Convergence of products of stochastic matrices with positive diagonals and the opinion dynamics background

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Abstract. We present a convergence result for infinite products of stochastic matrices with positive diagonals. We regard infinity of the product to the left. Such a product converges partly to a fixed matrix if the minimal positive entry of each matrix does not converge too fast to zero and if either zero-entries are symmetric in each matrix or the length of subproducts which reach the maximal achievable connectivity is bounded.

Variations of this result have been achieved independently in [1], [2] and [3]. We present briefly the opinion dynamics context, discuss the relations to infinite products where infinity is to the right (inhomogeneous Markov processes) and present a small improvement and sketch another.

1 Introduction

Consider $n$ persons that discuss an issue which can be represented as a real number. Assume further that the persons revise their opinions if they hear the opinions of others. Each person finds his new opinion as a weighted arithmetic mean of the opinions of others. This model of opinion dynamics has been analyzed for the possibilities of consensus by DeGroot [4]. If these weights change over time we have an inhomogeneous consensus process.

While the homogeneous process has strong similarities with a homogeneous Markov chain, things get different when inhomogeneity comes in. While a consensus process relies on row-stochastic matrices multiplied from the left, a Markov process relies on row-stochastic matrices multiplied from the right. And infinity to the right is unfortunately not the same as infinity to the left if we consider row-stochastic matrices. But nevertheless, both processes fit in the common framework of infinite products of row-stochastic matrices.

Consensus processes are only briefly touched in the context of Markov chains [5]. Besides the early approaches of opinion dynamics [4,6] some results have been made in the context of decentralized computation [7]. Recently there have been some independent works that study consensus processes and
the underlying matrix-products in the context of opinion dynamics [8, 9, 10], multiagent systems where agents try to coordinate [2] and flocking where birds try to find agreement about their headings [3].

In this paper we want to analyze the structure that positive diagonals deliver in inhomogeneous consensus and Markov processes.

2 Consensus and Markov processes

For $n \in \mathbb{N}$ we define $\mathbb{N} := \{1, \ldots, n\}$.

Let $A(0), A(1), \ldots$ be a sequence of row-stochastic matrices. Unless otherwise stated we regard every matrix to be square and to have dimension $n$.

For natural numbers $s < t$ we define a forward accumulation $A(s, t) = A(s) \ldots A(t - 1)$ and a backward accumulation $A(t, s) = A(t - 1) \ldots A(s)$. Thus $A(s, s + 1) = A(s + 1, s) = A(s)$ and $A(s, s)$ is the identity.

We briefly explain the two paradigmatic example processes where forward and backward accumulations play a role.

Let $p(0)$ be a stochastic row vector and $p_i(0)$ is the $i$th component of $p(0)$, which is equal to the proportion of probability mass or population which is in state $i$ at the beginning. The sequence of row vectors $p(t) := p(0)A(0, t)$ is thus an inhomogeneous Markov process and $A(t)[i,j]$ determines the transition from state $i$ to $j$ at time step $t$. In that context $A(t)$ is called transition matrix.

Let $x(0)$ be a real column vector of opinions and $x_i(0)$ stands for the initial opinion of person $i$. The sequence of vectors $x(t) = A(t, 0)x(0)$ is an inhomogeneous consensus process and $A(t)[i,j]$ stands for the weight person $i$ gives to the opinion of agent $j$ at time step $t$. In that context $A(t)$ is called confidence matrix.

To understand the convergence behavior of inhomogeneous Markov and consensus processes the infinite products $A(0, \infty)$ and $A(\infty, 0)$ are of interest.

In this paper we focus on transitions and confidence matrices with positive diagonals. Thus, we regard Markov processes where we have always a positive probability to stay in one state and consensus processes where persons have at least a little bit of self-confidence.

In the next section we will see that the positive diagonal together with the Gantmacher form of nonnegative matrices will give us a good overview on the zero and positivity structure of the processes.

In section 4 we go on with a convergence theorem that is built on this structure and conclude in section 5 with a small improvement and discussion on how to fulfill the conditions of the theorem.

3 The positive diagonal

We regard two nonnegative matrices $A, B$ to be of the same type $A \sim B$ if $a_{ij} > 0 \Leftrightarrow b_{ij} > 0$. Thus, if their zero-patterns are equal.
Let \( A \) be a nonnegative matrix with a positive diagonal. For indices \( i, j \in \mathbb{N} \) we say that there is a path \( i \to j \) if there is a sequence of indices \( i = i_1, \ldots, i_k = j \) such that for all \( l \in k-1 \) it holds \( a_{i_l,i_{l+1}} > 0 \). We say \( i, j \in \mathbb{N} \) communicate if \( i \to j \) and \( j \to i \), thus \( i \leftrightarrow j \). In our case with positive diagonals there is always a path from an index to itself, which we call self-communicating and thus \( i \leftrightarrow i \) is an equivalence relation. An index \( i \in \mathbb{N} \) is called essential if for every \( j \in \mathbb{N} \) with \( i \to j \) it holds \( j \to i \). An index is called inessential if it is not essential.

Obviously, \( \mathbb{N} \) divides into disjoint self-communicating equivalence classes of indices \( I_1, \ldots, I_p \). Thus, in one class all indices communicate and do not communicate with other indices. The terms essential and inessential thus extend naturally to classes.

If we renumber indices with first counting the essential classes and second the inessential classes with a class \( I \) before a class \( J \) if \( J \to I \) then we can bring every row-stochastic matrix \( A \) to the Gantmacher form \( \text{(1)} \)

\[
\begin{bmatrix}
A_1 & 0 \\
& \ddots \\
0 & A_g \\
A_{g+1,1} & \ldots & A_{g+1,g} & A_{g+1} \\
& \vdots & \vdots & \ddots \\
A_{p,1} & \ldots & A_{p,g} & A_{p+1} \\
& & & & \ddots \\
& & & & \ddots
\end{bmatrix}
\]

by simultaneous row and column permutations. The diagonal Gantmacher blocks \( A_1, \ldots, A_p \) in \( \text{(1)} \) are square and irreducible. Irreducibility induces primitivity in the positive diagonal case. For the nondiagonal Gantmacher blocks \( A_{k,l} \) with \( k = g+1, \ldots, p \) and \( l = 1, \ldots, k-1 \) it holds that for every \( k \in \{g+1, \ldots, p\} \) at least one block of \( A_{k,1}, \ldots, A_{k,k-1} \) contains at least one positive entry.

The following proposition shows that an infinite backward or forward accumulation of nonnegative matrices can be divided after a certain time step into subaccumulations with a common Gantmacher form.

**Proposition 1.** Let \( (A(t))_{t \in \mathbb{N}_0} \) be a sequence of nonnegative matrices with positive diagonals. Then for the backward accumulation there exists a sequence of natural numbers \( 0 < t_0 < t_1 < \ldots \) such that for all \( i \in \mathbb{N}_0 \) it holds

\[
A(t_{i+1}, t_i) \sim A(t_1, t_0).
\]

Thus, \( A(t_{i+1}, t_i) \) can be brought to the same Gantmacher form for all \( i \in \mathbb{N} \). Further on, all Gantmacher diagonal blocks are positive and all nondiagonal Gantmacher-Blocks are either positive or zero.

The same holds for another sequences \( 0 < s_0 < s_1 < \ldots \) for the forward accumulation and \( A(s_i, s_{i+1}) \).
Proof. (In sketch, for more details see [1].)

The proof works with a double monotonic argument on the positivity of entries: While more and more (or exactly the same) positive entries appear in $A(t,0)$ monotonously increasing with rising $t$, we reach a maximum at $t_0^*$. We cut $A(t_0^*,0)$ of and find $t_1^*$ when $A(t,t_0^*)$ reaches maximal positivity again with rising $t$. We go on like this and get the sequence $(A(t_{i+1},t_i))_{i \in \mathbb{N}_0}$. Obviously, less and less (or exactly the same) positive entries appear monotonously decreasing with rising $i$ and we reach a minimum at $k$. We relabel $t_j^* := t_{k+j}$ and thus have the desired sequence $(t_i)_{i \in \mathbb{N}_0}$ with $A(t_{i+1},t_i)$ having the same zero-pattern.

Positivity of Gantmacher blocks follows for all blocks $A(t_{i+1},t_i)[J,I]$ where we have a path $J \rightarrow I$. If we have such a path, then there is a path from each index in $J$ to each index in $I$ and thus every entry must be positive in a long enough accumulation. Thus, the block has to be positive already, otherwise $(t_i)_{i \in \mathbb{N}_0}$ is chosen wrong.

To prove the result for forward accumulations, we can use the same arguments. $\square$

The next section regards the convergence behavior of the Gantmacher diagonal blocks.

4 Convergence

A row-stochastic matrix $K$ which has rank 1 and thus equal rows is called a consensus matrix because for a real vector $x$ it holds that $Kx$ is a vector with equal entries and thus represents consensus among persons in a consensus process. Suppose that $A(t) := K$ is a consensus matrix. It is easy to see that for all $u \geq t$ it holds for the backward accumulation that $A(u,0) = K$ while for the forward accumulation it only holds that $A(0,u)$ is a consensus matrix but may change with $u$.

We define the coefficient of ergodicity of a row-stochastic matrix $A$ according to Hartfiel [5] as

$$\tau(A) := 1 - \min_{i,j \in \mathbb{Z}} \sum_{k=1}^{n} \min\{a_{ik}, a_{jk}\}.$$ \hspace{1cm} (3)

The coefficient of ergodicity of a row-stochastic matrix can only be zero, if all rows are equal, thus if it is a consensus matrix.

The coefficient of ergodicity is submultiplicative (see [5]) for row-stochastic matrices $A_0, \ldots, A_i$

$$\tau(A_i \cdots A_1 A_0) \leq \tau(A_i) \cdots \tau(A_1) \tau(A_0).$$

If $\lim_{t \to \infty} \tau(A(0,t)) = 0$ we say that $A(0,t)$ is weakly ergodic. Weakly ergodic means that the $A(0,t)$ gets closer and closer to the set of consensus
matrices and thus the Markov process gets totally independent of the initial distribution $p(0)$.

For $M \subset \mathbb{R}_{\geq 0}$ we define $\min^+ M$ as the smallest positive element of $M$. For a stochastic matrix $A$ we define $\min^+ A := \min_{i,j \in \mathbb{N}} a_{ij}$. We call $\min^+$ the positive minimum.

For the positive minimum of a set of row-stochastic matrices $A_0, \ldots, A_i$ it holds

$$\min^+(A_i \cdots A_0) \geq \min^+ A_i \cdots \min^+ A_0. \quad (4)$$

**Theorem 1.** Let $(A(t))_{t \in \mathbb{N}_0}$ be a sequence of row-stochastic matrices with positive diagonals, $0 < t_0 < t_1 < \ldots$ be the sequence of time steps defined by proposition $\mathbb{I}_1, \ldots, \mathbb{I}_g$ be the essential and $\mathcal{J}$ be the union of all inessential classes of $A(t_1, t_0)$.

If for all $i \in \mathbb{N}_0$ it holds $\min^+ A(t_{i+1}, t_i) \geq \delta_i$ and $\sum_{i=1}^{\infty} \delta_i = \infty$, then

$$\lim_{t \to \infty} A(t, 0) = \begin{bmatrix}
K_1 & 0 & 0 \\
\ddots & \ddots & 0 \\
0 & \ddots & K_g \\
\text{not converging} & & 0
\end{bmatrix} A(t_0, 0)$$

where $K_1, \ldots, K_g$ are consensus matrices. (The matrices have to be sorted by simultaneous row and column permutations according to $\mathbb{I}_1, \ldots, \mathbb{I}_g, \mathcal{J}$.)

**Proof.** The interesting blocks are the diagonal blocks. It is easy to see due to the lower block triangular Gantmacher form of $A(t_{i+1}, t_i)$ for all $i \in \mathbb{N}_0$, that all diagonal blocks only interfere with themselves when matrices are multiplied.

Let us regard the essential class $\mathbb{I}_k$ and abbreviate $A_i := A(t_{i+1}, t_i)_{[\mathbb{I}_k, \mathbb{I}_k]}$.

We show that the minimal entry in a column $j$ of a row-stochastic matrix $B$ cannot sink when multiplied from the right with another row-stochastic matrix $A$,

$$\min_{i \in \mathbb{N}} (AB)_{ij} = \min_{i \in \mathbb{N}} \sum_{k=1}^{n} a_{ik} b_{kj} \geq \min_{i \in \mathbb{N}} b_{ij}.$$
Now it remains to show that the \([J,J]\)-diagonal block of the inessential classes converges to zero.

Let us define \(\|\cdot\|\) as the row-sum-norm for matrices. It holds \(\|A_{[J,J]}(t_{i+1}, t_i)\| \leq (1 - \delta_i)\) and thus like above it holds

\[
\|A_{[J,J]}(\infty, t_0)\| \leq \prod_{i=1}^{\infty} \|A_{[J,J]}(t_{i+1}, t_i)\| \leq \prod_{i=1}^{\infty} (1 - \delta_i) \leq 0.
\]

This proves that \(\lim_{t \to \infty} A_{[J,J]}(t, 0) = 0\). \(\Box\)

An inhomogeneous consensus process \(A(t, 0)x(0)\) with persons who have some self-confidence stabilizes (under weak conditions) such that we have \(g\) consensual subgroups (the essential classes) which have internal consensus, while all other persons (the inessential indices) may hop still around building opinions as convex combinations of the values reached in the consensual groups.

We will not treat the Markov case in detail. But a similar result can be made, but not with fixed matrices \(K_1, \ldots, K_g\) but with weak ergodicity after the time step \(s_0\) within the independent subgroups \(I_1, \ldots, I_g\).

In an inhomogeneous Markov process \(p(0)A(0, t)\) a certain number of independent absorbing classes evolve which get independent of their initial conditions after a certain time step.

5 Discussion on conditions for \(\min^+ A(t_{i+1}, t_i) \geq \delta_i\)

One thing where theorem \(\square\) stays unspecific is that it demands lower bounds for the positive minimum of the accumulations \(A(t_{i+1}, t_i)\). But, what properties of the single matrices may ensure the assumption \(\min^+ A(t_{i+1}, t_i) \geq \delta_i\) with \(\sum \delta_i = \infty\)?

The first idea would be to assume a uniform lower bound for the positive minimum \(\delta < \min^+ A(t)\) for all \(t\). But this is not enough.

Recent independent research \([2, 3, 10]\) has shown that either bounded intercommunication intervals \((t_{i+1} - t_i < N\) for all \(i \in \mathbb{N}_0\)) or type-symmetry \((A \sim A^T)\) of all matrices \(A(t)\) can be assumed additional to the uniform lower bound for the positive minimum to ensure the assumptions of theorem \(\square\). But improvements are possible.

**Bounded intercommunication intervals** Let us regard \(\delta < \min^+ A(t)\) for all \(t \in \mathbb{N}_0\). If \(t_{i+1} - t_i \leq N\) it holds by \(\square\) that \(\min^+ A(t_{i+1}, t_i) \geq \delta^N\) and thus \(\sum_{i=0}^{\infty} \delta^N = \infty\) and thus theorem \(\square\) holds. But \(t_{i+1} - t_i\) may slightly rise as the next two propositions show.

**Proposition 2.** Let \(0 < \delta < 1\) and \(a \in \mathbb{R}_{>0}\) then

\[
\sum_{n=1}^{\infty} \delta^{a \log(n)} < \infty \iff \delta < e^{-1}.
\]
Proof. We can use the integral test for the series \( \sum_{n=1}^{\infty} \delta^n \log(n) \) because \( f(x) := \delta^n \log(x) \) is positive and monotonously decreasing on \( [1, \infty[ \).

With substitution \( y = \log(x) \) (thus \( dx = e^y dy \)) it holds
\[
\int_1^\infty \delta^n \log(x) dx = \int_1^\infty e^{\log(\delta)\log(x)} dx = \int_1^\infty e^{\log(\delta)y} e^y dy
\]
\[
= \int_1^\infty e^{ay(\log(\delta)+1)} dy
\]
The integral is finite if and only if \( \log(\delta) + 1 < 0 \) and thus if \( \delta < e^{-1} \). \( \Box \)

**Proposition 3.** Let \( 0 < \delta < 1 \) and \( a \in \mathbb{R}_{>0} \) then
\[
\sum_{n=3}^{\infty} \delta^n \log(\log(n)) = \infty. \quad (6)
\]

Proof. We can use the integral test for the series \( \sum_{n=1}^{\infty} \delta^n \log(\log(n)) \) because \( f(x) := \delta^n \log(\log(x)) \) is positive and monotonously decreasing on \( [3, \infty[ \).

With substitution \( y = \log(\log(x)) \) (thus \( dx = e^{(y+e^y)} dy \)) it holds
\[
\int_3^\infty \delta^n \log(\log(x)) dx = \int_3^\infty e^{\log(\delta)\log(\log(x))} dx = \int_1^\infty e^{\log(\delta)y} e^{y+e^y} dy
\]
\[
= \int_1^\infty e^{ay(\log(\delta)+1)+e^y} dy
\]
The integral diverges because \( ay(\log(\delta)+1) + e^y \rightarrow \infty \) as \( y \rightarrow \infty \). \( \Box \)

Thus, assuming \( \min^+ A(t) > \delta > 0 \) for all \( t \in \mathbb{N}_0 \) we can allow a slow growing of \( t_{i+1} - t_i \) to fulfill the assumptions of theorem 1. Acceptable is a growing as quick as \( \log(\log(i)) \). If \( t_{i+1} - t_i \) grows as \( \log(i) \) then it must hold \( \delta > e^{-1} > \frac{1}{3} \). This can only hold if each row of \( A(t) \) contains only two positive entries (due to row-stochasticity).

**Type-symmetry** Another way to ensure the assumptions of theorem 1 is to demand all matrices to be type-symmetric and have a positive minimum uniformly bounded form below by \( \delta \). Perhaps a small improvement can be made with a very slowly sinking positive minimum approaching zero. But giving a precise form is a task for future work.

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