the Atiyah exact sequence in (2.12) (see Definition 2.3) is a holomorphic homomorphism of vector bundles

\[
h : TX \otimes O_X(-S) \rightarrow \text{At}(E^*)
\]
such that

\[
\sigma \circ h = \text{Id}_{TX \otimes O_X(-S)},
\]

(2.20)

where \( \sigma \) is the projection in (2.12).

**Lemma 2.4.** Giving a holomorphic splitting \( h \) of the Atiyah exact sequence for \( E^* \) (see (2.20)) is equivalent to giving a logarithmic connection \( D \) on \( E \) satisfying the following condition: for every \( x_i \in S \),

\[
\text{Res}(D, x_i)(E_{i,j}) \subset E_{i,j}
\]

for all \( 1 \leq j \leq l_i \) (see (2.2)), where \( \text{Res}(D, x_i)(E_{i,j}) \) is constructed in (2.16).

**Proof.** Let \( h \) be a holomorphic splitting of the Atiyah exact sequence in (2.12). In other words, \( h : TX \otimes O_X(-S) \rightarrow \text{At}(E^*) \) is a holomorphic homomorphism such that (2.20) holds. Recall from (2.11) that \( \text{At}(E^*) \subset \text{Diff}^1_p(E, E) \). The composition of homomorphisms

\[
TX \otimes O_X(-S) \xrightarrow{h} \text{At}(E^*) \hookrightarrow \text{Diff}^1_p(E, E)
\]

will be denoted by \( \tilde{h} \). Let \( \tilde{h}' \) denote the composition of homomorphisms

\[
TX \otimes O_X(-S) \xrightarrow{\tilde{h}} \text{Diff}^1_p(E, E) \xrightarrow{\iota_0} \text{Diff}^1(E, E)
\]

(2.21)

(see (2.8) for the inclusion map \( \iota_0 \)). Since (2.20) holds, we conclude that

\[
\sigma_0 \circ \tilde{h}' = \text{Id}_{TX \otimes O_X(-S)},
\]
where $\sigma_0$ is the projection in (2.7). Therefore, $\tilde{h}'$ is a logarithmic connection on $E$; this logarithmic connection on $E$ will be denoted by $D$. Since

$$\tilde{h}' = \iota_0 \circ \tilde{h}$$

(see (2.21)), the residues of the above defined logarithmic connection $D$ satisfy the following condition: for every $x_i \in S$,

$$\text{Res}(D, x_i)(E_{i,j}) \subset E_{i,j}$$

for all $1 \leq j \leq l_i$.

To prove the converse, let $D$ be a logarithmic connection on $E$ such that

$$\text{Res}(D, x_i)(E_{i,j}) \subset E_{i,j}$$

(2.22)

for all $x_i \in S$ and all $1 \leq j \leq l_i$. So $D$ gives a holomorphic homomorphism

$$h : TX \otimes O_X(-S) \longrightarrow \text{At}(E, S)$$

(see (2.19) for $\text{At}(E, S)$) such that the composition of homomorphisms

$$TX \otimes O_X(-S) \xrightarrow{h} \text{At}(E, S) \longrightarrow TX \otimes O_X(-S)$$

coincides with the identity map of $TX \otimes O_X(-S)$; see (2.19) for the above projection $\text{At}(E, S) \longrightarrow TX \otimes O_X(-S)$. The given condition in (2.22) implies that

$$h(TX \otimes O_X(-S)) \subset \text{Diff}_p(E, E) \bigcap \text{At}(E, S) = \text{At}(E_*) ;$$

note that from (2.11) we have

$$\text{At}(E_*) := \text{Diff}_p(E, E) \bigcap \sigma_0^{-1}(TX \otimes O_X(-S)) \subset \sigma_0^{-1}(TX \otimes O_X(-S)) = \text{At}(E, S).$$

Consequently, the homomorphism $h$ gives a holomorphic splitting, as in (2.20), of the Atiyah exact sequence for $E_*$.

**Corollary 2.5.** *Giving a connection on the parabolic bundle $E_*$ is equivalent to giving a holomorphic splitting $h$ of the Atiyah exact sequence for $E_*$ (see (2.12)) such that the logarithmic connection $D$ on $E$ associated to $h$ (see Lemma 2.4) satisfies the following condition: for every $x_i \in S$, the residue $\text{Res}(D, x_i)$ induces the endomorphism $\alpha_{i,j} \cdot \text{Id}_{E_{i,j}/E_{i,j+1}}$ of the quotient space $E_{i,j}/E_{i,j+1}$ for all $1 \leq j \leq l_i$.*

**Proof.** Note that from Lemma 2.4 we know that

$$\text{Res}(D, x_i)(E_{i,j}) \subset E_{i,j}$$

for all $x_i \in S$ and all $1 \leq j \leq l_i$. Therefore, $\text{Res}(D, x_i)$ induces an endomorphism of the quotient space $E_{i,j}/E_{i,j+1}$.

The result follows from Lemma 2.4 and the definition of a connection on $E_*$. □
2.3. **A homomorphism associated to a connection.** Let $E_* = (E, \{E_{i,j}\}, \{\alpha_{i,j}\})$ be a parabolic vector bundle on $X$ with parabolic divisor $S$. Let $D$ be a connection on $E_*$. Using $D$ we will construct a first order holomorphic differential operator

$$
\mathcal{D}_0 : \text{End}^\mathbb{P}(E_*) \longrightarrow \text{End}^n(E_*) \otimes K_X \otimes \mathcal{O}_X(S) \tag{2.23}
$$

(see (2.5) and (2.6) for $\text{End}^\mathbb{P}(E_*)$ and $\text{End}^n(E_*)$ respectively).

To construct $\mathcal{D}_0$, take any holomorphic section $\Phi \in \Gamma(U, \text{End}(E))$, where $U \subset X$ is any open subset. Let

$$
\mathcal{A}_U : \Gamma(U, E) \longrightarrow \Gamma(U, E \otimes K_X \otimes \mathcal{O}_X(S))
$$

be the homomorphism defined by

$$
\mathcal{A}_U(s) = D(\Phi(s)) - (\Phi \otimes \text{Id}_{K_X \otimes \mathcal{O}_X(S)})(D(s)).
$$

This $\mathcal{A}_U$ is evidently $\mathcal{O}_U$–linear. Hence we have

$$
\mathcal{A}_U \in \Gamma(U, \text{End}(E) \otimes K_X \otimes \mathcal{O}_X(S)).
$$

From the properties of $D$ it follows that

$$
\mathcal{A}_U \in \Gamma(U, \text{End}^n(E_*) \otimes K_X \otimes \mathcal{O}_X(S)) \subset \Gamma(U, \text{End}(E) \otimes K_X \otimes \mathcal{O}_X(S))
$$

if $\Phi \in \Gamma(U, \text{End}^\mathbb{P}(E_*))$. The homomorphism $\mathcal{D}_0$ in (2.23) is defined by $\Phi \longmapsto \mathcal{A}_U$.

Recall from (2.12) that $\text{End}^\mathbb{P}(E_*)$ is a holomorphic subbundle of $\text{At}(E_*)$. We will now extend $\mathcal{D}_0$ in (2.23) to a first order holomorphic differential operator

$$
\mathcal{D} : \text{At}(E_*) \longrightarrow \text{End}^n(E_*) \otimes K_X \otimes \mathcal{O}_X(S). \tag{2.24}
$$

To construct $\mathcal{D}$, take holomorphic sections

$$
\Phi \in \Gamma(U, \text{At}(E_*)) \quad \text{and} \quad \nu \in \Gamma(U, TX \otimes \mathcal{O}_X(-S)),
$$

where $U \subset X$ is any open subset. Denote

$$
w := \sigma(\Phi) \in \Gamma(U, TX \otimes \mathcal{O}_X(-S)), \tag{2.25}
$$

where $\sigma$ is the projection in (2.12). Take any $s \in \Gamma(U, E)$. So

$$
\Phi(s) \in \Gamma(U, E)
$$

(recall from (2.11) that $\text{At}(E_*) \subset \text{Diff}^1(E_*, E)$), and hence

$$
D(\Phi(s)) \in \Gamma(U, E \otimes K_X \otimes \mathcal{O}_X(S)).
$$

Therefore, we have

$$
\langle D(\Phi(s)), \nu \rangle \in \Gamma(U, E), \tag{2.26}
$$

where $\langle -, - \rangle$ is the natural duality pairing

$$
(K_X \otimes \mathcal{O}_X(S)) \otimes (TX \otimes \mathcal{O}_X(-S)) \longrightarrow \mathcal{O}_X. \tag{2.27}
$$

We have $\langle D(s), \nu \rangle \in \Gamma(U, E)$, so

$$
\Phi(\langle D(s), \nu \rangle) \in \Gamma(U, E), \tag{2.28}
$$

where $\langle -, - \rangle$ is the pairing in (2.27). Consider the Lie bracket of vector fields

$$
[v, w] \in \Gamma(U, TX \otimes \mathcal{O}_X(-S)),
$$
where $w$ is defined in (2.25). We have

$$\langle D(s), [v, w] \rangle \in \Gamma(U, E).$$

Let $B_U : \Gamma(U, E) \rightarrow \Gamma(U, E)$ be the homomorphism defined by

$$s \mapsto \langle D(\Phi(s)), v \rangle - \Phi(\langle D(s), v \rangle) - \langle D(s), [v, w] \rangle$$

(see (2.26), (2.28) and (2.29)). This homomorphism $B_U$ is evidently $O_U$–linear, and hence we have

$$B_U \in \Gamma(U, \text{End}(E)).$$

It is now straightforward to check that $B_U \in \Gamma(U, \text{End}^n(E)) \subset \Gamma(U, \text{End}(E))$.

The homomorphism $D$ in (2.24) is uniquely defined by the following property: For any open subset $U \subset X$ and sections

$$\Phi \in \Gamma(U, \text{At}(E_*)) \quad \text{and} \quad v \in \Gamma(U, TX \otimes \mathcal{O}_X(-S)),$$

the equality

$$\langle D(\Phi), v \rangle = B_U$$

holds, where $B_U$ is constructed in (2.30) from $\Phi$ and $v$ using $D$, and $\langle -, - \rangle$ is the pairing in (2.27).

From the constructions of $D$ and $D_0$ (see (2.23)) it follows immediately that the restriction of $D$ to $\text{End}^p(E_*) \subset \text{At}(E_*)$ actually coincides with $D_0$.

3. Infinitesimal deformations and isomonodromy

3.1. Infinitesimal deformations. Let $E_*$ be a parabolic vector bundle over $X$ with parabolic divisor $S$. Then the infinitesimal deformations of $E_*$, keeping the pair $(X, S)$ fixed, are parametrized by $H^1(X, \text{End}^p(E_*))$, where $\text{End}^p(E_*)$ is defined in (2.5) [MS]. We recall that the infinitesimal deformations of the pair $(X, S)$ are parametrized by $H^1(X, TX \otimes \mathcal{O}_X(-S))$.

Lemma 3.1. The infinitesimal deformations of the triple $(X, S, E_*)$

are parametrized by $H^1(X, \text{At}(E_*))$, where $\text{At}(E_*)$ is defined in (2.11).

Proof. The proof is very similar to that of [Ch2] p. 1413, Proposition 4.3] (see also [Ch1]).

The lemma actually follows from [Ch2] p. 1413, Proposition 4.3] once we invoke the correspondence between the parabolic bundles and the orbifold bundles. This is explained below.

For any $x_i \in S$, let $N_i$ be the smallest positive integer such that for all $1 \leq j \leq l_i$,

$$\alpha_{i,j} = \frac{m_{i,j}}{N_i},$$

where $m_{i,j}$ are nonnegative integers; see (2.3) and Assumption 2.1. There is a ramified Galois covering

$$\varphi : Y \rightarrow X$$

(3.1)
satisfying the following two conditions:

- \( \varphi \) is unramified over the complement \( X \setminus S \), and
- for every \( x_i \in S \) and one (hence every) point \( y \in \varphi^{-1}(x_i) \), the order of ramification of \( \varphi \) at \( y \) is \( N_i \).

Such a covering \( \varphi \) exists; see [Na, p. 26, Proposition 1.2.12] and Assumption 2.1.

Let \( \Gamma_{\varphi} = \text{Gal}(\varphi) \subset \text{Aut}(Y) \) be the Galois group for the ramified covering \( \varphi \), so \( X = Y/\Gamma_{\varphi} \). An equivariant vector bundle over \( Y \) is a holomorphic vector bundle \( V \to Y \) equipped with a lift of the action of \( \Gamma_{\varphi} \). In other words,

- \( \Gamma_{\varphi} \) acts holomorphically on the total space of \( V \), and
- the action of any \( g \in \Gamma_{\varphi} \) on \( V \) is a holomorphic automorphism of the vector bundle \( V \) over the automorphism \( g \) of \( Y \). In particular, the projection map \( V \to Y \) is \( \Gamma_{\varphi} \)-equivariant.

There is a natural equivalence of categories between the parabolic vector bundles on \( X \) whose parabolic weights at each \( x_i \) are integral multiples of \( \frac{1}{N_i} \) and the equivariant vector bundles on \( Y \) [Bi1, Bo1, Bo2].

Let \( F^* \) be a parabolic vector bundle on \( X \) whose parabolic weights at each \( x_i \) are integral multiples of \( \frac{1}{N_i} \). The holomorphic vector bundle underlying \( F^* \) will be denoted by \( F \). Let \( V \) be the equivariant vector bundle on \( Y \) corresponding to \( F \). The action \( \Gamma_{\varphi} \) on \( V \) produces a homomorphism from \( \Gamma_{\varphi} \) to the group \( \text{Aut}(\varphi_*V) \) of holomorphic automorphisms of the direct image \( \varphi_*V \), over the identity map of \( X \). Then we have

\[
F = (\varphi_*V)^{\Gamma_{\varphi}} \subset \varphi_*V,
\]

where \( (\varphi_*V)^{\Gamma_{\varphi}} \) denotes the invariant part for the action of \( \Gamma_{\varphi} \) on \( \varphi_*V \).

Take a holomorphic family of compact Riemann surfaces equipped with \( n \) ordered marked points. Assume that this family is parametrized by \( T \), and that there is a point \( t_0 \in T \) such that the fiber over \( t_0 \) is the given pair \( (X, S) \). Then the construction of the ramified Galois covering of \( X \), done in [Na, p. 26, Proposition 1.2.12], extends to produce a family of ramified Galois coverings of all Riemann surfaces over an open neighborhood of \( t_0 \) in \( T \).

Let \( V \) be the equivariant bundle on \( Y \) corresponding to the parabolic vector bundle \( E^* \) in the lemma. Then \( \text{End}^{\mathcal{P}}(E^*) \) is the holomorphic vector bundle underlying the parabolic bundle that corresponds to the equivariant vector bundle \( \text{End}(V) \) on \( Y \). The action of \( \Gamma_{\varphi} \) on \( Y \) produces an action of \( \Gamma_{\varphi} \) on \( TY \), and \( TX \otimes \mathcal{O}_X(-S) \) is the holomorphic line bundle underlying the corresponding parabolic line bundle on \( X \). The actions of \( \Gamma_{\varphi} \) on \( V \) and \( Y \) together produce an action of \( \Gamma_{\varphi} \) on \( \text{At}(V) \). The Atiyah bundle \( \text{At}(E^*) \) is the holomorphic vector bundle underlying the parabolic bundle corresponding to the equivariant vector bundle \( \text{At}(V) \) on \( Y \). This implies that

\[
H^1(X, \text{At}(E^*)) = H^1(Y, \text{At}(V))^{\Gamma_{\varphi}}.
\]  

(3.2)

In view of (3.2), the lemma follows immediately from [Ch2, p. 1413, Proposition 4.3].
Let
\[ 0 = H^0(X, TX \otimes \mathcal{O}_X(-S)) \longrightarrow H^1(X, \text{End}^p(E_*)) \xrightarrow{p_2} H^1(X, \text{At}(E_*)) \quad (3.3) \]
be the long exact sequence of cohomologies associated to the short exact sequence in \((2.12);\) we have \(H^0(X, TX \otimes \mathcal{O}_X(-S)) = 0\) by Assumption \((2.1)\). The projection \(p_2\) in \((3.3)\) is the forgetful map that sends an infinitesimal deformation of the triple \((X, S, E_*)\) to the infinitesimal deformation of the pair \((X, S)\) obtained from it by simply forgetting \(E_*\). The injective homomorphism \(p_1\) in \((3.3)\) sends an infinitesimal deformation of \(E_*\) to the infinitesimal deformation of \((X, S, E_*)\) obtained from it by keeping the pair \((X, S)\) fixed.

**Lemma 3.2.** Assume that the parabolic bundle \(E_*\) has a connection \(D\) such that the local monodromy of \(D\) around each point of \(S\) is semisimple (meaning diagonalizable). Then the local monodromy of every connection on \(E_*\) around each point of \(S\) is also semisimple.

**Proof.** Take a ramified Galois covering
\[ \varphi : Y \longrightarrow X, \]
as in \((3.1)\), satisfying the following two conditions:
- \(\varphi\) is unramified over the complement \(X \setminus S\), and
- for every \(x_i \in S\) and one (hence every) point \(y \in \varphi^{-1}(x_i)\), the order of ramification of \(\varphi\) at \(y\) is \(N_i\).

Let \(V\) be the equivariant vector bundle on \(Y\) corresponding to \(E_*\). Then \(D\) corresponds to an equivariant holomorphic connection on \(V\). The space of all connections on \(E_*\) is an affine space for the vector space \(H^0(X, \text{End}^a(E_*) \otimes K_X \otimes \mathcal{O}_X(S))\), where \(\text{End}^a(E_*)\) is defined in \((2.6)\). On the other hand the space of all equivariant holomorphic connections on \(V\) is an affine space for the vector space \(H^0(Y, \text{End}(V) \otimes K_Y)^\Gamma\), where \(\Gamma\) is the Galois group for the covering map \(\varphi\).

Since we have
\[ H^0(X, \text{End}^a(E_*) \otimes K_X \otimes \mathcal{O}_X(S)) = H^0(Y, \text{End}(V) \otimes K_Y)^\Gamma, \]
we conclude that every connection \(D'\) on \(E_*\) is given by an equivariant connection on \(V\). This implies that the order of the local monodromy of \(D'\) around any point \(x_i \in S\) is a sub-multiple of \(N_i\). This implies that the local monodromy of \(D'\) around every point of \(S\) is semisimple. \(\square\)

Let \(D\) be a connection on the parabolic bundle \(E_*\). We assume that the local monodromy of \(D\) around each point of \(S\) is semisimple. As mentioned above, the space of all connections on \(E_*\) is an affine space for the vector space \(H^0(X, \text{End}^a(E_*) \otimes K_X \otimes \mathcal{O}_X(S))\), where \(\text{End}^a(E_*)\) is defined in \((2.6)\). This implies that the infinitesimal deformations of the connection \(D\), keeping \((X, S, E_*)\) fixed, are parametrized by \(H^0(X, \text{End}^a(E_*) \otimes K_X \otimes \mathcal{O}_X(S))\).

Let \(\mathcal{A}_*\) be the following two-term complex of sheaves on \(X\)
\[ \mathcal{A}_* : \mathcal{A}_0 = \text{End}^p(E_*) \xrightarrow{D_0} \mathcal{A}_1 = \text{End}^a(E_*) \otimes K_X \otimes \mathcal{O}_X(S), \quad (3.4) \]
where \( \mathcal{D}_0 \) is the differential operator in (2.23), and \( \mathcal{A}_i \) is at the \( i \)-th position.

The following lemma is a standard fact.

**Lemma 3.3.** The infinitesimal deformations of the pair \((E_*, D)\), keeping \((X, S)\) fixed, are parametrized by the first hypercohomology \( H^1(\mathcal{A}_*) \), where \( \mathcal{A}_* \) is the complex in (3.4).

The complex \( \mathcal{A}_* \) in (3.4) fits in the following short exact sequence of complexes of sheaves on \( X \)

\[
\begin{array}{cccccccccc}
& & & & & & 0 & & & & \\
& & & & & & \downarrow & & & & \\
& & & & & & 0 & \rightarrow & \text{End}^n(E_*) \otimes K_X \otimes \mathcal{O}_X(S) & & \\
0 & \rightarrow & & & & & & & & & \\
& & & & & & \downarrow & & & & \\
& & & & & & id & \rightarrow & \text{End}^n(E_*) \otimes K_X \otimes \mathcal{O}_X(S) & & \\
& & & & & & \downarrow & & & & \\
& & & & \mathcal{A}_* : & \rightarrow & & & & & \\
& & & \downarrow & & & & & & & \\
& & & id & \rightarrow & 0 & & & & \\
& & & \downarrow & & & & & & & \\
& & & 0 & \rightarrow & 0 & & & & \\
\end{array}
\]

Let

\[
\rightarrow H^0(X, \text{End}^n(E_*) \otimes K_X \otimes \mathcal{O}_X(S)) \xrightarrow{\alpha_1} H^1(\mathcal{A}_*) \xrightarrow{\alpha_2} H^1(X, \text{End}^n(E_*)) \rightarrow (3.5)
\]

be the long exact sequence of hypercohomologies associated to the above short exact sequence of complexes. The homomorphism \( \alpha_2 \) in (3.5) sends an infinitesimal deformation of the pair \((E_*, D)\) to the infinitesimal deformation of \( E_* \) obtained from it by simply forgetting the connection. The homomorphism \( \alpha_1 \) in (3.5) sends an infinitesimal deformation of the connection \( D \) to the infinitesimal deformation of the pair \((E_*, D)\) obtained from it by keeping the parabolic bundle \( E_* \) fixed.

Let \( \mathcal{B}_* \) denote the following two-term complex of sheaves on \( X \)

\[
\mathcal{B}_* : \mathcal{B}_0 = \text{At}(E_*) \xrightarrow{\mathcal{D}} \mathcal{B}_1 = \text{End}^n(E_*) \otimes K_X \otimes \mathcal{O}_X(S), \quad (3.6)
\]

where \( \mathcal{D} \) is the homomorphism in (2.24), and \( \mathcal{B}_i \) is at the \( i \)-th position.

**Lemma 3.4.** The infinitesimal deformations of the quadruple 

\((X, S, E_*, D)\)

are parametrized by the first hypercohomology \( H^1(\mathcal{B}_*) \), where \( \mathcal{B}_* \) is the complex in (3.6).

**Proof.** The proof is very similar to the proof of [Ch2, p. 1415, Proposition 4.4] (see also [Ch1]). In fact, it can also be deduced from [Ch2, p. 1415, Proposition 4.4], as done in the proof of Lemma 3.1. This is elaborated below.

Take the ramified Galois covering \((Y, \varphi)\) in (3.1). As in the proof of Lemma 3.1 \( V \) denotes the equivariant vector bundle on \( Y \) corresponding to the parabolic vector bundle \( E_* \). The connection \( D \) on \( E_* \) corresponds to a \( \Gamma_\varphi \)-invariant holomorphic connection on \( V \), where \( \Gamma_\varphi \)
is, as before, the Galois group for $\varphi$; this $\Gamma_\varphi$–invariant holomorphic connection on $V$ will be denoted by $D'$. Let

$$\mathcal{B}' : \mathcal{B}'_0 := \operatorname{At}(V) \xrightarrow{D'} \mathcal{B}_1 := \operatorname{End}(V) \otimes K_Y$$

be the complex in [Ch2, p. 1415, Proposition 4.4] for $D'$; it is the same complex as in (3.6) when there is no parabolic structure (meaning the parabolic structure is trivial). We note that the differential operator $D'$ in (3.7) is $\Gamma_\varphi$–equivariant, because the connection $D'$ on $V$ is $\Gamma_\varphi$–invariant. The holomorphic vector bundle $\operatorname{End}^n(E_*) \otimes K_X \otimes \mathcal{O}_X(S)$ coincides with the holomorphic vector bundle underlying the parabolic bundle that corresponds to the $\Gamma_\varphi$–equivariant bundle $\operatorname{End}(V) \otimes K_Y$ on $Y$. It was noted in the proof of Lemma 3.1 that $\operatorname{At}(E_*)$ is the holomorphic vector bundle underlying the parabolic bundle corresponding to the equivariant bundle $\operatorname{At}(V)$ on $Y$. Moreover, the operator $D$ in (3.6) coincides with the one given by $D'$ on the $\Gamma_\varphi$–invariant part of the direct image. These together imply that

$$H^1(\mathcal{B}_*) = H^1(\mathcal{B}'_*)^{\Gamma_\varphi},$$

where $H^1(\mathcal{B}'_*)^{\Gamma_\varphi} \subset H^1(\mathcal{B}')$ is the invariant part for the action of $\Gamma_\varphi$.

In view of (3.8), the lemma follows from [Ch2, p. 1415, Proposition 4.4]. \hfill $\square$

The complex $\mathcal{B}_*$ in (3.7) fits in the following short exact sequence of complexes of sheaves on $X$

$$0 \rightarrow \mathcal{A}_* : \xrightarrow{\partial} \mathcal{B}_* : \xrightarrow{D} \mathcal{B}_1 : \xrightarrow{\operatorname{id}} 0$$

where the vertical exact sequence in the left is the one in (2.12); see (3.4) and (3.6) for $\mathcal{A}_*$ and $\mathcal{B}_*$ respectively. Let

$$\rightarrow H^1(\mathcal{A}_*) \xrightarrow{\beta_1} H^1(\mathcal{B}_*) \xrightarrow{\beta_2} H^1(X, TX \otimes \mathcal{O}_X(-S)) \rightarrow$$

be the long exact sequence of hypercohomologies associated to the above short exact sequence of complexes. The homomorphism $\beta_2$ in (3.9) sends an infinitesimal deformation of the quadruple $(X, S, E_*, D)$ to the infinitesimal deformation of $(X, S)$ obtained from it by simply forgetting the pair $(E_*, D)$. The homomorphism $\beta_1$ in (3.9) sends an infinitesimal deformation of the pair $(E_*, D)$ to the infinitesimal deformation of the quadruple $(X, S, E_*, D)$ obtained from it by keeping the pair $(X, S)$ fixed.
3.2. Character variety and the monodromy map. Fix an integer \( r \geq 1 \). Fix a base point \( b_0 \in X \setminus S \), and denote by \( \Hom(\pi_1(X \setminus S, b_0), \GL(r, \mathbb{C})) \) the space of all homomorphisms from \( \pi_1(X \setminus S, b_0) \) to \( \GL(r, \mathbb{C}) \). Given any \( \rho \in \Hom(\pi_1(X \setminus S, b_0), \GL(r, \mathbb{C})) \), we consider \( \mathbb{C}^r \) as a \( \pi_1(X \setminus S, b_0) \)-module by combining \( \rho \) with the standard \( r \)-dimensional representation of \( \GL(r, \mathbb{C}) \). We recall that \( \rho \) is called completely reducible if the \( \pi_1(X \setminus S, b_0) \)-module \( \mathbb{C}^r \) corresponding to \( \rho \) is a direct sum of irreducible \( \pi_1(X \setminus S, b_0) \)-modules. Let

\[
\Hom(\pi_1(X \setminus S, b_0), \GL(r, \mathbb{C})) \subset \widetilde{\Hom}(\pi_1(X \setminus S, b_0), \GL(r, \mathbb{C}))
\]

be the space of all completely reducible representations. The adjoint action of \( \GL(r, \mathbb{C}) \) on itself produces an action of \( \GL(r, \mathbb{C}) \) on \( \Hom(\pi_1(X \setminus S, b_0), \GL(r, \mathbb{C})) \). The quotient space

\[
\mathcal{R}_X(r) = \Hom(\pi_1(X \setminus S, b_0), \GL(r, \mathbb{C}))/\GL(r, \mathbb{C})
\]

(3.10)

is a normal quasiprojective variety defined over \( \mathbb{C} \). (See [Si], [LS], and references therein, for \( \mathcal{R}_X(r) \).)

For another base point \( b_0' \in X \setminus S \), the two groups \( \pi_1(X \setminus S, b_0) \) and \( \pi_1(X \setminus S, b_0') \) are identified up to inner automorphisms, and hence \( \mathcal{R}_X(r) \) does not depend on the choice of the base point \( b_0 \). The complex structure of \( X \) does not play any role in the construction of \( \mathcal{R}_X(r) \); the space \( \mathcal{R}_X(r) \) depends only on the topological surface underlying \( X \setminus S \).

A connection \( D \) on a parabolic vector bundle \( E_\ast \) is called completely reducible if it is a direct sum of irreducible logarithmic connections.

Fix the dimensions of the subspaces \( E_{i,j} \) and fix the parabolic weights \( \alpha_{i,j} \); see (2.2), (2.3). Let \( \mathcal{M}_X(r) \) denote the moduli space of pairs \( (E_\ast, D) \), where \( E_\ast \) is a parabolic vector bundle of rank \( r \) on \( X \) having the given parabolic type and \( D \) is a completely reducible connection on \( E_\ast \); see [In], [IIS1], [IIS2], [Iw], [BBP] and references therein for the moduli space.

Since any logarithmic connection on a holomorphic vector bundle on \( X \) is flat, we can associate a monodromy representation to any logarithmic connection. Consequently, we have a holomorphic map

\[
\mathbb{M}_X : \mathcal{M}_X(r) \longrightarrow \mathcal{R}_X(r),
\]

(3.11)

where \( \mathcal{R}_X(r) \) is constructed in (3.10), that sends any connection to its monodromy.

Let

\[
\varpi : X_T \longrightarrow T
\]

(3.12)

be a holomorphic family of compact connected Riemann surfaces parametrized by a simply connected complex manifold \( T \). For any \( t \in T \), the fiber \( \varpi^{-1}(t) \) will be denoted by \( X_t \). For \( 1 \leq i \leq n \), let

\[
\phi_i : T \longrightarrow X_T
\]

(3.13)

be a holomorphic section such that \( \phi_i(T) \cap \phi_j(T) = \emptyset \) for all \( i \neq j \). Fix a base point \( t_0 \in T \). Denote \( X_{t_0} \) by \( X \), and also denote \( s_i(t_0) \) by \( x_i \) for every \( 1 \leq i \leq n \). As before, denote the subset \( \{x_1, \ldots, x_n\} \subset X \) by \( S \). For any \( t \in T \), the subset \( \{\phi_1(t), \ldots, \phi_n(t)\} \subset X_t := \varpi^{-1}(t) \) will be denoted by \( S_t \).

Since the parameter space \( T \) is simply connected, \( \pi_1(X \setminus S, b_0) \) and \( \pi_1(X_t \setminus S_t, b_t) \) are identified up to inner automorphisms. This implies that the character variety \( \mathcal{R}_X(r) \) constructed as in (3.10) is canonically identified with \( \mathcal{R}_X(r) \).
Let 
\[ M_T(r) \to T \] 
be the relative moduli space of parabolic bundles with connection for the family \( X_T \) in (3.12). In view of the above observation that \( R_{X_t}(r) \) is identified with \( R_{X}(r) \), the monodromy maps \( M_X \) in (3.11) for points of \( T \) actually fit together to produce a holomorphic map 
\[ M : M_T(r) \to R_X(r). \] (3.15)

Let \( E_T^* \) be a holomorphic family of parabolic bundles on \( X_T \), and let \( D_T \) be a relative connection \( E_T^* \). In other words, the pair \((E_T^*, D_T)\) corresponds to a holomorphic section 
\[ \Psi : T \to M_T(r) \] (3.16)

of the holomorphic family of moduli spaces in (3.14).

A holomorphic family of the above type \((E_T^*, D_T)\), of parabolic bundles with connection, is called isomonodromic if the composition \( M \circ \Psi \) is a constant map, where \( M \) is the monodromy map in (3.15) and \( \Psi \) is the map in (3.16). This condition of being isomonodromy defines a holomorphic foliation on \( M_T(r) \) which is transversal to the holomorphic foliation given by the projection \( M_T(r) \to T \) in (3.14). In other words, the direct sum of these two distributions coincides with the full tangent bundle \( TM_T(r) \to M_T(r) \), on the smooth locus of \( M_T(r) \).

Let \( D \) be a connection on a parabolic vector bundle \( E_* \) over \( X \) with parabolic structure over \( S \). We assume that the local monodromy of \( D \) around each point of \( S \) is semisimple. The foliation given by isomonodromy produces a homomorphism from the space of infinitesimal deformations of the pair \((X, S)\) to the space of infinitesimal deformations of the quadruple \((X, S, E_*, D)\). In view of Lemma 3.4, this homomorphism, from the space of infinitesimal deformations of \((X, S)\) to the space of infinitesimal deformations of \((X, S, E_*, D)\), is given by a homomorphism 
\[ \gamma : H^1(X, TX \otimes O_X(-S)) \to H^1(B_*) \] (3.17)

where \( B_* \) is the complex in (3.6).

From Corollary 2.5 we know that the connection \( D \) produces a homomorphism 
\[ h : TX \otimes O_X(-S) \to \text{At}(E_*) \]

such that \( \sigma \circ h = \text{Id}_{TX \otimes O_X(-S)} \), where \( \sigma \) is the projection in (2.12). This homomorphism \( h \) produces a homomorphism of complexes of sheaves on \( X \)
\[
\begin{array}{ccc}
TX \otimes O_X(-S) & \to & 0 \\
\downarrow h & & \downarrow \\
B_* & \to & \text{End}^n(E_*) \otimes K_X \otimes O_X(S)
\end{array}
\] (3.18)

We note that \( D \circ h = 0 \) because the connection \( D \) is flat. The homomorphism of complexes in (3.18) produces a homomorphism of hypercohomologies 
\[ \delta : H^1(X, TX \otimes O_X(-S)) \to H^1(B_*). \] (3.19)

**Lemma 3.5.** The homomorphism \( \gamma \) constructed in (3.17) coincides with the homomorphism \( \delta \) constructed in (3.19).
Proof. The proof of Lemma 3.5 is similar to the proof of [Ch2, p. 1417, Proposition 5.1] (see also [Ch1]). In fact, as done in the proofs of Lemma 3.1 and Lemma 3.4 the lemma can be deduced from [Ch2, p. 1417, Proposition 5.1] using the correspondence between the parabolic bundles and the orbifold bundles. We omit the details. \hfill \Box

4. Infinitesimal deformations of parabolic opers

We recall from [BDP] the definition of parabolic $\text{SL}(r, \mathbb{C})$-opers.

Consider the subset $S \subset X$ in (2.1); assume that the integer $n = \# S$ is even. Fix a holomorphic line bundle $\mathbb{L}$ on $X$ of degree $-\frac{n}{2}$ such that $\mathbb{L} \otimes 2$ is isomorphic to $\mathcal{O}_X(-S)$; also fix a holomorphic isomorphism

$$\varphi_0 : \mathbb{L} \otimes 2 \longrightarrow \mathcal{O}_X(-S)$$

of line bundles. Fix a holomorphic line bundle $K_X^{1/2}$ on $X$ of degree genus($X$) $- 1$ such that $(K_X^{1/2}) \otimes 2$ is isomorphic to $K_X$, in other words, $K_X^{1/2}$ is a theta characteristic on $X$; fix a holomorphic isomorphism

$$I_X : (K_X^{1/2}) \otimes 2 \longrightarrow K_X$$

of line bundles. Since

$$H^1(X, \text{Hom}(K_X^{-1/2} \otimes \mathbb{L}, K_X^{1/2} \otimes \mathbb{L})) = H^1(X, K_X) = H^0(X, \mathcal{O}_X)^* = \mathbb{C}$$

(the isomorphism $H^1(X, K_X) = H^0(X, \mathcal{O}_X)^*$ is given by Serre duality), we have a nontrivial extension

$$0 \longrightarrow K_X^{1/2} \otimes \mathbb{L} \longrightarrow \mathcal{E} \quad \longrightarrow \quad (K_X^{1/2}) \otimes 2 \otimes \mathbb{L} \longrightarrow 0 \quad (4.1)$$

corresponding to $1 \in H^1(X, \text{Hom}(K_X^{-1/2} \otimes \mathbb{L}, K_X^{1/2} \otimes \mathbb{L})) = \mathbb{C}$.

We will put a parabolic structure on the rank two vector bundle $\mathcal{E}$ in (4.1). Fix a function

$$c : S \longrightarrow \{ t \in \mathbb{Z} \mid t \geq 2 \} \quad (4.2)$$

the integer $c(x_i)$ will also be denoted by $c_i$.

Equip the vector bundle $\mathcal{E}$ in (4.1) with the following parabolic structure over $S$: For any $x_i \in S$, the quasiparabolic filtration of $\mathcal{E}_{x_i}$ is

$$(K_X^{1/2} \otimes \mathbb{L})_{x_i} \subset \mathcal{E}_{x_i},$$

where $K_X^{1/2} \otimes \mathbb{L}$ is the line subbundle in (4.1). The parabolic weight of $(K_X^{1/2} \otimes \mathbb{L})_{x_i}$ is $\frac{2q_i - 1}{2c_i}$ and the parabolic weight of $\mathcal{E}_{x_i}$ is $\frac{1}{2c_i}$ (see (4.2)). The resulting parabolic vector bundle of rank two on $X$ will be denoted by $\mathcal{E}_s$. The parabolic degree of $\mathcal{E}_s$ is zero. We note that the parabolic exterior product $\bigwedge^2 \mathcal{E}_s$ of $\mathcal{E}_s$ is the trivial holomorphic line bundle on $X$ equipped with the trivial holomorphic structure; see [BDP, Section 5], [Bi2] for the parabolic exterior product and the parabolic symmetric product.

**Remark 4.1.** It can be shown that the above rank two parabolic bundle $\mathcal{E}_s$ is indecomposable. To prove this assume that $\mathcal{E}_s = L_s \oplus L'_s$, where $L_s$ and $L'_s$ are parabolic line bundles. Suppose that $\text{par} - \deg(L_s) \geq \text{par} - \deg(L'_s)$. Consider the composition of homomorphism

$$L_s \longrightarrow \mathcal{E}_s \longrightarrow (K_X^{1/2} \otimes \mathbb{L})_s,$$
Lemma 4.2. For any parabolic $E$ where $1$. BISWAS, S. DUMITRESCU, S. HELLER, AND C. PAULY

First note that the parabolic $F$ such that the connection on $\bigwedge S$ around any point of $X$ does not split. In view of this contradiction we conclude that the parabolic bundle $E$ is indecomposable.

We note that a connection on a parabolic vector bundle $V$ induces connections on $\bigwedge^i V$ and $\text{Sym}^i(V)$ for every $i$.

Take any integer $r \geq 2$. Let $\text{Sym}^{r-1}(E)$ be the parabolic symmetric product of $E$. We have rank($\text{Sym}^{r-1}(E)$) = $r$, and the parabolic exterior product $\bigwedge^r \text{Sym}^{r-1}(E)$ is the trivial holomorphic line bundle on $X$ equipped with the trivial parabolic structure.

A parabolic $\text{SL}(r, \mathbb{C})$–oper on $X$ is a connection $D$ on the parabolic vector bundle $\text{Sym}^{r-1}(E)$ such that the connection on $\bigwedge^r \text{Sym}^{r-1}(E)$ induced by $D$ coincides with the connection on $O_X$ given by the de Rham differential $d$; see [BDP, p. 511, Definition 5.2].

**Lemma 4.2.** For any parabolic $\text{SL}(r, \mathbb{C})$–oper connection $D$, the local monodromy of $D$ around any point of $S$ is semisimple.

**Proof.** First note that the parabolic $E$ admits a connection, because $E$ is indecomposable (this was shown in Remark 1.1 and its parabolic degree is zero [BL, p. 594, Theorem 1.1]. Since the two parabolic weights at every $x_i \in S$ do not differ by an integer, we conclude that for any connection on the parabolic $E$, the local monodromy around any point of $S$ is semisimple (see (2.17)). Therefore, the connection on $\text{Sym}^{r-1}(E)$ induced by a connection on $E$ also has the property that the local monodromy around every point of $S$ is semisimple. From this it follows that every connection on $\text{Sym}^{r-1}(E)$ has the property that the local monodromy around every point of $S$ is semisimple; see Lemma 3.2. 

The subbundle $K^{1/2}_X \otimes \mathbb{L} \subset E$ in (1.1) equipped with the induced parabolic structure will be denoted by $F$. So the holomorphic line bundle underlying $F$ is $K^{1/2}_X \otimes \mathbb{L}$, and the parabolic weight of $F$ and any $x_i \in S$ is $2\frac{c_i-1}{c_i}$. The parabolic subbundle

$$F \xrightarrow{\iota_F} E$$

(4.3)

produces a filtration of parabolic subbundles of $\text{Sym}^{r-1}(E)$ in the following way. For any $0 \leq j \leq r-1$, consider the parabolic vector bundle $(F) \otimes (r-1-j) \otimes \text{Sym}^j(E)$; by definition, $(F) \otimes^0$ and $\text{Sym}^0(E)$ coincide with the trivial holomorphic line bundle with the trivial parabolic structure (see [Bi2, BDP] for the tensor product of parabolic vector bundles). So the rank of $(F) \otimes (r-1-j) \otimes \text{Sym}^j(E)$ is $j + 1$. This parabolic vector bundle $(F) \otimes (r-1-j) \otimes \text{Sym}^j(E)$ will be denoted by $F^{(j+1)}$ (since $j + 1$ is its rank). We note that

$$F^{(1)} \subset F^{(2)} \subset \cdots \subset F^{(r-1)} \subset F^{(r)} = \text{Sym}^{r-1}(E)$$

(4.4)

is a filtration of parabolic subbundles of $\text{Sym}^{r-1}(E)$. For any $1 \leq i \leq r-1$, the inclusion map

$$F^{(i)} \hookrightarrow F^{(i+1)}$$

where $q$ is the projection in (1.1) and $(K^{-1/2}_X \otimes \mathbb{L})_s$ is the quotient line bundle $K^{-1/2}_X \otimes \mathbb{L}$ in (1.1) equipped with the parabolic structure induced by $E$. This composition of homomorphisms vanishes identically, because par $- \text{deg}(L_s) > \text{par} - \text{deg}((K^{-1/2}_X \otimes \mathbb{L})_s)$. This implies that the short exact sequence in (1.1) splits. But the short exact sequence in (1.1) does not split. In view of this contradiction we conclude that the parabolic bundle $E$ is indecomposable.
in $(4.4)$ is constructed, in a straightforward way, using the inclusion map $F_* \hookrightarrow \mathcal{E}_*$ in $(4.3)$ together with the natural projection $\mathcal{E}_* \otimes \text{Sym}^r(\mathcal{E}_*) \longrightarrow \text{Sym}^{i+1}(\mathcal{E}_*)$. More precisely, we have

$$(t_F)^{(r-i)} : (F_*)^{(r-i)} \hookrightarrow (F_*)^{(r-i-1)} \otimes \mathcal{E}_* ,$$

where $t_F$ is the inclusion map in $(4.3)$. This implies that

$$\mathcal{F}_*^{(i)} = (F_*)^{(r-i)} \otimes \text{Sym}^{i-1}(\mathcal{E}_*) \hookrightarrow (F_*)^{(r-i-1)} \otimes \mathcal{E}_* \otimes \text{Sym}^{i-1}(\mathcal{E}_*)$$

$$\longrightarrow (F_*)^{(r-i-1)} \otimes \text{Sym}^i(\mathcal{E}_*) = \mathcal{F}_*^{(i+1)} .$$

The above composition of homomorphisms produces the inclusion map $\mathcal{F}_*^{(i)} \subset \mathcal{F}_*^{(i+1)}$ in $(4.4)$.

For any $1 \leq j \leq r$, the holomorphic vector bundle of rank $j$ underlying the parabolic vector bundle $\mathcal{F}_*^{(j)}$ will be denoted by $\mathcal{F}^{(j)}$. So the filtration in $(4.4)$ produce a filtration of holomorphic subbundles

$$\mathcal{F}^{(1)} \subset \mathcal{F}^{(2)} \subset \cdots \subset \mathcal{F}^{(r-1)} \subset \mathcal{F}^{(r)}$$

$(4.5)$

of $\mathcal{F}^{(r)}$. Note that $\mathcal{F}^{(r)}$ is the holomorphic vector bundle underlying the parabolic vector bundle $\text{Sym}^{r-1}(\mathcal{E}_*)$.

**Remark 4.3.** It should be clarified that although the holomorphic vector bundle underlying the parabolic bundle $\mathcal{E}_*$, namely the holomorphic vector bundle $\mathcal{E}$, does not depend on the function $c$ in $(4.2)$, the vector bundle $\mathcal{F}^{(r)}$ depends on $c$ in general. It should also be mentioned that the fact that the parabolic exterior product $\bigwedge^r \text{Sym}^{r-1}(\mathcal{E}_*)$ is the trivial holomorphic line bundle equipped with the trivial parabolic structure does not imply that $\bigwedge^r \mathcal{F}^{(r)}$ is the trivial holomorphic line bundle. In fact we have degree $(\bigwedge^r \mathcal{F}^{(r)}) < 0$. Note that degree($\mathcal{E}$) = $-n$.

**Remark 4.4.** It is a straightforward computation to check that

$$\bigwedge^r \mathcal{F}^{(r)} = \mathcal{O}_X(−\sum_{i=1}^n d_i x_i) ,$$

where

$$d_i = \sum_{k=0}^{r-1} \left( \frac{2k(c_i - 1) + r - 1}{2c_i} - \left\lfloor \frac{2k(c_i - 1) + r - 1}{2c_i} \right\rfloor \right) ;$$

the integral part of $b \in \mathbb{Q}$ is denoted $\lfloor b \rfloor$, so $\lfloor b \rfloor \in \mathbb{Z}$ and $0 \leq b - \lfloor b \rfloor < 1$. It is straightforward to check that each $d_i$ is an integer.

Consider the Atiyah bundle

$$\text{At}(\text{Sym}^{r-1}(\mathcal{E}_*)) \subset \text{Diff}^1(\mathcal{F}^{(r)}, \mathcal{F}^{(r)})$$

constructed as in $(2.11)$ for the parabolic bundle $\mathcal{E}_* = \text{Sym}^{r-1}(\mathcal{E}_*)$, where $\mathcal{F}^{(r)}$ is the vector bundle in $(4.5)$ and $\text{Sym}^{r-1}(\mathcal{E}_*)$ is the parabolic vector bundle in $(4.4)$. Let

$$\text{At}_X^r(r) \subset \text{At}(\text{Sym}^{r-1}(\mathcal{E}_*)) \subset \text{Diff}^1(\mathcal{F}^{(r)}, \mathcal{F}^{(r)})$$

$(4.6)$
be the holomorphic subbundle of \( \text{At} (\text{Sym}^{r-1}(\mathcal{E}_s)) \) constructed as follows: The space of all holomorphic sections of \( \text{At}'_X(r) \) over any open subset \( U \subset X \) is the space of all first order differential operations

\[
D_U : \Gamma (U, \text{Sym}^{r-1}(\mathcal{E}_s)) \longrightarrow \Gamma (U, \text{Sym}^{r-1}(\mathcal{E}_s))
\]
such that

- \( D_U \in \Gamma (U, \text{At} (\text{Sym}^{r-1}(\mathcal{E}_s))) \), and
- \( D_U (\mathcal{F}^{(j)}) \subset \mathcal{F}^{(j)} \) for all \( 1 \leq j \leq r \), where \( \mathcal{F}^{(j)} \) is the subbundle in (4.5).

Consider the subbundle

\[
\mathcal{O}_X \subset \text{Hom} (\mathcal{F}^{(r)}, \mathcal{F}^{(r)}) = \text{Diff}^0 (\mathcal{F}^{(r)}, \mathcal{F}^{(r)}) \subset \text{Diff}^1 (\mathcal{F}^{(r)}, \mathcal{F}^{(r)})
\]
given by pointwise multiplication. We note that

\[
\mathcal{O}_X \subset \text{At}'_X(r),
\]
where \( \text{At}'_X(r) \) is the subsheaf in (4.6). Let

\[
\text{At}_X(r) := \text{At}'_X(r)/\mathcal{O}_X
\]
(4.7)

be the quotient bundle.

Construct the holomorphic vector bundle \( \text{End}^n (\text{Sym}^{r-1}(\mathcal{E}_s)) \) by substituting, in (2.6), the parabolic vector bundle \( \text{Sym}^{r-1}(\mathcal{E}_s) \) in place of \( E_s \). Let

\[
\text{ad}^n (\text{Sym}^{r-1}(\mathcal{E}_s)) \subset \text{End}^n (\text{Sym}^{r-1}(\mathcal{E}_s))
\]
be the holomorphic subbundle of co-rank one given by the intersection of \( \text{End}^n (\text{Sym}^{r-1}(\mathcal{E}_s)) \) with the sheaf of endomorphisms of \( \mathcal{F}^{(r)} \) of trace zero. So using the natural inclusion map \( \mathcal{O}_X(-S) \hookrightarrow \text{End}^n (\text{Sym}^{r-1}(\mathcal{E}_s)) \) defined by pointwise multiplication, we have

\[
\text{End}^n (\text{Sym}^{r-1}(\mathcal{E}_s)) = \text{ad}^n (\text{Sym}^{r-1}(\mathcal{E}_s)) \oplus \mathcal{O}_X(-S) .
\]
(4.8)

Let

\[
\text{End}^n_1 (\text{Sym}^{r-1}(\mathcal{E}_s)) \subset \text{End}^n (\text{Sym}^{r-1}(\mathcal{E}_s))
\]
be the subbundle defined by imposing the condition that the subbundle \( \mathcal{F}^{(i)} \) in (4.5) is mapped to \( \mathcal{F}^{(i+1)} \) for all \( 1 \leq i \leq r-1 \). In other words, a locally defined holomorphic section \( s \) of the vector bundle \( \text{End}^n (\text{Sym}^{r-1}(\mathcal{E}_s)) \) is a locally defined section of \( \text{End}^n_1 (\text{Sym}^{r-1}(\mathcal{E}_s)) \) if and only if \( s (\mathcal{F}^{(i)}) \subset \mathcal{F}^{(i+1)} \) for every \( 1 \leq i \leq r-1 \). Define the intersection

\[
\text{ad}^n_1 (\text{Sym}^{r-1}(\mathcal{E}_s)) := \text{ad}^n (\text{Sym}^{r-1}(\mathcal{E}_s)) \bigcap \text{End}^n_1 (\text{Sym}^{r-1}(\mathcal{E}_s)) ;
\]
(4.9)
this intersection is taking place inside \( \text{End}^n (\text{Sym}^{r-1}(\mathcal{E}_s)) \). From (4.8) we have

\[
\text{End}^n_1 (\text{Sym}^{r-1}(\mathcal{E}_s)) = \text{ad}^n_1 (\text{Sym}^{r-1}(\mathcal{E}_s)) \oplus \mathcal{O}_X(-S) .
\]
(4.10)

Now let \( D \) be a connection on the parabolic vector bundle \( \text{Sym}^{r-1}(\mathcal{E}_s) \) defining a parabolic \( \text{SL}(r, \mathbb{C}) \)–oper on \( X \). In other words, the connection on \( \text{Sym}^{r-1}(\mathcal{E}_s) \) induced by \( D \) coincides with the connection on \( \mathcal{O}_X \) given by the de Rham differential \( d \). Consider the first order holomorphic differential operator

\[
D : \text{At} (\text{Sym}^{r-1}(\mathcal{E}_s)) \longrightarrow \text{End}^n (\text{Sym}^{r-1}(\mathcal{E}_s)) \otimes K_X \otimes \mathcal{O}_X(S)
\]
(4.11)
constructed as in (2.24) from \( D \); substitute \((\text{Sym}^{r-1}(\mathcal{E}_s), D)\) in place of \((E_s, D)\) in (2.24). The restriction of the differential operator \( D \) to the subbundle \( \text{At}'_X(r) \) constructed in (4.6) has its image contained in
\[
\text{End}^n_1(\text{Sym}^{r-1}(\mathcal{E}_s)) \otimes K_X \otimes \mathcal{O}_X(S) \subset \text{End}^n(\text{Sym}^{r-1}(\mathcal{E}_s)) \otimes K_X \otimes \mathcal{O}_X(S)
\]
(see (4.10)), in other words,
\[
\mathcal{D}(\text{At}'_X(r)) \subset \text{End}^n_1(\text{Sym}^{r-1}(\mathcal{E}_s)) \otimes K_X \otimes \mathcal{O}_X(S).
\]
Furthermore, the differential operator \( \mathcal{D} \) takes the subbundle \( \mathcal{O}_X \subset \text{At}'_X(r) \) (see (4.7)) to
\[
K_X = \mathcal{O}_X(-S) \otimes K_X \otimes \mathcal{O}_X(S) \subset \text{End}^n_1(\text{Sym}^{r-1}(\mathcal{E}_s)) \otimes K_X \otimes \mathcal{O}_X(S);
\]
see (4.10) for the subbundle \( \mathcal{O}_X(-S) \subset \text{End}^n_1(\text{Sym}^{r-1}(\mathcal{E}_s)) \). In fact, the restriction of \( \mathcal{D} \) to \( \mathcal{O}_X \subset \text{At}'_X(r) \) coincides with the de Rham differential \( d \). Consequently, using (4.12) and the decomposition in (4.10), the differential operator \( \mathcal{D}|_{\text{At}'_X(r)} \) in (4.11) produces a differential operator
\[
\mathcal{D}_B : \text{At}_X(r) \longrightarrow \text{ad}^n_1(\text{Sym}^{r-1}(\mathcal{E}_s)) \otimes K_X \otimes \mathcal{O}_X(S).
\]

Let \( \mathcal{C}_* \) denote the following two-term complex of sheaves on \( X \)
\[
\mathcal{C}_* : \mathcal{C}_0 = \text{At}_X(r) \xrightarrow{\mathcal{D}_B} \mathcal{C}_1 = \text{ad}^n_1(\text{Sym}^{r-1}(\mathcal{E}_s)) \otimes K_X \otimes \mathcal{O}_X(S),
\]
where \( \mathcal{D}_B \) is the homomorphism in (4.13), and \( \mathcal{C}_i \) is at the \( i \)-th position.

**Theorem 4.5.** The space of all infinitesimal deformation of the triple \((X, S, D)\), where \( D \) is a parabolic \( \text{SL}(r, \mathbb{C}) \)-operator on \( X \), is given by the hypercohomology
\[
H^1(\mathcal{C}_*),
\]
where \( \mathcal{C}_* \) is the complex in (4.14).

**Proof.** As in (3.1), take a ramified Galois covering
\[
\varphi : Y \longrightarrow X
\]
satisfying the following two conditions:
- \( \varphi \) is unramified over the complement \( X \setminus S \), and
- for every \( x_i \in S \) and each point \( y \in \varphi^{-1}(x_i) \), the order of ramification of \( \rho \) at \( y \) is \( c_i = c(x_i) \) (see (4.2)).

Such a covering \( \varphi \) exists by [Na] p. 26, Proposition 1.2.12. As before, denote by \( \Gamma_\varphi \) the Galois group \( \text{Gal}(\varphi) \) of the map \( \varphi \). From [BDP] p. 514, Theorem 6.3 we know that the parabolic \( \text{SL}(r, \mathbb{C}) \)-operator \( D \) on \( X \) corresponds to a \( \Gamma \)-invariant \( \text{PSL}(r, \mathbb{C}) \)-operator \( \mathbb{D} \) on \( Y \). We note that there is a natural bijection between the \( \text{PSL}(r, \mathbb{C}) \)-opers on \( Y \) and each connected component of the \( \text{SL}(r, \mathbb{C}) \)-opers on \( Y \). We will recall a description of this \( \text{PSL}(r, \mathbb{C}) \)-oper \( \mathbb{D} \) on \( Y \) corresponding to \( D \).

We first note that the parabolic vector bundle \( \text{Sym}^{r-1}(\mathcal{E}_s) \) defines a holomorphic parabolic principal \( \text{SL}(r, \mathbb{C}) \)-bundle on \( X \), because \( \bigwedge^r \text{Sym}^{r-1}(\mathcal{E}_s) \) is the trivial parabolic line bundle;
see [BBN1], [BBN2], [BBP] for the parabolic analog of principal bundles. Using the quotient map \( \text{SL}(r, \mathbb{C}) \rightarrow \text{PSL}(r, \mathbb{C}) \), a parabolic principal \( \text{SL}(r, \mathbb{C}) \)-bundle produces parabolic principal \( \text{PSL}(r, \mathbb{C}) \)-bundle. Let
\[
\tilde{P}_* \longrightarrow X
\]
(4.16)
denote the parabolic principal \( \text{PSL}(r, \mathbb{C}) \)-bundle given by the parabolic principal \( \text{SL}(r, \mathbb{C}) \)-bundle on \( X \) defined by \( \text{Sym}^{r-1}(E_*) \).

Fix a Borel subgroup
\[
B \subset \text{PSL}(r, \mathbb{C}).
\]
(4.17)
The filtration of parabolic vector bundles \( \{F^*_j\}_{j=1}^r \) in (4.4) produces a reduction of structure group of the parabolic principal \( \text{PSL}(r, \mathbb{C}) \)-bundle \( \tilde{P}_* \) in (4.16) to the subgroup \( B \) of \( \text{PSL}(r, \mathbb{C}) \) in (4.17). Let
\[
\tilde{P}(B)_* \subset \tilde{P}_*
\]
(4.18)
be this reduction of structure group to \( B \).

The parabolic principal \( \text{PSL}(r, \mathbb{C}) \)-bundle \( \tilde{P}_* \) in (4.16) corresponds to an equivariant holomorphic principal \( \text{PSL}(r, \mathbb{C}) \)-bundle on \( Y \) [BBN2], [BBN1]; let
\[
\mathbb{P}(r) \longrightarrow Y
\]
(4.19)
be the equivariant holomorphic principal \( \text{PSL}(r, \mathbb{C}) \)-bundle corresponds to \( \tilde{P}_* \). We note that \( \mathbb{P}(r) \) is the unique holomorphic principal \( \text{PSL}(r, \mathbb{C}) \)-bundle on \( Y \) underlying the \( \text{PSL}(r, \mathbb{C}) \)-opers on \( Y \). We will briefly recall a description of \( \mathbb{P}(r) \).

Take a theta characteristic \( K_{Y^{1/2}} \) on \( Y \). Let
\[
0 \longrightarrow K_{Y^{1/2}} \longrightarrow W \longrightarrow \left(K_{Y^{1/2}}\right)^* \longrightarrow 0
\]
(4.20)
be the nontrivial extension corresponding to
\[
1 \in \mathbb{C} = H^0(Y, \mathcal{O}_Y)^* = H^1(Y, K_Y) = H^1(Y, \text{Hom}(K_{Y^{1/2}}^*, K_{Y^{1/2}})).
\]
The holomorphic principal \( \text{PSL}(r, \mathbb{C}) \)-bundle \( \mathbb{P}(r) \) in (4.19) coincides with the holomorphic principal \( \text{PSL}(r, \mathbb{C}) \)-bundle defined by the projective bundle \( \mathbb{P} \left( \text{Sym}^{r-1}(W) \right) \). (By \( \mathbb{P}(V) \) we denote the projective bundle defined by the spaces of lines in the fibers of \( V \).) While the vector bundle \( \text{Sym}^{r-1}(W) \) depends on the choice of the theta characteristic \( K_{Y^{1/2}} \) when \( r \) is even (the vector bundle \( \text{Sym}^{r-1}(W) \) does not depend on the choice of \( K_{Y^{1/2}} \) when \( r \) is odd), the projective bundle \( \mathbb{P} \left( \text{Sym}^{r-1}(W) \right) \) is actually independent of the choice of the theta characteristic \( K_{Y^{1/2}} \). Indeed, if we replace \( K_{Y^{1/2}} \) by \( K_{Y^{1/2}} \otimes \xi \), where \( \xi \) is a holomorphic line bundle on \( Y \) of order two, then \( W \) gets replaced by \( W \otimes \xi \), and hence \( \text{Sym}^{r-1}(W) \) gets replaced by \( \text{Sym}^{r-1}(W) \otimes \xi^{\otimes (r-1)} \).

Since any holomorphic automorphism of \( Y \) takes a theta characteristic on \( Y \) to a (possibly different) theta characteristic on \( Y \), from the above property of \( \mathbb{P} \left( \text{Sym}^{r-1}(W) \right) \) it follows immediately that \( \mathbb{P}(r) \) is an equivariant holomorphic principal \( \text{PSL}(r, \mathbb{C}) \)-bundle on \( Y \). In other words, the action of \( \Gamma_{\varphi} \) on \( Y \) lifts to an holomorphic action of \( \Gamma_{\varphi} \) on \( \mathbb{P}(r) \) that commutes with the action of \( \text{PSL}(r, \mathbb{C}) \) on the principal bundle \( \mathbb{P}(r) \).
In particular, \( \mathbb{P}(W) \) is an equivariant vector bundle. The action of \( \Gamma_\varphi \) on \( \mathbb{P}(W) \) preserves the holomorphic section of the projective bundle \( \mathbb{P}(W) \) given by the line subbundle \( K_{Y}^{1/2} \) in (4.20).

The connection \( D \) on the parabolic vector bundle \( \text{Sym}^{r-1}(E_{\ast}) \) produces a connection on the parabolic principal \( \text{PSL}(r, \mathbb{C}) \)-bundle \( \tilde{\mathbb{P}}_{*} \) on \( X \) given by \( \mathbb{P}(\text{Sym}^{r-1}(E_{\ast})) \). This connection on \( \tilde{\mathbb{P}}_{*} \) in turn produces a \( \Gamma_\varphi \)-invariant holomorphic connection on the principal \( \text{PSL}(r, \mathbb{C}) \)-bundle \( \mathbb{P}(r) \) on \( Y \). This \( \Gamma_\varphi \)-invariant holomorphic connection on \( \mathbb{P}(r) \) will be denoted by \( D_{Y} \).

(4.21)

Any holomorphic connection on \( \mathbb{P}(r) \) is a \( \text{PSL}(r, \mathbb{C}) \)-oper on \( Y \) [BD1], [BD2], [Fr1]. In particular, \( D_{Y} \) in (4.21) is a \( \text{PSL}(r, \mathbb{C}) \)-oper on \( Y \).

Given an oper, a complex of sheaves was constructed by Sanders in [Sa]. We will briefly recall his construction of complex of sheaves for the \( \text{PSL}(r, \mathbb{C}) \)-oper \( D_{Y} \) in (4.21).

Let

\[ W_{1} \subset W_{2} \subset \cdots \subset W_{r-1} \subset W_{r} = \text{Sym}^{r-1}(W) \]

be the filtration of holomorphic subbundles, where \( W_{j} = (K_{Y}^{1/2})^{\otimes (r-j)} \otimes \text{Sym}^{j-1}(W) \) (see (4.20)); in particular \( \text{rank}(W_{j}) = j \). Let

\[ \mathbb{P}(W_{1}) \subset \mathbb{P}(W_{2}) \subset \cdots \subset \mathbb{P}(W_{r-1}) \subset \mathbb{P}(W_{r}) = \mathbb{P}(\text{Sym}^{r-1}(W)) \]

be the corresponding filtration of projective bundles; recall that \( \mathbb{P}(V) \) denotes the projective bundle defined by the spaces of lines in the fibers of \( V \). The filtration in (4.22) produces a holomorphic reduction of structure group

\[ \mathbb{P}(r)_{B} \subset \mathbb{P}(r) \]

(4.23)

to the Borel subgroup \( B \) in (4.17), where \( \mathbb{P}(r) \) is the holomorphic principal \( \text{PSL}(r, \mathbb{C}) \)-bundle in (4.19).

Recall that \( \mathbb{P}(W) \) is an equivariant vector bundle, and the action of \( \Gamma_\varphi \) on \( \mathbb{P}(W) \) preserves the holomorphic section of \( \mathbb{P}(W) \) given by the line subbundle \( K_{Y}^{1/2} \) in (4.20). Therefore, from the construction of the filtration in (4.22) it follows immediately that the action of \( \Gamma_\varphi \) on \( \mathbb{P}(\text{Sym}^{r-1}(W)) \) preserves each projective subbundle \( \mathbb{P}(W_{j}) \). Consequently, the action of \( \Gamma_\varphi \) on \( \mathbb{P}(r) \) preserves the reduction of the structure group \( \tilde{\mathbb{P}}(r)_{B} \) in (4.23). The principal \( B \)-bundle \( \tilde{\mathbb{P}}(r)_{B} \) in (4.23) in fact corresponds to the reduction \( \tilde{\mathbb{P}}(B)_{*} \) in (4.18).

The holomorphic reduction \( \mathbb{P}(r)_{B} \) in (4.23) coincides with the holomorphic reduction of structure group of \( \mathbb{P}(r) \) to the subgroup \( B \subset \text{PSL}(r, \mathbb{C}) \) that appears in the definition of a \( \text{PSL}(r, \mathbb{C}) \)-oper on \( Y \). Let

\[ \text{At}(\mathbb{P}(r)_{B}) \longrightarrow Y \]

be the Atiyah bundle for the holomorphic principal \( B \)-bundle \( \mathbb{P}(r)_{B} \) (see [At]). Let

\[ \text{At}(\mathbb{P}(r)) \longrightarrow Y \]

be the Atiyah bundle for the principal \( \text{PSL}(r, \mathbb{C}) \)-bundle \( \mathbb{P}(r) \) in (4.19). We have

\[ \text{At}(\mathbb{P}(r)_{B}) \subset \text{At}(\mathbb{P}(r)) \]

(4.24)

because of the reduction of structure group in (4.23).
Let $\text{ad}(\mathbb{P}(r)) \rightarrow Y$ be the adjoint bundle of the holomorphic principal $\text{PSL}(r, \mathbb{C})$–bundle $\mathbb{P}(r)$. We recall that $\text{ad}(\mathbb{P}(r))$ is the holomorphic vector bundle associated to the principal $\text{PSL}(r, \mathbb{C})$–bundle $\mathbb{P}(r)$ for the adjoint action of $\text{PSL}(r, \mathbb{C})$ on its Lie algebra. We will describe $\text{ad}(\mathbb{P}(r))$ explicitly. Let 

$$\gamma : \mathbb{P}(\text{Sym}^{r-1}(W)) \rightarrow Y$$

be the natural projection. Let 

$$T_\gamma \subset T\mathbb{P}(\text{Sym}^{r-1}(W))$$

be the relative holomorphic tangent bundle for the projection $\gamma$, meaning $T_\gamma$ is the kernel of the differential $d\gamma : T\mathbb{P}(\text{Sym}^{r-1}(W)) \rightarrow \gamma^*TY$ of $\gamma$. Then we have 

$$\text{ad}(\mathbb{P}(r)) = \gamma_*T_\gamma. \quad (4.25)$$

Given any holomorphic connection on the principal $\text{PSL}(r, \mathbb{C})$–bundle $\mathbb{P}(r)$, there is a holomorphic differential operator of order one from $\text{At}(\mathbb{P}(r)) \rightarrow \text{ad}(\mathbb{P}(r)) \otimes K_Y$ [Ch2, p. 1415, (1)], [Ch2, p. 1415, Proposition 4.4]. Consider the differential operator 

$$\text{At}(\mathbb{P}(r)) \rightarrow \text{ad}(\mathbb{P}(r)) \otimes K_Y$$

corresponding to the connection $D^Y$ in (4.21). Let 

$$\tilde{D}^Y : \text{At}(\mathbb{P}(r)_B) \rightarrow \text{ad}(\mathbb{P}(r)) \otimes K_Y \quad (4.26)$$

be its restriction to the subbundle $\text{At}(\mathbb{P}(r)_B)$ (see (4.21)).

Let $\text{ad}(\mathbb{P}(r)_B) \rightarrow Y$ be the adjoint bundle for the holomorphic principal $B$–bundle $\mathbb{P}(r)_B$. Note that we have 

$$\text{ad}(\mathbb{P}(r)_B) \subset \text{ad}(\mathbb{P}(r))$$

because of the reduction of structure group in (4.23). Recall the description of $\text{ad}(\mathbb{P}(r))$ in (4.25). For any $y \in Y$, the subspace 

$$\text{ad}(\mathbb{P}(r)_B)_y \subset \text{ad}(\mathbb{P}(r))_y$$

consists of all holomorphic vector fields $v$ on $\mathbb{P}(\text{Sym}^{r-1}(W))_y$ satisfying the following condition: for any $1 \leq j \leq r - 1$, and any 

$$z \in \mathbb{P}(\mathcal{W}_j)_y \subset \mathbb{P}(\text{Sym}^{r-1}(W))_y$$

(see (4.22)), 

$$v(z) \in T_z\mathbb{P}(\mathcal{W}_j)_y,$$

note that $T_z\mathbb{P}(\mathcal{W}_j)_y \subset T_z\mathbb{P}(\text{Sym}^{r-1}(W))_y$ because $\mathbb{P}(\mathcal{W}_j)_y \subset \mathbb{P}(\text{Sym}^{r-1}(W))_y$. Let 

$$R_n(\text{ad}(\mathbb{P}(r)_B)) \subset \text{ad}(\mathbb{P}(r)_B)$$

be the subbundle given by the nilpotent radical bundle; so for any $y \in Y$, the subspace $R_n(\text{ad}(\mathbb{P}(r)_B))_y \subset \text{ad}(\mathbb{P}(r)_B)_y$ is the nilpotent radical. So, for any $y \in Y$, we have 

$$R_n(\text{ad}(\mathbb{P}(r)_B))_y = [\text{ad}(\mathbb{P}(r)_B)_y, \text{ad}(\mathbb{P}(r)_B)_y].$$

Let 

$$[R_n(\text{ad}(\mathbb{P}(r)_B), R_n(\text{ad}(\mathbb{P}(r)_B))] \subset R_n(\text{ad}(\mathbb{P}(r)_B))$$

be the commutator. For any $y \in Y$, we have 

$$[R_n(\text{ad}(\mathbb{P}(r)_B), R_n(\text{ad}(\mathbb{P}(r)_B))] = [R_n(\text{ad}(\mathbb{P}(r)_B)_y, R_n(\text{ad}(\mathbb{P}(r)_B)_y].$$
Let
\[ \text{ad}_1(\mathbb{P}(r)_B) := [R_n(\text{ad}(\mathbb{P}(r)_B), R_n(\text{ad}(\mathbb{P}(r)_B))]^\perp \subset \text{ad}(\mathbb{P}(r)) \quad (4.27) \]
be the annihilator of \([R_n(\text{ad}(\mathbb{P}(r)_B)), R_n(\text{ad}(\mathbb{P}(r)_B))]\) for the adjoint bundle \(\text{ad}(\mathbb{P}(r))\). The vector bundle \(\text{ad}_1(\mathbb{P}(r)_B)\) has the following description in terms of the isomorphism in (4.25). For any \(y \in Y\), a holomorphic vector field \(v\) on \(\mathbb{P}(\text{Sym}^{-1}(V))_y\) lies in the fiber \(\text{ad}_1(\mathbb{P}(r)_B)_y\) if and only if the following condition holds: for any \(1 \leq j \leq r - 1\) and any \(z \in \mathbb{P}(\mathcal{W}_j)_y\),
\[ v(z) \in T_z\mathbb{P}(\mathcal{W}_{j+1})_y. \]

Since the connection \(D^Y\) in (4.21) is a \(\text{PSL}(r, \mathbb{C})\)-oper on \(Y\), the image of the differential operator \(\widetilde{D}^Y\) in (4.26) is contained in the subbundle
\[ \text{ad}_1(\mathbb{P}(r)_B) \otimes K_Y \subset \text{ad}(\mathbb{P}(r)) \otimes K_Y \]
defined in (4.27). Therefore, \(\widetilde{D}^Y\) defines a differential operator
\[ D_1^Y : \text{At}(\mathbb{P}(r)_B) \rightarrow \text{ad}_1(\mathbb{P}(r)_B) \otimes K_Y ; \quad (4.28) \]
see [Sa] (5.8). Let \(\mathcal{H}_\bullet\) be the following two-term complex of sheaves on \(Y\)
\[ \mathcal{H}_\bullet : \mathcal{H}_0 = \text{At}((\mathbb{P}(r)_B) \xrightarrow{D_1^Y} \mathcal{H}_1 = \text{ad}_1(\mathbb{P}(r)_B) \otimes K_Y , \quad (4.29) \]
where \(D_1^Y\) is the homomorphism in (4.28), and \(\mathcal{H}_i\) is at the \(i\)-th position.

The space of all infinitesimal deformations of the \(\text{PSL}(r, \mathbb{C})\)-oper \((Y, D^Y)\) is given by the hypercohomology \(H^1(\mathcal{H}_\bullet)\), where \(\mathcal{H}_\bullet\) is the complex constructed in (4.29) [Sa, Theorem 5.9].

We noted earlier that the action of \(\Gamma_\varphi\) on \(\mathbb{P}(r)\) preserves the reduction of structure group \(\mathbb{P}(r)_B\) in (1.23). Since \(\mathbb{P}(r)_B\) is an equivariant bundle, we conclude that that both \(\text{At}(\mathbb{P}(r)_B)\) and \(\text{ad}(\mathbb{P}(r)_B)\) are equivariant vector bundles. Hence \(R_n(\text{ad}(\mathbb{P}(r)_B))\) in (1.27) is an equivariant subbundle of \(\text{ad}(\mathbb{P}(r)_B)\), which in turn implies that \(\text{ad}_1(\mathbb{P}(r)_B)\) in (4.27) is an equivariant subbundle of \(\text{ad}(\mathbb{P}(r))\); the fiberwise Killing form on \(\text{ad}(\mathbb{P}(r))\) is evidently \(\Gamma_\varphi\)-invariant. The operator \(D_1^Y\) in (4.28) is \(\Gamma_\varphi\)-equivariant, because the connection \(D^Y\) in (1.21) is invariant under the action of \(\Gamma_\varphi\) on \(\mathbb{P}(r)\). Consequently, the complex \(\mathcal{H}_\bullet\) in (4.29) is \(\Gamma_\varphi\)-equivariant. Therefore, the group \(\Gamma_\varphi\) acts on the hypercohomology \(H^1(\mathcal{H}_\bullet)\). Let
\[ H^1(\mathcal{H}_\bullet)^{\Gamma_\varphi} \subset H^1(\mathcal{H}_\bullet) \quad (4.30) \]
be the invariant part for the action of \(\Gamma_\varphi\).

We have
\[ (\varphi_\ast \text{At}((\mathbb{P}(r)_B)))^{\Gamma_\varphi} = \text{At}_X(r) \]
and
\[ (\varphi_\ast (\text{ad}_1(\mathbb{P}(r)_B) \otimes K_Y))^{\Gamma_\varphi} = \text{ad}_1^n \left( \text{Sym}^{-1}(\mathcal{E}_s) \right) \otimes K_X \otimes \mathcal{O}_X(S) \]
(see (4.14) for \(\text{At}_X(r)\) and \(\text{ad}_1^n(\text{Sym}^{-1}(\mathcal{E}_s)) \otimes K_X \otimes \mathcal{O}_X(S)\)). Moreover, the differential operator \(D_1^Y\) in (4.28) gives the differential operator \(D_B\) in (4.14). Therefore, we conclude that
\[ H^1(\mathcal{C}_\bullet) = H^1(\mathcal{H}_\bullet)^{\Gamma_\varphi} , \]
Consider the short exact sequence of holomorphic vector bundles on $X$
\[
0 \longrightarrow \mathcal{O}_X \longrightarrow \text{At}_X^r \longrightarrow \text{At}_X^r \longrightarrow 0
\] in (4.31). We will show that it splits holomorphically.

Take any $\delta \in \Gamma \left( U, \text{At}_X^r \left( \text{Sym}^{r-1}(E) \right) \right) \subset \Gamma \left( U, \text{Diff}^1 \left( \mathcal{F}^r, \mathcal{F}^r \right) \right)$ (see (4.6)), where $U \subset X$ is an open subset. Then $\delta$ produces a holomorphic differential operator
\[
\tilde{\delta} \in \Gamma \left( U, \text{Diff}^1 \left( \bigwedge^r \mathcal{F}^r, \bigwedge^r \mathcal{F}^r \right) \right)
\] (4.32) which is constructed as follows: Take any $s = s_1 \wedge \cdots \wedge s_r \in \Gamma \left( U, \bigwedge^r \mathcal{F}^r \right)$ (4.33) where $s_i \in \Gamma \left( U, \mathcal{F}^r \right)$ for all $1 \leq i \leq r$. Now define
\[
\tilde{\delta}(s) := \sum_{j=1}^r s_1 \wedge \cdots \wedge s_{j-1} \wedge \delta(s_j) \wedge s_{j+1} \wedge \cdots \wedge s_r \in \Gamma \left( U, \bigwedge^r \mathcal{F}^r \right).
\]

It is straightforward to check that $\tilde{\delta}(s)$ is indeed independent of the choice of the decomposition of the section $s$ in (4.33).

Let
\[
\eta : \text{At}_X^r \longrightarrow TX \otimes \mathcal{O}_X(-S)
\] be the restriction of the natural projection $\text{At} \left( \text{Sym}^{r-1}(E) \right) \longrightarrow TX \otimes \mathcal{O}_X(-S)$ (see 2.12) to the subbundle $\text{At}_X^r \subset \text{At} \left( \text{Sym}^{r-1}(E) \right)$ in (4.34).

We recall from Remark 4.4 that $\bigwedge^r \mathcal{F}^r = \mathcal{O}_X \left( - \sum_{i=1}^n d_i x_i \right)$. Therefore, the de Rham differential $d$ on $\mathcal{O}_X$ produces a logarithmic connection on $\mathcal{O}_X \left( - \sum_{i=1}^n d_i x_i \right) = \bigwedge^r \mathcal{F}^r$. Let
\[
\tilde{d} : \bigwedge^r \mathcal{F}^r \longrightarrow \bigwedge^r \mathcal{F}^r \otimes K_X \otimes \mathcal{O}_X \left( \sum_{i=1}^n x_i \right)
\] be the logarithmic connection on $\bigwedge^r \mathcal{F}^r$ given by the de Rham differential.

Let
\[
\text{At}_X^0 \subset \text{At}_X^r
\] be the holomorphic subbundle whose holomorphic sections over any open subset $U \subset X$ consist of all
\[
\delta \in \Gamma \left( U, \text{At}_X^r \right)
\] satisfying the following condition:
\[
\tilde{\delta}(s) = \langle \tilde{d}(s), \eta(\delta) \rangle
\] for all $s \in \Gamma \left( U, \bigwedge^r \mathcal{F}^r \right)$, where $\tilde{\delta}$ is constructed in (4.32) from $\delta$ and $\eta$ is the homomorphism in (4.34), while and $\langle - , - \rangle$ is the duality pairing in (2.27) and $\tilde{d}$ is constructed in (4.35).
The composition of homomorphisms
\[ \text{At}^0_X(r) \hookrightarrow \text{At}'_X(r) \twoheadrightarrow \text{At}_X(r) \]
(see (4.36) and (4.31) for these homomorphisms) is evidently an isomorphism. Therefore, the holomorphic subbundle \( \text{At}^0_X(r) \) in (4.36) produces a holomorphic splitting of the short exact sequence in (4.31).

Let \( D \) be a connection on \( \text{Sym}^{r-1}(E^*) \) defining a parabolic \( \text{SL}(r, \mathbb{C}) \)–oper on \( X \). Consider the differential operator \( D \) in (4.11) constructed from the \( \text{SL}(r, \mathbb{C}) \)–oper \( D \). Clearly, we have
\[ D(\text{At}^0_X(r)) \subset \text{ad}_{\text{Sym}^{r-1}(E^*)} \otimes K_X \otimes O_X(S), \]
where \( \text{ad}_{\text{Sym}^{r-1}(E^*)} \) and \( \text{At}^0_X(r) \) are constructed in (4.9) and (4.36) respectively; recall from (4.12) that
\[ D(\text{At}'_X(r)) \subset \text{End}_{\text{Sym}^{r-1}(E^*)} \otimes K_X \otimes O_X(S) \]
and \( \text{End}_{\text{Sym}^{r-1}(E^*)} \) decomposes as in (4.10).

Consequently, the complex of sheaves \( C' \) in (4.14) is equivalent to the following complex \( C'_* \) of sheaves on \( X \)
\[ C'_0 = \text{At}^0_X(r) \xrightarrow{D} C'_1 = \text{ad}_{\text{Sym}^{r-1}(E^*)} \otimes K_X \otimes O_X(S). \]
(4.37)
Hence Theorem 4.5 gives the following:

**Corollary 4.6.** The space of all infinitesimal deformation of the triple \((X, S, D)\), where \( D \) is a parabolic \( \text{SL}(r, \mathbb{C}) \)–oper on \( X \), is given by the hypercohomology
\[ H^1(C'_*), \]
where \( C'_* \) is the complex in (4.37).

### 5. Monodromy of parabolic opers

Consider a family of \( n \)–pointed Riemann surfaces
\[ \{(X_T \xrightarrow{\varpi} T), (\phi_1, \ldots, \phi_n)\} \]
as in (3.12) and (3.13). Assume that this family is locally universal. The relative canonical line bundle on \( X_T \) for the projection \( \varpi \) will be denoted by \( K_\varpi \).

Fix a holomorphic line bundle \( L \) on \( X_T \) such that \( L \otimes L \) is holomorphically isomorphic to \( K_\varpi \otimes O_{X_T}(-\sum_{i=1}^n \phi_i(T)) \); such a line bundle \( L \) exists locally with respect to \( T \). Fix a holomorphic isomorphism between \( L \otimes L \) and \( K_\varpi \otimes O_{X_T}(-\sum_{i=1}^n \phi_i(T)) \). Fix a function
\[ c : \{1, \ldots, n\} \rightarrow \{t \in \mathbb{Z} \mid t \geq 2\} \]
as in (4.2).

Fix the parabolic structure to be that of an \( \text{SL}(r, \mathbb{C}) \)–oper. As in (3.14),
\[ \beta : \mathcal{M}^T(r) \rightarrow T \]
is the corresponding relative moduli space of parabolic bundles with connection. Let
\[ \mathcal{O}^T(r) \hookrightarrow \mathcal{M}^T(r) \]
(5.3)
be the locus of parabolic $\text{SL}(r, \mathbb{C})$–opers.

Consider the monodromy map
\[
\mathbb{M} : \mathcal{M}^T(r) \rightarrow \mathcal{R}_X(r)
\]
in (3.15). Let
\[
\mathbb{M}^0 : \mathcal{O}^T(r) \rightarrow \mathcal{R}_X(r)
\]  
be the restriction of $\mathbb{M}$ to the subspace $\mathcal{O}^T(r)$ in (5.3).

We will prove that the holomorphic map $\mathbb{M}^0$ in (5.4) is an immersion. Our proof is modeled on the proof of Sanders that $\mathbb{M}^0$ is an immersion under the assumption that $n = 0$ (parabolic points are absent); see [Sa, Theorem 6.3].

Take a point $t_0 \in T$. Denote the Riemann surface $\varpi^{-1}(t_0) = X_{t_0}$ by $X$. Denote the divisor $\sum \phi_i(t_0)$ on $X$ by $S$. We have
\[
T_{t_0} T = H^1(X, TX \otimes \mathcal{O}_X(-S));
\]
recall that $(\varpi, \{\phi_i\}_{i=1}^n)$ is a locally complete family. Take any parabolic bundle with connection
\[
(E_*, D) \in \beta^{-1}(t_0)
\]
on $X$, where $\beta$ is the projection in (5.2). From Lemma 3.4 it follows that
\[
T_{(E_*, D)} \mathcal{M}^T(r) = \mathbb{H}^1(B_*)
\]  
where $B_*$ is the complex in (3.6). Let
\[
T_\beta \subset T \mathcal{M}^T(r)
\]
be the relative tangent bundle for the projection $\beta$ in (5.2). From Lemma 3.3 it follows that
\[
(T_{(E_*, D)} \mathcal{M}^T(r) = \mathbb{H}^1(B_*) \oplus \delta T_{t_0} T = H^1(X, TX \otimes \mathcal{O}_X(-S))
\]
in (3.19). Recall that the two holomorphic foliations on $\mathcal{M}^T(r)$, one given by the isomonodromy condition and the other given by the projection $\beta$ in (5.2), are transversal. Therefore, from Lemma 3.3 and (5.5) we know that
\[
\mathbb{H}^1(B_*) = (T_{\beta}(E_*, D)) \oplus \delta T_{t_0} T = \mathbb{H}^1(A_*) \oplus \delta (H^1(X, TX \otimes \mathcal{O}_X(-S))).
\]  
Let
\[
\Phi : \mathbb{H}^1(B_*) \rightarrow \mathbb{H}^1(A_*)
\]
be the projection corresponding to the decomposition in (5.6). We will now describe this homomorphism $\Phi$ explicitly.

The connection $D$ on $E_*$ gives a holomorphic decomposition
\[
\text{At}(E_*) = \text{End}^P(E_*) \oplus (TX \otimes \mathcal{O}_X(-S))
\]  
(see Lemma 2.4). As in (3.18), $h : TX \otimes \mathcal{O}_X(-S) \rightarrow \text{At}(E_*)$ is the homomorphism given by the decomposition in (5.8). Consider $D$ constructed in (2.24). Recall that
\[
D \circ h = 0
\]
in (3.18). Consequently, the decomposition in (5.8) produces a homomorphism $P$ of complexes

$$
\begin{align*}
\mathcal{B}_* : & \quad \text{At}(E_*) \xrightarrow{\mathcal{D}} \text{End}^n(E_*) \otimes K_X \otimes O_X(S) \\
\mathcal{P} : & \quad p \\
\mathcal{A}_* : & \quad \text{End}^P(E_*) \xrightarrow{\mathcal{D}_0} \text{End}^n(E_*) \otimes K_X \otimes O_X(S)
\end{align*}
$$

(5.9)

where $p$ is the projection given by the decomposition in (5.8). Let

$$
P_* : \mathbb{H}^1(\mathcal{B}_*) \rightarrow \mathbb{H}^1(\mathcal{A}_*)
$$

(5.10)

be the homomorphism of hypercohomologies corresponding to the homomorphism of complexes $P$ in (5.9). The homomorphism $P_*$ in (5.10) evidently coincides with the projection $\Phi$ in (5.7).

Now set $E_*$ to be the rank $r$ parabolic vector bundle $\text{Sym}^{r-1}(E_*)$ in (4.3), and let $D$ be a connection on $\text{Sym}^{r-1}(E_*)$ such that

$$
(\text{Sym}^{r-1}(E_*), D) \in \mathcal{O}^T(r) \bigcap \beta^{-1}(t_0)
$$

(see (5.3) and (5.2)). From Corollary 4.6 we know that

$$
T_{(\text{Sym}^{r-1}(E_*), D)} \mathcal{O}^T(r) = \mathbb{H}^1(C_*),
$$

where $C_*$ is the complex in (4.37). We have the following two homomorphisms $P$ and $Q$ of complexes

$$
\begin{align*}
\begin{array}{c}
\mathcal{C}_* : \\
\mathcal{B}_* : \\
\mathcal{A}_*
\end{array}
\begin{array}{c}
\mathcal{Q}_0 = \text{At}_X(r) \\
\tilde{\mathcal{B}}_0 = \text{At}(\text{Sym}^{r-1}(E_*)) \\
\tilde{\mathcal{A}}_0 = \text{End}^P(\text{Sym}^{r-1}(E_*))
\end{array}
\begin{array}{c}
\text{D} \quad \text{D} \quad \text{D}
\end{array}
\begin{array}{c}
\mathcal{C}_1 = \text{ad}^n(\text{Sym}^{r-1}(E_*)) \otimes K_X \otimes O_X(S) \\
\tilde{\mathcal{B}}_1 = \text{End}^n(\text{Sym}^{r-1}(E_*)) \otimes K_X \otimes O_X(S) \\
\tilde{\mathcal{A}}_1 = \text{End}^n(\text{Sym}^{r-1}(E_*)) \otimes K_X \otimes O_X(S)
\end{array}
\begin{array}{c}
\mathcal{Q}_1 \\
\mathcal{P}_0 \\
\mathcal{P}_1
\end{array}
\end{align*}
$$

(5.11)

where

- $\tilde{\mathcal{B}}_*$ is the complex in (3.6), with $(\text{Sym}^{r-1}(E_*), D)$ substituted in place in $(E_*, D)$,
- $\tilde{\mathcal{A}}_*$ is the complex in (3.4), with $(\text{Sym}^{r-1}(E_*), D)$ substituted in place in $(E_*, D)$,
- $\mathcal{C}_*$ is the complex in (4.37),
- for $i = 0, 1$, the homomorphism $\mathcal{C}_i \rightarrow \tilde{\mathcal{B}}_i$ in (5.11) is the natural inclusion map, and
- the homomorphism $P$ is the homomorphism $\mathcal{P}$ in (5.9) with $(\text{Sym}^{r-1}(E_*), D)$ substituted in place in $(E_*, D)$. So $P_1$ is the identity map.

Let

$$
(P \circ Q)_* : \mathbb{H}^1(\mathcal{C}_*) \rightarrow \mathbb{H}^1(\tilde{\mathcal{A}}_*)
$$

(5.12)

be the homomorphism of hypercohomologies induced by the homomorphism $P \circ Q$ in (5.11).

Since the homomorphism $\mathcal{P}_*$ in (5.10) coincides with the projection $\Phi$ in (5.7), to prove that the map $M^0$ in (5.4) is an immersion, it suffices to show that the homomorphism $(P \circ Q)_*$ in (5.12) is injective.
Let
\[ q : \text{End}^a (\text{Sym}^{-1}(\mathcal{E}_s)) \otimes K_X \otimes \mathcal{O}_X (S) \rightarrow \] 
\[ \left( \text{End}^a (\text{Sym}^{-1}(\mathcal{E}_s)) \otimes K_X \otimes \mathcal{O}_X (S) \right) / P_0 Q_1 (\text{ad}^a (\text{Sym}^{-1}(\mathcal{E}_s)) \otimes K_X \otimes \mathcal{O}_X (S)) \]
be the quotient map. From the commutativity of (5.11) it follows immediately that the composition
\[ \text{End}^a (\text{Sym}^{-1}(\mathcal{E}_s)) \rightarrow \text{End}^a (\text{Sym}^{-1}(\mathcal{E}_s)) / \text{ad}^a (\text{Sym}^{-1}(\mathcal{E}_s)) \] 
vanishes on the subbundle \( \text{At}_0 (r) \cap \text{End}^P (\text{Sym}^{-1}(\mathcal{E}_s)) \subset \text{At}_X (r) \) (see (2.12) for the subbundle \( \text{End}^P (\text{Sym}^{-1}(\mathcal{E}_s)) \) of \( \text{At}(\text{Sym}^{-1}(\mathcal{E}_s)) \)). Therefore, \( q \circ \mathcal{D}_0 \circ (P_0 \circ Q_0) \) produces a homomorphism
\[ \mathcal{S} : TX \otimes \mathcal{O}_X (-S) \rightarrow (\text{End}^a (\text{Sym}^{-1}(\mathcal{E}_s)) / \text{ad}^a (\text{Sym}^{-1}(\mathcal{E}_s))) \otimes K_X \otimes \mathcal{O}_X (S) . \]
This homomorphism \( \mathcal{S} \) coincides with the second fundamental form of the reduction of structure group \( \mathcal{P}(B)_* \) in (1.18) for the connection on \( \mathcal{P}_* \) given by \( D \). We know that this second fundamental form is everywhere nonzero, because \( D \) is a parabolic \( \text{SL}(r, \mathbb{C}) \)-oper. Consequently, \( \mathcal{S} \) is everywhere nonzero. This implies that the homomorphism \( P_0 \circ Q_0 \) in (5.11) is injective.

Since \( P_0 \circ Q_0 \) in (5.11) is injective, it follows that the kernel of the homomorphism \( (P \circ Q)_* \) in (5.12) is the quotient of a subspace of \( H^0 (X, \text{End}^P (\text{Sym}^{-1}(\mathcal{E}_s)) / P_0 Q_0 (\text{At}_X (r))) \). More precisely, let
\[ V \subset H^0 (X, \text{End}^P (\text{Sym}^{-1}(\mathcal{E}_s)) / P_0 Q_0 (\text{At}_X (r))) \]
be the subspace consisting of all sections \( s \) such that \( \mathcal{D}_0 (s) = 0 \); note that from the commutativity of the diagram in (5.11) it follows that \( \mathcal{D}_0 \) produces a homomorphism
\[ H^0 (X, \text{End}^P (\text{Sym}^{-1}(\mathcal{E}_s)) / P_0 Q_0 (\text{At}_X (r))) \rightarrow H^0 (X, (\text{End}^a (\text{Sym}^{-1}(\mathcal{E}_s)) / \text{ad}^a (\text{Sym}^{-1}(\mathcal{E}_s))) \otimes K_X \otimes \mathcal{O}_X (S)) . \]
Let
\[ W \subset V \]
be the subspace consisting of all sections \( s \) such that there is a section
\[ \tilde{s} \in H^0 (X, \text{End}^P (\text{Sym}^{-1}(\mathcal{E}_s))) \]
satisfying the following two conditions:
\[ \begin{array}{l}
\bullet \ \mathcal{D}_0 (\tilde{s}) = 0, \text{ where } \mathcal{D}_0 \text{ is the homomorphism in (5.11), and} \\
\bullet \ \tilde{s} \text{ projects to } s \text{ under the natural map} \\
H^0 (X, \text{End}^P (\text{Sym}^{-1}(\mathcal{E}_s))) \rightarrow H^0 (X, \text{End}^P (\text{Sym}^{-1}(\mathcal{E}_s)) / P_0 Q_0 (\text{At}_X (r))) .
\end{array} \]
Then we have
\[ \text{kernel}((P \circ Q)_*) = V / W , \] 
(5.13)
where \( (P \circ Q)_* \) is the homomorphism in (5.12).

Now
\[ \text{End}^P (\text{Sym}^{-1}(\mathcal{E}_s)) / P_0 Q_0 (\text{At}_X (r)) = \mathcal{O}_X \oplus W , \]
where $W$ admits a filtration of holomorphic subbundles such that every successive quotient is of the form $(TX \otimes \mathcal{O}_X(-S))^{\otimes m}$, $m \geq 1$. From Assumption 2.1 it follows that
\[ H^0(X, (TX \otimes \mathcal{O}_X(-S))^{\otimes m}) = 0 \]
for all $m \geq 1$. Hence
\[ H^0(X, \text{End}^P (\text{Sym}^{r-1}(E)) / P_0 Q_0 (\text{At}_X^0 (r))) = H^0(X, \mathcal{O}_X) . \]
But $\mathcal{O}_X \subset \text{End}^P (\text{Sym}^{r-1}(E))$, and $D_0 (H^0(X, \mathcal{O}_X)) = 0$. Consequently, we have
\[ H^0(X, \mathcal{O}_X) \subset W, \]
where $W$ is the subspace in (5.13). Therefore, from (5.13) it follows that
\[ \ker((P \circ Q)_*) = 0 . \tag{5.14} \]
In other words, the homomorphism $(P \circ Q)_*$ is injective.

As noted above, the map $M^0$ in (5.4) is an immersion if the homomorphism $(P \circ Q)_*$ in (5.12) is injective. Therefore, we have proved the following:

**Theorem 5.1.** The map $M^0$ in (5.4) is an immersion.
The purpose of this appendix is to recall some results by K. Yokogawa [Yo] on Hom-sheaves, tensor products and extension classes of parabolic bundles and to give an alternative definition of a parabolic SL($r$)-oper [BDP] which is conceptually closer to the definition of an ordinary SL($r$)-oper.

A.1. Correspondence: flags and $\mathbb{R}$-filtered sheaves. We first recall the correspondence between a parabolic vector bundle as defined in section 2.1 and an $\mathbb{R}$-filtered sheaf $\{E_t\}_{t \in \mathbb{R}}$ as introduced and studied in [MY], [Yo], [BY]. Using the notation of section 2.1 we define for $t \in [0, 1]$ the vector bundle $E_t$ by the following equalities

$$
E_t^i = E \quad \text{for } 0 \leq t \leq \alpha_{i,1},
$$

$$
E_t^i = \ker(E \to E_{x_i}/E_{i,j}) \quad \text{for } \alpha_{i,j-1} < t \leq \alpha_{i,j},
$$

$$
E_t^i = E(-x_i) \quad \text{for } \alpha_{i,l_i} < t \leq 1,
$$

$$
E_t = \bigcap_{i=1}^n E_t^i.
$$

We extend to $\mathbb{R}$ by the formula $E_{t+1} = E_t(-S)$. We also denote this $\mathbb{R}$-filtered sheaf by $E_\cdot$. Note that $E_t \subset E_{t'}$ for any $t \geq t'$.

We recall that the family $E_\cdot$ is left-continuous, meaning that for any $t \in \mathbb{R}$

$$
\lim_{s \to t} E_s = E_t.
$$

The family $E_\cdot$ is not right-continuous and we will denote for any $t \in \mathbb{R}$

$$
E_{t+} := \lim_{s \to t \uparrow} E_s.
$$

Then $E_{t+}$ is a subsheaf of $E_t$ and the quotient $E_t/E_{t+}$ is a torsion-sheaf supported at the parabolic divisor $S$.

A.2. Special structure and shifts. Every vector bundle $E$ can be considered as a parabolic vector bundle $E_\cdot$ with the special structure, defined either by the properties $l_i = 1, \alpha_{i,1} = 0$ for $1 \leq i \leq n$, or equivalently by the equalities

$$
E_t := E \quad \text{for } t \in ]-1, 0].
$$

Given a parabolic vector bundle $E_\cdot$ and an $n$-tuple $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{R}^n$ we define the shift $E[\beta]_\cdot$ by the equalities

$$
E[\beta]_t = \bigcap_{i=1}^n E_{t+\beta_i}^i \quad \text{for } t \in \mathbb{R}.
$$

Also, in order to simplify notation, we define for $\beta \in \mathbb{R}$ the shift of $E[\beta]_\cdot$ as $E[\beta]_\cdot = E[\underline{\beta}]_\cdot$, with $\beta_i = \beta$ for every $i$. 

\"\text{Appendix A. Parabolic opers}\"
A.3. **Tensor products, Hom-sheaves and duals.** Given two parabolic vector bundles $E_s$ and $E'_s$ we define their parabolic tensor product (see e.g. [Yo] section 3) by the formula

$$(E_s \otimes E'_s)_t := \sum_{s \in \mathbb{R}} E_s \otimes E'_{t-s} \subset E_0 \otimes E'_0(*S).$$

Here $E_0 \otimes E'_0(*S)$ denotes the (non-coherent) sheaf of rational sections of $E_0 \otimes E'_0$ admitting arbitrary poles at the parabolic divisor $S$. We note that it is enough to consider the sum for $s$ running over a subinterval of $\mathbb{R}$ of length 1 (because of the invariance of the tensor product $E_s \otimes E'_{t-s}$ under the shift $s \mapsto s + 1$) and that only a finite number of subsheaves $E_s \otimes E'_{t-s}$ occur.

Before defining the parabolic Hom-sheaf of two parabolic vector bundles $E_s$ and $E'_s$, we define the sheaf $\text{Hom}(E_s, E'_s)$ as the subsheaf of $\text{Hom}(E_0, E'_0)$ consisting of parabolic homomorphisms, i.e., homomorphisms $f : E_0 \rightarrow E'_0$ satisfying

$$f(E_t) \subset E'_t$$

for any $t \in [0, 1]$ and thus for any $t \in \mathbb{R}$. Note that if $E_s$ and $E'_s$ are vector bundles, then $\text{Hom}(E_s, E'_s)$ is also a vector bundle. Now, we define the parabolic Hom-sheaf $\text{Hom}(E_s, E'_s)_t$ by the formula

$$\text{Hom}(E_s, E'_s)_t := \text{Hom}(E_s, E'_s[t]_s)$$

for any $t \in \mathbb{R}$.

The parabolic dual $E'_s^\vee$ of a parabolic vector bundle $E_s$ is by definition

$$E'_s^\vee := \text{Hom}(E_s, \mathcal{O}_s)_s,$$

where $\mathcal{O}_s$ denotes the trivial bundle with the special structure.

The above definitions of parabolic tensor products, Hom-sheaves and duals extend the standard operations on vector bundles, when considering a vector bundle as a parabolic vector bundle with its special structure. Also, the following relations are easy to check :

$$E[\beta]_s \otimes E'[\beta']_s = E \otimes E'[\beta + \beta']_s,$$

$$E[\beta]_s^\vee = E'[\beta]_s^\vee,$$

$$E'_s \otimes E'_s = \text{Hom}(E_s, E'_s)_s.$$

A.4. **Cohomology of a parabolic bundle.** Given a parabolic vector bundle $E_s$ over the curve $X$ we define the cohomology of $E_s$ as the cohomology of the vector bundle $E_0$

$$H^i(X, E_s) = H^i(X, E_0).$$

A.5. **Parabolic subbundles and parabolic degree.** We say that $E'_s$ is a parabolic subbundle of $E_s$ if there is an injective parabolic homomorphism

$$E'_s \hookrightarrow E_s$$

with torsion-free cokernel, or equivalently, for any $t \in \mathbb{R}$, the subsheaf $E'_t$ is a subbundle of $E_t$. 
The parabolic degree of a parabolic bundle \( E_* \) is defined as
\[
\text{pardeg}(E_*) = \int_0^1 \deg(E_t)dt + n\text{rk}(E_*),
\]
where \( n \) is the number of parabolic points. We have the following formulae:
\[
\text{pardeg}(E_* \otimes E'_*) = \text{rk}(E'_*)\text{pardeg}(E_*) + \text{rk}(E_*)\text{pardeg}(E'_*),
\]
\[
\text{pardeg}(E[\beta]_*) = \text{pardeg}(E_*) - \text{rk}(E_*) \sum_{i=1}^{n} \beta_i.
\]

A.6. **Canonical injections and quasi-isomorphisms.** Given a parabolic line bundle \( L_* \) and a vector \( \gamma \in \mathbb{R}^n \) with \( \gamma_i \geq 0 \) for all \( i \), we have a canonical parabolic injection
\[
\iota : L_* \longrightarrow L[-\gamma]_*
\]
induced by the natural inclusions \( L_t \subset L_t[-\gamma_i] \).

**Definition A.1.** We say that a parabolic homomorphism between two parabolic line bundles
\[
\phi : L_* \longrightarrow M_ *
\]
is a quasi-isomorphism, if there exists a vector \( \gamma \in \mathbb{R}^n \) with \( 0 \leq \gamma_i < 1 \) and a parabolic isomorphism \( M_* \cong L[-\gamma]_* \) such that via this isomorphism \( \phi \) identifies with the canonical injection \( \iota \). In that case we say that \( \phi \) is a quasi-isomorphism of weight \( \gamma \in \mathbb{R}^n \).

A.7. **Extensions of parabolic bundles.** Given two parabolic vector bundles \( E_* \) and \( E'_* \) we say that the parabolic vector bundle \( F_* \) is an extension of \( E_* \) by \( E'_* \) if there exists a short exact sequence of parabolic homomorphisms
\[
0 \longrightarrow E'_* \longrightarrow F_* \longrightarrow E_* \longrightarrow 0.
\]
By [Yo] Lemma 1.4 and Lemma 3.6 the isomorphism classes of extensions of \( E_* \) by \( E'_* \) are in one-to-one correspondence with the cohomology space \( \text{Ext}^1(E_*,E'_*) = H^1(X,\text{Hom}(E_*,E'_*)) \).

A.8. **Connections on parabolic bundles.** Given a parabolic vector bundle \( E_* \) with parabolic divisor \( S \subset X \) we define a connection \( \nabla_* \) on \( E_* \) as a \( \mathbb{C} \)-linear homomorphism between the parabolic bundles \( E_* \) and \( E_* \otimes K[-1]_* \)
\[
\nabla_* : E_* \longrightarrow E_* \otimes K[-1]_*
\]
such that for every \( t \in \mathbb{R} \) the map \( \nabla_t : E_t \rightarrow (EK[-1])_t = E_tK(S) \) is a logarithmic connection with poles at the parabolic divisor \( S \) and for any \( t \leq t' \) we have a commutative diagram
\[
\begin{array}{ccc}
E_{tt'} & \longrightarrow & E_{t'}K(S) \\
\downarrow & & \downarrow \\
E_t & \longrightarrow & E_tK(S)
\end{array}
\]
where the vertical maps are the natural inclusions.
On the trivial parabolic bundle $\mathcal{O}_*$ there is a natural connection given by the de Rham differentiation and which is denoted by $d_*$

$$d_* : \mathcal{O}_* \to K[-1]_*$$

and, fixing an integer $n$, is given for $t \in [n-1, n]$ by differentiation of regular functions having zeros or poles of order $n$ at $S$

$$d_t : \mathcal{O}_t = \mathcal{O}(-nS) \to K[t]_t = K(-(n-1)S).$$

We now describe the properties of $\nabla_*$ in terms of the parabolic structure given by the flags at the parabolic divisor.

**Lemma A.2.** Consider a connection $\nabla_*$ on $E_*$ as defined in (A.1). Then the logarithmic connection $\nabla_0$ on $E_0$ obtained by putting $t = 0$ satisfies

$$\text{Res}(\nabla_0, x_i)(E_{i,j}) \subset E_{i,j}$$

(A.2)

for any $i = 1, \ldots, n$ and any $j = 1, \ldots, l_i$. Conversely, any logarithmic connection $\nabla_0$ on $E_0$ satisfying (A.2) gives rise to a connection $\nabla_*$ on $E_*$. The proof of this lemma is standard and therefore left to the reader.

**A.9. Parabolic connections on parabolic bundles.** Consider a connection $\nabla_*$ on a parabolic bundle $E_*$ as defined in (A.1). Then for any $t \in \mathbb{R}$ we can consider the residue at $x_i \in S$ of the logarithmic connection $\nabla_t$

$$\text{Res}(\nabla_t, x_i) \in \text{End}_\mathbb{C}((E_t)_{x_i}).$$

Since $\nabla_t$ preserves all subsheaves $E_{t'}$ for $t' \geq t$ we obtain by passing to the quotient $(E_t/E_{t+})_{x_i}$ a linear map

$$\overline{\text{Res}}(\nabla_t, x_i) \in \text{End}_\mathbb{C}((E_t/E_{t+})_{x_i}).$$

and therefore an endomorphism, which we simply denote by $\overline{\text{Res}}(\nabla_t)$, of the torsion sheaf $E_t/E_{t+}$.

With this notation, we can now define a parabolic connection on a parabolic bundle.

**Definition A.3.** Let $E_*$ be a parabolic bundle. We say that a connection $\nabla_*$ on $E_*$ is a parabolic connection if for any $t \in \mathbb{R}$

$$\overline{\text{Res}}(\nabla_t) = t\text{Id}.$$
• the parabolic symmetric power $\text{Sym}^m E_*$ is naturally equipped with the symmetric power connection $(\text{Sym}^m \nabla)_*$, which is also parabolic.
• the de Rham differentiation $d_*$ on the trivial parabolic bundle $\mathcal{O}_*$ is a parabolic connection.

**Remark A.4.** Note that, when considering parabolic bundles from the “flag”-point of view, the relation between the parabolic weights of $E_*$ and those of its symmetric powers $\text{Sym}^m E_*$ is quite complicated (as one needs to take fractional parts and reorder them in order to obtain an increasing sequence of parabolic weights in the interval $[0, 1[$). Therefore the “$\mathbb{R}$-filtered sheaf”-point of view is more adapted when considering symmetric powers, as we will need to do in the sequel.

With this notation we can reformulate the following existence theorem.

**Theorem A.5 ([BL]).** The parabolic bundle $E_*$ admits a parabolic connection $\nabla_*$ if and only if any direct summand of $E_*$ has parabolic degree equal to 0.

**A.10. Parabolic SL(2)-opers.** We now define the parabolic analogue of the Gunning bundle. Given the parabolic weights $\alpha_{i,1}, \alpha_{i,2}$ for $1 \leq i \leq n$ satisfying the inequalities (2.3) and the additional assumption $\alpha_{i,1} + \alpha_{i,2} = 1$ for all $i$, we introduce the real numbers

$$\beta_i = \alpha_{2,i} - \alpha_{1,i} \in [0, 1[ \quad \text{and we define}$$

$$\underline{\beta} = (\beta_1, \beta_2, \ldots, \beta_n) \in \mathbb{R}^n.$$  

Let $K$ be the canonical bundle of the curve $X$. Then we define the canonical parabolic bundle by

$$K_{\text{par}}^* = K[\underline{-\beta}]_*,$$

where we equip $K$ with the special structure. We also define a parabolic theta-characteristic $\theta_{\text{par}}^*$ as a parabolic line bundle satisfying

$$\theta_{\text{par}}^* \otimes \theta_{\text{par}}^* = K_{\text{par}}^*.$$

One can easily check that the parabolic line bundle $\theta_{\text{par}}^*$ (resp. $(\theta_{\text{par}}^*)^{-1}$) corresponds via the above correspondence to a line bundle $M$ (resp. $Q$) satisfying $M^2 = K(-S)$ (resp. $Q^2 = K^{-1}(-S)$) with parabolic weight $\alpha_{i,2}$ (resp. $\alpha_{i,1}$) at the parabolic point $x_i$ for $1 \leq i \leq n$.

We define the parabolic Gunning bundle $\mathcal{G}_{\text{par}}^*$ as the unique non-split parabolic extension

$$0 \longrightarrow \theta_{\text{par}}^* \longrightarrow \mathcal{G}_{\text{par}}^* \longrightarrow (\theta_{\text{par}}^*)^{-1} \longrightarrow 0.$$  

(A.3)

We note that the space of parabolic extensions of $(\theta_{\text{par}}^*)^{-1}$ by $\theta_{\text{par}}^*$ is one-dimensional, since

$$\dim \text{Ext}^1((\theta_{\text{par}}^*)^{-1}, \theta_{\text{par}}^*) = \dim H^1(\theta_{\text{par}}^* \otimes \theta_{\text{par}}^*) = \dim H^1(K_{\text{par}}) = 1.$$  

Clearly, the parabolic bundle $\mathcal{G}_{\text{par}}^*$ has trivial parabolic determinant since

$$\det \mathcal{G}_{\text{par}}^* = \theta_{\text{par}}^* \otimes (\theta_{\text{par}}^*)^{-1} = \mathcal{O}_*.$$  

It is easy to check that the exact sequence (A.3) equals the exact sequence (3.4) in [BDP] defining the underlying parabolic bundle of a parabolic SL(2)-oper.
Furthermore, any parabolic connection $\nabla_*$ on $\mathcal{G}_*^{\text{par}}$ induces a second fundamental form, which is a $\mathcal{O}_X$-linear parabolic homomorphism

$$\psi : \theta^{\text{par}}_* \longrightarrow (\theta^{\text{par}}_*)^{-1} \otimes K[-1]_* \cong \theta^{\text{par}}_*(-1 + \beta).$$

The same argument as in the non-parabolic case shows that $\psi \neq 0$, since $\psi = 0$ would imply the existence of a parabolic connection on the parabolic bundle $\theta^{\text{par}}_*$ having parabolic degree $g - 1 + \frac{1}{2}(\sum_{i=1}^n \beta_i) > 0$, which contradicts Theorem [A.5]. Thus $\psi$ is a quasi-isomorphism of weight $1 - \beta$.

### A.11. Parabolic SL(r)-opers.

With the above introduced notation we give a new definition of a parabolic SL(r)-oper.

**Definition A.6.** A parabolic SL(r)-oper is a triple $(E_*, E_{\bullet *}, \nabla_*)$ consisting of a rank-$r$ parabolic vector bundle $E_*$, a filtration $E_{\bullet *}$ of $E_*$ by parabolic subbundles

$$0 = E_{0*} \subset E_{1*} \subset E_{2*} \subset \ldots \subset E_{r-1*} \subset E_{r*} = E_*$$

with $\text{rk}(E_{i*}) = i$ and a parabolic connection $\nabla_*$ on $E_*$ satisfying the following conditions

- $\det(E_*, \nabla_*) = (O_*, d_*)$
- $\nabla_*(E_{i*}) \subset E_{i+1*} \otimes K[-1]_*$ for any $i = 1, \ldots, r - 1$
- There exists a vector $\beta \in \mathbb{R}^n$ with $0 < \beta_i < 1$ such that for any $i = 1, \ldots, r - 1$ the parabolic homomorphisms induced by $\nabla_*$ between parabolic line bundles

$$\frac{(E_{i*}/E_{i-1*})}{(E_{i+1*}/E_{i*})} \otimes K[-1]_*$$

are quasi-isomorphisms of weight $1 - \beta$ (see Definition [A.11]).

**Remark A.7.** If $r = 2$ one can easily show that any parabolic SL(2)-oper is of the form $(\mathcal{G}_*^{\text{par}}, \mathcal{G}_*^{\text{par}, \bullet *}, \nabla_*)$, where $\mathcal{G}_*^{\text{par}}$ is the parabolic Gunning bundle introduced in section [A.10]. $\mathcal{G}_*^{\text{par}}$ is given by the exact sequence [A.3] and $\nabla_*$ is any parabolic connection satisfying $\det \nabla_* = d_*$.

We now show that, similar to the non-parabolic case, the underlying parabolic bundle of a parabolic SL(r)-oper is a parabolic symmetric power of the parabolic Gunning bundle.

**Theorem A.8.** Let $(E_*, E_{\bullet *}, \nabla_*)$ be a parabolic SL(r)-oper associated to the vector $\beta \in \mathbb{R}^n$. Then, up to tensor product with an $r$-torsion parabolic line bundle, we have an isomorphism between parabolic bundles

$$E_* \cong \text{Sym}^{r-1} \mathcal{G}_*^{\text{par}},$$

where $\mathcal{G}_*^{\text{par}}$ is the parabolic Gunning bundle associated to the vector $\beta \in \mathbb{R}^n$. Moreover, under this isomorphism the filtration $E_{\bullet *}$ corresponds to the natural filtration of $\text{Sym}^{r-1} \mathcal{G}_*^{\text{par}}$.

**Proof.** In order to simplify the notation we introduce the parabolic line bundles $Q_{i*} = E_{i*}/E_{i-1*}$ for $i = 1, \ldots, r$. Then the quasi-isomorphisms $Q_{i*} \rightarrow Q_{i+1*} \otimes K[-1]_*$ correspond to isomorphisms

$$Q_{i*} = Q_{i+1*} \otimes K[-i\beta_*],$$

for $i = 1, \ldots, r - 1$. Iterating these formulae we can express all line bundles in terms of $Q_{r*}$

$$Q_{r-1*} = Q_{r*} \otimes K[i\beta_*].$$
Since \( \det E_* = \mathcal{O}_* \), we obtain that the parabolic tensor product of all \( Q_\ast \) equals \( \mathcal{O}_* \), which leads to the isomorphism

\[
Q_{r*} = K^{r(r-1)/2} \left[ \frac{r(r-1)}{2} \beta \right].
\]

We choose a parabolic theta-characteristic \( \theta_{par} \), i.e., a parabolic line bundle satisfying \((\theta_{par})^2 = K_{par} = K[-\beta] \). Then the above equality is equivalent to saying that \( Q_{r*} \) and \((\theta_{par})^{-(r-1)}\) differ by an \( r \)-torsion line bundle. So, after tensorizing \( E_* \) and consequently all quotient line bundles \( Q_i \) by this \( r \)-torsion line bundle, we can assume that \( Q_{r*} = (\theta_{par})^{-(r-1)} \). From the above formulae, we immediately obtain that

\[
Q_{r-i*} = (\theta_{par})^{-(r-1)+2i}
\]

for \( i = 0, \ldots, r - 1 \). Next we will show that the natural exact sequence

\[
0 \longrightarrow Q_{i*} \longrightarrow E_{i+1*}/E_{i-1*} \longrightarrow Q_{i+1*} \longrightarrow 0
\]

is the unique non-split parabolic extension of \( Q_{i+1*} \) by \( Q_{i*} \). By section \([A.7]\) parabolic extensions are parameterized by \( \text{Ext}^1(Q_{i+1*}, Q_{i*}) \) and we have

\[
\dim \text{Ext}^1(Q_{i+1*}, Q_{i*}) = \dim H^1(Q_{i+1*} \otimes Q_{i*}) = \dim H^1(K[-\beta], K) = 1.
\]

We now show that \( E_{i+1*}/E_{i-1*} \) is non-split. Suppose on the contrary that we have a direct sum decomposition

\[
E_{i+1*}/E_{i-1*} = Q_{i*} \oplus Q_{i+1*}.
\]

We claim that this splitting implies that the exact sequence

\[
0 \longrightarrow E_{i*} \longrightarrow E_{i+1*} \longrightarrow Q_{i+1*} \longrightarrow 0 \tag{A.4}
\]

also splits. To see that, we consider the long exact sequence

\[
\cdots \longrightarrow \text{Ext}^1(Q_{i+1*}, E_{i-1*}) \longrightarrow \text{Ext}^1(Q_{i+1*}, E_{i*}) \xrightarrow{\mu} \text{Ext}^1(Q_{i+1*}, Q_{i*}) \longrightarrow \cdots
\]

where \( \mu \) is induced by the push-out under the map \( E_{i*} \rightarrow Q_{i*} \). Thus, to show that the exact sequence \((A.4)\) splits, it will be enough to show that \( \mu \) is injective. But \( \text{Ext}^1(Q_{i+1*}, E_{i-1*}) = 0 \), since \( E_{i-1*} \) can be constructed by a series of a successive extensions of \( Q_j \)'s for \( j \leq i - 1 \) and we have

\[
\text{Ext}^1(Q_{i+1*}, Q_{j*}) = 0 \quad \text{for all } j \leq i - 1.
\]

Thus \( E_{i+1*} = E_{i*} \oplus Q_{i+1*} \) and after projecting from \( E_{i+1*} \) onto \( E_{i*} \) the connection \( \nabla_* \) restricts to a connection on \( E_{i*} \). But, for \( i \leq r - 1 \) we have \( \text{pardeg}(E_{i*}) > 0 \), which is the desired contradiction by Theorem \([A.5]\).

Finally, we invoke Theorem 4.7 \([JP]\) to conclude that, since the rank-2 parabolic bundles \( E_{i+1*}/E_{i-1*} \) are the unique non-split parabolic extensions of \( Q_{i+1*} \) by \( Q_{i*} \) for all \( i = 1, \ldots, r - 1 \), the underlying parabolic vector bundle \( E_* \) is unique up to isomorphism. On the other hand, it is easily checked that the parabolic symmetric power \( \text{Sym}^{-1}G_{par}^r \) also satisfies these properties, hence by uniqueness both parabolic vector bundles \( E_* \) and \( \text{Sym}^{-1}G_{par}^r \) are isomorphic.

Note that Theorem 4.7 \([JP]\) deals with non-parabolic opers, but its extension to parabolic opers is straightforward. \(\square\)
Remark A.9. The last theorem shows that the above definition of parabolic $\text{SL}(r)$-oper coincides with [BDP] Definition 5.2.

References

[At] M. F. Atiyah, Complex analytic connections in fibre bundles, *Trans. Amer. Math. Soc.* **85** (1957), 181–207.

[BBN1] V. Balaji, I. Biswas and D. S. Nagaraj, Principal bundles over projective manifolds with parabolic structure over a divisor, *Tohoku Math. J.* **53** (2001), 337–367.

[BBN2] V. Balaji, I. Biswas and D. S. Nagaraj, Ramified $G$-bundles as parabolic bundles, *J. Ramanujan Math. Soc.* **18** (2003), 123–138.

[BBP] V. Balaji, I. Biswas and Y. Pandey, Connections on parahoric torsors over curves, *Publ. Res. Inst. Math. Sci.* **53** (2017), 551–585.

[BD1] A. Beilinson and V. G. Drinfeld, Opers, arXiv:0501398.

[BD2] A. Beilinson and V. G. Drinfeld, Quantization of Hitchin’s integrable system and Hecke eigensheaves, (1991).

[BF] D. Ben-Zvi and E. Frenkel, Spectral curves, opers and integrable systems, *Publ. Math. Inst. Hautes Études Sci.* **94** (2001), 87–159.

[Bh] U. N. Bhosle, Parabolic sheaves on higher-dimensional varieties, *Math. Ann.* **293** (1992), 177–192.

[Bi1] I. Biswas, Parabolic bundles as orbifold bundles, *Duke Math. J.* **88** (1997), 305–325.

[Bi2] I. Biswas, Parabolic ample bundles, *Math. Ann.* **307** (1997), 511–529.

[BDP] I. Biswas, S. Dumitrescu and C. Pauly, Parabolic $\text{SL}(r)$-opers, *Illinois J. Math.* **64** (2020), 493–517.

[BL] I. Biswas and M. Logares, Connection on parabolic vector bundles over curves, *Inter. Jour. Math.* **22** (2011), 593–602.

[BSY] I. Biswas, L. P. Schaposnik and M. Yang, Generalized B-opers, *Symmetry Integrability Geom. Methods Appl.* **16** (2020), Article 041.

[BY] H. Boden and K. Yokogawa, Moduli spaces of parabolic Higgs bundles and parabolic $K(D)$ pairs over smooth curves: I, *Inter. Jour. Math.* **7** (1996), 573–598.

[Bo1] N. Borne, Fibrés paraboliques et champ des racines, *Int. Math. Res. Not. IMRN*, **16**, Art. ID rnm049, 38, (2007).

[Bo2] N. Borne, Sur les représentations du groupe fondamental d’une variété privée d’un diviseur à croisements normaux simples, *Indiana Univ. Math. Jour.* **58** (2009), 137–180.

[Ch1] T. Chen, The associated map of the nonabelian Gauss–Manin connection, Thesis (Ph.D.)–University of Pennsylvania (2012), https://www.math.upenn.edu/grad/dissertations/ChenThesis.pdf.

[Ch2] T. Chen, The associated map of the nonabelian Gauss–Manin connection, *Cent. Eur. Jour. Math.* **10** (2012), 1407–1421.

[CS] B. Collier and A. Sanders, (G,P)-opers and global Slodowy slices, *Adv. Math.* **377** (2021), Paper No. 107490, 43 pp.

[De] P. Deligne, *Équations différentielles à points singuliers réguliers*, Lecture Notes in Mathematics, Vol. 163, Springer-Verlag, Berlin-New York, 1970.

[DFK+] O. Dumitrescu, L. Fredrickson, G. Kydonakis, R. Mazzeo, M. Mulase and A. Neitzke, From the Hitchin section to opers through nonabelian Hodge, *J. Differential Geom.* **117** (2021), 223–253.

[Fr1] E. Frenkel, Gaudin model and opers, *Infinite dimensional algebras and quantum integrable systems*, 1–58, Progr. Math., 237, Birkhäuser, Basel, 2005.

[Fr2] E. Frenkel, Lectures on the Langlands program and conformal field theory, *Frontiers in number theory, physics, and geometry. II*, 387–533, Springer, Berlin, 2007.

[FG1] E. Frenkel and D. Gaitsgory, Local geometric Langlands correspondence and affine Kac-Moody algebras, *Algebraic geometry and number theory*, 69–260, Progr. Math., 253, Birkhäuser Boston, Boston, MA, 2006.

[FG2] E. Frenkel and D. Gaitsgory, Weyl modules and opers without monodromy, *Arithmetic and geometry around quantization*, 101–121, Progr. Math., 279, Birkhäuser Boston, Boston, MA, 2010.
[FT] E. Frenkel and C. Teleman, Geometric Langlands correspondence near opers, *J. Ramanujan Math. Soc.* **28** (2013), 123–147.

[GH] P. Griffiths and J. Harris, *Principles of algebraic geometry*, Pure and Applied Mathematics, Wiley-Interscience, New York, 1978.

[In] M. Inaba, Moduli of parabolic connections on a curve and Riemann-Hilbert correspondence, *J. Algebraic Geom.* **22** (2013), 407–480.

[IIS1] M. Inaba, K. Iwasaki and M.-H. Saito, Dynamics of the sixth Painlevé equation, *Théories asymptotiques et équations de Painlevé*, 103–167, Sémin. Congr., 14, Soc. Math. France, Paris, 2006.

[IIS2] M. Inaba, K. Iwasaki and M.-H. Saito, Moduli of stable parabolic connections, Riemann-Hilbert correspondence and geometry of Painlevé equation of type VI. I., *Publ. Res. Inst. Math. Sci.* **42** (2006), 987–1089.

[Iw] K. Iwasaki, Fuchsian moduli on a Riemann surface—its Poisson structure and Poincaré-Lefschetz duality, *Pacific J. Math.* **155** (1992), 319–340.

[JP] K. Joshi and C. Pauly, Opers of higher types, Quot-schemes and Frobenius instability loci, *Épijournal de Géométrie Algébrique* **4** (2020), Article No. 7.

[KSZ] P. Koroteev, D. S. Sage and A. M. Zeitlin, (SL(N), q)-Opers, the q-Langlands correspondence, and quantum/classical duality, *Comm. Math. Phys.* **381** (2021), 641–672.

[LS] S. Lawton and A. S. Sikora, Varieties of characters, *Algebr. Represent. Theory* **20** (2017), 1133–1141.

[MY] M. Maruyama and K. Yokogawa, Moduli of parabolic stable sheaves, *Math. Ann.* 293 (1992), 77–99.

[MR] D. Masoero and A. Raimondo, Opers for higher states of quantum KdV models, *Comm. Math. Phys.* **378** (2020), 1–74.

[MS] V. B. Mehta and C. S. Seshadri, Moduli of vector bundles on curves with parabolic structures, *Math. Ann.* **248** (1980), 205–239.

[Na] M. Namba, *Branched coverings and algebraic functions*, Pitman Research Notes in Mathematics Series, 161, Longman Scientific & Technical, Harlow; John Wiley & Sons, Inc., New York, 1987.

[Sa] A. Sanders, The pre-symplectic geometry of opers and the holonomy map, preprint, arXiv:1804.04716.

[Si] A. S. Sikora, Character varieties, *Trans. Amer. Math. Soc.* **364** (2012), 173–208.

[Yo] K. Yokogawa, Infinitesimal Deformation of Parabolic Higgs sheaves, *Inter. Jour. Math.* **6** (1995), 125–148.