Abstract. In this paper, we revisit the proof of the existence of a solution to the semilinear heat equation in one space dimension with a flat blow-up profile, already proved by Bricmont and Kupainen together with Herrero and Velázquez. Though our approach relies on the well-celebrated method, based on the reduction of the problem to a finite-dimensional one, then the use of a topological "shooting method" to solve the latter, the novelty of our approach lays in the use of a modulation technique to control the projection of the zero eigenmode arising in the problem. Up to our knowledge, this is the first time where modulation is used with this kind of profiles. We do hope that this simplifies the argument.

1. Introduction

We consider the following nonlinear heat equation (NLH)

\[
\begin{aligned}
&u_t = \Delta u + |u|^{p-1} u, \\
&u(.,0) = u_0 \in L^\infty(\mathbb{R}^N, \mathbb{R}),
\end{aligned}
\]

where \( p > 1 \) and \( u(x,t) : \mathbb{R}^N \times [0,T) \rightarrow \mathbb{R} \). Equation (1.1) is considered as a model for many physical situations, such as heat transfer, combustion theory, thermal explosion, etc. (see more in Kapila [14], Kassoy and Poland [15, 16], Bebernes and Eberly [2]). Firstly, note that equation (1.1) is well-posed in \( L^\infty \). More precisely, for each \( u_0 \in L^\infty(\mathbb{R}^N) \), one of the following statements holds:

- either the solution is global in time;
- or the maximum existence time is finite i.e. \( T_{\text{max}} < +\infty \) and

\[
\lim_{t \rightarrow T_{\text{max}}} \|u(\cdot,t)\|_{L^\infty} = +\infty.
\]

In particular, \( T_{\text{max}} > 0 \) is called the blowup time of the solution and a point \( a \in \mathbb{R}^N \) is called a blowup point of the solution if there exists sequence \( (a_n, t_n) \rightarrow (a, T) \) as \( n \rightarrow +\infty \) such that

\[
|u(a_n, t_n)| \rightarrow +\infty \quad \text{as} \quad n \rightarrow +\infty.
\]

Blowup for equation (1.1) has been studied intensively by many mathematicians and no list can be exhaustive. This is the case for the question of deriving blowup profiles, which is completely understood in one space dimension (see in particular Herrero and Velázquez [10, 8, 11, 12]), unlike...
the higher dimensional case, where much less is known (see for example Velázquez [20, 21, 22], Zaag [23, 24, 26, 25] together with the recent contributions by Merle and Zaag [17, 18]).

In the one dimensional case, Herrero and Velázquez proved the following, unless the solution is space independent (see also Filippas and Kohn [7]):

(i) Either
\[
\sup_{|x-a|\leq K(T-t)|\ln(T-t)|} \left| (T-t)^{\frac{1}{p-1}} u(x,t) - \varphi \left( \frac{x-a}{(T-t)^{\frac{1}{p}} \ln(T-t)} \right) \right| \to 0,
\]
where \( \varphi(z) = (p-1 + b|z|^{2k})^{\frac{1}{p-1}} \) and \( b = \frac{(p-1)^2}{4p} \) is unique. (note that Herrero and Velázquez proved that this behavior is generic in [12, 9]).

(ii) Or
\[
\sup_{|x-a|\leq K(T-t)^{\frac{1}{2k}}} \left| (T-t)^{\frac{1}{p-1}} u(x,t) - \varphi_k \left( \frac{x-a}{(T-t)^{\frac{1}{2k}}} \right) \right| \to 0,
\]
where \( \varphi_k(z) = (p-1 + b|z|^{2k})^{\frac{1}{p-1}} \), where \( b \) is an arbitrary positive number.

In particular, we are interested in constructing blowup solution with a prescribed behaviors, via a “generic approximation”, called the blowup profile of the solution.

The existence of such solutions was observed by Velázquez, Galaktionov and Herrero [13] who indicated formally how one might find these solution. Later, Bricmont and Kupiainen [3], will give a rigorous proof of construction of such profiles (see also Herrero and Velázquez [8] for the profile \( \varphi_4 \)). In [1], Angenent and Velázquez gives a construction of blow up solution for the mean curvature flow inspired by the construction of (ii). Most of the constructions are made in one dimension \( N = 1 \). In higher dimension \( N \geq 2 \), recently Merle and Zaag give the construction of a new profile of type I with a superlinear power in the Sobolev subcritical range, for more details see [18].

In this paper we revisit the construction of ii) given in Section 4 of [3]. Our construction has the advantage that it uses the modulation parameter. We shall use a topological ”shooting” argument to prove existence of Solutions constructed in Theorem 1.1. The construction is essentially an adaptation of Wazewski’s principle (see [6], chapter II and the references given there). The use of topological methods in the analysis of singularities for blow-up phenomena seems to have been introduced by Bressan in [5].

The following is the main result in the paper

**Theorem 1.1.** Let \( p > 1 \) and \( k \in \mathbb{N}, k \geq 2 \), then there exist \( \delta_0 \) and \( \tilde{T} > 0 \) such that for all \( \delta \in (0, \delta_0) \) and \( T \in (0, \tilde{T}) \), we can construct initial datum \( u_0 \in L^\infty(\mathbb{R}) \) such that the corresponding solution to equation (1.1) blowup in finite time \( T \) and only at the origin. Moreover, there exists the flow \( b(t) \in C^1(0,T) \) such that the following description is valid

(i) For all \( t \in [0,T] \), it holds that
\begin{equation}
\left\| (T-t)^{\frac{1}{p-1}} u(\cdot,t) - f_{b(t)}\left( \frac{1}{T-t} \right) \right\|_{L^\infty(\mathbb{R})} \lesssim (T-t)^{\frac{4}{2} \left( 1 - \frac{k}{p} \right)} \text{ as } s \to \infty.
\end{equation}

(ii) There exists \( b^* > 0 \) such that \( b(t) \to b^* \) as \( t \to T \) and
\begin{equation}
|b(t) - b^*| \lesssim (T-t)^{\delta \left( 1 - \frac{k}{p} \right)}, \forall t \in (0,T),
\end{equation}
where \( f_{b(t)} \) is defined by
\begin{equation}
f_{b(t)}(y) = \left( p - 1 + b(t)y^{2k} \right)^{\frac{1}{p-1}}.
\end{equation}
**Remark 1.1.** One of the most important steps of the proof is to project the linearized partial differential equation (2.8) on the $H_{20}$, given by (3.3). We note that this is technically different from the work of Bricomont and Kupiainen [3], where the authors project the integral equation. Consequently, we will have additional difficulty coming from the projection of the different terms, see for example Lemma 5.1 and Lemma 5.2.

**Remark 1.2.** We note that $0 < b_0 \leq b(t) \leq 2b_0$ and (1.4), it holds that

$$
\left\| \left( p - 1 + b(t) y^{2k} \right)^{-\frac{1}{p - 1}} - \left( p - 1 + b^* y^{2k} \right)^{-\frac{1}{p - 1}} \right\|_{L^\infty(\mathbb{R})} \lesssim |b(t) - b^*| \lesssim (T - t)^{\frac{1}{2}(1 - \frac{1}{p})},
$$

which yields

$$
\left\| (T - t)^{\frac{1}{p - 1}} \frac{\partial}{\partial s} u(\cdot, t) - f_b^* \left( \frac{|\cdot|^{2k}}{T - t} \right) \right\|_{L^\infty(\mathbb{R})} \lesssim (T - t)^{\frac{1}{2}(1 - \frac{1}{p})} \text{ as } s \to \infty. \quad (1.6)
$$

The paper is organised as follows. In Section 2 and 3, we give the formulation of the problem. In Section 4 we give the proof of the existence of the profile assuming technical details. In particular, we construct a shrinking set and give an example of initial data giving rise to the blow-up profile and at the end of the section we give the proof of Theorem 1.1. The topological argument of Section 4 uses a number of estimates given by Proposition 4.5, we give the proof of this proposition in Section 5.

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### 2. Formulation of the problem

Let us consider $T > 0$, and $k \in \mathbb{N}, k \geq 2$, and $u$ be a solution to (1.1) which blows up in finite time $T > 0$. Then, we introduce the following *blow-up variable*:

$$w(y, s) = (T - t)^{-\frac{1}{p - 1}} u(x, t), \quad y = \frac{x}{(T - t)^{\frac{1}{4k}}}, \quad s = -\ln(T - t). \quad (2.1)$$

Since $u$ solves (1.1), for all $(x, t) \in \mathbb{R}^N \times [0, T)$, then $w(y, s)$ reads the following equation

$$
\frac{\partial w}{\partial s} = I^{-2}(s) \Delta w - \frac{1}{2k} y \cdot \nabla w - \frac{1}{p - 1} w + |w|^{p - 1} w,
$$

where $I(s)$ is defined by

$$
I(s) = e^{\frac{s}{2}(1 - \frac{1}{4})}. \quad (2.3)
$$

Adopting the *setting* investigated by [3], we consider $C^1$-flow $b$ and introduce

$$w(y, s) = f_b(y) \left( 1 + e_b(y)q(y, s) \right) \quad (2.4)$$

where $f_b$ and $e_b$ respectively defined as

$$f_b(y) = \left( p - 1 + by^{2k} \right)^{-\frac{1}{p - 1}}, \quad (2.5)$$

and

$$e_b(y) = \left( p - 1 + b|y|^{2k} \right)^{-1}. \quad (2.6)$$

and the flow $b$ will arise as an unknown functions that will be constructed together with the linearized solution $q$. Since $f_be_b = f_b^p$, by (2.4) $q$ can be written as follows

$$q = w f_b^{-p} - (p - 1 + by^{2k}). \quad (2.7)$$

In the following we consider $(q, b)(s)$ which satisfies the following equation
\begin{align}
\partial_s q &= \mathcal{L}_s q + \mathcal{N}(q) + \mathcal{D}_s(\nabla q) + \mathcal{R}_s(q) + b'(s)\mathcal{M}(q), \quad (2.8)
\end{align}

where

\begin{align}
\mathcal{L}_s q &= I^{-2}(s)\Delta q - \frac{1}{2k} y \cdot \nabla q + q, \quad I(s) = e^{\frac{s}{2}(1 - \frac{1}{k})}, \quad (2.9)
\mathcal{N}(q) &= |1 + e_bq|^p - (1 + e_bq) - 1 - pe_bq \quad (2.10)
\mathcal{D}_s(\nabla q) &= -\frac{4pkb}{p-1}I^{-2}(s)e_by^{2k-1}\nabla q, \quad (2.11)
\mathcal{R}_s(q) &= I^{-2}(s)y^{2k-2}\left(\alpha_1 + \alpha_2y^{2k}e_b + (\alpha_3 + \alpha_4y^{2k})e_bq\right), \quad (2.12)
\mathcal{M}(q) &= \frac{p}{p-1}y^{2k}(1 + e_bq) \quad (2.13)
\end{align}

and the constants \(\alpha_i\) are given by

\begin{align}
\alpha_1 &= -2k(2k-1)\frac{b}{p-1}; \quad \alpha_2 = 4pk^2\frac{b^2}{(p-1)^2}; \quad \alpha_3 = -2pk(2k-1)\frac{b}{p-1}; \quad \alpha_4 = 4p(2p-1)k^2\frac{b^2}{(p-1)^2}. \quad (2.14)
\end{align}

### 3. Decomposition of the solution

#### 3.1. Fundamental solution involving to \(\mathcal{L}_s\)

Let us define Hilbert space \(L^2_{\rho_s}(\mathbb{R})\) by

\begin{align}
L^2_{\rho_s}(\mathbb{R}) &= \{f \in L^2(\mathbb{R}), \int_{\mathbb{R}} f^2 \rho_s dy < \infty\}, \quad (3.1)
\end{align}

where

\begin{align}
\rho_s &= \frac{I(s)}{\sqrt{4\pi}} e^{-\frac{y^2}{4}}, \quad (3.2)
\end{align}

and \(I(s)\) is defined by (2.3).

In addition, we denote

\begin{align}
H_m(y, s) = I^{-m}(s)h_m(I(s)y) = \sum_{\ell=0}^{[m]} \frac{m!}{\ell!(m-2\ell)!}(-I^{-2}(s))^\ell y^{m-2\ell} \quad (3.3)
\end{align}

where \(h_m(z)\) be Hermite polynomial (physic version)

\begin{align}
h_m(z) = \sum_{\ell=0}^{[m]} \frac{m!}{\ell!(m-2\ell)!}(-1)^\ell z^{m-2\ell}. \quad (3.4)
\end{align}

In particular, it is well known that

\[ \int h_n h_m \rho_s dy = 2^n n! \delta_{nm}, \]

which yields

\[ (H_n(., s), H_m(., s))_s = \int h_n(y)H_m(y)\rho_s(y)dy = I^{-2m}2^n n! \delta_{nm}. \quad (3.5) \]

**Jordan block’s decomposition of \(\mathcal{L}_s\)**

By a simple computation (relying on fundamental identities of Hermite polynomials), we have

\begin{align}
\mathcal{L}_s H_m(y, s) &= \begin{cases}
(1 - \frac{m}{2k}) H_m(y, s) + m(m-1)(1 - \frac{1}{k})I^{-2}(s)H_{m-2} & \text{if } m \geq 2 \\
(1 - \frac{m}{2k}) H_m(y, s) & \text{if } m = \{0, 1\} \end{cases} \quad (3.6)
\end{align}
We define $K_{s,\sigma}$ as the fundamental solution to
\[ \partial_s K_{s,\sigma} = L_{s} K_{s,\sigma} \text{ for } s > \sigma \text{ and } K_{\sigma,\sigma} = \text{Id}. \] (3.7)
By using the Mehler’s formula, we can explicitly write its kernel as follows
\[ K_{s,\sigma}(y,z) = e^{s-\sigma} \mathcal{F} \left( e^{-\frac{s-\sigma}{2k} y - z} \right) \] (3.8)
where
\[ \mathcal{F}(\xi) = \frac{L(s,\sigma)}{\sqrt{4\pi \sigma}} e^{-\frac{L^2(s,\sigma)}{4\pi}} \] (3.9)
\[ L(s,\sigma) = I(\sigma) \sqrt{1 - e^{-(s-\sigma)}}, \] (3.9)
\[ I(\sigma) = e^{s} \frac{2}{(1 - k)}. \] (3.9)
In addition, we have the following equalities
\[ K_{s,\sigma} H_n(\cdot,\sigma) = e^{(s-\sigma)(1 - \frac{2k}{k})} H_n(\cdot, s), n \geq 0. \] (3.10)

### 3.2. Decomposition of $q$

For the sake of controlling the unknown function $q \in L^2_{\rho_s}$, we will expand it with respect to the polynomials $H_m(y,s)$. We start by writing
\[ q(y,s) = \sum_{m=0}^{[M]} q_m(s) H_m(y,s) + q_-(y,s) \equiv q_+(y,s) + q_-(y,s), \] (3.11)
where constant $[M]$ be the largest integer less than $M$ with
\[ M = \frac{2kp}{p-1}. \] (3.12)
From (3.5), we have
\[ q_m(s) = P_m(q) = \langle q, H_m \rangle_{L^2_{\rho_s}} \] (3.13)
as the projection of $q$ on $H_m$. In addition, $q_-(y,s)$ can be seen as the projection of $q$ onto $\{H_m, m \geq [M] + 1\}$ and we also denote as follow
\[ q_- = P_- (q). \] (3.14)

### 3.3. Equivalent norms

Let us consider $L^\infty_M$ defined by
\[ L^\infty_M(\mathbb{R}) = \{ g \text{ such that } (1 + |y|^M)^{-1} g \in L^\infty(\mathbb{R}) \}, \] (3.15)
and $L^\infty_M$ is complete with the norm
\[ \| g \|_{L^\infty_M} = \| (1 + |y|^M)^{-1} g \|_{L^\infty}, \] (3.16)
we introduce
\[ \| q \|_s = \sum_{m=0}^{[M]} |q_m| + |q_-|_s, \] (3.17)
where
\[ |q_-|_s = \sup_y \frac{|q_-(y,s)|}{I(s)^{-M} + |y|^{3M}}. \] (3.18)
It is straightforward to check that
\[ C_1(s) \| q \|_{L^\infty_M} \leq \| q \|_s \leq C_2(s) \| q \|_{L^\infty_M} \] where $C_i, i=1,2(s) > 0$.

Finally, we derive that $L^\infty_M(\mathbb{R})$ is also complete with the norm $\| \cdot \|_s$. 
4. The existence assuming some technical results

As mentioned before, we only give the proof in the one dimensional case. This section is devoted to the proof of Theorem 1.1. We proceed in five steps, each of them making a separate subsection.

- In the first subsection, we define a shrinking set \( V_{\delta,b_0}(s) \) and translate our goal of making \( g(s) \) go to 0 in \( L^\infty_M(\mathbb{R}) \) in terms of belonging to \( V_{\delta,b_0}(s) \).
- In the second subsection We exhibit a \( k \) parameter initial data family for equation (2.8) whose coordinates are very small (with respect to the requirements of \( V_{\delta,b_0}(s) \)) except for the \( k+1 \) first parameter \( q_0, ..., q_{2k-1} \).
- In the third subsection, we solve the local in time Cauchy problem for equation (2.8) coupled with some orthogonality condition.
- In the fourth subsection, using the spectral properties of equation (2.8), we reduce our goal from the control of \( q(s) \) (an infinite dimensional variable) in \( V_{\delta,b_0}(s) \) to the control of its \( 2k \) first components \( (q_0, ..., q_{2k-1}) \) (a \( (k) \)-dimensional variable) in \( [-I(s)^{-\delta}, I(s)^{-\delta}]^{2k} \).
- In the last subsection, we solve the finite dimensional problem using the shooting lemma and conclude the proof of Theorem 1.1.

4.1. Definition of the shrinking set \( V_{\delta,b_0}(s) \)

In this part, we introduce the shrinking set that controls the asymptotic behaviors of our solution

**Definition 4.1.** Let us consider an integer \( k > 1 \), the reals \( \delta > 0 \), \( b_0 > 0 \) and \( M \) given by (3.12), we define \( V_{\delta,b_0}(s) \) be the set of all \( (q,b) \in L^\infty_M \times \mathbb{R} \) satisfying

\[
|q_m| \leq I^{-\delta}(s), \quad \forall \, 0 \leq m \leq [M], \quad m \neq 2k, \tag{4.1}
\]

\[
|q_{2k}| \leq I^{-2\delta}(s), \tag{4.2}
\]

\[
|q^-|_s \leq I^{-\delta}(s), \tag{4.3}
\]

and

\[
\frac{b_0}{2} \leq b \leq 2b_0, \tag{4.4}
\]

where \( q_m \) and \( q^- \) defined in (3.11), \( I(s) \) defined as in (2.3) and \( |\cdot|_s \) norm defined in (3.18).

4.2. Preparation of Initial data

In this part, we aim to give a suitable family of initial data for our problem. Let us consider \((d_0, d_1, ..., d_{2k-1}) \in \mathbb{R}^{2k}, \delta > 0 \) and \( b_0 > 0 \), we then define

\[
\psi(d_0, ..., d_{2k-1}, y, s_0) = \sum_{i=0}^{2k-1} d_i I^{-\delta}(s_0) y^i, \tag{4.5}
\]

then, we have the following result

**Lemma 4.2** (Decomposition of initial data in different components). Let us consider \((d_i)_{0 \leq i \leq 2k-1} \in \mathbb{R}^{2k} \) satisfying \( \max_{0 \leq i \leq 2k-1} |d_i| \leq 1 \) and \( b_0 > 0 \) given arbitrarily. Then, there exists \( \delta_1(b_0) \) such that for all \( \delta \leq \delta_1 \), there exists \( s_1(\delta_1, b_0) \geq 1 \) such that for all \( s_0 \geq s_1 \), the following properties are valid with \( \psi(d_0, ..., d_{2k-1}) \) defined in (4.5):

(i) There exits a quadrilateral \( \mathbb{D}_{s_0} \subset [-2,2]^{2k} \) such that the mapping

\[
\Gamma : \mathbb{D}_{s_0} \to \mathbb{R}^{2k}, \quad (d_0, ..., d_{2k-1}) \mapsto (\psi_0, ..., \psi_{2k-1})', \tag{4.6}
\]
is linear one to one from $\mathbb{D}_{s_0}$ to $\hat{\mathcal{V}}(s_0)$, with
\[
\hat{\mathcal{V}}(s) = \left[-I(s)^{-\delta}, I(s)^{-\delta}\right]^{2k},
\]
where $(\psi_0, \ldots, \psi_{2k-1})$ are the coefficients of initial data $\psi(d_0, \ldots, d_{2k-1})$ given by the decomposition (3.11). In addition to that, we have
\[
\Gamma|_{\partial \mathbb{D}_{s_0}} \subset \partial \hat{\mathcal{V}}(s_0) \text{ and } \deg\left(\Gamma|_{\partial \mathbb{D}_{s_0}}\right) \neq 0.
\]

(ii) For all $(d_0, \ldots, d_{2k-1}) \in \mathbb{D}_{s_0}$, the following estimates are valid
\[
|\psi_0| \leq I^{-\delta}(s_0), \ldots, |\psi_{2k-1}| \leq I^{-\delta}(s_0), \quad \psi_{2k} = \psi_M = 0 \text{ and } \psi_- \equiv 0.
\]

Proof. The proof of the Lemma is quite the same as [19, Proposition 4.5].
- The proof of item (i): From (4.5), definition’s $H_n$ in (3.3) and (3.13), we get
\[
|\psi_n(s_0) - d_n I^{-\delta}(s_0)| \leq C(d_0, \ldots, d_{2k-1}) I^{-\delta-2}(s_0)
\]
which concludes item (i).
- The proof of item (ii): From (4.5), $\psi(d_0, \ldots, d_{2k-1}, s_0)$ is a polynomial of order $2k - 1$, so it follows that
\[
\psi_n = 0, \forall n \in \{2k, \ldots, M\} \text{ and } \psi_- \equiv 0.
\]
In addition, since $(d_0, \ldots, d_{2k-1}) \in \mathbb{D}_{s_0}$, we use item (i) to deduce that $(\psi_0, \ldots, \psi_{2k-1}) \in \hat{\mathcal{V}}(s_0)$ and
\[
|\psi_n| \leq I^{-\delta}(s_0), \forall n \in \{0, \ldots, 2k - 1\}
\]
which concludes item (ii) and the proof Lemma 4.2.

Remark 4.3. Note that $s_0 = -\ln(T)$ is the master constant. In almost every argument in this paper it is almost to be sufficiently depending on the choice of all other constants ($\delta_0$ and $b_0$). In addition, we denote $C$ as the universal constant that is independent to $b_0$ and $s_0$.

4.3. Local in time solution for the problem (2.8) & (4.11)

As we setup in the beginning, besides main solution $q$, modulation flow $b$ plays an important role in our analysis, since it helps us to disable the perturbation of the neutral mode corresponding to eigenvalue $\lambda_{2k} = 0$ of linear operator $L_s$. In particular, the modulation flow is one of the main contributions of our paper. The uniqueness of the flow $b$ is defined via the following orthogonal condition
\[
\langle q, H_{2k} \rangle_{L^2_{\rho \psi}} = 0.
\]
As a matter of fact, the cancellation ensures that $q_{2k} = 0$, the projection of the solution on $H_{2k}$, corresponding to eigenvalue $\lambda_{2k} = 0$, since the neutral issues to the control of our solution. Consequently, our problem given by (2.8) is coupled with the condition (4.11). In the following, we aim to establish the local existence and uniqueness.

Proposition 4.4 (Local existence of the coupled problem (2.8) & (4.11)). Let $(d_i)_{0 \leq i \leq 2k-1} \in \mathbb{R}^{2k}$ satisfying $\max_{0 \leq i \leq 2k-1} |d_i| \leq 2$ and $\delta > 0, b_0 > 0$, there exists $s_2(\delta, b_0) \geq 1$, such that for all $s_0 \geq s_2$, the following property holds: If we choose initial data $\psi$ as in (4.5), then, there exists $s^* > s_0$ such that the coupled problem (2.8) & (4.11) uniquely has solution on $[s_0, s^*]$. Assume furthermore that the solution $(q, b)(s) \in V_{\delta, b_0}(s)$ for all $s \in [s_0, s^*]$, then, the solution can be extended after the time $s^*$ i.e. the existence and uniqueness of $(q, b)$ are valid on $[s_0, s^* + \epsilon]$, for some $\epsilon > 0$ small.
Proof. Let us consider initial \( w_0 \) defined as in (2.4) with \( q(s_0) = \psi(d_0, d_1, ..., d_{2k-1}) \) given as in (4.5), since equation (1.1) is locally well-posed in \( L^\infty \), then, the solution \( w \) to equation (2.2) exists on \([s_0, \bar{s}]\) for some \( \bar{s} > s_0 \). Next, we need to prove that \( w \) is uniquely decomposed as in (2.4) and \((q, b)(s)\) solves (2.8) and (4.11). The result follows the Implicit function theorem. Let us define the functional \( \mathcal{F} \) by

\[
\mathcal{F}(s, b) = \left( w f_b^{-p} - \left( p - 1 + b y^{2k} \right), H_{2k} \right)_{L^b_p}.
\]

(4.12)

For \( b_0 > 0 \), and at \( s = s_0 \), from \( \psi(d_0, ..., d_1)'s \) definition in (4.5), it directly follows that

\[
\mathcal{F}(s_0, b_0) = 0.
\]

(4.13)

Next, we aim to verify

\[
\frac{\partial \mathcal{F}}{\partial b}(s_0, b_0) \neq 0.
\]

(4.14)

From (4.12), we obtain

\[
\frac{\partial \mathcal{F}}{\partial b}(s, b) = \left( w p y^{2k} - \frac{p - 1}{p - 1} f_b^{-1} - y^{2k}, H_{2k} \right)_{L^b_p}.
\]

(4.15)

According to (2.4), we express \( w(s_0) \) as follows

\[
w(y, s_0) = f_{b_0} \left( 1 + I^{-\delta}(s_0) f_{b_0}^{p-1}(y, s_0) \sum_{i=0}^{2k-1} d_i y^i \right).
\]

Then, we have

\[
\frac{\partial \mathcal{F}}{\partial b}(s_0, b_0) = I^{-\delta}(s_0) \left( \frac{p}{p - 1} \left( f_{b_0}^{p-1}(y, s_0) \sum_{i=0}^{2k-1} d_i y^{i+2k}, H_{2k} \right)_{L_{p,s_0}^2} \right) + \frac{1}{p - 1} \left( y^{2k}, H_{2k} \right)_{L_{p,s_0}^2} := A + B.
\]

(4.16)

Using (3.3) and (3.5), we immediately have

\[
B = \frac{2^{4k}(2k)!}{p - 1} I^{-4k}(s_0).
\]

In addition, we use (2.6) to get the following expression

\[
e_{b_0}(y) = (p - 1)^{-1} \left( \sum_{l=0}^{L} \left( -\frac{b y^{2k}}{p - 1} \right)^l + \left( -\frac{b y^{2k}}{p - 1} \right)^{L+1} \right),
\]

(4.17)

for \( L \in \mathbb{N}, L \geq 2 \) arbitrarily.

Now, we decompose the part \( A \) in (4.16) by

\[
A = I^{-\delta}(s_0) \left( \frac{p}{p - 1} \left( f_{b_0}^{p-1}(y, s_0) \sum_{i=0}^{2k-1} d_i(s_0) y^{i+2k}, H_{2k} \right)_{L_{p,s_0}^2} \right)
\]

\[
= I^{-\delta}(s_0) \sum_{i=0}^{2k-1} d_i \left( \int_{|y| \leq 1} e_{b_0}(y, s_0) y^{i+2k} H_{2k \rho s_0}(y) dy + \int_{|y| \geq 1} e_{b_0}(y, s_0) y^{i+2k} H_{2k \rho s_0}(y) dy \right)
\]

\[
= A_1 + A_2.
\]

Since \( e_{b_0} y^{2k} \) is bounded, we apply Lemma A.2 to get

\[
|A_2| \lesssim I^{-4k-\delta}(s_0),
\]
provided that $s_0 \geq s_{2,2}(\delta)$. Besides that, we use (4.17) with $L \geq 2$ arbitrarily and we write $A_1$ as follows

$$A_1 = (p - 1)^{-1} I^{-\delta}(s_0) \sum_{i=0}^{2k-1} d_i \int_{|y| \leq \delta} \left[ \sum_{j=0}^{L} \left( - \frac{b \zeta^{2k}}{p - 1} \right)^j + \left( - \frac{b \zeta^{2k}}{p - 1} \right)^L \right] y^{i+2k} H_{2k}(s_0) \rho(s_0) dy.$$ 

Using Lemmas A.1 and A.2, we get

$$|A_1| \lesssim I^{-4k-\delta}(s_0).$$ 

By adding all related terms, we obtain

$$\frac{\partial F}{\partial b}(s_0, b_0) = I^{-4k}(s_0) 2^{4k}(2k)! \left( 1 + O(I^{-\delta}(s_0)) \right) \neq 0,$$

provided that $s_0 \geq s_{2,3}(\delta, b_0)$. Thus, (4.14) follows.

By equality (4.13) and (4.14) and using the Implicit function Theorem, we obtain the existence of a unique $s^* > 0$ and $b \in C^1(s_0, s^*)$ such that $q$ defined as in (2.7), verifies (2.8), and the orthogonal condition (4.11) hold. Moreover, if we assume furthermore that $(q, b)$ is shrunk in the set $V_{A,b,b_0}(s)$ for all $s \in [s_0, s^*]$, then, we can repeat the computation for (??) in using the bounds given in Definition (4.1) and we obtain

$$\frac{\partial F}{\partial b}(s_0, b_0) = I^{-4k}(s^*) 2^{4k}(2k)! \left( 1 + O(I^{-\delta}(s^*)) \right) \neq 0.$$ 

Then, we can apply the Implicit function theorem to get the existence and uniqueness of $(q, b)$ on the interval $[s^*, s^* + \varepsilon]$ for some $\varepsilon > 0$ small and the conclusion of the Lemma completely follows.

4.4. Reduction to a finite dimensional problem

As we defined shrinking set $V_{A,b,b_0}$ in Definition 4.1, it is sufficient to prove there exists a unique global solution $(q, b)$ on $[s_0, +\infty)$ for some $s_0$ large that

$$(q, b)(s) \in V_{A,b,b_0}(s), \forall s \geq s_0,$$

In particular, we show in this part that the control of infinite problem is reduced to a finite dimensional one. To get this key result, we first show the following priori estimates.

**Proposition 4.5** (A priori estimates). Let $b_0 > 0$ and $k \in \mathbb{N}, k \geq 2, b_0 > 0$, then there exists $\delta_3(k, b_0) > 0$ such that for all $\delta \in (0, \delta_3)$, there exists $s_3(\delta, b_0)$ such that for all $s_0 \geq s_3$, the following properties hold: Assume $(q, b)$ is a solution to problem (2.8) & (4.11) that $(q, b)(s) \in V_{A,b,b_0}(s)$ for all $s \in [\tau, \bar{\tau}]$ for some $\bar{\tau} \geq s_0$, and $q_{2k}(s) = 0$ for all $s \in [\tau, \bar{\tau}]$, then for all $s \in [\tau, s_1], s_0 \leq \tau \leq \bar{\tau}$, the following properties hold:

(i) (ODEs on the finite modes). For all $j \in \{0, ..., [M]\}$, we have

$$\left| q_j(s) - \left( 1 - \frac{j}{2k} \right) q_j(s) \right| \leq CI^{-2\delta}(s).$$

(ii) (Smallness of the modulation $b(s)$). It satisfies that

$$|b'(s)| \leq CI^{-\delta}(s) \text{ and } \frac{3}{4}b_0 \leq b(s) \leq \frac{5}{4}b_0.$$ 

(iii) (Control of the infinite-dimensional part $q_-$): We have the following a priori estimate

$$|q_-(s)|_s \leq e^{-\frac{\tau - s}{b_0}} |q_-(\tau)|_\tau + C \left( I^{-\frac{3}{4}\delta}(s) + e^{-\frac{s - \tau}{b_0}} I^{-\frac{3}{4}\delta}(\tau) \right).$$
Proof of Proposition 4.5. This result plays an important role in our proof. In addition to that, the proof based on a long computation which is technical. To help the reader in following the paper, we will give the complete proof in Section 4.5.

Consequently, we have the following result

**Proposition 4.6** (Reduction to a finite dimensional problem). Let $b_0 > 0$ and $k \in \mathbb{N}, k \geq 2$, then there exists $\delta_1(b_0)$ such that for all $\delta \in (0, \delta_1)$, there exists $s_4(b_0, \delta)$ such that for all $s_0 \geq s_4$, the following property holds: Assume that $(q, b)$ is a solution to (2.8) & (4.11) corresponding to initial data $(q, b)(s_0) = (\psi(d_0, ..., d_{2k-1}), s_0)$ where $\psi(d_0, ..., d_{2k-1}, s_0)$ defined as in (4.5) with $\max_{0 \leq i \leq 2k-1} |d_i| \leq 2$; and $(q, b)(s) \in V_{\delta, b_0}(s)$ for all $s \in [s_0, \bar{s}]$ for some $\bar{s} > s_0$ that $(q, b)(\bar{s}) \in \partial V_{\delta, b_0}(\bar{s})$, then the following properties are valid:

(i) *(Reduction to finite modes)*: Consider $q_0, ..., q_{2k-1}$ be projections defined as in (3.13) then, we have

$$
(q_0, ..., q_{2k-1})(\bar{s}) \in \partial \hat{V}(\bar{s}),
$$

where $I(s)$ is given by (2.3).

(ii) *(Transverse crossing)* There exists $m \in \{0, ..., 2k - 1\}$ and $\omega \in \{-1, 1\}$ such that

$$
\omega q_m(s_1) = I(s_1)^{-\delta} \quad \text{and} \quad \omega \frac{dq_m}{ds} > 0.
$$

**Remark 4.7.** In (ii) of Proposition 4.6, we show that the solution $q(s)$ crosses the boundary $\partial V_{\delta, b_0}(s_1)$ at $s_1$ with positive speed, in other words, that all points on $\partial V_{\delta, b_0}(s_1)$ are strict exit points in the sense of [6, Chapter 2].

Proof. Let us start the proof Proposition 4.6 assuming Proposition 4.5. Let us consider $\delta \leq \delta_3$ and $s_0 \geq s_3$ that Proposition 4.5 holds.

- **Proof of item (i)** To get the conclusion of this item, we aim to show that for all $s \in [s_0, \bar{s}]$

$$
|q_j(s)| \leq \frac{1}{2} I^{-\delta}(s), \forall j \in \{2k + 1, ..., [M]\} \text{(note that } q_{2k} \equiv 0),
$$

and

$$
|q_-(s)|_s \leq \frac{1}{2} I^{-\delta}(s),
$$

+ For (4.18): From item (i) of Proposition 4.5, we have

$$
\left[ q_j(s) \pm \frac{1}{2} I^{-\delta}(s) \right]' = \left(1 - \frac{j}{2k}\right) q_j(s) \pm \frac{\delta}{2} \left( \frac{1}{2k} \right) I^{-\delta}(s) + O(I^{-2\delta}(s)),
$$

Hence, with $j > 2k, \delta \leq \delta_4, 1$ and initial data $q_j(s_0) = 0$ that $q_j(s_0) \in (-\frac{1}{2} I^{-\delta}(s_0), \frac{1}{2} I^{-\delta}(s_0))$, it follows that

$$
q_j(s) \in \left(-\frac{1}{2} I^{-\delta}(s), \frac{1}{2} I^{-\delta}(s) \right), \forall s \in [s_0, \bar{s}_0],
$$

which concludes (4.18).

+ For (4.19): Let consider $\sigma \geq 1$ fixed later. We divide into two cases that $s - s_0 \leq s_0$ and $s - s_0 \geq s_0$. According to the first case, we apply item (iii) with $\tau = s_0$ that

$$
|q_-(s)|_s \leq C \left(I^{-\frac{3}{2}d}(s) + e^{\frac{s-s_0}{\mu+1} I^{-\frac{3}{2}d}(s_0)} \right) \leq \frac{1}{2} I^{-\delta}(s),
$$
provided that $\delta \leq \delta_{42}$ and $s_0 \geq s_4(\delta)$. In the second case, we use item (iii) again with $\tau = s - s_0$, and we obtain
\[
|q_-(s)|_s \leq e^{-\frac{m}{p-1}}I^{-\delta}(\tau) + C\left(I^{-\frac{2s}{3}} + e^{-\frac{m}{p-1}}I^{-\frac{2s}{3}}(\tau)\right)
\leq C(e^{-\frac{m}{p-1}}I^{-\frac{2s}{3}}(s)I^{-\frac{2s}{3}}(\tau) + I^{-\frac{2s}{3}}(s))i^{-\delta}(s) \leq \frac{1}{2}I^{-\delta}(s).
\]
Thus, (4.19) follows. Finally, using the definition of $W_{b_0}(s)$, the fact $(q,b)(s) \in \partial W_{b_0}(s)$, estimates (4.18), (4.19), and item (ii) of Proposition 4.6, we get the conclusion of item (ii).

- Proof of item (ii): From item (ii) of Proposition 4.6, there exist $m = 0, \ldots, 2k - 1$ and $\omega = \pm 1$ such that $q_m(s_1) = \omega I(s_1)^{-\delta}$. By (ii) of Proposition 4.5, we see that for $\delta > 0$
\[
\omega q_m'(s_1) \geq (1 - m_{2k})\omega q_m(s_1) - CI^{-2\delta}(s_1) \geq C\left((1 - m_{2k})I^{-\delta}(s_1) - I^{-2\delta}(s_1)\right) > 0,
\]
which concludes the proof of Proposition 4.6. It remains to prove Proposition 4.5. This will be done in Section 5. \qed

4.5. Topological “shooting method“ for the finite dimension problem and proof of Theorem 1.1

In this part we aim to give the complete proof to Theorem 1.1 by using a topological shooting method:

The proof of Theorem 1.1. Let us consider $\delta > 0$, $T > 0$, $(T = e^{-s_0})$, $(d_0, \ldots, d_{2k-1}) \in \mathbb{D}$ such that problem (2.8) & (4.11) with initial data $\psi(d_0, \ldots, d_{2k-1}, s_0)$ defined as in (4.5) has a solution $(q(s), b(s))_{d_0, \ldots, d_{2k-1}}$ defined for all $s \in [s_0, +\infty)$ such that
\[
\|q(s)\|_{L^\infty} \leq CI^{-\delta}(s) \text{ and } |b(s) - b_*| \leq CI^{-2\delta}(s), \tag{4.20}
\]
for some $b_*>0$.

Let $b_0, \delta$ and $s_0$ such that Lemma 4.5, Propositions 4.6 and Proposition 4.5 hold, and we denote $T = e^{-s_0}$ (positive since $s_0$ is large enough). We proceed by contradiction, from (ii) of Lemma 4.5, we assume that for all $(d_0, \ldots, d_{2k-1}) \in \mathbb{D}$ there exists $s_* = s_*(d_0, \ldots, d_{2k-1}) < +\infty$ such that
\[
q_{d_0, \ldots, d_{2k-1}}(s) \in V_{b_0}(s), \quad \forall s \in [s_0, s_*],
q_{d_0, \ldots, d_{2k-1}}(s_*) \in \partial V_{b_0}(s_*).
\]

By using item (i) of Proposition 4.6, we get $(q_0, \ldots, q_{2k-1})_{s_*} \in \partial V(s_*)$ and we introduce $\Phi$ by
\[
\Phi : \mathbb{D} \to [0, 1]^{2k},
(d_0, \ldots, d_{2k-1}) \to \Phi(s)(q_0, \ldots, q_{2k-1})(s_*),
\]
which is well defined and satisfies the following properties:

(i) $\Phi$ is continuous from $\mathbb{D}$ to $[0, 1]^{2k}$ thanks to the continuity in time of $q$ on the one hand, and the continuity of $s_*$ in $(d_0, \ldots, d_{2k-1})$ on the other hand, which is a direct consequence of the transversality in item (ii) of Proposition 4.6.

(ii) It holds that $\Phi|_{\partial\mathbb{D}}$ has nonzero degree. Indeed, for all $(d_0, \ldots, d_{2k-1}) \in \partial\mathbb{D}$, we derive from item (i) of Lemma 4.2 that $s_*(d_0, \ldots, d_{2k-1}) = s_0$ and
\[
\deg \left(\Phi \big|_{\partial\mathbb{D}}\right) \neq 0.
\]

From Wazewski’s principle in degree theory such a $\Phi$ cannot exist. Thus, we can prove that there exists $(d_0, \ldots, d_{2k-1}) \in \mathbb{D}$ such that the corresponding solution $(q, b)(s) \in V_{b_0}(s), \forall s \geq s_0$. In particular, we derive from (2.4), $M = \frac{2kp}{p-1}$, and the following estimate
\[
|f_b|_b = |f_b|_b \leq C(1 + |y|^{-\frac{2kp}{p-1}}) = C(1 + |y|^{-M})
\]
that
\[ \|w(y,s) - f_b\|_{L^\infty} = \|f_{b\psi_b q}\|_{L^\infty} \leq CI^{-\delta}(s). \]
So, we conclude item (i) of Theorem 1.1.

The proof of item (ii): From (ii) of Proposition 4.5, it immediately follows that there exists \( b^* \in \mathbb{R}^*_+ \) such that
\[ b(s) \to b^* \text{ as } s \to +\infty, \]
which is equivalent to
\[ b(t) \to b^* \text{ as } t \to T. \]
In particular, by integrating the first inequality given by \( b \) between \( s \) and \( \infty \) and using the fact that
\[ b(s) \to b^* \text{ (see (5.36))}, \]
we obtain
\[ |b(s) - b^*| \leq Ce^{-\delta s(1 - \frac{1}{k})}. \]
Note that \( s = -\ln(T - t) \) then, (4.20) follows and the conclusion of item (ii) of Theorem 1.1.

5. Proof to Proposition 4.5

In this section, we prove Proposition 4.5. We just have to project equation (2.8) to get equations satisfied by the different coordinates of the decomposition (3.11). More precisely, the proof will be carried out in 2 subsections,

- In the first subsection, we write equations satisfied by \( q_j \), \( 0 \leq j \leq M \), then, we prove (i), (ii) of Proposition 4.5.
- In the second subsection, we first derive from equation (2.8) an equation satisfied by \( q_- \) and prove the last identity in (iii) of Proposition 4.5.

5.1. The proof to items (i) and (ii) of Proposition 4.5

- In Part 1, we project equation (2.8) to get equations satisfied by \( q_j \) for \( 0 \leq j \leq [M] \).
- In Part 2: We will use the precise estimates from part I to conclude items (i) and (ii) of Proposition 4.5.

Part 1: The projection of equation (2.8) on the eigenfunctions of the operator \( \mathcal{L}_s \).

Let \( (q, b) \) be solution to problem (2.8) \& (4.11) trapped in \( V_{\delta,b_0}(s) \) for all \( s \in [s_0, \bar{s}] \) for some \( \bar{s} > s_0 \). Then, we have the following:

a) First term \( \partial_s q \): In this part, we aim to estimate the error between \( \partial_s q_n(s) \) and \( P_n(\partial_s q) \) by the following Lemma

**Lemma 5.1.** For all \( n \in \{0, 1, \ldots, [M]\} \), it holds that
\[ P_n(\partial_s q) = \partial_s q_n(s) - \left(1 - \frac{1}{k}\right)(n + 1)(n + 2)I^{-2}(s)q_{n+2}(s), \forall s \in [s_0, \bar{s}]. \]

**Proof.** We only give the proof when \( n \geq 2 \), for \( n = 0, 1 \) it is easy to derive the result. Using (3.13), we have the following equality
\[ \langle H_n, H_n \rangle_{L^2_{\rho_s}} q_n(s) = \langle q, H_n(s) \rangle_{L^2_{\rho_s}}, \]
which implies
\[ \langle H_n, H_n \rangle_{L^2_{\rho_s}} \partial_s q_n(s) = \langle \partial_s q, H_n \rangle_{L^2_{\rho_s}} + \langle q, \partial_s H_n(s) \rangle_{L^2_{\rho_s}} + \left\langle q, H_n(s) \partial_s \rho_s \right\rangle_{L^2_{\rho_s}} - \partial_s \langle H_n, H_n \rangle_{\rho_s} q_n, \]
which yields

\[
P_n(\partial_s q) = \partial_s q_n - \langle q, \partial_s H_n(s) \rangle_{L^2_{\rho_s}} (H_n, H_n)^{-1}_{L^2_{\rho_s}}
- \left( q, H_n(s) \frac{\partial_s \rho_s}{\rho_s} \right)_{L^2_{\rho_s}} (H_n, H_n)^{-1}_{L^2_{\rho_s}} + \partial_s \langle H_n, H_n \rangle_{\rho_s} (H_n, H_n)^{-1}_{L^2_{\rho_s}} q_n.
\]

Thus, we can write

\[
\partial_s q_n = P_n(\partial_s q) + \tilde{L},
\]

where

\[
\tilde{L} = \langle q, \partial_s H_n(s) \rangle_{L^2_{\rho_s}} (H_n, H_n)^{-1}_{L^2_{\rho_s}} + \left( q, H_n(s) \frac{\partial_s \rho_s}{\rho_s} \right)_{L^2_{\rho_s}} (H_n, H_n)^{-1}_{L^2_{\rho_s}} - \partial_s \langle H_n, H_n \rangle_{\rho_s} (H_n, H_n)^{-1}_{L^2_{\rho_s}} q_n.
\]

We now aim to estimate \( \tilde{L} \) provided that \( (q(s), b(s)) \in V_{A, b_0, \delta}(s) \) and we also recall that

\[
q = \sum_{j=1}^{M} q_j H_j + q_-.
\]

+ For \( \partial_s \langle H_n, H_n \rangle_{\rho_s} (H_n, H_n)^{-1}_{L^2_{\rho_s}} q_n \): We have the facts that

\[
\langle H_n, H_n \rangle_{\rho_s} = I^{-2n(s)}2^n n!, \text{ and } I(s) = e^s(1 - \frac{1}{s}),
\]

which implies

\[
\partial_s \langle H_n, H_n \rangle_{L^2_{\rho_s}} = -n \left( 1 - \frac{1}{k} \right) \langle H_n, H_n \rangle_{L^2_{\rho_s}}.
\]

So, we obtain

\[
\partial_s \langle H_n, H_n \rangle_{L^2_{\rho_s}} = -n \left( 1 - \frac{1}{k} \right) \langle H_n, H_n \rangle_{L^2_{\rho_s}} q_n(s).
\]

\[
+ \text{ For } \left( q, H_n(s) \frac{\partial_s \rho_s}{\rho_s} \right)_{L^2_{\rho_s}} (H_n, H_n)^{-1}_{L^2_{\rho_s}} q_n(s) = -n \left( 1 - \frac{1}{k} \right) q_n(s).
\]

\[
\partial_s \rho_s = \frac{1}{2} \left( 1 - \frac{1}{k} \right) \rho_s - \frac{1}{4} \left( 1 - \frac{1}{k} \right) I^2(s)y^2 \rho_s,
\]

which yields

\[
\left( q, H_n(s) \frac{\partial_s \rho_s}{\rho_s} \right)_{L^2_{\rho_s}} = \frac{1}{2} \left( 1 - \frac{1}{k} \right) \left( q, H_n(s) \right)_{L^2_{\rho_s}} - \frac{1}{4} \left( 1 - \frac{1}{k} \right) \langle q, I^2(s)y^2 H_n(s) \rangle_{L^2_{\rho_s}}
\]

\[
= \frac{1}{2} \left( 1 - \frac{1}{k} \right) q_n \langle H_n, H_n \rangle_{L^2_{\rho_s}} - \frac{1}{4} \left( 1 - \frac{1}{k} \right) \langle q, I^2(s)y^2 H_n(s) \rangle_{L^2_{\rho_s}}.
\]

Thus, we derive

\[
\left( q, H_n(s) \frac{\partial_s \rho_s}{\rho_s} \right)_{L^2_{\rho_s}} (H_n, H_n)^{-1}_{L^2_{\rho_s}} = \frac{1}{2} \left( 1 - \frac{1}{k} \right) q_n - \frac{1}{4} \left( 1 - \frac{1}{k} \right) \langle q, I^2(s)y^2 H_n(s) \rangle_{L^2_{\rho_s}} (H_n, H_n)^{-1}_{L^2_{\rho_s}}.
\]

Using the polynomial Hermite identities, we obtain

\[
z^2 h_n = z h_{n+1} + 2nz h_{n-1} = h_{n+2} + 2(n+1)h_n + 4n(n-1)h_{n-2},
\]
and we find the following identity
\[ y^2 H_n(y, s) = H_{n+2}(y, s) + (4n + 2)I^{-2}(s)H_n(y, s) + 4n(n - 1)I^{-4}(s)H_{n-2}(y, s) \]
This implies that
\[
\langle q, I^2(s)y^2 H_n(s) \rangle_{L^2_{\rho s}} = I^2(s) \left[ q_{n+2} \| H_{n+2} \|_{L^2_{\rho s}}^2 + I^{-2}(s)q_n \| H_n \|_{L^2_{\rho s}}^2 + 4n(n - 1)q_{n-2}I^{-4}(s)\| H_{n-2} \|_{L^2_{\rho s}}^2 \right],
\]
which yields
\[
\begin{align*}
\left\langle q, H_n(s) \frac{\partial_s \rho_s}{\rho_s} \right\rangle_{L^2_{\rho s}} & = -n \left( 1 - \frac{1}{k} \right) q_n - n(n - 1) \left( 1 - \frac{1}{k} \right) q_{n-2}I^{-2}(s) \| H_{n-2} \|_{L^2_{\rho s}}^2 \\
& + \left( 1 - \frac{1}{k} \right) (n + 2)(n + 1)I^{-2}(s)q_{n+2},
\end{align*}
\]
for all \( n \in \{0, ..., [M]\} \) and \( \forall s \in [s_0, s^*] \) (with convention that \( q_j = 0 \) if \( j < 0 \)) and for some \( \tilde{c}_n \in \mathbb{R} \).

Let us recall the following identify on Hermite’s polynomial
\[ yH_{n-1}(y, s) = H_n(y, s) + I^{-2}(s)2(n - 1)H_{n-2}(y, s). \quad (5.2) \]
So, we can rewrite \( \partial_s H_n \) as follows
\[ \partial_s H_n(y, s) = n(n - 1) \left( 1 - \frac{1}{k} \right) I^{-2}(s)H_{n-2}(y, s). \quad (5.3) \]
Thus, we obtain
\[
\langle q, \partial_s H_n(s) \rangle_{L^2_{\rho s}} \langle H_n, H_n^{-1} \rangle_{L^2_{\rho s}} = n(n - 1) \left( 1 - \frac{1}{k} \right) I^{-2}(s)q_{n-2} \| H_{n-2} \|_{L^2_{\rho s}}^2.
\]
Finally, we obtain
\[ \partial_s q_n = P_n(\partial_s q) + \left( 1 - \frac{1}{k} \right)(n + 1)(n + 2)q_{n+2}, \forall n \in \{0, 1, ..., [M]\}, \]
which concludes the proof of the Lemma.

**b) Second term \( L_s(q) \)**

**Lemma 5.2.** For all \( 0 \leq n \leq [M] \), it holds that
\[ P_n(L_s q) = \left( 1 - \frac{n}{2k} \right)q_n + \left( 1 - \frac{1}{k} \right)(n + 1)(n + 2)I^{-2}q_{n+2}. \quad (5.4) \]
Proof. As in the proof of Lemma 5.1, we only give the proof when \( n \geq 2 \), for \( n = 0, 1 \) it is easy to derive the result. We write \( P_n(\mathcal{L}_s q) \) as follows:

\[
P_n(\mathcal{L}_s q) = \int \left( I^{-2}(s)\Delta q - \frac{1}{2} y \cdot \nabla q + q \right) H_n \rho_s dy + \int \frac{1}{2} (1 - \frac{1}{k}) y \nabla q H_n \rho_s dy
\]

\[
= A_1 + \frac{1}{2} (1 - \frac{1}{k}) A_2.
\]

In the following we will use Hermite polynomial identity (A.5) given by Lemma A.3. Using integration by part and polynomial identities we obtain

\[
A_1 = \int \left( I^{-2}(s)\Delta q - \frac{1}{2} y \cdot \nabla q + q \right) H_n \rho_s dy
\]

\[
= \int I^{-2} \text{div} (\nabla q \rho_s) + q H_n \rho_s dy,
\]

\[
= -I^{-2} \int \nabla q H_n \rho_s dy + q_n \| H_n \|_{L^2_{\rho_s}}^2,
\]

\[
= n(n-1)I^{-2} \int q H_n^{-2} \rho_s - \frac{n}{2} \int y q H_n \rho_s dy + q_n \| H_n \|_{L^2_{\rho_s}}^2
\]

\[
= - \left( 1 - \frac{n}{2} \right) q_n \| H_n \|_{L^2_{\rho_s}}^2.
\]

By a similar computation, using the change of variable \( z = I y \) and we introduce \( \rho(z) = I^{-1} \rho_s(y) \), we get

\[
A_2 = \int y \nabla q H_n \rho_s dy
\]

\[
= \left( - \int q H_n \rho_s dy - \int q ny H_n^{-1} \rho_s dy + \frac{I^2}{2} \int q y^2 H_n \rho_s dy \right)
\]

\[
= -q_n \| H_n \|_{L^2_{\rho_s}}^2 - I^{-n} n \int q z h_{n-1} \rho dz + \frac{1}{2} I^{-n} \int z^2 q \rho dz
\]

\[
- q_n \| H_n \|_{L^2_{\rho_s}}^2 - I^{-n} n \int q (h_n + (n-1) h_{n-2}) \rho dz + \frac{1}{2} I^{-n} \int z^2 q \rho dz
\]

Using the polynomial Hermite identities that

\[
z^2 h_n = z h_{n+1} + 2 n z h_{n-1} = h_{n+2} + 2(2n+1)h_n + 4n(n-1)h_{n-2},
\]

which yields

\[
A_2 = \left( -q_n \| H_n \|_{L^2_{\rho_s}}^2 - I^{-n} n \int q z h_{n-1} \rho dz + \frac{1}{2} I^{-n} \int z^2 q \rho dz \right)
\]

\[
= -q_n \| H_n \|_{L^2_{\rho_s}}^2 - I^{-n} n \int q (h_n + 2(n-1) h_{n-2}) \rho dz
\]

\[
+ \frac{1}{2} I^{-n} \int q h_{n+2} + 2(2n+1)h_n + 4n(n-1)h_{n-2} \rho dz
\]

\[
= -q_n \| H_n \|_{L^2_{\rho_s}}^2 - q_n \| H_n \|_{L^2_{\rho_s}}^2 - 2n(n-1)I^{-2} q_{n-2} \| H_{n-2} \|_{L^2_{\rho_s}}^2 + \frac{1}{2} q_{n+2} I^2 \| H_{n+2} \|_{L^2_{\rho_s}}^2
\]

\[
= (nq_n + 2(n+2)(n+1)I^{-2} q_{n+2}) \| H_n \|_{L^2_{\rho_s}}^2.
\]

Thus, we obtain by adding all related terms that

\[
P_n(\mathcal{L}_s q) = \left( 1 - \frac{n}{2k} \right) q_n + (1 - \frac{1}{k})(n + 2)(n + 1)I^{-2} q_{n+2},
\]

which concludes the proof of Lemma 5.2. 

\[\square\]

c) Third term, the nonlinear term \( N(q) \)

In this part, we aim to estimate to the projection of \( N(q) \) on \( H_n \), for some \( n \in \{0, 1, \ldots, [M]\} \). More precisely, we have the following Lemma:
Lemma 5.3. Let $b_0 > 0$, then, there exists $\delta_5(b_0) > 0$ such that for all $\delta \in (0, \delta_5)$ there exists $s_5(b_0, \delta) \geq 1$ such that for all $s_0 \geq s_5$, the following property is valid: Assume $(q, b)(s) \in V_{b_0}^\delta(s)$ for all $s \in [s_0, \bar{s}]$ for some $\bar{s} > s_0$, then, we have

$$|P_n(N)| \leq C I^{-2\delta}(s), \forall \text{ and } n \in \{0, 1, \ldots, [M]\},$$

(5.5)

for all $s \in [s_0, \bar{s}]$ and $0 \leq n \leq [M]$.

Proof. We argue as in [3]. First, let us recall the nonlinear term $N$ and $P_n(N)$ defined as in (2.10) and (3.13), respectively. The main goal is to use the estimates defined in $V_{b_0}^\delta(s)$ to get an improved bound on $P_n(N)$. Firstly, we recall the following identity

$$e_b(y) = (p - 1)^{-1} \left( \sum_{\ell=0}^{L} \left( -\frac{by^{2k}}{p - 1} \right)^\ell + \left( -\frac{br^{2k}}{p - 1} \right)^{L+1} e_b(y) \right), \forall L \in \mathbb{N}^*.$$  (5.6)

From the fact that $(q, b)(s) \in V_{b_0}^\delta(s)$ for all $s \in [s_0, s_1]$, then get the following

$$|e_b(y)q(y)| = |e_b(y)| \left( \sum_{m=0}^{M} q_m(s)H_m(y, s) + q_-(y, s) \right) \leq CI^{-\delta}(s)(1 + |y|^M),$$

(5.7)

which implies

$$|N(q)(y, s)| \leq C |1 + e_b(y)q(y, s)|^p \leq C[1 + L^{-p-\delta}(s)(1 + |y|^M)].$$

(5.8)

By applying Lemma A.2 with $f(y) = N(y)$ and $K = pM, \delta = 0$, we obtain

$$\left| \int_{|y| \geq 1} N(y, s)H_n(y, s)\rho_s(y)dy \right| \leq Ce^{-\frac{L}{M}}, \forall n \in \{0, 1, \ldots, [M]\},$$

(5.9)

then it follows

$$\left| \int_{|y| \geq 1} N(y, s)H_n(y, s)\rho_s(y)dy \right| \leq CI^{-2\delta-2n}, \forall n \in \{0, 1, \ldots, [M]\}.$$  (5.10)

provided that $s_0 \geq s_1, (\delta, M)$. We here claim that the following estimate

$$\left| \int_{|y| \leq 1} N(y, s)H_n(y, s)\rho_s(y)dy \right| \leq CI^{-2\delta-2n}, \forall n \in \{0, 1, \ldots, [M]\},$$

(5.11)

directly concludes the proof of Lemma 5.3. Indeed, let us assume that (5.10) and (5.11) hold, then we derive

$$\left| \langle N, H_n(y, s) \rangle_{L^2} \right| \leq CI^{-2\delta-2n}(s), \forall n \in \{0, 1, \ldots, [M]\},$$

which implies

$$|P_n(N)| \leq CI^{-\delta}(s), \forall s \in [s_0, s_1] \text{ and } n \in \{0, 1, \ldots, [M]\},$$

since and it concludes (5.3) and also Lemma 5.3. Now, it remains to prove (5.11). From (5.7), we have

$$|e_b(s)(y)q(y, s)| \leq CI^{-\delta}(s), \forall s \in [s_0, s_1] \text{ and } |y| \leq 1.$$

(5.12)

then, we apply Taylor expansion to function $N(q)$ in the variable $z = qe_b$ (here we usually denote $b$ standing for $b(s)$) and we get

$$N(q) = [1 + e_bq]^{p-1}(1 + e_bq) - 1 - pe_bq = \sum_{j=2}^{K} c_j(e_bq)^{j} + R_K,$$

(5.13)

where $K$ will be fixed later and the reader should bear in mind that we only consider $|y| \leq 1$ in this part. For the remainder $R_K$, we derive from

$$|R_K(y, s)| \leq C |e_b(y)q(y, s)|^{K+1} \leq CI^{-\delta(K+1)}(s).$$

(5.14)
Besides that, we recall from (3.11) that \( q = q_+ + q_- \) and we have then express
\[
\sum_{j=2}^{K} c_j(e_0q)^j = \sum_{j=2}^{K} d_{j,j}(e_0q_+)^j + \sum_{j=2}^{K} \sum_{\ell=0}^{i-1} d_{j,\ell}e_{\ell}(q_+)^\ell(q_-)^{j-\ell} = A + S,
\]
where
\[
A = \sum_{j=2}^{K} d_{j}(e_0q_+)^j \text{ and } S = \sum_{j=2}^{K} \sum_{\ell=0}^{i-1} \tilde{d}_{j,\ell}e_{\ell}(q_+)^\ell(q_-)^{j-\ell}, \text{ for some } d_j, \tilde{d}_{j,\ell} \in \mathbb{R}. \tag{5.15}
\]

From the above expressions, we can decompose \( \mathcal{N} \) by
\[
\mathcal{N} = A + S + R_K,
\]
and we also have
\[
\int_{|y| \leq 1} \mathcal{N}(y, s)H_n(y, s)\rho(y)dy = \int_{|y| \leq 1} AH_n(y, s)\rho(y)dy + \int_{|y| \leq 1} SH_n(y, s)\rho(y)dy + \int_{|y| \leq 1} R_KH_n(y, s)\rho(y)dy.
\]

- The integral for \( R_K \) Note that \( H_n \) defined in (3.3) satisfies
\[
|H_n(y, s)| \leq C(1 + |y|^n) \leq C, \forall |y| \leq 1,
\]
hence, it follows from (5.14) that
\[
\int_{|y| \leq 1} R_K(y, s)H_n(y, s)\rho_s(y)dy \leq C I^{-\delta(K+1)}(s) \int_{|y| \leq 1} e^{-\frac{I(\rho_s)^2}{4}} I(s)dy \leq C I^{-\delta(K+1)}(s) \leq C I^{-2\delta-2n}(s), \forall s \in [s_0, s_1], \tag{5.16}
\]
provided that \( K \geq K_1(\delta, M) \).

- The integral for \( S \): Since \( (q, b)(s) \in V_{0,b_0}(s) \), for all \( s \in [s_0, \bar{s}] \), we can estimate as follows
\[
|q_+(y, s)|^\ell + |q_-(y, s)|^\ell = \left| \sum_{m=0}^{M} m_n(s)H_m(y, s) \right|^\ell + C I^{-\ell\delta}(s)(I^{-M}(s) + |y|^M)^\ell \leq C I^{-\ell\delta}(s),
\]
for all \( |y| \leq 1 \) and \( \ell \in \mathbb{N} \). Regarding to (5.15), we can estimate as follows
\[
|S(y, s)| \leq C \left( |q_+(y, s)| + |q_-(y, s)|^2 \right) \leq C I^{-2\delta}(s)(I^{-M}(s) + |y|^M),
\]
provided that \( s_0 \geq s_{1.3}(K) \). Thus, we derive
\[
\left| \int_{|y| \leq 1} S(y, s)H_n(y, s)\rho_s(y)dy \right| \leq C I^{-2\delta}(s) \int_{|y| \leq 1} (I^{-M}(s) + |y|^M) |H_n(y, s)| e^{-\frac{I(\rho_s)^2}{4}} I(s)dy.
\]
Accordingly to (3.3) and changing of variable \( z = I(s)y \), we have
\[
\int_{|y| \leq 1} (I^{-M}(s) + |y|^M) |H_n(y, s)| e^{-\frac{I(\rho_s)^2}{4}} I(s)dy = I^{-M-n}(s) \int_{|z| \leq I(s)} (1 + |z|^M)|h_n(z)| e^{-\frac{|z|^2}{4}} dz \leq C I^{-M-n}(s), \tag{5.17}
\]
Finally, we have
\[
\left| \int_{|y| \leq 1} S(y, s)H_n(y, s)\rho_s(y)dy \right| \leq C I^{-2\delta-2n}(s), \forall n \leq M, \forall s \in [s_0, s_1], \tag{5.18}
\]
provided that \( s_0 \geq s_{1.3}(K) \).
- The integral for $A$: From (3.11) and (4.17), we write

$$
(e_q q_+)^j = \left( \sum_{k=0}^{K-1} E_\ell^f y^{2k} \right)^j \left( \sum_{m=0}^{[M]} q_m H_m \right)^j + O(|q_+|^2 y^{K(2k)}), \forall j \geq 2.
$$

By using the technique in (5.17) (changing variable $z = I(s)y$), we obtain

$$
\int_{|y| \leq 1} |y|^{K(2k)} |q_+|^2(y) \rho_s(y) dy \leq CI^{-\delta}(s) \int_{|y| \leq 1} |y|^{K(2k)} \left( \sum_{m=0}^{[M]} |H_m(y, s)| \right)^2 \rho_s dy
$$

$$
\leq I^{-2\delta - K(2k)}(s) \leq CI^{-2\delta - 2n}(s),
$$

provided that $K \geq K_2(\delta, M)$ large enough. In addition, we derive from $H_m$'s definition defined in (3.3) that

$$
\left( \sum_{\ell=0}^{K-1} E_\ell^f y^{(2k)\ell} \right)^j \left( \sum_{m=0}^{[M]} q_m H_m \right)^j = \sum_{k=0}^{L} A_k(s)y^k \text{ where } L = j ([M] + (K - 1)(2k)),
$$

and $A_j$ satisfying

$$
|A_j(s)| \leq CI^{-\delta}(s).
$$

Now, we apply Lemmas A.1 and A.2 to deduce

$$
\left| \int_{|y| \leq 1} \left( \sum_{\ell=0}^{K-1} E_n b^n y^{(2k)\ell} \right)^j \left( \sum_{m=0}^{[M]} q_m H_m \right)^j H_n(y, s) \rho_s(y) dy \right| \leq CI^{-2\delta - 2n}(s).
$$

(5.20)

Thus, we get

$$
\left| \int_{|y| \leq 1} A(y, s)H_n(y, s) \rho_s(y) dy \right| \leq CI^{-2\delta - 2n}(s), \forall n \leq M, \forall s \in [s_0, s_1].
$$

(5.21)

According to (5.16), (5.18) and (5.21), we have

$$
\left| \int_{|y| \leq 1} N(q)H_n(y, s) \rho_s(y) dy \right| \leq CI^{-2\delta - 2n}(s),
$$

(5.22)

provided that $s_0 \geq s_{1,3}(K)$, and $K \geq K_2$. Thus, (5.11) follows which concludes the conclusion of the Lemma.

**d) Fourth term $b'(s)\mathcal{M}(q)$.** Let us consider $\mathcal{M}$'s definition that

$$
\mathcal{M}(q) = \frac{p}{p-1} y^{2k} (1 + e_q q),
$$

we have then the following result:

**Lemma 5.4.** Let $b_0 > 0$, then there exists $\delta_0(b_0)$ such that for all $\delta \in (0, \delta_0)$, then there exists $s_0(\delta, b_0) \geq 1$ such that for all $s_0 \geq s_0$ the following holds: Assume $(q, b)(s) \in V_{\delta(b_0)}(s), \forall s \in [s_0, \bar{s}]$ for some $\bar{s}$ arbitrary, then it holds that

$$
P_n(\mathcal{M}(q)(s)) = \begin{cases} 
\frac{p}{p-1} + O(I^{-\delta}(s)) & \text{if } n = 2k \\
O(I^{-\delta}(s)) & \text{if } n \neq 2k, n \in \{0, 1, \ldots, [M]\}
\end{cases}
$$

(5.23)

for all $s \in [s_0, \bar{s}]$. 
Proof. We firstly decompose as follows
\[
\langle \mathcal{M}, H_n(y, s) \rangle_{L^2_{\rho_s}} = \left\langle \frac{p}{p-1} y^{2k}, H_n(y, s) \right\rangle_{L^2_{\rho_s}} + \left\langle \frac{p}{p-1} y^{2k} e_b(y) q, H_n(y, s) \right\rangle_{L^2_{\rho_s}}.
\]
From (3.3), we get the following
\[
\left\langle \frac{p}{p-1} y^{2k}, H_n(y, s) \right\rangle_{L^2_{\rho_s}} = \frac{p}{p-1} \begin{cases} 
\| H_{2k} \|_{L^2_{\rho_s}}^2 & \text{if } n = 2k \\
O(1^{2k-2}(s)) & \text{if } n < 2k \\
0 & \text{if } n > 2k 
\end{cases}, \tag{5.24}
\]
Now we focus on the scalar product
\[
\left\langle \frac{p}{p-1} y^{2k} e_b(y) q, H_n(y, s) \right\rangle_{L^2_{\rho_s}}.
\]
We decompose
\[
\left\langle \frac{p}{p-1} y^{2k} e_b(y) q, H_n(y, s) \right\rangle_{L^2_{\rho_s}} = \int_{|y| \leq 1} \frac{p}{p-1} y^{2k} e_n(y) qH_n(y, s) \rho_s(y) dy + \int_{|y| \geq 1} \frac{p}{p-1} y^{2k} e_n(y) qH_n(y, s) \rho_s(y) dy.
\]
Since \( q \in V_{\delta, b_0}(s) \) for all \( s \in [s_0, s^*] \), the following estimate holds
\[
\left| \frac{p}{p-1} y^{2k} e_b(y) q \right| \leq C I^{-\delta}(s) |y|^{2k}(1 + |y|^M).
\]
Using Lemma A.2, we conclude
\[
\left| \int_{|y| \geq 1} \frac{p}{p-1} y^{2k} e_b(y) qH_n(y, s) \rho_s(y) dy \right| \leq C I^{-\delta} e^{-\frac{1}{8} t(s)} \leq C I^{-2\delta}(s), \forall s \in [s_0, s^*],
\]
provided that \( s_0 \geq s_3(\delta) \).

Let us decompose
\[
\frac{p}{p-1} y^{2k} e_b(y) q = \frac{p}{p-1} y^{2k} e_b(y) q_+ + \frac{p}{p-1} y^{2k} e_b(y) q_-.
\]
Since \( q \in V_{\delta, b_0}(s) \) and \( e_b \) bounded, we get
\[
\left| \frac{p}{p-1} y^{2k} e_b(y) q_- \right| \leq C I^{-\delta}(s) |y|^{2k}(I^{-M}(s) + |y|^M).
\]
By the same technique in (5.17), we obtain
\[
\left| \int_{|y| \leq 1} \frac{p}{p-1} y^{2k} e_b(y) q_- H_n(y, s) \rho_s(y) dy \right| \leq C I^{-2\delta-2n}(s), \forall s \in [s_0, s^*] \text{ and } n \in \{0, 1, \ldots, [M]\}. \tag{5.26}
\]
In addition, using (3.11) and (4.17), we write
\[
\frac{p}{p-1} y^{2k} e_b(y) q_+ = \sum_{i=0}^{M} \sum_{j=1}^{K} m_{ij} b^j q_i(s) y^{2kj} H_i(y, s) + O \left( I^{-\delta}(s) y^{(K+1)2k}(1 + |y|^M) \right).
\]
Repeating the technique in (5.17) (changing variable \( z = I(s)y \)), we obtain
\[
\left| \int_{|y| \leq 1} I^{-\delta}(s) y^{(K+1)2k}(1 + |y|^M) H_n(y, s) \rho_s(y) dy \right| \leq C I^{-2\delta-2n}(s), \forall s \in [s_0, s^*], n \in \{0, 1, \ldots, M\},
\]
provided that $K$ large enough. Besides that, we use the fact that $q \in V_{\delta,b_0}(s)$ to get
\[ |q_j(s)| \leq CI^{-\delta}(s), \]
and $H_i$ can be written by a polynomial in $y$, we apply Lemma A.1 and Lemma A.2, we derive
\[ \left| \int_{|y| \leq 1} \left( \sum_{i=0}^{M} \sum_{j=1}^{K} m_{i,j} b_{i}(s) y^{2kj} H_i(y, s) \right) H_n(y, s) \rho_s(y) dy \right| \leq CI^{-\delta-2n}(s), \forall s \in [s_0, s^*] \text{ and } n \in \{0, 1, \ldots, [M]\}. \]
Finally, we get
\[ \left| \int_{|y| \leq 1} \frac{p}{p-1} y^{2k} e_b(y) q H_n(y, s) \rho_s(y) dy \right| \leq CI^{-\delta-2n}(s), \forall s \in [s_0, s^*] \text{ and } n \in \{0, 1, \ldots, [M]\}. \quad (5.27) \]
Now, we combine (5.26) with (5.27) to imply
\[ \left| \int_{|y| \leq 1} \frac{p}{p-1} y^{2k} e_b(y) q H_n(y, s) \rho_s(y) dy \right| \leq CI^{-\delta-2n}(s), \forall s \in [s_0, s^*] \text{ and } n \in \{0, 1, \ldots, [M]\}. \quad (5.28) \]
We use (5.25) and (5.28) to conclude
\[ \left| \left\langle \frac{p}{p-1} y^{2k} e_b(y) q H_n(y, s) \right\rangle_{L^2_{\rho_s}} \right| \leq CI^{-\delta-2n}(s), \forall s \in [s_0, s^*] \text{ and } n \in \{0, 1, \ldots, [M]\}. \quad (5.29) \]
Finally, by (5.24) and (5.29) we conclude the proof of the Lemma. \( \blacksquare \)

e) Fifth term $\mathcal{D}_s(q)$

**Lemma 5.5** (Estimation of $P_n(\mathcal{D}_s)$). Let $b > 0$, then there exists $\delta = \delta(b_0) > 0$ such that for all $\delta \in (0, \delta_7)$, there exists $s_7(\delta, b_0)$ such that for all $s_0 \geq s_7$, the following property holds: Assume $(q, b)(s) \in V_{\delta,b_0}(s)$ for all $s \in [s_0, \tilde{s}]$ for some $\tilde{s} \geq s_0$, then we have
\[ |P_n(\mathcal{D}_s(q))| \leq CI^{-2\delta}(s), \text{ for all } s \in [s_0, \tilde{s}], \quad (5.30) \]
for all $0 \leq n \leq M$.

**Proof.** Let us now recall from (2.11) that
\[ \mathcal{D}_s(\nabla q) = -\frac{4pkb}{p-1} y^{2k-1} e_b \nabla q. \]
From (3.13) and (3.5), it is sufficient to estimate to
\[ \left\langle \mathcal{D}_s, H_n(y, s) \right\rangle_{L^2_{\rho_s}} = \int_{\mathbb{R}} \left( -\frac{4pkb}{p-1} y^{2k-1} e_b \nabla q H_n(y, s) \rho_s(y) dy \right) \]
From the fact that $\nabla (H_n) = n H_{n-1}, \rho_s(y) = \frac{I(s)}{4\pi} e^{-\frac{r^2(s)}{4}}$, we use integration by parts to derive
\[ \left\langle \mathcal{D}_s, H_n(y, s) \right\rangle_{L^2_{\rho_s}} = \frac{4pkb}{p-1} I^{-2}(s) \left( \int \nabla (y^{2k-1} e_b) q H_{n-1} \rho_s(y) dy, + n \int y^{2k-1} e_b q H_{n-1} \rho_s dy \right) - \frac{1}{2} I^2(s) \int y^{2k} e_b q y H_n \rho_s dy. \]
Then, we explicitly write the scalar product by four integrals as follows
\[
\langle D_s, H_n(y,s) \rangle_{L^2_{\rho_s}} = \frac{4pkb}{p-1}I^{-2}(s) \left\{ (2k-1) \int y^{2k-2}e_{yq}H_n\rho_s(y)dy - 2kb \int y^{4k-2}e_{yq}^2H_n\rho_s(y)dy + n \int y^{2k-1}e_{yq}H_{n-1}\rho_sdy - \frac{1}{2}I^2(s) \int y^{2k}e_{yq}H_n\rho_sdy \right\}.
\]
By the technique established in Lemma 5.4, we can prove
\[
\left| \langle D_s, H_n(y,s) \rangle_{L^2_{\rho_s}} \right| \leq CI^{-2\delta}(s), \quad \forall s \in [s_0, s^*], \quad \text{and} \quad n \in \{0, 1, ..., [M]\}.
\]
which concludes (5.30) and the conclusion of the Lemma follows. ■

f) Sixth term \( R_s(q) \)

Lemma 5.6 (Estimation of \( P_n(R_s) \)). Let \( b_0 > 0 \), then there exists \( \delta_s(b_0) > 0 \) such that for all \( \delta \in (0, \delta_s) \) there exists \( s_s(b_0, \delta) \geq 1 \) such that for all \( s_0 \geq s_s \), the following holds
\[
|P_n(R_s(q))| \leq CI^{-2\delta}(s),
\]
for all \( s \in [s_0, \bar{s}] \) and \( 0 \leq n \leq M \).

Proof. The technique is quite the same as the others terms in above. Firstly, we write \( R_s \)'s definition given in (2.12) as follows
\[
R_s(q) = I^{-2}(s)y^{2k-2} \left( \alpha_1 + \alpha_2 y^{2k}e_b + (\alpha_3 + \alpha_4 y^{2k}e_b)q \right),
\]
then, we have the following
\[
P_n(R_s) = \frac{\langle R_s, H_n(y,s) \rangle_{L^2_{\rho_s}}}{\left\| H_n(s) \right\|^2_{L^2_{\rho_s}}},
\]
where \( \left\| H_n(s) \right\|^2_{L^2_{\rho_s}} \) computed in (3.5). In particular, we observe that (5.31) immediately follows by
\[
\left| \langle R_s, H_n(y,s) \rangle_{L^2_{\rho_s}} \right| \leq CI^{-2\delta}(s), \quad \forall s \in [s_0, \bar{s}] \quad \text{and} \quad \forall n \in \{0, 1, ..., [M]\}.
\]
Besides that the technique of the proof of (5.32) is proceed as in Lemma 5.4. For that reason, we kindly refer the reader to check the details and we finish the proof of the Lemma. ■

Part 2: Proof of (i) and (ii) of Proposition 4.5:
- Proof of (i) of Proposition 4.5:
Combining Lemma 5.1-5.6 the estimates defined in \( V_{\delta,b_0}(s) \), we obtain (i) of Proposition 4.5
\[
\forall n \in \{0, ..., [M]\}, \quad \partial_s q_n - \left( 1 - \frac{n}{2k} \right) q_n \leq CI^{-2\delta}(s), \quad \forall s \in [s_0, \bar{s}],
\]
provided that \( \delta \leq \delta_3 \) and \( s_0 \geq s_3(\delta, b_0) \). Thus, we conclude item (i).

- Proof of (ii) of Proposition 4.5: Smallness of the modulation parameter.
Let us recall the equation satisfied by \( q \):
\[
\partial_s q = L_s q + b'(s)M(q) + N(q) + D_s(\nabla q) + R_s(q),
\]
this part aims to obtain an estimation of the modulation parameter \( b(s) \). For this we will project the equation (5.33) on \( H_{2k} \) and take on consideration that \( q_{2k} = 0 \), we obtain
\[
0 = \frac{p}{p-1}b'(s) \left( 1 + P_{2k}(y^{2k}e_bq) \right) + P_{2k}(N) + P_{2k}(D_s) + P_{2k}(R_s),
\]
(5.34)
Using estimations given by equation (5.5) and Lemmas 5.4, 5.5 and 5.6, we obtain
\[ |b'(s)| \leq C I(s)^{-2\delta} = C e^{\frac{1-k}{k}s}, \] (5.35)
where \(0 < \delta \leq \min(\delta_j, 5 \leq j \leq 8)\) is a strictly positive real, which gives us the smallness of the modulation parameter in i) of Proposition 4.5 and we obtain
\[ b(s) \to b^* \text{ as } s \to \infty, \quad (t \to T). \] (5.36)
Integrating inequality (5.35) between \(s_0\) and infinity, we obtain
\[ |b^* - b_0| \leq C e^{\frac{1-k}{k}s_0}, \]
we conclude that there exist \(s_9\) such that dor for \(s_0 \geq s_9\) big enough, we have
\[ \frac{3}{4} b_0 \leq b^* \leq \frac{5}{4} b_0, \]
which is (ii) of Proposition 4.5.

5.2. The proof to item (iii) of Proposition 4.5

Here, we prove the last identity of Proposition 4.5. As in the previous subsection, we proceed in two parts:

- In Part 1, we project equation (2.8) using projector \(P_-\) defined in (3.14).
- In Part 2, we prove the estimate on \(q_-\) given by (iii) of Proposition 4.5.

Part 1: The projection of equation (2.8) using the projector \(P_-\). Let \((q, b)\) be solution to problem (2.8) & (4.11) trapped in \(V_{\delta,b_0}(s)\) for all \(s \in [s_0, \bar{s}]\) for some \(\bar{s} > s_0\). Then, we have the following results:

First term \(\partial_s q\).

Lemma 5.7. For all \(s \in [s_0, \bar{s}]\), it holds that
\[ P_- (\partial_s q) = \partial_s q_- - I^{-2}(1 - \frac{1}{k}) \sum_{n=[M]-1}^{[M]} (n+1)(n+2)q_{n+2}(s)H_n. \] (5.37)

Proof. We firstly have
\[ P_- (\partial_s q) - \partial_s q_- = - (\partial_s q - P_- (\partial_s q)) + (\partial_s q - \partial_s q_-), \]
\[ = - \sum_{n=0}^{[M]} P_n(\partial_s q)H_n - \sum_{n=0}^{[M]} \partial_s(q_nH_n), \]
\[ = - \sum_{n=0}^{[M]} P_n(\partial_s q)H_n + \sum_{n=0}^{[M]} \partial_s q_nH_n + \sum_{n=2}^{[M]} q_n\partial_s H_n, \]
we recall by (5.3) that for all \(n \geq 2\)
\[ \partial_s H_n(y,s) = n(n-1) \left(1 - \frac{1}{k}\right) I^{-2}(s) H_{n-2}(y,s), \]
then by Lemma 5.1, we obtain the desired result
\[ P_- (\partial_s q) = \partial_s q_- - I^{-2} \left(1 - \frac{1}{k}\right) \sum_{n=[M]-1}^{[M]} (n+1)(n+2)q_{n+2}(s)H_n. \]
Second term \( L_s q \).
By the spectral properties given in Section 3, we can write

**Lemma 5.8.** For all \( s \in [s_0, \bar{s}] \), it holds that

\[
P_-(L_s q) = L_s q - I^{-2} (1 - \frac{1}{k^n}) \sum_{n=[M]-1}^{[M]} (n+1)(n+2)q_{n+2}H_n.
\]

**Proof.** We write

\[
P_-(L_s q) - L_s q_- = -(L_s q - P_-(L_s q) -) + (L_s q - L_s q_-),
\]

\[
= - \sum_{n=0}^{[M]} P_n(L_s q) H_n + L_s (q - q_-),
\]

\[
= - \sum_{n=0}^{[M]} P_n(L_s q) H_n + \sum_{n=0}^{[M]} q_n L_s(H_n).
\]

From (3.6), we obtain

\[
\sum_{n=0}^{[M]} q_n L_s(H_n) = q_0 + (1 - \frac{n}{2k})q_1 H_1 + \sum_{n=2}^{[M]} q_n \left[ (1 - \frac{n}{2k})H_n + I^{-2}n(n-1)(1 - \frac{1}{k^n})H_{n-1} \right],
\]

\[
= \sum_{n=0}^{M} (1 - \frac{n}{2k})q_n H_n + I^{-2}(1 - \frac{1}{k^n}) \sum_{n=0}^{M}(n+1)(n+2)q_{n+2} H_n
\]

We deduce from Lemma 5.2 that

\[
P_-(L_s q) - L_s q_- = -I^{-2}(1 - \frac{1}{k^n}) [M(M+1)q_{M+1}H_{M-1} - (M+1)(M+2)q_{M+2}H_M].
\]

Third term \( N \).

**Lemma 5.9.** Let \( b_0 > 0 \), then there exists \( \delta_{10}(b_0) \) such that for all \( \delta \in (0, \delta_{10}) \), then there exists \( s_{10}(\delta, b_0) \geq 1 \) such that for all \( s_0 \geq s_{10} \) the following holds: Assume \( (q,b)(s) \in V_{\delta,b_0}(s), \forall s \in [s_0, \bar{s}] \) for some \( \bar{s} \) arbitrary, then it holds that

\[
|P_-(N)| \leq C \left( I(s)^{-2\delta} + I(s)^{-p\delta} \right) (I(s)^{-M} + |y|^M).
\]

**Proof.** We argue as in [3]. We recall from (2.10) that

\[
N(q) = |1 + e_b q|^{p-1} (1 + e_b q) - 1 - pe_b q.
\]

We proceed in a similar fashion as in the projection \( P_n(N) \), we will give estimations in the outer region \( |y| \geq 1 \) and the inner region \( |y| \leq 1 \). Let us first define \( \chi_0 \) a \( C_0^\infty(\mathbb{R}^+, [0, 1]) \), with \( \text{supp}(\chi) \subset [0, 2] \) and \( \chi_0 = 1 \) on \([0, 1]\), we define

\[
\chi(y) = \chi_0(|y|).
\]

Using the fact that

\[
N = \chi N + \chi^c N,
\]

we claim the following:
Claim 5.10.  

\[(i) \quad |P_-(\chi^cN)| \leq CI(s)^{-\delta p} (I(s)^{-M} + |y|^M), \quad (5.39)\]

\[(ii) \quad |P_-(\chi N)| \leq CI(s)^{-2\delta} (I(s)^{-M} + |y|^M). \quad (5.40)\]

Proof. First, we will estimate \(P_-(\chi^cN)\), then \(P_-(\chi N)\) and conclude the proof of the lemma.

(i) Let us first write \[P_-(\chi^cN) = \chi^cN - \sum_{n \leq \lceil M \rceil + 1} P_n(\chi^cN)H_n \]

\[= \chi^cN - \sum_{n \leq \lceil M \rceil + 1} \frac{\int_{|y| \geq 1} \mathcal{N}H_n \rho_s dy}{\|H_n\|_{L^2(s)}} H_n, \]

using the definition of the shrinking set we can write

\[|\chi^c(N)| \leq |\chi^c(CI^{-\delta} e_b |y|^M)^p| = |\chi^c(CI^{-\delta} e_b y^{2k})| |y|^\frac{2k}{p-1} |, \]

by the fact that \(|e_b y^{2k}| \leq C\) and \(M = \frac{2kp}{p-1}\), we have

\[|\chi^c(N)| \leq CI^{-\delta p} |y|^M \]

Then using (5.9) we deduce (i) of Claim 5.10:

\[|P_-(\chi^cN)| \leq CI(s)^{-\delta p} (I(s)^{-M} + |y|^M). \quad (5.41)\]

(ii) In the inner region \(|y| \leq 1\), we proceed as in the proof of Lemma 5.3. For \(|y| \leq 1\), using the Taylor expansion as in (5.13), we write

\[\chi N = \chi (A + S + R_K), \]

where \(A\) and \(S\) are given by (5.15)

\[A = \chi \sum_{j=2}^{K} d_j (e_b q_+)^j \quad \text{and} \quad S = \chi \sum_{j=2}^{K} \sum_{\ell=0}^{j-1} \tilde{d}_j, \ell e_b^\ell (q_+)^\ell (q_-)^{j-\ell}, \quad \text{for some} \quad d_j, \tilde{d}_j, \ell \in \mathbb{R}. \]

We get for \(K\) large,

\[|\chi R_K| \leq \mathcal{T}_s^{-\delta} I(s)^{-M}. \quad (5.42)\]

We proceed in a similar fashion as in the proof of Lemma 5.3, we write \(A\) as

\[A = \chi \sum_{n,p}^{K} \sum_{j=2}^{K} d_j (e_b q_+)^j = \chi \sum_{n,p}^{K} c_{n,p} b(s) y^{n_p} y^{\Pi_{i=1}^{\lceil M \rceil} q_i} H_i^{n_i} + I(s)^{-2\delta} b(s) \frac{2k(L+1)}{2k} y^{2k(L+1)} \chi Q, \quad (5.43)\]

where \(\chi Q\) is bounded. Then, we divide the sum \(A_1\) as follows

\[A_1 = \chi \sum_{n,p}^{K} c_{n,p} b(s) y^{n_p} y^{\Pi_{i=1}^{\lceil M \rceil} q_i} H_i^{n_i}, \]

\[= \chi \sum_{n,p}^{K} c_{n,p} b(s) y^{n_p} y^{\Pi_{i=1}^{\lceil M \rceil} q_i} H_i^{n_i} + \chi \sum_{n,p}^{K} c_{n,p} b(s) y^{n_p} y^{\Pi_{i=1}^{\lceil M \rceil} q_i} H_i^{n_i} = \quad (5.44)\]

where \(\chi Q\) is bounded. Then, we divide the sum \(A_1\) as follows

\[A_1 = \chi \sum_{n,p}^{K} c_{n,p} b(s) y^{n_p} y^{\Pi_{i=1}^{\lceil M \rceil} q_i} H_i^{n_i}, \]

\[= \chi \sum_{n,p}^{K} c_{n,p} b(s) y^{n_p} y^{\Pi_{i=1}^{\lceil M \rceil} q_i} H_i^{n_i} + \chi \sum_{n,p}^{K} c_{n,p} b(s) y^{n_p} y^{\Pi_{i=1}^{\lceil M \rceil} q_i} H_i^{n_i} = \quad (5.44)\]

In the first sum, \(A_{1,1}\), we replace \(\chi = 1 - \chi^c\) by \(-\chi^c\), since 1 will not contribute to \(A_\). Using the fact that \(|y| \geq 1\) and by (4.4), we get

\[\chi^c \left| y^{\Pi_{i=1}^{\lceil M \rceil} H_i^{n_i}} \right| \leq C|y|^M. \]
Since $H_m$ is bounded as follows
\[ |H_m(y, s)| \leq C(I(s)^{-m} + |y|^m), \]
we obtain by (4.4)
\[ \chi |y^p \prod_{i=1}^M H_i^{n_i}| \leq C(I(s)^{-M} + |y|^M). \]

We conclude by the definition of the shrinking set given by (4.1), that
\[ |A_{1,2}| \leq CI(s)^{-2\delta} \chi(y) \left(I(s)^{-M} + |y|^M \right). \]  
(5.45)

By the properties of the shrinking set and the bound for $q_-$, we obtain the bound for the term $A_2$, defined by (5.43), more precisely we have
\[ |A_2| \leq CI(s)^{-2\delta} \chi(y) \left(I(s)^{-M} + |y|^M \right). \]

Then, we conclude that
\[ |P_-(A)| = |A_-| \leq CI(s)^{-2\delta} (I(s)^{-M} + |y|^M), \]  
(5.46)

which yields the conclusion of item (ii). □

Now, we return to the proof of the Lemma. We deduce by (5.41), (5.42) and (5.46) the following estimation for $P_-(\mathcal{N})$
\[ |P_-(\mathcal{N})| = |\mathcal{N}_-| \leq C(I(s)^{-2\delta} + I(s)^{-p\delta})(I(s)^{-M} + |y|^M), \]  
(5.47)

thus end the proof of Lemma 5.9. □

**Fourth term $b'(s)\mathcal{M}(q)$.**

**Lemma 5.11.** Let $b_0 > 0$, then there exists $\delta_{11}(b_0)$ such that for all $\delta \in (0, \delta_{11})$, then there exists $s_10(\delta, b_0) \geq 1$ such that for all $s_0 \geq s_{11}$ the following folds: Assume $(q, b)(s) \in V_{\delta, b_0}(s), \forall s \in [s_0, \bar{s}]$ for some $\bar{s}$ arbitrary, then it holds that
\[ |P_-(\mathcal{M})| \leq CI(s)^{-\delta} \left(I(s)^{-M} + |y|^M \right). \]

We recall that
\[ \mathcal{M} = \frac{p}{p-1} y^{2k} (1 + e_b q), \]
then, we can write
\[ P_-(\mathcal{M}(q)) = \frac{p}{p-1} P_-(y^{2k} e_b q). \]

Let us write
\[ P_-(y^{2k} e_b q) = P_-(\chi y^{2k} e_b q) + P_-(\chi y^{2k} e_b q), \]
we claim the following:

**Claim 5.12.**

(i) \[ |P_-(\chi y^{2k} e_b q)| \leq CI(s)^{-\delta} \left(I(s)^{-M} + |y|^M \right), \]  
(5.48)

(ii) \[ |P_-(\chi y^{2k} e_b q)| \leq CI(s)^{-\delta} \left(I(s)^{-M} + |y|^M \right), \]  
(5.49)
Proof. Let us first write
\[
\begin{align*}
P_-(\chi^c y^{2k}e_bq) &= \chi^c y^{2k}e_bq - \sum_{n \leq |M|+1} P_n(\chi^c y^{2k}e_bq)H_n \\
&= \chi^c y^{2k}e_bq - \sum_{n \leq |M|+1} \int_{|y| \geq b(s)} \frac{y^{2k}e_bqH_n}{E_n} dy H_n,
\end{align*}
\]
When \(|y| \geq 1\), using (4.1), we can write
\[
|y^{2k}e_bq| \leq C|q| \leq CI(s)^{-\delta} \frac{1}{b(s)} |y|^{M} \leq CI(s)^{-\delta} |y|^M.
\]

ii) As for i), we Write
\[
P_-(\chi^c y^{2k}e_bq) = \chi^c y^{2k}e_bq - \sum_{n \leq |M|+1} P_n(\chi^c y^{2k}e_bq).
\]

By Lemma 5.4 we have \(\sum_{n \leq |M|+1} P_n(\chi^c y^{2k}e_bq)\|H_n\|_{L^2_{\rho_s}}^{-2} \leq CI(s)^{-\delta}\),

We conclude using the definition of the shrinking set and we obtain the following estimation
\[
|\chi^c y^{2k}e_bq| \leq CI(s)^{-\delta}.
\]

\[\]

**Fifth term** \(D_s(\nabla q)\)

**Lemma 5.13.** Let \(b_0 > 0\), then there exists \(\delta_{12}(b_0)\) such that for all \(\delta \in (0, \delta_{12})\), then there exists \(s_{12}(\delta, b_0) \geq 1\) such that for all \(s_0 \geq s_{12}\) the following folds: Assume \((q, b)(s) \in V_{b, b_0}(s), \forall s \in [s_0, \bar{s}]\) for some \(\bar{s}\) arbitrary, then it holds that
\[
P_-(D_s) \leq CI^{-\delta} (I(s)^{-M} + |y|^M).
\]

**Proof.** Let us first write
\[
P_-(D_s) = D_s - \sum_{n=0}^{[M]} P_n(D_s)H_n.
\]

Since we are using the properties given by the shrinking set in Definition 4.1, it will be more convenient to estimate
\[
d = \int_{\sigma}^s d\tau K_{s, \tau}(y, z)D_s(\nabla q). \quad (5.50)
\]
Using integration by parts, we obtain
\[
d = 4pkb(p - 1)^{-1} \int_{\sigma}^s d\tau I(\tau)^{-2} \int dz \partial_z \left( K_{s, \tau}(y, z)e_b(z)z^{2k-1} \right) q(z, \tau),
\]
\[
= 4pkb(p - 1)^{-1} \int_{\sigma}^s d\tau I(\tau)^{-2} \int dz K_{s, \tau}(y, z)\partial_z \left( e_b(z)z^{2k-1} \right) q(z, \tau)
\]
\[
+ pkb(p - 1)^{-1} \int_{\sigma}^s d\tau I(\tau)^{-2} \int dz \partial_z (K_{s, \tau}(y, z))e_b(z)z^{2k-1}q(z, \tau),
\]
\[
= d_1 + d_2. \quad (5.51)
\]

For the estimation of the first term \(d_1\), we argue in a similar fashion as in the projection of \(P_n(\mathcal{M})\), see Lemma 5.4. For the second term, we argue as in Bricomont Kupiainen [3]. Indeed, we need to bound \(\partial_z K_{s, \tau}\). From equations (3.8) we obtain
\[
|\partial_z (K_{s, \tau}(y, z))| \leq C LF_{+L^2} \left( e^{\frac{s-z}{2k}} y - z \right) \leq \frac{CI(s)}{\sqrt{s - \tau}} F_{+L^2} \left( e^{\frac{s-z}{2k}} y - z \right), \quad (5.52)
\]
where \( L = \frac{I(s)^2}{(1-e^{-(s-r)})} \), \( \mathcal{F} \) defined by (3.9) and \( I(s) = e^{\frac{2}{3}(1-\frac{1}{3})} \). Then, by Definition 4.1, we obtain
\[
|d_2| \leq I(s)^{-1}I(s)^{-\delta}
\]
and we conclude that there exist \( \delta_\tau \) such that for all \( 0 < \delta \leq \delta_\tau \),
\[
|d| \leq C I^{-2\delta} (I(s)^{-M} + |y|^M).
\]

(5.53)

On the other hand by Lemma 5.30, we obtain
\[
|\sum_{n=0}^{[M]} P_n(D_s)H_n| \leq C I^{-2\delta} (I(s)^{-M} + |y|^M).
\]

(5.54)

We conclude from (5.53), (5.53) that
\[
P_-(D_s) \leq C I^{-2\delta} (I(s)^{-M} + |y|^M).
\]

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Sixth term \( \mathcal{R}_s(q) \)

Lemma 5.14. Let \( b_0 > 0 \), then there exists \( \delta_{13}(b_0) \) such that for all \( \delta \in (0, \delta_{13}) \), then there exists \( s_{13}(\delta, b_0) \geq 1 \) such that for all \( s_0 \geq s_{13} \) the following holds: Assume \( (q, b)(s) \in \mathcal{V}_{\delta, b_0}(s), \forall s \in [s_0, \bar{s}] \) for some \( \bar{s} \) arbitrary, then it holds that
\[
|P_-(\mathcal{R}_s(q))| \leq CI(s)^{-2\delta} (I(s)^{-M} + |y|^M).
\]

(5.55)

Proof. By (2.12)
\[
\mathcal{R}_s(q) = I(s)^{-2} y^{2k-2} \left( \alpha_1 + \alpha_2 y^{2k} q \right),
\]
we proceed as for the estimation of \( P_-(\mathcal{M}) \).

Part 2: Proof of the identity (iii) in Proposition 4.5 (estimate on \( q_- \)) If we apply the projector \( P_- \) to the equation of (2.8), we obtain
\[
\partial_s q_- = \mathcal{L}_s q_- + P_- (\mathcal{N}(q) + \mathcal{D}_s(q) + \mathcal{R}_s(q) + b'(s)\mathcal{M}(q))
\]
Using the kernel of the semigroup generated by \( \mathcal{L}_s \), we get for all \( s \in [\tau, s_1] \) The integral equation of the equation above is
\[
q_-(s) = K_{\tau \tau} q_-(\tau) + \int_{\tau}^{s} K_{s \tau} \left( P_- [\mathcal{N}(q) + \mathcal{D}_s(\nabla q) + \mathcal{R}_s(q) + b'(s)\mathcal{M}(q)] \right) ds'.
\]
Using Lemma A.5, we get
\[
|q_-(s)| \leq e^{-\frac{1}{p-1} (s-\tau)|q_-(\tau)|_\tau} + \int_{\tau}^{s} e^{-\frac{1}{p-1} (s-s') |P_- [\mathcal{N}(q) + \mathcal{D}_s(\nabla q) + \mathcal{R}_s(q) + b'(s)\mathcal{M}(q)]|_s} ds'.
\]
By Lemma 5.9, Lemma 5.11, Lemma 5.14, equations (5.53), (5.54) and the smallness of the modulation parameter \( b(s) \) given by (ii) of Proposition 4.5, we obtain
\[
|q_-(s)| \leq e^{-\frac{1}{p-1} (s-\tau)|q_-(\tau)|_\tau} + \int_{\tau}^{s} e^{-\frac{1}{p-1} (s-s') I(s')^{-\delta_{min}(p,2)+1}} ds'.
\]
Then, for \( \delta \leq \delta_\tau \), it holds that
\[
|q_-(s)| \leq e^{-\frac{1}{p-1} |q_-(\tau)|_\tau} + C \left( I^{-\frac{3}{2}\delta}(s) + e^{-\frac{1}{p-1}} I^{-\frac{3}{2}\delta}(\tau) \right).
\]
which concludes the proof of the last identity of Proposition 4.5.

A. Computation on Hermite polynomials

Lemma A.1. Let us consider $H_m$ defined as in (3.3), for some $m \geq 0$, let us consider

$$f(y) = \sum_{j=0}^{\ell} f_j y^j, f_j \in \mathbb{R}.$$ 

Then, the following holds

$$|\langle f, H_m \rangle_{L_{\rho_s}^2}| \leq \begin{cases} CI^{2m} & \text{if } m \leq \ell, \\ 0 & \text{if } m > \ell \end{cases} \quad (A.1)$$

Proof. We note first that for all integer $j$

$$\langle y^j, H_m \rangle_{L_{\rho_s}^2} = I^{-m} \int y^j h_m(yI)\rho_s dy, \quad I^{-m-j} \int z^j h_m(z)\rho dz \quad \text{where } z = Iy.$$ 

then we conclude that

$$\langle y^j, H_m \rangle_{L_{\rho_s}^2} = C I^{m-j}$$

and

$$\frac{|\langle f, H_m \rangle_{L_{\rho_s}^2}|}{\|H_m\|_{L_{\rho_s}^2}} \leq \begin{cases} C & \text{if } \ell \geq m, \\ 0 & \text{if } \ell < m, \end{cases}$$

which end the proof of Lemma.

Lemma A.2. Let us consider $m \in \mathbb{N}$, $I(s)$ defined as in (2.3) and $f \in L_K^\infty$ for some $K > 0$, where $L_K^\infty$ is defined by (3.15). Then, we have

$$\left| \int_{|y| \geq 1} f H_m \rho_s dy \right| \leq C(m, K)\|f\|_{L_M^\infty} e^{-\frac{1}{8}I(s)}, \quad \text{with } s \geq s_0. \quad (A.2)$$

consequently, if $f$ has the form

$$f(y) = \sum_{j=0}^{\ell} f_j y^j, f_j \in \mathbb{R},$$

then, we have

$$\left| \int_{|y| \leq 1} f(y)H_m(y, s)\rho_s(y) dy \right| \leq \begin{cases} CI^{-2m(s)} & \text{if } \ell \geq m, \\ Ce^{-\frac{I(s)}{8}} & \text{if } \ell < m. \end{cases} \quad (A.3)$$

Proof. Let us decompose as follows

$$\int_{\mathbb{R}} f H_m \rho_s dy = \int_{|y| \leq 1} f H_m \rho_s dy + \int_{|y| \geq 1} f H_m \rho_s dy.$$ 

From (3.2), we get

$$|H_m(y)| \leq C (1 + y^m).$$

Then, we get

$$\left| \int_{y \geq 1} f H_m \rho_s dy \right| \leq C\|f\|_{L_M^\infty} \int_{y \geq 1} (1 + y^{m+K})e^{-\frac{j^2(s)y^2}{4}} I(s) dy.$$
By changing variable \( z = I(s)y \), we obtain
\[
\int_{y \geq 1} (1 + y^{m+K})e^{-\frac{I^2(s)y^2}{4}}I(s)dy = \int_{z \geq I(s)} (1 + z^{m+K}I^{-m-K}(s)) e^{-\frac{z^2}{4}}dz \\
\leq \int_{z \geq I(s)} (1 + z^{2m+M}) e^{-\frac{z^2}{4}}dz \\
\leq e^{-\frac{I^2(s)}{8}} \int_{0}^{\infty} (1 + z^{m+K}) e^{-\frac{z^2}{8}}dz \\
\leq Ce^{-\frac{I^2}{8}},
\]
which concludes the proof of the Lemma. \( \blacksquare \)

**Lemma A.3** (Some scaled Hermite polynomial identities). Let us consider \( H_n, n \in \mathbb{N} \) defined as in (3.3) be scaled Hermite polynomials and \( \ell \in \mathbb{N} \), then, we have
\[
y^{\ell}H_n(y, s) = \sum_{j=0}^{n} c_{j,\ell,n}(s)H_{n+\ell-2j}(y, s).
\]
(A.4)
In particular, when \( \ell = 1 \) and \( \ell = 2 \), we have
\[
y^2H_n(y, s) = H_{n+2}(y, s) + (4n + 2)I^{-2}(s)H_n(y, s) + 4n(n - 1)I^{-4}(s)H_{n-2}(y, s) \quad \text{(A.5)}
yH_{n-1}(y, s) = H_n(y, s) + I^{-2}(s)2(n - 1)H_{n-2}(y, s). \quad \text{(A.6)}
\]

**Proof.** The result immediately follows the fact that \( \{H_n, n \geq 0\} \) is a basis of \( L^2_{\rho} \) and \( y^{\ell}H_n(y, s) \) is a polynomial of order \( n + \ell \). In addition to that, we also have the property that \( \langle y^{\ell}H_n, H_k \rangle_{L^2_{\rho}} = 0 \) whenever \( k + \ell < n \), then, the terms of \( H_{n+\ell-2j}, j > \left[ \frac{\ell+1}{2} \right] \) don’t appear in the sum (A.4). Thus, the result of (A.4) completely follows. In addition to that, we also specify the constants \( c_{j,\ell,n} \) by
\[
c_{j,\ell,n} = \frac{\langle y^{\ell}H_n, H_{n+\ell-2j} \rangle_{L^2_{\rho}}}{\|H_{n+\ell-2j}\|_{L^2_{\rho}}^2}.
\]
Finally, for the special cases \( \ell = 1, 2 \), we get (A.5) and (A.6), and we concludes the proof of the Lemma. \( \blacksquare \)

**Lemma A.4.** It holds that
\[
|K_{\tau \sigma}q_{-}|_{r} \leq Ce^{-\frac{1}{r-1}(\tau - \sigma)} |q_{-}|_{\sigma}. 
\]
**Proof.** We proceed as in the proof of Lemma 1 in Section 4, page 569 from [4]. We start by recalling the change of variable \( z = yI(s) \) and we note
\[
\theta(z) = q_{-}(y) = q_{-} \left( \frac{z}{\sqrt{I}} \right) \hat{\theta}(z) = (K_{\tau \sigma}\theta) \left( \frac{y}{\sqrt{I}} \right),
\]
then we have
\[
\hat{\theta} = e^{(\tau - \sigma)\mathcal{L}}\theta,
\]
where \( e^{s\mathcal{L}} \) is the semigrup generated by the operator \( \mathcal{L}q = \Delta q - \frac{y}{2}q + q \)
\[
e^{s\mathcal{L}}(z, z') = \frac{1}{[4\pi(1 - e^{-t})]^2} \exp \left[ -\frac{(z' - ze^{-s/2})^2}{4(1 - e^{-s})} \right].
\]
From the definition of the shrinking set \( \mathcal{V}_{\delta, s} \) given by Definition 4.1, we have
\[
|\theta(z)| \leq I^M(1 + |z|^M)|q_{-}|_{\sigma},
\]
with 
\[(\theta, h_m) = \int \theta h_m(z) \rho(z) dz \quad \text{for} \quad m \leq [M] \quad \text{and} \quad \rho(z) = \frac{e^{-z^2/4}}{(4\pi)^{N/2}},\]

where \( M = \frac{2kp}{p-1} \).

Proceeding as in the derivation of equation (66) in Section 3 and Lemma 4, in section 3 from [4] and using Definition 4.1 we get for \( \tau - \sigma \geq 1 \),

\[|\tilde{\theta}(z)| \leq Ce^{-(\tau-\sigma)}e^{-([M]+1)\frac{z^2}{4}}I^{-M}(1 + |z|^M)|q_\sigma| \quad \text{for} \quad 0 \leq m \leq [M] + 1.\]

We recall that \( M = \frac{2kp}{p-1} \) and using the fact that

\[e^{-\frac{M}{2}(\tau-\sigma)}I^{-M}(\sigma) = e^{-\frac{\tau}{p-1}(\tau-\sigma)}I^{-M}(\tau),\]

then we obtain the desired result

\[|K_{\tau\sigma}q_-(y)| \leq Ce^{-\frac{(\tau-\sigma)}{p-1}(I^{-M}(\tau) + |y|^M)|q_\sigma|.\]

\[\square\]

**Lemma A.5.** There exists a constant \( C \) such that if \( \phi \) satisfies

\[\forall x \in \mathbb{R}, \quad |\phi(x, \tau)| \leq (I^{-M}(\tau) + |x|^M),\]

then for all \( y \in \mathbb{R} \) and \( s \geq s_0 \), we have

\[|K_{\tau\sigma}(\phi)(y, \tau)| \leq Ce^{-\frac{(\tau-\sigma)}{p-1}(I^{-M}(\tau) + |y|^M)}.\]

**Proof.** The proof is similar to the proof of Lemma A.4 and we use the fact that for \( z = yI(s) \), if

\[|\Phi(z, \tau)| \leq (1 + |z|^M),\]

then

\[|e^{sL}P_-(\Phi)(z)| \leq Ce^{\frac{(M+1)}{2}I(1 + |z|^M)}.\]

\[\square\]

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