REAPING NUMBERS OF BOOLEAN ALGEBRAS

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Abstract. A subset $A$ of a Boolean algebra $B$ is said to be $(n, m)$-reaped if there is a partition of unity $P \subset B$ of size $n$ such that $|\{b \in P : b \land a \neq 0\}| \geq m$ for all $a \in A$. The reaping number $r_{n,m}(B)$ of a Boolean algebra $B$ is the minimum cardinality of a set $A \subset B \setminus \{0\}$ such which cannot be $(n, m)$-reaped. It is shown that, for each $n \in \omega$, there is a Boolean algebra $B$ such that $r_{n+1,2}(B) \neq r_{n,2}(B)$. Also, $\{r_{n,m}(B) : \{n, m\} \subseteq \omega\}$ consists of at most two consecutive integers. The existence of a Boolean algebra $B$ such that $r_{n,m}(B) \neq r_{n',m'}(B)$ is equivalent to a statement in finite combinatorics which is also discussed.

Keywords: Boolean algebra, reaping number, tree, colouring

1. Introduction

A subset $A$ of a Boolean algebra $B$, is said to be reaped by $b \in B$, if, for all $a \in A$, both $a \land b$ and $a - b$ are non-zero. For any Boolean algebra $B$, the cardinal invariant $r(B)$ is defined to be the least cardinality of a subset of $B \setminus \{\nu\}$ which can not be reaped. The more familiar cardinal invariant of the continuum known as the reaping number, $r$, is therefore nothing more than $r(\mathcal{P}(\omega)/[\omega]^{<\aleph_0})$. This cardinal invariant was introduced by A. Beslagić and E. van Douwen [3] who showed that the equality $r = c$ implies that $\omega^* \setminus \{p\}$ is non-normal for each ultrafilter $p \in \omega^*$. The reaping number appears again in a paper of B. Balcar and P. Simon [1] concerning the $\pi$-character of points in $\omega^*$, the Stone space of $\mathcal{P}(\omega)/[\omega]^{<\aleph_0}$. The $\pi$-character of a point $x$ in a topological space $(X, \tau)$ is the least cardinality of a set $A \subset \tau$ such that for every neighbourhood $V$ of $x$ there is $A \in A$ such that $A \subset V$ — it is not required that $A$ consist of neighbourhoods of $x$. In [1] $r$ is characterised as the minimum of all $\pi$-characters of points in $\omega^*$. This leads naturally to the question of whether or not $r(B)$ is the minimum of all $\pi$-characters of ultrafilters on $B$ when these are considered as points in the Stone space of $B$.

Given a Boolean algebra $B$ and an integer $n$, an $n$-partition of $B$ is a set $R \in [B]^\kappa$ such that $\bigvee R = 1$ and $a \land b = 0$ for $\{a, b\} \in [R]^2$. A
set $A \subseteq \mathbb{B}$ will be said to be $(n, k)$-reaped by $R$ if $R$ is an $n$-partition of $\mathbb{B}$ such that $|\{b \in R : b \land a \neq 0\}| \geq k$ for all $a \in A$. Therefore a set $A$ is reaped by $b$ if and only if it is $(2, 2)$-reaped by $\{b, 1 - b\}$. The cardinal $r_{n,k}(\mathbb{B})$ is the least cardinality of a subset of $\mathbb{B} \setminus \{\varnothing\}$ which cannot be $(n, k)$-reaped by some $n$-partition of $\mathbb{B}$. Define $r_{n}(\mathbb{B})$ to be $r_{n,2}(\mathbb{B})$ and the phrase $n$-reaped will be used to mean $(n, 2)$-reaped. Let $r_{\omega}(\mathbb{B})$ denote the least cardinal of a set $A \subseteq \mathbb{B} \setminus \{0\}$ which cannot be $n$-reaped for any $n$. The sequence of inequalities
\[
r_2 \leq r_3 \leq r_4 \leq \cdots \leq r_\omega
\]
is easily verified. With these definitions, the results of [1] can now be stated. The first establishes a connection with topology.

**Proposition 1.1.** For every Boolean algebra $\mathbb{B}$, $r_{\omega}(\mathbb{B})$ is equal to the least cardinal $\kappa$ such that there is an ultrafilter on $\mathbb{B}$ with $\pi$-character $\kappa$.

**Proof:** Note that a $\pi$-base for an ultrafilter cannot be $n$-reaped for any $n \in \omega$ so it suffices to show that there is some ultrafilter on $\mathbb{B}$ whose $\pi$-character is $r_{\omega}(\mathbb{B})$. Let $A \subseteq \mathbb{B}$ be a set which cannot be $n$-reaped for any $n \in \omega$ and such that $|A| = r_{\omega}(\mathbb{B})$. Then $G = \{b \in \mathbb{B} : (\forall \mathcal{A} \in \mathbb{A})(\mathcal{A} \neq 0)\}$ generates a proper ideal because otherwise there is, for some $n \in \omega$, an $n$-partition consisting of elements of $G$ and this contradicts that $A$ is not $n$-reaped. This ideal can be extended to a maximal one and its dual filter is the desired filter. It can be checked that every member of this dual filter contains some member of $A$. ■

While Proposition 1.1 illustrates the importance of the cardinal invariants $r_n(\mathbb{B})$, the next result from [1] indicates that, in some cases, they are all equal. It also shows that the equivalence of $r$ and the least $\pi$-character of a point in $\omega^*$ is due to the incidental fact that $\mathcal{P}(\omega)/[\omega]^\text{cof}(\tau)$ is homogeneous.

**Proposition 1.2.** If $\mathbb{B}$ is homogenous or complete, then $r_2(\mathbb{B}) = r_{\omega}(\mathbb{B})$.

**Proof:** First suppose that $\mathbb{B}$ is homogeneous. In fact, simply assume that for all $b \in \mathbb{B}$, $r_2(\mathbb{B}) = r_2(\mathbb{B} \mid b)$. Proceed by induction on $n$ to show that if $\mathbb{B}$ is any Boolean algebra satisfying these assumptions then $r_n(\mathbb{B}) = r_n(\mathbb{B} \mid b)$ — this clearly suffices. The case $n = 2$ is trivial. Let $A \subseteq \mathbb{B}$ be a set of cardinality $r_2(\mathbb{B})$ which can not be $2$-reaped. It may, without loss of generality be assumed that $A$ is closed under finite products since closing it off under this operation will not change its cardinality. If $R$ is an $n + 1$-partition of $\mathbb{B}$ which $n + 1$-reaps $A$ fix any $r \in R$. Since $A$ is not $2$-reaped, there is an $a \in A$, such that
a \leq r \text{ or } a \land r = 0. \] The first case contradicts that $A$ is $n + 1$-reaped by $R$. Therefore $A \cap \mathbb{B} \upharpoonright \emptyset$ is $n$-reaped by $R \setminus \{r\}$. This contradicts the induction hypothesis and the homogeneity of $\mathbb{B}$.

Now suppose that $\mathbb{B}$ is complete and choose $A \subseteq \mathbb{B}$ which cannot be 2-reaped and is of minimal cardinality. Find a set $C \subseteq \mathbb{B}$ such that

- $\bigvee C = 1$
- $c \land c' = 0$ for all $c$ and $c'$ in $C$ provided that $c \neq c'$
- if $c \in C$ and $b \leq c$ then $r_2(\mathbb{B} \upharpoonright) = r_2(\mathbb{B} \upharpoonright)$

If $r_2(\mathbb{B} \upharpoonright) = r_2(\mathbb{B})$ for some $c \in C$ then the homogeneity assumptions of the first part are satisfied. Otherwise, for each $c \in C$, choose $b_c < c$ such that $A \cap \mathbb{B} \upharpoonright$ is 2-reaped by $\{b_c, c - b_c\}$. It is easily seen that the partition $\{\bigvee_{c \in C} b_c, 1 - \bigvee_{c \in C} b_c\}$ also 2-reaps $A$. ■

The question of whether $r_2(\mathbb{B}) = r_\omega(\mathbb{B})$ for all Boolean algebras was raised by the authors of $[1]$. It will be shown that the answer is no. The same result was obtained independently by B. Balcar and P. Simon in $[2]$. The next section contains results which yield this as a corollary and establish the best possible upper bound for $r_n(\mathbb{B})$ for $n \leq \omega$. The more general cardinal invariant $r_{i,j}(\mathbb{B})$ which is introduced in this paper plays a central role in this analysis.

2. Reaping Numbers and a Finite Combinatorial Property

This section is devoted to establishing an equivalence between an assertion about trees and the statement that $r_{i,j}(\mathbb{B}) \leq r_{k,n}(\mathbb{B})$ for all Boolean algebras $\mathbb{B}$. Throughout this paper the term tree will be reserved for sets of sequences of integers which are closed under initial segments. The canonical examples of trees are the sets $T(D, m) = \cup_{\xi \in D}^{\xi \in D}^{\xi \in D} m$ where $D$ is a set of ordinals — in other words, $T(D, m)$ is the set of all $m$-valued functions with domain an initial segment of $D$. All trees considered in this paper will be subsets of some $T(D, m)$. The ordering on a tree is the restriction to an initial segment relation; in other words, if $\sigma$ and $\theta$ belong to $T(D, m)$ then $\sigma \leq \theta$ if and only if there is some $\delta \in D$ such that $\sigma = \theta \upharpoonright \delta$. The set of maximal elements of a tree $T$ will be denoted by $\mu(T)$. A tree $T$ is said to be $k$-branching if every $\sigma \in T \setminus \mu(T)$ has exactly $k$ successors.

**Definition 2.1.** The quadruple of integers $(i, j, k, m)$ will be said to satisfy the property $P(i, j, k, m)$ if and only if for every finite, $i$-branching tree $T$ and every colouring $\chi : \mu(T) \to j$ of the maximal members of $T$ by $j$ colours, there is a $k$-branching subtree $T' \subseteq T$ such that $\mu(T') \subseteq \mu(T)$ and the range of $\chi \upharpoonright \mu(T')$ has no more than $m$ colours.
The main result of this section is Theorem 2.1 which establishes a relationship between the assertion that $\tau_{t_{11}}(B) \leq \tau_{t_{1m}}(B)$ for every Boolean algebra $B$ and the property $P(i, k, j, m)$. The next lemma is the key to establishing this theorem.

**Lemma 2.1.** For all pairs of integers $i$ and $k$ such that $k \leq i$ there are $k$-branching trees $\{T_n : n \in \omega\}$ such that for each $F \in [\omega_1]^{<\aleph_0}$ and for each $k$-branching subtree $T \subseteq T(F, i)$ there are infinitely many $n \in \omega$ such that $T = \{\psi \upharpoonright F : \psi \in T_n\}$.

**Proof:** It is possible to prove this result by constructing the natural partial order to force the trees $T_n$ with finite conditions and then appeal to D. Velleman’s Martin’s Axiom type equivalence to $\omega$-morasses [3]. For the reader’s convenience however, a self contained proof will be presented. The trees $T_n$ will be chosen to form a dense subset of a certain product space whose points can be interpreted as trees. The fact that the product of no more than $2^{\aleph_0}$ separable spaces is separable will play an important role [4].

For each $\alpha \in \omega_1$ let $S(\alpha)$ be the set of all colourings $\chi : \mu(T(F_\chi, k)) \rightarrow [i]^k$ where $F_\chi \in [\alpha]^{<\aleph_0}$ with the discrete topology and let $S = \prod_{\alpha \in \omega_1} S(\alpha)$. Since each $S(\alpha)$ is countable it follows that $S$ is separable. Moreover each element of $S$ can be considered to be a $k$-branching subtree of $T(\omega_1, i)$. In order to do this let $a^\epsilon : a \rightarrow k$ be the unique, order preserving bijection from $a$ to $k$ for each $a \in [i]^k$. Then, if $x \in S$ define a $k$-branching subtree $x^\theta_\theta \subseteq T(\theta, i)$ by induction on $\theta$. If $\theta$ is limit of countable cofinality then

$$x^\theta_\theta = \{\sigma : (\forall \zeta \in \theta)(f \upharpoonright \zeta \in x^\zeta_\zeta)\}$$

and $x^\theta$ will be defined to be $\cup_{\theta \in \omega_1} x^\theta_\theta$. If $\theta = \zeta + 1$ then let $C_\zeta : x^\zeta_\zeta \rightarrow T(\zeta, k)$ be the canonical collapse of $x^\zeta_\zeta$; namely, letting $a(\sigma)$ be the set of successors of $\sigma$ in $x^\zeta_\zeta$, $C_\zeta(\sigma)(\beta) = a(\sigma \upharpoonright \beta)^*(\sigma(\beta))$. Then $x^\theta$ is defined to be the set of all $\sigma \in T(\theta, i)$ such that $\sigma \upharpoonright \zeta \in x^\zeta_\zeta$ and $\sigma(\zeta) \in x(\zeta)(C_\zeta(\sigma \upharpoonright \zeta) \upharpoonright F(\zeta))$.

All that has to be checked is that if $F \in [\omega_1]^{<\aleph_0}$ and $T \subseteq T(F, i)$ is a $k$-branching subtree then

$$\{x \in S : (\forall \sigma \in x^\theta)(\sigma \upharpoonright F \in T)\}$$

is a nonempty open set. This is a routine exercise.

**Theorem 2.1.** The property $P(i, j, k, m - 1)$ is satisfied if and only if there is a Boolean algebra $B$ such that $\tau_{t_{1i}}(B) > \tau_{t_{1m}}(B)$. 

Proof: If \( P(i, j, k, m - 1) \) fails, then there is some integer \( K \) and a colouring \( \chi : \mu(T(K, i)) \to j \) such that every \( k \) branching subtree is coloured by at least \( m \) colours. Let \( \mathbb{B} \) be a Boolean algebra and \( F \subseteq \mathbb{B} \) be of cardinality less than \( r_{1, \mathbb{B}}(\mathbb{B}) \). It must be shown that \( F \) can be \((j, m)\)-reaped, so suppose that this is not possible.

For each \( a \in \mathbb{B} \) it is possible to choose an \( i \)-partition \( \{p_a(s) : s \in i\} \) which \((i, k)\)-reaps \( \{f \cap a : f \in F \text{ and } f \cap a \neq 0\} \). For each \( \sigma \in T(K, i) \) it is possible to define inductively a function \( P : T(K, i) \to \mathbb{B} \) as follows: \( P(\emptyset) = 1_{\mathbb{B}} \) and, if the domain of \( \sigma \) is \( L \), then

\[
P(\sigma \cup \{(L, s)\}) = p_s(P(\sigma)) \cap P(\sigma)
\]

Define \( b_c = \cup\{P(\sigma) : \sigma : K \to i \text{ and } \chi(\sigma) = c\} \) for \( c \in j \). It must be that the partition \( \{b_c : c \in j\} \) does not \((j, m)\)-reap \( F \) and so there is some \( f \), an element of \( F \), which witnesses this fact. Using the fact that \( \{p_a(s) : s \in i\} \) is a partition which \((i, k)\)-reaps \( F \) it follows that there is a \( k \) branching subtree \( T' \subseteq T(K, i) \) such that \( f \cap P(\sigma) \neq 0 \) for each \( \sigma \in \mu(T') \). But \( \mu(T') \) is coloured by \( \chi \) with at least \( m \) colours and \( f \cap b_c \neq 0 \) if some member of \( \mu(T') \) is coloured with colour \( c \). It follows that \( \{c \in j : b_c \cap f \neq 0\} \) has at most \( m \) elements and this is a contradiction to the choice of \( f \).

Suppose \( P(i, j, k, m) \) holds. It will be shown that there is a Boolean algebra in which a countable family cannot be \((j, m + 1)\) reaped but every countable family can be \((i, k)\) reaped. Let \( \mathbb{B} = \mathbb{C}_\omega \oplus \mathbb{F}_{\omega_1} \) where \( \mathbb{C}_\omega \) is the Boolean subalgebra of \( \mathcal{P}(\omega) \) generated by the finite sets and \( \mathbb{F}_{\omega_1} \) is the free algebra of size \( \aleph_1 \) represented as \( \prod_{\omega_1} \mathcal{P}(i) \). The algebra \( \mathbb{B} \) is generated by rectangles of the form \( F \times [\sigma] \) where \( F \) is a finite or cofinite subset of \( \omega \) and \( \sigma \) is a finite partial function from \( \omega_1 \) into \( i \); these rectangles will be denoted by \( F \otimes [\sigma] \). The trees \( T_n \) of Lemma 2.1 will be used to generate a quotient algebra \( \mathbb{A} \). Let \( \mathcal{G}_{\omega_1} = \{\sigma \upharpoonright F : \sigma \in T_n \upharpoonright F \in [\omega_\infty]^{<\aleph_1}\} \). Let \( \mathcal{I} \) be the ideal in \( \mathbb{B} \) generated \( \{n \otimes [\psi] : \psi \notin \mathcal{G}_n\} \) and define \( \mathbb{A} = \mathbb{B}/\mathcal{I} \).

To see that \( r_{i, \mathbb{A}}(\mathbb{A}) = \aleph_\kappa \) let \( \mathbb{B}_\alpha \) be the subalgebra of \( \mathbb{B} \) generated by sets of the form \( \{n\} \otimes [\psi] \) where \( \text{dom}(\psi) \subseteq \alpha \) and define \( \mathbb{A}_\alpha = \mathbb{B}_\alpha/\mathcal{I} \). It suffices to show that \( \mathbb{A}_\alpha \) is \((i, k)\)-reaped by \( \{\omega \otimes [(\alpha, \ell)] : \ell \in i\} \). Now, if the equivalence class of \( \{n\} \otimes [\psi] \) is not empty in \( \mathbb{A}_\alpha \) then \( \psi \in \mathcal{G}_n \). By Lemma 2.1 given this \( \psi \) and \( n \) and \( \alpha \), it follows that there is \( \sigma \in T_n \) such that \( \psi \subseteq \sigma \) and there is \( a \in [i]^k \) such that \( \{\sigma \cup \{(\alpha, t) \in a\} \subseteq T_n \). Hence \( \{\psi \cup \{(\alpha, t) \in a\} \subseteq \mathcal{G}_n \). From this it will follow that \( \mathbb{A}_\alpha \) is \((i, k)\)-reaped provided that it can be shown that every nonempty member of \( \mathbb{A}_\alpha \) contains something of the form \( \{n\} \otimes [\psi] \) such that \( \psi \in \mathcal{G}_n \).
To this end notice that every nonempty member of $\mathbb{A}_\kappa$ contains something of the form $C \otimes [\psi]$. If $C$ is finite then $C \otimes [\psi] = \bigcup_{n \in C} \{n\} \otimes [\psi]$ and the fact that $C \otimes [\psi] \neq 0$ implies that there is some $n \in C$ such that $\{n\} \otimes [\psi] \notin I$. If $C$ is infinite then it contain $\omega \setminus J$ for some $J \in \omega$. Let $\Gamma$ be the domain of $\psi$ and let $T$ be any $k$-branching subtree of $T(\Gamma, i)$ such that $\psi \in \mu(T)$. From the properties of $\{T_n : n \in \omega\}$ stated in Lemma 2.1 it follows that there is some $j > J$ such that $\psi \in T \subseteq T_i$.

Hence $\{j\} \otimes [\psi] \subseteq C \otimes [\psi]$. To see that $a \in B$ which is not zero in $A$ can be represented as

$$a = \bigvee_{\psi \in T(F, i)} C(\psi) \otimes [\psi]$$

where $F \in [\omega]^{<\aleph_0}$ and where $C(\psi)$ is a finite or cofinite subset of $\omega$. It can therefore be deduced that any partition of unity $\{a_z : z \in J\}$ can be represented as

$$a_z = \bigvee_{\psi \in T(F, i)} C_z(\psi) \otimes [\psi]$$

for some $F \in [\omega]^{<\aleph_0}$ where $\{C_z(\psi) : z \in J\}$ is a partition of $\omega$ into finite and cofinite sets, for each $\psi$. Given such a partition of unity it is possible to choose an integer $K$ such that for every $z \in J$ and $\psi \in T(F, i)$ either $C_z(\psi) \subseteq K$ or $\omega \setminus K \subseteq C_z(\psi)$.

Now define a colouring $\chi$ of $\mu(T(F, I))$ by the rule $\chi(\psi) = z$ if and only if $\omega \setminus K \subseteq C_z(\psi)$. Now, using $P(i, j, k, m)$, there is a $k$-branching subtree $\Gamma$ of $T(F, i)$ such that the range of $\chi \upharpoonright \mu(\Gamma)$ has size at most $m$. By the key property of the family of trees stated in Lemma 2.1 it follows that there is an integer $n > K$ such that $\{\sigma \upharpoonright F : \sigma \in T_n\} = \mu(\Gamma)$.

Since $n > K$ it follows that $n \in C_z(\psi)$ if and only if $\chi(\psi) = z$. Also, if $\{n\} \otimes [\psi] \notin I$ and the domain of $\psi$ is $F$ then $\psi \in \mu(\Gamma)$. Hence, if $a_z \wedge \{n\} \otimes [\psi] \neq \emptyset$ then $\chi(\psi) = z$ for some $\psi \in \mu(\Gamma)$. Since the range $\chi \upharpoonright \mu(\Gamma)$ has at most $m$ elements it follows that $z$ can take on at most $m$ values and the proof is complete. $\blacksquare$

Note that the Boolean algebra constructed in this section has the countable chain condition. Note also that it is possible to generalise the construction for any cardinal $\kappa$ to obtain a Boolean algebra $A$ such that $\mathfrak{t}_{i+1}(A) = \kappa^+$ while $\mathfrak{t}_{i,m+1}(A) = \kappa$. 
3. Some Inequalities

Certain monotonicity results for the cardinal invariants \( r_{n,m}(\mathbb{B}) \) are easily established.

**Lemma 3.1.** If \( \mathbb{B} \) is any Boolean algebra and \( n \) and \( m \) are integers with \( m \geq n \geq 2 \), then the following inequalities hold

1. \( r_{m,n}(\mathbb{B}) \geq r_{m,n+1}(\mathbb{B}) \)
2. \( r_{m+1,n}(\mathbb{B}) \geq r_{m,n}(\mathbb{B}) \)
3. \( r_{m+1,n+1}(\mathbb{B}) \leq r_{m,n}(\mathbb{B}) \) and, more generally, \( r_{i,j}(\mathbb{B}) \leq r_{m,n}(\mathbb{B}) \) provided that there is some integer \( k \) such that \( m \leq i \leq mk \) and \( j > (n-1)k \)
4. \( r_{m,n}(\mathbb{B}) \geq r_{2}(\mathbb{B}) \) for all integers \( m \) and \( n \)
5. \( r_{n,m}(\mathbb{B}) = r_{2}(\mathbb{B}) \) for all integers \( n \) and \( m \) provided that \( n/(m-1) \geq 2 \).

**Proof:** The first two assertions are immediate. To verify the last clause of (3) suppose that \( \{a_\ell : \ell \in i\} \) is an \( i \)-partition which \((i,j)\)-reaps \( A \subseteq \mathbb{B} \). Then it is possible to find \( \{u_\ell : \ell \in m\} \subseteq [i]^{\leq k} \) which partition \( i \) into nonempty sets. Let \( b_\ell = \bigvee\{a_x : x \in u_\ell\} \). It follows that \( \{b_\ell : \ell \in m\} \) \((m,n)\)-reaps \( A \) because of the fact that \( j > (n-1)k \) and so, if less than \( n \) members of \( \{b_\ell : \ell \in m\} \) meet some member of \( A \) then less than \( j \) members of \( \{a_\ell : \ell \in i\} \) meet that same element.

To prove (4) suppose that \( A \subseteq \mathbb{B} \) can not be \((i,j)\)-reaped and \( |A| < r_{2}(\mathbb{B}) \). It follows that there is a 2-partition \( \{a_0, a_1\} \) which 2-reaps \( A \). Moreover

\[
|\{a \wedge a_0 : a \in A\}| \leq |A| < r_{2}(\mathbb{B})
\]

and \( |\{a \wedge a_1 : a \in A\}| \leq |A| < r_{2}(\mathbb{B}) \). Hence there are partitions \( \{a_{i,j} : i \in 2\} \) of \( a_j \) which 2-reap \( \{a \wedge a_j : a \in A\} \) for each \( j \in 2 \). Consequently, \( \{a_{i,j} : \{i,j\} \subseteq 2\} \) is a 4-partition which \((4,4)\)-reaps \( A \). Continuing in this manner one obtains an \( i \)-partition which \((i,i)\)-reaps \( A \) and hence, also \((i,j)\)-reaps \( A \).

Statement (5) is an immediate consequence of (4) and (3). In particular, (4) implies that \( r_{n,n}(\mathbb{B}) \geq r_{2}(\mathbb{B}) \) while (3) yields that \( r_{n,m}(\mathbb{B}) \leq r_{2}(\mathbb{B}) \) for every Boolean algebra \( \mathbb{B} \) provided that \( n/(m-1) \geq 2 \). ☐

The question of whether it is possible to have a Boolean algebra with three different reaping invariants was unresolved for some time. The main theorem of this section, Theorem 3.1, will show that this is not possible and that even more restrictions apply. On the road to establishing this it will prove to be useful to have the following definition of polarized partition relation.
Definition 3.1. The collection of symbols

\[
\binom{m}{n} \not\rightarrow \binom{j}{i}_{k,q}^{1,1}
\]

is defined to mean that there is a function, \( h \), from \( n \times m \) into \( k \) such that, for all \( a \in \binom{n}{i} \) and \( b \in \binom{m}{j} \) the function \( h \upharpoonright a \times b \) takes on more than \( q \) values.

Lemma 3.2. For each prime integer \( n \) and \( k < n \)

\[
\binom{n}{n} \not\rightarrow \binom{2}{k}_{n,k}^{1,1}
\]

Proof: Define the function \( h : n \times n \rightarrow n \) by \( h(i,j) = i + j \mod n \).

Lemma 3.3. Suppose that \( i, j, k, m, n \) and \( q \) are integers such that

\[
\binom{m}{n} \not\rightarrow \binom{j}{i}_{k,q}^{1,1}
\]

Then any Boolean algebra \( B \) which satisfies that \( r_{n,i}(B) > r_{\ell,q+1}(B) \) also satisfies that \( r_{m,j}(B) \leq r_{\ell,q+1}(B) \).

Proof: Let \( h : m \times n \rightarrow k \) witness that

\[
\binom{m}{n} \not\rightarrow \binom{j}{i}_{k,q}^{1,1}
\]

Let \( B \) be a Boolean algebra and assume that \( r_{n,i}(B) \geq \kappa = r_{\ell,q+1}(B) \). Choose a set \( A \in [B]^{r_{\ell,q+1}(B)} \) which cannot be \( (k, q + 1) \)-reaped. Let \( B_{\beta} \) be the algebra generated by \( A \). Recursively choose \( n \)-partitions \( \{b(\alpha, \ell) : \ell < n\} \) which \((n, i)\)-reap \( B_{\alpha} \) where, for \( \beta < \kappa \), \( B_{\beta+\kappa} \) is the algebra generated by \( B_{\beta} \cup \{(\gamma, \ell) : \ell < \kappa\} \) and if \( \beta \) is a limit ordinal, then \( B_{\beta} = \bigcup_{\gamma < \beta} B_{\gamma} \). Finally, assuming that \( r_{m,j}(B) > \kappa \), choose an \( m \)-partition, \( \{b(\kappa, \ell) : \ell < m\} \), which \((m, j)\)-reaps \( B_{\kappa} \).

For each \( \alpha < \kappa \) and \( \ell < k \) define

\[
c(\alpha, \ell) = \bigvee \{b(\alpha, \xi) \land b(\kappa, \zeta) : h(\xi, \zeta) = \ell\}.
\]

Since \( \{c(\alpha, \ell) : \ell < k\} \) is a \( k \)-partition of \( B \), there are \( \{\ell_{e}^{\alpha} : e \in q\} \subseteq k \) and an element \( a_{\alpha} \in A \) such that \( a_{\alpha} \leq \lor c(\alpha, \ell_{e}^{\alpha}) : e \in q\} \). Now there are \( \alpha_{1} < \alpha_{2} < \ldots < \alpha_{i} < \kappa \) so that \( \ell_{e}^{\alpha_{1}} = \ell_{e}^{\alpha_{2}} = \ldots = \ell_{e}^{\alpha_{i}} = \ell_{e}^{\alpha} \), for each \( e \in q \), and \( a_{\alpha_{1}} = a_{\alpha_{2}} = \ldots = a_{\alpha_{i}} = a \).

Now choose inductively \( I_{q} < n \) so that
\[ I_q \neq I_{q'} \text{ if } q \neq q' \]
\[ a \land b(\alpha_1, I_1) \land b(\alpha_2, I_2) \land \ldots \land b(\alpha_i, I_i) > 0 \]

This is easily done because if \( I_1, I_2, \ldots, I_q \) have been chosen so that \( q < i \) and \( a \land b(\alpha_1, I_1) \land b(\alpha_2, I_2) \land \ldots \land b(\alpha_q, I_q) > 0 \) then there is some \( w \in [n]^i \) such that \( a \land b(\alpha_1, I_1) \land b(\alpha_2, I_2) \land \ldots \land b(\alpha_q, I_q) \land b(\alpha_{q+1}, y) > 0 \) for each \( y \in w \) by virtue of the fact that \( B_{\alpha_{q+1}} \) is \((n, i)\)-reaped by \( \{b(\alpha_{q+1}, t) : t \in n\} \). It follows that it is possible to choose \( I_{q+1} \in w \setminus \{I_1, I_2, \ldots, I_q\} \).

Since \( a \leq \lor \{c(\alpha_i, \ell \in) : e \in q\} \) it follows that \( 0 < a \land b(\alpha_1, I_1) \land b(\alpha_2, I_2) \land \ldots \land b(\alpha_i, I_i) \land b(\kappa, y) > 0 \) for each \( y \in u \) by virtue of the fact that \( B_{\kappa} \) is \((m, j)\)-reaped by \( \{b(\kappa, y) : y \in m\} \).

It will now be shown that \( h(x, y) \in \{\ell \in e \in q\} \) for any \( x \in \{I_1, I_2, \ldots, I_i\} \) and \( y \in u \). To see this let \( x = I_1 \) and notice that
\[ a \land b(\alpha_1, I_1) \land b(\kappa, y) \geq a \land b(\alpha_1, I_1) \land b(\alpha_2, I_2) \land \ldots \land b(\alpha_i, I_i) \land b(\kappa, y) > 0 \]

Since \( a \leq \lor \{c(\alpha_i, \ell \in) : e \in q\} \) it follows that \( 0 < a \land b(\alpha_1, I_1) \land b(\kappa, y) < \lor \{c(\alpha_i, \ell \in) : e \in q\} \) and hence that \( h(I_1, y) = \ell \in e \) for some \( e \in q \). This is a contradiction to the hypothesis on \( h \) that its range on any \( i \times j \) rectangle has more than \( q \) points in it.

**Corollary 3.1.** If \( 2 \leq i \leq n \leq k \) then \( r_{n,i}(B) \leq r_{1,1}(B)^+ \).

**Proof:** If \( n = i \) then this follows from (4) and (5) of Lemma 3.1 — hence it may be assumed that \( i < n \). Therefore it follows from Lemma 3.2 that
\[
\binom{n}{i} \not\prec \binom{2}{i} \quad \text{and consequently} \quad \binom{n}{i} \not\prec \binom{i}{i} \quad \text{because of simple monotonicity properties of polarized partition relations.}
\]
Finally let \( n = m \) and \( i = j \) and \( q = 1 \) in Lemma 3.3.

**Theorem 3.1.** The inequality \( r_n(B) \leq r_2(B)^+ \) holds for every integer \( n \) and Boolean algebra \( B \).

**Proof** From Lemma 3.1 (2) it suffices to show this only in the case when \( n \) is a prime. Let \( B \) be a Boolean algebra. If \( r_{n,2}(B) = r_2(B) \) there is nothing to prove so it may be assumed that there is some \( k \) such that \( r_{n,k}(B) \neq r_2(B) \) — let \( k \) be the greatest such integer.
Lemma 3.1 (5), it follows that \( k < n \) and from (4) of that same lemma it follows that \( r_{n,1}(\mathcal{B}) > r_{2}(\mathcal{B}) \). \( \square \)

and hence, from Lemma 3.3 it may be concluded that

\[
\binom{n}{n} \not\rightarrow \binom{2}{k}_{n,k}
\]

and hence, from Lemma 3.3 it may be concluded that \( r_{n,2}(\mathcal{B}) \leq r_{n,1}(\mathcal{B}) + 1 \) because the other hypothesis of Lemma 3.3 — namely that \( r_{n,1}(\mathcal{B}) > r_{n,k+1}(\mathcal{B}) \) — follows from the maximality assumption on \( k \). But, again because of the maximality of \( k \), \( r_{2}(\mathcal{B}) = r_{n,1}(\mathcal{B}) \) and hence \( r_{n,2}(\mathcal{B}) \leq r_{2}(\mathcal{B}) \). \( \square \)

The following immediate corollary to Theorem 3.1, together with Proposition 1.1, shows that the minimum \( \pi \)-character of a point in the Stone space of any Boolean algebra \( \mathcal{B} \) is bounded above by \( r_{2}(\mathcal{B}) \).}

**Corollary 3.2.** For every Boolean algebra \( \mathcal{B} \), \( r_{\omega}(\mathcal{B}) \leq r_{2}(\mathcal{B})^+ \).

4. Finite Combinatorics

It is the purpose of this section to provide some evidence for the following conjecture: For any pair of integers \( i \) and \( j \) there is an integer \( n \) such that \( r_{n}(\mathcal{B}) = r_{i,j}(\mathcal{B}) \) for any Boolean algebra \( \mathcal{B} \). Indeed it is reasonable to conjecture that for any pair of integers \( i \geq j \geq 2 \) and any Boolean algebra \( \mathcal{B} \)

\[
r_{n}(\mathcal{B}) = r_{i,j}(\mathcal{B})
\]

where \( n \) is the least integer greater than or equal to \( i/(j-1) \). Although this conjecture remains unproved it will be shown in this section to be true when \( n = 2 \) or \( i \leq 8 \) and in many other cases. Lemmas 4.1 and 4.2 will be used to do this.

**Lemma 4.1.** If \( \mathcal{B} \) is any Boolean algebra then \( r_{3,2}(\mathcal{B}) = r_{5,3}(\mathcal{B}) = r_{6,3}(\mathcal{B}) \).

**Proof:** \( \square \)From Theorem 2.1 it follows that in order to show that \( r_{6,3}(\mathcal{B}) \leq r_{3,2}(\mathcal{B}) \) it must be shown that \( P(6,3,3,1) \) fails. This is easy since the colouring of \( \mu(T(1,6)) \) which partitions \( \mu(T(1,6)) \) into three pairs is a counterexample to \( P(6,3,3,1) \).

To show that \( r_{3,2}(\mathcal{B}) \leq r_{5,3}(\mathcal{B}) \) it must be shown that \( P(3,5,2,2) \) fails. Define

\[
\chi : \mu(T(3,3)) \rightarrow 5
\]

by \( \chi(\sigma) = \sum_{i \in 3} \sigma(i) \mod 5 \) and check that this is a counterexample. \( \square \)
Lemma 4.2. If $\mathbb{B}$ is any Boolean algebra then $r_{3,2}(\mathbb{B}) = r_{7,4}(\mathbb{B}) = r_{8,4}(\mathbb{B}) = r_{9,4}(\mathbb{B})$.

Proof: As in Lemma 4.1, to show that $r_{9,4}(\mathbb{B}) \leq r_{3,2}(\mathbb{B})$ it must be shown that $P(9,3,4,1)$ fails. This is easy since the colouring of $\mu(T(1,9))$ which partitions $\mu(T(1,9))$ into three triples is a counterexample to $P(9,3,4,1)$.

To show that $r_{3,2}(\mathbb{B}) \leq r_{9,4}(\mathbb{B})$ it must be shown that $P(3,7,2,3)$ fails. Define

$$\chi : \mu(T(3,3)) \rightarrow 7$$

by $\chi(\sigma) = \sum_{i \in 3} \sigma(i) \mod 7$ and check that this is a counterexample.

That $r_{7,4}(\mathbb{B}) \leq r_{8,4}(\mathbb{B}) \leq r_{9,4}(\mathbb{B})$ follows from Lemma 3.1.■

The truth of the following conjecture, which was mentioned at the beginning of this section can now be established in the cases $n = 2$ or $i \leq 8$: For any pair of integers $i \geq j \geq 2$ and any Boolean algebra $\mathbb{B}$

$$r_n(\mathbb{B}) = r_{i,j}(\mathbb{B})$$

where $n$ is the least integer greater than or equal to $i/(j - 1)$. First, if $n = 2$ this is a direct consequence of Lemma 3.1 (5). If $j = 2$ then the conjecture is a consequence of the definition of $r_{i,2}(\mathbb{B})$. So, if $i \leq 8$ then from Lemma 4.2 it follows that

$$r_{8,4}(\mathbb{B}) = r_{7,4}(\mathbb{B}) = r_{3,2}(\mathbb{B}) = r_{3}(\mathbb{B})$$

while from Lemma 4.1 it follows that

$$r_{5,3}(\mathbb{B}) = r_{6,3}(\mathbb{B}) = r_{3,2}(\mathbb{B}) = r_{3}(\mathbb{B})$$

so only $r_{8,3}(\mathbb{B})$ and $r_{7,3}(\mathbb{B})$ need be considered. The following result of C. Laflamme [5] deals with this case.

Theorem 4.1. If $k \geq 2m - 1$ then $P(m, k, 2, 2)$ fails and hence $r_{m+1}(\mathbb{B}) \leq r_{e,3}(\mathbb{B})$.

The point is that for any Boolean algebra $\mathbb{B}$ the reaping numbers $r_{7,3}(\mathbb{B})$ and $r_{8,3}(\mathbb{B})$ are both less than or equal to $r_{4,2}(\mathbb{B})$ by Lemma 3.1 (5). The opposite inequalities are consequences of Theorem 4.1. Hence $r_{8,3}(\mathbb{B}) = r_{4}(\mathbb{B}) = r_{7,3}(\mathbb{B})$. This completes the first seven row of the following table in which the entry $n$ in row $i$ and column $j$ signifies that $r_{i,j}(\mathbb{B}) = r_{n}(\mathbb{B})$ for every Boolean algebra $\mathbb{B}$.
By using techniques similar in spirit to those of Lemmas 4.1 and 4.2 it is possible to prove the following.

**Lemma 4.3.** If $\mathbb{B}$ is a Boolean algebra then $r_{3,2}(\mathbb{B}) = r_{9,5}(\mathbb{B}) = r_{12,5}(\mathbb{B})$.

This, together with Theorem 4.1 and Lemma 3.1, allow all but one of the entries in the last row of the table to be filled in. The question mark indicates an open problem.

**Question 4.1.** Does there exist a Boolean algebra $\mathbb{B}$ such that $r_{9,4}(\mathbb{B}) \neq r_{3}(\mathbb{B})$?

It is also possible to establish a simple arithmetic condition which is equivalent to $P(m, k, n, 1)$.

**Lemma 4.4.** The property $P(m, k, n, 1)$ fails if and only if $m < kn - k + 1$.

**Proof:** Colour $\mu(T(1, m))$ with $k$ colours in such a way that each colour gets used at most $n - 1$ times. It is possible to do this because $k(n - 1) \geq m$. Any $n$ branching subtree gets at least 2 colours.

Suppose $kn - k + 1 \leq m$. It will be shown by induction on $h \in \omega$ that for any colouring $\chi : \mu(T(h, m)) \to k$ there is a monochromatic $n$-branching subtree; in other words, an $n$-branching subtree $S \subseteq T(h, m)$ such that $\mu(S) \subseteq \mu(T(h, m))$ and $\chi \upharpoonright \mu(S)$ is constant. If $h = 1$ a simple pigeonhole argument can be applied. Otherwise $h = h' + 1$ and the induction hypothesis can be used to find $c(i) \in k$ and an $n$-branching subtree $S_i \subseteq T(h \setminus 1, m)$ such that $\mu(S_i) = \mu(T(h \setminus 1, m))$ and $\chi(\sigma) = c(i)$ for each $\sigma \in \mu(S_i)$. Again use a pigeonhole argument to find $X \in [m]^n$ and $c \in k$ such that $c(i) = c$ for $i \in X$ and then let $S = \{\sigma : \sigma(0) \in X \text{ and } \sigma \upharpoonright h \setminus 1 \in S_{\sigma(0)}\}$.

**Corollary 4.1.** For any integer $k \geq 2$ there is some Boolean algebra $\mathbb{B}$ such that $r_{t+1,2}(\mathbb{B}) \not\leq r_{t,2}(\mathbb{B})$. 
Proof: Let \( n = 2 \) in Lemma 4.4. ■

**Corollary 4.2.** For any integer \( k \geq 2 \) there is some Boolean algebra \( B \) such that \( r_{2k+1,3}(B) \not\leq r_{k,2}(B) \).

Proof: Let \( n = 3 \) in Lemma 4.4. ■

**Corollary 4.3.** There are Boolean algebras \( B_1, B_2 \) and \( B_3 \) such that

- \( r_{9,5}(B) \not\leq r_{2,2}(B) \).
- \( r_{10,4}(B) \not\leq r_{5,2}(B) \).
- \( r_{9,3}(B) \not\leq r_{4,2}(B) \).

Proof: Use Lemma 4.4. ■

It is possible to introduce a partial order \( \prec \) on \( \omega \) by defining \((n, m) \prec (i, j)\) if and only if \( r_{n,m}(B) \leq r_{i,j}(B) \) for every Boolean algebra \( B \). Obviously \( \prec \) is a transitive relation and so it induces a partial order on equivalence classes. As an example, Lemma 4.3 shows that \((3, 2), (5, 3)\) and \((6, 3)\) are all \( \prec \) equivalent, Lemma 3.3 shows that \((2, 2)\) is \( \prec \) minimal and Corollary 4.3 shows that \((9, 3) \not\prec (4, 2)\). There are various open questions about the partial order induced by \( \prec \).

**Question 4.2.** Is the \( \prec \) interval between \((2, 2)\) and \((3, 2)\) empty?

**Question 4.3.** What is the order type of \((\omega \times \omega, \prec)\)?

It is not even known if \((\omega \times \omega, \prec)\) is dense, well founded or linear. The simplest open question about the property \( P(i, j, k, m) \) can be phrased as follows.

**Question 4.4.** Does \((11, 6) \prec (3, 2)\) hold?

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