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Abstract. We exhibit a non-hyperelliptic curve $C$ of genus 3 such that the class of the Ceresa cycle $[C] - [-C]$ in the intermediate Jacobian of $JC$ is torsion.

1. Introduction

Let $C$ be a complex curve of genus $g \geq 3$, and $p$ a point of $C$. We embed $C$ into its Jacobian $J$ by the Abel–Jacobi map $x \mapsto [x] - [p]$. The Ceresa cycle $\tilde{z}_p(C)$ is the cycle $[C] - [(1) \ast C]$ in the Chow group $CH_1(J)_{\text{hom}}$ of homologically trivial 1-cycles. The Ceresa class $c_p(C)$ is the image of $\tilde{z}_p(C)$ in the intermediate Jacobian $J_1(J)$ parameterizing 1-cycles under the Abel–Jacobi map $CH_1(J)_{\text{hom}} \to J_1(J)$.

When $C$ is general, $\tilde{z}_p(C)$ is not algebraically trivial [2]. On the other hand, if $C$ is hyperelliptic $\tilde{z}_p(C)$ is algebraically trivial – in fact it is zero if one chooses for $p$ a Weierstrass point. Not much is known besides these two extreme cases. There are few curves for which $\tilde{z}_p(C)$ is known to be not algebraically trivial: Fermat curves of degree $\leq 1000$ [4], and the Klein quartic [5]. An essential ingredient of these results is the fact that $c_p(C)$ is not a torsion class.

It is an open question whether there are non-hyperelliptic curves with $\tilde{z}_p(C)$ algebraically trivial. As observed in [3, Remark 2.4], this condition is equivalent to a number of interesting properties: in particular the existence of a multiplicative Chow–Künneth decomposition modulo algebraic equivalence, or the fact that the class $[C] \in CH_1(J) \otimes \mathbb{Q}$ is algebraically equivalent to the minimal class $\theta^{g-1}_{(g-1)\mathbb{P}}$, where $\theta \in CH^1(J)$ is the class of the principal polarization.

In this note we exhibit a curve $C$ of genus 3 with the weaker property that the Ceresa class $c_p(C)$ is torsion (under the Bloch–Beilinson conjectures, this actually implies the algebraic triviality of $\tilde{z}_p(C)$ up to torsion). The construction is very simple: the curve $C$ has an automorphism $\sigma$ which
fixes a point $p$, and therefore preserves $c_p(C)$; we just have to check that the fixed point set of $\sigma$ acting on $\mathcal{J}_1(J)$ is finite.

A similar example, based on a much more sophisticated approach, appears in [1, Remark 3.6].

2. The result

**Proposition 1.** Let $C \subset \mathbb{P}^2$ be the genus 3 curve defined by $X^4 + XZ^3 + Y^3Z = 0$, and let $p = (0, 0, 1)$. The Ceresa class $c_p(C)$ is torsion.

**Proof.** Let $\omega$ be a primitive 9th root of unity. We consider the automorphism $\sigma$ of $C$ defined by $\sigma(X, Y, Z) = (X, \omega^2 Y, \omega^3 Z)$. We have $\sigma(p) = p$; therefore $\sigma$ preserves the Ceresa cycle $\mathcal{J}_0(C)$, and also its class $c_p(C)$ in $\mathcal{J} := J_1(J)$.

Thus it suffices to prove that $\sigma$ has finitely many fixed points on $\mathcal{J}$; equivalently, that the eigenvalues of $\sigma$ acting on the tangent space $T_0(J)$ are $\neq 1$.

Now $T_0(J)$ is identified with $H^{0,3}(J) \oplus H^{1,2}(J) = \Lambda^3 V^* \oplus (\Lambda^2 V^* \otimes V)$, where $V = H^{1,0}(J) = H^0(C, K_C)$. We first compute the eigenvalues of $\sigma$ on $V$. The elements of $V$ are of the form $L \cdot X dZ - Z dX + Y^2 Z$, with $L \in H^0(\mathbb{P}^2, O_{\mathbb{P}^2}(1))$; it follows that the eigenvalues of $\sigma$ on $V$ are $\omega^5, \omega^7, \omega^8$. Therefore the eigenvalue on $\Lambda^3 V^*$ is $\omega^7$, and the eigenvalues on $\Lambda^2 V^*$ are $\omega^5, \omega^5, \omega^6$. Thus each product of an eigenvalue on $\Lambda^2 V^*$ and one on $V$ is $\neq 1$, hence the Proposition. $\square$

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