Convergence to $\alpha$-stable Lévy motion for chaotic billiards with several cusps at flat points

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Abstract

We consider billiards with several possibly non-isometric and asymmetric cusps at flat points; the case of a single symmetric cusp was studied previously in Zhang (2017 *Dynamical Systems, Ergodic Theory, and Probability: in Memory of Kolya Chernov* (Contemporary Mathematics vol 698) (Providence, RI: American Mathematical Society) pp 287–316) and Jung and Zhang (2018 *Ann. Henri Poincaré* 19 3815–53). In particular, we show that properly normalized Birkhoff sums of Hölder observables, with respect to the billiard map, converge in Skorokhod’s $M_1$-topology to an $\alpha$-stable Lévy motion, where $\alpha$ depends on the ‘curvature’ of the flattest points and the skewness parameter $\xi$ depends on the values of the observable at those same points. Previously, Jung and Zhang (2018 *Ann. Henri Poincaré* 19 3815–53) proved convergence of the one-point marginals to totally skewed $\alpha$-stable distributions for a symmetric cusp. The limits we prove here are stronger, since they are in the functional sense, but also allow for more varied behaviour due to the presence of multiple cusps. In particular, the general limits we obtain allow for any skewness parameter, as opposed to just the totally skewed cases. We also show that convergence in the stronger $J_1$-topology is not possible.

Keywords: dynamical systems, billiards, stable law, statistical properties

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1. Introduction

The origins of the modern theory of hyperbolic dynamical systems lie in classical and statistical mechanics through the study of ergodic and statistical properties. Indeed understanding statistical properties and proving various probability limit theorems are vital in the study of statistical mechanics. For example, such studies shed light on the important issue of fluctuations in entropy production. This paper is a contribution in this direction by showing a mechanism by which an hyperbolic dynamical system can converge on the process level to something other than a Brownian motion.

Begin by letting $T$ be a measure-preserving transformation on $(M, \mu)$ and $\{X_n = f \circ T^n\}$ the stochastic process generated by the system for an observable $f$. One typical goal is to prove the (functional) central limit theorem for the normalized partial-sum process associated to $\{X_n\}$ which is itself generated by a mixing hyperbolic system, i.e. to prove that, weakly

$$\left(\frac{1}{\sqrt{n}} \sum_{j=1}^{[n]} X_j\right)_{t \geq 0} \to (\sigma B(t))_{t \geq 0}$$

where $(B(t))_{t \geq 0}$ is a standard Brownian motion. In other words, the (normalized) partial-sum process converges in distribution, in $C[0, 1]$, to a Brownian motion with diffusion parameter $\sigma > 0$. The finiteness of the variance/diffusion parameter requires that the covariances $\text{Cov}(X_1, X_n)$ are summable in $n \in \mathbb{N}$. However, when the covariance sequence is not summable, the usual central limit theorem for the partial-sum process fails; and there are very few results, for hyperbolic systems, which investigate what happens under such failure.

Billiards with cusps were first introduced in the early 80s in [Mac83], but it was not until [CM07] that a rigorous polynomial decay of correlations of $O(1/n)$ was proved for Machta’s original model (where cusps are formed by tangential circles). This polynomial decay of correlations led to a nonstandard central limit theorem where a normalization of $\sqrt{\pi \log n}$ was used instead of $\sqrt{n}$ (see also [BCD11]).
In [Zha17], the author was able to construct a hyperbolic billiard model with arbitrarily slow decay rates of correlations, of order
\[
\text{Cov}(X_1, X_n) = O(n^{-a}),
\]
with \(a \in (0, 1)\) depending on the curvature at the cusp’s vertex. In [JZ18], the first and last authors were able to prove that due to the slow decay of correlations, instead of a central limit theorem, rather, a stable limit theorem holds for the partial-sum process generated by a billiard with a single symmetric cusp at a flat point. In particular, they showed that the limiting distribution is a totally skewed \(\alpha\)-stable law. In this paper, we first investigate the convergence to a stable law, for the case when the billiard table has several, possibly asymmetric, cusps at flat points. We then extend these results to a functional limit theorem in Skorokhod space, i.e. convergence to an \(\alpha\)-stable Lévy motion. In the dynamical systems literature, functional convergence to Lévy motion has been shown for expanding maps (see for instance [TK10]), however this is the first result of this kind for 2d hyperbolic billiards that we are aware of (after finishing this work, we were made aware of the concurrent work [MV19] where a functional limit theorem is also proved for the model in [JZ18]).

Let us now describe in detail the billiards with cusps at flat points that we consider here. See figure 1. The dispersing billiard table \(Q\) is a bounded domain of \(\mathbb{R}^2\), the boundary of which consists of \(q \geq 3\) dispersing \(C^3\) smooth curves \(\partial Q = \bigcup_{i \in \mathbb{Z}/q} \Gamma_i\), numbered in clockwise order. The intersections \(P_i = \Gamma_i \cap \Gamma_{i+1}\) of two consecutive curves \(\Gamma_i\) and \(\Gamma_{i+1}\) consist of either standard corner points (i.e. the tangent lines at \(P_i\) to \(\Gamma_i\) and \(\Gamma_{i+1}\) do not coincide) or ‘cusps’ at flat points. This means that, in an appropriate euclidean coordinate system \((s, z)\) originated at some point \(P_i\), the two curves \(\Gamma_i\) and \(\Gamma_{i+1}\) can be represented respectively as \(z = z_{i+}(s)\) with \(z_{i+}\) differentiable and satisfying
\[
z_{i+}(s) = \pm \frac{C_i}{\beta_i} s^{\beta_i} + O(s^{2\beta_i - 1})
\]
with \(\epsilon > 0\) some small fixed number, \(C_i, C_{i+} \geq 0\) not both null and with \(\beta_i \geq 2\). We will say that such a cusp at \(P_i\) has flatness \(\beta_i\). We also define
\[
C_i := \frac{(C_{i+} + C_{i-})}{2}.
\]
We assume moreover that, for every cusp at some \(P_i\), the (unique) tangent trajectory coming out of the cusp at \(P_i\) hits \(\partial Q\) outside of any another cusp. Also, all boundary components are assumed to be dispersing and have curvature bounded away from zero except at the cusps \(P_i\)’s. Throughout the paper, we assume that
\[
\beta_* := \max(\beta_i : P_i \text{ is a cusp}) > 2.
\]
The functional limit theorem we prove here will depend only on the observable near the ‘maximally flat’ points $P_i$ (the heuristic reason for this is that the limit theorem is dominated by long sequences of collisions inside the various cusps, and the flattest cusps trap the longest sequences). Let $J$ be the set of labels $i$ such that $P_i$ is a cusp and let $J_*$ be the set of labels $i \in J$ such that $\beta_i = \beta_\ast$.

The usual billiard flow is defined on the unit sphere bundle $Q \times S^1$ and preserves the Liouville measure, see [CM06] for details. We consider the natural cross section $\mathcal{M} \subset Q \times S^1$ made of all post-collision vectors based at the boundary of the table $\partial Q$. Any post-collision vector $x \in \mathcal{M}$ can be represented by $x = (r, \varphi)$, where $r \in \mathbb{R}/|\partial Q|\mathbb{Z}$ is the clockwise curvilinear abscissa along $\partial Q$ (where $|\partial Q|$ denotes the length of $\partial Q$), and $\varphi \in [0, \pi]$ is the angle formed by the tangent line of the boundary and the collision vector in the clockwise direction. Therefore the collision space $\mathcal{M}$ is identified with $\mathbb{R}/|\partial Q|\mathbb{Z} \times [0, \pi]$. The corresponding billiard map $T : \mathcal{M} \to \mathcal{M}$ takes a vector $x \in \mathcal{M}$ to the next post-collision vector along the trajectory of $x$. Let the set $S_0$ consist of all grazing collision vectors with walls as well as all collision vectors at corner points/cusps. Then $S_0 := S_0 \cup T^{-1}S_0$ is the singular set of $T$. The billiard map $T : \mathcal{M} \setminus S_0 \to \mathcal{M} \setminus T S_0$ is a local $C^2$ diffeomorphism and preserves a natural absolutely continuous probability measure

$$d\mu = \frac{1}{2|\partial Q|} \sin \varphi \, dr \, d\varphi$$

on the collision space $\mathcal{M} = \{(r, \varphi) \in \mathbb{R}/|\partial Q|\mathbb{Z} \times [0, \pi]\}$.

A recently popular approach for studying the statistical properties of $(T, \mathcal{M})$ is to use an inducing scheme as introduced in [Mar04, CZ05]. By removing spots with weak hyperbolicity from the phase space, one considers the first return map on some subspace $M \subset \mathcal{M}$. For any $x \in M$, the first return time function is defined by

$$\mathcal{R}(x) := \min\{n \geq 1 : T^n(x) \in M\}$$

and the (first) return map $F : M \to M$ is defined by

$$F(x) := T^{\mathcal{R}(x)}(x), \quad \text{for all } x \in M. \quad (1.3)$$

The return map $F$ preserves the conditional measure $\bar{\mu} := \frac{1}{\mu(M)}\mu|_M$ and we define an induced function by

$$\bar{f}(x) := \sum_{k=0}^{\mathcal{R}(x) - 1} f(T^k x), \quad x \in M. \quad (1.4)$$

Here we define the subset $M$ to be the subset of $\mathcal{M}$ which consists of all collisions occurring on

$$\partial Q \setminus (\bigcup_{i \in J} B_i(P_i)) \quad (1.5)$$

where $B_i(P_i) \cap \partial Q$ admits a representation as in (1.1) and where $\epsilon$ is small enough so that a billiard trajectory cannot go directly from one $B_i(P_i)$ to another $B_j(P_i)$ without hitting $\partial Q \setminus (\bigcup_{k \in J} B_k(P_k))$. We decompose

$$M = M_0 \cup \bigcup_{i \in J} M_i$$

so that for $i \in J$, $M_i$ is made up of those points in $M$ whose first collision inside $\bigcup_{k \in J} B_k(P_i)$ (see (1.5)), under the map $F$, occurs inside $B_i(P_i)$. Moreover, $M_0 = M \setminus \bigcup_{i \in J} M_i$ consists of the remaining points which do not immediately enter any cusps.
Rigorous bounds on the decay of correlations for billiards with flat points were derived in [Zha17], where a more detailed description of billiards with flat points is also given. It was shown that if $f, g$ are Hölder continuous functions on the collision space $\mathcal{M}$, then for all $n \in \mathbb{Z}$,

$$\mu(f \circ T^n \cdot g) - \mu(f)\mu(g) = O(n^{-\beta^*}).$$

(1.6)

Here we use the standard notation $\mu(f) = \int_{\mathcal{M}} f \, d\mu$, which we henceforth assume to be 0 for $f$ and $g$. As already mentioned, it is the above slow decay of correlations that led to a stable limit theorem in [JZ18], rather than the standard central limit theorem, for a normalized version of the Birkhoff sums

$$S_n f := f + f \circ T + \ldots + f \circ T^{n-1},$$

where $f$ is a Hölder function on $\mathcal{M}$ and $f \neq 0$ at the cusp’s vertex. Our results will show that for a general dispersing billiard table with multiple cusps, only the $|J^*| \geq 1$ maximally flat cusps will contribute to the limit of properly normalized Birkhoff sums. In other words, we show that cusps with smaller order flatness will not ‘contribute’ to the limiting stable law (nor to the functional-level convergence to a Lévy process). Although the analysis in [JZ18] can be carried to a general table to study statistical properties for observables supported on any individual symmetric cusp, the convergence to a stable law jointly for multiple asymmetric cusps here is rather different and requires significant adjustments to the original arguments.

In order to consider a functional limit theorem, denote the process $\{W_n(t) : t \geq 0\}, n \geq 1$ by

$$W_n(t) := \sum_{j=0}^{[nt]-1} f \circ T^j / n^{1/\alpha}, \quad \text{for } t \geq 0, \quad \text{where } \alpha := \frac{\beta_*}{\beta_* - 1}. $$

(1.7)

One can check that $\alpha \in (1, 2)$ since $\beta_* > 2$. In [MZ15], a general result was proved for obtaining functional convergence of dynamical systems to an $\alpha$-stable Lévy motion by ‘lifting’ such a limit law from an induced dynamical system to the original system. Our functional limit theorem for $\{W_n(t) : t \geq 0\}, n \geq 1$, will be proved in two steps: first showing that the induced system satisfies the functional limit theorem, utilizing standard methods from [DR78], and then applying the lifting principle of [MZ15].

The rest of the paper is organized as follows. In the next section we state our main result. In section 3 we give the top-level proof of our main result as just described in the previous paragraph. In section 4 we provide the details of the limit theorem for the induced system via the mechanism of [DR78]. In the appendix we present the additional technical details needed to handle the general setting of asymmetric cusps we consider here, and also provide (for any reader who is seeing these for the first time) a very brief introduction to the Skorokhod $J_1$ and $M_1$ topologies.

2. Main result

A function $f$ with $\mu(f) = 0$ is said to be in the domain of attraction of a (strictly) $\alpha$-stable law if there exists $\{b_n\}$ such that $\{S_n \over b_n\}$ converges in distribution to a random variable with an $\alpha$-stable law. Here, strictly simply means that $\mu(f) = 0$, and we shall henceforth just say $\alpha$-stable. Contrary to central limit behavior and the Gaussian case of $\alpha = 2$, it is well-known that even though we have a mean of zero, the limiting distribution may not be symmetric. In
particular, there exists a skewness parameter $\xi \in [-1, 1]$ and $s > 0$ such that the limit stable random variable $\mathcal{Y}$ satisfies

$$\lim_{x \to \infty} x^\alpha \mathcal{P}(\mathcal{Y} > x) = C_\alpha \alpha \frac{1 + \xi}{2}$$

and

$$\lim_{x \to -\infty} x^\alpha \mathcal{P}(\mathcal{Y} < x) = C_\alpha \alpha \frac{1 - \xi}{2},$$

with $C_\alpha$ given by

$$C_\alpha = \frac{1}{\Gamma(1 - \alpha) \cos(\pi \alpha/2)},$$

see [ST94, p 17]. We will henceforth denote by $\mathcal{Z}_{\alpha, \xi, s}$ a stable random variable with tail distribution satisfying (2.1), i.e. with characteristic function

$$\mathbb{E}(e^{iu\mathcal{Z}_{\alpha, \xi, s}}) = \exp(-|u|^{\alpha} \left(1 - i\xi \text{sign}(u) \tan \frac{\pi \alpha}{2}\right)), \quad u \in \mathbb{R}. \quad (2.2)$$

For any $\gamma \in (0, 1)$, we denote $\mathcal{H}_\gamma$ as the class of all Hölder continuous functions $f : \mathcal{M} \setminus S_0 \to \mathbb{R}$, with Hölder exponent $\gamma$. Let us write $\tilde{r}_i$ for the curvilinear abscissa of the cuspid position $P_i$. We define

$$\forall \varphi \in [0, \pi], \quad \tilde{f}_{i+}(\varphi) := \lim_{r \to \tilde{r}_i +} f(r, \varphi) \quad \text{and} \quad \tilde{f}_{i-}(\varphi) := \lim_{r \to \tilde{r}_i -} f(r, \varphi),$$

so that $\tilde{f}_{i-}$ (resp. $\tilde{f}_{i+}$) corresponds to the limit function on $P_i \times [0, \pi]$ of $f|_{\Gamma_i}$ (resp. $f|_{\Gamma_{i+}}$). We also define

$$I_{f,i} = I_{f,i,\alpha} := \frac{1}{4} \int_0^\pi (\tilde{f}_{i-}(\varphi) + \tilde{f}_{i+}(\varphi)) \sin^{\frac{\alpha}{2}} \varphi \, d\varphi. \quad (2.3)$$

**Theorem 2.1 (Stable limit theorem for billiard with cusps).** Let $Q$ be a billiard table as described above, with cusps defined by (1.1). Suppose $f \in \mathcal{H}_\gamma$ for some $\gamma > 0$ satisfying $\mu(f) = 0$ and suppose there exists some $i \in \mathcal{J}_s$ such that $I_{f,i} \neq 0$. Then as $n \to \infty$,

$$\frac{S_{df}}{n^{1/\alpha}} \to \sum_{i \in \mathcal{J}_s, I_{f,i} \neq 0} \mathcal{Z}_{\alpha, \xi, s} \frac{\sigma_{i,\alpha}}{c_i^{\alpha}}, \quad (2.4)$$

where the limit is a sum of independent stable variables with

$$\sigma_{f,i}^{\alpha} := \frac{2}{|\partial Q|} \cdot \frac{I_{f,i}^2}{\beta c_i^{\alpha - 1}} \quad \text{and} \quad \xi_{f,i} := \text{sign}(I_{f,i}).$$

**Theorem 2.2 (Functional $\alpha$-stable limit theorem).** Under the assumptions of theorem 2.1 if, for every $i \in \mathcal{J}_s$, there exists a neighbourhood $\mathcal{U}_i$ of $P_i \times [0, \pi]$ such that either $f|_{\mathcal{U}_i} \geq 0$ or $f|_{\mathcal{U}_i} \leq 0$, then, in addition to the convergence in (2.4),

$$\left(\{W_n(t)\}_{t \in [0,1]}\right)_{n},$$

defined in (1.7), converges in distribution in the Skorokhod space $\mathcal{D}([0, 1])$ using the $M_1$-metric, to an $\alpha$-stable Lévy motion $(Y_t)_{t \in [0, 1]}$ (an $\alpha$-stable process with stationary and independent increments) such that $Y_1$ has the same distribution as the right side of (2.4). Moreover, this convergence does not hold for the $J_1$-metric.

Note that the values of $f$ at the maximally flat cusps determine the value $\xi$ and $\sigma$ and in fact one can compute that the right side of (2.4) is just a random variable $\mathcal{Z}_{\alpha, \xi, s} \frac{1}{c_i^{\alpha - 1}} \sigma_i$ with
\[ \sigma_f^\alpha := \sum_{i \in J} \frac{2|I_f|_i^\alpha}{\partial C_1^\alpha \partial Q_i} \quad \text{and} \quad \xi_f := \frac{\sum_{i \in J} \text{sign}(I_f) \partial C_1^{-\alpha}|I_f|_i^\alpha}{\sum_{i \in J} \partial C_1^{-\alpha}|I_f|_i^\alpha}. \quad (2.5) \]

Let us remark that, as in [JZ18], theorem 2.2 (as well as theorem 3.1 below) holds also for \( f \) bounded and piecewise \( \gamma \)-Hölder, for some \( \gamma > 0 \), with discontinuities contained in the singular set of \( T \) and such that \( f \) is \( \gamma \)-Hölder in a neighborhood of the region in \( \mathcal{M} \) corresponding to each of the cusps.

3. Proofs of theorems 2.1 and 2.2 using an induced map

3.1. Convergence for the \( M_1 \)-metric

Recall that
\[ F(x) := T^{R(x)}(x), \]
with \( R(x) := \min\{n \geq 1 : T^n(x) \in \mathcal{M}\} \). For every \( g : \mathcal{M} \to \mathbb{R} \), we denote the Birkhoff sums on the induced space by
\[ S_n g(x) := \sum_{k=0}^{n-1} g \circ F^k(x), \quad x \in \mathcal{M}, \quad n \geq 0. \]

Recall also that, given \( f : \mathcal{M} \to \mathbb{R} \), we define \( \tilde{f} : \mathcal{M} \to \mathbb{R} \) by setting
\[ \tilde{f}(x) := S_{R(x)} f(x). \quad (3.1) \]

We similarly define the process \( \tilde{W}_n := \tilde{W}_n(t)_{t \in [0,1]} \) as follows:
\[ \tilde{W}_n(t) := n^{-1/\alpha} \tilde{S}_{[0,t]} \tilde{f}. \quad (3.2) \]

An intermediate functional limit theorem for the induced map is as follows.

**Theorem 3.1.** Let \( f : \mathcal{M} \to \mathbb{R} \) be as in theorem 2.1. Then \( (\tilde{W}_n) \) converges in distribution, in the Skorokhod \( J_1 \)-topology, to an \( \alpha \)-stable Lévy motion with \( \alpha = \frac{\beta}{\beta-1} \), skewness parameter \( \xi_f \), and scale parameter
\[ s = C_\alpha^{-1/\alpha} \tilde{\sigma}_f := (C_\alpha \mu(\mathcal{M}))^{-1/\alpha} \sigma_f, \]
with \( \sigma_f \) and \( \xi_f \) defined in (2.5).

Both theorems 2.1 and 2.2 are consequences of theorem 3.1.

**Proof of theorem 2.1.** By considering the first hitting time of the \( M \) (in lieu of the first return time) we may extend the definitions of \( R \) and \( F \) to all of \( \mathcal{M} \), so that in particular \( f \circ F \) is defined on all of \( \mathcal{M} \). Due to theorem 3.1,
\[ \left( (n^{-1/\alpha} \tilde{S}_{[0,t]} \tilde{f} \circ F)_{t \geq 0} \right)_n \]
converges in distribution, with respect to \( \mu \) and to the metric \( J_1 \), to a Lévy process \( (Y_t)_{t \geq 0} \). Moreover, for every positive integer \( m \), we define \( \tau_m(x) \) as the number of visits of \( (T^k(x))_{k=1,\ldots,m} \) to \( M \) so that

\[ \]
\[ \sum_{k=0}^{\tau_n(x)-1} R \circ F^k(x) \leq m < \sum_{k=0}^{\tau_n(x)} R \circ F^k(x). \]

Due to the ergodic theorem \( (\frac{\tau_m}{m})_{m \geq 1} \) converges almost surely to \( \mu(M) \) and so \( \left( (n^{-\frac{1}{\alpha}} S_{\tau_m} - \tilde{f} \circ F)^n \right) \) converges in distribution, with respect to \( \mu \) and to \( J_1 \), to \( (\mathcal{Y}_{\mu(M)})_{t \geq 0} \).

Moreover,
\[ \left| S_{mf}(x) - S_{\tau_n(x)-1} \tilde{f}(F(x)) \right| \leq \|f\|_\infty (R(x) + R_\tau(T^m(x)) \), \]

where \( R_\tau \) corresponds to the length of time since the last visit to \( M \). Therefore by reversibility of the dynamics, for every \( \epsilon > 0 \),
\[ \mu \left( n^{-\frac{1}{\alpha}} \left| S_{[m]}f - S_{\tau_m - 1} \tilde{f} \circ F \right| > \epsilon \right) \leq 2 \mu \left( n^{-\frac{1}{\alpha}} \|f\|_\infty R > \epsilon/2 \right). \]

This implies the convergence in probability of \( (n^{-\frac{1}{\alpha}} (S_{[m]}f(x) - S_{\tau_m} - 1 \tilde{f}(F(x))))_n \) to 0 for every \( t \geq 0 \) and so the convergence of the finite-dimensional distributions (with respect to \( \mu \)) of \( (n^{-\frac{1}{\alpha}} S_{[m]}f)_n \) to the ones of \( (\mathcal{Y}_{\mu(M)})_{t \geq 0} \).

The result of convergence in distribution of theorem 2.2 will follow directly from theorem 3.1 and proposition 3.2 below. The latter is a version of [MZ15, theorem 2.2], the statement of which requires the following notion:
\[ f^*(x) := \left( \max_{0 \leq t', t \in R(x)} (S_{t'}f(x) - S_{t}f(x)) \right) \wedge \left( \max_{0 \leq t', t \in R(x)} (S_{t}f(x) - S_{t'}f(x)) \right). \]

**Proposition 3.2 (Lifting a weak invariance principle, Melbourne and Zweimüller).** Under the above assumptions on \( f \) and \( \tilde{W}_n \) if \( (\tilde{W}_n(t), t \in [0, 1]) \) converges weakly in the Skorokhod \( M_1 \)-topology, as \( n \to \infty \), to an \( \alpha \)-stable Lévy motion \( (W(t), t \in [0, 1]) \) and
\[
\begin{align*}
n^{-1/\alpha} \left( \max_{0 \leq k \leq n} f^* \circ F^k \right) \xrightarrow{d} 0 \tag{3.3}
\end{align*}
\]
then \( (W_n(s), s \geq 0) \xrightarrow{d} (W(s \mu(M)), s \geq 0) \) in the Skorokhod \( M_1 \)-topology.

Note that the version of this result in [MZ15] assumes strong distributional convergence of the induced process, but proposition 2.8 there allows us to state it as we did above.

**Proof of theorem 2.2.** We need only check the so-called weak monotonicity condition (3.3) in our setting. Since \( f \) takes a single sign in the neighborhood around each cusp \( P_i \) with \( i \in \mathcal{J}_s \), we have nearly full monotonicity of the ergodic sums during an excursion, with the only exception possibly coming from the first step. Thus
\[
\begin{align*}
f^* &\leq \|f\|_\infty \mathcal{R} 1_{\cup_{i \in \mathcal{J} \setminus \mathcal{J}_s} M_i} + \|f\|_\infty \mathcal{R} 1_{\cup_{i \in \mathcal{J}_s} M_i} \\
&\leq \|f\|_\infty (\mathcal{R} 1_{\cup_{i \in \mathcal{J} \setminus \mathcal{J}_s} M_i} + 1).
\end{align*}
\]

Now let \( \tilde{\alpha} := \min \{ \frac{\beta}{\pi_i^1} : i \in \mathcal{J} \setminus \mathcal{J}_s \} \), which by assumption satisfies \( \tilde{\alpha} > \alpha \). Due to section 4,
\[ \left( Y_n(t) := n^{-1/\alpha} S_{[n]} (\mathcal{R} 1_{\cup_{i \in \mathcal{J} \setminus \mathcal{J}_s}} M_i - \tilde{\mu}(\mathcal{R} 1_{\cup_{i \in \mathcal{J} \setminus \mathcal{J}_s} M_i})) \right) \]

converges in the $J_1$-metric, as $n$ goes to infinity, to the 0 function (here $\tilde{\mu} := \mu(|M|)$). Therefore

$$n^{-\frac{1}{\alpha}} \left( \max_{0 \leq k \leq n} f^* \circ F^k \right) \leq \|f\|_{\infty} n^{-\frac{1}{\alpha}} \max_{0 \leq k \leq n} \left( (\mathcal{R}1_{J_1 \cup \mathcal{I} \cup \mathcal{J}} - \tilde{\mu}(\mathcal{R}1_{J_1 \cup \mathcal{I} \cup \mathcal{J}})) \circ F^k + 1 + \tilde{\mu}(\mathcal{R}) \right),$$

which converges in distribution to 0.

This checks that (3.3) holds which completes the proof of theorem 2.2 up to a proof of theorem 3.1 which is given in section 4. □

3.2. Non-convergence for the $J_1$-metric

In this subsection, we explain why the convergence in theorem 2.2 does not hold in the $J_1$-topology (see [TK10, example 2.1] or [AT92] for other redactions of this argument).

Let $(w_n(t))$, be the continuous process obtained from $(W_n(t))$, by linearization:

$$w_n(t) := W_n(t) + \left( nt - \lfloor nt \rfloor \right) f \circ T^\lfloor nt \rfloor, \quad t \in [0, 1].$$

Since $f$ is uniformly bounded, we also have

$$\sup_{t \in [0, 1]} |w_n(t) - W_n(t)| \leq \|f\|_{\infty} n^{1/\alpha} \rightarrow 0, \quad \text{as} \quad n \rightarrow \infty.$$

Therefore, the convergence in distribution of $(W_n(t))_n$ in the $J_1$-metric would imply the same convergence in distribution for $(w_n(t))_n$ in the $J_1$-metric and so for the uniform metric on every compact interval, this would contradict the fact that the limiting process is discontinuous.

Note that the reason this argument does not apply to the induced process in theorem 3.1 (or to $J_1$-convergence for general Lévy processes), is because in that setting $\|\tilde{f}\|_{\infty} = \infty$.

4. Proof of theorem 3.1 (functional limit theorem for the induced system)

Assume the assumptions of theorem 2.1. Recall that $\tilde{\mu} = \mu(|M|)$. We define

$$\tilde{f}_i := \tilde{f} \cdot 1_{M_i} - \tilde{\mu}(\tilde{f} \cdot 1_{M_i}).$$

Note that, due to the Kac theorem,

$$\tilde{\mu}(\mathcal{R}) < \infty. \quad (4.1)$$

4.1. Simplification

In order to prove theorem 3.1, we show that we can replace $\tilde{f}$ by the simplest function $\tilde{R}$ given by:

$$\tilde{R}(x) := \sum_{i \in \mathcal{I}} \frac{I_{1,\alpha}}{I_{1,\alpha}} (\mathcal{R}_i(x) - \tilde{\mu}(\mathcal{R}_i)), \quad \text{with} \quad \mathcal{R}_i := \mathcal{R}1_{M_i}. \quad (4.2)$$
with

\[ I_{1,\alpha} := I_{1,\alpha} = \int_0^{\pi/2} \sin^{1/2} \varphi \, d\varphi \quad \text{and} \quad \alpha_i = \frac{\beta_i^*}{\beta_i^* - 1}. \]

We simply write \( I_1 \) for \( I_{1,\alpha} \). The goal of this section is to prove the following result.

**Proposition 4.1.** We have

\[ \sup_{k \leq n} S_k(\tilde{f} - \tilde{R}) \frac{1}{n^{1/\alpha}} \to 0, \]

in probability, as \( n \) goes to infinity.

This proposition will come from the following lemmas which require some notations. Fix a small \( \delta > 0 \), and split \( M_1 \) according to the low, intermediate, and high regions of the index \( m \) for the sets \( M_{1,m} := \{ x \in M_1 : R(x) = m \} \):

\[ M_{1,1} := \bigcup_{m < \delta^n} M_{1,m}, \quad M_{1,0} := \bigcup_{\delta^n \leq m < \delta^{n+1}} M_{1,m} \quad \text{and} \quad M_{1,\infty} := \bigcup_{m \geq \delta^{n+1}} M_{1,m} \]  

(4.3)

which all depend implicitly on \( n \) and \( \delta \). We also denote

\[ \tilde{f}_{i,\zeta} := \tilde{f} \cdot 1_{M_{i,\zeta}^c} - \tilde{\mu}(\tilde{f} \cdot 1_{M_{i,\zeta}^c}), \]

with \( \zeta \in \{L, I, H\} \).

Similarly to [JZ18, lemma 4.3], \( \tilde{f}_i \) is essentially determined by \( \tilde{f}_{i,L} \). Indeed, by considering the induced system restricted to each \( M_1 \), the proof of lemma 4.3 in [JZ18] gives us the following result:

**Lemma 4.2 (Vanishing of truncated portions).** For every \( i \in \mathcal{J} \), we have, in probability,

\[ \lim_{\delta \to 0} \sup_{k \leq n} S_k 1_{M_{1,\infty}^c} \frac{1}{n^{1/\alpha}} = 0. \]  

(4.4)

Moreover, there exist \( C_0 > 0, \theta \in (0, 1) \) and \( \kappa > 0 \) such that, for every \( i \in \mathcal{J} \), and every \( k \geq 1 \), we have

\[ \text{Cov} \left( \tilde{f}_{i,L}, \tilde{f}_{i,L} \circ \Phi^t \right) \leq C_0 \theta^k n^{\frac{\kappa}{\alpha} - 1 - \kappa}. \]  

(4.5)

For \( \zeta \in \{L, I, H\} \), set \( \mathcal{R}_{i,\zeta} := \mathcal{R}|_{M_{i,\zeta}^c} \) and define \( E_{i,\zeta} \) as:

\[ E_{i,\zeta} := \tilde{f}_{i,\zeta} - \frac{I_{1,\alpha}}{I_1} (\mathcal{R}_{i,\zeta}(x) - \tilde{\mu}(\mathcal{R}_{i,\zeta})). \]  

(4.6)

Again we note that the truncation to the intermediate region, denoted by \( I \) in the subscripts/superscript, implicitly depends on \( n \). Similar to [JZ18], lemma 4.4 and proposition 4.2 in [JZ18] implies that \( f \) is \( \gamma \)-Hölder as the following lemma indicates.

**Lemma 4.3.** There exist \( C = C(\delta) > 0 \) and \( \psi \in (0, 1) \) such that

\[ |E_{i,J}| \leq C |\mathcal{R}_{i,J} - \tilde{\mu}(\mathcal{R}_{i,J})|^{1 - \frac{\gamma}{\psi - 1}} \quad \text{on} \ M_{i,J}. \]
\[
\text{Cov} \left( E_{i,j}, E_{i,j} \circ F^k \right) \leq C n^{\frac{1}{\alpha_i} - \frac{1}{i(\alpha_i + 1)}} \theta^k.
\]

The proof of the above lemma is similar to arguments in [JZ18]; we indicate which estimates in the proof need updates and/or adjustments at the end of appendix A.2.

**Proof of proposition 4.1.** Lemma 4.3 implies that there exists \( K > 0 \) such that for all \( m \),
\[
\tilde{\mu}(\bar{S}_m E_{i,j}) = \text{Var}(S_m E_{i,j}) \leq 2 \sum_{k=0}^{m-1} (m - k) \text{Cov}(E_{i,j}, E_{i,j} \circ F^k) \leq K m n^{\frac{1}{\alpha_i} - \frac{1}{i(\alpha_i + 1)}}.
\]

The exponent of \( n \) can be rewritten
\[
-\alpha - \frac{1}{\alpha_i} - \frac{1}{i(\alpha_i + 1)} \leq - \frac{\alpha - 1}{\alpha_i (\alpha_i + 1)} \text{ since } \alpha_i \geq \alpha. \text{ So, by [Bil68, exercise 12.5] (with } \gamma = 2, \alpha = 1 \text{ there, see also [Ser70, equation (2.3)])}, \text{ for all } m,
\]
\[
\tilde{\mu} \left( \sup_{k=1,...,m} |S_k E_{i,j}|^2 \right) \leq (\log_2 4m)^2 K m n^{-\frac{\alpha - 1}{\alpha_i (\alpha_i + 1)}}.
\]

there exist \( K_0 = K_0(\delta) > 0 \) and \( \varepsilon > 0 \) such that
\[
\text{for all } n \geq 0, \quad \tilde{\mu} \left( \sup_{k=1,...,n} |S_k E_{i,j}|^2 \right) \leq K_0 n^{\frac{1}{2} - \varepsilon}.
\]

Proving analogously, due to lemma 4.2 applied to \( \tilde{f} - \bar{R} \) instead of \( \tilde{f} \), there exist \( K'_0 > 0 \) and \( \kappa > 0 \) such that
\[
\text{for all } n \geq 0, \quad \tilde{\mu} \left( \sup_{k=1,...,n} |S_k E_{i,j}|^2 \right) \leq K'_0 \delta^n n^{\frac{1}{2} - \varepsilon}.
\]

Fix \( \varepsilon > 0 \). Due to the Markov inequality, we can find \( \delta > 0 \) small enough so that for all \( n \)
\[
\tilde{\mu} \left( \sup_{k \leq n} \left| S_k \left( \sum_{i \in J} E_{i,j} \right) \right| > \varepsilon \right) < \varepsilon.
\]

Also, by (4.4), we can choose \( \delta > 0 \) small enough so that for all \( n \)
\[
\tilde{\mu} \left( \sup_{k \leq n} \left| S_k \left( \sum_{i \in J} E_{i,j} \right) \right| > \varepsilon \right) < \varepsilon.
\]

Now, due to (4.7) and Markov’s inequality, for the minimum of these choices of \( \delta \), and for \( n \) large enough,
\[
\tilde{\mu} \left( \sup_{k \leq n} \left| S_k \left( \sum_{i \in J} E_{i,j} \right) \right| > \varepsilon \right) < \frac{\varepsilon}{2}.
\]

**4.2. Convergence in distribution**

Due to the simplification in the previous subsection, it remains now to prove the convergence in distribution as \( n \to \infty \) for the following functions of \( t \) in the \( J_1 \)-metric
\[
\left( Z_n(t) := \frac{S_{[nt]} \sum_{i \in J} A_i (R_i - \tilde{\mu}(R_i))}{n^{1/\alpha}} \right)_t \bigg|_n
\]
for any fixed real numbers \( \{A_i, i \in \mathcal{J}_+\} \). To this end, we consider the family of point processes \((N_n)_{n \in \mathbb{N}}\) on \((0, +\infty) \times (\mathbb{R} \setminus \{0\})\) given by

\[
N_n := \sum_{j=1}^n \delta_{\left(\frac{Z_j}{\varepsilon_n}\right)}, \quad \text{with} \quad Z_j := \left[ \sum_{i \in \mathcal{J}_+} A_i (\mathcal{R}_i - \bar{\mu}(\mathcal{R}_i)) \right] \circ F^i,
\]

and we employ standard methods for proving functional limit theorems from [DR78, section 4]. Here we state a version from [TK10, theorem 1.2].

**Proposition 4.4.** Assume the following two conditions.

**Condition I.** (Point process convergence). The sequence of point processes \((N_n)_{n \in \mathbb{N}}\) converges in distribution to a Poisson point process \(N\) on \((0, +\infty) \times (\mathbb{R} \setminus \{0\})\) with intensity \(\gamma\) having density \(\psi\) with respect to the Lebesgue measure on \((0, +\infty) \times (\mathbb{R} \setminus \{0\})\), with

\[
\psi(t, y) = \sum_{A \in \mathcal{J}_+} \frac{2I^\alpha_{A}}{\beta|C_A|\mu(M)|\partial Q|} 1_{\{A \neq 0\}} |y|^{-\alpha - 1}.
\]

**Condition II (Vanishing small values).** For all \(\gamma > 0\)

\[
\lim_{\varepsilon \to 0} \limsup_{n \to \infty} \max_{0 < |y| < \varepsilon} \left[ \sum_{i \in \mathcal{J}_+} \left| \frac{Z_i}{n^{1/\alpha}} \cdot 1_{|Z_i|/n^{1/\alpha} < \varepsilon} \right| - k\bar{\mu} \left( \frac{Z_i}{n^{1/\alpha}} \cdot 1_{|Z_i|/n^{1/\alpha} < \varepsilon} \right) \right] > \gamma = 0.
\]

Then \(((Z_n(t))_{t > 0}, (S_{|A|})_{A \in \mathcal{J}_+})\) converges in distribution (in the \(J_1\)-metric) to an \(\alpha\)-stable Lévy motion \((Y_t)_{t \geq 0}\), such that \(Y_t\) has the same distribution as \(Z_{\alpha, \xi, \sigma}\) with

\[
\sigma^\alpha := \frac{2I^\alpha_{A}}{\beta|C_A|\mu(M)|\partial Q|} \quad \text{and} \quad \xi := \frac{\sum_{i \in \mathcal{J}_+} \text{sign}(A_i) \frac{2I^\alpha_{A}}{\beta|C_A|\mu(M)|\partial Q|}}{\sum_{i \in \mathcal{J}_+} \frac{2I^\alpha_{A}}{\beta|C_A|\mu(M)|\partial Q|}}.
\]

**4.3. A key estimate for Condition I**

In order to prove the convergence in distribution \(N_n \to N\), we use the Kallenberg theorem or a method based on this theorem. A key result in this study is the following lemma, which was proved as lemma 2.2 [JZ18] only for one specific symmetric cusp. The generalization of this in our context is the key estimate that shows that only cusps of maximal flatness are relevant. The modifications to the proof given in [JZ18] is nontrivial, thus we provide the proof of this lemma in appendix A.2.

**Lemma 4.5.** The return time function satisfies

\[
\forall i \in \mathcal{J}_+, \quad \lim_{y \to +\infty} y^{\alpha_i} \bar{\mu} (x \in M_i : \mathcal{R}(x) > y) = \frac{2I^\alpha_{A_i}}{\beta|C_A|\mu(M)|\partial Q|},
\]

(4.8)

with \(\alpha_i := \frac{\beta^\alpha}{\beta^\alpha - 1}\).

Since \(\alpha_i > \alpha\) for \(i \notin \mathcal{J}_+\), this lemma ensures in particular that

\[
\lim_{y \to +\infty} y^{\alpha_i} \bar{\mu} (x \in M_i : \mathcal{R}(x) > y) > 0 \quad \Leftrightarrow \quad i \in \mathcal{J}_+.
\]

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Corollary 4.6. For any \( \epsilon > 0 \),
\[
\lim_{y \to +\infty} y^\alpha \tilde{\mu}(e^{1/\alpha} Z_{-1} > y) = \lim_{y \to +\infty} y^\alpha \sum_{i \in \mathcal{J}} \tilde{\mu}\left(M_i \cap \{ A_i \mathcal{R}_i - \sum_j A_j \tilde{\mu}(\mathcal{R}_j) > e^{-\frac{\alpha}{2}} y \}\right) = \epsilon \sum_{i \in \mathcal{J}, A_i > 0} 2^{I_i^a} A_i^a \beta C_i^{\alpha - 1} \mu(M) |\partial Q|
\]
and analogously
\[
\lim_{y \to -\infty} y^\alpha \tilde{\mu}(e^{1/\alpha} Z_{-1} < -y) = \epsilon \sum_{i \in \mathcal{J}, A_i < 0} 2^{I_i^a} |A_i|^a \beta C_i^{\alpha - 1} \mu(M) |\partial Q|.
\]

4.4. Condition I: point process convergence

In order to prove the convergence in distribution \( N_n \to N \), due to the Kallenberg theorem [Kal73] (see also [Res87, proposition 3.22]), it is enough to prove that
\[
\lim_{n \to +\infty} \tilde{\mu}(N_n(R)) = \eta(R) \tag{4.9}
\]
and
\[
\lim_{n \to +\infty} \tilde{\mu}(N_n(R) = 0) = e^{-\eta(R)}, \tag{4.10}
\]
for every \( R \) of the form
\[
R = \bigcup_{i=1}^{m} (a_i, b_i) \times I_{c_i, c'_i},
\]
with \( I_{c_i, c'_i} = (-\infty, -c) \cup (c', +\infty), 0 < a_i < b_i < 1 \) and \( c_i, c'_i > 0 \) for every \( i \). In \( \mathbb{R}\setminus\{0\} \), we fix a subset \( \mathcal{I} = I_{c, c'} \) with \( c, c' > 0 \). Our correlation bounds will depend on sets of the form
\[
D_{n,i} := \{ x \in M : \frac{1}{\sqrt{n}} (\mathcal{R}_j - c) \circ F_j(x) \in \mathcal{I} \}.
\]

Lemma 4.7 (Exponential decay of correlations for \( q \)-point marginals, [CZ09, JZ18]). For every \( \mathcal{I} \), there is a constant \( C > 0 \) and \( \theta \in (0, 1) \) such that
\[
\tilde{\mu}(D_{n,1} \cap \cdots \cap D_{n,q} \cap D_{n,q+k+1} \cap \cdots \cap D_{n,2q+k}) - \left( \tilde{\mu}(D_{n,1} \cap \cdots \cap D_{n,q}) \right)^2 \leq C \theta^k
\]
for all \( k, n, q \in \mathbb{N} \) satisfying \( 2q + k \leq n \). Also, there exists \( \theta_0 > 0 \) such that for all \( 1 \leq i < j \leq n \)
\[
\tilde{\mu}(D_{n,i} \cap D_{n,j}) \leq o \left( \frac{1}{n^{1+\theta_0}} \right). \tag{4.12}
\]

The first part above, (4.11) follows from theorem 4 in [CZ09] which covers the setting we are in. For (4.12), the proof of lemma 3.2 in [JZ18] can be used by replacing lemma 2.1 there with the estimates we provide in proposition in the appendix.
4.4.1. Proof of (4.9). Let $m \geq 1$, and real numbers $a_1, b_1, c_1, c_1', \ldots, a_m, b_m, c_m, c_m'$ be such that, for every $i$, $0 < a_i < b_i$ and $c_i < c_i'$ with $c_i c_i' > 0$. We assume without any loss of generality that $b_i \leq a_{i+1}$.

Let $R := \bigcup_{i=1}^{m} (a_i, b_i) \times I_{c_i,c_i'}$, so that $N_n = (\tilde{C}_n, \tilde{m})$ as defined above lemma 4.7.

Let $\mu := \tilde{C}_n \cdot \tilde{m}$, and we replace the set $R$ by the splitting into a product measure in terms of a union of disjoint rectangles, and the projection onto the second coordinate, i.e. onto an empirical measure on $\mathbb{R}$

Due to corollary 4.6,

$$\tilde{\mu} (N_n(R)) = \sum_{i=1}^{m} \sum_{j \neq i} \sum_{k \in J} \tilde{\mu} \left( M_k; (A_k R_k - \sum_{i} A_i \tilde{\mu}(R_i)) \circ F^j \in I_{c_i,c_i'} \right).$$

Due to corollary 4.6,

$$\tilde{\mu} (N_n(R)) \sim \frac{2I_1}{\beta \tilde{C}_n^{\alpha-1} \tilde{\mu}(M) \partial Q} \sum_{i=1}^{m} \sum_{k \in J} (b_i - a_i) 1_{\{A_k > 0\}} |A_k|^\alpha.$$

Therefore, we have proved (4.9) with $\eta$ of density as in proposition 4.4.

4.4.2. Proof of (4.10). In order to ease the notation below, we will prove only the case where $m = 1, a = 0$ and $b = 1$. In this special case we can consider the canonical projection of $N_n$ onto its second argument, i.e. onto an empirical measure on $\mathbb{R} \setminus \{0\}$. Let us call this projection $\hat{N}_n$, and we replace the set $R$ with $I$ as defined above lemma 4.7.

In the general case, one simply needs to write $R$ in terms of a union of disjoint rectangles, and consider each of rectangles in the union separately; also the projection onto the second coordinate for each of these rectangles must be scaled by the Lebesgue measure of the time interval.

As in [JZ18, proposition 3.4] we use Bernstein’s block method [Ber27]. Condition (4.10) would follow if the random variables $\{Z_j\}$ were independent since then we have for $Z_j = ((R_{\tilde{\mu}}(R_j)) \sum_{i \in J} A_i [\tilde{\mu}] \circ F^j$, the splitting into a product measure

$$\tilde{\mu}(\hat{N}_n(I) = 0) = \prod_{j=1}^{n} \tilde{\mu} \left( n^{-1/\alpha} Z_j \notin I \right)$$

which easily implies the required convergence. Since we do not have independence, we instead use asymptotic independence together with Bernstein’s block method to control the dependence between the $\{Z_j\}$. Choose

$$0 < v < w < \theta_0,$$

where $\theta_0$ is as in (4.12), and for any $n \geq 1$, divide $\{1, \ldots, n\}$ into a sequence of pairs of alternating big intervals (blocks) of length $n^v$ and small blocks of length $n^w$. The number of pairs of big and small blocks is $B = [n/(n^v) + [n^w])$ so that $\lim_{n \to \infty} B n^{1-w} = 1$. There may be a leftover partial block $L$ in the end which is negligible since

$$\sum_{j \in L} \tilde{\mu}(n^{-1/\alpha} Z_j \in I) \leq \frac{C n^w}{n}.$$

Thus we may henceforth assume $n = ([n^v] + [n^w])B$.

We denote by $B_k$ and $S_k$ for $k = 1, \ldots, B$, the elements of $\{1, \ldots, n\}$ in $k$th big block and small block, respectively. Let

$$Y_{n,k} = \sum_{j \in B_k} 1_{\{n^{-1/\alpha} Z_j \in I\}}, \quad V_{n,k} = \sum_{j \in S_k} 1_{\{n^{-1/\alpha} Z_j \in I\}}.$$

For each $n$, both $\{Y_{n,k}\}$ and $\{V_{n,k}\}$ are sequences of identically distributed random variables. Let $S''_n = \sum_{k=1}^{B} Y_{n,k}$ and $S'''_n = \sum_{k=1}^{B} V_{n,k}$. 
Similar to the argument for $L$, we have
\[
\bar{\mu}(S_n^\nu) \leq C \frac{Bn^\nu}{n} \leq C \frac{1}{n^{w-\sigma}},
\]
so that we can ignore small blocks. Thus, it is enough to show that $\bar{\mu}(S_n^\nu = 0)$ converges to $\exp(-\lambda(I))$. Since big blocks are separated by small blocks, we can peel off one factor at a time in the product $\prod_{(Y_{n,k}=0)}$ as follows:
\[
\bar{\mu}(S_n^\nu = 0) = \bar{\mu}(Y_{n,1} = 0, Y_{n,2} = 0, \ldots, Y_{n,B} = 0)
\leq \bar{\mu} \left( \prod_{k=1}^{B \cdot 1} 1_{Y_{n,k}=0} \right) \cdot \bar{\mu}(I_{Y_{n,0}=0}) + C\theta^\nu
\leq \left( \mu \left( \prod_{k=1}^{B \cdot 2} 1_{Y_{n,k}=0} \right) \bar{\mu}(I_{Y_{n,0}=0}) + C\theta^\nu \right) \cdot \bar{\mu}(I_{Y_{n,0}=0}) + C\theta^\nu
\]
where we used (4.11) in the last two lines. Repeating this, we obtain for some $\theta_1 \in (\theta, 1)$,
\[
\bar{\mu}(S_n^\nu = 0) = (\bar{\mu}(Y_{n,1} = 0))^{B} + O(\theta_1^\nu). \tag{4.14}
\]
It remains to estimate $\bar{\mu}(Y_{n,1} = 0)$.
\[
\bar{\mu}(Y_{n,1} = 0) \leq 1 - \sum_{j=1}^{w} \left( \bar{\mu}(n^{-1/\alpha} Z_j \in I) - \sum_{k=1}^{w} \bar{\mu}(\{ n^{-1/\alpha} Z_j \in I \} \cap \{ n^{-1/\alpha} Z_k \in I \}) \right)
\leq 1 - \sum_{j=1}^{w} \left( \bar{\mu}(n^{-1/\alpha} Z_j \in I) - o\left( \frac{1}{n} \right) \right)
\leq 1 - n^w \bar{\mu}(n^{-1/\alpha} Z_1 \in I) + n^w \cdot o\left( \frac{1}{n} \right)
\]
where the second inequality follows from (4.12) since $w < \theta_0$. An even easier lower bound is given by
\[
\bar{\mu}(Y_{n,1} = 0) \geq 1 - \sum_{j=1}^{w} \bar{\mu}(n^{-1/\alpha} Z_j \in I).
\]
Putting things together we have
\[
\bar{\mu}(S_n^\nu = 0) = \left( 1 - [n^w] \bar{\mu}(n^{-1/\alpha} Z_1 \in I) + o(n^{w-1}) \right)^B + O(\theta_1^\nu),
\]
which is what we need since $B/n^{1-w} \to 1$ and $n\bar{\mu}(n^{-1/\alpha} Z_1 \in I) \to \lambda(I)$.

4.5. Condition I: alternative approach

It is enough to prove the convergence of $(N_n)_n$ to $N$ as a point process on $(0, +\infty) \times (\mathbb{R} \setminus [-a, a])$ for every $a > 0$. We fix $a > 0$ and set $A_\epsilon := \{ |Z| > a\epsilon^{-1/\alpha} \}$. Due to corollary 4.6, $\bar{\mu}(A_\epsilon) \sim \epsilon \eta_0((\mathbb{R} \setminus [-a, a])$ with $\eta_0$ the measure on $\mathbb{R}$ with density $\psi(1, \cdot)$ with respect to...
the Lebesgue measure. The convergence of \( \{N_n \}_n \) to \( N \) on \( (0, +\infty) \times (\mathbb{R} \setminus [-a, a]) \) follows directly from the following result.

**Lemma 4.8.** For every \( a > 0 \), the family of point processes \( \left\{ \sum_{i \geq 1} : T_i \in A, \delta_{\left( \mu_i(A_i), e^{1/\alpha}Z_{i-1} \right)} \right\}_\epsilon \) converges in distribution, as \( \epsilon \to 0 \), to a Poisson process of intensity \( \gamma/\eta \circ \left( \mathbb{R} \setminus [-a, a] \right) \) with respect to the Lebesgue measure on \( (0, +\infty) \times (\mathbb{R} \setminus [-a, a]) \).

**Proof.** We will apply [PS18, theorem 2.1] (which uses the Kallenberg theorem) with \( A \), as above, with \( H_c : A_c \to V : = \mathbb{R} \setminus [-a, a] \) given by \( H_c(x) := e^{1/\alpha}Z_{-1}(x) \), with \( m := 9\eta(\mathbb{R} \setminus [-a, a]) \) and with

\[
W := \{(c, c'); a < c < c' < \infty \} \cup \{(-c', -c); a < c' < \infty \}.
\]

To apply [PS18, theorem 2.1], we have to prove first that \( \mu(H_c^{-1}(\cdot)|A_c) \) converges in distribution to \( m \) (this is ensured by corollary 4.6) and second that, for every \( K \geq 1 \) and every choice of intervals \( W_1, ..., W_K \in W \),

\[
\Delta_{c, 1} = o(\mu(A_c)) \quad \text{(i.e. in } o(\epsilon)), \quad (4.15)
\]

with

\[
\Delta_{c, n} := \sup_{A, B \in A : A \in \mathcal{G}_c, B \in \sigma(\bigcup_{j \geq n} F^{-i}(\mathcal{G}_c))} |\mu(B \cap A) - \mu(A)\mu(B)|,
\]

and with

\[
\mathcal{G}_c := \mathcal{G}_{c, W_1, ..., W_K} := \{H^{-1}_c(W_i); \ i = 1, ..., K\}.
\]

As in the proof of [PS18, proposition 4.2], the fact that \( \Delta_{c, 1} = o(\epsilon) \) will follow from the following lemmas.

Set

\[
\tau_{A_c} := \min\{n \geq 1 : F^n(\cdot) \in A_c\}.
\]

We will use a general argument given by the next general lemma and its general corollary.

**Lemma 4.9 (General result).** For any positive integer \( p \) and any \( \epsilon > 0 \),

\[
\Delta_{c, 1} \leq \Delta_{c, p+1} + \mu(A_c) \left( \mu(\tau_{A_c} \leq p|A_c) + \mu(\tau_{A_c} \leq p) \right).
\]

**Proof.** Let \( A \in \mathcal{G}_c \) and \( B \in \sigma \left( \bigcup_{j \geq 1} H^{-1}_c(\mathcal{G}_c) \right) \). Note that there exists a function \( g : (\{0, 1\}^K)^\mathbb{N} \to \{0, 1\} \) such that \( 1_B = g(X_1, ...) \), where \( X_i = \left( 1_{H^{-1}_c(W_i)} \right)_{j=1, ..., K} \circ F^i \). Set \( C \) such that

\[
1_C := g(0, ..., 0, X_{p+1}, ...) \in \sigma \left( \bigcup_{j \geq p+1} F^{-i}(\mathcal{G}_c) \right) \quad (4.16)
\]

and observe that \( |1_B - 1_C| \leq 1_{(\tau_{A_c} \leq p)} \) so that

\[
|\text{Cov}(1_A, 1_B) - \text{Cov}(1_A, 1_C)| \leq \mu(A, \tau_{A_c} \leq p) + \mu(A)\mu(\tau_{A_c} \leq p). \quad (4.17)
\]
Finally, due to (4.16), $|\text{Cov}(I_A, I_C)| \leq \Delta_{\epsilon, p_r+1}$. This combined with (4.17) and $A \subset A_{\epsilon}$ ends the proof of the lemma.

**Corollary 4.10 (General result when $\mu(A_{\epsilon}) = O(\epsilon)$).** If there exists a family of positive integer $(p_r)_r$ such that $p_r = o(\epsilon^{-1})$ and $\bar{\mu}(\tau_{A_{\epsilon}} \leq p_r \mid A_{\epsilon}) = o(1)$, then

$$\Delta_{\epsilon, 1} \leq \Delta_{\epsilon, p_r+1} + o(\epsilon).$$

**Proof.** This corollary comes directly from lemma 4.9 combined with $\mu(A_{\epsilon}) = O(\epsilon)$ and

$$\bar{\mu}(\tau_{A_{\epsilon}} \leq p_r) = \bar{\mu} \left( \bigcup_{k=1}^{p_r} F^{-k}(A_{\epsilon}) \right) \leq \sum_{k=1}^{p_r} \bar{\mu}(F^{-k}(A_{\epsilon})) = p_r \bar{\mu}(A_{\epsilon}) = o(1).$$

We fix $\theta \in (0, \theta_0)$ with $\theta_0 = \epsilon$ as in (4.12) and set $p_{\epsilon} = \epsilon^{-\theta}$ so that $p_{\epsilon} \ll \epsilon^{-1}$ and

**Lemma 4.11.** We have $\bar{\mu}(\tau_{A_{\epsilon}} \leq p_r \mid A_{\epsilon}) = o(1)$.

**Proof.** Due to (4.12),

$$\bar{\mu}(\tau_{A_{\epsilon}} \leq p_r \mid A_{\epsilon}) \leq \sum_{k=1}^{p_r} \frac{\bar{\mu}(A_{\epsilon} \cap F^{-k}A_{\epsilon})}{\bar{\mu}(A_{\epsilon})} = O \left( \frac{p_r \epsilon^{1+\theta_0}}{\epsilon} \right) = O \left( \epsilon^{-\theta+\theta_0} \right) = o(1).$$

**Lemma 4.12 (Decorrelation).** $\Delta_{\epsilon, p_r+1} = o(\epsilon)$.

**Proof.** Let $A \in G_{\epsilon}$ and $B \in \sigma \left( \bigcup_{j > p_r+1} F^{-j}G_{\epsilon} \right)$. Observe that $A \in G_{\epsilon}$ is a finite union of level sets of $R \circ F^{-1}$ intersected by some $F(M)$. Therefore $A$ can be smoothly foliated by a union of unstable curves. Moreover, since $F^{-j}(G_{\epsilon})$ can be smoothly foliated by stable curves, the set $B$ can be rewritten $B = F^{-p_{\epsilon}}B'$ with $B'$ smoothly foliated by a union of stable curves. Therefore, due to [CZ09, theorem 3], there exists $z \in (0, 1)$ and $C_0 > 0$ such that

$$\Delta_{\epsilon, p_r+1} \leq C_0 \epsilon^{p_{\epsilon}}.$$

Note that $C_0$ does not depend on $A$ and $B$, i.e. it is a uniform constant.

Corollary 4.10 combined with lemmas 4.11 and 4.12 ensures (4.15) for every $a > 0$, for every $K \geq 1$ and every $W_1, \ldots, W_k \in W$ and so lemma 4.8.

### 4.6. Condition II: vanishing small values

Recall that

$$Z_{\epsilon} := \left[ \sum_{J \in J} \text{A}_J \left( R_J - \bar{\mu}(R_J) \right) \right] \circ F.$$

Condition II will follow immediately from [Bil99, theorem 10.1] together with lemma 4.14 below. Given $\epsilon \in (0, 1)$ and $n \geq 1$, we set

$$U_{n,a}(x) = U_{n,a}(x) := \sum_{i=0}^{n-1} (V_i(x) - \bar{\mu}(V_0)), \quad \text{with} \quad V_i = V_{i,a} := \frac{Z_{\epsilon}(x)}{n^{1/\alpha}} \cdot 1_{\{|Z_{\epsilon}| / n^{1/\alpha} < \epsilon\}}.$$
We will also set $W_i := V_i - \tilde{\mu}(V_i)$. In order to prove lemma 4.14, we need an estimate of the variance of $U_m$.

**Lemma 4.13.** There exists $\tilde{C} > 0$ and $\theta_2 > 0$ such that, for every $\epsilon \in (0, 1)$ and for all $n$ large enough (more precisely, there exists $n_\epsilon$ such that for all $n > n_\epsilon$),

$$\sum_{k \geq 0} |\text{Cov}_{W_k}(V_0, V_k)| = \sum_{k \geq 0} |\tilde{\mu}(W_k)| \leq \tilde{C} \left( \epsilon^{2-n-1} + \epsilon^2 n^{-1-\theta_2} \right).$$

**Proof.** Due to [CZ09, theorem 3],

$$|\tilde{\mu}(W_k)| \leq C^2 \theta_1^t.$$  \hspace{1cm} (4.18)

This estimate will be useful for big $k (k \geq 2 \log n/|\log \theta_1|)$. For $k = 0$, due to lemma 4.5, there exists $\epsilon' > 0$ such that

$$\tilde{\mu}(W_0^2) \leq \tilde{\mu} \left( \left( Z_0 / n^{\frac{3}{2}} \right)^2 > r \right) \leq \epsilon' \int_0^r r^{-\frac{3}{2}} n^{-1} dr = C \epsilon' n^{-1}.$$

Fix $\gamma \in (0, \frac{5}{n} - 1)$. Noticing that

$$|W_k| \leq C n^{\gamma - \frac{1}{2}} + \max \left( |W_k| - C n^{\gamma - \frac{1}{2}}, 0 \right)$$

we obtain that

$$|\tilde{\mu}(W_k)| \leq \tilde{\mu} \left( \left( |W_k| - C n^{\gamma - \frac{1}{2}} \right) W_k \right) \leq C n^{\gamma - \frac{1}{2}} \tilde{\mu} \left( |W_k| \right) \left( \max \left( |W_k| - C n^{\gamma - \frac{1}{2}}, 0 \right) \right)$$

Due to the definition of $W_i, V_j$ and $Z_v$, we have

$$C n^{\gamma - \frac{1}{2}} \tilde{\mu} \left( |W_0| \right) \leq 2 C n^{\gamma - \frac{1}{2}} \tilde{\mu} \left( |V_0| \right) \leq 4 C n^{\gamma - \frac{1}{2}} \tilde{\mu} \left( |R_0 \cap \tilde{R}_0| \right),$$

where $K_0 := \max_{i \in J} |A_i|$. Moreover,

$$\tilde{\mu} \left( \max \left( |W_0| - C n^{\gamma - \frac{1}{2}}, 0 \right) \right) \leq 4 C n^{\gamma - \frac{1}{2}} \tilde{\mu} \left( |W_0| - C n^{\gamma - \frac{1}{2}}, 0 \right)$$

$$= \int_{\left( C n^{\gamma - \frac{1}{2}}, \infty \right]} \tilde{\mu} \left( |W_0| > r, |W_k| > s \right) dr ds \leq 4 C n^{\gamma - \frac{1}{2}}.$$ 

since $|\tilde{\mu}(V_0)| \leq \frac{2 C \tilde{\mu}(R)}{n^{\frac{1}{2}}}$. Assuming $\epsilon$ and $n$ are such that $\frac{2 C \tilde{\mu}(R)}{n^{\frac{1}{2}}} < \frac{C n^{\gamma - \frac{1}{2}}}{2}$, we get

$$\tilde{\mu} \left( \max \left( |W_0| - C n^{\gamma - \frac{1}{2}}, 0 \right) \right) \leq 4 C n^{\gamma - \frac{1}{2}} \tilde{\mu} \left( |W_0| - C n^{\gamma - \frac{1}{2}}, 0 \right) \leq 2 C n^{\gamma - \frac{1}{2}}.$$


Now, using the inequality
\[ \hat{\mu}(R > a, R \circ F^k > b) \leq \min(\hat{\mu}(R > \min(a, b), R \circ F^k > \min(a, b)), \hat{\mu}(R > \max(a, b))) \]
we obtain the existence of \( K'_0 > 0 \) such that, for \( \epsilon \) and \( n \) as above and for every \( k \in [1, (en')^\alpha] \),
\[ \hat{\mu}\left( \max\left( |W_0| - en' - \frac{\epsilon}{2}, 0 \right), \max\left( |W_k| - en' - \frac{\epsilon}{2}, 0 \right) \right) \leq \int_{(en')^\alpha - \frac{\epsilon}{2}}^{en' + \frac{\epsilon}{2}} K'_0 \min\left( (\min(r, s))^{-\alpha(1+\theta_0)} n^{-(1+\theta_0)}, (n \max(r, s)^\alpha)^{-1} \right) \, dr \, ds, \]
using (4.12) (applied with \( n \) of (4.12) equal to \( n(\min(r, s))^\alpha \geq k \) and \( I = I_{\epsilon, c'} \) with \( c' = c' = 1/(2K_0) \) for the first term in the last line and lemma 4.5 for the second term in the last line. Therefore
\[ \int_{en'}^{en' + \frac{\epsilon}{2}} \left| W_0 \right| - en' - \frac{\epsilon}{2}, 0 \right) \leq 2K'_0 \int_{en'}^{en' + \frac{\epsilon}{2}} \min\left( (n\alpha)^{-1-\theta_0}, (ns^\alpha)^{-1} \right) \, dr \, ds. \]
We assume from now on without loss of generality (up to restricting the value of \( \theta_0 \) if necessary) that \( (\alpha - 1)(1+\theta_0) < 1 \).
Observe that \((ns^\alpha)^{-1-\theta_0} < (ns^\alpha)^{-1}\) happens if and only if \( s < n^{-\frac{\theta_0}{\alpha}} r^{1+\theta_0} \) and that \((ns^\alpha)^{-1} < (ns^\alpha)^{-1-\theta_0}\) happens if and only if \( r < n^{-\frac{\theta_0}{\alpha}} r^{1+\theta_0} \), which leads to
\[ \hat{\mu}\left( \max\left( |W_0| - en' - \frac{\epsilon}{2}, 0 \right) \leq 2K'_0 \left( \int_0^e (n\alpha)^{-1-\theta_0} \frac{n^\theta_0}{\alpha} r^{1+\theta_0} \, dr + \int_0^e (n\alpha)^{-1} n^{-\frac{\theta_0}{\alpha}} r^{1+\theta_0} \, dr \right) \]
\[ \leq 2K'_0 \left( n^{-\theta_0 + \frac{\theta_0}{\alpha}} \int_0^e r^{-(\alpha - 1)(1+\theta_0)} \, dr + n^{-\frac{\theta_0}{\alpha}} \int_0^e s^{1+\theta_0} - \alpha \, ds \right) \]
\[ = O\left( e^{-(\alpha - 1)(1+\theta_0)} n^{-\theta_0 + \frac{\theta_0}{\alpha}} + e^{-\frac{1-(\alpha - 1)(1+\theta_0)}{1+\theta_0}} n^{-1-\frac{\theta_0}{\alpha}} \right) = O\left( e^{\theta_0 n^{-1-\theta_0}} \right), \]
(4.22)
with \( \theta_0 > 0 \). Fix \( \epsilon > 0 \) and consider \( n \) large enough so that \( K_0 \frac{\hat{\mu}(R)}{n^{\theta_0}} < \frac{\epsilon}{2} \) and \( 2 \log n / |\log \theta_1| \leq (en')^\alpha \). Putting together estimate (4.18) (for the sum over \( k \geq 2 \log n / |\log \theta_1| \)), combined with (4.19)–(4.22) (for the sum over \( k < 2 \log n / |\log \theta_1| \)), we obtain that there exists \( \tilde{K} > 0 \) such that for every \( \epsilon > 0 \), for every \( n \) large enough,
\[ \sum_{k \geq 0} |\text{Cov}(\hat{\mu}(V_0, V_k))| \leq \tilde{K}\left( e^{\epsilon n^{-2} + e^{-\alpha} n^{-1} + \log n \left( (en^{-\frac{\theta_0}{\alpha}} + e^{\theta_0 n^{-1-\theta_0}} \right) \right), \]
(4.23)
which ends the proof of the lemma.

Armed with the above preliminary result, we can now prove a lemma which combined with [Bi99, equation (10.12) and theorem 10.1] will directly implies Condition II.

**Lemma 4.14.** There exists \( \tilde{C}_0 > 0 \) and \( \tilde{\theta}, \tilde{\theta}_0 > 0 \) such that, for every \( \epsilon \in (0, 1) \) and for every \( n \) large enough (more precisely, there exists \( n_\epsilon \) such that for all \( n > n_\epsilon \)), and for every
0 < m_1 < m_2 \leq n$, we have
\[
\tilde{\mu} \left( U_{m_1}^2 (U_{m_2} - U_{m_1})^2 \right) \leq \tilde{C}_0 \left( \frac{m_2}{n} \right)^{1 + \tilde{\theta}} \left( \epsilon^{4 - 2\alpha} + \epsilon^\alpha n^{-\tilde{\theta}_\nu} \right).
\]

**Proof.** We have
\[
\tilde{\mu} \left( U_{m_1}^2 (U_{m_2} - U_{m_1})^2 \right) = \sum_{k_1, k_2 = 0}^{m_1 - 1} \sum_{k_3 = m_1}^{m_2 - 1} \tilde{\mu} \left( \prod_{j=1}^4 W_{k_j} \right)
\]
\[
\leq 4 \sum_{0 \leq k_1 \leq k_2 < m_1 \leq k_3 \leq k_4} \left| \tilde{\mu} \left( \prod_{j=1}^4 W_{k_j} \right) \right|.
\]
(4.24)
But, due to [CZ09, theorem 3], for every \( j_0 \in \{1, 2, 3 \} \),
\[
\left| \tilde{\mu} \left( \prod_{j=1}^4 W_{k_j} \right) \right| \leq \tilde{\mu} \left( \prod_{j \neq j_0} W_{k_j} \right) \tilde{\mu} \left( \prod_{j > j_0} W_{k_j} \right) + C' \epsilon^{4} \theta_1^{k_{j_0} + 1 - k_{j_0}}.
\]
(4.25)
Notice that the first part of the right-hand side of (4.25) vanishes unless \( j_0 = 2 \). We apply (4.25) to the sum on the right-hand side of (4.24), by splitting up this sum according to the largest of the three gaps between the four indices \( k_1 \leq k_2 < k_3 \leq k_4 \). We apply (4.25) with \( j_0 = 0 \) (respectively 2 or 3) when the largest gap occurs between \( k_1 \) and \( k_2 \) (respectively \( k_2 \) and \( k_3 \) or \( k_3 \) and \( k_4 \)), providing this gap is larger than or equal to \( k \). We conclude that, for every \( k \geq 1 \),
\[
\tilde{\mu} \left( U_{m_1}^2 (U_{m_2} - U_{m_1})^2 \right)
\]
\[
\leq 4 \sum_{\ell_k} \left( \tilde{\mu} \left( \prod_{j=1}^4 W_{k_j} \right) \right) + \left( \sum_{0 \leq k_1 \leq k_2 < m_1 \leq k_3 \leq k_4} \left| \tilde{\mu} \left( \prod_{j=1}^4 W_{k_j} \right) \right| \right)^2 + C' \epsilon^{4} \sum_{\ell \geq k} \epsilon^{4} \theta_1^{\ell},
\]
where \( \ell_k \) is the set of \((k_1, k_2, k_3, k_4)\) satisfying \( 0 \leq k_1 \leq k_2 < m_1 \leq k_3 \leq k_4 < m_2 \) such that \( \max(k_2 - k_1, k_3 - k_2, k_4 - k_3) \leq k \).

Here \( \ell \) in the final sum represents the largest gap, i.e. it is \( \max(k_2 - k_1, k_3 - k_2, k_4 - k_3) \), and the coefficient \( \epsilon^4 \) in the final sum comes from varying the \( k_i \) subject to the constraint \( 0 \leq k_1 \leq k_2 < m_1 \leq k_3 \leq k_4 < m_2 \).

Now, lemma 4.13 ensures that, for every \( \epsilon > 0 \), there exists \( n_\epsilon \) such that, for every \( n \geq n_\epsilon \) and every \( 0 \leq m_1 < m_2 \), we have
\[
\tilde{\mu} \left( U_{m_1}^2 (U_{m_0} - U_{m_1})^2 \right) \leq 4 \sum_{\ell_k} \left( \tilde{\mu} \left( \prod_{j=1}^4 W_{k_j} \right) \right) + \epsilon^{4n} (m_2)^2 (\epsilon^4 - 2\alpha n^{-2} + \epsilon^{2\alpha} n^{-2-2\alpha}) + 4C' \epsilon^{4} \sum_{\ell \geq k} \epsilon^{4} \theta_1^{\ell},
\]
(4.26)
Fix \( \gamma \in (0, \frac{1}{2} - 1) \). Due to (4.20)–(4.22), there exists \( K_0 > 0 \) and \( \theta_0' > 0 \) such that, for every \( \epsilon \) and \( n \) satisfying \( K_0 \tilde{\mu}(R) < \frac{\epsilon^4}{2} \), for every \((k_1, k_2, k_3, k_4)\) such that \( \max j_{i=1,2,3} \left(k_{j+1} - k_j \right) \in [1, (\epsilon n^\gamma)^\alpha] \),
\[ \tilde{\mu} \left( \prod_{j=1}^{4} W_k \right) \leq 4e^2 \tilde{\mu} (|W_k W_{k+1}|) \leq K_0 \epsilon^\theta n^{-1-\theta \epsilon} \tag{4.27} \]

Fix \( \epsilon > 0 \) and consider \( n \) large enough so that \( K_0 \tilde{\mu}(\mathcal{R}) < \frac{\epsilon^\theta}{2} \) and such that \( k = 3 \log n/\log \theta_1 \leq (\epsilon n)\alpha \). Putting together (4.26) with (4.27), we obtain that
\[ \tilde{\mu} (U_m^2 (U_m - U_m)^2) \leq C'' \left( e^4 n^{-2} + \frac{m_2^2}{n^2} (e^{4-2\alpha} + n^{-2\beta}) + (\log n)^4 n^{-1-\theta \epsilon} \right) \]
with \( \tilde{\theta} := \min(1, \frac{\theta l}{2}) \) since \( n^{-1-\theta \epsilon} \leq \frac{m_1 + \theta}{n^{1+\theta}} \leq \left( \frac{m_2}{n} \right)^{1+\theta} n^{-\frac{\theta l}{2}} \). This ends the proof of lemma 4.14. \( \square \)

**Proof of Condition II.** Due to the Markov inequality and to lemma 4.14, for every \( 0 \leq i \leq j \leq k \leq n \),
\[ \tilde{\mu} (|U_j - U_i| \land |U_k - U_j| \geq \lambda) \leq \lambda^{-4} E_{\tilde{\mu}} [U_j - U_i|^4 \land |U_k - U_j|^4] \]
\[ \leq \lambda^{-4} E_{\tilde{\mu}} [U_j - U_i|^2 \land |U_k - U_j|^2] \]
\[ \leq \lambda^{-4} E_{\tilde{\mu}} [U_{j-i}^2 |U_{k-i} - U_{j-i}|^2] \]
\[ \leq \lambda^{-4} \tilde{C}_0 \left( \frac{k-i}{n} \right)^{1+\theta} \left( e^{4-2\alpha} + \epsilon n^{-\frac{\theta l}{2}} \right). \]

Therefore, due to theorem [Bil99, theorem 10.1] (with \( \beta = 1 \) and \( \alpha = (1 + \tilde{\theta})/2 \) and \( u_t = n^{-1}(\tilde{C}_0(e^{4-2\alpha} + n^{-\tilde{\theta}_0}))^{1/(1+\tilde{\theta})} \)), there exists \( K > 0 \) depending only on \( (\alpha, \beta) \) such that
\[ \tilde{\mu} (L_m \geq \lambda) \leq \lambda^{-4} K \tilde{C}_0 \left( e^{4-2\alpha} + \epsilon n^{-\tilde{\theta}_0} \right), \tag{4.28} \]
with \( L_m := \max_{0 \leq i < j < k \leq n} (|U_j - U_i| \land |U_k - U_j|) \). As noticed in [Bil99, (10.3)],
\[ \max_{k \leq n} |U_k| \leq 3L_n + \max_{k \leq n} |V_k - \tilde{\mu}(V_0)|. \]

Therefore, for every \( \gamma > 0 \),
\[ \tilde{\mu} \left( \max_{0 \leq k \leq n} |U_k| \geq \gamma \right) \leq \tilde{\mu} \left( L_0 \geq \frac{\gamma}{6} \right) + \tilde{\mu} \left( \max_{k \leq n} |V_k - \tilde{\mu}(V_0)| \geq \frac{\gamma}{2} \right) \]
\[ \leq K 6^4 \gamma^{-4} \tilde{C}_0 \left( e^{4-2\alpha} + \epsilon n^{-\tilde{\theta}_0} \right) + \tilde{\mu} \left( \max_{k \leq n} |V_k| \geq \frac{\gamma}{2} \right) + \tilde{\mu} \left( \epsilon \geq \frac{\gamma}{2} - \frac{\tilde{\mu}(Z_0)}{n^\pi} \right). \]

Consequently, for every \( \gamma > 0 \),
\[ \lim_{\epsilon \rightarrow 0} \limsup_{n \rightarrow +\infty} \tilde{\mu} \left( \max_{0 \leq k \leq n} |U_k| > \gamma \right) \leq \lim_{\epsilon \rightarrow 0} \left( K 6^4 \gamma^{-4} \tilde{C}_0 e^{4-2\alpha} + \tilde{\mu} \left( \epsilon \geq \frac{\gamma}{2} \right) \right) = 0. \tag{4.29} \] \( \square \)
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Appendix A. Asymmetric cusps with flatness $\beta$

In this section, we concentrate on one cusp $P_i$ with flatness $\beta$ for $i \in J$. To simplify notation, we ignore the index $i$, and consider here the cusp formed by $\Gamma, \Gamma'$ with a common tangent line at the end point $P$. To simplify notation, we assume that the flat point $P$ has curvilinear abscissa $r = 0$ and $r' = |\partial Q|$. We choose a Cartesian coordinate system $(s, z)$ with origin at $P$, and with the horizontal $s$-axis being the tangent line to the boundary of the billiard table. By (1.1), for some small $\epsilon_0 > 0$, the boundary pair $\Gamma$ and $\Gamma'$ adjacent to $P$ can be represented in the $\epsilon_0$-neighborhood of the cusp $P$ as: $\{(s, z_0(s)), s \in [0, \epsilon_0]\} \cup \{(s, -z_1(s)), s \in [0, \epsilon_0]\}$, with

$$z_i(s) = c_i \beta^{-1} s^\beta + O(s^{\beta-1}), \quad z'_i(s) = c_i s^{\beta-1} + O(s^{2\beta-2}), \quad \forall s \in [0, \epsilon_0],$$

(A.1)

where $\epsilon_0, c_i \geq 0$ not both equal to 0. We also set $\bar{\epsilon} = \frac{\alpha_0 + \epsilon_0}{2}$ and $\alpha := \beta/(\beta - 1)$.

We write $\mathcal{M}$ for the set of vectors in $\mathcal{M}$ that are in the cusp area $B(P)$ and such that the previous reflection off of $\partial Q$ was outside the cusp area $B_i(P)$. Fix $N_0$. For any $N \geq N_0$, we define $M_N$ to be the set of points in $\mathcal{M}$ whose forward trajectories explore the cusp at $P$ over $N$ reflections off of the first curve, either $\Gamma$ or $\Gamma'$, that it hits, before leaving the cusp.

A.1. The corner series

In this subsection, we investigate the geometry of corner series, which correspond to certain billiard trajectories entering an asymmetric cusp of flatness $\beta > 0$ and experience a large number of reflections there before exiting. For a large $N \geq N_0$, we consider a corner sequence entering the cusp at $P$ with $N$ reflection off the first curve it hits before leaving the cusp (so making either $2N$ or $2N + 1$ reflections in the cusp).

We assume moreover (up to permuting the roles played by $\Gamma$ and by $\Gamma'$) that the first reflection is on $\Gamma$. We denote $(x_n, y_n) = ((r_n, \varphi_n))_n$ as the consecutive sequence colliding with the boundary $\Gamma$, and $(x'_n, y'_n) = ((r'_n, \varphi'_n))_n$ the sequence on $\Gamma'$. Let $s_n$ (resp. $s'_n$) to be the $s$-coordinate of the base point of $x_n$ (resp. $x'_n$), for $n = 1, \cdots, N$. By the smoothness of the boundary curves,

$$|r_n| = \int_0^{s_n} \sqrt{1 + (z'_0(u))^2} \, du = s_n + O(s_n^{2\beta-1}), \quad |r'_n| = s'_n + O((s'_n)^{2\beta-1}).$$

(A.2)

To estimate the tail distribution of $\mu_M(\mathcal{R} \geq n)$, we will fix $N_0$ (as above), and only consider those corner series, such that $N \geq N_0$. We will also work with more convenient coordinates:

$\gamma_n = \min(\varphi_n, \pi - \varphi_n)$, \hspace{1cm} \gamma'_n = \min(\varphi'_n, \pi - \varphi'_n) \hspace{1cm}$ and \hspace{1cm} $\alpha_n = \arctan(z'_0(s_n)), \quad \alpha'_n = \arctan(z'_0(s'_n)).$

Note that by (A.1), the tangent vector of $\partial Q$ at $(s_n, z_n(s_n))$ is $(1, z'_0(s_n))$, which implies, by Taylor-expanding, that

$$a_n = \arctan(z'_0(s_n)) = c_0 s_n^{\beta-1} + O(s_n^{2\beta-2}), \quad \alpha'_n = \arctan(z'_0(s'_n)) = c_1 s'_n^{\beta-1} + O((s'_n)^{2\beta-2}),$$

(A.3)
where $\alpha_n$ (resp. $\alpha'_n$) stands for the angle in $[0, \frac{\pi}{2}]$ of the tangent line to $\Gamma$ at $(s_n, z_0(s_n))$ (resp. $\Gamma'$ at $(s'_n, -z_1(s'_n))$) made with the horizontal axis, or equivalently, with the tangent line through the flat point $P$. Note that both $\alpha_n$ and $\gamma_n$ are positive for $1 \leq n \leq N - 1$. The sequences $(\alpha_n)_n$ and $(\alpha'_n)_n$ are decreasing and take small values if $N$ is large enough. While the $\gamma_n$ are initially small, they slowly grow to about $r/2$ for $n \sim N/2$, and then again decrease and get small. We use notation similar to that of [CM07], and define $N_2$ such that

$$\alpha_{N_2} := \min\{\alpha_n : 1 \leq n \leq N\}.$$  

Comparing the trajectory $(T^{N_2-2j}x)_{j=0,\ldots,\lfloor N_2/2\rfloor}$ with the outgoing trajectory $(T^{N_2+2j}x)_{j=0,\ldots,\lfloor (N-N_2)/2\rfloor}$, we conclude that only two cases can occur

either $s_{N_2-1} < s_{N_2+1} < s_{N_2-2} < s_{N_2+2} < \ldots$ or $s_{N_2+1} < s_{N_2-1} < s_{N_2-2} < s_{N_2+2} < \ldots$

This implies that $|N_2 - N/2| \leq 2$. We further subdivide the corner series into three segments. We fix a small enough $\bar{\gamma}$ and set

$$N_1 := \max\{n \leq N_2 : \gamma_n < \bar{\gamma}\}, \quad N_3 := \max\{n \geq N_2 : \gamma_n > \bar{\gamma}\}.$$  

We call the segment on $[1, N_1]$ the ‘entering period’ in the corner series, the segment $[N_1 + 1; N_3 - 1]$ the ‘turning period’, and the segment $[N_3, N]$ the ‘exiting period’. The same argument as above for $N_1$ shows that $|N_3 - N_2| \leq 2$ so that

$$|N_3 + N_1 - N| \leq |(N_3 - N_2) - (N_2 - N_1)| + |2N_2 - N|$$

implies $|N_3 + N_1 - N| \leq 6$. By the reversibility of the billiard dynamics, it is enough to consider the first half of the series, $1 \leq n \leq N_2$.

Using the relations above, we collect various estimates in the following proposition for a corner series of length $N$ generated by any $x \in M_N$. We first denote an important function $H$ defined on $\mathcal{M}$ in the cusp by

$$H(r, \varphi) := |r|^\beta \sin \varphi.$$  

Recall that here $|r|$ represents the curvilinear distance on $\partial Q$ between the cusp $P$ and the point $x = (r, \varphi)$ in the cusp that we are interested in. For every $n = 1, \ldots, N$, we set

$$H_n := H(r_n, \varphi_n) = |r_n|^\beta \sin \varphi_n \quad \text{and} \quad H'_n := H(r'_n, \varphi'_n) = |r'_n|^\beta \sin \varphi'_n.$$  

**Proposition A.1.** Assume $\beta_0 = \beta_1$ or $\max(\beta_0, \beta_1) \geq 2\beta - 1$. The following are true\(^5\):

1. $N_1 \approx N_2 = N_1 \approx N_1 - N_1 \approx N - N_1 \approx N$, i.e. all three segments in the corner series have length of order $N$;
2. $s_2 \approx s'_2 \approx N^{-\frac{\beta}{\beta - 1}}$, $s_{n} \approx s'_{n} \approx n^{-\frac{1}{\beta - 1}} \sim N^{-\frac{\beta}{\beta - 1}}$, for $n \in [N_1, N_2]$;
3. $s_n \approx s'_n \approx (nN^{-\frac{\beta}{\beta - 1}})^{-\frac{1}{\beta - 1}}$, for $n \in [2, N_2]$;
4. $\gamma_1 \approx \gamma'_1 = O(N^{-\frac{\beta}{\beta - 1}})$, $\gamma_2 \approx \gamma'_2 \approx N^{-\frac{\beta}{\beta - 1}}$;
5. $v_n \approx v'_n \approx \gamma_n \approx \gamma'_n \approx (nN^{-\frac{\beta}{\beta - 1}})^{-\frac{1}{\beta - 1}}$, for $n \in [2, N_2]$;
6. For $N$ sufficiently large, the quantity $\{H_n((r_n, \varphi_n)), n = 1, \ldots, N - 1\}$ and $\{H'_n((r'_n, \varphi'_n)), n = 1, \ldots, N - 1\}$ are both almost invariant along a corner series of length $N$:

$$\forall n = 1, \ldots, N_2, \quad H_n = C_N + O(s_n^{2\beta - 1}) \quad \text{and} \quad H'_n = C'_N + O(s'_n^{2\beta - 1}).$$

\(^5\)Where $\alpha_n \approx \alpha'_n$ means that, for $n$ large enough, there exist $\tilde{c}, \tilde{C} > 0$ (independent of the corner series and of $n$) such that $\tilde{c} \alpha_n \leq \alpha'_n \leq \tilde{C} \alpha_n$.  

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More precisely, we will see in appendix A.2 that \( C_N = \tilde{c}^{-\alpha} I_N^\alpha N^{-\frac{\beta}{2\pi}} + O\left(N^{-\frac{\beta}{2\pi}} \ln N\right) \)
and that \( C'_N = \tilde{c}^{-\alpha} I'_N^\alpha N^{-\frac{\beta}{2\pi}} + O\left(N^{-\frac{\beta}{2\pi}} \ln N\right) \) uniformly in \( x \in M_N \).

**Proof.** For \( n = 1, \ldots, N_2 \), both \( \{\alpha_n\} \) and \( \{\alpha'_n\} \) are decreasing sequences, and \( \{\gamma_n\} \) and \( \{\gamma'_n\} \) are increasing sequences. Observe that

\[
\gamma_{n+1} = \gamma'_n + \alpha_n + \alpha_{n+1} \quad \text{and} \quad \gamma'_n = \gamma_n + \alpha_n + \alpha'_n.
\]

(A.4)

Now we denote \( \nu_n := \gamma_n + \alpha_n \) and \( \nu'_n := \gamma'_n + \alpha'_n \). Observe that \( \nu_n \) and \( \nu'_n \) correspond to the angles between the horizontal line (i.e. the tangent line to the cusp) and the reflected directions corresponding to \( (r_n, \varphi_n) \) and \( (r'_n, \varphi'_n) \), respectively. With this notation, (A.4) leads to

\[
\nu_{n+1} = \nu'_n + 2\alpha_{n+1}, \quad \nu'_n = \nu_n + 2\alpha'_n.
\]

(A.5)

This also implies that

\[
\nu_n = \nu_{n-1} + 2\alpha_n + 2\alpha'_{n-1}, \quad \nu'_n = \nu'_{n-1} + 2\alpha_n + 2\alpha'_n.
\]

(A.6)

By (A.5) and (A.6), we have

\[
\nu_2 > 2\alpha_2, \quad \sum_{n=1}^{N_2} \alpha_n \leq \nu_{N_2}/2 \leq \pi/4
\]

(A.7)

and

\[
(n-1)2\tilde{c}(s^{\beta-1}_n + O(s^{\beta-2}_n)) \leq (n-1)(\alpha_n + \alpha'_{n-1}) \leq \sum_{k=2}^{n} (\alpha_k + \alpha'_{k-1}) \leq \nu_n/2 \leq \min\left(\frac{\pi}{4} \sin \varphi_n, \frac{\tan \varphi_n}{2}\right).
\]

(A.8)

This implies in particular that

\[
s_n = O\left(n^{-\frac{\beta}{2\pi}}\right).
\]

(A.9)

We denote

\[
\tau_n := \frac{z_0(s_n) + z_1(s'_n)}{\sin \nu_n}, \quad \tau'_n := \frac{z_0(s_{n+1}) + z_1(s'_{n+1})}{\sin \nu'_{n+1}}.
\]

(A.10)

Here \( \tau_n \) is the free path between two collisions based at \( x_n \) and \( x'_n \), while \( \tau'_n \) is the free path between two collisions based at \( x'_n \) and \( x_{n+1} \). Observe that

\[
s_n = s'_{n-1} - \tau'_{n-1} \cos \nu'_{n-1} = s'_{n-1} - \frac{z_0(s_n) + z_1(s'_{n-1})}{\tan \nu'_{n-1}}, \quad s'_n = s_n - \tau_n \cos \nu_n = s_n - \frac{z_0(s_n) + z_1(s'_{n})}{\tan \nu_n}.
\]

(A.11)

This implies that

\[
s_{n+1} - s_n = O\left(\frac{s^{\beta}_n}{\tan \nu_n}\right)
\]

(A.12)

which combined with (A.8) implies that \( s_{n+1}/s_n = 1 + O(s^{\beta-1}_n/\tan \nu_n) = 1 + O(1/n) \)
and that

\[
s_n/s_{n+1} = O(1).
\]

(A.13)
More precisely, we obtain that
\[ s_{n+1} - s_n = -\frac{z_0(s_{n+1}) + z_0(s_n)}{\tan v_n} - \frac{2\gamma(s_n)}{\tan v_n} + O\left(\frac{s_n^{2\beta-1}}{(\sin v_n)^2}\right) \tag{A.14} \]
\[ = -\frac{4\gamma s_n^\beta}{\beta \tan v_n} + O\left(\frac{s_n^{2\beta-1}}{(\sin v_n)^2}\right) \tag{A.15} \]
where we used the fact that \(\frac{1}{\tan v_n} = \frac{1}{\tan v_n'} = O\left(\frac{\sin v_n'}{\sin v_n}\right)\) combined with (A.5) and (A.1). We denote
\[ A_n := s_n^\beta \sin v_n, \quad n = 1, \ldots, N_2. \]
The next lemma tells that \(A_n \sim A_{N_2}\) is almost invariant, as long as \(N\) is large. We set \(\tilde{N}_2 := \max\{n < N_2 : v_n < \frac{\pi}{2} - \tilde{\eta}_1\}\) (for any fixed \(\tilde{\eta}_1 > 0\) and prove the following estimates.

**Lemma A.2.**
\[ A_{n+1} - A_n = O\left(\frac{s_n^{3\beta-2}}{\sin v_n}\right). \tag{A.16} \]
Moreover
\[ \forall n = 1, \ldots, \tilde{N}_2, \quad A_n = A_{\tilde{N}_2} + O(s_n^{2\beta-1}), \tag{A.17} \]
and for every \(\varepsilon \in (0, 1),\)
\[ \forall n = 1, \ldots, N_2, \quad A_n = A_{\tilde{N}_2} + O(s_n^{2\beta-1-\varepsilon}). \tag{A.18} \]

**Proof.** We start by writing
\[ A_{n+1} - A_n = (s_{n+1}^\beta - s_n^\beta)(\sin v_n + \sin v_{n+1} - \sin v_n) + s_n^\beta(\sin v_{n+1} - \sin v_n). \]
Taylor expansions at the first order combined with (A.15) and with (A.6) lead to
\[ s_{n+1}^\beta - s_n^\beta = \beta s_n^{\beta-1}(s_{n+1} - s_n) + O\left(s_n^{\beta-2}(s_{n+1} - s_n)^2\right) = -\frac{4\gamma s_n^{\beta-1}}{\tan v_n} + O\left(\frac{s_n^{3\beta-2}}{(\sin v_n)^2}\right), \]
and
\[ \sin v_{n+1} - \sin v_n = \cos v_n (v_{n+1} - v_n) + O\left(v_n (v_{n+1} - v_n)^2\right) = 2 \cos v_n (\alpha_{n+1} + \alpha_n') + O\left(s_n^{2\beta-2}\right) = 4\gamma s_n^{\beta-1} \cos v_n + O\left(s_n^{2\beta-2}/\tan v_n\right). \]
Therefore
\[ A_{n+1} - A_n = O\left(\frac{s_n^{3\beta-2}}{\sin v_n}\right). \tag{A.19} \]
Using (A.15) and (A.13), it comes that
\[ s_n^{2\beta-1} - s_n^{2\beta-1} = -\frac{4\gamma (2\beta - 1)}{\beta} s_n^{3\beta-2} + O\left(\frac{s_n^{4\beta-3}}{(\sin v_n)^3}\right) = -\frac{4\gamma (2\beta - 1)}{\beta} s_n^{3\beta-2} + O\left(s_n^{2\beta-1}/n^2\right). \tag{A.20} \]
due to \((A.8)\), which implies that
\[
\forall n = 1, \ldots, N_2, \quad \sum_{k=n}^{N_2} \frac{s_k^{\beta-2}}{\tan v_k} = O\left(s_n^{2\beta-1}\right),
\]
and so that
\[
\forall n = 1, \ldots, N_2, \quad |A_n - A_{N_2}| \leq \sum_{k=n}^{N_2-1} |A_{k+1} - A_k| = O\left(s_n^{2\beta-1}\right).
\]

Let \(u \in (0, 1)\) be such that \(\beta + (\beta - 1)(1 - u) = 2\beta - 1 - \varepsilon\). For every \(n = 1, \ldots, N_2\),
\[
A_n - A_{N_2} = O\left(\frac{\sum_{k=n}^{N_2} s_k^{\beta-2}}{\sin v_k}\right) = O\left(\frac{s_n^{\beta+(\beta-1)(1-u)}}{\sum_{k=n}^{N_2} s_k^{\beta+(\beta-1)(1-u)}}\right) = O\left(s_n^{\beta+(\beta-1)(1-u)}\right),
\]
using \((A.8)\).

We define
\[
C_N := A_{N_2} = s_{N_2}^{\beta} \sin v_{N_2}.
\]

Now let us prove the key estimates

- For every \(n = N_1, \ldots, N_2\), \(\sin v_n \approx 1\) and so \(s_n^{\beta} \approx s_{N_2}^{\beta} \approx C_N\). Therefore
  \[
  \forall n = N_1, \ldots, N_2, \quad s_n \approx C_N^{1/\beta}.
  \]
  \((A.22)\)

- \(1 \approx v_{N_1} - v_{N_2} \approx \sum_{k=N_2}^{N_1} s_k^{\beta-1} \approx (N_2 - N_1)C_N^{\beta-1}\). Therefore
  \[
  N_2 - N_1 \approx C_N^{\beta-1}.
  \]
  \((A.23)\)

- Let us see that \(v_1 = O(s_1^{\beta-1})\). This is obviously true if \(s_1 \geq \varepsilon_0/2\). Assume now that \(s_1 < \varepsilon_0/2\), then by definition of \(s_1\) the incident line at \((x_1, z(x_1))\) intersects \([(x_1, z_0(2x_1)), (x_1, -z_1(2x_1))]\), which implies that \(\tan(v_1 - \alpha_1) \leq O(s_1^{\beta})/s_1 = O(s_1^{\beta-1})\), and so \(v_1 = O(s_1^{\beta-1})\).

This combined with \((A.6)\) and \((A.13)\) implies that \(v_2 \approx s_2^{\beta-1}\) and so \(C_N \approx s_2^{\beta} v_2 \approx s_2^{\beta-1}\), and so
\[
s_2 \approx C_N^{\frac{1}{\beta-1}}.
\]
  \((A.24)\)

- For every \(n = 2, \ldots, N_1, \) \(s_{n+1} - s_n \approx s_n^{\beta} \approx \frac{s_n^{\beta}}{C_N s_n} \approx s_n^{\beta} \frac{1}{C_N}\) and so
  \[
  s_{n+1}^{2\beta+1} - s_n^{2\beta+1} \approx s_n^{-2\beta}(s_{n+1} - s_n) \approx C_N^{-1}.
  \]
Moreover, due to \((A.24)\), since \(s_2^{-2\beta+1} \approx C_N^{-1}\). Therefore
\[
\forall n = 2, \ldots, N_1, \quad s_n \approx \left(s_n^{2\beta+1}\right)^{-\frac{1}{2\beta+1}} \approx \left(\frac{n}{C_N}\right)^{-\frac{1}{2\beta+1}} \approx \left(\frac{C_N}{n}\right)^{\frac{1}{2\beta+1}}.
\]
  \((A.25)\)
• Due to (A.25), \( s_{N_1} \approx (C_N/N_1)^{\frac{1}{\beta-1}} \). This and (A.22) leads to (\( C_N/N_1 \))^{\frac{1}{\beta-1}} \approx C_N^{\frac{1}{\beta-1}}. Therefore

\[
N_1 \approx C_N^{\frac{\beta-1}{\beta}}. \tag{A.26}
\]

Combining this with (A.23), we obtain

\[
N_1 \approx N_2 - N_1 \approx N \approx C_N^{\frac{\beta-1}{\beta}} \tag{A.27}
\]

and so in particular

\[
C_N \approx N^{\frac{\beta-1}{\beta-1}}. \tag{A.28}
\]

• Combining (A.28) with respectively (A.25) and (A.22), we obtain

\[
\forall n = 2, ... , N_1, \quad s_n \approx (nN^\frac{\beta}{\beta-1})^{-\frac{1}{\beta-1}}. \tag{A.29}
\]

\[
\forall n = N_1, ... , N_2, \quad s_n \approx N^{\frac{\beta-1}{\beta}} \approx (nN^\frac{\beta}{\beta-1})^{-\frac{1}{\beta-1}}, \tag{A.30}
\]

since \( N_1 \approx N_2 \approx N \).

This further implies that for \( n \in [2, N_2] \), we have

\[
s_n^{\beta-1} \approx (\alpha_n + \beta_n^{-1}) \approx (n^{\beta-1}N^\beta)^{-\frac{1}{\beta-1}}, \quad \gamma_n \approx v_n \approx (nN^{\frac{\beta}{\beta-1}})^{-\frac{1}{\beta-1}}. \tag{A.31}
\]

In particularly, for \( n \in [N_1, N_2] \), using the above fact that \( N_1 \approx N_2 \approx N \), we have

\[
(\alpha_n + \beta_n^{-1}) \approx s_n^{\beta-1} \approx n^{-1} \approx N^{-1}, \quad v_n \approx 1. \tag{A.32}
\]

Now by (A.17), we know that

\[
\forall n = 1, ... , N_2, \quad A_n = s_n^\beta \sin v_n = C_N + \mathcal{O}(s_n^{2\beta-1}).
\]

Due to (A.16) and to (A.32),

\[
\forall n = N_2, ... , N_2, \quad A_{N_2} - A_n = \mathcal{O} \left( \sum_{k = N_2}^{N_1} s_k^{3\beta-2} \right) = \mathcal{O} \left( s_n^{1-\beta}s_n^{3\beta-2} \right) = \mathcal{O} \left( s_n^{2\beta-1} \right).
\]

Due to (A.2), we have

\[
H_n = |r_n|^\beta \sin \phi_n = s_n^\beta \sin \gamma_n + \mathcal{O}(s_n^{3\beta-2} \sin \gamma_n). \tag{A.33}
\]

The above estimation implies that

\[
H_n = A_n - s_n^\beta (\sin \gamma_n - \sin v_n) + \mathcal{O}(s_n^{3\beta-2} \sin \gamma_n)
\]

\[
= A_n + 2s_n^\beta \sin \frac{\alpha_n}{2} \cos (\gamma_n + \frac{\alpha_n}{2}) + \mathcal{O}(s_n^{3\beta-2} \sin \gamma_n)
\]

\[
= A_n + \mathcal{O}(s_n^{2\beta-1}) = C_N + \mathcal{O}(s_n^{2\beta-1}),
\]

where we used \( \alpha_n = \mathcal{O}(s_n^{\beta-1}) \). We also observe that

\[
H_n' = |r_n'|^\beta \sin \phi_n' = (s_n')^{\beta} \sin (\gamma_n') + \mathcal{O}(s_n^{2\beta} \sin \gamma_n). \tag{A.34}
\]

The above estimation implies that \( H_n' = C_N + \mathcal{O}(s_n^{2\beta-1}) \).
Due to the time reversibility of billiard dynamics, all the asymptotic formulas obtained for the entering period remain valid for the exiting period.

A.2. Proof of lemma 4.5

Recall that $\alpha = \frac{\beta}{\beta - 1}$. We set $\nu := 2|\partial Q|\mu$ for the non normalized measure on $M$ simply given by $d\nu = \sin \varphi \, d\varphi \, d\varphi$. Write $\tilde{M}$ for the set of vectors in $M$ that are in the cusp area $B_\epsilon(P)$ and such that the previous reflection off $Q$ was outside the cusp area $B_\epsilon(P)$. The purpose of this section is to prove that

$$\lim_{y \to \infty} y^n \nu \left( \tilde{M} : R > y \right) = \frac{4}{\beta} e^{-\frac{n}{\beta-1}} T^n,$$  \hspace{1cm} (A.35)

which will imply lemma 4.5 since $\nu$ is $T$-invariant and since $\tilde{\mu} = \frac{\nu|\nu}{\mu(M)}$, Recall that we denote $T^k(x) = (r_n, \varphi_n)$ and $\gamma_n = \min(\varphi_n, \pi - \varphi_n)$. Due to proposition A.1, for $N$ large enough the following sequences are almost constant for $n = 1, \ldots, N$:

$$H_n = C_N + O(s_n^{2\beta - 1}), \quad H'_{n+1} = C'_N + O(s_n^{2\beta - 1}).$$ \hspace{1cm} (A.36)

where $C_N = O(N^{-\alpha})$, $C'_N = O(N^{-\alpha})$. In order to estimate $C_N$ and $C'_N$, we use an elliptic integral and introduce

$$w_n := \int_0^{\gamma_n} (\sin u)^{1-\frac{1}{\beta}} \, du,$$

for $n = 1, \ldots, N$. Then

$$w_{n+1} - w_n = \int_{\gamma_n}^{\gamma_{n+1}} (\sin u)^{1-\frac{1}{\beta}} \, du = (\sin \gamma_n)^{1-\frac{1}{\beta}} (\gamma_{n+1} - \gamma_n)$$ \hspace{1cm} (A.37)

for some $\gamma_n \in [\gamma_n, \gamma_{n+1}]$. Due to (A.33) and (A.36), we have

$$\sin \gamma_n = \frac{H_n}{r_n} = \frac{H'_n}{s_n} + O\left(s_n^{2\beta - 2}\right) = C_N s_n + O(s_n^{\beta - 1}).$$ \hspace{1cm} (A.38)

By (A.4) and (A.12),

$$\gamma_{n+1} - \gamma_n = 2\alpha_n + \alpha_n + \alpha_{n+1} = 4\varepsilon_s s_n^{\beta - 1} + O\left(s_n^{2\beta - 2} / \tan \gamma_n\right)$$ \hspace{1cm} (A.39)

and $\gamma_{n+1} - \gamma_n = O(s_n^{\beta - 1})$. Now combining the above and recalling that $\alpha = \beta / (\beta - 1)$, we rewrite (A.37) as

$$w_{n+1} - w_n = (\sin \gamma_n^{\frac{1}{\beta - 1}}) (\gamma_{n+1} - \gamma_n) + O \left(\sin^{-\frac{2}{\beta - 1}} \gamma_n (\gamma_{n+1} - \gamma_n)^2\right)$$

$$= \left[ \left( \frac{C_N}{s_n^{\beta - 1}} \right)^{\frac{1}{\beta - 1}} + O \left(\sin^{-\frac{2}{\beta - 1}} \gamma_n s_n^{\beta - 1}\right) \right] \left[ 4\varepsilon_s s_n^{\beta - 1} + O\left(s_n^{2\beta - 2} / \tan \gamma_n\right) \right] + O \left(\gamma_n^{\frac{2}{\beta - 1}} s_n^{2\beta - 2}\right)$$

$$= 4\varepsilon(C_N)^{\frac{1}{\beta - 1}} + O\left(\gamma_n^{\frac{2}{\beta - 1}} s_n^{2\beta - 2}\right)$$

$$= 4\varepsilon(C_N)^{\frac{1}{\beta - 1}} + O(N^{-1} n^{-1}),$$ \hspace{1cm} (A.40)

due to proposition A.1. Recalling (A.36), if we use a dummy variable and sum (A.40) from 1 to $n$, we get

$$w_n = 4\varepsilon n C_N^{1/\beta} + O(\ln n / N).$$ \hspace{1cm} (A.41)
In particular, for \( n = N_2 = N/2 + \mathcal{O}(1) \) we get
\[
\int_0^{\pi/2} (\sin u)^{1-\frac{\pi}{2}} \, du = \int_0^{\gamma_n} (\sin u)^{1-\frac{\pi}{2}} \, du + \mathcal{O} \left( \gamma_n - \frac{\pi}{2} \right)
\]
\[
= w_{N_2} + \mathcal{O}(s_{N_2}^{\alpha-1}) = w_{N_2} + \mathcal{O}(N^{-1}) = 2cNC^{\frac{1}{N}} + \mathcal{O}(\ln N/N).
\]
Thus
\[
C_N^{\alpha} = (2c)^{-\alpha} \left( \int_0^{\pi/2} (\sin u)^{1-\frac{\pi}{2}} \, du \right)^{\alpha} + \mathcal{O}(N^{-\alpha-2} \ln N) = \frac{I_n}{(2c)^{\alpha}} + \mathcal{O}(\ln N/N), \tag{A.42}
\]
and so
\[
w_n = \frac{2I_1 n}{N} + \mathcal{O}(\ln n/N).
\]
Similarly, one can show that
\[
C_N^{\alpha'} = (2c)^{-\alpha'} I_{\alpha'} + \mathcal{O}(\ln N/N). \tag{A.43}
\]
Let \( N' \) be the number of reflections in the cusp (both on \( \Gamma \) and on \( \Gamma' \)). Note that either \( N' = 2N \) or \( N' = 2N + 1 \), so that
\[
C_N^{\alpha} = (2c)^{-\alpha} I_{\alpha} + \mathcal{O}(\ln N'/N') \quad \text{and} \quad C_N^{\alpha'} = (2c)^{-\alpha'} I_{\alpha'} + \mathcal{O}(\ln N'/N'). \tag{A.44}
\]
Let \( \tilde{M}_m \) be the set of points in \( \tilde{M} \) whose forward trajectories explore the cusp at \( P \) during \( m \) reflections off \( \Gamma \cup \Gamma' \), before leaving the cusp. Since \( \nu \) is \( T \)-invariant,
\[
\nu \left( \tilde{M}_m \right) = \frac{1}{m} \nu \left( \bigcup_{n=1}^{m} T^n \tilde{M}_m \right). \tag{A.45}
\]
Let \( y \) be an integer. Observe that
\[
\sum_{m' \geq y} 1_{\{N' \geq m' \}} = \sum_{m' \geq y} \sum_{m \geq m'} 1_{\{N' = m\}} = \sum_{m \geq m'} \sum_{m' \geq y} 1_{\{N' = m\}} = \sum_{m \geq m'} (m - y + 1) 1_{\{N' = m\}},
\]
and so
\[
\sum_{m' \geq y} \nu \left( x \in \tilde{M} : N' \geq m' \right) = \sum_{m \geq y} (m - y + 1) \nu \left( \tilde{M}_m \right) = -y \nu \left( x \in \tilde{M} : N' \geq y \right) + \nu \left( \bigcup_{m \geq y} \bigcup_{n=0}^{m} T^n \tilde{M}_m \right). \tag{A.46}
\]
But (A.36) and (A.42) together imply that the set \( \bigcup m \geq y \bigcup_{n=1}^{m} T^n \tilde{M}_m \) corresponds to the set of points \( (r, \varphi) \) based in the cusp area \( B_x(P) \) such that
\[
H(r, \varphi) = |r|^\alpha \sin \varphi \leq e^{-\alpha} I_{\alpha} y^{-\alpha} + \mathcal{O}(r^{2\beta-1}) + \mathcal{O}(y^{-1-\alpha} \ln y).
\]
Therefore the set \( \bigcup m \geq y \bigcup_{n=1}^{m} T^n \tilde{M}_m \) corresponds to the set of points \( (r, \varphi) \) based in the cusp area \( B_x(P) \) such that
\[
r \leq \frac{e^{-\frac{\alpha}{2}} I_{\alpha}^{\frac{1}{2}} y^{-\frac{\alpha}{2}}}{\sin^{\frac{\alpha}{2}} \varphi} + \mathcal{O} \left( \frac{y^{-\alpha(\frac{1}{2} - 1)}}{\sin^{\frac{\alpha}{2}} \varphi} (\ln y/y + r^{2\beta-1}) \right).
\]
Therefore
\[
\nu \left( \bigcup_{m \geq y} \bigcup_{n = 0}^{m} \mathcal{T}^n \mathcal{M}_m \right) = 2 \int_0^\pi \frac{e^{-y} \tilde{T}_m^2 \tilde{T}_n^2 \sin^2 \varphi}{\sin^2 \varphi} \, d\varphi + \mathcal{O}(y^{-1})
\]
(\text{A.47})

This tells us that
\[
\sum_{m \geq y} (m + 1) \nu(\mathcal{M}_m) \sim 4e \alpha \nu_y^{-1},
\]
as $y$ goes to infinity. Unfortunately, we cannot apply directly the classical Tauberian theorem because we do not know if $m \nu(\mathcal{M}_m)$ is decreasing or not. But we do not want to estimate $\nu(\mathcal{M}_m)$, we just want to estimate
\[
\nu_y := \nu \left( x \in \mathcal{M} : N' \geq y \right) = \sum_{m \geq y} \nu(\mathcal{M}_m).
\]
To this end, we adapt the Tauberian theorem argument as follows. Due to (A.46) and (A.47) (considering the difference between the term of order $[y]$ and the term of order $[(1 + \varepsilon)y]$), we know that, for every $\varepsilon > 0$,
\[
[y]p_{[y]} - [(1 + \varepsilon)y]p_{[(1 + \varepsilon)y]} + \sum_{m = [y]}^{[(1 + \varepsilon)y]} p_m \sim 4e \alpha \nu_y^{-1} (1 - (1 + \varepsilon)^{1-\alpha}) y^{1-\alpha},
\]
as $y$ goes to infinity. Now using the fact that $(p_m)_m$ is decreasing, we obtain that
\[
[y] \{ p_{[y]} - p_{[(1 + \varepsilon)y]} \} \leq [y]p_{[y]} - [(1 + \varepsilon)y]p_{[(1 + \varepsilon)y]} + \sum_{m = [y]}^{[(1 + \varepsilon)y]} p_m \leq [(1 + \varepsilon)y] \{ p_{[y]} - p_{[(1 + \varepsilon)y]} \}.
\]
(\text{A.49})

Fix an arbitrary $\vartheta \in (0, 1)$. Choose $\varepsilon \in (0, \vartheta)$ small enough so that
\[
\alpha - 1 < (1 - \vartheta) \leq 1 - (1 + \varepsilon)^{1-\alpha} \leq (\alpha - 1) \varepsilon (1 + \vartheta)
\]
and
\[
\alpha \varepsilon (1 - \vartheta) \leq 1 - (1 + \varepsilon)^{-\alpha} \leq \alpha \varepsilon (1 + \vartheta).
\]
Due to (A.48), for every $y$ large enough, we know that
\[
(\alpha - 1)e(1 - \vartheta) 4e \alpha \nu_y^{-1} \leq [y]p_{[y]} - [(1 + \varepsilon)y]p_{[(1 + \varepsilon)y]} + \sum_{m = [y]}^{[(1 + \varepsilon)y]} p_m \leq (\alpha - 1)\varepsilon (1 + \vartheta) 4e \alpha \nu_y^{-1} y^{1-\alpha},
\]
and so, due to (A.49), we obtain that
\[
(\alpha - 1)e(1 - \vartheta)^2 4e \alpha \nu_y^{-1} y^{1-\alpha} \leq p_{[y]} - p_{[(1 + \varepsilon)y]} \leq (\alpha - 1)\varepsilon (1 + \vartheta)^2 4e \alpha \nu_y^{-1} y^{1-\alpha},
\]
for every $y$ large enough. Now using the fact that $p_y = p_{[y]} = \sum_{k \geq 0} \left( p_{[(1 + \varepsilon)^k y]} - p_{[(1 + \varepsilon)^{k+1} y]} \right)$, it follows that, for every $y$ large enough,
\[
(\alpha - 1)e(1 - \vartheta)^2 4e \alpha \nu_y^{-1} \sum_{k \geq 0} (1 + \varepsilon)^{-k} y^{-\alpha} \leq p_y \leq (\alpha - 1)\varepsilon (1 + \vartheta)^2 4e \alpha \nu_y^{-1} \sum_{k \geq 0} (1 + \varepsilon)^{-k} y^{-\alpha}
\]
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which can be rewritten
\[(\alpha - 1) \frac{\varepsilon}{1 - (1 + \varepsilon)^{-\alpha}} \cdot \frac{(1 - \vartheta)^2}{1 + \vartheta} 4e^{1-\alpha} P_1 y^{-\alpha} \leq p_y \leq (\alpha - 1) \frac{\varepsilon}{1 - (1 + \varepsilon)^{-\alpha}} \cdot (1 + \vartheta)^2 4e^{1-\alpha} P_1 y^{-\alpha}.\]

Using our second condition on \(\varepsilon\), this leads to
\[\frac{\alpha - 1}{\alpha} \frac{(1 - \vartheta)^2}{(1 + \vartheta)^2} 4e^{1-\alpha} P_1 y^{-\alpha} \leq p_y \leq \frac{\alpha - 1}{\alpha} \frac{(1 + \vartheta)^2}{1 - \vartheta} 4e^{1-\alpha} P_1 y^{-\alpha},\]
for every \(y\) large enough. Since \(\vartheta\) is arbitrary, we have proved that
\[p_y \sim 4 \frac{\alpha - 1}{\alpha} e^{1-\alpha} P_1 y^{-\alpha} = 4e^{-1} e^{1-\alpha} P_1 y^{-\alpha},\]
as \(y\) goes to infinity, which completes the proof of lemma 4.5.

**Proof of lemma 4.3.** We also now obtain the proof of lemma 4.3 by using (A.41) and (A.44) in place of (6.8) and (6.9) in the proof of lemma 4.4 in [JZ18], and then making the appropriate obvious adjustments.

**Appendix B. Skorokhod \(J_1\) and \(M_1\) topologies**

Here, we review some basics about the Skorokhod \(J_1\) and \(M_1\) topologies. More details can be found in [Bil99] or [Whi02]. A stronger result than the limit theorem in (2.4) is its functional version, called a functional limit theorem or weak invariance principle. The \(W_n\)’s are elements in the Skorokhod space \(D[0, \infty]\), i.e. the space of all functions \(\varphi\) on \([0, \infty]\) that are right-continuous and have left-hand limits \(\varphi(t^-)\) for every \(t > 0\).

We will consider two different topologies on \(D[0, \infty]\). The most commonly used topology in the literature is Skorokhod’s \(J_1\)-topology which is described as follows: if \(\varphi_n, \varphi \in D[0, \infty]\), then \(\varphi_n \to \varphi\) in the \(J_1\)-topology if and only if there exists a sequence of \(\{\lambda_m\} \subset A\), such that
\[\sup_t |\lambda_m(s) - s| \to 0, \quad \sup_s |\varphi_n(\lambda_m(s)) - \varphi(s)| \to 0\]
for all \(m \in \mathbb{N}\), where \(A\) is the family of strictly increasing, continuous mappings \(\lambda\) of \([0, \infty)\) onto itself such that \(\lambda(0) = 0\) and \(\lambda(\infty) = \infty\).

In contrast, the \(M_1\)-topology allows a function \(\phi_1\) with a jump at \(t\) to be approximated arbitrarily well by some continuous \(\phi_2\) (with large slope near \(t\)). The metric \(d_{M_1}\) that generates the \(M_1\)-topology on \(D[0, \infty]\) is defined using completed graphs. For \(\phi \in D[0, \infty]\) the completed graph of \(\phi\) is the set
\[\Gamma(\phi) := \{(t, z) \in [0, \infty) \times \mathbb{R} : z = \lambda \phi(t^-) + (1 - \lambda) \phi(t) \text{ for some } \lambda \in [0, 1]\}\]
where \(\phi(t^-)\) is the left limit of \(\phi\) at \(t\). Besides the points of the graph \(\{(t, \phi(t)) : t \in [0, \infty)\}\), the completed graph of \(\phi\) also contains the vertical line segments joining \((t, \phi(t^-))\) and \((t, \phi(t^-))\) for all discontinuity points \(t\) of \(\phi\). We define an order on the graph \(\Gamma(\phi)\) by saying that \((t_1, z_1) \leq (t_2, z_2)\) if either (i) \(t_1 < t_2\) or (ii) \(t_1 = t_2\) and \(|\phi(t_1^-) - z_1| \leq |\phi(t_2^-) - z_2|\). A parametric representation of the completed graph \(\Gamma(\phi)\) is a continuous nondecreasing function \((s, y)\) mapping \([0, \infty)\) onto \(\Gamma(\phi)\), with \(s\) being the time component and \(y\) being the spatial
component. Let $\Lambda(\phi)$ denote the set of parametric representations of the graph $\Gamma(\phi)$. For $\phi_1, \phi_2 \in \mathbb{D}[0, \infty)$ define
\[
d_{M_1}(\phi_1, \phi_2) := \inf\{\|s_1 - s_2\|_{[0, \infty)} \vee \|u_1 - u_2\|_{[0, \infty)} : (s_i, u_i) \in \Lambda(\phi_i), i = 1, 2\},
\]
where $||\phi||_{[0, \infty)} = \sup\{|\phi(t)| : t \in [0, \infty)\}$. This definition introduces $d_{M_1}$ as a metric on $\mathbb{D}[0, \infty)$. The induced topology is called Skorokhod’s $M_1$-topology and is weaker than the more frequently used $J_1$-topology.

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**References**

[AT92] Avram F and Taqqu M 1992 Weak convergence of sums of moving averages in the $\alpha$-stable domain of attraction Ann. Probab. 20 483–503

[BCD11] Bálint P, Chernov N and Dolgopyat D 2011 Limit theorems for dispersing billiards with cusps Commun. Math. Phys. 308 479

[Ber27] Bernstein S 1927 Sur l’extension du théorème limite du calcul des probabilités aux sommes de quantités dépendantes Math. Ann. 97 1–59

[Bil68] Billingsley P 1968 Convergence of Probability Measures (New York: Wiley)

[Bil99] Billingsley P 1999 Convergence of Probability Measures (New York: Wiley)

[CM06] Chernov N and Markarian R 2006 Chaotic Billiards (Providence, RI: American Mathematical Society)

[CM07] Chernov N and Markarian R 2007 Dispersing billiards with cusps: slow decay of correlations Comm. Math. Phys. 270 727–58

[CZ05] Chernov N and Zhang H-K 2005 Billiards with polynomial mixing rates Nonlinearity 18 1527

[CZ09] Chernov N and Zhang H-K 2009 On statistical properties of hyperbolic systems with singularities J. Stat. Phys. 136 615–42

[DR78] Durrett R and Resnick S I 1978 Functional limit theorems for dependent variables Ann. Probab. 6 829–46

[JZ18] Jung P and Zhang H-K 2018 Stable laws for chaotic billiards with cusps at flat points Ann. Henri Poincaré 19 3815–53

[Kal73] Kallenberg O 1973 Characterization and convergence of random measures and point processes Probab. Theory Relat. Fields 27 9–21

[Mac83] Machta J 1983 Power law decay of correlations in a billiard problem J. Stat. Phys. 32 555–64

[Mar04] Markarian R 2004 Billiards with polynomial decay of correlations Ergod. Theor. Dyn. Syst. 24 177–97

[MV19] Melbourne I and Varandas P 2019 Convergence to a Lévy process in the Skorohod $M_1$ and $M_2$ topologies for nonuniformly hyperbolic systems, including billiards with cusps Commun. Math. Phys. (accepted)

[MZ15] Melbourne I and Zweimüller R 2015 Weak convergence to stable Lévy processes for nonuniformly hyperbolic dynamical systems Ann. l’Inst. Henri Poincaré 51 545–56

[PS18] Pène F and Saussol B 2018 Spatio-temporal Poisson processes for visits to small sets (arXiv:1803.06865)

[Res87] Resnick S I 1987 Extreme Values, Regular Variation, and Point Processes (New York: Springer)

[Ser70] Serfling R J 1970 Moment inequalities for the maximum cumulative sum Ann. Math. Stat. 41 1227–34

[ST94] Samorodnitsky G and Taqqu M 1994 Stable non-Gaussian Random Processes: Stochastic Models with Infinite Variance (London: Chapman and Hall)
[TK10] Tyran-Kamińska M 2010 Weak convergence to lévy stable processes in dynamical systems Stoch. Dyn. 10 263–89

[Whi02] Whitt W 2002 Stochastic-Process Limits: an Introduction to Stochastic-Process Limits and their Application to Queues (New York: Springer)

[Zha17] Zhang H-K 2017 Decay of correlations for billiards with flat points II: cusps effect Dynamical Systems, Ergodic Theory, and Probability: in Memory of Kolya Chernov (Contemporary Mathematics vol 698) (Providence, RI: American Mathematical Society) pp 287–316