Utility Maximization Problem with Transaction Costs: Optimal Dual Processes and Stability

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Abstract
In this paper, we investigate the stability problem of the numéraire-based utility maximization problem in markets with transaction costs, where the stock price is not necessarily a semimartingale. Precisely, the static stability of primal and dual value functions as well as the convergence of primal and dual optimizers are presented when perturbations occur in the utility function and in the physical probability. Furthermore, this study focuses on the optimal dual process (ODP), which induces the dual optimizer and attains optimality for a dynamical dual problem. Properties of ODPs are discussed which are complement of the duality theory for this utility maximization problem. When the parameters of the market and the investor are slightly perturbed, both the dual optimizer and the associated optimal dual process are stable. Thus, a shadow price process is constructed based on the sequence of ODPs corresponding to problems with small misspecified parameters.

Keywords Utility maximization problem · Transaction costs · Stability · Optimal dual processes · Shadow price processes

Mathematics Subject Classification 91G80 · 93E15 · 60G48

1 Introduction
The utility maximization problem with constant proportional transaction costs has been thoroughly studied. In this paper, we restrict ourselves to consider such a problem with a numéraire-based general model, in which an investor trades the stock using admissible strategies and aims to maximize the expected utility for terminal wealth:
In frictionless market models beyond semimartingales, arbitrage opportunities exist, preventing the utility maximization problem from being well-posed. The presence of transaction costs could exclude such strategies (cf. Guasoni [25]). In this case, optimal strategies can be found for the maximization problem with a finite indirect utility function [24]. With proportional transaction costs $\lambda$, the duality theory for utility maximization dates to the seminal works [12,13], in which $S$ is driven by an Itô-process, extended by [14,17] into a general framework when the utility function supports only the positive half-line. Besides, we refer the reader to [7–9,19,47] in similar context but with multivariate utility functions.

In [14], the solution pair $\hat{\varphi} = (\hat{\varphi}_0, \hat{\varphi}_1)$ for the primal problem is established by first solving the corresponding dual problem, in particular, under the assumption of existence of consistent price systems introduced in [27] (see Definition 2.2) which takes the similar role as equivalent local martingale measures in a frictionless market. Instead of the Fatou limit argument in [31], we formulate the domain $B(1)$ for the dual problem of (1.1) as the collection of all optional strong supermartingales, which are the limits of consistent price systems in the sense of [15],

$$Z_n^\tau \longrightarrow Y_\tau, \quad \text{in } P, \quad \text{for all } 0 \leq \tau \leq T. \quad (1.2)$$

With the help of the above notion, the dual problem then can be written as

$$\mathbb{E}[V(yY_0^\tau T)] \rightarrow \min!, \quad Y = (Y^0, Y^1) \in B(1). \quad (1.3)$$

In the framework of [14,17], the present paper addresses the following question:

How do small changes in input parameters (e.g., initial wealth and the investor’s preferences) influence the optimal strategy and the optimality of the problem (1.1)?

Without transaction costs, research on the stability of a problem similar to (1.1) was conducted in [28] (and [11]) in a market driven by an Itô-process. This result was generalized by Larsen [34] for a similar problem with continuous semimartingales. We also refer the readers to [2,3,22,30,35–37,39,45,46] for more research in this direction with various settings.

Still in the frictionless market, the sensitivity analysis of utility maximization problem (cf. [32,33]) also plays an important role. More recently, Backhoff-Veraguas and Silva [1] have studied the sensitivity of the expected power utility maximization in incomplete Brownian markets under weak and strong perturbations. Mostovyi and Sirbu [40] then have studied such a problem in a continuous semimartingale setting with respect to small variations in the market price of risk.

On the other hand, the stability problem in markets with transaction costs is explored in [4,26]. Herdegen and Muhle-Karbe have studied the stability with respect to the asymptotic expansions. They formulated the equilibrium models with small proportional transaction costs, then focused on the frictionless equilibrium and the asymptotic equilibrium with small transaction costs. Recently, assuming extended weak convergence of the underlying processes, Bayraktar et al. [4] proved the stability of the
corresponding utility maximization problem and established a limit theorem for the optimal trading strategies in a financial market with proportional transaction costs.

This paper starts from the usual formulation of utility maximization problem (1.1) and (1.3). As far as we know, this is the first stability result in the setting with transaction costs. The present paper is organized as follows. Section 2 introduces the formulation of the model and the duality result for solving the problem (1.1) with a càdlàg price process $S$. Section 3 presents a stability analysis: $L^0$-stability of primal and dual optimizers and continuity of the primal and dual value functions and their derivatives, in which the framework of [30] is adapted to the context of transaction costs when initial wealth, investor’s preferences and market’s subjective probabilities are perturbed. This result is called static stability result in the sequel.

We point out that the duality result in [14,17] is both static and dynamic; thus one can find an optimal dual process (ODP) $\hat{Y} = (\hat{Y}^0, \hat{Y}^1)$ in $\mathcal{B}(1)$ whose first component $\hat{Y}^0$ attains the optimality of (1.3). Section 4.1 devotes to the study on the dual domain of the problem (1.3) and the properties of ODPs, complementing the results of [14,17]. In particular, it is proven that for $Y \in \mathcal{B}(1)$, the second component $Y^1$ is a local martingale if and only if $Y^0$ is a local martingale. Additionally, a simple explanation is provided to the fact that the ODP must be a local martingale in [17] when $S$ is continuous and the running liquidation value of the optimal wealth process is bounded away from 0, see (4.2). Another important observation is that the optimal dual process is not unique, which is in contrast to the classical frictionless theory. A counterexample can be constructed with a simple time-changed geometric Brownian motion.

In addition to the static stability result in Sect. 3, the so-called dynamic stability is investigated in Sect. 4.2 based on the studies on ODPs presented in the previous subsection. Mathematically, if the sequence $\{\hat{Y}^n\}_{n \in \mathbb{N}}$ are ODPs associated with the perturbed problems, then a convex combination of $\{\hat{Y}^n\}_{n \in \mathbb{N}}$ exists that admits a limit $\hat{Y}(x; U, P)$ in the sense of (1.2). This limit must be an ODP of the limiting problem.

It is stated in [14] that if $\hat{Y}^0$ in (1.3) is a local martingale, then a shadow market can be defined via $\hat{S} := \frac{\hat{Y}^1}{\hat{Y}^0}$. Conceptually, the shadow market is a frictionless market driven by a price process $\hat{S}$ between the bid-ask spread $[(1 - \lambda)S, S]$. If the investor trades with $\hat{S}$ frictionlessly instead of with $S$ under transaction costs, the utility maximization problem (1.1) is solved by the trading strategy which yields the same optimality (cf. [6,12,16–18,29]). We provide a possible way for constructing a shadow price process for the limiting problem from those of perturbed problems in Sect. 4.3.

### 2 Formulation of the Utility Maximization Problem

In this section, we introduce the market model with transaction costs and recall existing results on duality obtained in [14].

Fix a finite time horizon $T > 0$. We consider a scalar-valued strictly positive, càdlàg, adapted price process $S = (S_t)_{0 \leq t \leq T}$ based on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})$ satisfying the usual conditions. Additionally, we assume $\mathcal{F}_{T^-} = \mathcal{F}_T$. The market is completed by a risk-free bond $B = (B_t)_{0 \leq t \leq T}$ with $B_t = 1$ for all $t$. The utility function $U \in \mathcal{U}$ is a $\mathcal{F}_T$-measurable random variable, which is to be maximized subject to the constraints $0 < \omega_0 < \omega_T < \infty$. The transactions are subject to a proportional transaction cost $\beta > 0$. Let $\lambda > 0$ denote the bid-ask spread, and let $\lambda S = (\lambda S_t)_{0 \leq t \leq T}$ be the bid-ask spread process.

Let $\mathcal{M} := \{\mu \in L^1(\Omega, \mathcal{F}_{T^{-}}, \mathbb{P}) : \mu\text{ is adapted and }\mu(\omega) > 0\text{ for all }\omega\}$ be the set of admissible measures. The investor’s problem is to find a strategy $\mu \in \mathcal{M}$ such that

$$\mathbb{E}(U(\omega_T) - \beta \int_0^T \lambda S_t \mu_t dt)$$

is maximized. The optimal strategy is a càdlàg process $\mu^* = (\mu^*_t)_{0 \leq t \leq T}$ that solves the dual problem

$$\sup_{\mu \in \mathcal{M}} \mathbb{E}(U(\omega_T) - \beta \int_0^T \lambda S_t \mu_t dt)$$

subject to

$$\mathbb{E}(\mu_T) = 1.$$
$\mathcal{F}_T$ and $S_T = S_T$. The market involves proportional transaction costs $0 < \lambda < 1$, i.e., an investor buys stock shares at the higher ask price $S_t$ and only receives a lower bid price $(1 - \lambda)S_t$ when selling them.

Fix $0 < \lambda < 1$. A self-financing trading strategy, modeling holdings in units of the bond and of stock, is an $\mathbb{R}^2$-valued, predictable process $\varphi = (\varphi^0_t, \varphi^1_t)_{0 \leq t \leq T}$ of finite variation such that $\int_s^t d\varphi^0_u \leq -\int_s^t S_u d\varphi^1_u + \int_s^t (1 - \lambda)S_u d\varphi^1_u$, for all $0 \leq s < t \leq T$, where $\varphi^1 = \varphi^{1\uparrow} - \varphi^{1\downarrow}$ denotes the canonical decompositions of $\varphi^1$ into the difference between two increasing processes. The liquidation value $V^\text{liq}_T(\varphi)$ of a trading strategy $\varphi$ at time $t \in [0, T]$ is defined by $V^\text{liq}_T(\varphi) := \varphi^0_t + (\varphi^1_t)^+(1 - \lambda)S_t - (\varphi^1_t)^-S_t$. A trading strategy $\varphi$ is admissible, if $V^\text{liq}_T(\varphi) \geq 0$ for all $0 \leq t \leq T$. For $x > 0$, $\mathcal{A}(x)$ denotes the set of all admissible, self-financing trading strategies $\varphi$ starting with the initial endowment $(\varphi^0_0, \varphi^1_0) = (x, 0)$, and $\mathcal{C}(x)$ denotes the convex subset of terminal liquidation values as

$$C(x) := \left\{ V^\text{liq}_T(\varphi) : \varphi \in \mathcal{A}(x) \right\} = \left\{ \varphi^0_T : \varphi = (\varphi^0, \varphi^1) \in \mathcal{A}(x), \varphi^1_T = 0 \right\} \subseteq L^0_+(\mathbb{P}).$$

Given the initial endowment $x > 0$, the investor wants to maximize her expected utility at terminal time $T$:

$$\mathbb{E}[U(V^\text{liq}_T(\varphi))] \to \max!, \quad \varphi \in \mathcal{A}(x), \quad (2.1)$$

where $U : \mathbb{R}_+ \to \mathbb{R}$ is a strictly increasing, strictly concave and smooth function, satisfying the Inada conditions as well as the condition of reasonable asymptotic elasticity introduced in [31]. To avoid a trivial case, the following is assumed.

**Assumption 2.1** Suppose that $u(x) := \sup_{g \in \mathcal{C}(x)} \mathbb{E}[U(g)] < \infty$, for some $x > 0$.

The duality theorem for utility maximization in [14] requires an assumption on the existence of consistent price systems, that is similar to the existence of equivalent martingale measures or similar weaker conditions for frictionless markets.

**Definition 2.2** Fix $0 < \lambda < 1$. A $\lambda$-consistent price system is a two-dimensional strictly positive process $Z = (Z^0_t, Z^1_t)_{0 \leq t \leq T}$ with $Z^0_0 = 1$, consisting of a martingale $Z^0$ and a local martingale $Z^1$ under $\mathbb{P}$, such that

$$\tilde{S}_t := \frac{Z^1_t}{Z^0_t} \in [(1 - \lambda)S_t, S_t], \quad \text{a.s.,} \quad 0 \leq t \leq T.$$ 

Denote by $\mathcal{Z}^\lambda(S)$ the set of $\lambda$-consistent price systems, and we say that $S$ satisfies the condition $(\text{CPS}^\lambda)$ of admitting a $\lambda$-consistent price system if $\mathcal{Z}^\lambda(S)$ is nonempty.

We say that $S$ satisfies the condition $(\text{CPS}^\infty)$ locally if there exists a strictly positive process $Z$ and a sequence $\{\tau_n\}_{n \in \mathbb{N}}$ of $[0, T] \cup \{\infty\}$-valued stopping times, increasing

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1 These assumptions avoid special notations for possible trading at the terminal time $T$. In this case, we may assume without loss of generality that the agent liquidates her position in the stock shares at time $T$. For more details see e.g. [10, Remark 4.2] or [14, p. 1895].
to infinity, such that each stopped process $Z_{\tau}$ defines a $\lambda$-consistent price system for the stopped process $S_{\tau}$. This process $Z$ is called a local $\lambda$-consistent price system, and $\mathcal{Z}^{loc,\lambda}$ denotes the set of all such processes.

**Assumption 2.3** The stock price process $S$ satisfies $(CPS^{\mu})$ locally for all $0 < \mu < \lambda$.

We denote by $B(y)$ the set of all optional strong supermartingale deflectors, which are pairs of nonnegative optional strong supermartingales\(^2\) $Y = (Y^0, Y^1)_{0 \leq t \leq T}$, such that $Y^0_0 = y$, $Y^1 = Y^0 \mathcal{S}$ for some $[(1 - \lambda)S, S]$-valued process $S = (S_t)_{0 \leq t \leq T}$, and $Y^0(\varphi^0 + \varphi^1 \mathcal{S}) = Y^0\varphi^0 + Y^1\varphi^1$ is a nonnegative optional strong supermartingale for all $(\varphi^0, \varphi^1) \in \mathcal{A}(1)$. Accordingly, we define

$$\mathcal{D}(y) = \left\{ Y^0_T : (Y^0, Y^1) \in B(y) \right\}, \quad \text{for } y > 0.$$  

As usual we denote by $V$ the conjugate function of $U$: $V(y) := \sup_{x > 0} \{ U(x) - xy \}$, $y > 0$. Then, the dual problem is given by

$$v(y) := \inf_{h \in \mathcal{D}(y)} \mathbb{E}[V(h)]. \quad (2.2)$$

The polarity between sets $\mathcal{C} := \mathcal{C}(1)$ and $\mathcal{D} := \mathcal{D}(1) \subseteq L^0_+(\mathcal{P})$ is established in [14, Lemma A.1]. As a result, the duality result in [14] is straightforward by following the lines of [31], which is summarized as follows.

**Theorem 2.4** [14, Theorem 3.2] Under Assumptions 2.1 and 2.3, the following statements hold true.

(i) For all $x, y > 0$, the solutions $\hat{g}(x) \in \mathcal{C}(x)$ and $\hat{h}(y) \in \mathcal{D}(y)$ exist and are unique. There exist $(\hat{\varphi^0}(x), \hat{\varphi^1}(x)) \in \mathcal{A}(x)$ and $(\hat{Y^0}(y), \hat{Y^1}(y)) \in B(y)$ such that

$$V^{liq}_T(\hat{\varphi}(x)) = \hat{\varphi}^0_T = \hat{g}(x) \quad \text{and} \quad \hat{Y}^0_T(y) = \hat{h}(y). \quad (2.3)$$

(ii) For $x, y > 0$ with $u'(x) = y$, or equivalently by $x = -u'(y)$, $\hat{g}(x)$ and $\hat{h}(y)$ are related, i.e.,

$$\hat{h}(y) = U'(\hat{g}(x)) \quad \text{and} \quad \hat{g}(x) = -V'(\hat{h}(y)), \quad (2.4)$$

and we have

$$\mathbb{E}[\hat{g}(x)\hat{h}(y)] = xy. \quad (2.5)$$

In particular, $\hat{\varphi^0}(x)\hat{Y^0}(y) + \hat{\varphi^1}(x)\hat{Y^1}(y)$ is a $\mathcal{P}$-martingale for all $(\hat{\varphi^0}(x), \hat{\varphi^1}(x)) \in \mathcal{A}(x)$ and $(\hat{Y^0}(y), \hat{Y^1}(y)) \in B(y)$ satisfying (2.3).

\(^2\) A real-valued optional process $X = (X_t)_{0 \leq t \leq T}$ is called optional strong supermartingale if

- $X_\tau$ is integrable for every $[0, T]$-valued stopping time $\tau$;
- For all stopping times $\sigma$ and $\tau$ with $0 \leq \sigma < \tau \leq T$, we have $X_\sigma \geq \mathbb{E}[X_\tau | \mathcal{F}_\sigma]$.

It is a generalisation of càdlàg supermartingales. See [38] and [21, Appendix I] for more properties.
(iii) Finally, we have

\[ v(y) = \inf_{(Z^0, Z^1) \in Z^{loc, \lambda}} \mathbb{E} \left[ V(y Z^0_T) \right]. \]

For reader’s convenience, the notations for stability in the following sections are clarified. For the utility maximization problem in the market under the physical probability \( P \), the following simplified notations are employed

\[ B(y) = B(y, P), \quad D(y) = D(y, P), \quad Z^\lambda = Z^\lambda(P), \quad Z^{loc, \lambda} = Z^{loc, \lambda}(P). \]

Moreover, an optional strong supermartingale deflator \((\hat{Y}^0(y), \hat{Y}^1(y)) \in B(y)\) inducing \( \hat{Y}^0_T(y) = \hat{h}(y) \) is called an optimal dual process (ODP). Sometimes, the normalized process \( \hat{Y} := (\hat{Y}^0, \hat{Y}^1) \in B(1) \) is used, and by abuse of definition, still called ODP. Similarly, a trading strategy \((\hat{\varphi}^0(x), \hat{\varphi}^1(x)) \in A(x)\) satisfying \( V_{liq}^T(\hat{\varphi}(x)) = \hat{g}(x) \) is called an optimal primal process (OPP).

### 3 Static Stability

For a fixed physical probability measure \( P \) and a fixed utility function \( U \), and for any initial endowment \( x \), by Theorem 2.4, the primal problem (2.1) can be solved by some \( \hat{\varphi}^0_T(x; U, P) \in C(x; P) \), and the optimal value is denoted by \( u(x; U, P) \). For the dual input \( y > 0 \), the dual problem (2.2) parameterised by \((V, P)\) admits a solution \( \hat{Y}^0_T(y; V, P) \in D(y; P) \), and the optimal value is denoted by \( v(y; V, P) \). This section discusses how the optimality of the primal and dual problems is affected by

- perturbations of the initial endowment and of the dual input;
- variations in the investor’s utility;
- misspecification of the underlying market model.

Suppose that variations in the investor’s utility are represented by a sequence of perturbed utility functions \((U_n)_{n \in \mathbb{N}}\). In addition, assume that the variation in the underlying market model is described by a sequence of probabilities \((P_n)_{n \in \mathbb{N}}\). Now, introduce the strict mathematical formulation of this problem, in which assumptions are made in accordance with [30] for considering a frictionless case with random endowments.

**In order to avoid redundancy, here we emphasis that the assumptions needed for Theorem 2.4 are made for each problem \( u(x_n, U_n, P_n), n \in \mathbb{N} \) as well as for the problem \( u(x, U, P) \) throughout this paper.**

**Assumption 3.1** For each \( n \in \mathbb{N} \), \( P_n \sim P \) and

\[ \lim_{n \to \infty} P_n = P, \text{ in total variation, and } \lim_{n \to \infty} U_n = U, \text{ pointwisely,} \]

and \( x_n \to x > 0, \ y_n \to y > 0 \).

**Remark 3.2** We note that:
(1) For each \( n \), the Radon-Nikodym derivative \( \frac{dP_n}{dP} \) exists. Denote
\[
\tilde{Z}_t^n := \mathbb{E}^P \left[ \frac{dP_n}{dP} \Big| \mathcal{F}_t \right],
\]
which is a \( P \)-martingale. Note that \( Z_{loc, \lambda}(P) \neq \emptyset \) implies \( Z_{loc, \lambda}(P_n) \neq \emptyset \) for each \( n \in \mathbb{N} \). In fact, for any \( Z \in Z_{loc, \lambda}(P) \), \((\tilde{Z}_n)^{-1} Z \in Z_{loc, \lambda}(P_n) \) and for any \( Z' \in Z_{loc, \lambda}(P_n) \), \( \tilde{Z}_n Z' \in Z_{loc, \lambda}(P) \). Moreover, we define \( \tilde{Z}_n := \tilde{Z}_n T = \frac{dP_n}{dP} \).

As in [30], we only consider misspecifications of probability measures that are equivalent, which correspond to misspecifications of drifts. As pointed out in [30, Remark 2.5], the stability in the general nonequivalent case can only be studied in the distributional sense. Furthermore, the structure of the dual sets (the sets of consistent price systems) in the limit and that in the prelimit models differ in nonequivalent cases, which limits the applicability of the duality method.

(2) That \( \lim_{n \to \infty} U_n = U \) pointwisely implies the pointwise convergence of the sequence of their Legendre-Fenchel transforms, i.e., \( \lim_{n \to \infty} V_n = V \) pointwise. Indeed, \( \{U_n\}_{n \in \mathbb{N}} \) is a family of concave functions in a finite-dimensional space. Then, pointwise convergence is equivalent to epi-convergence on the interior of the domain of the limiting function, and the epi-convergence of \( U_n \) to \( U \) is equivalent to the convergence of the conjugate sequence \( V_n \) to \( V \) (for a more general analysis, see [42,43]).

(3) Due to convexity, the sequences \( \{U_n\}_{n \in \mathbb{N}}, \{V_n\}_{n \in \mathbb{N}} \) and their derivatives \( \{U'_n\}_{n \in \mathbb{N}}, \{V'_n\}_{n \in \mathbb{N}} \) converge uniformly on compact subsets of \((0, \infty)\) to their respective limits \( U, V, U' \) and \( V' \) (see [41, Theorem 10.8 and 25.7] for a general statement).

In the frictionless case, Larsen mentioned in [34, Sect. 2.6] that the pointwise convergence of \( \{U_n\}_{n \in \mathbb{N}} \) is not sufficient to prove the upper semi-continuity of the value function \( v \) and that more structure conditions on the converging sequence of \( \{U_n\}_{n \in \mathbb{N}} \) should be imposed. In the present paper, the following assumption is introduced, for which an analogue in the frictionless case can be found in [30, Sect. 2.3.2].

**Assumption 3.3** Define
\[
Z_T := \left\{ Z_T^0 : (Z^0, Z^1) \in Z_{loc, \lambda} \right\}.
\]
There exists \( Z_T^0 \in Z_T \), such that for all \( y > 0 \) the family \( \left\{ Z_n V^+_n \left( y \frac{Z^0}{Z_n} \right) \right\}_{n \in \mathbb{N}} \) is \( P \)-uniformly integrable.

**Remark 3.4** We refer the interested readers to [30, Sect. 2.2.3] for a more detailed discussion about the corresponding condition in the frictionless setting.

**Example 3.5** Consider a market driven by \( dS_t/S_t = dW^P_t \), \( S_0 = 1 \). For a fixed \( \lambda \), the pair \((1, S)\) is a consistent price system. Let \( \frac{dP_n}{dP} := \mathcal{E}(f_0 - \mu_n dW^P_T) = \tilde{Z}_n \) and \( U_n(x) = \log \left( \frac{n}{n+1} (x) \right) \). Suppose that \( \mu_n \to \mu \) is a sequence of positive numbers and we note \( \mu_n \leq \tilde{\mu} \). Without loss of generality, we only consider \( y = 1 \). Then
\[ 0 \leq \tilde{Z}_n V_n^\phi \left( \frac{Z_T^0}{Z_n} \right) = \tilde{Z}_n \left( - \log \left( \frac{1}{Z_n} \right) - \frac{n+1}{n} \right)^\phi \leq \tilde{Z}_n \left( - \log \left( \frac{1}{Z_n} \right) \right)^\phi \leq \tilde{Z}_n \mathbb{E} \left[ |W^P_T| \right]. \]

It follows from \( \mathbb{E}^P[|\tilde{Z}_n|^2] \leq 4 \exp(\tilde{\mu}^2 T) \) that Assumption 3.3 is satisfied in this example.

**Theorem 3.6** Under Assumptions 3.1 and 3.3, the following limiting relationship is given for value functions and optimal solutions:

\[
\lim_{n \to \infty} u(x_n; U_n, P_n) = u(x; U, P), \quad \lim_{n \to \infty} v(y_n; V_n, P_n) = v(y; V, P); \tag{3.2}
\]

\[
\lim_{n \to \infty} \frac{\partial}{\partial x} u(x_n; U_n, P_n) = \frac{\partial}{\partial x} u(x; U, P), \quad \lim_{n \to \infty} \frac{\partial}{\partial y} v(y_n; V_n, P_n) = \frac{\partial}{\partial y} v(y; V, P); \tag{3.3}
\]

\[
\lim_{n \to \infty} \phi_T^0(x_n; U_n, P_n) = \phi_T^0(x; U, P), \quad \lim_{n \to \infty} \tilde{h}(y_n; V_n, P_n) = \tilde{h}(y; V, P), \text{ a.s.} \tag{3.4}
\]

To prove this theorem, the dual problem \( v_n \) must be reformulated using the following proposition.

**Proposition 3.7** If \( h \in D(y, P_n) \), then \( h\tilde{Z}_n \in D(y, P) \). Conversely, if \( \tilde{h} \in D(y, P) \), then for each \( n \in \mathbb{N} \), \( \tilde{h} \in D(y, P_n) \).

**Proof** Without loss of generality, let \( h \in D(1, P_n) \). By definition of \( D(1, P_n) \), there is a nonnegative \( P_n \)-optional strong supermartingale \( (\phi^0, \phi^1) \) with \( \phi^0_0 = 1 \), \( \phi^0_T = h \), satisfying that \( \phi^0 T \in [(1-\lambda)S, S] \), and \( \phi^0 Y^0 + \phi^1 Y^1 \) is a nonnegative \( P_n \)-optional strong supermartingale for all \( \phi \in A(1) \). We adopt the definition of \( \tilde{Z}_n^\mu \) in (3.1). Clearly, the density process \( (\tilde{Z}_n^\mu)_{0 \leq t \leq T} \) is a \( P \)-martingale and \( \tilde{Z}_n = \tilde{Z}_n^\mu \).

It suffices to show the supermartingale property of the process \( \phi^0 T \tilde{Z}_n^\mu + \phi^1 Y^1 \tilde{Z}_n^\mu \) under \( P \) for all \( \phi \in A(1) \). For \( 0 \leq s < t \leq T \) and \( \phi = (\phi^0_t, \phi^1_t)_{0 \leq t \leq T} \in A(1) \), the Bayes’ formula gives

\[
(\phi^0_s T \tilde{Z}_s^\mu + \phi^1_s Y^1_s) \tilde{Z}_s^\mu \geq \mathbb{E}^P \left[ \left( \phi^0_T T^0 + \phi^1_T Y^1_T \right) \tilde{Z}_T^\mu \bigg| \mathcal{F}_s \right].
\]

The second assertion can be shown analogously. \( \Box \)

**Proof of Theorem 3.6** The proof, which is similar as that in [30] by setting the random endowment \( q = 0 \), is divided into four steps. To avoid redundancy, the proof is presented in detail in the following steps only with the needs for transaction costs. Readers are referred to [30, Sect. 3] for more details.

1. The first step is to show that the dual value function has a lower-semicontinuity property. By Proposition 3.7, the dual problem \( v(\cdot; V_n, P_n) \) corresponding to the
utility maximization problem parameterised by \((U_n, P_n)\) can be reformulated with the parameters \((U, P)\), as

\[
v(y; V_n, P_n) = \inf_{h \in \mathcal{D}(y, P_n)} \mathbb{E}^{P_n}[V_n(h)] = \mathbb{E}^{P_n}[V_n(\hat{h}(y, P_n))]
\]

\[
= \inf_{\tilde{h} \in \mathcal{D}(y, P)} \mathbb{E}^{P}[\tilde{Z}_n V_n(\frac{\tilde{h}}{\tilde{Z}_n})] = \mathbb{E}^{P}[\tilde{Z}_n V_n(\frac{\hat{h}_n(y, P)}{\tilde{Z}_n})], \tag{3.5}
\]

where \(\tilde{Z}_n\) converges to 1 in \(L^1(P)\) and thus in \(L^0(P)\); the dual optimizers \(\hat{h}(y; P_n) \in \mathcal{D}(y, P_n)\) and \(\hat{h}_n(y; P) \in \mathcal{D}(y, P)\) attain the infimum \(v(y; V_n, P_n)\). We replace (3.1) in [30] by the above definition (3.5). Then, by recalling the properties of functions \(\{V_n\}_{n \in \mathbb{N}}\) and \(V\) as well as explained in Remark 3.2 and Assumption 3.1, we could follow from line to line in [30, Sect. 3.2] to find the semi-continuity

\[
v(y; V, P) \leq \liminf_{n \to \infty} v(y; V_n, P_n), \quad \text{for } y > 0.
\]

(2) The second step is to show that the dual value function has an upper-semicontinuity property. Indeed, for any \(Z_T^0 \in \mathcal{Z}_T, y > 0\), define

\[
\tilde{\mathcal{D}}(y, Z_T^0) := \left\{ h \in \mathcal{D}(y, P) : \frac{Z_T^0}{h} \in L^\infty \right\},
\]

then we have \(\tilde{\mathcal{D}}(y, Z_T^0) \neq \emptyset\), because \(\frac{1}{k} y Z_T^0 + (1 - \frac{1}{k}) h \in \tilde{\mathcal{D}}(y, Z_T^0)\) for all \(h \in \mathcal{D}(y, P)\) and \(k \in \mathbb{N}\). The counterparts of Lemmas 3.3 and 3.4 from [30, Sect. 3.3] in the context of transaction costs can be formulated as follows. For the convenience of the reader, we give the proof with the notation in the present paper.

**Lemma 3.8** (compare [30, Lemma 3.3]) Fix \(y > 0\), and let \(Z_T^0 \in \mathcal{Z}_T\) be such that \(V^+(y Z_T^0) \in L^1(P)\). Then,

\[
v(y; V, P) = \inf_{\hat{h} \in \tilde{\mathcal{D}}(y, Z_T^0)} \mathbb{E}^{P}[V(\hat{h})].
\]

**Proof** Let \(\hat{h} \in \mathcal{D}(y, P)\) satisfy \(v(y; V, P) = \mathbb{E}[V(\hat{h})]\). Define \(h_k := \frac{1}{k} y Z_T^0 + (1 - \frac{1}{k}) \hat{h} \in \tilde{\mathcal{D}}(y, Z_T^0)\) for all \(k \in \mathbb{N}\). Moreover, \(V^+(y Z_T^0) \in L^1(P)\) and \(V(y Z_T^0) \geq -y Z_T^0 \in L^1(P)\) imply \(V(y Z_T^0) \in L^1(P)\). Then, it follows from the convexity of \(V\) that

\[
\mathbb{E}[V(h_k)] \leq \frac{1}{k} \mathbb{E}\left[V(y Z_T^0)\right] + \left(1 - \frac{1}{k}\right) \mathbb{E}[V(\hat{h})].
\]

Letting \(k \to \infty\), we have \(v(y; V, P) = \lim_{k \to \infty} \mathbb{E}[V(h_k)]\).
Lemma 3.9 (compare [30, Lemma 3.4]) Suppose that $f \in L^0_+$ makes the collection \( \{ \tilde{Z}_n V_n^+(f/\tilde{Z}_n) \}_{n \in \mathbb{N}} \) \( \mathbb{P} \)-uniformly integrable. Let $h \in L^1(\mathbb{P})$ be such that $h \geq f$ a.s. Then,

\[
\lim_{n \to \infty} \tilde{Z}_n V_n(h/\tilde{Z}_n) = V(h) \text{ in } L^1(\mathbb{P}).
\]

Proof Since $\tilde{Z}_n$ converges to 1 in $L^1(\mathbb{P})$, $\tilde{Z}_n$ converges to 1 in $L^0(\mathbb{P})$. Moreover, $V_n \to V$ pointwisely, so we have

\[
\lim_{n \to \infty} \tilde{Z}_n V_n(h/\tilde{Z}_n) = V(h) \text{ in } L^0(\mathbb{P}).
\]

We only need to show \( \{ \tilde{Z}_n V_n(h/\tilde{Z}_n) \}_{n \in \mathbb{N}} \) is \( \mathbb{P} \)-uniformly integrable. Each $V_n$ is decreasing, thus $\tilde{Z}_n V_n^+(h/\tilde{Z}_n) \leq \tilde{Z}_n V_n^+(f/\tilde{Z}_n)$, and \( \{ \tilde{Z}_n V_n^+(f/\tilde{Z}_n) \}_{n \in \mathbb{N}} \) is \( \mathbb{P} \)-uniformly integrable. Then, we obtain the \( \mathbb{P} \)-uniform integrability of \( \{ \tilde{Z}_n V_n^+(h/\tilde{Z}_n) \}_{n \in \mathbb{N}} \). For the negative part of the $V_n$, because $V_n(x) \geq -x$, \( \forall n \in \mathbb{N} \), we obtain $\tilde{Z}_n V_n^-(h/\tilde{Z}_n) \leq h \in L^1(\mathbb{P})$, which shows the uniform integrability of the \( \{ \tilde{Z}_n V_n^-(h/\tilde{Z}_n) \}_{n \in \mathbb{N}} \).

To show the upper semicontinuity of the function $v$, we pick up a $Z^0_T \in Z_T$ such that Assumption 3.3 is satisfied, i.e., \( \{ \tilde{Z}_n V_n^+(y Z^0_T) \}_{n \in \mathbb{N}} \) is \( \mathbb{P} \)-uniformly integrable for all $y > 0$. Since \( \{ \tilde{Z}_n V_n^+(y \frac{Z^0_T}{\tilde{Z}_n}) \}_{n \in \mathbb{N}} \) converges to $V^+(y Z^0_T)$ in $L^0(\mathbb{P})$, then it converges to $V^+(y Z^0_T)$ in $L^1(\mathbb{P})$. Moreover, $V^+(y Z^0_T) \in L^1(\mathbb{P})$ and by Lemma 3.8, we have

\[
v(y; V, \mathbb{P}) = \inf_{\tilde{h} \in \tilde{D}(y, Z^0_T)} \mathbb{E}^\mathbb{P} \left[ V(\tilde{h}) \right].
\]

For any $\tilde{h} \in \tilde{D}(y, Z^0_T) \subseteq D(y, \mathbb{P})$, we have $\frac{Z^0_T}{\tilde{h}} \in L^\infty$. By Lemma 3.9,

\[
\lim_{n \to \infty} \mathbb{E} \left[ \tilde{Z}_n V_n(h/\tilde{Z}_n) \right] = \mathbb{E} \left[ V(\tilde{h}) \right].
\]

Then, together with the expression of $v(y_n; V_n, \mathbb{P}_n)$ and $v(y; V, \mathbb{P})$, for any $\tilde{h} \in \tilde{D}(y, Z^0_T) \subseteq D(y, \mathbb{P})$, we have

\[
v(y; V, \mathbb{P}) \geq \lim_{n \to \infty} \sup v(y; V_n, \mathbb{P}_n), \quad \text{for } y > 0.
\]

(3) From the results in (1) and (2) and the fact that $v(\cdot; V_n, \mathbb{P}_n)$ and $v(\cdot; V, \mathbb{P})$ are convex functions, it can be concluded that

\[
\lim_{n \to \infty} v(y_n; V_n, \mathbb{P}_n) = v(y; V, \mathbb{P}).
\]

On the other hand, since $u(\cdot; U, \mathbb{P})$ (resp. $u(\cdot; U_n, \mathbb{P}_n)$) is the Legendre-Fenchel transform of $v(\cdot; V, \mathbb{P})$ (resp. $v(\cdot; V_n, \mathbb{P}_n)$) and similar to Remark 3.2 (2) and (3), we have
\[
\lim_{n \to \infty} u(x_n; U_n, P_n) = u(x; U, P).
\]

Thus, (3.2) is proven. For (3.3), it suffices to apply the epi-convergence properties of \(u(\cdot; U_n, P_n) \to u(\cdot; U, P)\) and of \(v(\cdot; V_n, P_n) \to v(\cdot; V, P)\) and to proceed the argument on the graphical convergence of subdifferentials, which is the same as [30, Sect. 3.4].

(4) Finally, to prove the \(L^0\)-convergence of the dual optimizer \(\lim_{n \to \infty} \hat{h}(y_n; V_n, P_n) = \hat{h}(y; V, P)\) in (3.4), first define an auxiliary sequence by 

\[
\begin{align*}
fn &:= n^{-1} y Z_0^T + (1 - n^{-1}) \hat{h}(y; V, P) \\
\hat{h}(y_n; V_n, P_n) &\in \tilde{D}(y, Z_0^T),
\end{align*}
\]

for some \(Z_0^T \in Z_T\), such that \(V^+(Z_0^T) \in L^1(P)\). Obviously, \(f_n \to \hat{h}(y; V, P)\) in \(L^0\). Then, we can prove by following the procedures in [30, Sect. 3.5.2] the existence of a subsequence indexed by \(\{n_k\}_{k \in \mathbb{N}}\), such that

\[
\lim_{k \to \infty} P_{n_k} \left\{ \hat{h}(y_{n_k}; V_{n_k}, P_{n_k}) \in \left[ \frac{1}{k}, k \right], \quad \frac{1}{k} \leq f_{n_k} \frac{n_{n_k}}{Z_T} \leq \frac{1}{k} \right\} = 0,
\]

where a more elaborate version of the method used in the proof of [20, Lemma A1.1] plays a key role. The other part of (3.4) can be obtained in view of (2.4), (3.3) and Remark 3.2.

\(\square\)

4 Dynamic Stability

In the previous section, only the stability of the terminal values of the wealth processes and of the optional supermartingale deflators attaining the dual optimality are considered. The present section focuses on the dynamics of the whole process inducing the optimizer. First, the properties of optimal dual processes are investigated and so-called dynamic stability is discussed.

When solving a utility maximization problem in the frictional market, we often wonder whether this market can be replaced by a frictionless market that yields the same optimal strategy and utility. Such a frictionless market is called a shadow market for the utility maximization problem.

**Definition 4.1** A semimartingale \(\tilde{S} = (\tilde{S}_t)_{0 \leq t \leq T}\) is called a shadow price process for the optimization problem (2.1) if \(\tilde{S} \in [(1 - \lambda)S, S]\) and the solution \(\tilde{\varphi} = (\tilde{\varphi}^0, \tilde{\varphi}^1)\) to the corresponding frictionless utility maximization problem

\[
\mathbb{E}[U(x + \varphi^1 \cdot \tilde{S}_T)] \to \max!, \quad (\varphi^0, \varphi^1) \in A(x; \tilde{S})
\]

exists and coincides with the solution \(\hat{\varphi} = (\hat{\varphi}^0, \hat{\varphi}^1)\) to (2.1) with transaction costs, where \(A(x; \tilde{S})\) denotes the set of all admissible trading strategies for \(\tilde{S}\) without transaction costs, as in [31].
In the setting of Theorem 2.4, if an ODP for the dual problem of utility maximization satisfies appropriate conditions, then a shadow price process can be constructed by this ODP.

**Proposition 4.2** [14, Proposition 3.7, 3.8] Assume that the dual optimizer \( \hat{h}(y) \) equals \( \hat{Y}_T(y) \), where \( u'(x) = y \), and \( \hat{Y}(y) \in B(y; \mathbf{P}) := (\hat{Y}^0(y), \hat{Y}^1(y)) \) is a pair of \( \mathbf{P} \)-local martingales. Then, the strictly positive semimartingale \( \hat{S} := \frac{\hat{Y}^1(y)}{\hat{Y}^0(y)} \) is a shadow price process for the optimization problem (2.1).

Conversely, if a shadow price \( \hat{S} \) exists, it can be represented by \( \hat{S} = \frac{\hat{Y}^1(y)}{\hat{Y}^0(y)} \) with an ODP \( (\hat{Y}^0(y), \hat{Y}^1(y)) \in B(y, \mathbf{P}) \) to the dual problem (2.2).

In the second part of this section, a shadow price process for the limiting problem parameterised by \( (x; \mathbf{U}, \mathbf{P}) \) is constructed using the sequence of shadow prices corresponding to the problems parameterised by \( \{(x_n; \mathbf{U}_n, \mathbf{P}_n)\}_{n \in \mathbb{N}} \). The following assumption is necessary for the existence of shadow price processes (see [17]).

**Assumption 4.3** Additionally, assume that the stock price is driven by a continuous process.

### 4.1 Properties of Optional Strong Supermartingale Deflators

In this subsection, the properties of optional strong supermartingale deflators, which are necessary for the argument on the dynamic stability, are discussed. For the simplicity of notation, only \( Y := (Y^0, Y^1) \in B(1) \) is considered. Since the processes \( Y^0 \) and \( Y^1 \) are optional strong supermartingales, according to [38], the Doob–Meyer–Mertens decomposition, which is an analogue of the Doob–Meyer type decomposition for càdlàg supermartingales, is \( Y^i = M^i - A^i, i = 0, 1 \), where \( M^i \) is a càdlàg local martingale, and \( A^i \) is a nondecreasing càdlàg predictable process. Then, the following proposition shows the relation between the nondecreasing parts \( A^0 \) and \( A^1 \), symbolically given as \( (1 - \lambda)S_t dA^0_t \leq dA^1_t \leq S_t dA^0_t \).

**Proposition 4.4** Let Assumption 4.3 hold and \( Y := (Y^0, Y^1) \in B(1) \). The processes \( Y^0 \) and \( Y^1 \) admit the unique Doob–Meyer–Mertens decomposition. Let \( \varepsilon + \lambda \leq 1 \), and let \( \sigma \) be a stopping time taking values in \([0, T]\). Define

\[
\tau_{\varepsilon} := \inf \left\{ t \geq \sigma \left| \frac{S_t}{S_\sigma} = (1 + \varepsilon) \text{ or } (1 - \varepsilon) \right. \right\} \wedge T.
\]

Then, for all stopping times \( \tau \) satisfying \( \sigma \leq \tau \leq \tau_{\varepsilon} \),

\[
(1 - \varepsilon)(1 - \lambda)S_\sigma \mathbb{E}\left[ A^0_\tau - A^0_\sigma \mid \mathcal{F}_\sigma \right] \leq \mathbb{E}\left[ A^1_\tau - A^1_\sigma \mid \mathcal{F}_\sigma \right] \leq (1 + \varepsilon)S_\sigma \mathbb{E}\left[ A^0_\tau - A^0_\sigma \mid \mathcal{F}_\sigma \right].
\]

**Remark 4.5** A similar lemma can be found in [17, Lemma 3.5], which gives the property (4.1) for a particular supermartingale deflator constructed as the Fatou limiting process of a sequence of local consistent price systems. The simple proof below of
the above proposition will not depend on the construction of $Y$, but only on the supermartingale property of the process $\varphi^0 Y^0 + \varphi^1 Y^1$ for all $\varphi \in A(1)$.

**Proof** Consider the following trading strategy $\varphi = (\varphi_t^0, \varphi_t^1)_{0 \leq t \leq T} \in A(1)$ defined by

$$(\varphi_t^0, \varphi_t^1) := \begin{cases} (1 - (1 - \lambda)(1 - \varepsilon), 0), & 0 \leq t < \sigma; \\ (-1 - \lambda)(1 - \varepsilon), & \sigma \leq t < \tau_\varepsilon; \\ V_{\tau_\varepsilon}^{liq}(\varphi), & \tau_\varepsilon \leq t \leq T. \end{cases}$$

This trading strategy is $\lambda$-self-financing and admissible. Indeed, only the worst case needs to be considered, i.e., the stock price attains $(1 - \varepsilon)S_{\sigma}$. Then, for $\sigma < t \leq T$, we have the liquidation value

$$V_{t}^{liq}(\varphi) \geq - (1 - \lambda)(1 - \varepsilon) + (1 - \lambda)\frac{1}{S_{\sigma}}(1 - \varepsilon)S_{\sigma} = 0.$$ 

By definition of the optional strong supermartingale deflator, for each $\tau \in [\sigma, \tau_\varepsilon]$,

$$-(1 - \lambda)(1 - \varepsilon)(M^0_{\sigma} - A^0_{\sigma}) + \frac{1}{S_{\sigma}}(M^1_{\sigma} - A^1_{\sigma}) \geq \mathbb{E}\left[ -(1 - \lambda)(1 - \varepsilon)\left(M^0_{\tau} - A^0_{\tau}\right) + \frac{1}{S_{\sigma}}\left(M^1_{\tau} - A^1_{\tau}\right) \, | \, \mathcal{F}_{\sigma} \right].$$

We assume without loss of generality that $M^0$ and $M^1$ are true martingales; otherwise, a stopping technique applies. Therefore, we obtain

$$\mathbb{E}\left[ A^1_{\tau} - A^1_{\sigma} \, | \, \mathcal{F}_{\sigma} \right] \geq (1 - \varepsilon)(1 - \lambda)S_{\sigma}\mathbb{E}\left[ A^0_{\tau} - A^0_{\sigma} \, | \, \mathcal{F}_{\sigma} \right].$$

For the other inequality, define another trading strategy $\varphi = (\varphi_t^0, \varphi_t^1)_{0 \leq t \leq T} \in A(1)$ by

$$(\varphi_t^0, \varphi_t^1) := \begin{cases} (\lambda + \varepsilon, 0), & 0 \leq t < \sigma; \\ ((1 - \lambda) + \lambda + \varepsilon, -\frac{1}{S_{\sigma}}), & \sigma \leq t < \tau_\varepsilon; \\ V_{\tau_\varepsilon}^{liq}(\varphi), & \tau_\varepsilon \leq t \leq T. \end{cases}$$

Using the same argument, for each $\sigma \leq \tau \leq \tau_\varepsilon$, we have

$$(1 + \varepsilon)S_{\sigma}\mathbb{E}\left[ A^0_{\tau} - A^0_{\sigma} \, | \, \mathcal{F}_{\sigma} \right] \geq \mathbb{E}\left[ A^1_{\tau} - A^1_{\sigma} \, | \, \mathcal{F}_{\sigma} \right],$$

which ends the proof. 

The following corollary is straightforward.

**Corollary 4.6** For each optional strong supermartingale deflator $(Y^0, Y^1) \in B(1)$, if the first component $Y^0$ is a local martingale, then the second component $Y^1$ is also a local martingale.
Remark 4.7. We may directly show the assertion of Corollary 4.6 without using Proposition 4.4. By defining an admissible trading strategy $(\varphi_0^t, \varphi_1^t)_{0 < t \leq T} = (1 + \frac{1}{n}, -\frac{1}{n})$ and $(\hat{\varphi}_0^t, \hat{\varphi}_1^t)_{0 < t \leq T} = (1 - \frac{1}{n}, \frac{1}{n})$, $n \in \mathbb{N}$, the proof can be completed.

Next, the local martingale property of the ODP in the framework of [17] is studied. Actually, Czichowsky et al. found in [17] a sufficient condition that guarantees the local martingale property of OPPs when $S$ is continuous, that is, the liquidation value processes of OPPs are a.s. strictly positive, i.e., $\inf_{0 \leq t \leq T} \hat{V}^{liq}_t(\hat{\varphi}) > 0$, a.s. This result from [17] is summarized in the following proposition.

Proposition 4.8 [17, Proposition 3.3] Fix a level $0 < \lambda < 1$ of transaction costs and assume that Assumption 4.3 holds. In addition, suppose that the liquidation value process of the optimal primal process $\hat{\varphi} = (\hat{\varphi}_0^t, \hat{\varphi}_1^t)_{0 \leq t \leq T} \in \mathcal{A}(x)$ is strictly positive, given as

$$\inf_{0 \leq t \leq T} V^{liq}_t(\hat{\varphi}) := \inf_{0 \leq t \leq T} \left\{ \hat{\varphi}_0^t + (1 - \lambda)(\hat{\varphi}_1^t)^+ S_t - (\hat{\varphi}_1^t)^- S_t \right\} > 0, \text{ a.s.} \quad (4.2)$$

Let $y = u'(x)$; then, there exists a local $\lambda$-consistent price system $\hat{Z} \in \mathcal{Z}^{loc,\lambda}$, such that $y \hat{Z}^0_T = \hat{h}$.

Remark 4.9. The continuity of the price process $S$ is essential to the above proposition. Otherwise, counterexamples can be found, e.g., in [16]. Very recently, it has been proved in [18] that (4.2) holds if the condition of “two way crossing” (TWC) is satisfied (see Bender [5] for a definition). This condition implies that $S$ satisfies $(CPS^\mu)$ locally for all $0 < \mu < 1$. (See the proof of Theorem 2.3 in [18].)

The proof of the proposition above in [17] is based on the strict positivity of the liquidation value of the OPP and [17, Lemma 3.5] (see also Lemma 4.4). In the present paper, a proposition slightly stronger than [17, Proposition 3.3] is given; however, its proof is much reduced. The result of Proposition 4.8 constructs such an ODP that is a pair of local martingales for the purpose of constructing a shadow price process, whereas the proposition below states that all ODPs are local martingales in the same framework. This statement is meaningful in a frictional context, since unlike the frictionless case, the ODP is no longer unique (a counterexample will be discussed in Example 4.12).

Proposition 4.10 Fix a level $0 < \lambda < 1$ of transaction costs and assume that Assumption 4.3 and (4.2) hold. Let $y = u'(x)$ and an optional supermartingale deflator $\hat{Y} \in \mathcal{B}(y)$ be associated with the dual optimizer as $\hat{Y}^0_T = \hat{h}$ a.s. Then, $\hat{Y}$ is a local martingale. In other words, each $\lambda$-optional supermartingale deflator associated with the dual optimizer is a local $\lambda$-consistent price system.

Proof. The stock price process $S$ is continuous; thus $S$ is locally bounded. We may assume without loss of generality that the liquidation value of the OPP $(\hat{V}^{liq}_t)_{0 \leq t \leq T}$ is predictable (otherwise, consider the càglàd version of the trading strategy). Then, from $\inf_{0 \leq t \leq T} \hat{V}^{liq}_t > 0$, a sequence of a.s. increasing and diverging stopping times.
\( \{q_m\}_{m=1}^{\infty} \) can be found, such that, for each \( m \in \mathbb{N} \) and \( t \in [0, T] \), we have \( \hat{V}^{hq}_{t \wedge q_m} \geq \frac{1}{m} \), and

\[
\frac{1}{m} \leq (1 - \lambda) S_{t \wedge q_m} \leq S_{t \wedge q_m}.
\]

Fix an optimal trading strategy \((\hat{\varphi}^0(x), \hat{\varphi}^1(x)) \in A(x)\). It is easy to verify that \((\hat{\varphi}^0_{t \wedge q_m} - \frac{1}{m}, \hat{\varphi}^1_{t \wedge q_m}) \in A(x)\). Then,

\[
\left( \hat{\varphi}^0_{t \wedge q_m} - \frac{1}{m} \right) \hat{Y}^0 + \hat{\varphi}^1_{t \wedge q_m} \hat{Y}^1
\]

is an optional strong supermartingale. Since \( \hat{\varphi}^0 \hat{Y}^0 + \hat{\varphi}^1 \hat{Y}^1 \) is a martingale, \( \hat{Y}^0 \) is an optional strong submartingale up to \( q_m \). As \( \hat{Y} \in \mathcal{B}(y) \), \( \hat{Y}_{t \wedge q_m} \) must be a martingale. Thus, \( \hat{Y}^0 \) is a local martingale, then by Corollary 4.6, \( \hat{Y}^1 \) is again a local martingale. \( \square \)

**Remark 4.11** Similar to the frictionless case, an ODP always includes a pair of local martingales when \( S \) is continuous. Whenever the liquidation value is strictly positive, the first coordinate of the ODP does not “lose” its mass. This can be seen if a similar argument as that of [23, Appendix] is used to prove the above proposition. It suffices to replace the wealth process by the liquidation value process.

Unlike the frictionless case, we do NOT have the uniqueness for the ODPs, even for the first coordinate. The non-uniqueness of the second coordinate is an easy observation because it already occurs in the setting of finite \( \Omega \) (cf. [44, Example 2.4]). For the non-uniqueness of the first coordinate of ODPs, we gratefully acknowledge Walter Schachermayer for the following counterexample (see also [44]).

**Example 4.12** (Walter Schachermayer) Let \( W = (W_t)_{t \geq 0} \) be an \((\mathcal{F}^W_t)_{t \geq 0}\)–Brownian motion, where \((\mathcal{F}^W_t)_{t \geq 0}\) is the natural filtration generated by \( W \), which satisfies the usual conditions. Assume that the investor’s utility is characterised by the logarithmic function \( U(x) = \log(x) \). Then, construct two frictionless markets driven by two different processes \( \hat{S} \) and \( \tilde{S} \), and eventually find a market \( S \) with transaction costs, such that \( \hat{S} \) and \( \tilde{S} \) are both shadow price processes for this logarithmic utility maximization problem when trading with \( S \). Furthermore, we prove that \( \hat{Z} = (\hat{S}^{-1}, 1) \) and \( \tilde{Z} = (\tilde{S}^{-1}, 1) \) are two ODPs for such a problem, but \( \hat{Z} \neq \tilde{Z} \). The construction is divided into three steps:

**Step 1** Define

\[
N_t = \exp \left( W_t + \frac{t}{2} \right), \quad t \geq 0.
\]

Fix the level of transaction costs \( 0 < \lambda < \frac{1}{2} \) and define

\[
\tau^\lambda = \inf \left\{ t : N_t = 2(1 - \lambda) \right\}.
\]

Let \( \hat{S} \) be a time-changed restriction of \( N \) on the stochastic interval \([0, \tau^\lambda]\), i.e.,

\[
\hat{S}_t = N_{\tan \left( \frac{\pi}{2} t \right) \wedge \tau^\lambda}, \quad 0 \leq t \leq 1.
\]
Consider a frictionless market driven by the process \( \hat{S} \), which is adapted to the time-changed filtration \( \mathcal{F} \) defined by \( \mathcal{F}_t := \mathcal{F}_t^{\tan(\frac{\pi}{2}) \wedge t^+} \). Then, \( \hat{Z}^0 := (\hat{S})^{-1} \) defines a local martingale deflator for \( \hat{S} \). Due to Merton’s rule, the logarithmic utility maximization problem is solved by a strategy consisting of buying one stock at time \( t = 0 \) at price \( \hat{S}_0 = 1 \) and selling it at time \( t = 1 \) at price \( \hat{S}_1 = 2(1-\lambda) \), i.e., a buy-hold-sell strategy.

**Step 2** Next, define a perturbation of the process \( \hat{S} \), denoted by \( \hat{S} \). To do so, first define a perturbation of \( \hat{Z}^0 = (\hat{S})^{-1} \), then decompose this local martingale into \( \hat{Z}^0 = \hat{M} + \hat{P} \), where

\[
\hat{M}_t := \mathbb{E}\left[ \hat{Z}_t^0 \mid \mathcal{F}_t \right] = \frac{1}{2(1-\lambda)}, \quad \hat{P}_t := \hat{Z}_t - \hat{M}_t.
\]

Note that \( \hat{P}_0 = \frac{1-2\lambda}{2(1-\lambda)} \) and \( \hat{P}_1 = 0 \). Therefore, \( \hat{P} \) is a potential. Let \( \sigma := \inf \{ t : \hat{P}_t = 1 \} \), then the stopped local martingale \( \hat{P}^\sigma \) is bounded and thus is a martingale. Moreover, \( \mathbb{P}[\sigma < \infty] = \frac{1-2\lambda}{2(1-\lambda)} \). For \( \delta > 0 \), choose an arbitrary \( \mathcal{F}^\sigma \)-measurable function \( f \) taking values in \( [1-\delta, 1+\delta] \), such that

\[
\mathbb{E}[f1_{\sigma < \infty}] = \frac{1-2\lambda}{2(1-\lambda)}
\]

and \( f \) is not identically equal to 1 on \( \sigma < \infty \). Define the potential \( \hat{P} \) by

\[
\hat{P}_t = \begin{cases} \mathbb{E}[f1_{\sigma < \infty} \mid \mathcal{F}_t \wedge \sigma], & 0 \leq t \leq \sigma, \\ f \hat{P}_t, & \sigma \leq t \leq 1, \end{cases}
\]

which is again a local martingale starting at \( \hat{P}_0 = \frac{1-2\lambda}{2(1-\lambda)} \) and ending at \( \hat{P}_1 = 0 \). Note that \( 0 \leq \hat{P}_t \in [(1-\delta)\hat{P}_t, (1+\delta)\hat{P}_t], 0 \leq t < 1 \), a.s. Define \( \hat{Z}^0 := \hat{M} + \hat{P} \) and \( \hat{S} := (\hat{Z}^0)^{-1} \). Then, the ratio \( \frac{\hat{S}}{\hat{S}_t} \in [(1+\delta)^{-1}, (1-\delta)^{-1}] \). It is clear that the frictionless market driven by \( \hat{S} \) has a local martingale deflator \( \hat{Z} \). For this market, the log-optimal strategy is the buy-hold-sell strategy mentioned above.

**Step 3** Define

\[
m_t = \max(\hat{S}_t, \hat{S}_t), \quad M_t = (1-\lambda)^{-1} \min(\hat{S}_t, \hat{S}_t).
\]

Assume that \( (1-\lambda)(1+\delta) < (1-\delta) \) to have \( m_t < M_t, 0 \leq t \leq 1 \), a.s. Define \( S \) as

\[
S_t = (1-t)m_t + tM_t, \quad 0 \leq t \leq 1,
\]

which is continuous and adapted, starting at \( S_0 = 1 \) and ending at \( S_1 = 2 \). Both \( \hat{S} \) and \( \hat{S} \) remain in the bid-ask spread \( [(1-\lambda)S, S] \). Therefore, it is clear that both \( (\hat{Z}^0, 1) \) and \( (\hat{Z}^0, 1) \) are elements in \( \mathcal{Z}^{loc.,\lambda}(S) \). Since trading for \( S \) with transaction costs yields no more than trading for \( \hat{S} \) or \( \hat{S} \) frictionlessly, the trading strategy to buy one stock at \( t = 0 \) and to sell it only when \( t = 1 \) is optimal for the frictional logarithmic utility
maximization problem with $S$. By (2.4), we can verify that both $(\hat{Z}^0, 1)$ and $(\hat{Z}^0, 1)$ induce the dual optimizer $\hat{h} = \frac{1}{2(1-\lambda)}$. However, $\vec{Z}^0 \neq \hat{Z}^0$.

### 4.2 Convergence of Optimal Dual Processes

Thanks to the previous subsection, we are at the point to discuss the dynamic stability, which means the stability of the ODPs.

**Proposition 4.13** Let Assumption 3.3 and 4.3 hold. Moreover, the condition (4.2) holds for each problem $u(x_n, U_n, P_n)$, $n \in \mathbb{N}$ as well as for the problem $u(x, U, P)$. Let $U_n \rightarrow U$ pointwisely and $x_n \rightarrow x > 0$. Suppose that for each $n \in \mathbb{N}$, $P_n \sim P$ and $\lim_{n \rightarrow \infty} P_n = P$ in total variation. For a sequence of ODPs $(\hat{Y}^0(y_n; V_n, P_n), \hat{Y}^1(y_n; V_n, P_n)) \in y_n Z_{loc, \lambda}^1(P_n)$, define $(\hat{Y}^0, \hat{Y}^1) := (\hat{Y}^0(y_n; V_n, P_n)\hat{Z}^n, \hat{Y}^1(y_n; V_n, P_n)\hat{Z}^n)$. Then, there exists a pair of $P$-local martingales $\hat{Y} := (\hat{Y}^0, \hat{Y}^1)$ and a subsequence of convex combinations of $(\hat{Y}^0)_n \subseteq B(y_n, P)$, denoted still by $(\hat{Y}^0)_n$, such that for every $[0, T]$-valued stopping time $\tau$, $(\hat{Y}^0_\tau, \hat{Y}^1_\tau) \xrightarrow{P} (\hat{Y}^0, \hat{Y}^1)_\tau$.

Moreover, the pair $\hat{Y} \in y Z_{loc, \lambda}^1(P)$ is an ODP corresponding to the limiting dual problem $v(y; V, P)$ with $\hat{Y}^0_T(P) = \hat{h}(P)$.

**Proof** We note that an ODP $(\hat{Y}^0(y_n; V_n, P_n), \hat{Y}^1(y_n; V_n, P_n)) \in y_n Z_{loc, \lambda}^1(P_n)$ is a pair of $P_n$-local martingales. By change of the measure, we obtain

$$\hat{Y}^n = (\hat{Y}^0, \hat{Y}^1) := (\hat{Y}^0(y_n; V_n, P_n)\hat{Z}^n, \hat{Y}^1(y_n; V_n, P_n)\hat{Z}^n) \in y_n Z_{loc, \lambda}^1(P)$$

$$\subseteq B(y_n, P), \quad (4.3)$$

which is a pair of $P$-local martingales. Because of $\hat{Z}^n_T \rightarrow 1$ in $L^1(P)$ and Theorem 3.6,

$$\hat{Y}^0_T(y_n; V_n, P_n)\hat{Z}^n \xrightarrow{P} \hat{h}(y, V, P) = \hat{Y}^0_T(y; V, P),$$

where the right-hand side is the dual optimizer for $v(y, V, P)$. Thus, again by [15, Theorem 2.7], there exists a subsequence of convex combinations of $(\hat{Y}^0, \hat{Y}^1)_n$, denoted still by $(\hat{Y}^0, \hat{Y}^1)_n$, such that for every stopping time $\tau$ taking values in $[0, T]$,

$$(\hat{Y}^0_\tau, \hat{Y}^1_\tau) \xrightarrow{P} (\hat{Y}^0, \hat{Y}^1)_\tau, \quad (4.4)$$

which implies that $(\hat{Y}^0, \hat{Y}^1) \in y Z_{loc, \lambda}^1(P)$ is an ODP for $v(y; V, P)$. From Proposition 4.10, we know $\hat{Y}$ is a pair of $P$-local martingales. \hfill $\Box$

**Remark 4.14** If only the conditions in Theorem 2.4 are assumed for each problem $u(x_n; U_n)$ and the limiting problem $u(x; U)$, then the results in the above proposition hold without the local martingale property.
Remark 4.15 Recall that we assume the usual conditions for the filtration throughout this paper, then every ODP discussed in this subsection has a càdlàg modification. This property is crucial for the construction of a shadow market.

Remark 4.16 In [11], a convergence result for the trading strategy is obtained. However, in this setting with transaction costs, it is always difficult to have a similar result, since for each $n$, we are working with a two-dimensional process of the trading strategy $(\varphi_n^0, \varphi_n^1)$ and there is no uniqueness for the optimal trading strategy (cf. Example 4.12). In order to get a reasonable convergence result of optimal trading strategies, we need some appropriate condition to guarantee the convex compactness of the set of optimal trading strategies, see e.g. [10, Propositions 3.3, 3.4]. It is left for future research.

4.3 Construction of Shadow Price Processes

This subsection is devoted to the construction of shadow price processes for the limiting utility maximization problem based on a sequence of shadow price processes corresponding to the problem with perturbations. We keep all the notation and assumption in the previous subsection.

Proposition 4.17 For each $n$, there exists a shadow price given by

$$\hat{S}_n(x_n; U_n, P_n) = \frac{\hat{Y}_1^1(y_n; V_n, P_n)}{\hat{Y}_0^0(y_n; V_n, P_n)},$$

where $(\hat{Y}_0^0(y_n; V_n, P_n), \hat{Y}_1^1(y_n; V_n, P_n))$ are ODPs for the problem $(x_n; U_n, P_n)$. Moreover, define $(\tilde{Y}_n^0, \tilde{Y}_n^1) := (\hat{Y}_0^0(y_n; V_n, P_n)\tilde{Z}_n, \hat{Y}_1^1(y_n; V_n, P_n)\tilde{Z}_n)$, then there exists a subsequence of convex combinations of $\{\tilde{Y}_n\}_{n=0}^\infty$, denoted again by $\{\tilde{Y}_n\}_{n=0}^\infty$, and a limiting process $\tilde{Y} \in B(y, P)$, such that $\tilde{Y}_n$ converges to $\tilde{Y}$ in the sense of (4.4). Moreover, $\hat{S} := \frac{\hat{Y}_1^1}{\hat{Y}_0^0}$ defines a shadow price process for the limiting problem $u(x; U, P)$.

Proof This proposition can be proved by applying Propositions 4.13 and 4.2. □

5 Conclusion

In a market with proportional transaction costs, where the price process admits a consistent price system, the problem of maximizing expected utility from terminal wealth can be solved using the convex duality method. When the price process is continuous and a No-Arbitrage-type condition applies, the optimal dual process is a local martingale, which is the foundation for constructing a shadow price process. The proof of this result is much simplified in the present paper. In the context of transaction costs, the optimal dual process is no longer unique.

The convergence of initial wealth, the preference of the investor and the market probability model induce the convergence of the optimal expected utility and the
value of its dual problem. This means that the optimal solutions of both primal and dual problems are stable. In addition, the process associated with the dual optimizer is stable under perturbation, which suggests a way to construct the shadow price process using the processes corresponding to the perturbed problems.

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