A FACTORIZATION OF THE S–MATRIX INTO JOST MATRICES

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abstract

An effective algebraic approach to $S$–matrix factorization into Jost matrices is developed in the case of coupled channels. The Jost matrix is given as a solution of boundary value Riemann – Hilbert problem. A rational form is assumed for tangent of mixing angle, while there is no limitations for approximation of phase shifts.

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Introduction

As it is well known, the Jost function plays an important role in scattering theory [1], and particularly in the inverse scattering problem [2]–[3]. One of the methods of obtaining Jost function $F(k)$ is to solve Riemann – Hilbert problem for the half–plane. In the frame of scattering theory the solution of this problem is reduced to the $S$–matrix factorization in terms of Jost matrices [1]. The Jost matrix is also used to obtain the solutions of nonlinear differential equations (see, e.g. [6], [7]).

At present time a number of different approaches to the Jost matrix construction exists, e.g. [8]–[14]. However, while the boundary value problem solution for one channel elastic scattering can always be represented explicitly in terms of phase shift, this is not so for two–channel scattering. In general case the Jost matrix can not be constructed effectively. It is known only that the matrix boundary value problem can be reduced to a system of singular integral equations [15]. While solving it one needs to take into account the following important fact. Jost matrix arises at an intermediate stage in the investigation of various problems, so its explicit form must be the most adequate for analytical and numerical calculations.

In this connection, one is interested to obtain the solution avoiding integral equations.

Let us note that the procedure of obtaining the Jost matrix can be effectively simplified if the $S$–matrix is factorized previously [9]. This enables one to write the result in a compact form, as can be seen below from the concrete example.

In this paper we shall use the approximate form for mixing parameter. Some of preliminary results were presented in [16].

This paper is organized as follows. In Sec. I the boundary value Riemann – Hilbert problem for $S$–matrix is formulated. The special features of the problem in matrix case are discussed in Sec. II. Our algebraic approach consists of three steps. In Sec. III the first step $S$–matrix factorization — prefactorization is realized. The ”correcting matrix” concept needed to solve the boundary value problem is
discussed in Sec. IV. In Sec. V the final result for Jost matrix is obtained. The effectiveness of our approach is demonstrated in Sec. VI for the well–known effective radius approximation.

I The formulation of the problem

The general mathematical formulation of the problem is the following. One needs to find the piecewise holomorphic matrix $F(k)$ (i.e. its matrix elements are piecewise holomorphic functions) in the upper and lower half–planes. Boundary values of $F(k)$ satisfy the following condition on the real axis:

$$F_+(k) = S^{-1}(k) F_-(k), \quad \text{Im} \ k = 0, \quad -\infty < k < \infty$$

(1)

Here $F_+(k)$ is a matrix holomorphic in the upper half–plane, $F_-(k) —$ in the lower one, $k$ is momentum in the center–of–mass system, $S(k)$ is the scattering matrix which is non–singular:

$$\text{det} \ S(k) \neq 0, \quad \text{Im} \ k = 0$$

(2)

and Hölder condition is valid for its matrix elements $s_{ij}(k), \ i, j = 1, 2$ on the real axis

$$|s_{ij}(k_1) - s_{ij}(k_2)| \leq A |k_1 - k_2|^{\mu}, \ A > 0, \ 0 < \mu \leq 1, \ i, j = 1, 2.$$  

(3)

Standard conditions are imposed on the scattering matrix $S(k)$ and on the $F(k)$ matrix (the boundary values of the matrix $F(k)$ are the Jost matrices) [1]:

$$S^\dagger(k) = S^{-1}(k) = S^*(k) = S(-k), \quad \text{Im} \ k = 0$$

(4)

$$\lim_{k \to \pm \infty} S(k) = I,$$

(5)

$$F_+^*(k) = F_+(k) = F_+(k), \quad \text{Im} \ k = 0$$

(6)

$I$ is the unity matrix. As usual the physical formulation of the problem requires, in addition, the asymptotic condition

$$\lim_{k \to \pm \infty} F_+(k) = I$$

(7)
and the condition

\[
\det F_+(i\kappa_j) = 0, \quad \kappa_j > 0, \ j = 1, 2, \ldots, m, \quad m < +\infty, \quad (8)
\]
to be satisfied \((m)\) is the number of bound states).

II Boundary value problem in the matrix case

In the scalar (non–matrix) case of uncoupled partial waves the solution of the problem formulated above can be given explicitly \([17]\):

\[
F_\pm(k) = \Pi_\pm(k) \exp\left(\frac{1}{2\pi i} \int_{-\infty}^{+\infty} \ln\left(S^{-1}(k)\Pi_\pm^2(k)\right) dk'\right), \quad (9)
\]

\[
\Pi_\pm(k) = \prod_{j=1}^{m} \frac{k \mp i\kappa_j}{k \pm i\kappa_j}, \quad (10)
\]

\[
\Pi_\pm(k) \equiv 1, \quad m = 0. \quad (11)
\]

However, in general case the matrices given by the Eqs.\((8)–(11)\) do not satisfy the boundary value condition \((\Pi)\). In fact, using the well–known equation

\[
\frac{1}{k' - k \mp i0} = \frac{1}{k' - k} \pm i\pi \delta(k' - k)
\]

and substituting \((8)\) in \((\Pi)\), one obtains

\[
e^{\frac{1}{2}g(k)+h(k)} = e^{g(k)} e^{-\frac{1}{2}g(k)+h(k)},
\]

with

\[
g(k) = \ln(S^{-1}(k)\Pi_\pm^2(k)), \quad h(k) = \text{P.V.} \int_{-\infty}^{+\infty} \frac{g(k')}{k' - k} dk',
\]

P.V. stays for the principal value of the integral. So if we define the matrices \(A\) and \(B\) by

\[
A = g(k), \quad B = -\frac{1}{2} g(k) + h(k)
\]
then the rule
\[ e^{A+B} = e^A e^B \]
is to be fulfilled. The Eq. (12) is valid only if
\[ S(k_1) S(k_2) = S(k_2) S(k_1), \quad k_1 \neq k_2. \] (13)
The condition (13) means, from the physical point of view, that the potential in Schröedinger equation is central. If the interaction is of tensor character, then the condition (13) and, consequently, the Eq. (12), are not fulfilled. This means that (9) can not be used for solving the boundary value problem of Sec. [1].
To construct the Jost matrix in this case let us begin with the first stage — preliminary factorization of the \( S \)-matrix — ”prefactorization”.

### III The S-matrix prefactorization

Now the problem of finding of Jost matrix in the coupled channels case is reduced to the boundary value problem (1) with conditions (2)–(8). \( S \)-matrix can be diagonalized by the use of an orthogonal transformation
\[ S(k) = U(k^2) \begin{pmatrix} e^{i2\delta_1(k)} & 0 \\ 0 & e^{i2\delta_2(k)} \end{pmatrix} U^{-1}(k^2), \quad \text{Im} k = 0, \] (14)
where \( \delta_j(k), j=1,2 \) are the real–valued phase shifts,
\[ \delta_j(-k) = -\delta_j(k), \quad \delta(\pm\infty) = 0, \quad j=1,2. \]
The matrix \( U(k^2) \) as usually can be written in the form:
\[ U(k^2) = \begin{pmatrix} \cos \varepsilon(k^2) & \sin \varepsilon(k^2) \\ -\sin \varepsilon(k^2) & \cos \varepsilon(k^2) \end{pmatrix} \]
where \( \varepsilon(k^2) \) is the real–valued mixing angle. We shall approximate the tangent of mixing angle \( \varepsilon(k^2) \) by a rational function:
\[ \tan \varepsilon(k^2) = \frac{P_M(k^2)}{Q_N(k^2)}. \] (15)
Here $P_M(k^2)$ and $Q_N(k^2)$ are polynomials in $k$ of the degree $2M$ and $2N$, respectively. Without loss of generality one can suppose that the polynomials have no zeros simultaneously. The diagonal part of the $S$–matrix (14) can be factorized:

$$
\begin{pmatrix} e^{2i\delta_1(k)} & 0 \\ 0 & e^{2i\delta_2(k)} \end{pmatrix} = \begin{pmatrix} f_1^-(k) & 0 \\ 0 & f_2^-(k) \end{pmatrix} \begin{pmatrix} f_1^-(k) & 0 \\ 0 & f_2^-(k) \end{pmatrix},
$$

(16)

where $f_{j\pm}(k), j=1,2$ are the solutions

$$
f_{j\pm}(k) = e^{-2i\delta_j(k)} f_j^-(k), \quad j = 1, 2,
$$

of scalar boundary value problems given by the Eqs.(9)–(11). Let us note, that if the number of bound states in the first channel (for $j=1$) is equal to $m_1$ then one must change $m$ for $m_1$ in (9)–(11). Similarly one has to change $m$ for $m_2$ for $j=2$; $m_1 + m_2 = m$. Taking into account (15), (16) the problem (1) can be written in the form:

$$
F^{(0)}(k) + (k) = S^{-1}(k) F^{(0)}(k)
$$

(17)

with

$$
F^{(0)}_\pm(k) = \begin{pmatrix} Q_N(k^2) & P_M(k^2) \\ -P_M(k^2) & Q_N(k^2) \end{pmatrix} \begin{pmatrix} f_{1\pm}(k) & 0 \\ 0 & f_{2\pm}(k) \end{pmatrix}.
$$

(18)

So the preliminary factorization of $S$–matrix is realized.

**IV The analytical properties of the matrix $F_+^{(0)}(k)$**

The matrix $F_+^{(0)}(k)$ has a number of properties which are characteristic for the Jost matrix. As it follows from Eq. (18) $F_+^{(0)}(k)$ is analytical in the upper half–plane and the following equations take place:

$$
F_+^{(0)*}(k) = F_+^{(0)}(-k) = F_-^{(0)}(k), \quad \text{Im} \ k = 0.
$$

(19)

Nevertheless, $F_+^{(0)}(k)$ is not a solution to the boundary value problem of Sec. I because: 1) the condition (7) is not satisfied; 2) det $F_+^{(0)}(k)$ has nonphysical extra zeros originated by the roots of the equations:

$$
Q_N(k^2) + iP_M(k^2) = 0,
$$

(20)
\[ Q_N(k^2) - iP_M(k^2) = 0. \] (21)

Using (18) one can write

\[ \det F^{(0)}_+(k) = (Q^2_N(k^2) + P^2_M(k^2)) f_1+(k) f_2+(k) \]

so that the condition (8) is not satisfied.

Let us emphasize one particularly important feature of the roots of the Eqs. (20)–(21). The polynomials are even functions of \( k \) and are not equal to zeros simultaneously (see Sec. III), so that the Eqs. (20), (21) have no roots if \( \text{Im} \, k = 0 \), or if \( \text{Re} \, k = 0 \). Let us also notice that for real \( k \) the Eqs. (20), (21) are complex conjugate. Now we can denote the roots of the Eqs. (20) and (21) by \( \pm i\lambda_j, \pm i\lambda^*_j, j = 1, 2, \ldots, L, L = \max\{M, N\} \). For definiteness let us suppose \( \text{Im}(i\lambda_j) > 0, j = 1, 2, \ldots, L \). It is easy to see that the positions of all roots are symmetrical relatively to real and imaginary axes. We shall refer to these roots as to nonphysical zeros of the \( F^{(0)}_+(k) \)–matrix determinant. The symmetry described in this Sec. is of great importance for the solution of the boundary value problem.

Let us consider first the case of simple zeros. In order to obtain the Jost matrix, one needs only to remove the nonphysical zeros from the \( \det F^{(0)}_+(k) \), but in such a way that the analytical properties of the matrix \( F^{(0)}_+(k) \) remain unchanged and the condition (7) is fulfilled. Let us multiply the boundary value condition (17) from the right by a rational real matrix \( W(k^2) \)

\[ F^{(0)}_+(k) W(k^2) = S^{-1}(k) F^{(0)}_-(k) W(k^2). \]

If the matrix \( W(k^2) \) is such, that conditions on \( F^{(0)}_+(k) \) mentioned above are fulfilled, then the Jost matrix \( F_+(k) \) is of the form

\[ F_+(k) = F^{(0)}_+(k) W(k^2). \] (22)

We will refer to the matrix \( W(k^2) \) as the correcting matrix. The method of obtaining the explicit form of \( W(k^2) \) is based on the ideas of [8], [9], [18].
V The construction of the correcting matrix $W(k^2)$

Let us make use of the symmetry property of the nonphysical zeros (Sec. [LV]) and let us fix four of them: $\pm i\lambda_1, \pm i\lambda_1^*$. We shall now remove these nonphysical zeros from $\det F_+^{(0)}(k)$. To do this let us consider first the matrix

$$W_1(k^2) = P_1 \frac{1}{(k - i\lambda_1)(k + i\lambda_1)} + (I - P_1) \frac{1}{(k - i\lambda_1^*)(k + i\lambda_1^*)}, \quad (23)$$

where $P_1$ is a projection matrix

$$P_1^2 = P_1. \quad (24)$$

Using the fact that (24) implies

$$\det P_1 = 0, \quad \text{Sp} P_1 = 1,$$

it is easy to check that

$$\det W_1(k^2) = \frac{1}{(k + \lambda_1^2)(k + \lambda_1^{2*})}. \quad (25)$$

Now we can obtain a new matrix:

$$F_+^{(1)}(k) = F_+^{(0)}(k) W_1(k^2).$$

As can be seen from Eq. (23) the $\det F_+^{(1)}(k)$ has no zeros at the points $k = \pm i\lambda_1, \pm i\lambda_1^*$.

Now we need to verify the analyticity of $F_+^{(1)}(k)$ in the upper half-plane and the properties analogous to (19). This second condition will be satisfied if

$$P_1^* = I - P_1. \quad (26)$$

The explicit form of $P_1$ must be chosen in such a way that the matrix $F_+^{(1)}(k)$ will have no poles at the points $k = i\lambda_1, i\lambda_1^*$. Taking into account the Eqs. (24), (26), we can define

$$P_1 = \frac{1}{\text{Sp} Y_1} Y_1, \quad (27)$$
with
\[ Y_1 = \begin{pmatrix} f_{2+}(i\lambda_1) & 0 \\ 0 & f_{1+}(i\lambda_1) \end{pmatrix} \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix} \begin{pmatrix} f_{1+}^*(i\lambda_1) & 0 \\ 0 & f_{2+}^*(i\lambda_1) \end{pmatrix}. \tag{28} \]

Now \( F_+^{(1)}(k) \) has no zeros in the upper half-plane. To check this fact it is sufficient to show that the following equations are satisfied:
\[ F_+^{(1)}(i\lambda_1) P_1 = 0, \tag{29} \]
\[ F_+^{(1)}(i\lambda_1^*) (I - P_1) = 0. \tag{30} \]

This can be done using the explicit form of \( F_+^{(1)}(k) \) and \( P_1 \) \cite{18} \( (27), (28), \text{ respectively}) \) and the fact that \( k = i\lambda_1, i\lambda_1^* \) are the roots of the Eqs. \( (24), (21) \).

After removing of \( 4(n-1) \), \( n \leq L \) nonphysical zeros we obtain for \( F_+^{(n)}(k) \):
\[ F_+^{(n)}(k) = F_+^{(n-1)}(k) W_n(k^2), \tag{31} \]
\[ W_n(k^2) = P_n \frac{1}{(k - i\lambda_n)(k + i\lambda_n)} + (I - P_n) \frac{1}{(k - i\lambda_n^*)(k + i\lambda_n^*)}, \tag{32} \]

where \( P_n \) has properties analogous to the Eqs. \( (24), (26) \) and is of the form
\[ P_n = \frac{1}{\text{Sp} Y_n} Y_n, \tag{33} \]
\[ Y_n = W_{n-1}^{-1}(-\lambda_n^2) \times \cdots \times W_1^{-1}(-\lambda_1^2) \times \left( \begin{array}{cc} f_{2+}(i\lambda_1) & 0 \\ 0 & f_{1+}(i\lambda_1) \end{array} \right) \left( \begin{array}{cc} 1 & -i \\ i & 1 \end{array} \right) \left( \begin{array}{cc} f_{1+}^*(i\lambda_1) & 0 \\ 0 & f_{2+}^*(i\lambda_1) \end{array} \right) \]
\[ \times [W_1(-\lambda_1^2) \times \cdots \times W_{n-1}(-\lambda_n^2)]^*. \tag{34} \]

It is easy to show that \( \text{det} F_+^{(n)}(k) \) has no nonphysical zeros \( k = i\lambda_j, i\lambda_j^* \), \( j = 1, 2, \ldots, n \), and the matrix \( F_+^{(n)}(k) \) is analytical in the upper half-plane.
When all nonphysical zeros are removed and \( L = \max\{M, N\} \) projection matrices are obtained, then taking into account the Eq. (7) one has for the correcting matrix:

\[
W(k^2) = W_1(k^2) \times \cdots \times W_L(k^2) \ C, \tag{35}
\]

\[
C = \begin{cases} 
\frac{1}{q_N} I, & N > M, \\
\frac{1}{p_M} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, & N < M, \\
\begin{pmatrix} q_N & p_M \\ -p_M & q_N \end{pmatrix}^{-1}, & N = M,
\end{cases} \tag{36}
\]

where \( p_M, q_N \) are the highest degree coefficients in the polynomials \( P_M(k^2), Q_N(k^2) \), respectively. The Eqs. (22), (23)–(36) give the solution of the boundary value problem of Sec. I in the case of the approximation (15) for the mixing angle.

VI The example

As an example let us consider the elastic two–channel neutron–proton scattering with the phase shifts \( ^3S_1, ^3D_1 \) in the effective radius approximation. In this case the scattering phase shifts and mixing angle are of the form:

\[
e^{i2\delta_S(k)} = \frac{(k + i\kappa)(k + i\varphi)}{(k - i\kappa)(k - i\varphi)}, \quad e^{i2\delta_D(k)} = 1, \quad \tan \varepsilon(k^2) = \frac{k^2}{2\chi^2},
\]
where \( k = i\kappa \) is the bound state point, so that at \( m = 1 \) the condition (8) is to be taken into account.

The \( S \)-matrix is a rational function and has the form:

\[
S(k) = \frac{1}{k^4 + 4\chi^4} \begin{pmatrix} 2\chi^2 & k^2 \\ -k^2 & 2\chi^2 \end{pmatrix} \begin{pmatrix} \frac{(k+i\kappa)(k+i\varphi)}{(k-i\kappa)(k-i\varphi)} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2\chi^2 & -k^2 \\ k^2 & 2\chi^2 \end{pmatrix}.
\]

Let us perform the \( S \)-matrix prefactorization in terms of matrices \( F_\pm(0)(k) \) (see (17), (18))

\[
F_\pm(0)(k) = \begin{pmatrix} 2\chi^2 & k^2 \\ -k^2 & 2\chi^2 \end{pmatrix} \begin{pmatrix} \frac{k+i\kappa}{k+i\varphi} & 0 \\ 0 & 1 \end{pmatrix}.
\]

In our case the polynomials \( P_M(k^2), Q_N(k^2) \) (see (15)) are of the form:

\[
P_M(k^2) = k^2, \quad Q_N(k^2) = 2\chi^2,
\]

with \( M = 1, N = 0 \). The number of projection operators \( P_j, j = 1, 2, \ldots \) is \( L \), \( L = \max\{M, N\} \) (see Sec. [V]). In our case \( L = 1 \), so to obtain the Jost matrix we need to construct only one projection matrix \( P_1 \). The nonphysical zeros of \( \det F_\pm(0)(k) \) (the roots of Eqs. (20), (21)) are \( k = \pm \lambda_1, \pm \lambda_1^* \), with \( \lambda = \chi(1 - i) \).

Following (27), (28) it is easy to obtain the projection matrix \( P_1 \) in the form:

\[
P_1 = \frac{1}{2\eta} \begin{pmatrix} \eta + i\chi(\varphi + \kappa) & -i(\chi^2 + (\chi + \varphi)^2) \\ i(\chi^2 + (\chi - \kappa)^2) & \eta - i\chi(\varphi + \kappa) \end{pmatrix},
\]

with

\[
\eta = \chi^2 + (\chi + \varphi)(\chi - \kappa).
\]

Using the condition (7) one has for the Jost matrix:

\[
F_+(k) = \begin{pmatrix} 2\chi^2 & k^2 \\ -k^2 & 2\chi^2 \end{pmatrix} \begin{pmatrix} \frac{k-i\kappa}{k+i\varphi} & 0 \\ 0 & 1 \end{pmatrix} \times \left[ P_1 \frac{1}{k^2 + \lambda_1^2} + (I - P_1) \frac{1}{k^2 + \lambda_1^*} \right] \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.
\]

Instead of the \( P_1 \) matrix we can use the \( P \) matrix defined by

\[
P_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} P \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.
\]
Now

\[ F_+(k) = \left( \begin{array}{c}
\frac{k^2}{2\chi^2} \\
2k^2 - 2\chi \frac{k-i\kappa}{k+i\varphi}
\end{array} \right) \left[ P \frac{1}{k^2 + \lambda^2_1} + (I - P) \frac{1}{k^2 + \lambda^2_1^*} \right]. \tag{37} \]

This simple case considered as an example was investigated in \cite{19} and one can show that (37) is nothing else that the well known result of \cite{19} for the Jost matrix, but in more compact form. Our projection matrix \( P \) is just \( P_3 \) — one of the projection matrices of \cite{19}. However, the \( S \)-matrix prefactorization and the use of the symmetry property for nonphysical zeros make our algebraic method more effective and allows one to apply it in more complicated cases.

\section{VII Conclusion}

We have developed an effective algebraic method of the explicit factorization of the \( S \)-matrix in terms of the Jost matrices in the case of tensor potential. The only approximation used is the rational form for the mixing angle tangent. One of the main advantages of the method is the fact that the number of projection matrices involved does not depend on the phase shifts approximations. The obtained Jost matrices have simple compact explicit form and can be used in a number of physical situations (see e.g. \cite{4}).

Our result, as a mathematical result, is of independent interest by its own rights, too. We have enlarged the class of unitary matrices for which the solution of the Riemann – Hilbert boundary value problem can be obtained in a way avoiding systems of singular integral equations.

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