Phenomenological description of quantum gravity inspired modified classical electrodynamics

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We discuss a large class of phenomenological models incorporating quantum gravity motivated corrections to electrodynamics. The framework is that of electrodynamics in a birefringent and dispersive medium with non-local constitutive relations, which are considered up to second order in the inverse of the energy characterizing the quantum gravity scale. The energy-momentum tensor, Green functions and frequency dependent refraction indices are obtained, leading to departures from standard physics. The effective character of the theory is also emphasized by introducing a frequency cutoff $\Omega$. The analysis of its effects upon the standard notion of causality is performed, showing that in the radiation regime ($\Omega R >> 1$) the expected corrections of the order $(\omega/\Omega)^n$ get further suppressed by highly oscillating terms proportional to $\sin(\Omega R)$, $\cos(\Omega R)$, thus forbidding causality violations to show up in the corresponding observational effects.

I. INTRODUCTION

Lorentz covariance in local inertial frames is a well established symmetry at the energies of present day experiments. However, its validity at high energies is subject to test. Possible Lorentz invariance violations may arise from dynamical modifications induced by quantum gravity (QG). The effects of such violations in the range well below the Planck energy ($E_P \sim 10^{19}$ GeV) have been recently the object of intense scrutiny \cite{1, 2, 3, 4, 5}. This issue is closely linked to theoretical and experimental research, based on the Standard Model Extension \cite{6}, concerning Lorentz and CPT violations \cite{7}. Heuristic loop QG derivations of such effects \cite{2, 3} make it clear that a better understanding of the corresponding semiclassical limit is required \cite{8}. String theory has also provided models for explaining such QG induced corrections \cite{9}. Moreover, effective field theory models have been constructed that include higher dimension Lorentz invariance violating (LIV) operators \cite{10}. Synchrotron radiation arising from the model in \cite{10} has been extensively analyzed in Ref. \cite{11}. These effective theories use a reduced number of degrees of freedom to describe the physics at a low energy scale, ignoring the detailed dynamics inherent to Planck energies. In other words, if QG dominates at a scale $E_{QG}$, usually considered of the order of $E_P$, a corresponding low energy effective theory can be visualized as an expansion in powers of $\xi \simeq E_{QG}^{-1}$, truncated at a given finite order. In this way it will be a good description, hopefully simpler than the original one, for energies $E \ll E_{QG}$. This restricted validity relaxes some of the constraints usually required for physical theories, such as renormalizability. Stability and causality, perhaps of more essential status than the Lorentz symmetry itself, are assumed to remain valid at the low energy regime \cite{14}. Nevertheless, fine tuning problems arise when considering radiative corrections \cite{12}, which can be circumvented by extending the notion of dimensional regularization \cite{13}.

In fact, one of the possible manifestations of QG at low energy is the appearance of correction terms related to the

* Dedicated to Octavio Obregón on his sixtieth birthday
scale \( E_{QG} \) in the standard particle propagation and interaction properties. The most direct interpretation of such corrections, though not the only one \[15\], is in terms of a spontaneous breaking of Lorentz covariance at high energies. If this is so, the effective theory will be covariant under Lorentz transformation between inertial frames (passive or observer transformations), and the observable Lorentz symmetry violations will be associated with rotations and boosts of the fields in a given inertial frame (active or particle transformations). In this case the space-time coordinates at low energy remain commutative. We focus here on a general analysis of QG effects in electrodynamics in these models, where we can introduce the full usual mathematical framework of field theory, specially Fourier transformations. QG may also modify the space-time itself so that the coordinates become noncommutative, as in the case of Double Special Relativity models, for example. In this situation, which will not be discussed here, ordering ambiguities preclude a direct use of such transformations.

Considering now the electromagnetic field, the models proposed to describe low energy effects of QG can usually be expressed in terms of modified dispersion relations, with a polynomial dependence in energy and momentum. Such modifications include standard Lorentz invariance violations as well as possible extensions of Lorentz covariance \[17\]. Most of these approaches can be unified in the description

\[
i k \cdot D = 4 \pi \rho, \quad k \cdot B = 0, \quad (1)
\]

\[
k \times E - \omega B = 0, \quad i k \times H + i \omega D = 4 \pi j, \quad (2)
\]

where the auxiliary fields

\[
D^i = \alpha^{ij} E_j + \rho^{ij} B_j, \quad H^i = \beta^{ij} B_j + \sigma^{ij} E_j, \quad (3)
\]

are such that the coefficients \( \alpha^{ij}, \beta^{ij}, \rho^{ij} \) and \( \sigma^{ij} \) depend on the energy \( \omega \) and the momentum \( k \) of the electromagnetic field. These equations correspond to a higher order linear dynamics. Equations \(1\) and \(2\) strongly resemble the usual description for an electromagnetic field in a medium \[18\], where the fields \( D \) and \( H \) are characterized by constitutive relations of the form \[19\]. In terms of electrodynamics in media, we can interpret the low energy QG corrections in terms of a dispersive bianisotropic media. From a heuristic point of view, as shown below, these effective media are non-local in space and time, which can be interpreted as a footprint of the granularity induced by QG.

Strictly speaking, the effective models are characterized by Eqs.\(1,2\), but under certain restrictions it is also possible to pose an action from which they derive. Although not essential, this is a useful approach to visualize general features of the dynamics. Let us recall the Lagrangian for the electromagnetic field in a local medium

\[
L = -\frac{1}{4} F_{\mu \nu} \chi^{[\mu \nu]} \chi^{[\alpha \beta]} F_{\alpha \beta} - 4 \pi j_{\mu} A^{\mu}, \quad (4)
\]

where \( F_{\mu \nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu} \), and \( \chi^{[\mu \nu]} \chi^{[\alpha \beta]} \) contains the information about the medium. This structure warrants gauge invariance and hence charge conservation. The dynamics is given by the equations of motion

\[
\partial_{\mu} H^{\mu \nu} = 4 \pi j^{\nu}, \quad (5)
\]
together with the constitutive relations

\[ H^{\mu\nu} = \chi^{[\mu\nu]\alpha\beta} F_{\alpha\beta}. \] (6)

Defining the electric and magnetic fields as \( F_{0i} = E_i \) and \( F_{ij} = -\epsilon_{ijk}B_k \) respectively, and the corresponding components of \( H^{\mu\nu}, H^{0i} = D^i \) and \( H^{ij} = -\epsilon^{ijk}H^k \), the constitutive relations become

\[
D^i = 2\chi^{[0i][0j]}E_j - \chi^{[0i][mn]}\epsilon_{mnj}B_j, \quad H^i = \epsilon_{ijk}\chi^{[lk][0j]}E_j - \frac{1}{2}\epsilon_{ijk}\chi^{[lk][mn]}\epsilon_{mnj}B_j.
\] (7)

In our notation the components of any three-dimensional vector \( \mathbf{V} \) are given by those with subindices \( V_i \).

The equations of motion incorporating QG corrections acquire a similar form, but with one important difference arising from the nonlocal character of the effective medium. The \( \chi^{[\mu\nu]\alpha\beta} \) is now a non-local tensor, such that

\[
L = -\frac{1}{4} \int d^4\tilde{x} F_{\mu\nu}(x^\sigma) \chi^{[\mu\nu][\alpha\beta]}(x^\sigma - \tilde{x}^\sigma) F_{\alpha\beta}(\tilde{x}^\sigma) - 4\pi j_\mu(x^\sigma) A^\mu(x^\sigma),
\] (8)

instead of (14). As usual if \( F_{\mu\nu} \) and \( L \) are real, the reality of \( \chi^{[\mu\nu]\alpha\beta}(x^\sigma - \tilde{x}^\sigma) \) and subsequently of

\[
H^{\mu\nu}(x) = \int d^4\tilde{x} \chi^{[\mu\nu][\alpha\beta]}(x^\sigma - \tilde{x}^\sigma) F_{\alpha\beta}(\tilde{x}^\sigma),
\] (9)

are implied. Writing \( \chi^{[\mu\nu][\alpha\beta]}(x^\sigma - \tilde{x}^\sigma) \) in terms of its Fourier transform

\[
\chi^{[\mu\nu][\alpha\beta]}(x^\sigma - \tilde{x}^\sigma) = \int d^4k \ e^{-ik(x-\tilde{x})} \chi^{[\mu\nu][\alpha\beta]}(k^\sigma),
\] (10)

we can easily demonstrate that (8) can also be written a

\[
L = -\frac{1}{4} F_{\mu\nu}(x^\sigma) \left[ \chi^{[\mu\nu][\alpha\beta]}(i\partial_\sigma) F_{\alpha\beta}(x^\sigma) \right],
\] (11)

with \( \chi^{[\mu\nu][\alpha\beta]} \) being a derivative operator. In terms of the Fourier transform, the reality of \( \chi^{[\mu\nu][\alpha\beta]}(x^\sigma - \tilde{x}^\sigma) \) is stated as

\[
\left[ \chi^{[\mu\nu][\alpha\beta]}(k^\sigma) \right]^* = \chi^{[\mu\nu][\alpha\beta]}(-k^\sigma),
\] (12)

which holds similarly for the transformed fields \( F_{\alpha\beta}(k^\sigma) \) and \( H_{\alpha\beta}(k^\sigma) \). If \( \chi^{[\mu\nu][\alpha\beta]} \) is symmetric, in the sense that for each set of index values \( (\mu,\nu,\alpha,\beta) \) (no sum with respect to repeated indices)

\[
\int d^4x \ F_{\mu\nu} \left[ \chi^{[\mu\nu][\alpha\beta]} \right] F_{\alpha\beta} = \int d^4x \ F_{\alpha\beta} \left[ \chi^{[\alpha\beta][\mu\nu]} \right] F_{\mu\nu}
\] (13)

is satisfied, then it is possible to perform integrations by parts making the equations of motion of the same form as in the usual non-operator case. Thus the Fourier transform of the equations of motion and constitutive relations acquire the structure of (14) and (15) respectively. In the following we assume that this property is satisfied. When the components of \( \chi^{[\mu\nu][\alpha\beta]} \) do not correspond to a standard Lorentz tensor, this Lagrangian describes a model where the Lorentz symmetry is broken by the medium.
We use this approach, where the QG modifications are described phenomenologically by constitutive relations, to discuss the main properties of QG induced effects in electrodynamics. A low energy expansion is developed in terms of the parameter $\xi \simeq E_{QG}^{-1}$. Working to order $\xi^2$ allows us to present $\hat{\chi}^{[\mu\nu][\alpha\beta]}$ in the form

$$\hat{\chi}^{[\mu\nu][\alpha\beta]} = \chi_0^{[\mu\nu][\alpha\beta]} + \chi_1^{[\mu\nu][\theta[\alpha\beta]} \partial_0 + \chi_2^{[\mu\nu][\theta[\alpha\beta]} \partial_0 \partial_0 \partial_0,$$

(14)

where the constant coefficients $\chi_1^{[\mu\nu][\theta[\alpha\beta]}, \chi_2^{[\mu\nu][\theta[\alpha\beta]}$ are proportional to $\xi, \xi^2$ respectively. They are antisymmetric in the indices inside square brackets and symmetric in the indices inside curly brackets. In this way we are considering a Lagrangian depending up to third derivatives in the basic electromagnetic potential $A_\mu$. Moreover, the integration conditions (13) require the following symmetry properties

$$\chi_0^{[\mu\nu][\alpha\beta]} = \chi_0^{[\alpha\beta][\mu\nu]}, \quad \chi_1^{[\mu\nu][\theta[\alpha\beta]} = -\chi_1^{[\alpha\beta][\theta[\mu\nu]} , \quad \chi_2^{[\mu\nu][\theta[\alpha\beta]} = \chi_2^{[\alpha\beta][\theta[\mu\nu]}.$$

(15)

Once the coefficients of the constitutive relations have been promoted to derivative operators we obtain the relations

$$\hat{\chi}^{[0i][0j]} = \frac{1}{2} \hat{\alpha}^{ij}, \quad \hat{\chi}^{[0i][mn]} = -\frac{1}{2} \epsilon_{mnj} \hat{\beta}^{ij},$$

$$\hat{\chi}^{[mn][0j]} = +\frac{1}{2} \epsilon_{mni} \hat{\sigma}^{ij}, \quad \hat{\chi}^{[lk][mn]} = +\frac{1}{2} \epsilon_{kli} \hat{\theta}^{ij} \epsilon_{jmn},$$

(16)

(17)

by comparing Eqs. (14) and (13). The expansion (14) induces the corresponding form in the coefficients of the constitutive relations

$$\hat{\alpha}^{ij} = \alpha_0^{(ij)} + \hat{\xi} \alpha_1^{(ij)} \psi \partial_0 + \hat{\xi}^2 \alpha_2^{(ij)} \psi \partial_0 \partial_0 \partial_0,$$

(18)

and similarly for $\hat{\beta}^{ij}, \hat{\sigma}^{ij}$ and $\hat{\theta}^{ij}$, in an obvious notation. Here $\alpha_0^{(ij)}, \alpha_1^{(ij)}$ and $\alpha_2^{(ij)}$ are constant coefficients.

To analyze the propagation of the fields and to define the corresponding refraction index, the dependence of the constitutive relations on $\omega$ and $k$ has to be made explicit. To achieve this we first expand the coefficients of the constitutive relations in space derivatives, maintaining covariance under rotations. Considering that these models can be understood as perturbative descriptions in terms of the parameter $\xi$ we have, up to order $\xi^2$,

$$\hat{\alpha}^{ij} = \alpha_0 (\partial_i \eta^{ij} + \alpha_1 (\partial_i \hat{\xi}^{ijr} \partial_r + \alpha_2 (\partial_i \hat{\xi}^2 \partial^r \partial^i),$$

(19)

with analogous expansions for $\hat{\beta}^{ij}, \hat{\sigma}^{ij}$ and $\hat{\theta}^{ij}$. Here $\alpha_A, \beta_A, \sigma_A$, and $\rho_A$, with $A = 0, 1, 2$, are $SO(3)$ scalar operators. This approach can be generalized to models with preferred spatial directions. The symmetry of $\hat{\chi}$ in Eq. (13) implies that the terms in $\hat{\alpha}^{ij}, \hat{\beta}^{ij}$ and $\hat{\sigma}^{ij}$ with an even number of derivatives are symmetric under $i \leftrightarrow j$, while the terms with an odd number of derivatives are antisymmetric. In the case of $\rho^{ij}$ and $\sigma^{ij}$ Eq. (18) leads to

$$\rho_A = -\sigma_A.$$

(20)

Furthermore, we can also consistently expand the coefficients $\alpha_A, \beta_A, \sigma_A$, and $\rho_A$ in powers of $\hat{\xi}$, according to

$$\zeta_A = \zeta_{A0} + \zeta_{A1} \hat{\xi} \partial_0 + \zeta_{A2} \hat{\xi}^2 \partial_0 \partial_0 \partial_0,$$

$$\zeta_A = \{\alpha_A, \beta_A, \rho_A, \sigma_A\},$$

(21)
where we denote

\[ \alpha_0^{(ij)} = \beta_0^{(ij)} = \eta^{ij}, \quad \rho_0^{(ij)} = 0, \quad \alpha_1^{(ij)} = \beta_1^{(ij)} = \rho_1^{(ij)} = 0, \]

\[ \alpha_1^{(ij)} = \epsilon^{ijr} \alpha_{10}, \quad \beta_1^{(ij)} = \epsilon^{ijr} \beta_{10}, \quad \rho_1^{(ij)} = \epsilon^{ijr} \rho_{10}, \]

\[ \alpha_2^{(ij)00} = \eta^{ij} \alpha_{200}, \quad \beta_2^{(ij)00} = \eta^{ij} \beta_{200}, \quad \rho_2^{(ij)00} = \eta^{ij} \rho_{200}, \quad \alpha_2^{(ij)0p} = \beta_2^{(ij)0p} = \rho_2^{(ij)0p} = 0, \]

\[ \alpha_2^{(ij)mn} = \frac{1}{2} \left( \delta^{im} \delta^{jn} + \delta^{in} \delta^{jm} \right) \alpha_{20}, \quad \beta_2^{(ij)mn} = \frac{1}{2} \left( \delta^{im} \delta^{jn} + \delta^{in} \delta^{jm} \right) \beta_{20}, \quad \rho_2^{(ij)mn} = \frac{1}{2} \left( \delta^{im} \delta^{jn} + \delta^{in} \delta^{jm} \right) \rho_{20}. \] (22)

The above partition is consistent with the requirement of covariance under rotations. We have taken \( \alpha_{00} = \beta_{00} = 1 \) and \( \rho_{00} = \sigma_{00} = 0 \) to recover the usual vacuum as the background for \( \tilde{\xi} = 0 \).

The fact that this theory does not hold at high energies will be coded by cutoff \( \Omega \ll E_{\text{QG}} \). We provide a general description of such modified electrodynamics including expressions for the equations of motion, the energy-momentum tensor and the Green functions as well as the corresponding refraction indexes, up to second order in \( \tilde{\xi} \).

### II. EQUATIONS OF MOTION

Equations (23), together with the corresponding ones for the remaining coefficients of the constitutive relations give

\[ \mathbf{D} = \left( \alpha_0 + \alpha_2 \tilde{\xi}^2 k^2 \right) \mathbf{E} - \left( \sigma_0 + i \alpha_1 \tilde{\xi} \omega \right) \mathbf{B} + \left( i \sigma_1 \tilde{\xi} + \omega \alpha_2 \tilde{\xi}^2 \right) \left( \mathbf{k} \times \mathbf{B} \right), \] (23)

\[ \mathbf{H} = \left( \beta_0 - i \omega \sigma_1 \tilde{\xi} \right) \mathbf{B} - i \beta_1 \tilde{\xi} \left( \mathbf{k} \times \mathbf{B} \right) + \sigma_0 \mathbf{E}. \] (24)

In the approximation to order \( \tilde{\xi}^2 \) here considered, we have \( \tilde{\xi}^2 k^2 \simeq \tilde{\xi}^2 \omega^2 \) and we can write

\[ \mathbf{D} = d_1(\omega) \mathbf{E} + id_2(\omega) \mathbf{B} + d_3(\omega) \tilde{\xi} \left( \mathbf{k} \times \mathbf{B} \right), \] (25)

\[ \mathbf{H} = h_1(\omega) \mathbf{B} + ih_2(\omega) \mathbf{E} + ih_3(\omega) \tilde{\xi} \left( \mathbf{k} \times \mathbf{B} \right), \] (26)

where the functions \( d_i(\omega) \) and \( h_i(\omega) \) depend only on \( \omega \) and admit a series expansion in powers of \( \tilde{\xi} \omega \), characterizing each specific model. From Eqs. (23) we get the equations for \( \mathbf{E} \) and \( \mathbf{B} \)

\[ id_1 \left( \mathbf{k} \cdot \mathbf{E} \right) = 4\pi \rho(\omega, \mathbf{k}), \] (27)

\[ i\omega d_1 \mathbf{E} + \left( h_3 k^2 - g(\omega) \right) \tilde{\xi} \mathbf{B} + \left( \omega d_3 \tilde{\xi} + h_1 \right) \left( \mathbf{i} \mathbf{k} \times \mathbf{B} \right) = 4\pi \mathbf{j}(\omega, \mathbf{k}), \] (28)

where we denote

\[ (d_2 + h_2) \omega = g(\omega) \tilde{\xi}. \] (29)

The expressions (23), (24), (25), (26) indeed indicate that the above combination is of order \( \tilde{\xi} \). We thus see that in fact there are only three independent functions of \( \omega \) and \( k \) which determine the dynamics

\[ P = d_1, \quad Q = h_1 + \omega d_3 \tilde{\xi}, \quad R = \left( h_3 k^2 - g(\omega) \right) \tilde{\xi}. \] (30)
Using the homogeneous equation \( \omega \mathbf{B} = \mathbf{k} \times \mathbf{E} \) that yields \( \omega (\mathbf{k} \times \mathbf{B}) = (\mathbf{k} \cdot \mathbf{E}) \mathbf{k} - k^2 \mathbf{E} \), and charge conservation \( \omega \rho - \mathbf{k} \cdot \mathbf{J} = 0 \), we decouple the equations for fields \( \mathbf{E} \) and \( \mathbf{B} \). Finally, we introduce the standard potentials \( \Phi \) and \( \mathbf{A} \)

\[
\mathbf{B} = i \mathbf{k} \times \mathbf{A}, \quad \mathbf{E} = i \omega \mathbf{A} - i \Phi.
\]

in the radiation gauge, \( \mathbf{k} \cdot \mathbf{A} = 0 \), in which case we have

\[
\Phi = 4\pi (k^2 P)^{-1} \rho,
\]

\[
(k^2 Q - \omega^2 P) \mathbf{A} + i R (\mathbf{k} \times \mathbf{A}) = 4\pi \left[ \mathbf{j} - \left( \mathbf{j} \cdot \mathbf{k} \right) \mathbf{k} \right] = 4\pi \mathbf{j}_T,
\]

from Eqs. (27-28). The presence of birefringence depends on the parity violating term proportional to \( R \). It is clear that a diagonalization is obtained in a circular polarization basis. Decomposing the vector potential and the current in such a basis

\[
\mathbf{A} = \mathbf{A}^+ + \mathbf{A}^-, \quad \mathbf{j}_T = \mathbf{j}_T^+ + \mathbf{j}_T^-,
\]

and recalling the basic properties \( \mathbf{k} \times \mathbf{A}^+ = -i \mathbf{A}^+, \mathbf{k} \times \mathbf{A}^- = i \mathbf{A}^- \), we separate (33) into the decoupled equations

\[
[k^2 Q - \omega^2 P + \lambda k R] \mathbf{A}^\lambda = 4\pi \mathbf{j}_T^\lambda, \quad \lambda = \pm 1.
\]

In terms of the basic functions introduced in the constitutive relations (25-26), the factor in (35) is rewritten as

\[
k^2 Q - \omega^2 P + \lambda k R = \lambda h_3 k^3 \xi + \left( h_1 + d_3 \omega \xi \right) k^2 - \lambda g_4 k - d_1 \omega^2.
\]

This is the key expression to obtain the Green functions and the refraction indices.

### III. GENERALIZED ENERGY-MOMENTUM TENSOR

Any application of this modified electrodynamics related to radiation and its properties requires the construction of the corresponding energy momentum tensor. This section is devoted to such a construction. The theories under consideration are of higher order in the field derivatives, and thus call for an extension of the standard Noether theorem. The manipulations are highly simplified by proceeding in a covariant notation, the point being that the tensorial operator \( \hat{\chi}^{[\mu\nu][\alpha\beta]} \) is constructed in a given reference frame and satisfies only passive Lorentz covariance. We assume that active Lorentz invariance is violated while active translation invariance is maintained, so that there is an energy momentum tensor given by the Noether theorem.

Before constructing this tensor in our particular case we recall the general formalism for a Lagrangian including up to three derivatives in the fields. This can be useful also when consistently including dimension five and six operators in the matter field coupled to the above electrodynamics. We start from an action of the form

\[
S = \int d^4x L(\Phi_A, \Phi_{A,\mu}, \partial_\nu \Phi_{A,\mu}, \partial_\nu \partial_\rho \Phi_{A,\mu}),
\]

(37)
where we consider $\Phi_A$, $\Phi_{A,\mu}$, $\partial_\nu \Phi_{A,\mu}$, $\partial_\nu \partial_\rho \Phi_{A,\mu}$, to be independent fields, i.e., at this level we take for example that $\partial_\nu \Phi_{A,\mu} \neq \partial_\mu \Phi_{A,\nu}$. Applying the standard action principle one derives the Euler-Lagrange (EL) equations

$$0 = \frac{\delta L}{\delta \Phi_A} - \partial_\mu \left( \frac{\delta L}{\delta \Phi_{A,\mu}} \right) + \partial_\nu \partial_\sigma \left( \frac{\delta L}{\delta \Phi_{A,\nu,\sigma}} \right) - \partial_\mu \partial_\nu \partial_\rho \left( \frac{\delta L}{\delta \partial_\rho \partial_\nu \partial_\mu \Phi_{A,\nu,\sigma}} \right), \tag{38}$$

Assuming that translations generated by $x'^\mu = x^\mu + a^\mu$ are a symmetry of the action, Noether’s theorem leads to the energy-momentum tensor

$$T^\tau_\sigma = -\delta^\tau_\sigma L + \Phi_{A,\sigma} \left[ \frac{\delta L}{\delta \Phi_{A,\tau}} \right] + \partial_\nu \partial_\sigma \left( \frac{\delta L}{\delta \Phi_{A,\nu,\sigma}} \right) + \partial_\mu \partial_\nu \partial_\rho \left( \frac{\delta L}{\delta \partial_\rho \partial_\nu \partial_\mu \Phi_{A,\nu,\sigma}} \right), \tag{39}$$

whose conservation $\partial_\nu T^\tau_\sigma = 0$ can be directly verified via the equations of motion (38). Next we apply the above general results to our Lagrangian (11) together with the realization (14), where $\Phi_A = A_\alpha$. The corresponding derivatives are

$$\frac{\delta L}{\delta A_\alpha} = 0, \quad \frac{\delta L}{\delta A_{\alpha,\tau}} = -F_{\mu\nu} \chi^0 \left[ \mu^\nu [\tau \alpha] \right] - \frac{1}{2} \chi_1 \left[ \tau [\rho \sigma] \partial_\nu \chi^{\rho \sigma \alpha} \right] - \frac{1}{2} \chi_2 \partial_\nu \partial_\rho \chi^{\nu \rho \sigma \alpha}, \tag{40}$$

$$\frac{\delta L}{\delta \partial_\mu A_{\alpha,\tau}} = -\frac{1}{2} F_{\theta \sigma} \chi^1 \left[ \theta \sigma [\tau \alpha] \right], \quad \frac{\delta L}{\delta \partial_\mu \partial_\nu A_{\alpha,\tau} \partial_\rho A_{\alpha,\tau}} = \frac{1}{2} F_{\theta \sigma} \chi^2 \left[ \theta \sigma [\nu \rho \tau \alpha] \right]. \tag{41}$$

The equations of motion outside the sources can be written as

$$0 = \partial_\nu H^{\tau \alpha}, \tag{42}$$

where

$$H^{\tau \alpha} = -\left( \frac{\delta L}{\delta A_{\alpha,\tau}} \right) + \partial_\nu \left( \frac{\delta L}{\delta \partial_\nu A_{\alpha,\tau}} \right) - \partial_\nu \partial_\rho \left( \frac{\delta L}{\delta \partial_\rho \partial_\nu \partial_\mu A_{\alpha,\tau}} \right) + \chi^{\tau \alpha [\theta \psi]} F_{\alpha \beta}. \tag{43}$$

Now let us consider the energy-momentum tensor (39). Let us observe that the second term in this equation is precisely $-A_{\alpha,\sigma} H^{\tau \alpha}$ which is not directly gauge invariant. It can be rewritten as

$$-A_{\alpha,\sigma} H^{\tau \alpha} = -F_{\sigma \alpha} H^{\tau \alpha} - A_{\sigma,\alpha} H^{\tau \alpha} = -F_{\sigma \alpha} H^{\tau \alpha} - \partial_\alpha \left( A_\sigma H^{\tau \alpha} \right), \tag{44}$$

by using the equations of motion. The last term is identically conserved and does not contribute to the corresponding charges. The remaining contributions are

$$\frac{\partial L}{\partial \partial_\mu A_{\alpha,\nu,\sigma}} - \partial_\rho \left( \frac{\partial L}{\partial \partial_\rho \partial_\mu A_{\alpha,\nu,\sigma}} \right) = -\frac{1}{2} \chi^1 \left[ \theta \psi [\tau \sigma \mu \alpha] \right] F_{\theta \psi} + \frac{1}{2} \chi_2 \left[ \theta \psi [\nu \rho \tau \alpha] \right] \partial_\rho F_{\theta \psi},$$

$$\left( \partial_\sigma \partial_\nu \partial_\mu \Phi_{A,\nu,\sigma} \right) \left( \frac{\partial L}{\partial \partial_\rho \partial_\nu \partial_\mu \Phi_{A,\nu,\sigma}} \right) = -\frac{1}{4} \left( \partial_\sigma \partial_\nu F_{\mu \alpha} \right) F_{\theta \sigma} \chi^2 \left[ \theta \sigma [\nu \rho \tau \alpha] \right], \tag{45}$$

which are naturally gauge invariant. The final gauge invariant, non-symmetric energy-momentum tensor is

$$T^\tau_\sigma = -\delta^\tau_\sigma L - F_{\sigma \alpha} H^{\tau \alpha} - \frac{1}{4} \left( \partial_\sigma F_{\mu \alpha} \right) \chi^1 \left[ \theta \psi [\tau \sigma \mu \alpha] \right] F_{\theta \psi},$$

$$+ \frac{1}{4} \left( \partial_\sigma F_{\mu \alpha} \right) \chi^2 \left[ \theta \psi [\nu \rho \tau \alpha] \right] \left( \partial_\rho F_{\theta \psi} \right) - \frac{1}{4} F_{\theta \psi} \chi^2 \left[ \theta \psi [\nu \rho \tau \alpha] \right] \left( \partial_\rho \partial_\nu F_{\mu \alpha} \right). \tag{46}$$
A direct but rather long calculation allows to verify the conservation \( \partial_t T^r_\sigma = 0 \), via the equations of motion \( \Box \).

To express the energy-momentum components in terms of the fields \( \mathbf{E} \) and \( \mathbf{B} \) we use the following \((3 + 1)\) splitting

\[
W_{\mu\nu} \chi_1^{\mu\nu} \tau^{[\alpha\beta]} U_{\alpha\beta} = 4W_{0\alpha} \chi_1^{[0i][0m]} U_{0m} + 2 \chi_1^{[0s][m]} [W_{0s} U_{mn} - W_{mn} U_{0s}] + W_{ij} \chi_1^{[ij][mn]} U_{mm},
\]

\[
W_{\mu\nu} \chi_2^{\mu\nu} \tau^{[\alpha\beta]} U_{\alpha\beta} = 4W_{0\alpha} \chi_2^{[0i][m]} U_{0m} + 2 \chi_2^{[0s][m]} [W_{0s} U_{mn} + W_{mn} U_{0s}] + W_{ij} \chi_2^{[ij][mn]} U_{mm},
\]

for antisymmetric fields \( W_{\mu\nu}, U_{\alpha\beta} \). We also recall the relation

\[
- L - F_{0\alpha} H^{0\alpha} = \frac{1}{2} (\mathbf{E} \cdot \mathbf{D} + \mathbf{B} \cdot \mathbf{H}),
\]

which is useful in calculating the energy density.

Let us illustrate the above construction by writing the energy density \( u = T^0_0 \) and the Poynting vector \( S_t = T^t_0 \) in terms of the fields \( \mathbf{E} \) and \( \mathbf{B} \) to first order in \( \tilde{\xi} \):

\[
S = \mathbf{E} \times \mathbf{B} + \frac{1}{2} \tilde{\epsilon}_{\alpha 10} \mathbf{E} \times \partial_t \mathbf{E} + \tilde{\xi}_{\beta 10} \left[ \frac{1}{2} (\partial_t \mathbf{B}) \times \mathbf{B} - \mathbf{E} \times (\nabla \times \mathbf{B}) \right] + \sigma_{10} \tilde{\xi} \partial_t [\mathbf{E} \times \mathbf{B}],
\]

\[
u = \frac{1}{2} (\mathbf{E}^2 + \mathbf{B}^2) - \frac{1}{2} \beta_{10} \tilde{\xi} \mathbf{B} \cdot \nabla \times \mathbf{B} - \frac{1}{2} \alpha_{10} \tilde{\xi} \mathbf{E} \cdot \nabla \times \mathbf{E} - \frac{1}{2} \sigma_{10} \tilde{\xi} \nabla \cdot [\mathbf{E} \times \mathbf{B}].
\]

The terms proportional to \( \sigma_{10} \) correspond to the liberty of modifying the energy momentum tensor as

\[
\tilde{T}^r_\sigma = T^r_\sigma + \partial_\rho V^r_\sigma, \quad V^r_\sigma = -V^r_\rho,
\]

which was previously used in Eq. \( \Box \). In the \((3 + 1)\) partition the above means

\[
\tilde{u} = u - \nabla \cdot \mathbf{Q}, \quad \tilde{S} = S + \partial_t \mathbf{Q} + \nabla \times \mathbf{W},
\]

with \( Q_i = V^i_0 \) and \( W_i = \frac{1}{2} \epsilon_{ijk} V^j_0 V^k_0 \). The last terms in Eqs. \( \Box \) and \( \Box \) correspond to the choice

\[
\mathbf{Q} = \frac{1}{2} \sigma_{10} \tilde{\xi} (\mathbf{E} \times \mathbf{B}), \quad \mathbf{W} = 0.
\]

Thus, the contributions proportional to \( \sigma_{10} \) can be eliminated and one recovers the corresponding expressions that can be obtained directly from Maxwell’s equations. Using the fields \( \mathbf{E}, \mathbf{B}, \mathbf{D} \) and \( \mathbf{H} \), together with the equations of motion \( \Box \) in vacuum and the freedom given by the energy-momentum tensor transformation \( \Box \), these magnitudes can also be written to first order in \( \tilde{\xi} \) in a much more compact form as

\[
S = \frac{1}{2} (\mathbf{E} \times \mathbf{B} + \mathbf{D} \times \mathbf{H}) - \frac{1}{2} (\alpha_{10} - \beta_{10}) \tilde{\xi} (\mathbf{B} \times (\nabla \times \mathbf{E}) - \mathbf{E} \times (\nabla \times \mathbf{B})),
\]

\[
u = \frac{1}{2} (\mathbf{E} \cdot \mathbf{D} + \mathbf{B} \cdot \mathbf{H}).
\]

### IV. GREEN FUNCTIONS

The exact retarded Green function for the potential \( \mathbf{A} \) in the circular polarization basis is \( \Box \)

\[
G_{ij}^{ret}(\omega, \mathbf{R}) = \int \frac{d^3k}{(2\pi)} e^{ik \cdot r} \tilde{G}_{ij}^{ret}(\omega, \mathbf{k}) = \frac{1}{2} \int \frac{d^3k}{(2\pi)} e^{ik \cdot R} \sum_{\lambda} G^{\lambda}(\omega, \mathbf{k}) \left( \delta_{ik} - \frac{k_i k_R}{k} + i\lambda \epsilon_{irk} \frac{k_r}{k} \right),
\]

\( \Box \).
where \( R = r - r' \), \( \hat{k}_i = k_i/|k| \), \( k = |k| \), and \( G^\lambda(\omega, k) \) is obtained from Eq. (55),

\[
G^\lambda(\omega, k) = \frac{1}{k^2 Q - \omega^2 P + \lambda k R}, \quad \lambda = \pm 1.
\] (57)

Taking the analytic continuation \( \omega \to \omega + i\epsilon \) to obtain the causal Green functions, only the poles in the upper half plane of \( k \) make a contribution to the integration. By successive rescalings, the denominator in Eq. (57) can be written in a more convenient form

\[
Qk^2 - P\omega^2 + \lambda k R = -n_0^2\omega^2 Q_a(M^\lambda - M_0)(M^\lambda - M_+)(M^\lambda - M_-),
\] (58)

where we introduce the notation

\[
Q = h_1 + d_3\omega\xi, \quad a = h_3n_0\chi, \quad c = \frac{g}{\omega^2n_0}\chi,
\] (59)

\[
\chi = \frac{\xi\omega}{h_1 + d_3\xi\omega}, \quad M^\lambda = \frac{\lambda k}{n_0\omega}, \quad n_0^2 = \frac{d_1}{h_1 + \omega\xi d_3}.
\] (60)

To study the modifications to the dynamics it is enough to expand each root in powers of the small parameter \( \chi \)

\[
M_0 \simeq \frac{1}{\beta_1}\chi^{-1}, \quad M_\pm \simeq \pm \left[ 1 + \frac{1}{2} \left( \tilde{\beta}_1 - \tilde{\alpha}_1 \right) \left( \lambda\chi + \frac{1}{4} \left( 5\tilde{\beta}_1 - \tilde{\alpha}_1 \right) \chi^2 \right) \right],
\] (61)

where \( \tilde{\beta}_1 = h_3n_0 \) and \( \tilde{\alpha}_1 = g/(\omega^2n_0) \). Since the parameter \( \lambda \) and the momentum \( k \) appear only in the combination \( \lambda k \), we have the symmetry property

\[
G^\lambda(\omega, k) = G^{-\lambda}(\omega, -k),
\] (62)

which will be useful in the final calculation of the Green functions \( G^\lambda(\omega, R) \). In the radiation approximation, the integral in (56) produces

\[
G_{ik}^{ret}(\omega, R) = -\frac{i}{(2\pi)^2 R} \sum_n \frac{1}{\lambda} \left( \delta_{ik} - n_in_k + i\lambda\epsilon_{ipkn_p} \right) \int_{-\Omega}^{\Omega} kdke^{ikR} G^\lambda(\omega, k),
\] (63)

where \( n_i = x_i/r \) and \( R = |r - r'| \). From now on we set \( R = r \) in all denominators and understand that \( R = r - n \cdot r' \) in the exponentials. We also neglect terms of order higher than \( 1/r \). The cutoff \( \Omega < E_{QG} \) defines the low energy domain of the effective theory. In this way we identify \( G^\lambda(\omega, R) \) as

\[
G^\lambda(\omega, R) = -\frac{i}{(2\pi)^2 R} \int_{-\Omega}^{\Omega} kdke^{ikR} G^\lambda(\omega, k).
\] (64)

The factor \( e^{ikR} \) forces us to close the integration contour in the upper half complex plane, choosing for example a semicircle with radius \( k = \Omega \), picking up the poles in this region. Our description is valid only for momenta \( k << \Omega \).

According to Eqs. (61), the pole at \( M_0^\lambda \) corresponds to the momentum value \( |k_0| = |Qh_3^{-1}| \xi^{-1} \). In the present approximation the contribution to the integral of this pole, together with the one of the semicircle in the upper half complex plane, can be neglected. The two remaining poles, which are the ones that contribute to the integral, are located at small displacements with respect to \( |k_\pm| = n_0\omega << E_{QG} \). In this way we take

\[
G^\lambda(\omega, k) = \frac{1}{n_0^2\omega^2 QaM_0 (M^\lambda - M_+)(M^\lambda - M_-)}.
\] (65)
From the leading order expressions in (61) we conclude that the pole that contributes in the case \( \lambda = +1 \) is
\( (\omega + i \epsilon) n_0 M_+ \), while for \( \lambda = -1 \) it is \( - (\omega + i \epsilon) n_0 M_- \). The resulting integral is
\[
G^\lambda(\omega, R) = \frac{1}{4\pi Q} \frac{2n_\lambda}{n_- + n_+} e^{i\omega_n \lambda R},
\]
where we have considered that the dominant term in \( M_0 \) yields \( aM_0 = 1 \). The refraction indices are
\[
n_\lambda(\omega) = \lambda n_0 M_\lambda. \tag{67}
\]
The minus sign in \( n_- \) is because \( M_- \) starts with a \( -1 \). Up to the order considered, the refraction indices are given by the expressions
\[
n_\lambda(\omega) = n_0 \left[ 1 + \lambda \left( \tilde{\beta}_1 - \tilde{\alpha}_1 \right) \frac{\lambda}{2} + \left( \tilde{\beta}_1 - \tilde{\alpha}_1 \right) \left( 5\tilde{\beta}_1 - \tilde{\alpha}_1 \right) \frac{\lambda^2}{8} \right]. \tag{68}
\]
From Eq. (68), and using Eqs. (18), (22), (27-29), (60) and (61), we can obtain the second order expansion for \( n_\lambda \)
\[
n_\lambda(\omega) \simeq 1 + \lambda \left( \tilde{\xi}_1 \omega \right) n_1 + \left( \tilde{\xi}_2 \omega \right)^2 n_2, \tag{69}
\]
with the real coefficients \( n_1 \) and \( n_2 \) given by
\[
n_1 = \frac{1}{2} \left( \alpha_{10} - \beta_{10} \right), \quad n_2 = \frac{1}{8} \left[ (\alpha_{10} - \beta_{10}) (\alpha_{10} - 5\beta_{10}) + 4(\beta_{02} - \alpha_{02}) \right]. \tag{70}
\]
According to this the phase velocity is not 1. Due to the dispersive character of the background it becomes
\[
v_{ph}(\omega) \simeq 1 - \lambda \left( \tilde{\xi}_1 \omega \right) n_1 + \left( \tilde{\xi}_2 \omega \right)^2 \left( (n_1)^2 - n_2 \right).
\]
Thus, the Green function in terms of space-time coordinates is
\[
G^\lambda(\tau, R) = \frac{1}{4\pi R} \int_{-\Omega}^{\Omega} d\omega \frac{2n_\lambda}{Q (n_- + n_+)} e^{i\omega_n \lambda R} e^{-i\omega\tau} \\
= \frac{1}{4\pi R} \int_{-\Omega}^{\Omega} d\omega \left[ 1 + \lambda \frac{\omega}{2} \left( \tilde{\xi}_1 \omega \right) (\alpha_{10} - \beta_{10}) - \left( \tilde{\xi}_2 \omega \right)^2 (\alpha_{20} - \beta_{02}) \right] e^{i\omega \left[ 1 + \lambda (\tilde{\xi}_1 \omega) n_1 + (\tilde{\xi}_2 \omega)^2 n_2 \right]} R e^{-i\omega\tau}, \tag{71}
\]
where \( \tau = t - t' \).

If \( \Omega \to \infty \) the choice of the poles warrants the causal behavior of the Green function. But the frequency cutoff could introduce some violation of causality. To investigate this possibility, we compute the Fourier transform of the Green function to obtain its time dependent expression, by expanding the integrand in powers of \( \tilde{\xi} \)
\[
G^\lambda(\tau, R) \simeq \frac{1}{2\pi R} \int_{-\Omega}^{\Omega} d\omega \left[ 1 + \lambda n_1 (1 + i\omega R) \omega \tilde{\xi} - \left[ \alpha_{20} - \beta_{02} - i \left( n_2 + n_1^2 \right) R \omega + \frac{1}{2} n_1^2 R^2 \omega^2 \right] \omega^2 \tilde{\xi}^2 \right] e^{i\omega (R - \tau)} \\
= \frac{1}{2\pi R} \left\{ 1 - i\lambda n_1 \tilde{\xi} (1 + R\partial_R) \partial_R + \tilde{\xi}^2 \left[ \alpha_{20} - \beta_{02} - \left( n_2 + n_1^2 \right) R \partial_R - \frac{1}{2} n_1^2 R^2 \partial_R^2 \right] \partial_R^2 \right\} \frac{\sin (R - \tau) \Omega}{R - \tau}. \tag{72}
\]
This shows that the main effect of the cutoff is to spread the propagating field around the light cone, within a wedge defined by \( R \simeq \tau \pm \pi/2\Omega \). Returning to \( G^\lambda(\omega, R) \), Eq. (66), we can characterize the effect of the cutoff in
the causal behavior of the Green function using the generalized susceptibility theorem \[21\], a generalization of the Kramers-Kronig relations. Its real and imaginary parts as functions of the frequency \(\omega\) are

\[
\text{Re} \, G^\lambda(\omega, R) \simeq \frac{1}{4\pi r} \left\{ 1 + \xi^2 \left( \alpha_{20} - \beta_{02} - (n_2 + n_1^2) R \partial_R - \frac{1}{2} n_1^2 R^2 \partial_R^2 \right) \partial_R^2 \right\} \cos \omega R + \lambda n_1 \xi (1 + R \partial_R) \partial_R \sin (\omega R),
\]

(73)

\[
\text{Im} \, G^\lambda(\omega, R) \simeq \frac{1}{4\pi r} \left\{ 1 + \xi^2 \left( \alpha_{20} - \beta_{02} - (n_2 + n_1^2) R \partial_R - \frac{1}{2} n_1^2 R^2 \partial_R^2 \right) \partial_R^2 \right\} \sin \omega R - \lambda n_1 \xi (1 + R \partial_R) \partial_R \cos \omega R.
\]

(74)

To have a causal behavior they must satisfy the Kramers-Kronig relation

\[
\text{Im} \, G(\omega)|_{KK} = -\frac{1}{\pi} P \int_{-\Omega}^\Omega d\omega' \frac{\text{Re} \, G(\omega') - \text{Re} \, G(\Omega)}{\omega' - \omega}
\]

which gives

\[
\text{Im} G^\lambda(\omega)|_{KK} = -\frac{1}{4\pi^2 r} \left\{ 1 + \xi^2 \left( \alpha_{20} - \beta_{02} - (n_2 + n_1^2) R \partial_R - \frac{1}{2} n_1^2 R^2 \partial_R^2 \right) \partial_R^2 \right\} P \int_{-\Omega}^\Omega d\omega' \frac{\cos (\omega' R) - \cos (\Omega R)}{\omega' - \omega}
\]

\[
+ \lambda n_1 \xi (1 + R \partial_R) \partial_R P \int_{-\Omega}^\Omega d\omega' \frac{\sin (\omega' R) - \sin (\Omega R)}{\omega' - \omega}
\]

(76)

For \(\omega/\Omega \ll 1\) the integrals reduce to

\[
P \int_{-\Omega}^\Omega d\omega' \frac{\cos (\omega' R) - \cos (\Omega R)}{\omega' - \omega} \simeq 2 (1 - \cos \omega R) \frac{\omega R}{\Omega} \cos (\Omega R) - \left[ \pi + \left[ (\Omega R) (\cos \Omega R) - \sin \Omega R \right] \left( \frac{\omega}{\Omega} \right)^2 \right] \sin \omega R,
\]

\[
P \int_{-\Omega}^\Omega d\omega' \frac{\sin (\omega' R) - \sin (\Omega R)}{\omega' - \omega} \simeq 2 (1 - \sin \omega R) \frac{\omega R}{\Omega} \cos (\Omega R) + \left[ \pi + \left[ (\Omega R) (\cos \Omega R) - \sin \Omega R \right] \left( \frac{\omega}{\Omega} \right)^2 \right] \cos \omega R.
\]

(77)

Furthermore, in the case of a radiation field \(\Omega R \gg 1\) and hence the factors \(\cos \Omega R\) and \(\sin \Omega R\) become strongly oscillating, nullifying the contributions of the terms where they appear (which also have a factor \((\omega/\Omega)^n\), with \(n \geq 1\)).

Thus we can take

\[
P \int_{-\Omega}^\Omega d\omega' \frac{\cos (\omega' R) - \cos (\Omega R)}{\omega' - \omega} \simeq -\pi \sin \omega R,
\]

\[
P \int_{-\Omega}^\Omega d\omega' \frac{\sin (\omega' R) - \sin (\Omega R)}{\omega' - \omega} \simeq \pi \cos \omega R.
\]

(78)

Replacing these integrals in (76), we finally get that if \(\text{Re} \, G^\lambda(\omega)\) is given by Eq. (73), the imaginary part of the Green function, \(\text{Im} \, G^\lambda(\omega)\), must be

\[
\text{Im} G^\lambda(\omega)|_{KK} = \frac{1}{4\pi r} \left\{ 1 + \xi^2 \left( \alpha_{20} - \beta_{02} - (n_2 + n_1^2) R \partial_R - \frac{1}{2} n_1^2 R^2 \partial_R^2 \right) \partial_R^2 \right\} \sin \omega R - \lambda n_1 \xi (1 + R \partial_R) \partial_R \cos \omega R,
\]

(79)

which coincides with Eq. (74), obtained by direct computation.

V. FINAL REMARKS

To summarize, in the preceding sections we have presented a general description for a large class of effective models for the electromagnetic field incorporating dynamical corrections motivated by QG and leading to departures from
standard physics. The main features characterizing the models to which such a description is applicable are: (1) the validity of gauge invariance and charge conservation, (2) the use of standard commuting space-time coordinates together with the corresponding Fourier transform methods (which is not the case of Double Special Relativity models, for example), (3) the assumption that effective field theories constitute an appropriate tool for describing the low energy behavior of remnant effects which could arise from quantum gravity, (4) the assumption that low energy dynamics is linear in the potential field, (5) the inclusion of non-local effects via the operator character of the generalized susceptibilities. This description makes it also possible to include anisotropic corrections in the constitutive relations, for example via additional non-dynamical tensors arising from spontaneous Lorentz symmetry breaking, a case which is not considered in this work.

The proposed formalism is quite similar to the usual electrodynamics in a medium, except for the non-local character of the effective QG corrections, mirroring the granularity of the space-time induced by quantum fluctuations of the metric. This feature leads to an electrodynamics with non-local constitutive relations, which contains terms connecting $D$ and $H$ with both $E$ and $B$. Thus the QG modifications can be modelled by a dispersive bianisotropic medium, where the propagation of the electromagnetic field is characterized by a refraction index, whose first order term in the perturbative parameter $\tilde{\xi}$ is directly related to vacuum birrefringence. We have considered the models from the point of view of active transformations, i.e. observable Lorentz symmetry violations associated with boosts in a given reference frame.

The effective models correspond in fact to high order theories. Hence we used an adequate generalization of the Noether theorem to find the energy-momentum tensor to second order in the LIV parameter $\tilde{\xi}$. Next we determined the density of energy and momentum carried by the electromagnetic field, for which we give explicit expressions to order $\tilde{\xi}$. They acquire a simple form when written in terms of the fields $E$, $B$ and $D$, $H$, which shows the convenience of the latter for describing the dynamics, in an analogous way to the usual electrodynamics in media.

This theory is valid only for low energies. We have also studied the consequences of this fact by using an explicit cutoff $\Omega \ll E_{QG} \sim \tilde{\xi}^{-1}$. In fact, the results in Eqs. (77) and (78) show that the introduction of the cutoff does not produce any significant causality violation in the radiation regime ($\Omega R \gg 1$) because the expected modifications proportional to $(\omega/\Omega)^n$ in (77), which are the subject of possible signals of new physics in these approaches, are further suppressed by highly oscillating terms proportional to $\sin(\Omega R)$, $\cos(\Omega R)$ thus nullifying the impact of causality violation upon the corresponding observational effects. The most outstanding manifestation of the cutoff is a spreading of the propagation of the electromagnetic field around the light cone. In fact there are two sources for such a spreading. One is due to the cutoff and the other arises from the dispersive character of the effective medium, which leads to an $\omega$-dependent phase velocity. The relation between both effects depends on the relative value of $\Omega$ and $\tilde{\xi}^{-1}$. In any case, for distances large enough from the source ($\omega R \gg 1$), the dispersive effect will finally dominate.

There remains to discuss the causal behavior of the full theory. There are two possible sources of acausality. One is related to the dispersive character of the effective medium, while the other is related to the existence of velocities
\( v > 1 \) that leads to photons propagating to the past in highly boosted reference frames, and hence to the possibility of acausal loops. This issue is beyond the scope of the present work, and will be discussed in detail elsewhere.

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