Macrosopic Universality :
Why QCD in Matter is Subtle ?

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We use a chiral random matrix model with \(2N_f\) flavors to mock up the QCD Dirac spectrum at finite chemical potential. We show that the \(1/N\) approximation breaks down in the quenched state with spontaneously broken chiral symmetry. The breakdown condition is set by the divergence of a two-point function, that is shown to follow the general lore of macroscopic universality. In this state, the fermionic fluctuations are not suppressed in the large \(N\) limit.

1. In the presence of a chemical potential, the lattice QCD Dirac operator is non-hermitean. As a result, the fermionic determinant (probability measure) of the QCD Dirac operator is a complex number. Since Monte-Carlo simulations demand a positive definite measure, straightforward algorithms have failed \([\text{1}]\). Unconventional QCD algorithms have been devised, leading to a chiral phase transition at finite chemical potential and strong coupling \((g = \infty)\) \([\text{6,7}]\). In the quenched approximation, however, the restoration of chiral symmetry was found to set in at surprisingly small chemical potentials of the order of half the pion mass, thereby vanishing in the chiral limit \([\text{8}]\). The origin of the discrepancy was traced back to the phase of the Dirac operator \([\text{8}]\), as demonstrated by in \([\text{8}]\) using lattice simulations.

While many of the basic issues related to the quenched calculations are known on the lattice \([\text{8,9}]\), we feel the need for a better understanding using simple models. The spontaneous breaking of chiral symmetry in QCD reflects the spectral distribution of quark eigenvalues near zero-virtuality \([\text{10}]\). For sufficiently random gauge configurations, the Dirac operator can be approximated by a chiral matrix with random entries \([\text{11}]\), in qualitative agreement with a number of lattice simulations for both the bulk eigenvalue distributions \([\text{12}]\), and correlations \([\text{13,14}]\).

Chiral random matrix models with finite chemical potential have been discussed recently in \([\text{15,16}]\) using various methods. For zero chemical potential, the ground state breaks spontaneously chiral symmetry. The low-lying spectrum is that of constituent quarks with no Fermi surface. Naively one would expect that for a chemical potential of the order of the constituent quark mass, chiral symmetry would be restored. This expectation is not borne out by direct numerical calculations which show that chiral symmetry is restored for any nonzero chemical potential in the chiral limit and for large matrix sizes \([\text{15}]\), in qualitative agreement with the quenched lattice simulations. The numerical results have been shown to follow from a modified replica trick \([\text{16,17}]\), in agreement with \([\text{16}]\).

In this letter, we reanalyze the spectral density with \(2N_f\) conventional quarks. In section 2, we introduce the model and discuss the \(1/N\) expansion around the quenched state with spontaneously broken chiral symmetry. In section 3, we evaluate a pertinent two-point correlation in the same state, and show that it diverges in the eigenvalue plane in the small eigenvalue region. The support for the ensuing spectral distribution is in agreement with a recent result \([\text{18}]\) using different arguments. We explain why. Our result for the two-point correlator, extends the macroscopic universality argument in random matrix models \([\text{19}]\) to the chiral case. In section 4 we discuss the difference between the unquenched and quenched spectral densities. We argue that the quenched state with spontaneous symmetry breaking does not suppress the fermionic fluctuations. Our conclusions are summarized in section 5.

2. A model form of the QCD Hamiltonian in the chiral basis with finite chemical potential \(\mu\) and zero current quark mass is

\[
Q_5^{QCD} \equiv M = \begin{pmatrix} 0 & \mu \\ -\mu & 0 \end{pmatrix} + \begin{pmatrix} 0 & -iA \\ iA^\dagger & 0 \end{pmatrix}
\]

(1)

where only constant quark modes have been retained \([\text{11}]\). In the context of four dimensional field-theories \(Q_5\) is the Hamiltonian of a Dirac particle in five dimensions. The translation of this operator to chiral random matrix models is achieved by assuming that the gluonic configurations \(A\) are sufficiently random so that \(A \rightarrow R\), where \(R\) is an \(N \times N\) complex random matrix with Gaussian weight. With this in mind, \([\text{20}]\) is the sum of a deterministic \(D\) and random (chiral) piece \(R\). The deterministic piece is non-hermitian for \(\mu \neq 0\), with complex \(\pm i|\mu|\) eigenvalues.

Now, consider the partition function associated with \([\text{21}]\) for \(2N_f\) flavors.
\[ Z(2N_f, \mu) = \langle \det^{2N_f}(z - M) \rangle \]  
with \( V_{2N_f} = -\log Z/2N_f \) playing the role of a complex potential. For \( z = \text{im} \), \( \mu \) mocks up the partition function of \( 2N_f \) quarks of equal mass \( m \). The averaging in \( \langle \rangle \) is carried using a gaussian weight

\[ \langle \ldots \rangle = \frac{1}{Z} \int \ldots e^{-\Sigma \text{tr} RR^\dagger} dR \]  
with \( \Sigma \) setting the scale of chiral symmetry breaking (throughout \( \Sigma \) is set to 1 for simplicity, unless indicated otherwise). The gaussian weight \( \langle \rangle \) is an idealization of the gluon action, and defines the quenched averagings to be discussed below. For sufficiently small quark eigenvalues (here constant modes), the gluon interaction may be thought as random \( \langle \rangle \). The gaussian weight follows from the principle of maximum entropy.

First, let us consider the spectral distribution associated to \( \langle \rangle \) in the quenched approximation. With \( V_{2N_f} \) playing the role of a complex potential, the spectral distribution follows from Gauss law \( \langle \rangle \)

\[ \nu(z, \bar{z}) = \lim_{N_f \to 0} \nu(z, \bar{z}) = \lim_{N_f \to 0} -\frac{1}{\pi N} \partial_z \partial_{\bar{z}} V_{2N_f} \]  
Explicitly, \( \pi \nu(z, \bar{z}) = \partial G/\partial \bar{z} \), with the resolvent

\[ G(z, \bar{z}) = \frac{1}{N} \langle \text{Tr}(z - M)^{-1} \rangle \]  
We have retained \((z, \bar{z})\) in both \( \langle \rangle \) and \( \langle \rangle \) to allow for the possibility that holomorphic symmetry may be broken in the thermodynamical limit by the quenched averaging \( \langle \rangle \). The factor of \( 1/N \) in \( \langle \rangle \) implies that \( \nu(z, \bar{z}) \) is normalized to one on its pertinent support.

For large \( z \) and to leading order in \( 1/N \), \( G \) obeys Pastur’s equation (planar approximation) \( \langle \rangle \)

\[ G((z - G)^2 + \mu^2) - z + G = 0 \]  
The solution to \( \langle \rangle \) is a holomorphic function of \( z \), except for line discontinuities, with end-points given by the zeroes of the discriminant

\[ 4\mu^2 z^4 + z^2(8\mu^4 - 20\mu^2 - 1) + 4(\mu^2 + 1)^3 = 0 \]  
For \( \mu = 0 \), there is a cut along the real axis between \( -2 \) and \( +2 \). The discontinuity along the cut is Wigner’s semicircle, with a quenched and stable state that breaks spontaneously chiral symmetry. With increasing \( \mu^2 \to 1/8 \), the two roots approach each other, followed by the emergence of two new roots from real infinity. For \( \mu^2 = 1/8 \) they coalesce pairwise, and move to the complex plane for \( \mu^2 > 1/8 \). This behavior suggests that the quenched state with \( \nu(+i0, -i0) \neq 0 \), supports a chiral condensate up to \( \mu^2 = 0.125 \), after which a first order transition is observed. This conclusion, however, does not hold since the \( 1/N \) approximation breaks down for small eigenvalues in the present model, as we now show.

3. The breakdown can be probed either by looking at the nonplanar corrections to \( \langle \rangle \), or by testing the stability content of the quenched state against local correlations. One pertinent correlation function is given by the two-point function

\[ N^2C(z, \bar{z}) = \langle \text{Tr}(z - M)^{-1} \text{Tr}(\bar{z} - M)^{-1} \rangle \]  
in the quenched state. The lower script in \( \langle \rangle \) is for connected. The physical interpretation of \( \langle \rangle \) will be clarified in the next section. For large values of \( z, \bar{z} \) may be analyzed using again a \( 1/N \) expansion \( \langle \rangle \). Following the arguments in \( \langle \rangle \), the result is \( \langle \rangle \)

\[ N^2C(z, \bar{z}) = -\frac{1}{4} \partial_z \partial_{\bar{z}} \log \{ (H - \mu^2)^2 - |z - G|^2 \}/H^2 \]  
where \( H = |z - G|^2/|G|^2 \) and \( G \) is a solution to Pastur’s equation \( \langle \rangle \). For \( \mu = 0 \) and \( z = \bar{z} \) outside the cut, \( \langle \rangle \) reduces to \( N^2C(z, \bar{z}) = 1/2(z^2 - \bar{z}^2)^2 \), which is different from the hermitean (nonchiral) result of \( \langle \rangle \), but in agreement with the one of \( \langle \rangle \). The result \( \langle \rangle \) shows that in our case the two-point correlation function is also amenable to the one-point function, thereby extending the macroscopic universality argument discussed originally in \( \langle \rangle \) to the chiral case.

From \( \langle \rangle \) we observe that the two-point correlation function diverges in the eigenvalue-plane in the domain prescribed by the zero of the logarithm,

\[ |z - G|^2(1 - |G|^2) - \mu^2 |G|^2 = 0 \]  
For \( \mu = 0 \), this condition is fulfilled for \( G = z \), which is a trivial (unphysical) solution to Pastur’s equation, and for \( |G|^2 = 1 \) a nontrivial (physical) solution to Pastur’s equation along the cut. The latter when supplemented with the condition \( \lim_{z \to \infty} G \sim 1/z \) in the outside, yields Wigner’s distribution for the eigenvalue density.

Figure \( \langle \rangle \) shows the envelope in dashed line for which the condition \( \langle \rangle \) is met in the \( w = iz \) plane, for several values of the chemical potential \( \mu \). A structural change occurs at \( \mu^2 = 1 \). The solid lines are the mean-field solution following from \( \langle \rangle \). They lie in the eigenvalue-domain where the \( 1/N \) fluctuations are dominant, a signal that chiral symmetry is restored. Since \( \langle \rangle \) mixes \( z \) and \( \bar{z} \), holomorphic symmetry is spontaneously broken in the quenched state. One can recast \( \langle \rangle \) using \( \langle \rangle \) into

\[ (\mu^2 - \bar{z}^2)^2 y^2 = [4\mu^4(1-\mu^2)(1 + 4\mu^2 - 8\mu^4)x^2 - 4\mu^2 x^4], \]

The result \( \langle \rangle \) is in agreement with the one in \( \langle \rangle \), where the quenched spectral density was evaluated in the inside (small eigenvalues) using a pair of conjugate quarks in the quenched approximation (essentially \( z - M \) in \( \langle \rangle \)). Conventional quenched QCD in the \( 1/N \) approximation works as well outside. The two approaches agree on \( \langle \rangle \) which sets their domain of validity. To be able to describe the spectral distribution inside using the quenched
version of (2) requires a different method than the $1/N$ approximation we have used.

![Graph](image)

**FIG. 1.** The envelope (dashed lines) is from eq. (10) in the plane $w = iz$ while the cuts (solid lines) are from eq. (7), for different $\mu$. Shaded islands represent “other vacuum”.

Figure 2 shows the analytical behavior (solid lines) for the imaginary part of the resolvent (4) and the correlation function (8) for two values of $w = iz$ and fixed $\mu^2 = 2$, that is $w = i4 - y$ (left) and $w = i0.02 - y$ (right) with $y \geq 0$. The dashed curves are a comparison to a numerical estimate using an ensemble of 200 chiral gaussian plus deterministic matrices with dimension $100 \times 100$. Note that in the right figures, two different solutions to (8) have to be used while crossing the region of non-analyticity (shaded region). In the “bounded region” between the islands of Figure 1 the asymptotic condition $\lim_{z_2 \to \infty} G \sim 1/z$ is no longer required. We remark that $\text{Im}G = 0$ for $z = 0$, and chiral symmetry is restored.

![Graph](image)

**FIG. 2.** Imaginary part of the resolvent (upper figures) and the correlation function (lower figures) for $w = ix - y$ with $x = 4, 0.02$ and $y \geq 0$ at $\mu^2 = 2$. The solid curves are analytical while the dashed curves are numerical (see text).

4. A dequenching of the spectral density (4) can be simply achieved by noting that since $(z - M)$ is non-hermitean, we can split it into a phase and a modulus through

$$ (z - M)^2 = |z - M|^2 \times \left( \frac{z - M}{z - M} \right) \quad (12) $$

In terms of (2) and (12), the unquenched spectral density for finite $N$ and $N_f$ is now given by $\rho(z, \bar{z})$. From (2) the contribution of the modulus is

$$ \rho_{\text{mod}}(z, \bar{z}) = \nu(z, \bar{z}) - \frac{N_f N}{2\pi} C(z, \bar{z}) + \frac{2N_f}{N} \text{Tr} \delta(z - M) \ln \det \left| z - M \right| > c + \ldots \quad (13) $$

while the contribution of the phase is

$$ \rho_{\text{ph}}(z, \bar{z}) = \frac{N_f N}{2\pi} C(z, \bar{z}) + \ldots \quad (14) $$

The density $\rho(z, \bar{z})$ is the sum of (13) and (14) modulo an extra contribution from the crossed terms, essentially the last term in (13) to the order quoted. We note that the connected two-point function (2) is essentially the fluctuation of the phase of the fermion determinant in the quenched state. It appears with opposite signs in the modulus and the phase, and cancels in the sum. The difference $\rho(z, \bar{z}) - \nu(z, \bar{z})$ accounts for the effects of the sea fermions (unquenching). This difference involves non-local correlation functions of $z$ and $\bar{z}$, of which (2) is a generic example. The fermionic effects are down by $1/N$, provided that the correlation functions are stable. This is not the case in the quenched state with spontaneous symmetry breaking.

Further insights to the above results can be achieved, if were to note that (8) may be rewritten as

$$ N^2 C(z, \bar{z}) = \langle q^\dagger q Q^\dagger Q \rangle_c \quad (15) $$

with

$$ qq^\dagger = -\frac{1}{z - M} \quad \text{and} \quad QQ^\dagger = -\frac{1}{\bar{z} - M^\dagger} \quad (16) $$

playing the role of “propagators”. For $\mu = 0$, $\langle q^\dagger q \rangle$ and $\langle Q^\dagger Q \rangle$ are nonzero for $z \sim i0$, and the vacuum contributions cancel out. Typically, $\rho(z, \bar{z}) - \nu(z, \bar{z}) \sim O(N_f/N)$ for $z \sim i0$. For $\mu = 0$ and large $N$, the fermionic fluctuations are suppressed in the quenched state. The latter breaks spontaneously chiral symmetry, and preserves holomorphic symmetry $(qq^\dagger Q)$. For small $\mu$, (8) through (11) diverges when closing on the dashed curve of Figure 1 from the outside. This is a signal that (13) is receiving increasingly large contributions from the “mixed condensates” $(Q^\dagger Q)$ and their conjugates (other “vacuum”), so the fluctuations are not suppressed any more. As a result $\rho(z, \bar{z}) - \nu(z, \bar{z}) \sim O(N_f N^0)$ for small $z$. In the quenched state with spontaneous symmetry breaking, the fermionic fluctuations are not suppressed in the large $N$ limit.

5. Using a chiral random matrix model with a finite chemical potential $\mu$, we have shown that the small and quenched eigenvalue distribution of the Dirac operator is fluctuation driven, with a size in the complex...
plane conditioned by the divergence of the connected two-point correlation function $\langle \hat{\alpha} \rangle$. In this domain the $1/N$ approximation breaks down and the effects from the fermionic fluctuations do not decouple in the thermodynamical limit. The fluctuation driven phase breaks spontaneously holomorphic symmetry and preserves chiral symmetry, as originally shown in [12].

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