A free boundary problem with nonlocal diffusion and unbounded initial range

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Abstract. We consider a free boundary problem with nonlocal diffusion and unbounded initial range, which can be used to model the propagation phenomenon of an invasion species whose habitat is the interval \((-\infty, h(t))\) with \(h(t)\) representing the spreading front. Since the spatial scale is unbounded, a different method from the existing works about nonlocal diffusion problem with free boundary is employed to obtain the well-posedness. Then we prove that the species always spreads successfully, which is very different from the free boundary problem with bounded range. We also show that there is a finite spreading speed if and only if a threshold condition is satisfied by the kernel function. Moreover, the rate of accelerated spreading and accurate estimates on longtime behaviors of solution are derived.

Mathematics Subject Classification. 35K57, 35R09, 35R20, 35R35, 92D25.

Keywords. Nonlocal diffusion, Free boundary, Unbounded range, Accelerated spreading.

1. Introduction

Using reaction–diffusion equations with free boundary to model propagation has been increasingly accepted by many researchers after the pioneering work \([1]\), where the Stefan boundary condition was first incorporated into the reaction–diffusion equation arising from ecology. Specifically, to investigate the propagation of an invasive species, Du and Lin \([1]\) proposed the following problem

\[
\begin{aligned}
&v_t = Dv_{xx} + v(a - bv), & t > 0, \ s_1(t) < x < s_2(t), \\
v(t, x) = 0, & t > 0, \ x = s_i(t), \ i = 1, 2, \\
\dot{s}_i(t) = -\mu v_x(t, s_i(t)), & t > 0, \ i = 1, 2, \\
-s_1(0) = s_2(0) = \tilde{s}_0, & v(0, x) = v_0(x), \quad -\tilde{s}_0 \leq x \leq \tilde{s}_0.
\end{aligned}
\]

In this model, the invasive species, whose density is denoted by \(v\), initially occupies the spatial domain \((\tilde{s}_0, \tilde{s}_0)\) with initial density \(v_0(x)\). Since the species will move instinctively for survival as time goes on, its habitat will evolve naturally and further is represented by \((s_1(t), s_2(t))\). The spreading frontiers \(s_i(t)\) are assumed to satisfy the Stefan boundary condition, and in other words they are expanding at a rate proportional to the population of the invasive species across them. Du and Lin \([1]\) found that the species either spreads successfully in the sense that

\[
\lim_{t \to \infty} -s_1(t) = \lim_{t \to \infty} s_2(t) = \infty, \quad \lim_{t \to \infty} v(t, x) = \frac{a}{b} \quad \text{locally uniformly in } \mathbb{R},
\]

or fails to spread in the sense that

\[
\lim_{t \to \infty} [s_2(t) - s_1(t)] < \infty, \quad \lim_{t \to \infty} \max_{x \in [s_1(t), s_2(t)]} v(t, x) = 0.
\]

This work was supported by NSFC Grants 12171120, 11901541.
This phenomenon is called spreading-vanishing dichotomy in [1] and is very different from the dynamics of the corresponding Cauchy problem

\[ v_t = Dv_{xx} + v(a - bv), \quad t > 0, \quad x \in \mathbb{R}, \]

where the species will always establish itself in the whole space, please see [2]. When giving a nontrivial and supported compact initial function to (1.2), one can see from [3, 4] that \(2\sqrt{Da}\) is the spreading speed of the level set of solution to (1.2).

Similarly to the Cauchy problem (1.2), if the species spreads successfully in (1.1), Du and Lin [1] proved

\[
\lim_{t \to \infty} \frac{-s_1(t)}{t} = \lim_{t \to \infty} \frac{s_2(t)}{t} = k_0,
\]

where \(k_0\) is uniquely given by the semi-wave problem

\[
\begin{cases}
Dq'' - kq' + q(a - bq) = 0 \text{ in } (0, \infty), \\
q(0) = 0, \quad q(\infty) = a/b, \quad k = \mu q'(0).
\end{cases}
\]

For more results about propagation modeled by reaction–diffusion equations, which has a long history, one can refer to [5–8] and the references therein, and for the results derived from reaction–diffusion equations with free boundary, one can refer to the expository article [9].

As is well-known to us, long-distance dispersal usually happens, such as wind-dispersed seeds, which is inappropriate to be modeled by the Laplacian operator, also called random diffusion or local diffusion operator, please see, e.g., [10, 11]. Therefore, a class of new operators which can take long-distance dispersal into account are derived, please see [12] for the details of the derivation. Such kind of operator is usually referred to nonlocal diffusion operator, and one of them often takes the following form

\[
d \left( \int_{-\infty}^{\infty} J(x - y)v(t, y)dy - v(t, x) \right),
\]

where \(d\) is the dispersal rate, and \(J(x)\) is the kernel function which is usually nonnegative, bounded and unit integrable.

The dynamics of (1.2) with \(Dv_{xx}\) replaced by (1.3) have already been known, please see [13–17] and the references therein. Recently, Cao et al. [18] proposed the nonlocal diffusion version of (1.1). More precisely, they considered the following problem

\[
\begin{cases}
v_t = d \int_{g(t)}^{h(t)} J(x - y)v(t, y)dy - dv + f(v), & t > 0, \quad g(t) < x < h(t), \\
v(t, x) = 0, & t > 0, \quad x \notin (g(t), h(t)), \\
h'(t) = \mu \int_{g(t)h(t)}^{h(t)} J(x - y)v(t, x)dydx, & t > 0, \\
g'(t) = -\mu \int_{g(t)h(t)}^{h(t)g(t)} J(x - y)v(t, x)dydx, & t > 0, \\
h(0) = -g(0) = \ell_0 > 0, \quad v(0, x) = \tilde{v}_0(x), \quad |x| \leq \ell_0,
\end{cases}
\]

where nonlinear term \(f\) is of the Fisher-KPP type, i.e.,

\[(F) \quad f \in C^1, \quad f'(0) > 0 = f(0), \quad f'(u^*) < 0 = f(u^*) \text{ for some } u^* > 0, \text{ and } f(u)/u \text{ is strictly decreasing in } u > 0.\]
Similarly to (1.1), Cao et al. proved that the dynamics of (1.4) was also govern by a spreading-vanishing dichotomy in [18]. When spreading occurs, the spreading speed was later obtained by Du et al. [19]. Particularly, there are two differences worth emphasizing between the dynamics of (1.1) and (1.4). Firstly, from [18], if the dispersal rate $d$ is no more than the growth rate $f'(0)$, then spreading always happens. While in [1], no matter how small the dispersal rate $d$ is, vanishing always can occur if the initial habitat is suitably small. Secondly, when spreading happens, the spreading speed of (1.1) is always finite, while that of (1.4) can be infinite, which is usually called the accelerated spreading, if $J$ does not satisfy a threshold condition.

Since the work [18], there have been lots of related researches on the free boundary problem with nonlocal diffusion, a sample of which can be referred to [20–29] and the references therein.

Motivated by the above works, we here consider the case where the species occupying the initial habitat $(-\infty, h_0)$ propagates with the strategy of nonlocal diffusion as in [18], and $h(t)$ stands for its spreading front. Therefore, same as the derivation of the problem (1.4), the concerned model can be written as

$$
\begin{aligned}
&\begin{cases}
    u_t = d \int_{-\infty}^{h(t)} J(x - y)u(t, y)dy - du + f(u), & t > 0, \quad -\infty < x < h(t), \\
    u(t, x) = 0, & t > 0, \quad h(t) \leq x < \infty,
\end{cases} \\
&h'(t) = \mu \int_{-\infty}^{h(t)} \int_{\infty}^{x} J(x - y)u(t, x)dydx, \quad t > 0, \\
h(0) = h_0, \quad u(0, x) = u_0(x), \quad -\infty < x \leq h_0,
\end{aligned}
$$

where $f(u)$ satisfies the above (F), and $u_0(x)$ meets

(H) $u_0(x) \in C((-\infty, h_0]) \cap L^\infty((-\infty, h_0]) \cap L^1((-\infty, h_0]), u_0(h_0) = 0 < u_0(x)$ in $(-\infty, h_0)$.

The kernel function $J$ satisfies

(J) $J \in C(\mathbb{R}) \cap L^\infty(\mathbb{R})$, $J(x) \geq 0$, $J(0) > 0$, $J$ is even, and $\int_{\mathbb{R}} J(x)dx = 1$.

If $f(u) \equiv 0$, $\int_{-\infty}^{h_0} |x|u_0(x)dx < \infty$ and $J$ are compactly supported, Cortázar et al. [30] proved that the solution $(u, h)$ of (1.5) satisfies

$$
\lim_{t \to -\infty} h(t) \leq h_0 + \int_{-\infty}^{h_0} u_0(x)dx \quad \text{and} \quad ||u(t, x)||_{L^\infty((-\infty, h(t))]} = O(t^{-1}).
$$

However, our main result shows that under the assumptions on $f$, $J$ and $u_0(x)$, the solution $(u, h)$ of (1.5) has very different longtime behaviors.

Throughout this paper, we always suppose that conditions (F), (H) as well as (J) hold. For convenience, we first give some notations. For any given $h_0 \in \mathbb{R}$ and $T > 0$, we define

$$
\mathbb{H}^T_{h_0} = \left\{ h \in C([0, T]) : h(0) = h_0, \inf_{0 \leq t_1 < t_2 \leq T} \frac{h(t_2) - h(t_1)}{t_2 - t_1} > 0 \right\},
$$

and for $h \in \mathbb{H}^T_{h_0}$, define

$$
D^T_h = \{(t, x) : 0 < t \leq T, \quad -\infty < x < h(t)\}, \\
\mathcal{X}^T_{u_0, h} = \left\{ \varphi \in C(D^T_h) : \varphi(0, x) = u_0(x), \varphi(t, h(t)) = 0 \right\}.
$$
In the following, $\overline{D}_h^T$ represents the closure of $D_h^T$, $A := \max\{\|u_0\|_{L^\infty((-\infty, h_0))}, u^*\}$, and we say $s(t) = o(r(t))$ if $\lim_{t \to \infty} s(t)/r(t) = 0$. We use $C, C_1$ and $C_2$ to represent the generic positive constants, which may vary with the place they appear. Below are our main results.

**Theorem 1.1.** Problem (1.5) has a unique global solution $(u, h)$. Moreover, $u(t, \cdot), u_t(t, \cdot) \in L^1((-\infty, h(t)))$ for all $t > 0$, $u, u_t \in C(\overline{D}_h^T)$, $0 \leq u(t, x) \leq A$ in $\overline{D}_h^T$ and $h \in C^1([0, T])$ for any $T > 0$.

**Remark 1.1.** If the initial value $u_0(x) \notin L^1((-\infty, h_0))$, then problem (1.5) may not have a solution $(u, h)$ satisfying $u \in C(\overline{D}_h^T)$ and $h \in C^1([0, T])$. In fact, we assume that $J(x) \geq C|x|^{-2}$ when $|x| \gg 1$ and $u_0(x) = 1$ when $x \leq -L$ for some $L \gg 1$. If the problem (1.5) has a solution $(u, h)$ satisfying $u \in C(\overline{D}_h^T)$ and $h \in C^1([0, T])$ for some $T > 0$, define $u(t, x) = \xi u_0(x) e^{-dt}$ with $\xi > 0$ sufficiently small such that $\xi\|u_0\|_{L^\infty((-\infty, h_0))} \leq u^*$. By a comparison argument, we have $u(t, x) \geq u(t, x)$ for $t \geq 0$ and $x \leq h_0$. Then for every $t \in (0, T]$ and $L \gg 1$, we have

$$\infty > h'(t) = \mu \int_{-\infty}^{h(t)} \int_{-\infty}^{h_0} J(x - y) u(t, x) dy dx \geq \mu \int_{-\infty}^{h(t)} \int_{-\infty}^{h_0} J(x - y) u(t, x) dy dx \geq \mu \int_{-\infty}^{h(t)} \int_{-\infty}^{h_0} J(x - y) u(t, x) dy dx \geq \mu \xi C e^{-dt} \int_{-\infty}^{h(t) - x} \int_{-\infty}^{h_0} y^2 u_0(x) dy dx = \mu \xi C e^{-dt} \int_{-\infty}^{h(t) - x} \int_{-\infty}^{h_0} u_0(x) dx = \infty.$$  

This contradiction indicates that problem (1.5) may not have a solution when $u_0(x) \notin L^1((-\infty, h_0))$.

Next we show the longtime behavior of solution $(u, h)$ to (1.5). We start by giving a proposition that is crucial to the later discussions.

**Proposition 1.1.** ([19, Theorem 1.2]). The problem

$$\begin{cases}
    d \int_{-\infty}^{0} J(x - y) \phi(y) dy - d \phi + c \phi' + f(\phi) = 0, & -\infty < x < 0, \\
    \phi(-\infty) = u^*, \phi(0) = 0, & c = \mu \int_{-\infty}^{0} J(x - y) \phi(x) dy dx
\end{cases} \quad (1.6)$$

has a unique solution pair $(c_0, \phi^{c_0})$ with $c_0 > 0$ and $\phi^{c_0}$ nonincreasing in $(-\infty, 0]$ if and only if $J$ satisfies
(J1) \[ \int_0^\infty xJ(x)dx < \infty. \]

**Theorem 1.2.** Let \((u, h)\) be the unique solution of (1.5). Then
\[ \lim_{t \to \infty} h(t) = \infty \text{ and } \lim_{t \to \infty} u(t, x) = u^* \text{ locally uniformly in } \mathbb{R}. \] (1.7)

Moreover, the following more accurate estimates hold.
1. If (J1) holds, then
\[ \lim_{t \to \infty} \frac{h(t)}{t} = c_0 \text{ and } \lim_{t \to \infty} \max_{t \leq |x|} |u(t, x) - u^*| = 0 \text{ for all } c \in [0, c_0), \]
where \(c_0\) is uniquely determined by the semi-wave problem (1.6). Moreover, if we further suppose that \(J\) is compactly supported and \(f \in C^2\), then
\[ h(t) - c_0t = O(1). \] (1.8)

2. If (J1) does not hold, then
\[ \lim_{t \to \infty} \frac{h(t)}{t} = \infty \text{ and } \lim_{t \to \infty} \max_{t \leq |x|} |u(t, x) - u^*| = 0 \text{ for all } c \geq 0. \]

3. If \(J(x) \geq C|x|^{-\gamma}\) for \(\gamma \in (1, 2)\) and \(|x| \gg 1\), then when \(t \gg 1\),
\[ h(t) \geq C_1 t^{\frac{1}{\gamma - 1}} \text{ for some } C_1 > 0 \text{ and } \lim_{t \to \infty} \max_{t \leq |x| \leq s(t)} |u(t, x) - u^*| = 0 \text{ for } s(t) = o(t^{\frac{1}{\gamma - 1}}). \]

4. If \(J(x) \geq C|x|^{-2}\) for \(|x| \gg 1\), then when \(t \gg 1\),
\[ h(t) \geq C_1 t \ln t \text{ for some } C_1 > 0 \text{ and } \lim_{t \to \infty} \max_{t \leq |x| \leq s(t)} |u(t, x) - u^*| = 0 \text{ for } s(t) = o(t \ln t). \]

5. The free boundary \(h(t)\) will not spread faster than exponential, that is,
\[ h(t) \leq \frac{\mu \|u_0(x)\|_{L^\infty((\infty, h_0))} (e^{f'(0)t} - 1) + h_0}{f'(0)} \text{ for all } t \geq 0. \]

**Remark 1.2.** We can improve the conclusion (5) in Theorem 1.2 if extra assumptions are imposed on the initial value \(u_0(x)\) and kernel \(J\). More precisely, the assumptions we need are as follows:

(i) The Fourier transform \(\hat{u}_0\) of \(u_0(x) \equiv 0\) for \(x > h_0\) belongs to \(L^1(\mathbb{R})\);

(ii) There exist \(B > 0\) and \(\alpha \in (0, 2]\) such that
\[ \hat{J}(\zeta) = 1 - B|\zeta|\alpha + o(|\zeta|\alpha) \text{ as } \zeta \to 0, \]
where \(\hat{J}(\zeta)\) is the Fourier transform of \(J\). Moreover,
\[ \int_0^\infty \left( \int_{-\infty}^{\infty} J(x - y)dy \right)^q dx < \infty \text{ for some } q \in (1, \infty). \]

By the above assumptions and \([31, \text{Theorem 1.1}]\), we can obtain the decay rate of \(L^p\) norm of solution \(v\) to the later problem (3.12), that is, \(\|v(t, \cdot)\|_{L^p(\mathbb{R})} \leq Ct^{-\frac{1}{p}(1 - \frac{1}{q})}\) for every \(p \in (1, \infty)\) and some \(C > 0\).
Then we have the following improvement of the conclusion (5) in Theorem 1.2:

\[
h'(t) = \mu \int_{-\infty}^{h(t)} \int J(x-y)u(t,x)dydx \\
\leq \mu e^{f'(0)t} \int_{-\infty}^{h(t)} \int J(x-y)v(t,x)dydx \\
\leq \mu e^{f'(0)t} \left( \int_{-\infty}^{h(t)} v \frac{1}{\pi(1+x^2)}(t,x)dx \right)^{\frac{q-1}{q}} \left[ \int_{-\infty}^{h(t)} \left( \int_{-\infty}^{\infty} J(x-y)dy \right)^q dx \right]^{\frac{1}{q}} \\
\leq C \mu e^{f'(0)t} t^{-\frac{1}{pq}} \left[ \int_{-\infty}^{0} \left( \int_{0}^{\infty} J(x-y)dy \right)^q dx \right]^{\frac{1}{q}}.
\]

We give an example of kernel \( J \):

\[
J(x) = \frac{1}{\pi(1+x^2)}
\]

Obviously, this kernel \( J \) satisfies (J) and the above assumption (ii) with \( \alpha = B = 1 \) and all \( q \in (1, \infty) \), but doesn’t meet (J1).

This paper is arranged as follows. In Sect. 2, we show the maximum principle and comparison principle. In Sect. 3, we give the proofs of the above two theorems. Note that the spatial range of (1.5) is unbounded. Therefore, a new method, which is different from that of [18], is adopted to prove the well-posedness of (1.5). Concretely, we construct a sequence of free boundary problems defined in bounded domain, then prove that the solutions of this sequence converge to a solution of (1.5), and at last show the uniqueness by a comparison argument and the contraction mapping principle. Then a brief discussion is given in Sect. 4.

2. The maximum principle and comparison principle

This section is devoted to show the maximum principle and comparison principle.

**Lemma 2.1.** (Maximum principle) Assume that \( s_0 \in \mathbb{R}, T > 0 \) and \( z_0 \in C((-\infty, s_0]) \). Let \( s \in \mathbb{H}_s^T \), \( z, z_t \in C(D_s^T) \) and \( z, c(t,x) \in L^\infty(D_s^T) \). If \( z \) satisfies

\[
\begin{cases}
    z_t \geq d \int_{-\infty}^{s(t)} J(x-y)z(t,y)dy + c(t,x)z, & 0 < t \leq T, \; -\infty < x < s(t), \\
    z(t, s(t)) \geq 0, & 0 < t \leq T, \\
    z(0, x) = z_0(x) \geq 0, & -\infty < x \leq s_0,
\end{cases}
\]

then \( z \geq 0 \) in \( D_s^T \). Moreover, if \( z_0(x) \not\equiv 0 \) in \( (-\infty, s_0] \), then \( z > 0 \) in \( D_s^T \).
Proof. For small $\varepsilon > 0$ and $B > d + 1 + \|c\|_{L^\infty(D_T)}$, define $w = z + \varepsilon e^{Bt}$. Then $w \geq \varepsilon$ in the parabolic boundary of $D_T^e$. Moreover, for $(t, x) \in (0, T) \times (-\infty, s(t))$,

$$w_t - d \int_{-\infty}^{s(t)} J(x - y) w(t, y) dy - c(t, x) w \geq \left( -d \int_{-\infty}^{s(t)} J(x - y) dy - c(t, x) + B \right) \varepsilon e^{Bt} \geq \varepsilon e^{Bt}.$$

(2.1)

Hence, there is $t_0 > 0$ such that $w > 0$ for $(t, x) \in [0, t_0] \times (-\infty, s(t))$. We further can define $T_0 = \sup \{s \in (0, T) : w > 0 \text{ in } [0, s] \times (-\infty, s_0]\}$, and clearly $T_0 \geq t_0$.

We now claim $T_0 = T$. Arguing indirectly, if $T_0 < T$, then $\inf_{x \in (-\infty, s_0)} w(T_0, x) = 0$. Since if it does not hold, noting that $w(t_0, x_0)$ has a finite lower bound independent of $x \in (-\infty, s_0]$, we can find some $0 < \delta \ll 1$ such that $w(t, x) > 0$ for $(t, x) \in [T_0, T_0 + \delta] \times (-\infty, s_0]$. Obviously, this is a contradiction to the definition of $T_0$. Thus, for any sequence $\{\varepsilon_n\}$ with $\varepsilon_n > 0$ and $\lim_{n \to \infty} \varepsilon_n = 0$, there exists a sequence $\{x_n\} \subset (-\infty, s_0]$ such that $w(T_0, x_n) < \varepsilon_n$.

Since $w$ is bounded below and continuous in $[0, T_0] \times \mathbb{R}^+$, it follows from Ekeland’s variational principle that for such $\varepsilon_n$, $(T_0, x_n)$ and $\sigma = \min\{T_0/2, 1\}$, there exists a sequence $\{(t_n, y_n)\} \subset [0, T_0] \times (-\infty, s_0]$ such that

$$\begin{cases} w(t_n, y_n) \leq w(T_0, x_n) < \varepsilon_n, & |T_0 - t_n| + |x_n - y_n| < \sigma, \\ w(t_n, y_n) - w(t, y_n) \leq \frac{|t_n - t|\varepsilon_n}{\sigma} & \text{for } t \in [0, T_0]. \end{cases}$$

Due to the choice of $\sigma$, we have $t_n > 0$, and thus,

$$w(t_n, y_n) \leq \varepsilon_n/\sigma \to 0 \quad \text{as} \quad n \to \infty. \quad (2.2)$$

On the other hand, for the later discussion, we now show that $w \geq 0$ for $(t, x) \in [0, T_0] \times [s_0, s(t)]$. Let $v = w e^{-Bt}$. In view of $v \geq 0$ for $(t, x) \in [0, T_0] \times ((-\infty, s_0] \cup \{s(t)\})$, if $\min_{[0, T_0] \times [s_0, s(t)]} v(t, x) < 0$, then there must exist some $(t_*, x_*) \in (0, T_0] \times (s_0, s(t))$ such that $\min_{[0, T_0] \times [s_0, s(t)]} v(t, x) = v(t_*, x_*) < 0$. Clearly, $v_t(t_*, x_*) \leq 0$ and

$$d \int_{-\infty}^{s(t)} J(x_* - y) v(t_*, y) dy - d \int_{-\infty}^{s(t)} J(x_* - y) dy v(t_*, x_*) \geq 0.$$

Noticing (2.1), we have

$$0 \geq v_t(t_*, x_*) - d \int_{-\infty}^{s(t)} J(x_* - y) v(t_*, y) dy - d \int_{-\infty}^{s(t)} J(x_* - y) dy v(t_*, x_*)$$

$$\geq v(t_*, x_*) \left( d \int_{-\infty}^{s(t)} J(x_* - y) dy v(t_*, x_*) + c(t_*, x_*) - B \right) > 0,$$

which implies that $w \geq 0$ for $(t, x) \in [0, T_0] \times [s_0, s(t)]$. Therefore, $w \geq 0$ for $(t, x) \in [0, T_0] \times (-\infty, s(t)]$. Combining this with (2.1), we have

$$w_t(t_n, y_n) \geq c(t_n, y_n) w(t_n, y_n) + \varepsilon e^{Bt_n} \geq -\|c\|_{\infty} \varepsilon_n + \varepsilon e^{Bt_n} \geq \frac{1}{2} \varepsilon \quad \text{for all large } n,$$
which contradicts (2.2). So $T_0 = T$ and $w \geq 0$ for $(t, x) \in [0, T] \times (-\infty, s_0]$. Furthermore, we similarly can show $w \geq 0$ for $(t, x) \in [0, T] \times (s_0, s(t)]$. Thus, $w \geq 0$ in $(t, x) \in [0, T] \times (-\infty, s(t)]$. By letting $\varepsilon \to 0$ we derive $\nu \geq 0$ in $[0, T] \times (-\infty, s(t)]$.

If $z_0(x) \neq 0$ in $(-\infty, s_0]$, by [32, Lemma 2.2] we have $z > 0$ in $[0, T] \times (-\infty, s_0]$. If there exists $(t^*, x^*) \in [0, T] \times (s_0, s(t))$ such that $z(t^*, x^*) = 0$, then by continuity of $z$ we can find a $\hat{x} \in (s_0, x^*)$ such that $z(t^*, \hat{x}) = 0$ and $z(t^*, x) > 0$ for $x \in [s_0, \hat{x})$. Clearly, $z(t^*, x) \leq 0$. Hence, by (J), we see

$$0 \geq z_t(t^*, x) \geq d \int_{-\infty}^{s(t^*)} J(\hat{x} - y) z(t, y) dy > 0.$$ 

This contradiction indicates that $z > 0$ in $(0, T] \times (s_0, s(t))$. The proof is complete.

**Lemma 2.2.** (Comparison principle) Suppose that (J1) holds. For $T \in (0, \infty)$, let $\bar{h} \in C([0, T])$, and $\bar{u}, \bar{u}_t \in C(\bar{D}_\bar{h}^T) \cap L^\infty(D_\bar{h}^T)$ satisfy

$$\begin{cases}
\bar{u}_t \geq d \int_{-\infty}^{\bar{h}(t)} J(x - y) \bar{u}(t, y) dy - d\bar{u} + f(\bar{u}), & 0 < t \leq T, -\infty < x < \bar{h}(t), \\
\bar{u}_t(\bar{h}(t)) \geq 0, & 0 < t \leq T, \\
\bar{h}'(t) \geq \mu \int_{-\infty}^{\bar{h}(t)} J(x - y) \bar{u}_t(t, x) dy dx, & 0 < t \leq T, \\
\bar{h}(0) \geq h_0, \ \bar{u}(0, x) \geq u_0(x), \ x \in (-\infty, h_0]; \ \bar{u}(0, x) > 0, \ x < \bar{h}(0).
\end{cases}$$

Then the solution $(u, h)$ of (1.5) satisfies

$$u(t, x) \leq \bar{u}(t, x), \ h(t) \leq \bar{h}(t) \text{ for } 0 < t \leq T, -\infty < x \leq \bar{h}(t).$$

**Proof.** We start by showing $\bar{u}(t, x) > 0$ in $(0, T] \times (-\infty, \bar{h}(t))$. The above third inequality and the dominated convergence theorem imply

$$\bar{h}'(0) \geq \mu \int_{-\infty}^{\bar{h}(0)} J(x - y) \bar{u}(0, x) dy dx.$$ 

By (J) and $\bar{u}(0, x) > 0$ in $(-\infty, \bar{h}(0))$, we have $\bar{h}'(0) > 0$. Thus, there exists some $t_0 \in (0, T]$ such that $\bar{h}'(t) > 0$ for $t \in [0, t_0]$. Then we can use Lemma 2.1 to $(\bar{u}, \bar{h})$ over $[0, t_0] \times (-\infty, \bar{h}(t))$, and obtain $\bar{u}(t, x) > 0$ in $(0, t_0] \times (-\infty, \bar{h}(t))$. If $t_0 = T$, then we derive the desired result. Otherwise, we can define

$$t_* = \sup\{s \in (0, T) : \bar{u}(t, x) > 0 \text{ in } (0, s) \times (-\infty, \bar{h}(s))\}.$$ 

We claim that $t_* = T$. If $t_* < T$, by Lemma 2.1 we have $\bar{u}(t, x) > 0$ in $(0, t_*) \times (-\infty, \bar{h}(t))$. Then we can think of $t_*$ as the initial time, and arguing as above obtain $\bar{u}(t, x) > 0$ in $(0, t_1] \times (-\infty, \bar{h}(t))$ with some $t_1 > t_*$, which contradicts the definition of $t_*$. Thus, $t_* = T$, and further the conclusion we wanted follows.

For small $\varepsilon > 0$, let $(u_{\varepsilon}, h_{\varepsilon})$ be the unique solution of (1.5) with $(\mu, h_0, u_0(x))$ replaced by $(\mu_{\varepsilon}, h_{0\varepsilon}, u_{0\varepsilon}(x))$, where

$$\mu_{\varepsilon} = \mu(1 - \varepsilon), \ h_{0\varepsilon} = h_0 - \varepsilon, \ u_{0\varepsilon} = (1 - \varepsilon)u_0.$$ 

Clearly,

$$u_{0\varepsilon} \to u_0 \text{ in } L^\infty((-\infty, h_{0\varepsilon}]) \cap L^1((-\infty, h_{0\varepsilon}]) \text{ as } \varepsilon \to 0.$$
We claim $\bar{h}(t) \geq h_\varepsilon(t)$ for $t \in [0,T]$. Arguing indirectly, due to $\bar{h}(0) \geq h_0 > h_0 - \varepsilon = h_\varepsilon(0)$ and the continuity, there exists the largest $t_0 \in (0,T)$ such that $\bar{h}(t_0) = h_\varepsilon(t_0)$ and $\bar{h}(t) > h_\varepsilon(t)$ for all $t \in [0,t_0)$. Then $\bar{h}'(t_0) \leq h'_\varepsilon(t_0)$. Since $\bar{u}(t,x) > 0$ in $(0,T) \times (-\infty, \bar{h}(t))$, we have

$$
\begin{align*}
\frac{d}{dt} \bar{u}(t, x) &\geq d \int_{-\infty}^{h_\varepsilon(t)} J(x - y) \bar{u}(t, y) dy - d\bar{u} + f(\bar{u}), & 0 < t \leq t_0, \quad -\infty < x < h_\varepsilon(t), \\
\bar{u}(t, h_\varepsilon(t)) &\geq 0, & 0 < t \leq t_0, \\
\bar{u}(0, x) &> u_{0\varepsilon}(x), & -\infty < x \leq h_{0\varepsilon}.
\end{align*}
$$

By Lemma 2.1, we have $\bar{u}(t, x) > u_\varepsilon(t, x)$ for $(t, x) \in (0, t_0) \times (-\infty, h_\varepsilon(t))$. Thus,

$$
0 \geq \bar{h}'(t_0) - h'_\varepsilon(t_0) = \mu \int_{-\infty}^{h_\varepsilon(t_0)} \int_{h_\varepsilon(t_0)} \cdots > 0,
$$

which clearly implies that our claim is true. Then using Lemma 2.1 again, we directly derive $\bar{u}(t, x) > u_\varepsilon(t, x)$ in $(0,T) \times (-\infty, h_\varepsilon(t))$. Note that $(u_\varepsilon, h_\varepsilon)$ depends on parameter $\varepsilon$ continuously. Letting $\varepsilon \to 0$ leads to the desired conclusions. The proof is thus complete. \hfill \square

### 3. Proofs of Theorems 1.1 and 1.2

This section is devoted to the proofs of Theorems 1.1 and 1.2.

The proof of Theorem 1.1 will be carried out by four steps.

**Proof of Theorem 1.1.** **Step 1:** The construction of a sequence of free boundary problems defined in bounded domain. For any given $l < h_0$, we consider problem

$$
\begin{align*}
\begin{cases}
\frac{d}{dt} u(t, x) = d \int_{l}^{h(t)} J(x - y) u(t, y) dy - du + f(u), & t > 0, \quad l \leq x < h(t), \\
u(t, x) = 0, & t > 0, \quad x \geq h(t) \\
h(t) &\to \infty \quad t > 0, \\
h'(t) &\to \mu \int_{l}^{h(t)} \cdots > 0, \\
h(0) = h_0, \quad u(0, x) = u_0(x), & x \in [l, h_0].
\end{cases}
\end{align*}
$$

(3.1)

Define $\bar{u}(t, x) = u(t, x + l)$. Then (3.1) is equivalent to problem

$$
\begin{align*}
\begin{cases}
\frac{d}{dt} \bar{u}(t, x) = d \int_{0}^{h(t) - l} J(x - y) \bar{u}(t, y) dy - d\bar{u} + f(\bar{u}), & t > 0, \quad 0 \leq x < h(t) - l, \\
\bar{u}(t, h(t) - l) = 0, & t > 0, \\
h'(t) &\to \mu \int_{0}^{h(t) - l} \cdots > 0, \\
h(0) = h_0, \quad \bar{u}(0, x) = u_0(x + l), & x \in [0, h_0 - l].
\end{cases}
\end{align*}
$$

(3.2)

By [32, Theorem 3.1], for any $l < h_0$, problem (3.2) has a unique solution $(\bar{u}_l, h_l(t) - l)$, and $(\bar{u}_l(t, x - l), h_l(t))$ solves (3.1). Hence, problem (3.1) has a unique solution $(u_l, h_l)$ defined for all $t \geq 0$. Additionally,
$u_l \in C([0, \infty) \times [0, h_l(t)])$, $h_l \in C^1([0, \infty))$, and $0 < u_l(t, x) \leq A$ for $(t, x) \in [0, \infty) \times [0, h_l(t))$. Thus, this step is complete.

**Step 2:** The monotonicity of $(u_i, h_i)$ on $l$. For any $l_1 < l_2 < h_0$, denote the corresponding solution of (3.1) by $(u_i, h_i)$ for $i = 1, 2$, respectively. Then $(u_1, h_1)$ satisfies

$$
\begin{align*}
\partial_t u_1 & \geq d \int_{l_2}^{h_1(t)} J(x - y)u_1(t, y)dy - du_1 + f(u_1), \quad t > 0, \quad l_2 \leq x < h_1(t), \\
u_1(t, h_1(t)) & = 0, \quad t > 0, \\
h_1'(t) & \geq \mu \int_{l_2}^{h_1(t)} \int_{l_2}^{h_1(t)} J(x - y)u_1(t, x)dydx, \quad t > 0, \\
h(0) & = h_0, \quad u_1(0, x) = u_0(x), \quad x \in [l_2, h_0].
\end{align*}
$$

It then follows from a comparison argument that $h_1(t) > h_2(t)$ and $u_1(t, x) > u_2(t, x)$ for $t > 0$ and $x \in [l_2, h_2(t)]$. The step 2 is thus finished.

**Step 3:** The existence of solution to (1.5). For any $l < h_0$, let $(u_l, h_l)$ be the unique global solution of (3.1). Consider the following problem

$$
\begin{align*}
U_l & = d \int_{R} J(x - y)U(t, y)dy - dU + f(U), \quad t > 0, \quad x \in \mathbb{R}, \\
U(0, x) & = u_0(x), \quad x \in (-\infty, h_0]; \quad U(0, x) \equiv 0, \quad x \in (h_0, \infty).
\end{align*}
$$

Notice that $U(0, x) \in C(\mathbb{R}) \cap L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$. It then follows from the Cauchy–Lipschitz theorem that problem (3.3) has a unique solution $U \in C^1([0, \infty), C(\mathbb{R}) \cap L^\infty(\mathbb{R}) \cap L^1(\mathbb{R}))$. By a comparison consideration, we see that $u_l(t, x) \leq U(t, x)$ for $(t, x) \in [0, \infty) \times [l, h_l(t)]$ and $l < h_0$. Hence

$$
\begin{align*}
h_l'(t) & = \mu \int_{l}^{h_l(t)} \int_{l}^{h_l(t)} J(x - y)u_l(t, x)dydx \\
& \leq \mu \int_{l}^{h_l(t)} \int_{l}^{h_l(t)} J(x - y)U(t, x)dydx \\
& \leq \mu \|U(t, \cdot)\|_{L^1(\mathbb{R})}.
\end{align*}
$$

Thus, for any $0 < T < \infty$,

$$
h_t(t) \leq h_0 + T \mu \max_{t \in [0, T]} \|U(t, \cdot)\|_{L^1(\mathbb{R})}, \quad t \in [0, T].
$$

Then we can define $h_\infty(t) = \lim_{l \to -\infty} h_l(t)$ and $u_\infty(t, x) = \lim_{l \to -\infty} u_l(t, x)$ for $t \geq 0$ and $x \in (-\infty, h_\infty(t))$. Clearly, $h_0 \leq h_\infty(t) \leq h_0 + T \max_{t \in [0, T]} \|U(t, \cdot)\|_{L^1(\mathbb{R})}$ and $0 < u_\infty(t, x) \leq U(t, x) \leq A$. By the equation of $h_l'$, we have that for $t \in (0, T]$,

$$
h_l(t) - h_0 = \mu \int_{0}^{h_l(t)} \int_{l}^{h_l(t)} J(x - y)u_l(t, x)dydx.\tau.
$$
By the dominated convergence theorem, letting \( l \to -\infty \) we have
\[
\begin{align*}
    h_\infty(t) - h_0 &= \mu \int_0^t \int_{-\infty}^{h_\infty(\tau)} \int_0^l J(x-y \Delta) u_\infty(\tau, x)dydx, \quad t \in (0, T],
\end{align*}
\]
which yields
\[
\begin{align*}
    h'_\infty(t) &= \mu \int_{-\infty}^{h_\infty(t)} \int_0^l J(x-y \Delta) u_\infty(t, x)dy, \quad t \in (0, T].
\end{align*}
\]
On the other hand, for any \((t, x) \in (0, T] \times (-\infty, h_\infty(t))\), there is a \( l_1 \) such that \((t, x) \in (0, T] \times (l, h_1(t))\) for \( l \leq l_1 \). If \( x \leq h_0 \), for any \( l \leq l_1 \) we have
\[
\begin{align*}
    u(t, x) - u_0(x) &= d \int_0^t \int_0^{h_1(\tau)} \int_0^l J(x-y \Delta) u_\tau(y, \tau)dyd\tau - d \int_0^t \int_0^{h_1(\tau)} \int_0^l f(u_\tau(\tau, x))d\tau.
\end{align*}
\]
Using the dominated convergence theorem again, we have
\[
\begin{align*}
    u_\infty(t, x) - u_0(x) &= d \int_0^t \int_{-\infty}^{h_\infty(\tau)} \int_0^l J(x-y \Delta) u_\infty(\tau, y)dy - d \int_0^t \int_{-\infty}^{h_\infty(\tau)} \int_0^l f(u_\infty(\tau, x))d\tau.
\end{align*}
\]
Differentiating the above equation by \( t \) leads to
\[
\begin{align*}
    \partial_t u_\infty(t, x) &= d \int_{-\infty}^{h_\infty(t)} J(x-y \Delta) u_\infty(t, y)dy - d \int_{-\infty}^{h_\infty(t)} \int_0^l f(u_\infty(t, x))d\tau.
\end{align*}
\]
Therefore, \((u_\infty, h_\infty)\) satisfies the first equation of (1.5) for \((t, x) \in (0, T] \times (-\infty, h_0]\). When \( x \in (h_0, h_1(t))\) for \( l \leq l_1 \), there exists a \( s \in (0, t) \) such that \( h_1(s) = x < h_1(t) \) for any \( l < l_1 \). Thus, for any \( l < l_1 \), we have
\[
\begin{align*}
    u(t, x) - u(s, x) &= d \int_0^t \int_{h_1(s)}^{h_1(\tau)} \int_0^l J(x-y \Delta) u_\tau(y, \tau)dy - d \int_0^t \int_{h_1(s)}^{h_1(\tau)} \int_0^l f(u_\tau(\tau, x))d\tau.
\end{align*}
\]
Similarly to the above, we obtain
\[
\begin{align*}
    \partial_t u_\infty(t, x) &= d \int_{-\infty}^{h_\infty(t)} J(x-y \Delta) u_\infty(t, y)dy - d \int_{-\infty}^{h_\infty(t)} \int_0^l f(u_\infty(t, x))d\tau.
\end{align*}
\]
Thus, \((u_\infty, h_\infty)\) satisfies the first equation of (1.5) for \((t, x) \in (0, T] \times (-\infty, h_\infty(t))\). Moreover, it is easy to see that \( h_\infty \) is increasing in \( t \in [0, T] \) and \( h_\infty \in C^1([0, T]) \). Define
\[
\begin{align*}
    \tilde{u}_0(x) &= \begin{cases} 
    0, & x > h_0, \\
    u_0(x), & x \leq h_0,
    \end{cases} \\
    t_x &= \begin{cases} 
    t_x, & \text{if } x \in [h_0, h_\infty(T)), \\
    h_\infty(t_x, h_\infty), & \text{if } x = h_\infty(t_x, h_\infty), \\
    0 & \text{if } x \leq h_0.
    \end{cases}
\end{align*}
\]
Obviously, \( t_x \) is continuous in \( x < h_\infty(T) \). By the above analysis, for any \( x \in (h_0, h_\infty(T)) \), we see \( u(\cdot, x), \tilde{u}_t(\cdot, x) \in L^\infty((t_x, T)) \). Thus, we can define \( u_\infty(t_x, x) = \lim_{t \to t_x} u_\infty(t, x) \). Next we show \( u_\infty(t_x, x) = 0 \)
for \( x \in (h_0, h_\infty(T)) \), which implies \( u_\infty(t, h_\infty(t)) = 0 \) for \( t \in (0, T) \). Obviously, for any \( t \in (t_\times, T) \)

\[
    u_\infty(t, x) - u_\infty(t_\times, x) = d \int_{t_\times}^{t} \left( \int_{-\infty}^{h_\infty(\tau)} J(x-y)u_\infty(\tau, y)dy - du_\infty(\tau, x) + f(u_\infty(\tau, x)) \right) d\tau. \tag{3.4}
\]

Moreover, since \( x \in (h_0, h_\infty(T)) \), there is a \( l_2 \) such that \( x \in (h_0, h_l(T)) \) for any \( l < l_2 \). Thus, for such \( l \) we can find a \( t_l \in (0, T) \) such that \( h_l(t_l) = x \). Since \( h_l \) is increasing in \( l \), \( t_l \) is decreasing in \( l \). Thus, we can define \( t = \lim_{l \to -\infty} t_l \). Clearly, \( t \geq t_\times \). If \( t > t_\times \), one can choose a \( t_l \in (t_\times, t) \). Therefore, there is a large \( L > 0 \) such that \( t_l > t_\times \) for any \( l < -L \). Furthermore, for such \( l \), we have that \( 0 = u_l(t_1, x) - u_\infty(t_1, x) > 0 \) as \( l \to -\infty \). This contradiction implies that \( \lim_{l \to -\infty} t_l = t_\times \).

For some \( t \) close to \( T \), we have

\[
    u_l(t, x) = u_l(t, x) - u_l(t_1, x)
    = d \int_{t_1}^{t} \left( \int_{-\infty}^{h_l(\tau)} J(x-y)u_l(\tau, y)dy - du_l(\tau, x) + f(u_l(\tau, x)) \right) d\tau.
\]

By the dominated convergence theorem again, we see

\[
    u_\infty(t, x) = d \int_{t_\times}^{T} \left( \int_{-\infty}^{h_\infty(\tau)} J(x-y)u_\infty(\tau, y)dy - du_\infty(\tau, x) + f(u_\infty(\tau, x)) \right) d\tau,
\]

which, compared with (3.4), immediately yields \( u_\infty(t, h_\infty(t)) = 0 \) for \( t \in (0, T) \). Thanks to (1.5) and \( u_\infty(t, h_\infty(t)) = 0 \) for \( t \in (0, T) \), we have that for any \( x \in (h_0, h_\infty(T)) \),

\[
    u_\infty(T, x) = d \int_{t_\times}^{T} \left( \int_{-\infty}^{h_\infty(\tau)} J(x-y)u_\infty(\tau, y)dy - du_\infty(\tau, x) + f(u_\infty(\tau, x)) \right) d\tau
    \leq d \left( A + \max_{u \in [0, A]} |f'(u)| \right) (T - t_\times) \to 0 \text{ as } x \to h_\infty(T),
\]

which indicates that \( \lim_{x \to h_\infty(T)} u_\infty(T, x) = 0 \). So we can let \( u_\infty(T, h_\infty(T)) = 0 \). We now show that \( u_\infty(t, x) \) is continuous in \([0, T] \times (-\infty, h_\infty(t))\) with \( u_\infty(t, x) := 0 \) for \( t \in [0, T] \) and \( x = h_\infty(t) \). By our previous analysis, we have

\[
    \begin{cases}
    \partial_t u_\infty = -du_\infty + f(u_\infty) + F(t, x), & t_\times < t \leq T, \ x \in (-\infty, h_\infty(T)), \\
    u_\infty(t_\times, x) = u_0(x), & x \in (-\infty, h_0); \ u_\infty(t_\times, x) \equiv 0, & x \in (h_0, h_\infty(T)),
    \end{cases}
\]

where

\[
    F(t, x) = d \int_{-\infty}^{h_\infty(t)} J(x-y)u_\infty(t, y)dy.
\]

Since \( J \) satisfies (J), \( h_\infty(t) \) and \( u_\infty(t, x) \) are continuous in \( t \in [0, T] \), \( F(t, x) \) is continuous in \([0, T] \times (-\infty, h_\infty(t))\). Then it follows from the fundamental theory of ODEs that \( u_\infty(t, x) \) is continuous in \([0, T] \times (-\infty, h_\infty(t))\).

Therefore, \((u_\infty, h_\infty)\) solves (1.1). Moreover, for any \( T > 0 \), \( h_\infty \in C^1([0, T]) \), \( u_\infty \in C([0, T] \times (-\infty, h_\infty(t))] \) and \( 0 < u_\infty(t, x) \leq A \) in \((t, x) \in (0, T] \times (-\infty, h_\infty(t))) \).
**Step 4:** The uniqueness of solution to (1.5). Now we show the uniqueness of solution to problem (1.5). To this aim, we first give some estimates for the solution of (1.5). Let \((u, h)\) and \(U\) be a solution of (1.5) and (3.3), respectively. By a comparison consideration, we have \(u(t, x) \leq U(t, x)\) with \(t \geq 0\) and \(x \leq h(t)\). As before, we have
\[
h(t) \leq h_0 + T \mu \max_{t \in [0, T]} \|U(t, \cdot)\|_{L^1(\mathbb{R})}\text{ for any } T \in (0, \infty).
\]
Furthermore, by (J) there exist small \(\varepsilon_0\) and \(\delta_0\) such that \(J(x) \geq \delta_0\) for \(|x| \leq \varepsilon_0\). Thus, there is small \(T_1\) depending only on initial data such that when \(T \leq T_1\), one has
\[
h(t) \leq h_0 + \frac{\varepsilon_0}{4}\text{ for } t \in [0, T].
\]
Noticing that \((u, h)\) solves (1.5) and letting \(L(A) = \max_{x \in [0, A]} |f'(u)|\), we have
\[
\begin{align*}
\mu &\int_{-\infty}^{h(t)} \int_{-\infty}^{h(t)} J(x - y) u(t, x) dy dx \geq \mu \int_{h(t) - \frac{\varepsilon_0}{4}}^{h(t)} \int_{h(t)}^{h(t) + \frac{\varepsilon_0}{4}} J(x - y) u(t, x) dy dx \\
&\geq \mu \int_{h_0 - \frac{\varepsilon_0 h_0}{4}}^{h_0} \int_{h_0}^{h_0 + \frac{\varepsilon_0}{4}} J(x - y) u(t, x) dy dx \\
&\geq \frac{1}{4} \varepsilon_0 \delta_0 \mu e^{-(d+L(A))T_1} \int_{h_0 - \frac{\varepsilon_0}{4}}^{h_0} u_0(x) dx
\end{align*}
\]
which arrives at
\[
u(t, x) \geq e^{-(d+L(A))T} u_0(x),\quad (t, x) \in (0, T] \times (-\infty, h_0].
\]
Combining the above estimates with the third equation of (1.5), we have
\[
\mu \int_{-\infty}^{h(t)} \int_{-\infty}^{h(t)} J(x - y) u(t, x) dy dx \geq \mu \int_{h(t) - \frac{\varepsilon_0}{4}}^{h(t)} \int_{h(t)}^{h(t) + \frac{\varepsilon_0}{4}} J(x - y) u(t, x) dy dx \\
\geq \mu \int_{h_0 - \frac{\varepsilon_0 h_0}{4}}^{h_0} \int_{h_0}^{h_0 + \frac{\varepsilon_0}{4}} J(x - y) u(t, x) dy dx \\
\geq \frac{1}{4} \varepsilon_0 \delta_0 \mu e^{-(d+L(A))T_1} \int_{h_0 - \frac{\varepsilon_0}{4}}^{h_0} u_0(x) dx
\]
\[
:= M.
\]
Therefore, for \(T \in (0, T_1]\), we have
\[
h(t) \leq h_0 + \frac{\varepsilon_0}{4}\text{ and } h'(t) \geq M,\quad t \in [0, T].\tag{3.5}
\]
Now we verify the uniqueness of solution of (1.1). Let \((u_1, h_1)\) and \((u_2, h_2)\) be the solution of (1.1) with \(h_i \in C^1([0, T]), u_i \in C([-\infty, h_i(t)])\) and \(0 < u_i(t, x) \leq A\) in \((t, x) \in (0, T] \times (-\infty, h_i(t))\) for any \(T > 0\) and \(i = 1, 2\). In the following, we first show that there is a small \(T_2 < T_1\), which relies only on initial data, such that \(h_1(t) \equiv h_2(t)\) for \(t \in [0, T_2]\). Thus, we can define
\[
T_* = \sup\{T > 0 : h_1(t) \equiv h_2(t) \text{ for } t \in [0, T]\}\]
Clearly, \(T_2 \leq T_*\). On the other hand, using Lemma 2.1 we have \(u_1(t, x) \equiv u_2(t, x)\) for \(t \in [0, T_2]\) and \((0, T_1)\). If \(T_* = \infty\), then the uniqueness is obtained. Assume that \(T_* < \infty\). By continuity and Lemma 2.1, we have \(h_1(T_*) = h_2(T_*)\) and \(u_1(T_*, x) \equiv u_2(T_*, x)\) for \(x \in (-\infty, h_1(T_*))\). Thus, we can think of \(T_*\) as the initial time, and then argue as above to derive that there is a \(\delta > 0\) such that \(h_1(t) \equiv h_2(t)\) for \(t \in [T_*, T_* + \delta]\), which obviously contradicts the definition of \(T_*\). Then the uniqueness is obtained.
Therefore, to our purpose, it remains to show that there is a small $T_2 < T_1$ depending only on initial data such that $h_1(t) \equiv h_2(t)$ for $t \in [0, T_2]$. Recall that $(u_1, h_1)$ and $(u_2, h_2)$ satisfy (1.5). Then for $t \in (0, T]$, we have

$$|h_1(t) - h_2(t)| \leq \mu \int_0^t \int_{-\infty}^{h_1(\tau)} \int_{-\infty}^{h_2(\tau)} J(x - y)u_1(\tau, x)dydx - \int_{-\infty}^{h_2(\tau)} J(x - y)u_2(\tau, x)dydx + \mu \int_0^t \int_{-\infty}^{h_1(\tau)} \int_{-\infty}^{h_2(\tau)} J(x - y)u_1(\tau, x)dydx |d\tau|

\leq \mu \int_{-\infty}^{h_2(\tau)} \int_{-\infty}^{h_1(\tau)} J(x - y)u_1(\tau, x) - u_2(\tau, x)dydx + \mu \int_0^t \int_{-\infty}^{h_1(\tau)} \int_{-\infty}^{h_2(\tau)} J(x - y)u_1(\tau, x)dydx |d\tau|

\leq \mu T \max_{t \in [0, T]} \|u_1(t, \cdot) - u_2(t, \cdot)\|_{L^1(\mathbb{R})} + 2\mu TA \|h_1(t) - h_2(t)\|_{C([0, T])}.

Hence, choosing $T$ sufficiently small arrives at

$$\|h_1(t) - h_2(t)\|_{C([0, T])} \leq CT \max_{t \in [0, T]} \|u_1(t, \cdot) - u_2(t, \cdot)\|_{L^1(\mathbb{R})}.$$ (3.6)

Now we estimate $\max_{t \in [0, T]} \|u_1(t, \cdot) - u_2(t, \cdot)\|_{L^1(\mathbb{R})}$. For $t \in (0, T]$, we assume that $h_1(t) \geq h_2(t)$. Then

$$\int_{\mathbb{R}} |u_1(t, x) - u_2(t, x)|dx = \left\{ \int_{-\infty}^{h_2(t)} \int_{h_2(t)}^{h_1(t)} + \int_{-\infty}^{h_2(t)} \int_{h_1(t)}^{\infty} \right\} |u_1(t, x) - u_2(t, x)|dx

\leq \int_{-\infty}^{h_2(t)} |u_1(t, x) - u_2(t, x)|dx + 2A \|h_1(t) - h_2(t)\|_{C([0, T])}

\leq \int_{-\infty}^{h_2(t)} |u_1(t, x) - u_2(t, x)|dx

+ \int_{h_0}^{h_2(t)} |u_1(t, x) - u_2(t, x)|dx + 2A \|h_1(t) - h_2(t)\|_{C([0, T])}

= I_1 + I_2 + 2A \|h_1(t) - h_2(t)\|_{C([0, T])}

with

$$I_1 := \int_{-\infty}^{h_0} |u_1(t, x) - u_2(t, x)|dx \quad \text{and} \quad I_2 := \int_{h_0}^{h_2(t)} |u_1(t, x) - u_2(t, x)|dx.$$
(i) The estimate of $I_1$. Recalling the equation satisfied by $u_1$ and $u_2$, we have

$$I_1 = \int_{-\infty}^{t} |u_1(t, x) - u_2(t, x)| dx$$

$$\leq \int_{-\infty}^{h_0} d \int_{0}^{h_1(\tau)} J(x-y)u_1(\tau, y)dy d\tau - d \int_{0}^{h_2(\tau)} J(x-y)u_2(\tau, y)dy d\tau$$

$$+ d \int_{0}^{h_0} \int_{h_1(\tau)}^{h_2(\tau)} |u_1(\tau, x) - u_2(\tau, x)| dx d\tau + \int_{0}^{h_0} |f(u_1(\tau, x)) - f(u_2(\tau, x))| dx d\tau$$

$$\leq d \int_{-\infty}^{h_0} \int_{h_1(\tau)}^{h_2(\tau)} J(x-y)u_1(\tau, y)dy d\tau$$

$$+ d \int_{-\infty}^{h_0} \int_{h_1(\tau)}^{h_2(\tau)} |u_1(\tau, x) - u_2(\tau, x)| dx d\tau + \int_{0}^{h_0} |f(u_1(\tau, x)) - f(u_2(\tau, x))| dx d\tau$$

$$\leq d A T \|h_1(t) - h_2(t)\|_{C([0, T])} + (2d + L(A)) T \max_{t \in [0, T]} \|u_1(t, \cdot) - u_2(t, \cdot)\|_{L^1(\mathbb{R})}.$$

(ii) The estimate of $I_2$. Notice $h_1(t) \geq h_2(t)$ and monotonicity of $h_1$. For every $x \in (h_0, h_2(t))$, from (3.5) there are the unique $t_1$ and $t_2 \in (0, T)$ such that $h_1(t_1) = h_2(t_2) = x$.

$$I_2 = \int_{h_0}^{h_2(t)} |u_1(t, x) - u_2(t, x)| dx$$

$$\leq \int_{h_0}^{h_2(t)} d \int_{t_1}^{t} J(x-y)u_1(\tau, y)dy d\tau - \int_{t_2}^{h_2(t)} J(x-y)u_2(\tau, y)dy d\tau$$

$$+ d \int_{t_2}^{h_2(t)} \int_{h_1(\tau)}^{h_2(\tau)} |u_1(\tau, x) - u_2(\tau, x)| dx d\tau + \int_{h_0}^{h_2(t)} |f(u_1(\tau, x)) - f(u_2(\tau, x))| dx d\tau$$

$$\leq [2d + L(A)] T \max_{t \in [0, T]} \|u_1(t, \cdot) - u_2(t, \cdot)\|_{L^1(\mathbb{R})}$$

$$+ \left[ \frac{2d \varepsilon_0 A + L(A) \varepsilon_0 + dT \varepsilon_0 A M}{4M} \right] \|h_1(t) - h_2(t)\|_{C([0, T])}.$$

Therefore, letting $T$ small enough leads to

$$\max_{t \in [0, T]} \|u_1(t, \cdot) - u_2(t, \cdot)\|_{L^1(\mathbb{R})} \leq C_1 \|h_1(t) - h_2(t)\|_{C([0, T])},$$

where $C_1$ depends only on initial data. Then substituting the above inequality into (3.6) and choosing $T$ appropriately small, we have

$$\|h_1(t) - h_2(t)\|_{C([0, T])} \leq \frac{1}{2} \|h_1(t) - h_2(t)\|_{C([0, T])},$$
which clearly implies $h_1(t) = h_2(t)$ for every $t \in [0, T]$. So the uniqueness is proved. The properties of solution $(u, h)$ clearly follow from the above arguments. Therefore, we complete the proof of Theorem 1.1. □

Next we prove Theorem 1.2 by using some properly upper and lower solutions as well as some comparison arguments.

**Proof of Theorem 1.2.** Consider problem

$$\bar{u}_t = f(\bar{u}); \quad \bar{u}(0) = \|u_0\|_{L^\infty((-\infty, h_0))}.$$  

Clearly, $\lim_{t \to \infty} \bar{u}(t) = u^*$. By Lemma 2.1, we have $u(t, x) \leq \bar{u}(t)$ for $t \geq 0$ and $x \leq h(t)$, which implies

$$\limsup_{t \to \infty} u(t, x) \leq u^* \text{ uniformly in } \mathbb{R}. \quad (3.7)$$

On the other hand, for $l < h_0$, we consider the following problem

\[
\begin{aligned}
& u_t = d \int_{g(t)}^{h(t)} J(x - y)u(t, y)dy - du + f(u), & t > 0, & g(t) < x < h(t), \\
& u(t, g(t)) = u(t, h(t)) = 0, & t > 0, \\
& h'(t) = \mu \int_{-\infty}^{g(t)h(t)} J(x - y)u(t, x)dydx, & t > 0, \\
& g'(t) = -\mu \int_{g(t)h(t)}^{\infty} J(x - y)u(t, x)dydx, & t > 0, \\
& u(0, x) = u_0(x), & h(0) = -g(0) = h(1) - l, & |x| \leq h(1) - l,
\end{aligned}
\]

where $u_0(x) \in C([-h(1) + l, h(1) - l])$, $u_0(x) > 0 = u_0(-h(1) + l) = u_0(h(1) - l)$ and $u_0(x) \leq u(1, x + l)$ for $|x| \leq h(1) - l$. From [18, Theorem 1.3] and [33, Theorem 1.2], the following results hold:

(a) If $l$ is sufficiently small, then spreading happens for (3.8), i.e., $\lim_{t \to \infty} \frac{h(t)}{t} = -\lim_{t \to \infty} \frac{g(t)}{t} = \infty$ and $\lim_{t \to \infty} u(t, x) = u^*$ locally uniformly in $\mathbb{R}$.

(b) When spreading happens,

\[
\begin{aligned}
& \lim_{t \to \infty} \frac{h(t)}{t} = \lim_{t \to \infty} \frac{-g(t)}{t} = c_0 \text{ and } \lim_{t \to \infty} \max_{|x| \leq ct} |u(t, x) - u^*| = 0 \text{ for any } c \in [0, c_0] \text{ if } (J1) \text{ holds}, \\
& \lim_{t \to \infty} \frac{h(t)}{t} = \lim_{t \to \infty} \frac{-g(t)}{t} = \infty \text{ and } \lim_{t \to \infty} \max_{|x| \leq ct} |u(t, x) - u^*| = 0 \text{ for any } c \geq 0 \text{ if } (J1) \text{ is violated},
\end{aligned}
\]

where $c_0$ is uniquely determined by Proposition 1.1.
Define \( u(t, x) = u(t, x - l) \) with \( t \geq 0 \) and \( g(t) + l \leq x \leq h(t) + l \). Obviously,

\[
\begin{align*}
\frac{h(t)}{t} + l & \geq d \int_{g(t)}^{h(t)} J(x - y)u(t, y)dy - du(t + 1, x) \\
& \quad + f(u(t + 1, x)), \\
\frac{u(t, g(t) + l)}{h(t) + l} & > 0, \quad u(t, h(t) + l) = 0,
\end{align*}
\]

with \( t > 0 \), \( g(t) + l < x < h(t) + l \).

Then from a comparison argument, it follows that

\[
\begin{align*}
\left\{ \begin{array}{ll}
\frac{h(t)}{t} + l & = \mu \int_{g(t)}^{h(t)} J(x - y)u(t, y)dydx, \\
\frac{g(t)}{t} + l - l & = -\mu \int_{g(t)}^{h(t)} J(x - y)u(t, x)dydx,
\end{array} \right.
\]

with \( t > 0 \), \( g(t) + l < x < h(t) + l \).

Since \( l < h_0 \) and \( g(t) \) is decreasing in \([0, \infty)\), we obtain \( h(t + 1) > g(t) \) for \( t \geq 0 \). Thus, \((u(t + 1, x), h(t + 1))\) satisfies

\[
\begin{align*}
\frac{h(t)}{t} + l & \geq d \int_{g(t)}^{h(t)} J(x - y)u(t + 1, y)dy - du(t + 1, x) \\
& \quad + f(u(t + 1, x)), \\
\frac{u(t, g(t) + l)}{h(t) + l} & > 0, \quad u(t + 1, h(t) + l) = 0,
\end{align*}
\]

with \( t > 0 \), \( g(t) + l < x < h(t) + l \).

Then from a comparison argument, it follows that \( u(t + 1, x) \geq u(t, x) = u(t, x - l) \) and \( h(t + 1) \geq h(t) + l \) with \( t \geq 0 \) and \( g(t) + l \leq x \leq h(t) + l \). Thus, recalling the above results (a) and (b), we have the followings:

(i) If (J1) holds, then

\[
\liminf_{t \to \infty} \frac{h(t)}{t} \geq c_0 \quad \text{and} \quad \liminf_{t \to \infty} u(t, x) \geq u^* \quad \text{uniformly in } [-ct, ct] \quad \text{for any } c \in [0, c_0].
\]

(ii) If (J1) does not hold, then

\[
\lim_{t \to \infty} \frac{h(t)}{t} = \infty \quad \text{and} \quad \liminf_{t \to \infty} u(t, x) \geq u^* \quad \text{uniformly in } [-ct, ct] \quad \text{for any } c \geq 0.
\]

Combining these with (3.7), we directly obtain (1.7) and conclusion (2) in Theorem 1.1. We now prove

\[
\limsup_{t \to \infty} \frac{h(t)}{t} \leq c_0. \tag{3.9}
\]

For small \( \varepsilon > 0 \) and \( L > 0 \) to be determined later, define

\[
\tilde{h}(t) = (1 + \varepsilon)c_0 + L \quad \text{and} \quad \tilde{u}(t, x) = (1 + \varepsilon)\phi^0(x - \tilde{h}(t)).
\]
where \((c_0, \phi^{c_0})\) is the unique solution pair of the semi-wave problem (1.6). Next we show that there exist suitably \(T > 0\) and \(L > 0\) such that

\[
\begin{cases}
\bar{h}(t) \\
\bar{u}_t \geq d \int_{-\infty}^{\bar{h}(t)} J(x-y)\bar{u}(t,y)dy - d\bar{u} + f(\bar{u}), & t > 0, \quad -\infty < x < \bar{h}(t), \\
u(t, \bar{h}(t)) \geq 0, & t > 0, \\
\bar{h}'(t) \geq \mu \int_{-\infty}^{\bar{h}(t)} J(x-y)\bar{u}(t,x)dy, & t > 0, \\
\bar{u}(0, x) \geq u(T, x), \quad \bar{h}(0) \geq h(T), & x \leq h(T).
\end{cases}
\]

(3.10)

Once it is done, using Lemma 2.2 we derive that \(u(t + T, x) \leq \bar{u}(t, x)\) and \(\bar{h}(t) \geq h(t + T)\) for \(t \geq 0\) and \(-\infty < x \leq h(t + T)\). Thus,

\[
\lim_{t \to \infty} \sup \frac{\bar{h}(t)}{t} \leq \lim_{t \to \infty} \frac{\bar{h}(t) - T}{t} = (1 + \varepsilon)c_0,
\]

which together with the arbitrariness of \(\varepsilon\) yields (3.9).

Now we begin to prove (3.10). Direct computations show

\[
\bar{u}_t = -(1 + \varepsilon)^2 c_0 \phi^{c_0'}(x - \bar{h}(t)) \geq -(1 + \varepsilon)c_0 \phi^{c_0'}(x - \bar{h}(t))
\]

\[
= (1 + \varepsilon) \left( d \int_{-\infty}^{\bar{h}(t)} J(x-y)\phi^{c_0}(y - \bar{h}(t))dy - d\phi^{c_0}(x - \bar{h}(t)) + f(\phi^{c_0}(x - \bar{h}(t))) \right)
\]

\[
\geq d \int_{-\infty}^{\bar{h}(t)} J(x-y)\bar{u}(t,y)dy - d\bar{u} + f(\bar{u}),
\]

which indicates that the first inequality of (3.10) holds. Moreover,

\[
(1 + \varepsilon)\mu \int_{-\infty}^{\bar{h}(t)} \int_{-\infty}^{\bar{h}(t)} J(x-y)\phi^{c_0}(x - \bar{h}(t))dydx = (1 + \varepsilon)\mu \int_{0}^{\infty} \int_{-\infty}^{0} J(x-y)\phi^{c_0}(x)dydx
\]

\[
= (1 + \varepsilon)c_0 = \bar{h}'(t),
\]

which implies that the third one in (3.10) is valid. In view of (3.7) and \(\phi^{c_0}(-\infty) = u^*\), there exist \(T > 0\) and sufficiently large \(L > 0\) such that \(u(T, x) \leq (1 + \frac{\varepsilon}{2})u^* \leq (1 + \varepsilon)\phi^{c_0}(h(T) - L) \leq (1 + \varepsilon)\phi^{c_0}(x - L) = \bar{u}(0, x)\) for \(x \leq h(T)\). So the last two inequalities in (3.10) hold. Since the second one is obvious, we immediately obtain (3.10).

Next we prove (1.8). Note that \(h(t + 1) \geq h(t) + l\) for \(t \geq 0\). From [21, Lemma 4.8], there exists \(C_1 > 0\) such that \(h(t) - c_0t \geq -C_1\) with \(t \geq 0\). Then for our purpose, it suffices to prove that there is \(C_2 > 0\) such that \(h(t) - c_0t \leq C_2\) for \(t \geq 0\). Inspired by [21], for some positive constants \(\theta\) and \(K\), we define

\[
\begin{align*}
\hat{h}(t) &= c_0t + \delta(t), \quad t \geq 0, \\
\hat{u}(t, x) &= (1 + \varepsilon(t))\phi^{c_0}(x - \hat{h}(t)), \quad t \geq 0, \quad x \leq \hat{h}(t),
\end{align*}
\]

where \((c_0, \phi^{c_0})\) is uniquely given by (1.6), and

\[
\varepsilon(t) = (t + \theta)^{-2}, \quad \delta(t) = K - c_0[(t + \theta)^{-1} - \theta^{-1}].
\]
As before, we are going to prove that there exists a $T_1 > 0$ such that the following inequalities hold
\[
\begin{align*}
\dot{u}_t &\geq d \int_{-\infty}^{h(t)} J(x-y) \dot{u}(t,y) dy - d\dot{u} + f(\dot{u}), \quad t > 0, \quad -\infty < x < \hat{h}(t), \\
\dot{u}(t,\hat{h}(t)) &\geq 0, \quad t > 0, \\
\dot{h}'(t) &\geq \mu \int_{-\infty}^{h(t)} J(x-y) \dot{u}(t,x) dx, \quad t > 0, \\
\dot{u}(0,x) &\geq u(T,x), \quad \hat{h}(0) \geq h(T_1), \quad x \leq h(T_1).
\end{align*}
\] (3.11)

Once it is done, from a comparison method and the definition of $\hat{h}(t)$ we immediately obtain that $h(t) - c_0 t \leq C_2$ for some $C_2 > 0$ and $t \geq 0$.

Let’s start by verifying the first inequality in (3.11). Direct computations show
\[
\dot{u}_t = -((1+\varepsilon(t))(c_0 + \delta'(t))(\phi^{c_0})'(x - \hat{h}(t)) + \varepsilon'(t)\phi^{c_0}(x - \hat{h}(t))
\]
\[
= (1+\varepsilon(t)) \left( d \int_{-\infty}^{h(t)} J(x-y) \phi^{c_0}(y - \hat{h}(t)) dy - d\phi^{c_0}(x - \hat{h}(t)) + f(\phi^{c_0}(x - \hat{h}(t))) \right)
\]
\[
\quad - (1+\varepsilon(t))\delta'(t)(\phi^{c_0})'(x - \hat{h}(t)) + \varepsilon'(t)\phi^{c_0}(x - \hat{h}(t))
\]
\[
\geq d \int_{-\infty}^{h(t)} J(x-y) \dot{u}(t,y) dy - d\dot{u} + f(\dot{u}) + \alpha(t,x),
\]
where
\[
\alpha(t,x) = (1+\varepsilon(t))f(\phi^{c_0}(x - \hat{h}(t))) - f((1+\varepsilon(t))\phi^{c_0}(x - \hat{h}(t)))
\]
\[
\quad - (1+\varepsilon(t))\delta'(t)(\phi^{c_0})'(x - \hat{h}(t)) + \varepsilon'(t)\phi^{c_0}(x - \hat{h}(t)).
\]

It is sufficient to show that $\alpha(t,x) \geq 0$ for $t \geq 0$ and $x \in (-\infty, \hat{h}(t))$. Let $C = \max_{[0, 2u^*]} |f''(u)|$. By the Taylor expansion, we have, with $\bar{u} \in [u, u^*],$
\[
(1+\varepsilon)f(u) - f((1+\varepsilon)u) = -f((1+\varepsilon)u^* + (1+\varepsilon)[f'(\bar{u}) - f'((1+\varepsilon)\bar{u})](u-u^*)
\]
\[
\geq -\varepsilon u^* f'(u^*) + o(\varepsilon) - (1+\varepsilon)C\varepsilon u^*(u^*-u) \quad \text{for all } u \in [0, u^*].
\]

Since $\phi^{c_0}(-\infty) = u^*$, for any small $\varepsilon_0 > 0$ there exists $K_1 > 0$ such that $\phi^{c_0}(-K_1) \geq (1-\varepsilon_0)u^*$. So $(1-\varepsilon_0)u^* \leq \phi^{c_0}(x - \hat{h}(t)) \leq u^*$ for $x \in (-\infty, \hat{h}(t) - K_1)$. When $\theta \gg 1$, $K > K_1$ and $0 < \varepsilon_0 \ll 1$, we have
\[
\alpha(t,x) \geq (1+\varepsilon(t))f(\phi^{c_0}(x - \hat{h}(t))) - f((1+\varepsilon(t))\phi^{c_0}(x - \hat{h}(t))) - \frac{2}{(t+\theta)^3} \phi^{c_0}(x - \hat{h}(t))
\]
\[
\geq -\varepsilon(t)u^* f'(u^*) + o(\varepsilon(t)) - (1+\varepsilon(t))C\varepsilon(t)\varepsilon_0(u^*)^2 - \varepsilon(t)\frac{2}{t+\theta} u^*
\]
\[
\geq \varepsilon(t) \left[-u^* f'(u^*) + o(1) - 2C\varepsilon_0(u^*)^2 - 2u^*/\theta \right]
\]
\[
\geq 0 \quad \text{for } x \in (-\infty, \hat{h}(t) - K_1),
\]
and
\[
\alpha(t, x) \geq -(1 + \varepsilon(t))\delta'(t)(\phi^{\varepsilon_0})(x - \hat{h}(t)) - \frac{2}{(t + \theta)^3} \phi^{\varepsilon_0}(x - \hat{h}(t)) \\
\geq c_0 \varepsilon_1 \varepsilon(t) - \frac{2}{(t + \theta)^3} u^* \\
\geq (t + \theta)^{-3}(c_0 \varepsilon_1 \theta - 2u^*) \\
\geq 0 \text{ for } x \in [\hat{h}(t) - K_1, \hat{h}(t)],
\]
where \( \varepsilon_1 = \inf_{x \in [-K_1, 0]} \{-(\phi^{\varepsilon_0})(x)\} > 0 \). Hence, the first inequality of \((3.11)\) holds. The second one is obvious. We next prove the third one of \((3.11)\). Simple calculations yield
\[
\mu \int_{-\infty}^{\hat{h}(t)} \int J(x - y) \hat{u}(t, x) dy dx = \mu(1 + \varepsilon(t)) \int J(x - y) \phi^{\varepsilon_0}(x - \hat{h}(t)) dy dx \\
\leq \mu(1 + \varepsilon(t)) \int_{-\infty}^{\infty} J(x - y) \phi^{\varepsilon_0}(x) dy dx \\
= (1 + \varepsilon(t))c_0 \\
= \hat{h}'(t),
\]
which indicates that the third inequality of \((3.11)\) holds.

Since \( \limsup_{t \to \infty} u(t, x) \leq u^* \) uniformly in \((-\infty, h(t)]\), for \( \theta \) chosen as above, there is \( T_1 > 0 \) such that \( u(T_1, x) \leq (1 + \varepsilon(0)/2)u^* \) for \( x \in (-\infty, h(T_1)] \). Together with \( \phi^{\varepsilon_0}(-\infty) = u^* \), one may choose \( K \) sufficiently large if necessary, such that \( \hat{h}(0) = K > h(T_1) \), and
\[
\hat{u}(0, x) = (1 + \varepsilon(0))\phi^{\varepsilon_0}(x - K) \geq (1 + \varepsilon(0)/2)u^* \geq u(T_1, x) \text{ for } x \in (-\infty, h(T_1)].
\]
Therefore, \((3.11)\) holds. Then in order to complete the proof of Theorem 1.2, it remains to prove conclusions (3) and (4) in Theorem 1.2.

Let’s now prove the conclusions (3) and (4) in Theorem 1.2. It follows from \([21, \text{Theorem 1.3}]\) and \([33, \text{Theorem 1.2}]\) that

(1) if \( J(x) \geq C|x|^{-\gamma} \) for \( \gamma \in (1, 2) \) and \( |x| \gg 1 \), then \( \hat{h}(t), -g(t) \geq C_1 t^{\frac{1}{\gamma-1}} \) for some \( C_1 > 0 \) and \( \liminf_{t \to \infty} u(t, x) \geq u^* \) uniformly in \( |x| \leq o(t^{\frac{1}{\gamma-1}}) \);
(2) if \( J(x) \geq C|x|^{-2} \) for \( |x| \gg 1 \), \( \hat{h}(t), -g(t) \geq C_1 \ln t \) for some \( C_1 > 0 \) and \( \liminf_{t \to \infty} u(t, x) \geq u^* \) uniformly in \( |x| \leq o(t \ln t) \).

Note that \( u(t + 1, x) \geq u(t, x - l) \) and \( h(t + 1) \geq h(t) + l \) with \( t \geq 0 \) and \( g(t) + l \leq x \leq h(t) + l \), which clearly indicates

(1) if \( J(x) \geq C|x|^{-\gamma} \) for \( \gamma \in (1, 2) \) and \( |x| \gg 1 \), then \( \hat{h}(t) \geq C_2 t^{\frac{1}{\gamma-1}} \) for some \( C_2 > 0 \) and \( \liminf_{t \to \infty} u(t, x) \geq u^* \) uniformly in \( |x| \leq o(t^{\frac{1}{\gamma-1}}) \);
(2) if \( J(x) \geq C|x|^{-2} \) for \( |x| \gg 1 \), then \( h(t) \geq C_2 \ln t \) for some \( C_2 > 0 \) and \( \liminf_{t \to \infty} u(t, x) \geq u^* \) uniformly in \( |x| \leq o(t \ln t) \).

Together with \((3.7)\), we obtain the conclusions (3) and (4) in Theorem 1.2.
To complete the proof of Theorem 1.4, it remains to show its conclusion (5). Let \( v \) be the solution of problem
\[
\begin{aligned}
&v_t = d \int_{-\infty}^{\infty} J(x - y)v(t, y)dy - dv, & t > 0, \ x \in \mathbb{R}, \\
v(0, x) = u_0(x), \ x \in (\infty, h_0); \ v(0, x) \equiv 0, \ x \in [h_0, \infty).
\end{aligned}
\]
(3.12)

It is easy to see that \( \|v(t, \cdot)\|_{L^1(\mathbb{R})} = \|u_0(x)\|_{L^\infty((\infty, h_0))} \) for all \( t \geq 0 \). By a comparison argument, we have \( u(t, x) \leq e^{f'(0)t}v(t, x) \) for \( t \geq 0 \) and \( x \leq h(t) \). Thus, for \( t > 0 \), we see
\[
\begin{aligned}
h'(t) &= \mu \int_{-\infty}^{h(t)} \int_{-\infty}^{h(t)} J(x - y)u(t, x)dydx \\
&\leq \mu e^{f'(0)t} \int_{-\infty}^{h(t)} \int_{-\infty}^{h(t)} J(x - y)v(t, x)dydx \\
&\leq \mu e^{f'(0)t} \|u_0(x)\|_{L^\infty((\infty, h_0))},
\end{aligned}
\]
which clearly implies the conclusion (5). The proof is ended. \( \square \)

4. Discussion

In this paper, we study the model (1.5), including the longtime behaviors, spreading speed, and rate of accelerated spreading. Compared to the models in the existing works concerning nonlocal diffusion problem with free boundary, such as [18, 32], our model (1.5) is defined in a unbounded range of spatial variable \( x, (\infty, h(t)) \), which causes some extra difficulties on the discussions of the well-posedness, longtime behaviors as well as spreading speed.

Here we would like mention that spreading always happens for model (1.5), which is very different from the spreading-vanishing dichotomy in [18, 26, 27, 32]. Moreover, similarly to [19, 32], model (1.5) has a finite spreading speed if and only if the condition (J1) holds. However, since the spatial range of model (1.5) is unbounded, we need some more subtly upper solution than those in [21] to obtain the accurate estimates on the rate of accelerated spreading as in [21]. We conjecture that the rate of accelerated spreading of (1.5) may be bigger than those in [21, 23]. This challenging problem will be considered in a future work.

Acknowledgements

The authors would like to thank the anonymous referees for their helpful comments and suggestions.

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(Received: June 16, 2022; revised: July 21, 2022; accepted: July 31, 2022)