A Global, Continuous, and Exponentially Convergent Observer for Attitude and Gyro Bias

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Abstract—We propose a 12-dimensional, global, continuous, and exponentially convergent observer for attitude and gyro bias of a rigid body. The observer is designed in the set of 3 by 3 real matrices, thus making the topological obstruction on the special orthogonal group irrelevant.

Index Terms—Attitude, gyro bias, observer, estimation.

I. INTRODUCTION

Estimating the attitude of a rigid body from vector measurements has been for decades a problem of interest, because of its importance for a variety of technological applications such as satellites or unmanned aerial vehicles [8]. The method of attitude estimation from vector measurements can be divided into three categories [8]: 1) optimization-based methods; 2) stochastic filtering; and 3) nonlinear deterministic observers. In this paper, we focus on the third category because it is the only category in which convergence can be proven. The third category is again divided into two subcategories [8]: i) observers on SO(3) or unit quaternions [3], [5]–[7], [9] and ii) observers in \( \mathbb{R}^{3 \times 3} \) or some \( \mathbb{R}^n \) [1], [4], [8], the latter of which is sometimes called “geometry-free” because it is free from the configuration space SO(3) of a rigid body. The observers built on SO(3) are mathematically elegant, but have the critical drawback that the region of convergence is not global due to the topological property of SO(3) that it is not a contractible space. Hence, the geometry-free approach is getting more popular because it has been successful in achieving global or semi-global convergence, e.g. [1], [4], [8].

The most representative geometry-free observers for attitude and constant gyro bias are those proposed in [1], [4], [8], and they have the following limitations. The dimension of the observer in [1] is \( 3N + 12 \) when there are measurements of \( N \) vectors, so it undesirably leads to a high-dimensional observer for several vector measurements. The observer of [4] uses the knowledge of an upper bound of the magnitude of unknown constant gyro bias, and the region of convergence of the observer depends on the bound. Hence, their observer is not really global but only semi-global. The observer in [8] is proven to be only uniformly globally asymptotically convergent and locally exponentially convergent, but not globally exponentially convergent. Furthermore, the observer in [8], which was developed for the case of two vector measurements, may encounter the same high-dimension problem with several vector measurements as that in [1].

In this paper, we propose a continuous and globally exponentially convergent observer for attitude and gyro bias for a rigid body, which is constructed on \( \mathbb{R}^{3 \times 3} \times \mathbb{R}^3 \) instead of \( \text{SO}(3) \times \mathbb{R}^3 \). The dimension of our observer is always 12 irrespective of the number of vector measurements. It assumes as usual that the unknown bias vector is constant, but it does not require any knowledge of an upper bound of the magnitude of the bias. It is truly global and exponentially convergent without any hybrid switching rule. The observer in this paper is proposed in a unified form first, and then two kinds of observers are derived from it: one that estimates from vector measurements like the explicit complementary filter in [5] and the other that estimates from reconstructed attitude like the direct complementary filter and the passive complementary filter in [5]. So, it unifies all the three kinds of filters in the seminal paper [5]. We also make a remark on the case of measurement of time-varying inertial vectors which was addressed in [3], [4]. A preliminary result has been submitted to the 2018 IEEE Conference on Control and Decision [2].

II. MAIN RESULTS

We first invite the reader to read the Appendix to get acquainted with the mathematical preliminaries that will be used throughout the paper. The kinematic equation of a rigid body is given by

\[
\dot{\mathbf{R}} = \mathbf{R} \hat{\Omega},
\]

where \( \mathbf{R} \in \text{SO}(3) \) is the rotation or attitude of a rigid body, \( \hat{\Omega} \in \mathbb{R}^3 \) is the body angular velocity, and the symbol \( \hat{\cdot} \) over \( \Omega \) denotes the hat map, \( \hat{\mathbf{R}} : \mathbb{R}^3 \to \text{so}(3) \), defined in the Appendix. We make the following three assumptions.

Assumption II.1. A matrix-valued signal \( \mathbf{A} \in \mathbb{R}^{3 \times 3} \) is available and can be expressed as

\[
\mathbf{A} = \mathbf{GR},
\]

where \( \mathbf{G} \) is a constant invertible matrix in \( \mathbb{R}^{3 \times 3} \) and \( \mathbf{R} \in \text{SO}(3) \) is the attitude of the rigid body.

Assumption II.2. A measured angular velocity \( \Omega_y \) with bias is available and related to the angular velocity \( \Omega \) of the rigid body as follows:

\[
\Omega_y = \Omega + \mathbf{b},
\]

where \( \mathbf{b} \) is an unknown bias vector.

Assumption II.3. The trajectory of angular velocity \( \tilde{\Omega}(t) \) is bounded, and the bias vector \( \mathbf{b} \) is constant.
We propose the following observer:
\[
\hat{A} = \hat{A}_0 - \hat{A}b + k_P(A - \hat{A}), \quad (2a)
\]
\[
\hat{b} = k_I \text{Skew}(AT \hat{A})^\Lambda, \quad (2b)
\]
with \(k_P > 0\) and \(k_I > 0\), where \((\hat{A}, \hat{b}) \in \mathbb{R}^{3 \times 3} \times \mathbb{R}^3\) is an estimate of \((A, b)\). So, \((G^{-1} \hat{A}, \hat{b})\) becomes an estimate of \((R, b)\) by Assumption II.1. The global and exponentially convergent property of this observer is proven in the following theorem.

**Theorem II.4.** Let
\[
E_A = A - \hat{A}, \quad e_b = b - \hat{b}.
\]
Under Assumptions II.1—II.3 for any \(k_P > 0\) and \(k_I > 0\) there exist numbers \(a > 0\) and \(C > 0\) such that
\[
\|E_A(t)\| + \|e_b(t)\| \leq C(\|E_A(0)\| + |e_b(0)|)e^{-at} \quad (3)
\]
for all \(t \geq 0\) and all \((\hat{A}(0), \hat{b}(0)) \in \mathbb{R}^{3 \times 3} \times \mathbb{R}^3\).

**Proof.** By Assumption II.2 the observer can be written as
\[
\hat{A} = \hat{A}_0 + \hat{A}b + k_P E_A, \quad (4a)
\]
\[
\hat{b} = -k_I \text{Skew}(AT E_A)^\Lambda, \quad (4b)
\]
since \(\text{Skew}(AT \hat{A}) = -\text{Skew}(AT E_A)\). By Assumption II.3 there are numbers \(M_\Omega > 0\) and \(M_b > 0\) such that \(\|\Omega(t)\| \leq M_\Omega\) for all \(t \geq 0\) and \(\|b\| \leq M_b\). Let \(M = \max\{M_\Omega, M_b\}\). Then, there is a number \(\epsilon\) such that
\[
0 < \epsilon < \frac{1}{\|G\| \sqrt{k_I}}
\]
and
\[
0 < \epsilon < \frac{4k_P \lambda_{\min}(G^T G)}{\|G\|^2 (4k_I \lambda_{\min}(G^T G) + (k_P + 3\sqrt{2}M)^2)},
\]
where \(\lambda_{\min}(G^T G)\) denotes the smallest eigenvalue of \(G^T G\), which is positive since \(G\) is invertible. The following three quadratic functions of \((\|E_A\|, \|e_b\|)\) are then all positive definite:
\[
V_1(\|E_A\|, \|e_b\|) = \frac{1}{2} \|E_A\|^2 + \frac{1}{k_I} \|e_b\|^2 - \sqrt{2} \|G\| \|E_A\| \|e_b\|,
\]
\[
V_2(\|E_A\|, \|e_b\|) = \frac{1}{2} \|E_A\|^2 + \frac{1}{k_I} \|e_b\|^2 + \sqrt{2} \|G\| \|E_A\| \|e_b\|,
\]
\[
V_3(\|E_A\|, \|e_b\|) = (k_P - \epsilon k_I \|G\|^2) \|E_A\|^2 + 2 \epsilon \lambda_{\min}(G^T G) \|e_b\|^2 - \epsilon (\sqrt{2}k_P + 6M) \|G\| \|E_A\| \|e_b\|.
\]
Hence, there are numbers \(\alpha > 0\) and \(\beta > 0\) such that
\[
V_2 \leq \alpha V_1, \quad \beta V_2 \leq V_3. \quad (5)
\]
Let
\[
V(E_A, e_b) = \frac{1}{2} \|E_A\|^2 + \frac{1}{k_I} \|e_b\|^2 + \epsilon \langle A, \hat{A}e_b \rangle,
\]
which satisfies
\[
V_1(\|E_A\|, \|e_b\|) \leq V(E_A, e_b) \leq V_2(\|E_A\|, \|e_b\|) \quad (6)
\]
for all \((E_A, e_b) \in \mathbb{R}^{3 \times 3} \times \mathbb{R}^3\) by the Cauchy-Schwarz inequality, statements 3 and 5 in Lemma A.1 in the Appendix, and \(|A| = \|GR\| = \|G\|\) since \(R \in \text{SO}(3)\). From (1), (4), Assumption II.1 and the assumption of the bias \(b\) being constant, it follows that the estimation error \((E_A, e_b)\) obeys
\[
\dot{E}_A = E_A (\hat{\Omega} + \hat{b}) - A\hat{e}_b - k_P E_A, \quad (7)
\]
\[
e_{\hat{b}} = k_I \text{Skew}(AT E_A)^\Lambda. \quad (8)
\]
Along any trajectory of the composite system consisting of the rigid body (1) and the observer (2),
\[
\frac{dV}{dt} = \langle E_A, E_A(\hat{\Omega} + \hat{b}) - A\hat{e}_b - k_P E_A \rangle + 2 \langle e_b, \text{Skew}(AT E_A)^\Lambda \rangle
\]
\[+ \epsilon \langle E_A, A\hat{e}_b \rangle + \epsilon k_I \langle E_A, A\hat{e}_b \rangle \leq -\epsilon \|k_P - \epsilon k_I \|G\|^2 \|E_A\|^2 - 2\epsilon \lambda_{\min}(G^T G) \|e_b\| \|G\| \|E_A\| \|e_b\|,
\]
\[+ \epsilon (\sqrt{2}k_P + 6M) \|G\| \|E_A\| \|e_b\| = -V_3 \leq -\beta V_2 \leq -\beta V,
\]
where the following have been used:
\[
\langle E_A, E_A(\hat{\Omega} + \hat{b}) \rangle = \langle E_A^T E_A, (\hat{\Omega} + \hat{b}) \rangle = 0,
\]
\[
\langle E_A, A\hat{e}_b \rangle = \langle \text{Skew}(AT E_A), \hat{e}_b \rangle = 2 \langle \text{Skew}(AT E_A)^\Lambda, e_b \rangle,
\]
\[
\|A\hat{e}_b, A\hat{e}_b\| \geq \lambda_{\min}(G^T G) \| \hat{e}_b \|^2 = 2\lambda_{\min}(G^T G) \|e_b\|^2,
\]
\[
\langle E_A, A\text{Skew}(AT E_A) \rangle = \langle \text{Skew}(AT E_A)^\Lambda, \hat{e}_b \rangle \leq \|AT E_A\|^2 \leq \|AT E_A\|^2 \|G\|^2 \|E_A\|^2.
\]
Hence, \(V(t) \leq V(0)e^{-\beta t}\) for all \(t \geq 0\) and all \((\hat{A}(0), \hat{b}(0)) \in \mathbb{R}^{3 \times 3} \times \mathbb{R}^3\). It follows from (5) and (6) that
\[
V_1(t) \leq V(t) \leq V(0)e^{-\beta t} \leq \alpha V_1(0)e^{-\beta t} \quad (9)
\]
for all \(t \geq 0\) and all \((\hat{A}(0), \hat{b}(0)) \in \mathbb{R}^{3 \times 3} \times \mathbb{R}^3\). Since \(0 < \epsilon < 1/(\|G\| \sqrt{k_I})\), the map defined by
\[
(x_1, x_2) \mapsto \sqrt{\frac{1}{2} \frac{x_1^2}{k_I} + \frac{1}{k_I} x_2^2 - \sqrt{2} \|G\| \|x_1, x_2\|}
\]
is a norm on \(\mathbb{R}^2\), where \((x_1, x_2) \in \mathbb{R}^2\), which is equivalent to the 1-norm on \(\mathbb{R}^2\) since all norms are equivalent on a finite-dimensional vector space. Hence, \(V_1(t) \leq \alpha V_1(0)e^{-\beta t}\) implies that there exists \(C > 0\) such that (3) holds for all \(t \geq 0\) and all \((\hat{A}(0), \hat{b}(0)) \in \mathbb{R}^{3 \times 3} \times \mathbb{R}^3\), where \(a = \beta/2\).

Notice in the proof that the numbers \(C\) and \(a\) in (3) may depend on \(M_\Omega\) and \(M_b\), which has not prevented us from showing the exponential convergence of the observer. Moreover, the choice of \(k_P\) and \(k_I\) is totally independent of \(M_\Omega\) and \(M_b\).

**Corollary II.5.** Suppose that Assumptions II.1—II.3 hold, and let
\[
E_R = R - G^{-1}\hat{A}, \quad e_b = b - \hat{b}.
\]
Then, there exist numbers \(a > 0\) and \(C > 0\) such that
\[
\|E_R(t)\| + \|e_b(t)\| \leq C(\|E_R(0)\| + \|e_b(0)\|)e^{-at} \quad (9)
\]
for all \(t \geq 0\) and all \((\hat{A}(0), \hat{b}(0)) \in \mathbb{R}^{3 \times 3} \times \mathbb{R}^3\).

**Proof.** Use \(|\|E_R\|/\|G\|^2| \leq \|E_A\| \leq \|G\| \|E_R\|\) and (3) with the constant \(C\) redefined appropriately. □
In particular, if \( G = I \), then the observer (2) reduces to
\[
\begin{align*}
\dot{R} &= \bar{R}\hat{\Omega}_g - \bar{R}\hat{b} + k_P(R - \bar{R}), \\
\dot{\hat{b}} &= k_I \text{Skew}(R^T\bar{R})^\vee,
\end{align*}
\] (10a)
(10b)
where \((\bar{R}, \hat{b}) \in \mathbb{R}^{3 \times 3} \times \mathbb{R}^3\) is an estimate of \((R, b)\). This form of observer would correspond to the direct complementary filter and the passive complementary filter proposed by Mahony et al. that appear in (12) and (13) in [5]. We now derive from (2) an observer that estimates \((R, b)\) from vector measurements. Assume that there is a set \(\mathcal{S} = \{s_i, 1 \leq i \leq n\} \) of \(n\) known fixed inertial directions, where each \(s_i \in \mathcal{S}\) is a unit vector in \(\mathbb{R}^3\). Assume also that measurements of the directions are made in the body-fixed frame and the set of the measured vectors is denoted by \(\mathcal{C} = \{c_i, 1 \leq i \leq n\}\) and related to \(\mathcal{S}\) as follows:
\[
c_i = R^Ts_i, \quad i = 1, \ldots, n,
\]
where \(R\) is the orientation of the rigid body. Define a matrix \(G \in \mathbb{R}^{3 \times 3}\) by
\[
G = \sum_{i=1}^{n} w_i s_i s_i^T \quad (11)
\]
with \(w_i \in \mathbb{R}\setminus\{0\}, 1 \leq i \leq n\). Assume \(\text{rank}(G) = 3\). Let
\[
A = \sum_{i=1}^{n} w_i s_i c_i^T, \quad (12)
\]
which consists of measured signals. Then, \(A \) and \(R\) satisfy the relationship, \(A = GR\), so Assumption II.1 is satisfied. With \(G\) and \(A\) given in (11) and (12), the observer (2) can be written as
\[
\begin{align*}
\dot{\hat{A}} &= \bar{A}\hat{\Omega}_g - \bar{A}\hat{b} + k_P(A - \bar{A}), \\
\dot{\hat{b}} &= -k_I \sum_{i=1}^{n} w_i c_i \times \bar{A}^Ts_i, \quad (13a)
\end{align*}
\] (13b)
where statement 6 in Lemma A.4 has been used in the derivation of (13b), and \(k_I/2\) has been replaced with \(k_I\) to make (13b) look simple. Here, \((\hat{A}, \hat{b}) \in \mathbb{R}^{3 \times 3} \times \mathbb{R}^3\) is an estimate of \((GR, b)\). This form of observer corresponds to the explicit complementary filter in (32) in [5]. By Theorem II.4 \((A(t), b(t))\) converges exponentially to \((GR(t), b)\) for all \((\hat{A}(0), \hat{b}(0)) \in \mathbb{R}^{3 \times 3} \times \mathbb{R}^3\) as \(t \to \infty\). It follows that \((G^{-1}\hat{A}(t), \hat{b}(t))\) converges exponentially to \((R(t), b)\) for all \((\hat{A}(0), \hat{b}(0)) \in \mathbb{R}^{3 \times 3} \times \mathbb{R}^3\) as \(t \to \infty\). The case of rank \(G = 2\) can be converted to the case of rank \(G = 3\) by choosing any two linearly independent vectors \(s_i, s_j \in \mathcal{S}\) and adding \((s_i \times s_j)/\|s_i \times s_j\| \) and \((c_i \times c_j)/\|c_i \times c_j\|\) to \(\mathcal{S}\) and \(\mathcal{C}\), respectively.

Remark II.6. Our observer, whose dynamics evolve globally in Euclidean space, is straightforward to numerically integrate, whereas most observers on SO(3) would require an operation of projection onto SO(3) at each step of numerical integration, which is computationally expensive. The use of unit quaternions for numerical integration is not free from the projection requirement, either.

Remark II.7. Putting the numerical integration issue aside, we can always approximate the trajectory of estimates \(\hat{R}(t)\) ∈ \(\mathbb{R}^{3 \times 3}\) with a curve of rotation matrices using polar decomposition as explained in Proposition 7 in [5]. However, this approximation may not be even necessary when \(\hat{R}(t)\) is directly used in feedback control. Suppose that we have an \(\mathbb{R}^3\)-valued control law \(u(R, \Omega)\) for a rigid body system, where \(R \in \text{SO}(3)\) is the attitude of the body and \(\Omega \in \mathbb{R}^3\) the body angular velocity. We can naturally extend the function \(u(R, \Omega)\) to \(\mathbb{R}^{3 \times 3} \times \mathbb{R}^3\) by treating \(R\) as a \(3 \times 3\) matrix after replacement of any occurrence of \(R^{-1}\) with \(R^T\) in the expression of \(u(R, \Omega)\). Then, as far as an estimate \((\hat{R}, \hat{\Omega}) \in \mathbb{R}^{3 \times 3} \times \mathbb{R}^3\) is close to \((R, \Omega) \in \text{SO}(3) \times \mathbb{R}^3\) in \(\mathbb{R}^{3 \times 3} \times \mathbb{R}^3\), \(u(\hat{R}, \hat{\Omega})\) will be close to \(u(R, \Omega)\) in \(\mathbb{R}^3\), so the purpose of observer is squarely served.

Remark II.8. Instead of (2), we can consider the following form of observer:
\[
\begin{align*}
\dot{\hat{A}} &= \bar{A}\hat{\Omega}_g - \bar{A}\hat{b} + k_P(A - \bar{A}), \\
\dot{\hat{b}} &= k_I \text{Skew}(A^{-1}\bar{A})^\vee
\end{align*}
\] (14a)
(14b)
with \(k_P > 0\) and \(k_I > 0\), where the only difference between (2) and (14) is in the equation for \(\hat{b}\). It is not difficult to prove that this observer also enjoys the property of global and exponential convergence for any \(k_P > 0\) and \(k_I > 0\), whose proof is left to the reader.

Remark II.9. We can relax Assumption II.1 by allowing the matrix \(G\) to be time-varying. More specifically, we make the following assumption: there are numbers \(\ell_{\min} > 0\) and \(\ell_{\max} > 0\) such that
\[
\begin{align*}
\ell_{\min} &\leq \lambda_{\min}(G^T(t)G(t)) \leq \lambda_{\max}(G^T(t)G(t)) \leq \ell_{\max} \quad (15)
\end{align*}
\]
for all \(t \geq 0\). In this case, we propose the following observer:
\[
\begin{align*}
\dot{\hat{A}} &= \bar{A}\hat{\Omega}_g - \bar{A}\hat{b} + k_P(A - \bar{A}) + GG^{-1}A, \\
\dot{\hat{b}} &= k_I \text{Skew}(A^T\bar{A})^\vee
\end{align*}
\]
with \(k_P > 0\) and \(k_I > 0\), where \((\hat{A}, \hat{b}) \in \mathbb{R}^{3 \times 3} \times \mathbb{R}^3\) is an estimate of \((A, b)\). It is not difficult to show that Theorem II.4 and Corollary II.3 still hold for this observer with the relaxed assumption on \(G(t)\) as above. The proof would involve a small modification of the proof of Theorem II.4 which is left to the reader. Furthermore, the observer in (13) is modified to
\[
\begin{align*}
\dot{\hat{A}} &= \bar{A}\hat{\Omega}_g - \bar{A}\hat{b} + k_P(A - \bar{A}) + GG^{-1}A, \\
\dot{\hat{b}} &= -k_I \sum_{i=1}^{n} w_i c_i \times \bar{A}^Ts_i
\end{align*}
\]
with \(G\) in (11) where \(s_i\)'s are allowed to vary in \(t\) such that (15) holds.

We now run a simulation to compare our observer given in (13) with the explicit complementary filter on SO(3) proposed by Mahony et al. that appears in (32) in [5]. We choose the following true attitude and angular velocity:
\[
\begin{align*}
R(t) &= \exp(t\hat{c}_1) \exp(t\hat{c}_2) \exp(t\hat{c}_1), \\
\Omega(t) &= (1 + \cos t, \sin t - \sin t \cos t, \cos t + \sin^2 t),
\end{align*}
\] (16)
(17)
symmetrization operator, respectively, on square matrices, which are defined by
this inner product, which is called the Frobenius or Euclidean
Skew continuous, and exponentially convergent observer for attitude
matrices in this paper, i.e., $\hat{\Omega} = \Omega - \varphi$ for all $\Omega \in \mathbb{R}^3$ and $\varphi \in \mathbb{R}^3$. The inverse map of the hat map is called the vee map and denoted $\vee$ such that $(\Omega)\vee = \Omega$ for all $\Omega \in \mathbb{R}^3$ and $(\lambda\vee)\hat{\lambda} = \lambda$ for all $\lambda \in \mathfrak{so}(3)$.

**Lemma A.1.**
1. $(\vect{RA}, RB) = (\vect{AB}, B)$ for all $R \in \mathbf{SO}(3)$ and $A, B \in \mathbb{R}^{3\times3}$.
2. $\min_{\lambda}(\min_{\lambda}(\lambda R^T A) ||B||^2) \leq (\lambda A B) \leq \max_{\lambda}(\lambda R^T A) ||B||^2$ for all $A \in \mathbb{R}^{n\times m}$ and $B \in \mathbb{R}^{m\times t}$.
3. $\langle \hat{x}, \hat{y} \rangle = 2xy$ for all $x, y \in \mathbb{R}^3$.
4. $\|A\| = \|\text{Sym}(A)\|^2 + \|\text{Skew}(A)\|_2$ for all $A \in \mathbb{R}^{n\times n}$.
5. $\|A B\| = \|A\| ||B||$ for all $A \in \mathbb{R}^{n\times m}$ and $B \in \mathbb{R}^{m\times t}$.
6. $x \times y = (xy^T - yx^T)$ for all $x, y \in \mathbb{R}^3$.
7. $\max_{R_1, R_2 \in \mathbf{SO}(3)} \|R_1 - R_2\| = 2\sqrt{2}$.

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where $\{e_1, e_2, e_3\}$ is the standard basis for $\mathbb{R}^3$. Assume that the unknown constant gyro bias $b$ is given by
$$b = (1, 0.5, -1), \quad (18)$$
Assume that there are three inertial direction vectors given by
$$s_1 = e_1, \quad s_2 = \frac{e_1 + e_2}{\|e_1 + e_2\|}, \quad s_3 = \frac{e_2 - e_3}{\|e_2 - e_3\|}$$
with the weights $w_1 = 1/3$, $w_2 = 1/3$, and $w_3 = 1/3$. Our observer and the Mahony filter both use the gains $k_p = 4$ and $k_t = 20$, and start from the initial state,
$$\hat{R}(0) = \begin{bmatrix} 0.2440 & 0.9107 & -0.3333 \\ 0.9107 & -0.3333 & -0.2440 \\ -0.3333 & -0.2440 & -0.9107 \end{bmatrix}$$
and $\tilde{b}(0) = 0.999999b$, where $b$ is the unknown true constant bias given in (13). The simulation results are plotted in Figure 1. It can be observed that the observer proposed in (14) converges fast to the true value whereas the filter by Mahony et al. converges slowly and has an undesirably large overshoot in the estimation. This demonstrates the excellent global and exponentially convergent property of our observer.

**III. Conclusion**

We have successfully designed a 12-dimensional, global, continuous, and exponentially convergent observer for attitude and gyro bias of a rigid body.

**APPENDIX**

This appendix contains mathematical preliminaries to help the reader understand the main results of the paper. The usual Euclidean inner product is exclusively used for vectors and matrices in this paper, i.e., $\langle A, B \rangle = \sum_{i,j} A_{ij} B_{ij} = \text{tr}(A^T B)$ for any two matrices of equal size. The norm induced from this inner product, which is called the Frobenius or Euclidean norm, is exclusively used for vectors and matrices. Let $\text{Sym}$ and $\text{Skew}$ denote the symmetrization operator and the skew-symmetrization operator, respectively, on square matrices, which are defined by
$$\text{Sym}(A) = \frac{1}{2}(A + A^T), \quad \text{Skew}(A) = \frac{1}{2}(A - A^T)$$
for any square matrix $A$. Then,
$$A = \text{Sym}(A) + \text{Skew}(A), \quad \langle \text{Sym}(A), \text{Skew}(A) \rangle = 0.$$

Namely,
$$\mathbb{R}^{n \times n} = \text{Sym}(\mathbb{R}^{n \times n}) \oplus \text{Skew}(\mathbb{R}^{n \times n})$$
with respect to the Euclidean inner product. Let $\mathbf{SO}(3)$ denote the set of all $3 \times 3$ rotation matrices, which is defined as $\mathbf{SO}(3) = \{ R \in \mathbb{R}^{3 \times 3} \mid R^T R = I, \det R = 1 \}$. Let $\mathfrak{so}(3)$ denote the set of all $3 \times 3$ skew symmetric matrices, which is defined as $\mathfrak{so}(3) = \{ A \in \mathbb{R}^{3 \times 3} \mid A^T + A = 0 \}$. The hat map $\wedge : \mathbb{R}^3 \rightarrow \mathfrak{so}(3)$ is defined by
$$\hat{\Omega} = \begin{bmatrix} 0 & -\Omega_3 & \Omega_2 \\ \Omega_3 & 0 & -\Omega_1 \\ -\Omega_2 & \Omega_1 & 0 \end{bmatrix}$$
for $\Omega = (\Omega_1, \Omega_2, \Omega_3) \in \mathbb{R}^3$. The inverse map of the hat map is called the vee map and denoted $\vee$ such that $(\Omega)\vee = \Omega$ for all $\Omega \in \mathbb{R}^3$ and $(\lambda\vee)\hat{\lambda} = \lambda$ for all $\lambda \in \mathfrak{so}(3)$.

Fig. 1. The attitude estimation error $\|R(t) - \hat{R}(t)\|$ and the gyro bias estimation error $\|b - \tilde{b}(t)\|$ with the initial state $\hat{R}(0)$ in (14) and $\tilde{b}(0) = 0.999999b$ for our observer (13) proposed in this paper (solid) and the MHP explicit complementary filter by Mahony et al. (dashed).