Rotating Kaluza-Klein Black Holes

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Abstract

All regular four-dimensional black holes are constructed in the theory obtained by Kaluza-Klein reduction of five-dimensional Einstein gravity. They are interpreted in string theory as rotating bound states of D0- and D6-branes. In the extremal limit the solutions are stable, due to angular momentum conservation. The thermodynamics, the duality symmetries, and the near-horizon limit are explored.
1 Introduction

Black holes in Kaluza-Klein theories have attracted tremendous attention over the last decades. A driving force of this interest is their role as low-energy approximations to string theory. The simplest Kaluza-Klein theory is obtained by starting with pure Einstein gravity in five dimensions, and dimensionally reduce to four dimensions. The field content of the resulting theory is gravity, the dilaton $\Phi_4$, and the gauge field $A_\mu$; the Lagrangian is:

$$L = \frac{1}{16\pi G_4} \int d^4x \left[ R - 2\partial_\mu \Phi_4 \partial^\mu \Phi_4 - \frac{1}{4} e^{-2\sqrt{3}\Phi_4} F_{\mu\nu} F^{\mu\nu} \right],$$  \hspace{1cm} (1)

in the Einstein frame. The most general stationary black hole solutions to this theory are parametrized by the mass $M$, the angular momentum $J$, and the electric/magnetic charges of the gauge field, $Q/P$. The purpose of this paper is to construct this four-parameter family of black holes.

The electromagnetic field created by electric and magnetic sources, charged with respect to the same $U(1)$ field, carry angular momentum satisfying the universal bound:

$$J \geq \frac{n Q^n P^n}{2},$$  \hspace{1cm} (2)

where $n_{Q,P}$ are the quantized values of the charges. Thus the properties of dyonic black holes are intimately related to rotation. It is ultimately this feature which makes it interesting to construct the full family of black holes, including angular momentum.

As the spin of a black hole is increased the inner and outer horizons approach each other, eventually merging, and then exhibiting a naked singularity. Very rapidly spinning black holes are thus singular. The precise regularity bound on the angular momentum will be determined in the course of the present investigation. It is:

$$J < \frac{n Q^n P^n}{2},$$  \hspace{1cm} (3)

in the extremal limit. This is an interesting result, in view of (2). It shows that the regular black holes are precisely those which cannot form from widely separated sources carrying electric and magnetic charges. Stated another way, the regular black holes are prevented from decaying into constituent parts by angular momentum conservation. This is significant because the mass of the black hole exceeds the sum of
its constituent masses; so energy conservation is not sufficient to stabilize the black holes against spontaneous fragmentation.

The black hole is thus stable in the extremal limit; so it is reasonable to describe it as the ground state in a conformal field theory. Moving away from extremality, the black holes exhibit thermal properties, as expected. These are interpreted in terms of perturbations of the conformal field theory and should remain under control, as long as they are small. The prospects of a precise microscopic description of the black holes, even though they are not supersymmetric, is the ultimate goal of the inquiry. However, this will be pursued elsewhere; the discussion in this paper focusses on the classical properties of the black holes.

The present investigation is motivated by several relations to string theory. For example, add six additional toroidally compactified dimensions, and interpret the compact Kaluza-Klein direction as the M-theory circle. Then the solutions are charged with respect to the “electric” charge of $D0$-branes and the “magnetic” charge of $D6$-branes, fully wrapping the six inert dimensions. Thus the solutions can be interpreted as rotating bound states of $D0$- and $D6$-branes. As discussed above, angular momentum conservation implies that such bound states are stable, even though they are not supersymmetric. The argument may explain the stability to the leading order noted in the string theory description of the $D0 - D6$ system [1].

For another application, recall that general four dimensional black holes in $N = 4, 8$ string theory are generated by black holes depending on 5 parametric charges [2]. By now it is standard to consider four of these charges [3, 4], but the fifth charge is difficult and our understanding is incomplete, even at the level of classical solutions; for discussion see e.g [5]. The problematic fifth charge parametrizes the inner product of the electric and magnetic charge vectors; when it is absent, the four charges refer to independent $U(1)$’s, up to duality. In the present work there is only one $U(1)$ field, so electric and magnetic charges are necessarily parallel. In a sense, there is only the fifth charge, and therefore good opportunity to study its properties.

The paper is organized as follows. In section 2, we review the solution generating technique, following Sen [6]. The resulting solution is presented in its five-dimensional form, along with necessary notation. There is also a discussion of various special cases, and the relation to previously known solutions. Subsequently, in section 3
the properties of the corresponding black holes in four dimensions are derived. The various subsections are quasi-independent, each considering e.g. duality symmetries, thermodynamic properties, dipole moments, or the near-horizon geometry.

2 The Solution

In this section we review the solution generating technique, following Sen [7, 6]. This leads to the solution, of course; but it also serves to describe a specific embedding into string theory, and to explain how a potential Taub-NUT singularity is avoided. Sen considered the heterotic string theory but for the present purposes it is sufficient to consider the five dimensional vacuum sector. This truncation simplifies the problem and further gives a universal embedding, valid for all the string theories.

2.1 The Solution Generating Technique

The idea of the solution generating technique is to note that, being interested in stationary solutions, time can be assumed compact, for the purpose of the equations of motions and their symmetries. This procedure yields an effective theory in three dimensions. In the present context the compactification from five to three dimensions gives an $SO(2, 2)$ T-duality symmetry, with transformations acting as conjugations on the moduli matrix:

$$M = \begin{pmatrix} G^{-1} & G^{-1}B \\ -BG^{-1} & G \end{pmatrix},$$

where $G$ and $B$ are $2 \times 2$ matrices with t/y indices. The labels in the explicit computations below are chosen so that $M = \text{diag}[g_{yy}^{-1}, g_{yy}, g_{tt}^{-1}, g_{tt}]$ for diagonal metrics.

The components of the metric and the B-field having one azimuthal index, and the other in the t/y directions, correspond to magnetic fields in three dimensions. They are represented as four pseudoscalar potentials $\Psi$ after the dualization:

$$- \sqrt{g_3} e^{-2\Phi_3} g^{\mu\nu'} g^{\rho\sigma'} (ML)_{ab} F^{(b)}_{\mu
u'} = \epsilon^{\mu
u\rho} \partial_\rho \Psi^{(a)}.$$

Here $L = \text{diag}[\sigma_1, \sigma_1]$, where:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$
The four scalars fields $\Psi$ and the dilaton $\Phi_3$ are combined with the matrix $M$ to form an extended moduli matrix $\mathcal{M}$, of size $6 \times 6$:

$$\mathcal{M} = \begin{pmatrix}
    M - e^{2\Phi_3}\Psi\Psi^T & e^{2\Phi_3}\Psi & M\tilde{\Psi} - \frac{1}{2}e^{2\Phi_3}\Psi\Psi^T\tilde{\Psi} \\
    e^{2\Phi_3}\Psi & -e^{2\Phi_3} & -\frac{1}{2}e^{2\Phi_3}\Psi^T\tilde{\Psi} \\
    \tilde{\Psi}^TM + \frac{1}{2}e^{2\Phi_3}\Psi^T\tilde{\Psi} & \frac{1}{2}e^{2\Phi_3}\Psi^T\tilde{\Psi} - e^{-2\Phi_3} + \tilde{\Psi}^TM\tilde{\Psi} - \frac{1}{4}e^{2\Phi_3}(\Psi^T\tilde{\Psi})^2
\end{pmatrix}$$

where $\tilde{\Psi} \equiv L\Psi$. Note that $\mathcal{M}$ is symmetric $\mathcal{M} = \mathcal{M}^T$.

The group of solution generating transformations is formed by repeated application of T-duality and S-duality; the resulting group is $SO(3, 3)$. The solution generating matrices $\Omega \in SO(3, 3)$ satisfy:

$$\Omega L\Omega^T = L ,$$

where $L = \text{diag}[^{\sigma_1, \sigma_1, \sigma_1}]$. They leave the effective three dimensional metric invariant, and transforms the extended moduli matrix as:

$$\mathcal{M}' = \Omega \mathcal{M} \Omega^T .$$

The solutions are required to have canonical moduli at infinity; i.e. the asymptotic moduli matrix must be $\mathcal{M}_{\text{as}} = \text{diag}[I_2, -I_2, -I_2]$, where $I_2$ is the $2 \times 2$ identity matrix. This reduces the group of solution generating transformations $SO(3, 3) \to SO(2, 1) \times SO(2, 1)$. The unbroken group includes the compact $U(1)$ subgroup of the $SL(2, \mathbb{R})$ S-duality group, which leaves invariant the four dimensional solution used as starting point; so the space of solutions becomes a five-dimensional coset:

$$(SO(2, 1) \times SO(2, 1))/SO(2) .$$

The five parameters are interpreted as electric/magnetic Kaluza-Klein charge, electric/magnetic B-field charge, and a Taub-NUT parameter. Removing the charges associated with the B-field, i.e. the FS- and NS5-charges, we are left with the diagonal $SO(2, 1)$ of the coset above. Three generating elements of this group are:

$$\Omega_E = \begin{pmatrix}
    \cosh \alpha & \sinh \alpha & I_2 & 0 \\
    \sinh \alpha & \cosh \alpha & I_2 & 0 \\
    0 & 0 & 0 & I_2
\end{pmatrix},$$

$$(11)$$
When acting on a background Schwarzschild or Kerr black hole, each of these transformations create a solution with the designated charge. The three infinitesimal generators satisfy the $SO(2,1)$ algebra, so together they generate a three parameter family of solutions, parametrized by the three charges.

In the present work only regular solutions are of interest; so the Taub-NUT charge must vanish. This can be cumbersome to achieve, because the transformations (11-13) do not commute. For example, generating first a KK-monopole using $\Omega_M$, and then applying the boost $\Omega_E$, gives a solution with magnetic and electric KK-charge, but also Taub-NUT charge. The standard remedy is to further act with some $\Omega_{TN}$, with the parameter chosen to cancel the unwanted Taub-NUT charge. Solutions constructed this way tend to be quite unwieldy, due to the constraint of vanishing Taub-NUT charge. In particular, the symmetry between electric and magnetic charges is generally obscured. The strategy in the present construction is to solve the Taub-NUT constraint early on, implementing the result directly in the $SO(2,1)$ matrix. This leads to the two-parameter family of transformations:

$$\Omega = \begin{pmatrix} \sqrt{\frac{qp}{4m^*}} I_2 & \frac{(q-2m)(p+2m)}{8m^*} I_2 & \frac{(q+2m)(p-2m)}{8m^*} \sigma_1 \\ \sqrt{\frac{p^2-4m^*}{4m^2}} \sigma_1 & \frac{q(-2+2m)}{8m^2} I_2 & \frac{q(-2-2m)}{8m^2} \sigma_1 \\ \sqrt{\frac{q^2-4m^*}{4m^2}} \sigma_1 & \frac{p(2-2m)}{8m^2} \sigma_1 & \frac{p(2+2m)}{8m^2} I_2 \end{pmatrix}. \quad (14)$$

These matrices indeed belong to the $SO(2,1)$ of interest and, as shown below, further do not lead to Taub-NUT charge. When only electric or magnetic charges are present the correspondence with the parameters in $\Omega_E$ and $\Omega_M$ is $q = 2m \cosh^2 \alpha$ and $p = 2m \cosh^2 \beta$, respectively; but in general there is no simple relation to the familiar “boost” parameters.
2.2 The Explicit Computation

We now turn to the explicit computation. The starting point is the standard Kerr black hole in four dimensions. In three dimensional form it is described by the metric:

\[ ds^2_3 = H_3 \left( \frac{dr^2}{\Delta} + d\theta^2 + \frac{\Delta}{H_3} \sin^2 \theta d\phi^2 \right), \]  
(15)

where:

\[ H_3 = r^2 + a^2 \cos^2 \theta - 2mr, \]  
(16)

\[ \Delta = r^2 + a^2 - 2mr, \]  
(17)

and the moduli:

\[
\mathcal{M} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & -f^{-1} & 0 & 0 & -g \\
0 & 0 & 0 & -f(1+g^2) & g & 0 \\
0 & 0 & 0 & g & -f^{-1} & 0 \\
0 & 0 & -g & 0 & 0 & -f(1+g^2)
\end{pmatrix}, \]  
(18)

where:

\[ f = \frac{r^2 + a^2 \cos^2 \theta - 2mr}{r^2 + a^2 \cos^2 \theta}, \]  
(19)

\[ g = \frac{2ma \cos \theta}{r^2 + a^2 \cos^2 \theta - 2mr}. \]  
(20)

It is useful to note:

\[ f(1+g^2) - 1 = -2m(r-2m)H_3^{-1}, \]  
(21)

\[ f^{-1} - 1 = 2mrH_3^{-1}. \]  
(22)

It follows without detailed computation from the structure of the various matrices that the components \( \mathcal{M}'_{15}, \mathcal{M}'_{35}, \mathcal{M}'_{23} \) vanish identically. This implies that the fields carrying axion-, NS5- and FS-charges are consistently set to zero. More importantly, the leading term in \( \mathcal{M}'_{45} \) cancels, ensuring a vanishing Taub-NUT charge (the sub-leading terms are related to angular momentum.) An explicit computation further shows the relation:

\[ \mathcal{M}'_{11} \mathcal{M}'_{33} - (\mathcal{M}'_{13})^2 = \mathcal{M}'_{55} , \]  
(23)
which implies that the five-dimensional dilaton $e^{2\Phi_5} = 1$. The remaining components of $\mathcal{M}'$ can now be interpreted consistently in terms of the five-dimensional geometry. It can be extracted from just a few elements of $\mathcal{M}'$, e.g:

\begin{align}
e^{2\Phi_3} &= -M'_{55}, \\ G^{-1}_{tt} &= M'_{33}, \\ A_t &= -\frac{M'_{13}}{M'_{33}}, \\ G_{yy} &= \frac{M'_{33}}{M'_{55}}, \\ \Psi^{(2)} &= -\frac{M'_{25}}{M'_{55}}, \\ \Psi^{(4)} &= -\frac{M'_{45}}{M'_{55}}. \end{align}

The next step is to invert the definitions of $\Psi^{(2,4)}$ from (5), and find the corresponding gauge fields $A^{(1,3)}_\phi$. This is a lengthy computation. The $A^{(3)}_\phi$ gives the azimuthal part of the four-dimensional metric (denoted $B_\phi$ below). To find the azimuthal part of the gauge potential one must further transform from the effective three-dimensional metric to the five-dimensional form using:

\begin{equation}
A_\phi = A^{(1)}_\phi + A_t A^{(3)}_\phi. \tag{30}
\end{equation}

### 2.3 The Result

The explicit computations lead to the five-dimensional metric:

\begin{equation}
ds_5^2 = \frac{H_2}{H_1} (dy + A)^2 - \frac{H_3}{H_2} (dt + B)^2 + H_1 \left(\frac{dr^2}{\Delta} + d\theta^2 + \frac{\Delta}{H_3} \sin^2 \theta d\phi^2\right), \tag{31}
\end{equation}

where:

\begin{align}
H_1 &= r^2 + a^2 \cos^2 \theta + r(p - 2m) + \frac{p}{p+q} \frac{(p-2m)(q-2m)}{2} - \\
&\quad - \frac{p}{2m(p+q)} \sqrt{(q^2 - 4m^2)(p^2 - 4m^2)} \ a \cos \theta, \tag{32}
\end{align}

\begin{align}
H_2 &= r^2 + a^2 \cos^2 \theta + r(q - 2m) + \frac{q}{p+q} \frac{(p-2m)(q-2m)}{2} + \\
&\quad + \frac{q}{2m(p+q)} \sqrt{(q^2 - 4m^2)(p^2 - 4m^2)} \ a \cos \theta, \tag{33}
\end{align}

\begin{align}
H_3 &= r^2 + a^2 \cos^2 \theta - 2mr, \tag{34}
\end{align}

\begin{align}
\Delta &= r^2 + a^2 - 2mr. \tag{35}
\end{align}
are quadratic functions of the Boyer-Lindquist type radial variable \( r \) (we repeated \( H_3, \Delta \) for easy reference); and the 1-forms:

\[
A = - \left[ 2Q \left( r + \frac{p - 2m}{2} \right) + \frac{q^3(p^2 - 4m^2)}{4m^2(p + q)} a \cos \theta \right] H_2^{-1} dt
- \left[ 2P \left( H_2 + a^2 \sin^2 \theta \right) \cos \theta + \frac{p(q^2 - 4m^2)}{4m^2(p + q)^3} \times \right.
\times \left. (p + q)(pr - m(p - 2m)) + q(p^2 - 4m^2) \right] a \sin^2 \theta \right] H_2^{-1} d\phi ,
\]

\[
B = \sqrt{pq} \frac{(pq + 4m^2)r - m(p - 2m)(q - 2m)}{2m(p + q)H_3} a \sin^2 \theta d\phi ,
\]

play the role of gauge potentials in the effective three-dimensional theory, obtained by compactifying \( t \) as well as \( y \). It is sometimes an advantage to write \( A_\phi \) in the alternative form:

\[
A_\phi = - \left[ 2P \left( \Delta + rq + \frac{q(p - 2m)(q - 2m)}{2(p + q)} \right) \cos \theta + \frac{2Pq \sqrt{(q^2 - 4m^2)(p^2 - 4m^2)}}{2m(p + q)} \right.
\left. + \frac{p(q^2 - 4m^2)}{4m^2(p + q)} (pr - m(p - 2m)) a \sin^2 \theta \right] H_2^{-1} .
\]

The four parameters \((m, a, q, p)\) appearing in the solution are related to the physical mass \((M)\), angular momentum \((J)\), electric charge \((Q)\), and magnetic charge \((P)\) through:

\[
2G_4 M = \frac{p + q}{2} ,
\]

\[
G_4 J = \frac{\sqrt{pq}(pq + 4m^2)}{4m(p + q)} a ,
\]

\[
Q^2 = \frac{q(q^2 - 4m^2)}{4(p + q)} ,
\]

\[
P^2 = \frac{p(p^2 - 4m^2)}{4(p + q)} .
\]

The charge parameters \( Q, P \) were used already in writing the solution above. Note that \( q, p \geq 2m \), with equality corresponding to the absence of electric or magnetic charge, respectively.
2.4 Relation to Previous Work

The black holes presented above are related to many different families of black holes considered in previous work. The non-rotating black holes were found by Gibbons and Wiltshire [8]. They appear in a string theory context in [8, 9].

The purely electric, or the purely magnetic, rotating black holes are special cases of the string theory black holes considered in [11, 10]. Rotating black holes with both magnetic and electric charges, but with respect to different $U(1)$ fields, were found in a string theory context in [12, 11, 13]. Sen [6] further describes the construction of a very general black hole, including the ones considered here, but found it unpractical to carry out the details in full generality. Other strategies, which could lead in principle to the class of black holes considered here, were described in [14, 15].

The “diagonal” case where electric and magnetic charges are equal $P = Q$, and rotation is absent, is the standard Reissner-Nordström solution in Einstein-Maxwell theory. In [14, 11] it was found as a string theory solution, in this form. This is possible, despite the coupling between the gauge field and the dilaton in the Lagrangean (1), because the dual field strength has the inverse coupling so that, when the electric and magnetic charges are equal, it is consistent to take a constant dilaton in four dimensions. The more general solution found here shows that this phenomenon does not generalize to the rotating case: the standard charged, rotating Kerr-Newman black hole is not a special case of the family constructed here; in particular, it is not the diagonal case $P = Q$.

The truncated theory defined by the Lagrangian (1) is not closed under electric-magnetic duality. One may attempt to make the Lagrangian duality invariant by including additional fields, for example the axion. The derivation of the solution given above shows that it is consistent to set these fields to zero. However, doing so fixes the duality orbit so duality does not act as a solution generating transformation within the family constructed here. The electric and magnetic charges therefore appear as genuinely independent parameters in the metric. Other ways to obtain electric-magnetic duality involve the introduction of further gauge fields; for examples, see the literature discussed above.
3 Black Holes in Four Dimensions

After dimensional reduction the five dimensional black string becomes a four dimensional black hole with the metric:

\[
 ds_{4,E}^2 = -\frac{H_3^2}{\sqrt{H_1H_2}}(dt + B)^2 + \sqrt{H_1H_2} \left( \frac{dr^2}{\Delta} + d\theta^2 + \frac{\Delta}{H_3} \sin^2 \theta d\phi^2 \right) .
\]  

(43)

The matter fields are the gauge field \( A \) given in (36), and the dilaton:

\[
 e^{-2\Phi_4} = \frac{\sqrt{H_2}}{H_1} .
\]  

(44)

The black hole has inner and outer horizons at \( \Delta = 0 \), i.e.:

\[
 r_\pm = m \pm \sqrt{m^2 - a^2} .
\]  

(45)

The outer horizon is surrounded by an ergosphere, and shields a ring singularity. These and other features of the causal structure are similar to those of Kerr black holes, see e.g. [18]. The equations describing them are even the same, when written in terms of the parametric variables \( m, a \).

3.1 Duality and other Symmetries

After embedding into string theory the solutions are related to others by duality. As an example, consider the embedding into type II string theory, compactified to four dimensions on \( T^6 \). Then the electric and magnetic charges can be interpreted as the KK-momentum and KK-monopole around any of the compact dimensions. Alternatively, as mentioned in the introduction, they are interpreted as the D0-D6 system, or any of its obvious dual pairs of D-branes.

There are much more general possibilities. The subgroup of the general \( E_{7(7)} \) duality group leaving the asymptotic Minkowski space invariant is the maximal compact subgroup \( SU(8) \). A general strategy to investigate the orbit of these transformations is to consider the central charge matrix \( Z_{AB} \) of \( N = 8 \) supergravity in four dimensions, transforming as an antisymmetric tensor under \( SU(8) \). From general formulae, e.g. in [19], it follows that in the present case the central charge matrix is particularly simple, of the form:

\[
 Z = \lambda \, \text{diag} [\epsilon, \epsilon, \epsilon, \epsilon] ,
\]  

(46)
where $\lambda = P + iQ$ and:

$$\epsilon = i\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$  \hspace{1cm} (47)

The four skew-eigenvalues of the central charge matrix are thus identical and equal to $\lambda$. $SU(8)$ duality transformations can at most change the skew-eigenvalues by phases. Moreover, the overall phase must be left invariant, because the duality group is $SU(8)$, rather than $U(8)$. It is therefore clear that, for generic $Q, P$, at least some of the skew-eigenvalues of the central charge matrix remain complex throughout the duality orbit.

The canonical family of black holes considered in the string theory literature depends on four independent charges [3, 4]. The charges can be identified up to duality as the skew-eigenvalues of the central charge matrix. The eigenvalues of the central charge matrix can thus be chosen real for the standard black holes; so these are distinct from the ones considered here, even after general dualities are taken into account.

The Einstein-Maxwell theory has a $U(1)$ duality symmetry acting on the field strength. In view of the various dualities in string theory it is natural to expect an analogous symmetry in the present context. Such a symmetry, exchanging electric and magnetic charges in a continuous fashion, would multiply the four skew-eigenvalues by an identical phase. In general, this is evidently not a symmetry, since the phases do not multiply to unity. Thus there are no continuous duality symmetries that act within the family of solutions considered in this paper. The significance of this result is that the parameters $p$ and $q$ are genuinely independent.

Acting on the skew-eigenvalues with a phase is not generally a symmetry, but $i^4 = 1$ so this specific value generates a discrete duality symmetry, interchanging the electric and magnetic charges. More precisely it takes $Q \rightarrow P$, $P \rightarrow -Q$, leaving the four-dimensional geometry invariant. A more explicit route to this symmetry starts from the truncated Lagrangean (48), rewriting the Maxwell field in dual variables. This computation also shows that $\Phi_4 \rightarrow -\Phi_4$ under the discrete duality.

Now, let us inspect the discrete symmetries of the solutions. One symmetry has the parametric form:

$$p \leftrightarrow q \ ; \ a \rightarrow -a.$$  \hspace{1cm} (48)

11
It interchanges the functions $H_1 \leftrightarrow H_2$, inverting the dilaton $\Phi_4 \to -\Phi_4$; and it leaves the four dimensional geometry invariant, except for $B \to -B$. It also interchanges the electric potential $A_t$ with the magnetic potential $\Psi^{(2)}$ appearing in intermediate stages of the explicit computations in section (2.2).

The discrete symmetry under time-reversal is superficially similar, yet different in nature. It leaves the four dimensional geometry invariant except for $B \to -B$; again, this takes the angular momentum $J \to -J$. It also inverts the electric potential $A_t \to -A_t$, but not the magnetic one $A_\phi \to A_\phi$. This implies $Q \to -Q$, $P \to P$.

The product of the two discrete symmetries described above is also a symmetry, taking $Q \to P$, $P \to -Q$, leaving $J$ invariant. It is this combined symmetry which is the discrete duality.

For another discussion of duality, emphasizing non-supersymmetric orbits, see [20].

### 3.2 The Extremal Limit and Stability

For a given value of the conserved charges $Q, P, J$, the lowest possible value of the mass occurs for $p, q \gg m, a$. This three parameter family of solutions is referred to as extremal. In the extremal limit the parametric relations (39-42) can be inverted with the result:

\[
2G_4 M = \left( \frac{Q^2}{3} + \frac{P^2}{3} \right)^{3/2}. \tag{49}
\]

\[
\frac{a}{m} = \frac{G_4 J}{PQ}. \tag{50}
\]

Note that the mass is independent of angular momentum; but nontrivial dependence on the ratio (50) can nevertheless be retained in the extremal geometry.

A natural benchmark for the mass is the BPS inequality, written in the present units as:

\[
2G_4 M \geq \sqrt{Q^2 + P^2}. \tag{51}
\]

This condition is always satisfied and cannot be saturated, when both electric and magnetic charges are present. The extremal black holes can therefore not be supersymmetric.

A stronger benchmark is the comparison with the energy of two widely separated
fragments, each carrying either the electric or the magnetic charge:

\[ 2G_4 M \geq Q + P. \]  

(52)

This inequality is also satisfied for all ranges of parameters. This shows that spontaneous fragmentation of the black hole into two parts is consistent with energy conservation. However, as discussed in the introduction, the two constituents in the final state must satisfy Dirac’s bound on the angular momentum:

\[ J \geq \frac{PQ}{G_4}. \]  

(53)

It is therefore only rapidly spinning black holes that are unstable towards this decay.

There are many other potential fragmentations, and angular momentum conservation may not generally rule such decays out. Fragmentation in identical lumps, each with charge assignments \((Q/2, P/2)\), is particularly worrisome, because two identically charged fragments can have vanishing angular momentum. A nonrotating black hole with mutually prime quantized charges is not subject to this concern, but others are. It is nevertheless interesting that the most obvious decay channel is forbidden.

Independently of these considerations a large degree of stability is expected on entropy grounds. As discussed below, the black holes have considerable entropy, even in the extremal limit, and any fragmentation would significantly lower the total entropy of the system.

The extremal family of solutions allow arbitrary value of \(m/a \leq 1\), so the inner and outer horizons \((43)\) do not in general coincide. A computation of the inner and outer horizon area, given below, finds that those do agree, so there remains a sense that the horizons are “close” in the extremal case. In general relativity, the term extremality usually refers to a specific property of the causal structure, the appearance of a bifurcate Killing horizon. This is not the terminology used here. It seems to require \(r_+ = r_-\), and so \(m = a\). This condition is satisfied by a class of solutions parametrized by \(q, p, m\), with no further constraints (beyond \(q, p \geq 2m\)). It would clearly be interesting to study the causal structure in more detail, and particularly to investigate the extremal limit closer.

### 3.3 Thermodynamics

We next turn to the evaluation of the thermodynamical variables of the black hole.
The area of the black hole is determined from the four dimensional Einstein metric as:

\[ A = \int \sqrt{g_{\theta\theta} g_{\phi\phi}} d\theta d\phi = \int \sqrt{-H_3 B_3^2} d\theta d\phi . \]  \hspace{1cm} (54)

This gives the black hole entropy:

\[ S = \frac{A}{4G_4} = \frac{\pi \sqrt{pq}}{G_4} \left[ m + \frac{pq + 4m^2}{2m(p + q) \sqrt{m^2 - a^2}} \right] \]

\[ = 2\pi \left[ \frac{m\sqrt{pq}}{2G_4} + \sqrt{\frac{pq}{16G_4^2} \left( \frac{pq + 4m^2}{p + q} \right)^2 - J^2} \right] . \]  \hspace{1cm} (55)

In the extremal limit the entropy simplifies to:

\[ S = 2\pi \sqrt{\frac{P^2 Q^2}{G_4^2} - J^2} . \]  \hspace{1cm} (56)

Dirac’s bound on the angular momentum (53) is precisely such that it forces the expression under the square root less than or equal to zero. Inspecting the solution, this corresponds to a singularity in the geometry outside the event horizon. Such solutions are usually discarded. The remaining regular black holes are precisely the ones that are stabilized by angular momentum conservation.

The limit \( m \to a \) leads to the bizarre entropy formula:

\[ S = \frac{4\pi m(p + q)}{pq + 4m^2} J , \]  \hspace{1cm} (57)

which does not simplify further.

It is simplest to compute the inverse temperature at \( \theta = 0 \), where the event horizon and the ergosphere meet. Here \( g_{tt} = -g_{rr}^{-1} \) and:

\[ \beta_H = \frac{4\pi}{|g_{tt}|} , \]  \hspace{1cm} (58)

where the prime denotes differentiation with respect to the coordinate \( r \). The result is:

\[ \beta_H = \frac{\pi \sqrt{pq}}{m} \left[ \frac{pq + 4m^2}{p + q} + \frac{2m^2}{\sqrt{m^2 - a^2}} \right] . \]  \hspace{1cm} (59)

The Hawking temperature vanishes in the extremal limit \( p, q \gg m, a \), and also in the limit \( m \to a \).

The rotational velocity of the black hole horizon is also determined at \( \theta = 0 \) where:

\[ \Omega_H = -\frac{g_{\phi t}}{g_{tt}} = -\frac{1}{B_\phi} . \]  \hspace{1cm} (60)
It can be written as:

\[ \Omega_H = \frac{2m(p+q)}{\sqrt{pq}\left(\frac{1}{(p+q)^2} + \frac{4m^2}{\sqrt{m^2-a^2}}\right)} a \].  

(61)

Note that the product of the inverse temperature and the rotational velocity is particularly simple:

\[ \beta H \Omega_H = \frac{2\pi a}{\sqrt{m^2-a^2}} \].  

(62)

It can be verified that the first law of black hole thermodynamics:

\[ dM = T_H dS + \Phi_E dQ + \Phi_M dP + \Omega dJ \].  

(63)

is consistent with the results above. This computation also gives the potentials conjugate to electric and magnetic charge:

\[ \beta H \Phi_E = \frac{\pi}{2mG_4} \sqrt{\frac{p(q^2-4m^2)}{p+q}} \left( p + \frac{2m^2}{\sqrt{m^2-a^2}} \right) \],  

(64)

\[ \beta H \Phi_M = \frac{\pi}{2mG_4} \sqrt{\frac{q(p^2-4m^2)}{p+q}} \left( q + \frac{2m^2}{\sqrt{m^2-a^2}} \right) \].  

(65)

Finally, homogeneity of the thermodynamic potentials gives the generalized Smarr formula:

\[ 2S = \beta H M - \beta H \Phi_E Q - \beta H \Phi_M P - 2\beta H \Omega J \].  

(66)

It is a good check on the computations that this formula is satisfied.

### 3.4 Quantization Rules

The electric and magnetic charges are quantized according to:

\[ Q = 2G_4 M_0 n_Q \],  

(67)

\[ P = 2G_4 M_6 n_P \],  

(68)

where \( n_Q \) and \( n_P \) are integral. The mass parameters satisfy \( 8G_4 M_0 M_6 = 1 \) so that:

\[ \frac{2PQ}{G_4} = n_Q n_P \].  

(69)

The discussion in this paper uses (69), essentially Dirac’s quantization rule, several times. However, the precise quantization rules on the independent charges are less
important. It may nevertheless be useful to note that, if the electric and magnetic objects are interpreted in string theory as $D0$- and $D6$-branes, the masses are:

\[
M_0 = \frac{1}{l_s g_s},
\]

\[
M_6 = \frac{V_6}{(2\pi)^6 l_s^7 g_s},
\]

where the string units are defined so $l_s = \sqrt{\alpha'}$ and $V_6$ is the volume of the six compact dimensions wrapped by the $D6$-brane.

The extremal entropy (56) can be rewritten as:

\[
S = 2\pi \sqrt{\frac{n_P^2 n_Q^2}{4} - J^2},
\]

using the quantization rule (69). It is interesting that this expression involves only the quantized charges, the moduli cancel out. This is sometimes interpreted as a signal that a clear connection to the microscopic theory is possible [21]. For supersymmetric black holes, the cancellation of moduli is understood as a consequence of enhanced supersymmetry at the horizon [22]. However, this result does not seem to apply in the present circumstances.

Taking the cancellation of moduli into account, the duality group is enlarged to the full noncompact group. In $N = 8$ supergravity the extremal entropy formula for the full orbit is therefore:

\[
S = 2\pi \sqrt{\frac{1}{4} J_4 - J^2},
\]

where the quartic invariant of $E_7(7)$ satisfies $J_4 > 0$. The corresponding formula for the orbit of extremal black holes with a supersymmetric limit is:

\[
S = 2\pi \sqrt{J^2 - \frac{1}{4} J_4},
\]

with $J_4 < 0$. The analogous entropy formulae for the $N = 4$ theory are identical, except that the $J_4$ is replaced by the quartic $S$- and $T$- duality invariant.

### 3.5 Comments on the Nonextreme Entropy

The family of black holes considered here are all far from extremality, and so it is difficult to establish a firm connection with microscopic ideas for the entropy. The extremal (and diagonal) case was interpreted using the correspondence principle in [17]; for a brane-antibrane interpretation of the general nonextreme case see [19].
The purpose of this section is to apply the observations of 23, 24 in the present case. The idea is to interpret the black hole as a state in some underlying conformal field theory, with the effective levels of the state as the quantities that control the thermodynamics. Now, the proposal gives a geometric determination of the levels as:

$$S^{(\pm)} = 2\pi(\sqrt{N_L} \pm \sqrt{N_R}) ,$$

(75)

where $S^{(+)}$ is the usual entropy, computed from the area of the event horizon, and $S^{(-)}$ is computed similarly, but from the area of the inner horizon. In the present case:

$$S^{(-)} = 2\pi \left[ \sqrt{\frac{pq}{16G_4^2}} \left( \frac{pq + 4m^2}{p + q} \right)^2 - J^2 - \frac{m\sqrt{pq}}{2G_4} \right],$$

(76)

so the prescription leads to the levels:

$$N_L = \frac{pq}{16G_4^2} \left( \frac{pq + 4m^2}{p + q} \right)^2 - J^2 ,$$

$$N_R = \frac{m^2pq}{4G_4^2} .$$

(77)

(78)

According to (73) we have $N_L > N_R$ by convention. In black holes with a supersymmetric limit this implies that the right movers are supersymmetric, but not the left movers. In the present context there is no supersymmetric limit; here the convention implies that the left movers are supersymmetric, but not the right movers.

In general, it is not known what conformal field theory has these expressions as effective levels and no independent verification is possible. However, general principles of conformal field theories nevertheless suffice for some consistency checks. An important requirement is that the difference of the levels must be an integer, by modular invariance. In the present case a simple computation gives:

$$N_L - N_R = \frac{Q^2P^2}{G_4^2} - J^2 = \frac{1}{4} n_Q^2 n_P^2 - J^2 .$$

(79)

This is indeed an integer, for all values of the charges and the angular momentum. Note that this result is quite delicate: $J$ may be half-integer and $n_Q n_P$ may be odd, but the angular momentum quantization rules for dyons ensure that these possibilities are correlated so that neither happen, or both happen at once. This is precisely what is needed.
In conclusion, we find that $N_R - N_L$ is integral, confirming a rule noted in many previous examples. This may be an indication that both $N_R$ and $N_L$ are levels in some conformal field theory.

### 3.6 The Magnetic Dipole Moment

The magnetic dipole moment is a good indicator of the interplay between the angular momentum and the electric and magnetic charges. It can also be useful in establishing the connection to various microscopic models, see e.g. [25, 26, 27].

The magnetic dipole moment can be read off from the gauge potential $A$, given in (36). The coefficient is determined by the precise analogy with the relation between angular momentum and the effective gauge potential $B$. This procedure gives:

$$
\mu = \sqrt{\frac{p^3(q^2 - 4m^2)}{(p + q)^3} \frac{a}{4m}}.
$$

A natural benchmark for the electric dipole moment is the classical value:

$$
\mu_0 = \frac{QJ}{2M} = \sqrt{\frac{p(q^2 - 4m^2)}{(p + q)^3} (pq + 4m)q \frac{a}{4m}}.
$$

The gyromagnetic ratio $g$, defined through $\mu = g\mu_0$, becomes:

$$
g = \frac{p(p + q)^2}{q(pq + 4m^2)}.
$$

To interpret this formula, consider some special cases.

The purely electric black hole has $p = 2m$ and so the gyromagnetic ratio:

$$
g = \frac{q + 2m}{q}.
$$

This agrees with previous results [11, 6]. The function decreases monotonically from Dirac’s quantum value $g = 2$, in the limit of vanishing electric charge ($q = 2m$), to the classical value $g = 1$, for very large charge. The large charge limit is extreme and the result can be compared successfully with our microscopic understanding [27]. The appearance of Dirac’s value in a natural limit of parameter space is intriguing, and reminiscent of the fact that this also happens for Kerr-Newman black holes. As in that case, it is not clear why Dirac’s value should appear.
A more illuminating special case may be that of vanishing electric charge \( p = 2m \), giving:

\[
g = \frac{p}{2m} (\frac{p}{2m} + 1) \ . \tag{84}
\]

The gyromagnetic ratio is always larger that Dirac’s value \( g = 2 \), obtained in the limit of vanishing magnetic charge \( q = 2m \). It increases monotonically with the magnetic charge, with no upper bound. Thus, the rotating magnetic background is very efficient at creating an magnetic dipole moment. This property suggests some sort of phase separation of the microscopic constituents; however, the details are puzzling.

As a last special case, consider the general extreme limit \( p, q \gg m \) where:

\[
g = \frac{(q + p)^2}{q^2} = \left[ 1 + \left( \frac{P}{Q} \right)^{2/3} \right]^2 \ . \tag{85}
\]

This attains the classical value for purely electric black holes, and increases without bound as the magnetic charge is turned on and becomes large. Similar qualitative behavior was noticed in [13], but for black holes that are not obviously related to the present ones. The full function (85) characterizes properties of extremal black holes, and may be a good target for a microscopic analysis.

The consideration of the electric dipole moment is completely parallel to that of the magnetic dipole moment. Here one starts with the magnetic potentials, already computed in section (2.2). The resulting expressions for the magnetic variables can be found from the corresponding electric ones, via the interchange of variables \( p \leftrightarrow q \). The associated discussion is therefore analogous.

### 3.7 The Near-horizon Geometry

In some cases the AdS/CFT correspondence [28] offers a direct avenue from the near-horizon geometry of black holes to aspects of the underlying microscopic theory. As a first step in this strategy, consider the nonrotating black holes \( a = 0 \), and take the limit \( p, q \gg r, m \). The metric becomes:

\[
d s^2 = \frac{q}{p} dy^2 - \frac{2(p + q)}{q^2 p} (r^2 - 2mr) dt^2 + \frac{p^2 q}{2(p + q)} (\frac{dr^2}{r^2 - 2mr} + d\Omega^2) \ . \tag{86}
\]

The radius of the compact dimension is constant, so the geometry is effectively four-dimensional. In fact, for \( r \gg m \) it is precisely \( AdS_2 \times S^2 \), the near-horizon limit of
extreme Reissner-Nordström black holes. A general value of $m$ parametrizes a deformation away from the $AdS_2$ limit which preserves the asymptotic behavior. These results are promising, but the conformal quantum mechanics dual to $AdS_2$ spaces is not well understood [29], and the decoupling limit which defines it as a theory without gravity is more subtle than for other $AdS$-spaces [30].

Another decoupling limit for the $D0/D6$ system was considered in [31]. In this limit the effective near-horizon geometry is the five-dimensional Kerr black hole. The precise relation with the $AdS_2 \times S^2$ geometry is not clear.

When the rotational parameter is included the decoupling limit has $p, q \gg r, m, a$. In this limit the geometry is (31) with:

$$H_1 = \frac{p^2 q}{2(p+q)} \left(1 - \frac{a \cos \theta}{m}\right), \quad (87)$$

$$H_2 = \frac{q^2 p}{2(p+q)} \left(1 + \frac{a \cos \theta}{m}\right), \quad (88)$$

$$B_t = \frac{\sqrt{p^2 q^3}}{2(p+q)} (r - m) H_3^{-1} \frac{a}{2m} \sin^2 \theta, \quad (89)$$

$$A_t = -\sqrt{\frac{p+q}{q}}, \quad (90)$$

$$A_\phi = -2P \frac{m \cos \theta + a}{m + a \cos \theta}, \quad (91)$$

and $H_3, \Delta$ retained in full. The parameter $a/m$ must be less than unity for the geometry to be regular, in agreement with the discussion in section 3.2. The scale of the compact dimension is set by $H_2/H_1$ and is independent of $r$, but dependent of $\theta$. Thus the compact direction effectively becomes part of the angular $S^2$.

The azimuthal coordinate mixes with the time coordinate, as expected for rotating spacetimes. However, $G_{\phi,t} \sim r$ for large $r$ so the $AdS_2 \times S^2$ geometry is not recovered asymptotically. To interpret this result, recall that rotation similarly modifies black holes with near-horizon geometry $AdS_3 \times S^2$ by mixing the $AdS_3$ with the $S^2$, but in this case the asymptotic geometry is unaffected by the rotation [32]. The field theory interpretation is that the rotational parameter characterizes a specific excitation, rather than a deformation of the theory. In the present case we reach the opposite conclusion: the deformation due to rotation is so large that it modifies the dual theory.

One may take further limits between the small parameters, in an attempt to find the geometry corresponding to the vacuum of the dual CFT. The relation $r \gg m, a$
is natural because it retains the ratio $a/m$ characterizing the rotation. However, no special simplification seems to occur in the limit.

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