Properties of the Zeros of the Sum of two Polynomials

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Some properties—including relations having a Diophantine character—of the zeros of the sum of two polynomials are reported.

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1. Introduction

Let \( p_N(z) \) respectively \( q_M(z) \) be two arbitrary (for definiteness, monic) polynomials of the independent (complex) variable \( z \), of degree \( N \) respectively \( M \) with, for definiteness and simplicity, \( N > M \); and let us define as follows the, also monic, polynomial \( \psi_N(z,t) \) of degree \( N \) in \( z \) and depending on the parameter \( t \):

\[
\psi_N(z,t) = p_N(z) + t q_M(z).
\]

(1.1)

Here and hereafter \( N \) is an arbitrary positive integer (say, larger than unity, to avoid that some of the formulas written below become too degenerate and trivial), and \( t \) is an a priori arbitrary parameter.

Hereafter we denote as \( z_n \equiv z_n(t) \) the \( N \) (generally complex) zeros of this polynomial \( \psi_N(z,t) \):

\[
\psi_N(z,t) = \prod_{n=1}^{N} [z - z_n(t)].
\]

(1.2)

For simplicity, we hereafter assume these \( N \) zeros to be all different among themselves; the formulas displayed below remain generally valid when this assumption is violated, but possibly only after taking an appropriate limit. Likewise for the possibility that the two polynomials \( p_N(z) \) and \( q_M(z) \) have a common zero \( \hat{z} \), \( p_N(\hat{z}) = q_M(\hat{z}) = 0 \); entailing of course that both their wronskian and the
derivative of their wronskian also vanish at \( \xi \) (see below (2.1)); although this case might be excluded without significant loss of generality, since the common zero can be simply factored out and it will then remain fixed (i.e., independent of the parameter \( t \)).

In this paper we review some properties of these \( N \) zeros \( z_n(t) \). Some of these properties are so easily identified that they might be considered already known, others are a bit less easily shown to hold so they might be considered new: in any case I have not seen any of them explicitly highlighted in the literature—although it is hard to be certain since the literature on polynomials and their zeros is too large to allow an exhaustive survey. The survey that has indeed been performed without finding a display of the findings reported below is too vast to justify listing it here; we merely quote the recent preprint [1] on a somewhat related topic—polynomial pencils—with thanks to professors Dimitri Leites and Boris Shapiro for bringing this paper to my attention.

In the following Section 2 these properties are displayed, and they are then proven in Section 3. A terse Section 4 mentions possible extensions of these findings.

### 2. Results

Hereafter we denote as \( w(z) \) the wronskian of the two (monic) polynomials \( p_N(z) \) and \( q_M(z) \).

\[
w(z) = p'_N(z) \, q_M(z) - q'_M(z) \, p_N(z), \quad (2.1a)
\]

\[
w'(z) = p''_N(z) \, q_M(z) - q''_M(z) \, p_N(z). \quad (2.1b)
\]

Here and hereafter appended primes denote differentiations with respect to the variable \( z \). Clearly \( w(z) \) is a polynomial of degree \( N + M - 1 \) (obviously related to the monic polynomial \( \hat{w}(z) \) as follows: \( w(z) = (N - M) \, \hat{w}(z) \)).

**Proposition 2.1.** The \( N \) zeros \( z_n \equiv z_n(t) \) of the polynomial \( \psi_N(z,t) \), see (1.1) and (1.2), satisfy the following system of \( N \) nonlinear algebraic equations:

\[
\sum_{\ell=1}^{N} \frac{1}{z_n - z_{\ell}} = \frac{w'(z_n)}{2 \, w(z_n)},
\]

\[
= \frac{p''_N(z_n) \, q_M(z_n) - q''_M(z_n) \, p_N(z_n)}{2 \, [p'_N(z_n) \, q_M(z_n) - q'_M(z_n) \, p_N(z_n)]}, \quad n = 1, \ldots, N. \quad (2.2)
\]

Here (and hereafter) the notation \( f'(z_n) \) indicates of course the derivative of the function \( f(z) \) with respect to \( z \), \( f'(z) \equiv df(z)/dz \), evaluated at \( z = z_n = z_n(t) \).

Note that, for notational simplicity, we did not display in (2.2) the \( t \)-dependence of the \( N \) zeros \( z_m \equiv z_m(t) \) of \( \psi_N(z,t) \), see (1.2) and (1.1), which are of course generally quite different from the \( (t \)-independent) zeros of \( w(z) \), see (2.1a).

**Corollary 2.1.** The \( N \) zeros \( z_n \equiv z_n(t) \) of the polynomial \( \psi_N(z,t) \), see (1.1) and (1.2), satisfy the following sum rule:

\[
\sum_{n=1}^{N} \frac{w'(z_n)}{w(z_n)} = 0. \quad (2.3)
\]
Again, let us emphasize that we did not display in this formula the \( t \)-dependence of the \( N \) zeros \( z_n \equiv z_n(t) \).

**Proposition 2.2.** Let the \( N \times N \) matrix \( A \equiv A(t) \) be defined, componentwise, as follows in terms of the \( N \) zeros \( z_n \equiv z_n(t) \) of the polynomial \( \psi_N(z, t) \), see (1.1) and (1.2):

\[
A_{nn} = \frac{z_n w'(z_n)}{2 w(z_n)}, \quad n = 1, \ldots, N, \tag{2.4a}
\]
\[
A_{nm} = \frac{z_n}{z_n - z_m}, \quad n, m = 1, \ldots, N, \quad n \neq m. \tag{2.4b}
\]

Then this matrix features the first \( N \) nonnegative integers (from \( 0 \) to \( N - 1 \)) as its eigenvalues,

\[
A^{(m)} = (m - 1) v^{(m)}, \quad m = 1, \ldots, N, \tag{2.5a}
\]

with the \( N \) eigenvectors \( v^{(m)} \equiv v^{(m)}(t) \) defined componentwise as follows:

\[
v_n^{(m)} = z_n^{m-1} v_n^{(0)}, \quad v_n^{(0)} = \prod_{\ell=1}^{N} (z_n - z_\ell)^{-1}, \quad m, n = 1, \ldots, N. \tag{2.5b}
\]

Note that neither the \( N \times N \) matrix \( A \), nor its \( N \) eigenvectors \( v^{(m)} \), feature explicitly in their definitions the parameter \( t \); but they of course do depend on \( t \) via the dependence of the \( N \) zeros \( z_n \equiv z_n(t) \) on that parameter. While instead the \( N \) eigenvalues of \( A \) are independent of \( t \) and have a neat Diophantine connotation.

These findings imply several consequences: see for instance Section 2.4 entitled “Finite-dimensional representation of differential operators, Lagrangian interpolation, and all that” of [2]. Here we only report the following

**Corollary 2.2.** The \( N \) zeros \( z_n \equiv z_n(t) \) of the polynomial \( \psi_N(z, t) \), see (1.1) and (1.2), satisfy the following relations:

\[
\sum_{n=1}^{N} \left( \frac{z_n w'(z_n)}{w(z_n)} \right) = N (N - 1); \tag{2.6a}
\]
\[
\det A = 0, \quad \det [I + A] = N!. \tag{2.6b}
\]

Here \( I \) is the \( N \times N \) unit matrix.

Again, at the cost of being repetitive, let us emphasize that in the left-hand side of these formulas the \( N \) zeros \( z_n \equiv z_n(t) \) of \( \psi_N(z, t) \), see (1.1) and (1.2)—which appear explicitly in (2.6a), and in (2.6b) via the definition (2.4) of the \( N \times N \) matrix \( A \)—depend on the parameter \( t \), while the right-hand sides of these formulas are clearly independent of that parameter and have a Diophantine connotation.
Proposition 2.3. The \(N\) zeros \(z_n(t)\) of \(\psi_N(z, t)\), see (1.1) and (1.2), satisfy the system of \(N\) ODEs

\[
\ddot{z}_n = \sum_{\ell \neq n}^{N} \left( \frac{2 \dot{z}_n \dot{z}_\ell}{z_n - z_\ell} \right), \quad n = 1, \ldots, N, \tag{2.7}\]

with the polynomials \(p_N(z)\) and \(q_M(z)\) defined as follows in terms of the “initial” values of the \(N\) zeros \(z_n(t)\) and their \(t\)-derivatives:

\[
p_N(z) = \psi_N(z, 0) = \prod_{n=1}^{N} [z - z_n(0)], \tag{2.8a}\]

\[
q_M(z) = -p_N(z) \sum_{n=1}^{N} \left[ \frac{\dot{z}_n(0)}{z - z_n(0)} \right] = \sum_{n=1}^{N} \left\{ \dot{z}_n(0) \prod_{\ell \neq n}^{N} [z - z_\ell(0)] \right\}. \tag{2.8b}\]

Here and hereafter superimposed dots denote differentiations with respect to \(t\), \(\dot{z}_n(t) \equiv d z_n(t)/d t\), \(\ddot{z}_n(t) \equiv d^2 z_n(t)/d t^2\); and note that the second of the two formulas (2.8) implies that, for generic “initial” data \(z_n(0), \dot{z}_n(0)\), there holds the relation \(M = N - 1\).

Note that this system of ODEs, (2.7), coincides—via the identification of the parameter \(t\) with “time”—with the equations of motion of Newtonian type (“accelerations equal forces”, with velocity-dependent two-body forces)—of the “goldfish” \(N\)-body problem [3,4,2,5]. Clearly for a generic assignment of the polynomials \(p_N(z)\) and \(q_M(z)\) (and \(t\) real) the \(N\) zeros \(z_n(t)\) coincide with the \(N\) zeros of the polynomial \(p_N(z)\) at \(t = 0\) (see (1.1) and (2.8a)), while \(M\) of them tend to the \(M\) zeros of the polynomial \(q_M(z)\) as \(t \to \pm \infty\) and the remaining \(N - M\) ones escape to infinity (see the more detailed description of this phenomenology in Section 4.2.4 entitled “The simplest model: explicit solution (the game of musical chairs), Hamiltonian structure” in the book [2]). It is also clear that, for generic initial data, the motion of the \(N\) zeros \(z_n(t)\) in the complex \(z\)-plane as functions of time (i.e., of real \(t\)) does not entail any “particle collision”, corresponding of course to a singularity of the equations of motion (2.7); but such singularities are possible for a lower-dimensional set of initial data, and in such cases it is easily seen that the typical behavior of the interparticle distance of the two colliding particles, say \(z_n(t) - z_m(t)\), is to vanish proportionally to \(\delta = |t - t_c|^{-1/2}\) (where \(t_c\) is the time at which the collision occurs), with the corresponding velocities respectively accelerations diverging proportionally to \(\delta^{-1}\) respectively \(\delta^{-2}\). This phenomenology of course also accounts for the initial behavior at \(t = 0\) in the case when the polynomial \(p_N(z)\) features a double zero.

3. Proofs

The proof of Proposition 2.1 is rather close to the treatment given in [6] (see in particular Remark 2.5 there); but its presentation, see below, does not require reading that paper. The point of departure of this proof is the well-known observation that, up to its normalization and an appropriate identification of the parameter \(t\) (both specified above, see (1.1)), the polynomial \(\psi_N(z, t)\) can be identified as
the solution of the second-order ODE

$$\det \begin{pmatrix} \psi''_N(z,t) & p''_N(z) & q''_M(z) \\ \psi'_N(z,t) & p'_N(z) & q'_M(z) \\ \psi_N(z,t) & p_N(z) & q_M(z) \end{pmatrix} = 0,$$

(3.1a)

or equivalently (see (2.1a))

$$w(z) \psi''_N(z,t) - w'(z) \psi'_N(z,t) + \left[p''_N(z) q'_M(z) - q''_M(z) p'_N(z)\right] \psi_N(z,t) = 0.$$  

(3.1b)

At $z = z_n \equiv z_n(t)$ (where $\psi_N(z,t)$ vanishes, see (1.2)) the last formula clearly entails that

$$w(z_n) \psi''_N(z_n,t) - w'(z_n) \psi'_N(z_n,t) = 0,$$

(3.2a)

hence

$$\frac{\psi''_N(z_n,t)}{\psi'_N(z_n,t)} = \frac{w'(z_n)}{w(z_n)},$$

(3.2b)

On the other hand logarithmic $z$-differentiation of (1.2) implies

$$\psi''_N(z_n,t) = \psi_N(z_n,t) \sum_{m=1}^{N} \left( \frac{1}{z - z_m} \right),$$

(3.3)

and a second $z$-differentiation yields (via (3.3), and a neat cancellation of the terms featuring double poles)

$$\psi''_N(z,t) = \psi_N(z,t) \sum_{m, \ell=1}^{N} \left( \frac{1}{z - z_m} \right) \left( \frac{1}{z - z_\ell} \right),$$

(3.4)

so that

$$\frac{\psi''_N(z,t)}{\psi'_N(z,t)} = \left\{ \sum_{m, \ell=1}^{N} \left( \frac{1}{z - z_m} \right) \left( \frac{1}{z - z_\ell} \right) \right\}^{-1} \left\{ \sum_{m=1}^{N} \left( \frac{1}{z - z_m} \right) \right\}^{-1}. $$

(3.5)

And clearly for $z = z_n$ this formula yields

$$\frac{\psi''_N(z_n,t)}{\psi'_N(z_n,t)} = 2 \sum_{\ell=1}^{N} \left( \frac{1}{z_n - z_\ell} \right),$$

(3.6)

hence, via (3.2b),

$$\sum_{\ell=1}^{N} \left( \frac{1}{z_n - z_\ell} \right) = \frac{w'(z_n)}{2 w(z_n)}.$$  

(3.7)

The first equality (2.2) is thereby proven. As for the second, it is clearly an immediate consequence of (2.1).
Corollary 2.1 is then an immediate consequence of Proposition 2.1, since the sum over \( n \) from 1 to \( N \) of the left-hand side of (2.2) clearly vanishes due to the antisymmetry of the summand under the exchange of the two dummy indices \( n \) and \( \ell \).

The proof of Proposition 2.2 is also an immediate consequence of Proposition 2.1, due to the fact—which should by now be well-known, see for instance Section 2.4 (entitled “Finite-dimensional representations of differential operators, Lagrangian interpolation, and all that”) of [2]—that the \( N \times N \) matrix \( \tilde{A} \) defined componentwise as follows in terms of \( N \) arbitrary numbers \( \tilde{z}_n \),

\[
\tilde{A}_{nn} = \sum_{\ell \neq n}^{N} \left( \frac{\tilde{z}_n}{\tilde{z}_n - \tilde{z}_\ell} \right), \quad n = 1, \ldots, N, \tag{3.8a}
\]

\[
\tilde{A}_{nm} = \frac{\tilde{z}_n}{\tilde{z}_n - \tilde{z}_m}, \quad n, m = 1, \ldots, N, \quad n \neq m, \tag{3.8b}
\]

has the first \( N \) nonnegative integers (from 0 to \( N - 1 \)) as its eigenvalues,

\[
\tilde{A} \sum^{(m)} = (m - 1) \sum^{(m)}, \quad m = 1, \ldots, N, \tag{3.9a}
\]

with the \( N \) eigenvectors \( \tilde{v}^{(m)} \) defined componentwise as follows,

\[
\tilde{v}_n^{(m)} = z_n^{m-1} \tilde{v}_n^{(0)} \quad \tilde{v}_n^{(0)} = \prod_{\ell \neq n}^{N} (\tilde{z}_n - \tilde{z}_\ell)^{-1}, \quad m, n = 1, \ldots, N. \tag{3.9b}
\]

As for Corollary 2.2, clearly the first formula, (2.6a), corresponds merely to the identification of the trace of the \( N \times N \) matrix \( A \), see (2.4), with the sum of its eigenvalues, see (2.5a); and likewise the other two, (2.6b), correspond to the identification of the determinants of the two \( N \times N \) matrices \( A \) and \( I + A \) with the product of their eigenvalues, see (2.5a).

As for Proposition 2.3, it is an immediate consequence of the well-known fact—first proven in [4], and see also [3,2,5]—that the solution of the initial-value problem for the goldfish \( N \)-body system characterized by the Newtonian equations of motion (2.7) is provided by the \( N \) zeros \( z_n(t) \) of the polynomial (1.1) with the identifications (2.8). Note that the first of these formulas, (2.8a), is an immediate consequence of (1.1), while the second obtains by the evaluation of the logarithmic \( t \)-derivative of \( \psi_N(z,t) \), see (1.1) and (1.2), yielding

\[
q_M(z) = -\psi_N(z,t) \sum_{m=1}^{N} \left[ \frac{\dot{z}_m(t)}{z - \dot{z}_m(t)} \right]. \tag{3.10}
\]

Indeed the obvious \( t \)-independence of the left-hand side of this formula implies the less obvious fact that the right-hand side is as well \( t \)-independent—a reflection of the integrable character of the goldfish \( N \)-body model (2.7). And of course at \( t = 0 \) this formula, via (2.8a), implies (2.8b).

4. Outlook
The extension of these findings to the zeros of sums of more than two polynomials shall be reported in subsequent papers if these results will be deemed sufficiently novel and neat to justify their publication.
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References

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