Information theoretical formulation of anyonic entanglement

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Anyons are modeled by topologically protected Hilbert spaces which obey complex superselection rules restricting possible operations. As a result, entanglement associated with anyonic degrees of freedom follows distinguished principles. In particular, a decomposition of local subsystems into tensor factors is impossible, whereas the tensor product structure is a foundation of conventional entanglement theory. We formulate bipartite entanglement theory for pure anyonic states and analyze its properties as a non-local resource for quantum information processing. We introduce a new entanglement measure, asymptotic entanglement entropy (AEE), and show that it characterizes distillable entanglement and entanglement cost similarly to entanglement entropy in conventional systems. AEE depends not only on the Schmidt coefficients but also on the quantum dimensions of the anyons shared by the local subsystems. Moreover, it turns out that the AEE coincides with the entanglement gain by anyonic excitations in certain topologically ordered phases.

Introduction.— The discovery of topologically ordered phases (TOP) as in the fractional quantum hall effect [1–3] has revealed a new kind of quantum phases not described by the conventional symmetry breaking picture. In TOP, energy eigenstates have distinct properties, namely, the ground states have topological degeneracy and the statistics of quasiparticle excitations are not necessary fermionic or bosonic as usual particles, but anyonic. Many developments have recently been made in understanding quantum properties of TOP (see e.g., [4] and references therein). In particular, the analysis by means of entanglement - a standard procedure in quantum information theory [5] - revealed new properties distinguishing them from conventional phases. In fact, TOP can be characterized through specific patterns of long-range entanglement of their ground state, whereas no local order parameter exists [6, 7]. Subsequently, also the entanglement of excited states of TOP have been investigated [8–10].

Another discovery that anyonic excitations can be utilized for fault-tolerant quantum computation by employing their topologically protected Hilbert space [11–13] have brought interests on TOP as a resource for quantum information processing. The topologically protected Hilbert space or also called fusion space in the following is generated by braiding of the localized quasiparticle excitations and it can be described by an abstract anyon model. A key role in quantum information plays entanglement due to its power as a resource for tasks as, e.g., quantum teleportation [14]. A quantitative analysis of entanglement of anyonic systems described by the topologically protected Hilbert space is necessary for thoroughly understanding the potential of TOP as a resource in quantum information processing.

However, standard entanglement theory cannot be directly applied to anyonic systems since the topologically protected Hilbert space describing anyons cannot be decomposed into tensor products of local subsystems. This property can be understood as the existence of superselection rules (SSR) restricting the physically possible operations. Entanglement properties under SSR induced by group symmetries or SSR of fermions have been recently studied in [8, 15, 17], but their techniques cannot be applied to anyons.

In this paper, we formulate an entanglement theory for anyonic systems in pure bipartite settings from an information theoretical viewpoint, and find a qualitatively different behavior compared to ordinary systems with tensor product structure. In standard entanglement theory, the uniqueness theorem [15] states that many entanglement measure operational in an asymptotic situation of infinite identical copies coincide to the entanglement entropy (EE) due to its additivity. Distillable entanglement and entanglement cost are examples of such operational measures characterizing the asymptotic conversion rates from a given state to the maximally entangled state (MESs) and vice versa under local operations and classical communications (LOCC) [7], respectively.

However, it turns out that for anyonic systems the corresponding EE is not additive but super-additive. We then show that the asymptotic rate of the EE which we call the asymptotic entanglement entropy (AEE) is given by a contribution from EE plus a non-negative correction term depending on the quantum dimension that describes the asymptotic growth rate of the Hilbert space of anyons. Moreover, we show that the AEE takes over the role of the EE in anyonic systems by establishing its operational meaning as distillable entanglement and entanglement cost.

Interestingly, the anyonic correction term is similar to the EE increase due to a single anyonic excitation in certain TOP [3, 10], suggesting that in the asymptotic situation all the entanglement that is not contained in the ground state can be distilled by only braiding operations. We finally remark that a similar expression to the AEE also appeared in a different context related to topological quantum field theory [20].

Anyon Models.— A general anyon model is determined by the possible charges (i.e., anyon types), the fusion rules and the braiding statistics. In the following we as-
sume that the possible charges are labeled by a finite set \( \mathcal{L} = \{a, b, c, \ldots\} \) where 1 denotes the unique vacuum. The number of ways for \( a \) and \( b \) fusing to \( c \) is given by \( N_{ab}^c \in \mathbb{N} \) and we simply write \( a \times b = \sum_{c} N_{ab}^c c \) to indicate the possible fusion channels of \( a \) and \( b \). For any \( a \in \mathcal{L} \), the vacuum satisfies \( 1 \times a = a \) and there exists a unique anti-label \( \bar{a} \) with \( a \times \bar{a} \) can be 1. A charge \( a \) is said to be abelian if the fusion rules for any \( b \) is unique \( \sum_{c} N_{ab}^c = 1 \), and otherwise non-abelian.

The Hilbert spaces associated to an anyon model are called fusion spaces and spanned by the possible fusion ways. In particular, the fusion space of two charges \( a \) and \( b \) fusing to \( c \) is given by

\[
V_{ab}^c = \text{span}\{\{ab : c, \mu\} | \mu = 1, 2, \ldots, N_{ab}^c\}. \tag{1}
\]

Note that the order of the anyons and their arrangement on the two dimensional manifold is crucial because of the braiding statistics (c.f. [2]). In this paper, we use the convention that any anyon chain \( \{a_1, a_2, \ldots, a_n\} \) with total charge \( c \) is assumed to be consecutively aligned from left to right on a two dimensional plane. The corresponding fusion space of such a chain can be constructed from two-anyon fusion spaces \( V \) as

\[
V_{a_1 \ldots a_n}^c = \bigoplus_{b_1 \ldots b_{n-2}} V_{a_1 a_2}^{b_1} \otimes V_{a_2 a_3}^{b_2} \otimes \cdots \otimes V_{a_{n-2} a_n}^{b_{n-2}}. \tag{2}
\]

It is convenient to illustrate the decomposition of the fusion space by tree diagrams (FIG. 1).

The scaling behavior of the dimension of the fusion space \( V_{a_n}^c \) of \( n \) \( a \)-anyons \( \{a, a, \ldots, a\} \) is determined by the quantum dimension \( d_a \). For example, the quantum dimension of an abelian anyon \( a \) is \( d_a = 1 \), and \( \dim V_{a_n}^c = 1 \) if \( a^n \) can fuse to \( c \) and 0 else. If \( a \) is non-abelian and primitive \([21]\), \( \dim V_{a \ldots a}^c \approx d^a_a D^2 \) for large \( n \) and all \( c \in \mathcal{L} \) for which a fusion way exists \([2]\). Here, \( D \) denotes the total quantum dimension given by \( D = \sqrt{\sum_a d^a_a} \). One of the most studied primitive anyon due to its simplicity is the Fibonacci anyon which allows for universal quantum computation \([12]\). We note further that the so-called Ising anyon is non-abelian but not primitive and does not enable universal quantum computation \([12]\).

Interchanging neighboring anyons acts as a unitary transformation \( R_{ab}^c : V_{ab}^c \rightarrow V_{ba}^c \) and is called the R-matrix. Due to the associativity of the fusion rule \( (a \times b) \times c = a \times (b \times c) \), there exist isomorphisms \( F_{bc}^a \) relating the fusion spaces \( \bigoplus_z V_{ab}^z \otimes V_{xc}^d \) and \( \bigoplus_z V_{ad}^z \otimes V_{bc}^x \) referred to as the F-matrix. An arbitrary braiding operation is then fully described via R- and F-matrix.

**Entangled States in Anyon Systems.** In the following, we are interested in the bipartite entanglement of a pure state of an anyon chain \( \{a_1, a_2, \ldots, a_n\} \). Let us consider the bipartite spatial splitting into \( A = \{a_1, \ldots, a_m\} \) and \( B = \{a_{m+1}, \ldots, a_n\} \). We denote by \( \mathcal{H}_A^a \) and \( \mathcal{H}_B^b \) the fusion space of the anyon chain \( A \) and \( B \) with total charge \( x \). Then, the joint fusion space \( \mathcal{H}_{AB} := V_{a_1 \ldots a_n}^c \) with total charge \( c \) can be decomposed as

\[
\mathcal{H}_{AB} = \bigoplus_{a,b \in \mathcal{L}} \mathcal{H}_A^a \otimes \mathcal{H}_B^b \otimes V_{ab}^c. \tag{3}
\]

with a tree diagram as depicted on the right hand side of FIG. [1]. Hence, the joint Hilbert space can generally not be written as the tensor product of the individual subsystems \( A \) and \( B \).

We are now interested in the state restricted to the local subsystem \( A \). The SSR restricts the possible operations on the subsystem \( A \) such that the total charge of \( A \) is conserved. Hence, the Hilbert space corresponding to \( A \) decays into superselection (SS) sectors each determined by the local total charge \( a \), and the reduced state can be assumed to have block structure with respect to \( a \). This motivates to define a partial trace on anyonic systems in the following way. We first embed \( \mathcal{H}_{AB} \) in the larger (non-physical) Hilbert space \( \mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B \), where \( \mathcal{H}_A = \bigoplus_a \mathcal{H}_A^a \) and \( \mathcal{H}_A = \bigoplus_{ab} \mathcal{H}_B^b \otimes V_{ab}^c \). This canonical embedding and its natural extension to operators will be denoted by \( J \). We then define the partial trace on the anyonic system mapping states on \( \mathcal{H}_{AB} \) to states on \( \mathcal{H}_A \) by \( \tau_A = \text{tr}_A \circ J \), where \( \text{tr}_A \) denotes the usual partial trace over the system \( \mathcal{H}_A \). By construction it is clear that the corresponding reduced states \( \rho_A \) have a block structure \( \rho_A = \bigoplus_a \rho_a^A \) what will be imported in the sequel. In particular, it is important to be aware that only degrees of freedom in \( \mathcal{H}_A \) which respect the SSR sectors \( \mathcal{H}_A^a \) are physical. We finally emphasize that if the total charge \( c \) is the vacuum 1, we have that \( \mathcal{H}_A^a = \mathcal{H}_B \).

We define operations on an anyon chain as any combination of (i) adding ancillary anyons with total charge 1, (ii) tracing out a part of the anyon chain, (iii) applying unitary transformations by braiding neighboring anyons and (iv) projective measurements which respect the SSR. Local operations on \( A \) and \( B \) are given by operations only including anyons from \( A \) and \( B \), respectively. In analogy to tensor product systems, we then define a separable state as a state which can be created from \( |\psi_A^a\rangle |\psi_B^b\rangle \) by using LOCC, where \( |\psi_A^a\rangle (|\psi_B^b\rangle) \) is a pure state on \( \mathcal{H}_A^a (\mathcal{H}_B^b) \). Otherwise, a state is called entangled. For pure states, a state \( |\psi_{AB}^c\rangle \in \mathcal{H}_{AB}^c \) is separable if it can be written as \( |\psi_{AB}^c\rangle = |\psi_A^a\rangle |\psi_B^b\rangle \), where \( |\psi_A^a\rangle \in \mathcal{H}_A^a \), \( |\psi_B^b\rangle \in \mathcal{H}_B^b \) and at least one of the labels \( a \) or \( b \) is abelian, thus, \( V_{ab}^c \) is one-dimensional.

**Asymptotic Entanglement Entropy.** Given a qudit sys-
tem $\mathcal{H}_A \otimes \mathcal{H}_B$, the entanglement entropy $E_A$ between $A$ and $B$ of a pure state $\rho$ is defined as the von Neumann entropy of the reduced state $\rho_A = \text{tr}_B \rho$, that is, $E_A(\rho) = -\text{tr} \rho \log \rho_A$. In anyonic systems, we define the generalization of the qudit entanglement entropy for a pure state $\rho = |\psi\rangle\langle \psi|$ on $\mathcal{H}_{AB}$ by

$$E_A^1(\rho) = -\text{tr} \rho_A \log \rho_A,$$

(4)

where $\rho_A = \tau_A(\rho(\psi|\psi))$. In the following we use the notation $E_A^1(\rho)$ interchangeably with $E_A^1(\rho)$. In the case of $c = 1$, any pure state can be written as $|\psi\rangle = \sum_a \sqrt{p_a} |\psi_a\rangle$, where $|\psi_a\rangle \in \mathcal{H}_A^a \otimes \mathcal{H}_B^a$, such that

$$E_A^1(|\psi\rangle) = H(\{p_a\}) + \sum_a p_a E_A^1(|\psi_a\rangle)$$

(5)

where $H(\{p_a\})$ denotes the Shannon entropy of the probability distribution $p_a$. Note that it is crucial here that $\rho_A$ has block structure with respect to the charges $a$.

In an anyon model, independent and identical copies correspond to independent preparation of identical states in the same anyonic system. But as discussed before, the fusion space of $N$ such copies of an anyonic state cannot be written as a tensor product of the single copy spaces. Moreover, in order to unambiguously define the $N$-copy version of an anyonic state $\rho$, we have to assume that its total charge is vacuum (or abelian) since otherwise no unique fusion way between $N$ copies of $\rho$ exists (see Appendix VI for a possible extension to arbitrary charges). Since then the total charge of the system is vacuum as well, the $N$-copy anyon Hilbert space is given by $\mathcal{H}_{A B N} = \bigoplus_a \mathcal{H}_A^a \otimes \mathcal{H}_B^a$ where we omit the trivial vector spaces $V_1^a$. We then define the $N$-copy state of $\rho$ by $\rho^N = \rho(\psi \otimes N)$ where $\rho$ denotes the corresponding embedding of $\mathcal{H}_{AB}^N$ into $\mathcal{H}_{A B N}$ (and similarly its extension to the state space).

The fact that $\rho^N$ is not equivalent to the $N$-fold tensor product of $\rho$, and thus, neither is $\rho^N_A$ of $\rho_A = \tau_B \rho$, implies that $E_A^1(\rho^N)$ is generally not equal to $N E_A^1(\rho)$. This motivates to define the asymptotic entanglement entropy (AEE) of a pure state $\rho$ by

$$E_A^\infty(\rho) = \lim_{N \to \infty} \frac{E_A^1(\rho^N)}{N}.$$  

(6)

The AEE can be expressed in the following closed form.

**Theorem 1.** For a pure state $\rho = |\psi\rangle\langle \psi|$ given by $|\psi\rangle = \sum_a \sqrt{p_a} |\psi_a\rangle \in \mathcal{H}_A^a \otimes \mathcal{H}_B^a$ follows that

$$E_A^\infty(\rho) = E_A^1(\rho) + \sum_a p_a \log d_a.$$  

(7)

In contrast to $E_A$ and $E_A^1$, $E_A^\infty$ depends not only on the Schmidt coefficients of the state but also the quantum dimensions of the anyon charges shared by $A$ and $B$. As expected the additional contribution $\log d_a$ vanishes only if $a$ is abelian, i.e., $d_a = 1$. Theorem 1 further shows that $E_A^\infty$ coincides with the entanglement increase induced by simple anyonic excitations in 2-dimensional topologically ordered spin systems [9] and conformal field theories [10].

In order to derive (7), we have to express the $N$-copy state (see top part of FIG. 2)

$$\bar{\rho}(\psi) = \sum_{a_1, \ldots, a_N} \sqrt{p_{a_1}} a_1 \cdots \sqrt{p_{a_N}} a_N |a_1 a_2 \cdots a_N; 1\rangle \cdots |a_N a_N; 1\rangle,$$

\[
\rho = p_a \cdots p_{a_N} \times |a_1 a_2 \cdots a_N; 1\rangle \cdots |a_N a_N; 1\rangle,
\]

with $p_a = p_{a_1} \cdots p_{a_N}$, in a basis respecting the global charges shared by Alice and Bob, i.e., $\mathcal{H}_{AB}^N = \bigoplus_c \mathcal{H}_A^c \otimes \mathcal{H}_B^c$. This is accomplished by means of F-matrices as illustrated in FIG. 2. Using that $(F_{\alpha a})_{bc} = \sqrt{d_a d_b} \delta_{\alpha b}$, it is then easy to see that

$$\bar{\rho}(\psi) = \sum_{a, b, c} \sqrt{p_{a} d_a} a \ldots c |a_1 a_2 \cdots a_N; c\rangle, \cdots |a_N a_N; c\rangle,$$

(8)

where $d_a = d_{a_1} \cdots d_{a_N}$ and $|a, b, c\rangle = |a_1 a_2; b_1\rangle \cdots |b_{N-2} a_N; c\rangle$ denotes an orthonormal basis of $\mathcal{H}_A^c$ (similar for $|\bar{a}, \bar{b}, \bar{c}\rangle$ and $\mathcal{H}_B^c$). Hence, the reduced state $\rho_A^N$ is block diagonal w.r.t. $(a, b, c)$, and the probability and state associated to the corresponding subspace are $p_{a, b, c} = p_{a} d_c / d_a$ and $\text{tr}_B(|\psi_{a, b, c}\rangle \cdots \text{tr}_B(|\psi_{a, b, c}\rangle)$. A simple computation (see Section 11 of Appendix for details) shows that $H(p_{a, b, c}) = NH(|p_a\rangle) + N \sum_a p_a \log d_a + O(1)$, and thus,

$$N(E_A^1(|\psi\rangle) + H(|p_a\rangle)) = N \sum_a p_a \log d_a + O(1).$$

(9)

Dividing by $N$ and taking the limit results finally in (7).

The AEE fulfills the necessary properties of an entanglement measure for anyonic bipartite pure states with total charge 1. That is, $E_A^\infty$ is a non-negative function which is 0 if and only if the state is separable and it is non-increasing under anyonic LOCC operations. The proof of these properties can be found in Appendix 11.

Let us now address the question which states maximize the AEE. For qudit systems, the MES $|\Psi_{\text{max}}\rangle$ in a $d \times d$ system is a state with uniform Schmidt coefficients, i.e., $|\Psi_{\text{max}}\rangle = \sum_{i=1}^d \frac{1}{\sqrt{d}} |ii\rangle$ for some basis $\{|ii\rangle\}$.
and $E_A(\ket{\Psi_{\text{max}}}) = \log d$. However, since the AEE also depends on the quantum dimensions, the anyonic MES maximizing the AEE is different and has to follow a particular distribution over the SSR sectors. For simplicity, let us restrict to the situation where both systems $A$ and $B$ are given by $n$ anyons with charges $\{x_1, \ldots, x_n\}$ and $\{\bar{x}_1, \ldots, \bar{x}_n\}$ where all $x_i$ are primitive. Then, the maximum of $E_A^\infty$ over states in the corresponding fusion space is attained for

$$\ket{\Psi_{\text{max}}^x} = \sum_a \sqrt{\frac{\dim(V^a_{x_1, \ldots, x_n})}{\prod d_{x_i}}} \ket{\psi^a_{\text{max}}}, \quad (10)$$

where $\ket{\psi^a_{\text{max}}}$ denotes the usually maximally entangled state in $\mathcal{H}_A^a \otimes \mathcal{H}_B^a$. Moreover, it holds that $E_A^\infty(\ket{\Psi_{\text{max}}^x}) = \sum_i \log d_{x_i}$ (see Appendix [V] for details).

**Entanglement Distillation and Cost**—In the following we provide an operational meaning to the AEE by relating it to the optimal asymptotic rates of entanglement distillation and entanglement dilution. An entanglement distillation protocol converts copies of $\ket{\psi}$ to copies of the anyonic MES by LOCC operations. The rate of an entanglement distillation protocol is defined as the ratio of anyonic MES per copy of $\ket{\psi}$ in the limit of infinite number of copies. The reverse task of generating copies of $\ket{\psi}$ from anyonic MES is called entanglement dilution protocol. The rate is defined similarly as for entanglement distillation.

We then define the distillable entanglement $E_D$ as the optimal rate of any entanglement distillation protocol and the entanglement cost $E_C$ as the optimal rate of any entanglement dilution protocol.

In order to ensure that the anyonic MES is well defined, we impose the same restrictions as used to specify the MES in [10]. We further assume that we can individually perform any special unitary on each sector by braiding with arbitrary accuracy. This is for instance satisfied for the Fibonacci anyon [12]. We then obtain the equivalence of the AEE with $E_D$ and $E_C$.

**Theorem 2.** For any pure bipartite state $\ket{\psi} = \sum_{a} \sqrt{p_a} \ket{\psi_a}$ with a primitive label $a \in \mathcal{L}$ satisfying $p_a \neq 0$, holds that

$$E_D(\ket{\psi}) = \frac{E_A^\infty(\ket{\psi})}{E_A^\infty(\ket{\Psi_{\text{max}}^x})} = E_C(\ket{\psi}). \quad (11)$$

We first discuss the optimal entanglement distillation protocol and show achievability. Plugging in the Schmidt decomposition $\ket{\psi_a} = \sum_{i_a} \sqrt{\lambda_{i_a}} \ket{i_a} \ket{i_a}$ in [8], we see that $\iota(\ket{\psi^a})$ can be written as

$$\sum_{a, b, c, \lambda_i} \sqrt{p_a \lambda_i d_c} \ket{a, b, c, i_a} \ket{\bar{a}, \bar{b}, \bar{c}, \bar{i_a}}, \quad (12)$$

where $\lambda_{i_a} = \lambda_{i_1} \cdots \lambda_{i_n}$. Note now that the amplitudes follow an identical and independent distribution according to $p_a \lambda_i$ with weight $1/d_a$. We can thus project for every $c$ onto the $\delta$-typical subspace corresponding to $p_a \lambda_i$ and bring the success probability arbitrary close to 1 by choosing a sufficiently large $N$. In a next step we project onto the different type-classes in the $\delta$-typical subspace such that all the amplitudes for fixed $c$ are equal. Using standard typicality properties we know that the dimension of the resulting typical subspace is about $2^{NH(p_a \lambda_i)} = 2^N E_A^\infty(\ket{\psi})$ for any $c$. Moreover, the additional fusion dimensions labeled by $b$ count up for given $c$ to $\dim V_{a}^c$ where $a$ is a typical sequence. Using the scaling behavior of the fusion space and that $a^c$ is typical, we find that $\dim V_{a}^c \approx (d_c/\mathcal{D})^\frac{p_a}{d_d}$. Hence, the total dimension of the non-trivial subspace in sector $c$ is about $(d_c/\mathcal{D})^{2NH(p_a \lambda_i)} \dim V_{a}^c \approx (d_c/\mathcal{D})^{2N E_A^\infty(\ket{\psi}) + \sum_i p_a \log d_i} = (d_c/\mathcal{D})^{2 N E_A^\infty(\ket{\psi})}.$

Note now that on each sector $c$, $\iota(\ket{\Psi_{\text{max}}^x})$ corresponds to the MES in the fusion space $V_{x, c}^L$ with dimension $\dim V_{x, c}^L = (d_c/\mathcal{D})^L$. Hence, the dimensions of the subspaces match for every $c$ if we choose $L \approx N E_A^\infty(\ket{\psi}) \log d_c = N E_A^\infty(\ket{\psi}) / E_A^\infty(\ket{\Psi_{\text{max}}^x})$. However, this is also sufficient such that a simple local basis change converts the state from the protocol into $L$ anyonic MES. For the technical details including convergence questions see Appendix [V].

The optimal entanglement dilution protocol is based on LOCC convertibility of each individual sector. For that we use similar methods as for the distillation protocol to show that on each sector the amplitudes of about $L = N E_A^\infty(\ket{\psi}) / E_A^\infty(\ket{\Psi_{\text{max}}^x})$ copies of $\ket{\Psi_{\text{max}}^x}$ majorize the amplitudes of the typical part of $N$ copies of $\ket{\psi}$. Optimality of the two protocols can be argued by cascadability. For example, if there would exist an entanglement distillation protocol which performs strictly better, we could convert $N$ copies of $\ket{\psi}$ into $N' > N$ copies of $\ket{\psi}$, which leads to a contradiction as entanglement cannot be increased by LOOC.

**Conclusion.**—We have presented an information-theoretical approach to analyze bipartite entanglement in pure anyonic systems and established an asymptotic resource theory for entanglement distillation and dilution. It turned out that the operationally relevant entanglement measure is the AEE which in contrast to the usual EE includes an anyonic contribution depending on the quantum dimension. The same contribution was identified as the entanglement gain by the transition from ground to first excited states in particular models of TOP [8][10][20]. This indicates a deep connection between the entanglement contained in the braiding degrees of freedom of an anyonic quasiparticle excitation and its actual entanglement generated in the TOP. We finally point out that our result may apply to other interesting situations obeying similar SSRs as anyons, as e.g., angular momenta with no shared reference frame [21].

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[25] We call a label a primitive if the corresponding matrix $(N_{ai})_{ij} = N_{ai}^{ij}$ is primitive, i.e., for large $n$, $(N_{ai}^{n})_{ij} > 0$ for all $i$ and $j$ (see Section II of Appendix for details).
Appendix

I. THE QUANTUM DIMENSION OF AN ANYON

We consider an anyon model determined by the finite set of labels \( \mathcal{L} = \{1, a, b, c, \ldots\} \) and the fusion rules \( a \times b = \sum N^c_{ab}c \). In general, the fusion space of multiple anyons corresponding to the possible fusion ways has no simple decomposition into tensor products of individual subspaces corresponding to each particle. It is therefore not possible to assign a Hilbert space dimension to the different particle types. But instead, there is a different characteristic that can be assigned to each anyon type called the quantum dimension.

**Definition 3.** (Quantum Dimension) For any anyon model with charges in \( \mathcal{L} \) exists positive real numbers \( d_a \) for \( a \in \mathcal{L} \) such that

\[
d_a d_b = \sum_c N^c_{ab} d_c.
\]

We call \( d_a \) the quantum dimension of the anyon type \( a \). Furthermore, \( D := \sqrt{\sum_a d_a^2} \) is called the total quantum dimension of the anyon model.

The existence and uniqueness of the quantum dimension is guaranteed by the Perron-Frobenius theorem [1]. The quantum dimension of an abelian anyon is 1, and strictly larger than 1 for any non-abelian anyon. We call the matrix \( N_a \) defined by \((N_a)_{bc} \equiv N^c_{ab}\) the fusion matrix of the anyon type \( a \). Note that (13) implies that \( d_a \) is an eigenvalue of \( N_a \) to the eigenvector \( d = (d_c)_{c \in \mathcal{L}} \).

The following classifications of the fusion matrix will be useful in the sequel.

**Definition 4.** (Irreducible matrix) A \( n \times n \) matrix \( A = (A_{ij}) \) with \( A_{ij} \geq 0 \) is called irreducible if for any fixed \( 1 \leq i, j \leq n \), there exists \( m \in \mathbb{N} \) such that \((A_m)_{ij} > 0\).

**Definition 5.** (Primitive matrix) An irreducible \( n \times n \) matrix \( A = (A_{ij}) \) is called primitive if there exists a common \( m \in \mathbb{N} \) such that \((A^m)_{ij} > 0\) for all \( 1 \leq i, j \leq n \).

Henceforth, we call a label \( a \) irreducible or primitive if the corresponding fusion matrix \( N_a \) is irreducible or primitive.

**Example 6.** Abelian anyons are not irreducible. The fusion matrix of Ising anyon \( N_\sigma \) is

\[
N_\sigma = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}
\]

in the \((1, \sigma, \psi)\) basis. \( N_\sigma \) is an irreducible matrix but not primitive, so that the Ising anyon is non-primitive. The fusion matrix of the Fibonacci anyon \( N_\tau \) is

\[
N_\tau = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}
\]

in the \((1, \tau)\) basis. Since \( N_\tau \) is a primitive matrix, the Fibonacci anyon is a primitive anyon.

We proceed by relating the quantum dimensions to the exponential growth of the fusion space in the number of anyons. The dimension of \( V_{a_1 a_2 \cdots a_n}^b \) is given by

\[
\dim V_{a_1 a_2 \cdots a_n}^b = \sum_{b_1, b_2, \cdots, b_{n-2}} N^{b_1}_{a_1 a_2} N^{b_2}_{b_1 a_3} \cdots N^{b_n}_{b_{n-2} a_n}.
\]

In the following, we use the notation \( a^N \) for \( N \) anyons of the same type \( a \). The following statement shown in [2, 3] relates the quantum dimension to the dimension of the fusion space of many anyons of the same type.

**Lemma 7.** [2, 3] If \( a \) is a primitive label, then it holds that

\[
\dim V_{a}^b = \frac{d_a^N d_b}{D^2} \left( 1 + O(c_b^N) \right),
\]

where \( |c_b| < 1 \).
More generally if at least $a$ is primitive, it holds that
\begin{equation}
\dim V_{\gamma \delta}^{x} \cap \gamma^{a} \delta^{b} \cdots \gamma^{c} = \frac{d_{a}^{N_{a}} d_{b}^{N_{b}} \cdots d_{c}^{N_{c}}}{D^{2}} (1 + \mathcal{O}(c^{N_{a}})) \quad (|c| < 1).
\end{equation}

The proof is similar to the one of Lemma 7. We emphasize that permittivity of the fusion matrix is essential since the proof of these relations requires the Perron-Frobenius theorem.

We conclude this section with another relation between the quantum dimension and the dimension of the fusion space.

**Lemma 8.** For all $n \in \mathbb{N}$ holds that
\begin{equation}
d_{a_{1}} \cdots d_{a_{N}} = \sum_{b} \dim V_{a_{1} \cdots a_{N}}^{b} d_{b}
\end{equation}

**Proof.** Using the definition of the quantum dimension, we can compute
\begin{equation}
d_{a_{1}} d_{a_{2}} \cdots d_{a_{N}} = \sum_{b_{1}} N_{a_{1}a_{2}}^{b_{1}} d_{b_{1}} d_{a_{3}} \cdots d_{a_{N}} = \sum_{b_{1}, b_{2}} N_{a_{1}a_{2}}^{b_{1}} N_{b_{1}a_{3}}^{b_{2}} d_{b_{2}} d_{a_{4}} \cdots d_{a_{N}} = \sum_{b_{1}, \ldots, b_{N-2}, b} N_{a_{1}a_{2}}^{b_{1}} N_{b_{1}a_{3}}^{b_{2}} \cdots N_{b_{N-2}a_{N}}^{b} d_{b}.
\end{equation}
Hence, inserting relation (16) completes the proof.

\section{II. PROOF OF THEOREM 1}

The Theorem 1 in the main text gives the quantitative relation between $E_{A}^{\infty}$ and $E_{A}^{1}$ and reads as follows.

**Theorem 9.** For a pure state $\rho = |\psi \rangle \langle \psi|$ given by $|\psi \rangle = \sum_{a} \sqrt{p_{a}} |\psi_{a} \rangle$ with $|\psi_{a} \rangle \in \mathcal{H}_{A}^{a} \otimes \mathcal{H}_{B}^{a}$ follows that
\begin{equation}
E_{A}^{\infty}(\rho) = E_{A}^{1}(\rho) + \sum_{a} p_{a} \log d_{a}.
\end{equation}

**Proof.** As discussed in the main text, we first apply a basis transformation to the state
\begin{equation}
\mathcal{I}(\rho) = \sum_{a} \sqrt{p_{a}} |\psi_{a} \rangle \cdots |\psi_{a_{N}} \rangle |a_{1} \bar{a}_{1}; 1 \rangle \cdots |a_{N} \bar{a}_{N}; 1 \rangle,
\end{equation}
in order to be able to split it into local parts $A$ and $B$. The explicit transformation is given by
\begin{equation}
|a_{1} \bar{a}_{1}; 1 \rangle \cdots |a_{N} \bar{a}_{N}; 1 \rangle = \sum_{b, c} (F_{a_{2}a_{3}1}^{a_{2}}) b_{1} \cdots (F_{a_{N}b_{N-2}1}^{a_{N}}) c_{1} |a, b, c \rangle \langle \bar{a}, \bar{b}, \bar{c}|,
\end{equation}
and by using $(F_{a_{2}a_{1}1}^{a_{2}})_{1} = \sqrt{\frac{d_{a}}{d_{a_{1}}}}$, we arrive at
\begin{equation}
\mathcal{I}(\rho) = \sum_{a, b, c} \sqrt{\frac{p_{a_{1}} \cdots p_{a_{N}} \delta_{c}}{d_{a_{1}} \cdots d_{a_{N}}}} |\psi_{a_{1}} \rangle \cdots |\psi_{a_{N}} \rangle \otimes |a, b, c \rangle \langle \bar{a}, \bar{b}, \bar{c}|.
\end{equation}

Recall that we denote $a = (a_{1}, \ldots, a_{N})$ and $b = (b_{1}, \ldots, b_{N-2})$, and that $|a, b, c \rangle = |a_{1}a_{2}b_{1} \cdots b_{N-2}a_{N}; c \rangle$ are mutual orthogonal. Thus, using the definition of the partial trace $\tau_{B}$, the reduced density matrix $\rho_{A}^{N} = \tau_{B}(\rho^{N})$ can be written as
\begin{equation}
\rho_{A}^{N} = \sum_{a, b, c} \frac{p_{a_{1}} \cdots p_{a_{N}} \delta_{c}}{d_{a_{1}} \cdots d_{a_{N}}} \tau_{B}(|\psi_{a_{1}} \rangle \langle \psi_{a_{1}}|) \otimes \cdots \otimes \tau_{B}(|\psi_{a_{N}} \rangle \langle \psi_{a_{N}}|) \otimes |a, b, c \rangle \langle a, b, c|.
\end{equation}
The reduced state $\rho_{A}^{N}$ can be seen as a classical quantum state where the classical degrees are associated to $a, b, c$ which are distributed according to $\frac{p_{a_{1}} \cdots p_{a_{N}} \delta_{c}}{d_{a_{1}} \cdots d_{a_{N}}}$. Using standard properties of the von Neumann entropy, it directly follows that
\begin{equation}
E_{A}^{\infty}(\rho) = H\left(\left\{ \frac{p_{a_{1}} \cdots p_{a_{N}} \delta_{c}}{d_{a_{1}} \cdots d_{a_{N}}} \right\} \right) + \sum_{a, b, c} \frac{p_{a_{1}} \cdots p_{a_{N}} \delta_{c}}{d_{a_{1}} \cdots d_{a_{N}}} \left( E_{A}^{1}(|\psi_{a_{1}} \rangle) + \cdots + E_{A}^{1}(|\psi_{a_{N}} \rangle) \right),
\end{equation}
where $H(\cdot)$ denotes the Shannon entropy. The first term on the right hand side of the equation (26) can be computed to

$$H \left( \left\{ \frac{p_{a_1} \cdots p_{a_N} d_c}{d_{a_1} \cdots d_{a_N}} \right\} \right) = - \sum_{a,b,c} \frac{p_{a_1} \cdots p_{a_N} d_c}{d_{a_1} \cdots d_{a_N}} \log \frac{p_{a_1} \cdots p_{a_N} d_c}{d_{a_1} \cdots d_{a_N}}$$

$$= - \sum_{a,c} \frac{p_{a_1} \cdots p_{a_N} \dim V_{a_1}^c \cdots d_{a_N}}{d_{a_1} \cdots d_{a_N}} \log \frac{p_{a_1} \cdots p_{a_N} d_c}{d_{a_1} \cdots d_{a_N}}$$

$$= - \sum_{a} p_{a_1} \cdots p_{a_N} \left( \log \frac{p_{a_1}}{d_{a_1}} + \cdots + \log \frac{p_{a_N}}{d_{a_N}} \right)$$

$$- \sum_{c} \left( \sum_{a} p_{a_1} \cdots p_{a_N} \dim V_{a_1}^c \cdots d_{a_N} \right) \log d_c$$

$$= N \left( H(\{p_a\}) + \sum_{a \in \mathcal{L}} p_a \log d_a \right) - \sum_{c \in \mathcal{L}} q_c \log d_c,$$

with $q_c := \sum_a p_{a_1} \cdots p_{a_N} \dim V_{a_1}^c \cdots d_{a_N}/d_{a_1} \cdots d_{a_N}$. For the second equality, we used that for fixed charges $a$ and $c$, $\sum_b 1 = \dim V_a^c$ which follows from (16). And the third equality follows from Lemma 8. Note further that Lemma 8 implies that $q_c$ is a probability distribution, i.e., $q_c \geq 0$ and

$$\sum_c q_c = \sum_a p_{a_1} \cdots p_{a_N} \sum_c \frac{\dim V_{a_1}^c \cdots d_{a_N}}{d_{a_1} \cdots d_{a_N}} = \sum_a p_{a_1} \cdots p_{a_N} = 1. \quad (32)$$

The second term in (26) can be simplified to

$$\sum_a p_{a_1} \cdots p_{a_N} \frac{\dim V_{a_1}^c \cdots a_N d_c}{d_{a_1} \cdots d_{a_N}} = \sum_a p_{a_1} \cdots p_{a_N}. \quad (33)$$

Since $E_A^1(\ket{\psi}) = H(\{p_a\}) + \sum_a p_a E_A^1(\ket{\psi_a})$, we finally arrive at

$$E_A^\infty(\rho) = \lim_{N \to \infty} \frac{1}{N} E_A^1(\rho N) = E_A^1(\ket{\psi}) + \sum_{a \in \mathcal{L}} p_a \log d_a - \lim_{N \to \infty} \frac{1}{N} \sum_{c \in \mathcal{L}} q_c \log d_c = E_A^1(\ket{\psi}) + \sum_{a \in \mathcal{L}} p_a \log d_a. \quad (34)$$

\[\square\]

III. PROPERTIES OF THE AEE

In the following, we show that the AEE satisfies the usual properties of an entanglement measure of pure states. The non-negativity follows directly from the definition.

a) Non-negativity. For all $\ket{\psi} \in \bigoplus_a \mathcal{H}_A^a \otimes \mathcal{H}_B^a$ holds that

$$E_A^\infty(\ket{\psi}) \geq 0. \quad (35)$$

b) Vanishing on separable states. For all $\ket{\psi} \in \bigoplus_a \mathcal{H}_A^a \otimes \mathcal{H}_B^a$ holds that

$$E_A^\infty(\ket{\psi}) = 0 \quad (36)$$

if and only if $\ket{\psi}$ is separable.

Proof. By definition, $E_A^1(\ket{\psi}) = 0$ if $\ket{\psi}$ is separable. If the total charge is the vacuum, a separable state $\ket{\psi}$ is a state on $\mathcal{H}_A^0 \otimes \mathcal{H}_B^0$ with $a$ and $\bar{a}$ abelian, such that $d_a$ is 1. Therefore, $E_A^\infty(\ket{\psi}) = 0$ follows from Theorem 1. \[\square\]

c) Monotonicity. If $\ket{\psi} \in \bigoplus_a \mathcal{H}_A^a \otimes \mathcal{H}_B^a$ can be converted to $\ket{\phi_j} \in \bigoplus_a \mathcal{H}_A^a \otimes \mathcal{H}_B^a$ with probability $q_j$ by LOCC, then

$$E_A^\infty(\ket{\psi}) \geq \sum_j q_j E_A^\infty(\ket{\phi_j}). \quad (37)$$
Proof. For any fixed \( N \), the function \( E_A^1(\rho^N) \) is computed by first embedding the state \( \rho^N \) by \( \mathcal{J} \) into a tensor product Hilbert space \( \mathcal{H}_{A_N} \) and then evaluating the entanglement entropy of the qubit. Under this embedding \( \mathcal{J} \), any LOCC operation on the anyonic system transforms to an LOCC operation on the bigger space \( \mathcal{H}_{A_N} \). Since the entanglement entropy cannot increase under LOCC neither can the asymptotic entanglement entropy. \( \square \)

d) Additivity. For any two states \( |\psi_1\rangle, |\psi_2\rangle \) with \( |\psi_i\rangle \in \bigoplus_a \mathcal{H}^a_{A} \otimes \mathcal{H}^b_{B} \) holds that

\[
E_{A_1A_2}(\iota(|\psi_1\rangle \otimes |\psi_2\rangle)) = E_{A_1}(|\psi_1\rangle) + E_{A_2}(|\psi_2\rangle).
\] (38)

Proof. Assume that \( |\psi_i\rangle = \sum_a \sqrt{p_a} |\psi^{a}_i\rangle \) with \( |\psi^{a}_i\rangle \in \mathcal{H}^a_{A} \otimes \mathcal{H}^b_{B} \) where \( i = 1, 2 \). The embedded state \( \iota(|\psi_1\rangle \otimes |\psi_2\rangle) \) is then given by

\[
\iota(|\psi_1\rangle \otimes |\psi_2\rangle) = \sum_{a_1a_2} \sqrt{p_{a_1}p_{a_2}} d_c \langle \psi^{a_1}_{a_2} | \langle \psi^{a_2}_{a_2} | \otimes | a_2a_1; c \rangle | \bar{a}_2\bar{a}_2; c \rangle,
\] (39)

By using Theorem 1, we then obtain

\[
E^\infty_{A_1A_2}(\iota(|\psi_1\rangle \otimes |\psi_2\rangle)) = E^1_{A_1}(|\psi_1\rangle) + E^1_{A_2}(|\psi_2\rangle) + \sum_{abc} \frac{p_a p_b p_c}{d_a d_b} \log d_c.
\] (40)

We can directly calculate the first term in (40) as

\[
E^1_{A_1}(|\psi_1\rangle) + E^1_{A_2}(|\psi_2\rangle) = E_{A_1}(|\psi_1\rangle) + E_{A_2}(|\psi_2\rangle) + \sum_a p_a \log d_a - \sum_{abc} \frac{p_a p_b p_c}{d_a d_b} \log d_c.
\] (41)

By plugging Eq. (42) into (40) we see that additivity is satisfied. \( \square \)

IV. ANYONIC MAXIMALLY ENTANGLED STATES.

Assume that both systems \( A \) and \( B \) are given by anyon chains \( \{x_1, \ldots, x_n\} \) and \( \{x_1, \ldots, \bar{x}_n\} \) with all labels \( x_i \) primitive. Then, the maximum of \( E^\infty_A \) over states in the corresponding fusion space is attained for

\[
|\Psi^\infty_{\text{max}}\rangle = \sum_b \sqrt{\dim(V^b_{x_1 \ldots x_n})} d_b |\psi^b_{\text{max}}\rangle,
\] (43)

where \( |\psi^b_{\text{max}}\rangle \) denotes the usual maximally entangled state in \( \mathcal{H}^b_A \otimes \mathcal{H}^b_B \). Moreover, it holds that

\[
E^\infty_A(|\Psi^\infty_{\text{max}}\rangle) = \sum_i \log d_{x_i}.
\] (44)

To see this, let us set \( D^a_N = \dim V^a_{x_1 \ldots x_N} \) and \( D^a_{N} = \dim V^a_{\bar{x}_1 \ldots \bar{x}_N} \). We first derive an upper bound on \( E^\infty_A \) and then show that the state in (43) attains the upper bound. The embedded \( N \) copy state \( \mathcal{J}(\rho^N) \) is a state on the Hilbert space

\[
\left( \bigoplus_{a,d} \mathcal{H}^a_{A^1} \otimes \cdots \otimes \mathcal{H}^a_{A^N} \otimes V^d_{a^N \ldots a_1} \right) \otimes \left( \bigoplus_{b,d} \mathcal{H}^b_{B^1} \otimes \cdots \otimes \mathcal{H}^b_{B^N} \otimes V^d_{b^N \ldots b_N} \right).
\] (45)

We denote \( \mathcal{H}_{A^N} = \bigoplus_{a,d} \mathcal{H}^a_{A^1} \otimes \cdots \otimes \mathcal{H}^a_{A^N} \otimes V^d_{a^N \ldots a_N} \) and \( \mathcal{H}_{B^N} = \bigoplus_{b,d} \mathcal{H}^b_{B^1} \otimes \cdots \otimes \mathcal{H}^b_{B^N} \otimes V^d_{b_N \ldots b_N} \). The subspaces \( \mathcal{H}_{A^N} \) and \( \mathcal{H}_{B^N} \) contain \( N \) \( x \)-anyons, and due to Lemma 7 we know that for sufficiently large \( N \), \( D^a_{x^N} \approx d^a_{x_N} d_{a}/D^2 \).

Hence,

\[
\dim \mathcal{H}_{A^N} = \dim \mathcal{H}_{B^N} = \sum_{a \ldots b} \sum_i D^b_{x^N} \dim V^a_{b_1 \ldots b_N} \approx \prod_i d^N_a \sum_{a \ldots b} \frac{d_{a_1} \cdots d_{a_N}}{D^{2n}} \dim V^a_{b_1 \ldots b_N},
\] (46)
which implies that
\[
\frac{1}{N} E^1_A (\iota(\ket{\psi}^{\otimes N})) \leq \frac{1}{N} \log \dim \mathcal{H}_{AN} = \frac{1}{N} \left( \sum_i \log d_i^N + O(1) \right).
\] (47)

Taking the limit \( N \) to infinity in Eq. (47), we obtain the upper bound \( E^\infty_A (\ket{\psi}) \leq \sum_i \log d_i \). In order to see that \( \ket{\psi_{\text{max}}} \) attains the optimal value, we compute
\[
E^\infty_A (\ket{\psi_{\text{max}}}) = H \left( \left\{ \frac{D_a^2 d_a}{\prod_i d_{x_i}} \right\} \right) + \sum_a \frac{D_a^2 d_a}{\prod_i d_{x_i}} \left( E^1_A (\ket{\psi_{\text{max}}}) + \log d_a \right)
\]
\[
= \sum_a \frac{D_a^2 d_a}{\prod_i d_{x_i}} \left( - \log \frac{D_a^2 d_a}{\prod_i d_{x_i}} + \log D_a + \log d_a \right)
\]
\[
= \sum_{a,j} \frac{D_a^2 d_a}{\prod_i d_{x_i}} \log d_{x_i} = \sum_i \log d_{x_i}.
\] (48)

The last equality holds by Lemma 8. One example of the anyonic maximally entangled state is given if all pairs of anyons \( x_i \) in A and \( \bar{x}_i \) in B are created from the vacuum.

V. PROOF OF THEOREM 2: OPTIMAL ENTANGLEMENT DISTILLATION AND DILUTION PROTOCOL

A. Entanglement Distillation

In the following, we first present a distillation protocol and show that its rate is given by the AEE. Assume that both systems \( A \) and \( B \) are given by chains of anyons with primitive labels \( x = \{x_1, \ldots, x_n\} \) and \( \bar{x} = \{\bar{x}_1, \ldots, \bar{x}_n\} \). This is sufficient to have a well defined maximally entangled state (c.f. Section II.C). In order to simplify notation, we further assume that \( x = x_1 = \ldots = x_n \) which can be generalized straightforward to arbitrary primitive labels. At the beginning of the protocol, Alice and Bob share many identical copies of a state \( \ket{\psi} = \sum_a \sqrt{p_a} \ket{\psi_a} \) with \( \ket{\psi_a} \in \mathcal{H}_A^a \otimes \mathcal{H}_B^a \). By the Schmidt decomposition, each \( \ket{\psi_a} \) can be written as
\[
\ket{\psi_{a,j}} = \sum_{i_{a,j}} \sqrt{\lambda_{i_{a,j}}} \ket{i_{a,j}},
\] (49)
where \( \{\lambda_{i_{a,j}}\} \) denotes the Schmidt coefficients of \( \ket{\psi_{a,j}} \). According to (24), the N-copy state \( \ket{\psi^N} \equiv \iota(\ket{\psi}^{\otimes N}) \) can then be written as
\[
\ket{\psi^N} = \sum_{a, b^c, c, i_{a}} \sqrt{p_{a, b^c, c, i_{a}} d_{a, b^c, c, i_{a}}} \lambda_{i_{a, 1}} \cdots \lambda_{i_{a, n}} \ket{a, b^c, c, i_{a}}_{AN} \ket{\bar{a}, \bar{b}^c, c, i_{a}}_{BN},
\] (50)
where \( i_{a} = (i_{a, 1}, \ldots, i_{a, n}) \). Our goal is to obtain \( L \) copies of the maximally entangled state
\[
\iota(\ket{\psi_{\text{max}}}^\otimes L) \equiv \sum_{c} \sqrt{\frac{\dim V^c_{\otimes L} d_c}{\prod_{x \in \Lambda} d_x^{nL}}} \left[ \frac{1}{\dim V^c_{\otimes L}} \sum_{y \in \Lambda} \ket{y^c}_{AN} \ket{\bar{y^c}}_{BN} \right],
\] (51)
where \( L \) should be maximal.

Let us now focus on the \( \delta \)-strongly typical set \( T^N_{\delta} \) induced by the distribution \( p_a \lambda_{i_{a}} \) which is defined as
\[
T^N_{\delta} := \{ (a, i_{a}) : \frac{1}{N} \left| N(a, i_{a}) - p_a \lambda_{i_{a}} \right| \leq \delta \},
\] (52)
where \( N(a, i_{a}) \) is the number of \( (a, i_{a}) \) in the sequence \( a, i_{a} \). Performing the projective measurement on the corresponding typical subspace, we obtain the state
\[
\ket{\psi^N_{\text{typ}}} = \frac{1}{\sqrt{P^N_{\delta}}} \sum_{c} \sum_{a, i_{a} \in T^N_{\delta}} \sqrt{p_{a, b^c, c, i_{a}}} \lambda_{i_{a, 1}} \cdots \lambda_{i_{a, n}} \ket{a, b^c, c, i_{a}}_{AN} \ket{\bar{a}, \bar{b}^c, c, i_{a}}_{BN}
\] (53)
where for any \( \epsilon > 0 \), we can choose \( N \) large enough that the success probability \( P_N^N \) is at least \( 1 - \epsilon \). This follows due to the typicality properties for independent and identical distributed variables (see, e.g., [4]). Next, we consider the type class \( T_i^N \) which is given by

\[
T_i^N := \left\{ (a, i_a) : \frac{1}{N} N(a, i_a|a, i_a) = t^{a,i_a} \right\},
\]

where \( t^{a,i_a} \) is the \((a, i_a)\) component of the probability distribution \( t \). By defining the set of types in \( T_{\delta}^N \) by

\[
\tau_{\delta} = \left\{ t : |t^{a,i_a} - p_a \lambda_{i_a}| \leq \delta \right\},
\]

\( T_{\delta}^N \) can be decomposed as

\[
T_{\delta}^N = \bigcup_{t \in \tau_{\delta}} T_i^N.
\]

Note that the cardinality of the set \( \tau_{\delta} \) is bounded by \( |\tau_{\delta}| < (N + 1)^d \), where \( d = \sum_a \dim V^a \). Expanded into the different type classes, \( |\psi_{typ}^N\rangle \) can be written as

\[
|\psi_{typ}^N\rangle = \frac{1}{\sqrt{P_{\delta}^N}} \sum_{t \in \tau_{\delta}} \sum_c \sum_{a, i_a \in T_i^N} \sum_{b^c_a} \sqrt{p_{\lambda} \cdots p_{\lambda} \lambda_{i_{a_1}} \cdots \lambda_{i_{a_N}} |a, b^c_a, c, i_a\rangle_A^N |\bar{a}, \bar{b}^c_a, \bar{c}, i_a\rangle_B^N} = \frac{1}{\sqrt{P_{\delta}^N}} \sum_{t \in \tau_{\delta}} \sum_c \sqrt{d_c} \sum_{a, i_a \in T_i^N} \sum_{b^c_a} |a, b^c_a, c, i_a\rangle_A^N |\bar{a}, \bar{b}^c_a, \bar{c}, i_a\rangle_B^N,
\]

where \( t^a = \sum_{t \in \tau_{\delta}} t^{a,i_a} \).

In the next step of the protocol, we measure the type \( t \) and obtain the state

\[
\sqrt{\frac{\prod_{a, i_a} (p_{\lambda} \lambda_{i_a})^{N(t^{a,i_a})}}{P_{\delta}^N q_t^N \prod_a d_{a}^{N_{t^{a,i_a}}}}} \sum_c \sqrt{d_c} \sum_{a, i_a \in T_i^N} \sum_{b^c_a} |a, b^c_a, c, i_a\rangle_A^N |\bar{a}, \bar{b}^c_a, \bar{c}, i_a\rangle_B^N,
\]

with probability \( q_t^N \). The probability \( q_t^N \) is given by

\[
q_t^N = \frac{|T_t^N| \prod_{a, i_a} (p_{\lambda} \lambda_{i_a})^{N(t^{a,i_a})}}{P_{\delta}^N}
\]

and bounded by [5]

\[
(N + 1)^{-d} 2^{-D(t||p\lambda)} \leq q_t^N \leq 2^{-N D(t||p\lambda)} ,
\]

where \( D(t||p\lambda) \) is the relative entropy of \( t \) and \( p\lambda \).

In the following, we denote the dimension of the subspace corresponding to the type \( t \) in sector \( c \) by

\[
N_c^t = \sum_{a, i_a \in T_i^N} \sum_{b^c_a} = |T_i^N| \dim V^c_a.
\]

Moreover, for every \( c \) and fixed \( t \), we relabel \((a, i_a, b^c_a)\) by \( \alpha^c \in \{1, \ldots, N_c^t\} \). We can then write the state in (59) as

\[
\sqrt{\frac{\prod_{a, i_a} (p_{\lambda} \lambda_{i_a})^{N(t^{a,i_a})}}{P_{\delta}^N q_t^N \prod_a d_{a}^{N_{t^{a,i_a}}}}} \sum_c \sqrt{d_c} \sum_{\alpha^c = 1}^{N_c^t} |\alpha^c\rangle_A^N |\bar{\alpha}^c\rangle_B^N
\]

\[
= \sqrt{\frac{|T_t^N| \prod_{a, i_a} (p_{\lambda} \lambda_{i_a})^{N(t^{a,i_a})}}{P_{\delta}^N q_t^N \prod_a d_{a}^{N_{t^{a,i_a}}}}} \sum_c \sqrt{d_c \dim V^c_a} \sum_{\alpha^c = 1}^{N_c^t} \frac{1}{\sqrt{N_c^t}} |\alpha^c\rangle_A^N |\bar{\alpha}^c\rangle_B^N.
\]
As shown in [6], for all $t \in \tau_\delta$ we can bound
\[
|T^N_t| \geq 2^N|H(p\lambda) - \eta(d\delta) - \frac{\delta}{N} \log(N+1)|,
\]
where $\eta(d\delta)$ is a function such that $\eta(d\delta) \to 0$ ($\delta \to 0$). Therefore, using equation (18), we obtain a lower bound on $N^c_t$ via
\[
N^c_t = |T^N_t| \dim V^c_a \
\geq 2^N|H(p\lambda) - \eta(d\delta) - \frac{\delta}{N} \log(N+1)| \times \prod_{a \in L} d^{N(p_a - \delta)}_a \left(1 + \mathcal{O}(v^N_{tN})\right)
\]
\[
= 2^N(p\alpha + \sum a \log d_a - \eta(d\delta) - \frac{\delta}{N} \log(N+1) - \delta \log d_a) \frac{d^c}{D^2} \left(1 + \mathcal{O}(v^N_{tN})\right),
\]
where $|v_c|, |v'_c| < 1$ for all $c \in L$. Note that by using equation (18), we have to make the assumption that $p_a \neq 0$ for at least one primitive label $a$. Let us set
\[
M^c = 2^N(p\alpha + \sum a \log d_a - \eta(d\delta) - \frac{\delta}{N} \log(N+1) - \delta \log d_a - \frac{\chi_N}{N} \log d^c_a) \frac{d^c}{D^2},
\]
where $\chi_N$ is some constant such that $0 \leq \chi_N < 1$. Then, we divide each $N^c_t$ dimensional sector into $M^c$ dimensional subspaces and the rest. Since the dimension of the rest subspace is strictly smaller than $M^c$, the probability that the projection onto the $M^c$-dimensional subspaces fails is less than $\frac{M^c}{N^c}$. This error probability is thus bounded by
\[
\frac{M^c}{N^c} \leq 2^{-\log(N+1) - \chi_N \log d^c_a}.
\]
Hence, we obtain
\[
\frac{1}{\sqrt{P^N_{\delta}}} \sum_c \sqrt{d_c \dim V^c_a} \sum_{a \in 1}^{M^c} \frac{1}{\sqrt{M^c}} |\alpha^c_A^N |\alpha^c_B^N .
\]
with success probability at least
\[
1 - 2^{-\log(N+1) - \chi_N \log d^c_a}.
\]
By performing the local unitary which transform $|\alpha^c_A^N$ to $|\gamma^c_B^N$, we obtain
\[
|\tilde{\psi}^N = \frac{1}{\sqrt{P^N_{\delta}}} \sum_c \sqrt{d_c \dim V^c_a} \sum_{a \in 1}^{M^c} \frac{1}{\sqrt{M^c}} |\gamma^c_A^N |\gamma^c_B^N .
\]
Recall that the L copies of the maximally entangled state are given by
\[
\ell(|\Psi^\infty_{\max}^L\rangle) = \sum_c \sqrt{\dim V^c_{x=L} d_c} \left[\frac{1}{\sqrt{\dim V^c_{x=L}}} \sum_{y} |\gamma^c_y A^N |\gamma^c_y B^N \right].
\]
We consider the set $\{L^c\}$ such that for all $c \in L$, $\dim V^c_{x=L} = M^c$. Then, since $\dim V^c_{x=L} = \frac{d^c_{x=L} d^c_{x}}{D^2} (1 + \mathcal{O}(w^L_{x}))$ with $|w_x| < 1$, we have that
\[
L^c = \frac{N(E^\infty_{A}(|\psi\rangle) - \eta(d\delta) - \frac{d+1}{N} \log(N+1) - \delta \log d_a) - \log(1 + \mathcal{O}(w^N_{x})) - \chi_N}{\log d^c_a}.
\]
We choose the number of copies $L$ as
\[
L = \frac{N(E^\infty_{A}(|\psi\rangle) - \eta(d\delta) - \frac{d+1}{N} \log(N+1) - \delta \log d_a) - \log(1 + K|w^N|) - \chi_N}{\log d^c_a}.
\]
where we choose $0 < K < \infty$ and $w$ to bound the function $O(w_c^N)$ by $|O(w_c^N)| \leq K|w|^N$ for all $c$ and large enough $N$. Thus $L \leq L^c$ and $dim \psi_{xN}^c \leq M^c$ for all $c$ in this setting. Note that we can choose the constant $0 \leq \chi_N < 1$ to make $L$ a natural number beforehand.

In order to see that $|\tilde{\psi}^N|$ converges to the maximally entangled state $\langle \psi_{\max}^\otimes \rangle$, we consider the inner product between $|\tilde{\psi}^N|$ and $\langle \psi_{\max}^\otimes \rangle$ given by

$$\langle \tilde{\psi}^N | \langle \psi_{\max}^\otimes \rangle = \sum_c \sqrt{\frac{d_c^2 \dim \psi_{xN}^c \dim \psi_{xN}^c}{P_d N^M^c \prod_i d_i^{N^c}}} \frac{\dim \psi_{xN}^c}{\sqrt{\dim \psi_{xN}^c M^c}} \tag{78}$$

$$= \sum_c d_c \sqrt{\frac{d_c^2}{D^2} \left(1 + O(w_c^N) \right) \left(1 + O(z_c^N) \right)} \frac{\dim \psi_{xN}^c}{M^c} \tag{79}$$

$$= \sum_c d_c \sqrt{\frac{d_c^2}{D^2} \left(1 + O(w_c^N) \right) \left(1 + O(z_c^N) \right)} \frac{1 + O(w_c^N)}{1 + K|w|^N}. \tag{80}$$

Here we used $\dim \psi_{xN}^c = \prod_i d_i^{N^c} d_c \left(1 + O(z_c^N) \right)$, where $|z_c| < 1$. Hence, since $|w_c|, |z_c|, |w| < 1$, we find that the absolute value of the inner product goes to $1$ for $N \to \infty$.

The asymptotic rate of this distillation protocol is then given by

$$\frac{L}{N} \to \infty, \delta \to 0 \Rightarrow \frac{E_{\tilde{\psi}}^N(|\tilde{\psi}|)}{E_{\tilde{\psi}}^N(|\psi_{\max}|)}. \tag{81}$$

and the success probability is $(1 - \epsilon)(1 - 2^{-\log(\chi)} - \chi N \log d_c^2)$ and goes to 1 for $N \to \infty$ and $\epsilon \to 0$.

### B. Entanglement Dilution

Usual entanglement dilution protocols for qubits are based on quantum teleportation [7]. However, quantum teleportation of anyonic systems has not been established yet. In the following, we thus derive the rate of a dilution protocol without using teleportation but instead, majorization and LOCC convertibility.

We start from $L$ copies of the maximally entangled state

$$\langle \psi_{\max}^\otimes \rangle = \sum_c \sqrt{\frac{\dim \psi_{xN}^c}{(d_c^2)(d_{N^c})}} \left[ \frac{1}{\sqrt{\dim \psi_{xN}^c M^c}} \sum_{y} \langle y^f | A_N^c \langle y^f | B_N^c \right]. \tag{82}$$

Our purpose is to make $N$ copies of $|\tilde{\psi}\rangle$ by using LOCC. Let us consider the normalized state obtained by the projection onto the typical subspace corresponding to $T_\delta^N$ from $[53]$

$$|\psi_{\typ}^N\rangle = \sum_c \sqrt{Q^c} \left[ \sum_{a,b\in T_\delta^N} \sum_{b_\delta} \langle p_a \cdots p_{a_{N^c}} | d_c \cdots d_{a_{N^c}} | \lambda_{a_{1}} \cdots \lambda_{i_{N^c}} | a, b_\delta^c, c, i_\delta \rangle A_N^c | \tilde{a}, b_\delta^c, c, i_\delta \rangle B_N^c \right]. \tag{83}$$

where $Q^c = \sum_{a, i_\delta} \sum_{b_\delta} \langle p_a \cdots p_{a_{N^c}} | d_c \cdots d_{a_{N^c}} | \lambda_{a_{1}} \cdots \lambda_{i_{N^c}} \rangle$. The schmidt rank of each sector $c$ is bounded by

$$\sum_{a, i_\delta} \sum_{b_\delta} \dim \psi_{a}^c \leq \sum_{a, i_\delta} \dim \psi_{a}^c \sum_{b_\delta} \prod_{e'=1}^{p_{a_{N^c}}} \frac{d_{a_{e'}}}{d_{a_{e'}}} (1 + O(w_e^N)) = |T_\delta^N| \prod_{e=1}^{p_{a_{N^c}}} \frac{d_{a_{e'}}}{d_{a_{e'}}} (1 + O(w_e^N)), \tag{84}$$

where the first inequality is due to $[18]$ and the fact that $(a, i_\delta) \in T_\delta^N$. Since $|T_\delta^N|$ can be bounded by $2^{N(H(p) + \delta)}$ (see, e.g., [5]), we can further bound

$$\sum_{a, i_\delta} \sum_{b_\delta} 1 \leq 2^{N(H(p) + \delta)} 2^{N(p_a + \delta \log d_a)} \frac{d_c}{D^2} (1 + O(w_c^N)) \tag{85}$$

$$= 2^{N(E_{\tilde{\psi}}^N(|\tilde{\psi}|) + \delta (1 + \sum_{a} \log d_a))} \frac{d_c}{D^2} (1 + O(w_c^N)) \tag{86}.$$
If we choose now $L$ as

$$L = \left\lfloor \frac{N (E^\infty_A(|\psi\rangle) + \delta(1 + \sum a \log d_a))}{\log d_x^a} + 1 \right\rfloor$$  \hspace{1cm} (87)$$

we find by using (86) for large $N$ that

$$\dim V_{x,L}^c = \frac{d_c 2^{L \log d_c} \mathcal{O}(z^N_c))}{\mathcal{D}^2} \geq \sum_{a,i_c \in T^N_c} \dim V^c_a.$$  \hspace{1cm} (88)$$

Therefore, for all $c \in \mathcal{L}$ and large $N$,

$$\left\{ \frac{1}{\dim V_{x,L}^c} \right\} \leq \left\{ \frac{p_{a_1} \cdots p_{a_N} d_c}{P^n_d Q^c d_{a_1} \cdots d_{a_N}} \lambda_{i_{a_1}} \cdots \lambda_{i_{a_N}} \right\},$$  \hspace{1cm} (89)$$

where the left hand side is a uniform distribution and $\preceq$ indicates that the distribution on the right hand side majorizes the one on the left hand side. Moreover, we extended trivially the size of the domain of the probability distribution on the right hand side to match the left hand side. By the LOCC convertibility theorem under SSRs [8], we can convert $L$ copies of the maximally entangled state to $|\psi^N_{\text{typ}}\rangle$ deterministically. There is no error in this protocol and for any $\epsilon > 0$, we obtain that

$$|\tilde{\psi}^N_c\rangle = \sum_c \sqrt{\dim V_{x,L}^c} d_c \left[ \sum_{a,i_c \in T^N_c} \sum_{b^c_a} \frac{p_{a_1} \cdots p_{a_N} d_c}{P^n_d Q^c d_{a_1} \cdots d_{a_N}} \lambda_{i_{a_1}} \cdots \lambda_{i_{a_N}} |\bar{a},b^c_a,c,i_c\rangle_{A_N}|\bar{a},b^c_a,c,i_c\rangle_{B_N} \right].$$  \hspace{1cm} (90)$$

By simple calculation, we can bound the fidelity between $|\tilde{\psi}^N_c\rangle$ and $|\psi^N\rangle$ by

$$F(|\tilde{\psi}^N_c\rangle, |\psi^N\rangle) \geq (1 - \epsilon)^2 (1 - K^c)(1 - K^c)|\tilde{\psi}^N_c\rangle \xrightarrow{N \to \infty, \epsilon \to 0} 1.$$  \hspace{1cm} (91)$$

The asymptotic rate of this protocol is given as claimed by

$$\frac{L}{N} \xrightarrow{N \to \infty, \delta \to 0} \frac{E^\infty_A(|\psi\rangle)}{E^\infty_A(|\psi^\text{max}\rangle)}.$$  \hspace{1cm} (92)$$

C. Optimality

Finally, we show that the asymptotic rate of the previous distillation and dilution protocol is optimal. The argument is the same as to show optimality for the qubit case [7]. Let us replace $L$ in (77) by $L_D$ and in (87) by $L_C$. In the previous distillation protocol, we obtain $L_D$ copies of the maximally entangled state from $N$ copies of $|\psi\rangle$ (with small errors). Let us assume that there exists a distillation protocol which performs strictly better and obtains $|L_D + \xi N\rangle (\xi > 0)$ copies of the maximally entangled state from $N$ copies of $|\psi\rangle$. By using the previous dilution protocol, we can obtain at least $N'$ copies of $|\psi\rangle$, where $N'$ is given by

$$N' = \left\lfloor \frac{|L_D + \xi N\rangle}{L_C} \right\rfloor N.$$  \hspace{1cm} (93)$$

Therefore,

$$\lim_{N' \to N} \frac{N'}{N} \geq \frac{E^\infty_A(|\psi\rangle) + \xi \log d_x^a}{E^\infty_A(|\psi\rangle)} > 1,$$  \hspace{1cm} (94)$$

and thus, in the limit of $N \to \infty$, $\delta$, $\epsilon \to 0$, we can increase the amount of entanglement by LOCC. This conflicts the property of LOCC implying that such a protocol does not exist. Therefore, the asymptotic rate of the distillation protocol is optimal. Similarly, one can prove that the asymptotic rate of the dilution protocol is optimal.
VI. NON-ABELIAN TOTAL CHARGE AND ANYONIC PURIFICATION

In the main text, we restricted ourselves to system with total charge 1 in which the Hilbert space can be written as $\mathcal{H}^1 = \bigoplus_{\mu \in \mathcal{L}} \mathcal{H}_A^\mu \otimes \mathcal{H}_B^\mu$. Let us now give an outlook what happens if we relax this condition and allow a general total charge $c$. In this situation, the Hilbert space is given by

$$\mathcal{H}^c = \bigoplus_{a,b \in \mathcal{L}} \mathcal{H}_A^a \otimes \mathcal{H}_B^b \otimes V_{ab}^c,$$

where $\dim V_{ab}^c$ can be strictly larger than 1. There are several problems arising in this situation among them the impossibility to generally define a state on the combination of two systems with charge not equal 1 by knowing only the state on the partial systems. Or more physically, there must have been a joint preparation of the combined system since the splitting into subsystems with charge $c$ and $\bar{c}$ does not have tensor product structure, and thus, does not allow local preparations.

In order to deal with this problems, it is convenient to complement the system by a reference system with total charge given by the anti-label $\bar{c}$ such that the combined system has again charge 1. We call this the anyonic purification and it leads to a total Hilbert space

$$\tilde{\mathcal{H}}^1 = \bigoplus_{a,b \in \mathcal{L}} \mathcal{H}_A^a \otimes \mathcal{H}_B^b \otimes V_{ab}^c \otimes V_{\bar{c}\bar{c}}^1,$$

including the 1-dimensional Hilbert space $V_{\bar{c}\bar{c}}^1$. A similar method to treat non-trivial total charges has already been discussed in [9]. Graphically, the anyonic purification is illustrated in FIG. 3. Note that since $V_{\bar{c}\bar{c}}^1$ is only one-dimensional every state on the system with total charge $c$ allows a unique extension to the purified system $\tilde{\mathcal{H}}^1$ up to a global phase which can be neglected.

![FIG. 3: The right hand side illustrates the anyonic purification of the system on the left hand side with total charge $c$.](image)

By using this purified system the total charge is now 1. Thus, two purified systems can be combined in the same way as described in the main text and the joint state is uniquely defined. Note that this construction corresponds exactly to the requirement that the two states are independently prepared. From an operational perspective, this local preparation can only be achieved by the creation of a state from vacuum and then discarding part of it. However, the operations and manipulations on the restricted system are independent on the extension so that the anyonic purification is always sufficient.

This then allows one to define multiple copies of states of systems with arbitrary total charge such that the AEE can be defined. But in general $A \neq B$ because of the existence of $V_{\bar{c}a}^c$. Therefore, if the total charge $c$ is nontrivial, the AEE can be asymmetric even if the state is pure, that is,

$$E_A^\infty(\ket{\psi}) \neq E_B^\infty(\ket{\psi}).$$

Clearly in this case, Theorem 2 does not hold anymore. In particular, two independent pure states with non-abelian total charge can behave like a mixed state. This is due to the fact that we cannot access the reference system of the anyonic purification and thus, the total charge of the combined state is a superposition of different labels. For this reason, the analysis of bipartite entanglement for these “anyonically mixed” states is more subtle.

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