Full-analytic frequency-domain 1pN-accurate gravitational wave forms from eccentric compact binaries

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The article provides ready-to-use 1pN-accurate frequency-domain gravitational wave forms for eccentric nonspinning compact binaries of arbitrary mass ratio including the first post-Newtonian (1pN) point particle corrections to the far-zone gravitational wave amplitude, given in terms of tensor spherical harmonics. The averaged equations for the decay of the eccentricity and growth of radial frequency due to radiation reaction are used to provide stationary phase approximations to the frequency-domain wave forms.

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The post-Newtonian (pN) description of the dynamics of compact binary systems is the topic of actual research. Due to the strong nonspherically symmetric gravitational interactions, those objects are supposed to be “secure” sources for the detection of gravitational waves. Currently, LIGO, VIRGO, and GEO600 search for the last seconds or minutes in the life of those sources when the gravitational wave (GW) emission frequency enters the bandwidth of the mentioned detectors.

The computer resources – in contrast – are currently unable to create numerical GW templates for the early stage of the binary inspiral (this case is important for LISA), where hundreds or even thousands of GW cycles have to be simulated which makes it necessary to propose an analytical prescription of the orbital evolution, and in the case of spinning compact binaries, the spin evolution as well. An equally essential ingredient of the GW data analysis is the transformation of the GW signal into the frequency domain. There exist numerous more or less optimized numerical routines to convert the time domain signal into the frequency or Fourier domain. To economize on computer resources, it is also reasonable and desirable to provide analytical Fourier domain wave forms for the data analysis community. Numerous authors have investigated the performance of circular inspiral templates and their analytical Fourier-domain pendant, which have been set up to a certain standard in the literature thanks to the work of Damour, Blanchet and Iyer to compute higher pN order corrections to the GW energy loss [1–4].

This work is inspired by [5] and [6] and is a direct sequel of [7]. Galtsov et al. provided a formal frequency-domain decomposition in multiples of two irreducible frequencies: the radial frequency $f_r$, describing the elapsed time form one periastron passage to the next, and the frequency associated to the periastron advance parameter, $f_\phi$, but used this decomposition only for the computation of the far-zone energy flux, where some mathematical aspects made it unnecessary to compute the Fourier coefficients directly.

Ref. [7] provided numerical insights to the case of no radiation reaction (RR) with the use of essentially the same frequencies that appeared in [5].

Yunes et al. [6] established analytic Fourier-domain inspiral templates for eccentric binary emission for the exemplary case of Newtonian equations of motion (EOM) and the leading-order GW amplitude. Those authors used the expansion of the Kepler equation and related trigonometric function combinations of the eccentric anomaly in terms of Bessel functions and applied this to a series expansion of the leading-order GW amplitude in the orbital eccentricity.

What we like to do in this article is the generalisation of Yunes et al. to 1pN order of the conservative orbital dynamics. The work will include, in addition to Yunes et al., the effect of periastron advance and also the GW amplitude corrections to 1pN order relative to the leading order quadrupole approximation. We present the results in terms of irreducible positive frequencies $f_r$ and $f_\phi$. Afterwards, we compute the Fourier domain for the case when RR is taken into account.

The paper is organized in the following way. Section I gives an overview over the solution to the 1pN-accurate orbital motion. Section II summarizes the transverse-traceless (TT) projection of the far-zone GW field up to 1pN order corrections to the leading-order quadrupole field. The vital components of the Fourier decomposition of the GW field are summarised in Section III. Section IV depicts how to decompose the terms of the multipole moments of equal structure. The Fourier domain multipoles themselves, incorporating purely 1pN conservative dynamics, are given in V. The effect of RR is incorporated in Section VI using the method of the stationary phase approximation (SPA).

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I. ORBITAL MOTION OF COMPACT BINARIES

Before we start computing the GW forms, we give an overview on the solution to the orbital EOM in a quasi-Keplerian parameterization (QKP). The QKP for non-spinning compact binaries up to and including 3pN point particle (PP) contributions can be found in \cite{9} and references therein. We restrict ourselves to the 1pN part thereof. The QKP for nonspinning compact binaries to 1pN is the following:

\[ r = a_r(1 - e_r \cos u), \]
\[ M := n (t - t_0) = u - e_r \sin u, \]
\[ \phi - \phi_0 = (1 + k) v, \]
\[ v = 2 \arctan \sqrt{\frac{1 + e_r \tan \frac{u}{2}}{1 - e_r \tan \frac{u}{2}}}, \]
\[ n = 2\sqrt{2} |E|^{3/2} + e^2 |E|^{5/2}(\eta - 15), \]
\[ a_r = \frac{1}{2 |E|} \left( 1 + e^2 |E| \left( \frac{\eta - 7}{4} \right) \right), \]
\[ e_r = e_t \left( 1 + e^2 |E| (8 - 3\eta) \right), \]
\[ e_\phi = e_t \left( 1 + e^2 |E| (-2(\eta - 4)) \right), \]
\[ k = e^2 \frac{\eta^{2/3}}{1 - e^2} = e^2 \frac{6 |E|}{1 - e^2}. \]

In the above Eqs. (1-9), \( \epsilon \) counts the inverse order of the speed of light \( c \), \( \epsilon^2 = c^{-2} \), \( r \) is the scaled relative separation of the two bound objects, \( \phi \) is the orbital phase, \( v \) is the true anomaly, \( k \) is the periastron advance parameter, \( e_r \) and \( e_t \) are some “radial” and “time” eccentricity. The latter appears in the Kepler equation (KE), Eq. (2), connecting the mean anomaly \( M \) (which is directly proportional to the elapsed time \( t \) with the proportionality factor \( n = 2\pi/P_r = 2\pi f_r \), where \( P_r \) is the time between two consecutive periastron passages) to the eccentric anomaly \( u \), see \cite{8} for geometrical insight. The shorthands \( \eta, \mu \), and \( |E| \) denote the symmetric mass ration \( \eta = (m_1m_2)/(m_1 + m_2)^2 \) (\( m_1 \) and \( m_2 \) are the individual masses with \( m_t = m_1 + m_2 \)), the reduced mass, and the absolute value of the orbital binding energy.

This parameterization is essential for the computation of the higher-order time derivatives of the radiative multipole moments. We can easily obtain them using the KE. How the orbital parameterization effects the far-zone gravitational field will be depicted in the next sections.

II. THE 1PN-ACCURATE TRANSVERSE AND TRACELESS FAR-ZONE RADIATION FIELD FOR ECCENTRIC COMPACT BINARIES

We start collecting the expressions of the transverse-traceless far-zone field \( h_{TT} \) in Junker and Schäfer \cite{9}. They are given in tensor-spherical harmonics and it is remarkable to note their structure as they appear as \( I_{lm}^{(i)} = e^{-i\phi} S_{lm}^{(i)}(u(t)) \) and \( S_{lm}^{(i)} = e^{-i\phi} g_{lm}(u(t)) \). Taking as basis

\[ h_{TT}^{(i)} = \epsilon^4 G R \left\{ \sum_{m=-2}^{2} \left( I_{2m}^{(2)} T_{E2,2m}^{E2,2m} + e \sum_{m=-2}^{2} \left( I_{2m}^{(2)} S_{2m}^{B2,2m} T_{ij}^{B2,2m} + 3 \sum_{m=-3}^{3} I_{3m}^{(3)} T_{ij}^{E2,3m} \right) \right) + e^2 \sum_{m=-3}^{3} \left( 3 I_{3m}^{(3)} T_{ij}^{E2,3m} + 4 I_{4m}^{(4)} T_{ij}^{E2,4m} \right) \right\}, \]

where \( R \) is the distance from the observer to the source, \( T_{E2,lm}^{E2,lm} \) and \( T_{B2,lm}^{B2,lm} \) are the “electric” and “magnetic” tensor spherical harmonics \cite{10}, the superscript \( (i) \) allowing \( i = 2, 3, 4 \) denotes the \( i \)th time differentiation. They read – up to the required overall order of \( \epsilon^6 = c^{-6} \) in Eq. (10) –

\[ (2) I_{22} = 8 \frac{2\pi}{5} E \mu e^{-2i\phi} \left( 2 - 2e_t^2 \right) A^2 + 1 + |E| e^2 \left( \frac{2(15\eta - 82)(e_t^2 - 1)}{21A^3} + \frac{2}{3}(3\eta - 1)e_t^2 + 7\eta - 15 \right). \]
where $\mu$ is the reduced mass and $A(u) := 1 - e_t \cos u$. The above multipole moments contain terms with certain equal structure. Evaluating the exponential functions $e^{-im\phi}$ with $m = 1, 2, 3, 4$ to 1pN accuracy, one obtains factors of the form $A(u)\text{ or }\sin(u)/A(u)$. These terms can be expanded first in $\sin$ and $\cos$ series of $u$ and afterwards

\[ \sin u \text{ and } \cos u \text{ of higher power than one in the denominator can always be expressed again in } A(u) \text{ and } \sin u. \]
in terms of $M$ with the help of Eqs. (16) and (18). We will calculate the expansion coefficients in the next section. Some calculation steps will be required repeatedly and we will collect them as ingredients for a frequency-domain GW recipe.

III. INGREDIENTS FOR FOURIER DECOMPOSITION

The Fourier-domain GW form requires computation of relatively simple structures which will combine in the tensor-spherical harmonics. Let us start with the most fundamental quantities which we compute from the scratch on the one hand and some which we collect from the literature on the other.

1. The inverse scaled radial separation with an arbitrary integer exponent $n > 0$:

$$\frac{1}{(1 - e \cos u)^n} = 1 + b_0^{(n)} + \sum_{j=1}^{\infty} b_j^{(n)} \cos j u ,$$

$$b_0^{(n)} = \sum_{i=1}^{\infty} \beta_{2i,i}^{(n)},$$

$$b_j^{(n)} = \sum_{i=0}^{\infty} \beta_{j+2i,i}^{(n)} + \beta_{j+2i,i}^{(n)},$$

$$\beta_{m,k}^{(n)} := \frac{(n + m - 1)!}{(n - 1)!} \frac{1}{m! 2^m} \frac{e^m(m)}{k} \prod_{i=0}^{m-1} (n + i).$$

Equation (12) is proven in Appendix A. Note that from here onwards, $e$ can be set $e_\phi$ or $e_t$, depending on the context.

2. Trigonometric functions of multiples of the eccentric anomaly $u$ in terms of Bessel functions (see [11], p. 555),

$$\sin n u = m \sum_{n=1}^{\infty} \frac{1}{n} \left( J_{n-m}(me_t) + J_{n+m}(me_t) \right) \sin(n M) = \sum_{n=1}^{\infty} \bar{\sigma}_n^m \sin n M ,$$

$$\bar{\sigma}_n^m := \frac{m}{n} \left( J_{n-m}(me_t) + J_{n+m}(me_t) \right) ,$$

$$\cos n u = m \sum_{n=1}^{\infty} \frac{1}{n} \left( J_{n-m}(me_t) - J_{n+m}(me_t) \right) \cos(n M) = \sum_{n=1}^{\infty} \bar{\gamma}_n^m \cos n M ,$$

$$\bar{\gamma}_n^m := \frac{m}{n} \left( J_{n-m}(me_t) - J_{n+m}(me_t) \right) - \frac{e_t}{2} \delta_{m1} \delta_{n0} .$$

Note that the above Eqs. (16) - (20) are only valid taking $e_t$ because of the structure of the KE.

3. The decomposition of $v$ in $M$ (see [12], p. 33),

$$v(e=\varepsilon_t) = M + \sum_{m=1}^{\infty} G_m(\varepsilon_t) \sin m M ,$$

with $G_m$ defined as

$$G_m(e) = \frac{2}{m} J_m(me) + \sum_{s=1}^{\infty} \alpha^s [J_{m-s}(me) - J_{m+s}(me)] ,$$

for $m > 0$. 


and $\alpha$ extractable from
\[ e = \frac{2\alpha}{1 + \alpha^2}. \]  
(24)

4. The decomposition of the 1pN accurate Kepler equation itself in terms of the mean anomaly (see \[11\], p. 553),
\[ u = M + \sum_{n=1}^{\infty} \frac{2}{n} J_n(n\varepsilon_t) \sin(nM), \]  
(25)

which converges for all $\varepsilon_t < 1$. These inputs are sufficient to obtain a 1pN-accurate F-domain decomposition of the 1pN far-zone GW field including 1pN-accurate orbital dynamics. More complicated expressions will be based on those above and can be computed more or less laboriously. A detailed appendix of this article will give some proves for resummation rules for these terms. The next section will deal with certain exponential functions including the 1pN-accurate phase. They will be combined with the fast oscillatory terms in $f_{lm}$ and $g_{lm}$ and have individual Fourier decompositions as shown below.

**IV. DECOMPOSITION OF THE ELEMENTS APPEARING IN THE MULTPOLES**

This section is subject to the Fourier decomposition of single terms in the multipole field expressions. We will apply the aforementioned steps to obtain the individual Fourier decompositions. The “magnetic” number (the factor of $-i \phi$ in the phase exponentials of Eqs. \[11n\] - \[11s\]) which we shall call $m$ in the multipoles dictates terms of the form $e^{-im\phi}$, which have to be expanded in $\varepsilon$.

\begin{align*}
v_{e_t} & := 2 \arctan \left[ \sqrt{\frac{1 + \varepsilon_t}{1 - \varepsilon_t}} \tan \frac{u}{2} \right], \\
v_{e\phi} & := 2 \arctan \left[ \sqrt{\frac{1 + e\phi}{1 - e\phi}} \tan \frac{u}{2} \right], \quad e^{im\phi} = e^{-im\phi_0} e^{-im(1+k)\phi_0} = e^{-im\phi_0} e^{-imv_{e_t}} e^{-imv_{e\phi}}, \\
& = e^{-im\phi_0} \left( \cos u - e\phi \cos u - i \sqrt{1 - e^2\phi} \sin u \frac{\sin u}{1 - e\phi \cos u} \right) e^{-imv_{e\phi}}, \\
& = e^{-im\phi_0} \left( \cos u - e\phi \cos u - i \sqrt{1 - e^2\phi} \sin u \frac{\sin u}{1 - e\phi \cos u} \right) \exp \left[ -imk \left( M + \sum_{j=1}^{\infty} G_j(e_t) \sin jM \right) \right] \times \\
& \exp [-imkM] \left( 1 - imk \sum_{j=1}^{\infty} G_j(e_t) \sin jM \right). \quad (28)
\end{align*}

For the first line we took the definition of $v$ as a function of $u$ and $e_t$, Eq. \[11\], and Eq. \[12\]. In the last line above we truncated the exponential of the sin series after the linear order in $k$, because $k$ is already of 1pN order and used the Fourier-domain expansion Eq. \[22\] for $v$. We are allowed to do this because there is no secular contribution coming up in the exponent due to its periodicity of $2\pi$ in $M$ and its average to zero in the interval $[0, 2\pi]$.

In the multipole moments the numbers $m = (0, 1, 2, 3, 4)$ appear, so we preserve all contributions with these $m$ up to the order of $\varepsilon$ required.

\begin{align*}
e^{-iv_{e_t}} & = \frac{1}{e_t} \left\{ i \sqrt{1 - e_t^2} \sin u - e_t^2 + 1 \right\} \frac{1}{A(u)} + O(e^2), \quad (29) \\
e^{-iv_{e\phi}} & = \frac{1}{e_t^2} \left( 2 - e_t^2 + \frac{4(e_t^2 - 1)}{A(u)} + 2ie_t \sqrt{1 - e_t^2} \sin u \right) + \frac{2(e_t^2 - 1)^2 - 2ie_t \sqrt{1 - e_t^2} \sin u}{A(u)^2}.
\end{align*}
The help of Eq. (12) we get

$$e^{-i3w_\phi} = \frac{1}{e_t^3} \left\{ -8(\eta - 4)(e_t^2 - 1)^2 - 8ie_t\sqrt{1-e_t^2}(\eta - 4)e_t(e_t^2 - 1)\sin(u) \right\} + O(e^4),$$

$$e^{-i4w_\phi} = \frac{1}{e_t^4} \left\{ 3e_t^2 - 4 + \frac{3(e_t^4 - 5e_t^2 + 4) - 4ie_t\sqrt{1-e_t^2}e_t(1 - e_t^2)}{A(u)^3} - \frac{12(e_t^2 - 1)^2 + 8ie_t\sqrt{1-e_t^2}\sin(u)}{A(u)^2} \right\} + O(e^2),$$

Those terms will mix with $f_{lm}$ and $g_{lm}$. Let us get more precise now and start decomposing the $A(u)^{-n}$ term. With the help of Eq. (12) we get

$$\frac{1}{(1-e_t\cos u)^n} = 1 + \sum_{m=1}^{\infty} \sum_{k=0}^{m} \beta_{m,k}^{(n)} \cos((m-2k)u)$$

$$= 1 + b_0^{(n)} + \sum_{k=1}^{\infty} b_k^{(n)} \cos ku,$$

where we have collected those terms with the same positive frequency. Appendix A will give deeper explanation how to select the frequencies with the same absolute value. The same contribution – being multiplied with $\sin u$ – will be decomposed analogously, but having $\sin mu$ this time as an odd function of $u.$

$$\frac{\sin u}{(1-e_t\cos u)^n} = \left( 1 + b_0^{(n)} + \sum_{j=1}^{\infty} b_j^{(n)} \cos j u \right) \sin u$$

$$= \left( 1 + \sum_{k=1}^{\infty} \beta_{2k,k}^{(n)} \right) \sin u + \frac{1}{2} \left( \sum_{m=2}^{\infty} \left( b_{m-1}^{(n)} - b_{m+1}^{(n)} \right) \sin mu \right)$$

$$= \sum_{j=1}^{\infty} S_j^{(n)} \sin ju,$$

$$S_1^{(n)} := 1 + \sum_{k=1}^{\infty} \beta_{2k,k}^{(n)} - \frac{1}{2} b_2^{(n)} ,$$

$$S_{j>1}^{(n)} := \frac{1}{2} \left( b_{j-1}^{(n)} - b_{j+1}^{(n)} \right) ,$$

Remembering Eqs. (16) and (18), we get

$$\frac{1}{(1-e_t\cos u)^n} = 1 + b_0^{(n)} + \sum_{k=1}^{\infty} b_k^{(n)} \cos ku = 1 + b_0^{(n)} + \sum_{k=1}^{\infty} b_k^{(n)} \left( \sum_{j=1}^{\infty} \gamma_j^{(n)} \cos jM \right)$$
\[
\begin{align*}
&= 1 + b_0^{(n)} + \sum_{j=1}^{\infty} \left( \sum_{k=1}^{\infty} \tilde{\gamma}_j b_k^{(n)} \right) \cos jM \\
&= 1 + b_0^{(n)} + \sum_{n=1}^{\infty} A_j^{(n)} \cos jM, \\
\sin u \quad \text{on} \quad (1 - e_t \cos u)^n &= \sum_{k=1}^{\infty} S_k^{(n)} \sin ku = \sum_{k=1}^{\infty} S_k^{(n)} \left( \sum_{j=1}^{\infty} \tilde{\sigma}_j^k \sin jM \right) = \sum_{j=1}^{\infty} \left( \sum_{k=1}^{\infty} \tilde{\sigma}_j^k S_k^{(n)} \right) \cos jM \\
&= \sum_{j=1}^{\infty} S_j^{(n)} \sin jM, \\
A_j^{(n)} &:= 1 + b_0^{(n)}, \\
A_{j>0}^{(n)} &:= \left( \sum_{m=1}^{\infty} \tilde{\gamma}_m b_m^{(n)} \right), \\
S_j^{(n)} &:= \left( \sum_{m=1}^{\infty} \tilde{\sigma}_m G_m^{(n)} \right). 
\end{align*}
\]

The abbreviations \(A_j^{(n)}\) and \(S_j^{(n)}\) stand for the \(j\)th contribution of \(A(u)^{-n}\) and \(A(u)^{-n} \sin u\). Of course, both expressions will not take their place alone, but will be multiplied with the expansion of the \(e^{i n k v_x}\) term due to Eq. (26). We Fourier decompose the following products of series as they appear in the field multipole moments. The first quantity consists of two pure \(\sin\) series. Factor 1 is the expansion of \(\sin(u) A(u)^{-n}\) in \(M\) and factor 2 is the non-secular part of the exponential of the \(v(l)\)-term.

\[
\left( \sum_{k=1}^{\infty} S_k^{(n)} \sin kM \right) \left( \sum_{m=1}^{\infty} G_m \sin mM \right) = \sum_{m=1}^{\infty} \sum_{k=1}^{\infty} \frac{1}{2} S_k^{(n)} G_m (\cos[k-m]M - \cos[k+m]M) \\
= \sum_{j=0}^{\infty} P_j^{SS,[n]} \cos jM. 
\]

The coefficients \(P_j^{SS,[n]}\) ("SS" stands for the product of two \(\sin\) series) are defined as follows.

\[
P_0^{SS,[n]} := \frac{1}{2} \sum_{k=1}^{\infty} S_k^{(n)} G_k, \\
P_j^{SS,[n]} := \frac{1}{2} \left\{ \sum_{k=1}^{\infty} \left( S_k^{(n)} G_{k+j} + S_k^{(n)} G_{k-j} \right) - \sum_{k=1}^{j-1} S_k^{(n)} G_{j-k} \right\} \quad \text{for } j > 1. 
\]

The second quantity consists of a \(\sin\) and a \(\cos\) series. Factor 1 is the decomposition of \(A(u)^{-n}\) in \(M\) and factor 2 is – again – the non-secular part of the exponential of the \(v(l)\)-term.

\[
\left( \sum_{k=1}^{\infty} A_k^{(n)} \cos kM \right) \left( \sum_{m=1}^{\infty} G_m \sin mM \right) = \frac{1}{2} \sum_{m=1}^{\infty} \sum_{k=1}^{\infty} A_k^{(n)} G_m (\sin[k+m]M - \sin[k-m]M) \\
= \sum_{j=1}^{\infty} P_j^{CS,[n]} \sin jM. 
\]

As well as for \(P_j^{SS,[n]}\), we provide the definition of the \(P_j^{CS,[n]}\) as combinations of the elements of the two series.

\[
P_j^{CS,[n]} := \frac{1}{2} \left\{ \sum_{k=1}^{j-1} \left( A_k^{(n)} G_{j-k} - A_k^{(n)} G_{k+j} \right) \right\} \quad \text{for } j > 1. 
\]
The decomposition coefficients do not become zero, or else, these elements are set to zero automatically.

We will perform the equivalent steps for

\[
\sum_{j=1}^{\infty} A_j^{(n)} \cos jM \left[ 1 + b_0^{(n)} \right] + \sum_{m=1}^{\infty} G_m \sin mM \left[ e^{-i(qk)M} \right]
\]

for a variety of reasons:

1. The decomposition coefficients do not become zero, or else, these elements are set to zero automatically.
2. The two above summation formulas will also be proven in Appendices D and C. Please keep in mind that the \((n)\) is always present in those terms and reminds the user of keeping the right exponential of \(A(u)^{-n}\) and that those functions always depend on the value of \(e\). Additionally note that the statements “for \(j > 1\)” and “for \(k > j\)” ensure that, for example, the first term in (4.19) is not able to contribute to the first harmonic in \(l\) and that the indices of the decomposition coefficients do not become zero, or else, these elements are set to zero automatically.

The next step is to sum up and summarise the terms appearing in the multipole moments. Using Eq. (22), they also contain absolute static elements and will be listed below.

\[
\frac{1}{1 - e \cos u} e^{-i(qk)\nu_{\varepsilon}} = \left\{ 1 + b_0^{(n)} + \sum_{j=1}^{\infty} A_j^{(n)} \cos jM \right\} \left\{ 1 + (-i(qk) \sum_{m=1}^{\infty} G_m \sin mM \right\} \left\{ e^{-i(qk)M} \right\}
\]

\[
= \left\{ 1 + (-i(qk) \sum_{j=1}^{\infty} G_j \sin jM \right\} \left( 1 + b_0^{(n)} \right) + \left\{ \sum_{j=1}^{\infty} A_j^{(n)} \cos jM \right\} \left\{ e^{-i(qk)M} \right\}
\]

\[
+ (-i(qk) \sum_{j=1}^{\infty} P_j^{CS[n]} \sin jM \right\} \left\{ e^{-i(qk)M} \right\}
\]

\[
= \left\{ 1 + b_0^{(n)} \right\} + (-i(qk) \sum_{j=1}^{\infty} \left[ 1 + b_0^{(n)} \right] G_j + P_j^{CS[n]} \right\} \sin jM \left\{ e^{-i(qk)M} \right\}
\]

\[
+ \sum_{j=1}^{\infty} A_j^{(n)} \cos jM \right\} \left\{ e^{-i(qk)M} \right\}.
\]

(47)

We will perform the equivalent steps for

\[
\frac{\sin u}{1 - e \cos u} e^{-i(qk)\nu_{\varepsilon}} = \left\{ \sum_{j=1}^{\infty} S_j^{(n)} \sin jM \right\} \left\{ 1 + (-i(qk) \sum_{m=1}^{\infty} G_m \sin mM \right\} \left\{ e^{-i(qk)M} \right\}
\]

\[
= \left\{ \sum_{j=1}^{\infty} S_j^{(n)} \sin jM + (-i(qk) \sum_{j=0}^{\infty} P_j^{SS[n]} \cos jM \right\} \left\{ e^{-i(qk)M} \right\}.
\]

(48)

This will be shortened by writing

\[
\frac{1}{1 - e \cos u} e^{-i(qk)\nu_{\varepsilon}} = \left\{ (A_{\varepsilon})_{j=0}^{[n,q]} + \sum_{j=1}^{\infty} (A_{\varepsilon})_{j}^{[n,q]} \cos jM + \sum_{j=1}^{\infty} (A_s)_{j}^{[n,q]} \sin jM \right\} e^{-i(qk)M},
\]

(49)

\[
\frac{\sin u}{1 - e \cos u} e^{-i(qk)\nu_{\varepsilon}} = \left\{ (S_{\varepsilon})_{j=0}^{[n,q]} + \sum_{j=1}^{\infty} (S_{\varepsilon})_{j}^{[n,q]} \cos jM + \sum_{j=1}^{\infty} (S_s)_{j}^{[n,q]} \sin jM \right\} e^{-i(qk)M},
\]

(50)

\[
(A_{\varepsilon})_{j=0}^{[n,q]} := (-i(qk) \left[ 1 + b_0^{(n)} \right] G_j + P_j^{CS[n]} \right],
\]

(51)

\[
(A_{\varepsilon})_{j}^{[n,q]} := \left[ 1 + b_0^{(n)} \right],
\]

(52)

\[
(A_s)_{j=0}^{[n,q]} := A_j^{(n)},
\]

(53)

\[
(S_{\varepsilon})_{j=0}^{[n,q]} := P_j^{SS[n]},
\]

(54)

\[
(S_s)_{j=0}^{[n,q]} := S_j^{(n)}.
\]

(55)

We are now in the lucky position to decompose all point particle contributions to the radiation field amplitude in terms of the above defined \((A_{\varepsilon})_{j}^{[n,q]}, (A_s)_{j}^{[n,q]}, (S_{\varepsilon})_{j=0}^{[n,q]}, \) and \((S_s)_{j}^{[n,q]}\). For simplicity, we introduce

\[
\frac{1}{1 - e \cos u} e^{-i(qk)\nu_{\varepsilon}} =: \mathcal{F}_{[n,q]}(u) e^{-i(qk)M},
\]

(56)

\[
\frac{\sin u}{1 - e \cos u} e^{-i(qk)\nu_{\varepsilon}} =: \mathcal{F}_{S}[n,q](u) e^{-i(qk)M},
\]

(57)
and use them from now on in the multipoles, where \( n \) and \( q \) will also be allowed to be 0. We finally rewrite our wave forms in the following manner,

\[
e^{-im\phi} f_{ml} = e^{-im\phi_0} e^{-im\nu \cdot \mathbf{e}} e^{-imk \cdot \mathbf{v}} f_{ml} = e^{-im\phi_0} \left[ e^{-im\nu \cdot \mathbf{e}} f_{ml} + \mathcal{O}(\epsilon^2) \right] e^{-imk \cdot \mathbf{v}}
\]

\[
= e^{-im\phi_0} \left[ e^{-im\nu \cdot \mathbf{e}} f_{ml} + \mathcal{O}(\epsilon^2) \right] e^{-imk \cdot \mathbf{v}} \left( 1 + (-imk) \sum_{j=1}^{\infty} G_j \sin jM \right)
\]

\[
= e^{-im\phi_0} \sum_j \left[ \kappa_{jm} F_{jm}(u) + \tilde{\kappa}_{jm} F_{Sjm}(u) \right] e^{-imkM},
\]

with some \( \kappa_{jm} \) and \( \tilde{\kappa}_{jm} \) to be determined.

V. DECOMPOSITION OF THE MULTIPOLAR MOMENTS

We will decompose the multipoles in terms of the functions \( F_{mq}(u) \) introduced above next. These functions will have prefactors \( \alpha_n^{(m)}(\phi_0) \), \( \beta_n^{(m)}(\phi_0) \), ..., and \( \alpha_n^{(m)}(\nu_0) \), \( \beta_n^{(m)}(\nu_0) \), ... for the different multipole types, where \( m \) denotes again the magnetic number and \( n \) is the same exponent as in Defs. \((55)\) and \((57)\).

\[
I_{22}^{(2)} = 8 \sqrt{\frac{2\pi}{5}} e^2 \epsilon_\nu^2 e^{-2iM} e^{-i2\phi_0} \left\{ \sum_{k=0}^{5} \alpha_{[2k]} F_{[2k]}(u) + \sum_{k=1}^{5} \beta_{[2k]} F_{S[2k]}(u) \right\},
\]

\[
\alpha_{[02]} := [2 - \epsilon_\nu^2] + \epsilon_\nu^2 \frac{1}{14} |E| \left[ (9\eta - 3)\epsilon_\nu^2 + 94\eta - 442 \right],
\]

\[
\alpha_{[12]} := \epsilon_\nu^2 - 2 + \epsilon_\nu^2 \frac{1}{14} |E| \left[ (285\eta - 781)\epsilon_\nu^2 - 346\eta + 1450 \right],
\]

\[
\alpha_{[22]} := 2 (\epsilon_\nu^2 - 1) + \epsilon_\nu^2 |E| \frac{1}{21} \left[ 84(3\eta - 8)\epsilon_\nu^4 + (3713 - 1458\eta)\epsilon_\nu^2 + 981\eta - 2861 \right],
\]

\[
\alpha_{[32]} := 2 (\epsilon_\nu^2 - 1)^2 + \epsilon_\nu^2 |E| \left\{ -\frac{1}{21} (\epsilon_\nu^2 - 1) \left( (663\eta - 1537)\epsilon_\nu^2 - 1005\eta + 1861 \right) \right\},
\]

\[
\alpha_{[42]} := \epsilon_\nu^2 |E| \left\{ \frac{2}{7} (21\eta - 262) (\epsilon_\nu^2 - 1)^2 \right\},
\]

\[
\alpha_{[52]} := \epsilon_\nu^2 |E| \left\{ \frac{4}{7} (3\eta - 1) (\epsilon_\nu^2 - 1)^3 \right\},
\]

\[
\beta_{[12]} := \left[ 2i\epsilon_\nu \sqrt{1 - \epsilon_\nu^2} \right] + \epsilon_\nu^2 \left[ \frac{i|E|\epsilon_\nu \left( (109 - 19\eta)\epsilon_\nu^2 + 47\eta - 221 \right)}{7\sqrt{1 - \epsilon_\nu^2}} \right],
\]

\[
\beta_{[22]} := \epsilon_\nu^2 \left[ -\frac{2|E|\epsilon_\nu \left( 2(3\eta - 8)\epsilon_\nu^4 + (46 - 15\eta)\epsilon_\nu^2 + 9(\eta - 4) \right)}{\sqrt{1 - \epsilon_\nu^2}} \right],
\]

\[
\beta_{[32]} := \left[ -2i\epsilon_\nu \left( 1 - \epsilon_\nu^2 \right)^{3/2} \right] + \epsilon_\nu^2 \left[ \frac{1}{21} |E|\epsilon_\nu \sqrt{1 - \epsilon_\nu^2} \left( (1531 - 645\eta)\epsilon_\nu^2 + 603\eta - 1349 \right) \right],
\]

\[
\beta_{[42]} := i\epsilon_\nu^2 \left[ -\frac{2}{21} |E| (201\eta - 256)\epsilon_\nu \left( 1 - \epsilon_\nu^2 \right)^{3/2} \right],
\]

\[
\beta_{[52]} := i\epsilon_\nu^2 \left\{ \frac{4}{7} |E| (3\eta - 1)\epsilon_\nu \left( 1 - \epsilon_\nu^2 \right)^{5/2} \right\},
\]

\[
I_{21}^{(2)} = 0,
\]

\[
I_{20}^{(2)} = -16 \sqrt{\frac{\pi}{15}} \epsilon_\nu^2 \left\{ \alpha_{[00]} F_{[00]}(u) + \alpha_{[10]} F_{[10]}(u) + \alpha_{[20]} F_{[20]}(u) + \alpha_{[30]} F_{[30]}(u) \right\},
\]

\[
\alpha_{[00]} := 1 - \frac{3}{14} |E|\epsilon_\nu^2 (3\eta - 1),
\]

\[
\alpha_{[10]} := \frac{1}{14} |E|\epsilon_\nu^2 (51\eta - 115) - 1,
\]
\[ \alpha_{[20]} := \frac{1}{7} |E| \epsilon^2 (4 - 19 \eta), \]
\[ \alpha_{[30]} := \frac{2}{7} |E| \epsilon^2 (\eta - 26) (\epsilon_t^2 - 1), \]
\[ (2) \]
\[ (2) S_{22} = 0, \]
\[ (2) S_{21} = \frac{32}{3} \sqrt{\frac{2}{5}} \sqrt{1 - e_t^2 \eta (-E)^{3/2}} (m_1 - m_2) e^{-ikM} e^{-i\phi} \times \left\{ \beta_{[21]} F_{[21]}(u) + \beta_{[31]} F_{[31]}(u) + \tilde{\beta}_{[31]} S_{[31]}(u) \right\}, \]
\[ \beta_{[21]} := \frac{1}{e_t}, \]
\[ \beta_{[31]} := \frac{1}{e_t - e_t}, \]
\[ \tilde{\beta}_{[31]} := -i \sqrt{1 - e_t^2}, \]
\[ (2) S_{20} = 0, \]
\[ (3) I_{33} = \frac{8}{21} \frac{2\pi}{2} \eta (m_1 - m_2) (-E)^{3/2} e^{-ikM} e^{-i\phi} \left\{ \sum_{k=0}^{5} \gamma_{[k3]} F_{[k3]}(u) + \sum_{k=1}^{5} \tilde{\gamma}_{[k3]} S_{[k3]}(u) \right\}, \]
\[ \gamma_{[03]} := i \sqrt{1 - e_t^2} (\epsilon_t^2 - 4), \]
\[ \gamma_{[13]} := -i \sqrt{1 - e_t^2} (\epsilon_t^2 - 4), \]
\[ \gamma_{[23]} := i \frac{(12 - 7\epsilon_t^2)}{2\epsilon_t^2} \sqrt{1 - e_t^2}, \]
\[ \gamma_{[33]} := i \frac{\sqrt{1 - e_t^2}}{2\epsilon_t^2} (\epsilon_t^2 - 1) (\epsilon_t^2 + 4), \]
\[ \gamma_{[43]} := -\frac{10i}{\epsilon_t^2} \sqrt{1 - e_t^2} (\epsilon_t^2 - 1)^2, \]
\[ \gamma_{[53]} := -\frac{6i}{\epsilon_t^2} \sqrt{1 - e_t^2} (\epsilon_t^2 - 1)^3, \]
\[ \tilde{\gamma}_{[13]} := \frac{4}{\epsilon_t^2} - 3, \]
\[ \tilde{\gamma}_{[23]} := 0, \]
\[ \tilde{\gamma}_{[33]} := \frac{5\epsilon_t^2}{2} - \frac{6}{\epsilon_t^2} + \frac{17}{2}, \]
\[ \tilde{\gamma}_{[43]} := -\frac{4(\epsilon_t^2 - 1)^2}{\epsilon_t^2}, \]
\[ \tilde{\gamma}_{[53]} := -\frac{6(\epsilon_t^2 - 1)^3}{\epsilon_t^2}, \]
\[ (3) I_{32} = 0, \]
\[ (3) I_{31} = e^{-ikM} e^{-i\phi} \frac{8}{21} \frac{2\pi}{2} (-E)^{3/2} (m_1 - m_2) \eta \times \left\{ \sum_{k=0}^{4} \gamma_{[k1]} F_{[k1]}(u) + \sum_{k=1}^{4} \tilde{\gamma}_{[k1]} S_{[k1]}(u) \right\}, \]
\[ \gamma_{[01]} := \frac{i}{\epsilon_t} \sqrt{1 - e_t^2}, \]
\[ \gamma_{[11]} := -\frac{i}{\epsilon_t} \sqrt{1 - e_t^2}, \]
\gamma_{[21]} := -\frac{5i\sqrt{1-e_t^2}}{6e_t}, \quad (99)
\gamma_{[31]} := -\frac{5i\sqrt{1-e_t^2}}{6e_t}(e_t^2 - 1), \quad (100)
\hat{\gamma}_{[11]} := -1, \quad (101)
\hat{\gamma}_{[21]} := 0, \quad (102)
\hat{\gamma}_{[31]} := -\frac{5}{6}(e_t^2 - 1), \quad (103)

\begin{align*}
I_{30} & = 0, \quad (104) \\
S_{33} & = 0, \quad (105) \\
S_{32} & = e^{-i2\phi_0} e^{-i2\phi_0} \frac{8}{3} \sqrt{\frac{2\pi}{7}} E^2 \sqrt{1-e_t^2} \mu(1 - 3\eta) \times \left\{ \sum_{k=2}^{5} \delta_{[k2]\bar{\mathcal{F}}_{[k2]}(u)} + \sum_{k=3}^{5} \hat{\delta}_{[k2]\bar{\mathcal{F}}_{[k2]}(u)} \right\}, \quad (106)
\delta_{[22]} & := -\frac{2i}{e_t}, \quad (107)
\delta_{[32]} & := \left(2i - \frac{2i}{e_t} \right) \sqrt{1-e_t^2}, \quad (108)
\delta_{[42]} & := \left(\frac{10i}{e_t} - 10i \right) \sqrt{1-e_t^2}, \quad (109)
\delta_{[52]} & := -\frac{6i\sqrt{1-e_t^2}(e_t^2 - 1)^2}{e_t}, \quad (110)
\hat{\delta}_{[32]} & := \frac{2}{e_t} - e_t, \quad (111)
\hat{\delta}_{[42]} & := \frac{4}{e_t} - 4e_t, \quad (112)
\hat{\delta}_{[52]} & := -\frac{6(e_t^2 - 1)^2}{e_t}, \quad (113)
\end{align*}

\begin{align*}
S_{31} & = 0, \quad (114) \\
S_{30} & = -16 \sqrt{\frac{\pi}{105}} E^2 e_t \sqrt{1-e_t^2} \mu(1 - 3\eta) \hat{\delta}_{[30]} \bar{\mathcal{F}}_{[30]}(u), \quad (115)
\hat{\delta}_{[30]} & := 1, \quad (116) \\
I_{44} & = e^{-i4\phi_0} e^{-i4\phi_0} \frac{4}{9} \sqrt{\frac{2\pi}{7}} E^2 \mu(1 - 3\eta) \times \left\{ \sum_{k=0}^{7} \zeta_{[k4]\bar{\mathcal{F}}_{[k4]}(u)} + \sum_{k=1}^{7} \hat{\zeta}_{[k4]\bar{\mathcal{F}}_{[k4]}(u)} \right\}, \quad (117)
\zeta_{[04]} & := \frac{6(e_t^4 - 8e_t^2 + 8)}{e_t^4}, \quad (118)
\zeta_{[14]} & := -\frac{6(e_t^4 - 8e_t^2 + 8)}{e_t^4}, \quad (119)
\zeta_{[24]} & := -\frac{29e_t^4 - 112e_t^2 + 88}{e_t^4}, \quad (120)
\zeta_{[34]} & := \frac{3e_t^2(e_t^4 + 8) - 8}{e_t^4}, \quad (121)
\zeta_{[44]} & := -\frac{32(e_t^2 - 6)(e_t^2 - 1)^2}{e_t^4}, \quad (122)
\zeta_{[54]} & := \frac{16(e_t^2 - 1)^3(3e_t^2 - 4)}{e_t^4}, \quad (123)\end{align*}
\[ \zeta_{[64]} := -\frac{280\left(e_t^2 - 1\right)^4}{e_t^4}, \quad (124) \]
\[ \zeta_{[74]} := -\frac{120\left(e_t^2 - 1\right)^5}{e_t^4}, \quad (125) \]
\[ \tilde{\zeta}_{[14]} := -\frac{24i\sqrt{1 - e_t^2}\left(e_t^2 - 2\right)}{e_t^4}, \quad (126) \]
\[ \zeta_{[24]} := 0, \quad (127) \]
\[ \tilde{\zeta}_{[34]} := -\frac{2i\sqrt{1 - e_t^2}\left(9e_t^4 - 46e_t^2 + 44\right)}{e_t^3}, \quad (128) \]
\[ \tilde{\zeta}_{[44]} := -\frac{8i\sqrt{1 - e_t^2}\left(5e_t^4 - 17e_t^2 + 12\right)}{e_t^3}, \quad (129) \]
\[ \tilde{\zeta}_{[54]} := -\frac{12i\left(1 - e_t^2\right)^{5/2}\left(e_t^2 - 8\right)}{e_t^4}, \quad (130) \]
\[ \tilde{\zeta}_{[64]} := \frac{160i\left(1 - e_t^2\right)^{7/2}}{e_t^3}, \quad (131) \]
\[ \tilde{\zeta}_{[74]} := -\frac{120i\left(1 - e_t^2\right)^{9/2}}{e_t^4}, \quad (132) \]

\[ I_{43} = 0, \quad (133) \]

\[ I_{42} = e^{-i^2 \mathbf{K}_0} e^{-i^2 \phi_0} \frac{8}{63} \sqrt{2\pi E^2 \mu (1 - 3\eta)} \times \left\{ \sum_{k=0}^{3} \zeta_{[k2]} \mathcal{F}_{[k2]}(u) + \sum_{k=1}^{5} \tilde{\zeta}_{[k2]} \mathcal{F}_{[k2]}(u) \right\}, \quad (134) \]
\[ \zeta_{[02]} := 6 - \frac{12}{e_t^2}, \quad (135) \]
\[ \zeta_{[12]} := \frac{12}{e_t^2} - 6, \quad (136) \]
\[ \zeta_{[22]} := \frac{16}{e_t^2} - 11, \quad (137) \]
\[ \zeta_{[32]} := \frac{-(e_t - 1)(e_t + 1)(3e_t^2 - 8)}{e_t^2}, \quad (138) \]
\[ \zeta_{[42]} := -\frac{20(e_t - 1)^2(e_t + 1)^2}{e_t^4}, \quad (139) \]
\[ \zeta_{[52]} := -\frac{12(e_t - 1)^3(e_t + 1)^3}{e_t^4}, \quad (140) \]
\[ \tilde{\zeta}_{[12]} := -\frac{12i\sqrt{1 - e_t^2}}{e_t}, \quad (141) \]
\[ \tilde{\zeta}_{[22]} := 0, \quad (142) \]
\[ \tilde{\zeta}_{[32]} := -\frac{i(3e_t - 4)(3e_t + 4)}{e_t} \sqrt{1 - e_t^2}, \quad (143) \]
\[ \tilde{\zeta}_{[42]} := \frac{8i\left(1 - e_t^2\right)^{3/2}}{e_t}, \quad (144) \]
\[ \tilde{\zeta}_{[52]} := \frac{8i\left(1 - e_t^2\right)^{3/2}}{e_t}, \quad (145) \]

\[ I_{41} = 0, \quad (146) \]

\[ I_{40} = \frac{8}{21} \sqrt{\frac{\pi}{5}} E^2 \mu (1 - 3\eta) \left\{ \sum_{k=0}^{3} \zeta_{[k2]} \mathcal{F}_{[k2]}(u) \right\}, \quad (147) \]
\[ \zeta_{[00]} := 6, \quad (148) \]
\[ \zeta_{[10]} := -6, \]
\[ \zeta_{[20]} := -5, \]
\[ \zeta_{[30]} := 5 - 5e_i^2, \]
\[ \gamma \]

It seems that there are divergences for \( e_i \to 0 \), but the sum of the divergent terms is finite. Each decomposition term will contribute to \( \sin jM \) and to \( \cos jM \), allowing for \( j = 0, \infty \), with its own values \([nm]\) in boxy brackets for the multipole moment having the same special “magnetic” number \( m \) and \( n \) labeling contributions with \( A(u)^{-n} \). Using this and, not less importantly, Eqs. (149) and (50) to separate \( \exp(-imkM) \) from the products of \( A(u)^{-n} \) and \( \sin u A(u)^{-n} \) with \( \exp(-im\phi) \), the moments have the Fourier decomposition

\[
I_{22}^{(2)} = e^{-2i\phi_0} e^{-2i\pi k M} \left[ 8 \sqrt{\frac{2\pi}{5}} E \mu e_i^{-2} + \left( \sum_{j=1}^{\infty} \sin jM \left\{ \sum_{k=0}^{5} \alpha_{[k,2]}(\bar{A}_s)_{[j,2]}^{[k]} + \sum_{k=1}^{5} \bar{\alpha}_{[k,2]}(\bar{S}_s)_{[j,2]}^{[k]} \right\}, \right. \\
+ \left. \sum_{j=0}^{\infty} \cos jM \left\{ \sum_{k=0}^{5} \alpha_{[k,2]}(\bar{A}_c)_{[j,2]}^{[k]} + \sum_{k=1}^{5} \bar{\alpha}_{[k,2]}(\bar{S}_c)_{[j,2]}^{[k]} \right\} \right], \quad (152)
\]

\[
I_{21}^{(2)} = 0, \quad (153)
\]

\[
I_{20}^{(2)} = -16 \sqrt{\frac{\pi}{15}} E \mu \left( \sum_{j=1}^{\infty} \sin jM \left\{ \sum_{k=0}^{3} \alpha_{[k,2]}(\bar{A}_s)_{[j,2]}^{[k]} \right\} + \sum_{j=0}^{\infty} \cos jM \left\{ \sum_{k=0}^{5} \alpha_{[k,2]}(\bar{A}_c)_{[j,2]}^{[k]} \right\} \right), \quad (154)
\]

\[
S_{22}^{(2)} = 0, \quad (155)
\]

\[
S_{21}^{(2)} = e^{-i\phi_0} e^{-i\pi k M} \left[ \frac{32}{3} \sqrt{\frac{2\pi}{5}} \sqrt{1 - e_i^2 \eta} \left( -E \right)^{3/2} (m_1 - m_2) \times \left( \sum_{j=1}^{\infty} \sin jM \left\{ \sum_{k=2}^{3} \beta_{[k,1]}(\bar{A}_s)_{[j,1]}^{[k]} + \bar{\beta}_{[3,1]}(\bar{S}_s)_{[j,1]}^{[3]} \right\} \right), \quad (156)
\]

\[
S_{20}^{(2)} = 0, \quad (157)
\]

\[
I_{33}^{(3)} = e^{-3i\phi_0} e^{-3i\pi k M} \left[ 8 \sqrt{\frac{2\pi}{53}} (-E)^{3/2} (m_1 - m_2) \eta \times \left( \sum_{j=1}^{\infty} \sin jM \left\{ \sum_{k=0}^{5} \gamma_{[k,3]}(\bar{A}_s)_{[j,3]}^{[k]} + \sum_{k=1}^{5} \bar{\gamma}_{[k,3]}(\bar{S}_s)_{[j,3]}^{[k]} \right\} \right), \quad (158)
\]

\[
I_{32}^{(3)} = 0, \quad (159)
\]

\[
I_{31}^{(3)} = e^{-i\phi_0} e^{-i\pi k M} \left[ 8 \sqrt{\frac{2\pi}{33}} (-E)^{3/2} (m_1 - m_2) \eta \times \left( \sum_{j=1}^{\infty} \sin jM \left\{ \sum_{k=0}^{3} \gamma_{[k,1]}(\bar{A}_s)_{[j,1]}^{[k]} + \sum_{k=1}^{3} \bar{\gamma}_{[k,1]}(\bar{S}_s)_{[j,1]}^{[k]} \right\} \right), \quad (160)
\]

\[
I_{30}^{(3)} = 0, \quad (161)
\]
\( S_{33} = \begin{cases} 0, \\ e^{-2i\phi_0}e^{-2i} k M \frac{8}{3} \sqrt{\frac{2\pi}{7}} E^2 \sqrt{1 - e_t^2} \mu (1 - 3\eta) \times \left( \sum_{j=1}^{5} \sin j M \left\{ \sum_{k=2}^{5} \delta_{k1} (\tilde{A}_x)_{[k]} + \sum_{k=3}^{5} \delta_{k1} (\tilde{S}_x)_{[k]} \right\} + \sum_{j=0}^{5} \cos j M \left\{ \sum_{k=2}^{5} \delta_{k1} (\tilde{A}_c)_{[k]} + \sum_{k=3}^{5} \delta_{k1} (\tilde{S}_c)_{[k]} \right\} \right), \end{cases} \) (162)

\( S_{32} = \begin{cases} 0, \\ e^{-4i\phi_0} e^{-4i} k M \frac{4}{5} \sqrt{\frac{2\pi}{7}} E^2 \mu (1 - 3\eta) \times \left( \sum_{j=1}^{7} \sin j M \left\{ \sum_{k=0}^{7} \tilde{\zeta}_{k4} (\tilde{A}_x)_{[k]} + \sum_{k=1}^{7} \tilde{\zeta}_{k4} (\tilde{S}_x)_{[k]} \right\} + \sum_{j=0}^{7} \cos j M \left\{ \sum_{k=0}^{7} \tilde{\zeta}_{k4} (\tilde{A}_c)_{[k]} + \sum_{k=1}^{7} \tilde{\zeta}_{k4} (\tilde{S}_c)_{[k]} \right\} \right), \end{cases} \) (164)

\( S_{31} = \begin{cases} 0, \\ e^{-2i\phi_0} e^{-2i} k M \frac{8}{63} \sqrt{\frac{2\pi}{7}} E^2 \mu (1 - 3\eta) \times \left( \sum_{j=1}^{5} \sin j M \left\{ \sum_{k=0}^{5} \tilde{\zeta}_{k2} (\tilde{A}_x)_{[k]} + \sum_{k=1}^{5} \tilde{\zeta}_{k2} (\tilde{S}_x)_{[k]} \right\} + \sum_{j=0}^{5} \cos j M \left\{ \sum_{k=0}^{5} \tilde{\zeta}_{k2} (\tilde{A}_c)_{[k]} + \sum_{k=1}^{5} \tilde{\zeta}_{k2} (\tilde{S}_c)_{[k]} \right\} \right), \end{cases} \) (166)

\( S_{30} = \begin{cases} 0, \\ e^{-4i\phi_0} e^{-4i} k M \frac{8}{2} \sqrt{\frac{\pi}{7}} E^2 \mu (1 - 3\eta) \times \left( \sum_{j=1}^{3} \sin j M \left\{ \sum_{k=0}^{3} \tilde{\zeta}_{k0} (\tilde{A}_x)_{[k]} + \sum_{k=1}^{3} \tilde{\zeta}_{k0} (\tilde{S}_x)_{[k]} \right\} + \sum_{j=0}^{3} \cos j M \left\{ \sum_{k=0}^{3} \tilde{\zeta}_{k0} (\tilde{A}_c)_{[k]} + \sum_{k=1}^{3} \tilde{\zeta}_{k0} (\tilde{S}_c)_{[k]} \right\} \right), \end{cases} \) (168)

The Fourier transformation of the above gravitational waveforms is easily done. We take the Fourier transformation of the \( \sin j M \) and \( \cos j M \) terms,

\[ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-imkM} \sin j M e^{i\omega t} dt = i \sqrt{\frac{\pi}{2}} \delta(jn + kmn - \omega) - i \sqrt{\frac{\pi}{2}} \delta(jn - kmn + \omega), \] (171)

\[ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-imkM} \cos j M e^{i\omega t} dt = \sqrt{\frac{\pi}{2}} \delta(jn + kmn - \omega) + \sqrt{\frac{\pi}{2}} \delta(jn - kmn + \omega), \] (172)

and apply this to the sums over \( j \). How many terms of the infinite sum combinations are necessary to determine the multipoles up to a certain arbitrary order in the eccentricity of the binary system is discussed in detail in Appendix F. Note that in Eqs. (153) to (170), the expansion coefficients are not longer dependent on \( M \). They are functions of \( e_t, n \) and the “magnetic” number \( m \) and include multiple sums as we keep in mind the previous sections. In 1PN-accurate orbital dynamics, they are constants. Due to RR, they will become slowly varying functions of time. All the terms, where this condition is satisfied and which will appear together with an explicit phase factor, can be treated with the help of the SPA. All other ones which do not have a phase factor have to be treated individually.
VI. THE EFFECT OF RADIATION REACTION AND THE STATIONARY PHASE APPROXIMATION OF THE GW FIELD

A. Radiative dynamics

Having the conservative evolution under control, we can apply the Fourier decomposition to the radiative evolution. This is done by separating the full orbital evolution into two time scales: the orbital and the reactive timescale. The latter is – for our calculation – assumed to be much larger than the orbital scale. This is equivalent to the assumption that the rate of change of the orbital frequency is small measured over one orbit. The time dependence of the orbital frequency is closely related to the loss of energy and orbital angular momentum due to the GW emission. Peters and Mathews [13, 14] proposed a relatively simple model of the binary inspiral in the approximation of slow motion and weak gravitational interaction for arbitrary eccentricities $0 \leq e < 1$, which has been standard for some time. Blanchet, Damour, Iyer, and Thorne made endeavors to obtain higher-order corrections to the quadrupole formula [1, 10, 15, 16]. The far-zone fluxes of energy and angular momentum to $2pN$ order can be found in [2]. These corrections have shown up to be unrenounceable for the data analysis community. For the time being, we will restrict ourselves to the Peters-Mathews model for an exemplary calculation. Higher-order corrections can be included in a forthcoming publication.

In the conservative case, $n$ and $e_t$ were constants of motion, but when the orbit shrinks due to RR, both elements will follow coupled EOM connected to the loss of orbital energy and angular momentum [17, 18],

\[ \dot{n} = \mathcal{N}(n, e_t), \]
\[ \dot{e_t} = \mathcal{E}(n, e_t), \]  

symbolically. Note that above Eqs. hold after averaging over one orbit. Thus, $n$ will not simply be $M = n(t - t_0)$, but will satisfy

\[ M(t) = \int_{t_0}^{t} n(t') dt', \]
\[ \dot{M}(t) = n(t). \]

The Fourier-domain wave forms will not longer follow Eqs. [153] – [170] and will require some more considerations. We will apply the SPA at the first order to approximate the frequency-domain wave form first to each single frequency separately and will later sum up all the terms of the discrete decomposition as a virtue of the linearity of the Fourier integral with respect to the integrand.

B. SPA of the GW signal

Suppose an integral of the form

\[ \tilde{h}(f) = \int_{-\infty}^{+\infty} A(t) e^{i(2\pi ft - \phi(t))} dt. \]  

Then, having found a stationary point $t^*$, where the phase defined as $\Phi := 2\pi f - \phi(t)t$ has zero ascent, $\dot{\Phi}(t_0) = 0$, with the assumption of the right behavior at the boundary, the integral takes the form

\[ \tilde{h}(f) \approx \sqrt{2\pi} \frac{A(t^*)}{\sqrt{\dot{\phi}(t^*)}} e^{-i(\phi(t^*) - 2\pi ft^* + \pi/4)}. \]  

In the case of our GW, the integral turns out to be

\[ \tilde{h}(f)_{nm} = \frac{1}{\sqrt{2\pi}} e^{-i\phi(0)} \int_{-\infty}^{+\infty} \sum_{j=0}^{\infty} (S_j \sin jM + C_j \cos jM) e^{-imkM} e^{i2\pi ft} dt, \]  

\[ \tilde{h}(f)_{C,nm} = \frac{1}{\sqrt{2\pi}} e^{-i\phi(0)} \int_{-\infty}^{+\infty} C_2 \left(e^{ijM-imkM} + e^{-ijM-imkM} \right) e^{i2\pi ft} dt. \]
where the subscript “C” shall denote the function of cos $j\mathbf{M}$. The reader should carefully note that $C$ and $S$ are functions of the elapsed time, as the orbit decays due to RR. We assumed that the binary system evolves far away from the last stable orbit, such that the periastron advance parameter is much smaller than unity, $k \ll 1$. Thus, $nk \ll j$ with $j = 1, 2, 3, \ldots$. The case $j = 0$ will be discussed in one of the upcoming subsections. In the first part of Eq. (180), therefore, there exist no points of stationary phase for $j > 0$ and this term vanishes. The second term in Eq. (180), having the exponential argument $i(-j\mathbf{M}+2\pi ft-mk\mathbf{M})$ will contribute, since the phase $\Phi_{mj}$ defined as

$$\Phi_{mj} := 2\pi ft - j\mathbf{M} - mk\mathbf{M}$$

has a stationary point at $t^*$ where

$$\dot{\Phi}_{mj} = 2\pi f - jn - mkn = 0.$$  \hspace{1cm} (182)

Here we note that $k = \mathcal{O}(e^7)$ and will be consistently neglected. We want to compute $t^*$ and its contribution in the following lines. The question arises how to find out $t^*$ without solving Eqs. (173) and (174) numerically. It is answered in a simple manner. We search for the solution $n = n(e_t, e_{t0})$, $e_{t0}$ being the eccentricity at $t = t_0$, of

$$\frac{dn}{de_t} = \dot{n},$$  \hspace{1cm} (183)

insert it into Eq. (174),

$$\dot{e}_t = \mathcal{E}(n(e_t), e_t, e_{t0}),$$  \hspace{1cm} (184)

$$\Rightarrow e_t = e_t(e_{t0}, t - t_0),$$  \hspace{1cm} (185)

and invert Eq. (180) to find $(t - t_0)$ as a function of $e_t$ and $e_{t0}$, where for the stationary point the value $e^*_t$ corresponds to the solution of the equation

$$0 = 2\pi f - jn(e^*_t, e_{t0}) - mk(n(e^*_t, e_{t0}), e^*_t) n(e^*_t, e_{t0}).$$  \hspace{1cm} (186)

We see that $e^*_t$ and thus $t^*$ depends only on the magnetic number $m$, the summation index $j$, the eccentricity and mean motion at $t_0$, $e_{t0}$ and $n_0$, and of course the frequency $f$. The sin term of all those calculations will have the same stationary point $t^*$, such that the full Fourier-domain waveform will have the following appearance:

$$\begin{align*}
\tilde{h}^{(n)}(f)_{n,m>0} &= e^{i\frac{G}{R} \sum_{j=0}^\infty \frac{1}{2} \left(iS^*_{m(j>0)} + (1 + \delta_{0j})C^*_{mj}\right)} \left[\frac{1}{\sqrt{\tilde{\phi}_{mj}(t^*_m)}} e^{i(\phi_{mj}(t^*_m)-\pi/4)} \right] e^{-im\phi_0}, \hspace{1cm} (187) \\
\tilde{h}^{(n)}(f)_{n,m=0} &= e^{i\frac{G}{R} \sum_{j=1}^\infty \frac{1}{2} \left(iS^*_{m(j=0)} + C^*_{0j}\right)} \left[\frac{1}{\sqrt{\tilde{\phi}_{0j}(t^*_0)}} e^{i(\phi_{0j}(t^*_0)-\pi/4)} \right], \hspace{1cm} (188) \\
\tilde{\phi}_{mj}(t^*_m) &:= \dot{n}(j + mk)|_{t=t^*_m}, \hspace{1cm} (189) \\
S^*_{mj} &:= S_{mj}|_{t=t^*_m}, \hspace{1cm} (190) \\
C^*_{mj} &:= C_{mj}|_{t=t^*_m}. \hspace{1cm} (191)
\end{align*}$$

The addend $\delta_{0j}$ appears only at one special place in Eq. (187) because in Eq. (180) both terms, the first and the second, contribute to the cos terms for $j = 0$ and the sin terms cancel each other. For computing the energy $E$ which is a usual prefactor with some exponent in all the multipoles, use of Eq. (180) has to be made when the “stationary” $e_t$ and $n$ are determined. This approximation is justified as we simply use the leading-order SPA and is valid in a regime where the orbit evolves slowly towards coalescence. It is, thus, an approximation around the point $(n_0, e_{t0})$ with the aforementioned requirements.

These are the ideas so far. Here are the explicit equations up to 1pN in conservative and only to leading-order in radiative dynamics, where $n$ is unscaled (that means it has unit second$^{-1}$),
\[
\dot{n} = \mathcal{N}(n, e_t) = e^5 n^2 \eta \left(37e_t^4 + 292e_t^2 + 96\right) (Gm_t n)^{5/3} / 5 \left(1 - e_t^2\right)^{7/2}, \\
\dot{e}_t = \mathcal{E}(n, e_t) = -e^5 \eta e_t \left(121e_t^2 + 304\right) (Gm_t n)^{5/3} / 15 \left(1 - e_t^2\right)^{5/2}.
\]

(192)  

(193)

In terms of combined quantities, related formulas for the simple Peters and Matthews approach have been published by Pierro and Pinto [19] and Appell's 2-variable hypergeometric function \( \text{AppellF}_1 \) has come to use. We will keep our own expressions for \( n \) and \( e_t \) and will express unsealed elapsed times as functions of the latter. Note that by an appropriate scaling the factor \( Gm_t \) is re-absorbed in the time unit.

The solution to \( \frac{dn}{de_t} = (dn/dt)/(de_t/dt) \) reads

\[
n(n_0, e_t, e_{t0}) = n_0 \left(\frac{e_t^2 - 1}{e_{t0}^2 - 1}\right)^{3/2} \left(\frac{e_{t0}}{e_t}\right)^{18/19} \left(\frac{121e_{t0}^2 + 304}{121e_t^2 + 304}\right)^{1305/2299},
\]

(194)

with \( e_{t0} \) and \( n_0 \) as the value of \( e_t \) and \( n \) at the initial instant of time \( t_0 \), respectively. The elapsed time as a function of \( e_t \) and \( e_{t0} \) reads

\[
t - t_0 = e^{-5} \frac{9519^{1181/2299}}{2299^{8/3} \eta (Gm_t)^{5/3}} \times \\
\left\{ e_{t0}^{48/19} \text{AppellF}_1 \left[\frac{24}{19}, 3, 2; \frac{1181}{2299}, 19; e_{t0}^2, -\frac{121e_{t0}^2}{304}\right] \\
- e_t^{48/19} \text{AppellF}_1 \left[\frac{24}{19}, 3, 2; \frac{1181}{2299}, 19; e_t^2, -\frac{121e_t^2}{304}\right] \right\},
\]

(195)

\[
c_0 := e_{t0}^{18/19} \left(\frac{121e_{t0}^2 + 304}{121e_t^2 + 304}\right)^{1305/2299} / (1 - e_{t0}^2)^{3/2}.
\]

(196)

A check will show that the right hand side has the dimension of time.

C. Solution to the SPA condition equation, \( j \neq 0 \)

It showed up that it is easier to solve Eq. (180) for \( n \) instead of \( e_t \) and then to express \( e_t - e_{t0} \) in terms of \( n - n_0 \). Expressed fully in terms of \( n \), it reads

\[
0 = 2\pi f - j n^* - m k(n^*, e_t(n^*)) n^*.
\]

(197)

It is solved in two steps: first, solve the Newtonian and second: solve the 1pN equation with the help of step 1,

Step I:

\[
n_N^* = \frac{2\pi f}{j},
\]

(198)

Step II:

\[
n_{1pN}^* = \frac{2\pi f - m k(N^*, e_t(N^*)) n_N^*}{j}.
\]

(199)

Step II drops out in case \( m = 0 \). The Newtonian “stationary eccentricity” for the frequency \( f \), i.e. \( e_t(n_N^*) \), can be found numerically or with the help of a perturbative solution scheme. It is a rather numerical issue to apply fixpoint-method-like iterative algorithms a la Danby and Burkhards' method [20] to solve the Kepler equation, and a detailed analysis could indicate how many steps are necessary and reasonable towards the solution. Anyway, we promised to give analytical results and this will be done below. We are aware that there may exist better algorithms and refer the reader to the common literature of approximative solving methods. The function to be inverted for \( e_t - e_{t0} \) is the following,

\[
n(n_0, e_t, e_{t0}) = n_0 \left(\frac{e_t^2 - 1}{e_{t0}^2 - 1}\right)^{3/2} \left(\frac{e_{t0}}{e_t}\right)^{18/19} \left(\frac{121e_{t0}^2 + 304}{121e_t^2 + 304}\right)^{1305/2299}.
\]

(200)
we obtain

\[ g^{(p)} := \frac{1}{p!} \frac{\partial^p}{\partial t^p} n(n_0, e_t, e_{t0})|_{e_{t0}}, \]

having introduced some smallness parameter \( \tilde{\epsilon} \) which will be set 1 after the calculation. The solution algorithm reads (defining \( \kappa \) as the difference of \( e_t^p \) and \( e_{t0} \) and leaving out the “star”)

\[ \kappa^{[N]} := (e_t - e_{t0})^{[N]}, \]

\[ \kappa^{[1]} = \frac{n(n_0)}{g^{(1)}}, \]

\[ \kappa^{[2]} = \frac{1}{g^{(1)}} \left\{ (n(n_0) - \tilde{\epsilon} g^{(2)}(\kappa^{[1]}))^2 \right\}, \]

\[ \kappa^{[N]} = \frac{1}{g^{(1)}} \left\{ (n(n_0) - \sum_{p=2}^{N} \tilde{\epsilon}^{p-1} g^{(p)}(\kappa^{[N+1-p]})^p \right\}, \]

with some current solution order \([N]\). For convenience, we will give the first four orders of \( e_t - e_{t0} \) in terms of \( n - n_0 \). With the definitions

\[ f_1 := -121e_{t0}^4 - 183e_{t0}^2 + 304, \]

\[ f_2 := 37e_{t0}^4 + 292e_{t0}^2 + 96, \]

we obtain

\[ \kappa^{[4]} = -(n(n_0) \frac{f_1 e_{t0}}{3 f_2 n_0}) \]

\[-\tilde{\epsilon} (n(n_0))^2 \frac{f_1 e_{t0}}{18 f_2^2 n_0} \left\{ f_1 e_{t0} \right\} \]

\[-\tilde{\epsilon}^2 (n(n_0))^3 \left\{ f_1 e_{t0} \right\} \]

\[-\tilde{\epsilon}^3 (n(n_0))^4 \left\{ f_1 e_{t0} \right\} \]

\[ + 7597950000e_{t0}^{22} + 2119273605000e_{t0}^{20} - 1850 (96940 f_2 - 10211818689) e_{t0}^{20} + (459468074902815 - 62450686800 f_2) e_{t0}^{18} + 90 (135716 f_2^2 - 42030994737 f_2 - 75142160154162) e_{t0}^{16} + 8 (755339319 f_2^3 + 6763450991635 f_2 + 3192333977427390) e_{t0}^{14} - 3 (31474158721 f_2^4 + 56138837764240 f_2 + 8745121825435200) e_{t0}^{12} + 24 (7251390883 f_2^5 + 4696313592840 f_2 + 143898051631680) e_{t0}^{10} - 8 (35529727041 f_2^6 + 21737777348800 f_2 + 5492015496046080) e_{t0}^{8} - 384 (94956913 f_2^7 - 19783158080 f_2 - 34649027328480) e_{t0}^{6} + 30720 (2707169 f_2^8 + 277826496 f_2 - 683225807616) e_{t0}^4. \]
We have expansion of Eq. (210) in Burkhardt [20] to the fourth order. We need to have a nice initial guess for solution to this equation analytically, we find it convenient to consider the perturbation algorithm from Danby & Danby [19] of Eq. (11a) – (11c).

This should also note that the solution to Eq. (186) will introduce new 1pN correction terms to the multipole moments of Eqs. (195). This in turn can be inserted into Eq. (181) to get the value of the phase at the stationary point $t^*$. The reader should also note that the solution to Eq. (180) will introduce new 1pN correction terms to the multipole moments of Eqs. (11a) – (11c).

The stationary phase condition for the pure periastron-dependent terms (those with $j = 0$) reads

$$
\Phi = 2\pi f - m \nu
$$

$$
= 2\pi f - mn(e^*_t) \frac{3n(e^*_t)^{2/3}}{1 - (e^*_t)^2} = 0,
$$

$$
g(e_t) := 2\pi f - mn(e^*_t) \frac{3n(e^*_t)^{2/3}}{1 - (e^*_t)^2} = 0,
$$

with $n(e_t)$ taken from Eq. (194). Here, we proceed presenting all quantities expressed in terms of $e_t$. To find the solution to this equation analytically, we find it convenient to consider the perturbation algorithm from Danby & Burkhardt [20] to the fourth order. We need to have a nice initial guess for $e_t$, which we take from the first-order expansion of Eq. (210) in $e_t - e_{t0}$,

$$
e_t^{[0]} = \frac{\pi f}{3\pi n_0^{3/2}} \left( 242e_{t0}^7 + 124e_{t0}^5 - 974e_{t0}^3 + 608e_{t0} - 178e_{t0} - 669e_{t0}^3 - 784e_{t0}^5 \right)
$$

$$
\left( 3(19e_{t0}^4 - 284e_{t0}^2 - 160) \right),
$$

noting that the case $m = 0$ is excluded. This can be inserted into an iterative solution algorithm, which solves for $\delta$ in the expression

$$
e_t^* = e_t^{[0]} + \delta,
$$

$$
g(e_t^{[0]} + \delta) = 0.
$$

This $\delta$ is found with the help of the following procedure,

$$
\delta_1 = -\frac{g}{g'},
$$

$$
\delta_2 = -\frac{g}{g' + \frac{1}{2} \delta_1 g''},
$$

$$
\delta_3 = -\frac{g}{g' + \frac{1}{2} \delta_2 g'' + \frac{1}{2} \delta_1 g''},
$$

$$
\delta_4 = -\frac{g}{g' + \frac{1}{2} \delta_3 g'' + \frac{1}{2} \delta_2 g'' + \frac{1}{2} \delta_1 g''},
$$

$$
g^\prime := \frac{\partial}{\partial e_t} g(e_t).
$$

We have $e_t^{[4]} = e_t^{[0]} + \delta_4$ as the fourth-order solution to Eq. (210) with quintic convergence, and again extract $n(e_t^{[4]})$, $t - t_0$ and so on. The case $m = 0$ will be discussed below.

The case $j = 0$ and $m \neq 0$. Fourier transformation of a slow-in-time signal

In Eq. (177), there is no fast oscillating term but only a slow variable of time to be Fourier transformed,

$$
\tilde{h}(f) = \int_{-\infty}^{+\infty} A(t) e^{i(-2\pi ft)} dt.
$$
The term $A(t)$ depends on time only due to RR. Those terms are nontrivially dependent on time and have to be treated individually when they are requested analytically. In principle, one would have to express $|E|$, $n$ and $e_t$ as explicit functions of time. That would include inversion of Appell functions or perturbation theory.

However, for the case of inspiralling compact binaries, they will not be able to significantly contribute to frequencies in comparison to those with fast oscillating exponents as we compare typical time scales for one orbit and for the inspiral. We will therefore impose the following relation.

$$\tilde{h}(f)_{\text{static}} \ll \tilde{h}(f)_{\text{stationary}}, \quad (221)$$

where $\tilde{h}_{\text{static}}$ means all Fourier integrals over terms where the mean anomaly – or equivalently, the time – does not appear explicitly. We state that the $(j = 0, m = 0)$ Fourier domain terms almost vanish:

$$\tilde{h}(f)_{[j=0,m=0]} \approx 0, \quad (222)$$

for the frequency domain of interest for the regarded detector. LISA for example, will hardly see those terms operating near $n_0 \approx 0.001$Hz. Let us give an exemplary number to support this statement. The rate of change of the GW frequency over one year will be

$$\Delta f_{RR} \sim 1.6 \times 10^{-9} \left( \frac{m_t}{2.8 M_\odot} \right)^2 \left( \frac{\eta}{0.25} \right) \left( \frac{f_r}{10^{-4} \text{Hz}} \right)^{\frac{7}{4}} \left( 1 + \frac{73}{24} e_t^2 + \frac{37}{96} e_t^4 \right) \text{Hz}, \quad (223)$$

where $f_r$ is the radial frequency, given by $f_r = n(2\pi)^{-1}$. Let $n = 10^{-3}$, $m_1 = m_2 = 1.4 M_\odot$ and $e_t = 0.1$. Then, $\Delta f_{RR} \sim 2 \times 10^{-12}$Hz and the scaled energy loss is $\Delta E_{RR} \sim 3 \times 10^{-13}$.

F. The limit $e_t \to 0$: the quasi-circular case

The quasi-circular inspiral has been discussed extensively in the literature, especially in [6] which we have oftenly cited. For further information, see e.g. [21, 22]. In the limit $e_t \to 0$, all elements of our calculation simplify drastically. The following equations,

$$M \to u, \quad (224)$$

$$A(u) \to 1, \quad (225)$$

$$f_{lm}(u) \to F_{lm}(E), \quad (226)$$

$$g_{lm}(u) \to G_{lm}(E), \quad (227)$$

$$\phi - \phi_0 \to (1 + k)M, \quad (228)$$

$$e^{-im\phi} \to e^{-im\phi} e^{-im(1+k)M} \quad (229)$$

show that we have a simple prototype for the SPA for all the multipole moments. Eq. (224) is Kepler’s equation for quasi-circular orbits. The infinite summation series in Eq. (187) shrink to one single term, where the phase term has to be replaced by the one in Eq. (229). This is because the $\sin u$ terms will always have a factor $e_t$ and vanish in the case in question. The value of the phase and the angular velocity, the elapsed time and the resulting SPA integral can be taken one-to-one from [6].

G. Some concluding remarks for the eccentric inspiral templates

It is interesting to note how many parameters are included in the wave form. For the relatively simple case of circular inspiral, the templates used to have

- $t_c$, the time to coalescence,
- $\phi_c$, some phase instant,
- $M_c$, the “chirp mass”,
- $\eta$, the symmetric mass ratio (rather important for higher-order corrections to the RR effects), and
- $i$, the inclination angle of the orbital plane.
Because for eccentric orbits, both $e_t$ and $n$ will dictate the contribution to infinitely many frequencies already on the purely conservative level of EOM, both have to be regarded as parameters for the template. Thus, we have

- $\phi_0$,
- $m$, the total mass of the system,
- $\eta$, as before,
- $i$ as well,
- $n_0$ as the value of $n$ at $t_0$,
- $e_{t_0}$ as the value of $e_t$ at $t_0$.

Both $n_0$ and $e_{t_0}$ will contribute to the time to coalescence, see [19] for the value of what is called “lifetime”. The parameter space has grown by one dimension, but the good news is that, for data analysis considerations, the ambiguity function is still maximized in view of $\phi_0$ in a considerably simple way (see [23] how to do this).

VII. CONCLUSIONS AND OUTLOOK

In this article, we provided the far-zone GW form, including 1pN corrections to the orbital dynamics as well as to the amplitude. This was done by applying the 1pN accurate QKP to the conservative dynamics first and to decompose each term in Fourier modes and in a second step to solve the Fourier integrals, modified by the leading-order effect of radiation reaction, with the help of the SPA method. The GW field is given in terms of tensor spherical harmonics and all terms are given in a purely analytical form, at least as they are solved to some required order. The inclusion of the fully analytically solved Kepler equation in terms of Bessel functions implies that single and double infinite summations appear. It is up to the user to restrict those summations to finite ones, as far as the required accuracy demands a minimal $j_{\text{max}}$ and eccentricity expansion, due to the detector and other requirements in question.

It is interesting to note that this approximation scheme is easily applicable to the case of spinning compact binaries with aligned spins and orbital angular momentum, including the leading-order spin-orbit interaction [24]. For non-aligned spins, the calculation of the Fourier domain will be structured more complicatedly, since the precession of the orbital plane will introduce another typical frequency in addition to the orbital, the periastron precession and the RR time scale frequencies.

For a future publication it is intended to include higher-order RR terms to the “quadrupolar” contribution and, as it is highly demanded, to include the 2pN and 3pN point particle Hamiltonians into the dynamics.

VIII. ACKNOWLEDGMENTS

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Appendix A: Proof of some summation formulas

In this appendix, we like to provide some proves of formulas we only listed in the previous sections. Throughout the remaining sections the eccentricity $e_t$ we are using is simply called $e$. Let us start with the inverse scaled relative separation with some arbitrary positive integer exponent $n$. First, we perform a Taylor series expansion in $e$,

$$\frac{1}{(1 - e \cos u)^n} = 1 + \sum_{m=1}^{\infty} \frac{(n + m - 1)!}{(n - 1)!} \frac{e^m}{m!} \cos^m u$$

$$= 1 + \sum_{m=1}^{\infty} \frac{(n + m - 1)!}{(n - 1)!} \frac{e^m}{m!} \left( \frac{1}{2^m} \sum_{l=0}^{m} \binom{m}{l} \cos(u[m - 2l]) \right),$$

and list the “factorial” function of the integer number $n$ as

$$\frac{(n + m - 1)!}{(n - 1)!} \equiv \prod_{k=1}^{m} (n + k - 1).$$
To optically simplify this equation, we summarize the terms before \( \cos \sim u \) with \( \beta_{m,k}^{(n)} \) as follows,

\[
\beta_{m,k}^{(n)} := \frac{(n + m - 1)!}{(n - 1)!} \frac{1}{m! \frac{e^m}{m} \binom{m}{k}},
\]

(A3)

and write the sum with this definition:

\[
\frac{1}{(1 - e \cos u)^n} = 1 + \sum_{m=1}^{\infty} \sum_{k=0}^{m} \beta_{m,k}^{(n)} \cos([m - 2k]u).
\]

(A4)

To further markedly reduce the complexity of this double sum, it is the task to find out which pairs of \((m, k)\) lead to the same frequency \(ju\) and which \(\beta_{m,k}^{(n)}\) have to be added to this frequency contribution:

\[
|m - 2k| = j,
\]

(A5)

\[
\Rightarrow m_1 = 2k + j,
\]

(A6)

\[
\Rightarrow m_2 = 2k - j.
\]

(A7)

A small table for the \(\cos\) function argument may help (note that \(k \leq m\) always holds):

| \(m : k\) | 0 | 1 | 2 | 3 | 4 | 5 |
|-------|---|---|---|---|---|---|
| 1 | 1-1 |
| 2 | 2 | 0-2 |
| 3 | 3 | 1-1-3 |
| 4 | 4 | 2 | 0-2-4 |
| 5 | 5 | 3 | 1-1-3-5 |

The zero mode \(j = 0\) always appears at even numbers \(m\), \(m = 2i\) with \(i = (1, 2, ...)\), and \(m - 2k = 2i - 2k = 0\) is satisfied by \(k = i\). Thus, we choose

\[
b_{0}^{(n)} = \sum_{i=1}^{\infty} \beta_{2i,i}^{(n)}.
\]

(A8)

The other frequencies \(j > 0\), appearing at \(m = j + 2i\) with \(i = (0, 1, 2, ...)\) lead to \(k = i\) as well as \(k = i + j\), such that

\[
b_{j}^{(n)} = \sum_{i=0}^{\infty} \beta_{j+2i,i}^{(n)} + \beta_{j+2i,j+i}^{(n)}.
\]

(A9)

Summarizing the elements for the first term, we have

\[
\frac{1}{(1 - e \cos u)^n} = 1 + b_{0}^{(n)} + \sum_{j=1}^{\infty} b_{j}^{(n)} \cos ju
\]

(A10)

with the \(b_{j}^{(n)}\) defined in Eqs. (A8) and (A9). Having this at hands, it is easy to compute the associated decomposition of

\[
\frac{\sin u}{(1 - e \cos u)^n} = \left(1 + b_{0}^{(n)} + \sum_{j=1}^{\infty} b_{j}^{(n)} \cos ju\right) \sin u
\]

\[
= (1 + b_{0}^{(n)}) \sin u + \frac{1}{2} \sum_{j=1}^{\infty} b_{j}^{(n)} (\sin[(1 - j)u] + \sin[(1 + j)u]).
\]

(A11)

This decomposition demands collecting the terms having the same frequency as well. Having \(m\) times \(u\) in the sin argument, following \(j\) will lead to \(\sin(mu)\) and \(-\sin(mu)\) for \(\sin(1 - j)u\) in the first two lines and for \(\sin(1 + j)u\) in the third:

\[
m = (1 - j) \rightarrow j = 1 - m,
\]

(A12)

\[
-m = (1 - j) \rightarrow j = 1 + m.
\]

(A13)
$m = (1 + j) \rightarrow j = m - 1$. \hfill (A14)

Again, a small table may help:

| $j$ | $1$ | $2$ | $3$ | $4$ |
|-----|-----|-----|-----|-----|
| $1 + j$ | $2$ | $3$ | $4$ | $5$ |
| $1 - j$ | $0$ | $-1$ | $-2$ | $-3$ |

We clearly see that the first harmonic in this sum is only realized by $j = 2$ for the $\sin(1 - j)u$ term, whereas $j = 0$ will not contribute. Thus,

$$\sum_{j=1}^{\infty} b_j^{(n)} (\sin[(1 - j)u] + \sin[(1 + j)u]) = \sum_{m=2}^{\infty} b_{m-1}^{(n)} \sin mu - \sum_{m=2}^{\infty} b_{m+1}^{(n)} \sin mu - b_2^{(n)} \sin u. \hfill (A15)$$

Summarizing these terms, we write the lhs of Eq. (A11) as a simple sum.

$$\sin u \over (1 - e \cos u)^n = \sum_{j=1}^{\infty} S_j^{(n)} \sin j u, \hfill (A16)$$

$$S_1^{(n)} := 1 + b_0^{(n)} - {1 \over 2} b_2^{(n)}, \hfill (A17)$$

$$S_{j>1}^{(n)} := \frac{1}{2} \left( b_{j+1}^{(n)} - b_{j-1}^{(n)} \right). \hfill (A18)$$

Appendix B: Fourier representation of the $\sin mu$ and $\cos mu$ terms

In the previous appendix, we have proven some formulas for terms appearing in the multipole expansion of the far-zone gravitational field. Now, we have to express simple series in $\sin mu$ and $\cos mu$ as trigonometric functions in $ml$. Using Eq. (18), we have

$$\frac{1}{(1 - e \cos u)^n} = 1 + b_0^{(n)} + \sum_{m=1}^{\infty} b_m^{(n)} \cos mu = 1 + b_0^{(n)} + \sum_{m=1}^{\infty} b_m^{(n)} \left( \sum_{j=1}^{\infty} \tilde{\gamma}_j^{(m)} \cos j l \right)$$

$$= 1 + b_0^{(n)} + \sum_{j=1}^{\infty} \left( \sum_{m=1}^{\infty} \tilde{\gamma}_j^{(m)} b_m^{(n)} \right) \cos j l$$

$$= \mathcal{A}_0^{(n)} + \sum_{j=1}^{\infty} \mathcal{A}_j^{(n)} \cos j l, \hfill (B1)$$

$$\mathcal{A}_0^{(n)} := 1 + b_0^{(n)}, \hfill (B2)$$

$$\mathcal{A}_j^{(n)} := \left( \sum_{m=1}^{\infty} \tilde{\gamma}_j^{(m)} b_m^{(n)} \right). \hfill (B3)$$

Similarly, getting help from Eq. (16), we obtain

$$\frac{\sin u}{(1 - e \cos u)^n} = \sum_{m=1}^{\infty} S_m^{(n)} \sin mu = \sum_{m=1}^{\infty} S_m^{(n)} \left( \sum_{j=1}^{\infty} \tilde{\sigma}_j^{(m)} \sin j l \right) = \sum_{j=1}^{\infty} \left( \sum_{m=1}^{\infty} \tilde{\sigma}_j^{(m)} S_m^{(n)} \right) \cos j l$$

$$= \sum_{j=1}^{\infty} S_j^{(n)} \sin j l, \hfill (B4)$$

$$S_{j>0}^{(n)} := \left( \sum_{m=1}^{\infty} \tilde{\sigma}_j^{(m)} S_m^{(n)} \right). \hfill (B5)$$
Appendix C: Fourier representation of products of two sin series

In section IV we provided a simple series representation of the term

\[
\left( \sum_{k=1}^{\infty} S_k^{(n)} \sin k M \right) \left( \sum_{m=1}^{\infty} G_m \sin m M \right) = \sum_{k=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{2} S_k^{(n)} G_m \left( \cos[k - m] M - \cos[k + m] M \right)
\]

\[
= \sum_{j=0}^{\infty} P_{j}^{SS, [n]} \cos j M.
\] (C1)

Here we want to explain how the coefficients \( P_{j}^{SS, [n]} \) come up. First, we take a look at \( \cos[k - m] M \). The zero mode appears only at \( k = m \). Thus, the absolute static part will be given by

\[
P_{0}^{SS, [n]} := \frac{1}{2} \sum_{m=1}^{\infty} S_m^{(n)} G_m.
\] (C2)

The higher harmonics \( j > 0 \) will be realized by \( k - m = j \) and \( k - m = -j \) and these equations solved for \( m \) give \( m = k - j \) (which only nonzero for \( k > j \)) and \( m = k + j \). The first part, \( \cos[k - m] M \), will give

\[
[1] P_{j>0}^{SS, [n]} = \frac{1}{2} \sum_{k=1}^{\infty} \left( S_k^{(n)} G_{k-j} + S_k^{(n)} G_{k+j} \right).
\] (C3)

The second part, \( -\cos([k + m] M) \), gives \( -\cos j M \) for \( j = k + m \), viz. \( m = j - k \), such that it is equal to

\[
[2] P_{j>1}^{SS, [n]} = -\frac{1}{2} \sum_{k=1}^{j-1} \left( S_k^{(n)} G_{j-k} \right),
\] (C4)

together giving

\[
P_{j>0}^{SS, [n]} = [1] P_{j>0}^{SS, [n]} + [2] P_{j>1}^{SS, [n]} = \frac{1}{2} \sum_{k=1}^{\infty} \left( S_k^{(n)} G_{k-j} + S_k^{(n)} G_{k+j} \right) - \frac{1}{2} \sum_{k=1}^{j-1} \left( S_k^{(n)} G_{j-k} \right).
\] (C5)

Additionally, we will provide the tables for part 1 and 2.

| k:m | 1 | 2 | 3 | 4 |
|-----|---|---|---|---|
| 1   | 0 | -1| -2| -3|
| 2   | 1 | 0 | -1| -2|
| 3   | 2 | 1 | 0 | -1|
| 4   | 3 | 2 | 1 | 0 |

Numbers for part 1

| k:m | 1 | 2 | 3 | 4 |
|-----|---|---|---|---|
| 1   | 0 | 2 | 3 | 4 |
| 2   | 1 | 3 | 4 | 5 |
| 3   | 2 | 4 | 5 | 6 |
| 4   | 3 | 5 | 6 | 7 |

Numbers for part 2
Appendix D: Fourier representation of products of a sin and a cos series

What we did in the previous appendix is also necessary for the following term,
\[
\left( \sum_{k=1}^{\infty} A_k^{(n)} \cos kM \right) \left( \sum_{k=1}^{\infty} G_k \sin kM \right) = \frac{1}{2} \sum_{k=1}^{\infty} \sum_{m=1}^{\infty} (\sin[k + m]M - \sin[k - m]M) = \sum_{j=1}^{\infty} P_{j}^{CS,[n]} \sin jM.
\]

(D1)

Part 1, \(\sin[k + m]l\) will contribute to frequencies \(j = k + m\) for \(k, m > 0\). We already know the result from part 2 of the last section.

\[
|1| P_{j > 1}^{CS,[n]} = \sum_{k=1}^{j-1} A_k^{(n)} G_{j-k}
\]

(D2)

Part 2, \(- \sin[k - m]l\) will also have positive and negative contributions, for \(m = k - j\) (implying \(k > j\)) \(m = k + j\), giving

\[
|2| P_{j > 1}^{CS,[n]} = -\frac{1}{2} \sum_{k=1}^{\infty} \left( A_k^{(n)} G_{k-j} - A_k^{(n)} G_{k+j} \right)
\]

(D3)

Summation of part 1 and 2 yields

\[
P_{j}^{CS,[n]} := \frac{1}{2} \left\{ \sum_{k=1}^{j-1} \left( A_k^{(n)} G_{j-k} \right) - \sum_{k=1}^{\infty} \left( \underbrace{A_k^{(n)} G_{k-j} - A_k^{(n)} G_{k+j}}_{\text{for } k > j} \right) \right\}.
\]

(D4)

Appendix E: Appell’s integral formula in the solution for the elapsed time

The integral in Eq. (195) in section VI can be solved with the help of the following integral representation of the AppellF1 function (see http://dlmf.nist.gov/ or [25] for further information).

\[
\int_0^1 du \frac{u^{\alpha-1}(1-u)^{\gamma-\alpha-1}}{(1-ux)^{\beta_1}(1-uy)^{\beta_2}} = \frac{\Gamma(\alpha)\Gamma(\gamma-\alpha)}{\Gamma(\gamma)} \text{AppellF}_1(\alpha; \beta_1, \beta_2; \gamma; x, y).
\]

(E1)

Appendix F: Accuracy of finite sums

In Sections III and IV we provided decompositions of functions of some “elementary” type in terms of \(u\) which contain infinite summations. Naturally, for numerics it is important to know how many terms are needed to reach some desired accuracy. Considering compact binaries with small eccentricities only, one is allowed to expand the elementary expressions in powers of \(e\) and then to look how many terms are needed for the error to be shifted to \(\mathcal{O}(e^{M+1})\) with some finite \(M\).

1. Accuracy of basic elements

We start with the basic definitions. The upper limits of the summation has to give a term of order \(\mathcal{O}(e^M)\), thus the individual limit has to be matched appropriately,

\[
\beta^{(n)}_{m,k} = \frac{(n + m - 1)!}{(n - 1)!} e^m \frac{1}{m!} \binom{m}{k} = \mathcal{O}(e^m),
\]

(F1)
\[ b_0^{(n)} = 1 + \sum_{i=1}^{\infty} \beta_{2i,i}^{(n)} = 1 + \sum_{i=0}^{M/2} \beta_{2i,i}^{(n)} + \mathcal{O}(e^{M+1}), \]  

(F2)

\[ b_j^{(n)}_{j>0} = \sum_{i=0}^{\infty} \left( \beta_{j+2i,i}^{(n)} + \beta_{j+2i,j+i}^{(n)} \right) = \frac{(M-j)/2}{M} \sum_{i=0}^{M/2} \left( \beta_{2i,i}^{(n)} + \beta_{j+2i,j+i}^{(n)} \right) + \mathcal{O}(e^{M+1}), \]  

(F3)

\[ \Rightarrow b_j^{(n)}_{j>0} = \mathcal{O}(e^j), \]  

(F4)

\[ A(u)^{-n} = \sum_{j=0}^{M} b_j^{(n)} \cos ju + \mathcal{O}(e^{M+1}). \]  

(F5)

In the last line we have used that the summation in Eq. (F3) starts with \( i = 0 \) and leaves no term if \( j > M \). The same quantity with \( \sin u \) will also be truncated in the \( u \) domain, keeping in mind the definitions (F5) and (F6) and their dependency on the summation index \( j \),

\[
\frac{\sin u}{A(u)^n} = \sum_{j=1}^{\infty} S_j^{(n)} \sin ju,
\]  

(F6)

\[
S_1^{(n)} = \left( 1 + b_0^{(n)} \right) - \frac{1}{2} b_2^{(n)} = \left( 1 + \sum_{i=1}^{M/2} \beta_{2i,i}^{(n)} \right) - \frac{1}{2} \sum_{i=0}^{(M-2)/2} \left( \beta_{2i,i}^{(n)} + \beta_{j+2i,j+i}^{(n)} \right) + \mathcal{O}(e^{M+1}),
\]  

(F7)

\[
S_j^{(n)}_{j>1} = \frac{1}{2} \left( \sum_{i=0}^{(M-j+1)/2} \left( \beta_{j+1+2i,i}^{(n)} + \beta_{j-1+2i,j-1+i}^{(n)} \right) - \sum_{i=0}^{(M-j-1)/2} \left( \beta_{j+1+2i,i}^{(n)} + \beta_{j+1+2i,j+1+i}^{(n)} \right) \right) + \mathcal{O}(e^{M+1}),
\]  

(F8)

\[
\Rightarrow S_j^{(n)}_{j>1} = \mathcal{O}(e^{j-1}) + \mathcal{O}(e^{j+1}) = \mathcal{O}(e^j),
\]  

(F9)

\[
\frac{\sin u}{A(u)^n} = \sum_{j=1}^{M+1} S_j^{(n)} \sin ju + \mathcal{O}(e^{M+1}).
\]  

(F10)

For the Fourier representation, we remember Eq. (F7) and take the expansion of the Bessel coefficients (12),

\[
J_n(x) = x^n \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{2^{2k+n} k! (k + n)!},
\]  

(F11)

for the determination of the limit for our finite sums,

\[
\hat{\gamma}_j^m = \frac{m}{j} (J_{j-m}(je) - J_{j+m}(je)) = \mathcal{O}(e^{j-m}),
\]  

(F12)

\[
\Rightarrow \cos mu = \sum_{j=1}^{M+m} \hat{\gamma}_j^m \cos jM = \mathcal{O}(e^{M+1}),
\]  

(F13)

\[
A_j^{(n)} = \sum_{m=1}^{M} \gamma_j^m b_m^{(n)} = \mathcal{O}(e^{j-m}) \mathcal{O}(e^m) = \mathcal{O}(e^j),
\]  

(F14)

\[
A(u)^{-n} = (1 + b_0^{(n)}) + \sum_{j=1}^{M} A_j^{(n)} \cos jM + \mathcal{O}(e^{M+1}).
\]  

(F15)

In the third line above we used Eq. (F5) to truncate the number \( j \) in \( \cos ju \). For the other relevant term with \( \sin u \) we consider

\[
\tilde{\gamma}_j^m = \frac{m}{j} (J_{j-m}(je) + J_{j+m}(je)) = \mathcal{O}(e^{j-m}),
\]  

(F16)

\[
\Rightarrow \sin mu = \sum_{j=1}^{M+m} \tilde{\gamma}_j^m \sin jM + \mathcal{O}(e^{M+1}),
\]  

(F17)
The product of two sin series can be truncated taking into account the individual series’ terms. In the following we will use the fact that those coefficients could already be truncated to some finite order see Eqs. (F14) and (F18), and that they are at least of some order of $j$

In the third line again, we used (F9) for the index $v$ from Eq. (F24). With the regular solution to Eq. (24) at $e = 0$,

$$
\alpha = \frac{1 - \sqrt{1 - e^2}}{e} = \mathcal{O}(e^1),
$$

their order is calculated to be

$$
G_m(e) = \frac{2}{m} J_m(me) + \sum_{s=1}^{\infty} \alpha^s [J_{m-s}(me) - J_{m+s}(me)]
$$

$$
= \mathcal{O}(e^m) + \sum_{s=1}^{\infty} \mathcal{O}(e^s) [\mathcal{O}(e^{m-s}) - \mathcal{O}(e^{m+s})]
$$

$$
= \mathcal{O}(e^m).
$$

2. Accuracy of finite sum products

The product of two sin series can be truncated taking into account the individual series’ terms. In the following we will use the fact that those coefficients could already be truncated to some finite order $M$ in the previous subsection, see Eqs. (F14) and (F18), and that they are at least of some order of $e$ themselves,

$$
\left( \sum_{k=1}^{\infty} S_k^{(n)} \sin kM \right) \left( \sum_{m=1}^{\infty} G_m \sin mM \right) = \sum_{j=0}^{\infty} P_j^{SS,[n]} \cos jM.
$$

For clarity, we again write down the definitions and make use of the elaborated orders, where some abuse of notation is made,

$$
P_0^{SS,[n]} := \frac{1}{2} \sum_{k=1}^{\infty} S_k^{(n)} G_k,
$$

$$
A_k^{(n)} = \mathcal{O}(e^k),
$$

$$
S_k^{(n)} = \mathcal{O}(e^{k-1}),
$$

$$
G_k = \mathcal{O}(e^k),
$$

$$
\Rightarrow P_0^{SS,[n]} = \frac{1}{2} \sum_{k=1}^{\infty} \mathcal{O}(e^{k-1}) \mathcal{O}(e^k) = \frac{1}{2} \sum_{k=1}^{\infty} \mathcal{O}(e^{2k-1}) = \frac{1}{2} \sum_{k=1}^{M+1} S_k^{(n)} G_k + \mathcal{O}(e^{M+1}) = \mathcal{O}(e^1),
$$

$$
P_j^{SS,[n]} := \frac{1}{2} \left\{ \sum_{k=1}^{\infty} \left( S_k^{(n)} G_{k+j} + S_k^{(n)} G_{k-j} \right) \right\}
$$

$$
= \frac{1}{2} \left\{ \sum_{k=1}^{\infty} \mathcal{O}(e^{k-1}) \mathcal{O}(e^{k+j}) + \sum_{k=j+1}^{\infty} \mathcal{O}(e^{k-1}) \mathcal{O}(e^{k-j}) + \sum_{k=1}^{j-1} \mathcal{O}(e^{k-1}) \mathcal{O}(e^{j-k}) \right\}
$$

$$
= \frac{1}{2} \left\{ \sum_{k=1}^{\infty} \mathcal{O}(e^{2k+j-1}) + \sum_{k=j+1}^{\infty} \mathcal{O}(e^{2k-j-1}) - \sum_{k=1}^{j-1} \mathcal{O}(e^{j-1}) \right\}
$$
Because of Eqs. (F28) and (F30), one can truncate the summations over the index \( j \) for

1. each \( A^{(n)}_j \) term at \( j M = M \),
2. each \( S^{(n)}_j \) term at \( j M = M + 1 \),
3. each \( P^{SS,[n]}_j \) term at \( j M = M + 1 \), and
4. each \( P^{CS,[n]}_j \) term at \( j M = M \).

in Eqs. (153) to (170).

\[
\begin{align*}
&\frac{1}{2} \left\{ \sum_{k=1}^{M+1} A^{(n)}_k G_{k+1} + \sum_{k=j+1}^{M+1} S^{(n)}_k G_{k-j} - \sum_{k=1}^{j-1} S^{(n)}_k G_{j-k} \right\} + O(e^{M+1}) \\
&= O(e^{j-1}).
\end{align*}
\]  

The individual final indices are evaluated when the maximal exponent of \( e \) reaches \( M \). The last line is evaluated when one takes the smallest index \( k \). The other product of \( \cos \) and \( \sin \) series is the following,

\[
\left( \sum_{k=1}^{\infty} A^{(n)}_k \cos kM \right) \left( \sum_{m=1}^{\infty} G_m \sin mM \right) = \sum_{j=1}^{\infty} P^{CS,[n]}_j \sin jM.
\]  

The definitions are worked through immediately, again with some minor abuse of notation,

\[
P^{CS,[n]}_j := \frac{1}{2} \left\{ \sum_{k=1}^{j-1} A^{(n)}_k G_{j-k} - \sum_{k=j+1}^{\infty} A^{(n)}_k G_{k-j} + \sum_{k=1}^{\infty} A^{(n)}_k G_{k+j} \right\}
\]

\[
= \frac{1}{2} \left\{ \sum_{k=1}^{j-1} O(e^k)O(e^{-k}) - \sum_{k=j+1}^{\infty} O(e^k)O(e^{-k}) + \sum_{k=1}^{\infty} O(e^k)O(e^{k+j}) \right\}
\]

\[
= \frac{1}{2} \left\{ \sum_{k=1}^{j-1} O(e^k) - \sum_{k=j+1}^{\infty} O(e^{2k-j}) + \sum_{k=1}^{\infty} O(e^{2k+j}) \right\}
\]

\[
= \frac{1}{2} \left\{ \sum_{k=1}^{j-1} A^{(n)}_k G_{j-k} - \sum_{k=j+1}^{\infty} A^{(n)}_k G_{k-j} + \sum_{k=1}^{\infty} A^{(n)}_k G_{k+j} \right\}
\]

\[
= O(e^j).
\]  

Because of Eqs. (F28) and (F30), one can truncate the summations over the index \( j \) for

1. each \( A^{(n)}_j \) term at \( j M = M \),
2. each \( S^{(n)}_j \) term at \( j M = M + 1 \),
3. each \( P^{SS,[n]}_j \) term at \( j M = M + 1 \), and
4. each \( P^{CS,[n]}_j \) term at \( j M = M \).

in Eqs. (153) to (170).

[1] L. Blanchet and T. Damour, Phil. Trans. R. Soc. A 320, 379 (1986).
[2] L. Blanchet, T. Damour, and B. R. Iyer, Phys. Rev. D 51 (1995), gr-qc/9501029.
[3] L. Blanchet, G. Faye, B. R. Iyer, and B. Joguet, Phys. Rev. D 65, 061501(R) (2002), gr-qc/0105099.
[4] L. Blanchet, G. Faye, B. R. Iyer, and B. Joguet, Phys. Rev. D 71, 129902(E) (2005).
[5] D. V. Gal’tsov, A. A. Matiukhin, and V. I. Petukhov, Phys. Lett. A 77, 387 (1980).
[6] N. Yunes, K. G. Arun, E. Berti, and C. M. Will, Phys. Rev. D 80, 084001 (2009), 0906.0313.
[7] M. Tessmer and A. Gopakumar, Mon. Not. R. Astron. Soc. 374, 721 (2007), gr-qc/0610139.
[8] R.-M. Memmesheimer, A. Gopakumar, and G. Schäfer, Phys. Rev. D 70, 104011 (2004), gr-qc/0407049.
[9] W. Junker and G. Schäfer, Mon. Not. R. Astron. Soc. 254, 146 (1992).
[10] K. S. Thorne, Rev. Mod. Phys. 52, 299 (1980).
[11] G. N. Watson, A treatise on the theory of Bessel functions (Cambridge University Press, Cambridge, 1980), 2nd ed., ISBN 0-521-09382-1.
[12] P. Colwell, Solving Kepler’s equation over three centuries (Willman-Bell, Inc., Richmond, VA 23235, 1993), ISBN 0-943964-09-9.
[13] P. C. Peters and J. Mathews, Phys. Rev. 131, 435 (1963).
[14] P. C. Peters, Phys. Rev. 136, B1224 (1964).
[15] L. Blanchet, Proc. R. Soc. A 409, 383 (1987).
[16] T. Damour and B. R. Iyer, Phys. Rev. D 43, 3259 (1991).
[17] T. Damour, A. Gopakumar, and B. R. Iyer, Phys. Rev. D 70, 064028 (2004), gr-qc/0404128.
[18] C. Königsdörffer and A. Gopakumar, Phys. Rev. D 73, 124012 (2006), gr-qc/0603056.
[19] V. Pierro and I. M. Pinto, Nuovo Cim. B 111, 631 (1996).
[20] J. M. A. Danby and T. M. Burkhardt, Celestial Mechanics 31, 95 (1983).
[21] K. G. Arun, B. R. Iyer, B. S. Sathyaprakash, and P. A. Sundararajan, Phys. Rev. D 71, 084008 (2005), gr-qc/0411146.
[22] T. Damour, B. R. Iyer, and B. S. Sathyaprakash, Phys. Rev. D 62, 084036 (2000), gr-qc/0001023.
[23] K. Martel and E. Poisson, Phys. Rev. D 60, 124008 (1999), gr-qc/9907006.
[24] M. Tessmer, J. Hartung, and G. Schäfer (2010), 1003.2735.
[25] F. W. J. Olver, D. W. Lozier, R. F. Boisvert, and C. W. Clark, NIST Handbook of Mathematical Functions (Cambridge University Press, Cambridge, 2010), 1st ed., ISBN 9780521140638.