ON CIRCULAR FLOWS: LINEAR STABILITY AND DAMPING

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ABSTRACT. In this article we establish linear inviscid damping with optimal decay rates around 2D Taylor-Couette flow and similar monotone flows in an annular domain \( B_{r_2}(0) \setminus B_{r_1}(0) \subset \mathbb{R}^2 \). Following recent results by Wei, Zhang and Zhao [10], we establish stability in weighted norms, which allow for a singularity formation at the boundary, and additional provide a description of the blow-up behavior.

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1. Introduction

In this article we consider the linear stability and long-time asymptotic behavior of circular flows in an annular domain \( (x, y) \in B_{r_2}(0) \setminus B_{r_1}(0) \). Such two-dimensional flows can for example be established experimentally in rotating cylinders, where the rotation is sufficiently slow as to not cause a (three-dimensional) Taylor-Couette instability.

In this setting, radial vorticities

\[
\omega(x, y) = \omega(r),
\]

\[
v(x, y) = \partial_r \psi e_\theta = \left(-\frac{y}{x}\right) \frac{\psi'(r)}{r},
\]

\[
\psi''(r) + \frac{1}{r} \psi'(r) = \omega(r),
\]

are stationary solutions of the incompressible 2D Euler equations.

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Considering a small perturbation to Taylor-Couette flow,
\[ \frac{\phi'(r)}{r} = A + \frac{B}{r^2}, \]
we observe in Figure 1 that for \( B = 0 \), i.e. constant angular velocity, perturbations are rotated while keeping their shape. However, in the general case when \( B \neq 0 \), \( \frac{\phi'(r)}{r} \) is strictly monotone and the perturbation is sheared in way reminiscent of plane Couette flow, as is depicted in Figure 2. This mixing behavior underlies the phenomenon of (linear) inviscid damping.

**Figure 1.** Transport with constant angular velocity. We consider the Taylor-Couette flow \( r \) in an annulus. The time 1 flow-lines are drawn as arrows. A perturbation initially concentrated on a line stays concentrated on a line. On the right this behavior is expressed in polar coordinates.

**Figure 2.** Transport by a monotone flow. We consider the Taylor-Couette flow \( r + \frac{1}{r} \), which we observe to be mixing. As time tends to infinity this mixing results in weak convergence to an averaged quantity.
Considering polar coordinates, the linearized Euler equations around these stationary solutions are given by

\[
\partial_t f + U(r) \partial_\theta f = b(r) \partial_\theta \phi,
\]

\[
(\partial_r^2 + \frac{1}{r} \partial_r + \frac{1}{r^2} \partial_\theta^2) \phi = f,
\]

\[
\partial_\theta \phi |_{r=r_1, r_2} = 0,
\]

\[
(t, \theta, r) \in \mathbb{R} \times T \times [r_1, r_2],
\]

where \( U \) and \( b \) are given by

\[
U(r) = \frac{\phi'(r)}{r},
\]

\[
b(r) = -\frac{1}{r} \partial_r (\partial_r^2 \phi(r) + \frac{1}{r} \partial_r \phi(r)),
\]

and \( b(r) \equiv 0 \) if and only if one considers Taylor-Couette flow, \( U(r) = A + \frac{B}{r} \).

As suggested by our notation, these equations share strong similarities with the linearized Euler equations around a shear flow \( (U(y), 0) \) in a plane finite periodic channel, \( T \times [0, 1] \):

\[
\partial_t \omega + U(y) \partial_x \omega - U''(y) \partial_x \phi = 0,
\]

\[
(\partial_y^2 + \partial_x^2) \phi = \omega,
\]

\[
\partial_x \phi |_{y=0, 1} = 0,
\]

\[
(t, x, y) \in \mathbb{R} \times T \times [0, 1].
\]

Here, various different approaches have been used to study this and related settings.

- In [9], Stepin studies the asymptotic stability of monotone shear flows using spectral methods. Under the assumption that the associated Rayleigh boundary value problem possesses no eigenvalues, he obtains an asymptotic description of the stream function and non-optimal decay rates.

- In [6], Bouchet and Morita provide heuristic results which suggest that the algebraic decay rates of Couette flow should hold for general monotone flows as well. However, their methods are not rigorous and do not provide sufficient error and stability estimates, especially in higher Sobolev regularity, in order to prove decay with optimal rates.

- In [13] and [11], the author establishes linear inviscid damping and scattering for monotone shear flows in an infinite and finite periodic channel. In the latter setting, we restrict to perturbations in \( H^2 \cap H^1_0 \) in order to obtain the optimal decay rates. Conversely, in the setting without vanishing Dirichlet boundary values, the sharp stability threshold is shown to be given by \( H^{s}, s = 3/2 \) due to asymptotic singularity formation at the boundary.

- In [10], Wei, Zhang and Zhao follow similar methods as in [9] and establish linear inviscid damping with optimal decay rates for monotone shear flows under the condition of there being no embedded eigenvalues. In particular, they remove the requirement of vanishing Dirichlet data and note that, using the boundary conditions of the velocity field and Hardy’s inequality, one may allow for some blow-up at the boundary and still attain optimal decay rates.

- In a seminal work [3], [4] Bedrossian and Masmoudi establish nonlinear inviscid damping for Couette flow in an infinite periodic channel. There perturbations are required to be extremely regular, more precisely of Gevrey 2 class, in order to control nonlinear resonances. In particular, due to the singularity formation at the boundary and the associated blow-up of
relatively low Sobolev norms, the question of linear inviscid damping for settings with boundary remains open.

- In addition to the inviscid setting, Bedrossian, Germain and Masmoudi also consider Couette flow as a solution of the Navier-Stokes equation in a two and three-dimensional infinite periodic channel. There, in addition to inviscid damping, the interaction between the mixing and viscous behavior yields additional stabilization by enhanced dissipation. Nonlinear inviscid damping is then established in Gevrey regularity [2] and more recently in Sobolev regularity [1], [5], where the threshold for stability results depends on $\nu > 0$.

- In the circular setting, research has focused on instability results, such as Taylor-Couette instability, bifurcation and turbulence. For an introduction we refer to the book of Chossat and Iooss [7].

As the main results of this article we prove linear inviscid damping and scattering for a general class of circular flows, satisfying suitable monotonicity and smallness assumptions. In comparison to our previous results, we note the following changes and improvements:

- We obtain optimal decay rates also for perturbations without vanishing Dirichlet data.
- We show that $\partial_\theta W$ splits into a bulk part $\Gamma$, which is stable also in un-weighted higher Sobolev spaces, and a boundary correction $\beta$, which is stable in a suitably weighted $H^1$ space, but exhibits blow-up in $L^\infty$.
- The smallness condition is strongly reduced for results in higher regularity.
- In this circular setting, periodicity in $\theta$ is a natural condition, unlike in the setting of a plane periodic channel.
- We obtain a finer description of the boundary layer in terms of only the Dirichlet boundary values of the initial data.

1.1. **Main results.** Our main results are summarized in the following theorem.

**Theorem 1.1** (Linear inviscid damping with optimal decay rates). Let $0 < r_1 < r_2 < \infty$ and let $U : (r_1, r_2) \to (a, b)$ be bilipschitz and suppose that $h(\cdot) = b(U^{-1}(\cdot)) \in W^{3, \infty}((a, b))$ and that $\|h\|_{W^{1, \infty}}$ is sufficiently small. Then, for any $f_0 \in H_\theta^{-1}H_r^2$ there exists $v_\infty(r)$ such that the solution $f$ of (3) satisfies

\[
\|v(t, \theta, r) - v_\infty(r)e_\theta\|_{L^2} \lesssim t > 1 \|f_0\|_{H_\theta^{-1}H_1^1},
\]

\[
\|v(t, \theta, r)e_r\|_{L^2} \lesssim t > 2 \|f_0\|_{H_\theta^{-1}H_2^2},
\]

as $t \to \infty$. There exists $f_\infty \in L^2_0H^1_r$ such that

\[
f(t, \theta - tU(r), r) \to f_\infty \text{ in } L^2,
\]

and

\[
\|f(t, \theta - tU(r), r) - f_\infty(\theta, r)\|_{L^2_\theta} \lesssim t > 1 \|f_0\|_{H_\theta^{-1}H_2^2}.
\]

Furthermore, $f$ satisfies

\[
\|f(t, \theta - tU(r), r)\|_{H^{-1}H^1} + \|(r - r_1)(r - r_2)\frac{d^2}{dr^2} f(t, \theta - tU(r), r)\|_{H^{-1}H^1} \lesssim \|f_0\|_{H^{-1}H^2}.
\]

However, unless $bf|_{r=r_1,r_2}$ is constant,

\[
\sup_{t \geq 0} \|f(t, \theta - tU(r), r)\|_{H^{-1}H^1} = \infty,
\]
for any $s > 3/2$. More precisely, there exists an (explicit) function $\nu(t, \theta, r)$ determined solely by $f_0|_{r=r_1, r_2}$ and $U$ such that

$$\| \frac{d^2}{dt^2} f(t, \theta - rU(r), r) - \nu \|_{L^2L^2} \leq \| f_0 \|_{L^2H^2},$$

and such that

$$\|(r - r_1)(r - r_2)\nu\|_{L^2} \leq |f_0|_{r=r_1,r_2}.$$

**Remark 1.**

- While $h = b(U^{-1})$ is required to be regular, the smallness assumption is only imposed on the $W^{1, \infty}$ norm.
- This theorem summarizes the main results of Proposition 2.2 and Theorems 3.1, 4.1 and 4.2 in terms of common norms in the variables $t, \theta, r$. In Section 3 we introduce a scattering formulation, which is used throughout the article.
- The function $\nu$ is introduced in Section 4.2.2.
- In [10] it has been observed that, by a use of Hardy’s inequality, the second derivative of $W$ can be allowed to form a singularity as $t \to \infty$ while still attaining the optimal $t^{-2}$ decay rate. Here, we stress that stability in $H^2$ indeed does not hold due to singularity formation at the boundary as $t \to \infty$, as quantified in $\nu$ and $\beta$ (c.f. Section 4).
- As we discuss in Section 3, our method of proof does not rely on cancellations or conserved quantities. Hence, the results extend to complex-valued $b(r)$ and various modified equations in a straightforward manner. In the case of the linearized Euler equations in a plane finite periodic channel, however, Wei, Zhang and Zhao [10] have shown, using different methods, that weaker assumptions suffice to obtain damping.

Similarly to [11] our strategy is to first establish the damping and scattering result, assuming stability in higher Sobolev norms. We stress that the damping estimate necessarily loses regularity. Hence, usual Duhamel fixed point iteration approaches or energy methods can not yield stability results. Instead we employ a finer study of the damping mechanism, which allows us to construct a Lyapunov functional using the mode-wise decay to avoid the necessary loss of regularity of uniform damping estimates.

The remainder of the article is organized as follows:

- In Section 2, we show that regularity of the vorticity in coordinates moving with the flow can be exchanged for uniform damping estimates and that the problem of linear inviscid damping thus reduces to a stability problem. As motivating examples, we discuss the specific cases of Taylor-Couette flow, a point vortex and of Couette flow in a plane channel, where explicit solutions are available and, in a sense, trivial.
- In Section 3, we introduce several reductions and changes of variables to arrive at a scattering formulation of the linearized Euler equations. Subsequently, we analyze the structure of the equation and establish $L^2$ stability.
- Section 4 considers higher regularity and singularity formation at the boundary. Compared to [12], in addition to considering a circular setting, we introduce a splitting $\partial_y W = \Gamma + \beta$, where $\Gamma$ is shown to be stable in higher regularity, regardless of Dirichlet boundary data. On the other hand, $\beta$ is determined solely by the underlying circular flow and the Dirichlet boundary data of the initial perturbation and provides an explicit characterization of the boundary layer. Subsequently, we further split $\partial_y \beta$ to obtain an explicit characterization of the $H^2$ blow-up in the form of $\nu$ and stability.
in weighted spaces. Here, we rely on a new approach based on Duhamel’s principle and an iterative estimate in order to control the evolution of the weighted quantities.

- The Appendices A and B provide a description of boundary evaluations for elliptic ODEs and a variant of Duhamel’s formula adapted to a time-dependent right-hand-side of the equation.

2. Damping by mixing, the role of regularity and examples

As in the case of inviscid damping in a plane channel or Landau damping, decay of the velocity/force field and regularity of the solution in a coordinate system moving with the flow are closely linked. More precisely, in this section we show that uniform damping estimates closely correspond to a control of the regularity of

$$W(t, \theta, r) := f(t, \theta - tU(r), r)$$

with respect to $r$ and that such a control is necessary. The problem of linear inviscid damping with optimal decay rates thus turns out to be a stability problem, studied in Section 4, which is the main focus of this article.

We consider the linearized Euler equations

$$\begin{align*}
\partial_t f + U(r)\partial_\theta f &= b(r)\partial_\theta \phi, \\
\left(\partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\theta^2\right)\phi &= f, \\
\partial_\theta \phi|_{r=r_1, r_2} &= 0,
\end{align*}$$

(8)

as a perturbation around the transport problem

$$\partial_t g + U(r)\partial_\theta g = 0,$$

(9)

$$(t, \theta, r) \in \mathbb{R} \times \mathbb{T} \times [r_1, r_2].$$

Based on this view, we measure the deviation of these equations by introducing the scattered vorticity

$$W(t, \theta, r) := f(t, \theta - tU(r), r).$$

Assuming regularity of $W$ uniformly in time, damping results for (8) then reduce to estimates for (9). Here, we it has recently been observed by Wei, Zhang and Zhao \[10\] that quadratic decay rates only require control of a weighted $H^2$ norm

$$\|W\|_{H^1} + \|y(1-y)\partial_y^2 W\|_{L^2}$$

(11)

by using a Hardy inequality in the duality estimate.

The following two propositions provide damping estimates in terms of regularity of $W$ in the case of a plane channel and a circular domain, respectively.

**Proposition 2.1** (Damping by regularity for plane channel \[10\], \[12\], \[8\]). Let $-\infty < a < b < \infty$ and let $U : (a, b) \to \mathbb{R}$ be locally $C^1$ and suppose that $U'(y) \neq 0$ for almost every $y \in (a, b)$. Let $W \in H^{-1}_x H^1_y (\mathbb{T} \times (a, b))$ with $\int_{\mathbb{T}} W \, dx = 0$ and let

$$\omega(t, x, y) = W(t, x - tU(y), y).$$
Let further the associated velocity field $v$ be defined by
\begin{align*}
v_1 &= -\partial_y \phi, \\
v_2 &= \partial_x \phi, \\
\Delta \phi &= \omega, \\
\partial_x \phi|_{y=a,b} &= 0, \\
\phi &\in \dot{H}^1.
\end{align*}

Then $v$ satisfies
\begin{equation}
\|v(t)\|_{L^2} \lesssim \min \left( \frac{\|W\|_{H_x^{-1}L_y^2}}{U^2}, t^{-1} \frac{\|W\|_{H_x^{-1}L_y^2}}{H_x^{-1}H_y^1} \right). \tag{12}
\end{equation}

Furthermore, suppose that $\partial_y^2 W$ exists. Then, $v_2$ additionally satisfies
\begin{align*}
\|v_2(t)\|_{L^2} &\leq t^{-2} \left( \|W\|_{(U^2)^2} \frac{1}{H_x^{-1}H_y^1} + \left\| W \partial_y \frac{1}{(U^2)^2} \right\|_{H_x^{-1}L_y^2} \right) + \min \left( \left\| \left( y-a \right) \left( y-b \right) \frac{\partial_y^2 W}{(U^2)^2} \right\|_{H_x^{-1}L_y^2}, \left\| \frac{\partial_y^2 W}{(U^2)^2} \right\|_{H_x^{-1}L_y^2} \right) \tag{14}.
\end{align*}

Proof of Proposition 2.1. We note that, by integration by parts,
\begin{align*}
\|v\|_{L^2}^2 = \|\nabla \phi\|_{L^2}^2 = -\iint \phi \omega \, dx \, dy.
\end{align*}

Applying Plancherel’s theorem with respect to $x$ and noting that,
\begin{align*}
\mathcal{F}_x(\omega(t, \cdot, y))(k) = e^{iktU(y)} \hat{W}(t,k),
\end{align*}

this equals
\begin{align*}
\sum_{k \neq 0} \int \hat{\phi}(t,k,y) e^{iktU(y)} \hat{W}(t,k,y).
\end{align*}

Integrating
\begin{align*}
e^{iktU} = \frac{1}{iktU^2} \partial_y e^{iktU(y)}
\end{align*}

by parts, we further obtain
\begin{align*}
\sum_{k \neq 0} \frac{1}{t} \int e^{iktU(y)} \partial_y \left( \frac{\tau}{ikU^2} \hat{W} \right)
\end{align*}

which is controlled by
\begin{align*}
t^{-1} \|\phi\|_{L^2 H^1} \left\| \frac{W}{U^2} \right\|_{H_x^{-1}H_y^1}.
\end{align*}

The estimate (12) thus follows by noting that
\begin{align*}
\|\phi\|_{L^2 H^1} \leq \|v\|_{L^2}.
\end{align*}

In order to prove (14), we note that
\begin{align*}
\Delta v_2 &= \partial_y \omega
\end{align*}
and define \( \psi \) s.t.
\[
\Delta \psi = v_2, \\
\partial_x \psi \big|_{y=a,b} = 0, \\
\psi \in H^1.
\]

Then, using integration by parts, we obtain
\[
\|v\|_{L^2}^2 = \int \int \psi \partial_x \omega = \sum_{k \neq 0} \hat{\omega}_k e^{iktU(y)} \int \psi \hat{W} dy \\
= \frac{1}{U^2} \sum_{k \neq 0} \int e^{iktU(y)} \partial_y \left( \frac{1}{U} \partial_y \left( \frac{1 - \psi}{k} \right) \right) dy \\
+ \frac{1}{U^2} \sum_{k \neq 0} e^{iktU(y)} \frac{1}{U} \partial_y \left( \frac{1 - \psi}{k} \right) \big|_{y=a}^{b}
\]

The result hence follows by the Cauchy-Schwarz inequality, the trace map and by using the estimates
\[
\| \phi \|_{H^1} \lesssim \|v_2\|_{L^2},
\]
\[
\| \phi \|_{(y-a)(y-b)} \|_{L^2} \lesssim \| \phi \|_{H^1},
\]
The first estimate here follows by standard elliptic regularity theory, while the second one is given Hardy’s inequality, as observed in \([10]\].

The following proposition adapts these results to the setting of circular flows.

**Proposition 2.2 (Damping for circular flows; \([10]\), \([12]\), \([8]\)).** Let \(0 < r_1 < r_2 < \infty\) and let \(U : (r_1, r_2) \rightarrow \mathbb{R}\) be locally \(C^1\) with \(U'(r) \neq 0\) for almost every \(r \in (r_1, r_2)\).

Let \(W(t, \theta, r) \in H^{-1}L^1(\mathbb{T} \times (r_1, r_2), r dr d\theta)\) with \(\int_\mathbb{T} W d\theta = 0\) and let
\[
f(t, \theta, r) = W(t, \theta - tU(r), r).
\]

Let further the associated velocity field be by defined by
\[
\begin{align*}
v_r(t, \theta, r) &= \frac{1}{r} \partial_\theta \phi(t, \theta, r), \\
v_\theta(t, \theta, r) &= \partial_r \phi(t, \theta, r), \\
(\partial_r^2 + \frac{1}{r} \partial_r + \frac{1}{r^2} \partial_\theta^2) \phi &= f, \\
\partial_\theta \phi \big|_{r=r_1,r_2} &= 0, \\
v &\in L^2(r dr d\theta).
\end{align*}
\]

Then \(v\) satisfies
\[
\|v(t)\|_{L^2(r dr d\theta)} \lesssim \min \left( \|W\|_{L^2(r dr d\theta)}, t^{-1} \frac{\|W(t)\|_{L^2(r dr d\theta)}}{r} \|H^{-1}L^2_0(r dr d\theta)\| \right),
\]
(16)
\[
\|v(t)\|_{L^2(r dr d\theta)} \lesssim \min \left( \|W\|_{L^2(r dr d\theta)}, t^{-1} \frac{\|W(t)\|_{L^2(r dr d\theta)}}{r} \|H^{-1}L^2_0(r dr d\theta)\| \right).
\]
(17)
\[
\left( \frac{\|W(t)\|_{L^2_0}}{r} + W r \partial_r \frac{1}{r} \|L^2_0(\mathbb{T} \times (r_1, r_2)) \| \right) + \| \partial_r \partial_\theta W\|_{H^{-1}L^2(r dr d\theta)}
\]

and let
\[
\Delta \psi = v_2, \\
\partial_x \psi \big|_{y=a,b} = 0, \\
\psi \in H^1.
\]

Then, using integration by parts, we obtain
\[
\|v\|_{L^2}^2 = \int \int \psi \partial_x \omega = \sum_{k \neq 0} \hat{\omega}_k e^{iktU(y)} \int \psi \hat{W} dy \\
= \frac{1}{U^2} \sum_{k \neq 0} \int e^{iktU(y)} \partial_y \left( \frac{1}{U} \partial_y \left( \frac{1 - \psi}{k} \right) \right) dy \\
+ \frac{1}{U^2} \sum_{k \neq 0} e^{iktU(y)} \frac{1}{U} \partial_y \left( \frac{1 - \psi}{k} \right) \big|_{y=a}^{b}
\]

The result hence follows by the Cauchy-Schwarz inequality, the trace map and by using the estimates
\[
\| \phi \|_{H^1} \lesssim \|v_2\|_{L^2},
\]
\[
\| \phi \|_{(y-a)(y-b)} \|_{L^2} \lesssim \| \phi \|_{H^1},
\]
The first estimate here follows by standard elliptic regularity theory, while the second one is given Hardy’s inequality, as observed in \([10]\].

The following proposition adapts these results to the setting of circular flows.

**Proposition 2.2 (Damping for circular flows; \([10]\), \([12]\), \([8]\)).** Let \(0 < r_1 < r_2 < \infty\) and let \(U : (r_1, r_2) \rightarrow \mathbb{R}\) be locally \(C^1\) with \(U'(r) \neq 0\) for almost every \(r \in (r_1, r_2)\).

Let \(W(t, \theta, r) \in H^{-1}L^1(\mathbb{T} \times (r_1, r_2), r dr d\theta)\) with \(\int_\mathbb{T} W d\theta = 0\) and let
\[
f(t, \theta, r) = W(t, \theta - tU(r), r).
\]

Let further the associated velocity field be by defined by
\[
\begin{align*}
v_r(t, \theta, r) &= \frac{1}{r} \partial_\theta \phi(t, \theta, r), \\
v_\theta(t, \theta, r) &= \partial_r \phi(t, \theta, r), \\
(\partial_r^2 + \frac{1}{r} \partial_r + \frac{1}{r^2} \partial_\theta^2) \phi &= f, \\
\partial_\theta \phi \big|_{r=r_1,r_2} &= 0, \\
v &\in L^2(r dr d\theta).
\end{align*}
\]

Then \(v\) satisfies
\[
\|v(t)\|_{L^2(r dr d\theta)} \lesssim \min \left( \|W\|_{L^2(r dr d\theta)}, t^{-1} \frac{\|W(t)\|_{L^2(r dr d\theta)}}{r} \|H^{-1}L^2_0(r dr d\theta)\| \right),
\]
(16)
\[
\|v(t)\|_{L^2(r dr d\theta)} \lesssim \min \left( \|W\|_{L^2(r dr d\theta)}, t^{-1} \frac{\|W(t)\|_{L^2(r dr d\theta)}}{r} \|H^{-1}L^2_0(r dr d\theta)\| \right).
\]
(17)
Furthermore, suppose that $\partial^2 r W$ exists. Then, $v_r$ additionally satisfies
\begin{equation}
\|v_r(t)\|_{L^2(rdr)} \lesssim t^{-2} \left( \left\| \frac{W}{(U')^2} \right\|_{H^{-1}(rdrd\theta)} + \left\| W \partial_r \frac{1}{(U')^2} \right\|_{H^{-1}(rdrd\theta)} \right) + \min \left( \left\| \partial^2 r W \right\|_{H^{-1}(rdrd\theta)} , \left\| (r - r_1)(r - r_2) \partial^2 r W \right\|_{H^{-1}(rdrd\theta)} \right).
\end{equation}

Remark 2. We note that for any given $0 < r_1 < r_2 < \infty$ we could replace $rdr$ by just $dr$ in the above estimates at the cost of a constant $C(r_1, r_2)$. In this way the result and its proof can be made more similar to the setting of a finite channel. However, the above formulation also allows us to pass to the limits $r_1 \downarrow 0$ and $r_2 \uparrow \infty$.

Proof of Proposition 2.2. In order to obtain a more tractable stream function formulation of Euler’s equations, in this proof we consider conformal coordinates, i.e.
\[(r, \theta) = (e^s, \theta), \quad s \in (\log(r_1), \log(r_2)) =: (s_1, s_2).\]

With respect to these coordinates, the stream function $\psi$ and the velocity field are given by
\[e^{-2s}(\partial_s^2 + \partial_{\theta}^2)\psi = \omega, \quad v_r = e^{-s}\partial_{\theta}\psi, \quad v_\theta = e^{-s}\partial_s\psi.\]

Furthermore, the kinetic energy satisfies
\[
\int |v_r|^2 rdrd\theta = \int e^{-2s}|\partial_{\theta}\psi|^2 e^{2s} ds d\theta = \int |\partial_{\theta}\psi|^2 ds d\theta,
\]
\[
\int |v_\theta|^2 rdrd\theta = \int |\partial_s\psi|^2 ds d\theta,
\]
\[
\int |v|^2 rdrd\theta = - \int \psi e^{2s} ds d\theta.
\]

Applying a Fourier transform in $\theta$ and using the definition of $W$, we hence obtain
\[
\int |\partial_s\psi|^2 + |\partial_{\theta}\psi|^2 ds d\theta = - \sum_k \int \overline{\psi} e^{2s} e^{iktU(e^s)} \hat{W} ds.
\]

Integrating
\[e^{iktU(e^s)} = e^{-s} \frac{1}{i k t U'(e^s)} \partial_s e^{iktU(e^s)}\]
by parts, we further compute
\[
\int |\partial_s\psi|^2 + |\partial_{\theta}\psi|^2 ds d\theta = \sum_{k \neq 0} \int \overline{\psi} e^{iktU(e^s)} \partial_s \left( \frac{1}{i k t U'(e^s)} \overline{\psi} e^{k t U(e^s)} \hat{W} \right) ds.
\]

In order to estimate this integral, we use various different tools:
- If $\partial_s$ does not fall on $\psi$, we control
\[
\sum_{k \neq 0} \int |\overline{\psi} \frac{1}{k} X| ds \leq \|\psi\|_{L^2(ds d\theta)} \|X\|_{H^{-1}L^2(ds d\theta)}
\]
and use Poincaré’s inequality to further estimate
\[
\|\psi\|_{L^2(ds d\theta)} \leq C \|\partial_{\theta}\psi\|_{L^2}.
\]
• Alternatively, instead of Poincaré’s inequality, duality yields an estimate by
\[ \| \partial_\theta \psi \|_{L^2(ds d\theta)} \| X \|_{H^{-2}L^2(ds d\theta)}. \]

• Since \( \psi \) has zero boundary values, we can also use Hardy’s inequality to control by
\[ \| \psi \|_{L^2(ds d\theta)} \| (e^s - e^{s_1})(e^s - e^{s_2})X \|_{H^{-1}L^2(ds d\theta)} \leq \| \partial_s \psi \|_{L^2(ds d\theta)} \| (e^s - e^{s_1})(e^s - e^{s_2})X \|_{H^{-1}L^2(ds d\theta)}. \]

In the case of a fixed annulus \( T \times (r_1, r_2), 0 < r_1 < r_2 < \infty \), the precise choice of estimate is not essential. However, when considering a non-periodic setting, e.g. \( \mathbb{R} \times [a, b] \), or a point vortex, i.e. \( r_1 = 0 \), or initial data with singularities at the boundary, all these estimates can yield improvements.

In order to obtain the quadratic decay estimate for \( v_r = e^{-s} \partial_\theta \psi \), we note that
\[(\partial^2_s + \partial^2_\theta) \partial_\theta \psi = e^{2s} \partial_\theta \omega.\]

Thus, we define a potential \( \gamma \) by
\[(\partial^2_s + \partial^2_\theta) \gamma = \partial_\theta \psi,\]
\[\gamma \big|_{s=s_1,s_2} = 0,\]
\[\nabla \gamma \in L^2,\]
and compute
\[\int |v_r|^2 r dr d\theta = \int \| \partial_\theta \psi \|^2 ds d\theta = \int \gamma (\partial^2_s + \partial^2_\theta) \partial_\theta \psi = \int \gamma e^{2s} \partial_\theta \omega ds d\theta = \sum_{k \neq 0} ik \int \gamma e^{ikU(e^s)} \hat{\omega} ds.\]

The result hence follows by integrating \( e^{iktU(e^s)} \) by parts twice and using the Dirichlet data of \( \gamma \) and \( \partial_\theta \psi \), the trace inequality and a variant of Hardy’s inequality. That is, since \( \gamma \) has zero Dirichlet boundary values,
\[\| \frac{\gamma}{(e^s - e^{s_1})(e^s - e^{s_2})} \|_{L^2(ds d\theta)} \leq \| \frac{\gamma}{(s - s_1)(s - s_2)} \|_{L^2(ds d\theta)} \lesssim \| \partial_\theta \gamma \|_{L^2(ds d\theta)} = \| v_r \|_{L^2(r dr d\theta)}.\]

We stress that these uniform damping estimates necessarily lose regularity, since the associated change of coordinates is a unitary operator. Thus, the operator norm of \( f \mapsto v \) considered as a mapping from \( L^2 \) to \( L^2 \) does not improve in time. Hence, it is not possible to derive stability of (8) using a common Duhamel-type approach or a fixed point mapping. Instead, in Sections 3.4 and 4 we have to make use of finer properties of the dynamics and the mode-wise decay of the principal symbol of the evolution operator. Before that, in the following we discuss some examples for which explicit computations are possible.
2.0.1. Taylor-Couette flow. As an application of the damping results, we discuss some exceptional cases for which $W$ can be trivially computed in terms of the initial datum.

**Corollary 2.1 (Couette flow).** Let $U(y) = y$ on $T \times [a, b]$ with $a, b \in [-\infty, \infty]$, then the linearized Euler equations reduce to the free transport equations. Furthermore, if $\omega_0 \in H^{-1}_x H^2_y$, then the associated velocity field satisfies

\[ \|v(t) - (v|_{t=0})_0\|_{L^2} \leq t^{-1}\|\omega_0\|_{H^{-1}_x H^1_y}, \]
\[ \|v_2(t)\|_{L^2} \leq t^{-2}\|\omega_0\|_{H^{-1}_x H^2_y}. \]

**Corollary 2.2 (Taylor-Couette flow; Point vortex).** Let $A, b \in \mathbb{R}$ and let $0 \leq r_1 < r_2 \leq \infty$, then the linearized Euler equations around Taylor-Couette flow

\[ U(r) = \frac{Ar}{r} + \frac{B}{r} \omega, \]

are given by

\[ \partial_t f + (A + \frac{B}{r^2})\partial_r f = 0, \text{ on } (0, \infty) \times T \times (r_1, r_2) \]
\[ f|_{t=0} = f_0 \text{ on } T \times (r_1, r_2). \]

Furthermore, the associated velocity field $v$ satisfies

\[ \|v\|_{L^2((r, r+\epsilon) drd\theta)} \leq C t^{-1} B^{-1} \|f_0\|_{H^{-1}_x H^1_y((r, r+\epsilon) drd\theta)}, \]
\[ \|v_r\|_{L^2((r, r+\epsilon) drd\theta)} \leq C t^{-2} B^{-2} \|f_0\|_{H^{-1}_x H^2_y((r, r+\epsilon) drd\theta)}. \]

Here, the case the case $r_1 = 0, A = 0, B \neq 0$ corresponds to a point vortex.

**Proof of Corollary 2.2.** We note that $(A + \frac{B}{r^2}) = -B \frac{2}{r^2}$. Hence, by direct computation

\[ \|\frac{\partial v}{U}\|_{H^{-1}_x H^1_y((r, r+\epsilon) drd\theta)} + \|\frac{\partial \omega}{U}\|_{L^2((r, r+\epsilon) drd\theta)} \leq 2B^{-1}\|\omega_0\|_{H^{-1}_x H^2_y((r, r+\epsilon) drd\theta)} + \|\partial_r \omega_0\|_{H^{-1}_x L^2((r, r+\epsilon) drd\theta)}. \]

\[ \square \]

3. Scattering formulation and $L^2$ stability

As established in Section 2, the core problem of (linear) inviscid damping consists of establishing a control of higher Sobolev norms of the vorticity moving with the flow:

\[ (20) \quad W(t, \theta, r) := f(t, \theta - tU(r), r). \]

Here, we largely follow a similar approach as in the plane setting considered in [12]. As key improvements we obtain a less restrictive smallness condition and develop a splitting of $\partial_t W$ into a well-behaved and more regular part $\Gamma$ and a (relatively) explicit boundary layer $\beta$. This then allows us to deduce damping with optimal decay rates and a detailed stability in suitable weighted Sobolev spaces, such as the ones considered in Proposition 2.2.

In order simplify our analysis, in this section we introduce several changes of variables as well as useful auxiliary functions.

3.1. Scattering formulation. Expressing the linearized Euler equations

\[ \partial_t f + U(r)\partial_r f = b(r)\partial_\theta \phi, \]
\[ (\partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\theta^2)\phi = f, \]
\[ \partial_\theta \phi|_{r=r_1, r_2} = 0, \]
\[ (t, \theta, r) \in \mathbb{R} \times T \times [r_1, r_2], \]
in terms of the scattered quantities
\begin{align}
F(t, \theta, r) &= f(t, \theta - tU(r), r), \\
\Upsilon(t, \theta, r) &= \phi(t, \theta - tU(r), r),
\end{align}
we obtain
\begin{align}
\partial_t F &= b(r) \partial_\theta \Upsilon, \\
\left( (\partial_r - tU'(r) \partial_\theta)^2 + \frac{1}{r} (\partial_r - tU'(r) \partial_\theta) + \frac{1}{r^2} \partial_\theta^2 \right) \Upsilon &= F, \\
\partial_\theta \Upsilon|_{r=r_1, r_2} &= 0, \\
(t, \theta, r) &\in \mathbb{R} \times \mathbb{T} \times [r_1, r_2],
\end{align}
As none of the coefficient functions depend on \( \theta \), our system decouples with respect to Fourier modes \( k \) in \( \theta \).
\begin{align}
\partial_t \hat{F} &= b(r) ik \hat{\Upsilon}, \\
\left( (\partial_r - iktU'(r))^2 + \frac{1}{r} (\partial_r - iktU'(r)) - \frac{k^2}{r^2} \right) \hat{\Upsilon} &= \hat{F}, \\
ikt \hat{\Upsilon}|_{r=r_1, r_2} &= 0, \\
(t, k, r) &\in \mathbb{R} \times 2\pi \mathbb{Z} \times [r_1, r_2],
\end{align}
We in particular note that the mode \( k = 0 \), which corresponds to a purely circular flow, is conserved in time. Using the linearity of our equations, in the following we hence without loss of regularity consider \( k \in 2\pi (\mathbb{Z} \setminus \{0\}) \) as a given parameter.
In view of the structure of the differential equation for \( \Phi \), it is further advantageous to use that \( U \), as a strictly monotone function, is invertible. Introducing a change of coordinates
\begin{align}
r \mapsto y = U(r),
\end{align}
as well a denoting
\begin{align}
h(y) &= \frac{(\omega_0)}{r}|_{r=U^{-1}(y)}, \\
g(y) &= U'(r)|_{r=U^{-1}(y)}, \\
W(t, y, k) &= \hat{F}(t, r, k)|_{r=U^{-1}(y)}, \\
\Phi(t, y, k) &= \frac{1}{ik^2} \hat{\Upsilon}(t, r, k)|_{r=U^{-1}(y)},
\end{align}
our system is then given by the following definition.
\begin{definition}[Euler’s equations in scattering formulation] Let \( U : [r_1, r_2] \to \mathbb{R} \) be strictly monotone and let \( h(y) = b|_{r=U^{-1}(y)} \) and \( g = U'(U^{-1}(y)) \). Then Euler’s equations in scattering formulation are given by
\begin{align}
\partial_t W &= \frac{ih(y)}{k} \Phi =: \frac{ih(y)}{k} L_t W, \\
\mathcal{E}_t \Phi := \left( (g(y) \frac{\partial_y}{k} - it)^2 + \frac{g(y)}{kr(y)} \frac{\partial_y}{k} - it - \frac{1}{r^2(y)} \right) \Phi &= W, \\
\Phi|_{y=a,b} &= 0, \\
(t, k, y) &\in \mathbb{R} \times 2\pi (\mathbb{Z} \setminus \{0\}) \times [a, b],
\end{align}
where \( a = \min(U^{-1}(r_1), U^{-1}(r_2)) \), \( b = \max(U^{-1}(r_1), U^{-1}(r_2)) \) and \( k \in 2\pi (\mathbb{Z} \setminus \{0\}) \).

Remark 3.  
- Our methods do not rely on the specific form of $h$ or $g$ in terms of $U$. For example, we can allow for $h$ to be an arbitrary complex valued $W^{1,\infty}$ function.
- Here the notation $L_t W$ is used to stress that the mapping $W \mapsto \Phi$ is a linear operator in $W$.
- As this system decouples with respect to $k$, we will often treat $k \neq 0$ as a fixed given external parameter and with slight abuse of notation use $W(t,k,y)$ to refer to $W(t,k,y)$ for the given $k$.

3.2. Shifted elliptic regularity and modified spaces. We note that in this scattering formulation $E_t$ is obtained from an elliptic operator by conjugation with $e^{ikty}$ and hence define suitable replacements of the $H^1$ and $H^{-1}$ energies:

**Definition 3.2** ($\tilde{H}_t^1$ and $\tilde{H}_t^{-1}$ energies). Let $u \in H^1([a,b])$ and let $k \in 2\pi(\mathbb{Z}\setminus \{0\})$ be given, then for every $t \in \mathbb{R}$, we define

$$\|u\|_{\tilde{H}_t^1}^2 := \|e^{ikty}u\|_{H^1}^2 = \|u\|_{L^2}^2 + \|\frac{\partial}{\partial t}u - itu\|_{L^2}^2.$$  

Furthermore, we define a dual quantity in the following way. Let $v \in L^2$ and let $\Psi[v]$ be the unique solution of

$$(−1 + \frac{\partial}{\partial t} - it)^2 \Psi[v] = v,$$

$$\Psi[v]|_{y=a,b} = 0.$$  

Then we define

$$\|v\|_{\tilde{H}_t^{-1}} := \|\Psi[v]\|_{\tilde{H}_t^1}.$$  

**Lemma 3.1** (Duality). Let $W \in L^2$ and let $k \in (\mathbb{Z}\setminus \{0\})$ be given. Then

$$\|W\|_{\tilde{H}_t^{-1}} = sup\{\langle W, \alpha \rangle_{L^2} : \alpha \in H^1_0, ||\alpha||_{H^1} \leq 1\},$$

e.g. $\tilde{H}_t^{-1}$ is dual to $\tilde{H}_t^1$.

**Proof of Lemma 3.1.** Since multiplication by $e^{ikty}$ is a unitary operation and preserves zero Dirichlet boundary values and

$$\Psi_t[v] = e^{-ikty}\Psi_0[e^{ikty}v],$$

it suffices to consider the case $t = 0$, which is given by the usual $H^1$ and $H^{-1}$ norms (where we use $\partial_y$ instead of $\partial_y$).

The result then follows using integration by parts:

$$-(W, \alpha) = \langle (1 - \frac{\partial_y^2}{k}) \Psi[W], \alpha \rangle$$

$$= \langle \Psi[W], \alpha \rangle + \langle \frac{\partial_y}{k} \Psi[W], \frac{\partial_y}{k} \alpha \rangle \leq \|W\|_{H^{-1}} \|\alpha\|_{H^1},$$

with equality if $\alpha = -\frac{1}{\|\Psi[W]\|_{H^1}} \Psi[W]$. Taking the supremum over all $\alpha$ with $\|\alpha\|_{H^1}$ we hence obtain the result. $\square$

3.3. Heuristics and obstructions. On a heuristic level, in order to establish stability in $L^2$, we use that

$$\frac{d}{dt} \|W(t)\|_{L^2}^2 = 2Re(W, \frac{i\kappa}{k} L_t W) \lesssim C(h,k)\|W(t)\|_{H_t^{-1}}^2,$$

and that for fixed functions $u \in L^2$, which do not depend on time,

$$\int_0^\infty \|u\|_{H_t^{-1}}^2 dt \leq C\|u\|_{L^2}^2,$$
as can be computed from a Fourier characterization. Hence, it seems reasonable to expect that solutions $W(t)$ of (27) satisfy an estimate of the form

$$\|W(t)\|_{L^2} \leq \exp(C\|h\|_{L^\infty} |k|^{-1})f_0\|_{L^2},$$

also for complex valued $h$, which is the case for some explicit model problems (c.f. [11]).

However, we stress that this heuristic is very rough and does not account for several obstructions:

- We note that integrability in time in general fails for time-dependent $u \in L^\infty_t(L^2)$. For example, choosing $u(t,k,y) = e^{ikt}u_0(k,y)$, we observe that
  $$\int_0^T \|u\|^2_{\tilde{H}^{-1}_t} dt = T\|u_0\|^2_{\tilde{H}^{-1}},$$
  which diverges as $T \to \infty$ despite $\|u(t,y)\|_{L^2} = \|u_0\|_{L^2}$ being uniformly bounded.

- Since the first estimate does not account for antisymmetric operators in $\frac{d}{dt}W$ it is not sufficient to establish $L^2$ stability. For example, this estimate is satisfied by solutions $u(t,y)$ to
  $$\partial_t u + iu = \Phi,$$
  $$(-1 + (\partial_y - it)^2)\Phi = u,$$
  $$\Phi|_{y=a,b} = 0.$$

Considering $v(t,y) = e^{ity}u(t,y)$, we observe that $v$ solves

$$\partial_t v = \phi,$$
$$\partial_y^2\phi = v,$$
$$\phi|_{y=a,b} = 0.$$

Hence, choosing $u|_{t=0}$ to be an eigenfunction of $(1 - \partial_y^2)$, we obtain an exponentially growing solution.

### 3.4. $L^2$ stability

As the main result of this section, we adapt the Lyapunov functional approach of [13] to this circular setting and prove stability of (27). In the following we formulate the main ingredients of our approach as a series of Lemmata, which are then used to prove $L^2$ stability in Theorem 3.1. Subsequently, we elaborate on the theorem’s statement and assumptions in comparison to existing results and prove the lemmata. Here, the lemmata are formulated in a general way in order to facilitate their use for higher regularity estimates in later sections.

**Lemma 3.2.** Let $L_t$ be given by (27) and let $\kappa \in W^{1,\infty}$. Then, for any $u, v \in L^2$

$$\langle u, \kappa L_t v \rangle \leq \left(\|\kappa\|_{L^\infty} + \frac{1}{|k|}\|\partial_y \kappa\|_{L^\infty} \right)\|u\|_{\tilde{H}^{-1}_t} \|L_t v\|_{\tilde{H}^{-1}_t}.$$  

(32)

**Lemma 3.3.** Let $L_t$ be as in (27). Then there exists a constant $C = C(a,b,g)$ such that for any $u \in L^2$ and any $t \geq 0$

$$\|L_t u\|_{\tilde{H}^{1}_t} \leq C\|u\|_{\tilde{H}^{-1}_t}.$$  

(33)
Lemma 3.4 ([12, Lemma 4.5]). Let \( u \in L^2([a, b]) \) and let \( \sum_{n \in (b-a)\mathbb{N}} u_n \sin(ny) \) be its series expansion. Define the symmetric, positive definite, non-increasing operator \( A \) by

\[
\langle u, Au \rangle := \sum_n \exp(\arctan(\frac{n}{C} - t))|u_n|^2.
\]

Then \( A \) is symmetric, positive definite, non-increasing, \( C^1 \) in time and comparable to the identity, i.e.

\[
e^{-\pi}||u||_{L^2} \leq \langle u, Au \rangle \leq e^{\pi}||u||_{L^2},
\]

for all \( u \in L^2 \).

Furthermore, there exists a constant \( e^{-\pi} \leq C_2 \leq e^\pi \) and \( \delta > 0 \) such that

\[
||u||_{\dot{H}^{-1}} ||Au||_{\dot{H}^{-1}} \leq -C_2 \langle u, \dot{A}u \rangle - C_2 ||u||_{L^2}.
\]

Using the preceding lemmata, we can establish \( L^2 \) stability.

Theorem 3.1 (\( L^2 \) stability). Let \( A \) and \( C_2 \) be given by Lemma 3.4 and let \( C \) be as in Lemma 3.3. Further suppose that there exists \( \delta > 0 \) such that

\[
|k|^{-1} (||h||_{L^\infty} + |k|^{-1}||\partial_y h||_{L^\infty}) \leq \frac{1}{C} \left( \frac{1}{C_2} - \delta \right).
\]

Then for any solution \( W \) to the Euler equations in scattering formulation (27) the functional

\[
I(t) := \langle W, A(t)W \rangle.
\]

is non-increasing and satisfies

\[
\partial_t I(t) \leq -\delta ||W(t)||_{\dot{H}^{-1}} ||A(t)W(t)||_{\dot{H}^{-1}} \leq 0.
\]

In particular, this implies

\[
e^{-\pi}||W(t)||_{L^2}^2 \leq I(t) \leq I(0) \leq e^\pi ||W_0||_{L^2}^2.
\]

Proof of Theorem 3.1. Using Lemma 3.2, we estimate

\[
\partial_t I(t) \leq \langle W, \dot{A}W \rangle + 2|k|^{-1} (||h||_{L^\infty} + |k|^{-1}||\partial_y h||_{L^\infty}) ||AW||_{\dot{H}^{-1}} ||L_t W||_{\dot{H}^{-1}}.
\]

Applying Lemma 3.4 and Young’s inequality, we further control

\[
||AW||_{\dot{H}^{-1}} ||L_t W||_{\dot{H}^{-1}} \leq \frac{C}{2} ||AW||_{\dot{H}^{-1}} ||W||_{\dot{H}^{-1}}.
\]

The result then follows by an application of Lemma 3.4 and noting that, by our smallness assumption,

\[
\partial_t I(t) \leq \langle W, \dot{A}W \rangle + |k|^{-1} (||h||_{L^\infty} + |k|^{-1}||\partial_y h||_{L^\infty}) C ||AW||_{\dot{H}^{-1}} ||W||_{\dot{H}^{-1}} \leq \langle W, \dot{A}W \rangle + \frac{1}{C_2} - \delta ||AW||_{\dot{H}^{-1}} ||W||_{\dot{H}^{-1}} \leq -\delta ||AW||_{\dot{H}^{-1}} ||W||_{\dot{H}^{-1}} \leq 0.
\]

\( \square \)

Let us briefly remark on this result and its assumptions:

- We require a smallness condition on \( \frac{iK}{h} L_t \) in order to rule out the obstacles mentioned in Section 3.3.
- Since \( h \) is allowed to be complex-valued, we do not rely on conserved quantities or classical stability results such as the ones of Rayleigh, Fjortoft or Arnold.
• In the setting of a plane finite periodic channel, in [10] Wei, Zhang and Zhao use a spectral approach to establish linear stability and decay with optimal rates for monotone shear flows under the assumption that the strictly monotone shear flow $U(y)$ possesses no embedding eigenvalues. In comparison, our smallness assumption is more restrictive, but extends to related problems such as stability in fractional Sobolev spaces, complex valued functions $h$ and fractional operators $L_t$ in a straightforward way.

• In Section 4, we show that $\partial_y^2 W$ can be split into a very regular, stable part $\Gamma$ and a boundary layer part $\beta$ which develops a singularity at the boundary. Here, $\beta$ is determined solely by the Dirichlet boundary data of the initial datum, $\omega_0$, and allows for a detailed study of the stability properties of the evolution.

It remains to prove Lemmata 3.2 and 3.3.

Proof of Lemma 3.2. Let $u,v \in L^2$ and let $\Psi[u]$ be the unique solution of

$$(-1 + (\partial_y - ikt)^2)\Psi[u] = u,$$

$$\Psi[u]|_{y=a,b} = 0.$$  

Then we directly compute

$$|\langle u, \kappa L_t v \rangle| = \left|\langle (1 - (\partial_y - ikt)^2)\Psi[u], \kappa L_t v \rangle\right|$$

$$\leq ||\Psi[u]||_{L^2}||\kappa L_t||_{L^2} + ||(\partial_y - it)\Psi[u]||_{L^2}||(|\partial_y - it)\kappa L_t v||_{L^2},$$

$$\leq ||\Psi[u]||_{\tilde{H}_1^t} (||\kappa||_{L^\infty} + \frac{1}{|k|}||\partial_y \kappa||_{L^\infty}) ||L_t v||_{\tilde{H}_1^t}.$$  

Here, we used that $L_t v$ by definition satisfies zero Dirichlet boundary conditions and hence no boundary contributions appear when integrating by parts. □

Proof of Lemma 3.3. We recall that $L_t u$ is the solution of

$$\mathcal{E}_t L_t u = \left( g(y) \left(\frac{\partial_y}{k} - it\right) \right)^2 + \frac{g(y)}{kr(y)} \left(\frac{\partial_y}{k} - it\right) L_t u = 0,$$

$$L_t u|_{y=a,b} = 0,$$

and that $g(y)$ and $\frac{r(y)}{r'(y)}$ are bounded from below (and above).

Hence $\mathcal{E}_t$ is a shifted elliptic operator and testing by $L_t u$ (or $\frac{1}{\rho} L_t u$) we obtain that

$$\|L_t u\|_{\tilde{H}_1^t}^2 \leq -C(L_t u, u),$$

for some $C > 0$. Applying Lemma 3.1, we thus obtain

$$\|L_t u\|_{\tilde{H}_1^t} \leq C\|L_t u\|_{\tilde{H}_1^t} ||\Psi[u]||_{\tilde{H}_1^t},$$

$$\Leftrightarrow \|L_t u\|_{\tilde{H}_1^t} \leq C\|u\|_{\tilde{H}_1^{-1}}.$$  

□

Having introduced the basic tools of our approach, in the following section we consider higher stability of $W$, i.e. control of $\partial_y W$. Here, boundary effects qualitatively change the dynamics and necessitate a modification of the weight $A(t)$.
4. Higher Stability and Boundary Layers

In this section we show that the $L^2$ stability result can be extended to higher Sobolev regularity. However, unlike in the setting of an infinite periodic channel, boundary effects cannot be neglected and result in the formation of singularities. As the main improvements over our previous work for the plane channel in [12], we provide an explicit splitting into a more regular good parts and a boundary layer exhibiting blow-up as well as an improved smallness condition. This splitting then allows to provide a more detailed description of the blow-up also in weighted Sobolev spaces. For this purpose we also introduce a different method of proof.

Let thus $W$ be a solution to (27)
\[
\partial_t W = \frac{i\hbar}{k} L_t W, \\
\mathcal{E}_t L_t W = W, \\
L_t|_{y=a,b} = 0, \\
(t, k, y) \in \mathbb{R} \times 2\pi(\mathbb{Z} \setminus \{0\}) \times [a, b].
\]

We begin by studying $\partial_y W$, which satisfies
\[
\begin{align*}
\partial_t \partial_y W &= \frac{i\hbar}{k} L_t \partial_y W + \frac{i\hbar'}{k} L_t W + \frac{i\hbar}{k} \mathcal{E}_t \partial_y L_t W + \frac{i\hbar}{k} H^{(1)}, \\
\mathcal{E}_t H^{(1)} &= 0, \\
H^{(1)}_{y=a,b} &= \partial_y L_t W|_{y=a,b}.
\end{align*}
\]

In contrast to the $L^2$ setting (or a setting without boundary such as $\mathbb{T} \times \mathbb{R}$) we hence obtain a correction $H^{(1)}$ due to $\partial_y L_t W$ not satisfying zero Dirichlet boundary conditions.

As a main result of Appendix A, we study the boundary behavior of $\partial_y L_t W$ (also confer [12]) and obtain the following description of $H^{(1)}$:

**Lemma 4.1.** Let $W$ be a solution of (27) and let $H^{(1)}$ be the unique solution of
\[
\begin{align*}
\mathcal{E}_t H^{(1)} &= 0, \\
H^{(1)}_{y=a,b} &= \partial_y L_t W|_{y=a,b}.
\end{align*}
\]

Then there exist functions $u_1, u_2, \tilde{u}_1, \tilde{u}_2 \in H^2$ (depending on $a, b, k$ and $g$ but not on $t$) and constants $c_1, c_2$ such that
\[
H^{(1)}(t, y) = c_1 \langle W, e^{ikt(y-a)} \tilde{u}_1 \rangle e^{ikt(y-a)} u_1 + c_2 \langle W, e^{ikt(y-b)} \tilde{u}_2 \rangle e^{ikt(y-b)} u_2.
\]

Furthermore, for instance for $u_1$ for any $t > 0$
\[
\langle W, e^{ikt(y-a)} \tilde{u}_1 \rangle = \frac{\omega_0(a)}{ikt} - \frac{1}{ikt} \langle W, e^{ikt(y-a)} \partial_y \tilde{u}_1 \rangle - \frac{1}{ikt} \langle \partial_y W, e^{ikt(y-a)} \tilde{u}_1 \rangle.
\]

Based on this characterization of $H^{(1)}$, we introduce a splitting of $\partial_y W$ into a function $\beta$ depending only on $\omega_0|_{y=a,b}$ and $\Gamma = \partial_y W - \beta$. As we show in Theorem 4.1, $\Gamma$ is stable also in higher regularity. In contrast, unless $\omega_0|_{y=a,b}$ is trivial, $\beta$ asymptotically develops singularities at the boundary and exhibits blow-up in $H^s, s > 1/2$. If one however considers weighted spaces, it is possible to compensate for these singularities by vanishing weights and hence establish sufficient control for damping with optimal decay rates.
Lemma 4.2. Let $W$ be a solution of (27) and let $\Gamma$ be the solution of (56)
\[
\partial_t \Gamma = \frac{i h}{k} L_t \Gamma - \frac{h}{k^2 t} \langle \Gamma, e^{ikt(y-a)} u_1 \rangle e^{ikt(y-a)} u_1 - \frac{h}{k^2 t} \langle \Gamma, e^{ikt(y-b)} \tilde{u}_2 \rangle e^{ikt(y-b)} u_2 \\
+ \frac{i h'}{k} L_t W + \frac{i h}{k} L_t [\tilde{\epsilon}_t, \partial_y] L_t W - c_1 \frac{h}{k^2 t} \langle W, e^{ikt(y-a)} \partial_y \tilde{u}_2 \rangle e^{ikt(y-a)} u_1 \\
- c_2 \frac{h}{k^2 t} \langle W, e^{ikt(y-b)} \partial_y \tilde{u}_2 \rangle e^{ikt(y-b)} u_2,
\]
and let $\beta$ be the solution of (57)
\[
\partial_t \beta = \frac{i h}{k} L_t \beta - \frac{h}{k^2 t} \langle \beta, e^{ikt(y-a)} u_1 \rangle e^{ikt(y-a)} u_1 - \frac{h}{k^2 t} \langle \beta, e^{ikt(y-b)} \tilde{u}_2 \rangle e^{ikt(y-b)} u_2 \\
- c_1 \frac{h}{k^2 t} \langle \beta, e^{ikt(y-a)} u_1 \rangle e^{ikt(y-a)} u_1 - c_2 \frac{h}{k^2 t} \langle \beta, e^{ikt(y-b)} \tilde{u}_2 \rangle e^{ikt(y-b)} u_2,
\]
and $\beta(t) \equiv 0$. Then $\partial_t W = \Gamma + \beta$. The function $\beta$ is called the boundary layer.

Theorem 4.1 ($H^2$ regularity of $\Gamma$). Suppose that $g,h$ satisfy the assumptions of Theorem 3.1.

1. Suppose that additionally $g \in W^{2,\infty}$ and $h \in W^{2,\infty}$. Then there exists a constant $C_1$ such that for all $\omega_0 \in H^1$ and any $t \geq 0$, the solution $\Gamma$ of (56) satisfies
\[
\|\Gamma(t)\|_{L^2} \leq C \|\omega_0\|_{H^1}.
\]

2. Suppose that additionally $g \in W^{3,\infty}$ and $h \in W^{3,\infty}$, then there exists a second constant $C_2$ such that for any $\omega_0 \in H^2$ and for any $t \geq 0$, $\|\Gamma(t)\|_{H^1} \leq C_2 \|\omega_0\|_{H^2}$.

Theorem 4.2 ($H^2$ regularity of $\beta$). Suppose $g,h$ satisfy the assumptions of Theorem 3.1.

1. Then there exists a constant $C_1$ such that for all $t \geq 0$, the solution $\beta$ of (57) satisfies
\[
\|\beta(t)\|_{L^2} \leq C_1 (|\omega_0(a)| + |\omega_0(b)|).
\]

2. Suppose that additionally $g,h \in W^{2,\infty}$, then there exists a second constant $C_2$ such that
\[
\|(y-a)(y-b)\partial_y \beta(t)\|_{L^2} \leq C_2 (|\omega_0(a)| + |\omega_0(b)|).
\]
However, if for instance $|\omega_0(a)| > 0$, then
\[
|\beta(t,a)| \gtrsim \log(t)
\]
as $t \to \infty$ (similarly for $b$). In particular, by the Sobolev embedding, we obtain blow-up in $H^s, s > 1/2$.

Remark 4.
- Combining Theorems 3.1, 4.1 and 4.2 and Proposition 2.2, we obtain Theorem 1.1.
- It is possible to further split $\Gamma$ into functions controlled solely in terms of $\|\omega_0\|_{L^2}, \|\partial_\theta \omega_0\|_{L^2}$ and $\|\partial_\theta^2 \omega_0\|_{L^2}$, if finer control is desired.
- Like Theorem 3.1, in addition to these stability results we obtain Lyapunov functionals. As a key difference, these functionals are however in general only decreasing for times $t \geq T > 0$. Control up time $T$ is hence provided by a Gronwall-type argument, which determines the constants $C_1,C_2$. 

• We stress that we do not require higher norms of \( g, h \) to be small but only finite, so that derivatives of the equation are well-defined as mappings in \( L^2 \).

• When considering a setting without boundary contributions such as \( \mathbb{T} \times \mathbb{R} \) or \( \mathbb{T} \times \mathbb{T} \), no boundary correction \( \beta \) is needed. Thus (a suitable modification of) this result already yields the desired stability for decay with optimal rates. Furthermore, this result generalizes to higher derivatives in a straightforward way, where again only finiteness of higher norms has to be required.

### 4.1. Stability of \( \Gamma \)

As the main result of this subsection we provide a proof of Theorem 4.1. Here, the \( L^2 \) stability result is self-contained, while the \( H^1 \) estimate presupposes the \( L^2 \) stability of \( \beta \), which is established in the following subsection. Furthermore, we briefly discuss the implications of Theorem 4.1 for settings without boundary and provide an improved stability result for the setting of an infinite plane periodic channel, \( \mathbb{T}_L \times \mathbb{R} \).

We recall that \( \Gamma \) is the solution of

\[
\partial_t \Gamma = \frac{ih}{k} L_t \Gamma - \frac{h}{k^2 t} \langle \Gamma, e^{ikt(y-a)} \tilde{u}_1 \rangle e^{ikt(y-a)} u_1 - \frac{h}{k^2 t} \langle \Gamma, e^{ikt(y-b)} \tilde{u}_2 \rangle e^{ikt(y-b)} u_2 + \frac{ih'}{k} L_t W + \frac{ih}{k} L_t [\mathcal{E}_t, \partial_y] L_t W - c_1 \frac{h}{k^2 t} \langle W, e^{ikt(y-a)} \partial_y \tilde{u}_1 \rangle e^{ikt(y-a)} u_1 - c_2 \frac{h}{k^2 t} \langle W, e^{ikt(y-b)} \partial_y \tilde{u}_2 \rangle e^{ikt(y-b)} u_2,
\]

\[
\Gamma|_{t=0} = \hat{\partial_y \omega}_0,
\]

In addition to the estimates for \( L_t \) derived in Section 3.4, we hence need to control contributions of the form

\[
\frac{1}{ikt} \langle \Gamma, e^{ikt(y-a)} \tilde{u}_1 \rangle \langle A\Gamma, e^{ikt(y-b)} u_2 \rangle,
\]

which can not be controlled by the previous choice of \( A(t) \).

Instead, we construct a modified weight \( A_1(t) \), which is introduced in the following Lemmata (cf. [13] for a similar construction adapted to fractional Sobolev spaces).

**Lemma 4.3.** Let \( u \in H^1 \), then for \( 0 < \mu < 1/2 \) and for every \( v = \sum_n v_n e^{iny} \in L^2 \)

\[
|\langle v, e^{ikt(y-a)} u \rangle|^2 \leq C_\mu \| u \|_{H^1}^2 \sum_n < n - kt >^{-2\mu} |v_n|^2.
\]

**Proof of Lemma 4.3.** By expanding the \( L^2 \) inner product in a basis, we obtain that

\[
\langle v, e^{ikt(y-a)} u \rangle = \sum_n v_n \langle e^{iny}, e^{ikt(y-a)} u \rangle.
\]

Integrating by parts and using the trace inequality, we further estimate

\[
|\langle e^{iny}, e^{ikt(y-a)} u \rangle| \leq < n - kt >^{-1} \| u \|_{H^1}.
\]

The result hence follows by an application of the Cauchy-Schwarz inequality:

\[
|\langle v, e^{ikt(y-a)} u \rangle| \leq \sum_n v_n < n - kt >^{-\mu} < n - kt >^{1-\mu}
\]

\[
\leq \| v_n < n - kt >^{-\mu} \|_2 \| < n - kt >^{1-\mu} \|_2
\]

\[
\leq 2 \| v_n < n - kt >^{-\mu} \|_2 \| < n >^{-1} \|_2
\]

\[
= C_\mu \| v_n < n - kt >^{-\mu} \|_2,
\]

where we used that \( < n >^{-1} \in L^2 \) if \( \mu < 1/2 \).
Lemma 4.4. Let \( 0 < \lambda, \mu < 1 \) with \( \lambda + 2\mu > 1 \) and let \( \epsilon > 0 \) and define the symmetric operator \( A_1(t) \) by its action on the basis:

\[
A_1(t) : e^{iny} \mapsto \exp \left( \arctan \left( \frac{y}{k} - t \right) - \epsilon \int t < \tau >^{-\lambda} < n - k\tau >^{-2\mu} \, d\tau \right).
\]

Then for every \( u \in L^2 \) and every \( t \in \mathbb{R} \)

\[
C\|u\|_{L^2}^2 \leq \langle u, A_1(t)u \rangle \leq C^{-1}\|u\|_{L^2}^2,
\]

\[
\langle u, A_1(t)u \rangle \leq -C_1\|u\|_{H_t^{-1}}^2 + C_2 \sum_n \langle t >^{-\lambda} < n - k\tau >^{-2\mu} |u_n|^2 \leq 0.
\]

Proof of Lemma 4.4. We note that \( \langle t >^{-\lambda} < n - k\tau >^{-2\mu} \in L^1(\mathbb{R}) \) and that

\[
-\epsilon \int t < \tau >^{-\lambda} < n - k\tau >^{-2\mu} \, d\tau
\]

is monotonically decreasing. The properties of \( A_1(t) \) hence follow by direct computation, where

\[
C = \exp(-\pi - \epsilon \| \cdot >^{-\lambda} < n - k\cdot >^{-2\mu} \|_{L^1(\mathbb{R})}).
\]

and \( C_1 \) is determined by \( C \) and Lemma 3.4. \( \square \)

Lemma 4.5. Let \( g \in W^{2,\infty} \), \( g \geq c > 0 \), then for every \( u \in L^2 \) and for every \( t \geq 0 \),

\[
\|L_t[\mathcal{E}_t, \partial_y]L_tu\|_{H_t^{-1}} \lesssim \|u\|_{H_t^{-1}}.
\]

Proof of Lemma 4.5. By Lemma 3.3, we obtain that

\[
\|L_t[\mathcal{E}_t, \partial_y]L_tu\|_{H_t^{-1}} \lesssim \|\mathcal{E}_t, \partial_y\|_{L^1(\mathbb{R})} \lesssim \|u\|_{H_t^{-1}}.
\]

We further note that

\[
[\mathcal{E}_t, \partial_y] = e^{-ikt\gamma}[\mathcal{E}_0, \partial_y - ikt\gamma]e^{ikt\gamma} = e^{-ikt\gamma}[\mathcal{E}_0, \partial_y]e^{ikt\gamma},
\]

and that, by direct computation, \([\mathcal{E}_0, \partial_y]\) is a second-order operator. Hence, using integration by parts, we further estimate

\[
\|\mathcal{E}_t, \partial_y\|_{L^1(\mathbb{R})} \lesssim \|L_tu\|_{H_t^{-1}} \lesssim \|u\|_{H_t^{-1}}.
\]

\( \square \)

Using these results, we can now provide a proof of Theorem 4.1 and thus establish \( L^2 \) stability.

Proof of Theorem 4.1, part (1). Fix \( 0 < \lambda, \mu < 1 \) with \( 2\mu + \lambda > 1 \) and let \( A_1 \) be given by Lemma 4.4, where

\[
0 < \epsilon < \frac{1}{100} \| n - k\cdot >^{-2\mu} \|_{L^1(\mathbb{R})}^{-1}.
\]

Then we define

\[
I(t) := \langle \Gamma, A_1(t)\Gamma \rangle = C_1(W, A(t)W),
\]

where \( C_1 \gg 0 \) is to be chosen later. We then claim that there exists \( T > 0 \) such that for all initial data and for all \( t \geq 0 \), \( I(t) \) satisfies

\[
\frac{d}{dt} I(t) \leq C \epsilon^{2(1-\mu/2)} \|\omega_0\|_{L^2}^2 \in L^1(\mathbb{R}).
\]

Using Gronwall’s inequality, we further obtain that

\[
I(T) \leq \exp(CT)I(0),
\]

which concludes the proof.
It remains to prove the claim. Using Theorem 3.1 and Lemma 4.4, we directly compute

\[
\frac{d}{dt} I(t) \leq -C \|\Gamma\|_{\tilde{H}^{-1}}^2 - C\epsilon \sum_n < t > -\lambda < n - kt > -2\mu \|\Gamma_n\|^2
\]

\[
- C_1 \delta \|W(t)\|_{\tilde{H}^{-1}}^2 + 2\Re \left( \frac{d}{dt} \Gamma, A_1(t)\Gamma \right).
\]

Using Lemma 4.3 and Lemma 3.2 and recalling (56), we further estimate

\[
2\Re \left( \frac{d}{dt} \Gamma, A_1(t)\Gamma \right) \leq C(h, k)\|\Gamma\|_{\tilde{H}^{-1}} \|A_1\|_{\tilde{H}^{-1}} + C(h, k, \mu) \frac{1}{t} \left( \sum_n < n - kt > -2\mu \|\Gamma_n\|^2 \right)
\]

\[
+ C(h, h', k) \|A_1\|_{\tilde{H}^{-1}} \left( \|L_t W\|_{\tilde{H}^1} + \|L_t [E_t, \partial_y] L_t\|_{\tilde{H}^1} \right)
\]

\[
+ C(h, k, g) t^{-1} \|\omega_0\|_{L^2} \sqrt{\sum_n < n - kt > -2\mu \|\Gamma_n\|^2}.
\]

Splitting \( t = t^{-(1-\mu)} t^{-\mu} \) and using Young’s inequality and Lemmata 3.3 and 4.5, we further control

\[
\frac{1}{t} \left( \sum_n < n - kt > -2\mu \|\Gamma_n\|^2 \right) = t^{-(1-\lambda)} \sum_n t^{-\lambda} < n - kt > -2\mu \|\Gamma_n\|^2,
\]

\[
\|A_1\|_{\tilde{H}^{-1}} \left( \|L_t W\|_{\tilde{H}^1} + \|L_t [E_t, \partial_y] L_t\|_{\tilde{H}^1} \right) \leq \sigma \|A_1\|_{\tilde{H}^{-1}}^2 + \sigma^{-1} \|W(t)\|_{\tilde{H}^{-1}}^2,
\]

\[
t^{-1} \|\omega_0\|_{L^2} \sqrt{\sum_n < n - kt > -2\mu \|\Gamma_n\|^2} \leq \sigma (\Gamma, \dot{A}_1(t)\Gamma) + \sigma^{-1} t^{-2(1-\mu/2)} \|\omega_0\|_{L^2}^2.
\]

Choosing \( \sigma \) sufficiently small and letting \( T > 0 \) be sufficiently large and using the smallness assumption of Theorem 3.1, we observe that

\[
- C \|\Gamma\|_{\tilde{H}^{-1}}^2 - C\epsilon \sum_n < t > -\lambda < n - kt > -2\mu \|\Gamma_n\|^2 + C(h, k)\|\Gamma\|_{\tilde{H}^{-1}} \|A_1\|_{\tilde{H}^{-1}}
\]

\[
+ (C(h, k, \mu) t^{-(1-\mu)}) \sum_n < t > -\lambda < n - kt > -2\mu \|\Gamma_n\|^2
\]

\[
+ \sigma \|A_1\|_{\tilde{H}^{-1}}^2 + \sigma (\Gamma, \dot{A}_1(t)\Gamma) \leq 0.
\]

Similarly, choosing \( C_1 \) sufficiently large, we observe that

\[
- C_1 \delta \|W(t)\|_{\tilde{H}^{-1}}^2 + \sigma^{-1} \|W(t)\|_{\tilde{H}^{-1}}^2 \leq 0.
\]

Hence, we conclude that for \( t \geq T > 0 \), \( I(t) \) satisfies

\[
\frac{d}{dt} I(t) \leq -\sigma^{-1} t^{-2(1-\mu/2)} \|\omega_0\|_{L^2}^2.
\]

which finishes the proof of the claim and hence of the \( L^2 \) stability result, (1). \( \square \)
Next, we consider the evolution of $\partial_y \Gamma$:

\begin{align}
\partial_t \partial_y \Gamma = \frac{ih}{k} L_t \partial_y \Gamma + \frac{ih'}{k} L_t \Gamma + \frac{ih}{k} L_t [\mathcal{E}_t, \partial_y] L_t \Gamma - (\partial_y L_t \Gamma) (a) e^{ikt(y-a)} u_1 - (\partial_y L_t \Gamma) (b) e^{ikt(y-b)} u_1 \\
+ \partial_y \left( \frac{h}{k^2 t} \langle \Gamma, e^{ikt(y-a)} \tilde{u}_1 \rangle e^{ikt(y-a)} u_1 - \frac{h}{k^2 t} \langle \Gamma, e^{ikt(y-b)} \tilde{u}_2 \rangle e^{ikt(y-b)} u_2 \right) \\
- c_1 \frac{h}{k^2 t} \langle W, e^{ikt(y-a)} \partial_y \tilde{u}_1 \rangle e^{ikt(y-a)} u_1 - c_2 \frac{h}{k^2 t} \langle W, e^{ikt(y-b)} \partial_y \tilde{u}_2 \rangle e^{ikt(y-b)} u_2 \\
+ \partial_y \left( \frac{ih'}{k} L_t W + \frac{ih}{k} L_t [\mathcal{E}_t, \partial_y] L_t W \right),
\end{align}

\begin{align}
\partial_y \Gamma|_{t=0} = \partial_y \omega_0.
\end{align}

Since we here also have to compute $\partial_y W = \Gamma + \beta$ in order to control $\|\partial_y \Gamma\|_{L^2}$, we require $L^2$ estimates on $\beta$. Before continuing with the proof of Theorem 4.1, we hence prove the first part of Theorem 4.2 as well as some further properties of the evolution of $\beta$, which are formulated in the following proposition.

**Proposition 4.1.** Suppose $g, h$ satisfy the assumptions of Theorem 4.2. Let $\beta$ be the solution of (57) and let $A_1(t)$ be given by Lemma 4.4. Then there exists $T > 0$ such that for all $t \geq 0$

\[ I_2(t) = \langle \beta, A_1(t) \beta \rangle \]

satisfies

\[ \frac{d}{dt} I_2(t) \leq \delta(\beta, \hat{A}_1(t) \beta) + C t^{-2(1-\mu/2)} |\omega_0|_{y=a,b}^2. \]

**Proof of Proposition 4.1.** Using the same weight $A_1$, we observe that

\[ \Re \langle A_1(t) \beta, \omega_0(a) \frac{1}{ikt} e^{ikt y} u \rangle \lesssim C_\lambda \|\beta_n\| < n - kt \lesssim \|\beta_n\|_{y=a,b} \frac{1}{|kt|}. \]

Using Young’s inequality and choosing $\sigma$ sufficiently small, we thus obtain that

\[ \frac{d}{dt} \langle \beta, A_1 \beta \rangle \leq \delta(\beta, \hat{A}_1 \beta) + C \sigma^{-1} t^{-2(1-\mu/2)} |\omega_0|_{y=a,b}^2. \]

The first part of Theorem 4.2 then follows by integrating this inequality and using a Gronwall-type estimate to control the growth up to time $T$. 

Additionally, we make use of the following estimates for boundary evaluations of $L_t \Gamma, W$ and $\Gamma$, which are obtained as an application of the results of Appendix A.
Lemma 4.6. Let $g, h, k$ satisfy the assumptions of the second part of Theorem 4.1. Then,

$$
(\partial_y L_t \Gamma)(a) = c_1 \langle \Gamma, e^{ikt(y-a)} \hat{u}_1 \rangle,
$$

$$
(\partial_y L_t \Gamma)(b) = c_2 \langle \Gamma, e^{ikt(y-b)} \hat{u}_2 \rangle,
$$

and the following estimates hold:

$$
\left| \langle \Gamma, e^{ikt(y-a)} \hat{u}_1 \rangle \right| \lesssim \frac{C_k}{kt} \sqrt{\sum_n |(\partial_y \Gamma)|^2} \lesssim n - kt > -2k + \frac{C_k}{kt} |\Gamma(a, t)|,
$$

$$
\left| \langle W, e^{ikt(y-a)} \hat{u}_1 \rangle \right| \lesssim \frac{C_k}{kt} (||\Gamma||_{L^2} + ||\beta(t)||_{L^2}) + \frac{C_k}{kt} |\omega_0(a, t)|,
$$

$$
|\Gamma(a, t)| \leq \log(t) (|\omega_0(a)| + ||\omega_0||_{L^2}).
$$

Proof of Lemma 4.6. The evaluations of $\partial_y L_t \Gamma$ at the boundary are obtained as an application of Lemma A.2. The first two estimates follow by integration by parts. In order to show the last estimate, we restrict (56) to the boundary and obtain that

$$
|\partial_t \Gamma(a, t)| \lesssim \frac{1}{kt} (||\Gamma||_{L^2} + ||W(t)||_{L^2}),
$$

where we used that $L_t$ enforces zero Dirichlet data. The result hence follows by using Theorem 3.1 and the first part of Theorem 4.1 to control

$$
||\Gamma(t)||_{L^2} + ||W(t)||_{L^2} \lesssim |\omega_0(a)| + ||\omega_0||_{L^2},
$$

and then integrating the inequality. □

Lemma 4.7. Let $W$ be the solution of (27) with initial datum $\omega_0 \in H^1$ and let $\Gamma$ and $\beta$ be as in Lemma 4.2. Then, for any $\sigma > 0$,

$$
\Re \langle A_1 \partial_y \Gamma, \left[ \left( \frac{ih}{k} L_t + \frac{ih}{k} L_t [\mathcal{E}_t, \partial_y L_t] \right), \partial_y \right] W \rangle \leq \sigma |(\partial_y \Gamma, \hat{A_1 \partial_y \Gamma})| + C \sigma^{-1} ||W||_{H_t^{\sigma-1}}.
$$

Proof of Lemma 4.7. The contribution due to $\frac{ih}{k} L_t$ can be estimated as in Lemma 4.5. In the following we thus focus on the commutator and decompose the commutator into the cases where $\partial_y$ falls on $h$,

$$
\frac{ih}{k} L_t W + \frac{ih}{k} L_t [\mathcal{E}_t, \partial_y L_t] W,
$$

the terms solving an elliptic equation with vanishing Dirichlet data,

$$
\frac{ih}{k} L_t [\mathcal{E}_t, \partial_y] L_t W + \frac{ih}{k} L_t [\mathcal{E}_t, \partial_y] L_t [\mathcal{E}_t, \partial_y] L_t W,
$$

and the homogeneous corrections,

$$
\frac{ih}{k} ((\partial_y L_t W)(a, t) e^{ikt(y-a)} u_1 + (\partial_y L_t W)(b, t) e^{ikt(y-b)} u_2),
$$

$$
\frac{ih}{k} ((\partial_y L_t [\mathcal{E}_t, \partial_y] L_t W)(a, t) e^{ikt(y-a)} u_1 + (\partial_y L_t [\mathcal{E}_t, \partial_y] L_t W)(b, t) e^{ikt(y-b)} u_2),
$$

In the first and second case, we use Lemmata 3.3 and 3.2 to estimate by

$$
||\partial_y \Gamma||_{H_t^{-1}} ||W||_{H_t^{-\sigma}},
$$

which is of the desired form by Young’s inequality.
It hence only remains to consider the homogeneous corrections. Here, we estimate
\[
\Re\left(A_1(t)\partial_y \Gamma \left(\frac{i\hbar}{k}((\partial_y L_t W)(a, t)e^{ikt(a-y)}u_1 + (\partial_y L_t W)(b, t)e^{ikt(b-y)}u_2)
+ \frac{i\hbar}{k}(\partial_y L_t[E_t, y]L_t W)(a, t)e^{ikt(a-y)}u_1 + (\partial_y L_t[E_t, y]L_t W)(b, t)e^{ikt(b-y)}u_2)
\right)
\leq C_\mu \left(\sum_n |\partial_y \Gamma_n|^2 < n - kt >^{-2\mu} \left(|\partial_y L_t W(a, t)| + |\partial_y L_t W(b, t)|
+ |\partial_y L_t[E_t, y]L_t W(a, t)| + |\partial_y L_t[E_t, y]L_t W(b, t)|\right)\right).
\]
We further recall from Section A that boundary evaluations can be obtained by testing with suitable homogeneous solution to the adjoint problem. Hence,
\[
|\partial_y L_t W(a, t)| + |\partial_y L_t W(b, t)| \lesssim t^{-1} \|\partial_y W\|_{L^2} \lesssim t^{-1} \|W\|_{H^1},
\]
\[
|\partial_y L_t[E_t, y]L_t W(a, t)| + |\partial_y L_t[E_t, y]L_t W(b, t)| \lesssim t^{-1} \|\partial_y L_t W\|_{H^1}.
\]
We can thus conclude the proof, if we can show that
\[
||E_t, \partial_y L_t W||_{H^1} \lesssim ||W||_{H^1}.
\]
Expressing \(\partial_y E_t, \partial_y L_t W = [E_t, \partial_y L_t \partial_y W + [E_t, \partial_y L_t, \partial_y]W\), this estimate follows from elliptic regularity theory for \([E_t, \partial_y]L_t\mid_{t=0}\) and using that multiplication by \(e^{ikt}y\) is an isometry. \(\square\)

Building on these results, we can now complete the proof of Theorem 4.1.

**Proof of Theorem 4.1, part 2.** Following a similar strategy as in the previous part, we consider
\[
I_2(t) := \langle \partial_y \Gamma, A_1(t)\partial_y \Gamma \rangle + C_1 \langle \Gamma, A_1(t)\Gamma \rangle + C_2 \langle \beta, A_1(t)\beta \rangle + C_3 \langle W, A(t)W \rangle,
\]
where \(C_1, C_2, C_3 > 0\) are to be chosen later.

Using the preceding results and strategy, it suffices to study
\[
\Re\langle \partial_y \partial_y \Gamma, A_1 \partial_y \Gamma \rangle.
\]
Following the same strategy as in the previous part of the proof and using Lemma 4.6, we estimate
\[
\Re\left(A_1 \partial_y \Gamma \left(\frac{\hbar}{k^2}L_t \partial_y \Gamma + \frac{\hbar}{k}L_t \Gamma + \frac{i\hbar}{k}L_t [E_t, y]L_t \Gamma
- (\partial_y L_t \Gamma)(a)e^{ikt(y-a)}u_1 - (\partial_y L_t \Gamma)(b)e^{ikt(y-b)}u_1\right)\right)
\leq (C + \sigma + c(1-\mu) \log(t))\|\partial_y \Gamma\|_{H^1}^2 \leq \sigma^{-1}|\langle \Gamma, A\Gamma \rangle|,
\]
which can be absorbed.

Furthermore, applying Lemma 4.3, we can control
\[
\Re\left(A_1 \partial_y \Gamma \left(\frac{h}{k^2} \langle \Gamma, e^{ikt(y-a)}\tilde{u}_1 \rangle e^{ikt(y-a)}u_1 - \frac{h}{k^2} \langle \Gamma, e^{ikt(y-a)}\tilde{u}_2 \rangle e^{ikt(y-b)}u_2
- \frac{h}{k^2} (W, e^{ikt(y-a)}\partial_y \tilde{u}_1) e^{ikt(y-a)}u_1 - \frac{h}{k^2} (W, e^{ikt(y-a)}\partial_y \tilde{u}_2) e^{ikt(y-b)}u_2\right)\right)
\leq C(\mu, \hbar, k) \left(\sum_n |\Gamma_n|^2 < n - kt >^{-2\mu} \left(\|\Gamma, e^{ikt(y-a)}\tilde{u}_1\|
+ \|W, e^{ikt(y-a)}\partial_y \tilde{u}_1\| + \|W, e^{ikt(y-b)}\partial_y \tilde{u}_2\|\right)\right).
\]
Applying the estimates of Lemma 4.6 and using Young's inequality, these contributions can hence again be partially absorbed provided $\sigma$ is sufficiently small and $T > 0$ is sufficiently large. The remaining non-absorbed terms can be estimated by $t^{-2(1-\mu/2)}(\|\Gamma(a,t)\| + \|\Gamma(b,t)\| + \|\partial_y W(t)\|_{L^2} + \|\omega_0(a)\| + \|\omega_0(b)\| \leq t^{-2(1-\mu/2)}\|\omega_0\|_{H^1}$, where we used Theorem 3.1, the first part of Theorem 4.1 and the Sobolev embedding.

It remains to estimate

$$\Re \left\langle A_1(t) \partial_y \Gamma, \partial_y \left( \frac{ih'}{k} L_t W + \frac{ih}{k} L_t [\mathcal{E}_t, \partial_y] L_t W \right) \right\rangle.$$ 

Recalling the definition of $\Gamma$ and $\beta$, we express the right function as

$$\left( \frac{ih'}{k} L_t \cdot + \frac{ih}{k} L_t [\mathcal{E}_t, \partial_y] L_t \right) (\Gamma + \beta) + \left[ \left( \frac{ih'}{k} L_t \cdot + \frac{ih}{k} L_t [\mathcal{E}_t, \partial_y] L_t \right), \partial_y \right] W.$$

We then estimate

$$\Re \left\langle A_1(t) \partial_y \Gamma, \left( \frac{ih'}{k} L_t \cdot + \frac{ih}{k} L_t [\mathcal{E}_t, \partial_y] L_t \right) (\Gamma + \beta) \right\rangle \lesssim \|\partial_y \Gamma\|_{\bar{H}^{-1}} (\|\Gamma\|_{\bar{H}^{-1}} + \|\beta\|_{\bar{H}^{-1}}).$$

Using Young's inequality, the respective terms can then again be controlled, given a suitable choice of $\sigma$. Finally, using Lemma 4.7,

$$\Re \langle A_1 \partial_y \Gamma, \frac{ih'}{k} L_t \cdot + \frac{ih}{k} L_t [\mathcal{E}_t, \partial_y] L_t, \partial_y \rangle W \leq \sigma \|\partial_y \Gamma, A_1 \partial_y \Gamma\| + C \sigma^{-1} \|W\|_{\bar{H}^{-1}}^2,$$

which can again be absorbed and hence concludes the proof. □

4.2. Weighted stability of $\partial_y \beta$ and boundary blow-up. In this section we consider the evolution of $\partial_y \beta$. Since the behavior at both boundary points is similar and separates, we for simplicity of notation consider the case $\omega_0(a) \neq 0, \omega_0(b) = 0$. The general case can then be obtained by switching $a$ and $b$ and using the linearity of the equation. The function $\beta$ then satisfies (57):

$$\partial_t \beta - \frac{ih}{k} L_t \beta - \frac{h}{k^2t} \beta e^{ik(y-a)u} e^{ik(y-a)u} = \omega_0(a) \frac{h}{k^2t} e^{ik(y-a)u},$$

$$\beta|_{t=0} = 0.$$

We note that, if $\omega_0|_{y=x,b} = 0$, then $\beta$ identically vanishes.

We recall that by Proposition 4.1 under suitable assumptions on $h, g$ and $k$, $\beta$ is stable in $L^2$. However, stability in $H^s$ or, indeed in $H^s, s > 1/2$, does not hold due to the asymptotic formation of singularities at the boundary.

**Lemma 4.8** (Boundary blow-up). *Suppose that for some $s > 0$,

$$\sup_{t>0} \|\beta(t)\|_{H^s} = C < \infty.$$

Then $\beta(a,t)$ satisfies

$$|\beta(a,t) - h(a)\omega_0(a) k^{-2} \log(t)| \leq C_* C,$$

as $t \to \infty$. In particular, if $\omega_0(a) \neq 0$, then

$$\sup_{t} \|\beta(t)\|_{C^0} = \infty.$$

Hence, by the Sobolev embedding, in that case,

$$\sup_{t} \|\beta(t)\|_{H^s} \geq \sup_{t} \log(t) = \infty,$$
Applying one □ which is hence identified as the term driving the blow-up. Based on this reasoning we use a different method of proof based Duhamel’s formula, the details of which where we used that

\[
\beta(H^s) \leq Ct^{-s}\|\beta\|_{H^s}.
\]

Hence, \(\beta(a, t)\) satisfies

\[
\partial_t \beta(a) + \omega_0(a) h(a) = \frac{h(a) \omega_0(a)}{k^2 t} \partial_t \log(t) = t^{-1} O(t^{-s}) \in L^1_t.
\]

The result hence follows by integrating in time. \( \square \)

Letting \( s = 1 \) in the preceding Lemma, we in particular note that in general \( H^1 \) stability of \( \beta \) fails. Following a similar approach as in [13], one can further show that \( s = 1/2 \) is indeed critical in the sense that stability holds for \( H^s, s < 1/2 \). As this is however not sufficient for optimal decay rates in the damping estimate of Section 2, in the following we prove weighted \( H^1 \) stability as formulated in Theorem 4.2. Here, we use a different method of proof based Duhamel’s formula, the details of which can be found in Appendix B.

4.2.1. Splitting \( \partial_y \beta \). We recall that \( \beta \) solves

\[
\partial_t \beta - \frac{i h}{k} L_t \beta - \frac{h}{k^2 t} \beta, e^{ik(t(y-a))u} = \omega_0(a) \frac{h}{k^2 t} e^{ik(t(y-a))u},
\]

\( \beta|_{t=0} = 0 \).

Applying one \( y \) derivative to this equation, we obtain

\[
\partial_t \partial_y \beta - \frac{i h}{k} L_t \partial_y \beta + \frac{h}{k^2 t} (\partial_y \beta, e^{ik(t(y-a))u}) e^{ik(t(y-a))u} + \frac{h}{k^2 t} (\partial_y, e^{ik(t(y-a))u}) e^{ik(t(y-a))u} - \omega_0(a) \frac{h'}{k^2 t} e^{ik(t(y-a))u} + \frac{i \omega_0(a) h}{k} e^{ik(t(y-a))u}.
\]

where we used that

\[
\frac{h}{k^2 t} (\beta, e^{ik(t(y-a))u}) \partial_y (e^{ik(t(y-a))u}) = \frac{h}{k^2 t} (\partial_y (\beta u), e^{ik(t(y-a))}) - \beta u e^{ik(t(y-a))|_{y=0}} e^{ik(t(y-a))u}.
\]

We note that most terms in (67) are very similar to ones in equation (59) satisfied by \( \partial_y \Gamma \), with the exception of

\[
\frac{i \omega_0(a) h}{k} e^{ik(t(y-a))u},
\]

which is hence identified as the term driving the blow-up. Based on this reasoning the following lemma introduces a splitting of \( \partial_y \beta \).
Lemma 4.9. Let \( \beta_I \) be the solution of
\begin{equation}
\partial_t \beta_I - \frac{ih}{k} L \beta_I + \frac{1}{kt} \langle \beta_I, e^{ikt} u \rangle e^{ikt} u = \frac{ih}{k} L \beta_I, \partial_y \beta + \frac{h}{k^2 t} \langle \beta, e^{ikt} u \rangle e^{ikt} u
+ \frac{h}{k^2 t} \beta(a,t) e^{ikt} u - \frac{h}{k^2 t} \langle \beta, e^{ikt} u \rangle e^{ikt} u
- \omega_0(a) \frac{h}{k^2 t} e^{ikt} \partial_y u + \omega_0(a) \frac{h}{k^2 t} e^{ikt} \partial_y u,
\end{equation}
and let \( \beta_{II} \) be the solution of
\begin{equation}
\partial_t \beta_{II} - \frac{ih}{k} L \beta_{II} + \frac{h}{k^2 t} \langle \beta_{II}, e^{ikt} u \rangle e^{ikt} u = \omega_0(a) e^{ikt} u,
\end{equation}
Then \( \partial_y \beta = \beta_I + \beta_{II} \).

Following the same strategy as in Section 4.1, we obtain \( L^2 \) stability of \( \beta_I \).

Proposition 4.2. Suppose the assumptions of Theorem 4.2 are satisfied, then
\[ \| \beta_I(t) \|_{L^2} \lesssim | \omega_0 |_{y=a,b} \].

Proof. Following the same strategy as in the proof of Theorem 4.1, we show that,
\[ \frac{d}{dt} \langle \beta_I, A_1(t) \beta_I \rangle \leq \langle \beta_I, A_1(t) \beta_I \rangle + C \| \beta_I \|_{H^\sigma}^2 \]
\[ + (C t^{-(1-\mu)} + \sigma) \sum_n |(\beta_I)_n|^2 < n - kt > - 2^\lambda t^{-\mu} \]
\[ + C \sigma^{-1} t^{-(1-\mu/2)} (|\beta(a,t)|^2 + \| \beta \|_{L^2}^2 + |\omega_0(a)|^2) \].

Hence, restricting to \( t \geq T > 0 \) and choosing \( \sigma \) sufficiently small,
\[ \frac{d}{dt} \langle \beta_I, A_1(t) \beta_I \rangle \leq C \sigma^{-1} t^{-(1-\mu/2)} (|\beta(a,t)|^2 + \| \beta \|_{L^2}^2 + |\omega_0(a)|^2) \]
\[ \leq C \sigma^{-1} t^{-(1-\mu/2)} \log(t)^2 |\omega_0(a)|^2 \]
where we used Proposition 4.1 and that, by equation (57),
\[ |\beta(a,t)| \lesssim \int^t \tau^{-1} \| \beta(\tau) \|_{L^2} d\tau \lesssim \log(t) |\omega_0(a)| \].

For later reference, we note that we have thus also proven the following proposition.

Proposition 4.3. Suppose that \( g, h, k \) satisfy the assumptions of the second part of Theorem 4.1. Then, for any \( \omega_0 \in L^2 \), the solution \( W \) of (27) satisfies
\[ \| W(t) \|_{H^1} + \| \partial_y^2 W(t) - \beta_{II}(t) \|_{L^2} \lesssim \| \omega_0 \|_{H^1} \]
where \( \beta_{II} \) is given by Lemma 4.9.

Proof. This result combines Theorems 3.1 and 4.1 and Propositions 4.1 and 4.2. \( \square \)
4.2.2. Weighted stability of \( \beta_{II} \). In order to complete the proof of Theorem 4.2, it only remains to study the stability of

\[ \partial_t \beta_{II} - \frac{ih}{k} L_t \beta_{II} + \frac{h}{k^2 t} (\beta_{II}, e^{ikty}u) e^{ikty} u = \frac{ih}{k} \omega_0(a) e^{ikty} u, \]

\[ \beta_{II}|_{t=0} = 0. \]

While it would be possible to study this equation directly, we instead build on our previous analysis of \( \partial_t - \frac{ih}{k} L_t \) (69) and introduce an additional boundary layer \( \nu \) (c.f. Theorem 1.1) solving

\[ (\partial_t - \frac{ih}{k} L_t) \nu = \frac{h}{k} \omega_0(a) e^{ikty}, \]

\[ \nu|_{t=0} = 0, \]

and also define \( \beta_V = \beta_{II} - \nu \). Then \( \beta_V \) solves

\[ \partial_t \beta_V - \frac{ih}{k} L_t \beta_V + \frac{\langle \beta_V, e^{ikty}u \rangle}{ikt} e^{ikty} u = \frac{\langle \nu, e^{ikty}u \rangle}{ikt} e^{ikty} u, \]

\[ \beta_V|_{t=0} = 0. \]

**Remark 5.** Instead of \( \nu \) one might attempt to choose the explicit function

\[ \int \frac{ih}{k} \omega_0(a) e^{ikty} d\tau = \frac{ih}{k} \omega_0(a) e^{ikty} - \frac{1}{iky} =: \chi. \]

However, we note that part of this function oscillates like \( e^{ikty} \) and that

\[ L_t \chi = e^{ikty} L_0 \frac{h}{k^2 y} \omega_0(a) + L_t \frac{h}{k^2 y}, \]

where \( L_0 \frac{h}{k^2 y} \omega_0(a) \) is independent of \( t \). Hence, even for a constant function \( u \)

\[ \langle u, L_t \chi \rangle \]

would not decay or oscillate rapidly enough to be an integrable perturbation.

As the main result of this section we establish the following proposition, which concludes the proof of Theorem 4.2.

**Proposition 4.4.** Suppose the assumptions of Theorem 4.2 are satisfied. Then the functions \( \beta_V \) and \( \nu \) satisfy

\[ \| \beta_V(t) \|_{L^2} \lesssim |\omega_0(a)|, \]

\[ \| (y - a)(y - b) \nu(t) \|_{L^2} \lesssim |\omega_0(a)|. \]

As the evolution of \( \beta_V \) depends on \( \nu \) via

\[ \langle \nu, e^{ikt(y-a)u} \rangle \]

and as our estimates of \( \nu \) rely on properties of the solution operator of (69) (and hence \( W \)), we follow a multi-step approach:

1. Using Propositions 4.2 and 2.2, we show that (75) grows at most like \( \sqrt{t} \).
2. By direct computation, we show that \( \| (y - a)(y - b) \nu \|_{L^2} \) grows at most like \( \log(t) \).
3. This yields a weaker form of Proposition 4.4 with an estimate by \( \sqrt{t} |\omega_0(a)| \).
4. Combining this estimate with the damping result of Section 2, the estimate of (75) improves to \( \log(t) \) and we obtain a uniform bound of \( \| (y - a)(y - b) \nu \|_{L^2} \).
5. Finally, we establish \( L^2 \) stability of \( \beta_V \) and thus conclude the proof of Proposition 4.4.
Lemma 4.10. Assume that the assumptions of Theorem 4.2 are satisfied. Then (75) satisfies
\[(76) \quad |\langle e^{ikt} \tilde{u}, \nu(t) \rangle| \lesssim \sqrt{t} |\omega_0(a)| \]
as \(t \to \infty\).

Lemma 4.11. Assume that the assumptions of Theorem 4.2 are satisfied. Then \(\nu(t)\) satisfies
\[(77) \quad \|(y-a)(y-b)\nu(t)\|_{L^2} \lesssim \log(t) |\omega_0(a)| \]
as \(t \to \infty\).

Lemma 4.12. Assume that the assumptions of Theorem 4.2 are satisfied. Then, as \(t \to \infty\), \(\beta_V\) and \(\nu\) satisfy
\[
\|\beta_V(t)\|_{L^2} \lesssim \sqrt{t} |\omega_0(a)|, \\
\|(y-a)(y-b)\nu(t)\|_{L^2} \lesssim \log(t) |\omega_0(a)|.
\]
In particular, we conclude that the solution operator
\[
S(t,0) : H^2(dy) \to H^2((y-a)(y-b)dy), \\
\omega_0 \mapsto W(t),
\]
satisfies
\[
|||S(t,0)||| \lesssim \sqrt{t}.
\]

Lemma 4.13. Assume that the assumptions of Theorem 4.2 are satisfied. Then \(\nu\) satisfies
\[(78) \quad \|(y-a)(y-b)\nu(t)\|_{L^2} \lesssim |\omega_0(a)| \]
as \(t \to \infty\).

Lemma 4.14. Assume that the assumptions of Proposition 4.4 are satisfied. Then \(\beta_V\) satisfies
\[(79) \quad \|\beta_V(t)\|_{L^2} \lesssim |\omega_0(a)| \]
as \(t \to \infty\).

In our proof of Lemmata 4.10 to 4.14, we rely on more detailed, (semi-explicit) characterization of \(\nu(t)\) via Duhamel’s formula, which is established in Appendix B.

Proof of Lemma 4.10. We directly compute
\[
\langle e^{ikt} \tilde{u}, \int_0^t e^{ikt} S(t,\tau)e^{ik\tau y}ud\tau \rangle
\]
\[(80) \quad = \langle \tilde{u}, \int_0^t e^{ik(t-\tau)y} S(t-\tau,0)ud\tau \rangle.
\]
Next, we integrate
\[
e^{ik(t-\tau)y} = \partial_\tau \frac{e^{ik(t-\tau)y} - 1}{iky}
\]
by parts in \(\tau\). Here, we obtain a boundary term
\[(81) \quad \langle \tilde{u}, e^{ikt} - \frac{1}{iky} S(t,0)u \rangle
\]
and an integral term
\[
\langle \tilde{u}, \int_0^t e^{ik(t-\tau)y} - \frac{1}{iky} \partial_\tau S(t-\tau,0)ud\tau \rangle.
\]
For (83) we apply Hölder’s inequality and control by
\[ \| \tilde{u} \|_{L^\infty} \left\| e^{ikt}u - \frac{1}{iky} \right\|_{L^2} \| S(t,0)u \|_{L^\infty} \lesssim \log(t) \| u \|_{H^1}. \]

In the integral term we use the damping estimate, Proposition 2.2, to control by
\[ \int_0^t \| \tilde{u} \|_{L^\infty} \left\| e^{ikt(-\tau)} - \frac{1}{iky} \right\|_{L^2} \| \partial_r S(t - \tau,0)u \|_{L^2} \, d\tau \]
\[ \lesssim \int_0^t \sqrt{|t-\tau|} \| S(t - \tau,0)u \|_{H^1} \, d\tau \]
\[ \lesssim \int 0 \sqrt{t - \tau} > -1/2 \, d\tau \lesssim \sqrt{t}. \]

\[ \square \]

**Proof of Lemma 4.11.** Using Lemmata B.1 and B.2, we obtain that
\[ \nu(t) = \int_0^t e^{ikt(-\tau)}(y-a) S(t - \tau,0)u \, d\tau \]
Multiplying with \((y-a),\) we use that
\[ -\partial_r e^{ikt(-\tau)(y-a)} - \frac{1}{ik} = (y-a)e^{ikt(-\tau)(y-a)} \]
and hence control
\[ \|(y-a)\nu(t)\|_{L^2} \leq \| e^{ikt(-\tau)(y-a)} - \frac{1}{ik} S(t - \tau,0)u \|_{L^2} \]
\[ + \int_0^t \left\| e^{ikt(-\tau)(y-a)} - \frac{1}{ik} \partial_r S(t - \tau,0)u \right\|_{L^2} \, d\tau \]
\[ \lesssim |k|^{-1} \| u \|_{L^2} + |k|^{-2} \| h \|_{L^\infty} \int_0^t \left\| L_{t-\tau} S(t - \tau,0)u \right\|_{L^2} \]
\[ \lesssim |k|^{-1} \| u \|_{L^2} + |k|^{-2} \| h \|_{L^\infty} \int_0^t < t - \tau > ^{-1} \| S(t - \tau,0)u \|_{H^1} \, d\tau \]
\[ \lesssim |k|^{-1} \| u \|_{L^2} + |k|^{-2} \| h \|_{L^\infty} \| u \|_{H^1} \log(t), \]
where we used Proposition 2.2 and Theorem 4.1. \[ \square \]

**Proof of Lemma 4.12.** Using our Lyapunov functional approach on \( \beta_V, \) we need to estimate
\[ \langle A_1 \beta_V, e^{ikt}u \rangle \frac{\langle \nu, e^{ikt}u \rangle}{ikt}. \]
By Lemma 4.10, we control
\[ \left| \frac{\langle \nu, e^{ikt}u \rangle}{ikt} \right| \lesssim t^{-1/2}, \]
and using Lemma 4.4, we estimate.
\[ \left| \langle A_1 \beta_V, e^{ikt}u \rangle \right| \leq C_\lambda \| (\beta_V) \|_{H^0} < n - kt > ^{-\lambda} \| u \|_{L^2}, \]
where \( 0 < \lambda < \frac{1}{2}. \)
Hence, using Young’s inequality, we can control (90) by
\[ \epsilon \| (\beta_V) \|_{H^0} < n - kt > ^{-\lambda} \| u \|_{L^2}^{-1/2} + C(\epsilon, \lambda) t^{-1/2}. \]
Here, for \( \epsilon \) sufficiently small, the first term can be absorbed by
\[ \langle \beta_V, A_1 \beta_V \rangle \]
and in summary we obtain
\[ \partial_t \langle \beta_V, A_1 \beta_V \rangle \leq C(\epsilon, \lambda)t^{-1/2}. \]
Integrating this inequality then yields the result. \(\square\)

We remark that already in step 3 we could obtain a better growth bound by optimizing in \(\lambda\) and the splitting of \(t^{-1/2}\) in Young’s inequality. However, since \(t^{-1/2} \notin L^2\) this would only yield a non-uniform bound and our multi-step proof only requires a better than linear growth bound.

**Proof of Lemma 4.13.** Following the proof of Lemma 4.11 it suffices to show that
\[ (91) \int_{0}^{t} \| \partial_\tau S(t - \tau, 0)u \|_{L^2} d\tau \lesssim 1, \]
uniformly in \(t\). Using Hölder’s inequality and Proposition 2.2, we estimate
\[ \| \partial_\tau S(t - \tau, 0)u \|_{L^2} = \frac{ih}{k} L_{t-\tau} S(t - \tau, 0)u \|_{L^2} \leq \| h \|_{L^\infty} |k|^{-1} \| L_{t-\tau} S(t - \tau, 0)u \|_{L^2} \]
\[ \leq \| h \|_{L^\infty} |k|^{-1} < t - \tau >^{-\lambda} (\| (y - a)(y - b) \beta^2 \partial_\tau S(t - \tau, 0)u \|_{L^2} + \| S(t - \tau, 0)u \|_H^1) \leq \| h \|_{L^\infty} |k|^{-1} < t - \tau >^{-\lambda} \| S(t - \tau, 0)u \|_{H^2}, \]
the operator norm of \(S(t - \tau, 0)\) is given by Lemma 4.12. Hence, we obtain that
\[ \| \partial_\tau S(t - \tau, 0)u \|_{L^2} \lesssim \| h \|_{L^\infty} |k|^{-1} \| u \|_{H^2} < t - \tau >^{-\lambda} \sqrt{t - \tau}, \]
which is integrable in \(\tau\) and thus concludes the proof. \(\square\)

**Proof of Lemma 4.14.** We claim that
\[ (92) \| (e^{ikty}\tilde{u}, \nu(t)) \| \lesssim \log(t) |\omega_0(a)|. \]
Following the proof of Lemma 4.12, this implies that
\[ (93) \| (A_1 \beta_V, e^{ikty}u) \| \leq \| (\beta_V) < n - kt >^{-\lambda} \| t \|_{L^2} \| \log(t) \| t \]
\[ \leq \epsilon \| (\beta_V) < n - kt >^{-\lambda} \| t \|_{L^2}^{2}t^{-2\mu} + C(\epsilon) \log(t)^{2}t^{-2(1 - \mu)}, \]
where \(C(\epsilon)\) is given by Young’s inequality and \(0 < \mu < 1\) is chosen such that \(2\lambda + 2\mu > 1\) and \(2(1 - \mu) > 1\). Choosing \(\epsilon\) sufficiently small, we thus obtain
\[ \partial_t \langle \beta_V, A_1 \beta_V \rangle \leq \langle \beta_V, \dot{A}_1 \beta_V \rangle + \epsilon \| (\beta_V) < n - kt >^{-\lambda} \| t \|_{L^2}^{2}t^{-2\mu} + C(\epsilon) \log(t)^{2}t^{-2(1 - \mu)} \lesssim C(\epsilon) \log(t)^{2}t^{-2(1 - \mu)} \in L^1_t([1, \infty)). \]
Integrating this inequality then yields the desired result.

It remains to prove the claim (92). Here, we estimate
\[ \| (e^{ikty}\tilde{u}, \nu(t)) \| \lesssim \log(t) \| u \|_{H^1} + \int_{0}^{t} \| \tilde{u} \|_{L^\infty} \| e^{ik(t-\tau)y} \|_{L^2} \| \partial_\tau S(t - \tau, 0)u \|_{L^2} d\tau. \]
Using Lemma 4.12 and Proposition 2.2 we control
\[ \| \partial_\tau S(t - \tau, 0)u \|_{L^2} \leq < t - \tau >^{-\lambda} \| S(t - \tau, 0)u \|_{H^2} \lesssim < t - \tau >^{-3/2} \| u \|_{H^2}, \]
and we directly compute that
\[ \| e^{ik(t-\tau)y} \|_{L^2} \lesssim \sqrt{t - \tau}. \]
Hence, we control
\[
\int_0^t \| \tilde{u} \|_{L^\infty} \| S(t - \tau, 0) e^{-iky (t - \tau)} \| \| y - 1 \|_{L^2} d\tau \\
\lesssim \| \tilde{u} \|_{L^\infty} \| u \|_{H^2} \int_0^t < t - \tau >^{-1} d\tau \leq \| \tilde{u} \|_{L^\infty} \| u \|_{H^2} \log(t),
\]
which proves the claim. □

**Appendix A. Auxiliary functions and boundary evaluations**

In this section we introduce several auxiliary functions, which can be used to compute boundary evaluations of derivatives of $L_t W$ and related quantities.

**Lemma A.1.** Let $u_1, u_2$ be solutions of
\[
E_t u = 0, \\
z \in [a, b],
\]
with boundary values
\[
(96)
\begin{pmatrix}
 u_1(a) \\
 u_2(a)
\end{pmatrix} = \begin{pmatrix}
 1 \\
 0
\end{pmatrix}, \\
\begin{pmatrix}
 u_1(b) \\
 u_2(b)
\end{pmatrix} = \begin{pmatrix}
 0 \\
 1
\end{pmatrix}.
\]
Let further $\tilde{u}_1, \tilde{u}_2$ be solutions to the adjoint problem
\[
(97)
\begin{pmatrix}
 E_t \tilde{u} = \left( \left( \frac{\partial y}{k} - it \right) g(y) \right)^2 - \left( \frac{\partial y}{k} - it \right) \frac{g(y)}{kr(y)} \right) \tilde{u} = 0, \\
y \in [a, b],
\end{pmatrix}
\]
with boundary values
\[
(98)
\begin{pmatrix}
 \tilde{u}_1(a) \\
 \tilde{u}_1(b)
\end{pmatrix} = \begin{pmatrix}
 1 \\
 0
\end{pmatrix}, \\
\begin{pmatrix}
 \tilde{u}_2(a) \\
 \tilde{u}_2(b)
\end{pmatrix} = \begin{pmatrix}
 0 \\
 1
\end{pmatrix}.
\]
Then $u_1, u_2, \tilde{u}_1, \tilde{u}_2$ satisfy
\[
(99)
\begin{pmatrix}
 u_1(t, r, k) = e^{ikt(y - a)} u_1(0, r, k), \\
 u_2(t, r, k) = e^{ikt(y - b)} u_2(0, r, k), \\
 \tilde{u}_1(t, r, k) = e^{ikt(y - a)} \tilde{u}_1(0, r, k), \\
 \tilde{u}_2(t, r, k) = e^{ikt(y - b)} \tilde{u}_2(0, r, k).
\end{pmatrix}
\]

**Proof of Lemma A.1.** We note that the operators in equations (95) and (97) are obtained by conjugating by $e^{iktz}$ and are complex linear. The result hence follows by noting that multiplication by $e^{ikt(y - a)}$ or $e^{ikt(y - b)}$ is compatible with the boundary conditions (96) and (98). □

**Lemma A.2.** Let $W$ be a given function and let $\Phi$ be a solution of
\[
E_t \Phi = W, \\
\Phi|_{y=a,b} = 0,
\]
and let $u_1, u_2, \tilde{u}_1, \tilde{u}_2$ be as in Lemma A.1. Define
\[
(101)
H^{(1)} = \frac{k^2}{g^2(a)} \langle \Phi, \tilde{u}_1 \rangle_{L^2} u_1 + \frac{k^2}{g^2(b)} \langle \Phi, \tilde{u}_2 \rangle_{L^2} u_2, \\
\Phi^{(1)} = \partial_r \Phi - H^{(1)}.
Then $\Phi$ satisfies
\begin{equation}
\langle W, \tilde{u}_1 \rangle_{L^2} = \frac{g^2(a)}{k^2} \partial_r \Phi(t, k, a),
\end{equation}
\begin{equation}
\langle W, \tilde{u}_2 \rangle_{L^2} = \frac{g^2(b)}{k^2} \partial_r \Phi(t, k, b),
\end{equation}
and $\Phi^{(1)}$ solves
\begin{equation}
\mathcal{E}_t \Phi^{(1)} = \partial_y W + [\mathcal{E}_t, \partial_y] \Phi,
\end{equation}
\begin{equation}
\Phi^{(1)}|_{y=a,b} = 0.
\end{equation}

The function $H^{(1)}$ is a solution of (95) and is called the (first) homogeneous correction.

**Proof.** Testing the equation (100) with the homogeneous solutions of Lemma A.1, the results follow by integration by parts and direct calculations. □

**Lemma A.3.** Let $\Phi, W$ as in Lemma A.2 and let $u_1, u_2, \tilde{u}_1, \tilde{u}_2$ as in Lemma A.1. Then $\Phi$ satisfies
\begin{equation}
\frac{g^2(a)}{k^2} \partial_r^2 \Phi(t, k, a) = -\frac{g(a)g'(a)}{k^2} \partial_r \Phi(t, k, a) - \frac{g(y)}{k^2 r(y)} \partial_y \Phi(t, k, a) + W(t, k, a),
\end{equation}
\begin{equation}
\frac{g^2(a)}{k^2} \partial_r^2 \Phi(t, k, b) = -\frac{g(b)g'(b)}{k^2} \partial_r \Phi(t, k, b) - \frac{g(y)}{k^2 r(y)} \partial_y \Phi(t, k, b) + W(t, k, b).
\end{equation}

Define
\begin{equation}
H^{(2)} = \partial_y^2 \Phi(t, k, a) u_1 + \partial_y^2 \Phi(t, k, b) u_2,
\end{equation}
\begin{equation}
\Phi^{(2)} = \partial_y^2 \Phi - H^{(2)},
\end{equation}
then $\Phi^{(2)}$ satisfies
\begin{equation}
\partial_y^2 W + \left[ (g(y)(\frac{\partial_y}{k} - it))^2 + \frac{g(y)}{kr(y)} (\frac{\partial_y}{k} - it) - \frac{1}{r^2(y)} \right] \Phi^{(2)} = 0.
\end{equation}

The function $H^{(2)}$ is a solution of (95) and is called the (second) homogeneous correction.

**Proof.** Direct computation. □

**Appendix B. Duhamel’s formula and shearing**

**Lemma B.1** (Time dependent Duhamel). Let $(L(t))_{t \in \mathbb{R}}$ be a given family of linear operators and denote by $S(t, t')$ the solution operator of
\begin{equation}
(\partial_t + \frac{i}{k} L_t) a = 0,
\end{equation}
mapping a prescribed $a(t')$ to $a(t)$. Then for any given function $F$ the unique solution of
\begin{equation}
(\partial_t + \frac{i}{k} L_t) u = F,
\end{equation}
\begin{equation}
u(0) = u_0,
\end{equation}
is given by
\[ u(t) = S(t,0)u_0 + \int_0^t S(t,t')F(t')dt'. \]

Proof. Since \( S(0,0) = Id \), we observe that the such defined \( u(t) \) satisfies \( u(0) = u_0 \).

It remains to show that \( u \) satisfies the equation. We directly compute
\[
(\partial_t + \frac{ih}{k} L_t) u(t) = (\partial_t + \frac{ih}{k} L_t) S(t,0) u_0 + \int_0^t (\partial_t + \frac{ih}{k} L_t) S(t,t') F(t') dt' + S(t,t) F(t) \\
= 0 + \int_0^t 0 S(t,t') F(t') dt' + \text{Id} F(t) = F(t).
\]
Here we used that for any \( t' \)
\[
(\partial_t + \frac{ih}{k} L_t) S(t,t') = 0.
\]
We stress that
\[
(\partial_t + \frac{ih}{k} L_t) S(t,t')
\]
do not vanish in general for any \( \tilde{t} \neq t \). □

Applying Lemma B.1 to (70), we obtain that
\[
\nu(t) = \omega_0(a) \int_0^t S(t,\tau) e^{ik\tau y} u d\tau,
\]
where \( S(t,\tau) \) is the solution operator corresponding to (69). Since \( L_t \) was defined by a conjugation of \( L_0 \) with \( e^{ikty} \), we can also conjugate \( S(t,\tau) \).

Lemma B.2. Let \( \sigma > 0 \), then for any \( 0 \leq s \leq \tau \leq t \) the solution operator \( S \) satisfies
\[
S(t,\tau) e^{ik\tau y} f = e^{ik\sigma y} S(t - \sigma, \tau - \sigma) f
\]
for any \( f \in L^2 \).

Proof. We note that for any \( t \)
\[
e^{-ik\sigma y} e_{t,\sigma} e^{ik\sigma y} = e_{t-\sigma}
\]
and that also
\[
e^{ik\sigma y} (e^{ik\sigma y} f, e^{ik\sigma y} u) e^{ik\sigma y} u = (f, e^{ik(t-\sigma)y} u) e^{ik(t-\sigma)y} u.
\]
Hence, conjugating the equation by \( e^{ik\tau y} \) is equivalent to a shift in time, which yields the desired result. □

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