LAGRANGIAN KLEIN BOTTLES IN $\mathbb{R}^{2n}$

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The $n$-dimensional Klein bottle $K^n$, $n \geq 2$, is obtained by gluing the ends of the cylinder $S^{n-1} \times [0, 1]$ via an orientation reversing isometry of the standard $(n-1)$-sphere $S^{n-1} \subset \mathbb{R}^n$.

**Theorem.** The $n$-dimensional Klein bottle $K^n$ admits a Lagrangian embedding into the standard symplectic $2n$-space $(\mathbb{R}^{2n}, \omega_0)$ if and only if $n$ is odd.

The existence of Lagrangian embeddings of odd-dimensional Klein bottles into $(\mathbb{R}^{2n}, \omega_0)$ was proved by Lalonde [10]. As observed in [14], an explicit embedding is suggested by Picard–Lefschetz theory. Indeed, the antipodal map

$$\mathbb{R}^{2k+1} \ni S^k \ni (x_1, x_2, \ldots, x_{2k+1}) \mapsto (-x_1, -x_2, \ldots, -x_{2k+1}) \in S^{2k} \subset \mathbb{R}^{2k+1}$$

reverses the orientation on $S^{2k} \subset \mathbb{R}^{2k+1}$ and therefore the formula

$$S^{2k} \times [0, 1] \ni (x_1, \ldots, x_{2k+1}, t) \mapsto (e^{\pi it} x_1, \ldots, e^{\pi it} x_{2k+1}) \in \mathbb{C}^{2k+1}$$

defines an embedding of the odd-dimensional Klein bottle $K^{2k+1}$ into $\mathbb{C}^{2k+1} = \mathbb{R}^{4k+2}$. It is easy to check that this embedding is Lagrangian with respect to the standard symplectic form $\omega_0 = \frac{i}{2} \sum dz_\ell \wedge d\bar{z}_\ell$.

Thus, the main task is to prove that an even-dimensional Klein bottle does not admit a Lagrangian embedding into $(\mathbb{R}^{2n}, \omega_0)$. For the usual Klein bottle $K^2$, this problem was proposed by Givental’ [6] and resolved by Shevchishin [10]. The argument in the general case follows the lines of the author’s alternative proof of Shevchishin’s result [15] (cf. also [4]). Namely, self-linking invariants introduced by Rokhlin and Viro are used to show that a suitable Luttinger-type surgery along a Lagrangian $K^{2k} \subset \mathbb{R}^{4k}$ would produce an impossible symplectic manifold.

1. **Rokhlin and Viro indices for totally real Klein bottles.** Let us fix $n$ and denote the $n$-dimensional Klein bottle simply by $K$. Let $m \subset K$ be a fibre of the natural fibre bundle $K \to S^1$. Then $m$ is an embedded $(n-1)$-dimensional sphere in $K$. Note that $m$ is co-orientable and choose a non-vanishing normal vector field $\nu_{m,K}$ on $m$.

Consider now a totally real embedding $K \hookrightarrow \mathbb{C}^n$, i.e., an embedding such that $T_p K$ is transversal to $iT_p K$ at every point $p \in K$. (For a Lagrangian embedding, the subspaces $T_p K$ and $iT_p K$ would be orthogonal with respect to the standard metric on $\mathbb{C}^n$.) Let $m^\sharp$ be the pushoff of $m$ in the direction of the vector field $i\nu_{m,K}$. The mod 2 homology class $[m^\sharp] \in H_{n-1}(\mathbb{C}^n \setminus K; \mathbb{Z}/2)$ is independent of the choice of $\nu_{m,K}$. The linking number

$$V = \text{lk}(K, m^\sharp) \in \mathbb{Z}/2$$

is called the **Viro index** of $m \subset K$ (cf. [15], §1.2).

In order to compute $V$, we choose an immersed $n$-ball $M = \iota(B^n) \subset \mathbb{C}^n$ such that

a) $\partial M = \iota(\partial B^n) = m$, and $M$ is normal to $K$ along $m$;

b) the self-intersections of $M$ and the intersections of its interior with $K$ are transverse double points;

c) the tangent (half-)space of $M$ at a point $p \in m = \partial M$ is spanned by $T_p m$ and $i\nu_{m,K}$;

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d) the $\mathbb{R}C$-singular points of $M$ are generic (see the definitions in §3 below). (Immersions satisfying (a) and (b) are called membranes spanned by $m$.) Note that by (c) the pushoff of $m$ inside $M$ is precisely $m^\sharp$, and hence

$$V = \#(M \cap K) \mod 2,$$

(2)

where $\#(M \cap K)$ denotes the number of interior intersection points of $M$ and $K$.

Suppose now that $n$ is even and consider the Rokhlin index of $M$ defined by the formula

$$R = n(M, \nu_{m,K}) + \#(M \cap K),$$

(3)

where $n(M, \nu_{m,K}) \in \mathbb{Z}$ is the obstruction to extending $\nu_{m,K}$ to a non-vanishing normal vector field on $M$, i.e., the algebraic number of zeroes of a generic normal extension. (For odd $n$, this number is defined only mod 2.)

**Lemma 1.** $R = 0 \mod 2$.

This will be proved in §2 using nothing much. Note, however, that this is the only place where the assumption that $n$ is even will be used in a crucial way (see Remark 5).

**Lemma 2.** $n(M, \nu_{m,K}) = 1 \mod 2$.

This will be proved in §4 using a topological count of $\mathbb{R}C$-singularities recalled briefly in §3 following Domrin [2].

**Lemma 3.** $V = 1 \mod 2$.

This follows immediately from formulas (2) and (3) and the preceding two lemmas and will play a key role in the proof of the main theorem in §5 and §6.

2. **Proof of Lemma 1.** Cut $K$ along $m$ and glue two copies of $M$ into the resulting ‘holes’ to obtain an $n$-sphere $S$. Choose an orientation on $S$ and note that it induces the same orientation on each of the two copies of $M$. (If we had $S^{n-1} \times S^1$ instead of the Klein bottle, the orientations would be opposite.) Let $\nu$ be a generic normal extension of $\nu_{m,K}$ to $M$. Transform $S$ into a generically immersed sphere by pushing the two copies of $M$ apart in the direction of $\nu$ and then smoothing the result.

Now we can compute the normal Euler number of $S$ and the algebraic number of its double points. Namely,

$$n(S) = n(K) + 2n(M, \nu_{m,K}) = 2n(M, \nu_{m,K}),$$

where we have used the fact that for a totally real embedding of $K$ the normal Euler number is equal to the Euler characteristic of $K$ which is zero. Similarly,

$$\#_{alg}(S) = n(M, \nu_{m,K}) + 2 \#_{alg}(M \cap K) + 4 \#_{alg}(M),$$

where the signs in $\#_{alg}(M \cap K)$ and $\#_{alg}(M)$ are given by the induced orientations on $M$ and $K$ as subsets of $S$.

On the other hand, by the usual formula for the homological self-intersection index of an oriented immersed submanifold, we have

$$[S] \cdot [S] = n(S) + 2 \#_{alg}(S) = 4\left(n(M, \nu_{m,K}) + \#_{alg}(M \cap K) + 2 \#_{alg}(M)\right).$$

The homology class $[S]$ is obviously trivial in $\mathbb{C}^n$, hence

$$n(M, \nu_{m,K}) + \#_{alg}(M \cap K) + 2 \#_{alg}(M) = 0,$$

and the result follows from (3) because $\#_{alg}(M \cap K) = \#(M \cap K) \mod 2$. □
Remark 4. The above argument and the result for \( n = 2 \) go back to Rokhlin (see [8] and the proof of Lemma 1.12 in [13]). Note that we are actually proving a congruence modulo 8 using a trivial case of van der Blij’s lemma to conclude that \([S] \cdot [S] = 0 \mod 8\).

Remark 5. For odd \( n \), the residue \( R \mod 2 \) is well-defined but the lemma is false. (Our proof does not work because the intersection index is not symmetric.) For instance, for the embedding given by (1), the totally real \( n \)-ball \( \{(z_1, \ldots, z_n) \in \mathbb{C}^n \mid z_j \in \mathbb{R}, |z| \leq 1\} \) is a membrane satisfying conditions (a)-(d) and such that \( R = 1 \mod 2 \).

3. \( \mathbb{R}C \)-singularities and characteristic classes. Here’s a digression needed for the proof of Lemma 2. The material is mostly taken from [2] (cf. also [9] and [17]).

Let \( j : N \to \mathbb{C}^n \) be an immersion of a real oriented \( n \)-dimensional manifold. (Note that the real dimension of \( N \) is equal to the complex dimension of \( \mathbb{C}^n \).) A point \( p \in N \) is called \( \mathbb{R}C \)-singular if the dimension of the maximal complex subspace in \( j_\ast T_p N \subset \mathbb{C}^n \) is positive (i.e., larger than expected). This dimension is called the order of an \( \mathbb{R}C \)-singular point. Denote by \( C_\mu(N) \) the set of \( \mathbb{R}C \)-singular points of order \( \mu \) and by \( C(N) \) the set of all \( \mathbb{R}C \)-singular points.

Let \( j_C^\ast : TN \otimes \mathbb{C} \to j_\ast T\mathbb{C}^n \) be the complex vector bundle map given by \( j_C^\ast(v \otimes \lambda) := j_\lambda(v) \). Its kernel at a point \( p \in N \) is isomorphic to the maximal complex subspace of \( j_\ast T_p N \). Thus, \( C_\mu(N) \) coincides with the singularity set \( \Sigma_\mu = \{ p \in N \mid \text{rk}_C j_C^\ast = n - \mu \} \). If the immersion \( j \) is generic, then by \[2\], Lemma 1.3, the bundle map \( j_C^\ast \) is generic in the sense of [13]. Hence, each \( C_\mu(N) \) is an oriented \( (n - 2\mu^2) \)-dimensional submanifold, \( C(N) = \overline{C_1(N)} \), and there exists a canonical desingularisation \( \widetilde{\Sigma}_1 \to \Sigma_1 = C(N) \) such that the complex line bundle \( \ker j_C^\ast|_{\Sigma_1} \) extends to \( \widetilde{\Sigma}_1 \). (Explicitly, \( \widetilde{\Sigma}_1 \) is the closure of the image of \( \ker j_C^\ast|_{\Sigma_1} \) in the projectivisation of \( TN \otimes \mathbb{C} \) and the extension of \( \ker j_C^\ast|_{\Sigma_1} \) is given by the tautological line bundle.) The extended bundle lies in the kernel of the pull-back of \( j_C^\ast \) to \( \widetilde{\Sigma}_1 \) and therefore corresponds to a complex line subbundle \( Z \) of the pull-back of \( TN \).

Assume now that the manifold \( N \) is compact without boundary and its dimension \( n \) is even. Define \([C(N)]\) as the fundamental class of the oriented manifold \( \widetilde{\Sigma}_1 \). Then from Theorem 3 and Remark 1.4 in [2] one obtains the formula

\[
\langle c_1(Z)^{(n/2 - 1)}, [C(N)] \rangle = \langle c_{n/2}(-TN \otimes \mathbb{C}), [N] \rangle,
\]

where \(-TN \otimes \mathbb{C}\) denotes the \( K \)-theoretic inverse of \( TN \otimes \mathbb{C} \). In the statements of the results in [2] it is assumed that \( C(N) = C_1(N) \) but the proofs carry over to the general case with formal changes. (\( \widetilde{\Sigma}_1 \) has to be used instead of the set \( \Sigma \) introduced on p. 910 of [2].)

Remark 6. The \( \mathbb{R}C \)-singular points of a generic immersed surface in \( \mathbb{C}^2 \) are isolated (and more often referred to as ‘complex points’ or ‘complex tangencies’). Formula (1) reduces in this case to the elementary formula \( I_+ - I_- = 0 \), where the Laplace indices \( I_\pm \) of an oriented immersed surface are defined by counting its complex points with suitable signs (see [2], §3).

4. Proof of Lemma 2 (cf. [15], Proof of Lemma 1.13). Let us construct a normal extension of \( \nu_{m,K} \) to \( M \) in the following way. Consider the vector field \( i\nu_{m,K} \). It is tangent to \( M \) and transverse to \( \partial M \) by the choice of \( M \). Let \( \tau \) be an extension of this vector field to a tangent vector field on \( M \) with a single transverse zero. (Recall that \( M \) is a ball.) Then \(-i\tau \) gives a normal extension of \( \nu_{m,K} \) that vanishes at the zero of \( \tau \) and at the points where \( \tau \) lies in a non-trivial complex subspace contained in \( T_p M \). For a sufficiently generic \( \tau \), the latter points lie in \( C_1(M) \). In other words, we have to count the zeroes of a (generic) section of the quotient bundle \( E = \widetilde{T}M/Z \), where \( \widetilde{T}M \) is the pull-back of \( TM \) to \( \widetilde{\Sigma}_1 \). As we only need the answer mod 2, it is given by the evaluation of the top Stiefel–Whitney class \( w_{n-2}(E) \) on the fundamental class \([C(M)] := [\widetilde{\Sigma}_1]\).
Since $TM$ is trivial, we have
\[ 1 = (1 + w_2(Z))(1 + w_1(E) + \cdots + w_{n-2}(E)) \] (5)
by the Whitney formula. It follows immediately that
\[ w_{n-2}(E) = w_2(Z)^{(n/2-1)} \]
and hence
\[ \langle w_{n-2}(E), [C(M)] \rangle = \langle w_2(Z)^{(n/2-1)}, [C(M)] \rangle = \langle c_1(Z)^{(n/2-1)}, [C(M)] \rangle \mod 2. \]

In order to show that the latter quantity vanishes (already as an integer), we apply formula (4) to an immersed sphere $S$ similar to the one used in the proof of Lemma 1 above. Namely, we glue two copies of $M$ to $K$ cut along $m$ but this time only smoothen the result near $m$. Condition (c) in (4) ensures that this smoothing can be done so that no additional $\mathbb{R}C$-singularities are created and hence the set $C(S)$ consists of two copies of $C(M)$ with the same orientation and the same line bundle $Z$. Thus,
\[ 2\langle c_1(Z)^{(n/2-1)}, [C(M)] \rangle = \langle c_1(Z)^{(n/2-1)}, [C(S)] \rangle \equiv \langle c_{n/2}(-TS \otimes \mathbb{C}), [S] \rangle = 0. \]

It follows that $\langle w_{n-2}(E), [C(M)] \rangle = 0 \mod 2$ and hence the normal projection of $-i\tau$ has an odd number of zeroes, which proves that $n(M, \nu_{m,K}) = 1 \mod 2$. \hfill $\square$

\textbf{Remark 7.} For odd $n$, the vanishing of $w_{n-2}(E)$ follows already from (5) without any appeal to (4). Thus Lemma 2 is true in that case as well.

5. **Dehn surgery.** Let $U \supset K$ be a tubular neighbourhood of a totally real embedded Klein bottle $K \subset \mathbb{C}^n$. We consider two distinguished classes in the homology group $H_{n-1}(\partial U; \mathbb{Z}/2)$. Firstly, the fibre class $[\delta]$ generating the kernel of the inclusion homomorphism $H_{n-1}(\partial U; \mathbb{Z}/2) \to H_{n-1}(U; \mathbb{Z}/2)$ and, secondly, the class $[m^2]$ of the $\mathbb{C}$-normal pushoff of $m$ introduced in (4).

\textbf{Lemma 8 (cf. [15], Theorem 2.2).} Consider a surgery $X = \overline{U} \cup_f (\mathbb{C}^n \setminus U)$ defined by a diffeomorphism $f : \partial U \to \partial U$ such that
\[ f_*[\delta] = [\delta] + [m^2]. \] (6)

If $n$ is even, then $K$ is homologically non-trivial in $X$. In particular, $H_n(X; \mathbb{Z}/2) \neq 0$.

\textbf{Proof.} As $n$ is even, we know that $lk(K, m^2) = 1 \mod 2$ by Lemma 3 and the definition of the Viro index. Since $lk(K, \delta) = 1 \mod 2$ by definition, it follows that the sum $[\delta] + [m^2]$ bounds a mod 2 chain in $\mathbb{C}^n - U$. By property (6), this chain and the $n$-ball bounded by $\delta$ in $\overline{U}$ are glued into a mod 2 cycle in $X$ whose intersection index with $K$ is 1 mod 2. \hfill $\square$

\textbf{Lemma 9.} If $X$ is orientable, then $H_2(X; \mathbb{R}) = 0$.

\textbf{Proof.} Note first that $H_2(X; \mathbb{R}) = H^{2n-2}_c(X; \mathbb{R})$ by Poincaré(-Lefschetz) duality. Since $X \setminus K = \mathbb{C}^n \setminus K$, an inspection of the cohomology long exact sequences
\[ \cdots \to H^{2n-3}_c(K; \mathbb{R}) \to H^{2n-2}_c(\mathbb{C}^n \setminus K; \mathbb{R}) \to H^{2n-2}_c(\mathbb{C}^n; \mathbb{R}) \cong 0 \]
\[ \cdots \to H^{2n-2}_c(X \setminus K; \mathbb{R}) \to H^{2n-2}_c(X; \mathbb{R}) \to H^{2n-2}_c(K; \mathbb{R}) \cong 0 \]
shows that $\dim_{\mathbb{R}} H^{2n-2}_c(X; \mathbb{R}) = \dim_{\mathbb{R}} H^{2n-3}_c(K; \mathbb{R})$. Thus, $\dim_{\mathbb{R}} H^{2n-2}_c(X; \mathbb{R})$ is zero for all $n \geq 3$ and does not exceed one for $n = 2$. In the latter case, however, it follows from Euler characteristic additivity that the dimension of $H^2_c(X; \mathbb{R})$ is even and hence also equals zero. \hfill $\square$
6. Symplectic rigidity. Proof of the main result. If the surgery in Lemma 8 were symplectic (i.e., there were a symplectic form on \(X\) restricting to \(\omega_0\) on \(U\) and \(\mathbb{C}^n \setminus \mathcal{U}\)), then the conclusions of Lemmas 8 and 9 for an even \(n\) would contradict the following result:

**Theorem 10** (Eliashberg–Floer–McDuff [12], [3]). Let \((X, \omega)\) be a symplectic manifold symplectomorphic to \((\mathbb{R}^{2n}, \omega_0)\), \(n \geq 2\), outside of a compact subset. Assume that \([\omega]\) vanishes on all spherical elements in \(H_2(X; \mathbb{R})\). Then \(X\) is diffeomorphic to \(\mathbb{R}^{2n}\).

**Remark 11.** If \(n = 2\), then \(X\) is actually symplectomorphic to \((\mathbb{R}^4, \omega_0)\) by Gromov’s classical result [7]. Note, however, that we only need to know that \(X\) must have the \(\mathbb{Z}/2\)-homology of the ball, which is proved in all dimensions by a basic application of pseudoholomorphic curves (see [12], §3.8).

Thus, to prove the main theorem it remains to show that for a Lagrangian embedding of the Klein bottle \(K\) there exists a symplectic surgery having property \(\mathcal{G}\). This can be done in all dimensions by the following elementary construction.

Represent \(K\) as the quotient of \(\mathbb{R}^n \setminus \{0\}\) by the \(\mathbb{Z}\)-action generated by the transformation

\[x \mapsto 2\sigma(x),\]

where \(\sigma \in O_-(\mathbb{R}^n)\) is a reflection (in particular, \(\sigma = \sigma^T = \sigma^{-1}\)). The cotangent bundle \(T^*K\) is the quotient of \(T^*(\mathbb{R}^n \setminus \{0\}) \cong (\mathbb{R}_x^* \setminus \{0\}) \times \mathbb{R}_y^n\) by the \(\mathbb{Z}\)-action generated by

\[(x, y) \mapsto (2\sigma(x), \frac{1}{2}\sigma(y)).\]

Note that the Riemannian metric \(g = \frac{1}{|x|^2} \sum dx_i^2\) on \(\mathbb{R}^n \setminus \{0\}\) is invariant with respect to \((\mathbb{R}^n \setminus \{0\})\) and equip \(K\) with the induced metric. Note further that the unit sphere bundle \(ST^*(\mathbb{R}^n \setminus \{0\}) \subset T^*(\mathbb{R}^n \setminus \{0\})\) with respect to \(g\) is the hypersurface \(\{\|y\|^2 = 1/\|x\|^2\}\).

On \(T^*(\mathbb{R}^n \setminus \{0\})\) with the zero section removed, consider the map

\[(x, y) \mapsto (-y, x).\]

Obviously, this map preserves the unit sphere bundle \(ST^*(\mathbb{R}^n \setminus \{0\})\) and the canonical symplectic form on \(T^*(\mathbb{R}^n \setminus \{0\})\). Furthermore, it maps the orbits of the action \((\mathbb{R}^n \setminus \{0\})\) into orbits. Hence, it defines a symplectomorphism of \(T^*K\) with the zero section removed that maps \(ST^*K\) into itself.

Let us check that the action of the map \((\mathbb{R}^n \setminus \{0\})\) on \(H_{n-1}(ST^*K; \mathbb{Z}/2)\) satisfies condition \((\mathcal{G})\). The fibre class \([\delta]\) is represented by the ‘vertical’ \((n - 1)\)-sphere \(\{x = \text{const}, \|y\| = 1\}\) and its image is obviously the class of the ‘horizontal’ \((n - 1)\)-sphere \(\{\|x\| = 1, y = \text{const}\}\). Choose \(m = \{\|x\| = 1\} \subset K\) and \(\nu_{m,K}(x) = x\). For any almost complex structure on \(T^*K\) compatible with the canonical symplectic form, the isotopy class of the \(\mathbb{C}\)-normal pushoff \(m^2 = m + J\nu_{m,K}\) is the same as for the standard complex structure, i.e., it is given by the ‘diagonal’ \((n - 1)\)-sphere \(\{y = x, \|x\| = 1\} \subset ST^*K\). It follows immediately that the image of \([\delta]\) with respect to \((\mathbb{R}^n \setminus \{0\})\) is \([\delta] + [m^2]\), as required.

Finally, if \(K\) is an embedded Lagrangian Klein bottle in a symplectic manifold, we can identify its closed tubular neighbourhood \(U\) with the unit disc bundle \(DT^*K\) by a conformally symplectic diffeomorphism and define the gluing map \(f : \partial U \to \partial U\) as the restriction of the symplectomorphism constructed above to \(ST^*K\).

**Remark 12.** Replacing the action \((\mathbb{R}^n \setminus \{0\})\) by \(x \mapsto 2x\), one obtains a completely analogous surgery construction for the product \(S^{n-1} \times S^1\). Further symplectic surgeries along a Lagrangian Klein bottle or \(S^{n-1} \times S^1\) can be defined by taking the gluing map from the group generated by the map \(f\) induced by \(\mathcal{G}\) and the map \(\tau\) induced by the co-differential of the topologically non-trivial \(g\)-isometry \(x \mapsto \frac{x}{\|x\|^2}\).
Remark 13 (Comparison with Luttinger surgery). (i) In the case of the product $S^{n-1} \times S^1$, the surgeries found by Luttinger [11] for $n = 2$ and by Borrelli [1] for $n = 4$ and $n = 8$ correspond to the gluing maps $(f \circ \tau)^k$, where $k \in \mathbb{Z}$ and the maps $f$ and $\tau$ are defined as in Remark 12. (ii) The surgery used in [15] in the case of the usual Klein bottle $K^2$ corresponds to the gluing map $(f \circ \tau)^{-1}$. In the notation of [15], one has

$$f(\varphi, \psi, \theta) = (-\varphi, \psi + \theta + \pi, -\theta - \pi) \quad \text{and} \quad \tau(\varphi, \psi, \theta) = (-\varphi, \psi, -\theta - \pi)$$

so that $f \circ \tau = f_{0,-1}$. (iii) There is an alternative description of these surgeries in terms of regluing Lefschetz pencils via fibrewise symplectic Dehn twists (see the first draft of this paper, arxiv:0712.1760v1, and the references therein).

Remark 14 (Totally real embeddings). It is perhaps worth mentioning that totally real embeddings $K^n \hookrightarrow \mathbb{C}^n$ exist for all $n$. Indeed, Lalonde [10] constructed Lagrangian immersions $K^n \hookrightarrow \mathbb{C}^n$ that are regularly homotopic to embeddings. The existence of totally real embeddings follows in this situation from Gromov’s $h$-principle (see, e.g., [5], §19.3).

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