Triviality Bounds
in
the Next to Minimal Supersymmetric Standard Model

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*This work was supported in part by the Director, Office of Energy Research, Office of High Energy and Nuclear Physics, Division of High Energy Physics of the U.S. Department of Energy under Contract DE-AC03-76SF00098 and in part by the National Science Foundation under grant PHY90-21139.
Abstract

We study the implications of the triviality problem for the Higgs masses and other relevant parameters in the Next to Minimal Supersymmetric Standard Model (NMSSM). By means of triviality, a new way to constrain parameters is proposed, and therefore we are able to derive triviality bounds on the heaviest-Higgs mass, the lightest-Higgs mass, the soft SUSY-breaking parameters, and the vacuum expectation value of the Higgs gauge singlet through a thorough examination of the parameter space. The triviality upper bound on the lightest-Higgs mass predicted by NMSSM is indeed larger than the upper bound predicted by MSSM (the Minimal Supersymmetric Standard Model).
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1 Introduction

There have been many studies about the upper bound on the Higgs mass of the standard model (or its supersymmetric extension) \[1, 2, 3\]. One of the approaches is based on triviality of the $\phi^4$ theory \[4\]. Due to triviality, the standard model is inconsistent as a fundamental theory but is a reasonable effective theory with momentum cut-off $\Lambda$. Furthermore, by requiring that $\Lambda$ be larger than the Higgs mass in order to maintain the consistency of the standard model as an effective theory, Dashen and Neuberger were the first to derive the triviality upper bound (about 800 GeV) on the Higgs mass in the minimal standard model \[4\]. Improvements on this triviality upper bound have also been made, including the non-perturbative calculations, or the contributions of gauge couplings and the top Yukawa coupling \[5, 6, 7, 8\]. So far, supersymmetry is the only viable framework where the Higgs scalar is natural \[9, 10\]. Therefore, it is important to understand how this triviality upper bound on the Higgs mass behaves in the supersymmetric extension of the standard model, especially the issue of the lightest-Higgs mass.

The Minimal Supersymmetric Standard Model (MSSM) is the most studied supersymmetric extension of the standard model. Another possible extension is the Next to Minimal Supersymmetric Standard Model (NMSSM) with two SU(2)×U(1) Higgs doublets and one Higgs singlet \[10, 11, 12\]. The inclusion of the Higgs gauge singlet in NMSSM provides an explanation to the $\mu$ problem of MSSM \[13\]. In addition, the existence of the Higgs singlet is suggested in many superstring models \[14, 15\] and grand unified supersymmetric models \[13\]. These features make NMSSM an appealing alternative to MSSM. An important issue about MSSM is the upper bound on the lightest-Higgs mass \[16\]. Because there is no guarantee that there will be the signal of Higgs particles before we reach the upper bound predicted by MSSM, it is definitely interesting to investigate whether the upper bound on the lightest-Higgs mass predicted by NMSSM can be larger than that predicted by MSSM or not. A pioneering work \[12\] has been done in this respect.

Similar to the standard model, it is suggested that triviality still persists in NMSSM when the Higgs couplings are strong. In short, triviality means that, given the low-energy values of the Higgs couplings, the Higgs couplings will
eventually blow up at some momentum scale \( \Lambda_L \) (the Landau pole) if it is scaled upward, where \( \Lambda_L \) is determined essentially by the low-energy values of the Higgs couplings. The stronger the low-energy Higgs couplings are, the smaller \( \Lambda_L \) is. This observation certainly implies an upper bound on the Higgs mass. In order to establish an upper bound on the lightest-Higgs mass by means of triviality, one of the possible approaches is to treat NMSSM as an effective theory with momentum cut-off \( \Lambda \), and then require that the Higgs couplings remain finite beneath the cut-off \( \Lambda \). We will call this approach the Finite Coupling Constant Formulation (FCCF). In FCCF, based on triviality associated with the RGE’s, the upper bounds on the low-energy Higgs couplings can be easily computed once the cut-off \( \Lambda \) is specified. Therefore, the corresponding upper bound on the lightest-Higgs mass can be obtained directly from these triviality upper bounds on the Higgs couplings. The cut-off \( \Lambda \) has to be specified by assuming certain underlying grand unification scheme \((\Lambda = 10^{15} \sim 10^{17} \text{ GeV in most cases.})\)

There have been several works done in this approach [17, 18, 19, 20]. They all arrived at the same conclusion that the upper bound on the lightest-Higgs mass of NMSSM is indeed larger than that of MSSM. For example, W.T.A. ter Veldhuis [17] reported that the upper bound on the lightest-Higgs mass of NMSSM will be 25 GeV larger than that of MSSM if the top-quark mass is 150 GeV.

However, according to the spirit of the paper by Dashen and Neuberger [1], the approach of FCCF is not sufficient and the result is model-dependent. The formulation of triviality constraints proposed by Dashen and Neuberger is based on the requirement that \( \Lambda_L \geq m_{HH} \) \((m_{HH}: \) the heaviest-Higgs mass) in order to ensure the consistency of NMSSM as an effective theory with momentum cut-off \( \Lambda \leq \Lambda_L \). We will call this approach the Effective Theory Consistency Formulation (ETCF). In ETCF, the requirement of FCCF is always met because \( \Lambda_L \) is constructed in such a way that Higgs couplings blow up at \( \Lambda_L \). In this sense, ETCF is stronger and more reasonable than FCCF since ETCF ensures not only the requirement of FCCF but also the consistency of NMSSM. ETCF treats \( \Lambda_L \) as a function of Higgs couplings and the constraint \( \Lambda_L \geq m_{HH} \) represents a constraint on the full parameter space, including the Higgs couplings and the soft SUSY-breaking parameters because \( m_{HH} \) depends on the full parameter space in general. ETCF does extend the non-trivial implications of triviality
to the full parameter space, whereas the implication of FCCF is limited to
the Higgs couplings. Therefore, ETCF is able to constrain every parameter of
the full parameter space. These constraints will be computed in Section 4. In
addition, there is no need to introduce certain GUT scheme in ETCF because
the cut-off $\Lambda$ is determined dynamically by the triviality constraint through a
thorough search of the parameter space. In this sense, the approach of ETCF is
more general and model-independent. In conclusion, FCCF is a special case of
ETCF. ETCF provides a more reasonable basis for us to extend the triviality
constraint to the full parameter space. Since ETCF and FCCF are different in
nature, it is worth studying the triviality bounds of NMSSM based on ETCF.

The purpose of this paper is to describe how to establish triviality bounds
on the full parameter space based on ETCF. The computation of the NMSSM
effective potential in this paper includes the tree-level contributions only. There-
fore, all the triviality bounds obtained in Sections 4 and 5 are tree-level results.
However, it has been pointed out in several works [17, 18, 20] that the top and
stop loop contributions are quite substantial when the stop mass is much larger
than the top-quark mass. Hence, the present computations are not very precise.
One-loop contributions, including those of the top and stop, must be included
in future computations in order to make the predictions of the triviality bounds
more precise.

In Section 2, the relevant NMSSM lagrangian and renormalization group
equations are given. Triviality is observed in the case of strong Higgs couplings,
which implies the Landau-pole behavior of the Higgs couplings. To facilitate
the computations of triviality bounds, an analytic expression of the Landau
pole $\Lambda_L$ is also derived. In Section 3, the parametrization of the NMSSM Higgs
mass spectrum over the full parameter space is done. The determination of
the full parameter space is non-trivial because the minimization of the scalar
potential leads to several constraints on the parameters. In Section 4, ETCF is
established and the triviality bound is solved through the full parameter space
by requiring $\Lambda_L \geq m_{HH}$, where $\tan \beta = 1$ is chosen for the sake of simplicity.
Combined with the present experimental lower bound on the Higgs mass, this
analysis indicates that a very large portion of the parameter space is excluded.
For example, the VEV of the Higgs gauge singlet $v_3$ can be constrained to:
$0.24M_W \leq |v_3| \leq 0.749M_W$, where $M_W$ is the mass of W gauge boson. The soft
SUSY-breaking parameters are constrained from above. Furthermore, the above constraints will become stronger if the experimental lower bound on the Higgs mass is raised, which implies a better understanding of the correct parameter ranges. In Section 5, an absolute upper bound of \(2.8M_W\) on the lightest-Higgs mass is established by a search through the full parameter space. This absolute upper bound is beyond the reach of LEP.

2 Indication of Triviality in NMSSM

The supersymmetric Higgs scalar potential of the Next to Minimal Supersymmetric Standard Model (NMSSM) at tree level can be written as follows \[10, 21\]:

\[
V = |hN|^2(\Phi_1^\dagger \Phi_1 + \Phi_2^\dagger \Phi_2) + |h\Phi_1 \Phi_2 + \lambda N^2|^2 + \frac{1}{8}g_1^2(\Phi_1^\dagger \Phi_1 - \Phi_2^\dagger \Phi_2)^2
\]

\[
+ \frac{1}{8}g_2^2[(\Phi_1^\dagger \Phi_1 + \Phi_2^\dagger \Phi_2)^2 - 4(\Phi_1 \Phi_2)(\Phi_2 \Phi_1)]
\]

(1)

\(\Phi_1 = (\phi_1^0, \phi_1^0)\) and \(\Phi_2 = (\phi_2^0, \phi_2^0)\) are two SU(2)×U(1) Higgs doublets, and \(N\) is a complex singlet. \(h\) and \(\lambda\) are the Higgs couplings. It is assumed that only \(\phi_1^0, \phi_2^0\) and \(N\) acquire non-trivial VEV’s \(v_1, v_2\) and \(v_3\) respectively. The scalar-quarks and scalar-leptons do not acquire VEV’s, and we can ignore their contributions to the scalar potential when studying the Higgs mass spectrum. Note that the superpotential corresponding to (1) does not contain linear and bilinear terms \[10\] because these terms lead to naturalness problems. Besides, these terms do not appear in a large class of superstring-inspired models. As for the soft SUSY-breaking terms, a particular \(V_{\text{soft}}\) is chosen:

\[
V_{\text{soft}} = m_1^2\Phi_1^\dagger \Phi_1 + m_2^2\Phi_2^\dagger \Phi_2 - m_{12}^2\Phi_1^\dagger \Phi_2 - m_{12}^*\Phi_2^\dagger \Phi_1
\]

(2)
in order to have predictive power. However, this particular choice of \(V_{\text{soft}}\) does not destroy the generality of the conclusions obtained in this paper. The most general \(V_{\text{soft}}\) will be considered in the last section.

We assume three generations of quarks and leptons together with their supersymmetric partners. As for the renormalization group equations relevant to the Higgs couplings, all the Yukawa couplings are neglected except for the
top Yukawa coupling $f_t$. The relevant one-loop RGE’s of NMSSM are given in [13, 21]:

$$8\pi^2 \frac{d}{dt} g_1^2 = 11 g_1^4$$

$$8\pi^2 \frac{d}{dt} g_2^2 = g_2^4$$

$$8\pi^2 \frac{d}{dt} g_3^2 = -3 g_3^4$$

$$8\pi^2 \frac{d}{dt} f_t^2 = f_t^2(6 f_t^2 + h^2 - \frac{13}{9} g_1^2 - 3 g_2^2 - \frac{16}{3} g_3^2)$$

$$8\pi^2 \frac{d}{dt} h^2 = h^2(4 h^2 + 2 \lambda^2 + 3 f_t^2 - g_1^2 - 3 g_2^2)$$

$$8\pi^2 \frac{d}{dt} \lambda^2 = 6 \lambda^2 (\lambda^2 + h^2)$$

where $g_1$, $g_2$ and $g_3$ are the gauge couplings associated with SU(3), SU(2) and U(1) gauge groups respectively. The parameter $t$ is defined as:

$$t = \frac{1}{2} \ln(\frac{-q^2}{M_W^2})$$

where $q^2$ is the space-like effective square of the momentum at which these couplings are defined. At $t=0$, the gauge couplings can be determined from the experimentally derived inputs [22]:

$$g_1^2 = 0.126, \quad g_2^2 = 0.446, \quad g_3^2 = 1.257$$

To understand the triviality problem, strong Higgs couplings are assumed, and therefore the top Yukawa coupling $f_t$ and all the gauge couplings can be ignored. This is not an unreasonable assumption considering the following observations: The RGE’s (3)-(8) have been studied numerically by Babu and Ma [21]. Their results indicate that gauge couplings are negligible if the ratio $\frac{g_2^2}{g_1^2} \geq 5$ or $\frac{\lambda^2}{g_1^2} \geq 2$ holds, where $\tan \beta = 1$ and the top-quark mass $m_t = 40 \sim 400$ GeV. (See Fig.1 in [21] for a more precise description.) Therefore, it’s certainly reasonable to expect that $\frac{h^2}{g_1^2} \geq 5$ or $\frac{\lambda^2}{g_1^2} \geq 2$ holds in the case of strong Higgs couplings. For the sake of self-consistency, the assumption of negligible gauge couplings in the case of strong Higgs couplings will be verified *a posteriori* by the numerical
results of Sections 4 and 5. The assumption of negligible Yukawa coupling $f_t$ is a little obscure. There has been some numerical evidence in the standard model [5] that the determination of triviality bounds on the Higgs mass is insensitive to the top-quark mass $m_t$ if $m_t \leq 200$ GeV, and it may still be true in NMSSM. The assumption of negligible $f_t$ will be checked by the computations of Sections 4 and 5, and it turns out that $f_t$ is important only in certain extreme situations. A detailed discussion will be given in Section 4.

In the case of strong Higgs couplings with negligible $f_t$ and $g_i$, the one-loop RGE’s for the Higgs couplings $h$ and $\lambda$ are:

\[
8\pi^2 \frac{d h^2}{dt} = h^2(4h^2 + 2\lambda^2) \quad (11)
\]

\[
8\pi^2 \frac{d \lambda^2}{dt} = 6\lambda^2(\lambda^2 + h^2) \quad (12)
\]

There is only one fixed point, the infrared stable fixed point at $h^2 = 0$, $\lambda^2 = 0$. Therefore, similar to the Landau pole [3, 10] of the pure $\phi^4$ theory, $h^2(t)$ and $\lambda^2(t)$ diverge at some finite $t = t_L$ (the Landau pole) unless the Higgs couplings vanish. Triviality is clearly indicated at one-loop. Notice that the above conclusion is still valid even if the top Yukawa coupling $f_t$ is included. The general solution of the Landau pole $t_L$ can be found by means of the method of integrating factor [23]:

\[
t_L = \frac{2\pi^2 \lambda_0^{-\frac{2}{3}} \sqrt{1 + C\lambda_0^{-\frac{2}{3}}}}{C} - \frac{2\pi^2 \ln(\sqrt{1 + C\lambda_0^{-\frac{2}{3}}} + \sqrt{C\lambda_0^{-\frac{2}{3}}})}{C^\frac{2}{3}} \quad (13)
\]

\[
C = h_0^4 \lambda_0^{-\frac{8}{3}} + 2 h_0^2 \lambda_0^{-\frac{2}{3}} \quad (14)
\]

\[
h_0 = h(t = 0), \quad \lambda_0 = \lambda(t = 0), \quad \Lambda_L \equiv M_W \cdot \exp(t_L) \quad (15)
\]

where $\Lambda_L$ is the momentum corresponding to the Landau pole $t_L$. (13)-(15) will be useful to the computations of triviality bounds in Section 4.

Notice that the treatment of triviality here is of perturbative nature. Although the problem of triviality should be of non-perturbative nature, several non-perturbative numerical simulations have been performed [24, 25] and the results indicated that the renormalized perturbative calculation gives essentially the correct triviality upper bound on the Higgs mass. This observation may justify our approach as a first approximation.
3 Parameter Space and Higgs Mass Spectrum

Consider the tree-level scalar potential $V' = V + V_{soft}$, and the relevant parameters are $h$, $\lambda$, $v_1$, $v_2$, $v_3$, $m_1^2$, $m_2^2$, $m_{12}^2$. Without loss of generality, our convention is to take $h$, $\lambda$, $m_1^2$, $m_2^2$ to be real, and $v_1$, $v_2$, $v_3$, $m_{12}^2$ to be complex, i.e., $v_1 = \tilde{v}_1 e^{i\delta_1}$, $v_2 = \tilde{v}_2 e^{i\delta_2}$, $v_3 = \tilde{v}_3 e^{i\delta_3}$, $m_{12}^2 = \tilde{m}_{12}^2 e^{i\delta_m}$. The minimization of the scalar potential $V'$ leads to three complex vacuum constraints on these parameters:

$$h^2(|v_1|^2 + |v_2|^2)v_3 + 2\lambda(hv_1^*v_2 + \lambda v_3^2)v_3^* = 0$$  \hspace{1cm} (16) $$\frac{1}{4}(g_1^2 + g_2^2)(|v_1|^2 - |v_2|^2) + h^2(|v_2|^2 + |v_3|^2) + m_1^2 = \frac{v_2}{v_1}(m_{12}^2 - h\lambda v_3^2)$$  \hspace{1cm} (17) $$\frac{1}{4}(g_1^2 + g_2^2)(|v_2|^2 - |v_1|^2) + h^2(|v_1|^2 + |v_3|^2) + m_2^2 = \frac{v_1}{v_2}(m_{12}^2 - h\lambda v_3^2)$$  \hspace{1cm} (18)

In addition, one has the following physical constraint:

$$M_W^2 = \frac{1}{2}g_2^2(|v_1|^2 + |v_2|^2)$$  \hspace{1cm} (19)

Imaginary parts of the constraints (16)-(18) fix the phases among the complex parameters $v_1$, $v_2$, $v_3$, $m_{12}^2$, and (19) reduces ($\tilde{v}_1$, $\tilde{v}_2$) to a single parameter $\tan \beta = \frac{\tilde{v}_2}{\tilde{v}_1}$. Furthermore, ($h$, $\lambda$, $\tilde{v}_3$) can be expressed in terms of other parameters by means of the real parts of the constraints (16)-(18). The parameter space under study is then defined as the set of parameters ($\phi$, $\tan \beta$, $m_1^2$, $m_2^2$, $\tilde{m}_{12}^2$), where

$$0 \leq \phi < 4\pi, \hspace{0.5cm} -\infty < \tan \beta, \hspace{0.5cm} m_1^2, \hspace{0.5cm} m_2^2, \hspace{0.5cm} \tilde{m}_{12}^2 < \infty$$  \hspace{1cm} (20)

The other parameters can be expressed in terms of (20):

$$\tilde{v}_3 = \frac{M_W}{g_2} \sqrt{\frac{2AB\tan \beta - B^2(1 + \tan^2 \beta)}{A^2(1 + \tan^2 \beta)}}$$

$$A = \tilde{m}_{12}^2(\tan \beta - \frac{1}{\tan \beta}) - m_1^2 + m_2^2 + M_W^2(1 + \frac{g_1^2}{g_2^2})\frac{\tan^2 \beta - 1}{\tan^2 \beta + 1}$$

$$B = \frac{1}{\tan \beta}\{m_1^2 + \frac{1}{2}(1 + \frac{g_1^2}{g_2^2})M_W^2\} - \tan \beta\{m_2^2 + \frac{1}{2}(1 + \frac{g_1^2}{g_2^2})M_W^2\}$$  \hspace{1cm} (21)

$$v_1 = \frac{M_W}{g_2} \sqrt{\frac{\sqrt{2}}{\tan^2 \beta + 1}}, \hspace{0.5cm} v_2 = \frac{M_W}{g_2} \frac{\sqrt{2}\tan \beta}{\sqrt{\tan^2 \beta + 1}} e^{i\phi}, \hspace{0.5cm} v_3 = \tilde{v}_3 e^{i\frac{\phi}{2}}$$  \hspace{1cm} (22)
\[ m_{12}^2 = \tilde{m}_{12}^2 e^{-i\phi} \]  

\[ h^2 = \frac{\tilde{m}_{12}^2 \tan \beta - m_1^2 + \frac{M_W^2}{2} (1 + \frac{g_2^2}{g_2^2} \frac{\tan^2 \beta - 1}{\tan^2 \beta + 1})}{\tilde{v}_3^2 + \tan \beta \frac{M_W^2}{g_2^2} (\frac{\tan \beta}{\tan^2 \beta + 1}) \pm \sqrt{\frac{\tan^2 \beta}{(\tan^2 \beta + 1)^2} - \frac{g_2^2 \tilde{v}_3^2}{M_W^2}}} \]  

\[ \lambda = h \frac{M_W^2}{g_2^2 \tilde{v}_3^2} (\frac{\tan \beta}{\tan^2 \beta + 1} \pm \sqrt{\frac{\tan^2 \beta}{(\tan^2 \beta + 1)^2} - \frac{g_2^2 \tilde{v}_3^2}{M_W^2}}) \]  

\[ \tilde{v}_3, \ h, \ \text{and} \ \lambda \ \text{are real.} \]  

"±" in (24) and (25) indicates \((h, \ \lambda)\) has two solutions, where the solution with "+" is denoted as \((h_A, \ \lambda_A)\) and the solution with "-" is denoted as \((h_B, \ \lambda_B)\).

(20)-(26) specify the full parameter space. (26) is non-trivial because square roots are involved in (21), (24) and (25). Together with (25), the reality condition of \(\lambda\) leads to:

\[ \tilde{v}_3 \leq \frac{|\tan \beta|}{g_2(\tan^2 \beta + 1)} M_W \]  

\[ \frac{|\tan \beta|}{\tan^2 \beta + 1} \] has its maximum=\(\frac{1}{2}\) at \(\tan \beta = \pm 1\). Therefore, we are able to establish an absolute upper bound on \(\tilde{v}_3\):

\[ \tilde{v}_3 \leq \frac{1}{2g_2} \ M_W \]  

Given \(M_W = 80 \text{ GeV}\) and \(g_2^2 = 0.446\) from (10), this absolute upper bound on the magnitude of \(v_3\) is 60 GeV. As \(\tilde{v}_3 \to \frac{M_W}{2g_2}\), (27) implies \(|\tan \beta| \to 1\). Therefore, in NMSSM, \(\tan \beta\) can be constrained by means of \(|v_3|\), and vice versa.

For the sake of simplicity, we choose \(\tan \beta = 1\) when studying (20)-(26). When \(\tan \beta = 1\), \(\tilde{v}_3\) becomes a free parameter, and \(m_1^2 = m_2^2\) is required by the minimization of the scalar potential \(V': (17)\) and (18). On the whole, the number of free parameters is unchanged. The full parameter space (\(\tan \beta = 1\)) is then defined as the set of parameters \((\phi, \ \tilde{v}_3, \ m_1^2 = m_2^2, \ \tilde{m}_{12}^2)\) plus the following constraints:

\[ 0 \leq \phi < 4\pi, \ 0 < \tilde{v}_3 \leq \frac{1}{2g_2} M_W \]  

\[ -\infty < m_1^2 = m_2^2 < \tilde{m}_{12}^2 < \infty \]  

There is no essential change to the expressions of the other parameters except
for \( h \) and \( \lambda \):

\[
h^2 = \frac{\tilde{m}_{12}^2 - m_1^2}{\tilde{v}_3^2 + M_W^2 \left( \frac{1}{2} \pm \sqrt{\frac{1}{4} - \frac{g_2^2 \tilde{v}_3^2}{M_W^2}} \right)} \tag{31}
\]

\[
\lambda = \frac{h M_W^2}{g_2 \tilde{v}_3} \left(-\frac{1}{2} \pm \sqrt{\frac{1}{4} - \frac{g_2^2 \tilde{v}_3^2}{M_W^2}} \right) \tag{32}
\]

(29)-(32) specify the full parameter space and will be studied later. Notice that (31) and (32) imply:

\[
h^2, \lambda^2 \propto (\tilde{m}_{12}^2 - m_1^2) \tag{33}
\]

and \( m_1^2 = m_2^2 < \tilde{m}_{12}^2 \) in (30) is the consequence of the reality condition of \( h \).

Based on the parameter space described in (20) or (29), it is trivial to work out the Higgs squared-mass spectrum from the tree-level potential \( V' \). In general, \( \{\phi_0^1, \phi_0^2, N\} \) does not mix with \( \{\phi_1^1, \phi_1^2\} \). The squared-mass matrix \( [M_n^2] \) for \( \{\phi_0^1, \phi_0^2, N\} \) is a 6x6 matrix, and the squared-mass matrix \( [M_c^2] \) for \( \{\phi_1^1, \phi_1^2\} \) is a 4x4 matrix. As expected, \( [M_n^2] \) contains five neutral Higgs bosons and one massless particle. \( [M_c^2] \) contains two charged Higgs bosons and two massless particles. The detailed expressions of \( [M_n^2] \) and \( [M_c^2] \) have been given in [20] and won’t be repeated here. However, there are several symmetries of \( [M_n^2] \) and \( [M_c^2] \):

\[
[M_n^2] \text{ and } [M_c^2] \text{ are periodic in } \phi, \text{ of periodicity } \pi. \tag{34}
\]

In addition, \( [M_n^2] \) and \( [M_c^2] \) are invariant under two discrete symmetries on the parameter space:

\[
h \rightarrow -h, \ \lambda \rightarrow \lambda, \ \tan \beta \rightarrow -\tan \beta, \ m_{12}^2 \rightarrow -m_{12}^2 \tag{35}
\]

\[
h \rightarrow -h, \ \lambda \rightarrow -\lambda \tag{36}
\]

As for the Higgs squared-mass spectrum, we are interested in these two quantities: \( m_{HH} \) (the heaviest-Higgs mass) and \( m_{LH} \) (the lightest-Higgs mass). In Sections 4 and 5, the triviality bounds \( B_{HH} \) (the upper bound on \( m_{HH} \)) and \( B_{LH} \) (the upper bound on \( m_{LH} \)) will be derived. In general, \( m_{HH} \) and \( m_{LH} \) have to be computed from \( [M_n^2] \) numerically, and there are no simple analytic expressions. However, the understanding of the dependence of the lightest-Higgs
mass $m_{LH}$ on the Higgs coupling $h$ will be very useful in establishing the triviality bounds. Using the fact that the smallest eigenvalue is smaller than the smallest diagonal term and choosing an appropriate basis for $[M^2_n]$, Binétruy and Savoy \cite{19} have derived an upper bound on $m_{LH} (\tan \beta = 1)$:

$$B_{LH}^{(BS)} = \frac{h}{\sqrt{2}}v \geq m_{LH}, \quad v \approx 250 \text{ GeV}$$  \hspace{1cm} (37)

Notice that (37) is the result of tree-level computations. This upper bound is denoted as $B_{LH}^{(BS)}$ in order to be distinguished from $B_{LH}$ (the triviality upper bound on $m_{LH}$). Therefore, the dependence of $m_{LH}$ on $h$ can be understood as: $m_{LH} \leq B_{LH}^{(BS)} \propto h$, where $B_{LH}^{(BS)}$ is proportional to the Higgs coupling $h$. Combined with the present experimental lower bound on the Higgs mass, $m_{LH} \leq B_{LH}^{(BS)} = h \sqrt{2}v$ implies that $h$ must be bounded from below. Notice that $B_{LH}^{(BS)}$ is introduced for illustrative purpose only. In practice, the precise determination of this lower bound on $h$ is made by solving $m_{LH}$ from $[M_n]$ and requiring $m_{LH} \geq$ the experimental lower bound. By means of triviality, it will be argued in Section 4 that $h$ decreases with the soft SUSY-breaking parameter $m_1$, which enables us to establish an upper bound on the soft SUSY-breaking parameter $m_1$ based on the lower bound on $h$. Details will be given in Section 4.

4 Constraints on the Higgs Mass and the Soft SUSY-Breaking Parameters

Based on the parameter space $(\phi, \tilde{v}_3, m_1^2 = m_2^2, \tilde{m}_{12}^2)$ specified by (29)-(32), $\Lambda_L \geq m_{HH}$ ($m_{HH}$: the heaviest-Higgs mass) is required by ETCF, and $B_{HH}$ (the triviality upper bound on the heaviest-Higgs mass $m_{HH}$) is established by $\Lambda_L = m_{HH} \equiv B_{HH}$, where $\Lambda_L$ is defined in (13)-(15) and $m_{HH}$ is computed from $[M^2_n]$ numerically. Geometrically, the triviality upper bound $B_{HH}$ defines a surface in the parameter space by means of $\Lambda_L = m_{HH}$, and our convention is to parametrize this triviality surface in terms of $(\phi, \tilde{v}_3, m_1^2 = m_2^2)$, where $\tilde{m}_{12}^2$ depends on $(\phi, \tilde{v}_3, m_1^2 = m_2^2)$ through $\Lambda_L = m_{HH}$. At any point $(\phi, \tilde{v}_3, m_1^2 = m_2^2)$ of this triviality surface, the seven non-zero eigenvalues of $[M_n]$ and $[M_c]$ define the seven triviality upper bounds on the seven physical Higgs masses respectively. For example, the triviality upper bound on $m_{LH}$ ($m_{LH}$: the lightest-Higgs
mass) is defined as the $m_{LH}$ evaluated on the triviality surface. Therefore, care should be taken in distinguishing $m_{LH}$ from the triviality upper bound on $m_{LH}$. Next, we will study the general features of the triviality surface.

A typical example is chosen as: $(h, \lambda) = (h_A, \lambda_A)$, $\phi = 0$, $\tilde{v}_3 = 0.7 M_W$. Its triviality surface is computed, and $B_{HH}$ versus $m_1^2 (= m_2^2)$ is plotted in Fig.1. Fig.1 corresponds to a line on the triviality surface. Several universal features of Fig.1 are important. First, when $m_1^2$ is small, the soft SUSY-breaking terms are not important and therefore the determination of $B_{HH}$ is insensitive to $m_1^2$. Second, the curve in Fig.1 ends at $m_1^2 \approx -M^2_W$ because the squared-mass matrix $[M_n^2]$ or $[M_c^2]$ will develop negative eigenvalues if $m_1^2$ becomes too negative. Together with (30), this observation indicates that $m_1^2$, $m_2^2$, $\tilde{m}_{12}$ are bounded from below. Since nothing interesting happens when $m_1^2 < 0$, we will assume $0 \leq m_1 = m_2 < \tilde{m}_{12}$ from now on.

The last but most important universal feature of Fig.1 is: When $m_1^2$ is large, the linear relation $B_{HH} = \sqrt{2} m_1$ is a good approximation. This observation can be understood as follows. (13) implies that, on the triviality surface, $h^2$ and $\lambda^2 \rightarrow 0$ if $B_{HH} (= \Lambda_L) \rightarrow \infty$. The structure of $[M_n^2]$ also implies $B_{HH} \rightarrow \infty$ if $m_1 = m_2 \rightarrow \infty$. The above two observations lead to:

On the triviality surface, $h^2$ and $\lambda^2 \rightarrow 0$ if $m_1 = m_2 \rightarrow \infty$  \hspace{1cm} (38)

Therefore, in the limit $m_1 = m_2 \rightarrow \infty$ on the triviality surface, $[M_n^2]$ and $[M_c^2]$ can be solved up to order $O(h^2)$ exactly:

Eigenvalues of $[M_n^2] = \begin{bmatrix} 2m_1^2 + O(h^2), & 2m_1^2 + O(h^2), & O(h^2), & O(h^2), & 0 \end{bmatrix}$

Eigenvalues of $[M_c^2] = \begin{bmatrix} 2m_1^2 + O(h^2), & 2m_1^2 + O(h^2), & 0, & 0 \end{bmatrix}$  \hspace{1cm} (39)

The square roots of the seven non-zero eigenvalues in (39) are just the seven triviality upper bounds on the seven Higgs masses respectively, including the triviality upper bound on $m_{LH}$. (39) together with (38) explains why $B_{HH} = \sqrt{2} m_1$ is valid up to order $O(h^2)$ when $m_1^2$ is large. (39) also implies that there are exactly three light neutral Higgs bosons when $m_1^2$ is large. As for the lightest-Higgs mass $m_{LH}$, (39) indicates that the triviality upper bound on $m_{LH}$ is of order $O(h)$, which is consistent with the upper bound $B_{LH}^{(BS)}: m_{LH} \leq B_{LH}^{(BS)} = \frac{h}{\sqrt{2} v}$ in (37). Due to (38), the triviality upper bound on $m_{LH}$ decreases to zero.
monotonically as $m_1 = m_2 \rightarrow \infty$. According to the present experimental lower bound on the Higgs mass \cite{20, 27}, we require that the triviality upper bound on $m_{LH}$ be larger than $1\, M_W$, and this requirement leads to an upper bound on $m_1 (= m_2)$ due to the fact that the triviality upper bound on $m_{LH}$ decreases to zero as $m_1 = m_2 \rightarrow \infty$. An explicit realization of this idea is given in Fig.2, where $(h, \lambda) = (h_A, \lambda_A), \phi = 0.6$, and $(\tilde{v}_3, m_1)$ is examined thoroughly. The enclosed region of Fig.2 is the allowed range of $\tilde{v}_3$ versus $m_1$. An interesting quantity $B_{soft}$ can be defined in such a way that the allowed range of $\tilde{v}_3$ shrinks to a single point at $m_1 = B_{soft}$ and there is no solution for $m_1 > B_{soft}$. In Fig.2, $B_{soft} = 138\, M_W$. The meaning of $B_{soft}$ is clear: For given $\phi$, $B_{soft}$ is the absolute upper bound on $m_1$ for $0 < \tilde{v}_3 \leq \frac{M_W}{2g_2}$. That is, $B_{soft}$ is the upper bound on $m_1$ when $\tilde{v}_3 = \frac{M_W}{2g_2}$, and the upper bound on $m_1$ is smaller than $B_{soft}$ when $\tilde{v}_3 < \frac{M_W}{2g_2}$. In fact, $B_{soft}$ can be interpreted as the absolute upper bound (with $\phi$ fixed) on all the soft SUSY-breaking parameters ($m_1, m_2, \tilde{m}_{12}$) because $m_1^2 \approx \tilde{m}_{12}^2$ is true on the triviality surface when $m_1^2$ is large. The fact that $m_1^2 \approx \tilde{m}_{12}^2$ on the triviality surface when $m_1^2$ is large can be explained by the observations that $h^2 \propto (\tilde{m}_{12}^2 - m_1^2)$ and that $h^2$ is negligible when $m_1^2$ is large.

The dependence of $B_{soft}$ on $\phi$ is displayed in Fig.3, where $(h_A, \lambda_A)$ and $(h_B, \lambda_B)$ have identical results. In Fig.3, it is also required that the triviality upper bound on $m_{LH}$ be larger than $1\, M_W$. Fig.3 and Fig.2 form the complete picture of the triviality upper bound on the soft SUSY-breaking parameters. For example, $B_{soft} = 2380\, M_W$ at $\phi = 0$, and $B_{soft} = 84.6\, M_W$ at $\phi = \frac{\pi}{2}$. Furthermore, all the conclusions about $B_{soft}$ can be re-interpreted as the absolute triviality upper bound on the heaviest-Higgs mass (with $\phi$ fixed) by means of $B_{HH} \simeq \sqrt{2}\, B_{soft}$. Therefore, Fig.3 also provides the complete picture of the absolute triviality upper bound on the heaviest-Higgs mass $m_{HH}$. The computations of Fig.3 are very sensitive to the experimental lower bound on the Higgs mass. Fig.3 is obtained by requiring that the triviality upper bound on $m_{LH}$ be larger than $1\, M_W$, and $B_{soft} = 2380\, M_W$ is obtained when $\phi = 0$. However, $B_{soft} = 4.88\, M_W$ at $\phi = 0$ will be obtained if we require that the triviality upper bound on $m_{LH}$ be larger than $2\, M_W$.

Inspired by Fig.2, we can define the absolute triviality lower bound $B_N$ on $\tilde{v}_3$ for given $\phi$. For example, $B_N = 0.7\, M_W$ in Fig.2. $B_N$ gives a modest
measure of the constraint on $\tilde{v}_3$. Fig.4 displays the dependence of $B_N$ on $\phi$ for $(h_A, \lambda_A)$ and $(h_B, \lambda_B)$. Besides, it is required that the triviality upper bound on $m_{LH}$ be larger than $1$ $M_W$. The dotted straight line corresponds to the absolute upper bound of $0.749$ $M_W$ on $\tilde{v}_3$, (28). For $\phi = 0$ (i.e., no CP-violation in the scalar sector), we have $0.24$ $M_W \leq \tilde{v}_3 \leq 0.749$ $M_W$. For $\phi = \frac{\pi}{2}$, $0.65$ $M_W \leq \tilde{v}_3 \leq 0.749$ $M_W$. Therefore, triviality is very helpful for a better understanding of $\tilde{v}_3$. In addition, if the experimental lower bound on the Higgs mass is raised in the future, all the bounds involved in Fig.3 and Fig.4 will become stronger, which implies a better understanding of the heaviest-Higgs mass, the VEV of the Higgs singlet, and the soft SUSY-breaking parameters. However, NMSSM is not consistent with an unlimited raise of the experimental lower bound on the Higgs mass. In Section 5, we will derive an absolute upper bound of $2.8$ $M_W$ on the lightest-Higgs mass.

Finally, let’s check the assumptions of negligible $f_t$ and $g_i$. With the help of [13], all the computations involved in Figures 1-4 do satisfy the assumption of negligible $g_i$. To check the assumption of negligible $f_t$, take the mass of top quark $m_t = 170$ GeV. Generally speaking, this assumption is reasonable when $m_1^2$ is small, but it needs modifications when $m_1^2$ is large because $h^2$ and $\lambda^2$ are small according to (38). As for Fig.4, the computation of $B_N$ indicates that $f_t$ is negligible. However, the computation of $B_{soft}$ in Fig.3 indicates that $f_t^2$ is as important as $h^2$ and $\lambda^2$. To understand the effect of $f_t^2$ on $B_{soft}$, refer to (6)-(8). Because all the coefficients of $f_t^2$-terms in (6)-(8) are positive (assuming negligible $g_i$), the inclusion of $f_t$ makes triviality even stronger. That is, the Landau pole $t_L$ will be smaller if $f_t$ is included. Qualitatively, it implies that $B_{soft}$ should be smaller (i.e., a stronger upper bound) if $f_t$ is included. In general, all the triviality bounds will become stronger if $f_t$ is included. In other words, the results of Fig.3 should be regarded as a weak absolute upper bound on the soft SUSY-breaking parameters and the heaviest-Higgs mass.

### 5 Absolute Upper Bound on the Lightest-Higgs Mass

With the inclusion of the Higgs singlet in NMSSM, the tree-level upper bound on the lightest-Higgs mass of MSSM is no longer valid. Therefore, it is of considerable importance to study the triviality upper bound on the lightest-
Higgs mass in order to devise effective search strategies for the detection of Higgs particles. For given \( \phi \), we can define a new quantity \( B_{LH} \), the absolute triviality upper bound \( B_{LH} \) on the lightest-Higgs mass, as the largest triviality upper bound on the lightest-Higgs mass with respect to all the possible values of \( m_1 (= m_2) \) and \( \tilde{v}_3 \). However, (39) implies that a search in the small-\( m_1^2 \) regime is enough, and the dependence of \( B_{LH} \) on \( \phi \) is displayed in Fig.5, where line A and line B correspond to \( (h_A, \lambda_A) \) and \( (h_B, \lambda_B) \) respectively. It is verified that Fig.5 satisfies the assumptions of negligible \( f_t \) and \( g_i \).

When \( \phi = 0 \) (i.e., no CP-violation in the scalar sector), the absolute upper bound \( B_{LH} = 2.8 \, M_W \) for line B. When \( \phi = \frac{\pi}{2} \), \( B_{LH} = 1.75 \, M_W \) for line B. Therefore, the absolute triviality upper bound on the lightest-Higgs mass does lie outside the range of LEP.

6 Conclusion

With a complete study of the triviality surface, we are able to derive the triviality bounds on the heaviest-Higgs mass, the soft SUSY-breaking parameters, the VEV of the Higgs singlet in Section 4, and the absolute upper bound on the lightest-Higgs mass in Section 5. Essentially, all the triviality bounds are derived based on the observations (38) and (39), where the triviality upper bound on the lightest-Higgs mass decreases to zero as \( m_1 = m_2 \to \infty \).

The particular choice of the soft SUSY-breaking potential \( V_{soft} \) in (2) can be viewed as an unsatisfactory feature of the present formulation. However, the triviality bounds derived in Sections 4 and 5 persist even if a more general \( V_{soft} \) is considered. We begin the argument of the above statement with the most general \( V_{soft} \) [11, 21]:

\[
V_{soft} = m_1^2 \Phi_1 \Phi_1 + m_2^2 \Phi_2 \Phi_2 - m_{12}^2 \Phi_1 \Phi_2 - m_{12}^* \Phi_1 \Phi_2
+ m_4^2 N^* N + m_5^2 N^2 + m_{5}^* N^* N
+ h m_3 (A_1 \Phi_1 \Phi_2 N + A_1^* \Phi_2 \Phi_1 N^*)
+ \frac{1}{3} \lambda m_3 (A_2 N^3 + A_2^* N^3)
\] (40)
Now, the relevant parameter space consists of:

\[(h, \lambda, v_1, v_2, v_3, m^2_1, m^2_2, m_3, m^2_4, m^2_5, m^2_{12}, A_1, A_2)\]  \hspace{1cm} (41)

plus three complex vacuum constraints on the parameters derived from the minimization of \(V' = V + V_{\text{soft}}\) and (19). Choosing \(\tan \beta = 1\), we have \(m^2_1 = m^2_2\) from the minimization of \(V'\) again. In the limit of large \(\Lambda_L (= B_{HH})\), the Landau pole (13) always implies:

\[h^2 \to 0 \text{ if } \Lambda_L (= B_{HH}) \to \infty\]  \hspace{1cm} (42)

In the large-\(m_{HH}\) limit (e.g., in the large-\(m_1\) limit) on the triviality surface, (42) implies that \([M^2_n]\) and \([M^2_c]\) can be solved up to order \(O(h)\) exactly:

Eigenvalues of \([M^2_n]\) = \[2m^2_1 + O(h), 2m^2_1 + O(h), m^{(+)} + O(h), m^{(-)} + O(h), O(h), 0]\]

Eigenvalues of \([M^2_c]\) = \[2m^2_1 + O(h), 2m^2_1 + O(h), 0, 0]\]

\[m^{(\pm)} = m^2_3 + 4\lambda^2 |v_3|^2 \pm 2m^2_5 + \lambda m_3 v_3 A_2 + \lambda^2 v^2_3\]  \hspace{1cm} (43)

With the most general \(V_{\text{soft}}\) (40), there is, in general, exactly one Higgs boson staying light in the large-\(m_{HH}\) limit. According to (42) and (43), the triviality upper bound on \(m_{LH}\) is of order \(O(h^{1/2})\), and decreases to zero as \(m_{HH} \to \infty\). This observation implies that the analyses of Sections 4 and 5 still apply to the most general \(V' = V + V_{\text{soft}}\). That is, the triviality bounds on the heaviest-Higgs mass, the lightest-Higgs mass, the VEV of the Higgs singlet, and the soft SUSY-breaking parameters will not be lost even if the largest parameter space of NMSSM is considered.

Finally, two aspects of the present computations can be improved. First, the computation of the effective potential in this paper is performed only at tree level. Because the contributions of the top and stop loops are important, it is necessary for future works to include one-loop contributions. Second, \(\tan \beta = 1\) is chosen in this paper for the sake of simplicity. However, this particular choice has no physical motivation. Therefore, choices different from \(\tan \beta = 1\) should be considered and the discussion of the \(\tan \beta\)-dependence may be a point of interest in future works.
Acknowledgement

I would like to thank Professor Mary K. Gaillard for her support and nice advice. I also thank Dr. H.-C. Cheng for discussions about MSSM. This work was supported in part by the Director, Office of Energy Research, Office of High Energy and Nuclear Physics, Division of High Energy Physics of the U.S. Department of Energy under Contract DE-AC03-76SF00098 and in part by the National Science Foundation under grant PHY90-21139.
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Fig.1: A plot of the triviality upper bound $B_{HH} (M_W)$ on the heaviest-Higgs mass versus $m_1^2 (M_W^2)$ for $\phi = 0$, $\tilde{v}_3 = 0.7 M_W$, and $(h, \lambda) = (h_A, \lambda_A)$, where the unit $M_W = 80$ GeV.

Fig.2: A plot of the allowed range (the enclosed region) of $\tilde{v}_3 (M_W)$ versus $m_1 (M_W)$ for $\phi = 0.6$, $(h, \lambda) = (h_A, \lambda_A)$, where the unit $M_W = 80$ GeV. The allowed range of $\tilde{v}_3$ shrinks to a point at $m_1 = 138$ GeV $\equiv B_{soft}$.

Fig.3: The plot of the absolute triviality upper bound $B_{soft} (M_W)$ versus $\phi$ (the unit $M_W = 80$ GeV), where the two solutions $(h_A, \lambda_A)$ and $(h_B, \lambda_B)$ have identical results. $B_{soft}$ is periodic in $\phi$ with period $\pi$.

Fig.4: The plot of $B_N (M_W)$, the absolute triviality lower bound on $\tilde{v}_3$, versus $\phi$ (the unit $M_W = 80$ GeV), where the dashed line corresponds to $(h_A, \lambda_A)$ and the solid line corresponds to $(h_B, \lambda_B)$. $B_N$ is periodic in $\phi$ with period $\pi$. The dotted line corresponds to the absolute upper bound of $\frac{M_W}{2g_2}$ on $\tilde{v}_3$.

Fig.5: The plot of $B_{LH} (M_W)$, the absolute triviality upper bound on the lightest-Higgs mass versus $\phi$ (the unit $M_W = 80$ GeV), where the dashed line corresponds to $(h_A, \lambda_A)$ and the solid line corresponds to $(h_B, \lambda_B)$. $B_{LH}$ is periodic in $\phi$ with period $\pi$. 