Verification of the Cabaret schemes for the shallow water equations based on the conservation of angular momentum

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Abstract. To solve the problems of large-scale dynamics of the atmosphere and the ocean, a system of single-layer two-dimensional shallow water equations on a sphere is often used [1, 2]. Such a system is the basis for solving complete baroclinic systems of prognostic equations and is used to assess the accuracy and efficiency of computational algorithms for solving direct and inverse problems. Important issues here are the choice of a suitable form of writing differential equations and effective algorithms for their numerical solution.

In [3], instead of using the equation for the conservation of momentum in a spherical coordinate system, the authors proposed a technique based on the use of the equation for the conservation of angular momentum in a Cartesian coordinate system. This technique allows us to get rid of the degeneracy of the coordinate system near the poles, while preserving only two equations of motion. The semi-discrete finite-volume scheme written in [3] should be supplemented with a method for determining flows on the edges of a grid. The authors used the Cabaret technique [4] for this purpose, which ensures time reversibility on flows where the characteristics of one family do not intersect.

This paper presents the original shallow-water equations in an integral form, a semi-discrete finite-volume scheme, and a brief description of the closure of this scheme using the Cabaret technique. The main attention is paid to the verification of the scheme on a series of traditional tests.

1. The scheme

Let \( a \) be the radius of a sphere centered at the origin of a Cartesian coordinate system. Let \( h(\hat{r},t) = \zeta(\hat{r},t) - b(\hat{r}) \) be the thickness of the fluid at the point \( \hat{r} \) of the sphere, \( \zeta \) be the surface level, \( b \) be the topography and \( g \) be the gravitational acceleration. The equations describing the dynamics of such a thin layer are derived from the fundamental conservation laws. On the sphere, such laws are the law of conservation of mass and angular momentum.
\[\int \frac{\partial h}{\partial t} dS + \left[ \int h(\vec{w} \cdot \vec{n}) dL \right] = 0 \]  
\((1.1)\)

\[\int \frac{\partial}{\partial t} \left[ h(\vec{r} \times \vec{w}) \right] dS + \int h(\vec{r} \times \vec{w})(\vec{w} \cdot \vec{n}) dL + \frac{g}{2} \int \zeta^2 (\vec{r} \times \vec{n}) dL = 0 \]  
\(= g \int b(\vec{r} \times \nabla \zeta) dS + 2 \int h \cdot \vec{r} \times (\vec{\Omega} \times \vec{w}) dS \]  
\((1.2)\)

Here, \(\vec{w}\) is the velocity vector, \(\vec{n}\) is the outer normal vector, \(dS\) is the area element of the sphere surface, \(dL\) is the length element of the sphere surface. The equation of motion written in this way is the basis for the so-called “well balanced” scheme.

Consider an unstructured quadrangular grid on the sphere, the cell boundaries of which form arcs of large circles (geodesic grid). In [3], for equations \((1.1) - (1.2)\), a finite-volume semi-discrete scheme was constructed, which has the following form:

\[\frac{\partial h_c}{\partial t} + \frac{1}{S_C} \sum_{m=1}^{4} h_m w_m^t \Delta L_m = 0 \]  
\((1.3)\)

\[\frac{\partial}{\partial t} \left[ h(\vec{r} \times \vec{w}) \right]_c + \frac{1}{S_C} \sum_{m=1}^{4} h_m w_m^n \left[ w_m^t (\vec{r}_m \times \vec{n}_m) \Delta t_m + w_m^t (\vec{r}_m \times \vec{n}_m) \Delta L_m \right] + \]  
\[+ \frac{g}{2S_C} \sum_{m=1}^{4} h_m^2 (\vec{r}_m \times \vec{n}_m) \Delta L_m = \frac{gb_c}{S_C} \sum_{m=1}^{4} \zeta_m (\vec{r}_m \times \vec{n}_m) \Delta L_m + \frac{\Phi_c}{S_C} \]  
\((1.4)\)

Here, subscript \(C\) refers to conservative variables defined at the centers of the cells, and \(m\) refers to the flow variables specified at the centers of the faces, \(w_m^n\) and \(w_m^t\) are the normal and tangential components of the flow velocity vector \(\vec{w}_m\), \(\Delta L_m\) and \(\Delta L_m\) are the length of the arc of the face and the length of the chord, based on the ends of the face. \(\Phi_c\) is an approximation of the Coriolis force. For example:

\[2 \int_h h \cdot \vec{r} \times (\vec{\Omega} \times \vec{w}) ds \approx -2 \cdot a \cdot \vec{r} \cdot \vec{w}_c \cdot \sin \phi_c \cdot S_c = \Phi_c \]  
\((1.5)\)

The system of equations \((1.3) - (1.4)\) describes the evolution of conservative variables defined in the cells. To close the system we need to determine the flows on the faces and the time discretization. For this, various approaches can be used (see, for example, [5]). In [3], the Cabaret method was used, according to which conservative variables are changed in accordance with the balance equations based on \((1.3) - (1.4)\):

\[\frac{U_c^{t+1} - U_c^t}{\Delta t} + \frac{1}{S_C} \sum_{m=1}^{4} F_m^{t+1/2} \Delta L_m = Q \]  
\((1.6)\)

where \(U_c = \left\{ h_c, \left[ h(\vec{r} \times \vec{w}) \right]_c \right\} \), \(F_m^{t+1/2} = \left( F_m^t + F_m^{t+1} \right) / 2 \). The form of the flows follows from \((1.4)\).

Balance equations in the form of \((1.6)\) provide the reversibility of the scheme over time, provided that the calculation of the flow variables is reversible.

To find the flow variables on the time layer \(t+1\), a characteristic approach is used, according to which local invariants are defined in the cells adjacent to the faces, which are extrapolated to the
center of the face on the time layer $t+1$, and then the flow values of the variables $(h_m)^{t+1}, (w_m^u)^{t+1}, (w_m^v)^{t+1}$ are restored based on the values of the invariants.

2. Numerical tests

Below will be presented the results of several tests that are important for the verification of the Cabaret scheme.

2.1. The stationary vortex

As noted above, the Cabaret is time reversible in an area where the characteristic lines do not intersect. This means that in such areas the scheme is non-dissipative. To confirm this property, the problem of a stationary vortex with a radius $r_0 \leq a$ on a fixed sphere was solved. The initial data in spherical coordinates are as follows:

$$
\begin{align*}
 h(\lambda, \phi) &= h_i - \frac{\Theta^2}{4 g \beta}, \quad u(\lambda, \phi) = \Theta \cdot \frac{\phi - \phi_0}{r_0}, \quad v(\lambda, \phi) = -\Theta \cdot \frac{\lambda - \lambda_0}{r_0}, \quad \Theta = \alpha \cdot \exp \left[ \beta \left( 1 - \frac{r^2}{r_0^2} \right) \right] 
\end{align*}
$$

(2.1)

Here $h$ is depth, $\lambda$ and $\phi$ are longitude and latitude, $u$ and $v$ are velocity components, $r = |\vec{r} - \vec{r}_c|$ is a distance to the vortex center, $\alpha > 0$ and $\beta > 0$ are some parameters. The calculation was carried out on the “cubosphere” grid with the cells number $6 \cdot 90^2 = 48600$ up to a time corresponding to 50 vortex revolutions for points $r = r_0$. The vortex radius corresponds to the size of 3 cells, the background depth is $h_0 = 1000$, the depth in the center of the vortex is $h_{r=0} = 758.853$.

Figure 1a. Part of the grid and colorized initial value of the depth $h$

Figure 1b. $\varepsilon_{h_{\text{min}}} (t)$ and $\varepsilon_{h_{\text{max}}} (t)$

In the fig.1 the part of the grid and the initial distribution of the depth are shown, as well as the relative deviation of the minimum and maximum depth of the liquid from the initial one during the calculation time. On the horizontal axis of the fig. 1b marked the number of revolutions of the vortex, on the vertical - the value $\varepsilon = (h - h_{i=0}) / h_{i=0}$. From the above results it follows that during the first few turns the grid solution changes slightly, and then remains unchanged.

2.2. Advection of Cosine Bell over the Pole

This test is included in the set of tests proposed by Williamson in [6]. A incomplete system of equations is used. The continuity equation is left active, and the velocity field is assumed to be given and frozen so as to transfer the initial surface perturbation around the sphere without any distortions.
\[ u = u_0 \left( \cos \phi \cos \alpha + \sin \phi \cos \lambda \sin \alpha \right), \quad v = -u_0 \sin \lambda \sin \alpha \]

The parameter \( 0 \leq \alpha \leq \pi / 2 \) is the angle between the axis rotation and the polar axis of the spherical coordinate system. The initial cosine bell pattern is given by

\[ h(\lambda, \phi) = \begin{cases} (h_0 / 2)(1 + \cos(\pi r / R)) & \text{if } r < R \\ 0 & \text{if } r \geq R \end{cases} \]

Following parameters were used:

\[ u_0 = 2 \pi a / (12 \text{ days}); \quad h_0 = 1000 \text{ m}; \quad R = a / 3; \quad a = 6.37122 \cdot 10^8 \text{ m}. \]

The grid is “cubosphere” with the cell number \( 6 \cdot 90^2 = 48600 \). With these parameters, in the analytical case, the perturbation of the surface will move along the direction determined by the parameter \( \alpha \), and after 12 days will return to its original position. Below are the results of the transfer in directions \( \alpha = 0 \) and \( \alpha = 0.05 \). Due to the symmetry of the cubosphere grid, the results of calculations along directions \( \alpha = \pi / 2, \alpha = \pi / 2 - 0.05 \) are equal to \( \alpha = 0, \alpha = 0.05 \).

**Figure 2.** Surface level for \( t=12 \text{ days} \)

**Figure 3.** Error for \( t=12 \text{ days} \)

Figure 2. shows the isolines of the surface level after one revolution of the perturbation around the sphere. The isolines are constructed in 100 m increments, the dotted lines correspond to the exact
solution, the solid lines correspond to the result of the calculation for a time \( t = 12 \) days. Figure 3. shows the isolines of the error \( \varepsilon = h_{t-0} - h_{t-12} \). The maximum error does not exceed 3%.

2.3. Development of instability in the zonal flow

This problem is taken from [7]. The problem is solved on a uniform rotating sphere. The background current in the analytical case is stationary. The speed is set by two components:

\[
u(\phi) = \begin{cases}
0 & \text{for } \phi \leq \phi_0 \\
\frac{u_{\text{max}}}{\exp[-4(\phi - \phi_0)^2]} \cdot \exp\left[\frac{1}{(\phi - \phi_0)(\phi - \phi_1)}\right] & \text{for } \phi_0 \leq \phi < \phi_1 \\
0 & \text{for } \phi \geq \phi_1
\end{cases}
\]

Here \( \phi_0 = \pi / 7, \phi_1 = \pi / 2 - \phi_0, u_{\text{max}} = 80 \, m / s \). The depth is given by

\[
h(\phi) = h_0 - \frac{1}{g} \int_0^\phi a u(\varphi) \left[2\Omega \sin(\varphi) + \frac{\tan(\varphi)}{a} u(\varphi)\right] d\varphi
\]

The constant \( h_0 \) is defined so the middle depth is 10 km. \( a = 6.37122 \cdot 10^6 \, m, \Omega = 7.292 \cdot 10^3 \, s^{-1}, g = 9.80616 \, m / s \). The stationary flow is perturbed by adding small value to depth:

\[
h'(\lambda, \phi) = \tilde{h} \cdot \cos(\phi) \cdot \exp[-(\lambda / \alpha)^2] \cdot \exp[-(\phi_2 - \phi)^2 / \beta^2]
\]

where \( \phi_2 = \pi / 4, \alpha = 1 / 3, \beta = 1 / 15, \tilde{h} = 120 \, m \). This perturbation leads to flow instability. Below are the results of calculations by the Cabaret scheme on the G360 grid for a time of 144 hours in comparison with the results of calculations using the spectral method FSM-SWM on the T341 grid taken from [7]. From the comparison it is clear that the results are close, but the Cabaret circuit for this task generates less high-frequency noise.

2.4. Rossby-Haurwitz Wave

The problem is taken from [8]. The solution is a structure extending eastward without change, making one revolution in about 12 days. Below are the depth contours calculated by the Cabaret scheme on the G360 grid (left) and, for comparison, on the right, the results of the calculation from [8].

Comparing the results with [8], it is possible to conclude that the distortions are small and in general the flow calculated by the Cabaret scheme corresponds to the reference calculation.
Figure 5. Isolines of density for 6 and 12 days. Left column – Cabaret scheme, right - [8].

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