The connection between Jackson and Hausdorff derivatives in the context of generalized statistical mechanics

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Abstract

In literature one can find many generalizations of the usual Leibniz derivative, such as Jackson derivative, Tsallis derivative and Hausdorff derivative. In this article we present a connection between Jackson derivative and recently proposed Hausdorff derivative. On one hand, the Hausdorff derivative has been previously associated with non-extensivity in systems presenting fractal aspects. On the other hand, the Jackson derivative has a solid mathematical basis because it is the \(\eta\)-analog of the ordinary derivative and it also arises in quantum calculus. From a quantum deformed \(\eta\)-algebra we obtain the Jackson derivative and then address the problem of \(N\) non-interacting quantum oscillators. We perform an expansion in the quantum grand partition function from which we obtain a relationship between the parameter \(\eta\), related to Jackson derivative, and the parameters \(\zeta\) and \(q\) related to Hausdorff derivative and Tsallis derivative, respectively.

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I. INTRODUCTION

The observation of nature has always motivated the human curiosity about understanding the world around us. In particular, from a mathematical perspective, new concepts and ideas have been developed in the last centuries to model and describe a wide variety of complex physical systems. For example, it is not possible to associate Euclidean geometry with many forms found in nature, such as the shape of clouds, coastlines, etc. This difficulty in describing some physical systems in nature is basically due to non-standard irregularities and patterns that are present. In spite of the enormous complexity, most of these forms and trajectories present relatively simple scaling laws. The understanding of these complex and irregular physical systems was the motivation for the development of the concept of fractal geometry [1–4].

Fractal objects have been identified in many physical situations and play an important role in very distinct areas, such as topology, Brownian motion, fluid turbulence, surface roughness, porosity of rocks, dynamics systems, number theory, population of species, applications in medicine, quasicrystals [3–5], etc. There is a close connection between fractal geometry and chaotic systems, which has allowed the possibility of seeing order and patterns where previously only the random and chaotic were observed, ranging from problems of aggregation to the behavior of chaotic dynamical systems [10].

Fractals have peculiar properties and characteristics. As a consequence of the irregularities found in systems presenting fractal behavior, the tools of classical calculus may not be adequate to address natural phenomena described by functions associated to those systems. In particular, in the last decades several proposals for extending the concept of calculus, with great potential of application for the study of systems presenting fractal behavior, have appeared. Among these proposals we can cite many generalizations of the usual Leibniz derivative, such as the Jackson derivative, the Tsallis derivative, fractional derivatives and the Hausdorff derivative, which can be applied in a large set of functions. We highlight the Hausdorff derivative, with applications in mapping of fractal domains into the continuous [11–13] (for properties of coherent states of dissipative systems and in relations developed with generalized algebras for the study of complex systems in statistical mechanics [14–17]).

It is known from the literature that a possible mechanism to generate a deformed version of the classical statistical mechanics is to replace the usual Boltzmann-Gibbs distribution by
a deformed version. To do so, a form of deformed entropy is postulated that implies a theory of
generalized thermodynamics \[18\text{--}25\]. We can find in literature a number of generalizations of
well-known Boltzmann-Gibbs statistical mechanics obtained from \(q\)-deformed algebras \[26\text{--}35\]. In particular, \(q\)-calculus in the Tsallis version of non-extensive statistical mechanics
has been applied to a wide class of different systems.

In a recent work Weberszpil et al. showed a connection between \(q\)-deformed algebra, in Tsallis version of non-extensive statistical mechanics, with Hausdorff derivative, mapped
into a continuous medium with a fractal measure \[15\]. In this paper we show a relationship between Jackson derivative, Hausdorff derivative and Tsallis \(q\)-derivative.

Our interest in the Jackson derivative stems from the fact that it is the \(\overline{q}\)-analog of the
ordinary derivative. Unlike more recent deformations of the geometric and hypergeometric
series, the theory of \(\overline{q}\)-series has centuries of solid mathematical underpinning. The study
of \(\overline{q}\)-series began with some theorems of Gauss (e.g., for theta functions), Euler (e.g., the
combinatorial version of the pentagonal number theorem) and Cauchy (e.g., \(\overline{q}\)-analog of the
binomial theorem). At the turn of the 19th century and the beginning of the 20th century,
the English mathematicians F. H. Jackson (known for the JD) and L. J. Rogers (known
for the Rogers-Ramanujan identities) built the foundations of this important research area
of mathematics. It is often said that the person who most contributed to \(\overline{q}\)-series was
S. Ramanujan. Given this long mathematical tradition, the \(\overline{q}\)-analogs have a far more
solid theoretical basis in comparison to other deformations of the standard algebra. Our
motivation here is thus to take the HD and to try to replace it with the better-understood
JD.

This article is organized as follows. In Sec. \[\text{II}\] we review the connection between Hausdorff
derivative and Tsallis \(q\)-derivative. In Sec. \[\text{III}\] we obtain Jackson derivative from deformed
quantum algebra. Sec. \[\text{IV}\] is devoted to develop the relationship between Jackson and
Hausdorff derivatives. Finally in Sec. \[\text{V}\] we make our final comments.

\section{Connection between Hausdorff and Tsallis \(q\)-derivatives}

Statistical mechanics is one of the areas of physics that deals with the so-called complex
systems, which have been attracting a lot of attention in the last decades. In order to
approach such complex systems, in the last years several non-conventional formalisms have
been proposed, among them we can point out Tsallis, Kaniadakis and Abe \[18–20\], which are
generalizations of Boltzmann-Gibbs statistics \[41, 42\] through modified entropies. On the
other hand, formulations of deformed derivatives have shown great potential of applications
in the study of complex systems and phenomena presenting fractal properties. In this sense,
the concept of Hausdorff derivative (HD) of a function, with respect to a fractal measure
\[11\], has demonstrated possible connections with the q-derivatives in the framework of non-
extensive statistics \[15–17\], leading to a better understanding of both formalisms. It is worth
to remark that HD differs from the standard fractional derivative because it does not involve
the convolution integral and it is local in nature.

Let us consider HD derivative which is defined by \[11\],
\[
\frac{d^H}{dx^\zeta} f(x) = \lim_{x \to y} \frac{f(y) - f(x)}{y^\zeta - x^\zeta},
\]
with the local fractional differential operators being described as \[12\],
\[
\frac{d^H}{dx^\zeta} f(x) = \left(1 + \frac{x}{l_0}\right)^{1-\zeta} \frac{d}{dx} f = \frac{l_0^{\zeta-1}}{c_1} \frac{d}{dx} f = \frac{d}{d^\zeta x} f.
\]
Here \(l_0\) is the lower cutoff along the cartesian \(x\)-axis and the scaling exponent \(\zeta\) characterizes
the density of states along the direction of the normal to the intersection of the fractal
continuum with the plane (see Ref. \[12\] for details). Following the lines of Ref. \[15\], we
perform an expansion in \(x\) to the first order with exponent \((1 - \zeta)\). We get,
\[
\frac{d^H}{dx^\zeta} f(x) = \left(1 + \frac{x}{l_0}\right)^{1-\zeta} \frac{d}{dx} f(x) \approx \left[1 + (1 - \zeta) \frac{x}{l_0} + \cdots\right] \frac{d}{dx} f(x).
\]

Let us now consider the \(q\)-deformed derivative. In the Tsallis non-extensive framework,
the \(q\)-deformed difference operator is defined by \[21\],
\[
x \ominus_q z = \frac{x - z}{1 + (1 - q)x}, \quad \text{with} \quad z \neq \left(\frac{1}{q-1}\right).
\]
Therefore, we may write the Tsallis \(q\)-derivative as
\[
D_{(q)} f(x) = \lim_{z \to x} \frac{f(x) - f(z)}{x \ominus_q z} = \left[1 + (1 - q)x\right] \frac{d}{dx} f(x).
\]

By comparing Eqs. (5) and (3), Weberszpil and collaborators \[15\] obtained a relationship
between Tsallis \(q\)-derivative and HD derivative as follows,
\[
1 - q = \frac{1 - \zeta}{l_0}.
\]
Those authors conclude that the deformed Tsallis \(q\)-derivative is the first order expansion
of the Hausdorff derivative and that there is a strong connection between those formalisms
by means of a fractal metric.
III. q-DEFORMED QUANTUM ALGEBRA

It is worth to remark that the q-deformed quantum algebra, considered in the present section to introduce Jackson derivative, is a generalization of the Heisenberg quantum algebra. Therefore, it is defined in a completely different context from the Tsallis q-algebra (which is defined in the non-extensive statistical mechanics context) presented in the previous section. In order to avoid misunderstanding, in this work we use the notation q for Tsallis algebra and \( \bar{q} \) for the deformed quantum algebra.

At the beginning of the 20th Century, Jackson developed a number of works which have in many aspects played an important role in the understanding and developing of deformed quantum algebra [26]. The appearance of the deformation is made through the key ingredient - the deformation parameter \( \bar{q} \), which is introduced in the commutation relations that define the Lie algebra of the system. The original Lie algebra, not deformed, is recovered when \( \bar{q} \rightarrow 1 \).

Let us illustrate Jackson derivative by considering \( N \) non-interacting quantum oscillators in order to define a generalized \( \bar{q} \)-deformed dynamics in a \( \bar{q} \)-commutative phase space. The algebraic symmetry of the quantum oscillator is defined by the Heisenberg algebra in terms of annihilation and creation operators \( c, c^\dagger \), respectively, and the number operator \( N \) as follows [31, 43],

\[
[c, c]_\kappa = [c^\dagger, c^\dagger]_\kappa = 0, \quad cc^\dagger - \kappa \bar{q} c^\dagger c = 1, \tag{7}
\]

\[
[N, c^\dagger] = c^\dagger, \quad [N, c] = -c. \tag{8}
\]

Here the deformation parameter \( \bar{q} \) is real; the constant \( \kappa = 1 \) for \( \bar{q} \)-bosons (with commutators) and \( \kappa = -1 \) for \( \bar{q} \)-fermions (with anticommutators). We also define the basic number as

\[
[x_i] = c_i^\dagger c_i = \frac{\bar{q}^x - 1}{\bar{q} - 1}, \tag{9}
\]

with

\[
[x, y]_\kappa = xy - \kappa yx, \quad cc^\dagger = [1 + \kappa N]. \tag{10}
\]

Note that for \( \bar{q} \neq 1 \) the basic number \([x]\) does not meet additivity, i.e.,

\[
[x + y] = [x] + [y] + (\bar{q} - 1)[x][y]. \tag{11}
\]
It is also easy to observe that as \( q \to 1 \) the basic number \( [x] \) is reduced to an ordinary number \( x \).

The \( q \)-Fock space spanned by the orthonormalized eigenstates \( |n\rangle \) is constructed according to

\[
|n\rangle = \frac{(c^\dagger)^n}{\sqrt{[n]!}} |0\rangle, \quad c|0\rangle = 0,
\]

where the factorial of the basic number \( [n] \) is defined as

\[
[n]! = [n][n-1], \ldots, [1].
\]

The applications of \( c, c^\dagger \) and \( N \) to a state \( |n\rangle \) in the \( q \)-Fock space are known to provide

\[
c|n\rangle = [n]^{1/2}|n-1\rangle, \quad (15)
\]

and

\[
N|n\rangle = n|n\rangle. \quad (16)
\]

We may perform a linear transformation from the \( q \)-Fock space to the configuration space (Bargmann Holomorphic representation) \[29\] as follows,

\[
c^\dagger = x, \quad c = \partial^q_x. \quad (17)
\]

Here \( \partial^q_x \) is the Jackson derivative (JD) \[26\],

\[
\partial^q_x f(x) = \frac{f(qx) - f(x)}{x(q - 1)}. \quad (18)
\]

Note that it becomes an ordinary derivative as \( q \to 1 \). Therefore, JD naturally occurs in quantum deformed systems and, as we will show later, it plays an important role in the \( q \)-generalization of thermodynamics relations.

Let us address, for example, the Hamiltonian of \( q \)-deformed non-interacting quantum oscillators,

\[
\mathcal{H} = \sum_i (\epsilon_i - \mu) N_i = \sum_i (\epsilon_i - \mu) c_i^\dagger c_i. \quad (19)
\]

Here \( \mu \) is the chemical potential, \( \epsilon_i \) is the kinetic energy in state \( i \) associated to the operator number \( N_i \). The Hamiltonian is deformed and implicitly depends on \( q \) through the basic
number defined by Eq. (9). The mean value of the $q$-deformed occupation number $n_i^{(q)}$ can be calculated by

$$[n_i^{(q)}] \equiv \langle n_i^{(q)} \rangle = \frac{tr(\exp(-\beta H)c_i^\dagger c_i)}{\Xi}. \quad (20)$$

Here $\beta = (\kappa_B T)^{-1}$, $\kappa_B$ is the Boltzmann constant, $T$ is the temperature and $\Xi = tr[\exp(-\beta H)]$ is the partition function of the system. From Eqs. (7), (9) and (20), using the cyclic property of the trace [31], we get

$$n_i^{(q)} = \frac{1}{\ln(q)} \ln \left( \frac{z^{-1} \exp(\beta \epsilon_i) - 1}{z^{-1} \exp(\beta \epsilon_i) - q} \right), \quad (21)$$

where $z = \exp(\beta \mu)$ is the fugacity of the system. From the algebra of non-deformed quantum oscillators we know that the average occupation number is given by

$$n_i = \frac{1}{z^{-1} \exp(\beta \epsilon_i) - 1}, \quad (22)$$

with

$$N = \sum_i n_i \quad \text{and in a similar way} \quad N^{(q)} = \sum_i n_i^{(q)}. \quad (23)$$

We can obtain the total number of particles $N$ from the logarithm of the grand partition function $\Xi$, i.e.,

$$\ln \Xi = -\kappa \sum_i \ln [1 - z \kappa \exp(-\beta \epsilon_i)], \quad (24)$$

so that

$$N = z \frac{\partial}{\partial z} \ln \Xi = \sum_i \frac{z \exp(-\beta \epsilon_i)}{1 - z \exp(-\beta \epsilon_i)} = \sum_i \frac{1}{z^{-1} \exp(\beta \epsilon_i) - 1} = \sum_i n_i. \quad (25)$$

However, the total number of particles in the formalism of the $q$-deformed oscillators, $N^{(q)}$, can not be obtained by using usual thermodynamics. On the other hand, we can establish a relationship between $N^{(q)}$ and $N$ by performing an expansion considering $z \ll 1$, corresponding to high temperature or diluted gas limit, in Eqs. (21) and (22), i.e.,

$$n_i^{(q)} = \frac{q - 1}{\ln(q)} z \exp(-\beta \epsilon_i) \quad \text{and} \quad n_i = z \exp(-\beta \epsilon_i), \quad (26)$$

which means

$$n_i^{(q)} = \frac{q - 1}{\ln(q)} n_i \quad (27)$$
and
\[ \sum_i n_i^{(q)} = \frac{\bar{q} - (q)^{-1}}{2 \ln(q)} \sum_i n_i \Rightarrow N^{(q)} = \frac{\bar{q} - 1}{\ln(q)} N. \] (28)

Therefore the deformed version of Eq. (25) may be written as
\[ N^{(q)} = zD_z^{(q)} \ln \Xi = \frac{\bar{q} - (q)^{-1}}{2 \ln(q)} \frac{\partial}{\partial z} \ln \Xi. \] (29)

Here \( D_z^{(q)} \) is the so-called deformed differential operator defined as
\[ D_z^{(q)} = \frac{\bar{q} - 1}{\ln(q)} \partial_z^{(q)} \quad \text{with limit} \quad \bar{q} \to 1 \quad \Rightarrow \quad D_z = \frac{\partial}{\partial z}. \] (30)

It is quite clear from Eq. (30) the connection between deformed derivative \( D_z^{(q)} \) and the usual derivative defined by Leibniz \( \frac{\partial}{\partial z} \).

IV. CONNECTION BETWEEN JD AND HD

In order to obtain a connection between JD and HD we follow an approximation similar to that made in Eq. (3) of Sec. II. Thus, we will carry out a second order expansion of Eq. (18), with \( f(x) = \exp[-\beta(\epsilon - \mu)] \). We get,
\[ \partial_x^{(q)} f(x) \approx \left[ 1 + \frac{(q+1)}{2} x + \cdots \right] \frac{df}{dx}(x). \] (31)

By comparing Eqs. (3) and (31) we can infer that
\[ \frac{\bar{q} + 1}{2} = 1 - \frac{\zeta}{l_0}. \] (32)

From Ref. [15] we know that
\[ 1 - q = 1 - \frac{\zeta}{l_0}, \] (33)

so that we can relate both deformation parameters \( q \) and \( \bar{q} \), as follows
\[ \bar{q} = 1 - 2q. \] (34)

Therefore, now we have the deformed parameter \( \bar{q} \) connected to the fractal metric, via \( \zeta \), and to the non-extensive Tsallis framework, via \( q \). This result provides the recipe for using the JD to substitute for the HD, which has the advantages previously discussed of having a solid theoretical foundation. It is also a clear indication that the HD does not have any
unique property that renders it fundamental importance, rather the JD is able to play the same role.

Let us discuss about our result. It is well known that the Boltzmann factor $e^{-\beta \varepsilon}$ invariably arises in the treatment of the thermodynamic limit of isolated systems in equilibrium whose Hamiltonians have sufficiently short-range interactions. However, if the systems have finite size or are not in equilibrium or else if the Hamiltonian has long-range interactions, then there is no reason to expect that the Boltzmann factor will properly describe the energy distribution. In this context, alternative formulations that generalize the Boltzmann factor have been proposed, whereby the exponential function suffers a deformation. Recall that the exponential function $\exp(\lambda x)$ is an eigenfunction of the differential operator $d/dx$ with eigenvalue $\lambda$. In contrast, a deformed exponential will not, in general, be an eigenfunction of $d/dx$. For example, the Tsallis $q$-exponential $\exp_q(x)$ is not an eigenfunction of $d/dx$ except when $q = 1$. It is thus natural to expect that the ordinary derivative should be replaced by other derivatives, such as the Hausdorff derivative \[15, 17\]. But there is more than one way to deform the exponential function. For example, in addition to the Tsallis $q$-exponential function, we can cite the Kaniadakis \[19\] $\kappa$-exponential function $\exp_\kappa(x)$. One of the oldest deformations is the $q$-analog $e_q(x)$ of the exponential function (note that $e_q(x) \neq \exp_q(x)$, the latter being the Tsallis $q$-exponential). It is in this context that we can use the JD, because the JD is the $\mathcal{F}$-analog of the ordinary derivative. Our results show that the JD, with its deeper mathematical origins, can be just as useful as the HD. Since the HD has been applied to study fractal and other systems where the Boltzmann-Gibbs treatment is expected to not be applicable, we expect that the JD will be similarly useful.

V. CONCLUSIONS

Fractal systems are relevant in many areas of science. Their statistical description and phenomenological modeling lead to a new understanding of complexities and irregularities of nature. In order to address systems presenting fractal geometry, Hausdorff derivative (HD) has been considered and applied. In a similar way, Tsallis $q$-derivative has been considered and applied in the study of non-extensive statistical mechanics systems. Recently, Weber-szpil and collaborators \[15\] obtained a relationship between $q$-derivative and HD derivative, $1 - q = \frac{1 - \zeta}{\alpha}$, showing that there is a strong connection between those formalisms by means
of a fractal metric. On the other hand, Jackson derivative (JD) is a generalization of the Heisenberg quantum algebra and it is connected to the study of $\mathfrak{q}$-deformed quantum systems. In the present work we showed a relationship between $\mathfrak{q}$-derivative and HD derivative, $\frac{\mathfrak{q}+1}{2} = \frac{1-\zeta}{t_0}$, and between $\mathfrak{q}$-derivative and $q$-derivative, $1 - q = \frac{1-\zeta}{t_0}$. In particular, the deformation parameters $q$ and $\mathfrak{q}$ are connected by $\mathfrak{q} = 1 - 2q$. We can conclude that the deformed parameter $\mathfrak{q}$ is connected to the fractal metric, via $\zeta$, and to the non-extensive Tsallis framework, via $q$. Our results show that the JD, with its deeper mathematical origins, can be just as useful as the HD. Since the HD has been applied to study fractal and other systems where the Boltzmann-Gibbs treatment is expected to not be applicable, we expect that the JD will be similarly useful. We believe that our results may open new perspectives on the understanding of complex systems.

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[1] A. Bunde and S. Havlin (Eds.), Fractals in Science, Springer - Verlag, Berlin, (1995).
[2] K. Falconer, Fractal Geometry - Mathematical Foundations and Applications, Wiley, New York, (1990).
[3] B.B. Mandelbrot, The Fractal Geometry of Nature, Freeman, New York, (1982).
[4] D. Bercioux and A.Iniguez, Nature Physics 15, 111-112 (2019).
[5] R. Lopes and N. Betrouni, Medical Image Analysis 13, 634-649 (2009).
[6] B. Buard, G. Mahé, F. Chapeau-Blondeau, D. Rousseau, P. Abraham, A. Humeau, Med. Phys. 37, 2827-2836 (2010).
[7] D.L. Azevedo, Kleber A.T. da Silva, P.W. Mauriz, G.M. Viswanathan, F.A. Oliveira, Physica A 445, 27-34 (2016).
[8] I.P. Coelho, M.S. Vasconcelos, C.G. Bezerra, Phys. Lett. A 374, 1574-1578 (2010).
[9] C.G. Bezerra, E.L. Albuquerque, A.M. Mariz, L.R. da Silva, C. Tsallis, Physica A 294, 415-423 (2001).
[10] H.O. Peitgen, H.Jürgens, D. Saupe, *Chaos and Fractals - New Frontiers of Science*, Springer - Verlag, New York, (2004).

[11] W. Chen, Chaos, Sol. and Fractals 28, 923-929 (2006).

W. Chen, H. Sun, X. Zhang, D. Korosak, Comp. Math. Appl. 59, 1754-1758 (2010).

[12] A.S. Balankin and B.E. Elizarraraz, Phys. Rev. E 85, 056314 (2012);
A.S. Balankin and B.E. Elizarraraz, Phys. Rev. E 85, 025302(R) (2012).

[13] A.S. Balankin, J.B. Reyes and M. Shapiro, Physica A 444, 345-359 (2016).

[14] M.L. Lyra and C. Tsallis, Phys. Rev. Lett. 80, 53-56 (1998).

[15] J. Weberszpil, M.J. Lazo and J.A.Helayël-Neto, Physica A 436, 399-404 (2015).

[16] J. Weberszpil and J.A.Helayël-Neto, Physica A 450, 217-227 (2016);
J. Weberszpil, W. Chen, Entropy 19, 407 (2017);
W. Rosa and J. Weberszpil, Chaos, Solitons and Fractals 117, 137-141 (2018).

[17] G. Vitiello, Phys. Lett. A 376, 2527-2532 (2012);
G. Vitiello, J. Phys.: Conf. Series 380, 012021 (2012);
G. Vitiello, J. Phys.: Conf. Series 670, 012052 (2016).

[18] C. Tsallis, J. Stat. Phys. 52, 479 (1988).

[19] G. Kaniadakis, Physica A 296, 405-425 (2001);
G. Kaniadakis, Phys. Rev. E 66, 056125 (2002).

[20] S. Abe, Phys. Lett. A 224, 326 (1997).

[21] E.P. Borges, Physica A 340, 95-101 (2004).

[22] M.S. Stankovic, S.D. Marinkovic and P.M. Rajkovic, Appl. Math. Comp. 218, 2439-2448 (2011);
S.D. Marinkovic, P.M. Rajkovic and M.S. Stankovic, Appl. Math. Inf. Mech 5, 2, 69-77 (2013).

[23] A.P. Santos, R. Silva, J.S. Alcaniz, D.H.A.L. Anselmo, Phys. Lett. A 375, 352-355 (2011).

[24] A.M. Filho, D.A. Moreira, R. Silva, Luciano R. da Silva, Phys. Lett. A 377, 842-846 (2013).

[25] N.T.C.M Souza, D.H.A.L. Anselmo, R. Silva, M.S. Vasconcelos, V. D. Mello, Eur. Phys. Lett. 108, 38004 (2014);
N.T.C.M Souza, D.H.A.L. Anselmo, V.D. Mello, R. Silva, Phys. Lett. A 378, 1691-1694 (2014).

[26] F.H. Jackson, Mess. Math. 38, 57 (1909).

[27] L. Biedenharn, J. Phys. A: Math. Gen. 22, L873 (1989).
[28] A. Macfarlane, J. Phys. A: Math. Gen. **22**, 4581 (1989).

[29] E.G. Floratos, J. Phys. Math. **24**, 4739 (1991).

[30] M. Arik, E. Demircan, T. Turgut, L. Ekinci, M. Mungan, Z. Phys. C **55**, 89-95 (1992).

[31] A. Lavagno, N.P. Swamy, Phys. Rev. E **61**, 1218 (2000); A. Lavagno, N.P. Swamy, Phys. Rev. E **65**, 036101 (2002); A. Lavagno, P.N. Swamy, Chaos Solitons. Frac. **13**, 437-444 (2002).

[32] O.K. Pashaev and S. Nalci, J. Phys. A: Math. Theor. **47**, 045204 (2014).

[33] A. Algin, J. Stat. Mech. Theor. Exp. **P10009**, 10 (2008); A. Algin, E. Arslan, J. Phys. A: Math. Theor. **41**, 365006 (2008); A. Algin, E. Arslan, Phys. Lett. A **372**, 2767-2773 (2008); A. Algin, M. Arik, D. Kocabacakoglu, Int. J. Theor. Phys. **47**, 1322-1332 (2008); A. Algin, J. Stat. Mech. Theor. Exp. **P04007**, 04 (2009); A. Algin, J. CNSNS **15**, 1372-1377 (2010); A. Algin and A.S. Arikan, J. Stat. Mech., 043105 (2017); A. Algin and A. Olkun, Annals of Phys. **383**, 239-256 (2017).

[34] W.S. Chung and A. Algin, Phys. Lett. A **381**, 3266-3271 (2017).

[35] A.I. Olemskoi, S.S. Borysov and I. Shuda, Eur. Phys. J. B **77**, 219-231 (2010).

[36] J.M. Ziman, *Electron and Phonons - The Theory of Transport Phenomena in Solids*, Oxford Univ. Press, (1960).

[37] H. Umezawa, H. Matsumoto, M. Tachiki, *Thermo Field Dynamics and Condensed States*, North-Holland Publ. Co., Amsterdam, (1982).

[38] C. Kittel, *Introduction to Solid State Physics*, John Wiley & Sons, (1996).

[39] Y.M. Bunkov, H. Godfrin (Eds.), *Topological Defects and the Nonequilibrium Dynamics of Symmetry Breaking Phase Transitions*, Kluwer Academic Publ., Dordrecht, (2000).

[40] F.A. Brito, A.A. Marinho, Physica A **390**, 2497-2503 (2011); D. Tristant, F.A. Brito, Physica A **407**, 276-286 (2014); A.A. Marinho, F.A. Brito, J. Math. Phys. **60**, 012102 (2019).

[41] S.R.A. Salinas, *Introduction to Statistical Physics*, Springer-Verlag, NY (2001).

[42] R.K. Patthria, *Statistical Mechanics*, Pergamon press, Oxford (1972).

[43] J.J. Sakurai, *Modern Quantum Mechanics*, Late-Univ. of California, LA (1985).