Solitons in Schrödinger-Maxwell equations

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Abstract

In this paper we study the Nonlinear Schrödinger-Maxwell equations (NSM). We are interested to analyse the existence of solitons, namely of finite energy solutions which exhibit stability properties. This paper is divided in two parts. In the first, we give an abstract definition of soliton and we develop an abstract existence theory. In the second, we apply this theory to NSM.

Key words: Maxwell equations, Nonlinear Schrödinger equation, solitary waves, Hylomorphic solitons, variational methods.

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1 Introduction

In this paper we study a system of equations obtained by coupling the Schrödinger equation with the Maxwell equations (NSM) (see eq. (56), (57)). This system, usually called the Schrödinger-Poisson system or Schrödinger Maxwell system, describes many interesting physical situations (see e.g. [44] and its references). We are interested to analyse the existence of solitons, namely of finite energy solutions which exhibit a strong form of stability. In particular we are interested in a class of solitons which, following [3], [4], [7], [9], are called hylomorphic. The existence of such solitons is due to the interplay between two constants of the motion: the energy and the charge.

This paper is divided in two parts.

In the first part, following [13], we give an abstract definition of soliton and we develop an abstract existence theory for hylomorphic solitons. This theory is based on concentration-compactness type arguments (see [33], [34]).

In the second part this theory has been used to prove the existence of hylomorphic solitons for NSM (see Theorems 26 and 27) when the coupling constant $q$ is sufficiently small. If $q = 0$ the NSM reduce to the Schrödinger equation. So Theorems 26 and 27 extend to the case of NSM some of the well known stability results stated for the Schrödinger equation (see e.g. [10], [16], [28], [27], [4], [45] and its references). NSM has been largely studied by many authors and under various assumptions on the nonlinear term. There is a huge bibliography on this subject and the list of our references is far to be complete. For the existence of solutions we refer to [1], [2], [6], [8], [15], [19], [17], [21], [24], [25], [31], [38], [43], [41], [44]. However we know only few results ([6], [32]) proving the existence of stable solitary waves (namely solitons) for such equations. For the study of some qualitative properties of the solutions, like the presence of concentration phenomena and the study of semiclassical limits, we refer to [22], [23], [20], [30], [40], [42].

Our approach to NSM presents the following novelties:

- The proof of the existence result is based on a new abstract framework.
- The nonlinear term is not assumed to be homogeneous.
- The stability of the solutions is proved.
- The presence of a "lattice type" potential $V(x)$ is allowed.

2 Solitary waves and solitons: abstract theory

In this section, following [13] and [14], we introduce a functional abstract framework which allows to define solitary waves, solitons and hylomorphic solitons. Then, we will state some abstract existence theorems. These theorems are based on a general minimization principle related to the concentration compactness techniques.
2.1 Basic definitions

Solitary waves and solitons are particular states of a dynamical system described by one or more partial differential equations. Thus, we assume that the states of this system are described by one or more fields which mathematically are represented by functions

\[ u : \mathbb{R}^N \to V \]

where \( V \) is a vector space with norm \( | \cdot |_V \) which is called the internal parameters space. We assume the system to be deterministic; this means that it can be described as a dynamical system \((X, \gamma)\) where \( X \) is the set of the states and \( \gamma : \mathbb{R} \times X \to X \) is the time evolution map. If \( u_0(x) \in X \), the evolution of the system will be described by the function

\[ u(t, x) := \gamma_t u_0(x). \]  

(1)

We assume that the states of \( X \) have "finite energy" so that they decay at \( \infty \) sufficiently fast and that

\[ X \subset L^1_{loc}(\mathbb{R}^N, V). \]

(2)

Using this framework, we give the following definitions:

**Definition 1** A dynamical system \((X, \gamma)\) is called of FT type (field-theory-type) if \( X \) is a Hilbert space of functions satisfying (2).

For every \( \tau \in \mathbb{R}^N \), and \( u \in X \), we set

\[ (T_\tau u)(x) = u(x - \tau). \]

(3)

Clearly, the group

\[ \mathcal{T} = \{T_\tau \mid \tau \in \mathbb{R}^N\}; \]

(4)

is a representation of the group of translations.

**Definition 2** A set \( \Gamma \subset X \) is called compact up to space translations or \( \mathcal{T} \)-compact if for any sequence \( u_n(x) \in \Gamma \) there is a subsequence \( u_{n_k} \) and a sequence \( \tau_k \in \mathbb{R}^N \) such that \( u_{n_k}(x - \tau_k) \) is convergent.

Now, we want to give an abstract definition of solitary wave. Roughly speaking a solitary wave is a field whose energy travels as a localized packet and which preserves this localization in time. For example, consider a solution of a field equation having the following form:

\[ u(t, x) = u(x - vt - x_0)e^{i(v \cdot x - \omega t)}; \ u \in L^2(\mathbb{R}^N). \]

(5)

The field \( u \) is a solitary wave depending on the constants \( x_0, v \) and \( \omega \). The evolution of a solitary wave is a translation plus a mild change of the internal parameters (in this case the phase).

This situation can be formalized by the following definition:
**Definition 3** If \( u \in X \), we denote the closure of the orbit of \( u \) by
\[
O(u) := \overline{\{ \gamma_t u(x) \mid t \in \mathbb{R} \}}.
\]

A state \( u \in X \) is called solitary wave if
- (i) \( 0 \not\in O(u) \);
- (ii) \( O(u) \) is \( T \)-compact.

Clearly, (5) describes a solitary wave according to the definition above. The standing waves, namely objects of the form
\[
\gamma_t u = u(t, x) = u(x)e^{-i\omega t}, \quad u \in L^2(\mathbb{R}^N), \quad u \neq 0
\]
probably are the "simplest" solitary waves. In this case the orbit \( O(u) \) is compact.

Take \( X = L^1(\mathbb{R}^N) \) and \( u \in X \); if \( \gamma_t u = u(e^t x) \), \( u \) is not a solitary wave since \( \| \gamma_t u \|_X \to 0 \) as \( t \to +\infty \) and (i) is clearly violated. If \( \gamma_t u = e^t u(e^t x) \), \( u \) is not a solitary wave since (ii) of Def. 3 does not hold. Also, according to our definition, a "couple" of solitary waves is not a solitary wave: in fact
\[
\gamma_t u = [u(x - vt) + u(x + vt)] e^{i(v \cdot x - \omega t)}, \quad u \in L^2(\mathbb{R}^N)
\]
is not a solitary wave since (ii) is violated.

The solitons are solitary waves characterized by some form of stability. To define them at this level of abstractness, we need to recall some well known notions in the theory of dynamical systems.

**Definition 4** A set \( \Gamma \subset X \) is called invariant if \( \forall u \in \Gamma, \forall t \in \mathbb{R}, \gamma_t u \in \Gamma \).

**Definition 5** Let \((X, d)\) be a metric space and let \((X, \gamma)\) be a dynamical system. An invariant set \( \Gamma \subset X \) is called stable, if \( \forall \varepsilon > 0, \exists \delta > 0, \forall u \in X, \)
\[
d(u, \Gamma) \leq \delta,
\]
implies that
\[
\forall t \geq 0, \quad d(\gamma_t u, \Gamma) \leq \varepsilon.
\]

Now we are ready to give the definition of soliton:

**Definition 6** A state \( u \in X \) is called soliton if \( u \in \Gamma \subset X \) where
- (i) \( \Gamma \) is an invariant, stable set
- (ii) \( \Gamma \) is \( T \)-compact
- (iii) \( 0 \notin \Gamma \).

The set \( \Gamma \) is called soliton manifold.
The above definition needs some explanation. First of all notice that every \( u \in \Gamma \) is a soliton and that every soliton is a solitary wave. Now for simplicity, we assume that \( \Gamma \) is a manifold\(^1\). Then (ii) implies that \( \Gamma \) is finite dimensional. Since \( \Gamma \) is invariant, \( u \in \Gamma \Rightarrow \gamma_t u \in \Gamma \) for every time. Thus, since \( \Gamma \) is finite dimensional, the evolution of \( u \) is described by a finite number of parameters. The dynamical system \((\Gamma, \gamma)\) behaves as a point in a finite dimensional phase space. By the stability of \( \Gamma \), a small perturbation of \( u \) remains close to \( \Gamma \). However, in this case, its evolution depends on an infinite number of parameters. Thus, this system appears as a finite dimensional system with a small perturbation.

We now assume that the dynamical system \((X, \gamma)\) has two constants of motion: the energy \( E \) and the hylenic charge \( C \). At this level of abstraction, the name energy and hylenic charge are conventional, but in the applications, \( E \) and \( C \) will be the energy and the hylenic charge as defined in section 3.1.

**Definition 7** A state \( u_0 \in X \) is called *hylomorphic soliton* if it is a soliton according to Def. 6 and if the soliton manifold \( \Gamma \) has the following structure

\[
\Gamma = \Gamma (e_0, c_0) = \{ u \in X \mid E(u) = e_0, \ |C(u)| = c_0 \} \tag{7}
\]

where

\[
e_0 = \min \{ E(u) \mid |C(u)| = c_0 \}. \tag{8}
\]

Notice that, by (8), we have that a hylomorphic soliton \( u_0 \) satisfies the following nonlinear eigenvalue problem:

\[
E'(u_0) = \lambda C'(u_0).
\]

In general, a minimizer \( u_0 \) of \( E \) on

\[
\mathcal{M}_{c_0} := \{ u \in X \mid |C(u)| = c_0 \},
\]

is not a soliton; in fact, according to Def. 6 it is necessary to check the following facts:

- (i) the set \( \Gamma (e_0, c_0) \) is stable.
- (ii) the set \( \Gamma (e_0, c_0) \) is \( T \)-compact (i.e. compact up to translations).
- (iii) \( 0 \notin \Gamma (e_0, c_0) \) since otherwise, some \( u \in \Gamma (e_0, c_0) \) is not even a solitary wave (see Def. 6 (i)).

In concrete cases, the point (i) is the most delicate point to prove. If (i) does not hold, according to our definitions, \( u_0 \) is a solitary wave but not a soliton.

\(^1\)Actually, in many concrete models, this is the generic case; this is the reason why \( \Gamma \) is called soliton manifold even if it might happen that it is not a manifold.
2.2 An abstract minimization theorem

In the previous section, we have seen that the existence of hylomorphic soliton is related to the existence of minimizers of the energy. So in this section we assume that \( X \) is a Hilbert space and that \( E \) and \( C \) are two differentiable functionals defined on it and we will investigate the following minimization problem

\[
\min_{u \in \mathcal{M}_c} E(u) \quad \text{where} \quad \mathcal{M}_c := \{ u \in X \mid |C(u)| = c \} .
\]

(9)

2.2.1 Preliminary notions

We need a few abstract definitions some of which have been introduced in [13]. In the following \( G \) will denote a group with a unitary action on \( X \).

Definition 8 A subset \( \Gamma \subset X \) is called \( G \)-invariant if

\[
\forall u \in \Gamma, \forall g \in G, g u \in \Gamma.
\]

In many concrete situations, \( G \) will be a subgroup of the translations group \( T \).

Definition 9 A sequence \( u_n \) in \( X \) is called \( G \)-compact if there is a subsequence \( u_{n_k} \) and a sequence \( g_k \in G \) such that \( g_k u_{n_k} \) is convergent. A subset \( \Gamma \subset X \) is called \( G \)-compact if every sequence in \( \Gamma \) is \( G \)-compact.

If \( G = \{Id\} \) or more in general it is a compact group, \( G \)-compactness implies compactness. If \( G \) is not compact such as the translation group \( T \), \( G \)-compactness is a weaker notion than compactness.

Definition 10 A \( G \)-invariant functional \( J \) on \( X \) is called \( G \)-compact if any minimizing sequence \( u_n \) is \( G \)-compact.

Clearly a \( G \)-compact functional has a \( G \)-compact set of minimizers.

Definition 11 We say that a functional \( F \) on \( X \) has the splitting property if given a sequence \( u_n = u + w_n \in X \) such that \( w_n \) converges weakly to \( 0 \), we have that

\[
F(u_n) = F(u) + F(w_n) + o(1)
\]

Remark 12 Every continuous quadratic form satisfies the splitting property; in fact, in this case, we have that \( F(u) := \langle Lu, u \rangle \) for some continuous selfadjoint operator \( L \); then, given a sequence \( u_n = u + w_n \) with \( w_n \to 0 \) weakly, we have that

\[
F(u_n) = \langle Lu, u \rangle + \langle L w_n, w_n \rangle + 2 \langle Lu, w_n \rangle
= F(u) + F(w_n) + o(1)
\]

Definition 13 A sequence \( u_n \in X \) is called vanishing sequence if it is bounded and if for any sequence \( g_n \in G \) the sequence \( g_n u_n \) converges weakly to \( 0 \).
So, if $u_n \to 0$ strongly, $u_n$ is a vanishing sequence. However, if $u_n \to 0$ weakly, it might happen that it is not a vanishing sequence; namely it might exist a subsequence $u_{n_k}$ and a sequence $g_k \in G$ such that $g_k u_{n_k}$ is weakly convergent to some $\bar{u} \neq 0$. Let see an example: if $u_0 \in X \subset L^1(\mathbb{R}^N)$ and $x_n \to +\infty$, then the sequence $T_{x_n} u_0 = u_0(x - x_n)$ is not vanishing. Clearly, in this example $G$ contains the group of translations $[4]$.

Now, we set

$$\Lambda (u) := \frac{E(u)}{|C(u)|},$$

(10)

$\Lambda$ will be called **hylenic ratio**.

The notions of vanishing sequence and of hylenic ratio allow to introduce the following (important) definition:

**Definition 14** We say that the hylomorphy condition holds if

$$\inf_{u \in X} \frac{E(u)}{|C(u)|} < \Lambda_0,$$

(11)

where

$$\Lambda_0 := \inf \{ \liminf \Lambda(u_n) \mid u_n \text{ is a vanishing sequence} \}$$

(12)

Moreover, we say that $u_0 \in X$ satisfies the hylomorphy condition if,

$$\frac{E(u_0)}{|C(u_0)|} < \Lambda_0.$$  

(13)

So, if $u_n$ is a bounded sequence, we have the following:

$$\liminf \Lambda(u_n) < \Lambda_0 \Rightarrow \exists u_{n_k}, g_k \in G : g_k u_{n_k} \to \bar{u} \neq 0.$$  

In order to apply the existence theorems of the next subsection, it is necessary to estimate $\Lambda_0$; the following propositions may help to do this.

**Proposition 15** Assume that there exists a seminorm $\|\cdot\|_\ast$ on $X$ such that

$$\{ u_n \text{ is a vanishing sequence} \} \Rightarrow \left( \|u_n\|_\ast \to 0 \right)$$

(14)

Then

$$\liminf_{\|u\|_\ast \to 0} \Lambda(u) \leq \Lambda_0 \leq \liminf_{\|u\| \to 0} \Lambda(u).$$

(15)

**Proof.** By definition [13] and by [14] we have

$$\left( \|u_n\| \to 0 \right) \Rightarrow \left( u_n \text{ vanishing sequence} \right) \Rightarrow \left( \|u_n\|_\ast \to 0 \right)$$

(16)

Then, by [12] and [16], we get [15].

\[\square\]
2.2.2 The minimization result

We shall make the following assumptions on the functionals $E$ and $C$:

- **(EC-0) (Values at 0)**
  
  
  $E(0) = C(0) = 0$; $E'(0) = C'(0) = 0$.

- **(EC-1) (Invariance)** $E(u)$ and $C(u)$ are $G$-invariant.

- **(EC-2) (Splitting property)** $E$ and $C$ satisfy the splitting property (see Definition 11).

- **(EC-3) (Coercivity)** We assume that there exists $a > 0$ and $s > 1$ such that
  
  - (i) $\forall u \neq 0$, $C(u) > 0$ and $E(u) + aC(u)^s > 0$;
  - (ii) if $\|u\| \to \infty$, then $E(u) + aC(u)^s \to \infty$;
  - (iii) for any bounded sequence $u_n$ in $X$ such that $E(u_n) + aC(u_n)^s \to 0$, we have that $u_n \to 0$.

Now we can state the main results. We start with a technical lemma.

**Lemma 16** Assume that (EC-2) and (EC-3) (i) are satisfied. Let $u_n = u + w_n \in X$ be a sequence such that $u \neq 0$, $w_n \neq 0$ and $w_n$ converges weakly to 0. Then, up to a subsequence, we have

$$
\lim \Lambda (u + w_n) \geq \min (\Lambda (u), \lim \Lambda (w_n)) \quad (17)
$$

**Proof.** The proof is contained in [13]. We shall repeat it for completeness. Given four real numbers $A, B, a, b$, (with $B, b > 0$), we have that

$$
A + a \frac{B}{b} \geq \min \left( \frac{A}{B}, \frac{a}{b} \right) \quad (18)
$$

In fact, suppose that $\frac{A}{B} \geq \frac{a}{b}$; then

$$
A + a \frac{B}{b} = \frac{A}{B} B + \frac{a}{b} \geq \frac{a}{b} B + \frac{a}{b} = \frac{a}{b} \geq \min \left( \frac{A}{B}, \frac{a}{b} \right)
$$

Notice that the equality holds if and only if

$$
\frac{A}{B} = \frac{a}{b} \quad (19)
$$

Since $u \neq 0$ and $w_n \neq 0$, by (EC-3) (i), we have $C(u) + C(w_n) > C(u) > 0$.

Now, using the splitting property and [13], we have that

$$
\Lambda (u + w_n) = \frac{E(u) + E(w_n) + o(1)}{C(u) + C(w_n) + o(1)} \geq \min \left( \frac{E(u) + o(1)}{C(u) + o(1)}, \frac{E(w_n)}{C(w_n)} \right).
$$
Then, up to a subsequence, we get (17).

Now we set
\[ \Phi(u) = E(u) + 2aC(u)^s \]  
and define
\[ \delta = \sup \{ \delta > 0 \mid \exists v : \Lambda(v) + \delta \Phi(v) < \Lambda_0 \} \]  
By (11), we have that \( \delta > 0 \).

**Lemma 17** For any \( \delta \geq 0 \), \( J_{\delta}(u) \geq \frac{\delta}{2} \Phi(u) - M_{\delta} \) where
\[ M_{\delta} = -a \min_{t \geq 0} \left( \frac{\delta}{2} t^s - t^{s-1} \right) \]

**Proof:** By assumption (EC-3)(i) we get
\[ J_{\delta}(u) = \frac{E(u)}{C(u)} + \delta \Phi(u) \geq \frac{\delta}{2} [E(u) + 2aC(u)^s] + \frac{\delta}{2} \Phi(u) \]
\[ \geq \frac{\delta}{2} [E(u) + 2aC(u)^s] + \frac{\delta}{2} \Phi(u) \]
\[ = -aC(u)^{s-1} + \frac{\delta}{2} E(u) + \frac{\delta}{2} \Phi(u) \geq \frac{\delta}{2} \Phi(u) - M_{\delta} \]
where
\[ M_{\delta} = -a \min_{t \geq 0} \left( \frac{\delta}{2} t^s - t^{s-1} \right) \]

**Theorem 18** Assume that \( E \) and \( C \) satisfy (EC-0),..., (EC-3) and the hylo-

morphic condition (11). Then, for every \( \delta \in (0, \delta_\infty) \) (see (21)), \( J_{\delta} \) is G-

compact and it has a minimizer \( u_{\delta} \neq 0 \). Moreover \( u_{\delta} \) is a minimizer of \( E \) on \( \mathcal{M}_\delta := \{ u \in X \mid C(u) = c_\delta \} \) where \( c_\delta = C(u_{\delta}) \).

**Proof.** Let \( \delta \in (0, \delta_\infty) \), where \( \delta_\infty \) is defined in (21), and set
\[ j_\delta := \inf_{u \in X} J_{\delta}(u) \]
By lemma 17 and since \( \Phi(u) \geq 0 \), we have \( j_{\delta} > -\infty \). Then, since \( \delta \in (0, \delta_\infty) \), we have
\[ -\infty < j_{\delta} < \Lambda_0 \]  
(22)

Now let \( u_n \) be a minimizing sequence of \( J_{\delta} \). Let us prove that \( u_n \) is G-

compact. To this end we shall first prove that
\[ u_n \] is bounded.
Arguing by contradiction assume that, up to a subsequence, \( \|u_n\| \to +\infty \). Then, by (EC-3)(ii), we have
\[
\Phi(u_n) = E(u_n) + 2aC(u_n)^s \to +\infty.
\] (23)

By Lemma 17 and (23) we get
\[
J_\delta(u_n) \to +\infty.
\]

This contradicts the fact that \( u_n \) is a minimizing sequence of \( J_\delta \) and hence \( u_n \) is bounded.

Let us prove that \( u_n \) is not vanishing.

By (22) and since \( u_n \) is a minimizing sequence for \( J_\delta \), for large \( n \) we have
\[
\Lambda(u_n) \leq J_\delta(u_n) < \Lambda_0 - \eta, \quad \eta > 0.
\] (24)

Then, by definition of \( \Lambda_0 \), \( u_n \) is not a vanishing sequence. Hence, by Def. 13 we can extract a subsequence \( u_{n_k} \) and we can take a sequence \( g_k \subset G \) such that \( u_{n_k}' := g_k u_{n_k} \) is weakly convergent to some
\[
u_\delta \neq 0.
\] (25)

We can write
\[
u_{n_k}' = u_\delta + w_n
\]
with \( w_n \rightharpoonup 0 \) weakly. In order to show that \( J_\delta \) is \( G \)-compact we need to prove that, up to a subsequence, we have
\[
w_n \to 0 \text{ strongly}
\]

Clearly we can assume that \( w_n \neq 0 \) for all \( n \).

By the splitting property of \( E \) and \( C \) and lemma (10), we have that
\[
\begin{align*}
\delta &= \lim J_\delta(u_\delta + w_n) = \lim \{ \Lambda(u_\delta + w_n) + \delta \Phi(u_\delta + w_n) \} \\
&\geq \min \{ \Lambda(u_\delta), \lim \Lambda(w_n) \} + \delta \lim \Phi(u_\delta + w_n)
\end{align*}
\] (26)

(27)

By the splitting property (EC-2) and since \( s \geq 1 \), we have that
\[
\begin{align*}
\lim \Phi(u_\delta + w_n) &= \lim (E(u_\delta + w_n) + aC(u_\delta + w_n)^s) \\
&= E(u_\delta) + \lim E(w_n) + a \lim (C(u_\delta) + C(w_n))^s \\
&\geq E(u_\delta) + \lim E(w_n) + a \lim (C(u_\delta)^s + C(w_n)^s) \\
&= E(u_\delta) + aC(u_\delta)^s + \lim E(w_n) + a \lim C(w_n)^s \\
&= \Phi(u_\delta) + \lim \Phi(w_n).
\end{align*}
\] (28)

Then by (27) and by (28) we have
\[
\begin{align*}
\delta \geq \min \{ \Lambda(u_\delta), \lim \Lambda(w_n) \} + \delta \Phi(u_\delta) + \delta \lim \Phi(w_n).
\end{align*}
\] (29)
Now there are two possibilities:

(a) \[ \min \{ \Lambda(u_\delta), \lim \Lambda(w_n) \} = \lim \Lambda(w_n), \]

(b) \[ \min \{ \Lambda(u_\delta), \lim \Lambda(w_n) \} = \Lambda(u_\delta). \]

We will show that the possibility (a) cannot occur. In fact, if it holds, by (29), we have that

\[ j_\delta \geq \lim \Lambda(w_n) + \delta \Phi(u_\delta) \]

and hence, we get that \( \Phi(u_\delta) \leq 0; \) this, by (EC-3)(i), implies that \( u_\delta = 0, \) contradicting (25). Then the possibility (b) holds and, by (29), we have that

\[ j_\delta \geq \Lambda(u_\delta) + \delta \Phi(u_\delta) + \delta \lim \Phi(w_n) \]

Then, \( \lim \Phi(w_n) \to 0 \) and by (EC-3)(iii), \( w_n \to 0 \) strongly. We conclude that \( J_\delta \) is \( G \)-compact and \( u_\delta \) is a minimizer of \( J_\delta. \) Then \( u_\delta \) minimizes also the functional

\[ \frac{E(u)}{c_\delta} + \delta [E(u) + ac_\delta] = \left( \frac{1}{c_\delta} + \delta \right) E(u) + \delta ac_\delta \]

on the set \( \mathfrak{M}_\delta = \{ u \in X | C(u) = c_\delta \} \) and hence \( u_\delta \) minimizes also \( E|_{\mathfrak{M}_\delta}. \]

In the following \( u_\delta \) will denote a minimizer of \( J_\delta. \)

**Lemma 19** Let the assumptions of Theorem 18 be satisfied. Let \( \delta_1, \delta_2 \in (0, \delta_\infty) \)

\( \delta_1 < \delta_2 \) and let \( u_{\delta_1}, u_{\delta_2} \) be minimizers of \( J_{\delta_1}, J_{\delta_2} \) respectively. Then the following inequalities hold:

- (a) \( J_{\delta_1}(u_{\delta_1}) < J_{\delta_2}(u_{\delta_2}) \)
- (b) \( \Phi(u_{\delta_1}) \geq \Phi(u_{\delta_2}), \)
- (c) \( \Lambda(u_{\delta_1}) \leq \Lambda(u_{\delta_2}), \)
- (d) \( C(u_{\delta_1}) \geq C(u_{\delta_2}). \)

**Proof.** We prove first the inequality (a)

\[ J_{\delta_1}(u_{\delta_1}) = \Lambda(u_{\delta_1}) + \delta_1 \Phi(u_{\delta_1}) \leq \Lambda(u_{\delta_2}) + \delta_1 \Phi(u_{\delta_2}) \]

(since \( u_{\delta_1} \) minimizes \( J_{\delta_1} \))

\[ < \Lambda(u_{\delta_2}) + \delta_2 \Phi(u_{\delta_2}) \]

(since \( \Phi \) is positive and \( \delta_1 < \delta_2 \))

\[ = J_{\delta_2}(u_{\delta_2}). \]
In order to prove inequalities (b) and (c) we set

\[ \Lambda(u_{\delta_1}) = \Lambda(u_{\delta_2}) + a \\
\Phi(u_{\delta_1}) = \Phi(u_{\delta_2}) + b \]

We need to prove that \( b \geq 0 \) and \( a \leq 0 \). We have

\[
J_{\delta_2}(u_{\delta_2}) \leq J_{\delta_1}(u_{\delta_1}) \Rightarrow \\
\Lambda(u_{\delta_2}) + \delta_2 \Phi(u_{\delta_2}) \leq \Lambda(u_{\delta_1}) + \delta_2 \Phi(u_{\delta_1}) \Rightarrow \\
\Lambda(u_{\delta_2}) + \delta_2 \Phi(u_{\delta_2}) \leq (\Lambda(u_{\delta_2}) + a) + \delta_2 (\Phi(u_{\delta_2}) + b) \Rightarrow \\
0 \leq a + \delta_2 b. \tag{30}
\]

On the other hand,

\[
J_{\delta_1}(u_{\delta_2}) \geq J_{\delta_1}(u_{\delta_1}) \Rightarrow \\
\Lambda(u_{\delta_2}) + \delta_1 \Phi(u_{\delta_2}) \geq \Lambda(u_{\delta_1}) + \delta_1 \Phi(u_{\delta_1}) \Rightarrow \\
\Lambda(u_{\delta_2}) + \delta_1 \Phi(u_{\delta_2}) \geq (\Lambda(u_{\delta_2}) + a) + \delta_1 (\Phi(u_{\delta_2}) + b) \Rightarrow \\
0 \geq a + \delta_1 b. \tag{31}
\]

From (30) and (31) we get

\[(\delta_2 - \delta_1) b \geq 0\]

and hence \( b \geq 0 \).

Moreover (30) and (31) give also

\[
\left( \frac{1}{\delta_2} - \frac{1}{\delta_1} \right) a \geq 0
\]

and hence \( a \leq 0 \).

Finally we prove inequality (d). Arguing by contradiction we assume that

\[ C(u_{\delta_1}) < C(u_{\delta_2}). \tag{32} \]

Then

\[ aC(u_{\delta_1})^* < aC(u_{\delta_2})^*. \tag{33} \]

By (c) and (32) we get

\[ C(u_{\delta_1}) \Lambda(u_{\delta_1}) < C(u_{\delta_2}) \Lambda(u_{\delta_2}). \tag{34} \]

Taking the sum in (33) and (34) we get

\[ \Phi(u_{\delta_1}) < \Phi(u_{\delta_2}) \]

and this contradicts (b).

\[ \square \]
Lemma 20 Let the assumptions of Theorem 18 be satisfied and assume that also (41) is satisfied. Let \( \delta_1, \delta_2 \in (0, \delta_\infty) \) \( \delta_1 < \delta_2 \) and let \( u_{\delta_1}, u_{\delta_2} \) be non zero minimizers of \( J_{\delta_1}, J_{\delta_2} \) respectively. The following inequalities hold:

- (a) \( \Phi(u_{\delta_1}) > \Phi(u_{\delta_2}) \)
- (b) \( \Lambda(u_{\delta_1}) < \Lambda(u_{\delta_2}) \)
- (c) \( C(u_{\delta_1}) > C(u_{\delta_2}) \)

**Proof:** Let \( \delta_1, \delta_2 \in (0, \delta_\infty) \) \( \delta_1 < \delta_2 \). By Lemma 19 there exist \( u_{\delta_1}, u_{\delta_2} \) non zero minimizers of \( J_{\delta_1}, J_{\delta_2} \).

By Lemma 19 we know that \( \Phi(u_{\delta_1}) > \Phi(u_{\delta_2}) \), so in order to prove (a) we need only to show that \( \Phi(u_{\delta_1}) \neq \Phi(u_{\delta_2}) \). We argue indirectly and assume that \( \Phi(u_{\delta_1}) = \Phi(u_{\delta_2}) \). (35)

By the previous lemma, we have that

\[
\Lambda(u_{\delta_1}) = \Lambda(u_{\delta_2})
\]

Also, we have that

\[
\Lambda(u_{\delta_2}) + \delta_2 \Phi(u_{\delta_2}) \leq \Lambda(u_{\delta_1}) + \delta_2 \Phi(u_{\delta_1}) \quad \text{(since \( u_{\delta_2} \) minimizes \( J_{\delta_2} \))}
\]

and so

\[
\Lambda(u_{\delta_2}) \leq \Lambda(u_{\delta_1})
\]

and by (35) we get

\[
\Lambda(u_{\delta_1}) = \Lambda(u_{\delta_2}).
\] (37)

Then, it follows that \( u_{\delta_1} \) is also a minimizer of \( J_{\delta_2} \); in fact, by (37) and (35)

\[
J_{\delta_2}(u_{\delta_1}) = \Lambda(u_{\delta_1}) + \delta_2 \Phi(u_{\delta_1}) = \Lambda(u_{\delta_2}) + \delta_2 \Phi(u_{\delta_2}) = J_{\delta_2}(u_{\delta_2}).
\]

Then, we have that \( J'_{\delta_2}(u_{\delta_1}) = 0 \) as well as \( J_{\delta_1}(u_{\delta_1}) = 0 \) which explicitely give

\[
\Lambda'(u_{\delta_1}) + \delta_2 \Phi'(u_{\delta_1}) = 0
\]

\[
\Lambda'(u_{\delta_1}) + \delta_1 \Phi'(u_{\delta_1}) = 0.
\]

The above equations imply that

\[
\Phi'(u_{\delta_1}) = 0
\]

\[
\Lambda'(u_{\delta_1}) = 0.
\]

Since \( \Lambda(u) = \frac{E(u)}{C(u)} \) and \( \Phi(u) = E(u) + 2aC(u)^s \), the above system of equations becomes

\[
\frac{E'(u_{\delta_1})}{C(u_{\delta_1})} - \frac{E(u_{\delta_1})}{C(u_{\delta_1})}C'(u_{\delta_1}) = 0
\]

\[
E'(u_{\delta_1}) + 2asC(u_{\delta_1})^{s-1}C'(u_{\delta_1}) = 0.
\] (38)
Eliminating $E'(u_{\delta_1})$, we get

$$
(2asC(u_{\delta_1})^s + E(u_{\delta_1})) \frac{C'(u_{\delta_1})}{C(u_{\delta_1})^2} = 0
$$

and, using (20), we get

$$
\frac{\Phi(u_{\delta_1}) + 2a(s - 1)C(u_{\delta_1})^s}{C(u_{\delta_1})^2}C'(u_{\delta_1}) = 0.
$$

By assumption (EC-3) (i) and since $s > 1$, we have

$$
\frac{\Phi(u_{\delta_1}) + 2a(s - 1)C(u_{\delta_1})^s}{C(u_{\delta_1})^2} > 0,
$$

then $C'(u_{\delta_1}) = 0$, and hence, by (38), also $E'(u_{\delta_1}) = 0$. Finally by (11) $u_{\delta_1} = 0$, and we get a contradiction.

In order to prove (b) we argue indirectly and assume that

$$
\Lambda(u_{\delta_1}) = \Lambda(u_{\delta_2}).
$$

By (a), we have that

$$
\Phi(u_{\delta_1}) > \Phi(u_{\delta_2}).
$$

Also, we have that

$$
\Lambda(u_{\delta_1}) + \delta_1 \Phi(u_{\delta_1}) \leq \Lambda(u_{\delta_2}) + \delta_1 \Phi(u_{\delta_2}) \text{ (since } u_{\delta_1} \text{ minimizes } J_{\delta_1})
$$

$$
\leq \Lambda(u_{\delta_1}) + \delta_1 \Phi(u_{\delta_2}) \text{ (by (39))}
$$

and so

$$
\Phi(u_{\delta_1}) \leq \Phi(u_{\delta_2})
$$

and this contradicts (40).

Let us prove the inequality (c).

Since

$$
\Lambda(u_{\delta_i})C(u_{\delta_i}) = E(u_{\delta_i}), \ i = 1, 2
$$

we have

$$
\Phi(u_{\delta_i}) = \Lambda(u_{\delta_i})C(u_{\delta_i}) + 2aC(u_{\delta_i})^s, \ i = 1, 2
$$

and the conclusion easily follows from inequalities (a) and (b).

□
2.3 The stability result

In the previous subsection 2.2 we have proved the existence of minimizers, namely that $\Gamma(e,c) \neq \emptyset$ (see (7)). In this subsection, we prove the stability of $\Gamma(e,c)$ namely that the minimizers are hylomorphic solitons. More exactly we will prove the following two theorems:

**Theorem 21** Assume that $E$ and $C$ satisfy (EC-0), ..., (EC-2), (EC-3). Assume also that the hylomorphy condition of Def. 14 is satisfied. Then for any $\delta \in (0, \delta_\infty)$ ($\delta_\infty > 0$ defined in (21)) there exists a hylomorphic soliton $u_\delta$. Moreover assume that

$$\|E'(u)\| + \|C'(u)\| = 0 \iff u = 0.$$  \hfill (41)

Then, if $\delta_1 < \delta_2$, the corresponding solitons $u_{\delta_1}, u_{\delta_2}$ are distinct, and we have that

- (a) $\Lambda(u_{\delta_1}) < \Lambda(u_{\delta_2})$
- (b) $C(u_{\delta_1}) > C(u_{\delta_2})$.
- (c) $E(u_{\delta_1}) + aC(u_{\delta_1})^s > E(u_{\delta_2}) + aC(u_{\delta_2})^s$

**Remark 22** Variants of the above results have been stated in [12] and [13].

Before proving Theorem 21, we need to recall some result.

**Theorem 23** Let $\Gamma$ be an invariant set and assume that there exists a differentiable real function $V$ (called a Liapunov function) defined on a neighborhood of $\Gamma$ such that

- (a) $V(u) \geq 0$ and $V(u) = 0 \iff u \in \Gamma$
- (b) $\partial_t V(\gamma_t(u)) \leq 0$
- (c) $V(u_n) \to 0 \iff d(u_n, \Gamma) \to 0$.

Then $\Gamma$ is stable.

**Proof.** This is a classical result. A proof of it in this form can be found in [12] or [13].

We shall need also the following Lemma

**Lemma 24** Let $V \geq 0$ be $G$-compact functional and let $\Gamma = V^{-1}(0)$ be the set of minimizers of $V$. If $\Gamma \neq \emptyset$, then $\Gamma$ is $G$-compact and $V$ satisfies the point (c) of the previous lemma.
Proof: A proof can be found in [12] or [13].

\[ \square \]

Proof of Th. 21. Let \( u_\delta \) be a minimizer of \( E \) on

\[ \mathcal{M}_\delta = \{ u \in X \mid C(u) = c_\delta \} \]
as in Theorem [18]. It remains to show that

\[ \Gamma (e_\delta, c_\delta) = \{ u \in X \mid C(u) = c_\delta, E(u) = e_\delta \}, \quad (e_\delta = E(u_\delta)) \]
is \( G \)-compact and stable.

- \( \Gamma (e_\delta, c_\delta) \) is \( G \)-compact.

To this end, by Lemma [24], it will be enough to show that

\[ V(u) = (E(u) - e_\delta)^2 + (C(u) - c_\delta)^2 \]
is \( G \) compact.

Let \( w_n \) be a minimizing sequence for \( V \), then \( V(w_n) \to 0 \) and consequently \( E(w_n) \to e_\delta \) and \( C(w_n) \to c_\delta \). Now, since

\[ \inf J_\delta = \frac{e_\delta}{c_\delta} + \delta [e_\delta + ac_\delta] , \]
we have that \( w_n \) is a minimizing sequence also for \( J_\delta \). Then, since by Theorem [18] \( J_\delta \) is \( G \)-compact, we get that

\[ w_n \text{ is } G\text{-compact.} \quad (42) \]

So we conclude that \( V \) is \( G \)-compact.

- \( \Gamma (e_\delta, c_\delta) \) is stable.

In fact, since \( V \) is \( G \)-compact, by Lemma [24] we deduce that \( V^{-1}(0) = \Gamma (e_\delta, c_\delta) \) satisfies the point (c) in Theorem [23]. Moreover clearly \( V \) satisfies also the points (a) and (b) in Theorem [23]. So, by Theorem [24], we conclude that \( \Gamma (e_\delta, c_\delta) \) is stable.

Finally, if we assume [11], we can use Lemma [20] to get different solitons for different values of \( \delta \). Namely for \( \delta_1 < \delta_2 \) we have \( \Lambda(u_{\delta_1}) < \Lambda(u_{\delta_2}) \) and \( C(u_{\delta_1}) > C(u_{\delta_2}) \).

\[ \square \]

3 The nonlinear Schrödinger Maxwell equation

In this section we derive a system of equations (NSM) obtained by coupling the nonlinear Schrödinger equation with Maxwell equations and we prove the existence of a family of stable solitary waves.
3.1 General features

The Schrödinger equation for a particle which moves in a potential \( V(x) \) is given by

\[
i\frac{\partial \psi}{\partial t} = -\frac{1}{2} \Delta \psi + V(x)\psi
\]

where \( \psi : \mathbb{R} \times \mathbb{R}^3 \rightarrow \mathbb{C} \) and \( V : \mathbb{R}^3 \rightarrow \mathbb{R} \).

We are interested in the nonlinear Schrödinger equation:

\[
i\frac{\partial \psi}{\partial t} = -\frac{1}{2} \Delta \psi + \frac{1}{2} W'(\psi) + V(x)\psi \quad (43)
\]

where \( W : \mathbb{C} \rightarrow \mathbb{R} \) and

\[
W'(\psi) = \frac{\partial W}{\partial \psi_1} + i \frac{\partial W}{\partial \psi_2}. \quad (44)
\]

We assume that \( W \) depends only on \(|\psi|\), namely

\[
W(\psi) = F(|\psi|) \quad \text{and so} \quad W'(\psi) = F'(|\psi|) \frac{\psi}{|\psi|}.
\]

for some smooth function \( F : [0, \infty) \rightarrow \mathbb{R} \). In the following we shall identify, with some abuse of notation, \( W \) with \( F \).

If \( V(x) = 0 \), then we get the equation

\[
i\frac{\partial \psi}{\partial t} = -\frac{1}{2} \Delta \psi + \frac{1}{2} W'(\psi); \quad (\text{NS})
\]

Equation (43) is the Euler-Lagrange equation relative to the Lagrangian density

\[
\mathcal{L}_s = \text{Re} \left( i \partial_t \bar{\psi} \psi \right) - \frac{1}{2} |\nabla \psi|^2 - W(\psi) - V(x) |\psi|^2 \quad (45)
\]

Now we want to couple the Schrödinger equation with the Maxwell equations. We recall that the use of the covariant derivative provides a very elegant procedure to combine relativistic field equations (Dirac, Klein-Gordon etc.) with the Maxwell equations (see e.g. \[17\], \[39\], \[11\], \[10\]). It is possible to use this procedure also to couple Schrödinger and Maxwell equations. This situation describes the interaction between a charged "matter field" with the electromagnetic field when the relativistic effects are negligible (see \[11\] and its references).

Let us see how this procedure works. We denote by \( E, H \) the electric and the magnetic field and by \( \varphi : \mathbb{R}^3 \rightarrow \mathbb{R} \) and \( A : \mathbb{R}^3 \rightarrow \mathbb{R}^3, A = (A_1, A_2, A_3) \) their gauge potentials, namely fields such that

\[
E = -\frac{\partial A}{\partial t} - \nabla \varphi, \quad H = \nabla \times A.
\]

Now couple (43) with Maxwell equations by means of the covariant derivatives. So \( \mathcal{L}_s \) becomes

\[
\mathcal{L}_c = \text{Re} \left( i D_t \bar{\psi} \psi \right) - \frac{1}{2} |D_x \psi|^2 - W(\psi) - V(x) |\psi|^2 ,
\]

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where $D_x, D_t$ denote the covariant derivatives

$$D_x \psi = (D_1 \psi, D_2 \psi, D_3 \psi)$$

$$D_t = \frac{\partial}{\partial t} + iq\varphi, \quad D_j = \frac{\partial}{\partial x^j} - iqA_j$$

and $q$ denotes a positive coupling constant which represents the "strength" of the interaction. Adding to $L_c$ the Lagrangian related to the Maxwell equations

$$L_M = \frac{1}{2} \left( |\nabla \varphi| - |\nabla \times A|^2 \right),$$

we get the total Lagrangian

$$L = L_c + L_M.$$  \hspace{1cm} (46)

So the total action is

$$S = \int L \, dxdt.$$  \hspace{1cm} (47)

If we write $\psi$ in polar form

$$\psi(x, t) = u(x, t) e^{iS(x,t)}, \quad u \geq 0, \quad S \in \mathbb{R}/2\pi \mathbb{Z}$$

the action (47) takes the following form

$$S(u, S, \varphi, A) = - \int \int \left[ \frac{1}{2} |\nabla u|^2 + V(x)u^2 + W(u) \right] \, dxdt +$$

$$- \int \int \left[ \left( \frac{\partial S}{\partial t} + q\varphi \right) + \frac{1}{2} |\nabla S - qA|^2 \right] u^2 \, dxdt$$

$$+ \frac{1}{2} \int \int \left( \left| \frac{\partial A}{\partial t} + \nabla \varphi \right|^2 - |\nabla \times A|^2 \right) \, dxdt.$$  \hspace{1cm} (48)

Making the variations of $S$ with respect $u, S, \varphi, A$ we get respectively the equations:

$$- \frac{1}{2} \Delta u + \frac{1}{2} W'(u) + \left[ \frac{1}{2} |\nabla S - qA|^2 + \left( \frac{\partial S}{\partial t} + q\varphi + V(x) \right) \right] u = 0$$  \hspace{1cm} (49)

$$\frac{\partial(u^2)}{\partial t} + \nabla \cdot [(\nabla S - qA) u^2] = 0$$  \hspace{1cm} (50)

$$- \nabla \cdot \left( \frac{\partial A}{\partial t} + \nabla \varphi \right) = qu^2$$  \hspace{1cm} (51)

$$\nabla \times (\nabla \times A) + \frac{\partial}{\partial t} \left( \frac{\partial A}{\partial t} + \nabla \varphi \right) = q(\nabla S - qA) u^2.$$  \hspace{1cm} (52)
The last two equations (51) and (52) are the second couple of the Maxwell equations (Gauss and Ampere laws) with respect to a matter distribution whose electric charge and current density are respectively $\rho$ and $j$ defined by:

$$\rho = qu^2$$  \hspace{1cm} (53)

$$j = q(\nabla S - qA) u^2.$$  \hspace{1cm} (54)

Notice that equation (50) is a continuity equation which gives rise to the conservation of the hylenic charge $C$

$$C = \int u^2.$$  \hspace{1cm} (55)

and hence also to the conservation of the electric charge $qC = q \int u^2$.

Moreover (50) is easily derived from (51) and (52). In conclusion our system of equations is reduced to (49), (51), (52).

Observe that in the electrostatic case, i.e. when

$$\frac{\partial u}{\partial t} = 0, \ S = \omega t, \ \omega \ real, \ A = 0,$$

the system (49), (51), (52) reduces to the system

$$- \frac{1}{2} \Delta u + \frac{1}{2} W'(u) + (q\varphi + \omega + V(x)) u = 0$$  \hspace{1cm} (56)

$$- \Delta \varphi = qu^2.$$  \hspace{1cm} (57)

System (56), (57) is called nonlinear Schrödinger-Maxwell system or nonlinear Schrödinger-Poisson system and it will be denoted by NSM.

Observe that, if we consider $\varphi$ as a scalar field (and not the time-component of a 4-vector) the Schrödinger-Poisson equations are invariant under the Galileo group if $V$ is constant.

Now we compute the energy $E$ related to the system (49) ... (52).

**Theorem 25** If $(u, S, \varphi, A)$ satisfy the Gauss equations (51), the energy $E$ related to the system (49) ... (52) takes the following form:

$$E = \int \left[ \frac{1}{2} |\nabla u|^2 + V(x) u^2 + W(u) \right] dx$$

$$+ \frac{1}{2} \int \left[ |\nabla S - qA|^2 u^2 \right] dx + \frac{1}{2} \int \left[ \frac{\partial A}{\partial t} + \nabla \varphi \right]^2 + |\nabla \times A|^2 \right] dx.$$
Proof. The Lagrangian $L$ related to the system (49, 50) is

$$L = -\frac{1}{2} |\nabla u|^2 - V(x)u^2 - W(u) - \left(\frac{\partial S}{\partial t} + q\varphi\right) u^2 - \frac{1}{2} |\nabla S - qA|^2 u^2$$

$$+ \frac{1}{2} \left( \left| \frac{\partial A}{\partial t} + \nabla \varphi \right|^2 - |\nabla \times A|^2 \right).$$

This Lagrangian does not depend on $\frac{\partial u}{\partial t}$ and $\frac{\partial \varphi}{\partial t}$. Then the related energy is (see [20] chapter 7)

$$E = \int \left[ \frac{\partial L}{\partial \left( \frac{\partial S}{\partial t} \right)} \cdot \frac{\partial S}{\partial t} + \frac{\partial L}{\partial \left( \frac{\partial A}{\partial t} \right)} \cdot \frac{\partial A}{\partial t} - L \right] dx.$$

So, by a direct calculation, we get

$$E = \int \left( \frac{\partial A}{\partial t} + \nabla \varphi \right) \cdot \frac{\partial A}{\partial t} + q\varphi u^2 + \frac{1}{2} \left( |\nabla u|^2 + |\nabla S - qA|^2 u^2 \right) + V(x)u^2 + W(u) - \frac{1}{2} |\nabla S - qA|^2 u^2.$$

By the Gauss equation (51), multiplying by $\varphi$ and integrating, we get

$$\int q\varphi u^2 = \int \nabla \cdot \left( \frac{\partial A}{\partial t} + \nabla \varphi \right).$$

The above equality (60) easily implies that

$$\int q\varphi u^2 + \left( \frac{\partial A}{\partial t} + \nabla \varphi \right) \cdot \frac{\partial A}{\partial t} - \frac{1}{2} |\nabla S - qA|^2 u^2 =$$

$$\frac{1}{2} \int \left| \frac{\partial A}{\partial t} + \nabla \varphi \right|^2. \quad (61)$$

Inserting (61) into (59) we get the conclusion.

\[\square\]

3.2 Statement of the results

We make the following assumptions on $W$ and $V$

- **(W-i)** $W$ is a $C^2$ function s.t.

$$W(0) = W'(0) = 0 \text{ and } W''(0) = 2E_0 > 0; \quad (62)$$
\( W(s) = E_0 s^2 + N(s), \) 

then

\[ \exists s_0 \in \mathbb{R}^+ \text{ such that } N(s_0) < -V_0 s_0^2 \]  

where

\[ V_0 = \max V; \]

- **(W-iii)** there exist \( q, r \) in \((2,6)\), s. t.

\[ |N'(s)| \leq c_1 s^{r-1} + c_2 s^{q-1} \]

- **(W-iv)**

\[ N(s) \geq -cs^p, \ c \geq 0, \ 2 < p < 2 + \frac{4}{3} \text{ for } s \text{ large} \]

\[ V : \mathbb{R}^3 \rightarrow \mathbb{R} \] being a potential function satisfying the assumptions:

- **(V-i)** \( V \) continuous and

\[ V(x) \geq 0, \ x \in \mathbb{R}^3 \]

- **(V-ii)** \( V \) is a lattice potential, namely it satisfies the periodicity condition:

\[ V(x) = V(x + Az) \text{ for all } x \in \mathbb{R}^3 \text{ and } z \in \mathbb{Z}^3 \]

where \( A \) is a \( 3 \times 3 \) invertible matrix.

If we set

\[ E = -\frac{\partial A}{\partial t} - \nabla \varphi, \ H = \nabla \times A, \ \Theta = (\nabla S - q A) u, \]

the energy \( E \) takes the form

\[ E = \int \left( \frac{1}{2} |\nabla u|^2 + V(x) u^2 + W(u) + \frac{1}{2} (\Theta^2 + E^2 + H^2) \right) dx. \]

Instead of using the variables \((u, S, E, H)\), we will use the variables \((u, \Theta, E, H)\) so that the generic point in the phase space is given by

\[ u = (u, \Theta, E, H) \]

and the phase space is given by

\[ X = \left\{ u = (u, \Theta, E, H) \in H^1 (\mathbb{R}^3) \times L^2 (\mathbb{R}^3)^9 : \nabla \cdot E = qu^2 \right\} \]

where \( H^1 (\mathbb{R}^3) \) is the usual Sobolev space.
We equip $X$ with the norm related to the quadratic part of the energy, namely:
\[
\|u\|^2 = \int |\nabla u|^2 + 2E_0 u^2 + \Theta^2 + E^2 + H^2
\]
(71)
where $E_0$ is defined by (63). Then the energy $E$ can be written as follows:
\[
E = \frac{1}{2} \|u\|^2 + \int V(x)u^2 + \int N(u).
\]
(72)

We notice that the new variables do not change the expression for the charge, namely $C$ keeps the form (55). Finally, as usual
\[
\Lambda = \frac{E(u)}{C(u)}
\]
will denote the hylenic ratio.

In the following we shall assume that the Cauchy problem for the system (49, 51, 52) is well posed in $X$ and we refer to [29], [35] and [36] for some results in this direction.

We shall prove the following existence results of hylomorphic solitons for NSM.

**Theorem 26** Let $W$ and $V$ satisfy the assumptions (WB-i),...,({WB-iv}) and (V-i),(V-ii). Then, if $q > 0$ is sufficiently small, there exists $\delta_\infty > 0$ such that the dynamical system described by the system (49), (51) (52) has a family $u_\delta = (u_\delta, \Theta_\delta, E_\delta, H_\delta)$ ($\delta \in (0, \delta_\infty)$) of hylomorphic solitons (Definition 7). Moreover if $\delta_1 < \delta_2$ we have that

- (a) $\Lambda(u_{\delta_1}) < \Lambda(u_{\delta_2})$
- (b) $\|u_{\delta_1}\|_{L^2} > \|u_{\delta_2}\|_{L^2}$

**Theorem 27** The solitons $u_\delta = (u_\delta, \Theta_\delta, E_\delta, H_\delta)$ in Theorem 26 are stationary solutions of (49), (51) (52), this means that $\Theta_\delta = H_\delta = 0$, $E_\delta = -\nabla \varphi_\delta$, $u_\delta, \varphi_\delta$ do not depend on $t$ and they solve, for suitable real numbers $\omega$, the nonlinear Schrödinger-Poisson system
\[
-\frac{1}{2}\Delta u_\delta + V(x)u_\delta + \frac{1}{2}W'(u_\delta) + q\varphi_\delta u_\delta = -\omega u_\delta
\]
(73)
\[
-\Delta \varphi_\delta = qu_\delta^2.
\]
(74)

**Remark 28** If the coupling constant $q = 0$ equations (49), (71), (72) reduce to the Schrödinger equation and Theorem 26 becomes in this case a variant of well known stability results (see [10], [27] and its references).

The proof of Theorem 26 is based on the abstract Theorem ?? and its references. First of all observe that, since $V$ satisfies (68), the energy $E$ is invariant under the representation $T_z$ of the group $G := \mathbb{Z}^3$
\[
T_z u(x) = u(x + Az), \quad z \in \mathbb{Z}^3
\]
where $A$ is as in (68).
3.3 Proof of the results

In this section we shall prove that $E$ and $C$ satisfy assumptions (EC-2) (splitting), (EC-3) (coercivity) and the hylomorphy assumption.

**Lemma 29** Let the assumptions of theorem 26 be satisfied. Then $E$ and $C$, defined by (72) and (55) satisfy the splitting property (EC-2).

**Proof.** For any $u = (u, \Theta, E, H) \in X$ the energy $E(u)$ in (72) can be written

$$E(u) = A(u, u) + K(u)$$

where

$$A(u, u) = \frac{1}{2} \|u\|^2 + \int V(x) u^2$$

and

$$K(u) = \int N(u) dx. \quad (75)$$

The hylenic charge $C(u) = \int u^2$ and $A(u, u)$ are quadratic forms, then, by remark 12, they satisfy the splitting property. So, in order to show that also the energy $E(u)$ satisfies (EC-2), we have only to show that $K(u)$ in (75) satisfies the splitting property. Let $H^1(\mathbb{R}^3)$ denote the usual Sobolev space, then for any measurable $A \subset \mathbb{R}^3$ and any $u \in H^1(\mathbb{R}^3)$, we set

$$K_A(u) = \int_A N(u) dx.$$ 

Now consider any sequence

$$u_n = u + w_n \in H^1(\mathbb{R}^3)$$

where $w_n$ converges weakly to 0.

Choose $\varepsilon > 0$ and $R = R(\varepsilon) > 0$ such that

$$|K_{B_R^c}(u)| < \varepsilon \quad (76)$$

where

$$B_R^c = \mathbb{R}^3 - B_R \text{ and } B_R = \{x \in \mathbb{R}^3 : |x| < R\}.$$ 

Since $w_n \to 0$ weakly in $H^1(\mathbb{R}^3)$, by usual compactness arguments, we have that

$$K_{B_R}(w_n) \to 0 \text{ and } K_{B_R}(u + w_n) \to K_{B_R}(u). \quad (77)$$

Then, by (76) and (77), we have
Lemma 30

for $z$

So we have

bounded. Then, by (78) and (79), we easily get

\[
\lim_{n \to \infty} |K(u + w_n) - K(u) - K(w_n)|
\]

= \lim_{n \to \infty} |K_{B_n^R}(u + w_n) + K_{B_n^R}(u + w_n) - K_{B_n^R}(u) - K_{B_n^R}(w_n) - K_{B_n^R}(w_n)|

= \lim_{n \to \infty} |K_{B_n^R}(u + w_n) - K_{B_n^R}(u) - K_{B_n^R}(w_n)|

\leq \lim_{n \to \infty} |K_{B_n^R}(u + w_n) - K_{B_n^R}(w_n)| + \varepsilon. \quad (78)

Now, by the intermediate value theorem, there exists $\zeta \in (0, 1)$ such that for $z_n = \zeta_n u + (1 - \zeta_n) w_n$, we have that

\[
|K_{B_n^R}(u + w_n) - K_{B_n^R}(w_n)| = \left| \langle K_{B_n^R}(\zeta_n), u \rangle \right|
\]

\leq \int_{B_n^R} |N'(z_n) u| \leq (by 65)

\leq \int_{B_n^R} c_1 |z_n| \frac{r^{-1}}{L^1(B_n^R)} \|u\| \frac{r^{-1}}{L^q(B_n^R)} + c_2 \|z_n\| \frac{r^{-1}}{L^q(B_n^R)} \|u\| \frac{r^{-1}}{L^q(B_n^R)}

(if $R$ is large enough)

\leq c_3 \left( \|z_n\| \frac{r^{-1}}{L^1(B_n^R)} + \|z_n\| \frac{r^{-1}}{L^q(B_n^R)} \right) \varepsilon.

So we have

\[
|K_{B_n^R}(u + w_n) - K_{B_n^R}(w_n)| \leq c_3 \left( \|z_n\| \frac{r^{-1}}{L^1(B_n^R)} + \|z_n\| \frac{r^{-1}}{L^q(B_n^R)} \right) \varepsilon. \quad (79)

Since $z_n$ is bounded in $H^1(\mathbb{R}^3)$, the sequences $\|z_n\| \frac{r^{-1}}{L^1(B_n^R)}$ and $\|z_n\| \frac{r^{-1}}{L^q(B_n^R)}$ are bounded. Then, by (78) and (79), we easily get

\[
\lim_{n \to \infty} |K(u + w_n) - K(u) - K(w_n)| \leq \varepsilon + M \cdot \varepsilon. \quad (80)
\]

where $M$ is a suitable constant.

Since $\varepsilon$ is arbitrary, from (80) we get

\[
\lim_{n \to \infty} |K(u + w_n) - K(u) - K(w_n)| = 0.
\]

\]

In order to prove the coercivity properties we need the following lemma:

**Lemma 30** Let the assumptions of Theorem 20 be satisfied. Then $E$ and $C$ defined by (65) and (72) satisfy the coercivity assumption (EC-3).
**Proof.** By the Gagliardo-Nirenberg interpolation inequalities (see e.g. [37]) there exists $b > 0$ such that for any $u \in H^1(\mathbb{R}^3)$

$$\|u\|_{L^p}^p \leq b \|u\|_{L^r}^r \|\nabla u\|_{L^2}^q$$

(81)

where $q = 3p \left(\frac{1}{p} - \frac{1}{r}\right)$ and $r = p - q$. By (66) $2 < p < \frac{10}{3}$, then $q < 2$ and $r > 0$.

Then by Hölder inequality we have for $M > 0$

$$\|u\|_{L^p}^p \leq bM \|u\|_{L^r} \frac{1}{M} \|\nabla u\|_{L^2}^q \leq \frac{1}{\gamma} (bM \|u\|_{L^r})^\gamma + \frac{1}{\gamma} \left(\frac{1}{M} \|\nabla u\|_{L^2}^q\right)^\gamma = \frac{(bM)^\gamma}{\gamma} \|u\|_{L^r}^\gamma \frac{1}{\gamma} \|\nabla u\|_{L^2}^q.$$

Now chose $\gamma = \frac{2}{q}$ and $M = \left(\frac{2c}{\gamma}\right)^{1/\gamma}$, where $c$ is the constant in assumption (66), so that

$$\|u\|_{L^p}^p \leq \frac{(bM)^\gamma}{\gamma} \|u\|_{L^r}^\gamma \frac{1}{\gamma} \|\nabla u\|_{L^2}^q.$$  

Then

$$c \|u\|_{L^p}^p \leq a \|u\|_{L^2}^{2s} + \frac{1}{2} \|\nabla u\|_{L^2}^2$$

(82)

where

$$a = \frac{c (bM)^\gamma}{\gamma} ; \quad s = \frac{r \gamma'}{2}.$$  

So using (66), (82) and setting

$$F^2 = \Theta^2 + E^2 + H^2,$$

(83)

we have for any $u = (u, \Theta, E, H) \in X$

$$E(u) + aC(u)^s = \frac{1}{2} \|u\|^2 + \int V(x)u^2 + \int N(u) + a \|u\|_{L^2}^{2s}$$

(84)

$$\geq \frac{1}{2} \|\nabla u\|_{L^2}^2 + \int \left( E_0u^2 + \frac{F^2}{2} \right) + \int N(u) + a \|u\|_{L^2}^{2s}$$

$$\geq \frac{1}{2} \|\nabla u\|_{L^2}^2 + \int \left( E_0u^2 + \frac{F^2}{2} \right) - c \int |u|^p + a \|u\|_{L^2}^{2s}$$

(85)

$$\geq \int \left( E_0u^2 + \frac{F^2}{2} \right).$$

(86)

Observe that, since $p > 2$, we have $s > 1$. So (EC-3)(i) is satisfied. Now we prove that also (EC-3)(ii) is satisfied.
Let \( u_n = (u_n, \Theta_n, E_n, H_n) \in X \) be a sequence such that
\[
\|u_n\|^2 = \int |\nabla u_n|^2 + \int (2E_0 u^2 + F_n^2) \to \infty. \tag{87}
\]

Now distinguish two cases:

- Assume first that \( \int (2E_0 u^2 + F_n^2) \) is unbounded. Then by (86), we have (up to a subsequence)
  \[
  E(u_n) + aC(u_n)^s \to \infty
  \]
  So in this case (EC-3) (ii) is satisfied.

- Assume now that \( \int (2E_0 u_n^2 + F_n^2) \) is bounded and set
  \[
  d = \sup \|u_n\|_{L_2}^p.
  \]

So by (81) we have
\[
\|u_n\|_{L_p}^p \leq c_1 \|\nabla u_n\|_{L_2}^q \text{ where } c_1 = bd. \tag{88}
\]

Since \( \int (2E_0 u^2 + F_n^2) \) is bounded, by (87) we get
\[
\int |\nabla u_n|^2 \to \infty. \tag{89}
\]

On the other hand by (85), we have
\[
E(u_n) + aC(u_n)^s \geq \frac{1}{2} \|\nabla u_n\|_{L_2}^2 - c \int |u_n|^p \geq (\text{by } 88)
\]
\[
\frac{1}{2} \|\nabla u_n\|_{L_2}^2 - c_2 \text{ where } c_2 = cc_1. \tag{90}
\]

Clearly (89) and (90) prove that (EC-3)(ii) holds.

Now let us prove (EC-3)(iii). Let \( u_n = (u_n, \Theta_n, E_n, H_n) \in X \) be a bounded sequence such that \( E(u_n) + aC(u_n)^s \to 0 \), then by (80) we have
\[
\int (2E_0 u^2 + F_n^2) \to 0 \tag{91}
\]
and hence
\[
\int u^2 \to 0 \tag{92}
\]

Then, in order to show that \( \|u_n\| \to 0 \) it remains to prove that
\[
\|\nabla u_n\|_{L_2}^2 \to 0. \tag{93}
\]

Since \( u_n \) is bounded in \( H^1(\mathbb{R}^3) \), by (81) and (91), we get
\[
\int |u_n|^p \to 0. \tag{94}
\]
Since \( E(u_n) + aC(u_n)^s \to 0 \) and by assumption (66), we have

\[
0 = \lim (E(u_n) + aC(u_n)^s) \tag{95}
\]
\[
\geq \limsup \left[ \frac{1}{2} ||\nabla u_n||_{L^2}^2 + E_0 \int |u_n|^2 - c \int |u_n|^p + a ||u_n||_{L^2}^{2s} \right] \tag{96}
\]
\[
= \limsup \left( \frac{1}{2} ||\nabla u_n||_{L^2}^2 + D_n \right) \tag{97}
\]

where

\[
D_n = E_0 \int |u_n|^2 - c \int |u_n|^p + a ||u_n||_{L^2}^{2s}. \tag{98}
\]

By (92) and (94) \( D_n \to 0 \). So by (95) we deduce (93).

\[ \square \]

In the following we will verify that the hylomorphy condition (11) is satisfied.

For \( u_n = (u, \Theta, E, H) \in X \), we set

\[ ||u||_\# = ||(u, \Theta, E, H)||_\# = ||u||_{L^t} , \ 2 < t < 6 \tag{99} \]

\[ \Lambda_0 := \inf \{ \lim \inf \Lambda(u_n) \mid u_n \text{ is a vanishing sequence} \} , \ \Lambda_\# = \lim \inf \Lambda(u) . \]

First of all we prove the following:

**Lemma 31** The seminorm \( ||u||_\# \) defined by (99) satisfies the property (14), namely, if \( u_n = (u_n, \Theta_n, E_n, H_n) \) is vanishing (see Definition 13), then \( ||u_n||_\# = ||u_n||_{L^t} \to 0 \).

**Proof.** For \( j \in \mathbb{Z}^3 \) we set

\[ Q_j = A (j + Q^0) = \{ Aj + Aq : q \in Q^0 \} \]

where \( Q^0 \) is now the cube defined as follows

\[ Q^0 = \{ (x_1, \ldots, x_n) \in \mathbb{R}^3 : 0 \leq x_i < 1 \} . \]

Now let \( x \in \mathbb{R}^3 \) and set \( y = A^{-1}(x) \). Clearly there exist \( q \in Q^0 \) and \( j \in \mathbb{Z}^3 \) such that \( y = j + q \). So

\[ x = Ay = A(j + q) \in Q_j , \]

Then we conclude that

\[ \mathbb{R}^3 = \bigcup_j Q_j . \]

Let \( u_n \) be a bounded sequence in \( H^1(\mathbb{R}^3) \) such that, up to a subsequence, \( ||u_n||_{L^t} \geq a > 0 \). We need to show that \( u_n \) is not vanishing. Then, if \( L \) is the
constant for the Sobolev embedding $H^1(Q_j) \subset L^t(Q_j)$ and $\|u_n\|_{H^1}^2 \leq M$, we have

$$0 < a^t \leq \int |u_n|^t \leq \sum_j \int_{Q_j} |u_n|^t = \sum_j \|u_n\|_{L^t(Q_j)}^{t-2} \|u_n\|_{L^t(Q_j)}^2 \leq \left( \sup_j \|u_n\|_{L^t(Q_j)}^{t-2} \right) \cdot \sum_j \|u_n\|_{L^t(Q_j)}^2 \leq L \left( \sup_j \|u_n\|_{L^t(Q_j)}^{t-2} \right) \|u_n\|_{H^1}^2 \leq LM \left( \sup_j \|u_n\|_{L^t(Q_j)}^{t-2} \right).$$

Then

$$\left( \sup_j \|u_n\|_{L^t(Q_j)} \right) \geq \left( \frac{a^t}{LM} \right)^{1/(t-2)}$$

Then, for any $n$, there exists $j_n \in \mathbb{Z}^3$ such that

$$\|u_n\|_{L^t(Q_{j_n})} \geq \alpha > 0. \quad (100)$$

Then, if we set $Q = AQ^0$, we easily have

$$\|T_{j_n} u_n\|_{L^t(Q)} = \|u_n\|_{L^t(Q_{j_n})} \geq \alpha > 0. \quad (101)$$

Since $u_n$ is bounded, also $T_{j_n} u_n$ is bounded in $H^1(\mathbb{R}^3)$. Then we have, up to a subsequence, that $T_{j_n} u_n \rightharpoonup u_0$ weakly in $H^1(\mathbb{R}^3)$ and hence strongly in $L^t(Q)$. By $(101)$, $u_0 \neq 0$. □

By $(69)$ and $(83)$ the hylenic ratio takes the following form:

$$\Lambda(u) = \frac{\int \left( \frac{1}{2} |\nabla u|^2 + E_0 u^2 + V(x) |u|^2 + \frac{F^2}{t} \right) dx + \int N(u)}{\int |u|^2 dx} \quad (102)$$

**Lemma 32** If the assumptions of Theorem $26$ are satisfied, then for $2 < t < 6$, we have

$$\liminf_{u \in H^1, \|u\|_{L^t} \to 0} \Lambda(u) \geq E_0$$

**Proof.** Clearly by $(102)$

$$\liminf_{u \in H^1, \|u\|_{L^t} \to 0} \Lambda(u) = \liminf_{u \in H^1, \|u\|_{L^t} = 1, \varepsilon \to 0} \Lambda(\varepsilon u) \geq E_0 + \liminf_{u \in H^1, \|u\|_{L^t} = 1, \varepsilon \to 0} \varepsilon \int N(\varepsilon \psi).$$

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So the proof of Lemma will be achieved if we show that
\[ \liminf_{u \in H^1, \|u\|_{L^t} = 1, \varepsilon \to 0} \frac{\int N(\varepsilon u)}{\varepsilon^2 \int |u|^2} = 0. \] (103)

By (65) and (66) we have
\[ -c s^p \leq N(s) \leq \bar{c} (s^q + s^r) \] (104)
where \( c, \bar{c} \) are positive constants and \( q, r \) belonging to the interval \((2, 2^*)\). Then by (104) we have
\[ -c A \varepsilon^{p-2} \leq \inf_{\|u\|_{L^t} = 1} \frac{\int N(\varepsilon u)}{\varepsilon^2 \int |u|^2} \leq \bar{c} B (\varepsilon^{q-2} + \varepsilon^{r-2}) \] (105)
where
\[ A = \inf_{u \in H^1, \|u\|_{L^t} = 1} \frac{\int |u|^p}{\int |u|^2}, \quad B = \inf_{u \in H^1, \|u\|_{L^t} = 1} \frac{\int (|u|^q + |u|^r)}{\int |u|^2}. \]

By (105) we easily get (103).
\[ \Box \]

Now we can give an estimate of \( \Lambda_0 \) (see (12)).

**Corollary 33** If the assumptions of Theorem 26 are satisfied, then
\[ E_0 \leq \Lambda_0. \]

**Proof.** By Proposition 15, Lemma 31 and Lemma 32
\[ \Lambda_0 \geq \Lambda_\sharp = \liminf_{\|u\|_{L^t} \to 0} \Lambda(u) \geq E_0. \]
\[ \Box \]

**Lemma 34** Let \( W \) and \( V \) satisfy assumptions (62), (64), (65), (66), (67), (68). Then, if \( q \) is sufficiently small, the hylomorphy condition (11) holds, namely
\[ \inf_{u \in X} \Lambda(u) < \Lambda_0. \] (106)

**Proof.** Clearly, by corollary 33, in order to prove (106) it will be enough to show that for \( q \) sufficiently small we have
\[ \inf_{u \in X} \Lambda(u) < E_0. \] (107)
Taking $q$ sufficiently small, we will be able to construct $u \in X$ such that $\Lambda(u) < E_0$.

Let $R > 0$ and take $u_R = (u_R, 0, -\nabla \varphi_R, 0)$, where $u_R$ is defined by

$$u_R = \begin{cases} 
  s_0 & \text{if } |x| < R \\
  0 & \text{if } |x| > R + 1 \\
  \frac{|x|}{R} s_0 - \left( |x| - R \right) \frac{R+1}{R} s_0 & \text{if } R < |x| < R + 1
\end{cases}$$

and $\varphi_R$ solves the equation

$$-\Delta \varphi_R = qu_R^2.$$  \hfill (108)

Take $u_R = (u_R, 0, -\nabla \varphi_R, 0)$. Clearly, by definition of $X$ (see (70)) we have $u_R \in X$. Then

$$\int |\nabla u_R|^2 \, dx = O(R^2), \quad \int |u_R|^2 \, dx = O(R^4),$$

so that

$$\frac{\int \left[ \frac{1}{2} |\nabla u_R|^2 + (E_0 + V) u_R^2 \right] \, dx}{\int u_R^2 \, dx} \leq E_0 + V_0 + O \left( \frac{1}{R} \right).$$  \hfill (109)

Moreover

$$\int N(u_R) \, dx = N(s_0) m(B_R) + \int_{B_R \setminus B_{R+1}} N(u_R).$$

where $m(A)$ denotes the measure of $A$. So

$$\frac{\int N(u_R) \, dx}{\int u_R^2 \, dx} \leq \frac{N(s_0) m(B_R) + c_1 R^2}{\int u_R^2 \, dx} \leq \left( \text{ since } N(s_0) < 0 \right) \leq \frac{N(s_0) m(B_R)}{s_0^2 m(B_{R+1})} + \frac{c_1 R^2}{s_0^2 m(B_R)} = \frac{N(s_0)}{s_0^2} \left( \frac{R}{R+1} \right)^3 + \frac{c_2}{R}. \hfill (110)$$

Then, since $\Theta = H = 0$, by (102), (109) and (110) we get

$$\Lambda(u_R) \leq \int \left( \frac{1}{2} |\nabla u_R|^2 + (E_0 + V(x)) u_R^2 \right) \, dx \leq \frac{\int N(u_R) \, dx}{\int u_R^2 \, dx} + \frac{1}{2} \frac{\int |\nabla \varphi_R|^2 \, dx}{\int u_R^2 \, dx} \leq E_0 + V_0 + \frac{N(s_0)}{s_0^2} \left( \frac{R}{R+1} \right)^3 + \frac{c_2}{R} + \frac{1}{2} \frac{\int |\nabla \varphi_R|^2 \, dx}{\int u_R^2 \, dx}. \hfill (111)$$

Now we will estimate the term containing $\varphi_R$ in (112). Observe that $u_R^2$ has radial symmetry and that the electric field outside any spherically symmetric charge distribution is the same as if all of the charge were concentrated into a point. So $|\nabla \varphi_R(r)|$ corresponds to the strength of an electrostatic field at distance $r$, created by an electric charge given by

$$|C_{el}| = \int_{|x| \leq r} qu_R^2 \, dx = 4\pi \int_0^r qu_R^2 v^2 \, dv$$

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and located at the origin. So we have

\[ |\nabla \varphi_R(r)| = \frac{|C_{el}|}{r^2} \begin{cases} = \frac{4}{3} \pi qs_0^2 r & \text{if } r < R \\ \leq \frac{4}{3} \pi qs_0^2 (R+1)^3 & \text{if } r \geq R \end{cases} \]

Then

\[
\int |\nabla \varphi_R|^2 \, dx \leq c_3 q^2 s_0^4 \left( \int_{r<R} r^2 \, dr + \int_{r>R} \frac{(R+1)^6}{r^2} \, dr \right) \\
\leq c_4 q^2 s_0^4 \left( R^3 + \frac{(R+1)^6}{R} \right) \leq c_5 q^2 s_0^4 R^5.
\]

So

\[
\frac{\frac{1}{2} \int |\nabla \varphi_R|^2}{\int u_R^2} \leq \frac{c_6 \int |\nabla \varphi_R|^2}{s_0^2 R^3} \leq c_7 q^2 s_0^3 R^2. \tag{113}
\]

By (113) and (112), we get

\[
\Lambda(u_R) \leq E_0 + V_0 + \frac{N(s_0)}{s_0^2} \left( R \frac{R}{R+1} \right)^3 + \frac{c_2}{R} + c_7 q^2 s_0^2 R^2. \tag{114}
\]

Since by our assumptions

\[
\frac{N(s_0)}{s_0^2} < -V_0
\]

for \( R \) large we get

\[
V_0 + \frac{N(s_0)}{s_0^2} \left( R \frac{R}{R+1} \right)^3 + \frac{c_2}{R} < 0 \tag{115}
\]

So, if \( q \) is small enough, by (114) and (115) we get

\[
\Lambda(u_R) < E_0.
\]

\[ \square \]

**Proof of Theorem 26** We shall show that all the assumptions of theorem 21 are satisfied. Assumptions (EC-0), (EC-1), are clearly satisfied. By Lemma 29 and Lemma 30 also the splitting property (EC-2) and the coercivity property (EC-3) hold. By Lemma 33 the hylomorphy condition (11) holds. Finally also the assumption (41) is satisfied. In fact it is immediate to see that

\[
E'(u, \Theta, E, H) = 0 \implies \Theta = E = H = 0
\]

\[
C'(u, \Theta, E, H) = 0 \implies C'(u) = 0 \implies u = 0.
\]

\[ \square \]

**Proof of Theorem 27** Let \( u_\delta = (u_\delta, \Theta_\delta, E_\delta, H_\delta) \) be an hylomorphic soliton for NSM. So there exists a constant \( \sigma \) such \( u_\delta \) minimizes the energy \( E \) (see (69)) on the manifold

\[
\mathcal{M}_\sigma = \left\{ u = (u, \Theta, E, H) \in X : C(u) = \int u^2 \, dx = \sigma \right\}
\]

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where

\[
X = \left\{ \mathbf{u} = (u, \Theta, \mathbf{E}, \mathbf{H}) \in H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)^9 : \nabla \cdot \mathbf{E} = qu^2 \right\}.
\]

Since \( \mathbf{u}_\delta = (u_\delta, \Theta_\delta, \mathbf{E}_\delta, \mathbf{H}_\delta) \) minimizes the energy \( E \) on \( \mathcal{M}_\sigma \), we have \( \Theta_\delta = \mathbf{H}_\delta = 0 \), then

\[
\mathbf{u}_\delta = (u_\delta, 0, \mathbf{E}_\delta, 0).
\]

If we set \( \mathbf{E} = -\nabla \varphi \), the constraint \( \nabla \cdot \mathbf{E} = qu^2 \) becomes

\[
-\Delta \varphi = qu^2.
\]

So \( \mathbf{u}_\delta \) is a critical point of \( E \) on the manifold made up by those \( \mathbf{u} = (u, 0, -\nabla \varphi, 0) \) satisfying the constraints (116) and

\[
C(\mathbf{u}) = \int u^2 \, dx = \sigma.
\]

Therefore, for suitable Lagrange multipliers \( \omega \in \mathbb{R}, \xi \in D^{1,2} \) (\( D^{1,2} \) is the closure of \( C^\infty_0 \) with respect to the norm \( \| \nabla \varphi \|_{L_2} \)), we have that \( \mathbf{u}_\delta \) is a critical point of the free functional

\[
E_{\omega, \xi}(\mathbf{u}) = E(\mathbf{u}) + \omega \left( \int u^2 - \sigma \right) + \langle \xi, \Delta \varphi + qu^2 \rangle
\]

\[
= \int \left( \frac{1}{2} |\nabla u|^2 + V(x)u^2 + W(u) + \frac{1}{2} |\nabla \varphi|^2 \right) \, dx
\]

\[
+ \omega \left( \int u^2 - \sigma \right) + \langle \xi, \Delta \varphi + qu^2 \rangle
\]

where \( \langle \cdot, \cdot \rangle \) denotes the duality map in \( D^{1,2} \) and \( \mathbf{u} \) can be identified with \( (u, \varphi) \in H^1(\mathbb{R}^3) \times D^{1,2} \). \( E_{\omega, \xi}(\mathbf{u}_\delta) = 0 \) gives the equations

\[
\forall v \in H^1(\mathbb{R}^3), \quad \left\langle \frac{\partial E_{\omega, \xi}(\mathbf{u})}{\partial u}, v \right\rangle = 0
\]

\[
\forall \chi \in D^{1,2}(\mathbb{R}^3), \quad \left\langle \frac{\partial E_{\omega, \xi}(\mathbf{u})}{\partial \varphi}, \chi \right\rangle = 0
\]

namely

\[
\forall v \in H^1(\mathbb{R}^3), \quad \int \nabla u \cdot \nabla v + 2 \left[ V(x)u + \frac{1}{2} W'(u) + \omega u + q \xi u \right] v = 0
\]

\[
\forall \chi \in D^{1,2}, \quad \int \nabla \varphi \cdot \nabla \chi + (\xi, \Delta \chi) = 0.
\]

So, \( u_\delta, \varphi_\delta \) are weak solutions of the following equations:

\[
-\frac{1}{2} \Delta u_\delta + V(x)u_\delta + \frac{1}{2} W'(u_\delta) + \omega u_\delta + q \xi u_\delta = 0 \quad (119)
\]

\[
\Delta \varphi_\delta = \Delta \xi. \quad (120)
\]
From (120) we get $\xi = \varphi_\delta$, so (119) becomes

$$-\Delta u_\delta + 2V(x)u_\delta + W'(u_\delta) + 2\omega u_\delta + 2q\varphi_\delta u_\delta = 0.$$ 

This equation and the constraint (116) give the system (73) and (74). □

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