Effective Potential and Thermodynamics for a Coupled Two-Field Bose Gas Model

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We study the thermodynamics of a two-species homogeneous and dilute Bose gas that is self-interacting and quadratically coupled to each other. We make use of field theoretical functional integral techniques and evaluate the one-loop finite temperature effective potential for this system considering the resummation of the leading order temperature dependent as well as infrared contributions. The symmetry breaking pattern associated to the model is then studied by considering different values of self and inter-species couplings. We pay special attention to the eventual appearance of re-entrant phases and/or shifts in the observed critical temperatures as compared to the mono atomic (one-field Bose) case.

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I. INTRODUCTION

Due to a larger region of parameters, multi-field theories may exhibit a much richer phase structure in comparison to single field theories. Phenomena that are completely absent in single field theories may arise in some regions of the space parameter related to multi-field theories. From a qualitative point of view one expects that, at finite temperatures and/or densities, multi-field theories will display phase transition patterns which cannot occur in the single field case (see for instance Refs. [1, 2] and references therein). Quantitatively, on the other hand, the actual value of critical quantities may be different in the two cases. For example, consider a scalar multi-field theory composed by two different fields, Φ and χ, which self interact via quartic interactions (e.g., gΦΦ2 and gχχ2). The two fields may also interact quadratically with each other, e.g., with an interaction term gΦχ2. In general, these type of interactions lead to an O(NΦ) × O(Nχ) invariant theory, where NΦ and Nχ represent the number of components of each field. In the relativistic case one may think of Φ and χ as representing an extended Higgs sector of the standard model, e.g. the Kibble-Higgs sector of a SU(5) grand-unified theory, with NΦ = 90 and Nχ = 24 [5]. In the non-relativistic case one can associate Φ and χ to two different species of bosonic atoms in an homogeneous dilute Bose gas with NΦ = Nχ = 2. To assure stability the numerical values of the couplings need to observe certain constraints whose general form is gΦ gχ ≥ b g2, where the self-couplings satisfy gΦ > 0, gχ > 0, whereas g can be either positive or negative. b is some positive real number whose value depends on the way the interactions are normalized in the Lagrangian density. At the same time, stability (boundness of the potential energy) only requires that the coupling constant be positive in the one field case. It is then easy to understand what was said above about the numerical values of critical quantities, like the critical temperature, Tc, in which many cases depends on the masses and couplings, causing Tc = Tmono ≠ Tmult. Also, the way the transitions occur can be highly influenced by the presence of the crossed interaction term proportional to g, as first noted by Weinberg [3]. This fact arises as a consequence of g assuming either positive or negative values and still keep the theory bounded from below. In his work, Weinberg used perturbation theory to analyze a relativistic O(NΦ) × O(Nχ) model at finite temperature. He found that, for g < 0, one may find regions of the parameter space where unexpected phenomena arise. In particular, it was found that a symmetry that was broken at T = 0 could remain broken at arbitrarily high temperatures, in what was called symmetry nonrestoration (SNR). Also, a symmetry that was unbroken at T = 0 could be broken at some finite Tc in a manifestation of inverse symmetry breaking (ISB). It is worth recalling that these two phenomena never show up in the mono-field O(N) theories where one always reaches the symmetric phase at some finite temperature.

As phenomena happening at finite temperatures, one could argue that SNR and ISB are just artifacts of perturbation theory, which although used in Ref. [1], is well known to be inadequate to treat high temperature field theories [3, 4]. However, powerful non-perturbative techniques [1, 5, 8] that include resummations of leading order and infrared terms and thermal effects on the couplings, have confirmed the possibility of these phenomena showing up in the relativistic case. The nonrelativistic case, where the appearance of such phenomena is highly counter intuitive, has been treated very recently by some of the present authors [2, 6, 10]. There, it was shown that SNR and ISB cannot manifest themselves when thermal effects on the couplings are taken into account. A broken symmetry at T = 0 is always restored at some finite Tc while an unbroken symmetry (at T = 0) can be broken at a finite TISB only to be restored at a higher critical temperature characterizing a re-entrant phase that is typical of many condensed matter systems, like the Rochelle salt, spin glasses, compounds known as the manganites, liquid crystals and many others materials, as recently reviewed in [11]. In the work carried out in Ref. [2] one body terms proportional to κΦΦ2 (and κχχ2) have been introduced to drive symmetry breaking. Then, apart from the masses and couplings one has the parameters κ
which may, for example, represent external fields that are temperature independent at the tree level.

The aim of the present work is also to analyze a multi-field nonrelativistic theory but this time we want to make contact with the case that is relevant for dilute homogenous weakly interacting Bose gases. Therefore, we will choose a $U(1) \times U(1)$ version (or equivalently, $O(2) \times O(2)$) taking the one body parameters as representing the chemical potentials. As we shall see this choice makes the finite temperature treatment very complex and here we will use functional techniques to evaluate the effective potential (or free energy). This will allow us to investigate the possibility of SNR/ISB occurring in coupled homogeneous Bose gases. At the same time we will be in position to check if the coupling of two different species of Bose gases shifts the critical temperatures in relation to the case where this coupling does not exist. Due to the non-perturbative nature of such evaluation we will consider a resummation of the leading order temperature dependent and infrared contributions to the effective potential to one-loop. As we shall see, the present investigation excludes the possibility of re-entrant phases when the nonrelativistic models studied in Ref. [1] contain the chemical potentials needed to represent Bose gases. Moreover, the values of the critical temperatures for the two different species seem to be insensitive to the existence of the (new) cross coupling. As in the mono atomic case, their values only depend on the density (and mass) of each specie coinciding with the ideal gas situation. Although the conclusion about the non existence of re-entrant phases is very plausible the statement about the critical temperature should be taken with care. This is because here we are only resumming one-loop (or direct contributions) which do not contribute at the critical point due to the Hugenholtz-Pines theorem. On the other hand, our non-perturbative calculation can reveal the differences in the thermal behavior of uncoupled and coupled direct contributions) which do not contribute at the critical point due to the Hugenholtz-Pines theorem. On the other hand, our non-perturbative calculation can reveal the differences in the thermal behavior of uncoupled and coupled Bose gases at least in the region $0 \leq T < T_c$. The point $T = T_c$ can be fully exploited only by using non-perturbative techniques which go beyond one-loop [12].

The experimental realization of the Bose-Einstein condensation (BEC) in dilute atomic gases has greatly stimulated an enormous number of theoretical studies in this field. For recent reviews on the theory and experiments, see for instance Ref. [13]. Most of this interest comes from the fact that in these experiments a great deal of control can be achieved in almost every parameter of the system. Thus, experiments in dilute atomic Bose gases provide a perfect ground to test numerous models and field theory methods applied to these models, as for example finite temperature quantum field theory methods as commonly used to study phase transition properties of relativistic models. This makes BEC one of the most attractive and promising systems in which one can use models and can test schemes and approximations that could also prove useful in very different environments such as in the early universe, heavy-ion collisions, etc, where the use of one or multi-field models may have importance in their understanding. Therefore, the extension to BEC systems of the analysis performed in previous works in the phase structure of multi-field models is particularly interesting given the possibility of using experiments in dilute atomic Bose gases as an analog system to model and test these finite temperature quantum field theory systems.

This work is organized as follows. In Sec. II we present the model representing two, self-interacting, coupled Bose gases. In Sec. III we evaluate the finite temperature effective potential to one-loop order. Section IV, which is divided in two subsections, presents the evaluation of thermodynamic quantities as well as self-energies. In the first subsection we consider the one-field case whereas the two-field case is considered in the next subsection. In Sec. V we consider the high temperature (symmetry restored) phases. Our conclusions and final remarks are presented in Sec. VI.

II. THE TWO SELF-INTERACTING COUPLED MODEL FOR BOSE GASES

Let us now consider the case of two coupled Bose gases in the presence of self-interactions. The model we consider is similar to the ones used in other theoretical studies of homogeneous dilute coupled Bose gases [14], that consists of a hard core sphere gas model described by nonrelativistic interacting (complex) scalar fields, with an overall repulsive potential. This system can be described by the following global $U_\Phi(1) \times U_\Phi(1)$ invariant Lagrangian model for two-species nonrelativistic complex scalar field $\Phi$ and $\Psi$, with self-couplings $g_\Phi$ and $g_\Psi$ and inter-species coupling $g$,

$$
\mathcal{L}(\Phi^*, \Phi, \Psi^*, \Psi) = \Phi^* \left( i \partial_t + \frac{1}{2m_\Phi} \nabla^2 \right) \Phi + \mu_\Phi \Phi^* \Phi - \frac{g_\Phi}{2} (\Phi^* \Phi)^2 + \Psi^* \left( i \partial_t + \frac{1}{2m_\Psi} \nabla^2 \right) \Psi + \mu_\Psi \Psi^* \Psi - \frac{g_\Psi}{2} (\Psi^* \Psi)^2 - g (\Phi^* \Phi) (\Psi^* \Psi),
$$

where the associated chemical potentials are represented by $\mu_i$ ($i = \Psi$ or $\Phi$) while $m_i$ represent the masses. For the hard core sphere self-interactions we take the phenomenological coupling constants as being the ones normally used in the absence of cross interactions and which are valid in the dilute gas approximation [15]. In terms of the corresponding s-wave scattering lengths, $a_i$, they can be written as $g_i = 4\pi a_i/m_i$, while the cross-coupling is chosen as $g = 4\pi a_{\Phi,\Phi}/m_{\Phi,\Phi}$ where $m_{\Phi,\Phi} = m_{\Phi}m_{\Phi}/(m_{\Phi} + m_{\Psi})$ represents a reduced mass.
The use of analog condensed matter systems to study multi-field theory models like \( (2.1) \), can be envisaged by the use of a system composed by a mixture of coupled atomic gases, like the ones originally produced in Ref. [15] in which one has the same chemical element in two different hyperfine states and that may be treated as “effectively distinguishable”, or just consider the mixing of two different mono-atomic Bose gases.

The equivalent finite temperature Euclidean \((\tau = it)\) spacetime action to Eq. (2.1) is given by

\[
S_E(\beta) = \int_0^\beta d\tau \int d^3x \left[ \Psi^* \left( \frac{\partial}{\partial \tau} - \frac{\nabla^2}{2m_\Psi} - \mu_\Psi \right) \Psi + \frac{g_\Psi}{2} (\Psi^* \Psi)^2 + \Phi^* \left( \frac{\partial}{\partial \tau} - \frac{\nabla^2}{2m_\Phi} - \mu_\Phi \right) \Phi + \frac{g_\Phi}{2} (\Phi^* \Phi)^2 + g (\Psi^* \Psi)(\Phi^* \Phi) \right],
\]

(2.2)

where \( T = 1/\beta \) (we are considering throughout this paper all quantities in natural unities, where \( \hbar = k_B = 1 \)). Let us initially consider \( \Psi \) and \( \Phi \) as pure condensates, in the absence of fluctuations (pure \( c \)-numbers). By varying the action Eq. (2.2) with respect to \( \Psi \) and \( \Phi \) we obtain the equations of motion (in Euclidean time), analogous to the Gross-Pitaevskii equation in the case of a monoatomic gas, given by

\[
\partial_\tau \Psi = -\frac{1}{2m_\Psi} \nabla^2 \Psi - \mu_\Psi \Psi + \left( g_\Psi |\Psi|^2 + g |\Phi|^2 \right) \Psi, \\
\partial_\tau \Phi = -\frac{1}{2m_\Phi} \nabla^2 \Phi - \mu_\Phi \Phi + \left( g_\Phi |\Phi|^2 + g |\Psi|^2 \right) \Phi.
\]

(2.3)

The minimization of the potential part of the action Eq. (2.2) leads to the relations involving the condensates for \( \Psi \) and \( \Phi \), \( \rho_{\Psi,c} = |\Psi|^2 \) and \( \rho_{\Phi,c} = |\Phi|^2 \), respectively, and the chemical potentials:

\[
\mu_\Psi = g_\Psi \rho_{\Psi,c} + g \rho_{\Phi,c}, \\
\mu_\Phi = g_\Phi \rho_{\Phi,c} + g \rho_{\Psi,c}.
\]

(2.4)

In addition, for Eq. (2.4) to represent a local minimum of the potential, we still need to make the requirement that it be bounded from below, which requires the coupling constants to satisfy

\[
g_\Psi g_\Phi - g^2 > 0,
\]

(2.5)

with repulsive self-interactions, \( g_\Psi > 0, \ g_\Phi > 0 \). In the experimental situation of a coupled binary system of Bose atoms, the condition \( (2.5) \) is required for the mixture of condensates to be stable. In the case where Eq. (2.5) is violated, de-mixing of the condensates happens, so that the mixture of condensates will tend to separate spatially, as actually observed in the experiment of the second reference in Ref. [15].

Equation (2.4) also gives the condensate densities in terms of the chemical potentials:

\[
\rho_{\Psi,c} = \frac{g_\Psi \mu_\Psi - g_\Phi \mu_\Phi}{g_\Psi g_\Phi - g^2},
\]

(2.6)

and

\[
\rho_{\Phi,c} = \frac{g_\Phi \mu_\Phi - g_\Psi \mu_\Psi}{g_\Psi g_\Phi - g^2}.
\]

(2.7)

### III. THE ONE-LOOP FINITE TEMPERATURE EFFECTIVE POTENTIAL

Let us now consider the effect of fluctuations. Here we choose to study the system at equilibrium and will include fluctuations in the system by means of the field theoretical method of the effective potential. We will follow closely the functional integration derivation used by the authors in Ref. [16] to the usual one field self-interacting Bose gas (for other field theoretic methods applied to BEC problems, see also Ref. [17, 18] and for a review Ref. [19]). As usual in the computation of the effective potential, we start by decomposing the fields \( \Phi \) and \( \Psi \) in Eq. (2.1) in terms...
of (constant) background fields (which, without loss of generality, can be taken as real fields, c-numbers) \( \phi_0 \) and \( \psi_0 \), respectively, and \((q\text{-number})\) fluctuations \( \Phi \) and \( \Psi \), which in terms of real components, become

\[
\Phi = \frac{1}{\sqrt{2}} (\phi_0 + \phi_1 + i\phi_2), \quad \Phi = \frac{1}{\sqrt{2}} (\psi_0 + \psi_1 + i\psi_2).
\]

(3.1)

Considering the leading order in the fluctuations (which is equivalent to keep terms up to order \( \hbar \) in the effective potential, or one-loop order), when substituting Eqs. (3.1) and (3.2) in Eq. (2.1) we only need to keep the quadratic terms in the fluctuation fields for the computation of the one-loop potential for the background fields, \( \phi_0 \) and \( \psi_0 \). We then obtain the functional partition function to one-loop order

\[
Z[\beta] = \int D\phi_1 D\phi_2 D\phi_0 D\phi_2 \exp (-S_2),
\]

(3.3)

where the functional integral is restricted over fields satisfying the periodic boundary conditions (the Kubo-Martin-Schwinger condition) \( \Psi(x, \tau) = \Psi(x, \tau + \beta) \) and \( \Phi(x, \tau) = \Phi(x, \tau + \beta) \). \( S_2 \) is the Euclidean action to quadratic order in the fluctuation fields,

\[
S_2 = \int_0^\beta d\tau \int d^3x \left[ -\frac{\mu_\Phi}{2} \phi_0^2 + \frac{g_\Phi}{8} \phi_0^4 - \frac{\mu_\Psi}{2} \psi_0^2 + \frac{g_\Psi}{8} \psi_0^4 + \frac{g_\Phi}{4} \phi_0^2 \phi_2^2 + \frac{1}{2} \chi \cdot \hat{M} \cdot \chi \right],
\]

(3.4)

where we have defined the vector \( \chi = (\phi_1, \phi_2, \psi_1, \psi_2) \) and \( \hat{M} \) is the matrix operator for the quadratic terms in the fluctuations,

\[
\hat{M} = \begin{pmatrix}
\frac{-\Sigma^2}{2m_\Phi} - \mu_\Phi + \frac{3g_\Phi}{2} \phi_0^2 + \frac{g_\Psi}{2} \phi_0^2 & i\partial_\tau & 0 & g_\Phi \phi_0 \psi_0 \\
-i\partial_\tau & -\frac{\Sigma^2}{2m_\Phi} - \mu_\Phi + \frac{g_\Phi}{2} \phi_0^2 + \frac{g_\Psi}{2} \phi_0^2 & 0 & 0 \\
g_\Phi \phi_0 \psi_0 & 0 & -\frac{\Sigma^2}{2m_\Phi} - \mu_\Phi + \frac{3g_\Phi}{2} \phi_0^2 + \frac{g_\Psi}{2} \phi_0^2 & \frac{1}{2} i\partial_\tau \\
0 & 0 & -\frac{\Sigma^2}{2m_\Phi} - \mu_\Phi + \frac{g_\Phi}{2} \phi_0^2 + \frac{g_\Psi}{2} \phi_0^2 & -i\partial_\tau \\
\end{pmatrix}
\]

(3.5)

The partial time derivative in (3.5) is over Euclidean time: \( \partial_\tau = \partial / \partial \tau; \tau = it \). As usual, the effective potential is defined from the functional partition function by

\[
V_{\text{eff}}(\psi_0, \phi_0) = -\frac{1}{\beta V} \ln Z[\beta],
\]

(3.6)

where \( V \) is the volume. By performing the functional integration in the quadratic fluctuations \( \chi \), the one-loop effective potential \( V_{\text{eff}}(\phi_0, \psi_0) \) obtained from Eq. (3.4) is then given by

\[
V_{\text{eff}}(\phi_0, \psi_0) = -\frac{\mu_\Phi}{2} \phi_0^2 + \frac{g_\Phi}{8} \phi_0^4 - \frac{\mu_\Psi}{2} \psi_0^2 + \frac{g_\Psi}{8} \psi_0^4 + \frac{g_\Phi}{4} \phi_0^2 \phi_0^2 + \frac{1}{2} \ln \det \hat{M},
\]

(3.7)

where the last term on the RHS of Eq. (3.7) comes from the functional integral over the components of \( \chi \),

\[
\frac{1}{2} \ln \det \hat{M} = -\frac{1}{\beta V} \int D\phi_1 D\phi_2 D\psi_1 D\psi_2 \exp \left[ -\int_0^\beta d\tau \int d^3x \left( \frac{1}{2} \chi \cdot \hat{M} \cdot \chi \right) \right].
\]

(3.8)

Expressing Eqs. (3.3) and (3.7) in the space-time momentum Fourier transform form, we obtain

\[
\frac{1}{2} \ln \det \hat{M} = \frac{1}{2} \sum_{n=-\infty}^{\infty} \int \frac{d^d q}{(2\pi)^d} \ln \left\{ \left[ \omega_n^2 + E^2_{\Phi}(q) \right] \left[ \omega_n^2 + E^2_{\Psi}(q) \right] \right. \\
- g_\Phi^2 \phi_0^2 \psi_0^2 \left[ \omega_\Phi(q) + \frac{g_\Phi}{2} \phi_0^2 + \frac{g_\Psi}{2} \phi_0^2 \right] \left[ \omega_\Psi(q) + \frac{g_\Psi}{2} \phi_0^2 + \frac{g_\Phi}{2} \phi_0^2 \right] \left[ \omega_\Psi(q) + \frac{g_\Psi}{2} \phi_0^2 + \frac{g_\Phi}{2} \phi_0^2 \right],
\]

(3.9)
with
\[ E_\psi(q) = \sqrt{\omega_\psi(q) + \frac{3g_\psi\phi_0^2 + \frac{g}{2}\phi_0^2}{2}} \left[ \omega_\psi(q) + \frac{g_\psi\phi_0^2 + \frac{g}{2}\phi_0^2}{2} \right], \quad (3.10) \]
and
\[ E_\Phi(q) = \sqrt{\omega_\Phi(q) + \frac{3g_\Phi\phi_0^2 + \frac{g}{2}\phi_0^2}{2}} \left[ \omega_\Phi(q) + \frac{g_\Phi\phi_0^2 + \frac{g}{2}\phi_0^2}{2} \right], \quad (3.11) \]

where \( \omega_i(q), i = \Phi, \Psi \) is given by
\[ \omega_i(q) = \frac{q^2}{2m_i} - \mu_i, \quad (3.12) \]
and \( \omega_n \) in Eq. 3.9 represents the Matsubara frequencies, \( \omega_n = 2\pi n/\beta, n = 0, \pm 1, \pm 2, \cdots \).

Considering the classical condensates densities in the absence of fluctuations, Eqs. 2.6 and 2.7, noticing that \( \rho_{\psi,c} = \psi_0^2/2 \) and \( \rho_{\Phi,c} = \phi_0^2/2 \) and substituting them in Eqs. 3.10 and 3.11 we recover the Bogoliubov dispersion relations for the gases in the broken phase
\[ E_\psi(q) = \sqrt{\omega_\psi(q) + 3g_\psi\rho_{\psi,c} + g_\psi\rho_{\psi,c}} \left[ \omega_\psi(q) + g_\psi\rho_{\psi,c} + g_\psi\rho_{\psi,c} \right] 
= \sqrt{\frac{q^2}{2m_\psi} \left( \frac{q^2}{2m_\psi} + 2g_\psi\rho_{\psi,c} \right)}, \quad (3.13) \]
and
\[ E_\Phi(q) = \sqrt{\omega_\Phi(q) + 3g_\Phi\rho_{\Phi,c} + g_\Phi\rho_{\Phi,c}} \left[ \omega_\Phi(q) + g_\Phi\rho_{\Phi,c} + g_\Phi\rho_{\Phi,c} \right] 
= \sqrt{\frac{q^2}{2m_\Phi} \left( \frac{q^2}{2m_\Phi} + 2g_\Phi\rho_{\Phi,c} \right)}, \quad (3.14) \]
which are consequences of the breaking of the two continuous symmetries of the model and Goldstone’s theorem. We can then recognize in Eqs. 3.13 and 3.14 a Higgs and Goldstone modes like terms, that we denote respectively by \( H_i \) and \( G_i \), given by
\[ H_\psi(q, \phi_0, \psi_0) = \omega_\psi(q) + \frac{3g_\psi\phi_0^2 + \frac{g}{2}\phi_0^2}{2}, \quad (3.15) \]
\[ G_\psi(q, \phi_0, \psi_0) = \omega_\psi(q) + \frac{g_\psi\phi_0^2 + \frac{g}{2}\phi_0^2}{2}, \quad (3.16) \]
\[ H_\Phi(q, \phi_0, \psi_0) = \omega_\Phi(q) + \frac{3g_\Phi\phi_0^2 + \frac{g}{2}\phi_0^2}{2}, \quad (3.17) \]
\[ G_\Phi(q, \phi_0, \psi_0) = \omega_\Phi(q) + \frac{g_\Phi\phi_0^2 + \frac{g}{2}\phi_0^2}{2}. \quad (3.18) \]

In terms of Eqs. 3.10-3.18, the contribution of fluctuations to the classical potential in 3.77, \( \Delta V \), becomes
\[ \Delta V = \frac{1}{2} \ln \det \hat{M} = \frac{1}{2} \sum_n \int \frac{d^3q}{(2\pi)^3} \ln \left[ (\omega_n^2 + H_\psi G_\psi) (\omega_n^2 + H_\Phi G_\Phi) - g^2 \psi_0^2 \phi_0^2 G_\psi G_\Phi \right] 
= \frac{1}{2} \sum_n \int \frac{d^3q}{(2\pi)^3} \ln \left[ (\omega_n^2 + A^2) (\omega_n^2 + B^2) \right] 
= \frac{1}{2} \sum_n \int \frac{d^3q}{(2\pi)^3} \ln \left( \omega_n^2 + A^2 \right) + \frac{1}{2} \sum_n \int \frac{d^3q}{(2\pi)^3} \ln \left( \omega_n^2 + B^2 \right), \quad (3.19) \]
where the terms $A$ and $B$ in Eq. (3.19) are given by

\[
A^2 = \frac{H_G G \phi + H_G G \phi}{2} - \frac{1}{2} \left[ (H_G G \phi - H_G G \phi)^2 + 4g^2 \psi^2 \Phi^2 G \Phi G \Phi \right]^{1/2},
\]

and

\[
B^2 = \frac{H_G G \phi + H_G G \phi}{2} + \frac{1}{2} \left[ (H_G G \phi - H_G G \phi)^2 + 4g^2 \psi^2 \Phi^2 G \Phi G \Phi \right]^{1/2}.
\]

The one-loop correction to the classical potential when expressed in the form of the last term in the RHS of Eq. (3.19) is a suitable form that allows to easily perform the sum over the Matsubara frequencies by using the formula,

\[
\frac{1}{\beta} \sum_{n=-\infty}^{+\infty} \frac{1}{\omega_n^2 + \omega^2} = \frac{1}{2\omega} \left( 1 + \frac{2}{e^{\beta\omega} - 1} \right).
\]

Defining the quantity $v(\omega)$,

\[
v(\omega) = \frac{1}{\beta} \sum_{n=-\infty}^{+\infty} \ln(\omega_n^2 + \omega^2),
\]

we have

\[
\frac{\partial v(\omega)}{\partial \omega} = \frac{1}{\beta} \sum_{n=-\infty}^{+\infty} \frac{2\omega}{\omega_n^2 + \omega^2},
\]

which, from Eq. (3.22), becomes

\[
\frac{\partial v(\omega)}{\partial \omega} = 1 + \frac{2}{e^{\beta\omega} - 1},
\]

and so

\[
v(\omega) = \omega + \frac{2}{\beta} \ln \left( 1 - e^{-\beta\omega} \right) + \text{terms independent of } \omega.
\]

Neglecting the constant terms, we then obtain for Eq. (3.19) the result

\[
\Delta V = \frac{1}{2} \int \frac{d^3q}{(2\pi)^3} (A + B) + \frac{1}{\beta} \int \frac{d^3q}{(2\pi)^3} \left[ \ln (1 - e^{-\beta A}) + \ln (1 - e^{-\beta B}) \right].
\]

IV. THERMODYNAMIC QUANTITIES AND SELF-ENERGIES

Given the effective potential, from Eqs. (3.27) and (3.20),

\[
V_{\text{eff}}(T, \phi_0, \psi_0) = -\frac{\mu \phi^2}{2} + \frac{g \phi^4}{8} - \frac{\mu \psi^2}{2} + \frac{g \psi^4}{8} + \frac{g}{4} \phi_0^2 \psi_0^2 + \frac{1}{2} \int \frac{d^3q}{(2\pi)^3} (A + B) + \frac{1}{\beta} \int \frac{d^3q}{(2\pi)^3} \left[ \ln (1 - e^{-\beta A}) + \ln (1 - e^{-\beta B}) \right],
\]

with the functions $A \equiv A(q, \phi_0, \psi_0)$ and $B \equiv B(q, \phi_0, \psi_0)$, given by Eqs. (3.20) and (3.21), respectively, we can compute all relevant thermodynamical functions. In particular, we have that the pressure is defined as the negative of the effective potential computed at its minima (which is the thermodynamical free energy of the system),
\[ P \equiv P(T, \mu, \mu, \mu) = -V_{\text{eff}}(T, \phi_0, \psi_0) = -V_{\text{eff}}(T, \phi_0, \psi_0) \bigg|_{\phi_0=\phi_m, \psi_0=\psi_m} , \]  

where \( \phi_m \) and \( \psi_m \) are the values of \( \phi_0 \) and \( \psi_0 \) that extremizes (corresponding to a minimum of) the effective potential.

\[
\left. \frac{\partial V_{\text{eff}}(T, \phi_0, \psi_0)}{\partial \phi_0} \right|_{\phi_0=\phi_m, \psi_0=\psi_m} = 0 , \quad \left. \frac{\partial V_{\text{eff}}(T, \phi_0, \psi_0)}{\partial \psi_0} \right|_{\phi_0=\phi_m, \psi_0=\psi_m} = 0 .
\]  

For the tree-level potential, \( \psi_m \) and \( \phi_m \) are given by Eqs. \ref{eq:2.6} and \ref{eq:2.7}, respectively (with \( \rho_{\psi,c} = \psi_m^2 / 2 \) and \( \rho_{\Phi,c} = \phi_m^2 / 2 \)). As we will see below, interactions will change these tree-level expressions and, consequently, modifications will have to be implemented in the effective potential so as to preserve the Goldstone's theorem in the presence of interactions and finite temperature effects. From the pressure the total number of particles follows as

\[
\rho_{\phi} = \frac{\partial P(T, \mu, \mu)}{\partial \mu} , \quad \rho_{\psi} = \frac{\partial P(T, \mu, \mu)}{\partial \mu} .
\]  

The corrections to the tree-level (zero temperature) densities change Eqs. \ref{eq:2.6} and \ref{eq:2.7} according to Eq. \ref{eq:4.3} and the obtained expression for the effective potential. Let us see this with some more details starting with the uncoupled gas case.

A. The One-Field Case

For \( g = 0 \), Eqs. \ref{eq:3.20} and \ref{eq:3.21} become

\[
A^2(g = 0) = \left( \omega_\Phi(q) + \frac{3\Phi}{2} \phi_0^2 \right) \left( \omega_\Phi(q) + \frac{g_\Phi}{2} \phi_0^2 \right) ,
\]  

and

\[
B^2(g = 0) = \left( \omega_\Psi(q) + \frac{3\Psi}{2} \psi_0^2 \right) \left( \omega_\Psi(q) + \frac{g_\Psi}{2} \psi_0^2 \right) ,
\]  

and the contributions from the \( \Phi \) and \( \Psi \) fields to the effective potential, Eq. \ref{eq:4.1}, decouple. Let us consider then, e.g., the \( \Phi \) field contribution. In this case

\[
V_{\text{eff}, \Phi}(T, \phi_0) = -\frac{\mu_\Phi}{2} \phi_0^2 + \frac{g_\Phi}{8} \phi_0^4 + \frac{1}{2} \int \frac{d^3q}{(2\pi)^3} \left[ \left( \omega_\Phi(q) + \frac{3\Phi}{2} \phi_0^2 \right) \left( \omega_\Phi(q) + \frac{g_\Phi}{2} \phi_0^2 \right) \right] + \frac{1}{\beta} \int \frac{d^3q}{(2\pi)^3} \ln \left( 1 - \exp \left[ -\beta \sqrt{\left( \omega_\Phi(q) + \frac{3\Phi}{2} \phi_0^2 \right) \left( \omega_\Phi(q) + \frac{g_\Phi}{2} \phi_0^2 \right)} \right] \right) .
\]  

The zero temperature contribution in Eq. \ref{eq:4.7} is divergent and require proper renormalization. It is easier to do it by performing the momentum integral in \( d = 3 - \epsilon \) dimensions and the resulting integral is found to be finite in dimensional regularization (when taking \( \epsilon \to 0 \) at the end). This is so since the divergence in Eq. \ref{eq:4.7} is a power-law one and, therefore, in dimensional regularization, the regularized integral results to be finite. We then obtain the minimum of the effective potential as

\[
\left. \frac{\partial V_{\text{eff}, \Phi}(T, \phi_0)}{\partial \phi_0} \right|_{\phi_0=\phi_m} = 0 \Rightarrow \frac{\phi_m^2}{2} = \frac{\mu_\Phi}{g_\Phi} - \frac{1}{2} \int \frac{d^3q}{(2\pi)^3} \left[ \frac{2\omega_\Phi(q) + \frac{3g_\Phi}{2} \phi_m^2}{\left( \omega_\Phi + \frac{3\Phi}{2} \phi_m^2 \right) \left( \omega_\Phi + \frac{g_\Phi}{2} \phi_m^2 \right)} \right] [1 + 2n_\Phi(q)] .
\]  

where \( n_\Phi \) is the number density of \( \Phi \).
where
\[
n_{\Phi}(q) = \frac{1}{\exp\left[\beta \sqrt{\left(\omega_{\Phi}(q) + \frac{3g_{\Phi}^2}{2} \phi_m^2\right)\left(\omega_{\Phi}(q) + \frac{2g_{\Phi}^2}{2} \phi_m^2\right)}\right] - 1},
\] (4.9)
is the Bose-Einstein distribution for the single self-interacting field and, in Eq. (4.8), \(\mu_{\Phi}/g_{\Phi}\) is the tree-level condensate density (which also follows from Eq. (2.7) for \(g = 0\)). By demanding that the spectrum for the single self-interacting gas at finite temperature is still gapless (Goldstone’s theorem), we can define the condensate density like for instance \(\phi_m^2/2 = \bar{\mu}_{\Phi}/g_{\Phi}\), where \(\bar{\mu}_{\Phi}\) denotes an effective chemical potential (we here will be using an analogous definition as taken by the authors of Ref. [16] in their study of the effective potential for a single self-interacting Bose field). This is expected, since including the finite temperature contributions implies that the original chemical potential must be changed accordingly. In terms of these new definitions Eq. (4.8) becomes
\[
\phi_m^2/2 = \frac{\mu_{\Phi}}{g_{\Phi}} - \frac{1}{2} \int \frac{d^3q}{(2\pi)^3} \left\{ \frac{2\phi_m^2 + \mu_{\Phi}}{\phi_m^2 + 2\mu_{\Phi}} \left[ 1 + \exp\left[\beta \sqrt{\left(\phi_m^2 + 2\mu_{\Phi}\right)}\right] - 1 \right] \right\}. 
\] (4.10)

Replacements \(\mu_{\Phi}\) by the effective chemical potential in the fluctuation terms of the effective potential corresponds to replacing the tree-level field propagators by the self-energy dressed ones. In this case, the equivalent of the matrix \(\hat{M}\), Eq. (3.5), for the one-field case, in the basis \((\phi_1, \phi_2)\) and in momentum space, becomes
\[
\hat{M}_{\Phi}(\omega_n, q) = \begin{pmatrix}
\frac{q^2}{2m_{\Phi}} - \mu_{\Phi} + 3g_{\Phi}^2\phi_0^2 + \Sigma_{\phi_1, \phi_1} & -\omega_n + \Sigma_{\phi_2, \phi_1} \\
\omega_n + \Sigma_{\phi_2, \phi_1} & \frac{q^2}{2m_{\Phi}} - \mu_{\Phi} + \frac{g_{\Phi}^2}{2}\phi_0^2 + \Sigma_{\phi_2, \phi_2}
\end{pmatrix}. 
\] (4.11)

One-loop diagrams that contribute to the self-energies are shown in Fig. 1.

![Fig. 1: Typical one-loop Feynman diagrams contributing to the field self-energy. External lines stand for \(\phi_i\) and the internal propagators \(D_{jk}\) are defined by the inverse of the matrix of quadratic fluctuations \(\hat{M}\).](image)

Since \(\Sigma_{\phi_i, \phi_j}\) for \(i \neq j\) are linear in \(\omega_n\) (see, e.g. Ref. [16]), they are identically zero, \(\Sigma_{\phi_1, \phi_2} = \Sigma_{\phi_2, \phi_1} = 0\). From Eq. (4.11), the equivalent of the Higgs and Goldstone mode terms, Eqs. (3.17) and (3.18) becomes
\[
H_\Phi = \frac{q^2}{2m_{\Phi}} - \mu_{\Phi} + \frac{3g_{\Phi}^2}{2}\phi_0^2 + \Sigma_{\phi_1, \phi_1},
\] (4.12)
\[
G_\Phi = \frac{q^2}{2m_{\Phi}} - \mu_{\Phi} + \frac{g_{\Phi}^2}{2}\phi_0^2 + \Sigma_{\phi_2, \phi_2}.
\] (4.13)

Therefore, Goldstone’s theorem is preserved if we define the effective potential \(\bar{\mu}_{\Phi}\) as

---

1 The same effect happens in thermal field theory, where instead of a thermodynamic chemical potential, we would now talk about a constant mass term for the field. However, finite temperature contributions entering via the self-energies change this mass such as to make it temperature dependent. This is equivalent to change the original mass by a “dressed” one, where self-energy corrections are taken into account in the definitions of the field propagators.
Comparing Eq. (4.16) with (4.10), we get the result given previously for the condensate density at high temperatures, \( \rho_{(4.17)} \), we determine the total density, \( \rho_{\Phi} \), as in Eq. (4.15), since these terms are higher order than the ones considered in the one-loop level. From Eq. (4.11), it can be checked that, without any further modifications, the change produced by Eq. (4.11) starts overcounting diagrams at the two-loop level (like two-bubble vacuum and self-energy diagrams in perturbation theory). This overcounting can be eliminated completely by writing the original Lagrangian density, in terms of the field components \( \phi_1 \) and \( \phi_2 \) like:

\[
\mathcal{L} \to \mathcal{L} [\mu_\Phi \Phi_i^2 \to \bar{\mu}_\Phi \Phi_i^2 = (\mu_\Phi - \Sigma_{\phi_1,\phi_2}) \Phi_i^2] + \frac{1}{2} \sum_i \Sigma_{\phi_1,\phi_2} \Phi_i^2,
\]

where the self-energy terms are at the same time added and subtracted in the original action. While the added self-energies dress the field propagators, the subtracted terms are treated as additional interaction terms and they here act in the sense of subtracting the extra contributions coming from the dressing of the Higgs and Goldstone modes, Eqs. (4.12) and (4.13). The procedure in (4.15) is common in various other instances of studies involving resummation of quantum and temperature correction terms in quantum field theory \[20\].

From the self-energies, one can check that for temperatures \( T \gg \bar{\mu}_{\Phi} \), which is the regime of temperatures we are interested in, the exchange diagrams (like the two-vertex one-loop diagrams of the form of Fig. 1b) are subleading compared to the tadpole diagrams, Fig. 1a. (see also Ref. [16].) In this case, \( \Sigma_{\phi_1,\phi_1} \approx \Sigma_{\phi_2,\phi_2} \) and they are given, at the minimum of the effective potential, by

\[
\Sigma_{\phi_1,\phi_1}(T) \approx \Sigma_{\phi_2,\phi_2}(T) \approx \frac{g_{\Phi}}{2} \int \frac{d^3q}{(2\pi)^3} \frac{2 \frac{q^2}{2m_{\Phi}} + \bar{\mu}_{\Phi}}{\sqrt{\frac{q^2}{2m_{\Phi}} + \frac{2q^2}{2m_{\Phi}} + 2\bar{\mu}_{\Phi}}} \left[ 1 + \frac{2}{\exp \left( \sqrt{\frac{q^2}{2m_{\Phi}} + \frac{2q^2}{2m_{\Phi}} + 2\bar{\mu}_{\Phi}} \right) - 1} \right].
\]

Comparing Eq. (4.16) with (4.10), we get the result given previously for the condensate density at high temperatures, \( \phi_m^2/2 = \bar{\mu}_{\Phi}/g_{\Phi} \).

In terms of Eq. (4.11), the dressed effective potential therefore becomes just like in Eq. (4.7), but with \( \mu_{\Phi} \) exchanged by \( \bar{\mu}_{\Phi} \) in the correction terms for the tree-level potential. With this self-energy “improved” effective potential computed at the minimum \( \phi_0 = \phi_m \), we obtain the pressure,

\[
P(T, \mu_\Phi) = \frac{\mu_{\Phi}^2 - \Sigma_{\phi_2,\phi_2}(T)}{2g_\Phi} - \frac{1}{2} \int \frac{d^3q}{(2\pi)^3} \sqrt{\frac{q^2}{2m_{\Phi}} + \frac{2q^2}{2m_{\Phi}} + 2\bar{\mu}_{\Phi}} \left[ 1 - \frac{1}{\beta} \ln \left( 1 - \exp \left[ -\beta \sqrt{\frac{q^2}{2m_{\Phi}} + \frac{2q^2}{2m_{\Phi}} + 2\bar{\mu}_{\Phi}} \right] \right) \right],
\]

where we have neglected the corrections coming from the new interaction terms given by the change of the Lagrangian density as in Eq. (4.15), since these terms are higher order than the ones considered in the one-loop level. From Eq. (4.17), we determine the total density, \( \rho_{\Phi} \), as

\[\text{Note that here and also for the two-field case discussed below we are discarding contributions coming from the derivatives of the self-energy, both with relation to the fields and chemical potential, since these terms result to be higher order than the one-loop order being considered. For instance } \partial \Sigma/\partial \phi_0 \text{ is already of order } O(g_{\Phi}^3) \text{ and same order corrections at the two-loop order should be considered for consistency as well.}\]
\[
\rho_{\Phi} = \frac{\partial P(T, \mu_{\Phi})}{\partial \mu_{\Phi}} = \frac{\mu_{\Phi}}{g_{\Phi}} - \frac{1}{2} \int \frac{d^3 q}{(2\pi)^3} \frac{q^2}{2m_{\Phi}} \left( \frac{q^2}{2m_{\Phi}} + 2\mu_{\Phi} \right) \left\{ 1 + \frac{2}{\exp \left[ \beta \left( \frac{q^2}{2m_{\Phi}} + 2\mu_{\Phi} \right) \right] - 1} \right\}, \quad (4.18)
\]

or, also using Eq. (4.10) to express \( \mu_{\Phi} \) in term of \( \bar{\mu}_{\Phi} \),

\[
\rho_{\Phi} = \frac{\bar{\mu}_{\Phi}}{g_{\Phi}} + \frac{1}{2} \int \frac{d^3 q}{(2\pi)^3} \frac{q^2}{2m_{\Phi}} + \bar{\mu}_{\Phi} \left\{ 1 + \frac{2}{\exp \left[ \beta \left( \frac{q^2}{2m_{\Phi}} + 2\bar{\mu}_{\Phi} \right) \right] - 1} \right\}. \quad (4.19)
\]

The \( T = 0 \) term in the RHS of Eq. (4.19) gives the quantum depletion of the condensate, while the finite temperature term gives the thermal depletion, as usual. The \( T = 0 \) momentum integral term can be easily computed using dimensional regularization. Converting the momentum integral to arbitrary \( d = 3 - \epsilon \) dimensions and taking \( \epsilon \to 0 \), we obtain the result,

\[
\rho_{\Phi} = \int \frac{d^3 q}{(2\pi)^3} \frac{1}{\epsilon^{3/2}} \Rightarrow T_{\Phi,c} = \frac{2}{m_{\Phi}} \left\{ \frac{\rho_{\Phi}}{\zeta(3/2)} \right\}^{2/3}, \quad (4.21)
\]

where \( \zeta(3/2) \approx 2.612 \). The result Eq. (4.21) is the standard one for the critical temperature for an homogeneous Bose gas.

It is instructive to see, from Eq. (4.19), how the condensate density \( \rho_{\Phi,c}(T) \) changes with temperature. Writing the coupling constant \( g_{\Phi} \) in terms of the \( s \)-wave scattering length \( a_s \), \( g_{\Phi} = 4\pi a_s/m_{\Phi} \) and defining the dimensionless quantities \( \bar{\rho}_{\Phi,c} = \rho_{\Phi,c}/\rho_\Phi, \bar{T}_\Phi = T/T_{\Phi,c} \), where \( T_{\Phi,c} \) is given by Eq. (4.21), we obtain for Eq. (4.19), the implicit equation for \( \bar{\rho}_{\Phi,c}(\bar{T}_\Phi) \),

\[
1 = \bar{\rho}_{\Phi,c} + \frac{8}{3\pi^{1/2}} n_{\Phi}^{1/2} \bar{\rho}_{\Phi,c}^{3/2} + \frac{4}{\pi^{1/2}\zeta(3/2)} \bar{T}_\Phi^{3/2} \int_0^{\infty} dx \frac{x^2 + 2\zeta(3/2)^2/3 n_{\Phi}^{1/3} \bar{\rho}_{\Phi,c}/\bar{T}_\Phi}{x^2 + 4\zeta(3/2)^2/3 n_{\Phi}^{1/3} \bar{\rho}_{\Phi,c}/\bar{T}_\Phi} \sqrt{\frac{1}{e^x [x^2 + 4\zeta(3/2)^2/3 n_{\Phi}^{1/3} \bar{\rho}_{\Phi,c}/\bar{T}_\Phi]^{1/2} - 1}}, \quad (4.22)
\]

where we have made the change of integration variable in Eq. (4.19), \( x^2 = q^2/(2m_{\Phi}T) \) and also used the quantum depletion result Eq. (4.20). In Eq. (4.22) we have defined the diluteness parameter \( n_{\Phi} = \rho_{\Phi}a_s^3 \) \[13\]. The integration in \( x \) in (4.22) can easily be performed numerically producing the standard result shown in Fig. 2 for different values of \( n_{\Phi} \).

\[3\] Note that at this level of approximation that we are considering, \( T_c \) is the same as that for the ideal Bose gas. Corrections due to the self-interactions are only accessible through non-perturbative methods (beyond one-loop) requiring at least second order corrections in the self-energy, see, e.g., Refs. \[12\] and \[13\, 21\] for recent reviews.
B. The Two-Field Case

Let us now return to the two-field case model. The field propagators in the non-vanishing self-energy contributions are found by the inverse of the dressed matrix of quadratic terms, $\hat{M}$, which generalizes Eq. (4.11) of the one-field case and Eq. (3.5) for the free (inverse of) propagator terms in the $(\phi_1,\phi_2,\psi_1,\psi_2)$ basis. In momentum space $\hat{M}$ is represented by the matrix

$$
\hat{M}_{\Phi,\Psi}(\omega_n, q) = 
\begin{pmatrix}
H_{\Phi} + \Sigma_{\phi_1,\phi_1} & -\omega_n & g_{\phi_0}\psi_0 + \Sigma_{\phi_1,\psi_1} & 0 \\
\omega_n & G_{\Phi} + \Sigma_{\phi_2,\phi_2} & 0 & 0 \\
g_{\phi_0}\psi_0 + \Sigma_{\phi_1,\psi_1} & 0 & H_{\Psi} + \Sigma_{\phi_1,\psi_1} & -\omega_n \\
0 & 0 & \omega_n & G_{\Psi} + \Sigma_{\psi_2,\psi_2}
\end{pmatrix}, \tag{4.23}
$$

where the functions $H_i, G_i, i = \Psi, \Phi$, are given by Eqs. (3.15) - (3.18), respectively. In obtaining the self-energies it is easy to show that all cross-like self-energies vanish identically except by $\Sigma_{\phi_1,\psi_1}$, which is given by exchange-like diagrams (like in Fig. 1b). The introduction of the self-energy terms in (4.23) can again be implemented self-consistently already at the Lagrangian level by an analogous procedure as the one shown in Eq. (4.15) for the one-field case.

In complete analogy with the one-field case we now have, from Eq. (4.23), that in order to preserve Goldstone’s theorem in both $\Phi$ and $\Psi$ field directions in the broken (condensed) phase, the inclusion of fluctuations must change the tree-level chemical potentials to

$$
\bar{\mu}_\Phi = \mu_\Phi - \Sigma_{\phi_2,\phi_2}, \tag{4.24}
$$

$$
\bar{\mu}_\Psi = \mu_\Psi - \Sigma_{\psi_2,\psi_2}. \tag{4.25}
$$

As in the one-field case, we restrict our main analysis to the high temperature regime, for which $T \gg \bar{\mu}_i$. In this regime it can again be checked that the exchange diagrams are subleading compared to the tadpole diagrams. In this case $\Sigma_{\phi_1,\psi_1}$ can be neglected compared to the tree level term in Eq. (4.23) and $\Sigma_{\phi_1,\phi_1} \simeq \Sigma_{\phi_2,\phi_2}, \Sigma_{\psi_1,\psi_1} \simeq \Sigma_{\psi_2,\psi_2}$. They are given, at the minima of the effective potential $(\phi_m, \psi_m)$, by

$$
\Sigma_{\phi_1,\phi_1} \simeq \int \frac{d^3q}{(2\pi)^3} \left[ \frac{\partial A}{\partial \phi_0^2} (1 + 2n_A) + \frac{\partial B}{\partial \phi_0^2} (1 + 2n_B) \right] \bigg|_{\phi_0 = \phi_m, \psi_0 = \psi_m}, \tag{4.26}
$$

and

$$
\Sigma_{\psi_1,\psi_1} \simeq \int \frac{d^3q}{(2\pi)^3} \left[ \frac{\partial A}{\partial \psi_0^2} (1 + 2n_A) + \frac{\partial B}{\partial \psi_0^2} (1 + 2n_B) \right] \bigg|_{\phi_0 = \phi_m, \psi_0 = \psi_m}, \tag{4.27}
$$
with

\[ n_A = \frac{1}{e^{\beta \mu_A} - 1}, \quad n_B = \frac{1}{e^{\beta \mu_B} - 1}, \tag{4.28} \]

where \( \tilde{A} \) and \( \tilde{B} \), which follows from the determinant of Eq. (4.23) are the analogous of Eqs. (3.24) and (3.24) with \( \mu_\phi \) and \( \mu_\psi \) in those expressions replaced by the effective chemical potentials \( \mu_\phi \) and \( \mu_\psi \), given by Eqs. (4.24) and (4.25), respectively. The minima of the effective potential, \( \phi_m \) and \( \psi_m \), are determined from the effective potential \( V_{\text{eff}}(T, \phi_0, \psi_0) \), Eq. (4.1), with \( A \to \tilde{A} \) and \( B \to \tilde{B} \). Note that in this case, just like in the one-field case, the condensate densities in the presence of fluctuations, \( \rho_{\phi,c}(T) \equiv \psi_m^2 / 2 \) and \( \rho_{\psi,c}(T) \equiv \phi_m^2 / 2 \), are given by the same equations (2.6) and (2.7), but with chemical potentials given by the effective ones,

\[ \rho_{\phi,c}(T) = \frac{g_\phi \mu_\phi - g \bar{\mu}_\phi}{g_\phi g_\psi - g^2}, \tag{4.29} \]

and

\[ \rho_{\psi,c}(T) = \frac{g_\psi \mu_\psi - g \bar{\mu}_\psi}{g_\psi g_\phi - g^2}. \tag{4.30} \]

Let us now compute the total densities, \( \rho_\phi \) and \( \rho_\psi \), which are obtained from Eq. (4.1), with the pressure given by

\[ P(T, \mu_\phi, \mu_\psi) = \frac{1}{2(g_\phi g_\psi - g^2)} \left[ g_\psi \left( \mu_\phi^2 - \Sigma_{\phi_2} \right) + g_\phi \left( \mu_\psi^2 - \Sigma_{\psi_2} \right) + 2g \left( \Sigma_{\phi_2} + \Sigma_{\psi_2} - \mu_\phi \mu_\psi \right) \right] - \int \frac{d^3q}{(2\pi)^3} \left( \tilde{A} + \tilde{B} \right) \left[ \psi_0 = \psi_m, \phi_0 = \phi_m \right] - \frac{1}{\beta} \int \frac{d^3q}{(2\pi)^3} \left[ \ln \left( 1 - e^{-\beta \tilde{A}} \right) + \ln \left( 1 - e^{-\beta \tilde{B}} \right) \right] \left| \psi_0 = \psi_m, \phi_0 = \phi_m \right. . \tag{4.31} \]

If we now express \( \tilde{A} \) as \( \tilde{A} \equiv \tilde{A}[\phi_m(\mu_\phi, \mu_\psi, T), \phi_m(\mu_\psi, \mu_\phi, T), \mu_\phi, \mu_\psi] \) and \( \tilde{B} \) as \( \tilde{B} \equiv \tilde{B}[\psi_m(\mu_\phi, \mu_\psi, T), \phi_m(\mu_\phi, \mu_\psi, T), \mu_\phi, \mu_\psi] \) it follows, from Eq. (4.31), that we can write for \( \rho_\phi \) and \( \rho_\psi \) the following expressions:

\[ \rho_\phi = \frac{g_\phi \mu_\phi - g \bar{\mu}_\phi}{g_\phi g_\psi - g^2} - \int \frac{d^3q}{(2\pi)^3} \left[ \frac{g_\phi}{g_\phi g_\psi - g^2} \frac{\partial \tilde{A}}{\partial \psi_m^2} - \frac{g}{g_\phi g_\psi - g^2} \frac{\partial \tilde{B}}{\partial \phi_m^2} + \frac{1}{2} \frac{\partial \tilde{A}}{\partial \mu_\psi} \right] (1 + 2n_A), \tag{4.32} \]

and

\[ \rho_\psi = \frac{g_\psi \mu_\psi - g \bar{\mu}_\psi}{g_\psi g_\phi - g^2} - \int \frac{d^3q}{(2\pi)^3} \left[ \frac{g_\psi}{g_\psi g_\phi - g^2} \frac{\partial \tilde{A}}{\partial \phi_m^2} - \frac{g}{g_\psi g_\phi - g^2} \frac{\partial \tilde{B}}{\partial \psi_m^2} + \frac{1}{2} \frac{\partial \tilde{A}}{\partial \mu_\phi} \right] (1 + 2n_B). \tag{4.33} \]

We now make use of the expressions for the condensate densities at finite temperature, Eqs. (4.29) and (4.30), with Eqs. (4.24) and (4.25) together with the self-energies expressions Eqs. (4.26) and (4.27), to express Eqs. (4.32) and (4.33) completely in terms of the temperature dependent condensate densities (e.g., in terms of \( \phi_m \) and \( \psi_m \)) instead of the tree-level \( (T = 0) \) condensate densities \( \phi_0 \) and \( \psi_0 \). This process is analogous to the one used to obtain Eq. (4.19) for the one-field case. After some straightforward algebra, this then results in the coupled equations expressing \( \psi_m \) and \( \phi_m \) in terms of the total densities \( \rho_\phi \) and \( \rho_\psi \),

\[ \rho_\phi = \frac{\psi_m^2}{2} - \frac{1}{2} \int \frac{d^3q}{(2\pi)^3} \left[ \frac{\partial \tilde{A}}{\partial \mu_\psi} (1 + 2n_A) + \frac{\partial \tilde{B}}{\partial \mu_\phi} (1 + 2n_B) \right], \tag{4.34} \]

and
Using Eqs. (4.34) and (4.35) in Eqs. (4.34) and (4.35) we can compute the quantum depletion terms (the $T = 0$ terms) appearing in the coupled system of equations, Eqs. (4.34) and (4.35). Like in the one field case, we use again dimensional regularization to compute the zero temperature momentum integrals in Eqs. (4.34) and (4.35) to obtain the results,

$$- \frac{1}{2} \int \frac{d^3q}{(2\pi)^3} \left( \frac{\partial A}{\partial \mu_\phi} + \frac{\partial B}{\partial \mu_\phi} \right) = \frac{(2m_+)^{3/2} + (2m_-)^{3/2}}{12\pi^2}$$

with $m_+ = m_\phi = m$, the expressions for $A$ and $B$ simplify to

$$A^2 = \frac{q^2}{2m} \left( \frac{q^2}{2m} + \alpha_- \right),$$
$$B^2 = \frac{q^2}{2m} \left( \frac{q^2}{2m} + \alpha_+ \right),$$

with

$$\alpha_\pm = \frac{g_\phi \psi_m^2}{2} + \frac{g_\phi \phi_m^2}{2} \pm \left[ \frac{g_\phi \psi_m^2}{2} - \frac{g_\phi \phi_m^2}{2} \right]^2 + \frac{g^2 \psi_m^2 \phi_m^2}{2}$$

Using Eqs. (4.34) and (4.35) in Eqs. (4.34) and (4.35) we can compute the quantum depletion terms (the $T = 0$ terms) appearing in the coupled system of equations, Eqs. (4.34) and (4.35). Like in the one field case, we use again dimensional regularization to compute the zero temperature momentum integrals in Eqs. (4.34) and (4.35) to obtain the results,

$$\rho_\phi = \frac{\phi_m^2}{2} - \frac{1}{2} \int \frac{d^3q}{(2\pi)^3} \left[ \frac{\partial A}{\partial \mu_\phi} (1 + 2n_A) + \frac{\partial B}{\partial \mu_\phi} (1 + 2n_B) \right],$$
and where the temperature independent (quantum depletion) part is given by chemical potentials, \( \phi \) term in (4.36) always appears multiplying both \( \rho \) and the monoatomic case, and the dimensionless integral measure, (1 dimensionless temperatures \( \tilde{\Psi} \) and \( \tilde{\Phi} \) case both the condensate density and the effective chemical potential, for definition, vanish at the critical point. In From either Eq. (4.40) or Eq. (4.41), it can easily be checked that for \( g = 0 \) we re-obtain the result (4.20) for either the \( \Psi \) or the \( \Phi \) fields. Note also that by taking \( g = 0 \) Eqs. (4.34) and (4.35) decouple and we recover the one-field expression (4.19) for each of the fields individually. It is interesting to point out that, at this level of approximation we are considering, the above equations show that if any of the fields go above the transition point (either \( \phi_m = 0 \) or \( \psi_m = 0 \)) the two equations (4.34) and (4.35) decouple, becoming independent of each other, since the cross-coupling term in (4.35) always appears multiplying both \( \phi_m \) and \( \psi_m \). As far as SNR/ISB are concerned one may conclude, based on the above equations, that these phenomena do not arise for this theory since the cross coupling always appears as \( g^2 \) so the relevant physical quantities are insensitive to the sign of \( g \).

Equations (4.34) and (4.35) are the two-field analogous of the one-field case, Eq. (4.19). Note that here it is more convenient to express the resulting densities for the completeness finally in terms of \( \phi_m \) and \( \psi_m \). In the one-field case both the condensate density and the effective chemical potential, for definition, vanish at the critical point. In the two-field case, as Eqs. (4.29) and (4.30) show, at the critical points for \( \Psi \) and \( \Phi \) we have \( \phi_m(T = T_\Phi) = 0 \) and \( \psi_m(T = T_\Psi) = 0 \). However, the same does not necessarily (as it should not, actually) happens with the effective chemical potentials, \( \bar{\mu}_\Phi \) and \( \bar{\mu}_\Psi \). The critical temperature for transitions in the \( \Phi \) and \( \Psi \) directions are to be determined from the numerical solution of the coupled equations (4.34) and (4.35). For this, it is useful to express Eqs. (4.34) and (4.35) into dimensionless quantities, like in Eq. (4.22) for the one-field case. We start by making the definitions \( \rho_\Psi = \theta_\rho_\Phi, \ m_\Psi = m_\Phi = m, \ a^2_{\Phi, \Psi} = \gamma a_\Psi a_\Phi \) and \( a_i = (n_i/\rho_i)^{1/3} \) where \( n_i (i = \Psi, \Phi) \) are the diluteness parameters for the \( \Psi \) and \( \Phi \). It then follows that \( g_i = (4\pi/m)(n_i/\rho_i)^{1/3}, \ g = \gamma(8\pi/m)^2(n_\Psi/\rho_\Psi)^{1/3}(n_\Phi/\rho_\Phi)^{1/3} \), whereas the boundness condition now reads just \( \gamma < 1/4 \). As in the monoatomic case, \( x^2 = \bar{q}^2/(2mT) \). Other useful quantities are the dimensionless temperatures \( \bar{T}_i = T/T_{i,c}, \) where \( T_{i,c} = (2\pi/m)(\rho_i/\zeta(3/2))^{2/3} \) represents the critical temperature for the monoatomic case, and the dimensionless integral measure, \((1/\rho_i)[d^3q/(2\pi)^3] = (1/\zeta(3/2))(4/\sqrt{\pi})\bar{T}^{3/2}x^2dx \). Then, Eqs. (4.34) and (4.35) can be written as

\[
1 = \bar{\rho}_{c,\Psi} - \frac{1}{2\bar{\rho}_i} \int \frac{d^3q}{(2\pi)^3} \left( \frac{\partial \hat{A}}{\partial \mu_i} + \frac{\partial \hat{B}}{\partial \mu_i} \right) - \frac{1}{\bar{\rho}_i} \int \frac{d^3q}{(2\pi)^3} \left( \frac{\partial \hat{A}}{\partial \mu_i} n_{\hat{A}} + \frac{\partial \hat{B}}{\partial \mu_i} n_{\hat{B}} \right),
\]

(4.42)

where the temperature independent (quantum depletion) part is given by
where the dimensionless Bose factors are

\[
\tilde{\alpha}_{\pm} = (4\pi) \left\{ n_{\psi}^{1/3} \tilde{\rho}_{\psi,c} + n_{\psi}^{1/3} \tilde{\rho}_{\psi,c} / \theta^{2/3} \right\} \pm \left[ \left( n_{\psi}^{1/3} \tilde{\rho}_{\psi,c} - n_{\psi}^{1/3} \tilde{\rho}_{\psi,c} / \theta^{2/3} \right)^2 + 16 \left( \gamma / \theta^{2/3} \right) n_{\psi}^{1/3} \tilde{\rho}_{\psi,c} \tilde{\rho}_{\psi,c} \right]^{1/2}.
\]

Then, using \( \rho_{\psi} = \rho_{\psi} / \theta \) one gets

\[
- \frac{1}{2\rho_{\psi}} \int \frac{d^3q}{(2\pi)^3} \left( \frac{\partial A}{\partial \mu_{\psi}} + \frac{\partial B}{\partial \mu_{\psi}} \right) = \frac{\theta}{12\pi^2} [(2\tilde{\alpha}_+)^{3/2} + (2\tilde{\alpha}_-)^{3/2}]
\]

\[
+ \frac{1}{12\pi^2} \left\{ \left[ n_{\psi}^{1/3} \tilde{\rho}_{\psi,c} - n_{\psi}^{1/3} \tilde{\rho}_{\psi,c} / \theta^{2/3} \right]^2 + 16 \left( \gamma / \theta^{2/3} \right) n_{\psi}^{1/3} \tilde{\rho}_{\psi,c} \tilde{\rho}_{\psi,c} \right\}^{1/2}
\]

\[
- \frac{1}{\pi} \left( n_{\psi}^{1/3} \tilde{\rho}_{\psi,c} \right) [(2\tilde{\alpha}_+)^{1/2} + (2\tilde{\alpha}_-)^{1/2}]
\]

\[
- \frac{1}{\pi} \left\{ \left[ n_{\psi}^{1/3} \tilde{\rho}_{\psi,c} - n_{\psi}^{1/3} \tilde{\rho}_{\psi,c} / \theta^{2/3} \right]^2 + 16 \left( \gamma / \theta^{2/3} \right) n_{\psi}^{1/3} \tilde{\rho}_{\psi,c} \tilde{\rho}_{\psi,c} \right\}^{1/2} \times [(2\tilde{\alpha}_+)^{1/2} - (2\tilde{\alpha}_-)^{1/2}],
\]

where

\[
\tilde{T}_\psi = T / T_{\psi,c}
\]

as a reference temperature to define the temperature dependent parts, starting with

\[
- \frac{1}{\rho_{\psi}} \int \frac{d^3q}{(2\pi)^3} \left( \frac{\partial A}{\partial \mu_{\psi}} n_A + \frac{\partial B}{\partial \mu_{\psi}} n_B \right) = - \frac{4}{\sqrt{\pi(3/2)}} \tilde{T}^{3/2}_\psi \int_0^{\infty} dx x^2 \left[ \frac{\partial A(x)}{\partial \mu_{\psi}} n_{A(x)} + \frac{\partial B(x)}{\partial \mu_{\psi}} n_{B(x)} \right],
\]

where the dimensionless Bose factors are \( n_{A(x)} = 1 / (e^{A(x)} - 1) \) and \( n_{B(x)} = 1 / (e^{B(x)} - 1) \), with

\[
A(x) = x \left[ x^2 + \left( \tilde{\alpha} - \zeta(3/2)^{2/3} / 2\pi T_{\psi} \right) \right]^{1/2},
\]

and

\[
B(x) = x \left[ x^2 + \left( \tilde{\alpha} + \zeta(3/2)^{2/3} / 2\pi T_{\psi} \right) \right]^{1/2}.
\]

Let us further define
\[ F_\Psi(x) = \left[ x^2 + 2\zeta(3/2)^{2/3} \frac{n_\Psi^{1/3} \rho_{\Psi,c}}{T_\Psi} \right], \]  
(4.49)  

\[ G(x) = x^2 \left[ x^2 + 4\zeta(3/2)^{2/3} \frac{n_\Psi^{1/3} \rho_{\Psi,c}}{T_\Psi} \right] - x^2 \left[ x^2 + 4 \left( \frac{\zeta(3/2)}{\theta} \right)^{2/3} \frac{n_\Psi^{1/3} \rho_{\Psi,c}}{T_\Psi} \right], \]  
(4.50)  

and  
\[ H(x) = \left[ G(x)^2 + 64\gamma \frac{\zeta(3/2)^{4/3}}{\theta^{2/3}} n_\Psi^{1/3} \rho_{\Psi,c} \rho_{\Phi,c} x^2 T_\Psi \right]^{1/2}. \]  
(4.51)  

Then,  
\[ \frac{\partial \tilde{A}}{\mu_\Psi} = - \frac{1}{2A(x)} \left[ F_\Psi(x) - \frac{1}{H(x)} \left( F_\Psi(x) G(x) + 16\gamma \frac{\zeta(3/2)^{4/3}}{\theta^{2/3}} n_\Psi^{1/3} \rho_{\Psi,c} \rho_{\Phi,c} x^2 \right) \right], \]  
(4.52)  

and  
\[ \frac{\partial \tilde{B}}{\mu_\Psi} = - \frac{1}{2B(x)} \left[ F_\Psi(x) + \frac{1}{H(x)} \left( F_\Psi(x) G(x) + 16\gamma \frac{\zeta(3/2)^{4/3}}{\theta^{2/3}} n_\Psi^{1/3} \rho_{\Psi,c} \rho_{\Phi,c} x^2 \right) \right]. \]  
(4.53)  

Finally, let us write down  
\[ - \frac{1}{\rho_\Psi} \int \frac{d^3q}{(2\pi)^3} \left( \frac{\partial \tilde{A}}{\mu_\Phi} n_A + \frac{\partial \tilde{B}}{\mu_\Phi} n_B \right) = - \frac{4}{\sqrt{\pi} \zeta(3/2) \theta^{3/2}} \int_0^\infty dx x^2 \left( \frac{\partial A(x)}{\mu_\Phi} n_A(x) + \frac{\partial B(x)}{\mu_\Phi} n_B(x) \right), \]  
(4.54)  

using  
\[ F_\Phi(x) = \left[ x^2 + 2\zeta(3/2)^{2/3} \frac{n_\Phi^{1/3} \bar{\rho}_{\Phi,c}}{T_\Psi} \right], \]  
(4.55)  

and the above definitions for \( G \) and \( H \). Then,  
\[ \frac{\partial \tilde{A}}{\mu_\Phi} = - \frac{1}{2A(x)} \left[ F_\Phi(x) + \frac{1}{H(x)} \left( F_\Phi(x) G(x) - 16\gamma \frac{\zeta(3/2)^{4/3}}{\theta^{2/3}} n_\Psi^{1/3} \rho_{\Psi,c} \rho_{\Phi,c} x^2 \right) \right], \]  
(4.56)  

and  
\[ \frac{\partial \tilde{B}}{\mu_\Phi} = - \frac{1}{2B(x)} \left[ F_\Phi(x) - \frac{1}{H(x)} \left( F_\Phi(x) G(x) - 16\gamma \frac{\zeta(3/2)^{4/3}}{\theta^{2/3}} n_\Psi^{1/3} \rho_{\Psi,c} \rho_{\Phi,c} x^2 \right) \right]. \]  
(4.57)  

We have carried out a careful numerical analysis finding that the cross coupling (now characterized by the dimensionless parameter \( \gamma \)) does not affect the critical temperature that each gas observes when the system is uncoupled. As shown in Fig. 3 when the temperature is increased each type of condensate returns to the symmetric (gas) phase at a critical temperature whose value coincides with the one obtained in the one field case (see Eq. (4.24)). That is, the two distinct critical temperatures displayed in Fig. 3 are insensitive to \( \gamma \) (i.e. \( g \)) and seem to depend separately on each density \(^4\) that are now related by \( \theta \). Although the critical temperatures for each gas in the coupled

\(^4\) Actually, it is very plausible that the critical temperatures should also depend on the different masses as in Eq. (4.51). However, here we had to use the approximation \( m_\Phi \approx m_\Psi = m \) which does not allow us to fully confirm this fact.
FIG. 3: The quantities $\tilde{\rho}_{c,\psi}$ and $\tilde{\rho}_{c,\phi}$ plotted as functions of the reference dimensionless temperature $\tilde{T}_\psi$ for $\gamma = 1/6$, $\theta = 1.5$ and $n_\psi = n_\phi = 10^{-5}$. The solid lines correspond to the coupled case whereas the dashed lines are the corresponding curves for the one field case. Note how the observed $T_c$’s agree with the value predicted for the one field case, Eq. (4.21). The symmetry $O(2)$ referring to the less dense gas, $\Phi$, is restored at a lower temperature $\tilde{T}_\psi \simeq 0.763$ (which corresponds to $\tilde{T}_\phi = \theta \tilde{T}_\psi = 1$) than the one observed by the other species, $\Psi$ ($\tilde{T}_\phi = 1$).

FIG. 4: The quantity $\tilde{\rho}_{c,\psi}$ as a function of the dimensionless temperature $\tilde{T}_\psi$ for $\theta = 1.0$ and $n_\psi = n_\phi = 10^{-5}$. The solid line corresponds to a strongly coupled case, $\gamma = 2/9$, whereas the dotted line corresponds to a weakly coupled case, $\gamma = 1/10$. The dashed line corresponds to the uncoupled case.

case coincide with the ones observed in the monoatomic case the temperature dependence of the quantities $\tilde{\rho}_{c,i}$ are influenced by the cross-coupling.

To investigate how the cross coupling influences the general temperature behavior of $\tilde{\rho}_{c,i}$, we consider a large, a small and the null value for $\gamma$. Figure 4 shows the situation for the $\Psi$ sector displaying the fact that, for $0 < \tilde{T}_\psi < 1$, the density $\tilde{\rho}_{c,\psi}$ assumes smaller values for larger values of the cross coupling. However, as noticed before, the critical value for the temperature corresponds to the monoatomic case that in fact is the critical temperature for a non interacting gas. This is expected since at this (one-loop) level of approximation the Hugenholtz-Pines theorem washes out all nontrivial contributions to $T_c$ (see Ref. [12]) so that our results for this quantity become trivial. Nevertheless Fig. 4 suggests that the critical temperature value may be influenced by the cross coupling interaction in a computation that includes higher corrections.
V. THE SYMMETRY RESTORED, HIGH TEMPERATURE PHASES

Let us now study the two-coupled field system in the symmetry restored phase. In this case the spectrum for both \( \Phi \) and \( \Psi \) field changes, since \( \phi_m \) and \( \psi_m \) both vanish. The functions \( H_\Phi \) and \( G_\Phi \) defined by Eqs. (4.12) and (4.13), and similarly for \( H_\Psi \) and \( G_\Psi \), become

\[
H_\Phi = G_\Phi = \frac{q^2}{2m_\Phi} - \bar{\mu}_\Phi, \quad (5.1)
\]

\[
H_\Psi = G_\Psi = \frac{q^2}{2m_\Psi} - \bar{\mu}_\Psi, \quad (5.2)
\]

where \( \bar{\mu}_\Phi = \mu_\Phi - \Sigma_{\Phi,\Phi} \) and \( \bar{\mu}_\Psi = \mu_\Psi - \Sigma_{\Psi,\Phi} \), with the self-energies \( \Sigma \), in the normal phase and at one-loop order, given by

\[
\Sigma_{\Phi,\Phi} = \Sigma_{\Phi_1,\phi_1} = \Sigma_{\Phi_2,\phi_2} = g_\Phi \int \frac{d^3q}{(2\pi)^3} \left\{ 1 + \frac{2}{\exp \left( \beta \left( \frac{q^2}{2m_\Phi} - \bar{\mu}_\Phi \right) \right) - 1} \right\} + \frac{g}{2} \int \frac{d^3q}{(2\pi)^3} \left\{ 1 + \frac{2}{\exp \left( \beta \left( \frac{q^2}{2m_\Psi} - \bar{\mu}_\Psi \right) \right) - 1} \right\} = 2g_\Phi \left( \frac{m_\Phi T}{2\pi} \right)^{3/2} \text{Li}_{3/2} \left( e^{\beta \bar{\mu}_\Phi} \right) + g \left( \frac{m_\Phi T}{2\pi} \right)^{3/2} \text{Li}_{3/2} \left( e^{\beta \bar{\mu}_\Phi} \right), \quad (5.3)
\]

and similarly for \( \Sigma_{\Psi,\Psi} \),

\[
\Sigma_{\Psi,\Psi} = \Sigma_{\Psi_1,\psi_1} = \Sigma_{\Psi_2,\psi_2} = 2g_\Psi \left( \frac{m_\Psi T}{2\pi} \right)^{3/2} \text{Li}_{3/2} \left( e^{\beta \bar{\mu}_\Psi} \right) + g \left( \frac{m_\Psi T}{2\pi} \right)^{3/2} \text{Li}_{3/2} \left( e^{\beta \bar{\mu}_\Psi} \right), \quad (5.4)
\]

where \( \text{Li}_{3/2}(z) \) is a polylogarithmic function,

\[
\text{Li}_\alpha(z) = \sum_{l=1}^{\infty} \frac{z^l}{l^\alpha}. \quad (5.5)
\]

The pressure in this case, \( P(T, \mu_\Phi, \mu_\Psi) = -V_{\text{eff}}(T, \phi_0 = 0, \psi_0 = 0) \), becomes just

\[
P(T, \mu_\Phi, \mu_\Psi) = \left( \frac{m_\Phi}{2\pi} \right)^{3/2} T^{5/2} \text{Li}_{5/2} \left( e^{\beta \bar{\mu}_\Phi} \right) + \left( \frac{m_\Psi}{2\pi} \right)^{3/2} T^{5/2} \text{Li}_{5/2} \left( e^{\beta \bar{\mu}_\Psi} \right) + \frac{1}{2} \Sigma_{\Phi,\Phi} \left( \frac{m_\Phi T}{2\pi} \right)^{3/2} \text{Li}_{3/2} \left( e^{\beta \bar{\mu}_\Phi} \right) + \frac{1}{2} \Sigma_{\Psi,\Psi} \left( \frac{m_\Psi T}{2\pi} \right)^{3/2} \text{Li}_{3/2} \left( e^{\beta \bar{\mu}_\Psi} \right), \quad (5.6)
\]

where the two last terms in Eq. (5.6) come from the sum of two-bubble vacuum diagrams made of the quartic self-interaction and cross-interaction vertices with the two additional terms, proportional to the self-energies, added to the original Lagrangian and regarded as additional interaction terms (the last term in Eq. (4.13) and the similar contribution for the components of \( \Psi \)). Using Eqs. (5.3) and (5.4) in (5.6), we obtain

\[
P(T, \mu_\Phi, \mu_\Psi) = \left( \frac{m_\Phi}{2\pi} \right)^{3/2} T^{5/2} \text{Li}_{5/2} \left( e^{\beta \bar{\mu}_\Phi} \right) + \left( \frac{m_\Psi}{2\pi} \right)^{3/2} T^{5/2} \text{Li}_{5/2} \left( e^{\beta \bar{\mu}_\Psi} \right) + g_\Phi \left( \frac{m_\Phi T}{2\pi} \right)^{3} \text{Li}_{3/2} \left( e^{\beta \bar{\mu}_\Phi} \right) + g_\Psi \left( \frac{m_\Psi T}{2\pi} \right)^{3} \text{Li}_{3/2} \left( e^{\beta \bar{\mu}_\Psi} \right) + g(m_\Phi m_\Psi)^{3/2} \left( \frac{T}{2\pi} \right)^{3} \text{Li}_{3/2} \left( e^{\beta \bar{\mu}_\Phi} \right) \text{Li}_{3/2} \left( e^{\beta \bar{\mu}_\Psi} \right). \quad (5.7)
\]
From Eq. \((5.7)\) we can now fix the chemical potentials from the expressions giving the densities,

\[
\rho_\Phi = \frac{\partial P}{\partial \mu_\Phi} = \left(\frac{m_\Phi T}{2\pi}\right)^{3/2} \text{Li}_{3/2}(e^{\beta \mu_\Phi}) \left[ (1 + 2g_\Phi A_\Phi) \frac{\partial \tilde{\mu}_\Phi}{\partial \mu_\Phi} + gA_\Phi \frac{\partial \tilde{\mu}_\Phi}{\partial \mu_\Phi} \right] + \left(\frac{m_\Phi T}{2\pi}\right)^{3/2} \text{Li}_{3/2}(e^{\beta \tilde{\mu}_\Phi}) \left[ (1 + 2g_\Phi A_\Phi) \frac{\partial \tilde{\mu}_\Phi}{\partial \mu_\Phi} + gA_\Phi \frac{\partial \tilde{\mu}_\Phi}{\partial \mu_\Phi} \right], \tag{5.8}
\]

and

\[
\rho_\Psi = \frac{\partial P}{\partial \mu_\Psi} = \left(\frac{m_\Psi T}{2\pi}\right)^{3/2} \text{Li}_{3/2}(e^{\beta \mu_\Psi}) \left[ (1 + 2g_\Psi A_\Psi) \frac{\partial \tilde{\mu}_\Psi}{\partial \mu_\Psi} + gA_\Psi \frac{\partial \tilde{\mu}_\Psi}{\partial \mu_\Psi} \right] + \left(\frac{m_\Psi T}{2\pi}\right)^{3/2} \text{Li}_{3/2}(e^{\beta \tilde{\mu}_\Psi}) \left[ (1 + 2g_\Psi A_\Psi) \frac{\partial \tilde{\mu}_\Psi}{\partial \mu_\Psi} + gA_\Psi \frac{\partial \tilde{\mu}_\Psi}{\partial \mu_\Psi} \right], \tag{5.9}
\]

where

\[
A_i = \frac{1}{T} \left(\frac{m_i T}{2\pi}\right)^{3/2} \text{Li}_{1/2}(e^{\beta \tilde{\mu}_i}). \tag{5.10}
\]

The derivatives involving the chemical potentials in \((5.8)\) and \((5.9)\), are defined by

\[
\frac{\partial \tilde{\mu}_\Phi}{\partial \mu_\Phi} = 1 - 2g_\Phi A_\Phi \frac{\partial \tilde{\mu}_\Phi}{\partial \mu_\Phi} - gA_\Phi \frac{\partial \tilde{\mu}_\Phi}{\partial \mu_\Phi},
\]

\[
\frac{\partial \mu_\Phi}{\partial \mu_\Phi} = -2g_\Phi A_\Phi \frac{\partial \mu_\Phi}{\partial \mu_\Phi} - gA_\Phi \frac{\partial \mu_\Phi}{\partial \mu_\Phi},
\]

\[
\frac{\partial \tilde{\mu}_\Psi}{\partial \mu_\Psi} = 1 - 2g_\Psi A_\Psi \frac{\partial \tilde{\mu}_\Psi}{\partial \mu_\Psi} - gA_\Psi \frac{\partial \tilde{\mu}_\Psi}{\partial \mu_\Psi},
\]

\[
\frac{\partial \mu_\Psi}{\partial \mu_\Psi} = -2g_\Psi A_\Psi \frac{\partial \mu_\Psi}{\partial \mu_\Psi} - gA_\Psi \frac{\partial \mu_\Psi}{\partial \mu_\Psi}. \tag{5.11}
\]

Eq. \((5.11)\) represents a set of equations for the derivatives of the effective chemical potentials that can be easily solved and then the results substituted back in Eqs. \((5.8)\) and \((5.9)\). The resulting expressions are just

\[
\rho_\Phi = \left(\frac{m_\Phi T}{2\pi}\right)^{3/2} \text{Li}_{3/2}(e^{\beta \mu_\Phi}) , \tag{5.12}
\]

and

\[
\rho_\Psi = \left(\frac{m_\Psi T}{2\pi}\right)^{3/2} \text{Li}_{3/2}(e^{\beta \mu_\Psi}) . \tag{5.13}
\]

Eqs. \((5.12)\) and \((5.13)\) can also be used in the equations (at the one-loop level) defining the self-energies in the normal phase, Eqs. \((5.3)\) and \((5.4)\). Then, by expanding Eqs. \((5.12)\) and \((5.13)\) in the high temperature limit \((\bar{\mu}_i/T \ll 1)\) one obtains the expression for the chemical potentials in terms of the densities (after inverting Eqs. \((5.12)\) and \((5.13)\)),

\[
\mu_\Phi \simeq 2g_\Phi \rho_\Phi + g\rho_\Phi - \frac{T}{4\pi} \left[ \left(\frac{2\pi}{m_\Phi T}\right)^{3/2} \rho_\Phi - \zeta(3/2) \right]^2 , \tag{5.14}
\]

and

\[
\mu_\Psi \simeq 2g_\Psi \rho_\Psi + g\rho_\Psi - \frac{T}{4\pi} \left[ \left(\frac{2\pi}{m_\Psi T}\right)^{3/2} \rho_\Psi - \zeta(3/2) \right]^2 . \tag{5.15}
\]

The above results, for the uncoupled two Bose gas \((g = 0)\) can be shown to agree with the results of Ref. \([16]\) obtained for the one Bose gas case and similar analysis that also follow form our results \((5.14)\) and \((5.15)\).
VI. CONCLUSIONS

We have considered a nonrelativistic model suitable to describe a system of homogeneous dilute Bose gas composed by two different types of atoms. A survey of the literature \[1\] shows that there is a growing interest in this type of systems. In general, coupled systems show a richer phase structure in comparison with uncoupled ones due to the presence of a cross coupling \[1, 2, 3, 10\]. For example, coupled nonrelativistic systems under the influence of external fields, represented by one body terms, may exhibit re-entrant phases \[2, 3\]. The appearance of such phenomena depends on the sign of the inter-species coupling, being completely ruled out in single species models. Apart from this fact, one also observes that the uncoupled and coupled models produce different values for critical quantities such as \(T_c\).

In this work, our main motivation was to check how the presence of an inter-species coupling would affect the qualitative and quantitative behavior of the transition regarding homogeneous coupled Bose gases. With this purpose we have evaluated the effective potential at finite temperature in a nonperturbative fashion to one-loop. Due to the complexity of zero temperature contributions, the complete evaluation was only possible in the approximation where both atomic masses are approximately the same. Our results show a dramatic difference concerning the case studied in Ref. \[2\] where the one body term represents, e.g., external fields and the Bose gas case, considered here, where these terms represent chemical potentials. Our present results exclude the possibility of exotic transition patterns such as inverse symmetry breaking and re-entrant phases arising in models relevant for BEC. This is rather satisfactory since, intuitively, one expects that the BEC transition for coupled gases should also observe the same simple pattern observed by monoatomic gases. That is, the system smoothly goes from an unsymmetric phase to a symmetric phase as soon as the critical temperature is reached. This is so because the value of cross coupling always appears squared eliminating the possible occurrence of ISB. Then, our second concern was to check the numerical values for the critical temperatures at which symmetry restoration occurs. Numerically, we found that the \(T_c\) values are insensitive to the coupling and coincide with the monoatomic results (which at this level of approximation corresponds to the standard ideal gas result). This can be understood in the view of the Hugenholtz-Pines theorem, in which case our one-loop nonperturbative approximations cannot probe the effects of interactions at \(T = T_c\). Nevertheless, in the region \(0 \leq T < T_c\) (where our approximation is more reliable) the curve describing the transition is seen to be influenced by the cross coupling. Taking the two species as having different densities, we have started at \(T = 0\) increasing the temperature and observing a first transition in which the more dilute species always reaches the non-condensed phase. At this first critical point the system decouples, the denser species remains in the condensed phase while the less dense acts as a thermal cloud. Then, a second transition occurs with the denser species reaching the gas phase exactly at the \(T_c\) observed in the corresponding monoatomic case. If the two species have the same density the unbroken phase is reached at once, which is not surprising, remembering that we took the atomic masses as being approximately the same. In summary, as in the monoatomic case and due to the Hugenholtz-Pines theorem, only the shapes of the curves describing the thermal behavior of the effective potential minima are sensitive to the numerical values of the couplings. The cross coupling, in particular, has a non negligible effect in this parameterization hinting in the possibility that a shift will appear if one computes, at higher orders, the critical temperatures for interacting uncoupled and coupled Bose gases.

Though in this work we have considered a rather idealized case, in comparison to real experiments performed in BEC, by only working with a homogeneous model and neglecting any non-homogeneity effects, we hope that the approach developed here may still be useful in the analysis of the thermodynamics in realistic Bose-Einstein condensation experiments with coupled atomic gases. The results and field theory methods used here could be considered, for instance, to be applicable to trapped atomic gases in the central region of wide traps.

Finally we remark that the coupled field model studied here, Eq. (2.1), could actually be viewed as the nonrelativistic limit of a two complex scalar fields \(\Phi\) and \(\Psi\), both with conserved charges, with chemical potentials \(\mu_{r,i}\), \(i = \Phi, \Psi\). In the nonrelativistic limit, the chemical potentials appearing in Eq. (2.1) should then be identified with \(\mu_i \equiv \mu_{r,i} - m_i\), which is the correct identification of a nonrelativistic chemical potential \[2\]. Some of the details of the nonrelativistic limit of the corresponding relativistic action of a two-scalar field model were already given in Refs. \[2, 4\], while for the one field model this was previously discussed in \[22\]. The calculations and analysis performed in the present work could then be easily extended to the relativistic problem and the combined effects of temperature and finite densities for the corresponding phase diagram of the model be studied. This would be of particular relevance for studies related to the early universe phase transition problems as well as the current heavy-ion collision experiments. We will report on the extension of the results presented in this work to the relativistic model in a forthcoming publication.
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