FEKETE-SZEGÖ INEQUALITY OF BI-STARLIKE AND BI-CONVEX FUNCTIONS OF ORDER $b$ ASSOCIATED WITH SYMMETRIC $q$-DERIVATIVE IN CONIC DOMAINS

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Abstract. In this paper, two new subclasses of bi-univalent functions related to conic domains are defined by making use of symmetric $q$-differential operator. The initial bounds for Fekete-Szegö inequality for the functions $f$ in these classes are estimated.

1. Introduction

Let $A$ denotes the set of all functions which are analytic in the unit disc $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ with Taylor’s series expansion of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are normalized by $f(0) = 0$, $f'(0) = 1$.

The subclass of $A$ consisting of all univalent functions is denoted by $S$. That is

$$S = \{f \in A : f \text{ is univalent in } \Delta\}.$$ 

A function $f \in A$ is said to be a starlike function if

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > 0, \ z \in \Delta.$$ 

A function $f \in A$ is said to be a convex function if

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0, \ z \in \Delta.$$ 

Goodman [8–10] introduced the classes uniformly starlike and uniformly convex functions as subclasses of starlike and convex functions. A starlike function (or convex function) is said to be uniformly starlike (or uniformly convex) if the image of every circular arc $\zeta$ contained in $\Delta$, with center at $\xi$ also in $\Delta$ is starlike (or convex) with respect to $f(\xi)$. The class of uniformly starlike functions is represented by $UST$ and the class of uniformly convex functions is represented by $UCV$. The class of parabolic starlike functions is represented by $S_p$. Rønning [22] and Minda [16,17] independently gave the characterization for the classes $S_p$ and $UCV$ as follows.

A function $f \in A$ is said to be in the class $S_p$ if and only if

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \left|\frac{zf'(z)}{f(z)} - 1\right|, \ z \in \Delta.$$ 

2010 Mathematics Subject Classification. Primary 30C45, 30C50.

Key words and phrases. Analytic function, Bi-univalent function, Bi-starlike function, Bi-convex function, Conic domain, $q$-differential operator, Fekete-Szegö inequality.

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A function \( f \in \mathcal{A} \) is said to be in the class \( UCV \) if and only if
\[
\Re \left( 1 + \frac{zf''(z)}{f'(z)} \right) > k \left| \frac{zf''(z)}{f'(z)} \right|, \quad z \in \Delta.
\]

Also, it is clear that \( f \in UCV \iff zf'(z) \in S_p \).

Kanas and Wisniowska [13, 14] introduced \( k \)-uniformly starlike functions and \( k \)-uniformly convex functions as follows.
\[
k - ST = \left\{ f : f \in S \text{ and } \Re \left( \frac{zf'(z)}{f(z)} \right) > k \left| \frac{zf'(z)}{f(z)} - 1 \right|, z \in \Delta, k \geq 0 \right\}
\]
\[
k - UCV = \left\{ f : f \in S \text{ and } \Re \left( 1 + \frac{zf'(z)}{f(z)} \right) > k \left| \frac{zf''(z)}{f'(z)} \right|, z \in \Delta, k \geq 0 \right\}.
\]

Bharati, et al. [6], defined \( k - ST(\beta) \) and \( k - UCV(\beta) \) as follows. A function \( f \in \mathcal{A} \) is said to be in the class \( k - ST(\beta) \) if and only if
\[
\Re \left( \frac{zf'(z)}{f(z)} \right) - \beta > k \left| \frac{zf'(z)}{f(z)} - 1 \right|, \quad z \in \Delta.
\]

A function \( f \in \mathcal{A} \) is said to be in the class \( k - UCV(\beta) \) if and only if
\[
\Re \left( 1 + \frac{zf''(z)}{f'(z)} \right) - \beta > k \left| \frac{zf''(z)}{f'(z)} \right|, \quad z \in \Delta.
\]

Sim et al. [24], generalized above classes and introduced \( k - ST(\alpha, \beta) \) and \( k - UCV(\alpha, \beta) \) as below.

**Definition 1.** A function \( f \in \mathcal{A} \) is said to be in the class \( k - ST(\alpha, \beta) \) if and only if
\[
\Re \left\{ \frac{zf'(z)}{f(z)} \right\} - \beta > k \left| \frac{zf'(z)}{f(z)} - \alpha \right|, \quad z \in \Delta,
\]
where \( 0 \leq \beta < \alpha \leq 1 \) and \( k(1 - \alpha) < 1 - \beta \).

**Definition 2.** A function \( f \in \mathcal{A} \) is said to be in the class \( k - UCV(\alpha, \beta) \) if and only if
\[
\Re \left( 1 + \frac{zf''(z)}{f'(z)} \right) - \beta > k \left| 1 + \frac{zf''(z)}{f'(z)} - \alpha \right|, \quad z \in \Delta.
\]
where \( 0 \leq \beta < \alpha \leq 1 \) and \( k(1 - \alpha) < 1 - \beta \).

In particular, for \( \alpha = 1, \beta = 0 \) the classes \( k - ST(\alpha, \beta) \) and \( k - UCV(\alpha, \beta) \) reduces to \( k - ST \) and \( k - UCV \) respectively. Further, for \( \alpha = 1 \) these classes coincides with the classes studied by Nishiwaki et al. [18] and Shams et al. [23]. In 2017, Annamalai et al. [5], obtained second Hankel determinant of analytic functions involving conic domains.

**Geometric Interpretation:** A function \( f \in k - ST(\alpha, \beta) \) and \( k - UCV(\alpha, \beta) \) if and only if \( z \frac{f'(z)}{f(z)} \) and \( 1 + z \frac{f''(z)}{f'(z)} \), respectively takes all the values in the conic domain \( \Omega_{k, \alpha, \beta} \).

\[
\Omega_{k, \alpha, \beta} = \{ \omega: \omega \in \mathbb{C} \text{ and } k|\omega - \alpha| < \Re(\omega) - \beta \}
\]
or
\[
\Omega_{k, \alpha, \beta} = \{ \omega: \omega \in \mathbb{C} \text{ and } k\sqrt[4]{|\Re(\omega) - \alpha|^2 + |\Im(\omega)|^2} < \Re(\omega) - \beta \},
\]
where $0 \leq \beta < \alpha \leq 1$ and $k(1 - \alpha) < 1 - \beta$. Clearly $1 \in \Omega_k, \alpha, \beta$ and $\Omega_k, \alpha, \beta$ is bounded by the curve

$$\partial \Omega_k, \alpha, \beta = \{ \omega : \omega = u + iv \text{ and } k^2(u - \alpha)^2 + k^2v^2 = (u - \beta)^2 \}.$$  

**Definition 3.** The Caratheodory functions $p \in P$ is said to be in the class $\mathcal{P}(p_k, \alpha, \beta)$ if and only if $p$ takes all the values in the conic domain $\Omega_k, \alpha, \beta$. Analytically it is defined as follows:

$$\mathcal{P}(p_k, \alpha, \beta) = \{ p : p \in \mathcal{P} \text{ and } p(\Delta) \subset \Omega_k, \alpha, \beta \},$$

$$\mathcal{P}(p_k, \alpha, \beta) = \{ p : p \in \mathcal{P} \text{ and } p(z) < p_k, \alpha, \beta, z \in \Delta \}.$$  

Note that $\partial \Omega_k, \alpha, \beta$ represents conic section about real axis. In particular, $\Omega_k, \alpha, \beta$ represents an elliptic domain for $k > 1$, parabolic domain for $k = 1$, hyperbolic domain for $0 < k < 1$. Sim et al. [24] obtained the functions $p_k, \alpha, \beta(z)$ which play the role of extremal functions of $\mathcal{P}(p_k, \alpha, \beta)$ as

$$p_k, \alpha, \beta(z) = \left\{ \begin{array}{ll} 1 + (1 - 2\beta)z, & \text{for } k = 0; \\ \frac{1 - z}{\alpha + \frac{2(\alpha - \beta)}{\pi^2}\log^2\left(1 + \frac{\sqrt{u_k(z)}}{1 - \sqrt{u_k(z)}}\right)}, & \text{for } k = 1; \\ \frac{(\alpha - \beta)}{1 - k^2}\cosh\{u(k)\log\left(\frac{1 + \sqrt{u_k(z)}}{1 - \sqrt{u_k(z)}}\right)\} + \frac{\beta - \alpha k^2}{1 - k^2}, & \text{for } 0 < k < 1; \\ \frac{(\alpha - \beta)}{k^2 - 1}\sin^2\left(\frac{\pi}{2K(k)}\int_0^\omega dt\right) + \frac{\alpha k^2 - \beta}{k^2 - 1}, & \text{for } k > 1; \end{array}\right.$$

where $u(k) = \frac{2}{\pi}\cos^{-1}k$, $u_k(z) = \frac{z + \rho_k}{1 + \rho_k z}$ and

$$\rho_k = \left\{ \begin{array}{ll} \frac{(e^A - 1)^2}{(e^A + 1)^2}, & \text{for } k = 1; \\ \exp\left(-\frac{1}{u_k(z)}\arccosh B\right) - 1, & \text{for } 0 < k < 1; \\ \exp\left(-\frac{1}{u_k(z)}\arccosh B\right) + 1, & \text{for } k > 1; \end{array}\right.$$

with $A = \sqrt{\frac{1 - \alpha}{2(\alpha - \beta)}}, B = \frac{1}{\alpha - \beta}(1 - k^2 - \beta + \alpha k^2)$, $C = \frac{1}{\alpha - \beta}(k^2 - 1 + \beta - \alpha k^2)$. Also

$$K(k) = \frac{\pi}{\sqrt{1 - t^2}K(1 - t^2)},$$

$$K'(\kappa) = K(\sqrt{1 - \kappa^2}),$$

$$\kappa = \cosh\left(\frac{\pi K'(\kappa)}{4K(k)}\right).$$

According to Koebe’s $\frac{1}{4}$ theorem, every analytic and univalent function $f$ in $\Delta$ has an inverse $f^{-1}$ and is defined as

$$f^{-1}(f(z)) = z, (z \in \Delta), f(f^{-1}(w)) = w\left(|w| < r_0(f); r_0(f) \geq \frac{1}{4}\right).$$
Also the function $f^{-1}$ can be written as

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \ldots.$$ (1.6)

**Definition 4.** A function $f \in \mathcal{A}$ is said to be bi-univalent if both $f$ and analytic extension of $f^{-1}$ in $\Delta$ are univalent in $\Delta$. The class of all bi-univalent functions is denoted by $\Sigma$. That is a function $f$ is said to be bi-univalent if and only if

1. $f$ is an analytic and univalent function in $\Delta$.
2. There exists an analytic and univalent function $g$ in $\Delta$ such that $f(g(z)) = g(f(z)) = z$ in $\Delta$.

The class of bi-univalent functions was introduced by Lewin [15] in 1967. Recently many researchers ([1], [4], [12], [19], [20], [25], [26], [27], [28], [29], [30], [31], [32]) have introduced and investigated several interesting subclasses of the bi-univalent functions and they have found non-sharp estimates of two Taylor-Maclaurin coefficients $|a_2|, |a_3|$, Fekete-Szegö inequality and second Hankel determinants. In 2017, Şahsene Altinkaya, Sibel Yalçın [2], [3] estimated the coefficients and Fekete-Szegö inequality for some subclasses of bi-univalent functions involving symmetric $q$-derivative operator subordinate to the generating function of Chebyshev polynomial.

**Definition 5.** [11] Jackson defined $q$-derivative operator $D_q$ of an analytic function $f$ of the form (1.1) as follows:

$$D_qf(z) = \begin{cases} \frac{f(qz) - f(z)}{(q-1)z}, & \text{for } z \neq 0, \\ f'(0), & \text{for } z = 0 \end{cases}$$

$$D_qf(0) = f'(0) \text{ and } D_q^2 = D_q(D_qf(z)).$$

If $f(z) = z^n$ for any positive integer $n$, the $q$-derivative of $f(z)$ is defined by

$$D_qz^n = \frac{(q^nz^n - z^n)}{qz - z} = [n]_q z^{n-1},$$

where $[n]_q = \frac{q^n - 1}{q - 1}$. As $q \to 1^-$ and $k \in \mathbb{N}$, we have $[n]_q \to n$ and $\lim_{q \to 1}(D_qf(z)) = f'(z)$ where $f'$ is normal derivative of $f$. Therefore

$$D_qf(z) = 1 + \sum_{n=2}^{\infty} [n]_q a_n z^{n-1}.$$ (1.1)

**Definition 6.** [7] The symmetric $q$-derivative operator $\tilde{D}_q$ of an analytic function $f$ is defined as follows:

$$\tilde{D}_qf(z) = \begin{cases} \frac{f(qz) - f(q^{-1}z)}{(q - q^{-1})z}, & \text{for } z \neq 0, \\ f'(0), & \text{for } z = 0 \end{cases}.$$ (1.2)

It is clear that $\tilde{D}_qz^n = [n]_q z^{n-1}$ and $\tilde{D}_qf(z) = 1 + \sum_{n=2}^{\infty} [n]_q a_n z^{n-1}$, where $[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}$. The relation between $q$-derivative operator and symmetric $q$-derivative operator is given by

$$(\tilde{D}_qf)(z) = D_q^2 f(q^{-1}z).$$
If \( g \) is the inverse of \( f \) then

\[
\tilde{(D_q g)}(w) = \frac{g(qw) - g(q^{-1}w)}{(q - q^{-1})w} = 1 - [2]_q a_2 w + [3]_q (2a_2^2 - a_3)w^2 - [4]_q (5a_2^3 - 5a_2a_3 + a_4)w^3 + \ldots
\]

The \( q \)-calculus has so many applications in various branches of mathematics and physics. Jackson [11] developed \( q \)-integral and \( q \)-derivative in a systematic way. The fractional \( q \)-calculus is an important tool used to study various families of analytic functions. In recent years, several subclasses of analytic functions involving fractional \( q \)-integral and fractional \( q \)-derivative operators were constructed and coefficient inequality, Fekete-Szegö inequality and Hankel determinant were estimated for the functions in these classes.

Motivated by the above mentioned work, in this paper, bi-starlike functions of order \( b \) and bi-convex functions of order \( k \) involving \( q \)-derivative operator subordinate to the conic domains are defined and the Fekete-Szegö inequality for the function in these classes are obtained.

**Definition 7.** A function \( f \in \Sigma \) is said to be in the class \( k-ST_{\Sigma, b}(\alpha, \beta) \); where \( 0 \leq \beta < \alpha \leq 1 \) and \( k(1 - \alpha) < 1 - \beta \), and \( b \) is a non-zero complex number, if it satisfies the following conditions:

\[
1 + \frac{1}{b} \left[ \frac{z\tilde{D}_q f(z)}{f(z)} - 1 \right] < p_k, \alpha, \beta(z) \quad \text{and} \quad 1 + \frac{1}{b} \left[ \frac{w\tilde{D}_q g(w)}{g(w)} - 1 \right] < p_k, \alpha, \beta(w)
\]

where \( g \) is an extension of \( f^{-1} \) to \( \Delta \).

**Definition 8.** A function \( f \in \Sigma \) is said to be in the class \( k-UCV_{\Sigma, b}(\alpha, \beta) \); where \( 0 \leq \beta < \alpha \leq 1 \) and \( k(1 - \alpha) < 1 - \beta \), and \( b \) is a non-zero complex number, if it satisfies the following conditions:

\[
1 + \frac{1}{b} \left[ \frac{z\tilde{D}_q(\tilde{D}_q f(z))}{\tilde{D}_q(f(z))} \right] < p_k, \alpha, \beta(z) \quad \text{and} \quad 1 + \frac{1}{b} \left[ \frac{w\tilde{D}_q(\tilde{D}_q g(w))}{\tilde{D}_q(g(w))} \right] < p_k, \alpha, \beta(w)
\]

where \( g \) is an extension of \( f^{-1} \) to \( \Delta \).

2. **Main Results**

In this section, Fekete-Szegö inequality for the functions in the \( f \) classes \( k-ST_{\Sigma, b}(\alpha, \beta) \) and \( k-UCV_{\Sigma, b}(\alpha, \beta) \) are estimated.

**Theorem 1.** If \( f \in k-ST_{\Sigma, b}(\alpha, \beta) \) and is of the form (1.1) then

\[
|a_2| \leq \frac{P_1 \sqrt{P_1} b^2}{\sqrt{[P_1 b([3]_q - [2]_q) + 2(P_1 - P_2)([2]_q - 1)^2]}} \quad \text{and} \quad |a_3| \leq \frac{b^2 P_1}{([2]_q - 1)^2} + \frac{b P_1}{([3]_q - 1)}
\]

and

\[
|a_3 - \mu a_2^2| \leq \begin{cases} \frac{P_1 b}{([3]_q - 1)} & \text{if } 0 \leq |s(\mu)| \leq 1 \\ \frac{P_1 |s(\mu)|}{([3]_q - 1)} & \text{if } |s(\mu)| \geq 1, \end{cases}
\]

where

\[
s(\mu) = \frac{P_1 b (1 - \mu)}{4([P_1 b([3]_q - [2]_q) + 2(P_1 - P_2)([2]_q - 1)^2])}.
\]
\textbf{Proof.} Let $f \in k - ST_{\Sigma, b}(\alpha, \beta)$ and $g$ be an analytic extension of $f^{-1}$ in $\Delta$. Then there exist two Schwarz functions $u, v$ in $\Delta$ such that

\begin{equation}
1 + \frac{1}{b} \left[ \frac{z \tilde{D}_q f(z)}{f(z)} - 1 \right] = P_{k, \alpha, \beta}(u(z)),
\end{equation}
\begin{equation}
1 + \frac{1}{b} \left[ \frac{w \tilde{D}_q g(w)}{g(w)} - 1 \right] = P_{k, \alpha, \beta}(v(w)).
\end{equation}

Define two functions $h, q \in P$ such that

$$h(z) = \frac{1 + u(z)}{1 - u(z)} = 1 + h_1 z + h_2 z^2 + h_3 z^3 + \ldots$$

and

$$q(w) = \frac{1 + v(w)}{1 - v(w)} = 1 + q_1 w + q_2 w^2 + q_3 w^3 + \ldots$$

Then

\begin{align*}
P_{k, \alpha, \beta} \left( \frac{h(z) - 1}{h(z) + 1} \right) &= 1 + \frac{P_1 h_1 z}{2} + \left( \frac{P_1}{2} h_2 - \frac{h_1^2}{4} + \frac{P_2 h_1^2}{4} \right) z^2 \\
&+ \left( \frac{P_1}{2} \left( \frac{h_3}{4} - h_1 h_2 + h_3 \right) + \frac{P_2}{4} (2h_1 h_2 - h_1^3) + \frac{P_3}{8} h_1^3 \right) z^3 + \ldots \\
P_{k, \alpha, \beta} \left( \frac{v(w) - 1}{v(w) + 1} \right) &= 1 + \frac{P_1 q_1 w}{2} + \left( \frac{P_1}{2} q_2 - \frac{q_1^2}{4} + \frac{P_2 q_1^2}{4} \right) w^2 \\
&+ \left( \frac{P_1}{2} \left( \frac{q_3}{4} - q_1 q_2 + q_3 \right) + \frac{P_2}{4} (2q_1 q_2 - q_1^3) + \frac{P_3}{8} q_1^3 \right) w^3 + \ldots.
\end{align*}

Then the above equations become

\begin{align}
1 + \frac{1}{b} \left[ \frac{z \tilde{D}_q f(z)}{f(z)} - 1 \right] &= P_{k, \alpha, \beta} \left( \frac{h(z) - 1}{h(z) + 1} \right), \\
1 + \frac{1}{b} \left[ \frac{w \tilde{D}_q g(w)}{g(w)} - 1 \right] &= P_{k, \alpha, \beta} \left( \frac{v(w) - 1}{v(w) + 1} \right).
\end{align}

Comparing the coefficients of similar powers of $z$ in equations (2.5) and (2.6), we get

\begin{align}
\frac{1}{b} \left( [2]_q - 1 \right) a_2 &= \frac{P_1 h_1}{2}, \\
\frac{1}{b} \left( [3]_q - 1 \right) a_3 - ( [2]_q - 1 ) a_2^2 &= \frac{P_1}{2} (h_2 - \frac{h_1^2}{4} + \frac{P_2 h_1^2}{4},
\end{align}

and

\begin{align}
-\frac{1}{b} \left( [2]_q - 1 \right) a_2 &= \frac{P_1 q_1}{2}, \\
\frac{1}{b} \left( [3]_q - 1 \right) (2a_2 - a_3) - ( [2]_q - 1 ) a_2^2 &= \frac{P_1}{2} (q_2 - \frac{q_1^2}{2} + \frac{P_2 q_1^2}{4}.
\end{align}

From the equations (2.5) and (2.7)

\begin{equation}
h_1 = -q_1.
\end{equation}
Now squaring and adding the equations (2.5) from (2.7), we get
\[ h_1^2 + q_1^2 = \frac{8(\tilde{[2]}_q - 1)^2 a_2^2}{P_1^2 b^2}. \]  
(2.10)

Now adding (2.6) and (2.8), use the equation (2.10), one can get
\[ a_2^2 = \frac{P_1^3(h_2 + q_2)b^2}{4[P_1^2 b([3]_q - [2]_q) + 2(P_1 - P_2)([2]_q - 1)^2]}. \]  
(2.11)

Now subtract the equation (2.8) from (2.6),
\[ a_3 = a_2^2 + \frac{bP_1(h_2 - q_2)}{4([3]_q - 1)}. \]  
(2.12)

Then using the equation (2.10), we get
\[ a_3 = \frac{P_1^3 b^2(h_1^2 + q_1^2)}{8([2]_q - 1)^2} + \frac{bP_1(h_2 - q_2)}{4([3]_q - 1)}. \]  
(2.13)

Then using the equations (2.11) and (2.12), we get
\[ a_3 - \mu a_2^2 = \frac{bP_1}{4([3]_q - 1)} \left[ h_2(1 + s(\mu)) + q_2(-1 + s(\mu)) \right], \]  
(2.14)

where
\[ s(\mu) = \frac{P_2^2 b(1 - \mu)}{4[P_1^2 b([3]_q - [2]_q) + 2(P_1 - P_2)([2]_q - 1)^2]}. \]

By applying the modulus for the equations (2.11), (2.13) and (2.14), we get the required results.

\[ |a_2| \leq \frac{P_1 \sqrt{P_1 b}}{\sqrt{2[2]_q([3]_q - [2]_q)bP_1^2 + [2]_q^2(P_1 - P_2)}} \quad \text{and} \quad |a_3| \leq \frac{P_2^2 b^2}{[2]_q} + \frac{bP_1}{[2]_q [3]_q} \]

and
\[ |a_3 - \mu a_2^2| \leq \begin{cases} \frac{P_1 b}{[2]_q [3]_q}, & \text{if } 0 \leq |s(\mu)| \leq 1 \\ \frac{P_2 b |s(\mu)|}{[2]_q [3]_q}, & \text{if } |s(\mu)| \geq 1, \end{cases} \]

where
\[ s(\mu) = \frac{P_2^2 b(1 - \mu)}{4[2]_q([3]_q - [2]_q)bP_1^2 + [2]_q^2(P_1 - P_2)}. \]

**Theorem 2.** If \( f \in k - UCV_{\Sigma, \nu}(\alpha, \beta) \) and is of the form (1.1) then

\[ |a_2| \leq \frac{P_1 \sqrt{P_1 b}}{\sqrt{2[2]_q([3]_q - [2]_q)bP_1^2 + [2]_q^2(P_1 - P_2)}} \quad \text{and} \quad |a_3| \leq \frac{P_2^2 b^2}{[2]_q} + \frac{bP_1}{[2]_q [3]_q} \]

and

\[ |a_3 - \mu a_2^2| \leq \begin{cases} \frac{P_1 b}{[2]_q [3]_q}, & \text{if } 0 \leq |s(\mu)| \leq 1 \\ \frac{P_2 b |s(\mu)|}{[2]_q [3]_q}, & \text{if } |s(\mu)| \geq 1, \end{cases} \]

where
\[ s(\mu) = \frac{P_2^2 b(1 - \mu)}{4[2]_q([3]_q - [2]_q)bP_1^2 + [2]_q^2(P_1 - P_2)}. \]

**Proof.** If \( f \in k - UCV_{\Sigma, \nu}(\alpha, \beta) \) and \( g \) is an analytic extension of \( f^{-1} \) in \( \Delta \), then there exist two Schwarz functions \( u, v \) in \( \Delta \) such that
\[ 1 + \frac{1}{b} \left[ z\tilde{D}_q(\tilde{D}_q f(z)) \right] = p_{k, \alpha, \beta}(u(z)), \]  
(2.15)
Comparing the coefficients of similar powers of \( z \),

\[ 1 + \frac{1}{b} \left[ \frac{w \tilde{D}_q(\tilde{D}_q g(w))}{\tilde{D}_q(g(w))} \right] = p_{k, \alpha, \beta}(v(w)). \]  

(2.16)

Define two functions \( h, q \) such that

\[ h(z) = \frac{1 + u(z)}{1 - u(z)} = 1 + h_1 z + h_2 z^2 + h_3 z^3 + \ldots \]

and

\[ q(w) = \frac{1 + v(w)}{1 - v(w)} = 1 + q_1 w + q_2 w^2 + q_3 w^3 + \ldots . \]

Then

\[ P_{k, \alpha, \beta} \left( \frac{h(z)}{h(z) + 1} \right) = 1 + \frac{P_1 h_1 z}{2} + \left( \frac{P_1}{2}(h_2 - \frac{h_1^2}{2}) + \frac{P_2 h_1^2}{4} \right) z^2 \]

\[ + \left( \frac{P_1}{2} \left( \frac{h_3}{4} - h_1 h_2 + h_3 \right) + \frac{P_2}{4}(2h_1 h_2 - h_3) + \frac{P_3}{8} h_1^3 \right) z^3 + \ldots \]

and

\[ P_{k, \alpha, \beta} \left( \frac{v(w)}{v(w) + 1} \right) = 1 + \frac{P_1 q_1 w}{2} + \left( \frac{P_1}{2}(q_2 - \frac{q_1^2}{2}) + \frac{P_2 q_1^2}{4} \right) w^2 \]

\[ + \left( \frac{P_1}{2} \left( \frac{q_3}{4} - q_1 q_2 + q_3 \right) + \frac{P_2}{4}(2q_1 q_2 - q_3) + \frac{P_3}{8} q_1^3 \right) w^3 + \ldots \]

Then the above equations reduces to

\[ 1 + \frac{1}{b} \left[ \frac{z \tilde{D}_q(\tilde{D}_q f(z))}{\tilde{D}_q(f(z))} \right] = P_{k, \alpha, \beta} \left( \frac{h(z)}{h(z) + 1} \right), \]

(2.17)

\[ 1 + \frac{1}{b} \left[ \frac{w \tilde{D}_q(\tilde{D}_q g(w))}{\tilde{D}_q(g(w))} \right] = P_{k, \alpha, \beta} \left( \frac{v(w)}{v(w) + 1} \right). \]

(2.18)

Comparing the coefficients of similar powers of \( z \) in equations (2.17) and (2.18)

\[ \frac{1}{b} \left[ \tilde{a}_q \right] \left[ \tilde{a}_q \right] a_2 = \frac{P_1 h_1}{2}, \]

(2.19)

\[ \frac{\left[ \tilde{a}_q \right] \left[ \tilde{a}_q \right] \left[ \tilde{a}_q \right] a_3 - \left[ \tilde{a}_q \right] \left[ \tilde{a}_q \right] a_2^2}{b} = \frac{P_1}{2}(h_2 - \frac{h_1^2}{2}) + \frac{P_2 h_1^2}{4}, \]

(2.20)

and

\[ -\frac{1}{b} \left[ \tilde{a}_q \right] \left[ \tilde{a}_q \right] \left[ \tilde{a}_q \right] a_2 = \frac{P_1 q_1}{2}, \]

(2.21)

\[ \frac{1}{b} \left[ \tilde{a}_q \right] \left[ \tilde{a}_q \right] \left[ \tilde{a}_q \right] \left( 2a_2 - a_3 \right) - \left[ \tilde{a}_q \right] \left[ \tilde{a}_q \right] \left[ \tilde{a}_q \right] a_2^2 = \frac{P_1}{2}(q_2 - \frac{q_1^2}{2}) + \frac{P_2 q_1^2}{4}. \]

(2.22)

From the equations (2.19) and (2.21), we get

\[ h_1 = -q_1. \]

(2.23)

Squaring and adding the equations (2.19) from (2.21), we get

\[ h_1^2 + q_1^2 = \frac{4 \left[ \tilde{a}_q \right] \left[ \tilde{a}_q \right] ^2 a_2^2}{P_1^2 b^2}. \]

(2.24)
Adding (2.20) and (2.22), and using the equation (2.24), one can get
\[ a_2^2 = \frac{P_1^3(h_2 + q_2)b^2}{4[2]_{q^3}([3]_q - [2]_q)P_1^2 + ([2]_q)^2(P_1 - P_2)}. \] (2.25)
Subtracting the equation (2.22) from (2.20), we get
\[ a_3 = a_2^2 + \frac{bP_1(h_2 - q_2)}{4([2]_q[3]_q)}. \] (2.26)
Using the equation (2.24), we obtain
\[ a_3 = \frac{P_1^2b^2(h_1^2 + q_1^2)}{8[2]_q^2} + \frac{bP_1(h_2 - q_2)}{4([2]_q[3]_q)}. \] (2.27)
Then using the equations (2.25) and (2.26), we get
\[ a_3 - \mu a_2^2 = \frac{bP_1}{4([2]_q[3]_q)} \left[ h_2(1 + s(\mu)) + q_2(-1 + s(\mu)) \right], \] (2.28)
where
\[ s(\mu) = \frac{bP_1^2(1 - \mu)}{4[2]_{q^3}([3]_q - [2]_q)bP_1^2 + [2]_q^2(P_1 - P_2)^2}. \]
By applying modulus for the equations (2.25), (2.27) and (2.28) on both sides we get the required results. \(\square\)

Acknowledgement: The work presented in this paper is partially supported by DST-FIST-Grant No.SR/FST/MSI-101/2014, dated 14/1/2016.

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