A concentration inequality for the excess risk in least-squares regression with random design and heteroscedastic noise

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Abstract

We prove a new and general concentration inequality for the excess risk in least-squares regression with random design and heteroscedastic noise. No specific structure is required on the model, except the existence of a suitable function that controls the local suprema of the empirical process. So far, only the case of linear contrast estimation was tackled in the literature with this level of generality on the model. We solve here the case of a quadratic contrast, by separating the behavior of a linearized empirical process and the empirical process driven by the squares of functions of models.

keywords: regression, least-squares, excess risk, empirical process, concentration inequality, margin.

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1 Introduction

The excess risk of a M-estimator is a fundamental quantity of the theory of statistical learning. Consequently, a general theory of rates of convergence as been developed in the nineties and early 2000 ([Mas07, Kol11]). However, it has been recently identified that some theoretical descriptions of learning procedures need finer controls than those brought by the classical upper bounds of the excess risk. In this case, the derivation of concentration inequalities for the excess risk is a new and exiting axis of research, of particular importance for obtaining satisfying oracle inequalities in various contexts, especially linked to high dimension.

In the field of model selection, it has been indeed remarked that such concentration inequalities allow to discuss the non-asymptotic optimality of model selection procedures ([BM07, AM09, Let11, San12]). More precisely, concentration inequalities for the excess risk and for the excess empirical risk ([BM11]) are central tools to access the optimal constants in the oracle inequalities describing the model selection accuracy. Such results have put to evidence the optimality of the so-called slope heuristics ([BM07, BMM12]) and more generally of selection procedures based on the estimation of the minimal penalty ([AB09]), in statistical frameworks linked to regularized quadratic estimation. Under similar assumptions, it is also possible to discuss optimality of resampling and cross-validation type procedures ([Arl08, AL15, Cel14, NS17]).

In high dimension, convex methods allow to design and compute efficient estimators. This is the reason why Chatterjee [Cha14] has recently focused on the estimation of the mean of a high dimensional Gaussian vector under convex constraints. By getting a concentration inequality for the excess risk of a projected least-squares estimator, Chatterjee [Cha14] has proved the universal admissibility of this estimator. The concentration inequality has then been sharpened and extended to the excess risk of estimators minimizing penalized convex criteria ([MvdG15, vdGW17]).

It is also well known (see for instance [BLT17]) that a weakness of the theory of regularized estimators in a sparsity context is that classical oracle inequalities, such as in [BRT09], describe the performance of estimators with an amount of regularization that actually depends on the confidence level considered in the oracle inequality. This does not correspond to any practice with this kind of estimators, whose regularization parameter is usually fixed using a cross-validation procedure. Recently, Bellec and Tsybakov [BT17], building on [BLT18], have established more satisfying oracle inequalities, describing the performance of regularized estimators such as LASSO, group LASSO and SLOPE with a confidence level independent of the regularization
We will also denote \( f \) where \( P \) projection called the subset of \( R^L \) in \( L \).

We consider the least-squares estimator \( \hat{g} \) where \( P \) where \( G \) subset of \( \mathbb{R}^G \) formula holds for the excess risk on \( \mathbb{R}^G \).

In this paper, we extend the technology developed in [Cha14, MvdG15, vdGW17] in order to establish a new concentration inequality for the excess risk in least-squares regression with random design and homoscedastic noise, while [vdGW17] have to assume that the law of the design is known in order to perform a "linearized least-squares regression" (see Section 6.2 of [vdGW17]).

Our strategy is as follows. We first remark that the empirical process of interest splits into two parts: a linear process and a quadratic one. Then we prove that the linear process achieves a second order margin condition as defined in [vdGW17] and put meaningful conditions on the quadratic process in order to handle it. Techniques from empirical process theory such as Talagrand’s type concentration inequalities and contraction arguments are at the core of our approach.

The paper is organized as follows. The regression framework as well as some properties linked to margin relations are described in Section 2. Then we state our main result in Section 3. The proofs are deferred to Section 4.

2 Least-squares heteroscedastic regression with random design

2.1 Setting

Let \( \{(X_i, Y_i)\}_{i=1}^{n} \) be an i.i.d. sample taking values in \( \mathcal{X} \times \mathbb{R} \), where \( \mathcal{X} \) is a measurable space - typically a subset of \( \mathbb{R}^p \). We assume that the following relation holds

\[
Y_i = g* (X_i) + \sigma (X_i) \varepsilon_i \quad \text{for } i = 1, ..., n ,
\]

where \( g* \) is the regression function, \( \sigma \) is the heteroscedastic noise level and \( \mathbb{E} [\varepsilon_i | X_i] = 0, \mathbb{E} [\varepsilon_i^2 | X_i] = 1 \).

We take a closed convex model \( \mathcal{G} \subset L_2 \left(P^X\right) \), where \( P^X \) is the common distribution of the \( X_i \)'s, and set

\[
g^0 = \arg \min_{g \in \mathcal{G}} \{ P (\gamma (g)) \} ,
\]

where \( P \) is the common distribution of the pairs \( (X_i, Y_i) \) and \( \gamma \) is the least-squares contrast, defined by

\[
\gamma (g) (x, y) = (y - g(x))^2 .
\]

We will also denote \( f_\gamma := \gamma (g) , f_* = \gamma (g_* ) , f^0 = \gamma (g^0) , \hat{f} = \gamma (\hat{g}) \) and \( 0 : = \gamma (\mathcal{G}) \). The function \( g^0 \) will be called the projection of the regression function onto the model \( \mathcal{G} \). Indeed, if we denote \( \| \cdot \| \) the quadratic norm in \( L_2 \left(P^X\right) \), it holds

\[
\| g^0 - g_* \| = \min_{g \in \mathcal{G}} \| g - g_* \| .
\]

We consider the least-squares estimator \( \hat{g} \) over \( \mathcal{G} \), defined to be

\[
\hat{g} \in \arg \min_{g \in \mathcal{G}} \{ P_n (\gamma (g)) \} ,
\]

where \( P_n := 1/n \sum_{i=1}^{n} \delta_{(X_i, Y_i)} \) is the empirical measure associated to the sample.

We want to assess the concentration of the quantity \( P \left( \hat{f} - f^0 \right) = P \left( \gamma (\hat{g}) - \gamma (g^0) \right) \), called the excess risk of the least-squares estimator on \( \mathcal{G} \), around a single deterministic point. To this end, it is easy to see ([Sau12, Remark 1, NS17, Section 5.2 or van de Geer and Wainwright [vdGW17]]) that the following representation formula holds for the excess risk on \( \mathcal{G} \) in terms of empirical process,

\[
\hat{s} := \sqrt{P \left( \hat{f} - f^0 \right)} = \arg \min_{s \geq 0} \left\{ s^2 - \hat{E}_n (s) \right\} , \tag{1}
\]

where

\[
\hat{E}_n (s) = \max_{f \in \mathcal{F}} \left\{ (P_n - P) \left( f^0 - f \right) \right\} ,
\]
with $F_s = \{f \in F; P(f - f^0) \leq s^2\}$. It is shown in [vdGW17] for various settings that include linearized regression that the quantity $\hat{s}$ actually concentrates around the following point,

$$s_0 := \arg \min_{s \geq 0} \{s^2 - E(s)\}, \quad E(s) := E\left[\hat{E}_n(s)\right].$$

(2)

2.2 On a margin-like relation pointed on the projection of the regression function

In order to prove concentration inequalities for the excess risk on $G$, we will need to check the following margin-like relation, also called "quadratic curvature condition" in [vdGW17]: there exists a constant $C > 0$ such that

$$P(f - f^0) \geq \frac{\sigma^2(f - f^0)}{C^2} \text{ for all } f \in F,$$

(3)

where

$$\sigma^2(f) := \mathbb{E}f^2(X_1) - (\mathbb{E}f(X_1))^2$$

is the variance of $f$ with respect to $P_X$.

A very classical relation in statistical learning, called margin relation, consists in assuming that (3) holds with $f^0$ is replaced by the image of the target $f_* = \gamma(g_*)$. Such relation is satisfied in least-squares regression whenever the response variable $Y$ is uniformly bounded. Here we do not assume that $f_*$ belongs to $F$ thus $f^0$ may be different from $f_*$.  

**Condition 1** There exists $A_1 > 0$ such that $|Y| \leq A_1$ a.s.

From Condition 1 we deduce that $\|g_*\|_\infty \leq A_1$ and $\|\sigma\|_\infty \leq 2A_1$.

**Condition 2** There exists $A_2 > 0$ such that

$$\sup_{g \in \mathcal{G}} \|g\|_\infty \leq A_2 < \infty.$$

From Conditions 1 and 2 we deduce that the image model $F$ is also uniformly bounded: there exists $K > 0$ such that

$$\sup_{f \in F} \|f - f^0\|_\infty \leq K < \infty.$$

More precisely, $K = 2(A_1 + A_2)$ is convenient.

The following proposition shows that relation (3) is satisfied in our regression setting whenever the response variable is bounded and the model $G$ is convex and uniformly bounded. It can also be found in [Lec11], Proposition 1.3.3.

**Proposition 3** If the model $G$ is convex and Conditions 1 and 2 hold, then there exists a constant $C > 0$ such that

$$P(f - f^0) \geq \frac{\sigma^2(f - f^0)}{C^2} \text{ for all } f \in F,$$

(4)

Furthermore, $C = 2(A_1 + A_2)$ is convenient.

The major gain brought by Proposition 3 over the classical margin relation is that the bias of the model, that is the quantity $P(f^0 - f_*)$ that is implicitly contained in the excess risk appearing in the classical margin relation, is pushed away from Inequality (4). Proposition 3 is thus a refinement over the classical notion of margin relation. It is stated for the least-squares contrast but it is easy to see that it can be extended to more general situations, where the contrast $\gamma$ is convex and regular in some sense (see Proposition 2.1, Section 2.2.3 in [Sau10]). For completeness, the proof of Proposition can be found in Section 4.1.
2.3 Second order quadratic margin condition

First notice that the arguments of the empirical process of interest can be decomposed into a linear and a quadratic part. It holds, for any \( f = f_g \in \mathcal{F} \) and any \((x, y) \in \mathcal{X} \times \mathbb{R}\),

\[
   f_g (x, y) - f^0 (x, y) = \gamma (g) (x, y) - \gamma (g^0) (x, y) = \psi (x, y) \cdot (g - g^0) (x) + (g - g^0)^2 (x),
\]

where \( \psi (x, y) = -2 (g - g^0 (x)) \).

To this contrast expansion around the projection \( g^0 \) of the regression function onto \( \mathcal{G} \), we can associate two empirical processes, that we will call respectively the linear and the quadratic empirical process, and we will be more precisely interested by their local maxima on \( \mathcal{G}_s := \{ g \in \mathcal{G} : \| g - g^0 \| \leq s \}, s \geq 0 \).

\[
   \hat{E}_{n, \ell} (s) = \max_{g \in \mathcal{G}_s} \{ (P_n - P) (\psi \cdot (g - g^0)) \} \quad \text{and} \quad \hat{E}_{n, q} (s) = \max_{g \in \mathcal{G}_s} \{ (P_n - P) (g - g^0)^2 \}.
\]

In what follows, we will not directly show that the excess risk concentrates around \( s_0 \) defined in (2), but rather around a point \( \hat{s}_0 \), defined to be,

\[
   \hat{s}_0 = \arg \min_{s \geq 0} \{ s^2 - \mathbb{E}_\ell (s) \}, \quad \mathbb{E}_\ell (s) = \mathbb{E} \left[ \hat{E}_{n, \ell} (s) \right].
\]

It holds around \( \hat{s}_0 \) a relation of the type of a second order margin relation, introduced in \cite{vdGW17}, as proved in the following Lemma, which proof is available in Section 4.1.

Lemma 4 For any \( s \geq 0 \), it holds

\[
   s^2 - \mathbb{E}_\ell (s) - \left[ \hat{s}_0^2 - \mathbb{E}_{\ell} (\hat{s}_0) \right] \geq (s - \hat{s}_0)^2,
\]

and also

\[
   s^2 - \hat{E}_{n, \ell} (s) - \left[ \hat{s}_0^2 - \hat{E}_{n, \ell} (\hat{s}_0) \right] \geq (s - \hat{s}_0)^2.
\]

The fundamental difference with the second order margin relation stated in \cite{vdGW17} is that we require in Lemma 4 conditions on the linear part of empirical process and not on the empirical process of origin, that takes in arguments contrasted functions. Indeed, it seems that for the latter empirical process, the second order margin relation does not hold or is hard to check in least-squares regression in general. This difficulty indeed forced van de geer and Wainwright \cite{vdGW17}, Section 6.2, to work in a "linearized least-squares regression" context, under the quite severe restriction that the distribution of design is known from the statistician. On contrary, our main result stated in Section 3 below is stated for a general regression situation, where the distribution of the design is unknown and the noise level is heteroscedastic.

3 Main result

Before stating our new concentration inequality, we describe the required assumptions. In order to state the next condition, let us denote

\[
   \hat{E}_{n, 1} (s) = \max_{g \in \mathcal{G}_s} \{ (P_n - P) (g - g^0) \}, \quad \mathbb{E}_1 (s) = \mathbb{E} \left[ \hat{E}_{n, 1} (s) \right].
\]

Condition 5 There is a sequence \( m_n \) and a strictly increasing function \( \mathcal{J}_1 \) such that the function

\[
   \Phi_{\mathcal{J}_1} (u) := \left[ \mathcal{J}_1^{-1} (u) \right]^2, \quad u > 0,
\]

is strictly convex and such that

\[
   \mathbb{E}_1 (s) \leq \frac{\mathcal{J}_1 (s)}{m_n}, \quad s \geq 0.
\]
**Condition 6** Take $K > 0$. For any $s > 0$, there exists a constant $D(s) \in (0, K]$ such that

$$
\sup_{g \in G} \|g - g^0\|_\infty \leq D(s).
$$

Notice that if Conditions 1, 2 and 5 hold, then by the use of classical symmetrization and contraction arguments, for all $s \geq 0$,

$$
E_{\ell}(s) \leq C_{A_1, A_2} \frac{J_1(s)}{m_n} \quad \text{and} \quad E_q(s) \leq 4D(s) \frac{J_1(s)}{m_n},
$$

where $C_{A_1, A_2}$ is a positive constant that only depends on $A_1$ and $A_2$. From now on, we set

$$
\mathcal{J} := \max \{C_{A_1, A_2}, 4(K \vee 1)\} J_1,
$$

so that $\max\{E_{\ell}(s); E_q(s)\} \leq \mathcal{J}(s)/m_n$ for any $s > 0$ and also $E_q(s) \leq D(s) \mathcal{J}(s)/m_n$.

We are now able to state our main result.

**Theorem 7** If

$$
\mathcal{J}(s) = A_\mathcal{J}s; \quad D(s) = A_\infty s; \quad m_n = \sqrt{n}; \quad \tilde{s}_0 = \frac{A_0}{\sqrt{n}}
$$

then it holds, for any $t \geq 0$,

$$
\mathbb{P} \left( \left\| \hat{g} - g^0 \right\| - \tilde{s}_0 \right\| \geq \frac{\sqrt{A_\mathcal{J} A_\infty \tilde{s}_0}}{n^{1/4}} \vee c_0 \left( \sqrt{\frac{t + \ln (1 + K \sqrt{n})}{n}} + \frac{t + \ln (1 + K \sqrt{n})}{n} \right) \right) \leq e^{-t}. \quad (8)
$$

Hence, if moreover $\sqrt{\ln n} \ll A_0 \ll \sqrt{n}$ and $A_\mathcal{J} A_\infty \ll \sqrt{n}$ then,

$$
\left\| \hat{g} - g^0 \right\| \leq \frac{\sqrt{A_\mathcal{J} A_\infty \tilde{s}_0}}{n^{1/4}} \vee \frac{\sqrt{\ln n}}{A_0} = o_p(1).
$$

Inequality (8) of Theorem 7 is a new concentration inequality related least-squares regression with random design and heteroscedastic noise on a convex, uniformly bounded model. In particular, it extends results of VdGW17 related to linearized regression, which is a simplified framework for regression, to the classical and general least-squares framework described in Section 2.1 above. The proof of Theorem 7 is detailed in Section 4.2.

The following corollary provides an generic example entering into the assumptions of Theorem 7 and that is related to linear aggregation via empirical risk minimization.

**Corollary 8** Define $\mathcal{M} = \text{Span}(\varphi_1, ..., \varphi_D)$ the linear span generated by $(\varphi_k)_{k=1}^D$, an orthonormal dictionary in $L_2(P^n)$. Take $G = B_{\mathcal{M}, L_\infty}(g^0, 1) \subset \mathcal{M}$ the unit ball in sup-norm of $\mathcal{M}$ centered on $g^0$, the projection of $g^*$ onto $\mathcal{M}$. Assume that

$$
\sup_{g \in \mathcal{M}, \|g\|_2 = 1} \|g\|_\infty \leq c_{\mathcal{M}} \sqrt{D} \quad \text{(9)}
$$

and

$$
(ln n)^2 \leq D \leq \frac{\sqrt{n}}{ln n}.
$$

Then, if $\tilde{s}_0 \asymp \sqrt{D/n}$,

$$
\left\| \hat{g} - g^0 \right\| \leq \frac{\sqrt{D}}{n^{1/4}} \vee \frac{\sqrt{\ln n}}{D} = o_p(1).
$$

Note that Inequality (9) relating the sup-norm to the quadratic norm of the functions in the linear span of the dictionary, is classical in non-parametric estimation and is satisfied for the usual functional bases such as the Fourier basis, wavelets or piecewise polynomials over a regular partition (including histograms), see for instance BBM99. In particular, Corollary 8 extends a concentration inequality recently obtained in Sau18, for the excess risk of the ERM in the linear aggregation problem when the dictionary at hand is the Fourier dictionary.
4 Proofs

4.1 Proofs related to Section 2

Proof of proposition 3. Take \( f = f_g \in \mathcal{F} \). Then, on the one hand,

\[
\sigma^2 (f - f^0) \leq P (f - f^0)^2 = \mathbb{E} \left[ (\gamma (g) - \gamma (g^0))^2 (X, Y) \right] = \mathbb{E} \left[ (g - g^0)^2 (X) (2Y - g(X) - g_0(X))^2 \right] \leq 4 (A_1 + A_2)^2 \| g - g^0 \|^2 .
\]

On the other hand,

\[
P (f - f^0) = \mathbb{E} \left[ (\gamma (g) - \gamma (g^0)) (X, Y) \right] = \mathbb{E} \left[ (g^0 - g)(X) (2Y - g(X) - g_0(X)) \right] = \mathbb{E} \left[ (g^0 - g)(X) (2Y - g_0(X) + g_0(X) - g(X)) \right] = \| g - g^0 \|^2 - 2 \mathbb{E} [(Y - g^0(X)) (g - g^0)(X)] = \| g - g^0 \|^2 - 2 \mathbb{E} [(g_0 - g^0)(X) (g - g^0)(X)] \geq \| g - g^0 \|^2 .
\]

The latter inequality, which corresponds to \( \mathbb{E} [(g_0 - g^0)(X) (g - g^0)(X)] \leq 0 \) comes from the fact that \( \mathcal{G} \) is convex and so, \( g^0 \) being the projection of \( g_* \) onto \( \mathcal{G} \), the scalar product in \( L_2 (P^X) \) between the functions \( g_* - g^0 \) and \( g - g^0 \) is nonpositive. Combining \([10] \) and \([11] \) now gives the result. \( \blacksquare \)

Proof of Lemma 4. Inequality \([4] \) derives from \([7] \) by taking expectation on both sides. Concerning the proof of \([4] \), it is easily seen that the function \( s \mapsto \hat{\mathbb{E}}_{n, \ell} (s) \) is concave. Indeed, take for \( s_i \geq 0 \), \( i \in \{1, 2\} \),

\[
g_{n,s_i} = \arg \max_{g \in G_{s_i}} \{ (P_n - P) (\psi_1 \cdot (g - g^0)) \} .
\]

For any \( \alpha, \beta \in (0, 1) \), \( \alpha + \beta = 1 \), if \( g_b = \alpha g_{n,s_1} + \beta g_{n,s_2} \), then by the triangular inequality, \( g_b \in G_{\alpha s_1 + \beta s_2} \), which gives,

\[
\hat{\mathbb{E}}_{n, \ell} (\alpha s_1 + \beta s_2) \geq (P_n - P) (\psi_1 \cdot (g_b - g^0)) = \alpha \hat{\mathbb{E}}_{n, \ell} (s_1) + \beta \hat{\mathbb{E}}_{n, \ell} (s_2) .
\]

Now, from the concavity of \( s \mapsto \hat{\mathbb{E}}_{n, \ell} (s) \), we deduce that the function \( s \mapsto s^2 - \hat{\mathbb{E}}_{n, \ell} (s) \) is 1-strongly convex, which implies \([3] \). \( \blacksquare \)

4.2 Proofs related to Section 3

Proof of Theorem 7. We prove the concentration of \( \| \hat{g} - g^0 \| \) at the right of \( \hat{s}_0 \) and arguments will be of the same type for the deviations at the left. Take \( t, \varepsilon > 0 \), \( J := \lceil K/\varepsilon \rceil \) and set, for any \( j \in \{1, ..., J\} \), the intervals

\[
I_j := ((j - 1) \varepsilon + \delta + \hat{s}_0, \hat{s}_0 + \delta + j \varepsilon] .
\]

We also set

\[
z(t) = 2C\hat{s}_0 \sqrt{\frac{t}{n}} + r_0 \sqrt{\frac{t}{n} + \frac{Kt}{n}} .
\]

It holds, for any \( \delta > 0 \) (to be chosen later),

\[
\mathbb{P} (\hat{s}_0 + \delta < \hat{s} \leq K) \leq \mathbb{P} \left( \exists s \in (\hat{s}_0 + \delta, K), \; s^2 - \hat{\mathbb{E}}_n (s) \leq \hat{s}_0^2 - \hat{\mathbb{E}}_n (\hat{s}_0) \right) \leq \mathbb{P} \left( \exists s \in (\hat{s}_0 + \delta, K), \; s^2 - \hat{\mathbb{E}}_n (s) \leq \hat{s}_0^2 - \mathbb{E} (\hat{s}_0) + z(t) \right) + e^{-t} ,
\]

(10)
where in the last inequality we used Lemma 11. Furthermore, by setting for all \( j \in \{1, \ldots, J\} \),
\[
P_j := P\left( \exists s \in I_j, \ s^2 - \hat{E}_n (s) \leq \tilde{s}_0^2 - E (\tilde{s}_0) + z (t) \right),
\]
a union bound gives,
\[
P\left( \exists s \in (\tilde{s}_0 + \delta, K), \ s^2 - \hat{E}_n (s) \leq \tilde{s}_0^2 - E (\tilde{s}_0) + z (t) \right) \leq \sum_{j=1}^{J} P_j.
\]
Now, for each index \( j \) and for all \( s \in I_j \), we have
\[
s^2 - \hat{E}_n (s) \geq ((j - 1) \varepsilon + \delta + \tilde{s}_0)^2 - \hat{E}_n (j \varepsilon + \delta + \tilde{s}_0).
\]
Furthermore, it holds for all \( u > 0 \), with probability \( 1 - e^{-u} \),
\[
\hat{E}_n (j \varepsilon + \delta + \tilde{s}_0) \leq E (j \varepsilon + \delta + \tilde{s}_0) + 2C (\delta + j \varepsilon) \sqrt{\frac{u}{n}} + z (u)
\leq E_n (j \varepsilon + \delta + \tilde{s}_0) + E_q (j \varepsilon + \delta + \tilde{s}_0)
+ 2C (\delta + j \varepsilon) \sqrt{\frac{u}{n}} + z (u),
\]
where the first inequality comes from Lemma 11. By Lemma 13, we then have
\[
((j - 1) \varepsilon + \delta + \tilde{s}_0)^2 - E_n (j \varepsilon + \delta + \tilde{s}_0) - E_q (j \varepsilon + \delta + \tilde{s}_0)
\geq \tilde{s}_0^2 - E (\tilde{s}_0) + (\delta + j \varepsilon)^2 - D \left( (j \varepsilon + \delta + \tilde{s}_0) \mathcal{J} (j \varepsilon + \delta + \tilde{s}_0) / m_n \right)
+ ((j - 1) \varepsilon + \delta + \tilde{s}_0)^2 - (j \varepsilon + \delta + \tilde{s}_0)^2
\geq \tilde{s}_0^2 - E (\tilde{s}_0) + (\delta + j \varepsilon)^2 - D \left( (j \varepsilon + \delta + \tilde{s}_0) \mathcal{J} (j \varepsilon + \delta + \tilde{s}_0) / m_n \right)
- 2 \varepsilon (\tilde{s}_0 + \delta + j \varepsilon) + \varepsilon^2.
\]
Putting the previous estimates in (12), we get, for all \( s \in I_j \),
\[
s^2 - \hat{E}_n (s) \geq \tilde{s}_0^2 - E (\tilde{s}_0) + (\delta + j \varepsilon)^2 - D \left( (j \varepsilon + \delta + \tilde{s}_0) \mathcal{J} (j \varepsilon + \delta + \tilde{s}_0) / m_n \right)
- 2C \left( \sqrt{\frac{u}{n}} + \varepsilon \right) (\delta + j \varepsilon) - 2 \varepsilon \tilde{s}_0 + \varepsilon^2 - z (u)
\geq \tilde{s}_0^2 - E (\tilde{s}_0) + \frac{1}{2} (\delta + j \varepsilon)^2 - D \left( (j \varepsilon + \delta + \tilde{s}_0) \mathcal{J} (j \varepsilon + \delta + \tilde{s}_0) / m_n \right)
- 2C \left( \sqrt{\frac{u}{n}} + \varepsilon \right)^2 - 2 \varepsilon \tilde{s}_0 + \varepsilon^2 - z (u),
\]
with probability \( 1 - e^{-u} \). We require that
\[
(\delta + j \varepsilon)^2 \geq 4D \left( (j \varepsilon + \delta + \tilde{s}_0) \mathcal{J} (j \varepsilon + \delta + \tilde{s}_0) / m_n \right).
\]
Using the assumptions, it is equivalent to require
\[
(\delta + j \varepsilon)^2 \geq 4 \sqrt{A_{\mathcal{J} A_{\infty}} (j \varepsilon + \delta + \tilde{s}_0)^2 / n}.
\]
Whenever \( A_{\mathcal{J} A_{\infty}} \leq \sqrt{n} \), the last display is true if \( \delta \sim A_{\mathcal{J} A_{\infty}} \tilde{s}_0 / n^{1/4} \). We also require that
\[
(\delta + j \varepsilon)^2 \geq 4 \left( 2 \left( C \sqrt{\frac{u}{n}} + \varepsilon \right)^2 + 2 \varepsilon \tilde{s}_0 - \varepsilon^2 + z (u) + z (t) \right).
\]
To finish the proof, we fix \( \varepsilon = 1 / \sqrt{n} \) and \( u = t + \ln (1 + K \sqrt{n}) \). In particular, \( J \leq 1 + K \sqrt{n} \) and
\[
\sum_{j=1}^{J} \mathbb{P}_j \leq e^{-t}.
\]

Our conditions on \( \delta = \delta(t) \) become, for a constant \( c_0 \) only depending on \( C \) and \( K \),
\[
\delta(t) \geq \frac{\sqrt{A_J A_{\infty} s_0}}{n^{1/4}} + c_0 \left( \sqrt{\frac{t + \ln (1 + K \sqrt{n})}{n}} + \frac{t + \ln (1 + K \sqrt{n})}{n} \right),
\]
since \( s_0 \lor r_0 \ll 1 \).

**Proof of Corollary 8.** Under Assumption (9), we have
\[
\mathbb{E}_1(s) \leq \mathbb{E}^{1/2} \left[ \left( \max_{g \in G} \{ (P_n - P) (g - g^0) \} \right)^2 \right] \leq s \sqrt{T_n},
\]
where we used two times Cauchy-Schwarz inequality. Hence, \( m_n = \sqrt{n} \) and \( J(s) = s \sqrt{D} \) are convenient. Furthermore, by Assumption (9), we have
\[
\sup_{g \in G} \| g - g^0 \|_\infty \leq c M s \sqrt{D}.
\]
We can thus apply Theorem [7] with \( D(s) = c M s \sqrt{D} \). Consequently, condition \( \sqrt{\ln n} \ll A_0 \ll \sqrt{n} \) turns into
\[
\ln n \ll D \ll n
\]
and condition \( A_J A_{\infty} \ll \sqrt{n} \) is satisfied whenever \( D \ll \sqrt{n} \).

In the following theorem (see Theorem 8.1 in [vdGW17]), the right-tail inequalities are direct applications of Bousquet [Bou02] and the left-tail inequalities are deduced from Klein and Rio [KR05].

**Theorem 9.** If Conditions 1 and 2 are satisfied and we set
\[
K := \sup_{f \in F} \| f - f^0 \|_\infty, \quad \sigma_s := \sup_{f \in F} \sigma (f - f^0),
\]
then it holds
\[
\mathbb{P} \left( \hat{E}_n(s) \geq \mathbf{E}(s) + \sqrt{\frac{(8K \mathbf{E}(s) + 2\sigma_s^2) t}{n}} + \frac{2Kt}{3n} \right) \leq e^{-t},
\]
\[
\mathbb{P} \left( \hat{E}_n(s) \leq \mathbf{E}(s) - \sqrt{\frac{(8K \mathbf{E}(s) + 2\sigma_s^2) t}{n}} - \frac{Kt}{n} \right) \leq e^{-t}.
\]

Using Proposition 3 and Conditions 1, 2, and 5 we can simplify the bounds given in Theorem 9 as follows.

**Lemma 10.** If Conditions 1, 2, and 5 are satisfied, then with the same notations as in Theorem 9 and also
\[
r_0^2 := 2C^2 \Phi \left( \frac{8K}{C^2 n} \right),
\]
we have
\[
\mathbb{P} \left( \hat{E}_n(s) \geq \mathbf{E}(s) + 2Cs \sqrt{\frac{t}{n}} + r_0 \sqrt{\frac{t}{n} + \frac{2Kt}{3n}} \right) \leq e^{-t},
\]
\[
\mathbb{P} \left( \hat{E}_{n, \ell}(s) \leq \mathbf{E}(s) - 2Cs \sqrt{\frac{t}{n}} - r_0 \sqrt{\frac{t}{n} - \frac{Kt}{n}} \right) \leq e^{-t},
\]

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Proof. We have $\sigma^2_s \leq C^2 s^2$, where the constant $C$ is defined in proposition. Furthermore, using the fact that $uv \leq \Phi_J(u) + \Phi_J^*(v)$ for any $u, v > 0$, we get

$$8K E(s) \leq \frac{16K}{m_n} J(s) \leq 2C^2 s^2 + 2C^2 \Phi_J^*\left(\frac{8K}{m_n C^2}\right) = 2C^2 s^2 + r^2_0.$$ 

The conclusion is then easy to obtain by using $\sqrt{a + b} \leq \sqrt{a} + \sqrt{b}$. ■

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