SUBGROUPS OF INERTIA GROUPS ARISING FROM ABELIAN VARIETIES

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ABSTRACT. Given an abelian variety over a field with a discrete valuation, Grothendieck defined a certain open normal subgroup of the absolute inertia group. This subgroup encodes information on the extensions over which the abelian variety acquires semistable reduction. We study this subgroup, and use it to obtain information on the extensions over which the abelian variety acquires semistable reduction.

1. INTRODUCTION

Suppose $X$ is an abelian variety over a field $F$, and $v$ is a discrete valuation on $F$. Fix an extension $\bar{v}$ of $v$ to a separable closure $F^s$ of $F$, and write $I_v$ for the inertia subgroup in $\text{Gal}(F^s/F)$ for $\bar{v}$. In [6] (see pp. 354–355), Grothendieck defined a subgroup $I'_v$ of $I_v$ with the property that $X$ has semistable reduction at the restriction $w$ of $\bar{v}$ to a finite separable extension of $F$ if and only if $I'_w \subseteq I'_v$. In particular, if $F_{nr}^v$ denotes the maximal unramified extension of the completion of $F$ at $v$, then $I'_v$ cuts out the smallest Galois extension of $F_{nr}^v$ over which $X$ has semistable reduction. We denote the group $I'_v$ by $I_v,X$ because of its dependence on $X$ and $v$.

In §4 we give some properties of the group $I_v,X$. We show that the Zariski closure of its image under the $\ell$-adic representation (for $\ell$ different from the residue characteristic) coincides with the identity connected component of the Zariski closure of the image of $I_v$. The proofs of the results in §4 are in the spirit of [18], where we dealt with connectedness questions for Zariski closures of images of $\ell$-adic representations.

In §5 we show that the finite group $G_{v,X} = I_v/I_v,X$ injects into $\text{GL}_{t_v}(\mathbb{Z}) \times \text{Sp}_{2(a-a_v)}(\mathbb{Z})$ for all but finitely many primes $\ell$, where $t_v$ and $a_v$ (respectively, $t$ and $a$) are the toric and abelian ranks of the special fiber of the Néron model of $X$ at $v$ (respectively, at an extension of $v$ over which $X$ has semistable reduction). Here, the projection onto the first factor is independent of $\ell$, and the characteristic polynomial of the projection onto the second factor has integer coefficients independent of $\ell$. The group $G_{v,X}$ was introduced by Serre in the case of elliptic curves in §5.6 of [10] (where it was called $\Phi_p$).
In §6 we obtain divisibility bounds on the order of $G_{v,X}$, and in §7 we deduce results on semistable reduction of abelian varieties. Bounds on the prime divisors and the exponent of $\#G_{v,X}$ were obtained by Lorenzini (see Proposition 3.1 of [7]). In particular, the bound on $Q_{v,X}$ in Corollary 6.3 was essentially obtained by Lorenzini.

The paper continues our earlier work on semistable reduction of abelian varieties (see [17] and [19]). The proofs are heavily influenced by the fundamental results of Grothendieck and Serre.

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2. Notation and preliminaries

If $K$ is a field, write $K^s$ for a separable closure. If $G$ is an algebraic group, let $G^0$ denote its identity connected component. Let $\varphi$ denote the Euler $\varphi$-function, let $\zeta_M$ denote a primitive $M$-th root of unity, and let $F_\ell$ denote the finite field with $\ell$ elements.

If $Y$ is a commutative algebraic group over a field $K$ (e.g., an abelian variety or an algebraic torus), let $Y_n$ denote the kernel of multiplication by $n$ in $Y(K^s)$, let $T_\ell(Y) = \varprojlim Y_{\ell^n}$, and let $V_\ell(Y) = T_\ell(Y) \otimes \mathbf{Q}_\ell$. For example, if $G_m$ is the multiplicative group, then $\mathbf{Z}_\ell(1) := T_\ell(G_m)$ is a free $\mathbf{Z}_\ell$-module of rank 1.

Throughout this paper, $X$ is a $d$-dimensional abelian variety over a field $F$, $v$ is a discrete valuation on $F$ of residue characteristic $p \geq 0$, and $\ell$ is a prime not equal to $p$. Fix an extension $\bar{v}$ of $v$ to $F^s$. If $w$ is the restriction of $\bar{v}$ to a finite separable extension $L$ of $F$, let $I_w$ denote the inertia subgroup in $\text{Gal}(F^s/L)$ for $\bar{v}$, let $X_w$ denote the special fiber of the Néron model of $X$ at $w$, and let $T_w$ denote the maximal subtorus of $X_w$. Let

$$\rho_\ell : \text{Gal}(F^s/F) \to \text{GL}(T_\ell(X))$$

denote the $\ell$-adic representation. Let $\mathcal{G}$ denote the Zariski closure of $\rho_\ell(I_v)$ in $\text{GL}(V_\ell(X))$. We will make repeated use of the following result.

**Theorem 2.1** (Galois Criterion for Semistable Reduction). $X$ has semistable reduction at $v$ if and only if $I_v$ acts unipotently on $V_\ell(X)$.

**Proof.** See Proposition 3.5 and Corollaire 3.8 of [8] and Theorem 6 on p. 184 of [1].

**Theorem 2.2.** The Zariski closure $\mathcal{G}$ of $\rho_\ell(I_v)$ is connected if and only if $X$ has semistable reduction at $v$.

**Proof.** See Theorem 5.2 of [18]; see also Remarque 1 on p. 396 of [4].
Suppose \( \lambda \) is a polarization on \( X \). Then \( \lambda \) gives rise to a non-degenerate, alternating, \( \text{Gal}(F^s/F) \)-equivariant, \( \mathbb{Z}_\ell(1) \)-valued pairing on \( T_\ell(X) \) (see §§1.0 and 2.5 of [6]). Since \( I_v \) acts trivially on \( \mathbb{Z}_\ell(1) \), we obtain a non-degenerate, alternating, \( I_v \)-invariant pairing

\[
E_\lambda : T_\ell(X) \times T_\ell(X) \rightarrow \mathbb{Z}_\ell.
\]

Then \( E_\lambda \) is perfect if and only if \( \deg(\lambda) \) is not divisible by \( \ell \) (see (2.5.1) and §1.0 of [6]). Let \( \perp \) denote the orthogonal complement with respect to \( E_\lambda \).

**Proposition 2.3.**

(i) We can identify

\[
T_\ell(X_v) = T_\ell(X)^{T_v} \quad \text{and} \quad V_\ell(X_v) = V_\ell(X)^{T_v}.
\]

(ii) \( T_\ell(T_v) = T_\ell(X)^{T_v} \cap (T_\ell(X)^{T_v})^\perp = T_\ell(X_v) \cap T_\ell(X_v)^\perp \).

(iii) \( T_\ell(X_v)/T_\ell(T_v) \) is a free \( \mathbb{Z}_\ell \)-module.

**Proof.** For (i), see Lemma 2 on p. 495 of [15]. For (ii), use (i) and Grothendieck’s Orthogonality Theorem (see (2.5.2) of [6]). For (iii), see (2.1.6) of [6]. \( \square \)

**Proposition 2.4.** Suppose \( L \) is a finite separable extension of \( F \), \( w \) is the restriction of \( \overline{v} \) to \( L \), and \( X \) has semistable reduction at \( w \). Then

\[
T_\ell(T_w) = T_\ell(X_w)^\perp \subseteq T_\ell(X)^{T_w} = T_\ell(X_w).
\]

**Proof.** See Proposition 3.5 of [6]. \( \square \)

### 3. Linear Algebra

We will use the following linear algebra facts in Theorem 5.2 below.

**Definition 3.1.** Suppose \( R \) is a principal ideal domain, and \( \mathcal{M}_1 \) and \( \mathcal{M}_2 \) are free \( R \)-modules. A bilinear form \( e : \mathcal{M}_1 \times \mathcal{M}_2 \rightarrow R \) is called a perfect pairing if the natural homomorphisms \( \mathcal{M}_1 \rightarrow \text{Hom}(\mathcal{M}_2, R) \) and \( \mathcal{M}_2 \rightarrow \text{Hom}(\mathcal{M}_1, R) \) are bijective. If \( \mathcal{L} \) is a submodule of \( \mathcal{M}_1 \) (resp., \( \mathcal{M}_2 \)), we write \( \mathcal{L}^\perp \) for the orthogonal complement of \( \mathcal{L} \) with respect to \( e \) in \( \mathcal{M}_2 \) (resp., \( \mathcal{M}_1 \)).

**Remark 3.2.** Suppose \( R \) is a principal ideal domain, \( \mathcal{M} \) is a free \( R \)-module of rank \( 2n \), and \( e : \mathcal{M} \times \mathcal{M} \rightarrow R \) is an alternating bilinear form. If \( e \) is perfect, then \( \text{Aut}(\mathcal{M}, e) \cong \text{Sp}_{2n}(R) \), where \( \text{Sp}_{2n}(R) \) denotes the group of \( 2n \times 2n \) symplectic matrices over \( R \) (see §5 of [2]).

**Proposition 3.3.** Suppose \( \ell \) is a prime number, \( G \) is a finite group whose order is not divisible by \( \ell \), \( V \) is a finite-dimensional \( \mathbb{Q}_\ell \)-vector space with a linear \( G \)-action, and \( e : V \times V \rightarrow \mathbb{Q}_\ell \) is a \( G \)-invariant non-degenerate alternating (resp., symmetric) bilinear form. Suppose \( \mathcal{M} \) is a \( G \)-stable \( \mathbb{Z}_\ell \)-lattice in \( V \) (there always exist such). Then there exists a perfect \( G \)-invariant alternating (resp., symmetric) bilinear form \( e' : \mathcal{M} \times \mathcal{M} \rightarrow \mathbb{Z}_\ell \).
We obtain a V (isomorphic) image in \(V\) and let symmetric) form \(e\). Corollary 1 of §14.4 of [11]). Denote by \(e\) a proper subset of \(M\), there is a \(G\)-invariant subspace of \(M\), which can be lifted to a \(G\)-invariant splitting of \(Z_\ell\)-lattices \(M = M_1 \oplus M_2\) with \(M_1/\ell M_1 = M_1\) and \(M_2/\ell M_2 = M_2\) (see §15.5 and Corollary 1 of §14.4 of [11]). Denote by \(e_2\) the restriction of \(e\) to \(V_2 := M_2 \otimes \mathbb{Z}_\ell,\mathbb{Q}_\ell\), and let \(V_1\) denote the orthogonal complement of \(V_2\) in \(V\) with respect to \(e\). The restriction of \(e\) to \(V_1\) is non-degenerate, and the restriction of \(e\) to \(M_2\) is perfect. We obtain a \(G\)-invariant orthogonal splitting \(V = V_1 \oplus V_2\). Replace \(M_1\) by its (isomorphic) image in \(V_1\) under the projection map from \(V_1 \oplus V_2\) to \(V_1\). Since \(\dim(V_1) < \dim(V)\), we obtain inductively a perfect \(G\)-invariant alternating (resp., symmetric) form \(e_1\) on \(M_1\). Let \(M = M_1 \oplus M_2\) and \(e' = e_1 \oplus e_2\).

**Lemma 3.4.** Suppose \(R\) is a principal ideal domain, \(M_1\) and \(M_2\) are free \(R\)-modules, \(L\) is a submodule of \(M_1\), \(M_1/L\) is torsion-free, and \(e : M_1 \times M_2 \to R\) is a perfect pairing. Then:

(i) \((L^\perp)^\perp = L\),

(ii) the natural map \(\text{Hom}(M_1, R) \to \text{Hom}(L, R)\) is surjective,

(iii) the induced form \(L \times M_2/L^\perp \to R\) is a perfect pairing.

**Proof.** Clearly, \(L \subseteq (L^\perp)^\perp\). Let \(K\) denote the fraction field of \(R\). By dimension arguments we have \(L \otimes K = ((L \otimes K)^\perp)^\perp = (L^\perp)^\perp \otimes K\). Therefore, \((L^\perp)^\perp/L\) is torsion. Since \(M_1/L\) is torsion-free, we obtain (i). Further, since \(M_1/L\) is torsion-free, \(L\) is a direct summand of \(M_1\), and therefore we have (ii).

Since \(e\) is perfect, we can identify \(\text{Hom}(M_2, R)\) with \(M_1\). Under this identification, \(\text{Hom}(M_2/L^\perp, R) = (L^\perp)^\perp = L\), by (i). Since \(e\) is perfect we can identify \(M_2\) with \(\text{Hom}(M_1, R)\). The natural injection \(M_2/L^\perp \to \text{Hom}(L, R)\) is surjective since \(M_2 = \text{Hom}(M_1, R) \to \text{Hom}(L, R)\) is surjective by (ii).

**Proposition 3.5.** Suppose \(R\) is a principal ideal domain, \(M\) is a free \(R\)-module of finite rank, \(e : M \times M \to R\) is an alternating (resp., symmetric) perfect pairing, \(N\) is a submodule of \(M\), and \(M/N\) is torsion-free. Assume that \(N^\perp \subseteq N\). Then \(\tilde{e}(a + N^\perp, b + N^\perp) = e(a, b)\)

defines an alternating (resp., symmetric) perfect pairing \(\tilde{e} : N/N^\perp \times N/N^\perp \to R\).
Proof. Applying Lemma 3.4 iii with $M_1 = M_2 = M$ and $L = N$ gives an alternating (resp., symmetric) perfect pairing $e' : N \times M/N^\perp \to R$. Now applying Lemma 3.4 iii to $e'$ with $M_1 = M/N^\perp$, $M_2 = N$, and $L = N/N^\perp$, we obtain the desired result.

4. Properties of $I_{v,X}$

We will give a different definition for $I_{v,X}$ than Grothendieck did, and will then show that the two definitions are equivalent. Grothendieck’s definition coincides with (ii) of Theorem 4.2 below.

Definition 4.1. Define $I_{v,X}$ to be the kernel of the natural surjective homomorphism $I_v \to G/G_0$. Define $G_{v,X} = I_v/I_{v,X}$.

Theorem 4.2. $I_{v,X}$ is an open normal subgroup of $I_v$ which enjoys the following properties:

(i) $I_{v,X}$ is the largest open subgroup of $I_v$ such that the Zariski closure of its image under $\rho_\ell$ is $\mathfrak{G}^0$.
(ii) $I_{v,X} = \{ \sigma \in I_v : \sigma$ acts unipotently on $V_\ell(X) \}$.
(iii) If $L$ is a finite separable extension of $F$ and $w$ is the restriction of $\bar{v}$ to $L$, then the following are equivalent:
   (a) $X$ has semistable reduction at $w$,
   (b) $I_w \subseteq I_{v,X}$,
   (c) the Zariski closure of $\rho_\ell(I_w)$ is $\mathfrak{G}^0$.
(iv) $I_{v,X} = I_v$ if and only if $X$ has semistable reduction at $v$.
(v) $I_{v,X}$ is independent of the choice of $\ell$.

Proof. By the definition of $I_{v,X}$, it is an open normal subgroup of $I_v$, and it is the largest open subgroup of $I_v$ such that the Zariski closure of its image under $\rho_\ell$ is $\mathfrak{G}^0$.

Suppose $L$ is a finite separable extension of $F$, and $w$ is the restriction of $\bar{v}$ to $L$. Let $\mathfrak{G}_w$ denote the Zariski closure of $\rho_\ell(I_w)$, and let $V = V_\ell(X)^I_w$.

Suppose first that $X$ has semistable reduction at $w$. Then $\mathfrak{G}_w$ is connected, by Theorem 2.2. Therefore, $I_w \subseteq I_{v,X}$ and $\mathfrak{G}_w = \mathfrak{G}^0$. Since $\mathfrak{G}^0$ is a normal subgroup of $\mathfrak{G}$, it follows that $V$ is stable under $I_{v,X}$. Next we will show that

$$I_{v,X} = \{ \sigma \in I_v : \sigma$ acts unipotently on $V_\ell(X) \}$$
$$= \{ \sigma \in I_v : \sigma$ acts unipotently on $V \}.$$

Since $\mathfrak{G}^0 = \mathfrak{G}_w$, $\mathfrak{G}^0$ acts as the identity on $V$ and on $V_\ell(X)/V$. Therefore, every element of $I_{v,X}$ acts unipotently on $V_\ell(X)$, and therefore on $V$. To show the reverse inclusions, suppose $g \in I_v$ and $g$ acts unipotently on $V$. By Proposition 2.4 (and
after tensoring with \( Q_\ell \), \( V^\perp \subseteq V \). Since \( V^\perp \) is the dual of \( V_\ell(X)/V_\ell \), it follows that \( g \) acts unipotently on \( V_\ell(X)/V_\ell \), and therefore acts unipotently on \( V_\ell(X) \). By Proposition 2.5 of [18], \( g \in \mathcal{I}_{v,X} \). We therefore obtain the desired equalities.

By (i), if \( \mathfrak{G}^0 = \mathfrak{G}_w \), then \( \mathcal{I}_w \subseteq \mathcal{I}_{v,X} \). By (ii) and Theorem 2.1, if \( \mathcal{I}_w \subseteq \mathcal{I}_{v,X} \) then \( X \) has semistable reduction at \( w \). We therefore have (iii). We easily deduce (iv) from (iii).

By Théorème 4.3 of [3], if \( \sigma \in \mathcal{I}_v \) then the characteristic polynomial of \( \rho_\ell(\sigma) \) is independent of \( \ell \). By (ii),

\[
\mathcal{I}_{v,X} = \{ \sigma \in \mathcal{I}_v : \text{the characteristic polynomial of } \rho_\ell(\sigma) \text{ is } (x - 1)^{2d} \}.
\]

Therefore, \( \mathcal{I}_{v,X} \) is independent of \( \ell \).

\[ \square \]

**Proposition 4.3.** If \( L \) is a finite separable extension of \( F \), and \( X \) has semistable reduction at the restriction \( w \) of \( \bar{v} \) to \( L \), then

1. \( \mathcal{I}_{v,X} = \{ \sigma \in \mathcal{I}_v : \sigma \text{ acts unipotently on } V_\ell(X)^{\mathfrak{G}_w} \} \),
2. \( G_{v,X} \) acts faithfully on \( T_\ell(X)^{\mathfrak{G}_w}/T_\ell(X)^{\mathfrak{G}_w} \),
3. \( T_\ell(X)^{\mathfrak{G}_w} = T_\ell(X)^{\mathfrak{G}_w} \).

**Proof.** The proof of Theorem 4.2 included a proof of (i), and easily implies (ii). For (iii), let \( \mathfrak{G}_w \) denote the Zariski closure of \( \rho_\ell(\mathcal{I}_w) \), and note that \( T_\ell(X)^{\mathfrak{G}_w} = T_\ell(X)^{\mathfrak{G}_w} = T_\ell(X)^{\mathfrak{G}_w} \), by Theorem 4.2. \[ \square \]

5. **Restrictions on \( G_{v,X} \)**

**Remark 5.1.** Note that \( G_{v,X} \) is isomorphic to the group of connected components of \( \mathfrak{G} \). If \( p = 0 \) then \( G_{v,X} \) is cyclic, and if \( p > 0 \) then \( G_{v,X} \) is an extension of a cyclic group of order prime to \( p \) by a \( p \)-group (as can be seen by replacing \( F \) by the maximal unramified extension of the completion of \( F \) at \( v \), looking at the extension cut out by \( \mathcal{I}_{v,X} \), taking its maximal tamely ramified subextension, and applying §8 of [3]). In particular, \( G_{v,X} \) is solvable, and has a unique Sylow-\( p \)-subgroup if \( p > 0 \).

The group \( G_{v,X} \) does not change if we replace \( F \) by an extension unramified at \( v \). Therefore, in this section we can and do replace \( F \) by the maximal unramified extension of the completion of \( F \) at \( v \). Then \( \mathcal{I}_{v,X} \) cuts out the smallest Galois extension \( L \) of \( F \) over which \( X \) has semistable reduction. Let \( w \) denote the restriction of \( \bar{v} \) to \( L \). Then \( \mathcal{I}_{v,X} = \mathcal{I}_w \). Therefore, \( \mathcal{I}_w \) is an open normal subgroup of \( \mathcal{I}_v \) of finite index, and \( T_\ell(X_w) = T_\ell(X)^{\mathfrak{G}_w} \) is \( \mathcal{I}_w \)-stable. By Proposition 2.3 we have

\[
T_\ell(X_v) = T_\ell(X_v)^{\mathfrak{G}_w} \subseteq T_\ell(X)^{\mathfrak{G}_w} = T_\ell(X_w)
\]
as \( \mathcal{I}_v \)-modules.

Over an algebraic closure of the residue field, there are exact sequences

\[
0 \to T_w \to X_w^0 \to B_w \to 0, \quad 0 \to U_v \times T_v \to X_v^0 \to B_v \to 0,
\]
where \(B_w\) and \(B_v\) are abelian varieties and \(U_v\) is a unipotent group (see §2.1 of [3]). Base change for Néron models induces a homomorphism \(\iota : X_v \to X_w\) such that if \(n\) is a positive integer not divisible by \(p\), then the restriction of \(\iota\) to the \(n\)-torsion \((X_v)_n\) is injective (see Lemma 2 of [15]; see also (3.1.1) of [3]). Here, \(X_v\) and \(X_w\) are viewed as commutative algebraic groups over an algebraic closure of the residue field at \(w\). The map \(\iota\) induces homomorphisms \(T_v \to T_w\) and \(B_v \to B_w\) whose kernels are finite group schemes killed by appropriate powers of \(p\). The image of \(T_v\) (resp., \(B_v\)) is a subtorus (resp., abelian subvariety) in \(T_w\) (resp., \(B_w\)), and we let \(T\) (resp., \(B\)) denote the corresponding quotient. Let \(a\) and \(t\) (respectively, \(a_v\) and \(t_v\)) denote the abelian and toric ranks of \(X_w\) (respectively, \(X_v\)). Note that \(a\) and \(t\) are independent of the valuation \(w\) above \(v\) at which \(X\) has semistable reduction. We have \(\text{rk}(T_v(X_v)) = 2a + t\) and \(\text{rk}(T_v(X_v)) = 2a_v + t_v\). By the functoriality of Néron models, \(G_{v,X}\) acts on \(X_w\) (see §4.2 of [3]), and therefore acts on \(T_w\) and on \(B_w\). One may easily check that this action is trivial on the image of \(X_v \to X_w\). It follows that \(G_{v,X}\) acts on \(T\) and on \(B\).

Fix a polarization \(\lambda\) on \(X\). Let \(W_\ell\) (respectively, \(S_\ell\)) denote the orthogonal complement of \(V_\ell(X_v)/V_\ell(T_v)\) (respectively, \(T_\ell(X_v)/T_\ell(T_v)\)) with respect to the pairing \(e_\lambda\) on \(V_\ell(X_w)/V_\ell(T_w)\) (respectively, \(T_\ell(X_w)/T_\ell(T_w)\)) induced by \(E_\lambda\). Then \(W_\ell\) is a \(G_{v,X}\)-stable \(\mathbb{Q}_\ell\)-vector space of dimension \(2a - 2a_v\) and \(S_\ell\) is a \(G_{v,X}\)-stable \(\mathbb{Z}_\ell\)-sublattice of rank \(2a - 2a_v\).

Recall that \(\ell\) is always a prime not equal to \(p\).

**Theorem 5.2.**

(i) The form \(e_\lambda : W_\ell \times W_\ell \to \mathbb{Q}_\ell\) is non-degenerate, alternating, and \(G_{v,X}\)-invariant. The vector space \(W_\ell\) and the lattice \(S_\ell\) do not depend on the choice of polarization \(\lambda\). The natural actions of \(G_{v,X}\) on \(T\), \(W_\ell\), and \(e_\lambda\) induce an injection

\[
G_{v,X} \hookrightarrow \text{Aut}(T) \times \text{Aut}(W_\ell, e_\lambda) \cong \text{GL}_{\ell - t_v}(\mathbb{Z}) \times \text{Sp}_{2(a - a_v)}(\mathbb{Q}_\ell)
\]

such that the projection onto the first factor is independent of \(\ell\), and the characteristic polynomial of the projection onto the second factor has integer coefficients independent of \(\ell\).

(ii) If \(\ell\) does not divide \(\deg(\lambda) \# G_{v,X}\), then \(e_\lambda : S_\ell \times S_\ell \to \mathbb{Z}_\ell\) is perfect and the above injection takes values in

\[
\text{Aut}(T) \times \text{Aut}(S_\ell, e_\lambda) \cong \text{GL}_{\ell - t_v}(\mathbb{Z}) \times \text{Sp}_{2(a - a_v)}(\mathbb{Z}_\ell).
\]

(iii) Suppose \(\ell\) does not divide \(\# G_{v,X}\). Then for every \(G_{v,X}\)-stable \(\mathbb{Z}_\ell\)-lattice \(M\) in \(W_\ell\), there exists a perfect \(\mathbb{Z}_\ell\)-valued \(G_{v,X}\)-invariant alternating pairing \(e'\) on \(M\). I.e., there is an injection

\[
G_{v,X} \hookrightarrow \text{Aut}(T) \times \text{Aut}(M, e') \cong \text{GL}_{\ell - t_v}(\mathbb{Z}) \times \text{Sp}_{2(a - a_v)}(\mathbb{Z}_\ell).
\]
Proof. By Proposition 4.3i, $G_{v,X}$ acts faithfully on $T_{\ell}(X_v)$. The natural homomorphism

$$\varphi_{\ell} : G_{v,X} \to \text{Aut}(T_{\ell}(T_w)) \times \text{Aut}(T_{\ell}(X_w)/T_{\ell}(T_w))$$

is injective, since its kernel consists of unipotent operators of finite order in characteristic zero. The map $\varphi_{\ell}$ factors through $\text{Aut}(T) \times \text{Aut}(T_{\ell}(X_w)/T_{\ell}(T_w))$, since $G_{v,X}$ acts trivially on $T_v$. By Theorem 2.3i and Proposition 2.4,

$$T_{\ell}(T_v) \subseteq T_{\ell}(X_v) \cap T_{\ell}(T_w) = T_{\ell}(X_v) \cap T_{\ell}(X_w)^{\perp} \subseteq T_{\ell}(X_v) \cap T_{\ell}(X_w)^{\perp} = T_{\ell}(T_v).$$

Therefore,

$$T_{\ell}(T_v) = T_{\ell}(X_v) \cap T_{\ell}(T_w) = T_{\ell}(X_v) \cap T_{\ell}(X_w)^{\perp}.$$

By Proposition 2.4, $T_{\ell}(T_w) = T_{\ell}(X_w)^{\perp}$. It follows that $e_\lambda$ is non-degenerate on $T_{\ell}(X_w)/T_{\ell}(T_w)$ and on $T_{\ell}(X_v)/T_{\ell}(T_v)$, and therefore on $V_{\ell}(X_w)/V_{\ell}(T_w)$ and on $V_{\ell}(X_v)/V_{\ell}(T_v)$. Further, $e_\lambda$ is $G_{v,X}$-invariant.

Let $r = \#G_{v,X}$ and let $u = \frac{1}{r} \sum_{g \in G_{v,X}} g \in Q_{\ell}[G_{v,X}]$. Then

$$u \left( V_{\ell}(X_w) \right) = V_{\ell}(X_w)^{G_{v,X}} = V_{\ell}(X_v), \quad u \left( V_{\ell}(T_w) \right) = V_{\ell}(T_w)^{G_{v,X}} = V_{\ell}(T_v),$$

$$W_{\ell} = (1 - u)(V_{\ell}(X_w)/V_{\ell}(T_w)), \quad S_{\ell} = W_{\ell} \cap (T_{\ell}(X_w)/T_{\ell}(T_w)).$$

Therefore $W_{\ell}$ and $S_{\ell}$ are independent of the choice of $\lambda$, and we have a $G_{v,X}$-invariant $e_\lambda$-orthogonal splitting

$$V_{\ell}(X_w)/V_{\ell}(T_w) \cong V_{\ell}(X_v)/V_{\ell}(T_v) \oplus W_{\ell}.$$ 

Since $G_{v,X}$ acts trivially on $V_{\ell}(X_v)/V_{\ell}(T_v)$, the map $\varphi_{\ell}$ factors through $\text{Aut}(T) \times \text{Aut}(W_{\ell})$. Further, since $e_\lambda$ is non-degenerate, alternating, and $G_{v,X}$-invariant, $\varphi_{\ell}$ factors through $\text{Aut}(T) \times \text{Aut}(W_{\ell}, e_\lambda) \cong \text{GL}_{r-1}(\mathbb{Z}) \times \text{Sp}_{2(a-\alpha_\ell)}(\mathbb{Q}_{\ell})$. See p. 360 of [2] for the integrality and $\ell$-independence.

If $r$ is not divisible by $\ell$, then $u \in \mathbb{Z}_{\ell}[G_{v,X}]$,

$$u \left( T_{\ell}(X_w) \right) = T_{\ell}(X_w)^{G_{v,X}} = T_{\ell}(X_v), \quad u \left( T_{\ell}(T_w) \right) = T_{\ell}(T_w)^{G_{v,X}} = T_{\ell}(T_v),$$

$$S_{\ell} = (1 - u)(T_{\ell}(X_w)/T_{\ell}(T_w)),$$

and we have a $G_{v,X}$-invariant $e_\lambda$-orthogonal splitting

$$T_{\ell}(X_w)/T_{\ell}(T_w) \cong T_{\ell}(X_v)/T_{\ell}(T_v) \oplus S_{\ell}.$$ 

To derive (iii), apply Proposition 1.3 and Remark 3.2. To derive (ii), suppose $\deg(\lambda)$ is not divisible by $\ell$. Then $E_\lambda$ is perfect. Applying Proposition 3.3 with $\mathcal{M} = T_{\ell}(X), \mathcal{N} = T_{\ell}(X_w), \mathcal{R} = \mathbb{Z}_{\ell}$, and $e = E_\lambda$, we deduce that $e_\lambda$ is perfect, giving (ii).}

Next we give a variation of Theorem 5.2, whose proof and statement are of independent interest. Retain the notation from the proof of Theorem 5.2. Since $G_{v,X}$ is finite, there exists a $G_{v,X}$-invariant polarization $\delta$ on the abelian variety $B$. 


Theorem 5.3. If $\ell$ does not divide $\deg(\delta)$, then the action of $G_{v,X}$ on $T$ and on $B$ induces injections

$$G_{v,X} \hookrightarrow \text{Aut}(T) \times \text{Aut}(B, \delta) \hookrightarrow \text{GL}_{t-t_v}(Z) \times \text{Sp}_{2(a-a_v)}(Z_\ell)$$

where the projections onto the first factors are independent of $\ell$, and the characteristic polynomials of the projections onto the second factors have integer coefficients independent of $\ell$. In particular, the conclusion holds for all $\ell \neq p$, whenever $a-a_v = 0$ or $1$ (e.g., if $X$ or $B$ is an elliptic curve).

Proof. As $\mathbb{Z}_\ell$-modules,

$$T_\ell(B_w) \cong T_\ell(X_w)/T_\ell(T_w), \quad T_\ell(B_v) \cong T_\ell(X_v)/T_\ell(T_v),$$

$$T_\ell(B) \cong T_\ell(B_w)/T_\ell(B_v), \quad \text{and} \quad T_\ell(T) \cong T_\ell(T_w)/T_\ell(T_v),$$

and we have $\dim(T) = t - t_v$ and $\dim(B) = a - a_v$. By considering the pairing $E_\delta$ on $T_\ell(B)$ induced by $\delta$, it follows that the image of $G_{v,X}$ in $\text{Aut}(T_\ell(B))$ lies in $\text{Aut}(T_\ell(B), E_\delta)$. Since $\ell$ does not divide $\deg(\delta)$, then $E_\delta$ is perfect. The actions of $G_{v,X}$ on $T$ and $B$ therefore induce a homomorphism

$$\eta_\ell : G_{v,X} \to \text{Aut}(T) \times \text{Aut}(T_\ell(B), E_\delta) \cong \text{GL}_{t-t_v}(Z) \times \text{Sp}_{2(a-a_v)}(Z_\ell).$$

By Proposition 4.3ii, $G_{v,X}$ acts faithfully on $T_\ell(X_w)$. The kernel of $\eta_\ell$ consists of elements in $G_{v,X} \subset \text{Aut}(T_\ell(X_w))$ which act as the identity on $T_\ell(T_w)$ and on $T_\ell(X_w)/T_\ell(T_w)$. Therefore, these elements act as unipotent operators on $T_\ell(X_w)$. Since they also have finite order, it follows that $\eta_\ell$ is injective. As before, see p. 360 of [3] for integrality and $\ell$-independence. When $\dim(B) = 0$ or $1$, we may suppose that $\deg(\delta) = 1$. \qed

Remark 5.4. If $\ell \neq 2$, then reducing the second factor modulo $\ell$ in Theorem 5.3ii or 5.3 gives

$$G_{v,X} \hookrightarrow \text{GL}_{t-t_v}(Z) \times \text{Sp}_{2(a-a_v)}(F_\ell).$$

If either $p \nmid G_{v,X}$ or $p \deg(\delta)$ is odd, then

$$G_{v,X} \hookrightarrow \text{GL}_{t-t_v}(Z) \times \text{Sp}_{2(a-a_v)}(Z/4Z).$$

Remark 5.5. It follows from Corollary 6.3 that the statement in Theorem 5.3ii holds true whenever $p \neq \ell > 1 + \max\{t - t_v, 2(a-a_v)\}$ or $p \neq \ell > 2d + 1$.

Remark 5.6. Theorems 5.2 and 5.3 and Remark 5.4 have led us to the question of when a finite inertia group embedded in $\text{Sp}_{2D}(Q_\ell)$ can also be embedded in $\text{Sp}_{2D}(Z_\ell)$ or in $\text{Sp}_{2D}(F_\ell)$ in such a way that the characteristic polynomials are preserved. Suppose $p$ is a prime number, and $G$ is a finite group with a normal Sylow-$p$ subgroup $P$ such that $G/P$ is cyclic. Suppose $\ell$ is a prime number, $\ell \neq p$, $V$ is a $2D$-dimensional $Q_\ell$-vector space, $e : V \times V \to Q_\ell$ is a non-degenerate
alternating bilinear form, and $\varphi : G \to \text{Aut}(V, e)$ is an injection. Does there always exist a $G$-stable $\mathbb{Z}_\ell$-lattice $S$ in $V$ with a perfect $G$-invariant alternating $\mathbb{Z}_\ell$-valued pairing? The answer is no. However, the answer is yes if $\ell > D + 1$, and this bound is sharp. Does there always exist an injection $\eta : G \to \text{Sp}_{2D}(\mathbb{F}_\ell)$ such that for all $g \in G$, the characteristic polynomial of $\eta(g)$ is equal to the characteristic polynomial of $\varphi(g) \mod \ell$? The answer is no, but is yes if $\ell > 3$, and this bound is sharp. Proofs and counterexamples (including counterexamples with inertia groups of the form $G_{v,X}$ for abelian surfaces $X$) will appear in a later paper.

**Remark 5.7.** Note that if $\# G_{v,X}$ is not divisible by $p$, then $X$ acquires semistable reduction over a tamely ramified extension; see [3] for a study of Néron models in this important case.

### 6. Bounds on the Order of $G_{v,X}$

One can use Theorem 5.2, Remark 5.1, and group theory to obtain more precise information about the finite group $G_{v,X}$. In Corollary 6.3 below we give one such result. The next two results, along with Theorem 5.2, will be used to prove Corollary 6.3.

**Lemma 6.1.** If $\ell$ is a prime number, $\ell \equiv 5 \pmod{8}$, $r$ and $m$ are positive integers, and $\text{Sp}_{2m}(\mathbb{F}_\ell)$ has an element of (exact) order $2^r$, then $2^{r-1} \leq 2m$.

**Proof.** We may assume $r \geq 3$. Let $\zeta$ be a primitive $2^r$-th root of unity in $\bar{\mathbb{F}}_\ell$. It is easy to check that the condition $\ell \equiv 5 \pmod{8}$ implies that $[\mathbb{F}_\ell(\zeta) : \mathbb{F}_\ell] = 2^{r-2}$ and that $\zeta^{-1}$ is not a conjugate of $\zeta$. It follows that in $\mathbb{F}_\ell[x]$ we have $\Phi_{2^r} = fg$, where $\Phi_{2^r}$ is the $2^r$-th cyclotomic polynomial, $f$ and $g$ are irreducible polynomials in $\mathbb{F}_\ell[x]$ of degree $2^{r-2}$, and the roots of $g$ are the inverses of the roots of $f$. Let $\gamma$ denote an element of order $2^r$ in $\text{Sp}_{2m}(\mathbb{F}_\ell)$. If $\alpha$ is an eigenvalue of $\gamma$, then so is $\alpha^{-1}$. It follows that the characteristic polynomial of $\gamma$ is divisible by $\Phi_{2^r}$. Therefore, $2^{r-1} \leq 2m$. \qed

**Proposition 6.2.** If $q$ is a prime number, $r$ and $m$ are positive integers, and for all prime numbers $\ell$ in a set of density 1 the group $\text{Sp}_{2m}(\mathbb{F}_\ell)$ contains an element of (exact) order $q^r$, then $\varphi(q^r) \leq 2m$.

**Proof.** If $q = 2$ the result follows from Lemma 6.1. Suppose $q$ is odd. Then $(\mathbb{Z}/q^r\mathbb{Z})^\times$ is cyclic, and by the Chebotarev density theorem there is a set of primes $\ell$ of positive density such that $\text{Gal}(\mathbb{F}_\ell(\zeta_{q^r})/\mathbb{F}_\ell) \cong (\mathbb{Z}/q^r\mathbb{Z})^\times \cong \text{Gal}(\mathbb{Q}(\zeta_{q^r})/\mathbb{Q})$. If $q \neq \ell$, then $\text{GL}_n(\mathbb{F}_\ell)$ contains an element of order $q^r$ if and only if $[\mathbb{F}_\ell(\zeta_{q^r}) : \mathbb{F}_\ell] \leq n$ (see [20]). Therefore, $\varphi(q^r) \leq 2m$. \qed
An explicit description of all possible orders of elements of general linear groups over arbitrary fields is given in [20].

Let \([\ ]\) denote the greatest integer function, let \(s(n, q) = \sum_{j=0}^{\infty} \left\lfloor \frac{n}{q^j(q-1)} \right\rfloor\), and let \(J(n) = \prod q^{s(n, q)}\), where \(q\) runs over the prime numbers. Note that the prime divisors of \(J(n)\) are the primes \(q \leq n + 1\). For example, \(J(0) = 1, J(1) = 2,\) and \(J(2) = 24\). For \(n \geq 1\), Theorem 3.2 of [16] shows that \(J(n) < (4.462n)^n\) if \(n\) is even and \(J(n) < \sqrt{2}(4.462n)^n\) if \(n\) is odd. The method of Minkowski and Serre ([8] and pp. 119–121 of [12]; see also Formula 3.1 of [16]) shows that, for all \(N \geq 3\), \(J(2m)\) is equal to the greatest common divisor of the orders of the groups \(\text{Sp}_{2m}(F_\ell)\), for primes \(\ell \geq N\). Further ([14]), \(J(n)\) is the least common multiple of the orders of the finite subgroups of \(\text{GL}_n(Q)\) (or equivalently, of \(\text{GL}_n(Z)\)). While \(J(n)\) is optimal from the point of view of divisibility, there are sharper upper bounds on the orders of finite subgroups of \(\text{GL}_n(Q)\). The determination of the finite subgroups of maximum order for general linear groups over \(Q\) and over cyclotomic fields is given in [4].

Let

\[r_p = s(t - t_v, p) + s(2(a - a_v), p), \quad M_{v, X} = \max\{t - t_v, 2(a - a_v)\},\]

and for all primes \(q\) such that \(p \neq q \leq M_{v, X} + 1\) let

\[r_q = 1 + \left\lfloor \log_q \left( \frac{M_{v, X}}{q - 1} \right) \right\rfloor.\]

If \(X\) has semistable reduction at \(v\) let \(N_{v, X} = 1\), and otherwise let \(N_{v, X} = \prod q^{s_q}\), where the product runs over all prime numbers \(q \leq M_{v, X} + 1\) (this might include \(q = p\)). Let \(Q_{v, X}\) denote the largest prime divisor of \(#G_{v, X}\) (let \(Q_{v, X} = 1\) if \(I_v = I_{v, X}\)).

**Corollary 6.3.** The order of \(G_{v, X}\) divides \(N_{v, X}\). In particular, \(Q_{v, X} \leq M_{v, X} + 1 \leq 2d + 1\), and \(#G_{v, X}\) divides \(J(t - t_v)J(2(a - a_v))\) and divides \(J(2d)\).

**Proof.** By Theorem 3.2i, \(#G_{v, X}\) divides \(J(t - t_v)J(2(a - a_v))\), and therefore \(Q_{v, X} \leq M_{v, X} + 1\). Note that \(J(t - t_v)J(2(a - a_v))\) divides \(J(2d)\), since \(t - t_v + 2(a - a_v) \leq t + 2a = 2d - t \leq 2d\). As noted in Remark 5.3, the prime-to-\(p\) part of \(G_{v, X}\) is cyclic. Suppose \(q\) is a prime divisor of \(#G_{v, X}\), and \(q \neq p\). Then \(\text{GL}_n(Q)\) contains an element of order \(q^r\) if and only if \(\varphi(q^r) \leq n\), i.e., if and only if \(r \leq 1 + \left\lfloor \log_q \left( \frac{n}{q-1} \right) \right\rfloor\). The result now follows from Proposition 6.2. \(\square\)

7. Applications

Retain the notation of the previous sections. The next result follows immediately from Corollary 6.3 and Theorem 4.2ii,iv.
Corollary 7.1. Suppose $L$ is a finite separable extension of $F$, and $w$ is the restriction of $\bar{v}$ to $L$. Suppose $X$ has semistable reduction at $w$ but not at $v$. Then $[\mathcal{I}_v : \mathcal{I}_w]$ has a prime divisor $q$ such that $q \leq Q_{v,X} \leq M_{v,X} + 1 \leq 2d + 1$.

Remark 7.2. Suppose $L$ is a finite separable extension of $F$, and $w$ is the restriction of $\bar{v}$ to $L$. Let $k_w$ and $k_v$ denote the residue fields and let $e(w/v) = [w(L^\times) : v(F^\times)]$ (= the ramification degree). Then $[\mathcal{I}_v : \mathcal{I}_w] = e(w/v)[k_w : k_v]$, where the subscript $i$ denotes the inseparable degree (see Proposition 21 on p. 32 of [9] for the case where $L/F$ is Galois. In the non-Galois case, take a Galois extension $L'$ of $F$ which contains $L$, and apply the result to $L'/L$ and $L'/F$, to obtain the result for $L/F$). Taking completions, then $[L_w : F_v] = e(w/v)[k_w : k_v] = [\mathcal{I}_v : \mathcal{I}_w][k_w : k_v]$, where the subscript $s$ denotes the separable degree. Therefore, $[\mathcal{I}_v : \mathcal{I}_w]$ divides $[L_w : F_v]$.

Corollary 7.3. Suppose $L$ is a finite separable extension of $F$. Suppose in addition that either $F$ is complete with respect to $v$, or $L/F$ is Galois. Suppose $X$ has semistable reduction at the restriction $w$ of $\bar{v}$ to $L$, but does not have semistable reduction at $v$. Then $[L : F]$ has a prime divisor $q$ such that $q \leq Q_{v,X}$. In particular, if $[L : F]$ is a power of a prime $q$, then $q \leq Q_{v,X}$.

Proof. Under our assumptions on $L/F$, its degree is divisible by $[\mathcal{I}_v : \mathcal{I}_w]$. The result now follows from Corollary 7.1.

Recall that there exists a finite Galois extension $L$ of $F$ such that $X$ has semistable reduction at the extensions of $v$ to $L$ (see Proposition 3.6 of [3]).

Corollary 7.4. Let $r = \#G_{v,X}$ and let $\zeta_r$ denote a primitive $r$-th root of unity. Suppose that $r$ is not divisible by $p$. Then there is a cyclic degree $r$ extension $L$ of $F(\zeta_r)$ such that $X$ acquires semistable reduction over every extension of $v$ to $L$. If either $F = F(\zeta_r)$ or $p > Q_{v,X}$, then there exists a finite Galois extension $L$ of $F$ of degree prime to $p$ such that $X$ acquires semistable reduction over every extension of $v$ to $L$.

Proof. Let $L$ be the field obtained by adjoining an $r$-th root of a uniformizing parameter to $F(\zeta_r)$. Then $L/F$ is Galois, $F(\zeta_r)/F$ is unramified above $v$, $L/F(\zeta_r)$ is totally ramified, and $\text{Gal}(L/F(\zeta_r)) \cong G_{v,X} \cong \mathbb{Z}/r\mathbb{Z}$. Let $w$ be the restriction of $\bar{v}$ to $L$. By the construction of $L$, we have $\mathcal{I}_w \cong \mathcal{I}_{v,X}$. By Theorem 4.3ii, $X$ has semistable reduction at $w$. Since $L/F$ is Galois, $X$ has semistable reduction at every extension of $v$ to $L$ by Theorem 4.1. If $p > Q_{v,X}$, then $p$ does not divide $\varphi(r)$ by Corollary 6.3, and therefore $p$ does not divide $[F(\zeta_r) : F]$.

By Corollary 6.3, $Q_{v,X}$ can be replaced by $2d + 1$ (or by $M_{v,X} + 1$) in Corollaries 7.3 and 7.4.
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