POSITIVE STRUCTURES IN LIE THEORY

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0.1. In late 19th century and early 20th century, a new branch of mathematics was born: Lie theory or the study of Lie groups and Lie algebras (Lie, Killing, E.Cartan, H.Weyl). It has become a central part of mathematics with applications everywhere. More recent developments in Lie theory are as follows.

- Analogues of simple Lie groups over any field (including finite fields where they explain most of the finite simple groups): Chevalley 1955;
- Infinite dimensional versions of the simple Lie algebras/simple Lie groups: Kac and Moody 1967, Moody and Teo 1972;
- Theory of quantum groups: Drinfeld and Jimbo 1985.

0.2. In Lie theory to any Cartan matrix one can associate a simply connected Lie group \( G(\mathbb{C}) \); Chevalley replaces \( \mathbb{C} \) by any field \( k \) and gets a group \( G(k) \). In [L94] we have defined the totally positive (TP) submonoid \( G(\mathbb{R}_{>0}) \) of \( G(\mathbb{R}) \) and its “upper triangular” part \( U^+(\mathbb{R}_{>0}) \). In this lecture we will review the TP-monoids \( G(\mathbb{R}_{>0}), U^+(\mathbb{R}_{>0}) \) attached to a Cartan matrix, which for simplicity is assumed to be simply-laced. In [L94] the nonsimply laced case is treated by reduction to the simply laced case.

0.3. The total positivity theory in [L94] was a starting point for

- A solution of Arnold’s problem for real flag manifolds, Rietsch 1997;
- The theory of cluster algebras, Fomin, Zelevinsky 2002;
- A theory of TP for the wonderful compactifications, He 2004;
- Higher Teichmüller theory, Fock, Goncharov 2006;
- The use of the TP Grassmannian in physics, Postnikov 2007, Arkani-Hamed, Trnka 2014;
- A theory of TP for the loop group of \( GL_n \), Lam, Pylyavskyy 2012;
- A theory of TP for certain nonsplit real Lie groups, Guichard-Wienerbhard 2018;
- A new approach to certain aspects of quantum groups, Goncharov, Shen.

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0.4. Schoenberg (1930) and Gantmacher-Krein (1935) (after initial contributions of Fekete and Pólya (1912)) defined the notion of TP matrix in $GL_n(R)$: a matrix all of whose $s \times s$ minors are $\geq 0$ for any $s$. Gantmacher and Krein showed that if for any $s$, all $s \times s$ minors of a matrix $A$ are $> 0$ then the eigenvalues of $A$ are real, distinct and $> 0$. For example, the Vandermonde matrix $(A_{ij})$, $A_{ij} = x_i^j - 1$ with $x_1 < x_2 < \cdots < x_n$ real and $> 0$ is of this type. According to Pólya and Szegö, the matrix $(A_{ij})$, $A_{ij} = \exp(x_i y_j)$ with $x_1 > x_2 > \cdots > x_n$, $y_1 > y_2 > \cdots > y_n$ real is also of this type.

The TP matrices in $GL_n(R)$ form a monoid under multiplication. This monoid is generated by diagonal matrices with $> 0$ entries on diagonal and by matrices which have 1 on diagonal and a single nonzero entry off diagonal which is $> 0$ (Whitney, Loewner, 1950’s). Our definition [L94] of the TP part of any $G(R)$ was inspired by the work of Whitney, Loewner.

However, to prove properties of the resulting monoid (such as the generalization of the Gantmacher-Krein theorem) I had to use the canonical bases in quantum groups (discovered in [L90]) and their positivity properties. The role of $s \times s$ minors is played in the general case by the canonical basis of [L90]. Unlike in [L94], here we define $G(R_{>0})$ by generators and relations, independently of $G(R)$. Surprisingly, this definition of $G(R_{>0})$ is simpler than that of $G(R)$ (see [ST]). From it one can recover the Chevalley groups $G(k)$ for any field $k$. Namely, the relations between the generators of $G(R_{>0})$ involve only rational functions with integer coefficients. They make sense over $k$ and they give rise to a “birational form” of a semisimple group over $k$. This is the quotient field of the coordinate ring of $G(k)$; then $G(k)$ itself appears as a subgroup of the automorphism group of this field. In this approach the form $G(R_{>0})$ is the most basic, all other forms are deduced from it.

0.5. We now describe the content of various sections. In §1 we define a positive structure on a set. Such structures have appeared in [L90], [L94] in connection with various objects in Lie theory. In §2 we define the monoid $U^+(R_{>0})$. In §3 we define the monoid $G(R_{>0})$. In §4 we use this monoid to recover the Chevalley groups over a field. In §5 we define the non-negative part of a flag manifold.

1. Positive structures

1.1. The TP monoid can be defined not only over $R_{>0}$ but over a structure $K$ in which addition, multiplication, division (but no subtraction) are defined. In [L94] three such $K$ were considered.

(i) $K = R_{>0}$;

(ii) $K = R(t)_{>0}$, the set of $f \in R(t)$ of form $f = te f_0/f_1$ for some $f_0, f_1$ in $R[t]$ with constant term in $R_{>0}$, $e \in Z$ ($t$ is an indeterminate);

(iii) $K = Z$.

In case (i) and (ii), $K$ is contained in a field $R$ or $R(t)$ and the sum and product
are induced from that field. In case (iii) we consider a new sum \((a, b) \mapsto \min(a, b)\) and a new product \((a, b) \mapsto a + b\). A 4th case is

(iv) \(K = \{1\}\) with \(1 + 1 = 1, 1 \times 1 = 1\).

In each case \(K\) is a semifield (a terminology of Berenstein, Fomin, Zelevinsky 1996): a set with two operations, \(+\), \(\times\), which is an abelian group with respect to \(\times\), an abelian semigroup with respect to + and in which \((a + b)c = ac + bc\) for all \(a, b, c\). We fix a semifield \(K\). There is an obvious semifield homomorphism \(K \rightarrow \{1\}\). We denote by \((1)\) the unit element of \(K\) with respect to \(\times\).

1.2. In [L94] we observed that there is a semifield homomorphism \(\alpha : \mathbb{R}(t)_{>0} \rightarrow \mathbb{Z}\) given by \(t^e f_0 / f_1 \mapsto e\) which connects geometrical objects over \(\mathbb{R}(t)_{>0}\) with piecewise linear objects involving only integers. I believe that this was the first time that such a connection (today known as tropicalization) was used in relation to Lie theory.

1.3. 

For any \(m \in \mathbb{Z}_{>0}\) let \(P_m\) be set of all nonzero polynomials in \(m\) indeterminates \(X_1, X_2, \ldots, X_m\) with coefficients in \(\mathbb{N}\).

A function \((a_1, a_2, \ldots, a_m) \mapsto (a'_1, a'_2, \ldots, a'_m)\) from \(K^m\) to \(K^m\) is said to be admissible if for any \(s\) we have \(a'_s = P_s(a_1, a_2, \ldots, a_m) / Q_s(a_1, a_2, \ldots, a_m)\) where \(P_s, Q_s\) are in \(P_m\). (This ratio makes sense since \(K\) is a semifield.) In the case where \(K = \mathbb{Z}\), such a function is piecewise-linear. If \(m = 0\), the unique map \(K^0 \rightarrow K^0\) is considered to be admissible (\(K^0\) is a point.)

1.4. A positive structure on a set \(X\) consists of a family of bijections \(f_j : K^m \rightarrow X\) (with \(m \geq 0\) fixed) indexed by \(j\) in a finite set \(J\), such that \(f_j^{-1} f_j : K^m \rightarrow K^m\) is admissible for any \(j, j'\) in \(J\); the bijections \(f_j\) are said to be the coordinate charts of the positive structure. The results of [L94], [L97], [L98], can be interpreted as saying that various objects in Lie theory admit natural positive structures.

2. The monoid \(U^+(K)\)

2.1. The Cartan matrix. We fix a finite graph; it is a pair consisting of two finite sets \(I, H\) and a map which to each \(h \in H\) associates a two-element subset \([h]\) of \(I\). The Cartan matrix \(A = (i : j)_{i,j \in I}\) is given by \(i : i = 2\) for all \(i \in I\) while if \(i, j\) in \(I\) are distinct then \(i : j\) is \(-1\) times the number of \(h \in H\) such that \([h] = \{i, j\}\).

An example of a Cartan matrix is:

\[
I = \{i, j\}, A = \begin{pmatrix}
2 & -1 \\
-1 & 2
\end{pmatrix}.
\]

We fix a Cartan matrix \(A\). For applications to Lie theory \(A\) is assumed to be positive definite. But several of the subsequent definitions make sense without this assumption.

We attach to \(A\) and a field \(k\) a group \(G(k)\). When \(A\) is positive definite, \(G(k)\) is the group of \(k\)-points of a simply connected semisimple split algebraic group of
type $A$ over $k$. Without the assumption that $A$ is positive definite, the analogous group $G(k)$ (with $k$ of characteristic 0) has been defined in [MT], [Ma], [Ti].

We will associate to $A$ and $K$ a monoid $G(K)$ and a submonoid $U^+(K)$ of $G(K)$. In the case where $K = \mathbb{R}_{>0}$ (resp. $K = \mathbb{R}(t)_{>0}$), $G(K)$ and $U^+(K)$ can be viewed as submonoids of $G(k)$ where $k = \mathbb{R}$ (resp. $k = \mathbb{R}(t)$). In the case where $K = \mathbb{R}_{>0}, k = \mathbb{R}, G(R) = SL_n(\mathbb{R}), U^+(K)$ is the monoid of TP matrices in $G(\mathbb{R})$ which are upper triangular with 1 on diagonal. We first define $U^+(K)$.

2.2. Let $U^+(K)$ be the monoid (with 1) with generators $i^a$ with $i \in I$, $a \in K$ and relations

\[ i^a i^b = i^a+b \quad \text{for } i \in I, \; a, b \in K; \]

\[ i^a j^b i^c = j^{bc/(a+c)} i^{a+c} j^{ab/(a+c)} \quad \text{for } i, j \in I \text{ with } i : j = -1, \; a, b, c \in K; \]

\[ i^a j^b = j^b i^a \quad \text{for } i, j \in I \text{ with } i : j = 0, \; a, b \in K. \]

(In the case where $K = \mathbb{Z}$, relations of the type considered above involve piecewise-linear functions; they first appeared in [L90] in connection with the parametrization of the canonical basis.) The definition of $U^+(K)$ is reminiscent of the definition of the Coxeter group attached to $A$. In the case where $K = \mathbb{Z}$ and $A$ is positive definite the definition of $U^+(K)$ given above first appeared in [L94, 9.11].

2.3. When $A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$, $K = \mathbb{R}_{>0}$, we can identify $U^+(K)$ with the submonoid of $SL_3(\mathbb{R})$ generated by

\[
\begin{pmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix},
\]

with $a, b$ in $\mathbb{R}_{>0}$.

2.4. Let $W$ be the Coxeter group attached to $A$. It has generators $i$ with $i \in I$ and relations $ii = 1$ for $i \in I$; $iji = jij$ for $i, j \in I$, $i : j = -1$; $ij = ji$ for $i, j \in I$, $i : j = 0$. Let $\mathcal{O}_w$ be the set of reduced expressions of $w$ that is the set of sequences $(i_1, i_2, \ldots, i_m)$ in $I$ such that $i_1 i_2 \ldots i_m = w$ in $U^+(\{1\})$ where $m$ is minimum. We write $m = |w|$ (=length of $w$).

When $K = \{1\}, U^+(K)$ is the monoid (with 1) with generators $i^1$ with $i \in I$ and relations $i^1 i^1 = i^1$ for $i \in I$; $i^1 j^1 i^1 = j^1 i^1 j^1$ for $i, j \in I$, $i : j = -1$; $i^1 j^1 = j^1 i^1$ for $i, j \in I$, $i : j = 0$. By a lemma of Iwahori and Matsumoto we have can identify (as sets) $W = U^+(\{1\})$ by $w = i_1 i_2 \ldots i_m \leftrightarrow i_1^1 i_2^1 \ldots i_m^1$ for any $(i_1, i_2, \ldots, i_m) \in \mathcal{O}_w$. This bijection is not compatible with the monoid structures.

2.5. The semifield homomorphism $K \to \{1\}$ induces a map of monoids $U^+(K) \to U^+(\{1\})$. Let $U_w^+(K)$ be the fibre over $w \in U^+(\{1\})$. We have $U^+(K) = \sqcup_{w \in W} U_w^+(K)$.

We now fix $w \in W$. It turns out that the set $U_w^+(K)$ can be parametrized by $K^m$, in fact in many ways, indexed by $\mathcal{O}_w$. For $i = (i_1, i_2, \ldots, i_m) \in \mathcal{O}_w$ we define $\phi_i : K^m \to U_w^+(K)$ by
\[ \phi_i(a_1, a_2, \ldots, a_m) = i_1^{a_1}i_2^{a_2} \cdots i_m^{a_m}. \]

This is a bijection. Now \( U_+^w(K) \) together with the bijections \( \phi_i : K^m \to U_+^w(K) \) is an example of a positive structure. (We will see later other such structures.)

2.6. Let \( w \in W, m = |w| \). In the case \( K = \mathbb{Z} \), \( U_+^w(N) := \phi_i(N^m) \subset U_+^w(\mathbb{Z}) \) is independent of \( i \in O_w \). We set \( U^+(N) = \sqcup_{w \in W} U_+^w(N) \); this is a subset of \( U^+(\mathbb{Z}) \).

When \( W \) is finite, let \( w_I \) be the element of maximal length of \( W \). Let \( \nu = |w_I| \). Now \( U_+^w(N) \) was interpreted in \([L90]\) as an indexing set for the canonical basis of the plus part of a quantized enveloping algebra. A similar interpretation holds for \( U^+_w(N) \) when \( W \) is not necessarily finite and \( w \) is arbitrary, using \([L96, 8.2]\).

3. The monoid \( G(K) \)

3.1. In order to define the monoid \( G(K) \) we consider besides \( I \), two other copies \(-I = \{ -i ; i \in I \}, \ L = \{ i; i \in I \} \) of \( I \), in obvious bijection with \( I \). For \( \epsilon = \pm 1 \), \( i \in I \) we write \( e_i = i \) if \( \epsilon = 1 \), \( e_i = -i \) if \( \epsilon = -1 \).

Let \( G(K) \) be the monoid (with 1) with generators \( i^a, (-i)^a, \hat{i}^a \) with \( i \in I, a \in K \) and the relations below.

\begin{enumerate}
  \item \( (e_i)^a(e_i)^b = (e_i)^{a+b} \) for \( i \in I, \epsilon = \pm 1, a, b \) in \( K \);
  \item \( (e_i)^a(e_j)^b(e_i)^c = (e_j)^{b/(a+c)}(e_i)^{a+c}(e_j)^{ab/(a+c)} \)
    for \( i, j \in I \) with \( i : j = -1, \epsilon = \pm 1, a, b, c \) in \( K \);
  \item \( (e_i)^a(e_j)^b = (e_j)^b(e_i)^a \)
    for \( i, j \in I \) with \( i : j = 0, \epsilon = \pm 1, a, b \) in \( K \);
  \item \( (e_i)^a(-e_i)^b = (e_i)^{b/(1+ab)}(1+ab)^i(1+ab)^a \)
    for \( i \in I, \epsilon = \pm 1, a, b \) in \( K \);
  \item \( \hat{i}^{a+b} = \hat{i}^{ab}, \hat{i}^{(1)} = 1 \) for \( i \in I, a, b \) in \( K \);
  \item \( \hat{i}^{a+b} = \hat{j}^{b,a} \)
    for \( i, j \in I, a, b \) in \( K \);
  \item \( \hat{j}^{a}(e_i)^b = (e_i)^{a(i)}(j)^b \hat{i}^a \)
    for \( i, j \in I, \epsilon = \pm 1, a, b \) in \( K \);
  \item \( (e_i)^a(-e_j)^b = (-e_j)^b(e_i)^a \)
    for \( i \neq j \in I, \epsilon = \pm 1, a, b \) in \( K \).
\end{enumerate}

3.2. When \( A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \), \( K = \mathbb{R}_{>0} \), we can identify \( G(K) \) with the submonoid of \( SL_3(\mathbb{R}) \) generated by:

\[
\begin{pmatrix}
1 & a & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix},
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & b \\
0 & 0 & 1
\end{pmatrix},
\begin{pmatrix}
1 & 0 & 0 \\
c & 1 & 0 \\
0 & 0 & 1
\end{pmatrix},
\begin{pmatrix}
e & 0 & 0 \\
0 & (1/e) & 0 \\
0 & 0 & 1
\end{pmatrix},
\begin{pmatrix}
e & 0 & 0 \\
0 & f & 0 \\
0 & 0 & (1/f)
\end{pmatrix},
\]

with \( a, b, c, d, e, f \) in \( \mathbb{R}_{>0} \).
3.3. The assignment \( i^a \mapsto i^a \) (with \( i \in I, a \in K \)) defines a monoid isomorphism of \( U^+(K) \) onto a submonoid of \( G(K) \); when \( K = \{1\} \), we denote by \( w \in G(\{1\}) \) the image of \( w \in U(\{1\}) \) under this imbedding. The assignment \( i^a \mapsto (-i)^a \) (with \( i \in I, a \in K \)) defines a monoid isomorphism of \( U^+(K) \) onto a submonoid of \( G(K) \); when \( K = \{1\} \), we denote by \(-w \in G(\{1\}) \) the image of \( w \in U(\{1\}) \) under this imbedding. The map \( W \times W \to G(\{1\}) \), \((w, w') \mapsto w(-w') \) is a bijection of sets (not of monoids).

3.4. Tits has said that \( W \) ought to be regarded as the Chevalley group \( G(k) \) where \( k \) is the (non-existent) field with one element. But \( G(\{1\}) \) is defined for the semifield \( \{1\} \). The bijections \( W \xrightarrow{\sim} U^+(\{1\}) \), \( W \times W \xrightarrow{\sim} G(\{1\}) \) almost realizes the wish of Tits.

3.5. For general \( K \), the semifield homomorphism \( K \to \{1\} \) induces a monoid homomorphism \( G(K) \to G(\{1\}) \). Let \( G_{w,-w'}(K) \) be the fibre over \( w(-w') \) of this homomorphism. We have \( G(K) = \bigsqcup_{(w, w') \in W \times W} G_{w,-w'}(K) \). We now fix \( (w, w') \in W \times W \). Let \( M = |w| + |w'| + r \). It turns out that the set \( G_{w,-w'}(K) \) can be parametrized by \( K^M \), in fact in many ways, indexed by a certain finite set \( \mathcal{O}_{w,-w'} \). Let \( \mathcal{O}_{w,-w'} \) be the set of sequences \((-i_1, -i_2, \ldots, -i_{|w'|})\) in \(-I\) such that \((i_1, i_2, \ldots, i_{|w'|}) \in \mathcal{O}_{w'}\). Let \( \mathcal{O}_{w,-w'} \) be the set of sequences \((h_1, h_2, \ldots, h_M)\) in \( I \subseteq (-I) \sqcup I \) such that the subsequence consisting of symbols in \( I \) is in \( \mathcal{O}_{w} \), the subsequence consisting of symbols in \(-I \) is in \( \mathcal{O}_{-w'} \), the subsequence consisting of symbols in \( I \) contains each symbol \( \hat{i} \) (with \( i \in I \)) exactly once.

For \( h = (h_1, h_2, \ldots, h_M) \in \mathcal{O}_{w,-w'} \) we define \( \psi_h : K^M \to G_{w,-w'}(K) \) by

\[
\psi_h(a_1, a_2, \ldots, a_M) = h_1^{a_1} h_2^{a_2} \ldots h_M^{a_M}.
\]

This is a bijection. The bijections \( \psi_h : K^M \to G_{w,-w'}(K) \) (for various \( h \in \mathcal{O}_{w,-w'} \)) define a positive structure on \( G_{w,-w'}(K) \).

In the case where \( K = \mathbf{R}_{>0} \) or \( K = \mathbf{R}(t)_{>0} \), the statements above are proved by using Bruhat decomposition in the group \( G(k) \) considered in 2.1 with \( k = \mathbf{R} \) or \( \mathbf{R}(t) \). (When \( W \) is finite this is done in [L19]. See also the proof of [L94, Lemma 2.3] and [L94, 2.7].) The case where \( K = \mathbf{Z} \) follows from the case where \( K = \mathbf{R}(t)_{>0} \), using \( \alpha : \mathbf{R}(t)_{>0} \to \mathbf{Z} \) in 1.2.

4. Chevalley groups

4.1. In this section we assume that \( K = \mathbf{R}_{>0} \) and that \( I \neq \emptyset \). Let \( k_0 \) be a field and let \( k \) be an algebraic closure of \( k_0 \).

Let \( w \in W, w' \in W \). Let \( M = |w| + |w'| + r \). For \( h, h' \) in \( \mathcal{O}_{w,-w'} \), \( \psi_h^{-1} \psi_{h'} : K^M \to K^M \) (see 3.5) is of the form \((a_1, a_2, \ldots, a_M) \mapsto (a_1', a_2', \ldots, a_M')\) where \( a_i' = (P_h')_s(a_1, a_2, \ldots, a_M)/(Q_h')_s(a_1, a_2, \ldots, a_M) \) and each of \((P_h')_s, (Q_h')_s \) is a nonzero polynomial in \( \mathbf{N}[X_1, X_2, \ldots, X_M] \) (independent of \( K \)) such that the g.c.d. of its \( \neq 0 \) coeff. is 1.
Applying the obvious ring homomorphism \( \mathbb{Z} \to k_0 \) to the coefficients of these polynomials we obtain \( \neq 0 \) polynomials \((\bar{P}_h')_s, (\bar{Q}_h')_s \) in \( k_0[X_1, X_2, \ldots, X_M] \). We define a rational map \( \psi_h' : k^M \to k^M \) by
\[
(z_1, z_2, \ldots, z_M) \mapsto (z'_1, z'_2, \ldots, z'_M),
\]
\[
z'_s = (\bar{P}_h')_s(z_1, z_2, \ldots, z_M)/(\bar{Q}_h')_s(z_1, z_2, \ldots, z_M)
\]
This is a birational isomorphism. It induces an automorphism \([\psi_h']\) of the quotient field \([k^M]\) of the coordinate ring of \( k^M \). We have \([\psi_h'][\psi_h'''] = [\psi_h''']\) for any \( h, h', h'' \). Hence there is a well defined field \([G_w, -w'(k)] \) containing \( k \) with \( k \)-linear field isomorphisms \([\psi_h] : [G_w, -w'(k)] \to [k^M] \) (for \( h \in O_{w, -w'} \)) such that
\[
[\psi_h'] = [\psi_h][\psi_h']^{-1} : [k^M] \to [k^M]
\]
for all \( h, h' \).

4.2. We now assume that \( W \) is finite. Let \( w, \nu \) be as in 2.6. Let \( M = 2\nu + r \).

Let \( i \in I, \epsilon = \pm 1, z \in k_0 \). We can choose \( h = (h_1, h_2, \ldots, h_M) \in O_{w, -w} \) such that \( h_1 = \epsilon i \). The isomorphism \( k^M \to k^M, (z_1, z_2, \ldots, z_M) \mapsto (z_1 - z, z_2, \ldots, z_M) \) induces a field isomorphism \( \tau_z : [k^M] \to [k^M] \). Let \( A \) be the group of all \( k \)-linear field automorphisms of \([G_w, -w(k)]\). We define \((\epsilon i)^z \in A \) as the composition
\[
[G_w, -w(k)] \xrightarrow{[\psi_h]} [k^M] \xrightarrow{\tau_z} [k^M] \xrightarrow{[\psi_h]^{-1}} [G_w, -w(k)].
\]
Now \((\epsilon i)^z \) is independent of the choice of \( h \). Let \( G(k_0) \) be the subgroup of \( A \) generated by \((\epsilon i)^z \) for various \( i \in I, \epsilon = \pm 1, z \in k_0 \). Then \( G(k_0) \) is the Chevalley group associated to \( k_0 \) and our Cartan matrix.

5. Flag manifolds

5.1. In this section \( W \) is not assumed to be finite. We assume that \( K \) is \( R_{>0} \).

Let \( G(R) \) be the group considered in 2.1. Let \( V \) be an \( R \)-vector space which is an irreducible highest weight integrable representation of \( G(R) \) whose highest weight takes the value 1 at any simple coroot. Let \( \eta \) be a highest weight vector of \( V \). Let \( B \) be the canonical basis of \( V \) (see [L93, 11.10]) containing \( \eta \). Let \( P \) be the set of lines in the \( R \)-vector space \( V \). Let \( P_{\geq 0} \) be the set of all \( L \in P \) such that for some \( x \in L - \{0\} \) all coordinates of \( x \) with respect to the basis \( B \) are \( \geq 0 \). The flag manifold \( B \) of \( G(R) \) is defined as the subset of \( P \) consisting of lines in the \( G(R) \)-orbit of the line spanned by \( \eta \). We define \( B(K) = B \cap P_{\geq 0} \). By a positivity property [L93, 22.1.7] of \( B \) (stated in the simply laced case but whose proof remains valid in our case), the obvious \( G(R) \)-action on \( B \) restricts to a \( G(K) \)-action on \( B(K) \).

(As mentioned in 2.1, \( G(K) \) can be viewed as a submonoid of \( G(R) \).) When \( W \) is finite, \( B(K) \) is the same as the subset \( B_{\geq 0} \) defined in [L94, §8]. (This follows from [L94, 8.17].)

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