Random Semicomputable Reals Revisited

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To Cristian Calude on the occasion on his 60th birthday

Abstract. The aim of this expository paper is to present a nice series of results, obtained in the papers of Chaitin [3], Solovay [8], Calude et al. [2], Kučera and Slaman [5]. This joint effort led to a full characterization of lower semicomputable random reals, both as those that can be expressed as a “Chaitin Omega” and those that are maximal for the Solovay reducibility. The original proofs were somewhat involved; in this paper, we present these results in an elementary way, in particular requiring only basic knowledge of algorithmic randomness. We add also several simple observations relating lower semicomputable random reals and busy beaver functions.

1 Lower Semicomputable Reals and the ⪯1-Relation

Recall that a real number α is computable if there is a computable sequence of rationals an that converges to α computably: for a given ε > 0 one may compute N such that |an − α| ≤ ε for all n > N. (One can assume without loss of generality that the an are increasing.)

A weaker property is lower semicomputability. A real number α is lower semicomputable if it is a limit of a computable increasing sequence of rational numbers. Such a sequence is called approximation of α from below in the sequel.

Equivalent definition: α is lower semicomputable if the set of all rational numbers less than α is enumerable. One more reformulation: if α = ∑i≥0 di where di is computable series of rational numbers, and all di with i > 0 are non-negative. (We let d0 be negative, since lower semicomputable α can be negative.)

It is easy to see that α is computable if and only if α and −α are lower semicomputable. There exist lower semicomputable but non-computable reals. Corresponding sequences of rational numbers have non-computable convergence. (Recall that convergence of a sequence ai to some α means that for every rational ε > 0 there exist some integer N such that |ai − α| < ε as soon as i > N. Noncomputable convergence means that there is no algorithm that produces some N with this property given ε.)
We want to classify computable sequences according to their convergence speed and formalize the intuitive idea “one sequence converges better (i.e., not worse) than the other one”.

**Definition 1.** Let \( a_i \to \alpha \) and \( b_j \to \beta \) be two computable strictly increasing sequences converging to lower semicomputable \( \alpha \) and \( \beta \) (approximations of \( \alpha \) and \( \beta \) from below). We say that \( a_n \to \alpha \) converges “better” (not worse) than \( b_n \to \beta \) if there exists a total computable function \( h \) such that

\[
\alpha - a_{h(i)} \leq \beta - b_i
\]

for every \( i \).

In other terms, we require that for each term of the second sequence one may algorithmically find a term of the first one that approaches the limit as close as the given term of the second sequence. Note that this relation is transitive (take the composition of two reducing functions).

In fact, the choice of specific sequences that approximate \( \alpha \) and \( \beta \) is irrelevant: any two increasing computable sequences of rational numbers that have the same limit, are equivalent with respect to this quasi-ordering. Indeed, we can just wait to get a term of a second sequence that exceeds a given term of the first one. We can thus set the following definition.

**Definition 2.** Let \( \alpha \) and \( \beta \) be two lower semicomputable reals, and let \( (a_n) \), \( (b_n) \) be approximations of \( \alpha \) and \( \beta \) respectively. If \( (a_n) \) converges better than \( (b_n) \), we write \( \alpha \preceq_1 \beta \) (by the above paragraph, this does not depend on the particular approximations we chose).

This definition can be reformulated in different ways. First, we can eliminate sequences from the definition and say that \( \alpha \preceq_1 \beta \) if there exists a partial computable function \( \varphi \) defined on all rational numbers \( r < \beta \) such that

\[
\varphi(r) < \alpha \quad \text{and} \quad \alpha - \varphi(r) \leq \beta - r
\]

for all of them. Below, we refer to \( \varphi \) as the reduction function.

The following lemma is yet another characterization of the order (perhaps less intuitive but useful).

**Lemma 1.** \( \alpha \preceq_1 \beta \) if and only if \( \beta - \alpha \) is lower semicomputable (or said otherwise, if and only if \( \beta = \alpha + \rho \) for some lower semicomputable real \( \rho \)).

**Proof.** To show the equivalence, note first that for every two lower semicomputable reals \( \alpha \) and \( \rho \) we have \( \alpha \preceq_1 \alpha + \rho \). Indeed, consider approximations \( (a_n) \) to \( \alpha \), \( (r_n) \) to \( \rho \). Now, given a rational \( s < \alpha + \rho \), we wait for a stage \( n \) such that \( a_n + r_n > s \). Setting \( \varphi(s) = a_n \), it is easy to check that \( \varphi \) is a suitable reduction function witnessing \( \alpha \preceq_1 \alpha + \rho \).

It remains to prove the reverse implication: if \( \alpha \preceq_1 \beta \) then \( \rho = \beta - \alpha \) is lower semicomputable. Indeed, if \( (b_n) \) is a computable approximation (from below) of \( \beta \) and \( \varphi \) is