STARLIKENESS OF CERTAIN ANALYTIC FUNCTIONS

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Abstract. Let \( f \) and \( g \) be analytic functions on the open unit disk of the complex plane with \( f/g \) belonging to the class \( \mathcal{P} \) of functions with positive real part consisting of functions \( p \) with \( p(0) = 1 \) and \( \text{Re} \, p(z) > 0 \) or to its subclass consisting of functions \( p \) with \( |p(z) - 1| < 1 \). We obtain the sharp radius constants for the function \( f \) to be starlike of order \( \alpha \), parabolic starlike, etc. when \( g/k \in \mathcal{P} \) where \( k \) denotes the Koebe function defined by \( k(z) = z/(1-z)^2 \).

1. Motivation and Radii Results

Let \( \mathcal{A} \) be the class of all analytic function \( f \) on the open unit disk \( \mathbb{D} \) normalized by \( f(0) = 0 \) and \( f'(0) = 1 \), and let \( \mathcal{S} \) be the subclass of all univalent function in \( \mathcal{A} \). For two arbitrary subclasses \( \mathcal{F} \) and \( \mathcal{G} \) of \( \mathcal{A} \), the \( \mathcal{G} \) radius of \( \mathcal{F} \), denoted by \( R_{\mathcal{G}}(\mathcal{F}) \), is defined as the largest number \( R_{\mathcal{G}} \) such that \( r^{-1}f(rz) \in \mathcal{G} \) for all \( r \) with \( 0 < r < R_{\mathcal{G}} \), and for all \( f \in \mathcal{F} \). Whenever the class \( \mathcal{G} \) is characterized by a geometric property \( \mathbf{P} \) the number \( R_{\mathcal{G}} \) is called as the radius of the property \( \mathbf{P} \) of the class \( \mathcal{F} \). Although there are variety of radius problems considered in literature, we investigate the functions \( f \) characterized by the ratio of \( f \) with another function \( g \in \mathcal{A} \); these kinds of problems were considered by MacGregor [10, 11, 12]. Ali et al. determined various radii results for functions \( f \) satisfying the following conditions: (i) \( \text{Re} \, (f(z)/g(z)) > 0 \) where \( \text{Re} \, (g(z)/z) > 0 \) or \( \text{Re} \, (g(z)/z) > 1/2 \) (ii) \( |(f(z)/g(z) - 1) - 1| < 1 \) where \( \text{Re} \, (f(z)/g(z)) > 0 \) or \( g \) is convex. All these classes are associated to class of functions with positive real part; this class, denoted by \( \mathcal{P} \), consists of all analytic functions \( p : \mathbb{D} \to \mathbb{C} \) with \( p(0) = 1 \) and \( \text{Re} \, (p(z)) > 0 \) for all \( z \in \mathbb{D} \). Asha and Ravichandran [15] investigated several radii for the functions \( f/g \in \mathcal{P} \) and \((1+z)g/z \in \mathcal{P} \), belonging to some subclasses of starlike functions (see [6, 7] for further works). For \( 0 \leq \alpha < 1 \), we let \( \mathcal{P}(\alpha) := \{ p \in \mathcal{P} : \text{Re} \, p(z) > \alpha \} \). Let \( k \) be the Koebe function defined by \( k(z) = z/(1-z)^2 \). In this paper, we consider the two subclasses \( \Pi_1 \) and \( \Pi_2 \) of analytic functions given below:

\[
\Pi_1 := \{ f \in \mathcal{A} : f/g \in \mathcal{P} \text{ for some } g \in \mathcal{A} \text{ with } g/k \in \mathcal{P} \},
\]

and

\[
\Pi_2 := \{ f \in \mathcal{A} : |f/g - 1| < 1 \text{ for some } g \in \mathcal{A} \text{ with } g/k \in \mathcal{P} \}.
\]

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We determine radii for functions in these two classes to belong to several subclasses of starlike functions which we discuss below.

In 1985, Padmanabhan and Parvatham [14] used the Hadamard product (convolution) and subordination to introduce the class of functions \( f \in A \) satisfying 
\[
z(k_0 \ast f)'/(k_0 \ast f) < h
\]
where \( k_0(z) = z/(1-z)^\alpha, \alpha \in \mathbb{R} \), and \( h \) is convex. When \( h \) is the normalized mapping of \( \mathbb{D} \) onto the right half-plane, this class reduces to the usual classes of starlike and convex functions respectively for \( \alpha = 1 \) and \( \alpha = 2 \). Later, in 1989, Shanmugam [16] studied the class \( S^*_\alpha(\varphi) =: \{ f \in A : z(f \ast g)'/(f \ast g) < \varphi \} \) where \( g \) is fixed and \( \varphi \) a convex function, respectively; this class includes several classes defined by means of linear operator such as Rucheweyh differential operator and Salagean operator. When \( g(z) = z/(1-z) \) and \( g(z) = z/(1-z)^2 \), the subclass \( S^*_\alpha(\varphi) \) is denoted respectively by \( S^*(\varphi) \) and \( K(\varphi) \). In 1992, Ma and Minda [9] studied growth, distortion, covering theorems and coefficient problems for the classes \( S^*(\varphi) \) and \( K(\varphi) \) when \( \varphi \in P \) is just a univalent function mapping unit disk \( \mathbb{D} \) onto domain symmetric with respect to the real line and starlike with respect to \( \varphi(0) = 1 \) and \( \varphi'(0) > 0 \). For \( \varphi(z) = (1 + (1 - 2\alpha)z)/(1-z) \) with \( 0 \leq \alpha < 1 \), the classes \( S^*(\varphi) \) and \( K(\varphi) \) reduce to the class \( S^*(\alpha) \) of starlike functions of order \( \alpha \) and the class \( K(\alpha) \) of convex functions of order \( \alpha \) respectively. For more work in this direction, see [3] [5]. When \( \varphi \) equals \( 1 + (2/\pi)^2(\log((1 + \sqrt{z})/(1 - \sqrt{z})))^2, \sqrt{1+z}, e^z, 1 + (4/3)z + (2/3)z^2, \sin z, z + \sqrt{1+z^2} \) and \( 1 + (zk + z^2/(k^2 - k)z) \) where \( k = \sqrt{2} + 1 \), we denote the class \( S^*(\varphi) \) respectively by \( S_P, S^*_L, S^*_c, S^*_s, S^*_t, S^*_q, \) and \( S^*_k \). The class \( S^*_L \) was introduced by Sokół and Stankiewicz [19]. For information about the other classes, we refer to the recent articles [15] [6] [7].

The functions \( f_0, f_1 : \mathbb{D} \to \mathbb{C} \) defined by

\[
f_0(z) = \frac{z(1+z)^2}{(1-z)^3} \quad \text{and} \quad f_1(z) = \frac{z}{(1+z)^2}
\]

(1.1)

belongs to the class \( \Pi_1 \) and therefore the class \( \Pi_1 \) is non-empty. They satisfy the required conditions with the functions \( g_0, g_1 : \mathbb{D} \to \mathbb{C} \) defined by

\[
g_0(z) = \frac{z(1+z)}{(1-z)^3} \quad \text{and} \quad g_1(z) = \frac{z}{1-z^2},
\]

indeed, we have

\[
\text{Re} \left( \frac{f_i(z)}{g_i(z)} \right) > 0 \quad \text{and} \quad \text{Re} \left( \frac{(1-z)^2 g_i(z)}{z} \right) > 0
\]

for \( i = 0, 1 \). The function \( f_0 \) is an extremal function for the radius problem that we consider. However, the function \( f_1 \) is univalent, but the function \( f_0 \) is not univalent as the coefficients of the Taylor’s series \( f_0(z) = z + 6z^2 + 19z^3 + 44z^4 + \cdots \) do not satisfy the de Branges theorem (previously the Bieberbach conjecture). The derivative of \( f_0 \) is given by

\[
f_0'(z) = \frac{(1+6z+z^2)(1+z)}{(1-z)^3}.
\]

Since \( f_0'(-3 + 2\sqrt{2}) = 0 \) and, by Theorem [14] (1), the radius of starlikeness of the class \( \Pi_1 \) is \( 3 - 2\sqrt{2} \), it follows that the radius of univalence of this class is also
Theorem 1.1. The following radius results hold for the class $\Pi_1$:

1. The $S^*(\alpha)$ radius is $R_{S^*(\alpha)} = (1 - \alpha)/(3 + \sqrt{8 + \alpha^2})$, \quad 0 \leq \alpha < 1.
2. The $S_L^*$ radius is $R_{S_L^*} = (\sqrt{2} - 1)(\sqrt{10} - 3) \approx 0.067217$.
3. The $S_P^*$ radius is $R_{S_P^*} = (6 - \sqrt{33})/3 \approx 0.0851$.
4. The $S_e^*$ radius is $R_{S_e^*} = (e - 1)/(3e + \sqrt{8e^2 + 1}) \approx 0.1080$.
5. The $S_c^*$ radius is $R_{S_c^*} = (9 - \sqrt{73})/4 \approx 0.1140$.
6. The $S_{sin}^*$ radius is $R_{S_{sin}^*} = \sin(1)/(\sqrt{9 + \sin^2(1) + 2\sin(1) + 3}) \approx 0.1320$.
7. The $S_{q}^*$ radius is $R_{S_{q}^*} = 3/\sqrt{2} - \sqrt{1/2} (11 - 2\sqrt{2}) \approx 0.09999$.
8. The $S_{R}^*$ radius $R_{S_{R}^*} = (3 - 2\sqrt{5} - 2\sqrt{2})/(2\sqrt{2} - 1) \approx 0.0289$.

The functions $f_2, f_3 : \mathbb{D} \to \mathbb{C}$ defined by

$$
    f_2(z) = \frac{z}{1 - z} \quad \text{and} \quad f_3(z) = \frac{z(1 + z)^2}{(1 - z)^3},
$$

satisfy the conditions $|f_i(z)/g_i(z) - 1| < 1$ and $\text{Re}((1 - z)^2g_i(z)) > 0$ for $i = 2, 3$ with $g_2, g_3 : \mathbb{D} \to \mathbb{C}$ defined by

$$
    g_2(z) = \frac{z}{1 - z^2} \quad \text{and} \quad g_3(z) = \frac{z(1 + z)}{(1 - z)^3},
$$

and hence $f_2, f_3 \in \Pi_2$. This proves that the class $\Pi_2$ is non-empty. The Taylor series $f_3(z) = z + 5z^2 + 13z^3 + 25z^4 + \cdots$ shows that it is not univalent. It is an extremal function for the radius problems we consider. The derivative of $f_3$ is given by

$$
    f_3'(z) = \frac{(1 + 5z)(1 + z)}{(1 - z)^4}.
$$

Since $f_3'(-1/5) = 0$ and, by Theorem 1.2 (1), the radius of starlikeness of the class $\Pi_1$ is $1/5$, it follows that the radius of univalence of this class is also $1/5$. The other radius results for class $\Pi_2$ are given in the following theorem.

Theorem 1.2. The following radius results hold for the class $\Pi_2$:

1. The $S^*(\alpha)$ radius is $R_{S^*(\alpha)} = 2(1 - \alpha)/(5 + \sqrt{4\alpha^2 - 4\alpha + 25})$, \quad 0 \leq \alpha < 1.
2. The $S_L^*$ radius is at least $R_{S_L^*} = (\sqrt{4\sqrt{2} + 25} - 5)/(2(\sqrt{2} + 2)) \approx 0.0786$.
3. The $S_P^*$ radius is $R_{S_P^*} = 5 - 2\sqrt{5} \approx 0.1010$.
4. The $S_e^*$ radius is $S_{e}^* = (2(e - 1))/(5e + \sqrt{25e^2 - 4e + 4}) \approx 0.1276$.
5. The $S_c^*$ radius is $R_{S_c^*} = (15 - \sqrt{217})/2 \approx 0.1345$.
6. The $S_{sin}^*$ radius is at least $S_{sin}^* = (\sqrt{25 + 4\sin(1)} + \sqrt{15})/(2(3 + \sin(1))) \approx 0.1508$.
7. The $S_{q}^*$ radius is $R_{S_{q}^*} = (5 - \sqrt{41 - 12\sqrt{2}})/(2(\sqrt{2} - 1)) \approx 0.1183$.
8. The $S_{R}^*$ radius $R_{S_{R}^*} = (5 - \sqrt{81 - 40\sqrt{2}})/(4(\sqrt{2} - 1)) \approx 0.0345$.

It is worth to point out that $R_{S_{P}^*} = R_{S^*(1/2)}$ and $R_{S_{c}^*} = R_{S^*(1/\epsilon)}$ in both theorems.
2. Proofs of theorems

We need the following lemmas to prove our results.

**Lemma 2.1.** [1] Lemma 2.2, p.4] For $0 < \alpha < \sqrt{2}$, let $r_a$ be given by

$$r_a = \begin{cases} \alpha - a^2 - (1 - a^2)^{\frac{3}{2}}, & 0 < \alpha \leq 2\sqrt{2}/3 \\ \sqrt{2} - a, & 2\sqrt{2}/3 \leq \alpha < \sqrt{2}. \end{cases}$$

Then $\{\omega : |\omega - a| < r_a\} \subseteq \{\omega : |\omega^2 - 1| < 1\}$.

**Lemma 2.2.** [17] Lemma 1, p. 321] For $a > \frac{1}{2}$, let $r_a$ be given by

$$r_a = \begin{cases} a - \frac{1}{2}, & \frac{1}{2} < a \leq \frac{3}{2} \\ \sqrt{2a - 2}, & a \geq \frac{3}{2} \end{cases}$$

Then $\{w : |w - a| < r_a\} \subseteq \{w : \text{Re} w > |w - 1|\}$.

**Lemma 2.3.** [13] Lemma 2.2, p.368] For $e^{-1} < a < e$, let $r_a$ be given by

$$r_a = \begin{cases} a - e^{-1}, & e^{-1} < a \leq \frac{e + e^{-1}}{2} \\ e - a, & \frac{e + e^{-1}}{2} \leq a < e. \end{cases}$$

Then $\{w : |w - a| < r_a\} \subseteq \{w : |\log w| < 1\} = \Omega_e$.

**Lemma 2.4.** [18] Lemma 2.2, p. 926] For $\frac{1}{3} < a < 3$, let $r_a$ be given by

$$r_a = \begin{cases} a - \frac{1}{3}, & \frac{1}{3} < a \leq \frac{5}{3} \\ 3 - a, & \frac{5}{3} \leq a < 3. \end{cases}$$

Then $\{w : |w - a| < r_a\} \subseteq \Omega_e$, where $\Omega_e$ is the region bonded by the cardioid given $\{x + iy : (9x^2 + 9y^2 - 18x + 5)^2 - 16(9x^2 + 9y^2 - 6x + 1) = 0\}$.

**Lemma 2.5.** [2] Lemma 3.3, p.7] For $1 - \sin 1 < a < 1 + \sin 1$, let $r_a = \sin 1 - |a - 1|$. Then $\{w : |\omega - a| < r_a\} \subseteq \Omega_\omega$; $\Omega_\omega$ is the image of the unit disk $\mathbb{D}$ under $1 + \sin z$.

**Lemma 2.6.** [4] Lemma 2.1, p. 3.] For $\sqrt{2} - 1 < a < \sqrt{2} + 1$, let $r_a = 1 - |\sqrt{2} - a|$. Then

$$\{w : |w - a| < r_a\} \subseteq \{w : |w^2 - 1| < 2|w|\}.$$  \(2.1\)

**Lemma 2.7.** [8] Lemma 2.2, p. 202] For $2(\sqrt{2} - 1) < a < 2$, let $r_a$ be given by

$$r_a = \begin{cases} a - 2(\sqrt{2} - 1), & 2(\sqrt{2} - 1) < a \leq \sqrt{2} \\ 2 - a, & \sqrt{2} \leq a < 2. \end{cases}$$

Then $\{w : |w - a| < r_a\}$, where $\Omega_r$ is the image of the disk $\mathbb{D}$ under the function $1 + (zk + z^2)/(k^2 - kz)$, $k = \sqrt{2} + 1$.

**Proof of Theorem 2.1.** Let the function $f \in \Pi_1$. Then there is a function $g : \mathbb{D} \to \mathbb{C}$ satisfying

$$\text{Re} \left( \frac{f(z)}{g(z)} \right) > 0 \quad \text{and} \quad \text{Re} \left( \frac{(1-z)^2 g(z)}{z} \right) \quad \forall z \in \mathbb{D}.$$  \(2.1\)
Define functions $p_1, p_2 : \mathbb{D} \to \mathbb{C}$ as the following.

$$p_1(z) = \frac{(1 - z)^2 g(z)}{z} \quad \text{and} \quad p_2(z) = \frac{f(z)}{g(z)}. \quad (2.2)$$

By using (2.1) and (2.2), we have $p_1, p_2 \in \mathcal{P}$, and $f(z) = z p_1(z) p_2(z)/(1 - z)^2$. Then it follows that

$$\frac{zf'(z)}{f(z)} = \frac{zp'_1(z)}{p_2(z)} + \frac{zp'_2(z)}{p_2(z)} + \frac{1 + z}{1 - z}. \quad (2.3)$$

The bilinear transformation $(1 + z)/(1 - z)$ maps the disk $|z| \leq r$ onto the disk

$$\left| \frac{1 + z}{1 - z} - \frac{1 - r^2}{1 + r^2} \right| \leq \frac{2r}{1 - r^2}. \quad (2.4)$$

For $p \in \mathcal{P}(\alpha)$, we have

$$\left| \frac{zp'(z)}{p(z)} \right| \leq \frac{2(1 - \alpha)r}{(1 - r)(1 + (1 - 2\alpha)r)}, \quad |z| \leq r. \quad (2.5)$$

By using (2.3), (2.4) and (2.5), function $f$ maps disk $|z| \leq r$ onto disk

$$\left| \frac{zf'(z)}{f(z)} - \frac{1 + r^2}{1 - r^2} \right| \leq \frac{6r}{1 - r^2}. \quad (2.6)$$

From (2.6), it follows that

$$\text{Re} \frac{zf'(z)}{f(z)} \geq \frac{1 - 6r + r^2}{1 - r^2} \geq 0, \quad (2.7)$$

for all $0 \leq r \leq 3 - 2\sqrt{2}$. Therefore, the function $f \in \Pi_1$ is starlike in $|z| \leq 3 - 2\sqrt{2} \approx 0.171573$. Hence, all our radii found here must be less than $3 - 2\sqrt{2}$.

1. The number $\rho = R_{S^*}(\alpha)$, is the smallest positive root of the equation $(1 + \alpha)r^2 - 6r + 1 - \alpha = 0$ in $[0, 1]$. For $0 < r \leq R_{S^*}(\alpha)$, from (2.7), it follows that

$$\text{Re} \frac{zf'(z)}{f(z)} \geq \frac{1 - 6\rho + r^2}{1 - r^2} \geq \frac{1 - 6\rho + \rho^2}{1 - \rho^2} = \alpha.$$

This shows that the radius of starlikeness of order $\alpha$ is at least $R_{S^*}(\alpha)$. To show that it is sharp, consider the function $f_0 \in \Pi_1$ given in (1.1). For this function $f_0$, we have

$$\frac{zf_0'(z)}{f_0(z)} = \frac{1 + 6z + z^2}{1 - z^2}.$$

At $z = -\rho$, we have

$$\text{Re} \frac{zf_0'(z)}{f_0(z)} = \frac{1 - 6\rho + \rho^2}{1 - \rho^2} = \alpha,$$

proving the sharpness of the radius.

2. We can give a proof using Lemma 2.1 but we give a different proof here. The number $\rho := R_{S_L}$ is the smallest positive root of the equation $(1 + \sqrt{2})r^2 + 6r + 1 - \sqrt{2} = 0$ in interval $(0, 1)$, and, from (2.6), it is clear that, for $0 \leq r \leq \rho$,

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq \left| \frac{zf'(z)}{f(z)} - \frac{1 + r^2}{1 - r^2} \right| + \frac{2r^2}{1 - r^2} \leq \frac{6r + 2r^2}{1 - r^2} \leq \frac{6\rho + \rho^2}{1 - \rho^2} = \sqrt{2} - 1 \quad (2.8)$$

and
\[ \left| \frac{zf'(z)}{f(z)} + 1 \right| \leq 2 + \left| \frac{zf'(z)}{f(z)} - 1 \right| \leq \sqrt{2} + 1. \tag{2.9} \]

Thus, from (2.8) and (2.9), it follows that, for \(0 \leq r \leq \rho\),
\[ \left( \frac{zf'(z)}{f(z)} \right)^2 - 1 = \left| \frac{zf'(z)}{f(z)} - 1 \right| \left| \frac{zf'(z)}{f(z)} + 1 \right| \leq (\sqrt{2} + 1)(\sqrt{2} - 1) = 1. \]

For the function \(f_0 \in \Pi_1\) given in (1.1), we have, at \(z = \rho\),
\[ \frac{zf'_0(z)}{f_0(z)} = 1 + \frac{6\rho + 2\rho^2}{1 - \rho^2} = \sqrt{2} \]
and so, at \(z = \rho\),
\[ \left| \left( \frac{zf'_0(z)}{f_0(z)} \right)^2 - 1 \right| = 1. \]

This proves the sharpness.

(3) For \(\rho := R_{SP} = (6 - \sqrt{33})/3\), we have
\[ \frac{1}{2} < 1 - a = \frac{1 + r^2}{1 - r^2} \leq \frac{1 + \rho^2}{1 - \rho^2} = \frac{3\sqrt{33} - 1}{16} \approx 1.0146 < 3/2. \]

Also, for \(\rho = R_{SP}\), we have
\[ \frac{6\rho}{1 - \rho^2} \leq \frac{1 + \rho^2}{1 - \rho^2} - \frac{1}{2} \]
and the disk in (2.6) for \(r = \rho\) becomes
\[ \left| \frac{zf'(z)}{f(z)} - a \right| = \left| \frac{zf'(z)}{f(z)} - \frac{1 + \rho^2}{1 - \rho^2} \right| \leq \frac{1 + \rho^2}{1 - \rho^2} - \frac{1}{2} = a - \frac{1}{2}. \]

By Lemma 2.2, it follows that the disk in (2.6) lies inside region \(\Omega_{PAR}\). This proves that the radius of parabolic starlikeness is at least \(R_{SP}\).

The radius is sharp for the function \(f_0 \in \Pi_1\). At the point \(z = -\rho = -R_{SP}\), we have
\[ \text{Re} \left( \frac{zf'_0(z)}{f_0(z)} \right) = \frac{1 - 6\rho + \rho^2}{1 - \rho^2} = \frac{1}{2} = \frac{6\rho - 2\rho^2}{1 - \rho^2} = \left| \frac{zf'_0(z)}{f_0(z)} - 1 \right|. \]

(4) For \(e^{-1} < a \leq \frac{e + e^{-1}}{2}\), Lemma 2.3 gives
\[ \{ w \in \mathbb{C} : |w - a| < a - e^{-1} \} \subseteq \{ w \in \mathbb{C} : |\log w| < 1 \} =: \Omega_e, \tag{2.10} \]

For \(\rho = R_{Sz}\), we have
\[ e^{-1} < a := \frac{1 + \rho^2}{1 - \rho^2} = \frac{1 + 9e^2}{e(1 + 3\sqrt{1 + 8e^2})} \approx 1.0236 \leq \frac{e + e^{-1}}{2} \approx 1.5430 \]
and, \(\rho\) being smallest positive root of the equation \((1 + e)r^2 - 6er + e - 1 = 0\),
\[ \frac{6\rho}{1 - \rho^2} \leq \frac{1 + \rho^2}{1 - \rho^2} - \frac{1}{e} = a - e^{-1}. \]
Consequently, the disk in (2.6) for $r = \rho$ becomes
\[
\left| \frac{zf'(z)}{f(z)} - a \right| = \left| \frac{zf'(z)}{f(z)} - \frac{1 + \rho^2}{1 - \rho^2} \right| \leq \frac{1 + \rho^2}{1 - \rho^2} - \frac{1}{e} = a - e^{-1}.
\]

By (2.10) the above disk is inside $\Omega_e$ proving that the $S_e^*$ radius for the class $\Pi_1$ is at least $R_{S_e^*}$. The result is sharp for the function $f_0$ given in (1.1). Indeed, at $z = -\rho$ where $\rho = R_{S_e^*}$, we have
\[
\left| \log \left( \frac{zf'(z)}{f(z)} \right) \right| = \left| \log \left( \frac{1 - 6\rho + \rho^2}{1 - \rho^2} \right) \right| = 1.
\]

(5) For $\frac{1}{3} < a \leq \frac{5}{3}$, by an application of Lemma 2.4 it follows that
\[
\left\{ w \in \mathbb{C} : |w - a| < a - \frac{1}{3} \right\} \subseteq \Omega_c,
\]
where $\Omega_c$ is the domain bounded by the cardioid $\{x + iy : (9x^2 + 9y^2 - 18x + 5)^2 - 16(9x^2 + 9y^2 - 6x + 1) = 0\}$. For $\rho = R_{S_e^*}$, we have
\[
\frac{1}{3} < a := \frac{1 + \rho^2}{1 - \rho^2} = \frac{3\sqrt{3} - 1}{24} \approx 1.0263 \leq \frac{5}{3}
\]
and, $\rho$ being the smallest positive root of the equation $2r^2 - 9r + 1 = 0$,
\[
\frac{6\rho}{1 - \rho^2} = \frac{1 + \rho^2}{1 - \rho^2} - \frac{1}{3}.
\]

Therefore, the disk in (2.6) becomes
\[
\left| \frac{zf'(z)}{f(z)} - a \right| = \left| \frac{zf'(z)}{f(z)} - \frac{1 + \rho^2}{1 - \rho^2} \right| \leq \frac{1 + \rho^2}{1 - \rho^2} - \frac{1}{3} = a - \frac{1}{3}
\]
and this disk is inside $\Omega_c$. This shows that $S_e^*$ radius is at least $R_{S_e^*}$.

For the function $f_0$ given in (1.1), at $z = \rho = R_{S_e^*}$, we have
\[
\frac{zf'(z)}{f(z)} = \frac{1 - 6\rho + \rho^2}{1 - \rho^2} = \frac{1}{3} = \varphi_c(-1) \in \partial \varphi_c(\mathbb{D})
\]
where $\varphi_c(z) = 1 + 4z/3 + 2z^2/3$.

(6) For $\rho = R_{S_{\sin}^*}$, and $a := (1 + r^2)/(1 - r^2)$, we have
\[
|a - 1| = \frac{2\rho^2}{1 - \rho^2} \approx 0.13199 < \sin 1 \approx 0.8414.
\]
and
\[
\frac{6\rho}{1 - \rho^2} \leq \sin 1 - \frac{2\rho^2}{1 - \rho^2}.
\]

The disk in (2.6) for $r = \rho$ becomes
\[
\left| \frac{zf'(z)}{f(z)} - a \right| = \left| \frac{zf'(z)}{f(z)} - \frac{1 + \rho^2}{1 - \rho^2} \right| \leq \sin 1 - \frac{2\rho^2}{1 - \rho^2} = \sin 1 - |1 - a|.
\]

Lemma 2.5 shows that the disk in (2.6) is inside $\Omega_s$ where $\Omega_s := \varphi_s(\mathbb{D})$ is the image of the unit disk $\mathbb{D}$ under the mapping $\varphi_s(z) = 1 + \sin z$. This proves that
the $S_{\sin}^*$ radius is at least $R_{S_{\sin}^*}$. For the function $f_0$ given in (1.1), with $\rho = R_{S_{\sin}^*}$, we have
\[
\left( \frac{zf'(z)}{f(z)} \right) = \frac{1 + 6\rho + \rho^2}{1 - \rho^2} = 1 + \sin 1 \in \varphi_s(1) \in \partial \varphi_s(\mathbb{D}).
\]

(7) For $\rho = R_{S_{\sin}^*}$, we have
\[
a := \frac{1 + \rho^2}{1 - \rho^2} \approx 1.0202 \in (\sqrt{2} - 1, \sqrt{2} + 1)
\]
and
\[
\frac{1 - 6\rho + \rho^2}{1 - \rho^2} = \sqrt{2} - 1.
\]
The disk in (2.6) becomes
\[
\left| \frac{zf'(z)}{f(z)} - a \right| \leq 1 - |\sqrt{2} - a|
\]
and by Lemma 2.6 it lies inside $\{w : |w^2 - 1| < 2|w|\}$. This shows that $S_{\sin}^*$ radius is at least $R_{S_{\sin}^*}$. The sharpness follows as the function $f_0$ defined in (1.1) satisfies, at $z = -\rho = -R_{S_{\sin}^*}$,
\[
\left| \left( \frac{zf_0'(z)}{f_0(z)} \right)^2 - 1 \right| = \left| \left( \frac{1 - 6\rho + \rho^2}{1 - \rho^2} \right)^2 - 1 \right| = 2(\sqrt{2} - 1)
\]
\[
= 2 \frac{1 - 6\rho + \rho^2}{1 - \rho^2} = 2 \left| \frac{zf_0'(z)}{f_0(z)} \right|.
\]

(8) For $\rho = R_{S_R^*}$, we have
\[
2(\sqrt{2} - 1) < a := \frac{1 + \rho^2}{1 - \rho^2} \approx 1.00167 \leq \sqrt{2} < 2,
\]
and
\[
\frac{1 - 6\rho + \rho^2}{1 - \rho^2} = 2 - 2\sqrt{2}.
\]
The disk (2.6) becomes
\[
\left| \frac{zf'(z)}{f(z)} - a \right| < a - 2(\sqrt{2} - 1). \tag{2.12}
\]
By Lemma 2.7, this disk lies inside the domain $\Omega_r$. This proves that $S_{\sin}^*$ radius is at least $R_{S_R^*}$.

To prove sharpness, consider the function $f_0 \in \Pi_1$ given in (1.1). At $z = -\rho = -R_{S_R^*}$, we have
\[
\frac{zf'(z)}{f(z)} = \frac{1 - 6\rho + \rho^2}{1 - \rho^2} = 2(\sqrt{2} - 1) = \varphi_r(-1) \in \partial \varphi_r(\mathbb{D})
\]
where $\varphi_r(z) = 1 + (kz + z^2)/(k^2 - kz)$, $k = \sqrt{2} + 1$. \qed
Proof of Theorem 1.2. Since $|w - 1| < 1$ is equivalent to $\text{Re}(1/w) > 1/2$, the condition $|f(z)/g(z) - 1| < 1$ is the same as the condition $\text{Re}(g(z)/f(z)) > 1/2$. Let the function $f \in \Pi_2$. Let $g : \mathbb{D} \to \mathbb{C}$ be chosen such that

$$\text{Re}\left(\frac{g(z)}{f(z)}\right) > \frac{1}{2} \quad \text{and} \quad \text{Re}\left(\frac{(1-z)^2}{z} g(z)\right).$$

(2.13)

Define $p_1, p_2 : \mathbb{D} \to \mathbb{C}$ as

$$p_1(z) = \frac{(1-z)^2}{z} g(z), \quad p_2(z) = \frac{g(z)}{f(z)}.$$  

(2.14)

From (2.13) and (2.14) it follows that $p_1 \in \mathcal{P}$, and $p_2 \in \mathcal{P}(1/2)$, and $f(z) = z/(1 - z)^2 p_1(z)/p_2(z)$. A calculation shows that

$$\frac{zf'(z)}{f(z)} = \frac{zp_1'(z)}{p_1(z)} - \frac{zp_2'(z)}{p_2(z)} + \frac{1+z}{1-z}.$$  

(2.15)

The bilinear transformation $\omega = (1+z)/(1-z)$ maps the disk $|z| \leq r$ onto disk

$$\left|\frac{1+z}{1-z} - \frac{1+r^2}{1-r^2}\right| \leq \frac{2r}{1-r^2}.$$  

(2.16)

Recall that for $p \in \mathcal{P}(\alpha)$, we have

$$\left|\frac{zp'(z)}{p(z)}\right| \leq \frac{2(1-\alpha)r}{(1-r)(1+(1-2\alpha)r)}, \quad |z| \leq r.$$  

(2.17)

Using (2.16) and (2.17) in (2.15), we get

$$\left|\frac{zf'(z)}{f(z)} - \frac{1+r^2}{1-r^2}\right| \leq \frac{5r + r^2}{1-r^2}.$$  

(2.18)

From (2.18), it follows that

$$\text{Re}\left(\frac{zf'(z)}{f(z)}\right) \geq \frac{1-5r}{1-r^2} \geq 0$$  

(2.19)

for $0 \leq r \leq 1/5$. For the function $f_3$ given in (1.3), we have

$$\frac{zf_3'(z)}{f_3(z)} = \frac{1+5z}{1-z^2} = 0$$

for $z = -1/5$. Thus, the radius of starlikeness of the class $\Pi_2$ is $1/5$. All radius values to be computed here will be less than $1/5$.

(1) The number $\rho := R_{S^*}(\alpha)$ is the smallest positive root of the equation $\alpha r^2 - 5r + 1 - \alpha = 0$. For $0 < r \leq R_{S^*}(\alpha)$, from (2.19), we have

$$\text{Re}\left(\frac{zf'(z)}{f(z)}\right) \geq \frac{1-5\rho}{1-r^2} \geq \frac{1-5\rho}{1-\rho^2} = \alpha.$$  

For the function $f_2 \in \Pi_2$ given in (1.3), we have, at $z = -\rho = -R_{S^*}(\alpha)$,

$$\frac{zf_2'(z)}{f_3(z)} = \frac{1-5\rho}{1-\rho^2} = \alpha.$$  

This proves that the radius of starlikeness of order $\alpha$ is $R_{S^*}(\alpha)$. 
(2) From (2.18), it follows that
\[
\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq \left| \frac{zf'(z)}{f(z)} - \frac{1 + r^2}{1 - r^2} \right| + \frac{2r^2}{1 - r^2} \leq \frac{5r + 3r^2}{1 - r^2}. \tag{2.20}
\]

The number \( \rho = R_{S^*_L} \), is the positive root of the equation \( 5r + 3r^2 - (1 - r^2)(\sqrt{2} - 1) = 0 \). For \( 0 < r \leq \rho = R_{S^*_L} \), we have
\[
\frac{5r + 3r^2}{1 - r^2} \leq \frac{\sqrt{2} + 5}{\sqrt{2} - 1} = 1. \tag{2.21}
\]

Therefore, by (2.20), (2.21), and for \( 0 < r \leq \rho = R_{S^*_L} \), it follows that
\[
\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq \sqrt{2} - 1, \tag{2.22}
\]

and
\[
\left| \frac{zf'(z)}{f(z)} + 1 \right| \leq \sqrt{2} + 1. \tag{2.23}
\]

The last two inequalities immediately yields
\[
\left( \frac{zf'(z)}{f(z)} \right)^2 - 1 \leq \frac{zf'(z)}{f(z)} + 1 \left| \frac{zf'(z)}{f(z)} - 1 \right| \leq (\sqrt{2} + 1)(\sqrt{2} - 1) = 1.
\]

This proves that \( S^*_L \) is at least \( R_{S^*_L} \).

(3) For \( 0 \leq r \leq \rho := R_S = 5 - 2\sqrt{6} \), we have for
\[
\frac{1}{2} < 1 \leq a = \frac{1 + \rho^2}{1 - \rho^2} = \frac{5\sqrt{6}}{12} \leq 3/2
\]

and, \( \rho \) being the smallest positive root of the equation \( r^2 - 10r + 1 = 0 \),
\[
\frac{5\rho + \rho^2}{1 - \rho^2} \leq \frac{1 + \rho^2}{1 - \rho^2} - \frac{1}{2}.
\]

The disk in (2.18) becomes
\[
\left| \frac{zf'(z)}{f(z)} - \frac{1 + \rho^2}{1 - \rho^2} \right| \leq \frac{1 + \rho^2}{1 - \rho^2} - \frac{1}{2}.
\]

By Lemma 2.2, the disk in (2.18) is inside the region \( \Omega_{PAR} \). Thus, the radius of parabolic starlikeness of the class \( \Pi_2 \) is at least \( R_{S^*_p} \).

For the function \( f_3 \) given in (1.3) at \( z = -\rho \) where \( \rho = R_{S^*_p} \), we have
\[
\Re \left( \frac{zf_3'(z)}{f_3(z)} \right) = \frac{1 - 5\rho}{1 - \rho^2} = \frac{5\rho - \rho^2}{1 - \rho^2} = \left| \frac{zf_3'(z)}{f_3(z)} - 1 \right|.
\]

(4) For \( \rho = R_{S^*_z} \), we have \( 1/e < a := (1 + \rho^2)/(1 - \rho^2) \approx 1.0331 \leq (e + e^{-1})/2 \) and
\[
\frac{5\rho + \rho^2}{1 - \rho^2} = \frac{1 + \rho^2}{1 - \rho^2} - \frac{1}{e}.
\]

The disk in (2.18) becomes
\[
\left| \frac{zf'(z)}{f(z)} - \frac{1 + \rho^2}{1 - \rho^2} \right| \leq \frac{1 + \rho^2}{1 - \rho^2} - \frac{1}{e}.
\]
By Lemma 2.3 this disk is inside the region $\Omega_\epsilon$, proving that $S^*_c$ radius is at least $R_{S^*_c}$.

The result is sharp for the function $f_3$ given in (1.3). For this function, we have, at $z = -\rho$ where $\rho = R_{S^*_c}$,

$$\left| \log \left( \frac{zf'_3(z)}{f_3(z)} \right) \right| = \left| \log \left( \frac{1 - 5\rho}{1 - \rho^2} \right) \right| = |\log(e^{-1})| = 1.$$

(5) For $\rho = R_{S^*_c}$, we have $1/3 < a := (1 + \rho^2)/(1 - \rho^2) = \frac{1}{42}(1 + 5\sqrt{21}) \approx 1.03686 \leq 5/2$ and, $\rho$ being the smallest positive root of $r^2 - 15r + 2 = 0$,

$$\frac{5\rho + \rho^2}{1 - \rho^2} = \frac{1 + \rho^2}{1 - \rho^2} - \frac{1}{3}.$$ 

The disk in (2.18) becomes

$$\left| \frac{zf''(z)}{f(z)} - \frac{1 + \rho^2}{1 - \rho^2} \right| \leq \frac{1 + \rho^2}{1 - \rho^2} - \frac{1}{3}.$$ 

By Lemma 2.3 this disk is inside the region $\Omega_c$, proving that $S^*_c$ radius is at least $R_{S^*_c}$.

The radius is sharp for the function $f_3$ given in (1.3). At $z = -\rho$ where $\rho = R_{S^*_c}$, we have

$$\frac{zf'_3(z)}{f_3(z)} = \frac{1 - 5\rho}{1 - \rho^2} = \frac{1}{3} = \varphi_c(-1) \in \partial \varphi_c(\mathbb{D})$$

where $\varphi_c(z) = 1 + 4z/3 + 2z^2/3$.

(6) For $\rho = R_{S^*_{\sin}}$, and $a := (1 + \rho^2)/(1 - \rho^2)$, we have

$$|a - 1| = \frac{2\rho^2}{1 - \rho^2} \approx 0.0465396 < \sin 1 \approx 0.8414.$$ 

and

$$\frac{5\rho + \rho^2}{1 - \rho^2} \leq \sin 1 - \frac{2\rho^2}{1 - \rho^2}.$$ 

The disk in (2.6) for $r = \rho$ becomes

$$\left| \frac{zf'(z)}{f(z)} - a \right| = \left| \frac{zf'(z)}{f(z)} - \frac{1 + \rho^2}{1 - \rho^2} \right| \leq \sin 1 - \frac{2\rho^2}{1 - \rho^2} = \sin 1 - |1 - a|.$$ 

Lemma 2.5 shows that the disk in (2.18) is inside $\Omega_s$ where $\Omega_s =: \varphi_s(\mathbb{D})$ is the image of the unit disk $\mathbb{D}$ under the mapping $\varphi_s(z) = 1 + \sin z$. This proves that the $S^*_{\sin}$ radius is at least $R_{S^*_{\sin}}$.

(7) For $\rho = R_{S^*_q}$, we have

$$a := \frac{1 + \rho^2}{1 - \rho^2} \approx 1.02839 \in (\sqrt{2} - 1, \sqrt{2} + 1)$$ 

and

$$\frac{5\rho + \rho^2}{1 - \rho^2} = \frac{1 + \rho^2}{1 - \rho^2} + 1 - \sqrt{2}.$$ 

The disk in (2.18) becomes

$$\left| \frac{zf'(z)}{f(z)} - a \right| \leq 1 - |\sqrt{2} - a|.$$
and by Lemma 2.6 it lies inside \( \{ w : |w^2 - 1| < 2|w| \} \). This shows that \( S^*_L \) radius is at least \( R_{S^*_L} \). The sharpness follows as the function \( f_3 \) defined in (1.1) satisfies,

\[
\left| \left( \frac{zf_3'(z)}{f_3(z)} \right)^2 - 1 \right| = \left| \left( \frac{1 - 5\rho}{1 - \rho^2} \right)^2 - 1 \right| = 2(\sqrt{2} - 1) = 2 \left| \frac{zf_3'(z)}{f_3(z)} \right|.
\]

(8) For \( \rho = R_{S^*_L} \), we have

\[
2(\sqrt{2} - 1) < a := \frac{1 + \rho^2}{1 - \rho^2} \approx 1.00238 \leq \sqrt{2} < 2,
\]

and

\[
\frac{5\rho + \rho^2}{1 - \rho^2} = \frac{1 + \rho^2}{1 - \rho^2} - 2(\sqrt{2} - 1).
\]

The disk (2.18) becomes

\[
\left| \frac{zf'(z)}{f(z)} - a \right| < a - 2(\sqrt{2} - 1).
\]

By Lemma 2.7, this disk lies inside the domain \( \Omega_r \). This proves that \( S^*_L \) radius is at least \( R_{S^*_L} \).

To prove sharpness, consider the function \( f_3 \in \Pi_1 \) given in (1.1). At \( z = -\rho = -R_{S^*_L} \), we have

\[
\left| \frac{zf_3'(z)}{f_3(z)} \right| = \frac{1 - 5\rho}{1 - \rho^2} = 2(\sqrt{2} - 1) = \varphi_r(-1) \in \partial \varphi_r(\mathbb{D})
\]

where \( \varphi_r(z) = 1 + (kz + z^2)/(k^2 - k^2) \), \( k = \sqrt{2} + 1 \). □

We have only obtained lower bounds for the \( S^*_L \) and \( S^*_\sin \) radii for the class \( \Pi_2 \) and we believe the bounds are sharp but unable to prove it.

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