Non-Abelian Conversion and Quantization of Non-scalar Second-Class Constraints

I. Batalin, M. Grigoriev, and S. Lyakhovich

Abstract. We propose a general method for deformation quantization of any second-class constrained system on a symplectic manifold. The constraints determining an arbitrary constraint surface are in general defined only locally and can be components of a section of a non-trivial vector bundle over the phase-space manifold. The covariance of the construction with respect to the change of the constraint basis is provided by introducing a connection in the “constraint bundle”, which becomes a key ingredient of the conversion procedure for the non-scalar constraints. Unlike in the case of scalar second-class constraints, no Abelian conversion is possible in general. Within the BRST framework, a systematic procedure is worked out for converting non-scalar second-class constraints into non-Abelian first-class ones. The BRST-extended system is quantized, yielding an explicitly covariant quantization of the original system. An important feature of second-class systems with non-scalar constraints is that the appropriately generalized Dirac bracket satisfies the Jacobi identity only on the constraint surface. At the quantum level, this results in a weakly associative star-product on the phase space.
1. **Introduction**

The quantization problem is usually understood as that of constructing a quantum theory for a given classical system, at the same time preserving important properties of the system such as locality and global symmetries. This additional requirement is crucial. Indeed, formally one can always find a representation such that all the constraints are solved, gauge symmetries are just shift symmetries, and the Poisson bracket has the canonical form. But in doing so one usually destroys locality and global symmetries. It is the problem of quantization of relativistic local field theories that initiated the development of sophisticated quantization methods applicable to systems with non-Abelian and open gauge algebras \[1, 2, 3, 4\].

From this point of view, the problem of quantizing curved phase space appears as a problem of constructing quantization in a way that is explicitly covariant with respect to arbitrary change of phase-space coordinates. Given such a method (at the level of deformation quantization at least) one can always find quantization in each coordinate patch and then glue everything together. Similar to the curved phase-space quantization problem is the one of quantizing arbitrary constraint surface. Any surface can be represented by independent equations (constraints) but in general only locally. In fact, one can always assume that the surface is the zero locus of a section of a vector bundle over the phase
space. The quantization problem for arbitrary constrained systems can then be reformulated as the problem of constructing quantization that is explicitly covariant with respect to the basis of constraints. In this paper, we restrict ourselves to the case of second-class constraints and address the problem of constructing a quantization scheme which is explicitly covariant with respect to the change of phase-space coordinates and constraint basis.

A general framework that allows to quantize second-class constraints at the same footing as first-class ones is the well-known conversion – the procedure that converts the original second-class constraints into first-class ones by introducing extra variables known as conversion variables. At least locally in the phase space, any second-class constraints can be converted into Abelian ones, and therefore the Abelian conversion is sufficient for most applications. The situation changes drastically if one wants the quantization to be explicitly covariant with respect to the change of the constraint basis. Indeed, by changing the constraint basis one can always make the converted constraints non-Abelian. Additional price one has to pay for covariance is the appearance of a connection in the vector bundle associated with the constraints. This is reminiscent of the quantization of systems with curved phase space, where phase-space covariance requires introducing a symplectic connection on the phase space. In fact, this is more than a coincidence.

The coordinate and constraint basis covariance appear to be intimately related within the quantization methods developed in [5, 6] (see also [7, 8]). Indeed, the key ingredient of these methods is the embedding of the system into the cotangent bundle over its phase space. In the natural coordinate system $x^i, p_i$, the embedding constraints $p_i = 0$ are non-scalar [7]. In this example, the reparametrization covariance in the original phase space translates into the covariance with respect to the basis of constraints $p_i$. In [8], this approach was extended to general second-class constrained systems with constraints being scalar functions.

In this paper, we extend the method in [8] to the case where the second-class constraint surface is an arbitrary symplectic submanifold of the phase space, not necessarily defined by zero locus of the set of any independent scalar functions. Considering the quantization problem for the constrained systems whose classical dynamics evolves on the constraint surface, one has to take care of the geometry of the tubular neighborhood of the constrained submanifold. The geometry of the entire phase space is irrelevant for this problem. In its turn, any tubular neighborhood of the submanifold can be identified with the normal bundle over the submanifold. For coisotropic submanifolds (first-class constrained systems), the corresponding approach to quantization was considered in [9]. It then follows that arbitrary constrained submanifold can be considered as a zero locus of a section of the appropriate vector bundle over the phase space. Moreover, in practical physical problems, the second-class constraints can appear from the outset as components
of a section of some bundle over the original phase space rather than scalar functions. This leads naturally to the concept of constrained systems with non-scalar constraints.

By considering the original non-scalar constraints $\theta_\alpha$ at the same footing with the constraints $p_i$ determining the embedding into $T^*\mathcal{M}$, we achieve a globally defined description for general second-class systems. Using the appropriate non-Abelian conversion procedure and subsequent BRST quantization, we then arrive at the formulation of the quantum theory (at the level of deformation quantization) that is explicitly covariant under the reparametrizations of the original phase space and under the changing the constraint basis. We note that the non-scalar first-class constraints were also considered in [10] in a different framework.

The conventional approach to second-class systems is based on the Dirac bracket – a Poisson bracket on the entire phase space, which is determined by constraints and for which the constraint surface is a symplectic leaf. This allows considering the Poisson algebra of observables as a Dirac bracket algebra of phase-space functions modulo those vanishing on the constrained submanifold. From this point of view, the quantization problem can be understood as that of quantizing a degenerate Poisson bracket. However, outside the constraint surface, the Dirac bracket is not invariant under the change of the constraint basis and therefore is not well-defined in the case of non-scalar constraints. The Dirac bracket bivector can be invariantly continued from the constraint surface under certain natural conditions, although the price is that the Jacobi identity is in general satisfied only in the weak sense, i.e., on the constrained submanifold. In the non-Abelian conversion framework such a covariant generalization of the Dirac bracket is naturally determined by the Poisson bracket of observables of the converted system. We note that weak brackets were previously studied in various contexts in [11, 12, 13, 14].

At the quantum level, the lack of Jacobi identity for the covariant Dirac bracket results in a phase-space star-product that is not associative in general. Within the non-Abelian conversion approach developed in the paper, this star-product naturally originates from the quantum multiplication of BRST-invariant extensions of phase-space functions. In the BRST cohomology, we obtain an associative star product which is identified with the quantum deformation of the classical algebra of observables (functions on the constraint surface). In particular, the associativity of the phase-space star-product is violated only by the terms vanishing on the constraint surface.

The quantization method developed in this paper can be viewed as an extension of the Fedosov quantization scheme [15, 6] to systems whose constrained submanifolds are defined by non-scalar constraints and whose phase spaces, as a result, carry a weak Poisson structure. We note that gauge systems with a weak Poisson structure can be alternatively quantized [14] using the Kontsevich formality theorem.
2. GEOMETRY OF CONSTRAINED SYSTEMS WITH LOCALLY DEFINED CONSTRAINTS

We consider a constrained system on a general symplectic manifold \((\omega, M)\). The constrained system is defined on \(M\) by specifying a submanifold \(\Sigma \subset M\) such that the restriction \(\omega|_\Sigma\) of the symplectic form to the constraint surface has a constant rank. If \(\Sigma\) is coisotropic, the constrained system is called first-class. A constrained system is called second-class if the restriction \(\omega|_\Sigma\) of the symplectic form is invertible on \(\Sigma\).

Let us assume for a moment that \(\Sigma\) is determined by constraints \(\theta_\alpha = 0\) which are globally defined functions on \(M\), then \(\{\theta_\alpha, \theta_\beta\}|_\Sigma = 0\) (\(\{\theta_\alpha, \theta_\beta\}|_\Sigma\) is invertible) iff the system is first- (respectively second-) class. The converse is also true, but only locally: if a constrained system is first- (respectively second-) class then locally there exist independent functions \(\theta_\alpha\) determining constraint surface \(\Sigma\) by \(\theta_\alpha = 0\) and any such functions satisfy \(\{\theta_\alpha, \theta_\beta\}|_\Sigma = 0\) (respectively \(\{\theta_\alpha, \theta_\beta\}|_\Sigma\) is invertible).

The dynamics of a constrained system on \(M\) is assumed evolving on the constraint surface \(\Sigma \subset M\). At the quantum level, a tubular neighborhood of \(\Sigma\) gets involved in describing dynamics. In its turn, it is a standard geometrical fact that any such neighborhood is diffeomorphic to a vector bundle over \(\Sigma\). Indeed, in each neighborhood \(U^{(i)}\) of a point of \(\Sigma\) one can pick a coordinate system \(x^a_{(i)}, \theta^{(i)}\) such that \(\Sigma \cap U^{(i)}\) is singled out by \(\theta^{(i)}_\alpha = 0\) and on the intersection of two such neighborhoods \(U^{(i)}\) and \(U^{(j)}\)

\[
x^a_{(i)} = X^a_{(ij)}(x^{(j)}), \quad \theta^{(i)}_\alpha = (\phi^{(ij)})^\beta_\alpha \theta^{(j)}_\beta
\]

with some functions \(X^a_{(ij)}(x)\) and \(\phi^{(ij)}(x)\). Functions \((\phi^{(ij)})^\beta_\alpha\) can be identified with transition functions of a vector bundle \(V^*(\Sigma)\) over \(\Sigma\) (we use the notation for a dual bundle to make notations convenient in what follows). Under the identification of an open neighborhood of \(\Sigma\) with the vector bundle \(V^*(\Sigma)\), coordinates \(\theta_\alpha\) are identified with constraint functions on \(M\). In particular, \(\Sigma\) goes to the zero section of \(V^*(\Sigma)\). Note that the constraints \(\theta_\alpha\), being understood as functions on \(M\), are defined only locally. If there exist globally defined constraints then \(V^*(\Sigma)\) is trivial.

It can be useful to pull back the vector bundle \(V^*(\Sigma)\) to the vector bundle \(V^*(M)\) over \(M\). Functions \(\theta_\alpha\) are then naturally identified with the components of a globally defined section \(\theta\) of \(V^*(M)\). At the same time \(\Sigma\) is nothing else than a submanifold of points where \(\theta\) vanishes. These arguments motivate the following concept of a constrained system:

**Definition 2.1.** A constrained system with non-scalar constraints is a triple \((M, V^*(M), \theta)\) where \(M\) — symplectic manifold with a symplectic form \(\omega\), \(V^*(M)\) — vector bundle over \(M\), and \(\theta\) is a fixed section of \(V^*(M)\). It is assumed that vanishing points of \(\theta\) are regular and form a submanifold \(\Sigma \subset M\) (constraint surface) such that \(\omega|_\Sigma\) has a constant rank.
The definitions of first- and second-class constrained systems still stand because they are formulated entirely in the intrinsic terms of the constraint surface $\Sigma$, making use only of the rank of $\omega|_\Sigma$ irrespectively to the way of defining $\Sigma$.

Several comments are in order:

(i) Another possibility to consider arbitrary constraint surface keeping at the same time constraints globally defined functions is to use overcomplete sets of constraints (i.e. reducible constraints, in different terminology). However, depending on a particular system this can be a complicated task. Moreover, even if the constraints are reducible it can also be useful to allow them to be non-scalar.

(ii) As we have seen, any submanifold $\Sigma \subset M$ can be represented as a surface of regular vanishing points of a section of a vector bundle over $M$. Note, however, that by taking arbitrary constraints $\theta^{(i)}_\alpha$ in each neighborhood $U^{(i)}$ one does not necessarily arrive at a vector bundle. Indeed, in the intersection $U^{(i)} \cap U^{(j)}$ one still has

$$\theta^{(i)}_\alpha = (\phi^{(ij)})^{\beta}_\alpha \theta^{(j)}_\beta$$

(2.2)

But functions $(\phi^{(ij)})^{\beta}_\alpha$ are defined only up terms of the form $(\chi^{(ij)})^{\beta\gamma}_\alpha \theta_\gamma$ with $(\chi^{(ij)})^{\beta\gamma}_\alpha = -(\chi^{(ij)})^{\gamma\beta}_\alpha$. As a consequence, functions $(\phi^{(ij)})^{\beta}_\alpha$ satisfy the cocycle condition also up to terms proportional to $\theta$

$$\phi^{(ik)}_\beta(\phi^{(kj)})^{\gamma}_\beta = (\phi^{(ij)})^{\beta}_\alpha + \ldots.$$  

(2.3)

This means that only appropriately chosen constraints can be identified with components of a section of a vector bundle over $M$. What differential geometry tells us is that such a choice always exists.

3. Connections and symplectic structures on vector bundles

In what follows we need some geometrical facts on the connections and symplectic structures on the appropriately extended cotangent bundle over a symplectic manifold. Let now $M$ be a symplectic manifold and $\mathcal{W}(M) \to M$ be a symplectic vector bundle over $M$. Let also $e^A$ be a local frame (locally defined basic sections of $\mathcal{W}(M)$) and $\mathcal{D}$ be the symplectic form on the fibers of $\mathcal{W}(M)$. The components of $\mathcal{D}$ with respect to $e^A$ are determined by $D_{AB} = \mathcal{D}(e_A, e_B)$.

It is well known (see e.g. [6]) that any symplectic vector bundle admits a symplectic connection. Let $\Gamma$ and $\nabla$ denote a symplectic connection and the corresponding covariant differential in $\mathcal{W}(M)$. The compatibility condition reads as

$$\nabla \mathcal{D} = 0,$$

(3.1)

$$\partial_i D_{AB} - \Gamma^C_{iA} D_{CB} - \Gamma^C_{iB} D_{AC} = 0,$$

where the coefficients $\Gamma^C_{iA}$ of $\Gamma$ are determined as:

$$\nabla e^A = dx^i \Gamma^C_{iA} e^C.$$  

(3.2)
It is useful to introduce the following connection 1-form:
\begin{equation}
\Gamma_{AB} = dx^i \Gamma_{AiB}, \quad \Gamma_{AiB} = D_{AC} \Gamma_{iB}^C.
\end{equation}

Then compatibility condition (3.1) rewrites as
\begin{equation}
dD_{AB} = \Gamma_{AB} - \Gamma_{BA}, \quad \partial_i D_{AB} - \Gamma_{AiB} + \Gamma_{BiA} = 0.
\end{equation}

As a consequence of the condition one arrives at the following property of the connection 1-form \( \Gamma_{AB} \):
\begin{equation}
d\Gamma_{AB} = d\Gamma_{BA}.
\end{equation}

Consider the following direct sum of vector bundles:
\begin{equation}
E_0 = W(M) \oplus T^* M,
\end{equation}
where \( T^* M \) denotes a cotangent bundle over \( M \). Let \( x^i, p_j \) and \( Y^A \) be standard local coordinates on \( E_0 \) \( (x^i) \) are local coordinates on \( M \), \( p_j \) are standard coordinates on the fibers of \( T^* M \), and \( Y^A \) are coordinates on the fibers of \( W(M) \) corresponding to the local frame \( e_A \). Assume in addition that \( M \) is equipped with a closed 2-form \( \omega \) (not necessarily nondegenerate).

Considered as a manifold, \( E_0 \) can be equipped with the following symplectic structure
\begin{equation}
\omega^{E_0} = \pi^* \omega + 2 dp_i \wedge dx^i + D_{AB} dY^A \wedge dY^B + d \Gamma_{AB} Y^A Y^B - 2 \Gamma_{AB} \wedge dY^A Y^B,
\end{equation}
where \( \pi^* \omega \) is the 2-form \( \omega \) on \( M \) pulled back by the bundle projection \( \pi : E_0 \to M \). One can directly check that 2-form (3.7) is well defined. Indeed, it can be brought to the standard explicitly covariant form, similar to that of the supersymplectic manifolds \[16\]
\begin{equation}
\omega^{E_0} = \pi^* \omega + 2 dp_i \wedge dx^i + D_{AB} \nabla Y^A \wedge \nabla Y^B + R_{AB} Y^A Y^B,
\end{equation}
Here, \( \nabla Y^A = d Y^A + \Gamma^A_C Y^C \), and \( R_{AB} = R_{ij;AB} dx^i \wedge dx^j \) denotes the curvature of \( \Gamma \):
\begin{equation}
R_{ij;AB} = D_{AC} R_{ijB}^C = D_{AC} \left( \partial_i \Gamma_{jB}^C - \partial_j \Gamma_{iB}^C + \Gamma_{iD}^C \Gamma_{jB}^D - \Gamma_{jD}^C \Gamma_{iB}^D \right) = \partial_i \Gamma_{AjB} - \partial_j \Gamma_{AiB} + \Gamma_{CiA} D^{CD} \Gamma_{DjB} - \Gamma_{CiA} D^{CD} \Gamma_{DjB}.
\end{equation}
The last equality follows from nondegeneracy of \( D_{AB} \) and compatibility condition (3.4).

Also, it is straightforward to show that, the 2-form (3.7) is exact, besides the first term:
\begin{equation}
\omega^e = \pi^* \omega + d \left[ 2 p_i dx^i + Y^A D_{AB} \nabla Y^B \right]
\end{equation}
Analyzing the structure in the r.h.s. of (3.10) one can see that an arbitrary (not necessarily symplectic) connection \( \tilde{\Gamma} \) can be taken to construct the close 2-form on \( \mathcal{E} \) in (3.10).
It turns out that the resulting 2-form still has the structure (3.7) with $\Gamma$ given by

$$\Gamma_{AB} = \frac{1}{2} (dD_{AB} + \Gamma^0_{AB} + \Gamma^0_{BA}).$$

It is easy to see that connection $\Gamma$ is by construction compatible with the symplectic structure $D$ for any connection $\Gamma^0$. In addition, if $\Gamma^0$ was taken symplectic it would bring $\Gamma = \Gamma^0$.

The Poisson bracket on $\mathcal{E}_0$ corresponding to the symplectic form (3.7) is determined by the following basic relations:

$$\{p_i, x^j\}_{\mathcal{E}_0} = -\delta^j_i, \quad \{p_i, p_j\}_{\mathcal{E}_0} = \omega_{ij}(x) + \frac{1}{2} R_{ij;AB}(x) Y^A Y^B,$$

$$\{Y^A, Y^B\}_{\mathcal{E}_0} = D^{AB}(x), \quad \{p_i, Y^A\}_{\mathcal{E}_0} = \Gamma^A_{iB}(x) Y^B,$$

with all the others vanishing: $\{x^i, x^j\}_{\mathcal{E}_0} = 0$.

4. Embedding and Conversion at the Classical Level

4.1. Embedding. Consider a second-class constrained system $(\mathcal{M}, V^*(\mathcal{M}), \theta)$ with locally-defined constraints $\theta_\alpha$ (i.e. $\theta_\alpha$ are components of a section $\theta$ of $V^*(\mathcal{M})$ with respect to a local frame $e^\alpha$). Let $T^*_\omega \mathcal{M}$ be a cotangent bundle equipped with the modified symplectic structure $2dp_i \wedge dx^i + \pi^*_0 \omega$, where $\omega$ is a symplectic form on $\mathcal{M}$ and $\pi_0: T^*_\omega \mathcal{M} \to \mathcal{M}$ is the canonical projection.

The embedding of $\mathcal{M}$ into $T^*_\omega \mathcal{M}$ as a zero section is a symplectic map, i.e. a restriction of symplectic form $2dp_i \wedge dx^i + \pi^*_0 \omega$ to the submanifold $\mathcal{M}$ is $\omega$. Moreover, constrained system $(\mathcal{M}, V^*(\mathcal{M}), \theta)$ is equivalent to the constrained system $(T^*_\omega \mathcal{M}, W^*(\mathcal{M}), \Theta)$, where $W^*(\mathcal{M})$ is a direct sum $W^*(\mathcal{M}) = T^*\mathcal{M} \oplus V^*(\mathcal{M})$ considered as a vector bundle over $T^*_\omega (\mathcal{M})$ and components of $\Theta$ with respect to the local frame $dx^i, e^\alpha$ are $-p_i, \theta_\alpha$ (in other words, locally, the constraints are given by $-p_i = 0$ and $\theta_\alpha = 0$). Indeed, by solving constraints $-p_i = 0$ one arrives at the starting point constrained system. At this stage the construction here repeats the one from [8] with the only difference that constraints $\theta_\alpha$ are now defined only locally.

4.2. Non-Abelian conversion. Given second-class constraints $\Theta_A$ one can always find an appropriate extension of the phase space by introducing conversion variables $Y^A$ whose Poisson bracket relations have the form $\{Y^A, Y^B\} = D^{AB}$ with $D^{AB}$ invertible. Then one can find converted constraints $T_A$ in the extended phase space, satisfying

$$\{T_A, T_B\} = U^C_{AB} T_C, \quad T_A|_{Y=0} = \Theta_A.$$

The resulting first-class system with constraints $T_A$ is equivalent to the original second-class one and is called converted system. For second-class constraints that are scalar functions on the phase space one can always assume the conversion to be Abelian, i.e.
with vanishing functions $U^C_{AB}$ (see [17] for a detailed discussion of the conversion and the existence theorem for the Abelian conversion).

For the non-scalar constraints one naturally wants to build a converted constraints in the invariant way, i.e. independently of a particular choice of the constraint basis. As we will see momentarily this forces one to consider, in general, a non-Abelian conversion.

To see this, one first needs to introduce conversion variables in a geometrically covariant way. It is useful to take as conversion variables the coordinates on the fibers of the bundle $W(\mathcal{M})$ dual to the bundle $W^*(\mathcal{M}) = T^*\mathcal{M} \oplus V^*(\mathcal{M})$ associated to constraints $\theta_\alpha, -p_i$. The phase space is then

$$E_0 = T^*\mathcal{M} \oplus W(\mathcal{M}), \quad W(\mathcal{M}) = V(\mathcal{M}) \oplus TM,$$

We introduce unified notation $e_A$ and $Y^A$ for the local frame and coordinates on the fibers of $W(\mathcal{M})$ respectively. In the adapted basis $Y^A$ split into $Y^i$ and $Y^\alpha$.

Given a connection $\bar{\Gamma}$ in $V(\mathcal{M})$ one can equip $W(\mathcal{M})$ with the following fiberwise symplectic structure

$$D_{ij} = \omega_{ij}, \quad D_{i\alpha} = -D_{\alpha i} = \bar{\nabla}_i \theta_\alpha = \partial_i \theta_\alpha - \bar{\Gamma}_i^\beta \theta_\beta, \quad D_{\alpha\beta} = 0.$$

In what follows we also need the explicit form of its inverse $D^{AC}$, $D^{AC}D_{CB} = \delta^A_B$

$$D^{\alpha\beta} = \Delta^{\alpha\beta}, \quad D^{i\beta} = -\omega^{ij} D_{l\gamma} \Delta^{\gamma\beta}, \quad D^{ij} = \omega^{ij} - \omega^{ik} D_{l\alpha} \Delta^{\alpha\beta} D_{l\beta} \omega^{lj},$$

where we introduced $\Delta^{\alpha\beta}$ as follows:

$$\Delta^{\alpha\gamma} \Delta^{\gamma\beta} = \delta^{\alpha}_\beta, \quad \Delta_{\alpha\beta} = D_{\alpha\beta} \omega^{ij} D_{j\beta}.$$

$\Delta$ is invertible on $\Sigma$ by assumption (recall that its invertibility is a part of the defining property of second-class constraints). It is then invertible in some neighborhood of $\Sigma$ and we assume that it is invertible on the entire $\mathcal{M}$.

Note that $D^{ij}$ determines a bivector field on $\mathcal{M}$ which coincides on $\Sigma$ with the conventional Dirac bracket. The latter bracket is not well-defined beyond $\Sigma$ if the constraints are not scalars. The bracket determined by $D^{ij}$ in (4.4) can therefore be understood as a covariant generalization of the Dirac bracket to the case of non-scalar constraints. It is straightforward to check that the covariant Dirac bracket satisfies Jacobi identity modulo terms vanishing on $\Sigma$.

Furthermore, one can equip $W(\mathcal{M})$ with the symplectic connection compatible with the fiberwise symplectic form. This is achieved as follows. First one picks a linear symplectic connection $\bar{\nabla}_{^0}$ on the symplectic manifold $\mathcal{M}$ and equips $W(\mathcal{M})$ with the direct sum connection $\bar{\nabla}$ determined by

$$\bar{\nabla}_{^0} e_i = (\bar{\Gamma}_{^0})^i_j e_j, \quad \bar{\nabla}_{^0} e_\alpha = (\bar{\Gamma}_{^0})^\alpha_\beta e_\beta,$$
where $\nabla^0$ denotes the covariant differential determined by $\Gamma^0$. Given “bare” connection $\nabla^0$ in $W(M)$ one then arrives at the symplectic connection $\Gamma$ using (3.11). In its turn the symplectic connection in $W(M)$ determines a symplectic structure $\omega_{E_0}$ on $E_0$ in accordance to the general formula (3.7). The associated Poisson bracket reads as

$$\{p_i, x^j\} = -\delta^j_i, \quad \{p_i, p_j\} = \omega_{ij}(x) + \frac{1}{2} R_{ij;AB}(x) Y^A Y^B,$$

$$\{Y^A, Y^B\} = \mathcal{D}^{AB}(x), \quad \{p_i, Y^A\} = \Gamma^A_i(x) Y^B,$$

Here and in what follows we drop the superscript of the Poisson bracket on the extended phase space whenever it can not lead to confusions. Note that embedding of $T^*_\omega M$ into $E_0$ is symplectic. This implies that coordinates $Y^A$ can be treated as second-class constraints (they can also be understood as gauge conditions for the converted system). Considered together with constraints $\Theta_A$ they determine a constrained system on $E_0$ that is equivalent to the original constrained system on $M$.

Since we are interested in the non-Abelian conversion, it is preferable to work in terms of the BFV-BRST formalism from the very beginning. To this end we introduce ghost variables $C^A$ and $P_A$ with the transformation law determined by that of components of a section of $W(M)$ and $W^*(M)$ respectively. One can consistently assume canonical Poisson bracket relations

$$\{P_A, C^B\} = -\delta^B_A,$$

and brackets between $C^A$ and $P_A$ and all other variables vanishing. Note that in order for Poisson brackets between ghosts and other variables to remain vanishing when passing from one neighborhood to another momenta $p_i$ should transform inhomogeneously. This means that the extended phase space is $E = T^*_\omega (\Pi W(M)) \oplus W(M)$ with $W(M)$ in the second summand considered as a vector bundle over $\Pi W(M)$. Here and below $\Pi$ indicates that the Grassmann parity of the fibers of a vector bundle is reversed. Note also that the extended phase space $E$ is not anymore a vector bundle over $M$ because $p_i$ transform in an inhomogeneous way.

In the BRST language the conversion problem can be formulated as follows. Given “bare” generating function $\bar{\Omega}$ whose expansion with respect to the ghosts variables starts with given second-class constraints $\Theta_A$

$$\bar{\Omega} = C^A \Theta_A + \ldots, \quad \text{gh}(\bar{\Omega}) = 1.$$  

The conversion implies finding BRST charge satisfying

$$\{\Omega, \bar{\Omega}\} = 0, \quad \text{gh}(\bar{\Omega}) = 1, \quad \Omega|_{Y=0} = \bar{\Omega}.$$

Note that $\Omega$ and $\bar{\Omega}$ are assumed to be a globally defined functions on the entire extended phase space and its submanifold determined by $Y^A = 0$ respectively.
Now we describe conversion of the second-class constraints $\Theta_A = \{-p_i, \theta_\alpha\}$. Taking into account their transformation properties a natural anzatz for a generating function $\bar{\Omega}$ is as follows
\begin{equation}
\bar{\Omega} = -C^i p_i + C^\alpha \theta_\alpha + C^i (\bar{\Gamma})^{\alpha}_{ij} C^j \mathcal{P}_\alpha.
\end{equation}
Indeed, the nonlinear in ghosts term coming from the transformation law for $p_i$ is compensated by the term coming from inhomogeneous contribution in the transformation law for the connection coefficients. This is exactly the point. In order for the generating function $\bar{\Omega}$ as well as BRST charge $\Omega$ to be globally defined functions, one needs to introduce the terms nonlinear in ghosts. In terms of constraints, this implies that the conversion is non-Abelian.

**4.3. Existence and construction of the classical BRST charge.** In the standard BFV-BRST formalism the BRST charge and BRST invariant observables are constructed by expanding in homogeneity degree in ghost momenta. The existence of a nilpotent BRST charge is ensured by Homological Perturbation Theory \cite{18} with the relevant operator being Koszul-Tate differential associated with the constraints. At the same time, within the Abelian conversion procedure the effective first-class constraints, BRST charge, and BRST-invariant observables are constructed by expanding in homogeneity degree in conversion variables and all these quantities are to be found order by order in these variables.

In the case of non-Abelian conversion it is then natural to take as an expansion degree the total homogeneity in ghost momenta $\mathcal{P}_A$ and conversion variables $Y^A$:
\begin{equation}
\deg Y^A = \deg \mathcal{P}_A = 1, \quad \deg x^i = \deg p_i = \deg C^A = 0.
\end{equation}
Accordingly, $\Omega$ decomposes as
\begin{equation}
\Omega = \sum_{s=0} \Omega_s, \quad \Omega_0 = C^\alpha \theta_\alpha - C^i p_i, \quad \Omega_1 = C^i (\bar{\Gamma})^{\alpha}_{ij} C^j \mathcal{P}_\alpha + \ldots,
\end{equation}
where we have explicitly kept the term from the first order contribution which is needed for covariance. The required BRST charge satisfying \eqref{eq:master_eq} is to be constructed order by order in the degree. To this end one first needs to satisfy the master equation to zeroth order which implies finding $\Omega_1$. A “minimal” form of $\Omega_1$ which satisfies master equation to the zeroth order can be taken as
\begin{equation}
\Omega_1 = C^i (\bar{\Gamma})^{\alpha}_{ij} C^j \mathcal{P}_\alpha - C^A D_{AB} Y^B.
\end{equation}

In constructing BRST charge it is also useful to restrict ourselves to the following class of phase-space functions: let $\mathcal{A}^0$ be the space of formal power series in $Y^A$, ghosts $C^A$, and ghost momenta $\mathcal{P}_\alpha$ with coefficients being smooth functions in $x^i$. In other words we forbid dependence on $p_i$ and $\mathcal{P}_i$. The space $\mathcal{A}^0$ is closed under the multiplication and the Poisson bracket (both operations can be naturally defined for formal power series).
Algebra $\mathcal{A}^0$ decomposes with respect to the degree (4.12) as $\mathcal{A}^0 = \oplus_{s \geq 0} \mathcal{A}_s^0$ so that an element of $\mathcal{A}_s^0$ has the form

$$a = \sum_{p \geq 0, q \geq 0}^{p+q=s} (a_{pq})_{A_1 \ldots A_p} Y^{A_1} \ldots Y^{A_p} P_{\alpha_1} \ldots P_{\alpha_q}, \quad a_{pq} = a_{pq}(x, C).$$

Since the BRST charge and BRST invariant observables are to be constructed by expanding in the degree (4.12) the lowest degree term $-\delta$ in the expansion of $\{\Omega, \cdot \}$ plays a role of the nilpotent operator determining homological perturbation theory. Considered acting on elements from $\mathcal{A}^0$, operator $\delta$ is completely determined by $\bar{\Omega}$ and is given by

$$\delta = C^A \frac{\partial}{\partial Y^A} + \theta_{\alpha} \frac{\partial}{\partial P_{\alpha}}.$$ 

It is therefore a sum of standard Koszul–Tate operator $\delta_K = \theta_{\alpha} \frac{\partial}{\partial P_{\alpha}}$ associated with original constraints $\theta_{\alpha}$ and the operator $C^A \frac{\partial}{\partial Y^A}$ which determines a homological perturbation theory in the Abelian conversion framework and in the Fedosov quantization.

To proceed with the conversion we need to introduce a version of the contracting homotopy operator determined by

$$\delta^{{\star}} f_{pq} = \frac{1}{p + q} Y^A \frac{\partial}{\partial C^A} f_{pq}, \quad p + q \neq 0, \quad \delta^{{\star}} f_{00} = 0, \quad \delta^{{\star}} 2 \equiv 0$$

for an element $f_{pq} \in \mathcal{A}^0$ which is homogeneous in $C^A$ and $Y^A$ of orders $p$ and $q$ respectively. Operators $\delta$ and $\delta^{{\star}}$ satisfy

$$\delta^{{\star}} \delta a + \delta \delta^{{\star}} a = a - a|_{C = Y = 0}.$$

**Proposition 4.1.** There exists a classical BRST charge $\Omega$, $\mathfrak{gl}(\Omega) = 1$ satisfying master equation $\{\Omega, \Omega\} = 0$, boundary conditions (4.13), (4.14), and such that $\Omega_s \in \mathcal{A}_s^0$ for $s \geq 2$. In addition, given $\Omega_0$ and $\Omega_1$ such a BRST charge is unique provided $\delta^{{\star}} \Omega_s = 0$ for all $s \geq 2$.

**Proof.** The Poisson bracket on $\mathcal{E}$ can also be expanded with respect to the degree as

$$\{\cdot, \cdot, \cdot\} = \{\cdot, \cdot\}_2 + \{\cdot, \cdot\}_{-1} + \{\cdot, \cdot\}_0 + \{\cdot, \cdot\}_2$$

(terms with other degrees vanish) where each term is a bilinear first order differential operator of definite degree. In particular

$$\{f, g\}_{-2} = f \frac{\partial}{\partial Y^A} D^{AB} \frac{\partial}{\partial Y^B} g.$$

The master equation at order $n$ in degree implies,

$$\{\Omega_0, \Omega_{n+2}\}_{-2} + \{\Omega_0, \Omega_{n+1}\}_{-1} + \{\Omega_1, \Omega_{n+1}\}_{-2} + B_n = 0.$$
where $B_n$ depends on $\Omega_s$ with $s \leq n$ only and is given explicitly by

\[
B_n = \sum_{0 \geq p, q \geq n}^{p+q+s=n} \{\Omega_p, \Omega_q\}_s.
\]

In fact, the first term in (4.21) vanishes because $\Omega_0$ doesn’t depend on $Y$ and the equation takes the form

\[
\delta \Omega_{n+1} = B_n.
\]

This equation can always be solved by $\Omega_{n+1} = \delta^* B_n$ using (4.18), $B_n|_{C=Y=0} = 0$, and the consistency condition $\delta B_n = 0$. The later is fulfilled provided the master equation holds to lowest orders, i.e. that $\Omega_s$ for $s \leq n$ are such that

\[
\{\Omega^{(n)},\Omega^{(n)}\}_s \in \bigoplus_{s \geq n} \mathbb{A}_s^0,
\]

\[
\Omega^{(n)} = \sum_{s=0}^{n} \Omega^{(n)}_s.
\]

Indeed, consider the following identity

\[
\{\Omega^{(n)},\Omega^{(n)}\}_s = 0.
\]

Next, observe that $\{\Omega^{(n)},\Omega^{(n)}\} = B_n + \ldots$ with dots denoting terms from $\mathbb{A}_{n+1}^0$ and finally, check that to order $n − 1$ in the degree this identity gives $\delta B_n = 0$.

This solution for $\Omega^{n+1}$ obviously belongs to $\mathbb{A}_0^0$ and satisfies $\delta^* \Omega^{n+1} = 0$. Conversely, equation (4.23) has a unique solution $\Omega^{n+1}$ satisfying $\Omega^{n+1} \in \mathbb{A}_{n+1}^0$, $\delta^* \Omega^{n+1} = 0$, and $\text{gh}(\Omega^{n+1}) = 1$. \hfill \Box

4.4. Classical observables and weak Dirac bracket. We show that observables of the original system on $\mathcal{M}$ are isomorphic to observables of the BFV-BRST system on $\mathcal{E}$. The latter are understood as cohomology of the adjoint action

\[
Q = \{\Omega, \cdot\}
\]

of the BRST charge.

**Proposition 4.2.** Let $f_0 = f_0(x,C)$ be any $Y$ and $\mathcal{P}$-independent function. Then there exists $f \in \mathbb{A}_0^0$ such that

\[
\{\Omega, f\} = 0, \quad f|_{Y = \mathcal{P} = 0} = f_0, \quad \text{gh}(f) = \text{gh}(f_0).
\]

If in addition $\delta^* (f - f_0) = 0$ and $\text{gh}(f) \geq 0$ then $f$ is a unique BRST invariant extension of $f_0$. Moreover, if $f, \tilde{f} \in \mathbb{A}_0^0$ both satisfy (4.27) with the same function $f_0$, then $f - \tilde{f} = Qh$ for some function $h \in \mathbb{A}_0^0$. 


Proof. The proof is standard and follows by expanding $Qf = 0$ with respect to degree (4.12) and using the fact that $\delta$-cohomology is trivial in nonzero degree. The later statement obviously holds provided cohomology of the standard Koszul–Tate operator $\delta_K = \theta_\alpha \frac{\partial}{\partial P_\alpha}$ vanishes in nonzero degree in $P_\alpha$. Locally, operator $\delta_K$ is known to have vanishing cohomology in nonzero degree provided constraints $\theta_\alpha$ satisfy standard regularity assumptions. This also holds globally as can be shown by using suitable partition of unity. □

Let $f_0$ and $g_0$ be two inequivalent observables of the original system, i.e. $f_0|_\Sigma - g_0|_\Sigma \neq 0$. It then follows from the explicit form of $\Omega$ that their BRST invariant extensions $f$ and $g$ determined by Proposition 4.2 are not equivalent, i.e., $f - g \neq \{\Omega, h\}$ for any $h$. This means that observables of the original system are observables of the BFV-BRST system. In fact, one can show that these systems are equivalent in the sense that the Poisson algebra of inequivalent observables of the original system (i.e. the algebra of functions on $\Sigma$ equipped with the Poisson bracket) is isomorphic to the Poisson algebra of ghost number zero BRST cohomology of the BFV-BRST system. Now we restrict ourselves to a little bit weaker equivalence statement. Namely we show that this holds for BRST cohomology evaluated in $A^0$ ($Q$ obviously maps $A^0$ to itself).

Proposition 4.3. Let $f$ be an arbitrary function from $A^0$ satisfying $Qf = 0$. Then $f = Qh$ for some $h$ iff $f|_\Sigma = f|_{\theta_\alpha = C^A = Y^A = P_\alpha = 0} = 0$.

Proof. Let $f_0 = f|_{Y = P = 0}$. Condition $f|_\Sigma = f_0|_\Sigma = 0$ implies that there exist $f_0^\alpha(x)$ and $f_0A(x, C)$ such that

\begin{equation}
(4.28) \quad f_0 = \theta_\alpha f_0^\alpha + C^A f_0A
\end{equation}

and their transformation properties can be assumed to be those of sections of $V(M)$ and $W^*(M)$ respectively. One can then check that

\begin{equation}
(4.29) \quad (Qh)|_{Y = P = 0} = f_0, \quad h = -P_\alpha f_0^\alpha - Y^A f_0A
\end{equation}

because $f_0 = -\delta h$ and $(Qh)|_{A^0} = -\delta h$ for $h \in A^1$. Proposition 4.2 then implies that there exists $h' \in A^0$ such that $f = Q(h + h')$. □

To summaries we have

Theorem 4.1. The BRST cohomology of $Q = \{\Omega, \cdot\}$ evaluated in $A^0$ are given by

\begin{equation}
H^n(Q, A^0) = C^\infty(\Sigma) \quad n = 0,
\end{equation}

\begin{equation}
H^n(Q, A^0) = 0 \quad n \neq 0.
\end{equation}

The fact that all the physical observables can be taken elements of $A^0$ suggests to consider $A^0$ as a fundamental object replacing algebra of functions on the entire extended phase space. This can be consistently done in spite of the fact that the BRST charge $\Omega$
and the ghost charge $G = C^AP_A$ do not belong to $\mathfrak{A}^0$. Indeed, from a more general point of view, a classical BFV-BRST system is determined by (i) Poisson algebra with not necessarily nondegenerate Poisson structure, which is also graded with the ghost degree (ii) Odd nilpotent BRST differential $Q$ of ghost number 1 that differentiates both the product of functions and the Poisson bracket and (iii) differential $V$ (determining evolution) of zero ghost number which differentiates both the product and the Poisson bracket and satisfies $[Q, V] = 0$. The standard Hamiltonian BFV-BRST system fits this definition with $Q = \{\Omega, \cdot\}$ and $V = \{H, \cdot\}$ with $H$ denoting Hamiltonian. Such a generalization of the Hamiltonian BFV-BRST theory was recently studied in \cite{14}. Note also that in the Lagrangian context this corresponds to theories described by BRST differential not necessarily generated by a master action and an antibracket. Theories of this type were recently considered in \cite{19}.

From this slightly more general point of view, the Poisson algebra $\mathfrak{A}^0$ is a BFV-BRST system because $Q$ and ghost number operator preserve $\mathfrak{A}^0$. The notion of generalized BFV-BRST system can be extended to the quantum case by replacing the Poisson algebra with the star-product algebra. It can also be generalized further in the sense that the bracket can be allowed to satisfy Jacobi identity only up to $Q$-exact terms as well as $V$ can preserve the bracket only weakly \cite{14}.

Let us give some further comments concerning the Poisson bracket of BRST observables. In the case where $\Omega$ is Abelian (see \cite{8} for detailed discussion of this case) Proposition 4.2 establishes an isomorphism between the algebra of functions of $x^i$ and functions of $x_i, Y^A$ satisfying $\{\Omega, \cdot\} = 0$ and $\delta^* \cdot = 0$. The later algebra (understood as a subalgebra in $\mathfrak{A}^0$) is closed under the Poisson bracket in $\mathfrak{A}^0$. The Poisson bracket in this algebra determines a Poisson bracket on $\mathcal{M}$ that can be easily seen to coincide with the Dirac bracket associated to second-class constraints $\theta_\alpha$.

In the present case $\Omega$ explicitly depends on $P_\alpha$ and one is forced to consider $\delta^*$ and $\{\Omega, \cdot\}$-closed functions from $\mathfrak{A}^0$ which are now allowed to depend also on $C^A$ and $P_\alpha$. However, this algebra is not anymore closed under the Poisson bracket and therefore a direct counterpart of the Dirac bracket fails to satisfy Jacobi identity outside $\Sigma$ in this case. Indeed, finding unique lifts $f, g \in \mathfrak{A}^0$ of two phase-space functions $f_0$ and $g_0$, evaluating their Poisson bracket, and putting $Y = P = 0$ one finds a bracket on $\mathcal{M}$ which coincides with the standard Dirac bracket when $\theta_\alpha = 0$. Explicitly, the bracket reads

$$\{f_0, g_0\}_D = \partial_i f_0 \omega^{ij} \partial_j g_0 = \partial_i f_0 \omega^{ij} \partial_j g_0 - \partial_i f_0 \omega^{il} \bar{\nabla}_l \theta_\alpha \Delta^{\alpha\beta} \bar{\nabla}_k \theta_\beta \omega^{kj} \partial_j g_0,$$

where

$$\bar{\nabla}_i \theta_\alpha = \partial_i \theta_\alpha - \bar{\Gamma}^\beta_{\alpha\beta} \theta_\beta,$$

$$\Delta^{\alpha\gamma} \Delta_{\gamma\beta} = \delta^\alpha_\beta, \quad \Delta_{\alpha\beta} = \bar{\nabla}_i \theta_\alpha \omega^{ij} \bar{\nabla}_j \theta_\beta.$$

This bracket can be considered as a direct generalization of the standard Dirac bracket. Unlike the later this generalized bracket does not depend on the choice of constraint basis
and therefore is well-defined outside $\Sigma$ in the case of non-scalar constraints. The Jacobi identity for the bracket (4.31) is violated by the terms proportional to the curvature $\bar{R}^\alpha_{ij\beta}$ of the connection $\bar{\nabla}$ and to the constraints $\theta_\alpha$. So it is inevitably a weak bracket if the bundle $V(M)$ does not admit flat connection.

4.5. Dirac connection. As we have seen the construction imposes no constraints on the connection $\Gamma^A_{iB}$ entering the Poisson bracket on $E$ but the compatibility with the symplectic form $D_{AB}$. Symplectic connection always exists and can be obtained starting from arbitrary connection in $W(M)$, e.g., using (3.11) Let us, nevertheless, give an explicit form of the particular symplectic connection which as we are going to see also has some additional properties.

To this end let us consider an explicit form of the compatibility condition $\nabla D = 0$

$$\partial_i \omega_{jk} - \Gamma_{jik} + \Gamma_{kij} = 0,$$
$$\partial_i \bar{\nabla}_j \theta_\alpha - \Gamma_{ji\alpha} + \Gamma_{\alpha ij} = 0,$$
$$\Gamma_{\alpha i\beta} - \Gamma_{\beta i\alpha} = 0.$$

The solution which is compatible with the transformation properties and which is in some sense a minimal choice reads as

$$\Gamma_{\alpha i\beta} = 0,$$
$$\Gamma_{ji\alpha} = D_{j\beta}^\alpha (\bar{\Gamma})_{\beta i\alpha},$$
$$\Gamma_{\alpha ij} = -\nabla_i \bar{\nabla}_j \theta_\alpha,$$
$$\Gamma_{\beta i\alpha} = \bar{\Gamma}_{\beta i\alpha},$$

where $\nabla_i \bar{\nabla}_j \theta_\alpha = \nabla_i D_{ja} = \partial_i D_{ja} - \Gamma_{ia\beta} D_{j\beta} - \bar{\Gamma}^\beta_{\beta i\alpha} D_{j\beta}.$

It is easy to see that if $V(M)$ is trivial and one takes $\bar{\Gamma} = 0$ then (4.34) coincides with the Dirac connection introduced in [8]. In fact, connection (4.34) possesses similar properties with respect to the weak Dirac bracket. To see this let us write down this connection in terms of the coefficients with upper indices

$$\Gamma^j_{ik} = \omega^{jl} \left( (\bar{\Gamma})_{lijk} + D_{l\gamma} \Delta^{\gamma\alpha} \nabla_i \hat{\nabla}_k \theta_\alpha \right),
\Gamma^j_{ia} = 0,$$

where $\hat{\nabla}_i \theta_\alpha = \nabla_i \theta_\alpha$ and $\hat{\nabla}_i \bar{\nabla}_j \theta_\alpha = \partial_i D_{ja} - \bar{\Gamma}^{\beta}_{ia\beta} D_{j\beta} - (\bar{\Gamma}^\beta_{i\alpha})_{ji} D_{k\alpha}$. Connection $\Gamma$ in $W(M)$ determines a connection $\Gamma_D$ in $TM$ whose coefficients are $\Gamma^j_{ik}$. It follows from $\nabla D^{AB} = 0$ and $\Gamma^j_{ia} = 0$ that

$$\nabla_D (D^{jk})_{il} = \partial_i D_{jkl} + \Gamma^j_{dl} D^{lk} + \Gamma^k_{dl} D^{jl} = 0,$$

which means that the Dirac bivector is covariantly constant with respect to the connection $\Gamma_D$. One then concludes that $\Gamma_D$ can be considered as a generalization of the Dirac connection introduced in [8].

Note that $\Gamma_D$ is in general not symmetric and its torsion is proportional to the curvature of $\bar{\Gamma}$. On the constraint surface this connection coincides with the Dirac connection.
in [8]. Similar arguments then show that \( \Gamma_M \) can be restricted to \( \Sigma \) and its restriction is a symplectic connection on \( \Sigma \) considered as a symplectic manifold.

In the construction of \( \Gamma \) there is an ambiguity described by an arbitrary 1-form with values in symmetric tensor product of the bundle \( W^* \)(M) (i.e. arbitrary connection has the form \( \Gamma_{AiB} = \Gamma_{AiB}^{\text{fixed}} + \gamma_{AiB} \), with \( \gamma_{AiB} - \gamma_{BjA} = 0 \)). One can try to find additional conditions in order to fix the ambiguity in the connection. In particular, to find an invariant criterion which allows to separate connections compatible with the Dirac bracket.

It turns out that it is possible to formulate a condition of this type by analyzing the conversion procedure. To see this we note that the term in \( \Omega_2 \) of the form \( C^i \gamma_{AiB} Y^A Y^B \) can be absorbed into the redefinition of \( p_i \) which in turn leads to the adjustment of the symplectic connection \( \Gamma_{AiB} \rightarrow \Gamma_{AiB} + \gamma_{AiB} \). It is then natural to choose the connection such that the respective contribution to \( \Omega_2 \) vanishes, i.e. the connection which is not modified by the conversion. For \( \Omega \) satisfying conditions of the second part of Proposition 4.1 this implies that

\[
\left( \delta^* \left\{ - C^i p_i + C^i \tilde{\Gamma}_\alpha^\beta \mathcal{P}_\beta, C^A D_{AB} Y^B \right\} \right) \bigg|_{\mathcal{C}^\alpha=0} = 0 .
\]

This gives the following conditions on \( \Gamma \)

\[
\begin{align*}
\partial_i \omega_{jk} - \Gamma_{jik} + \partial_k \omega_{ji} - \Gamma_{jki} + \Gamma_{ijk} + \Gamma_{kji} &= 0 , \\
\partial_i D_{j\alpha} - D_{ij} \tilde{\Gamma}_j^\beta - \Gamma_{jia} + \Gamma_{ija} + \Gamma_{\alpha ji} &= 0 , \\
\Gamma_{\alpha ji} + \Gamma_{\beta jia} &= 0 ,
\end{align*}
\]

If one takes \( \Gamma_{ijk} = (\tilde{\Gamma}_M)_{ijk} \) where \( (\tilde{\Gamma}_M)_{ijk} \) are coefficients of a fixed symmetric symplectic connection on \( M \) then equations (4.33) and (4.38) have a unique solution with \( \Gamma_{ijk} = (\tilde{\Gamma}_M)_{ijk} \). It is given explicitly by

\[
\begin{align*}
\Gamma_{\alpha i\beta} &= 0 , & \Gamma_{j\alpha i} &= D_{j\beta} (\tilde{\Gamma})^\beta_{i\alpha} + \frac{1}{3} \tilde{R}_{j\alpha i}^\beta \theta^\beta , \\
\Gamma_{ijk} &= (\tilde{\Gamma}_M)_{ijk} , & \Gamma_{\alpha ij} &= - \nabla_i \nabla_j \theta_{\alpha} - \frac{1}{3} \tilde{R}_{ij\alpha}^\beta \theta^\beta ,
\end{align*}
\]

Note that consistency of (4.33) and (4.38) together with \( d\omega = 0 \) requires \( \Gamma_{ijk} - \Gamma_{ikj} = 0 \).

This connection differs from the one in (4.34) by the terms proportional to \( \tilde{R}_{ij\alpha}^\beta \theta^\beta \). It also determines a connection \( \Gamma'_D \) on \( M \) which coincides with \( \Gamma_D \) on \( \Sigma \). In particular, \( \Gamma'_D \) is compatible with the Dirac bracket only weakly in general.

5. Conversion at the Quantum Level

5.1. Quantization of the extended phase space. At the quantum level we concentrate on the algebra \( \hat{\mathcal{A}}^0 = \mathcal{A}^0 \otimes [\hbar] \) and its extension \( \hat{\mathcal{A}} \) obtained by allowing dependence on \( p_i \) and \( P_i \) through the following combinations (see [8] for details)

\[
P = C^i (-p_i + \tilde{\Gamma}_i^\alpha \mathcal{P}_\alpha ) , \quad G = C^i \mathcal{P}_i .
\]
A general element of $\hat{\mathcal{A}}$ has the form
\begin{equation}
(a = \mathbf{P}^r \mathbf{G}^s a_0, r = 0, 1, \ s = 0, 1, \ldots, \dim(M), \ a_0 \in \hat{\mathcal{A}}^0)
\end{equation}

The algebra $\hat{\mathcal{A}}$ is closed under ordinary multiplication and the Poisson bracket. Moreover, it can be directly quantized. To this end first quantize $\hat{\mathcal{A}}^0$ by introducing Weyl star-product according to
\begin{equation}
(a \ast b)(x, Y, C, \mathcal{P}, \hbar) = (a(x, Y_1, C_1, \mathcal{P}_2, \hbar) \exp(-\frac{i\hbar}{2} (D^{AB}_{\partial Y_1 \partial Y_2} - \frac{\partial}{\partial C_1} \frac{\partial}{\partial \mathcal{P}_1} - \frac{\partial}{\partial C_2} \frac{\partial}{\partial \mathcal{P}_2}))) b(x, Y_2, C_2, \mathcal{P}_2, \hbar) \bigg|_{Y_1 = Y_2 = Y, C_1 = C_2 = C, \mathcal{P}_1 = \mathcal{P}_2 = \mathcal{P}},
\end{equation}
where $\mathcal{P}$ stands for $\mathcal{P}_\alpha$ only. This star product can also be extended from $\hat{\mathcal{A}}^0$ to $\hat{\mathcal{A}}$. Here we give only those formulas which we really need in what follows (we refer to [8, 7] for further details of such an extensions):
\begin{align}
\frac{i}{\hbar} [\mathcal{P}, a] &= \mathcal{C}^j (\frac{\partial}{\partial x^j} - \bar{\Gamma}^j_{i\alpha} C^\alpha \frac{\partial}{\partial C^i} + \bar{\Gamma}^\alpha_{i\beta} C^\beta \frac{\partial}{\partial \mathcal{P}_\beta} - \Gamma^B_{i\alpha} Y^B \frac{\partial}{\partial Y^A}) a, \quad a \in \hat{\mathcal{A}}^0,
\frac{i}{\hbar} [\mathcal{P}, \mathcal{P}] &= -i\hbar \mathcal{C}^j (\omega_{ij} + \bar{R}^j_{i\alpha} C^\alpha \mathcal{P}_\beta + \frac{1}{2} \bar{R}^{j}_{ij A B} Y^A Y^B),
\frac{i}{\hbar} [\mathcal{G}, a] &= -\mathcal{C}^i \frac{\partial}{\partial C^i} + \mathcal{P}_i \frac{\partial}{\partial \mathcal{P}_i}, \quad a \in \hat{\mathcal{A}}^0.
\end{align}

Note that these relations are enough to consistently consider $\hat{\mathcal{A}}^0$ as a star-product algebra underlying the BFV-BRST system at the quantum level in the sense described in [4, 4].

On $\hat{\mathcal{A}}$ we introduce the following degree
\begin{equation}
\deg Y^A = \deg \mathcal{P}_A = \deg p_i = 1, \quad \deg C^A = \deg x^i = 0, \quad \deg \hbar = 2.
\end{equation}

One then decomposes $\hat{\mathcal{A}}^0$ and $\hat{\mathcal{A}}$ with respect to the degree as
\begin{equation}
\hat{\mathcal{A}}^0 = \bigoplus_{s=0} \hat{\mathcal{A}}^0_s
\end{equation}
and similarly for $\hat{\mathcal{A}}$. The star product also decomposes into homogeneous components with respect to degree
\begin{equation}
\ast = \ast_0 + \ast_1 + \ast_2 + \ldots
\end{equation}
In particular, $\ast_0$ contains ordinary product, Weyl product in the sector of $Y$ variables, and the component of the product which takes $\mathcal{P}$ with itself into $-i\hbar \mathcal{C}^j C^j \omega_{ij}$.

Let us note that the choice of the degree is not unique. The one we are using is convenient for general proofs but perhaps is not the most suitable for computations because it is not preserved by the star product in $\hat{\mathcal{A}}^0$. From this point of view one can consider another degree for which $\deg C^A = 1$ and gradings of other variables left unchanged.
5.2. Quantum BRST charge. Now we are going to show the existence of the quantum BRST charge satisfying

\[ [\hat{\Omega}, \hat{\Omega}] = 0, \quad \text{gh}(\hat{\Omega}) = 1, \]

together with the condition \( \hat{\Omega}|_{\hbar = 0} = \Omega \). Here and below \([\cdot, \cdot]\) stands for the graded commutator with respect to the star-multiplication in \( \hat{A} \), which is also decomposed into homogeneous component with respect to the degree. A degree \( s \) component of the commutator is denoted by \([\cdot, \cdot]_s\).

It follows from the standard deformation theory and the vanishing of \( Q \)-cohomology in nonzero ghost number that quantum BRST charge exists. However, instead of deforming the classical BRST charge we construct the quantum one from scratch. To this end we show that the quantum master equation (5.8) has a solution satisfying the following boundary conditions

\[ (5.9) \quad \hat{\Omega}_0 = C^\alpha \theta_\alpha, \quad \hat{\Omega}_1 = P - C^A \mathcal{D}_{AB} Y^B = -C^i p_i + C^i \bar{\Gamma}_{i\beta} C^\beta \mathcal{P}_\alpha - \mathcal{C}^A \mathcal{D}_{AB} Y^B. \]

**Proposition 5.1.** Equation (5.8) has a solution satisfying boundary condition (5.9) and \( \hat{\Omega}_s \subset \hat{A}_0^0 \) for \( s \geq 2 \). Under the additional condition \( \delta \hat{\Omega}_s = 0, s \geq 2 \) the solution is unique.

**Proof.** Prof is completely standard once degree is prescribed. The only thing to check is that with boundary conditions (5.9), the master equation holds at orders 0, 1 and 2 which is straightforward. The rest follows by induction using that

\[ (5.10) \quad [\hat{\Omega}_0, a]_0 = 0, \quad \frac{1}{i\hbar} \left( [\hat{\Omega}_0, a]_1 + [\hat{\Omega}_1, a]_0 \right) = \delta a, \]

for any \( a \in \hat{A}_0^0 \). Here, \([\cdot, \cdot]_s\) denotes the degree \( s \) component of the star-commutator. \( \square \)

5.3. Quantum BRST observables and non-associative star-product on \( \mathcal{M} \). Given a nilpotent quantum BRST charge one can consider the cohomology group of its adjoint action \( \hat{Q} = \frac{1}{\hbar}[\hat{\Omega}, \cdot] \). It follows from the standard deformation theory and Theorem 4.1 that any classical BRST cohomology class determines a quantum one. In fact it also follows that

\[ (5.11) \quad H^n(\hat{Q}, \hat{A}_0^0) \cong H^n(Q, A_0^0) \otimes [\hbar]. \]

It is nevertheless useful to explicitly construct representatives of the quantum BRST cohomology classes. Similarly to the classical case this is achieved by finding a lift of functions of \( x^i, C^A \) to BRST invariant elements of \( \hat{A}_0^0 \). We have the following

**Proposition 5.2.** For any \( f_0 = f_0(x, C, \hbar) \) there exists \( f \in \hat{A}_0^0 \) such that

\[ (5.12) \quad [\Omega, f] = 0, \quad \text{gh}(f) = \text{gh}(f_0), \quad f|_{Y=P=0} = f_0. \]
If in addition $f$ is such that $\delta^*(f - f_0) = 0$ and $gh(f) \geq 0$ then $f$ is a unique quantum BRST-invariant extension of $f_0$. Moreover, if $f$ and $\tilde{f}$ both satisfy (5.12) with the same $f_0$ then $f - \tilde{f} = [\tilde{\Omega}, h]$ for some $h \in \hat{A}_0$.

Proof. The proof is standard once degree is prescribed. That equation holds to lowest order follows from $\delta f_0 = 0$. \hfill $\square$

If $f, g \in \hat{A}_0$ are unique BRST invariant extensions of functions $f_0(x)$ and $g_0(x)$ determined by Proposition 5.2 then one can define a bilinear operation

(5.13) \[ f_0 \star_D g_0 = (f \star g)|_{y = p = 0}. \]

This operation is not an associative product in general. However, it determines the associative star-product on $\Sigma$. Indeed, BRST cohomology can be identified with functions on $\Sigma$ while quantum multiplication in $\hat{A}_0$ determines a quantum multiplication in the cohomology. By choosing different lifts from functions on $\mathbb{M}$ to $\hat{A}_0$ one can describe different extensions of the associative star products on $\Sigma$ to in general non-associative product on $\mathbb{M}$.

As a final remark, we comment on the emergence of weak Poisson brackets and weak star-products in the context of constrained systems. In dynamics, and especially in what concerns the deformations and quantization of classical dynamical systems, the Poisson geometry is stereotypically considered the most fundamental structure of the theory. But whenever a constrained or a gauge system is considered that does not allow explicitly solving constraints nor taking the quotient over the gauge symmetry, the dynamics, as such, does not require a Poisson algebra to exist for all functions on the entire phase-space manifold. Only the space of physical quantities has to carry a Poisson structure, and hence the geometry of the entire manifold turns out to have a weaker structure than the Poisson one. As we have seen, this is the case with non-scalar second-class constraints. The BRST theory was originally worked out as a tool for quantizing systems with gauge symmetries defined by weakly integrable distributions. Now, as is seen, the idea of BRST cohomology allows one to quantize systems whose Poisson algebra is also weak.

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