FINITE DIFFERENCE OPERATORS WITH A FINITE–BAND
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Abstract. We study the correspondence between almost periodic difference operators and algebraic curves (spectral surfaces). An especial role plays the parametrization of the spectral curves in terms of, so called, branching divisors. The multiplication operator by the covering map with respect to the natural basis in the Hardy space on the surface is the \(2d+1\)-diagonal matrix; the \(d\)-root of the product of the Green functions (counting their multiplicities) with respect to all infinite points on the surface is the symbol of the shift operator. We demonstrate an application of our general construction to a particular covering, which generate widely discussed almost periodic CMV matrices. We discuss an important theme: covering of one spectral surface by another one and related to this operation transformations on the set of multidirectional operators (so called Renormalization Equations). We proof several new results dealing with Renormalization Equations for periodic Jacobi matrices (polynomial coverings) and the case of a rational double covering.

1. Introduction

1.1. Ergodic finite difference operators and associated Riemann surfaces. The standard (three–diagonal) finite–band Jacobi matrices [9, 26] can be defined as almost periodic or even ergodic Jacobi matrices with absolutely continuous spectrum that consists of a finite system of intervals. We wish to find a natural extension of this class of finite difference operators to the multi-diagonal case.

The correspondence between periodic difference operators and algebraic curves (spectral surfaces) was discussed in detail in [17]. Let us generalize this construction to show (at least on a speculative level) how almost periodic or ergodic operators give rise to a corresponding spectral surface.

Recall the definition of an ergodic operator [7, 18], see also [28]. Let \((\Omega, \mathfrak{A}, d\chi)\) be a separable probability space and let \(\mathcal{T} : \Omega \to \Omega\) be an invertible ergodic transformation, i.e., \(\mathcal{T}\) is measurable, it preserves \(d\chi\), and every measurable \(\mathcal{T}\)–invariant set has measure 0 or 1. Let \(\{q^{(k)}\}_{k=0}^{d}\) be functions from \(L_{d\chi}^{\infty}\), with \(q^{(d)}\) positive–valued and \(q^{(0)}\) real–valued. Note that in the periodic case \(\Omega = \mathbb{Z}/N\mathbb{Z}\), where \(N\) is the period and \(\mathcal{T}\{n\} = \{n+1\}, \{n\} \in \mathbb{Z}/N\mathbb{Z}\).

Then with almost every \(\omega \in \Omega\) we associate a self–adjoint \(2d+1\)–diagonal operator \(J(\omega)\) as follows:

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\[
(J(\omega)x)_n = \sum_{k=-d}^{d} q_n^{(k)}(\omega)x_{n+k}, \quad x = \{x_n\}_{n=-\infty}^{\infty} \in l^2(\mathbb{Z}),
\]

where \(q_n^{(k)}(\omega) := q^{(k)}(T^n \omega)\) and \(q^{(-k)}(\omega) := q^{(k)}(T^{-k} \omega)\).

Note that the structure of \(J(\omega)\) is described by the following identity
\[
J(\omega)S = SJ(\omega),
\]
where \(S\) is the shift operator in \(l^2(\mathbb{Z})\). The last relation indicates strongly that one can associate with the family of matrices \(\{J(\omega)\}_{\omega \in \Omega}\) a natural pair of commuting operators (in the periodic case one just uses the fact that \(J\) and \(S^{N}\) commute). Namely, let \(L^2_{d\chi}(l^2(\mathbb{Z}))\) be the space of \(l^2(\mathbb{Z})\)-valued vector functions, \(x(\omega) \in l^2(\mathbb{Z})\), with the norm
\[
||x||^2 = \int_{\Omega} ||x(\omega)||^2 d\chi.
\]

Define
\[
(\hat{J}x)(\omega) = J(\omega)x(\omega), \quad (\hat{S}x)(\omega) = Sx(\mathcal{T}\omega), \quad x \in L^2_{d\chi}(l^2(\mathbb{Z})).
\]

Then (2) implies
\[
(\hat{J}\hat{S}x)(\omega) = J(\omega)Sx(\mathcal{T}\omega) = SJ(\mathcal{T}\omega)x(\mathcal{T}\omega) = (\hat{S}\hat{J}x)(\omega).
\]

Further, \(\hat{S}\) is a unitary operator and \(\hat{J}\) is self-adjoint. The space \(L^2_{d\chi}(l^2(\mathbb{Z}^{+}))\) is an invariant subspace for \(\hat{S}\). It is not invariant with respect to \(\hat{J}\) but it is invariant with respect to \(\hat{J}\hat{S}^d\). Put
\[
\hat{S}^+ = \hat{S}|L^2_{d\chi}(l^2(\mathbb{Z}^{+})), \quad (\hat{J}\hat{S}^d)^+ = \hat{J}\hat{S}^d|L^2_{d\chi}(l^2(\mathbb{Z}^{+})).
\]

**Definition 1.1** (local functional model). We say that a pair of commuting operators \(A_1 : H \rightarrow H\) and \(A_2 : H \rightarrow H\) has a (local) functional model if there is a unitary embedding \(i : H \rightarrow H_{O}\) in a space \(H_{O}\) of functions \(F(\zeta)\) holomorphic in some domain \(O\) with a reproducing kernel \((F \mapsto F(\zeta_0), \zeta_0 \in O,\) is a bounded functional in \(H_{O}\)) such that operators \(i_\ast A_1\) and \(i_\ast A_2\) become a pair of operators of multiplication by holomorphic functions, say
\[
A_1 x \mapsto a_1(\zeta)F(\zeta), \quad A_2 x \mapsto a_2(\zeta)F(\zeta).
\]

Existence of a local functional model implies a number of quite strong consequences. In what follows \(a(\zeta)\) and \(b(\zeta)\) denote the functions (symbols) related to the operators \((\hat{J}\hat{S}^d)^+\) and \(\hat{S}^+\). Let \(k_\zeta\) be the reproducing kernel in \(H_{O}\) and let \(\hat{k_\zeta}\) be its preimage \(i^{-1}k_\zeta\) in \(L^2_{d\chi}(l^2(\mathbb{Z}^{+}))\). Then
\[
\langle \hat{S}^+_\hat{k_\zeta}, x \rangle = \langle k_\zeta, \hat{S}^+_x \rangle = \langle k_\zeta, bF \rangle.
\]

By the reproducing property
\[
\langle k_\zeta, bF \rangle = \overline{b(\zeta)F(\zeta)} = \langle \overline{b(\zeta)}k_\zeta, F \rangle.
\]

Hence,
\[
\langle \hat{S}^+_\hat{k_\zeta}, x \rangle = \langle \overline{b(\zeta)}k_\zeta, x \rangle.
\]

That is \(\hat{k_\zeta}\) is an eigenvector of \(\hat{S}^+\) with the eigenvalue \(\overline{b(\zeta)}\). In the same way, \(\hat{k_\zeta}\) is an eigenvector of \((\hat{J}\hat{S})^+_\ast\) with the eigenvalue \(\overline{a(\zeta)}\).
Thus, if a functional model exists then the spectral problem
\[
\begin{cases}
\hat{S}_+^* \hat{k}_\zeta = \overline{b(\zeta) \hat{k}_\zeta} \\
\overline{(JS^d)^*} \hat{k}_\zeta = a(\zeta) \hat{k}_\zeta
\end{cases}
\tag{4}
\]
has a solution \( \hat{k}_\zeta \) antiholomorphic in \( \zeta \). Moreover, linear combinations of all \( \hat{k}_\zeta \) are dense in \( L^2_{d\chi}(l^2(\mathbb{Z}_+)) \). Vice versa, if (4) has a solution of such kind then we define
\[
F(\zeta) := \langle x, \hat{k}_\zeta \rangle, \quad ||F||^2 := ||x||^2.
\]
This provides a local functional model for the pair \( \hat{S}_+, (\hat{J}S)_+ \).

The following proposition is evident.

**Proposition 1.2.** Let \( \mathcal{U} : L^2_{d\chi} \to L^2_{d\chi} \) be the unitary operator associated with the ergodic transformation \( \mathcal{T} : (\mathcal{U}c)(\omega) = c(\mathcal{T}\omega), \ c \in L^2_{d\chi} \). We denote by the same letter \( q \) both a function \( q \in L^2_{d\chi} \) and the multiplication operator \( q(e.g., (qc)(\omega) := q(\omega)c(\omega)) \). Problem (4) is equivalent to the following spectral problem
\[
\sum_{k=-d}^{d} \mathcal{U}^k q^{(k)} b^k(\zeta) c_\zeta = z(\zeta) c_\zeta,
\tag{5}
\]
where \( z(\zeta) := a(\zeta)/b^d(\zeta) \) and \( c_\zeta \) is an anti–holomorphic \( L^2_{d\chi} \)–valued vector function. Moreover \( \{c_\zeta\} \) is complete in \( L^2_{d\chi} \) if and only if \( \{\hat{k}_\zeta\} \) is complete in \( L^2_{d\chi}(l^2(\mathbb{Z}_+)) \).

We may hope to glue the local functional realizations to a global functional model on a Riemann surface \( X_0 = \mathbb{D}/\Gamma_0 \) formed by functions \((z, b)\). Of course, this model does not necessarily exist (even existence of a local model requires some additional assumptions on the ergodic map and the coefficients functions).

But, in particular, in the periodic case, when \( L^2_{d\chi} = \mathbb{C}^N \), we have
\[
\mathcal{U} \begin{bmatrix} c_0 \\ \vdots \\ c_{N-1} \end{bmatrix} = \begin{bmatrix} c_1 \\ \vdots \\ c_0 \end{bmatrix}, \quad \mathcal{U} = \begin{bmatrix} 0 & 1 \\ \vdots & \vdots \\ 1 & 0 \end{bmatrix},
\]
where \( c_\gamma := c(\{n\}) \) and the \( q^{(k)} \)'s became the diagonal matrices. Thus (5) means that the \( N \times N \) matrix has a nontrivial annihilating vector and we arrive at the curve \( X_0 \) in the form:
\[
\det \left[ \sum_{k=-d}^{d} \mathcal{U}^k q^{(k)} b^k - \tau I \right] = 0
\tag{6}
\]
and the restriction \( |b| < 1 \).

The surface \( X_0 \) is in generic case of infinite genus. However we can reduce it because \( X_0 \) possesses a family of automorphisms. Let \( c_{\{\gamma\}} \) be an eigenvector of \( \mathcal{U} \) with an eigenvalue \( \mu_{\{\gamma\}} \). The systems of eigenfunctions and eigenvalues form both Abelian groups with respect to multiplication. Using (5) we get immediately that \( \{\gamma\} : (z, b) \mapsto (z, \mu_{\{\gamma\}} b) \) is an automorphism of \( X_0 \). Taking a quotient of \( X_0 \) with respect to these automorphisms we obtain a much smaller surface \( X = \mathbb{D}/\Gamma, \ \Gamma = \{\gamma \Gamma_0\} \). In periodic case this means that (6) is actually a polynomial expression in \( z, \lambda \) and \( \lambda^{-1} \) where \( \lambda := b^N \) (since it is invariant with respect to the substitutions \( b \mapsto e^{2\pi i \frac{k}{N}} b, \ 0 < k < N \)), see [17], see also Sect. 6.
Note that $z$ is still a function on $X$ but $b$ becomes a character automorphic function. Finally, using $z$ we may glue the boundary of $X$, remove punched points (where $z = \infty$) and get in this way a compact Riemann surface $X_c$, such that

$$X = (X_c \setminus \{ P : z(P) = \infty \}) \setminus E.$$  

The simplest assumption is that the boundary $E$ is a finite system of cuts on $X_c$. In this case we get that the triple $\{ X_c, z, E \}$ characterizes the spectrum of a finite difference operator. We came to this triple basically due to heuristic arguments, but the opposite direction is already a solid mathematical fact: every triple of this kind gives rise to a family of ergodic finite difference operators [17, Sect. 6]. In Sect. 2 and 3 we give details on constructions of such operators using the theory of Hardy spaces on Riemann surfaces.

Now we would like to note that one can describe all triples of a given type up to a natural equivalence relation. This natural parametrization of the triples is one of the main point of the current paper. It allows us to put consideration from the pure algebraic points of view to a wider setting and to recruit very much analytic tools.

1.2. Parametrization of the spectral curves in terms of branching divisors. Probably the best known result in spectral theory of the nature we want to discuss deals with the description of the spectrum of periodic Jacobi matrices. Such a set $E$ should have a form of an inverse polynomial image $E = T^{-1}([-1, 1])$. The polynomial $T$ should have all critical points real $\{ c_k : T'(c_k) = 0 \} \subset \mathbb{R}$, moreover all critical values $t_k := T(c_k)$ should have modulus not less than 1 and their signs should alternate, i.e.: $|T(c_k)| \geq 1$ and

$$T(c_{k-1})T(c_k) < 0 \quad \text{for} \quad \ldots < c_{k-1} < c_k < \ldots .$$

The claim is that the system of numbers $\{ t_k : |t_k| \geq 1, t_{k-1}t_k < 0 \}$, determines a polynomial with the prescribed and ordered critical values $t_k$ uniquely, modulo a change of the independent variable $z \mapsto az + b$, $a > 0, b \in \mathbb{R}$. The proof uses a special representation of $T$:

$$T(z) = \pm \cos \phi(z),$$

where $\phi$ is the conformal map of the upper half plane onto the half strip with a system of cuts:

$$\Pi = \{ w = u + iv : 0 \leq v \leq \pi d \} \setminus \bigcup_k \{ v = \pi k, u \leq h_k \},$$

where $d = \deg T$ and $\cosh(h_k) := |t_k|$, $\phi(\infty) = \infty$.

We mention a more general theorem of MacLane [15] and Vinberg [27] on the existence and uniqueness of real polynomials (actually, and entire functions) with prescribed (ordered!) sequences of critical values. In the case of polynomials, this theorem says that there is a one to one correspondence between finite “up-down” real sequences

$$\ldots \leq t_{k-1} \leq t_k \leq t_{k+1} \geq \ldots ,$$

and real polynomials whose all critical points are real, also, modulo a change of the independent variable $z \mapsto az + b$, $a > 0, b \in \mathbb{R}$. The MacLane–Vinberg theorem is based on an explicit description of the Riemann surfaces spread over the plane of the inverse functions $T^{-1}$.

To be more precise in the general case we start with the following
Definition 1.3. We say that two triples \((X_{c1}, z_1, E_1)\) and \((X_{c2}, z_2, E_2)\) are equivalent if there exists a holomorphic homeomorphism \(h : X_{c1} \to X_{c2}\) such that \(z_1 = h^*(z_2)\) and \(E_2 = h(E_1)\).

Note, that for any triple \((X_c, z, E)\) the holomorphic function \(z : X_c \to \overline{\mathbb{C}}, \overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}\), is a ramified covering of \(\overline{\mathbb{C}}\). The fact is that it is possible and very convenient to describe equivalence classes of ramified coverings in terms of branching divisor. Namely, a point \(P \in X_c\) such that \(\frac{dz}{d\zeta}|_P = 0\) where \(\zeta\) is a local holomorphic coordinate in a neighborhood of \(P\) is called ramification point (or, critical point). Its image \(z(P)\) is called a branching point (or, critical value). The set of all branching points of function \(z\) form a branching divisor of \(z\). Note that infinity also can be a branching point. Since in our consideration it plays an exclusive role it is convenient in what follows to denote by \(\mathcal{Z} := \{z_i\}_{i=1}^N\) all branching points from the finite part of the complex plane \(\mathbb{C}\).

Clearly, branching divisors of equivalent functions are the same. Moreover, the compact holomorphic curve \(X_c\) is also uniquely determined by the branching divisor and some additional ramification data (of combinatorial type). Namely, assume that \(z\) has degree \(d\). Let \(w_0 \in \mathbb{C}\) be a non branching point. Fix a system of non-intersecting paths \(\gamma_i = \{\gamma_i\}_{i=1,...,N}\). The \(i\)th path \(\gamma_i\) connects \(w_0\) and \(z_i\). We want to construct a system of loops \(l_i \subset \mathbb{C}\). To construct \(l_i\) we start from \(w_0\) and follow first \(\gamma_i\) almost to \(z_i\), then encircle \(z_i\) counterclockwise along a small circle and finally go back to \(w_0\) along \(-\gamma_i\). Using \(l_i\) we associate with each branching point an element of the permutation group \(\sigma_i \in \Sigma_d\).

The point \(w_0\) has exactly \(d\) preimages. Let us label them by integers \(\{1,\ldots, d\}\). Let us follow the loop \(l_i\) and lift this loop to \(X_c\) starting from each of the preimages of \(w_0\). The monodromy along path \(l_i\) gives us a permutation \(\sigma_i \in \Sigma_d\) of preimages. We add to this system of loops one more \(l_\infty\), related to infinity (starting from \(w_0\) we go sufficiently far then make a big circle in the clockwise direction and go back to \(w_0\)). This loop gives us one more permutation \(\sigma_\infty\).

Note that the product \(\sigma_1 \cdot \ldots \cdot \sigma_N\) times \(\sigma_\infty\) is the identity operator. Therefore, the function \(z\), including its behavior at infinity, determines \(N\) branching points and \(N\) permutations. These permutations are not uniquely defined, they depend on the labeling of preimages of \(w_0\). Therefore, they are determined, up to a conjugacy by the elements of \(\Sigma_d\).

Given a set of branching points \(\mathcal{Z}(z) = \{z_i\}_{i=1,...,N} \subset \mathbb{C}\) and a system of permutations \(\sigma(z) = (\sigma_1,\ldots,\sigma_N) \in \Sigma_d \times \cdots \times \Sigma_d/\Sigma_d\), where the last quotient is taken with respect to diagonal conjugation, we can restore by Riemann theorem the surface \(X_c\) and the function \(z\). Actually there is one more topological condition: the surface should be connected. Throughout the paper we assume that the system of permutations guaranteed this condition to be hold.

Hence, the triple \((X_c, z, E)\) is equivalent to the triple \((\mathcal{Z}, \sigma, E)\). We use this triple as free parameters determining the spectral surface of a \(2d+1\)-diagonal matrix.

Summary. Comparably with the case of Jacobi matrices, where we have only system of cuts (spectral intervals) in the complex plane, in the multidiagonal case we have a new additional system of parameters. We have to fix in \(\mathbb{C}\) a system of critical points \(\mathcal{Z}\), and associate to them a system of permutation \(\sigma\), which actually depends on the base point \(w_0 \in \mathbb{C} \setminus \mathcal{Z}\) and the system of paths \(\gamma\). They define a Riemann surfaces \(X_c\) and a covering \(z\). Then on the set \(z^{-1}(\mathbb{R}) \subset X_c\) we chose a system of cuts \(E\), and thus \((X_c, z, E)\) is restored up to the equivalence 1.3.
1.4. Remark. As it was mentioned by the branching divisors language one can extend consideration from the pure algebraic level. In particular, we are very interested in infinite dimensional generalizations: the point is that starting from the 5–diagonal case we have the branching divisor \( Z \) as a completely new system of parameters characterizing the spectral surface. What is its influence on the properties of the corresponding surface? Consider, indeed, the simplest case of 5–diagonal matrices. Then we have just to specify the point set \( Z \) (all permutations are of the form \( \sigma = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \)). What is the speed of accumulation of an infinite system of points \( Z \) to, say, a finite system of cuts \( E \) so that \( X_c \setminus E \) is of Widom type, or of Widom type with the Direct Cauchy Theorem (for the definition of these type of surfaces see [12], see also [25] for their role in the spectral theory)? We provide here the following example. Let \( W(z) \) be the infinite Blaschke product in the upper half plane with zeros at \( \{ z_k \}_{k \geq 0} \). Define \( R = \{ P = (z, w) : w^2 = W, \Im z \geq 0 \} \).

Then \( \sqrt{\frac{z - z_0}{z - \overline{z_0}}} \) is the Green function in \( R \). Note that \( C_k = (z_k, 0), k \geq 1 \) are its critical points, and therefore the Widom function is:

\[
\Delta = \sqrt{Wz - \overline{z_0} z - z_0}.
\]

Note that the Carleson condition for this function on \( R \) is the standard Carleson condition for \( \{ z_k \}_{k \geq 1} \) in the upper half plane. Thus the Blaschke and Carleson conditions on the zero set in the half plane guarantee that \( R \) is of Widom type and of Widom type with the Direct Cauchy Theorem respectively.

1.3. Structure of the paper and main results. In Sect. 2, having the spectral surface fixed, we define a system of Hardy spaces on it (natural counterparts of the Hardy space in the unit disk). In that section we assume that there is only one "infinity" \( P_0 \in X_c \setminus E, \ z(P_0) = \infty \), that is, the product of permutations \( \sigma_1 \cdot \cdots \cdot \sigma_N \) is a cycle. There is an intrinsic basis in the Hardy space: each next basis elements has at \( P_0 \) a zero of bigger and bigger multiplicity. Of course, this is a counterpart of the standard basis system \( \{ \zeta^n \}_{n \geq 0} \) in the standard \( H^2 \). Extending this basis to the negative integers (the system extends in the direction of functions having a pole at \( P_0 \) with growing multiplicity) one gets a basis in the whole \( L^2 \). Finally the multiplication operator by the covering map with respect to this basis is the 2\( d+1 \)–diagonal matrix (to this end it is important to note that \( z \) has pole of multiplicity \( d \) at \( P_0 \)). The complex Green function with zero at \( P_0 \) is playing the role of the symbol of the shift operator. Then we study the question of uniqueness of such a model for an ergodic operator (see Theorem 2.3 and the example right after it).

The general case (several "infinities") is considered in Sect. 3. The \( d \)–root of the product of the Green functions (counting their multiplicities) with respect to all infinite points on the surface becomes the symbol of the shift. We have the ordering of the infinities as one more parameter, defining the basis system and the corresponding multidiagonal matrix.

In Sect. 4 we demonstrate, how an application of our general construction to a particular covering, generate the well known and now widely discussed almost periodic CMV matrices.
Starting from Sect. 5 we discuss an important theme: covering of one spectral surface by another one and related to this operation transformations on the set of multidiagonal operators (so called Renormalization Equations). Let \( \pi : Y_c \to X_c \) be a \( d \)-sheeted covering. For a system of cuts \( E \) define \( F = \pi^{-1}(E) \). Then we have \( \pi : Y_c \setminus F \to X_c \setminus E \). The study is based on the relation between the Hardy spaces on these Riemann surfaces.

The Renormalization Equations generated by polynomial coverings have played an important role in studying of almost periodic Jacobi matrices with a singular continuous spectrum, [5], see also [20], [21]. They act in the most natural way on periodic Jacobi matrices, see Sect. 6.

In Sect. 7 we proof several new results dealing with Renormalization Equations for periodic Jacobi matrices: describe the complete set of their solutions; show their relation with the Ruelle operators. Finally, we give a possible generalization of the constructions from Sect. 6 for a wider class of almost periodic Jacobi matrices with a singular continuous spectrum. In particular, we prove the Lipschitz property of the Darboux transform.

Having in mind importance of the paper [3], where the Renormalization Equation generated by "just" quadratic polynomial was used, we investigate in Sect. 8 the case of rational double covering \( \pi(v) = \tau v - \frac{1}{\tau}, \tau > 1 \). As usual, the renormalization procedure is simpler to formulate for operators acting on the (integer) half axis.

**Definition 1.5.** Let \( A \) be a self-adjoint operator acting in \( l^2_+ = l^2(\mathbb{Z}_+) \) with a cyclic vector \( |0\rangle \) and the spectrum on \([-1, 1]\). We define its transform \( \pi^*(A) \) in the following steps. First we define the upper triangular matrix \( \Phi \) (with positive diagonal entries) by the condition
\[
A^2 + 4\tau(\tau - 1) = \Phi^*\Phi. \tag{7}
\]
Introduce \( A_* := \Phi A \Phi^{-1} \) and define the operator
\[
\begin{bmatrix}
A & \Phi^* \\
\Phi & A_*
\end{bmatrix}, \tag{8}
\]
acting in \( l^2_+ \oplus l^2_+ \). Finally, using the unitary operator \( U : l^2_+ \to l^2_+ \oplus l^2_+ \), such that
\[
U|2k\rangle = |k\rangle \oplus 0, \quad U|2k + 1\rangle = 0 \oplus |k\rangle \tag{9}
\]
we construct
\[
\pi^*(A) := \frac{1}{2\tau} U^* \begin{bmatrix}
A & \Phi^* \\
\Phi & A_*
\end{bmatrix} U : l^2_+ \to l^2_+. \tag{10}
\]

We have a theorem (see Theorem 8.8) on the weak convergence of the iterative procedure \( A_{n+1} = \pi^*(A_n) \) to an operator with a simple singular continuous spectrum supported on the Julia set of the given expanding mapping and pose here a question on the contractivity of the renorm operator, at least for big values of \( \tau \). The main general conjecture deals with contractivity of all renormalizations, generated by a covering with sufficiently big critical values.

2. The Global Functional Model (single infinity case). Uniqueness

**Theorem**

2.1. **Hardy spaces and basises.** There are different ways to define Hardy spaces on the the Riemann surfaces, – the spaces of vector bundles, multivalued functions
or forms. These definitions are equivalent. We start from 1–forms, the most natural object with this respect from our point of view.

Let \( \pi(\zeta) : \mathbb{D} \to X \) be a uniformization of the surface \( X = X_\gamma \setminus E \). Thus there exists a discrete subgroup \( \Gamma \) of the group \( SU(1,1) \) consisting of elements of the form
\[
\gamma = \begin{bmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{bmatrix}, \quad \gamma_{11} = \overline{\gamma_{22}}, \quad \gamma_{12} = \overline{\gamma_{21}}, \quad \det \gamma = 1,
\]
such that \( \pi(\zeta) \) is automorphic with respect to \( \Gamma \), i.e., \( \pi(\gamma(\zeta)) = \pi(\zeta), \forall \gamma \in \Gamma \), and any two preimages of \( P \in X \) are \( \Gamma \)–equivalent. We normalize \( Z(\zeta) := (z \circ \pi)(\zeta) \) by the conditions \( Z(0) = \infty, (\zeta^d Z)(0) > 0 \).

Note that \( \Gamma \) acts dissipatively on \( \mathbb{T} \) with respect to the Lebesgue measure \( dm \), that is there exists a measurable (fundamental) set \( E \), which does not contain any two \( \Gamma \)–equivalent points, and the union \( \bigcup_{\gamma \in r} \gamma(E) \) is a set of full measure. In fact \( E \) can be chosen as a finite union of intervals, – the \( \mathbb{T} \)–part of the boundary of the fundamental domain. For the space of square summable functions on \( E \) (with respect to \( dm \)), we use the notation \( L^2_{dm}(\mathbb{R}) \).

A character of \( \Gamma \) is a complex–valued function \( \alpha : \Gamma \to \mathbb{T} \), satisfying
\[
\alpha(\gamma_1 \gamma_2) = \alpha(\gamma_1) \alpha(\gamma_2), \quad \gamma_1, \gamma_2 \in \Gamma.
\]
The characters form an Abelian compact group denoted by \( \Gamma^* \). The further Hardy spaces on \( X \) are marked by characters of \( \Gamma \).

Let \( f \) be an analytic function in \( \mathbb{D} \), \( \gamma \in \Gamma \). Then we put
\[
f[|\gamma]_k = \frac{f(\gamma(\zeta))}{(\gamma_{21} \zeta + \gamma_{22})^k} \quad k = 1, 2.
\]
Notice that \( f[|\gamma]_2 = f \) for all \( \gamma \in \Gamma \), means that the form \( f(\zeta) d\zeta \) is invariant with respect to the substitutions \( \zeta \to \gamma(\zeta) \) \( (f(\zeta) d\zeta \) is an Abelian integral on \( \mathbb{D}/\Gamma \)).

Analogously, \( f[|\gamma] = \alpha(\gamma) f \) for all \( \gamma \in \Gamma, \alpha \in \Gamma^* \), means that the form \( |f(\zeta)|^2 |d\zeta| \) is invariant with respect to these substitutions.

We recall, that a function \( f(\zeta) \) is of Smirnov class, if it can be represented as a ratio of two functions from \( H^\infty \) with an outer denominator. The following spaces related to the Riemann surface \( \mathbb{D}/\Gamma \) are counterparts of the standard Hardy spaces \( H^2 \ (H^1) \) on the unit disk.

**Definition 2.1.** The space \( A^2_1(\Gamma, \alpha) \ (A^2_2(\Gamma, \alpha)) \) is formed by functions \( f \), which are analytic on \( \mathbb{D} \) and satisfy the following three conditions
\[
1) f \text{ is of Smirnov class} \\
2) f[|\gamma] = \alpha(\gamma) f \quad (f[|\gamma]_2 = \alpha(\gamma) f) \quad \forall \gamma \in \Gamma \\
3) \int_E |f|^2 \, dm < \infty \quad (\int_E |f| \, dm < \infty).
\]

\( A^2_1(\Gamma, \alpha) \) is a Hilbert space with the reproducing kernel \( k^\alpha(\zeta, \zeta_0) \), moreover
\[
0 < \inf_{\alpha \in \Gamma^*} k^\alpha(\zeta_0, \zeta_0) \leq \sup_{\alpha \in \Gamma^*} k^\alpha(\zeta_0, \zeta_0) < \infty. \quad (11)
\]

Put
\[
k^\alpha(\zeta) = k^\alpha(\zeta, 0) \quad \text{and} \quad K^\alpha(\zeta) = K^\alpha_\zeta(0) = \frac{k^\alpha(\zeta)}{\sqrt{k^\alpha(0)}}.
\]
We need one more special function. The Blaschke product
\[ b(\zeta) = \zeta \prod_{\gamma \in \Gamma, \gamma \neq 12} \frac{\gamma(0) - \zeta}{1 - \gamma(0)\zeta} \]
is called the Green’s function of \( \Gamma \) with respect to the origin. It is a character–
automorphic function, i.e., there exists \( \mu \in \Gamma^* \) such that \( b(\gamma(\zeta)) = \mu(\gamma)b(\zeta) \). Note,
if \( G(P) = G(P, P_0) \) denotes the Green’s function of the surface \( X \), then
\[ G(\pi(\zeta)) = -\log |b(\zeta)|. \]

We are ready to construct the basis in \( A^2_\Gamma(\Gamma, \alpha) \). Consider the following subspace
of this space
\[ \{ f \in A^2_\Gamma(\Gamma, \alpha) : f(0) = 0 \}. \]
The following two facts are evident
1) \( \{ f \in A^2_\Gamma(\Gamma, \alpha) : f(0) = 0 \} = \{ b\tilde{f} : \tilde{f} \in A^2_\Gamma(\Gamma, \mu^{-1}\alpha) \} = bA^2_\Gamma(\Gamma, \mu^{-1}\alpha), \]
2) \( A^2_\Gamma(\Gamma, \alpha) = \{ K^\alpha \} \oplus \{ f \in A^2_\Gamma(\Gamma, \alpha) : f(0) = 0 \}. \)

Thus
\[ A^2_\Gamma(\Gamma, \alpha) = \{ K^\alpha \} \oplus \{ bA^2_\Gamma(\Gamma, \mu^{-1}\alpha) \} = \{ K^\alpha \} \oplus \{ bK^\alpha^{\mu^{-1}} \} \oplus \{ b^2A^2_\Gamma(\Gamma, \mu^{-2}\alpha) \}, \]
and so on.

Basically we proved the following theorem, note, however, that the second state-
ment is not a direct consequence of the first one.

**Theorem 2.2.** Given \( \alpha \in \Gamma^* \), the system of functions \( \{ b^nK^{\alpha\mu^{-n}} \}_{n \geq 0} \) forms an orthonormal basis in \( A^2_\Gamma(\Gamma, \alpha) \); the system \( \{ b^nK^{\alpha\mu^{-n}} \}_{n \in \mathbb{Z}} \) is an orthonormal basis in \( L^2_{dm|\mathbb{D}} \).

**2.2. The Global Functional Model.** Of course the constructions in this section
and our speculation in Sect. 1 are closely related and of mutual influence. In
this subsection we close the construction by proving the Global Functional Model
Theorem.

Let \( \Gamma_0 := \ker \mu \), that is \( \Gamma_0 = \{ \gamma \in \Gamma : \mu(\gamma) = 1 \} \). Evidently, \( b(\zeta) \) and \( (zb^d)(\zeta) \)
are holomorphic functions on the surface \( X_0 = \mathbb{D}/\Gamma_0 \).

Assume that \( \alpha_0 \in \Gamma_0 \) can be extended to a character on \( \Gamma \), i.e.,
\[ \Omega_{\alpha_0} = \{ \alpha \in \Gamma^* : \alpha|\Gamma_0 = \alpha_0 \} \neq \emptyset. \]

Note that the set of characters
\[ \Omega_\iota = \{ \alpha \in \Gamma^* : \alpha|\Gamma_0 = \iota \} \]
where \( \iota(\gamma) = 1 \) for all \( \gamma \in \Gamma_0 \) is isomorphic to the set \( (\Gamma/\Gamma_0)^* \).

Let us fix an element \( \hat{\alpha}_0 \in \Omega_{\alpha_0} \). Since
\[ \{ \alpha \in \Gamma^* : \alpha|\Gamma_0 = \alpha_0 \} = \{ \hat{\alpha}_0\beta : \beta \in \Gamma^* : \beta|\Gamma_0 = \iota \} \]
we can define a measure \( d\chi_{\alpha_0}(\alpha) \) on \( \Omega_{\alpha_0} \) by the relation
\[ d\chi_{\alpha_0}(\alpha) = d\chi_{\alpha_0}(\hat{\alpha}_0\beta) = d\chi_{\iota}(\beta), \]
where \( d\chi_{\iota}(\beta) \) is the Haar measure on \( (\Gamma/\Gamma_0)^* \) (the measure \( d\chi_{\alpha_0}(\alpha) \) does not depend
on a choice of the element \( \hat{\alpha}_0 \)).
Obviously, $T\alpha := \mu^{-1}\alpha$ is an invertible ergodic measure–preserving transformation on $\Omega = \Omega_\alpha$, with respect to the measure $d\chi = d\chi_\alpha$.

The following Theorem is a slightly modified version of Theorem 2.2 from [28].

**Theorem 2.3.** With respect to the basis from Theorem 2.2, the multiplication operator by $z$ is a $2d + 1$–diagonal ergodic finite difference operator with $\Omega = \Omega_\alpha$, $d\chi = d\chi_\alpha$, $T\alpha := \mu^{-1}\alpha$ and $\alpha_0 = \alpha|\Gamma_0$. Moreover, the operators $\tilde{S}_\alpha$ and $(\tilde{S}^d)_+$ are unitarily equivalent to multiplication by $b$ and $(b^d Z)$ in $A^2_1(\Gamma_0, \alpha_0)$ respectively. This unitary map is given by the formula

$$
\sum_{\{\gamma\} \in \Gamma/\Gamma_0} f[\gamma] \alpha^{-1}(\gamma) = \sum_{n \in \mathbb{Z}_+} x_n(\alpha) b^n K^{\alpha\mu - n}, \quad f \in A^2_1(\Gamma_0, \alpha_0),
$$

where the vector function $x(\alpha) := \{x_n(\alpha)\}$ belongs to $L^2_{\alpha}(L^2(\mathbb{Z}_+))$.

**2.3. Uniqueness Theorem.** The natural question up to which extend our functional realization is unique?

**Theorem 2.4.** Assume that a finite difference ergodic operator has a finite band functional model that is there exist a triple $\{X_c, \tilde{z}, E\}$, a character $\alpha_0 \in \Gamma^*$ and a map $F$ from $\Omega$ to $\tilde{\Omega} := \Omega_\alpha$, such that $F \tilde{\omega} = \mu^{-1} F \omega, \chi(F^{-1}(A)) = \chi(A), A \subset \tilde{\Omega}$, with $d\tilde{\chi} := d\chi_\alpha$, and $\mu$ is the character of the Green’s function $\tilde{b}$ on $X_c \setminus E$. Moreover $\tilde{q}(k)(\omega) = \hat{\tilde{q}}(k)(F \omega)$, where the coefficients $\hat{\tilde{q}}(k)(\alpha)$ are generated by the multiplicative operator $\tilde{z}$ with respect to the orthonormal basis $\{b^n K^{\alpha\mu - n}\}_{n \in \mathbb{Z}}$.

If the functions $\tilde{z}$ and $\{d \log \tilde{b}/d\tilde{z}\}$ separate points on $X_c \setminus E$ then any local functional model is generated by one of the branches of the function $\tilde{b}$.

For proof see [22].

The following example shows that in the case when these two functions $\tilde{z}$ and $\{d \log \tilde{b}/d\tilde{z}\}$ do not separate points on $X_c \setminus E$ one can give different global functional realizations for the same ergodic operator.

**Example [22].** Let $J = S^d + S^{-d}$. There exist a ”trivial” functional model with $X_c \setminus E \sim \mathbb{D}$. In this case $J$ is the multiplication operator by $z = \zeta^d + \zeta^{-d}$ with respect to the standard basis $\{\zeta^i\}$ in $L^2_\Delta$. Note that $b = \zeta$, thus

$$
w := \frac{d \log \tilde{b}}{d\tilde{z}} = \frac{1}{\zeta^d - \zeta^{-d}} \frac{1}{d},
$$

that is $z^2 + (wd)^{-2} = 4, |(wd)^{-1} + z| < 2$.

On the other hand let us fix any polynomial $T(u)$, deg $T = d$, with real critical values on $\mathbb{R} \setminus [-2, 2]$ and define $X_c \setminus E = T^{-1}(\mathbb{C} \setminus [-2, 2]) \sim \mathbb{C} \setminus T^{-1}[-2, 2]$. As we discussed the last set is the resolvent set for a $d$–periodic Jacobi matrix, say $J_0$. Moreover $T(J_0) = J$, and $-\log |b|$ is just the Green’s function of this domain in the complex plain. So, using the standard functional model for $J_0$ (see Sect. 7 for details) with the symbols $u$ and $b$ we get a functional model for $J$ with $z = T(u)$ and the same $b$. Note that as before $z^2 + (wd)^{-2} = 4, |(wd)^{-1} + z| < 2$ with $w := \frac{d \log b}{d\tilde{z}}$.

**Remark 2.5.** In [8] the identity $T(J_0) = S^d + S^{-d}$ that holds for a periodic Jacobi matrix $J_0$ with the spectrum on $T^{-1}[-2, 2]$ is called the Magic Formula. There it plays an important role in proving counterparts of Denisov–Rakhmanov and Killip–Simon Theorems for perturbations of periodic Jacobi matrices.
3. Several infinities case. Existence Theorem

Now we examine the situation in which the reduced surface $X_c$ has several "infinities" that is the covering function $z$ (the symbol of an almost periodic operator) equals infinity at several (distinct) points on $X_c$.

We start with a simple example.

3.1. A five diagonal matrix of period two. Assume that $z$ is a two sheeted covering with only two branching points, say, $z_1 = -2$, $z_2 = 2$. With necessity corresponding substitutions are $\sigma_1 = \sigma_2 = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$. In this case $X_c$ is equivalent to the complex plane $\mathbb{C}$, moreover we can put $z = v + \frac{1}{v}$, $v \in \mathbb{C}$. (12)

Thus, on this surface we have two "infinities" $v = \infty$ and $v = 0$.

Note that $z^{-1}(\mathbb{R}) = T \cup \mathbb{R}$. We cut $\mathbb{C}$ over the interval $[a,b]$, where $0 < a < b < 1$, that is we consider $X_c \setminus E$ of the form $\mathbb{C} \setminus [a,b]$. We uniformize $X_c \setminus E$ by

$$v = \frac{a + b}{2} + \frac{b - a}{2} \frac{1}{\zeta} + \frac{1}{\zeta}, \quad \zeta \in \mathbb{D}.$$

(13)

For $v = \infty$ we have $\zeta_0 = 0$. Solving

$$\zeta^2 + 2\frac{b + a}{b - a} \zeta + 1 = 0$$

we get the image of the second infinity in $\mathbb{D}$, $\zeta_1 = -\frac{\sqrt{b^2 - 4} - \sqrt{a}}{2(b - a)}$. As a result we get a symbol function $z : \mathbb{D} \to \mathbb{C}$ (see (12), (13)) for the forthcoming operator with two infinities $\zeta_0, \zeta_1$.

Next point is the symbol $b$ for the shift operator. Recall that the Green function in $\mathbb{D}$ is the Blaschke factor

$$b_{\zeta_0} = \zeta, \quad b_{\zeta_1} = \frac{\zeta - \zeta_1}{1 - \zeta \zeta_1}.$$

The product $b_{\zeta_0}b_{\zeta_1}$ is the smallest unimodular multiplier that cancels poles of $z$. Since $b^2$ is of the same nature, $b^2z$ is holomorphic with a unimodular function $b$ on $T$, we have $b^2 = b_{\zeta_0}b_{\zeta_1}$.

Finally, we need a certain functional space and an intrinsic basis in it that generalize the construction in Theorem 2.2. Recall $b$ should be related to the shift $S$, and we are going to define the periodic operator $J$ as the multiplication operator with respect to this basis. To this end we define the following functional spaces. Given $\alpha_k \in \mathbb{T}, k = 0, 1$, we associate the space $H^2(\alpha_0, \alpha_1)$ of analytic multivalued functions $f(\zeta)$, $\zeta \in \mathbb{D} \setminus \{\zeta_0, \zeta_1\}$, such that $|f(\zeta)|^2$ has a harmonic majorant and

$$f \circ \gamma_i = \alpha_i f,$$

where $\gamma_i$ is a small circle around $\zeta_i$. Such a space can be reduced to the standard Hardy space $H^2$, moreover

$$H^2(\alpha_0, \alpha_1) = b_{\zeta_0}b_{\zeta_1}H^2, \quad \alpha_k = e^{2\pi i \tau_k}, \quad 0 \leq \tau_k < 1.$$

Lemma 3.1. The space $bH^2(-1,1)$ is a subspace of $H^2(1, -1)$ having a one dimensional orthogonal compliment, moreover

$$H^2(1, -1) = \{\sqrt{b_{\zeta_1}k_{\zeta_0}} \} \oplus bH^2(-1,1),$$

(14)
where \( k_\zeta \) is the reproducing kernel of the standard \( H^2 \) with respect to \( \zeta \).

This lemma allows us to repeat construction of subsection 3.1. Iterating, now, the decomposition (14)

\[
H^2(1, -1) = \{ \sqrt{b_c} k_\zeta \} \oplus b H^2(-1, 1) \\
= \{ \sqrt{b_c} k_\zeta \} \oplus b \{ \sqrt{b_c} k_\zeta \} \oplus b^2 H^2(1, -1) = \ldots
\]

one gets an orthogonal basis in \( H^2(1, -1) \) consisting of vectors of two sorts

\[
b^{2m} \{ \sqrt{b_c} k_\zeta \} \quad \text{and} \quad b^{2m+1} \{ \sqrt{b_c} k_\zeta \}.
\]

Note that this orthogonal system can be extended on negative integers \( m \) so that we obtain a basis in \( L^2(1, -1) \).

**Theorem 3.2.** With respect to the orthonormal basis

\[
e_n = \begin{cases} 
  b^{2m} \sqrt{b_c} k_\zeta / ||k_\zeta||, & n = 2m \\
  b^{2m+1} \sqrt{b_c} k_\zeta / ||k_\zeta||, & n = 2m + 1
\end{cases}
\]

the multiplication operator by \( z \) is a 5-diagonal matrix of period 2.

3.2. **General case.** Let \( z : X_c \to \mathbb{C} \) be \( d \)-sheeted covering with \( d \) (distinct) infinities. Further, let \( \mathbb{D} / \Gamma \) be a uniformization of \( X_c \setminus E \). A given character \( \alpha \in \Gamma^* \) and a fixed ordering of infinities \( P_1, P_2, \ldots, P_d \) define

- \( b_j = b_{P_j} \) the Green function with respect to \( P_j \), \( b_j \circ \gamma = \mu_j(\gamma)b_j, \mu_j \in \Gamma^* \);
- \( k_{\alpha}^j = k_{\alpha}^j \) the reproducing kernel of \( A^2_1(\Gamma, \alpha) \) with respect to \( P_j \), \( K_{\alpha}^j := k_{\alpha}^j/||k_{\alpha}^j|| \).

**Theorem 3.3.** Let \( b = (b_1, \ldots, b_d)^{\frac{1}{2}} \). With respect to the orthonormal basis

\[
e_n = \begin{cases} 
  b^{dm} b_1^{\frac{1}{2}} \cdots b_{d-1}^{\frac{1}{2}} K_1^{\alpha(\mu_1, \ldots, \mu_d) - m}, & n = dm \\
  b^{dm+1} b_2^{\frac{1}{2}} \cdots b_{d-1}^{\frac{1}{2}} K_1^{\alpha(\mu_1, \ldots, \mu_d) - m}, & n = dm + 1 \\
  \ldots \\
  b^{dm+d-1} b_d^{\frac{1}{2}} \cdots b_{d-2}^{\frac{1}{2}} \cdots b_1^{\frac{1}{2}} K_{d-1}^{\alpha(\mu_1, \ldots, \mu_{d-1}, \mu_d) - m}, & n = dm + d - 1
\end{cases}
\]

the multiplication operator by \( z \) is a \((2d + 1)\)-diagonal almost periodic matrix.

4. **Five-diagonal almost periodic self-adjoint matrices and OPUC**

We start again with two-sheeted covering (12). We have \( X_c \cong \mathbb{C} \) and \( z^{-1}(\mathbb{R}) = \mathbb{R} \cup \mathbb{T} \), but let us, in this case, cut \( \mathbb{C} \) on an arbitrary finite union of (necessary non-degenerate) arcs on the unit circle, \( E \cong \{ T \setminus \cup_{j=0}^d (a_j, b_j) \} \).

The domain \( X_c \setminus E \) is conformally equivalent to the quotient of the unit disk by the action of a discrete group \( \Gamma = \Gamma(E) \). Let

\[
v : \mathbb{D} \to \{ \mathbb{C} \setminus \mathbb{T} \} \cup \{ \cup_{j=0}^d (a_j, b_j) \}
\]

be a covering map, \( v \circ \gamma = v, \forall \gamma \in \Gamma \). In what follows we assume the following normalization to be hold \( v : (-1, 1) \to (a_0, b_0) \), so that one can chose a fundamental domain \( \mathfrak{F} \) and a system of generators \( \{ \gamma_j \}_{j=1}^\infty \) of \( \Gamma \) such that they are symmetric with respect to the complex conjugation:

\[
\mathfrak{F} = \mathfrak{F}, \quad \gamma_j = \gamma_j^{-1}.
\]
Denote by $\zeta_0 \in \mathfrak{F}$ the preimage of the origin, $v(\zeta_0) = 0$, then $v(\overline{\zeta_0}) = \infty$. Thus the action of the group $\Gamma$, $P_0 = \{\gamma(\zeta_0)\}_{\gamma \in \Gamma}$ and $P_1 = \{\gamma(\overline{\zeta_0})\}_{\gamma \in \Gamma}$. These are two infinities that we have in the case under consideration.

Thus, to define the function $b$ (the symbol of the shift operator) we have to introduce the Green functions $B(\zeta, \zeta_0)$ and $B(\zeta, \overline{\zeta_0})$. It is convenient to normalize them by $B(\zeta_0, \zeta_0) > 0$ and $B(\zeta_0, \overline{\zeta_0}) > 0$. Then

$$v(\zeta) = e^{ic} \frac{B(\zeta, \zeta_0)}{B(\zeta, \overline{\zeta_0})}. \quad (18)$$

Also, we can rotate (if necessary) $v$–plane so that $c = 0$. Note that $B(\zeta, \zeta_0)$ is a character–automorphic function

$$B(\gamma(\zeta), \zeta_0) = \mu(\gamma)B(\zeta, \zeta_0), \quad \gamma \in \Gamma,$$

with a certain $\mu \in \Gamma^*$. By (18), $B(\zeta, \overline{\zeta_0})$ has the same factor of automorphy,

$$B(\gamma(\zeta), \overline{\zeta_0}) = \mu(\gamma)B(\zeta, \overline{\zeta_0}), \quad \gamma \in \Gamma.$$

By the definition $b^2 = B(\zeta, \zeta_0)B(\zeta, \overline{\zeta_0})$, we get a multivalued analytic function $b$ on the punched surface $\mathbb{D}/\Gamma \setminus \{P_0, P_1\}$.

In this case we have only two possibilities for ordering of infinities: $\{P_0, P_1\}$ and $\{P_1, P_0\}$. According to Theorem 3.3, to any of them, say the first one, and to an arbitrary $\alpha \in \Gamma^*$ we can associate the operator $J = J(-1, 1; \alpha)$ by fixing the space $H^2(-1, 1; \alpha) = \sqrt{B(\zeta, \zeta_0)A^2_1(\alpha)}$ and a natural basis in it. Up to a common multiplier $\sqrt{B(\zeta, \zeta_0)}$ it is a basis in $A^2_1(\alpha)$ of the form

$$K^\alpha(\zeta, \zeta_0), B(\zeta, \zeta_0)K^{\alpha-1}(\zeta, \zeta_0), B(\zeta, \zeta_0)B(\zeta, \overline{\zeta_0})K^{\alpha-2}(\zeta, \overline{\zeta_0}), \ldots$$

Thus we get the same system of functions that we used describing almost periodic Verblunsky coefficients [19]. The last one we can define by

$$a(\alpha) = a = \frac{K^\alpha(\zeta_0, \overline{\zeta_0})}{K^\alpha(\zeta_0, \zeta_0)}.$$

In [19] they appear in the following recursion

\[
\begin{align*}
K^\alpha(\zeta, \zeta_0) &= a(\alpha)K^\alpha(\zeta, \zeta_0) + \rho(\alpha)B(\zeta, \zeta_0)K^{\alpha-1}(\zeta, \zeta_0), \\
K^\alpha(\zeta, \zeta_0) &= \overline{a(\alpha)K^\alpha(\zeta, \overline{\zeta_0})} + \rho(\alpha)B(\zeta, \overline{\zeta_0})K^{\alpha-1}(\zeta, \zeta_0),
\end{align*}
\]

where

$$\rho(\alpha) = \rho = \sqrt{1 - |a|^2} = B(\zeta_0, \zeta_0) \frac{K^{\alpha-1}(\overline{\zeta_0}, \zeta_0)}{K^\alpha(\zeta_0, \zeta_0)}.$$

The goal of this section is to represent $J$ in terms of Verblunsky coefficients.

**Lemma 4.1.** With respect to the basis

$$..., K^{\alpha\mu}(\zeta, \zeta_0)/B(\zeta, \zeta_0), K^\alpha(\zeta, \zeta_0), B(\zeta, \zeta_0)K^{\alpha\mu-1}(\zeta, \zeta_0), \ldots \quad (20)$$
the multiplication operator by \( v \) is a matrix having at most two non vanishing diagonals over the main diagonal. Moreover,

\[
v \sim \begin{bmatrix}
\rho(\alpha)\rho(\alpha^2) & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & -\rho(\alpha\mu)a(\alpha^2) & 0 \\
\cdot & \cdot & -a(\alpha^2)a(a^2) \\
\cdot & \cdot & \cdot & \rho(\alpha)a(a^2) & 0 \\
\cdot & \cdot & \cdot & \cdot & -a(\alpha^2)a(a^2) \\
\end{bmatrix}.
\tag{21}
\]

Similarly, the multiplication operator by \( 1/v \) is of the form

\[
1/v \sim \begin{bmatrix}
0 & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \rho(\alpha\mu)a(\alpha) & \rho(\alpha)\rho(\alpha^2) \\
\cdot & -a(\alpha^2)a(a^2) & -\rho(\alpha)\rho(\alpha^2) \\
\cdot & \cdot & -a(\alpha^2)a(a^2) \\
\cdot & \cdot & \cdot & \cdot \\
\end{bmatrix}.
\tag{22}
\]

**Proof.** We give a proof, say, for (21). Recall (18), from which we can see that the decomposition of the vector \( v(\zeta)K^\alpha(\zeta, \zeta_0) \) begins with

\[
v(\zeta)K^\alpha(\zeta, \zeta_0) = c_0 \frac{K^{\alpha\mu}(\zeta, \zeta_0)}{B(\zeta, \zeta_0)B(\zeta, \zeta_0)} + c_1 \frac{K^{\alpha \nu}(\zeta, \zeta_0)}{B(\zeta, \zeta_0)} + c_2 K^\alpha(\zeta, \zeta_0) + \ldots.
\]

Multiplying by the denominator \( B(\zeta, \zeta_0)B(\zeta, \zeta_0) \) we get

\[
B^2(\zeta, \zeta_0)K^\alpha(\zeta, \zeta_0) = c_0 \frac{K^{\alpha\mu}(\zeta, \zeta_0)}{B(\zeta, \zeta_0)B(\zeta, \zeta_0)} + c_1 K^{\alpha \nu}(\zeta, \zeta_0)B(\zeta, \zeta_0) + c_2 K^\alpha(\zeta, \zeta_0)B(\zeta, \zeta_0) + \ldots.
\tag{23}
\]

First we put \( \zeta = \zeta_0 \). By the definition of \( \rho(\alpha) \) we have

\[
c_0 = B^2(\zeta_0, \zeta_0) \frac{K^{\alpha\mu}(\zeta_0, \zeta_0)}{K^{\alpha\mu^2}(\zeta_0, \zeta_0)} = \rho(\alpha)\rho(\alpha^2).
\]

Putting \( \zeta = \zeta_0 \) in (23) and using the definition of \( a(\alpha) \), we have

\[
c_1 = -c_0 \frac{K^{\alpha\mu}(\zeta_0, \zeta_0)}{K^{\alpha \nu}(\zeta_0, \zeta_0)B(\zeta_0, \zeta_0)} = -\rho(\alpha)\rho(\alpha^2) \frac{a(\alpha^2)}{\rho(\alpha^2)} = -\rho(\alpha)\rho(\alpha^2)a(\alpha^2).
\]

Doing in the same way we can find a representation for \( c_2 \) that would involve derivatives of the reproducing kernels. However, we can find \( c_2 \) in terms of \( a \) and \( \rho \) calculating the scalar product

\[
c_2 = \langle B^2(\zeta, \zeta_0)K^\alpha(\zeta, \zeta_0), B(\zeta, \zeta_0)B(\zeta, \zeta_0)K^\alpha(\zeta, \zeta_0) \rangle.
\]

Since \( B(\zeta, \zeta_0) \) is unimodular, using (19), we get

\[
c_2 = \langle \frac{K^{\alpha \nu}(\zeta, \zeta_0) - a(\alpha^2)K^{\alpha \mu}(\zeta, \zeta_0)}{\rho(\alpha^2)}, B(\zeta, \zeta_0)K^\alpha(\zeta, \zeta_0) \rangle.
\]

Recall that \( k^\alpha(\zeta, \zeta_0) = K^\alpha(\zeta, \zeta_0)K^\alpha(\zeta_0, \zeta_0) \) is the reproducing kernel. Thus

\[
c_2 = -\frac{a(\alpha^2)B(\zeta, \zeta_0)K^\alpha(\zeta, \zeta_0)}{\rho(\alpha^2)} = -\frac{a(\alpha^2)}{\rho(\alpha^2)} \frac{a(\alpha^2)}{\rho(\alpha^2)} = -\rho(\alpha)\rho(\alpha^2)a(\alpha).
\]
To find the decomposition of the vector \( v(\zeta)B(\zeta, \zeta_0)K^{\alpha\mu^{-1}}(\zeta, \zeta_0) \) is even simpler since only two leading terms are involved. Note that all other columns of the matrix in (21), starting from these two, can be obtain by the character’s shift by \( \mu^{-2} \) along diagonals.

Now let us remind the CMV representation for operators related to OPUC [23]. A given sequence of numbers from \( D \)

\[\ldots, a_{-1}, a_0, a_1, a_2, \ldots\]  

(24)

define unitary matrices

\[ A_k = \begin{bmatrix} a_k & \rho_k \\ \rho_k & -a_k \end{bmatrix}, \quad \rho_k = \sqrt{1 - |a_k|^2}, \]

and unitary operators in \( l^2(\mathbb{Z}) \) given by block-diagonal matrices

\[ \mathfrak{A}_0 = \begin{bmatrix} \ddots & & & \\ & A_{-2} & & \\ & & A_{-1} & \\ & & & A_0 \end{bmatrix}, \quad \mathfrak{A}_1 = S \begin{bmatrix} \ddots & & & \\ & A_{-1} & & \\ & & A_1 & \\ & & & \ddots \end{bmatrix} S^{-1}. \]

The CMV operator \( \mathfrak{A} \), related to the sequence (24), is the product

\[ \mathfrak{A} = \mathfrak{A}(\{a_k\}) := \mathfrak{A}_0 \mathfrak{A}_1. \]  

(25)

**Theorem 4.2.** Define the sequence \( a_k = a(\alpha \mu^{-k}) \). Then \( v \sim \mathfrak{A}(\{a_k\}) \), see (25).

**Proof.** Note that operators in (21) and (22) are mutually conjugated, therefore under-diagonal entries of both operators are also known. The rest is an easy direct computation.

**Theorem 4.3.** With respect to the basis (20) the multiplication operator by \( z \),

\[ z \sim \begin{bmatrix} q^{(-2)}(\alpha) & 0 \\ \vdots & \ddots & q^{(-1)}(\alpha) & q^{(-2)}(\alpha \mu^{-1}) \\ q^{(0)}(\alpha) & \{q^{(-1)}(\alpha \mu^{-1})\} & \{q^{(0)}(\alpha \mu^{-1})\} \\ \ast & \ast & \ast & \ddots \end{bmatrix}, \]

is defined by functions

\[ q^{(-2)}(\alpha) = \rho(\alpha \mu) \rho(\alpha \mu^2), \quad q^{(-1)}(\alpha) = \rho(\alpha \mu) \{a(\alpha) - a(\alpha \mu^2)\}, \]

and

\[ q^{(0)}(\alpha) = -2\Re\{a(\alpha) a(\alpha \mu)\}. \]

It is worth to mention that the second column is not only the shift by \( \mu^{-1} \), there is also the conjugation.

**Proof.** Recall that \( z = v + 1/v \) and use the previous lemma.

Using general constructions from [17] one can define and integrate the flows hierarchy given by

\[ \hat{\mathfrak{A}} = [(\mathfrak{A}^n + \mathfrak{A}^{-n})_+ \cdot \mathfrak{A}] \]  

(26)

Note that for \( n = 1 \), (26) gives the Schur flow [11].
5. Coverings

Discussing this subject we prefer to use a functional version of presentation of Hardy spaces. First we introduce these spaces and then will remark how they are related to the spaces of forms.

Let $\omega_{P_0}$ denote the harmonic measure on an open surface $X_c \setminus E$ with respect to $P_0 \in X_c \setminus E$. Note that $\omega_{P_0}$ is the restriction of the differential $\frac{1}{2\pi i} d \log b(P, P_0)$ on $E$. By $H^2(\alpha, \omega_{P_0})$ we denote the closure of $H^\infty(\alpha)$ in $L^2$ with respect to the measure $\omega_{P_0}$. The natural question is how this space is related to the space with another point fixed, say $P_1 \in X_c \setminus E$, or, more generally, with $H^2(\alpha, \omega)$, which denotes the closure of $H^\infty(\alpha)$ with respect to an equivalent norm given by a measure of the form $\omega = \rho \omega_{P_0}$, where $0 < C_1 \leq \rho \leq C_2 < \infty$. (By the Harnack Theorem $\omega_{P_i}$ and $\omega_{P_0}$ satisfy this property).

To answer it, let us define an outer function $\phi$, such that $\rho = |\phi|^2$. This function belongs to $H^\infty(\beta)$ with a certain $\beta \in \Gamma^*$. In this case

$$f \mapsto \phi f$$

is a unitary map from $H^2(\alpha, \omega)$ to $H^2(\beta, \omega_{P_0})$. Then the equality

$$\langle (\phi f)(P), k^\alpha_Q(P; \omega)\phi(P)\phi(Q)\rangle_{H^2(\alpha, \omega_{P_0})} = \langle f(P), k^\alpha_Q(P; \omega)\phi(Q)\rangle_{H^2(\alpha, \omega)} = f(Q)\phi(Q)$$

shows that the reproducing kernels of $H^2(\alpha, \omega_{P_0})$ and $H^2(\alpha, \omega)$ are related by

$$k^\alpha_{Q}(P; \omega_{P_0}) = k^\alpha_{Q}(P; \omega)\phi(P)\phi(Q).$$

For the normalized kernels we have

$$K^\alpha_Q(P; \omega_{P_0}) = K^\alpha_Q(P; \omega)\frac{\phi(Q)}{|\phi(Q)|}\phi(P).$$

Therefore the matrix of a multiplication operator in $H^2(\alpha, \omega)$, with respect to the reproducing kernels basis, actually can be obtain by a character’s shift for the matrix with the same symbol related to the chosen space $H^2(\alpha, \omega_{P_0})$. (Let us mention in brackets a specific normalization of a basis vector given by the unimodular factor $\frac{\phi(Q)}{|\phi(Q)|}$).

The relations between $A^2(T)$ and $H^2(T)$ are of the same nature. Indeed, let $\rho : \mathbb{D}/\Gamma \to X_c \setminus E$, $\rho(0) = P_0$, be the uniformization of the given surface. Then $H^2(\alpha)$ is a subspace of the standard $H^2$ in $\mathbb{D}$: $f(\rho(\zeta)) \in H^2$ for $f \in H^2(\alpha)$, moreover

$$\|f\|^2 = \int_T |f(\rho(t))|^2 dm(t),$$

where $dm$ is the Lebesgue measure. Fix a fundamental set $E$ for the action of $\Gamma$ on $T$, $T = \cup_{\gamma \in \Gamma^*} E(\gamma)$. Then

$$\|f\|^2 = \int_E |f(\rho(t))|^2 |\psi(t)|^2 dm(t),$$

where

$$|\psi(t)|^2 := \sum_{\gamma \in \Gamma^*} |\gamma'(t)|.$$  

Again, we can consider $\psi$ as an outer function and then

$$f \mapsto (f \circ \rho)\psi.$$
is the unitary map from $H^2(\alpha) \rightarrow A^2(\alpha\beta)$, where the character $\beta$ is generated by the 1-form $\psi$. The $\beta$ here is a particular character, so when $\alpha$ runs on the whole group $\Gamma^*$ $\alpha\beta$ covers also all characters, and we have one to one correspondence between two ways of writing of the Hardy spaces.

But, as we noted above, working with coverings, it will be convenient to use character automorphic $H^2$–spaces with respect to the following specific measure

$$\omega = \frac{1}{d} \sum_{i=1}^{l} \omega_{P_i},$$

(27)

associated with a system of points $\{P_i\}$ on $X_c \setminus E$. Naturally in what follows $P_i$’s are infinities on $X_c \setminus E$.

Now we can go back to the coverings. Let $\Gamma_X$ (respectively $\Gamma_Y$) be the fundamental group on $X_c \setminus E$ (respectively $Y_c \setminus E$). We have $\pi_* : \Gamma_Y \rightarrow \Gamma_X$ ($\pi_*(\gamma)$ is the image of a contour $\gamma \in \Gamma_Y$) and $\pi^* : \Gamma_X \rightarrow \Gamma_Y$ ($\pi^*(\gamma)$ is the full preimage of a contour $\gamma \in \Gamma_X$). Note that $\pi^*\pi_* = d\text{Id}$. The maps $\pi_* : \Gamma_Y \rightarrow \Gamma_X$ and $\pi^* : \Gamma_X \rightarrow \Gamma_Y$ are defined by duality.

For a system of points $\{P_i\}_{i=1}^l$, $P_i \in X_c \setminus E$, define the measure $\omega_*$ on $E$ by (27). Let $\{Q_k^{(i)}\}_{i=1}^d$ be the unitary map from $H^2(\alpha) \rightarrow A^2(\alpha\beta)$, where $\beta$ is a particular character, so when $\alpha$ runs on the whole group $\Gamma^*$ $\alpha\beta$ covers also all characters, and we have one to one correspondence between two ways of writing of the Hardy spaces.

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For a system of points $\{P_i\}_{i=1}^l$, $P_i \in X_c \setminus E$, define the measure $\omega_*$ on $E$ by (27). Let $\{Q_k^{(i)}\}_{i=1}^d = \pi^{-1}(P_i)$. Define

$$\omega^* = \frac{1}{d} \sum_{i=1}^{l} \omega_{Q_k^{(i)}}.$$

In this case

$$\int_F (f \circ \pi) \omega^* = \int_E f \omega_*.$$

Moreover,

$$\int_F f \omega^* = \int_E (\mathcal{L}f) \omega_*,$$

where $(\mathcal{L}f)(P) = \frac{1}{d} \sum_{\pi(Q) = P} f(Q)$. (28)

As a direct consequence of (28), (29) we get

**Lemma 5.1.** The map $V : H^2(\alpha, \omega_*) \rightarrow H^2(\pi^*\alpha, \omega^*)$, defined by

$$Vf = f \circ \pi, \quad f \in H^2(\alpha, \omega_*),$$

(30)

is an isometry with

$$(V^*f)(P) = (\mathcal{L}f)(P), \quad f \in H^2(\pi^*\alpha, \omega^*).$$

(31)

Also,

$$Vk_{P_0}^{\alpha} = \frac{1}{d} \sum_{\pi(Q^{(0)}) = P_0} k_{Q^{(0)}}^{\alpha}.$$  

(32)

for the reproducing kernel $k_{P_0}^{\alpha} \in H^2(\alpha, \omega_*)$.

**Theorem 5.2.** Let $z_* : X_c \setminus E \rightarrow \overline{\mathbb{C}}$. Using notations introduced above, assume that

$$z_* : E \rightarrow \mathbb{R}, \quad z_*^{-1}(\infty) \subset \{P_i\}_{i=1}^l.$$  

Let $z_*^{(\alpha)}$ be the multiplication operator by $z_*$ with respect to the basis (16). For an arbitrary ordering of $\{Q_k^{(i)}\}_{1 \leq k \leq l, 1 \leq i \leq l}$ subordinated to the ordering of $\{P_i\}$ consider the operator $z^*(\pi^*\alpha)$ with the symbol $z^* := z_* \circ \pi$ and the related by (30) isometry $V$. Then, the following relations hold

$$V^*z^*(\pi^*\alpha) = z_*^{(\alpha)}V^*, \quad V^*S^d = SV^*.$$  

(33)
Equations (33) are very close to the so called Renormalization Equations that we start to discuss now.

6. The renormalization of periodic matrices

We recall some basic facts from the spectral theory of periodic Jacobi matrices. The spectrum $E$ of any periodic matrix $J$ is an inverse polynomial image

$$E = U^{-1}[-1,1]$$

the polynomial $U$ of degree $g + 1$ should have all critical points $\{c_U\}$ real and for all critical values $|U(c_U)| \geq 1$. For simplicity we assume that $|U(c_U)| > 1$. Then the spectrum of $J$ consists of $g$ intervals

$$E = [b_0, a_0] \cup \bigcup_{j=1}^{g} (a_j, b_j).$$

Also it would be convenient for us to normalize $U$ by a linear change of the variable such that $b_0 = -1$ and $a_0 = 1$. In this case $U$ is a so called expanding polynomial.

Having the set $E$ of the above form fixed, let us describe the whole set of periodic Jacobi matrices $J(E)$ with the given spectrum. To this end we associate with $U$ the hyper-elliptic Riemann surface (the surface is given by (6) with $\lambda = b^N, N = g + 1$)

$$X = \{Z = (z, \lambda) : \lambda - 2U(z) + \lambda^{-1} = 0\}.$$

The involution on it we denote by $\tau$,

$$\tau Z := \left(z, \frac{1}{\lambda}\right) \in X.$$  (35)

The set

$$X_+ = \{Z \in X : |\lambda(Z)| < 1\}$$

we call the upper sheet of $X$. Note $X_+ \simeq \mathbb{C} \setminus E$, in fact, $z(Z) \in \mathbb{C} \setminus E$ if $Z \in X_+$.

The following well known theorem describes $J(E)$ in terms of real divisors on $X$. The Jacobian variety of $X$, $\text{Jac}(X)$, is a $g$ dimensional complex torus, $\text{Jac}(X) \simeq \mathbb{C}^g/L(X)$, where $L$ is a lattice (that can be chosen in the form $L = \mathbb{Z}^g + \Omega\mathbb{Z}^g$ with $\Im \Omega > 0$). Consider the $g$ dimensional real subtorus consisting of divisors of the form

$$D(E) = \{D = D_+ - D_C, D_+ := \sum_{i=1}^{g} Z_i : Z_i \in X, z(Z_i) \in [a_i, b_i]\},$$

here $D_C$ is a point of normalization that we choose of the form

$$D_C := \sum_{i=1}^{g} C_i : C_i \in X, z(C_i) = (c_U)_i, |\lambda(C_i)| > 1,$$

— the collections of the points on the lower sheet with the $z$–coordinates at the critical points. (At least topologically, it is evident $D(E) \simeq \mathbb{R}^g/\mathbb{Z}^g$).

Theorem 6.1. For given $E$ of the form (34) there exists an one-to-one correspondence between $J(E)$ and $D(E)$.

Let now $\hat{U}$ and $T$ be polynomials of the described above form, and we define $U = \hat{U} \circ T$. Then we have a covering $\pi$ of the Riemann surface $\hat{X}$ associated to $\hat{U}$ by the surface $X$ associated to $U$:

$$\pi(z, \lambda) = (T(z), \lambda),$$  (36)
According to the general theory, this covering generates different natural mappings \[\pi^* : \text{Jac}(\tilde{X}) \rightarrow \text{Jac}(X).\] Thus, in combination with Theorem 6.1, we get the map

\[
\begin{array}{ccc}
D(\tilde{E}) & \rightarrow & D(E) \\
\downarrow & & \downarrow \\
J(\tilde{E}) & \rightarrow & J(E)
\end{array}
\] (39)

We study this map in terms described in the previous section.

The differential \(\frac{1}{2\pi i}d\log b\), being restricted on \(\partial X_+\), is the harmonic measure \(d\omega\) of the domain \(\mathbb{C} \setminus E\) with pole at infinity. The space \(L^p(\partial X_+)\), in a sense, is the \(L^p\) space with respect to the harmonic measure, but it should be mentioned that \(\partial X_+ = (E - i0) \cup (E + i0)\), i.e., an element \(f\) of \(L^p(\partial X_+)\) may have different values \(f(x + i0)\) and \(f(x - i0), x \in E\), and \(H^p(X_+)\) is the closure of the set of holomorphic functions uniformly bounded in \(X_+\) with respect to this norm.

Since \(\tilde{\lambda} \circ \pi = \lambda\) we have the relation

\[
(40)
\]

Thus, we get

\[
\int_{\partial X_+} f \, d\omega = \int_{\partial \tilde{X}_+} \frac{1}{d} \left( \sum_{\pi(Z) = \tilde{Z}} f(Z) \right) (\tilde{Z}) \, d\tilde{\omega}
\] (41)

for every \(f \in L^1(\partial X_+)\).

As it follows directly from (41), the covering (36) generates an isometrical enclosure

\[
v_+ : H^2(\tilde{X}) \rightarrow H^2(X_+)
\] (42)

acting in a natural way

\[
(v_+ f)(Z) = f(\pi(Z)).
\] (43)

**Remark.** As it was mentioned, the function \(b\) is not single valued but \(|b(z)|\) is a single valued function. We define the character \(\mu \in \Gamma^*\) by

\[
b(\gamma z) = \mu(\gamma)b(z).
\]

Let \(\gamma_j\) be the contour, that starts at infinity (or any other real point bigger than 1), go in the upper half–plane to the gap \((a_j, b_j)\) and then go back in the lower half–plane to the initial point. Assuming that \(b_0 < \ldots < a_j < b_j < a_{j+1} < \ldots < a_0\), we have \(\mu(\gamma_j) = e^{-2\pi i \frac{a_{j+1} - a_j}{a_j - a_{j+1}}},\) equivalently \(\omega(b_j, a_0) = \frac{a_j - a_{j+1}}{a_j - a_0}\). Note that the system of the above contours \(\gamma_j\) is a generator of the free group \(\Gamma^*(E)\). In other words a character \(\alpha\) is uniquely defined by the vector

\[
[\alpha(\gamma_1), \alpha(\gamma_2), \ldots, \alpha(\gamma_g)] \in \mathbb{T}^g.
\]

This sets an one–to–one correspondence between \(\Gamma^*(E)\) and \(\mathbb{T}^g\).
Recall the key role of the reproducing kernels $k^{\alpha}$ in our construction. In this particular case they especially well studied [10]. First of all, they have analytic continuation (as multivalued functions) on the whole $X$, so we can write $k^{\alpha}(Z)$.

**Theorem 6.2.** For every $\alpha \in \Gamma^*$ the reproducing kernel $k^{\alpha}(Z)$ has on $X$ exactly $g$ simple poles that do not depend on $\alpha$ and $g$ simple zeros. The divisor $D_+ = \sum_{j=1}^{g} Z_j$ of zeros

$$k^{\alpha}(Z_j) = 0$$

with the divisor of poles form the divisor

$$\text{div}(k^{\alpha}) = D_+ - D_C$$

that belongs to $D(E)$, moreover (45) sets an one–to–one correspondence between $D(E)$ and $\Gamma^*(E)$.

The functions $k^{\alpha}$ possess different representations, in particular, in terms of theta–functions [16], and the map $D \mapsto \alpha$ can be written explicitly in terms of abelian integrals (the Abel map).

**Summary.** The three objects $J(E)$, $D(E)$ and $\Gamma^*(E)$ are equivalent. Both maps $\Gamma^*(E) \to D(E)$ and $\Gamma^*(E) \to J(E)$ can be defined in terms of the reproducing kernels of the spaces $H^2(X_+, \alpha)$, $\alpha \in \Gamma^*(E)$. The first one is given by (45). It associates to the given $k^{\alpha}(Z)$ the sets of its zeros and poles (the poles are fixed and the zeros vary with $\alpha$). The matrix $J(\alpha) \in J(E)$ is defined as the matrix of the multiplication operator by $z(Z)$

$$z(Z)e_s^\alpha(Z) = p_s^\alpha e_{s+1}^\alpha(Z) + q_s^\alpha e_s^\alpha(Z) + p_{s+1}^\alpha e_{s+1}^\alpha(Z), \ Z \in X, \ s \in \mathbb{Z},$$

with respect to the basis

$$e_s^\alpha(Z) = b^\alpha(Z)K^{\alpha\mu^{-s}}(Z).$$

It’s really easy to see that $J(\alpha)$ is periodic: just recall that $b^{s+1}$ is single valued, that is, $\mu^{s+1} = 1$, and therefore the spaces $H^2(X_+, \alpha)$ and $H^2(X_+, \alpha\mu^{-(s+1)})$ (and their reproducing kernels) coincide.

Now we can go back to the Renormalization Equation. Note that $\pi$ acts naturally on $\Gamma(E)$:

$$\pi_{\gamma} = \{\pi(Z), \ Z \in \gamma\} \in \Gamma(\bar{E}), \ \text{for} \ \gamma \in \Gamma(E) .$$

The map $\pi^* : \Gamma^*(\bar{E}) \to \Gamma^*(E)$ is defined by duality:

$$(\pi^* \hat{\alpha})(\gamma) = \hat{\alpha}(\pi_{\gamma}).$$

**Theorem 6.3.** Let $T$, $T^{-1} : [-1, 1] \to [-1, 1]$, be an expanding polynomial. Let $\tilde{J}$ be a periodic Jacobi matrix with spectrum $\bar{E} \subset [-1, 1]$, and therefore there exists a polynomial $\tilde{U}$ such that $\bar{E} = \tilde{U}^{-1}[-1, 1]$ and a character $\hat{\alpha} \in \Gamma^*(\bar{E})$ such that $\tilde{J} = J(\tilde{\alpha})$. Then

$$J := J(\pi^* \alpha) = J_T(\tilde{J})$$

is the periodic Jacobi matrix with spectrum $E = U^{-1}[-1, 1]$, $U := \tilde{U} \circ T$, that satisfies the Renormalization Equation

$$V^*(z - J)^{-1}V = \frac{T'(z)}{d}(T(z) - J)^{-1},$$

(50)
where the isometry matrix $V$ is defined by $V|k| := |kd|$. 

**Remark 6.4.** Let us mention that the Renormalization Equation can be rewritten equivalently in the form of polynomials equations, as it should be since we have the map from one algebraic variety, $\text{Jac}(\tilde{X})$, in the another one, $\text{Jac}(X)$. Equation (50) is equivalent to, see [20],

$$V^*T(J) = \hat{J}V^*, \quad (51)$$

$$V^*\frac{T(z) - T(J)}{z - J}V = T'(z)/d. \quad (52)$$

**Proof of Theorem 6.3.** First we note, that for the operator multiplication by $z(Z)$ in $L^2(\partial X_+)$, the operator multiplication by $\tilde{z}(Z)$ in $L^2(\partial \tilde{X}_+)$, the spectral parameter $z_0$ and the isometry 

$$(vf)(Z) = f(\pi(Z)), \quad v : L^2(\partial X_+) \to L^2(\partial X_+),$$

we have

$$\int_{\partial X_+} \frac{1}{z_0 - z(Z)}|vf(Z)|^2 \, d\omega = \int_{\partial \tilde{X}_+} \left( \frac{1}{d} \sum_{\pi(Z) = \tilde{z}} \frac{1}{z_0 - z(Z)} \right) |f(\tilde{Z})|^2 \, d\tilde{\omega}. \quad (53)$$

It is evident, that

$$\frac{1}{d} \sum_{T(y) = x} \frac{1}{z_0 - y} = \frac{T'(z_0)/d}{T(z_0) - x}. \quad (54)$$

Thus

$$v^*(z_0 - z(Z))^{-1} v = (T'(z_0)/d)(T(z_0) - \tilde{z}(\tilde{Z}))^{-1}. \quad (55)$$

It remains to show that $\pi$ transforms the basis vector $\tilde{e}_n^\alpha = \hat{\tilde{b}}^n K^\alpha \tilde{n}$ into

$$e_n^{\pi \alpha} = \hat{b}^n K^{(\pi^* \alpha)\mu - nd} = (\hat{b}^n \circ \pi) K^{\pi^* (\tilde{\alpha}) \mu - nd}. \quad (56)$$

Or, what is the same, that $K^\alpha \circ \pi = K^{\pi^* \alpha}$ for all $\tilde{\alpha} \in \Gamma^*(\hat{E})$. Note that both functions are of norm one in the same space $H^2(X_+, \pi^* \alpha)$, in particular, they have the same character of automorphy $\pi^* \alpha \in \Gamma^*(\hat{E})$. Note, finally, that the divisor

$$\text{div}(k^\alpha \circ \pi) = \pi^{-1}(\hat{D}_+) - \pi^{-1}(\hat{D}_C),$$

where $\text{div}(k^\alpha) = \hat{D}_+ - \hat{D}_C$, belongs to $D(E)$, therefore $k^\alpha \circ \pi$ is the reproducing kernel and the theorem is proved. \hfill \square

Probably it would be better to call (50) the Renormalization Identity in the above theorem. The idea is that one can try to define $J$ as the solution of (50) with the given $\hat{J}$. Indeed, in the case of one–sided matrices such equation has the unique solution. Now we demonstrate that in two sided case for the given periodic $\hat{J}$ we can find $2^{d-1}$ solutions.

To find all this solutions of (50) let us look a bit more carefully at the above proof. Note that the same identity (53) holds for any isometry $v$ of the form

$$vf = v_0f = \theta(f \circ \pi),$$

where $\theta$ is a unimodular ($|\theta| = 1$) function on $\partial X_+$.

Concerning the second part of the proof, let us mention that the set of critical points of $U$ splits in two sets:

$$\{c_U\} = T^{-1}\{c_{\tilde{U}}\} \cup \{c_T\}.$$
Correspondingly,
\[ \sum (C_U)_j = \sum_k \sum_{\pi(C_U)_{k,j} = (C_U)_k} (C_U)_{k,j} + \sum (C_T)_j, \]
and the divisor of $k^{\tilde{a}} \circ \pi$ consists of two parts, that one that depends on $\tilde{a}$
$\pi^{-1}(\hat{D})$,
and that part that corresponds to the critical points of the polynomial $T$
\[ \{(C_T)_j\}_{j=1}^{d-1}, \]
since
\[ D = \text{div}(k^{\tilde{a}} \circ \pi) = \pi^{-1}(\hat{D}) + \sum_{j=1}^{d-1} (C_T)_j - \pi^{-1}(\hat{D}_C) - \sum_{j=1}^{d-1} (C_T)_j. \]
Thus we can fix an arbitrary system of points $\{Z_{c,j}\}_{j=1}^{d-1}$ such that $z(Z_{c,j})$ belongs
to the same gap in the spectrum $E$ as the critical point $(c_T)_j$. If $\theta$ is the canonical
product on $X$ with the divisor
\[ \text{div}(\theta) = \sum_{j=1}^{d-1} Z_{c,j} - \sum_{j=1}^{d-1} (C_T)_j, \]
then $\theta k^{\tilde{a}} \circ \pi$ is the reproducing kernel simultaneously for all $\tilde{a} \in \Gamma^*(\hat{E})$. But to
make $\theta$ unimodular (zeros and poles are symmetric) our choice is restricted just to
$Z_{c,j} = (C_T)_j$ or $Z_{c,j} = \tau(C(T)_j)$. Note that $\tau(C_T)_j - (C_T)_j$ is the divisor of the
Complex Green function $b_{(C_T)_j}$. In this way we arrive at

**Theorem 6.5.** For an expanding polynomial $T$, and a periodic Jacobi matrix
$\hat{J} = J(\tilde{a}), \tilde{a} \in \Gamma^*(\hat{E})$ as in Theorem 6.3 there exist $2^{d-1}$ solutions of the Renor-
malization Equation (50). Denote by $\mu_{(c_T)_j}$ the character generated by the Green’s
function $b_{(c_T)_j}, b_{(c_T)_j} \circ \gamma = \mu_{(c_T)_j}(\gamma) b_{(c_T)_j}$. Then these solutions are of the form

\[ J := J(\eta_b \pi^* \tilde{a}), \quad \eta_b := \prod_{j=1}^{d-1} \mu_{(c_T)_j}^{(1+\delta_{(c_T)_j})}, \]

as before
\[ \delta = \{\delta_{(c_T)_j}\}, \quad \delta_{(c_T)_j} = \pm 1. \]

**Proof.** We define the isometry
\[ (v f)(Z) = \left( \prod_{j=1}^{d-1} b_{(c_T)_j}^{(1+\delta_{(c_T)_j})} (Z) \right) f(\pi(Z)) \]
and then repeat the arguments of the proof of Theorem 6.3. \( \square \)

7. **The Renormalization Equation for Two–sided Jacobi Matrices**

**GENERAL CASE**

In this section we assume that
\[ T(z) = z^d - qdz^{d-1} + ... \]
is a monic expanding polynomial. Under this normalization $T^{-1} : [-\xi, \xi] \to [-\xi, \xi]$ for a certain $\xi > 0$. It was proved in [20] that the Renormalization Equation has
$2^{d-1}$ solutions for every two sided Jacobi matrix $\hat{J}$ with the spectrum in $[-\xi, \xi]$. 
not only for periodic one. Moreover, they are the only possible solutions. For the reader convenience we formulate these theorems here.

By \( l^2_\pm(s) \) we denote the spaces which are formed by \( \{s + k\} \) with \( k \leq 0 \) and \( k \geq 0 \) respectively, that is \( l^2(\mathbb{Z}) = l^2_+(s) \oplus l^2_-(s + 1) \). Correspondingly to these decompositions we set \( \tilde{J}_\pm(s) = P_{l^2_+(s)} J [l^2_+(s)]. \) Recall that a (finite or infinite) one–sided Jacobi matrix is uniquely determined by its so called resolvent function

\[
\tilde{r}_\pm(z, s) = \langle s | (\tilde{J}_\pm(s) - z)^{-1} | s \rangle.
\]  

This set of solutions we parametrize by a collections of vectors

\[
\delta := \{\delta_c\}_c,
\]

where each component \( \delta_c \) can be chosen as plus or minus one.

**Theorem 7.1.** Fix a vector \( \delta \) of the form (57). For a given \( \tilde{J} \) with the spectrum on \([-\xi, \xi]\) define the Jacobi matrix \( J \) according to the following algorithm:

For \( s \in \mathbb{Z} \) we put

\[
\frac{1}{T^{(s)}(c)} = -\tilde{r}_-(T(c), s), \quad \text{if } \delta_c = -1,
\]

and

\[
T^{(s)}(c) = -\tilde{r}_+(T(c), s + 1), \quad \text{if } \delta_c = 1,
\]

where the functions \( \tilde{r}_\pm(z, s) \) are defined by (56). Then define the monic polynomial \( T^{(s)}(z) \) of degree \( d \) by the interpolation formula

\[
T^{(s)}(z) = (z - q)T'(z)/d + \sum_{c \in T^{(s)}(c) = 0} \frac{T'(z)}{(z - c)T''(c)}T^{(s)}(c).
\]

Define the block

\[
J^{(s)} = \begin{bmatrix}
q_{sd} & P_{sd+1} & \\
P_{sd+1} & q_{sd+1} & P_{sd+2} \\
& \ddots & \ddots \\
& & P_{sd+d-2} & q_{sd+d-2} & P_{sd+d-1} \\
& & & P_{sd+d-1} & q_{sd+d-1}
\end{bmatrix},
\]

by its resolvent function

\[
\langle 0 | (z - J^{(s)})^{-1} | 0 \rangle = \frac{T'(z)/d}{T^{(s)}(z)},
\]

where \( T^{(s)}(z) \) is a monic polynomial of degree \( d \). Finally define the entry \( p_{sd+d} \) by \( p_{sd+1} \cdots p_{sd+d} = \tilde{p}_{s+1} \).

We claim that the matrix \( J = J(\delta, \tilde{J}) \), combined with such blocks and entries over all \( s \), satisfies (50).

**Theorem 7.2.** Theorem 7.1 describes the whole set of solutions of the Renormalization Equation.

In [20] we concentrate only on one of the solutions of (50), namely that one that related to the vector

\[
\delta_- = \{-1, \ldots, -1\},
\]

that is all \( T^{(s)}(c) \) are defined by (58). Note, \( J(\delta_-, \tilde{J}) = J_T(\tilde{J}) \) for a periodic \( \tilde{J} \). Precisely for this solution we proved (main Theorem 1.1 in [20]):
Theorem 7.3. Let $\tilde{J}$ be a Jacobi matrix with the spectrum on $[-\xi, \xi]$. Then the Renormalization Equation (50) has a solution $J = J(\delta, \tilde{J})$ with the spectrum on $T^{-1}([-\xi, \xi])$. Moreover, if $\min |t_j| \geq 10\xi$ then
\[
\|J(\delta, \tilde{J}_1) - J(\delta, \tilde{J}_2)\| \leq \kappa \|\tilde{J}_1 - \tilde{J}_2\|
\]
(63)
with an absolute constant $\kappa < 1$.

Let us emphasize the especial role of the position of the critical values: the transform $\mathcal{J}_T(\tilde{J})$ is a contraction as soon as critical values are distant sufficiently far from the spectrum.

In this work we add certain remarks concerning other solutions of (50).

7.1. The duality $\delta \mapsto -\delta$. At least for one more solution of the Renormalization Equation is a contraction.

Theorem 7.4. The dual solution of the Renormalization Equation $J(\tilde{J}, -\delta)$, possesses the contractibility property simultaneously with $J(\tilde{J}, \delta)$.

It deals with the following universal involution acting on Jacobi matrices
\[
J \rightarrow J_\tau := U_\tau J U_\tau, \quad \text{where } U_\tau |l| = |1 - l|.
\]
(64)
Obviously $VU_\tau = U_\tau S^{1-d}V$. Thus, having $J$ as a solution of the renormalization equation corresponding to $\tilde{J}$ we have simultaneously that $S^{d-1}J_\tau S^{1-d}$ solves the equation with the initial $J_\tau$. The following lemma describes which branch corresponds to which in this case.

Lemma 7.5. Let $J = J(\tilde{J}, \delta)$ then
\[
S^{d-1}J_\tau S^{1-d} = J(\tilde{J}_\tau, -\delta).
\]
(65)

Proof. We give a proof using the language of Sect. 5, so formally we prove the claim only for periodic matrices.

Note that the involution (64) is strongly related to the standard involution $\tau$ (35) on $X$. Indeed, the function $K(\tau Z, \alpha)$ has the divisor
\[
\tau D_+ - \tau D_C = (\tau D_+ - D_C) - (\tau D_C - D_C),
\]
that is,
\[
K(\tau Z, \alpha) = \frac{K(Z, \beta)}{b_{c_1}(Z) \ldots b_{c_d}(Z)},
\]
and $\beta = \nu \alpha^{-1}$, where $\nu = \mu_{c_1} \ldots \mu_{c_d}$. Due to this remark and the property $z(\tau Z) = z(Z)$ we have
\[
(J(\alpha))_\tau = J(\nu \mu \alpha^{-1}).
\]
(66)
Now we apply (66) to prove (65). Let $\tilde{J}_\tau = J(\tilde{\alpha})$ with $\tilde{\alpha} \in \Gamma^+(\tilde{X}_+)$. Or, in other words, $\tilde{J} = J(\mu \tilde{\alpha})^{-1}$. Then by (55)
\[
J(\tilde{J}, \delta) = J(\eta_\delta \pi^*(\mu \tilde{\alpha}^{-1})), \quad \eta_\delta := \prod_{j=1}^{d-1} \mu_{(c_j)}^{1+1}(1+\delta(\tau Z)_j).
\]
But $\pi^* \mu = \mu^d$ and $\pi^*(\tilde{\nu}) = \nu \eta_\delta^{-1}$ (just to look at the characters of the corresponding Blaschke products). Thus, having in mind that $\eta_\delta \eta_{-\delta} = \eta_{\delta +}$, we obtain
\[
J(\tilde{J}, \delta) = J(\mu^d \nu \eta_{-\delta}^{-1} \pi^*(\tilde{\alpha}^{-1})).
\]
Using again (66) we get
\[ (J(\tilde{J}, \delta))_\tau = J(\mu^{1-d} \eta_{-d} \pi^*(\tilde{\alpha})) = S^{1-d} J(\eta_{-d} \pi^*(\tilde{\alpha})) S^{-1}, \]
and the lemma and Theorem 7.4 are proved. □

Having two different contractive branches of solutions of the renormalization equation, following [14], to an arbitrary sequence
\[ \epsilon = \{\epsilon_0, \epsilon_1, \epsilon_2 \ldots\}, \quad \epsilon_j = \delta_{\pm}. \]
we can associate a limit periodic matrix \( J \) with the spectrum on Jul(\( T \)). For a fixed sufficiently hyperbolic polynomial \( T \), we define \( J \) as the limit of
\[ J_n := J(\eta_{\epsilon_0} \pi^* \eta_{\epsilon_1} \ldots \pi^* \eta_{\epsilon_{n-1}}). \quad (67) \]

### 7.2. Other solutions of the Renormalization Equation and the Ruelle operators.

Iterating the renormalization transform
\[ J_{n+1} := J(J_n, \delta_-), \]
we obtain almost periodic operator \( J = \lim J_n \) with the spectrum on the Julia set of the sufficiently expanding polynomial \( T \) (the key point here, of course property (63)). This operator possesses the following structure [20]: it is the direct sum of two (one–sided) Jacobi matrices \( J_{\pm} \). Moreover, the spectral measure of the operator \( J_+ \) is the balanced measure \( \mu \) on the Julia set. That is it is the eigen–measure of the Ruelle operator \( L^* \), where
\[ (Lf)(x) = \frac{1}{d} \sum_{Tn(y) = x} f(y). \]
The spectral measure \( \nu \) of \( J_- \) is the so called Bowen–Ruelle measure. It is the eigen–measure for \( L_2^* \),
\[ (L_2f)(x) = \sum_{T(y) = x} \frac{f(y)}{T'(x)^2}, \]
We conjecture that actually all branches of solutions of the renormalization equation are contractions for sufficiently hyperbolic \( T \). At least the previous subsection looks as a quite strong indication in this direction: considering, instead of initial \( T \), \( T^2 = T \circ T \) or its bigger powers, we get, as in (67), several \( \delta \)'s,
\[ \eta_\delta = \eta_{\epsilon_0} \pi^* \eta_{\epsilon_1} \ldots \pi^* \eta_{\epsilon_{n-1}}, \]
possessing the contractibility property with respect to the polynomial \( T^n \) and different from \( \delta_{\pm} \) (related to \( \pi^* \)).

Similarly to the above statement we formulate

**Conjecture.** Let \( T(z) \) be an expanding polynomial. For a given \( \delta \) we factorize \( T'(z) = A_1(z) A_2(z) \) putting in the first factor all critical points related to \( \delta_e = 1 \). Denote by \( \sigma_{1,2} \), the (nonnegative) eigen–measures, corresponding to the Ruelle operators
\[ (L_{A_i} f)(x) = \sum_{T(y) = x} \frac{f(y)}{A_i(y)^2}, \quad (68) \]
i.e., \( L_{A_i}^* \sigma_i = \rho_i \sigma_i \). Finally let \( J_{1,2} \) be the one-sided Jacobi matrices associated with \( \sigma_{1,2} \). We conjecture that the iterations \( J_{n+1} := J(J_n, \delta) \), converges to the block matrix \( J = J_- \oplus J_+ \) with \( J_- = J_1 \) and \( J_+ = J_2 \). In particular this means that all such operators are limit periodic.
Note that the conjecture holds true also, say, for \( T^2(z) \) and \( A_1(z) = T'(z), \) \( A_2 = T'(T(z)) \).

### 7.3. Shift transformations with the Lipschitz property.

We say that the direction \( \eta \in \Gamma^* \) has the Lipschitz property with a constant \( C(\eta) \) if for all \( \alpha, \beta \in \Gamma^* \)

\[
\| J(\eta \alpha) - J(\eta \beta) \| \leq C(\eta) \| J(\alpha) - J(\beta) \|. \tag{69}
\]

Then, one can get the contractibility of the map \( \eta \pi^* \) in two steps:

\[
\| J(\eta \pi^* \hat{\alpha}) - J(\eta \pi^* \hat{\beta}) \| \leq C(\eta) \| J(\pi^* \hat{\alpha}) - J(\pi^* \hat{\beta}) \|
\leq C(\eta) \kappa \| J(\hat{\alpha}) - J(\hat{\beta}) \|. \tag{70}
\]

Note, that in fact the situation is a bit more involved because we should be able to compare Jacobi matrices with different spectral sets, for example, when \( E_i = T^{-1} E_i, \) \( \hat{E}_1 \neq \hat{E}_2. \) But we just want to indicate the general idea. Note, in particular, that for directions \( \eta \delta \) of the form (55) such a comparison is possible. Of course, for our goal the constant \( C(\eta) \) should be uniformly bounded when we increase the level of sufficient hyperbolicity of \( T \) making \( \kappa \) smaller.

However the key point of this remark (this way of proof) is that, actually, we do not need to constrain ourselves by the form of the vector \( \eta. \) Combining a “Lipschitz” shift by \( \eta \) (the direction is restricted just by this property) with a sufficiently contractive pull--back \( \pi^* \) we arrive at an iterative process that produces a limit periodic Jacobi matrix with the spectrum on the same \( \text{Jul}(T). \)

In the next subsection we give examples of directions with the required property, see Corollary 7.8.

We do not have a proof of the Lipschitz property of \( \eta \delta \)'s, but there is a good chance to generalize the result of the next subsection in a way that at least some of the directions \( \eta \delta \) will be also available.

Finally, we would be very interested to know, whether there is in general a relation between the form of the “weight” vector \( \eta \) and the corresponding weights of the Ruelle operators (if any exists).

### 7.4. Quadratic polynomials and the Lipschitz property of the Darboux transform.

Consider the simplest special case \( T(z) = \rho(z^2 - 1) + 1, \rho > 2. \) Note that the spectral set \( E = T^{-1} \hat{E} \) is symmetric, moreover the matrix related to \( H^2(\pi^* \hat{\alpha}) \) has zero main diagonal (as well as a one--sided matrix related to a symmetric measure). Now we introduce a decomposition of \( H^2(\pi^* \hat{\alpha}) \) which is very similar to the standard decomposition into even and odd functions.

We define the two--dimensional vector--function representation of \( f \in H^2(\pi^* \hat{\alpha}) \)

\[
f \mapsto \frac{1}{\sqrt{2}} \begin{bmatrix} f(Z_1(\hat{Z})) \\ f(Z_2(\hat{Z})) \end{bmatrix} \mapsto \begin{bmatrix} g_1(\hat{Z}) \\ g_2(\hat{Z}) \end{bmatrix}, \tag{71}
\]

where

\[
\begin{bmatrix} g_1(\hat{Z}) \\ g_2(\hat{Z}) \end{bmatrix} = \frac{1}{2} \begin{bmatrix} f(Z_1(\hat{Z})) + f(Z_2(\hat{Z})) \\ f(Z_1(\hat{Z})) - f(Z_2(\hat{Z})) \end{bmatrix},
\]

the first component, in a sense, is even and the second is odd. To be more precise, let us describe analytical properties of this object in \( \partial \hat{X}_+. \)

Note that due to

\[
\int_{\partial \hat{X}_+} |f|^2 d\omega = \int_{\partial \hat{X}_+} \frac{1}{2} \sum_{\pi(\hat{Z}) = \hat{Z}} |f|^2 d\omega
\]
metrically it is of $L^2$ with respect to $\tilde{\omega}$, moreover the transformation is norm-preserved.

It is evident that the function $g_1$ belongs to $H^2(\tilde{X}_+, \tilde{\alpha})$. Consider the second function. Note that the critical points of $T$ are zero and infinity. For a small circle $\gamma$ around the point $T(0) = -\rho + 1$ we have $g_2 \circ \gamma = -g_2$ and the same property for a contour $\gamma$ that surrounds infinity. Let us introduce

$$\Delta^2 := b_{T(0)} b.$$ 

Note that for the above contours we have $\Delta \circ \gamma = -\Delta$. We are going to represent $g_2$ in the form $g_2 = \Delta \hat{g}_2$ and to claim that $\hat{g}_2$ has nice automorphic properties in $\tilde{X}_+$. Let us note that

$$\tilde{b} \tilde{z} - T(0) \tilde{b} = \tilde{b} \tilde{z} - T(0),$$

is an outer function in the domain $\mathbb{C} \setminus \tilde{E} \simeq \tilde{X}_+$. So, the square root of this function is well defined. We put

$$\tilde{b} \phi := \sqrt{\tilde{b} \tilde{z} - T(0)}$$

and denote by $\tilde{\eta}$ the character generated by $\phi$, $\phi \circ \gamma = \eta(\gamma) \phi$. Thus (72) reduces the ramification of the function $\Delta$ to the function $\phi$, which is well defined in the domain, and to the elementary function $\sqrt{\tilde{z} - T(0)}$.

**Theorem 7.6.** The transformation $f \mapsto g_1 \oplus \hat{g}_2$ given by (95) is a unitary map from $H^2(\pi^* \tilde{\alpha})$ to $H^2(\tilde{\alpha}) \oplus H^2(\tilde{\alpha} \tilde{\eta})$. Moreover with respect to this representation

$$zf \mapsto \begin{bmatrix} 0 & \phi \\ \phi & 0 \end{bmatrix} \begin{bmatrix} g_1 \\ \hat{g}_2 \end{bmatrix}$$

and

$$v_+ f \mapsto f \oplus 0, \quad f \in H^2(\tilde{\alpha}),$$

where the isometry $v_+ : H^2(\tilde{\alpha}) \to H^2(\pi^* \tilde{\alpha})$ is defined by (43).

**Proof.** By the definition of $\Delta$ we have

$$g_2 = \Delta \hat{g}_2, \text{ where } \hat{g}_2 \in H^2(\tilde{\alpha} \tilde{\eta}).$$

Further, since

$$z_{1,2} = \pm \sqrt{\frac{\tilde{z} - T(0)}{\rho}},$$

we have, say for the second component,

$$\frac{1}{\Delta} \frac{(zf)(Z(\tilde{Z}_1)) - (zf)(Z(\tilde{Z}_2))}{2} = \frac{\tilde{z} - T(0) f(Z(\tilde{Z}_1)) + f(Z(\tilde{Z}_2))}{\rho \Delta^2} = \phi g_1.$$  

(75)

Since on the boundary of the domain

$$\phi^2 \Delta^2 = \frac{\tilde{z} - T(0)}{\rho} = |\phi|^2$$

(the second expression is positive on $\partial \tilde{X}_+$) we have

$$\phi \Delta^2 = \overline{\phi} \text{ on } \tilde{E}.$$  

(76)

Using this relation, similarly to (75), we prove the identity of the first components in (98).
Theorem 7.7. The multiplication operator \( \phi : L^2(\partial \tilde{X}_+) \to L^2(\partial \tilde{X}_+) \) with respect to the basis systems (47) related to \( \tilde{\alpha} \) and \( \tilde{\eta} \), respectively, is a two diagonal matrix \( \Phi \). Moreover,

\[
\Phi^* \Phi = \frac{J(\tilde{\alpha}) - T(0)}{\rho}, \quad \Phi \Phi^* = \frac{J(\tilde{\eta} \tilde{\alpha}) - T(0)}{\rho}.
\]

In other words, the transformation \( J(\tilde{\alpha}) \to J(\tilde{\eta} \tilde{\alpha}) \) is the Darboux transform.

Proof. First of all \( \phi \) is a character–automorphic function with the character \( \tilde{\eta} \) with a unique pole at infinity \( \hat{\phi} \) is an outer function. Therefore the multiplication operator acts from \( \hat{b} H^2(\tilde{\alpha})^{-1} \) to \( H^2(\tilde{\eta} \tilde{\alpha}) \). Therefore, the operator \( \Phi \) has only one non–trivial diagonal above the main diagonal. The adjoint operator has the symbol \( \tilde{\phi} \). According to (99) it has holomorphic continuation from the boundary inside the domain. Thus \( \Phi^* \) is a lower triangular matrix. Combining these two facts we get that \( \Phi \) has only two non–trivial diagonals. Then, just comparing symbols of operators on the left and right parts of (100), we prove these identities. \( \square \)

Corollary 7.8. Let \( \tilde{J}_{1,2} \) be periodic Jacobi matrices with the spectrum on \([-1,1]\). Let \( \text{Drb}(\tilde{J}_{1,2}, \rho) \) be their Darboux transforms. Then

\[
\|\text{Drb}(\tilde{J}_1, \rho) - \text{Drb}(\tilde{J}_2, \rho)\| \leq C(\rho)\|\tilde{J}_1 - \tilde{J}_2\|.
\]

Proof. For the given \( \tilde{J}_{1,2} \) we define \( J_{1,2} \) via the quadratic polynomial \( T(z) = \rho(z^2 - 1) + 1 \). Being decomposed into even and odd indexed subspaces they are of the form

\[
J_{1,2} = \begin{bmatrix} 0 & \Phi_{1,2}^* \varepsilon_2 \\ \Phi_{1,2} \varepsilon_2 & 0 \end{bmatrix}.
\]

Due to the main theorem, that gives the uniform estimate for \( \|J_1 - J_2\| \), we have

\[
\|\Phi_1 - \Phi_2\| \leq \kappa(\rho)\|\tilde{J}_1 - \tilde{J}_2\|,
\]

with \( \kappa(\rho) = \frac{\rho C}{2} \rho^{-2} \). C is an absolute constant. Using (100) we get (80) with \( C(\rho) = \frac{2}{\rho^2 - 2} \). \( \square \)

8. Double covering \( \pi(v) = \tau v - 1/v \)

8.1. Over the simply–connected domain. The simplest expanding double coverings are \( T_1 = v^2 - \lambda \), \( \lambda > 2 \), and \( T_2 = \tau v - 1/v \), \( \tau > 1 \). Denote by \( \xi_{1,2} \) the fixed points \( T_{1,2}(\xi_{1,2}) = \xi_{1,2} \), \( \xi_{1,2} > 0 \). To start with we give a complete description of finite difference operators related to these coverings over the simply–connected domain \( \mathbb{C} \setminus E \), where \( E = [-\xi_{1,2}, \xi_{1,2}] \).

More precisely, we start with a Jacobi matrix \( J_0 \) with constant coefficients. Under the normalization \( \sigma(J_0) = E \) we have \( J = \frac{\xi_{1,2}}{2} (S + S^{-1}) \). That is the symbols of this operator \( (z_*, b_*) \) are related by

\[
z_* = \frac{\xi_{1,2}}{2} \left( \frac{1}{b_*} + b_* \right),
\]

\( (b_*) \) is the Green function of the domain \( \mathbb{C} \setminus E \). For \( \pi = T_1 \) or \( \pi = T_2 \) we consider the open Riemann surfaces \( Y_c \setminus F \) with \( Y_c \simeq \mathbb{C} / \mathbb{Z} \), \( F = \pi^{-1}(E) \) and describe operators with the symbols \( (z^*, b^*) \):

\[
z^* = z_* \circ \pi \quad \text{and} \quad (b^*)^2 = b_* \circ \pi.
\]
The main difference between these two cases is that in the first one we have only one infinity on $Y$ ($\infty \in \mathbb{C}$) and in the second case there are two infinities: $0, \infty \in \mathbb{C}$. Correspondingly an intrinsic basis contains the reproducing kernels related only to one fixed point in the first case and to two specific points in the second case. As result the multiplication operator by $v$ with respect to this basis is a Jacobi matrix in the first case and a five diagonal matrix (of a special form, see Lemma 2.1) in the second.

8.1.1. $\pi = v^2 - \lambda$. Due to (81), (82) we have

$$v^2 - \lambda = \frac{\xi_1}{2} \left( \frac{1}{(b^*)^2} + (b^*)^2 \right).$$

(83)

Since $v$ is the symbol of a Jacobi matrix,

$$v \sim S\Lambda_1 + \Lambda_0 + \Lambda_1 S^{-1},$$

(84)

where $\Lambda_{0,1}$ are diagonal of period two matrices, and $(b^*)^2 \sim S^2$, we have from (83)

$$\lambda_0^{(1)} \lambda_1^{(1)} = \xi_1/2,$$

$$\lambda_0^{(0)} + \lambda_1^{(0)} = 0,$$

$$\lambda_0^{(0)} + \lambda_1^{(0)} + (\lambda_0^{(0)})^2 + (\lambda_1^{(0)})^2 = \lambda.$$  

(85)

8.1.2. $\pi = \tau v - 1/v$. Repeating arguments (82), (81) we get instead of (83)

$$\tau v - 1/v = \frac{\xi_2}{2} \left( \frac{1}{(b^*)^2} + (b^*)^2 \right).$$

or

$$\tau v^2 - 1 = \frac{\xi_2}{2} \left( \frac{1}{(b^*)^2} + (b^*)^2 \right) v.$$ 

(86)

In this case we have a five diagonal matrix,

$$v \sim S^2\Lambda_2 + SA_1 + \Lambda_0 + \Lambda_1 S^{-1} + \Lambda_2 S^{-2},$$

(87)

but of a specific structure (see Lemma 2.1). Depending of ordering of infinities one of coefficients $\lambda_0^{(2)}, \lambda_1^{(2)}$ vanishes. Say, $\lambda_1^{(2)} = 0$, respectively $\lambda_0^{(2)} \neq 0$. Using $(b^*)^2 \sim S^2$ we get from (86)

$$\tau S^2\Lambda_2 S^2\Lambda_2 = \frac{\xi_2}{2} S^4\Lambda_2,$$

$$\tau(S^2\Lambda_2 SA_1 + SA_1 S^2\Lambda_2) = \frac{\xi_2}{2} S^3\Lambda_1,$$

$$\tau(S^2\Lambda_2 A_0 + SA_1 S\Lambda_2 + \Lambda_0 S^2\Lambda_2) = \frac{\xi_2}{2} S^2\Lambda_0,$$  

(88)

$$\tau(S^2\Lambda_2 A_1 S^{-1} + SA_1 A_0 + \Lambda_0 S\Lambda_1 + A_1 S\Lambda_2) = \frac{\xi_2}{2} S^2\Lambda_1 S^{-1},$$

$$\tau(S^2\Lambda_2 S^{-2} + SA_1 S^{-1} + \Lambda_0 S^{-1} + A_1 S^{-2} + \Lambda_2 S^{-2}) - I = \frac{\xi_2}{2} (S^2\Lambda_2 S^{-2} + \Lambda_2).$$

That is $\tau\lambda_0^{(2)} = \xi_2/2$ and the second relation in (88) is an identity. Further,

$$\tau\lambda_0^{(1)} = -\frac{\xi}{2} \lambda_0^{(0)} = \frac{\xi}{2} \lambda_1^{(0)}$$ 

(89)
and the fourth relation is an identity. Finally, from the last equation we get

\[(\lambda_{0}^{(0)})^2 + (\lambda_{0}^{(1)})^2 + (\lambda_{1}^{(1)})^2 = 1/\tau.\]  

(90)

Thus (89), (90) are counterparts of (85) in this case.

8.2. The Renormalization Equation in the general case. For \(\pi : \mathbb{C} \to \mathbb{C}\) given by \(T_2\) (that is \(X_c \simeq \mathbb{C}\) and \(Y_c \simeq \mathbb{C}\)) let \(E\) be a system of intervals, a subset of \([-\xi_2, \xi_2]\). As usual \(F = \pi^{-1}(E)\). For a system of points on \(X_c \setminus E\)

\[P_1 = \infty, P_2, ..., P_l;\]  

(91)

we define

\[b^l_s = b_{P_1}...b_{P_l}.\]

We consider \(z_s\) given by the identical map \(X_c \to \mathbb{C}\). The finite difference operator \(\tilde{J}\) related to this symbol, a character \(\alpha \in \Gamma^*_X\) and the ordering system of infinities (91) is of the form

\[
\tilde{J} = 
\begin{bmatrix}
* & * & \ldots & * \\
* & * & \ldots & * \\
* & \ldots & * \\
\vdots & \vdots & \vdots & \vdots \\
* & * & \ldots & * \\
* & \ldots & * \\
\vdots & \vdots & \vdots & \vdots \\
* & & & \\
\end{bmatrix},
\]

(92)

because, actually, \(z_s(P_j) = \infty\) only for \(j = 1\).

The system of infinities on \(Y_c\) is given by

\[Q_1^{(1)} = \infty, Q_2^{(1)} = 0, ..., Q_1^{(l)}, Q_2^{(l)},\]

(93)

where \(\{Q_1^{(j)}, Q_2^{(j)}\} = \pi^{-1}(P_j), j = 1, ..., l\). Respectively,

\[z^* = z_s \circ \pi = \tau v - \frac{1}{v^l}, \quad (b^*)^2 = b_s \circ \pi.\]

We are interested to find a relation between the initial operator \(\tilde{J}\) and the finite difference operator \(J\) with the symbol \(v\) related to the character \(\pi^*\alpha \in \Gamma_Y^*\) and the ordering system of infinities (93). Let us point out that the symbol of \(J\) is \(v\), not \(z^*\). It gives us an opportunity to iterate the procedure, which appears to be a certain rescaling.

Let us prove the following lemma.

**Lemma 8.1.** Let \(T(z) = \frac{Q(z)}{P(z)}\). Define

\[(\mathcal{L} f)(x) = \frac{1}{d} \sum_{T(y) = x} f(y).\]

Then

\[\mathcal{L} \frac{1}{z - x} = \frac{1}{d} \frac{P'(z)}{P(z)} + \frac{1}{d} \frac{T'(z)}{T(z) - x}.\]
where $X$ is singlevalued in $\mathbb{E}$.

We have a certain representation of $f$. Let us describe analytical properties of this object (metrically it is of $L^2$ with respect to $\omega_*$, moreover the transformation is norm–preserved).

Theorem 8.2. The following relation holds

$$V^*(z - J)^{-1} V = \frac{1}{d} \frac{P'(z)}{P(z)} + \frac{1}{d} T'(z) (T(z) - \bar{J})^{-1},$$

where $\bar{J} = \pi_*(\alpha)$ and $J = \pi(\pi^* \alpha)$.

Proof. We use the previous lemma and the functional representation of all operators involved in (94).

8.3. A vector representation of $H^2(\pi^* \alpha)$. Due to

$$\int_F |f|^2 \omega_* = \int_E \frac{1}{2} \sum_{\pi(Q) = P} |f|^2 \omega_*$$

we have a certain representation of $f \in H^2(\pi^* \alpha)$

$$f \mapsto \frac{1}{\sqrt{2}} \begin{bmatrix} f(Q_1(P)) \\ f(Q_2(P)) \end{bmatrix} \mapsto \begin{bmatrix} g_1(P) \\ g_2(P) \end{bmatrix},$$

where

$$\begin{bmatrix} g_1(P) \\ g_2(P) \end{bmatrix} = \frac{1}{2} \begin{bmatrix} f(Q_1(P)) + f(Q_2(P)) \\ f(Q_1(P)) - f(Q_2(P)) \end{bmatrix},$$

as a two–dimensional vector–function. Let us describe analytical properties of this object (metrically it is of $L^2$ with respect to $\omega_*$, moreover the transformation is norm–preserved).

It is evident that the function $g_1$ belongs to $H^2(\alpha)$. Consider the second function. Let $c_\pm$ be critical points

$$\pi'(c_\pm) = 0 \Rightarrow c_\pm = \pm \frac{i}{\sqrt{\tau}}.$$

For a small circle $\gamma$ around the point $\pi(c_\pm) = \pm 2i/\sqrt{\tau}$ we have $g_2 \circ \gamma = -g_2$. Let us introduce

$$\Delta^2 := b_{\pi(c_+)} b_{\pi(c_-)}.$$

Note that $\Delta \circ \gamma = -\Delta$. Further, since the ratio

$$\frac{b_{\pi(c_+)}(z)}{b_{\pi(c_-)}(z)} = e^{iC} \frac{z - \pi(c_+)}{z - \pi(c_-)}$$

is singlevalued in $X_h \setminus E$ the both Green functions have the same character of automorphy, which we denote by $\nu \in \Gamma_h^*$. Thus, we get

$$g_2 = \Delta \tilde{g}_2,$$

where $\tilde{g}_2 \in H^2(\alpha \nu^{-1})$.

We can summarize the result of this subsection as

$$\sum_{Q(y) - xP(y) = 0} \frac{1}{z - y} = \frac{Q'(z) - xP'(z)}{Q(z) - xP(z)}$$

and then collect corresponding terms

$$\frac{Q'(z) - xP'(z)}{Q(z) - xP(z)} = \frac{Q(z) - Q(z)P'(z)/P(z) + \{Q(z) - xP(z)\}P'(z)/P(z)}{Q(z) - xP(z)}$$

$$= \frac{P'(z)}{P(z)} + \frac{Q'(z)P(z) - Q(z)P'(z)}{P^2(z)} \frac{1}{T(z) - x}.$$
Theorem 8.3. The transformation $f \mapsto g_1 \oplus g_2$ given by (95) is a unitary map from $H^2(\pi^*\alpha)$ to $H^2(\alpha) \oplus \Delta H^2(\alpha \nu^{-1})$. Moreover,

$$V f \mapsto f \oplus 0, \quad f \in H^2(\alpha),$$

where the isometry $V : H^2(\alpha) \to H^2(\pi^*\alpha)$ is defined by (30). Also,

$$k_{Q \pm}^{\nu^{-1}} = k_p^{\nu^{-1}} \oplus (\pm \Delta \Delta(P)k_p^{\nu^{-1}})$$

for the reproducing kernel $k_{Q \pm}^{\nu^{-1}} \in H^2(\pi^*\alpha)$, $\pi(Q \pm) = P$.

Extending this vector representation onto $L^2$–spaces we get immediately

Theorem 8.4. The transformation $f \mapsto g_1 \oplus \tilde{g}_2$ given by (95) and (96) is a unitary map from $L^2(\pi^*\alpha)$ to $L^2(\alpha) \oplus L^2(\alpha \nu^{-1})$. With respect to this representation the multiplication operator by $z^* = z^* \circ \pi$ is of the form

$$z^*(\pi^*\alpha) \mapsto \begin{bmatrix} z_*(\alpha) & 0 \\ 0 & z_*(\alpha \nu^{-1}) \end{bmatrix}.$$  \hspace{1cm} (98)

To get a similar representation for the multiplication operator by $v$ we need to introduce the following notations. Let us note that $b_\infty z^2 + 4\tau$ is an outer function in the domain $C \setminus E$. So, the square root of this function is well defined. We put

$$b_\infty \phi := \sqrt{b_\infty z^2 + 4\tau}.$$

Since on the boundary of the domain

$$\phi^2 \Delta^2 = z^2 + 4\tau = |\phi|^2$$

(the second expression is positive on $E$) we have

$$\phi \Delta^2 = \overline{\phi} \quad \text{on } E.$$  \hspace{1cm} (99)

Lemma 8.5. The multiplication operator $\phi : L^2(\alpha) \to L^2(\alpha \nu^{-1})$ with respect to bases systems related to the infinities $\{P_1, \ldots, P_l\}$ has as many diagonals as $z_*(\alpha)$ and $z_*(\alpha \nu^{-1})$. Moreover,

$$\phi z_*(\alpha) = z_*(\alpha \nu^{-1}) \phi,$$

$$\phi^* \phi = z_*(\alpha) + 4\tau,$$

$$\phi \phi^* = z_*(\alpha \nu^{-1}) + 4\tau.$$  \hspace{1cm} (100)

Proof. First of all $\phi$ is a character–automorphic function with the character $\nu^{-1}$, therefore the multiplication operator acts from $L^2(\alpha)$ to $L^2(\alpha \nu^{-1})$. Since $b_\infty \phi$ is an outer function, $\phi$ has a unique pole at infinity, and, therefore, the operator $\phi$ has the same structure over diagonal as the operator multiplication by $z$. The adjoint operator has the symbol $\overline{\phi}$. According to (99) it has analytic continuation from the boundary inside the domain with the only pole at infinity. Thus $\phi^*$ is also of the same structure over the main diagonal as $z_*(\alpha)$ or $z_*(\alpha \nu^{-1})$. Combining these two facts we get that the whole structure of $\phi$ coincide with the structure of the matrix $z_*(\alpha)$. Then, just comparing symbols of operators on the left and right parts of (100), we prove these identities. \hfill $\square$
**Theorem 8.6.** With respect to the decomposition \(L^2(\pi^*\alpha) \simeq L^2(\alpha) \oplus L^2(\alpha\nu^{-1})\) the multiplication operator by \(v\) is of the form

\[
v(\pi^*\alpha) \simeq \frac{1}{2\tau} \begin{bmatrix} z_*(\alpha) & \phi^* \\ \phi & z_*(\alpha\nu^{-1}) \end{bmatrix}.
\]  

(101)

Let us mention that according to (100), the operators given in (98) and (101) commute and satisfy the identity, which is generated by the symbols identity \(z^* = \tau v - \frac{1}{v}\).

8.4. **One sided matrices and the expanding transform** \(\pi(v) = \tau v - \frac{1}{v}\).

Note that in this normalization \(v = 1\) is the positive fixed point, \(\pi(1) = 1\). Let \(E_0 = [-1, 1]\). For a continuous function \(f\) on \(E_1 = \pi \cdot [-1, 1] = [-1, -1 + \frac{1}{\tau}] \cup [1 - \frac{1}{\tau}, 1]\)

we define

\[
(Lf)(x) = \frac{1}{2} \sum_{\pi(v) = x} f(v).
\]  

(102)

The conjugate operator acts on measures

\(L^* : C(E_0)^* \to C(E_1)^*\).

Let \(f_0, f_1, f_2, \ldots\) be a certain orthonormal system with respect to a (positive) measure \(\nu \in C(E_0)^*\) then

\(f_0 \circ \pi, f_1 \circ \pi, f_2 \circ \pi, \ldots\)

is orthonormal with respect to \(\mu := L^*\nu\). Note that if the first system form basis in \(L^2_{d\nu}\), the second one form basis in the set of "even" functions from \(L^2_{d\nu}\), the functions that are invariant with respect to the substitution \(f(\tau v - \frac{1}{v}) = f(v)\).

**Example.** Let \(f_0, f_1, f_2, \ldots\) be orthonormal polynomials in \(L^2_{d\nu}\). \(f_0 \circ \pi, f_1 \circ \pi, f_2 \circ \pi, \ldots\)

is a certain orthonormal system in \(L^2_{d\nu}\) consisting of "polynomials" of \(v\) and \(1/v\), similarly to the systems that generate CMV matrices:

\(1, v, 1/v, v^2, 1/v^2, \ldots\)

Making a small modification in this procedure, we orthogonalize

\[1, \tau v + \frac{\tau - 1}{v}, \tau v - \frac{\tau - 1}{v}, (\tau v)^2 - \left(\frac{\tau - 1}{v}\right)^2, (\tau v)^2 + \left(\frac{\tau - 1}{v}\right)^2, \ldots\]

and denote the orthonormal system by \(e_0, e_1, e_2, \ldots\).

It is evident that

\(e_{2k} = f_k \circ \pi\)

and

\(e_{2k+1}(v) = \left(\tau v + \frac{\tau - 1}{v}\right) g_k(\pi(v))\),

where \(g_k\) is also orthonormal system of polynomials but with respect to the measure \((x^2 + 4\tau(\tau - 1))d\nu(x)\), since

\[
\left(\tau v + \frac{\tau - 1}{v}\right)^2 = x^2 + 4\tau(\tau - 1), \quad \text{for} \ x = \tau v - \frac{1}{v}.
\]

Let \(J\) be the Jacobi matrix, corresponding to the multiplication operator in \(L^2_{d\nu}\) with respect to the basis of the orthonormal polynomials.
The given $J$ we want to describe the multiplication operator in $L^2_{d\mu}$ with respect to $\{e_k\}$.

We decompose $L^2_{d\mu}$ onto even and odd subspaces. Then

$$
\tau v - \frac{\tau - 1}{v} \mapsto \begin{bmatrix} J & 0 \\ 0 & J_* \end{bmatrix},
$$

where $J_*$ is the Jacobi matrix corresponding to the measure $(x^2 + 4\tau(\tau - 1))d\nu(x)$.

Further,

$$
\tau v + \frac{\tau - 1}{v} \mapsto \begin{bmatrix} 0 & \Phi^* \\ \Phi & 0 \end{bmatrix}.
$$

It is quite evident that $\Phi$ is an upper triangular matrix.

We get that

$$
v \mapsto \frac{1}{2\tau} \begin{bmatrix} J & \Phi^* \\ \Phi & J_* \end{bmatrix},
$$

and

$$
-1/v \mapsto \frac{1}{2(\tau - 1)} \begin{bmatrix} J & -\Phi^* \\ -\Phi & J_* \end{bmatrix}.
$$

Therefore

$$
\begin{bmatrix} J^2 - \Phi^*\Phi & -J\Phi^* + \Phi^*J_* \\ \Phi J - J_*\Phi & J_*^2 - \Phi^*\Phi \end{bmatrix} = \begin{bmatrix} -4\tau(\tau - 1) & 0 \\ 0 & -4\tau(\tau - 1) \end{bmatrix}.
$$

Thus $\Phi$ can be found in the upper–lower triangular factorization

$$
\Phi^*\Phi = J^2 + 4\tau(\tau - 1),
$$

and for $J_*$ we have $J_* = \Phi J \Phi^{-1}$.

Thus for

$$
J = \begin{bmatrix} 0 & p_1 & 0 & p_2 \\ p_1 & 0 & 0 & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{bmatrix}
$$

we have

$$
\pi^*(J) = \frac{1}{2\tau} \begin{bmatrix} 0 & \lambda_0 & 0 & 0 \\ p_1 & 0 & 0 & \lambda_1 \\ 0 & p_2 \frac{\lambda_1}{\lambda_0} & 0 & 0 \\ 0 & 0 & 0 & p_3 \frac{\lambda_2}{\lambda_1} \lambda_2 \\ \vdots & \ddots & \ddots & \ddots \end{bmatrix}.
$$

The matrix is selfadjoint and $\lambda_n$ are defined by the recursion

$$
\lambda_n^2 = 4\tau(\tau - 1) + p_{n+1}^2 + p_n^2 - \frac{p_n^2 p_{n-1}^2}{\lambda_{n-2}^2}.
$$

with the initial data

$$
\lambda_0^2 = p_1^2 + 4\tau(\tau - 1), \quad \lambda_1^2 = p_1^2 + p_2^2 + 4\tau(\tau - 1).
$$

□
Theorem 8.7. Let $\nu$ be the spectral measure of $A$.
\[
\int \frac{d\nu(x)}{x - z} = \langle 0 | (A - z)^{-1} | 0 \rangle. \tag{106}
\]
Then $\pi^*(A)$ is a self-adjoint operator with the cyclic vector $|0\rangle$ and the spectral measure $\mu = \mathcal{L}^* \nu$.

Proof. By Definition 1.5 and (106) we have
\[
\langle 0 | (\pi^*(A) - z)^{-1} | 0 \rangle = 2\tau \left[ \left[ A - 2\tau z \Phi \Phi^* A - 2\tau z \right]^{-1} \left[ 0 \right], \left[ 0 \right] \right]
= 2\tau \langle 0 | (A - 2\tau z - \Phi^* (A - 2\tau z) (A - 2\tau z)^{-1})^{-1} | 0 \rangle
= 2\tau \langle 0 | (A - 2\tau z - A^2 + 4\tau (1 - \tau)) (A - 2\tau z)^{-1} | 0 \rangle \tag{107}
= \int \frac{2\tau d\nu(x)}{x - 2\tau z - (x^2 + 4\tau (1 - \tau))(x - 2\tau z)^{-1}}
= \int \frac{2\tau z^2 - 2xz - 2(1 - \tau)}{2\tau z^2 - 2xz - 2(1 - \tau)}.
\]
Since
\[
\left( \mathcal{L} \frac{1}{v - z} \right)(x) = \frac{1}{2} \sum_{\tau v - x \epsilon = x} \frac{1}{v - z} = \frac{x - 2\tau z}{2\tau z^2 - 2xz - 2(1 - \tau)},
\]
we get
\[
\langle 0 | (\pi^*(A) - z)^{-1} | 0 \rangle = \int \left( \mathcal{L} \frac{1}{v - z} \right)(x) d\nu(x) = \int \frac{1}{v - z} d(\mathcal{L}^* \nu)(v)
\]
and the theorem is proved. \hfill \square

Using Ruelle’s Theorem with respect to the map $\pi(v)$ we can summarize our considerations with the following theorem.

Theorem 8.8. The iterative procedure
\[
A_{n+1} = \pi^*(A_n)
\]
converges to the operator $A = \lim_{n \to \infty} A_n$ with the spectral measure $\mu$ which is the eigen-measure for the Ruelle operator $\mathcal{L}^* \mu = \mu$. The operator $A$ is the multiplication operator by independent variable in $L^2_\mu$ with respect to the following basis
\[
e_{2k}(v) = e_k(\pi(v)),
\]
\[
e_{2k+1}(v) = (1 + \tau - \frac{1}{v}) \sum_{j=0}^{k} c_j^k e_j(\pi(v)), \quad e_0(v) = 1,
\]
where the coefficients $c_j^k$ with $c_k^k > 0$ are uniquely determined due to the orthogonality condition $\langle e_{2k+1}, e_l \rangle = \delta_{2k+1,l}$, $l \leq 2k + 1$. Moreover, $e_k(v)$ is a rational function of $v$ such that $e_k(A) |0\rangle = |k\rangle$, and
\[
\begin{pmatrix}
0 & c_0 & c_0^2 & \cdots \\
0 & c_1 & c_1^2 & \cdots \\
& \ddots & \ddots & \ddots
\end{pmatrix} = \Phi^{-1}. \tag{108}
\]
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