Vector Potential Electromagnetic Theory with Generalized Gauge for Inhomogeneous Anisotropic Media

W. C. Chew

June 19, 2014 at 01:36

Abstract

Vector and scalar potential formulation is valid from quantum theory to classical electromagnetics. The rapid development in quantum optics calls for electromagnetic solutions that straddle quantum physics as well as classical physics. The vector potential formulation is a good candidate to bridge these two regimes. Hence, there is a need to generalize this formulation to inhomogeneous media. A generalized gauge is suggested for solving electromagnetic problems in inhomogenous media which can be extended to the anistropic case. The advantages of the resulting equations are their absence of low-frequency catastrophe. Hence, usual differential equation solvers can be used to solve them over multi-scale and broad bandwidth. It is shown that the interface boundary conditions from the resulting equations reduce to those of classical Maxwell’s equations. Also, classical Green’s theorem can be extended to such a formulation, resulting in similar extinction theorem, and surface integral equation formulation for surface scatterers. The integral equations also do not exhibit low-frequency catastrophe as well as frequency imbalance as observed in the classical formulation using E-H fields. The matrix representation of the integral equation for a PEC scatterer is given.

1 Introduction

Electromagnetic theory has been guided by Maxwell’s equations for 150 years now [1]. The formulation of electromagnetic theory based on \( E, H, D, \) and \( B \), simplified by Heaviside [2], offers physical insight that results in the development of myriads of electromagnetic-related technologies. However, there are certain situations where the \( E-H \) formulation is not ideal. This is in the realm of quantum mechanics where the \( A-\Phi \) formulation is needed. In certain situations where \( E-H \) are zero, but \( A \) is not zero, and yet, the effect of \( A \) is felt in quantum mechanics. This is true of the Aharonov-Bohm effect [3, 4]. Moreover, the quantization of electromagnetic field can be done more expediently with the vector potential \( A \) rather than \( E \) and \( H \). More importantly, when the electromagnetic effect needs to be incorporated in

\(^1\)U of Illinois, Urbana-Champaign; visiting professor, HKU.
Schrödinger equation, vector and scalar potentials are used. This will be important in many quantum optics studies [5][11].

Normally, electromagnetic equations formulated in terms of \( \mathbf{E} - \mathbf{H} \) have low-frequency breakdown or catastrophe. Hence, many numerical methods based on \( \mathbf{E} - \mathbf{H} \) formulation are unstable at low frequencies or long wavelength. Therefore, the \( \mathbf{E} - \mathbf{H} \) formulation is not truly multi-scale, as it has catastrophe when the dimension of objects are much smaller than the wavelength. Different formulations using tree-cotree, or loop-tree decomposition [12][16], have to be sought when the frequency is low or the wavelength is long. Due to the low-frequency catastrophe encountered by \( \mathbf{E} - \mathbf{H} \) formulation, the vector potential formulation has been very popular for solving low frequency problems [17][27].

This work will arrive at a general theory of vector potential formulation for inhomogeneous anisotropic media, together with the pertinent integral equations. This vector potential formulation does not have apparent low-frequency catastrophe of the \( \mathbf{E} - \mathbf{H} \) formulation and it is truly multi-scale. It can be shown that with the proper gauge, which is the extension of the simple Lorentz gauge to inhomogeneous anisotropic media, the scalar potential equation is decoupled from the vector potential equation.

\section{Pertinent Equations–Inhomogeneous Isotropic Case}

The vector potential formulation for homogeneous medium has been described in most text books [28][31]. We derive the pertinent equations for the inhomogeneous isotropic medium case first. To this end, we begin with the Maxwell’s equations:

\begin{align}
\nabla \times \mathbf{E} &= -\partial_t \mathbf{B}, \\
\nabla \times \mathbf{H} &= \partial_t \mathbf{D} + \mathbf{J}, \\
\nabla \cdot \mathbf{B} &= 0, \\
\nabla \cdot \mathbf{D} &= \rho.
\end{align}

From the above we let

\begin{align}
\mathbf{B} &= \nabla \times \mathbf{A}, \\
\mathbf{E} &= -\partial_t \mathbf{A} - \nabla \Phi
\end{align}

so that the first and third of four Maxwell’s equations are satisfied. Then, using \( \mathbf{D} = \varepsilon \mathbf{E} \) for isotropic, non-dispersive, inhomogeneous media, we obtain that

\begin{align}
- \partial_t \nabla \cdot \varepsilon \mathbf{A} - \nabla \cdot \varepsilon \nabla \Phi &= \rho, \\
\nabla \times \mu^{-1} \nabla \times \mathbf{A} &= -\varepsilon \partial_t^2 \mathbf{A} - \varepsilon \partial_t \nabla \Phi + \mathbf{J}.
\end{align}
For homogeneous medium, the above reduce to

\[- \partial_t \nabla \cdot \mathbf{A} - \nabla^2 \Phi = \rho/\varepsilon, \quad (9)\]

\[\mu^{-1} (\nabla \nabla \cdot \mathbf{A} - \nabla^2 \mathbf{A}) = -\varepsilon \partial_t^2 \mathbf{A} - \varepsilon \partial_t \nabla \Phi + \mathbf{J}. \quad (10)\]

By using the simple Lorentz gauge

\[\nabla \cdot \mathbf{A} = -\mu \varepsilon \partial_t \Phi \quad (11)\]

we have the usual

\[\nabla^2 \Phi - \mu \varepsilon \partial_t^2 \Phi = -\rho/\varepsilon, \quad (12)\]
\[\nabla^2 \mathbf{A} - \mu \varepsilon \partial_t^2 \mathbf{A} = -\mu \mathbf{J} \quad (13)\]

Lorentz gauge is preferred because it treats space and time on the same footing as in special relativity \[28\].

For inhomogeneous media, we can choose the generalized Lorentz gauge. This gauge has been suggested previously, for example in \[24\].

\[\varepsilon^{-1} \nabla \cdot \varepsilon \mathbf{A} = -\mu \varepsilon \partial_t \Phi. \quad (14)\]

However, we can decouple \[7\] and \[8\] with an even more generalized gauge, namely

\[\nabla \cdot \varepsilon \mathbf{A} = -\chi \partial_t \Phi \quad (15)\]

Then we get from \[7\] and \[8\] that

\[\nabla \cdot \varepsilon \nabla \Phi - \chi \partial_t^2 \Phi = -\rho, \quad (16)\]
\[- \nabla \times \mu^{-1} \nabla \times \mathbf{A} - \varepsilon \partial_t^2 \mathbf{A} + \varepsilon \nabla \chi^{-1} \nabla \cdot \varepsilon \mathbf{A} = -\mathbf{J}. \quad (17)\]

It is to be noted that \[16\] can be derived from \[17\] by taking the divergence of \[17\] and then using the generalized gauge and the charge continuity equation that \(\nabla \cdot \mathbf{J} = -\partial_t \rho\). In general, we can choose

\[\chi = \alpha \varepsilon^2 \mu \quad (18)\]

where \(\alpha\) can be a function of position. When \(\alpha = 1\), it reduces to the generalized Lorentz gauge used in \[14\].

For homogeneous medium, \[16\] and \[17\] reduce to \[12\] and \[13\] when we choose \(\alpha = 1\) in \[18\], which is the case of the simple Lorentz gauge. Unlike the vector wave equations for inhomogeneous electromagnetic fields, the above do not have apparent low-frequency breakdown when \(\partial_t = 0\). Hence, the above equations can be used for electrodynamics as well as electrostatics when the wavelength tends to infinity.
We can rewrite the above as a sequence of three equations, namely,

\[ \nabla \cdot \varepsilon \mathbf{A} = -\chi \partial_t \Phi \]  
\[ \nabla \times \mathbf{A} = \mu \mathbf{H} \]  
\[ \nabla \times \mathbf{H} + \varepsilon \partial_t^2 \mathbf{A} + \varepsilon \nabla \partial_t \Phi = \mathbf{J} \]

The last equation can be rewritten as

\[ \nabla \times \mathbf{H} - \varepsilon \partial_t (-\partial_t \mathbf{A} - \nabla \Phi) = \mathbf{J} \]

which is the same as solving Ampere’s law. Hence, solving (17) is similar to solving Maxwell’s equations.

It is to be noted that (17) resembles the elastic wave equation in solids where both longitudinal and transverse waves can exist [35]. Furthermore, these two waves can have different velocities in a homogeneous medium if \( \alpha \neq 1 \) in (18). The longitudinal wave has the same velocity as the scalar potential, which is \( 1/\sqrt{\alpha \mu \varepsilon} \), while the transverse wave has the velocity of light, or \( 1/\sqrt{\mu \varepsilon} \). If we choose \( \alpha = 0 \), or \( \chi = 0 \), we have the Coulomb gauge where the scalar potential has infinite velocity.

3 Boundary Conditions for the Potentials

Many problems can be modeled with piecewise homogeneous medium. In case, the solutions can be sought in each of the piecewise homogenous region, and then sewn together using boundary conditions. The above equations, (16) and (17), are the governing equations for the scalar and vector potentials \( \Phi \) and \( \mathbf{A} \) for inhomogeneous media. The boundary conditions at the interface of two homogeneous media are also embedded in these equations. By eyeballing equation (17), we see that \( \nabla \times \mathbf{A} \) must be finite at an interface. This induces the boundary condition that

\[ \hat{n} \times \mathbf{A}_1 = \hat{n} \times \mathbf{A}_2 \]
across an interface. Assuming that $J$ is finite, we also have
\[ \hat{n} \times \frac{1}{\mu_1} \nabla \times A_1 = \hat{n} \times \frac{1}{\mu_2} \nabla \times A_2. \]  
(24)
The above is equivalent to
\[ \hat{n} \times H_1 = \hat{n} \times H_2 \]  
(25)
at an interface. When a surface current sheet is present, we have to augment the above with the current sheet as is done in standard electromagnetic boundary conditions. Furthermore, due to the finiteness of $\nabla \cdot \varepsilon A$ at an interface, it is necessary that
\[ \hat{n} \cdot \varepsilon_1 A_1 = \hat{n} \cdot \varepsilon_2 A_2. \]  
(26)
It can be shown that if a surface dipole layer exists at an interface, we will have to augment the above with the correct discontinuity or jump condition. A surface current with a normal component to the surface will constitute a surface dipole layer.

By the same token, we can eyeball the scalar potential equation (16), and notice that
\[ \hat{n} \cdot \varepsilon_1 \nabla \Phi_1 = \hat{n} \cdot \varepsilon_2 \nabla \Phi_2. \]  
(27)
The above will be augmented with the necessary jump or discontinuity condition if a surface charge layer exists at an interface. Equations (26) and (27) together mean that
\[ \hat{n} \cdot \varepsilon_1 E_1 = \hat{n} \cdot \varepsilon_2 E_2. \]  
(28)
where we have noted that $E = -\frac{\partial}{\partial t} A - \nabla \Phi$ from (6). This is the usual boundary condition for the normal component of the electric field.

Equation (16) also implies that
\[ \Phi_1 = \Phi_2, \]  
(29)
or
\[ \hat{n} \times \nabla \Phi_1 = \hat{n} \times \nabla \Phi_2. \]  
(30)
Equations (27) and (30) imply that
\[ \hat{n} \times E_1 = \hat{n} \times E_2. \]  
(31)
This is the normal boundary condition for the tangential component of the electric field.

If $\varepsilon_2$ is a perfect electric conductor (PEC), $\varepsilon_2 \rightarrow \infty$. From (17), it implies that $A_2 = 0$, if $\omega \neq 0$ or $\partial_t \neq 0$. Then (23) for a PEC surface becomes
\[ \hat{n} \times A_1 = 0. \]  
(32)
Also, by eyeballing (16), we see that for a PEC, $\Phi_2 = 0$. This together with (29), (30), and (32) imply that $\hat{n} \times E_1 = 0$ on a PEC surface. For a perfect magnetic conductor (PMC), $\mu_2 \rightarrow \infty$, from (24) and (25)
\[ \hat{n} \times H_1 = 0. \]  
(33)
When $\omega = 0$ or $\partial_t = 0$, $\mathbf{A}$ does not contribute to $\mathbf{E}$. But from (27), when $\varepsilon_2 \to \infty$, we deduce that $\hat{n} \cdot \nabla \Phi_2 = 0$, implying that $\Phi_2$ is constant for $r_2 \in V_2$ for arbitrary $S$. Hence, from (29) and (30), $\hat{n} \times \mathbf{E}_1 = 0$ on a PEC surface even when $\omega = 0$.

4 General Anisotropic Media Case

For inhomogeneous, dispersionless, anisotropic media, the generalized gauge becomes

$$\nabla \cdot \varepsilon \cdot \mathbf{A} = -\chi \partial_t \Phi$$

(34)

In the above, $\chi$ is arbitrary, but we can choose

$$\chi = \alpha |\varepsilon \cdot \mu \cdot \varepsilon|$$

(35)

where the vertical bar means determinant. When the medium is inhomogeneous and isotropic, the above gauge reduces to the generalized gauge previously discussed. When $\alpha = 1$, the above reduces to the generalized Lorentz gauge for inhomogeneous isotropic medium. In general, (16) and (17) become

$$\nabla \cdot \varepsilon \cdot \nabla \Phi - \chi \partial^2_t \Phi = -\rho$$

(36)

$$\nabla \times \mu^{-1} \nabla \times \mathbf{A} + \varepsilon \cdot \partial^2_t \mathbf{A} - \varepsilon \cdot \nabla \chi^{-1} \nabla \cdot \varepsilon \cdot \mathbf{A} = \mathbf{J}$$

(37)

The above can be rewritten in the manner of (19) to (22), showing that solving the above is the same as solving the original Maxwell’s equations. The boundary condition (23) remains the same. Boundary condition (24) becomes

$$\hat{n} \times \mu_1^{-1} \nabla \times \mathbf{A}_1 = \hat{n} \times \mu_2^{-1} \nabla \times \mathbf{A}_2$$

(38)

and boundary condition (25) remains the same. Similarly, boundary condition (26) becomes

$$\hat{n} \cdot \varepsilon_1 \cdot \mathbf{A}_1 = \hat{n} \cdot \varepsilon_2 \cdot \mathbf{A}_2.$$

(39)

The boundary condition (27) becomes

$$\hat{n} \cdot \varepsilon_1 \cdot \nabla \Phi_1 = \hat{n} \cdot \varepsilon_2 \cdot \nabla \Phi_2$$

(40)

while the other boundary conditions, similar to the isotropic case, can be similarly derived.

5 Green’s Theorem—Time Harmonic Case

As mentioned previously, for inhomogeneous media consisting of piecewise homogeneous regions, it is best to seek the solution in each region first, and then the solutions sewn together by boundary conditions. Consequently, surface (boundary) integral equations can be derived to solve such problems where unknowns only need to be assigned to the interfaces or boundaries.
between regions. In this manner, a 3D problem is reduced to a problem on a 2D manifold, beating the tyranny of dimensionality. Moreover, in recent years, fast algorithms have been developed to solve these surface integral equations rapidly \cite{32,33,34}, greatly underscoring their importance.

To this end, we need to derive the equivalence of the Green’s theorem for vector potential formulation. In the following, we assume a simple Lorentz gauge so that the equations for homogeneous region greatly simplify. In other words, we need to derive Green’s theorem’s equivalence for

$$\nabla^2 + k^2 A(r) = -\mu J(r).$$ \hspace{1cm} (41)

where \(k^2 = \omega^2 \mu \varepsilon\) and the time dependence is \(\exp(-i\omega t)\). It is more expedient to write the above as

$$\nabla \times \nabla \times A(r) - \nabla \nabla \cdot A(r) - k^2 A(r) = \mu J(r).$$ \hspace{1cm} (43)

We can define a dyadic Green’s function that satisfies

$$\nabla \times \nabla \times G(r, r') - \nabla \nabla \cdot G(r, r') - k^2 G(r, r') = I\delta(r - r').$$ \hspace{1cm} (44)

The solution to the above is simply

$$G(r, r') = I \frac{e^{ik|r-r'|}}{4\pi|r-r'|} = I g(r, r').$$ \hspace{1cm} (45)

From the above, using methods outlined in \cite{36}, Chapter 8, as well as in Appendix A we have

If \(\alpha \neq 1\), the ensuing equation is of the form

$$\nabla \times \nabla \times A(r) - \alpha^{-1} \nabla \nabla \cdot A(r) - k^2 A(r) = \mu J(r).$$ \hspace{1cm} (42)

But the dyadic Green’s function of such an equation can still be found using methods outlined in \cite{35}.

Figure 2: The figure used to derive the Green’s theorem for vector potential equation.
for region 1,

\[ \mathbf{r} \in V_1, \quad \mathbf{A}_1(\mathbf{r}) = \mathbf{A}_{\text{inc}}(\mathbf{r}) + \int_S dS' \left\{ \mu_1 \mathbf{G}_1(\mathbf{r}, \mathbf{r}') \cdot \mathbf{n}' \times \mathbf{H}_1(\mathbf{r}') - \left[ \nabla' \times \mathbf{G}_1(\mathbf{r}, \mathbf{r}') \right] \cdot \mathbf{n}' \times \mathbf{A}_1(\mathbf{r}') \right\} \]

\[ + \int_S dS' \mathbf{n}' \cdot \left\{ \mathbf{G}_1(\mathbf{r}, \mathbf{r}') \nabla' \cdot \mathbf{A}_1(\mathbf{r}') - \mathbf{A}_1(\mathbf{r}') \nabla' \cdot \mathbf{G}_1(\mathbf{r}, \mathbf{r}') \right\}. \quad (46) \]

We can rewrite the above using scalar Green’s function as

\[ \mathbf{r} \in V_1, \quad \mathbf{A}_1(\mathbf{r}) = \mathbf{A}_{\text{inc}}(\mathbf{r}) + \int_S dS' \left\{ \mu_1 g_1(\mathbf{r}, \mathbf{r}') \mathbf{n}' \times \mathbf{H}_1(\mathbf{r}') - \nabla' g_1(\mathbf{r}, \mathbf{r}') \times \mathbf{n}' \times \mathbf{A}_1(\mathbf{r}') \right\} \]

\[ + \int_S dS' \left\{ - \mathbf{n}' g_1(\mathbf{r}, \mathbf{r}') \nabla' \cdot \mathbf{A}_1(\mathbf{r}') + (\mathbf{n}' \cdot \mathbf{A}_1(\mathbf{r}')) \nabla' g_1(\mathbf{r}, \mathbf{r}') \right\}. \quad (47) \]

A similar equation can be derived for region 2. These equations can be used to formulate surface integral equations for scattering. The lower parts of the above equations are known as the extinction theorem [36, 37].

As a side note, one can use the scalar Green’s theorem directly on (41) and obtain

\[ \mathbf{r} \in V_1, \quad \mathbf{A}_1(\mathbf{r}) = \mathbf{A}_{\text{inc}}(\mathbf{r}) - \int_S dS' \left\{ g_1(\mathbf{r}, \mathbf{r}') \mathbf{n}' \cdot \nabla' \mathbf{A}_1(\mathbf{r}') - \mathbf{n}' \cdot \nabla' g_1(\mathbf{r}, \mathbf{r}') \mathbf{A}_1(\mathbf{r}') \right\}. \quad (48) \]

After some lengthy manipulations, (47) becomes (48) as shown in Appendix A. In the above derivation, there is a surface integral at infinity that can be shown to vanish as in [36, Chapter 8] when radiation condition is invoked.

6 PEC Scatterer Case

For a PEC scatterer, we have proved that \( \mathbf{n} \times \mathbf{A}_1 = 0 \). Since \( \nabla \cdot \mathbf{A}_1 = i\omega \mu_1 \varepsilon \mathbf{\Phi} \) and that \( \mathbf{\Phi} = 0 \) on a PEC surface. Hence, for surface sources that satisfy the PEC scattering solution, the above becomes

\[ \mathbf{r} \in V_1, \quad \mathbf{A}_1(\mathbf{r}) = \mathbf{A}_{\text{inc}}(\mathbf{r}) + \int_S dS' \left\{ \mu_1 g_1(\mathbf{r}, \mathbf{r}') \mathbf{n}' \cdot \mathbf{H}_1(\mathbf{r}') + \mathbf{n}' \cdot \mathbf{A}_1(\mathbf{r}') \nabla' g_1(\mathbf{r}, \mathbf{r}') \right\}. \quad (49) \]

The first term in the integral comes from the induced surface current flowing on the PEC surface. It will be interesting to ponder the meaning of the second term. It is to be noted that the surface charge on the PEC surface is given by

\[ \mathbf{n} \cdot \varepsilon_1 \mathbf{E}_1 = \mathbf{n} \cdot \varepsilon (i\omega \mathbf{A}_1 - \nabla \mathbf{\Phi}_1) \]

(50)

The scalar potential \( \mathbf{\Phi} \) can be obtained from the vector potential using Lorentz gauge, namely, \( \nabla \cdot \mathbf{A} = i\omega \mu_\varepsilon \mathbf{\Phi} \). Hence, one can view that \( \mathbf{n} \cdot \mathbf{A} \) as the contribution to the surface charge from the
vector potential $\mathbf{A}$. In fact, using the Lorentz gauge, and that $\mathbf{E} = i\omega \mathbf{A} - \nabla \Phi$, one can recover from the above that the electric field in region 1 outside the PEC is given by (see Appendix B)

$$\mathbf{E}_1 = \mathbf{E}_{\text{inc}} + \int_{S'} dS' \left\{ i\omega \mu_1 g_1(r, r') \mathbf{J}_1(r') - \nabla g_1(r, r') \frac{\sigma_1(r')}{\varepsilon_1(r')} \right\}$$

(51)

where $\mathbf{J}_1 = \mathbf{n} \times \mathbf{H}_1$ and $\nabla \cdot \mathbf{J}_1 = i\omega \sigma_1$. The above is just the traditional relationship between the $\mathbf{E}$ field in region 1 and the sources on the PEC surface.

We can rewrite (49) in terms of two integral equations

$$\mathbf{A}_1(r) = \mathbf{A}_{\text{inc}}(r) + \int_S dS' \left\{ \mu_1 g_1(r, r') \mathbf{J}_1(r') + \Sigma_1(r') \nabla' g(r, r') \right\}$$

(52)

$$\sigma_1(r) = \Sigma_{\text{inc}}(r) + \int_S dS' \left\{ \mu_1 g_1(r, r') \mathbf{n} \cdot \mathbf{J}_1(r') + \Sigma_1(r') \mathbf{n} \cdot \nabla' g(r, r') \right\}$$

(53)

where the second equation is obtained by $\mathbf{n} \cdot$ the first equation. Also, the boundary condition is such that $\mathbf{n} \times \mathbf{A}_1(r) = 0$ on $S$. The above can be solved by the subspace projection method such as the Galerkin’s [38] or moments methods [39, 40]. The unknowns are $\mathbf{A}$ and $\Sigma$ that span the subspaces of $\mathbf{J}$ such as the Galerkin’s or moments methods. The unknowns are $\mathbf{J}$ and $\Sigma$ while $\mathbf{A}_{\text{inc}}$ and $\Sigma_{\text{inc}}$ are known. We expand the unknowns in terms of basis functions $\mathbf{J}_n$ and $\sigma_m$ that span the subspaces of $\mathbf{J}_1$ and $\Sigma_1$, respectively. Namely,

$$\mathbf{J}_1(r') = \sum_{n=1}^N j_n \mathbf{J}_n(r')$$

(54)

$$\Sigma_1(r') = \sum_{m=1}^M s_m \sigma_m(r')$$

(55)

We choose $\mathbf{J}_n(r')$ to be divergence conforming tangential current so that the vector potential $\mathbf{A}_1$ that it produces is also divergence conforming [41]. In the above, $\sigma_m(r')$ can be chosen to approximate a surface charge well. After expanding the unknowns, we project the field that they produce onto the subspace spanned by the same unknown set as in the process of testing in the Galerkin’s method. Consequently, (52) and (53) become

$$0 = \langle \mathbf{J}_n'(r), \mathbf{A}_{\text{inc}}(r) \rangle + \mu_1 \sum_{n=1}^N \langle \mathbf{J}_n'(r), g_1(r, r'), \mathbf{J}_n(r) \rangle j_n$$

$$+ \sum_{m=1}^M s_m \langle \mathbf{J}_n'(r), \nabla' g_1(r, r'), \sigma_m(r') \rangle$$

(56)

$$\sum_{m=1}^M s_m \langle \sigma_m'(r), \sigma_m(r) \rangle = \langle \sigma_m'(r), \Sigma_{\text{inc}}(r) \rangle$$

$$+ \mu_1 \sum_{n=1}^N \langle \sigma_m'(r), \mathbf{n} g_1(r, r'), \mathbf{J}_n(r') \rangle j_n$$

$$+ \sum_{m=1}^M s_m \langle \sigma_m'(r), \mathbf{n} \cdot \nabla' g_1(r, r'), \sigma_m(r') \rangle s_m$$

(57)

The above is a matrix system of the form

$$0 = a_{\text{inc}} + \mathbf{f}, j + \mathbf{f}_s$$

(58)
\[ \mathbf{B} \cdot \mathbf{s} = \Sigma_{\text{inc}} + \mathbf{\Gamma}_{1,\sigma,j} \cdot \mathbf{j} + \mathbf{\Gamma}_{1,\sigma,\sigma} \cdot \mathbf{s} \] (59)

where \( \mathbf{j} \) and \( \mathbf{s} \) are unknowns, while \( \mathbf{a}_{\text{inc}} \) and \( \Sigma_{\text{inc}} \) are known. In detail, elements of the above matrices and vectors are given by

\[ [\mathbf{a}_{\text{inc}}]_{n'} = \langle \mathbf{J}_{n'}(\mathbf{r}), \mathbf{A}_{\text{inc}}(\mathbf{r}) \rangle \] (60)

\[ [\mathbf{\Gamma}_{1,j,j}]_{n',n} = \mu \langle \mathbf{J}_{n'}(\mathbf{r}), \mathbf{g}_1(\mathbf{r}, \mathbf{r}'), \mathbf{J}_n(\mathbf{r}') \rangle \] (61)

\[ [\mathbf{\Gamma}_{1,j,\sigma}]_{n',m} = \langle \mathbf{J}_{n'}(\mathbf{r}), \nabla' g_1(\mathbf{r}, \mathbf{r}'), \sigma_m(\mathbf{r}') \rangle \] (62)

\[ [\mathbf{B}]_{m',m} = \langle \sigma_{m'}(\mathbf{r}), \sigma_m(\mathbf{r}) \rangle \] (63)

\[ [\Sigma_{\text{inc}}]_{m'} = \langle \sigma_{m'}(\mathbf{r}), \Sigma_{\text{inc}}(\mathbf{r}) \rangle \] (64)

\[ [\mathbf{\Gamma}_{1,\sigma,j}]_{n',m} = \mu_1 \langle \sigma_{m'}(\mathbf{r}), \mathbf{n} \cdot \mathbf{g}_1(\mathbf{r}, \mathbf{r}'), \mathbf{J}_n(\mathbf{r}') \rangle \] (65)

\[ [\mathbf{\Gamma}_{1,\sigma,\sigma}]_{m',m} = \langle \sigma_{m'}(\mathbf{r}), \mathbf{n} \cdot \nabla' g_1(\mathbf{r}, \mathbf{r}'), \sigma_m(\mathbf{r}') \rangle \] (66)

\[ [\mathbf{j}]_{n} = \mathbf{j}_n, \quad [\mathbf{s}]_{m} = \mathbf{s}_m \] (67)

Furthermore, in the above,

\[ \langle \mathbf{f}(\mathbf{r}), \mathbf{h}(\mathbf{r}) \rangle = \int_S dS \mathbf{f}(\mathbf{r}) \cdot \mathbf{h}(\mathbf{r}) \] (68)

\[ \langle \mathbf{f}(\mathbf{r}), \gamma(\mathbf{r}, \mathbf{r}'), \mathbf{h}(\mathbf{r}) \rangle = \int_S dS' \mathbf{f}(\mathbf{r}) \cdot \int_S dS' \gamma(\mathbf{r}, \mathbf{r}') \mathbf{h}(\mathbf{r}') \] (69)

where \( \mathbf{f}(\mathbf{r}) \) and \( \mathbf{h}(\mathbf{r}) \) can be replaced by scalar functions, and \( \gamma(\mathbf{r}, \mathbf{r}') \) can be replaced by a vector function with the appropriate inner products between them.

The \( \mathbf{\Gamma} \) matrices above are different matrix representations of the scalar Green’s function and its derivative. It is to be noted that all the \( \mathbf{\Gamma} \) matrices above do not have low-frequency catastrophe as in the matrix representation of the dyadic Green’s function. Hence, the above behaves like the augmented electric field integral equation (A-EFIE) [42].

7 Vector Potential Plane Wave

A time-harmonic vector potential plane wave is the solution to the equation

\[ (\nabla^2 + k^2) \mathbf{A} = 0 \] (70)
But it seems odd that $A_x$, $A_y$, and $A_z$ are decoupled from each other. To dispel this notion, we should think of $A$ as the solution to

$$\left(\nabla^2 + k^2\right) A = -\mu J$$

(71)

The vector potential above satisfies the Lorentz gauge via the charge continuity equation. By taking the divergence of the above, we have

$$\left(\nabla^2 + k^2\right) \nabla \cdot A = -\mu \nabla \cdot J = -i\omega \mu \rho$$

(72)

where $\nabla \cdot A = i\omega \mu \varepsilon \Phi$.

If $J$ is due to a Hertzian dipole source

$$J(r) = I\ell \hat{\ell} \delta(r)$$

(73)

the corresponding vector potential $A$ is

$$A(r) = \mu I\ell e^{ikr} \frac{e^{i\kappa_0 \cdot s}}{4\pi r_0}$$

(74)

We can produce a locally plane wave by letting $r = r_0 + s$ where $|r_0| \gg |s|$. Then the above spherically wave can be approximated by a locally plane wave:

$$A(r) \approx \mu I\ell e^{ikr_0} e^{i\kappa_0 \cdot s} = a e^{i\kappa_0 \cdot s}$$

(75)

where $\kappa_0 = kr_0$ and $\hat{r}_0$ is a unit vector that points in the direction of $r_0$. It is seen that the components of $A$ generated this way satisfies the gauge condition and are not independent of each other. We have to keep this notion in mind when we generate a vector potential plane wave.

Hence, for a plane wave incident,

$$A_{\text{inc}}(r) = (a_\perp + a_\parallel) e^{ik \cdot r}$$

(76)

where $a_{\parallel i} = a_0 \hat{k}_i$, and $\hat{k}_i \cdot a_\perp = 0$. Therefore

$$\nabla \cdot A_{\text{inc}} = i k_i \cdot a_{\parallel i} e^{ik \cdot r} = i a_0 k_i e^{ik \cdot r}$$

(77)

$$B_{\text{inc}} = \nabla \times A_{\text{inc}} = i k_i \times A_{\text{inc}} = i k_i \times a_\perp e^{ik \cdot r}$$

(78)

and

$$E_{\text{inc}} = \frac{\nabla \times B_{\text{inc}}}{-i\omega \mu \varepsilon} = \frac{i k_i \times (k_i \times a_\perp) e^{ik \cdot r}}{\omega \mu \varepsilon}$$

(79)

$$= i \left[k_i^2 a_\perp - k_i (k_i \cdot a_\perp) \right] e^{ik \cdot r} / \omega \mu \varepsilon$$

$$= i \omega \left[I - \hat{k}_i \hat{k}_i \right] \cdot a_\perp e^{ik \cdot r}$$

(80)

It is to be noted that if $A_{\text{inc}}$ has only the longitudinal component, then both $E$ and $B$ are zero even though $A$ is not zero. This could happen to leading order along the axial direction of a Hertzian dipole.
A Derivations of (46), (47), and (48)

We will ignore the source term $J$ in order to derive some identities similar to Green’s theorem. We begin with the following equations:

$$\nabla \times \nabla \times \mathbf{A} - \nabla \nabla \cdot \mathbf{A} - k^2 \mathbf{A} = 0 \quad (A.1)$$

$$\nabla \times \nabla \times \mathbf{G} - \nabla \nabla \cdot \mathbf{G} - k^2 \mathbf{G} = \bar{\mathbf{I}} \delta(\mathbf{r} - \mathbf{r}') \quad (A.2)$$

In the above, $\mathbf{A} = \mathbf{A}(\mathbf{r})$ and $\mathbf{G} = \mathbf{G}(\mathbf{r}, \mathbf{r}')$, but we will suppress these dependent variables for the time being in the following. First, we dot-multiply (A.1) from the right by $\mathbf{A} \cdot \mathbf{a}$ where $\mathbf{a}$ is an arbitrary vector, and then dot-multiply (A.2) from the left by $\mathbf{A}$ and the right by $\mathbf{a}$. We take their difference, and ignoring the $\nabla \nabla$ term for the time being, to get

$$\nabla \times \nabla \times \mathbf{A} \cdot \mathbf{G} \cdot \mathbf{a} - \mathbf{A} \cdot \nabla \times \nabla \times \mathbf{G} \cdot \mathbf{a} = \nabla \cdot \left( \nabla \times \mathbf{A} \times \mathbf{G} \cdot \mathbf{a} + \mathbf{A} \times \nabla \times \mathbf{G} \cdot \mathbf{a} \right) \quad (A.3)$$

Integrating right-hand side of the above over $V$, we have

$$I_1 \cdot \mathbf{a} = \int_S \hat{n} \cdot (\nabla \times \mathbf{A} \times \mathbf{G} \cdot \mathbf{a} + \mathbf{A} \times \nabla \times \mathbf{G} \cdot \mathbf{a}) dS$$

$$= \int_S [\hat{n} \times (\nabla \times \mathbf{A}) \cdot \mathbf{G} \cdot \mathbf{a} + (\hat{n} \times \mathbf{A}) \cdot \nabla \times \mathbf{G} \cdot \mathbf{a}] dS \quad (A.4)$$

Including now the $\nabla \nabla$ term gives

$$- \nabla \nabla \cdot \mathbf{A} \cdot \mathbf{G} \cdot \mathbf{a} + \mathbf{A} \cdot \nabla \nabla \cdot \mathbf{G} \cdot \mathbf{a} = \nabla \cdot (- \nabla \cdot \mathbf{A} \mathbf{G} \cdot \mathbf{a} + \mathbf{A} \cdot \nabla \cdot \mathbf{G} \cdot \mathbf{a}) \quad (A.5)$$

Integrating the right-hand side of the above over $V$, we have

$$I_2 \cdot \mathbf{a} = \int_S \hat{n} \cdot (- \nabla \cdot \mathbf{A} \mathbf{G} \cdot \mathbf{a} + \mathbf{A} \cdot \nabla \cdot \mathbf{G} \cdot \mathbf{a}) dS$$

$$= \int_S [\hat{n} \times (\nabla \times \mathbf{A}) \cdot \mathbf{G} \cdot \mathbf{a} + (\hat{n} \times \mathbf{A}) \cdot \nabla \times \mathbf{G} \cdot \mathbf{a}] dS \quad (A.6)$$

Letting $\mathbf{G} = g \bar{\mathbf{I}}$, where $g = g(\mathbf{r}, \mathbf{r}')$, the scalar Green’s function, the above becomes

$$I_1 \cdot \mathbf{a} = \int_S [\hat{n} \times (\nabla \times \mathbf{A} g \cdot \mathbf{a} + \hat{n} \times \mathbf{A} \cdot \nabla g \times \mathbf{a})] dS$$

$$= \int_S [\hat{n} \times (\nabla \times \mathbf{A}) g + (\hat{n} \times \mathbf{A}) \times \nabla g] dS \cdot \mathbf{a} \quad (A.7)$$

or

$$I_1 = \int_S [\hat{n} \times (\nabla \times \mathbf{A}) g + (\hat{n} \times \mathbf{A}) \times \nabla g] dS \quad (A.8)$$

Similarly, we have

$$I_2 = \int_S [\hat{n} \times (\nabla \times \mathbf{A}) g + \hat{n} \times \mathbf{A} \cdot \nabla g] dS \quad (A.9)$$

Using the above, we get (47).

To get (48), more manipulations are needed. Using $\hat{n} \times (\nabla \times \mathbf{A}) = (\nabla \mathbf{A}) \cdot \hat{n} - (\hat{n} \cdot \nabla) \mathbf{A}$, $\nabla g \times (\hat{n} \times \mathbf{A}) = \hat{n} (\mathbf{A} \cdot \nabla g) - (\hat{n} \cdot \nabla g) \mathbf{A}$

$$I_1 = \int_S [- (\hat{n} \cdot \nabla \mathbf{A}) g + (\nabla \mathbf{A}) \cdot \hat{n} g - \hat{n} (\mathbf{A} \cdot \nabla g) + (\hat{n} \cdot \nabla g) \mathbf{A}] dS \quad (A.10)$$
First, we look at

\[
I_3 = \int_S \hat{n}[-\nabla \cdot \vec{A} \cdot g - \vec{A} \cdot \nabla g]dS
\]

\[
= \int_V \nabla(-\nabla \cdot \vec{A} \cdot g - \vec{A} \cdot \nabla g)
\]

\[
= \int_V \nabla [-\nabla \nabla \cdot \vec{A} \cdot g - \vec{A} \nabla \cdot \nabla \nabla g - \nabla \cdot \nabla \vec{A} \cdot \nabla g]
\]

(A.11)

\[
I_4 = \int_S [g \nabla \cdot \vec{A} \cdot \nabla \hat{n} + \hat{n} \cdot \vec{A} \cdot \nabla \nabla g]
\]

\[
= \int_V \nabla \cdot [(g \nabla \vec{A})^t + \vec{A} \cdot \nabla g]
\]

(A.12)

Furthermore, with the knowledge that

\[
\nabla \cdot [(g \nabla \vec{A})^t - \vec{A} \cdot \nabla g] = \partial_i [g \partial_k \vec{A}_i + \vec{A}_i \partial_j g]
\]

(A.13)

\[
= \partial_i [g \partial_k \vec{A}_i + \vec{A}_i \partial_j g]
\]

(A.14)

\[
= \nabla \vec{A} \cdot \nabla g + g \nabla \nabla \cdot \vec{A} + \vec{A} \cdot \nabla g + \vec{A} \cdot \nabla \nabla g
\]

(A.15)

it is seen that \(I_3 + I_4 = 0\). Using this fact, we can show (48), or that

\[
I_1 + I_2 = \int_S [\nabla \cdot (\vec{A} \cdot \nabla g)] dS
\]

(A.16)

### B Derivation of (51)

\[
\vec{A}_1 = \vec{A}_{\text{inc}} + \int_S dS' \left\{ \mu_1 g_1 J_1 + (\hat{n}' \cdot \vec{A}_1) \nabla' g_1 \right\}
\]

(B.1)

\[
\nabla \cdot \vec{A}_1 = \nabla \cdot \vec{A}_{\text{inc}} + \int_S dS' \left\{ \mu_1 g_1 \nabla' \cdot J_1 - (\hat{n}' \cdot \vec{A}_1) \nabla^2 g_1 \right\}
\]

(B.2)

Since \(\nabla \cdot \vec{A} = i \omega \mu \varepsilon \Phi\), the above becomes

\[
i \omega \mu_1 \varepsilon \Phi_1 = i \omega \mu_1 \varepsilon \Phi_{\text{inc}} + \int_S dS' \left\{ \mu_1 g_1 i \omega \sigma_1 + (\hat{n}' \cdot \vec{A}_1) k_1^2 g_1 \right\}
\]

(B.3)

or

\[
\Phi_1 = \Phi_{\text{inc}} + \int_S dS' \left\{ g_1 \frac{\sigma_1}{\varepsilon_1} - i \omega (\hat{n}' \cdot \vec{A}_1) g_1 \right\}
\]

(B.4)

Since \(\vec{E} = i \omega \vec{A} - \nabla \Phi\), using (B.1) and (B.4), we have

\[
\vec{E}_1 = \vec{E}_{\text{inc}} + \int_S dS' \left\{ i \omega \mu_1 g_1 J_1 - \nabla g_1 \frac{\sigma_1}{\varepsilon_1} + i \omega (\hat{n}' \cdot \vec{A}_1) \nabla' g_1 + i \omega (\hat{n}' \cdot \vec{A}_1) \nabla g_1 \right\}
\]

(B.5)

The last two terms cancel each other.
Acknowledgements

This work was supported in part by the USA NSF CCF Award 1218552, SRC Award 2012-IN-2347, at the University of Illinois at Urbana-Champaign, by the Research Grants Council of Hong Kong (GRF 711609, 711508, and 711511), and by the University Grants Council of Hong Kong (Contract No. AoE/P-04/08) at HKU. The author is grateful to M. Wei, H. Gan, C. Ryu, T. Xia, Y. Li, Q. Liu, and L. Meng for helping to typeset the manuscript.

References

[1] Maxwell, J. Clerk. “A dynamical theory of the electromagnetic field.” Philosophical Transactions of the Royal Society of London. (1865): 459-512. (First presented to the British Royal Society in 1864).

[2] Heaviside, Oliver. Electromagnetic theory. Vol. 3. Cosimo, Inc., 2008.

[3] Aharonov, Yakir, and David Bohm. “Significance of electromagnetic potentials in the quantum theory.” Physical Review. 115.3 (1959): 485.

[4] Gasiorowicz, Stephen. Quantum physics. John Wiley & Sons, 2007.

[5] Cohen-Tannoudji, Claude, Jacques Dupont-Roc, and Gilbert Grynberg. Atom-photon interactions: basic processes and applications. New York: Wiley, 1992.

[6] Mandel, Leonard, and Emil Wolf. Optical coherence and quantum optics. Cambridge university press, 1995.

[7] Scully, Marlan O. and M. Suhail Zubairy. Quantum optics. Cambridge university press, 1997.

[8] Loudon, Rodney. The quantum theory of light. Oxford university press, 2000.

[9] Gerry, Christopher, and Peter Knight. Introductory quantum optics. Cambridge university press, 2005.

[10] Fox, Mark. Quantum Optics: An Introduction. Vol. 15. Oxford University Press, 2006.

[11] Garrison, John, and Raymond Chiao. Quantum Optics. Oxford University Press, USA, 2014.

[12] Manges, John B., and Zoltan J. Cendes. “A generalized tree-cotree gauge for magnetic field computation.” Magnetics, IEEE Transactions on. 31.3 (1995): 1342-1347.

[13] Lee, Shih-Hao, and Jian-Ming Jin. “Application of the tree-cotree splitting for improving matrix conditioning in the full-wave finite-element analysis of high-speed circuits.” Microwave and Optical Technology Letters. 50.6 (2008): 1476-1481.

[14] Wilton, D. R., and A. W. Glisson. “On improving the electric field integral equation at low frequencies.” Proc. URSI Radio Sci. Meet. Dig. 24 (1981).
[15] Vecchi, Giuseppe. “Loop-star decomposition of basis functions in the discretization of the EFIE.” *Antennas and Propagation, IEEE Transactions on*. 47.2 (1999): 339-346.

[16] Zhao, Jun-Sheng, and Weng Cho Chew. “Integral equation solution of Maxwell’s equations from zero frequency to microwave frequencies.” *Antennas and Propagation, IEEE Transactions on*. 48.10 (2000): 1635-1645.

[17] Chawla, B. R., S. S. Rao, and H. Unz. “Potential equations for anisotropic inhomogeneous media.” *Proceedings of the IEEE*. 55.3 (1967): 421-422.

[18] Geselowitz, David B. “On the magnetic field generated outside an inhomogeneous volume conductor by internal current sources.” *Magnetics, IEEE Transactions on*. 6.2 (1970): 346-347.

[19] Demerdash, N. A., F. A. Fouad, T. W. Nehl, and O. A. Mohammed. “Three dimensional finite element vector potential formulation of magnetic fields in electrical apparatus.” *Power Apparatus and Systems, IEEE Transactions on*. 8 (1981): 4104-4111.

[20] Biro, Oszkar, and Kurt Preis. “On the use of the magnetic vector potential in the finite-element analysis of three-dimensional eddy currents.” *Magnetics, IEEE Transactions on*. 25.4 (1989): 3145-3159.

[21] MacNeal, B. E., J. R. Brauer, and R. N. Coppolino. “A general finite element vector potential formulation of electromagnetics using a time-integrated electric scalar potential.” *Magnetics, IEEE Transactions on*. 26.5 (1990): 1768-1770.

[22] Dyczij-Edlinger, Romanus, and O. Biro. “A joint vector and scalar potential formulation for driven high frequency problems using hybrid edge and nodal finite elements.” *Microwave Theory and Techniques, IEEE Transactions on*. 44.1 (1996): 15-23.

[23] Dyczij-Edlinger, Romanus, Guanghua Peng, and J-F. Lee. “A fast vector-potential method using tangentially continuous vector finite elements.” *Microwave Theory and Techniques, IEEE Transactions on*. 46.6 (1998): 863-868.

[24] De Flaviis, F, M.G. Noro, R.E. Diaz, G. Franceschetti, and N.G. Alexopoulos, “A time-domain vector potential formulation for the solution of electromagnetic problems,” *Microwave Guided Wave Lett., IEEE*. 8(9), 310-312, 1998.

[25] Biro, Oszkar. “Edge element formulations of eddy current problems.” *Computer methods in applied mechanics and engineering*. 169.3 (1999): 391-405.

[26] Dular, Patrick, et al. “A 3-D magnetic vector potential formulation taking eddy currents in lamination stacks into account.” *Magnetics, IEEE Transactions on*. 39.3 (2003): 1424-1427.

[27] Zhu, Yu, and Andreas C. Cangellaris, eds. *Multigrid finite element methods for electromagnetic field modeling*. Vol. 28. John Wiley & Sons, 2006.
[28] Jackson, John David. *Classical Electrodynamics*. 3rd Edition, Wiley-VCH, July 1998.

[29] Harrington, Roger F. *Time-harmonic electromagnetic fields*. (1961): 224.

[30] Kong, Jin Au. *Theory of electromagnetic waves*. New York, Wiley-Interscience, 1975. 348 p. 1 (1975).

[31] Balanis, Constantine A. *Advanced engineering electromagnetics*. Vol. 111. John Wiley & Sons, 2012.

[32] Greengard, L. and V. Rokhlin, “A Fast Algorithm for Particle Simulations,” *Journal of Computational Physics*, vol. 73, pp. 325–348, 1987.

[33] Coifman, R., V. Rokhlin, and S. Wandzura, “The fast multipole method for the wave equation: A pedestrian prescription,” *IEEE Ant. Propag. Mag.*, vol. 35, no. 3, pp. 7–12, Jun. 1993.

[34] Chew, W. C., J. M. Jin, E. Michielssen, and J. M. Song (eds.), *Fast and Efficient Algorithms in Computational Electromagnetics*. Boston, MA: Artech House, 2001.

[35] Chew, Weng Cho, Mei-Song Tong, and Bin Hu. *Integral equation methods for electromagnetic and elastic waves*. Morgan & Claypool Publishers, 2008.

[36] Chew, Weng Cho. *Waves and fields in inhomogeneous media*. Vol. 522. New York: IEEE press, 1995. (First published in 1990 by Van Nostrand Reinhold.)

[37] Ishimaru, Akira. *Electromagnetic wave propagation, radiation, and scattering*. Englewood Cliffs, NJ: Prentice Hall, 1991.

[38] Galerkin, Boris Grigoryevich. “Series solution of some problems of elastic equilibrium of rods and plates.” Vestn. Inzh. Tekh 19 (1915): 897-908.

[39] Kravchuk, M. F. “Application of the method of moments to the solution of linear differential and integral equations.” *Ukrain. Akad. Nauk*, Kiev (1932).

[40] Harrington, R. F. *Field Computation by Moment Method*, NY: Macmillan, 1968.

[41] Dai, Q. I., W. C. Chew, Y. H. Lo, and L. J. Jiang, “Differential forms motivated discretizations of differential and integral equations,” *Antennas Wireless Propag. Lett.*, IEEE, to appear, 2014.

[42] Qian, Zhi-Guo, and Weng Cho Chew. “Fast full-wave surface integral equation solver for multiscale structure modeling.” *Antennas and Propagation, IEEE Transactions on*. 57.11 (2009): 3594-3601.