EXACT EXPONENTIAL BOUNDS FOR THE RANDOM FIELD MAXIMUM DISTRIBUTION VIA THE MAJORING MEASURES (GENERIC CHAINING)

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In this paper non-asymptotic exact exponential estimates are derived for the tail of maximums distribution of random field in the terms of majoring measures or, equally, generic chaining.

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1. Introduction. Notations. Statement of problem.

Let \((\Omega, F, \mathbb{P})\) be a probability space, \(\Omega = \{\omega\}\), \(T = \{t\}\) be arbitrary set, \(\xi(t), t \in T\) be centered: \(E\xi(t) = 0\) separable random field (or process). For arbitrary subset \(S \subset T\) we denote

\[
Q(S, u) = P(\sup_{t \in S} \xi(t) > u), \quad u \geq 2. \tag{1}
\]

\[
Q_+(S, u) = P(\sup_{t \in S} |\xi(t)| > u), \quad u \geq 2. \tag{2}
\]

Our aim is obtaining an exponentially exact as \(u \to \infty\) estimation for the probability \(Q(u) \overset{\text{def}}{=} Q(T, u)\) in the terms of majoring measures or equally in the terms of generic chaining.

Definitions and some important results about \(E\sup_{t \in T} \xi(t)\) in the terms of majoring measures see, for example, in [2], [3], [6], p. 309 - 330, [10], [11], [12], [13]. In the so-called entropy terms this problem was considered in [3], [4]. See also [8].

Note that the majoring measures method is more general in comparison to the entropy technique ([6], p. 309 - 330, [10], [12]).

2. Auxiliary facts. In order to formulate our result, we need to introduce some addition notations and conditions. Let \(\phi = \phi(\lambda), \lambda \in (-\lambda_0, \lambda_0), \lambda_0 = const \in (0, \infty]\) be some even strong convex which takes positive values for positive arguments twice
continuous differentiable function, such that
\[ \phi(0) = 0, \quad \phi'/(0) > 0, \quad \lim_{\lambda \to \lambda_0} \phi(\lambda)/\lambda = \infty. \] (3)

We denote the set of all these function as \( \Phi; \Phi = \{ \phi(\cdot) \} \).

We say that the centered random variable (r.v) \( \xi = \xi(\omega) \) belongs to the space \( B(\phi) \), if there exists some non-negative constant \( \tau \geq 0 \) such that
\[ \forall \lambda \in (-\lambda_0, \lambda_0) \Rightarrow E \exp(\lambda \xi) \leq \exp[\phi(\lambda \tau)]. \] (4)
The minimal value \( \tau \) satisfying (4) is called a \( B(\phi) \) norm of the variable \( \xi \), write
\[ ||\xi||_{B(\phi)} = \inf\{\tau, \tau > 0 : \forall \lambda \Rightarrow E \exp(\lambda \xi) \leq \exp(\phi(\lambda \tau))\}. \]

This spaces are very convenient for the investigation of the r.v. having a exponential decreasing tail of distribution, for instance, for investigation of the limit theorem, the exponential bounds of distribution for sums of random variables, non-asymptotical properties, problem of continuous of random fields, study of Central Limit Theorem in the Banach space etc.

The space \( B(\phi) \) with respect to the norm \( || \cdot ||_{B(\phi)} \) and ordinary operations is a Banach space which is isomorphic to the subspace consisted on all the centered variables of Orlichs space \( (\Omega, F, P), N(\cdot) \) with \( N \) – function
\[ N(u) = \exp(\phi^*(u)) - 1, \quad \phi^*(u) = \sup_{\lambda} (\lambda u - \phi(\lambda)). \]
The transform \( \phi \to \phi^* \) is called Young-Fenchel transform. The proof of considered assertion used the properties of saddle-point method and theorem of Fenchel-Moraux:
\[ \phi^{**} = \phi. \]

The next facts about the \( B(\phi) \) spaces are proved in [4], [8, p. 19-40]:

1. \( \xi \in B(\phi) \Leftrightarrow E\xi = 0, \text{ and } \exists C = const > 0, \)
\[ U(\xi, x) \leq \exp(-\phi^*(Cx)), x \geq 0, \]
where \( U(\xi, x) \) denotes in this article the tail of distribution of the r.v. \( \xi \):
\[ U(\xi, x) = \max (P(\xi > x), P(\xi < -x)), x \geq 0, \]
and this estimation is in general case asymptotically exact.

Here and further \( C, C_j, C(i) \) will denote the non-essentially positive finite ”constructive” constants.

More exactly, if \( \lambda_0 = \infty \), then the following implication holds:
\[ \lim_{\lambda \to \infty} \phi^{-1}(\log E \exp(\lambda \xi))/\lambda = K \in (0, \infty) \]
if and only if
\[
\lim_{x \to \infty} (\phi^*)^{-1}(|\log U(\xi, x)|)/x = 1/K.
\]
Here and further \(f^{-1}(\cdot)\) denotes the inverse function to the function \(f\) on the left-side half-line \((C, \infty)\).

The function \(\phi(\cdot)\) may be constructive introduced by the formula
\[
\phi(\lambda) = \phi_0(\lambda) \overset{df}{=} \log \sup_{t \in T} \mathbb{E} \exp(\lambda \xi(t)),
\]
if obviously the family of the centered r.v. \(\{\xi(t), t \in T\}\) satisfies the *uniform* Kramers condition:
\[
\exists \mu \in (0, \infty), \sup_{t \in T} U(\xi(t), x) \leq \exp(-\mu x), \quad x \geq 0.
\]

In this case, i.e. in the case the choice the function \(\phi(\cdot)\) by the formula (5), we will call the function \(\phi(\lambda) = \phi_0(\lambda)\) a natural function.

2. We define \(\psi(p) = p/\phi^{-1}(p), \quad p \geq 2\). Let us introduce a new norm (the so-called moment norm) on the set of r.v. defined in our probability space by the following way: the space \(G(\psi)\) consist, by definition, on all the centered r.v. with finite norm
\[
||\xi||_{G(\psi)} \overset{df}{=} \sup_{p \geq 2} |\xi|_p/\psi(p), \quad |\xi|_p = \mathbb{E}^{1/p}|\xi|^p.
\]

It is proved that the spaces \(B(\phi)\) and \(G(\psi)\) coincides: \(B(\phi) = G(\psi)\) (set equality) and both the norm \(|| \cdot ||_{B(\phi)}\) and \(|| \cdot ||\) are equivalent: \(\exists C_1 = C_1(\phi), C_2 = C_2(\phi) = const \in (0, \infty), \forall \xi \in B(\phi)\)
\[
||\xi||_{G(\psi)} \leq C_1 ||\xi||_{B(\phi)} \leq C_2 ||\xi||_{G(\psi)}.
\]

3. The definition (6) is correct still for the non-centered random variables \(\xi\). If for some non-zero r.v. \(\xi\) we have \(||\xi||_{G(\psi)} < \infty\), then for all positive values \(u\)
\[
\mathbb{P}(||\xi|| > u) \leq \exp(-u/(C_3 ||\xi||_{G(\psi)})).
\]
and conversely if a r.v. \(\xi\) satisfies (7), then \(||\xi||_{G(\psi)} < \infty\).

We suppose in this article that there exists a function \(\phi \in \Phi\) such that \(\forall t \in T \Rightarrow \xi(t) \in B(\phi)\) and
\[
\sup_{t} [||\xi(t)||_{B(\phi)}] < \infty,
\]
or equally \(\mathbb{E}\xi(t) = 0, \quad t \in T,\) and for all non-negative values \(x\)
\[
\sup_{t} \max [\mathbb{P}(\xi(t) > x), \mathbb{P}(\xi(t) < -x)] \leq \exp(-\phi^*(x)).
\]

Note that if for some \(C = const \in (0, \infty)\)
\[
Q_+(T, u) \leq \exp(-\phi^*(Cu)),
\]
then the condition (8) is satisfied (the necessity of the condition (8)).

M. Talagrand [10] [13], W. Bednorz [2], X. Fernique [3] etc. write instead our function \( \exp (-\phi^*(x)) \) some Youngs function \( \Psi(x) \) and used as a rule a function

\[ \Psi(x) = \exp(-x^2/2) \]

( the so-called subgaussian case).

Without loss of generality we can and will suppose

\[ \sup_{t \in T} ||\xi(t)||B(\phi) = 1, \]

(this condition is satisfied automatically in the case of natural choosing of the function \( \phi: \phi(\lambda) = \phi_0(\lambda) \) ) and that the metric space \((T,d)\) relatively the so-called natural distance (more exactly, semi-distance)

\[ d(t,s) \overset{\text{def}}{=} ||\xi(t) - \xi(s)||B(\phi) \]

is complete.

Recall that the semi-distance \( d = d(t,s) \), \( s,t \in T \) is, by definition, non-negative symmetrical numerical function, \( d(t,t) = 0 \), \( t \in T \), satisfying the triangle inequality, but the equality \( d(t,s) = 0 \) does not means (in general case) that \( s = t \).

For example, if \( \xi(t) \) is a centered Gaussian field with covariation function

\[ D(t,s) = \text{Var}[\xi(t) - \xi(s)] = \sqrt{D(t,t) - 2D(t,s) + D(s,s)}. \]

There are many examples of martingales, e.g., in the article [7], \((\xi(n), F(n))\), \( T = \{1, 2, 3, \ldots, n, \ldots\} \) satisfying the following modification \((8a)\) of the condition (8):

\[ \sup_n U(\xi(n)/\sigma(n), x) \leq \exp[-\phi^*(x)], \quad (8a), \]

in particular, there are many examples with

\[ \phi^*(x) \sim x^r L^{1/r}(x)/r, \quad x \to \infty; \quad r = \text{const} \geq 1, \quad (9) \]

where as usually \( f(x) \sim g(x), \quad x \to \infty \) denotes

\[ \lim_{x \to \infty} f(x)/g(x) = 1; \]

and with

\[ n^\beta L_1(n) \leq \sigma(n) \leq n^\beta L_2(n), \quad \beta = \text{const} > 0, \quad (10) \]

\( L_1(x), L_2(x), L(x) \) are some positive continuous slowly varying as \( x \to \infty \) functions,

\[ \sigma(n) = \sqrt{\text{Var}(\xi(n))}. \]

It is known ( [4], [8], p. 22 - 25) that (9) is equivalent in the case \( r > 1 \) (under some simple assumption) to the following equality:

\[ \lambda \to \infty \Rightarrow \phi(\lambda) \sim \lambda^s L^{-1/s}(\lambda^s)/s, \quad s = r/(r-1). \]
Let us introduce for any subset $V, V \subset T$ the so-called entropy $H(V, d, \epsilon) = H(V, \epsilon)$ as a logarithm of a minimal quantity $N(V, d, \epsilon) = N(V, \epsilon) = N$ of a balls $S(V, t, \epsilon), \ t \in V$:

$$S(V, t, \epsilon) \overset{\text{def}}{=} \{s, s \in V, \ d(s, t) \leq \epsilon\},$$

which cover the set $V$:

$$N = \min\{M : \exists\{t_i\}, i = 1, 2, \ldots, M, \ t_i \in V, \ V \subset \bigcup_{i=1}^{M} S(V, t_i, \epsilon)\},$$

and we denote also

$$H(V, d, \epsilon) = \log N; \ S(t_0, \epsilon) \overset{\text{def}}{=} S(T, t_0, \epsilon), \ H(d, \epsilon) \overset{\text{def}}{=} H(T, d, \epsilon).$$

It follows from Hausdorf’s theorem that $\forall \epsilon > 0 \Rightarrow H(V, d, \epsilon) < \infty$ iff the metric space $(V, d)$ is precompact set, i.e. is the bounded set with compact closure.

Now we recall, modify and rewrite some definition of generic chaining theory, belonging to X.Fernique [3] and M.Talagrand [10] - [13]. Let the ball $S(t_0, \epsilon) = S(t_0, \delta), \ t_0 \in T, \ \delta \in (0, 1]$ be a given. The sequence $R$ of finite subsets of $S(t_0, \delta)$ $T_m, m = 0, 1, 2, \ldots, T_m \subset S(t_0, \delta), R = \{T_m\}$ such that $T_0 = \{t_0\}$; here and further a symbol $|V|$ will denote the number of elements of a finite set $V$: $|V| = \text{card}(V)$, and such that the set $\bigcup_{n=0}^{\infty} T_n$ is dense in $T$ with respect to the semi-distance $d$, is called generic chaining of $S(t_0, \delta)$. The set of all generic chaining we will denote $W$:

$$W = W(S(t_0, \delta)) = W(t_0, \delta) \overset{\text{def}}{=} \{R\}.$$

For any element $t \in S(t_0, \delta)$ we denote arbitrary, but fixed (non-random) element $\pi_n(t)$ of a subset of $T_n : \pi_n(t) \in T_n$ such that

$$d(t, \pi_n(t)) = \min_{s \in T_n} d(t, s). \ (11)$$

Let $\gamma = \{\gamma_n\}, \ n = 1, 2, \ldots$ be arbitrary fixed non-random sequence of a positive numbers such that

$$\sum_{n=1}^{\infty} 1/\gamma_n = 1, \ \gamma_1 \geq 3; \ (12)$$

for example, $1/\gamma_n = \rho^{n-1}(1-\rho), \ \rho = \text{const} \in (2/3, 1)$. Let us introduce the following important function

$$L(t_0, \delta, R, \gamma) = L(t_0, \delta, R) = \sup_{t \in B(t_0, \delta)} \sum_{m=1}^{\infty} d(\pi_m(t), \pi_{m-1}(t))/\gamma_m. \ (13)$$

We will consider only the so-called admissible random fields (in the terms of M.Talagran) $\xi(\cdot)$, i.e. which satisfied the following conditions.

Let us denote

$$K(\xi, \phi, \delta) = K(\delta) = \inf_{R \in A} \inf_{\gamma} \sup_{t_0 \in T} L(t_0, \delta, R, \gamma),$$

if the set $A$ is not empty, and $K(\delta) = +\infty$ in the other case.
The following conclusions will be interest only in the case if for some function \( \phi(\cdot) \in \Phi \), for example for the natural function \( \phi_0(\cdot) \)

\[
\lim_{\delta \to 0^+} K(\xi, \phi, \delta) = 0. \tag{14}
\]

We will suppose moreover that the condition (14), which will called the uniform generic chaining condition, write: \( \xi(\cdot), \phi \in UA \), is satisfied.

Let us introduce also the events \( D, E(n), n \geq 1 \) as follow: \( E(n) = E(u; n, t_0, \delta, R) = \)

\[
\cup_{t \in S(t_0, \delta)} [\xi(\pi_n(t)) - \xi(\pi_{n-1}(t)) > u d(\pi_n(t), \pi_{n-1}(t))/\gamma_n] =
\]

\[
\cup_{t \in T_{n-1}} [\xi(\pi_n(t)) - \xi(\pi_{n-1}(t)) > u d(\pi_n(t), \pi_{n-1}(t))/\gamma_n] =
\]

\[
\{ \omega : \max_{t \in T_{n-1}} \frac{\xi(\pi_{n-1}(t)) - \xi(\pi_n(t))}{d(\pi_{n-1}(t), \pi_n(t))} > \frac{u}{\gamma_n} \},
\]

if we define \( 0/0 = 0 \) (in the case if \( d(\pi_{n-1}(t), \pi_n(t)) = 0 \));

\[
D = D(u; t_0, \delta, R) = \cup_{n=1}^{\infty} E(u; n, t_0, \gamma, R).
\]

We denote also

\[
Z_n = Z_n(u; t_0, \delta, \gamma, R) = P[E(u; n, t_0, \gamma, R)],
\]

\[
Y(u) = Y(u; t_0, \delta, R) = P[D(u; t_0, \delta, R)].
\]

It is evident that

\[
Z_n \leq |T_n| |T_{n-1}| \exp (-\phi^*(u/\gamma_n)),
\]

\[
Y(u) \leq \sum_{n=1}^{\infty} |T_n| |T_{n-1}| \exp (-\phi^*(u/\gamma_n)) \overset{\text{def}}{=} X(u; t_0, \delta, \gamma, R), \tag{15}
\]

since

\[
P[\xi(\pi_n(t)) - \xi(\pi_{n-1}(t)) > u d(\pi_n(t), \pi_{n-1}(t))] \leq \exp (-\phi^*(u)), \ u > 0.
\]

The random field \( \xi(t) \) and the function \( \phi(\cdot) \) satisfies, by definition, the uniform generic chaining condition, and write \( \xi(\cdot) \in A \), or more simple: there exists the set of generic chaining \( R \) (depending on the \( \xi(\cdot) \)) belonging to \( A \), \( R \in A \), if for all \( \delta \in (0, 1) \) and for arbitrary ball \( S(t_0, \delta) \) there exists (for some sequence \( \{\gamma\} \)) a generic chaining \( R \) in \( S(t_0, \delta) \) for which

\[
Y(u; t_0, \delta, \gamma, R) \leq \exp (-\phi^*(u/2)), \tag{16}
\]
if, for example,  

\[ X(u; t_0, \delta, \gamma, R) \leq \exp \left( -\phi^* \left( \frac{u}{2} \right) \right). \]  

\[ (16a) \]

The existence of such a generic chaining it follows from our next assumptions.

**Lemma 1.** We have under the conditions (14) and (16) (or (16a)) for all the values \( \delta \in (0, 1) \):

\[
\sup_{t_0 \in T} \mathbb{P} \left[ \sup_{t \in S(t_0, \delta)} (\xi(t) - \xi(t_0)) > u K(\delta) \right] \leq \exp \left( -\phi^* \left( \frac{u}{2} \right) \right).
\]

**Proof.** The proof of this assertion is alike to the original proof of Talagran ([10], [12, chapter 1, pp. 9 - 14]) for the probability \( Q(u) = Q(T; u) \) estimation. Namely, let \( t_0 \) be arbitrary element of \( T \), \( \delta \in (0, 1] \). Let also \( R = \{T_0, T_1, T_2, \ldots\} \), \( R \in W \) be arbitrary chaining into the ball \( S(t_0, \delta) \). We rewrite the Talagran's decomposition ([12], chapter 1, p. 10) for the ball \( S(t_0, \delta) \):

\[
\xi(t) - \xi(t_0) = \sum_{n=1}^{\infty} [\xi(\pi_n(t)) - \xi(\pi_{n-1}(t))].
\]

Recall that \( \pi_0(t) = t_0 \) and that \( \forall t \in T \)

\[
\lim_{n \to \infty} \pi_n(t) = t, \quad \lim_{n \to \infty} \xi(\pi_n(t)) = \xi(t)
\]

in the sense of convergence in probability.

We get analogously to the works [10], [11] and taking into account the inclusion \( R \in A \):

\[
G(u; t_0, \delta) \overset{\text{def}}{=} \mathbb{P} \left( \sup_{t \in S(t_0, \delta)} (\xi(t) - \xi(t_0)) > u K(\delta) \right) \leq Y(u; t_0, \delta, \gamma, R) \leq \exp \left( -\phi^* \left( \frac{u}{2} \right) \right).
\]

\[ (17) \]

Note that it follows from conclusion of Lemma 1 the continuity of \( \xi(t) \) with probability one in the semi-distance \( d \):

\[
\mathbb{P}(\xi(\cdot) \in C(T, d)) = 1;
\]

\( C(T, d) \) denotes as usually the space of all continuous with respect to the semi-distance \( d \) functions \( f : T \to R \).

The conditions (14) and (16) is equivalent to the so-called condition of the uniform convergence of the majoring integral, see [10], [11].

**3. Main result.** Let us denote for \( h \in \left( 0, \sup_{\delta \in (0, 1)} K(\delta) \right) \overset{\text{def}}{=} (0, K_0) \)

\[
K^{-1}(h) = \inf \{\delta, \delta \in (0, 1), \ K(\delta) \geq h\},
\]

\[
\Delta(C, u) = \Delta_\phi(u) = K^{-1} \left[ 0.5 \ C / (u \ \phi^* (u)/du) \right],
\]

\[
\Delta_\phi(u) = K^{-1} \left[ 0.5 \ C / (u \ \phi^* (u)/du) \right],
\]
where $d/du$ denotes the right derivative; it is obvious that the derivative $d\phi^*(u)/du$ there exists, is continuous and the function $u \to \Delta(u)$ tends monotonically to zero as $u \to \infty$. Therefore, for arbitrary constant $C \in [1, \infty)$ there is a positive value $u_0 = u_0(C)$, for which $u \geq u_0 \Rightarrow \Delta(C, u) \leq 0.5K_0$.

**Theorem 1.** Suppose for any function $\phi(\cdot) \in \Phi$

$$\lim_{\delta \to 0^+} K(\xi(\cdot), \phi, \delta) = 0$$

and suppose the condition (16), or, more generally, (16a) is also satisfied.

Then for arbitrary constant $C \in (0, \infty)$ and for all the values $u \geq u_0(C)$

$$Q(u) \leq [\exp(C) + 1] N(T, d, C\Delta\phi(u)) \exp(-\phi^*(u)). \quad (18)$$

As a consequence:

$$Q_+(u) \leq 2 [\exp(C) + 1] N(T, d, C\Delta\phi(u)) \exp(-\phi^*(u)).$$

**Proof.** Step 1. Let $C$ be arbitrary positive constant,

$$u \geq u_0(C), \delta_0 = \delta_0(u) \overset{\text{def}}{=} K^{-1}(0.5 C \Delta(u)),$$

We consider at first the probability

$$Q(W, u) = P \left( \sup_{t \in S(t_0, \delta_0)} \xi(t) > u \right),$$

$W = S(t_0, \delta_0)$. Denote $\beta = C \Delta(u), \alpha = 1 - \beta$; then $\alpha, \beta > 0, \alpha + \beta = 1$. We obtain:

$$Q(W, u) \leq P(\xi(t_0) > \alpha u) + P(\sup_{t \in W}(\xi(t) - \xi(t_0)) > \beta u) \overset{\text{def}}{=} I_1 + I_2.$$ 

For the first member is true the simple estimation:

$$I_1 \leq \exp(-\phi^*(\alpha u)).$$

As long as the function $x \to \phi^*(x)$ is convex and twice differentiable,

$$\phi^*(\alpha u) \geq \phi^*(u) - (\phi^*)'(u) C u \Delta(u) = \phi^*(u) - C;$$

therefore

$$I_1 \leq \exp(C) \exp(-\phi^*(u)).$$

Further, since

$$K(\delta_0) \leq 0.5 C \Delta(u),$$

we conclude using the inequality (17)

$$I_2 \leq \exp(-\phi^*(u)).$$
Summing, we receive:
\[ Q(W, u) \leq C_1 \exp(-\phi^*(u)), \quad C_1 = 1 + \exp C. \]

**Step 2.** Let \( \epsilon = C \Delta(u) \) and \( \{ t_i \}, \ i = 1, 2, \ldots, N \), where \( N = N(T, d, \epsilon) \) be a centers of a balls \( B(T, d, \epsilon) \) forming a minimal (not necessary to be unique) \( \epsilon \) – net of \( T \) with respect to the semi-distance \( d \).

Since the probability \( Q(S, u) \) has a property
\[ Q(S_1 \cup S_2, u) \leq Q(S_1, u) + Q(S_2, u), \quad S_1, S_2 \subset T, \]
we conclude:
\[ Q(T, u) \leq \sum_{i=1}^{N} Q(S(t_i, C\Delta(u))). \]

The last probabilities was estimated in (18).

The low bounds for probabilities \( Q(T, u), \ Q_+(T, u) \) was obtained in ([8], 105 - 117); see also [9].

**Corollary.** We explain here the exponential exactness of the estimation of theorem 1.

In many practical cases (statistics, method Monte-Carlo etc.) the entropy \( N(T, d, C\Delta_\phi(u) \) satisfies the inequality:  \( \forall \epsilon \in (0, 1/2) \ \exists U = U(\epsilon) \in (0, \infty) \ \Rightarrow \ \forall u \geq U(\epsilon) \)
\[ N(T, d, C\Delta_\phi(u) \leq \exp(\phi^*(\epsilon u)), \]
for example,
\[ N(T, d, C\Delta_\phi(u)) \leq C(u + 1)^\kappa, \quad \kappa \in (0, \infty), \ u \geq 0. \]

Therefore in this cases
\[ Q(T, u) \leq C_1(\epsilon) \exp(-\phi^*((1 - \epsilon))u). \]

But there exists a random variable \( \xi, \ \xi \in B(\phi), \ ||\xi||B(\phi) = 1 \) for which for \( u \geq U(\epsilon) \)
\[ P(\xi > u) \geq C_2(\epsilon) \exp(-\phi^*((1 + \epsilon))u). \]

**4. Exponential bounds for the sums of random fields.** Let \( \{ \xi_i(t) \}, \ i = 1, 2, \ldots \) be an independent copies of \( \xi(t), \)
\[ \eta_n(t) = n^{-1/2} \sum_{i=1}^{n} \xi_i(t), \]
\[ Q_n(S, u) = P\left( \sup_{t \in S} \eta_n(t) > u \right), \quad Q_n(u) = Q_n(T, u), \]
\[ 9 \]
\( Q_\infty(S, u) = \sup_n Q_n(S, u), \; Q_\infty(u) = Q_\infty(T, u). \)

We obtain in this section using (18) the exponentially exact as \( u \to \infty \) in the aforementioned sense uniform and non-uniform estimations for the probabilities \( Q_n(u), Q_\infty(T, u) \) again in the terms of generic chaining.

In the entropy terms this estimations are obtained in [1], [5].

Let us denote for \( \lambda \in (-\lambda_0, \lambda_0) \)
\[
\phi_n(\lambda) = n \phi(\lambda/\sqrt{n}), \; \zeta(\lambda) = \sup_n \phi_n(\lambda),
\]
and introduce some new semi-distances:
\[
d_n(t, s) = ||\xi(t) - \xi(s)||B(\phi_n), \quad r(t, s) = ||\xi(t) - \xi(s)||B(\zeta).
\]
As long as there exists a limit \( \lim_{n \to \infty} n \phi(\lambda/\sqrt{n}) = \sigma^2 \lambda^2 / 2, \; \sigma^2 = \text{const} \in (0, \infty), \)
we conclude that the function \( \zeta(\cdot) \) exists, is non-trivial and convex.

**Theorem 2.**

**A.** Suppose for some function \( \phi(\cdot) \in \Phi \)
\[
\lim_{\delta \to 0^+} K(\eta_n(\cdot), \phi_n, \delta) = 0.
\]

Then for arbitrary constant \( C \in (0, \infty) \) and for all the values \( u \geq u_0(C) \) the following inequality holds:
\[
Q_n(u) \leq [\exp(C) + 1] \; N(T, d_n, C\Delta_{\phi_n}(u)) \; \exp(-\phi_n^*(u)).
\]

**B.** Suppose for some function \( \phi(\cdot) \in \Phi \)
\[
\lim_{\delta \to 0^+} \sup_n K(\eta_n(\cdot), \zeta, \delta) = 0.
\]

Then for arbitrary constant \( C \in (0, \infty) \) and for all the values \( u \geq u_0(C) \)
\[
Q_\infty(u) \leq [\exp(C) + 1] \; N(T, r, C\Delta_{\zeta}(u)) \; \exp(-\zeta^*(u)).
\]

The conclusion of theorem 2 it follows trivially from the theorem 1 and the following elementary fact: if \( \theta \in B(\phi), \; \phi \in \Phi, \) and \( \theta(i) \) are independent copies of \( \theta, \)
\[
\nu_n \overset{\text{def}}{=} n^{-1/2} \sum_{i=1}^n \theta(i),
\]
then
\[
E \exp(\lambda \nu_n) \leq \exp(\phi_n(\lambda)).
\]

Note that under the conditions of theorem 2 **B** the sequence of the random fields \( \{\xi_i(t)\} \) satisfies the Central Limit Theorem (CLT) in the Banach space \( C(T, r) \) of all continuous in the semi-distance \( r \) functions \( f : T \to R. \)
Recall that the CLT in the considered space means that for all continuous bounded functional \( F : C(T, r) \to R \)

\[
\lim_{n \to \infty} E F(\eta_n(\cdot)) = F(\eta_\infty(\cdot))
\]
or equally that for all continuous functional \( F : C(T, r) \to R \)

\[
\lim_{n \to \infty} \text{Law}(F(\eta_n(\cdot))) = \text{Law}(F(\eta_\infty(\cdot))).
\]

Indeed, the convergence of the finite-dimensional distributions \( \{\eta_n(t)\} \) as \( n \to \infty \) to the finite-dimensional distributions of a Gaussian random centered continuous with probability one relative to the distance \( r \) field \( \eta_\infty(t) \) with covariance function

\[
E \eta_\infty(t) \eta_\infty(s) = E \xi_1(t) \xi_1(s)
\]
is evident; the tightness of the family of measures induced by the random fields \( \{\eta_n(t)\}, t \in T \) in the space \( C(T, r) \) it follows from the equality (15) for the random fields \( \eta_n(t) \).

Thus, we can write, e.g., for each positive values \( u \):

\[
\lim_{n \to \infty} \mathbf{P}(\sup_{t \in T} |\eta_n(t)| > u) \to \mathbf{P}(\sup_{t \in T} |\eta_\infty(t)| > u).
\]

The exponential estimation (and the exact asymptotic) for the last probability is known ([9], chapter 3).

The last equality play very important role in the Monte-Carlo method and in statistics ([9], chapter 4).

5. Examples. We will consider in this section a two examples random fields where a so-called entropy integral (some generalization of Dudley’s integral, see ([6], p. 310)

\[
I = \int_0^1 \psi^{-1}(\exp H(T, d, \epsilon)) \, d\epsilon, \quad \psi(x) = \exp(-\phi^*(x))
\]
diverges. We intend to obtain in these examples the exponential exact estimation for tail of maximum distribution using our methods.

The first example belongs to M.Talagrand [13].

A. Subgaussian random field. Let \( \{\epsilon(n)\}, n = 1, 2, \ldots \), i.e. \( T = Z_\epsilon \), be a sequence of independent symmetrically distributed subgaussian r.v.:

\[
\mathbf{P}(|\epsilon(i)| > x) = \exp(-x^2/2), \quad x \geq 0,
\]
and let \( u \geq 2, \)

\[
\xi(n) = \epsilon(n)/\sqrt{\log(n + e - 1)}.
\]

It follows from estimation of theorem 1 after the optimization over \( C \):

\[
Q(u) \leq \exp(-0.5 \, u^2 + C_0),
\]
where \( C_0 \) is some absolute constant, in the comparison to the real value of \( Q(u) \), for which

\[
\exp(-0.5\ u^2) + \exp\left(-(0.5 + C_2)u^2\right) \leq \exp(-0.5\ u^2) + \exp\left(-(0.5 + C_1)\ u^2\right),
\]

(asympotical exponential exactness).

**B. Exponential bounds of distribution in the LIL for martingales.**

Assume here that the martingale \((\xi(n), F(n))\) satisfies the conditions (8a), (9) and (10). Let us choose

\[
v(n) = v_r(n) = \log(\log(n + 3))^{1/r},
\]

or equally

\[
v(n) = v_r(n) = \log(\log(\sigma(n) + 3))^{1/r},
\]

then we obtain after some calculation on the basis of theorem 1 under condition (16a) instead (16) and choosing the partition over the balls, more exactly, closed intervals \( R = \{[A(k), A(k + 1) - 1]\} = \{[A(k), B(k)]\} \) of a view:

\[
A(k) = Q^{k-1},
\]

where \( Q = \lfloor (1 + \epsilon)^k \rfloor \) for \( k \geq k_0 \), \( \epsilon = \text{const} > 0 \) and \( [Z] \) denotes here the integer part of \( Z \); \( t_0 = A(k) \), \( \delta = B(k) - A(k) \) :

\[
P\left(\sup_n \frac{\xi(n)}{\sigma(n) v_r(n)} > u\right) \leq \exp\left[-C\ u^r\ L^{1/r}(u)\right], \ u > 2. \tag{20}
\]

In the considered case the entropy integral in general case, i.e. if

\[
\sup_n P\left(\frac{\xi(n)}{\sigma(n) v_r(n)} > u\right) \geq \exp\left[-C_0\ u^r\ L^{1/r}(u)\right], \ u > 2,
\]

divergent. In detail, suppose \( \exists n_0 = 1, 2, \ldots \Rightarrow \)

\[
P\left(\frac{\xi(n_0)}{\sigma(n_0)} > u\right) \geq \exp\left[-C_2\ u^r\ L^{1/r}(u)\right], \ u > 2,
\]

and let us introduce the random process (sequence)

\[
\chi(n) = \frac{\xi(n)}{\sigma(n) v_r(n)},
\]

and we must add to the set \( T \) the infinite point \( \{\infty\} \) and define for the completeness of the set \( T : \chi(\infty) = 0. \)

We have for the natural function \( \phi^*_r(\cdot) \) for the process \( \chi(n) : \)

\[
\phi_r(\lambda) \overset{\text{def}}{=} \log E \sup_n \exp(\lambda \chi(n))
\]
the "tail" inequality: \( x \geq 2 \Rightarrow \)

\[
C_1 x^r L^{1/r}(x) \leq \phi_r^*(x) \leq C_2 x^r L^{1/r}(x).
\]

The natural distance \( d_\chi(n, m) \) for the process \( \chi(n) \) is calculated by the formula

\[
d_\chi(n, m) = ||\chi(n) - \chi(m)|| B(\phi_r).
\]

Put \( m = \infty \); then we have for the amount \( N = N(T, d_\chi, \varepsilon) \) of optimal \( \varepsilon \)-net the inequality

\[
\varepsilon \geq d_\chi(n, \infty) \geq C/v_r(N).
\]

We find solving the last inequality relatively the variable \( N \):

\[
H(T, d_\chi, \varepsilon) \geq \exp(C(r) /\varepsilon^r), \, \varepsilon \in (0, 1/2].
\]

The inequality [20] is in general case exact: for all the values \( r = 2/d, \beta = d/2, \, d = 1, 2, \ldots \) there exists a polynomial martingale \((\xi(n), F(n))\) satisfying the conditions (9) and (10) with \( L_1(x) = L_2(x) = L(x) = 1 \) and such that

\[
P \left( \sup_n \frac{\xi(n)}{\sigma(n) v_r(n)} > u \right) \geq \exp [-C_3 u^r], \, u > 2, \quad (21a)
\]

and

\[
P \left[ \lim_{n \to \infty} \frac{\xi(n)}{\sigma(n) v_r(n)} > 0 \right] > 0. \quad (21b)
\]

In detail, let us consider the Rademacher sequence \( \{\varepsilon(i)\}, \, i = 1, 2, \ldots \) i.e. where \( \{\varepsilon(i)\} \) are independent and \( P(\varepsilon(i) = 1) = P(\varepsilon(i) = -1) = 0.5 \).

It is known that that the r. v. \( \{\varepsilon(i)\} \) belongs to the \( B(\phi_2) \) space with the corresponding function

\[
\phi_2(\lambda) = 0.5 \lambda^2, \, \lambda \in (-\infty, \infty).
\]

Indeed,

\[
E(\exp(\lambda \varepsilon(i))) = \cosh(\lambda) \leq \exp(0.5\lambda^2).
\]

Let us denote for \( d = 1, 2, 3, \ldots \) \( \xi(n) = \xi_d(n) = \)

\[
\sum \sum \ldots \sum_{1 \leq i(1) < i(2) \ldots < i(d) \leq n} \varepsilon(i(1)) \varepsilon(i(2)) \varepsilon(i(3)) \ldots \varepsilon(i(d))
\]

under natural filtration \( F(n) = \sigma(\{\varepsilon(j), \, j \leq n\}) \).

It is easy to verify that \( (\xi(n), F(n)) \) is a martingale and that

\[
0 < C_1 \leq \sigma^2(n) / n^d \leq C_2 < \infty.
\]

We will prove the following inequality:

\[
P \left( \lim_{n \to \infty} \frac{\xi(n)}{n \log (\log(n + 3))^{d/2}} > 0 \right) > 0.
\]
It is enough to consider only the case \( d = 2 \), i.e. when
\[
\xi(n) = \sum_{1 \leq i < j \leq n} \epsilon(i) \epsilon(j).
\]

We observe:
\[
2 \xi(n) = \left( \sum_{k=1}^{n} \epsilon(k) \right)^2 - \sum_{m=1}^{n} (\epsilon(m))^2 \overset{\text{def}}{=} \Sigma_1(n) - \Sigma_2(n).
\]

From the classical LIL on the form belonging to Hartman-Wintner it follows that there exist a finite non-trivial non-negative random variables \( \theta_1, \theta_2 \) for which
\[
|\Sigma_2(n)| \leq n + \theta_2 \sqrt{n \log \log(n + 3)}
\]
and
\[
\Sigma_1(n_m) \geq \theta_1 n_m \log(n_m + 3)
\]
for some (random) integer positive subsequence \( n_m, m \to \infty \).

This completes the proof of inequality of (21b); the relation (21a) may be proved by means of more fine considerations.

More exactly, by means of considered method may be proved the following relation:
\[
\lim_{n \to \infty} \frac{\xi(n)}{(n \log \log(n + 3))^{d/2}} \overset{\text{a.e.}}{=} \frac{2^{d/2}}{d!}.
\]

Note that we use in the martingale case in order to estimate the variable \( Y(u; t_0, \delta, \gamma, R) \) inside from the generic chaining method some classical properties of martingales and \( B(\phi) \) spaces: Doob’s inequality, moment estimations, connection with \( G(\psi) \) norms in order to calculate the value \( Y(u; t_0, \delta, R) \).

Namely, let us denote \( E(k) = [A(k), B(k)] \). But we write instead the estimation (17) for the probability \( G(u; t_0, \delta) \) the following estimation: \( Y_k(u) \overset{\text{def}}{=} \)
\[
Y(u; t_0, \delta, \gamma, R) \leq P \left( \max_{n \in E(k)} \xi(n) > u \sigma(A(k)) \nu_r(A(k))/\sigma(B(k)) \right),
\]
as long as both the functions \( \sigma(\cdot) \) and \( \nu_r(\cdot) \) are monotonically increasing.

It follows from the Doob’s inequality
\[
| \max_{n \in E(k)} \xi_n |_p \leq C \sigma(B(k)) \cdot (p/\phi^{-1}(p)) \cdot (p/(p - 1)) \leq 2 C' \sigma(B(k)) \cdot (p/\phi^{-1}(p))
\]
as long as \( p \geq 2 \). Therefore
\[
Y_k(u) \leq \exp \left( -\phi^*(Cu \sigma(A(k)) \nu_r(A(k))/\sigma(B(k)) \right). \tag{22}
\]
The assertion (20) it follows from (22) after the summing over \( k \).
Moreover, if the martingale \((\xi(n), F(n))\) satisfies the conditions (8a), (9) and (10), then with probability one

\[
\lim_{n \to \infty} \frac{\xi(n)}{\sigma(n) \nu_r(n)} \leq 1,
\]

and the last inequality is exact, e.g., for the polynomial martingales [7].

Note that in this case the condition of "convergence of majoring integral" is not satisfied.
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