SYMMETRY PROPERTIES IN SYSTEMS OF FRACTIONAL LAPLACIAN EQUATIONS

ZHIGANG WU
Department of Applied Mathematics, Donghua University
Shanghai 201620, China

HAO XU*
Department of Applied Mathematics, University of Colorado Boulder
Boulder, CO 80302, USA

Abstract. We consider the systems of fractional Laplacian equations in a domain (bounded or unbounded) in $\mathbb{R}^n$. By using a direct method of moving planes, we show that $u_i(x)$ ($i = 1, 2, \ldots, m$) are radial symmetric about the same point and strictly decreasing in the radial direction with respect to this point. Comparing with Zhuo-Chen-Cui-Yuan [38], our results not only include subcritical case and critical case but also include supercritical case, and we need not the nonlinear terms to be homogenous. In addition, we completely remove the nonnegativity of $\frac{\partial f_i}{\partial u_i}$. Above all, to the best of our knowledge, it is the first result on the symmetric property of the system containing the gradient of the solution in the nonlinear terms.

1. Introduction. This paper is devoted to investigate the symmetry and monotonicity properties for positive solutions of fractional Laplacian equations. Especially, we consider the following fractional Laplacian system with homogeneous Dirichlet condition for $\alpha \in (0, 2)$ and $i = 1, 2, \ldots, m$:

\[
\begin{aligned}
&\left\{
\begin{array}{l}
(-\Delta)^{\frac{\alpha}{2}} u_i = f_i(x, u_1, u_2, \ldots, u_m, \nabla u_i) \text{ in } \Omega, \\
u_i > 0 \text{ in } \Omega, \quad \text{and } u_i = 0 \text{ on } \Omega^c,
\end{array}
\right.
\end{aligned}
\]

where $\Omega$ is a domain (bounded or unbounded) in $\mathbb{R}^n$ which is convex in $x_1$-direction, and the given functions $f_i(i = 1, 2, \ldots, m)$ satisfy the following assumption (call that the system is cooperative)

\[
(H1): \quad \frac{\partial f_i}{\partial u_k} \geq 0, \quad \text{for } k \neq i, \quad 1 \leq i, k \leq m.
\]

We call a domain $\Omega$ is convex in $x_1$-direction if and only if $(x_1, x')$, $(\bar{x}_1, x') \in \Omega$ implies that $(tx_1 + (1-t)\bar{x}_1, x') \in \Omega$ for $0 < t < 1$. $(-\Delta)^{\frac{\alpha}{2}}$ is the fractional Laplacian defined as

\[
(-\Delta)^{\frac{\alpha}{2}} u(x) = C_{n, \alpha} P.V. \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+\alpha}} dy, \quad 0 < \alpha < 2,
\]

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* Corresponding author: Hao Xu.
where $P.V.$ stands for the Cauchy principle value. Define

$$\mathcal{L}_\alpha = \left\{ u : \mathbb{R}^n \to \mathbb{R} | \int_{\mathbb{R}^n} \frac{|u(y)|}{1 + |y|^{n+\alpha}} \, dy < +\infty \right\}.$$  

Then it is easy to see that for $u \in C^{1,1}_{\text{loc}}(\Omega) \cap \mathcal{L}_\alpha$, $(-\Delta)^{\frac{\alpha}{2}} u(x)$ is well-defined for all $x \in \Omega$. In the following, we use $u$ to denote $(u_1, u_2, \cdots, u_m)$ and $p$ to denote $(\partial_{x_1} u_i, \partial_{x_2} u_i, \cdots, \partial_{x_n} u_i)$.

During the last decades, nonlinear equations involving general integrodifferential operators, especially, fractional Laplacian, have been extensively studied since the work of Caffarelli and Silvestre [8]. For other results on fractional Laplacian equations, we refer readers to [6, 7] for regularity, maximum principles, and Hamiltonian estimates, [10] for regularity of radial extremal solutions, [16] for existence and symmetry results of a Schrödinger type problem, [31] for regularity up to the boundary, [33] for mountain pass solutions, [34] for regularity of the obstacle problem, and [37] for a Liouville theorem in half space. Recently, Li and coauthors [23] established Böcher theorems for both Laplacian and fractional Laplacian cases, and as an important application, they also developed several maximum principles on a punctured ball. Which is basic and easy to be applied to the related problems for singular solutions. In fact, by using these maximum principle for the new variables after using Kelvin transform, and the method of moving planes, Li and Wu [24] derived the symmetry properties of the system of fractional Laplacian under some quite general structural conditions.

For the convenience of readers, we first review the following system with the nonlinear terms

$$\begin{cases}
(-\Delta)^{\frac{\alpha}{2}} u_i = f_i(u_1, u_2, \cdots, u_m) \text{ in } \Omega, \\
u_i > 0 \text{ in } \Omega, \quad \text{and } u_i = 0 \text{ on } \partial \Omega,
\end{cases} \quad \text{for } \alpha \in (0, 2) \text{ and } i = 1, 2, \cdots, m, \quad (4)$$

since the previous related results mainly focus on the system (4), which does not contain the gradient of the solution in the nonlinear terms. When $\alpha = 2$ and $m \geq 2$, the first work on symmetric properties for the system (4) of Laplacian equations in a bounded ball is Troy [36] under the key assumption (2). Then it was extended by Figueiredo [18] and Sirakov [35] for other kinds of domains like cones, paraboloids, cylinders. Later on, for the whole space, the first result is in Busca-Sirakov [5] under the assumption (2), some assumptions on the decay conditions of the solution at infinity and the derivatives of the nonlinear terms to guarantee the system cannot be reduced to two independent systems. Chen-Li [13] used the equivalence between the system (4) and a corresponding integral system to deduce the symmetric properties of the system when the nonlinear terms are homogeneous and critical. In addition, they introduce a natural condition on nonlinear terms that the system is \textit{non-degeneracy} if

$$(H2): \quad (f_{i_1}(u), f_{i_2}(u), \cdots, f_{i_k}(u)) \neq (f_{i_1}(v), f_{i_2}(v), \cdots, f_{i_k}(v))$$

whenever

$$(u_{i_1}, u_{i_2}, \cdots, u_{i_k}) = (v_{i_1}, v_{i_2}, \cdots, v_{i_k})$$

and

$$u_{i_{k+1}} > v_{i_{k+1}}, \quad u_{i_{k+2}} > v_{i_{k+2}}, \cdots, u_{i_m} > v_{i_m}.$$  

Here $i_1, i_2, \cdots, i_m$ is a permutation of $1, 2, \cdots, m$. This assumption can also guarantee that the system contains no independent subsystem. It is worth mentioning that due to our nonlinear terms containing $x$ and $\nabla u_i$, the \textit{non-degeneracy} assumption
in the present paper is just like (5) except replacing $f_i(u)$ by $f_i(x, u, \nabla u_i)$. Recently, Zhuo-Chen-Cui-Yuan [38] showed the equivalence between the system (4) with $m \geq 2$ and a corresponding integral system for $0 < \alpha < n$. Then once it’s done, under the assumption (5), the rest immediately follows from the result in Chen-Li-Ou [14], where this integral equation has been well studied. However, both [38] and [13] need the assumption
\[
\frac{\partial f_i}{\partial u_k} \geq 0, \quad \text{for } 1 \leq i, k \leq m.
\]
(6)
Additionally, the equivalence method in [38] cannot easily be applied to our system (1).
In terms of equations of fractional Laplacian, There are also some efforts on the symmetry and monotonicity results for equations in the unit ball or in $\mathbb{R}^n$ recently. These results are mainly based on the method of moving planes which was introduced by Aleksandrov [1] and developed by Serrin [32], Gidas-Ni-Nirenberg [20, 21] and many others. The first one is the extension method in Caffarelli-Silvestre [8] by transforming the non-local problem to a local one. The second one is based on the method of moving planes in the integral form established in Chen-Li [14] when $m = 1$, which was extended to the case of systems by Liu-Zhou in [29]. Recently, basing on the work of Jarohs-Weth [22] in bounded domains, Chen-Li-Li [11] developed a direct method of moving planes to treat the non-local problems in general domains by establishing some interesting maximum principles for antisymmetric functions. Then, Cheng-Huang-Li [15] also obtained a new version of the maximum principle in [11] (Lemma 2.1 below), and used it to the scalar case of (1) for both bounded domain and unbounded domain in $\mathbb{R}^n$. When $m = 2$ and $\alpha = 2$, there are many results on the symmetry of the solutions. Figueiredo-Felmer [19] considered Lane-Emden system by using the Kelvin transform and the particular expression of the nonlinear term. Busca-Sirakov [5] studied (4) with nonlinear terms as $f(u, v)$ and $g(u, v)$ in $\mathbb{R}^n$, under the assumption $(u, v) \rightarrow 0$ as $|x| \rightarrow \infty$, and several conditions on the derivatives on $f, g$ as the point $(0, 0)$. Later, Ma-Liu [30] relax some condition in [5] on the derivatives of $f, g$ for the price of supposing exact solution of the solutions at infinity, and of the nonlinearities at zero. We remark that by using the method in Chen-Li-Li [11], Liu-Ma [27] studied the radial symmetry properties of the system
\[
\begin{aligned}
(-\Delta)^{\alpha/2} u(x) &= f(u, v), \quad x \in \mathbb{R}^n, \\
(-\Delta)^{\alpha/2} v(x) &= g(u, v), \quad x \in \mathbb{R}^n, \\
u(x) > 0, \quad v(x) > 0, \quad x \in \mathbb{R}^n, \quad \alpha \in (0, 2),
\end{aligned}
\]
(7)
under the assumption $\frac{\partial f}{\partial u} > 0$ and $\frac{\partial g}{\partial u} > 0$, some decay conditions as $|x| \rightarrow \infty$ and growth conditions on the nonlinear terms $f$ and $g$. Later, Liu [28] also considered the similar problem for logarithmic Laplacian system in bounded domains. However, as mentioned in [5, 13], comparing with the case $m = 2$, there are their own difficulties for the large system $m \geq 3$. Moreover, our nonlinear terms contain the gradient of the solutions. Finally, for the method of moving planes applied to the fractional Laplacian with negative powers, we refer readers to [9] and references therein.

The motivation of the present paper is to extend the results in [15] to the system (1) with $m \geq 2$. Due to the good form of the pointwise estimate of $(-\Delta)^{\alpha/2} u$ at the minimum point in Lemma 2.1, we can directly obtain both the symmetry and monotonicity properties for the positive solution in bounded domain and unbounded domain in $\mathbb{R}^n$. On the one hand, we extent the case $\alpha = 2$ in Troy [36], Figueiredo [18] and Sirakov [35] to the fractional case $0 < \alpha < 2$. On the other hand, compared
Theorem 1.2. Let $\Omega$ be a bounded domain in $\mathbb{R}^n$ which is convex in $x_1$-direction and symmetrical about $x_1 = 0$. Suppose that $u_i \in C_{\text{loc}}^{1,1}(\Omega) \cap C(\mathbb{R}^n) (i = 1, 2, \cdots, m)$ solve (1) and $f_i(x, u, p)$ are Lipschitz continuous with respect to $u$. If $f_i(x, u, p) (i = 1, 2, \cdots, m)$ satisfy (2) and

$$
\begin{align*}
\left\{ \begin{array}{l}
\quad f_i(x_1, x', u, p_1, p_2, \cdots, p_n) \leq f_i(\bar{x}_1, x', u, -p_1, p_2, \cdots, p_n), \\
\quad \forall x_1, p_1 \leq 0, \ x_1 \leq \bar{x}_1 \leq -x_1,
\end{array} \right.
\end{align*}
$$

(8)

then $u_i(x_1, x') (i = 1, 2, \cdots, m)$ are strictly increasing in the left half of $\Omega$ in $x_1$-direction and

$$
u_i(x_1, x') \leq u_i(-x_1, x'), \ \forall x_1 < 0, \ (x_1, x') \in \Omega.
$$

(9)

Moreover, if $f_i(x_1, x', u, p) = f_i(-x_1, x', u, p)$, then

$$
u_i(x_1, x') = u_i(-x_1, x').
$$

Before state our result for unbounded domains, we impose a growth condition on $f(x, u, p)$:

When $u_i, \bar{u}_i \to 0^+ (i = 1, 2, \cdots, m)$, there exists a constant $C_1 > 0$ such that

$$
\begin{align*}
&|f(\cdots, u_{i-1}, u_i, u_{i+1}, \cdots, u_m, p) - f(\cdots, u_{i-1}, \bar{u}_i, u_{i+1}, \cdots, u_m, p)|
\leq C_1 \prod_{1 \leq i \leq m} \max \left\{ |u_i|^{s_i} + |\bar{u}_i|^{s_i}, (|u_i| + |\bar{u}_i|)^{s_i} \right\},
\end{align*}
$$

(10)

for some $s_i \geq 0$ and $\sum_{i=1}^m s_i > 0$.

Theorem 1.2. Let $\Omega$ be an unbounded domain in $\mathbb{R}^n$ which is convex in $x_1$-direction and symmetrical about $x_1 = 0$. Suppose that $u(x) \in C_{\text{loc}}^{1,1}(\Omega) \cap C(\mathbb{R}^n) \cap \mathcal{L}_\alpha$ solves (1) for $\alpha \in (0, 2)$. If $f_i(x, u, p) (i = 1, \cdots, m)$ satisfy (2), (8), (10) and (5), and $u_i(x)$ have the following asymptotic as $|x| \to +\infty$ that

$$
\begin{align*}
u_i(x) = o(|x|^{-a_i}), \ a_i > 0, \ i = 1, 2, \cdots, m
\end{align*}
$$

(11)

and $a_i$ and $s_i$ satisfy

$$
\sum_{i=1}^m a_i s_i \geq \alpha,
$$

(12)

where the constant $s_i$ is defined in (10). Then there exists some constant $\mu_0$ such that $u_i(x) (i = 1, 2, \cdots, m)$ are strictly increasing in the left half of $\Omega$ in $x_1$-direction when $x_1 < \mu_0$, and

$$
u_i(x_1, x') = u_i(2\mu_0 - x_1, x'), \ (x_1, x') \in \Omega.
$$

Then, because of the nonlinear terms containing the gradient of the solutions in the theorems above, we can immediately get many new radial symmetry results for the system with fractional Laplacian.
Corollary 1. Let \((u, v, w) \in (L_\alpha \cap C^{1,1}(\mathbb{R}^n))^3\) be a positive solution of system

\[
\begin{cases}
(-\Delta)_{\alpha}^s u(x) = u^{p_1}v^{q_1}w^{r_1} + f(|\nabla u|), & x \in \mathbb{R}^n, \\
(-\Delta)_{\alpha}^s v(x) = u^{p_2}v^{q_2}w^{r_2} + g(|\nabla v|), & x \in \mathbb{R}^n, \\
(-\Delta)_{\alpha}^s w(x) = u^{p_3}v^{q_3}w^{r_3} + h(|\nabla w|), & x \in \mathbb{R}^n,
\end{cases}
\tag{13}
\]

where \(p_i, q_i, r_i \geq 0\) \((i = 1, 2, 3)\) and the functions \(f, g, h\) are positive and in \(C^1(\mathbb{R}^+)\). Suppose that:

1. \(u(x) \leq C|x|^{-a}, \ v(x) \leq C|x|^{-b}, \ w(x) \leq C|x|^{-c}\), \(a, b, c > 0\), as \(|x| \to \infty\);
2. \(\min\{a(p_i-1)+bq_i+cr_i, ap_i+bq_i+cr_i, ap_i+bq_i+c(r_i-1)\} > \alpha,\)

then \((u, v, w)\) are radially symmetric about some point in \(\mathbb{R}^n\).

Corollary 2. Let \((u, v, w) \in (L_\alpha \cap C^{1,1}(\mathbb{R}^n))^3\) be a positive solution of the system

\[
\begin{cases}
(-\Delta)_{\alpha}^s u(x) = (\pm u^{p_1} + v^{q_1} + w^{r_1}) \cdot f(|\nabla u|), & x \in \mathbb{R}^n, \\
(-\Delta)_{\alpha}^s v(x) = (u^{p_2} + v^{q_2} + w^{r_2}) \cdot g(|\nabla v|), & x \in \mathbb{R}^n, \\
(-\Delta)_{\alpha}^s w(x) = (u^{p_3} + v^{q_3} + w^{r_3}) \cdot h(|\nabla w|), & x \in \mathbb{R}^n,
\end{cases}
\tag{14}
\]

where \(p_i, q_i, r_i > 1\) and the functions \(f, g, h\) are positive and in \(C^1(\mathbb{R}^+)\). Suppose that:

1. \(u(x) \leq C|x|^{-a}, \ v(x) \leq C|x|^{-b}, \ w(x) \leq C|x|^{-c}\), \(a, b, c > 0\), as \(|x| \to \infty\);
2. \(\min\{a(p_i-1)+bq_i+cr_i, ap_i+bq_i+cr_i, ap_i+bq_i+c(r_i-1)\} > \alpha,\)

then \((u, v, w)\) are radially symmetric about some point \(x^0\) in \(\mathbb{R}^n\).

Corollary 3. Let \((u, v, w) \in (L_\alpha \cap C^{1,1}(\mathbb{R}^n))^3\) be a positive solution of fractional Hénon system

\[
\begin{cases}
(-\Delta)_{\alpha}^s u(x) = |x|^{\sigma_1}v^{p} \cdot f(|\nabla u|), & x \in \mathbb{R}^n, \\
(-\Delta)_{\alpha}^s v(x) = |x|^{\sigma_2}w^{q} \cdot g(|\nabla v|), & x \in \mathbb{R}^n, \\
(-\Delta)_{\alpha}^s w(x) = |x|^{\sigma_3}u^{r} \cdot h(|\nabla w|), & x \in \mathbb{R}^n,
\end{cases}
\tag{15}
\]

where \(p, q, r \geq \frac{\alpha+n}{n-\alpha}, \ 0 \leq \sigma_1, \sigma_2, \sigma_3 < \alpha\) and the functions \(f, g, h\) are positive and in \(C^1(\mathbb{R}^+)\). If further

\[
\lim_{|x| \to +\infty} |x|^{n-\alpha}(u(x), v(x), w(x)) = +\infty,
\tag{16}
\]

then \((u, v, w)\) are radially symmetric about the same point in \(\mathbb{R}^n\).

The paper is organized as follows. Section 2 and Section 3 are devoted to the monotonicity and symmetry properties in bounded and unbounded domains respectively. We also apply Theorem 1.2 to two types of unbounded domains to get two corollaries.

2. Bounded domains. First, we define some frequently used notation for this paper:

\[
x = (x_1, x') \in \mathbb{R}^n, \ x^\lambda = (2\lambda - x_1, x'), \ T_\lambda = \{(x_1, x') \in \mathbb{R}^n | x_1 = \lambda\},
\]

\[
\Sigma_\lambda = \{(x_1, x') \in \mathbb{R}^n | x_1 < \lambda\}, \ \Sigma_\lambda = \{(x_1, x') \in \mathbb{R}^n | x_1 > \lambda\}.
\]
The following pointwise estimate of \((-\Delta) \tilde{\omega}\) at the minimum point is key for our main results. In the rest of the paper, we call a function \(\omega(x)\) is \(\lambda\)-antisymmetric function if and only if

\[
\omega(x_1, x_2, \ldots, x_n) = -\omega(2\lambda - x_1, x_2, \ldots, x_n).
\]  

(17)

**Lemma 2.1.** [15] Let \(\omega(y) \in \mathcal{L}_\alpha\) be a \(\lambda\)-antisymmetric function defined in (17). Suppose there exists \(x \in \Sigma_\lambda\) such that

\[
\omega(x) = \inf_{\Sigma_\lambda} \omega(y) \leq 0.
\]

If \(\omega\) is \(C^{1,1}\) at \(x\), we have

\[
(-\Delta) \tilde{\omega}(x) \leq \tilde{C}_{n,\alpha} \left( \delta^{-\alpha} \omega(x) - \delta \int_{\Sigma_\lambda} \frac{(\omega(y) - \omega(x))(\lambda - y_1)}{|x - y|^{n+\alpha+2}} \, dy \right),
\]

where \(\alpha \in (0, 2), \delta = d(x, T_\lambda) = |x_1 - \lambda|,\) and \(\tilde{C}_{n,\alpha}\) is a positive constant depending on \(n\) and \(s\) only.

With the above preparations, we now can prove Theorem 1.1.

**The proof of Theorem 1.1.** Since \(\Omega\) is bounded, without loss of generality, we may assume \(\Omega \subset \{|x_1| \leq 1\}\) and \(\partial \Omega \cap \{x_1 = -1\} \neq \emptyset\). Set

\[
u_i^0(x) = u_i(x), \quad \omega_i^0(x) = u_i^0(x) - u_i(x), \quad \forall x \in \Sigma_\lambda.
\]

**Step 1.** We claim that there exists \(\delta_0 \geq 0\) small enough such that

\[
\omega_i^0(x) \geq 0, \quad \forall x \in \Sigma_\lambda, \quad \forall \lambda \in [-1, -1 + \delta_0], \quad i = 1, 2, \ldots, m.
\]

(19)

Suppose not, we set

\[
A_0 = \inf_{x \in \Sigma_\lambda, 1 \leq i \leq m} \omega_i^0(x) < 0.
\]

(20)

Since \(0 \leq u_i(x) \in C(\mathbb{R}^n)\) and \(u_i(x) \equiv 0\) in \(\mathbb{R}^n / \Omega\), \(A_0\) can be obtained for some \(i_0 \in \{1, 2, \ldots, m\}\) and \((\lambda_0, x_0) \in \{(\lambda, x) | (\lambda, x) \in [-1, -1 + \delta_0] \times \Sigma_\lambda \cap \Omega\} \). Notice that \(\omega_{i_0}^0 \geq 0\) on \(\partial \Omega \cap \Sigma_{\lambda_0}\), we have \(x_0 \in \Sigma_{\lambda_0} \cap \Omega\). From the fact that \(\Sigma_{-1} \cap \Omega = \partial \Sigma_{-1} \cap \Omega\) for \(\lambda = -1\), one gets \(\omega_{i_0}^{-1}(x) \geq 0\) in \(\Sigma_{-1} \cap \Omega\), this yields that \(\lambda_0 > -1\). Since \((\lambda_0, x_0)\) is a minimizing point, we have

\[
(a) \quad \frac{\partial \omega_{i_0}^0(x)}{\partial \lambda} \big|_{\lambda = \lambda_0} \leq 0, \quad \text{which implies that} \quad \partial_{x_i} u_{i_0}(x_0)^{\lambda_0} \leq 0.
\]

(21)

\[
(b) \quad \nabla_x \omega_{i_0}^0(x_0) = 0, \quad \text{which implies that} \quad \nabla_x u_{i_0}^{\lambda_0}(x_0) = \nabla_x u_{i_0}(x_0).
\]

Now \((-\Delta) \tilde{\omega}_{i_0}^{\lambda_0}(x_0)\) can be estimated as

\[
(-\Delta) \tilde{\omega}_{i_0}^{\lambda_0}(x_0) = f_{i_0}(x_0, u_i^{\lambda_0}(x_0), \nabla u_{i_0}^{\lambda_0}(x_0)) - f_{i_0}(x_0, u_i(x_0), \nabla u_{i_0}(x_0))
\]

\[
= f_{i_0}(x_0, u_i^{\lambda_0}(x_0), \ldots, u_m^{\lambda_0}(x_0), \nabla u_{i_0}^{\lambda_0}(x_0)) - f_{i_0}(x_0, u_i^{\lambda_0}(x_0), \ldots, u_m^{\lambda_0}(x_0), \nabla u_{i_0}^{\lambda_0}(x_0))
\]

\[
+ \cdots + f_{i_0}(x_0, u_1(x_0), \ldots, u_{i_0-1}(x_0), u_{i_0}(x_0), u_{i_0+1}^{\lambda_0}(x_0), \ldots, u_m^{\lambda_0}(x_0), \nabla u_{i_0}^{\lambda_0}(x_0))
\]

\[
+ f_{i_0}(x_0, u_1(x_0), \ldots, u_{i_0-1}(x_0), u_{i_0}(x_0), u_{i_0}^{\lambda_0}(x_0), \ldots, u_m^{\lambda_0}(x_0), \nabla u_{i_0}^{\lambda_0}(x_0)) + \cdots
\]

\[
\geq c_1(x_0) \omega_1^{\lambda_0}(x_0) + \cdots + c_{i_0}(x_0) \omega_{i_0}^{\lambda_0}(x_0) + \cdots + c_m(x_0) \omega_m^{\lambda_0}(x_0),
\]

(22)
where
\[ c_k(x_0) = \frac{f_{x_0}(x_0, \cdot \cdot \cdot, u_{k-1}(x_0), u_k^{\lambda_0}(x_0), u_{k+1}(x_0), \cdot \cdot \cdot) - f_{x_0}(x_0, \cdot \cdot \cdot, u_{k-1}(x_0), u_k(x_0), u_{k+1}(x_0), \cdot \cdot \cdot)}{u_k^{\lambda_0}(x_0) - u_k(x_0)} \]

are uniformly bounded since \( u_i(x) \in C(\mathbb{R}^n) \) with compact support and \( f_{x_0} \) are Lipschitz continuous with respect to \( u \). The estimate (22) is from the assumption (8) and the following fact from (21) that
\[ \partial_{x_i} u_{i_0}(x_0^{\lambda_0}) = \partial_{x_i} u_{i_0}(x_0) \text{ for } i = 2, 3, \cdot \cdot \cdot, n, \]
and
\[ 0 \geq \partial_{x_1} u_{i_0}(x_0^{\lambda_0}) = -\partial_{x_1} u_{i_0}(x_0). \]

On the other hand, from the assumption (2) we know \( c_k(x_0) \geq 0 \) for \( k \neq i_0 \), which yields
\[
\sum_{k=1}^{m} c_k(x_0) u_k^{\lambda_0}(x_0) \\
\geq \sum_{l \neq i_0} c_l(x_0) u_l^{\lambda_0}(x_0) + c_{i_0}(x_0) u_{i_0}^{\lambda_0}(x_0), \text{ with } l \neq i_0, \quad u_i^{\lambda_0}(x_0) \geq u_{i_0}^{\lambda_0}(x_0) \\
\geq (\sum_{l} c_l(x_0) + c_{i_0}(x_0)) \omega_{i_0}^{\lambda_0}(x_0) := \tilde{c}(x_0) \omega_{i_0}^{\lambda_0}(x_0).
\]

Now by Lemma 2.1, we have
\[
0 \leq (-\Delta)^{\frac{\alpha}{2}} \omega_{i_0}^{\lambda_0}(x_0) - \tilde{c}(x_0) \omega_{i_0}^{\lambda_0}(x_0) \leq (\tilde{C}_{n,\alpha} \delta_0^{-\alpha} - \tilde{c}(x_0)) \omega_{i_0}^{\lambda_0}(x_0) < 0,
\]

if we choose \( \delta_0 \) small enough. This yields a contradiction and hence proves the claim (19).

**Step 2.** Set
\[
\lambda_0 = \sup_{-1 \leq \lambda \leq 0} \left\{ \lambda \mid \min_{1 \leq i \leq m} \omega_i^{\mu}(x) \geq 0, \forall x \in \Sigma_\mu, \forall \mu \leq \lambda \right\}.
\]

By the definition of \( \lambda_0 \) and the continuity of \( u_i(x) \), we have \( \omega_i^{\lambda_0}(x) \geq 0 \) for all \( x \in \Sigma_{\lambda_0} \).

It remains to show \( \lambda_0 = 0 \). First, we have the following fact:
\[ \text{If } \lambda_0 < 0, \text{ then } \omega_i^{\lambda_0}(x) \neq 0, \forall x \in \Sigma_{\lambda_0}, \ i = 1, 2, \cdot \cdot \cdot, m. \]

Then, we give a claim:
\[ \text{There exists } \epsilon_0 \geq 0 \text{ small enough such that } \]
\[ \text{if } \lambda_0 < 0, \text{ then } \omega_i^{\lambda_0}(x) \geq 0 \ (i = 1, 2, \cdot \cdot \cdot, m), \text{ for } \forall x \in \Sigma_{\lambda}, \forall \lambda \leq \lambda_0 + \epsilon_0. \]

Suppose (25) is not true, one has
\[ A_k = \inf_{\Sigma_{\lambda}} \omega_i^{\lambda}(x) < 0, \text{ for a sequence of } \lambda_k \downarrow \lambda_0, \text{ as } k \to \infty. \]

The minimum \( A_k \) can be obtained for some \( i_0 \in \{1, 2, \cdot \cdot \cdot, m\} \), \( \mu_k \in (\lambda_0, \lambda_k], \ x_k \in \Sigma_{\mu_k} \cap \Omega \), that is, we can set \( \omega_{i_0}^{\mu_k}(x_k) = A_k \). Then we denote \( d_k = d(x_k, T_{\mu_k}) \). As in step 1, by Lemma 2.1, we have
\[
0 \leq (-\Delta)^{\frac{\alpha}{2}} \omega_{i_0}^{\mu_k}(x_k) - \tilde{c}(x_k) \omega_{i_0}^{\mu_k}(x_k) \\
\leq (\tilde{C}_{n,\alpha} d_k^{-\alpha} - \tilde{c}(x_k)) \omega_{i_0}^{\mu_k}(x_k) - \tilde{C}_{n,\alpha} d_k \int_{\Sigma_{\mu_k}} \frac{\omega_{i_0}^{\mu_k}(y)(\mu_k - y_k)}{|x_k - y_k|^{n+\alpha+2}} dy,
\]
where \(\hat{c}(\cdot)\) is the same as (23). From (26) we know
\[
d_k^\alpha \geq \frac{\bar{C}_{n,\alpha}}{|\hat{c}(x^k)|} \geq c_0 > 0.\]
Then along a suitable subsequence of \(k \to +\infty\), we may assume that \(x^k \to x^0 \in \Omega \cap \Sigma\lambda_0\) and \(d(x^0, T_{\lambda_0}) \geq c_0 > 0\). From the Lebesgue dominated convergence theorem, we get
\[
0 \leq \int_{\Sigma_{\lambda_0}} \omega_{\lambda_0}^{\lambda \omega_0} (y) (\lambda - y_1) \frac{dy}{|x^0 - y_{\lambda_0}|^{n+\alpha+2}} \leq 0,
\]
which implies that \(\omega_{\lambda_0}^{\lambda \omega_0} (x) \equiv 0\) for \(x \in \Sigma_{\lambda_0}\) and ultimately yields a contradiction with the above fact (24). This proves the claim (25), i.e., \(\lambda_0 = 0\).

**Step 3.** We shall prove the strictly monotonicity property. We claim:
\[
\text{if } \lambda < 0, \text{ then } \omega_{\lambda_i}^{\lambda} (x) > 0, \text{ for } i = 1, 2, \cdots, m. \tag{27}
\]
If (27) is not true, there exists some \(x_0 \in \Omega \cap \Sigma\lambda\) and \(i_0 \in \{1, 2, \cdots, m\}\) such that \(\omega_{\lambda_{i_0}}^{\lambda} (x_0) = 0\). Then as in step 1, we have
\[
(-\Delta)^\frac{n}{2} \omega_{\lambda_{i_0}}^{\lambda} (x_0) \geq \sum_{k=1}^m c_k(x_0) \omega_{\lambda_{i_0}}^{\lambda} (x_0) \geq c_{i_0} (x_0) \omega_{\lambda_{i_0}}^{\lambda} (x_0),
\]
which together with Lemma 2.1 and the fact \(\omega_{\lambda_{i_0}}^{\lambda} (x) \neq 0\) yields
\[
0 \leq (-\Delta)^\frac{n}{2} \omega_{\lambda_{i_0}}^{\lambda} (x_0) - c_{i_0} (x_0) \omega_{\lambda_{i_0}}^{\lambda} (x_0)
\leq -\bar{C}_{n,\alpha} |x_{0,1} - \lambda| \int_{\Sigma_{\lambda}} \omega_{\lambda_{i_0}}^{\lambda} (y) (\lambda - y_1) \frac{dy}{|x_0 - y_{\lambda_0}|^{n+\alpha+2}} < 0.
\]
Hence, this contradiction suffices to prove the claim (27).

Then, by taking \(\lambda = \frac{x_{1,1} + x_1}{2}\), and choosing \((x_1, x'), (\bar{x}_1, x') \in \Omega\) with \(0 < x_1 < \bar{x}_1\), we have \(u_i(x_1, x') > u_i(\bar{x}_1, x')\) for \(i = 1, 2, \cdots, m\). We complete the proof of Theorem 1.1. \(\square\)

### 3. Unbounded domains

This section is devoted to prove Theorem 1.2.

**Step 1.** We first claim:

There exists \(R_0 > 0\) large enough such that \(\omega_i^\lambda(x) \geq 0\), \(\forall x \in \Sigma\lambda, \forall \lambda \leq -R_0\). \(\tag{28}\)

If not, there exists \(\lambda_k \to -\infty\) and
\[
A_k = \inf_{x \in \Sigma\lambda} \omega_i^\lambda(x) < 0. \tag{29}\]
Since \(\omega_i^\lambda(x) = u_i^\lambda(x) - u_i(x) \geq -u_i(x) \geq \frac{A_k}{2}\), if \(x \leq -R_k\) for some \(R_k > 0\) large enough. Hence we must have
\[
\omega_{i_0}^{\mu_k} (x) = A_k, \text{ for some } \mu_k \in [-|x^k|, \lambda_k], \ |x^k| \leq R_k. \tag{30}\]
As in the proof of Theorem 1.1, a direct calculation gives
\[
(-\Delta)^{\frac{\alpha}{2}} \omega_i^a(x^k) \geq \sum_{r=1}^{m} c_r(x^k) w_r^\mu(x^k) \\
\geq \sum_i c_i(x^k) w_i^\mu(x^k) + c_{i_0}(x^k) \omega_i^a(x^k), \text{ with } i \neq i_0, \ w_i^\mu(x^k) < 0 \\
\geq \left( \sum_i c_i(x^k) + c_{i_0}(x^k) \right) \omega_i^a(x^k) := \sum_i c_i(x^k) \omega_i^a(x^k) \\
\geq 2C_1 \sum_i \prod_{i=1}^{m} |u_i(x^k)|^{s_i} \omega_i^a(x^k),
\]
where we have used the expression
\[
c_i(x^k) = f_0(\cdots, u_{i-1}(x^k), u_i^\mu(x^k), u_{i+1}^\mu(x^k), \cdots) - f_0(\cdots, u_{i-1}(x^k), u_i(x^k), u_{i+1}^\mu(x^k), \cdots) \\
\omega_i^a(x^k) - u_i(x^k),
\]
and the fact that from the assumption (10)
\[
c_i(x^k) \leq C_1 \prod_{i=1}^{m} \max\{|u_i^\mu(x^k)|^{s_i} + |u_i(x^k)|^{s_i}, (|u_i^\mu(x^k)| + |u_i(x^k)|)^{s_i}\} \\
\leq 2C_1 \prod_{i=1}^{m} |u_i(x^k)|^{s_i}.
\]

On one hand, by using Lemma 2.1, we have
\[
(-\Delta)^{\frac{\alpha}{2}} \omega_i^a(x^k) \leq \frac{\tilde{C}_{n, \alpha}}{|x_i^k - \mu_k|^\alpha} \omega_i^a(x^k).
\]  

From (31) and (32), it need the following inequality
\[
\tilde{C}_{n, \alpha} \leq 2C_1 |x_i^k - \mu_k|^\alpha \sum_i \left( \prod_{i=1}^{m} |u_i(x^k)|^{s_i} \right).
\]

On the other hand, from the assumptions (11)-(12) and the fact $|x^k| \geq |\lambda_k| \to +\infty$ as $k \to +\infty$, we have
\[
2C_1 |x_i^k - \mu_k|^\alpha \sum_i \left( \prod_{i=1}^{m} |u_i(x^k)|^{s_i} \right) < 2C_1 |x^k|^\alpha \left( \prod_{i=1}^{m} |u_i(x^k)|^{s_i} \right) \\
\leq 2mC_1 |x^k|^\alpha (|x^k|^{-\sum_i s_i}) \leq \tilde{C}_{n, \alpha},
\]
which together with (33) leads to a contradiction. This proves the claim (28).

**Step 2.** Set
\[
\lambda_0 = \sup_{\lambda \leq 0} \left\{ \lambda | \min_{1 \leq i \leq m} \omega^\mu_i (x) \geq 0, \forall x \in \Sigma_\mu, \forall \mu \leq \lambda \right\}.
\]

By the definition of $\lambda_0$ and the continuity of $u_i(x)$, we have $\omega^\lambda_i(x) \geq 0$ $(1 \leq i \leq m)$ for all $x \in \Sigma_{\lambda_0}$.

Suppose that some of $u_i(x)(i = 1, 2, \cdots, m)$ are not symmetric in the $x_1$-direction at $\lambda_0$ when $x \in \Sigma_{\lambda_0}$. Without loss of generality, we assume that
\[
\omega^\lambda_i(x) \neq 0, \ i = 1, 2, \cdots, k, \ k \geq 1; \ \omega^\lambda_i(x) \equiv 0, \ i = k + 1, \cdots, m.
\]

Then, we claim that
\[
\omega^\lambda_i(x) > 0, \ i = 1, 2, \cdots, m.
\]
Step 2.1. We conclude that
\[ \omega^k_i(x) > 0, \quad i = 1, 2, \ldots, k, \quad x \in \Sigma_{\lambda_0}. \] (37)

Otherwise, there exists some \( x_0 \in \Sigma_{\lambda_0} \) such that \( \omega^k_i(x_0) = 0 \) from some \( i_0 \in \{1, 2, \ldots, k\} \). By using Lemma 2.1 and the condition (2), and the assumption \( \omega^k_i(x) \not\equiv 0, \quad i = 1, 2, \ldots, k \), we have
\[
0 \leq (-\Delta)^{\frac{m}{2}} \omega^k_i(x_0) - \sum_{l} c_l(x_0) \omega^k_l(x_0) = (-\Delta)^{\frac{m}{2}} \omega^k_i(x_0) - \sum_{l \leq k} c_l(x_0) \omega^k_l(x_0)
\]
\[
\leq (-\Delta)^{\frac{m}{2}} \omega^k_i(x_0) \leq -C_{n,\alpha} |x_0 - \lambda_0| \int_{\Sigma_{\lambda_0}} \frac{\omega^{\lambda_0+\epsilon}(y)(\lambda_0 - y_1)}{|x_0 - y^{\lambda_0}|^{n+\alpha+2}}\,dy < 0,
\] (38)

which yields a contradiction.

Step 2.2. If \( k = m \), the claim (36) is proved. Thus we assume \( k < m \). We set \( \bar{x} \in \Sigma_{\lambda_0} \), then
\[
0 = \sum_{l = k+1}^{m} (-\Delta)^{\frac{m}{2}} \omega^k_l(\bar{x}) = \sum_{l = k+1}^{m} \left[ f_l(\bar{x}^{\lambda_0}, u^{\lambda_0}(\bar{x}), p^{\lambda_0}(\bar{x})) - f_l(\bar{x}, u(\bar{x}), p(\bar{x})) \right]
\]
\[
\geq \sum_{l = k+1}^{m} \left[ f_l(\bar{x}, u^{\lambda_0}(\bar{x}), p(\bar{x})) - f_l(\bar{x}, u(\bar{x}), p(\bar{x})) \right] > 0,
\] (39)

where we used the assumption that the system is non-degeneracy. This yields that \( k = m \), and hence we have proved the claim (36).

Step 2.3. We want to deduce another contradiction from the claim (36). In fact, we need to show the following holds,
\[
\omega^k_i(x) \geq 0, \quad \forall x \in \Sigma_{\lambda}, \quad \lambda \in (-\infty, \lambda_0 + \epsilon_0), \quad \text{for some } \epsilon_0 > 0.
\] (40)

Suppose (40) is not true, one has
\[
B_k = \inf_{\lambda_0 \leq \lambda \leq \lambda_k} \omega^k_i(x) < 0, \quad \text{for a sequence of } \lambda_k \downarrow \lambda_0, \quad \text{as } k \to \infty.
\]

As in Step 1, \( B_k \) can be obtained, i.e. \( \omega_{j_0}^k(x^k) = B_k \), for some \( j_0 \in \{1, 2, \ldots, m\} \), \( \mu_k \in (\lambda_0, \lambda_k] \) and \( x^k \in \Sigma_{\mu_k} \cap \Omega \). By using Lemma 2.1 and (33) again, we have
\[
\tilde{C}_{n,\alpha} \leq 2C_1 |x^k_1 - \mu_k|^\alpha \sum_{l} \left( \prod_{i} |u_i(x^k)|^{s_i} \right),
\] (41)

which together with (11) implies that \( |x^k| \) is uniformly bounded independent of \( k \).

Set \( d_k = d(x^k, T_{\mu_k}) \). As in step 2 in the proof of Theorem 1.1, we have from Lemma 2.1
\[
0 \leq (-\Delta)^{\frac{m}{2}} \omega_{j_0}^{\mu_k}(x^k) - \sum_{l} c_l(x^k) \omega_{j_0}^{\mu_k}(x^k)
\]
\[
\leq \left( \tilde{C}_{n,\alpha} d_k^{-\alpha} - \sum_{l} c_l(x^k) \right) \omega_{j_0}^{\mu_k}(x^k) - \tilde{C}_{n,\alpha} d_k \int_{\Sigma_{\mu_k}} \frac{\omega_{j_0}^{\mu_k}(y)(\mu_k - y_1)}{|x^k - y^{\mu_k}|^{n+\alpha+2}}\,dy,
\] (42)

where \( c_l(x^k) \) is the same as in (31). Hence, \( f_i(x, u, \nabla u_i) \in A \) and \( u \in C(\mathbb{R}^n) \) imply that \( \sum_{l} c_l(x^k) \) is uniformly bounded independent of \( k \). From (42) we know
\[
d_k^{\alpha} \geq \frac{\tilde{C}_{n,\alpha}}{\sum_{l} c_l(x^k)} \geq c_0 > 0.
\]
Then along a suitable subsequence of \( k \to +\infty \), we may assume that \( x^k \to x^0 \in \Omega \cap \Sigma_{\lambda_0} \) and \( d(x^0, T_{\lambda_0}) \geq c_0 > 0 \). From the Lebesgue dominated convergence theorem, we get

\[
0 \leq \int_{\Sigma_{\lambda_0}} \frac{\omega_0^{\lambda_0}(y)(\lambda_0 - y_1)}{|x^0 - y|^{n+\alpha}} \, dy \leq 0,
\]

which implies that \( \omega_0^{\lambda_0}(x) \equiv 0 \) for \( x \in \Sigma_{\lambda_0} \) and yields a contradiction with the claim (36). This proves the claim (40), which contradicts the definition of \( \lambda_0 \). At last, the strictly monotonicity follows from \( \omega_i^{\lambda}(x) > 0 \) in \( \Sigma_{\lambda} \cap \Omega \) for \( \lambda < \lambda_0 \), which can be proved by repeating the Step 2.1 and Step 2.2 above. This completes the proof of Theorem 1.2.

As an application of Theorem 1.2, we present two corollaries below.

**Corollary 4.** Suppose that \( 0 \leq u_i(x) \in C^{1,1}({\mathbb R}^n) \cap {\mathcal L}_\alpha \) solve the system

\[
(\Delta)^{\frac{\alpha}{2}} u_i = f_i(|x|, u_1, u_2, \ldots, u_m, |\nabla u_i|) \quad \text{in} \quad {\mathbb R}^n, \quad \text{for} \quad i = 1, 2, \ldots, m, \quad \text{and} \quad \alpha \in (0, 2),
\]

where \( f_i(|x|, u_1, \ldots, u_m, |\nabla u_i|) \) satisfy (2), (5) and for \( 0 \leq u_i, \bar{u}_i < 1 \) that

\[
|f_i(|x|, u_1, \ldots, u_m, |\nabla u_i|) - f_i(|x|, u_1, \ldots, u_m, |\nabla u_i|)| \leq C \prod_{1 \leq i \leq m} \max\{|u_i|^{\gamma_i} + |\bar{u}_i|^{\gamma_i}, (|u_i| + |\bar{u}_i|)^{\gamma_i}\}, \quad \gamma_i > 0.
\]

Moreover, for \( 1 \leq i \leq m \), we assume that

\[
u_i(x) = o\left(\frac{1}{|x|^{\alpha_i}}\right), \quad \sum_{i=1}^{m} a_i \gamma_i = \alpha.
\]

Then \( u_i(x) \) are radially symmetric about some point in \( {\mathbb R}^n \) and monotone decreasing in the radial direction with respect to this point.

Since \( f_i \) satisfy (8) in any direction in \( {\mathbb R}^n \) and all the assumptions in Theorem 1.2 are fulfilled in Corollary 4, the proof is obvious.

Another typical example is the so-called infinite cylinder \( C = (-\infty, +\infty) \times \Omega' \) where \( \Omega' \subset {\mathbb R}^{n-1} \) is a bounded domain.

**Corollary 5.** Suppose that \( 0 \leq u_i(x) \in C^{1,1}({\mathbb R}^n) \cap {\mathcal L}_\alpha \) solve the system

\[
\left\{ \begin{array}{l}
(\Delta)^{\frac{\alpha}{2}} u_i = f_i(|x|, u_1, \ldots, u_m, |\nabla u_i|) \quad \text{in} \quad C, \\
u_i > 0 \quad \text{in} \quad C, \quad \text{and} \quad u_i = 0 \quad \text{on} \quad \partial C \cap {\mathcal L}_\alpha,
\end{array} \right. \quad \text{for} \quad \alpha \in (0, 2),
\]

where \( f_i(|x|, u_1, \ldots, u_m, |\nabla u_i|) \) satisfy (2), (5) and for \( 0 \leq u_i, \bar{u}_i < 1 \) that

\[
|f_i(|x|, u_1, \ldots, u_m, |\nabla u_i|) - f_i(|x|, u_1, \ldots, u_m, |\nabla u_i|)| \leq C \prod_{1 \leq i \leq m} \max\{|u_i|^{\gamma_i} + |\bar{u}_i|^{\gamma_i}, (|u_i| + |\bar{u}_i|)^{\gamma_i}\}, \quad \gamma_i > 0.
\]

Moreover, for \( 1 \leq i \leq m \), we assume that

\[
u_i(x) = o\left(\frac{1}{|x|^{\alpha_i}}\right), \quad \text{and} \quad \sum_{i=1}^{m} a_i \gamma_i = \alpha.
\]

Then \( u_i(x) \) are radially symmetric about some point in \( {\mathbb R}^n \) and monotone decreasing in the radial direction with respect to this point.
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E-mail address: zgwu@dhu.edu.cn
E-mail address: Hao.Xu-1@colorado.edu