We investigate the influence of the measure in the path integral for Euclidean quantum gravity in four dimensions within the Regge calculus. The action is bounded without additional terms by fixing the average lattice spacing. We set the length scale by a parameter $\beta$ and consider a scale invariant and a uniform measure. In the low $\beta$ region we observe a phase with negative curvature and a homogeneous distribution of the link lengths independent of the measure. The large $\beta$ region is characterized by inhomogeneous link lengths distributions with spikes and positive curvature depending on the measure.

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General relativity is the profound classical theory of gravitation. However, a fundamental description of gravity needs a synthesis with quantum theory. A direct route from classical to quantum gravity is the sum-over-histories formulation that considers the Euclidean path integral

$$Z = \int Dg e^{-I_E(g)}$$

(1)
as the starting point for investigations of non-perturbative quantum gravity \[1, 2, 3\]. The functional integration extents over a class of four-geometries \(g\) with gravitational action

$$-I_E(g) = L_P^{-2} \int d^4 x g^{\frac{1}{2}} R,$$

(2)

where \(L_P\) is the Planck length, \(R\) the curvature scalar and \(g\) the determinant of the metric \(g\). In the following we consider only geometries with the topology of a four-torus and therefore the Einstein-Hilbert action is used without surface terms. The main difficulties of integral (1) are well known. (i) A unique definition of the measure \(Dg\) does not exist \[4\]. (ii) The action \(I_E\) is not bounded due to fluctuations of the conformal factor \[5\].

The aim of this work is to investigate the influence of the measure using the Regge calculus to approximate the path integral in a systematic way \[3, 6\]. For this purpose we perform computer simulations on a simplicial lattice, \(i.e.\) four-simplices which are glued together to form a piecewise flat four-geometry \[7, 8, 9\].

We use a construction derived from a triangulation of the four-torus which has the advantage that the coordination numbers of the lattice are close to those of a random lattice \[7\]. Given a Euclidean configuration \(\{q_l\}\) where \(q_l\) is the squared length of the link \(l\) we can calculate the area \(A_t\) of every triangle \(t\) and the deficit angle defined by

$$\delta_t = 2\pi - \sum_{s \supset t} \Theta_{s,t},$$

(3)
The sum runs over all four-simplices \(s\) sharing the same triangle \(t\) and \(\Theta_{s,t}\) denotes the dihedral angle between the two tetrahedras in simplex \(s\) which have triangle \(t\) in common \[3, 6\].
The Einstein-Hilbert action can be replaced then by a sum over all triangles

\[- I_E \rightarrow L_p^{-2} \sum_t 2A_t \delta_t =: + L_p^{-2} S_R, \quad (4)\]

where the Regge action \(S_R\) is defined with a minus sign. We can now use the path integral of simplicial quantum gravity to calculate expectation values within the Regge calculus

\[
\langle O \rangle = \frac{\int D\mu O e^{L_p^{-2} S_R} \int D\mu e^{L_p^{-2} S_R}}{\int D\mu e^{L_p^{-2} S_R}}, \quad (5)
\]

where \(D\mu\) denotes an integration over different simplicial lattices [3]. Following Berg [7] and Hamber [8] we hold the incidence matrices of the lattice fixed and vary the \(q_l\), reducing the integration (5) to a summation over different configurations \(\{q_l\}\). The difficulties (i) and (ii) appear now within the Regge calculus.

(i) The measure \(D\mu\) has to be specified but as in the continuum case a unique definition does not exist. In this study we compare the scale invariant measure

\[
M1 : \quad D\mu = (\prod_l \frac{dq_l}{q_l}) \mathcal{F}(q_1, ..., q_{N_1}) \quad (6)
\]

with the uniform measure

\[
M2 : \quad D\mu = (\prod_l dq_l) \mathcal{F}(q_1, ..., q_{N_1}), \quad (7)
\]

where \(\mathcal{F}\) is one for configurations fulfilling the Euclidean triangle inequalities in four dimensions and zero otherwise.

(ii) The sum \(S_R\) in Eq. (4) is not bounded and we distinguish two types of divergences. The first type is due to a rescaling \(q_l \rightarrow \lambda q_l\) leading to \(S_R \rightarrow \lambda S_R\). The second occurs when some of the four-simplices collapse leading to triangles \(t\) with \(\delta_t \rightarrow 2\pi\) and \(A_t \rightarrow \infty\). The introduction of a cosmological constant term removes only the first type of divergences [3, 8]. However, for a lattice with a finite number of links \(N_1\) the action is bounded if we require the average lattice spacing to stay finite. The prescription

\[
\bar{a}^2 = \beta L_p^2 = N_1^{-1} \sum_l q_l \quad (8)
\]
defines the average lattice spacing $\bar{a}$ in units of the Planck length and introduces the coupling parameter $\beta$.

In our study of the scale invariant measure M1 we use the constraint $\bar{a} = \text{const}$ to limit the action. The incorporation of the measure and the necessary rescaling have been described by Berg [7].

The above constraint (8) implies a cutoff since $q_l < N_1 \beta L_P^2$ for Euclidean configurations. This suggests for computations with the uniform measure M2 to impose a cutoff $q_l < \text{const}$ for every link.

For system M1 the parameter $\beta$ is the expectation value $\langle q_l \rangle$ in units of $L_P$. To allow a comparison of the two measures we computed various expectation values as a function of $\langle q_l \rangle$ on lattices with $4^4$ vertices.

In Fig. 1 the action density $\langle S_R \rangle / \langle V \rangle$ is depicted versus $\langle q_l \rangle^{\frac{1}{2}}$ where $V$ is the total volume of the lattice. One sees clearly that both measures agree for small $\beta$. Increasing $\beta$ stepwise a sudden jump at a critical $\beta_0$ to positive values occurs for the scale invariant measure whereas the transition from negative to positive curvature seems to be smooth in the case of the uniform measure. We define $\beta_0$ as the transition point from negative to positive action which can depend on the measure.

It is interesting to investigate separately the behavior of the areas $A_t$ and of the deficit angles $\delta_t$. Fig. 2 shows $\langle A_t \rangle^{\frac{1}{2}}$ versus $\langle q_l \rangle^{\frac{1}{2}}$ and again M1 and M2 yield the same results for small $\beta$. However, for M2 the areas grow linearly with the squared link lengths for all $\beta$ values whereas M1 shows a clear deviation from the linear behavior for $\beta > \beta_0$. The reason is the growth of spikes across the transition point for the invariant measure which will be discussed below. This gives also rise for a fractal dimension $d < 4$ of the simplicial lattice.

In Fig. 3 both $\langle \delta_t \rangle$ and $\langle \delta_t^2 \rangle$ are drawn versus $\langle q_l \rangle^{\frac{1}{2}}$. Surprisingly the average of the deficit angles stays negative for all $\beta$. The change of sign in $S_R$ must be due to a correlation of a few positive $\delta_t$ with large $A_t$ and many negative $\delta_t$ with small $A_t$. The value $\langle \delta_t^2 \rangle$ turns
out to decrease in the small $\beta$ region where M1 and M2 give (nearly) the same results and shows an increasing behavior in the right phase.

The transition at $\beta_0$ is clearly seen also in the histograms of single configurations $\{q_l\}$. In the small $\beta$ region the equilibrium distributions of the link lengths are rather homogeneous having a small width. As seen in Fig. 4 a characteristic change of the distributions occurs for larger $\beta$ leading to inhomogeneous configurations. For M1 we observe a few very large links and a drastical increase of the number of short links. For M2 both the number of small links and the number of links near the cutoff increases.

To better understand this behavior we calculated the next neighbor distances $\bar{q}_v$ by averaging $q_l$ over all links meeting at the same vertex $v$. These distances are plotted in Fig. 5 both for the low and high $\beta$ region. One observes the formation of isolated "spikes" in the large $\beta$ region for M1. The site-to-site fluctuations of $\bar{q}_v$ for M2 indicate a crumpled lattice without extremely large spikes.

A few remarks about our experience with numerical convergence are in order. Using inhomogeneous start configurations it is difficult within M1 to reach an equilibrium even in the small $\beta$ region and very long runs are necessary. For M2 the calculated observables converge rather fast and show no dependence on the start configuration for all $\beta$ values.

To conclude, we have investigated the influence of the measure in the Regge discretization of Euclidean quantum gravity. Comparing a scale invariant and a uniform measure it turned out that there exists a phase of short link lengths in units of the Planck scale where all observables under consideration were independent of the two measures. The differences between the expectation values and the link lengths distribution for larger $\beta$ seem to indicate an influence of the measure. Investigations of a system with a scale invariant measure and fixed mean volume yield essentially the same results as fixed mean link length. At present simulations with a measure $\prod dq_l^{(s-1)}, 0 < s \leq 1$, interpolating between the uniform and the scale invariant measure are in progress. Preliminary results show again a phase independent
of the measure indicating a universality of the Regge calculus.

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References

[1] S. Hawking, in General Relativity - An Einstein Centenary Survey, edited by S. Hawking and W. Israel (Cambridge U.P., Cambridge, 1979).

[2] J. Hartle and S. Hawking, Phys. Rev. D28, 2960 (1983).

[3] J. Hartle, J. Math. Phys. 26, 804 (1985); J. Hartle, J. Math. Phys. 27, 287 (1986);
    J. Hartle, J. Math. Phys. 30, 452 (1989).

[4] P. Menotti, Nucl. Phys. B (Proc. Suppl.) 17, 29 (1990).

[5] S. Giddings, Int. J. Mod. Phys. A5, 3811 (1990); P. Mazur, E. Mottola, Nucl. Phys.
    B341, 187 (1990); J. Greensite, M. Halpern, Nucl. Phys. B242, 167 (1984).

[6] T. Regge, Nuovo Cimento 19, 558 (1961).

[7] B. Berg, Phys. Rev. Lett. 55, 904 (1985); B. Berg, Phys. Lett. 176B, 39 (1986).

[8] H. Hamber, in Critical Phenomena, Random Systems, Gauge Theories, Proceedings of
    the Les Houches Summer School, Les Houches, France, 1984, edited by K. Osterwalder
    and R. Stora, Les Houches Summer School Proceedings Vol. XLIII (North Holland,
    Amsterdam, 1986) Vol. 1; H. Hamber, Phys. Rev. D45, 507 (1992).

[9] W. Beirl, E. Gerstenmayer and H. Markum, in Proceedings of Lattice 91, International
    Symposium on Lattice Field Theory, Tsukuba, Japan, 1991, edited by M. Fukugita, to
    be published in Nucl. Phys. B (Proc. Suppl.).
FIG. 1. Action density versus link length in units of $L_P$ for the scale invariant measure M1 and the uniform measure M2 on lattice size $4^4$. Averages are taken over the whole lattice and over 1500 iterations for every $\beta$. Error bars due to mean standard deviation are smaller than the symbols. In the small $\beta$ region for both measures a stable regime with negative action density is found. With increasing $\beta$ M1 exhibits a sudden jump to positive values whereas M2 shows a smooth behavior.
FIG. 2. Square root of average area versus average link length in units of $L_P$ computed for the scale invariant (M1) and the uniform (M2) measure. M2 shows a linear increase in the entire $\beta$ regime whereas M1 exhibits a sudden deviation from the linear behavior at $\beta_0$. 
FIG. 3. Average deficit angle and squared deficit angle versus link length. \( \langle \delta_t \rangle \) stays negative even for \( \beta \)'s with positive curvature and \( \langle \delta^2_t \rangle \) reaches its minimum at the transition point \( \beta_0 \).
FIG. 4. Histograms of squared link lengths in units of the average link length in double logarithmic scale. The shapes are nearly independent of the measure for low $\beta$. For higher $\beta$ the histograms differ significantly. In M1 the formation of a few large and a lot of small links occurs while for M2 the lack of small links is a characteristics of the measure.
FIG. 5. Average squared distance $\bar{q}_v$ of the vertices from their next neighbors in units of the average link length. The pictures correspond to the configurations of Fig. 4. For higher $\beta$ the scale invariant measure develops spikes whereas a crumpled lattice without significant spikes is observed for the uniform measure.