A Complex Version of G-Expectation and Its Application to Conformal Martingale

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Abstract

This paper is concerned with the connection between G-Brownian Motion and analytic functions. We introduce the complex version of sublinear expectation, and then do the stochastic analysis in this framework. Furthermore, the conformal G-Brownian Motion is introduced together with a representation, and the corresponding conformal invariance is shown.

Key words: G-expectation, complex sublinear expectation, G-Brownian motion, conformal martingale, conformal invariance

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1 Introduction

G-expectation was first introduced by Peng to handle risk measures, stochastic volatility and model uncertainty (see [4],[5]). The idea of G-expectation lies in the new explanation of independence and distribution. The main difference between G-expectation and linear expectation is, of course, sublinearity, which is mathematically explained by a powerful tool in PDE, viscosity solution theory. A typical G-expectation is constructed on linear space of random variables $L^1_G(\Omega)$, the completion of $\text{Lip}(\Omega)$ of Lipschitz cylinder functions on d-dimensional continuous path space $\Omega = C^0_d[0,\infty)$. The corresponding G-Brownian Motion on this space is the canonical process, i.e. $B_t(\omega) = \omega(t), \omega \in \Omega$. A sublinear expectation $\hat{E}$ on $(\Omega, \text{Lip}(\Omega))$ is defined as a viscosity solution of a heat equation, which is determined by a sublinear monotone function G. Then the random variable space $\text{Lip}(\Omega)$ is extended to completion by the norm $\hat{E}[\cdot]$, and we denote the completion as $L^1_G(\Omega)$. By such construction, a time consistent conditional expectation is introduced naturally on $L^1_G(\Omega)$.

G-expectation theory is thriving in recent years. (see [2],[6],[7]) In classical case, we know there are many interesting connection between Brownian Motion and analytic functions. A fundamental one is Paul Lévy’s theorem, that is, if

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$f(z)$ is analytic nonconstant function, and $B_t$ is a two dimensional Brownian Motion, the process $f(B_t), t \geq 0$ is again Brownian Motion, probably moving at a variable speed. (see [1]) Lévy’s result leads to the possibility to study analytic functions probabilistically. On the other hand, a beauty of linear expectation is that a distribution can be uniquely described by its characteristic function. However, there is little work on G-expectation under complex framework.

In this paper, we give the stochastic analysis of G-expectation in complex case, and furthermore, the conformal G-Brownian Motion and conformal martingales. We find that a special G-Brownian Motion, namely the conformal G-Brownian Motion still holds conformal invariance, i.e. it is still conformal after a transformation by a nonconstant analytic function. We should notice that conformal G-Brownian Motion is much more complicated than the classical case.

This paper is organized as follows. In section 2, we recall some basic results of G-Brownian. In section 3 and 4, we define the sublinear expectation under complex framework and the complex G-normal distribution and G-Brownian Motion. In the fifth part, we do the stochastic analysis in this framework and give Itô’s formula. In the last part, we introduce the conformal martingale and prove conformal invariance.

2 Preliminaries

We start from some basic notes and results of G-expectation theory. More details can be read from [4], [5], [6].

2.1 Sublinear Expectation and G-Expectation

Let $\Omega$ be a given set and $\mathcal{H}$ be a linear space of real valued functions on $\Omega$ containing constants. Furthermore, suppose $|X| \in \mathcal{H}$ if $X \in \mathcal{H}$. The space $\Omega$ is viewed as sample space and $\mathcal{H}$ is the space of random variables.

Definition 2.1. A sublinear expectation $\hat{E}$ is a functional $\hat{E} : \mathcal{H} \to \mathbb{R}$ satisfying

(i) constant preserving:

$$\hat{E}(c) = c, \quad \forall \ c \in \mathbb{R}$$

(ii) positive homogeneity:

$$\hat{E}(\lambda X) = \lambda \hat{E}(X), \quad \lambda \geq 0 \quad X \in \mathcal{H}$$

(iii) constant transferability:

$$\hat{E}(X + c) = \hat{E}(X) + c, \quad c \in \mathbb{R} \quad X \in \mathcal{H}$$

(iv) monotonicity:

$$\hat{E}(X_1) \geq \hat{E}(X_2) \quad \text{if} \ X_1 \geq X_2$$

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The triple $(\Omega, \mathcal{H}, \hat{E})$ is called a sublinear expectation space.

**Definition 2.2.** Let $X_1$ and $X_2$ be two $n$-dimensional random vectors defined respectively in sublinear expectation spaces $(\Omega_1, \mathcal{H}_1, \hat{E}_1)$ and $(\Omega_2, \mathcal{H}_2, \hat{E}_2)$. They are called identically distributed, denoted by $X_1 \overset{d}{=} X_2$, if $\hat{E}_1[\varphi(X_1)] = \hat{E}_2[\varphi(X_2)]$, for all $\varphi \in C_{b,Lip}(\mathbb{R}^n)$.

**Definition 2.3.** In a sublinear expectation space $(\Omega, \mathcal{H}, \hat{E})$, a random vector $Y = (Y_1, \ldots, Y_n)$, $Y_i \in \mathcal{H}$, is said to be independent of another random vector $X = (X_1, \ldots, X_m)$, $X_i \in \mathcal{H}$ under $\hat{E}$, denoted by $Y \perp X$, if for every test function $\varphi \in C_{b,Lip}(\mathbb{R}^m \times \mathbb{R}^n)$ we have $\hat{E}[\varphi(X, Y)] = \hat{E}[\hat{E}[\varphi(x, y)]_{x \in X}]$.

**Definition 2.4.** (maximal distribution) A $d$-dimensional random vector $\eta = (\eta_1, \ldots, \eta_d)$ on a sublinear expectation space $(\Omega, \mathcal{H}, \hat{E})$ is called maximal distributed if $\eta$ satisfies:

$$a\eta + b\bar{\eta} \overset{d}{=} (a + b)\eta, \ a, b \geq 0,$$

where $\bar{\eta}$ is an independent copy of $\eta$, i.e., $\bar{\eta} \overset{d}{=} \eta$ and $\bar{\eta} \perp \eta$.

**Remark 2.5.** By the definition of maximal distribution, we can get that there exists a bounded, closed and convex subset $\Gamma \subset \mathbb{R}^d$ such that for any $\varphi \in C_{b,Lip}(\mathbb{R}^d)$, we have:

$$\hat{E}[\varphi(\eta)] = \max_{y \in \Gamma} \varphi(y).$$

When $d = 1$, we have $\Gamma = [\mu, \bar{\mu}]$, where $\bar{\mu} = \hat{E}[\eta]$ and $\mu = -\hat{E}[-\eta]$. Furthermore, in this case we denote $\eta \overset{d}{=} \mathcal{N}(\{\mu, \bar{\mu}\} \times 0)$.

In the classical case, the maximal distributed random is a constant.

**Definition 2.6.** (G-normal distribution) A $d$-dimensional random vector $X = (X_1, \ldots, X_d)$ in a sublinear expectation space $(\Omega, \mathcal{H}, \hat{E})$ is called G-normally distributed if $\hat{E}[|X|^2] < \infty$ and for each $a, b \geq 0$

$$aX + b\bar{X} \overset{d}{=} \sqrt{a^2 + b^2}X,$$

where $\bar{X}$ is an independent copy of $X$, and

$$G(A) := \frac{1}{2}\hat{E}[(AX, X)] : \mathbb{S}_d \rightarrow \mathbb{R},$$

Here $\mathbb{S}_d$ denotes the collection of $d \times d$ symmetric matrices.

By [6] we know that $X = (X_1, \ldots, X_d)$ is G-normal distributed iff $u(t, x) := \hat{E}[\varphi(x + \sqrt{t}X)]$, $(t, x) \in [0, \infty) \times \mathbb{R}^d$, $\varphi \in C_{b,Lip}(\mathbb{R}^d)$, is the viscosity solution of the following G-heat equation:

$$\partial_t u - G(D_x^2 u) = 0, \ u(0, x) = \varphi(x).$$

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The function $G(\cdot): S_d \to \mathbb{R}$ is a monotonic, sublinear functional on $S_d$, from which we can deduce that there exists a bounded, convex and closed subset $\Sigma \subset S_d^+$ such that

$$G(A) = \frac{1}{2} \sup_{B \in \Sigma} tr[AB],$$

where $S_d^+$ denotes the collection of nonnegative matrixes in $S_d$.

Here is the typical construction of G-expectation from $[5]$. For simplicity, we only consider the one dimensional condition.

Let $\Omega = C_0(\mathbb{R}^+)$ be the space of all $\mathbb{R}$-valued continuous paths $(\omega_t)_{t \in \mathbb{R}^+}$, with $\omega_0 = 0$, equipped with the distance

$$\rho(\omega^1, \omega^2) := \sum_{i=1}^{\infty} 2^{-i} [\max_{t \in [0,i]} |\omega^1_t - \omega^2_t|] \wedge 1.$$ 

and the canonical process is defined by $B_t(\omega) = \omega_t$, $t \in [0, \infty)$, for $\omega \in \Omega$. Then we define

$$L_{ip}(\Omega) := \{ \varphi(B_{t_1}, \ldots, B_{t_n}) : n \geq 1, t_1, \ldots, t_n \in [0, \infty), \varphi \in C_b,L_{ip}(\mathbb{R}^n) \}.$$ 

Give a monotonic and sublinear function $G : S_d \to \mathbb{R}$. The corresponding G-expectation $\hat{E}$ is a sublinear expectation on $L_{ip}(\Omega)$ satisfying:

$$\hat{E}[X] = \hat{E}[\varphi(\sqrt{t_1 - t_2}, \ldots, \sqrt{t_m - t_{m-1}})]$$

for all $X = \varphi(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \ldots, B_{t_m} - B_{t_{m-1}})$ with $0 \leq t_0 < t_1 < \cdots < t_m < \infty$, where $\xi_1, \ldots, \xi_n$ are identically distributed $d$-dimensional $G$-normally distributed random vectors in a sublinear expectation space $(\hat{\Omega}, \hat{\mathcal{H}}, \hat{E})$ such that $\xi_{i+1}$ is independent of $(\xi_1, \ldots, \xi_i)$ for every $i = 1, \ldots, m - 1$.

**Definition 2.7.** Let $\Omega_t = \{ \omega_{\Lambda t} : \omega \in \Omega \}$ for $t \geq 0$. For each $\xi = \varphi(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \ldots, B_{t_m} - B_{t_{m-1}})$, the conditional G-expectation of $\xi$ under $\Omega_t$, is defined by

$$\hat{E}_{t_\cdot}(\varphi(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \ldots, B_{t_m} - B_{t_{m-1}}))$$

$$= \hat{\varphi}(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \ldots, B_{t_i} - B_{t_{i-1}}),$$

where

$$\hat{\varphi}(x_1, \ldots, x_i) = \hat{E}[\varphi(x_1, \ldots, x_i, B_{t_{i+1}} - B_{t_i}, \ldots, B_{t_m} - B_{t_{m-1}})].$$

For each fixed $T \geq 0$, we define

$$L_{ip}(\Omega_T) := \{ \varphi(B_{t_1}, \ldots, B_{t_n}) : n \geq 1, t_1, \ldots, t_n \in [0, T], \varphi \in C_b,L_{ip}(\mathbb{R}^n) \}.$$ 

It is simple that $L_{ip}(\Omega_{T_1}) \subset L_{ip}(\Omega_{T_2}) \subset L_{ip}(\Omega)$ for $T_1 < T_2$. Furthermore, we denote $L_{ip}^p(\Omega)$ and $L_{ip}^p(\Omega_T)$, $p \geq 1$ as the completion of $L_{ip}(\Omega)$ and $L_{ip}(\Omega_T)$.
under the norm \( \| \xi \|_p = (\mathbb{E}[|\xi|^p])^{1/p} \). Consequently, \( L^p_G(\Omega) \subset L^q_G(\Omega) \) for \( p_1 \geq p_2 \geq 1 \).

By the above construction we can get that the G-expectation \( \hat{\mathbb{E}}[\cdot] \) can be continuously extended to a sublinear expectation on \((\Omega, L^1_G(\Omega))\) and it is still denoted by \( \hat{\mathbb{E}}[\cdot] \). For each given \( t \geq 0 \), the conditional G-expectation \( \hat{\mathbb{E}}_t[\cdot] : L^p_G(\Omega_t) \rightarrow L^p_G(\Omega_t) \) can be continuously extended as a mapping \( \hat{\mathbb{E}}_t[\cdot] : L^1_G(\Omega) \rightarrow L^1_G(\Omega_t) \) and satisfies the following properties:

(i) If \( X, Y \in L^1_G(\Omega), X \geq Y, \) then \( \hat{\mathbb{E}}_t[X] \geq \hat{\mathbb{E}}_t[Y] \);
(ii) If \( X \in L^1_G(\Omega_t), Y \in L^1_G(\Omega), \) then \( \hat{\mathbb{E}}_t[X + Y] = X + \hat{\mathbb{E}}_t[Y] \);
(iii) If \( X, Y \in L^1_G(\Omega), \) then \( \hat{\mathbb{E}}_t[X + Y] \leq \hat{\mathbb{E}}_t[X] + \hat{\mathbb{E}}_t[Y] \);
(iv) If \( X \in L^1_G(\Omega_t) \) is bounded, \( Y \in L^1_G(\Omega), \) then \( \hat{\mathbb{E}}_t[XY] = X^+\hat{\mathbb{E}}_t[Y] + X^-\hat{\mathbb{E}}_t[-Y] \);
(v) If \( X \in L^1_G(\Omega), \) then \( \hat{\mathbb{E}}_s[\hat{\mathbb{E}}_t[X]] = \hat{\mathbb{E}}_{s\wedge t}[X], \) in particular, \( \hat{\mathbb{E}}[\hat{\mathbb{E}}_t[X]] = \hat{\mathbb{E}}[X] \).

### 2.2 Stochastic calculus under G-Expectation

For a partition \( \Pi = \{t_0, t_1, \cdots, t_N\} \) of \([0, T]\), we define

\[
\eta_t(\omega) := \sum_{k=0}^{N-1} \xi_k(\omega)I_{[t_k, t_{k+1})}(t)
\]

where \( \xi_k \in L^p_G(\Omega_{t_k}), k = 0, 1, 2, \cdots, N-1 \) are given. We denote these processes by \( M^p_G(0, T) \).

**Definition 2.8.** For any \( \eta \in M^p_0(0, T) \), with the form \( \eta_t(\omega) = \sum_{k=0}^{N-1} \xi_k(\omega)I_{[t_k, t_{k+1})}(t) \), the related Bochner integral on \([0, T]\) is defined:

\[
\int_0^T \eta_t(\omega)dt := \sum_{k=0}^{N-1} \xi_k(\omega)(t_{k+1} - t_k)
\]

We then complete the space \( M^p_0(0, T) \) under norm \( \| \cdot \|_p := (\mathbb{E}[\int_0^T |\cdot|^p dt])^{1/p} \) and denote the completion as \( M^p(0, T) \). It is clear that \( M^p_0(0, T) \subset M^p(0, T) \), if \( 1 \leq q \leq p \). Here is the definition of Itô’s integral.

**Definition 2.9.** For a \( \eta \in M^{2,0}_G(0, T) \) of the form

\[
\eta_t(\omega) = \sum_{k=0}^{N-1} \xi_k(\omega)I_{[t_k, t_{k+1})}(t)
\]

We define the Itô integral as the following operator \( I(\cdot) : M^{2,0}_G(0, T) \rightarrow L^2_G(\Omega_T) \) as following:

\[
I(\eta) = \int_0^T \eta_t dB_t := \sum_{j=0}^{N-1} \xi_j(B_{t_{j+1}} - B_{t_j}),
\]

where \( B_t \) is a G-Brownian Motion, and \( G(\alpha) = \frac{1}{2}(\tilde{\sigma}^2 \alpha^+ + \tilde{\sigma}^2 \alpha^-), 0 \leq \tilde{\sigma}^2 \leq \infty \)
Lemma 2.10. For the mapping $I : M^2_G(0, T) \to L^2_G(\Omega_T)$, we have:

$$\hat{E} \left[ \int_0^T \eta_t dB_t \right] = 0,$$

$$\hat{E} \left[ | \int_0^T \eta_t dB_t |^2 \right] \leq \bar{\sigma}^2 \hat{E} \left[ \int_0^T |\eta_t|^2 dt \right].$$

Thus we can continuously extend $I$ to a mapping from $M^2_G(0, T)$ to $L^2_G(\Omega_T)$, which is also denoted as $I$.

Definition 2.11. For a $\eta \in M^2_G(0, T)$, the stochastic integral is defined as

$$\int_0^T \eta_t dB_t := I(\eta),$$

$$\int_s^t \eta_u dB_u := \int_0^T I_{[s,t]}(u) \eta_u dB_u.$$  

Here is some basic properties of the integral of G-Brownian motion, and the proof is omitted.

Proposition 2.12. For $\eta, \zeta \in M^2_G(0, T)$, and $0 \leq s \leq r \leq t \leq T$, we have:

(i) $\int_s^t \eta_u dB_u = \int_s^r \eta_u dB_u + \int_r^t \eta_u dB_u$.

(ii) $\int_s^t (\alpha \eta_u + \zeta_u) dB_u = \alpha \int_s^t \eta_u dB_u + \int_s^t \zeta_u dB_u$, where $\alpha$ is bounded in $L^1_G(\Omega)$.

(iii) $\hat{E}[X + \int_s^T \eta_u dB_u | \Omega_s] = \hat{E}[X | \Omega_s]$, for $X \in L^1_G(\Omega_T)$

Now we consider the quadratic variation process of G-Brownian motion $(B_t)_{t \geq 0}$ with $B_1 \overset{d}{=} N(\{0\} \times [\sigma^2, \bar{\sigma}^2])$.

For a G-B.M. $B_t$ and a partition $\Pi_N$ of $[0, t]$: $0 = t_0 \leq t_1 \leq \cdots \leq t_N = t$, notice

$$B^2_t = \sum_{j=0}^{N-1} 2B_{t,j}^N(B_{t,j+1}^N - B_{t,j}^N) + \sum_{j=0}^{N-1} (B_{t,j}^N - B_{t,j+1}^N)^2.$$  

As $||\Pi_N|| \to 0$, we can show that

$$\sum_{j=0}^{N-1} 2B_{t,j}^N(B_{t,j+1}^N - B_{t,j}^N) \overset{L^2_G(\Omega)}{\to} 2 \int_0^T B_u dB_u,$$

so by the completeness of $L^2_G(\Omega)$, $\sum_{j=0}^{N-1} (B_{t,j}^N - B_{t,j+1}^N)^2$ must also converge in $L^2_G(\Omega)$.

Definition 2.13. By the argument above, we define

$$\langle B \rangle_t := \lim_{||\Pi_N|| \to 0} \sum_{j=0}^{N-1} (B_{t,j}^N - B_{t,j+1}^N)^2$$

$$= B^2_t - 2 \int_0^t B_u dB_u,$$

and call $\langle B \rangle$ the quadratic variation process of G-Brownian Motion.
We now define the integral of a process \( \eta \in M_G^1(0, T) \) with respect to \( \langle B \rangle \).

Firstly, we define a mapping:

\[
Q_{0,T}(\eta) = \int_0^T \eta_t \langle \langle B \rangle \rangle_t := \sum_{j=0}^{N-1} \xi_j (\langle B \rangle_{t_{i+1}} - \langle B \rangle_{t_i}) : M_G^{1,0}(0, T) \to L^2_G(\Omega_T). 
\]

Furthermore, we have the following lemma, which allows us to extend this mapping to \( M_G^1(0, T) \).

**Lemma 2.14.** For each \( \eta \in M_G^{1,0}(0, T) \),

\[
\hat{E}[\int_0^T \eta_t \langle \langle B \rangle \rangle_t] \leq \sigma^2 \hat{E}[\int_0^T |\eta_t| dt].
\]

Here two properties w.r.t \( \langle B \rangle \) and its integral which would be used in the next part. The proof can be found in [6].

**Proposition 2.15.** (i) For \( \eta \in M_G^2(0, T) \), we have

\[
\hat{E}[\int_0^T \eta_t dB_t^2] = \hat{E}[\int_0^T \eta_t^2 dB_t].
\]

(ii) For each fixed \( s, t \geq 0 \), \( \langle B \rangle_{t+s} - \langle B \rangle_s \) is identically distributed with \( \langle B \rangle_t \) and independent from \( \Omega_s \). In addition, \( \langle B \rangle_t \) is \( N(\sigma^2 t, \sigma^2 t) \times 0 \) - distributed.

For the multi-dimensional case. Suppose \( \langle B \rangle \), \( \Omega \) to be a d-dimensional G-Brownian motion. For each \( a \in \mathbb{R}^d \), we can easily check that \( \langle B^a \rangle_{t \geq 0} := \langle \langle B^a \rangle \rangle_{t \geq 0} \) is a 1-dimensional Gd-Brownian Motion, with \( Gd(\alpha) = \frac{1}{2}\sigma^{2}_{aa}t \alpha^+ - \sigma^{2}_{-aa}t \alpha^- \), where \( \sigma^{2}_{aa} = 2G(aaT) \) and \( \sigma^{2}_{-aa} = -2G(aaT) \). Then the related integrals with respect to \( B^a_t \) and \( \langle B^a \rangle_t \) are same as the one dimensional case. Furthermore, for any \( a, \bar{a} \in \mathbb{R}^d \), we define the mutual variation process by

\[
\langle B^a, B^\bar{a} \rangle_t := \frac{1}{4} \langle \langle B^a + B^\bar{a} \rangle \rangle_t - \langle B^a - B^\bar{a} \rangle_t = \frac{1}{4} \langle \langle B^a+ B^\bar{a} \rangle \rangle_t - \langle B^{a-\bar{a}} \rangle_t.
\]

We can prove that for a sequence of partition \( \pi^n_t, n = 1, 2, \ldots \) of \( [0, t] \) with \( ||\pi^n_t|| \to 0 \), we have

\[
\lim_{n \to \infty} \sum_{k=0}^{n-1} (B^a_{t_{k+1}} - B^a_{t_k})(B^\bar{a}_{t_{k+1}} - B^\bar{a}_{t_k}) = \langle B^a, B^\bar{a} \rangle_t.
\]

Now we give the famous Itô’s formula in G-framework. More recent progresses can be read from [3].

**Theorem 2.16.** (G-Itô’s Formula) Let \( \Phi \) be a twice continuous function on \( \mathbb{R}^n \) with polynomial growth growth for the first and second order derivatives. \( X \) is a Itô process, i.e.,

\[
X^\nu_t = X^\nu_0 + \int_0^t \alpha^\nu_s ds + \int_0^t \eta^\nu_i d\langle B^i \rangle + \int_0^t \beta^\nu_j dB^j_s.
\]

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where \( \nu = 1, \ldots, n \), \( i, j = 1, \ldots, d \), \( \alpha^\nu_{ij}, \beta^\nu_{ij} \) are bounded processes in \( M^2_G(0, T) \).
Here repeated indices means summation over the same indices. Then for each \( t \geq s \geq 0 \) we have in \( L^2_G(\Omega_t) \):

\[
\Phi(X_t) - \Phi(X_s) = \int_s^t \partial_{x^\nu} \Phi(X_u) \beta^\nu_{ij} d\langle B^i, B^j \rangle_u + \int_s^t \partial_{x^\nu} \Phi(X_u) \alpha^\nu_{ij} du + \int_s^t \left[ \partial_{x^\nu} \Phi(X_u) \eta^\nu_{ij} + \frac{1}{2} \partial^2_{x^\nu x^\nu} \Phi(X_u) \beta^\nu_{ij} \beta^\nu_{ij} \right] d\langle B^i, B^j \rangle_u
\]

3 Complex Sublinear Expectation

Here we try to define a sublinear expectation under complex case, which means we have to decide what’s “sublinear” here. From here on, when we compare two complex numbers, we compare them as real vectors. What we do here is to connect the G-expectation with complex analysis, so we define complex sublinear expectation in the following heuristic way.

Given a set \( \Omega \) and a linear space \( \mathcal{H} \) consisting of real valued functions on \( \Omega \), we suppose \( \mathcal{H} \) is a vector lattice and consider the following set of functions on \( \Omega \):

\[
\mathcal{H}_C = \{ X + iY | X, Y \in \mathcal{H} \}, \text{ where } i^2 = -1
\]

**Definition 3.1.** Suppose \( (\Omega, \mathcal{H}, \mathbb{E}) \) is a sublinear space, we define a function \( \mathbb{E}_C \) from \( \mathcal{H}_C \) to complex field as:

\[
\mathbb{E}_C : \mathcal{H}_C \mapsto \mathbb{C}
\]

\[
X + iY \mapsto \mathbb{E}_C(X) + i\mathbb{E}_C(Y)
\]

**Remark 3.2.** Notice that for any \( Z \in \mathcal{H}_C \), \( Z \) has a unique expression as \( X + iY \), so \( \mathbb{E}_C \) is well defined.

Here are some basic properties of \( \mathbb{E}_C \):

(i) constant preserving:

\[
\mathbb{E}_C(z) = z, \quad \forall \ z \in \mathbb{C}
\]

(ii) positive homogeneity:

\[
\mathbb{E}_C(\lambda Z) = \lambda \mathbb{E}_C(Z), \quad \lambda \geq 0 \quad Z \in \mathcal{H}_C
\]

(iii) constant transferability:

\[
\mathbb{E}_C(Z + c) = \mathbb{E}_C(Z) + c, \quad \forall \ c \in \mathbb{C} \quad Z \in \mathcal{H}_C
\]

(iv) monotonicity:

\[
\mathbb{E}_C(Z_1) \geq \mathbb{E}_C(Z_2) \quad \text{if } Z_1 \geq Z_2
\]
(v) convexity:

\[ \hat{E}_C[\alpha Z_1 + (1 - \alpha)Z_2] \leq \alpha \hat{E}_C(Z_1) + (1 - \alpha)\hat{E}_C(Z_2), \quad \alpha \in [0, 1] \]

**Remark 3.3.** Conversely, given some properties of the complex expectation, we can define the sublinear expectation in a more general version:

\[ \hat{E}_C : \mathcal{H}_C \to \mathbb{C} \]

satisfying

\[ \hat{E}_C[X] \in \mathbb{R} \quad \text{for any } X \in \mathcal{H} \]

and

(i) \( \hat{E}_C(c) = c, \forall c \in \mathbb{C} \)
(ii) \( \hat{E}_C(Z_1) \geq \hat{E}_C(Z_2), Z_1 \geq Z_2, Z_1, Z_2 \in \mathcal{H}_C \)
(iii) \( \hat{E}_C(Z_1 + Z_2) \leq \hat{E}_C(Z_1) + \hat{E}_C(Z_2) \)
(iv) \( \hat{E}_C(\lambda Z) = \lambda \hat{E}_C(Z), \lambda \geq 0, \lambda \in \mathbb{R} \)
(v) \( \hat{E}_C(Z) \leq \hat{E}_C(X) + \hat{E}_C(Y), Z = X + iY \)

If (v) in the above is changed to an equation, we get the same sublinear expectation as definition (3.1) by the representation theorem in the real case.

**Remark 3.4.** For each \( \varphi \in C_{l.lip}(\mathbb{C}) \), we suppose \( \varphi(x + iy) = \varphi_1(x, y) + i\varphi_2(x, y) \). Then we have \( \varphi_i(x, y) \in C_{l.lip}(\mathbb{R}^2) \). Also, if \( \varphi_i(x, y) \in C_{l.lip}(\mathbb{R}^2), i = 1, 2 \), we have \( \varphi(x + iy)(:= \varphi_1(x, y) + i\varphi_2(x, y)) \) belongs to \( C_{l.lip}(\mathbb{C}) \).

**Definition 3.5.** Suppose \( Z = (Z_1, \cdots, Z_n) \) a n-dimensional vector on the complex sublinear space \((\Omega, \mathcal{H}_C, \hat{E}_C)\). The functional from \( C_{l.lip}(\mathbb{C}^n) \) to \( \mathbb{C} \) defined by

\[ F_Z(\varphi) := \hat{E}_C[\varphi(Z)] = E[\varphi_1(X, Y)] + E[\varphi_2(X, Y)], \quad (3) \]

where \( \varphi \in C_{l.lip}(\mathbb{C}^n) \) and \( \varphi(x + iy) := \varphi_1(x, y) + i\varphi_2(x, y) \), is called the distribution of \( Z \) under \( \hat{E}_C \). Also, the triple \( (\mathbb{C}^n, C_{l.lip}(\mathbb{C}^n), F_Z) \) forms a complex sublinear space.

**Definition 3.6.** Two n-dimensional random vectors \( Z_1 \) and \( Z_2 \) defined on two complex sublinear expectation spaces \((\Omega_1, \mathcal{H}_C^1, \hat{E}_C^1) \) and \((\Omega_2, \mathcal{H}_C^2, \hat{E}_C^2) \) respectively, are called identically distributed if

\[ \hat{E}_C^1[\varphi(Z_1)] = \hat{E}_C^2[\varphi(Z_2)], \quad \text{for any } \varphi \in C_{l.lip}(\mathbb{C}^n). \]

Such relation is denoted by \( Z_1 \overset{d}{=} Z_2 \).
Proposition 3.7. Let \((\Omega, \mathcal{H}_C, \hat{\mathbb{E}}_C)\) be a complex sublinear expectation. \(Z_1, Z_2\) are two random variables and \(Z_2\) satisfies
\[
\hat{\mathbb{E}}_C[Z_2] = -\hat{\mathbb{E}}_C[-Z_2]
\]
Then we have
\[
\hat{\mathbb{E}}_C[Z_1 + cZ_2] = \hat{\mathbb{E}}_C[Z_1] + c \hat{\mathbb{E}}_C[Z_2], \quad \text{for any } c \in \mathbb{C}.
\]
Proof. Suppose \(Z_1 = X_1 + iY_1, \ Z_2 = X_2 + iY_2, \ c = x + iy, \ where \ X_i, Y_i \in \mathcal{H}, \ and \ x, y \in \mathbb{R}. \) Since \(\hat{\mathbb{E}}_C[Z_2] = -\hat{\mathbb{E}}_C[-Z_2], \) we have
\[
E[X_2] = -E[-X_2], \quad E[Y_2] = -E[-Y_2]
\]
This leads to
\[
\hat{E}[X + aX_2] = \hat{E}[X] + a \hat{E}[X_2], \quad \text{for } a \in \mathbb{R}, \ X \in \mathcal{H}
\]
and similar result for \(Y_2.\)

\[
\hat{\mathbb{E}}_C[Z_1 + cZ_2] = \hat{\mathbb{E}}_C[(X_1 + xX_2 - yY_2) + i(Y_1 + xY_2 + yX_2)]
\]
\[
= \hat{E}[X_1] + x\hat{E}[X_2] - y\hat{E}[Y_2] + i\hat{E}[Y_1] + ix\hat{E}[Y_2] + iy\hat{E}[X_2]
\]
\[
= \hat{\mathbb{E}}_C[Z_1] + c\hat{\mathbb{E}}_C[Z_2]
\]

Another important definition in sublinear expectation is the independence. Since our main purpose is to connect real sublinear expectation with complex analysis, we find the following definition does this job.

Definition 3.8. In a complex sublinear space \((\Omega, \mathcal{H}_C, \hat{\mathbb{E}}_C)\), a random vector \(Z_2 \in \mathcal{H}_C^n\) is said to be independent from another one \(Z_1 \in \mathcal{H}_C^m\) if for each test function \(\varphi \in C_{l.h.p}(\mathbb{C}^{m+n})\) we have
\[
\hat{\mathbb{E}}_C[\varphi(Z_1, Z_2)] = \hat{\mathbb{E}}_C[\hat{\mathbb{E}}_C[\varphi(z, Z_2)]_{z=Z_1}]
\]
Here is an important lemma to connect the complex case with the real one.

Lemma 3.9. Let \(Z_1 = X_1 + iY_1, \ Z_2 = X_2 + iY_2, \ where \ X_1, Y_1 \in \mathcal{H}^n, \ X_2, Y_2 \in \mathcal{H}^m. \) Then \(Z_2\) is independent of \(Z_1\) if and only if \((X_2, Y_2)\) is independent of \((X_1, Y_1)\) under \(\hat{E}.\)

Proof. If \(Z_2\) is independent of \(Z_1,\) we need to show that for any \(\varphi \in C_{l.h.p}(\mathbb{R}^{2n+2m})\) we have
\[
\hat{E}[\varphi(X_1, Y_1, X_2, Y_2)] = \hat{E}[\hat{E}[\varphi(x, y, X_2, Y_2)]_{(x,y)=(X_1, Y_1)}]
\]
For any fixed \(\varphi \in C_{l.h.p}(\mathbb{R}^{2n+2m}), \) we take \(\psi := \varphi + i \cdot 0 = \varphi, \) which belongs to \(C_{l.h.p}(\mathbb{C}^{n+m}).\)
By the definition of independence under complex case, we have
\[
\hat{\mathbb{E}}[\varphi(X_1, Y_1, X_2, Y_2)] = \hat{\mathbb{E}}_C[\psi(Z_1, Z_2)] = \hat{\mathbb{E}}_C[\hat{\mathbb{E}}_C[\psi(z, Z_2)]_{z = Z_1}]
\]
\[
= \hat{\mathbb{E}}_C[\hat{\mathbb{E}}[\varphi(x, y, X_2, Y_2)]_{(x, y) = (X_1, Y_1)}] = \hat{\mathbb{E}}[\hat{\mathbb{E}}[\varphi(x, y, X_2, Y_2)]_{(x, y) = (X_1, Y_1)}]
\]

For the backward implication, if \(\varphi \in C_{t, l, p}(\mathbb{R}^{n+2m})\), we suppose \(\varphi = \varphi_1 + i\varphi_2\), where \(\varphi_1, \varphi_2 \in C_{t, l, p}(\mathbb{R}^{2n+2m})\)
\[
\hat{\mathbb{E}}_C[\varphi(Z_1, Z_2)] = \hat{\mathbb{E}}_C[\varphi_1(X_1, Y_1, X_2, Y_2)] + i\hat{\mathbb{E}}_C[\varphi_2(X_1, Y_1, X_2, Y_2)]
\]
\[
= \hat{\mathbb{E}}[\hat{\mathbb{E}}[\varphi_1(x, y, X_2, Y_2)]_{(x, y) = (X_1, Y_1)}] + i\hat{\mathbb{E}}[\hat{\mathbb{E}}[\varphi_2(x, y, X_2, Y_2)]_{(x, y) = (X_1, Y_1)}]
\]
\[
= \hat{\mathbb{E}}_C[\hat{\mathbb{E}}_C[\varphi(z, Z_2)]_{z = Z_1}]
\]

Remark 3.10. (Completion of Complex Sublinear Expectation Space) For a real sublinear expectation space \((\Omega, \mathcal{H}, \hat{\mathbb{E}})\), we denote \((\Omega, \mathcal{H}_p, \hat{\mathbb{E}})\) the completion of the real sublinear expectation space under norm \((\hat{\mathbb{E}}[\cdot]_p)^{1/p}\). Then we define \(\mathcal{H}_p = \{X + iY \mid X, Y \in \mathcal{H}_p\}\), and it is also a banach space under norm \((\hat{\mathbb{E}}[\cdot]_p)^{1/p}\) by the completeness of \(\mathcal{H}_p\).

4 Complex G-normal distribution and G-Brownian Motion

Here we define the complex G-normal distribution and G-Brownian Motion.

Definition 4.1. (Complex G-normal distribution) A \(d\)-dimensional random variable \(Z = (Z_1, \cdots, Z_d)\) on a complex sublinear expectation space \((\Omega, \mathcal{H}_C, \hat{\mathbb{E}}_C)\) is called complex G-normal distributed if
\[
aZ + b\hat{Z} \overset{d}{=} \sqrt{a^2 + b^2}Z, \text{ for } a, b \geq 0
\]
where \(\hat{Z}\) is an independent copy of \(Z\).

Example 4.2. Suppose \(Z\) is a complex G-normal distributed variable. By the definition of the complex G-normal distribution, \(e^{i\theta}Z\) is also normal distributed, which means a change of the argument of a complex G-normal distributed variable is also G-normal distributed.

Remark 4.3. By the definition of identically distributed and lemma(3.9), \(Z = X + iY\) is complex G-normal distributed if and only if \((X, Y)\) is G-normal distributed in \((\Omega, \mathcal{H}, \hat{\mathbb{E}})\)
Remark 4.4. If we define $\omega(t,z) := \hat{\mathbb{E}}_C[\varphi(z + \sqrt{t}Z)]$, where $\varphi \in C_{b.lip}(\mathbb{C}^d)$, $Z \in \mathcal{H}_C^d$ and $Z$ is complex $G$-normal distributed, we have

$$
\omega(t+s,z) = \hat{\mathbb{E}}_C[\varphi(z + \sqrt{t} + \sqrt{s}Z + \sqrt{t}Z)] = \hat{\mathbb{E}}_C[\varphi(z + \sqrt{s}c + \sqrt{t}Z)_{c=Z}] = \hat{\mathbb{E}}_C[\omega(t,z + \sqrt{s}Z)]
$$

If $\omega(t,z)$ is continuously differentiable on $t$, complex differentiable on $z$, and has at most polynomially growth at infinity, thanks to proposition (3.7), we could get a $G_C$ heat equation:

$$
\partial_t \omega(t,z) - G_C(D_2 \omega(t,z)) = 0
$$

where $G_C(c) = \frac{1}{2} \hat{\mathbb{E}}_C[cZ^2]$. Here we should notice that $G$, which has 4 parameters, could determine $G_C$, a 3-parameter function and the converse is not usually true. However, there are conditions under which $G$ and $G_C$ are mutually determined, such as $Z = X + i\hat{X}$, where $X$ is the independent copy of the $G$ normal distributed variable $X$.

Now we can turn to stochastic analysis. Let’s start with complex G-Brownian Motion.

**Definition 4.5.** complex G-Brownian Motion A $d$-dimensional process $(B_t)_{t \geq 0}$ is called a complex G-Brownian Motion if the following are satisfied:

(i) $B_0(\omega) = 0$

(ii) For each $s, t \geq 0$, the increment $B_{t+s} - B_t$ is complex $G$-normal distributed and is independent from $(B_{t_1}, B_{t_2}, \cdots, B_{t_n})$, for any $n \in \mathbb{N}$ and $0 \leq t_1 \leq t_2 \leq \cdots \leq t_n \leq t$.

5 Stochastic Integral and Related Stochastic Calculus

**Definition 5.1.** For each $T \in [0, \infty)$, we set Lipschitz cylinder functions as:

$$
\text{Lip}_C(\Omega_T) := \{\varphi(B_{t_1 \wedge T}, B_{t_2 \wedge T}, \cdots, B_{t_n \wedge T}) : n \in \mathbb{N}, t_1, \cdots, t_n \in [0, \infty), \varphi \in C_{l.lip}(\mathbb{C}^{d \times n})\}
$$

where $(B_t)_{t \geq 0}$ is a complex G-Brownian Motion. It is clear that $\text{Lip}_C(\Omega_T) \subseteq \text{Lip}_C(\Omega_T)$ for $t \leq T$. Also, we let $\text{Lip}_C(\Omega) := \bigcup_{n=1}^{\infty} \text{Lip}_C(\Omega_n)$. Then we have a complex sublinear expectation called complex $G$-expectation as:

$$
\hat{\mathbb{E}}_C[\cdot] : \text{Lip}_C(\Omega) \rightarrow \mathbb{C}.
$$

**Remark 5.2.** The corresponding conditional expectation also explains itself as the real case.

To complete $\text{Lip}_C(\Omega_T)$, we need a small lemma to define a norm on it.
Lemma 5.3. \((\Omega, \mathcal{H}, \hat{E})\) is a real sublinear space. Then \(\| \cdot \| := (\hat{E}, \cdot \| p \|^p)^\frac{1}{p}, p > 1\), defines a norm on \((\Omega, \mathcal{H}_C, \hat{E}_C)\).

Proof. The proof is similar as the classical one, so we omit it. \(\square\)

Remark 5.4. We define a norm \(\hat{E}[\cdot \| p]^{\frac{1}{p}}\) on \(\text{Lip}_C(\Omega)\) and denote \(L^p_G(\Omega)_C\) as the completion of linear space \(\text{Lip}_C(\Omega)\). Also, \(L^p_G(\Omega_T)_C\) explains itself. Then we have the following.

Proposition 5.5. The completion \(L^p_G(\Omega_T)_C\) of \(\text{Lip}_C(\Omega_T)\) under norm \(\hat{E}[\cdot \| p]^{\frac{1}{p}}\) have the following expression:

\[
L^p_G(\Omega_T)_C = \{\xi_1 + i\xi_2 | \xi_1, \xi_2 \in L^p_G(\Omega_T)\}
\]

under norm \(\hat{E}[\cdot \| p]^{\frac{1}{p}}\), where \(L^p_G(\Omega_T)\) is the completion of

\[
\text{Lip}(\Omega_T) := \{\varphi(B^{(1)}_{t_1 \wedge T}, B^{(1)}_{t_2 \wedge T}, \ldots, B^{(2)}_{t_n \wedge T}, \ldots, B^{(2)}_{t_n \wedge T})| n \in \mathbb{N}, \varphi \in C_{\text{l.lip}}(\mathbb{R}^{d \times 2n})\}
\]

and \(B_i = B^{(1)}_i + iB^{(2)}_i\) is the complex \(G\)-Brownian Motion.

Proof. For any \(\xi_1, \xi_2 \in \text{Lip}(\Omega_T)\), without loss of generality, we can suppose they have the same form as \(\xi_i = \varphi_i(B^{(1)}_{t_1 \wedge T}, B^{(1)}_{t_2 \wedge T}, \ldots, B^{(2)}_{t_1 \wedge T}, \ldots, B^{(2)}_{t_n \wedge T})\), where \(\varphi_i \in C_{\text{l.lip}}(\mathbb{R}^{d \times 2n})\), \(i = 1, 2\). Then we can take \(\varphi = \varphi_1 + i\varphi_2\) and conclude that \(\xi_1 + i\xi_2 \in L^p_G(\Omega_T)_C\). The forward inclusion follows from facts (i) \(\text{Lip}(\Omega_T)\) is dense in \(L^p_G(\Omega_T)\) and (ii) \(L^p_G(\Omega_T)_C\) is complete.

For the converse inclusion, notice that for any \(\varphi \in C_{\text{l.lip}}(\mathbb{C}^{d \times n})\), \(\varphi\) has a unique decomposition \(\varphi = \varphi_1 + i\varphi_2\) where \(\varphi_1, \varphi_2 \in C_{\text{l.lip}}(\mathbb{R}^{2d \times n})\). It means \(\text{Lip}_C(\Omega_T) \subseteq \{\xi_1 + i\xi_2 | \xi_1, \xi_2 \in L^p_G(\Omega_T)_C\}\) by the completeness of \(\{\xi_1 + i\xi_2 | \xi_1, \xi_2 \in L^p_G(\Omega_T)\}\). \(\square\)

Now we can define the å’s integral by starting from simple processes as the following: for a partition \(\Pi = \{t_0, t_1, \ldots, t_N\}\) of \([0, T]\), we define

\[
\eta_k(\omega) := \sum_{k=0}^{N-1} \xi_k(\omega) I_{(t_k, t_{k+1})}(t)
\]

where \(\xi_k \in L^p_G(\Omega_{t_k})_C\), \(k = 0, 1, 2, \ldots, N-1\) are given. We denote these processes by \(M^{p,0}_G(0, T)_C\).

Definition 5.6. For any \(\eta \in M^{p,0}_G(0, T)_C\), with the form \(\eta(\omega) = \sum_{k=0}^{N-1} \xi_k(\omega) I_{(t_k, t_{k+1})}(t)\), the related Bochner integral on \([0, T]\) is defined as natural:

\[
\int_0^T \eta(\omega) dt := \sum_{k=0}^{N-1} \xi_k(\omega)(t_{k+1} - t_k)
\]
Then we can complete the space $M^p_G(0,T)_C$ under norm

$$\| \cdot \|_p^M := \{ \hat{E}[\int^T_0 |\cdot|^p dt] \}^{\frac{1}{p}}$$

and denote the completion as $M^p_G(0,T)_C$. Similar as $L^p_G(\Omega_T)_C$, $M^p_G(0,T)_C$ has the expression:

$$M^p_G(0,T)_C = \{ \eta^{(1)}_t + i\eta^{(2)}_t \mid \eta^{(1)}_t, \eta^{(2)}_t \in M^p_G(0,T) \}$$

where $M^p_G(0,T)$ is the corresponding space in real case.

**Definition 5.7.** For a $\eta \in M^{2,0}_G(0,T)_C$ of the form

$$\eta_t(\omega) = \sum_{k=0}^{N-1} \xi_k(\omega) I_{[t_k, t_{k+1}]}(t)$$

We define the Itô integral as the following operator $I(\cdot) : M^{2,0}_G(0,T)_C \mapsto L^2_G(\Omega_T)_C$ as following:

$$I(\eta) = \int_0^T \eta_t dB_t := \sum_{j=0}^{N-1} \xi_j (B_{t_{j+1}} - B_{t_j}).$$

where $B_t = B^{(1)}_t + iB^{(2)}_t$ is a Brownian Motion.

**Proposition 5.8.** For the mapping $I : M^{2,0}_G(0,T)_C \mapsto L^2_G(\Omega_T)_C$, we have:

1. $\hat{E}_C[\int_0^T \eta_t dB_t] = 0$, (4)
2. $\hat{E}[|\int_0^T \eta_t dB_t|^2] \leq 16K \hat{E}[\int_0^T |\eta_t|^2 dt]$, (5)

where $K = (\bar{\sigma}^2_1 + \bar{\sigma}^2_2)$, $\bar{\sigma}^2_1 := \hat{E}[(B^{(1)}_t)^2]$, $\bar{\sigma}^2_2 := \hat{E}[(B^{(2)}_t)^2]$.

**Proof.** We denote

$$\Delta B_j = B_{t_{j+1}} - B_{t_j},
\Delta B^{(i)}_{j} = B^{(i)}_{t_{j+1}} - B^{(i)}_{t_j}, i = 1, 2$$

For (4), we notice $\hat{E}_C[\int_0^T \eta_t dB_t] = \hat{E}_C[\sum_{j=0}^{N-1} \xi_j \Delta B_j]$, and

$$\hat{E}_C[\xi_j \Delta B_j] = \hat{E}_C[(\xi^{(1)}_j + i\xi^{(2)}_j)(\Delta B^{(1)}_j + i\Delta B^{(2)}_j)]$$

$$= \hat{E}[\xi^{(1)}_j \Delta B^{(1)}_j - \xi^{(2)}_j \Delta B^{(2)}_j] + i\hat{E}[\xi^{(1)}_j \Delta B^{(2)}_j + \xi^{(2)}_j \Delta B^{(1)}_j].$$

Also, we have $\hat{E}[\xi^{(i)}_j \Delta B^{(k)}_j] = 0, i, k = 1, 2$. By proposition (3.7), these equations imply the conclusion.
For (5), firstly notice

\[\hat{E}[\int_0^T \eta_t dB_t^2] = \hat{E}[\int_0^T \sum_{j=0}^{N-1} [i(\xi_j^{(1)} \Delta B_j^{(1)} - \xi_j^{(2)} \Delta B_j^{(2)}) + i(\xi_j^{(1)} \Delta B_j^{(2)} + \xi_j^{(2)} \Delta B_j^{(1)})] dt]\]

\[\leq \hat{E}[(\sum_{j=0}^{N-1} (\xi_j^{(1)} \Delta B_j^{(1)} - \xi_j^{(2)} \Delta B_j^{(2)})^2] + \hat{E}[(\sum_{j=0}^{N-1} (\xi_j^{(1)} \Delta B_j^{(2)} + \xi_j^{(2)} \Delta B_j^{(1)})^2] \]

\[\leq 2\hat{E}[(\sum_{j=0}^{N-1} (\xi_j^{(1)} \Delta B_j^{(1)}))^2) + \hat{E}((\sum_{j=0}^{N-1} (\xi_j^{(2)} \Delta B_j^{(2)}))^2] + \hat{E}((\sum_{j=0}^{N-1} (\xi_j^{(1)} \Delta B_j^{(2)}))^2] \]

Also, we have

\[\hat{E}[\xi_j^{(1)} B_j^{(k)} \xi^{(m)}_l \Delta B_l^{(n)}] = \hat{E}[\xi_j^{(1)} B_j^{(k)} \xi^{(m)}_l \hat{E}[\Delta B_l^{(n)} | \Omega_t]] = 0,\]

where \(t \geq j, and 1, k, m, n = 1, 2,\)

and

\[\int_0^T |\eta_t|^2 dt = \sum_{j=0}^{N-1} ((\xi_j^{(1)})^2 + (\xi_j^{(2)})^2)(t_{j+1} - t_j).\]

We only need to show

\[\hat{E}[\sum_{j=0}^{N-1} (\xi_j^{(1)} \Delta B_j^{(1)})^2] \leq 2K \hat{E}[(\sum_{j=0}^{N-1} ((\xi_j^{(1)})^2 + (\xi_j^{(2)})^2)(t_{j+1} - t_j),\]

and the others are similar.

Notice that

\[\hat{E}[(\xi_j^{(1)})^2 (\Delta B_j^{(1)})^2 + (\xi_j^{(1)})^2 (\Delta B_j^{(1)})^2] = \hat{E}[(\xi_j^{(1)})^2 (\Delta B_j^{(1)})^2]^2 + \hat{E}[(\xi_j^{(1)})^2 (\Delta B_j^{(1)})^2]^2\]

\[= \hat{E}[(\xi_j^{(1)})^2 (\Delta B_j^{(1)})^2 + \sigma_j^2 (\xi_j^{(1)})^2(t_{j+2} - t_j + 1)]\]

\[\leq \hat{E}[(\xi_j^{(1)})^2 (\Delta B_j^{(1)})^2 | \Omega_t] + \hat{E}[\sigma_j^2 (\xi_j^{(1)})^2(t_{j+2} - t_j + 1)]\]

\[= \hat{E}[(\xi_j^{(1)})^2 (\Delta B_j^{(1)})^2 | \Omega_t] + \hat{E}[\sigma_j^2 (\xi_j^{(1)})^2(t_{j+2} - t_j + 1)]\]

\[= \hat{E}[(\xi_j^{(1)})^2 (\Delta B_j^{(1)})^2 | \Omega_t] + \hat{E}[\sigma_j^2 (\xi_j^{(1)})^2(t_{j+2} - t_j + 1)].\]
Then we have
\[
\mathbb{E} \left[ \sum_{j=0}^{N-1} (\xi_j^{(1)} \triangle B_j^{(1)})^2 \right] \leq 2 \mathbb{E} \left[ \sum_{j=0}^{N-1} (\xi_j^{(1)})^2 (\triangle B_j^{(1)})^2 \right] + 2 \mathbb{E} \left[ \sum_{j=0}^{N-1} (\xi_j^{(2)})^2 (\triangle B_j^{(2)})^2 \right]
\]
\[
\leq 2 \sigma_2^2 \mathbb{E} \left[ \sum_{j=0}^{N-1} \left[ (\xi_j^{(1)})^2 + (\xi_j^{(2)})^2 \right] (t_{j+1} - t_j) \right]
\]
\[
+ 2 \sigma_2^2 \mathbb{E} \left[ \sum_{j=0}^{N-1} \left[ (\xi_j^{(1)})^2 + (\xi_j^{(2)})^2 \right] (t_{j+1} - t_j) \right]
\]
\[
\leq 2 K \mathbb{E} \left[ \sum_{j=0}^{N-1} \left[ (\xi_j^{(1)})^2 + (\xi_j^{(2)})^2 \right] (t_{j+1} - t_j) \right]
\]
\[
\]
\[
\]

So we can extend the stochastic integral mapping \( I \) from \( M_{G}^{2,0}(0,T) \) to \( M_{G}^{2}(0,T) \).

**Definition 5.9.** For a \( \eta \) in \( M_{G}^{2}(0,T) \), the stochastic integral is defined as
\[
\int_0^T \eta_t dB_t := I(\eta),
\]
\[
\int_s^t \eta_u dB_u := \int_0^T I_{[s,t]}(u) \eta_u dB_u,
\]
where \( I \) is the extension of the former integral mapping from \( M_{G}^{2}(0,T) \) to \( L_{G}^{2}(\Omega_T) \) by Hahn-Banach extension theorem.

Naturally, we have the following basic properties. The proof is trivial.

**Proposition 5.10.** For \( \eta, \zeta \in M_{G}^{2}(0,T) \), and \( 0 \leq s \leq r \leq t \leq T \), we have:
(i) \( \int_s^r \eta_u dB_u = \int_r^s \eta_u dB_u + \int_s^r \eta_u dB_u \).
(ii) \( \int_s^r (\alpha \eta_u + \zeta_u) dB_u = \alpha \int_s^r \eta_u dB_u + \int_s^r \zeta_u dB_u \), where \( \alpha \) is bounded in \( L_{G}^{2}(\Omega_T) \).
(iii) \( \mathbb{E}_{C}[Z + \int_0^T \eta_u dB_u] = \mathbb{E}_{C}[Z] + \mathbb{E}_{C} [\int_0^T \eta_u dB_u] \), for \( Z \in L_{G}^{2}(\Omega_T) \).

Now we turn to Quadratic Variation Process of complex G-Brownian Motion.

For a complex G-B.M. \( B_t \), notice
\[
B_t^2 = \sum_{j=0}^{N-1} 2B_{j,N}(B_{j+1,N} - B_{j,N}) + \sum_{j=0}^{N-1} (B_{j,N} - B_{j+1,N})^2
\]
for a partition \( \Pi_N \) of \( [0,t] \): \( 0 = t_0 \leq t_1 \leq \cdots \leq t_N = t \). As \( ||\Pi_N|| \to 0 \), by independence under complex framework, we can show that
\[
\sum_{j=0}^{N-1} 2B_{j,N}(B_{j+1,N} - B_{j,N}) \overset{L_{G}^{2}(\Omega_T)}{\to} 2 \int_0^t B_u dB_u,
\]
\[
\]
so by the completeness of $L^2_G(\Omega)_C$, $\sum_{j=0}^{N-1} (B_{t_j}^{N^j} - B_{t_j}^N)^2$ must also converge in $L^2_G(\Omega)_C$.

**Definition 5.11.** By the argument above, we define

$$\langle B \rangle_t := \lim_{\|B_N\| \to 0} \sum_{j=0}^{N-1} (B_{t_j}^{N^j} - B_{t_j}^N)^2$$

$$= B_t^2 - 2 \int_0^t B_s dB_s,$$

and call $\langle B \rangle$ the quadratic variation process of complex G-Brownian Motion.

**Remark 5.12.** Since $L^p_G(\Omega_T)_C = \{ \xi_1 + i \xi_2 | \xi_1, \xi_2 \in L^p_G(\Omega_T) \}$, we can similarly show that

$$\int_0^T \eta_t dB_t = \int_0^T (\eta_t^{(1)} + i \eta_t^{(2)}) dB_t^{(1)} + i dB_t^{(2)}$$

Furthermore, an algebraic calculation tells:

$$\langle B \rangle_t = (B_t^{(1)} + i B_t^{(2)}))^2 - 2 \int_0^T (B_s^{(1)} + i B_s^{(2)}) dB_s^{(1)} + i dB_s^{(2)}$$

$$= (B_t^{(1)})^2 - (B_t^{(2)})^2 + 2i B_t^{(1)} B_t^{(2)}$$

$$- 2i \int_0^T B_s^{(1)} dB_s^{(1)} - \int_0^T B_s^{(2)} dB_s^{(2)} + i \int_0^T B_s^{(1)} dB_s^{(2)} + i \int_0^T B_s^{(2)} dB_s^{(1)}$$

$$= [(B_t^{(1)})^2 - 2 \int_0^T B_s^{(1)} dB_s^{(1)}] - [(B_t^{(2)})^2 - 2 \int_0^T B_s^{(2)} dB_s^{(2)}]$$

$$+ 2i [B_t^{(1)} B_t^{(2)} - \int_0^T B_s^{(1)} dB_s^{(2)} - \int_0^T B_s^{(2)} dB_s^{(1)}]$$

$$= \langle B^{(1)} \rangle_t - \langle B^{(2)} \rangle_t + 2i \langle B^{(1)}, B^{(2)} \rangle_t$$

Here are some basic properties of $\langle B \rangle$. For simplicity, we denote

$$\overrightarrow{B}_t = (B_t^{(1)}, B_t^{(2)})$$

$$\overrightarrow{\langle B \rangle}_t = \left( \begin{array}{c} \langle B^{(1)} \rangle_t \\ \langle B^{(2)} \rangle_t \\ \langle B^{(1)}, B^{(2)} \rangle_t \\ \langle B^{(1)}, B^{(2)} \rangle_t \end{array} \right)$$

**Remark 5.13.** Since $\overrightarrow{B}_t$ is a two dimensional G-Brownian Motion, $\overrightarrow{\langle B \rangle}_t$ is maximal distributed, so are $\text{Re} \langle B \rangle_t$ and $\text{Im} \langle B \rangle_t$. Then we can similarly define maximal distribution under complex framework in a trivial way. Then, of course, $\langle B \rangle_t$ is maximal distributed.
Lemma 5.14. For $s, t \geq 0$, $\langle B \rangle_{t+s} - \langle B \rangle_t$ is identically distributed with $\langle B \rangle_s$ and independent from $\Omega_s$.

Proof. Notice $\langle B \rangle_t = \langle B^{(1)} \rangle_t - \langle B^{(2)} \rangle_t + 2i\langle B^{(1)}, B^{(2)} \rangle_t$, and by the definition of independence we have the conclusion. \qed

Then we need to define integral with respect to $\langle B \rangle_t$. Firstly, we define the map $J$ from $M^1(0,T)_c$ to $L^1_\Omega(\Omega_T)_c$:

$$J(\eta) = \int_0^T \eta_t d\langle B \rangle_t := \sum_{j=0}^{N-1} \xi_j(\langle B \rangle_{t_{j+1}} - \langle B \rangle_{t_j}),$$

where $\eta_t(\omega) = \sum_{k=0}^{N-1} \xi_k(\langle 1 \rangle_{t_k} - \langle 1 \rangle_{t_k+1})(t)$ and then by the following lemma, we could have $\int_0^T \eta_t d\langle B \rangle_t$ for $\eta_t \in M^1(0,T)_c$.

Lemma 5.15. For $\eta_t = \sum_{k=0}^{N-1} \xi_k(\langle 1 \rangle_{t_k} - \langle 1 \rangle_{t_k+1})(t) \in M^1(0,T)_c$,

$$\hat{E}[|J(\eta)|] \leq 4(\hat{E}[\sigma_1^2] + \hat{E}[\sigma_2^2] + 2\hat{E}[\sigma_3^2]) \int_0^T |\eta_t dt|,$$

where $\sigma_i^2 = \sup_{\Lambda \in \Sigma} |\sigma_i|^2$, $i = 1, 2, 3$, and $G(A) = \frac{1}{2} \sup_{\Lambda \in \Sigma} (A, \Lambda)$, with $\Sigma$ a bounded closed convex subset of $2 \times 2$ symmetric matrix and $\Lambda = \left( \begin{array}{cc} \sigma_1 & \sigma_2 \\ \sigma_2 & \sigma_3 \end{array} \right)$

Proof. We denote

$$\Delta t_j = t_{j+1} - t_j,$$

$$\Delta \langle B \rangle_j = \langle B \rangle_{t_{j+1}} - \langle B \rangle_{t_j},$$

$$\Delta \langle B^{(i)} \rangle_j = \langle B^{(i)} \rangle_{t_{j+1}} - \langle B^{(i)} \rangle_{t_j}, i = 1, 2,$$

$$\Delta \langle B^{(1)}, B^{(2)} \rangle_j = \langle B^{(1)}, B^{(2)} \rangle_{t_{j+1}} - \langle B^{(1)}, B^{(2)} \rangle_{t_j}.$$

We have $\hat{E}[|\xi_j \Delta \langle B \rangle_j|] \leq \hat{E}[|Re(\xi_j \Delta \langle B \rangle_j)|] + \hat{E}[|Im(\xi_j \Delta \langle B \rangle_j)|]$ and then

$$\hat{E}[|Re(\xi_j \Delta \langle B \rangle_j)|]$$

$$= \hat{E}[|\xi_j \Delta \langle B^{(1)} \rangle_j - \xi_j \Delta \langle B^{(2)} \rangle_j| - 2\xi_j \Delta \langle B^{(1)}, B^{(2)} \rangle_j|$$

$$\leq (\sigma_1^2 + \sigma_3^2) \hat{E}[\Delta t_j |\xi_j^{(1)}| + |\xi_j^{(2)}|].$$

The similar result holds for $Im(\xi_j \Delta \langle B \rangle_j)$, so we have

$$\hat{E}[\sum_{j=0}^{N-1} \xi_j \Delta \langle B \rangle_j] \leq 2(\hat{E}[\sigma_1^2] + \hat{E}[\sigma_3^2] + 2\hat{E}[\sigma_2^2]) \hat{E}[\Delta t_j |\xi_j^{(1)}| + |\xi_j^{(2)}|].$$

The inequality $|a| + |b| \leq 2\sqrt{a^2 + b^2}$ would finish the proof. \qed
Proposition 5.16. Here are some properties for the integral with respect to $\langle B \rangle$.

(i) For $0 \leq s \leq t$, $\xi \in L^2_G(\Omega_s)c$, $Z \in L^1_G(\Omega_s)c$, we have

$$\mathbb{E}_C[Z + \xi(\langle B \rangle_t - \langle B \rangle_s)] = \mathbb{E}_C[Z + \xi(B_t - B_s)^2]$$

(ii) For $\eta \in M^2_G(0,T)c$, we have

$$\mathbb{E}_C[\left(\int_0^T \eta_t dB_t\right)^2] = \mathbb{E}_C[\left(\int_0^T \eta_t^2 d\langle B \rangle_t\right)]$$

Proof. For (i), by the definition of $\langle B \rangle_t$ and proposition (3.7), we have

$$\mathbb{E}_C[Z + \xi((\langle B \rangle_t - \langle B \rangle_s))] = \mathbb{E}_C[Z + \xi(B_t^2 - B_s^2 + \int_s^t B_r dB_r)]$$

$$= \mathbb{E}_C[Z + \xi(B_t^2 - B_s^2)]$$

$$= \mathbb{E}_C[Z + \xi((B_t - B_s)^2 + 2B_s(B_t - B_s))]$$

$$= \mathbb{E}_C[Z + \xi(B_t - B_s)^2]$$

For (ii), notice $\mathbb{E}_C[Z + 2\xi_j(B_{t,i+1} - B_{t,i})\xi_i(B_{t,i+1} - B_{t,i})] = \mathbb{E}_C[Z]$ and then

$$\mathbb{E}_C[\left(\int_0^T \eta_t dB_t\right)^2] = \mathbb{E}_C[\left(\sum_{j=0}^{N-1} \xi_j(B_{t,i+1} - B_{t,i})\right)^2].$$

By (i), we would have the conclusion.

\[\square\]

6 Complex Itô Formula and Conformal G-Brownian Motion

Now we turn to Itô formula under this framework. Firstly we need to set some basic definitions.

Definition 6.1. For $f = u + iv$ differentiable with real part $u$ and imaginary part $v$, we define operator $\partial f$ and $\bar{\partial} f$ as:

$$\partial f := \frac{\partial f}{\partial z} := \frac{1}{2}(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}) = \frac{1}{2}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + i\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\right)$$

$$\bar{\partial} f := \frac{\partial f}{\partial \bar{z}} := \frac{1}{2}(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}) = \frac{1}{2}\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} + i\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\right)$$

Remark 6.2. For $B_i = \alpha_i B^{(1)}_i + \beta_i B^{(2)}_i$, $i = 1, 2$, where $\alpha_i, \beta_i \in \mathbb{R}$ and $B^{(1)}_i + iB^{(2)}_i$ is a complex G-Brownian Motion, we have $(\alpha_i B^{(1)}_i, \beta_i B^{(2)}_i)$ is also
Theorem 6.3. (Complex Version of \( \alpha, \beta, \eta \) where \( \hat{\text{Motion}} \)) Then we can define

\[
\langle \hat{B}_1, \hat{B}_2 \rangle_t := \frac{1}{4}(\langle \hat{B}_1 + \hat{B}_2 \rangle_t - \langle \hat{B}_1 - \hat{B}_2 \rangle_t)
\]

and we have

\[
\langle \hat{B}_1, \hat{B}_2 \rangle_t = \langle \alpha_1 B^{(1)} + i\beta_1 B^{(2)}, \alpha_2 B^{(1)} + i\beta_2 B^{(2)} \rangle_t
\]

\[
= \frac{1}{4}[(\langle \alpha_1 + \alpha_2 \rangle B^{(1)} + i(\beta_1 + \beta_2) B^{(2)})_t - (\langle \alpha_1 - \alpha_2 \rangle B^{(1)} + i(\beta_1 - \beta_2) B^{(2)})_t]
\]

\[
= \frac{1}{4}[(\langle \alpha_1 + \alpha_2 \rangle B^{(1)})_t - (\langle \beta_1 + \beta_2 \rangle B^{(2)})_t + 2i(\langle \alpha_1 + \alpha_2 \rangle B^{(1)}, (\beta_1 + \beta_2) B^{(2)})_t - (\langle \alpha_1 - \alpha_2 \rangle B^{(1)})_t - (\langle \beta_1 - \beta_2 \rangle B^{(2)})_t]
\]

\[
= \alpha_1\alpha_2(B^{(1)})_t - \beta_1\beta_2(B^{(2)})_t + i(\alpha_1\beta_2 + \beta_1\alpha_2)(B^{(1)}, B^{(2)})_t.
\]

In particular, \( \langle B, \hat{B} \rangle_t = \langle B^{(1)} \rangle_t + \langle B^{(2)} \rangle_t \).

Furthermore, \( \int_0^T \eta_s \langle B_1, B_2 \rangle_s \) also explains itself.

**Theorem 6.3.** (Complex Version of Ito’s Lemma) For

\[
Z_t = Z_0 + \int_0^t \alpha_s ds + \int_0^t \eta_s d\langle B \rangle_s + \int_0^t \beta_s dB_s,
\]

where \( \alpha, \beta, \eta \) are all bounded processes in \( M^2_G(0,T) \), and any function \( f \), which is twice continuously differentiable and satisfies polynomial growth condition for the second order derivatives, we have the following equation in \( L^2_G(\Omega_t) \), \( \forall t \geq 0 : \)

\[
f(Z_t) - f(Z_0) = \int_0^t \partial f(Z_s) dZ_s + \int_0^t \partial f(Z_s) d\bar{Z}_s + \int_0^t \partial f(Z_s) dB_s + \int_0^t \bar{\partial f}(Z_s) d\bar{B}_s
\]

\[
+ \int_0^t \bar{\partial f}(Z_s) d\bar{B}_s + 2 \int_0^t \partial f(Z_s) d\langle B, \bar{B} \rangle_s
\]

\[
= \int_0^t (\partial f\alpha_s + \bar{\partial f}\bar{\alpha}_s) ds + \int_0^t \partial f\beta_s dB_s + \int_0^t \bar{\partial f}\bar{\beta}_s d\bar{B}_s + \int_0^t \partial f\eta_s d\langle B \rangle_s + \int_0^t \bar{\partial f}\bar{\eta}_s d\langle \bar{B} \rangle_s
\]

\[
+ \int_0^t \bar{\partial f} \beta^2_s d\langle B \rangle_s + \int_0^t \bar{\partial f} \bar{\beta}^2_s d\langle \bar{B} \rangle_s + 2 \int_0^t \bar{\partial f} |\beta_s|^2 d\langle B, \bar{B} \rangle_s.
\]

**Proof.** Notice \( \langle B \rangle_s = \langle B^{(1)} \rangle_s - \langle B^{(2)} \rangle_s - 2i\langle B^{(1)}, B^{(2)} \rangle_s \). The proof can be done by a review of itô formula in the real case and a careful algebraic calculation. \( \square \)

**Remark 6.4.** The first equation in the above theorem is formal to avoid the complex structure of the second equation. See [9], [10].
Remark 6.5. If $f$ is twice continuously differentiable, we have $\frac{\partial}{\partial x} \frac{\partial}{\partial y} f = \frac{\partial}{\partial y} \frac{\partial}{\partial x} f$, and further $\partial \bar{\partial} f = \bar{\partial} \partial f = \frac{1}{4} \Delta f$, where $\Delta$ is the laplace operator, i.e. $\Delta := \left( \frac{\partial}{\partial x} \right)^2 + \left( \frac{\partial}{\partial y} \right)^2$.

We know that a complex G-Brownian Motion can be viewed as a two-dimensional real G-Brownian Motion, which includes too many elements to get better properties. Here is a special kind of complex G-Brownian Motion, called conformal G-Brownian Motion, which is the main object of complex stochastic analysis.

Definition 6.6. A complex G-Brownian Motion is called a conformal G-Brownian Motion if $\langle B \rangle_t = 0$ in $L^2_G(\Omega_t)$ for any $t \geq 0$.

Remark 6.7. Notice $\langle B \rangle_t = \langle B^{(1)} \rangle_t - \langle B^{(2)} \rangle_t + 2i\langle B^{(1)}, B^{(2)} \rangle_t$, so $\langle B \rangle_t \equiv 0$ if and only if $\langle B^{(1)} \rangle_t = \langle B^{(2)} \rangle_t$ and $\langle B^{(1)}, B^{(2)} \rangle_t = 0$. This means the real part and the imaginary part moves as the same rate (identically distributed) and they are irrelevant. In the classical case, a complex Brownian Motion is surely conformal, since its real part and imaginary part are independent. However, things are a little different under G-framework. We cannot say $B^{(1)}$ and $B^{(2)}$ are independent under G-framework.

Example 6.8. For a random vector $X = (X_1, X_2)$, where $X_1$ is a real G-normal distributed variable with $\bar{\sigma}^2 > \bar{\sigma}^2$, and $X_2$ is an independent copy of $X_1$, we can claim that $X$ fails to be a real G-normal distributed vector, which can be easily checked by the definition of real G-normal distribution. In fact, if $\bar{X} = (\bar{X}_1, \bar{X}_2)$ is an independent copy of $X$, $\bar{X}_2$ is independent of $\bar{X}_1$ by the definition of independence, so for $\varphi(x, y) = x^2y$, we have

$$\mathbb{E} \left[ \varphi(X + \bar{X}) \right] = \mathbb{E} \left[ (X_1 + \bar{X}_1)^2 (X_2 + \bar{X}_2) \right] = \mathbb{E} [X_2^2 \sigma^2 - X_2^2 \bar{\sigma}^2] = (\bar{\sigma}^2 - \sigma^2) \mathbb{E} [X_2^2] = \frac{1}{2}(\bar{\sigma}^2 - \sigma^2) \mathbb{E} [||X_2||] > 0$$

while $\mathbb{E} [\varphi(\sqrt{2}X)] = 0$. This means a nontrivial two dimensional G-normal distributed vector fails to have independent elements, so does a complex G-normal distributed variable.

Before giving a description of complex G-conformal Brownian Motion, we need a simple fact.

Lemma 6.9. If $X$ is a real maximal distributed $n$-dimensional vector and satisfies

$$\mathbb{E} [\varphi(X)] = \varphi(0)$$

for any $\varphi \in C_c^{lip}(\mathbb{R}^n)$. Then we have $X = 0$, q.s.
Proof. Take $\varphi(x) = |x|^2$. Since $X$ is maximal distributed,

$$\hat{E}[\varphi(X)] = \sup_{(X_1, \ldots, X_n) \in V} (X_1^2 + \cdots + X_n^2) = 0$$

where $V$ is a convex closed subset of $\mathbb{R}^n$ (see remark(2.5)). It must be $V = \{0\}$, and we have the conclusion.

Here is a description of conformal G-Brownian Motion.

**Theorem 6.10.** A complex G-Brownian Motion $B_t$ is conformal if and only if $G(\cdot)$ has the following expression:

$$G(A) := \hat{E}[\overrightarrow{AB_t^T}] = \frac{1}{2} \sup_{\Lambda \in \Sigma} (A, \Lambda)$$

where $\Sigma = \{(\sigma_1^2, 0, \sigma_2^2) | \sigma^2 \in [\sigma_1^2, \bar{\sigma}^2], \text{ and } \sigma^2 \geq \bar{\sigma}^2 \geq 0 \}$.

Proof. According to remark (6.7), the only if part is simple. In fact, if we denote

$$\Lambda = \left( \begin{array}{cc} \sigma_1^2 & \sigma_2^2 \\ \sigma_2^2 & \sigma_3^2 \end{array} \right),$$

since $(B^{(1)})_t = (B^{(2)})_t$ and $(B^{(1)}, B^{(2)})_t = 0$, we have

$$\hat{E}[f((B^{(1)})_t - (B^{(2)})_t)] = \sup_{\Lambda \in \Sigma} f(\sigma_1^2 - \sigma_3^2) = f(0)$$

and

$$\hat{E}[g((B^{(1)}, B^{(2)})_t)] = \sup_{\Lambda \in \Sigma} g(\sigma_2^2) = g(0),$$

for any $f, g \in C_{l, lip}(\mathbb{R})$. It follows that $\Sigma$ must have the above expression.

For the if part, notice $B_t$ is uniquely determined by $G(A)$, and

$$\hat{E}[\varphi((B)_t)] = \sup_{\Lambda \in \Sigma} \varphi(\Lambda).$$

We have

$$\hat{E}[\varphi((B^{(1)})_t - (B^{(2)})_t)] = \sup_{\Lambda \in \Sigma} \varphi(\sigma_1^2 - \sigma_3^2) = \varphi(0)$$

and

$$\hat{E}[\varphi((B^{(1)}, B^{(2)})_t)] = \sup_{\Lambda \in \Sigma} \varphi(0) = \varphi(0),$$

for any $\varphi \in C_{l, lip}(\mathbb{R})$. By lemma (6.9), we get the conclusion.

We give the definition of G-martingale under complex case in a trivial way.
Definition 6.11. For a complex process \((M_t)_{t \geq 0}\), it is called a complex \(G\)-martingale if \(M_t \in L^1_G(\Omega_t)_C\), and
\[
\hat{E}_C[M_t | \Omega_s] = M_s,
\]
for \(0 \leq s \leq t\).

Here is a property of analytic function, which we will use in the next.

Lemma 6.12. \(f\) is continuously differentiable complex function. Then \(f\) is analytic if and only if \(\partial f = 0\). In this case,
\[
f'(z) = \partial f(z),
\]
here \(f'(z)\) means the derivative of \(f\) in the complex sense.

Proof. Notice that \(\partial f = 0\) if and only if \(\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}\), when \(f = u + iv\). The conclusion follows from Cauchy-Riemann equation.

Corollary 6.13. Suppose \(B_t\) is a conformal \(G\)-Brownian Motion, and \(f\) is analytic. Then \(f(B_t)\) is a symmetric complex \(G\)-martingale.

Proof. In this case, by ito’s formula and the above lemma, we have
\[
f(B_t) - f(B_0) = \int_0^t f'(B_s)dB_s.
\]

Example 6.14. Suppose \(B_t\) is conformal. Then \(B_t^2\) is a martingale. In face, \(B_t^2 = (B_t^{(1)})^2 - (B_t^{(2)})^2 + 2iB_t^{(1)}B_t^{(2)}\), so we have
\[
\hat{E}_C[B_t^2 | \Omega_s] = \hat{E}[(B_t^{(1)})^2 - (B_t^{(2)})^2 | \Omega_s] + 2i\hat{E}[B_t^{(1)}B_t^{(2)} | \Omega_s]
\]
\[
= \hat{E}[(B_t^{(1)})^2 - (B_t^{(2)})^2] + (B_t^{(1)})_t - (B_t^{(2)})_t + (B_t^{(1)})_t - (B_t^{(1)})_t + (B_t^{(2)})_t - (B_t^{(2)})_t | \Omega_s]
\]
\[
+ 2iB_s^{(1)}B_s^{(2)}
\]
\[
= (B_t^{(1)})^2 - (B_t^{(2)})^2 + 2iB_s^{(1)}B_s^{(2)}
\]
\[
= B_s^2
\]

In fact, we can take this conclusion further to get the conformal invariance by considering martingale with the form \(M_t = \int_0^t \eta_u dB_u\), where \(\eta_u \in M^2_G(0,T)_C\). Since \(M_t\) is a symmetric martingale, we can define the quadratic variation in the old fashion way: the limit point under norm of \(L^2_G(\Omega_T)_C\), and then we would have
\[
\langle M \rangle_t = \int_0^t \eta_s^2 d\langle B \rangle_s.
\]

Definition 6.15. A complex martingale \(M_t\) is conformal if \(\langle M \rangle_t = 0\).
Obviously, a conformal G-Brownian motion is a conformal martingale. Furthermore, we suppose $\eta_s \in M^2_G(0,T_\mathbb{C})$. If $B_t$ is conformal, by Itô’s lemma and B-D-G inequity, we have $M^2_t = 2 \int_0^t M_s \eta_s dB_s$, so $\langle M^2 \rangle_t = 4 \int_0^t M^2_s \eta^2_s d\langle B \rangle_s$. Notice $d\langle M \rangle_s = \eta^2_s d\langle B \rangle_s$, and we get $\langle M^2 \rangle_t = 4 \int_0^t M^2_s d\langle M \rangle_s$. In conclusion, we get:

**Corollary 6.16.** $B_t$ is conformal and $M_t$ is defined as above. Then $M_t, M^2_t$ are conformal martingales.

**Corollary 6.17.** If $M_t = \int_0^t \eta_s dB_s$ is conformal, with values in an open set $E$, and $f$ is a bounded analytic function with bounded first order derivative on $E$, polynomial growth of the second order derivative, then $f(M_t)$ is conformal. Furthermore,

$$\langle f(M), f(M) \rangle_t = \int_0^t f'(M_s) f'(M_s) d\langle M, M \rangle_s$$

**Proof.** By Itô’s lemma, $f(M_t) = \int_0^t f'(M_s) \eta_s dB_s$, so

$$\langle f(M) \rangle_t = \int_0^t (f'(M_s))^2 \eta^2_s d\langle B \rangle_s$$

$$= \int_0^t (f'(M_s))^2 d\langle M \rangle_s$$

$$= 0$$

The equation follows from the fact that $f(M)$ is also a symmetric martingale. 

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