On the stratifications of 2-qubits X-state space

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Abstract. The 7-dimensional family $\Psi_X$ of so-called mixed X-states of 2-qubits is considered. Two types of stratification of 2-qubits $X$-state space, i.e., partitions of $\Psi_X$ into orbit types with respect to the adjoint group actions, one of the global unitary group $G_X \subset SU(4)$ and another one under the action of the local unitary group $LG_X \subset G_X$, is described. The equations and inequalities in the invariants of the corresponding groups, determining each stratification component, are given.

1 Introduction

The understanding of a symmetry that a physical system possesses, as well as this symmetry’s breaking pattern allows us to explain uniquely a wide variety of phenomena in many areas of physics, including elementary particle physics and condensed matter physics \cite{1}. The mathematical formulation of symmetries related to the Lie group action consists of the detection of the stratification of the representation space of corresponding symmetry group. Dealing with closed quantum systems the symmetries are realized by the unitary group actions and the quantum state space plays the role of the symmetry group representation space. Below, having in mind these observations, we will outline examples of the stratifications occurring for a quantum system composed of a pair of 2-level systems, two qubits. We will analyze symmetries associated with two subgroups of the special unitary group $SU(4)$. More precisely, we will consider the 7-dimensional subspace $\Psi_X$ of a generic 2-qubit state space, the family of $X$-states (for definition see \cite{2}, \cite{3} and references therein) and reveal two types of its partition into the set of points having the same symmetry type. The primary stratification originates from the action of the invariance group of $X$-states, named the global unitary group $G_X \subset SU(4)$, whereas the secondary one is due to the action of the so-called local group $LG_X \subset G_X$ of the $X$-states.

2 X-states and their symmetries

The mixed 2-qubit $X$-states can be defined based on the purely algebraic consideration. The idea is to fix the subalgebra $g_X := su(2) \oplus su(2) \oplus u(1) \in su(4)$ of the algebra $su(4)$ and define the density matrix of $X$-states as

$$g_X = \frac{1}{4} (I + i g_X) \ .$$

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In order to coordinatize the $X$–state space we use the tensorial basis for the $\mathfrak{su}(4)$ algebra, $\sigma_{\mu\nu} = \sigma_{\mu} \otimes \sigma_{\nu}$, $\mu, \nu = 0, 1, 2, 3$. It consists of all possible tensor products of two copies of Pauli matrices and a $2 \times 2$ matrix, $\sigma_{\mu} = (I, \sigma_1, \sigma_2, \sigma_3)$, which we order as follows (see the details in [3]):

$$\lambda_1, \ldots, \lambda_{15} = \frac{i}{2} \left( \sigma_{x0}, \sigma_{y0}, \sigma_{z0}, \sigma_{0x}, \sigma_{0y}, \sigma_{0z}, \sigma_{xx}, \sigma_{xy}, \sigma_{xz}, \sigma_{yy}, \sigma_{yz}, \sigma_{zx}, \sigma_{yz}, \sigma_{zy}, \sigma_{zz} \right). \quad (2)$$

In this basis the $7$-dimensional subalgebra $\mathfrak{g}_X$ is generated by the subset $\alpha_X = (\lambda_3, \lambda_6, \lambda_7, \lambda_8, \lambda_{10}, -\lambda_{11}, \lambda_{15})$, and thus the unit norm $X$–state density matrix is given by the decomposition:

$$\varrho_X = \frac{1}{4} \left( I + 2i \sum_{\lambda_i \in \alpha_X} h_k \lambda_k \right). \quad (3)$$

The real coefficients $h_k$ are subject to the polynomial inequalities ensuring the semi-positivity of the density matrix, $\varrho_X \geq 0$:

$$\Psi_X = \{ h_i \in \mathbb{R}^7 | (h_3 \pm h_6)^2 + (h_8 \pm h_{10})^2 + (h_7 \pm h_{11})^2 \leq (1 \pm h_{15})^2 \}. \quad (4)$$

Using the definition (1) one can conclude that the $X$–state space $\Psi_X$ is invariant under the 7-parameter group, $G_X := \exp(\mathfrak{g}_X) \in SU(4)$:

$$g \varrho_X g^\dagger \in \Psi_X, \quad \forall g \in G_X. \quad (5)$$

Group $G_X$ plays the same role for the $X$–states as the special unitary group $SU(4)$ plays for a generic 4-level quantum system, and thus is termed the global unitary group of $X$–states. According to [3], group $G_X$ admits the representation:

$$G_X = P_\pi \left( e^{-i\omega_15 SU(2)} \begin{pmatrix} 0 \\ 0 \\ \rho^{i\omega_15 SU(2)'} \end{pmatrix} \right) P_\pi, \quad \text{with} \quad P_\pi = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \quad (6)$$

Correspondingly, the local unitary group of the $X$–states is

$$LG_X = P_\pi \exp(i \frac{\phi_1}{2}) \otimes \exp(i \frac{\phi_2}{2}) P_\pi \subset G_X. \quad (7)$$

### 3 Global orbits and state space decomposition

Now we give a classification of the global $G_X$–orbits according to their dimensionality and isotropy group. Every density matrix $\varrho_X$ can be diagonalized by some element of the global $G_X$ group. In other words, all global $G_X$–orbits can be generated from the density matrices, whose eigenvalues form the partially ordered simplex $\Delta_X$, depicted on Figure [4]

The tangent space to the $G_X$–orbits is spanned by the subset of linearly independent vectors, built from the vectors: $t_k = [\lambda_k, \varrho_X], \lambda_k \in \alpha_X$. The number of independent vectors $t_k$ determines the dimensionality of the $G_X$–orbits and is given by the rank of the $7 \times 7$ Gram matrix:

$$G_{kl} = \frac{1}{2} \text{Tr}(t_k t_l). \quad (8)$$

The Gram matrix (8) has three zero eigenvalues and two double multiplicity eigenvalues:

$$\mu_\pm = -\frac{1}{8} \left( (h_3 \pm h_6)^2 + (h_8 \pm h_{10})^2 + (h_7 \pm h_{11})^2 \right). \quad (9)$$
Correspondingly, the $G_X$-orbits have dimensionality of either 4, 2 or 0. The orbits of maximal dimensionality, $\dim(O)_{\text{gen}} = 4$, are characterized by non-vanishing $\mu_\pm \neq 0$ and consist of the set of density matrices with a generic spectrum, $\sigma(q_x) = (r_1, r_2, r_3, r_4)$. If the density matrices obey the equations

$$h_6 = \pm h_3, \ h_{10} = \pm h_8, \ h_{11} = \pm h_7,$$  

(10)

they belong to the so-called degenerate orbits, $\dim(O)_\pm = 2$. The latter are generated from the matrices which have the double degenerate spectrum of the form, $\sigma(q_x) = (p, p, r_3, r_4)$ and $\sigma(q_x) = (r_1, r_2, q, q)$ respectively. Finally, there is a single orbit $\dim(O)_0 = 0$, corresponding to the maximally mixed state $q_x = \frac{1}{4}I$.

Considering the diagonal representative of the generic $G_x$-orbit one can be convinced that its isotropy group is

$$H = P_\pi \begin{pmatrix} e^{i\omega} & 0 \\ 0 & e^{-i\omega} \end{pmatrix} P_\pi,$$  

(11)

while for a diagonal representative with a double degenerate spectrum the isotropy group is given by one of two groups:

$$H_+ = P_\pi \begin{pmatrix} 0 & 0 \\ e^{i\omega} & 0 \end{pmatrix} P_\pi, \quad H_- = P_\pi \begin{pmatrix} 0 & 0 \\ e^{-i\omega} & 0 \end{pmatrix} P_\pi.$$  

For the single, zero dimensional orbit the isotropy group $H_0$ coincides with the whole invariance group, $H_0 = G_X$. Therefore, the isotropy group of any element of $G_x$-orbits belongs to one of these conjugacy classes: $[H], [H_\pm]$ or $[H_0]$. Moreover, a straightforward analysis shows that $[H_+] = [H_-]$. Hence, any point $q \in \Psi_X$ belongs to one of three above-mentioned types of $G_x$-orbits denoted afterwards as $[H_t], t = 1, 2, 3$. For a given $H_t$, the associated stratum $\Psi_{[H_t]}$, defined as the set of all points whose stabilizer is conjugate to $H_t$:

$$\Psi_{[H_t]} := \{ y \in \Psi_X | \text{isotropy group of } y \text{ is conjugate to } H_t \}$$

determines the sought-for decomposition of the state space $\Psi_X$ into strata according to the orbit types:

$$\Psi_X = \bigcup_{\text{orbit types}} \Psi_{[H_t]}.$$  

(12)

1 The orbit type $[q]$ of a point $q \in \Psi_X$ is given by the conjugacy class of the isotropy group of point $q$, i.e., $[q] = [G_{q_x}]$. 
The strata $\Psi_{\{H\}}$ are determined by this set of equations and inequalities:

\begin{align}
(1) \quad \Psi_{\{H\}} & := \{ h_i \in \Psi_X \mid \mu_+ > 0, \mu_- > 0 \}, \\
(2) \quad \Psi_{\{H_{1-}\}} & \cup \Psi_{\{H_{-1}\}} := \{ h_i \in \Psi_X \mid \mu_+ = 0, \mu_- > 0 \} \cup \{ h_i \in \Psi_X \mid \mu_+ > 0, \mu_- = 0 \}, \\
(3) \quad \Psi_{\{H_0\}} & := \{ h_i \in \Psi_X \mid \mu_+ = 0, \mu_- = 0 \}.
\end{align}

4 Local orbits and state space decomposition

Analogously, one can build up the $X-$ state space decomposition associated with the local group $LG_x$ action. For this action the dimensionality of $LG_x$-orbits is given by the rank of the corresponding $2 \times 2$ Gram matrix constructed out of vectors $t_3$ and $t_6$. Since its eigenvalues read:

\[ \mu_1 = -\frac{1}{8} \left( (h_8 + h_{10})^2 + (h_7 + h_{11})^2 \right), \quad \mu_2 = -\frac{1}{8} \left( (h_8 - h_{10})^2 + (h_7 - h_{11})^2 \right), \]

the $LG_x$-orbits are either generic ones with the dimensionality of $\dim (O_L)_{\text{Gen}} = 2$, or degenerate $\dim (O_L) = 1$, or exceptional ones, $\dim (O_L) = 0$. The $LG_x$-orbits can be collected into the strata according to their orbit type. There are three types of strata associated with the “local” isotropy subgroups of $LH \in LG_X$. Correspondingly, one can define the following “local” strata of state space:

- the generic stratum, $\Psi_{\{I\}}$, which has a trivial isotropy type, $[I]$, and is represented by the inequalities: $\Psi_{\{I\}} := \{ h_i \in \Psi_X \mid \mu_1 > 0, \mu_2 > 0 \}$,
- the degenerate stratum, $\Psi_{\{H_{\pm}\}}$, collection of the orbits whose type is $[H_{\pm}]$, with the subgroup either $H_\pm = I \times \exp (i\alpha \varpi_3)$, or $H_\pm = \exp (i\alpha \varpi_3) \times I$. The stratum defining equations read respectively:
  \[ h_{10} = \pm h_8, \quad h_{11} = \pm h_7, \]
- the exceptional stratum, $\Psi_{\{LG_x\}}$ of the type $[LG_x]$, determined by the equations: $h_{11} = h_{10} = h_8 = h_7 = 0$.

Therefore, the local group action prescribes the following stratification of 2-qubit $X-$state space:

\[ \Psi_X = \Psi_{\{I\}} \cup \Psi_{\{H_{\pm}\}} \cup \Psi_{\{H_{\pm}\}} \cup \Psi_{\{LG_x\}}. \]

5 Concluding remarks

In the present article we describe the stratification of 2-qubit $X-$state space associated with the adjoint action of the global and local unitary groups. The global unitary symmetry is related to the properties of a system as a whole, while the local symmetries comprise information on the entanglement, cf. [4]. In an upcoming publication, based on the introduced stratification of state space, we plan to analyze an interplay between these two symmetries and particularly determine the entanglement/separability characteristics of every stratum.

References

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