A GLOBAL IN TIME PARABOLIC EQUATION FOR SYMMETRIC LÉVY OPERATORS

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Abstract. The overreaching goal of this paper is to investigate the existence and uniqueness of weak solutions to a semilinear parabolic equation involving symmetric integrodifferential operators of Lévy type and a term called the interaction potential, that depends on the time-integral of the solution over the entire interval of solving the problem. The existence and uniqueness of a weak solution of the nonlocal complement value problem is proven under fair conditions on the interaction potential.

1. Introduction

Let Ω be an open bounded domain in \( \mathbb{R}^N \) \((N \geq 1)\). For \( T > 0 \), we consider the following parabolic nonlocal problem with Dirichlet complement value:

\[
\begin{aligned}
\partial_t u + \mathcal{L}u + \varphi\left( \int_0^T u(\cdot, \tau) \, d\tau \right) u &= 0 \quad \text{in} \quad \Omega_T := \Omega \times (0, T), \\
u &= 0 \quad \text{in} \quad \Sigma := (\mathbb{R}^N \setminus \Omega) \times (0, T), \\
u(\cdot, 0) &= u_0 \quad \text{in} \quad \Omega,
\end{aligned}
\] (1.1)

where \( u = u(x, t) \) is an unknown scalar function, and \( \varphi \) a scalar function that will be specified below. The initial state \( u_0 : \Omega \to \mathbb{R} \) is prescribed. Here, we restrict ourselves on a purely integrodifferential operator of Lévy type \( \mathcal{L} \), which is a particular type of nonlocal operators acting on a smooth measurable function \( u : \mathbb{R}^N \to \mathbb{R} \) as follows

\[
\mathcal{L}u(x) := \text{p.v.} \int_{\mathbb{R}^N} (u(x) - u(y)) \nu(x - y) \, dy, \quad (x \in \mathbb{R}^N),
\] (1.2)

whenever the right hand side exists and makes sense. Here and henceforward, the function \( \nu : \mathbb{R}^N \setminus \{0\} \to [0, \infty] \) is the density of a symmetric Lévy measure. In other words, \( \nu \) is positive and measurable such that

\[
\nu(-h) = \nu(h) \quad \text{for all} \quad h \in \mathbb{R}^N \quad \text{and} \quad \int_{\mathbb{R}^N} (1 \wedge |h|^2) \nu(h) \, dh < \infty.
\] (1.3)

Notationally, for \( a, b \in \mathbb{R} \), we write \( a \wedge b \) to denote \( \min(a, b) \). In addition, we will also assume that \( \nu \) does not vanish on sets of positive measure. In view of the nonlocal character of the operator \( \mathcal{L} \), it appears natural tout prescribe the Dirichlet condition \( u = 0 \) the complement set \((\mathbb{R}^N \setminus \Omega) \times (0, T)\).

A prototypical example of an operator \( \mathcal{L} \) is the fractional Laplacian \( (-\Delta)^s \), which is obtained by letting \( \nu(h) = C_{N,s} |h|^{-2s-N} \) with \( h \neq 0 \) and fixed \( s \in (0, 1) \). The constant \( C_{N,s} \) (c.f [19] Chapter 2) for an elementary computation) given by

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\[ C_{N,s} := \left( \int_{\mathbb{R}^N} \frac{1 - \cos(h_1)}{|h|^N+2s} \, dh \right)^{-1} = \frac{2^{2s} \Gamma\left(s + \frac{N}{2}\right)}{\pi^\frac{N}{2} \Gamma(-s)}, \]

is chosen so that the Fourier transform gives the relation \((\hat{-\Delta})^s u(\xi) = |\xi|^{2s} \hat{u}(\xi), \xi \in \mathbb{R}^N\), holds for all \(u \in C_\infty^\infty(\mathbb{R}^N)\). The fractional Laplacian is one of the most heavily studied integrodifferential operators. Some basics notions related to the fractional Laplacian can be found, for example, in the references [4, 7, 17, 26] and many other references therein, for further related subjects, see also [1, 5, 6, 9, 34]. The operator \(L\) in (1.2) arises naturally in the study of pure Lévy stochastic processes with the jump interaction measure \(\nu(h)dh\). Analogous to the classical case, attempts have been made recently (see, e.g., [19]), where elliptic and parabolic problems related to this type of nonlocal operators are studied, see also [12, 13, 14, 15] where various other typical studies related to this of nonlocal operator are considered.

Our main result (see Theorem 4.2 and Theorem 4.4) consist into proving the existence and uniqueness of a weak solution to the problem (1.1). It is noteworthy emphasizing that, we are able to obtain the uniqueness for \(T\) sufficiently small by imposing some a boundedness condition on the initial value \(u_0\). As a side remark, we point out that under appropriate setting (see Remark 4.5), analogue achievements can be obtain replacing the homogeneous Dirichlet complement condition in problem (1.1) with the homogeneous Neumann complement condition. We do this provided that the potential \(\varphi\) satisfies the following crucial assumption.

**Assumption 1.1.** The potential \(\varphi : \mathbb{R} \to [0, \infty)\) is a continuous non-negative function such that \(\varphi(0) = 0\) and \(\tau \mapsto \varphi(\tau)\tau\) is a non-decreasing differentiable function whose derivative is bounded on every compact subset of \(\mathbb{R}\).

This assumption admits functions \(\varphi\) that are not convex and not increasing as its argument tends to \(+\infty\). Besides that, we do not impose any restrictions on the growth of \(\varphi\) at infinity.

An interesting feature of the problem under consideration is that main equation in (1.1) contains a non-local in time term that depends on the integral over the whole interval \((0, T)\) on which the problem is being solved and a nonlocal operator of Lévy type in space. The mixture of nonlocal terms (spacial and time variables) appearing in (1.1) render the problem somehow fully nonlocal. Thus, the reason (1.1) is called nonlocal and global in time. It is noteworthy emphasizing that, similar analysis has been carry out in [30] where the classical Laplace operator \(-\Delta\) is used in place of the nonlocal operator \(L\). There are several works that study problems with memory for parabolic equations which includes the integral of the solution from the initial to the current time and it is not difficult to find appropriate works on this subject. The problems with memory differ from ours. As pointed out in [30], we need to know the “future” in order to determine the coefficient in equation (1.1) and we need to deal with the two nonlocality terms in both spacial and time variables. It is worth emphasizing that the problem in (1.1) cannot be reduced to known ones by any transformation. However, in the paper [29] in the classical scenario that study problems where the “future” stands in the boundary and initial data. We point that the classical problem appears while modelling a biological nano-sensor in the chaotic dynamics of a polymer chain or, as it is also called, a polymer chain in an aqueous solution see for instance [30, 31, 32] and references therein. In [31], the weak solvability of the problem is proven for the case where \(u\) is a positive bounded function and \(\varphi\) is the so called Flory–Huggins potential. The positiveness is a natural requirement since \(u\) is a density of probability. The Flory–Huggins potential is a convex increasing function that tends to infinity as its argument approaches a certain positive value.
The original work of [30, 31] demonstrate that, in the case where only the Laplace operator is involved, the problem (1.1) makes sense. In parallel with the article [30], we take this work to the next stage by using the generator of a pure jump stochastic process of Lévy type, which is a symmetric nonlocal type operator of the form $L$, to prove further results on weak solvability for this type of problem.

The rest of the paper is structured as follows. In Section 2, we provide some preliminaries well-known results and functions spaces which are useful in this paper. In Section 3, we prove auxiliary results which are the milestones to prove our core result. Finally, Section 4 is devoted to the proof of the existence and uniqueness of a weak solution to the problem (1.1) thereby constituting the main goal of this article. We prove the existence with the aid of the Tychonoff fixed-point theorem and prove the uniqueness for sufficiently small $T$.

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2. Preliminaries Notions

The purpose of this section is to introduce notations and some preliminary results. Let us collect some basics on nonlocal Sobolev-like spaces in the $L^2$ setting that are generalizations of Sobolev–Slobodeckij spaces and which will very helpful in the sequel. Let us emphasize that, those function spaces are tailor made elliptic complement value problem involving symmetric Lévy operators of type $L$. We refer the reader to [19] more extensive discussions on this topic.

From now on, unless otherwise stated, $\Omega \subset \mathbb{R}^N$ is an open bounded set. We also assume that $\nu : \mathbb{R}^N \setminus \{0\} \to [0, \infty]$ has full support, satisfies the Lévy integrability condition, i.e., $\nu \in L^1(\mathbb{R}^N, 1 \wedge |h|^2 dh)$ and is symmetric, i.e., $\nu(h) = \nu(-h)$ for all $h \in \mathbb{R}^N$. We define the space

$$V_\nu(\Omega|\mathbb{R}^N) := \left\{ u : \mathbb{R}^N \to \mathbb{R} \text{ meas : } \mathcal{E}(u, u) < \infty \right\},$$

where $\mathcal{E}(\cdot, \cdot)$ is the bilinear form defined by

$$\mathcal{E}(u, v) := \frac{1}{2} \iint_{\Omega(\Omega)} (u(x) - u(y))(v(x) - v(y))\nu(x - y) \, dy \, dx$$

and $\Omega(\Omega)$ is the cross-shaped set on $\Omega$ given by

$$\Omega(\Omega) := (\Omega \times \Omega) \cup (\Omega \times (\mathbb{R}^N \setminus \Omega)) \cup ((\mathbb{R}^N \setminus \Omega) \times \Omega).$$

We endow the space $V_\nu(\Omega|\mathbb{R}^N)$ with the norm

$$\|u\|_{V_\nu(\Omega|\mathbb{R}^N)} := \left( \int_{\Omega} |u(x)|^2 \, dx + \mathcal{E}(u, u) \right)^{1/2}.$$
where \( V_\nu(\Omega|\mathbb{R}^N) \) is defined as in (2.1). The space \( X_\nu(\Omega|\mathbb{R}^N) \) is clearly a closed subspace of \( V_\nu(\Omega|\mathbb{R}^N) \). Furthermore, we have that

\[
\|u\|_{X_\nu(\Omega|\mathbb{R}^N)} = \left( \iint_{\mathbb{R}^N \times \mathbb{R}^N} (u(x) - u(y))^2 \nu(x - y) \, dy \, dx \right)^{\frac{1}{2}}
\]

defines an equivalent norm on \( X_\nu(\Omega|\mathbb{R}^N) \). Indeed, in virtue of the Poincaré-Friedrichs inequality on \( X_\nu(\Omega|\mathbb{R}^N) \), there exists a constant \( C = C(N, \Omega, \nu) > 0 \) depending only on \( N, \Omega \) and \( \nu \) such that

\[
\|u\|_{L^2(\Omega)} \leq C \|u\|_{X_\nu(\Omega|\mathbb{R}^N)} \quad \text{for every} \quad u \in X_\nu(\Omega|\mathbb{R}^N). \tag{2.3}
\]

This can be verified by observing that \( \mathbb{R}^N \setminus B_R(x) \subset \mathbb{R}^N \setminus \Omega \) for all \( x \in \Omega \), where \( R = \text{diam}(\Omega) > 0 \) is the diameter of \( \Omega \). For \( u \in X_\nu(\Omega|\mathbb{R}^N) \), we recall that \( u = 0 \) a.e. on \( \mathbb{R}^N \setminus \Omega \). Hence we have

\[
\|u\|^2_{X_\nu(\Omega|\mathbb{R}^N)} \geq 2 \int_{\Omega} |u(x)|^2 \, dx \int_{\mathbb{R}^N \setminus \Omega} \nu(x - y) \, dy \geq 2 \int_{\Omega} |u(x)|^2 \, dx \int_{\mathbb{R}^N \setminus B_R(x)} \nu(x - y) \, dy = 2\|\nu_R\|_{L^1(\mathbb{R}^N)} \|u\|^2_{L^2(\Omega)}.
\]

It suffices to take \( C = (2\|\nu_R\|_{L^1(\mathbb{R}^N)})^{-1/2} \) with \( \nu_R = \nu 1_{\mathbb{R}^N \setminus B_R(0)} \). According to [22], as in the classical case, the Poincaré-Friedrichs inequality (2.3) remains true if \( \Omega \) is only bounded in one direction.

Now, we define \( T_\nu(\mathbb{R}^N \setminus \Omega) \) the trace space of \( V_\nu(\Omega|\mathbb{R}^N) \), i.e., the space of restrictions to \( \mathbb{R}^N \setminus \Omega \) of functions of \( V_\nu(\Omega|\mathbb{R}^N) \). To be more precise, we have

\[
T_\nu(\mathbb{R}^N \setminus \Omega) = \left\{ v : \mathbb{R}^N \setminus \Omega \to \mathbb{R} \text{ meas. such that} \quad v = u|_{\mathbb{R}^N \setminus \Omega} \text{ with } u \in V_\nu(\Omega|\mathbb{R}^N) \right\}.
\]

We equip \( T_\nu(\mathbb{R}^N \setminus \Omega) \) with its natural norm

\[
\|v\|_{T_\nu(\Omega|\mathbb{R}^N)} = \inf \left\{ \|u\|_{V_\nu(\Omega|\mathbb{R}^N)} : u \in V_\nu(\Omega|\mathbb{R}^N) \text{ with } u|_{\mathbb{R}^N \setminus \Omega} = v \right\}.
\]

Next, we consider the weighted \( L^2 \)-spaces on \( \mathbb{R}^N \setminus \Omega \) denoted by \( L^2(\mathbb{R}^N \setminus \Omega, \nu_K) \) where for a given measurable set \( K \subset \Omega \) with \( 0 < |K| < \infty \), we define

\[
\nu_K(x) := \inf_{y \in K} \nu(x - y) \quad \text{and} \quad \nu_K(x) = \int_K 1 \wedge \nu(x - y) \, dy.
\]

The aforementioned spaces are Hilbert spaces and are somewhat linked. Let \((V_\nu(\Omega|\mathbb{R}^N))^*\) and \((X_\nu(\Omega|\mathbb{R}^N))^*\) be the dual spaces of \( V_\nu(\Omega|\mathbb{R}^N) \) and \( X_\nu(\Omega|\mathbb{R}^N) \) respectively. Therefore, we have the following Gelfand evolution triple embeddings

\[
X_\nu(\Omega|\mathbb{R}^N) \hookrightarrow L^2(\Omega) \hookrightarrow (X_\nu(\Omega|\mathbb{R}^N))^* \quad \text{and} \quad V_\nu(\Omega|\mathbb{R}^N) \hookrightarrow L^2(\Omega) \hookrightarrow (V_\nu(\Omega|\mathbb{R}^N))^*.
\]

In addition we have the continuous embeddings

\[
X_\nu(\Omega|\mathbb{R}^N) \hookrightarrow V_\nu(\Omega|\mathbb{R}^N) \hookrightarrow T_\nu(\Omega|\mathbb{R}^N) \hookrightarrow L^2(\mathbb{R}^N \setminus \Omega, \nu_K). \tag{2.4}
\]

It is worth of noticing that these interactions in (2.4) between the spaces \( V_\nu(\Omega|\mathbb{R}^N) \), \( X_\nu(\Omega|\mathbb{R}^N) \), \( T_\nu(\Omega|\mathbb{R}^N) \), and \( L^2(\mathbb{R}^N \setminus \Omega, \nu_K) \) respectively are analogous to the ones between the classical Sobolev spaces \( H^1(\Omega) \), \( H^1_0(\Omega) \), \( H^1(\partial \Omega) \), and \( L^2(\partial \Omega) \). Alternatively, the embeddings in (2.4) remain true if the weight \( \nu_K \) is replaced with \( \nu_K \). The next result borrowed from [19] [22] [11], provides sufficient conditions under which the spaces \( X_\nu(\Omega|\mathbb{R}^N) \) and \( V_\nu(\Omega|\mathbb{R}^N) \) are compactly embedded in \( L^2(\Omega) \).
**Theorem 2.1.** Assume \( \nu \in L^1(\mathbb{R}^N, 1 \wedge |h|^2) \) and \( \Omega \subset \mathbb{R}^N \) is open bounded. If \( \nu \notin L^1(\mathbb{R}^N) \) then the embedding \( \mathcal{X}_\nu(\Omega; \mathbb{R}^N) \hookrightarrow L^2(\Omega) \) is compact. Furthermore, the embedding \( V_\nu(\Omega; \mathbb{R}^N) \hookrightarrow L^2(\Omega) \) is also compact if \( \Omega \) has a Lipschitz boundary, \( \nu \notin L^1(\mathbb{R}^N) \) and

\[
\lim_{\delta \to 0} \frac{1}{\delta^N} \int_{B_\delta(0)} |h|^2 \nu(h) \, dh = \infty.
\]

(2.5)

It is worthwhile noticing that we have the natural continuous and dense embeddings \( L^2(0, T; \mathcal{X}_\nu(\Omega; \mathbb{R}^N)) \hookrightarrow L^2(0, T; L^2(\Omega)) \hookrightarrow L^2(0, T; (\mathcal{X}_\nu(\Omega; \mathbb{R}^N))^*) \).

We recall from [25], since \( \mathcal{X}_\nu(\Omega; \mathbb{R}^N) \) is a real Hilbert space, if we set

\[
H_\nu(0, T) = \left\{ \zeta \in L^2(0, T; \mathcal{X}_\nu(\Omega; \mathbb{R}^N)) : \partial_t \zeta \in L^2(0, T; ((\mathcal{X}_\nu(\Omega; \mathbb{R}^N))^*) \right\},
\]

then, \( H_\nu(0, T) \) is a Hilbert space endowed with the norm given by

\[
||\zeta||^{2}_{H_\nu(0,T)} = ||\zeta||^{2}_{L^{2}(0,T;\mathcal{X}_{\nu}(\Omega;\mathbb{R}^{N}))} + ||\partial_{t}\zeta||^{2}_{L^{2}(0,T;((\mathcal{X}_{\nu}(\Omega;\mathbb{R}^{N}))^{*)}).
\]

**Proposition 2.2.** With the assumptions of Theorem 2.1 in force, the following assertions are true.

(i) **Lions-Magenes Lemma** [2, Theorem II.5.12]: the following embedding is continuous

\[
H_\nu(0, T) \hookrightarrow C([0, T]; L^2(\Omega)).
\]

(ii) **Lions-Aubin Lemma** [2, Theorem II.5.16]: the following embedding is compact

\[
H_\nu(0, T) \hookrightarrow L^2(0, T; L^2(\Omega)).
\]

(2.6)

(2.7)

Now we state the integration by parts formula contained in [19] for smooth functions. Precisely for every \( \phi, \psi \in C_0^\infty(\mathbb{R}^N) \) following nonlocal Gauss-Green formula holds true

\[
\mathcal{E}(\phi, \psi) = \int_{\Omega} \psi \mathcal{L}(\phi) \, dx + \int_{\mathbb{R}^N \setminus \Omega} \psi(y) N\phi(y) \, dy
\]

(2.8)

where, the bilinear form \( \mathcal{E}(\cdot, \cdot) \) is defined in [22] and \( N\phi \) denotes the nonlocal normal derivative of \( \phi \) across the boundary of \( \Omega \) with respect to \( \nu \) and is defined by

\[
N\phi(x) := \int_{\Omega} (\phi(x) - \phi(y)) \nu(x - y) \, dy, \quad x \in \mathbb{R}^N \setminus \Omega.
\]

(2.9)

With the aforementioned function spaces at hand, and motivated by the Gauss-Green formula (2.8), we are now in position to define the notion solution of a weak to the problem (1.1).

**Definition 2.3.** A function \( u : \mathbb{R}^N \times (0, T) \to \mathbb{R} \) will be a weak solution of problem (1.1) if

(i) \( u \in L^2(0, T; \mathcal{X}_\nu(\Omega; \mathbb{R}^N)) \) and \( \partial_t u \in L^2(0, T; (\mathcal{X}_\nu(\Omega; \mathbb{R}^N))^*) \);

(ii) For every \( \psi \in L^2(0, T; \mathcal{X}_\nu(\Omega; \mathbb{R}^N)) \), \( u \) satisfies \( u(\cdot, 0) = u_0 \) and

\[
\int_{\Omega} \partial_t u \psi \, dx \, dt + \mathcal{E}(u, \psi) \, dt + \int_{\Omega} \varphi(v) u \psi \, dx \, dt = 0 \quad \text{for all } 0 \leq t \leq T.
\]

(2.10)

In particular, we have

\[
\int_{0}^{T} \int_{\Omega} \partial_t u \psi \, dx \, dt + \int_{0}^{T} \mathcal{E}(u, \psi) \, dt + \int_{0}^{T} \int_{\Omega} \varphi(v) u \psi \, dx \, dt = 0.
\]

Our proof of the existence of a weak solution to the problem (1.1) relies upon the following Tychonoff fixed-point Theorem [2, 3] which is a generalization of the Brouwer and Schauder fixed-point theorems.
Theorem 2.4 (Tychonoff [33]). Let $X$ be a reflexive separable Banach space and $G \subset X$ be a closed convex bounded set. If a mapping $\pi : G \to G$ is weakly sequentially continuous, then $\pi$ has at least one fixed point in $G$.

It is noteworthy emphasizing that for the particular case where $G$ is compact and convex, Theorem 2.4 is known as the Schauder fixed-point theorem while the finite dimension case dim $X < \infty$ is known as the Brouwer fixed-point theorem.

3. Nonlocal elliptic and parabolic problem

The overreaching goal of this section is to investigate weak solutions to two specific nonlocal problems which is of interest in the proof of our main result. The first problem is an elliptic nonlocal problem and the second one is a parabolic nonlocal problem.

3.1. Nonlocal elliptic problem. Given a measurable function $f : \Omega \to \mathbb{R}$, we consider the elliptic problem consisting into finding a function $v : \mathbb{R}^N \to \mathbb{R}$ satisfying of the following problem:

$$
\begin{cases}
\mathcal{L}v + \varphi(v)v = f & \text{in } \Omega, \\
v = 0 & \text{on } \mathbb{R}^N \setminus \Omega.
\end{cases}
$$

(3.1)

Heuristically, the problem (3.1) results from the evolution problem (3.1) by integrating with respect to $t$ from 0 to $T$. In a sense, the functions $v$ and $f$ correspond to $\int_0^T u(\cdot, t) \, dt$ and $u_0 - u(\cdot, T)$, respectively. Problems of type (3.1) are considered in the classical scenario in [3, 18, 20, 30] with the operator $\mathcal{L}$ replaced with $-\Delta$. There, the difficulties with the integrability of the term $\varphi(v)v$ were handled. In our case, we consider the function $f \in L^2(\Omega)$, so that we expect more from the solution of the problem such as $\varphi(v) \in L^2(\Omega)$. We follow the strategy of the proof presented in [30]. We use the following notation

$$
\chi(\tau) = \varphi(\tau)\tau.
$$

A function $v \in \mathbb{X}_\nu(\Omega; \mathbb{R}^N)$ is said to be a weak solution of problem (3.1) if $\chi(v) \in L^2(\Omega)$ and

$$
\mathcal{E}(v, \psi) + (\chi(v), \psi) = (f, \psi) \quad \text{for all } \psi \in \mathbb{X}_\nu(\Omega; \mathbb{R}^N).
$$

(3.2)

Next, we want to show that the above variational problem (3.2) is well-posed in the sense of Hadamard. In other words, it possesses a unique solution which continuously depends upon the initial data. Let us start with the following stability lemma.

Lemma 3.1. Let $f_i \in L^2(\Omega), i = 1, 2$. Assume that $v_i \in \mathbb{X}_\nu(\Omega; \mathbb{R}^N)$ satisfies

$$
\mathcal{E}(v_i, \psi) + (\chi(v_i), \psi) = (f_i, \psi) \quad \text{for all } \psi \in \mathbb{X}_\nu(\Omega; \mathbb{R}^N).
$$

Then for some constant $C = C(N, \Omega, \nu) > 0$ only depending only $N, \Omega$ and $\nu$ such that

$$
\|v_1 - v_2\|_{\mathbb{X}_\nu(\Omega; \mathbb{R}^N)} \leq C\|f_1 - f_2\|_{(\mathbb{X}_\nu(\Omega; \mathbb{R}^N))^\ast}.
$$

Proof. Combining both equation and testing with $\psi = v_1 - v_2$ yields

$$
\mathcal{E}(v_1 - v_2, v_1 - v_2) + (\chi(v_1), v_1 - v_2)_{L^2(\Omega)} = (f_1 - f_2, v_1 - v_2)_{L^2(\Omega)}.
$$

Observing that $\tau \mapsto \chi(\tau) = \varphi(\tau)\tau$ is non-decreasing, is equivalent to saying that

$$
(\chi(\tau_1) - \chi(\tau_2))(\tau_1 - \tau_2) \geq 0 \quad \text{for all } \tau_1, \tau_2 \in \mathbb{R},
$$

(3.3)

the above relation implies

$$
\|v_1 - v_2\|_{\mathbb{X}_\nu(\Omega; \mathbb{R}^N)} \leq \|f_1 - f_2\|_{(\mathbb{X}_\nu(\Omega; \mathbb{R}^N))^\ast} \|v_1 - v_2\|_{\mathbb{X}_\nu(\Omega; \mathbb{R}^N)}.
$$

The desired estimate follows from the Poincaré-Friedrichs inequality (2.3). \[\square\]
Theorem 3.2. Let Assumption 1.1 be in force and let \( f \in L^2(\Omega) \). Then the problem (3.1) has a unique weak solution \( v \in X_\nu(\Omega; \mathbb{R}^N) \). Moreover, the following estimates hold true:

(i) \( \mathcal{E}(v, v) \leq C \|f\|^2_{L^2(\Omega)} \);

(ii) \( \|\varphi(v)\|_{L^2(\Omega)} \leq C \|f\|_{L^2(\Omega)} \), with \( C > 0 \) only depending on \( N, \Omega, \text{ and } \nu \);

(iii) \( \|\varphi(v)\|^2_{L^2(\Omega)} \leq \frac{1}{\delta^2} \|f\|^2_{L^2(\Omega)} + |\Omega| \), with \( \delta > 0 \) only depending on \( \varphi \).

Proof. Note that the uniqueness immediately follows from Lemma 3.1. We prove the remaining results of Theorem 3.2 in several steps by adapting the strategy of the proof of [30] Lemma 3.1.

Step 1: We are interested in establishing the well-posedness of problem (3.1) using the Galerkin method which consists into projecting the latter on suitable finite dimensional space. First of all, we mention that bounded functions are dense in \( V_\nu(\Omega; \mathbb{R}^N) \) and hence in \( X_\nu(\Omega; \mathbb{R}^N) \). Thus, there is an orthonormal basis \( \{\phi_\ell\} \) of \( X_\nu(\Omega; \mathbb{R}^N) \) whose elements are bounded, i.e., \( \phi_\ell \in L^\infty(\Omega) \).

We emphasize that the inner product in \( X_\nu(\Omega; \mathbb{R}^N) \) is defined as \( \langle \psi_1, \psi_2 \rangle_{X_\nu(\Omega; \mathbb{R}^N)} = \mathcal{E}(\psi_1, \psi_2) \) for \( \psi_1, \psi_2 \in X_\nu(\Omega; \mathbb{R}^N) \). Let \( V_k \) be the subspace of \( X_\nu(\Omega; \mathbb{R}^N) \) spanned by the basis functions \( \{\phi_1, \ldots, \phi_k\} \).

For each \( k \in \mathbb{N} \), we claim the existence of a function \( v_k \in V_k \) such that

\[
\mathcal{E}(v_k, \psi) + (\varphi(v_k), \psi) = (f, \psi) \quad \text{for all } \psi \in V_k.
\]

We prove this in two different ways. First, note that (5.4) is equivalent to the minimization problem

\[
\mathcal{J}(v_k) = \min_{w \in V_k} \mathcal{J}(w) \quad \text{with} \quad \mathcal{J}(w) := \frac{1}{2} \mathcal{E}(w, w) + \int_\Omega G(w) \, dx + \int_\Omega f w \, dx
\]

where we define the function \( G(w) = \int_0^w \chi(\tau) \, d\tau = \int_0^w \varphi(\tau) \, d\tau \). Note that \( G \) is non-negative since \( \varphi(\tau) \geq 0 \) and that the mapping \( w \mapsto \mathcal{J}(w) \) is continuous on \( V_k \). Furthermore, with the aid of the Poincaré-Friedrichs inequality [23] we find that \( \mathcal{J}(w) \to \infty \), as \( \|w\|_{X_\nu(\Omega; \mathbb{R}^N)} \to \infty \) and \( w \in V_k \).

Alternatively, as highlighted in [30], we obtain the existence of \( v_k \) using the Brouwer fixed-point theorem as follows. Let \( w \in V_k \), necessarily \( \varphi(w) \) is a bounded function since \( \phi_\ell \)'s are also bounded. The Lax-Milgram lemma implies there is a unique function \( \tilde{w} \in V_k \) such that

\[
\mathcal{E}(\tilde{w}, \psi) + (\varphi(w) \tilde{w}, \psi) = (f, \psi) \quad \text{for all } \psi \in V_k.
\]

In particular, the Poincaré–Friedrichs inequality [23] yields

\[
\mathcal{E}(\tilde{w}, \tilde{w}) + \int_\Omega \varphi(w) \tilde{w}^2 \, dx \leq \|f\|_{L^2(\Omega)} \|\tilde{w}\|_{L^2(\Omega)} \leq C \|f\|_{L^2(\Omega)} \|\tilde{w}\|_{X_\nu(\Omega; \mathbb{R}^N)}
\]

Thus, letting \( R = C \|f\|_{L^2(\Omega)} \), since \( \varphi \geq 0 \) we obtain the following estimates

\[
\|\tilde{w}\|_{X_\nu(\Omega; \mathbb{R}^N)} \leq R \quad \text{and} \quad \int_\Omega \varphi(w) \tilde{w}^2 \, dx \leq R^2.
\]

We let \( B_R = \{w \in V_k : \|w\|_{X_\nu(\Omega; \mathbb{R}^N)} \leq R\} \), be the closed ball in \( V_k \) of radius \( R \) centered at the origin. Clearly, [23] implies that the mapping \( T : V_k \to B_R \) with \( Tw = \tilde{w} \) is well defined. It remains to prove that \( T \) is a continuous mapping. Indeed, let \( \{w_n\} \) be a sequence in \( V_k \) with \( w_n = \lambda_1 \phi_1 + \cdots + \lambda_k \phi_k \) converging in \( V_k \) to a function \( w = \lambda_1 \phi_1 + \cdots + \lambda_k \phi_k \), i.e., \( \lambda_{\ell, n} \xrightarrow{n \to \infty} \lambda_\ell \), \( \ell = 1, 2, \ldots, k \). By continuity we have \( \varphi(w_n) \xrightarrow{n \to \infty} \varphi(w) \) almost everywhere. In addition, the convergence in \( L^2(\Omega) \) also holds, i.e., \( \|\varphi(w_n) - \varphi(w)\|_{L^2(\Omega)} \xrightarrow{n \to \infty} 0 \) since the continuity gives \( \sup_{n \geq 0} \|\varphi(w_n)\|_{L^\infty(\Omega)} < \infty \) because \( \sup_{n \geq 0} \|w_n\|_{L^\infty(\Omega)} < \infty \). On the other side, in virtue of the first
estimate in (3.5), the sequence \( \{Tw_n\} \) is bounded in finite dimensional space \( \mathcal{V}_k \) and thus converges in \( \mathcal{V}_k \) up to a subsequence to some \( w_* \in \mathcal{V}_k \). Altogether, it follows that, for all \( \psi \in \mathcal{V}_k \subset L^\infty(\Omega) \)

\[
(f, \psi) = \lim_{n \to \infty} \mathcal{E}(\hat{w}_n, \psi) + (\varphi(w_n)\hat{w}_n, \psi) = \mathcal{E}(w_*, \psi) + (\varphi(w)w_*, \psi).
\]

The uniqueness of \( \hat{w} \) entails \( w_*=\hat{w}=Tw \) and hence the whole sequence \( \{Tw_n\} \) converges in \( Tw \) in \( \mathcal{V}_k \), which gives the continuity of \( T \). Therefore, by the Brouwer fixed-point theorem, \( T \) has a fixed point \( v_\in \mathcal{V}_k \), i.e., \( v_k=Tv_k \) which clearly satisfies (3.4) as announced.

Furthermore, recalling \( R=C\|f\|_{L^2(\Omega)} \), from (3.5) we get the following estimates for all \( k \in \mathbb{N} \),

\[
\|v_k\|_{X_\nu(\Omega|\mathbb{R}^N)} \leq R \quad \text{and} \quad \int_\Omega \chi(v_k) v_k \, dx \leq R^2. \tag{3.6}
\]

Therefore, the sequence \( \{v_k\} \) is clearly bounded in \( X_\nu(\Omega|\mathbb{R}^N) \). The compactness Theorem 2.1 yields the existence of a subsequence, still denoted by \( \{v_k\} \), weakly converging in \( X_\nu(\Omega|\mathbb{R}^N) \) and strongly converging in \( L^2(\Omega) \) to a function \( v \). Wherefore, due to the continuity of the function \( \chi \), we get

\[
\chi(v_k) \to \chi(v) \quad \text{almost everywhere in } \Omega. \tag{3.7}
\]

**Step 2:** Next, we prove that the functions \( \{\chi(v_k)\} \) are uniformly integrable. In view of the estimate (3.6), for each measurable set \( \Gamma \subset \Omega \) and each \( \Lambda > 0 \), we let \( \Gamma^k_\Lambda = \{x \in \Gamma : |v_k(x)| \geq \Lambda\} \) so that

\[
\frac{1}{\Lambda} \int_\Omega \chi(v_k) v_k \, dx \leq R^2,
\]

where \( \chi \) is non-decreasing, putting \( \gamma(\Lambda) = \Lambda \max\{\varphi(-\Lambda), \varphi(\Lambda)\} \), we get

\[
|\chi(\tau)| \leq \gamma(\Lambda) \quad \text{for all} \quad \tau \in [-\Lambda, \Lambda].
\]

Therefore, the following relation holds

\[
\int_{\Gamma \setminus \Gamma^k_\Lambda} |\chi(v_k)| \, dx \leq \gamma(\Lambda) |\Gamma|,
\]

where \( |\Gamma| \) is the Lebesgue measure of the set \( \Gamma \). These inequalities imply that

\[
\int_\Omega |\chi(v_k)| \, dx \leq R^2 + \gamma(\Lambda) |\Gamma|.
\]

Patently, for an arbitrary \( \varepsilon > 0 \), we take \( \Lambda = 2R^2/\varepsilon \) and \( \delta = \varepsilon/(2\gamma(\Lambda)) \). Therefore, we find that

\[
\sup_{k \geq 1} \int_\Omega |\chi(v_k)| \, dx < \varepsilon
\]

for an arbitrary measurable set \( \Gamma \subset \Omega \) such that \( |\Gamma| < \delta \). This, is precisely the uniform integrability of \( \chi(v_k) \). This fact together with (3.7) and the Vitali convergence theorem (see, e.g., [19 Theorem A.19]) enable us to conclude that \( \chi(v) \in L^1(\Omega) \) and \( \chi(v_k) \rightharpoonup \chi(v) \) in \( L^1(\Omega) \) as \( k \to \infty \). Now passing to the limit in (3.4) as \( k \to \infty \) we find that \( v \) satisfies (3.2), which along with Lemma 3.1 means that \( v \) is a unique weak solution of problem (3.1).

**Step 3:** Next, we prove the estimates appearing in (i) – (iii). The first estimate follows from (3.6). In order to prove the second one, we introduce the truncated function for every \( \ell \in \mathbb{N} \) as follows:

\[
v_\ell(x) = \begin{cases} 
  \ell, & v(x) \geq \ell, \\
  v(x), & -\ell < v(x) < \ell, \\
  -\ell, & v(x) \leq -\ell.
\end{cases}
\]

Since \( \chi(v)\chi(v_\ell) \geq \chi^2(v_\ell) \) and \( \mathcal{E}(v_\ell, \chi(v_\ell)) \geq 0 \) by (3.3), then taking \( \psi = \chi(v_\ell) \) in (3.2) yields
\[ \|X(v_\ell)\|_{L^2(\Omega)}^2 \leq E(v_\ell, X(v_\ell)) + \|X(v_\ell)\|_{L^2(\Omega)}^2 \leq \|f\|_{L^2(\Omega)} \|X(v_\ell)\|_{L^2(\Omega)}. \]

Which, as \( \{X^2(v_\ell)\} \to X^2(v) \) a.e. in \( \Omega \), in virtue of Fatou lemma implies the second estimate
\[ \|X(v)\|_{L^2(\Omega)} \leq \liminf_{\ell \to \infty} \|X(v_\ell)\|_{L^2(\Omega)} \leq \|f\|_{L^2(\Omega)}. \]

Finally, by continuity of \( \varphi \), there exists \( \delta > 0 \) such that \( \varphi^2(\tau) \leq 1 \) for all \( \tau \in [-\delta, \delta] \). Hence, letting \( \Gamma_\delta = \{ x \in \Omega : |v(x)| \geq \delta \} \), the second estimate implies the third one as follows
\[
\int_\Omega \varphi^2(v) \, dx = \int_{\Gamma_\delta} \varphi^2(v) \, dx + \int_{\Omega \setminus \Gamma_\delta} \varphi^2(v) \, dx \leq \frac{1}{\delta^2} \int_\Omega \varphi^2(v) \, v^2 \, dx + \int_{\Omega \setminus \Gamma_\delta} \varphi^2(v) \, dx \leq \frac{1}{\delta^2} \|f\|_{L^2(\Omega)}^2 + |\Omega|.
\]

Next, define the mapping \( V : L^2(\Omega) \rightarrow X_\nu(\Omega; \mathbb{R}^N) \) such that, for \( f \in L^2(\Omega) \),
\[ v = V(f) \]

is the unique weak solution of problem (3.1). (3.8)

**Lemma 3.3.** Let \( \{f_k\} \) be a sequence in \( L^2(\Omega) \) that weakly converges to \( f \) in \( L^2(\Omega) \). If \( v_k = V(f_k) \) and \( v = V(f) \), then as \( k \to \infty \) we have

(i) \( v_k \to v \) strongly in \( X_\nu(\Omega; \mathbb{R}^N) \);

(ii) \( \varphi(v_k) \to \varphi(v) \) weakly in \( L^2(\Omega) \).

**Proof.** Let us identify \( f_k - f \) in \( (X_\nu(\Omega; \mathbb{R}^N))^* \) with the linear form
\[ w \mapsto \int_\Omega (f_k(x) - f(x)) w(x) \, dx. \]

Since, the space \( X_\nu(\Omega; \mathbb{R}^N) \) is reflexive, for each \( k \geq 1 \) there exists \( w_k \in X_\nu(\Omega; \mathbb{R}^N) \) such that (c.f. [21] Theorem 2 or [24] Chapter 6)), \( \|w_k\|_{X_\nu(\Omega; \mathbb{R}^N)} \leq 1 \) and
\[ \|f_k - f\|_{(X_\nu(\Omega; \mathbb{R}^N))^*} = \int_\Omega (f_k(x) - f(x)) w_k(x) \, dx. \]

According to the compactness Theorem [21] we may assume that \( \{w_k\} \) strongly converges to some \( w \) in \( L^2(\Omega) \). Therefore, the weakly convergence of \( \{f_k\} \) implies that
\[ \|f_k - f\|_{(X_\nu(\Omega; \mathbb{R}^N))^*} = \int_\Omega (f_k(x) - f(x)) w_k(x) \, dx \xrightarrow{k \to \infty} 0. \]

The convergence in \( X_\nu(\Omega; \mathbb{R}^N) \) follows immediately from Lemma 3.1 since
\[ \|v_k - v\|_{X_\nu(\Omega; \mathbb{R}^N)} \leq C \|f_k - f\|_{(X_\nu(\Omega; \mathbb{R}^N))^*} \xrightarrow{k \to \infty} 0. \]

On the other hand, we also have the strong convergence of \( \{v_k\} \) in \( L^2(\Omega) \) and the continuity of \( \varphi \) imply that \( \{\varphi(v_k)\} \) converges almost everywhere to \( \varphi(v) \) up a subsequence. Furthermore, since \( \{f_k\} \) is bounded, as in the proof of Lemma 3.1 one easily gets that
\[ \|\varphi(v_k)\|_{L^2(\Omega)} \leq C \quad \text{for all } k \geq 1, \]

for a constant \( C > 0 \) independent on \( k \). Thus, \( \{\varphi(v_k)\} \) has a further subsequence weakly converging in \( L^2(\Omega) \). The Banach-Saks theorem [27] Appendix A or [28] Proposition 10.8 infers the existence of a further subsequence whose Césaro mean converges strongly in \( L^2(\Omega) \) and almost everywhere in \( \Omega \) to the same limit. Necessarily, since \( \{\varphi(v_k)\} \) converges almost everywhere to \( \varphi(v) \), the entire sequence \( \{\varphi(v_k)\} \) weakly converges in \( L^2(\Omega) \) to \( \varphi(v) \).
3.2. Nonlocal parabolic problem. We consider the following parabolic problem:

\[
\begin{cases}
\partial_t u + \mathcal{L} u + \zeta u = 0 & \text{in } \Omega_T, \\
u = 0 & \text{in } \Sigma, \\
 u(\cdot, 0) = u_0 & \text{in } \Omega,
\end{cases}
\]

where \( u_0, \zeta \in L^2(\Omega) \) with \( \zeta \geq 0 \). We also assume \( \nu \not\in L^1(\mathbb{R}^N) \) so that by Theorem 2.1, the embedding \( X_\nu(\Omega; \mathbb{R}^N) \hookrightarrow L^2(\Omega) \) is compact. Therefore, by the standard Galerkin superposition method (see for instance [19 Section 4.6]), a weak solution \( u \) of the problem (3.9) can be easily obtained in \( L^2(0, T; X_\nu(\Omega; \mathbb{R}^N) \cap L^2(\Omega, \zeta)) \). Here \( L^2(\Omega, \zeta) \) is the Hilbert space with the norm

\[
\|u\|_{L^2(\Omega, \zeta)}^2 = \int_\Omega |u(x)|^2 \zeta(x) dx.
\]

We omit the proof as well as various justifications (see also [11 16]). Another possibility, is to observe that [23] there exists a unique semigroup with generator \( A \) on \( L^2(\Omega) \) associated to the closed bilinear form \( a(u, v) = (u, v)_{X_\nu(\Omega; \mathbb{R}^N)} + \langle u, v \rangle_{L^2(\Omega, \zeta)} \), with \( u, v \in X_\nu(\Omega; \mathbb{R}^N) \cap L^2(\Omega, \zeta) \), such that \( a(u, v) = \langle Au, v \rangle \). Thus \( u(x, t) = e^{-tA}u_0(x) \), \( 0 \leq t \leq T \), is the unique weak solution to (3.9).

The weak solution of problem (3.9) satisfies the energy estimate:

\[
\frac{1}{2} \|u(\cdot, t)\|_{L^2(\Omega)}^2 + \int_0^t \mathcal{E}(u, u) \, dt + \int_0^t \int_\Omega \zeta u^2 \, dx \, d\tau \leq \frac{1}{2} \|u_0\|_{L^2(\Omega)}^2 \tag{3.10}
\]

for all \( t \in [0, T] \). Besides that, \( \partial_t u \) belongs to the space \( L^2(0, T; (X_\nu(\Omega; \mathbb{R}^N) \cap L^2(\Omega, \zeta))^*) \), where \((X_\nu(\Omega; \mathbb{R}^N) \cap L^2(\Omega, \zeta))^* \) is the conjugate space to \( X_\nu(\Omega; \mathbb{R}^N) \cap L^2(\Omega, \zeta) \). As a consequence of this fact, from (2.6), we find that \( u \in C(0, T; L^2(\Omega)) \). Thus, the function \( u_T = u(\cdot, T) \) is well defined as an element of \( L^2(\Omega) \) and (3.10) holds for all \( t \in [0, T] \).

For each \( \zeta \in L^2(\Omega), \zeta \geq 0 \), define the mapping \( \mathcal{U} : \zeta \mapsto \mathcal{U}(\zeta) \) where \( \mathcal{U}(\zeta) \in L^2(0, T; X_\nu(\Omega; \mathbb{R}^N) \cap L^2(\Omega, \zeta)) \) is the unique weak solution of problem (3.9).

Define also \( \mathcal{U}_T(\zeta)(\cdot) = \mathcal{U}(\zeta)(\cdot, T) \). We now investigate the dependence of \( \mathcal{U} \) and \( \mathcal{U}_T \) on \( \zeta \).

**Lemma 3.4.** Let \( u_0 \in L^2(\Omega) \) and \( \{\zeta_k\} \) be a sequence of non-negative functions converging weakly in \( L^2(\Omega) \) to a function \( \zeta \). Then \( \mathcal{U}_T(\zeta_k) \rightharpoonup \mathcal{U}_T(\zeta) \) weakly in \( L^2(\Omega) \) as \( k \to \infty \).

**Proof.** For brevity, we denote \( u_k = \mathcal{U}(\zeta_k) \) and \( u = \mathcal{U}(\zeta) \). Let \( \psi : \mathbb{R}^N \times (0, T) \to \mathbb{R} \) be an arbitrary smooth function such that \( \psi = 0 \) in \( \mathbb{R}^N \setminus \Omega \times (0, T) \). As it follows from (3.10), for all \( k \in \mathbb{N} \),

\[
\frac{1}{2} \|u_k(\cdot, T)\|_{L^2(\Omega)}^2 + \int_0^T \mathcal{E}(u_k, u_k) \, dt + \int_0^T \int_\Omega \zeta_k u_k^2 \, dx \, dt \leq \frac{1}{2} \|u_0\|_{L^2(\Omega)}^2.
\]

On the other hand, using the weak formulation of \( u_k \) we get

\[
\left| \int_0^T \langle \partial_t u_k, \psi \rangle \, dt \right|^2 \leq - \int_0^T \mathcal{E}(u_k, \psi) \, dt - \int_0^T \int_\Omega \zeta_k u_k \psi \, dx \, dt \\
\leq \left( \int_0^T \mathcal{E}(u_k, u_k) \, dt \right)^{1/2} \left( \int_0^T \mathcal{E}(\psi, \psi) \, dt \right)^{1/2} + \left( \int_0^T \int_\Omega \zeta_k u_k^2 \, dx \, dt \right)^{1/2} \left( \int_0^T \int_\Omega \zeta_k \psi^2 \, dx \, dt \right)^{1/2} \\
\leq \frac{1}{2} \|u_0\|_{L^2(\Omega)}^2 \left( \int_0^T \mathcal{E}(\psi, \psi) \, dt + \int_0^T \int_\Omega \zeta_k \psi^2 \, dx \, dt \right)^{1/2}.
\]
The boundedness of \( \{\zeta_k\} \) implies that for a constant \( C_\psi > 0 \) only depending on \( \psi \) and \( u_0 \) such that

\[
\sup_{k \geq 1} \left| \int_0^T (\partial_t u_k, \psi) dt \right| \leq C_\psi.
\]

The uniform boundedness principle implies that \( \{\partial_t u_k\} \) is bounded in \( L^2(0, T; (\mathcal{X}_\nu(\Omega)\mathbb{R}^N)^*) \) and thus \( \{u_k\} \) is bounded in \( H_\nu(0, T) \). Therefore, taking into account the compactness result from Proposition 2.2, the sequence \( \{u_k\} \) has a subsequence still denoted by \( \{u_k\} \) such that, as \( k \to \infty \),

\[
\begin{align*}
  u_k(\cdot, T) &\to h \text{ weakly in } L^2(\Omega), \\
  u_k &\rightharpoonup w \text{ weakly in } L^2(0, T; \mathcal{X}_\nu(\Omega)\mathbb{R}^N), \\
  u_k &\to w \text{ strongly in } L^2(0, T; L^2(\Omega)).
\end{align*}
\]

It turns out that \( h = w(\cdot, T) \) and the strong convergence of \( \{u_k\} \) in \( L^2(0, T; L^2(\Omega)) \) implies that

\[
\int_0^T \int_\Omega \zeta_k u_k \psi dx dt \to \int_0^T \int_\Omega \zeta w \psi dx dt \quad \text{as } k \to \infty.
\]

For each \( k \geq 1 \), by definition of \( u_k \), we get

\[
\int_0^T \int_\Omega u_k \partial_t \psi dx dt + \int_0^T \int_\Omega \mathcal{E}(u_k, \psi) dt + \int_0^T \int_\Omega \zeta u_k \psi dx dt dt = -\int_\Omega (u_k(\cdot, T)\psi(\cdot, T) - u_0\psi(\cdot, 0)) dx.
\]

Finally, letting \( k \to \infty \), we get \( w(\cdot, 0) = u_0 \) and

\[
\int_0^T \int_\Omega u_\partial_t w \psi dx dt + \int_0^T \int_\Omega \mathcal{E}(w, \psi) dt + \int_0^T \int_\Omega \zeta w \psi dx dt = 0.
\]

Thus \( w \) is a weak solution to (3.9) and by uniqueness we have \( w = u \). The desired result follows.

\[\square\]

4. WEAK SOLVABILITY AND UNIQUENESS OF THE SOLUTION

Armed with the above auxiliaries results, let us turn our attention to the proof of the weak solvability of problem (1.1). In order to apply the Tychonoff fixed-point Theorem 2.4 we take \( X = L^2(\Omega) \), \( G = \{w \in L^2(\Omega) : \|w\|_{L^2(\Omega)} \leq \|u_0\|_{L^2(\Omega)}\} \) which is clearly closed, convex and bounded. The next result provides the existence of a weak solution to the problem (1.1).

**Theorem 4.1.** Let \( u_0 \in L^2(\Omega) \) and \( T > 0 \). Let the mapping \( \pi : G \to G \) be defined for \( w \in G \), by \( \pi(w) = \mathcal{U}_T(\varphi(v)) \), where \( v = V(u_0 - w) \) (defined as in (3.8)) is the unique weak solution to (3.1) with \( f = u_0 - w \). Then \( \pi \) has a fixed point \( u_T \) that is

\[
u_T = \pi(u_T) = \mathcal{U}_T(\varphi(v)) \quad \text{with } v = V(u_0 - u_T).
\]

Moreover, \( v = \int_0^T u dt \) and

\[
u = \mathcal{U}(\varphi(v)) \quad \text{is a weak solution of the problem (1.1)}.
\]

**Proof.** Let \( w \in G \) then from Lemma 3.3 we know that \( \varphi(v) \in L^2(\Omega) \) with \( v = V(u_0 - w) \). For the non-negative function \( \zeta = \varphi(v) \in L^2(\Omega) \), the function \( \mathcal{U}(\zeta) \) satisfies (3.11) which implies that \( \|\mathcal{U}_T(\zeta)\|_{L^2(\Omega)} \leq \|u_0\|_{L^2(\Omega)} \). In particular, \( \|\pi(w)\|_{L^2(\Omega)} \leq \|u_0\|_{L^2(\Omega)} \) for all \( w \in G \) and thus, \( \pi(G) \subseteq G \). It remains to prove the weak sequential continuity of \( \pi \). Let \( \{w_k\} \) be an arbitrary sequence in \( G \) that converges to \( w \in G \) weakly in \( L^2(\Omega) \). We need to prove that \( \pi(w_k) \to \pi(w) \) weakly in \( L^2(\Omega) \) as \( k \to \infty \). In virtue of Lemma 3.3 \( v_k = V(u_0 - w_k) \to v = V(u_0 - w) \) strongly in \( \mathcal{X}_\nu(\Omega)\mathbb{R}^N \) and \( \varphi(v_k) \to \varphi(v) \) weakly in \( L^2(\Omega) \) as \( k \to \infty \), where \( v_k = V(u_0 - w_k) \) and \( v = V(u_0 - w) \). In turn, Lemma 3.3 implies that \( \pi(w_k) = \mathcal{U}_T(\varphi(v_k)) \to \mathcal{U}_T(\varphi(v)) = \pi(w) \) weakly in...
L^2(\Omega) as k \to \infty. Thus, according to Theorem 2.1, \pi has a fixed point \( u_T = \pi(u_T) \). Next, knowing that \( u_T \) is a fixed point of the mapping \( \pi \), we show that \( u = \mathcal{U}(\varphi(v)) \), with \( v = V(u_0 - u_T) \), is a weak solution to the problem (1.1). Indeed, recall that \( u = \mathcal{U}(\varphi(v)) \) is the unique weak solution to the problem (3.9) with \( \zeta = \varphi(v) \), i.e., \( u(\cdot,0) = u_0 \) and for all \( \psi \in L^2(0,T;X_\nu(\Omega|\mathbb{R}^N)) \),

\[
\int_0^T \int_\Omega \partial_t u \psi \, dx \, dt + \int_0^T E(u, \psi) \, dt + \int_0^T \int_\Omega \varphi(v) u \psi \, dx \, dt = 0 \tag{4.1}
\]

in particular for \( \psi \in X_\nu(\Omega|\mathbb{R}^N) \) (time independent) we get

\[
E \left( \int_0^T u \, dt, \psi \right) + \int_\Omega \varphi(v) \psi \int_0^T u \, dt \, dx = \int_\Omega [u_0 - u_T] \psi \, dx.
\]

Thus, according to Lemma 3.1, \( v = \int_0^T u(x,t) \, dt \) is the unique weak solution to the elliptic problem

\[
\mathcal{L} v + \varphi(v) v = u_0 - u_T \quad \text{in } \Omega \quad \text{and} \quad v = 0 \quad \text{on } \mathbb{R}^N \setminus \Omega. \tag{4.2}
\]

We have shown that \( v = V(u_0 - u_T) = \int_0^T u \, dt \). Therefore, we obtain

\[
u = \mathcal{U} \left( \varphi \left( \int_0^T u \, dt \right) \right) \tag{4.3}
\]

which, according to the relation (4.1), implies that \( u \) is a weak solution to the problem (1.1). \( \square \)

The main result of the paper is the following theorem.

**Theorem 4.2.** Let \( u_0 \in L^2(\Omega) \), \( T > 0 \) and \( \varphi \) satisfies Assumption 7.1. The problem (1.1) has a weak solution \( u \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;X_\nu(\Omega|\mathbb{R}^N)) \) such that

\[
\varphi(v) \in L^2(\Omega), \quad \varphi(v) v \in L^2(\Omega), \quad \varphi(v) u^2 \in L^1(\Omega_T), \quad \text{and} \quad u \in C(0,T;L^2(\Omega)),
\]

where \( v = \int_0^T u \, dt \). Moreover, the following estimates hold true

\[
\frac{1}{2} \left\| u \right\|^2_{L^2(0,T;\mathcal{L}^2(\Omega))} + \left\| u \right\|^2_{L^2(0,T;X_\nu(\Omega|\mathbb{R}^N))} + \int_0^T \int_\Omega \varphi(v) u^2 \, dx \, dt \leq \frac{1}{2} \left\| u_0 \right\|^2_{L^2(\Omega)}
\]

\[
\left\| \partial_t u \right\|^2_{L^2(0,T;X_\nu(\Omega|\mathbb{R}^N) \cap \mathcal{L}^2(\Omega,\varphi(v)^*) \cap \mathcal{L}^2(\Omega))} \leq \frac{1}{2} \left\| u_0 \right\|^2_{L^2(\Omega)}.
\]

**Proof.** The existence of a weak solution to the problem (1.1) follows immediately from Theorem 4.1. Furthermore, mimicking the estimate (3.10) yields

\[
\frac{1}{2} \left\| u \right\|^2_{L^2(0,T;\mathcal{L}^2(\Omega))} + \left\| u \right\|^2_{L^2(0,T;X_\nu(\Omega|\mathbb{R}^N))} + \int_0^T \int_\Omega \varphi(v) u^2 \, dx \, dt \leq \frac{1}{2} \left\| u_0 \right\|^2_{L^2(\Omega)} \tag{4.4}
\]

Now, each for \( \psi \in L^2(0,T;X_\nu(\Omega|\mathbb{R}^N) \cap L^2(\Omega,\varphi(v)) \), by definition of \( u \), we have

\[
\left| \int_0^T (\partial_t u, \psi) \, dt \right|^2 = \left| \int_0^T \mathcal{E}(u, \psi) \, dt - \int_0^T \int_\Omega \varphi(v) u \psi \, dx \, dt \right|^2
\]

\[
\leq \left( \int_0^T \mathcal{E}(u, \psi) \, dt \right) \left( \int_0^T \mathcal{E}(\psi, \psi) \, dt \right) + \left( \int_0^T \int_\Omega \varphi(v) u^2 \, dx \, dt \right) \left( \int_0^T \int_\Omega \varphi(v) \psi^2 \, dx \, dt \right)
\]

\[
\leq \frac{1}{2} \left\| u_0 \right\|^2_{L^2(\Omega)} \left( \int_0^T \mathcal{E}(\psi, \psi) \, dt + \int_0^T \int_\Omega \varphi(v) \psi^2 \, dx \, dt \right).
\]

This implies that
Therefore, we have \( u \in L^2(0, T; \mathbb{X}_\nu(\Omega)) \cap L^2(\Omega) \) and \( \partial_t u \in L^2(0, T; (\mathbb{X}_\nu(\Omega)) \cap L^2(\Omega, \zeta)) \) with \( \zeta = \varphi \left( \int_0^T u(t) \, dt \right) \) which implies that \( \varphi(v) u \in L^1(\Omega_T) \). By Proposition 2.2 we get \( u \in C(0, T; L^2(\Omega)) \). On the other hand, we know that \( v = \int_0^T u(t) \, dt \) is the unique weak solution to the problem (1.2) and hence from Theorem 3.2 we have \( \varphi(v), \varphi(v) v \in L^2(\Omega) \). This ends the proof. \( \square \)

Next, we prove that problem (1.1) has a unique solution, provided that the initial condition \( u_0 \) is bounded. Before, we need to establish the following maximum principle result.

**Lemma 4.3.** Let \( u = u(x, t) \) be a weak solution of the problem (1.1), i.e., satisfies (2.10) with \( u_0 \in L^2(\Omega) \cap L^\infty(\Omega) \) then \( \|u\|_{L^\infty(0, T; L^\infty(\Omega))} \leq \|u_0\|_{L^\infty(\Omega)} \) on \( \Omega_T \), i.e., \( |u| \leq \|u_0\|_{L^\infty(\Omega)} \) on a.e. \( \Omega_T \).

**Proof.** Set \( \Lambda = \|u_0\|_{L^\infty(\Omega)} \) and consider the convex function \( F : \mathbb{R} \rightarrow [0, \infty) \) defined by

\[
F(\tau) = \begin{cases} 
(\tau + \Lambda)^2 & \text{if } \tau < -\Lambda \\
0 & \text{if } |\tau| \leq \Lambda \\
(\tau - \Lambda)^2 & \text{if } \tau > \Lambda. 
\end{cases}
\]

So that, \( F(\tau) = 0 \) if and only if \( |\tau| \leq \Lambda \), in particular \( F(u_0) = 0 \) a.e. on \( \Omega \). By convexity, \( F' \) is non-decreasing, i.e., \( (F'(\tau_1) - F'(\tau_2))(\tau_1 - \tau_2) \geq 0 \) for all \( \tau_1, \tau_2 \in \mathbb{R} \) in particular, since \( F'(0) = 0 \), we have \( F'(\tau_1) \geq 0 \) for all \( \tau_1 \in \mathbb{R} \). Furthermore, \( F'(u(\cdot, t)) \in \mathbb{X}_\nu(\Omega) \) because \( u(\cdot, t) \in \mathbb{X}_\nu(\Omega) \) and one can check that \( F' \) is Lipschitz since \( F'' \) is bounded. Therefore, testing the equation (2.10) against \( \zeta = F'(u) \) gives

\[
\frac{d}{dt} \int_\Omega F(u(x, t)) \, dx = -E(u, F'(u)) - \int_\Omega \varphi(v)(x) F'(u(x, t)) u(x, t) \, dx \leq 0.
\]

Since \( F(u_0) = 0 \) almost everywhere on \( \Omega \), integrating the inequality gives

\[
\int_\Omega F(u(x, t)) \, dx \leq 0 \quad \text{for all} \quad 0 \leq t \leq T.
\]

Thus, \( F(u(x, t)) = 0 \) a.e. on \( \Omega_T \), and hence \( |u(x, t)| \leq \Lambda \) a.e. on \( \Omega_T \). \( \square \)

**Theorem 4.4.** Assume that \( \varphi \) satisfies Assumption (1.1), \( u_0 \in L^2(\Omega) \cap L^\infty(\Omega) \) and that for \( \Lambda = \|u_0\|_{L^\infty(\Omega)} \) we have \( |\varphi'(\tau)| \leq \kappa \) for \( \tau \in [-\Lambda T, \Lambda T] \) for some constant \( \kappa > 0 \). Then the weak solution of problem (1.1) is unique provided that \( \kappa \Lambda T^2 < 1 \).

**Proof.** Suppose that problem (1.1) has two weak solutions \( u_1 \) and \( u_2 \), and put \( v_i(x) = \int_0^T u_i(x, t) \, dt, \) \( i = 1, 2 \). Then \( u = u_1 - u_2 \) is a weak solution to

\[
\begin{cases} 
\partial_t u + \mathcal{L} u + \varphi(v_1) u_1 - \varphi(v_2) u_2 = 0 & \text{in } \Omega_T, \\
u = 0 & \text{in } \Sigma, \\
u(\cdot, 0) = 0 & \text{in } \Omega.
\end{cases}
\]

The maximum principle in Lemma 4.3 implies that \( |u_1| \leq \Lambda \) a.e. in \( \Omega_T \) and hence \( |v_i| \leq \Lambda T, \) \( i = 1, 2, \) a.e. in \( \Omega \). Testing the above equation with \( u \) leads to the following equality:

\[
\frac{1}{2} \frac{d}{dt} \|u(\cdot, t)\|_{L^2(\Omega)}^2 + E(u, u) + \int_\Omega \varphi(v_1) u_1^2 \, dx + \int_\Omega (\varphi(v_1) - \varphi(v_2)) u_2 \, dx = 0
\]
which implies that
\[
\frac{1}{2} \|u(\cdot, t)\|_{L^2(\Omega)}^2 + \int_0^t \mathcal{E}(u, u) \, d\tau \leq \kappa \Lambda \int_0^t \int_{\Omega} |v(x)| \, |u(x, \tau)| \, dx \, d\tau
\]
for all \( t \in [0, T] \), where \( v = v_1 - v_2 = \int_0^T u(\cdot, \tau) \, d\tau \). Noticing that,
\[
\|v\|_{L^2(\Omega)}^2 = \int_{\Omega} \left( \int_0^T u(x, \tau) \, d\tau \right)^2 \, dx \leq T \int_0^T \|u(\cdot, \tau)\|_{L^2(\Omega)}^2 \, d\tau,
\]
we get
\[
\int_0^t \int_{\Omega} |v(x)| \, |u(x, \tau)| \, dx \, d\tau \leq T \left( \int_0^T \|u(\cdot, \tau)\|_{L^2(\Omega)}^2 \, d\tau \right)^{1/2} \left( \int_0^T \|u(\cdot, \tau)\|_{L^2(\Omega)}^2 \, d\tau \right)^{1/2},
\]
Therefore, we obtain the following inequality for all \( t \in [0, T] \)
\[
\|u(\cdot, t)\|_{L^2(\Omega)}^2 \leq 2\kappa \Lambda T \left( \int_0^T \|u(\cdot, \tau)\|_{L^2(\Omega)}^2 \, d\tau \right)^{1/2} \left( \int_0^T \|u(\cdot, \tau)\|_{L^2(\Omega)}^2 \, d\tau \right)^{1/2}.
\]
In short we rewrite the above inequality as follows
\[
\varrho'(t) \leq 2\kappa \Lambda T \varrho^{1/2}(T) \varrho^{1/2}(t) \quad \text{with} \quad \varrho(t) = \int_0^t \|u(\cdot, \tau)\|_{L^2(\Omega)}^2 \, d\tau.
\]
A routine integration yields that \( \varrho^{1/2}(t) \leq \kappa \Lambda T^2 \varrho^{1/2}(T) \) and, in particular, \( \varrho^{1/2}(T) \leq \kappa \Lambda T^2 \varrho^{1/2}(T) \). The latest inequality holds true only if \( \varrho(T) = 0 \) since \( \kappa \Lambda T^2 < 1 \), which implies that \( u = 0 \). \( \square \)

We now point out the following the closing remark which shows how the function spaces considered in this note extends our studies to a slightly different type of problems.

**Remark 4.5.** Analogous results to those obtained in this notes can be established replacing the Dirichlet complement condition \( u = 0 \) in \((\mathbb{R}^N \setminus \Omega) \times (0, T)\), the problem \( \square \) with the Neumann complement condition \( \partial u = 0 \) in \((\mathbb{R}^N \setminus \Omega) \times (0, T)\), where \( \partial u \) represents the nonlocal normal derivative of \( u \) across as defined in \( \square \). To this end, it is decisive to taking into account the setting of Theorem \( \square \), namely that \( \Omega \) is bounded and Lipschitz and that \( \nu \) satisfies the asymptotic condition \( \square \), in such a way that the compactness of the embedding \( V_{\nu}(\Omega; \mathbb{R}^N) \hookrightarrow L^2(\Omega) \) holds true. Wherefrom, one readily obtains (see \( \square \)) the Poincaré type inequality
\[
\|u\|_{L^2(\Omega)}^2 \leq C \mathcal{E}(u, u) \quad \text{for all} \quad u \in V_{\nu}(\Omega; \mathbb{R}^N),
\]
for some constant \( C > 0 \) and where \( V_{\nu}(\Omega; \mathbb{R}^N) = \{ V_{\nu}(\Omega; \mathbb{R}^N) : \int_{\Omega} u \, dx = 0 \} \). These observations, alongside of our procedure, allow to replace the space \( X_{\nu}(\Omega; \mathbb{R}^N) \) with the space \( V_{\nu}(\Omega; \mathbb{R}^N) \).

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