Three-dimensional dynamics of four-dimensional topological BF theory with boundary

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Abstract. We consider the four-dimensional (4D) abelian topological BF theory with a planar boundary, following Symanzik’s method. We find the most general boundary conditions compatible with the field equations broken by the boundary. The residual gauge invariance is described by means of two Ward identities which generate a current algebra. We interpret this algebra as canonical commutation relations of fields, which we use to construct a 3D Lagrangian. As a remarkable by-product, we find a (unique) boundary condition which can be read as a duality relation between 3D dynamical variables.
1. Introduction

In this paper, we consider the four-dimensional (4D) BF theory, aiming at the study of the dynamics induced on a planar 3D boundary. The boundary is treated according to Symanzik’s method [1], which strictly observes the quantum field theory (QFT) rules of locality and power counting, together with the idea of separability which is a request concerning the propagators, i.e.

$$\Delta_{\phi_1,\phi_2}(x_1, x_2) = 0, \quad \text{if } x_1 \cdot x_2 < 0,$$

(1.1)

where $\phi_1(x_1)$ and $\phi_2(x_2)$ are a couple of fields of the theory. In a sense, the constraint (1.1) represents an operative definition of a QFT with boundary: the boundary separates the world into two parts in such a way that nothing propagates from one side to the other.

This simple idea has been very fruitful for the analysis of the physics on a planar boundary in several circumstances. The original Symanzik’s motivation was the study of the Casimir effect [1], but later on it has been exploited in topological quantum field theories (TQFTs), which, as is well known, do not have local dynamics but their observables are globally defined and mostly deal with geometrical properties of the manifolds they are built on [2, 3]. A boundary breaks Lorentz invariance and also the topological character of the theory is lost; for this reason, only in the presence of a boundary might a local dynamics appear in TQFTs. This is the case for the 3D Chern–Simons (CS) theory [4–7] which, on the boundary, displays conserved chiral currents, which are the insertions of the fields on the two sides of the boundary obeying a Kač–Moody (KM) algebra whose central charge depends on the CS coupling constant. Similarly, the 3D BF theory presents, on the boundary, an algebraic structure carried by two sets of conserved chiral currents [8, 9]. The BF algebraic structure is more complex than the CS case, but we shall come to this point later.

However, in recent years, it is in the field of condensed matter physics that TQFTs with a boundary has received much attention. This is mainly due to the discovery in the early 1980s of a new state of matter which cannot be described in terms of a symmetry breaking mechanism but in terms of topological order. In contrast to the quantum Hall effect (QHE), where the magnetic field breaks time-reversal ($T$) symmetry, a new class of $T$ invariant systems, called topological insulators (TI), has been predicted [10, 11] and experimentally observed [12] in 3D, leading to the quantum spin Hall effect (QSHE). At the boundary of these systems, one has helical states, namely electrons with opposite spin propagating in opposite directions [13]. The low-energy sector of these materials is well described in terms of the CS model and the BF theory.
Moreover, it is important to study these models in the presence of a boundary in order to analyze the dynamics of the edge states. In fact, it is well known that the edge dynamics of the QHE is successfully described by the abelian CS theory with a boundary both in the integer and in the fractional regimes [14]. In the case of 3D TI, an abelian doubled CS [11], which is equivalent to a 3D BF theory with cosmological constant [8], has been introduced [9, 15]. Moreover, the authors of [15] argued that the abelian 4D BF theory with a boundary could describe some features of TI in 3D, in particular their edge fermionic degrees of freedom.

The paper is organized as follows. In section 2, the bulk 4D BF theory is described. The field equations and the Ward identities describing the residual gauge invariance typical of the axial gauge are derived. In section 3, the boundary is introduced following Symanzik’s approach. The most general boundary action and the corresponding boundary conditions on the fields are written, together with the Ward identities of the theory, modified by the boundary. In section 4, the boundary algebra is computed, which gives rise to a surprising electromagnetic structure on the boundary, which is to be understood from both the geometrical \((x_3 = 0)\) and the field theoretical (mass-shell) point of view. An ‘electric’ scalar and a ‘magnetic’ vector potential are identified, by means of which we write a 3D boundary Lagrangian. A remarkable ‘duality’ relation between the electromagnetic potentials is obtained. All the results are summarized and discussed in detail in section 5, and the propagators of the full theory, including the boundary, are explicitly given in the appendix.

2. The classical theory

In the abelian case, the action of the 4D BF model \([3, 16]\), which describes the interaction between the two-form \(B_{\mu\nu}\) and the gauge field \(A_\mu\), is given by

\[
S_{bf} = \frac{\kappa}{2} \int d^4x e^{\mu\nu\rho\sigma} F_{\mu\nu} B_{\rho\sigma},
\]

where \(F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\). It is well known that the BF theories, in any spacetime dimensions, both in the abelian and the non-abelian case, do not depend on a coupling constant. Here, \(\kappa\) is a constant which we have introduced in order to distinguish the boundary terms from the bulk terms; it can eventually be set equal to 1 at the end of the computation.

The action (2.1) is invariant under the symmetries

\[
\delta^{(1)} A_\mu = -\partial_\mu \theta, \quad \delta^{(1)} B_{\mu\nu} = 0,
\]

and

\[
\delta^{(2)} A_\mu = 0, \quad \delta^{(2)} B_{\mu\nu} = -(\partial_\mu \phi_\nu - \partial_\nu \phi_\mu),
\]

where \(\theta\) and \(\phi_\mu\) are local parameters.

We recall that the action (2.1) is not the most general one compatible with the symmetries (2.2) and (2.3). Indeed, a Maxwell term \(\int d^4x F_{\mu\nu} F^{\mu\nu}\) could be added, coupled to an additional parameter. In this paper, we consider the action (2.1) alone, because we are interested in the 3D dynamics on the edge of a TQFT. We are allowed to do that if we think of the action (2.1) as the abelian limit of the non-abelian 4D BF theory [17], which, like any other TQFT, is protected from the occurrence of non-topological terms by an additional symmetry, called ‘vector supersymmetry’ in [18]. Nevertheless, the non-abelian case, which is much richer from the fields theoretical point of view [19, 20], and the addition of a Maxwell term are interesting extensions, which we shall present elsewhere [21].

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Table 1. Canonical mass dimensions of the quantum fields.

| $A_\mu$ | $B_{\mu
u}$ | $b$ | $d^i$ |
|---------|--------------|-----|-------|
| Dimension | 1 | 2 | 3 | 2 |

Lorentz invariance will be broken with the introduction of a planar boundary. Consequently, a convenient choice for the gauge conditions on the two fields involved is the axial one:

$$A_3 = 0, \quad B_{i3} = 0,$$

where Latin letters run over 0, 1, 2. The axial choice is implemented by adding to the action (2.1) the gauge fixing term

$$S_{gf} = \int d^4x \{ bA_3 + d^i B_{i3} \},$$

where $b$ and $d^i$ are, respectively, the Lagrange multipliers for the fields $A_3$ and $B_{i3}$. As usual, in the abelian case the ghost fields are decoupled from other fields. One of the main differences compared with the non-abelian case is in the structure of the gauge fixing term, which, because of the reducible symmetry (2.3), involves ghosts for ghosts and therefore is highly non-trivial \cite{19,20}.

Summarizing, the classical action is given by

$$\Gamma_c[J_\phi] = \int d^4x (\kappa \epsilon^{ijk}[2 \partial_i A_j B_{k3} + (\partial_i A_3 - \partial_3 A_i) B_{jk}] + bA_3 + d^i B_{i3}$$

$$+ J_{bij}^i B_{ij} + 2J_{B,i3}^i B_{i3} + J_{A,i}^i A_i + J_{A,3}^i A_3 + J_b b + J_{d}^i d^i \},$$

where $J_\phi$ are the external sources coupled to the quantum fields $\Phi$. From (2.6), we obtain the bulk field equations:

$$J_{A,i}^i + \kappa \epsilon^{ijk}[2 \partial_j B_{k3} + \partial_3 B_{jk}] = 0,$$

$$J_{B,i}^i + \kappa \epsilon^{ijk}(\partial_k A_3 - \partial_3 A_k) = 0,$$

$$J^3 + b - \kappa \epsilon^{ijk} \partial_j B_{jk} = 0,$$

$$2J_{B,i3}^i + d^i + 2\kappa \epsilon^{ijk} \partial_j A_k = 0,$$

$$A_3 + J_b = 0,$$

$$B_{i3}^i + J_d^i = 0.$$ (2.7)

It is well known \cite{22} that the axial gauge is not a complete gauge fixing. A residual gauge invariance remains on the plane $x_3 = 0$, which is described by two (one for each gauge symmetry (2.2) and (2.3)) local Ward identities:

$$W(x)\Gamma_c[J_\phi] = \partial_t J_{A,i}^i + \partial_3 J_{A,3}^i + \partial_3 \frac{\delta \Gamma_c}{\delta J_b} = 0,$$

$$W'(x)\Gamma_c[J_\phi] = \partial_{t'} J_{B,i}^i + \partial_3 J_{B,i3}^i + \frac{1}{2} \partial_3 \frac{\delta \Gamma_c}{\delta J_d^i} = 0.$$ (2.8, 2.9)

In table 1, we list the canonical mass dimensions of the fields of the theory.
3. The boundary

To introduce a planar boundary in the theory, we adopt Symanzik’s method [1], which basically consists in writing the most general boundary Lagrangian, according to the general QFT principles of locality and power counting, and then computing, for the modified theory, the propagators on which the constraint of ‘separability’ is imposed. This corresponds to requesting that the propagators between points on opposite sides of the boundary $x = 0$ vanish:

$$\Delta_{\phi_1\phi_2}(x_1, x_2) = 0, \quad \text{if } x_1 \cdot x_2 < 0,$$

where $\phi_1(x_1)$ and $\phi_2(x_2)$ are two generic fields of the theory. No other assumption is required. In particular, as we shall see, no boundary condition is imposed. Rather, we shall find out which are the most general ones compatible with the presence of a separating boundary.

The most general boundary Lagrangian must respect locality, power counting and covariance in the plane $x_3 = 0$. An additional, more subtle, constraint comes from the fact that the field equations (2.7) appear to be first order differential equations. This originates from the BF action (2.1), which contains only first derivatives. Hence, in order to be solved, only one boundary condition on any field is needed, and no more than one is allowed. This obvious consideration translates into the request that the boundary term in the action should not contain differentiated fields (with respect to $x_3$). In the present case, only a term proportional to $\delta(x_3) A^i \partial_3 A_i(x)$ would matter, resulting in a boundary condition on the differentiated fields $\partial_3 A_i(x)|_{x_3=0}$, in addition to the one on the undifferentiated fields $A_i(x)|_{x_3=0}$, which is already present. This would be incompatible with the first-order differential equations (2.7). Hence, such a term will not be considered. Two more remarks on this point: the first concerns the fact that this is a peculiarity of the 4D (or higher) since in the lower dimensional cases (CS and 3D BF) such a term violates the power counting constraint. The second concerns the possible presence in the bulk of a Maxwell term, which, being of second order in the derivatives, would spoil the previous argument, and the only possible term we are talking of, namely $\delta(x_3) A^i \partial_3 A_i(x)$, should be included in that case.

Summarizing, the most general boundary Lagrangian is

$$L_{bd} = \delta(x_3) \left[ a_1 A_i \tilde{B}^i + a_3 m A^i \tilde{A}^i + a_5 b \tilde{J}^i + \frac{a_4}{2} \epsilon^{ijk} \partial_j A_k \tilde{A}^i + \frac{a_3}{2} \delta A^i \right],$$

where $\tilde{B}^i = \epsilon^{ijk} B_{jk}$ (and correspondingly $\tilde{J}_{\tilde{B}^i}^{ji} = \epsilon^{ijk} J_{B^j}^{ki}$), and $a_\alpha, \alpha = 1, \ldots, 5$ are constant parameters to be determined. In order to have all the $a$-parameters massless, $a_2$ has been given an explicit $m$-mass dependence, which will turn out to be important in the following.

The separability condition (3.1) allows for a considerable simplification, by considering only one of the two sides of the boundary. In fact, basically, the constraint (3.1) means that the opposite sides of the boundary are completely decoupled. Hence, we can consider only one side of the boundary, say the ‘+’ side, forgetting about the opposite side, which can be obtained from the ‘+’ side by parity.

Keeping this in mind, the boundary Lagrangian $L_{bd}$ (3.2) modifies the bulk field equations (2.7) as follows:

$$J^i_{\tilde{A}^i} + 2 \kappa \epsilon^{ijk} \partial_j B_{k3} + \alpha \partial_3 \tilde{B}^i = -\delta(x_3) [a_1 \tilde{B}^i + a_3 m \tilde{A}^i + a_5 b^i + a_4 \epsilon^{ijk} (\partial_j A_k)^i + a_3 d^i],$$

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\[\epsilon^{ijk} J_{\hat{B}^i} + \kappa \epsilon^{ijk} (\partial_k A_3 - \partial_3 A_k) = -a_1 \delta(x^3) \epsilon^{ijk} A_3^+, \]
\[J^3 + b - \kappa \partial_i \tilde{B}^i = 0, \quad (3.3)\]
\[2J_{\hat{B}^3} + d^i + 2\kappa \epsilon^{ijk} \partial_j A_k = 0, \quad (3.4)\]
\[A_3 + J_b = -\delta(x^3) a_3, \quad (3.5)\]

where the superscript + denotes the insertions of the fields of the theory on the ‘+’ side of the boundary \(x_3 = 0\).

Consequently, the boundary term \(L_{bd} (3.2)\) breaks the local Ward identities:
\[\partial_i J^i_{A^i} + \partial_3 J^3_{A^3} + \partial_3 b = -\delta(x^3)[a_1 \partial_i \tilde{B}^{i+} + a_2 \partial_i A^{i+} + a_5 \partial_i d^{i+}], \quad (3.4)\]
\[\epsilon^{ijk} \partial_j J_{\hat{B}^k} + \epsilon^{ijk} \partial_j J_{\hat{B}^3} + \frac{1}{4} \partial_3 d^i = -\delta(x^3) a_1 \epsilon^{ijk} \partial_j A_k^+ \quad (3.5)\]

We stress that the breaking terms at the rhs of (3.4) and (3.5) are linear in the quantum fields, and hence a non-renormalization theorem ensures that they are present at the classical level only and do not acquire quantum corrections \([23]\). For this reason, linearly broken symmetries are perfectly allowed by the general principles of QFT.

Postulating that \(b(x_3 = \pm \infty) = d^i (x_3 = \pm \infty) = 0\), the Ward identities (3.4) and (3.5) can be integrated as follows:
\[\int_{-\infty}^{\infty} dx_3 \partial_i J^i_{A^i} = -[a_1 \partial_i \tilde{B}^{i+} + a_2 \partial_i A^{i+} + a_5 \partial_i d^{i+}], \quad (3.6)\]
\[\int_{-\infty}^{\infty} dx_3 \epsilon^{ijk} \partial_j J_{\hat{B}^k} = -a_1 \epsilon^{ijk} \partial_j A_k^+. \quad (3.7)\]

### 3.1. The boundary conditions

When dealing with a boundary, the question arises as to the boundary conditions on the fields and their derivatives. Symanzik’s approach allows us to find out the most general ones, without imposing any additional constraint. In fact, the separability condition (3.1) and the boundary Lagrangian (3.2) can be used to calculate the propagators everywhere, in the bulk and on its boundary. Studying the limit of the propagators on the different sides of the boundary, one can infer which are the possible boundary conditions on the fields.

This program has been carried out for the 4D scalar QFT \([1, 5]\), for the CS theory \([5, 6]\) and for the 3D BF model \([8]\). But it might happen that the explicit calculation of the propagators of the whole theory (bulk and boundary) may become very difficult (this is the case for the 3D Maxwell–Chern–Simons theory with boundary \([24]\), for instance). Nonetheless, even in that case it is possible to find out the boundary conditions and the physics on the boundary (typically the existence of conserved currents and the algebra they form), avoiding the explicit calculation of the propagators.

In this paper, we are mainly interested in the dynamics of the 3D boundary of the 4D BF theory, so we proceed directly to determine which are the possible boundary conditions that characterize the physics on the boundary. The complete propagators of the full theory are given in the appendix.
Table 2. Algebraic solutions of equation (3.11).

| $a_2$ | $a_4$ | $A^+$ | $\tilde{B}^+$ |
|-------|-------|-------|---------------|
| 1     | $\neq 0$ | 0     | $\neq 0$ | $-\frac{a_2 m}{2\kappa} A^+$ |
| 2     | 0     | 0     | $\neq 0$ | 0 |
| 3     | 0     | $\neq 0$ | $\neq 0$ | $-\frac{a_2}{2\kappa} \epsilon^{ijk} \partial_j A_k^+$ |
| 4     | $\neq 0$ | $\neq 0$ | $\neq 0$ | $-\frac{1}{2\kappa} (a_2 m A^+ + a_4 \epsilon^{ijk} \partial_j A_k^1)$ |

In order to find out the most general boundary conditions, we integrate the broken equations of motion (3.3) in an infinitesimal interval around $x_3 = 0$ [24]. Because of the separability condition (3.1), we obtain the following algebraic system, involving quantities lying on the ‘+’ side of the boundary (the opposite one being obtained by parity), i.e. the fields on that side of the boundary and the $a$-parameters:

\[
\begin{align*}
(\kappa + a_1) \tilde{B}^+ &= -a_2 m A^+ + a_3 b^+ - a_4 \epsilon^{ijk} \partial_j A_k^+ - a_5 d^+, \\
(\kappa - a_1) A_j^+ &= 0, \quad a_3 = 0, \quad a_5 A^+ = 0.
\end{align*}
\tag{3.8}
\]

The solutions of the system (3.8) are acceptable if the rhs of the Ward identities (3.6) and (3.7) do not vanish. Indeed, in such a case, differentiating the Ward identity (3.6) (or (3.7)) with respect to $J_A$ (or $J_{\tilde{B}}$), we obtain the inconsistency

\[
\int d\mathbf{x}_3 \partial \delta^3(x - \mathbf{x}') = 0.
\tag{3.9}
\]

The interesting fact we learn from this simple remark is that the boundary Lagrangian $L_{bd}$ (3.2) acts as a kind of gauge fixing for the residual gauge invariance, in the sense that its presence is necessary (and sufficient) to calculate the propagators on the boundary.

We see that if we request that the boundary term of the Ward identity (3.7) does not vanish, we must impose the condition $A^+ \neq 0$. Consequently, it must be

\[
a_5 = 0, \quad a_1 = \kappa.
\tag{3.10}
\]

Having done that, the system (3.8) reduces to the single equation:

\[
2\kappa \tilde{B}^+ = -a_2 m A^+ - a_4 \epsilon^{ijk} \partial_j A_k^+,
\tag{3.11}
\]

which has four different algebraic solutions, listed in table 2.

We note that solutions 2 and 3 lead to ill-defined Ward identities (the rhs of (3.6) vanishes), and for this reason, they are not acceptable.

We are then left with two sets of boundary conditions which satisfy the system (3.11):

In both cases the parameter $a_2$ must be different from zero: the presence of the massive term $m$ is necessary in order to make the theory consistent. In other words, it is necessary that the boundary Lagrangian should not be scale-invariant. This fact is very important for the study of the physics on the boundary, as we shall see later.

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Table 3. Acceptable solutions of equation (3.11).

|   | $a_2$ | $a_4$ | $\Lambda^+$ | $\hat{B}^{i+}$ |
|---|---|---|---|---|
| I | $\neq 0$ | 0 | $\neq 0$ | $-\frac{a_2 m}{\kappa} \Lambda^+$ |
| II | $\neq 0$ | $\neq 0$ | $\neq 0$ | $-\frac{1}{\kappa}(a_2 m \Lambda^+ + a_4 \epsilon^{i j k} \partial_j \Lambda^+_k)$ |

4. The algebra and the physics on the boundary

In this section, we derive the algebra of local observables which is generated on the boundary, due to the residual gauge invariance of the theory, functionally described by the Ward identities (2.8) and (2.9). Next, we shall argue that it is possible to describe the physics on the boundary in terms of two fields: a gauge field $\zeta'(X)$ and a scalar massless field $\Lambda(X)$ (we recall that $X \equiv (x_0, x_1, x_2)$, while $x \equiv (X, x_3)$). We shall identify the Lagrangian which describes the physics on the boundary by interpreting the boundary algebra as a set of canonical commutation relations for the fields $\zeta'$ and $\Lambda$.

4.1. The boundary algebra

Despite the fact that the solutions listed in table 3 appear to be different, and depending on free parameters, the broken Ward identities for both of them are

$$\int_{-\infty}^{\infty} dx_3 \partial_t J_{\lambda_i}^i = \kappa \partial_t \hat{B}^{i+},$$

$$\int_{-\infty}^{\infty} dx_3 \epsilon^{i j k} \partial_j J_{\lambda_k} = -\kappa \epsilon^{i j k} \partial_j \Lambda^+_k. \tag{4.1}$$

We therefore remark that, as a matter of fact, a unique solution exists, which does not depend on any free parameter. We recall that the constant $\kappa$ was introduced in order to keep track of the bulk dependence, but it is not a true coupling constant. We can therefore freely put $\kappa = 1$ in what follows. Evaluating the previous relations at vanishing sources, i.e. on the mass shell, we find that

$$\partial_t \hat{B}^{i+} = 0, \tag{4.3}$$

$$\epsilon^{i j k} \partial_j \Lambda^+_k = 0. \tag{4.4}$$

We now differentiate (4.1) with respect to $J_{\lambda_i}^i(x')$, with $x'$ lying on the ‘+’ side of the boundary $x_3 = 0$, obtaining

$$\delta_t^i \partial_t \delta^{(3)}(X' - X) = \partial_t \left( \Delta_{\lambda_i \hat{B}^i}(x', x) \right)_{x_3 = x'_3 = 0^+}. \tag{4.5}$$

Next, we express the propagator in (4.5) as follows:

$$\left( \Delta_{\lambda_i \hat{B}^i}(x', x) \right)_{x_3 = x'_3 = 0^+} = \theta(t - t') \langle \hat{B}^{i+}(X) \Lambda^+_k(X') \rangle + \theta(t' - t) \langle \Lambda^+_k(X') \hat{B}^{i+}(X) \rangle. \tag{4.6}$$

Substituting the previous identity into (4.5), we find that

$$\delta_t^i \partial_t \delta^{(3)}(X' - X) = \delta(t - t') \left( \Delta_{\Lambda^+_k \lambda_i \hat{B}^i} \right)_{x_3 = x'_3 = 0^+} + \theta(t - t') \langle \partial_i \hat{B}^{i+}(X) \Lambda^+_k(X') \rangle + \theta(t' - t) \langle \Lambda^+_k(X') \partial_i \hat{B}^{i+}(X) \rangle. \tag{4.7}$$
Recalling (4.3), we obtain that
\[ \delta(t - t') [\tilde{B}^{0*}(X), A_{\alpha}(X')] = \delta^i_j \delta^0(X' - X). \] (4.8)

For \( l = 1, 2 \), it is possible to factorize \( \delta(t - t') \), finding that
\[ [\tilde{B}^{0*}(X), A_{\alpha}(X')]_{l=t'} = \delta_{\alpha} \delta^{(2)}(X' - X), \] (4.9)
where \( \alpha \) denotes the indices 1, 2. From now on, Greek letters will denote spatial coordinates on the plane \( x_3 = 0 \).

Next, differentiating (4.1) with respect to \( J^j_{\tilde{B}}(x') \), with \( x' \) lying on the ‘+’ side of the boundary, we obtain
\[ \partial_{t_j}(\Delta_{\tilde{B}x}(x', x))_{x_3 = x'_3 = 0^+} = 0. \] (4.10)

Following the same reasoning that led to (4.9), we obtain that
\[ [\tilde{B}^{0*}(X), \tilde{B}^j_{x'}(X')]_{l=t'} = 0. \] (4.11)

In particular, if \( l = 0 \) the previous commutation relation becomes
\[ [\tilde{B}^{0*}(X), \tilde{B}^0_{x'}(X')]_{l=t'} = 0. \] (4.12)

Let us now consider (4.2). The differentiation of this identity with respect to \( J_{A_{\bar{A}l}}(x') \) (with \( x'_3 = 0^+ \)) leads to
\[ \epsilon^{ijk}(\Delta_{A_{\bar{A}l}}(x', x))_{x_3 = x'_3 = 0^+} = 0. \] (4.13)

Taking into account (4.4), the previous identity yields the following commutation relation:
\[ [A^+_{\bar{A}l}(X), A^{\bar{A}+}_{\bar{A}l}(X')]_{l=t'} = 0. \] (4.14)

Next, differentiating (4.2) with respect to \( J^j_{\tilde{B}}(x') \), we find that
\[ (\partial_j \delta^j_k - \partial_k \delta^j_j) \delta^{(3)}(X' - X) = -\partial_j \left( \Delta_{\tilde{B}A}(x', x) \right)_{x_3 = x'_3 = 0^+} + \partial_k \left( \Delta_{\tilde{B}A}(x', x) \right)_{x_3 = x'_3 = 0^+}, \] (4.15)
which does not provide new commutation relations.

In conclusion, the commutation relations (4.9), (4.12) and (4.14) form the following algebra of local boundary observables:
\[
\begin{align*}
[\tilde{B}^{0*}(X), A_{\alpha}(X')]_{l=t'} &= \delta_{\alpha} \delta^{(2)}(X' - X), \\
[\tilde{B}^{0*}(X), \tilde{B}^0_{x'}(X')]_{l=t'} &= 0, \\
[A^+_{\bar{A}l}(X), A^{\bar{A}+}_{\bar{A}l}(X')]_{l=t'} &= 0,
\end{align*}
\] (4.16)
which will be discussed in detail in the last section of this paper, together with the other results.

4.2. The physics on the boundary

Let us consider again the broken Ward identities (4.1) and (4.2). As mentioned, they describe the residual gauge invariance of the 4D BF theory on the planar boundary \( x_3 = 0 \). Going on the mass shell, i.e. at vanishing external sources, we find equations (4.3) and (4.4) for the fields on
the boundary, for which the conditions listed in table 3 hold. Equations (4.3) and (4.4) are easily recognized as the electromagnetic Maxwell equations for an electric ($\leftrightarrow \tilde{B}^i$) and a magnetic ($\leftrightarrow A^i$) field. In other words, the 4D BF theory, which, like any other TQFT, does not have local observables and has a vanishing Hamiltonian, when dimensionally reduced on a 3D planar boundary, acquires a rich, physical, electromagnetic structure.

But we can push this further. Conditions (4.3) and (4.4) allow us to express the fields $\tilde{B}^i$ and $A^i$ in terms of the potentials $\Lambda$ and $\zeta$:

$$\partial_i \tilde{B}^i = 0 \Rightarrow \tilde{B}^i = \epsilon^{ijk} \partial_j \zeta_k,$$

$$\epsilon^{ijk} \partial_j A_k^i = 0 \Rightarrow A_k^i = \partial_k \Lambda,$$  \hspace{1cm} (4.17)

where $\Lambda(X)$ and $\zeta^i(X)$ have canonical dimensions 0 and 1, respectively. The fields $A^i$ and $\tilde{B}^i$ are left invariant by translational and gauge transformations of the potentials as follows:

$$\delta \Lambda = c, \hspace{0.5cm} \delta \zeta_i = \partial_i \theta,$$  \hspace{1cm} (4.18)

where $c$ is a constant and $\theta(X)$ is a local parameter.

Let us now consider the boundary condition I in table 3. With a suitable choice of the free parameter $a_2$, we can rewrite this condition in terms of the fields $\zeta$ and $\Lambda$:

$$\epsilon^{ijk} \partial_j \zeta_k = m \partial^i \Lambda.$$  \hspace{1cm} (4.19)

The massive parameter $m$ in equation (4.19) allows for a rescaling of the fields $\zeta^i$ and $\Lambda$ as follows:

$$\Lambda \rightarrow \frac{\Lambda}{\sqrt{m}}, \hspace{0.5cm} \zeta^i \rightarrow \sqrt{m} \zeta^i.$$  \hspace{1cm} (4.20)

So, thanks to the massive parameter $m$, the rescaled fields can be given the standard canonical dimensions of a gauge field and of a scalar field in three space–time dimensions ($[\zeta^i] = [\Lambda] = \frac{1}{2}$). Therefore, equation (4.19) becomes

$$\epsilon^{ijk} \partial_j \zeta_k = \partial^i \Lambda,$$  \hspace{1cm} (4.21)

which is exactly the duality relation between a scalar field and a gauge field that is required for constructing massless fermionic fields in three dimensions via the tomographic representation [25]. This could be interpreted as the sign that the actual degrees of freedom of the 3D theory obtained on the boundary are fermionic rather than bosonic. We shall come back to this point in the conclusive section 5. We now consider solution II in table 3:

$$\tilde{B}^i = -(a_2 m A^i + a_4 \epsilon^{ijk} \partial_j A_k^i).$$  \hspace{1cm} (4.22)

It is evident that, if we evaluate the previous condition on the mass-shell, the term proportional to $a_4$ vanishes due to condition (4.4), and the previous equation is equivalent to the boundary condition I:

$$\tilde{B}^i = -a_2 m A^i.$$  \hspace{1cm} (4.23)

In other words, the duality condition (4.21) always holds, and the 3D physics we are discussing here is therefore uniquely determined.

We are now able to find a 3D Lagrangian for the fields $\zeta^i$ and $\Lambda$ which describes the physics on the boundary and which is compatible with the duality condition (4.21). In what follows, we shall interpret the algebra (4.16) as a set of canonical commutation relations for the fields $\zeta^i$ and $\Lambda$, and we shall find the corresponding Lagrangian, doing the contrary of what is
commonly done, which is to find the canonical variables and their commutation relations from a given Lagrangian.

Let us consider equation (4.8) with \( l = 0 \):

\[
\delta(t - t') [\tilde{B}^{0\alpha}(X), A_0(X')] = \delta'(t - t') \delta^{(2)}(X - X').
\]  

(4.24)

Writing this identity in terms of the fields \( \zeta^i \) and \( \Lambda \), we obtain that

\[
\delta(t - t') \delta_0^i [\epsilon^{\alpha\beta} \partial_\alpha \xi_\beta(X), \Lambda(X')] = \delta'(t - t') \delta^{(2)}(X - X'),
\]  

(4.25)

where we have factorized the operator \( \delta_0^i \) on the rhs since it acts only on the field \( \Lambda \). It is easy to see that \( \delta(t - t') \delta_0^i = -\delta'(t - t') \) and, consequently, we can factor out the \( \delta'(t - t') \), finding the following commutation relation:

\[
[\Lambda(X'), \epsilon^{\alpha\beta} \partial_\alpha \xi_\beta(X)] |_{t = t'} = \delta^{(2)}(X - X').
\]  

(4.26)

Consider then the first commutation relation in (4.16):

\[
[\tilde{B}^{0\alpha}(X), A_\alpha(X')] |_{t = t'} = \partial_\alpha \delta^{(2)}(X' - X).
\]  

(4.27)

If we express the previous identity in terms of the fields \( \Lambda \) and \( \zeta^i \), we find that

\[
\partial_\alpha [\epsilon^{\alpha\beta} \xi_\beta(X), \Lambda(X')] |_{t = t'} = \delta_\gamma^i \partial_\eta \delta^{(2)}(X' - X),
\]  

(4.28)

which yields

\[
[\epsilon^{\alpha\beta} \xi_\beta(X), \Lambda(X')] |_{t = t'} = \delta^{a\beta} \delta^{(2)}(X' - X).
\]  

(4.29)

We are now ready to construct the Lagrangian. The commutation relations (4.26) and (4.29) allow us to interpret the fields \( \Pi_{(\Lambda)} \equiv \epsilon^{\alpha\beta} \partial_\alpha \xi_\beta \) and \( \Pi_{(\zeta)} \equiv \partial_\alpha \Lambda \) as the conjugate momenta of the fields \( \Lambda \) and \( \zeta^i \equiv \epsilon^{\alpha\beta} \xi_\beta \), respectively.

With these assumptions, the Lagrangian of the system \( \mathcal{L} = \sum \Pi \Phi - H \), where \( H \) is the Hamiltonian of the system, is given by

\[
\mathcal{L} = \epsilon^{\alpha\beta} \partial_\alpha \Lambda \partial_\beta \Lambda + \partial_\alpha \Lambda \epsilon^{\alpha\beta} \partial_\beta \Lambda - (\epsilon^{\alpha\beta} \partial_\alpha \Lambda)^2 - (\partial_\alpha \Lambda)^2,
\]  

(4.30)

which is equivalent to the Lagrangian postulated in [15] for the study of the TI. We stress that the Lagrangian (4.30) is the most general one compatible with power counting and respecting the symmetries (4.18). Furthermore, the coefficients of the terms appearing in (4.30) are fixed by making the field equations of motion compatible with the duality relation (4.21). Moreover, if we omit the kinetic term, \( \mathcal{L} \) is equivalent to the Lagrangian considered in [26] to study the edge states of the 4D BF theory.

5. Summary and discussion

The main results presented in this paper are

1. **Boundary as gauge fixing**

   According to Symanzik’s approach, the boundary conditions on the fields are not imposed, but rather are derived from the form of the propagators. Now, this is not always feasible and is almost always quite difficult (we are talking about computing the propagators of the theory, including the boundary and satisfying the separability condition (1.1)). In a previous

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2 We thank the referee for the suggestion that we should clarify the issue of the uniqueness of (4.30).
paper [24], we already succeeded in finding out the boundary conditions (which are the starting point for the study of the physics on the boundary) without explicitly computing the propagators. In the end, one is faced with a nonlinear algebraic system whose unknowns are the parameters on which the boundary Lagrangian depends, and the fields (and its derivatives) on the boundary. Most of these solutions are inconsistent or unacceptable for some reasons, and in [24] these unphysical solutions were ruled out one by one. Here, we found a nice, general criterion to obtain the same result: the rhs of the Ward identities (5.1) and (5.2), i.e. the linear breaking due to the boundary term in the action, must always be different from zero. Otherwise, the propagators cannot be defined. A nice interpretation of this statement is that the boundary term in the action plays the role of a gauge fixing of the residual gauge invariance on the boundary. This observation immediately leads to solutions I and II listed in table 3.

2. Ward identities in the presence of a boundary

$$\int_{-\infty}^{+\infty} dx_3 \partial_3 J^i\!_\Lambda = \partial_i \tilde{B}^{i+},$$

(5.1)

$$\int_{-\infty}^{+\infty} dx_3 \epsilon^{ijk} \partial_j \tilde{J}_k = -\epsilon^{ijk} \partial_j A^+_k.$$  

(5.2)

Quite remarkably, the apparently distinct solutions I and II of table 3 physically coincide, since they lead to the same Ward identities on the boundary. This is the first evidence for the striking electromagnetic structure which determines the physics on the boundary, as we shall discuss soon. In addition, despite the fact that the solutions depend on free parameters, when put into the Ward identities (5.1) and (5.2), which contain all the physical information, these disappear. The separability condition isolates a unique dynamics on the boundary, without any dependence on free parameters.

3. Electromagnetism on the boundary

$$\partial_i \tilde{B}^{i+} = 0 \Rightarrow \tilde{B}^{i+} = \epsilon^{ijk} \partial_j \xi_k,$$

(5.3)

$$\epsilon^{ijk} \partial_j A^+_k = 0 \Rightarrow A^+_k = \partial_k \Lambda.$$  

(5.4)

On the boundary $x_3 = 0$, and on the mass shell, $J_\phi = \frac{\partial S}{\partial \phi} |_{J=0} = 0$ (we stress this double constraint defining the boundary), the 4D topological BF theory displays Maxwell equations for an electric field and a magnetic field, to be identified with the boundary insertions $\tilde{B}^{i+}$ and $A^+_i$, respectively. This is a direct consequence of result 2. Consequently, two potentials can be introduced: an electric scalar potential $\Lambda(X)$ and a magnetic vector potential $\xi^i(X)$, depending on the 3D coordinates on the plane $x_3 = 0$: $X = (x_0, x_1, x_2)$.

4. Duality

$$\epsilon^{ijk} \partial_j \xi_k = \partial^i \Lambda.$$  

(5.5)

The solutions of table 3, i.e. the possible boundary conditions on the fields, translate into the ‘duality’ condition between the potentials (5.5). This confirms the fact that the dynamics on the boundary is uniquely determined by the Ward identities (5.1) and (5.2). We find here, in a well-defined field theoretical framework, a strong motivation for a relation that is known since a long time ago [25], where this duality (or ‘tomographic’)

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relation was introduced to give a Bose description of fermions in 3D. Here, this condition appears as the unique boundary condition on the fields $A^i = \tilde{B}^i$, written in terms of electromagnetic potentials defined by the boundary Maxwell equations (5.3) and (5.4). This strongly suggests that the actual degrees of freedom of the dimensionally reduced 3D theory are fermionic, confirming recent developments concerning the edge states of TI, which seem to be described in terms of fermion fields [27].

5. 3D boundary algebra

\begin{align}
[\tilde{B}^0(X), A^\alpha(X')] \big|_{t=t'} &= \delta_\alpha^\beta (X' - X), \\
[\tilde{B}^0(X), \tilde{B}^\alpha_0(X')] \big|_{t=t'} &= 0, \\
[A^\alpha_0(X), A^\beta_0(X')] \big|_{t=t'} &= 0. \\
\end{align}  \tag{5.6}

On the boundary, the above algebra is found. It is formed by a vectorial, conserved current, whose 3D components are the insertions of the fields on (one side of) the boundary ($\tilde{B}^0(X)$ and $A^\alpha_0(X)$, $\alpha = 1, 2$, related by the duality-boundary condition (4.23)). We stress that the conservation of the current is obtained on the mass-shell, i.e. at vanishing external sources $J_\phi$. This is in perfect analogy with what happens in the 3D CS and the 3D BF theory. In all cases, the conservation of the currents comes from the Ward identities of the residual gauge invariance broken by the most general boundary term respecting Symanzik’s separability condition (1.1), going on the mass-shell, and exploiting the boundary condition previously found on the quantum fields. The physical interpretation of the current conservation is different, since in the 3D CS and BF cases, it leads, thanks to the boundary conditions, to the chirality of the currents. In the 4D BF case the current conservation (5.3) (again, together with the duality-boundary condition (5.5)) is tightly related to the electromagnetic structure and the consequent determination of the electromagnetic potentials. One more comment on the algebra (5.6) is in order. The 3D BF theory shows two types of algebraic structures on its 2D planar boundary, whether the cosmological constant $\lambda$, whose presence is peculiar of the 3D case, is vanishing or not [8]. For $\lambda \neq 0$, one finds the direct sum of two KM algebras, and this is not surprising because the 3D BF theory with cosmological constant can be written in terms of two CS theories with opposite coupling constants, each of which shows a boundary KM algebra. In contrast, the algebra found for vanishing cosmological constant is truly BF-like. Indeed, in that case the BF theory, in any dimension, cannot be rephrased in terms of CS actions. A remarkable check of our results is that we find exactly the same algebraic structure as that found for the 3D BF case for $\lambda = 0$, whose interesting features and relationships with other bulk theories are discussed elsewhere [21].

In a different framework and language the same algebra has been found in [26], written in terms of the same dynamical variables we treated in this paper, i.e. the edge states of 4D BF theory.

6. Canonical commutation relations and dimensional reduction

\begin{align}
[\Lambda(X), \Pi_{(\Lambda)}(X')] \big|_{t=t'} &= \delta^{(2)}(X - X'), \\
[\tilde{\zeta}^\alpha(X), \Pi_{(\tilde{\zeta})}\beta(X')] \big|_{t=t'} &= \delta^\alpha_\beta \delta^{(2)}(X - X'), \\
\end{align}  \tag{5.7}

where $\Pi_{(\Lambda)} \equiv \epsilon^{\alpha\beta} \partial_\alpha \zeta_\beta$ and $\Pi_{(\tilde{\zeta})}\alpha \equiv \partial_\alpha \Lambda$ are the conjugate momenta of the fields $\Lambda$ and $\tilde{\zeta}^\alpha \equiv \epsilon^{\alpha\beta} \zeta_\beta$ respectively. The point to stress here is that, written in terms of the
electromagnetic potentials (5.1) and (5.2), the boundary algebra (5.6) can be interpreted as a set of canonical commutation relations for the canonically conjugate variables. Once we have realized this, it is almost immediate to write down the corresponding 3D Lagrangian, which is uniquely determined by our procedure. Indeed, this analysis can be viewed as a systematic way of finding \((D-1)\)-dimensional Lagrangians out of \(D\)-dimensional bulk theories. It is a surprising and welcome result that this new way of dimensionally reducing \(D\)-dimensional theories originates from the algebraic structure found on the boundary, interpreted as a set of canonical commutation relations, and which comes from the Ward identities describing the residual gauge invariance on the boundary and broken (by the boundary itself) in the most general (and unique) way compatible with Symanzik’s simple criterion of separability.

7. 3D Lagrangian

\[
\mathcal{L} = \sum \Pi \Phi - H = \epsilon^{\alpha\beta} \partial_\alpha \zeta_\beta \partial_t \Lambda + \partial_\alpha \Lambda \epsilon^{\alpha\beta} \partial_\beta \zeta_\rho - (\epsilon^{\alpha\beta} \partial_\beta \zeta_\rho)^2 - (\partial_\alpha \Lambda)^2. \tag{5.9}
\]

This is the 3D Lagrangian obtained on the mass-shell boundary of the 4D topological BF theory. It is the unique solution compatible with the QFT request of locality and power counting and with Symanzik’s criterion of separability (1.1). It is left invariant by gauge and translational transformations. It is non-covariant, and its dynamical variables (scalar and vector potentials) are coupled in a non-trivial way. Quite remarkably, this action, uniquely derived here by very general QFT principles, coincides with that studied in [26] for the edge states of the 4D BF theory, where the same algebraic origin is stressed. In a completely different theoretical framework, the action (5.9) is employed to study the surface of 4D (3+1) TI [15]. The duality relation (5.5) is exploited there to extract the desired fermionic degrees of freedom.

There are several interesting developments in the results presented in this paper. The most obvious is the non-abelian extension, with particular attention to the duality relation (5.5), to the boundary algebra (5.6) and to the boundary Lagrangian (5.9). Moreover, it is of interest to apply our method to dimensionally reduce 5D bulk theories, in order to find out the resulting 4D actions.

Appendix. The propagators

In this appendix, we shall derive the propagators of the 4D BF model (2.1) with boundary (3.2), taking into account the boundary conditions in table 3.

The separability condition (3.1) is satisfied by propagators with the following form:

\[
\Delta_{\phi_1\phi_2}(x, x') = \theta(x_3)\theta(x'_3)\Delta_{\phi_1\phi_2}^+(x, x') + \theta(-x_3)\theta(-x'_3)\Delta_{\phi_1\phi_2}^-(x, x'), \tag{A.1}
\]

where \(\Delta_{\phi_1\phi_2}^\pm\) are the propagators on the ‘\pm’ side of the boundary. They are solutions of the system of equations obtained by differentiating the bulk equations of motion (2.7) with respect to the sources of the fields. They must be compatible with the boundary equations of motion (3.3) and with the Ward identities (3.4) and (3.5). Since \(\Delta_{\phi_1\phi_2}^+\) and \(\Delta_{\phi_1\phi_2}^-\) are transformed into each other by a parity transformation, in this appendix we derive a solution for \(\Delta_{\phi_1\phi_2}^+\), where \(x_3, x'_3 \geq 0\). In what follows, we shall omit the index +.

If we differentiate the equations of motion (2.7) with respect to the sources of the fields and evaluate the expressions obtained at vanishing sources, we obtain a system of equations for
the propagators of the theory:

\[
\begin{align*}
\Delta_{A_3 \psi}(x, x') &= 0, \quad \forall \psi(x') \neq b(x'), \\
\Delta_{A_3 b}(x, x') &= -\delta^4(x - x'), \\
\Delta_{B^3 \psi}(x, x') &= 0, \quad \forall \psi(x') \neq d'(x'), \\
\Delta_{B^3 d}(x, x') &= -\delta_i^j\delta^4(x - x'), \\
\partial_3 \Delta_{A_i \bar{B}^j}(x', x) &= -\delta_i^j\delta^4(x - x'), \\
\partial_3 \Delta_{\bar{B}^i B_i}(x', x) &= 0, \\
\partial_3 \Delta_{d_i \bar{B}^j}(x', x) &= 2\epsilon_{ijk} \partial_j \delta^4(x - x'), \\
\partial_3 \Delta_{A_i A^i}(x', x) &= 0, \\
\partial_3 \Delta_{B_i A^i}(x', x) &= \delta_i^j\delta^4(x - x'), \\
\partial_3 \Delta_{d_i A^i}(x', x) &= -\partial_i\delta^4(x' - x), \\
\Delta_{A_3 b}(x', x) &= \partial_i \Delta_{A_i \bar{B}^i}(x', x), \\
\Delta_{\bar{B}^i b}(x', x) &= \partial_i \Delta_{\bar{B}^i\bar{B}^i}(x', x), \\
\Delta_{d_i b}(x', x) &= \partial_i \Delta_{d_i \bar{B}^j}(x', x), \\
\Delta_{A_i d^i} &= -2\epsilon_{ijk} \partial_j \Delta_{A_i A^i}(x', x), \\
\Delta_{B_i d^i} &= -2\epsilon_{ijk} \partial_j \Delta_{B_i A^i}(x', x), \\
\Delta_{d_i d^i} &= -2\epsilon_{ijk} \partial_j \Delta_{d_i A^i}(x', x), \\
\Delta_{b d^i} &= -2\epsilon_{ijk} \partial_j \Delta_{b A^i}(x', x). 
\end{align*}
\]

\hspace{1cm} (A.2)

Note that it follows directly from the gauge conditions, i.e. from the last two equations in (2.7), that the Green functions containing \(A_3\) and \(B_{i3}\) vanish except \(\Delta_{A_3 b}(x, x') = -\delta^4(x - x')\) and \(\Delta_{B^{3}_{i3}}(x, x') = -\delta^i_3\delta^4(x - x')\) and, for this reason, we do not list these propagators in the following.

The most general solution of the previous system is

\[
\begin{align*}
\Delta_{\phi_1 \phi_2}(x', x) = \left(\begin{array}{cccc}
\Xi^i_j(X, X') & \delta^i_j T_{c_1}(x, x') & -2\epsilon_{ijk} \partial_j \Xi^i_k(X', X) & -\partial_i T_{c_1}(x, x') \\
-\delta^i_j T_{c_2}(x', x) & \Omega^i_j(X, X') & 2\epsilon_{ijk} \partial_j T_{c_2}(x', x) & \partial_i \Omega^i_j(X, X') \\
-2\epsilon_{jk} \partial_j \Xi^i_k(X', X) & 2\epsilon_{ijk} \partial_j T_{c_2}(x', x) & 4\epsilon_{ijk} \partial_j \epsilon_{kpq} \partial^p \Xi^q(X, X') & 0 \\
\delta^i T_{c_3}(x', x) & \delta^i \Omega^i_j(X', X) & 0 & \delta^i \partial_i \Omega^i_j(X', X')
\end{array}\right).
\end{align*}
\]

\hspace{1cm} (A.3)
The labels $\phi_i$ and $\phi_{i+}$ run over the set of fields $\{A, \tilde{B}, d, b\}$. $T_{c_i}(x, x')$ is the tempered distribution $(\delta(x_3 - x'_3) + c_i)\delta^3(X - X')$, $\Xi^i(X, X')$ and $\Omega_i^j(X, X')$ are generic functions of the transverse coordinates $X \equiv (x_0, x_1, x_2)$, and $c_i, i = 1, \ldots, 4$, are constant parameters.

Let us now consider boundary conditions I and II in Table 3. Remarkably, for both these solutions the Ward identities (3.6) and (3.7) take the following form:

$$
\int_{-\infty}^{\infty} dx_3 \partial_i J_{A_i} = \partial_i \tilde{B}_i^+, \quad \int_{-\infty}^{\infty} dx_3 \epsilon^{ijk} \partial_j \tilde{B}_{k}^+ = -\epsilon^{ijk} \partial_j A_i^+.
$$

(A.4)

Differentiating equations (A.4) with respect to the sources $J_{A_{i}}(x')$, $J_{B_{i}}(x')$, $J_{d}^i(x')$ and $J_{b}(x')$, we obtain eight differential equations for the propagators:

$$
\partial_i \delta_{i}^j \delta^3(X' - X) = \partial_i \left( \Delta_{A_i A_i'}(x', x) \right)_{x_3 = 0},
$$

(A.5)

$$
\partial_i \left( \Delta_{B_i B_i'}(x', x) \right)_{x_3 = 0} = 0,
$$

(A.6)

$$
\partial_i \left( \Delta_{d_i B_i'}(x', x) \right)_{x_3 = 0} = 0,
$$

(A.7)

$$
\partial_i \left( \Delta_{b_i B_i'}(x', x) \right)_{x_3 = 0} = 0,
$$

(A.8)

$$
\epsilon^{ijk} \partial_j \left( \Delta_{A_i A_i'}(x', x) \right)_{x_3 = 0} = 0,
$$

(A.9)

$$
\epsilon^{ijk} \partial_j \delta_{kl} \delta^3(X' - X) = -\epsilon^{ijk} \partial_j \left( \Delta_{B_l A_k}(x', x) \right)_{x_3 = 0},
$$

(A.10)

$$
\epsilon^{ijk} \partial_j \left( \Delta_{d_i A_k}(x', x) \right)_{x_3 = 0} = 0,
$$

(A.11)

$$
\epsilon^{ijk} \partial_j \left( \Delta_{b_i A_k}(x', x) \right)_{x_3 = 0} = 0.
$$

(A.12)

Substituting the propagators (A.3) into the above system of differential equations, we obtain the following constraints on the parameters $c_1$ and $c_2$:

$$
c_1 = -1, \quad c_2 = 0.
$$

(A.13)

Additional constraints come from the request that the Lagrange multipliers $b(x)$ and $d^i(x)$ vanish at $x_3 \to \infty$. Consequently, the propagators involving these fields must satisfy

$$
\lim_{x_3 \to +\infty} 2\epsilon^{ij} \partial_j T_{c_i}(x, x') = 0, \quad \lim_{x_3 \to +\infty} \partial_i T_{c_3}(x', x) = 0,
$$

(A.14)

which yield

$$
c_3 = -1, \quad c_4 = 0.
$$

(A.15)

Moreover, the propagators $\Delta_{A_i A_i'}(x', x)$ and $\Delta_{B_i B_i'}(x', x)$ must be symmetric for the exchange $\{x, i\} \leftrightarrow \{x', l\}$. As a consequence, taking into account (A.6) and (A.11), we obtain that the functions $\Xi_i^j(X, X')$ and $\Omega_i^j(X, X')$ take the following form

$$
\Xi_i^j(X, X') = \delta^i \delta_l \eta(X - X'),
$$

$$
\Omega_i^j(X, X') = \epsilon^{ijk} \partial_j \epsilon^{rl} \partial_r \phi_{kl}(X - X'),
$$

(A.16)
where \( \eta(X - X') \) and \( \phi_{\alpha \beta}(X - X') \) are generic functions of the transverse coordinates \( X - X' \) and have canonical mass dimensions 0 and 2, respectively. In terms of these functions, the matrix of propagators (A.3) finally reads

\[
\Delta_{AB}(x', x) = \begin{pmatrix}
\partial^i \partial_i \eta(X - X') & -\delta_i^j T_{-1}(x, x') & 0 & -\partial_j T_{-1}(x, x') \\
-\delta_i^j T_0(x', x) & \epsilon^{ijk} \partial_j \epsilon_{rs} \partial_r \phi_{\alpha \beta}(X - X') & 2\epsilon^{ij}_l \partial_j T_0(x', x) & 0 \\
0 & 2\epsilon^{ij}_l \partial_j T_0(x, x') & 0 & 0 \\
\partial^i T_{-1}(x', x) & 0 & 0 & 0
\end{pmatrix}.
\] (A.17)

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