Generalized Chromatic Functions

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We define vertex-colourings for edge-partitioned digraphs, which unify the theory of $P$-partitions and proper vertex-colourings of graphs. We use our vertex-colourings to define generalized chromatic functions, which merge the chromatic symmetric and quasisymmetric functions of graphs and generating functions of $P$-partitions. Moreover, numerous classical bases of symmetric and quasisymmetric functions, both in commuting and noncommuting variables, can be realized as special cases of our generalized chromatic functions. We also establish product and coproduct formulas for our functions. Additionally, we construct the new Hopf algebra of $r$-quasisymmetric functions in noncommuting variables, and apply our functions to confirm its Hopf structure, and establish natural bases for it.

1 Introduction

The story of the theory of $P$-partitions started with MacMahon’s work \cite{18} on plane partitions at the start of the 20th century, and 60 years later Stanley, in his Ph.D. thesis \cite{25}, extended the notion of plane partitions to $P$-partitions (for a more complete history, see \cite[pp. 169–188]{11}). The problem that MacMahon considered in his work on plane partitions was the same as counting the number of fillings of Young diagrams with nonnegative integers with a given sum such that the entries are weakly decreasing in...
Stanley, in his work on the theory of $P$-partitions [25], generalized MacMahon’s idea and replaced Young diagrams with posets and the weakly decreasing relation with weakly and strictly decreasing relations. In this paper, we generalize Stanley’s $P$-partitions to certain vertex-colourings of digraphs whose edges are coloured with three colours. Roughly speaking, we replace posets in the theory of $P$-partitions with digraphs, and in addition to the weakly and strictly decreasing relations, we have another relation related to the proper colourings of digraphs. More precisely, we define a proper vertex-colouring of a digraph whose edges are coloured by three colours, identified by $\rightarrow$, $\to$, and $\Rightarrow$, to be a function $\kappa : V(G) \to \mathbb{P}$, where $\mathbb{P}$ is the set of positive integers such that:

(i) If $a \rightarrow b$ in $G$, then $\kappa(a) \neq \kappa(b)$.
(ii) If $a \to b$ in $G$, then $\kappa(a) < \kappa(b)$.
(iii) If $a \Rightarrow b$ in $G$, then $\kappa(a) \leq \kappa(b)$.

Let $C(G)$ denote the set of all proper vertex-colourings of the edge-coloured digraph $G$.

In the above definition, by replacing the infinite paintbox of colours $\mathbb{P}$ with a finite paintbox of colours $\mathbb{P} = \{1, 2, \ldots, p\}$, we generalize the classic proper vertex-colouring and weak/strong proper colouring in [22]. The generalized chromatic number of an edge-coloured digraph $G$ is the smallest number of colours we need to make a proper vertex-colouring of $G$. Let $P(G, p)$ denote the number of ways that one can properly colour the vertices of $G$ with $p$ colours. We show that $P(G, p)$ is a polynomial in $p$ (see Theorem 7.2) and call it the generalized chromatic polynomial of $G$.

Stanley in [24] defined the chromatic symmetric function of a finite simple graph. This symmetric function gives Birkhoff’s chromatic polynomial by setting the first $p$
variables to 1 and all others to 0. There are two main conjectures regarding chromatic
symmetric functions that have been open for more than 25 years—the Tree Conjecture
[24] and the $(3+1)$-free Conjecture [26]. Later on, refining the conjectures and seeking
classical properties, other chromatic symmetric and quasisymmetric functions emerged,
such as the chromatic quasisymmetric functions of graphs and digraphs [7, 21], the
extended chromatic symmetric function of a graph [6], and the $k$-balanced chromatic
quasisymmetric function of a graph [16].

Let $\mathbb{Q}[[x_1, x_2, \ldots]]$ be the set of all power series in commuting variables
$x = \{x_1, x_2, \ldots\}$. We define the generalized chromatic function of an edge-coloured digraph
$G$ to be the bounded degree power series

$$\mathcal{X}_G(x) = \sum_{\kappa \in \mathcal{C}(G)} x_\kappa,$$

where $x_\kappa = \prod_{a \in V(G)} x_{\kappa(a)}$. Note that the generalized chromatic polynomial $P(G, p)$ is equal
to the generalized chromatic function of $G$ after setting the first $p$ variables to 1 and all
others to 0. In the theory of $P$-partitions, a labelled poset $P$ corresponds to a quasisym-
metric function $F_p$ [23]. Since each $P$-partition is a proper vertex-colouring of an edge-
coloured digraph (see Section 3), we can view $F_p$ as a generalized chromatic function.
Moreover, Stanley’s chromatic symmetric functions [24], extended chromatic symmetric
functions [6], and chromatic quasisymmetric functions in [7, 21] are the generalized
chromatic functions of certain edge-coloured digraphs (see Section 5). Therefore, we can
merge all of these chromatic functions and the generating functions of $P$-partitions, $F_p$,
into one object.

As we will see, generalized chromatic functions are quasisymmetric functions.
There are many well-known bases for the Hopf algebras of symmetric and quasisymmet-
ric functions. Any of these bases can be expressed as a family of generalized chromatic
functions. This offers a significant advantage because if one finds a product, coproduct,
antipode, etc., formula for generalized chromatic functions, then this impacts knowledge
about all these bases of the Hopf algebras of symmetric and quasisymmetric functions.
For example, we give a generic coproduct formula for generalized chromatic functions.

We also study a chain of Hopf algebras starting from symmetric functions and
ending with quasisymmetric functions. We present different bases for the Hopf algebras
in the chain using generalized chromatic functions.

We then change gears to the noncommuting world and define $\mathcal{Y}_G(x)$, the
generalized chromatic function of a labelled edge-coloured digraph $G$ in noncommuting
variables $\mathbf{x} = \{x_1, x_2, \ldots\}$. We extend most of the results in commuting variables to noncommuting variables.

Moreover, we naturally expand the fundamental basis of the Malvenuto-Reutenauer Hopf algebra to the fundamental basis of the Hopf algebra of quasisymmetric functions in noncommuting variables.

The Hopf algebra of $r$-quasisymmetric functions was defined in [14] by Hivert. In [9], Garsia and Wallach showed that the algebra of $r$-quasisymmetric functions is free over symmetric functions. In the last section, we introduce the Hopf algebra of $r$-quasisymmetric functions in noncommuting variables.

More precisely, our paper is structured as follows. In Section 2, we recall the background of symmetric, quasisymmetric, and $r$-quasisymmetric functions. In Section 3, we present some basic definitions in graph theory and then define some operators between edge-coloured digraphs. The vertex-colouring of an edge-coloured digraph is defined in Definition 3.1. We conclude Section 3 by showing that every $P$-partition is the proper vertex-colouring of an edge-coloured digraph in Proposition 3.3. In Section 4, we introduce generalized chromatic functions in Definition 4.1 and show that they are quasisymmetric functions, and then in Section 5, show that other chromatic symmetric and quasisymmetric functions are special cases of generalized chromatic functions. In Section 6, the product and coproduct formulas for generalized chromatic functions are presented in Proposition 6.1 and Theorem 6.3. In Section 7, we show that many well-known bases of symmetric, quasisymmetric, and $r$-quasisymmetric functions can be realized as special cases of generalized chromatic functions of edge-coloured digraphs. Moreover, in Theorem 7.2, we show that the generalized chromatic polynomial of an edge-coloured digraph is indeed a polynomial. In Section 8, we recall the background of symmetric and quasisymmetric functions in noncommuting variables and then define $r$-quasisymmetric functions in noncommuting variables. In Section 9, we introduce generalized chromatic symmetric functions in noncommuting variables in Definition 9.1, and the product and coproduct formulas for them are presented in Propositions 10.1 and 10.2. In Sections 11 and 12, we show that several bases for symmetric functions in noncommuting variables are the symmetrizations of certain generalized chromatic functions and give several bases for quasisymmetric functions in noncommuting variables, including its fundamental basis, which contains the fundamental basis of the Malvenuto-Reutenauer Hopf algebra. We conclude by showing that the set of $r$-quasisymmetric functions in noncommuting variables is a Hopf algebra in Theorem 13.1 and constructing the $r$-dominant monomial and upper-fundamental bases of the Hopf algebra of $r$-quasisymmetric functions in noncommuting variables in Proposition 13.2.
2 Symmetric Functions and Generalizations

This section introduces the Hopf algebras of symmetric, quasisymmetric, and \( r \)-quasisymmetric functions. The bases of these Hopf algebras are indexed by partitions, compositions, and \( r \)-compositions, respectively. We begin by recalling the definitions and notation related to these combinatorial objects.

2.1 Partitions, compositions, and \( r \)-compositions

A composition \( \alpha \) of \( n \), denoted \( \alpha \models n \), is a list of positive integers whose sum is \( n \). Given a composition \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k) \), each \( \alpha_i \) is called a part of \( \alpha \), the size of \( \alpha \) is \(|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_k \), and the length of \( \alpha \) is \( k \). For convenience, we denote by \( \emptyset \) the unique composition of size and length zero. If \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k) \models n \), then we define

\[
\text{set}(\alpha) = \{\alpha_1, \alpha_1 + \alpha_2, \ldots, \alpha_1 + \cdots + \alpha_{k-1}\} \subseteq [n - 1].
\]

For example, \((2,1,2)\) is a composition of 5 with length 3 and \( \text{set}(\alpha) = \{2,3\} \). For compositions \( \alpha \) and \( \beta \) of \( n \), we write \( \alpha \leq \beta \) and say \( \alpha \) coarsens \( \beta \) (or \( \beta \) refines \( \alpha \)) if \( \text{set}(\alpha) \subseteq \text{set}(\beta) \).

A partition \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \) of \( n \), denoted \( \lambda \vdash n \), is a weakly decreasing composition. Let \( m_i \) be the number of parts of \( \lambda \) that are equal to \( i \). Let \( \lambda^! = m_1! m_2! \cdots m_n! \), and let \( \lambda^! = \lambda_1! \lambda_2! \cdots \lambda_k! \). We sometimes write

\[
\lambda = (n^{m_n}, (n - 1)^{m_{n-1}}, \ldots, 1^{m_1}).
\]

The dominance order of partitions is defined as follows. For partitions \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \) and \( \mu = (\mu_1, \mu_2, \ldots, \mu_l) \) of \( n \), we write \( \mu \leq \lambda \) if \( k \leq l \) and for every \( 1 \leq i \leq k \),

\[
\mu_1 + \mu_2 + \cdots + \mu_i \leq \lambda_1 + \lambda_2 + \cdots + \lambda_i.
\]

For example, \((3,1,1,1) \leq (3,2,1) \).

Let \( r \) be a positive integer or infinity. A pair \( (\beta, \mu) \) is called an \( r \)-composition of \( |\beta| + |\mu| \) when:

(i) \( \beta \) is a composition whose parts are at least \( r \).

(ii) \( \mu \) is a partition whose parts are strictly smaller than \( r \).

For example, \(((7,4,5), (3,2))\) is a 4-composition of 21.
2.2 Quasisymmetric functions

The Hopf algebra of quasisymmetric functions was formally introduced by Gessel [12] in 1984. From this concept, a whole research area emerged; a history can be found in [17, Introduction].

Recall that \( \mathbb{Q}[[x_1, x_2, \ldots]] \) is the algebra of formal power series in infinitely many commuting variables \( x = \{x_1, x_2, \ldots\} \) over \( \mathbb{Q} \). Let \( S_n \) be the group of all permutations of \( [n] \). Let \( S_\infty = \bigsqcup_{n \geq 0} S_n \). We identify a permutation \( \sigma \in S_n \subseteq S_\infty \) with a bijection of the positive integers by defining \( \sigma(i) = i \) if \( i > n \).

**Definition 2.1.** A quasisymmetric function is a formal power series \( f \in \mathbb{Q}[[x_1, x_2, \ldots]] \) such that:

(i) The degrees of the monomials in \( f \) are bounded.

(ii) For every composition \( (\alpha_1, \alpha_2, \ldots, \alpha_k) \), all monomials \( x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_k}^{\alpha_k} \) in \( f \) with indices \( i_1 < i_2 < \cdots < i_k \) have the same coefficient.

The set of all quasisymmetric functions is denoted by \( \text{QSym}(x) \).

The vector space \( \text{QSym}(x) \) is a Hopf algebra, where its product is the same as the product of the formal power series and its coproduct \( \Delta \) is defined as follows (for more details, see [13, p. 142]). Consider the linear order on two sets of commuting variables \( (x, y) = (x_1 < x_2 < \cdots < y_1 < y_2 < \cdots) \), and inject \( \text{QSym}(x) \otimes \text{QSym}(y) \) into \( \mathbb{Q}[[x, y]] \) by identifying every \( f \otimes g \in \text{QSym}(x) \otimes \text{QSym}(y) \) with \( fg \in \mathbb{Q}[[x, y]] \). We then have that

\[
\text{QSym}(x, y) \subseteq \text{QSym}(x) \otimes \text{QSym}(y).
\]

We can define \( \Delta : \text{QSym}(x) \to \text{QSym}(x) \otimes \text{QSym}(x) \) as the composite of the following maps:

\[
\text{QSym}(x) \cong \text{QSym}(x, y) \to \text{QSym}(x) \otimes \text{QSym}(y) \cong \text{QSym}(x) \otimes \text{QSym}(x)
\]

\[
f \mapsto f(x_1, x_2, \ldots, y_1, y_2, \ldots)
\]

For more details about the Hopf algebra of quasisymmetric functions and its well-known bases, see [23].

2.3 Symmetric functions

The Hopf algebra of symmetric functions is a Hopf subalgebra of \( \text{QSym}(x) \).

**Definition 2.2.** A symmetric function is a formal power series \( f \in \mathbb{Q}[[x_1, x_2, \ldots]] \) such that:

(i) The degrees of the monomials in \( f \) are bounded.
(ii) For any permutation \( \sigma \in \mathfrak{S}_\infty \),

\[ \sigma \cdot f(x_1, x_2, \ldots) = f(x_{\sigma(1)}, x_{\sigma(2)}, \ldots) = f(x_1, x_2, \ldots). \]

The set of all symmetric functions is denoted by \( \text{Sym}(x) \).

For more details about the Hopf algebra of symmetric functions and its well-known bases, see [23].

2.4 \( r \)-quasisymmetric functions

Hivert introduced the Hopf algebra of \( r \)-quasisymmetric functions in [14], which is a Hopf subalgebra of \( \text{QSym}(x) \).

For each \( r \)-composition \((\beta, \mu)\) where \( \beta = (\beta_1, \beta_2, \ldots, \beta_k) \) and \( \mu = (\mu_1, \mu_2, \ldots, \mu_l) \), define the \( r \)-dominant monomial function to be

\[ M_{(\beta, \mu)} = \sum x_{i_1}^{\beta_1} x_{i_2}^{\beta_2} \cdots x_{i_k}^{\beta_k} x_{i_{k+1}}^{\mu_1} x_{i_{k+2}}^{\mu_2} \cdots x_{i_{k+l}}^{\mu_l}, \]

where the sum is over all distinct positive integers \( i_1, i_2, \ldots, i_{k+l} \) such that only the first \( k \) indices are required to be in strictly increasing order, that is, \( i_1 < i_2 < \cdots < i_k \). Define

\[ \text{QSym}^r(x) = \bigoplus_{n \geq 0} \text{QSym}^r_n(x), \]

where

\[ \text{QSym}^r_n(x) = \mathbb{Q}\text{-span}\{M_{(\beta, \mu)} : (\beta, \mu) \text{ is an } r \text{-composition of } n\}. \]

We have that

\[ \text{QSym}(x) = \text{QSym}^1(x) \supset \text{QSym}^2(x) \supset \cdots \supset \text{QSym}^\infty(x) = \text{Sym}(x). \]

In Section 7, we present different bases for the Hopf algebras in this chain.

3 Proper Colourings

In graph theory, there are many families of graph colourings. These colourings are usually defined by setting specific constraints on the colours of vertices or edges of a graph. In this paper, we consider particular vertex-colourings of a digraph to unify several combinatorial constructions. In our colouring, the constraints on the colours of
the vertices are subject to an edge-colouring of the digraph. We recall the definitions and notation in graph theory that we need throughout this paper.

A simple digraph \( G = (V(G), E(G)) \) is a digraph with no loops, and for any distinct vertices \( a \) and \( b \), there can be at most one directed edge from \( a \) to \( b \). Note that for distinct vertices \( a \) and \( b \) of a simple digraph, we can have two edges one from \( a \) to \( b \) and one from \( b \) to \( a \). Throughout this paper, all digraphs are finite and simple, and we usually use \( G \) to denote a simple digraph. The underlying graph of \( G \) is the undirected graph whose vertex set is the vertex set of \( G \) and its edge set is \( \{ab : (a, b) \text{ or } (b, a) \text{ is an edge of } G \} \). A directed cycle is a digraph \( G \) with the vertex set \( V(G) = \{a_1, a_2, \ldots, a_n\} \) and the edge set \( E(G) = \{(a_i, a_{i+1}) : i = 1, 2, \ldots, n-1\} \cup \{(a_n, a_1)\} \). A directed path is a digraph \( G \) with the vertex set \( V(G) = \{a_1, a_2, \ldots, a_n\} \) and the edge set \( E(G) = \{(a_i, a_{i+1}) : i = 1, 2, \ldots, n-1\} \). A complete digraph is a digraph \( G \) with the vertex set \( V(G) = \{a_1, a_2, \ldots, a_n\} \) and the edge set \( E(G) = \{(a_i, a_j) : 1 \leq i < j \leq n\} \).

Let \( S \) and \( S' \) be subsets of \( \mathbb{P} \), the set of positive integers. We regard the elements in \( S \) and \( S' \) as colours. Let \( G \) be a digraph. An \( S \)-vertex-colouring of \( G \) is a function \( \kappa \) that assigns a colour in \( S \) to each vertex of the digraph \( G \). By a vertex-colouring, without mentioning the set \( S \), we mean a \( \mathbb{P} \)-vertex-colouring. An \( S' \)-edge-colouring of \( G \) is a function \( \kappa' \) that assigns a colour in \( S' \) to each edge of the digraph \( G \). An \( S' \)-edge-coloured digraph is a digraph \( G \) together with an \( S' \)-edge-colouring \( \kappa' \) of \( G \). Throughout this paper, we only consider \( S' \)-edge-colourings of digraphs where \( |S'| = 3 \), so we only have three types of coloured edges in a digraph, which are denoted by \( \rightarrow, \rightarrow', \text{and } \Rightarrow \).

Consider edge-coloured digraphs \( G_1 \) with the edge-colouring \( \kappa_1' \) and \( G_2 \) with the edge-colouring \( \kappa_2' \). The disjoint union of the edge-coloured digraphs \( G_1 \) and \( G_2 \), denoted \( G_1 \cup G_2 \), is an edge-coloured digraph \( G \) together with an edge-colouring \( \kappa' \) such that:

(i) The vertex set of \( G \) is the disjoint union of the vertex sets of \( G_1 \) and \( G_2 \).
(ii) The edge set of \( G \) is the disjoint union of the edge sets of \( G_1 \) and \( G_2 \).
(iii) \( \kappa'(e) = \kappa_j'(e) \) if \( e \in E(G_j) \) for \( j = 1, 2 \).
Also, the \textit{dashed (solid and double, respectively) sum} of the edge-coloured digraphs $G_1$ and $G_2$ is an edge-coloured digraph $G$ together with an edge-colouring $\kappa'$ such that:

(i) The vertex set of $G$ is the disjoint union of the vertex sets of $G_1$ and $G_2$.

(ii) The edge set of $G$ is the disjoint union of the edge set of $G_1$, edge set of $G_2$, and $\{(a, b) : a$ is a vertex of $G_1$ and $b$ is a vertex of $G_2\}$.

(iii) For every edge $e = (a, b)$ in $E(G)$, $\kappa'(e) = \kappa'_j(e)$ if $e \in E(G_j)$ for $j = 1, 2$, and if $a$ is a vertex of $G_1$ and $b$ is a vertex of $G_2$, then $e$ is a dashed (solid and double, respectively) edge.

The dashed, solid, and double sums of the edge-coloured digraphs $G_1$ and $G_2$ are denoted by

$$G_1 \bigcirc \bigcirc G_2, \quad G_1 \bigotimes G_2, \quad \text{and} \quad G_1 \boxdot G_2,$$

respectively.

We frequently use the edge-coloured digraphs in the following table.
Table 1  Useful edge-coloured digraphs.

| Notation | Expression                                      |
|----------|-------------------------------------------------|
| $C_n$    | The directed cycle with $n$ vertices and double edges |
| $P_n$    | The directed path with $n$ vertices and solid edges |
| $Q_n$    | The directed path with $n$ vertices and double edges |
| $K_n$    | The complete digraph with $n$ vertices and dashed edges |

Fig. 5. The edge-coloured digraphs $C_3, P_3, Q_3, K_3$.

Let $(P, <_P)$ be a poset. The *digraph associated to $P$*, denoted $G_P$, is a digraph whose vertex set is the elements of $P$ and $(a, b)$ is an edge of $G_P$ if and only if $b$ covers $a$ in $P$; that is, $a <_P b$ and there is no $c \in P$ such that $a <_P c <_P b$. Note that the underlying graph of $G_P$ gives the Hasse diagram of $P$.

We now define our vertex-colouring of an edge-coloured digraph $G$, which plays an essential role in this paper.

**Definition 3.1.** A proper vertex-colouring of an edge-coloured digraph $G$ is a function $\kappa$ from $V(G)$ to $\mathbb{P}$ such that for any edge $(a, b)$ in $E(G)$:

(i) If $a \rightarrow b$, then $\kappa(a) \neq \kappa(b)$.
(ii) If $a \rightarrow b$, then $\kappa(a) < \kappa(b)$.
(iii) If $a \Rightarrow b$, then $\kappa(a) \leq \kappa(b)$.

Let $C(G)$ denote the set of all proper vertex-colourings of the edge-coloured digraph $G$.

**Remark 3.2.** Note that for some edge-coloured digraphs $G$, $C(G)$ is empty.

We next see that each $P$-partition corresponds to a proper vertex-colouring of $G_P$ where its edges are solid or double.
Proper colourings and $P$-partitions. A *labelled poset* is a partially ordered set $P$ whose underlying set is some finite subset of positive integers. A *$P$-partition* is a function $f : P \rightarrow \mathbb{P}$ such that for $a, b \in P$ with $a <_P b$:

(i) If $a <_Z b$, then $f(a) \leq f(b)$.
(ii) If $a >_Z b$, then $f(a) < f(b)$.

The definitions of $P$-partitions and proper vertex-colourings of edge-coloured digraphs yield the following proposition.

**Proposition 3.3.** Let $P$ be a poset, and let $G_P$ be the digraph associated to $P$. Then there is a bijection between the set of all $P$-partitions and the set of all proper colourings of the edge-coloured digraph $G$ where the digraph $G$ is isomorphic to $G_P$ and moreover:

(i) $a \Rightarrow b$ in $G$ if $a <_P b$ and $a <_Z b$.
(ii) $a \rightarrow b$ in $G$ if $a <_P b$ and $a >_Z b$.

4 Generalized Chromatic Functions

Throughout this paper, if $\kappa$ is any type of colouring of a graph, digraph, or edge-coloured digraph with vertex set $V$, we define

$$x_{\kappa} = \prod_{a \in V} x_{\kappa(a)}. \quad (1)$$

We now introduce our main object of study.

**Definition 4.1.** (The generalized chromatic function of an edge-coloured digraph) Let $G$ be an edge-coloured digraph. Define

$$\mathcal{X}_G(x, t) = \sum_{\kappa \in \mathcal{C}(G)} t^{ASC(\kappa)} x_{\kappa},$$
where $x_\kappa$ is defined by Equation 1 and

$$\text{asc}(\kappa) = |\{(a, b) \in E(G) : \kappa(a) < \kappa(b)\}|.$$ 

The power series

$$\mathcal{X}(x) = \mathcal{X}(x, 1)$$

is called the generalized chromatic function of $G$.

**Remark 4.2.** Note that $\mathcal{X}(x, t) = 0$ when $C(G) = \emptyset$.

If $G$ is an edge-coloured digraph and $a \rightarrow b$ is an edge of $G$, then the generalized chromatic function of $G$ is the sum of the generalized chromatic functions of two edge-coloured digraphs, one is obtained by deleting $a \rightarrow b$ in $G$ and replacing it with $a \rightarrow b$ and the other is obtained by deleting $a \rightarrow b$ in $G$ and replacing it with $b \rightarrow a$; that is,

$$\mathcal{X}(x) = \mathcal{X}_{G-(a \rightarrow b)+(b \rightarrow a)}(x) + \mathcal{X}_{G-(a \rightarrow b)+(a \rightarrow b)}(x).$$

For vertices $a$ and $b$ of $G$, write $a \sim b$ if $a = b$ or there is a directed cycle $C$ with double edges in $G$ and $a, b \in V(C)$. The transitive closure of this relation gives an equivalence relation on the vertices of $G$. Let $[a_1], [a_2], \ldots, [a_s]$ be all equivalence classes. We have

$$\mathcal{X}(x) = \sum_{\kappa \in \mathcal{C}(G)} x^{[a_1]}_{\kappa(a_1)} x^{[a_2]}_{\kappa(a_2)} \cdots x^{[a_s]}_{\kappa(a_s)}.$$ 

By the above equation we can realize each generalized chromatic function as a sum of generating functions of weighted $P$-partitions [2, Section 3], and so the generalized chromatic function of an edge-coloured digraph is a quasisymmetric function, which we now state as a proposition.

**Proposition 4.3.** Let $G$ be an edge-coloured digraph. Then $\mathcal{X}(x) \in \text{QSym}(x)$.

For example, if $G$ is the following edge-coloured digraph

```
  \begin{center}
  \begin{tikzpicture}
    \node (a1) at (0,0) {$a_1$};
    \node (a2) at (1,0) {$a_2$};
    \node (a3) at (2,0) {$a_3$};
    \node (a4) at (3,0) {$a_4$};
    \node (a5) at (4,0) {$a_5$};
    \node (a6) at (5,0) {$a_6$};
    \node (a7) at (6,0) {$a_7$};
    \draw[->] (a1) to (a2);
    \draw[->] (a2) to (a3);
    \draw[->] (a3) to (a4);
    \draw[->] (a4) to (a5);
    \draw[->] (a5) to (a6);
    \draw[->] (a6) to (a7);
  \end{tikzpicture}
\end{center}
```
then \([a_1] = \{a_1, a_2\}, [a_3] = \{a_3\}, \text{and } [a_4] = \{a_4, a_5, a_6, a_7\}\) are the equivalence classes, and

\[
\mathcal{X}_G(x) = \mathcal{Y}_{G-(a_2 \rightarrow a_3) + (a_2 \rightarrow a_3)}(x) + \mathcal{Y}_{G-(a_2 \rightarrow a_3) + (a_3 \rightarrow a_2)}(x).
\]

5 Other Chromatic Functions

Let \(H = (V(H), E(H))\) be a finite simple graph. Throughout this paper, all graphs are finite and simple, and we usually use \(H\) to denote a simple graph. A proper vertex-colouring of \(H\) is a function \(\kappa\) from \(V(H)\) to \(\mathbb{P}\) such that if the vertices \(a\) and \(b\) are adjacent, then \(\kappa(a) \neq \kappa(b)\). The set of all proper vertex-colourings of \(H\) is denoted by \(C(H)\).

Stanley’s chromatic symmetric function [24]: For a graph \(H\), the chromatic symmetric function of \(H\) is

\[
X_H(x) = \sum_{\kappa \in C(H)} x_\kappa,
\]

where \(x_\kappa\) is defined by Equation 1.

By definition we have the following.

Proposition 5.1. Let \(H\) be a graph. Then,

\[
X_H(x) = \mathcal{Y}_G(x),
\]

where \(G\) is any edge-coloured digraph that has only dashed edges and its underlying graph is isomorphic to \(H\).

Crew-Spirkl’s extended chromatic symmetric function [6]: A weighted graph is a pair \((H, wt)\) where \(H\) is a graph and \(wt : V(H) \rightarrow \mathbb{P}\) is a vertex-weight function. The extended chromatic symmetric function of \((H, wt)\) is

\[
X_{(H, wt)}(x) = \sum_{\kappa \in C(H)} x_{\kappa}^{wt},
\]

where

\[
x_{\kappa}^{wt} = \prod_{a \in V(H)} x_{\kappa(a)}^{wt(a)}.
\]

By definition we have the following.
Proposition 5.2. Let \((H, wt)\) be a weighted graph. For each vertex \(a\) in \(V(H)\), consider a cycle \(C_a\) isomorphic to \(C_{wt(a)}\), and let \(v(a)\) be an arbitrary vertex of \(C_a\). Define \(G\) to be an edge-coloured digraph with the vertex set
\[
\bigcup_{a \in V(H)} V(C_a)
\]
and the edge set
\[
\bigcup_{a \in V(H)} E(C_a) \cup \{v(a) \rightarrow v(b) : ab \in E(H)\}.
\]
Then,
\[
X_{(H, wt)}(x) = \mathcal{X}_G(x).
\]

Shareshian-Wachs’ chromatic quasisymmetric function [21]: For a graph \(H\) with \(V(H)\) a subset of \(\mathbb{P}\), the chromatic quasisymmetric function of \(H\) is
\[
X_H(x, t) = \sum_{\kappa \in C(H)} t^{asc(\kappa)} x_\kappa,
\]
where \(x_\kappa\) is defined by Equation 1 and
\[
asc(\kappa) = |\{ab \in E(H) : a < b\ and\ \kappa(a) < \kappa(b)\}|.
\]

By definition we have the following:

Proposition 5.3. Let \(H\) be a graph such that \(V(H)\) is a subset of \(\mathbb{P}\). Then
\[
X_H(x, t) = \mathcal{X}_G(x, t),
\]
where \(G\) is any edge-coloured digraph that has only dashed edges with \(E(G) = \{(a, b) : ab \in E(H), a < b\}\) and its underlying graph is isomorphic to \(H\).

Ellzey’s chromatic quasisymmetric function [7]: Let \(G\) be a digraph. A proper vertex-colouring of \(G\) is a vertex-colouring of \(G\) such that the colours of adjacent vertices are different. Then the chromatic quasisymmetric function of \(G\) is
\[
Z_G(x, t) = \sum_{\kappa \in C(G)} t^{asc(\kappa)} x_\kappa,
\]
where \( x_\kappa \) is defined by Equation 1 and
\[
\text{asc}(\kappa) = |\{(a, b) \in E(G) : \kappa(a) < \kappa(b)\}|.
\]

By definition we have the following.

**Proposition 5.4.** Let \( G \) be a digraph. Then
\[
Z_G(x, t) = \mathcal{Z}_G(x, t),
\]
where \( G \) has been edge-coloured with only dashed edges.

**Remark 5.5.** It is known that Shareshian-Wachs’ chromatic quasisymmetric functions of graphs and Ellzey’s chromatic quasisymmetric functions of digraphs are symmetric functions when \( t = 1 \). However, the generalized chromatic function \( \mathcal{Z}_G(x) \) is a quasisymmetric function (not necessarily a symmetric function) as seen in Proposition 4.3. For instance, in Section 7, we show that many bases of QSym(\( x \)) can be realized by the generalized chromatic functions of edge-coloured digraphs.

Recall that an orientation of an undirected graph \( H \) is a digraph \( G \) with the same vertices, so that for every edge \( ab \) of \( H \), exactly one of \( (a, b) \) and \( (b, a) \) is an edge of \( G \). A weak cycle of an orientation \( G \) of \( H \) is a subgraph \( C \) of \( G \) such that the underlying graph of \( C, \overline{C} \), is a cycle in \( H \). For \( k \geq 1 \), \( G \) is \( k \)-balanced if for any weak cycle \( C \) of \( G \) with \( E(C) = \{a_i a_{i+1} : i = 1, 2, \ldots, n - 1\} \cup \{a_1 a_n\} \), there are at least \( k \) edges in \( G \) of the form \( (a_i, a_{i+1}) \) and at least \( k \) edges of the form \( (a_{i+1}, a_i) \) (subscripts are taken modulo \( n \)).

Let \( \kappa : V(H) \to P \) be a proper vertex-colouring of \( H \). Then the orientation induced by \( \kappa \) is the orientation \( G_\kappa \) where each edge is directed towards the vertex with the greater colour. If \( G_\kappa \) is \( k \)-balanced, then \( \kappa \) is called a \( k \)-balanced colouring.

**Humpert’s \( k \)-balanced chromatic quasisymmetric function [16]:** Given a graph \( H \) with \( n \) vertices and any positive integer \( k \), the \( k \)-balanced chromatic quasisymmetric function of \( H \) is
\[
X_H^k(x) = \sum x_\kappa,
\]
where the sum runs over all \( k \)-balanced colourings \( \kappa : V(H) \to P \) and \( x_\kappa \) is defined by Equation 1.
By definition we have the following.

**Proposition 5.6.** Let $H$ be a graph and $k$ be a positive integer. Then

$$X_H^k(x) = \sum X_G(x),$$

where the sum runs over all digraphs $G$ such that $G$ is a $k$-balanced orientation of $H$ that has been edge-coloured with only solid edges.

6 Product and Coproduct Formulas for Generalized Chromatic Functions

We now establish the product and coproduct formulas for generalized chromatic functions. The following is straightforward because any pair of independent proper colourings of edge-coloured digraphs $G_1$ and $G_2$ naturally corresponds to a proper colouring of $G_1 \uplus G_2$.

**Proposition 6.1.** Let $G_1$ and $G_2$ be edge-coloured digraphs. Then,

$$X_{G_1}^r(x, t)X_{G_2}^r(x, t) = X_{G_1 \uplus G_2}^r(x, t).$$

To give our coproduct formula, we first need to introduce some notation. Let $G$ be an edge-coloured digraph. An *induced subdigraph* $F = (V(F), E(F))$ of $G$ is a subdigraph of $G$ whose edge set $E(F)$ is the set of all edges $(a, b)$ of $G$ with $a, b \in V(F)$. For a subset $A$ of $V(G)$, let $G|_A$ be the induced subdigraph of $G$ with vertex set $A$ where the colours of edges of $G|_A$ are the same as their colours in $G$. An edge-coloured subdigraph $F$ of $G$ is called a $\{\to, \Rightarrow\}$-*induced subdigraph* of $G$ if $F$ is an induced subdigraph of $G$ such that if $a \in V(F)$ and either $a \to b$ or $a \Rightarrow b$ in $G$, then $b \in V(F)$.

**Example 6.2.** The $\{\to, \Rightarrow\}$-induced subdigraphs of the following edge-coloured digraph
are

\[
\begin{array}{cccc}
V_1 & \rightarrow & V_2 & \rightarrow \downarrow \\
\downarrow & V_3 & \rightarrow & \downarrow \\
V_4 & \rightarrow & V_3 & \rightarrow \\
V_2 & \rightarrow & V_2 & \rightarrow \\
V_3 & \rightarrow & V_3 & \rightarrow \\
V_4 & \rightarrow & V_4 & \rightarrow \\
\end{array}
\]

We now give the coproduct formula for $X_G(x)$.

**Theorem 6.3.** Let $G$ be an edge-coloured digraph. Then

\[
\Delta(X_G(x)) = \sum X_{G[V(G)\setminus V(F)]}(x) \otimes X_F(x),
\]

where the sum runs over all $\{\rightarrow, \Rightarrow\}$-induced subdigraphs $F$ of $G$.

**Proof.** Recall that the coproduct of $\text{QSym}(x)$ can be seen as the composite of the following functions:

\[
\text{QSym}(x) \cong \text{QSym}(x, y) \rightarrow \text{QSym}(x) \otimes \text{QSym}(y) \cong \text{QSym}(x) \otimes \text{QSym}(x),
\]

where $\text{QSym}(x) \cong \text{QSym}(x, y)$ is defined by $f(x_1, x_2, \ldots) \mapsto f(x_1, x_2, \ldots, y_1, y_2, \ldots)$. We have

\[
X_G(x, y) = \sum \prod_{a \in V(G)} x_{\kappa(a)},
\]

where the sum runs over all functions $\kappa$ from $V(G)$ to the alphabet

\[
\{x_1 < x_2 < \cdots < y_1 < y_2 < \cdots\}
\]

such that when $(a, b)$ is in $E(G)$:

(i) If $a \rightarrow b$, then $\kappa(a) \neq \kappa(b)$.

(ii) If $a \rightarrow b$, then $\kappa(a) < \kappa(b)$.

(iii) If $a \Rightarrow b$, then $\kappa(a) \leq \kappa(b)$. 

\[
\]
For the induced subdigraph $F$ with the vertex set \( \{ a : \kappa(a) \in \{ y_1, y_2, \ldots \} \} \), we see that if $a \in V(F)$ and either $a \to b$ or $a \Rightarrow b$ in $G$, then $\kappa(a) \leq \kappa(b)$, and so $b \in V(F)$. Therefore, $F$ is a \( \{\to, \Rightarrow\}\)-induced subdigraph of $G$. Also, note that the rest of vertices produce the edge-coloured digraph $G|_{V(G) - V(F)}$. Applying the above composite, we have

\[
\mathcal{X}_G(x) \mapsto \mathcal{X}_G(x, y) \mapsto \sum \mathcal{X}_{G|_{V(G) - V(F)}}(x) \otimes \mathcal{X}_F(y) \mapsto \sum \mathcal{X}_{G|_{V(G) - V(F)}}(x) \otimes \mathcal{X}_F(x),
\]

where the sums run over all \( \{\to, \Rightarrow\}\)-induced subdigraphs $F$ of $G$.\[\hfill \blacksquare\]

**Remark 6.4.** Theorem 6.3 is a generalization of [2, Proposition 3.4]. Moreover, when $G$ is an edge-coloured digraph that has only dashed edges, then the generalized chromatic function $\mathcal{X}_G(x)$ is equal to Stanley’s chromatic symmetric function of the underlying graph of $G$, so this coproduct formula will give a coproduct formula for Stanley’s chromatic symmetric functions.

**Example 6.5.**

\[
\begin{align*}
\Delta(\mathcal{X}^-)(v_1) &\rightarrow \mathcal{X}^- (v_2) = 1 \otimes \mathcal{X}^- (v_2) + \mathcal{X}^- (v_4) \otimes \mathcal{X}^- (v_3) + \mathcal{X}^- (v_4) \otimes \mathcal{X}^- (v_1) + \mathcal{X}^- (v_2) \otimes \mathcal{X}^- (v_1) + \mathcal{X}^- (v_3) \otimes \mathcal{X}^- (v_2) + \mathcal{X}^- (v_4) \otimes \mathcal{X}^- (v_4) + 1.
\end{align*}
\]
7 Bases for $Q\text{Sym}^r(x)$ Using Generalized Chromatic Functions

In this section, we realize different bases for the Hopf algebras in the following chain as special cases of the generalized chromatic functions of certain edge-coloured digraphs.

$$Q\text{Sym}(x) = Q\text{Sym}^1(x) \supset Q\text{Sym}^2(x) \supset \cdots \supset Q\text{Sym}^\infty(x) = \text{Sym}(x)$$

Given a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l) \vdash n$, the edge-coloured digraph $G_\lambda$ is an edge-coloured digraph such that:

(i) The vertex set of $G$ is

$$\{(i, j) : 1 \leq i \leq l, 1 \leq j \leq \lambda_i\}.$$

(ii) $(i, j) \rightarrow (i', j')$ in $G$ if and only if $i' = i + 1$ and $j' = j$.

(iii) $(i, j) \Rightarrow (i', j')$ in $G$ if and only if $i' = i$ and $j' = j + 1$.

Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k) \vdash [n]$ and $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l) = (n^{m_n}, (n - 1)^{m_{n-1}}, \ldots, 1^{m_1}) \vdash n$. In the following table, we see that many well-known bases of $\text{Sym}(x)$ and $Q\text{Sym}(x)$ are the generalized chromatic functions of some edge-coloured digraphs produced by the actions of the operators

$$\bigcup, \quad \bigcirc, \quad \bigtriangledown, \quad \bigtriangledown$$

on the edge-coloured digraphs in Table 1. For these well-known bases, the result follows immediately by definition, and hence readers unfamiliar with the classical definitions may take these results to be the definitions, or refer to [23]. For the upper-fundamental basis of $Q\text{Sym}(x)$ take the commutative image of the noncommutative upper-fundamental basis in [8].

Remark 7.1. If we generalize the edge-coloured digraph $G_\lambda$ for a partition $\lambda$ to $G_\alpha$ for a composition $\alpha$ in the natural way, but restrict the second condition to only the first column, then by definition we have

$$\mathcal{F}_{G_\alpha}(x) = \mathcal{S}_\alpha^\ast,$$

where $\mathcal{S}_\alpha^\ast$ is the dual immaculate function indexed by $\alpha$ [3]. Switching the $\rightarrow$ for $\Rightarrow$ and vice versa in $G_\alpha$ we obtain the row-strict dual immaculate function $\mathcal{R}\mathcal{S}_\alpha^\ast$ [19].
Let $G$ be an edge-coloured digraph. Recall that $P(G,p)$ is the number of ways that one can properly colour the vertices of $G$ with $p$ colours. The following theorem shows that $P(G,p)$ is a polynomial in $p$.

**Theorem 7.2.** For any edge-coloured digraph $G$, $P(G,p)$ is a polynomial in $p$.

**Proof.** Note that when we set the first $p$ variables of $\mathcal{X}_G(x)$ equal to 1 and all others to 0, we have $P(G,p)$. Since $\mathcal{X}_G(x)$ is a quasisymmetric function by Proposition 4.3, it is a linear combination of the monomial basis elements of QSym(x). Thus, we only need to show that for any composition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$,

$$M_\alpha(1,1,\ldots,1,0,0,\ldots)$$

$p$ times

is a polynomial in $p$. Note that this is equal to the number of monomials appearing in $M_\alpha(x_1,x_2,\ldots,x_p)$. The number of these monomials is the number of ways of choosing $k$ of the $p$ variables, which is $\binom{p}{k}$, a polynomial in $p$. 

\[\text{Table 2} \quad \text{Bases for Sym}(x) \text{ and QSym}(x) \text{ reinterpreted.}\]

| Basis | $G$ | $\mathcal{X}_G$ |
|-------|-----|----------------|
| Monomial basis of Sym(x) | $\bigodot_{j=1}^{n} \left( \bigodot_{i=1}^{m_j} C_j \right)$ | $m_\lambda$ |
| Augmented monomial basis of Sym(x) | $\bigodot_{i=1}^{l} C_{\lambda,i}$ | $\tilde{m}_\lambda = \lambda^1 m_\lambda$ |
| Elementary basis of Sym(x) | $\bigcup_{i=1}^{l} P_{\lambda,i}$ | $e_\lambda$ |
| Augmented elementary basis of Sym(x) | $\bigcup_{i=1}^{l} K_{\lambda,i}$ | $\lambda^1 e_\lambda$ |
| Complete homogeneous basis of Sym(x) | $\bigcup_{i=1}^{l} Q_{\lambda,i}$ | $h_\lambda$ |
| Power sum basis of Sym(x) | $\bigcup_{i=1}^{l} C_{\lambda,i}$ | $p_\lambda$ |
| Schur basis of Sym(x) | $G_\lambda$ | $s_\lambda$ |
| Monomial basis of QSym(x) | $\bigodot_{i=1}^{k} C_{\alpha,i}$ | $M_\alpha$ |
| Fundamental basis of QSym(x) | $\bigodot_{i=1}^{k} O_{\alpha,i}$ | $F_\alpha$ |
| Upper-fundamental basis of QSym(x) | $\bigodot_{i=1}^{k} C_{\alpha,i}$ | $\overline{F}_\alpha$ |
In [5], Cho and van Willigenburg constructed an infinite family of bases for \(\text{Sym}(x)\) using chromatic symmetric functions of graphs. In the following theorem, we construct an infinite family of bases for \(\text{QSym}(x)\) using generalized chromatic functions.

**Theorem 7.3.** Let \(F_i\) be an edge-coloured digraph with \(i\) vertices that has only double edges. For each \(\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k) \vdash n\), define

\[
\mathcal{F}_\alpha = F_{\alpha_1} \bigotimes F_{\alpha_2} \bigotimes \cdots \bigotimes F_{\alpha_k}.
\]

Then

\[
\{ \mathcal{F}_\alpha(x) : \alpha \vdash n \}
\]

is a basis for \(\text{QSym}_n(x)\).

**Proof.** Writing \(\mathcal{F}_\alpha(x)\) in terms of the monomial basis elements of \(\text{QSym}(x)\), we have

\[
\mathcal{F}_\alpha(x) = M_\alpha + \sum_{\beta > \alpha} c_{\alpha,\beta} M_\beta
\]

for some coefficients \(c_{\alpha,\beta}\). Since each \(\mathcal{F}_\alpha(x)\) has a unique leading term \(M_\alpha\) under the \(<\) order, we can conclude that \(\{ \mathcal{F}_\alpha(x) : \alpha \vdash n \}\) is a basis for \(\text{QSym}_n(x)\). \(\blacksquare\)

**Example 7.4.** If \(F_i = C_i\), then \(\{ \mathcal{F}_\alpha(x) \}\) is the monomial basis of \(\text{QSym}(x)\). If \(F_i = Q_i\), then \(\{ \mathcal{F}_\alpha(x) \}\) is the fundamental basis of \(\text{QSym}(x)\).

We now define a product on \(\mathbb{Q}[x_1, x_2, \ldots]\) that plays a crucial role in the rest of this section. Given a multiset \(I\) of positive integers, define

\[
x_I = \prod_{i \in I} x_i.
\]

Define a product on \(\mathbb{Q}[x_1, x_2, \ldots]\) by bilinearly extending the operator

\[
x_I \odot x_J = \begin{cases} 
    x_I x_J & \text{if } I \cap J = \emptyset, \\
    0 & \text{otherwise}.
\end{cases}
\]
Then,

\[ \mathcal{X}_{G_1 \boxtimes G_2} (x) = \mathcal{X}_{G_1} (x) \otimes \mathcal{X}_{G_2} (x). \]

Let \( r \) be a positive integer or infinity. We now establish several new bases for \( \text{QSym}^r(x) \). Let \((\beta, \mu) = ((\beta_1, \beta_2, \ldots, \beta_k), (\mu_1, \mu_2, \ldots, \mu_l))\) be an \( r \)-composition. By definition we have that

\[ M_{(\beta, \mu)} = \mathcal{X} \left( \bigotimes_{i=1}^{k} C_{\beta_i} \right) \otimes \left( \bigotimes_{j=1}^{l} C_{\mu_j} \right) (x) = M_\beta \otimes \tilde{m}_\mu. \]

Define

\[ S_{(\beta, \mu)} = \mathcal{X} \left( \bigotimes_{i=1}^{k} C_{\beta_i} \right) \otimes G_\mu (x) = M_\beta \otimes s_\mu, \]

\[ \bar{F}_{(\beta, \mu)} = \mathcal{X} \left( \bigotimes_{i=1}^{k} C_{\beta_i} \right) \otimes \left( \bigotimes_{j=1}^{l} C_{\mu_j} \right) (x) = \bar{F}_\beta \otimes \tilde{m}_\mu, \]

and

\[ \bar{S}_{(\beta, \mu)} = \mathcal{X} \left( \bigotimes_{i=1}^{k} C_{\beta_i} \right) \otimes G_\mu (x) = \bar{F}_\beta \otimes s_\mu. \]

Let \( K_{\mu, \nu} \) be the Kostka number indexed by partitions \( \mu \) and \( \nu \) \([23]\).

**Theorem 7.5.** Let \((\beta, \mu)\) be an \( r \)-composition. Then we have that:

(i) \( S_{(\beta, \mu)} = \sum_{\nu \leq \mu} K_{\mu, \nu} M_{(\beta, \nu)}. \)

(ii) \( \bar{F}_{(\beta, \mu)} = \sum_{\gamma \leq \beta} M_{(\gamma, \mu)}. \)

(iii) \( \bar{S}_{(\beta, \mu)} = \sum_{\gamma \leq \beta, \nu \leq \mu} K_{\mu, \nu} \frac{\bar{m}_\nu}{\nu!} M_{(\gamma, \nu)}. \)

**Proof.** Note that

\[ s_\mu = \sum_{\nu \leq \mu} K_{\mu, \nu} \frac{\bar{m}_\nu}{\nu!} \text{ and } \bar{F}_\beta = \sum_{\gamma \leq \beta} M_{\gamma}. \]
Therefore,

\[ S(\beta, \mu) = M_{\beta} \circ s_{\mu} = \sum_{\nu \leq \mu} \frac{K_{\mu, \nu}}{\nu!} (M_{\beta} \circ \tilde{m}_{\nu}) = \sum_{\nu \leq \mu} \frac{K_{\mu, \nu}}{\nu!} M_{(\beta, \nu)}, \]

\[ F(\beta, \mu) = F_{\beta} \circ \tilde{m}_{\mu} = \sum_{\gamma \leq \beta} M_{\gamma} \circ \tilde{m}_{\mu} = \sum_{\gamma \leq \beta} M_{(\gamma, \mu)}, \]

and

\[ S(\beta, \mu) = F_{\beta} \circ s_{\mu} = \sum_{\gamma \leq \beta, \nu \leq \mu} \frac{K_{\mu, \nu}}{\nu!} (M_{\gamma} \circ \tilde{m}_{\nu}) = \sum_{\gamma \leq \beta, \nu \leq \mu} \frac{K_{\mu, \nu}}{\nu!} M_{(\gamma, \nu)}. \]

Since \( K_{\mu, \mu} = 1 \), for any partition \( \mu \), we have the following corollary.

**Corollary 7.6.** Each of the following is a basis for \( \text{QSym}^r_n(x) \):

(i) \( \{ S(\beta, \mu) : (\beta, \mu) \text{ is an } r\text{-composition of } n \} \).

(ii) \( \{ F(\beta, \mu) : (\beta, \mu) \text{ is an } r\text{-composition of } n \} \).

(iii) \( \{ S(\beta, \mu) : (\beta, \mu) \text{ is an } r\text{-composition of } n \} \).

**Remark 7.7.** The sets

\[ \{ F_{\beta} \circ \tilde{m}_{\mu} : (\beta, \mu) \text{ is an } r\text{-composition of } n \} \]

and

\[ \{ F_{\beta} \circ s_{\mu} : (\beta, \mu) \text{ is an } r\text{-composition of } n \} \]

are not bases for \( \text{QSym}^r_n(x) \) since we do not necessarily have \( F_{\beta} \circ \tilde{m}_{\mu}, F_{\beta} \circ s_{\mu} \in \text{QSym}^r_n(x) \). As an example, \( ((2, 2), \emptyset) \) is a 2-composition, and

\[ F_{(2, 2)} \circ \tilde{m}_{\emptyset} = F_{(2, 2)} \circ s_{\emptyset} = F_{(2, 2)} = M_{(2, 2)} + M_{(2, 1, 1)} + M_{(1, 1, 2)} + M_{(1, 1, 1, 1)} \notin \text{QSym}^2_4(x). \]
8 Symmetric Functions and Generalizations in Noncommuting Variables

This section introduces the Hopf algebras of symmetric, quasisymmetric, and $r$-quasisymmetric functions in noncommuting variables. The bases of these Hopf algebras are indexed by set partitions, set compositions, and $r$-set-compositions, respectively. We recall the definitions and notation related to these combinatorial objects.

8.1 Set partitions, set compositions, and $r$-set-compositions

A set partition $\Pi$ of a finite set $A$ is a set consisting of mutually disjoint nonempty subsets $\Pi_1, \Pi_2, \ldots, \Pi_l$ of $A$ such that their union is $A$; this is denoted by $\Pi = \Pi_1/\Pi_2/\cdots/\Pi_l \vdash A$. Each $\Pi_i$ is called a block of the set partition $\Pi$, and the length of $\Pi$ is $l$. By convention, we denote by $\emptyset$ the unique empty set partition of $[0] = \emptyset$. Let $\lambda(\Pi) = (|\Pi_1|, |\Pi_2|, \ldots, |\Pi_l|)$ where we assume that $|\Pi_1| \geq |\Pi_2| \geq \cdots \geq |\Pi_l|$. We say $\Pi$ is of shape $\lambda$ if $\lambda(\Pi) = \lambda$. The standardization of $\Pi$, std($\Pi$), is the set partition of $|A|$ such that the $i$th smallest element of $A$ is replaced by $i$ in each block of $\Pi$ for all $i \in |A|$. For example, $\Pi = 35/67/9 \vdash \{3, 5, 6, 7, 9\}$ has length 3, $\lambda(\Pi) = (2, 2, 1)$, and std($\Pi$) = 12/34/5.

A set composition $\Phi$ of a finite set $A$ is a list of mutually disjoint nonempty subsets $\Phi_1, \Phi_2, \ldots, \Phi_k$ of $A$ such that their union is $A$; this is denoted by $(\Phi_1|\Phi_2|\cdots|\Phi_k) \vdash A$. Each $\Phi_i$ is called a block of the set composition $\Phi$, and the length of $\Phi$ is $k$. By convention, we denote by $\emptyset$ the unique empty set composition of $[0] = \emptyset$. Let $\alpha(\Phi) = (|\Phi_1|, |\Phi_2|, \ldots, |\Phi_k|)$. We say $\Phi$ is of shape $\alpha$ if $\alpha(\Phi) = \alpha$. The standardization of $\Phi$, std($\Phi$), is the set composition of $|A|$ such that the $i$th smallest element of $A$ is replaced by $i$ in each block of $\Phi$ for all $i \in |A|$. For example, $\Phi = (35|9|67) \vdash \{3, 5, 6, 7, 9\}$ has length 3, $\alpha(\Phi) = (2, 1, 2)$, and std($\Phi$) = 12/34/5. We write the elements within each block in increasing order and consider them ordered. We say a set composition $\Psi$ corrupts $\Phi$ if $\Psi$ is $\Phi$ with some bars removed, and say that $\Phi$ reforms $\Psi$ if $\Psi$ is $\Phi$ with some bars added. In particular, the numbers of both must be written in the same order. For example, (13|2|456) corrupts (13|2|4|56), and (1|3|2|4|56) reforms (13|2|4|56) but (3|1|2|4|56) does not reform (13|2|4|56).

Let $r$ be a positive integer or infinity. Let $A \subseteq B \subseteq \mathbb{P}$ with $B$ finite. A pair $(\Phi, \Pi)$ where $\Phi$ is a set composition of $A$ and $\Pi$ is a set partition of $B - A$ is called an $r$-set-composition of $B$ if $(\alpha(\Phi), \lambda(\Pi))$ is an $r$-composition. The standardization of an $r$-set-composition $(\Phi, \Pi)$ of $B$, std($\Phi, \Pi$) = ($\Psi, \Omega$), is the $r$-set-composition of $B$ such that the $i$th smallest element of $B$ is replaced by $i$ in each block of $\Phi$ and $\Pi$ for all $i \in |B|$. For example, ((36|29), 4/5/8) is a 2-set-composition of $\{2, 3, 4, 5, 6, 8, 9\}$ and we have std((36|29), 4/5/8) = (25|17, 3/4/6).
8.2 Quasisymmetric functions in noncommuting variables

The Hopf algebra of quasisymmetric functions in noncommuting variables first appeared in [15]. It is realized as power series in noncommuting variables, and the bases of this Hopf algebra are indexed by set compositions.

Let $\mathbb{Q}\langle\langle x_1, x_2, \ldots \rangle\rangle$ be the algebra of formal power series in infinitely many noncommuting variables $x = \{x_1, x_2, \ldots \}$ over $\mathbb{Q}$.

**Definition 8.1.** A quasisymmetric function in noncommuting variables is a formal power series $f \in \mathbb{Q}\langle\langle x_1, x_2, \ldots \rangle\rangle$ such that:

(i) The degrees of the monomials in $f$ are bounded.

(ii) For every set composition $\Phi = (\Phi_1 | \Phi_2 | \cdots | \Phi_k) \subseteq [n]$, all monomials $x_{i_1} x_{i_2} \cdots x_{i_n}$ in $f$ satisfying the following conditions have the same coefficient:

(a) $i_j = i_\ell$ if $j$ and $\ell$ are in the same block of $\Phi$.

(b) $i_j < i_\ell$ if $j \in \Phi_p$ and $\ell \in \Phi_q$ with $p < q$.

The set of all quasisymmetric functions in noncommuting variables is denoted by $\text{NCQSym}(x)$.

The vector space $\text{NCQSym}(x)$ is a Hopf algebra where its product is the same as the product of the formal power series in noncommuting variables, and its coproduct $\Delta$ is defined as follows (for more details, see [4, Section 5]). Evaluate an element $f(x) \in \text{NCQSym}(x) \cong \text{NCQSym}(x, y)$ as $f(x, y)$ using the linearly ordered noncommuting variables $(x, y) = (x_1 < x_2 < \cdots < y_1 < y_2 < \cdots)$. Denote by $\tilde{f}(x, y)$ the image of $f(x, y)$ after imposing the partial commutativity relations

$$x_i y_j = y_j x_i \quad \text{for every pair } (x_i, y_j) \in x \times y.$$

We have that $\tilde{f}(x, y)$ lies in a subalgebra isomorphic to $\text{NCQSym}(x) \otimes \text{NCQSym}(y)$. Let the image of $\tilde{f}(x, y)$ in $\text{NCQSym}(x) \otimes \text{NCQSym}(y)$ be

$$\sum \tilde{f}_1(x) \otimes \tilde{f}_2(y).$$

Applying the following isomorphism,

$$\text{NCQSym}(x) \otimes \text{NCQSym}(y) \cong \text{NCQSym}(x) \otimes \text{NCQSym}(x),$$
we have
\[ \sum \tilde{f}_1(x) \otimes \tilde{f}_2(y) \mapsto \sum \tilde{f}_1(x) \otimes \tilde{f}_2(x). \]

Now we define \( \Delta : \text{NCQSym}(x) \to \text{NCQSym}(x) \otimes \text{NCQSym}(x) \) such that
\[ \Delta(f(x)) = \sum \tilde{f}_1(x) \otimes \tilde{f}_2(x). \]

### 8.3 Symmetric functions in noncommuting variables

The Hopf algebra of symmetric functions in noncommuting variables is a Hopf subalgebra of \( \text{NCQSym}(x) \).

**Definition 8.2.** A symmetric function in noncommuting variables is a formal power series \( f \in \mathbb{Q} \langle \langle x_1, x_2, \ldots \rangle \rangle \) such that:

1. The degrees of the monomials in \( f \) are bounded.
2. For any permutation \( \sigma \in \mathfrak{S}_\infty \),
   \[ \sigma \cdot f(x_1, x_2, \ldots) = f(x_{\sigma(1)}, x_{\sigma(2)}, \ldots) = f(x_1, x_2, \ldots). \]

The set of all symmetric functions in noncommuting variables is denoted by \( \text{NCSym}(x) \).

### 8.4 \( r \)-quasisymmetric functions in noncommuting variables

We now introduce the new Hopf algebra of \( r \)-quasisymmetric functions in noncommuting variables.

For each \( r \)-set-composition \((\Phi, \Pi)\) of \( [n] \), define the \( r \)-dominant monomial function in noncommuting variables to be

\[ M_{(\Phi, \Pi)} = \sum_{(i_1, i_2, \ldots, i_n)} x_{i_1} x_{i_2} \cdots x_{i_n}, \]

where the sum runs over all tuples \((i_1, i_2, \ldots, i_n)\) such that:

1. \( i_j = i_\ell \) if and only if \( j \) and \( \ell \) are in the same block of either \( \Phi \) or \( \Pi \).
2. \( i_j < i_\ell \) if \( j \in \Phi_p \) and \( \ell \in \Phi_q \) with \( p < q \).
Let

\[ NCQSym^r(x) = \bigoplus_{n \geq 0} NCQSym_n^r(x), \]

where

\[ NCQSym_n^r(x) = \mathbb{Q}\text{-span}\{M_{(\Phi,\Pi)} : (\Phi,\Pi) \text{ is an } r\text{-set-composition of } [n]\}. \]

We have that

\[ NCQSym(x) = NCQSym^1(x) \supset NCQSym^2(x) \supset \cdots \supset NCQSym^\infty(x) = NCSym(x). \]

We will show later in Section 13 that NCQSym^r(x) is a Hopf algebra and give natural bases for the Hopf algebras in this chain realized as generalized chromatic functions in noncommuting variables, which are defined in the next section.

9 Generalized Chromatic Functions in Noncommuting Variables

A labelled edge-coloured digraph is an edge-coloured digraph whose vertex set is a subset of \( \mathbb{P} \). We usually denote a labelled edge-coloured digraph by \( G \).

We frequently use the labelled edge-coloured digraphs in the following table.
Table 3 Useful labelled edge-coloured digraphs.

| Notation | Expression |
|----------|------------|
| $C_A$    | $C_{|A|}$ with vertex set $A$ |
| $P_A$    | $P_{|A|}$ with vertex set $A$ such that if $a \to b$, then $a \preceq b$ |
| $Q_A$    | $Q_{|A|}$ with vertex set $A$ such that if $a \Rightarrow b$, then $a \preceq b$ |
| $K_A$    | $K_{|A|}$ with vertex set $A$ such that if $a \dashrightarrow b$, then $a \preceq b$ |

In the rest of the paper, if $\kappa$ is any type of colouring of a labelled graph, digraph, or edge-coloured digraph with vertex set $[n]$, we define

$$x_\kappa = x_\kappa(1)x_\kappa(2) \cdots x_\kappa(n),$$

Definition 9.1. (The generalized chromatic function of a labelled edge-coloured digraph in noncommuting variables) Let $G$ be a labelled edge-coloured digraph with vertex set $[n]$. Define

$$Y_G(x, t) = \sum_{\kappa \in \mathcal{C}(G)} t^{\text{asc}(\kappa)} x_\kappa,$$

where $x_\kappa$ is defined by Equation 2 and

$$\text{asc}(\kappa) = |\{(a, b) \in E(G) : \kappa(a) < \kappa(b)\}|.$$

The power series

$$Y_G(x) = Y_G(x, 1)$$

is called the generalized chromatic function of $G$.

Consider the following commutation map:

$$\rho : \mathbb{Q}[[x_1, x_2, \ldots]] \to \mathbb{Q}[[x_1, x_2, \ldots]]$$

$$x_i \mapsto x_i.$$
By definition we have the following.

**Proposition 9.2.** For any labelled edge-coloured digraph $G$, we have

$$\rho(\mathcal{Y}_G(x, t)) = \mathcal{Y}_G(x, t),$$

where $G$ is $G$ with labels removed.

**Gebhard-Sagan’s chromatic symmetric function in noncommuting variables [10]:** Let $H$ be a labelled graph with vertex set $[n]$. Define

$$Y_H(x) = \sum_{\kappa \in \mathcal{C}(H)} x_{\kappa},$$

where $x_{\kappa}$ is defined by Equation 2.

By definition we have the following.

**Proposition 9.3.** Let $H$ be a labelled graph with vertex set $[n]$. Then

$$Y_H(x) = \mathcal{Y}_G(x),$$

where $G$ is any labelled edge-coloured digraph that has only dashed edges and mapping the vertex $a$ of the underlying graph of $G$ to the vertex $a$ of $H$ gives an isomorphism.

Let $A$ be a nonempty subset of $\mathbb{P}$. Define $\mathcal{G}_A$ to be the set of all bijections from $A$ to itself. Let $G$ be a labelled edge-coloured digraph with vertex set $A$. Then for $\sigma \in \mathcal{G}_A$, define $\sigma \circ G$ to be the labelled edge-coloured digraph obtained by replacing each vertex $a$ of $G$ by $\sigma(a)$. The **symmetrized generalized chromatic function** of $G$ is

$$\mathcal{G}\mathcal{Y}_G(x) = \sum_{\sigma \in \mathcal{G}_A} \mathcal{Y}_{\sigma \circ G}(x).$$

### 10 Product and Coproduct Formulas for Generalized Chromatic Functions in Noncommuting Variables

We now establish the product and coproduct formulas for generalized chromatic functions in noncommuting variables.
Let $G$ be a labelled edge-coloured digraph, and let $n$ be a positive integer. Then $G + n$ is the labelled edge-coloured digraph obtained by replacing each vertex $a$ of $G$ by $a + n$.

**Proposition 10.1.** Let $G_1$ and $G_2$ be labelled edge-coloured digraphs. Then,

$$V_{G_1}(x, t)V_{G_2}(x, t) = V_{G_1 \cup (G_2 + |V(G_1)|)}(x, t).$$

Let $G$ be a labelled edge-coloured digraph. The *standardization* of $G$, denoted $\text{std}(G)$, is the labelled edge-coloured digraph obtained by replacing the $i$th smallest element of $V(G)$ by $i$.

By a proof analogous to that of Theorem 6.3, we have the following coproduct formula for generalized chromatic functions of labelled edge-coloured digraphs.
**Proposition 10.2.** Let $G$ be a labelled edge-coloured digraph. Then

$$
\Delta(\mathcal{V}_G(x)) = \sum \mathcal{V}_{\text{std}(G|_{\mathcal{V}(G) - \mathcal{V}(F)})}(x) \otimes \mathcal{V}_{\text{std}(F)}(x),
$$

where the sum runs over all $\{\to, \Rightarrow\}$-induced subdigraphs $F$ of $G$.

## 11 Bases for NCSym$(x)$ and NCQSym$(x)$ Using Generalized Chromatic Functions in Noncommuting Variables

In this section, we show that different bases for the Hopf algebras NCSym$(x)$ and NCQSym$(x)$ can be realized as generalized chromatic functions in noncommuting variables of certain labelled edge-coloured digraphs. Since these bases are less well-known than their commutative counterparts, we provide their classical definitions taken from [20] for NCSym$(x)$ and [4, 8] for NCQSym$(x)$.

**Monomial basis** $(m_\Pi)$ of NCSym$(x)$. Given $\Pi = \Pi_1/\Pi_2/\cdots/\Pi_l \vdash [n]$, define the *monomial symmetric function* in noncommuting variables to be

$$
m_\Pi = \sum_{(i_1, i_2, \ldots, i_n)} x_{i_1}x_{i_2}\cdots x_{i_n},
$$

where the sum is over all $n$-tuples $(i_1, i_2, \ldots, i_n)$ with $i_j = i_k$ if and only if $j$ and $k$ are in the same block of $\Pi$. For example,

$$
m_{13/24} = x_1x_2x_1x_2 + x_2x_1x_2x_1 + x_1x_3x_1x_3 + x_3x_1x_3x_1 + x_2x_3x_2x_3 + x_3x_2x_3x_2 + \cdots.
$$

By definition we have that $m_\Pi = \mathcal{V}_G(x)$ where

$$
G = C_{\Pi_1} \otimes \cdots \otimes C_{\Pi_l}.
$$

For example, if $G$ is the following labelled edge-coloured digraph

```
1 \to 3 \to 2 \to 4
```

then $\mathcal{V}_G(x) = m_{13/24}$. 
Power sum basis \( \{ p_\Pi \} \) of NCSym(\( x \)). Given \( \Pi = \Pi_1 / \Pi_2 / \cdots / \Pi_l \vdash [n] \), define the power sum symmetric function in noncommuting variables to be

\[
p_\Pi = \sum_{(i_1,i_2,\ldots,i_n)} x_{i_1}x_{i_2} \cdots x_{i_n},
\]

where \( i_j = i_k \) if \( j, k \) are in the same block of \( \Pi \). For example,

\[
p_{13/24} = x_1x_2x_1x_2 + x_2x_1x_2x_1 + x_1^4 + x_2^4 + \cdots.
\]

By definition we have that \( p_\Pi = \mathcal{Y}_G(x) \) where

\[
G = C_{\Pi_1} \cup C_{\Pi_2} \cup \cdots \cup C_{\Pi_l}.
\]

For example, if \( G \) is the following labelled edge-coloured digraph

\[
\begin{array}{c}
1 \rightarrow 3 \\
2 \rightarrow 4
\end{array}
\]

then \( \mathcal{Y}_G(x) = p_{13/24} \).

Remark 11.1. Note that the underlying edge-coloured digraphs for the augmented monomial basis elements of Sym(\( x \)) and the monomial basis elements of NCSym(\( x \)) are the same. Moreover, the underlying edge-coloured digraphs for power sum basis elements of Sym(\( x \)) and power sum basis elements in NCSym(\( x \)) are the same. Thus, by Proposition 9.2, we recover [20, Theorem 2.1 (i), (ii), (iii)], where the third part is recovered from the first interpretation below.

Elementary basis \( \{ e_\Pi \} \) of NCSym(\( x \)). Given \( \Pi = \Pi_1 / \Pi_2 / \cdots / \Pi_l \vdash [n] \), define the elementary symmetric function in noncommuting variables to be

\[
e_\Pi = \sum_{(i_1,i_2,\ldots,i_n)} x_{i_1}x_{i_2} \cdots x_{i_n},
\]

where \( i_j \neq i_k \) if \( j, k \) are in the same block of \( \Pi \). For example,

\[
e_{13/24} = x_1x_1x_2x_2 + x_2x_2x_1x_1 + x_1x_2x_2x_1 + x_2x_1x_1x_2 + x_1x_2x_3x_4 + \cdots.
\]
It follows by definition that the elementary symmetric functions in noncommuting variables can be written in two ways using labelled edge-coloured digraphs. The first is that \( e_{\Pi} = \mathcal{Y}_G(x) \) where

\[
G = K_{\Pi_1} \cup K_{\Pi_2} \cup \cdots \cup K_{\Pi_l}.
\]

For example, if \( G \) is the following labelled edge-coloured digraph

\[
\begin{array}{c}
1 \quad \cdots \quad 3 \\
2 \quad \longrightarrow \quad 4
\end{array}
\]

then \( \mathcal{Y}_G(x) = e_{13/24} \).

The second is that

\[
e_{\Pi} = \sum_{(\sigma_1, \ldots, \sigma_l) \in \mathcal{S}_{\Pi_1} \times \cdots \times \mathcal{S}_{\Pi_l}} \mathcal{Y}_{(\sigma_1, \ldots, \sigma_l \circ P_{\Pi_l})}(x).
\]

For example, \( e_{13/24} \) is equal to

\[
\begin{align*}
\mathcal{Y}_1 \quad \longrightarrow \quad 3 & \quad + \quad \mathcal{Y}_2 \quad \longrightarrow \quad 4 \quad (x) \\
2 \quad \longrightarrow \quad 4 & \quad + \quad \mathcal{Y}_3 \quad \longrightarrow \quad 1 \quad \longrightarrow \quad 2 \quad \longrightarrow \quad 4 \quad (x)
\end{align*}
\]

Complete homogeneous basis \( \{ h_{\Pi} \} \) of \( \text{NCSym}(x) \). For set partitions \( \Pi \) and \( \Omega \) of \([n]\), let \( \Omega \leq \Pi \) if each block of \( \Omega \) is contained in some block of \( \Pi \). The set of all set partitions of \([n]\) with this partial ordering gives a lattice; the meet (greatest lower bound) and join (least upper bound) operations of this lattice are denoted by \( \wedge \) and \( \vee \), respectively. The complete homogeneous symmetric function in noncommuting variables is

\[
h_{\Pi} = \sum_{\Omega \subseteq [n]} (\lambda(\Omega \wedge \Pi))! m_{\Omega}.
\]

For example,

\[
h_{13/24} = m_{1/2/3/4} + m_{12/3/4} + 2m_{13/2/4} + m_{14/2/3} + m_{1/23/4} + 2m_{1/24/3} + m_{1/2/34}
\]

\[
+ m_{12/34} + 4m_{13/24} + m_{14/23} + 2m_{123/4} + 2m_{124/3} + 2m_{134/2} + 2m_{1/234} + 4m_{1234}.
\]
Using labelled edge-coloured digraphs we now present the complete homogeneous basis. Let \( \Pi = \Pi_1/\Pi_2/\cdots/\Pi_l \) be a set partition of \([n]\). By [1, Lemma 2.14], we have that

\[
h_{\Pi} = \sum_{(\sigma_1, \ldots, \sigma_l) \in S_{\Pi_1} \times \cdots \times S_{\Pi_l}} Y_{(\sigma_1 \circ Q_{\Pi_1})}(x).
\]

For example, \( h_{13/24} \) is equal to

\[
+ Y_{1 \rightarrow 3 \rightarrow 4}(x) + Y_{3 \rightarrow 1 \rightarrow 2 \rightarrow 4}(x)
\]

**Remark 11.2.** Given a partition \( \lambda \), the generalized chromatic functions of \( P_{\lambda_i}'s \) give the elementary symmetric function in commuting variables, \( e_\lambda \). Likewise, given a set partition \( \Pi \), the symmetrized generalized chromatic functions of \( P_{\Pi_i}'s \) give the elementary symmetric function in noncommuting variables \( e_{\Pi} \). Similarly, by looking at \( Q_{\lambda_i}'s \) and \( Q_{\Pi_i}'s \) we can obtain the complete homogeneous symmetric functions \( h_\lambda \) and \( h_{\Pi} \).

**Rosas-Sagan Schur functions of NCSym(x).** Rosas and Sagan in [20], as an analogy for the monomial, power sum, elementary and homogeneous bases of \( \text{Sym}(x) \), introduced the above bases for \( \text{NCSym}(x) \), recalling the elementary basis from the work of Wolf [28]. Their proposed analogy for Schur functions did not produce enough distinct elements to make a basis for \( \text{NCSym}(x) \). However, their functions have a natural realization in terms of generalized chromatic functions, so we include them here.

Let \( \Pi = \Pi_1/\Pi_2/\cdots/\Pi_l \) be a set partition of \([n]\) with \( \lambda = \lambda(\Pi) \). Let \( G_\lambda \) denote the labelled edge-coloured digraph obtained by replacing the vertex \((i, j)\) in \( G_\lambda \) by \( \lambda_1 + \cdots + \lambda_{i-1} + j \). Now define

\[
S_{\Pi} = \sum_{\sigma \in S_n} Y_{\sigma \circ G_\lambda}(x).
\]

Then \( S_{\Pi} \in \text{NCSym}(x) \), and \( S_{\Pi} = S_\Omega \) if and only if \( \lambda(\Pi) = \lambda(\Omega) = \lambda \). Note that \( S_{\Pi} \) is the same as \( S_\lambda \) introduced in [20]. Moreover, \( \rho(S_{\Pi}) = n! s_\lambda \). However, the set

\[
\{ S_{\Pi} : \Pi \vdash [n] \} \]
is not a basis for $\text{NCSym}_n(\mathbf{x})$ since the dimension of the space spanned by this set is equal to the number of partitions, which is less than the number of set partitions for $n > 2$.

**Monomial basis** $\{M_\Phi\}$ of $\text{NCQSym}(\mathbf{x})$. Given $\Phi = (\Phi_1 | \Phi_2 | \cdots | \Phi_k) \subseteq [n]$, define the *monomial quasisymmetric function* in noncommuting variables to be

$$M_\Phi = \sum_{(i_1,i_2,\ldots,i_n)} x_{i_1}x_{i_2}\cdots x_{i_n}$$

where the sum runs over all tuples $(i_1,i_2,\ldots,i_n)$ such that:

(i) $i_j = i_\ell$ if $j$ and $\ell$ are in the same block of $\Phi$.
(ii) $i_j < i_\ell$ if $j \in \Phi_p$ and $\ell \in \Phi_q$ with $p < q$.

For example,

$$M_{(13|24)} = x_1x_2x_1x_2 + x_1x_2x_1x_3 + x_2x_2x_3x_3 + \cdots$$

By definition, we have that $M_\Phi = \mathcal{Y}_G(\mathbf{x})$, where

$$G = C_{\Phi_1} \bigoplus C_{\Phi_2} \bigoplus \cdots \bigoplus C_{\Phi_k}.$$  

For example, if $G$ is the following labelled edge-coloured digraph

![Graph](image)

then $\mathcal{Y}_G(\mathbf{x}) = M_{(24|378|6|15)}$.

To the best of our knowledge the following basis is not in the literature, however, similar bases exist [4, 8]. Therefore we will define the basis elements in terms of generalized chromatic functions in noncommuting variables first, before deriving an explicit formula for them, and establishing that they are a basis for $\text{NCQSym}(\mathbf{x})$. Since they can be defined using generalized chromatic functions in noncommuting variables their product and coproduct formulas follow from Propositions 10.1 and 10.2.
Fundamental basis $\{F_\Phi\}$ of $\text{NCQSym}(x)$. Given $\Phi = (\Phi_1|\Phi_2|\cdots|\Phi_k) \models [n]$, define the fundamental quasisymmetric function in noncommuting variables to be

$$F_\Phi = \mathcal{Y}_C(x),$$

where

$$G = Q_{\Phi_1} \bigoplus Q_{\Phi_2} \bigoplus \cdots \bigoplus Q_{\Phi_k}.$$  

For example, if $\Phi = (24|378|6|15)$ then $G$ is the following labelled edge-coloured digraph

\[
\begin{array}{c}
2 \rightarrow 4 \bigoplus 3 \rightarrow 7 \bigoplus 8 \bigoplus 6 \bigoplus 1 \rightarrow 5
\end{array}
\]

and $\mathcal{Y}_C(x) = F_{(24|378|6|15)}$.

By comparing the labelled edge-coloured digraphs in the realizations of $F_\Phi$ and $M_\Phi$ as generalized chromatic functions in noncommuting variables, we immediately get the following.

**Proposition 11.3.** Let $\Phi$ be a set composition. Then

$$F_\Phi = \sum_{\Psi \text{ reforms } \Phi} M_\Psi,$$

and hence $\{F_\Phi\}$ is a basis for $\text{NCQSym}(x)$.

For example,

$$F_{(13|24)} = M_{(13|24)} + M_{(1|3|24)} + M_{(13|2|4)} + M_{(1|3|2|4)}.$$  

Upper-fundamental basis $\{\overline{F}_\Phi\}$ of $\text{NCQSym}(x)$. This basis also appears in [8] as the $L$ basis, but we reinterpret it here as generalized chromatic functions. Given $\Phi = (\Phi_1|\Phi_2|\cdots|\Phi_k) \models [n]$, define the upper-fundamental quasisymmetric function in noncommuting variables to be

$$\overline{F}_\Phi = \sum_{\Psi \text{ corrupts } \Phi} M_\Psi.$$
For example,

\[
F_{(13|24)} = M_{(13|24)} + M_{(1324)} = M_{(13|24)} + M_{(1234)}.
\]

By definition, we have that \( F_\phi = \mathcal{Y}_G(x) \), where

\[
G = \left\{ C_{\phi_1}, C_{\phi_2}, \ldots, C_{\phi_k} \right\}
\]

For example, if \( G \) is the following labelled edge-coloured digraph

\[
\begin{array}{cccccccc}
2 & \rightarrow & 4 & \rightarrow & 3 & \rightarrow & 7 & \rightarrow & 8 & \rightarrow & 6 & \rightarrow & 1 & \rightarrow & 5 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow
\end{array}
\]

then \( \mathcal{Y}_G(x) = F_{(24|378|6|15)} \).

**Remark 11.4.** Note that the underlying edge-coloured digraphs of monomial (fundamental and upper-fundamental, respectively) bases of \( \text{QSym}(x) \) and \( \text{NCQSym}(x) \) are the same. Moreover, for any set composition \( \Phi \),

\[
\rho(M_\Phi) = M_{\alpha(\Phi)}, \quad \rho(F_\Phi) = F_{\alpha(\Phi)}, \quad \text{and} \quad \rho(F_\Phi) = F_{\alpha(\Phi)}.
\]

Let \( \Phi = (\Phi_1|\Phi_2|\cdots|\Phi_k) \) be a set composition, and \( \Pi = \Pi_1/\Pi_2/\cdots/\Pi_l \) be a set partition. In the following table, we summarize our results for this section using the operators

\[
\cup, \quad \cap, \quad \ominus, \quad \oplus
\]

on the labelled edge-coloured digraphs in Table 3.

**Remark 11.5.** The Schur function in noncommuting variables \( s_x \) in [1] cannot be written as a single generalized chromatic function of a labelled edge-coloured digraph. For example, the Schur function in noncommuting variables

\[
s_{12/34} = \frac{1}{12} \, m_{1/2/3/4} + \frac{1}{3} \, m_{1/2/34} - \frac{1}{12} \, m_{1/23/4} + \frac{1}{6} \, m_{1/234} + \frac{1}{12} \, m_{1/24/3} + \frac{1}{6} \, m_{12/3/4} - \frac{1}{2} \, m_{123/4} + \frac{1}{6} \, m_{123/4} - \frac{1}{12} \, m_{13/2/4} + \frac{1}{6} \, m_{13/24} + \frac{1}{12} \, m_{14/2/3} - \frac{1}{12} \, m_{14/23},
\]
Table 4  Bases for NCSym(\(x\)) and NCQSym(\(x\)) reinterpreted.

| Basis                                      | Expression | Notation |
|--------------------------------------------|------------|----------|
| Monomial basis of NCSym(\(x\))           | \(\bigotimes_{i=1}^{l} \frac{C_{\Pi_{i}}}{(x)}\) | \(m_{\Pi}\) |
| Elementary basis of NCSym(\(x\))         | \(\sum_{(\sigma_{1}, \ldots, \sigma_{l}) \in \Phi_{1} \times \cdots \times \Phi_{l}} (\sigma_{1} \circ P_{\Pi_{1}})^{x}_{(x)}\) | \(e_{\Pi}\) |
| Elementary basis of NCSym(\(x\))         | \(\bigotimes_{i=1}^{l} K_{\Pi_{i}}^{x}_{(x)}\) | \(e_{\Pi}\) |
| Complete homogeneous basis of NCSym(\(x\)) | \(\sum_{(\sigma_{1}, \ldots, \sigma_{l}) \in \Phi_{1} \times \cdots \times \Phi_{l}} (\sigma_{1} \circ Q_{\Pi_{1}})^{x}_{(x)}\) | \(h_{\Pi}\) |
| Power sum basis of NCSym(\(x\))          | \(\bigotimes_{i=1}^{k} C_{\Phi_{i}}^{x}_{(x)}\) | \(p_{\Pi}\) |
| Monomial basis of NCQSym(\(x\))          | \(\bigotimes_{i=1}^{l} C_{\Phi_{i}}^{x}_{(x)}\) | \(M_{\Phi}\) |
| Fundamental basis of NCQSym(\(x\))       | \(\bigotimes_{i=1}^{k} Q_{\Phi_{i}}^{x}_{(x)}\) | \(F_{\Phi}\) |
| Upper-fundamental basis of NCQSym(\(x\))  | \(\bigotimes_{i=1}^{k} C_{\Phi_{i}}^{x}_{(x)}\) | \(F_{\Phi}\) |

and so in the expansion of \(s_{12/34}\) in terms of monomials the coefficient of \(x_{1}x_{1}x_{1}x_{2}\) is negative, but the generalized chromatic function of any labelled edge-coloured digraph is a nonnegative linear combination of monomials. Nevertheless, the source Schur functions in [1] can be realized as the noncommutative determinant of a matrix where its entries are generalized chromatic functions in noncommuting variables.

12 Fundamental Basis to Fundamental Basis: an Injection of the Malvenuto-Reutenauer Hopf Algebra into NCQSym(\(x\))

Given a list \(i_{1}i_{2} \cdots i_{n}\) of positive integers, define \(\text{std}(i_{1}i_{2} \cdots i_{n}) = u_{1}u_{2} \cdots u_{n}\) where

\[
    u_{k} = |\{j \in [n] : j < i_{k}\}| + |\{j \in [n] : j \leq k, i_{j} = i_{k}\}|,
\]

that is the number of integers in \(i_{1}i_{2} \cdots i_{n}\) that are smaller than \(i_{k}\) plus the number of integers equal to \(i_{k}\) that are weakly to the left of \(i_{k}\). For example,

\[
    \text{std}(3544231) = 3756241.
\]
Given a permutation \( \sigma \in S_n \), let \( |\sigma| = n \). Define

\[ F_\sigma = \sum x_{i_1} x_{i_2} \cdots x_{i_n} \in \mathbb{Q}\langle\langle x_1, x_2, \ldots \rangle\rangle, \]

where the sum runs over all lists \( i_1 i_2 \cdots i_n \) with \( i_j \in \mathbb{P} \) such that \( \text{std}(i_1 i_2 \cdots i_n) = \sigma^{-1} \).

The Malvenuto-Reutenauer Hopf algebra is the graded Hopf algebra

\[ \mathcal{G}\text{Sym}(x) = \bigoplus_{n \geq 0} \mathcal{G}\text{Sym}_n(x), \]

where

\[ \mathcal{G}\text{Sym}_n(x) = \mathbb{Q}\text{-span}\{F_\sigma : \sigma \in S_n\}. \]

For more details about the Malvenuto-Reutenauer Hopf algebra, see [13, Section 8].

Given any permutation \( \sigma \), let \( \text{Des}(\sigma) = \{i : \sigma(i) > \sigma(i + 1)\} \). Let \( i_1 < i_2 < \cdots < i_t \) be all elements of \( \text{Des}(\sigma) \). By definition we have that \( F_\sigma = \mathcal{Y}_G(x) \) where

\[ G = Q_{\{\sigma(1), \ldots, \sigma(i_1)\}} \bigoplus Q_{\{\sigma(i_1 + 1), \ldots, \sigma(i_2)\}} \bigoplus \cdots \bigoplus Q_{\{\sigma(i_t + 1), \ldots, \sigma(n)\}}. \]

For the permutation \( \sigma \), define the set composition

\[ \Phi_\sigma = (\sigma(1) \cdots \sigma(i_1) | \sigma(i_1 + 1) \cdots \sigma(i_2) | \cdots | \sigma(i_{t-1} + 1) \cdots \sigma(i_t)). \]

For example, if \( \sigma = 836791524 \), then

\[ \text{Des}(\sigma) = \{1, 5, 7\}, \]

and \( F_\sigma = \mathcal{Y}_G \) where

\[ G = Q_{\{8\}} \bigoplus Q_{\{3,6,7,9\}} \bigoplus Q_{\{1,5\}} \bigoplus Q_{\{2,4\}}. \]

Moreover,

\[ \Phi_\sigma = (8|3679|15|24). \]

Considering the product and coproduct formulas for the fundamental bases of \( \mathcal{G}\text{Sym}(x) \) and \( \text{NCQSym}(x) \), we have the following injection:

\[ \mathcal{G}\text{Sym}(x) \to \text{NCQSym}(x) \]

\[ F_\sigma \mapsto F_{\Phi_\sigma}. \]
13 NCQSym$^r(x)$ and Its Bases

To conclude we prove that NCQSym$^r(x)$ is a Hopf algebra, where

$$\text{NCQSym}(x) = \text{NCQSym}^1(x) \supset \text{NCQSym}^2(x) \supset \cdots \supset \text{NCQSym}^\infty(x) = \text{NCSym}(x).$$

We then establish two natural bases for this Hopf algebra using generalized chromatic functions.

The embedding of NCQSym$^r(x)$ to NCQSym$^x(x)$ is given by

$$M_{(\Phi, \Pi)} = \sum_{\Psi \in \Phi \boxplus \Pi} M_{\Psi}$$

where $\boxplus$ is defined as follows. If $\Phi = (\Phi_1 | \Phi_2 | \cdots | \Phi_k)$ and $\Pi = \Pi_1 / \Pi_2 / \cdots / \Pi_l$, then $\Phi \boxplus \Pi$ is the set of set compositions $(\Psi_1 | \Psi_2 | \cdots | \Psi_{k+l})$ such that

$$\{\Psi_1, \Psi_2, \ldots, \Psi_{k+l}\} = \{\Phi_1, \Phi_2, \ldots, \Phi_k, \Pi_1, \Pi_2, \ldots, \Pi_l\}$$

and if $\Phi_i = \Psi_p$ and $\Phi_{i+1} = \Psi_q$, then we always have $p < q$.

In the following theorem, we show that NCQSym$^r(x)$ is a Hopf algebra by finding the product and coproduct formulas for the $r$-dominant monomial basis of NCQSym$^r(x)$.

**Theorem 13.1.** For any positive integer $r$, NCQSym$^r(x)$ is a Hopf algebra.

**Proof.** Since NCQSym$^r(x)$ is graded with NCQSym$^r_0(x) = \mathbb{Q}$, by Takeuchi’s formula [27], it is enough to show that NCQSym$^r(x)$ is closed under the product and coproduct of NCQSym(x). In NCSym(x), the coproduct on the monomial basis is taking subsets of blocks, followed by standardization. That is, given a set partition $\Pi = \Pi_1 / \Pi_2 / \cdots / \Pi_l$, then

$$\Delta(m_{\Pi}) = \sum_{|\{i_1, i_2, \ldots, i_l\}| = |\Pi|} m_{\text{std}(\Pi_1 / \Pi_2 / \cdots / \Pi_l)} \otimes m_{\text{std}(\Pi_{i+1} / \Pi_{i+2} / \cdots / \Pi_l)}.$$

Recall that for a set composition $\Phi = (\Phi_1 | \Phi_2 | \cdots | \Phi_k)$, in NCQSym(x), we have

$$\Delta(M_{\Phi}) = \sum_{t=0}^{k} M_{\text{std}(\Phi_1 | \Phi_2 | \cdots | \Phi_t)} \otimes M_{\text{std}(\Phi_{t+1} | \Phi_{t+2} | \cdots | \Phi_k)}.$$
Therefore, in NCQSym\(^r\)(x), if \((\Phi, \Pi) = (\Phi_1|\Phi_2|\cdots|\Phi_k, \Pi_1/\Pi_2/\cdots/\Pi_l)\), we have

\[
\Delta(M_{\Phi, \Pi}) = \sum_{t=0}^{k} \sum_{\{i_1, \ldots, i_l\} = \emptyset} M_{\text{std}(\Phi_1|\cdots|\Phi_t, \Pi_1/\cdots/\Pi_{i_1})} \otimes M_{\text{std}(\Phi_{t+1}|\cdots|\Phi_k, \Pi_{i_1+1}/\cdots/\Pi_{i_l})}.
\]

For example, when \(r = 2\),

\[
\Delta(M_{((24), 1/3)}) = 1 \otimes M_{((24), 1/3)} + M_{(\emptyset, 1)} \otimes M_{((13), 2)} + M_{(\emptyset, 1)} \otimes M_{((23), 1)} + M_{((23), 1)} \otimes M_{(\emptyset, 1)}
\]
\[
+ M_{((13), 2)} \otimes M_{(\emptyset, 1)} + M_{((12), \emptyset)} \otimes M_{(\emptyset, 1/2)} + M_{(\emptyset, 1/2)} \otimes M_{((12), \emptyset)} + M_{((24), 1/3)} \otimes 1.
\]

The product formula is a little more complicated. Before we continue we need to recall the product for the monomial basis of NCQSym\(x\). Let \(\Phi = (\Phi_1|\Phi_2|\cdots|\Phi_k)\) be a set composition of \([n]\), and let \(A\) be a subset of \([n]\). The restriction of \(\Phi\) to \(A\), \(\Phi|_A\), is the set composition obtained by dropping the empty parts of \((\Phi_1 \cap A|\Phi_2 \cap A|\cdots|\Phi_k \cap A)\). For example,

\[
(357|26|14)|_{[1, 3, 4]} = (3|14).
\]

Let \(\Phi \vdash [n]\) and \(\Psi \vdash [m]\). The shifted quasi-shuffle of \(\Phi\) and \(\Psi\), denoted \(\Phi \diamond \Psi\), is the set of set compositions \(\Gamma \vdash [n+m]\) such that \(\Gamma|_{[1,2,\ldots,n]} = \Phi\) and std\((\Gamma|_{[n+1,n+2,\ldots,n+m]} = \Psi\).

The product formula for the monomial basis of NCQSym\(x\) is

\[
M_{\Phi} \cdot M_{\Psi} = \sum_{\Gamma \in \Phi \diamond \Psi} M_{\Gamma}.
\]

Now for an \(r\)-set-composition \((\Phi, \Pi)\), we have

\[
M_{(\Phi, \Pi)} = \sum_{\Psi \in \Phi \diamond \Pi} M_{\Psi}.
\]

Conversely, given a set composition \(\Psi\), there is a unique \(r\)-set-composition \((\Phi, \Pi)\) such that \(\Psi \in \Phi \diamond \Pi\). We denote \(\Phi\) by \(\Psi|r_{\text{comp}}\) and \(\Pi\) by \(\Psi|r_{\text{par}}\).

Also, for set compositions \(\Gamma\) and \(\Theta\) we have

\[
M_{\Gamma} \cdot M_{\Theta} = \sum_{\Gamma \in \Gamma \diamond \Theta} M_{\Gamma}.
\]
Conversely, given a set composition $\Upsilon$ with $|\Upsilon| = n + m$, there is a unique pair of set compositions $\Gamma, \Theta$ such that $|\Gamma| = n$, $|\Theta| = m$ and $\Upsilon \in \Gamma \boxtimes \Theta$. We denote $\Gamma$ by $\Upsilon|_{1, \ldots, n}$ and $\Theta$ by $\Upsilon|_{n+1, \ldots, n+m}$.

Therefore,

$$M_{(\Phi, \Pi)} \cdot M_{(\Psi, \Omega)} = \sum_{\Gamma \in \Phi \boxtimes \Pi, \Theta \in \Psi \boxtimes \Omega} \left( \sum_{\Upsilon \in \Gamma \boxtimes \Theta} M_{\Upsilon} \right) = \sum_{\Upsilon} C_{(\Phi, \Pi), (\Psi, \Omega)}^{\Upsilon} M_{\Upsilon}.$$

Let $|(\Phi, \Pi)| = n$ and $|(\Psi, \Omega)| = m$. We first note that all coefficients $C_{(\Phi, \Pi), (\Psi, \Omega)}^{\Upsilon}$ are 0 or 1. Indeed, $C_{(\Phi, \Pi), (\Psi, \Omega)}^{\Upsilon} = 1$ if and only if:

(i) $\Phi = (\Upsilon|_{1, \ldots, n})_{r\text{-comp}}$,
(ii) $\Pi = (\Upsilon|_{1, \ldots, n})_{r\text{-par}}$,
(iii) $\Psi = (\Upsilon|_{n+1, \ldots, n+m})_{r\text{-comp}}$,
(iv) $\Omega = (\Upsilon|_{n+1, \ldots, n+m})_{r\text{-par}}$.

Let $C_{(\Phi, \Pi), (\Psi, \Omega)}^{\Upsilon'} = 1$ for some $\Upsilon'$, and let $\Phi' = \Upsilon'|_{r\text{-comp}}$ and $\Pi' = \Upsilon'|_{r\text{-par}}$. We want to show that for any $\Upsilon'' \in \Phi' \boxtimes \Pi'$, we have $C_{(\Phi, \Pi), (\Psi, \Omega)}^{\Upsilon''} = 1$. Then the product $M_{(\Phi, \Pi)} \cdot M_{(\Psi, \Omega)}$ is in $\text{NCQSym}^r(x)$.

Let $\Gamma' = (\Upsilon'|_{1, \ldots, n})$, $\Theta' = (\Upsilon'|_{n+1, \ldots, n+m})$, $\Gamma'' = (\Upsilon''|_{1, \ldots, n})$ and $\Theta'' = (\Upsilon''|_{n+1, \ldots, n+m})$. Note the fact that in a quasi-shuffle, the blocks of size less than $r$ can only be obtained from blocks of size less than $r$. Since $\Upsilon'_{r\text{-comp}} = (\Upsilon''|_{r\text{-comp}} = \Phi'$, we must have $\Gamma'_{r\text{-comp}} = \Gamma''_{r\text{-comp}} = \Phi$ and $\Theta'_{r\text{-comp}} = \Theta''_{r\text{-comp}} = \Psi$. And therefore, we must also have $\Gamma'_{r\text{-par}} = \Gamma''_{r\text{-par}} = \Pi$ and $\Theta'_{r\text{-par}} = \Theta''_{r\text{-par}} = \Omega$. Hence, $\Gamma'' \in \Phi \boxtimes \Pi$ and $\Theta'' \in \Psi \boxtimes \Omega$, that is, $C_{(\Phi, \Pi), (\Psi, \Omega)}^{\Upsilon''} = 1$.  

Finally we establish two bases for $\text{NCQSym}^r(x)$. Given an $r$-set-composition $(\Phi, \Pi) = ((\Phi_1|\Phi_2| \cdots |\Phi_k), \Pi_1/\Pi_2/ \cdots /\Pi_l)$ of $[n]$, by definition we have that

$$M_{(\Phi, \Pi)} = \mathcal{U} \left( \bigoplus_{i=1}^{k} C_{\Phi_i} \right) \circ \left( \bigoplus_{j=1}^{l} C_{\Pi_j} \right)(x).$$

Define

$$\overline{F}_{(\Phi, \Pi)} = \mathcal{U} \left( \bigoplus_{i=1}^{k} C_{\Phi_i} \right) \circ \left( \bigoplus_{j=1}^{l} C_{\Pi_j} \right)(x).$$
Proposition 13.2. Each of the following is a basis for $\text{NCQSym}_n^r(\mathbf{x})$.

(i) $\{M_{(\Phi, \Pi)} : (\Phi, \Pi) \text{ is an } r\text{-set-composition of } [n]\}$.

(ii) $\{\bar{F}_{(\Phi, \Pi)} : (\Phi, \Pi) \text{ is an } r\text{-set-composition of } [n]\}$.

Proof. The first set is the $r$-dominant monomial basis for $\text{NCQSym}_n^r(\mathbf{x})$. Now, consider that

$$\bar{F}_{(\Phi, \Pi)} = \sum_{\psi \text{ corrupts } \Phi} M_{(\psi, \Pi)}.$$ 

Therefore, the second set is also a basis for $\text{NCQSym}_n^r(\mathbf{x})$. ■

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References

[1] Aliniaeifard, F., S. Li, and S. van Willigenburg. “Schur functions in noncommuting variables.” Adv. Math. 406 (2022): 108536. https://doi.org/10.1016/j.aim.2022.108536.

[2] Aliniaeifard, F., V. Wang, and S. van Willigenburg. “P-partition power sums.” European J. Combin. 110 (2023): 103688. https://doi.org/10.1016/j.ejc.2023.103688.

[3] Berg, C., N. Bergeron, F. Saliola, L. Serrano, and M. Zabrocki. “A lift of the Schur and Hall-Littlewood bases to non-commutative symmetric functions.” Canad. J. Math. 66 (2014): 525–65. https://doi.org/10.4153/CJM-2013-013-0.

[4] Bergeron, N. and M. Zabrocki. “The Hopf algebras of symmetric functions and quasi-symmetric functions in non-commutative variables are free and cofree.” J. Algebra Appl. 08 (2009): 581–600. https://doi.org/10.1142/S0219498809003485.

[5] Cho, S. and S. van Willigenburg. “Chromatic bases for symmetric functions.” Electron. J. Combin. 23 (2016): 7 pp. https://doi.org/10.37236/5540.
[6] Crew, L. and S. Spirkl. “A deletion-contraction relation for the chromatic symmetric function.” European J. Combin. 89 (2020): 103–43.

[7] Ellzey, B. “Chromatic quasi-symmetric functions of directed graphs.” Sémin. Lothar. Combin. 78B Article 74 (2017): 12 pp.

[8] Féray, V. “Cyclic inclusion-exclusion.” SIAM J. Discrete Math. 29 (2015): 2284–311. https://doi.org/10.1137/140991364.

[9] Garsia, A. and N. Wallach. “r-OSym is free over Sym.” J. Combin. Theory Ser. A 114 (2007): 704–32. https://doi.org/10.1016/j.jcta.2006.08.009.

[10] Gebhard, D. and B. Sagan. “A chromatic symmetric function in noncommuting variables.” J. Algebraic Combin. 13 (2001): 227–55. https://doi.org/10.1023/A:1011258714032.

[11] Gessel, I. “A historical survey of P-partitions.” The Mathematical Legacy of Richard P. Stanley. Rhode Island: American Mathematical Society, 2016.

[12] Gessel, I. “Multipartite P-partitions and inner products of skew Schur functions.” Contemp. Math. 34 (1984): 289–301. https://doi.org/10.1090/conm/034/777705.

[13] Grinberg, D. and V. Reiner. “Hopf algebras in combinatorics.” (2020): preprint arXiv:1409.8356v7.

[14] Hivert, F. “Local action of the symmetric group and generalizations of quasi-symmetric functions.” FPSAC Proc. 2005 (2005) 16 pp.

[15] Hivert, F. Combinatoire et calculs symboliques dans les algèbres de Hopf. Habilitation thesis, Gaspard Monge Institute, 2004.

[16] Humpert, B. “A quasisymmetric function generalization of the chromatic symmetric function.” Electron. J. Combin. 18 (2011) 13 pp. https://doi.org/10.37236/518.

[17] Luoto, K., S. Mykytiuk, and S. van Willigenburg. An Introduction to Quasisymmetric Schur Functions. Hopf Algebras, Quasisymmetric Functions, and Young Composition Tableaux. New York: Springer, 2013, https://doi.org/10.1007/978-1-4614-7300-8.

[18] MacMahon, P. “Memoir on the theory of the partitions of numbers. Part V. Partitions in two dimensional space.” Proc. R. Soc. Lond. A 85 (1911): 304–5. https://doi.org/10.1098/rspa.1911.0044.

[19] Niese, E., S. Sundaram, S. van Willigenburg, J. Vega, and S. Wang. “Row-strict dual immaculate functions.” Adv. in Appl. Math. 149 (2023): 102540. https://doi.org/10.1016/j.aam.2023.102540.

[20] Rosas, M. and B. Sagan. “Symmetric functions in noncommuting variables.” Trans. Amer. Math. Soc. 358 (2004): 215–32. https://doi.org/10.1090/S0002-9947-04-03623-2.

[21] Shareshian, J. and M. Wachs. “Chromatic quasisymmetric functions.” Adv. Math. 295 (2016): 497–551. https://doi.org/10.1016/j.aim.2015.12.018.

[22] Sotskov, Y., V. Tanaev, and F. Werner. “Scheduling problems and mixed graph colorings.” Optimization 51 (2002): 597–624. https://doi.org/10.1080/023319302100004994.

[23] Stanley, R. and S. Fomin. Enumerative Combinatorics. Vol. 2. Cambridge: Cambridge University Press, 1999, https://doi.org/10.1017/CBO9780511609589.

[24] Stanley, R. “A symmetric function generalization of the chromatic polynomial of a graph.” Adv. Math. 111 (1995): 166–94. https://doi.org/10.1006/aima.1995.1020.

[25] Stanley, R. Ordered Structures and Partitions. Ph.D. thesis. Harvard University, 1971.
[26] Stanley, R. and J. Stembridge. “On immanants of Jacobi-Trudi matrices and permutations with restricted position.” *J. Combin. Theory Ser. A* 62 (1993): 261–79. https://doi.org/10.1016/0097-3165(93)90048-D.

[27] Takeuchi, M. “Free Hopf algebras generated by coalgebras.” *J. Math. Soc. Japan* 23 (1971): 561–82. https://doi.org/10.2969/jmsj/02340561.

[28] Wolf, M. “Symmetric functions of non-commutative elements.” *Duke Math. J.* 2 (1936): 626–37. https://doi.org/10.1215/S0012-7094-36-00253-3.