Majority out-dominating sets in digraphs

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Abstract

The concept of majority domination in graphs has been defined in at least two different ways: As a function and as a set. In this work we extend the latter concept to digraphs, while the former was extended in another paper. Given a digraph $D = (V, A)$, a set $S \subseteq V$ is a majority out-dominating set (MODS) of $D$ if $|N^+(S)| \geq \frac{n}{2}$. The minimum cardinality of a MODS in $D$ is the set majority out-domination number $\gamma^+_m(D)$ of $D$. In this work we introduce these concepts and prove some results about them, among which the characterization of minimal MODSs.

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Key words: Majority dominating set, majority out-dominating set, orientation of a graph.

1 Introduction

Although domination and other related concepts have been extensively studied for undirected graphs, the respective analogues on digraphs have not received much attention. A survey of results on domination in directed graphs is found in chapter 15 of [3], but it mostly focuses on kernels and solutions (that is, independent in- and out-dominating sets), and on domination in tournaments.

The notion of majority out-dominating set is fairly interesting from the mathematical point of view, since it is close enough to that of out-dominating set as to inherit several of its properties and allow the adaptation of some known results, and at the same time it is different enough as to open a new line of research.

This concept has interesting applications, specially related to democracy: The main idea of democracy is that of a representative group which is accepted by a majority of the population. In some way, this corresponds to majority dominating sets in undirected graphs. However, it is important to notice that the relation is actually directed: The representative group must be accepted by at least half of the population, but if the group itself accepts or not a particular sector of such population has no influence at all in the scope of simple democracy. Of course,
more complex systems exist, with the aim that every important minority has some acceptance from the representative group, and those systems are better fit for large populations, like that of a country. Nevertheless, simple democracy is still the best option for small groups, like the members of a club or those of a small company.

In the context of simple democracy, the concept of majority out-dominating set in digraphs works more accurately than that of majority dominating set in undirected graphs.

2 Fundamentals

Throughout this paper $D = (V, A)$ is a finite directed graph with neither loops nor multiple arcs (but pairs of opposite arcs are allowed) and $G = (V, E)$ is a finite undirected graph with neither loops nor multiple edges. Unless stated otherwise, $n$ denotes the order of $D$ (or $G$), that is, $n = |V|$. For basic terminology on graphs and digraphs we refer to [2].

Let $G = (V, E)$ be a graph. For any vertex $u \in V$, the set $N_G(u) = \{v : uv \in E\}$ is called the neighborhood of $u$ in $G$. $N_G[u] = N_G(u) \cup \{u\}$ is the closed neighborhood of $u$ in $G$. The degree of $u$ in $G$ is $d_G(u) = |N_G(u)|$. When the graph $G$ is clear from the context, we may write simply $N(u), N[u], \text{ and } d(u)$.

Let $D = (V, A)$ be a digraph. For any vertex $u \in V$, the sets $N^+_D(u) = \{v : vu \in A\}$ and $N^-_D(u) = \{v : uv \in A\}$ are called the out-neighborhood and in-neighborhood of $u$ in $D$, respectively. $N^+_D[u] = N^+_D(u) \cup \{u\}$ is the closed out-neighborhood of $u$ in $D$, and $N^-_D[u] = N^-_D(u) \cup \{u\}$ is the closed in-neighborhood of $u$ in $D$. When the digraph $D$ is clear from the context, we may write simply $N^+(u), N^-(u), N^+\{u\}, \text{ and } N^\{u\}$. The out-degree and in-degree of $u$ in $D$ are defined by $d^+_D(u) = |N^+_D(u)|$ and $d^-_D(u) = |N^-_D(u)|$, respectively. The maximum out-degree of $D$ is denoted by $\Delta^+_D$. When the digraph $D$ is clear from the context, we may write $d^-(u), d^+(u), \text{ and } \Delta^+$.

Given a set $X \subseteq V$ and $u \in X$, the set of external private out-neighbors of $u$ respect to $X$ is $pn^+(u, X) = \{v \in V \setminus X : N^-(v) \cap X = \{u\}\}$, and $pn^+[u, X] = pn^+(u, X) \cup \{u\}$. Moreover, $D[X]$ denotes the subdigraph of $D$ induced by $X$.

Let $G = (V, E)$ be a graph. A subset $S$ of $V$ is called a dominating set of $G$ if every vertex in $V \setminus S$ is adjacent to at least one vertex in $S$. The minimum cardinality of a dominating set of $G$ is called the domination number of $G$ and is denoted by $\gamma(G)$, or simply $\gamma$.

Let $D = (V, A)$ be a digraph. A subset $S$ of $V$ is called an out-dominating set of $D$ if for every vertex $v \in V \setminus S$ there exists at least one vertex $u \in S \cap N^+(v)$. The minimum cardinality of an out-dominating set of $D$ is the out-dominating number of $D$ and is denoted by $\gamma^+(D)$, or simply $\gamma^+$. In-dominating sets in digraphs are defined in a similar way, and the minimum cardinality of an in-dominating set of $D$ is called the in-dominating number of $D$, denoted by $\gamma^-(D)$.

Let $G = (V, E)$ be a graph. A majority dominating function [1] is a function $f : V \to \{-1, 1\}$ such that the set $S = \{v \in V : \sum_{u \in N[v]} f(u) \geq 1\}$ satisfies $|S| \geq \frac{n}{2}$; the weight of a majority dominating function is $w(f) = \sum_{v \in V} f(v)$, and $\min\{w(f) : f$ is a majority dominating function in $G\}$ is the majority domination number of $G$, denoted $\gamma_{maj}(G)$. A majority dominating set [3] is a set $M \subseteq V$ such that $|N[M]| \geq \frac{n}{4}$, and $\min\{|M| : M$ is a majority dominating set of $G\}$ is the set majority domination number of $G$, denoted $\gamma_m(G)$; a majority dominating set $M$ of $G$ such that $|M| = \gamma_m(G)$ is a $\gamma_m(G)$-set. It is straightforward that for every
graph $G$ and every majority dominating function $f$ of $G$, $\gamma_m(G) \leq |f^{-1}(1)|$, since $f^{-1}(1)$ is a majority dominating set of $G$.

Both concepts can be naturally extended to digraphs: Given a digraph $D$, a majority out-dominating function of $D$ is a function $f : V \rightarrow \{-1, 1\}$ such that the set $S = \{v \in V : \sum_{u \in N^+(v)} f(u) \geq 1\}$ satisfies $|S| \geq \frac{m}{n}$; the weight of $f$ is $w(f) = \sum_{v \in V} f(v)$, and $\min\{w(f) : f \text{ is a majority out-dominating function in } D\}$ is the majority out-dominating number of $D$, denoted $\gamma_{maj}^+(D)$. Similarly, a majority out-dominating set (MODS) of $D$ is a set $M \subseteq V$ such that $|N^+[M]| \geq \frac{m}{n}$, and $\min\{|M| : M \text{ is a MODS of } D\}$ is the set majority out-domination number of $D$, denoted $\gamma_m^+(D)$; a MODS $M$ of $D$ such that $|M| = \gamma_m^+(D)$ is a $\gamma_m^+(D)$-set.

However, it does not hold that for every digraph $D$ and every majority out-dominating function $f$ of $D$, $\gamma_m^+(D) \leq |f^{-1}(1)|$, since $f^{-1}(1)$ is a majority in-dominating set of $D$ (defined analogously), but not necessarily a MODS of $D$. For example, consider the digraph $D = (V, A)$ shown in Figure 1, where $V = \{u, v\} \cup S \cup T$, $|S| = k \geq 3$, $|T| = k + 2$, $d^-(x) = 0$ for every $x \in S \cup T$, $d^+(u) = 0$, $N^+(u) = S$, and $N^-(v) = S \cup T$. Then the function $g : V \rightarrow \{-1, 1\}$ such that $g(u) = g(v) = 1$ and $g(x) = -1$ for every $x \in S \cup T$ is a majority out-dominating function of $D$, but $\gamma_m^+(D) = k$.

![Figure 1](image)

In this article we focus on majority dominating sets, while we study majority out-dominating functions in another paper [6].

3 Majority out-dominating sets

**Observation 3.1.** If $H$ is a spanning subdigraph of a digraph $D$, then $\gamma_m^+(D) \leq \gamma_m^+(H)$.

*Proof.* The result follows immediately because any MODS of $H$ is also a MODS of $D$. \qed

**Observation 3.2.** For the directed path $P_n$, $\gamma_m^+(P_n) = \lceil \frac{n}{2} \rceil$, and for the directed cycle $C_n$ ($n \geq 3$), $\gamma_m^+(C_n) = \lceil \frac{n}{2} \rceil$.

**Observation 3.3.** For the directed path $P_n$, $\gamma_m^+(P_n) = \lceil \frac{n}{2} \rceil$, and for the directed cycle $C_n$ ($n \geq 3$), $\gamma_m^+(C_n) = \lceil \frac{n}{2} \rceil$.

**Observation 3.4.** For any digraph $D$ which has a hamiltonian circuit, $\gamma_m^+(D) \leq \lceil \frac{n}{2} \rceil$.

*Proof.* The result follows from Observation 3.3. \qed
Proposition 3.5. Let $l(D)$ denote the length of a longest directed path in $D$. Then $\gamma^+_m(D) \leq \left\lfloor \frac{2n-c(D)}{4} \right\rfloor + 1$, and the bound is sharp.

Proof. Let $P$ be a longest directed path in $D$. Let $S_1$ be a minimum MODS of $P$, and let $S_2 \subseteq V(D) \setminus V(P)$ such that $|S_2| = \left\lfloor \frac{|E(D)|}{V(P)} \right\rfloor$. Clearly $S = S_1 \cup S_2$ is a MODS of $D$ and hence $\gamma^+_m(D) \leq |S_1| + |S_2| = \frac{l(D)+1}{4} + \left\lfloor \frac{n-l(D)-1}{4} \right\rfloor = \left\lfloor \frac{2n-c(D)}{4} \right\rfloor + 1$.

The proof is similar to that of the previous proposition. The bound is trivially attained for directed paths.

Proposition 3.6. Let $c(D)$ denote the length of a longest directed cycle in $D$. Then $\gamma^+_m(D) \leq \frac{2n-c(D)}{4}$, and the bound is sharp.

Proof. The proof is similar to that of the previous proposition. The bound is trivially attained for directed cycles.

Theorem 3.7. For any digraph $D$, $\gamma^+_m(D) = \gamma^+(D)$ if, and only if, $\Delta^+(D) = n-1$.

Proof. Let $D$ be a digraph with $\Delta^+(D) \leq n-2$; then $\gamma^+(D) \geq 2$. Let $S$ be a $\gamma^+$-set of $D$, and let $S = S_1 \cup S_2$ with $|S_1| \geq 1$, $|S_2| \geq 1$, and $S_1 \cap S_2 = \emptyset$. Then $|N^+[S]| = n$, either $|N^+[S_1]| \geq \frac{n}{2}$ or $|N^+[S_2]| \geq \frac{n}{2}$. If follows that at least one of $S_1$ and $S_2$ is a MODS of $D$. Therefore, $\gamma^+_m(D) < |S| = \gamma^+(D)$. The converse is obvious.

Proposition 3.8. For any digraph $D$, $\gamma^+_m(D) \leq \left\lfloor \frac{n}{2(\Delta^+(D)+1)} \right\rfloor$. The bound is sharp.

Proof. (i) $\left\lfloor \frac{n}{2(\Delta^+(D)+1)} \right\rfloor \leq \gamma^+_m(D)$. The bound is sharp.

(ii) Either $\gamma^+_m(D) = 1$ or $\gamma^+_m(D) \leq \left\lfloor \frac{n}{2} \right\rfloor - \Delta^+(D)$. In the second case the bound is sharp.

Proof. (i). Let $S = \{v_1, \ldots, v_{\gamma^+_m(D)}\}$ be a $\gamma^+_m$-set of $D$. Then $\left\lfloor \frac{n}{2} \right\rfloor \leq |N^+[S]| \leq \sum_{v \in S} d^+(v) + \gamma^+_m(D) \leq \sum_{v \in S} \Delta^+(D) + \gamma^+_m(D) = \gamma^+_m(D)(\Delta^+(D) + 1)$. Equality is attained by double stars oriented so that the stem vertices have in-degree zero.

(ii). Suppose $\gamma^+_m(D) > 1$. Then $\Delta^+(D) < \left\lfloor \frac{n}{2} \right\rfloor - 1$. Let $v$ be a vertex with $d^+(v) = \Delta^+(D)$, and let $S \subseteq V(D) \setminus N^+(v)$ such that $v \in S$ and $|S| = \left\lfloor \frac{n}{2} \right\rfloor - \Delta^+(D)$. Since $v \in S$ and $S \cap N^+(v) = \emptyset$, it follows that $S$ is a MODS of $D$, so the result follows. Equality holds for the directed path $P_n$, among others.

Corollary 3.11. For every digraph $D$, $\gamma^+_m(D) \leq \frac{n-\Delta^+(D)+1}{2}$. The bound is sharp.

Proof. If $\Delta^+(D) = n-1$, then $\gamma^+_m(D) = 1 = \frac{n-\Delta^+(D)+1}{2}$. Otherwise, Theorem 3.10 (ii) implies that $\gamma^+_m(D) \leq \frac{n-\Delta^+(D)+1}{2} \leq \frac{n-\Delta^+(D)+1}{2}$. As already mentioned, equality holds for any digraph $D$ such that $\Delta^+(D) = n-1$. 

\[ \square \]
Proposition 3.12. Let $D$ be a digraph which is not a totally disconnected digraph of odd order. If $S$ is a minimal MODS of $D$, then $V \setminus S$ is a MODS of $D$.

Proof. Suppose $D$ is a totally disconnected (di)graph of even order. Then any minimal MODS of $D$ contains $\frac{n}{2}$ vertices and hence its complement is also a MODS of $D$. Suppose $D$ is not a totally disconnected digraph, and let $S$ be a minimal MODS of $D$. Therefore, $|S| \leq \frac{n}{2}$ and $|V \setminus S| \geq \lceil \frac{n}{2} \rceil$, which implies that $V \setminus S$ is a MODS of $D$.

Theorem 3.13. Let $S$ be a MODS of a digraph $D = (V, A)$. Then $S$ is minimal if, and only if, one of the following conditions hold:

(i) $|N^+[S]| > \left\lceil \frac{n}{2} \right\rceil$ and $\forall v \in S$, $|pm^+[v, S]| > |N^+[S]| - \left\lceil \frac{n}{2} \right\rceil$.

(ii) $|N^+[S]| = \left\lceil \frac{n}{2} \right\rceil$ and $\forall v \in S$, either $v$ is an isolate in $D[S]$ or $pm^+(v, S) \neq \emptyset$.

Proof. Let $D = (V, A)$ be a digraph. Let $S$ be a minimal MODS of $D$ and take $v \in S$. Assume $|N^+[S]| > \left\lceil \frac{n}{2} \right\rceil$. Since $S \setminus \{v\}$ is not majority out-dominating, $|N^+[S \setminus \{v\}]| = |N^+[S]| - |pm^+[v, S]| < \left\lceil \frac{n}{2} \right\rceil$. Hence $|pm^+[v, S]| > |N^+[S]| - \left\lceil \frac{n}{2} \right\rceil$. Now assume $|N^+[S]| = \left\lceil \frac{n}{2} \right\rceil$. Since $S \setminus \{v\}$ is not majority out-dominating, $|N^+[S \setminus \{v\}]| < \left\lceil \frac{n}{2} \right\rceil$. If $v$ is an isolate in $D[S]$, $|N^+[S \setminus \{v\}]| \leq \left\lceil \frac{n}{2} \right\rceil - 1 < \left\lceil \frac{n}{2} \right\rceil$.

We now consider the effect on $\gamma^+_m(D)$ of the removal of a vertex or an arc from $D$.

Theorem 3.14. Let $D$ be any digraph with $\gamma^+_m(D) = k$. Let $v \in V(D)$ and $e \in A(D)$. Then

(i) $k \leq \gamma^+_m(D - e) \leq k + 1$,

(ii) $k - 1 \leq \gamma^+_m(D - v) \leq \max\{k, k - 1 + d^+(v)\}$.

Proof. (i) If $S$ is a $\gamma^+_m$-set of $D - e$, then Observation 3.1 implies that $S$ is a MODS of $D$. Hence $k \leq \gamma^+_m(D - e)$.

Now, let $S$ be a $\gamma^+_m$-set of $D$ and let $e = uv \in A$. If $\{u, v\} \subseteq S, \{u, v\} \subseteq V \setminus S$, or $u \in V \setminus S$ and $v \in S$, then $S$ is a MODS of $D - e$. If $u \in S$ and $v \in V \setminus S$, then $S' = S \cup \{v\}$ is a MODS of $D - e$. Thus in all cases $\gamma^+_m(D - e) \leq k + 1$.

(ii) Let $S$ be a $\gamma^+_m$-set of $D - v$. Then $S' = S \cup \{v\}$ is a MODS of $D$, so $\gamma^+_m(D) \leq \gamma^+_m(D - v) + 1$. Thus $k - 1 \leq \gamma^+_m(D - v)$.

Now, let $S$ be a $\gamma^+_m$-set of $D$. If $v \in V \setminus S$, then $S$ is a MODS of $D - v$. If $v \in S$, then $S \cup N^+(v)$ is a MODS of $D - v$. Thus, $\gamma^+_m(D - v) \leq \max\{k, k - 1 + d^+(v)\}$.

Now we consider the effect on $\gamma^+_m(D)$ of adding an arc to $D$.

Proposition 3.15. Let $D$ be any digraph with $\gamma^+_m(D) = k, e \in A(D)$. Then

$\gamma^+_m(D + e) \leq \gamma^+_m(D)$.  

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Proof. Let $e = uv \in A(D)$. The fact that $\gamma_m^+(D + e) \leq \gamma_m^+(D)$ follows from Theorem 3.14. On the other hand, let $S$ be a $\gamma_m^+$-set of $D + uv$. Then $S \cup \{v\}$ is a $\gamma_m^+$-set of $D$, so $\gamma_m^+(D) - 1 \leq \gamma_m^+(D + e)$. □

Proposition 3.16. Let $D$ be a digraph, and let $D'$ be the digraph obtained by reversing the direction of a single arc of $D$. Then $|\gamma_m^+(D) - \gamma_m^+(D')| \leq 1$.

Proof. The proof is identical to the proof of the corresponding result for out-domination given in [3]. □

Definition 3.17. Let $D = (V, A)$ be any digraph. An arc $e \in A(D)$ is $\gamma_m^+$-critical if $\gamma_m^+(D - e) = \gamma_m^+(D) + 1$.

Theorem 3.18. An arc $e = uv$ of a digraph $D$ is $\gamma_m^+$-critical if, and only if, for every $\gamma_m^+(D)$-set $S$ we have that $u \in S$, $v \in pm^+(u, S)$, and $|N^+[S]| = \lceil \frac{m}{2} \rceil$

Proof. Let $e = uv$ be a $\gamma_m^+$-critical arc, and let $S$ be a $\gamma_m^+(D)$-set. If $u \notin S$, then $S$ is a $\gamma_m^+(D - e)$-set, which is a contradiction, so $u \in S$. If $v \notin pm^+(u, S)$, we have that $S$ is a $\gamma_m^+(D - e)$-set, again a contradiction. Moreover, if $|N^+[S]| > \lceil \frac{m}{2} \rceil$, then $S$ is as well a $\gamma_m^+(D - e)$-set. Therefore, for every $\gamma_m^+(D)$-set $S$ we have that $u \in S$, $v \in pm^+(u, S)$, and $|N^+[S]| = \lceil \frac{m}{2} \rceil$.

Conversely, suppose that for every $\gamma_m^+(D)$-set $S$ we have that $u \in S$, $v \in pm^+(u, S)$, and $|N^+[S]| = \lceil \frac{m}{2} \rceil$. It follows that for every $\gamma_m^+(D)$-set $S$, $|N^+_D[S]| = |N^+_D[S] - 1\lceil \frac{m}{2} \rceil - 1\lceil \frac{m}{2} \rceil - 1$, so no $\gamma_m^+(D)$-set is a MODS of $D - e$. Now suppose there is a $\gamma_m^+(D - e)$-set $S'$ with $|S'| = \gamma_m^+(D)$, then from Observation 3.14 it follows that $S'$ is a MODS of $D$, which is a contradiction. Therefore, $e = uv$ is a $\gamma_m^+$-critical arc. □

4 Oriented graphs

Let $G = (V, E)$ be a graph. An orientation of $G$ is a digraph $D = (V, A)$ such that $uv \in E \Leftrightarrow (uv \in A$ or $vu \in A)$, and $|E| = |A|$.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{CycleC5.png}
\caption{Two orientations of the cycle $C_5$}
\end{figure}

Two distinct orientations of a given graph can have different majority domination numbers, as shown in Figure 2 for the orientations $D_1$ and $D_2$ of the cycle $C_5$, where $\gamma_m^+(D_1) = 2$ and $\gamma_m^+(D_2) = 1$. This suggests the following definitions:

Definition 4.1. Let $G$ be a graph. The lower orientable set majority domination number of $G$ is $\text{dom}_m^+(G) = \min\{\gamma_m^+(D) : D$ is an orientation of $G\}$, and the upper orientable set majority domination number of $G$ is $\text{DOM}_m^+(G) = \max\{\gamma_m^+(D) : D$ is an orientation of $G\}$.

These concepts are inspired in the notions of lower orientable domination number $\text{dom}(G)$ and upper orientable domination number $\text{DOM}(G)$, introduced by Chartrand et al. in [3].

Theorem 4.2. For every graph $G$, $\text{dom}_m^+(G) = \gamma_m(G)$.
Proof. Let $G = (V, E)$ be a graph and let $S$ be a minimum majority dominating set of $G$. Now we consider the following orientation $D = (V, A)$ of $G$: For every two adjacent vertices $u \in S$ and $v \in V \setminus S$, let $uv \in A$; edges between vertices of $S$ and edges between vertices of $V \setminus S$ can be oriented arbitrarily. Then $S$ is a MODS of $D$ and so $\text{dom}_{m}^{+}(G) \leq \gamma_{m}^{+}(D) \leq |S| = \gamma_{m}(G)$.

Now, let $D'$ be an orientation of $G$ for which $\text{dom}_{m}^{+}(G) = \gamma_{m}^{+}(D')$, and let $S'$ be a $\gamma_{m}^{+}(D')$-set. It is clear that $S'$ is a majority dominating set of $G$. Therefore, $\gamma_{m}(G) \leq |S'| = \gamma_{m}^{+}(D') = \text{dom}_{m}^{+}(G)$.

We now proceed to determine the upper and lower orientable set majority domination numbers for several classes of graphs:

Proposition 4.3.

(i) For $n \geq 1$, we have $\text{DOM}^{-}_{m}(K_n) = \text{dom}_{m}^{+}(K_n) = 1$.

(ii) For $n \geq 1$, $\text{dom}_{m}^{+}(P_n) = \lceil \frac{n}{2} \rceil$.

(iii) For $n \geq 3$, $\text{dom}_{m}^{+}(C_n) = \lceil \frac{n}{2} \rceil$.

(iv) For any two integers $r, s$ with $r \leq s$, $\text{dom}_{m}^{+}(K_{r,s}) = 1$.

Proof. (i) Since for any tournament $T$, $\sum_{v \in V(T)} d^{+}(v) = \lceil \frac{n(n-1)}{2} \rceil$, it follows that there exists a vertex $u$ in $T$ with $d^{+}(u) \geq \lceil \frac{2n}{3} \rceil$. This completes the proof.

(ii) From (i) of Theorem 3.10 it follows that $\text{dom}_{m}^{+}(P_n) \geq \lceil \frac{n}{2} \rceil$. Now, consider the following orientation $D = (V, A)$ of $P_n$: We number the vertices of $V(P_n)$ in order, that is, $V(P_n) = \{v_1, ..., v_n\}$, with $N(v_i) = \{v_{i-1}, v_{i+1}\}$ for $i \in \{2, ..., n-1\}$, $N(v_1) = \{v_2\}$, and $N(v_n) = \{v_{n-1}\}$; we orient the edges of $P_n$ in such a way that for a vertex $v_i \in V$, $d^{+}(v_i) = 2$ if, and only if, $i \equiv 2 \pmod{3}$. Then $S = \{v_i : i \equiv 2 \pmod{3}, i \leq \lceil \frac{n}{2} \rceil \}$ is a MODS of $D$, and $|S| = \lceil \frac{n}{2} \rceil$. Therefore, $\text{dom}_{m}^{+}(P_n) = \lceil \frac{n}{2} \rceil$.

(iii) The proof is similar to that of (ii).

(iv) Let $K_{r,s} = (V, E)$, and let $V_1 = \{v_1, v_2, ..., v_r\}$ and $V_2 = \{u_1, u_2, ..., u_s\}$ be the bipartition of $V$. Let $D$ be an orientation of $G$ such that $d^{-}(v_1) = 0$. Then $\{v_1\}$ is a MODS of $D$, so $\text{dom}_{m}^{+}(G) = 1$.

Theorem 4.4. [3] For every integer $n \geq 3$, $\text{DOM}(P_n) = \text{DOM}(C_n) = \lceil \frac{n}{2} \rceil$.

Proposition 4.5. For every integer $n \geq 3$, $\text{DOM}^{+}_{m}(P_n) = \text{DOM}^{+}_{m}(C_n) = \lceil \frac{n}{2} \rceil$.

Proof. The result follows from Observation 3.3, Theorem 3.8 and Theorem 4.4.

Proposition 4.6. For $n \geq 3$, $\text{DOM}^{+}_{m}(K_{1,n-1}) = \lceil \frac{n}{2} \rceil$.

Proof. First notice that $\lceil \frac{n}{2} \rceil = \lceil \frac{n+1}{2} \rceil$. Let $v$ be the central vertex of the star $K_{1,n-1} = (V, E)$, and take any orientation $D$ of $K_{1,n-1}$. Let $X \subseteq V \setminus N^{+}_{D}[v]$, such that $|X| = \max\{0, \lceil \frac{n-2|N^{+}_{D}[v]|}{2} \rceil \}$. If $d^{+}_{D}(v) > 0$, then the set $S = \{v\} \cup X$ is a MODS of $D$, and $|S| \leq \lceil \frac{n}{2} \rceil$. If $d^{+}_{D}(v) = 0$, then $X$ is a $\gamma_{m}^{+}(D)$-set of cardinality $\lceil \frac{n-1}{2} \rceil$.

Theorem 4.7. For every double star $G$, $\text{dom}^{+}_{m}(G) = 1$. Moreover, if $n \geq 5$ then $\text{DOM}^{+}_{m}(G) = 2 + \max\{0, \lceil \frac{n-2}{2} \rceil \}$.

Proof. Let $G = (V, E)$ be a double star, and let $u$ and $v$ be the stem vertices of $G$, with $d(u) = d(v)$. If we take an orientation $D'$ of $G$ such that $d^{+}_{D'}(v) = 0$, then $\{v\}$ is a MODS of $D'$, so $\text{dom}^{+}_{m}(G) = 1$.

For $\text{DOM}^{+}_{m}(G)$, since there exist vertices $x \in N(u) \setminus \{v\}$ and $y \in N(v) \setminus \{u\}$, it follows that for any orientation $D$ of $G$, either $u \in N^{+}_{D}(x)$ or $x \in N^{+}_{D}(u)$, and
either $v \in N^+_D(y)$ or $y \in N^+_D(v)$. Therefore, there is always a set $S$ with $|S| = 2$ and $|N^+| \geq 4$. This means that for $m \leq 8$, $\text{DOM}^+_m(G) \leq 2$. Moreover, if $m > 8$ then $S \subseteq X$ is a MODS of $D$, where $X \subseteq V \setminus N^+|S|$ and $|X| = \max\{0, [\frac{m^2}{2m+1}]\}$. Then we have that for $m \geq 5$, $\text{DOM}^+_m(G) \leq 2 + \max\{0, [\frac{m^2}{2m+1}]\}$.

On the other hand, if $D'$ is an orientation of $G$ such that $d^+(u) = 1$ and $d^+(v) = 0$, then $\gamma^+_m(D') = 2 + \max\{0, [\frac{m^2}{2m+1}]\}$.

**Observation 4.8.** For every graph $G = (V, E)$ with $n \leq 4$ and such that $E \neq \emptyset$, $\text{DOM}^+_m(G) = 1$.

**Proposition 4.9.** Take two positive integers $r, s$ with $r \leq s$, then $\text{DOM}^+_m(K_{r,s}) = 1$ if, and only if, $r + s \leq 4$ or

- (i) $r = 2$, $s = 3$
- (ii) $r = 2$, $s = 4$
- (iii) $r = s = 3$

**Proof.** Consider a complete bipartite graph $K_{r,s}$. If $r + s \leq 4$, from Observation 4.8 it follows that $\text{DOM}^+_m(K_{r,s}) = 1$. Likewise, it is easy to check that in any orientation $D$ of $K_{2,3}$, $K_{2,4}$, and $K_{3,3}$, there is a vertex $v$ such that $d^+(v) \geq 2$, so $\{v\}$ is a MODS of $D$.

Now take $K_{r,s}$ which is not one of the cases mentioned above, and let $R$ and $S$ be the defining partite sets of $V$, with $|R| = r$ and $|S| = s$. If $s > r + 2$, the orientation $D$ of $K_{r,s}$ such that $d^+(v) = 0$ for every $v \in R$ satisfies $\gamma^+_m(D) > 1$. Now suppose $r \leq s \leq r + 2$. This implies $r > 2$, since otherwise the graph would be one of those mentioned earlier. Moreover, since $K_{3,3}$ is as well one of the cases already considered, we have that $r + s \geq 7$. Let $R = \{u_1, \ldots, u_r\}$ and $S = \{v_1, \ldots, v_s\}$, and consider the orientation $D = (V, A)$ of $K_{r,s}$ such that the only arcs whose tail is in $u_i$ are $\{u_i, v_j : j = 2i, j \neq 2i - 1\}$, where the product is taken modulo $s$. Then for every $u \in R$, $d^+(u) = 2$. If $r \neq s$, for every $v \in S$, $d^+(v) \leq r - 1$; since $r + s \geq 2r + 1$, this implies $\gamma^+_m(D) > 1$. If $r = s$, then for every $v \in S$, $d^+(v) = r - 2$; since $r + s \geq 2r$, we have as well that $\gamma^+_m(D) > 1$. Therefore, $\text{DOM}^+_m(K_{r,s}) > 1$ except for the cases mentioned above.

In general, it seems difficult to find $\text{DOM}^+_m(K_{r,s})$. However, we have the following conjecture:

**Conjecture 4.10.** Let $K_{r,s}$ be a complete bipartite graph with $r \leq s$, and such that $\text{DOM}^+_m(K_{r,s}) \neq 1$. Then:

$$\text{DOM}^+_m(K_{r,s}) = \begin{cases} 2 & \text{if } s \leq r + 2, \\ [\frac{m^2}{2m+1}] & \text{otherwise.} \end{cases}$$

**Theorem 4.11.** For $n \geq 4$, $\text{dom}^+_m(W_n) = 1$ and $\text{DOM}^+_m(W_n) = [\frac{n^2}{2n+1}]$.

**Proof.** Let $W_n = v + C_{n-1}$, and let $D$ be an orientation of $W_n$ such that $d^-(v) = 0$. Then $\{v\}$ is an out-dominating set of $D$, which in particular is a MODS of $D$, so $\text{dom}^+_m(W_n) = 1$.

On the other hand, consider $W_n = v + C_{n-1}$ and let $D = (V, A)$ be any orientation of $W_n$. Observe that for any two consecutive vertices $x, y$, $C_{n-1}$, one of the arcs $xy$ and $yx$ is in $A$. If $d^+_D(v) < [\frac{n^2}{2n+1}]$, then $\{v\}$ is a MODS of $D$, as mentioned earlier. Suppose $d^+_D(v) \geq [\frac{n^2}{2n+1}]$, and number the vertices of $C_{n-1}$ following the order of the cycle, that is, $V(C_{n-1}) = \{u_1, \ldots, u_{n-1}\}$, in such a way that $u_1 \in A$ and $u_1 u_2 \in A$ (notice that such a vertex $u_1$ will always exist, since $d^+_D(v) \geq [\frac{n^2}{2n+1}]$). Now from the set $S_1 = \{u_6, u_7, u_8, u_9\}$ take one vertex which out-dominates other vertex of $S_1$, and call it $z_1$; from the set $S_2 = \{u_{10}, u_{11}, u_{12}, u_{13}\}$ take one vertex which out-dominates other vertex of $S_2$, and call it $z_2$, and so on. The last set
$S_k = \{\ldots, u_{n-1}\}$ may have less than four vertices, but we take anyway a vertex $z_k$ which out-dominates another vertex of the set, unless $S_k = \{u_{n-1}\}$, in which case we take $z_k = u_{n-1}$. Then $S = \{u_1\} \cup \{z_1, \ldots, z_k\}$ is a MODS of $D$ of cardinality $\lceil \frac{n}{4} \rceil$. If $D$ is the orientation of $W_n$ such that $d^+(v) = 0$ and $d^-(x) = 1$ for every $x \in V \setminus \{v\}$ (i.e., $D - v$ is a directed cycle on $n - 1$ vertices), then $S$ is a $\gamma_m^+(D)$-set.

Finally, we note that an "Intermediate Value Theorem" for orientable majority out-domination holds:

**Theorem 4.12.** For every graph $G$ and every integer $c$ with $\text{dom}^+_m(G) \leq c \leq \text{DOM}^+_m(G)$, there exists an orientation $D$ of $G$ such that $\gamma_m^+(D) = c$.

**Proof.** The proof is identical to the proof of the corresponding result for out-domination given in [3].

5 Conclusions and scope

In this paper we extended the notion of majority dominating set to digraphs. In addition to its applications, the topic is of mathematical interest since the behavior of MODSs is somewhat different to that of their counterparts in graphs. This is only an introductory work, in which the concept is defined and some basic results are proven. We hope this paper will be helpful for people working in related topics, and perhaps it will encourage further research in the field.

It would be interesting to prove the NP-completeness of the decision problem Majority Dominating Set (MODS): Given a graph $G$ and a positive integer $k$, does $G$ have a MODS of cardinality $k$ or less?

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