Magnetic Monopoles, Vortices and the Topology of Gauge Fields\textsuperscript{a,b}

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Lattice calculations performed in Abelian gauges give strong evidence that confinement is realized as a dual Meissner effect, implying that the Yang-Mills vacuum consists of a condensate of magnetic monopoles. Alternative lattice calculations performed in the maximum center gauge give strong support that center vortex configurations are the relevant infrared degrees of freedom responsible for confinement and that the magnetic monopoles are mostly sitting on vortices. In this talk I study the continuum Yang-Mills-theory in Abelian and center gauges. In Polyakov gauge the Pontryagin index of the gauge field is expressed by the magnetic monopole charges. The continuum analogues of center vortices and the continuum version of the maximum center gauge are presented. It is shown that the Pontryagin index of center vortices is given by their self-intersection number, which vanishes unless magnetic monopole currents are flowing on the vortices.

1 Introduction

There are two fundamental essentially non-perturbative features of QCD: confinement and spontaneous breaking of chiral symmetry. The latter can be more or less understood in terms of instantons. By the Atiah-Singer theorem\textsuperscript{1} instanton fields having a non-trivial Pontryagin index give rise to zero modes of the quarks and after averaging over all gauge fields favour a non-zero quark level density $\rho(\lambda) \neq 0$ at zero virtuality $\lambda = 0$. By the Banks-Casher theorem\textsuperscript{2} $\langle \bar{q}q \rangle = \pi \rho(0)$ this implies a non-zero quark condensate, which is the order parameter of spontaneous breaking of chiral symmetry. However, one should stress that this explanation of spontaneous breaking of chiral symmetry does not really rely on instantons, i.e. on finite action solutions of the classical field equation but only on topologically non-trivial field configurations.

On the other hand the confinement mechanism is much less understood. However, lattice calculations performed over the last couple of years\textsuperscript{3,4} have accumulated evidence that confinement is realized either as dual Meissner effect\textsuperscript{5} implying a condensation of magnetic monopoles in the QCD vacuum or by a condensation of magnetic vortices\textsuperscript{6}. In fact, recent lattice calculations also indicate that the magnetic monopoles are related to the vortices\textsuperscript{4}.

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From the conceptional point of view it is not very appealing that spontaneous breaking of chiral symmetry and confinement are attributed to different types of field configurations. This is because lattice calculations indicate that the deconfinement phase transition is accompanied by the restoration of chiral symmetry. Moreover, the spontaneous breaking of chiral symmetry is determined by topologically non-trivial field configurations. Since magnetic monopoles are long-range fields they should be relevant for the global topological properties of gauge fields. One can therefore expect an intimate relation between magnetic monopoles and the topology of gauge fields. In this lecture I will show that in Abelian gauges the non-trivial topology of gauge fields is generated by magnetic monopoles. In these gauges instantons give rise to magnetic monopoles but the latter do exist without instantons. Furthermore, magnetic vortices can be topologically non-trivial only if they host magnetic monopoles. In this sense, magnetic monopoles have to be considered as the fundamental topological objects of gauge fields, at least in Abelian gauges.

2 Emergence of Magnetic Monopoles in Abelian Gauges

Magnetic monopoles arise in the so-called Abelian gauges which fix the coset $G/H$ of the gauge group $G$, but leave Abelian gauge invariance with respect to the Cartan subgroup $H$. Recent lattice calculations show that the dual Meissner effect is equally well realized in all Abelian gauges studied, while the Abelian and monopole dominance is more pronounced in the so-called maximum Abelian gauge. Here, for simplicity, we will use the Polyakov gauge, defined by a diagonalization of the Polyakov loop

$$\Omega(\vec{x}) = P \exp \left( - \int_0^T dx_0 A_0 \right) = V^\dagger \omega V \rightarrow \omega, \quad (1)$$

where $\omega \in H$ is the diagonal part of the Polyakov loop and the matrix $V \in G/H$ is obviously defined only up to an Abelian gauge transformation $V \rightarrow gV$, $g \in H$. This gauge is equivalent to the condition $A_0^{ch} = 0$, $\partial_0 A_0^n = 0$, where $A_0^n$ and $A_0^{ch}$ denote the diagonal (neutral with respect to $H$) and off-diagonal (charged) parts of the gauge field, respectively. For simplicity, we will assume $G = SU(2)$ below.

When the Polyakov loop $\Omega(\vec{x})$ becomes a center element of the gauge group at some isolated point $\vec{x}_i$ in 3-space

$$\Omega(\vec{x}_i) = (-1)^{n_i}, \quad (2)$$
there is a topological obstruction in the diagonalization and the induced gauge field
\[ \mathcal{A} = V \partial V^\dagger, \]
arising from the gauge transformation \( V \in G/H \) which makes the Polyakov loop diagonal, develops a magnetic monopole. An important property of the magnetic monopoles arising in the Abelian gauges is that their magnetic charge
\[ m[V] = \frac{1}{4\pi} \int_{S_2} d^3 B \]
is topologically quantized and given by \( m[V] \in \Pi_2(SU(2)/U(1)) \), i.e. by the winding number of the mapping \( V(\vec{x}) \in SU(2)/U(1) \). In the above equation \( S_2 \) is an infinitesimal 2-sphere around the monopole position with the piercing point of the Dirac string left out.

### 3 Magnetic Monopoles as Sources of Non-Trivial Topology

Since the magnetic monopoles are long-range fields, we expect that they are relevant for the topological properties of gauge fields. As is well known, the gauge fields \( A_\mu(x) \) are topologically classified by the Pontryagin index
\[ \nu = -\frac{1}{16\pi^2} \int d^4x \text{Tr} F_{\mu\nu} \tilde{F}_{\mu\nu}. \]
In the gauge \( \partial_0 A_0 = 0 \) (which is satisfied by the Polyakov gauge) and for temporally periodic spatial components of the gauge field \( \vec{A}(\vec{x}, T) = \vec{A}(\vec{x}, 0) \), the Pontryagin index \( \nu[A] \) equals the winding number of the Polyakov loop \( \Omega(\vec{x}) \):
\[ n[\Omega] = 1 \sum_i L_i m_i, \quad L_i = \Omega \partial_i \Omega^\dagger. \]
For this winding number to be well defined, \( \Omega(\vec{x}) \) has to approach at spatial infinity, \( |\vec{x}| \to \infty \), a value independent of the direction \( \hat{x} \), so that the 3-space \( \mathbb{R}^3 \) can be topologically compactified to the 3-sphere \( S_3 \). Without loss of generality we can assume that for \( |\vec{x}| \to \infty \), \( \Omega(\vec{x}) \) approaches a center element \( \Omega(\vec{x}) \to (-1)^{n_0} \) with \( n_0 \) integer. In ref. the following exact relation for the winding number \( n[\Omega] \) has been derived
\[ n[\Omega] = \sum_i \ell_i m_i. \]
Here the summation runs over all magnetic monopoles, $m_i$ being the monopole charges, and the integer

$$\ell_i = n_i - n_0$$

is defined by the center element $\Omega(\bar{x}_i) = (-1)^{n_i}$ which the Polyakov loop acquires at the monopole position $\bar{x}_i$ (cf. eq. (8)), and by the boundary condition to $\Omega(\bar{x})$ specified above. The quantity $\ell_i$ represents the invariant length of the Dirac string in group space, i.e. the distance in group space between the center elements taken by the Polyakov loop at the monopoles connected by the Dirac string.

Eq. (7) shows that in the Polyakov gauge the topology of gauge fields is exclusively determined by the magnetic monopoles. Therefore, magnetic monopoles should be also sufficient to trigger spontaneous breaking of chiral symmetry, which is usually attributed to instantons. Instantons in Abelian gauges give rise to magnetic monopoles, but monopoles can exist in the absence of instantons. The above considerations show that in the Abelian gauges both confinement and spontaneous breaking of chiral symmetry can be generated by the same field configurations: magnetic monopoles.

Although the above considerations have been explicitly carried out in Polyakov gauge, I strongly believe that also in other Abelian gauges the Pontryagin index is given by magnetic charges. In fact, a direct relation between the topological charge and the magnetic monopole charges has been also seen in lattice calculations in the maximum Abelian gauge. These lattice calculations show that the Pontryagin index vanishes when the monopole part of the Abelian gauge field is removed. Finally let me mention that a relation similar to eq. (7) has been subsequently derived in refs. [12,13]. The precise connection between refs. [12,13] and ref. [9] has been established in ref. [14].

4 Center vortices

The center $Z(N)$ of the gauge group is known to play a crucial role for the confinement of charges in the fundamental representation. In the so-called maximum center gauge, the gauge freedom is used to bring the link variables $U_\mu(x)$ as close as possible to a center element of the gauge group. After fixing the coset $SU(N)/Z(N)$, leaving the center symmetry $Z(N)$ untouched, center projection of the links implies for $SU(2)$ replacing the link variables $U_\mu(x)$ by their sign. Thereby vortices arise as strings (in $D = 3$) and sheets (in $D = 4$) of
plaquettes (−1), which are closed by the Bianchi identity. The remarkable lattice result is that center projection reproduces the full string tension, which has been referred to as center dominance and implies vortex dominance. In fact, removing the field configurations which after center projection result in center vortices removes completely the string tension. In this sense center vortices have been interpreted as the confiners of the theory. Furthermore, the center vortex picture gives also a natural explanation of the deconfinement phase transition. It has been also shown, that removal of the vortex configurations destroys the spontaneous breaking of chiral symmetry.

Center dominance without center gauge fixing is trivial. Therefore, the maximum center gauge fixing seems to accumulate the dominant infrared physics on the vortices. To illustrate the effect of the maximum center gauge fixing let us consider the following Abelian vortex field (in cylindric coordinates ρ, ϕ, z)

$$\vec{a} = \frac{1}{\rho} \vec{e}_\varphi T_3,$$

which represents a singular magnetic flux line on the z-axis. (Ignoring the group generator $T_3$ this field represents the gauge potential of a thin solenoid.) Putting this field configuration on the lattice the maximum center gauge concentrates the gauge potential on a sheet of plaquettes (−1) bounded by the vortex at $\rho = 0$ with all other links equal to 1. This configuration is not changed by center projection, while center projection before maximum center gauge fixing would remove the vortex totally. This illustrates the important role of the maximum center gauge fixing before center projection. In the continuum limit the maximal center gauge fixed field configuration corresponding to (9) becomes

$$A = 2\pi \delta(y) \Theta(x),$$

i.e. the gauge potential is concentrated after maximal center gauge fixing on a singular sheet (given here by the right half of the x-z-plane), which is bounded by the vortex. We will refer to this center vortex arising after maximum center gauge fixing as ideal center vortex.

In $D = 4$ the ideal center vortices arising after center projection are given by closed magnetic flux sheets $S = \partial \Sigma$ with all links in the enclosed 3-dimensional volume $\Sigma$ being (−1). In the continuum these ideal vortices are given by

$$A_\mu(k, \Sigma, x) = E(k) \int_{\Sigma} d^3\tilde{\sigma} \delta(x - \tilde{x}(\sigma)),$$
where $\bar{x}_\mu(\sigma)$ denotes a parametrization of the volume $\Sigma$ enclosed by the vortex sheet $S = \partial\Sigma$. Furthermore, $d^3\bar{\sigma}_\mu = \frac{1}{4}\epsilon_{\mu\alpha\beta\gamma}d^3\sigma_{\alpha\beta\gamma}$, $d^3\sigma_{\alpha\beta\gamma}$ being the 3-dimensional volume element and $E(k) = E_a(k)T_a$ denotes a vector of the root lattice of $SU(N)/Z(N)$, which lives in the Cartan subalgebra and whose exponent gives rise to a center element

$$\exp(-E(k)) = Z(k) \in Z(N), \quad k = 0, 1, 2, \ldots, N - 1 .$$

The ideal vortex field $A(k, \Sigma, x)$ indeed contributes the center element $Z(k)$ to each Wilson loop $C$ non-trivially linked to the vortex $S = \partial\Sigma$

$$W[A](C) = \exp \left[ - \oint_C dx_\mu A_\mu(k, \Sigma, x) \right] = \exp \left[ -E(k)I(C, \Sigma) \right] = Z(k)^I(C,\Sigma),$$

where

$$I(C, \Sigma) = \oint_C dx_\mu \int_{\Sigma} d^3\bar{\sigma}_\mu \delta^4(x - \bar{x}(\sigma))$$

is the intersection number between $C$ and $\Sigma$. Performing the Abelian gauge transformation $A_\mu \to A_\mu^{V(k,\Sigma)}$, $V(k, \Sigma) = \exp \left( -E(k)\Omega(\Sigma, x) \right)$, where $\Omega(\Sigma, x)$ is the solid angle in $D = 4$, the ideal vortex $A(k, \Sigma, x)$ is converted into the thin vortex

$$a_\mu(k, \partial\Sigma, x) = A_\mu + E(k)\partial_\mu\Omega = E(k) \int_{\partial\Sigma} d^2\bar{\sigma}_{\mu k} \partial_k D(x - \bar{x}(\sigma)) ,$$

where $-\partial_\mu\partial_\mu D(x - x') = \delta^4(x - x')$. Eq. (15) is the $D = 4$ generalization of eq. (10) for arbitrary vortex shapes $S$. Unlike the ideal center vortex field $A(k, \Sigma, x)$ the thin vortex field $a_\mu(k, \Sigma, x)$ does not depend on the precise shape of the hypersurface $\Sigma$ but depends only on its boundary $S = \partial\Sigma$, i.e. on the flux sheet of the vortex. Since $a_\mu(k, \Sigma, x)$ is gauge equivalent to $A(k, \Sigma, x)$ it yields the same Wilson loop as it is immediately seen by noticing that

$$\oint_C dx_\mu a_\mu(k, \partial\Sigma, x) = E(k)\cdot L(C, \partial\Sigma) ,$$

where $L(C, \partial\Sigma)$ is the linking number between $C$ and $S = \partial\Sigma$, which equals the intersection number $I(C, \Sigma)$.

A careful analysis shows that the continuum analogue of the maximum center gauge is given by the condition

$$-\int \text{tr} \left( A_\mu^0 + a_\mu(k, \partial\Sigma) \right)^2 \to \min ,$$

where
where the minimalization is performed with respect to all coset gauge transformations \( g \in SU(N)/Z(N) \) and with respect to all vortex fields \( a_\mu(k, \partial \Sigma, x) \). Note that the gauge condition depends only on the thin vortex \( a(k, \partial \Sigma) \). For fixed \( a_\mu(k, \partial \Sigma, x) \) minimalization with respect to all gauge transformations \( g \) leads to the background gauge condition

\[
[\partial_\mu - a_\mu(k, \partial \Sigma, x), A_\mu(x)] = 0
\]

with the thin vortex figuring as background field. To arrive at eq. (18) we have used that \( \partial_\mu a_\mu(k, \partial \Sigma, x) = 0 \).

The continuum version (17) shows that the maximum center gauge condition brings a given gauge potential as close as possible to a center vortex field, which we can either represent as a thin vortex \( a_\mu(k, \partial \Sigma, x) \) or as an ideal vortex \( A(k, \Sigma, x) \). The latter is the direct analogue of the center vortices arising on the lattice after maximum center gauge fixing and center projection. There is, however, an important difference between the ideal center vortices in the continuum and on the lattice. Contrary to \( A(k, \Sigma, x) \) the lattice center vortices defined after center projection by 3-dimensional volumes \( \Sigma \) of links \((-1)\) do not distinguish between the direction of the magnetic flux on \( S = \partial \Sigma \), i.e. the lattice vortex sheets \( S = \partial \Sigma \) are not oriented. As a consequence the center projection on the lattice removes topological properties of the vortices related to their orientations, which enter the Pontryagin index.

In the continuum theory the field strength of (thin or ideal) center vortices is given by

\[
F_{\mu\nu}[A] = E(k) \int d^2\bar{\sigma} \delta^4(x - \bar{x}(\sigma)), \quad d^2\bar{\sigma} = \frac{1}{2} \epsilon_{\mu\nu\kappa\lambda} d\sigma_{\kappa\lambda}.
\]

From this expression one finds for the Pontryagin index of a center vortex

\[
\nu[A] = \nu[a] = \frac{1}{4} I(S, S),
\]

where \( I(S, S) \) is the self intersection number of the vortex sheet \( S = \partial \Sigma \). A more detailed analysis shows that the Pontryagin index is indeed integer valued and vanishes unless there are magnetic monopole currents flowing on the vortex sheet. Thus even in the vortex picture the non-trivial topology is generated by magnetic monopole loops in agreement with the findings in the Polyakov gauge given above.

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