BOUNDS ON GROMOV HYPERBOLICITY CONSTANT

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Abstract. If $X$ is a geodesic metric space and $x_1, x_2, x_3 \in X$, a geodesic triangle $T = \{x_1, x_2, x_3\}$ is the union of the three geodesics $[x_1x_2], [x_2x_3]$ and $[x_3x_1]$ in $X$. The space $X$ is $\delta$-hyperbolic in the Gromov sense if any side of $T$ is contained in a $\delta$-neighborhood of the union of the two other sides, for every geodesic triangle $T$ in $X$. If $X$ is hyperbolic, we denote by $\delta(X)$ the sharp hyperbolicity constant of $X$, i.e. $\delta(X) = \inf\{\delta \geq 0 : X \text{ is } \delta\text{-hyperbolic}\}$. To compute the hyperbolicity constant is a very hard problem.

Then it is natural to try to bound the hyperbolicity constant in terms of some parameters of the graph. Denote by $\mathcal{G}(n, m)$ the set of graphs $G$ with $n$ vertices and $m$ edges, and such that every edge has length 1. In this work we estimate $A(n, m) := \min\{\delta(G) \mid G \in \mathcal{G}(n, m)\}$ and $B(n, m) := \max\{\delta(G) \mid G \in \mathcal{G}(n, m)\}$.

In particular, we obtain good bounds for $B(n, m)$, and we compute the precise value of $A(n, m)$ for all values of $n$ and $m$. Besides, we apply these results to random graphs.

Keywords: Gromov hyperbolicity, hyperbolicity constant, finite graphs, geodesic.

1. Introduction

Gromov hyperbolicity was introduced by the Russian mathematician Mikhail Leonidovich Gromov in the setting of geometric group theory [26, 25, 24, 19], but has played an increasing role in analysis on general metric spaces [10, 11, 4], with applications to the Martin boundary, invariant metrics in several complex variables [3] and extendability of Lipschitz mappings [37].

The theory of Gromov hyperbolic spaces was used initially for the study of finitely generated groups, where it was demonstrated to have an enormous practical importance. This theory was applied principally to the study of automatic groups (see [41]), which plays an important role in sciences of the computation. The concept of hyperbolicity appears also in discrete mathematics, algorithms and networking. Another important application of these spaces is the secure transmission of information by internet. In particular, the hyperbolicity plays an important role in the spread of viruses through the network (see [32, 33]). The hyperbolicity is also useful in the study of DNA data (see [12]).

The study of mathematical properties of Gromov hyperbolic spaces and its applications is a topic of recent and increasing interest in graph theory; see, for instance, [7, 12, 14, 16, 28, 32, 33, 34, 35, 38, 39, 41, 47, 49].

Last years several researchers have been interested in showing that metrics used in geometric function theory are Gromov hyperbolic. For instance, the Gehring-Osgood $j$-metric is Gromov hyperbolic; and the Vuorinen $j$-metric is not Gromov hyperbolic except in the punctured space (see [27]). The study of Gromov hyperbolicity of the quasi-hyperbolic and the Poincaré metrics is the subject of [1, 10, 28, 41, 43, 49]. In particular, the equivalence of the hyperbolicity of Riemannian manifolds and the hyperbolicity of a simple graph was proved in [41, 49], hence, it is useful to know hyperbolicity criteria for graphs.

Now, let us introduce the concept of Gromov hyperbolicity and the main results concerning this theory. For detailed expositions about Gromov hyperbolicity, see e.g. [1, 24, 19] or [50].

If $X$ is a metric space we say that the curve $\gamma : [a, b] \rightarrow X$ is a geodesic if we have $L(\gamma|[t, s]) = d(\gamma(t), \gamma(s)) = |t - s|$ for every $s, t \in [a, b]$ (then $\gamma$ is equipped with an arc-length parametrization). The metric space $X$ is said geodesic if for every couple of points in $X$ there exists a geodesic joining them; we denote by $[xy]$ any geodesic joining $x$ and $y$; this notation is ambiguous, since in general we do not have

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uniqueness of geodesics, but it is very convenient. Consequently, any geodesic metric space is connected. If
the metric space \( X \) is a graph, then the edge joining the vertices \( u \) and \( v \) will be denoted by \([u, v]\).

In order to consider a graph \( G \) as a geodesic metric space, identify (by an isometry) any edge \([u, v] \in E(G)\)
with the interval \([0, 1]\) in the real line; then the edge \([u, v]\) (considered as a graph with just one edge) is
isometric to the interval \([0, 1]\). Thus, the points in \( G \) are the vertices and, also, the points in the interior
of any edge of \( G \). In this way, any graph \( G \) has a natural distance defined on its points, induced by taking
the shortest paths in \( G \), and we can see \( G \) as a metric graph. Throughout this paper, \( G = (V, E) \) denotes
a simple connected graph such that every edge has length 1. These properties guarantee that any graph is a
geodesic metric space. Note that to exclude multiple edges and loops is not an important loss of generality,
since \([8\) Theorems 8 and 10\]) reduce the problem of computing the hyperbolicity constant of graphs with
multiple edges and/or loops to the study of simple graphs.

If \( X \) is a geodesic metric space and \( J = \{J_1, J_2, \ldots, J_n\} \) is a polygon with sides \( J_i \subseteq X \), we say that \( J \) is
\( \delta \)-thin if for every \( x \in J_i \) we have that \( d(x, \bigcup_{j \neq i} J_j) \leq \delta \). In other words, a polygon is \( \delta \)-thin if each of its sides is
contained in the \( \delta \)-neighborhood of the union of the other sides. We denote by \( \delta (J) \) the sharp thin constant
of \( J \), i.e., \( \delta (J) := \inf \{\delta \geq 0 \mid \text{\( J \) is \( \delta \)-thin}\} \). If \( x_1, x_2, x_3 \in X \), a geodesic triangle \( T = \{x_1, x_2, x_3\} \) is the union
of the three geodesics \([x_1x_2], [x_2x_3] \) and \([x_3x_1]\). The space \( X \) is \( \delta \)-hyperbolic (or satisfies the Rips condition
with constant \( \delta \)) if every geodesic triangle in \( X \) is \( \delta \)-thin. We denote by \( \delta (X) \) the sharp hyperbolicity
constant of \( X \), i.e., \( \delta (X) := \sup \{\delta (T) \mid T \text{ is a geodesic triangle in } X\} \). We say that \( X \) is hyperbolic if \( X \) is
\( \delta \)-hyperbolic for some \( \delta \geq 0 \). If \( X \) is hyperbolic, then \( \delta (X) = \inf \{\delta \geq 0 \mid \text{\( X \) is \( \delta \)-hyperbolic}\} \).

The following are interesting examples of hyperbolic spaces. Every bounded metric space \( X \) is \((\text{diam } X)\)-
hyperbolic. The real line \( \mathbb{R} \) is 0-hyperbolic due to any point of a geodesic triangle in the real line belongs to
two sides of the triangle simultaneously. The Euclidean plane \( \mathbb{R}^2 \) is not hyperbolic, since the midpoint of a
side on a large equilateral triangle is far from all points in the other two sides. A normed vector space \( E \) is
hyperbolic if and only if \( \text{diam } E = 1 \). Every simply connected complete Riemannian manifold with sectional
curvature verifying \( K \leq -k^2 \), for some positive constant \( k \), is hyperbolic (see, e.g., \([24\) p.52\]). The graph \( \Gamma \)
of the routing infrastructure of the Internet is also empirically shown to be hyperbolic (see \([5\).

The main examples of hyperbolic graphs are trees. In fact, the hyperbolicity constant of a geodesic metric
space can be viewed as a measure of how “tree-like” the space is, since those spaces \( X \) with \( \delta (X) = 0 \) are
precisely the metric trees. This is an interesting subject since, in many applications, one finds that the
borderline between tractable and intractable cases may be the tree-like degree of the structure to be dealt
with (see, e.g., \([17\).

For a general graph deciding whether or not a space is hyperbolic seems an unabordable problem. Therefore,
it is interesting to study the hyperbolicity of particular classes of graphs. The papers \([7, 12, 13, 15, 16, 39, 42, 43, 45, 48, 51\) study the hyperbolicity of, respectively, complement of graphs, chordal graphs, strong
product graphs, corona and join of graphs, line graphs, Cartesian product graphs, cubic graphs, tessellation
graphs, short graphs, median graphs and \( k \)-chordal graphs. In \([13, 15, 39\) the authors characterize the
hyperbolic product graphs (for strong product, corona and join of graphs, and Cartesian product) in terms
of properties of the factor graphs. In this work we study the hyperbolicity constant of the graphs with \( n \)
vertices and \( m \) edges.

Let \( \mathcal{G}(n, m) \) be the set of graphs \( G \) with \( n \) vertices and \( m \) edges, and such that every edge has length 1.
If \( m = n - 1 \), then every \( G \in \mathcal{G}(n, m) \) is a tree and \( \delta (G) = 0 \). On the other hand, the complete graph \( K_n \)
belongs to \( \mathcal{G}(n, m) \) with \( m = \binom{n}{2} \). Thus we consider \( n - 1 \leq m \leq \binom{n}{2} \).

Let us define

\[
A(n, m) := \min \{\delta (G) \mid G \in \mathcal{G}(n, m)\},
\]

\[
B(n, m) := \max \{\delta (G) \mid G \in \mathcal{G}(n, m)\}.
\]

Our aim in this paper is to estimate \( A(n, m) \) and \( B(n, m) \). In particular, we obtain good bounds for
\( B(n, m) \), and we compute the precise value of \( A(n, m) \) for all values of \( n \) and \( m \).
The structure of this paper is as follows. In the next section we consider some previous results regarding hyperbolicity. In Section 3 we prove an upper bound for $B(n, m)$ (see Theorem 2.13). Also, we find a lower bound for $B(n, m)$ in Section 4 (see Theorem 5.2). In Section 5 we give an estimation of the difference between the upper and the lower bounds of $B(n, m)$. One of the main results of this work is Theorem 5.11 which gives the precise value of $A(n, m)$. We conclude this paper with Section 7, where we discuss the applications of our previous results to random graphs.

2. Upper Bound of $B(n, m)$

First, our purpose is to find an upper bound for $B(n, m)$. In order to simplify this proof, we prove some technical lemmas. We begin by proving Lemma 2.3. In order to prove it, we will use Karush-Kuhn-Tucker necessary conditions for nonlinear optimization problems with inequality constraints.

Let $X$ be a non-empty open set of $\mathbb{R}^n$ and $f, g_j \ (j = 1, \ldots, k)$ functions of $X \subseteq \mathbb{R}^n$ in $\mathbb{R}$. Consider the problem:

$$
P: \min_{x \in V} f(x),$$

with $V := \{ x \in X \mid g_j(x) \leq 0 \ \ (j = 1, \ldots, k) \}$.

Given $x^* \in V$, let $I(x^*)$ be the set of subscripts $j$ for which $g_j(x^*) = 0$.

**Definition 2.1.** We say that a point $x^* \in V$ is regular if the vectors $\nabla g_j(x^*) \ (j \in I(x^*))$ are linearly independent.

**Theorem 2.2.** Let $x^*$ be a point in $V$. Suppose that $f, g_j \ (j \in I(x^*))$ are continuously differentiable functions and $g_j \ (j \notin I(x^*))$ are continuous functions at $x^*$. If $x^*$ is a regular point and a local minimum of $f$ in $V$, then there exist unique scalars $\mu_j \ (j \in I(x^*))$ such that:

$$
\nabla f(x^*) + \sum_{j \in I(x^*)} \mu_j \nabla g_j(x^*) = 0,
\mu_j \geq 0, \quad j \in I(x^*).
$$

The above conditions can be written as:

$$
\nabla f(x^*) + \sum_{j=1}^{k} \mu_j \nabla g_j(x^*) = 0,
\mu_j g_j(x^*) = 0, \quad j = 1, \ldots, k,
\mu_j \geq 0, \quad j = 1, \ldots, k.
$$

Consider $G \in \mathcal{G}(n, m)$. Fix $\text{diam} (V(G)) = r$ and choose $u, v \in V(G)$ such that $d(u, v) = r$. Let $k_j = \# \{ w \in V(G) : d(w, u) = j \} \ (0 \leq j \leq r)$. The number of edges that we must eliminate from the complete graph of $n$ vertices in order to obtain $G$ is at least

$$
f_r(k_1, k_2, \ldots, k_r) := \sum_{t=2}^{r} k_t \sum_{s=0}^{t-2} k_s,
$$

In the next result we compute the minimum value of $f_r$ such that $\text{diam} (V(G)) = r$ with $k_j \geq 2, \quad (0 \leq j \leq r - 1)$.

**Lemma 2.3.** Consider the following optimization problem:

$$
\Delta_r := \min_{x \in W} f_r, \quad \text{with} \quad f_r(k_1, k_2, \ldots, k_r) := \sum_{t=2}^{r} k_t \sum_{s=0}^{t-2} k_s, \quad 2 \leq r \leq \frac{n}{2},
$$

and $W := \{ k_0 = 1, \ k_j \geq 2, \ \text{if} \ 1 \leq j \leq r - 1, \ k_r \geq 1, \ 1 + k_1 + k_2 + \cdots + k_r = n \}$.

Then $\Delta_2 = 1, \Delta_3 = n - 1$ and $\Delta_r = 2n(r - 3) - 2r^2 + 6r + 5$ for $r \geq 4$. 


Proof. If \( r = 2 \), then \( f_2(k_1, k_2) = k_2 \), with \( k_2 \geq 1 \). Hence \( \Delta_2 = 1 \).

Consider now \( r \geq 3 \). The set \( W \) can be written as:

\[
W = \{ k_0 = 1, \ g_j = -k_j + 2 \leq 0, \text{ if } 1 \leq j \leq r - 1, \ g_r = -k_r + 1 \leq 0, \ h = 1 + k_1 + k_2 + \cdots + k_r - n = 0 \}.
\]

Note that if \( W \neq \emptyset \), then \( n = 1 + \sum_{i=1}^{r} k_i \geq 1 + 2(r - 1) + 1 \) and \( 2r \leq n \). Conversely, if \( 2r \leq n \), then \( W \neq \emptyset \). Hence, we are assuming \( 2r \leq n \).

We eliminate a variable of our problem by solving \( k_r \) in the equality restriction. Substituting the expression obtained in \( f_r \), the original problem is reduced to the following:

\[
\Delta_r = \min_{x \in W^1} f_r^1, \quad \text{with} \quad f_r^1(k_1, k_2, \ldots, k_{r-1}) := \sum_{t=2}^{r-1} k_t \sum_{s=0}^{t-2} k_s +
\]

\[
+ (n - \sum_{s=0}^{r-1} k_s) \sum_{s=0}^{r-2} k_s,
\]

and \( W^1 := \{ k_0 = 1, \ g_j = -k_j + 2 \leq 0, \text{ if } 1 \leq j \leq r - 1, \ g_r = -k_r + 1 = 2 - n + \sum_{s=0}^{r-1} k_s \leq 0 \} \).

Note that the vectors \( \{\nabla g_j(x^*), \ j = 1, \ldots, r\} \) are linearly dependent but become a linearly independent set by removing any of its elements. Therefore, it suffices to consider that at least one of the coefficients \( \mu_j \) is zero, so that the point is regular.

Let us consider first the case in which \( x^* \) is not a regular point (then \( g_j(x^*) = 0 \) for every \( 1 \leq j \leq r \)). Hence:

\[
h = 1 + 2(r - 1) + 1 - n = 0 \quad \Rightarrow \quad 2r = n.
\]

Therefore, \( x^* = (2, \ldots, 2) \), \( W^1 = \{x^*\} \) and evaluating \( f_r \) at \( x = (x^*, 1) = (2, \ldots, 2, 1) \) we get:

\[
f_r(x) = \sum_{t=2}^{r-1} k_t \sum_{s=0}^{t-2} k_s + (1 + \sum_{s=0}^{r-2} k_s)
\]

\[
= 2\sum_{t=2}^{r-1} k_t + 2r - 3
\]

\[
= (1 + 2r - 5)(r - 2) + 2r - 3
\]

\[
= 2r^2 - 6r + 5,
\]

and then \( \Delta_r = 2r^2 - 6r + 5 \).

Now assume that the minimum point is regular, then \( g_j \neq 0 \) for some \( 1 \leq j \leq r \) and we can apply Theorem 2.2. Since:

\[
\frac{\partial f_r^1}{\partial k_{r-1}} = \sum_{s=0}^{r-3} k_s - \sum_{s=0}^{r-2} k_s = -k_{r-2},
\]

we conclude that the following equality must be satisfied at a regular minimum point:

\[
\begin{pmatrix}
* \\
\vdots \\
* \\
\end{pmatrix} + \mu_1 \begin{pmatrix}
-1 \\
0 \\
0 \\
0 \\
\end{pmatrix} + \cdots + \mu_{r-1} \begin{pmatrix}
0 \\
0 \\
1 \\
-1 \\
1 \\
\end{pmatrix} + \mu_r \begin{pmatrix}
1 \\
0 \\
0 \\
\end{pmatrix} = \begin{pmatrix}
0 \\
0 \\
0 \\
\end{pmatrix}
\]

with \( \mu_j \geq 0 \) for \( j = 1, \ldots, r \).

Assuming that \( \mu_r = 0 \), from the previous expression we obtain that \( -k_{r-2} = \mu_r-1 \). The restriction \( g_{r-2} \leq 0 \) of the problem and the positivity of the coefficient \( \mu_r-1 \) implies that \( -2 \geq -k_{r-2} = \mu_r-1 \geq 0 \) and this is a contradiction, therefore \( \mu_r > 0 \).

Considering the condition \( \mu_{r-1} g_r(x^*) = 0 \) we deduce that \( g_r = -k_r + 1 = 0 \) and \( k_r = 1 \).

We write again the optimization problem, with \( k_r = 1 \):

\[
\Delta_r = \min_{x \in W^2} f_r^2, \quad \text{with} \quad f_r^2(k_1, k_2, \ldots, k_{r-1}) := \sum_{t=2}^{r-1} k_t \sum_{s=0}^{t-2} k_s + \sum_{s=0}^{r-2} k_s,
\]

and \( W^2 := \{k_0 = 1, \ k_j \geq 2, \text{ if } 1 \leq j \leq r - 1, \} \).
If \( r = 3 \), then \( f_3^3(k_1, k_2) = k_2 + 1 + k_1 \), with \( k_1, k_2 \geq 2 \) and \( k_1 + k_2 = n - 2 \). Hence, \( \Delta_3 = n - 1 \).

Consider now \( r \geq 4 \). Note that:

\[
f_r^3 = k_2 + \Sigma_{1 \leq t \leq r}^r k_t (1 + \Sigma_{s=3}^r k_s) + 1 + \Sigma_{t=1}^{r-1} k_t k_s - k_t \Delta_{t-1} = 2n - 3 - k_1 - k_{r-1} + \Sigma_{t=2}^{r-2} k_t k_s, \quad \text{with} \quad \Sigma_{t=1}^{r-1} k_t = n - 2.
\]

Consider now the expression \( (\Sigma_{t=1}^r k_t)^2 \):

\[
(\Sigma_{t=1}^r k_t)^2 = \Sigma_{t=1}^r k_t^2 + 2 \Sigma_{t=1}^{r-2} k_t k_s = \Sigma_{t=1}^r k_t^2 + 2 \Sigma_{t=1}^{r-1} k_t k_s + 2 \Sigma_{t=2}^{r-1} k_t k_s.
\]

Moreover, we can write:

\[
\Sigma_{t=1}^{r-2} k_t k_s = \frac{1}{2} (\Sigma_{t=1}^r k_t)^2 - \frac{1}{2} \Sigma_{t=1}^{r-1} k_t^2 - \Sigma_{t=2}^{r-1} k_t k_{t-1} \geq \frac{1}{2} (n - 2)^2 - \frac{1}{2} \Sigma_{t=1}^{r-1} k_t^2 - \Sigma_{t=2}^{r-1} k_t k_{t-1}.
\]

Thus we have deduced that \( \Delta_r = \min_{x \in W^3} f_r^3 \), with:

\[
f_r^3(k_1, k_2, \ldots, k_{r-1}) := \frac{1}{2} n^2 - 1 - k_1 - k_{r-1} - \frac{1}{2} \Sigma_{t=1}^{r-1} k_t^2 - \Sigma_{t=2}^{r-1} k_t k_{t-1},
\]

and \( W^3 := \{ k_j \geq 2 \text{ if } 1 \leq j \leq r - 1, \ k_1 + k_2 + \cdots + k_{r-1} = n - 2 \} \).

This formulation allows us to see that the problem is symmetric in the variables \( k_t \) and \( k_{r-t} \) for every \( 1 \leq t \leq r - 1 \).

Substituting \( k_r = 1 \) and \( k_{r-1} = n - 2 - \Sigma_{t=1}^{r-1} k_t \) in \( f_r \) we obtain \( \Delta_3 = \min_{x \in W^4} f_r^4 \), with:

\[
f_r^4(k_1, k_2, \ldots, k_{r-2}) := (n - 2 - \Sigma_{t=1}^{r-1} k_t) \Sigma_{s=0}^{r-2} k_s + \Sigma_{t=1}^{r-1} k_t \Sigma_{s=0}^{r-2} k_s + \Sigma_{t=1}^{r-1} k_t \Sigma_{s=0}^{r-2} k_s,
\]

and \( W^4 := \{ k_j \geq 2 \text{ if } 1 \leq j \leq r - 2, \ k_{r-1} = n - 2 - \Sigma_{t=1}^{r-1} k_t \geq 2 \} \).

Then \( k_1 \in [2, n - 4 - \Sigma_{t=1}^{r-2} k_t] \).

Computing the second derivative of \( f_r^4 \) with respect to \( k_1 \) we get:

\[
\frac{\partial^2 f_r^4}{\partial k_1^2} = -2 < 0.
\]

That is, the function is convex and the minimum is reached at the endpoints of the interval, \( k_1 = 2 \) or \( k_1 = n - 4 - \Sigma_{t=1}^{r-2} k_t \), i.e., \( k_1 = 2 \) or \( k_{r-1} = 2 \).

By iterating this argument one can check that if \( x^* = (k_1, k_2, \ldots, k_{r-1}) \) satisfies \( f_r^3(x^*) = \Delta_r \), then \( k_j = 2 \) except for one \( j_0 \) with \( 1 \leq j_0 \leq r - 1 \), and \( k_{j_0} = n - 2r + 2 \). By symmetry, the cases \( j_0 = 1 \) and \( j_0 = r - 1 \) provide the same value; furthermore, the cases \( 1 < j_0 < r - 1 \) provide the same value.

If \( j_0 = 1 \) or \( j_0 = r - 1 \), then:

\[
f_r^3(x^*) = \frac{1}{2} n^2 - 1 - n + 2r - 2 - 2 - \frac{1}{2} (n - 2r + 2)^2
\]

\[
= -\frac{1}{2} n^2 - (r - 2) - 2(n - 2r + 2) - 4(r - 3)
\]

\[
= n(2r - 5) - 2r^2 + 4r + 5.
\]

If \( 1 < j_0 < r - 1 \), substituting \( x^* = (2, \ldots, 2, n - 2r + 2, 2, \ldots, 2) \) in \( f_r^3 \) we get:

\[
f_r^3(x^*) = \frac{1}{2} n^2 - 5 - \frac{1}{2} (n - 2r + 2)^2 - \frac{1}{2} 4(r - 2) - 4(n - 2r + 2) - 4(r - 4)
\]

\[
= 2n(r - 3) - 2r^2 + 6r + 5.
\]

Then \( \Delta_r = 2n(r - 3) - 2r^2 + 6r + 5 \) for \( r \geq 4 \), since \( n \geq 2r \).
Theorem 2.4. In any graph \( G \) the inequality \( \delta(G) \leq \frac{1}{4} \operatorname{diam} G \) holds.

We say that a vertex \( v \) of a graph \( G \) is a cut-vertex if \( G \setminus \{v\} \) is not connected. A graph is two-connected if it is connected and it does not contain cut-vertices.

Given a graph \( G \), we say that a family of subgraphs \( \{G_s\} \) of \( G \) is a \( T \)-decomposition of \( G \) if \( \cup G_s = G \) and \( G_s \cap G_r \) is either a cut-vertex or the empty set for each \( s \neq r \). Every graph has a \( T \)-decomposition, as the following example shows. Given any edge in \( G \), let us consider the maximal two-connected subgraph containing it. We call to the set of these maximal two-connected subgraphs \( \{G_s\} \), the canonical \( T \)-decomposition of \( G \).

Note that every \( G_s \) in the canonical \( T \)-decomposition of \( G \) is an isometric subgraph of \( G \).

Given a graph \( G \), let \( \{G_s\} \) be the canonical \( T \)-decomposition of \( G \). We define the effective diameter as:

\[
\operatorname{effdiam} V(G) := \sup_s \operatorname{diam} V(G_s), \quad \operatorname{effdiam} G := \sup_s \operatorname{diam} G_s.
\]

The following result appears in [3, Theorem 3].

Lemma 2.5. Let \( G \) be a graph and \( \{G_s\} \) be any \( T \)-decomposition of \( G \), then \( \delta(G) = \sup_s \delta(G_s) \).

We will need the following result, which allows to obtain global information about the hyperbolicity of a graph from local information (see Lemma 2.5 and Theorem 2.4).

Lemma 2.6. Let \( G \) be any graph. Then

\[
\delta(G) \leq \frac{1}{2} \operatorname{effdiam}(G).
\]

We define \( M(n, r) := \left(\frac{n}{2}\right) - \Delta_r, \) for \( 2 \leq r \leq n/2 \).

We have the following expression for \( M(n, r) \):

\[
M(n, 2) = \frac{1}{2}[n^2 - n - 2].
\]

\[
M(n, 3) = \frac{1}{2}[n^2 - 3n + 2].
\]

\[
M(n, r) = \frac{1}{2}((n - 2r + 3)^2 + 5n - 19), \quad \text{if} \quad r \geq 4.
\]

Lemma 2.7. If \( G \in \mathcal{G}(n, m) \) and \( \operatorname{effdiam} V(G) = \operatorname{diam} V(G) = r \), then \( m \leq M(n, r) \).

Proof. Let us consider \( u, v \in V(G) \) such that \( d(u, v) = \operatorname{diam} V(G) = r \). Denote by \( k_j \) the cardinal of \( S_j := \{w \in V(G) \mid d(w, u) = j\} \) for \( 0 \leq j \leq r \). Since \( \operatorname{effdiam} V(G) = \operatorname{diam} V(G) = r \), we have \( k_0 = 1, k_j \geq 2 \) for \( 1 \leq j \leq r - 1 \), and \( k_r \geq 1 \).

Note that a vertex of \( S_j \) and a vertex of \( S_0 \cup S_1 \cup \cdots \cup S_{j-2} \) can not be neighbours for \( 2 \leq j \leq r \). Denote by \( x \) the minimum number of edges that can be removed from the complete graph with \( n \) vertices in order to obtain \( G \). Since the diameter of \( V(G) \) is \( r \), we have obtained the following lower bound for \( x \):

\[
x \geq k_2 + k_3(1 + k_1) + k_4(1 + k_1 + k_2) + \ldots + k_{r-1}(1 + k_1 + k_2 + \ldots + k_{r-3}) + k_r(1 + k_1 + k_2 + \ldots + k_{r-2}) = f_r.
\]

Then \( x \geq \Delta_r \) by Lemma 2.3 and \( m = \left(\frac{n}{2}\right) - x \leq \left(\frac{n}{2}\right) - \Delta_r = M(n, r) \).\( \square \)
Corollary 2.10. The inequality
\[ \frac{n-n_0+1}{2} \leq M(n, r) - M(n_0, r) \]
holds for \( 2 \leq r \leq n_0/2 \) and \( n > n_0 \).

Proof. If \( r \geq 4 \), then the inequality holds if and only if
\[
\frac{1}{2}(n-n_0+1)(n-n_0) \leq \frac{1}{2}n(n-1) - \frac{1}{2}n_0(n_0-1) - 2(n-n_0)(r-3)
\]

\( \Leftrightarrow \)
\[
(n-n_0+1)(n-n_0) \leq n^2 - n_0^2 - (n-n_0) - 4(n-n_0)(r-3)
\]

\( \Leftrightarrow \)
\[
n-n_0+1 \leq n+n_0-1-4(r-3) \quad \Leftrightarrow \quad 2r \leq n_0 + 5,
\]
and this holds since \( 2r \leq n_0 \).

If \( r = 3 \), then
\[
\frac{1}{2}(n-n_0+1)(n-n_0) \leq \frac{1}{2}n(n-1) - \frac{1}{2}n_0(n_0-1) - (n-1) - (n_0-1)
\]

\( \Leftrightarrow \)
\[
(n-n_0+1)(n-n_0) \leq n^2 - n_0^2 - (n-n_0) - 2(n-n_0)
\]

\( \Leftrightarrow \)
\[
n-n_0+1 \leq n+n_0-3 \quad \Leftrightarrow \quad n_0 \geq 2,
\]
and this holds since \( n_0 \geq 2r = 6 \).

If \( r = 2 \), then
\[
\frac{1}{2}(n-n_0+1)(n-n_0) \leq \frac{1}{2}n(n-1) - \frac{1}{2}n_0(n_0-1) \quad \Leftrightarrow \quad n-n_0+1 \leq n+n_0-1 \quad \Leftrightarrow \quad n_0 \geq 1.
\]

\[ \square \]

Lemma 2.9. If \( G \in G(n, m) \) and \( \text{effdiam} V(G) = r \), then \( m \leq M(n, r) \).

Proof. Given a graph \( G \in G(n, m) \) with canonical \( T \)-decomposition \( \{G_s\} \), let \( G_k \) be a subgraph with \( \text{effdiam} V(G_k) = \text{effdiam} V(G) = r \). If \( G_k \) has \( n_0 \) vertices and \( m_0 \) edges, then \( m_0 \leq M(n_0, r) \) by Lemma 2.7.

Note that \( 2r \leq n_0 \).

Completing \( G_k \) with the complete graph of \( n-n_0+1 \) vertices (one of the vertices belongs to \( G_k \)) we get that \( m \leq m_0 + \left(n-n_0+1\right)/2 \).

By Lemma 2.8 we have \( m \leq m_0 + M(n, r) - M(n_0, r) \) and, since \( m_0 \leq M(n_0, r) \), we conclude \( m \leq M(n, r) \).

\[ \square \]

Corollary 2.10. If \( G \in G(n, m) \), \( 2 \leq r \leq n/2 \) and \( m > M(n, r) \), then \( \text{effdiam} V(G) \neq r \).

We will show now that, in fact, this result can be improved.

Theorem 2.11. If \( G \in G(n, m) \), \( 2 \leq r \leq n/2 \) and \( m > M(n, r) \), then \( \text{effdiam} V(G) < r \).

Proof. By Corollary 2.10 it suffices to prove that \( M(n, r) \) is a decreasing function of \( r \). We have \( \Delta_2 \leq \Delta_3 \leq \Delta_4 \), since \( 1 \leq n-1 \leq 2n-3 \). Thus, \( M(n, 2) \geq M(n, 3) \geq M(n, 4) \). If \( r \geq 4 \), then \( M(n, r) \) decreases as a function of \( r \) since \( 2r \leq n \) gives
\[
\frac{\partial M(n, r)}{\partial r} = -2(n-2r+3) \leq 0.
\]

Since \( \text{effdiam} V(G) < r \) implies \( \text{effdiam} G \leq r \), Lemma 2.8 and Theorem 2.11 imply the following theorems.

Theorem 2.12. If \( G \in G(n, m) \), \( 2 \leq r \leq n/2 \) and \( m > M(n, r) \), then \( \delta(G) \leq r/2 \).

Define \( M(n, 1) := \frac{n(n-1)}{2} \).

Theorem 2.13. If \( n \geq 1 \) and \( m = n-1 \), then \( B(n, m) = 0 \). If \( n \geq 3 \) and \( n \leq m \leq n+3 \), then \( B(n, m) = n/4 \). If \( G \in G(n, m) \), \( 2 \leq r \leq n/2 \) and \( M(n, r) < m \leq M(n, r-1) \), then \( B(n, m) \leq r/2 \).
Proof. If $n \geq 1$ and $m = n - 1$, then every $G \in \mathcal{G}(n, m)$ is a tree and $\delta(G) = 0$; consequently, $B(n, m) = 0$.

If $n \geq 3$ and $n \leq m \leq n+3$, then [38, Theorem 30] gives that there exists $G_0 \in \mathcal{G}(n, m)$ with $\delta(G_0) = n/4$. Furthermore, $\delta(G) \leq n/4$ for every $n, m$ and $G \in \mathcal{G}(n, m)$ by [38, Theorem 30]. Hence, $B(n, m) = n/4$ for $3 \leq n \leq m \leq n + 3$.

The second part of the statement is a consequence of Theorem 2.13.

3. A LOWER BOUND FOR $B(n, m)$

**Theorem 3.1.** If $3 \leq n_0 \leq n$ and $n < m \leq n + \left(\frac{n_0 - 1}{2}\right)$, then $B(n, m) \geq (n - n_0 + 3)/4$.

**Proof.** Let us consider a cycle graph with $n$ vertices $C_n$. Given $n_0 \geq 3$, choose a path $\{v_1, ..., v_{n_0}\}$ in $C_n$ and add $\left(\frac{n_0}{2}\right) - (n_0 - 1) = \left(\frac{n_0 - 1}{2}\right)$ edges to $C_n$ if $n_0 < n$, or $\left(\frac{n}{2}\right) - n$ if $n_0 = n$, obtaining a graph $G_{n,n_0}$ such that the induced subgraph by $\{v_1, ..., v_{n_0}\}$ in $G_{n,n_0}$ is isomorphic to the complete graph with $n_0$ vertices.

Choose a path $\{v_1, ..., v_{n_0}\}$ in $C_n$ and add $m - n$ edges to $C_n$, obtaining a subgraph $G$ of $G_{n,n_0}$ with at least some $v_i$ verifying $\{v_i, v_1\}, \{v_i, v_{n_0}\} \in E(G)$. If $n_0 = 3$, then $n < m \leq n + 1$ and $m = n + 1$.

Note that $G \in \mathcal{G}(n, m)$. Let $\eta$ be the path in $C_n$ joining $v_1$ and $v_{n_0}$ with $v_2, ..., v_{n_0 - 1} \notin \eta$ and let $y$ be the midpoint of $\eta$. Define $x := v_i$, $\gamma_1 = [x, v_1] \cup [v_1 y]$ and $\gamma_2 = [x, v_{n_0}] \cup [v_{n_0} y]$. Then $\gamma_1$ and $\gamma_2$ are geodesics from $x$ to $y$ and

$$d_G(x, y) = 1 + \frac{n - (n_0 - 1)}{2} = \frac{n - n_0 + 3}{2}.$$ Consider the geodesic bigon $T = \{\gamma_1, \gamma_2\}$ and the midpoint $p$ of $\gamma_1$. Then

$$B(n, m) \geq \delta(G) \geq d_G(p, \gamma_2) = \frac{1}{2} L(\gamma_1) = \frac{n - n_0 + 3}{4}.$$

Theorems 2.13 and 3.1 have the following direct consequence.

**Theorem 3.2.** If $n \geq 1$ and $m = n - 1$, then $B(n, m) = 0$. If $n \geq 3$ and $n \leq m \leq n + 3$, then $B(n, m) = n/4$. If $5 \leq n_0 \leq n$ and $n + \left(\frac{n_0 - 2}{2}\right) < m \leq n + \left(\frac{n_0 - 1}{2}\right)$, then $B(n, m) \geq (n - n_0 + 3)/4$.

4. DIFFERENCE OF THE BOUNDS OF $B(n, m)$

Let $b_1(n, m)$ and $b_2(n, m)$ be the lower and upper bounds of $B(n, m)$ obtained in Theorems 5.2 and 2.13 respectively. In this section we prove that the difference between $b_2$ and $b_1$ is $O(\sqrt{n})$. This is a good estimate, since the sharp upper bound for graphs with $n$ vertices is $n/4$ (see [38, Theorem 30]).

**Lemma 4.1.** Given integers $n$ and $r$ with $2 \leq r \leq n/2$, let $n_0$ be the smallest natural number such that $3 \leq n_0 \leq n$ and $M(n, r) < n + \left(\frac{n_0 - 1}{2}\right)$. The following holds for $M(n, r) < m \leq n + \left(\frac{n_0 - 1}{2}\right)$:

- If $r = 2$, then $b_2(n, m) = b_1(n, m)$.
- If $r = 3$, then $b_2(n, m) - b_1(n, m) < 3/4$.
- If $4 \leq r \leq n/2$, then $b_2(n, m) - b_1(n, m) < \sqrt{3n}/4$.

**Remark 4.2.** Note that we always have $M(n, r) \leq \frac{1}{2} n(n - 1) < n + \left(\frac{n - 1}{2}\right)$, and this implies the existence of $n_0$. 

Proof. If \( r = 2 \), then \( M(n, r) = \binom{n}{2} - 1 \) and \( M(n, 2) < m \) implies \( m = \binom{n}{2} \). Hence, every graph \( G \in G(n, m) \) is isomorphic to the complete graph with \( n \) vertices, and \( \delta(G) = 1 \) since \( n \geq 4 \). Thus, \( A(n, m) = B(n, m) = 1 \) and \( b_1(n, m) = b_2(n, m) = 1 \).

If \( r = 3 \), then

\[
M(n, 3) < n + \frac{n_0 - 1}{2} \quad \Leftrightarrow \quad (\frac{n}{2}) - (n - 1) < n + \frac{n_0 - 1}{2} \quad \Leftrightarrow \quad n^2 - 5n < n_0^2 - 3n_0.
\]

Let us define \( \lambda_3 := n^2 - 5n \). Since \( n \geq 4 \), the smallest \( n_0 \) verifying the previous inequality is the smallest \( n_0 \) satisfying \( n_0 > \frac{3 + \sqrt{9 + 4\lambda_3}}{2} \). Thus \( n_0 \leq \frac{5 + \sqrt{9 + 4\lambda_3}}{2} =: \rho_0 \).

Then, the following holds

\[
\frac{r}{2} - \frac{n - n_0 + 3}{4} \leq \frac{3}{2} - \frac{n - n_0 + 3}{4} = \frac{11 - 2n + \sqrt{9 + 4\lambda_3}}{8} = \frac{11 - 2n + \sqrt{4(n - 5/2)^2 - 16}}{8}.
\]

Note that

\[
\frac{11 - 2n + \sqrt{4(n - 5/2)^2 - 16}}{8} < \frac{11 - 2n + \sqrt{4(n - 5/2)^2}}{8} = \frac{11 - 2n + 2(n - 5/2)}{8} = \frac{3}{4}.
\]

Therefore, for \( r = 3 \), we obtain \( b_2(n, m) - b_1(n, m) = \frac{r}{2} - \frac{n - n_0 + 3}{4} < \frac{3}{4} \).

Note that if \( r \geq 4 \), then

\[
M(n, r) < n + \frac{n_0 - 1}{2} \quad \Leftrightarrow \quad (\frac{n}{2}) - (2n(r - 3) - 2r^2 + 6r + 5) < n + \frac{n_0 - 1}{2} \quad \Leftrightarrow \quad n^2 + 9n - 4nr + 4r^2 - 12r - 12 < n_0^2 - 3n_0.
\]

Let us define \( \lambda_r := n^2 + 9n - 4nr + 4r^2 - 12r - 12 \). Then, the smallest \( n_0 \) verifying the previous inequality is the smallest \( n_0 \) satisfying \( n_0 > \frac{3 + \sqrt{9 + 4\lambda_r}}{2} \). Thus \( n_0 \leq \frac{5 + \sqrt{9 + 4\lambda_r}}{2} =: \rho_0 \).

Note that

\[
\frac{r}{2} - \frac{n - n_0 + 3}{4} \leq \frac{r}{2} - \frac{n - n_0 + 3}{4} = \frac{4r + \sqrt{9 + 4\lambda_r} - 2n - 1}{8}.
\]

Let us fix \( n \) and consider the function \( F(r) = 4r + \sqrt{9 + 4\lambda_r} \). It can be easily checked that \( F'(r) = 4 + \frac{2(-4n + 8r - 12)}{\sqrt{9 + 4\lambda_r}} > 0 \) for all \( r \in [4, n/2] \) if and only if \( n > 6 \).

Since \( r \geq 4 \), we have \( n \geq 8 \), \( F(r) \) is an increasing function and \( F(n/2) = 2n + \sqrt{9 + 4(3n - 12)} \) is the maximum value of \( F(r) \).

Then, the following inequalities hold

\[
b_2(n, m) - b_1(n, m) = \frac{r}{2} - \frac{n - n_0 + 3}{4} \leq \frac{F(n/2) - 2n - 1}{8} < \frac{\sqrt{9 + 4(3n - 12)}}{8} < \frac{2\sqrt{3n}}{4} = \frac{\sqrt{3n}}{2}.
\]

Lemma 4.3. Given integers \( n \) and \( r \) with \( 3 \leq r \leq n/2 \), let \( n_1 \) be the smallest natural number such that \( 3 \leq n_1 \leq n \) and \( M(n, r - 1) < n + \frac{n_1 - 1}{2} \). Consider \( n_0 \) defined as in Lemma 4.7. The following holds.

- If \( r = 3 \), \( r = 4 \) or \( r = n/2 \), then \( n_1 - n_0 \leq 2 \).
- If \( 5 \leq r < n/2 \), then \( n_1 - n_0 \leq 4 \).
Proof. If \( r = 3 \), then

\[ M(n, r - 1) < n + \binom{n_1 - 1}{2} \iff n^2 - 3n - 4 < n_1^2 - 3n_1. \]

Using the definition of \( \lambda_r \) in the proof of Lemma 4.1, we deduce that the smallest natural number \( n_1 \) verifying the previous inequality satisfies

\[ n_1 \leq \frac{5 + \sqrt{9 + 4\lambda_2}}{2} =: n'_1. \]

If \( r = 4 \), then

\[ M(n, r - 1) < n + \binom{n_1 - 1}{2} \iff n^2 - 5n < n_1^2 - 3n_1. \]

Therefore, the smallest \( n_1 \) verifying the previous inequality satisfies \( n_1 \leq \frac{5 + \sqrt{9 + 4\lambda_3}}{2} =: n'_1. \)

Note that if \( r \geq 5 \), then

\[ M(n, r - 1) < n + \binom{n_1 - 1}{2} \iff n^2 + 13n - 4nr + 4r^2 - 20r + 4 = n^2 + 9n - 4n(r - 1) + 4(r - 1)^2 - 12(r - 1) - 12 < n_0^2 - 3n_0. \]

Thus, the smallest \( n_1 \) verifying the previous inequality satisfies \( n_1 \leq \frac{5 + \sqrt{9 + 4\lambda_{r-1}}}{2} =: n'_1. \)

Now we estimate the difference between \( n_1 \) and \( n_0 \).

\[ n_1 - n_0 < n'_1 - (n'_0 - 1) = \frac{\sqrt{9 + 4\lambda_{r-1}} - \sqrt{9 + 4\lambda_r}}{2} + 1 = \frac{2(\lambda_{r-1} - \lambda_r)}{\sqrt{9 + 4\lambda_{r-1}} + \sqrt{9 + 4\lambda_r}} + 1 \leq \frac{\lambda_{r-1} - \lambda_r}{\sqrt{9 + 4\lambda_r}} + 1. \]

If \( r = 3 \), then \( n \geq 6 \) and

\[ n_1 - n_0 < \frac{\lambda_2 - \lambda_3}{\sqrt{9 + 4\lambda_3}} + 1 = \frac{2(n - 2)}{\sqrt{9 + 4\lambda_3}} + 1 < \frac{n - 2}{\sqrt{\lambda_3}} + 1. \]

The following holds

\[ \frac{n^2 - 4n + 4}{n^2 - 5n} = \frac{n^2 - 5n + n + 4}{n^2 - 5n} < 3 \Rightarrow \frac{n - 2}{\sqrt{\lambda_3}} < \sqrt{3}. \]

Therefore,

\[ n_1 - n_0 < \sqrt{3} + 1 \Rightarrow n_1 - n_0 \leq 2. \]

If \( r = 4 \), then \( n \geq 8 \) and

\[ n_1 - n_0 < \frac{\lambda_3 - \lambda_4}{\sqrt{9 + 4\lambda_4}} + 1 = \frac{2(n - 2)}{\sqrt{9 + 4\lambda_4}} + 1 < \frac{n - 2}{\sqrt{\lambda_4}} + 1. \]

The following holds

\[ \frac{n^2 - 4n + 4}{n^2 - 7n + 4} = \frac{n^2 - 7n + 4 + 3n}{n^2 - 7n + 4} \leq 3 \Rightarrow \frac{n - 2}{\sqrt{\lambda_4}} \leq \sqrt{3}. \]

Therefore,

\[ n_1 - n_0 < \sqrt{3} + 1 \Rightarrow n_1 - n_0 \leq 2. \]
If $r \geq 5$, then
\[
n_1 - n_0 < \frac{\lambda_{r-1} - \lambda_r}{\sqrt{9 + 4\lambda_r}} + 1 = \frac{4(n - 2r + 4)}{\sqrt{9 + 4\lambda_r}} + 1.
\]

Note that
\[
\lambda_r = (n - 2r)^2 + 9(n - 2r) + 6r - 12 \geq (n - 2r)^2 + 6r - 12 \geq (n - 2r)^2 + 18.
\]

If $r < n/2$, then
\[
\frac{4(n - 2r + 4)}{\sqrt{9 + 4\lambda_r}} \leq \frac{4(n - 2r + 4)}{\sqrt{81 + 4(n - 2r)^2}} < \frac{2n - 2r + 16}{n - 2r} < \frac{9}{4} < 4.
\]
Thus, $n_1 - n_0 < 5$ and $n_1 - n_0 \leq 4$.

If $r = n/2$, then
\[
\frac{4(n - 2r + 4)}{\sqrt{9 + 4\lambda_r}} \leq \frac{4(n - 2r + 4)}{\sqrt{81 + 4(n - 2r)^2}} = \frac{16}{9} < 2.
\]
Therefore $n_1 - n_0 < 3$ and $n_1 - n_0 \leq 2$.

The following result is a consequence of the two previous lemmas.

**Lemma 4.4.** Given integers $n$ and $r$ with $3 \leq r \leq n/2$, let $n_0$ be defined as in Lemma 4.1. Assume $M(n, r - 1) > n + \left\lfloor \frac{n_0 - 1}{2} \right\rfloor$. The following holds for $n + \left\lfloor \frac{n_0 - 1}{2} \right\rfloor < m \leq M(n, r - 1)$.

- If $r = 3$, then $b_2(n, m) - b_1(n, m) < \frac{5}{4}$.
- If $r = 4$ or $r = n/2$, then $b_2(n, m) - b_1(n, m) < \sqrt{3n}/4 + 1/2$.
- If $5 \leq r < n/2$, then $b_2(n, m) - b_1(n, m) < \sqrt{3n}/4 + 1$.

**Proof.** Let $n_1$ be defined as in Lemma 4.3.

On the other hand, $m \leq M(n, r - 1) < n + \left( \frac{n_1 - 1}{2} \right)$ and Theorem 3.1 gives $b_1(n, m) \geq (n - n_1 + 3)/4$.

On the other hand, $M(n, r) < n + \left( \frac{n_0 - 1}{2} \right) < m \leq M(n, r - 1)$ and Theorem 2.13 gives $b_2(n, m) = r/2$.

The following holds
\[
b_2(n, m) - b_1(n, m) = b_2(n, m) - \frac{n - n_0 + 3}{4} + \frac{n - n_0 + 3}{4} - b_1(n, m).
\]

Notice that
\[
\frac{n - n_0 + 3}{4} - b_1(n, m) \leq \frac{n - n_0 + 3}{4} - \frac{n - n_1 + 3}{4} = \frac{n_1 - n_0}{4}.
\]

Then, applying Lemmas 4.1 and 4.3 in order to bound $b_2(n, m) - (n - n_0 + 3)/4$ and $n_1 - n_0$, respectively, we obtain the desired upper bounds.

Lemmas 4.1 and 4.3 have the following consequence.

**Theorem 4.5.** The following holds for all $n \geq 3$.

\[
(4.1) \quad b_2(n, m) - b_1(n, m) < \frac{\sqrt{3n}}{4} + 1.
\]
Proof. If $m > M(n, 3)$, then $b_2(n, m) \leq 3/2$ by Theorem 2.12 and

$$b_2(n, m) - b_1(n, m) \leq b_2(n, m) \leq \frac{3}{2} - \frac{3}{4} + 1 \leq \frac{\sqrt{3n}}{4} + 1.$$

Consider now $r \geq 3$ and $n_0$ defined as in Lemma 4.1. If $M(n, r) < m \leq n + \left(\frac{n_0 - 1}{2}\right)$ or $M(n, r - 1) \leq m \leq n + \left(\frac{n_1 - 1}{2}\right)$, then Lemma 4.1 gives

$$b_2(n, m) - b_1(n, m) < \frac{\sqrt{3n}}{4}. \quad (4.2)$$

If $M(n, r - 1) \leq n + \left(\frac{n_0 - 1}{2}\right)$, then equation 4.2 holds for $M(n, r) < m \leq M(n, r - 1)$.

If $n + \left(\frac{n_0 - 1}{2}\right) < M(n, r - 1)$ and $n + \left(\frac{n_0 - 1}{2}\right) < m \leq M(n, r - 1)$, then Lemma 4.4 implies 4.1. Thus, equation 4.1 holds for every $m > M(n, \lfloor n/2 \rfloor)$.

Finally, assume that $n + 3 < m \leq M(n, \lfloor n/2 \rfloor)$.

First, note that if $M(n, \lfloor n/2 \rfloor) < m \leq \min \left\{ n + \left(\frac{n_0 - 1}{2}\right), M(n, \lfloor n/2 \rfloor - 1) \right\}$, then Lemma 4.1 implies

$$b_2(n, m) - b_1(n, m) = \frac{\lfloor \frac{n}{2} \rfloor}{2} - \frac{n - n_0 + 3}{4} < \frac{\sqrt{3n}}{4}.$$ 

Consider now $m \leq M(n, \lfloor n/2 \rfloor)$, then

$$b_2(n, m) - b_1(n, m) \leq \frac{n}{4} - \frac{n - n_0 + 3}{4} < \frac{2(\lfloor \frac{n}{2} \rfloor + 1)}{4} - \frac{n - n_0 + 3}{4} = \frac{\lfloor \frac{n}{2} \rfloor}{2} - \frac{n - n_0 + 3}{4} + \frac{1}{2} < \frac{\sqrt{3n}}{4} + \frac{1}{2}.$$ 

Hence, (4.2) holds for every $m \leq M(n, \lfloor n/2 \rfloor)$.

5. Computation of $A(n, m)$

Denote by $\Gamma_3$ the set of graphs such that every cycle has length 3 and every edge belongs to some cycle.

Proposition 5.1. Consider a graph $G \in \mathcal{G}(n, m) \cap \Gamma_3$. If $k$ denotes the number of cycles of $G$, then $n = 2k + 1$ and $m = 3k$.

Proof. Let us prove the result by induction on $k$.

If $k = 1$, then $G$ is isomorphic to $C_3$ and $n = m = 3$.

Assume that the statement holds for every graph $G_0$ with $k - 1$ cycles. Then $G_0$ has $n_0 = 2(k - 1) + 1$ vertices and $m_0 = 3(k - 1)$ edges. Any graph $G$ with $k$ cycles can be obtained by adding 2 vertices and 3 edges to some graph $G_0$ with $k - 1$ cycles, that is, $n = n_0 + 2 = 2k + 1$ and $m = m_0 + 3 = 3k$. \hfill \Box

We say that an edge $g$ of a graph $G$ is a cut-edge if $G \setminus \{g\}$ is not connected. Given a graph $G$, the $T$-edge-decomposition of $G$ is a $T$-decomposition such that each component $G_s$ is either a cut-edge or it does not contain cut-edges.

Proposition 5.2. Let $G \in \mathcal{G}(n, m)$ be a graph such that every cycle has length 3. Then $2m \leq 3n - 3$. 

Proof. The canonical T-edge-decomposition of $G$ has $r \geq 1$ graphs $\{G_1, \ldots, G_r\}$ in $\Gamma_3$ and $s \geq 0$ edges $\{G_{r+1}, \ldots, G_{r+s}\}$. For each component $G_i \in \Gamma_3$ we have, by \[5.1\]

$$n_i = 2k_i + 1, \quad m_i = 3k_i, \quad 1 \leq i \leq r,$$

where $n_i$, $m_i$ and $k_i$ denote the number of vertices, edges and cycles in $G_i$, respectively.

Let us denote by $k = \sum_{i=1}^r k_i$ the number of cycles of $G$. Let $n_0$ and $m_0$ be the number of vertices and edges we add to complete $G$, i.e., $n_0 = n - \sum_{i=1}^r n_i$, $m_0 = m - \sum_{i=1}^r m_i$. Then we have

$$n = \sum_{i=0}^r n_i = n_0 + \sum_{i=1}^r (2k_i + 1) = n_0 + 2k + r,$$

$$m = \sum_{i=0}^r m_i = m_0 + \sum_{i=1}^r (3k_i) = m_0 + 3k.$$

Hence,

$$n = n_0 + 2 \frac{m - m_0}{3} + r.$$

One can check that if $n_0 = 0$, then $m_0 = r - 1$ and if $n_0 \geq 1$, then $m_0 = n_0 + r - 1$. Therefore,

$$n = n_0 + 2 \frac{m - (n_0 + r - 1)}{3} + r \Rightarrow 2m = 3n - n_0 - r - 2 \Rightarrow 2m = 3n - m_0 - 3.$$

Then $2m \leq 3n - 3$. \hfill $\square$

The next result appears in [38].

**Theorem 5.3.** Let $G$ be any graph.

- $\delta(G) < 1/4$ if and only if $G$ is a tree.
- $\delta(G) < 1$ if and only if every cycle $g$ in $G$ has length $L(g) \leq 3$.

Furthermore, if $\delta(G) < 1$, then $\delta(G) \in \{0, 3/4\}$.

**Proposition 5.4.** If $m \geq n$ and $2m \leq 3n - 3$, then $A(n, m) = 3/4$.

**Remark 5.5.** Note that $n \leq m \leq (3n - 3)/2$ implies $n \geq 3$.

Proof. Since $m \geq n \geq 3$, if $G \in \mathcal{G}(n, m)$, then $G$ is not a tree. Hence Theorem [38] gives $\delta(G) \geq 3/4$ and $A(n, m) \geq 3/4$.

Fix $n, m$ verifying the hypotheses. Define $n_0 := m_0 := 3n - 3 - 2m$ and $k := m + 1 - n$. Then

$$n = 2k + 1 + n_0, \quad m = 3k + n_0.$$

Let us consider $k$ graphs $G_1, \ldots, G_k$ isomorphic to $C_3$ and $n_0$ graphs $\Gamma_1, \ldots, \Gamma_{n_0}$ isomorphic to $P_2$. Fix vertices $v_1 \in V(G_1), \ldots, v_k \in V(G_k), w_1 \in V(\Gamma_1), \ldots, w_{n_0} \in V(\Gamma_{n_0})$ and consider the graph $G$ obtained from $G_1, \ldots, G_k, \Gamma_1, \ldots, \Gamma_{n_0}$ by identifying $v_1, \ldots, v_k, w_1, \ldots, w_{n_0}$ in a single vertex. Then $G \in \mathcal{G}(n, m)$ and $\delta(G) = 3/4$. Therefore, $A(n, m) \leq 3/4$ and we conclude $A(n, m) = 3/4$. \hfill $\square$

**Definition 5.6.** Let $K_n$ be the complete graph with $n$ vertices and consider the numbers $N_i$, $i = 1, \ldots, s$, $(s \geq 1)$ such that $2 \leq N_1, \ldots, N_s < n$, $N_1 + \cdots + N_s \leq n$. Choose sets of vertices $V_1, \ldots, V_s \subset V(K_n)$ with $V_i \cap V_j = \emptyset$ if $i \neq j$ and $\#V_i = N_i$ for $i = 1, \ldots, s$. Let $K_n^{N_1, \ldots, N_s}$ be the graph obtained from $K_n$ by removing the edges joining any two vertices in $V_i$ for every $i = 1, \ldots, s$.

**Lemma 5.7.** We always have $\delta(K_n^{N_1, \ldots, N_s}) \leq 1$.

Proof. First of all, note that $\deltaiam(V(K_n^{N_1, \ldots, N_s})) = 2$. Hence, in order to prove $\deltaiam(K_n^{N_1, \ldots, N_s}) = 2$, it suffices to check that $d(x, y) \leq 2$ for every midpoint $x$ of any edge in $E(K_n^{N_1, \ldots, N_s})$ and every $y \in K_n^{N_1, \ldots, N_s}$.

Fix $i \in \{1, \ldots, s\}$ and $u \in V_i$. Then, $d(u, v) = 1$ for every $v \in V(K_n^{N_1, \ldots, N_s}) \setminus V_i$ and $d(u, v) = 2$ for every $v \in V_i \setminus \{u\}$.

Given a fixed vertex $u \in V_i$, let $x$ be the midpoint of the edge $[u, v]$ (then $v \notin V_i$). If $w \in V_i$, then there exists an edge joining $v$ with $w$. Therefore, we have $d(x, w) \leq d(x, v) + d(v, w) = 3/2$. If $w \notin V_i$, then
\[ u, w \in E(K^{N_1\ldots N_s}_n) \text{ and } d(x, w) \leq d(x, u) + d(u, w) = 3/2. \] Hence, \( d(x, v) \leq 3/2 \) for every \( v \in V(K^{N_1\ldots N_s}_n) \); thus, \( d(x, y) \leq 2 \) for every \( y \in K^{N_1\ldots N_s}_n \).

If \( N_1 + \cdots + N_s \leq n - 2 \), let \( x \) be the midpoint of \([v^1, v^2]\), where \( v^1, v^2 \notin \cup_i V_i \). If \( v \in V(K^{N_1\ldots N_s}_n) \), then there exists an edge joining \( v \) with \( v^1 \). Thus, we have \( d(x, v) \leq d(x, v^1) + d(v^1, v) = 3/2 \) for every \( v \in V(K^{N_1\ldots N_s}_n) \). Hence, \( d(x, y) \leq 2 \) for every \( y \in K^{N_1\ldots N_s}_n \).

Therefore \( \text{diam}(K^{N_1\ldots N_s}_n) = 2 \) and \( \delta(K^{N_1\ldots N_s}_n) \leq 1 \) by Theorem 2.4. \( \square \)

In order to prove our next result we need the following Combinatorial lemma.

**Lemma 5.8.** For all \( t \geq 3 \), \( t \neq 4, 5 \), there exist numbers \( t_i \geq 2, i = 1, \ldots, s \) \( (s \geq 1) \), such that

\[ \Sigma_i t_i \leq t \text{ and } \Sigma_i \left( \frac{t_i}{2} \right) = t. \]

**Proof.** If \( t = 3 \), then choose \( t_1 = 3, 3 \leq 3 \) and \( \left( \frac{3}{2} \right) = 3 \).

If \( t = 6 \), then choose \( t_1 = 4, 4 \leq 6 \) and \( \left( \frac{4}{2} \right) = 6 \).

If \( t = 7 \), then choose \( t_1 = 4, t_2 = 2, 4 + 2 \leq 7 \) and \( \left( \frac{4}{2} \right) + \left( \frac{2}{2} \right) = 7 \).

If \( t = 8 \), then choose \( t_1 = 4, t_2 = 2, t_3 = 2, 4 + 2 + 2 \leq 8 \) and \( \left( \frac{4}{2} \right) + \left( \frac{2}{2} \right) + \left( \frac{2}{2} \right) = 8 \).

If \( t = 9 \), then choose \( t_1 = 4, t_2 = 3, 4 + 3 \leq 9 \) and \( \left( \frac{4}{2} \right) + \left( \frac{3}{2} \right) = 9 \).

Let us prove the result by induction on \( t \).

We have seen that

\[ \Sigma_i t_i \leq t, \quad \Sigma_i \left( \frac{t_i}{2} \right) = t \]

holds for \( 6 \leq t \leq 9 \). Assume now that it holds for every value \( 3, 6, 7, \ldots, t - 1 \), with \( t > 9 \). Then it holds for \( t - 3 \geq 6 \) and there exist numbers \( t_i \geq 2, i = 1, \ldots, s \), such that \( \Sigma_i t_i \leq t - 3 \) and \( \Sigma_i \left( \frac{t_i}{2} \right) = t - 3 \).

Therefore, there exist numbers \( t'_i \geq 2, t'_i = t_i \) for \( i = 1, \ldots, s \), \( t'_{s+1} = 3 \) such that

\[ \Sigma_i t'_i = \Sigma_i t_i + 3 \leq t \]

and

\[ \Sigma_i \left( \frac{t'_i}{2} \right) = \Sigma_i \left( \frac{t_i}{2} \right) + \left( \frac{3}{2} \right) = t. \]

So we have shown that the statement holds at \( t \) when it is assumed to be true for \( 3, 6, 7, \ldots, t - 1 \). \( \square \)

**Corollary 5.9.** For all \( t \geq 1 \), there exist numbers \( t_i \geq 2, i = 1, \ldots, s \) \( (s \geq 1) \) such that \( \Sigma_i t_i \leq t + 2 \) and \( \Sigma_i \left( \frac{t_i}{2} \right) = t \).

**Proof.** If \( t \neq 1, 2, 4, 5 \), then Lemma 5.8 gives the result.

If \( t = 1 \), then choose \( t_1 = 2, 2 \leq 3 \) and \( \left( \frac{2}{2} \right) = 1 \).

If \( t = 2 \), then choose \( t_1 = 2, t_2 = 2, 2 + 2 \leq 4 \) and \( \left( \frac{2}{2} \right) + \left( \frac{2}{2} \right) = 2 \).
If \( t = 4 \), then choose \( t_1 = 3, t_2 = 2, 3 + 2 \leq 6 \) and \( \left( \frac{3}{2} \right) + \left( \frac{2}{2} \right) = 4 \).

If \( t = 5 \), then choose \( t_1 = 3, t_2 = 2, t_3 = 2, 3 + 2 + 2 \leq 7 \) and \( \left( \frac{3}{2} \right) + \left( \frac{2}{2} \right) + \left( \frac{2}{2} \right) = 5 \).

\( \blacksquare \)

**Proposition 5.10.** If \( m \geq n \) and \( 2m > 3n - 3 \), then \( A(n, m) = 1 \).

**Proof.** Consider any \( G \in \mathcal{G}(n, m) \). Proposition 5.2 gives that there exists at least one cycle in \( G \) with length greater or equal than 4. Then Theorem 5.3 gives \( \delta(G) \geq 1 \) for every \( G \in \mathcal{G}(n, m) \) and, consequently, \( A(n, m) \geq 1 \).

In order to finish the proof it suffices to find a graph \( G \in \mathcal{G}(n, m) \) with \( \delta(G) \leq 1 \).

Note that \( n \geq 4 \) since \( 2m > 3n - 3 \).

If \( m = n + 1 \), then consider a graph \( G_1 \) with 4 vertices and 5 edges and a path graph \( G_2 \) with \( n - 3 \) vertices and \( n - 4 \) edges. Fix vertices \( v_1 \in G_1 \) and \( v_2 \in G_2 \). Let \( G \) be the graph obtained by identifying \( v_1 \) and \( v_2 \) in a single vertex, then \( G \) has \( n \) vertices and \( m = n + 1 \) edges, and \( \delta(G) = \delta(G_1) = 1 \). Therefore \( A(n, m) \leq \delta(G) \leq 1 \) and we conclude \( A(n, m) = 1 \).

If \( m = \left( \frac{n}{2} \right) \) and \( G \in \mathcal{G}(n, m) \), the \( G \) is isomorphic to \( K_n \) and \( \delta(G) = 1 \). Therefore \( A(n, m) = 1 \).

Assume now that \( n + 2 \leq m < \left( \frac{n}{2} \right) \). Then \( m - 6 \geq n - 4 \) and we can define

\[
n_0 - 1 \defeq \max \left\{ 4 \leq j \leq n - 1 \mid m - \left( \frac{j}{2} \right) \geq n - j \right\}.
\]

Then \( 3 \leq n_0 \leq n \) and we have

\[
\left( \frac{n_0 - 1}{2} \right) + n - n_0 + 1 \leq m < \left( \frac{n_0}{2} \right) + n - n_0.
\]

Define \( T \defeq \left( \frac{n_0}{2} \right) + n - n_0 - m \). Notice that

\[
1 \leq T \leq \left( \frac{n_0}{2} \right) + n - n_0 - \left( \frac{n_0 - 1}{2} \right) - n + n_0 - 1 = n_0 - 2.
\]

It follows from Corollary 5.9 that there exist numbers \( t_i \geq 2, i = 1, \ldots, s \), such that \( \Sigma_i t_i \leq T + 2 \leq n_0 \) and \( \Sigma_i \left( \frac{t_i}{2} \right) = T \).

Choose sets of vertices \( V_1, \ldots, V_s \subset V(K_{n_0}) \) with \( V_i \cap V_j = \emptyset \) if \( i \neq j \) and \( \#V_i = t_i \) for \( i = 1, \ldots, s \). Let us denote by \( G_1 \) the graph obtained from \( K_{n_0} \) by removing the \( T = \Sigma_i \left( \frac{t_i}{2} \right) \) edges joining any two vertices in \( V_i \) for every \( i = 1, \ldots, s \). Then \( G_1 \in \mathcal{G}(n_0, m - n_0) \) and Lemma 5.7 implies \( \delta(G_1) = \delta(K_{n_0}^{t_1; \ldots; t_s}) \leq 1 \).

Let us define \( G_2 \) as a path graph with \( n - n_0 + 1 \) vertices and \( n - n_0 \) edges. Fix vertices \( v_1 \in G_1 \) and \( v_2 \in G_2 \). Let \( G \) be the graph obtained from \( G_1 \) and \( G_2 \) by identifying \( v_1 \) and \( v_2 \) in a single vertex, then \( G \in \mathcal{G}(n, m) \) and \( \delta(G) = \delta(G_1) = 1 \). Therefore \( A(n, m) \leq \delta(G) = 1 \) and we conclude \( A(n, m) = 1 \).

\( \blacksquare \)

The previous results have the following consequence.

**Theorem 5.11.** If \( m = n - 1 \), then \( A(n, m) = 0 \).

If \( m \geq n \) and \( 2m \leq 3n - 3 \), then \( A(n, m) = 3/4 \).

If \( m \geq n \) and \( 2m > 3n - 3 \), then \( A(n, m) = 1 \).
6. Random graphs

The field of random graphs was started in the late fifties and early sixties of the last century by Erdős and Rényi, see [20, 21, 30, 22]. At first, the study of random graphs was used to prove deterministic properties of graphs. For example, if we can show that a random graph has a certain property with a positive probability, then a graph must exist with this property. Lately there has been a great amount of work on the field. The practical applications of random graphs are found, for instance, in areas in which complex networks need to be modeled. See the standard references on the subject [9] and [29] for the state of the art.

Erdős and Rényi studied in [21] the simplest imaginable random graph, which is now named after them. Given \( n \) fixed vertices, the Erdős-Rényi random graph \( R(n, m) \) is characterized by \( m \) edges distributed uniformly at random among all possible \( \binom{n}{2} \) edges. However, in order to avoid disconnected graphs, which are not geodesic metric spaces, a random tree of order \( n \) is first generated and then the remaining \( m - (n - 1) \) edges are distributed uniformly at random over the remaining \( \binom{n}{2} - n + 1 \) possible edges. Call this new model \( R'(n, m) \). This modified Erdős-Rényi random graph \( R'(n, m) \) has a number of desirable properties as a model of a network, see [31].

We can apply the results obtained in this work to \( R'(n, m) \):

For all \( G \in R'(n, m) \) we have \( A(n, m) \leq \delta(G) \leq B(n, m) \), and Theorems 6.11 and 2.18 give the precise value for \( A(n, m) \) and an upper bound of \( B(n, m) \).

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