Structural characterization of Cayley graphs

Didier Caucal

1 CNRS, LIGM, University Paris-Est, France
didier.caucal@univ-mlv.fr

Abstract

We show that the directed labelled Cayley graphs coincide with the rooted deterministic vertex-transitive simple graphs. The Cayley graphs are also the strongly connected deterministic simple graphs of which all vertices have the same cycle language, or just the same elementary cycle language. Under the assumption of the axiom of choice, we characterize the Cayley graphs for all group subsets as the deterministic, co-deterministic, vertex-transitive simple graphs.

1 Introduction

A group is a basic algebraic structure that comes from the study of polynomial equations by Galois in 1830. To describe the structure of a group, Cayley introduced in 1878 [1] the concept of graph for any group \( G \) according to any generating subset \( H \). This is simply the set of labelled oriented edges \( g \xrightarrow{h} gh \) for every \( g \) of \( G \) and \( h \) of \( H \). Such a graph, called Cayley graph (or Cayley diagram), is directed and labelled in \( H \) (or an encoding of \( H \) by symbols called letters or colors). A characterization of unlabelled and undirected Cayley graphs was given by Sabidussi in 1958 [6]. These are the connected graphs whose automorphism group has a subgroup with a free and transitive action on the graph. So if one wants to know whether an unlabelled and undirected graph is a Cayley graph, we must know if we can extract a subgroup of the automorphism group that allows to define a free and transitive action on the graph. To better understand the structure of Cayley graphs, it is pertinent to look for characterizations by simple graph-theoretic conditions.

This approach was clearly stated by Hamkins in 2010: Which graphs are Cayley graphs? Every Cayley graph is a graph with high symmetry: it is vertex-transitive meaning that the action of its automorphism group is transitive, or equivalently that all its vertices are isomorphic meaning that we ‘see’ the same structure regardless of the vertex where we ‘look’. We can characterize the Cayley graphs as vertex-transitive graphs. By definition, any Cayley graph satisfies three basic graph properties. First is is simple: there are no two arcs of the same source and goal. It is also deterministic: there are no two arcs of the same source and label. Finally the identity element is a root. In this article, we show that these three conditions added to the condition of being vertex-transitive characterize exactly the Cayley graphs. This improves Sabidussi’s characterization that easily can be adapted to directed and labelled graphs: the Cayley graphs are the deterministic rooted simple graphs whose automorphism group has a subgroup with a free and transitive action on the graph. In other words, we reduce this last condition to the fact that the graph is vertex-transitive. For such a simplification, the key result is that every strongly connected, deterministic and co-deterministic graph is isomorphic to the canonical graph of any of its cycle languages. This is a fairly standard result in automata theory.

Precisely an automaton is just a directed labelled graph (finite or not) with input and output vertices. It recognizes the language of the labels of paths from an input to an output vertex. Any language \( L \) is recognized by its canonical automaton, namely the automaton whose graph is the set of transitions between the (left) residuals of \( L \), having \( L \) as its unique initial vertex, and the final vertices are the residuals of \( L \) containing the empty word. We minimize
an automaton by identifying its bisimilar vertices. Any minimal deterministic and reduced automaton is isomorphic to the canonical automaton of its recognized language. Moreover, any co-deterministic and reduced automaton is minimal. The previous key result follows from these last two properties: any deterministic and co-deterministic reduced automaton is isomorphic to the canonical automaton of its recognized language.

An equivalent characterization of Cayley graphs is obtained by strengthening the condition of being rooted by the strong connectivity, and simplify the condition of being vertex-transitive by the fact that all vertices have the same elementary cycle language. Finally, we consider the extension of Cayley graphs for all non-empty subsets of groups. Under the assumption of the axiom of choice, we show that these graphs are exactly the deterministic, co-deterministic, vertex-transitive simple graphs.

2 Automata

An automaton is just a directed labelled graph with input and output vertices. An accepting path is a path from an initial vertex to a final vertex. An automaton recognizes the language of accepting path labels. We recall the notions of minimal automaton of an automaton, and the notion of canonical automaton of a language. For any deterministic and reduced automaton, its minimal automaton is isomorphic to the canonical automaton of its recognized language. Moreover, any co-deterministic and co-accessible automaton is minimal. It follows that every strongly connected deterministic and co-deterministic graph is isomorphic to the canonical graph of the path language between any two vertices.

2.1 Definitions

We recall basic definitions for directed labelled graphs and automata.

Let \( A \) be an arbitrary (finite or infinite) set of symbols. We consider a graph as a set of directed edges labelled in \( A \). A directed A-graph \((V,G)\) is defined by a set \( V \) of vertices and a subset \( G \subseteq V \times A \times V \) of edges. Any edge \((s,a,t) \in G\) is from the source \( s \) to the goal \( t \) with label \( a \), and is also written by the transition \( s \xrightarrow{a} G t \) or directly \( s \xrightarrow{a} t \) if \( G \) is clear from the context. The sources and goals of edges form the set \( V_G \) of non-isolated vertices of \( G \) and we denote by \( A_G \) the set of edge labels:

\[
V_G = \{ s \mid \exists a, t \ (s \xrightarrow{a} t \lor t \xrightarrow{a} s) \} \quad \text{and} \quad A_G = \{ a \mid \exists s, t \ (s \xrightarrow{a} t) \}.
\]

Thus \( V - V_G \) is the set of isolated vertices. From now on, we assume that any graph \((V,G)\) is without isolated vertex \((i.e. \ V = V_G)\) hence the graph can be identified with its edge set \( G \). We also exclude the empty graph \( \emptyset \). Thus, every graph is a non-empty set of labelled edges. As any graph \( G \) is a set, there are no two edges with the same source, goal and label. We say that a graph is simple if there are no two edges with the same source and label: \((r \xrightarrow{a} s \land s \xrightarrow{b} t) \implies a = b\). We denote by \( G^{-1} = \{ (t,a,s) \mid (s,a,t) \in G \} \) the inverse of a graph \( G \). A graph is deterministic if there are no two edges with the same source and label: \((r \xrightarrow{a} s \land r \xrightarrow{b} s) \implies s = t\). A graph is co-deterministic if its inverse is deterministic: there are no two edges with the same goal and label. For instance, the graph \( \text{Even} = \{ (p,a,q), (p,b,p), (q,a,p), (q,b,q) \} \) represented as follows:

```
     b
     ↖
    p   a   b
    q
```
is deterministic and co-deterministic. The successor relation $\rightarrow_G$ is the unlabelled edge i.e. $s \rightarrow_G t$ if $s \xrightarrow{a} t$ for some $a \in A$. The accessibility relation $\xrightarrow{G}$ is the reflexive and transitive closure under composition of $\rightarrow_G$. A graph $G$ is accessible from $P \subseteq V_G$ if for any $s \in V_G$, there is $r \in P$ such that $r \xrightarrow{G} s$. A root is a vertex from which $G$ is accessible. A graph $G$ is co-accessible from $P \subseteq V_G$ if $G^{-1}$ is accessible from $P$. A graph $G$ is connected if every vertex of $G \cup G^{-1}$ is a root: $s \xrightarrow{G \cup G^{-1}} t$ for all $s, t \in V_G$.

Recall that a word $u = (a_1, \ldots, a_n)$ of length $n \geq 0$ over $A$ is a $n$-tuple of letters and denoted for simplicity $a_1 \ldots a_n$ i.e. $u$ is a mapping from $\{1, \ldots, n\}$ into $A$ associating with each position $1 \leq p \leq n$ its $p$-th letter $u(p)$. The length of a word $u$ is denoted by $|u|$ and for each label $a$, we denote $|u|_a = \{ |p| \leq |u| \text{ and } u(p) = a \}$ the number of positions or occurrences of $a$ in $u$. The word $( )$ of length $0$ is the empty word and is denoted by $\varepsilon$. Let $A^* = \{ (a_1, \ldots, a_n) \mid n \geq 0 \wedge a_1, \ldots, a_n \in A \}$ be the set of words over $A$ i.e. $A^*$ is the free monoid generated by $A$ for the concatenation operation.

A language $L$ is a set of words i.e. $L \subseteq A^*$ and $A_L = \{ a \in A \mid \exists u, v \in A^* (uav \in L) \}$ is its alphabet. For any $u \in A^*$, the language $u^{-1}L = \{ v \mid uw \in L \}$ is the left residual of $L$ by $u$. For all words $u, v \in A^*$, $uv$ is a conjugated word of $uv$.

A path $(s_0, a_1, s_1, \ldots, a_n, s_n)$ of length $n \geq 0$ in a graph is a sequence $s_0 \xrightarrow{a_1} s_1 \ldots \xrightarrow{a_n} s_n$ of $n$ consecutive edges, and we write $s_0 \xrightarrow{a_1 \ldots a_n} s_n$ for indicating the source $s_0$, the goal $s_n$ and the label word $a_1 \ldots a_n$ of the path. A cycle at a vertex $s$ is a path of source and goal $s$. A graph $G$ is strongly connected if every vertex is a root: $s \xrightarrow{G} t$ for all $s, t \in V_G$.

The set of words labelling the paths from $s$ to $t$ of a graph $G$ is

$$L_G(s, t) = \{ u \mid s \xrightarrow{u} t \}$$

the path language of $G$ from $s$ to $t$. For the previous graph Even, the path languages are

$$L_{\text{Even}}(p, p) = L_{\text{Even}}(q, q) = \{ u \in \{a, b\}^* \mid |u|_a \equiv 0 \ (\text{mod } 2) \} \text{ denoted } L_{\text{Even}}$$

$$L_{\text{Even}}(p, q) = L_{\text{Even}}(q, p) = \{ u \in \{a, b\}^* \mid |u|_a \equiv 1 \ (\text{mod } 2) \} \text{ denoted } L_{\text{Even}}.$$

The cycle language at vertex $s$ is $L_G(s, s) = \{ u \mid s \xrightarrow{u} s \}$ the set of labels of cycles at $s$; in particular $\varepsilon \in L_G(s, s)$. We say that a (non-empty) graph $G$ is a circular graph if $L_G(s, s) = L_G(t, t)$ for all $s, t \in V_G$ and in that case, we denote by $L_G$ this common cycle language. In other words, a graph is circular if we read the same cycle labels from any vertex. The graph Even is circular. Every acyclic graph $G$ is circular and of language $L_G = \{ \varepsilon \}$.

The path relation of a deterministic graph is a residual operation for recognized languages.

**Lemma 1.** For any $s \xrightarrow{u} t$ and $F \subseteq V_G$ with $G$ deterministic, $L_G(t, F) = u^{-1}L_G(s, F)$.

An automaton $A = (G, I, F)$ is a graph $G$ with a subset $I \subseteq V_G$ of initial vertices and a subset $F \subseteq V_G$ of final vertices. The language recognized by $A$ is

$$L(A) = L_G(I, F) = \bigcup_{i \in I, f \in F} L_G(i, f) = \{ u \mid \exists i \in I \exists f \in F (i \xrightarrow{u} f) \}.$$

An automaton $A = (G, I, F)$ is accessible (resp. co-accessible) if $G$ is accessible from $I$ (resp. co-accessible from $F$); $A$ is reduced if it is accessible and co-accessible. We say that $A$ is deterministic if $G$ is deterministic and $|I| = 1$. Similarly $A$ is co-deterministic if $G$ is co-deterministic and $|F| = 1$. Two automata $A$ and $B$ are equivalent if they recognize the same language: $L(A) = L(B)$.

### 2.2 Minimal automata

We reduce an automaton by identifying bisimilar vertices. For any deterministic co-accessible automaton, the bisimulation coincides with Nerode’s congruence. Any co-deterministic and
Lemma 2. A characterization of Cayley graphs

co-accessible automaton is minimal and its determinization remains minimal.

Let us consider automata $A = (G, I, F)$ and $A' = (G', I', F')$.

A simulation from $A$ into $A'$ is a relation $R \subseteq V_G \times V_{G'}$ such that

$$s \in V_G \implies \exists s' (s \sim s')$$

$$(s R s' \land s \overset{a}{\to} t) \implies \exists t' (s' \overset{a}{\to} t' \land t \in I')$$

$$s \in I \implies \exists s' (s \sim s' \land s' \in I')$$

$$(s R s' \land s \in F) \implies s' \in F'$$

we say that $A$ is simulated by $A'$ and then any word recognized by $A$ is recognized by $A'$:

if $A$ is simulated by $A'$ then $L(A) \subseteq L(A')$.

A morphism $h$ from $A$ into $A'$ is a mapping from $V_G$ into $V_{G'}$ which is a simulation:

$$s \overset{a}{\to} t \implies h(s) \overset{a}{\to} h(t) \quad \text{and} \quad s \in I \implies h(s) \in I' \quad \text{and} \quad s \in F \implies h(s) \in F'.$$

A simulation from $A$ into $A'$ whose the inverse relation is also a simulation is a bisimulation from $A$ on $A'$ and we say that $A$ and $A'$ are bisimilar; in this case, they recognize the same language. A bisimulation of $A$ is a bisimulation from $A$ on $A$.

A reduction $h$ from $A$ into $A'$ is a mapping from $V_G$ into $V_{G'}$ which is a bisimulation, and we write $A \sim h A'$ or directly $A \rightarrow A'$ if we do not specify a reduction. Thus, a reduction is a morphism whose inverse relation is a bisimulation.

Therefore two automata are bisimilar if and only if they are reducible into a same automaton. An injective reduction $h$ from $A$ into $A'$ is an isomorphism and we write $A \sim h A'$ or directly $A \rightarrow A'$.

A congruence of $A$ is an equivalence on $V_G$ which is a bisimulation of $A$.

The quotient of $A$ by a congruence $\sim$ is the automaton $([G]_\sim, [I]_\sim, [F]_\sim)$ with

$$[G]_\sim = \{ [s]_\sim \overset{a}{\to} [t]_\sim \mid s \overset{a}{\to} t \} \quad \text{and} \quad [S]_\sim = \{ [s]_\sim \mid s \in S \} \quad \text{for any} \ S \subseteq V_G.$$ 

which is reductive from $A$: $A \sim \{ A \}$. Thus $A$ and its quotient $[A]_\sim$ under a congruence $\sim$ recognize the same language. The family BiSim($A$) of bisimulations of $A$ is closed under arbitrary union, inverse and composition. Let

$$\approx_A = \bigcup \text{BiSim}(A)$$

be the greatest bisimulation of $A$ which is also the greatest congruence of $A$.

The minimal automaton $\text{Min}(A)$ of $A$ is the quotient of $A$ under its greatest bisimulation:

$$\text{Min}(A) = \{ A \approx \}.$$ 

Therefore two automata are bisimilar if and only if their minimal automata are isomorphic.

An automaton $A$ is minimal if $\approx_A$ is the identity i.e. $\text{Min}(A)$ is isomorphic to $A$.

For $A$ deterministic and co-accessible, its greatest bisimulation is Nerode’s congruence $[5]$.

Lemma 2. For any co-accessible automaton $A = (G, I, F)$ with $G$ deterministic,

$$s \approx_A t \iff L_G(s, F) = L_G(t, F) \quad \text{for all} \ s, t \in V_A.$$ 

For any graph $G$, we denote by $P = \{ t \mid \exists s \in P \ (s \overset{u}{\to} t) \}$ the set of vertices accessible from a vertex in $P \subseteq V_G$ by a path in $G$ labelled by $u \in A^*$. We determine any automaton $A = (G, I, F)$ into the following automaton Det($A$):

$$\{ (I \cdot u \overset{a}{\to} I \cdot ua \mid u \in A^* \land a \in A \land I \cdot ua \neq \emptyset ) \mid \{ I \} \}, \{ I \cdot u \mid u \in A^* \land I \cdot u \cap F \neq \emptyset \}$$
which is deterministic, accessible and recognizes \(L(A)\). Moreover \(\text{Det}(A)\) is co-accessible when \(A\) is co-accessible, and minimal if in addition \(A\) is co-deterministic.

**Lemma 3.** For any automaton \(A\) co-deterministic and co-accessible, \(A\) and \(\text{Det}(A)\) are minimal.

For any automaton \(A = (G, I, F)\), its inverse \(A^{-1} = (G^{-1}, F, I)\) recognizes the mirrors of the words of \(L(A)\). We co-determinize \(A\) into the equivalent automaton \(\text{CoDet}(A) = (\text{Det}(A^{-1}))^{-1}\) which is co-deterministic and co-accessible. Lemma 3 provides a fairly standard transformation of any automaton into a deterministic minimal equivalent automaton: we apply the co-determinization followed by the determinization.

**Proposition 4.** For any automaton \(A\), the automaton \(\text{Det}(\text{CoDet}(A))\) is minimal, deterministic, reduced and recognizes \(L(A)\).

### 2.3 Canonical automata

For any language \(L\) and up to isomorphism, there is a unique minimal, deterministic and reduced automaton recognizing \(L\). Such an automaton is given by the residual graph of \(L\) with the unique initial vertex \(L\) and the final vertices are the residuals of \(L\) containing the empty word. Any reduced, deterministic and co-deterministic automaton \(A\) is isomorphic to the canonical automaton of the language recognized by \(A\).

To every language \(L\) is associated its *canonical graph* or *residual graph*:

\[
\overrightarrow{L} = \{ u^{-1}L \rightarrow (ua)^{-1}L \mid u \in A^* \land a \in A \land (ua)^{-1}L \neq \emptyset \}.
\]

For instance \(a^{-1}L_{\text{Even}} = L'_{\text{Even}} = b^{-1}L'_{\text{Even}}\) and \(b^{-1}L_{\text{Even}} = L_{\text{Even}} = a^{-1}L'_{\text{Even}}\). Thus \(L_{\text{Even}} = L'_{\text{Even}}\) is the following graph which is isomorphic to the graph \(\text{Even}\):

```
   b  --a-- b
L_{Even}       L'_{Even}
```

The *canonical automaton* of any language \(L\) is the automaton

\[
\text{Can}(L) = (\overrightarrow{L}, \{L\}, \{u^{-1}L \mid u \in L\})
\]

which is the unique minimal, deterministic and reduced automaton recognizing \(L\).

**Lemma 5.** For any deterministic and reduced automaton \(A\), the automaton \(\text{Min}(A)\) is isomorphic to \(\text{Can}(L(A))\).

This Lemma 5 restricted to finite automata is the Myhill-Nerode theorem \([2, 3]\). Lemma 5 with Proposition 1 (or Lemma 3) imply the isomorphism of equivalent automata which are reduced, deterministic and co-deterministic.

**Proposition 6.** For any automaton \(A\) reduced, deterministic and co-deterministic, \(A\) is isomorphic to \(\text{Can}(L(A))\).

We just see that the graph \(\text{Even}\) is isomorphic to \(\overrightarrow{L_{\text{Even}}} \) or \(L'_{\text{Even}} \). This generalizes to any strongly connected, deterministic and co-deterministic graph \(G\) by applying Proposition 6 to the automaton \((G, s, t)\) for every vertices \(s, t\).
Corollary 7. For any graph $G$ strongly connected, deterministic and co-deterministic, $G$ is isomorphic to $L_G(s,t)$ for all $s, t \in V_G$.

This corollary implies that any strongly connected, deterministic and co-deterministic graph is minimal with respect to any of its cycle languages. It follows from Corollary 7 that two strongly connected deterministic and co-deterministic graphs are isomorphic if they have a same path language.

Corollary 8. For any graphs $G, H$ strongly connected, deterministic and co-deterministic, if $L_G(s,t) = L_H(p,q)$ for some $s, t \in V_G$ and $p, q \in V_H$ then $G \simeq H$.

This corollary is a key property to provide a structural characterization of Cayley graphs.

3 Cayley graphs

Sabidussi’s theorem characterizes the undirected and unlabelled Cayley graphs as the connected graphs having a free transitive action by a subgroup of the automorphism group. We simply adapt this theorem to directed labelled graphs by replacing the connectedness with the conditions of being rooted, deterministic and simple.

3.1 Cayley graphs and Sabidussi’s theorem

Let $(G, \cdot)$ be a group i.e. a set $G$ with an associative internal binary operation $\cdot$ such that there exists an identity element $1_G$ and each $g \in G$ has an inverse $g^{-1}$. Let $H$ be a non-empty generating subset of $G$: for any $g \in G$, there are $n \geq 0$ and $h_1, \ldots, h_n \in H$ such that $g = h_1 \cdot \ldots \cdot h_n$. Let $[,] : H \rightarrow A$ be an injective mapping coding each $h \in H$ by an element $[h] \in A$. The image of $[,]$ is the set $[H] = \{ [h] \mid h \in H \}$ of labels of $H$.

The Cayley graph of $(G, H, [\cdot])$ is the graph

$$C[G, H] = \{ g^{|h|} \cdot h | g \in G \land h \in H \}.$$ 

This graph is deterministic, co-deterministic, simple and strongly connected:

$$g^{|h_1|} \cdot C[G, H] \cdot g^{|h_2|} \cdot \ldots \cdot C[G, H] \cdot g^{|h_n|} \cdot h_1 \cdot \ldots \cdot h_n$$

for all $n \geq 0$ and $g, h_1, \ldots, h_n \in H$.

From this path, we deduce that any Cayley graph is circular and of language

$$L_{C[G, H]} = \{ [h_1] \cdot \ldots \cdot [h_n] \mid n \geq 0 \land h_1, \ldots, h_n \in H \land h_1 \cdot \ldots \cdot h_n = 1 \}.$$ 

By Corollary 7, $C[G, H]$ is isomorphic to the canonical graph $L_{C[G, H]}$. A well-known characterization of the unlabelled and non-oriented Cayley graphs was given by Sabidussi [3]. Let us recall this characterization.

First of all, a left action of $G$ on a set $V$ is a mapping $\cdot : G \times V \rightarrow V$ associating to each $(g, s) \in G \times V$ the image $g \cdot s$ such that for all $s \in V$ and $g, h \in G$,

$$1 \cdot s = s \quad \text{and} \quad h \cdot (g \cdot s) = (h \cdot g) \cdot s.$$ 

Note that for any $g \in G$, the mapping $g^{|\cdot|} : s \mapsto g \cdot s$ is a permutation of $V$. Thus, a group action of $G$ on $V$ may be seen as a group homomorphism from $G$ into the group of permutations of $V$. We say that the action $\cdot$ is transitive if

for all $s, t \in V$, there exists $g \in G$ such that $g \cdot s = t$.

We also say that the action $\cdot$ is free if

for all $g, h \in G$, if there exists $s \in V$ such that $g \cdot s = h \cdot s$ then $g = h$.

So a free and transitive action $\cdot$ means that
for all \( s, t \in V \), there exists a unique \( g \in G \) such that \( g \cdot s = t \).

Let \( G \) be a (directed and labelled) graph. An action of \( G \) on \( G \) is an action \( \cdot \) of \( G \) on \( V_G \) which is a morphism of \( G \) i.e.

\[
s \xrightarrow{a} G t \implies g \cdot s \xrightarrow{a} G g \cdot t \quad \text{for all } s, t \in V_G, \quad a \in A_G, \quad g \in G.
\]

Therefore, a group action of \( G \) on \( G \) may be seen as a group homomorphism from \( G \) into the group \( \text{Aut}(G) \) of automorphisms of \( G \) i.e. of isomorphisms from \( G \) to itself.

We say that vertices \( s, t \) of a graph \( G \) are isomorphic and we write \( s \cong_G t \) if there is an automorphism \( h \) of \( G \) such that \( t = h(s) \).

A graph \( G \) is vertex-transitive if there exists a transitive group action on \( G \). This means that \( \text{Aut}(G) \) acts transitively on \( G \) or equivalently that all its vertices are isomorphic: \( s \cong_G t \) for all \( s, t \in V_G \). In particular, any vertex-transitive graph is circular.

First, we adapt to all Cayley graphs a Sabidussi's characterization for a given group.

**Proposition 9.** A graph \( G \) is isomorphic to a Cayley graph of a group \( G \) if and only if \( G \) is a deterministic rooted simple graph with a free transitive action of \( G \) on \( G \).

**Proof.** \( \implies \): Assume that \( G \xrightarrow{\cdot} C[G, H] \) for some generating subset \( H \) of \( G \) and some coding \( [ ] \) of \( H \). The vertex set of \( C[G, H] \) is \( G \) whose group operation \( \cdot \) is a free transitive action of \( G \) on \( C[G, H] \).

In particular \( C[G, H] \) is vertex-transitive for any subset \( H \) of \( G \).

\( \Leftarrow \): Let \( \cdot \) be a free transitive action of \( G \) on \( G \).

Let us check that \( G \) is isomorphic to a Cayley graph of \( G \) by simply adapting the proof of the sufficient condition of Sabidussi’s theorem.

As \( G \) is rooted, we can pick a root \( r \) of \( G \).

For all \( s \in V_G \) there is a unique \( g_s \in G \) such that \( g_s \cdot r = s \). Thus \( h = g_h \cdot r \) for all \( h \in G \); in particular \( g_e = 1 \). We define \( H = \{ g_s \mid r \rightarrow_G s } \).

As \( G \) is simple and deterministic, we define the following injection \( [ ] \) from \( H \) into \( A_G \) by

\[
[g_s] = a \quad \text{for any } r \xrightarrow{a} G s.
\]

By renaming each vertex \( s \) of \( G \) by \( g_s \), we get \( G \) isomorphic to the graph

\[
\overline{G} = \{ g_s \xrightarrow{a} g_e \mid s \xrightarrow{a} G t } \}
\]

We check (see Appendix) that \( \overline{G} = C[G, H] \) and \( H \) is a generating subset of \( G \). \( \blacksquare \)

The determinism condition of this proposition is necessary. For instance, the following simple and strongly connected graph \( G \):

![Graph Diagram]

is not deterministic hence is not a Cayley graph, while this graph has a free transitive action of the (cyclic) group of order 3.

Proposition 9 is a restricted characterization of Cayley graphs since it is relative to a group \( G \). We say that a graph \( G \) has a free transitive action if there exists a group \( G \) with a free transitive action \( \cdot \) on \( G \); in that case \( \{ \overline{G} \mid g \in G \} \) is a subgroup of \( \text{Aut}(G) \) and its canonical action \( (\overline{G}, s) \mapsto \overline{G}(s) = g \cdot s \) is free and transitive on \( G \). Proposition 9 give a simple generalization of Sabidussi’s theorem to labelled directed graphs.
Proposition 10. A graph \( G \) is a Cayley graph if and only if \( G \) is deterministic, rooted, simple, with a free transitive action.

Proposition 10 characterizes the Cayley graphs using two conditions of different nature. The first condition is structural: the graph must be deterministic, simple and rooted. The second condition is algebraic namely the existence of a free transitive action. We now give a characterization that is only structural by restricting the algebraic condition to the vertex-transitivity: we no longer need to extract a subgroup of the automorphism group whose the canonical action is free and transitive.

3.2 Cayley graphs of languages

We briefly recall the definition of a group by a language whose letters form the set of generators and the words define the set of relators [4]. Let \( L \subseteq A^* \) be a language. The word operation of deleting a word of \( L \) is the rewriting according to \( L \times \{ \varepsilon \} \):

\[
xuy \rightarrow_L xy \quad \text{for any } u \in L \text{ and } x, y \in A^*
\]

and the inverse operation \((\rightarrow_L)^{-1}\) is the insertion of a word of \( L \).

The derivation \( \rightarrow_L \) and the Thue congruence \( \leftrightarrow_L \) of \( L \) are the reflexive and transitive closure under composition of respectively \( \rightarrow_L \) and \( \leftrightarrow_L = \rightarrow_L \cup (\rightarrow_L)^{-1} \).

The equivalence class of \( u \in A^* \) with respect to \( \leftrightarrow_L \) is denoted \([u]_L = \{ v \mid u \leftrightarrow_L v \}\).

We say that a language \( L \) is a group presentation language if

(i) \( A_L \neq \emptyset \) i.e. \( L \neq \emptyset \) and \( L \neq \{ \varepsilon \} \)

(ii) for all \( a \in A_L \), there exists \( u \in A_L^* \) such that \( au, ua \in [\varepsilon]_L \)

(iii) for all \( a \neq b \in A_L \), \([a]_L \neq [b]_L\).

In that case, the quotient \( \{ [u]_L \mid u \in A_L^* \} \) of \( A_L^* \) under the congruence \( \leftrightarrow_L \) is by (ii) a group for the operation

\[
[u]_L[v]_L = [uv]_L \quad \text{for all } u, v \in A_L^*
\]

and we define \( C(L) \) as being the Cayley graph of \( \{ [u]_L \mid u \in A_L^* \} \) generated by the subset \( \{ [a]_L \mid a \in A_L \} \):

\[
C(L) = C(\{ [u]_L \mid u \in A_L^* \}, \{ [a]_L \mid a \in A_L \})
\]

where \([a]_L\) is encoded by \([[a]_L] = a\) for all \( a \in A_L \); this makes sense from (iii) and this graph is non-empty by (i).

For instance \( C(\{a^6, b^3, (ab)^3\}) \) is represented by the following tiling plane:

where every simple (resp. double) arrow is labelled by \( a \) (resp. \( b \)), and \( C(\{a^6, b^3, aba\}) \) is
As \( C(L) \) is a Cayley graph, it is circular and its language is \([\varepsilon]_L\).

Lemma 11. For any group presentation language \( L \), \( L_{C(L)} = [\varepsilon]_L \) and \( C(L) = C([\varepsilon]_L) \).

We say that \( L \) is a stable language if

\[
vw \in L \iff uv \in L \text{ for any } v \in L
\]

meaning that \( L \) is preserved by insertion and deletion of factor in \( L \). By iterating these two word operations from \( \varepsilon \), it only gets all words of \( L \).

Lemma 12. A non-empty language \( L \) is stable if and only if \( L = [\varepsilon]_L \).

4 Graph characterizations of Cayley graphs

We begin with basic graph properties, especially for circul ar graphs. We then give a first characterization of Cayley graphs: they are the vertex-transitive and rooted deterministic simple graphs (Theorem 19). They are also the circular and strongly connected deterministic simple graphs (Theorem 24). We can also replace the circularity by the elementary circularity because every vertex-transitive graph is elementary circular which is then circular (Lemma 22). Another significant characterization concerns Cayley graphs for all subsets of groups: under ZFC, they are the deterministic, co-deterministic, vertex-transitive simple graphs (Theorem 28).

4.1 A first graph characterization

We consider the family \( F \) of deterministic, rooted, simple graphs which are vertex-transitive. We want to establish that these graphs are Cayley graphs. In particular, any graph of \( F \) should be strongly connected.

Lemma 13. Any rooted vertex-transitive graph is strongly connected.

Furthermore any graph of \( F \) should be co-deterministic.

Lemma 14. Any deterministic and strongly connected circular graph is co-deterministic.

By Corollary 7, Lemmas 13 and 14 any graph \( G \) of \( F \) is isomorphic to the canonical graph of its path languages, hence in particular to \( L_G \). This cycle language \( L_G \) is stable.

Lemma 15. For any deterministic circular graph \( G \), \( L_G \) is a stable language.

For any circular graph, the cycle language is closed under conjugacy, and any label of the graph is a letter of this language when the graph is strongly connected.

Lemma 16. For any strongly connected circular graph \( G \),

\[
L_G \text{ is closed under conjugacy and of letter set } A_{L_G} = A_G.
\]

Let us give a condition on a circular graph for its cycles to form a group presentation language.
Lemma 17. For any strongly connected, deterministic circular simple graph \( G \), 
\( L_G \) is a group presentation language.

Any vertex-transitive graph is circular. By Lemma 14 and Corollary 7, the converse is true when the graph is deterministic and strongly connected.

Lemma 18. For any deterministic and strongly connected graph \( G \),
\( G \) is vertex-transitive if and only if \( G \) is circular.

We are able to establish a first structural characterization of Cayley graphs.

Theorem 19. A graph is a Cayley graph if and only if it is deterministic, rooted, simple and vertex-transitive.

Proof. \( \implies \) By definition, a Cayley graph is strongly connected, deterministic, simple and circular. As already indicated in the proof of Proposition 9 (or by Lemma 18), \( G \) is vertex-transitive.

\( \Longleftarrow \) Let \( G \) be a deterministic, rooted, simple and vertex-transitive graph.
By Lemma 13 \( G \) is strongly connected. By Lemma 14 \( G \) is co-deterministic.
By Lemma 17 \( L_G \) is a group presentation hence \( C(L_G) \) is a Cayley graph.
By Lemma 11 we have \( L_C(L_G) = [e]_{L_G} \).
By Lemmas 14 and 12 \( L_G = [e]_{L_G} \).
By Corollary 8 \( G \) is isomorphic to \( C(L_G) \) hence \( G \) is a Cayley graph.

A deterministic graph of Petersen skeleton is given by the following two isomorphic representations:

Such a graph is not a Cayley graph because it is not circular (hence not vertex-transitive):
\[ ababa \in L_{Petersen}(1,1) - L_{Petersen}(2,2) \quad \text{and} \quad baaba \in L_{Petersen}(2,2) - L_{Petersen}(1,1). \]

Other condition such as the same in and out degrees is a consequence of Theorem 19.

Precisely a graph \( G \) is source-complete if for all vertex \( s \) and label \( a \), there is an edge from \( s \) labelled by \( a \):
\[ \forall s \in V_G \quad \forall a \in A_G \quad \exists t \ (s \overset{a}{\rightarrow} t). \]

A graph is co-complete if its inverse is complete.

Let us give weaker conditions than those of Theorem 19 to obtain a complete graph.

Lemma 20. Any strongly connected circular graph is complete and co-complete.

Note also that we can replace the vertex-transitivity in Theorem 19 with edge-transitivity. Recall that a graph \( G \) is edge-transitive if for all \( (s,a,t), (s',a,t') \in G \), there exists an automorphism \( h \) of \( G \) such that \( h(s) = s' \) and \( h(t) = t' \). The determinism with the completeness give the correspondence between the vertex-transitivity and the edge-transitivity.
Lemma 21. For any graph $G$ deterministic and complete, $G$ is vertex-transitive $\iff G$ is edge-transitive.

4.2 Other graph characterizations

Let us give another characterization of Cayley graphs by replacing the vertex-transitivity condition by the circularity or the elementary circularity which is now defined.

The elementary cycle language at vertex $s$ is the label set $E_G(s) = \{ a_1 \ldots a_n \mid n > 0 \land \exists s_0 \neq \ldots \neq s_{n-1} (s_0 \xrightarrow{a_1} s_1 \ldots \xrightarrow{a_{n-1}} s_{n-1} \xrightarrow{a_n} s_0 = s) \}$ of elementary cycles (passing only by distinct vertices) at $s$; in particular $\varepsilon \not\in E_G(s) \subset L_G(s, s)$. We say that the (non-empty) graph $G$ is an elementary circular graph if $E_G(s) = E_G(t)$ for all $s, t \in V_G$ and in that case, we denote by $E_G$ this common elementary cycle language. For instance $E_{\text{Pair}} = \{ aa, bb \}$.

Lemma 22. Any elementary circular graph $G$ is circular with $L_G \subseteq [\varepsilon]_{E_G}$.

Lemma 23. Any deterministic circular graph $G$ is elementary circular with $L_G = [\varepsilon]_{E_G}$.

Let us adapt Theorem 19 by applying Lemmas 18, 22, 23.

Theorem 24. The Cayley graphs are the deterministic, strongly connected, simple graphs which are equivalently vertex-transitive or circular or elementary circular.

Note that the strong connectedness condition in Theorem 24 can not be simplified to the rootedness condition since the semi-line $\{ n \xrightarrow{a} n + 1 \mid n \geq 0 \}$ is rooted, deterministic, simple, circular and elementary circular. However, this strongly connectedness may be replaced by the connectedness for finite graphs. This is based on the property below.

Lemma 25. Any finite connected vertex-transitive graph is strongly connected.

Corollary 26. The Cayley graphs of finite groups are the deterministic, connected, vertex-transitive, simple and finite graphs.

4.3 Other Cayley graphs

It is also sometimes defined the Cayley graph $C[G, H]$ for $H$ a weak generating subset of $G$ meaning that $H \cup H^{-1}$ is a generating subset of $G$ where $H^{-1} = \{ h^{-1} \mid h \in H \}$. In that case, we say that $C[G, H]$ is a weak generated Cayley graph. Theorem 24 allows to characterize this more general family of graphs.

Proposition 27. The weak generated Cayley graphs are the deterministic, co-deterministic, connected, simple and vertex-transitive graphs.

Proof. $\implies$: Let $G$ be a group and $H$ be a non-empty weak generating subset of $G$.

So $C[G, H \cup H^{-1}]$ is a deterministic, co-deterministic, strongly-connected, vertex-transitive simple graph. By removing the arrows labeled in $H^{-1}$, the graph $C[G, H]$ is connected and it remains deterministic, co-deterministic, simple and vertex-transitive.

$\impliedby$: Let $G$ be a deterministic, co-deterministic, connected, simple, vertex-transitive graph.

Let Inv be the set of labels $a$ such that $s \xrightarrow{a} G t \implies t \xrightarrow{a} G s$ for (all) $s, t \in V_G$ i.e.
A characterization of Cayley graphs

\[ \text{Inv} = \{ a \in A_G | \exists b \in A_G \ (ab \in L_G) \}. \]

To each \( a \in A_G - \text{Inv} \), we associate a new element \( \overline{a} \in A - A_G \) and we define the completion of \( G \) by the graph:

\[ H = G \cup \{ t \xrightarrow{a} s \mid s \xrightarrow{a} t \land a \not\in \text{Inv} \}. \]

So \( H \) is strongly connected and it remains simple, deterministic and vertex-transitive.

By Theorem 24, \( H \) is a Cayley graph:

\[ H = C[V_G, H] \]

for some generating subset \( H \) of a group \((V_G, \cdot)\) on its vertex set.

Therefore \( G \) is the weak generated Cayley graph:

\[ G = C[V_G, K] \quad \text{with} \quad K = \{ h \in H \mid [h] \in A_G \}. \]

We also consider the Cayley graph \( C[G, H] \) for every non-empty subset \( H \) of a group \( G \). Such graph is called here a generalized Cayley graph. These graphs form a more general class of graphs for which and under the assumption of the axiom of choice, we can give a simple graph characterization.

**Theorem 28.** Under ZFC set theory, the generalized Cayley graphs are the deterministic, co-deterministic, vertex-transitive simple graphs.

**Proof.** \( \implies \): By definition and for any subset \( H \) of \( G \), the graph \( C[G, H] \) is simple, deterministic and co-deterministic. As already indicated in the proof of Proposition 24, this graph is vertex-transitive.

\( \Leftarrow \): Let \( G \) be a deterministic, co-deterministic, vertex-transitive simple graph.

Let \( \text{Comp} \) be the set of connected components of \( G \).

Using ZFC set theory, there exists a binary operation \( \cdot \) such that \((\text{Comp}, \cdot)\) is a group (in fact in ZF set theory, the axiom of choice is equivalent to the property that any non-empty set has a group structure). We denoted by \( I \) the identity element of \((\text{Comp}, \cdot)\).

Note that \( I \) is connected, deterministic, co-deterministic, simple and vertex-transitive.

By Proposition 27, \( I = C[V_I, H] \) for some weak generating subset \( H \) of a group \((V_I, \cdot_I)\).

As \( G \) is vertex-transitive, there is an isomorphism from \( I \) to each \( C \in \text{Comp} \). By the axiom of choice, we take for each \( C \in \text{Comp} \) an isomorphism \( f_C \) from \( I \) to \( C: I \xrightarrow{f_C} C \).

It is assumed that \( f_I \) is the identity on \( V_I \).

We consider the group product \( V_I \times \text{Comp} \), its subset \( K = \{ (h, I) \mid h \in H \} \) and the mapping \([ \cdot ] \) defined by \([h, I] = [h] \) for any \( h \in H \).

We consider the bijection \( f: V_I \times \text{Comp} \to V_G \) defined by

\[ f(s, C) = f_C(s) \quad \text{for any} \ s \in V_I \text{ and} \ C \in \text{Comp}. \]

We check (see Appendix) that \( C[V_I \times \text{Comp}, K] \xrightarrow{f} G \).

\( \Box \)

5 Conclusion

In this article, we have given simple graph conditions to characterize the Cayley graphs as vertex-transitive graphs. This paper is only a first step in the study of symmetry to directed and labelled graphs.
References

1. A. Cayley, *The theory of groups: graphical representation*, American J. Math. 1, 174-76 (1878).
2. M. Harrison, *Introduction to formal language theory*, Addison-Wesley Publishing (1978).
3. J. Hopcroft and J. Ullman, *Introduction to automata theory, languages and computation*, Addison-Wesley Publishing (1979).
4. W. Magnus, A. Karrass and D. Solitar, *Combinatorial group theory: Presentations of groups in terms of generators and relations*, Pure Appl. Math. 13, Interscience (1966).
5. A. Nerode, *Linear automaton transformations*, Proceedings of the American Mathematical Society 9-4, 541-544 (1958).
6. G. Sabidussi, *On a class of fixed-point-free graphs*, Proceedings of the American Mathematical Society 9-5, 800-804 (1958).
We give here proofs and complementary results.

6 Standard results on automata

By adding two new symbols \( \iota, o \), every automaton \( \mathcal{A} = (G, I, F) \) can be seen as the colored graph \( G \cup \{ (\iota, i) \mid i \in I \} \cup \{ (o, f) \mid f \in F \} \); we write \( cs \) for any couple \( (c, s) \) with \( c \in \{ \iota, o \} \) and \( s \in V_G \). We also write \( V_\mathcal{A} \) for \( V_G \) and \( -\to_\mathcal{A} \) for \( -\to_G \).

Let us start with a basic property on the path relation of a deterministic graph.

**Lemma 30.** For any \( s -\to_G t \) and \( F \subseteq V_G \) with \( G \) deterministic, \( L_G(t, F) = u^{-1}L_G(s, F) \).

**Proof.** By union, it is sufficient to check the equality for \( F \) restricted to a single vertex \( f \).

\[ \subseteq: \text{Let } v \in L_G(t, f) \text{ i.e. } t -\to_G f. \text{ So } s -\to_G f \text{ hence } uv \in L_G(s, f) \text{ i.e. } v \in u^{-1}L_G(s, f). \]

\[ \supseteq: \text{Let } v \in u^{-1}L_G(s, f) \text{ i.e. } s -\to_G f. \text{ There is } t' \text{ such that } s -\to_G t' -\to_G f. \]

As \( G \) is deterministic, \( t = t' \) hence \( v \in L_G(t, f) \).

The simulation on automata involves the inclusion on the recognized languages.

**Lemma 29.** If \( \mathcal{A} \) is simulated by \( \mathcal{B} \) then \( L(\mathcal{A}) \subseteq L(\mathcal{B}) \).

**Proof.** Let \( R \) be a simulation from \( \mathcal{A} \) into \( \mathcal{B} \). Let \( u \in L(\mathcal{A}) \).

There are \( s, t \in V_\mathcal{A} \) such that \( s -\to_\mathcal{A} t \) and \( \iota s, o t \in \mathcal{A} \).

By definition of a simulation, there exists \( p \in V_\mathcal{B} \) such that \( \iota p, s R p \).

By induction on \( |u| \geq 0 \), there exists \( q \in V_\mathcal{B} \) such that \( p -\to_B q \) and \( t R q \).

As \( o t \in \mathcal{A} \) and \( t R q \), we have \( o q \in \mathcal{B} \). Thus \( u \in L(\mathcal{B}) \).

Every automaton is reduced in all quotient.

**Lemma 30.** For every congruence \( \sim \) of any automaton \( \mathcal{A} \), \( \mathcal{A} \to_{\sim_1} [\mathcal{A}]_{\sim} \).

**Proof.** The mapping \( [\ ]_{\sim}: s \in V_\mathcal{A} \mapsto [s] \in V_{[\mathcal{A}]_{\sim}} \) is surjective.

By definition of \( [\mathcal{A}]_{\sim} \), if \( s -\to_\mathcal{A} t \) then \( [s] -\to_{[\mathcal{A}]_{\sim}} [t] \).

Similarly, if \( cs \in \mathcal{A} \) then \( c[s] \in [\mathcal{A}]_{\sim} \).

Thus \( [\ ]_{\sim} \) is a surjective morphism. It remains to check that \( [\ ]_{\sim}^{-1} \) is a simulation.

If \( [s] -\to_{[\mathcal{A}]_{\sim}} [t] \) then there exists \( s' -\to_\mathcal{A} t' \) such that \( s \sim s' \) and \( t \sim t' \).

As \( \sim \) is a congruence, there is \( t'' \) such that \( s -\to_\mathcal{A} t'' \) and \( t'' \sim t' \), hence \( [t''] = [t] \).

If \( \iota [s] \in [\mathcal{A}]_{\sim} \) then there is \( s' \) such that \( s \sim s' \) and \( \iota s' \in \mathcal{A} \).

If \( o [s] \in [\mathcal{A}]_{\sim} \) then there is \( s' \) such that \( s \sim s' \) and \( os \in \mathcal{A} \), hence \( os \in \mathcal{A} \).

The similarity between two automata corresponds to the reduction to a same automaton, or to have the same minimal automaton.

**Lemma 31.** For all automata \( \mathcal{A} \) and \( \mathcal{B} \), we have the following equivalences:

a) \( \mathcal{A} \) and \( \mathcal{B} \) are bisimilar,

b) \( \mathcal{A} \) and \( \mathcal{B} \) are reducible to a same automaton,

c) \( \text{Min}(\mathcal{A}) \) and \( \text{Min}(\mathcal{B}) \) are isomorphic.

**Proof.**

a) \( \implies \) c): Let \( R \) be a bisimulation between \( \mathcal{A} \) and \( \mathcal{B} \). The relation \( h = \{ ([s]_{\sim_1} : [t]_{\sim_1}) \mid s R t \} = [\ ]^{-1}_{\sim_1} \circ R \circ [\ ]_{\sim_1} \)
is a bisimulation between \( \text{Min}(A) \) and \( \text{Min}(B) \).

For that we have \( \text{Min}(A) \approx_h \text{Min}(B) \), it remains to show that \( h \) are \( h^{-1} \) are injectives.

By symmetry, it is sufficient to show the injectivity of \( h \).

Let \( ([s]_{\approx_A}, [t]_{\approx_B}) \), \( ([s]_{\approx_A}, [t']_{\approx_B}) \in h \). There exists \( s_1 R t_1 \) and \( s_2 R t_2 \) such that
\[
[s_1]_{\approx_A} = [s_2]_{\approx_A} ; \quad [t_1]_{\approx_B} = [t]_{\approx_B} ; \quad [t_2]_{\approx_B} = [t']_{\approx_B} .
\]

Therefore
\[
(t_1, t_2) \in R^{-1} \circ [ \ ]_{\approx_A} \circ [ \ ]_{\approx_A}^{-1} \circ R \quad \text{is a bisimulation of } B.
\]

Thus \( t_1 \approx_B t_2 \) so \( [t_1]_{\approx_B} = [t_2]_{\approx_B} \) i.e. \( [t]_{\approx_B} = [t']_{\approx_B} \).

c) \( \implies \) b): Let \( \text{Min}(A) \approx_h \text{Min}(B) \). From Lemma 30 we have
\[
A \xrightarrow{\approx} h \approx_A \text{Min}(B) \quad \text{and} \quad B \xrightarrow{\approx} h \approx_B \text{Min}(B).
\]

b) \( \implies \) a): Let \( A \xrightarrow{\approx} g \approx \) and \( B \xrightarrow{\approx} h \approx \).

So \( g \cdot h^{-1} \) is a bisimulation between \( A \) and \( B \).

The largest bisimulation for any deterministic and co-accessible automaton coincides with the congruence of Nerode.

**Lemma 2** For any co-accessible automaton \( A = (G, I, F) \) with \( G \) deterministic,
\[
s \approx_A t \iff L_G(s, F) = L_G(t, F) \quad \text{for all } s, t \in V_A .
\]

**Proof.**

i) Let \( R \) be a bisimulation of any automaton \( A = (G, I, F) \). Let \( s R t \).

So the automata \( (G, s, F) \) and \( (G, t, F) \) are bisimilar. By Lemma 29 we get
\[
s R t \quad \implies \quad L_G(s, F) = L_G(t, F)
\]

hence the direct implication of this lemma for \( R \) equal to \( \approx_A \).

ii) Assume that \( A \) is co-accessible with \( G \) deterministic. We define the relation
\[
R = \{ (s, t) \mid s, t \in V_G \land L_G(s, F) = L_G(t, F) \} .
\]

For the property \( R \subseteq \approx_A \), it suffices to show that \( R \) is a bisimulation.

As \( R \) is an equivalence hence symmetric, it is sufficient to check that \( R \) is a simulation.

As \( R \) is reflexive, its domain is \( V_G \) and for \( s \in A \), we have \( s R s \). Furthermore
\[
os s \in A \iff \epsilon \in L_G(s, F) \quad \text{hence} \quad (s R t \wedge os \in A) \implies ot \in A .
\]

Finally, let \( s R t \) and \( s \xrightarrow{a} G s' \).

As \( G \) is deterministic and by Lemma 1 \( L_G(s', F) = a^{-1}L_G(s, F) = a^{-1}L_G(t, F) \).

As \( A \) is co-accessible, \( L_G(s', F) \neq \emptyset \) hence \( a^{-1}L_G(t, F) \neq \emptyset \).

So there is \( t' \in V_G \) such that \( t \xrightarrow{a} G t' \).

As \( G \) is deterministic and by Lemma 1 \( L_G(t', F) = a^{-1}L_G(t, F) = L_G(s', F) \) i.e. \( s' R t' \).

Every co-deterministic and co-accessible automaton is minimal and remains minimal by determinization.

**Lemma 3** For any automaton \( A \) co-deterministic and co-accessible,
\( A \) and \( \text{Det}(A) \) are minimal.
Lemma 32. Let $A = (G, I, \{f\})$ and $Det(A) = (H, \{I\}, F)$.

Proof. Let us denote $A = (G, I, \{f\})$ and $Det(A) = (H, \{I\}, F)$.

i) Let us check that $A$ is minimal. Let $R$ be a bisimulation of $A$. Let $s Rt$.

So $L_G(s,f) = L_G(t,f) \neq \emptyset$ since $A$ is co-accessible. As $A$ is co-deterministic, $s = t$.

ii) For any vertex $J$ of $H$ i.e. $J = I \cdot v \neq \emptyset$ for some $v \in A^*$, we have

$$L_H(J, F) = \{ u \in A^* \mid J \cdot u \in F \} = \{ u \in A^* \mid f \cdot J \cdot u \} = L_G(J, f).$$

In particular $L(Det(A)) = L_H(I, F) = L_G(I, F) = L(A)$.

iii) Let us check that $Det(A)$ is minimal. Consider two bisimilar vertices $I \cdot u, I \cdot v$ of $Det(A)$.

In particular $L_H(I \cdot u, F) = L_H(I \cdot v, F)$. Thus by (ii), $L_G(I \cdot u, f) = L_G(I \cdot v, f)$.

Let $s \in I \cdot u$. As $A$ is co-accessible, there is $w \in L_G(s,f) \subseteq L_G(I \cdot u,f) = L_G(I \cdot v,f)$.

There exists $t \in I \cdot v$ such that $w \in L_G(t,f)$.

As $A$ is co-deterministic, $s = t$. Thus $I \cdot u \subseteq I \cdot v$ and by symmetry $I \cdot u = I \cdot v$.

By definition, $Can(L)$ is deterministic and reduced; its recognized language is $L$.

**Lemma 32.** For any language $L$, the automaton $Can(L)$ is deterministic, reduced and recognizes $L$.

Proof. We just have to check that $L = L(Can(L))$.

$\subseteq$: Let $u \in L$. Then we have $L \xrightarrow{u_{Can(L)}} u^{-1}L$ i.e. $u \in L(Can(L))$.

$\supseteq$: Let $u \in L(Can(L))$. There is $v \in L$ such that $u \in L_{Can(L)}(L, v^{-1}L)$.

It follows that $\varepsilon \in v^{-1}L$ and $L \xrightarrow{u_{Can(L)}} v^{-1}L$.

As $Can(L)$ is deterministic, we get $u^{-1}L = v^{-1}L$. So $\varepsilon \in u^{-1}L$ i.e. $u \in L$.

The automaton $Can(L)$ is the unique minimal, deterministic and reduced automaton recognizing $L$.

**Lemma 4**. For any deterministic and reduced automaton $A$, the automaton $Min(A)$ is isomorphic to $Can(L(A))$.

Proof. Let $A = (G, I, F)$. As $A$ is deterministic and co-accessible, and by Lemma 2

$$s \approx_A t \iff L_G(s,F) = L_G(t,F) \quad \text{for all } s, t \in V_A.$$ 

Let $L = L(A) = L_G(I, F)$. As $A$ is deterministic and by Lemma 1

$$L_G(s,F) = u^{-1}L \quad \text{for all } i \xrightarrow{u} G s.$$ 

As $A$ is accessible and co-accessible, we can define the mapping from $V_{Min(A)}$ into $V_{Can(L(A))}$ by

$$[s]_{\approx_A} \mapsto u^{-1}L \quad \text{for } i \xrightarrow{u} G s.$$ 

which is an isomorphism from $Min(A)$ to $Can(L(A))$.

Due to the importance of Corollary 7, we give a direct proof.

**Corollary 7.** For any graph $G$ strongly connected, deterministic and co-deterministic, $G$ is isomorphic to $L_G(s,t)$ for all $s, t \in V_G$.

Proof. Let $G$ be a strongly connected, deterministic and co-deterministic graph.

Let $s, t \in V_G$ and $L = L_G(s,t)$. Let us check that for all $r \in V_G$ and $u \in A^*$,

$$L_G(r,t) = u^{-1}L \iff s \xrightarrow{u} G r. \quad (1)$$
\( \iff \): as \( G \) is deterministic, it suffices to apply Lemma 11.

\( \implies \): there exists \( v \) such that \( uv \in L \) and \( r \xrightarrow{u} t \).

As \( uv \in L \), there exists \( r' \) such that \( s \xrightarrow{u} r' \xrightarrow{v} t \).

As \( G \) is co-deterministic, we get \( r = r' \) hence \( s \xrightarrow{u} r \).

Thus Property (1) is checked. Let \( H = \overrightarrow{L}_G \).

We now show that \( G \) is isomorphic to \( H \) according to the following mapping:

\[
    h : V_G \rightarrow \{ u^{-1}L \mid u \in A^* \} - \{\emptyset\}
    \]

\[
    r \mapsto L_G(r, t)
    \]

Let us check that \( h \) is well-defined. Let \( r \in V_G \).

As \( G \) is strongly connected, \( L_G(r, t) \neq \emptyset \) and there exists \( u \in A^* \) such that \( s \xrightarrow{u} r \).

By 11, we have \( L_G(r, t) = u^{-1}L \).

Let us check that \( h \) is surjective. Let \( u \in A^* \) such that \( u^{-1}L \neq \emptyset \).

Thus there exists \( r \) such that \( s \xrightarrow{u} r \). By 11, we have \( L_G(r, t) = u^{-1}L \).

Let us check that \( h \) is injective.

Let \( p \neq q \in V_G \). As \( G \) is co-deterministic, we have

\[
    h(p) \cap h(q) = L_G(p, t) \cap L_G(q, t) = \emptyset.
    \]

As \( h(p), h(q) \neq \emptyset \), we get \( h(p) \neq h(q) \). So \( h \) is a bijection.

To show that \( h \) is an isomorphism from \( G \) to \( H \), it remains to verify that

\[
    \text{If } r \xrightarrow{a} G r' \iff h(r) \xrightarrow{a} H h(r').
    \]

If \( r \xrightarrow{a} G r' \) then by Lemma 11 \( L_G(r', t) = a^{-1}L_G(r, t) \) hence

\[
    h(r) = L_G(r, t) \xrightarrow{a} H L_G(r', t) = h(r').
    \]

If \( h(r) \xrightarrow{a} H h(r') \) then \( L_G(r', t) = a^{-1}L_G(r, t) \).

As \( G \) is strongly connected, there exists \( u \in A^* \) such that \( s \xrightarrow{u} r \). By 11

\[
    L_G(r', t) = a^{-1}L_G(r, t) = a^{-1}(u^{-1}L) = (ua)^{-1}L.
    \]

By 11, \( s \xrightarrow{ua} r' \) and as \( G \) is deterministic, we get \( r \xrightarrow{ua} r' \).

Co-determinism in the statement of Corollary 7 is required. Indeed, the following graph \( G \):

\[
    \begin{array}{c}
    \begin{array}{c}
    a \\
    c
    \end{array}
    \end{array}
    \begin{array}{c}
    \begin{array}{c}
    b \\
    c
    \end{array}
    \end{array}
    \begin{array}{c}
    \begin{array}{c}
    0
    \end{array}
    \end{array}
    \]

is deterministic, strongly connected and one of its path language is \( L_G(0, 0) = ((a + b)c)^* \) whose following canonical graph

\[
    \begin{array}{c}
    \begin{array}{c}
    a,b \\
    c
    \end{array}
    \end{array}
    \begin{array}{c}
    \begin{array}{c}
    L_G(0, 0) \\
    L_G(0, 0)
    \end{array}
    \end{array}
    \]

is not isomorphic to \( G \).
7 Cayley graphs

By removing labels and orientations of arrows of a graph $G$, we obtain its skeleton

$$\text{Ske}(G) = \{ \{ s, t \} \mid s \rightarrow_G t \}$$

which is a non-directed and unlabelled graph, or equivalently a bi-directed graph labelled by a single symbol $\#$: $\{ s \# \rightarrow t \mid \exists a (s \rightarrow_G t \lor t \rightarrow_G s) \}$. The skeleton of Even is represented by

![Skeleton of Even]

Sabidussi’s theorem gives a characterization of skeletons of the Cayley graphs.

**Theorem 33.** (Sabidussi) For any non-directed and unlabelled graph $G$,

- $G$ is the skeleton of a Cayley graph if and only if
  - $G$ is connected and there is a subgroup of $\text{Aut}(G)$ with a free transitive action on $G$.

We can removed the connectedness condition of this theorem by extending the definition of a Cayley graph for any non-empty subset $H$ of a group $G$. However, this condition is necessary when $H$ is a generating subset of $G$. For instance, consider the non-connected graph $G = \{1 \rightarrow 2, 3 \rightarrow 4\}$. A subgroup of $\text{Aut}(G)$ is the Klein group of permutation representation:

$$K_4 = \{(), (1,2)(3,4), (1,3)(2,4), (1,4)(2,3)\}$$

whose mapping $(f, s) \mapsto f(s)$ is a free and transitive action on $G$.

Note that the connectedness of a graph $G$ is a simple graph property and it is quite more difficult to know whether there exists a subgroup of $\text{Aut}(G)$ with a free and transitive action on $G$. For instance, the following Petersen skeleton:

![Petersen skeleton]

is connected, vertex-transitive but it has no free transitive group action.

Let us conclude the proof of Proposition 9.

**Proposition 9.** A graph $G$ is isomorphic to a Cayley graph of a group $G$ if and only if

$G$ is a deterministic rooted simple graph with a free transitive action of $G$ on $G$.

**Proof.**

Let us end this proof by checking that $\overline{G} = C[G, H]$ and $H$ is a generating subset of $G$.

Let $p \rightarrow_G q$. As $\bullet$ is transitive, there is $g \in G$ such that $g \bullet p = r$.

As $\bullet$ is a morphism, we get $r = g \bullet p \rightarrow_G g \bullet q$. We denote $s = g \bullet q$.

So $p = g^{-1} \bullet r$ i.e. $g^{-1} = g_p$.

Moreover $a = [g_s]$ and $q = g^{-1} \bullet s = g_p \bullet s = (g_p g_s) \bullet r$ i.e. $g_q = g_p g_s$.

Thus $A_G = [H]$ and was obtained the necessary condition of the following equivalence

$$p [g_s] q \iff (g_q = g_p g_s \land g_s \in H). \quad (2)$$
For the sufficient condition of (2), let \( g_q = g_pg_s \) with \( g_s \in H \).
So \( q = g_q \cdot r = g_pg_s \cdot r = g_p \cdot s \). As \( r \overset{[g_p]}{\rightarrow} s \), we get \( p = g_p \cdot r \overset{[g_q]}{\rightarrow} g_p \cdot s = q \).

Therefore and by (2), we get \( \bar{G} = \mathcal{C}[G, H] \) because
\[
p \overset{[g_q]}{\rightarrow} G q \iff (g_q = g_p g_s \land g_s \in H) \iff g_p \overset{[g_q]}{\rightarrow} \mathcal{C}[G, H] g_q.
\]
It remains to check that \( H \) is a generating subset of \( G \).

Let \( g \in G \). As \( r \) is a root of \( G \), there is a path from \( r \) to \( g \cdot r \):
\[
r = r_0 \overset{a_1}{\rightarrow} G r_1 \ldots r_{n-1} \overset{a_n}{\rightarrow} G r_n = g \cdot r.
\]
As \( A_G = [H] \), there are (unique) \( g_{s_1}, \ldots, g_{s_n} \in H \) such that \( a_1 = [g_{s_1}], \ldots, a_n = [g_{s_n}] \).

By (2), we have \( g_{r_1} = g_{r_0} g_{s_1}, \ldots, g_{r_n} = g_{r_{n-1}} g_{s_n} \) hence \( g = g_{r_n} = g_{s_1} \ldots g_{s_n} \).

The Cayley graph of a language \( L \) has for cycle language the closure of \( L \) by adding and deleting factors of \( L \).

**Lemma 11.** For any group presentation language \( L \), \( L_{\mathcal{C}(L)} = [e]_L \) and \( \mathcal{C}(L) = \mathcal{C}([e]_L) \).

**Proof.**
We have \( L_{\mathcal{C}(L)} = L(\mathcal{C}(L), [e]_L, [e]_L) \) and for any \( u \in A_L^* \),
\[
[e]_L \overset{\alpha}{\rightarrow} \mathcal{C}(L)[e]_L \iff [u]_L = [e]_L \iff u \in [e]_L.
\]

As \( \overset{\alpha}{\rightarrow} [e]_L \subseteq \overset{\alpha}{\rightarrow} L \) with \( L \subseteq [e]_L \), we get \( \overset{\alpha}{\rightarrow} [e]_L = \overset{\alpha}{\rightarrow} L \).

Thus \( [e]_L \) is a group presentation language and \( \mathcal{C}(L) = \mathcal{C}([e]_L) \).

The stability for a group presentation language \( L \) means that its Cayley graph \( \mathcal{C}(L) \) has for cycle language \( L \).

**Lemma 12.** A non-empty language \( L \) is stable if and only if \( L = [e]_L \).

**Proof.**
For every \( u \in L \), \( u \overset{\alpha}{\rightarrow} L \) hence \( u \in [u]_L = [e]_L \). Thus \( L \subseteq [e]_L \).
\( \Rightarrow \) Assume that \( L \) is a non-empty stable language. It remains to check that \( [e]_L \subseteq L \).

There exists \( u \in L \). We get by stability \( \varepsilon \in L \).

By induction on \( n \geq 0 \), (\( \varepsilon \overset{\alpha}{\rightarrow} L u_1 \ldots \overset{\alpha}{\rightarrow} L u_n \)) \( \Rightarrow u_n \in L \). Thus \( [e]_L \subseteq L \).

\( \Leftarrow \) Assume that \( L = [e]_L \).
Let \( u \overset{\alpha}{\rightarrow} L \) with \( u \in L \). It remains to verify that \( v \in L \).

By hypothesis \( u \in [e]_L \) hence \( v \in [u]_L = [e]_L = L \).

### 8 Elementary graph properties

We establish basic properties for vertex-transitive and circular graphs.

Note that the following graph:

![Graph](image)

is strongly connected, circular but is not vertex-transitive.

A first property for vertex-transitive graphs is that the strong connectedness is reduced to the existence of a root.

**Lemma 13.** Any rooted vertex-transitive graph is strongly connected.
Proof.
Let \( G \) be a vertex-transitive graph with root \( r \).
Let us check that \( G \) is strongly connected. Let \( s \in V_G \).
It is sufficient to check that \( s \rightarrow^* r \).
As \( G \) is vertex-transitive, there exists an automorphism \( h \) of \( G \) such that \( h(r) = s \).
As \( r \) is a root, we have \( r \rightarrow^* h^{-1}(r) \) hence \( s = h(r) \rightarrow^* r \).

\[ \blacksquare \]

In addition, any deterministic circular graph with a root is co-deterministic.

**Lemma 14.** Any deterministic and strongly connected circular graph is co-deterministic.

Proof.
Let \( G \) be a deterministic and strongly connected circular graph.
Let \( x \xrightarrow{a} z \) and \( y \xrightarrow{a} z \).
As \( G \) is strongly connected, there exists \( u \in A^* \) such that \( z \xrightarrow{u} x \).
As \( G \) is circular, we have \( au \in \mathcal{L}_G(x,x) = \mathcal{L}_G(y,y) \).
As \( G \) is deterministic, we get \( z \xrightarrow{u} y \). Thus \( z \xrightarrow{u} x \) and \( z \xrightarrow{u} y \).
As \( G \) is deterministic, \( x = y \).

\[ \blacksquare \]

The cycle language of any circular graph is stable when the graph is deterministic.

**Lemma 15.** For any deterministic circular graph \( G \), \( \mathcal{L}_G \) is a stable language.

Proof.
Let us show that \( \mathcal{L}_G \) is preserved by self-insertion.
Let \( uw, v \in \mathcal{L}_G \). One checks that \( uvw \in \mathcal{L}_G \).
There exists \( s, t \in V_G \) such that \( s \xrightarrow{u} t \xrightarrow{w} G s \).
As \( v \in \mathcal{L}_G \), we have \( t \xrightarrow{v} t \) therefore \( uvw \in \mathcal{L}_G(s,s) = \mathcal{L}_G \).
Let us show that \( \mathcal{L}_G \) is preserved by self-deletion.
Let \( uw, v \in \mathcal{L}_G \). One checks that \( uvw \in \mathcal{L}_G \).
There exists \( r, s, t \in V_G \) such that \( r \xrightarrow{u} G s \xrightarrow{u} G t \xrightarrow{w} G r \).
As \( v \in \mathcal{L}_G \) we have \( s \xrightarrow{v} G s \). As \( G \) is deterministic, we get \( s = t \).
So \( uvw \in \mathcal{L}_G(r,r) = \mathcal{L}_G \).

\[ \blacksquare \]

The cycle language of any circular graph is closed under conjugacy when the graph is strongly connected.

**Lemma 16.** For any strongly connected circular graph \( G \), \( \mathcal{L}_G \) is closed under conjugacy and of letter set \( A_{\mathcal{L}_G} = A_G \).

Proof.
Let us check that \( \mathcal{L}_G \) is closed under conjugacy. Let \( uv \in \mathcal{L}_G \).
There exists \( s, t \) such that \( s \xrightarrow{u} t \xrightarrow{v} s \). So \( vu \in \mathcal{L}_G(t,t) = \mathcal{L}_G \).
We have \( A_{\mathcal{L}_G} \subseteq A_G \). Let us check the reverse inclusion. Let \( a \in A_G \).
There exists \( s, t \) such that \( s \xrightarrow{a} t \).
As \( G \) is strongly connected, there exists \( u \) such that \( t \xrightarrow{u} s \).
Thus \( au \in \mathcal{L}_G \) hence \( a \in A_{\mathcal{L}_G} \).

\[ \blacksquare \]

The cycle language of any circular graph is a group presentation when the graph is deterministic, simple with a root.
Lemma [17] For any strongly connected, deterministic circular simple graph $G$, $L_G$ is a group presentation language.

Proof. By Lemma [16] $L_G$ is stable hence by Lemma [12] $[e]_{L_G} = L_G$. By Lemma [19] $L_G$ is closed under conjugacy and of letter set $A_{L_G} = A_G$. As $A_G \neq \emptyset$, $L_G$ satisfies condition (i) of a group presentation language. Let $a \in A_{L_G}$. There exists words $u, v$ such that $uav \in L_G$. Thus $ave, vua \in L_G = [e]_{L_G}$. So $L_G$ satisfies condition (ii) of a group presentation language. Let us check condition (iii). Let $a, b \in A_{L_G}$ such that $[a]_{L_G} = [b]_{L_G}$. So $a \not\rightarrow_{L_G} b$. By (ii), there exists $u \in A^*$ such that $ua \in [e]_{L_G}$. So $ub \in [e]_{L_G}$. Thus $ua, ub \in L_G$. Let $r \in V_G$. There are $s, t$ such that $r \rightarrow_G s \rightarrow_G r$ and $r \rightarrow_G t \rightarrow_G r$. As $G$ is deterministic, $s = t$. As $G$ is simple, $a = b$. Thus $L_G$ is a group presentation language. ▽

The concepts of circularity and vertex-transitivity coincide for deterministic and strongly connected graphs.

Lemma [18] For any deterministic and strongly connected graph $G$, $G$ is vertex-transitive if and only if $G$ is circular.

Proof. Let $G$ be a strongly connected, deterministic and circular graph. We have to check that $G$ is vertex-transitive. This follows from the fact that the equivalence of vertices corresponds to the equality of their cycle language:

$$s \simeq_G t \iff L_G(s, s) = L_G(t, t).$$

The necessary condition is immediate. Let us show the sufficient condition. Let $s, t \in V_G$ such that $L_G(s, s) = L_G(t, t)$.

We define the relation

$$k = \{ (s', t') \mid \exists u \in A^* (s \rightarrow_G s' \land \exists t \rightarrow_G t') \}$$

We have $(s, t) \in k$. Let us prove that $k$ is an automorphism of $G$.

Let us check that The domain of $k$ is $V_G$. Let $s' \in V_G$.

As $G$ is strongly connected, we have $s \rightarrow_G s'$ for some $u \in A_G$. As $G$ is deterministic, $L_G(s', s) = u^{-1}L_G(s, s) = u^{-1}L_G(t, t)$. As $G$ is strongly connected, $L_G(s', s) \neq \emptyset \iff u^{-1}L_G(t, t) \neq \emptyset$.

So there exists $t'$ such that $t \rightarrow t'$.

In addition, $k$ is a mapping since for any $(p, q), (p, r) \in k$, there exists $u, v \in A^*$ such that

$$s \rightarrow_G p \land t \rightarrow_G q \land s \rightarrow_G p \land t \rightarrow_G r.$$

As $G$ is strongly connected, there exists $w \in A^*$ such that $p \rightarrow_G s$. Also $uw, vw \in L_G(s, s) = L_G(t, t)$.

As $G$ is deterministic, we have $q \rightarrow t$ and $r \rightarrow t$.

As $G$ is co-deterministic, we get $q = r$.

By symmetry of $s, t$, the relation $k^{-1}$ is a mapping, hence $k$ is a bijection.

By symmetry of $s, t$, it only remains to check that $k$ is a morphism. Let $p \rightarrow_G q$.

As $G$ is strongly connected, there exists $u, v \in A^*$ such that $s \rightarrow_G p$ and $q \rightarrow_G s$. D. Cauca 21
Therefore $uav \in L_G(s,s) = L_G(t,t)$. So there exists $t', t''$ such that $t \xrightarrow{u} G t' \xrightarrow{a} G t''$. As $k$ is a bijection, we get $t' = k(p)$ and $t'' = k(q)$ hence $k(p) \xrightarrow{a} G k(q)$. △

Another basic property for strongly connected circular or vertex-transitive graphs is the completeness and the co-completeness.

**Lemma 20.** Any strongly connected circular graph is complete and co-complete.

**Proof.**
Let $G$ be a strongly connected circular graph.
Let $s \in V_G$ and $a \in A_G$. There exists an edge $p \xrightarrow{a} q$ of $G$.
As $G$ is strongly connected, there exists $u \in A^*$ such that $q \xrightarrow{u} p$.
As $G$ is circular, $au \in L_G(p,p) = L_G(s,s)$.
So there exists a vertex $t$ such that $s \xrightarrow{a} G t$. Thus $G$ is complete.
Furthermore $G^{-1}$ remains strongly connected and circular.
Therefore $G^{-1}$ is complete i.e. $G$ is co-complete. △

The vertex-transitivity coincides with the edge-transitivity for any deterministic and complete graphs.

**Lemma 21.** For any graph $G$ deterministic and complete, $G$ is vertex-transitive $\iff$ $G$ is edge-transitive.

**Proof.**
\(\Rightarrow\) : let $G$ be a vertex-transitive deterministic graph.
Let us check that $G$ is edge-transitive. Let $s \xrightarrow{a} t$ and $s' \xrightarrow{a} t'$.
As $G$ is vertex-transitive, there exists an automorphism $h$ of $G$ such that $h(s) = s'$.
As $h$ is a morphism, we have $s' = h(s) \xrightarrow{a} h(t)$.
As $G$ is deterministic, we get $h(t) = t'$.

\(\Leftarrow\) : let $G$ be an edge-transitive complete graph.
Let us check that $G$ is vertex-transitive. Let $s, s' \in V_G$ and $a \in A_G$.
As $G$ is complete, there exists $t, t'$ such that $s \xrightarrow{a} t$ and $s' \xrightarrow{a} t'$.
As $G$ is edge-transitive, there exists an automorphism $h$ of $G$ such that $h(s) = s'$. △

The elementary circularity involves the circularity.

**Lemma 22.** Any elementary circular graph $G$ is circular with $L_G \subseteq [\varepsilon]_{E_G}$.

**Proof.**
Let $G$ be an elementary circular graph.
i) Let us show that $G$ is circular.
It is sufficient to check that for every $n \geq 0$,

\[ L_G(s,s) \cap A^{\leq n} = L_G(t,t) \cap A^{\leq n} \text{ for all } s, t \in V_G \]

where $A^{\leq n} = \{ u \in A^* | |u| \leq n \}$. We prove this equality by induction on $n$.
$n = 0$ : we have $L_G(s,s) \cap A^0 = \{ \varepsilon \}$ for all $s \in V_G$.
$n \Rightarrow n + 1$ : Let $s, t \in V_G$.
By induction hypothesis and by symmetry of $s$ with $t$, it is sufficient to check that

\[ L_G(s,s) \cap A^{n+1} \subseteq L_G(t,t) \cap A^{n+1}. \]
Let \( u \in L_G(s, s) \) of length \(|u| = n + 1\). Let us show that \( u \in L_G(t, t) \).

We distinguish two complementary cases below.

**Case 1:** \( u \in E_G(s) \). By hypothesis, \( u \in E_G(t) \subset L_G(t, t) \).

**Case 2:** \( u \not\in E_G(s) \). There are \( x, u', y \in A^* \) and \( s' \in V_G \) such that

\[
\begin{align*}
  s &\xrightarrow{x} s' \xrightarrow{u'} s' \xrightarrow{y} s \\
\end{align*}
\]

with \( u = xu'y \) and \(|u'| > 0\) minimal.

Thus \( xy \in L_G(s, s) \) with \(|xy| < |u|\). By induction hypothesis, \( xy \in L_G(t, t) \). There is \( t' \) such that

\[
\begin{align*}
  t &\xrightarrow{x} t' \xrightarrow{y} t.
\end{align*}
\]

Furthermore \( u' \in E_G(s') = E_G(t') \) hence \( u = xu'y \in L_G(t, t) \).

**ii)** Let us show that \( u \in L_G \implies u \xrightarrow{\ast} E_G \varepsilon \).

By induction on \(|u| \geq 0\). For \(|u| = 0\) i.e. \( u = \varepsilon \), we have \( \varepsilon \xrightarrow{\ast} E_G \varepsilon \) (and \( \varepsilon \in L_G \)).

**Case 1:** \( u \in E_G \). Thus \( (u, \varepsilon) \in \tilde{E}_G \) hence \( u \xrightarrow{E_G} \varepsilon \).

**Case 2:** \( u \not\in E_G \). Let \( s \in V_G \). As \( u \in L_G \), we have \( s \xrightarrow{u} G \).

As \( u \not\in E_G \), there are \( x, u', y \in A^*_G \) and \( s' \in V_G \) such that

\[
\begin{align*}
  s &\xrightarrow{x} s' \xrightarrow{u'} s' \xrightarrow{y} G \quad \text{with} \quad u = xu'y \quad \text{and} \quad |u'| > 0 \text{ minimal}.
\end{align*}
\]

Therefore \( u' \in E_G(s') = E_G \) and \( xy \in L_G(s) = L_G \).

As \(|xy| < |u|\) and by induction hypothesis, \( xy \xrightarrow{\ast} E_G \varepsilon \).

As \( u' \in E_G \), we have \( u' \xrightarrow{E_G} \varepsilon \) hence \( u = xu'y \xrightarrow{E_G} xy \xrightarrow{\ast} E_G \varepsilon \).

The converse of Lemma 22 is not verified for the following graph \( G \):

![Graph Image]

since \( L_G = a^* \) while \( E_G(1) = E_G(2) = \{a, aa, aaa\} \) and \( E_G(3) = \{a, aaa\} \).

The circularity involves the elementary circularity for deterministic graphs.

**Lemma 23.** Any deterministic circular graph \( G \) is elementary circular with \( L_G = [\varepsilon]_{E_G} \).

**Proof.**

Let \( G \) be a deterministic circular graph.

**i)** Let us check that \( G \) is elementary circular.

We have \( L_G(s, s) = L_G(t, t) \) for all \( s, t \in V_G \).

Suppose that there are \( s, t \in V_G \) and \( u \in E_G(s) \subset E_G(t) \).

We have \( u \in E_G(s) \subset L_G(s, s) = L_G(t, t) \).

As \( u \not\in E_G(t) \), there are \( x, u', y \in A^* \) and \( t' \in V_G \) such that

\[
\begin{align*}
  t &\xrightarrow{x} t' \xrightarrow{u'} t' \xrightarrow{y} t \\
\end{align*}
\]

with \( u = xu'y \) and \(|u'| > 0\).

As \( u = xu'y \in L_G(s, s) \) and \( G \) is deterministic, there are single vertices \( s', s'' \) such that
s \xrightarrow{u} s' \xrightarrow{u'} s'' \xrightarrow{y} s.

As \( u' \in L_G(t', s') = L_G(s', s') \) and \( G \) is deterministic, we get \( s' = s'' \).

Thus \( u = xu'g \notin E_G(s) \) which is a contradiction.

ii) By Lemma 22, we have \( L_G \subseteq [\varepsilon] \subseteq E_G \subseteq [\varepsilon]L_G \).

By Lemma 15, \( L_G \) is a stable language. By Lemma 12, \( L_G = [\varepsilon]L_G \) hence the equality.

The connectedness for finite vertex-transitive graphs implies the strong connectedness.

Lemma 25. Any finite connected vertex-transitive graph is strongly connected.

Proof.
Let us check that \( G \) is strongly connected.
Let \( t \rightarrow s \). It is sufficient to show that \( s \rightarrow^* t \).

As \( G \) is finite, we take an elementary path \( r_0 \rightarrow \ldots \rightarrow r_n \) of maximal length \( n \).

As \( s \) is isomorphic to \( r_0 \), there is an elementary path \( s = s_0 \rightarrow \ldots \rightarrow s_n \).

By maximality of \( n \), we get \( t \in \{s_0, \ldots, s_n\} \) hence \( s \rightarrow^* t \).

The finiteness of Lemma 25 can not be removed since the graph
\[
\{ n + 1 \xrightarrow{a} n \mid n \in \mathbb{Z} \} \cup \{ (n, \varepsilon) \xrightarrow{a} n \mid n \in \mathbb{Z} \}
\]
\[
\cup \{ (n, u) \xrightarrow{a} (n, u) \mid n \in \mathbb{Z} \land u \in \{0, 1\}^* \land i \in \{0, 1\} \}
\]
represented as follows:

```
  a   a   a
  a   a   a
  a   a   a   a   a
  a   a   a   a   a
  a   a   a   a   a
```

is not rooted whereas it is deterministic, connected and vertex-transitive.

Let us conclude the proof of Theorem 28.

Theorem 28. The generalized Cayley graphs are the deterministic, co-deterministic, vertex-transitive simple graphs.

Proof.
It remains to check that \( C[V_I \times \text{Comp}, K] \triangleleft f \cdot f \).

Let us check that \( G \subseteq f(C[V_I \times \text{Comp}, K]) \).

Let \( s \xrightarrow{a} t \). There is a unique \( C \in \text{Comp} \) such that \( s \xrightarrow{a} C t \). So \( f^{-1}_C(s) \xrightarrow{a} f^{-1}_C(t) \).

As \( I = C[V_I, H] \), there is \( h \in H \) such that \([h] = a\) and \( f^{-1}_C(t) = f^{-1}_C(s) \cdot h\). So
\[
(f^{-1}_C(s), C) \xrightarrow{[h, I]} C[V_I, K_I] \cdot (f^{-1}_C(t), C) \quad \text{i.e.} \quad s \xrightarrow{a} f(C[V_I \times \text{Comp}, K], t).
\]

It remains to check that \( f(C[V_I \times \text{Comp}, K]) \subseteq G \).

Let \( (s, C) \xrightarrow{a} C[V_I \times \text{Comp}, K] \cdot (t, D) \).
There is $h \in H$ such that $[h] = a$ and $t = s \cdot h$ and $C = D$.
So $s \xrightarrow{[h]} C[V_I,K_J]$ and $s \xrightarrow{a} t$. Thus $f(s,C) = f_C(s) \xrightarrow{a} f_C(t) = f(t,D)$. □