INTRODUCTION

In the past two decades, circuit quantum electrodynamics (circuit QED) has become a field of growing interest for quantum information processing and also to realize new regimes in quantum optics [1-9]. The restriction to one dimensional (1D) waveguides in circuit and waveguide QED enhances directionality and reduces losses and therefore has a great advantage over higher dimensional systems to reach strong- and ultrastrong-coupling regimes [10-20]. A typical circuit QED set-up consists of a superconducting qubit coupled to a 1D transmission line (TL) [2, 6-8, 21]. Superconducting qubits are artificial atoms built with a non-linear Josephson Junction (JJ), that creates an anharmonic energy spectrum [9]. There are different kinds of superconducting qubits like flux qubits, phase qubits, and charge qubits [8, 22]. A 1D transmission line can be modelled by coupled LC oscillators, each having a characteristic impedance of $Z_0 = \sqrt{L_0/C_0} \approx 50 \Omega$, smaller than the quantum resistance $R_Q = \hbar/(2e)^2 \approx 1.0k\Omega$. But recent studies show that it is possible to reach impedances comparable to the quantum resistance or higher [23-29]. This can be realized by building circuits made of arrays of JJs [24, 26, 28, 32] or high-kineic-inductances materials, called superinductors [24, 27, 33-36]. High impedance JJ arrays and superinductors are for example used in the Fluxonium qubit [37-40], which has reduced charge noise sensitivity and can have relaxation times up to milliseconds [39, 41]. This also has an advantage for metrology, since the charge noise insensitivity makes it possible to measure the current very accurately [42]. Furthermore, high-impedance resonators make it possible for light-matter interaction to reach strong coupling regimes, due to strong coupling to vacuum fluctuations [43].

In this article, we investigate the spontaneous emission of a transmon [44] capacitively coupled to a 1D TL that is shorted at one end. This system is known as an "atom in front of a mirror" [45-49]. Instead of using a Markovian master equation approach, we are taking the photon traveling time fully into account, making the dynamics non-trivial [12, 50-59]. Furthermore, we explore the above-mentioned regime of a TL impedance exceeding the quantum resistance $Z_0 \gg R_Q$. We find that the system behaves qualitatively different, compared to the well studied low impedance regime $Z_0 \ll R_Q$. The atom reflects strongly at all frequencies, except its transition frequency. Together with the mirror, it thus forms a cavity and when the transition frequency is close to a cavity mode, we find a vacuum Rabi-splitting, resulting in Rabi oscillations in the spontaneous emission. In this regime, all dynamical timescales are independent of the coupling capacitance and instead depend on the intrinsic transmon capacitance and the TL impedance. Another cavity-free system that shows Rabi splitting is an artificial atom coupled surface acoustic waves [60].

CIRCUIT-QED MODEL

Our system consists of a transmon qubit capacitively coupled to a semi-infinite 1D TL at a distance $L$ from its grounded end (see Fig. 1). The transmon qubit consists of a superconducting anharmonic LC-oscillator, where the inductive ($L$) element is formed by a Josephson junction (JJ) with characteristic energy $E_J$ in parallel with a capacitor ($C$) with capacitance $C_J$. The sinusoidal current-phase relation of the JJ makes the energy spectrum of the transmon qubit anharmonic, allowing for excitation with a single microwave photon using standard harmonic microwave sources. The transmon is capacitively coupled to a microwave TL, characterised by its inductance per unit length $L_0$ and capacitance per unit length $C_0$. The velocity and impedance of the electromagnetic field inside the TL is given by $v = 1/\sqrt{L_0 C_0}$ and $Z_0 = \sqrt{L_0/C_0}$, respectively. Using the standard circuit quantization procedure [59, 61], we can derive...
the Heisenberg equations of motion for the charge \( p_0(t) \)
the circuit capacitance \( C_c \), the charge \( p_J(t) \) on \( C_j \)
and its conjugate flux \( \phi_J(t) \), giving the phase difference
\( 2\epsilon_0\phi_J(t) / \hbar \) over the JJ. Denoting the operators for
the voltages of the incoming/outgoing microwave fields to the
left/right of the transmon \( V_{L/R}^{\text{in/out}}(t) \) (see Fig. 1(a)),
these equations are

\[
\begin{align*}
\partial_t \phi_J(t) &= \frac{1}{C_J} (p_J(t) + p_0(t)), \\
\partial_t p_J(t) &= -E_J \frac{2e}{\hbar} \sin \left( \frac{2e}{\hbar} \phi_J(t) \right) \approx -\frac{\phi_J(t)}{L_J}, \\
\partial_t p_0(t) &= \frac{2p_0(t)}{Z_0 C_S} + \frac{2p_J(t)}{Z_0 C_J} - \frac{2}{Z_0} \left[ V_{L}^{\text{in}}(t) + V_{R}^{\text{in}}(t) \right], \quad (3) \\
V_{L/R}^{\text{out}}(t) &= V_{L/R}^{\text{in}}(t) - \frac{Z_0}{2} \partial_t p_0(t), \quad (4)
\end{align*}
\]

where we denoted the capacitance to ground seen by the JJ
as \( C_S = C_c C_J / (C_c + C_J) \) and in the second equation,
we introduced the Josephson inductance \( L_J = \hbar^2 / 4e^2 E_J \),
which describes the linearized dynamics of the Josephson
junction. This approximation is obviously good in
the weak excitation regime \( |\phi_J(t)| < \hbar/2e \) and will also
be sufficient to describe the spontaneous emission, where
the transmon is initially excited by a single microwave
photon.

FIG. 1. a) The circuit model of a transmon coupled through
the coupling capacitance \( C_j \) to a semi-infinite 1D TL with
impedance \( Z_0 \). The Josephson energy, flux, and capacitance
of the transmon are denoted by \( E_J, \phi_J, \) and \( C_j \). The flux
on the coupling capacitance \( C_j \) is denoted by \( \phi_0 \) with the
corresponding voltage \( V_0 = \phi_0 \).

b) A sketch of the system depicting an atom in front of a
mirror coupled to incoming/outgoing microwave fields to the
left/right characterised by their respective voltages \( V_{L/R}^{\text{in/out}} \)
at the transmon. The mirror couples the fields to the right
\( V_R^{\text{in}}(t) = -V_R^{\text{out}}(t - T) \), introducing the time of propagation
to the mirror and back \( T = 2L/v \).

FIG. 2. Reflection of a transmon in an open TL for different
ratios of the TL and qubit impedance \( Z_0 / Z_j \). The curves
show the reflection for \( C_c Z_0 / Z_j = 0.1 \) and \( Z_0 / Z_j = 0.1 \) (purple),
\( Z_0 / Z_j = 1 \) (blue), \( Z_0 / Z_j = 10 \) (green), \( Z_0 / Z_j = 100 \)
(yellow), and \( Z_0 / Z_j = 1000 \) (red) from low to high TL
impedance. For low impedance, the qubit reflects only at
the coupled qubit frequency \( \omega_0 \), but for high impedance, it
reflects everywhere but the uncoupled qubit frequency \( \omega_j \).

**REFLECTION**

**Open TL**

To characterise how the behaviour of the system
changes when we increase the TL impedance \( Z_0 \), we first
investigate the reflection of microwaves from the transmon
coupled to an open TL, i.e. without a mirror. Since
the equations of motion are linear, we can express the re-
lected field operator \( V_{L}^{\text{out}} \) in terms of the incoming probe
field operator \( V_{L}^{\text{in}} \) by Fourier transforming the equations
of motion (1)-(4), assuming no incoming field from the
right \( V_{R}^{\text{in}} = 0 \). The expression for the frequency depen-
dent reflection coefficient is given by

\[
\begin{align*}
\rho(\omega) &\equiv \frac{V_{L}^{\text{out}}(\omega)}{V_{L}^{\text{in}}(\omega)} = \frac{C_c Z_0 \omega \left( \frac{\omega^2}{\omega_j^2} - 1 \right)}{2i \left( 1 - \frac{\omega^2}{\omega_j^2} \right) + C_c Z_0 \omega \left( \frac{\omega^2}{\omega_j^2} - 1 \right)} \quad (5)
\end{align*}
\]

where \( \omega_0 = 1/\sqrt{L_J (C_c + C_J)} \) is the resonance frequency
of the coupled transmon and \( \omega_j = 1/\sqrt{L_J C_J} \) is the
resonance frequency of the bare (uncoupled) transmon. In
Fig. 2 the reflection around the transmon resonance
frequencies is shown for different values of \( Z_0 \). We see that
for low impedance \( Z_0 C_c \omega < 1 \), the reflection is weak ex-
cept at \( \omega_0 \) where it is unity, due to resonant reflection
from the transmon [2]. For high impedance \( Z_0 C_c \omega > 1 \)
we instead see strong reflection at all frequencies, except
around the "new" resonance frequency \( \omega_j \), where we find
zero reflection independent of \( Z_0 \). The crossover occurs
at \( Z_0 \sim Z_J C_J / C_c \), introducing the transmon impedance

\[
Z_J = \sqrt{\frac{L_J}{C_J}} = RQ\sqrt{\frac{2E_C}{E_J}},
\]

(6)

where \( E_C = e^2 / (2C_J) \) is the charging energy of the transmon.

In the high impedance regime, the strong scattering away from \( \omega_J \) occurs due to the comparably strong capacitive coupling to ground at the transmon without exciting the transmon. Close to \( \omega_J \), the resonantly excited transmon counteracts this capacitive coupling and effectively acts like an open circuit. This is the opposite behaviour compared to the low impedance regime, where the transmon is effectively an open circuit at all frequencies, except at its resonance frequency \( \omega_0 \), where it acts like a shorted circuit, giving full reflection. By fitting Lorentzians to Eq. (5), we can extract the capacitance incoming from the left by Fourier transform.

Similarly as before, we can find the response to a harmonic field incident on the transmon given by \( V_0 = V_0^\text{in} + V_0^\text{out} \). In the low impedance regime, we instead have large currents flowing through the TL, leading to the voltage at the coupling node close to zero, i.e. \( |V_0| \ll |V_J| \). In the high impedance regime, we have significant coupling through the strong reflection of the transmon and the energy dissipation scales with \( Z_0 \). In the following, we investigate how the mirror affects the scattering.

**Mirror**

The mirror couples the fields to the right of the transmon \( V_R^\text{in}(t) = -V_R^\text{out}(t-T) \), introducing the time of propagation to the mirror and back \( T = 2L/v \). Similarly as before, we can find the response to a harmonic field incoming from the left by Fourier transformation of the equations of motion. Since the absolute value of the reflection for the transmon in front of a mirror is always unity, we are now interested in the frequency dependence of the ratio between the trapped field (between the qubit and the mirror) and the incoming field, which is given by \( f(\omega) \equiv V_R^\text{out}(\omega) / V_R^\text{in}(\omega) \).

\[
f(\omega) = \frac{\omega^2\omega_0^2 - 1}{(\omega^2\omega_0^2 - 1) - iC_J Z_0 \omega_0 \left( \frac{\omega^2}{\omega_0^2} - 1 \right) (e^{i\omega T} - 1)},
\]

(7)

which is shown in Fig. 3. In the high impedance regime, we now find cavity resonances between the highly reflective atom and the mirror when the frequency is close to \( n \omega_c \) for \( n = 1, 2, \ldots \) and \( \omega_c = 2\pi / T \), as shown by the peaks in the inset of Fig. 3. These are broadened by the coupling to the TL by \( \gamma_c^n = |t(n \omega_c)|^2 / T \), where \( t(\omega) \) is the transmission across the transmon and \( |t(\omega)|^2 = 1 - |r(\omega)|^2 \). We find that the effect on the transmon resonance close to \( \omega_J \) is simply to reduce its broadening with a factor of two to \( \gamma_c^n = 1 / (2\omega_0 C_J) \). As shown in the main panel of Fig. 3, on resonance \( \omega_J \approx n \omega_c \), we find an avoided crossing with the coupling strength

\[
\Omega = \frac{2}{\sqrt{T C_J Z_0}} = \frac{2\omega_J}{\sqrt{2\pi n C_J Z_0 \omega_J}} = \frac{2\omega_J}{\sqrt{2\pi n \frac{\omega_0}{2}}},
\]

(8)

As we will see in the next section, where we investigate the spontaneous emission dynamics of the transmon, we find that this coupling indeed gives rise to vacuum Rabi-oscillations between the transmon and the cavity mode.

In the low impedance regime \( Z_0 C_c \omega < 1 \), \( f(\omega) \) is instead close to unity, indicating only little scattering from the transmon for all frequencies far from the transmon resonance \( \omega = \omega_0 \). Here, \( f(\omega_0) = 0 \), since the field is reflected by the transmon and does not reach the mirror. When the transmon is located at a distance corresponding to a node of the electromagnetic field at its resonance frequency, i.e. \( \omega_0 T = 2\pi n \), it is in a dark state and is thus completely invisible to the incoming field at frequency \( \omega_0 \), giving instead \( f(\omega_0) = 1 \). In the dark state, both the transmon and the field between the

![Graph](image-url)
The qubit energy variables are zero in the vacuum state. The Rabi-oscillations damped towards a finite dark state move in frequency towards the cavity frequency \( \omega_0 \) of the transmon (blue) and the flux periods of the Rabi oscillations. The Inset shows the energy of the transmon (blue), Field between the mirror and transmon (pink), outgoing field to the left (orange), sum of all the energies (red), approximated damping rate \( \gamma \), sum of all the energies (green), approximated damping rate \( \gamma \). For higher \( Z_0 \) slightly lower/higher than \( J_0 \), we see that this dark state resonance (1) at frequencies slightly lower/higher than \( \omega_0 \), see e.g. the purple line in Fig. 3. For higher \( Z_0 \), we see that this dark state resonance moves in frequency towards the cavity frequency \( \omega_c = 2\pi/T \). As shown in the supplemental material, for small \( C_c/C_J \ll 1 \), and \( \omega_0 = n \omega_c \), we can find vacuum Rabi-oscillations damped towards a finite dark state population.

In the following, we investigate how the high-impedance TL influences spontaneous emission of the transmon.

### Spontaneous Emission and Rabi Oscillations

We consider the case of a transmon initially excited at time \( t = 0 \) with a finite flux \( \phi_J(0) > 0 \), while the other qubit variables are zero \( p_J(0) = p_0(0) = 0 \) and the TL is in the vacuum state. The qubit energy

\[
E_q(t) = \frac{(p_J(t) + p_0(t))^2}{2C_J} + \frac{p_0(t)^2}{2C_c} + \frac{\phi_J(t)^2}{2L_J},
\]

is the sum of the capacitive energy on the two capacitances and the inductive energy in the JJ. The current amplitude emitted from the transmon into the TL is \( \partial_t p_0(t) \) and from this we can write the change of the energy \( E_R(t) \) of the field between the transmon and the mirror as

\[
\partial_t E_R(t) = \frac{Z_0}{4} \left[ (\partial_t p_0(t))^2 - (\partial_t p_0(t - T))^2 \right],
\]

where the first term corresponds to the instantaneous power emitted into the TL and the second term is the instantaneous power coming back from the mirror. The change of the energy of the field to the left of the transmon \( E_L(t) \) is given by the instantaneous left-moving power leaving the system,

\[
\partial_t E_L(t) = \frac{Z_0}{4} \left[ (\partial_t p_0(t) - \partial_t p_0(t - T))^2 \right],
\]

where the left-moving current amplitude is a sum of the current emitted by the transmon and the delayed current arriving from the mirror.

In Fig. 4 we plot these energies for \( Z_0/Z_J = 1000 \) for the case of resonance between transmon and the first cavity resonance \( \omega_J = \omega_c \). The system energies indeed perform damped Rabi-oscillations with the frequency \( \Omega = 2/\sqrt{TC_JZ_0} \) and half the off-resonance damping rate \( \gamma^2 / 2 = 1/2C_JZ_0 \), as indicated by the yellow line given by the expression \( e^{-\gamma^2 t/2} \cos^2(\Omega t/2) \), approximating the full numerical solution of the differential equations very well. We note that Laplace transforming the equations of motion (1)-(4) and calculating the residues of the system variables, gives similar expressions for the Rabi frequency and damping rate as the analysis of the resonances in the scattering amplitudes. More details and a comparison of the approximation to the numerical results can be seen in the supplemental materials.

### Effective Quantum Model: Atom in a Multimode Cavity Hamiltonian

We now go on to demonstrate that in the high-impedance regime the response function \( f(\omega) \) of the field trapped between the transmon and the mirror Eq. (7), reproduces the dynamics of an effective Hamiltonian of a single transition atom in a multimode cavity. This Hamiltonian is of the form:

\[
H = \omega_J \left( a^\dagger a + \frac{1}{2} \right) + \sum_{n=1}^{\infty} n\omega_c \left( c^\dagger_n c_n + \frac{1}{2} \right) \\
+ \frac{\Omega}{2} \sum_{n=1}^{\infty} \left( a^\dagger + a \right) \left( c^\dagger_n + c_n \right),
\]

where the operators \( a, a^\dagger \) are annihilation and creation bosonic operators \( \{a, a^\dagger\} = 1 \) associated with excitation in the transmon qubit, while \( c_n, c^\dagger_n \) annihilate/create photons in the cavity modes. When weakly excited, the
choice for bosonic excitations of the transmon is justified, while the orthogonality relations between the cavity modes is ensured by the high finesse of the latter, so that we have \[ [c_n, c_m^\dagger] = \delta_{nm}. \] Details of the diagonalisation of the Hamiltonian are shown in the supplemental materials. The response function \(|f(\omega)|\) and eigenfrequencies of Hamiltonian (12) are shown in Fig. 5. The eigenfrequencies are shown as superimposed white dashed lines. The parameters are chosen to be: \(Z_0/Z_J = 100\) and \(C_v/C_J = 0.3\).

![Graph showing response function](image)

**FIG. 5.** Response function (log color scale) versus normalized frequency \(\omega/\omega_J\) and linearly varied time delay \(\omega_JT\). The eigenvalues \(\Omega_n\) are computed for \(N = 8\) cavity modes and are shown as superimposed white dashed lines. The parameters are chosen to be: \(Z_0/Z_J = 100\) and \(C_v/C_J = 0.3\).

**DISCUSSION AND OUTLOOK**

We have made a first theoretical investigation of the properties of a transmon capacitively coupled to a high impedance transmission line, a system which is currently becoming experimentally accessible. By linearizing the Josephson junction, we could describe the low excitation dynamics, including spontaneous emission. We find qualitatively different behaviour, compared to the low impedance regime. In particular, the atom now forms its own cavity, and we can observe a vacuum Rabi splitting, giving rise to Rabi-oscillations in the spontaneous emission. The system is well described by a Hamiltonian for an atom weakly coupled to a multimode cavity. We hope that this analysis will inspire an experimental realization of this novel system.

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*Corresponding to the dark state is found for \(\omega/c\) cavity modes. Noticeably, a dip in the response function\(|\omega|\) is computed for \(N = 8\) cavity modes and are shown as superimposed white dashed lines. The parameters are chosen to be: \(Z_0/Z_J = 100\) and \(C_v/C_J = 0.3\).*

**DISCUSSION AND OUTLOOK**

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Rabi oscillations of an atom coupled to a high-impedance semi-infinite TL - additional parameter values

We can Laplace transform the equations of motion of the transmon shown in the main article and extract the exact poles numerically. With the following formulas we can calculate the inverse Laplace transform of the system variables and energy:

\[
\text{Res}_{s_{1,2}} \left[ p_J(s) \right] = \lim_{s \to s_{1,2}} p_J(s)(s - s_{1,2}^\pm),
\]

where \( k = s_{1,2}^\pm \) are the poles of \( p_J(s) \). Similarly, we calculate \( \phi_J(t) \). We show the results as an addition to Fig. 4 in the main article. Here we provide more figures for different system parameters. In all panels of Fig 6 the impedance is chosen to be \( Z_0/Z_J = 1000 \). In Fig. 6 a), the ratio of the coupling capacitance and the Josephson capacitance is fairly small \( C_c/C_J \approx 0.02 \) and the coupling to the TL is weak. The cavity frequency equals the transition frequency of the qubit for low impedance, \( \omega_C = \omega_J \). The decay is weak and the Rabi-oscillations are clearly visible. The parameters in Fig. 6 b) are similar to a), but now \( C_c/C_J + C_c = 0.02 \) and most importantly the cavity frequency equals the transition frequency of the qubit for low impedance, \( \omega_C = \omega_J \). The energy of the Rabi oscillations decays until it reaches the dark state, with energy \( E_\text{DS} = \frac{1}{2} \left( 1 + \frac{2}{1 + 2 \gamma_0^2} \right) \gamma_0 \omega_0^2 \omega C_c^2 / C_J \), where \( \gamma_0 = \frac{Z_0\omega_0^2}{2C_J^2} \). In Fig. 6 c), the coupling capacitance is much larger compared to a) and b), \( C_c/C_J + C_J = 0.3 \) and the transmon fulfills the dark state condition \( \omega_C = \omega_0 \). Anyhow, the system does not converge into a dark state and the Rabi oscillations are very weak. The main difference here is that, since \( C_c/C_J + C_J \) is rather large, it means that \( \omega_J \) is not close to \( \omega_C \) and the Rabi oscillations and coupling to the cavity are suppressed. In this parameter regime, the behaviour of the system seems to be independent of the position of the transmon with respect to the mirror. Similar to Fig. 6 c), in Fig. 6 d), the coupling capacitance is rather large too \( C_c/C_J = 0.3 \), but here the cavity frequency equals the frequency of the transmon \( \omega_C = \omega_J \), which means that the Rabi condition is fulfilled. We see clear Rabi oscillations and in addition the decay is much slower than in c).

As mentioned in the main article, we find an analytical expression for the oscillation frequency in the high impedance regime by analyzing the Laplace transform of the equations of motion. We find the Rabi frequency to be \( \Omega = \frac{\sqrt{C_J Z_0}}{2} \). In Fig. 7 we demonstrate the deviations of the approximation from numerically calculated values. We find that the higher the ratio \( C_c^2 Z_0^2 / C_J \), the closer the approximation resembles the numerical solution.

Analysis of the response functions

To analyze the solution of the Fourier transformation of the equations of motion in the main article in a convenient manner, we introduce the following functions:

\[
N(\omega) = R_0(\omega) - i R_J(\omega),
\]

\[
R_0(\omega) = \left( 1 - \frac{\omega^2}{\omega_0^2} \right),
\]

\[
R_J(\omega) = \frac{C_c Z_0}{2} \omega \left( 1 - \frac{\omega^2}{\omega_J^2} \right).
\]
FIG. 6. Energy of the transmon (blue), energy of the field trapped between the transmon and the mirror (magenta), outgoing field to the left side of the transmon (orange), total energy of the system (red) and semi-numerically calculated energy of the transmon using Eq. (14) as a function of period of the Rabi oscillations. The parameters for the panels are the following:

a) $Z_0/Z_J = 1000$, $C_c/C_J = 0.02$ and $\omega_J = \omega_C$. Since we chose $C_c/C_J = 0.02$, the decay is very slow.

b) $Z_0/Z_J = 1000$, $C_c/(C_J + C_c) = 0.02$ and $\omega_0 = \omega_C$. Here, the parameters are almost the same as in a), but now the dark state condition is fulfilled, $\omega_J = \omega_0$. The system converges into a dark state with a finite excitation probability of both the transmon and the field between the transmon and the mirror. The green line indicates the analytical value of the dark state energy $E_{DS} = \frac{1}{2} (1 + \frac{T_2 \gamma_0}{2})^2$ with $\gamma_0 = \frac{Z_0 \omega_0^2}{2} \frac{C_c^2}{C_J + C_c}$.

c) $Z_0/Z_J = 1000$, $C_c/(C_J + C_c) = 0.3$ and $\omega_0 = \omega_C$. Here despite $\omega_0 = \omega_C$ as in b), the system does not seem to converge into a dark state and the decay rate is given by $\gamma_J = 1/C_J Z_0$. The difference to b) is that the ratio between $\frac{C_c Z_0 \omega}{2 \gamma_0}$ here is much bigger than in b) which also means that $\omega_J$ is not close to $\omega_C$ and the Rabi oscillations are barely visible.

d) $Z_0/Z_J = 1000$, $C_c/C_J = 0.3$ and $\omega_J = \omega_C$. Here, as in c) $\frac{C_c Z_0 \omega}{2 \gamma_0} \gg 1$ but the "Rabi condition" $\omega_J = \omega_C$ is still fulfilled. We see clear Rabi oscillations and the decay is slower compared to c) and the decay rate is given by $\gamma_J^2/2 = 1/2C_J Z_0$.

With these definitions we are able to write the transmission and the reflection amplitudes of the qubit in the open transmission line as

$$r(\omega) = \frac{iR_J(\omega)}{N(\omega)}, \quad t(\omega) = \frac{R_0(\omega)}{N(\omega)}, \quad r(\omega) + 1 = t(\omega).$$  

(18)

We note that the high impedance regime corresponds to that $|R_J(\omega)| \gg |R_0(\omega)|$ away from resonances, i.e. $|C_c Z_0 \omega/2| \gg 1$, while the opposite ($|R_J(\omega)| \ll |R_0(\omega)|$) is true in the low impedance regime $|C_c Z_0 \omega/2| \ll 1$. 

Ω \left( \frac{C_c}{C_J} = 0.01 \right)
Ω \left( \frac{C_c}{C_J} = 0.05 \right)
Ω \left( \frac{C_c}{C_J} = 0.3 \right)
\Omega^A

Damping rate for the open TL

We analyse the scattering solution of the qubit excitation \( \phi_J \) to find the damping rate for the transmon in an open TL. The solution in frequency space reads

\[
\phi_J(\omega) = -\frac{C_c L_J \omega}{R_0(\omega) - R_J(\omega)}. \tag{19}
\]

In the high impedance regime, we perform an expansion around the bare qubit frequency \( \omega = \omega_J + \delta \omega \) and find to first order in \( \delta \omega \)

\[
\phi_J(\omega_J + \delta \omega) \approx \frac{i}{\omega_J} \frac{1}{1 - i 2 \delta \omega / \gamma_J}, \tag{20}
\]

where \( \gamma_J = 2 / Z_0 C_J \) is the energy damping rate for spontaneous emission.

In the low impedance regime, the qubit resonance is shifted to \( \omega_0 \) and we instead expand \( \omega = \omega_0 + \delta \omega \) to find

\[
\phi_J(\omega_0 + \delta \omega) = \frac{1}{C_c Z_0 \omega_0^2} \frac{1}{1 - i 2 \delta \omega / \gamma_0}, \tag{21}
\]

where \( \gamma_0 = \frac{Z_0 \omega_0^2}{2} \frac{C_c^2}{C_c + C_J} \) is the low-impedance damping rate \[59\].

Damping rates and Lamb shifts with a mirror

With a mirror, the solution for the Josephson flux \( \phi_J \) can be written as

\[
\phi_J(\omega) = \frac{-i C_c L_J \omega (1 - e^{i \omega T})}{R_0(\omega) - i R_J(\omega) (1 - e^{i \omega T})}. \tag{22}
\]

In the low impedance regime, we find that the qubit resonance is Lamb-shifted to \( \tilde{\omega}_0 = \omega_0 + \gamma_0 \sin(\omega_0 T / 2) \) and we can expand \( \phi_J(\omega) \) around the resonance and find

\[
\phi_J(\tilde{\omega}_0 + \delta \omega) \approx -i \frac{C_c}{C_J + C_c \gamma_0 \sin(\omega_0 T / 2)} \frac{e^{i \omega_0 T / 2}}{1 - i 2 \delta \omega / \gamma_0 m}, \tag{23}
\]
with the Purcell-modified damping rate $\gamma_j^\| = 2\gamma_0 \sin^2 (\omega_0 T/2)$.

In the high impedance regime, the resonance frequency is Lamb shifted to $\tilde{\omega}_J = \omega_J + \gamma_J \cot (\omega_J T/2)/4$. Expanding $\phi_J(\omega)$ around this frequency we find

$$\phi_J(\tilde{\omega}_J + \delta \omega) = -\frac{2i}{\omega_J} \frac{1}{1 + i2\delta \omega/\gamma_j^\|},$$

(24)

where $\gamma_j^\| = \gamma_J/2 = 1/C_J Z_0$ is the damping rate of an atom in front of a mirror in the high $Z_0$ regime. Here, we note that the expression for the Lamb-shift diverges when $\sin(\omega_J T/2) = 0$, i.e. when $\omega_J$ is close to a cavity resonance $\omega_0^C = 2\pi n/T$. This is when the single pole approximation is no longer valid and we find the vacuum Rabi splitting. Away from the Rabi condition, we also note that the damping rate is independent of the distance to the mirror, i.e. we see no Purcell effect. Away from the Rabi condition, we can also analyze the response function

$$f(\omega) = \frac{1}{1 - i R_J(\omega) / \pi n(\omega)(1 - e^{i\omega T})},$$

(25)

to extract the cavity modes. Due to the finite transmission through the transmon, they are slightly shifted from the perfect mirror frequencies to $\tilde{\omega}_C^n = \omega_0^C + R_0(\omega_0^C)/T R_J(\omega_0^C)$. Close to the resonances we can expand $\omega = \tilde{\omega}_C^n + \delta \omega$ and find

$$f(\tilde{\omega}_C^n + \delta \omega) \approx \frac{2}{t(\omega_0^C)} \frac{1}{1 - i2\delta \omega/\gamma_C^n},$$

(26)

where $\gamma_C^n \approx |t(\omega_0^C)|^2/T$ is the energy damping rate of cavity mode $n$.

**Hopfield diagonalization of the atom in a multimode cavity Hamiltonian**

To diagonalize the effective quantum optical Hamiltonian (Eq. (12) in the main article), we first introduce the new polaritonic operators:

$$\Pi_\alpha = x^\alpha a + y^\alpha a^\dagger + \sum_n \left( m_n^\alpha c_n + h_n^\alpha c_n^\dagger \right),$$

(27)

$x^\alpha, y^\alpha, m_n^\alpha, h_n^\alpha$ being the Hopfield coefficients associated to each bosonic operator $a, a^\dagger, c_n, c_n^\dagger$. To ensure the bosonicity of the polaritonic operators ($[\Pi_\alpha, \Pi_\beta^\dagger] = \delta_{\alpha\beta}$), the coefficients should satisfy the relation:

$$|x^\alpha|^2 - |y^\alpha|^2 + \sum_n \left( |m_n^\alpha|^2 - |h_n^\alpha|^2 \right) = 1.$$  

(28)

The new operators should satisfy the eigenvalue problem:

$$[\Pi_\alpha, H] = \Omega_\alpha \Pi_\alpha,$$

(29)

where $\Omega_\alpha$ are the eigenfrequencies labeled with a new index $\alpha$. Typically, if we couple the atom with $N$ cavity modes, then $\alpha$ runs from 1 to $N+1$. Expanding the commutator in the previous equation, it is possible to write the eigenvalue problem in a matrix form:

$$\begin{pmatrix}
\omega_J & 0 & g_1 & \cdots & g_N & -g_N \\
0 & -\omega_J & g_1 & \cdots & g_N & -g_N \\
g_1 & -g_1 & \omega_J & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\
g_N & -g_N & 0 & \cdots & \omega_N & 0 \\
g_N & -g_N & 0 & \cdots & 0 & -\omega_N
\end{pmatrix}
\begin{pmatrix}
x^\alpha \\
y^\alpha \\
m_1^\alpha \\
m_2^\alpha \\
m_3^\alpha \\
m_N^\alpha \\
h_1^\alpha \\
h_2^\alpha \\
h_3^\alpha \\
h_N^\alpha
\end{pmatrix}
= \Omega_\alpha
\begin{pmatrix}
x^\alpha \\
y^\alpha \\
m_1^\alpha \\
m_2^\alpha \\
m_3^\alpha \\
m_N^\alpha \\
h_1^\alpha \\
h_2^\alpha \\
h_3^\alpha \\
h_N^\alpha
\end{pmatrix}.$$  

(30)

This eigenvalue problem can be solved analytically for $N = 1$ but in general, one has to diagonalize it numerically. A comparison between the full response function and the eigenvalues are shown in Fig. 5 in the main article.